Junction Conditions of Friedmann-Robertson-Walker Space-Times

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Abstract

We complete a classification of junctions of two Friedmann-Robertson-Walker space-times bounded by a spherical thin wall. Our analysis covers super-horizon bubbles and thus complements the previous work of Berezin, Kuzmin and Tkachev. Contrary to sub-horizon bubbles, various topology types for super-horizon bubbles are possible, regardless of the sign of the extrinsic curvature. We also derive a formula for the peculiar velocity of a domain wall for all types of junction.

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Using the thin-wall formalism first devised by Israel [1], many authors studied bubble dynamics [2]-[7]. In the formalism, a (2+1) dimensional hypersurface of a trajectory of the bubble wall is embedded in two different homogeneous space-times: One describes the inside of the bubble and the other does the outside. Two space-times are pasted on the hypersurface with the junction conditions for metrics and extrinsic curvature tensors.

Berezin, Kuzumin and Tkachev [2] studied the junction conditions of two different Friedmann-Robertson-Walker (FRW) space-times bounded by a spherical thin wall. They considered only the case when bubbles are smaller than the horizon scale. From this restriction, they discussed the constraints on decaying-vacuum parameters, which are related to the size of a nucleated bubble, in an open or flat universe. Although Berezin et al. mentioned the possibility of bubbles which are larger than the horizon scale, they did not show the junction conditions explicitly. As long as the self-gravity of a bubble is ignored, a bubble is in fact nucleated within the horizon scale. However, in some inflationary models there may exist bubbles whose self-gravity is dominant [5] and, therefore, whose size could be larger than the horizon scale [6]. Further, vacuum parameters should be determined from microscopic physics but not from a macroscopic constraint such as the condition that a bubble should be smaller than the horizon size. In this paper, we re-examine junctions of FRW space-times and classify all possible spherical bubbles. Our study is a complement to that of Berezin et al., and will also be useful for analyzing super-horizon voids [8].

To begin, we define a unit space-like vector $N^{\mu}$ which is orthogonal to the world hypersurface $\Sigma$ denoting a trajectory of a bubble wall and which points from $V^-$ (inside of the bubble) to $V^+$ (outside). For convenience, we introduce a Gaussian normal coordinate system $(n, \tau, \theta, \phi)$, where $n = 0$ corresponds to $\Sigma$ and $\tau$ is the proper time of the wall. Then the angular component of the extrinsic curvature
The tensor is written as

\[ K_\theta^\theta \equiv N_\theta^\theta = -\Gamma^\theta_{\theta n} = \frac{1}{R} \left. \frac{\partial R}{\partial n} \right|_\Sigma, \]

where \( R(n, \tau) \) is the circumference radius of the circle of \( n = \text{constant}, \tau = \text{constant}, \theta = \pi/2 \) in the Gaussian normal coordinates, and \( R(0, \tau) = \bar{R}(\tau) \). As some authors have pointed out [3]-[4], the sign of \( K_\theta^\theta \) is a key to classify the global space-time structures, because (1) shows that its sign depends on whether or not the spherical area increases in the normal direction and the sign determines the global structure for static space-times. For example, space-time structures are classified by the signs of \( K_\theta^\theta \pm \) for junctions of Schwarzschild space-times or de Sitter space-times. In those cases, the signs of \( K_\theta^\theta \pm \) determine spatial topology. However, if either of two space-times is time-dependent, the relation between the signs of \( K_\theta^\theta \pm \) and the spatial topology is not so clear.

First, we shall investigate whether the signs of \( K_\theta^\theta \) fix the spatial topology in FRW space-time without using the Einstein equations. The FRW metric is written as

\[ ds^2 = -dt^2 + a^2(t)[d\chi^2 + r^2(\chi)(d\theta^2 + \sin^2 \theta d\varphi^2)], \]

where

\[ r(\chi) = \begin{cases} \sin \chi & (k = +1, \text{ closed universe}) \\ \chi & (k = 0, \text{ flat universe}) \\ \sinh \chi & (k = -1, \text{ open universe}) \end{cases} \]

In these coordinates (2) is written down as

\[ K_\theta^\theta = \frac{1}{R} \left. \frac{\partial R}{\partial n} \right|_\Sigma = \frac{1}{R} \left( \frac{\partial R}{\partial \chi} \frac{\partial \chi}{\partial n} + \frac{\partial R}{\partial t} \frac{\partial t}{\partial n} \right) \right|_\Sigma. \]

By giving \( \partial t/\partial n \) and \( \partial \chi/\partial n \) in terms of physical variables measured in \( V^\pm \), we finally get [3, 4]

\[ K_\theta^\theta = \zeta \frac{\gamma}{R} \left( \frac{dv}{d\chi} + vHR \right), \]

where \( H \equiv (da/dt)/a \) is the Hubble parameter, \( v \equiv a(d\chi/dt) \) is the peculiar velocity relative to the background expansion, \( \gamma \equiv \partial t/\partial \tau = 1/\sqrt{1 - v^2} \) is the Lorentz factor,
and \( \zeta \equiv \text{sign} \left( \partial \chi / \partial n \right) \). We have assumed \( \partial t / \partial \tau > 0 \) because both \( t \) and \( \tau \) are future directed.

From formula (5), we find that the signs of \( K^\theta_\theta \) and \( \zeta \) are independent. The reason is as follows: The topology of the space-time is determined by \( \zeta^\pm \) and \( k^\pm \) as listed in Fig.1. On the other hand, we can set up any sign of \( K^\theta_\theta \) for any value of \( \zeta^\pm \) and \( k^\pm \) as initial data, because the radius \( R \) and the velocity \( v \) are independent variables. Hence, all types of topology in Fig.1 are geometrically possible regardless of the signs of \( K^\theta_\theta^\pm \). The signs of \( K^\theta_\theta^\pm \) does not determine the spatial topology in general.

Next, by use of the Einstein equations we shall show the relation between matter energy density and the extrinsic curvature \( K^\theta_\theta \). The Einstein equations give the background expansion rate,

\[
H^2 + \frac{k}{a^2} \pm = \frac{8\pi G \rho^\pm}{3},
\]

and the junction condition at the wall [2]-[4],

\[
K^\theta_\theta^+ - K^\theta_\theta^- = -4\pi G \sigma,
\]

where \( \rho^\pm \) is energy density in \( V^\pm \) and \( \sigma \) is the surface energy density of the wall. From (3) with (3), (6) and

\[
\frac{dR}{d\tau} = \gamma \left( \frac{dr}{d\chi} v + HR \right) \pm,
\]

we obtain

\[
(K^\theta_\theta^\pm)^2 = \frac{1}{R^2} \left\{ 1 + \left( \frac{dR}{d\tau} \right)^2 - \frac{8\pi G \rho^\pm}{3} R^2 \right\},
\]

which coincides with the conventional expression of \( K^\theta_\theta \) [2]. Using (7) and (8), we have

\[
K^\theta_\theta^\pm \equiv \frac{(K^\theta_\theta^+)^2 - (K^\theta_\theta^-)^2 \pm (K^\theta_\theta^+ - K^\theta_\theta^-)^2}{2(K^\theta_\theta^+ - K^\theta_\theta^-)} = \frac{\rho^+ - \rho^- \mp 6\pi G \sigma^2}{3\sigma},
\]

where the signs \( \pm \) and \( \mp \) correspond to values of \( K^\theta_\theta \), respectively. If we assume \( \sigma \) to be positive, the signs of \( K^\theta_\theta^\pm \) are determined by the ratio of \( \rho^+ - \rho^- \) to \( \sigma^2 \).
We have summarized the classification of junctions of FRW space-times in Table 1. First, we find that the signs of $K^\theta_\theta \pm$ are determined from the value of $(\rho^+ - \rho^-)/\sigma^2$. The same relation has already been presented by Sato [3] and by Berezin et al. [4], although Sato considered only closed de Sitter space-times and Berezin et al. restricted their study to sub-horizon bubbles. Here we have extended the relation to the most general case, which covers any matter fluid, any spatial curvature, and any bubble size.

In the last column of Table 1, we present all possible junctions. To compare our results with the previous results for sub-horizon bubbles, we have listed only types with $\zeta^\pm = +1$ in Table 1. This is the case if we consider realistic bubble nucleation in the expanding universe (see Ref.[22] in Ref.[7]). The topology types which do not satisfy $\zeta^\pm = +1$ are with parentheses in Fig.1. For a static space-time, only the types written in italic characters in Table 1 are possible. New types, which appear in an expanding universe, are described by bold characters. An asterisk has been used for the types for which only super-horizon bubbles are possible. We should remark that Type II and III solutions are also possible even in a flat or open universe, contrary to the previous results for sub-horizon bubbles. The reason $\partial R/\partial n$ can be negative is simple. If the comoving radius of a bubble decreases in time, the normal vector $N^\mu$ points in the “past” direction in terms of the cosmic time $t^+$ in $V^+$ (i.e., $N^t < 0$), and accordingly $\partial t/\partial n$ is negative. When the second term in (4), which is negative, becomes large, $\partial R/\partial n$ could be negative. This condition needs that the size of the bubble must be larger than the horizon scale, because $K^\theta_\theta < 0$ in (3) implies $v < 0$ and $HR > (dr/d\chi)/|v| > 1$ for $k \leq 0$.

Finally, we shall investigate the dynamics of vacuum bubbles in FRW coordinates. Now, let us assume $\rho^\pm$ and $\sigma$ to be constant. Berezin et al. studied it only for Type I in Table 1, while our analysis below covers all types. From (2) with (3), we obtain
the solution of a bubble motion \(^2\):

\[ R(\tau) = R_0 \cosh \left( \frac{\tau}{R_0} \right) \text{ with } R_0 = \frac{3\sigma}{\sqrt{(\rho^+ + \rho^- + 6\pi G\sigma^2)^2 - 4\rho^+ \rho^-}}. \] (11)

This formula is valid for any spatial curvature \(k^\pm\), as long as a thin-wall approximation is valid. One may expect that \(dR/d\tau\) may vanish at nucleation time. This is true for the case of Type I, however, we cannot set \(dR/d\tau = 0\) for bubbles larger than the horizon scale in the case of Type II and III. We may understand this to be either that thin-wall approximation is no longer valid at nucleation or that the wall is expanding at nucleation time \((dR/d\tau > 0)\).

Now we derive the peculiar velocity of the domain wall observed in \(V^\pm\). From (5) and (8), we obtain

\[ v^\pm = \frac{K^g_\theta H R - \zeta (dr/d\chi) [(dR/d\tau)/R]}{\zeta H (dR/d\tau) - (dr/d\chi) K^g_\theta} \bigg|_{\pm}. \] (12)

This relation is applied to any equation of state for matter. For the case when \(\rho^\pm\) and \(\sigma\) are constant, using (11), we find the asymptotic value of the peculiar velocity \(v^\pm\) in the limit of \(\tau \to \infty\),

\[ v^\pm_\infty \equiv \lim_{\tau \to \infty} v^\pm = \zeta^\pm K^g_\theta R_0 = \zeta \frac{\rho^+ - \rho^- \mp 6\pi G\sigma^2}{\sqrt{(\rho^+ + \rho^- + 6\pi G\sigma^2)^2 - 4\rho^+ \rho^-}}. \] (13)

If we consider bubbles with \(\zeta^\pm = +1\), the signs of \(v^\pm_\infty\) coincide with the signs of \(K^g_\theta^\pm\). Then \(v^\pm_\infty\) is always negative for new types of junction (Type II, III). Even if \(v^\pm_\infty\) is negative, however, the physical size of such bubbles is still increasing in time.

To summarize, we re-examined the junctions of two FRW space-times and completed the classification. For vacuum bubbles, we presented the formula of the peculiar velocity in all range of parameters. One of the remarkable results is that false vacuum bubbles \((\rho^+ < \rho^-)\) can exist even in a homogeneous and flat universe, if the size of a bubble is larger than the horizon scale.

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Table 1. Classification of junctions of FRW space-times. The signs of $K_\theta^\pm$ are determined from the value of $(\rho^+ - \rho^-)/\sigma^2$, regardless of matter’s equation of state and spatial curvature. In the last column, we also present all possible junctions with $\zeta^\pm = +1$ (see Fig.1 for each letter). If a space-time is static, only the types written in italic characters are possible. New types are described by bold characters. We use an asterisk for the types for which only super-horizon bubbles are possible. We find that Type II and III solutions are also possible even in a flat or open universe, though they are impossible for sub-horizon bubbles.

**FIGURE CAPTIONS**

Fig.1. A list of topology types of spatial sections ($t_\pm = \text{const.}$) for all possible junctions. The types in parentheses do not satisfy $\zeta^\pm = +1$ and are not possible as long as the universe itself does not change its topology when a bubble is created.