The Soft Régime in NRQCD

Harald W. Grießhammer

Nuclear Theory Group, Department of Physics, University of Washington,
Box 351 560, Seattle, WA 98195-1560, USA

Abstract

A Lagrangean and a set of Feynman rules are presented for Non-Relativistic QCD with manifest power counting in the heavy quark velocity $v$. A régime is identified in which energies and momenta are of order $Mv$. It is neither identical to the ultrasoft régime corresponding to radiative processes with energies and momenta of order $Mv^2$, nor to the potential régime with on shell quarks and gluons providing the Coulomb binding. In this soft régime, gluons are on shell, and the quark propagator becomes static. Examples show that it contributes to one- and two-loop corrections of scattering and production amplitudes near threshold. The results are readily generalised to any effective field theory with more than one low energy scale.

1Talk presented at the workshop “Nuclear Physics with Effective Field Theories” at Caltech, 26th – 27th February 1998.

2Email: hgrie@phys.washington.edu
1 Introduction

Why a talk on Non-Relativistic QCD \[1, 2\] in a workshop dedicated to nuclear physics with effective field theories? First, a few words on the background: NRQCD (which in the following also includes NRQED) uses the fact that systems consisting of two or more heavy quarks with mass $M$ become very similar to the hydrogen atom: The Coulomb interaction rules the level spacing in Charmonium and Bottomium because $\alpha_s$ is small enough for perturbative calculations. The relative velocity of the quarks in such systems is $v \sim \alpha_s(Mv)$ by virtue of the virial theorem, where the scale at which the running coupling is to be taken is the inverse Bohr radius $Mv$ of the system. Although $\alpha_s$ increases with decreasing $Q^2 \sim (Mv)^2$, a window between the relativistic perturbative and the confinement régime remains in which both $\alpha_s$ and $v$ is small (for Bottomium, $\alpha_s(Mv) \approx 0.2$). In the resulting non-relativistic and perturbative framework, potentials and wave functions may be used, so that bound state physics is easily accounted for, and calculations of production cross sections, hyperfine splittings, lifetimes etc. are much facilitated.

Like in effective nuclear theories, the NRQCD Lagrangean in terms of the heavy quark (anti-quark) bi-spinors $Q (\bar{Q})$ and gluons ($D^a = \partial^a + igA^a$)

$$L_{NRQCD} = Q^\dagger (i\partial_0 - gA_0)Q + \frac{c_1}{2M} Q^\dagger \vec{D}^2 Q + \frac{c_2}{8M^3} Q^\dagger \vec{D}^4 Q + \ldots$$

$$+ \frac{gc_3}{2M} Q^\dagger \vec{\sigma} \cdot \vec{B} Q + \ldots - \frac{g^2 d_1}{4M^2} (C\bar{Q}^\dagger \vec{\sigma} Q) \cdot (Q^\dagger \vec{\sigma} C\bar{Q}) + \ldots$$

$$- \frac{e_1}{4} F_{\mu\nu} F^{\mu\nu,a} + \frac{g^2 e_2}{240\pi^2 M^2} T(Q) F_{\mu\nu} D^2 F^{\mu\nu,a} + \ldots + L_{GFix}$$

(1)

consists of infinitely many terms constrained only by the symmetries of the theory (here: gauge invariance) and is non-renormalisable. Predictive power is nonetheless established when only a finite number of terms contribute to a given order in the two expansion parameters $v$ and $\alpha_s$. Excitations with four-momenta bigger than $M$ are integrated out, giving rise e.g. to four-point interactions between quarks ($d_1 \neq 0$). So, the ultraviolet physics is encoded in the coefficients $c_i, d_i, e_i$. An advantage of NRQCD is that these can be determined by matching NRQCD matrix elements to their QCD counterparts, as both are perturbative in the coupling constant. At tree level, the Foldy-Wouthuysen transformation gives $c_i = 1$, and loop corrections are down by powers of $g$, the most famous example being the coefficient for the Pauli term related to the anomalous magnetic moment of the electron, $c_3 = 1 + \frac{\alpha_s}{\pi} + \ldots$. At one loop level, further coefficients enter, and $d_i = e_i = 1$. In contradistinction, the free parameters of effective nuclear theories have to be determined from data because it is still unclear how to match nuclear theory to QCD.

Another similarity between NRQCD and effective nuclear theory is the existence of three scales: Besides the heavy quark mass $M$, the typical energy and momentum scales in NRQCD are the bound state energy $Mv^2$ and the relative momentum of the two quarks $Mv$ (i.e. the inverse size of the bound state) \[1, 2\]. In $NN$ scattering, the pion production threshold $\sqrt{M_N m_\pi}$ and the pion mass $m_\pi$ supplement the nucleon mass $M_N$. As noted by Luke and Manohar \[3\], this allows one to formally identify a small parameter in nuclear theories, $\sqrt{\frac{m_\pi}{M_N}} \sim v$. In both theories, the effective Lagrangean does not exhibit the non-relativistic

\[1\]For clarity, the two will be distinguished in the following, although $v \sim \alpha_s$.\[2\]
expansion parameter explicitly, so that a power counting scheme has to be established which determines uniquely which terms in the Lagrangean must be taken into account to render consistent calculations and predictive power to a given order in \( v \). It is at this point that NRQCD can serve as a “toy model” for effective nuclear theory (although this grossly understates its value): It will establish what the relevant kinematic régimes and infrared variables are in a theory with three (or more) separate scales, and it will demonstrate how to count powers of \( v \).

Velocity power counting in NRQCD and identification of the relevant energy and momentum régimes has proven more difficult than previously believed. The first attempt by Lepage and co-workers \([1, 2]\) fell shot as working only in the Coulomb gauge and as being complicated and incomplete. In a recent article, Beneke and Smirnov \([4]\) pointed out that the much simpler velocity rescaling rules proposed by Luke and Manohar for Coulomb interactions \([3]\), and by Grinstein and Rothstein for bremsstrahlung processes \([5]\), as united by Luke and Savage \([6]\), and by Labelle’s power counting scheme in time ordered perturbation theory \([7]\), do not reproduce the correct behaviour of the two gluon exchange contribution to Coulomb scattering between non-relativistic particles near threshold. This has cast some doubt whether NRQCD, especially in its dimensionally regularised version \([8]\), can be formulated using a self-consistent low energy Lagrangean. A recent article \([8]\) has resolved this conflict in a toy model, and this contribution presents the extension to NRQCD.

It is organised as follows: In Sect. 2, the relevant régimes of NRQCD are identified, extending the formalism of Luke and Savage \([6]\) by the soft régime of Beneke and Smirnov \([4]\). Sect. 3 proposes the rescaling rules necessary for a Lagrangean with manifest velocity power counting and gives the vertex and loop velocity power counting rules. An example in Sect. 4 establishes the necessity of the new, soft régime introduced in Sect. 2. It is essential for the correct reproduction of the infrared behaviour of QCD in NRQCD scattering amplitudes. Summary and outlook conclude the article. There is also some overlap between the topic of this and Mike Luke’s talk at this workshop, although I tried to set slightly different priorities.

## 2 Idea of Dimensionally Regularised NRQCD

The NRQCD propagators are

\[
Q : \frac{i \text{ Num}}{T - \frac{k^2}{2M} + i\epsilon} \quad A^\mu : \frac{i \text{ Num}}{k^2 + i\epsilon},
\]

where \( T = p_0 - M = \frac{p^2}{2M} + \ldots \) is the kinetic energy of the quark. “Num” are numerators containing the appropriate colour, Dirac and flavour indices and the gauge fixing term for the gluons, all of which are unimportant for the considerations in this section.

Cuts and poles in scattering amplitudes close to threshold stem from bound states and on-shell propagation of particles in intermediate states. They give rise to infrared divergences, and in general dominate contributions to scattering amplitudes. With the two scales at hand, and energies and momenta being of either scale, three régimes are identified in which either the quark or the gluon in \((2)\) is on shell:

soft régime: \( A^\mu : \quad k_0 \sim |\vec{k}| \sim Mv \),
potential régime: \( Q_p : \ T \sim Mv^2 , \ |\vec{p}| \sim Mv \),
ultrasoft régime: \( A_u^\mu : \ k_0 \sim |\vec{k}| \sim Mv^2 \)

Ultrasoft gluons \( A_u^\mu \) are emitted as bremsstrahlung or from excited states in the bound system, and hence physical. Soft gluons \( A_s^\mu \) do not describe bremsstrahlung: Because in- and outgoing quarks \( Q_p \) are close to their mass shell, they have an energy of order \( Mv^2 \). Therefore, overall energy conservation forbids all processes with outgoing soft gluons but without ingoing ones, and vice versa, as their energy is of order \( Mv \).

The list of particles is not yet complete: In a bound system, one needs gluons which change the quark momenta but keep them close to their mass shell, relating the (instantaneous) Coulomb interaction:

\[ A_p^\mu : \ k_0 \sim Mv^2 , \ |\vec{k}| \sim Mv \] (4)

So far, only potential gluons and quarks, and ultrasoft gluons had been identified in the literature of power counting in NRQCD \([3, 5, 7]\). That the soft régime was overlooked cast doubts on the completeness of NRQCD after Beneke and Smirnov \([4]\) demonstrated its relevance near threshold in explicit one- and two-loop calculations. Here, the fields representing a non-relativistic quark or gluon came naturally by identifying all possible particle poles in the non-relativistic propagators, given the two scales at hand.

When a soft gluon \( A_s^\mu \) couples to a potential quark \( Q_p \), the outgoing quark is far off its mass shell and carries energy and momentum of order \( Mv \). Therefore, consistency requires the existence of quarks in the soft régime as well,

\[ Q_s : \ T \sim |\vec{p}| \sim Mv \] (5)

As the potential quark has a much smaller energy than either of the soft particles, it can – by the uncertainty relation – not resolve the precise time at which the soft quark emits or absorbs the soft gluon. So, we expect a “temporal” multipole expansion to be associated with this vertex. In general, the coupling between particles of different régimes will not be point-like but contain multipole expansions for the particle belonging to the weaker kinematic régime. For the coupling of potential quarks to ultrasoft gluons, this has been observed by Grinstein and Rothstein \([5]\), and by Labelle \([7]\).

Propagators will also be different from régime to régime. In order to clarify the relation of the presentation here and the work by Beneke and Smirnov \([4]\), let us consider a typical loop integral in NRQCD. One may expand the integrand about the various saddle points, i.e. about the values of the loop-momentum \( q \) where particles become on shell. For example, expanding about a saddle point coming from a physical gluon at the soft scale, a quark propagator may be expanded as \( (T_p \sim \frac{2\hat{\vec{p}}^2}{2M} \sim Mv^2 \ll q_0 \sim |\vec{q}| \sim Mv) \)

\[
\frac{i}{q_{0,s} + T_p - \frac{\vec{p} + \vec{q}}{2M}} \rightarrow \frac{i}{q_{0,s}} + \frac{i}{q_{0,s}} \left( T_p - \frac{\vec{p} + \vec{q}}{2M} \right) \frac{i}{q_{0,s}} + \ldots .
\] (6)

So, \( Q_s \) is expected to become static to lowest order, and the higher order terms in the expansion can be interpreted as insertions into the soft quark propagator, or (as mentioned above and to be confirmed below) as resulting from an energy multipole expansion which
modifies the vertex rules. As the energy of potential gluons is much smaller than their momentum, the \( A_p \)-propagator is expected to become instantaneous for similar reasons.

With these five fields \( Q_s, Q_p, A^\mu_s, A^\mu_p, A^\mu_u \) representing quarks and gluons in the three different non-relativistic régimes, soft, potential and ultrasoft, NRQCD becomes self-consistent. An ultrasoft quark (which would have a static propagator) is not relevant for this paper. It is hence not considered, as is a fourth (“exceptional”) régime in which momenta are of the order \( M v^2 \) and energies of the order \( M v \) or any régime in which one of the scales is set by \( M \). They do not derive from poles in propagators, and hence will be relevant only under “exceptional” circumstances. A future publication [9] has to prove that the particle content outlined is not only consistent but complete.

It is worth noticing that the particles of the soft régime can neither be mimicked by potential gluon exchange, nor by contact terms generated by integrating out the ultraviolet modes: Fields in the soft régime have momenta of the same order as the momenta of the potential régime, but much higher energies. Therefore, seen from the potential scale they describe instantaneous but non-local interactions, as pointed out by Beneke and Smirnov [4]. Integrating out the scale \( M v \), one arrives at soft gluons and quarks as point-like multi-quark interactions in the ultrasoft régime. The physics of potential quarks and gluons will still have to be described by spatially local, but non-instantaneous interactions. This suggests once more that there is no overlap between interactions and particles in different régimes.

Finally, the regularisation scheme must be chosen such that the three kinematic régimes still do not overlap, i.e. such that expansion around one saddle point in the loop integral does not obtain any contribution from other saddle points and régimes. One might use an energy and momentum cutoff separating the soft from the potential, and the potential from the ultrasoft régime, but the integrals encountered can in general not be performed analytically. Furthermore, cutoff regularisation usually jeopardises power counting and symmetries, and introduces unphysical power divergences as the (unphysical) cutoff is removed. In contradistinction, using dimensional regularisation after the saddle point expansion preserves power counting and gauge symmetry. Its homogeneity [4] guarantees that contributions from different saddle points and régimes do not overlap (A simple example can be found in Ref. [8]). Therefore, dimensional regularisation will be the method of choice in the example of Sect. 4.

3 Velocity Power Counting

3.1 Rescaling Rules and Propagators

In order to establish explicit velocity power counting in the NRQCD Lagrangean, one rescales the space-time coordinates such that typical momenta in either régime are dimensionless, as proposed by Luke and Manohar [3] for the potential régime, and by Grinstein and Rothstein [6] for the ultrasoft one:

\[
\text{soft:} \quad t = (Mv)^{-1} T_s \quad , \quad \vec{x} = (Mv)^{-1} \vec{X}_s \\
\text{potential:} \quad t = (Mv^2)^{-1} T_u \quad , \quad \vec{x} = (Mv)^{-1} \vec{X}_s \\
\text{ultrasoft:} \quad t = (Mv^2)^{-1} T_u \quad , \quad \vec{x} = (Mv^2)^{-1} \vec{X}_u .
\]
For the propagator terms in the NRQCD Lagrangean to be properly normalised, one sets for the representatives of the gluons in the three régimes

- **soft**: \( A_s^\mu(\vec{x}, t) = (Mv) A_s^\mu(\vec{X}_s, T_s) \),
- **potential**: \( A_p^\mu(\vec{x}, t) = (Mv^{\frac{3}{2}}) A_p^\mu(\vec{X}_s, T_u) \),
- **ultrasoft**: \( A_u^\mu(\vec{x}, t) = (Mv^2) A_u^\mu(\vec{X}_u, T_u) \),

and for the quark representatives

- **soft**: \( Q_s(\vec{x}, t) = (Mv)^{\frac{3}{2}} Q_s(\vec{X}_s, T_s) \),
- **potential**: \( Q_p(\vec{x}, t) = (Mv)^{\frac{5}{2}} Q_p(\vec{X}_s, T_u) \).

The rescaled free quark Lagrangean reads then

- **soft**: \( d^3X_s \ dT_s \ Q_s \left[ (i\partial_0 + \frac{v}{2} \vec{\partial}^2) \right] Q_s \),
- **potential**: \( d^3X_s \ dT_u \ Q_p \left[ (i\partial_0 + \frac{1}{2} \vec{\partial}^2) \right] Q_p \).

Here, as in the following, the positions of the fields have been left out whenever they coincide with the variables of the volume element. Derivatives are to be taken with respect to the rescaled variables of the volume element.

The gauge fixing term was included in the NRQCD Lagrangean (1) because the decomposition of the Lagrangean into a free and an interaction part is gauge dependent. Usually, the Coulomb gauge \( \vec{\partial} \cdot \vec{A} = 0 \) is chosen in NRQCD, but Luke and Savage [6] showed how to establish explicit velocity power counting in any gauge. Because of the difference between canonical and physical momentum, it is important to specify the gauge before identifying to which order in \( v \) a certain régime in the Lagrangean contributes, as seen shortly. Still, the classification of the three kinematic régimes itself relied only on the typical excitation energy and momentum, and hence on gauge invariant quantities, and the denominator in the gluon propagator (2) is gauge independent.

The rescaled free gluon Lagrangean in the Lorentz gauge reads for example

- **soft**: \( d^3X_s \ dT_s \frac{1}{2} A_s^\mu \left[ \partial^2 g_{\mu\nu} - (1 - \frac{1}{\alpha})\partial_\mu \partial_\nu \right] A_s^\nu \),
- **potential**: \( d^3X_s \ dT_u \frac{1}{2} A_p^\mu \left[ g_{\mu\nu}(v^2 \partial_0^2 - \vec{\partial}^2) - (1 - \frac{1}{\alpha})(v\delta_{\mu0}\partial_\nu + \delta_{\mu\nu}\partial_0) + v^2\delta_{\mu\nu} \right] A_u^\nu \),
- **ultrasoft**: \( d^3X_u \ dT_u \frac{1}{2} A_u^\mu \left[ \partial^2 g_{\mu\nu} - (1 - \frac{1}{\alpha})\partial_\mu \partial_\nu \right] A_u^\nu \),

(colour indices suppressed), while in the Coulomb gauge

- **soft**: \( d^3X_s \ dT_s \frac{1}{2} A_i,s \left[ (\vec{\partial}^2 - \partial_0^2)\delta_{ij} - \partial_i \partial_j \right] A_{j,s} \),
- **potential**: \( d^3X_s \ dT_u \frac{1}{2} \left[ A_{0,p} \vec{\partial}^2 A_{0,p} + A_{i,p} (\vec{\partial}^2 \delta_{ij} - \partial_i \partial_j - v^2 \partial_0^2 \delta_{ij}) A_{j,p} \right] \),
- **ultrasoft**: \( d^3X_u \ dT_u \frac{1}{2} A_{i,u} \left[ (\vec{\partial}^2 - \partial_0^2)\delta_{ij} - \partial_i \partial_j \right] A_{j,u} \).
The (un-rescaled) Coulomb gauge propagators are therefore (Dirac and colour indices suppressed, \(\delta_{ij}^{\tilde{u}} = \delta_{ij} - \frac{k^i k^j}{k^2}\))

\[
\text{soft: } Q_s : \quad \frac{(T, \bar{p})}{A_1} = \frac{i}{T + i\epsilon}, \quad \bar{A}_s : \quad \frac{\delta_{ij}^{\tilde{u}}}{k^2 + i\epsilon}, \quad (18)
\]

\[
\text{potential: } Q_p : \quad \frac{(T, \bar{p})}{A_2} = \frac{i}{T - \frac{p^2}{2M} + i\epsilon}, \quad \bar{A}_p : \quad \frac{i \delta_{ij}^{\tilde{u}}}{-k^2 + i\epsilon}, \quad (19)
\]

\[
\text{ultrasoft: } \bar{A}_u : \quad \frac{(T, \bar{p})}{A_6} = \frac{i \delta_{ij}^{\tilde{u}}}{k^2 + i\epsilon}. \quad (20)
\]

As expected, the soft quark becomes static and the potential gluon becomes instantaneous in both gauges. In order to maintain velocity power counting, corrections of order \(v^1\) or higher must be treated as insertions, represented in Coulomb gauge by the (un-rescaled) Feynman rules

\[
\frac{(T, \bar{p})}{A_7} = -i \frac{\bar{p}^2}{2M} = \mathcal{O}(v^1), \quad \frac{(T, \bar{p})}{A_8} = -i \frac{k^2}{k^2 + i\epsilon} = \mathcal{O}(v^2). \quad (21)
\]

The Lorentz gauge propagators and insertions are written down straightforwardly, too \([6]\), and look very similar to the Coulomb gauge result, especially for \(\alpha = 1\). As seen from \((15-17)\), the choice of the Coulomb gauge makes \(A_0\) instantaneous, and hence it contributes in the potential regime, only. Since in this gauge, \(A_0\) solely mediates the instantaneous Coulomb potential (physical fields are transverse by virtue of Gauss’ law), this result was to be expected. The field \(\bar{A}_p\) is associated with retardation effects like spin-orbit coupling and the Darwin term in \((1)\). The advantages of having couplings between \(A_0\) and the other fields only for potential \(A_0\) and of having no insertions in the \(A_0\) propagator is balanced by the demand for non-multiplicative renormalisation of the Coulomb gauge. The Lorentz gauge may hence facilitate some calculations although the number of diagrams is larger, as will be seen shortly.

Except for the physical gluons \(A_0^\mu\) and \(A_u^\mu\), there is no distinction between Feynman and retarded propagators in NRQCD: Antiparticle propagation has been eliminated by the field transformation from the relativistic to the non-relativistic Lagrangean, and both propagators have maximal support for on-shell particles, the Feynman propagator outside the light cone vanishing like \(e^{-M}\). Feynman’s perturbation theory becomes more convenient than the time-ordered formalism, as less diagrams have to be calculated.

### 3.2 Vertex Rules

By experience, particles in the various régimes couple: On-shell (potential) quarks radiate bremsstrahlung (ultrasoft) gluons. In general, one must allow all couplings between the various régimes which obey “scale conservation” for both energies and momenta. They must be conserved within each régime to the order in \(v\) one works. This will exclude for example
the coupling of two potential quarks \( T \sim M v^2 \) to one soft gluon \( q_0 \sim M v \), but not to two soft gluons via the \( Q^\dagger \vec{A} \cdot \vec{AQ} \) term of the Lagrangean (1).

As an example, consider a bremsstrahlung-like process: the radiation of a soft gluon off a soft quark, resulting in a potential quark. The rescaled interaction Lagrangean reads for the vector coupling

\[
d^3X_s \, dT_s \left[ -i g \, v \, Q_s^\dagger (\vec{X}_s, T_s) \, \vec{\partial} \cdot \vec{A}_s (\vec{X}_s, T_s) \, Q_p (\vec{X}_s, v T_s) \right].
\] (22)

Note that the scaling régime of the volume element is set by the particle with the highest momentum and energy. The Feynman rule for this vertex is hence

\[
\left( \frac{T, \vec{p}}{T', \vec{p}'} \right) \uparrow_{q,i} = -i g \, (\vec{p} - \vec{p}')_i \, (2\pi)^4 \, \delta^{(3)}(\vec{p} + \vec{p}' + \vec{q}) \times
\left[ \exp \left( \frac{T'}{\delta(T + q_0)} \right) \delta(T + q_0) \right] = \mathcal{O}(v \, e^v).
\] (23)

One sees that technically, the energy multipole expansion expected in Sect. 2 comes from the different scaling of \( \vec{x} \) and \( t \) in the three régimes. The factor \( e^v \) symbolises that the multipole expansion corresponds term by term to an expansion in \( v \). It should be truncated at the desired order in \( v \).

Amongst the fields introduced, six interactions are allowed within and between the various régimes for the vector coupling, and two (six) for the scalar coupling in the Coulomb (Lorentz) gauge. Their \( v \) counting is presented in tables 1 and 2. Note that – albeit both describing interactions with physical gluons – soft and ultrasoft couplings occur at different orders in \( v \), and obey different multipole expansion rules. On the level of the vertex rules, double counting is prevented by the fact that in addition to most of the propagators, all vertices are distinct because of different multipole expansions. Velocity power counting for other vertices is obtained again by rescaling and multipole expansion. For example, the rules for the Fermi term \( \mathcal{L}_{\text{int}} = g M Q^\dagger \vec{\sigma} \cdot \vec{A} Q \) are identical to those of table 1.

Table 1: Velocity power counting and vertices for the interaction Lagrangean \(- \frac{ig}{M} Q^\dagger \vec{\partial} \cdot \vec{AQ} \) in both Lorentz and Coulomb gauge.

| Vertex |  \( A \) |  \( A \) |  \( A \) |  \( A \) |  \( A \) |  \( A \) |
|--------|---------|---------|---------|---------|---------|---------|
| \( v \) power | \( \sqrt{v} \) | \( v \) | \( v \) | \( v \) | \( v^2 \) | \( v^2 \) |

Table 2: Velocity power counting and vertices for the interaction Lagrangean \(- g Q^\dagger A_0 Q \) in the Lorentz gauge. In the Coulomb gauge, only the first two diagrams exist.

| Vertex |  \( A_0 \) |  \( A_0 \) |  \( A_0 \) |  \( A_0 \) |  \( A_0 \) |  \( A_0 \) |
|--------|---------|---------|---------|---------|---------|---------|
| \( v \) power | \( \frac{1}{\sqrt{v}} \) | \( \sqrt{v} \) | \( v^0 \) | \( v^0 \) | \( v^0 \) | \( v \) |

Using the equations of motion, a temporal multipole expansion may be re-written such that the energy becomes conserved at the vertex. Now, both soft and potential or ultrasoft
energies are present in the propagators, making it necessary to expand it in ultrasoft and potential energies. An example would be to restate the vertex (23) as
\[ -i g (\vec{p} - \vec{p}')_i (2\pi)^4 \delta(T + T' + q_0) \delta^{(3)}(\vec{p} + \vec{p}' + \vec{q}) \]
and the soft propagator to contain insertions \( \mathcal{O}(v) \) for potential energies \( T' \)
\[ \frac{i}{q_0 + i \epsilon} \sum_{n=0}^{\infty} \left( \frac{-T'}{q_0} \right)^n. \]

This gives us the first part of the correspondence to Beneke and Smirnov’s threshold expansion in the example propagator (6), the second one coming from the soft quark insertion (21). The same can be shown for the momentum-non-conserving vertices, too.

It is also interesting to note that there is no choice but to assign one and the same coupling strength \( g \) to each interaction. Different couplings for one vertex in different régimes are not allowed. This is to be expected, as the connection to Beneke and Smirnov’s threshold expansion [4] demonstrated that the fields in the various régimes are representatives of one and the same non-relativistic particle, whose interactions are fixed by the non-relativistic Lagrangean (26).

In the renormalisation group approach, there is only one relevant coupling (i.e. only one which dominates at zero velocity): As expected, it is the \( Q_p Q_p A_{p,0} \) coupling providing the binding. In the Coulomb gauge, all other couplings and insertions are irrelevant, while the Lorentz gauge exhibits three marginal couplings: \( Q_p Q_p A_{u,0} \), \( Q_s Q_s A_{s,0} \) and \( Q_s Q_p A_{s,0} \).

### 3.3 Loop Rules

The velocity power counting is not yet complete. As one sees from the volume element used in (22), the vertex rules for the soft régime count powers of \( v \) with respect to the soft régime. One hence retrieves the velocity power counting of Heavy Quark Effective Theory [10, 11] (HQET), in which the interactions between one heavy (and hence static) and one or several light quarks are described. HQET becomes a sub-set of NRQCD, complemented by interactions between soft (HQET) and potential or ultrasoft particles.

In NRQCD with two potential quarks as initial and final states, the soft régime can occur only inside loops, as noted above. Therefore, the power counting in the soft sub-graph has to be transfered to the potential régime. Because soft loop momenta scale like \( [d^4 k_s] \sim v^4 \), while potential ones like \( [d^4 k_p] \sim v^5 \), each largest sub-graph which contains only soft quarks and no potential ones (a “soft blob”) is enhanced by an additional factor \( \frac{1}{v} \).

As an example, consider the graphs of Fig. 1: Using the Lorentz gauge, vertex power counting gives that the leading contribution is from the exchange of two potential gluons, coupled via \( Q_l Q A_0 \). There are four such vertices, so the diagram is \( \mathcal{O}(g^4 v^{-2}) \) (table 2). The next two diagrams are \( \mathcal{O}(g^4 v^0) \) and \( \mathcal{O}(g^6 v^0) \) from the vertex power counting, but another factor \( \frac{1}{v} \) must be included because there is one soft blob in the diagrams. The intermediate couplings in the third diagram take place in the soft régime and hence are counted in that

\(^2\) Usually, HQET counts inverse powers of mass in the Lagrangean, but because in the soft régime \( M v \sim \text{const.} \), the two approaches are actually equivalent.
régime. The last diagram, in which two soft blobs are separated by the propagation of two potential quarks is $\mathcal{O}(g^8v^0)$ from the vertices, and the loop counting gives a factor $\frac{1}{v^2}$. Each soft blob contributes at least four orders of $g$, but only one inverse power of $v \sim g^2$. Power counting is preserved. These velocity power counting rules in loops are verified in explicit calculations of the exemplary graphs (see also below), but a rigorous derivation is left for a future publication [9].

![Diagram](image)

Figure 1: Power counting with soft loops. The loops in the second and third diagram obtain an inverse power of $v$, the last diagram of $v^2$ in addition to the power counting following from the vertex rules [2].

With rescaling, multipole expansion and loop counting, the velocity power counting rules are established, and one can now proceed to check the validity of the proposed Lagrangean by matching NRQCD to the relativistic theory in an example given by Beneke and Smirnov [4].

4 A Model Calculation

By construction, NRQCD and QCD must agree in the infrared limit, and especially in the structure of collinear (infrared) divergences. Matching NRQCD to the low velocity limit of QCD will therefore confirm that the power counting proposed is correct and that the soft régime is relevant. In QCD, the lowest order infrared divergent contribution in the Coulomb

![Diagram](image)

Figure 2: Leading soft (left) and potential (right) planar contribution to quark-quark scattering in the Coulomb gauge.

gauge which comes from the soft régime is the one loop graph Fig. 2 of order $g^4v$, while the leading contribution at this order in $g$ comes from the two Coulomb gluon exchange and is order $v^{-2}$. The leading diagram mediating retardation effects ($A_p$ couplings) in this order in $g$ is $\mathcal{O}(v^2)$. At high enough order in $g$, soft physical gluon exchange will therefore be more significant than retardation effects. The number of diagrams to be considered in the case at hand is large enough to make one long for a simpler example.

3Actually, most of them are zero, but to establish a pattern which convinces one of that is not the topic of this contribution [9].
For the sake of simplicity, let us – following Beneke and Smirnov [4] – deal with a toy model NRQFT Lagrangean

\[ \mathcal{L}_{\text{NRQFT}} = \Phi^\dagger \left( i\partial_0 + \frac{\vec{\partial}^2}{2M} - gc_1 A \right) \Phi + \frac{1}{2} (\partial_\mu A)(\partial^\mu A) + c_2 (\Phi^\dagger \Phi)^2 + \ldots \]  

(26)

of a heavy, complex scalar field \( \Phi \) with mass \( M \) coupled to a massless, real scalar \( A \). The coupling constant \( g \) has been chosen dimensionless. In a slight abuse of language, \( \Phi \) will still be referred to as “quark” and \( A \) as “gluon”. The coefficients \( c_i \) are again to be determined by matching relativistic and non-relativistic scattering amplitudes. This Lagrangean is very similar to the \( A_0 \)-part of the NRQCD Lagrangean in Lorentz gauge. Especially, the vertex and loop power counting is identical to the one of table [4].

The first soft non-zero contribution in this toy model comes from the two gluon direct exchange diagram of Fig. 3 calculated by Beneke and Smirnov [4] using threshold expansion. The Mandelstam variable \( t = - (\vec{p} - \vec{p}')^2 \) describes the momentum transfer in the center of mass system, \( y = - (\vec{p})^2 \propto -v^2 \) the relative four-momentum squared of the ingoing quarks as indicator for the thresholdness of the process. The ultraviolet behaviour of this graph

Figure 3: Planar \( \mathcal{O}(g^4) \) contributions to Coulomb scattering in the toy model. The four-point interaction and insertion diagrams are not displayed.

is mimicked in NRQFT by the four-fermion exchange with \( i c_2 = \frac{-ig^4}{24\pi^2 M^2} = \mathcal{O}(t^0, y^0) \), which using the rescaling rules is seen to be \( \mathcal{O}(v^4) \).

The \( A_uA_u \)-diagram is of order \( v^3 \) (see table [2]) with a leading loop integral contribution (similar to Beneke and Smirnov’s [4] fl. (32))

\[
\int \frac{d^d k}{(2\pi)^d} \frac{1}{k_0^2 - \vec{k}^2} \frac{1}{k_0 - k_0 - \frac{\vec{p}^2}{2M}} \frac{1}{T + k_0 - \frac{\vec{p}^2}{2M}} \frac{1}{T - k_0 - \frac{\vec{p}^2}{2M}}.
\]  

(27)

The diagram is expected to be zero since the ultrasoft gluons do not change the quark momenta and therefore the scattering takes place only in the forward direction, \( \vec{p} = \vec{p}' \). As no scale is present, it indeed vanishes upon employing the on-shell condition for potential
quarks, $T = \frac{t^2}{2M}$ to leading order. The $A_uA_p$ and $A_pA_u$ contributions ($\mathcal{O}(\frac{1}{v^u})$) are zero for the same reason. The lowest order contribution to the $A_pA_p$ graph ($\mathcal{O}(\frac{1}{v^v})$) is:

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 - i\epsilon} \frac{1}{(\bar{p} - \bar{p}' + \bar{k})^2 - i\epsilon} T + k_0 - \frac{(\bar{k} + \bar{p})^2}{2M} + i\epsilon \quad T - k_0 - \frac{(\bar{k} + \bar{p})^2}{2M} + i\epsilon \quad (28)$$

In the light of the discussion at the end of Sect. 4, it is most consistent to perform the $k_0$ integration by dimensional regularisation, using $\int \frac{d^d k}{(2\pi)^d} = \int \frac{d^d k}{(2\pi)^d} \frac{\text{d}^d - d}{\text{d}^d - \sigma} = \sigma \rightarrow 1$. Split dimensional regularisation was introduced by Leibbrandt and Williams [13] to cure the problems arising from pinch singularities in non-covariant gauges. Here, it has the same effect as closing the $k_0$-contour and picking up the quark propagator poles prior to using dimensional regularisation in $d - 1$ Euclidean dimensions. Considering also one insertion at the potential gluon lines to achieve $\mathcal{O}(v^1)$ accuracy, the result,

$$\frac{i}{8\pi t} \frac{M + T}{\sqrt{y}} \left( \frac{2}{4 - d} - \gamma_E - \ln \frac{-t}{4\pi\mu^2} \right), \quad (29)$$

agrees with Beneke and Smirnov’s [4] fl. (31) when one keeps in mind that non-relativistic external lines were normalised differently, and that different conventions for dimensionally regularised integrals were chosen. Near threshold, the scale is set by the total threshold energy $4\pi\mu^2 = 4(M + T)^2$.

The soft gluon part is to lowest order ($\mathcal{O}(v^{-1})$) given by

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 - \bar{k}^2 + i\epsilon} \frac{1}{k_0^2 - (\bar{p} - \bar{p}' + \bar{k})^2 + i\epsilon} \frac{1}{k_0 + i\epsilon} \frac{1}{-k_0 + i\epsilon} \quad (30)$$

which corresponds to Beneke and Smirnov’s [4] fl. (33). Now, split dimensional regularisation must be used if no ad-hoc prescription for the pinch singularity at $k_0 = 0$ is to be invoked. That the pinch is accounted for by potential gluon exchange and hence must be discarded, agrees with the intuitive argument that zero four-momentum scattering in QED is mediated by a potential only, and no retardation or radiation effects occur. On the other hand, the model Lagrangean contains three marginal couplings as seen at the end of Sect. 3.2, which may give finite contributions as energies and momenta of the scattered particles go to zero. The result to $\mathcal{O}(v^1)$ exhibits another collinear divergence,

$$\frac{-i}{4\pi^2 t} \left( \frac{2}{4 - d} - \gamma_E - \ln \frac{-t}{4\pi\mu^2} \right) + \frac{i}{24\pi^2 M^2} \left[ 1 + \frac{2y}{t} \left( \frac{2}{4 - d} - \gamma_E - \ln \frac{-t}{4\pi\mu^2} \right) \right], \quad (31)$$

and agrees with fl. (36) given by Beneke and Smirnov [4]. The second term comes from insertions and multipole expansions to achieve $\mathcal{O}(v^1)$ accuracy.

It is easy to see that the power counting proposed works. As expected, the potential diagram is $\sqrt{y} \propto v$ stronger than the leading soft contribution, and $t\sqrt{y} \propto v^3$ stronger than the four-fermion interaction.

In conclusion, the proposed NRQCD power counting and Lagrangean with three different kinematic régimes [3] reproduces the collinear divergences of the planar graph of the relativistic theory only if the soft gluon and the soft quark are accounted for: The four-fermion

\footnote{Since $T - \frac{t^2}{2M} \sim M v^4 \ll k \sim M v (M v^2)$ in the potential régime, this is a legitimate expansion.}
contact interaction produces just a $\frac{1}{M^2}$-term, graphs containing ultrasoft gluons were absent, and the potential gluon gave no $O(y^0)$ contribution. The coupling strength of the $\Phi_s A_s \Phi_p$ vertex is also seen to be identical to the other vertex coupling strengths, $g$.

5 Conclusions and Outlook

The objective of this contribution was a presentation of the ideas behind explicit velocity power counting in dimensionally regularised NRQCD. The identification of three different régimes of scale for on-shell particles in NRQCD leads in a natural way to the existence of a new quark field and a new gluon field in the soft scaling régime $E \sim |\vec{p}| \sim M v$. In it, quarks are static and gluons on shell, and HQET becomes a sub-set of NRQCD. Neither of the five fields in the three régimes should be thought of as “physical particles”. Rather, they represent the “true” quark and gluon in the respective régimes as the infrared-relevant degrees of freedom. None of the régimes overlap. An NRQCD Lagrangean has been proposed which leads to the correct behaviour of scattering and production amplitudes. It establishes explicit velocity power counting which is preserved to all orders in perturbation theory, once dimensional regularisation is chosen to complete the theory. The reason is non-commutativity of the expansion in small parameters with dimensionally regularised integrals.

I would like to stress that the diagrammatic threshold expansion derived here allows for a more automatic and intuitive approach and makes it easier to determine the order in $v$ to which a certain graph contributes than Beneke and Smirnov’s way [4]. Also, the NRQCD Lagrangean can easily be applied to bound state problems. An investigation of the influence of soft quarks and gluons on bound state calculations in NRQED and NRQCD is important because – as seen at the beginning of Sect. [4] their contribution at $O(y^4)$ and higher becomes stronger than retardation effects. As the threshold expansion of Beneke and Smirnov starts in a relativistic setting, it may formally be harder to treat bound states there.

Coming back to the topic of this workshop, effective nuclear theories, NRQCD shows how to establish a power counting in any effective field theory with several low energy scales: First, identify the combinations of scales in which particles become on shell by looking at the denominators of the various propagators. This gives the scaling régimes. Then, the Lagrangean is rescaled to dimensionless fields in each régime to exhibit the vertex and loop power counting rules. A priori, all couplings obeying scale conservation are allowed.

The problem with effective $NN$ scattering is not that the three scales $M_N, \sqrt{M_N m_\pi}, m_\pi$ are separated only by powers of $\frac{\sqrt{M_N m_\pi}}{m_\pi} \approx 0.4$. Indeed, Kaplan, Savage and Wise [14, 15] obtain very promising results for scales much smaller than the pion mass (See also David Kaplan’s talk in this workshop, and Martin Savage’s contribution on the inclusion of what in the language of this article would be ultrasoft pions.). The difficulty is that the scale $\sqrt{M_N m_\pi} \approx 360\text{MeV}$ at which the soft régime becomes relevant is larger than the $NN$ scattering expansion parameter $\Lambda_{NN} \approx 300\text{MeV}$. How to overcome this is another interesting topic for the future.
Acknowledgments

It is my pleasure express my gratitude to J.-W. Chen, D. B. Kaplan, M. Luke and M. J. Savage for stimulating discussions. Cordial thanks also to the organisers and participants of this vivid workshop. The work was supported in part by a Department of Energy grant DE-FG03-97ER41014.

References

[1] W. E. Caswell and G. P. Lepage: Phys. Lett. B 167, 437 (1986).
[2] G. T. Bodwin, E. Braaten and G. P. Lepage: Phys. Rev. D 51, 1125 (1995); Phys. Rev. D 55, 5853 (1997).
[3] M. Luke and A. V. Manohar: Phys. Rev. D 55, 4129 (1997).
[4] M. Beneke and V. A. Smirnov: Asymptotic Expansion of Feynman Integrals near Threshold; CERN-TH/7-315, hep-ph/9711391, 1997 (to be published).
[5] B. Grinstein and I. Z. Rothstein: Effective Field Theories and Matching in Nonrelativistic Gauge Theories; UCSD-97-06, hep-ph/9703298, 1997 (to be published).
[6] M. Luke and M. J. Savage: Power Counting in Dimensionally Regularized NRQCD; UTPT-97-12, DOE/ER/41014-21-N97, hep-ph/9707313, 1997 (to be published).
[7] P. Labelle: Effective Field Theories for QED Bound States: Extending Nonrelativistic QED to Study Retardation Effects; MCGILL-96-33, hep-ph/9608491, 1996 (to be published).
[8] H. W. Grießhammer, Threshold Expansion and Dimensionally Regularised NRQCD; NT@UW-98-3, hep-ph/9712467, 1997 (to be published).
[9] H. W. Grießhammer, in preparation.
[10] N. Isgur and M. B. Wise: Phys. Lett. B 232, 113 (1989).
[11] N. Isgur and M. B. Wise: Phys. Lett. B 237, 527 (1990).
[12] J. Collins: Renormalization; Cambridge Monographs on Mathematical Physics, CUP, Chap. 4.1 (1984).
[13] G. Leibbrandt and J. Williams: Nucl. Phys. B 475, 469 (1996).
[14] D. B. Kaplan, M. J. Savage and M. B. Wise: A New Expansion for Nucleon-Nucleon Interactions; DOE/ER/40561-352-INT97-00-189, NT@UW-98-05, CALT-68-2155, nucl-th/9801034, 1998 (to be published).
[15] D. B. Kaplan, M. J. Savage and M. B. Wise: Two-Nucleon Systems from Effective Field Theory; DOE/ER/40561-357-INT98-00-5, NT@UW-98-08, CALT-68-2161, nucl-th/9802075, 1998 (to be published).