The Imani Periodic Functions:
Genesis and Preliminary Results

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Abstract

The Leah-Hamiltonian, \( H(x,y) = \frac{y^2}{2} + \frac{3x^{4/3}}{4} \), is introduced as a functional equation for \( x(t) \) and \( y(t) \). By means of a nonlinear transformation to new independent variables, we show that this functional equation has a special class of periodic solutions which we designate the Imani functions. The explicit construction of these functions is done such that they possess many of the general properties of the standard trigonometric cosine and sine functions. We conclude by providing a listing of a number of currently unresolved issues relating to the Imani functions.

Keywords: functional equations, periodic functions, Leah cosine and sine functions, Jacobi elliptic functions

AMS Subject Classification: 34K13, 35K57, 35K60, 39A23, 39B05

1 Introduction

The Leah differential equation is \[9, 10]\n
\[
\frac{d^2x}{dt^2} + x^{1/3} = 0.
\]

(1.1)

This equation models a “truly nonlinear oscillator” and arises in the study of the dynamics of nonlinear oscillatory systems \[5\]. The first-integral or Hamiltonian corresponding to Equation (1.1) is \[4, 5\]

\[
H(x,y) = \frac{y^2}{2} + \left( \frac{3}{4} \right) x^{4/3} = \frac{3}{4},
\]

(1.2)

where

\[
y(t) = \frac{dx(t)}{dt},
\]

(1.3)

and the following initial conditions are selected

\[
x(0) = 1, \quad y(0) = 0.
\]

(1.4)

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The equations of the trajectories, \( y = y(x) \), in the \((x,y)\) phase-space are determined by the first-order differential equation \[ 4, 5 \]

\[
\frac{dy}{dx} = -\frac{x^{1/3}}{y}. \tag{1.5}
\]

Using standard techniques from the qualitative theory of differential equations, it can be easily shown that the following statements are true:

(i) Both \( x(t) \) and \( y(t) \) are bounded and periodic, with a period \( T \) which can be explicitly calculated \[ 5, 9 \].

(ii) \( x(t) \) and \( y(t) \) are, respectively, even and odd, i.e.,

\[
x(-t) = x(t), \quad y(-t) = -y(t). \tag{1.6}
\]

(iii) The Taylor series for \( x(t) \) only exists over the interval

\[
-\frac{T}{4} < t < \frac{T}{4}. \tag{1.7}
\]

(iv) The Fourier series for \( x(t) \) and \( y(t) \) contain only odd harmonics and have the representations

\[
x(t) = \sum_{k=0}^{\infty} a_k \cos(2k + 1) \left( \frac{2\pi}{T} \right) t, \tag{1.8a}
\]

\[
y(t) = -\sum_{k=0}^{\infty} (2k + 1) \left( \frac{2\pi}{T} \right) a_k \sin(2k + 1) \left( \frac{2\pi}{T} \right) t. \tag{1.8b}
\]

Note that since the Hamiltonian, \( H(x,y) \), see Equation \[ 1.2 \], is constant, then upon taking its time derivative, it follows that

\[
\frac{dH(x,y)}{dt} = \frac{\partial H}{\partial x} \frac{dx}{dt} + \frac{\partial H}{\partial y} \frac{dy}{dt}
\]

\[
= x^{1/3} \frac{dx}{dt} + \frac{dy}{dt}
\]

\[
= x^{1/3} y + \frac{d^2 x}{dt^2}
\]

\[
= y \left( x^{1/3} + \frac{d^2 x}{dt^2} \right) = 0. \tag{1.9}
\]

Consequently, this provides a derivation of the equation of motion, as indicated in Equation \[ 1.1 \].

However, in general, the Hamiltonian does not provide a unique set of equations of motion. This can be seen easily by considering the first line of the expression given in Equation \[ 1.9 \], i.e., for

\[
H(x,y) = \text{constant}, \tag{1.10}
\]

then

\[
\frac{dH}{dt} = \frac{\partial H}{\partial x} \frac{dx}{dt} + \frac{\partial H}{\partial y} \frac{dy}{dt} = 0. \tag{1.11}
\]
It follows that the most general decomposition of this equation is
\[
\frac{dx}{dt} = \phi(x, y) \frac{\partial H}{\partial y}, \quad \frac{dy}{dt} = -\phi(x, y) \frac{\partial H}{\partial x},
\] (1.12)
where \(\phi(x, y)\) is an arbitrary function of \(x\) and \(y\). Thus, a priori, an equation of motion for a conservative system leads to a unique Hamiltonian, up to an additive constant. But, a Hamiltonian can correspond to an unlimited number of equations of motion.

These results can be generalized to \(2N - (x, y)\) variables.

Let us now return to Equation (1.2) and rewrite it to the form
\[
\left(\frac{2}{3}\right) y^2 + x^{4/3} = 1.
\] (1.13)
If we treat this as a functional equation [1], then the following question can be asked: What are possible solutions to Equation (1.13)? Clearly, these possibilities include the following four piece-wise continuous functions:

\begin{align*}
x(t) &= 1, \quad y(t) = 0, \quad \text{for } -\infty < t < +\infty; \quad (1.14) \\
x(t) &= 0, \quad y(t) = \sqrt[3]{3}, \quad \text{for } -\infty < t < +\infty; \quad (1.15) \\
x(t) &= \begin{cases} 
1, & t > 0, \\
0, & t < 0,
\end{cases} \quad y(t) = \begin{cases} 
0, & t > 0, \\
-\sqrt[3]{3}, & t < 0;
\end{cases} \quad (1.16) \\
\end{align*}

and

\begin{align*}
x(t) &= \begin{cases} 
1, & 0 < t < \frac{T}{2}, \\
0, & \frac{T}{2} < t < T,
\end{cases} \quad y(t) = \begin{cases} 
0, & 0 < t < \frac{T}{2}, \\
\sqrt[3]{3}, & \frac{T}{2} < t < T,
\end{cases} \quad (1.17a)
\end{align*}

where for \(T\) positive
\[
x(t + T) = x(t), \quad y(t + T) = y(t). \quad (1.17b)
\]

Our purpose, in this paper, is not to investigate the broad variety of solutions to Equation (1.13), which will be called the Leah-functional equation. The goal will be to investigate a particular class of periodic solutions to this equation, which follow from making a certain nonlinear transformation of the dependent variables \(x(t)\) and \(y(t)\). The properties of these new periodic functions, which we call the Imani-functions, will mimic in large part mathematical features of both the standard trigonometric cosine and sine functions [3], and the Jacobian elliptic functions [2].

In section 2, we demonstrate the construction of the Imani functions. Section 3 presents a summary of the basic properties of these periodic functions. Finally, in section 4, we give a brief summary of our results and several currently unresolved issues related to the Imani functions.
2 Construction of the Imani functions

Our task is to determine “some” of the periodic solutions to the Leah-functional equation

\[
\left(\frac{2}{3}\right) y(t)^2 + x(t)^{4/3} = 1,
\]

such that they satisfy the conditions

(a) \( x(0) = 1, \quad y(0) = 0; \)
(b) \( x(-t) = x(t), \quad y(-t) = -y(t); \)
(c) \( x(t + T) = x(t), \quad y(t + T) = y(t), \quad \text{for } T > 0. \)

To proceed, make the following nonlinear transformation of the dependent variables

\[
u(t)^2 = x(t)^{4/3}, \quad v(t)^2 = \left(\frac{2}{3}\right) y(t)^2.
\]

Note that \( u(t) \) and \( v(t) \) are not uniquely characterized by their definitions given in Equation (2.5); there are \((\pm)\) signs which can appear. However, this ambiguity will be used to enforce the condition specified in Equation (2.3).

From Equations (2.1) and (2.5), it follows that

\[
u(t)^2 + v(t)^2 = 1.
\]

Now make the following identification

\[
u(t)^2 = [\cos \psi(t)]^2, \quad v(t)^2 = [\sin \psi(t)]^2.
\]

To satisfy the conditions in Equations (2.3) and (2.4), \( \psi(t) \) is required to have the properties

\[
\psi(-t) = -\psi(t) \quad \text{(2.8a)}
\]
\[
\psi(t + T) = \psi(t) + 2\pi. \quad \text{(2.8b)}
\]

Consequently, \( x(t) \) and \( y(t) \) may be selected to be

\[
x(t) = [\text{sgn}(\cos \psi(t))] \cdot |\cos \psi(t)|^{3/2}
\]
\[
y(t) = \sqrt{\frac{3}{2}} \sin \psi(t).
\]

In the above expressions, “\( \cdot \)” denotes the absolute value and “\( \text{sgn} \)” is the sign-function \[6\], i.e.,

\[
\text{sgn}(t) \equiv \begin{cases} 
+1, & t > 0; \\
0, & t = 0; \\
-1, & t < 0.
\end{cases}
\]
Observe that \( \psi(t) \) satisfies the functional equation given in Equation (2.8) and its solutions is

\[
\psi(t) = A(t) + \left( \frac{2\pi}{T} \right) t,
\]

where

\[
A(-t) = -A(t), \quad A(t + T) = A(t).
\]

Assuming \( A(t) \) has a Fourier series, it follows that

\[
A(t) = \sum_{k=1}^{\infty} a_k \sin \left( \frac{2\pi k}{T} t \right).
\]

We name the functions given in Equations (2.9), respectively, the *Imani cosine and sine functions*, and denote them by the symbols “Ics” and “Isn”.

### 3 Properties of the Imani periodic functions

The Imani functions are a class of continuous, periodic solutions to the Leah functional equation. By construction, they satisfy the conditions specified in Equations (2.2), (2.3), and (2.4). Note that these functions may have a period \( T \), of any magnitude, for \( T > 0 \). Central to these representations, for Ics\((t)\) and Isn\((t)\), is the fact that \( \psi(t) \) has the mathematical form given in Equations (2.11a) and (2.12).

Overall, except for the complexity of the final representation for Ics\((t)\), the general properties of the Imani functions are similar to the standard trigonometric cosine and sine functions. It should also be clear that the construction of this paper could have been done using the Jacobi cosine and sine functions.

Figure 1 is a plot of \( \psi(t) \) vs \( t \), while Figure 2 is the corresponding plot of \( x(t) \) vs \( t \). Both of these results are for a single term in the Fourier expansion of \( A(t) \); see Equation (2.12).
4 Future work

In summary, we have shown how to construct a particular set of periodic solutions to the Leah functional equation. The following questions can be asked:

1) What other classes of continuous, periodic functions exist?
2) What is the full Fourier series representation for the Imani cosine function?
3) Do derivatives exist for $Ics(t)$ and $Isn(t)$? If so, what are they?
4) What are the integrals
   \[ \int Ics(t) dt \quad \text{and} \quad \int Isn(t) dt \]
   of these functions?
5) Can products of Imani functions
   \[ Ics(t_1)Ics(t_2), \quad Isn(t_1)Isn(t_2), \quad Isn(t_1)Ics(t_2), \]
   be calculated?
6) What are $Ics(t_1 + t_2)$ and $Isn(t_1 + t_2)$?
7) Are the Imani functions solutions of differential equations? If so, what are they? Is there a single differential equation for which both are special solutions?
8) Finally, can the Imani periodic functions be analyzed using geometrical methods such as those used for the elliptic functions [11] and the square periodic functions [7]?

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References

[1] J. Aczel, *On Applications and Theory of Functional Equations*, Academic Press, New York, 1969.

[2] P. F. Byrd and M. D. Friedman, *Handbook of Elliptic Integrals for Engineers and Physicists*, Springer-Verlag, Berlin, 1954.

[3] C. V. Durell and A. Robson, *Advanced Trigonometry*, Bell and Sons, London, 1930.

[4] H. Goldstein, *Classical Mechanics*, 2nd edition, Addison-Wesley, Reading, MA, 1980.

[5] R. E. Mickens, *Truly Nonlinear Oscillators*, World Scientific, Singapore, 2004.

[6] R. E. Mickens, *Difference Equations: Theory, Applications and Advanced Topics*, 3rd edition, CRC Press, Boca Raton, FL, 2015.

[7] R. E. Mickens, *Some properties of square (periodic) functions*, Proceedings of Dynamic Systems and Applications 7 (2016), pp. 282–286.

[8] R. E. Mickens, *Periodic solutions of the functional equation* \( f(t)^2 + g(t)^2 = 1 \), Journal of Difference Equations and Applications 22 (2016), pp. 67–74.

[9] S. A. Rucker, *Leah-cosine and sine-functions: Definitions and elementary properties*, pp. 265–280, in Abba B. Gumel (editor), *Mathematics of Continuous and Discrete Dynamical Systems*, American Mathematical Society, Providence, RI, 2014.

[10] S. A. Rucker and R. E. Mickens, *The Leah cosine and sine functions: Geometric definitions*, Proceedings of Dynamic Systems and Applications 7 (2016), pp. 317–320.

[11] W. Schwalm, *Elliptic functions sn, cn, dn as trigonometry*, [http://www.und.edu/instruct/schwalm/MAAPresentation_10-02/handout.pdf](http://www.und.edu/instruct/schwalm/MAAPresentation_10-02/handout.pdf), 2005.