MULTIPOLICY OF SOLUTIONS FOR A CLASS OF QUASILINEAR PROBLEMS INVOLVING THE 1-LAPLACIAN OPERATOR WITH CRITICAL GROWTH

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Abstract. The aim of this paper is to establish two results about multiplicity of solutions to problems involving the 1−Laplacian operator, with nonlinearities with critical growth. To be more specific, we study the following problem
\[
\begin{cases}
-\Delta_1 u + \xi \frac{|u|}{|u|} = \lambda |u|^{q-2}u + |u|^{1^*-2}u, & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega,
\end{cases}
\]
where \(\Omega\) is a smooth bounded domain in \(\mathbb{R}^N\), \(N \geq 2\) and \(\xi \in \{0, 1\}\). Moreover, \(\lambda > 0\), \(q \in (1, 1^*)\) and \(1^* = \frac{N}{N-1}\). The first main result establishes the existence of many rotationally non-equivalent and nonradial solutions by assuming that \(\xi = 1\), \(\Omega = \{x \in \mathbb{R}^N : r < |x| < r + 1\}\), \(N \geq 2\), \(N \neq 3\) and \(r > 0\). In the second one, \(\Omega\) is a smooth bounded domain, \(\xi = 0\), and the multiplicity of solutions is proved through an abstract result which involves genus theory for functionals which are sum of a \(C^1\) functional with a convex lower semicontinuous functional.

1. Introduction

In this work we are concerned with the existence of multiple solutions for the following class of problem
\[
\begin{cases}
-\Delta_1 u + \xi \frac{|u|}{|u|} = \lambda |u|^{q-2}u + |u|^{1^*-2}u, & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega,
\end{cases}
\]
where \(\Omega\) is a smooth bounded domain in \(\mathbb{R}^N\), \(q \in (1, 1^*)\), \(\xi \in \{0, 1\}\), \(\lambda > 0\) and \(1^* = \frac{N}{N-1}\) for \(N \geq 2\).

Problem (1.1) looks as the formal limit, as \(p \to 1^+\), of
\[
\begin{cases}
-\Delta_p u + \xi |u|^{p-2}u = \lambda |u|^{q-2}u + |u|^{p^*-2}u, & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega,
\end{cases}
\]
where \(p^* = \frac{Np}{N-p}\) for \(N \geq p\).

The interest in this sort of problem has started with the celebrated paper of Brézis and Nirenberg [16], in which the authors proved that, for \(p = q = 2\) and \(\xi = 0\), (1.2) admits a positive solution for every \(\lambda \in (0, \lambda_1)\) and \(N \geq 4\). Later, this result has been extended for \(p > 1\) by Egnell [26], García Azorero and Peral Alonso [31] and Gueda and Veron [32].

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As far as works involving the $1-$Laplacian operator are regarded, some of the pioneering works involving this operator were written by Andreu, Ballester, Caselles and Mazón in a series of papers (among them [8–10]), which gave rise to the monograph [11]. Indeed, in [8], the authors characterize the imprecise quotient $\frac{Du}{|Du|}$ (when $Du$ is just a Radon measure, rather than an $L^1$ function), through the Pairing Theory of Anzellotti (see [12] and also [11]). This theory allows them to introduce a vector field $z \in L^\infty(\Omega, \mathbb{R}^N)$ which plays the role of $\frac{Du}{|Du|}$. Among the very first works on this issue we could also cite the works of Kawohl [34] and also Demengel [22], where in the later, the author used the symmetry of the domain to get nodal solutions to problems involving the $1-$Laplacian operator and a nonlinearity with critical growth.

In [25], Degiovanni and Magrone studied the version of the Brézis-Nirenberg problem to the $1$-Laplacian operator, by applying a linking theorem. In that work, for compactness issues, they worked with an extension of the energy functional to the Lebesgue space $L^{1^*}(\Omega)$.

In [27], Figueiredo and Pimenta studied a problem related to (1.1), where the nonlinearity has a subcritical growth. In their main result, an approach based on the Nehari method has been developed in order to obtain ground-state solutions.

Regarding quasilinear problems of this type, the natural space to deal with it is the space of functions of bounded variation, $BV(\Omega)$. More specifically, when dealing with this sort of problems through variational methods, some difficulties related to the Palais-Smale condition arise. Moreover, other ones related to the lack of smoothness of the energy functional and to the lack of reflexiveness of $BV(\Omega)$, arise as well.

In this work, we exploit some facts and ideas from the above papers, especially from [22] to show the existence of multiple nontrivial solutions to (1.1). Our goal is twofold. First, we establish the existence of many rotationally non-equivalent and nonradial solutions for the above problem (1.1) with $\xi = 1$, involving a nonlinearity with critical growth in the case when

\begin{equation}
\Omega = \Omega_r = \{ x \in \mathbb{R}^N : r < |x| < r + 1 \}.
\end{equation}

Afterwards, we study (1.1) by assuming that $\Omega$ is a smooth bounded domain in $\mathbb{R}^N$ and $\xi = 0$. In this case, we prove the existence of multiple many solutions, by applying a version of an abstract result in [44].

The existence of many rotationally non-equivalent and nonradial solutions was considered in some problems involving the Laplacian operator. In Brézis and Nirenberg [16], it was proved the existence of a non-radial positive solution for the following problem

\begin{equation}
\begin{cases}
-\Delta u + u - u^p = 0, & \text{in } D, \\
u = 0, & \text{on } \partial D,
\end{cases}
\end{equation}

where

\[ D = \{ x \in \mathbb{R}^N : r < |x| < r + d \} \]

for some $d > 0$. In [21], Coffman proved that, if $p > 1$ and $N = 2$ or $1 < p < N/(N - 2)$ and $N \geq 3$, the number of nonradial and rotationally non-equivalent positive solutions of (1.4), tends to $+\infty$ as $r \to +\infty$. 
Motivated by the above papers, some authors have studied this class of problems. For the subcritical case, we can mention the papers of Li [35], Lin [36], Suzuki [45] and references therein. Related to the critical case, Wang and Willem [46] proved the existence of multiple solutions for the following problem

\begin{align}
-\Delta u = \lambda u + u^{2^*-1}, \quad &\text{in } \Omega_r, \\
u = 0, \quad &\text{on } \partial \Omega_r,
\end{align}

where \( \Omega_r \) is given in (1.3). The authors proved that for \( 0 < \lambda < \pi^2 \) and \( n \in \mathbb{N} \), there exists \( R(\lambda, n) \) such that for \( r > R(\lambda, n) \), (1.5) has at least \( n \) nonradial and rotationally non-equivalent solutions. Inspired by [46], de Figueiredo and Miyagaki [23] considered the following problem

\begin{align}
-\Delta u = f(|x|, u) + u^{2^*-1}, \quad &\text{in } \Omega_r, \\
u = 0, \quad &\text{on } \partial \Omega_r,
\end{align}

where \( f \) is a \( C^1 \) function with subcritical growth.

Still related to this class of problem, we would like to cite the papers of Alves and de Freitas [3], Byeon [15], Castro and Finan [17], Catrina and Wang [18], Mizoguchi and Suzuki [39], Hirano and Mizoguchi [33] and references therein.

Motivated by the works previously mentioned and more precisely, by [23], [25] and [46], our first main result is the following.

**Theorem 1.1.** For each \( n \in \mathbb{N} \) there is \( r_0 > 0 \) and \( \lambda_0 > 0 \) such that for all \( \lambda \geq \lambda_0 \) and \( r \geq r_0 \), (1.1) for \( \xi = 1 \) has at least \( n \) nonradial and rotationally non-equivalent solutions.

In what follows, according to Degiovanni and Magrone [25] and Kawohl and Schuricht [13], we say that \( u \in BV(\Omega_r) \) is a solution to (2.1) if there are \( z \in L^\infty(\Omega_r, \mathbb{R}^N) \) and \( \gamma \in L^\infty(\Omega_r, \mathbb{R}) \) such that

\begin{align}
|z|_\infty \leq 1, \quad &\text{div} \in L^N(\Omega_r), \quad -\int_{\Omega_r} u \text{div} z dx = \int_{\Omega_r} |Du| + \int_{\partial \Omega_r} |u| d\mathcal{H}^{N-1}, \\
|\gamma|_\infty \leq 1, \quad &\gamma |u| = u \quad \text{a.e. in } \Omega_r, \\
-\text{div} z + \gamma &= \lambda |u|^{q-2} u + |u|^{r-2} u, \quad \text{a.e. in } \Omega_r.
\end{align}

Our second main result was motivated by the study made in Wei and Wu [47], where the authors showed the existence of multiple solution for the following class of problems involving the \( p \)-Laplacian operator

\begin{align}
-\Delta_p u = f(x, u) + \lambda |u|^{p^*-2} u, \quad &\text{in } \Omega, \\
u = 0, \quad &\text{on } \partial \Omega,
\end{align}

where \( \Omega \) is a smooth bounded domain, \( \lambda \) is a positive parameter and \( f \) is continuous, with subcritical growth. In this work, the authors used a version of an abstract theorem due to Ambrosetti and Rabinowitz [7] which involves the genus theory for \( C^1 \) even functionals. Their main result proves that given \( n \in \mathbb{N} \), there is \( \lambda_0 = \lambda_0(n) > 0 \) such that problem (1.8) has at least \( n \) nontrivial solutions for \( \lambda \in (0, \lambda_0) \). In [42], Silva and Xavier improved the main results proved in [47].

Our main second result has the following statement.
Theorem 1.2. Given $n \in \mathbb{N}$, there is $\lambda_n > 0$ such that (1.1) for $\xi = 0$ has at least $n$ nontrivial solutions for $\lambda \geq \lambda_n$.

We would like to point out that in the proof of Theorems 1.1 and 1.2 we cannot use the classical variational methods for $C^1$ functionals, since problems involving the 1-Laplacian operator have energy functionals which are not $C^1$. This, in turn, brings a lot of difficulties for dealing with the problem. In order to overcome this difficulty, we use the minimax methods developed by Szulkin in [44], which works well for functionals that can be written as the sum of a $C^1$ functional with a convex lower semicontinuous one. Finally, we would like to point out that $\xi = 1$ in Theorem 1.1 is very important, because in our approach it was necessary to work with some sequences in $BV(\mathbb{R}^N)$ (see the proof of Lemma 2.5).

Before concluding this introduction, for those readers interested in problems involving the 1-Laplacian operator, we would like to cite Alves [1, 2], Alves and Pimenta [4], Alves, Figueiredo and Pimenta [5], Bellettini, Caselles and Novaga [14], Chang [20], Demengel [24], Figueiredo and Pimenta [29, 30], Mercaldo, Rossi, Segura de León and Trombetti [37], Mercaldo, Segura de León and Trombetti [38], Molino Salas and Segura de León [40], Ortiz Chata and Pimenta [41].

2. Existence of nonradial solutions

We say that $u \in BV(\Omega_r)$ is a solution of (2.1) if $0 \in \partial I_\lambda(u)$, where $\partial I_\lambda(u)$ denotes the generalized gradient of $I_\lambda$ in $u$, as defined in [19]. It is possible to prove that $0 \in \partial I_\lambda(u)$ if, and only if,

$$\|w\|_r - \|u\|_r \geq \int_{\Omega_r} (\lambda |u|^{q-2}u + |u|^{1^*-2}u)(w - u) dx, \quad \forall w \in BV(\Omega_r),$$

where $BV(\Omega_r)$ denotes the space of functions of bounded variation.

We say that $u \in L^1(\Omega_r)$ and its distributional derivative $Du$ is a vectorial Radon measure, i.e.,

$$BV(\Omega_r) = \{u \in L^1(\Omega_r); Du \in \mathcal{M}(\Omega_r, \mathbb{R}^N)\}.$$  

It can be proved that $u \in BV(\Omega_r)$ is equivalent to $u \in L^1(\Omega_r)$ and

$$\int_{\Omega_r} |Du| := \sup \left\{ \int_{\Omega_r} \text{div}\phi dx; \phi \in C^1_c(\Omega_r, \mathbb{R}^N), \text{s.t.} \|\phi\| \leq 1 \right\} < +\infty.$$
The space $BV(\Omega_r)$ is a Banach space endowed with the norm
\begin{equation}
\|u\| := \int_{\Omega_r} |Du| + |u|_{L^1(\Omega_r)}.
\end{equation}
Moreover, the Sobolev embeddings hold also for this space and its embedding into $L^r(\Omega)$ is continuous for all $r \in [1, 1^*)$ and compact for $r \in [1, 1^*)$.

In this section, we will consider the following norm on $BV(\Omega_r)$,
\begin{equation}
\|u\|_r = \int_{\Omega_r} |Du| \, dx + \int_{\Omega_r} |u| \, dx + \int_{\partial \Omega_r} |u| \, dH^{N-1},
\end{equation}
which is equivalent to the norm (2.3), where $H^{N-1}$ denotes the $(N - 1)$-dimensional Hausdorff measure.

As one can see in [13], the space $BV(\Omega_r)$ has different convergence and density properties when compared with the usual Sobolev spaces. For instance, $C^\infty(\overline{\Omega_r})$ is not dense in $BV(\Omega_r)$ with respect to the strong convergence. However, there is a weaker sense of convergence in $BV(\Omega_r)$, called intermediate convergence (or strict convergence), which makes $C^\infty(\overline{\Omega_r})$ dense on it. We say that $(u_n) \subset BV(\Omega_r)$ converges to $u \in BV(\Omega_r)$ in the sense of the intermediate convergence if
\begin{equation}
u_n \to u \quad \text{in} \quad L^1(\Omega_r)
\end{equation}
and
\begin{equation}
\int_{\Omega_r} |Du_n| \to \int_{\Omega_r} |Du|.
\end{equation}

In what follows, $O(N)$ denotes the group of $N \times N$ orthogonal matrices. For any integer $k \geq 1$, let us consider the finite rotational subgroup $O_k$ of $O(2)$ given by
\begin{equation}
O_k = \left\{ g \in O_2 : g(x) = \left( x_1 \cos \frac{2\pi l}{k} + x_2 \sin \frac{2\pi l}{k}, -x_1 \sin \frac{2\pi l}{k} + x_2 \cos \frac{2\pi l}{k} \right) \right\},
\end{equation}
where $x = (x_1, x_2) \in \mathbb{R}^2$ and $l \in \{0, 1, \ldots, k - 1\}$. We also consider the subgroups of $O(N)$
\begin{equation}
G_k = O_k \times O(N - 2), \quad 1 \leq k < \infty
\end{equation}
and
\begin{equation}
G_\infty = O(N).
\end{equation}

Now related to the above subgroups, we set the subspaces
\begin{equation}
BV_{G_k}(\Omega_r) = \{ u \in BV(\Omega_r) : u(x) = u(g^{-1}(x)), \quad \text{for all} \quad g \in G_k \},
\end{equation}
endowed with the norm $\| \cdot \|_r$.

From the compact embedding involving the space $BV(\Omega_r)$, it follows that the embedding
\begin{equation}
BV_{G_k}(\Omega_r) \hookrightarrow L^t(\Omega_r), \quad t \in [1, 1^*)
\end{equation}
is compact for $1 \leq k < +\infty$ and
\begin{equation}
BV_{G_\infty}(\Omega_r) \hookrightarrow L^t(\Omega_r), \quad t \in [1, +\infty)
\end{equation}
is compact, see [6, Lemma 2.1].
Moreover, Figueiredo and Pimenta [28] proves that the embedding
\[ BV_{G_{\infty}}(\mathbb{R}^N) \hookrightarrow L^t(\mathbb{R}^N), \quad t \in (1, 1^*) \]
is compact as well.

In the sequel, for each \(1 \leq k \leq \infty\), \(J_{\lambda,k,r}\) denotes the following real numbers
\[ J_{\lambda,k,r} = \inf_{\mathcal{N}_{k,r}} I_{\lambda}, \]
with
\[ \mathcal{N}_{k,r} = \{ BV_G(\Omega_r) \setminus \{0\}; \ E_\lambda(u) = 0 \}, \]
where
\[ E_\lambda(u) = \int_{\Omega_r} |Du| \, dx + \int_{\Omega_r} |u| \, dx + \int_{\partial \Omega_r} |u| \, d\mathcal{H}^{N-1} - \lambda \int_{\Omega_r} |u|^q \, dx - \int_{\Omega_r} |u|^{1^*} \, dx. \]

The set \(\mathcal{N}_{k,r}\) is called Nehari set associated with \(I_\lambda\) (see [27] for a detailed description of this set). It is possible to prove that \(J_{\lambda,k,r}\) is the mountain pass levels associated with \(I_\lambda\) on \(BV_{G_k}(\Omega_r)\) and \(BV_{G_{\infty}}(\Omega_r)\), respectively. Hence, there is a \((PS)\) sequence \((u_n)\) associated to \(J_{\lambda,k,r}\), i.e.,
\[ I_\lambda(u_n) \to J_{\lambda,k,r} \]
and
\[ \|v\|_r - \|u_n\|_r \geq \int_{\Omega_r} (\lambda|u_n|^{q-2}u_n + |u_n|^{1^*-2}u_n)(v - u_n) \, dx - \tau_n \|v - u_n\|_r, \quad \forall v \in BV_{G_k}(\Omega_r). \]
The last inequality implies that
\[ \|u_n\|_r = \int_{\Omega_r} (\lambda|u_n|^q + |u_n|^{1^*}) \, dx + o_n(1)\|u_n\|_r. \]

**Lemma 2.1.** The sequence \((u_n)\) is bounded.

**Proof.** From (2.8) and (2.9),
\[ J_{\lambda,k,r} + o_n(1) = I_\lambda(u_n) - \frac{1}{q}\|u_n\|_r + \frac{1}{q} \int_{\Omega_r} (\lambda|u_n|^q + |u_n|^{1^*}) \, dx + o_n(1)\|u_n\|_r. \]
Then
\[ J_{\lambda,k,r} + o_n(1) \geq \left( \frac{q - 1}{q} \right)\|u_n\|_r - o_n(1)\|u_n\|_r \geq \frac{(q - 1)}{2q}\|u_n\|_r \]
for \(n\) large enough, showing the boundedness of the sequence. \(\square\)

**Lemma 2.2.** For each \(\lambda > 0\) fixed, there is \(\eta = \eta(\lambda) > 0\) that is independent of \(k\) and \(r > 0\) such that \(J_{\lambda,k,r} \geq \eta\) for \(1 \leq k \leq \infty\).

**Proof.** For each \(u \in \mathcal{N}_{k,r}\) we have that
\[ \|u\|_r = \lambda \int_{\Omega_r} |u|^q \, dx + \int_{\Omega_r} |u|^{1^*} \, dx. \]
In what follows, we define the function \(\tilde{u} : \mathbb{R}^N \to \mathbb{R}\) given by
\[ \tilde{u}(x) = \begin{cases} u(x), & x \in \Omega_r, \\ 0, & x \in \Omega_r^c. \end{cases} \]
Hence,
\[
\int_{\Omega_r} |u|^q \, dx = \int_{\mathbb{R}^N} |\tilde{u}|^q \, dx, \quad \int_{\Omega_r} |u|^{1^*} \, dx = \int_{\mathbb{R}^N} |\tilde{u}|^{1^*} \, dx
\]
and by properties of \(BV(\mathbb{R}^N)\), \(\tilde{u} \in BV(\mathbb{R}^N)\) and
\[
\int_{\mathbb{R}^N} |D\tilde{u}| = \int_{\Omega_r} |Du| \, dx + \int_{\partial\Omega_r} |u| \, d\mathcal{H}^{N-1}.
\]
The definition of \(\tilde{u}\) combined with (2.10) gives
\[
\int_{\mathbb{R}^N} |D\tilde{u}| + \int_{\mathbb{R}^N} |\tilde{u}| \, dx = \lambda \int_{\mathbb{R}^N} |\tilde{u}|^q \, dx + \int_{\mathbb{R}^N} |\tilde{u}|^{1^*} \, dx.
\]
The function \(\| \cdot \| : BV(\mathbb{R}^N) \to \mathbb{R}\) given by
\[
\|w\| = \int_{\mathbb{R}^N} |Dw| + |w|_{L^1(\mathbb{R}^N)}.
\]
is a norm in \(BV(\mathbb{R}^N)\). Moreover, there are positive constants \(C_1, C_2 > 0\) such that
\[
\|w\|_{L^q(\mathbb{R}^N)} \leq C_1 \|w\| \quad \text{and} \quad \|w\|_{L^{1^*}(\mathbb{R}^N)} \leq C_2 \|w\|, \quad \forall w \in BV(\mathbb{R}^N).
\]
From (2.12)-(2.14), there is \(C_3 > 0\) such that
\[
1 \leq C_3(\lambda \|\tilde{u}\|^{q-1} + \|\tilde{u}\|^{1^*-1}).
\]
Therefore, there is \(\eta_1 = \eta_1(\lambda) > 0\) such that
\[
\|\tilde{u}\| \geq \eta_1,
\]
and so,
\[
\|u\| \geq \eta_1, \quad \forall u \in \mathcal{N}_{k,r}.
\]
From (2.10),
\[
\lambda \int_{\mathbb{R}^N} |u|^q \, dx + \int_{\mathbb{R}^N} |u|^{1^*} \, dx \geq \eta_1, \quad \forall u \in \mathcal{N}_{k,r},
\]
then,
\[
I_\lambda(u) = \frac{\lambda(q-1)}{q} \int_{\Omega_r} |u|^q \, dx + \frac{(1^*-q)}{q1^*} \int_{\Omega_r} |u|^{1^*} \, dx \geq \frac{(q-1)}{q} \eta_1,
\]
showing the result. \(\square\)

Hereafter, \(S\) denotes the following constant
\[
S = \inf_{\substack{u \in BV(\mathbb{R}^N) \setminus \{0\}}} \frac{\int_{\mathbb{R}^N} |Du|}{\|u\|_{L^{1^*}(\mathbb{R}^N)}}.
\]

**Lemma 2.3.** For each \(1 \leq k < \infty\), there is \(\lambda^*_k > 0\) such that
\[
J_{\lambda,k} < \frac{1}{2N}S^N, \quad \text{for all} \quad \lambda \geq \lambda^*_k.
\]
Proof. From the fact that \( \Omega_r \) is an open bounded domain, we may choose \( \sigma > 0 \) that is independent of \( r \), such that the ball \( B_\sigma = B_\sigma \left( \frac{2r+1}{2}, 0, ..., 0 \right) \subset \Omega_r \) verifies

\[
g^i B_\sigma \cap g^j B_\sigma = \emptyset, \text{ for } g^i \in G_k, \quad i \neq j, \quad i, j = 0, ..., k - 1.
\]

Let us choose \( \omega \in C^\infty_0(B_\sigma) \setminus \{0\} \) and define \( v := \Sigma_{g \in G_k} \omega \in BV_{G_k}(\Omega_r) \setminus \{0\} \). A simple computation implies that

\[
I_\lambda'(tv)tv > 0 \quad \text{for} \quad t \approx 0^+ \quad \text{and} \quad I_\lambda'(tv)tv \to -\infty, \quad \text{as} \quad t \to \infty.
\]

Hence, there exists \( t_v > 0 \) such that \( t_v \omega \in N_{k, r} \). From this, \( J_{\lambda, k, r} \leq I_\lambda(t_v \omega) \leq k \max_{t \geq 0} I_\lambda(t \omega) \). and so,

\[
J_{\lambda, k, r} \leq k \max_{t \geq 0} \left\{ t \|\omega\|_r - \frac{\lambda t^q}{q} \int_{B_\sigma} |\omega|^q \, dx \right\}.
\]

Putting \( g(t) = t \|\omega\|_r - \frac{\lambda t^q}{q} |\omega|^q \), this function attains its maximum at

\[
t_0 = \left( \frac{\|\omega\|_r}{\lambda |\omega|^q} \right)^{\frac{1}{q-1}}.
\]

Therefore,

\[
J_{\lambda, k, r} \leq k \frac{(q - 1)}{q} \left( \frac{\|\omega\|_r}{|\omega|^q} \right)^{\frac{q}{q-1}} \lambda^{\frac{1}{q-1}}.
\]

Taking \( \lambda_k^* > \left( \frac{2Nk(q-1)}{S_q} \right)^{q-1} \left( \frac{\|\omega\|_r}{|\omega|^q} \right)^q \) the proof is achieved. \( \square \)

Lemma 2.4. For each \( 1 \leq k \leq \infty \) the number \( J_{\lambda, k, r} \) is attained for \( \lambda \geq \lambda_k^* \).

Proof. The case \( k = \infty \) is immediate because of the compact embedding (2.5), then we will only show the case \( 1 \leq k < +\infty \). Recalling that \( J_{\lambda, k, r} \) is the mountain pass level of \( I_\lambda \) on the space \( BV_{G_k}(\Omega_r) \), we know that there is a \((PS)\) sequence \( (u_n) \) (see (2.8)), such that

\[
I_\lambda(u_n) \to J_{\lambda, k, r}
\]

and

\[
\|v\|_r - \|u_n\|_r \geq \int_{\Omega_r} (\lambda |u_n|^{q-2}u_n + |u_n|^{1^*-2}u)(v - u_n) \, dx - \tau_n \|v - u_n\|_r, \quad \forall v \in BV_{G_k}(\Omega_r),
\]

where \( \tau_n \to 0 \). Moreover, we also have the equality below

\[
(2.16) \quad \|u_n\|_r = \int_{\Omega_r} \left( \lambda |u_n|^q + |u_n|^{1^*} \right) \, dx + o_n(1) \|u_n\|_r.
\]

Since that \( (u_n) \) is bounded in \( BV_{G_k}(\Omega_r) \), for some subsequence, there is \( u \in BV_{G_k}(\Omega_r) \) such that

\[
\|u\|_r \leq \liminf_{n \to +\infty} \|u_n\|_r
\]

and

\[
u_n \to u \quad \text{in} \quad L^t(\Omega_r), \quad \forall t \in [1, 1^*).
\]
We claim that \( u \neq 0 \), otherwise if we argue as in the proof of Lemma 2.2, we would find a constant \( C > 0 \) such that
\[
1 \leq C(\lambda \|u_n\|_r^{q-1} + \|u_n\|_{r+1}^{-1}), \quad \forall n \in \mathbb{N}.
\] (2.17)

On the other hand, by (2.16),
\[
\int_{\Omega} |D u_n| + \int_{\Omega} |u_n| \, dx + \int_{\partial \Omega} |u_n| \, d\mathcal{H}^{N-1} = \int_{\Omega} |u_n|^{r^*} \, dx + o_n(1),
\]
then for some subsequence,
\[
\lim_{n \to +\infty} \left( \int_{\Omega} |D u_n| + \int_{\partial \Omega} |u_n| \, d\mathcal{H}^{N-1} \right) = \int_{\Omega} |u|^{r^*} \, dx = L \geq 0.
\]

We claim that \( L > 0 \), because otherwise we would have
\[
\lim_{n \to +\infty} \int_{\Omega} |u_n|^{r^*} \, dx = 0,
\]
and so,
\[
\lim_{n \to +\infty} \|u_n\|_r = 0,
\]
which contradicts (2.17). Since \( L > 0 \), we can assume that \( u_n \neq 0 \) for all \( n \in \mathbb{N} \).

Therefore, by definition of \( S \), see (2.15),
\[
S \leq \frac{\int_{\mathbb{R}^N} |\tilde{u}_n|^{1^*} \, dx}{\|\tilde{u}_n\|_{L^{1^*}(\mathbb{R}^N)}} = \frac{\int_{\Omega_r} |D u_n| + \int_{\partial \Omega_r} |u_n| \, d\mathcal{H}^{N-1}}{\|u_n\|_{L^{1^*}(\mathbb{R}^N)}}, \quad \forall n \in \mathbb{N},
\]
where \( \tilde{u}_n \) is defined as in (2.11). Letting \( n \to +\infty \), we get
\[
S \leq \frac{L}{L^{1^*}} = L^{\frac{N}{N+1}},
\]
that is,
\[
L \geq S^N.
\]

Now, using the fact that
\[
J_{\lambda,k,r} + o_n(1) = I_\lambda(u_n) = \frac{1}{N} \int_{\mathbb{R}^N} |\tilde{u}_n|^{1^*} \, dx + o_n(1) = \frac{1}{N}L \geq \frac{1}{N}S^N,
\]
which contradicts Lemma 2.3. Now, arguing as in [25], we also derive that
\[
\int_{\Omega_r} |Du| \, dx + \int_{\Omega_r} |u| \, dx + \int_{\partial \Omega_r} |u| \, d\mathcal{H}^{N-1} = \int_{\Omega_r} |u|^{q} \, dx + \int_{\Omega_r} |u|^{r^*} \, dx,
\]
from where it follows that $u \in \mathcal{N}_{k,r}$, and so,

$$J_{\lambda,k,r} \leq I_{\lambda}(u) = \frac{\lambda(q - 1)}{q} \int_{\Omega_r} |u|^q \, dx + \frac{(1^*-1)}{1^*} \int_{\Omega_r} |u|^{1^*} \, dx$$

$$\leq \frac{\lambda(q - 1)}{q} \lim_{n \to +\infty} \int_{\Omega_r} |u_n|^q \, dx + \liminf_{n \to +\infty} \frac{(1^*-1)}{1^*} \int_{\Omega_r} |u_n|^{1^*} \, dx$$

$$\leq \limsup_{n \to +\infty} (I(u_n) + o_n(1)) = \limsup_{n \to +\infty} I(u_n) = J_{\lambda,k,r},$$

from where it follows that $u \in \mathcal{N}_{k,r}$ and $I_{\lambda}(u) = J_{\lambda,k,r}$, showing the lemma. \hfill \Box

**Lemma 2.5.** There exists $r_0 = r_0(\lambda) > 0$ such that

$$J_{\lambda,\infty,r} \geq \frac{1}{2N} S^N, \text{ for } r > r_0.$$

**Proof.** Let us assume the opposite, i.e., that there exists a sequence $r_n \to \infty$ such that

$$J_{\lambda,\infty,r_n} < \frac{1}{2N} S^N, \text{ for } n \in \mathbb{N}.$$

By Lemma 2.4, $J_{\infty,r_n}$ is attained for all $n \in \mathbb{N}$, and so, there is $(u_n) \in BV_{G_\infty}(\Omega_{r_n}) \setminus \{0\}$ such that

$$E_{\lambda}(u_n) = 0 \quad \text{and} \quad I_{\lambda}(u_n) = J_{\lambda,\infty,r_n},$$

where $E_{\lambda}$ is given in (2.7).

The assumption $J_{\lambda,\infty,r_n} < \frac{1}{2N} S^N$ combined with the first equality above ensures that there is $M > 0$ such that

$$(2.18) \int_{\Omega_{r_n}} |Du_n| \, dx + \int_{\Omega_{r_n}} |u_n| \, dx + \int_{\partial \Omega_{r_n}} |u_n| \, d\mathcal{H}^{N-1} \leq M, \quad \forall n \in \mathbb{N}.$$ Setting

$$\tilde{u}_n(x) = \begin{cases} u_n(x), & x \in \Omega_{r_n}, \\ 0, & x \in \Omega^{c}_{r_n}, \end{cases}$$

we have that $\tilde{u}_n \in BV_{G_\infty}(\mathbb{R}^N)$,

$$\int_{\Omega_{r_n}} |u_n|^q \, dx = \int_{\mathbb{R}^N} |\tilde{u}_n|^q \, dx, \quad \int_{\Omega_{r_n}} |u_n|^{1^*} \, dx = \int_{\mathbb{R}^N} |\tilde{u}_n|^{1^*} \, dx$$

and

$$\|\tilde{u}_n\| = \int_{\Omega_{r_n}} |Du_n| \, dx + \int_{\Omega_{r_n}} |u_n| \, dx + \int_{\partial \Omega_{r_n}} |u_n| \, d\mathcal{H}^{N-1}.$$ The definition of $\tilde{u}_n$ combined with the fact that $E_{\lambda}(u_n) = 0$ gives

$$(2.19) \quad \|\tilde{u}_n\| \leq \lambda \int_{\mathbb{R}^N} |\tilde{u}_n|^q \, dx + \int_{\mathbb{R}^N} |\tilde{u}_n|^{1^*} \, dx.$$
From the definition of $\bar{u}_n$, it follows that $\bar{u}_n \to 0$ a.e in $\mathbb{R}^N$, and so,

\[(2.20) \quad \bar{u}_n \to 0 \text{ in } L^t(\mathbb{R}^N) \text{ for } t \in (1, 1^*)\]

Moreover, from (2.19), there is $t_n \in (0, 1] \cup (1, 1^*)$ such that

\[(2.21) \quad \int_{\mathbb{R}^N} |D\bar{u}_n| + \int_{\mathbb{R}^N} |\bar{u}_n| \, dx = \lambda t_n^{q-1} \int_{\mathbb{R}^N} |\bar{u}_n|^q + t_n^{1^*-1} \int_{\mathbb{R}^N} |\bar{u}_n|^{1^*} \, dx.

Arguing as in the proof of Lemma 2.2, there is $C > 0$ such that

\[(2.22) \quad 1 \leq C \left( \lambda ||t_n \bar{u}_n||^{q-1} + ||t_n \bar{u}_n||^{1^*-1} \right), \quad \forall n \in \mathbb{N}.

On the other hand, by (2.20) and (2.21),

\[
\int_{\mathbb{R}^N} |t_n D\bar{u}_n| + \int_{\mathbb{R}^N} |t_n \bar{u}_n| \, dx = \int_{\mathbb{R}^N} |t_n \bar{u}_n|^{1^*} \, dx + o_n(1),
\]

then for some subsequence,

\[
\lim_{n \to +\infty} ||t_n \bar{u}_n|| = \lim_{n \to +\infty} \int_{\mathbb{R}^N} |t_n \bar{u}_n|^{1^*} \, dx = L \geq 0.
\]

We claim that $L > 0$, since otherwise we would have

\[
\lim_{n \to +\infty} \int_{\mathbb{R}^N} |t_n \bar{u}_n|^{1^*} \, dx = 0,
\]

and so,

\[
\lim_{n \to +\infty} ||t_n \bar{u}_n|| = 0,
\]

which contradicts (2.22). Since $L > 0$, we can assume that $u_n \neq 0$ for all $n \in \mathbb{N}$. Therefore, by definition of $S$, see (2.15),

\[
S \leq \frac{\int_{\mathbb{R}^N} |D\bar{u}_n|}{||\bar{u}_n||^{1^*}} \leq \frac{||\bar{u}_n||}{||\bar{u}_n||^{1^*}}, \quad \forall n \in \mathbb{N}.
\]

Letting $n \to +\infty$, we get

\[
S \leq \frac{L}{L^{1^*}} = \frac{L}{L} = 1,
\]

that is,

\[
L \geq S^N.
\]

Now, using the fact that

\[
\frac{1}{N} S^N \leq \frac{1}{N} L = I_\lambda(t_n \bar{u}_n)(t_n \bar{u}_n) = \frac{1}{N} \int_{\mathbb{R}^N} |t_n \bar{u}_n|^{1^*} \, dx \leq \frac{1}{N} \int_{\mathbb{R}^N} |\bar{u}_n|^{1^*} \, dx,
\]

that is,

\[
\frac{1}{N} S^N \leq \frac{1}{N} \int_{\mathbb{R}^N} |\bar{u}_n|^{1^*} \, dx = \frac{1}{N} \int_{\Omega_{r_n}} |\bar{u}_n|^{1^*} \, dx = I_\lambda(u_n) = J_{\lambda, \infty, r_n},
\]

Hence,

\[
\frac{1}{N} S^N \leq \limsup_{n \to +\infty} J_{\lambda, \infty, r_n} \leq \frac{1}{2N} S^N,
\]

which is absurd. \qed

**Lemma 2.6.** For each $1 \leq k < \infty$ and $2 \leq m < \infty$, we have that $J_{\lambda, k, r} < J_{\lambda, km, r}$, for all $r \geq r_0$. 

Proof. Let \( u \in N_{\lambda,m,r} \) be such that \( I_\lambda(u) = J_{\lambda,km,r} \) and fix \( \varphi_n \subset C^\infty(\Omega) \cap BV_{\lambda,m,r}(\Omega_r) \) such that

\[
\varphi_n \to u \quad \text{in} \quad L^1(\Omega_r) \quad \text{as} \quad n \to +\infty
\]

and

\[
\int_{\Omega_r} |\nabla \varphi_n| \, dx \to \int_{\Omega_r} |Du| \quad \text{as} \quad n \to +\infty.
\]

Associated with \( \varphi_n \), we set \( v_n(\theta, \rho, |y|) = \varphi_n(\theta/m, \rho, |y|) \) where \((\theta, \rho)\) is the polar coordinates of \( x \in \mathbb{R}^2 \) and \( y \in \mathbb{R}^{N-2} \). Hence, \( v_n \in BV_{\lambda,m,r}(\Omega_r) \) and there is \( t_n > 0 \) such that \( \omega_n = t_n v_n \in N_{\lambda,k,r} \). A simple argument proves that \( (t_n) \) is bounded, then up to a subsequence, \( t_n \to t_0 \) as \( n \to +\infty \). Then

\[
\omega_n \to t_0 u \quad \text{in} \quad L^1(\Omega_r) \quad \text{as} \quad n \to +\infty
\]

and

\[
\int_{\Omega_r} |\nabla \omega_n| \, dx \to t_0 \int_{\Omega_r} |Du| \quad \text{as} \quad n \to +\infty.
\]

Thus,

\[
J_{\lambda,k,r} \leq I_\lambda(\omega_n) = \int_{\Omega_r} |\nabla \omega_n| \, dx \, dy + \int_{\Omega_r} |\omega_n| \, dx \, dy + \int_{\partial \Omega_r} |\omega_n| \, d\mathcal{H}^{N-1}
\]

\[
- \int_{\Omega_r} \left( \frac{\lambda}{q} |\omega_n|^q + \frac{1}{1^*} |\omega_n|^{1^*} \right) \, dx \, dy.
\]

Thereby,

\[
J_{\lambda,k,r} \leq \int_0^\pi \int_r^{r+1} \int_0^{2\pi} |\nabla \omega_n| \rho \, d\theta \, d\rho \, dy + \int_0^\pi \int_{\Omega_r} |\omega_n| \, dx \, dy + \int_{\partial \Omega_r} |\omega_n| \, d\mathcal{H}^{N-1}
\]

\[
- \int_{\Omega_r} \left( \frac{\lambda}{q} |\omega_n|^q + \frac{1}{1^*} |\omega_n|^{1^*} \right) \, dx \, dy.
\]

where \( |\nabla \omega_n| = \left( \frac{1}{\rho^2 m^*} (\omega_n)_\rho^2 + (\omega_n)_\rho^2 + |\nabla_y \omega_n|^2 \right)^{1/2} \). Using the fact that \( m > 1 \) we get

\[
\int_0^\pi \int_r^{r+1} \int_0^{2\pi} \frac{1}{m^2 \rho^2} (\omega_n)_\rho^2 \, d\theta \, d\rho \, dy < \int_0^\pi \int_r^{r+1} \int_0^{2\pi} \frac{1}{\rho^2} (\omega_n)_\rho^2 \, d\theta \, d\rho \, dy + \left( \frac{1}{m^2} - 1 \right) \int_0^\pi \int_r^{r+1} \int_0^{2\pi} \frac{1}{\rho^2} (\omega_n)_\rho^2 \, d\theta \, d\rho \, dy.
\]

We claim \( \liminf_{n \to +\infty} \int_0^\pi \int_r^{r+1} \int_0^{2\pi} \frac{1}{\rho^2} (\omega_n)_\rho^2 \, d\theta \, d\rho \, dy = \sigma > 0 \), otherwise we have that for some subsequence

\[
\lim_{n \to +\infty} \int_0^\pi \int_r^{r+1} \int_0^{2\pi} (\omega_n)_\rho^2 \, d\theta \, d\rho \, dy = 0.
\]

Since \( \omega_n \in W^{1,1}((0, \pi) \times (r, r+1) \times (0, 2\pi)) \) and \( \omega_n \to t_0 u \) in \( L^1(\Omega) \), the last limit implies that \( u(x) = u(|x|) \), that is \( u \in BV_{\infty,r}(\Omega_r) \), which is absurd, since \( J_{\lambda,k,m,r} < J_{\lambda,\infty,r} \) (see Lemmas 2.3 and 2.5). The previous analysis ensures that

\[
J_{\lambda,k,r} \leq I_\lambda(t_0 u) - \sigma < I_\lambda(t_0 u) \leq I_\lambda(u) = J_{\lambda,k,m,r}.
\]
2.1. **Proof of Theorem 1.1.** Given \( n \in \mathbb{N} \), by Lemmas 2.4 and 2.5 we know that \( J_{\lambda,2^m,r} \) are critical levels of \( I_\lambda \) with

\[
0 < J_{\lambda,2^m,r} < J_{\lambda,2^{m+1},r} < \ldots < J_{\lambda,2^n,r} < J_{\lambda,\infty,r}.
\]

Applying the Principle of Symmetric Criticality (see [43]), it follows that they are critical points of \( I_\lambda \) in \( BV(\Omega_r) \). This way, all minimizers of \( J_{\lambda,2^m,r} \) for \( m = 1, \ldots, n \) are nonradial, rotationally non-equivalent and non-negative solutions of (2.1).

3. **Existence of multiple solutions via genus**

In this section, we will prove Theorem 1.2, which implies in the existence of multiple solutions for the problem

\[
\begin{aligned}
-\Delta_1 u &= \lambda |u|^{q-2}u + |u|^{1^*-2}u, \quad \text{in } \Omega, \\
0 &= u, \quad \text{on } \partial \Omega, 
\end{aligned}
\]

(3.1)

where \( \Omega \subset \mathbb{R}^N \) is a smooth bounded domain in \( \mathbb{R}^N \) with \( N \geq 2 \), \( \lambda > 0 \) and \( q \in (1,1^*) \). Let us recall that \( u \in BV(\Omega) \) is a solution of (3.1) if there is \( z \in L^\infty(\Omega, \mathbb{R}^N) \) such that

\[
\begin{aligned}
|z|_{\infty} &\leq 1, \quad \text{div } z \in L^N(\Omega), \\
-\int_{\Omega} u \text{div } z \, dx &= \int_{\Omega} |Dz| + \int_{\partial \Omega} |z| \, d{\mathcal H}^{N-1}, \\
-\text{div } z &= \lambda |u|^{q-2}u + |u|^{1^*-2}u, \quad \text{a.e. in } \Omega.
\end{aligned}
\]

(3.2)

In this section, we will consider the energy functional

\[
I_\lambda : L^{1^*}(\Omega) \to (-\infty, +\infty],
\]

(3.3)

where

\[
\int_{\Omega} |Dz| \, dx + \int_{\partial \Omega} |z| \, d{\mathcal H}^{N-1} - \frac{\lambda}{q} \int_{\Omega} |u|^q \, dx - \frac{1}{1^*} \int_{\Omega} |u|^{1^*} \, dx.
\]

Hereafter, let us consider the functional

\[
f_0 : L^{1^*}(\Omega) \to [0, +\infty],
\]

(3.4)

which is convex and lower semicontinuous in \( L^{1^*}(\Omega) \). Moreover, let us define

\[
f_1 : L^{1^*}(\Omega) \to [0, +\infty]
\]

(3.5)

by

\[
\begin{aligned}
f_0(u) &= \begin{cases} 
\int_{\Omega} |Dz| \, dx + \int_{\partial \Omega} |z| \, d{\mathcal H}^{N-1}, & \text{if } u \in BV(\Omega) \\
+\infty, & \text{if } u \in L^{1^*}(\Omega) \setminus BV(\Omega),
\end{cases} \\
f_1(u) &= \frac{\lambda}{q} \int_{\Omega} |u|^q \, dx + \frac{1}{1^*} \int_{\Omega} |u|^{1^*} \, dx,
\end{aligned}
\]

which is a \( C^1 \) functional.

Then, the functional \( I_\lambda \) is written as the difference between a convex, proper and lower semicontinuous functional and a \( C^1 \) one. Hence, in the light of [44], we denote by \( \partial I_\lambda(u) \), the subgradient of \( I_\lambda \) at \( u \in L^{1^*}(\Omega) \), which is well defined as a subset of \( L^N(\Omega) \).

By [34, Proposition 4.23], we have the following result.

**Proposition 3.1.** Assume that \( u \in BV(\Omega) \) is a critical point of \( I_\lambda \), i.e., \( 0 \in \partial I_\lambda(u) \). Then \( u \in L^\infty(\Omega) \) and \( u \) is a solution of (3.1), in the sense of (3.2).
Proof. Note that

$$0 \in \partial I_{\lambda}(u)$$

if and only if

$$f_1'(u) \in \partial f_0(u).$$

On the other hand, the last inclusion implies that there exists \( w \in \partial f_0(u) \subset L^N(\Omega) \) such that

\[
(3.4) \quad f_1'(u) = w \quad \text{in} \quad L^N(\Omega).
\]

Taking into account the characterization of \( \partial f_0(u) \) given in [34, Proposition 4.23], there exists \( w \in \partial f_0(u) \subset L^N(\Omega) \) such that

\[
(3.5) \quad \begin{cases} 
-\text{div} z = w, & \text{a.e. in } \Omega, \\
\int_{\Omega} wudx = \int_{\Omega} |Du| + \int_{\partial \Omega} |u| d\mathcal{H}^{N-1}.
\end{cases}
\]

By (3.4) and (3.5), we also have that

\[-\text{div} z = \lambda |u|^{p-2}u + |u|^{r-2}u, \quad \text{a.e. in } \Omega.\]

Hence, \( u \) satisfies (3.2). The fact that \( u \in L^\infty(\Omega) \) is a regularity result which follows as in [25, Proposition 3.3].

□

Now let us define what we mean by a \((PS)\) sequence for \( I_{\lambda} \). We say that \((u_k) \subset L^{1*}(\Omega)\) is a \((PS)\) sequence for \( I_{\lambda} \) if there exist \( d \in \mathbb{R} \) and \((z_k) \subset L^N(\Omega)\) such that \( |z_k|_N \to 0 \) as \( k \to +\infty \),

\[
(3.6) \quad \lambda |u_k|^{q-2}u_k + |u_k|^{r-2}u_k + z_k \in \partial f_0(u_k)
\]

and

\[
I_{\lambda}(u_k) \to d, \quad \text{as } k \to +\infty.
\]

Next, we state an abstract result, whose proof follows as in Szulkin [44, Theorem 4.4]. Hereafter \( X \) denotes a Banach space. We say that a functional \( I : X \to (-\infty, +\infty] \) satisfies the condition \((H)\) if:

\[
(H) \quad I = \Phi + \psi, \text{ where } \Phi \in C^1(X, \mathbb{R}) \text{ and } \psi : X \to (-\infty, +\infty] \text{ is convex, proper (i.e. } \psi \not\equiv +\infty) \text{ and lower semicontinuous.}
\]

Moreover, for each \( c \in \mathbb{R} \) we denote

\[
I_c = \{ u \in X : I(u) \leq c \}
\]

and by \( \Sigma \) the collection of all symmetric subsets of \( X \setminus \{0\} \) which are closed in \( X \).

**Theorem 3.2.** Assume that \( I : X \to (-\infty, +\infty] \) satisfies \((H)\), \( I(0) = 0 \) and \( \Phi, \psi \) are even and there is \( d > 0 \) such that \( I \) has no critical points in \( I_{-d} \). Assume also that

a. there is \( M > 0 \) such that \( I \) satisfies \((PS)_c\) condition for \( 0 < c < M \).

b. there exist \( \alpha, \rho > 0 \) such that

\[
I(u) \geq \alpha \quad \text{for} \quad ||u|| = \rho.
\]
c. given $n \in \mathbb{N}$, there is a finite dimensional subspace $X_n \subset X$ and $R_n > \rho$ such that

$$I|_{\partial Q_n} \leq -d$$

where $Q_n = \overline{B}_{R_n} \cap X_n$.

Denoting by $\mathcal{F}$ the set

$$\mathcal{F} = \{ f \in C(Q_n, X) : f \text{ is odd and } f|_{\partial Q_n} \approx id_{\partial Q_n} \text{ in } I_{-d} \text{ by an odd homotopy} \},$$

we consider for each $j \in \mathbb{N}$ the sets $\Lambda_j'$ and $\Lambda_j$ given by,

$$\Lambda_j' = \left\{ f : Q_n - V \text{ is open in } Q_n \text{ and symmetric, } V \cap \partial Q_n = \emptyset, \text{ and for each } Y \subset V \text{ such that } Y \in \Sigma, \gamma(Y) \leq k - j \right\}$$

and

$$\Lambda_j = \left\{ A \subset X : A \text{ is compact, symmetric and for each open set } U \supset A, \text{ there is } A_0 \in \Lambda_j' \text{ such that } A_0 \subset U \right\}.$$

Using the above notation, the numbers

$$c_j = \inf_{A \in \Lambda_j} \sup_{u \in A} I(u)$$

are well defined for all $j \in \mathbb{N}$ and $0 < \alpha \leq c_1 \leq c_2 \leq \ldots \leq c_j \leq c_{j+1} \leq \ldots$ for all $j \in \mathbb{N}$. If $c_n < M$, then $c_j$ are critical values of $I$ for $j \in \{1, 2, \ldots, n\}$. Moreover, if there are $j_0 \in \{1, 2, \ldots, n\}$ and $p \in \mathbb{N}$ such that $c_j_0 = \ldots = c_{j_0+p} = c < M$, then $\gamma(K_c) \geq p + 1$.

Now, following the approach explored in [25], for each $h > 0$ we consider the functions $T_h, R_h : \mathbb{R} \to \mathbb{R}$ given by

$$T_h(s) = \min\{\max\{s, -h\}, h\} \quad \text{and} \quad R_h(s) = s - T_h(s).$$

A simple computation shows that for each $u \in L^1(\Omega)$,

$$T_h(u) \to u \quad \text{in } L^1(\Omega) \quad \text{as } h \to +\infty,$$

and so,

$$R_h(u) \to 0 \quad \text{in } L^1(\Omega) \quad \text{as } h \to +\infty.$$

Moreover, if $(u_k) \subset L^1(\Omega)$ is a sequence satisfying

$$u_k(x) \to u(x) \quad \text{a.e. in } \Omega \quad \text{as } k \to +\infty,$$

then, for each $h$ fixed, the Lebesgue Dominated Convergence Theorem ensures that

$$T_h(u_k) \to T_h(u) \quad \text{in } L^1(\Omega) \quad \text{as } k \to +\infty.$$

**Proposition 3.3.** Let $(u_k)$ be a sequence in $BV(\Omega)$ and $(w_k)$ be a sequence in $L^N(\Omega)$ such that, for $k \in \mathbb{N}$, $w_k \in \partial f_0(u_k)$ and, as $k \to +\infty$,

$$u_k \rightharpoonup u \quad \text{in } L^1(\Omega),$$

$$w_k \rightharpoonup w \quad \text{in } L^N(\Omega).$$

Then $u \in BV(\Omega)$ and $w \in \partial f_0(u)$.

**Proof.** See [25, Proposition 3.2]. \qed
Lemma 3.4. Let \((u_k)\) be a \((PS)\) sequence for \(I_\lambda\). Assume that \((u_k)\) is bounded in \(BV(\Omega)\) and
\[
\|u_k\|_{L^\infty} = \|u\|_{L^\infty}.
\]
Then,
\[
\lim_{k \to +\infty} (f_0(u_k) - |u_k|^{1^*}) = f_0(u) - |u|^{1^*},
\]
and, for \(h > 0\) fixed,
\[
\lim_{k \to +\infty} (f_0(R_h(u_k)) - |R_h(u_k)|^{1^*}) \leq f_0(R_h(u)) - |R_h(u)|^{1^*}.
\]

Proof. Since \((u_k)\) is a \((PS)\) sequence, there is \((z_k) \subset L^N(\Omega)\), where \(z_k = o_k(1)\) in \(L^N(\Omega)\) and
\[
f_0(v) - f_0(u_k) \geq \lambda \int_\Omega |u_k|^{q-2} u_k (v - u_k) \, dx + \int_\Omega |u_k|^{1^*-2} u_k (v - u_k) \, dx
\]
\[
+ \int_\Omega z_k (v - u_k) \, dx,
\]
for all \(v \in L^{1^*}(\Omega)\). From this,
\[
\lambda |u_k|^{q-2} u_k + |u_k|^{1^*-2} u_k + z_k \in \partial f_0(u_k), \quad \forall k \in \mathbb{N},
\]
and then there exists \(w_k \in \partial f_0(u_k)\) such that
\[
w_k = \lambda |u_k|^{q-2} u_k + |u_k|^{1^*-2} u_k + z_k, \quad \forall k \in \mathbb{N}.
\]
Moreover, by [34, Proposition 4.23],
\[
\int_\Omega w_k u_k \, dx = \int_\Omega |Du_k| \, dx + \int_{\partial \Omega} |u_k| \, d\mathcal{H}^{N-1}, \quad \forall k \in \mathbb{N}.
\]
Thus, from (3.10) and (3.11), for all \(k \in \mathbb{N}\),
\[
\int_\Omega |Du_k| \, dx + \int_{\partial \Omega} |u_k| \, d\mathcal{H}^{N-1} = \lambda \int_\Omega |u_k|^q \, dx + \int_\Omega |u_k|^{1^*} \, dx
\]
\[
+ \int_\Omega z_k u_k \, dx.
\]
Since \(q \in (1, 1^*)\), Hölder’s inequality implies that \(|u_k|^{q-2} u_k\) and \(|u_k|^{1^*-2} u_k\) are bounded in \(L^N(\Omega)\). Indeed, it is straightforward to see that
\[
\|u_k|^{1^*-2} u_k\|_N = \|u_k|^{1^*-2}\|_N
\]
and
\[
\|u_k|^{q-2} u_k\|_N^N = \int \|u_k|^{(q-1)N} \, dx
\]
\[
\leq \left( \int |u_k|^{1^*} \, dx \right)^{(N-1)(q-1)} |\Omega|^{1-(N-1)(q-1)},
\]
from where it follows that both these sequences are bounded in \(L^N(\Omega)\). Then
\[
|u_k|^{q-2} u_k \rightharpoonup |u|^{q-2} u
\]
and
\[(3.14)\quad |u_k|^{1^*-2}u_k \rightharpoonup |u|^{1^*-2}u\]
in $L^N(\Omega)$.

Hence, from (3.13) and (3.14),
\[(3.15)\quad w_k \rightharpoonup w \quad \text{in} \quad L^N(\Omega),\]
with
\[w = -\gamma + \lambda|u|^{q-2}u + |u|^{1^*-2}u.\]

Taking into account the hypothesis and (3.15), Proposition 3.3 yields that $u \in BV(\Omega)$,
\[\lambda|u|^{q-2}u + |u|^{1^*-2}u \in \partial f_0(u)\]
and then, by (3.11),
\[(3.16)\quad \int_\Omega |Du| \, dx + \int_{\partial \Omega} |u| \, dH^{N-1} = \lambda \int_\Omega |u|^q \, dx + \int_\Omega |u|^{1^*} \, dx.\]

Hence, from (3.12),
\[
\lim_{k \to +\infty} \left( f_0(u_k) - |u_k|^{1^*}\right) = \lim_{k \to +\infty} \left( \lambda \int_\Omega |u_k|^q \, dx + \int_\Omega z_k u_k \, dx \right) \\
= \lambda \int_\Omega |u|^q \, dx.
\]

Then, from the last equality and (3.16), it follows that
\[
\lim_{k \to +\infty} \left( f_0(u_k) - |u_k|^{1^*}\right) = (f_0(u) - |u|^{1^*}),
\]
showing (i). The item (ii) follows as in [25, Lemma 5.1].

\[\square\]

**Lemma 3.5.** Each $(PS)$ sequence for $I_\lambda$ is bounded in $BV(\Omega)$.

**Proof.** Let $(u_k)$ be a $(PS)_d$ sequence for $I_\lambda$, that is,
\[I_\lambda(u_k) \to d \quad \text{as} \quad k \to +\infty\]
and
\[f_0(v) - f_0(u_k) \geq \lambda \int_\Omega |u_k|^{q-2}u_k(v - u_k) \, dx \]
\[+ \int_\Omega |u_k|^{1^*-2}u_k(v - u_k) \, dx + \int_\Omega z_k(v - u_k) \, dx,\]
where $(z_k) \subset L^N(\Omega)$, with $z_k = o_k(1)$ in $L^N(\Omega)$, as $k \to +\infty$.

By Proposition [34, Proposition 4.23], for all $k \in \mathbb{N}$,
\[
\int_\Omega |Du_k| \, dx + \int_{\partial \Omega} |u_k| \, dH^{N-1} = \lambda \int_\Omega |u_k|^q \, dx + \int_\Omega |u_k|^{1^*} \, dx \\
+ \int_\Omega z_k u_k \, dx.
\]
Now, let us denote
\[
Q(u_k) = \int_\Omega |D u_k| \, dx + \int_\Omega |u_k| \, dx + \int_{\partial \Omega} |u_k| \, d\mathcal{H}^{N-1}
- \lambda \int_\Omega |u_k|^q \, dx - \int_\Omega |u_k|^{1^*} \, dx - \int_\Omega z_k u_k \, dx
\]
and note that
\[
(3.17) \quad Q(u_k) = 0, \quad \forall k \in \mathbb{N}.
\]
Thus, from (3.17),
\[
d + o_k(1) = I_\lambda(u_k)
\]
\[
= I_\lambda(u_k) - \frac{1}{q} Q(u_k)
\]
\[
\geq \left( 1 - \frac{1}{q} \right) f_0(u_k) + \left( 1 - \frac{1}{q} \right) |u_k|_1 + \left( \frac{1}{q} - \frac{1}{1^*} \right) |u_k|_{1^*} + \frac{1}{q} \int_\Omega z_k u_k \, dx
\]
\[
\geq \left( 1 - \frac{1}{q} \right) f_0(u_k) + \left( 1 - \frac{1}{q} \right) |u_k|_{1^*} - \frac{1}{q} |z_k|_{N} |u_k|_1.
\]
for \( k \) large enough. Since \( g : [0, +\infty) \to \mathbb{R} \), given by
\[
g(t) = t^{1^*} - t
\]
is bounded from below, there exists \( K > 0 \) such that
\[
g(t) \geq -K, \quad \forall t \in [0, +\infty).
\]
Then
\[
d + o_k(1) \geq \left( 1 - \frac{1}{q} \right) f_0(u_k) - \left( \frac{1}{q} - \frac{1}{1^*} \right) K,
\]
from where it follows that \((u_k)\) is bounded in \(BV(\Omega)\). \( \square \)

**Lemma 3.6.** For each \( \lambda > 0 \), the functional \( I_\lambda \) satisfies the \((PS)_c\) condition, for \( c < \frac{1}{N} S^N \).

**Proof.** Let \((u_k)\) be a \((PS)_c\) sequence for \( I_\lambda \) with \( c < \frac{1}{N} S^N \). Then,
\[
I_\lambda(u_k) \to c \quad \text{as} \quad k \to +\infty
\]
and
\[
f_0(v) - f_0(u_k) \geq \lambda \int_\Omega |u_k|^{q-2} u_k (v - u_k) \, dx
\]
\[
+ \int_\Omega |u_k|^{1^*-2} u_k (v - u_k) \, dx + \int_\Omega w_k (v - u_k) \, dx,
\]
for some \( w_k \in L^N(\Omega) \) and \( w_k = o_k(1) \) in \( L^N(\Omega) \). Moreover, we also have

\[
(3.18) \quad \int_{\Omega} |Du_k| \, dx + \int_{\partial \Omega} |u_k| \, d\mathcal{H}^{N-1} = \lambda \int_{\Omega} |u_k|^q \, dx + \int_{\Omega} |u_k|^r \, dx + \int_{\Omega} w_k u_k \, dx, \quad \forall k \in \mathbb{N}.
\]

Hence, from (3.18), for all \( k \in \mathbb{N} \),

\[
(3.19) \quad I_\lambda(u_k) = \lambda \left(1 - \frac{1}{q} \right) |u_k|^q + \left(1 - \frac{1}{1^*} \right) |u_k|^{1^*} + \int_{\Omega} w_k u_k \, dx.
\]

Since \((u_k)\) is bounded and \( w_k = o_k(1) \) in \( L^N(\Omega) \), (3.19) gives

\[
\lim_{k \to +\infty} |u_k|^{1^*} \leq Nc < S^N.
\]

Since \( f_0 \) is lower semicontinuous and

\[
f_0(u) = f_0(T_h(u)) + f_0(R_h(u)),
\]

(3.7) and (3.8) lead us to

\[
\limsup_{h \to +\infty} (f_0(R_h(u)) - |R_h(u)|^{1^*}) \leq 0.
\]

Therefore, given \( \epsilon > 0 \), there is \( h > 0 \) large enough such that

\[
(3.20) \quad f_0(R_h(u)) - |R_h(u)|^{1^*} < \epsilon \left(S - (Nc)^\frac{1}{N}\right).
\]

For \( h \) fixed above, the definition of \( R_h \) gives

\[
\limsup_{k \to +\infty} |R_h(u_k)|^{1^*-1} \leq \limsup_{k \to +\infty} |u_k|^{1^*-1} \leq (Nc)^\frac{1}{N}.
\]

Now, the inequality below

\[
(S - |R_h(u_k)|^{1^*-1}) |R_h(u_k)|_1^* \leq f_0(R_h(u_k)) - |R_h(u_k)|^{1^*},
\]

together with Lemma 3.4 and (3.20) leads to

\[
\limsup_{k \to +\infty} |R_h(u_k)|_1^* < \epsilon.
\]

Hence \( |R_h(u)|_1^* < \epsilon \). Moreover, since by (3.9), \((T_h(u_k))\) is strongly convergent to \( T_h(u) \), it follows that

\[
\limsup_{k \to +\infty} |u_k - u|_1^* \leq \limsup_{k \to +\infty} |T_h(u_k) - T_h(u)|_1^* + \limsup_{k \to +\infty} |R_h(u_k)|_1^* + |R_h(u)|_1^* \leq 2\epsilon.
\]

Since that \( \epsilon \) is arbitrary, the last inequality ensures that \( u_k \to u \) in \( L^{1^*}(\Omega) \). \( \square \)

**Lemma 3.7.** There are \( \alpha, \rho > 0 \) such that

\[
I_\lambda(u) \geq \alpha, \quad \text{for} \quad |u|_1^* = \rho.
\]
Proof. Note that, in order to verify this lemma, it suffices to consider \( u \in \text{BV}(\Omega) \), since otherwise we would have \( I_\lambda(u) = +\infty \). Then, if \( u \in \text{BV}(\Omega) \), from the continuous embedding \( \text{BV}(\Omega) \hookrightarrow L^{1^*}(\Omega) \) and Hölder inequality, we have that
\[
I_\lambda(u) \geq C_1 |u|_{1^*} - C_2 |u|^q_{1^*} - |u|_{1^*}^{1^*}.
\]
Since \( q > 1 \), the last inequality allows us to conclude that there are \( \alpha, \rho > 0 \) such that
\[
I_\lambda(u) \geq \alpha, \quad \text{for} \quad |u|_{1^*} = \rho.
\]
\( \square \)

3.1. Proof of Theorem 1.2. In what follows, we will assume that there is \( d > 0 \) such that \( I_\lambda \) has no critical point in \( I_{-d} \), otherwise \( I_\lambda \) has infinitely many critical points and Theorem 1.2 is proved.

Lemma 3.8. For \( n \in \mathbb{N} \) and a finite dimensional subspace \( X_n \subset X \), there exists \( R_n > \rho \) such that
\[
I_\lambda|_{\partial Q_n} \leq -d \quad \text{where} \quad Q_n = \overline{B}_{R_n} \cap X_n.
\]

Proof. Let \( X_n \subset L^{1^*}(\Omega) \) be a finite dimensional subspace, such that \( X_n \subset C_0^\infty(\Omega) \). Since in \( X_n \), all the norms are equivalent, there are positive constants \( a_n, d_n \) and \( b_n \) (which depend just on \( n \in \mathbb{N} \)), such that, for \( u \in X_n \),
\[
I_\lambda(u) \leq a_n |u|_{1^*} - d_n \lambda |u|^q_{1^*} - b_n |u|_{1^*}^{1^*}.
\]

The last inequality, in turn, implies that
\[
I_\lambda(u) \to -\infty \quad \text{as} \quad |u|_{1^*} \to +\infty.
\]
This proves the desired result. \( \square \)

Lemma 3.9. For each \( n \in \mathbb{N} \), there is \( \lambda_n > 0 \) such that if \( \lambda \geq \lambda_n \), then
\[
\sup_{u \in Q_n} I_\lambda(u) < \frac{1}{N^{S^N}}.
\]

Hence, \( c_n < \frac{1}{N^{S^N}} \).

Proof. Arguing as in the proof of Lemma 3.8, we get
\[
\sup_{u \in Q_n} I_\lambda(u) \leq \sup_{u \in Q_n} \{a_n |u|_{1^*} - d_n \lambda |u|^q_{1^*} - b_n |u|_{1^*}^{1^*} \} \leq \sup_{u \in Q_n} \{a_n |u|_{1^*} - d_n \lambda |u|^q_{1^*} \}
\]

Defining the function \( h : [0, +\infty) \to \mathbb{R} \) as
\[
h(t) = a_n t - d_n \lambda t^q,
\]
it is straightforward to see that
\[
\max_{t \geq 0} h(t) = C_n \left( \frac{1}{\lambda} \right)^{\frac{1}{1-q}},
\]
for some \( C_n \) which depends on \( n \). Thus, there is \( \lambda_n > 0 \) such that
\[
\max_{t \geq 0} h(t) < \frac{1}{N^{S^N}}, \quad \forall \lambda \geq \lambda_n.
\]
This ensures that 
\[ \sup_{u \in Q_n} I_\lambda(u) < \frac{1}{N} S^N, \quad \forall \lambda \geq \lambda_n. \]
Since \( Q_n \in \Lambda_n \), we have that 
\[ c_n \leq \sup_{u \in Q_n} I_\lambda(u) < \frac{1}{N} S^N, \quad \forall \lambda \geq \lambda_n. \]
\[ \square \]
Therefore, taking into account Lemmas 3.6, 3.7, 3.8 and 3.9, we see that \( I_\lambda \) satisfies all the conditions of Theorem 3.2, and so, Theorem 1.2 is proved.

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