Linear stable range for homology of congruence subgroups via FI-modules

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Abstract
We answer positively a question of Church, Miller, Nagpal and Reinhold on existence of a linear bound on the presentation degree of the homology of a complex of FI-modules. This implies a linear stable range for the homology of congruence subgroups of general linear groups.

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Introduction

Let $R$ be a ring. For any finite set $S$, let $R^S$ be the free right $R$-module on $S$ and let $\text{GL}_S(R)$ be the group of right $R$-module automorphisms of $R^S$. For any two-sided ideal $I$ of $R$, the congruence subgroup $\text{GL}_S(R, I)$ is defined to be the kernel of the natural group homomorphism $\text{GL}_S(R) \to \text{GL}_S(R/I)$. When $S$ is the finite set $[n] := \{1, \ldots, n\}$, we denote $\text{GL}_S(R, I)$ by $\text{GL}_n(R, I)$.

Suppose $T$ is a finite set and $S \subset T$. The natural group monomorphism $\text{GL}_S(R) \to \text{GL}_T(R)$ restricts to a group monomorphism $\text{GL}_S(R, I) \to \text{GL}_T(R, I)$. Taking group homology with coefficients in an abelian group $A$, we get a group homomorphism $H_k(\text{GL}_S(R, I); A) \to H_k(\text{GL}_T(R, I); A)$. For each $n \geq 0$ and $N \geq 0$, there is a canonical homomorphism:

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colim_{S \subseteq [n], |S| \leq N} H_k(\text{GL}_S(R, I); A) \longrightarrow H_k(\text{GL}_n(R, I); A)

where the colimit is taken over the poset of all subsets $S$ of $[n]$ such that $|S| \leq N$.

In this article, we prove the following result:

**Theorem 1** Suppose that $R$ satisfies Bass’s stable range condition $\text{SR}_{d+2}$ for some $d \geq 0$ and $I$ is a proper two-sided ideal of $R$. Then for each $k \geq 0$ and $n \geq 0$, one has a canonical isomorphism:

$$
\text{colim}_{S \subseteq [n], |S| \leq \omega(k)} H_k(\text{GL}_S(R, I); A) \sim \longrightarrow H_k(\text{GL}_n(R, I); A)
$$

(1)

where $\omega(k) = 4k + 2d + 6$.

We refer the reader to [1, (V, 3.1)] for the definition of condition $\text{SR}_{d+2}$. Bass proved in [1, (V, 3.5)] that any commutative Noetherian ring of Krull dimension $d$ satisfies $\text{SR}_{d+2}$. For example, any field satisfies $\text{SR}_2$ and any Dedekind domain satisfies $\text{SR}_3$.

It is trivial that (1) is an isomorphism when $n \leq \omega(k)$; the range $n > \omega(k)$ is called the stable range. Qualitatively, Theorem 1 says that there is a stable range starting at a linear function of $k$.

An excellent account of what was known about homology of the congruence subgroups is given in [15]. Let us recall the recent developments most relevant to our present article:

(i) Putman proved Theorem 1 with stable range $n > (d + 8)2^{k-1} - 4$ under the assumption that $A$ is a field whose characteristic is either 0 or at least $(d + 8)2^{k-1} - 3$; see [15, Theorem B].

(ii) Church–Ellenberg–Farb–Nagpal proved Theorem 1 without an explicit stable range in the special case where $R$ is the ring of integers of a number field and under the assumption that $A$ is a Noetherian ring; see [6, Theorems C and D].

(iii) Church–Ellenberg proved Theorem 1 with stable range $n > 2^{k-2}(2d + 9) - 1$; see [4, Theorem D’].

(iv) Church–Miller–Nagpal–Reinhold proved Theorem 1 with stable range $n > 4k^2 + (4d + 10)k + 5d + 6$; see [7, Application B].

Most importantly for us, Church–Miller–Nagpal–Reinhold reduced the proof of Theorem 1 to a question on $\text{FI}$-modules which may be of independent interest; see [7, Question 5.2]. We give a positive answer to their question in Theorem 5. Recently, Jeremy Miller and Jennifer Wilson applied our answer to this question to deduce a result similar to Theorem 1 for cohomology groups of ordered configuration spaces; for details, see [14].

This article is organized as follows. In Sect. 1, we recall the definitions and results on $\text{FI}$-modules that we need. In Sect. 2, we state and prove Theorem 5. In Sect. 3, we give the application of Theorem 5 to the proof of Theorem 1.

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1 See also [2, 8, 13, 16] for recent related work.
We thank Thomas Church and Jeremy Miller for answering our questions on [7]. Jeremy Miller and Rohit Nagpal informed us that they have recently proved Theorem 1 with a stable range starting at a linear function of $k$ in the special case where $R$ is the ring of integers of a number field. Their method of proof is completely different from the one we give here.

1 Generalities

We refer the reader to [4] for background on $\text{FI}$-modules; our notations will follow that of [7].

We work over a commutative ring $\mathbf{k}$. Let $\text{FI}$ be the category of finite sets and injective maps. By an $\text{FI}$-module (respectively, $\text{FI}$-group), we mean a functor from $\text{FI}$ to the category of $\mathbf{k}$-modules (respectively, groups); see [5]. Let $\text{FB}$ be the category of finite sets and bijective maps. An $\text{FB}$-module is a functor from $\text{FB}$ to the category of $\mathbf{k}$-modules. Suppose $M$ is an $\text{FI}$-module or $\text{FB}$-module. We write $M_S$ for the value of $M$ on a finite set $S$. We write $M_n$ for the value of $M$ on $[n]$. If $M$ is nonzero, its degree $\deg M$ is defined to be $\sup\{n \mid M_n \neq 0\}$; if $M$ is zero, we set $\deg M$ to be $-1$.

For any $\text{FB}$-module $V$, the induced $\text{FI}$-module $I(V)$ is defined by

$$I(V)_S = \bigoplus_{n \geq 0} \mathbf{k}[\text{Hom}_{\text{FI}}([n], S)] \otimes_{\mathbf{k}[S_n]} V_n,$$

where $S_n$ denotes the symmetric group on $[n]$. If $\alpha : S \to T$ is an injective map between finite sets, then $\alpha_* : I(V)_S \to I(V)_T$ is the $\mathbf{k}$-linear map defined by

$$\beta \otimes v \mapsto (\alpha \circ \beta) \otimes v$$

where $\beta \in \text{Hom}_{\text{FI}}([n], S)$ and $v \in V_n$ for any integer $n \geq 0$. The functor $V \mapsto I(V)$ is a left adjoint functor to the forgetful functor from the category of $\text{FI}$-modules to the category of $\mathbf{k}$-modules. The projective $\text{FI}$-modules are the induced $\text{FI}$-modules $I(V)$ where each $V_n$ is projective as a $\mathbf{k}[S_n]$-module.

Suppose $V$ is an $\text{FB}$-module. Then each $V_n$ can be regarded as an $\text{FB}$-module whose value on a finite set $S$ is $V_S$ if $|S| = n$, zero if $|S| \neq n$. There is a direct sum decomposition $V = \bigoplus_{n \geq 0} V_n$ in the category of $\text{FB}$-modules. Correspondingly, there is a direct sum decomposition

$$I(V) = \bigoplus_{n \geq 0} I(V_n)$$

in the category of $\text{FI}$-modules.

Suppose $V$ and $W$ are $\text{FB}$-modules. Since there are $\text{FI}$-module direct sum decompositions

$$I(V) = \bigoplus_{n \geq 0} I(V_n) \text{ and } I(W) = \bigoplus_{m \geq 0} I(W_m),$$
any FI-module homomorphism \( f : \mathcal{I}(V) \to \mathcal{I}(W) \) is given by a matrix \( (f^{m,n}) \) of FI-module homomorphisms

\[
f^{m,n} : \mathcal{I}(V_n) \to \mathcal{I}(W_m).
\]

If \( m > n \), then \( \mathcal{I}(W_m)_n = 0 \), so there is no nonzero homomorphism from \( \mathcal{I}(V_n) \) to \( \mathcal{I}(W_m) \). Thus, the matrix \( (f^{m,n}) \) is upper-triangular.

For any FI-module \( M \) and finite set \( S \), we write \( M \prec S \) for the FI-submodule of \( M \) generated by all \( M_T \) with \( |T| < |S| \). Define an FI-module \( H_{FI}^0(M) \) by

\[
H_{FI}^0(M)_S = (M/M_\prec S)_S.
\]

The functor \( H_{FI}^0 \) is right exact; let \( H_{FI}^i \) be its \( i \)-th left derived functor. For any FI-module \( M \), denote by \( t_i(M) \) the degree \( \deg H_{FI}^i(M) \). We call \( \max\{t_0(M), t_1(M)\} \) the presentation degree of \( M \).

If \( V \) is an FB-module, then it is plain that \( H_{FI}^0(\mathcal{I}(V)) = V \) and \( t_0(\mathcal{I}(V)) = \deg V \).

The following theorem is due to Church–Ellenberg [4, Theorem C]; see [10] for a very simple proof.

**Theorem 2** (Church–Ellenberg) Let \( M \) be an FI-module. If the presentation degree of \( M \) is at most \( N \), then there is a canonical isomorphism

\[
\colim_{S \subset \{1, \ldots, n\}, |S| \leq N} M_S \sim \to M_n \quad \text{for every } n \geq 0.
\]

The next theorem is also due to Church–Ellenberg [4, Theorem A]; alternative proofs can be found in [3,9] and [11].

**Theorem 3** (Church–Ellenberg) Let \( M \) be an FI-module. Then for each \( i \geq 2 \), one has:

\[
t_i(M) \leq t_0(M) + t_1(M) + i - 1.
\]

From Theorem 3, it is easy to deduce the following result.

**Corollary 4** Let \( P \) and \( Q \) be projective FI-modules. Let \( f : P \to Q \) be a homomorphism and \( Z = \ker(f) \). Then for each \( i \geq 0 \), one has:

\[
t_i(Z) \leq 2t_0(P) + i + 1.
\]

**Proof** The claim is trivial if \( t_0(P) = \infty \), so suppose \( t_0(P) < \infty \). Let \( N = t_0(P) \).

There are FB-modules \( V \) and \( W \) such that \( P = \mathcal{I}(V) = \bigoplus_{n \geq 0} \mathcal{I}(V_n) \) and \( Q = \mathcal{I}(W) = \bigoplus_{m \geq 0} \mathcal{I}(W_m) \). Since the matrix \( (f^{m,n}) \) is upper-triangular and \( V_n = 0 \) for every \( n > N \), the image of \( f \) is contained in \( \bigoplus_{m \leq N} \mathcal{I}(W_m) \). Therefore, replacing \( Q \) by \( \bigoplus_{m \leq N} \mathcal{I}(W_m) \), we may assume that \( t_0(Q) \leq N \).
From the short exact sequence $0 \to Z \to P \to P/Z \to 0$, we deduce that:

$$t_0(Z) \leq \max\{t_0(P), t_1(P/Z)\},$$
$$t_i(Z) = t_{i+1}(P/Z) \text{ if } i \geq 1.$$ 

Let $C$ be the cokernel of $f$. From the short exact sequence $0 \to P/Z \to Q \to C \to 0$, we deduce that $t_{i+1}(P/Z) = t_{i+2}(C)$; moreover, $t_0(C) \leq t_0(Q)$ and $t_1(C) \leq t_0(P/Z) \leq t_0(P)$. Applying Theorem 3, we have:

$$t_{i+2}(C) \leq t_0(C) + t_1(C) + i + 1 \leq t_0(Q) + t_0(P) + i + 1 \leq 2N + i + 1.$$ 

Putting the above inequalities together gives the corollary. $\square$

We write $H^{FI}_k$ for the $k$-th left hyper-derived functor of $H^{FI}_0$. If $M_\ast$ is a bounded-below chain complex of $FI$-modules and $P_\ast$ is a complex of projective (or induced\(^2\)) $FI$-modules quasi-isomorphic to $M_\ast$, then $H^{FI}_k(M_\ast)$ is, by definition, equal to the $k$-th homology of the chain complex $H^{FI}_0(P_\ast)$; denote by $t_k(M_\ast)$ the degree $\text{deg} \ H^{FI}_k(M_\ast)$.

### 2 Bounding presentation degree of homology

Let $M_\ast$ be a complex of $FI$-modules supported on non-negative homological degrees; its $k$-th homology $H_k(M_\ast)$ is an $FI$-module. The following theorem gives a positive answer to [7, Question 5.2].

**Theorem 5** For each $k \geq 0$, one has:

$$t_0(H_k(M_\ast)) \leq 2t_k(M_\ast) + 1,$$
$$t_1(H_k(M_\ast)) \leq 2 \max\{t_k(M_\ast), t_{k+1}(M_\ast)\} + 2.$$ 

In particular, if $a \geq 0$ and $t_k(M_\ast) \leq ak + b$ for every $k$, then one has:

$$t_0(H_k(M_\ast)) \leq 2ak + 2b + 1,$$
$$t_1(H_k(M_\ast)) \leq 2ak + 2a + 2b + 2.$$ 

To prove Theorem 5, let $P_\ast$ be the total complex of a projective Cartan-Eilenberg resolution of $M_\ast$. Then $P_\ast$ is a chain complex of projective $FI$-modules supported on non-negative degrees with $H_k(P_\ast) = H_k(M_\ast)$ for every $k$. Recall that, by definition, $t_k(M_\ast)$ is the degree of the $k$-th homology of the chain complex $H^{FI}_0(P_\ast)$.

For each $k$, there is an $FB$-module $V_k$ such that

$$P_k = \mathcal{I}(V_k) = \bigoplus_{n \geq 0} \mathcal{I}((V_k)_n),$$

\(^2\) Induced $FI$-modules are $FI$-homology acyclic; see [12, Theorem 1.3] or [17, Theorem B].
where \((V_k)_n\) is the value of \(V_k\) at the object \([n]\). To avoid confusion in notation, we shall always reserve \(k\) for homological degree and \(n\) for the order of the finite set \([n]\).

We write \(d\) for the differential map of the chain complex \(P_\bullet\). Then \(d : P_k \to P_{k-1}\) is an upper-triangular matrix \((d^{m,n})\) of homomorphisms

\[
d^{m,n} : \mathcal{I}((V_k)_n) \to \mathcal{I}((V_{k-1})_m).
\]

In particular, since \((\mathcal{I}((V_k)_n))_n = (V_k)_n\) and \((\mathcal{I}((V_{k-1})_n))_n = (V_{k-1})_n\), the homomorphism \(d^{n,n}\) at \([n]\) is a map \((V_k)_n \to (V_{k-1})_n\). One has \(H^0_{\text{FI}}(P_k) = V_k\). Thus, \(H^0_{\text{FI}}(P_\bullet)\) is the chain complex \(V_\bullet\), whose differential map \(\tilde{d} : V_k \to V_{k-1}\) is defined at \([n]\) to be the map \(d^{n,n}\) at \([n]\). One has:

\[
t_k(M_\bullet) = \deg(H_k(V_\bullet)).
\]

**Lemma 6** Suppose \(N > t_k(M_\bullet)\). Then the sequence

\[
\mathcal{I}((V_{k+1})_N) \xrightarrow{d^{N,N}} \mathcal{I}((V_k)_N) \xrightarrow{d^{N,N}} \mathcal{I}((V_{k-1})_N)
\]

is exact.

**Proof** Since \(N > t_k(M_\bullet)\), the sequence

\[
(V_{k-1})_N \xrightarrow{\tilde{d}} (V_k)_N \xrightarrow{\tilde{d}} (V_{k-1})_N
\]

is exact. Since \(\tilde{d}\) at \([N]\) is precisely \(d^{N,N}\) at \([N]\), and for every finite set \(S\), \(k[\text{Hom}_{\text{FI}}([N], S)]\) is a free right \(k[S_N]\)-module, the lemma follows. \(\square\)

Let \(Z_k\) be the kernel of \(d : P_k \to P_{k-1}\), and let \(B_k\) be the image of \(d : P_{k+1} \to P_k\). Thus, \(H_k(P_\bullet) = Z_k/B_k\). Let

\[
f : \bigoplus_{n \leq t_k(M_\bullet)} \mathcal{I}((V_k)_n) \to P_{k-1}
\]

be the restriction of \(d : P_k \to P_{k-1}\) to \(\bigoplus_{n \leq t_k(M_\bullet)} \mathcal{I}((V_k)_n)\). Let

\[
Z = \ker(f).
\]

**Lemma 7** The following composition is surjective:

\[
Z \twoheadrightarrow Z_k \twoheadrightarrow H_k(P_\bullet).
\]

**Proof** We need to prove that for every finite set \(S\) and \(x \in (Z_k)_S\), there exists \(y \in (B_k)_S\) such that \(x - y \in Z_S\). Let \(N\) be an integer such that

\[
x \in \bigoplus_{n \leq N} \mathcal{I}((V_k)_n)_S.
\]
If $N \leq t_k(M_\bullet)$, then $x \in Z(S)$ and we are done. Suppose that $N > t_k(M_\bullet)$. We write $x = x' + x''$ where
\[
x' \in \bigoplus_{n \leq N-1} I((V_k)_n)_S, \quad x'' \in I((V_k)_N)_S.
\]

Since the matrix $(d^{m,n})$ is upper-triangular, we have:
\[
d(x') \in \bigoplus_{n \leq N-1} I((V_{k-1})_n)_S, \quad d(x'') \in \bigoplus_{n \leq N} I((V_{k-1})_n)_S.
\]

But $d(x') + d(x'') = d(x) = 0$, so we must have:
\[
d(x'') \in \bigoplus_{n \leq N-1} I((V_{k-1})_n)_S.
\]

Hence, $d^{N,N}(x'') = 0$. Since $N > t_k(M_\bullet)$, by Lemma 6, there exists $w \in I((V_{k+1})_N)_S$ such that $d^{N,N}(w) = x''$. Since we also have
\[
d(w) \in \bigoplus_{n \leq N} I((V_k)_n)_S,
\]

it follows that
\[
x - d(w) \in \bigoplus_{n \leq N-1} I((V_k)_n)_S.
\]

If $N - 1 \leq t_k(M_\bullet)$, then we are done. If not, we repeat the above argument with $x$ replaced by $x - d(w)$.

Next, let
\[
\tilde{f} : \bigoplus_{n \leq \max\{t_k(M_\bullet), t_{k+1}(M_\bullet)\}} I((V_k)_n) \to P_{k-1}
\]

be the restriction of $d : P_k \to P_{k-1}$ to $\bigoplus_{n \leq \max\{t_k(M_\bullet), t_{k+1}(M_\bullet)\}} I((V_k)_n)$. Let
\[
\tilde{Z} = \ker(\tilde{f}).
\]

Then $Z \subset \tilde{Z}$, so by Lemma 7, the composition
\[
\tilde{Z} \hookrightarrow Z_k \twoheadrightarrow H_k(P_\bullet)
\]
is surjective. Since the kernel of $Z_k \twoheadrightarrow H_k(P_\bullet)$ is $B_k$, the kernel of $\tilde{Z} \twoheadrightarrow H_k(P_\bullet)$ is $\tilde{Z} \cap B_k$. 

\[\square\]
Lemma 8  One has: $t_0(\widetilde{Z} \cap B_k) \leq \max\{t_k(M_\bullet), t_{k+1}(M_\bullet)\}$.

Proof  Let 

$$g : \bigoplus_{n \leq \max\{t_k(M_\bullet), t_{k+1}(M_\bullet)\}} \mathcal{I}(V_{k+1})_n \to P_k$$

be the restriction of $d : P_{k+1} \to P_k$ to $\bigoplus_{n \leq \max\{t_k(M_\bullet), t_{k+1}(M_\bullet)\}} \mathcal{I}(V_{k+1})_n$. Let $\tilde{B}$ be the image of $g$. We claim that $\widetilde{Z} \cap B_k = \tilde{B}$, which would prove the lemma. It is clear that $\tilde{B} \subset \widetilde{Z} \cap B_k$. We need to prove that $\widetilde{Z} \cap B_k \subset \tilde{B}$.

Suppose $S$ is a finite set and $x \in (\widetilde{Z} \cap B_k)_S$. Then there exists an integer $N$ and $y \in \bigoplus_{n \leq N} \mathcal{I}(V_{k+1})_n$ such that $d(y) = x$.

If $N \leq \max\{t_k(M_\bullet), t_{k+1}(M_\bullet)\}$, then we are done. Suppose that $N > \max\{t_k(M_\bullet), t_{k+1}(M_\bullet)\}$. We write $y = y' + y''$ where 

$$y' \in \bigoplus_{n \leq N-1} \mathcal{I}(V_{k+1})_n, \quad y'' \in \mathcal{I}(V_{k+1})_N.$$

Then 

$$d(y') \in \bigoplus_{n \leq N-1} \mathcal{I}(V_k)_S, \quad d(y'') \in \bigoplus_{n \leq N} \mathcal{I}(V_k)_S.$$

But 

$$d(y) = x \in \widetilde{Z}_S \subset \bigoplus_{n \leq \max\{t_k(M_\bullet), t_{k+1}(M_\bullet)\}} \mathcal{I}(V_k)_S.$$

Hence, $d^{N,N}(y'') = 0$. Since $N > t_{k+1}(M_\bullet)$, by Lemma 6, there exists $w \in \mathcal{I}(V_{k+2})_S$ such that $d^{N,N}(w) = y''$. Since we also have 

$$d(w) \in \bigoplus_{n \leq N} \mathcal{I}(V_{k+1})_n,$$

it follows that 

$$y - d(w) \in \bigoplus_{n \leq N-1} \mathcal{I}(V_{k+1})_n.$$

One has $d(y - d(w)) = d(y) = x$. If $N - 1 \leq \max\{t_k(M_\bullet), t_{k+1}(M_\bullet)\}$, then we are done. If not, we repeat the above argument with $y$ replaced by $y - d(w)$.

We now prove Theorem 5.
**Proof of Theorem 5** By Lemma 7, we have \( t_0(H_k(P_\bullet)) \leq t_0(Z) \). By Corollary 4, we have \( t_0(Z) \leq 2t_k(M_\bullet) + 1 \). Hence,

\[ t_0(H_k(P_\bullet)) \leq 2t_k(M_\bullet) + 1. \]

Recall the short exact sequence:

\[ 0 \to \tilde{Z} \cap B_k \to \tilde{Z} \to H_k(P_\bullet) \to 0. \]

Lemma 8 says that \( t_0(\tilde{Z} \cap B_k) \leq \max\{t_k(M_\bullet), t_{k+1}(M_\bullet)\} \). By Corollary 4, we have

\[ t_1(\tilde{Z}) \leq 2\max\{t_k(M_\bullet), t_{k+1}(M_\bullet)\} + 2. \]

Hence,

\[ t_1(H_k(P_\bullet)) \leq 2\max\{t_k(M_\bullet), t_{k+1}(M_\bullet)\} + 2. \]

\[ \square \]

**Remark 9** For any \( \text{FI} \)-module \( L \), denote by \( \delta(L) \) the stable degree of \( L \); see [7, Definition 2.9]. It was proved in [7, Theorem 5.1] that for each \( k \geq 0 \), one has:

\[ \delta(H_k(M_\bullet)) \leq t_k(M_\bullet). \]

Let us give an alternative proof of this inequality. Lemma 7 implies that \( H_k(M_\bullet) \) is a subquotient of \( \bigoplus_{n \leq t_k(M_\bullet)} \mathcal{I}((V_k)_n) \). Therefore, by [7, Proposition 2.10], one has:

\[ \delta(H_k(M_\bullet)) \leq \delta(\bigoplus_{n \leq t_k(M_\bullet)} \mathcal{I}((V_k)_n)) = t_0(\bigoplus_{n \leq t_k(M_\bullet)} \mathcal{I}((V_k)_n)) \leq t_k(M_\bullet). \]

### 3 Homology of congruence subgroups

We now work over \( \mathbb{Z} \), so by \( \text{FI} \)-modules we mean functors from \( \text{FI} \) to the category of \( \mathbb{Z} \)-modules. Let \( R \) be a ring, let \( I \) be a two-sided ideal of \( R \), and denote by \( \text{GL}(R, I) \) the \( \text{FI} \)-group \( S \mapsto \text{GL}_S(R, I) \). Let \( A \) be an abelian group. Then \( H_k(\text{GL}(R, I); A) \) is an \( \text{FI} \)-module whose value on a finite set \( S \) is \( H_k(\text{GL}_S(R, I); A) \).

For any group \( G \), let \( E_\bullet G \) be the bar resolution of the trivial \( \mathbb{Z}G \)-module \( \mathbb{Z} \) and let \( C_\bullet(G; A) = E_\bullet G \otimes_G A \); so one has \( H_k(C_\bullet(G; A)) = H_k(G; A) \). Then \( C_\bullet(\text{GL}(R, I); A) \) is a complex of \( \text{FI} \)-modules such that

\[ H_k(C_\bullet(\text{GL}(R, I); A)) = H_k(\text{GL}(R, I); A). \]

We recall [7, Proposition 5.4]:
**Proposition 10** (Church–Miller–Nagpal–Reinhold) Suppose that $R$ satisfies Bass’s stable range condition $SR_{d+2}$ for some $d \geq 0$ and $I$ is a proper two-sided ideal of $R$. Then for each $k \geq 0$, one has:

$$t_k(C_\bullet(\text{GL}(R, I); A)) \leq 2k + d.$$ 

We deduce that:

**Theorem 11** Suppose that $R$ satisfies Bass’s stable range condition $SR_{d+2}$ for some $d \geq 0$ and $I$ is a proper two-sided ideal of $R$. Then for each $k \geq 0$, one has:

$$t_0(H_k(\text{GL}(R, I); A)) \leq 4k + 2d + 1,$$

$$t_1(H_k(\text{GL}(R, I); A)) \leq 4k + 2d + 6.$$ 

In particular, the presentation degree of $H_k(\text{GL}(R, I); A)$ is at most $4k + 2d + 6$.

**Proof** Immediate from Proposition 10 and Theorem 5. □

Theorem 1 now follows easily:

**Proof of Theorem 1** Immediate from Theorems 11 and 2. □

The following corollary strengthens [6, Theorem 1.4].

**Corollary 12** Let $R$ be the ring of integers of a number field. Let $I$ be a proper ideal of $R$. For any $k \geq 0$ and any field $k$, there exists a polynomial $P(T) \in \mathbb{Q}[T]$ such that

$$\dim_k H_k(\text{GL}_n(R, I); k) = P(n) \quad \text{for every } n > 8k + 10.$$ 

**Proof** We work over the field $k$. Since $R$ satisfies condition $SR_3$, by Theorem 11, we have:

$$t_0(H_k(\text{GL}(R, I); k)) \leq 4k + 3,$$

$$t_1(H_k(\text{GL}(R, I); k)) \leq 4k + 8.$$ 

It is known that $H_k(\text{GL}_n(R, I); k)$ is finite dimensional for every $n \geq 0$; see [6, Remark 1.6]. Hence, the $\text{FI}$-module $H_k(\text{GL}(R, I); k)$ is finitely generated. The corollary now follows from [11, Theorem 1.3]. □

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