Extracting long basic sequences from systems of dispersed vectors

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Abstract

Suppose $\{x_{\alpha}\}_{\alpha<\kappa} \subset X$ is a normalized suitably right separated (e.g. weakly null) sequence of a Banach space $X$. We consider the problem of finding a subsequence $\{x_{\alpha\beta}\}_{\beta<\kappa}$ which forms a monotone basic sequence, or something similar.

This problem differs considerably from the case with countable weakly null sequence. Also, monotonicity of a basic sequence is much weaker requirement than unconditionality. Countable weakly null sequences and the existence of unconditional subsequences have been studied in abundance in the literature. The conclusions and the techniques in our setting are quite different.

This talk is based on my recent paper available at ArXiv.
Some background and general remarks

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- The existence vs. non-existence of unconditional bases in different situations has been studied for some time, e.g. in connection to HI spaces and scarcity of operators on the space (References at the bottom).

It is a very natural question to look at a (countable or uncountable) sequence of vectors in a Banach space that is far from being constant, e.g. a weakly null sequence and trying to refine it further by selecting a subsequence to get a sequence of virtually ‘orthogonal’ vectors, e.g. an unconditional sequence.

In this refinement procedure one needs to control the ‘orthogonality’ of many subsets of the sequence. This can lead to heavily combinatorial considerations; Ramsey theory, cardinal invariants.
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$$\|x + z\| \geq C\|x\|, \quad x \in [x_\alpha: \alpha < \gamma], \ z \in [x_\alpha: \gamma \leq \alpha < \kappa].$$

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Therefore combinatorics is less involved in constructing these subsequences. And the conclusions are much weaker (probably in some cases known to some specialists).
Recall that the unit ball is compact in the weak topology iff the space is reflexive. However, many nice spaces are Lindelöf in the weak topology and thus long sequences \( \{ x_\alpha \}_{\alpha < \kappa} \) cluster. So, there is a trade-off in giving up reflexivity but considering uncountable sequences. The uncountable/WL case yields some phenomena that one would expect to see in the countable/reflexive case. This can be made to work at our advantage, so that extracting basic sequences in the uncountable setting is actually easier than in the countable setting.
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- The following condition is a kind of convex counterpart for the $\omega^*$-countable tightness of the dual space:

$$(C)$$ $X$ is said to have property $(C)$ (after Corson ’61), if each family of closed convex sets of $X$ with empty intersection has a countable subfamily with empty intersection.

$$(C)'$$ An equivalent reformulation of property $(C)$ (proved by Pol ’80): given a set $A \subset X^*$ and $f \in A^\omega^*$, there is a countable subset $A_0 \subset A$ such that $f \in \text{conv}^{\omega^*}(A_0)$. 

Recall that we have the following implications: $\text{WCG} = \Rightarrow \text{WLD} = \Rightarrow \text{weakly Lindelöf} = \Rightarrow \text{property } (C)$. 

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Some definitions

- Suppose that \( \{(x_\alpha, x_\alpha^*)\}_{\alpha<\lambda} \subset X \times X^* \) is a biorthogonal system, i.e. \( x_\alpha^*(x_\beta) = \delta_{\alpha,\beta} \).
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- Equivalently, \( \{x_\alpha\}_{\alpha<\lambda} \) is minimal, that is, \( x_\beta \notin [x_\alpha : \alpha \neq \beta] \) for all \( \beta \). The latter concept is sensible in topological vector spaces.
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- A biorthogonal system \( \{(x_{\alpha}, x_{\alpha}^*)\}_{\alpha} \) is bounded, if \( \sup_{\alpha} \|x_{\alpha}\| \cdot \|x_{\alpha}^*\| < \infty \).
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  $$\sup_\alpha \|x_\alpha\| \cdot \|x_\alpha^*\| < \infty.$$ 

- If $$[x_\alpha : \alpha < \lambda] = X$$ and $$[x_\alpha^* : \alpha < \lambda]^{\omega^*} = X^*$$, then $$\{(x_\alpha, x_\alpha^*)\}_{\alpha<\lambda}$$ is called a Markusevic basis or M-basis.
Weakly Lindelöf Determined spaces

- The WLD space are closely related to the topic; the spaces and conditions come very close to WLD here in several occasions.

\[ |\{ \alpha : f(x_\alpha) \neq 0 \}| \leq \aleph_0 \text{ for any } f \in X^*. \]
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The following equivalent formulation of WLD spaces (Kalenda 2000) is very convenient to work with: There is an M-basis $\{x_\alpha\}_\alpha$ of $X$ such that

$$|\{\alpha : f(x_\alpha) \neq 0\}| \leq \aleph_0 \text{ for any } f \in X^*.$$  \tag{1}
Some more tightness conditions for Banach spaces

We will apply the following structural/geometric assumption about Banach spaces:

Let \( \{Z_\alpha\}_{\alpha<\kappa} \) be a nested sequence of closed linear subspaces of \( X \) such that \( \bigcap_{\alpha<\kappa} Z_\alpha = \{0\} \). Then \( \bigcap_{\alpha<\kappa} B_{X+Z_\alpha} \) is bounded.

By applying an inverse limit space and the Banach open mapping principle the following condition implies (B).

For any uncountable, regular cardinal \( \kappa \) each nested sequence \( \{A_\alpha\}_{\alpha<\kappa} \) of closed affine subspaces of \( X \) has non-empty intersection.

This condition in turn follows from Corson's property (C).
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Dispersed sequences

We say that a sequence \( \{x_\alpha\}_{\alpha < \kappa} \) is dispersed if

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[x_\alpha : \gamma \leq \alpha < \kappa] \supset [x_\alpha : \nu \leq \alpha < \kappa], \quad \gamma < \nu < \kappa
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and strongly dispersed (SD) if dispersed and additionally

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- Clearly biorthogonal sequences (resp. weakly null, M-basic sequences) are examples of dispersed (resp. SD) sequences.
Theorem A

Let $X$ be a Banach space and $\{x_\alpha\}_{\alpha<\kappa}$ be a dispersed sequence of $X$. Then one can extract increasing subsequences $\{x_{\alpha_\sigma}\}_{\sigma<\kappa} \subset \kappa$ as follows.

(1) Suppose that $X$ satisfies $(C)$. Then there is a 2-bounded biorthogonal sequence $\{x_{\alpha_\sigma}\}_{\sigma<\kappa}$.

(2) If $X$ satisfies $(B)$ and $\{x_\alpha\}_{\alpha<\kappa}$ is SD, then there is a basic sequence $\{x_{\alpha_\sigma}\}_{\sigma<\kappa}$.

(3) If $Z \subset X^*$ is a norming subspace such that $\{x_\alpha\}_{\alpha<\kappa}$ is $\sigma(X,Z)$-null, then there exists a basic sequence $\{x_{\alpha_\sigma}\}_{\sigma<\kappa}$.

If, additionally, $X$ satisfies $(C)$ in (2), or $Z$ is 1-norming in (3), then the basic sequence can be chosen to be monotone.
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If, additionally, $X$ satisfies (C) in (2), or $Z$ is 1-norming in (3), then the basic sequence can be chosen to be monotone.
- After some considerations we obtain the following alternative.

**Corollary** Let $X$ be a Banach space with property $(C)$ and let $\{x_\alpha\}_{\alpha<\kappa} \subset X$. Then

- There is no subsequence $\{x_{\alpha_\sigma}\}_{\sigma<\kappa}$, which is dispersed,
- There is a 2-bounded biorthogonal sequence $\{x_{\alpha_\sigma}\}_{\sigma<\kappa}$.

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**Corollary**

Let $X$ be a Banach space with property (C) and let $\{x_\alpha\}_{\alpha<\kappa} \subset X$. Then exactly one of the following conditions hold:

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An angle lemma

Let $X$ be a Banach space, $Y \subset X$ a closed subspace with $\text{dens}(Y) < \kappa$, $\kappa$ an uncountable regular cardinal, and let $\{Z_\alpha\}_{\alpha < \kappa}$ be a nested sequence of closed subspaces of $X$ with trivial intersection. Suppose that $\bigcap_{\alpha < \kappa} B_X + Z_\alpha \subset rB_X$ for some $1 \leq r < \infty$. Then there exists $\beta < \kappa$ such that the angle between $Y$ and $Z_\beta$, $\text{dist}(S_Y, Z_\beta) \geq 1/r$. 

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An angle lemma

Lemma
Let $X$ be a Banach space, $Y \subset X$ a closed subspace with $\text{dens}(Y) < \kappa$, $\kappa$ an uncountable regular cardinal, and let $\{Z_\alpha\}_{\alpha<\kappa}$ be a nested sequence of closed subspaces of $X$ with trivial intersection. Suppose that

$$\bigcap_{\alpha<\kappa} \overline{B_X + Z_\alpha} \subset rB_X$$

for some $1 \leq r < \infty$. Then there exists $\beta < \kappa$ such that the angle between $Y$ and $Z_\beta$, $\text{dist}(S_Y, Z_\beta) \geq 1/r$. 
Proof of lemma

- First, observe that according to the assumption

\[
\bigcap_{\alpha<\kappa} (1-\epsilon)r^{-1}B_X + Z_\alpha = \bigcap_{\alpha<\kappa} (1-\epsilon)r^{-1}B_X + (1-\epsilon)r^{-1}Z_\alpha
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for \(0 \leq \epsilon \leq 1\).
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for \(0 \leq \epsilon \leq 1\).

- Assume to the contrary that \(\lim_{\alpha \to \kappa} \text{dist}(S_Y, Z_\alpha) = r^{-1}(1 - 2\epsilon)\) for some \(\epsilon > 0\). This reads \(\lim_{\alpha \to \kappa} \text{dist}(rS_Y, Z_\alpha) = 1 - 2\epsilon\).
Then there is a sequence $\{y_\alpha\}_{\alpha < \kappa} \subset rS_Y$ such that
\[ \text{dist}(y_\alpha, Z_\alpha) < 1 - \epsilon \]
for sufficiently large ordinals $\alpha < \kappa$. 

Bottom line: On cannot exit the set $rS_Y$ in sync with $\alpha \to \kappa$. 

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Then there is a sequence $\{y_\alpha\}_{\alpha<\kappa} \subset rS_Y$ such that $\text{dist}(y_\alpha, Z_\alpha) < 1 - \epsilon$ for sufficiently large ordinals $\alpha < \kappa$.

Since the Lindelöf number of $rS_Y$ is less than $\kappa$, we obtain that $\bigcap_{\beta<\kappa} \{y_\alpha : \beta < \alpha < \kappa\} \neq \emptyset$ and pick $y$ from this set.
Then there is a sequence \( \{y_\alpha\}_{\alpha < \kappa} \subset rS_Y \) such that 
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\[ \bigcap_{\beta < \kappa} \{y_\alpha : \beta < \alpha < \kappa\} \neq \emptyset \] 
and pick \( y \) from this set.

Observe that 
\[ y \in \bigcap_{\alpha < \kappa} (1 - \epsilon)B_X + Z_\alpha \subset (1 - \epsilon)rB_X. \]
Then there is a sequence \( \{y_\alpha\}_{\alpha<\kappa} \subset rS_Y \) such that 
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Observe that
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y \in \bigcap_{\alpha<\kappa} (1 - \epsilon)B_X + Z_\alpha \subset (1 - \epsilon)rB_X.
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Thus, we arrive at a contradiction, since \( y \in rS_Y \). \( \square \)
Then there is a sequence \( \{ y_{\alpha} \}_{\alpha < \kappa} \subset rS_Y \) such that \( \text{dist}(y_{\alpha}, Z_{\alpha}) < 1 - \epsilon \) for sufficiently large ordinals \( \alpha < \kappa \).

Since the Lindelöf number of \( rS_Y \) is less than \( \kappa \), we obtain that \( \bigcap_{\beta < \kappa} \{ y_{\alpha} : \beta < \alpha < \kappa \} \neq \emptyset \) and pick \( y \) from this set.

Observe that

\[
y \in \bigcap_{\alpha < \kappa} (1 - \epsilon)B_X + Z_{\alpha} \subset (1 - \epsilon)rB_X.
\]

Thus, we arrive at a contradiction, since \( y \in rS_Y \).

Bottom line: On cannot exit the set \( rS_Y \) in sync with \( \alpha \to \kappa \).
Sketch of the proof of Theorem A

- We will first consider the hardest case (2), where $X$ satisfies ($B$).
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- For each $\theta < \kappa$ let $\eta(\theta)$ be the infimum of numbers $C \geq 1$ such that there exists $\gamma < \kappa$ and a continuous linear projection $P : [x_\alpha : \alpha \in [0, \theta] \cup [\gamma, \kappa)] \to [x_\alpha : \alpha \in [0, \theta]]$

  given by $P(x + y) = x$ for $x \in [x_\alpha : \alpha \in [0, \theta]]$, $y \in [x_\alpha : \alpha \in [\gamma, \kappa)]$ with $\|P\| \leq C$ (and $\eta(\theta) = \infty$ if such $P$ does not exist).
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- Let $\epsilon > 0$. Suppose that $\theta_1 \leq \theta_2 < \kappa$ and
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- Then, putting $P_1 = P_2|_{[x_\alpha : \alpha \in [0, \theta_1] \cup [\gamma_2, \kappa])}$ defines a projection, which is admissible in the definition of $\eta(\theta_1)$ and again $\|P_1\| \leq \eta(\theta_2) + \epsilon$. 

Jarno Talponen (UEF)
April 21, 2013 16 / 26
Sketch of the proof of Theorem A

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We conclude that $\eta : [0, \kappa) \to \mathbb{R} \cup \{\infty\}$ is a non-decreasing function.
Next, we will show that $\eta(\theta) < \infty$ for each $\theta < \kappa$ under the hypothesis ($B$). Indeed, we scale the bounded set

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so that it does not intersect the unit sphere.
• Next, we will show that $\eta(\theta) < \infty$ for each $\theta < \kappa$ under the hypothesis $(\mathcal{B})$. Indeed, we scale the bounded set

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• It follows from Lemma that there is $\beta < \kappa$ such that the angle between $[x_\alpha : \alpha \leq \theta]$ and $[x_\alpha : \alpha \geq \beta]$ is strictly positive, which is equivalent to the statement that there is a continuous linear projection

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Next, we will show that $\eta(\theta) < \infty$ for each $\theta < \kappa$ under the hypothesis $(B)$. Indeed, we scale the bounded set

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Thus, the values of $\eta$ are finite.
By using the regularity of $\kappa$ and the fact that $\eta$ is non-decreasing we obtain that $\lim_{\theta \to \kappa} \eta(\theta)$ exists and is finite. Denote this limit by $1 \leq C < \infty$. 
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Let us define an increasing sequence $\Phi : [0, \kappa) \to [0, \kappa)$ by letting $\Phi(\theta)$ be the least $\phi$ such that there is a projection

$$P : [x_\alpha : \alpha \in [0, \theta] \cup [\phi, \kappa)] \to [x_\alpha : \alpha \in [0, \theta]]$$

with $\|P\| \leq C$. Indeed, this can be accomplished by the regularity of $\kappa$. 


The required basic sequence can be extracted by transfinite recursion as follows. Let $\alpha_0 = 0$ and

$$\alpha_\sigma = \Phi \left( \sup_{\gamma < \sigma} \alpha_\gamma \right) \lor \left( \sup_{\gamma < \sigma} \alpha_\gamma \right) + 1, \quad \sigma < \kappa.$$ 

The relevant basis projections are obtained by restriction from the projections provided by the definition of $\Phi$. It is clear that the basis constant is at most $C$. \qed
Inverse limits and tightness

- Given a SD sequence \( \{x_\alpha\}_{\alpha<\kappa} \) let us denote the ‘tail spaces’

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Z_\beta = [x_\alpha : \beta < \alpha < \kappa]
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- This space consists of sequences of quotient space elements \( \{\hat{z}_\beta\}_{\beta<\kappa} \) pairwise compatible in taking quotient mappings. The norm of an element is the sup = max of the quotient norms \( \|\hat{z}_\beta\|_{X/Z_\beta} \).
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- Moreover, \( \phi \) is an embedding iff the condition (\( B \)) holds.

- I intend to study the biduals of Banach space modeled by taking more general inverse limits over the lattice of all subspaces.
Theorem B

Let \( X \) be a topological vector space and let \( \{ x_\alpha \}_{\alpha < \kappa} \subset X \) be a sequence such that \( Y = \left[ x_\alpha : \alpha < \kappa \right] / \bigcap_{\beta < \kappa} \left[ x_\alpha : \beta < \alpha < \kappa \right] \) has density \( \kappa \) and suppose that the canonical inclusion \( \phi : Y \hookrightarrow \lim_{\leftarrow} (Y / \left[ x_\alpha : \alpha \geq \beta \right])_{\beta < \kappa} \) is a closed mapping. Then there exists a minimal sequence \( \{ x_{\alpha \beta} \}_{\beta < \kappa} \).
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Coseparable subspaces

Motivated by the Baire Category Theorem considerations, call a subspace $Y \subset X$ *coseparable* if $\text{dens}(X/Y) = \omega$. 

It follows easily from WLD of $X$: the coseparable subspaces are preserved in countable intersections ($\sigma$). Thus there are many good spaces with this property.

If $K$ is a non-metrizable separable compact space, then $\ell_2 \oplus C(K)$ fails this even for finite intersections.

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- By ($\sigma$) one can work with countably many ‘conditions’ simultaneously.
Theorem C
Suppose that $X$ is a nonseparable Banach space satisfying $(\sigma)$.

(i) Then $X$ has a monotone basic sequence of length $\omega_1$. Moreover, any basic sequence of $X$ having countable order type has an uncountable extension.

(ii) Given a separable subspace $A \subset X$ there exists a coseparable subspace $M \subset X$ such that $A$ is 1-complemented in $M$.

Proof. Let us check the latter claim in (i). This argument essentially covers both claims. Let $(x_n)_{n<\alpha}$, $\alpha<\omega_1$, be a countable basic sequence on $X$.

By using the separability of $\{x_n: n<\alpha\}$ we may let $(f_i)_{i<\omega}$ be a 1-norming sequence for $\{x_n: n<\alpha\}$.

According to $(\sigma)$ we have that $\bigcap_{i<\omega} \ker(f_i)$ is a coseparable subspace, in particular non-trivial. Hence we may pick $x_\alpha \in \bigcap_{i<\omega} \ker(f_i)$, $\|x_\alpha\|=1$.

Note that $\|x\| \leq \|x+tx_\alpha\|$ for any $x \in \{x_n: n<\alpha\}$ and $t \in \mathbb{R}$.

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Which ’large-density-related properties’ of a Banach space are inherited by the coseparable subspaces? For example, if $X$ is a non-WCG space then each coseparable subspace of $X$ is non-WCG (Valdivia 1989).
I would like to (ill-)pose the following problems:

- Which ’large-density-related properties’ of a Banach space are inherited by the coseparable subspaces? For example, if $X$ is a non-WCG space then each coseparable subspace of $X$ is non-WCG (Valdivia 1989).

- What can be said about properties of Banach spaces holding modulo separable subspaces?
Some future work

- It would be interesting to see how the coseparable intersection business can be applied further.
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Project

Let $X$ be a Banach space coseparable in its bidual. Suppose that $\{x_\alpha\}_{\alpha<\omega_1} \subset X$ is a weakly null sequence. Then there exist $\omega_1$-many countable successive blocks of ordinals,

$$\{\beta_\theta^{(\gamma)}\}_{\theta<\eta(\gamma)} \subset \omega_1, \quad 0 \leq \gamma < \omega_1$$

and a bimonotone basic sequence $\{z_\gamma\}_{\gamma<\omega_1} \subset X$ such that

$$z_\gamma = \sum_{\theta<\eta(\gamma)} a_\theta^{(\gamma)} x_{\beta_\theta^{(\gamma)}}, \quad 0 \leq \gamma < \omega_1$$

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The latter assumption above can be seen as a strengthening of Asplund space condition. An example of such a space is $\ell^2_\kappa(X)$ with $X$ coseparable in its bidual. In contrast, there is a reflexive space of density $\omega_1$ with a basis and without unconditional basic sequences (Argyros, Lopez-Abad, Todorcevic '03).
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