On Covariant Derivatives and Gauge Invariance in the Proper Time Formalism for String Theory

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March 28, 2022

Abstract

It is shown that the idea of “minimal” coupling to gauge fields can be conveniently implemented in the proper time formalism by identifying the equivalent of a “covariant derivative”. This captures some of the geometric notion of the gauge field as a connection. The proper time equation is also generalized so that the gauge invariances associated with higher spin massive modes can be made manifest, at the free level, using loop variables. Some explicit examples are worked out illustrating these ideas.
1 Introduction

A proper understanding of the massive modes in string theory is essential, not only from a conceptual standpoint, but also from a more practical, computational standpoint. In the first quantized (Polyakov) formalism, where the massless modes are represented by marginal operators, these massive modes are represented by irrelevant operators. The renormalization group equations of these theories and their connection with the equations of motion of the string modes has been the subject of a lot of papers [1-25]. Exact solutions of these equations have also been investigated [27]. The bulk of this work deals with massless modes. The massive modes are more difficult to deal with for two reasons: First, being massive, one has to deal with space-time dependent fields in order to satisfy the requirement that the corresponding vertex operators are marginal. This requirement arises because the usual $\beta$-function calculations are conveniently done only for marginal operators. Even if one is willing to deal with space-time dependent fields, the calculations are complicated because one typically has to sum an infinite number of diagrams to obtain non-trivial interactions [3, 28]. Second, the massive modes have higher spin and there are many extra symmetries associated with them. While the free equations have been obtained [14, 15], the subject of interactions has not been dealt with. At the present time, one has to resort to string field theory [34, 35, 36, 37, 38] for this. Indeed, the interaction of electromagnetic and massive modes, in the limit of uniform electromagnetic field strength, has been studied using string field theory [37].

Our aim is to apply the proper time formalism [10] to address the problem of massive modes. A prerequisite for understanding the massless case is the massless case. Some aspects of the massless case have been discussed in earlier papers [26, 27]. In this paper we discuss the “minimal” interaction of electromagnetism with other modes. We are interested in the general case of non-uniform fields, but with the restriction that they be close to their mass shell. In the case of massive higher spin modes the minimal interaction is not gauge invariant by itself and one has to include the direct coupling to the field strength for consistency. However we will not discuss the direct coupling to the field strength in this paper. Instead we will concentrate on getting a better understanding of how to implement the minimal interaction. This was worked out in some approximation in [30]. In this paper we show that something like the “covariant derivative”, an indispensable tool in gauge
theories, can also be introduced here. In [26] it was observed that in deriving
the covariant Klein Gordon equation each term arose as a surface term. This
will become manifest in the more general derivation given here.

There is another issue that needs to be resolved before one can begin to
discuss the interaction of massive modes in the proper time formalism. We
need to generalize the formalism to keep track of the extra gauge symmetries
associated with the massive modes. In [14, 15] the gauge invariant free equa-
tions of massive modes were worked out by requiring that a representation of
the string field, called a “loop variable”, not have any anomalous dependence
on the Liouville mode. This technique has some advantages, the principle
one probably being simplicity. We transcribe this into the framework of the
proper time equation, which is convenient for working out the interacting
equations of motion. This is the second main result of this paper. We also
give an explicit calculation by way of illustration. It is also interesting to
note that, in this form, the proper time equation looks very similar to the
equation written down by Witten in his background independent formalism
[39, 40, 41].

In section II we describe the minimal coupling of electromagnetism and
the covariant derivative, and illustrate it with the coupling to the tachyon.
In section III we generalize the proper time formalism to higher spin mas-
sive modes and show how the loop variable formalism can be incorporated.
Section IV contains some conclusions.
2 Minimal Coupling and the “Covariant Derivative”

The proper time equation, in its simplest form for the tachyon is:

\[
\int d^D k \left\{ \frac{d}{d \ln z} z^2 < V_p(z) V_k(0) > \right\}_{\ln z = 0} \phi(k) = 0. \tag{2.1}
\]

Here \( V_p = e^{i\nu X} \). We will use this equation to illustrate the minimal coupling.

The expectation value in (2.1) is calculated using the sigma model action appropriate to the problem. In the present case we will use the usual Polyakov action supplemented by an electromagnetic field interaction given by:

\[
\Delta S = \int dz \int d^D p A_\mu(p) \partial_z X^\mu e^{ipX(z)} \tag{2.2}
\]

The “minimal” part of the interaction can be isolated by using the following identity [30, 25]:

\[
A_\mu(k) \partial_z X^\mu e^{ikX(z)} = \int_0^1 d\alpha \partial_z (A_\mu X^\mu e^{i\alpha kX}) + i \int_0^1 d\alpha \partial_\mu A_\nu (X^\nu \partial_z e^{i\alpha kX}) \tag{2.3}
\]

When the RHS of (2.3) is substituted in (2.2), the first term represents the minimal interaction and the second term represents the direct coupling to the field strength \( F_{\mu\nu} \). In this paper we will discuss only the minimal coupling.

We have to evaluate

\[
\int d^D q \left\{ \frac{d}{d \ln z} z^2 < e^{ikX(z)} e^{i\Delta S e^{iqX(0)}} > \right\}_{\ln z = 0} \phi(q) = 0. \tag{2.4}
\]

If we insert the identity (2.3) in \( \Delta S \) we get (keeping only the first term of (2.3))

\[
\int d^D q \left\{ \frac{d}{d \ln z} z^2 < e^{ikX(z)} e^{i \int_0^1 d\alpha \int_0^z d\omega A_\mu(p) X^\mu e^{i\omega pX}} e^{iqX(0)} > \right\}_{\ln z = 0} \phi(q) = 0. \tag{2.5}
\]

We have suppressed the integral over \( p \). The integral over \( w \) gives a boundary term and we get

\[
\int d^D q \left\{ \frac{d}{d \ln z} z^2 < e^{i(k_\mu + \int_0^1 d\alpha A_\mu(p) e^{i\omega pX}) X^\mu(z)} e^{i(q_\mu - \int_0^1 d\alpha A_\mu(p) e^{i\omega pX}) X^\mu(0)} > \right\}_{\ln z = 0} \phi(q) = 0. \tag{2.6}
\]
One can easily see the effect of a gauge transformation $A_\mu \rightarrow A_\mu(k) + ik_\mu \Lambda$:

$$\int d^Dk \int_0^1 d\alpha A_\mu X^\mu e^{i\alpha kX(0)} \rightarrow \int d^Dk \int_0^1 d\alpha A_\mu X^\mu e^{i\alpha kX(0)} - i \int dk \Lambda(k) e^{ikX(0)} + i \int dk \Lambda(k) \quad (2.7)$$

On inserting (2.7) into the second exponential factor in (2.6) we find that the second term (of (2.7)) is cancelled by the variation $\phi \rightarrow \phi e^{i\Lambda(X(0))}$. The third piece is a space-time independent constant, i.e. a global phase, which is cancelled by a corresponding term from the first exponential factor at the point $z$. Thus (2.6) transforms by a phase factor $e^{i\Lambda(X(z))}$, which shows that the equation of motion that one obtains from (2.6) is guaranteed to be covariant and can therefore be expressed in terms of covariant derivatives. We will see this in an explicit calculation. In appearance, the exponent in (2.6) viz.

$$\left( k_\mu + \int_0^1 d\alpha A_\mu e^{i\alpha pX} X^\mu \right)$$

(2.8)

looks like $X^\mu D_\mu$. Of course at leading order this is guaranteed by the gauge invariance arguments given above and the fact that the $A$-independent part is $X^\mu \partial_\mu$. But \textit{a priori} it need not be true at the next order where there are several Lorentz invariant combinations possible. What is interesting is that, as we shall see, this is exact even at the next-to-leading order. From (2.5) and (2.6) we can also immediately see an explanation for the fact observed in [26] that in deriving the Klein Gordon equation only surface terms contribute.

Let us now turn to the actual evaluation of (2.6). We bring down one power of $A_\mu$ from each of the exponents in (2.6) to get:

$$\langle e^{i k \cdot X(z)} [1 + i \int_0^1 d\alpha A(p) X(z) e^{i\alpha pX(z)}] e^{i q \cdot X(0)} [1 - i \int_0^1 d\beta A(p') X(0) e^{i\beta p'X(0)}] \rangle \quad (2.9)$$

There are two points to note here. We have refrained from expanding $e^{i k X}$ as a power series in $k \cdot X$. This is only to make explicit the appearance of momentum conservation delta functions. As far as the actual algebra is concerned, one can just as well expand the exponential in a power series, keeping as many terms as necessary, and the answer would be the same. The second point concerns the integral over $\alpha$ and the presence of the exponential
$e^{iαpX}$. This has a momentum $αp$. But we would like $A_μ(p)$ to be associated with a vertex operator of momentum $p$. Therefore we will write

$$e^{iαpX} = e^{ipX+i(α-1)pX} = e^{ipX}[1 + i(α-1)pX + ...] \quad (2.10)$$

Thus at each order in $(α-1)p.X$ we have a vertex operator of momentum $p$. Of course for the purpose of doing the algebra $α$ can be kept in the exponent, as long as we remember that it actually represents a sum of vertex operators, each of momentum $p$.

Let us evaluate (2.9). Using $\frac{1}{i} < X^μ(z)X^ν(0) > = -\ln zδ^{μν} \quad (2.11)$ we get

$$e^{k.q ln δ(k+q)} + \int _0^1 dαA(p).q ln ze^{(k+αp).q ln z}δ(k+p+q)$$

$$- \int _0^1 dβA(p).k ln ze^{k.(q+βp)} ln zδ(k+p+q)$$

$$+ \int _0^1 dα \int _0^1 dβA(p).A(p')e^{k+αp}.(q+βp)ln zδ(k+p+p'+q) \quad (2.12)$$

Simplifying, we get on substituting into the proper time equation (2.4)

$$-(q^2-2)δ(q+k)+A(p).(2q+p)φ(q)δ(q+p+k)+A(p).A(p')φ(q)δ(k+p+p'+q) = 0 \quad (2.13)$$

In coordinate space this is

$$(D_μD^μ + 2)φ = 0 \quad (2.14)$$

Heuristically, if we think of expression (2.8) as standing for $X^μD_μ$, then to lowest non-trivial order (2.6) becomes

$$< (1 + X^μ(z)D_μ)(1 + X^ν(0)D_ν) > \quad (2.15)$$

Using (2.11) we get from the proper time equation, the result (2.14).

1All the calculations in this section are valid for point particles also. All we have to do is replace (2.11) by $< X(T)X(0) > = iT/2$
Let us turn to the next order to see if the heuristic identification of (2.8) with $X^\mu D_\mu$ has any value. If correct, one should obtain:

$$< \frac{D^\mu D^\nu X^\mu X^\nu D^\rho D^\sigma X^\rho X^\sigma}{2!} > $$

(2.16)

$$= \frac{D^\mu D^\nu D^\rho D^\sigma}{4} (\delta^{\mu\rho} \delta^{\nu\sigma} + \delta^{\mu\sigma} \delta^{\nu\rho}) (\ln z)^2$$

(2.17)

$$= \frac{(D^\mu D^\nu D_\nu D_\mu + D^\mu D^\nu D_\mu D_\nu)}{4} (\ln z)^2$$

(2.18)

A priori there are three possible structures at this order: $D^2 D^2$, $D^\mu D^2 D_\mu$ and $D^\mu D^\nu D_\mu D_\nu$. If we let $D \equiv \partial - iA$, so that $[D_\mu, D_\nu] = -iF_\mu\nu$, the following are easily proved:

$$D^\mu D^\nu D_\nu D_\mu = D^2 D^2 + i\partial_\mu F_\mu\nu D^\nu + F_\mu\nu F^{\mu\nu}$$

(2.19)

$$D^\mu D^\nu D_\mu D_\nu = D^2 D^2 + i\partial_\mu F_\mu\nu D^\nu + \frac{1}{2} F_\mu\nu F^{\mu\nu}$$

(2.20)

Let us now check if (2.18) is correct. At second order one has to evaluate:

$$< \{ 1 + i(k + \int_0^1 d\alpha A(p) e^{i\alpha pX(z)}) X(z) + \frac{i^2}{2!} [(k + \int_0^1 d\alpha A(p) e^{i\alpha pX(z)}) X(z)]^2 \} \{ 1 + i(q - \int_0^1 d\beta A(p') e^{i\beta pX(0)}) X(0) + \frac{i^2}{2!} [(q - \int_0^1 d\beta A(p') e^{i\beta pX(0)}) X(0)]^2 \} > $$

(2.21)

First of all, it is easy to see that the term with four derivatives and the term with four $A$'s are both consistent with $1/2D^2 D^2$ and therefore (using (2.13) and (2.20)) with (2.18). Next, let us calculate the piece that is linear in $A$. One obtains

$$-A(p).q[(p + q)^2 + q^2] - 1/2A(p).p[(p + q)^2 + q^2]$$

$$+ 1/2[A(p).qp^2 - A(p).pp.q]$$

(2.22)

Of the three terms in the above equation, the first two correspond to the linear (in $A$) part of $1/2D^2 D^2$. The third term is $i/2\partial_\mu F_\mu\nu \partial^\nu$ which is the linear (in $A$) part of $i/2\partial_\mu F_\mu\nu D^\nu$. Thus the result, (2.22), for the linear piece, is consistent with (2.18). However it is also consistent with any linear
combination of the two terms, (2.19) and (2.20), such that the sum of their coefficients add up to 1/2.

To determine the precise combination, we consider terms quadratic in $A$. Let us concentrate, for definiteness on terms of the type $A(p).p' A(p').p$ and $A(p) A(p') p.p'$. One finds from (2.21):

$$\frac{3}{4} A(p).p' A(p').p + \frac{1}{4} A(p) A(p') p'.p$$

$$= \frac{3}{4} [A(p).p' A(p').p - A(p) A(p') p'.p] + A(p).A(p') p'.p$$

(2.23)

The term in square brackets corresponds to $3/8 F_{\mu\nu} F^{\mu\nu}$ and the second term is the contribution from $1/2 D^2D^2$. This determines (using (2.19) and (2.20)) that it is precisely the linear combination given in (2.18) that is obtained.

We thus conclude that the heuristic identification of (2.8) with $X^\mu D_\mu$ is correct, at least to this order. If this is true to all orders, then we have an easy way of writing down the result at higher orders in $\ln z$ without the need to actually do a detailed calculation. It would certainly be interesting to know if this is true to all orders.

Thus we have shown in this section how one can isolate the minimal electromagnetic coupling in a simple way in the proper time equation. One of the advantages of this ($\sigma$ model) formalism is that it retains the (space-time) geometric notion of the gauge field being a connection. It should be possible to do something like this for the massive modes as well. In the next section we turn to the (free) massive modes.
3  Proper Time Equation for Massive Modes

In this section we will generalize the proper time equation to make it covariant under the extra gauge symmetries. Requiring that the coefficient of \( \ln z \) vanish is equivalent to imposing a dimensionality on the vertex operator, which is the condition \( L_0 = 1 \). For spin 1 and higher, we need to impose further conditions of the form \( L_n = 0 \; \forall n > 0 \), on the vertex operators. One thus starts by imposing a linear combination of these constrains. If the resulting equation has the required gauge symmetries, one can then impose all the constraints as gauge choices. To this end let us consider the following object:

\[
\langle \oint dt \lambda(t) T_{zz}(z + t) V_J(z) V_J(0) \rangle \bigg|_{\ln z = 0} \Phi^J(0) \tag{3.1}
\]

with

\[
\lambda(t) = \lambda_0 t + \lambda_{-1}t^2 + \lambda_{-2}t^3 + \ldots \tag{3.2}
\]

We will require that the coefficient of an appropriately chosen linear combination of \( \lambda_{-p} \) is zero. Note that we have evaluated (3.1) at \( \ln z = 0 \) (or \( \ln (z/a) = 0 \) where \( a \) is a short distance cutoff). Note also that \( \langle V_I(z) V_J(0) \rangle |_{z=a} \) is the Zamolodchikov metric \([42]\).

Before we apply it to the spin-2 case, as a warm up exercise let us apply it to derive the covariant Klein Gordon equation. Thus we consider:

\[
\langle \oint dt \lambda(t) \frac{1}{2} \partial_z X(z + t) \partial_z X(z + t) e^{i k.X(z)} e^{i \int_0^1 d\alpha A_\mu (p) e^{i \alpha p.X(0)} e^{i q.X(0)} \rangle \bigg|_{\ln z = 0} \phi(q). \tag{3.3}
\]

We have used the identity (2.3) to isolate the minimal interaction of the photon. The momentum integrals are suppressed for convenience. Doing the trivial \( w \) integral gives:

\[
\langle \oint dt \lambda(t) \frac{1}{2} \partial_z X(z + t) \partial_z X(z + t) e^{i (k_\mu + \int_0^1 d\alpha A_\mu e^{i \alpha p.X(0)}} \rangle \bigg|_{\ln z = 0} \phi(q) \tag{3.4}
\]

We can expand the exponent just as before, to get

\[
\langle \oint dt \lambda(t) \frac{1}{2} \partial_z X(z + t) \partial_z X(z + t) e^{i k.X(z)} + \int_0^1 d\alpha A_\mu(p) X^\mu(z) e^{i (k + \alpha p).X(z)} \rangle \tag{3.5}
\]
\[ + \int_0^1 d\alpha \int_0^1 d\beta A_\mu(p)A_\nu(p')X^\mu(z)X^\nu(z)e^{i(k+\alpha p+\beta p')X(z)}e^{i\mathbf{q}.X(0)} \bigg|_{\ln z=0} \phi(q) \]

Note that unlike what we did in Section II, we have kept two powers of \( A \) from the first exponential and none from the second. Any other contribution is higher order in \( \ln z \). We thus get

\[ -\int dt \frac{\lambda(t)}{t^2} < \left( \frac{k^2}{2} e^{ik.X(z)} + \int_0^1 d\alpha A(p)(k + \alpha p)(z)e^{i(k+\alpha p)X(z)} \right) \bigg|_{\ln z=0} \phi(q) \]

\[ = \lambda_0 \frac{k^2}{2} \delta(k + q) + (k + \frac{p}{2})A(p)\delta(k + p + q) - A(p)A(p')\delta(k + p + p' + q)\phi(q) \]

(3.6)

As explained in the last section, in the above calculation, \( e^{i(k+\alpha p)X(z)} \) is being interpreted as

\[ e^{i(k+p)X} (1 + i(\alpha - 1)pX + ...) \]

(3.8)

Setting the coefficient of \( \lambda_0 \) to zero in (3.7) gives the (covariant) Klein Gordon equation.

We turn now to the modes with gauge invariances. We will use the loop variables that were introduced for this purpose in [14]. We give below a brief review: The “loop variable” of [14] describes all the modes of the string and is the following:

\[ e^{i \int_c \alpha(t)k(t)\partial_\tau X(z+t)dt + ik_0X} \]

(3.9)

which can be rewritten as

\[ e^{i(k_0Y + k_1Y_1 + k_2Y_2 + ... + k_nY_n + ...)} \]

(3.10)

The \( k_n \) are defined by

\[ k(t) = k_0 + \frac{k_1}{t} + \frac{k_2}{t^2} + ... \]

(3.11)

\[ \text{Since we have to set } L_0 = 1, \text{ (and not } L_0 = 0) \text{ we should subtract } \lambda_0 e^{ikX} \text{ from (3.6). This will give the tachyon its mass.} \]
The $k_i$ define the fields as follows:

$$\int [dk_1 dk_2 dk_3 ... dk_n] \Psi [k_0, k_1, k_2, ... k_n, ...] = \phi(k_0)$$

$$\int [dk_1 dk_2 dk_3 ... dk_n] k_1^\mu \Psi [k_0, k_1, k_2, ... k_n, ...] = A_\mu(k_0)$$

$$\int [dk_1 dk_2 dk_3 ... dk_n] k_1^\mu k_1^\nu \Psi [k_0, k_1, k_2, ... k_n, ...] = S^{\mu\nu}(k_0)$$

$$\int [dk_1 dk_2 dk_3 ... dk_n] k_2^\mu \Psi [k_0, k_1, k_2, ... k_n, ...] = S^\mu(k_0)$$

Here, $\Psi$ is the string field and $\phi, A_\mu, S^{\mu\nu}, S^\mu$ are the tachyon, the massless vector and two massive modes. \footnote{The auxiliary fields are also included.}

If we define $\alpha_i$ by

$$\alpha(t) = \alpha_0 + \frac{\alpha_1}{t} + \frac{\alpha_2}{t^2} + ... + \frac{\alpha_n}{t^n} + ... \quad (3.12)$$

then

$$Y \equiv X + \alpha_1 \partial X + \alpha_2 \partial^2 X + \frac{\alpha_3 \partial^3 X}{3!} + ... + \frac{\alpha_n \partial^n X}{(n-1)!} + ... \quad (3.13)$$

Furthermore

$$Y_i \equiv \frac{\partial Y}{\partial x_i} \quad (3.14)$$

where $x_i$ are defined by

$$\alpha(t) \equiv e \sum_i x_i t^{-i} \quad (3.15)$$

Thus they satisfy

$$\frac{\partial \alpha_n}{\partial x_i} = \alpha_{n-i} \quad (3.16)$$

The $x_n$ are an infinite number of variables that describe reparametrizations of the boundary of the world sheet on which the loop variables are defined. In the Polyakov formalism these have to be integrated over. Thus the loop variables come with $\int [dx_n]$ attached to them. The gauge transformation on the string field is summarized by

$$k(t) \rightarrow k(t) \lambda(t) \quad (3.17)$$
This completes our review. For full details see [14, 15].

We would like to use these variables in our generalized proper time formalism. We thus need to know the action of \( \int dt \lambda(t) T_{zz}(z + t) \) on the loop variable (3.10). In fact the answer to this is known in closed form [38]. If we use the notation

\[ \tilde{Y} = \frac{\partial^m X}{(m - 1)!} \] (3.18)

then

\[ \sum_n k_n Y_n = \sum_m K_m \tilde{Y}_m \] (3.19)

where

\[ K_m \equiv k_0 \alpha_m + k_1 \alpha_{m-1} + k_2 \alpha_{m-2} + \ldots + k_m \] (3.20)

Thus

\[ e^{i \sum_n k_n Y_n} = e^{i \sum_m K_m \tilde{Y}_m} \] (3.21)

From [38] we know the exact expression for the operator product expansion \( e^{\sum_n \lambda_n L_n} e^{i K_m \tilde{Y}_m} \). Here we need it only to linear order, and only for \( n \geq 0 \).

The result is:

\[ e^{\sum_n \lambda_n L_n} : e^{i K_m \tilde{Y}_m} : = e^{-pq \alpha_{p-q} K_p K_q} : e^{i m K_m \lambda_{n-m} \tilde{Y}_n e^{i K_m \tilde{Y}_m + O(\lambda^2)}} : \] (3.22)

The first factor is the “quantum” or “anomalous” piece, whereas the second one is the “classical” piece. The latter can be ignored for our purposes since it is like a field redefinition [14]. Applying (3.22) to (3.10) we get (using (3.20))

\[ e^{\sum_n \lambda_n L_n} : e^{i \sum_n k_n Y_n} : = e^{\sum_{n,m,p} k_n k_m [\sum_q q(p-q) \alpha_{p-q} \alpha_{m-q}] \lambda_{p-m} + O(\lambda^2)} : \] (3.23)

Let us define a field \( \sigma \), a linear function of all the \( \lambda_{-p} \) and \( x_n \) by:

\[ \sigma \equiv \sum_q \sum_{p} q(p - q) \alpha_{p-q} \alpha_{p-m} \lambda_{-p} \] (3.24)

Then one can show that the RHS (3.23) can be rewritten as

\[ e^{-\sum_{n,m} k_n k_m \frac{\partial^2 \sigma}{\partial x_n \partial x_m} - \frac{\partial \sigma}{\partial x_{n+m}}} : e^{i \sum_n k_n Y_n} : \] (3.25)

\[ \text{In [38] we used the notation } Y_m \text{ for this. But we have already used } Y_m \text{ for } \frac{\partial Y}{\partial x_m}. \text{ If } \alpha(t) = 1 \text{ then } Y_m = \tilde{Y}_m. \]
In this form the result is exactly that of [14]. \( \sigma \) can be identified with the “new” Liouville mode introduced there. If \( \alpha(t) = 1 \) the “new” mode is just the usual Liouville mode and in fact (3.25) gives the Liouville mode dependence of a generalized vertex operator due to the Weyl anomaly.

We can now use all this in the proper time equation

\[
\frac{\delta}{\delta \sigma} \int [dx_n] < \oint dt \lambda(t) T_{zz}(z + t)V_I(z)V_J(0) > \bigg|_{\sigma=0} \Phi^I(0) = 0 \tag{3.26}
\]

In evaluating (3.26) we will use a two point function

\[
\Sigma(z) \equiv < Y(z)Y(0) > = \alpha' (\ln z + O(\alpha_1 z^{-1} \ldots)) \tag{3.27}
\]

Here we have introduced the string tension \( \alpha' \) to emphasize that higher orders in \( \Sigma \) are also higher orders in \( \alpha' \). When (3.26) is evaluated we will keep only terms linear in \( \Sigma \). Furthermore we have loop variables both at \( z \) and at \( 0 \), and each has a set of variables that describe reparametrizations. We denote the ones at \( z \) by \( x_1, x_2 \ldots \) and the ones at \( 0 \) by \( y_1, y_2 \ldots \). Thus \( \Sigma \) is a function of both \( x_n \) and \( y_n \).

Now, if we consider the following vertex operator and its associated \( \sigma \) dependence,

\[
e^{i(k_0 Y + k_1 Y_1 + k_2 Y_2) - k_0^2 \sigma - \frac{1}{2} \left( k_1 \frac{\partial^2 \sigma}{\partial x_1^2} - k_2 \frac{\partial \sigma}{\partial x_2} \right) - k_0 \frac{\partial \sigma}{\partial x_2} - k_1 \frac{\partial \sigma}{\partial x_1}} \tag{3.28}
\]

we have all we need to go to the second mass level.

For the photon we get:

\[
\int dx_1 \frac{\delta}{\delta \sigma} < (-k_1 k_0 \frac{\partial \sigma}{\partial x_1} + i k_1 \frac{\partial Y}{\partial x_1})e^{i(k_0 Y(z) - k_0^2 \sigma - k_1 \frac{\partial \sigma}{\partial x_1})} > \bigg|_{\sigma=0} = 0 \tag{3.29}
\]

\[
\Rightarrow \int dx_1 \frac{\delta}{\delta \sigma} (-k_1 k_0 \frac{\partial \sigma}{\partial x_1} + k_1 \frac{\partial Y}{\partial y_1})e^{ik_0 Y(0)} \bigg|_{\sigma=0} = 0 \tag{3.30}
\]

\[
\Rightarrow (k_1 k_0 k_0 q_1 - k_0^2 k_1 q_1) \frac{\partial^2 Y}{\partial x_1 \partial y_1} = 0 \tag{3.31}
\]

The coefficient of \( q_1^\mu \) is the equation of motion for the photon.

We can also apply this very easily to the next mass level. Since the calculation is very similar to that of [14, 15] we will merely write down the proper time equation and give a few results: The equation is

\[
\frac{\delta}{\delta \sigma} \int dx_1 dx_2 < \frac{1}{2} k_1 k_0 (\frac{\partial^2 Y}{\partial x_1^2} - \frac{\partial \sigma}{\partial x_2}) - k_1 k_0 \frac{\partial \sigma}{\partial x_2} - i k_1 k_0 \frac{\partial \sigma}{\partial x_1} k_1^\mu \frac{\partial Y^\mu}{\partial x_1} \tag{3.32}
\]

\]
\[ \frac{i k_2^\mu}{2} \frac{\partial Y^\mu}{\partial x_2} - \frac{k_1^\mu k_1^\nu}{2} \frac{\partial Y^\mu}{\partial x_1} \frac{\partial Y^\nu}{\partial x_1} \right) e^{ik_0 Y(z) - k_0^2 \sigma} \]

\[ e^{ik_0 Y(0)} (i l_2^\rho \frac{\partial Y^\rho}{\partial y_2} - \frac{l_1^\rho l_1^\sigma}{2} \frac{\partial Y^\rho}{\partial y_1} \frac{\partial Y^\sigma}{\partial y_1}) > \bigg|_{\sigma = 0} = 0 \quad (3.32) \]

One can evaluate each contraction and keep only those linear in \( \Sigma \). The coefficient of \( l_2^\mu \) gives the equation for the massive spin 1 (auxiliary) field \( S_\mu \) and that of \( l_1^\mu l_1^\nu \) gives the equation for the massive spin 2 field \( S_\mu \nu \). We will only give the equation for \( S_\mu \):

\[ \left[ -k_1 k_0 k_0^\mu + k_2 k_0 k_0^\mu + k_1 k_0 k_1^\mu - k_0^2 k_2^\mu \right] \frac{\partial^2 \Sigma}{\partial x_2 \partial y_2} = 0 \quad (3.33) \]

One can check that it is invariant under

\[ k_2 \rightarrow k_2 + \lambda_1 k_1 + \lambda_2 k_0 \quad k_1 \rightarrow k_1 + \lambda_1 k_0 \quad (3.34) \]

This is all exactly as in \[13\]. As shown there, the massive equations can be obtained by dimensional reduction and some identifications of the \( k_i \) with each other to reduce the number of degrees of freedom to that of conventional string theory. We will not repeat the details here.

What we have done so far is to generalize the proper time equation for higher modes of the string and we have done this by incorporating the loop variable approach of \[14, 15\]. We can now combine this with the results of the previous section and write down the minimal coupling of electromagnetism to massive modes. However this violates the gauge invariances \( (3.34) \) and is not consistent. To restore consistency one has to include the interaction of the \( F_\mu \nu \) part in \( (2.3) \). \[5\] In the general momentum dependent case, this can be done by a perturbative evaluation of the proper time equation. In order to do this one has to integrate over Koba-Nielsen parameters. But the presence of \( x_n \) complicates matters. In the BRST formalism, instead of \( x_n \), there are ghost fields - but these are amenable to standard conformal field theory techniques. Whether something analogous is possible here is an open question that requires further study.

\[5\] The full coupling has been worked out in the zero momentum limit using string field theory in \[37\].
4 Conclusions

In this paper two things have been done. First, we have introduced the equivalent of a covariant derivative and explained how minimal coupling of gauge fields can be implemented in a natural way. This captures the geometric idea of the gauge field being a connection. This kind of facility is perhaps one of the main attractions of the \( \sigma \)-model approach.

Second, the proper time equation has been generalized, by incorporating loop variables, to deal with higher spin modes. The formalism looks very similar to the background independent formalism of Witten \( [39, 40, 41] \). For massive modes we have done some explicit calculations at the free level.

The issue of interactions, in the case of massive modes, is not fully resolved. It is possible to introduce a minimal interaction with gauge fields, but by itself this is not consistent since it is not gauge invariant. As explained at the end of the last section, the crux of the problem is to understand how to do the Koba-Nielsen integrals in the presence of the \( x_n \)'s. We hope to return to this problem soon.

\footnote{The massless case has been worked out in \[25\].}
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