SYMPLECTIC STRUCTURES AND VOLUME ELEMENTS IN THE FUNCTION SPACE FOR THE CUBIC Schrödinger EQUATION

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Abstract. We consider various trace formulas for the cubic Schrödinger equation in the space of infinitely smooth functions subject to periodic boundary conditions. The formulas relate conventional integrals of motion to the periods of some Abelian differentials (holomorphic one-forms) on the spectral curve. We show that the periods of Abelian differentials are global coordinates on the moduli space of spectral curves. The exterior derivatives of the holomorphic one-forms are the basic and higher symplectic structures on the phase space. We write explicitly these symplectic structures in $QP$ coordinates. We compute the ratio of two symplectic volume elements in the infinite genus limit.

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1. Introduction. We consider the cubic Nonlinear Schrödinger equation (NLS)

\[ i\psi' = -\psi'' + 2|\psi|^2\psi \]

with periodic boundary conditions. It has an infinite series of conserved quantities – integrals of motion \( H_1, H_2, \cdots \). The first three are "classical" integrals

\[
H_1 = \frac{1}{2} \int |\psi|^2 dx = N = \text{number of particles},
\]

\[
H_2 = \frac{1}{2i} \int \psi' \bar{\psi} dx = P = \text{momentum},
\]

\[
H_3 = \frac{1}{2} \int |\psi'|^2 + |\psi|^4 dx = \mathcal{H} = \text{energy}.
\]

The others \( H_4, H_5, \cdots \) do not have classical names.

In order to express an invariant Gibbs State

\[ e^{-H} \prod_x d^{\infty} \psi(x) d^{\infty} \bar{\psi}(x) \]

in action-angle variables, [V1], one has to write \( H \) as a sum of actions

\[ H = \sum_n I_n. \]

The variables \( I \)'s depended, of course, on the integral \( H \). Such formulas for the NLS equation were obtain in [MCV1]. The \( I \)'s are the integrals of the various meromorphic 1-forms over the real ovals on the hyperelliptic curve associated with the spectral problem for NLS. In this paper we show that \( I \)'s entering into the trace formulas for \( H_1 \) and \( H_3 \) and supplemented by some other parameters are \textit{global moduli} for hyperelliptic Riemann surfaces with real branching points. From another side, they are the \textit{actions} in the mechanical sense [A]. They are associated with some symplectic structures on the phase space and we write the corresponding symplectic forms \textit{explicitly}.

As was demonstrated in [V2] the asymptotics for the ratio of two symplectic volumes in the limit of infinite genus determines the thermodynamic properties
of the Gibbs’ state of completely integrable system. There was conjectured a
form of the asymptotics for particles interacting via an elliptic potential. Here
we prove formulas of this type for hyperelliptic Riemann surfaces which arise in
the spectral problem for the NLS equation.

In the papers of Witten with coauthors, [SW,DW], the Bogomolny-Prasad-
Sommerfeld spectrum of $N = 2$ supersymmetric Yang-Mills theory was expressed
in terms of the periods of a similar one-form on a Riemann surface. The defining
property of the meromorphic 1-form is that the exterior derivative of it is a
holomorphic two-form. The subsequent paper of Krichever and Phong, [KP],
provides a firm algebro-geometrical framework for the constructions of physicists.
We essentially use the results of [KP] in this paper.

This paper is organized as follows. In sections 2 through 5 we derive various
auxiliary technical facts. The trace formulas are presented in the section 6.
In sections 7 through 9 we prepare the basic technical tool— the identity for
variations of the differential of quasimomentum entering into the trace formulas.
The variables $I$’s are studied as moduli in section 10. We write explicitly the
corresponding symplectic forms in sections 11 and 12. In section 14 we compute
the ratio of two symplectic volumes in the infinite genus limit.

2. **Zero curvature representation.** Throughout the paper the complex form
of the Nonlinear Schrödinger (NLS) equation*

$$i\psi^* = -\psi'' + 2|\psi|^2\psi,$$

where $\psi = Q + iP$ is replaced by the more convenient real form

$$Q^* = -P'' + 2(Q^2 + P^2)P,$$

$$P^* = Q'' - 2(Q^2 + P^2)Q.$$ 

The components $Q(x, t), P(x, t)$ are infinitely smooth real functions of the space
variable $x$, periodic with the period $2l_x$.

The equation can be written as a Hamiltonian system

$$Q^* = \{Q, \mathcal{H}\}, \quad P^* = \{P, \mathcal{H}\},$$

with the Hamiltonian $\mathcal{H} = 1/2 \int_{-l_x}^{l_x} Q'^2 + P'^2 + (Q^2 + P^2)^2 = \text{energy}$ and the
“classical” bracket

$$\{A, B\} = \int_{-l_x}^{l_x} \frac{\partial A}{\partial Q(x)} \frac{\partial B}{\partial P(x)} - \frac{\partial A}{\partial P(x)} \frac{\partial B}{\partial Q(x)} \, dx.$$ 

** denotes a derivative in the time variable $t$, ’ in the $x$ variable.
The equation has two other “classical” integrals of motion $N = 1/2 \int_{-l}^{l} Q^2 + P^2 = \# \text{of particles}$ and $\mathcal{P} = \int_{-l}^{l} Q P' = \text{momentum}$.

The NLS equation is a commutativity condition for $2 \times 2$ matrix differential operators

$$\left[ \frac{\partial}{\partial x} - U, \frac{\partial}{\partial t} - V \right] = 0,$$

where

$$U = -\frac{\lambda}{2} J + U_0 = -\frac{\lambda}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} Q & P \\ P & -Q \end{pmatrix},$$

$$V = \frac{\lambda^2}{2} J - \lambda U_0 + (Q^2 + P^2)J - JU'_0$$

$$= \frac{\lambda^2}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \lambda \begin{pmatrix} Q & P \\ P & -Q \end{pmatrix} + (Q^2 + P^2) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} -P' & Q' \\ Q' & P' \end{pmatrix}.$$ 

3. Monodromy matrix. Expansion at infinity. The monodromy matrix $M(x, x_0, \lambda)$ is a $2 \times 2$ solution of

$$M'(x, x_0, \lambda) = \left[ -\frac{\lambda}{2} J + U_0(x) \right] M(x, x_0, \lambda),$$

$$M(x, x_0, \lambda)|_{x=x_0} = I,$$

and it is given by the matrix exponent

$$M(x, x_0, \lambda) = \exp \int_{x_0}^{x} U(y, \lambda) \, dy.$$ 

For $QP \equiv 0$ the matrix $M$ can be easily computed

$$M_0(x, x_0, \lambda) = e^{-\lambda(x-x_0)/2}J = R \left( \lambda(x-x_0)/2 \right)$$

$$= \begin{pmatrix} \cos \lambda(x-x_0)/2 & -\sin \lambda(x-x_0)/2 \\ \sin \lambda(x-x_0)/2 & \cos \lambda(x-x_0)/2 \end{pmatrix}.$$ 

For $QP \neq 0$ and large $\lambda$ the matrix $M$ is a perturbation of this trivial case. It is a solution of the integral equation

$$M(x, x_0, \lambda) - \int_{x_0}^{x} R \left( \lambda(x-s)/2 \right) U_0(s) M(s, x_0, \lambda) \, ds = R \left( \lambda(x-x_0)/2 \right),$$

or in symbolic form $[I - A] M = R$.

Denote for any matrix $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ two traces $\text{tr} S \equiv a + d$, $\text{tr}_a S \equiv b + c$. 

Lemma 1. The matrix $M$ can be represented by the convergent series

$$M = [I - A]^{-1} R = \sum_{n=0}^{\infty} A^n R = M_0 + M_1 + M_2 + \cdots.$$ 

The first few terms are listed below

$$M_0(x, x_0, \lambda) = R\left(\lambda(x - x_0)/2\right),$$

$$M_1(x, x_0, \lambda) = R(\lambda(x_0 - x)/2) \left[-\frac{J}{\lambda} U_0(x) + \frac{1}{\lambda^2} U_0'(x)\right]$$

$$- R(\lambda(x_0 - x)/2) \left[-\frac{J}{\lambda} U_0(x_0) + \frac{1}{\lambda^2} U_0'(x_0)\right],$$

$$M_2(x, x_0, \lambda) = \frac{J}{\lambda} R(\lambda(x_0 - x)/2) \int_{x_0}^{x} Q^2 + P^2$$

$$+ \frac{1}{\lambda^2} R(\lambda(x_0 - x)/2) \left[I \int_{x_0}^{x} (QQ' + PP') + J \int_{x_0}^{x} (QP' - PQ')\right]$$

$$- R(\lambda(x_0 - x)/2) \frac{1}{\lambda^2} U_0(x) U_0(x_0) + R(\lambda(x - x_0)/2) \frac{1}{\lambda^2} U_0^2(x_0).$$

These are all up to the error $^*$

$$| | \leq O\left(\frac{e^{3\lambda(x-x_0)/2}}{\lambda^{5/2}}\right).$$

The whole tail $\sum_{n=3}^{\infty}$ can be estimated as

$$| | \leq o\left(\frac{e^{3\lambda(x-x_0)/2}}{\lambda}\right).$$

Moreover, $\text{tr} M_n = 0$ for $n$ odd and $\text{tr}_a M_n = 0$ for $n$ even.

Proof. The expression for each term $M_1$, $M_2$ etc., can be obtained from the formula $M_n = A M_{n-1}$ by integrating by parts. The estimate for the error is straightforward from this. The estimate for the whole tail is proved in [MCV1]. To prove the statement about traces we need an explicit expression for $M_n$:

$$M_n(x, x_0, \lambda) = \int_{x \geq x_1 \geq \cdots \geq x_n \geq x_0} d^n x RU_0(x_1) \cdots U_0(x_n).$$

* $| |$ stays for any multiplicative matrix norm.
The matrix $U_0$ can be written in the form

$$U_0(x) = \sqrt{Q^2 + P^2(x)} R(\varphi(x)) A,$$

where $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\varphi(x)$ is some real function. Therefore

$$M_n(x, x_0, \lambda) = \int_{x \geq x_1 \geq \cdots \geq x_n \geq x_0} d^n x \prod_{k=1}^{n} \sqrt{Q^2 + P^2(x_k)} R' A^n$$

with some $R'$. Therefore for $n$ even $\text{tr}_a M^n = \int d^n x \prod \sqrt{Q^2 + P^2} \text{tr}_a R' A^n = 0$. For $n$ odd $\text{tr}_a M^n = \int d^n x \prod \sqrt{Q^2 + P^2} \text{tr}_a R' A = 0$. □

4. Spectrum. Some products.

The discriminant is defined as $\Delta(\lambda) = \frac{1}{2} \text{tr} M(l_x, -l_x, \lambda)$. The roots of the equation $\Delta(\lambda)^2 - 1 = 0$ are the points of the periodic/antiperiodic spectrum. When $QP \equiv 0$ we have $\Delta_0(\lambda) = \cos \lambda l_x$ and double eigenvalues at the points $\lambda_n^\pm = \frac{\pi n}{l_x}$. If $n$ is even/odd, then the corresponding $\lambda_n^\pm$ belongs to the periodic/antiperiodic spectrum.

For generic $QP$, formal perturbation arguments show that

$$\lambda_n^\pm = \frac{\pi n}{l_x} \pm 2|\hat{\psi}(n)| + \cdots,$$

where

$$\hat{\psi}(n) = \frac{1}{2l_x} \int_{-l_x}^{+l_x} \psi(x) e^{-i \frac{\pi n}{l_x} x} dx.$$

The size of the gap is determined roughly speaking by the Fourier coefficients of the potential. In fact, for $QP \in C^\infty$ we have*

$$\lambda_n^+ - \lambda_n^- = O \left( \frac{1}{n^\infty} \right).$$

We denote by $\mu_n$ the roots of the equation $m_{12}(\lambda) = 0$ and by $\lambda_n^\bullet$ the roots of $\Delta^\bullet(\lambda) = 0$. These roots are always caught in the open gaps $[\lambda_n^-, \lambda_n^+]$. We can write Hadamard’s products for $\Delta^2 - 1$, $m_{12}(\lambda)$ and $\Delta^\bullet(\lambda)$ as

1. $$\Delta^2 - 1 = -l_x^2 (\lambda - \lambda_0^-)(\lambda - \lambda_0^+) \prod_{n \neq 0} \frac{(\lambda - \lambda_n^-)(\lambda - \lambda_n^+)}{\pi^2 n^2 l_x^2},$$

2. $$m_{12}(\lambda) = -l_x (\lambda - \mu_0) \prod_{n \neq 0} \frac{(\lambda - \mu_n)}{\pi n l_x},$$

3. $$\Delta^\bullet(\lambda) = -l_x^2 (\lambda - \lambda_0^\bullet) \prod_{n \neq 0} \frac{(\lambda - \lambda_n^\bullet)}{\pi n l_x}.$$  

*\(O \left( \frac{1}{n^k} \right) \) means \(O \left( \frac{1}{n^k} \right) \) for any \(k = 1, 2, \cdots \).
5. Floquet solutions. Differential of quasimomentum. The eigenvalues of the monodromy matrix $M(l_x, -l_x, \lambda)$ can be easily computed. They are the roots of the quadratic equation $w^2 - 2\Delta w + 1 = 0$, and given by the formula $w = \Delta \pm \sqrt{\Delta^2 - 1}$. They become single-valued on the Riemann surface (infinite-genus in the generic case) $\Gamma$:

$$(\lambda, y) \in \mathbb{C}^2 : \quad y^2 = \Delta^2(\lambda) - 1.$$  

We denote by $Q$ a point $(\lambda, y)$ on the curve $\Gamma$ and by $P_{\pm}$ the two infinities.

We introduce a multivalued function $p(Q)$ on the curve $\Gamma$ as $w(Q) = e^{ip(Q)2l_x}$. It is defined up to $\frac{\pi n}{l_x}$, where $n$ is an integer. Let $\tau_0$ be a holomorphic involution on the curve $\Gamma$ permuting sheets $\tau_0 : (\lambda, y) \mapsto (\lambda, -y)$, then $p(\tau_0 Q) + p(Q) = \frac{\pi n}{l_x}$.

The Floquet solution $e(x, Q)$ is defined as a solution which is an eigenvector of the matrix $M(l_x, -l_x, \lambda) : \quad M(l_x, -l_x, \lambda) e(l_x, Q) = w(Q) e(l_x, Q)$, normalized by the condition $e_1(-l_x, Q) = 1$. It is given by the formula

$$e(x, Q) = \begin{bmatrix} m_{11}(x, \lambda) \\ m_{21}(x, \lambda) \end{bmatrix} + \frac{w(Q) - m_{11}(l_x, \lambda)}{m_{12}(l_x, \lambda)} \begin{bmatrix} m_{12}(x, \lambda) \\ m_{22}(x, \lambda) \end{bmatrix}, \quad \lambda = \lambda(Q).$$  

For $QP \equiv 0$ the Floquet solution is $e(x, Q) = e^{\pm i\frac{x}{l_x}l_x} \begin{bmatrix} 1 \\ -i \end{bmatrix}$. If $QP \neq 0$ the function $e(x, Q)$ admits asymptotic expansion at infinities:

$$e(x, Q) = e^{+i\frac{x}{l_x}l_x} \left[ \sum_{s=0}^{\infty} a_s \lambda^{-s} \right], \quad Q \in (P_+) \quad \text{and} \quad a_0 = 1, \quad c_0 = -i;$$  

$$e(x, Q) = e^{-i\frac{x}{l_x}l_x} \left[ \sum_{s=0}^{\infty} b_s \lambda^{-s} \right], \quad Q \in (P_-) \quad \text{and} \quad b_0 = 1, \quad d_0 = i.$$  

In order to relate these two, consider

$$e(x, Q) + e(x, \tau_0 Q) = \begin{bmatrix} m_{11}(x, \lambda) \\ m_{21}(x, \lambda) \end{bmatrix} + \frac{m_{22}(l_x, \lambda) - m_{11}(l_x, \lambda)}{m_{12}(l_x, \lambda)} \begin{bmatrix} m_{12}(x, \lambda) \\ m_{22}(x, \lambda) \end{bmatrix},$$

which is a real function for real $\lambda$. This implies $a_s = \bar{b}_s$ and $c_s = \bar{d}_s$ for all $s = 0, 1, \ldots$.

Substituting the expansion for $e(x, Q)$ into $[\partial_x - U(x, \lambda)] e = 0$ we obtain the recurrent relation

$$(5) \quad - \begin{bmatrix} a_k' \\ c_k' \end{bmatrix} + U_0 \begin{bmatrix} a_k \\ c_k \end{bmatrix} = \left( \frac{i}{2} I + \frac{1}{2} J \right) \begin{bmatrix} a_{k+1} \\ c_{k+1} \end{bmatrix}, \quad k = 0, 1, \ldots.$$
For \( k = 0 \) the formula produces \( Q - iP = \frac{i}{2}a_1 + \frac{1}{2}c_1 \). Note that the matrix \( \frac{i}{2}I + \frac{1}{2}J \) is degenerate and the coefficients of the asymptotic expansion can not be computed recursively, but they can be computed with the aid of Lemma 1 and formula (4). For example,

\[
(6) \quad a_1 = -iQ - P + iQ(-l_x) + P(-l_x) - i \int_{-l_x}^{x} (Q^2 + P^2),
\]

\[
(7) \quad c_1 = Q - iP + Q(-l_x) - iP(-l_x) - \int_{-l_x}^{x} (Q^2 + P^2).
\]

Similarly, one can compute the asymptotic expansion for \( p(Q) \) at the infinities

\[
p(\lambda) = \pm \frac{\lambda}{2} + a_0^\pm + \frac{a_1^\pm}{\lambda} + \frac{a_2^\pm}{\lambda^2} \ldots.
\]

Indeed, from Lemma 1

\[
\Delta(\lambda) = \cos l_x \lambda + \frac{2N \sin l_x \lambda}{\lambda} + \ldots
\]

and using \( \Delta(\lambda) = \cos 2l_x p(\lambda) \) we obtain

\[
a_0^\pm = \frac{\pi k_\pm}{l_x},
\]

\[
a_1^\pm = \mp \frac{1}{2l_x} \int_{-l_x}^{l_x} Q^2 + P^2 = \mp \frac{1}{l_x} N,
\]

\[
a_2^\pm = \mp \frac{1}{2l_x} \int_{-l_x}^{l_x} QP' = \mp \frac{1}{l_x} P,
\]

\[
a_3^\pm = \mp \frac{1}{2l_x} \int_{-l_x}^{l_x} Q^2 + P'^2 + (Q^2 + P^2)^2 = \mp \frac{1}{l_x} H, \quad \text{etc.}
\]

The differential \( dp \) is of the second kind with double poles at the infinities of the form \( \pm dp = d \left( \frac{\lambda}{2} + O(1) \right) \) and zero \( a \)-periods. All these one can see from the formula

\[
dp = \pm \frac{1}{i2l_x} d \cosh^{-1} \Delta(\lambda) = \pm \frac{1}{i2l_x} \frac{\Delta^\bullet(\lambda)}{\sqrt{\Delta^2 - 1}} d\lambda.
\]

The \( b \)-periods of \( dp \) are \( \frac{\pi n_b}{l_x} \) \((\text{periodicity condition})\). This implies that \( w(Q) \) is single-valued on the curve.
6. Trace formulas. One-gap potentials. As in [MCV1] we have

\[ N = \sum_n I_n, \quad I_n = \frac{l_x}{2\pi i} \int_{a_n} p(\lambda) d\lambda, \]

(8)

\[ P = \sum_n I'_n, \quad I'_n = \frac{l_x}{2\pi i} \int_{a_n} \lambda p(\lambda) d\lambda, \]

(9)

\[ H = \sum_n I''_n, \quad I''_n = \frac{l_x}{2\pi i} \int_{a_n} \lambda^2 p(\lambda) d\lambda, \]

etc. (10)

We use these formulas here to determine positions of branching points in the case of a one-gap potential. Let

\[ \alpha = \frac{\lambda_n^- + \lambda_n^+}{2}, \quad \beta = \frac{\lambda_n^+ - \lambda_n^-}{2}. \]

Lemma 2. For \( \psi_n(x) = Ae^{i\frac{\pi n}{l_x}x} \) we have

i. \( \lambda_n^\bullet = \alpha \),

ii. \( \beta = 2|A| \),

iii. \( \alpha = \frac{\pi n}{l_x} \).

Proof. For such potential \( \lambda_k^- = \lambda_k^+ \) if \( k \neq n \). Using Hadamard’s products (1) and (3)

\[ dp = \frac{1}{i2l_x} \frac{\Delta^\bullet(\lambda) d\lambda}{\sqrt{\Delta^2 - 1}} = \frac{i}{2} \frac{(\lambda - \lambda_n^\bullet)}{\sqrt{\beta^2 - (\lambda - \alpha)^2}} d\lambda. \]

The condition \( \int_{a_n} dp = 0 \) implies \( \lambda_n^\bullet = \alpha \) and i. is proved.

If \( Q + iP = Ae^{i\frac{\pi n}{l_x}x} \), then \( N = l_x |A|^2 \). The integral in the right-hand side of the first trace formula can be computed

\[ N = \frac{l_x}{2\pi i} \int_{a_n} \lambda dp(\lambda) = \frac{l_x \beta^2}{4}. \]

This implies ii.

The trace formula for \( P \) can be used similarly to prove iii. \( \square \)

Remark. One can rescale the zero curvature representation as \( \lambda \to a\lambda \), where \( a \) is an arbitrary real number. With our choice of the scaling the potential with the frequency \( \frac{\pi n}{l_x} \) opens the n-th gap with middle point at \( \alpha = \frac{\pi n}{l_x} \).

7. Infinite hierarchy of commuting flows. BA function. This and the other two sections are preparatory. We obtain the formula for the variation of the differential \( dp \) which will be used later.
Consider a finite genus curve $\Gamma$ with arbitrary real branching points. The function $e(t, \zeta, Q)$ has asymptotics at infinities $P_\pm$

$$e(t, \zeta, Q) = e^{\pm i\left(\sum_{n=1}^{\infty} a_n \lambda^n t_n + \sum_{n=0}^{g} a'_n \lambda^n \zeta_n \right) \times \left[ \left( \frac{1}{\mp i} \right) + O \left( \frac{1}{\lambda} \right) \right]}$$

and poles at the points $\gamma_1, \cdots, \gamma_{g+1}$, located on the real ovals. As it is shown in [Kr1] for any $n = 1, 2, \cdots$ there exists a matrix $U_n(x, \lambda)$ which depends polynomially on the parameter $\lambda$ and such that

$$[\partial_{t_n} - U_n] e(t, \zeta, Q) = 0.$$ 

We limit ourself to the case when $e$ depends only on the two times, $t_1 = x, t_2 = t$,

$$e(x, t, \zeta, Q) = e^{\pm i\left(\frac{\lambda}{2} x - \frac{\lambda^2}{2} t + \sum_{n=0}^{g} a'_n \lambda^n \zeta_n \right) \times \left[ \left( \frac{1}{\mp i} \right) + O \left( \frac{1}{\lambda} \right) \right]}.$$ 

In this case we can simply write $U_1 = U, U_2 = V$. The quasiperiodic character of the solution $Q(x, t)$ and $P(x, t)$ has been known for a long time, see e.g. [I]. We need a particular form of it.

**Lemma 3.** [Kr2] The functions $Q(x, t)$ and $P(x, t)$ are quasiperiodic functions on the $g + 1$ dimensional real torus and the initial position on the torus is determined by the parameters $\zeta = (\zeta_0, \zeta_1, \cdots, \zeta_g) = (\zeta_0, \zeta_{1g})$

$$Q(x, t) = \tilde{Q}(\zeta_0, Ux + Wt + \zeta_{1g}) \quad P(x, t) = \tilde{P}(\zeta_0, Ux + Wt + \zeta_{1g}),$$

where $\tilde{Q}(z_0, z_1, \cdots, z_g), \tilde{P}(z_0, z_1, \cdots, z_g)$ are the functions with unit periods in all $z$’s and $U, W$ are vectors from $R^g$.

**Proof.** First, we introduce multi-valued functions $p(Q), E(Q)$ on the curve with singularities at infinities

$$p(Q) = \pm \frac{\lambda}{2} + a^\pm + O \left( \frac{1}{\lambda} \right), \quad \lambda = \lambda(Q), \quad Q \in (P_\pm).$$

Similar

$$E(Q) = \mp \frac{\lambda^2}{2} + b^\pm + O \left( \frac{1}{\lambda} \right), \quad \lambda = \lambda(Q), \quad Q \in (P_\pm).$$

The function $e(x, t, \zeta, Q)$ can be written in the form

$$e(x, t, \zeta, Q) = CA \Phi(Ux + Wt + \zeta_{1g}, Q) e^{ip(Q)x+iE(Q)t},$$

where $CA$ is a constant and $\Phi$ is a function defined on $R^g$ with period $1$ in all variables.
where
\[
C = \begin{pmatrix} 1 & 1 \\ -i & +i \end{pmatrix}, \quad A = \begin{pmatrix} e^{-ia^+x - ib^+t + ia_0'\zeta_0} & 0 \\ 0 & e^{-ia^-x - ib^-t - ia_0'\zeta_0} \end{pmatrix},
\]
\[
\Phi = \begin{pmatrix} \varphi_+ (Ux + Wt + \zeta_1g, Q) \\ \varphi_- (Ux + Wt + \zeta_1g, Q) \end{pmatrix}
\]
and \(\varphi_\pm (z_1, \cdots, z_g, Q)\) are the functions with unit periods in \(z\)'s and rational singularities in \(Q\).

Second, we expand \(e(x, t, \zeta, Q)\) at infinity similarly to section 5 and obtain explicit formula for the solution.

We present complete arguments only for the genus 0. The curve is rational and given by the equation (see section 6):
\[
y^2 = \beta^2 - (\lambda - \alpha)^2.
\]
The uniformizing parameter \(z\) is introduced as
\[
\lambda = z + \frac{\beta^2}{4(z - \alpha)}; \quad y = i(z - \alpha) + \frac{\beta^2}{4i(z - \alpha)}.
\]
It is easy to see \(z(\gamma_1) = \alpha + z_0 = \alpha + \frac{\beta}{2}e^{i\delta}\) and the sheet map \(\tau_0z = \alpha + \frac{\beta^2}{4(z - \alpha)}\).

On such a curve one can write the differentials \(dp, dE\) explicitly
\[
dp = \frac{i}{2} \frac{\lambda - \alpha}{\sqrt{\beta^2 - (\lambda - \alpha)^2}} d\lambda,
\]
\[
dE = \frac{i}{\sqrt{\beta^2 - (\lambda - \alpha)^2}} \frac{(\lambda - \alpha)^2 + \alpha(\lambda - \alpha) - \beta^2/2}{d\lambda}.
\]
The functions \(p(Q)\) and \(E(Q)\) are single-valued on the curve and given by the formulas
\[
p(z) = \frac{1}{2} \left( z - \frac{\beta^2}{4(z - \alpha)} \right),
\]
\[
E(z) = -\frac{1}{2} \left( z^2 + \frac{\beta^2}{2} \right) + \frac{1}{2} \left( \frac{\beta^2}{4(z - \alpha)} \right)^2 - \frac{2\alpha\beta}{4(z - \alpha)}.
\]
The parameters in the expansion of \(p(Q)\) and \(E(Q)\) are \(a^+ = b^+ = 0\) and \(a^- = \alpha\) and \(b^- = -(\alpha^2 + \frac{\beta^2}{2})\).
The functions $\varphi$ are given by the formulas $\varphi_{\pm} = h_{\pm}s_{\pm}$, where

$$h_{+}(z) = \frac{z - \alpha}{z - \alpha - z_0}, \quad h_{-}(z) = -\frac{z_0}{z - \alpha - z_0};$$

and $s_{\pm}(x, t, z) \equiv 1$. Expanding $e(x, t, \zeta_0, Q)$ at the infinity $P_{+}$, say, we obtain the formula for the solution

$$Q + iP = i\frac{\beta}{2}e^{-i\delta}e^{i2a'_0\zeta}e^{i\alpha - i(\alpha^2 + \beta^2)}t.$$

Choosing suitably the parameter $a'_0$ we prove the statement of the theorem.

The case of genus $> 0$ is considered in [Kr2]. □

8. Dual Baker-Akhiezer function. In this section we introduce the dual BA function. The differential $dp$ has $2g + 2$ zeros on the real ovals; $g + 1$ of them $\gamma_1, \cdots, \gamma_{g+1}$ are related to the other $\gamma_{g+1}^\dagger, \cdots, \gamma_1^\dagger$ by the involution permuting the sheets $\tau_0\gamma = \gamma^\dagger$. Now, place the poles of BA functions at the points $\gamma_1, \cdots, \gamma_{g+1}$ and define the function $e^\dagger(x, t, \zeta, Q)$ as

$$e^\dagger(x, t, \zeta, Q) = [e(x, t, \zeta, \tau_0Q)]^+. $$

The BA functions satisfy the equations

$$[J\partial_x - JU]e(x, t, \zeta, Q) = 0, \quad e^\dagger(x, t, \zeta, Q) [J\partial_x - JU] = 0$$

and similarly in the $t$ variable. The proof is by direct computation and the Riemann-Roch theorem. From the results of the previous section

$$e^\dagger(x, t, \zeta, Q) = e^{i\rho(\tau_0Q)x + iE(\tau_0Q)t} \Phi^+ A C^+. $$

The $2 \times 2$ matrix differential $e(t, \zeta, Q)e^\dagger(t, \zeta, Q)dp(Q)$ has singularities (at most) of rational character at the infinities. The sum of residues for each entry must be equal to zero. This requires something hidden from the potential. Namely, in the periodic case using formulas (2) and (3) we compute

$$(11) \quad P(-l_x) = 0.$$

This fact will be used later.

*The sign $+$ here stays for a transpose.

**The action of the differential operator $D = \sum_{j=0}^{k} \omega_j \partial^j$ on the row vector $f^\dagger$ is defined as $f^\dagger D = \sum_{j=0}^{k} (-\partial)^j(f^\dagger \omega_j)$. 
9. Variational identity. To follow the scheme developed in [Kr2] for 2+1 systems, we introduce an additional variable $\epsilon$ such that

$$e(\epsilon, x, t, \zeta, Q) = e(x, t, \zeta, Q) e^{\epsilon \lambda(Q)}, \quad e^\dagger(\epsilon, x, t, \zeta, Q) = e^{-\epsilon \lambda(Q)} e^\dagger(x, t, \zeta, Q).$$

The matrix polynomial $U_k = \sum_{n=0}^{k} u_n^k \lambda^n$ is replaced by the differential operator $U_k = \sum_{n=0}^{k} u_n^k \partial^\epsilon_n$. The "new" BA function satisfies

$$[J \partial_x - J U] e(\epsilon, x, t, \zeta, Q) = 0.$$

Let parameters $I_1, \ldots, I_N$ parametrize the set of spectral curves (not necessarily periodic) and $\zeta_0, \ldots, \zeta_g$ determine the point on the Jacobian (invariant manifold). For such curves we denote averaging in the $x$ variable as

$$< \bullet >_x = \lim_{L \to \infty} \frac{1}{2L} \int_{-L}^{L} dx' \bullet.$$

In the periodic case it produces the same result as the averaging over the period. Assume that all $I$’s, the point $Q$ on the curve depend on the parameter $\tau$ and so are $p(Q), a^\pm$, etc. Note that the variables $\zeta$’s are fixed. We are interested in the variation of $p(Q)$ in response to the variation of $\tau$.

**Lemma 4.** [Kr2] The following relation holds

$$i \partial_\tau p(Q) < e^\dagger J e >_x - i < e^\dagger J C \begin{pmatrix} \partial_\tau a^+ & 0 \\ 0 & \partial_\tau a^- \end{pmatrix} A C^{-1} e >_x$$

$$+ < e^\dagger J C A \sum_{j=1}^{g} \partial_\tau U_j \frac{\partial \Phi}{\partial \zeta_j} e^{i p(Q) + i E(Q) t} >_x$$

$$= -\partial_\tau \lambda(Q) < e^\dagger J U(1) e >_x + < e^\dagger J \partial_\tau U e >_x.$$

**Proof.** Let $e^\dagger = e^\dagger(\epsilon, x, t, \zeta, Q(\tau)), \quad e_1 = e(\epsilon, x, t, \zeta, Q(\tau_1)), \quad U = U(\tau), \quad V = V(\tau), \quad U_1 = U(\tau_1), \quad V_1 = V(\tau_1).$ They satisfy the identity*

$$\partial_x [e^\dagger J e_1] = -\partial_\epsilon \left[ e^\dagger J U(1) e_1 \right] + e^\dagger [J U_1 - J U] e_1.$$

*we use the formula $(f^\dagger D) g = \sum_{j=0}^{k} \partial_\epsilon^j (f^\dagger(D^j) g)$, where \(D^{(0)} = D, \quad D^{(1)} = -\sum_{i=1}^{k} j w_j \partial_\epsilon^{j-1}, \quad etc.$
Differentiating with respect to $\tau_1$ and assuming $\tau_1 = \tau$

\[ i \partial_\tau p(Q) \left[ e^\dagger Je \right] - i [e^\dagger JC \left( \begin{array}{cc} \partial_\tau a^+ & 0 \\ 0 & \partial_\tau a^- \end{array} \right) AC^{-1} e] \]

\[ + [e^\dagger JCA \sum_{j=1}^g \partial_\tau U_j \frac{\partial \Phi}{\partial \zeta_j} e^{ip(Q)+iE(Q)t}] \]

\[ = - \partial_\tau \lambda(Q) \left[ e^\dagger JU(1)e \right] + e^\dagger [J\partial_\tau U(\tau)] e + R, \]

where $R$ is the remainder term. Let $T_1(\zeta)$ is a subtorus of $T^{g+1}$ which contains the closure of the trajectory under $x$-dynamics. Integrate the previous identity with respect to the variable $\zeta$ over the flat measure on $T_1(\zeta)$. The error term vanishes. Due to the ergodicity of the $x$-dynamics on $T_1(\zeta)$ one can replace integration in $\zeta$ by $x$-averaging. The lemma is proved. □

10. Periods are moduli. Consider a deformation of the original curve $\Gamma$. Namely, assume that the branch points are differentiable functions of the parameter $\tau$:

\[ \Gamma(\tau) : y^2 = - \prod_{k=N}^{k=N} (\lambda - \lambda_k^+(\tau))(\lambda - \lambda_k^-(\tau)), \text{ where } \Gamma(\tau)|_{\tau=0} = \Gamma. \]

On the curve we have a multivalued function $p(\lambda, \tau)$, with the asymptotics at infinities

\[ p(\lambda, \tau) = \pm \frac{\lambda}{2} + a_0^+(\tau) + O \left( \frac{1}{\lambda} \right), \quad \lambda = \lambda(Q), \quad Q \in (P_\pm) \]

and normalized so that $a_0^+(\tau) = 0$.

We will show that the actions entering into the trace formula (8) and supplemented by the $b$-periods are global coordinates for such curves with real branching points. In [MCV1] it is proved under a periodicity assumption on the curve by direct computation of the Jacobian. The local version of such statement is presented in [KP]. Here we restrict the class of curves in consideration and therefore are able to obtain a global result. Our proof also can be easily generalized for the actions entering into the trace formula (10).

Lemma 5. The map from $\lambda_{-N}^- < \lambda_{-N}^+ < \cdots < \lambda_N^- < \lambda_N^+$ to

\[ I_1, \cdots, I_{g+1}, U_1, \cdots, U_g, a_0^- (\tau), \]

where

\[ I_k(\tau) = \frac{l_x}{2\pi i} \int_{a_k} p(\lambda, \tau) d\lambda, \quad k = 1, 2, \cdots, g + 1; \]
\[ U_k(\tau) = \int_{b_k} dp(\lambda, \tau), \quad k = 1, 2, \cdots, g. \]

has nonvanishing Jacobian.

**Proof.** Consider a deformation, such that \( \partial_\tau I_k(\tau)|_{\tau=0} = \partial_\tau U_k(\tau)|_{\tau=0} = 0 \) for all \( k \) and also \( \partial_\tau a^-_k(\tau)|_{\tau=0} = 0 \). This is equivalent to saying that the response of these parameters to the variation of \( \tau \) is of order \( O(\tau^2) \). Outside of the branching points the function \( p(\lambda, \tau) \) is defined up to

\[
\sum_{k=1}^{g} n_k b_k \left[ dp(\lambda, 0) \right] + O(\tau^2),
\]

where \( n_k \) are some integer numbers. Therefore, the derivative \( \partial_\tau p(\lambda, \tau) \) is a single-valued function on the curve.

In the vicinity of any branch point, say \( Q_k^+ = (\lambda_k^+, 0) \) the differential \( dp(\lambda, \tau) \) can be represented by the series

\[
dp(\lambda, \tau) = \sum_{k=0}^{\infty} \frac{\omega_k}{k!} (\lambda - \lambda_k^+(\tau))^{k-\frac{1}{2}} d\lambda.
\]

Then

\[
p(\lambda, \tau) = \int^\lambda dp(\lambda, \tau) = \sum_{k=0}^{\infty} \frac{\omega_k}{k!} \frac{1}{k + \frac{1}{2}} (\lambda - \lambda_k^+)^{k + \frac{1}{2}} + \text{regular part}.
\]

Therefore

\[
\partial_\tau p(\lambda, \tau) = -\omega_0 (\lambda - \lambda_k^+)^{-\frac{1}{2}} \partial_\tau \lambda_k^+ + \text{regular part},
\]

and

\[
\text{res } \partial_\tau p(\lambda, \tau) = -\omega_0 \partial_\tau \lambda_k^+(\tau).
\]

The rest of the proof consists of two parts. First, we show that \( \partial_\tau p \equiv 0 \). Second, we show that \( \omega_0 \neq 0 \) for all branch points. These lead to \( \partial_\tau \lambda_k^\pm = 0 \) and prove the result.

The condition \( \partial_\tau I_k(\tau)|_{\tau=0} = 0 \) implies

\[
(12) \quad \int_{Q_{k}^+} \partial_\tau p \, d\lambda = 0 \quad k = 1, \cdots, g + 1.
\]

\*\( a \)-periods of the differential \( dp \) are zero.
Therefore $\partial_{\tau}p$ has at least two zeros on each real oval. The condition $\partial_{\tau}a_{0}^{-} = 0$ makes $\partial_{\tau}p$ vanish at the infinities $P_{\pm}$. Therefore $\partial_{\tau}p$ has $2g + 2$ poles at the branching points, $2g + 4$ zeros and vanishes identically.

In order to show that zeros of $dp$ never match branch points $Q_{k}^{\pm}$ we use the variational identity of Lemma 4. We have $\partial_{\tau}U = \partial_{\tau}a_{0}^{\pm} = \partial_{\tau}U_{j} = 0$, $j = 1, \cdots, g$ and $U^{(1)} = \frac{1}{2} J$. Therefore

$$i\partial_{\tau}p(Q) < e^{\dagger}(x, Q)Je(x, Q) >_{x} = \frac{1}{2} \partial_{\tau}\lambda(Q) < e^{\dagger}(x, Q)e(x, Q) >_{x}$$

The points $Q_{k}^{\pm}$ are fixed points of the involution $\tau_{0}$ and $e^{\dagger}(x, Q) = e^{+}(x, Q)$ there. Moreover

$$< e^{\dagger}(x, Q)e(x, Q) >_{x} > 0 \text{ and } < e^{\dagger}(x, Q)Je(x, Q) >_{x} = 0 \text{ at } Q = Q_{k}^{\pm}.$$ 

This implies that $dp(Q_{k}^{\pm}) \neq 0$. The lemma is proved. □

**Remark.** The same statement is true for the actions entering into the trace formula (10), supplemented by periodicity conditions and $a_{0}^{-}$. The proof is exactly the same with identity (12) replaced by

$$\int_{Q_{k}^{\pm}}^{Q_{k}^{\pm}} \lambda^{2} \partial_{\tau}p d\lambda = 0 \quad k = 1, \cdots, g + 1.$$ 

**11. Classical symplectic structure.** In this section we express the algebro-geometrical 2-form

$$\omega = \sum_{k=1}^{g+1} i\delta_p(\gamma_k) \wedge \delta\lambda(\gamma_k)$$

entering into the trace formula (8) in $QP$ coordinates. The following identity, [KP], holds

$$\omega = \sum_{P_{\pm}} \text{res} \ 2i < \delta e^{\dagger} \wedge J\delta U_{0}e >_{x} dp.$$ 

The proof can be obtained following [KP] with the use of variational identity of Lemma 4. It is enough to compute contribution at one of the infinities, say at $P_{+}$. At another infinity, $P_{-}$, it is the same.

From the definition

$$e(x, Q) = e^{+i\frac{2}{\lambda}(x + l_{x})} \left[ \begin{pmatrix} 1 \\ -i \end{pmatrix} + z \begin{pmatrix} a_{1} \\ c_{1} \end{pmatrix} + \cdots \right],$$

$$\delta e^{\dagger}(x, Q) = e^{-i\frac{2}{\lambda}(x + l_{x})} \left[ z(\delta a_{1}, \delta c_{1}) + \cdots \right],$$

$$\delta U_{0} = \begin{pmatrix} \delta Q & \delta P \\ \delta P & -\delta Q \end{pmatrix}.$$
Computing the residue
\[ \frac{1}{2} \omega = \langle (\delta \bar{a}_1, \delta \bar{c}_1) \wedge \left( \frac{\delta P + i\delta Q}{\delta Q + i\delta P} \right) \rangle_x. \]

Using the formula \( Q - iP = \frac{i}{2}a_1 + \frac{1}{2}c_1 \) we finally obtain
\[ \omega = 4 \langle \delta Q \wedge \delta P \rangle_x. \]

**Remark.** This result was obtained in a different way in [MCV2], see also [MC].

13. Higher symplectic structures. Now we can write in \( QP \) coordinates the form
\[ \omega' = \sum_{k=1}^{g+1} i\delta p(\gamma_k) \wedge \delta \chi^2(\gamma_k). \]

entering into the trace formula (9). Using the identity
\[ \omega' = \sum_{P^\pm} \text{res} 2i \frac{\langle \delta e^\dagger \wedge J\delta U_0 e \rangle_x}{\langle e^\dagger e \rangle_x} \lambda(Q) dp \]

similar to the previous section we compute at \( P_+ \)
\[ \frac{1}{2} \omega' = i \langle (\delta \bar{a}_1, \delta \bar{c}_1) \wedge \left( \frac{\delta Pa_1 - \delta Qc_1}{-\delta Qa_1 - \delta Pc_1} \right) + (\delta \bar{a}_2, \delta \bar{c}_2) \wedge \left( \frac{\delta P + i\delta Q}{-\delta Q + i\delta P} \right) \rangle_x. \]

For \( k = 1 \) the formula (5) produces
\[ a_2 - ic_2 = Q' - iP' + X(Q - iP), \quad \text{where} \quad X = -ia_1 + c_1. \]

This leads to
\[ \frac{1}{2} \omega' = \langle \delta Q' \wedge \delta Q + \delta P' \wedge \delta P + \bar{X} \wedge (\delta Q^2 + \delta P^2) - 2i(\bar{X} - X)\delta Q \wedge \delta P \rangle_x. \]

Now using the formulas (6-7) and (11) we finally obtain
\[ \omega' = 2 \langle \delta Q' \wedge \delta Q + \delta P' \wedge \delta P + (\delta Q^2 + \delta P^2) \wedge \partial^{-1}(\delta Q^2 + \delta P^2) \rangle_x, \]

restricted to submanifold
\[ \int_{l_x}^{-l_x} Q^2 + P^2 = \text{const.} \]

The constant of integration in \( \partial^{-1} = \int_{-l_x}^{l_x} dx' \) is irrelevant since 1-form \( <Q\delta Q + P\delta P> \) vanishes on the vector fields tangent to the sphere (13).

**Remark.** This form of the second symplectic structure was conjectured in [MCV1].
14. **Symplectic volume elements.** We start this section with a simple example of the linear Schrödinger equation with 2$l_x$-periodic potential

\[ Q^* = -P'', \]
\[ P^* = Q''. \]

It has three classical integrals of motion, exactly like in the cubic case

\[ \mathcal{N} = \frac{1}{2} \int_{-l_x}^{l_x} Q^2 + P^2 = \sum_n I_n, \quad \text{with} \quad I_n = l_x |\hat{\psi}(n)|^2, \]
\[ \mathcal{P} = \int_{-l_x}^{l_x}QP' = \sum_n I'_n, \quad \text{with} \quad I'_n = l_x \left( \frac{\pi n}{l_x} \right) |\hat{\psi}(n)|^2, \]
\[ \mathcal{H} = \frac{1}{2} \int_{-l_x}^{l_x} Q'^2 + P'^2 = \sum_n I''_n, \quad \text{with} \quad I''_n = l_x \left( \frac{\pi n}{l_x} \right)^2 |\hat{\psi}(n)|^2. \]

The variables $I_n, I'_n$ and $I''_n$ are actions but relative to the different symplectic structures

\[ \omega = \int_{-l_x}^{l_x} dx \, dQ(x) \wedge dP(x), \]
\[ \omega' = \int_{-l_x}^{l_x} dx \, dQ'(x) \wedge dQ(x) + dP'(x) \wedge P(x), \]
\[ \omega'' = \int_{-l_x}^{l_x} dx \, dQ'(x) \wedge P'(x). \]

All the actions are conjugate to the same angles $\varphi_n = \text{phase } \hat{\psi}(n)$ in the corresponding bracket.

Now consider $\omega_N$, *etc.* the restriction of the forms on the subspace spanned by the first $2N$ harmonics $\psi_n(x) = e^{i \frac{2 \pi n}{l_x} x}$, $n = \pm 1, \cdots, \pm N$. Using

\[ \omega = dI_{-N} \wedge d\varphi_{-N} + \cdots + dI_N \wedge d\varphi_N, \quad \text{etc.} \]
we obtain in the limit of infinite dimension

\[
\frac{1}{2N} \log \frac{2N}{\omega_N} = \frac{1}{2N} \log \prod_{-N \leq n \leq N, \ n \neq 0} \frac{\pi n}{l_x} \\
= \log N + \left[ \log \frac{\pi}{l_x} - 1 \right] + \log \frac{2\pi N}{2N} + o \left( \frac{1}{N} \right),
\]

\[
\frac{1}{2N} \log \frac{2N}{\omega'_N} = \frac{1}{2N} \log \prod_{-N \leq n \leq N, \ n \neq 0} \left( \frac{\pi n}{l_x} \right)^2 \\
= 2 \log N + 2 \left[ \log \frac{\pi}{l_x} - 1 \right] + \log \frac{2\pi N}{N} + o \left( \frac{1}{N} \right).
\]

The picture in the case of cubic NLS is more complicated, but the shape of the formulas is similar.

Consider a submanifold in the function space with \(2N + 1\) gaps open and all other closed. Denote as before by \(\omega_N, \omega'_N\) the restriction of the symplectic forms \(\omega, \omega'\) on this submanifold.

**Lemma 7.** In the limit of infinite genus the asymptotic identity holds

\[
\frac{1}{2N} \log \frac{2N}{\omega_N} = \log N + \left[ \log \frac{\pi}{l_x} - 1 \right] + \log \frac{2\pi N}{2N} + \\
+ \frac{1}{2N} \left[ \log \int_{[0,1)^\infty} m_{12}(0) \ d\tilde{\theta} - \log l_x \right] + o \left( \frac{1}{N} \right).
\]

**Proof.** The strategy of the proof is similar to [MCV2]. The points of the divisor \(\gamma\)'s can be considered as a coordinates on the invariant torus. The canonical pairing implies

\[
\det \left[ \frac{\partial I_i'}{\partial I_j} \right] = \det \left[ \frac{\partial \theta_i}{\partial \theta_j'} \right] = \frac{\det \left[ \omega_i / d\lambda(\gamma_k) \right]}{\det \left[ \omega'_i / d\lambda(\gamma_k) \right]},
\]

where the angles \(\theta, \theta'\) are given by the Abel sum

\[
\theta_i = \sum_{-N \leq k \leq N, \ n \neq 0} \int_{Q_k^-} \omega_i, \quad \theta'_i = \sum_{-N \leq k \leq N, \ n \neq 0} \int_{Q_k^-} \omega'_i.
\]
where
\[ \omega_i = \sum_{j=1}^{2N} c_{ij} \frac{\lambda^{j-1}}{R} d\lambda, \quad \omega'_i = \sum_{j=1}^{2N} c'_{ij} \frac{\lambda^{j-1}}{R} d\lambda \]
are normalized differentials of the third kind. Using normalization \( a_k[\omega_j] = a_k[\lambda^{j'}] = \delta_{kj} \) we obtain
\[
\det \left[ \frac{\partial I'_i}{\partial I'_j} \right] = \frac{1}{l_x} \left( \prod_{1 \leq k \leq N} \frac{\pi n}{l_x} \right)^2 \int_{[0,1]^{2N+1}} \frac{1}{l_x} \left( \prod_{1 \leq k \leq N} \frac{\pi n}{l_x} \right)^2 d^{2N+1} \tilde{\theta}_k,
\]
where \( \tilde{\theta}_k = \theta_k/2\pi \) are normalized angles. Now, using the formula (2) we recognize under the integral \( m_{12}(0) \) averaged over the flat measure on the invariant torus. Taking the logarithm of both parts we obtain the statement. □

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