An Orthogonality Principle for Select-Maximum Estimation of Exponential Variables

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Abstract—It was recently proposed to encode the one-sided exponential source $X$ into $K$ parallel channels, $Y_1,\ldots,Y_K$, such that the error signals $X-Y_i, i=1,\ldots,K$, are one-sided exponential and mutually independent given $X$ [1], [2]. Moreover, it was shown that the optimal estimator $\hat{Y}$ of the source $X$ with respect to the one-sided error criterion, is simply given by the maximum of the outputs, i.e., $\hat{Y} = \max\{Y_1,\ldots,Y_K\}$. In this paper, we show that the distribution of the resulting estimation error $X - \hat{Y}$, is equivalent to that of the optimum noise in the backward test-channel of the one-sided exponential source, i.e., it is one-sided exponentially distributed and statistically independent of the joint output $Y_1,\ldots,Y_K$.

Index Terms—exponential source, multiple descriptions, independent encoding, estimation error.

I. INTRODUCTION

The rate-distortion function $R(D)$ for a source $X$ under the distortion measure $d(x,y)$ is defined as the infimum of the mutual information $I(X;Y)$ over the set $\{f_Y|X=y|x\}$ of conditional probability density functions, which satisfy a given distortion constraint $E[d(X,Y)] \leq D$ [3], [4]. The optimum distribution, say $f_Y|X=y|x$, can be expressed as

$$f_Y|X(y|x) = f_X^*(x|y) f_Y(y)/f_X(x)$$

where the conditional density $f_X^*(x|y)$ is often described via a so-called backward test channel [3], [4], where the reconstruction $Y$ acts as the input and the source $X$ is the output of the test channel, see Fig. 1.

For several sources and distortion measures, it is known that the backward channel is an additive channel in the sense that the channel noise $Z$ is statistically independent of $Y$, and the source is simply given as the sum $X = Z + Y$. Moreover, in some cases $X$ and $Z$ belongs to the same family of distributions. For example, in the case of a Gaussian source $X$ and under the mean squared error distortion measure, the optimum channel noise $Z$ in the backward test channel is Gaussian and independent of the reconstruction $Y$ [3], [4]. A similar relationship can be observed for the binary source under the hamming distortion measure [4] and for the one-sided exponential source under the one-sided error distortion measure [5].

If we have access to $K > 0$ different encodings of the Gaussians source, for example $Y_i = X + N_i, i=1,\ldots,K$, where $X$ and $N_1,\ldots,N_K$ are jointly Gaussian, then the estimation error $X - g(Y_1,\ldots,Y_K)$ due to optimally estimating $X$ from $Y_1,\ldots,Y_K$, in a minimum mean-squared error (MSE) sense, is also Gaussian. Moreover, this estimation error is independent of the outputs $Y_1,\ldots,Y_K$.

In this paper, we are interested in the case, where we have access to $K > 0$ outputs (reconstructions) $Y_i, i=1,\ldots,K$, of a one-sided exponential source $X$. Each reconstruction is by itself optimal under the one-sided error distortion measure and therefore satisfy the backward channel relationship $X = Z_i + Y_i$ where $Z_i, i=1,\ldots,K$, are one-sided exponentially distributed. We assume that the reconstructions $Y_1,\ldots,Y_K$ are independent encodings, which is a concept introduced in [1], [2], and which implies that the outputs $Y_i, i=1,\ldots,K$, are mutually independent given $X$. Thus, the following Markov chains apply: $Y_i - X - Y_j, \forall i,j$. It was shown in [1] that the optimum estimator $g(Y_1,\ldots,Y_K)$ of $X$ given $Y_1,\ldots,Y_K$, under the one-sided error distortion measure is the maximum of $Y_1,\ldots,Y_K$, i.e., $g(Y_1,\ldots,Y_K) = \max Y_i$. We show in this work, that the estimation error $X - g(Y_1,\ldots,Y_K)$ has a backward test channel formulation. Specifically, the estimation error $X - g(Y_1,\ldots,Y_K)$ is independent of the output vector $(Y_1,\ldots,Y_K)$ and is one-sided exponentially distributed.

If only a random subset of the $K$ encodings are used when forming the estimate of $X$, then the average estimation error is not exponentially distributed. We provide a closed-form expression for the distribution of the average estimation error in the case of using a random number of encodings for estimating the source.

Fig. 1. Backward additive test channel. The reconstruction $Y$ and the channel noise $Z$ are mutually independent.
A. Notation

We use upper case letters for random variables, and lower case letters for their realizations. For random variables $X, Y$, we use the notations $f_{Y|X}, F_{Y|X}, CF_{Y|X}$ for the conditional probability density function (PDF), conditional cumulative distribution function (CDF), and conditional complimentary CDF of $Y$ given $X$, respectively.

II. BACKGROUND

The one-sided exponential source with parameter $\lambda > 0$ is defined as [5]:

$$f_X(x) = \lambda e^{-\lambda x}, \ x \geq 0,$$

where $\mathbb{E}[X] = 1/\lambda$.

Let $y$ denote the reproduction of the coder, and let the one-sided error criterion be given by:

$$d(x, y) = \begin{cases} x - y, & \text{if } x \geq y \geq 0, \\ \infty, & \text{if } x < y. \end{cases}$$

The rate-distortion function (RDF) for the exponential source with one-sided error criterion is given by [5]:

$$R(D) = \begin{cases} -\log(\lambda D), & 0 \leq D \leq \frac{1}{\lambda}, \\ 0, & D > \frac{1}{\lambda}. \end{cases}$$

Let $X = Z + Y$ denote the backward test channel whose optimal conditional output distribution is given by [5, 6]:

$$f_{X|Y}(x|y) = \frac{1}{D} e^{-\frac{(x-y)}{\lambda D}}, \ x \geq y \geq 0.$$}

The channel noise, $Z$, is one-sided exponential distributed with parameter $1/D$. The distribution of $Z$ is easily obtained from [4] by inserting $Z = X - Y$.

The output distribution for $Y$ is a mixture distribution with an atom at zero and is given by [1]:

$$f_Y(y) = \begin{cases} \lambda D\delta(y) + (1 - \lambda D)\lambda e^{-\lambda y}, & 0 \leq y \leq x, \\ 0, & \text{otherwise}. \end{cases}$$

(5)

If we take the channel from $X$ to $Y$ to be the RDF-achieving forward test channel, then:

$$F_{Y|X}(y|x) = e^{\delta(y-x)},$$

where $\delta = \frac{1}{\lambda} - 1$, and $0 \leq y \leq x$.

III. DISTRIBUTION OF THE ESTIMATION ERROR

Let $X$ be one-sided exponentially distributed, and consider $K$ parallel test-channels, $Y_1, \ldots, Y_K$, where the errors $Z_i = X - Y_i, i = 1, \ldots, K$, are mutually independent given $X$. We will refer to the set $(Y_1, \ldots, Y_K)$ as parallel channels or independent encodings of the source $X$.

It was shown in [1], that the select-max estimator, i.e., the estimator that simply selects the maximum $\hat{Y} = \max\{Y_1, \ldots, Y_K\}$ of the $K$ channels as the estimate of $X$, is in fact an optimal estimator under the one-sided error distortion.

The following lemmas show that the estimation error is one-sided exponentially distributed and that it is independent of the outputs $(Y_1, \ldots, Y_K)$.

**Lemma 1 (Exponential estimation error):** Let $Y_1, \ldots, Y_K$, be $K > 0$ independent encodings of the one-sided exponential source $X$ with parameter $\lambda > 0$. Moreover, let $\delta = \frac{1}{\lambda} - 1, 0 \leq D \leq 1/\lambda$. Then, the estimation error $\tilde{Z} = X - Y_i$ due to estimating the source $X$ by the select-max estimator $\check{Y}$, is one-sided exponentially distributed with parameter $\lambda' = \lambda + K\delta$, i.e.:  

$$f(\tilde{z}) = (\lambda + K\delta) e^{-(\lambda + K\delta)\tilde{z}}, \ \tilde{z} \geq 0. \ (7)$$

**Proof of Lemma 1:** The sequence of outputs $Y_1, \ldots, Y_K$, are absolute continuous and conditionally independent given $X$. The conditional CDF of the select-max estimator from the $K$ channels is therefore given by [1]:

$$F_{Y|X}(\check{y}|x) = F_{Y|X}(y|x)^K,$$

where $F_{Y|X}(y|x)$ is the conditional CDF of each channel, respectively. We have here omitted the channel index $i$, since $Y_i, \forall i$, are identically distributed. The conditional PDF $f_{Y|X}(\check{y}|x)$ is given by:

$$f_{Y|X}(\check{y}|x) = K F_{Y|X}(y|x)^{K-1} f_{Y|X}(y|x). \ (9)$$

Using (5) it now follows that:

$$F_{Y|X}(\check{y}|x) = e^{K\delta(y-x)}, 0 \leq y \leq x. \ (10)$$

Since the estimation error is given by $\tilde{Z} = X - \check{Y}$, the probability of the event $\tilde{Z} \geq \xi$ is equal to that of the event $\check{Y} \leq \xi$, for some $\xi \in \mathbb{R}_+$. So the complementary CDF of $\tilde{Z}$ and the regular CDF of $\check{Y}$ are the same (up to a shift by $X$). It follows that the conditional complimentary CDF $CF_{Z|X}(\tilde{z}|x)$ is given by

$$CF_{Z|X}(\tilde{z}|x) = F_{Y|X}(\check{y} = x - \tilde{z}|x)$$

$$= e^{-K\delta\tilde{z}}, \ 0 \leq \tilde{z} \leq x, \ (11)$$

and the unconditional complimentary CDF $CF_{Z}$ is given by:

$$CF_{Z}(\tilde{z}) = \int_{x=\tilde{y}}^{\infty} CF_{Z|X}(\tilde{z}|x) f(x) dx$$

$$= \int_{x=\tilde{y}}^{\infty} e^{K\delta(\tilde{y}-x)} \lambda e^{-\lambda x} dx$$

$$= e^{-(\lambda + K\delta)\tilde{z}}. \ (15)$$

The unconditional CDF of the estimation error when using the select-max estimator on the $K$ channel outputs can now easily be obtained from the unconditional complimentary CDF in (15), that is:

$$F_{Z}(\tilde{z}) = 1 - e^{-(\lambda + K\delta)\tilde{z}}, \ \tilde{z} \geq 0, \ (16)$$

which implies that the error is exponentially distributed:

$$f_{Z}(\tilde{z}) = \frac{d}{d\tilde{z}} F_{Z}(\tilde{z}) = (\lambda + K\delta) e^{-(\lambda + K\delta)\tilde{z}}, \ \tilde{z} \geq 0, \ (17)$$
This proves the lemma. □

**Remark:** The distortion $D_K$ of the estimator $\hat{Y}$ inLemma 1 can be written as:

$$\frac{1}{D_K} = \lambda + K\left(\frac{1}{D} - \lambda\right),$$

(18)

which resembles the expression for the MMSE formula for $K$ measurements.

**Lemma 2 (Sufficient statistic):** Let $X$ be a one-sided exponential source, and let $X \rightarrow Y_1, \ldots, X \rightarrow Y_K$ be $K$ parallel (RDF achieving) test channels. Moreover, let $\hat{Y} = \max\{Y_1, \ldots, Y_K\}$. Then,

$$X - \hat{Y} - (Y_1, \ldots, Y_K)$$

(19)

form a Markov chain, i.e., $\hat{Y}$ is a sufficient statistic for $X$ from $(Y_1, \ldots, Y_K)$.

**Proof of Lemma 2:** In order to prove the lemma, we will derive an explicit expression for the conditional distribution of $X$ given $(Y_1, \ldots, Y_K)$, and observe that it only depends on the maximum value $\hat{Y}$ of the vector $(Y_1, \ldots, Y_K)$. Then we use this result to further show that the conditional distribution of $X$ given $\hat{Y}$ and $(Y_1, \ldots, Y_K)$ does not depend upon $(Y_1, \ldots, Y_K)$.

Let $y > 0$ be any positive number greater than $\tilde{y}_2, \ldots, \tilde{y}_K$, i.e., $y > \tilde{y}_2, \ldots, y > \tilde{y}_K$, where $\tilde{y}_i > 0, i = 2, \ldots, K$. The conditional density $f(x|Y_1 = y, Y_2 < \tilde{y}_2, \ldots, Y_K < \tilde{y}_K)$ of the source $X$ given the event $\{Y_1 = y, Y_2 < \tilde{y}_2, \ldots, Y_K < \tilde{y}_K\}$ on the $K$ outputs is nonzero only for $x \geq y$, since the test channels’ outputs are smaller than $X$ with probability 1. By the complete probability formula (Bayes law) this conditional density is equal to:

$$f(x|Y_1 = y, Y_2 < \tilde{y}_2, \ldots, Y_K < \tilde{y}_K) = f(x)f(y|x)F(\tilde{y}_2|x)\cdots F(\tilde{y}_K|x)$$

$$\times \left(\int_{-\infty}^{\tilde{y}_1} f(y)f(y|x)F(\tilde{y}_2|x)\cdots F(\tilde{y}_K|x)\right)^{-1},$$

(21)

where we used the fact that all the channels are identical and also conditional independent given the source $X$. Inserting the conditional cumulative distribution $F(y|x)$ of each channel, which is given by (5), makes it possible to rewrite the numerator (i.e., the terms outside the big brackets) of (21) as follows:

$$f(x)f(y|x)F(\tilde{y}_2|x)\cdots F(\tilde{y}_K|x)$$

$$= f(x)f(y|x)e^{-(x-\tilde{y}_2)\delta}\cdots e^{-(x-\tilde{y}_K)\delta}$$

$$= f(x)f(y|x)e^{-(K-1)\delta \tilde{y}_2}\cdots e^{K\delta \tilde{y}_K},$$

(22)

(23)

(24)

where the factor $e^{\tilde{y}_2\delta}\cdots e^{\tilde{y}_K\delta}$ is independent of $x$. We can rewrite the denominator (the terms inside the big brackets) of (21) in a similar manner, where the factor $e^{\tilde{y}_2\delta}\cdots e^{\tilde{y}_K\delta}$ also appears and goes outside the integral to be cancelled by the equivalent factor in the numerator. Hence the conditional density is simply given as:

$$f(x|Y_1 = y, Y_2 < \tilde{y}_2, \ldots, Y_K < \tilde{y}_K) = f(x)\int_{-\infty}^{\tilde{y}_1} f(y|x)e^{-(K-1)\delta \tilde{y}_2}d\tilde{y}_1^{-1},$$

(25)

$$= f(x)f(y|x)e^{-(\tilde{y}_2-\tilde{y})\delta}$$

(26)

which is indeed independent of $\tilde{y}_2, \ldots, \tilde{y}_K$ as desired.

If we now let the maximal output $Y_i = \bar{Y}$, where $\bar{y} = \arg\max_{Y_i}$ be equal to $y$, i.e., $Y_i = y$, and the remaining $K - 1$ outputs, $Y_1, \ldots, Y_{i-1}, Y_{i+1}, \ldots, Y_K$ be smaller than $\tilde{y}_2, \ldots, \tilde{y}_K$, respectively, then we can form the following $K$ non-intersecting events:

$$\{Y_1 = y, Y_2 < \tilde{y}_2, \ldots, Y_K < \tilde{y}_K\}$$

(27)

$$\{Y_2 = y, Y_1 < \tilde{y}_2, Y_3 < \tilde{y}_3, \ldots, Y_K < \tilde{y}_K\}$$

(28)

$$\vdots$$

$$\{Y_K = y, Y_1 < \tilde{y}_2, \ldots, Y_{K-1} < \tilde{y}_K\}.$$

(29)

Consider the first event given in (27), which is also the event occurring in (25)–(26). This event depends only on $x, y$ and $K$, but is independent of $(\tilde{y}_2, \ldots, \tilde{y}_K)$. Due to symmetry, the probabilities of the events are equal, and they are each individually independent of $(\tilde{y}_2, \ldots, \tilde{y}_K)$, which implies that so is the union of the events. This proves the lemma for the case where $Y_i > 0, \forall i$.

We have yet to consider the case where $Y_i = 0$ for some $i$. If $Y = 0$, then clearly all channel outputs are zero, and it easily follows that $X - \hat{Y} - (Y_1, \ldots, Y_K)$ is satisfied. Let $Y \neq 0$ but $Y_i = 0$ for some $i$, say $i = 1$. Then, in this case we have:

$$f_{Y|x}(y_1 = 0|x) = f_{X|Y}(x|y_1 = 0)f_Y(y_1 = 0)f_X(x)^{-1}$$

(30)

$$= \frac{1}{D} e^{-y/D\lambda}D f_X(x)^{-1}$$

(31)

$$= \lambda e^{-y/D\lambda}f_X(x)^{-1}$$

(32)

$$= \lambda e^{-(\delta + \lambda)\bar{y}} f_X(x)^{-1}$$

(33)

$$= e^{-\delta \bar{y}}$$

(34)

$$= F_{Y|X}(y = 0|x),$$

(35)

where we used $D^{-1} = \lambda + \delta$. The lemma is now proved since for $y = 0, f_{Y|x}(y|x) = f_{Y|x}(y|x)$, which means that (25) – (26) are unaffected if one changes the event from $Y_i < \tilde{y}_i$, where $\tilde{y}_i > 0$ into $Y_i = \tilde{y}_i = 0$.

□

**Remark:** It may be noticed that the conditional density $f(x|\hat{Y})$ simply describes the equivalent additive backward channel:

$$f(x|\hat{Y} = y) = f(x|\bar{y}) = f_{\bar{y}}(x - \bar{y})$$

(36)

because $\bar{y}$ is exponential with a parameter $\lambda + K\delta$. Thus, if the source is exponential, then the conditional density $f(x|\bar{y})$ becomes $f_{\bar{y}}(x - \bar{y})$, which is equivalent to the estimation error density computed at $\bar{z} = x - \bar{y}$. 
Lemma 3 (Orthogonality principle): Let $X$ be a one-sided exponential source, and let $X \to Y_1, \ldots, X \to Y_K$ be $K$ parallel (RDF achieving) test channels. Finally, let $Y = \max\{Y_1, \ldots, Y_K\}$. Then, the backward channel $X = Y + \tilde{Z}$ is additive, i.e., the estimation error $\tilde{Z}$ is independent of the estimator $\hat{Y}$. Moreover, $\tilde{Z}$ is independent of the joint output vector $(Y_1, \ldots, Y_K)$.

Proof of Lemma 3: That $\tilde{Z} = X - \hat{Y}$ is independent of $(Y_1, \ldots, Y_K)$ follows immediately due to $\hat{Y}$ being a sufficient statistics for $X$ from $(Y_1, \ldots, Y_K)$, see Lemma 2.

The first part of the lemma follows from the fact that the equivalent forward channel $X \to Y$ has the form:

$$F(\hat{y}|x) = F(y|x)^K = e^{-(y-x)K\delta},$$

which is the same form as $F(y|x)$, if we replace $\delta$ by $K\delta$. Thus, if the combination of an exponential source and the test channel $F(y|x)$ implies an additive exponential backward channel with distortion $1/(\lambda + \delta)$, then the combination of an exponential source and the channel $F(y|x)^K$ implies an exponential additive backward channel with distortion $1/((\lambda + K\delta))$. □

Remark: We note that the hypothetical rate $I(X;\hat{Y})$ is RD optimal w.r.t. the one-sided error distortion. This follows since the estimation error $X - \hat{Y}$ satisfies the "backward orthogonality" property, i.e., it is exponentially distributed and independent of $\hat{Y}$. Thus, we can write:

$$I(X;\hat{Y}) = h(X) - h(\hat{Y})$$

$$= h(X) - h(X - \hat{Y})$$

$$= h(X) - h(X - \tilde{Z})$$

$$= h(X) - h(\tilde{Z})$$

$$= R(D),$$

where $\tilde{Z}$ is exponentially distributed. Moreover, since $X - \hat{Y} = (Y_1, \ldots, Y_K) \implies I(X;Y_1, \ldots, Y_K) = I(X;\hat{Y})$. Thus, $I(X;Y_1, \ldots, Y_K)$ is also RD optimal w.r.t. the one-sided error distortion. Of course, the rate in independent encoding $I(X;\hat{Y}_1) + \cdots + I(X;\hat{Y}_K) = KI(X;\hat{Y})$ is generally not RD-optimal, except at very small coding rates $[7]$.

IV. THE CASE OF AN UNKNOWN NUMBER OF ENCODINGS

In the previous section, we assumed the availability of $K$ descriptions and formed the estimate $\hat{Y} = \max\{Y_1, \ldots, Y_K\}$. If the $K$ descriptions are individually transmitted over a packet-switched network, where packets could occasionally be dropped, then the number of received descriptions, say $\ell$, becomes a random variable, where $\ell \in \{1, \ldots, K\}$.

Let us consider a packet erasure channel with packet-loss probability $\theta$, and where the packet losses are independently distributed. In this case, the probability that $\ell$ out of $K$ packets are received is given by:

$$P(K = \ell) = \theta^{K-\ell} (1-\theta)^\ell.$$