SU(2) deformations of the minimal unitary representation of $OSp(8^*|2N)$ as massless 6D conformal supermultiplets

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Abstract: Minimal unitary representation of $SO^*(8) \simeq SO(6,2)$ realized over the Hilbert space of functions of five variables and its deformations labeled by the spin $t$ of an SU(2) subgroup correspond to massless conformal fields in six dimensions as was shown in arXiv:1005.3580. In this paper we study the minimal unitary supermultiplet of $OSp(8^*|2N)$ with the even subgroup $SO^*(8) \times USp(2N)$ and its deformations using quasiconformal methods. We show that the minimal unitary supermultiplet of $OSp(8^*|2N)$ admits deformations labeled uniquely by the spin $t$ of an SU(2) subgroup of the little group $SO(4)$ of lightlike vectors in six dimensions. We construct the deformed minimal unitary representations and show that they correspond to massless 6D conformal supermultiplets. The minimal unitary supermultiplet of $OSp(8^*|4)$ is the massless supermultiplet of $(2,0)$ conformal field theory that is believed to be dual to M-theory on $AdS_7 \times S^4$. We study its deformations in further detail and show that they are isomorphic to the doubleton supermultiplets constructed by using twistorial oscillators.

Keywords: AdS/CFT, Minimal Unitary Representations, Conformal Group.
1. Introduction

Motivated by the problem of constructing the relevant unitary representations of noncompact U-duality groups of extended supergravity theories, a general oscillator method was developed in [1–3]. The method, as formulated in [2], generalized and unified the special constructions that had previously appeared in the physics literature. The general oscillator construction was later extended to noncompact supergroups in [4] using bosonic as well as fermionic oscillators. In the generalized oscillator constructions of [2] and [4], one realizes the generators of noncompact groups or supergroups as bilinears of an arbitrary number $P_C$ ("colors") of sets of oscillators transforming in a definite representation (typically fundamental) of their maximal compact subgroups or subsupergroups. For symplectic groups $Sp(2n, \mathbb{R})$, the minimum
value of $P^C$ is one, and the resulting unitary representations are the singleton representations, which are referred to as metaplectic representations in the mathematics literature. Symplectic groups $Sp(2n, \mathbb{R})$ admit only two singleton irreducible representations (irreps). In general, the minimum allowed value of $P^C$ is two, and the resulting unitary representations of such noncompact groups were later called doubleton representations. For example, for the groups $SU(n, m)$ and $SO^*(2n)$, with maximal compact subgroups $SU(m) \times SU(n) \times U(1)$ and $U(n)$, respectively, one finds that $P^C_{min} = 2$. When $P^C_{min} = 2$, the noncompact group or supergroup admits an infinite number of doubleton irreps. The positive energy singleton or doubleton irreps of noncompact groups or supergroups do not belong to the discrete series representations. However, by tensoring them, one obtains positive energy unitary representations that, in general, belong to the holomorphic discrete series representations of the respective noncompact group or supergroup.

The Kaluza-Klein spectrum of IIB supergravity spontaneously compactified over the product space $AdS_5 \times S^5$ of 5D anti-de Sitter space $AdS_5$ with the five sphere $S^5$ was first obtained via the oscillator method by simple tensoring of the CPT self-conjugate doubleton supermultiplet of $N = 8$ $AdS_5$ superalgebra $PSU(2, 2 | 4)$ repeatedly with itself and restricting to the CPT self-conjugate short supermultiplets of $PSU(2, 2 | 4)$ [5]. The CPT self-conjugate doubleton supermultiplet of the symmetry superalgebra $PSU(2, 2 | 4)$ of $AdS_5 \times S^5$ solution of IIB supergravity does not have a Poincaré limit in five dimensions and decouples from the Kaluza-Klein spectrum as gauge modes. This led the authors of [5] to propose that the field theory of CPT self-conjugate doubleton supermultiplet of $PSU(2, 2 | 4)$ must live on the boundary of $AdS_5$, which can be identified with 4D Minkowski space on which $SO(4, 2)$ acts as a conformal group, and the unique candidate for this theory is the four dimensional $N = 4$ super Yang-Mills theory that was known to be conformally invariant.

The spectra of spontaneous compactifications of eleven dimensional supergravity over $AdS_4 \times S^7$ and $AdS_7 \times S^4$, that had been obtained by other methods previously, were fitted into supermultiplets of the symmetry superalgebras $OSp(8 | 4, \mathbb{R})$ and $OSp(8^* | 4)$ constructed by oscillator methods in [6] and [7], respectively. Furthermore, the entire Kaluza-Klein spectra of eleven dimensional supergravity over these two spaces were obtained by tensoring the singleton and scalar doubleton supermultiplets of $OSp(8 | 4, \mathbb{R})$ and $OSp(8^* | 4)$, respectively. The relevant singleton supermultiplet of $OSp(8 | 4, \mathbb{R})$ and scalar doubleton supermultiplet of $OSp(8^* | 4)$ do not have a Poincaré limit in four and seven dimensions, respectively, and decouple from the respective spectra as gauge modes. Again it was proposed that the field theories of the singleton and scalar doubleton supermultiplets live on the boundaries of $AdS_4$ and $AdS_7$ as superconformally invariant theories [6, 7].

These results have become an integral part of the work on AdS/CFT dualities in M-/superstring theory since the famous paper of Maldacena [8] and subsequent works of Witten [9] and of Gubser et al. [10].

Noncompact groups entered physics as spectrum generating symmetry groups during the 1960s. The work of physicists on spectrum generating symmetry groups motivated Joseph to introduce the concept of minimal unitary representations of Lie groups in [11]. These are
defined as unitary representations of corresponding noncompact groups over Hilbert spaces of functions of smallest possible (minimal) number of variables. Joseph gave the minimal realizations of the complex forms of classical Lie algebras and of the exceptional Lie algebra \( g_2 \) in a Cartan-Weyl basis. The minimal unitary representation of the split exceptional group \( E_{8(8)} \) was first identified within Langland’s classification by Vogan [12]. Later, Kostant studied the minimal unitary representation of \( SO(4,4) \) and its relation to triality in [13]. A general study of minimal unitary representations of simply laced groups was given by Kazhdan and Savin [14] and by Brylinski and Kostant [15, 16]. Pioline, Kazhdan and Waldron [17] reformulated the minimal unitary representations of simply laced groups given in [14] and determined the spherical vectors for the simply laced exceptional groups. The minimal unitary representations of quaternionic real forms of exceptional Lie groups were studied by Gross and Wallach in [18] and those of \( SO(p,q) \) in [19–22]. The relation of minimal representations of \( SO(p,q) \) to conformal geometry was studied rather recently in [23].

Over the last decade, there has been a great deal of progress made towards the goal of constructing physically relevant unitary representations of U-duality groups of extended supergravity theories. This was partly motivated by the proposals that certain extensions of U-duality groups act as spectrum generating symmetry groups of extremal black hole solutions in these theories. For example, the classification of the orbits of extremal black hole solutions in \( N = 8 \) supergravity and \( N = 2 \) Maxwell-Einstein supergravity theories with symmetric scalar manifolds led to the proposal that four dimensional U-duality groups act as spectrum generating conformal symmetry groups of corresponding five dimensional supergravity theories [24–29]. Extension of this proposal to corresponding spectrum generating symmetry groups of extremal black hole solutions of four dimensional supergravity theories with symmetric scalar manifolds led to the discovery of novel geometric quasiconformal realizations of three dimensional U-duality groups [25]. Quasiconformal extensions of four dimensional U-duality groups were then proposed as spectrum generating symmetry groups of the corresponding supergravity theories [25–29]. A concrete framework for the implementation of the proposal that three dimensional U-duality groups act as spectrum generating quasiconformal groups was given in [30–32]. This framework was based on the equivalence of equations of attractor flows of spherically symmetric stationary BPS black holes of four dimensional supergravity theories and the geodesic equations of a fiducial particle moving in the target space of three dimensional supergravity theories obtained by reduction of the 4D theories on a timelike circle [33].

Quasiconformal realization of three dimensional U-duality group \( E_{8(8)} \) of maximal supergravity in three dimensions is the first known geometric realization of any real form of \( E_8 \) [25]. As a quasiconformal group the action of \( E_{8(8)} \) leaves invariant a generalized light-cone with respect to a quartic distance function in 57 dimensions. Quasiconformal realizations exist for various real forms of all noncompact groups as well as for their complex forms [25, 34]. Furthermore, the quantization of geometric quasiconformal action of a noncompact group leads directly to its minimal unitary representation. This was first shown explicitly for the
maximally split exceptional group $E_{8(8)}$ with the maximal compact subgroup $SO(16)$ [35] and for the three dimensional U-duality group $E_{8(-24)}$ of the exceptional supergravity theory [36] in [37]. The minimal unitary representations of U-duality groups $F_4(4)$, $E_6(2)$, $E_7(-5)$, $E_8(-24)$ and $SO(d + 2, 4)$ of 3D $N = 2$ Maxwell-Einstein supergravity theories with symmetric scalar manifolds were studied in [34, 37]. A unified formulation of the minimal unitary representations of certain noncompact real forms of groups of type $A_2$, $G_2$, $D_4$, $F_4$, $E_6$, $E_7$, $E_8$ and $C_n$ was given in [38]. The minimal unitary representations of $Sp(2n, \mathbb{R})$ are simply the singleton representations. In [38], minimal unitary representations of noncompact groups $SU(m, n)$, $SO(m, n)$, $SO^*(2n)$ and $SL(m, \mathbb{R})$ obtained by quantization of their quasiconformal realizations were also given explicitly. Furthermore, this unified approach was generalized to define and construct the corresponding minimal representations of non-compact supergroups $G$ whose even subgroups are of the form $H \times SL(2, \mathbb{R})$ with $H$ compact.

In mathematics literature, the term minimal unitary representation, in general, refers to a unique representation of a noncompact group. Symplectic groups $Sp(2N, \mathbb{R})$ admit two singleton irreps whose quadratic Casimirs take on the same value. They are both minimal unitary representations, though in some of the mathematics literature only the scalar singleton is referred to as the minrep. Similarly the supergroups $OSp(M | 2N, \mathbb{R})$ with the even subgroup $SO(M) \times Sp(2N, \mathbb{R})$ admit two inequivalent singleton supermultiplets [6, 39, 40]. For noncompact groups or supergroups that admit only doubleton irreps, this raises the question as to whether any of the doubleton unitary representations can be identified with the minimal unitary representation, and if so, how the infinite set of doubletons are related to the minrep. This question was addressed for 5D anti-de Sitter or 4D conformal group $SU(2, 2)$ and corresponding supergroups $SU(2, 2 | N)$ in our earlier work [41]. We showed that the minimal unitary representation of the group $SU(2, 2)$ obtained by quantization of its quasiconformal realization coincides with the scalar doubleton representation corresponding to a massless scalar field in four dimensions. Furthermore the minrep of $SU(2, 2)$ admits a one-parameter ($\zeta$) family of deformations, and for a positive (negative) integer value of the deformation parameter $\zeta$, one obtains a positive energy unitary irreducible representation of $SU(2, 2)$ corresponding to a massless conformal field in four dimensions transforming in $\left( 0, \frac{\zeta}{2} \right)$ $\left( -\frac{\zeta}{2}, 0 \right)$ representation of the Lorentz subgroup, $SL(2, \mathbb{C})$. We showed that these representations are simply the doubletons of $SU(2, 2)$ that describe massless conformal fields in four dimensions [42, 43]. They were referred to as ladder (or most degenerate discrete series) unitary representations in some of the earlier literature on conformal group and it was shown by Mack and Todorov that they remain irreducible under restriction to the Poincaré subgroup [44]. Therefore the deformation parameter $\zeta$ can be identified with twice the helicity $h$ of the corresponding massless representation of the Poincaré group. We also extended these results to the minimal unitary representations of supergroups $SU(2, 2 | N)$ with the even subgroup $SU(2, 2) \times U(N)$ and their deformations. The minimal unitary supermultiplet of $SU(2, 2 | N)$ coincides with the CPT self-conjugate (scalar) doubleton supermultiplet, and for $PSU(2, 2 | 4)$ it is simply the four dimensional $N = 4$ Yang-Mills supermultiplet. We showed
that there exists a one-parameter family of deformations of the minimal unitary supermultiplet of $SU(2,2|N)$, and each integer value of the deformation parameter $\zeta$ leads to a unique unitary supermultiplet of $SU(2,2|N)$. The minimal unitary supermultiplet of $SU(2,2|N)$ and its deformations coincide with the unitary doubleton supermultiplets that were constructed and studied using the oscillator method earlier [5, 42, 43]. These results extend to the minreps of $SU(m,n)$ and of $SU(m,n|N)$ and their deformations in a straightforward manner.

More recently we gave a detailed study of the minimal unitary representation (minrep) of $SO(6,2) \simeq SO^*(8)$ over a Hilbert space of functions of five variables, obtained by quantizing its quasiconformal realization, and its deformations, and we constructed the minimal unitary supermultiplet of $OSp(8^*|2N)$ [45]. We showed that there exists a family of “deformations” of the minrep of $SO^*(8)$ labeled by the spin $t$ of an $SU(2)_T$ subgroup of the little group $SO(4)$ of lightlike vectors. These deformed minreps labeled by $t$ are positive energy unitary irreducible representations of $SO^*(8)$ that describe massless conformal fields in six dimensions. The $SU(2)_T$ spin $t$ is the six dimensional analog of $U(1)$ deformations of the minrep of $SU(2)$ labeled by helicity. The minimal unitary representation of $OSp(8^*|2N)$ describes a massless six dimensional conformal supermultiplet. In particular, the minimal unitary supermultiplet of $OSp(8^*|4)$ is the massless supermultiplet of $(2,0)$ conformal field theory that is believed to be dual to M-theory on $AdS_7 \times S^4$. It is simply the scalar doubleton supermultiplet of $OSp(8^*|4)$ first constructed in [7].

The oscillator construction of the positive energy unitary supermultiplets of $OSp(8^*|2N)$ was first given in [7]. The unitary supermultiplets of $OSp(8^*|2N)$ and their applications to $AdS_7/CFT_6$ dualities were further studied in [46, 47], where it was shown that the doubleton supermultiplets correspond to massless conformal supermultiplets in six dimensions. Construction of positive energy unitary supermultiplets of $OSp(8^*|2N)$ using harmonic super-space methods as well as their applications to $AdS_7/CFT_6$ dualities were studied in [48, 49]. A classification of positive energy unitary supermultiplets of 6D superconformal algebras using Cartan-Kac formalism was given in [50, 51]. The oscillator construction of positive energy unitary representations of general supergroups $OSp(2M^*|2N)$ with even subgroups $SO^*(2M) \times USp(2N)$ was given much earlier in [52].

In this paper we extend the results of [45] and show that the minimal unitary supermultiplet of $OSp(8^*|2N)$ admits deformations labeled uniquely again by the spin $t$ of an $SU(2)_T$ subgroup and construct all such deformed minimal unitary supermultiplets. In section 2, we review our results on the minimal unitary representation of $SO^*(8)$ and its deformations realized over the Hilbert space of functions of five variables. In particular we give a “particle basis” for these unitary representations over the tensor product of Fock space of four bosonic oscillators with the state space of a conformal (singular) oscillator. Their transformations under a distinguished $SO(4) \times U(1) \times U(1)$ subgroup are also given. In section 3, we present the deformations of the minimal unitary representation of $OSp(8^*|2N)$ labeled by the spin $t$ of an $SU(2)_T$ subgroup. Section 4 presents the compact 3-graded decomposition of the Lie superalgebra of $OSp(8^*|2N)$ with respect to the subsuperalgebra of $U(4|N)$. In section 5, we give
the general deformed minimal unitary representations of $OSp(8^*|2N)$ as $6D$ massless conformal supermultiplets. In section 6, we study the deformed minimal unitary supermultiplets of $OSp(8^*|4)$ which is the symmetry superalgebra of eleven dimensional supergravity compactified over $AdS_7 \times S^4$. We show that the minimal unitary supermultiplet of $OSp(8^*|4)$ and its deformations are precisely the doubleton supermultiplets that were constructed and studied using the twistorial oscillator construction [7,46,47]. Appendix A reviews the construction of relevant representations of $USp(2N)$ using “supersymmetry fermions.”

2. Deformations of the Minimal Unitary Representation of $SO^*(8)$

In our previous work [45], we gave a detailed study of the minimal unitary representation of $AdS_7$ or $Conf_6$ group $SO^*(8) \simeq SO(6,2)$ obtained by the quantization of its quasiconformal realization [38]. This minrep coincides with the scalar doubleton representation of $SO^*(8)$, which corresponds to a massless conformal scalar field in six dimensions [7,46,47]. There are infinitely many other doubleton representations of $SO^*(8)$, corresponding to $6D$ massless conformal fields of higher spin [7,46,47]. In the oscillator approach [1,2], all the doubleton representations can be constructed over the Fock space of two pairs of twistorial oscillators transforming in the spinor representation of $SO^*(8)$ [7,46]. In [45], we obtained all these higher spin doubleton representations from the minimal unitary representation via a “deformation” in a manner similar to what happens in the case of $4D$ conformal group $SU(2,2)$ [41].

In this section, we shall review how one deforms the minimal unitary representation of $SO^*(8)$ so as to obtain infinitely many irreducible unitary representations that are isomorphic to the irreducible doubleton representations of $SO^*(8)$.

2.1 The 5-grading of the deformed minrep of $SO^*(8)$ with respect to the subgroup $SO^*(4) \times SU(2) \times SO(1,1)$

The noncompact Lie algebra $so^*(8)$ has a 5-grading with respect to its subalgebra $g^{(0)} = so^*(4) \oplus su(2) \oplus so(1,1)$ [38]:

$$so^*(8) = g^{(-2)} \oplus g^{(-1)} \oplus [so^*(4) \oplus su(2) \oplus \Delta] \oplus g^{(+1)} \oplus g^{(+2)}$$

such that

$$\left[ \Delta, g^{(m)} \right] = m g^{(m)}$$

where $\Delta$ is the $SO(1,1)$ generator. In this decomposition, the subspaces $g^{(\pm 2)}$ are one-dimensional, and the subspaces $g^{(\pm 1)}$ transform in the $(4,2)$ dimensional representation of $SO^*(4) \times SU(2)$. Since $so^*(4) = su(1,1) \oplus su(2)$, the grade zero subalgebra can be written as

$$g^{(0)} = su(1,1)_N \oplus su(2)_A \oplus su(2)_T \oplus so(1,1)$$

We use the standard convention of denoting the groups with capital letters and the corresponding Lie algebras with small case letters.
where we denoted the $\mathfrak{su}(1,1)$ and $\mathfrak{su}(2)$ subalgebras of $\mathfrak{so}^*(4)$ as $\mathfrak{su}(1,1)_N$ and $\mathfrak{su}(2)_A$, respectively, and the $\mathfrak{su}(2)$ that commutes with $\mathfrak{so}^*(4)$ in equation (2.1) as $\mathfrak{su}(2)_T$. In the undeformed case, this $\mathfrak{su}(2)$ was denoted as $\mathfrak{su}(2)_S$ in [45]. In the deformation of the minimal unitary representation of $SO^*(8)$, the subalgebra $\mathfrak{su}(2)_S$ gets extended to the diagonal subalgebra

$$\mathfrak{su}(2)_S \subset \mathfrak{su}(2)_S \oplus \mathfrak{su}(2)_G$$

(2.4)

where the generators of $\mathfrak{su}(2)_S$ are realized as bilinears of bosonic oscillators and those of of $\mathfrak{su}(2)_G$ are realized in terms of fermionic oscillators.\(^2\)

To realize the minrep of $SO^*(8)$ and its deformations, one first introduces bosonic annihilation operators $a_m, b_m$ and their hermitian conjugates $a^m = (a_m)^\dagger, b^m = (b_m)^\dagger \ (m, n, \ldots = 1, 2)$ that satisfy the commutation relations:

$$[a_m, a^n] = [b_m, b^n] = \delta^n_m \quad [a_m, b_n] = [b_m, b_n] = 0 \quad (2.5)$$

and a single “central-charge coordinate” $x$ and its conjugate momentum $p$ such that

$$[x, p] = i \quad (2.6)$$

Now the generators of $\mathfrak{su}(2)_S$ are realized as follows:

$$S_+ = a^m b_m \quad S_- = (S_+)^\dagger = a_m b^m \quad S_0 = \frac{1}{2} (N_a - N_b) \quad (2.7)$$

where $N_a = a^m a_m$ and $N_b = b^m b_m$ are the respective number operators. They satisfy:

$$[S_+, S_-] = 2 S_0 \quad [S_0, S_\pm] = \pm S_\pm \quad (2.8)$$

The quadratic Casimir of $\mathfrak{su}(2)_S$ is

$$C_2 [\mathfrak{su}(2)_S] = S^2 = S_0^2 + \frac{1}{2} (S_+ S_- + S_- S_+)$$

$$= \frac{1}{2} (N_a + N_b) \left[ \frac{1}{2} (N_a + N_b) + 1 \right] - 2 a^m b^n a_{[m} b_{n]} \quad (2.9)$$

where square bracketing $a_{[m} b_{n]} = \frac{1}{2} (a_m b_n - a_n b_m)$ represents antisymmetrization of weight one.

To realize $SU(2)_G$, we introduce an arbitrary number $P$ pairs of fermionic annihilation operators $\xi^x$ and $\chi^x$ and their hermitian conjugates $\xi^x = (\xi^x)^\dagger$ and $\chi^x = (\chi^x)^\dagger \ (x = 1, 2, \ldots, P)$ that satisfy the usual anti-commutation relations:

$$\{\xi^x, \xi^y\} = \{\chi^x, \chi^y\} = \delta^x_y \quad \{\xi^x, \chi^y\} = \{\xi^x, \chi^y\} = 0 \quad (2.10)$$

The generators of $SU(2)_G$ are given by the following bilinears of these fermionic oscillators:

$$G_+ = \xi^x \chi^x \quad G_- = \chi^x \xi^x \quad G_0 = \frac{1}{2} (N_\xi - N_\chi) \quad (2.11)$$

\(^2\)We should note that in our previous paper [45], we added a “◦” above all deformed generators to distinguish them from the undeformed generators. In this paper, we drop those circles for the sake of simplicity.
where $N_\xi = \xi^x \xi_x$ and $N_\chi = \chi^x \chi_x$ are the respective number operators. They satisfy the commutation relations:\footnote{We should note that $SU(2)_G$, as defined in equation (2.11), commutes with the $USp(2P)$ group generated by the bilinears $\xi(x_\chi_y)$, $(\xi^x \xi_y - \chi_y \chi^x)$ and $\xi(x_\chi_y)$.}

\[
[G_+ , G_-] = 2 G_0 \quad \quad [G_0 , G_\pm] = \pm G_0
\]

Then the generators of $su(2)_T$ are simply:

\[
T_+ = S_+ + G_+ = a^m b_m + \xi^x \chi_x \\
T_- = S_- + G_- = b^m a_m + \chi^x \xi_x \\
T_0 = S_0 + G_0 = \frac{1}{2} (N_a - N_b + N_\xi - N_\chi)
\]

The $su(2)_G$ components realized in terms of fermions represent the deformations of the minrep. The quadratic Casimir of the subalgebra $su(2)_T$ is

\[
C_2 [su(2)_T] = T^2 = T_0^2 + \frac{1}{2} (T_+ T_- + T_- T_+) \, .
\]

The generators of $su(2)_A$ and $su(1,1)_N$, which we denote as $A_{\pm,0}$ and $N_{\pm,0}$, respectively, are realized purely in terms of bosonic oscillators:

\[
A_+ = a^1 a_2 + b^1 b_2 \\
N_+ = a^1 b_2 - a^2 b^1 \\
A_- = (A_+)^\dagger = a_1 a^2 + b_1 b^2 \\
N_- = (N_+)^\dagger = a_1 b_2 - a_2 b_1 \\
A_0 = \frac{1}{2} (a^1 a_1 - a^2 a_2 + b^1 b_1 - b^2 b_2) \\
N_0 = \frac{1}{2} (N_a + N_b) + 1
\]

and they do not get modified by $\xi$- and $\chi$-type fermionic oscillators under deformation. They satisfy the commutation relations:

\[
[A_+ , A_-] = 2 A_0 \quad \quad [N_-, N_+] = 2 N_0 \\
[A_0 , A_\pm] = \pm A_\pm \quad \quad [N_0 , N_\pm] = \pm N_\pm
\]

The quadratic Casimirs of these subalgebras

\[
C_2 [su(2)_A] = A^2 = A_0^2 + \frac{1}{2} (A_+ A_- + A_- A_+) \\
C_2 [su(1,1)_N] = N^2 = N_0^2 - \frac{1}{2} (N_+ N_- + N_- N_+)
\]

coincide and are equal to that of $su(2)_S$ in the minrep:

\[
S^2 = A^2 = N^2
\]

The generator $\Delta$ that defines the 5-grading is realized in terms of the “central charge coordinate” $x$ and its conjugate momentum $p$ as

\[
\Delta = \frac{1}{2} (xp + px) \, .
\]
The generator in grade $-2$ space is given by
\[ K_- = \frac{1}{2} x^2 \]  
(2.20)
and the eight generators in grade $-1$ subspace take the form:
\[
\begin{align*}
U_m &= x a_m \\
V_m &= x b_m \\
U^m &= x a^m \\
V^m &= x b^m
\end{align*}
\]  
(2.21)
Together with $K_-$, they form an Heisenberg algebra:
\[
\begin{align*}
[U_m, U^n] &= [V_m, V^n] = 2 \delta_m^n K_- \\
[U_m, U_n] &= [U_m, V_n] = [V_m, V_n] = 0
\end{align*}
\]  
(2.22)
The single generator in grade $+2$ subspace is realized as follows:
\[
K_+ = \frac{1}{2} p^2 + \frac{1}{4 x^2} \left( 8 T^2 + \frac{3}{2} \right)
\]  
(2.23)
The generators $\Delta, K_\pm$ form a distinguished $\text{su}(1,1)$ subalgebra, that we denote as $\text{su}(1,1)_K$:
\[
[K_-, K_+] = i \Delta \\
[\Delta, K_\pm] = \pm 2i K_\pm
\]  
(2.24)
Its quadratic Casimir operator turns out to be equal to that of $\text{su}(2)_T$:
\[
C_2[\text{su}(1,1)_K] = K^2 = \frac{1}{2} (K_+ K_- + K_- K_+) - \frac{1}{4} \Delta^2 = T^2
\]  
(2.25)
The generators in grade $+1$ subspace can be obtained by taking the commutators of the form $[\mathfrak{g}^{(-1)}, \mathfrak{g}^{(+2)}]$:
\[
\begin{align*}
\tilde{U}_m &= i [U_m, K_+] \\
\tilde{V}_m &= i [V_m, K_+] \\
\tilde{U}^m &= \left( \tilde{U}_m \right)^\dagger = i [U^m, K_+] \\
\tilde{V}^m &= \left( \tilde{V}_m \right)^\dagger = i [V^m, K_+]
\end{align*}
\]  
(2.26)
Explicitly they are given by:
\[
\begin{align*}
\tilde{U}_m &= -pa_m + \frac{2i}{x} \left[ \left( T_0 + \frac{3}{4} \right) a_m + T_- b_m \right] \\
\tilde{U}^m &= -pa^m - \frac{2i}{x} \left[ \left( T_0 - \frac{3}{4} \right) a^m + T_+ b^m \right] \\
\tilde{V}_m &= -pb_m - \frac{2i}{x} \left[ \left( T_0 - \frac{3}{4} \right) b_m + T_+ a_m \right] \\
\tilde{V}^m &= -pb^m + \frac{2i}{x} \left[ \left( T_0 + \frac{3}{4} \right) b^m - T_- a^m \right]
\end{align*}
\]  
(2.27)
They form an Heisenberg algebra with $K_+$ as its “central charge”:

\[
\begin{align*}
[U_m, \bar{U}^n] &= [\bar{V}_m, \bar{V}^n] = 2 \delta_m^n K_+ \\
[U_m, \bar{U}_n] &= [\bar{V}_m, \bar{V}_n] = [\bar{V}_m, \bar{V}_n] = 0
\end{align*}
\]  

(2.28)

The commutators $[\mathfrak{g}^{(-2)}, \mathfrak{g}^{(+1)}]$ close into grade $-1$ generators:

\[
\begin{align*}
[U_m, K_-] &= i U_m \\
[\bar{V}_m, K_-] &= i V_m
\end{align*}
\]  

(2.29)

The non-vanishing commutators of the form $[\mathfrak{g}^{(-1)}, \mathfrak{g}^{(+1)}]$ are:

\[
\begin{align*}
[U_m, \bar{U}^n] &= -\delta_m^n \Delta - 2i \delta_m^n N_0 - 2i \delta_m^n T_0 - 2i A^n_m \\
[V_m, \bar{V}^n] &= -\delta_m^n \Delta - 2i \delta_m^n N_0 + 2i \delta_m^n T_0 - 2i A^n_m \\
[U_m, \bar{V}^n] &= -2i \delta_m^n T_- \\
[V_m, \bar{U}^n] &= -2i \delta_m^n T_+
\end{align*}
\]  

(2.30)

where $\epsilon_{mn}$ is the Levi-Civita tensor ($\epsilon_{12} = +1$) and we have labeled the generators of $\mathfrak{su}(2)_A$ as $A^n_m$.

\[
A^1_1 = -A^2_2 = A_0 \quad A^1_2 = A_+ \quad A^2_1 = (A^1_2) = A_- \tag{2.31}
\]

With the generators defined above, the 5-graded decomposition of the deformed minimal unitary realization, which we denote as $\mathfrak{so}^* (8)_D$, takes the form:

\[
\mathfrak{so}^* (8)_D = \mathfrak{g}^{(-2)}_D \oplus \mathfrak{g}^{(-1)}_D \oplus \mathfrak{so}^* (4) \oplus \mathfrak{su}(2)_T \oplus \Delta \oplus \mathfrak{g}^{(+1)}_D \oplus \mathfrak{g}^{(+2)}_D
\]

\[= 1 \oplus (4, 2) \oplus [\mathfrak{su}(2)_A \oplus \mathfrak{su}(1, 1)_N \oplus \mathfrak{su}(2)_T \oplus \mathfrak{so}(1, 1)_\Delta] \oplus (4, 2) \oplus 1 \]

\[= K_- \oplus [U_m, U^m, V_m, V^m] \oplus [A_{\pm, 0} \oplus N_{\pm, 0} \oplus T_{\pm, 0} \oplus \Delta] \]

\[\oplus [\bar{U}_m, \bar{U}^m, \bar{V}_m, \bar{V}^m] \oplus K_+ \tag{2.32}
\]

The quadratic Casimir of $\mathfrak{so}^* (8)_D$ is given by

\[
\mathcal{C}_2 [\mathfrak{so}^* (8)_D] = \mathcal{C}_2 [\mathfrak{su}(2)_T] + C_2 [\mathfrak{su}(2)_A] + C_2 [\mathfrak{su}(1, 1)_N] + C_2 [\mathfrak{su}(1, 1)_K]
\]

\[- \frac{i}{4} \mathcal{F} (U, \bar{U}, V, \bar{V}) \tag{2.33}
\]

where

\[
\mathcal{F} (U, \bar{U}, V, \bar{V}) = \left( U_m \bar{U}^m + V_m \bar{V}^m + \bar{U}^m U_m + \bar{V}^m V_m \right) - \left( U^m \bar{U}_m + V^m \bar{V}_m + \bar{U}_m U^m + \bar{V}_m V^m \right) \tag{2.34}
\]
and reduces to
\[ C_2 [\mathfrak{so}^*(8)_D] = 2 G^2 - 4 \] (2.35)
where \( G^2 \) is the quadratic Casimir of \( \mathfrak{su}(2)_C \). Thus the quadratic Casimir of the deformed minrep of \( SO^*(8) \) depends only on the quadratic Casimir of \( SU(2)_C \) constructed out of \( \xi \)- and \( \chi \)-type fermionic oscillators used to deform the minrep.

### 2.2 The noncompact 3-grading of \( SO^*(8)_D \) with respect to the subgroup \( SU^*(4) \times SO(1,1) \)

Considered as the six dimensional conformal group, \( SO^*(8)_D \) has a noncompact 3-grading determined by the dilatation generator \( D \):
\[ \mathfrak{so}^*(8)_D = \mathfrak{m}_D^+ \oplus \mathfrak{m}_D^0 \oplus \mathfrak{m}_D^- \] (2.36)

where \( \mathfrak{m}_D^0 = \mathfrak{su}^*(4) \oplus \mathfrak{so}(1,1) \), with \( \mathfrak{su}^*(4) \simeq \mathfrak{so}(5,1) \) corresponding to the six dimensional Lorentz group. The \( \mathfrak{so}(1,1) \) dilatation generator is given by
\[ D = \frac{1}{2} [\Delta - i (N_+ - N_-)] . \] (2.37)

The generators that belong to \( \mathfrak{m}_D^\pm \) and \( \mathfrak{m}_D^0 \) subspaces are as follows:
\[
\begin{align*}
\mathfrak{m}_D^- &= K_- \oplus \left[ N_0 - \frac{1}{2} (N_+ + N_-) \right] \\
&\quad \oplus (U^1 - V_2) \oplus (U^2 + V_1) \oplus (V^1 + U_2) \oplus (V^2 - U_1) \\
\mathfrak{m}_D^0 &= D \oplus \frac{1}{2} [\Delta + i (N_+ - N_-)] \oplus T_{\pm,0} \oplus A_{\pm,0} \\
&\quad \oplus (U^1 + V_2) \oplus (U^2 - V_1) \oplus (V^1 - U_2) \oplus (V^2 + U_1) \\
&\quad \oplus (\bar{U}^1 - \bar{V}_2) \oplus (\bar{U}^2 + \bar{V}_1) \oplus (\bar{V}^1 + \bar{U}_2) \oplus (\bar{V}^2 - \bar{U}_1) \\
\mathfrak{m}_D^+ &= K_+ \oplus \left[ N_0 + \frac{1}{2} (N_+ + N_-) \right] \\
&\quad \oplus (\bar{U}^1 + \bar{V}_2) \oplus (\bar{U}^2 - \bar{V}_1) \oplus (\bar{V}^1 - \bar{U}_2) \oplus (\bar{V}^2 + \bar{U}_1)
\end{align*}
\]

In terms of the above operators, the Lorentz group generators \( \mathcal{M}_{\mu\nu} \) (\( \mu, \nu, \cdots = 0, 1, 2, \ldots, 5 \)) in six dimensions belonging to \( \mathfrak{m}_D^0 \) are given by:
\[
\begin{align*}
\mathcal{M}_{01} &= \frac{1}{4} \left[ (U^1 + V_2) + (V^2 + U_1) + i \left( \bar{U}^1 - \bar{V}_2 \right) + i \left( \bar{V}^2 - \bar{U}_1 \right) \right] \\
\mathcal{M}_{02} &= \frac{i}{4} \left[ (U^1 + V_2) - (V^2 + U_1) + i \left( \bar{U}^1 - \bar{V}_2 \right) - i \left( \bar{V}^2 - \bar{U}_1 \right) \right] \\
\mathcal{M}_{03} &= \frac{i}{4} \left[ (U^2 - V_1) + (V^1 - U_2) + i \left( \bar{U}^2 + \bar{V}_1 \right) + i \left( \bar{V}^1 + \bar{U}_2 \right) \right] \\
\mathcal{M}_{04} &= -\frac{1}{4} \left[ (U^2 - V_1) - (V^1 - U_2) + i \left( \bar{U}^2 + \bar{V}_1 \right) - i \left( \bar{V}^1 + \bar{U}_2 \right) \right]
\end{align*}
\] (2.39a)
\[ M_{15} = \frac{1}{4} \left[ (U^1 + V_2) + (V^2 + U_1) - i \left( \bar{U}^1 - \bar{V}_2 \right) - i \left( \bar{V}^2 - \bar{U}_1 \right) \right] \]
\[ M_{25} = \frac{i}{4} \left[ (U^1 + V_2) - (V^2 + U_1) - i \left( \bar{U}^1 - \bar{V}_2 \right) + i \left( \bar{V}^2 - \bar{U}_1 \right) \right] \]
\[ M_{35} = \frac{i}{4} \left[ (U^2 - V_1) + (V^1 - U_2) - i \left( \bar{U}^2 + \bar{V}_1 \right) - i \left( \bar{V}^1 + \bar{U}_2 \right) \right] \]
\[ M_{45} = -\frac{1}{4} \left[ (U^2 - V_1) - (V^1 - U_2) - i \left( \bar{U}^2 + \bar{V}_1 \right) + i \left( \bar{V}^1 + \bar{U}_2 \right) \right] \]
\[ M_{12} = T_0 + A_0 \]
\[ M_{14} = \frac{i}{2} \left( T_+ - T_- - A_+ + A_- \right) \]
\[ M_{24} = -\frac{1}{2} \left( T_+ + T_- - A_+ - A_- \right) \]
\[ M_{13} = \frac{1}{2} \left( T_+ + T_- + A_+ + A_- \right) \]
\[ M_{23} = \frac{i}{2} \left( T_+ - T_- + A_+ - A_- \right) \]
\[ M_{34} = T_0 - A_0 \]
\[ M_{05} = \frac{1}{2} \left[ \Delta + i \left( N_+ - N_- \right) \right] \]

They satisfy the commutation relations
\[ [M_{\mu\nu}, M_{\rho\tau}] = i \left( \eta_{\mu\rho} M_{\nu\tau} - \eta_{\mu\tau} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\tau} + \eta_{\nu\tau} M_{\mu\rho} \right) \]

where \( \eta_{\mu\nu} = \text{diag}(-, +, +, +, +) \). The six generators of grade +1 space are the momenta \( P_\mu \) that generate translations, and the six generators of grade −1 space are the special conformal generators \( K_\mu \) (\( \mu = 0, 1, 2, \ldots, 5 \)):

\[
\begin{align*}
P_0 &= K_+ + \left[ N_0 + \frac{1}{2} (N_+ + N_-) \right] \\
P_1 &= -\frac{i}{2} \left[ (\bar{U}^1 + \bar{V}_2) + (\bar{V}^2 + \bar{U}_1) \right] \\
P_2 &= -\frac{i}{2} \left[ (\bar{U}^1 + \bar{V}_2) - (\bar{V}^2 + \bar{U}_1) \right] \\
P_3 &= -\frac{i}{2} \left[ (\bar{U}^2 - \bar{V}_1) + (\bar{V}^1 - \bar{U}_2) \right] \\
P_4 &= \frac{1}{2} \left[ (\bar{U}^2 - \bar{V}_1) - (\bar{V}^1 - \bar{U}_2) \right] \\
P_5 &= K_+ - \left[ N_0 + \frac{1}{2} (N_+ + N_-) \right]
\end{align*}
\]

They satisfy the commutation relations:
\[ [D, P_\mu] = +i P_\mu \quad [D, K_\mu] = -i K_\mu \]
\[ [D, M_{\mu\nu}] = [P_\mu, P_\nu] = [K_\mu, K_\nu] = 0 \]
\[ [P_\mu, M_{\rho\nu}] = i \left( \eta_{\mu\rho} P_\nu - \eta_{\mu\nu} P_\rho \right) \]
\[ [K_\mu, M_{\rho\nu}] = i \left( \eta_{\mu\rho} K_\nu - \eta_{\mu\nu} K_\rho \right) \]
\[ [P_\mu, K_\nu] = 2i \left( \eta_{\mu\nu} D + M_{\mu\nu} \right) \]
Interestingly, none of the special conformal transformations \( \mathcal{K}_\mu (\mu = 0, 1, 2, \ldots, 5) \) receives any contributions from \( \xi \)- or \( \chi \)-type fermionic oscillators used to deform the minrep of \( SO^*(8) \). Furthermore, the six dimensional Poincaré mass operator vanishes identically:

\[
\mathcal{M}^2 = \eta_{\mu\nu} P^\mu P^\nu = 0
\]

for the deformed minimal unitary realization of \( SO^*(8) \) given above. Hence each deformed irreducible minrep corresponds to a massless conformal field in six dimensions.

### 2.3 The compact 3-grading of \( SO^*(8)_D \) with respect to the subgroup \( SU(4) \times U(1) \)

The Lie algebra of \( so^*(8)_D \) can be given a compact 3-grading

\[
so^*(8)_D = \mathfrak{c}_D^{-} \oplus \mathfrak{c}_D^{0} \oplus \mathfrak{c}_D^{+}
\]

with respect to its maximal compact subalgebra \( \mathfrak{c}_D^{0} = su(4) \oplus u(1) \), determined by the \( u(1) \) generator:

\[
H = N_0 + \frac{1}{2} (K_+ + K_-)
\]

This \( u(1) \) generator plays the role of the \( AdS \) energy or the conformal Hamiltonian when \( SO^*(8) \simeq SO(6,2) \) is taken as the seven dimensional \( AdS \) group or the six dimensional conformal group, respectively. In terms of the time components of momenta and special conformal generators defined in the noncompact 3-graded basis (equation (2.41)), we have

\[
H = \frac{1}{2} (K_0 + P_0).
\]

The grade \(-1\) operators in the compact basis are given by:

\[
Y_m = \frac{1}{2} \left( U_m - i \tilde{U}_m \right) = \frac{1}{2} (x + i p) a_m + \frac{1}{x} \left[ \left( T_0 - \frac{3}{4} \right) a_m + T_- b_m \right]
\]

\[
Z_m = \frac{1}{2} \left( V_m - i \tilde{V}_m \right) = \frac{1}{2} (x + i p) b_m - \frac{1}{x} \left[ \left( T_0 - \frac{3}{4} \right) b_m - T_+ a_m \right]
\]

\[
N_- = a_1 b_2 - a_2 b_1
\]

\[
B_- = \frac{i}{2} [\Delta + i (K_+ - K_-)] = \frac{1}{4} (x + i p)^2 - \frac{1}{x^2} \left( T^2 + \frac{3}{16} \right)
\]

and the grade \(+1\) operators are given by their hermitian conjugates:

\[
Y^m = \frac{1}{2} \left( U^m + i \tilde{U}^m \right) = \frac{1}{2} (x - i p) a^m + \frac{1}{x} \left[ \left( T_0 - \frac{3}{4} \right) a^m + T_+ b^m \right]
\]

\[
Z^m = \frac{1}{2} \left( V^m + i \tilde{V}^m \right) = \frac{1}{2} (x - i p) b^m - \frac{1}{x} \left[ \left( T_0 + \frac{3}{4} \right) b^m - T_- a^m \right]
\]

\[
N_+ = a_1^2 b_2^2 - a_2^2 b_1^2
\]

\[
B_+ = -\frac{i}{2} [\Delta - i (K_+ - K_-)] = \frac{1}{4} (x - i p)^2 - \frac{1}{x^2} \left( T^2 + \frac{3}{16} \right)
\]
The \( su(4) \) subalgebra has the maximal subalgebra

\[
\mathfrak{su}(4) \supset \mathfrak{su}(2)_T \oplus \mathfrak{su}(2)_A \oplus \mathfrak{u}(1)_J
\]

where the \( u(1)_J \) generator is given by:

\[
J = N_0 - \frac{1}{2} (K_+ + K_-) = \frac{1}{2} (\mathcal{K}_5 - \mathcal{P}_5)
\]  

(2.49)

The generators \( T_{\pm,0} \) and \( A_{\pm,0} \) of \( su(2)_T \) and \( su(2)_A \) were given in equations (2.13) and (2.15) and the generators belonging to the coset

\[
SU(4) / [SU(2)_T \times SU(2)_A \times U(1)_J]
\]

are as follows:

\[
C_{1m} = \frac{1}{2} \left( U_m + i \tilde{U}_m \right) = \frac{1}{2} (x - i p) a_m - \frac{1}{x} \left[ \left( T_0 + \frac{3}{4} \right) a_m + T_- b_m \right]
\]

\[
C^{1m} = \frac{1}{2} \left( U^m - i \tilde{U}^m \right) = \frac{1}{2} (x + i p) a^m - \frac{1}{x} \left[ \left( T_0 - \frac{3}{4} \right) a^m + T_+ b^m \right]
\]

\[
C_{2m} = \frac{1}{2} \left( V_m + i \tilde{V}_m \right) = \frac{1}{2} (x - i p) b_m + \frac{1}{x} \left[ \left( T_0 - \frac{3}{4} \right) b_m - T_+ a_m \right]
\]

\[
C^{2m} = \frac{1}{2} \left( V^m - i \tilde{V}^m \right) = \frac{1}{2} (x + i p) b^m + \frac{1}{x} \left[ \left( T_0 + \frac{3}{4} \right) b^m - T_- a^m \right]
\]

(2.50)

The \( u(1) \) generator \( H \) that defines the compact 3-grading is an operator whose spectrum is bounded from below. Hence the minrep of \( SO^*(8) \) and its deformations are all positive energy unitary (lowest weight) representations. The unitary lowest weight representations of \( SO^*(8)_D \) are uniquely labeled by a lowest energy K-type, that transforms irreducibly under the \( SU(4) \) subgroup, with the lowest energy eigenvalue with respect to the \( U(1) \) generator \( H \), and are annihilated by all the grade \(-1\) operators in \( \mathfrak{e}_D \). Since \( SU(2)_T \times SU(2)_A \times U(1)_J \) is a maximal subgroup of \( SU(4) \), one can label these lowest energy K-types by the \( SU(2)_T \times SU(2)_A \times U(1)_J \) quantum numbers of their highest weight vectors as irreps of \( SU(4) \).

2.4 Distinguished \( SU(1,1)_K \) subgroup of \( SO^*(8)_D \) generated by the isotonic (singular) oscillators

Note that the \( u(1) \) generator \( H \), given in equation (2.45), that determines the compact 3-grading of \( so^*(8)_D \) can be written as

\[
H = H_a + H_b + H_\odot
\]

(2.51)

where

\[
H_a = \frac{1}{2} (N_a + 2) \quad H_b = \frac{1}{2} (N_b + 2)
\]

(2.52)
are simply the Hamiltonians of standard bosonic oscillators of $a$- and $b$-type. On the other hand,

$$H_\otimes = \frac{1}{2} (K_+ + K_-) = \frac{1}{4} (x^2 + p^2) + \frac{1}{x^2} \left( T^2 + \frac{3}{16} \right)$$

$$= \frac{1}{4} \left( x^2 - \frac{\partial^2}{\partial x^2} \right) + \frac{1}{x^2} \left( T^2 + \frac{3}{16} \right)$$

is the Hamiltonian of a singular harmonic oscillator with a singular potential function

$$V_D (x) = \frac{G_D}{x^2} \quad \text{where} \quad G_D = 2 T^2 + \frac{3}{8}.$$  (2.54)

$H_\otimes$ also arises as the Hamiltonian of conformal quantum mechanics [53] with $G_D$ playing the role of the coupling constant [35]. In some literature it is also referred to as the isotonic oscillator [54, 55].

Together with the generators $B_\pm$ belonging to $\mathfrak{c}_D^\perp$ subspaces of $\mathfrak{so}^* (8)_D$ (equations (2.47) and (2.48)):

$$B_- = \frac{i}{2} [\Delta + i (K_+ - K_-)] = \frac{1}{4} (x + ip)^2 - \frac{1}{x^2} \left( T^2 + \frac{3}{16} \right)$$

$$= \frac{1}{4} \left( x + \frac{\partial}{\partial x} \right)^2 - \frac{1}{x^2} \left( T^2 + \frac{3}{16} \right)$$

$$B_+ = -\frac{i}{2} [\Delta - i (K_+ - K_-)] = \frac{1}{4} (x - ip)^2 - \frac{1}{x^2} \left( T^2 + \frac{3}{16} \right)$$

$$= \frac{1}{4} \left( x - \frac{\partial}{\partial x} \right)^2 - \frac{1}{x^2} \left( T^2 + \frac{3}{16} \right)$$

$H_\otimes$ generates the distinguished $\mathfrak{su}(1,1)_K$ subalgebra:\footnote{This is the $\mathfrak{su}(1,1)$ subalgebra generated by the longest root vector.}

$$[B_-, B_+] = 2 H_\otimes \quad \quad [H_\otimes, B_\pm] = \pm B_\pm$$  (2.56)

For a given eigenvalue $t (t + 1)$ of the quadratic Casimir $T^2$ of $\mathfrak{su}(2)_T$, the wave functions corresponding to the lowest energy eigenvalue of this singular harmonic oscillator Hamiltonian will be superpositions of functions of the form $\psi_0^{(\alpha t)} (x) \Lambda (t, m_t)$, where $\Lambda (t, m_t)$ is an eigenstate of $T^2$ and $T_0$, independent of $x$:

$$T^2 \Lambda (t, m_t) = t (t + 1) \Lambda (t, m_t) \quad \quad T_0 \Lambda (t, m_t) = m_t \Lambda (t, m_t)$$  (2.57)

and $\psi_0^{(\alpha t)} (x)$ is a function of $x$ that satisfies

$$B_- \psi_0^{(\alpha t)} (x) \Lambda (t, m_t) = 0$$  (2.58)

whose solution is given by [56]

$$\psi_0^{(\alpha t)} (x) = C_0 x^{\alpha t} e^{-x^2/2}$$  (2.59)
where $C_0$ is a normalization constant and
\[ \alpha_t = \frac{1}{2} + \sqrt{1 + 4t(t + 1)} = 2t + \frac{3}{2}. \] (2.60)

The normalizability of the state imposes the constraint
\[ \alpha_t \geq \frac{1}{2}. \] (2.61)

A state of the form $\psi_0^{(\alpha_t = 2t + 3/2)}(x) \Lambda(t, m_t)$ is an eigenstate of $H_{\odot}$ with eigenvalue $(t + 1)$:
\[ H_{\odot} \psi_0^{(2t+3/2)}(x) \Lambda(t, m_t) = (t + 1) \psi_0^{(2t+3/2)}(x) \Lambda(t, m_t) \] (2.62)

which is the lowest energy eigenvalue in the deformed case with the deformation parameter $t$. Higher energy eigenstates of $H_{\odot}$ can be obtained from $\psi_0^{(2t+3/2)}(x) \Lambda(t, m_t)$ by acting on it repeatedly with the raising generator $B_+$:
\[ \psi_n^{(2t+3/2)}(x) \Lambda(t, m_t) = C_n (B_+)^n \psi_0^{(2t+3/2)}(x) \Lambda(t, m_t) \] (2.63)

where $C_n$ are normalization constants. They correspond to energy eigenvalues $n + t + 1$:
\[ H_{\odot} \psi_n^{(2t+3/2)}(x) \Lambda(t, m_t) = (n + t + 1) \psi_n^{(2t+3/2)}(x) \Lambda(t, m_t) \] (2.64)

We shall denote the corresponding states as
\[ \ket{\psi_n^{(2t+3/2)}(x) \Lambda(t, m_t)} = \ket{\psi_n^{(2t+3/2)}(x)} \otimes \ket{\Lambda(t, m_t)} \]
and refer to them as the particle basis of the state space of the (isotonic) singular oscillator. The $(2t + 1)$ states belonging to the subspace corresponding to an irrep of $SU(2)_T$ labeled by spin $t$ will all have the same eigenvalue of $H_{\odot}$.

### 2.5 $SU(2)_T \times SU(2)_A \times U(1)_J \times U(1)_H$ basis of the deformed minimal unitary representations of $SO^\ast(8)$

The fermionic Fock vacuum $\ket{0}_F$ is chosen such that:
\[ \xi_x \ket{0}_F = \chi_x \ket{0}_F = 0 \quad x = 1, 2, \ldots, P \] (2.65)

A “particle basis” of states in the fermionic Fock space is provided by the action of creation operators $\xi^x$ and $\chi^y$ on the Fock vacuum $\ket{0}_F$. A state of the form\(^5\)
\[ \chi^{x_1 x_2 x_3 \ldots x_P} \ket{0}_F \]
has a definite eigenvalue $-\frac{P}{2}$ of $G_0$ and is annihilated by the lowering operator $G_-$. By repeatedly acting on this state with the raising operator $G_+$, one can obtain $P$ other states of the form:
\[ \xi^{x_1 x_2 x_3 \ldots x_P} \ket{0}_F \oplus \xi^{x_1 \xi x_2 x_3 \ldots x_P} \ket{0}_F \oplus \ldots \oplus \xi^{x_1 \xi \xi \xi \ldots x_P} \ket{0}_F \]

\(^5\)Note that square bracketing of fermionic indices implies complete anti-symmetrization of weight one.
We shall denote these $P + 1$ states as

$$\left| \frac{P}{2}, m_P \right>$$

where $m_P = -\frac{P}{2}, -\frac{P}{2} + 1, \ldots, + \frac{P}{2}$. (2.66)

They transform irreducibly under $su(2)_G$ in the spin $\frac{P}{2}$ representation. We shall denote the bosonic Fock vacuum annihilated by all bosonic oscillators $a_m, b_m$ ($m = 1, 2$) as $|0\rangle_B$:

$$a_m |0\rangle_B = b_m |0\rangle_B = 0$$

and the tensor product of fermionic and bosonic vacua simply as $|0\rangle$.

The tensor products of the states of the form $(a^m)^n_{a,m} |0\rangle_B$, $(b^m)^n_{b,m} |0\rangle_B$, $\xi^x |0\rangle_F$ and $\chi^x |0\rangle_F$, where $n_{a,m}$ and $n_{b,m}$ are non-negative integers, form the “particle basis” of states in the full Fock space. As the “particle basis” of the Hilbert space of the deformed minimal unitary representation of $SO^*(8)$, we shall take the following tensor products of the above states with the state space of the singular (isotonic) oscillator:

$$(a^1)^{n_{a,1}} (a^2)^{n_{a,2}} (b^1)^{n_{b,1}} (b^2)^{n_{b,2}} |0\rangle_B \otimes \xi^{x_1} \cdots \xi^{x_k} \chi^{x_{k+1}} \cdots \chi^{x_P} |0\rangle_F \otimes \psi_n^{(\alpha)}$$

where square brackets imply full anti-symmetrization with weight one. We denote them as

$$\left| \psi_n^{(\alpha)}; n_{a,1}, n_{a,2}, n_{b,1}, n_{b,2}; \frac{P}{2}, k - \frac{P}{2}\right>$$

or simply as

$$\left| \psi_n^{(\alpha)}; 0, 0, 0; \frac{P}{2}, k - \frac{P}{2}\right>$$

where $k = 0, 1, \ldots, P$. For a fixed $N = n_{a,1} + n_{a,2} + n_{b,1} + n_{b,2}$, these states transform in the $(\frac{N+P}{2}, \frac{N}{2})$ representation under the $SU(2)_T \times SU(2)_A$ subgroup. They are, in general, not eigenstates of $J$. The $(P + 1)$ states of the form

$$\left| \psi_0^{(P+\frac{3}{2})}; 0, 0, 0; \frac{P}{2}, k - \frac{P}{2}\right>$$

that transform in the $(\frac{P}{2}, 0)$ representation of $SU(2)_T \times SU(2)_A$ are, however, all eigenstates of $J$ with eigenvalue $\mathcal{J} = -\frac{P}{2}$. These $P + 1$ states are annihilated by grade $-1$ operators in $\mathfrak{c}_D^{\mathfrak{su}^*(8)}$ (given in equation (2.47)). The action of the coset generators

$$SU(4) / [SU(2)_T \times SU(2)_A \times U(1)_J]$$

given in equation (2.50) on the above states leads to a set of states transforming in an irreducible representation of $SU(4)$ with Dynkin labels $(2t, 0, 0) = (P, 0, 0)$. We denote this set of states as $\left| \Omega^{(P+\frac{3}{2})}(P, 0, 0)\right>$. They are all eigenstates of $H$ (AdS$_7$ energy), with the lowest eigenvalue of $E = t + 2 = \frac{P}{2} + 2$, and are annihilated by all grade $-1$ operators. Therefore, they form a lowest energy K-type and uniquely define a positive energy unitary irreducible representation of $SO^*(8)$, labeled by the $SU(2)_G$ spin $g = t = \frac{P}{2}$. The resulting unitary
irreducible representations correspond to deformations of the minimal unitary representation. With respect to the $SU(2)_T \times SU(2)_A \times U(1)_J$ subgroup of $SU(4)$, the states $|\Omega^{(P+\frac{3}{2})}(P,0,0)\rangle$ has the following decomposition:

$$|\Omega^{(P+\frac{3}{2})}(P,0,0)\rangle = \left(\begin{array}{c} \frac{P}{2},0 \end{array}\right)^{\frac{P}{2}} \oplus \left(\begin{array}{c} \frac{P}{2} - 1,\frac{1}{2} \end{array}\right)^{\frac{P}{2}+1} \oplus \left(\begin{array}{c} \frac{P}{2} - 1,1 \end{array}\right)^{\frac{P}{2}+2} \oplus \cdots \oplus \left(\begin{array}{c} 0,\frac{P}{2} \end{array}\right)^{\frac{P}{2}} \ . \ (2.69)$$

where we have labeled the irreps of $SU(2)_T \times SU(2)_A \times U(1)_J$ as $(t,a)^3$. All the other states of the “particle basis” of the deformed minrep can be obtained from $|\Omega^{(P+\frac{3}{2})}(P,0,0)\rangle$ by repeatedly acting on them with the six operators in $SU(2)$ and labeled by the $SU(2)$ irreducible representations correspond to deformations of the minimal unitary representation.

$|\Omega^{(P+\frac{3}{2})}(P,0,0)\rangle$ has the following decomposition:

$$|\Omega^{(P+\frac{3}{2})}(P,0,0)\rangle = \left(\begin{array}{c} \frac{P}{2},0 \end{array}\right)^{\frac{P}{2}} \oplus \left(\begin{array}{c} \frac{P}{2} - 1,\frac{1}{2} \end{array}\right)^{\frac{P}{2}+1} \oplus \left(\begin{array}{c} \frac{P}{2} - 1,1 \end{array}\right)^{\frac{P}{2}+2} \oplus \cdots \oplus \left(\begin{array}{c} 0,\frac{P}{2} \end{array}\right)^{\frac{P}{2}} \ . \ (2.69)$$

The generators $(Y^1, Z^1)$ and $(Y^2, Z^2)$ form two doublets under $SU(2)_T$, and the generators $(Y^1, Y^2)$ and $(Z^1, Z^2)$ form two doublets under $SU(2)_A$. $N_+$ and $B_+$ are both singlets under $SU(2)_T$ and $SU(2)_A$. The generators $Y^m$ and $Z^m$ have zero $J$-charge, while the generators $N_+$ and $B_+$ have $J$-charges $+1$ and $-1$, respectively.

In Table [1], we give the $SU(4) \times U(1)_H$ decomposition of the deformed minreps of $SO^*(8)$ uniquely determined by the $(P + 1)$ states

$$|\psi^{(P+\frac{3}{2})}_0; 0,0,0,0; \frac{P}{2}, k - \frac{P}{2}\rangle \qquad k = 0,1, \ldots, P \ . \ (2.70)$$

and labeled by the $SU(2)_G$ spin $g = t = \frac{P}{2}$.

Table 1: In this table, we give the $SU(4) \times U(1)_H$ decomposition of the deformed minimal unitary representation of $SO^*(8)$, defined by the “lowest weight state” $|\psi^{(P+\frac{3}{2})}_0; 0,0,0,0; \frac{P}{2}, -\frac{P}{2}\rangle$ for any non-negative $P$. These are massless representations of $SO^*(8)$, considered as the $6D$ conformal group. As massless $6D$ conformal fields, their $SU^*(4)$ transformations coincide with $SU(4)$ transformations of the lowest energy K-type $|\Omega^{(P+\frac{3}{2})}\rangle$ and whose eigenvalue $E$ of $H$ is the negative of the conformal dimension $\ell$. First column gives the states, second column gives the energy eigenvalues, and third column gives the $SU(4)$ Dynkin labels.

| States | $E$ | $SU(4)$ Dynkin |
|--------|-----|----------------|
| $|\Omega^{(P+\frac{3}{2})}(P,0,0)\rangle$ | $\frac{P}{2} + 2$ | $(P,0,0)$ |
The states with $SU(4)$ Dynkin labels $(P,n,0)$ and $AdS$ energy $\frac{P}{T} + n + 2$ decompose into the following $SU(2)_T \times SU(2)_A \times U(1)_J$ irreps labeled as $(t,a)^3$:

$$(P,n,0) = \left( \frac{P}{2},0 \right)_{P \geq 0, n \geq 0} - \frac{P}{T} - n + \left( \frac{P}{2} - 1, \frac{1}{2} \right)_{P \geq 1, n \geq 0} - \frac{P}{T} - n + 1 + \left( \frac{P}{2} + 1, \frac{1}{2} \right)_{P \geq 0, n \geq 1} - \frac{P}{T} - n + 1 + \left( \frac{P}{2} - 1, 1 \right)_{P \geq 2, n \geq 0} - \frac{P}{T} - n + 2 + \left( \frac{P}{2}, 0 \right)_{P \geq 1, n \geq 1} - \frac{P}{T} - n + 2 + \left( \frac{P}{2}, 1 \right)_{P \geq 1, n \geq 1} - \frac{P}{T} - n + 2 + \left( \frac{P}{2} + 1, 1 \right)_{P \geq 0, n \geq 2} - \frac{P}{T} - n + 2 + \left( 0, \frac{P}{2} \right)_{P \geq 0, n \geq 0} + \frac{P}{T} + n$$

(2.71)

with the subscripts denoting the allowed values of $P$ and $n$.

From the above table, it is clear that the deformed minrep of $SO^*(8)$ with deformation parameter $t$ is, in fact, the doubleton representation of $SO^*(8)$ whose lowest energy K-type has the $SU(4)$ Young tableau

$$|\Omega\rangle = \begin{array}{c}
\includegraphics[width=1cm]{young_tableau.png}
\end{array}$$

with $2t = P$.
previously constructed by the oscillator method in [7, 46]. We should stress again that in the oscillator construction of [7, 46], one realizes the generators of \(SO^*(8)\) as bilinears of two sets of twistorial bosonic oscillators transforming in the spinor representation of \(SO^*(8)\), and the Fock space of these oscillators decomposes into the direct sum of infinitely many doubleton irreps of \(SO^*(8)\). In the quasiconformal approach, each deformation labeled by \(SU(2)_G\) spin \((g = t)\) leads to a unique unitary irrep as explained above.

3. Deformations of the Minimal Unitary Representation of \(OSp(8^*|2N)\)

In our previous work [45], we constructed the “undeformed” minimal unitary supermultiplet of \(osp(8^*|2N)\), with particular emphasis on that of \(osp(8^*|4)\). The minimal unitary supermultiplet of \(osp(8^*|4)\) is simply the supermultiplet of the six dimensional \((2,0)\) conformal field theory that is believed to be dual to the M-theory on \(AdS_7 \times S^4\).

In this section, we will extend the deformed minimal unitary representations of \(SO^*(8)\), constructed in section 2, to deformed minimal unitary supermultiplets of \(OSp(8^*|2N)\). For this purpose, in addition to the \(\xi\) - and \(\chi\)-type fermionic oscillators introduced for deforming the minrep of \(SO^*(8)\), we introduce a new set of fermionic oscillators required by supersymmetry. More specifically to realize the compact Lie algebra \(usp(2N)\), we introduce \(N\) copies of \(\alpha\)- and \(\beta\)-type fermionic oscillators (with indices \(r = 1, 2, \ldots, N\), as reviewed in Appendix A (see equation (A.1)). We shall refer to the fermions \(\xi\) and \(\chi\) as deformation fermions and the fermions \(\alpha\) and \(\beta\) as supersymmetry fermions.

Recall that we labeled the \(su(2)\) subalgebra that commutes with \(so^*(4)\) subalgebra in the 5-graded decomposition of the minimal unitary realization of \(so^*(8)\) as \(su(2)_S\) such that

\[
g^{(0)} = su(2)_A \oplus su(1,1)_N \oplus su(2)_S \oplus so(1,1).\]

Under deformation of the minimal unitary realization of \(so^*(8)\), the subalgebra \(su(2)_S\) gets contributions from deformation fermions \(\xi\) and \(\chi\) and goes over to \(su(2)_T\), which is the diagonal subalgebra of \(su(2)_S\) and \(su(2)_G\) (as defined in equation (2.11)). Now in extending the deformed minimal unitary realization of \(so^*(8)\) to the deformed minimal unitary realization of \(osp(8^*|2N)\), it gets further contributions from supersymmetry fermions \(\alpha\) and \(\beta\). In particular the subalgebra \(su(2)_T\) gets extended to \(su(2)_T\), which is the diagonal subalgebra of \(su(2)_T\) and \(su(2)_F\), which involves only supersymmetry fermions \(\alpha\) and \(\beta\) (as given in equation (A.12)). The generators of \(su(2)_T\) turn out to be given by:

\[
\begin{align*}
T_+ &= S_+ + F_+ + G_+ = a^m b_m + \alpha^r \beta_r + \xi^x \chi_x \\
T_- &= S_- + F_- + G_- = b^m a_m + \beta^r \alpha_r + \chi^x \xi_x \\
T_0 &= S_0 + F_0 + G_0 = \frac{1}{2} (N_\alpha - N_\beta + N_\xi - N_\chi)
\end{align*}
\]

The quadratic Casimir of \(su(2)_T\) is

\[
C_2 [su(2)_T] = T^2 = T_0 T_0 + \frac{1}{2} (T_+ T_- + T_- T_+). \quad (3.2)
\]
Now $\text{osp}(8^*|2N)_D$ has a 5-graded decomposition
\[
\text{osp}(8^*|2N)_D = \mathfrak{g}^{(−2)}_D \oplus \mathfrak{g}^{(−1)}_D \oplus \mathfrak{g}^{(0)}_D \oplus \mathfrak{g}^{(+1)}_D \oplus \mathfrak{g}^{(+2)}_D
\]
with respect to the subsuperalgebra
\[
\mathfrak{g}^{(0)}_D = \text{osp}(4^*|2N) \oplus \text{su}(2)_\tau \oplus \text{so}(1,1)_\Delta
\]
such that grade ±2 subspaces are one-dimensional. In the deformed minimal unitary realization, generators of the subsuperalgebra $\text{osp}(4^*|2N)$ belonging to grade zero subspace do not get any contributions from the deformation fermions $\xi$ and $\chi$. Its generators have the same realization as in the undeformed case. The generators of $\text{so}^*(4) = \text{su}(2)_A \oplus \text{su}(1,1)_N$, those of $\text{osp}(2N)$, and the $8N$ supersymmetry generators are given by:
\[
\begin{align*}
A_+ &= a^1 a_2 + b^1 b_2 & N_+ &= a^1 b^2 - a^2 b^1 \\
A_- &= (A_+)^\dagger &= a_1 a^2 + b_1 b^2 & N_- &= (N_+)^\dagger &= a_1 b_2 - a_2 b_1 \\
A_0 &= \frac{1}{2} (a^1 a_1 - a^2 a_2 + b^1 b_1 - b^2 b_2) & N_0 &= \frac{1}{2} (N_a + N_b) + 1
\end{align*}
\]
\[
\begin{align*}
S_{rs} &= \alpha_r \beta_s + \alpha_s \beta_r & M^r_s &= \alpha^r \alpha_s - \beta_s \beta^r \\
S^{rs} &= \beta^r \alpha^s + \beta^s \alpha^r &= (S_{rs})^\dagger \\
\Pi_{mr} &= a_m \beta_r - b_m \alpha_r & \Pi^{mr} &= (\Pi_{mr})^\dagger &= a^m \beta^r - b^m \alpha^r \\
\Sigma^m_r &= a_m \alpha^r + b_m \beta^r & \Sigma^{m}_r &= (\Sigma^m_r)^\dagger &= a^m \alpha_r + b^m \beta_r
\end{align*}
\]

In the undeformed minimal unitary realization, the quadratic Casimir of the subsuperalgebra $\text{osp}(4^*|2N)$ turns out to be equal to the quadratic Casimir of the diagonal $\text{su}(2)$ subalgebra of $\text{su}(2)_S$ and $\text{su}(2)_F$, that commutes with $\text{so}^*(4)$, modulo an additive constant that depends on $N$ [45]. However, in the deformed realization, no such simple relation holds between the quadratic Casimir of $\text{osp}(4^*|2N)$ and that of $\text{su}(2)_\tau$. In the deformed minimal unitary realization, the quadratic Casimir of the subsuperalgebra $\text{osp}(4^*|2N)_D$ is given by
\[
C_2[\text{osp}(4^*|2N)] = C_2[\text{su}(2)_\tau] + C_2[\text{su}(2)_G] - 2 \left[ T_0 G_0 + \frac{1}{2} (T_+ G_- + T_- G_+) \right]
\]
\[
- \frac{N (N - 4)}{16}.
\]

The single bosonic generator in grade −2 subspace and 8 bosonic operators plus 4N supersymmetry generators in grade −1 subspace of the deformed minimal realization of $\text{osp}(8^*|2N)_D$ are also unchanged:
\[
\begin{align*}
\mathcal{K}_- = K_- &= \frac{1}{2} x^2
\end{align*}
\]
\[
\begin{align*}
\mathcal{U}_m = U_m = x a_m & \quad \mathcal{U}^m = U^m = x a^m \\
\mathcal{V}_m = V_m = x b_m & \quad \mathcal{V}^m = V^m = x b^m
\end{align*}
\]
They form a super Heisenberg algebra by (anti-)commuting into $\mathcal{K}_-$. However, since $\mathfrak{su}(2)_S$ has now been extended to $\mathfrak{su}(2)_T$, the grade +2 generator now depends on $T^2$ and is given by

$$\mathcal{K}_+ = \frac{1}{2} p^2 + \frac{1}{4 x^2} \left( 8 T^2 + \frac{3}{2} \right).$$

(3.12)

Therefore the generators in grade +1 subspace get modified, since they are obtained from the commutators of the form $[\mathfrak{g}_D^{(1)}, \mathfrak{g}_D^{(2)}]$:

$$\tilde{U}_m = i [U_m, \mathcal{K}_+] \quad \quad \tilde{U}^m = \left( \tilde{U}_m \right)^\dagger = i [U^m, \mathcal{K}_+]$$

$$\tilde{V}_m = i [V_m, \mathcal{K}_+] \quad \quad \tilde{V}^m = \left( \tilde{V}_m \right)^\dagger = i [V^m, \mathcal{K}_+]$$

$$\tilde{Q}_r = i [Q_r, \mathcal{K}_+] \quad \quad \tilde{Q}^r = \left( \tilde{Q}_r \right)^\dagger = i [Q^r, \mathcal{K}_+]$$

$$\tilde{S}_r = i [S_r, \mathcal{K}_+] \quad \quad \tilde{S}^r = \left( \tilde{S}_r \right)^\dagger = i [S^r, \mathcal{K}_+]$$

(3.13)

The explicit form of these 8 bosonic generators and 4\$N$ supersymmetry generators of grade +1 subspace are as follows:

$$\tilde{U}_m = -p a_m + \frac{2i}{x} \left[ \left( T_0 + \frac{3}{4} \right) a_m + T_- b_m \right]$$

$$\tilde{U}^m = -p a^m - \frac{2i}{x} \left[ \left( T_0 - \frac{3}{4} \right) a^m + T_+ b^m \right]$$

$$\tilde{V}_m = -p b_m - \frac{2i}{x} \left[ \left( T_0 - \frac{3}{4} \right) b_m - T_+ a_m \right]$$

$$\tilde{V}^m = -p b^m + \frac{2i}{x} \left[ \left( T_0 + \frac{3}{4} \right) b^m - T_- a^m \right]$$

$$\tilde{Q}_r = -p \alpha_r + \frac{2i}{x} \left[ \left( T_0 + \frac{3}{4} \right) \alpha_r + T_- \beta_r \right]$$

$$\tilde{Q}^r = -p \alpha^r - \frac{2i}{x} \left[ \left( T_0 - \frac{3}{4} \right) \alpha^r + T_+ \beta^r \right]$$

$$\tilde{S}_r = -p \beta_r - \frac{2i}{x} \left[ \left( T_0 - \frac{3}{4} \right) \beta_r - T_+ \alpha_r \right]$$

$$\tilde{S}^r = -p \beta^r + \frac{2i}{x} \left[ \left( T_0 + \frac{3}{4} \right) \beta^r - T_- \alpha^r \right]$$

(3.14)

These grade +1 generators (anti-)commute into the grade +2 generator and form a super Heisenberg algebra.
The anticommutators between the supersymmetry generators in $\mathfrak{g}^{(-1)}_D$ and $\mathfrak{g}^{(+1)}_D$ given above close into the bosonic generators in $\mathfrak{g}^{(0)}_D$:

\[
\begin{align*}
\left\{ Q_r, \bar{Q}_s \right\} &= 0 & \left\{ Q_r, \bar{S}_s \right\} &= -2i S_{rs} \\
\left\{ Q_r, \bar{Q}_s^g \right\} &= -\delta^g_r \Delta - 2i \delta^g_r T_0 + 2i M^g_r & \left\{ Q_r, \bar{S}_s^g \right\} &= -2i \delta^g_r T_-\\
\left\{ S_r, \bar{S}_s \right\} &= 0 & \left\{ S_r, \bar{Q}_s \right\} &= +2i S_{rs} \\
\left\{ S_r, \bar{Q}_s^g \right\} &= -\delta^g_r \Delta + 2i \delta^g_r T_0 + 2i M^g_r & \left\{ S_r, \bar{S}_s^g \right\} &= -2i \delta^g_r T_+
\end{align*}
\]

(3.17)

The commutators between the bosonic (even) and fermionic (odd) generators of $\mathfrak{g}^{(-1)}_D$ and $\mathfrak{g}^{(+1)}_D$ subspaces close into the fermionic (odd) generators of $\mathfrak{g}^{(0)}_D$:

\[
\begin{align*}
\left[ U_m, \bar{Q}_r \right] &= 0 & \left[ V_m, \bar{Q}_r \right] &= +2i \Pi_{mr} \\
\left[ U_m, \bar{Q}^g_r \right] &= -2i \Sigma^g_m & \left[ V_m, \bar{Q}^g_r \right] &= 0 \\
\left[ U_m, \bar{S}_r \right] &= -2i \Pi_{mr} & \left[ V_m, \bar{S}_r \right] &= 0 \\
\left[ U_m, \bar{S}^g_r \right] &= 0 & \left[ V_m, \bar{S}^g_r \right] &= -2i \Sigma^g_m \\
\left[ Q_r, \bar{U}_m \right] &= 0 & \left[ Q_r, \bar{V}_m \right] &= +2i \Pi_{mr} \\
\left[ Q^g_r, \bar{U}_m \right] &= -2i \Sigma^g_m & \left[ Q^g_r, \bar{V}_m \right] &= 0 \\
\left[ S_r, \bar{U}_m \right] &= -2i \Pi_{mr} & \left[ S_r, \bar{V}_m \right] &= 0 \\
\left[ S^g_r, \bar{U}_m \right] &= 0 & \left[ S^g_r, \bar{V}_m \right] &= -2i \Sigma^g_m
\end{align*}
\]

(3.18)

(3.19)

Thus the 5-grading of the Lie superalgebra $\mathfrak{osp}(8^*|2N)_D$, defined by the generator $\Delta$, takes the form:

\[
\mathfrak{osp}(8^*|4)_D = \mathfrak{g}^{(2)}_D \oplus \mathfrak{g}^{(1)}_D \oplus \left[ \mathfrak{osp}(4^*|2N) \oplus \mathfrak{su}(2)_T \oplus \mathfrak{so}(1,1)_D \right] \oplus \mathfrak{g}^{(+1)}_D \oplus \mathfrak{g}^{(+2)}_D
\]

\[
\mathcal{K}_- \oplus \left[ U_m, U^m, V_m, V^m, Q_r, Q^g_r, S_r, S^g_r \right] \\
\oplus \left[ A_{\pm,0}, N_{\pm,0}, S_{rs}, M_{rs}, r^{rs}, \Pi_{mr}, \Pi^{mr}, \Sigma^m_r, \Sigma^g_m, T_{\pm,0}, \Delta \right] \\
\oplus \left[ \bar{U}_m, \bar{U}^m, \bar{V}_m, \bar{V}^m, \bar{Q}_r, \bar{Q}^g_r, \bar{S}_r, \bar{S}^g_r \right] \oplus \mathcal{K}_+
\]

(3.20)

The quadratic Casimir of $\mathfrak{osp}(8^*|2N)_D$ is given by

\[
\mathcal{C}_2 [\mathfrak{osp}(8^*|2N)_D] = \mathcal{C}_2 [\mathfrak{osp}(4^*|2N)] - \mathcal{C}_2 [\mathfrak{su}(2)_T] + \mathcal{C}_2 [\mathfrak{su}(1,1)_{\mathcal{K}}] \\
- \frac{i}{8} \mathcal{F}_1 (U, V) + \frac{i}{8} \mathcal{F}_2 (Q, S)
\]

(3.21)
where

\[ F_1 (U, V) = \left( U_m \tilde{U}_m + V_m \tilde{V}_m + \tilde{U}_m U_m + \tilde{V}_m V_m \right) \]
\[ - \left( U_m \tilde{U}_m + V_m \tilde{V}_m + U_m \tilde{U}_m + \tilde{V}_m V_m \right) \]
\[ F_2 (Q, S) = \left( Q_r \tilde{Q}_r + S_r \tilde{S}_r - \tilde{Q}_r Q_r - \tilde{S}_r S_r \right) \]
\[ + \left( Q_r \tilde{Q}_r + S_r \tilde{S}_r - \tilde{Q}_r Q_r - \tilde{S}_r S_r \right) \]

and reduces to

\[ C_2 [osp(8^*|2N)_D] = G^2 - \frac{1}{16} (N^2 - 20 N + 32) \]

where \( G^2 \) is the quadratic Casimir of \( su(2)_G \). Hence each deformed irreducible minimal unitary supermultiplet of \( osp(8^*|2N) \) can be labeled by the eigenvalues \( g(g+1) \) of \( G^2 \) as we shall explicitly show later.

4. The Compact 3-Grading of \( OSp(8^*|2N)_D \)

The Lie superalgebra \( osp(8^*|2N)_D \) can be given a 3-graded decomposition with respect to its compact subsuperalgebra \( u(4|N) = su(4|N) \oplus u(1) \):

\[ osp(8^*|2N)_D = C_D^- \oplus C_D^0 \oplus C_D^+ \]

where

\[ C_D^- = \frac{1}{2} \left( U_m - i \tilde{U}_m \right) \oplus \frac{1}{2} \left( V_m - i \tilde{V}_m \right) \oplus N_- \oplus \frac{i}{2} \left[ \Delta + i \left( \kappa_+ - \kappa_- \right) \right] \oplus S_{rs} \]
\[ \oplus \frac{1}{2} \left( Q_r - i \tilde{Q}_r \right) \oplus \frac{1}{2} \left( S_r - i \tilde{S}_r \right) \oplus \Pi_{mr} \]
\[ C_D^0 = \left[ T_{\pm,0} \oplus A_{\pm,0} \left[ N_0 - \frac{1}{2} \left( \kappa_+ + \kappa_- \right) \right] \right] \oplus \frac{1}{2} \left( U_m + i \tilde{U}_m \right) \oplus \frac{1}{2} \left( V_m - i \tilde{V}_m \right) \]
\[ \oplus \frac{1}{2} \left( V_m + i \tilde{V}_m \right) \oplus \frac{1}{2} \left( V_m - i \tilde{V}_m \right) \oplus M_s \oplus \left[ \frac{1}{2} \left( \kappa_+ + \kappa_- \right) + \frac{2}{N} M_0 \right] \oplus \tilde{\gamma} \]
\[ C_D^+ = \frac{1}{2} \left( U_m + i \tilde{U}_m \right) \oplus \frac{1}{2} \left( V_m + i \tilde{V}_m \right) \oplus N_+ \oplus \frac{i}{2} \left[ \Delta - i \left( \kappa_+ - \kappa_- \right) \right] \oplus S^{rs} \]
\[ \oplus \frac{1}{2} \left( Q_r + i \tilde{Q}_r \right) \oplus \frac{1}{2} \left( S_r + i \tilde{S}_r \right) \oplus \frac{1}{2} \left( S^{rs} - i \tilde{S}^{rs} \right) \oplus \Sigma_m \oplus \Sigma_{mr} \]

The \( u(1) \) generator \( \tilde{\gamma} \) that defines the compact 3-grading of \( osp(8^*|2N)_D \) is given by

\[ \tilde{\gamma} = \frac{1}{2} \left( \kappa_+ + \kappa_- \right) + N_0 + M_0 \]
\[ = \frac{1}{4} \left( x^2 + p^2 \right) + \frac{1}{x^2} \left( T^2 + \frac{3}{16} \right) + \frac{1}{2} \left( N_a + N_b + N_a + N_\beta \right) + \frac{2 - N}{2} \]
and plays the role of the “total energy” operator for the deformed minrep of $\mathfrak{osp}(8^*|2N)$.

In the supersymmetric extension of the deformed minrep, the $\mathfrak{u}(1)$ generator that corresponds to the $\text{AdS}_7$ energy and determines a 3-grading of $\mathfrak{so}^*(8)_D$ is given by

$$
\mathcal{H} = \frac{1}{2} (\mathcal{K}_+ + \mathcal{K}_-) + N_0 \\
= \frac{1}{4} (x^2 + p^2) + \frac{1}{x^2} \left( T^2 + \frac{3}{16} \right) + \frac{1}{2} (N_a + N_b) + 1
$$

(4.4)

where $\mathcal{H}_\circ$ is the Hamiltonian of the singular oscillator:

$$
\mathcal{H}_\circ = \frac{1}{2} (\mathcal{K}_+ + \mathcal{K}_-) = \frac{1}{4} (x^2 + p^2) + \frac{1}{x^2} \left( T^2 + \frac{3}{16} \right)
$$

(4.5)

and $H_a$ and $H_b$ are the Hamiltonians corresponding to $a$- and $b$-type bosonic oscillators, respectively:

$$
H_a = \frac{1}{2} (N_a + 1) \quad H_b = \frac{1}{2} (N_b + 1)
$$

We shall label the bosonic operators that belong to the subspace $\mathfrak{C}^-_D$ of the deformed $\mathfrak{osp}(8^*|2N)_D$ in the compact 3-grading as follows:

$$
\mathcal{Y}_m = \frac{1}{2} \left( \mathcal{U}_m - i \tilde{\mathcal{U}}_m \right) = \frac{1}{2} (x + ip) a_m + \frac{1}{x} \left[ \left( T_0 + \frac{3}{4} \right) a_m + T_- b_m \right] \\
\mathcal{Z}_m = \frac{1}{2} \left( \mathcal{V}_m - i \tilde{\mathcal{V}}_m \right) = \frac{1}{2} (x + ip) b_m - \frac{1}{x} \left[ \left( T_0 - \frac{3}{4} \right) b_m - T_+ a_m \right] \\
N_- = a_1 b_2 - a_2 b_1 \\
\mathcal{B}_- = \frac{1}{2} \left[ \Delta + i (\mathcal{K}_+ - \mathcal{K}_-) \right] = \frac{1}{4} (x + ip)^2 - \frac{1}{x^2} \left( T^2 + \frac{3}{16} \right) \\
S_{rs} = \alpha_r \beta_s + \alpha_s \beta_r
$$

(4.6)

and the $4N$ supersymmetry generators in $\mathfrak{C}_D^-$ subspace as:

$$
\mathcal{Q}_r = \frac{1}{2} \left( \mathcal{Q}_r - i \tilde{\mathcal{Q}}_r \right) = \frac{1}{2} (x + ip) \alpha_r + \frac{1}{x} \left[ \left( T_0 + \frac{3}{4} \right) \alpha_r + T_- \beta_r \right] \\
\mathcal{S}_r = \frac{1}{2} \left( \mathcal{S}_r - i \tilde{\mathcal{S}}_r \right) = \frac{1}{2} (x + ip) \beta_r - \frac{1}{x} \left[ \left( T_0 - \frac{3}{4} \right) \beta_r - T_+ \alpha_r \right] \\
\Pi_{mr} = a_m \beta_r - b_m \alpha_r
$$

(4.7)

The generators that belong to $\mathfrak{C}_D^+$ subspace are the Hermitian conjugates of those in $\mathfrak{C}_D^-$. 

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Then the bosonic operators in $\mathcal{C}_D^+$ are:

\[ Y^m = \frac{1}{2} \left( U^m + i \tilde{U}^m \right) = \frac{1}{2} (x - i p) a^m + \frac{1}{x} \left[ \left( \mathcal{T}_0 - \frac{3}{4} \right) a^m + \mathcal{T}_+ b^m \right] \]

\[ Z^m = \frac{1}{2} \left( V^m + i \tilde{V}^m \right) = \frac{1}{2} (x - i p) b^m - \frac{1}{x} \left[ \left( \mathcal{T}_0 + \frac{3}{4} \right) b^m - \mathcal{T}_- a^m \right] \]

\[ N_+ = a^1 b^2 - a^2 b^1 \]

\[ B_+ = -i \frac{p}{2} \left[ \Delta - i (K_+ - K_-) \right] = \frac{1}{4} (x - i p)^2 - \frac{1}{x^2} \left( \mathcal{T}^2 + \frac{3}{16} \right) \]

\[ S^{rs} = \alpha^r \beta^s + \alpha^s \beta^r \]

and the $4N$ supersymmetry generators in $\mathcal{C}_D^+$ subspace are:

\[ \Omega^r = \frac{1}{2} \left( Q^r + i \tilde{Q}^r \right) = \frac{1}{2} (x - i p) \alpha^r + \frac{1}{x} \left[ \left( \mathcal{T}_0 - \frac{3}{4} \right) \alpha^r + \mathcal{T}_+ \beta^r \right] \]

\[ \Theta^r = \frac{1}{2} \left( S^r + i \tilde{S}^r \right) = \frac{1}{2} (x - i p) \beta^r - \frac{1}{x} \left( \mathcal{T}_0 + \frac{3}{4} \right) \beta^r - \mathcal{T}_- \alpha^r \]

\[ \Pi^{mr} = a^m \beta^r - b^m \alpha^r \]

(4.8)

Since the quadratic Casimir of $\text{OSp}(8^*|2N)_D$ depends only on the quadratic Casimir of $\text{SU}(2)_G$ constructed out of deformation fermions $\xi$ and $\chi$ (see equation (3.23)), just as the quadratic Casimir of $\text{SO}^*(8)_D$ depends only on the quadratic Casimir of $\text{SU}(2)_G$ (see equation (2.35)), one expects to obtain an irreducible unitary supermultiplet of $\text{OSp}(8^*|2N)$ for each spin $g$ labeling the irreps of $\text{SU}(2)_G$. Let us show that this indeed is the case. For each $\text{SU}(2)_T$ spin $t = g = \frac{P}{2} \neq 0$ there is a multiplet of states that are annihilated by all the operators in grade $-1$ subspace $\mathcal{C}_D^-$ and transforms irreducibly under the supersuperalgebra $u(4|N)$, which is the grade zero subspace $\mathcal{C}_D^0$. Let us call this supermultiplet of states the “lowest energy K-type” of $\text{OSp}(8^*|2N)_D$. To obtain this lowest energy K-type for a given $t = g$, consider the tensor product of states of the form

\[ |\psi_n^{(\alpha)}; n_{a,1}, n_{a,2}, n_{b,1}, n_{b,2}; \frac{P}{2}, k - \frac{P}{2} \rangle \]

constructed earlier, with the states created by the supersymmetry fermions

\[ \alpha^{\alpha_1} \ldots \alpha^{\alpha_n} \beta^{\alpha_{n+1}} \ldots \beta^{\alpha_{n+p}} |0\rangle_F \]

(4.9)

5. Deformed Minimal Unitary Representations of $\text{OSp}(8^*|2N)$ as $6D$ Massless Conformal Supermultiplets

Once again, we have the important relation

\[ Y^1 Z^2 - Y^2 Z^1 = N_+ B_+ \]

(4.10)
where \( |0\rangle_F \) is the fermionic Fock vacuum annihilated by all the fermionic annihilation operators \( \xi_x, \chi_x, \alpha_r, \) and \( \beta_r \) \((x = 1, \ldots, P \) and \( r = 1, \ldots, N \)), and denote them as

\[
|\psi_n^{(\alpha_1)}; n_{a,1}, n_{a,2}, n_{b,1}, n_{b,2}; n_{\alpha}, n_{\beta}; P/2, k - P/2\rangle.
\] (5.1)

The following \((P + 1)\) states of the form

\[
|\psi_0^{(\alpha)}; 0, 0, 0, 0; 0, 0; P/2, k - P/2\rangle \quad (k = 0, 1, \ldots, P)
\] (5.2)

transform in the \((P/2, 0)\) representation of \( SU(2)_T \times SU(2)_A \) with a definite eigenvalue \( J = -P/2 \) with respect to the \( U(1)_J \) generator \( J = N_0 - \frac{1}{2}(K_+ + K_-) \), and are annihilated by the six bosonic operators and \( N(N + 1)/2 \) supersymmetry generators in grade \(-1\) subspace \( \mathcal{C}_D^- \) of \( \mathfrak{osp}(8^*|2N)_D \) (given in equations (4.6) and (4.7)) for the choice

\[
\alpha_t = 2t + \frac{3}{2} = P + \frac{3}{2}.
\] (5.3)

These states uniquely define a positive energy unitary supermultiplet of \( OSp(8^*|2N) \), labeled by the \( SU(2)_G \) spin \( g = t = P/2 \), which corresponds to a deformation of the minimal unitary supermultiplet. By acting on the states in equation (5.2) with the coset generators

\[
SU(4|N)/[SU(2)_T \times SU(2)_A \times U(1)_J]
\]

one obtains a set of states transforming in an irreducible representation of \( SU(4|N) \) with the Young supertableau \( \begin{array}{c} \Lambda \end{array} \). We denote these states as

\[
|\Omega^{(P+3/2)}_{P=2t}; \begin{array}{c} \Lambda \end{array}\rangle
\]

For a given value of the deformation parameter \( t = g \), they have the lowest “total energy” \( \mathcal{H} = t + 1 = g + 1 = P/2 + 1 \) and are all annihilated by grade \(-1\) operators. Hence they form a lowest energy K-type of \( OSp(8^*|2N) \). By repeatedly acting on this set of states

\[
|\Omega^{(P+3/2)}_{P=2t}; \begin{array}{c} \Lambda \end{array}\rangle
\]

with the operators in grade \(+1\) subspace \( \mathcal{C}_D^+ \), one obtains an infinite set of states that form a basis of a unitary irreducible representation of \( OSp(8^*|2N) \). This infinite set of states can be decomposed into a finite number of irreducible representations of the even subgroup \( SO^*(8) \times USp(2N) \), with each irrep of \( SO^*(8) \) corresponding to a massless conformal field in six dimensions.

In Table 2, we present the general deformed minimal unitary supermultiplet of \( \mathfrak{osp}(8^*|2N) \) corresponding to the deformation parameter \( t = g \), obtained by starting from the lowest energy K-type

\[
|\Omega^{(P+3/2)}_{P=2t=2g}; \begin{array}{c} \Lambda \end{array}\rangle.
\]

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Table 2: The general deformed minimal unitary supermultiplet of osp(8∗|2N) corresponding to the deformation parameter \( t = g = P/2 \). First column gives the AdS energy \( \mathcal{H} \) which is equal to negative conformal dimension \( \ell \) of the corresponding conformal field. Last column gives the SU(4) Dynkin labels of the lowest energy K-type of SO∗(8), which coincides with the Dynkin labels of the corresponding conformal field under the Lorentz group SU∗(4), and the Dynkin labels with respect to USp(2N). The decomposition of SU(4) irreps with respect to SU(2)_T × SU(2)_A × U(1)_J is denoted by \((t, a)^3\) and is listed in the third column. Second column gives the eigenvalues of “total energy” \( \tilde{\mathcal{H}} \). Note that for \( P < N \) the states with negative entries in their Dynkin labels do not occur.

| \( \mathcal{H} = -\ell \) | \( \tilde{\mathcal{H}} \) | \((t, a)^3\) | \( SU^∗(4)_{\text{Dynkin}} = SU(4)_{\text{Dynkin}} \oplus \text{USp}(2N)_{\text{Dynkin}} \) |
|--------------------------|--------------------------|--------------------------|----------------------------------------------------------------------|
| \( t + 2 \) | \( t + 2 - \frac{N}{2} \) | \((t, 0)^{-t} \oplus (t - \frac{1}{2}, \frac{1}{2})^{-t+1} \oplus \cdots \oplus (0, t)^{t} \) | \((2t, 0, 0)_{SU(4)} \oplus (0, \ldots, 0, 1)_{USp(2N)}^{(N-1)} \) |
| \( t + \frac{5}{2} \) | \( t + 3 - \frac{N}{2} \) | \((t + \frac{1}{2}, 0)^{-t-\frac{1}{2}} \oplus (t, \frac{1}{2})^{-t+\frac{1}{2}} \oplus \cdots \oplus (0, t + \frac{1}{2})^{t+\frac{1}{2}} \) | \((2t + 1, 0, 0)_{SU(4)} \oplus (0, \ldots, 0, 1, 1)_{USp(2N)}^{(N-2)} \) |
| \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) |
| \( t + 2 + \frac{N}{2} \) | \( t + 2 + \frac{N}{2} \) | \((t + \frac{N}{2}, 0)^{-t-\frac{N}{2}} \oplus (t + \frac{N-1}{2}, \frac{1}{2})^{-t-\frac{N-2}{2}} \oplus \cdots \oplus (0, t + \frac{N}{2})^{t+\frac{N}{2}} \) | \((2t + N, 0, 0)_{SU(4)} \oplus (0, \ldots, 0)_{USp(2N)}^{N} \) |
| \( t + \frac{3}{2} \) | \( t + 2 - \frac{N}{2} \) | \((t - \frac{1}{2}, 0)^{-t+\frac{3}{2}} \oplus (t - 1, \frac{1}{2})^{-t+\frac{3}{2}} \oplus \cdots \oplus (0, t - \frac{1}{2})^{t-\frac{1}{2}} \) | \((2t - 1, 0, 0)_{SU(4)} \oplus (0, \ldots, 0, 1, 0)_{USp(2N)}^{(N-2)} \) |
| $\mathcal{H} = -\ell$ | $\delta$ | $(t, a)^3$ | $SU^*(4)_{\text{Dynkin}} = SU(4)_{\text{Dynkin}}$ $USp(2N)_{\text{Dynkin}}$ |
|---|---|---|---|
| $t + 2$ | $t + 3 - \frac{N}{2}$ | $(t, 0)^{-t} \oplus (t - \frac{1}{2}, \frac{1}{2})^{-t+1}$ $\oplus \cdots \oplus (0, t)^1$ | $(2t, 0, 0)_{SU(4)}$ $(0, \ldots, 0, 1, 0, 0)_{USp(2N)}^{(N-3)}$ |
| $t + 1 + \frac{N}{2}$ | $t + 1 + \frac{N}{2}$ | $(t + \frac{N-2}{2}, 0)^{-t}^{-\frac{N-2}{2}}$ $\oplus (t + \frac{N-3}{2}, \frac{1}{2})^{-t}^{-\frac{N-4}{2}}$ $\oplus \cdots \oplus (0, t + \frac{N-2}{2})^{1}^{1}^{-\frac{N-2}{2}}$ | $(2t - 2 + N, 0, 0)_{SU(4)}$ $(0, \ldots, 0)_{USp(2N)}^{N}$ |
| $t + 2 - \frac{n}{2}$ | $t + 2 - \frac{n}{2}$ | $(t - \frac{n}{2}, 0)^{-t}^{-\frac{n}{2}}$ $\oplus (t - \frac{n+1}{2}, \frac{1}{2})^{-t}^{-\frac{n+2}{2}}$ $\oplus \cdots \oplus (0, t - \frac{n}{2})^{1}^{1}^{-\frac{n}{2}}$ | $(2t - n, 0, 0)_{SU(4)}$ $(0, \ldots, 0, 1, 0, \ldots, 0)_{USp(2N)}^{(N-n-1)}$ |
| $t + \frac{5}{2} - \frac{n}{2}$ | $t + 3 - \frac{N}{2}$ | $(t - \frac{n-1}{2}, 0)^{-t}^{-\frac{n-1}{2}}$ $\oplus (t - \frac{n}{2}, \frac{1}{2})^{-t}^{-\frac{n+1}{2}}$ $\oplus \cdots \oplus (0, t - \frac{n-1}{2})^{1}^{1}^{-\frac{n-1}{2}}$ | $(2t - n + 1, 0, 0)_{SU(4)}$ $(0, \ldots, 0, 1, 0, \ldots, 0)_{USp(2N)}^{(N-n-2)}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
6. Deformed Minimal Unitary Supermultiplets of $OSp(8^*|4)$

Due to its importance as the symmetry superalgebra of the $S^4$ compactification of the eleven dimensional supergravity, we shall discuss the results for the case $N = 2$, i.e $OSp(8^*|4)$, in more detail. The indices $r, s, \ldots$ of $\alpha$- and $\beta$-type (supersymmetry) fermionic oscillators take the values 1,2 in this case.

Recall that the undeformed minimal unitary supermultiplet is obtained by taking the deformation parameter $t = 0$, which we present in Table 3.

| $H = -\ell$ | $\mathcal{F}$ | $(t, a)^3$ | $SU^*(4)_{\text{Dynkin}} = SU(4)_{\text{Dynkin}}$ $\oplus SP(2N)_{\text{Dynkin}}$ |
|-------------|--------------|-------------|---------------------------------|
| $t + 2 - n + \frac{N}{2}$ | $t + 2 - n + \frac{N}{2}$ | $(t - n + \frac{N}{2}, 0)^{-t+n-\frac{N}{2}}$ $\oplus (t - n + \frac{N-1}{2}, \frac{1}{2})^{-t+n-\frac{N+2}{2}}$ $\oplus \cdots \oplus (0, t - n + \frac{N}{2})^{t-n+\frac{N}{2}}$ | $(2t - 2n + N, 0, 0)_{SU(4)}$ $\oplus (0, \ldots, 0)_{USp(2N)}$ |
| $t + 2 - \frac{N}{2}$ | $t + 2 - \frac{N}{2}$ | $(t - \frac{N}{2}, 0)^{-t+\frac{N}{2}}$ $\oplus (t - \frac{N+1}{2}, \frac{1}{2})^{-t+\frac{N+2}{2}}$ $\oplus \cdots \oplus (0, t - \frac{N}{2})^{t-\frac{N}{2}}$ | $(2t - N, 0, 0)_{SU(4)}$ $\oplus (0, \ldots, 0)_{USp(2N)}$ |
Table 3: The minimal unitary supermultiplet of $\mathfrak{osp}(8'4)$ defined by the lowest weight vector $|\Omega^{(5/2)}, 1\rangle$, which corresponds to the deformation parameter $t = 0$. The decomposition of $SU(4)$ irreps with respect to $SU(2)_T \times SU(2)_A \times U(1)_J$ is denoted by $(t, a)^3$. $\mathcal{H}$ is the AdS energy (negative conformal dimension), and $\mathfrak{h}$ is the total energy. The Dynkin labels of the lowest energy $SU(4)$ representations of $SO^*(8)$ coincide with the Dynkin labels of the corresponding massless 6D conformal fields under the Lorentz group $SU^*(4)$. $USp(4)$ Dynkin labels of these fields are also given.

| $\mathcal{H} = -\ell$ | $\mathfrak{h}$ | $(t, a)^3$ | $SU(4) = SU^*(4)$ Dynkin | $USp(4)$ Dynkin |
|-----------------|----------|------------|---------------------|----------------|
| 2               | 1        | $(0, 0)^0$ | $(0,0,0)$           | $(0,1)$        |
| $\frac{5}{2}$   | 2        | $(\frac{1}{2}, 0)^{-\frac{1}{2}} \oplus (0, \frac{1}{2})^{\frac{1}{2}}$ | $(1,0,0)$       | $(1,0)$        |
| 3               | 3        | $(1, 0)^{-1} \oplus (\frac{1}{2}, \frac{1}{2})^0 \oplus (0, 1)^{+1}$ | $(2,0,0)$       | $(0,0)$        |

The simplest deformed case is when the deformation parameter $t = g = \frac{1}{2}$. This makes $\alpha_1 = \frac{5}{2}$. In this case, we must choose only one pair of deformation fermions $\xi^1$ and $\chi^1$ (i.e. $P = 1$). Then we act on states

$$\chi^1 |\psi_0^{(5/2)}\rangle \oplus \xi^1 |\psi_0^{(5/2)}\rangle$$

with the coset generators $SU(4|N) / \left[ SU(2)_T \times SU(2)_A \times U(1)_J \right]$ of grade zero subspace to obtain a set of lowest energy states $|\Omega^{(5/2)}\rangle$ transforming in an irreducible representation of $SU(4|2)$ with the Young supertableau $\square$. Repeatedly acting on these states with the supersymmetry generators in grade +1 subspace $\mathfrak{c}_D^+$, one obtains the supermultiplet given in Table 3.

This supermultiplet coincides with the doubleton supermultiplet given in [46,47] with the lowest energy K-type whose supertableau with respect to $SU(4|2)$ is $|\square\rangle$. 


Table 4: The deformed minimal unitary supermultiplet of $\mathfrak{osp}(8^*|4)$ corresponding to the deformation parameter $t = g = 1/2$. The decomposition of $SU(4)$ irreps with respect to $SU(2)_T \times SU(2)_A \times U(1)_J$ is denoted by $(t, a)^3$. $\mathcal{H}$ is the AdS energy (negative conformal dimension), and $\mathcal{H}$ is the total energy. The $SU(4)$ Dynkin labels and the $USp(4)$ Dynkin labels are also given. The Dynkin labels of the lowest energy $SU(4)$ representations of $SO^*(8)$ coincide with the Dynkin labels of the corresponding massless 6D conformal fields under the Lorentz group $SU^*(4)$.

| $\mathcal{H} = -\ell$ | $\mathcal{H}$ | $(t, a)^3$ | $SU(4)$ Dynkin | $USp(4)$ Dynkin |
|------------------------|--------------|------------|----------------|----------------|
| $\frac{5}{2}$ | $\frac{3}{2}$ | $(\frac{1}{2}, 0)^{-\frac{3}{2}} \oplus (0, \frac{1}{2})^{+\frac{3}{2}}$ | $(1,0,0)$ | $(0,1)$ |
| $3$ | $\frac{5}{2}$ | $(1, 0)^{-1} \oplus (\frac{1}{2}, \frac{1}{2})^0 (0, 1)^{+1}$ | $(2,0,0)$ | $(1,0)$ |
| $\frac{7}{2}$ | $\frac{7}{2}$ | $(\frac{3}{2}, 0)^{-\frac{3}{2}} \oplus (1, \frac{1}{2})^{-\frac{3}{2}} \oplus (\frac{1}{2}, 1)^{+\frac{3}{2}} \oplus (0, \frac{3}{2})^{+\frac{3}{2}}$ | $(3,0,0)$ | $(0,0)$ |
| $2$ | $\frac{3}{2}$ | $(0, 0)^0$ | $(0,0,0)$ | $(1,0)$ |
| $\frac{5}{2}$ | $\frac{5}{2}$ | $(\frac{1}{2}, 0)^{-\frac{3}{2}} \oplus (0, \frac{1}{2})^{+\frac{3}{2}}$ | $(1,0,0)$ | $(0,0)$ |

All higher spin doubleton supermultiplets can be obtained similarly as deformations of the minimal unitary supermultiplet by choosing the deformation parameter $t = g = P/2$ to take on all allowed values $t = 1/2, 1, 3/2, \ldots$. The resulting general deformed supermultiplet (higher spin doubleton) is given in Table 5.

This supermultiplet matches exactly the corresponding doubleton supermultiplet with lowest energy K-type $|\underbrace{\mathcal{J} \cdots \mathcal{J}}_{2t=2g=P}\rangle$ given in [46, 47].
Table 5: The general deformed minimal unitary supermultiplet of $\mathfrak{osp}(8^*|4)$ corresponding to the deformation parameter $t = g = P/2$. The decomposition of $SU(4)$ irreps with respect to $SU(2)_T \times SU(2)_A \times U(1)_J$ is denoted by $(t, a)^J$. $\mathcal{H}$ is the AdS energy (negative conformal dimension), and $H$ is the total energy. The $SU(4)$ Dynkin labels and the $USp(4)$ Dynkin labels are also given. The Dynkin labels of the lowest energy $SU(4)$ representations of $SO^*(8)$ coincide with the Dynkin labels of the corresponding massless 6D conformal fields under the Lorentz group $SU^*(4)$.

| $\mathcal{H} = -\ell$ | $H$ | $(t, a)^J$ | $SU(4)$ Dynkin | $USp(4)$ Dynkin |
|------------------------|-----|------------|----------------|----------------|
| $t + 2$                | $t + 1$ | $(t, 0)^{-1} \oplus \left(t - \frac{1}{2}, \frac{1}{2}\right)^{-t+1} \oplus \ldots \oplus (0, t)^{-1} \oplus \ldots$ | $(2t, 0, 0)$ | $(0, 1)$ |
| $t + \frac{5}{2}$     | $t + 2$ | $(t + \frac{1}{2}, 0)^{-t - \frac{1}{2}} \oplus \left(t, \frac{1}{2}\right)^{-t+\frac{1}{2}} \oplus \ldots \oplus (0, t + \frac{1}{2})^{-t+\frac{1}{2}}$ | $(2t + 1, 0, 0)$ | $(1, 0)$ |
| $t + 3$                | $t + 3$ | $(t + 1, 0)^{-t-1} \oplus \left(t + \frac{1}{2}, \frac{1}{2}\right)^{-t+\frac{1}{2}} \oplus \ldots \oplus (0, t + 1)^{-t+1} \oplus \ldots$ | $(2t + 2, 0, 0)$ | $(0, 0)$ |
| $t + \frac{3}{2}$     | $t + 1$ | $(t - \frac{1}{2}, 0)^{-t+\frac{1}{2}} \oplus \left(t - 1, \frac{1}{2}\right)^{-t+\frac{1}{2}} \oplus \ldots \oplus (0, t - \frac{1}{2})^{-t+\frac{1}{2}}$ | $(2t - 1, 0, 0)$ | $(1, 0)$ |
| $t + 2$                | $t + 2$ | $(t, 0)^{-1} \oplus \left(t - \frac{1}{2}, \frac{1}{2}\right)^{-t+1} \oplus \ldots \oplus (0, t)^{-1} \oplus \ldots$ | $(2t, 0, 0)$ | $(0, 0)$ |
| $t + 1$                | $t + 1$ | $(t - 1, 0)^{-t+1} \oplus \left(t - \frac{3}{2}, \frac{1}{2}\right)^{-t+2} \oplus \ldots \oplus (0, t - 1)^{-t+1} \oplus \ldots$ | $(2t - 2, 0, 0)$ | $(0, 0)$ |

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Appendix

A. Construction of Finite-Dimensional Representations of $USp(2N)$ in terms of Fermionic Oscillators

To realize the generators of the compact Lie algebra $\mathfrak{usp}(2N)$, we define two new sets of $N$ fermionic oscillators $\alpha_r, \beta_r$ and their hermitian conjugates $\alpha^r = (\alpha_r)^\dagger, \beta^r = (\beta_r)^\dagger$ ($r = 1, 2, \ldots, N$), such that they satisfy the usual anti-commutation relations:

$$\{\alpha_r, \alpha^s\} = \{\beta_r, \beta^s\} = \delta_r^s \quad \{\alpha_r, \alpha_s\} = \{\alpha_r, \beta_s\} = \{\beta_r, \beta_s\} = 0 \quad (A.1)$$

The Lie algebra $\mathfrak{usp}(2N)$ has a 3-graded decomposition with respect to its subalgebra $\mathfrak{u}(N)$ as follows:

$$\mathfrak{usp}(2N) = \mathfrak{g}^{(-1)} \oplus \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(+1)} = S_{rs} \oplus M^r_s \oplus S^{rs} \quad (A.2)$$

where the generators $M^r_s$ form the $\mathfrak{u}(N)$ subalgebra. They can be realized as bilinears of the fermionic oscillators:

$$S_{rs} = \alpha_r \beta_s + \alpha_s \beta_r$$
$$M^r_s = \alpha^r \alpha_s - \beta^r \beta_s$$
$$S^{rs} = \beta^r \alpha^s + \beta^s \alpha^r = (S_{rs})^\dagger \quad (A.3)$$

The $\mathfrak{usp}(2N)$ generators satisfy the following commutation relations:

$$[S_{rs}, S_{tu}] = -\delta^u_s M^r_t - \delta^t_s M^r_u - \delta^u_r M^t_s - \delta^t_r M^u_s$$
$$[M^r_s, S_{tu}] = -\delta^u_s S_{st} - \delta^t_s S_{su}$$
$$[M^r_s, S^{tu}] = \delta^u_s S^{rt} + \delta^t_s S^{ru}$$
$$[M^r_s, M^t_u] = \delta^u_s M^r_t - \delta^t_s M^r_u \quad (A.4)$$

The quadratic Casimir of $\mathfrak{usp}(2N)$ is given by

$$C_2[\mathfrak{usp}(2N)] = M^r_s M^r_s + \frac{1}{2} (S_{rs} S^{rs} + S^{rs} S_{rs})$$
$$= N (N + 2) - (N_\alpha + N_\beta) [(N_\alpha + N_\beta) + 2] - 8 \alpha^r (\beta^s \alpha^r \beta^s) \quad (A.5)$$

Note that realizing the generators of $USp(2N)$ as bilinears of a single pair of fermionic oscillators leads to a finite set of irreps, which are the compact analogs of "doubleton" irreps. To construct more general irreps of $USp(2N)$ one needs to take an arbitrary number (color) of pairs of these oscillators and sum over the color index. See [52] for a general treatment.
where “(rs)” represents symmetrization of weight one, \( \alpha (r \beta)_s = \frac{1}{2} (\alpha_r \beta_s + \alpha_s \beta_r) \).

We choose the Fock vacuum of these fermionic oscillators such that

\[
\alpha_r |0\rangle_F = \beta_r |0\rangle_F = 0.
\]  

(A.6)

To generate an irrep of \( USp(2N) \) in this Fock space in a \( U(N) \) basis, one chooses a set of states \( |\Omega\rangle \), transforming irreducibly under \( U(N) \) and is annihilated by all grade \(-1\) generators \( S_{rs} \), and act on it with grade \(+1\) generators \( S_{rs} \) [52].

The possible sets of states \( |\Omega\rangle \), that transform irreducibly under \( U(N) \) and are annihilated by \( S_{rs} \), are of the form

\[
\alpha^{r_1} \alpha^{r_2} \ldots \alpha^{r_m} |0\rangle_F
\]  

(A.7)

or of the equivalent form

\[
\beta^{r_1} \beta^{r_2} \ldots \beta^{r_m} |0\rangle_F.
\]  

(A.8)

where \( m \leq N \). They lead to irreps of \( USp(2N) \) with Dynkin labels [52]

\[
(0, \ldots, 0, 1, 0, \ldots, 0) \quad (m).
\]  

(A.9)

In addition, we have the following states

\[
\alpha^{[r} \beta^{s]} |0\rangle_F = \frac{1}{2} (\alpha^{r} \beta^{s} - \alpha^{s} \beta^{r}) |0\rangle_F.
\]  

(A.10)

that are annihilated by all grade \(-1\) generators \( S_{tu} \). They lead to the irrep of \( USp(2N) \) with Dynkin labels

\[
(0, \ldots, 0, 1, 0, 0) \quad (N-m-1).
\]  

(A.11)

Note that in the special case of \( usp(4) \), the states \( \alpha^{r} \alpha^{s} |0\rangle_F, \beta^{r} \beta^{s} |0\rangle_F \) and \( \alpha^{[r} \beta^{s]} |0\rangle_F \) all lead to the trivial representation.

Also note that the following bilinears of these \( \alpha \)- and \( \beta \)-type fermionic oscillators:

\[
F_+ = \alpha^r \beta_r, \\
F_- = \beta^r \alpha_r, \\
F_0 = \frac{1}{2} (N_\alpha - N_\beta)
\]  

(A.12)

where \( N_\alpha = \alpha^r \alpha_r \) and \( N_\beta = \beta^r \beta_r \) are the respective number operators, generate a \( usp(2) \) \( \cong \) \( su(2) \) algebra

\[
[F_+, F_-] = 2 F_0 \\
[F_0, F_\pm] = \pm F_\pm
\]  

(A.13)

that commutes with the \( usp(2N) \) algebra defined above. Nonetheless, the equivalent irreps of \( USp(2N) \) constructed from the states \( |\Omega\rangle \) involving only \( \alpha \)-type excitations or \( \beta \)-type excitations can form non-trivial representations of this \( USp(2) \).
$USp(2)_F$. The irrep of $USp(4)$ with Dynkin labels $(0, 1)$ defined by the vacuum state $|\Omega\rangle = |0\rangle$ is a singlet of $USp(2)_F$.

We should stress that the representations of $USp(2N)$ obtained above by using two sets of fermionic oscillators transforming in the fundamental representation of the subgroup $U(N)$ are the compact analogs of the doubleton representations of $SO^*(2M)$ constructed using two sets of bosonic oscillators transforming in the fundamental representation of $U(M)$ [52]. By realizing the generators of $USp(2N)$ in terms of an arbitrary (even) number of sets of oscillators, one can construct all the finite dimensional representations of $USp(2N)$ [7, 52].

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