LANDAU-LIFSHITZ-BLOCH EQUATION ON RIEMANNIAN MANIFOLD

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Abstract. In this article, we bring in Landau-Lifshitz-Bloch(LLB) equation on m-dimensional closed Riemannian manifold and prove that it admits a unique local solution. In addition, if \( m \geq 3 \) and \( L^\infty \)-norm of initial data is sufficiently small, the solution can be extended globally. Moreover, if \( m = 2 \), we can prove that the unique solution is global without assuming small initial data.

1. Introduction

Landau-Lifshitz-Gilbert equation describes physical properties of micromagnetic at temperatures below the critical temperature. The equation is as follows:

\[
\frac{\partial m}{\partial t} = \lambda_1 m \times H_{eff} - \lambda_2 (m \times H_{eff})
\]

where \( \times \) denotes the vector cross product in \( \mathbb{R}^3 \) and \( H_{eff} \) is effective field while \( \lambda_1 \) and \( \lambda_2 \) are real constants.

However, at high temperature, the model must be replaced by following Landau-Lifshitz-Bloch equation(LLB)

\[
\frac{\partial u}{\partial t} = \gamma u \times H_{eff} + L_1 \frac{1}{|u|^2} (u \cdot H_{eff}) u - L_2 \frac{1}{|u|^2} u \times (u \times H_{eff})
\]

where \( \gamma, L_1, L_2 \) are real numbers and \( \gamma > 0 \). \( H_{eff} \) is given by

\[
H_{eff} = \Delta u - \frac{1}{\chi \|} \left( 1 + \frac{3T}{5(T - T_c)} |u|^2 \right) u.
\]

where \( T > T_c > 0 \) and \( \chi \| > 0 \).

Now let us recall some previous results about LLB. In [6], Le consider the case that \( L_1 = L_2 =: \kappa_1 > 0 \). At that time, he rewrites (1.2) as

\[
\frac{\partial u}{\partial t} = \kappa_1 \Delta u + \gamma u \times \Delta u - \kappa_2 (1 + \mu |u|^2) u
\]

with \( \kappa_2 := \frac{\gamma}{\chi \|} \) and \( \mu := \frac{3T}{5(T - T_c)} \) and assume that \( \kappa_2, \gamma, \mu \) is positive. Le has proven that above equation with Neumann boundary value conditions has global weak solution(the weak solution here is different from ordinary one). Inspired by Le, in [5] Jia introduces following equation

\[
\begin{cases}
\frac{\partial u}{\partial t} = \kappa_1 \Delta u + \gamma \nabla F(u) \times \Delta u - \kappa_2 (1 + \mu \cdot F(u)) \nabla F(u) & \text{in } \Omega \times (0, \infty) \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega \times (0, \infty) \\
u(\cdot, 0) = u_0 & \text{in } \Omega
\end{cases}
\]

1
where $\Omega$ is a regular bounded domain of $\mathbb{R}^d (d \leq 3)$, $\nu$ is outer normal direction of $\partial \Omega$ and $F \in C^3(\mathbb{R}^3)$ is a known function. He calls it Generalized Landau-Lifshitz-Bloch equation (GLLB) and gets that (1.4) admits a local strong solution provided $u_0 \in W^{2,2}(\Omega, \mathbb{R}^3)$ and $\frac{\partial u_0}{\partial n} = 0$. In [4], Guo, Li and Zeng consider the coming LLB equation with initial condition

$$
\begin{aligned}
\left\{ \begin{array}{ll}
    u_t = \Delta u + u \times \Delta u - \lambda (1 + \mu |u|^2)u & \text{in } \mathbb{R}^d \times (0, T) \\
    u(0) = u_0 & \text{in } \mathbb{R}^d,
\end{array} \right.
\end{aligned}
$$

where the constant $\lambda, \mu > 0$. They prove the existence of smooth solutions of (1.5) in $\mathbb{R}^2$ or $\mathbb{R}^3$. And a small initial value condition should be added in the latter case.

In this paper, we would like to introduce an equation similar with (1.5) on Riemannian manifold. Before getting to this, we should make some preparation.

Let $\pi : (E, h, D) \rightarrow (M, g, \nabla)$ denote a smooth vector bundle over an $m-$dimensional smooth closed Riemannian manifold $(M, g, \nabla)$ with $\text{rank}(E) = 3$. $g$ means Riemannian metric of $M$ and $\nabla$ is its Levi-Civita connection. $h$ and $D$ are respectively metric and connection of $E$ such that $Dh = 0$. Sometimes we also write $h$ as $(\cdot, \cdot)$.

1.1. $k-$times continuously differentiable section. Suppose $\Gamma(E)$ is the set of all sections in $E$. Under arbitrary local frame $\{e_\alpha : 1 \leq \alpha \leq 3\}$, a section $s \in \Gamma(E)$ can be written in the form of $s = s^\alpha \cdot e_\alpha$. If $s^\alpha$ is $k-$times continuously differentiable, then we say $s$ is $k-$times continuously differentiable. Since $E$ is smooth, $k-$times continuous differentiability is independent of the choice of local frame. Define

$$
\Gamma^k(E) := \{ s \in \Gamma(E) : s \text{ is } k\text{-times continuously differentiable}\}.
$$

1.2. Orientable vector bundle. $E$ is called orientable if there exists an $\omega \in E^* \wedge E^* \wedge E^*$ such that $\omega$ is continuous and for all $p \in M$, $\omega(p) \neq 0$, where $E^*$ is dual bundle of $E$.

Suppose $\{e_1, e_2, e_3\}$ is a frame of $E$. It is called adapted to the orientation $\omega$ if

$$
\omega(e_1, e_2, e_3) > 0.
$$

From now on, we always assume that $E$ is orientable unless otherwise stated.

1.3. Cross product on orientable vector bundle. Suppose $\omega$ is an orientation of $E$. $\{e_\alpha : 1 \leq \alpha \leq 3\}$ is a local frame of $E$ which is adapted to $\omega$. For any $f_1, f_2 \in \Gamma(E)$, we assume that $f_1 := f_1^\alpha \cdot e_\alpha$, $f_2 := f_2^\alpha \cdot e_\alpha$. Their cross product $\times$ is defined as follow

$$
(f_1 \times f_2)(p) := f_1(p) \times f_2(p),
$$

where

$$
\begin{aligned}
(f_1 \times f_2)(p) & = (f_1^2(p) \cdot f_2^3(p) - f_2^2(p) \cdot f_1^3(p)) \cdot e_1(p) \\
& \quad + (f_2^1(p) \cdot f_3^3(p) - f_3^1(p) \cdot f_2^3(p)) \cdot e_2(p) \\
& \quad + (f_3^1(p) \cdot f_2^3(p) - f_2^1(p) \cdot f_3^3(p)) \cdot e_3(p).
\end{aligned}
$$

It is not hard to verify that $f_1(p) \times f_2(p)$ does not depend upon the choice of local frames which are adapted to $\omega$. 
1.4. **Laplace operator on vector bundle.** Define a functional \( \text{Energy} \) on \( \Gamma^2(E) \) which is given in the form of
\[
\text{Energy}(X) := \frac{1}{2} \int_M |D X|^2 dM.
\]
It is not hard to see that the Euler-Lagrange equation of \( \text{Energy} \) is
\[
\Delta X := \frac{\partial}{\partial x^i} \left( \frac{\partial}{\partial x^j} \left( D^2 X \right) \right) = 0,
\]
where \( g_{ij} := g \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) \) and \( (g^{ij}) \) is the inverse matrix of \( (g_{ij}) \). Then we say that \( \Delta \) is the Laplace operator on vector bundle \( E \).

1.5. **sections depending on time.** A section depending on time is a map
\[
V : I \rightarrow \Gamma(E),
\]
where \( I \) is an interval of \( \mathbb{R} \). Under arbitrary local frame \( \{e_\alpha : 1 \leq \alpha \leq 3\} \), \( V(t, x) := V^\alpha(t, x) \cdot e_\alpha(x) \). If \( V^\alpha \) is \( k \)-times continuously differentiable with respect to \( t \), we say \( V \) is \( k \)-times continuously differentiable with respect to \( t \) and use the symbol \( C^k(I, \Gamma(E)) \) to denote all such \( V \). Since \( E \) is smooth, differentiability with respect to time is independent of the choice of local frame. Moreover, we define
\[
(\partial_t^k V)(t, x) := (\partial_t^k V^\alpha)(t, x) \cdot e_\alpha(x).
\]

1.6. **Sobolev space on vector bundle.** Equip \( \Gamma^k(E) \) with a norm \( \| \cdot \|_{H^{k,p}}(p \geq 1) \) which is defined as follow
\[
\|s\|_{H^{k,p}} := \sum_{i=0}^{k} \int_M |D^i s|^p dM.
\]
The Sobolev space \( H^{k,p}(E) \) is the completion of \( \Gamma^k(E) \) with respect to the norm \( \| \cdot \|_{H^{k,p}} \). For convenience, we also denote \( H^{k,2} \) by \( H^k \) and \( \| \cdot \|_{H^{0,p}} \) by \( \| \cdot \|_p \).

Having the above preparation, we will give the definition of Landau-Lifshitz-Bloch equation (LLB) on Riemannian manifold.

For any \( T > 0, \lambda > 0 \) and \( \mu > 0 \), let us consider a section depending on time \( V \in C^1([0, T], \Gamma^2(E)), \) LLB is just the following equation
\[
\begin{aligned}
\partial_t V &= \Delta V + V \times \Delta V - \lambda \cdot (1 + \mu \cdot |V|^2) V & \text{in } (0, T] \times M \\
V(0, \cdot) &= V_0
\end{aligned}
\]
\hspace{1cm} (1.6)

Our main results are as follow:

**Theorem 1.1.** Let \( \pi : (E, h, D) \rightarrow (M, g, \nabla) \) denote a smooth vector bundle over an \( m \)-dimensional smooth closed Riemannian manifold \( (M, g, \nabla) \) with rank\( (E) = 3 \) and \( Dh = 0. \) \( E \) is orientable. Given \( l \geq m_0 + 1 \) (Here \( m_0 := \lceil \frac{m}{2} \rceil + 3 \) and \( [q] \) is the integral part of \( q \)) and \( V_0 \in H^1(E) \), there is a \( T^* = T^*([V_0]_{H^{m_0}}) > 0 \) and a unique solution \( V \) of (1.6) satisfying that for any \( 0 \leq j \leq \frac{1}{\tilde{m}} \) \( \tilde{m} := \max\{2, \lceil \frac{m}{2} \rceil + 1\} \) and \( \alpha \leq l - \tilde{m}j \),
\[
\partial_t^j D^\alpha V \in L^\infty([0, T^*], L^2(E)).
\]
\hspace{1cm} (1.7)
Furthermore, if \( V_0 \in \Gamma^\infty(E) \), then \( V \in C^\infty([0, T^*], \Gamma^\infty(E)) \).

**Theorem 1.2.** Let \( \pi : (E, h, D) \longrightarrow (M, g, \nabla) \) denote a smooth vector bundle over an \( m \)-dimensional smooth closed Riemannian manifold \( (M, g, \nabla) \) with \( \text{rank}(E) = 3 \), \( m \geq 3 \) and \( Dh = 0 \). \( E \) is orientable. For any \( T > 0 \) and \( N \geq m_0 + 1 \), there exists a \( \hat{B}_N > 0 \) such that for all \( V_0 \in H^N(E) \) with \( ||V_0||_\infty \leq \hat{B}_N \), there is a unique solution of (1.6) satisfying

\[
\partial_i^j D^\alpha V \in L^\infty([0, T], L^2(E)) \quad 0 \leq j \leq \left[ \frac{N}{m} \right] \quad \forall \alpha \leq N - mj
\]

and

\[
\partial_i^j D^\beta V \in L^2([0, T], L^2(E)) \quad 0 \leq i \leq \left[ \frac{N + 1}{m + 1} \right] \quad \forall \beta \leq N + 1 - (m + 1)i.
\]

Furthermore, if \( V_0 \in \Gamma^\infty(E) \), then \( V \in C^\infty([0, T], \Gamma^\infty(E)) \).

**Theorem 1.3.** Let \( \pi : (E, h, D) \longrightarrow (M, g, \nabla) \) denote a smooth vector bundle over an \( 2 \)-dimensional smooth closed Riemannian manifold \( (M, g, \nabla) \) with \( \text{rank}(E) = 3 \) and \( Dh = 0 \). \( E \) is orientable. For any \( T > 0 \), \( N \geq 5 \) and \( V_0 \in H^5(E) \), there is a unique solution of (1.6) satisfying

\[
\partial_i^j D^\alpha V \in L^\infty([0, T], L^2(E)) \quad 0 \leq j \leq \left[ \frac{N}{2} \right] \quad \forall \alpha \leq N - 2j
\]

and

\[
\partial_i^j D^\beta V \in L^2([0, T], L^2(E)) \quad 0 \leq i \leq \left[ \frac{N + 1}{3} \right] \quad \forall \beta \leq N + 1 - 3i.
\]

Furthermore, if \( V_0 \in \Gamma^\infty(E) \), then \( V \in C^\infty([0, T], \Gamma^\infty(E)) \).

### 2. Notation and Preliminaries

In the paper, we appoint that the same indices appearing twice means summing it. And \( Q_1 \lesssim Q_2 \) implies there is a universal constant \( C \) such that \( Q_1 \leq C \cdot Q_2 \).

#### 2.1. Riemannian curvature tensor on vector bundle.** Using the connection \( D \) on \( E \), we can define a tensor \( R^E \) called Riemannian curvature tensor. For any \( X, Y \in TM \) and \( s \in \Gamma^2(E) \),

\[
R^E(X, Y)s := D_X D_Y s - D_Y D_X s - D_{[X,Y]}s.
\]

Let \( R^M \) be the Riemannian curvature tensor of \( M \). Being going to represent \( R^M \) and \( R^E \) in local frame, we appoint \( \frac{\partial}{\partial x^i} \) as \( \partial_i \). Then,

\[
R^M(\partial_i, \partial_j)\partial_r := (R^M)^{h}_{ijr} \cdot \partial_h \quad \text{and} \quad R^E(\partial_i, \partial_j)e_\beta := (R^E)^{\alpha}_{ij\beta} \cdot e_\alpha.
\]

Now we give two tensors

\[
\mathcal{R}^M := (\mathcal{R}^M)^{ijkl} \cdot dx^i \otimes dx^j \otimes dx^k \otimes dx^l
\]

and

\[
\mathcal{R}^E := (\mathcal{R}^E)^{\alpha\beta}_{ij} \cdot dx^i \otimes dx^j \otimes e_\alpha \otimes e_\beta,
\]

where

\[
(\mathcal{R}^M)^{ijkl} := (R^M)^{h}_{ijkl} \cdot g_{hl} \quad \text{and} \quad (\mathcal{R}^E)^{\alpha\beta}_{ij} := (R^E)^{\alpha}_{ij\beta} \cdot h^{\beta\beta}.
\]

\( (h_{\alpha\beta}) \) is the metric matrix of \( h \) and \( (h^{\alpha\beta}) \) is its inverse matrix.
2.2. Cross product of tensors. We also want to introduce cross product between two tensors. Given $S \in \Gamma(T^* M \otimes^k \otimes E)$ and $T \in \Gamma(T^* M \otimes^l \otimes E)$, let us define

$$S \times T := (S_{i_1 \cdots i_k} \times T_{j_1 \cdots j_l}) \otimes dx^{i_1} \otimes \cdots \otimes dx^{i_k} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_l},$$

where

$$S_{i_1 \cdots i_k} := S(\partial_{i_1}, \ldots, \partial_{i_k}) \quad \text{and} \quad T_{j_1 \cdots j_l} := T(\partial_{j_1}, \ldots, \partial_{j_l}).$$

It is easy to check

$$|S \times T| \leq |S| \cdot |T| \quad (2.1)$$

2.3. Properties of cross product.

**Theorem 2.1.** For any $f_1, f_2 \in \Gamma^1(E)$, we have

$$D(f_1 \times f_2) = (Df_1) \times f_2 + f_1 \times (Df_2). \quad (2.2)$$

**Proof.** Take any $p \in M$. Then there exists a neighbourhood $U$ and a positive number $\delta$ such that the following map

$$\exp_p : N_{\delta} \triangleq \{ \dot{v} \in T_p M : ||\dot{v}|| < \delta \} \rightarrow U$$

is a diffeomorphism. Take $v \in T_p M$ such that $||v|| = 1$. Define $\gamma_v(t) := \exp_p(tv)$, where $t \in [0, \delta]$. Now take arbitrary orthonormal basis $\{e_{\alpha} : 1 \leq \alpha \leq 3\}$ in $E_p$ which is adapted to $\omega$ and let it move parallelly along $\gamma_v$ to get

$$\{e_{\alpha}(t, v) : t \in [0, \delta), 1 \leq \alpha \leq 3\}.$$ 

Clearly,

$$w(t) := \omega(e_1(t, v), e_2(t, v), e_3(t, v)) > 0, \quad \forall t \in [0, \delta)$$

since $w$ is a continuous function with respect to $t$. In the next, let $v$ range all the direction in $T_p M$ to obtain

$$\{e_{\alpha}(t, v) : t \in [0, \delta), v \in T_p M, ||v|| = 1, 1 \leq \alpha \leq 3\}.$$ 

It is an orthonormal frame on $U$ which is adapted to $\omega$ and

$$\gamma_v(p) = 0. \quad (2.3)$$

Assume that $f_1 = f_1^\alpha \cdot e_\alpha$ and $f_2 = f_2^\beta \cdot e_\beta$. Then, (2.3) yields

$$Df_1(p) = df_1^\alpha(p) \otimes e_\alpha(p) \quad \text{and} \quad Df_2(p) = df_2^\beta(p) \otimes e_\beta(p).$$

Recalling the definition of cross product, we have

$$f_1 \times f_2 := (f_1^2 \cdot f_2^3 - f_2^2 \cdot f_1^3) \cdot e_1 + (f_1^1 \cdot f_2^3 - f_1^3 \cdot f_2^1) \cdot e_2 + (f_1^1 \cdot f_2^2 - f_2^1 \cdot f_1^2) \cdot e_3.$$ 

Therefore, since of (2.3), one can get

$$[D(f_1 \times f_2)](p) = [df_1^2(p) \cdot f_2^3(p) + f_1^1(p) \cdot df_2^3(p) - df_2^2(p) \cdot f_1^3(p) - f_2^1(p) \cdot df_1^3(p)] \otimes e_1(p)$$

$$+ [df_2^1(p) \cdot f_1^3(p) + f_1^2(p) \cdot df_2^3(p) - df_1^1(p) \cdot f_2^3(p) - f_1^3(p) \cdot df_2^1(p)] \otimes e_2(p)$$

$$+ [df_1^1(p) \cdot f_2^2(p) + f_1^2(p) \cdot df_2^2(p) - df_2^1(p) \cdot f_1^2(p) - f_1^2(p) \cdot df_2^1(p)] \otimes e_3(p), \quad (2.4)$$

and

$$[f_1 \times (Df_2)](p) = f_1(p) \times (Df_2)(p)$$

$$= [df_1^2(p) \cdot f_2^3(p) - df_2^2(p) \cdot f_1^3(p)] \otimes e_1(p)$$

$$+ [df_2^1(p) \cdot f_1^3(p) - f_1^3(p) \cdot df_2^1(p)] \otimes e_2(p) \quad (2.5)$$


+ \left[ f_1^1(p) \cdot df_2^2(p) - df_1^1(p) \cdot f_2^1(p) \right] \otimes e_3(p),

and

\[ (Df_1) \times f_2(p) = (Df_1)(p) \times f_2(p) \]
\[ = \left[ df_1^1(p) \cdot f_2^1(p) - f_1^1(p) \cdot df_2^1(p) \right] \otimes e_1(p) \]
\[ + \left[ f_1^1(p) \cdot df_2^1(p) - df_1^1(p) \cdot f_2^1(p) \right] \otimes e_2(p) \]
\[ + \left[ df_1^1(p) \cdot f_2^1(p) - f_1^1(p) \cdot df_2^1(p) \right] \otimes e_3(p). \]

This theorem follows easily from combining (2.4) with (2.5) and (2.6).

Because of (2.2), it is easy to verify that

\[ D(S \times T) = (DS) \times T + S \times (DT), \]

provided \( S \in \Gamma^1(T^*M^\otimes k \otimes E) \) and \( T \in \Gamma^1(T^*M^\otimes l \otimes E). \)

### 2.4. Hamilton’s notation.

Suppose \( k, l, p, q \in \mathbb{N}, S \in T^*M^\otimes k \otimes E^\otimes p \) and \( T \in T^*M^\otimes l \otimes E^\otimes q, \) where

\[ E^\otimes p := E \otimes \cdots \otimes E, \]

we will write \( S * T, \) following Hamilton [2], to denote a tensor formed by contraction on some indices of \( S \otimes T \) using the coefficients \( g^{ij} \) or \( h_{\alpha\beta}. \)

**Theorem 2.2.**

\[ |S * T| \leq |S| \cdot |T| \]

**Proof.** We will get the above formula in an orthonormal basis of \( M \) and an orthonormal basis of \( E. \)

\[ |S * T|^2 = \sum_{\text{free indices}} \left( \sum_{\text{contracted indices}} S_{\alpha_1 \cdots \alpha_p}^{\alpha_{i_1} \cdots \alpha_k} \cdot T_{\beta_{j_1} \cdots j_l}^{\beta_{i_1} \cdots \beta_q} \right)^2 \]
\[ \leq \sum_{\text{free indices}} \left( \sum_{\text{contracted indices}} \left( S_{\alpha_1 \cdots \alpha_p}^{\alpha_{i_1} \cdots \alpha_k} \right)^2 \right) \cdot \left( \sum_{\text{contracted indices}} \left( T_{\beta_{j_1} \cdots j_l}^{\beta_{i_1} \cdots \beta_q} \right)^2 \right) \]
\[ \leq \left( \sum_{\text{free indices}} \left( \sum_{\text{contracted indices}} \left( S_{\alpha_1 \cdots \alpha_p}^{\alpha_{i_1} \cdots \alpha_k} \right)^2 \right) \right) \cdot \left( \sum_{\text{free indices}} \sum_{\text{contracted indices}} \left( T_{\beta_{j_1} \cdots j_l}^{\beta_{i_1} \cdots \beta_q} \right)^2 \right) \]
\[ = |S|^2 \cdot |T|^2. \]

Because we do not specifically illustrate which indices are contracted, we have to appoint that

\[ S_1 * T_1 - S_2 * T_2 := S_1 * T_1 + S_2 * T_2. \]
We will use the symbol \( q_s(T_1, \ldots, T_r) \) for a polynomial in the tensors \( T_1, \ldots, T_r \) and their iterated covariant derivatives with the * product like

\[
q_s(T_1, \ldots, T_r) := \sum_{j_1 + \cdots + j_r = s} c_{j_1 \cdots j_r} \cdot D^{j_1}T_1 \ast \cdots \ast D^{j_r}T_r,
\]

where for \( 1 \leq i \leq r \), \( T_i \in \Gamma^{j_i}(T^*M^\otimes t_i \otimes E^\otimes q^i) \) and \( c_{j_1 \cdots j_r} \) are some universal constants.

2.5. Ricci identity. Given \( s \in \Gamma^2(T^*M^\otimes k \otimes E) \), it is obvious to see that \( s \) can be written as follow

\[
s := s^\alpha_{i_1 \cdots i_k} \cdot dx^{i_1} \otimes \cdots \otimes dx^{i_k} \otimes e_\alpha.
\]

We denote \( Ds \) in the form of components

\[
Ds := s^\alpha_{i_1 \cdots i_k,p} \cdot dx^{i_1} \otimes \cdots \otimes dx^{i_k} \otimes dx^p \otimes e_\alpha.
\]

At some time, we also employ the coming convention

\[
Ds := s_{i_1 \cdots i_k,p} \cdot dx^{i_1} \otimes \cdots \otimes dx^{i_k} \otimes dx^p.
\]

Thanks to the above agreement, Ricci identity is conveniently represented in the next theorem.

**Theorem 2.3.**

\[
s^\alpha_{i_1 \cdots i_k,pq} - s^\alpha_{i_1 \cdots i_k,qp} = \sum_{l=1}^k s^\alpha_{i_1 \cdots i_{l-1}hi_{l+1} \cdots i_k} \cdot (R^M)^h_{pq_i} - s^\beta_{i_1 \cdots i_k} \cdot (R^E)^\alpha_{pq\beta} = k \cdot s \ast R^M + s \ast R^E.
\]

**Proof.** The proof is straightforward if one takes normal coordinates. So we omit it. \( \square \)

Given \( V \in \Gamma^{k+1}(E) \) and \( S \in \Gamma^{k+1}(T^*M \otimes E) \), by Theorem 2.3 and induction, the following formulas are easy.

**Formula 1.**

There exist \( a_{ij} \in \mathbb{Z} \) and \( b_{it} \in \mathbb{Z} \) such that

\[
V_{,pi_1 \cdots i_k} - V_{,i_1 \cdots i_k} = \sum_{i+j=k-1} a_{ij} \cdot D^iV \ast D^jR^E + \sum_{r+l=k-2} b_{it} \cdot D^{r+1}V \ast \nabla^lR^M = q_{k-1}(V, R^E) + q_{k-2}(DV, R^M)
\]

(2.9)

**Formula 2.**

There exist \( a_{ij} \in \mathbb{Z} \) and \( b_{it} \in \mathbb{Z} \) such that

\[
S_{p,qi_1 \cdots i_k} - S_{p,i_1 \cdots i_k} = \sum_{i+j=k-1} a_{ij} \cdot D^iS \ast D^jR^E + \sum_{r+l=k-1} b_{it} \cdot D^rS \ast \nabla^lR^M = q_{k-1}(S, R^E) + q_{k-1}(S, R^M)
\]

(2.10)
2.6. Interpolation for sections. We shall prove Gagliardo-Nirenberg inequality of sections on vector bundle.

**Theorem 2.4.** \((M, g)\) is a \(m\)–dimensional smooth closed Riemannian manifold. \((E, h, D)\) is a smooth vector bundle over \(M\) with \(Dh = 0\). rank\((E)\) may not be 3 and \(E\) may not be orientable. Let \(T\) be a smooth section of \(E\). Given \(s \in \mathbb{R}^+\) and \(j \in \mathbb{Z}^+\), we will have

\[
\|D^j T\|_{\mathbb{R}^s} \leq C(m, s, k, j) \cdot \|D^k T\|_{\mathbb{R}^{s \cdot \|\mathcal{T}\|_{1-k/4}}} \cdot \|\mathcal{T}\|_{1-k/4},
\]

provided \(k \in [j, \infty) \cap \mathbb{Z}\), \(l \in [1, s] \cap [j, s + j + 1 - k] \cap \mathbb{Z}\).

**Proof.** Apply induction for \(j\).

**Step 1:** When \(j = 1\), (2.11) is equivalent to

\[
\|D T\|_{\mathbb{R}^s} \leq C(m, s, k) \cdot \|D^k T\|_{\mathbb{R}^{s \cdot \|\mathcal{T}\|_{1-k/4}}} \cdot \|\mathcal{T}\|_{1-k/4},
\]

for all \(l \in [1, s] \cap [1, s + 2 - k] \cap \mathbb{Z}\). In order to show (2.12), we use induction for \(k\).

When \(k = 1\), (2.12) holds obviously.

When \(k = 2\), by 12.1 Theorem of [2] we know (2.12) holds.

Assume that for \(2 \leq \hat{k} \leq k\), we obtain

\[
\|D^l T\|_{\mathbb{R}^s} \leq C_1(m, s, \hat{k}) \cdot \|D^\hat{k} T\|_{\mathbb{R}^{s \cdot \|\mathcal{T}\|_{1-k/4}}} \cdot \|\mathcal{T}\|_{1-k/4},
\]

provided \(l \in [1, s] \cap [1, s + 2 - \hat{k}] \cap \mathbb{Z}\).

When \(\hat{k} = k + 1\), pick any \(l \in [1, s] \cap [1, s + 2 - (k + 1)] \cap \mathbb{Z}\). Clearly,

\[
l + 1 \in [1, s] \cap [1, s + 2 - k] \cap \mathbb{Z},
\]

since \(k \geq 2\). Using induction hypothesis, we get

\[
\|D^2 T\|_{\mathbb{R}^s} \leq C_2(m, s, k) \cdot \|D^k (D T)\|_{\mathbb{R}^{s \cdot \|\mathcal{T}\|_{1-k/4}}} \cdot \|DT\|_{1-k/4}.
\]

Because \(1 \leq l \leq s + 2 - (k + 1) < s\), using induction hypothesis for \(k = 2\) gives

\[
\|DT\|_{\mathbb{R}^s} \leq C(m, s) \cdot \|D^2 T\|_{\mathbb{R}^{s \cdot \|\mathcal{T}\|_{1-k/4}}} \cdot \|\mathcal{T}\|_{1-k/4}.
\]

Combining (2.13) with (2.14) yields

\[
\|DT\|_{\mathbb{R}^s} \leq C_3(m, s, k) \cdot \|D^{k+1} T\|_{\mathbb{R}^{s \cdot \|\mathcal{T}\|_{1-k/4}}} \cdot \|DT\|_{\mathbb{R}^{s \cdot \|\mathcal{T}\|_{1-k/4}}} \cdot \|\mathcal{T}\|_{1-k/4},
\]

which implies

\[
\|DT\|_{\mathbb{R}^s} \leq C(m, s, k + 1) \cdot \|D^{k+1} T\|_{\mathbb{R}^{s \cdot \|\mathcal{T}\|_{1-k/4}}} \cdot \|\mathcal{T}\|_{1-k/4}.
\]

**Step 2:** Suppose that for all the indices not greater than \(j\), (2.11) is true. Now we consider \(j + 1\). At this moment, we take any \(k \in [j + 1, \infty) \cap \mathbb{Z}\) and any \(l \in [1, s] \cap [j + 1, s + j + 2 - k] \cap \mathbb{Z}\). It is easy to deduce that

\[
k - 1 \in [j, \infty) \cap \mathbb{Z} \quad \text{and} \quad l \in [1, s] \cap [j, s + j + 1 - (k - 1)] \cap \mathbb{Z}.
\]

Using induction hypothesis leads to

\[
\|D^j (DT)\|_{\mathbb{R}^s} \leq C_1(m, s, k, j) \cdot \|D^{k-1} (DT)\|_{\mathbb{R}^{s \cdot \|\mathcal{T}\|_{1-k/4}}} \cdot \|DT\|_{1-k/4}.
\]
Since \( l - j \in [1, s] \cap [1, s + 2 - k] \cap \mathbb{Z} \), by Step 1 we have
\[
\|DT\|_{2s_j} \leq C(m, s, k) \cdot \|D^kT\|_{\frac{1}{2s_j + 2s_k}} \cdot \|T\|_{1 - \frac{1}{2s_j}}.
\]
Combining (2.15) with (2.16) gives
\[
\|D^{j+1}T\|_{2s_j} \leq C(m, s, k, j + 1) \cdot \|D^kT\|_{\frac{1}{2s_j + 2s_k + 1}} \cdot \|D^{k+1}T\|_{\frac{1}{2s_j + 2s_k + 2}} \cdot \|T\|_{1 - \frac{1}{2s_j}}.
\]
This completes the proof. \(\square\)

**Theorem 2.5.** \((M, g)\) is a \(m\)–dimensional smooth closed Riemannian manifold. \((E, h, D)\) is a smooth vector bundle over \(M\) with \(Dh = 0\). rank\((E)\) may not be 3 and \(E\) may not be orientable. Let \(T\) be a smooth section of \(E\). If \(r, q \geq 2\), then there is a universal constant \(C = C(m, r, q, j, k)\) such that
\[
\|D^j T\|_p \leq C \cdot \|D^k T\|_q^{\frac{1}{2}} \cdot \|T\|_q^{1 - \frac{1}{q}},
\]
provided
\[
1 \leq j \leq k \quad \text{and} \quad \frac{k}{p} = \frac{j}{r} + \frac{k - j}{q}.
\]

**Proof.** We consider 3 cases.

**Case 1:** When \(2 \leq r < q \leq \infty\), there exist \(s\) and \(l\) such that
\[
q = \frac{2s}{l - j} \quad \text{and} \quad r = \frac{2s}{l + k - j}.
\]
Since
\[
\frac{k}{p} = \frac{j}{r} + \frac{k - j}{q},
\]
we have \(p = \frac{2s}{t}\). From Theorem 2.4 it follows that
\[
\|D^j T\|_{\frac{2s}{l - j}} \leq C(m, s, j) \cdot \|D^k T\|_{\frac{2s}{l + k - j}} \cdot \|T\|_{1 - \frac{1}{l - j}}^{\frac{1}{2}},
\]
which means
\[
\|D^j T\|_p \leq C(m, r, q, j, k) \cdot \|D^k T\|_q^{\frac{1}{2}} \cdot \|T\|_q^{1 - \frac{1}{q}}.
\]

**Case 2:** When \(2 \leq q < r \leq \infty\), the proof is similar.

**Case 3:** When \(2 \leq q = r\), clearly we have \(p = q = r\). From 12.1 Theorem in [2] it follows that
\[
\|DT\|_p \leq C(m, p) \cdot \|D^2 T\|_p \cdot \|T\|_p^{\frac{1}{2}},
\]
which implies
\[
\|D^j T\|_p \leq C(m, p) \cdot \|D^{j+1} T\|_p^{\frac{1}{2}} \cdot \|D^{j-1} T\|_p^{\frac{1}{2}}.
\]
Let \(f(j) := \|D^j T\|_p\). It is easy to check that \(f\) meets the condition of 12.5 Corollary in [2]. Then we conclude this theorem. \(\square\)
3. Proof of Theorem 1.1

Given any \( T > 0 \), define an operator

\[
P : C^1([0, T], \Gamma^2(E)) \longrightarrow C([0, T], \Gamma(E))
\]

here

\[
P(V) := \partial_t V - \Delta V - V \times \Delta V - \lambda (1 + \mu |V|^2) V.
\]

It is not difficult to check that the leading coefficient of the linearised operator of \( P \) meets Legendre-Hadamard condition. By Main Theorem 1 in page 3 of \([1]\) we know \((1.6)\) admits a unique local smooth solution \( \hat{V} \) provided \( V_0 \in \Gamma^\infty(E) \).

In the sequel, we would like to know the lower bound of maximal existence time \( T_{\text{max}} \) of the above smooth solution. Our strategy is to deduce a Gronwall inequality. That is to say, we shall control \( \frac{d}{dt} ||V(t)||^2_{H^1} \). Before getting to this, it is important to obtain an upper bound of \( ||V(t)||_{\infty} \).

Taking inner product with \( |V|^{p-2}V (p > 2) \) in \((1.6)\), and integrating the result over \( M \), we get

\[
\int_M |V|^{p-2} \langle V, \partial_t V \rangle \, dM = \int_M |V|^{p-2} \langle V, \Delta V \rangle \, dM - \lambda \int_M (1 + \mu |V|^2) |V|^p \, dM
\]

\[
\leq - \int_M |V|^{p-2} \cdot |DV|^2 \, dM - (p - 2) \int_M |V|^{p-4} \cdot |\langle V, DV \rangle|^2 \, dM \leq 0.
\]

The left hand side of the above inequality is \( \frac{1}{p} \frac{d}{dt} ||V(t)||^p_{p} \), so this inequality means

\[
||V(t)||_{p} \leq ||V_0||_{p} \quad \forall t \in [0, T_{\text{max}}).
\]

Taking the limit \( p \to \infty \) leads to

\[
||V(t)||_{\infty} \leq ||V_0||_{\infty} \quad \forall t \in [0, T_{\text{max}}).
\]

Given \( k \geq 1 \), recalling our appointment \((2.8)\), we have the next identity

\[
\frac{1}{2} \frac{d}{dt} \int_M |D^k V|^2 \, dM = \int_M g^{pq} g^{ij_1} \ldots g^{i_k j_k} \langle V, j_1 \ldots j_k, V_{pq i_1 \ldots i_k} \rangle \, dM
\]

\[
+ \int_M g^{pq} g^{ij_1} \ldots g^{i_k j_k} \langle V, j_1 \ldots j_k, (V \times V_p)_{q_1 \ldots q_k} \rangle \, dM
\]

\[
- \lambda \cdot \mu \int_M g^{i_1 j_1} \ldots g^{i_k j_k} \langle V, j_1 \ldots j_k, (|V|^2 V)_{i_1 \ldots i_k} \rangle \, dM
\]

\[
- \lambda \int_M |D^k V|^2 \, dM
\]

Applying \((2.9)\) and \((2.10)\) to exchange the order of derivatives yields

\[
\frac{1}{2} \frac{d}{dt} \int_M |D^k V|^2 \, dM = - \int_M |D^{k+1} V|^2 \, dM + \int_M D^k V \ast q_k(V, R^E) \, dM
\]

\[
+ \int_M D^k V \ast q_{k-1}(DV, R^M) \, dM
\]

\[
- \int_M g^{pq} g^{i_1 j_1} \ldots g^{i_k j_k} \langle V, j_1 \ldots j_k, (V \times V_p)_{i_1 \ldots i_k} \rangle \, dM
\]
\[
- \int_M D^k V * q_{k-1}(V \times DV, \mathcal{R}^M) \, dM \\
- \int_M D^k V * q_{k-1}(V \times DV, \mathcal{R}^E) \, dM \\
- \lambda \int_M |D^k V|^2 \, dM - \lambda \mu \int_M |V|^2 \cdot |D^k V|^2 \, dM \\
- \lambda \mu \sum_{i+j=k-1} b_{ij} \cdot \int_M D^k V \ast D^{i+1}(|V|^2) \ast D^i V \, dM \\
+ \int_M q_k(V, \mathcal{R}^E) * D^{k-1}(V \times DV) \, dM \\
+ \int_M q_{k-1}(DV, \mathcal{R}^M) * D^{k-1}(V \times DV) \, dM,
\]

Here \(b_{ij} \in \mathbb{Z}^+\) are some universal constants. Note that

\[
g^{pq}g^{ij_1} \ldots g^{ij_k}(V, q_{j_1 \ldots j_k}, (V \times V_p)_{i_1 \ldots i_k}) = \sum_{i+j=k-1} a_{ij} \cdot D^{k+1}V \ast (D^{i+1}V \times D^{j+1}V)
\]

where \(a_{ij} \in \mathbb{Z}^+\) are some universal constants. Taking norms on the right hand side of (3.2) leads to

\[
\frac{1}{2} \frac{d}{dt} \int_M |D^k V|^2 \, dM \leq - \int_M |D^{k+1}V|^2 \, dM + \sum_{i=0}^k C_i \int_M |D^i V| \cdot |D^{i+1}V| \, dM \\
+ \sum_{i+j=k-1} a_{ij} \int_M |D^{k+1}V| \cdot |D^{i+1}V| \cdot |D^j V| \, dM \\
+ \sum_{i=0}^{k-1} \bar{C}_i \int_M |D^i V| \cdot |D^{i+1}(V \times DV)| \, dM \\
+ \lambda \mu \sum_{i+j=k-1} b_{ij} \int_M |D^k V| \cdot |D^{i+1}(|V|^2)| \cdot |D^j V| \, dM \\
+ \sum_{i=0}^k C_i \int_M |D^i V| \cdot |D^{k-1}(V \times DV)| \, dM,
\]

where \(C_i \) and \(\bar{C}_i \) depend upon \(\mathcal{R}^M, \mathcal{R}^E\) and their covariant differentiations. Applying (2.1) and (2.4) yields

\[
\frac{1}{2} \frac{d}{dt} \int_M |D^k V|^2 \, dM \leq - \int_M |D^{k+1}V|^2 \, dM + \sum_{i+j=k-1} a_{ij} \int_M |D^{k+1}V| \cdot |D^{i+1}V| \cdot |D^j V| \, dM \\
+ \bar{C}_k \{||D^k V||_2 \cdot ||V||_{H^k} + \sum_{0 \leq r \leq k-1} \sum_{0 \leq q \leq k-1} \int_M |D^k V| \cdot |D^r V| \cdot |D^q V| \cdot |D^j V| \, dM \}
\]

\[
(3.3) + \sum_{r+q+j=k} \int_M |D^k V| \cdot |D^r V| \cdot |D^q V| \cdot |D^j V| \, dM
\]
\[ + \sum_{i=0}^{k} \sum_{r+q+j=k-1} \int_{M} |D^i V| \cdot |D^r V| \cdot |D^{q+1} V| \, dM \],

where \( \tilde{C}_k \) depends upon \( R^M, R^F \) and their covariant differentiations.

**Lemma 3.1.** There is a \( C'_{m_0} > 0 \) depending on \( R^M, R^F \) and their covariant differentiations such that, for any \( t \in [0, T_{max}) \), we have

\[
\frac{d}{dt} \|V(t)\|_{H^{m_0}}^2 \leq C'_{m_0} \cdot (1 + \|V_0\|_{H^{m_0}}^2, \|V(t)\|_{H^{m_0}}^2 + \|V(t)\|_{H^{m_0}}^4). 
\]

**Proof.** Given \( 1 \leq k \leq m_0 \), we consider

\[
\sum_{r+q+j=k} \int_{M} |D^k V| \cdot |D^r V| \cdot |D^q V| \cdot |D^j V| \, dM
\]

\[
= \sum_{r+q+j=k \max\{r,q,j\}=k} \int_{M} |D^k V| \cdot |D^r V| \cdot |D^q V| \cdot |D^j V| \, dM
\]

\[
+ \sum_{r+q+j=k \max\{r,q,j\}=k-1} \int_{M} |D^k V| \cdot |D^r V| \cdot |D^q V| \cdot |D^j V| \, dM.
\]

Clearly,

\[
\sum_{r+q+j=k \max\{r,q,j\}=k} \int_{M} |D^k V| \cdot |D^r V| \cdot |D^q V| \cdot |D^j V| \, dM \leq \|D^k V\|_{L_2}^2 \cdot \|V\|_{\infty}^2 \leq \|D^k V\|_{L_2}^2 \cdot \|V_0\|_{\infty}^2.
\]

And we want to derive the following

\[
\sum_{r+q+j=k \max\{r,q,j\}=k-1} \int_{M} |D^k V| \cdot |D^r V| \cdot |D^q V| \cdot |D^j V| \, dM
\]

\[
\leq \sum_{r+q+j=k \max\{r,q,j\}=k-1} \|D^k V\|_{2} \cdot \|D^r V\|_{p_r} \cdot \|D^q V\|_{p_q} \cdot \|D^j V\|_{p_j}.
\]

where \( p_r, p_q \) and \( p_j \), belonging to \([1, \infty]\), will be determined later and satisfy

\[
(3.4) \quad \frac{1}{p_r} + \frac{1}{p_q} + \frac{1}{p_j} = \frac{1}{2}.
\]

And then we employ Theorem 2.1 due to [3] to obtain

\[
\|D^r V\|_{p_r} \leq \|V\|_{H^{m_0}}^{\alpha_{r}} \cdot \|V\|_{L_2}^{1-\alpha_{r}} \leq \|V\|_{H^{m_0}},
\]

\[
\|D^q V\|_{p_q} \leq \|V\|_{H^{m_0}}^{\alpha_{q}} \cdot \|V\|_{L_2}^{1-\alpha_{q}} \leq \|V\|_{H^{m_0}},
\]

and

\[
\|D^j V\|_{p_j} \leq \|V\|_{H^{m_0}}^{\alpha_{j}} \cdot \|V\|_{L_2}^{1-\alpha_{j}} \leq \|V\|_{H^{m_0}}.
\]
We hope $p_r$, $p_q$ and $p_j$ meet the next conditions:

**Condition 1.**
\[
\frac{1}{p_r} = \frac{r}{m} + \frac{1}{2} - a_r \cdot \frac{m_0}{m} \quad \text{with} \quad a_r \in \left[ \frac{r}{m_0}, 1 \right],
\]
which is equivalent to
\[
\left(3.5\right) \quad \frac{1}{p_r} \in \left( \frac{r - m_0}{m} + \frac{1}{2}, \frac{1}{2} \right).
\]

**Condition 2.**
\[
\frac{1}{p_q} = \frac{q}{m} + \frac{1}{2} - a_q \cdot \frac{m_0}{m} \quad \text{with} \quad a_q \in \left[ \frac{q}{m_0}, 1 \right],
\]
which is equivalent to
\[
\left(3.6\right) \quad \frac{1}{p_q} \in \left( \frac{q - m_0}{m} + \frac{1}{2}, \frac{1}{2} \right).
\]

**Condition 3.**
\[
\frac{1}{p_j} = \frac{j}{m} + \frac{1}{2} - a_j \cdot \frac{m_0}{m} \quad \text{with} \quad a_j \in \left[ \frac{j}{m_0}, 1 \right],
\]
which is equivalent to
\[
\left(3.7\right) \quad \frac{1}{p_j} \in \left( \frac{j - m_0}{m} + \frac{1}{2}, \frac{1}{2} \right).
\]

We claim there exist $p_r$, $p_q$ and $p_j$ which are in $[1, \infty]$ and satisfy (3.4), (3.5), (3.6) and (3.7). Obviously, that this claim holds is equivalent to
\[
\left(3.8\right) \quad (r - m_0 + \frac{1}{2}) + (q - m_0 + \frac{1}{2}) + (j - m_0 + \frac{1}{2}) < \frac{1}{2} \iff k < 3m_0 - m.
\]

Since $k \leq m_0$ and $m_0 > \frac{m}{2}$, (3.8) is true. In other words,
\[
\sum_{r+q+j=k} \int_M |D^kV| \cdot |D^rV| \cdot |D^qV| \cdot |D^jV| \, dM \lesssim ||D^kV||_2 \cdot ||V||_{H^{m_0}}^3
\]

In conclusion,
\[
\left(3.9\right) \quad \sum_{r+q+j=k} \int_M |D^kV| \cdot |D^rV| \cdot |D^qV| \cdot |D^jV| \, dM \lesssim ||D^kV||_2 \cdot ||V||_{H^{m_0}}^3 + ||D^kV||_2^2 \cdot ||V_0||_{\infty}^2.
\]

For the other terms of (3.3), using the same methods, we get similar estimations:

**Estimation 1.**
\[
\sum_{i+j=k-1} a_{ij} \int_M |D^{i+1}V| \cdot |D^{j+1}V| \cdot |D^{i+1}V| \, dM \lesssim ||D^{k+1}V||_2 \cdot ||D^kV||_2 \cdot ||DV||_{\infty} + \sum_{i+j=k-1} a_{ij} \cdot ||D^{k+1}V||_2 \cdot ||D^{i+1}V||_{p_i} \cdot ||D^{j+1}V||_{p_j} \lesssim ||D^{k+1}V||_2 \cdot ||D^kV||_2 \cdot ||DV||_{\infty} + ||D^{k+1}V||_2 \cdot ||V||_{H^{m_0}}^2.
\]
This completes the proof. \(\Box\)

**Estimation 2.**
\[
\sum_{0 \leq r+q \leq k-1} \int_M |D^rV| \cdot |D^qV| \cdot |D^{r+1}V| \, dM \lesssim \|V_0\|_{\infty} \cdot \|D^kV\|_2 \cdot \|V\|_{H^{m_0}}^2,
\]

**Estimation 3.**
\[
\sum_{i=0}^k \sum_{r+q=k-1} \int_M |D^rV| \cdot |D^qV| \cdot |D^{r+1}V| \, dM \lesssim \|V\|_{H^k} \cdot \|D^kV\|_2 \cdot \|V_0\|_{\infty} + \|V\|_{H^k} \cdot \|V\|_{H^{m_0}}^2.
\]

Summing \(k\) from 0 to \(m_0\) gives
\[
\frac{1}{2} \frac{d}{dt} \|V\|_{H^{m_0}}^2 \leq -\|V\|_{H^{m_0+1}}^2 + L_{m_0} \cdot \|V\|_{H^{m_0+1}} \cdot \|V\|_{H^{m_0}}^2
\]
\[
+ \tilde{L}_{m_0} \cdot \left\{ \|V\|_{H^{m_0}}^2 + \|V_0\|_{\infty} \cdot \|V\|_{H^{m_0}}^2
\]
\[
+ \|V\|_{H^{m_0}}^3 + \|V_0\|_{\infty}^2 \cdot \|V\|_{H^{m_0}}^2 \right\},
\]
where \(L_{m_0}\) is universal and \(\tilde{L}_{m_0}\) depends on \(\mathcal{R}^M, \mathcal{R}^E\) and their covariant differentiations. Then the result follows easily from Young’s inequality and Sobolev embedding
\[
\|V_0\|_{\infty} \lesssim \|V_0\|_{H^{m_0}}.
\]
This completes the proof. \(\Box\)

Consider an ODE
\[
\begin{aligned}
\frac{df}{dt} &= C_{m_0}' \cdot (1 + \|V_0\|_{H^{m_0}}^2) \cdot (f + f^2), \\
f(0) &= \|V_0\|_{H^{m_0}}^2.
\end{aligned}
\tag{3.10}
\]
Solving the above equation to get an expression of \(f\), we know that the maximal existence time of the solution \(f(\cdot, \|V_0\|_{H^{m_0}})\) to (3.10) is not smaller than
\[
T^* := \frac{1}{C_{m_0}' \cdot (1 + \|V_0\|_{H^{m_0}}^2)} \log \left( \frac{1 + 2\|V_0\|_{H^{m_0}}^2}{2\|V_0\|_{H^{m_0}}^2} \right).
\]
And \(f(t, \|V_0\|_{H^{m_0}})\) is monotone increasing with respect to \(t\). In other words, for all \(t \in [0, T^*]\),
\[
f(t, \|V_0\|_{H^{m_0}}) \leq f(T^*, \|V_0\|_{H^{m_0}}) = 1 + 2\|V_0\|_{H^{m_0}}^2.
\]
By comparison principle of ODE, we know that for any \(t \in [0, \min\{T_{\max}, T^*\})\),
\[
\|V(t)\|_{H^{m_0}} \leq \sqrt{1 + 2\|V_0\|_{H^{m_0}}^2}.
\]

In the sequel, we focus on the case that \(k\) is sufficiently big.

**Lemma 3.2.** When \(k \geq m_0 + 1\), there is a \(Q_k > 0\) depending on \(\mathcal{R}^M, \mathcal{R}^E\) and their covariant differentiations such that, for any \(t \in [0, T_{\max})\), we have
\[
\begin{aligned}
\frac{d}{dt} \|V(t)\|_{H^k}^2 &= \|V(t)\|_{H^{k+1}}^2 + \|V(t)\|_{H^{m_0}}^2 + \|V(t)\|_{H^{k-1}}^4 + \|V(t)\|_{H^k}^2 \cdot (1 + \|V_0\|_{\infty} + \|V_0\|_{\infty}^2)
\end{aligned}
\tag{3.11}
\]
\[
\leq Q_k \cdot \left\{ \|V(t)\|_{H^k} \cdot \|V(t)\|_{H^{m_0}} + \|V(t)\|_{H^{k-1}}^2 + \|V(t)\|_{H^k} \cdot (1 + \|V_0\|_{\infty} + \|V_0\|_{\infty}^2)\right\}
\]
\[ + \|V(t)\|_{H_{k-1}}^2 \cdot (\|V(t)\|_{H^{m_0}}^2 + \|V(t)\|_{H^{m_0}}^2 + \|V(t)\|_{H^{k-1}}^2). \]

**Proof.** Firstly, let us calculate one term of (3.3). Applying the same method of (3.9), one can see easily that there are \( p_i \) belonging to \([1, \infty]\) such that the following inequalities hold

\[
\sum_{i+j=k-1} a_{ij} \int_M |D^{k+1}V| \cdot |D^{i+1}V| \cdot |D^{j+1}V| dM \\
\lesssim \|D^{k+1}V\|_2 \cdot \|D^kV\|_2 \cdot \|DV\|_\infty + \|D^{k+1}V\|_2 \cdot \|D^{k-1}V\|_2 \cdot \|D^2V\|_\infty \\
\quad + \sum_{i+j=k-1} a_{ij} \cdot \|D^{k+1}V\|_2 \cdot \|D^{i+1}V\|_{p_i} \cdot \|D^{j+1}V\|_{p_j} \\
\lesssim \|D^{k+1}V\|_2 \cdot \|D^kV\|_2 \cdot \|V\|_{H^{m_0}} + \|D^{k+1}V\|_2 \cdot \|D^{k-1}V\|_2 \cdot \|V\|_{H^{m_0}} \\
\quad + \|D^{k+1}V\|_2 \cdot \|V\|_{H^{k-1}}^2.
\]

By the same procedure, we get the next estimations:

**Estimation 4.**

\[
\sum_{i+j=k-1} a_{ij} \int_M |D^{k+1}V| \cdot |D^rV| \cdot |D^{q+1}V| dM \\
\lesssim \|D^kV\|_2^2 \cdot \|V_0\|_\infty + \|D^kV\|_2 \cdot \|D^{k-1}V\|_2 \cdot \|V\|_{H^{m_0}} \\
\quad + \|D^kV\|_2 \cdot \|D^{k-2}V\|_2 \cdot \|V\|_{H^{m_0}} + \|D^kV\|_2 \cdot \|V\|_{H^{k-1}}^2.
\]

**Estimation 5.**

\[
\sum_{r+q+j=k} \int_M |D^kV| \cdot |D^rV| \cdot |D^qV| \cdot |D^jV| dM \\
\lesssim \int_M |D^kV|_2^2 \cdot \|V\|_2^2 dM + \int_M |D^kV| \cdot |D^{k-1}V| \cdot |DV| \cdot \|V\|_2 dM \\
\quad + \sum_{r+q+j=k} \int_M |D^kV| \cdot |D^rV| \cdot |D^qV| \cdot |D^jV| dM \\
\lesssim \|D^kV\|_2^2 \cdot \|V_0\|_\infty^2 + \|D^kV\|_2 \cdot \|D^{k-1}V\|_2 \cdot \|DV\|_\infty \cdot \|V\|_\infty + \|D^kV\|_2 \cdot \|V\|_{H^{k-1}}^3 \\
\lesssim \|D^kV\|_2^3 \cdot \|V_0\|_\infty^2 + \|D^kV\|_2 \cdot \|D^{k-1}V\|_2 \cdot \|V\|_{H^{m_0}}^2 + \|D^kV\|_2 \cdot \|V\|_{H^{k-1}}^3.
\]

**Estimation 6.**

\[
\sum_{i=0}^k \sum_{r+q=k-1} \int_M |D^iV| \cdot |D^rV| \cdot |D^{q+1}V| dM \\
\lesssim \|V\|_{H^k} \cdot \|V\|_{\infty} \cdot \|D^kV\|_2 + \|V\|_{H^k} \cdot \|D^{k-1}V\|_2 \cdot \|DV\|_\infty \\
\quad + \|V\|_{H^k} \cdot \|D^{k-2}V\|_2 \cdot \|D^2V\|_\infty + \|V\|_{H^k} \cdot \|V\|_{H^{k-1}}^2 \\
\lesssim \|V\|_{H^k} \cdot \|V_0\|_\infty \cdot \|D^kV\|_2 + \|V\|_{H^k} \cdot \|D^{k-1}V\|_2 \cdot \|V\|_{H^{m_0}} \\
\quad + \|V\|_{H^k} \cdot \|D^{k-2}V\|_2 \cdot \|V\|_{H^{m_0}} + \|V\|_{H^k} \cdot \|V\|_{H^{k-1}}^2.
\]
Substituting (3.12), (3.13), (3.14) and (3.15) into (3.3) and then summing $k$ lead to
\[
\frac{1}{2} \frac{d}{dt} ||V||^2_{H^k} \leq -||V||^2_{H^{k+1}} + \tilde{Q}_k \cdot (||V||_{H^{k+1}} \cdot ||V||_{H^k} \cdot ||V||_{H^m_0} + ||V||_{H^{k+1}} \cdot ||V||_{H^{k-1}}) \\
+ \tilde{Q}_k \cdot (||V||_{H^k} \cdot ||V||_{H^m_0} + ||V||_{H^{k+1}} \cdot ||V||_{H^m_0} \\
+ ||V||_{H^k} \cdot ||V||_{H^{k-1}} + ||V||_{H^m_0} + ||V||_{H^k} \cdot ||V||_{H^{k-1}} \cdot ||V||_{H^m_0} \\
+ ||V||_{H^k} \cdot ||V||_{H^{k-1}},
\]
where $\tilde{Q}_k > 0$ depends on $R^M$, $R^E$ and their covariant differentiations. Using Young’s inequality, we conclude this theorem.

\[ \square \]

Note that (3.11) is linear for $||V||^2_{H^k}$. It is now clear that inductively using (3.11) one can show the existence of $N_k = N(||V_0||_{H^k}, \tilde{Q}_k, Q_{k-1}, \cdots, Q_{m_0+1})$ for any $k \geq m_0 + 1$ such that
\[
||V(t)||_{H^k} \leq N_k \quad \forall t \in [0, \min\{T_{\max}, T^*\}),
\]
which implies
\[
T_{\max} \geq T^*.
\]

Now we return to prove Theorem 1.1. Define
\[
h(x) := \frac{1}{C_{m_0} \cdot (1 + x) \log \left(1 + \frac{2x}{2x}\right)}
\]
and we observe that it is a monotone decreasing function. Given $l \geq m_0 + 1$ and $V_0 \in H^l(E)$, there are $V_{i0} \in \Gamma^\infty(E)$ such that as $i \to \infty$,
\[
V_{i0} \rightarrow V_0 \quad \text{strongly in } H^l(E).
\]
By the above discussion we know there exist
\[
T^*_i \geq h(||V_{i0}||^2_{H^{m_0}}) > 0 \quad \text{and} \quad V_i \in C^\infty([0, T^*_i], \Gamma^\infty(E))
\]
such that
\[
\begin{align*}
\partial_t V_i &= \Delta V_i + V_i \times \Delta V_i - \lambda \cdot (1 + \mu|V_i|^2)V_i \\
V_i(0, \cdot) &= V_{i0},
\end{align*}
\]
here $T^*_i$ is the maximal existence time of $V_i$. Obviously, when $i$ is enough large,
\[
||V_{i0}||^2_{H^{m_0}} \leq ||V_0||^2_{H^{m_0}} + 1 \quad \text{and} \quad ||V_{i0}||_{H^l} \leq ||V_0||_{H^l} + 1,
\]
which imply
\[
T^*_i \geq h(||V_0||^2_{H^{m_0}} + 1) := 2\tilde{\delta} > 0
\]
and
\[
||V_i(t)||_{H^l} \leq N(||V_0||_{H^l} + 1, Q_k, Q_{k-1}, \cdots, Q_{m_0+1}) \quad \forall t \in [0, \tilde{\delta}].
\]
Then $V_i$ is a bounded sequence in $L^\infty([0, \tilde{\delta}], H^l(E))$. It is not hard to verify that $\partial_t V_i$ is a bounded sequence in $L^\infty([0, \tilde{\delta}], L^2(E))$. So there exists a $V \in L^\infty([0, \tilde{\delta}], H^l(E))$ and a subsequence which is still denoted by $\{V_i\}$ such that
\[
V_i \rightharpoonup V \quad \text{weakly * in } L^\infty([0, \tilde{\delta}], H^l(E)).
\]
By Aubin-Lions lemma, one can find a subsequence still denoted by \( \{V_i\} \) such that
\[
V_i \longrightarrow V \quad \text{strongly in } L^\infty([0, \tilde{\delta}], H^{l-1}(E)).
\]

Because \( l - 1 \geq m_0 \), \( H^{l-1}(E) \) can be embedded into \( \Gamma^2(E) \). In other words, \( V \) is a solution to (1.6). Using LLB to transform time derivatives into spatial derivatives gives that for all \( 0 \leq j \leq \left\lfloor \frac{l}{m_0} \right\rfloor \) and all \( \alpha \leq l - m_0 j \), we have
\[
\partial_t^j D^\alpha V \in L^\infty([0, \tilde{\delta}], L^2(E)).
\]

**Remark 3.3.** The proof of (3.17) is easy if one employs induction for \( j \).

At last, since \( l \geq \left\lceil \frac{m_0}{2} \right\rceil + 4 \), by the same method of Theorem 3 in [41] it is not difficult to know that the solution of (1.6) with initial data \( V_0 \in H^l(E) \) is unique. This completes the proof. \( \square \)

### 4. Proof of Theorem 1.2

Now we focus on global existence of LLB. Suppose that \( V \) is the local smooth solution of (1.6). Our trick is to deduce a uniform estimation for \( ||V||_{H^k} \). To this goal, firstly we should get a linear Gronwall inequality.

By (3.3) and Hölder inequality, we have
\[
\frac{1}{2} \frac{d}{dt} ||D^k V||_2^2 \leq -||D^{k+1} V||_2^2 + \sum_{i+j=k-1} a_{ij} \cdot ||D^{k+1} V||_2 \cdot ||D^i V| \cdot |D^j V||_2
\]
\[
+ \tilde{C}_k \cdot \left( ||V||^2_{H^k} + ||D^k V||_2 \sum_{0 \leq r+q \leq k-1} ||D^r V| \cdot |D^q V| \cdot |D^j V||_2 \right)
\]
\[
||D^k V||_2 \sum_{r+q+j=k} \left( ||D^r V| \cdot |D^q V| \cdot |D^j V||_2 \right) + ||V||_{H^k} \sum_{r+q=k-1} ||D^r V| \cdot |D^q V| \cdot |D^j V||_2 \right).
\]

For the second term on the right hand side of (4.1),
\[
||D^{i+1} V| \cdot |D^{j+1} V||_2 \leq ||D^{i+1} V||_{k+2} \cdot ||D^{k-i} V||_{k+2},
\]
since \( i + j = k - 1 \). Theorem 2.4 implies
\[
||D^{i+1} V||_{k+2} \lesssim ||D^{k+1} V||_{k+2} \cdot ||V||_{k+2}
\]
and
\[
||D^{k-i} V||_{k+2} \lesssim ||D^{k+1} V||_{k+2} \cdot ||V||_{k+2}.
\]

So
\[
\sum_{i+j=k-1} a_{ij} \cdot ||D^{k+1} V||_2 \cdot ||D^{i+1} V| \cdot |D^j V||_2
\]
\[
\leq B_k \cdot ||V||_{k+2} \cdot ||D^{k+1} V||_2 \leq B_k \cdot ||V||_{k+2} \cdot ||D^{k+1} V||^2_2.
\]
where $B_k$ is a universal constant. By the same way, we will get

\begin{equation}
\sum_{0 \leq r + q < k - 1} \left\| [D^r V] \cdot [D^{q+1} V] \right\|_2 \leq \sum_{0 \leq r + q < k - 1} \left\| [D^r V] \cdot [D^{q+1} V] \right\|_2 \leq B_k \cdot \{ |V_0|^2 \} \cdot \| D^{k+1} V \|_2 \tag{4.3} \end{equation}

Moreover, Theorem 2.4 yields

\begin{equation}
\sum_{0 \leq r + q < k - 1} \left\| [D^r V] \cdot [D^{q+1} V] \right\|_2 \leq \sum_{r + q + j = k} \left\| [D^r V] \cdot [D^{q+1} V] \right\|_2 \leq B_k \cdot \{ |V_0|^2 \} \cdot \| D^{k+1} V \|_2 \tag{4.4} \end{equation}

and

\begin{equation}
\sum_{r + q = k - 1} \left\| [D^r V] \cdot [D^{q+1} V] \right\|_2 \leq \| D^k V \|_2 \cdot \| V_0 \|_\infty \tag{4.5} \end{equation}

Substituting (4.2), (4.3), (4.4) and (4.5) into (4.1) leads to

\begin{equation}
\frac{1}{2} \frac{d}{dt} \| D^k V \|_2^2 + (1 - B_k \cdot \| V_0 \|_\infty) \cdot \| D^{k+1} V \|_2^2 \leq G_k \cdot \{ |V|^2 \} \cdot \| D^{k+1} V \|_2 \cdot \| V_0 \|_\infty \cdot \| D^k V \|_2 \cdot \| V_0 \|_\infty \tag{4.6} \end{equation}

where $G_k$ depends upon $\mathcal{R}^M$, $\mathcal{R}^E$ and their covariant differentiations. In the sequel, using Gronwall inequality gives the following theorem.

**Theorem 4.1.** Given $N \in \mathbb{N}$, there exists an $\tilde{B}_N > 0$ such that if $\| V_0 \|_\infty \leq \tilde{B}_N$, we will obtain

\begin{equation}
\| D^k V(t) \|_2^2 + \int_0^t \| D^{k+1} V(s) \|_2^2 ds \leq C_k (\| V_0 \|_{H^k}, \tilde{B}_N, t), \tag{4.7} \end{equation}

provided $0 \leq k \leq N$ and $t \in [0, T_{\text{max}})$. Here $C_k(x, y, t)$ is monotone increasing with respect to $x$ and $t$.

**Proof.** Employ induction for $N$.

In the case $N = 0$, let $\tilde{B}_0 := 1$. Taking inner product with $V$ in (1.6) and then integrating the result over $M$, we get

\begin{equation}
\frac{1}{2} \frac{d}{dt} \| V(t) \|_2^2 + \| D V(t) \|_2^2 + \lambda \int_M (1 + |V(t)|^2) \cdot |V(t)|^2 dM = 0 \tag{4.8} \end{equation}

which is equivalent to

\begin{equation}
\| V(t) \|_2^2 + 2 \int_0^t \| D V(s) \|_2^2 ds \tag{4.9} \end{equation}
\[+2\lambda \int_0^t ds \int_M (1 + \mu |V(s)|^2) \cdot |V(s)|^2 dM = ||V_0||_2^2.\]

Assume that for all the indices not larger than \(N\), (4.7) holds. Now we consider \(N + 1\).

Take \(\tilde{B}_{N+1} := \min\{\tilde{B}_N, \frac{1}{2B_{N+1}}\}\). If \(||V_0||_\infty \leq \tilde{B}_{N+1},\) (4.6) gives

\[
(4.9) \quad \frac{1}{2} \frac{d}{dt} ||D^{N+1}V||_2^2 + \frac{1}{2} ||D^{N+2}V||_2^2 \leq G_{N+1} \cdot (1 + \tilde{B}_{N+1} + \tilde{B}_{N+1}^2) \cdot \left\{ ||D^{N+1}V||_2^2 + \sum_{k=0}^{N} C_k(|||V_0|||_{H^k}, \tilde{B}_N, t) \right\}.
\]

Then this theorem follows easily from Gronwall inequality. This completes the proof. \(\square\)

Now we return to prove Theorem 1.2. Given \(T > 0\) and \(N \geq m_0 + 1 = \left[ \frac{m}{\pi} \right] + 4\), we take any \(V_0 \in H^N(E)\) with \(||V_0||_\infty \leq \frac{1}{2} \tilde{B}_N\) (This \(\tilde{B}_N\) is from Theorem 4.1). Then there are \(V_0 \in \Gamma^\infty(E)\) converging to \(V_0\) strongly in \(H^N(E)\).

Suppose \(V_i\) satisfies

\[
(4.10) \quad \begin{cases}
\partial_t V_i = \Delta V_i + V_i \times \Delta V_i - \lambda \cdot (1 + \mu \cdot |V_i|^2)V_i \\
V_i(0, \cdot) = V_{0i}
\end{cases}
\]

and its maximal existence time is \(T_i^*\). As \(i\) is large enough, we have

\[||V_{0i}||_\infty \leq 2||V_0||_\infty \leq \tilde{B}_N, \quad ||V_{0i}||_{H^{m_0}} \leq 2||V_0||_{H^{m_0}}\]

and \(||V_0||_{H^N} \leq 2||V_0||_{H^N}\).

If \(T_i^* < T\), then by Theorem 4.1,

\[||V_i(t)||_{H^{m_0}} + \int_0^t ||DV_i(s)||_{H^{m_0}} ds \leq C_{m_0}(||V_{0i}||_{H^{m_0}}, \tilde{B}_N, t) \leq C_{m_0}(2||V_0||_{H^{m_0}}, \tilde{B}_N, T),\]

provided \(t \in [0, T_i^*)\). Review that in the proof of Theorem 1.1 we have defined a monotone decreasing function

\[h(x) := \frac{1}{C''_{m_0} \cdot (1 + x)} \log \left( \frac{1 + 2x}{2x} \right).\]

So for arbitrary \(t \in [0, T_i^*)\), it is obvious to see

\[h(||V_i(t)||_{H^{m_0}}^2) \geq h(C_{m_0}(2||V_0||_{H^{m_0}}, \tilde{B}_N, T)) := \delta_0 > 0.\]

Now we bring in a new system

\[
(4.11) \quad \begin{cases}
\partial_t \hat{V}_i = \Delta \hat{V}_i + \hat{V}_i \times \Delta \hat{V}_i - \lambda \cdot (1 + \mu \cdot |\hat{V}_i|^2)\hat{V}_i \quad \text{in} \quad (T_i^* - \frac{\delta_0}{2}, \infty) \times M \\
\hat{V}_i(T_i^* - \frac{\delta_0}{2}, \cdot) = V_i(T_i^* - \frac{\delta_0}{2}, \cdot)
\end{cases}
\]

The maximal existence time of \(\hat{V}_i\) is not smaller than \(h(||V_i(T_i^* - \frac{\delta_0}{2})||_{H^{m_0}}^2)\) which is not smaller than \(\delta_0\). By the uniqueness we know that for any \(t \in [T_i^* - \frac{\delta_0}{2}, T_i^*)\), \(\hat{V}_i(t) = V_i(t)\).
It means that $V_i'$ can be extended to $[0, T_i^* + \frac{\delta}{2})$. Because $T_i^*$ is maximal, we get a contradiction. So $V_i \in C^\infty([0, T], \Gamma^\infty(E))$ and for all $t \in [0, T]$,
\[
||V_i(t)||_{L^2_{tH^N}} + \int_0^t ||D \overline{V_i}(s)||_{L^2_{tH^N}} ds \leq C_N(2||V_0||_{H^N}, \tilde{B}_N, T).
\]

By the same method we prove local well-posedness one can know there is a
\[
V \in L^\infty([0, T], H^N(E)) \cap L^2([0, T], H^{N+1}(E))
\]
such that $V_i$ converges to $V$ strongly in $L^\infty([0, T], H^{N-1}(E))$ (in the sense of picking subsequence). It means $V$ is a solution of LLB.

At last, we claim (1.8) and (1.9) are true. Since (1.8) is easy, we only prove (1.9).

**Proof.** Employ induction for $i$.

When $i = 0$, (1.9) holds.

Suppose that for all the indices not bigger than $i$, (1.9) is true.

Now we consider $i + 1$. Choose any $\beta \in [0, N+1-(\hat{m}+1)(i+1)] \cap \mathbb{Z}$. Applying $\partial_t^i D^\beta$ to both sides of (1.6), we get
\[
\partial_t^{i+1} D^\beta V = \partial_t^i D^\beta \Delta V - \partial_t^i D^\beta (V \times \Delta V) - \lambda \cdot \partial_t^i D^\beta \left[(1 + \mu |V|^2) V\right],
\]
which implies
\[
(4.12) \quad \int_0^T ||\partial_t^{i+1} D^\beta V(s)||_2^2 ds \lesssim \int_0^T ||\partial_t^i D^{\beta+2} V(s)||_2^2 ds + \int_0^T ||\partial_t^i D^\beta V(s)||_2^2 ds
\]
\[
+ \sum_{i', \beta'} \int_0^T \left| |\partial_t^{i'} D^\beta V(s)| \cdot |\partial_t^{i'-i'} D^{\beta'-2} V(s)| \right|_2^2 ds
\]
\[
+ \sum_{i_1+i_2+i_3=i, \beta_1+\beta_2+\beta_3=\beta} \int_0^T \left| |\partial_t^{i_1} D^{\beta_1} V(s)| \cdot |\partial_t^{i_2} D^{\beta_2} V(s)| \cdot |\partial_t^{i_3} D^{\beta_3} V(s)| \right|_2^2 ds.
\]

Because $\hat{m} := \max\{2, \left[\frac{m}{2}\right] + 1\}$,
\[
i' \leq i \leq \left[\frac{N+1}{\hat{m}} + 1\right] \leq \left[\frac{N}{\hat{m}}\right] \quad \text{and} \quad \beta' \leq \beta \leq N+1-(\hat{m}+1)(i+1) \leq N - \hat{m} \cdot i' - \hat{m},
\]
(1.8) yields
\[
||\partial_t^{i'} D^\beta V(t)||_2 \leq ||\partial_t^{i'} D^\beta V(t)||_{H^{\hat{m}}} < \infty \quad \forall t \in [0, T].
\]

And since $\hat{m}(i-i')+(\beta-\beta'+2) \leq \hat{m} \cdot i' + \beta + 2 \leq \hat{m} \cdot i + N+1-(\hat{m}+1)(i+1)+2 = N - \hat{m} + 2 \leq N - i \leq N$,
by (1.8) we have
\[
\int_0^T ||\partial_t^{i'-i} D^{\beta-\beta'+2} V(s)||_2^2 ds \leq \sup_{t \in [0, T]} \{||\partial_t^{i'-i} D^{\beta-\beta'+2} V(t)||_2^2\} \cdot T < \infty.
\]
Remark 5.1. Formula 3 easily implies 

\[ \int_0^T \left\| |\partial_t^\varepsilon D^{\beta'} V(s)| \cdot |\partial_t^{i-i^\varepsilon} D^{\beta-\beta'+2} V(s)| \right\|^2_2 ds \]

\[ \leq \sup_{t \in [0,T]} \{ |\partial_t^\varepsilon D^{\beta'} V(t)|^2 \} \cdot \int_0^T \left\| |\partial_t^{i-i^\varepsilon} D^{\beta-\beta'+2} V(s)| \right\|^2_2 ds < \infty \]

For the other terms on the right hand side of (4.12), using similar method we know all of them are strictly smaller than \( \infty \).

This completes the proof. \( \square \)

5. Proof of Theorem 1.3

In this section, we need some formulas. Their proofs are tedious. So we only list the results.

**Formula 3.** Suppose that \( V \in \Gamma^2(E) \). Then we will obtain 

\[ \|\Delta V\|^2_2 = \|D^2 V\|^2_2 + 2 \int_M \langle D^2 V, D^2 V \ast R^E \rangle dM \]

\[ + \int_M \langle D^2 V, V \ast D R^E \rangle dM + \int_M \langle D^2 V, D V \ast R^M \rangle dM. \]

**Remark 5.1.** Formula 3 easily implies

\[ \|D^2 V\|^2_2 \leq \|\Delta V\|^2_2 + \eta \cdot (\|D V\|^2_2 + \|V\|^2_2), \]

where \( \eta \) depends on \( R^M, R^E \) and their covariant derivatives.

**Formula 4.** Given \( V \in \Gamma^3(E) \),

\[ \|\Delta D V\|^2_2 = \|D^3 V\|^2_2 + 3 \int_M \langle D^2 V, D^2 V \ast R^M \rangle dM + 2 \int_M \langle D^2 V, D^2 V \ast R^E \rangle dM \]

\[ + \int_M \langle D^2 V, V \ast D R^M \rangle dM + \int_M \langle D^2 V, D V \ast D R^E \rangle dM. \]

**Remark 5.2.** From Formula 4 it follows that

\[ \|D^3 V\|^2_2 \leq \|\Delta D V\|^2_2 + \eta_2 \cdot (\|D^2 V\|^2_2 + \|D V\|^2_2), \]

where \( \eta_2 \) depends on \( R^M, R^E \) and their covariant derivatives. Since by (2.9) we have

\[ \Delta D V = D D V + q_1(V, R^E) + q_0(D V, R^M), \]

integration by parts and Hölder’s inequality yield

\[ \|\Delta D V\|^2_2 \leq \|D D V\|^2_2 + \eta_3 \cdot (\|D^2 V\|^2_2 + \|D V\|^2_2 + \|V\|^2_2), \]

where \( \eta_3 \) depends on \( R^M, R^E \) and their covariant derivatives. Substituting (5.3) into (5.2) gives

\[ \|D^3 V\|^2_2 \leq \|D D V\|^2_2 + (\eta_2 + \eta_3) \cdot (\|D^2 V\|^2_2 + \|D V\|^2_2 + \|V\|^2_2). \]
Formula 5. If $V \in \Gamma^4(E)$, then
\[
\|D^4 V\|_2^2 = \|\Delta^2 V\|_2^2 + \int_M \langle q_3(V, R^E), D^3 V \rangle dM + \int_M \langle q_4(DV, R^M), D^3 V \rangle dM
\]
\[+ \int_M \langle q_2(DV, R^E), D^2 V \rangle dM + \int_M \langle q_3(DV, R^E), q_1(DV, R^M) \rangle dM
\]
\[+ \int_M \langle q_1(DV, R^E), q_1(DV, R^M) \rangle dM.
\]

Remark 5.3. Formula 5, Hölder’s inequality and Young’s inequality lead to
\[
\|D^4 V\|_2^2 \leq \|\Delta^2 V\|_2^2 + \eta_5 \cdot (\|D^3 V\|_2^2 + \|D^2 V\|_2^2 + \|DV\|_2^2 + \|V\|_2^2)
\]
\[\leq \|\Delta^2 V\|_2^2 + \eta_6 \cdot (\|D\Delta V\|_2^2 + \|\Delta V\|_2^2 + \|DV\|_2^2 + \|V\|_2^2),
\]
where we have used (5.1), (5.4) and $\eta_5, \eta_6$ depend on $R^M, R^E$ and their covariant derivatives.

Formula 6. Assume that $V \in \Gamma^4(E)$. Then we get
\[
\|D^2 V\|_2^2 \leq \|\Delta^2 V\|_2^2 + \int_M \langle q_1(\Delta V, R^E), D\Delta V \rangle dM + \int_M \langle q_0(D\Delta V, R^M), D\Delta V \rangle dM
\]
\[\leq \|\Delta^2 V\|_2^2 + \eta_8 \cdot (\|D\Delta V\|_2^2 + \|\Delta V\|_2^2),
\]
where we have applied Hölder’s inequality, Young’s inequality and $\eta_8$ depends on $R^M, R^E$ and their covariant derivatives.

Now let us go on to prove Theorem [1.3]. Suppose that $V \in C^\infty([0, T^*], \Gamma^\infty(E))$ is the unique local smooth solution of (1.6), where $T^*$ is its maximal existence time. First of all, we shall estimate $\|DV(t)\|_\infty$ for all $t \in [0, T^*)$. By Sobolev embedding it is easy to see that we only need to get a uniform upper bound of $\|DV(t)\|_{H^2}$ (Note that in this section $m = 2$). Combining (5.1) and (5.4) one can know that we only need to estimate
\[
\|DV(t)\|_2^2 + \|\Delta V(t)\|_2^2 + \|D\Delta V(t)\|_2^2.
\]

Using the same method of (2.2) in [4] we can get
\[
\|DV(t)\|_2^2 + \int_0^t \|\Delta V(s)\|_2^2 ds \leq \lambda^2 \cdot \sqrt{1 + \mu} \|V_0\|_\infty^2 \cdot \|V_0\|_2^2 \cdot t + \|DV_0\|_2.
\]

For $\|\Delta V(t)\|_2^2$, our trick is to deduce a Gronwall’s inequality. (1.6) yields
\[
\frac{1}{2} \frac{d}{dt} \|\Delta V(t)\|_2^2 + \int_M |D\Delta V(t)|^2 dM + \lambda \cdot \|\Delta V(t)\|_2^2
\]
\[= - \int_M [DV(t) \times \Delta V(t)] \cdot D\Delta V(t) dM - \lambda_\mu \cdot \int_M \langle \Delta [V(t)]^2 \cdot V(t), \Delta V(t) \rangle dM
\]
\[\leq \int_M |DV(t)| \cdot |\Delta V(t)| \cdot |D\Delta V(t)| dM + C \cdot \|V(t)\|_\infty \cdot (\|\Delta V(t)\|_2^2 + \|DV(t)\|_2^2)
\]
\[\leq \|DV(t)\|_4 \cdot \|\Delta V(t)\|_4 \cdot \|D\Delta V(t)\|_2 + C \cdot \|V(t)\|_\infty \cdot (\|\Delta V(t)\|_2^2 + \|DV(t)\|_2^2),
\]
here $C$ is a universal constant. Theorem 2.1 of $[3]$, (5.1) and (5.4) give

$$
||DV(t)||_4 \lesssim ||DV(t)||_{H^2}^{\frac{1}{3}} \cdot ||DV(t)||_2^{\frac{2}{3}} 
$$

(5.9)

$$
\leq \eta_4 \cdot (||D\Delta V(t)||_4^{\frac{1}{2}} + ||\Delta V(t)||_4^{\frac{1}{2}} + ||DV(t)||_2^{\frac{1}{2}} \cdot ||DV(t)||_2^{\frac{3}{2}}) \cdot ||DV(t)||_2^{\frac{3}{2}}
$$

and

$$
||\Delta V(t)||_4 \lesssim ||\Delta V(t)||_{H^1}^{\frac{1}{3}} \cdot ||\Delta V(t)||_2^{\frac{2}{3}}
$$

(5.10)

$$
\lesssim \left( ||D\Delta V(t)||_4^{\frac{1}{2}} + ||\Delta V(t)||_4^{\frac{1}{2}} \right) \cdot ||\Delta V(t)||_2^{\frac{3}{2}},
$$

where $\eta_4$ depends upon $\eta_2$, $\eta_3$ and $\eta$. Thus we derive

$$
||DV(t)||_4 \cdot ||\Delta V(t)||_4 \cdot ||D\Delta V(t)||_2
\leq \eta_4 \cdot ||DV(t)||_2^{\frac{1}{3}} \cdot ||\Delta V(t)||_2^{\frac{2}{3}} \cdot ||D\Delta V(t)||_2^{\frac{3}{2}}
$$

$$
+ \eta_4 \cdot ||DV(t)||_2 \cdot ||\Delta V(t)||_4^{\frac{1}{2}} \cdot ||D\Delta V(t)||_2^{\frac{3}{2}}
$$

$$
+ \eta_4 \cdot ||DV(t)||_2 \cdot ||\Delta V(t)||_2 \cdot ||D\Delta V(t)||_2^{\frac{3}{2}}
$$

$$
+ \eta_4 \cdot ||DV(t)||_2 \cdot ||D\Delta V(t)||_2 \cdot ||\Delta V(t)||_2 \cdot ||D\Delta V(t)||_2^{\frac{1}{2}}
$$

$$
+ \eta_4 \cdot ||V(t)||_4^{\frac{1}{2}} \cdot ||DV(t)||_2^{\frac{1}{2}} \cdot ||\Delta V(t)||_2^{\frac{3}{2}} \cdot ||D\Delta V(t)||_2^{\frac{1}{2}}
$$

$$
+ \eta_4 \cdot ||V(t)||_4^{\frac{1}{2}} \cdot ||DV(t)||_2^{\frac{1}{2}} \cdot ||\Delta V(t)||_2 \cdot ||D\Delta V(t)||_2^{\frac{1}{2}}
$$

$$
+ ||\Delta V(t)||_2^{\frac{1}{2}} \cdot ||D\Delta V(t)||_2 \cdot ||\Delta V(t)||_2 \cdot ||D\Delta V(t)||_2
$$

$$
\leq \eta_4 \gamma_1 \cdot \left( ||\Delta V(t)||_{H^1}^{\frac{1}{3}} \cdot ||D\Delta V(t)||_2^{\frac{2}{3}} + ||\Delta V(t)||_2 \cdot ||D\Delta V(t)||_2^{\frac{1}{2}} \right)
$$

$$
+ ||\Delta V(t)||_2^{\frac{1}{2}} \cdot ||D\Delta V(t)||_2 \cdot ||\Delta V(t)||_2 \cdot ||D\Delta V(t)||_2^{\frac{1}{2}}
$$

$$
+ ||\Delta V(t)||_2 \cdot ||D\Delta V(t)||_2 \cdot ||\Delta V(t)||_2 \cdot ||D\Delta V(t)||_2
$$

where $\gamma_1$ relies on $||V_0||_\infty$, $||V_0||_2$, $||DV_0||_2$ and $t$. From Young's inequality it follows that

$$
||DV(t)||_4 \cdot ||\Delta V(t)||_4 \cdot ||D\Delta V(t)||_2
\leq \gamma_2 \cdot \left( ||\Delta V(t)||_2^4 + 1 \right) + \frac{1}{4} \cdot ||D\Delta V(t)||_2^2,
$$

where $\gamma_2$ is dependent of $\mathcal{R}^M$, $\mathcal{R}^E$ and their covariant derivatives, $||V_0||_\infty$, $||V_0||_2$, $||DV_0||_2$ and $t$. By the same way, we have

$$
||V(t)||_\infty^2 \cdot (||\Delta V(t)||_2^2 + ||DV(t)||_4^2)
\lesssim ||V(t)||_\infty^2 \cdot ||\Delta V(t)||_2^2
$$

$$
+ \frac{\eta_2}{4} \cdot ||V_0||_\infty^2 \cdot (||D\Delta V(t)||_2^{\frac{1}{2}} + ||\Delta V(t)||_2^{\frac{1}{2}} + ||DV(t)||_2^{\frac{1}{2}}) \cdot ||DV(t)||_2^{\frac{3}{2}}
$$

$$
\leq \gamma_3 \cdot \left( ||\Delta V(t)||_2^2 + 1 \right) + \frac{1}{4} \cdot ||D\Delta V(t)||_2^2,
$$

where $\gamma_3$ is dependent of $\mathcal{R}^M$, $\mathcal{R}^E$ and their covariant derivatives, $||V_0||_\infty$, $||V_0||_2$, $||DV_0||_2$ and $t$. By the same way, we have
where $\gamma_3$ is dependent of $\mathcal{R}^M$, $\mathcal{R}^E$ and their covariant derivatives, $\|V_0\|_\infty, \|V_0\|_2, \|DV_0\|_2$ and $t$. Substituting \eqref{5.11} and \eqref{5.12} into \eqref{5.8} and Young’s inequality lead to

$$\frac{1}{2} \frac{d}{dt} \|\Delta V(t)\|_2^2 + \frac{1}{2} \|D \Delta V(t)\|_2^2 + \lambda \|\Delta V(t)\|_2^2 \leq \gamma_4 \cdot (\|\Delta V(t)\|_2^4 + 1),$$

here $\gamma_4$ relies on $\gamma_2$ and $\gamma_3$. The generalized Gronwall’s inequality says that if

$$\frac{df}{dt} \leq C \cdot f \cdot g + C,$$

then

$$f \leq C \cdot \exp \left( \int_0^t g(s) \, ds \right) + C.$$ 

So if we replace $f$ and $g$ by $\|\Delta V(t)\|_2^2$ and note that \eqref{5.7} implies the boundedness of $\int_0^t g(s) \, ds$, then

$$\|\Delta V(t)\|_2^2 \leq \gamma_5$$

which implies

$$\int_0^t \|D \Delta V(s)\|_2^2 \, ds \leq 2 \gamma_4 \cdot (\gamma_5^2 + 1) \cdot t,$$

where $\gamma_5$ is dependent of $\mathcal{R}^M$, $\mathcal{R}^E$ and their covariant derivatives, $\|V_0\|_\infty, \|V_0\|_2, \|DV_0\|_2, \|\Delta V_0\|_2$ and $t$.

In the sequel, we are going to estimate $\|D \Delta V(t)\|_2$ for all $t \in [0, T^*)$. \eqref{1.6} gives

$$\frac{1}{2} \frac{d}{dt} \|D \Delta V(t)\|_2^2 = -\|\Delta^2 V(t)\|_2^2 - \int_M \langle \Delta^2 V(t), \Delta[V(t) \times \Delta V(t)] \rangle \, dM$$

$$+ \lambda \int_M \langle \Delta[(1 + \mu|V(t)|^2) \cdot V(t)], \Delta^2 V(t) \rangle \, dM$$

$$= -\|\Delta^2 V(t)\|_2^2 - 2 \int_M \Delta^2 V(t) \ast [DV(t) \times D \Delta V(t)] \, dM$$

$$+ \lambda \int_M \langle \Delta[(1 + \mu|V(t)|^2) \cdot V(t)], \Delta^2 V(t) \rangle \, dM$$

$$= -\|\Delta^2 V(t)\|_2^2 - 2 \int_M \Delta^2 V(t) \ast [DV(t) \times D \Delta V(t)] \, dM$$

$$+ \lambda \mu \int_M \langle \Delta\left[[V(t)]^2 \cdot V(t)\right], \Delta^2 V(t) \rangle \, dM - \lambda \|D \Delta V(t)\|_2^2.$$

On the other hand, Hölder inequality yields

$$\int_M \Delta^2 V(t) \ast [DV(t) \times D \Delta V(t)] \, dM \leq \|DV(t)\|_{\frac{16}{9}} \cdot \|D \Delta V(t)\|_{\frac{16}{9}} \cdot \|\Delta^2 V(t)\|_2$$

and

$$\int_M \langle \Delta\left[[V(t)]^2 \cdot V(t)\right], \Delta^2 V(t) \rangle \, dM$$
which implies

Furthermore, substituting (5.16), (5.17) and (5.18) into (5.15) we arrive at

Substituting the upper bounds of $\|V(t)\|_2, \|DV(t)\|_2$ and $\|\Delta V(t)\|_2$ into (5.19) leads to

By Sobolev Embedding, we have

Combining (5.1), (5.4) and (5.5) we arrive at

where $\eta_7$ is dependent of $R^M, R^E$ and their covariant derivatives. Moreover,

and

By (5.6) we are led to

which implies

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By (5.6) we are led to

which implies

Furthermore, substituting (5.16), (5.17) and (5.18) into (5.15) we arrive at

Substituting the upper bounds of $\|V(t)\|_2, \|DV(t)\|_2$ and $\|\Delta V(t)\|_2$ into (5.19) leads to
\[ + \eta_0 \cdot \left( \varepsilon \cdot \| \Delta^2 V(t) \|^2 \cdot \frac{1}{\varepsilon} \cdot \| D \Delta V(t) \|^2 \right) \]
\[ \lesssim \eta_0 \cdot \left( \varepsilon \cdot \| \Delta^2 V(t) \|^2 \cdot \frac{1}{\varepsilon} \cdot \| D \Delta V(t) \|^2 \right), \]

where \( \eta_0 \) depends on \( \mathcal{R}^M, \mathcal{R}^E \) and their covariant derivatives, \( \| V_0 \|_{C^2} \), \( \| V_0 \|_2 \), \( \| D V_0 \|_2 \), \( \| \Delta V_0 \|_2 \) and \( t \).

Moreover, there is a universal constant \( \kappa_1 \) such that
\[ \left| \int_M \langle \Delta [\| V(t) \|^2 \cdot V(t) \cdot \Delta^2 V(t) \rangle dM \right| \]
\[ \leq \frac{1}{4} \| \Delta^2 V(t) \|^2 + \kappa_1 \cdot \| V(t) \|^4 \cdot \| \Delta V(t) \|^2 + \kappa_1 \cdot \| V(t) \|^4 \cdot \| D V(t) \|^2. \]

Recalling (5.9) and (5.13) we obtain
\[ (5.20) \left| \int_M \langle \Delta [\| V(t) \|^2 \cdot V(t) \cdot \Delta^2 V(t) \rangle dM \right| \leq \frac{1}{4} \| \Delta^2 V(t) \|^2 + \kappa_1 \cdot \| V_0 \|^4 \cdot \gamma_5 \]
\[ + \kappa_1 \cdot \| V_0 \|^2 \cdot \eta_4 \cdot (\| D \Delta V(t) \|_2 + \| \Delta V(t) \|_2 + \| D V(t) \|_2 + \| V(t) \|_2) \cdot \| D V(t) \|^2. \]

Substituting the upper bounds of \( \| \Delta V(t) \|_2 \), \( \| D V(t) \|_2 \) and \( \| V(t) \|_2 \) into (5.20) gives
\[ \left| \int_M \langle \Delta [\| V(t) \|^2 \cdot V(t) \cdot \Delta^2 V(t) \rangle dM \right| \]
\[ \leq \frac{1}{4} \| \Delta^2 V(t) \|^2 + \kappa_2 \cdot (\| D \Delta V(t) \|_2 + 1) \]
\[ \leq \frac{1}{4} \| \Delta^2 V(t) \|^2 + \frac{\kappa_2}{2} \cdot (\| D \Delta V(t) \|^2 + 3), \]

where \( \kappa_2 \) relies on \( \mathcal{R}^M, \mathcal{R}^E \) and their covariant derivatives, \( \| V_0 \|_{C^2} \), \( \| V_0 \|_2 \), \( \| D V_0 \|_2 \), \( \| \Delta V_0 \|_2 \) and \( t \).

In conclusion,
\[ \frac{1}{2} \frac{d}{dt} \| D \Delta V(t) \|^2 + \| \Delta^2 V(t) \|^2 + \lambda \cdot \| D \Delta V(t) \|^2 \]
\[ \leq \eta_0 \cdot \left( \varepsilon \cdot \| \Delta^2 V(t) \|^2 \cdot \frac{1}{\varepsilon} \cdot \| D \Delta V(t) \|^2 \right) + \frac{1}{4} \| \Delta^2 V(t) \|^2 + \frac{\kappa_2}{2} \cdot (\| D \Delta V(t) \|^2 + 3), \]

where \( \eta_0 \) depends on \( \mathcal{R}^M, \mathcal{R}^E \) and their covariant derivatives, \( \| V_0 \|_{C^2} \), \( \| V_0 \|_2 \), \( \| D V_0 \|_2 \), \( \| \Delta V_0 \|_2 \) and \( t \). Let \( \varepsilon \) be small enough. From Young’s inequality it follows that
\[ \frac{1}{2} \frac{d}{dt} \| D \Delta V(t) \|^2 + \frac{1}{2} \| \Delta^2 V(t) \|^2 + \lambda \cdot \| D \Delta V(t) \|^2 \leq \eta_{10} \cdot (1 + \| D \Delta V(t) \|^2), \]

where \( \eta_{10} \) is dependent of \( \eta_0 \) and \( \kappa_2 \). Since of (5.14), the generalized Gronwall’s inequality implies
\[ \| D \Delta V(t) \|^2 \leq \gamma_6, \]
here \( \gamma_6 \) relies on \( \| D \Delta V_0 \|_2, \eta_{10}, \gamma_4, \gamma_5 \) and \( t \). Substituting the upper bounds of \( \| \Delta V(t) \|^2 \) and \( \| D \Delta V(t) \|^2 \) into (5.1) and (5.4) yields
\[ \| D V(t) \|_{H^2} \leq \gamma_7 \]
which implies
\[ \| D V(t) \|_{C^2} \leq \gamma_7, \]
where $\gamma_7$ relies on $R^M$, $R^E$ and their covariant derivatives, $\|V_0\|_\infty$, $\|V_0\|_2$, $\|DV_0\|_2$, $\|\Delta V_0\|_2$, $\|D\Delta V_0\|_2$ and $t$.

Now we return to prove Theorem 1.3. Reviewing (4.1), we know that the key is to estimate

$$\sum_{i+j=k-1} a_{ij} \cdot \|D^{k+1} V\|_2 \cdot \left| |D^{i+1} V| \cdot |D^{j+1} V| \right|_2.$$ 

Hölder inequality yields

$$\left| |D^{i+1} V| \cdot |D^{j+1} V| \right|_2 \leq \|D^{k+1} V\|^{\frac{k+i}{k}} \cdot \|DV\|^{\frac{k+j}{k}}.$$ 

From Theorem 2.4 it follows that

$$\|D^{i+1} V\|^{\frac{k+i}{k}} \leq \|D^{k+1} V\|^{\frac{k+i}{k}} \cdot \|DV\|^{\frac{k+j}{k}}.$$ 

and

$$\|D^{j+1} V\|^{\frac{k+j}{k}} \leq \|D^{k+1} V\|^{\frac{k+j}{k}} \cdot \|DV\|^{\frac{k+i}{k}}.$$ 

So one can get

$$(5.21) \quad \left| |D^{i+1} V| \cdot |D^{j+1} V| \right|_2 \leq \|D^{k+1} V\|_2 \cdot \|DV\|_\infty.$$ 

Substituting (5.21), (4.3), (4.4) and (4.5) into (4.1) we obtain

$$\frac{1}{2} \frac{d}{dt} \|D^k V\|_2^2 + \|D^{k+1} V\|_2^2 \leq \kappa_3 \cdot \|D^{k+1} V\|_2 \cdot \|D^{k+1} V\|_2 \cdot \|DV\|_\infty + G_k \cdot (1 + \|V_0\|_\infty + \|V_0\|_2^2) \cdot \|V\|_{H^k}^2$$

$$\leq \frac{1}{2} \cdot \|D^{k+1} V\|_2^2 + \kappa_3^2 \cdot \|D^{k+1} V\|_2^2 \cdot \|DV\|_\infty^2 + G_k \cdot (1 + \|V_0\|_\infty + \|V_0\|_2^2) \cdot \|V\|_{H^k}^2,$$

where $\kappa_3$ is a universal constant. Note the fact

$$\|DV(t)\|_\infty \lesssim \gamma_7 \quad \forall \ t \in [0, T^*).$$

Summing $k$ from 0 to $N$ and applying Gronwall’s inequality we are led to

$$\|V(t)\|_{H^N}^2 + \int_0^t \|V(s)\|_{H^{N+1}}^2 \, ds \leq C_N(\|V_0\|_{H^N}, t, \gamma_7, \|V_0\|_{\infty}, \kappa_3, G_0, \cdots, G_N).$$

The remaining part of the proof of Theorem 1.3 is as the same as that of Theorem 1.2. So we omit it. This completes the proof. \hfill \square

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