Topological Josephson $\phi_0$-junctions

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We study the effect of a magnetic field on the current-phase relation of a topological Josephson junction formed by connecting two superconductors through the helical edge states of a quantum spin-Hall insulator. We predict that the Zeeman effect along the spin quantization axis of the helical edges results in an anomalous Josephson relation that allows for a supercurrent to flow in the absence of superconducting phase bias. We relate the associated field-tunable phase shift $\phi_0$ in the Josephson relation of such a $\phi_0$-junction to the existence of a so-called helical superconductivity, which may result from the interplay of the Zeeman effect and spin-orbit coupling. We analyze the dependence of the magneto-supercurrent on the junction length and discuss its observability in suitably designed hybrid structures subject to an in-plane magnetic field.

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Introduction. The topological properties of quantum spin-Hall insulators (QSHI) manifest themselves in current-carrying helical edge states, characterized by a locking of the group velocity to the spin orientation of the edge excitations [1–5]. The effective one-dimensional (1D) superconductivity that may be induced in those edge states by the proximity effect due to conventional superconductors (S) is predicted to also be topological [6]. This is expected to lead to a fractional Josephson effect in S-QSHI-S junctions due to the presence of a topologically protected 4π-periodic Andreev bound state [7]. Recent experiments using HgTe/CdTe [9] or InAs/GaSb [10] as the QSHI, in contact with conventional superconducting leads, have demonstrated that the Josephson current is mainly carried by edge states in the QSH regime. However, a clear signature of the topological superconductivity induced in those states is still lacking.

The role of a Zeeman field transverse to the spin quantization axis of the edge states has been discussed intensively. In the “bulk” it may induce a transition from topologically trivial superconductivity [7–9] whereas a local field in the junction area acts as an effective barrier [6]. Much less is known about the role of a Zeeman field parallel to the spin quantization axis. In this Letter, we show that it induces an anomalous Josephson effect, that is, a supercurrent flow when no superconducting phase bias is applied to the junction. Thus, S-QSHI-S junctions realize so-called $\phi_0$-junctions, where the current-phase relation has a phase-shift $\phi_0$ that is tunable with an external magnetic field. We show that the phase shift depends both on the field in the superconductor and the field in the junction area via two different though related mechanisms. Note that $\phi_0$-junctions have been predicted previously to occur in topologically trivial systems, where both helicities are present, due to an interplay between spin-orbit coupling and a Zeeman field in the junction area [12–14]. However, in that case a partial compensation between the two helicities occurs and only a residual effect proportional to the mismatch in their densities of state remains. Such an effect has not been observed experimentally, so far. We show that the anomalous Josephson effect in S-QSHI-S junctions is special and directly probes the helical nature of the edge states [15].

In the following, we will first study the anomalous Josephson effect in a S-QSHI-S junction along a single edge. In particular, we will determine the current-phase relation as a function of the external magnetic field and the junction length. Then we will turn to the observability of the effect in S-QSHI-S junctions where both edges contribute. We will argue that Josephson junctions of unequal lengths should be realized in order to observe the effect. Furthermore, we will discuss that the anisotropy of the gyromagnetic tensor should allow for the observation of the effect with an in-plane magnetic field.

Model. A Josephson junction formed along one of the edges of a QSHI can be described by the 1D Bogoliubov-de Gennes Hamiltonian

$$H = (v_F p_x \sigma_z - \mu) \tau_z - h \sigma_z + \Delta(x) \tau_+ + \Delta^*(x) \tau_- .$$

Here, $v_F$ is the Fermi velocity, $x$ and $p_x$ are the position and momentum operators, respectively, $\mu$ is the Fermi energy, and $h > 0$ is a Zeeman field along the spin quantization axis. The superconducting gap induced by conventional superconducting leads is given as $\Delta(x) = \Delta_0 [e^{-i \phi/2} \Theta(x - L/2) + e^{i \phi/2} \Theta(-x - L/2)]$, where $\Delta_0$ is the magnitude of the gap, $\phi$ is the phase difference between the two leads, and $L$ is the junction length. Moreover, $\sigma_{i=x,y,z}$ and $\tau_{j=x,y,z}$ are Pauli matrices acting in spin and Nambu spaces, respectively, and $\tau_{\pm} = (\tau_x \pm i \tau_y)/2$. Note that we use units where $\hbar = k_B = 1$. 

\[ \]
The role of the magnetic field within the superconducting regions is most easily understood by considering first a 1D “bulk” superconductor, i.e., by setting $L = 0$ and $\phi = 0$ in Eq. (1). The Zeeman term induces a momentum mismatch $2h/eF$ between left- and right-moving states at the Fermi level which may be gauged out using the unitary transformation $H \rightarrow U^\dagger HU$ with $U = e^{i(\phi/eF)\sigma_z}$. However, this gauge transformation modifies the order parameter, $\Delta_0 \rightarrow \Delta_0 e^{-2i(h/eF)x}$, which means that, for a uniform order parameter, $\Delta_0 = \text{const}$, one obtains a current-carrying state. Such a state corresponds to an excited state of the system, whereas the ground state would require a spatially modulated order parameter, $\Delta_0(x) = \Delta_0 e^{iqx}$ with the modulation wavevector $q$.

Indeed, the free energy of the system can be easily computed and depends only on an effective field $h_q = h - v_F q/2$. At zero temperature, one finds $F(h,q) = F_0 + \Delta_0^2/(2\pi v_F) f(q/\Delta_0)$, where $F_0$ is the free energy at $h = q = 0$ and $f(x) = x^2 + \Theta(|x| - 1)[\text{arcosh} \, x - |x|\sqrt{x^2 - 1}]$. The supercurrent is obtained using the thermodynamic relation $I = -2e_i(\partial F/\partial q)$. One readily shows that the free energy is minimized and the current is zero for $h_q = 0$, corresponding to a modulation wavevector $q = 2h/eF$. Such a modulated or so-called “helical” superconductivity has been studied in higher dimensions [10].

By contrast, if superconductivity is induced by a conventional bulk superconductor with constant phase, the induced order parameter inherits the phase of the bulk and, thus, a modulation is not possible. Therefore, $q = 0$ and the superconductivity induced in the edge states carries a current, 

$$I(h) = \frac{e}{\pi} \left[ h - \Theta(h - \Delta_0) \sqrt{h^2 - \Delta_0^2} \right].$$

(2)

This is precisely the current $I(\phi = 0)$ that would flow in a short junction, $L \ll \xi$ with $\xi = v_F/\Delta_0$, at zero phase difference. Thus, the fact that the proximity-induced superconductivity forces the system into an excited state yields an anomalous Josephson effect. The anomalous current increases proportionally to $h$ at $h < \Delta_0$ and then decreases as $I \propto e\Delta_0^2/(2\pi h)$ at $h \gg \Delta_0$.

In the following, we extend the result (2) to arbitrary junction lengths and temperatures, and study the current-phase relation. Note that the fact that $\Delta_0$ is an induced gap also implies that there is no self-consistency condition and that fields $h > \Delta_0$ are possible as long as $\Delta_0$ is sufficiently smaller than the intrinsic gap of the superconducting leads.

**Current-phase relation.** We use the formalism of Refs. [17,19] to obtain the Josephson current corresponding to the Hamiltonian (1),

$$I_J = -4eT \frac{d}{d\phi} \sum_{\nu = 0}^{\infty} \ln \left[ 1 - a^2(\omega_\nu - ih)e^{-2(\omega_\nu - ih)/E_L} e^{i\phi} \right].$$

(3)

Here $\omega_\nu = (2\nu + 1)\pi T$ are Matsubara frequencies at temperature $T$, $a(\omega) = i(\omega - \sqrt{\omega^2 + \Delta_0^2})/\Delta_0$, and $E_L = v_F/L$ is the Thouless energy of the junction. Equation (3) accounts for the contributions of both the states in the continuum outside the superconducting gap and the Andreev bound states (ABS), whose subgap energies $E_n$ correspond to the poles of the r.h.s of Eq. (3) after analytic continuation, $\omega_\nu \rightarrow -iE + 0^+$. In particular, the ABS energies are given by the equation

$$2\arccos \left( \frac{E_n + h}{\Delta_0} \right) - 2\frac{(E_n + h)}{E_L} = \phi + 2\pi n$$

(4)

with $n \in \mathbb{Z}$.

Equation (3) can be used to numerically compute the current-phase relation at arbitrary junction lengths and temperatures. The results at low temperatures and various fields are shown in Fig. 1. The current-phase relation and the corresponding anomalous Josephson current as a function of the magnetic field are shown for a short junction [$L = 0.1\xi$, panels (a)-(b)] and a long junction [$L = 10\xi$, panels (c)-(d)], respectively. Below we analyze both short and long junctions further, starting with the limit of zero temperature.

In the short junction limit, $\Delta_0 \ll E_L$, we find that the continuum states are essential in determining the current-phase relation (in contrast with conventional junctions [17, where the supercurrent is carried by ABS only]. Evaluating Eq. (3) at $\phi = 0$, one readily recovers the result (2) which is a pure continuum contribution. At finite $\phi$, the junction accommodates for a single bound state with energy $E_A = \Delta_0 \cos(\phi/2) - h$. The unique zero-energy solution at $\phi^* = 2\arccos(h/\Delta_0)$ for $h < \Delta_0$ is a consequence of the topological nature of the junction. It leads to a jump in the current phase relation, cf. Fig. 1(a), which disappears at $h > \Delta_0$, signaling the transition to a topologically trivial state.

We now turn to the long junction limit, $\Delta_0 \gg E_L$. At $h < \Delta_0$, Eq. (4) yields a large number of ABS with energies

$$E_n = -\frac{E_L}{2} \left[ \phi + \phi_h + 2\pi (n + \frac{1}{2}) \right],$$

(5)

where $\phi_h = 2h/E_L + 2\arcsin(h/\Delta_0)$. Note that there are two different contributions to the phase shift $\phi_h$. The first term is proportional to the junction length and can be traced back to the magnetic field in the junction area. As $E_L \ll \Delta_0$, this is the dominant term. The second term stems from the bulk effect discussed in detail in the short junction limit.

Similar to long S-QSHI-S junctions in the absence of a magnetic field [20,21], we obtain the current-phase relation

$$I_J(\phi, h) = \frac{eE_L}{2\pi} \left[ \phi + \phi_h - 2\pi \text{Int} \left( \frac{\phi + \phi_h}{2\pi} \right) \right],$$

(6)
where \( \text{Int}(x) \) is the integer part of \( x \). The anomalous Josephson current, thus, displays a sawtooth behavior as a function of the applied magnetic field, which is visible in Fig. 1(d). This is reminiscent of the Little-Parks effect \cite{22}, though with a paramagnetic rather than orbital origin. As in the short junction limit, the topological nature of the junction manifests itself in a jump in the current-phase relation when the lowest ABS reaches zero energy, at \( \phi^* = (\pi - \phi_h) \mod 2\pi \).

At larger fields, \( h > \Delta_0 \), one may expand Eq. \( (3) \) in harmonics and evaluate each term using a steepest descent approximation. Summing up the series, one finds

\[
I_f(\phi, h) = -\frac{eE_L}{\pi} \arctan \left[ \frac{\sin(\phi + \frac{2h}{E_L})}{e^2 \arccosh \frac{2h}{E_L} - \cos(\phi + \frac{2h}{E_L})} \right].
\]

As expected, there is no more jump in the current-phase relation, and the anomalous Josephson current is suppressed with increasing field.

The behavior of short and long junctions is quite different. In short junctions, the anomalous Josephson effect stems from the magnetic fields in the leads. By contrast, in long junctions, the dominant contribution at small fields comes from the magnetic field in the junction area. In general, both the field in the leads and the normal part of the junction contribute. Note, however, that the slope of the anomalous Josephson current as a function of the field near \( h = 0 \) is the same in junctions of any length. Namely,

\[
\frac{\partial I_f(\phi = 0, h)}{\partial h} \bigg|_{h=0} = \frac{e}{\pi}.
\]

This universal result is a consequence of the helical nature of the QSHI edge states.

**Finite temperature effects.** Finite temperatures smear out the sharp features in the current-phase relation. At high temperatures, \( T \gg \min[\Delta_0, E_L] \), the current-phase relation becomes sinusoidal,

\[
I_f(\phi, h) = I_c(\Delta_0, h, E_L) \sin[\phi + \phi_0(h, T, E_L)],
\]

for both, short and long junctions. Thus, the anomalous Josephson effect is characterized by the phase shift \( \phi_0(h, T, E_L) \).

In short junctions, we find \( I_c = e\Delta_0^2 |\psi_1(z)|^2/(4\pi^2T) \) and \( \phi_0 = \arg\{\psi_1(z)\} \), where \( \psi_1 \) is the digamma function and \( z = 1/2 - i\hbar/(2\pi T) \). Thus, the phase shift increases from 0 to \( \pi/2 \) with increasing field. In long junctions, we find \( I_c = -4eT \exp[-\pi\Delta_0^2/4E_L]a(2\pi T - i\hbar)|^2 \) and \( \phi_0 = 2\hbar/E_L + 2 \arg\{a(2\pi T - i\hbar)\} \) for \( h \ll T \ll \Delta_0 \), one obtains \( |a(2\pi T - i\hbar)| = 1 \) and \( \arg\{a(2\pi T - i\hbar)\} = -\pi/2 \), whereas for \( T \ll \Delta_0 \ll h \), one obtains \( \arg\{a(2\pi T - i\hbar)\} = \Delta_0^2/(2\hbar)^2 \) and \( \arg\{a(2\pi T - i\hbar)\} = 0 \).

**Double Junction.** When creating a Josephson junction with a QSHI, typically both edges of the QSHI contribute to the Josephson current \cite{9,10}. If the width of the QSHI is sufficiently large, the system may be described as two junctions in parallel and their contributions may be computed separately. Then, as the two edges of the QSHI have opposite helicities, the contribution of the second edge can be accounted for by another copy of the Hamiltonian \( \mathcal{H}_J \) with \( h \to -h \). The corresponding current is given as \( I_f(\phi, -h) = -I_f(-\phi, h) \). As mentioned in the introduction in the case of conventional Josephson junctions, adding the current contributions of the two different helicities leads to a (partial) compensation of their anomalous Josephson currents. The same is true here. However, the spatial separation of the two helicities makes an important difference. Only if the two junctions on either side of the sample have the same length, the compensation is exact, and we obtain the conventional result, \( I_f(\phi = 0, h) = 0 \). However, if the two junctions have unequal lengths, as shown schematically in Fig. 2(a), the compensation is only partial and a residual effect remains. This residual effect is a signature that the Josephson current is carried by helical edge states. The dependence of the anomalous Josephson current at \( \phi = 0 \) is plotted as a function of \( h \) for various temperatures in Fig. 2(b).

**Discussion.** We now turn to the conditions of applicability of our model \cite{1}. According to the Bernevig-Hughes-Zhang (BHZ) model for inverted electron-hole
I/J(\phi = 0) = 0

In conclusion, we have demonstrated that the helical nature of the QSHI edge states leads to an anomalous Josephson effect or $\phi_0$-junction behavior in S-QSHI-S junctions subject to a magnetic field. The resulting anomalous supercurrent, flowing at zero phase difference between the two superconducting leads, is field tunable. We also discussed how to observe this effect using hybrid structures based on available QSHI realizations. Similarly, we expect a pronounced anomalous Josephson effect in junctions based on nanowires with strong spin-orbit coupling \cite{25,26} when they are in the topological regime.

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