On coverage and oracle radial rate of DDM-credible sets under excessive bias restriction

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For a general statistical model and an unknown parameter from a metric space, we introduce the notion of data dependent measure (DDM) on the parameter set. Typical examples of DDM are the posterior distributions with respect to priors on the parameter. Like for posteriors, the quality of a DDM is characterized by the contraction rate, which we allow to be local here; a local contraction rate depends on the value of the parameter. When applied appropriately, a local approach to contraction rates is more refined and powerful than global, it actually delivers many global results (minimax contraction rates for different scales) as consequence. We consider the problem of constructing confidence sets as DDM-credible sets. We are concerned with the optimality of such sets, which is basically a trade-off between the “size” of the set (the local radial rate) and its coverage probability. It is known that in general it is impossible to construct optimal (fully) adaptive confidence set in the minimax sense, only some limited amount of adaptivity can be achieved. Recently, some restrictions on the parameter set are proposed in the literature under which this is possible: self-similarity, polished tail condition. In the canonical normal sequence model, we construct a DDM, define a (default) DDM-credible set and propose the so called excessive bias restriction under which we establish the confidence optimality of our credible set with a strong local radial rate related to the so called projection oracle. This immediately implies the adaptive minimax optimality simultaneously over all scales that are covered by the considered local radial rate.

1 Introduction

Suppose we observe a random element $X^{(ε)} \sim \mathbb{P}^{(ε)}_0 \in \mathcal{P}^{(ε)}$, $X^{(ε)} \in \mathcal{X}^{(ε)}$ for some measurable space $(\mathcal{X}^{(ε)}, \mathcal{A}^{(ε)})$, where $\mathcal{A}^{(ε)}$ is a $\sigma$-algebra of $\mathcal{X}^{(ε)}$. In fact, we consider a sequence of observation models parametrized by $ε > 0$. Parameter $ε$ is assumed to be known, it reflects in some sense the influx of information in the data $X^{(ε)}$ as $ε \to 0$. For instance, $ε$ can

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be the variance of an additive noise, or $\varepsilon = n^{-1/2}$, where $n$ is the sample size. To avoid overloaded notations, we will often drop the dependence on $\varepsilon$; for example, $X = X^\varepsilon$ etc.

Consider an operator $A : \mathcal{P} \to \Theta$ and let $\theta_0 = A(\mathbb{P}_0)$ so that $\theta_0 \in \Theta \subseteq \mathcal{L}$ is an unknown parameter of interest belonging to some subset $\Theta$ of a linear space $\mathcal{L}$ equipped with a (semi-)metric $d(\cdot,\cdot) : \mathcal{L} \times \mathcal{L} \to [0, +\infty)$. A particular case is when $A$ is an identity operator $A(\mathbb{P}) = \mathbb{P}$, $\mathbb{P} \in \mathcal{P}$, i.e., $\Theta = \mathcal{P}$. Another important particular case is $\mathcal{P} = \{ \mathbb{P}_\theta, \theta \in \Theta \}$ and $A(\mathbb{P}_\theta) = \theta$, in this case $\mathbb{P}_0 = \mathbb{P}_{\theta_0}$. From now on, when we deal with probabilities of events in terms of the data $X \sim \mathbb{P}_\theta$, we write $\mathbb{P}_\theta$. Often we will denote by $\theta_0$ the so called “true” value of the parameter $\theta$ to distinguish it from the variable $\theta \in \Theta$.

The aim is to construct an optimal (to be defined later) confidence set for the parameter $\theta_0 \in \Theta$, $\theta_0 = A(\mathbb{P}_0)$, on the basis of observation $X \sim \mathbb{P}_0 \in \mathcal{P}$, with a prescribed coverage probability. The convention throughout this paper is that we measure the size of a set by the smallest possible radius of a ball containing that set. It is thus sufficient to consider only confidence balls as confidence sets. Let $l_0 \in \mathcal{L}$ and radius $r \geq 0$. Denote by $\mathcal{B}_\mathcal{L}$ the corresponding Borel $\sigma$-algebra on $\mathcal{L}$ and by $\mathcal{B}_{\mathbb{R}}$ the usual Borel $\sigma$-algebra on $\mathbb{R}$. A general confidence ball for the parameter $\theta$ is of the form $B(\hat{\theta}, \tilde{r}) = \{ \theta \in \mathcal{L} : d(\theta, \hat{\theta}) \leq \tilde{r} \}$, with some data dependent center (DDM-center) $\hat{\theta} = \hat{\theta}(X) = \hat{\theta}(X, \varepsilon)$, $\hat{\theta} : \mathcal{X} \to \Theta$, and some data dependent radius (DDM-radius) $\tilde{r}(X) = \tilde{r}(X, \varepsilon)$, $\tilde{r} : \mathcal{X} \to \mathbb{R}_+ = \{ a \in \mathbb{R} : a \geq 0 \}$. Quantities $\hat{\theta}$ and $\tilde{r}$ are such functions of the data $X$ that $d(\hat{\theta}, \theta)$ is $(\mathcal{B}_\mathcal{L} \times \mathcal{B}_{\mathbb{R}}, \mathcal{B}_{\mathbb{R}})$-measurable and $\tilde{r}$ is $(\mathcal{A}, \mathcal{B}_{\mathbb{R}})$-measurable.

Suppose we are given a random data dependent measure (DDM) $\mathbb{P}(\cdot|X)$ on $\Theta$. In order to settle the measurability issue for the rest of the paper, by DDM we will always mean a measurable probability measure in the sense that for all $x \in \mathcal{X}$ the quantity $\mathbb{P}(\cdot|X = x)$ is a probability measure on $(\mathcal{B}_\mathcal{L}, \Theta)$ (can be relaxed to $\mathbb{P}_{\theta_0}$-almost all $x \in \mathcal{X}$, for all $\theta_0 \in \Theta$) and $\mathbb{P}(B|X)$ is $\mathcal{A}$-measurable for each $B \in \mathcal{B}_\mathcal{L}$. Typically, such a DDM is obtained by using a Bayesian approach, as the resulting posterior (or empirical Bayes posterior) distribution with respect to some prior on $\Theta$. We slightly abuse the notation $\mathbb{P}(\cdot|X)$ because in general a DDM does not have to be a conditional distribution. Notice that empirical Bayes posteriors are, strictly speaking, not conditional distributions either. Given a DDM $\mathbb{P}(\cdot|X)$ on $\Theta$, we can take a DDM-credible set $C_\alpha(X)$ of level $\alpha \in [0, 1]$, i.e., $\mathbb{P}(\theta \in C_\alpha(X)|X) \geq \alpha$, as confidence set. The DDM approach has a conceptual advantage (also exploited in the Bayesian literature) that, no matter how complex the model and the setting of the problem is, there are always candidates for a confidence set, namely, DDM-credible sets. In general, a “good” DDM can be used for all kind of inference, e.g., also for estimation and testing problems. In this paper we focus on the following, for now loosely formulated, question:

When does DDM-credibility lead to confidence?

We would like to construct a confidence ball $B(\hat{\theta}, C \tilde{r})$ such that for any $\alpha_1, \alpha_2 \in (0, 1]$ and some functional $\mathcal{R}_\varepsilon(\mathcal{P})$, $\mathcal{R}_\varepsilon : \mathcal{P} \to \mathbb{R}_+$, there exists $C, c > 0$ such that

$$
\sup_{\mathbb{P} \in \mathbb{P}_0} \mathbb{P}(\theta_\varepsilon \notin B(\hat{\theta}, C \tilde{r})) \leq \alpha_1, \quad \sup_{\mathbb{P} \in \mathbb{P}_0'} \mathbb{P}(\tilde{r} \geq c \mathcal{R}_\varepsilon(\mathcal{P})) \leq \alpha_2,
$$

2
where \( \theta_0 = A(\mathbb{P}) \). \( P_0, P_0' \subseteq \mathcal{P} \). The functional \( \mathcal{R}_\varepsilon(\mathbb{P}) \), \( \mathcal{R}_\varepsilon : \mathcal{P} \to \mathbb{R}_+ \), has a meaning of the benchmark for the effective radius of the confidence ball \( B(\theta, \hat{r}) \) from the perspective of the “true” \( \mathbb{P} \)-measure. From now on, we will call the functional \( \mathcal{R}_\varepsilon(\mathbb{P}) \) the radial rate. Clearly, there are many possible radial rates, but it is desirable to find such a radial rate \( \mathcal{R}_\varepsilon(\mathbb{P}) \) that takes as small values as possible (whereas the second relation in the above display still holds true), with \( P_0, P_0' \) as big as possible, ideally \( P_0 = P_0' = \mathcal{P} \).

For each \( \theta \in \Theta \), introduce the inverse \( A^{-1}(\theta) = \{ \mathbb{P} \in \mathcal{P} : A(\mathbb{P}) = \theta \} \) and assume from now on that the radial rate \( \mathcal{R}_\varepsilon(\mathbb{P}) \) takes the same value for all \( \mathbb{P} \in A^{-1}(\theta) \). Without loss of generality, we will then use the notation \( \mathcal{R}_\varepsilon(\theta) \) instead of \( \mathcal{R}_\varepsilon(\mathbb{P}) \). In this paper we focus on the case \( \mathcal{P} = \{ \mathbb{P}_\theta, \theta \in \Theta \} \), \( P_0 = \{ \mathbb{P}_\theta, \theta \in \Theta_0 \} \); \( P_0' = \{ \mathbb{P}_\theta, \theta \in \Theta_0' \} \) with \( A(\mathbb{P}_\theta) = \theta \) and \( \Theta_0, \Theta_0' \subseteq \Theta \), then we can rewrite the last display in a more convenient form:

\[
\sup_{\theta \in \Theta_0} P_\theta(\theta \notin B(\hat{\theta}, C\hat{r})) \leq \alpha_1, \quad \sup_{\theta \in \Theta_0'} P_\theta(\hat{r} \geq c\mathcal{R}_\varepsilon(\theta)) \leq \alpha_2. \quad (1)
\]

In new terms, we want these relations to hold for the “fastest” possible radial rate \( \mathcal{R}_\varepsilon(\theta) \) and the “biggest” \( \Theta_0 \) and \( \Theta_0' \); ideally for \( \Theta_0 = \Theta_0' = \Theta \). The first relation in (1) is called coverage requirement and the second size requirement. Asymptotic formulation is also possible: \( \limsup_{\varepsilon \to 0} \) should be taken, constants \( \alpha_1, \alpha_2, C, c \) (possibly even sets \( \Theta_0 \) and \( \Theta_0' \)) can be allowed to depend on \( \varepsilon \). Thus the following optimality aspects are involved in the framework (1): the coverage, the radial rate and the uniformity subsets \( \Theta_0, \Theta_0' \). The optimality is basically a trade-off between these complementary aspects pushed to the utmost limits. At the utmost state, improving upon one aspect would lead to a deterioration in other aspects. Looking ahead, our approach is first to fix \( \Theta_0' = \Theta \) and some strong local radial rate \( \mathcal{R}_\varepsilon(\theta) \) and then establish (1) with \( \Theta_0 \subseteq \Theta \) as close to \( \Theta \) as possible. The meaning of ‘strong’ local radial rate is discussed later, for now: roughly, the one implying optimality in the minimax sense over some scales. Ideally \( \Theta_0 = \Theta \), but this turns out to be impossible for strong local radial rates \( \mathcal{R}_\varepsilon(\theta) \). In some papers, a confidence set satisfying the first relation in (1) is termed honest on \( \Theta_0 \). We will avoid this unfortunate terminology and say instead that \( B(\hat{\theta}, C\hat{r}) \) has coverage \( 1 - \alpha_1 \) uniformly over \( \Theta_0 \).

One approach to optimality is via minimax estimation framework. In the nonadaptive formulation, it is assumed that \( \theta \in \Theta_\beta \subseteq \Theta \) for some (known) “smoothness” parameter \( \beta \in \mathcal{B} \). The key notions here are the so called minimax rate \( R_\varepsilon(\Theta_\beta) \) (associated with some loss function) and the minimax estimator \( \hat{\theta} \) in the problem of estimating \( \theta \in \Theta_\beta \). In many concrete situations a minimax estimator \( \hat{\theta} \) and the minimax rate \( R_\varepsilon(\Theta_\beta) \) are derived, \( R_\varepsilon(\Theta_\beta) \) is then known. According to the minimax framework, one takes the radial rate \( \mathcal{R}_\varepsilon(\theta) = R_\varepsilon(\Theta_\beta) \) which is a global quantity as it is constant for all \( \theta \in \Theta_\beta \). By using lower bounds from the minimax estimation theory, it can be shown that the minimax rate \( R_\varepsilon(\Theta_\beta) \) is the best global radial rate (i.e., among all radial rates that are constant on \( \Theta_\beta \)); see Robins and van der Vaart (2006) and Appendix. But then the ball \( B(\hat{\theta}, CR_\varepsilon(\Theta_\beta)) \) satisfies (1) with \( \mathcal{R}_\varepsilon(\theta) = R_\varepsilon(\Theta_\beta) \) and \( \Theta_0 = \Theta_0' = \Theta_\beta \), for appropriate choices of involved constants. Thus the ball \( B(\hat{\theta}, CR_\varepsilon(\Theta_\beta)) \), with a deterministic radius, is optimal in the minimax sense. Knowledge \( \theta \in \Theta_\beta \) and the fact that radial rates are restricted to be global
lead to such a simplistic optimal solution. But this solution is of course not satisfactory (one would rather use a ball with a DD-radius), because even if we a priori know that \( \theta \in \Theta \), it is still possible that \( \theta \in \Theta_{\beta_1} \subset \Theta_{\beta} \), with \( \beta_1 > \beta \).

An adaptation problem arises when, for a given family of models \{\Theta_\beta, \beta \in \mathcal{B}\} (called functional scale), we only know that \( \theta \in \Theta_\beta \) for some unknown parameter \( \beta \in \mathcal{B} \). In fact, we assume \( \theta \in \bigcup_{\beta \in \mathcal{B}} \Theta_\beta \subseteq \Theta \) and the problem becomes in general more difficult. For \( \Theta_{0,\beta}, \Theta'_{\beta} \subseteq \Theta_\beta \), one wants to construct such a confidence ball \( B(\hat{\theta}, C\hat{r}) \) that

\[
\sup_{\theta \in \Theta_{0,\beta}} \mathbb{P}_\theta(\theta \not\in B(\hat{\theta}, C\hat{r})) \leq \alpha_1, \quad \sup_{\theta \in \Theta'_{\beta}} \mathbb{P}_\theta(\hat{r} \geq cR_e(\Theta_\beta)) \leq \alpha_2, \quad \text{for all } \beta \in \mathcal{B}, \quad (2)
\]

possibly in asymptotic setting: \( \limsup_{n \to 0} \) in front of both sup in (2). Ideally, \( \mathcal{B} \) is “big” and \( \Theta_{0,\beta} = \Theta'_{\beta} = \Theta_\beta \). However, in general it is impossible to construct optimal (fully) adaptive confidence set in the minimax sense, some limited amount of adaptivity can be achieved either by imposing some more structure on the set \( \mathcal{B} \) or by taking a smaller set \( \Theta_{0,\beta} \subset \Theta_\beta \), the remainder after certain “troublemakers” (we call them also “deceptive” parameters) are removed from \( \Theta_\beta \). Examples are \( \Theta_{0,\beta} = \Theta_\beta \cap \Theta_{ss} \) (related to Sobolev/Besov scales) considered by Picard and Tribouley (2000), Bull (2012), Bull and Nickl (2013), Szabó, van der Vaart and van Zanten (2014a, 2014b), Nickl and Szabo (2014), and \( \Theta_{0,\beta} = \Theta_\beta \cap \Theta_{pd} \) for a more general class of polished tail parameters \( \Theta_{pd} \) introduced by Szabó, van der Vaart and van Zanten (2014a).

For the classical many normal means model, there are negative results by Li (1989), Baraud (2004), Cai and Low (2006), this issue is also discussed by Robins and van der Vaart (2006). More literature on adaptive minimax confidence sets: Low (1997), Beran and Dümbgen (1998), Picard and Tribouley (2000), Juditsky and Lambert-Lacroix (2003), Genovese and Wasserman (2008), Giné and Nickl (2010), Hoffmann and Nickl (2011), Bull (2012), Bull and Nickl (2013), Szabó, van der Vaart and van Zanten (2014a, 2014b), Nickl and Szabo (2014).

In all the above mentioned papers global minimax radial rates \( R_e(\Theta_\beta) \) (as in (2)) were studied. In this paper we are going to allow local radial rates as in the framework (1). This approach is actually more powerful and flexible when applied appropriately. Namely, suppose that a local radial rate \( R_e(\theta) \) is such that \( R_e(\theta) \leq cR_e(\Theta_\beta) \) for all \( \theta \in \Theta_\beta \), all \( \beta \in \mathcal{B} \) and some uniform \( c > 0 \). In this case we say \( R_e(\theta) \) covers the scale \( \{\Theta_\beta, \beta \in \mathcal{B}\} \). If in addition \( \Theta_{0,\beta} \subseteq \Theta_0 \) and \( \Theta'_{\beta} \subseteq \Theta' \) for all \( \beta \in \mathcal{B} \), then the results of type (1) imply the results of type (2); in fact, this holds simultaneously for all scales \( \{\Theta_\beta, \beta \in \mathcal{B}\} \) that are covered by \( R_e(\theta) \). Clearly, the smaller a local radial rate, the better (then it covers more scales). But if it is too small, the size requirement in (1) may hold uniformly only over some “thin” set \( \Theta_0' \subset \Theta \) instead of the whole \( \Theta \). In this paper, we always require \( \Theta_0' = \Theta \) in (1). If one local radial rate is uniformly smaller than the other, preserving \( \Theta_0' = \Theta \) in (1) at the same time, we say it is stronger. Certainly, we wish to obtain results with strong local radial rates that cover many typical functional scales.

In Section 2 we consider a general setting and present two types of conditions on a DDM \( \mathbb{P}(\cdot|X) \) in terms of a given local radial rate \( R_e(\theta_0) \): the upper and lower bounds on the (local) contraction rate with which the DDM \( \mathbb{P}(\cdot|X) \) concentrates around the true \( \theta_0 \).
and a DD-center $\hat{\theta}$ from the $\mathbb{P}_{\theta_0}$-perspective. Roughly speaking, the upper bound means that the DDM $\mathbb{P}(\cdot|X)$ contracts at $\theta_0$ with the (local) rate at least $R_\varepsilon(\theta_0)$ from the $\mathbb{P}_{\theta_0}$-perspective (then one can construct a DD-center $\tilde{\theta}$ which is an estimator of $\theta_0$ with the rate $R_\varepsilon(\theta_0)$). The lower bound means that the DDM concentrates around the DD-center $\tilde{\theta}$ with the rate at most $R_\varepsilon(\theta_0)$. We show that the upper bound conditions allow to control the size of the $\mathbb{P}(\cdot|X)$-credible ball, whereas the lower bound is in some sense the minimal condition for providing its sufficient $\mathbb{P}_{\theta_0}$-coverage, thus establishing the relations (1) for appropriate choices of the involved quantities. For example, the Bernstein-von Mises property ensures (being even stronger than needed) the asymptotic versions of such conditions, thus leading to the well know fact that under BvM-property credible sets are asymptotically confidence sets. This is shortly discussed in Section 3.

In Section 4 we consider canonical Gaussian sequence model and implement the general approach of Section 2, with a certain strong local radial rate $R_\varepsilon(\theta_0)$ related to the projection oracle rate in the estimation problem (sometimes we call $R_\varepsilon(\theta_0)$ oracle rate). We construct a DDM $\mathbb{P}(\cdot|X)$, a DD-center $\tilde{\theta}$ and a $\mathbb{P}(\cdot|X)$-credible ball around $\tilde{\theta}$ (or alternatively, a default credible ball). For the proposed DDM, we prove the upper bound type conditions with the local radial rate $R_\varepsilon(\theta_0)$ uniformly over the whole set $\Theta = \ell_2$. These are of interest on its own and roughly mean the following: uniformly in $\theta_0 \in \ell_2$, the DDM $\mathbb{P}(\cdot|X)$ contracts, from the $\mathbb{P}_{\theta_0}$-perspective, at $\theta_0$ with the local rate $R_\varepsilon(\theta_0)$; the DD-center $\tilde{\theta}$ constructed by using the DDM $\mathbb{P}(\cdot|X)$ converges to $\theta_0$ with the rate $R_\varepsilon(\theta_0)$. In turn, these established properties imply the size relation in (1) uniformly over $\Theta = \ell_2$. The considered local radial rate $R_\varepsilon(\theta_0)$ is strong since (besides preserving $\Theta = \ell_2$) it covers a number of typical smoothness scales such as Sobolev and analytic ellipsoids, certain scales of Besov classes and $\ell_p$-bodies, Sobolev hyper-rectanges, tail classes etc.

On the other hand, the lower bound condition is fulfilled uniformly only over some $\Theta_0 \subset \ell_2$ which forms an actual restriction. This is in accordance with the above mentioned fact that it is impossible to construct optimal (fully) adaptive confidence set in the minimax sense. Actually, this is a genuine problem, not connected with an optimality framework used: global minimax (over certain scale) (2); or (1), i.e., in terms of local radial rate that covers the corresponding scale. Namely, the same problem must emerge for the local framework (1) as well (at least for strong local rates that cover typical scales), otherwise we would have constructed an adaptive confidence set in the minimax sense. We propose a set $\Theta_0 = \Theta_{eb}$ of (non-deceptive) parameters satisfying the so called excessive bias restriction and derive the lower bound uniformly over this set. Combining the obtained upper and lower bounds, we establish the optimality (1) of our credible ball with $\Theta_0 = \Theta_{eb}$, $\Theta_0 = \ell_2$ and the local radial rate $R_\varepsilon(\theta_0)$. The class $\Theta_{eb}$ is more general than the earlier mentioned self-similar and polished tail parameters, namely, $\Theta_{ss} \subseteq \Theta_{pt} \subseteq \Theta_{eb}$. Moreover, the optimality (1) with the local rate $R_\varepsilon(\theta_0)$ implies the optimality in the sense of adaptive minimaxity over all scales (some are mentioned above) simultaneously that are covered by the considered local rate. In this paper, we primarily interested in non-asymptotic assertions, asymptotic versions can be readily obtained. Section 5 contains the proofs. Some background information about minimax adaptive confidence balls is provided in Appendix.
2 General construction of confidence ball by using a DDM

2.1 Preliminaries

In this subsection we present the general construction of a confidence ball by using a DD-center \( \hat{\theta} = \hat{\theta}(X) \) and a DDM \( P(\theta|X) \). For a \( \kappa \in (0, 1) \), define first the DD-radius

\[
\hat{r}_\kappa = \hat{r}(\kappa, X, \hat{\theta}) = \inf \{r : P(d(\theta, \hat{\theta}) \leq r|X) \geq 1 - \kappa\}
\]

and then, for an \( M > 0 \), construct the confidence ball

\[
B(\hat{\theta}, M\hat{r}_\kappa) = \{\theta \in \Theta : d(\theta, \hat{\theta}) \leq M\hat{r}_\kappa\}.
\]

For \( M = 1 \), this is the smallest \( P(\theta|X) \)-credible ball around \( \hat{\theta} \) of level \( 1 - \kappa \). The multiplicative factor \( M \) is intended to trade-off the size of the ball against its coverage probability.

Typically, one gets a DDM on \( \theta \) by applying a Bayesian approach: put a prior \( \pi \) on \( \theta \) and regard \( P_\theta \) as conditional distribution of \( X \) given \( \theta \), i.e., \( X|\theta \sim P_\theta, \theta \sim \pi \). This leads to the posterior distribution \( \Pi(\theta|X) \). A DD-center \( \hat{\theta} = \hat{\theta}(X) \) can in turn be constructed by using \( \Pi(\theta|X) \).

Remark 1. In an adaptive inference context, one typically has a family of priors \( \{\pi_\beta, \beta \in B\} \), where parameter \( \beta \) models some additional structure on \( \theta \) (sometimes \( \beta \) has a meaning of “smoothness”). There are two basic approaches to derive a resulting adaptive posterior \( \Pi(\theta|X) \): pure Bayes or empirical Bayes. In the first case, we construct a hierarchical prior on \((\theta, \beta)\): regard \( \pi_\beta \) as a conditional prior on \( \theta \) given \( \beta \), and next we put a prior, say \( \lambda \), on \( \beta \in B \). This leads to the posterior \( \Pi(\theta|X) \) (also \( \lambda(\beta|X) \) that may also be useful in the inference). In the empirical Bayes approach, each prior \( \pi_\beta \) leads to the posterior \( \Pi_\beta(\theta|X) \). We then compute the marginal distribution \( \Pi_\beta \) of \( X \) and construct an estimator \( \hat{\beta} \) by using this marginal distribution (for example, marginal maximum likelihood). Next we plug in the obtained \( \hat{\beta} \) in the posterior \( \Pi_\beta(\theta|X) \), so that we get the so called empirical Bayes posterior \( \Pi(\theta|X) = \Pi_\beta(\theta|X) \). Both \( \Pi(\theta|X) \) and \( \Pi(\theta|X) \) can be used in the construction \((3) - (4)\). Also any combination of full Bayes and empirical Bayes approaches (with respect to different parameters) that leads to some resulting DDM \( P(\theta|X) \) can be used.

Our optimality benchmark is \((1)\) and in principle we want the radial rate \( R_\varepsilon(\theta_0) \) to be as small as possible. We allow local radial rates, the advantageous features of local radial rates (as compared to global ones) are already discussed in Introduction and will also be discussed below in Section \( \Box \) at length for the projection oracle rate in the Gaussian sequence framework. For the rest of this section, we are not concerned with specific choices for the radial rate \( R_\varepsilon(\theta_0) \) and simply suppose that we are given some (in general local) radial rate \( R_\varepsilon(\theta_0) \).

By \( c, C \) we denote generic constants which may be different in different expressions.

2.2 Assumptions

In this subsection we introduce some conditions which we will use for establishing some general statements about the confidence level and the size of the ball \( B(\hat{\theta}, M\hat{r}_\kappa) \).
For a point $\theta_0 \in \Theta$, some local radial rate $R_\varepsilon(\theta_0)$, a DDM $P(\cdot|X)$ and a DD-center $\hat{\theta} = \hat{\theta}(X)$, introduce the following conditions.

(A1) For some $\phi_1(M) = \phi_1(M, \varepsilon, \theta_0, \hat{\theta}) \geq 0$ such that $\phi_1(M) \downarrow 0$ as $M \uparrow \infty$,
\[ \mathbb{E}_{\theta_0}[P(d(\theta, \hat{\theta}) \geq M R_\varepsilon(\theta_0)|X)] \leq \phi_1(M). \]

(A2) For some $\psi(\delta) = \psi(\delta, \varepsilon, \theta_0, \hat{\theta}) \geq 0$ such that $\psi(\delta) \downarrow 0$ as $\delta \downarrow 0$,
\[ \mathbb{E}_{\theta_0}[P(d(\theta, \hat{\theta}) \leq \delta R_\varepsilon(\theta_0)|X)] \leq \psi(\delta). \]

(A3) For some $\phi_2(M) = \phi_2(M, \varepsilon, \theta_0, \hat{\theta}) \geq 0$ such that $\phi_2(M) \downarrow 0$ as $M \uparrow \infty$,
\[ P_{\theta_0}(d(\theta_0, \hat{\theta}) \geq M R_\varepsilon(\theta_0)) \leq \phi_2(M). \]

Remark 2. Of course, conditions (A1)–(A3) trivially hold for the functions
\[ \phi_1(M, \varepsilon, \theta_0, \hat{\theta}) = \mathbb{E}_{\theta_0}[P(d(\theta, \hat{\theta}) \geq M R_\varepsilon(\theta_0)|X)], \]
\[ \psi(\delta, \varepsilon, \theta_0, \hat{\theta}) = \mathbb{E}_{\theta_0}[P(d(\theta, \hat{\theta}) \leq \delta R_\varepsilon(\theta_0)|X)], \]
\[ \phi_2(M, \varepsilon, \theta_0, \hat{\theta}) = P_{\theta_0}(d(\theta_0, \hat{\theta}) \geq M R_\varepsilon(\theta_0)). \]

Conditions (A1)–(A3) become really useful when the functions $\phi_1, \psi, \phi_2$ do not depend on $\varepsilon \in (0, \varepsilon_0]$ and $\theta_0 \in \Theta_0$, for some $\varepsilon_0 > 0$ and $\Theta_0 \subseteq \Theta$ (preferably $\Theta_0 = \Theta$). Then, from the $P_{\theta_0}$-perspective, (A1) means that $P(\cdot|X)$ concentrates about $\hat{\theta}$ with the radial rate at least $R_\varepsilon(\theta_0)$, (A2) means that $P(\cdot|X)$ concentrates about $\hat{\theta}$ with the radial rate at most $R_\varepsilon(\theta_0)$ (in a way, no too fast “leakage” of $P(\cdot|X)$-mass through $\hat{\theta}$). Condition (A3) means that the DD-center $\hat{\theta}$ is an estimator of $\theta_0$ with the rate $R_\varepsilon(\theta_0)$. Together (A1) and (A2) imply that, from the $P_{\theta_0}$-perspective, $P(\cdot|X)$ concentrates on the spherical shell $\{ \theta : \delta R_\varepsilon(\theta_0) \leq d(\theta, \hat{\theta}) \leq MR_\varepsilon(\theta_0) \}$ for sufficiently small $\delta$ and large $M$.

Condition (A1) is reminiscent of the definition of the so called (global) posterior convergence rate $R_\varepsilon(\Theta)$ from the nonparametric Bayes literature: $\Pi(d(\theta, \hat{\theta}) \geq MR_\varepsilon(\Theta)|X)$ should be small for sufficiently large $M$ from the $P_{\theta_0}$-probability perspective. The following introduces a counterpart of a local convergence rate for a general DDM $P(\cdot|X)$.

(A1) For some $\varphi(M) = \varphi(M, \varepsilon, \theta_0) \geq 0$ such that $\varphi_1(M) \downarrow 0$ as $M \uparrow \infty$,
\[ \mathbb{E}_{\theta_0}[P(d(\theta_0, \hat{\theta}) \geq M R_\varepsilon(\theta_0)|X)] \leq \varphi(M). \]

Clearly, condition (A1) is implied by conditions (A1) and (A3) for an appropriate choice of function $\varphi_1(M)$: $\varphi_1(M) = \varphi_2(aM) + \varphi((1-a)M)$ for any $a \in (0, 1)$. We could therefore impose (A1), (A2) and (A3) instead of (A1)–(A3).

Introduce a strengthened version of condition (A2).

(A2) For some $\psi(\delta) = \psi(\delta, \varepsilon, \theta_0) \geq 0$ such that $\psi(\delta) \downarrow 0$ as $\delta \downarrow 0$ and any DD-center $\theta = \theta(X)$,
\[ \mathbb{E}_{\theta_0}[P(d(\theta, \hat{\theta}) \leq \delta R_\varepsilon(\theta_0)|X)] \leq \psi(\delta). \]
The difference between $\psi$ from (A2) and $\psi$ from (A2) is that in the latter does not depend on the DD-center. We keep however the same notation for the function $\psi$ in (A2) as in (A2) without confusion as we are never going to use both conditions simultaneously.

**Remark 3.** In the Bayesian framework, when the DDM $\mathbb{P}(\cdot|X)$ is the posterior (or empirical Bayes posterior) distribution on $\theta$ with respect to some prior, condition (A1) (and its asymptotic counterpart (A1) below) describes the so called posterior convergence rate $R_\varepsilon(\theta_0)$. To establish such assertions is an interesting and challenging problem nowadays, especially in nonparametric models when one wants to characterize the (frequentist) quality of Bayesian approach. Much recent research has been devoted to this topic. We will not dwell on this, but just mention that predominantly global posterior convergence rates are studied, i.e., $R_\varepsilon(\theta_0) = R_\varepsilon(\Theta)$ for all $\theta_0 \in \Theta$. To the best of our knowledge a local posterior convergence rate is considered only by Babenko and Belitser (2010).

**Remark 4.** Notice that a DD-center $\hat{\theta}$ satisfying conditions (A1)–(A3) is needed, while there is no $\theta$ involved in conditions (A1) and (A2). Actually, under condition (A1) there is a generic choice $\hat{\theta}$ (called default DD-center and studied in subsection 2.3) that would automatically satisfy conditions (A1) and (A3). Besides, (A2) certainly implies (A2) for any DD-center $\hat{\theta}$. This means that (A1) and (A2) imply (A1) and (A2) which in turn imply (A1)–(A3) for that generic DD-center $\hat{\theta}$. We consider this issue in the next section.

**Remark 5.** The smaller the radial rate $R_\varepsilon(\theta_0)$, the easier (A2) to satisfy, but the harder (A1), (A3) and (A1). Besides, one is interested in the smallest possible radial rate since this quantity will govern the size of the confidence ball. Thus, the right strategy would be first to determine the smallest radial rate $R_\varepsilon(\theta_0)$ for which (A1) (or (A1) and (A3)) holds, preferably for all $\theta_0 \in \Theta$. This would be the so called upper bound for the concentration rate of the DDM $\mathbb{P}(\cdot|X)$ around $\theta_0 \in \Theta$. Next, one needs to study whether (A2) holds as well with $R_\varepsilon(\theta_0)$ for all $\theta_0 \in \Theta$ (otherwise, for $\theta_0 \in \Theta_0$ with the “largest” $\Theta_0 \subset \Theta$), this is so called lower bound for the concentration rate of the DDM $\mathbb{P}(\cdot|X)$ around $\theta_0$.

Typically, under “good circumstances”, (A1) holds for all $\theta \in \Theta$, whereas requiring property (A2) leads to a condition on $\theta_0$. Looking ahead, we will see in the normal means model that although one can determine the smallest radial rate for which (A1) holds uniformly over $\theta_0 \in \Theta$, condition (A2) holds only for $\theta \in \Theta_0$, where $\Theta_0 \subset \Theta$.

The above non-asymptotic conditions may not be easy to check. Even in (regular) parametric models one typically verifies asymptotic versions instead. For a point $\theta_0 \in \Theta$, some radial rate $R_\varepsilon(\theta)$, a DDM $\mathbb{P}(\cdot|X)$ and a DD-center $\hat{\theta} = \hat{\theta}(X)$, introduce asymptotic versions of conditions (A1)–(A3), (A1) and (A2).

(AA1) For some positive $M_\varepsilon \to \infty$ as $\varepsilon \to 0$,

$$
E_{\theta_0}\left[\mathbb{P}(d(\theta, \hat{\theta}) \geq M_\varepsilon R_\varepsilon(\theta_0)|X)\right] \to 0 \quad \text{as} \quad \varepsilon \to 0.
$$

(AA2) For some $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$,

$$
E_{\theta_0}\left[\mathbb{P}(d(\theta, \hat{\theta}) \leq \delta_\varepsilon R_\varepsilon(\theta_0)|X)\right] \to 0 \quad \text{as} \quad \varepsilon \to 0.
$$
(AA3) For some $M'_ε \to \infty$ as $ε \to 0$,
\[ \mathbb{P}_{θ_0}(d(θ_0, \hat{θ}) \geq M'_ε R_ε(θ_0)) \to 0 \quad \text{as} \quad ε \to 0. \]

(AÂ1) For some $M_ε \to \infty$ as $ε \to 0$,
\[ \mathbb{E}_{θ_0}[\mathbb{P}(d(θ_0, θ) \geq M_ε R_ε(θ_0)|X)] \to 0 \quad \text{as} \quad ε \to 0. \]

(AÂ2) For some positive $δ_ε \to 0$ as $ε \to 0$ and any measurable $\tilde{θ} = \tilde{θ}(X)$,
\[ \mathbb{E}_{θ_0}[\mathbb{P}(d(θ, \tilde{θ}) \leq δ_ε R_ε(θ_0)|X)] \to 0 \quad \text{as} \quad ε \to 0. \]

**Remark 6.** An interested reader should be able to reproduce the asymptotic versions of the below assertions by using (AA1)–(AA3) instead of (A1)–(A3).

### 2.3 Default confidence ball based on DDM

For a $p \in (1/2, 1)$, define first
\[ \hat{r}^* = \hat{r}^*(p) = \inf \{ r : \mathbb{P}(d(θ, \hat{θ}) \leq r|X) \geq p \text{ for some } \hat{θ} \in Θ \}. \quad (5) \]

This is the smallest possible radius of $\mathbb{P}(θ|X)$-credible ball of level $p$. Next, for some $κ > 0$, take any $\hat{θ} ∈ Θ$ that satisfies
\[ \mathbb{P}(θ : d(θ, \hat{θ}) \leq (1 + κ)\hat{r}^*|X) \geq p. \quad (6) \]

We call the above defined $\hat{θ} = \hat{θ}(p, κ)$ default DD-center, with respect to the DDM $\mathbb{P}(·|X)$. In words, $\hat{θ} = \hat{θ}(p, κ)$ is the center of the ball of nearly the smallest radius subject to the constraint that its DDM $\mathbb{P}(·|X)$-mass is at least $p$.

**Remark 7.** Note that we also defined the default credible ball with respect to the DDM $\mathbb{P}(·|X)$.
\[ \hat{B} = \hat{B}_M = \hat{B}_{M, κ} = B(\hat{θ}, M\hat{r}_κ), \quad (7) \]
where $κ ∈ (0, 1)$, $B(\hat{θ}, M\hat{r}_κ)$ is defined by [3] and [4], and $\hat{θ}$ is defined by [3] and [5].

The following proposition claims that condition $(Â1)$ implies conditions $(A1)$ and $(A3)$ for the default DD-center $\hat{θ}$ with appropriate choices of functions $φ_1$ and $φ_2$.

**Proposition 1.** Let condition $(Â1)$ be fulfilled with function $φ(M)$ and let the default DD-center $\hat{θ}$ be defined by [3] and [4]. Then condition $(A1)$ holds with function $φ_1(M) = φ(aM/(2 + κ))/p + φ((1 - a)M)$ for any $a ∈ (0, 1)$, and condition $(A3)$ holds with function $φ_2(M) = φ(M/(2 + κ))/p$. 

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Proof. If (A3) holds true with $\phi_2(M) = \varphi(M/(2 + \varsigma))/p$, then, by using this and (A1),
\[
\mathbb{E}_{\theta_0} \left[ \mathbb{P}(d(\theta, \hat{\theta}) \geq M R_{\varphi}(\theta_0) | X) \right] \leq \mathbb{E}_{\theta_0} \left[ \mathbb{P}(d(\theta, \hat{\theta}) \geq a M R_{\varphi}(\theta_0) | X) \right] + \mathbb{E}_{\theta_0} \left[ \mathbb{P}(d(\theta, \hat{\theta}) \geq (1 - a) M R_{\varphi}(\theta_0) | X) \right]
\]
for any $a \in (0, 1)$, which implies (A1) with $\phi_1(M) = \varphi(a M/(2 + \varsigma))/p + \varphi((1 - a) M)$. Therefore, it remains to show (A3) with the function $\phi_2(M) = \varphi(M/(2 + \varsigma))/p$. From (A1) it follows by the Markov inequality that
\[
\mathbb{P}_{\theta_0}(\mathbb{P}(\theta \in B(\theta_0, M R_{\varphi}(\theta_0)) | X) \geq p) \geq 1 - \frac{\varphi(M)}{p}.
\]
By [3], the ball $B(\hat{\theta}, (1 + \varsigma) \hat{r}^*)$ has $\mathbb{P}(\cdot | X)$-probability at least $p$. If the ball $B(\theta_0, M R_{\varphi}(\theta_0))$ also has $\mathbb{P}(\cdot | X)$-probability at least $p$ (which happens with $\mathbb{P}_{\theta_0}$-probability at least $1 - \frac{\varphi(M)}{p}$), then, firstly, $\hat{r}^* \leq M R_{\varphi}(\theta_0)$ by virtue of the definition [5] of $\hat{r}^*$, and, secondly, the balls $B(\hat{\theta}, (1 + \varsigma) \hat{r}^*)$ and $B(\theta_0, M R_{\varphi}(\theta_0))$ must intersect, otherwise the total $\mathbb{P}(\cdot | X)$-mass would exceed $2p > 1$. Therefore, by the triangle inequality,
\[
d(\theta_0, \hat{\theta}) \leq (1 + \varsigma) \hat{r}^* + M R_{\varphi}(\theta_0) \leq (2 + \varsigma) M R_{\varphi}(\theta_0),
\]
with $\mathbb{P}_{\theta_0}$-probability at least $1 - \frac{\varphi(M)}{p}$. We thus see that, for the above defined default DD-center $\hat{\theta}$, condition (A3) holds with $\phi_2(M) = \varphi(M/(2 + \varsigma))/p$. \hfill \Box

Remark 8. Of course, $\hat{r}^*$ depends on $p$ and $\hat{\theta}$ depends on both $p$ and $\varsigma$. We however skip this dependences from the notations by assuming from now on that $p = 2/3$ and $\varsigma = 1/2$. We also take $a = 1/2$ in Proposition 4. Then, according to Proposition 4 if condition (A1) is fulfilled with function $\varphi(M)$, then conditions (A1) and (A3) hold for the default DD-center $\hat{\theta}$ with functions $\phi_2(M) = 3\varphi(2M/5)/2$ and $\phi_1(M) = 3\varphi(M/5)/2 + \varphi(M/2)$ respectively.

To put it short, (A1) and (A2) imply (A1)–(A3) with the default DD-center $\hat{\theta}$.

2.4 Coverage and size of the DDM-credible set

Recall that our main goal is to construct a confidence ball satisfying the benchmark optimality framework [1]. In this subsection we present some simple general (coverage and size) properties of the DDM-credible ball $B(\hat{\theta}, M \hat{r}_\kappa)$ defined by [1] with a DDM $\mathbb{P}(\cdot | X)$ and a DD-center $\hat{\theta}$ satisfying (A1)–(A3). Next we briefly outline how these properties can be used to establish the optimality framework [1] in concrete settings. Finally, we provide some insight about the minimality of conditions (A1)–(A3) for the framework [1].

The below assertion gives an upper (point-wise) bound for the coverage probability of the confidence ball [4].

Proposition 2. For a $\theta_0 \in \Theta$ and some radial rate $R_{\varphi}(\theta_0)$, let $\kappa \in (0, 1)$ and the ball $B(\hat{\theta}, M \hat{r}_\kappa)$ be defined by [4] with a DDM $\mathbb{P}(\cdot | X)$ and a DD-center $\hat{\theta}$ satisfying conditions (A2) and (A3). Then for any $M, \delta > 0$,
\[
\mathbb{P}_{\theta_0}(\theta_0 \not\in B(\hat{\theta}, M \hat{r}_\kappa)) = \mathbb{P}_{\theta_0}(d(\theta_0, \hat{\theta}) > M \hat{r}_\kappa) \leq \phi_2(M \delta) + \frac{\psi(\delta)}{1 - \kappa}.
\]
Proof. By the Markov inequality, \(3\) and conditions (A2) and (A3), we derive
\[
\mathbb{P}_{\theta_0}(d(\theta_0, \hat{\theta}) > M\hat{r}_\kappa) \\
\leq \mathbb{P}_{\theta_0}(d(\theta_0, \hat{\theta}) > M\hat{r}_\kappa, \hat{r}(\gamma) \geq \delta R_\varepsilon(\theta_0)) + \mathbb{P}_{\theta_0}(\hat{r}_\kappa < \delta R_\varepsilon(\theta_0)) \\
\leq \mathbb{P}_{\theta_0}(d(\theta_0, \hat{\theta}) > M\delta R_\varepsilon(\theta_0)) + \mathbb{P}_{\theta_0}(\mathbb{P}(d(\theta, \hat{\theta}) \leq \delta R_\varepsilon(\theta_0)) \leq 1 - \kappa) \\
\leq \phi_2(M\delta) + \frac{\mathbb{E}_{\theta_0}(\mathbb{P}(d(\theta, \hat{\theta}) \leq \delta R_\varepsilon(\theta_0)) | X)}{1 - \kappa} \leq \phi_2(M\delta) + \frac{\psi(\delta)}{1 - \kappa}.
\]

The following assertion gives some bound on the effective size (radius) of \(B(\hat{\theta}, M\hat{r}_\kappa)\) in terms of the local radial rate \(R_\varepsilon(\theta_0)\) from the \(\mathbb{P}_{\theta_0}\)-perspective.

Proposition 3. For a \(\theta_0 \in \Theta\), let a DDM \(\mathbb{P}(\cdot | X)\) and a DD-center \(\hat{\theta}\) satisfy condition (A1) for some radial rate \(R_\varepsilon(\theta_0)\). Let \(\hat{r}_\kappa\) be defined by \(3\). Then for any \(\kappa \in (0, 1)\), \(M > 0\),
\[
\mathbb{P}_{\theta_0}(\hat{r}_\kappa \geq M R_\varepsilon(\theta_0)) \leq \frac{\phi_1(M)}{\kappa}.
\]

Proof. By the conditional Markov inequality, \(3\) and condition (A1), we obtain
\[
\mathbb{P}_{\theta_0}(\hat{r}_\kappa \geq M R_\varepsilon(\theta_0)) \leq \mathbb{P}_{\theta_0}(\mathbb{P}(d(\theta, \hat{\theta}) \leq M R_\varepsilon(\theta_0)) \leq 1 - \kappa) \\
= \mathbb{P}_{\theta_0}(\mathbb{P}(d(\theta, \hat{\theta}) > M R_\varepsilon(\theta_0)) > \kappa) \\
\leq \frac{\mathbb{E}_{\theta_0}(\mathbb{P}(d(\theta, \hat{\theta}) > M R_\varepsilon(\theta_0)) | X)}{\kappa} \leq \frac{\phi_1(M)}{\kappa}.
\]

Remark 9. It is not difficult to see that condition (A2) guarantees that the rate \(R_\varepsilon(\theta_0)\) is actually sharp. Indeed, from (A2) it follows that
\[
\mathbb{P}_{\theta_0}(\hat{r}_\kappa \leq \delta R_\varepsilon(\theta_0)) \leq \mathbb{P}_{\theta_0}(\mathbb{P}(d(\theta, \hat{\theta}) \leq \delta R_\varepsilon(\theta_0)) \geq 1 - \kappa) \\
\leq \frac{\mathbb{E}_{\theta_0}(\mathbb{P}(d(\theta, \hat{\theta}) \leq \delta R_\varepsilon(\theta_0)) | X)}{1 - \kappa} \leq \frac{\psi(\delta)}{1 - \kappa}.
\]

Let us elucidate what else is needed in concrete situations to derive the optimality framework \(1\). Since the functions \(\phi_2, \psi\) and \(\phi_1\) in general depend on the unknown \(\theta_0\) (and \(\varepsilon\)), we cannot apply the above assertions directly for establishing \(1\). Suppose uniform bounds \(\phi_1(M, \varepsilon, \theta_0) \leq \bar{\phi}_1(M, \varepsilon) = \bar{\phi}_1(M)\) for all \(\theta_0 \in \Theta_0 \subseteq \Theta\), \(\phi_2(M, \varepsilon, \theta_0) \leq \bar{\phi}_2(M, \varepsilon) = \bar{\phi}_2(M)\) and \(\psi(M, \varepsilon, \theta_0) \leq \bar{\psi}(M, \varepsilon) = \bar{\psi}(M)\) for all \(\theta_0 \in \Theta_0 \subseteq \Theta\) and all \(\varepsilon \in (0, \varepsilon_0]\) are available, such that \(\bar{\phi}_1(M) \downarrow 0\), \(\bar{\phi}_2(M) \downarrow 0\) as \(M \uparrow \infty\) and \(\bar{\psi}(\delta) \downarrow 0\) as \(\delta \downarrow 0\). Clearly, then we can ensure \(1\) for the ball \(B(\hat{\theta}, M\hat{r}_\kappa)\) and the radial rate \(R_\varepsilon(\theta_0)\), by taking sufficiently large \(M\). In fact, we can optimize the choice of \(M\) as follows: first determine
\[
\min_{\delta > 0} \left\{ \frac{\bar{\phi}_2(M\delta) + \bar{\psi}(\delta)}{1 - \kappa} \right\} = \bar{\phi}(M, \kappa),
\]
where \( \tilde{\phi}(M, \kappa) \downarrow 0 \) as \( M \uparrow \infty \). Next, take constants \( M_1 \) and \( M_2 \) sufficiently large so that \( \tilde{\phi}(M_1, \kappa) \leq \alpha_1 \) and \( \tilde{\phi}(M_2)/\kappa \leq \alpha_2 \). Then the optimality framework (1) holds with \( C = M_1 \) and \( c = M_2 \).

According to Proposition 1 if condition (A1) holds, then conditions (A1) and (A3) are satisfied for the default DD-center \( \hat{\theta} \) defined by (3) and (6) with appropriate choices of functions \( \phi_2 \) and \( \phi_1 \). If \( \hat{\theta} \) happens to satisfy condition (A2), then all three conditions (A1)–(A3) are fulfilled. Certainly, under conditions (A1)–(A2), (A2) holds for the default DD-center \( \hat{\theta} \) and consequently the both above propositions hold for the default ball \( \tilde{B}(M, \kappa) \) defined by (7). By using Propositions 1, 2 and 3 we thus obtain the following corollary.

**Corollary 1.** Let a DDM \( \mathbb{P}(\cdot \mid X) \) satisfy conditions (A1)–(A2) with some radial rate \( R_c(\theta_0), \theta_0 \in \Theta \). Let \( \kappa \in (0, 1) \), the default ball \( B_{M, \kappa} \) be defined by (1) and \( \hat{r}_\kappa \) be its DD-radius defined by (5). Then for any \( M, \delta > 0 \),

\[
\mathbb{P}_{\theta_0}(\theta_0 \notin \tilde{B}_{M, \kappa}) \leq \frac{3\varphi(2M\delta/5)}{2} + \frac{\psi(\delta)}{1 - \kappa}, \quad \mathbb{P}_{\theta_0}(\hat{r}_\kappa \geq M R_c(\theta_0)) \leq \frac{3\varphi(M/5)}{2\kappa} + \frac{\varphi(M/2)}{\kappa}.
\]

This corollary can be used for establishing (1) exactly in the same way as before, provided the functions \( \varphi \) and \( \psi \) (in general depending on \( \theta_0 \) and \( \varepsilon \)) are bounded uniformly over corresponding sets.

Let us demonstrate that (A2) is in some sense the minimal condition for providing a sufficient \( \mathbb{P}_{\theta_0} \)-coverage of the \( \mathbb{P}(\cdot \mid X) \)-credible ball with the sharpest rate.

**Proposition 4.** For a DDM \( \mathbb{P}(\cdot \mid X) \) on \( \Theta \) and a DD-center \( \hat{\theta} \), let the ball \( B(\hat{\theta}, M \hat{r}_\kappa) \) be constructed according to (4) with any \( \kappa \in (0, 1) \) and \( M > 0 \). Further, for a \( \theta_0 \in \Theta \) and a radial rate \( R_c(\theta_0) \) denote

\[
\alpha(\delta) = \alpha(\delta, \varepsilon, \theta_0) = \mathbb{E}_{\theta_0} \left[ \mathbb{P}(d(\theta, \hat{\theta}) > \delta R_c(\theta_0) \mid X) \right],
\]

\[
\psi_2(\delta) = \psi_2(\delta, \varepsilon, \theta_0) = \mathbb{P}_{\theta_0}(d(\theta, \hat{\theta}) \leq \delta R_c(\theta_0)).
\]

Then

\[
\mathbb{P}_{\theta_0}(\theta_0 \in B(\hat{\theta}, M \hat{r}_\kappa)) \leq \psi_2(\delta M) + \alpha(\delta, \kappa)^{-1} \quad \text{for any } \delta > 0.
\]

**Proof.** In view of the definition (4), we derive

\[
\mathbb{P}_{\theta_0}(\theta_0 \in B(\hat{\theta}, M \hat{r}_\kappa)) = \mathbb{P}_{\theta_0}(\theta_0 \in B(\hat{\theta}, M \hat{r}_\kappa), \hat{r}_\kappa \leq \delta R_c(\theta_0)) + \mathbb{P}_{\theta_0}(\theta_0 \in B(\hat{\theta}, M \hat{r}_\kappa), \hat{r}_\kappa > \delta R_c(\theta_0)) \leq \mathbb{P}_{\theta_0}(\hat{r}_\kappa \leq \delta M R_c(\theta_0)) + \mathbb{P}_{\theta_0}(\hat{r}_\kappa > \delta R_c(\theta_0)) \leq \mathbb{P}_{\theta_0}(\hat{d}(\hat{\theta}, \theta_0) \leq \delta M R_c(\theta_0)) + \mathbb{P}_{\theta_0}(\mathbb{P}(d(\hat{\theta}, \theta) \leq \delta R_c(\theta_0) \mid X) \leq 1 - \kappa) \leq \mathbb{P}_{\theta_0}(\hat{d}(\hat{\theta}, \theta_0) \leq \delta M R_c(\theta_0)) + \frac{\mathbb{E}_{\theta_0}(\mathbb{P}(d(\hat{\theta}, \theta) > \delta R_c(\theta_0) \mid X)}{\kappa} \leq \psi_2(\delta M) + \frac{\alpha(\delta)}{\kappa}.
\]
One should interpret this proposition as follows. First, given a DD-center \( \hat{\theta} \), we determine a local radial rate \( R_{\epsilon}(\theta_0) \) such that \( \psi_2(\delta) \leq \hat{\alpha}(\delta) \) for all \( 0 < \delta \leq \delta_0 \), for some “small” \( \hat{\alpha}(\delta) \). This is nothing else but the sharpest rate for estimating \( \theta_0 \) by \( \hat{\theta} \). Next, \( \alpha(\delta) \) being small for small \( \delta \) means that the DDM \( \mathbb{P}(\cdot|X) \) concentrates around \( \hat{\theta} \) with a faster rate than the radial rate \( R_{\epsilon}(\theta_0) \), which can be regarded as negation of condition (A2). The above proposition says basically that, under negation of (A2), the coverage probability of the credible set \( B(\hat{\theta}, M\hat{r}_\kappa) \) is bounded from above. Thus, (A2) is the minimal condition if we want to have the sharpest rate and a good coverage. This quantifies the following simple intuitive idea: if the DDM \( \mathbb{P}(\cdot|X) \) contracts in the DD-center \( \hat{\theta} \) faster than \( R_{\epsilon}(\theta_0) \), then the resulting radius of the credible ball \( B(\hat{\theta}, M\hat{r}_\kappa) \) is going to be of a smaller order than \( R_{\epsilon}(\theta_0) \). But this is going to be (over-optimistically) too small if the convergence rate of the center \( \hat{\theta} \) to the truth \( \theta_0 \) is not faster than \( R_{\epsilon}(\theta_0) \). Then the credible ball \( B(\hat{\theta}, M\hat{r}_\kappa) \) will clearly miss the truth with some probability bounded from below.

3 Some simple examples

3.1 Normal case

Let us start with the simplest model: suppose we observe \( X = X^{(\epsilon)} \sim N(\theta_0, \epsilon^2) \), \( \theta_0 \in \Theta = \mathbb{R} \), information parameter \( \epsilon \to 0 \), but we derive also non-asymptotic relations. Take prior \( \theta \sim \pi = N(\mu, \tau^2) \), the estimator \( \hat{\theta} = X \) and the radial rate \( R_{\epsilon}(\theta_0) = \epsilon \). It is well known that

\[
\pi(\theta|X) = N\left(\frac{\epsilon^2 \mu + \tau^2 X}{\epsilon^2 + \tau^2}, \frac{\epsilon^2 \tau^2}{\epsilon^2 + \tau^2}\right).
\]

We take \( \mathbb{P}(\theta|X) = \pi(\theta|X)|_{\mu = \hat{\theta}} = N\left(X, \frac{\epsilon^2 \tau^2}{\epsilon^2 + \tau^2}\right) \), the empirical Bayes posterior with \( \hat{\mu} = X \), and construct the credible interval for \( \theta_0 \) according to \( \mathbb{I} \). Then (A1) and (A3) are satisfied with \( \phi_1(M) = \phi_2(M) = C\epsilon^{-eM^2}/M \). Indeed, for a \( \xi \sim N(0, 1) \),

\[
\mathbb{P}(|\hat{\theta} - \theta| \geq M R_{\epsilon}(\theta_0)|X) = \mathbb{P}\left(\frac{\epsilon \tau |\xi|}{\sqrt{\epsilon^2 + \tau^2}} \geq M\epsilon \right) \leq \mathbb{P}(\|\xi\| \geq M) \leq \frac{2 e^{-M^2/2}}{\sqrt{2\pi} M},
\]

\[
\mathbb{P}(|\hat{\theta} - \theta_0| \geq M R_{\epsilon}(\theta_0)) = \mathbb{P}(\|\xi\| \geq M\epsilon) = \mathbb{P}(\|\xi\| \geq M) \leq \frac{2 e^{-M^2/2}}{\sqrt{2\pi} M}.
\]

Assume \( \epsilon \leq \tau \), then condition (A2) is also satisfied with \( \psi(\delta) = \delta/\sqrt{\pi} \).

\[
\mathbb{P}(|\hat{\theta} - \theta| \leq \delta R_{\epsilon}(\theta_0)|X) = \mathbb{P}(\|\xi\| \leq \delta \sqrt{1 + \epsilon^2/\tau^2}) \leq \mathbb{P}(\|\xi\| \leq \delta \sqrt{2}) \leq \delta/\sqrt{\pi}.
\]

One can think of the above two properties of the normal distribution as “ring tightness”.

Suppose now we observe a sample \( X = X^{(n)} = (X_1, \ldots, X_n) \) from \( N(\theta_0, \sigma^2) \) and let \( X_n = \frac{1}{n} \sum_{i=1}^{n} X_i \). Then \( \hat{X}_n \sim N(\theta_0, \sigma^2/n) \) with \( \sigma^2_n = \sigma^2/n \), which is, by sufficiency, equivalent to the original model. Take again the prior \( \theta \sim \pi = N(\mu, \tau^2) \), then the posterior is

\[
\pi(\theta|X) = N\left(\frac{\sigma^2_n \mu + \tau^2 \hat{X}_n}{\sigma^2_n + \tau^2}, \frac{\sigma^2_n \tau^2}{\sigma^2_n + \tau^2}\right).
\]
Thus, this case reduces to the previous situation with \( \varepsilon = \sigma_n = \sigma n^{-1/2} \) and the estimator \( \hat{\theta} = \bar{X}_n \), so that the radial rate \( R_n(\theta_0) = \sigma n^{-1/2} \). Here the asymptotic regime is \( n \to \infty \), but we again have non-asymptotic relations too.

**Remark 10.** Of course, there is the classical (frequentist) confidence interval

\[
\bar{X}_n \pm z_{1-\alpha/2} \sigma / \sqrt{n},
\]

which has the same radial rate whose coverage may even be (non-asymptotically) better. In that respect, the above example is somewhat uninteresting and is provided only for the illustrative purposes. No wonder this resulting credible interval has a good coverage probability as the posterior, being normal, concentrates in a “ring-manner” around the truth. One can further consider multivariate normal case and construct confidence set as credible set on basis of an empirical Bayes posterior. Non-asymptotic improvement of the coverage probability is possible in this case (see Tseng and Brown (1997) and further references therein). We will not however pursue this problem here.

### 3.2 Bernstein-von Mises case

For the finite dimensional parameter, consider a general situation when some mild regularity conditions on the model and the prior lead to the resulting asymptotically normal posterior. This is the so called *Bernstein-von Mises* property as often termed in the literature. Suppose \( X = X^{(n)} \sim P_{\theta_0}, \theta \in \Theta \), information parameter \( \varepsilon = n^{-1/2} \), with a prior \( \theta \sim \pi \) on some \( \sigma \)-algebra \( B_\Theta \) on \( \Theta \) and a \( \sqrt{n} \)-consistent estimator \( \hat{\theta} \) such that (AA3) is satisfied with the radial rate \( R_n(\theta_0) = n^{-1/2} \) and in \( P_{\theta_0} \)-probability

\[
\sup_{B \in B_\Theta} \left| \pi(B|X) - N(\hat{\theta}, I(\theta_0))(B) \right| \to 0, \quad \text{as } n \to \infty.
\]

where \( N(\mu, \Sigma)(B) = P(Y \in B) \) with \( B \sim N(\mu, \Sigma) \) for some multivariate normal distribution with mean \( \mu \) and covariance matrix \( \Sigma \). Then, as (A1) and (A2) hold for the multivariate distribution \( N(\hat{\theta}, I(\theta_0)) \), the asymptotic conditions, with \( \varepsilon = n^{-1/2} \), (AA1)– (AA3) are satisfied. Asymptotic versions of Propositions 2 and 3 follow immediately, which yields us (asymptotically) a full coverage probability and the optimal global radial rate \( n^{-1/2} \). Interestingly, there is nothing special about normal distribution in the above arguments, any resulting limiting distribution with a “ring”-structure will do the same job. Ring-structure means negligible probability mass outside a ring, whose inner radius is a sufficiently small multiples of the radial rate and the outer radius is a sufficiently big multiples of the radial rate. In fact, the existence of an exact limiting distribution is also not decisive, “ring”-tightness (which is nothing else but (AA1)–(AA2)) would be enough. For example, the Bernstein-von Mises property is more than needed if we only want to make sure that a credible set serves as a proper confidence set.
4 Gaussian sequence framework

4.1 The model

Suppose we observe

\[ X^{(c)} = X = (X_i, i \in \mathbb{N}) \sim \mathbb{P}_\theta^{(c)} = \mathbb{P}_\theta = \bigotimes_{i \in \mathbb{N}} \mathcal{N}(\theta_i, \varepsilon^2), \tag{8} \]

which means \( X_i \overset{\text{ind}}{\sim} \mathcal{N}(\theta_i, \varepsilon^2), \ i \in \mathbb{N}. \) Here \( \theta = (\theta_i, i \in \mathbb{N}) \in \ell_2 \) is an unknown parameter of interest, \( \varepsilon^2 \) is the noise intensity, it reflects the increase of information in the data \( X^{(c)} \) as \( \varepsilon \to 0. \) Thus, in this case \( \Theta \) is the whole space \( \ell_2, \) with its usual norm \( \| \theta \| = (\sum_{i \in \mathbb{N}} \theta_i^2)^{1/2}. \)

From now on, all summations and products are over \( \mathbb{N} = \{1, 2, \ldots\}, \) unless otherwise specified; for instance, \( \bigotimes_{i} = \bigotimes_{i \in \mathbb{N}}. \) Introduce the notations: \( \mathbb{N}_k = \{1, \ldots, k\} \) for \( k \in \mathbb{N}, \)
\( S_1 \setminus S_2 = \{ s \in S_1 : s \notin S_2 \} \) for sets \( S_1, S_2, \mathbb{N}_k = \mathbb{N} \setminus \mathbb{N}_k, \) \( \varphi(x, \mu, \sigma^2) \) is the \( \mathcal{N}(\mu, \sigma^2) \)-density at point \( x, \) the indicator function \( \mathbb{1}(E) = 1 \) if the event \( E \) occurs and is zero otherwise. By \( Z \sim \mathcal{N}(c, 0) \) we mean \( \mathbb{P}(Z = c) = 1. \)

Model (8) is known to be the normal sequence framework. This model can be derived from the generalized linear Gaussian model as introduced by Birgé and Massart (2001): for some separable Hilbert space \( \mathbb{H} \) with scalar product \( \langle \cdot, \cdot \rangle, \)

\[ Y^{(c)}(x) = \langle y, x \rangle + \varepsilon W(x), \quad x \in \mathbb{H}, \]

where \( W \) is a so called isonormal process; see the exact definition in Birgé and Massart (2001). Take any orthonormal basis \( \{ b_i, i \in \mathbb{N} \} \) in \( \mathbb{H} \) and consider \( X_i = Y^{(c)}(b_i), i \in \mathbb{N}, \) to reduce the above model to (8).

The following model is known as the white noise model. We observe a stochastic process \( Y^{(c)}(t), t \in [0, 1], \) satisfying the stochastic differential equation

\[ dY^{(c)}(t) = f(t)dt + \varepsilon dW(t), \quad t \in [0, 1], \]

where \( f \in \mathbb{L}_2([0, 1]) \) is an unknown signal and \( W \) is a standard Brownian motion which represent the noise of intensity \( \varepsilon. \) If \( \{ b_i(t), i \in \mathbb{N} \} \) is an orthonormal basis in \( \mathbb{L}_2([0, 1]), \) then the white noise model can be translated into model (8) with observations \( X_i = \int_0^1 b_i(t)dY_1^{(c)}(t) \) and parameter \( \theta_i = \int_0^1 b_i(t) f(t)dt, i \in \mathbb{N}. \) While interesting in communication theory in its own right, the white noise model also provides a good approximation to a variety of curve estimation problems.

As the last example, we mention here the discrete regression model:

\[ Y_i = f(x_i) + e_i, \quad i \in \mathbb{N}_n, \tag{9} \]

where \( e_i \)’s are independent \( \mathcal{N}(0, \sigma^2), x_i \in [0, 1] \) are deterministic distinct points and \( f(t) \) is an unknown function. Let \( Y = (Y_1, \ldots, Y_n)^T, f = (f(x_i), i \in \mathbb{N}_n), \{ b_i, i \in \mathbb{N}_n \} \) be an orthonormal basis of \( \mathbb{R}^n, W = (b_1, \ldots, b_n)^T. \) Denote

\[ X = n^{-1/2}WY, \quad \theta = n^{-1/2}Wf, \quad \varepsilon = n^{-1/2} \tag{10} \]
to reduce (22) again to (8), with the convention that \( \theta = (\theta_1, \ldots, \theta_n, 0, 0 \ldots) \) in (8) has now zero coordinates starting from \((n + 1)\)-th position. Clearly, \( \| \bar{\theta} - \theta \|^2 = n^{-1} \| \tilde{f} - f \|^2 \) for \( \bar{\theta} = n^{-1/2} W \tilde{f} \).

Remark 11. If \( x_i = i/n, n = 2^{J+1} \) and \( f(t) \in L_2([0, 1]) \) in (8), we can choose a convenient wavelet basis (of regularity \( r > 0 \)) in \( L_2([0, 1]) \) and apply the corresponding discrete wavelet transform \( W \) in (10) to the original data \( Y \). Assume that the original curve \( f \) belongs to a certain scale of Besov balls (from Besov space \( B^s_{p,q} \), with \( \max\{0, 1/p - 1/2 \} < s < r, p, q \geq 1 \) from \( L_2([0, 1]) \), that include among others Hölder \( B^s_{\infty,\infty} \) and Sobolev \( B^s_{2,2} \) classes of smooth functions. Then the corresponding noiseless discrete wavelet transform \( n^{1/2} \theta = W f \) belongs to the corresponding scale of Besov balls in \( \ell_2 \). There is a dyadic indexing of vector \( n^{1/2} \theta \), but it can be reduced to the setting of (8) by an appropriate ordering; the details are nicely explained by Birgé and Massart (2001).

Remark 12. The model (8) captures many of the conceptual issues associated with non-parametric estimation, with a minimum of technical complication. Gaussian white noise models have become increasingly popular as a canonical type of model which serves as a purified approximation to some other statistical models such as nonparametric regression model, density estimation, spectral function estimation, by virtue of the so called equivalence principle; see Efromovich (1999). The statistical inference results for the generic model (8) can in principle be conveyed to other models, according to the above mentioned equivalence principle. However, in general the problem of establishing the equivalence in a precise sense is a delicate task. We will not go into this, but focus on the model (8).

4.2 Construction of a DDM \( \mathbb{P}(\theta|X) \)

First we introduce a DDM for \( \theta \) and subsequently use this for constructing confidence set as credible ball according to our general approach.

For some \( L > 0 \) define a family of DDM’s on \( \theta \) as follows:

\[
\mathbb{P}_I(\cdot|X) = \bigotimes_i N(X_i I\{i \in I_I\}, L\varepsilon^2 I\{i \in I_I\}, \quad I \in \mathbb{N}. \tag{11}
\]

Next, for some \( 0 \leq K_2 < K_1 \), define

\[
\tau_2^2(I) = \tau_2^2(I, \varepsilon, K_1, K_2) = K_1\varepsilon^2 I\{i \in I_I\} + K_2\varepsilon^2 I\{i \in I_{I'}\}, \quad i, I \in \mathbb{N}, \tag{12}
\]

and introduce a DDM on \( I \in \mathbb{N} \): with \( \lambda_I = C_\alpha e^{-\alpha I} \) and \( C_\alpha = e^\alpha - 1 \) for some \( \alpha > 0 \),

\[
\mathbb{P}(I = I|X) = \mathbb{P}_\alpha(I = I|X) = \frac{\lambda_I \bigotimes_i \varphi(X_i, 0, \varepsilon^2 + \tau_2^2(I)_i)}{\sum_j \lambda_j \bigotimes_i \varphi(X_i, 0, \varepsilon^2 + \tau_2^2(J)_i)}, \quad I \in \mathbb{N}. \tag{13}
\]

The random probability \( \mathbb{P}(I = I|X) \) originates from the following two-level hierarchical prior \( \pi = \pi_{\alpha,K_1,K_2} \) on \( \theta \):

\[
\theta \sim \pi \quad \iff \quad \theta|I = I \sim \bigotimes_i N(\mu_I(I), \tau_2^2(I)), \quad \mathbb{P}(I = I) = \lambda_I, \quad I \in \mathbb{N}, \tag{14}
\]

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with \( \mu_i(I) = 0 \) for \( i, I \in \mathbb{N} \), and \( \tau^2(I) \) defined by (12). Namely, the probability \( \mathbb{P}(I = I|X) \) defined by (13) is nothing but the posterior probability of \( I \) with respect to the prior \( \mathbb{P}_0 \), as it was introduced by Babenko and Belitser (2010). The right hand side of (13) means the \( \mathbb{P}_{\theta_0} \) almost sure limit

\[
\mathbb{P}(I = I|X) = \lim_{m \to \infty} \mathbb{P}(I = I|X_1, \ldots, X_m) = \lim_{m \to \infty} \frac{\lambda_I \bigotimes_{i=1}^{m} \varphi(0, \varepsilon^2 + \tau^2(I), X_i)}{\sum_J \lambda_J \bigotimes_{i=1}^{m} \varphi(0, \varepsilon^2 + \tau^2(J), X_i)},
\]

which exists by the martingale convergence theorem.

Finally, introduce a DDM on parameter \( \theta \) as follows:

\[
\mathbb{P}(\cdot|X) = \mathbb{P}_{K_1, K_2, \alpha, L}(\cdot|X) = \sum_I \mathbb{P}_{I}(\cdot|X) \mathbb{P}(I = I|X), \tag{15}
\]

where \( \mathbb{P}_I(\cdot|X) \) and \( \mathbb{P}(I = I|X) \) are define by (11) and (13) respectively. The idea of construction (13) is as follows. First, notice that \( \mathbb{P}(\cdot|X) \) is a random mixture over measures \( \mathbb{P}_I(\cdot|X), I \in \mathbb{N} \). From the \( \mathbb{P}_{\theta_0} \)-perspective, each \( \mathbb{P}_I(\cdot|X) \) contracts to the true \( \theta_0 \) with the quadratic local rate

\[
\mathcal{R}_\varepsilon^2(I, \theta_0) = \varepsilon^2 I + \sum_{i \in \mathbb{N}_I} \theta^2_{0,i}, \quad I \in \mathbb{N}. \tag{16}
\]

Indeed, denoting \( X(I) = (X_i|\{i \in \mathbb{N}_I\}, i \in \mathbb{N}) \), we obtain by the conditional Markov inequality

\[
\mathbb{E}_{\theta_0} \mathbb{P}_I(||\theta - \theta_0|| \geq \varepsilon \mathcal{R}_\varepsilon^2(I, \theta_0)|X) \leq \frac{\mathbb{E}_{\theta_0}(||X(I) - \theta_0||^2 + \varepsilon^2 I)}{M^2 \mathcal{R}_\varepsilon^2(I, \theta_0)} \leq \frac{2\varepsilon^2 I + \sum_{i \in \mathbb{N}_I} \theta^2_{0,i}}{M^2 \mathcal{R}_\varepsilon^2(I, \theta_0)} \leq \frac{2}{M^2}. \tag{17}
\]

For each \( \theta \in \ell_2 \), there is the best choice \( I_o = I_o(\theta) = I_o(\theta, \varepsilon) \) of parameter \( I \), called the oracle, which leads to the smallest possible rate called (quadratic) oracle rate:

\[
\mathcal{R}_\varepsilon^2(\theta) = \mathcal{R}_\varepsilon^2(I_o, \theta) = \min_{I \in \mathbb{N}} \mathcal{R}_\varepsilon^2(I, \theta) = \varepsilon^2 I_o + \sum_{i \in \mathbb{N}_{I_o}} \theta^2_i. \tag{18}
\]

Clearly, \( I_o(\theta, \varepsilon) \to \infty \) and \( \mathcal{R}_\varepsilon(\theta) \to 0 \) as \( \varepsilon \to 0 \).

Coming back to the idea of construction (13), the mission of the random mixing probabilities \( \mathbb{P}(I = I|X), I \in \mathbb{N} \), is to “mimic” the oracle \( I_o \); see Corollary [2] below. This property, combined with (17), assures the upper bound (A1) for the contraction rate of the DDM \( \mathbb{P}(\theta|X) \) defined by (14). In fact, such an upper bound has been established by Babenko and Belitser (2010) for the particular choices \( K_1 = 1, K_2 = 0, L = 1/2 \) and \( \alpha \in \left[ \frac{1}{4} - \log \left( \frac{2}{\sqrt{\varepsilon}} \right), \frac{1}{2} \right] \). Here we show that actually (A1) holds true also for \( \mathbb{P}_{K_1, K_2, \alpha, L}(\theta|X) \) with arbitrary \( 0 \leq K_2 < K_1 \) and \( \alpha > 0 \). This flexibility is needed for establishing the lower bound (A2): in order to prove (A2), we use some preliminary technical claims which hold only under some restrictions on the parameters \( 0 \leq K_2 < K_1 \) and \( \alpha > 0 \) (choice of \( L > 0 \) can improve the resulting constants). Unfortunately, the technical proofs in
Babenko and Belitser (2010) cannot be straightforwardly extended to arbitrary constants
\(0 \leq K_2 < K_1\) and \(L, \alpha > 0\). Besides, we will need more refined versions of some preliminary technical claims when proving the lower bound. We thus have to provide the proof of the upper bound from scratch.

4.3 DDM-contraction with the oracle radial rate

**Theorem 1** (oracle radial rate). Let DDM \(\mathbb{P}(\cdot | X)\) and the oracle radial rate \(R_\varepsilon(\theta)\) be defined by (15) and by (18) respectively, with \(0 \leq K_2 < K_1\) and \(L, \alpha > 0\). Then there exists a constant \(C_{or} = C_{or}(K_1, K_2, L, \alpha)\) such that, for any \(\theta_0 \in \ell_2\) and any \(M > 0\),

\[
\mathbb{E}_{\theta_0}\mathbb{P}(\|\theta - \theta_0\| \geq M R_\varepsilon(\theta_0)|X) \leq \frac{C_{or}}{M^2}.
\]

We provide a proof of this theorem in Section 5 below. One can in principle try to improve the constant \(C_{or}\) by optimizing the choices of \(K_1, K_2, L, \alpha\). As this is not our prime goal, we rather present a succinct proof than pursue the most accurate constant.

When proving the above theorem, we derive an assertion in passing which can be related to the variable selection problem. Define the \(\tau\)-proximity oracle set

\[
\mathcal{M}(\tau) = \mathcal{M}_\varepsilon(\tau, \theta_0) = \{I \in \mathbb{N} : R_\varepsilon^2(I, \theta_0) \leq \tau R_\varepsilon^2(\theta_0)\},
\]

where \(R_\varepsilon(I, \theta_0)\) and \(R_\varepsilon(\theta_0)\) are defined by (16) and (18) respectively. The following corollary follows from (44) and (48).

**Corollary 2.** There exist \(\tau > 1, C, c > 0\) (depending only on \(K_1, K_2\) and \(\alpha\)) such that, uniformly in \(\theta_0 \in \ell_2\),

\[
\mathbb{E}_{\theta_0}\mathbb{P}(I \notin \mathcal{M}(\tau)|X) \leq C \exp\{-c\varepsilon^{-2}R_\varepsilon^2(I, \theta_0)\}. \quad (19)
\]

The above statement about the variable selection can be further refined by using a technical lemma from the proof of the lower bound. The next theorem follows from Corollary 2 (applied to the case \(I > \tau I_0\)) and Lemma 3.

**Theorem 2.** Under the conditions of Lemma 3 there exist \(\varepsilon \in (0, 1), \tau > 1\) and \(C, c > 0\) (depending only on \(K_1, K_2\) and \(\alpha\)) such that, uniformly in \(\theta_0 \in \ell_2\),

\[
\mathbb{E}_{\theta_0}\mathbb{P}(I \notin [\varepsilon I_0, \tau I_0]|X) \leq C \exp\{-c I_0\}. \quad (20)
\]

In a way, this statement addresses the problem of recovering the oracle \(I_0(\theta_0)\). Very smooth \(\theta_0\)’s lead to small oracles \(I_0(\theta_0)\) so that the bound in (20) becomes not useful in this case. Thus, the theorem says basically that the DDM on \(I\) can identify a certain neighborhood of the oracle \(I_0(\theta_0)\) from the \(\mathbb{P}_{\theta_0}\)-perspective for not too smooth \(\theta_0\)’s.

**Remark 13.** Theorem 1 immediately implies the property (A1) for the DDM \(\mathbb{P}(\cdot | X)\) and \(\varphi(M) = C_{or}/M^2\), uniformly in \(\theta_0 \in \ell_2\). Proposition 1 ensures in turn that (A1) and (A3) hold as well for the default estimator \(\hat{\theta}\) defined by (13) and (18), with the measure (18) and functions \(\phi_1(M)\) and \(\phi_2(M)\) defined via \(\varphi(M)\) in Proposition 1.
Despite its seeming simplicity, Theorem 1 claims in fact a rather powerful property of the DDM \( \mathbb{P}(\cdot|X) \): from the \( \mathbb{P}_{\theta_0} \)-perspective, it contracts to \( \theta_0 \) with the oracle rate \( R_c(\theta_0) \), which is a local quantity as it depends on the true \( \theta_0 \). This means that \( \mathbb{P}(\cdot|X) \) is locally adaptive – the rate is fast for “smooth” \( \theta_0 \)-s and slow for “rough” ones. This is a stronger and more refined property than being globally adaptive. We explain this below.

To characterize the quality of a certain prior in Bayesian nonparametric theory, the notion of posterior convergence rate was introduced. Typically, the posterior convergence rate is related to the minimax risk \( R \) and more refined property than being globally adaptive. We explain this below.

\( \theta \) is related to the minimax risk \( R \) of the data \( X \) where the infimum is taken over all possible estimators \( \hat{\theta} \), measurable functions of the data \( X \).) The optimality of Bayesian procedures is then understood in the sense of adaptive minimax posterior convergence rate: given a prior, for a “true” \( \theta_0 \in \Theta \) for some \( \beta \in B \) (but this knowledge is not used in the prior), the resulting posterior contracts, from the \( \mathbb{P}_{\theta_0} \)-perspective, to \( \theta_0 \) at the minimax rate \( R_c(\Theta_\beta) \).

Let us elucidate the potential strength of a general oracle approach. Suppose we have a family of local rates \( R_c(\mathcal{A}) = \{ R_c(\alpha, \theta_0), \alpha \in \mathcal{A} \} \), e.g., in our case the family defined by (16). Let \( R_c(\theta_0) = \inf_{\alpha \in \mathcal{A}} R_c(\alpha, \theta_0) = R_c(\alpha_0, \theta_0) \) be the oracle rate and \( \alpha_0 \in \mathcal{A} \) the corresponding oracle. We say that the family of local rates \( R(\mathcal{A}) = \{ R_c(\alpha, \theta), \alpha \in \mathcal{A} \} \) covers a scale \( \Theta(B) = \{ \Theta_\beta, \beta \in B \} \) if for any \( \beta \in B \) there exists an \( \alpha = \alpha(\beta) \in \mathcal{A} \) such that \( R_c(\alpha(\beta), \theta_0) \leq c R_c(\Theta_\beta) \) for all \( \theta_0 \in \Theta_\beta \) and some uniform \( c \). Then, of course,

\[
\sup_{\theta_0 \in \Theta_\beta} R_c(\theta_0) \leq c R_c(\Theta_\beta), \quad \text{for all } \beta \in B.
\]

If \( R_c(\mathcal{A}) \) covers \( \Theta(B) \) and we have a result saying that some DDM \( \mathbb{P}(\cdot|X) \) contracts at the oracle rate \( R_c(\theta_0) \) for all \( \theta_0 \in \Theta \supseteq \Theta(B) \), then clearly \( \mathbb{P}(\cdot|X) \) also contracts with (at least) the minimax rate \( R_c(\Theta_\beta) \) for \( \theta_0 \in \Theta_\beta \). This immediately implies the adaptive (over the scale \( \Theta(B) \)) minimax contraction rate for \( \mathbb{P}(\cdot|X) \). Foremost, one oracle rate result implies adaptive minimax results simultaneously for all scales (!) that are covered by this oracle.

Thus, the usefulness and strength of the oracle approach stands or falls by, firstly, the availability of an oracle result for a family of local rates (like our Theorem 1), secondly, the range of scales that are covered by the oracle for which such a result is established. Finally we mention that the oracle (18) covers the following scales: Sobolev and analytic ellipsoids, certain scales of Besov classes and \( \ell_p \)-bodies, Sobolev hyper-rectangles, tail classes, etc. Details can be found in Birgé and Massart (2001), Cavalier and Tsybakov (2001) and Babenko and Belitser (2010).

In some sense, one could regard the DDM \( \mathbb{P}(\cdot|X) \) as an inference characteristics that “has it all”: it contracts to the true \( \theta_0 \) with the oracle local rate, it can be used for estimation (see the next subsection) and constructing confidence sets (see subsection 1.6) as \( \mathbb{P}(\cdot|X) \)-credible set, with strong optimality properties such as simultaneous adaptive minimaxity over a number of scales of nonparametric classes. It may also be used for testing, but we do not study this problem here.
4.4 Implication: the oracle property for the estimation

Recall the notation $X(I) = (X_i| i \in N_I, i \in \mathbb{N})$. Instead of default estimator $\hat{\theta}$, consider another estimator based on $\mathbb{P}(|X)$, namely,

$$\tilde{\theta} = \mathbb{E}(\theta|X) = \sum_I X(I)\mathbb{P}(I = I|X),$$  \hspace{1cm} (21)

which is nothing else but the (random) expectation with respect to the DDM $\mathbb{P}(|X)$ defined by (15). This estimator turns out to possess the oracle property for the estimation problem, as the following theorem claims.

**Theorem 3** (oracle estimation inequality). *Let $\theta_0 \in \ell^2$ and $\tilde{\theta}$ be defined by (21). Then there exist an absolute constant $C_{est} = C_{est}(K_1, K_2, L, \alpha) \geq 1$ such that

$$\mathbb{E}_{\theta_0} \|\tilde{\theta} - \theta_0\|^2 \leq C_{est} R^2_\varepsilon(I_0, \theta_0).$$  \hspace{1cm} (22)

We omit the proof of this theorem, it is based on the preliminary technical results for the previous theorem and follows the same lines as the corresponding result from Babenko and Belitser (2010).

**Remark 14.** Clearly, (A1) and (A3) follow from Theorems 1 and 3 for the estimator (21), $\phi_1(M) = (C_{or} + C_{est})/M^2$ and $\phi_2(M) = C_{est}/M^2$, uniformly in $\theta_0 \in \ell^2$. In fact, any other estimator that satisfy the oracle inequality in the above theorem will do the job, for example, a blockwise Stein’s estimator from Cavalier and Tsybakov (2001) or a penalized estimator from Birgé and Massart (2001).

The local rate $R_\varepsilon(I_0, \theta_0)$ is also the $\ell^2$-risk of the projection estimator $\hat{\theta}(I) = X(I)$:

$$\mathbb{E}_{\theta_0} \|\hat{\theta}(I) - \theta_0\|^2 = R^2_\varepsilon(I_0, \theta_0).$$

One can regard the oracle rate (18) as the smallest possible risk over the family of (projection) estimators $\hat{\theta}(N) = \{\hat{\theta}(I), I \in \mathbb{N}\}$, namely

$$R^2_\varepsilon(I_0, \theta_0) = \inf_{I_0 \in \mathbb{N}} \mathbb{E}_{\theta_0} \|\hat{\theta}(I_0) - \theta_0\|^2 = \mathbb{E}_{\theta_0} \|\hat{\theta}(I_0) - \theta_0\|^2 = \varepsilon^2 I_o + \sum_{i \in N_{I_0}} \theta_{0,i}^2.$$  \hspace{1cm} (22)

The corresponding estimator $\hat{\theta}(I_0)$ is called the *oracle estimator* over the family $\hat{\Theta}(N)$. One should remember that the oracle estimator is not really an estimator as it depends on the true $\theta_0$ through $I_o = I_o(\theta_0)$. The interpretation of the oracle $I_o(\theta_0)$ is that it selects the first $I_o$ most significant variables. Birgé and Massart (2001) call this *ordered variable selection*, this is a suitable strategy when the unknown $\theta_0$ possesses some relevant smoothness structure, e.g., belongs to some (unknown) ellipsoid. The estimation result of Theorem 3 was obtained earlier for a blockwise Stein’s estimator by Cavalier and Tsybakov (2001) and for a penalized estimator by Birgé and Massart (2001), both papers treat this problem within a more general framework.

By mimicking the oracle estimator (this is what Theorem 3 claims), we solve a number of minimax adaptive estimation problems simultaneously over the following scales: scales
of Sobolev ellipsoids, certain scales of Besov ellipsoids, some scales of hyper-rectangles, tail classes, etc.; examples of scales can be found in Birg´e and Massart (2001), Cavalier and Tsybakov (2001) and Babenko and Belitser (2010). It is instructive to give the complete description of all scales of classes for which the minimax adaptive estimation problem is solved by mimicking the ordered variable selection oracle: these are all scales \( \{ \Theta_\beta, \beta \in B \} \) that are covered by the estimators family \( \hat{\Theta}(N) \); cf. the discussion after Theorem 1. We say that a family of estimators \( \hat{\Theta}(A) = \{ \hat{\Theta}(\alpha), \alpha \in A \} \) covers a scale \( \Theta(B) = \{ \Theta_\beta, \beta \in B \} \) if for any \( \beta \in B \) there exists an \( \alpha = \alpha(\beta) \in A \) such that

\[
\sup_{\theta \in \Theta_\beta} E_{\theta} \| \hat{\Theta}(\alpha) - \theta \|_2 \leq C_\varepsilon \inf_{\hat{\theta}} \sup_{\theta \in \Theta_\beta} E_{\theta} \| \hat{\theta} - \theta \|_2,
\]

ideally with \( C_\varepsilon = 1 + o(1) \) as \( \varepsilon \to 0 \), otherwise with a uniform constant \( 1 \leq C_\varepsilon = C < \infty \). Basically, this means that the family \( \hat{\Theta}(A) \) contains the minimax estimators over the scale \( \Theta(B) \).

4.5 Some further remarks

Remark 15. Notice that the oracle (18) cannot take zero value, as we minimize over \( N \). Therefore, the oracle risk \( R^2_\varepsilon(\theta) \geq \varepsilon^2 \) and \( N_0 \geq 1 \). This is not restrictive since if we allow the oracle to be zero, Theorems 1 and 3 will hold only for the oracle rate with an additive penalty term, a multiple of the parametric rate \( \varepsilon^2 \). This boils down to the same resulting rate.

Remark 16. In the right hand side of the relation (17), a tighter exponential bound with respect to \( M \) is possible (based on the exponential large deviation bound for the \( \chi^2 \)-distribution), which would lead to exponential type functions \( \varphi, \phi_1 \) and \( \phi_2 \) in conditions (A1), (A1) and (A3). We however use the simplistic bound obtained by Markov inequality in (17) for the sake of a succinct presentation.

Remark 17. The data dependent distribution (15) is introduced and studied in Babenko and Belitser (2010) for values \( L = 1/2 \) and \( K_1 = 1 \) and \( \alpha \in \left[ \frac{1}{n} - \log \left( \frac{2}{\sqrt{m}} \right), \frac{1}{2} \right] \). For arbitrary \( K_1, \alpha > 0, K_2 = 0 \) and \( L = \frac{K_1+1}{K_1} > 1 \), (15) can be interpreted as the posterior distribution of \( L \theta \) with respect to the prior \( \pi_{\alpha,K_1,0} \) defined by (14). In view of Theorem 1, this means that from the \( \mathbb{P}_\theta \)-perspective the actual posterior of \( \theta \) contracts to \( \theta_0/L \) and not to \( \theta_0 \). This has to do with the over-shrinkage effect of (mixtures of) normal priors towards the prior mean. This is discussed at length by Silverman and Johnstone (2004), Babenko and Belitser (2010) and Castillo and van der Vaart (2012). The approaches in the first and third papers are based on (mixtures of) heavy-tailed priors instead of normal. However, as discussed by Babenko and Belitser (2010), there are several relatively easy ways to fix this issue whereas still keeping the normality in the first level of prior (14). For example, one can adjust \( \mathbb{P}_I(\cdot|X) \)-part by empirical Bayes approach. Yet another approach is to add one more level of hierarchy in (14) by putting a heavy-tailed prior on variances \( \tau^2_{I} \) (although the latter will of course destroy the normal conjugate structure of the prior).

Interestingly, the over-shrinkage effect is present only in the \( \mathbb{P}_I(\theta|X) \)-part in (15) and not in the \( \mathbb{P}(\mathcal{Z} = I|X) \)-part which turns out to be robust with respect to this effect. This is
because the oracle $I_o$ is the same for both the true parameter $\theta_0$ and its (in fact, arbitrary) multiple $L\theta_0$.

Remark 18. Other choices for $\mathbb{P}(I|X)$ and $\mathbb{P}(I = I|X)$ in (13) are possible. For example, $\mathbb{P}(I|X) = \bigotimes N(X_i|I(i \in N_I), \sigma_i^2(\varepsilon^2, I))$ with $c\varepsilon^2 \leq \sum_i \sigma_i^2(\varepsilon^2, I) \leq CI\varepsilon^2$ for some $0 < c < C$. If we only were interested in satisfying (A1) for the resulting $\mathbb{P}(\cdot|X)$, the best choice would be $\sigma_i^2(\varepsilon^2, I) = 0$ (or $L = 0$ in (11)), which would however make (A2) impossible to hold. Non-normal distributions in the construction of $\mathbb{P}(I|X)$ are also possible. However, when establishing (A2) below, we need to deal with a small ball probability, which is a relatively well studied problem for the Gaussian distribution. For a non-normal case, one would first have to derive small ball probability results.

Other choices for random mixing probabilities $\mathbb{P}(I = I|X)$ in (13) can do the job of mimicking the oracle as well. We mention here one more possible construction as an empirical Bayes posterior with respect to a certain family of normal priors in the scheme (14): for some $K > 0$, define $\rho_\varepsilon = \varepsilon^{-2}/(2 + 2K)$ and

$$\mathbb{P}(I = I|X) = \frac{e^{-\alpha I} \bigotimes I \in \mathbb{N}_I \exp\{-\rho_\varepsilon X_i^2\}}{\sum_J e^{-\alpha J} \bigotimes I \in \mathbb{N}_J \exp\{-\rho_\varepsilon X_i^2\}}, \quad I \in \mathbb{N}. \tag{23}$$

Indeed, take $\mu_i(I) = \mu_i \{i \in I\}$ and $\tau_i^2(I) = K\varepsilon^2$ in the prior (14), and let the corresponding posterior of $I$ be $\pi_\mu(I = I|X)$, $I \in \mathbb{N}$, with $\mu = (\mu_i(I), i, I \in \mathbb{N})$. Then, as is easy to see, $\mathbb{P}(I = I|X) = \pi_{\mu(I)}(I = I|X)$, where $\mu(I) = (\mu_i(I), i \in \mathbb{N})$ with $\mu_i(I) = X_i \{i \in \mathbb{N}_I\}$, the empirical Bayes estimator obtained by maximizing the marginal likelihood $\mathbb{P}_{X,\mu}$ with respect to $\mu$. The quantity (23) clearly exists as almost sure limit (as $n \to \infty$) of

$$\mathbb{P}_n(I = I|X) = \frac{e^{-\alpha I} \bigotimes I = I_{j+1} \exp\{-\rho_\varepsilon X_i^2\}}{\sum_{J=1}^n e^{-\alpha J} \bigotimes I \in \mathbb{N}_J \exp\{-\rho_\varepsilon X_i^2\}}$$

$$= \left(1 + \sum_{J=1}^{I-1} e^{\alpha(I-j)} \exp\left\{-\rho_\varepsilon \sum_{i=J+1}^{I} X_i^2\right\} \right)^{-1},$$

since $\mathbb{P}_n(I = I|X) \in [0, 1]$ and is decreasing in $n \geq I$ almost surely.

Remark 19. We can extend the idea of covering (introduced in subsection 4.3) to two different families of local rates. We say that a family of local rates $\mathcal{R}_\varepsilon(\mathcal{A}) = \{\mathcal{R}_\varepsilon^{\alpha}(\mathcal{A}, \alpha), \alpha \in \mathcal{A}\}$ covers another family of local rates $\mathcal{R}_\beta(\mathcal{B}) = \{\mathcal{R}_\varepsilon^{\beta}(\mathcal{B}, \beta), \beta \in \mathcal{B}\}$ over some $\Theta_0$ if for each $\theta \in \Theta_0$ and $\beta \in \mathcal{B}$ there exists an $\alpha = \alpha(\theta, \beta)$ such that for some uniform constant $c = c(\Theta_0, \mathcal{A}, \mathcal{B})$

$$\mathcal{R}_\varepsilon^{\alpha}(\mathcal{A}, \alpha) \leq c \mathcal{R}_\beta^{\beta}(\mathcal{B}, \beta).$$

If $\Theta_0$ contains the set of interest (e.g., $\Theta_0$ is the whole space), then a DDM-contraction result with an oracle rate with respect to the family $\mathcal{A}$ will imply the corresponding result for the family $\mathcal{B}$. For example, our family of local rates $\mathcal{R}_\varepsilon(\mathbb{N}) = \{\mathcal{R}_\varepsilon(I, \theta), I \in \mathbb{N}\}$ defined
by \([\ref{14}]\) covers the family \(\mathcal{R}_{1,\varepsilon}(\mathbb{R}_+) = \{\mathcal{R}_{1,\varepsilon}(\Lambda(\alpha), \theta), \alpha > 0\}\), where \(\Lambda(\alpha) = \{\lambda_i(\alpha), i \in \mathbb{N}\}\) and
\[
\mathcal{R}_{1,\varepsilon}^2(\Lambda(\alpha), \theta) = \sum_i (1 - \lambda_i(\alpha))^2 \theta_i^2 + \varepsilon^2 \sum_i \lambda_i^2(\alpha) \quad \text{with} \quad \lambda_i(\alpha) = \frac{i^{-2(\alpha+1)}}{\varepsilon^2 + i^{-2(\alpha+1)}}.
\]
This is the family of the risks of the minimax estimators over Sobolev balls of smoothness \(\alpha \in \mathbb{R}_+\). This is also the family of posterior convergence rates for the prior \(\pi_\alpha = \bigotimes_i N(0, i^{-2(\alpha+1)})\); cf. Belitser and Ghosal (2003), Szabó, van der Vaart and van Zanten (2014) (with \(p = 0\)). In fact, \(\mathcal{R}_{1,\varepsilon}(\mathbb{R}_+)\) covers even the richer family \(\mathcal{R}_{2,\varepsilon}(\Lambda) = \{\mathcal{R}_{2,\varepsilon}(\Lambda, \theta), \Lambda = (\lambda_i, i \in \mathbb{N} : \lambda_k \geq \lambda_{k+1}, k \in \mathbb{N}\}\). Indeed, for \(N_\lambda = \max\{i : \lambda_i \geq 1/2\}\),
\[
\mathcal{R}_{2,\varepsilon}^2(\Lambda, \theta) = \varepsilon^2 \sum_i \lambda_i^2 + \sum_i (1 - \lambda_i)^2 \theta_i^2 \geq \frac{N_\lambda \varepsilon^2}{4} + \frac{1}{4} \sum_{i=N_\lambda+1}^{\infty} \theta_i^2 = \frac{1}{4} \mathcal{R}_{2,\varepsilon}^2(N_\lambda, \theta).
\]
The family \(\mathcal{R}_{2,\varepsilon}(\Lambda)\) contains also the family of risks of the (asymptotically sharp) minimax Pinskers estimators and the family of risks of the Tikhonov regularization estimators (these correspond to spline estimators in the problem of curve estimation).

**Remark 20.** To give an idea how the proposed framework can be used in other (equivalent) models, we outline a possible approach to the discrete regression model \([\ref{9}]\) under conditions of Remark \([\ref{11}]\) 1) consider the discrete regression model \([\ref{9}]\) and assume that the unknown signal \(f\) belongs to a Besov ball \(B^{s}_{p,q}(Q)\) with an unknown smoothness \(s;\) 2) apply discrete wavelet transform to the data \(Y = (Y_i, i \in \mathbb{N}_n)\) from \([\ref{9}]\) to obtain the data \(X\) of form \([\ref{8}]\); 3) obtain the DDM \(\mathbb{P}(\theta|X)\) in terms of the data \(X;\) 4) transform this DDM back to the DDM \(\mathbb{P}(f|Y)\) for the signal \(f\) dependent on the data \(Y\) from \([\ref{9}]\).

Then this resulting random measure \(\mathbb{P}(f|Y)\) will concentrate around the true \(f_0\) from the \(\mathbb{P}_{f_0}\)-perspective at least with the optimal minimax rate corresponding to the smoothness \(s\). It will take a fair piece of effort to implement this outline approach in details, but conceptually it is a straightforward matter.

### 4.6 Fixing (A2) by the excessive bias restriction (EBR)

As we mentioned in the introduction, it has been realized by many researchers that in general it is impossible to construct optimal (fully) adaptive confidence set in the minimax sense with a prescribed high coverage probability. Clearly, the same problem should occur for the local approach \([\ref{1}]\), because otherwise we would have solved the minimax counterpart of the problem as well. The intuition behind this phenomenon is as follows. Informally, there are so called “deceptive” parameters \(\theta_0\) that “trick” the DDM \(\mathbb{P}(-|X)\) in the sense that the resulting random radius \(\hat{r}\) defined by \([\ref{23}]\) is overoptimistic, i.e., of a smaller order than the actual radial rate \(\mathcal{R}_{\varepsilon}(\theta_0)\), which makes the coverage probability too small.

Namely, one can prove that \(\hat{r}^2\) is always of the order \(\varepsilon^2 I_o(\theta_0)\) which is just the variance term of the oracle estimator \(\hat{\theta}(I_o(\theta_0))\) (see \([\ref{22}]\) or \([\ref{18}]\)), but not the whole oracle risk \(\mathcal{R}_{\varepsilon}^2(I_o, \theta_0)\). The DDM \(\mathbb{P}(-|X)\) can recover the oracle variance from the data, but not the
oracle bias and this is where the mismatch can occur. If the oracle bias $\sum_{i \in I_o(\theta)} \theta_{0,i}^2$ in (22) is of a bigger order than the oracle variance, $\hat{r}$ is then of a smaller order than it should be, which leads to a small coverage probability of the resulting confidence ball. A deceptive $\theta_0$ “pretends” to be smooth (small oracle variance, detected by the DDM), but it is not really (big oracle bias, not detected by the DDM).

One way to fix this problem is to remove a set (preferably, minimal) of deceptive parameters from the set $\Theta$ (in our case $\ell_2$) and derive the coverage relation in (11) for the remaining set of non-deceptive parameters. In a different framework, Picard and Tribouley (2000) introduced such a set, the so called self-similar parameters. Giné and Nickl (2010), Hoffmann and Nickl (2011), Bull (2012), Bull and Nickl (2013), Szabó, van der Vaart and van Zanten (2014a), Nickl and Szabó (2014) studied self-similar parameters in the wavelet context for different norms and different models. A somewhat restrictive feature of the self-similarity property is that it is linked to the Besov (Sobolev) functional scale.

Below is a version of the $\ell_2$ definition of self-similar parameters adopted to the Sobolev scale, taken from Szabó, van der Vaart and van Zanten (2014a). A parameter $\theta \in \Theta_\beta(Q) = \{\theta \in \ell_2 : \sum_i i^{2\beta} \theta_i^2 \leq Q\}$ is self-similar if, for some $\gamma, N_0 > 0$ and $\rho \geq 2$, $\theta \in \Theta_{ss}$, where

$$\Theta_{ss}(\beta) = \Theta_{ss}(\beta, Q, \gamma, N_0, \rho) = \left\{ \theta \in \Theta_\beta(Q) : \sum_{i=N}^{\rho N} \theta_i^2 \geq \gamma Q N^{-2\beta} \text{ for all } N \geq N_0 \right\}.$$ 

Next, we define $\Theta_{ss} = \bigcup_{\beta \in [\beta_{\min}, \beta_{\max}]} \Theta_{ss}(\beta)$ for some $0 < \beta_{\min} < \beta_{\max} < \infty$. In a way, the set $\Theta_{ss}$ consists of the “edges” of the Sobolev ellipsoids $\Theta_\beta(Q)$, $\beta \in [\beta_{\min}, \beta_{\max}]$.

**Remark 21.** Instead of Sobolev ellipsoids $\Theta_\beta(Q)$ one can use Sobolev hyper-rectangles $H_\beta(Q) = \{\theta \in \ell_2 : \sup_{i \in \mathbb{N}} i^{2\beta+1} \theta_i^2 \leq Q\}$, with the same conclusions, but different constants.

Szabó, van der Vaart and van Zanten (2014a) introduced a more general condition that is not linked to a particular smoothness classes scale, the polished tail (PT) condition: for $L_0, N_0 > 0$ and $\rho \geq 2$,

$$\Theta_{pt} = \Theta_{pt}(L_0, N_0, \rho) = \left\{ \theta \in \ell_2 : \sum_{i=N}^{\infty} \theta_i^2 \leq L_0 \sum_{i=N}^{\rho N} \theta_i^2, \text{ for all } N \geq N_0 \right\}.$$ 

For any $\gamma, N_0 > 0$ and $\rho \geq 2$ there exist $L_0, N_0' > 0$ and $\rho' \geq 2$ such that $\Theta_{ss}(\beta, Q, \gamma, N_0, \rho)$ is contained in $\Theta_{pt}(L_0, N_0', \rho')$. Clearly, one can take $N_0' = N_0, \rho' = \rho$ and $L_0 = \gamma^{-1}$. We express this property as $\Theta_{ss} \subseteq \Theta_{pt}$.

Introduce the excessive bias restriction (notation: EB or EBR): for a $\tau > 0$,

$$\Theta_{eb} = \Theta_{eb}(\tau) = \Theta_{eb}(\tau, \varepsilon) = \left\{ \theta \in \ell_2 : \sum_{i \in \mathbb{N}, \theta_i(\theta)} \theta_i^2 \leq \tau \varepsilon^2 I_0(\theta, \varepsilon) \right\},$$ 

where $I_0(\theta, \varepsilon)$ is defined by (18). Note that in principle $\Theta_{eb}$ also depends on $\varepsilon$ as we consider non-asymptotic setting. However for asymptotic considerations (as $\varepsilon \to 0$) we can introduce a uniform (in $\varepsilon$) version of EBR:

$$\Theta_{eb}(\tau, \varepsilon_0) = \left\{ \theta \in \Theta_{eb}(\tau, \varepsilon) \text{ for all } \varepsilon \in (0, \varepsilon_0) \right\} = \cap_{\varepsilon \in (0, \varepsilon_0]} \Theta_{eb}(\tau, \varepsilon).$$
We will not consider $\Theta_{eb}(\tau, \varepsilon_0)$ and $\Theta_{eb}(\tau, \varepsilon)$ separately and will always use the latter notation $\Theta_{eb}(\tau)$ for both in what follows, with the understanding that whenever one needs the uniform version, one can think of $\Theta_{eb}(\tau)$ as $\Theta_{eb}(\tau, \varepsilon_0)$, as all assertions below hold for the both versions of EBR.

Basically, EBR means that the bias term may not dominate (up to a constant factor) the variance term in the oracle risk. In a way, the parameters from $\Theta_{eb}$ are “typical” $\theta$'s: the oracle bias is of the same order as the oracle variance. Atypical ones have atypically excessive biases and these are removed from $\ell_2$ once we assume $\theta \in \Theta_{eb}$. In this case tricking the DDM is not possible anymore. Another way of looking at this condition is that the oracle risk is estimable from the data for $\theta \in \Theta_{eb}$.

Let us show that EB is less restrictive than PT, i.e., for any $L_0 > 0$, $N_0 \geq 1$ and $\rho \geq 2$, there exists a $\tau > 0$ such that $\Theta_{pt}(L_0, N_0, \rho) \subseteq \Theta_{eb}(\tau)$. Consider first $N_0 \leq 2$. If $\theta \in \bigcup_{N_0 \in \{1,2\}} \Theta_{pt}(L_0, N_0, \rho)$, then, according to the definition of the oracle $\sum_{i=L_o+1}^\infty \theta_i^2 \leq \varepsilon^2(I - I_o)$ for any $I > I_o$ so that

$$
\sum_{i=L_o+1}^\infty \theta_i^2 \leq L_0 \sum_{i=L_o+1}^\infty \rho(I_o + 1) - I_o \leq 2L_0\rho \varepsilon^2 I_o,
$$

(24)
as $I_o \geq 1$. We established that $\bigcup_{N_0=1}^2 \Theta_{pt}(L_0, N_0, \rho) \subseteq \Theta_{eb}(2L_0\rho)$. If $\theta \in \Theta_{pt}(L_0, N_0, \rho)$ for $N \geq 3$, a uniform inclusion is also possible:

$$
\sum_{i=L_o+1}^\infty \theta_i^2 = \sum_{i=L_o+1}^{N_0I_o-1} \theta_i^2 + \sum_{i=N_0I_o}^\infty \theta_i^2 \leq (N_0 - 1)\varepsilon^2 I_o + L_0 \sum_{i=N_0I_o}^{\rho N_0I_o} \theta_i^2
$$

$$
\leq (N_0 - 1)\varepsilon^2 I_o + L_0(\rho N_0 - N_0 + 1)\varepsilon^2 I_o \leq (N_0 - 1 + L_0((\rho - 1)N_0 + 1))\varepsilon^2 I_o,
$$

so that $\Theta_{pt}(L_0, N_0, \rho) \subseteq \Theta_{eb}(N_0 - 1 + L_0((\rho - 1)N_0 + 1))$ for any $N_0 \geq 3$.

Remark 22. Actually, for any $\theta \in \ell_2$, $I_o(\theta)$ becomes sufficiently large for sufficiently small $\varepsilon \leq \varepsilon_0$ so that $I_o(\theta) \geq N_0$ and (23) holds, but such $\varepsilon_0$ depends on $\theta$. Alternatively, one can define the oracle over the set $\{N_0, N_0 + 1, \ldots\}$ instead of $\mathbb{N}$, then there will be an additive penalty in the oracle risk of (parametric) order $N_0\varepsilon^2$.

To summarize the relations between three types of conditions describing non-deceptive parameters introduced above,

$$
\Theta_{ss} \subseteq \Theta_{pt} \subseteq \Theta_{eb}.
$$

Thus $\Theta_{eb}$ is the biggest set of non-deceptive parameters. As to the question how big (or “typical”) that set of non-deceptive parameters is, Szabó, van der Vaart and van Zanten (2014a) give three types of arguments for the PT-parameters, topological, minimax and Bayesian. Our set $\Theta_{eb} \supseteq \Theta_{pt}$, thus the same arguments certainly apply to $\Theta_{eb}$. We will not discuss this any further, but refer to the paper Szabó, van der Vaart and van Zanten (2014a).
Theorem 4 (small ball DDM-probability). Let the DDM $\mathbb{P}(\cdot | X) = \mathbb{P}_{K_1,K_2,\alpha,L}(\cdot | X)$ be given by (12) and let the constants $0 \leq K_2 < K_1$ and $\alpha > 0$ be chosen in such a way that

$$\rho \overset{\text{def}}{=} \frac{K_1 - K_2}{2(K_1 K_2 + 2 K_1 + 1)} - \frac{1}{2} \log \left( \frac{(K_1 + 1)^2}{K_1 K_2 + 2 K_1 + 1} \right) - \alpha > 0. \tag{25}$$

Then there exists a constant $C_{c_b} = C_{c_b}(K_1, K_2, \alpha, L) > 0$ such that, for every $\theta_0 \in \ell_2$, any DDM-center $\hat{\theta} = \hat{\theta}(X)$ and any $\delta \in (0, \min\{1, \bar{C}_\delta\}]$,

$$\mathbb{E}_{\theta_0} \mathbb{P}(\|\theta - \hat{\theta}\| \leq \delta \sqrt{\varepsilon^2 I_0 | X}) \leq C_{c_b} \delta \sqrt{\log(\delta^{-1})},$$

where the oracle $I_o = I_o(\theta_0)$ is defined by (18) and $\bar{C}_\delta = \left( \frac{L \rho}{2(\alpha + \rho) e} \right)^{1/2}$.

Remark 23. The effective radial rate in the above lower bound is determined by just the variance term $\varepsilon^2 I_o$ of the oracle risk rate $R^2_{\varepsilon}(\theta_0)$.

The EB restriction says basically that the variance term is the main term in the oracle risk. Under $\theta_0 \in \Theta_{c_b}(\tau)$, we have $R^2_{\varepsilon}(\theta_0) \leq (1 + \tau)\varepsilon^2 I_o$, where the oracle $I_o = I_o(\theta_0)$ is defined by (18). This leads to the following corollary.

Corollary 3. There exists a constant $C_{c_b} = C_{c_b}(K_1, K_2, \alpha, L, \tau) > 0$ such that

$$\sup_{\theta_0 \in \Theta_{c_b}(\tau)} \mathbb{E}_{\theta_0} \mathbb{P}(\|\theta - \hat{\theta}\| \leq \delta R_{\varepsilon}(\theta_0) | X) \leq C_{c_b} \delta \sqrt{\log(\delta^{-1})}$$

for any estimator $\hat{\theta} = \hat{\theta}(X)$ and any $0 < \delta \leq (1 + \tau)^{-1} \min\{1, C_\delta\}$, with $C_\delta = \left( \frac{L \rho}{2(\alpha + \rho) e} \right)^{1/2}$.

Clearly, the above assertion implies condition (A2) for the DDM $\mathbb{P}(\cdot | X)$ satisfying (22), with $\psi(\delta) = C_{c_b} \delta \sqrt{\log(\delta^{-1})}$, uniformly in $\theta_0 \in \Theta_{c_b}(\tau)$.

Remark 24. There is not much room for choosing constants $0 \leq K_2 < K_1$ and $\alpha > 0$ satisfying (24). For example, it is easy to show that $\rho \leq 0$ for any $\alpha > 0$ (i.e., (25) is impossible to fulfill) if $K_1 > K_2 \geq 1$. The largest value for $\rho + \alpha$ is approximately 0.0686 and attained in $K_1 = 0.39$ and $K_2 = 0$. One can choose, for example, $K_1 = 0.39$, $K_2 = 0$, $\alpha = 0.0346$, so that $\rho \approx 0.034$.

The larger the parameter $\tau$, the larger the constant $C_{c_b}$. On the other hand, $C_{c_b}$ can be made smaller by choosing a larger $L$. This in turn would increase the constant $C_{or}$ in Theorem 1 (cf. Remark 18 where we mentioned the best choice $L = 0$ to make the constant $C_{or}$ as small as possible, but this choice would make $C_{c_b}$ infinite).

4.7 Confidence ball under EBR

In this subsection we establish the main result of the paper. First we construct the DDM $\mathbb{P}(\cdot | X) = \mathbb{P}_{K_1,K_2,\alpha,L}(\cdot | X)$ given by (15), with constants $0 \leq K_2 < K_1$ and $\alpha > 0$ such that (25) is satisfied. Let us regard the constant $K_1, K_2, \alpha, L$ to be fixed throughout this subsection.
Next, by using this DDM, we construct the DD-center \( \hat{\vartheta} \) and the confidence ball \( B(\hat{\vartheta}, M\hat{r}_\kappa) \) defined by (21) and (4) respectively.

As we already mentioned in Section 4.4, conditions (A1) and (A3) are fulfilled for the estimator \( \hat{\vartheta} \) defined by (21), with \( \phi_1(M) = 4(C_{or} + C_{est})/M^2 \) and \( \phi_2(M) = C_{est}/M^2 \) and uniformly in \( \theta_0 \in \ell_2 \). Indeed, by Theorems 1 and 3

\[
\mathbb{E}_{\theta_0} \left[ \mathcal{P}(\| \theta - \hat{\vartheta} \| \geq M R_\varepsilon(\theta_0) | X) \right] \leq \mathbb{E}_{\theta_0} \left[ \mathcal{P}(\| \theta - \theta_0 \| \geq \frac{1}{2} M R_\varepsilon(\theta_0) | X) \right] + \mathbb{E}_{\theta_0} \left[ \mathcal{P}(\| \theta - \hat{\vartheta} \| \geq \frac{1}{2} M R_\varepsilon(\theta_0) | X) \right] \leq \frac{4(C_{or} + C_{est})}{M^2} = \phi_1(M),
\]

\[
\mathbb{P}_{\theta_0} \left( \| \theta_0 - \hat{\vartheta} \| \geq M R_\varepsilon(\theta_0) \right) \leq \mathbb{E}_{\theta_0} \left[ \| \theta_0 - \hat{\vartheta} \| \leq \frac{C_{est}}{M^2} = \phi_2(M). \right.
\]

**Remark 25.** Alternatively, since Theorem 1 implies (A1) with \( \varphi(M) = \frac{C_{or}}{M^2} \), we could use the default confidence ball \( \bar{B}_M \) defined by (7). Then by Theorem 1 and Proposition 1 (A1) and (A3) would be fulfilled for the default estimator \( \hat{\vartheta} \) given by (9), with (see Remark 8) \( \phi_1(M) = \frac{4C_{or}}{M^2} \) and \( \phi_2(M) = \frac{C_{est}}{M^2} \), uniformly in \( \theta_0 \in \ell_2 \).

Let us bound the coverage probability of the confidence ball \( B(\hat{\vartheta}, M\hat{r}_\kappa) \). In view of Corollary 4, we conclude that condition (A2) is met with \( \psi(\delta) = C_{eb}\delta\sqrt{\log(\delta^{-1})} \), uniformly in \( \theta_0 \in \Theta_{eb}(\tau) \). Recall that, according to (27), (A3) is also fulfilled with \( \phi_2(M) = C_{est}/M^2 \), uniformly in \( \theta_0 \in \ell_2 \). By applying Proposition 2 we derive that, for any \( M, \delta > 0 \)

\[
\sup_{\theta_0 \in \Theta_{eb}(\tau)} \mathbb{P}_{\theta_0} \left( \theta_0 \not\in B(\hat{\vartheta}, M\hat{r}_\kappa) \right) \leq \phi_2(M\delta) + \psi(\delta) \leq \phi_2(\kappa M^2) + C_{eb}\delta\sqrt{\log(\delta^{-1})}.
\]

For \( \alpha_1 \in (0,1) \) and \( C_\delta \) defined in Theorem 4 we take

\[
\delta = \max \{ \delta \in [0,1], \min\{1, C_\delta\} : C_{eb}(1 - \kappa)^{-1}\delta\sqrt{\log(\delta^{-1})} \leq \alpha_1/2 \}
\]

and \( M_1 = \min\{M \in \mathbb{N} : C_{est}/(M\delta)^2 \leq \alpha_1/2 \} \). Then, for all \( M \geq M_1 \),

\[
\sup_{\theta_0 \in \Theta_{est}(\tau)} \mathbb{P}_{\theta_0} \left( \theta_0 \not\in B(\hat{\vartheta}, M\hat{r}_\kappa) \right) \leq \alpha_1. \quad \text{(28)}
\]

Now, since condition (A1) takes the form (26), applying Proposition 3 yields that the size \( \hat{r}_\kappa \) (given by (3)) of the confidence ball \( B(\hat{\vartheta}, M\hat{r}_\kappa) \) is of the oracle rate order: for any \( M > 0 \) and every \( \theta_0 \in \ell_2 \),

\[
\mathbb{P}_{\theta_0} \left( \hat{r}_\kappa \geq M R_\varepsilon(\theta_0) \right) \leq \frac{\phi_1(M)}{\kappa} = \frac{4(C_{or} + C_{est})}{\kappa M^2}.
\]

For \( \alpha_1 \in (0,1) \), take \( M_2 = \min\{M \in \mathbb{N} : 4(C_{or} + C_{est})/(\kappa M^2) \leq \alpha_2 \} \). Then for all \( \theta_0 \in \ell_2 \) and \( M \geq M_2 \)

\[
\mathbb{P}_{\theta_0} \left( \hat{r}_\kappa \geq M R_\varepsilon(\theta_0) \right) \leq \alpha_2. \quad \text{(29)}
\]

Finally, by combining (28) and (29), we obtain the main result of the paper. 27
Theorem 5. Let the DDM \( \mathbb{P}(\cdot|X) = \mathbb{P}_{K_1,K_2,a,L}(\cdot|X) \) be given by (17), with constants \( 0 \leq K_2 < K_1 \) and \( \alpha > 0 \) such that (22) is fulfilled. Further, let the DD-center \( \hat{\theta} \) and the confidence ball \( B(\hat{\theta}, M_{\hat{r}_n}) \) be defined by (21) and (4) respectively. Then for any \( \alpha_1, \alpha_2 \in (0,1) \) there exist \( C_0 = C_0(\alpha_1, \tau) \) and \( c_0 = c_0(\alpha_2) \) such that, for all \( C \geq C_0 \) and \( c \geq c_0 \), the relations (11) hold for the ball \( B(\hat{\theta}, C\hat{r}) \), the radial rate \( R(x)(\theta_0) \) defined by (18), \( \Theta'_0 = \ell_2 \) and \( \Theta_0 = \Theta_{e_0}(\tau) \).

4.8 Alternative confidence set under EBR

In this subsection we consider another construction of confidence ball.

Define the estimator of the oracle

\[
\hat{I} = \min \{ I \in \mathbb{N} : \mathbb{P}(I = I|X) = \max_j \mathbb{P}(I = J|X) \}. \tag{30}
\]

The following assertion holds for the estimator \( \hat{I} \).

Proposition 5. For any \( \theta_0 \in \ell_2 \) and any \( I, I_0 \in \mathbb{N} \)

\[
\mathbb{P}_{\theta_0}(\hat{I} = I) \leq \frac{\lambda_I}{\lambda_{I_0}} \prod_{i=1}^{\infty} \left[ \frac{a_i(I)}{a_i(I_0)(1 + \varepsilon^2 a_i(I, I_0))} \right]^{1/2} \exp \left\{ -\frac{1}{2} \sum_{i=1}^{\infty} \frac{a_i(I, I_0)\theta_0^2}{1 + \varepsilon^2 a_i(I, I_0)} \right\},
\]

where \( a_i \)'s are defined by (33).

Proof. By the definition of \( \hat{I} \), we derive in the same way as in the proof of Lemma 1

\[
\mathbb{P}_{\theta_0}(\hat{I} = I) \leq \mathbb{P}_{\theta_0} \left( \mathbb{P}(I = I|X) \right) \leq \mathbb{E}_{\theta_0} \left[ \mathbb{P}(I = I|X) \right] 
= \frac{\lambda_I}{\lambda_{I_0}} \prod_{i=1}^{\infty} \left( \frac{a_i(I)}{a_i(I_0)} \right)^{1/2} \mathbb{E}_{\theta_0} \left[ \sum_{i=1}^{\infty} X_i a_i(I, I_0) \right] 
= \frac{\lambda_I}{\lambda_{I_0}} \prod_{i=1}^{m} \left[ \frac{a_i(I)}{a_i(I_0)(1 + \varepsilon^2 a_i(I, I_0))} \right]^{1/2} \exp \left\{ -\frac{1}{2} \sum_{i=1}^{m} \frac{a_i(I, I_0)\theta_0^2}{1 + \varepsilon^2 a_i(I, I_0)} \right\}. \tag{32}
\]

Remark 26. In view of Proposition 5 we can derive the assertions for \( \hat{I} \) analogous to (19) and (20). Namely, there exist \( \tau > 1, C, c > 0 \) such that, uniformly in \( \theta_0 \in \ell_2 \),

\[
\mathbb{P}_{\theta_0}(R^2_\varepsilon(I, \theta_0) > \tau R^2_\varepsilon(I_o) \) \leq C \exp \left\{ -c \varepsilon^{-2} R^2_\varepsilon(I, \theta_0) \right\}.
\]

Under the conditions of Lemma 3 there exist \( \varepsilon \in (0,1), \tau > 1 \) and \( C, c > 0 \) such that, uniformly in \( \theta_0 \in \ell_2 \),

\[
\mathbb{P}_{\theta_0}(\hat{I} \notin [\varepsilon I_o, \tau I_o]) \leq C \exp\{ -c I_o \}, \tag{31}
\]

where the oracle \( I_o = I_o(\theta_0) \) is defined by (18). All the above constants depend only on \( K_1, K_2, \alpha \).
The idea is that \( \tilde{r}^2 = \varepsilon^2 \hat{I} \) is a good estimator of \( \varepsilon^2 I_0 \) which in turn is of order \( \mathcal{R}_e^2(\theta_0) \) under EBR. This leads to the alternative confidence ball \( B(\hat{\theta}, M\tilde{r}) \), with sufficiently large \( M, \hat{\theta} \) and \( \tilde{r} \) defined by \( \eqref{1} \) and \( \eqref{2} \) respectively. We expect that \( \eqref{1} \) is satisfied also for the confidence ball \( B(\hat{\theta}, M\tilde{r}) \) and \( \Theta_0 = \Theta_{ss} \), possibly with some extra conditions on the constants \( K_1, K_2, \alpha \). We will not pursue this here, an interested reader can try to derive relevant results along the same lines as in the proof of Theorem 5, using the fact that \( \mathbb{P}_{\theta_0}(\hat{I} = I) \) is bounded from above by the same quantity as \( \mathbb{E}_{\theta_0}\mathbb{P}(\hat{I} = I|X) \) in Lemma \( \eqref{3} \).

Instead, we consider self-similar parameters \( \Theta_0 = \Theta_{ss} \) as we can derive some interesting implications for the problem of estimating the smoothness of self-similar \( \theta_0 \), cf. Szabó, van der Vaart and van Zanten (2014a). If \( \theta_0 \in \Theta_{ss}(\beta) \) for some \( \beta > 0 \), then it is easy to show that

\[
\begin{align*}
c_r(\varepsilon^2)^{2\beta/(2\beta + 1)} & \leq \mathcal{R}_e^2(\theta_0) \leq c_r(\varepsilon^2)^{2\beta/(2\beta + 1)}, \\
c_I(\varepsilon^2)^{-1/(2\beta + 1)} & \leq I_0 \leq c_I(\varepsilon^2)^{-1/(2\beta + 1)}.
\end{align*}
\]

(32)

The following follows immediately from Theorem 3, \( \eqref{1} \) and \( \eqref{2} \).

**Theorem 6.** For any \( \alpha_1, \alpha_2 \in (0,1) \) there exist positive \( C_0, c_0 \) and \( \varepsilon_0 \) depending on \( K_1, K_2, \alpha, L \) and the parameters of \( \Theta_{ss}(\beta) \) such that, for all \( C \geq C_0, c \geq c_0 \) and \( \varepsilon \leq \varepsilon_0 \), the relations \( \eqref{1} \) hold for the confidence ball \( B(\hat{\theta}, C\tilde{r}) \) and \( \Theta_0 = \Theta_{ss}(\beta) \), where \( \tilde{r}^2 = \varepsilon^2 \hat{I} \), \( \hat{\theta} \) and \( \tilde{r} \) are defined by \( \eqref{1} \) and \( \eqref{2} \) respectively.

**Remark 27.** Besides, from \( \eqref{1} \) and \( \eqref{2} \) it follows that the smoothness \( \beta \) of the parameter \( \theta_0 \in \Theta_{ss}(\beta) \) is actually estimable in the sense that, for \( \hat{\beta} = \frac{1}{2}(\log(\varepsilon^{-2}) - 1) \), uniformly in \( \theta_0 \in \ell_2 \),

\[
\mathbb{P}_{\theta_0}(\beta - \frac{c_1}{\log(\varepsilon^{-2})} \leq \hat{\beta} \leq \beta + \frac{c_2}{\log(\varepsilon^{-2})}) \leq C \exp\{-ce^{-2/(2\beta + 1)}\},
\]

for some \( c_1, c_2, C, c > 0 \) depending on \( K_1, K_2, \alpha \) and the parameters of \( \Theta_{ss}(\beta) \); cf. Szabó, van der Vaart and van Zanten (2014a) where a Bayesian approach is used with a prior that specifically targets the Sobolev scale structure. For the Sobolev scale, the result of Theorem 6 for our DDM-credible ball is similar to the corresponding result from Szabó, van der Vaart and van Zanten (2014a) for their credible ball in the case of direct observations. In that paper, a (mildly) ill-posed inverse signal-in-white-noise model was considered. It is possible to get the same results for our DDM-credible ball in the ill-posed case as in the above paper, but with an extra log factor and the proof is to be done essentially from scratch.

**Remark 28.** Define the estimator \( \hat{\theta} = X(\hat{I}) = (X_i I_i | i \in \mathbb{N}_J, i \in \mathbb{N}) \). This estimator can be related to the penalized projection estimator from Birgé and Massart (2001). Indeed, one can easily derive that

\[
\hat{I} = \arg \min_I \left\{ \sum_{i \in \mathbb{N}_J} \frac{X_i^2}{2\varepsilon^2(1 + K_1)} - \sum_{i \in \mathbb{N}_J} \frac{X_i^2}{2\varepsilon^2(1 + K_2)} + \alpha I \right\}
\]

\[
= \arg \min_I \left\{ -\sum_{i \in \mathbb{N}_J} X_i^2 + K\varepsilon^2 I \right\}, \quad \text{with} \quad K = \frac{2\alpha(1 + K_1)(1 + K_2)}{K_1 - K_2}.
\]
If the DDM $P(I = I|X)$ in (30) is defined by (23), we again obtain $\hat{I}$ as above, but with different constant $K$.

5 Proofs

5.1 Proof of Theorem 1

We prove Theorem 1 in several steps.

Step 1: a technical lemma. We start with a technical lemma, which concerns the DDM $P(I = I|X)$. For $I, I_0, i \in \mathbb{N}$, introduce the quantities

$$a_i(I) = (\tau_i^2(I) + \varepsilon^2)^{-1}, \quad a_i(I, I_0) = a_i(I) - a_i(I_0),$$

where $\tau_i^2(I)$ is defined by (12).

Lemma 1. For any $I, I_0 \in \mathbb{N}$ and any $\theta_0 \in \ell^2$,

$$\mathbb{E}_{\theta_0} P(I = I|X) \leq \frac{\lambda_I}{\lambda_{I_0}} \prod_i \left[ \frac{a_i(I)}{a_i(I_0)(1 + \varepsilon^2 a_i(I, I_0))} \right] \exp \left\{ -\frac{1}{2} \sum_{i=1}^{\infty} a_i(I, I_0) \theta_{0i}^2 \right\},$$

where $a_i(I)$ and $a_i(I, I_0)$ are defined by (33), and $\lambda_I$ is defined by (14).

Proof. The martingale convergence theorem and the dominated convergence theorem yield

$$\mathbb{E}_{\theta_0} P(I = I|X) = \lim_{m \to \infty} \mathbb{E}_{\theta_0} P(I = I|X_1, \ldots, X_m).$$

Since the conditional marginal $P(X_1, \ldots, X_m|I = I) = \otimes_{i=1}^{m} N(0, \tau_i^2(I) + \varepsilon^2)$,

$$P(I = I|X_1, \ldots, X_m) = \frac{\lambda_I \prod_{i=1}^{m} (\tau_i^2(I) + \varepsilon^2)^{-1/2} \exp \left\{ -\frac{X_i^2}{2(\tau_i^2(I) + \varepsilon^2)} \right\}}{\sum_{J} \lambda_J \prod_{i=1}^{m} (\tau_i^2(J) + \varepsilon^2)^{-1/2} \exp \left\{ -\frac{X_i^2}{2(\tau_i^2(J) + \varepsilon^2)} \right\}}.$$

Using this relation and the elementary identity

$$\mathbb{E} \left( \exp \left( -b Y^2 / 2 \right) \right) = (1 + b \sigma^2)^{-1/2} \exp \left\{ -\frac{\mu^2 b}{2(1 + b \sigma^2)} \right\}$$

for $Y \sim N(\mu, \sigma^2)$ and $b > -\sigma^{-2}$, we derive

$$\mathbb{E}_{\theta_0} P(I = I|X_1, X_2, \ldots, X_m) \leq \frac{\lambda_I}{\lambda_{I_0}} \prod_{i=1}^{m} \left[ \frac{a_i(I)}{a_i(I_0)(1 + \varepsilon^2 a_i(I, I_0))} \right] \exp \left\{ -\frac{1}{2} \sum_{i=1}^{m} X_i^2 a_i(I, I_0) \right\} \exp \left\{ -\frac{1}{2} \sum_{i=1}^{m} a_i(I, I_0) \theta_{0i}^2 \right\} \text{if } a_i(I, I_0) \geq -\frac{\varepsilon^2 (K_1 - K_2)}{(K_1 + 1)(K_2 + 1)} > -\varepsilon^2 = -\sigma^{-2}.$$
Step 2: corollaries from Lemma \[11\] Substituting \( \tau_1^2(I) \)'s defined by \[12\], \( \lambda_I = C_\alpha e^{-\alpha I} \) (with \( C_\alpha = e^{\alpha} - 1 \)) and \( I_0 = I_0(\theta_0) \) given by \[13\] in the right hand side of the inequality of Lemma \[1\] yields the following relations: for \( I < I_0 \),

\[
\mathbb{E}_{\theta_0} \mathbb{P}(I = I|X) \leq e^{-\alpha I} \exp \left\{ - A_1 \varepsilon^{-2} \sum_{i=I+1}^{I_0} \theta_{0,i}^2 + (A_2 + \alpha)I_o - A_2I \right\},
\]

with \( A_1 = K_1 - K_2 \)
\( \frac{2(K_1K_2 + 2K_1 + 1)}{2(K_1K_2 + 2K_2 + 1)} \)
\( A_2 = \frac{1}{2} \left( \frac{(K_1 + 1)^2}{K_1K_2 + 2K_1 + 1} \right) \); for \( I > I_0 \),

\[
\mathbb{E}_{\theta_0} \mathbb{P}(I = I|X) \leq e^{-\alpha I/2} \exp \left\{ B_1 \varepsilon^{-2} \sum_{i=I_0+1}^{I} \theta_{0,i}^2 - (B_2 + \alpha/2)I_o + (B_2 + \alpha)I_o \right\},
\]

with \( B_1 = K_1 - K_2 \)
\( \frac{2(K_1K_2 + 2K_2 + 1)}{2(K_1K_2 + 2K_2 + 1)} \)
\( B_2 = -\frac{1}{2} \log \left( \frac{(K_2 + 1)^2}{K_1K_2 + 2K_2 + 1} \right) \).

Note that \( A_1, A_2, B_1, B_2 + \alpha/2 > 0 \) since \( 0 \leq K_2 < K_1 \) and \( \alpha > 0 \).

Recall that \( e^{-2R_\varepsilon(I, \theta_0)} = I + e^{-2} \sum_{i \in \mathbb{N}_I} \theta_{0,i}^2 \). From \[36\] it follows that, for \( I < I_0 \),

\[
\mathbb{E}_{\theta_0} \mathbb{P}(I = I|X) \leq e^{-\alpha I} \exp \left\{ - \left( A_1 \varepsilon^{-2} \sum_{i=I_0+1}^{I} \theta_{0,i}^2 + A_2 I_o \right) + A_1 \sum_{i \in \mathbb{N}_I} \theta_{0,i}^2 + (A_2 + \alpha)I_o \right\}
\]

\[
\leq e^{-\alpha I} \exp \left\{ - A_3 \varepsilon^{-2} R_\varepsilon(I, \theta_0) + \max \{ A_1, A_2 + \alpha \} e^{-2} R_\varepsilon(I_o) \right\}
\]

\[
= e^{-\alpha I} \exp \left\{ - A_3 \varepsilon^{-2} (R_\varepsilon(I, \theta_0) - A_4 R_\varepsilon(I_o)) \right\},
\]

where \( A_3 = \min \{ A_1, A_2 \} \), \( A_4 = \max \{ A_1, A_2 + \alpha \} / A_3 \). Similarly, from \[37\] it follows that, for \( I > I_0 \),

\[
\mathbb{E}_{\theta_0} \mathbb{P}(I = I|X) \leq e^{-\alpha I/2} \exp \left\{ - B_3 \varepsilon^{-2} (R_\varepsilon(I, \theta_0) - B_4 R_\varepsilon(I_o)) \right\}.
\]

\( B_3 = \min \{ B_1, B_2 + \alpha / 2 \} \), \( B_4 = \max \{ B_1, B_2 + \alpha \} / B_3 \).

Step 3: introducing a partition. For any \( \tau_1, \tau_2 > 0 \), introduce a partition of \( \mathbb{N} = N^- (\tau_1) \cup O (\tau_1, \tau_2) \cup N^+ (\tau_2) \), where \( O (\tau_1, \tau_2) = O^-(\tau_1) \cup O^+(\tau_2) \),

\[
O^-(\tau_1) = \{ I \in \mathbb{N} : I \leq I_o, R_\varepsilon^2(I, \theta_0) \leq \tau_1 R_e^2(I_o) \}, \quad N^- (\tau_1) = N_{I_o} \setminus O^- (\tau_1),
\]

\[
O^+(\tau_2) = \{ I \in \mathbb{N} : I > I_o, R_\varepsilon^2(I, \theta_0) \leq \tau_2 R_e^2(I_o) \}, \quad N^+ (\tau_2) = N^c_{I_o} \setminus O^+ (\tau_2).
\]

Step 4: a bound by the sum of three terms. As \( \mathbb{P}_I (\cdot | X) = \bigotimes_{i \in \mathbb{N}_I} N(X_i, \mathbb{I} \{ i \in \mathbb{N}_I \}) \), we obtain by applying the Markov inequality that

\[
\mathbb{P}_I (\| \theta - \theta_0 \| \geq M R_e(I_o, \theta_0)) \leq \frac{\mathbb{E}_I (\| \theta - \theta_0 \|^2 | X)}{M^2 R_e^2(I_o)}
\]

\[
= \frac{L^2 + \varepsilon^2 \sum_{i \in \mathbb{N}_I} \theta_{0,i}^2 + \sum_{i \in \mathbb{N}_I} (X_i - \theta_{0,i})^2}{M^2 R_e^2(I_o)} \leq \frac{L_1 R_e^2(I, \theta_0) + \varepsilon^2 \sum_{i \in \mathbb{N}_I} \xi_i^2}{M^2 R_e^2(I_o)} = \psi_I,
\]

(40)
where \(L_1 = \max\{L, 1\}\) and \(\xi_i = \varepsilon^{-1}(X_i - \theta_{0,i})\) ind \(N(0, 1)\) from the \(P_{\theta_0}\)-perspective. Denote for brevity \(p_I = P(I = I|X)\), and \(v_I\) is defined by (40). In view of (15) and (10),

\[
\mathbb{P}(\|\theta - \theta_0\| \geq M\mathcal{R}_\varepsilon(I_0(\theta_0)),_X) \leq \sum_I v_I p_I = T_1 + T_2 + T_3,
\]

where \(T_1 = \sum_{I \in \mathcal{N}^{-}((\tau_1)} v_I p_I, T_2 = \sum_{I \in \mathcal{O}(\tau_1, \tau_2)} v_I p_I, T_3 = \sum_{I \in \mathcal{N}^{+}(\tau_2)} v_I p_I\).

**Step 5: handling the term \(T_2\).** Let \(I_{\max} = \max\{I \in \mathcal{O}(\tau_1, \tau_2)\} = \max\{I \in \mathcal{O}^+(\tau_2)\}\), then \(\mathcal{R}_\varepsilon^2(I_{\max}, \theta_0) = \varepsilon^2 I_{\max} + \sum_{I \in \mathcal{N}_{\max}} \theta_{0,i}^2 \leq \tau_2 \mathcal{R}_\varepsilon^2(I_0)\) so that \(\varepsilon^2 I_{\max} \leq \tau_2 \mathcal{R}_\varepsilon^2(I_0)\). Since \(\sum_I p_I = \sum_I P(I = I|X) = 1\),

\[
T_2 = \sum_{I \in \mathcal{O}(\tau_1, \tau_2)} v_I p_I \leq \max\{v_I : I \in \mathcal{O}(\tau_1, \tau_2)\} \leq \frac{L_1 \max\{\tau_1, \tau_2\}}{M^2} + \frac{\varepsilon^2 \sum_{i \in \mathcal{N}_{\max}} \xi_i^2}{M^2 \mathcal{R}_\varepsilon^2(I_0)}
\]

which implies

\[
\mathbb{E}_{\theta_0} T_2 \leq \frac{L_1 \max\{\tau_1, \tau_2\}}{M^2} + \frac{\varepsilon^2 I_{\max}}{M^2 \mathcal{R}_\varepsilon^2(I_0)} \leq \frac{L_1 \max\{\tau_1, \tau_2\} + \tau_2}{M^2}.
\]

**Step 6: handling the term \(T_1\).** Since \(I \leq I_0\) for \(I \in \mathcal{N}^{-}(\tau_1)\),

\[
T_1 = \sum_{I \in \mathcal{N}^{-}(\tau_1)} v_I p_I \leq \frac{L_1 \sum_{I \in \mathcal{N}^{-}(\tau_1)} \mathcal{R}_\varepsilon^2(I, \theta_0) p_I + \varepsilon^2 \sum_{i \in \mathcal{N}_{\max}} \xi_i^2}{M^2 \mathcal{R}_\varepsilon^2(I_0)}.
\]

Fix some \(h \in (0, 1)\) and take \(\tau_1 = \frac{A_4}{-h}\). If \(I \in \mathcal{N}^{-}(\tau_1)\), then \(\mathcal{R}_\varepsilon^2(I, \theta_0) \geq \frac{A_4}{1-h} \mathcal{R}_\varepsilon^2(I_0)\). Thus,

\[
\mathcal{R}_\varepsilon^2(I, \theta_0) - A_4 \mathcal{R}_\varepsilon^2(I_0) = h \mathcal{R}_\varepsilon^2(I, \theta_0) + (1-h) \mathcal{R}_\varepsilon^2(I, \theta_0) - A_4 \mathcal{R}_\varepsilon^2(I_0) \geq h \mathcal{R}_\varepsilon^2(I, \theta_0).
\]

From the last relation and (35), it follows that, for \(I \in \mathcal{N}^{-}(\tau_1)\),

\[
\mathbb{E}_{\theta_0} p_I \leq e^{-\alpha I} \exp\{-A_3\varepsilon^{-2} h \mathcal{R}_\varepsilon^2(I, \theta_0)\}.
\]

Using (43), \(\max_{x>0}\{xe^{-cx}\} \leq (ce)^{-1}\) (for any \(c > 0\)) and \(\sum I e^{-\alpha I} = C^{-1}\), we obtain

\[
\sum_{I \in \mathcal{N}^{-}(\tau_1)} \mathcal{R}_\varepsilon^2(I, \theta_0) \mathbb{E}_{\theta_0} p_I \leq \sum_{I \in \mathcal{N}^{-}(\tau_1)} e^{-\alpha I} \mathcal{R}_\varepsilon^2(I, \theta_0) \exp\{-A_3\varepsilon^{-2} h \mathcal{R}_\varepsilon^2(I, \theta_0)\}
\]

\[
\leq (eA_3h)^{-1}\varepsilon^2 \sum_{I \in \mathcal{N}^{-}(\tau_1)} e^{-\alpha I} \leq (eC\alpha A_3 h)^{-1}\varepsilon^2.
\]

Take \(h = 1/2\) so that \(\tau_1 = 2A_4\). Then the relations (15), (15), \(\varepsilon^2 \leq \varepsilon I_0 \leq \mathcal{R}_\varepsilon^2(I_0)\) imply

\[
\mathbb{E}_{\theta_0} T_1 \leq \frac{L_1 (eC\alpha A_3 h)^{-1}\varepsilon^2 + \varepsilon^2 I_0}{M^2 \mathcal{R}_\varepsilon^2(I_0)} \leq \frac{2L_1 (eC\alpha A_3 I_0)^{-1} + 1}{M^2} \leq \frac{2L_1 (eC\alpha A_3)^{-1} + 1}{M^2}.
\]
Step 7: handling the term $T_3$. We have

$$T_3 = \sum_{I \in \mathcal{N}^+(\tau_2)} v_I p_I \frac{L_1 \sum_{I \in \mathcal{N}^+(\tau_2)} \mathcal{R}_e^2(I, \theta_0) p_I + \varepsilon^2 \sum_{I \in \mathcal{N}^+(\tau_2)} p_I \sum_{i \in \mathcal{N}_I} \xi_i^2}{M^2 \mathcal{R}_e^2(I_0)}.$$  \hfill (47)

For some $h \in (0,1)$, let $\tau_2 = \frac{B_3}{1-h}$. Using (42) and $\sum_I e^{-aI/2} = C_{a/2}$, we derive in the same way as for (44) and (45) that, for $I \in \mathcal{N}^+(\tau_2)$,

$$\mathbb{E}_{\theta_0} p_I \leq e^{-aI/2} \exp \left\{ -B_3 \varepsilon^{-2h} \mathcal{R}_e^2(I, \theta_0) \right\} \leq \sum_{I \in \mathcal{N}^+(\tau_2)} \mathcal{R}_e^2(I, \theta_0) \mathbb{E}_{\theta_0} p_I \leq (eC_{a/2}B_3h)^{-1} \varepsilon^2.$$  \hfill (48)

For $\delta = 1/2$ in (48) and (49) so that $\tau_2 = 2B_4$. By using (48) and the fact $\sum_{k=m}^{\infty} (k+1) e^{-ck} = \frac{m e^{-cm}}{1-e^{-c}} + \frac{e^{-cm}}{(1-e^{-c})^2}$, we evaluate

$$\mathbb{E}_{\theta_0} \sum_{I \in \mathcal{N}^+(\tau_2)} p_I \sum_{i \in \mathcal{N}_I} \xi_i^2 \leq \sum_{I \in \mathcal{N}^+(\tau_2)} \left( \mathbb{E}_{\theta_0} p_I \right)^{1/2} \left[ \mathbb{E}_{\theta_0} \left( \sum_{i \in \mathcal{N}_I} \xi_i^2 \right)^2 \right]^{1/2} \leq \sum_{I \in \mathcal{N}^+(\tau_2)} (I + 1) \left( \mathbb{E}_{\theta_0} p_I \right)^{1/2} \leq \sum_{I = I_0}^{\infty} e^{-aI/4} (I + 1) \exp \left\{ -B_3 \varepsilon^{-2} \mathcal{R}_e^2(I, \theta_0) / 2 \right\} \leq \sum_{I = I_0}^{\infty} e^{-aI/4} (I + 1) \exp \left\{ -B_3 I / 2 \right\} = \sum_{I = I_0}^{\infty} (I + 1) \exp \left\{ -IB_3 \right\} \leq e^{-B_5 I_0} + \frac{e^{-B_5 I_0}}{(1 - e^{-B_5})^2} = g(I_0, \alpha, K_1, K_2) \leq \frac{(eB_5)^{-1}}{1 - e^{-B_5}} + \frac{e^{-B_5}}{(1 - e^{-B_5})^2} = B_6,$$  \hfill (50)

where $B_5 = (\alpha + 2B_3)/4$. Combining (47), (49), (50), $\varepsilon^{-2} \mathcal{R}_e^2(I, \theta_0) \geq I > I_0$ for $I \in \mathcal{N}^+(\tau_2)$, we evaluate

$$\mathbb{E}_{\theta_0} T_3 \leq \frac{2L_1 (eC_{a/2}B_3)^{-1} + B_6}{M^2}.$$  \hfill (51)

Step 8: finalizing the proof. Piecing together the relations (61), (72), (66) and (51),

$$\mathbb{E}_{\theta_0} \mathbb{P} \left( \| \theta - \theta_0 \| \geq M \mathcal{R}e(I_0, \theta_0) \right) \leq \mathbb{E}_{\theta_0} \left( T_1 + T_2 + T_3 \right) \leq \frac{C_{or}}{M^2}.$$  \hfill (61)

The constant $C_{or} = C_{or}(K_1, K_2, L, \alpha)$ can be calculated as follows:

$$C_{or} = 2L_1 \max\{ A_4, B_4 \} + 2B_4 + 2L_1 (eC_{a}A_3)^{-1} + 1 + 2L_1 (eC_{a/2}B_3)^{-1} + B_6,$$

where $L_1 = \max\{ L, 1 \}$, $C_{a} = e^a - 1$, $A_3 = \min\{ A_1, A_2 \}$, $A_4 = \max\{ A_1, A_2 + \alpha \}/A_3$, $B_3 = \min\{ B_1, B_2 + \alpha/2 \}$, $B_4 = \max\{ B_1, B_2 + \alpha \}/B_3$, $B_6$ is defined by (50), $A_1$ and $A_2$ are defined by (30), $B_1$ and $B_2$ are defined by (31).
5.2 Proof of Theorem 4

First we provide one basic technical result. Denote by \( \Lambda(S) \) the Lebesgue measure (or volume) of a set \( S \subseteq \mathbb{R}^k \) and by \( B_k(r) = \{ x \in \mathbb{R}^k : \| x \|^2 \leq r^2 \} \) (here \( \| \cdot \| \) is the usual Euclidean norm in \( \mathbb{R}^k \)) the Euclidean ball of radius \( r \) in space \( \mathbb{R}^k, k \in \mathbb{N} \).

Lemma 2.

\[
\Lambda(B_k(r)) \leq e\pi^{-1/2}r\pi^{k/2}k^{-(k+1)/2}(2e)^{k/2}.
\]

Proof of Lemma 3. By using Stirling’s approximation for the Gamma function \( \Gamma(x) = \sqrt{2\pi}x^{x-1/2}e^{-x}x^{1/12x} \) for all \( x \geq 1 \) and some \( 0 \leq \varrho \leq C \), we derive

\[
\Gamma(1 + k/2) = \sqrt{2\pi} \left(1 + \frac{k}{2}\right)^{(k+1)/2} e^{-1-k/2+\varrho/(6k+12)}
\]

\[
= \frac{(1+2/k)^{(k+1)/2}\sqrt{\pi} k^{(k+1)/2}}{e^{1-\varrho/(6k+12)}} (2e)^{-k/2}
\]

\[
= c_k k^{(k+1)/2}(2e)^{-k/2} \geq c k^{(k+1)/2}(2e)^{-k/2},
\]

where

\[
c_k = \frac{(1+2/k)^{(k+1)/2}\sqrt{\pi}}{e^{1-\varrho/(6k+12)}} > \frac{\sqrt{\pi}}{c} = c.
\]

(Actually, \( c_k \to \sqrt{\pi} \) as \( k \to \infty \) and more accurate estimates \( c_k > c \) are possible.) Combining the last relation with the well known fact

\[
\Lambda(B_k(r)) = r^k \Lambda(B_k(1)) = \frac{r^k \pi^{k/2}}{\Gamma(1 + k/2)}
\]

completes the proof of the lemma.

The next lemma is also needed in the proof of Theorem 4.

Lemma 3. Let the measure \( P(I = I[X], I \in \mathbb{N}, \) be given by \( (13) \) and let the constants \( 0 < K_2 < K_1 \) and \( \alpha > 0 \) be chosen in such a way that \( (25) \) is satisfied with some \( \rho = \rho(K_1, K_2, \alpha) > 0 \). Then for every \( \theta_0 \in \ell_2 \) and any \( h \in [0, 1] \)

\[
E_{\theta_0} P(I \leq \varepsilon h I_o | X) \leq C_\alpha^{-1} e^{-\rho I_o},
\]

where \( \varepsilon_h = (1-h)\rho/\alpha + \rho, C_\alpha = e^\alpha - 1 \), and the oracle \( I_o = I_o(\theta_0) \) is defined by \( (15) \).

Proof of Lemma 3. Recall the constants \( A_1, A_2 \) defined by \( (25) \), whereas \( (25) \) means that \( A_1 - A_2 = \alpha + \rho \). By the oracle definition \( (15) \), \( \mathcal{R}_2^2(I, \theta_0) \geq \mathcal{R}_2^2(I_o, \theta_0) \) for any \( \theta_0 \in \ell_2 \). For \( I < I_o \), this implies \( e^{-2} \sum_{i=I+1}^{I_o} \theta_{0,i}^2 \geq I_o - I \). In particular, for \( I \leq \varepsilon h I_o < I_o \) we obtain

\[
A_1 e^{-2} \sum_{i=I+1}^{I_o} \theta_{0,i}^2 - A_2(I_o - I) \geq (A_1 - A_2) (I_o - I) \geq (A_1 - A_2) (1 - \varepsilon_h) I_o
\]

\[= (\rho + \alpha)(1 - \varepsilon_h) I_o = (\rho h + \alpha) I_o,\]

34
where we used (25) and \( x_h = \frac{(1-h)\rho}{\alpha + \rho} \). The lemma follows from (32) and the fact that \( \sum I \lambda_I = 1: \)

\[
\mathbb{E}_{\theta_0} P(I \leq x_I | X) \leq \sum_{I \leq x_I} \frac{\lambda_I}{\lambda_I'} \exp \left\{ - A_1 \varepsilon^{-2} \sum_{i=I+1}^{I_0} \theta_{0,i}^2 + A_2 (I_0 - I) \right\}
\]

\[
\leq \sum_{I \leq x_I} \frac{\lambda_I}{\lambda_I'} \exp \left\{ -(\rho h + \alpha) I \right\} \leq C_{\alpha}^{-1} \sum_{I \leq x_I} \lambda_I \exp \left\{ -\rho \lambda I \right\} \leq C_{\alpha}^{-1} e^{-\rho h I_o}.
\]

\[\square\]

**Finalizing the proof of Theorem 4**. Let \( Z_1, \ldots, Z_I \) be independent \( N(0, 1) \) random variables and \( \xi = (\xi_i)_{i \in \mathbb{N}} \) with \( \xi_i = \varepsilon^{-1}(X_i - \theta_0) \) from the model (5), so that \( \xi_i \)'s are independent standard normal random variables under \( X \sim \mathbb{P}_{\theta_0} \). Further recall

\[
P_I(\theta | X) = \bigotimes_i N(X_i I \{ i \in \mathbb{N}_I \}, L \varepsilon^2 I \{ i \in \mathbb{N}_I \}), \quad I \in \mathbb{N}.
\]

Using this, Anderson's inequality and Lemma 2, we obtain that, with \( \mathbb{P}_{\theta_0} \)-probability 1,

\[
P_I(\| \theta - \hat{\theta} \| \leq \delta \sqrt{\varepsilon^2 I_o} | X) = P_I \left( \sum_{i \in \mathbb{N}_I} (X_i + \sqrt{L} \varepsilon Z_i - \hat{\theta}_{i})^2 + \sum_{i \in \mathbb{N}_I} \hat{\theta}_i^2 \leq \delta^2 \varepsilon^2 I_o | X \right)
\]

\[
\leq \mathbb{P} \left\{ L \varepsilon^2 \sum_{i \in \mathbb{N}_I} Z_i^2 \leq \delta^2 \varepsilon^2 I_o \right\} \leq \mathbb{P} \left\{ \sum_{i \in \mathbb{N}_I} Z_i^2 \leq \frac{\delta^2 I_o}{L} \right\} \leq (2\pi)^{-1/2} \lambda(B_l(\delta \sqrt{I_o/L}))
\]

\[
\leq (2\pi)^{-1/2} \frac{e}{\sqrt{\pi}} \left( \frac{\delta^2 I_o}{L} \right)^{1/2} \pi^{1/2} \Gamma(1/2)(2e)^{1/2} = \frac{e}{\sqrt{\pi L}} \left( \frac{\delta^2 I_o}{L} \right)^{1/2}.
\]

(52)

Consider two cases \( e^{-\rho I_o/2} \leq \delta \) and \( e^{-\rho I_o/2} > \delta \) (which are not void as \( 0 < \delta \leq 1 \)), where the oracle \( I_o = I_o(\theta_0) \) is defined by (15).

First consider the case \( e^{-\rho I_o/2} > \delta \). Then \( I_o < 2\rho^{-1} \log(\delta^{-1}) \). By using this and (52), we derive that, for \( e^{-\rho I_o/2} > \delta \),

\[
\mathbb{E}_{\theta_0} P\{ \| \theta - \hat{\theta} \| \leq \delta \sqrt{\varepsilon^2 I_o} | X \} = \mathbb{E}_{\theta_0} \sum_I P_I(\| \theta - \hat{\theta} \| \leq \delta \sqrt{\varepsilon^2 I_o} | X) \mathbb{P}(I = I | X)
\]

\[
\leq \sum_I \frac{e}{\sqrt{\pi I}} \left( \frac{\delta^2 I_o}{L I} \right)^{1/2} \mathbb{E}_{\theta_0} \mathbb{P}(I = I | X)
\]

\[
\leq C_2 \delta \sqrt{\log(\delta^{-1})} \sum_I \frac{(C_1 \delta^2 \log(\delta^{-1})^{(I-1)/2}}{I(I+1)/2} \mathbb{E}_{\theta_0} \mathbb{P}(I = I | X)
\]

\[
\leq C_3 \delta \sqrt{\log(\delta^{-1})},
\]

(53)

where \( C_1 = 2e/(\rho L) \), \( C_2 = e(C_1/\pi)^{1/2} \). Let us evaluate the constant \( C_3 \). For \( a > 0 \),

\[
h(a) = \max_{k \in \mathbb{N}} \frac{a^{(k-1)/2}}{k^{(k+1)/2}} \leq \mathbb{I}\{a \leq 8 \} + \frac{e a^{(2e)/2(2a)^{1/2}}}{(2a)^{1/2}} \mathbb{I}\{a > 8 \},
\]

(54)
for every \( \theta \) of the data where the infimum is taken over all possible estimators \( \hat{\theta} \), we can take

\[
C_4 = C_2 \left( \mathbb{I}\{C_1 \leq 8e\} + \frac{e^{a/(2e)}}{(2a)^{1/2}} \mathbb{I}\{C_1 > 8e\} \right) \leq C_2 \max \left\{ 1, \frac{e^{C_1/(2a^2) + 1/2}}{(2C_1)^{1/2}} \right\}.
\]

Now consider the case \( e^{-\rho L_o/2} \leq \delta \). Take \( \kappa = \frac{\rho}{2(\alpha + \rho)} \) (which is actually \( \kappa_{1/2} \) from Lemma 3), with \( \rho \) defined by (25). Applying Lemma 3 with \( h = 1/2 \), we obtain

\[
\mathbb{E}_{\theta_0} \mathbb{P}(I < \kappa I_o|X) \leq C_\alpha^{-1} e^{-\rho L_o/2}
\]

for every \( \theta_0 \in \ell_2 \). In view of (52) and (55),

\[
\mathbb{E}_{\theta_0} \mathbb{P}\left\{ \|\theta - \hat{\theta}\| \leq \delta \sqrt{\varepsilon^2 I_o|X} \right\} = \mathbb{E}_{\theta_0} \sum_I \mathbb{P}_I(\|\theta - \hat{\theta}\| \leq \delta \sqrt{\varepsilon^2 I_o|X}) \mathbb{P}(I = I|X)
\]

\[
\leq \sum_{I \geq \kappa I_o} \frac{e^{\frac{e^{\delta^2 I_o}}{\pi L}}}{\sqrt{\pi L}} \mathbb{E}_{\theta_0} \mathbb{P}(I = I|X) + \mathbb{E}_{\theta_0} \mathbb{P}(I < \kappa I_o|X)
\]

\[
\leq C_4 \delta \sum_{I} \left( \frac{e^{\frac{e^{\delta^2 L}}{\pi L}}}{\sqrt{\pi L}} \right)^{1/2} \mathbb{E}_{\theta_0} \mathbb{P}(I = I|X) + C_\alpha^{-1} e^{-\rho L_o/2} \leq (C_4 + C_\alpha^{-1}) \delta
\]

if \( e^{-\rho L_o/2} \leq \delta \) and \( \frac{e^{\delta^2 L}}{\pi L} \leq 1 \). Here \( C_4 = e\left( \frac{\delta^2 L}{\pi L} \right)^{1/2} = \left( \frac{e^{3(\alpha + \rho)}}{\pi L \rho} \right)^{1/2} \).

The last relation holds if \( e^{-\rho L_o/2} \leq \delta \leq \left( \frac{L_o}{\epsilon} \right)^{1/2} = \left( \frac{L_o}{\pi(\alpha + \rho)} \right)^{1/2} = C_\delta \) and the relation (53) holds if \( e^{-\rho L_o/2} > \delta \). Combining these two concludes the proof of the theorem:

\[
\mathbb{E}_{\theta_0} \mathbb{P}\left\{ \|\theta - \hat{\theta}\| \leq \delta \sqrt{\varepsilon^2 I_o|X} \right\} \leq \max\{C_3, C_4 + C_\alpha^{-1}\} \delta \sqrt{\log(\delta^{-1})},
\]

for \( 0 < \delta \leq \min\{1, C_\delta\} \).

6 Appendix: minimax adaptive confidence sets

Optimality is a well developed notion in the framework of minimax estimation theory and therefore our first approach to optimality of confidence sets will be based on the minimax convergence rates.

Suppose our prior knowledge about the model is formalized as follows: the unknown parameter (sometimes we call it signal or curve) \( \theta \in \Theta_\beta \subseteq \Theta, \beta \in \mathcal{B} \). Parameter \( \beta \in \mathcal{B} \) typically has a meaning of smoothness of \( \theta \). If we want to estimate a parameter \( \theta \in \Theta_\beta \) in a minimax setup, we measure the quality of an estimator \( \hat{\theta} = \hat{\theta}(X) \) by a risk function \( R_\varepsilon(\hat{\theta}, \theta) = \mathbb{E}_d(\hat{\theta}, \theta) \). Then a benchmark in this statistical problem is the minimax risk

\[
R_\varepsilon(\Theta_\beta) = \inf_{\hat{\theta}} R_\varepsilon(\hat{\theta}, \Theta_\beta) = \inf_{\hat{\theta}} \sup_{\theta \in \Theta_\beta} R_\varepsilon(\hat{\theta}, \theta),
\]

where the infimum is taken over all possible estimators \( \hat{\theta} = \hat{\theta}(X) \in \mathcal{L} \), measurable functions of the data \( X \). For example, in classical nonparametric regression model and density
estimation problem with Sobolev, Hölder or Besov classes $\Theta_\beta$ of $d$-variate functions of smoothness $\beta$ and the sample size $n$, the minimax risk $R_\varepsilon(\Theta_\beta) = \varepsilon^{-43/(2\beta+4)}$ with $\varepsilon = n^{-1/2}$.

Suppose further that we have constructed a so called minimax estimator $\hat{\theta}$, i.e., the one attaining the minimax risk:

$$R_\varepsilon(\hat{\theta}, \Theta_\beta) = \sup_{\theta \in \Theta_\beta} R_\varepsilon(\hat{\theta}, \theta) \leq c_\varepsilon R_\varepsilon(\Theta_\beta),$$

(56)

for some bounded $c_\varepsilon$, $1 \leq c_\varepsilon \leq c$. The above inequality becomes stronger as $c_\varepsilon$ gets closer to 1 (asymptotically as $\varepsilon \to 0$ or uniformly in $\varepsilon > 0$). Minimax estimators are constructed in a variety of model settings and nonparametric classes $\Theta_\beta$. Note also that, as the set $\Theta_\beta$ is known, $R_\varepsilon(\Theta_\beta)$ is in principle known as well. Then it is easy to construct an optimal (in the minimax sense) confidence set by using a minimax estimator $\hat{\theta}$ and the minimax risk $R_\varepsilon(\Theta_\beta)$: for some $C > 0$, we simply take $B(\hat{\theta}, CR_\varepsilon(\Theta_\beta))$. Indeed, its radius of the minimax risk order and its coverage is

$$\sup_{\theta \in \Theta_\beta} \mathbb{P}_\theta(\theta \notin B(\hat{\theta}, CR_\varepsilon(\Theta_\beta))) \leq \frac{\sup_{\theta \in \Theta_\beta} \mathbb{E}_\theta d(\theta, \hat{\theta})}{CR_\varepsilon(\Theta_\beta)} \leq \frac{c}{C},$$

which can be made small for sufficiently large $C$.

It can be shown by using lower bounds from the minimax estimation theory that the minimax rate $R_\varepsilon(\Theta_\beta)$ is in some sense the best global radial rate. It turns out that the ball $B(\hat{\theta}, CR_\varepsilon(\Theta_\beta))$ satisfies (asymptotic versions of) relations (2), for appropriate choices of involved constants. Below are the details.

Let $w : \mathbb{R}_+ \to \mathbb{R}_+$, be a loss function, i.e., nonnegative and nondecreasing on $\mathbb{R}_+$, $w(0) = 0$ and $w \neq 0$. The maximal risk of an estimator $\hat{\theta}$ over $\Theta_\beta$ is $r_\varepsilon(\Theta_\beta, \hat{\theta}) = r_\varepsilon(\Theta_\beta, \hat{\theta}, R_\varepsilon) = \sup_{\theta \in \Theta_\beta} \mathbb{E}_\theta[w(R_\varepsilon^{-1}d(\hat{\theta}, \theta))]$, and the minimax risk over $\Theta_\beta$ is $r_\varepsilon(\Theta_\beta) = \inf_{\hat{\theta}} r_\varepsilon(\Theta_\beta, \hat{\theta})$. We consider here only the asymptotic regime $\varepsilon \to 0$ as in the most part of the literature on minimax estimation theory. A positive sequence $R_\varepsilon = R_\varepsilon(\Theta_\beta)$ and an estimator $\hat{\theta}$ are called minimax rate and minimax estimator respectively if

$$0 < b \leq \liminf_{\varepsilon \to 0} r_\varepsilon(\Theta_\beta) \leq \limsup_{\varepsilon \to 0} r_\varepsilon(\Theta_\beta, \hat{\theta}) \leq B < \infty.$$  

(57)

The first inequality is called lower bound and the last one upper bound. There is vast literature on this topic, minimax rates and estimators are obtained in a variety of models, settings and smoothness classes $\Theta_\beta$. Now suppose there is a minimax estimator $\hat{\theta}$ satisfying (51) with $w(u) = u$ and the corresponding minimax rate $R_\varepsilon(\Theta_\beta)$ (known, since $\beta$ is known). If we use the minimax risk $R_\varepsilon(\Theta_\beta)$ as the benchmark for the effective radius of confidence balls, then the problem of constructing an optimal confidence ball satisfying (1) with $\mathcal{R}_\varepsilon(\theta) = R_\varepsilon(\Theta_\beta)$ is readily solved. Indeed, take the confidence ball $B(\hat{\theta}, CR_\varepsilon(\Theta_\beta))$, then

$$\limsup_{\varepsilon \to 0} \sup_{\theta \in \Theta_\beta} \mathbb{P}_\theta(\theta \notin B(\hat{\theta}, CR_\varepsilon(\Theta_\beta))) \leq \limsup_{\varepsilon \to 0} \frac{\sup_{\theta \in \Theta_\beta} \mathbb{E}_\theta d(\theta, \hat{\theta})}{CR_\varepsilon(\Theta_\beta)} \leq \frac{B}{C},$$

which can be made arbitrarily small for sufficiently large $C$, and the asymptotic version of the second relation in (1) is trivially satisfied for $c$ large enough. On the other hand,
suppose that a lower bound in (57) is established for \( w(u) = I\{u \geq c\} \) and a (minimax) rate \( R_{\varepsilon}(\Theta_\beta) \): for any \( \hat{\theta} \),
\[
\liminf_{\varepsilon \to 0} \sup_{\theta \in \Theta_\beta} P_{\theta}(\theta \notin B(\hat{\theta}, c R_{\varepsilon}(\Theta_\beta))) = \liminf_{\varepsilon \to 0} r_{\varepsilon}(\Theta_\beta, \hat{\theta}, R_{\varepsilon}(\Theta_\beta)) \geq b > 0. \tag{58}
\]

We claim that it is impossible for a confidence ball \( B(\hat{\theta}, \hat{r}) \) to have simultaneously a global radial rate of a smaller order than \( R_{\varepsilon}(\Theta_\beta) \) and its coverage probability being arbitrarily close to 1 uniformly in \( \theta \in \Theta_\beta \). There are two ways to establish lower bounds for the optimality of confidence sets: either assume the first relation in (1) and show that the second must fail or the other way around. In the literature, the former approach is commonly used for global minimax radial rates, cf. Robins and van der Vaart (2006). However, when we construct confidence sets as credible balls with respect to some DDM \( P(\cdot \mid X) \), it is more natural to use the latter approach since the DD-radius gets determined by the DDM and typically the size requirement in (1) holds true for the whole set \( \Theta \), whereas the coverage requirement fails to hold for the whole \( \Theta \).

More precisely, if we assume
\[
\liminf_{\varepsilon \to 0} \inf_{\theta \in \Theta} P_{\theta}(\hat{r} \leq c R_{\varepsilon}(\Theta_\beta)) \geq 1 - b/2, \tag{59}
\]
then
\[
P_{\theta}(\theta \notin B(\hat{\theta}, \hat{r})) = P_{\theta}(\theta \notin B(\hat{\theta}, \hat{r}), \hat{r} \leq R_{\varepsilon}(\Theta_\beta)) + P_{\theta}(\theta \notin B(\hat{\theta}, \hat{r}), \hat{r} > R_{\varepsilon}(\Theta_\beta))
\geq P_{\theta}(\theta \notin B(\hat{\theta}, R_{\varepsilon}(\Theta_\beta)), \hat{r} \leq R_{\varepsilon}(\Theta_\beta))
\geq P_{\theta}(\theta \notin B(\hat{\theta}, R_{\varepsilon}(\Theta_\beta))) + P_{\theta}(\hat{r} \leq R_{\varepsilon}(\Theta_\beta)) - 1.
\]
Combining this with (58) and (59), we obtain
\[
\liminf_{\varepsilon \to 0} \sup_{\theta \in \Theta_\beta} P_{\theta}(\theta \notin B(\hat{\theta}, \hat{r})) \geq b + 1 - b/2 - 1 \geq b/2,
\]
which gives a bound on the coverage probability of \( B(\hat{\theta}, \hat{r}) \), at least for some (worst representatives) \( \theta \in \Theta_\beta \).

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