THE f- AND h-VECTORS OF INTERVAL SUBDIVISIONS

IMRAN ANWAR AND SHAHEEN NAZIR

ABSTRACT. The interval subdivision $\text{Int}(\Delta)$ of a simplicial complex $\Delta$ was introduced by Walker. We give the complete combinatorial description of the entries of the transformation matrices from the $f$- and $h$-vectors of $\Delta$ to the $f$- and $h$-vectors of $\text{Int}(\Delta)$. We show that if $\Delta$ has non-negative $h$-vector then the $h$-polynomial of its interval subdivision has only real roots. As a consequence, we prove the Charney-Davis conjecture for $\text{Int}(\Delta)$, if $\Delta$ has non-negative reciprocal $h$-vector.

1. Introduction

In this paper, we study the behavior of the enumerative invariants of a simplicial complex under the interval subdivision, introduced by Walker [Wal88]. This work is motivated from the work of Brenti and Welker about the barycentric subdivision of simplicial complexes [BW08]. The enumeration data e.g., $f$-, $h$-, $\gamma$-, $g$-vectors of barycentric subdivision of a simplicial complex has been extensively studied in the literature, see [Sta92, BW08, KN09, Mur10, NPT11, Pet15]. Let $\Delta$ be a $(d-1)$-dimensional simplicial complex on the ground set $V$. The interval subdivision $\text{Int}(\Delta)$ of $\Delta$ is the simplicial complex on the ground set $I(\Delta \setminus \emptyset)$, where $I(\Delta \setminus \emptyset) := \{[A, B] \mid \emptyset \neq A \subseteq B \in \Delta\}$ as a partially ordered set ordered by inclusion defined as $[A, B] \subseteq [A', B'] \in I(\Delta \setminus \emptyset)$ if and only if $A' \subseteq A \subseteq B \subseteq B'$. By Walker [Wal88, Theorem 6.1(a)], the simplicial complex of all chains in the partially ordered set $I(\Delta \setminus \emptyset)$ is a subdivision of $\Delta$. It can be noted that this subdivision is the special case $N = 1$ of the simplicial complex considered in [CMS84, Fig. 1.2]. The potential aim of this article is to analyze the behavior of $f$- and $h$-vectors moving from $\Delta$ to $\text{Int}(\Delta)$. In the main result of this paper we show that if the $h$-vector of a simplicial complex is non-negative then the $h$-polynomial of its interval subdivision has only real roots. Moreover, it is shown that the refined $j$-Eulerian polynomials of type $B$ (defined in Section 2) are real-rooted.

The paper is organized as follows. In Section 2, we give the formula of $f$-vector of the interval subdivision $\text{Int}(\Delta)$ in terms of $f$-vector of the simplicial complex $\Delta$. In Section 3, we study the transformation of $h$-vector of the interval subdivision $\text{Int}(\Delta)$. We give the interpretation of the coefficients of the matrix transformation of $h$-vector in terms of refined

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Eulerian numbers of type $B$. It is well known that these coefficients are $j$-Eulerian numbers of type $A$ in the case of barycentric subdivision, see [Sta86, BW08]. Along the way, we also show that if $h$-vector of the simplicial complex $\Delta$ satisfies the Dehn-Sommerville relations, so does the $h$-vector of interval subdivision $\text{Int}(\Delta)$. Moreover, we investigate some simple results on the properties of $f$- and $h$-vector transformation matrices. In the sequel, we show that these transformation matrices are diagonalizable and similar. The Section 4 is devoted to the proof our main Theorem 4.1. It states that if $\Delta$ has a non-negative $h$-vector then the $h$-polynomial of its interval subdivision has only real zeros. Additionally, we prove that the refined $j$-Eulerian polynomials of type $B$ are real-rooted. As a consequence of the Theorem 4.1, we succeed to prove the Charney-Davis for the interval subdivision of a simplicial complex $\Delta$ with non-negative reciprocal $h$-vector in Corollary 4.11.

2. $f$-vector Transformation

Throughout from here, $\Delta$ represents a $(d-1)$-dimensional simplicial complex on the ground set $V = [n]$. In this section, we will describe the transformation sending $f$-vector of $\Delta$ to $f$-vector of $\text{Int}(\Delta)$. Recall that the vector $f(\Delta) = (f_{-1}(\Delta), f_0(\Delta), \ldots, f_{d-1}(\Delta))$, where $f_i(\Delta)$ is the number of $i$-dimensional faces of $\Delta$ is called the $f$-vector of $\Delta$ with $f_{-1}(\Delta) = 1$ (for $\dim \emptyset = -1$).

By the definition of $\text{Int}(\Delta)$, an $l$-dimensional face in $\text{Int}(\Delta)$ is a chain $[A_0, B_0] \subset [A_1, B_1] \subset \ldots \subset [A_l, B_l]$ of intervals of length $l$ in $I(\Delta \setminus \emptyset)$. As a warm-up we start with a description of $f_0(\text{Int}(\Delta))$.

$f_0(\text{Int}(\Delta))$: $f_0(\text{Int}(\Delta))$ is the number of intervals $[A, B]$, where $A \subseteq B$ for all $A, B \in \Delta \setminus \emptyset$. For any $B \in \Delta \setminus \emptyset$, all distinct subsets of $B$ (excluding $\emptyset$) give rise to distinct intervals terminating at $B$. Since there are $f_{l-1}(\Delta)$ choices for $B$ with $|B| = l$, and for a fixed $B \in \Delta$, the number of intervals of the form $[A, B]$ is $2^{|B|} - 1$. Therefore, the number of all possible intervals in $I(\Delta \setminus \emptyset)$ will be

$$(2^0 - 1)f_{-1}(\Delta) + (2^1 - 1)f_0(\Delta) + (2^2 - 1)f_1(\Delta) + (2^3 - 1)f_2(\Delta) + \cdots + (2^{d-1} - 1)f_{d-1}(\Delta).$$

Thus,

$$f_0(\text{Int}(\Delta)) = \sum_{k=0}^{d} (2^k - 1)f_{k-1}(\Delta). \quad (1)$$

Now we turn to the description of $f_k(\text{Int}(\Delta))$ in general.

$f_k(\text{Int}(\Delta))$: To compute $f_k(\text{Int}(\Delta))$, for $k \geq 0$, let us introduce some notations. It is easily seen that the number of chains of length $k$ terminating in $[A, B]$ only depends on $\alpha = |B \setminus A|$. Let $Q_k^\alpha$ denote the number of chains of intervals of length $k$ terminating at some fixed interval $[A, B]$, where $\alpha = |B \setminus A|$. By definition, $Q_k^\alpha = 0$ for $\alpha < k$ and $Q_0^\alpha = 1$ for all $\alpha$.

We group the $k$-chains in $I(\Delta \setminus \emptyset)$ according to the top element $[A, B]$ of the chain. For a fixed $[A, B]$, we have $Q_k^{t-1}$ chains of length $k$ terminating in $[A, B]$, where $t = |A|$ and
$l = |B|$. There are $f_l(\Delta)$ choices for $B$ with $|B| = l$ and for a fixed $B$ we have $\binom{l}{i}$ subsets $A \subseteq B$ with $|A| = t$. Hence, we have

$$f_k(\operatorname{Int}(\Delta)) = \sum_{i=0}^{d} \left[ \sum_{t=1}^{l} \binom{l}{t} Q^i_{k-t} \right] f_{l-1}(\Delta). \quad (2)$$

In the next lemma, we formulate the number $Q^\alpha_k$.

**Lemma 2.1.** The formula for $Q^\alpha_k$ is given as

$$Q^\alpha_k = \sum_{i=0}^{k} (-1)^i \binom{k}{i} (1 + 2(k - i))^\alpha. \quad (3)$$

**Proof.** We will prove (3) by induction on $k$. It is true for $k = 0$, follows from the definition. Now, suppose that (3) is true for $k - 1$. To compute $Q^\alpha_k$, we intend to count all $k$-chains of intervals terminating at some fixed interval $[A, B]$ with $|B \setminus A| = \alpha$. Let $B = A \cup \{a_1, a_2, \ldots, a_\alpha\}$. The intervals strictly contained in $[A, B]$ are of the form $[A \cup \{a_{t1}, \ldots, a_{ti}\}, A \cup \{a_{t1}, \ldots, a_{ti+s}\}]$ unless $t = 0$ and $s = \alpha$. There are $\binom{\alpha}{s} \binom{s}{t}$ choices for intervals of the form $[A \cup \{a_{t1}, \ldots, a_{ti}\}, A \cup \{a_{t1}, \ldots, a_{ti+s}\}]$ contained in $[A, B]$, and the number of all chains of length $k - 1$ terminating at $[A \cup \{a_{t1}, \ldots, a_{ti}\}, A \cup \{a_{t1}, \ldots, a_{ti+s}\}]$ is $Q^s_{k-1}$. Hence for fixed $\alpha$ and $k$, we have the following recurrence relation

$$Q^\alpha_k = \sum_{i=0}^{\alpha} \binom{\alpha}{i} \left[ \sum_{j=0}^{\alpha-i} \binom{\alpha-i}{j} Q^j_{k-1} \right] - Q^\alpha_{k-1}.$$

Since (3) is true for $k - 1$, so substitute the formula of $Q^\alpha_{k-1}$ in the above expression

$$Q^\alpha_k = \sum_{i=0}^{\alpha} \binom{\alpha}{i} \left[ \sum_{j=0}^{\alpha-i} \binom{\alpha-i}{j} \sum_{m=0}^{k-1} (-1)^m \binom{k-1}{m} (1 + 2(k - 1 - m))^\alpha \right]$$

$$- \sum_{m=0}^{k-1} (-1)^m \binom{k-1}{m} (1 + 2(k - 1 - m))^\alpha.$$

Using binomial formula twice (first taking sum over $j$ and then over $i$), we have

$$Q^\alpha_k = \sum_{m=0}^{k-1} (-1)^m \binom{k-1}{m} [(1 + 2k - 2m)^\alpha - (1 + 2k - 2(m+1))^\alpha]$$

Now, using the identity $\binom{k-1}{m} + \binom{k-1}{m-1} = \binom{k}{m}$, we get

$$Q^\alpha_k = (1 + 2k)^\alpha + \sum_{m=1}^{k-1} (-1)^m \binom{k}{m} (1 + 2k - 2m)^\alpha + (-1)^k$$

which gives the required form. \qed

Thus, we have the $f$-vector transformation as follows.
Theorem 2.2. Let $\Delta$ be a $(d - 1)$-dimensional simplicial complex. Then

$$f_k(\text{Int}(\Delta)) = \sum_{l=0}^{d} \sum_{i=0}^{k} (-1)^i \binom{k}{i} \left[ (2 + 2k - 2i)^l - (1 + 2k - 2i)^l \right] f_{l-1}(\Delta).$$

(4)

for $0 \leq k \leq d - 1$ and $f_{-1}(\text{Int}(\Delta)) = f_{-1}(\Delta) = 1$.

Proof. The result readily follows from the binomial expansion on equation (2). \qed

Remark 2.3. The formula (4) can be represented in terms of Stirling’s number of second kind $S(n, k)$ as:

$$f_k(\text{Int}(\Delta)) = \sum_{l=0}^{d} \sum_{i=0}^{l} \binom{k}{i} k! S(j, k) \left[ 2^l - 2^l \right] f_{l-1}(\Delta).$$

(5)

Here, we include an example to demonstrate the above computed transformation.

Example 2.4. Let $\Delta$ be a 2-simplex on the ground set $\{1, 2, 3\}$, then the $f(\Delta) = (3, 3, 1)$. The ground set of the $\text{Int}(\Delta)$ will be the set of all possible intervals in $\Delta \setminus \emptyset$. Therefore, $f_0(\text{Int}(\Delta)) = \sum_{l=0}^{2} (2^l - 1) f_{l-1}(\Delta) = 19$, $f_1(\text{Int}(\Delta)) = \sum_{l=0}^{2} \sum_{i=0}^{1} (-1)^i \binom{1}{i} \left[ (4 - 2i)^l - (3 - 2i)^l \right] f_{l-1}(\Delta) = 42$ and $f_2(\text{Int}(\Delta)) = \sum_{l=0}^{2} \sum_{i=0}^{2} (-1)^i \binom{2}{i} \left[ (6 - 2i)^l - (5 - 2i)^l \right] f_{l-1}(\Delta) = 24$.
3. \( h \)-vector Transformation

In this section, we represent the \( h \)-vector of an interval subdivision in term of \( h \)-vector of the given simplicial complex. Recall that \( h \)-vector \( h(\Delta) = (h_0(\Delta), \ldots, h_d(\Delta)) \) of \( (d-1) \)-simplicial complex \( \Delta \) is defined in terms of \( f \)-vector as

\[
h_k(\Delta) = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{k-i} f_{i-1}(\Delta).
\]

The \( h \)-polynomial of \( \Delta \) is defined as

\[
h(\Delta, x) = \sum_{i=0}^{d} h_i(\Delta) x^i.
\]

We need to recall some notions to give the combinatorial description of the entries of \( h \)-vector transformation matrix.

Signed Permutation Group \( B_d \). We present here some definitions and notations for the classical Weyl groups of type \( B \) (also known as the hyperoctahedral groups or the signed permutations group) and denoted as \( B_d \). It is the group consisting of all the bijections \( \sigma \) of the set \( \{\pm 1, \ldots, \pm d\} \) onto itself such that \( \sigma_i = -\sigma_{i+1} \) for all \( i \in \{\pm 1, \ldots, \pm d\} \). \( B_d \) can be viewed as a subgroup of \( S_{2d} \) and the element \( \sigma \in B_d \) is completely determined by \( \sigma_1, \ldots, \sigma_d \). In one-line notation, we write \( \sigma = \sigma_1 \ldots \sigma_d \). For \( \sigma \in B_d \), the descent set is defined as

\[
\text{Des}_{B}(\sigma) := \{ i \in [0, d-1] : \sigma_i > \sigma_{i+1} \},
\]

where \( \sigma_0 = 0 \) and the type \( B \) descent number is defined as \( \text{des}_{B}(\sigma) := |\text{Des}_{B}(\sigma)| \).

Let

\[
B^+_d := \{ \sigma \in B_d : \sigma_d > 0 \}
\]

and

\[
B^+_{d,j} := \{ \sigma \in B^+_d : \sigma_1 = j \}.
\]

Similarly, for the other half hyperoctahedral group \( B^-_d \), define

\[
B^-_d := \{ \sigma \in B_d : \sigma_d < 0 \}
\]

and

\[
B^-_{d,j} := \{ \sigma \in B^-_d : \sigma_1 = j \}.
\]

Let us define the \( j \)-Eulerian polynomials of type \( B^+ \) by

\[
B^+_{d,j}(t) := \sum_{\sigma \in B^+_{d,j}} t^{\text{des}_{B}(\sigma)} = \sum_{k=0}^{d-1} B^+(d,j,k) t^k,
\]

where \( B^+(d,j,k) \) be the number of elements in \( B^+_{d,j} \) with exactly \( k \) descents. Similarly, define the \( j \)-Eulerian polynomials of type \( B^- \) by

\[
B^-_{d,j}(t) = \sum_{\sigma \in B^-_{d,j}} t^{\text{des}_{B}(\sigma)} = \sum_{k=0}^{d-1} B^-(d,j,k) t^k,
\]
where \( B^-(d, j, k) \) be the number of elements in \( B_{d,j}^- \) with exactly \( k \) descents. Since \( B_{d,j} = B_{d,j}^+ \cup B_{d,j}^- \) so the \( j \)-Eulerian polynomial \( B_{d,j} \) of type \( B \) is

\[
B_{d,j}(t) = B_{d,j}^+(t) + B_{d,j}^-(t).
\]

Here, we list \( B_{d,j}^+(t) \) for \( d = 4 \) and \( 1 \leq s \leq 4 \):

\[
\begin{align*}
B_{4,1}^+(t) &= 1 + 16t + 7t^2 \\
B_{4,2}^+(t) &= 14t + 10t^2 \\
B_{4,3}^+(t) &= 10t + 14t^2 \\
B_{4,4}^+(t) &= 7t + 16t^2 + t^3
\end{align*}
\]

The transformation of \( f \)-vector of \( \Delta \) to \( f \)-vector of interval subdivision \( \text{Int}(\Delta) \) is given by the matrix:

\[
\mathcal{F}_d = [b_{k,l}]_{0 \leq k, l \leq d},
\]

where

\[
b_{0,l} = \begin{cases} 1, & l = 0; \\ 0, & l > 0. \end{cases}
\]

and for \( 1 \leq k \leq d \), we have

\[
b_{k,l} = \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} [(2k - 2j)^l - (2k - 2j - 1)^l]
\]

\[
(8)
\]

**h-Vector of Interval Subdivision.** Let \( \mathcal{H}_d \) be the transformation matrix from \( f \)-vector to \( h \)-vector, then

\[
\mathcal{H}_d = [(-1)^{i+j} \binom{d-i}{j-i}]_{0 \leq i, j \leq d}
\]

and the inverse transformation is

\[
\mathcal{H}_d^{-1} = [(d-i)_{j-i}]_{0 \leq i, j \leq d}
\]

Thus,

\[
h(\text{Int}(\Delta)) = \mathcal{H}_d \mathcal{F}_d \mathcal{H}_d^{-1} h(\Delta)
\]

Let's denote it by

\[
\mathcal{R}_d = \mathcal{H}_d \mathcal{F}_d \mathcal{H}_d^{-1}.
\]

For example,

\[
\mathcal{R}_4 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
61 & 46 & 32 & 22 & 15 \\
115 & 124 & 128 & 124 & 115 \\
15 & 22 & 32 & 46 & 61 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

In the following theorem, it has been shown that this transformation possesses a nice combinatorial description.
Theorem 3.1. The entries of the matrix $\mathcal{R}_d$ are given as:

$$\mathcal{R}_d = [B^+(d+1, s+1, r)]_{0 \leq s,r \leq d}$$

To prove the above theorem, we need the following lemma regarding the recurrence relation of entries $b_{r,l}$ of matrix $\mathcal{F}_d$.

Lemma 3.2. For $1 \leq r \leq d - 1$ and $1 \leq l \leq d$,

$$\sum_{i=1}^{l} 2^i \binom{l}{i} b_{r,l-i} = b_{r+1,l}.$$

Proof. Using (8), we have

$$\sum_{i=1}^{l} 2^i \binom{l}{i} b_{r,l-i} = \sum_{i=1}^{l} 2^i \binom{l}{i} \sum_{j=0}^{r-1} (-1)^j \binom{r-1}{j} [(2r - 2j)^{l-i} - (2r - 2j - 1)^{l-i}]$$

$$= \sum_{j=0}^{r-1} (-1)^j \binom{r-1}{j} \sum_{i=1}^{l} 2^i \binom{l}{i} [(2r - 2j)^{l-i} - (2r - 2j - 1)^{l-i}]$$

$$= \sum_{j=0}^{r-1} (-1)^j \binom{r-1}{j} [(2r - 2j + 2)^l - (2r - 2j + 1)^l - (2r - 2j)^l + (2r - 2j - 1)^l]$$

By re-summing, we get

$$(2r + 2)^l - (2r + 1)^l + \sum_{j=0}^{r-2} (-1)^{j+1} \binom{r-1}{j} + \binom{r-1}{j+1} [(2r - 2j)^l]$$

$$- (2r - 2j - 1)^l + (-1)^r (2^l - 1)$$

$$= \sum_{j=0}^{r} (-1)^j \binom{r}{j} [(2(r + 1) - 2j)^l - (2(r + 1) - 2j - 1)^l]$$

$$= b_{r+1,l}.$$

Proof of Theorem 3.1 Let $C_{r,s}$ be the $(r,s)$-entry of the matrix $C = \mathcal{F}_d \mathcal{H}_d^{-1}$. We have

$$C_{r,s} = \sum_{l=0}^{d} \binom{d-s}{l-s} b_{r,l} = \sum_{l=0}^{d} \binom{d-s}{d-l} b_{r,l}.$$

Let $C_{r,s}$ denote the set of all set partitions $A = A_0 | A_1 | \ldots | A_r$ of rank $r$ of $d+1$ elements among from $\{\pm 1, \pm 2, \ldots, \pm (d+1)\}$ for which exactly one of $\pm i$ appears in $A$ with $\min A_0 = s + 1$ and $\max A_r > 0$. To form such a set partition, we first choose $d - l$ elements from $\{s+2, \ldots, d+1\}$ to put in $A_0$ along with $\min A_0 = s + 1$. This can be done in $\binom{d-s}{d-l}$ ways. For $r = 0$, we have $\binom{d-s}{d-l} = \binom{d-s}{l-s} = C_{0,s}$ such set partitions. For $r > 0$, to form $A_1 | \ldots | A_r$, we need to create a set partition from the remaining $l$ elements, and this can be done in
\[ b_{r,l} \text{ ways. Let's prove this claim by induction on } r. \text{ For } r = 1, \text{ to form } A_1, \text{ we need to put } l \text{ elements from } \{\pm 1, \ldots, \pm (d + 1)\} \text{ such that } \max A_1 > 0. \text{ Thus we have } 2^l - 1 \text{ ways, which is the same as } b_{1,l}. \text{ Suppose that the number of such set partitions } \{A_1| \ldots |A_r \text{ of } l \text{ elements from } \{\pm 1, \ldots, \pm (d + 1)\} \text{ is } b_{r,l}. \text{ Now, to form such set partition } \{A_1| \ldots |A_{r+1} \text{ of } l \text{ elements, we first choose } i \text{ elements from } l \text{ remaining elements, where } i > 0. \text{ This can be done in } 2^i \binom{l}{i} \text{ ways; and the set partition } A_2 \ldots |A_{r+1} \text{ from remaining } l - i \text{ elements can be done in } b_{r,l-i} \text{ ways (by induction hypothesis). Thus we have } \sum_{i=1}^{l} 2^i \binom{l}{i} b_{r,l-i} \text{ ways to form the required set partitions of rank } r + 1 \text{ of } l \text{ elements. By Lemma [3.2] we have}
\]
\[
\sum_{i=1}^{l} 2^i \binom{l}{i} b_{r,l-i} = b_{r+1,l}.
\]

Thus, \( |C_{r,s}| = C_{r,s} \), the \((r, s)\)-entry of the matrix \( C \).

Let
\[
C_s = \bigcup_{r=0}^{d} C_{r,s}
\]
be the collection of all set partitions of \( d + 1 \) elements from \( \{\pm 1, \pm 2, \ldots, \pm (d + 1)\} \) for which exactly one of \( \pm i \) appears in \( A \) with \( \min A_0 = s + 1 \) and \( \max A_r > 0 \). Let \( C_s(t) \) denote the generating function counting these set partitions according to the number of bars,
\[
C_s(t) = \sum_{A \in C_s} t^{\text{rank } A} = \sum_{r=0}^{d} C_{r,s} t^r.
\]

It can be noted that \( C_s(t) \) is also the generating function for column \( s \) of the matrix \( C \).

But each set partition \( A = A_0|A_1| \ldots |A_r \) can be mapped to a permutation \( \sigma = \sigma(A) \) by removing bars and writing each block in increasing order. Since \( \min A_0 = s + 1 \), this means \( \sigma_1 = s + 1 \), and \( \max A_r > 0 \) means \( \sigma_{d+1} > 0 \). That is, \( \sigma \in B_{d+1,s+1}^+ \). Further, \( \text{Des}_B(\sigma) \subset D \), where \( D = D(A) = \{\{A_0\}, |A_0| + |A_1|, \ldots, |A_0| + |A_1| + \ldots + |A_{r-1}|\} \), i.e., there must be bars in \( A \) where there are descents in \( \sigma \). So we can write
\[
C_s(t) = \sum_{A \in C_s} t^{\text{rank } A}
\]
\[
= \sum_{I \subset \{1, \ldots, d\}} \sum_{A \in C_s, D(A)=I} t^{|I|}
\]
\[
= \sum_{I \subset \{1, \ldots, d\}} \sum_{\sigma \in B_{d+1,s+1}^+, D(\sigma) \subset I} t^{|I|}
\]
\[
= \sum_{\sigma \in B_{d+1,s+1}^+, D(\sigma) \subset I} t^{|I|}
\]
\[
= \sum_{\sigma \in B_{d+1,s+1}^+, \text{Des}_B(\sigma)} t^{|\text{des}_B(\sigma)} \cdot (1 + t)^{d - \text{des}_B(\sigma)}
\]
\[
= (1 + t)^d B_{d+1,s+1}^+ (t/(1 + t)).
\]
Since $C_s(t)$ reads the column $s$ of $F_d^{-1}$, the polynomial $H_d C_s(t) = B^+_d(t)$ reads the column $s$ of $R_d = H_d F_d^{-1}$. Thus the columns of $R_d$ are encoded by the $j$-Eulerian polynomials of type $B^+$.

\[ \square \]

**Corollary 3.3.** Let $\Delta$ be a $(d - 1)$-dimensional simplicial complex such that $h_r(\Delta) \geq 0$ for all $0 \leq r \leq d$. Then for $0 \leq r \leq d$,

\[ h_r(\text{Int}(\Delta)) \geq h_r(\Delta). \]

**Proof.** By Theorem 3.1 we have

\[ h_r(\text{Int}(\Delta)) = \sum_{s=0}^{d} B^+(d + 1, s + 1, r) h_s(\Delta). \]

Since $B^+(d + 1, s + 1, r) \geq 1$ and $h_r(\Delta) \geq 0$ for all $r$, therefore the result follows. \[ \square \]

**Example 3.4.** The above corollary is not true in general. Consider a 3-dimensional simplicial complex $\Delta = (1234, 125, 345)$ with $f$-vector $f(\Delta) = (1, 5, 10, 6, 1)$ and $h$-vector $h(\Delta) = (1, 1, 1, -3, 1)$. Then, $f(\text{Int}(\Delta)) = (1, 92, 380, 480, 192)$ and $h(\text{Int}(\Delta)) = (1, 88, 110, -8, 1)$.

Some basic facts about the numbers $B^+(d, s, r)$ are given in the following lemma:

**Lemma 3.5.** The following relations hold for $d \geq 1$, $1 \leq s \leq d$, $0 \leq r \leq d - 1$:

1. \[
   \sum_{r=0}^{d-1} B^+(d, s, r) = 2^{d-2}(d-1)!.
\]

2. \[
   B^+(d, s, r) = B^+(d, d - s + 1, d - r - 1).
\]

3. \[
   B^+(d, s, r) = \sum_{j=1}^{d-1} B^+(d - 1, -j, r) + \sum_{j=s}^{d-1} B^+(d - 1, j, r) + \sum_{j=1}^{s-1} B^+(d - 1, j, r - 1).
\]

Thus, the recurrence relation holds:

\[
B^+_{d,s}(t) = t \sum_{j=1}^{s-1} B^+_{d-1,j}(t) + \sum_{j=s}^{d-1} B^+_{d-1,j}(t) + \sum_{j=1}^{d-1} B^+_{d-1,-j}(t),
\]

with initial conditions $B^+_{1,1}(t) = 1$ and $B^+_{1,-1}(t) = 0$.

4. \[
   B^+(d, -s, r) = \sum_{j=1}^{d-1} B^+(d - 1, j, r - 1) + \sum_{j=s}^{d-1} B^+(d - 1, -j, r - 1) + \sum_{j=1}^{s-1} B^+(d - 1, -j, r).
\]

Thus, the recurrence relation holds:

\[
B^+_{d,-s}(t) = t \sum_{j=1}^{d-1} B^+_{d-1,j}(t) + \sum_{j=s}^{d-1} B^+_{d-1,-j}(t) + \sum_{j=1}^{s-1} B^+_{d-1,-j}(t).
\]
Proof. (1) follows from the definition of $B^+(d, s, r)$.
(2) Let $B^+(d, s, r)$ denote the set of all elements $\sigma \in B_{d,s}^+$ such that $\text{des}_B(\sigma) = r$. There is a bijection between the sets $B^+(d, s, r)$ and $B^+(d, d-s+1, d-r-1)$ given by $\sigma = (\sigma_1, \ldots, \sigma_d) \mapsto \bar{\sigma} = (\bar{\sigma}_1, \ldots, \bar{\sigma}_d)$, where

$$\bar{\sigma}_i := \begin{cases} d+1 - \sigma_i, & \sigma_i > 0; \\ -(d+1 + \sigma_i), & \sigma_i < 0. \end{cases}$$

Consider the following three possible cases:

- $\sigma_i > 0 > \sigma_{i+1}$ iff $\bar{\sigma}_i > 0 > \bar{\sigma}_{i+1}$
- $\sigma_i > \sigma_{i+1} > 0$ iff $\bar{\sigma}_{i+1} > \bar{\sigma}_i > 0$
- $0 > \sigma_i > \sigma_{i+1}$ iff $0 > \bar{\sigma}_{i+1} > \bar{\sigma}_i$

In the first case, $i \in \text{Des}_B(\sigma)$ iff $i \in \text{Des}_B(\bar{\sigma})$ and in other two cases, we have $i \in \text{Des}_B(\sigma)$ iff $i \notin \text{Des}_B(\bar{\sigma})$. It is clear that 0 is not descent of $\sigma$ and $\bar{\sigma}$. Thus, $\text{des}_B(\sigma) + \text{des}_B(\bar{\sigma}) = d-1$.

(3) The recursion formula follows from the effect of removing $\sigma_1 = s$ from the signed permutation $\sigma$ in $B_{d,s}^+$ with $\text{des}_B(\sigma) = r$. The proof of (4) is similar to (3). \qed

Recall that the $h$-vector $h(\Delta)$ of a $(d-1)$-dimensional simplicial complex $\Delta$ is reciprocal if $h_i(\Delta) = h_{d-1-i}(\Delta)$ for $0 \leq i \leq d$. This condition is equivalent to the $h$-vector satisfying the Dehn-Sommerville relations.

**Corollary 3.6.** Let $\Delta$ be a $(d-1)$-dimensional simplicial complex with reciprocal $h$-vector then $\text{Int}(\Delta)$ has also reciprocal $h$-vector.

**Proof.** The result follows from above Lemma 3.5(2) and Theorem 3.1. \qed

**Some Properties of Transformation Matrices.** In this subsection, we describe some properties of transformation matrices. We know from Theorem 2.2 and Theorem 3.1 that

$$f(\text{Int}(\Delta)) = F_d f(\Delta)$$

and

$$h(\text{Int}(\Delta)) = R_d h(\Delta).$$

**Lemma 3.7.** Let $d \geq 1$.

1. The matrices $F_d$ and $R_d$ are similar.
2. The matrices $F_d$ and $R_d$ are diagonalizable with eigenvalues 1 of multiplicity 2 and eigenvalues $2, 2!2, 3!, \ldots, 2^{d-1}.d!$ of multiplicity 1.

**Proof.** First assertion follows from the facts that the transformation from $f(\Delta)$ to $h(\Delta)$ is an invertible linear transformation and by Theorem 2.2 and Theorem 3.1. The second assertion follows as $F_d$ is an upper triangular matrix with diagonal 1, 1, 2, 2!, 3!, ..., $2^{d-1}.d!$; the first and the second unit vectors are eigenvectors for the eigenvalue 1. \qed

The next result holds from the fact that $F_{d+1}$ and $F_d$ are upper triangular and that if one deletes the $(d + 2)$-nd column and row from $F_{d+1}$ then one obtains $F_d$. 

Remark with non-negative coefficients, then

\[ B \]

Theorem 4.1. Let \( A \) be a descent set \( \text{Des} \) on symmetric group \( S_n \) unimodal.

Therefore, log-concave if there exists \( a \) with real roots. In particular, the \( Des \) for the eigenvalues \( 1, 2, 2!, \ldots, 2^{d-1}.d! \).

The following result holds due to Lemma 6 in [BW08] and above lemmas.

Lemma 3.9. Let \( \Delta \) be a \((d-1)\)-dimensional simplicial complex. Let \( w_1^1, w_2^1, w_2^2, \ldots, w_{2^{d-1}.d!} \) be a basis of eigenvectors of the matrix \( R_{d,i} \), where \( w_1^1, w_2^1 \) are eigenvectors for the eigenvalue \( 1 \) and \( w_{2^i} \) is an eigenvector for the eigenvalue \( 2^i \), \( 1 \leq i \leq d-1 \).

(1) If we expand \( h(\Delta) = a_1^1 w_1^1 + a_2^1 w_2^2 + \sum_{i=1}^{d-1} a_{2^i} w_{2^i} \) in terms of the eigenvectors, then \( a_{2^i} \neq 0 \).

(2) The first and last coordinate entry in \( w_2^1, \ldots, w_{2^{d-1}.d!} \) are zero.

(3) The vectors \( w_1^1 \) and \( w_2^2 \) can be chosen such that \( w_1^1 = (1, i_1, \ldots, i_{2^{d-2}.(d-1)!}, 0) \) and \( w_2^2 = (0, j_1, \ldots, j_{2^{d-2}.(d-1)!}, 1) \).

(4) The vector \( w_d \) can be chosen such that \( w_d = (0, b_1, \ldots, b_{2^{d-2}.(d-1)!}, 0) \) for strictly positive rational numbers \( b_{2^i} \), \( 0 \leq i \leq d-2 \).

Remark 3.10. It can be easily seen that the eigenvector \((v_1, \ldots, v_{d+1})\) of \( F_d \) for the eigenvalue different from \( 2^{d-1}.d! \) satisfies the identity \( \sum_{i=1}^{d+1} v_i = 0 \).

4. \( h \)-POLYNOMIAL IS REAL-ROOTED

We start this section with the description of the \( j \)-Eulerian polynomial of type \( B^+ \) and \( B^- \). Let's recall that a polynomial \( p(x) = \sum_{i=0}^{n} a_i x^i \) with real coefficients is unimodal if there exists \( j \) such that \( a_0 \leq a_1 \leq \cdots \leq a_j \geq a_{j+1} \geq \cdots \geq a_d \). The polynomial is said to be log-concave if \( a_i^2 \geq a_{i-1} a_{i+1} \) for \( 1 \leq i \leq d-1 \). It is well-known that if \( p(x) \) is real-rooted with non-negative coefficients, then \( p(x) \) is log-concave and unimodal. Here, we present the main result of this section.

Theorem 4.1. Let \( \Delta \) be a \((d-1)\)-dimensional simplicial complex such that \( h \)-vector \( h(\Delta) = (h_0(\Delta), \ldots, h_d(\Delta)) \) is non-negative. Then the \( h \)-polynomial

\[ h(\text{Int}(\Delta), t) = \sum_{i=0}^{d} h_i(\text{Int}(\Delta)) t^i \]

has only real roots. In particular, the \( h \)-polynomial \( h(\text{Int}(\Delta), t) \) is a log-concave, and hence unimodal.

To proceed toward its proof, we give some definitions and results about descent statistics on symmetric group \( A_d \) and the hyperoctahedral group \( B_d \). Recall that for \( \sigma \in A_d \), the descent set \( \text{Des}_A(\sigma) = \{ i \in [d-1] : \sigma_i > \sigma_{i+1} \} \) and the descent number \( \text{des}_A(\sigma) = |\text{Des}_A(\sigma)| \). Let's denote the \( j \)-Eulerian polynomial of type \( A \) as

\[ A_d, j(t) = \sum_{k=0}^{d-1} A(d, j, k) t^k, \]  

(9)
where $A(d, j, k)$ is the number of permutations $\sigma \in A_d$ with $\sigma_1 = j$ and $\text{des}_A(\sigma) = k$. The next result gives the recurrence relation for $A_{d,j}(t)$. For $j = 1$, it is already known due to [Car75] and [Gar79].

**Lemma 4.2.** For $d \geq 1$ and $1 \leq j \leq d$, we have

$$A_{d,j}(t) = (1 + t(d - 2))A_{d-1,j} + t(1 - t) \frac{d}{dt}(A_{d-1,j}(t)) \quad (10)$$

*Proof. Let $\sigma = j\sigma_2 \cdots \sigma_{d-1} \in A_{d-1,j}$. For $i \in \{2, \ldots, d\}$, define $\sigma^i$ in $A_{d,j}$ obtained from $\sigma$ by inserting $d$ at the $i$th position. For $2 \leq i \leq d$, we have $\sigma^i \in A_{d,j}$ if and only if $\sigma \in A_{d-1,j}$. Moreover, for $2 \leq i \leq d - 1$, we have

$$\text{des}_A(\sigma^i) = \begin{cases} 
\text{des}_A(\sigma), & \text{if } i - 1 \in \text{Des}_A(\sigma); \\
\text{des}_A(\sigma) + 1, & \text{otherwise}.
\end{cases}$$

and $\text{des}_A(\sigma) = \text{des}_A(\sigma^d)$. Therefore,

$$A_{d,j}(t) = \sum_{\sigma \in A_{d,1}} t^{\text{des}_A(\sigma)} = \sum_{i=2}^{d-1} \left( \sum_{\sigma \in A_{d-1,1}} t^{\text{des}_A(\sigma^i)} \right) + \sum_{\sigma \in A_{d-1,j}} t^{\text{des}_A(\sigma^d)}$$

$$= \sum_{\sigma \in A_{d-1,j}} (\text{des}_A(\sigma)t^{\text{des}_A(\sigma)} + (d - 2 - \text{des}_A(\sigma))t^{\text{des}_A(\sigma)+1}) + A_{d-1,j}(t)$$

$$= (d - 2) \sum_{\sigma \in A_{d-1,j}} t^{\text{des}_A(\sigma)+1} + (1 - t) \sum_{\sigma \in A_{d-1,j}} \text{des}_A(\sigma)t^{\text{des}_A(\sigma)} + A_{d-1,j}(t)$$

$$= (1 + (d - 2)t)A_{d-1,j}(t) + t(1 - t) \frac{d}{dt}(A_{d-1,j}(t)).$$

\qed

The next result gives the recurrence relations for $B^+_{d,j}(t)$ and $B^-_{d,j}(t)$. It has already been proved in [AS13] for $j = 1$.

**Lemma 4.3.** For $d \geq 1$ and $1 \leq j \leq d$, we have

$$B^+_{d,j}(t) = 2(d - 2)tB^+_{d-1,j}(t) + 2t(1 - t) \frac{d}{dt}(B^+_{d-1,j}(t)) + B_{d-1,j}(t) \quad (11)$$

and

$$B^-_{d,j}(t) = 2(d - 2)tB^-_{d-1,j}(t) + 2t(1 - t) \frac{d}{dt}(B^-_{d-1,j}(t)) + tB_{d-1,j}(t) \quad (12)$$

*Proof. We will prove the relation (11) and the proof of (12) follows from the symmetry. Let $\sigma = j\sigma_2 \cdots \sigma_{d-1} \in B_{d-1,j}$. For $i \in \{2, \ldots, d\}$, define $\sigma^i$ and $\sigma^{-i}$ in $B_{d,j}$ obtained from $\sigma$ by inserting $d$ and $-d$ respectively at the $i$th position. For $2 \leq i \leq d - 1$, we have $\sigma^i \in B^+_{d,j}$ (respectively, $\sigma^{-i} \in B^-_{d,j}$) if and only if $\sigma \in B^-_{d,j}$. On the other hand, $\sigma^{-d} \in B^-_{d,j}$ and $\sigma^d \in B^+_{d,j}$ for every $\sigma \in B_{d,j}$. Moreover, for $2 \leq i \leq d - 1$, we have

$$\text{des}_B(\sigma^{\pm i}) = \begin{cases} 
\text{des}_B(\sigma), & \text{if } i - 1 \in \text{Des}_B(\sigma); \\
\text{des}_B(\sigma) + 1, & \text{otherwise}.
\end{cases}$$
and des$_B(\sigma) = \text{des}_B(\sigma^{-d}) - 1$. Therefore,

$$B^{-}_{d,j}(t) = \sum_{\sigma \in B^{-}_{d,j}} t^{\text{des}_B(\sigma)} = \sum_{i=2}^{d-1} \left( \sum_{\sigma \in B^{-}_{d-1,j}} t^{\text{des}_B(\sigma^i)} + t^{\text{des}_B(\sigma^{i-1})} \right) + \sum_{\sigma \in B_{d-1,j}} t^{\text{des}_B(\sigma^{-d})}$$

$$= 2 \sum_{\sigma \in B^{-}_{d-1,j}} (\text{des}_B(\sigma)t^{\text{des}_B(\sigma)} + (d - 2 - \text{des}_B(\sigma))t^{\text{des}_B(\sigma)+1}) + tB_{d-1,j}(t)$$

$$= 2(d - 2) \sum_{\sigma \in B^{-}_{d-1,j}} t^{\text{des}_B(\sigma)+1} + 2(1 - t) \sum_{\sigma \in B_{d-1,j}} \text{des}_B(\sigma)t^{\text{des}_B(\sigma)} + tB_{d-1,j}(t)$$

$$= 2(d - 2)tB^{-}_{d-1,j}(t) + 2t(1 - t) \frac{d}{dt}(B^{-}_{d-1,j}(t)) + tB_{d-1,j}(t).$$

□

**Lemma 4.4.** For $d \geq 1$ and $1 \leq j \leq d$, we have

$$B^{+}_{d,j}(t) = t^d B^{-}_{d,j}(t^{-1})$$

(13)

and

$$B^{+}_{d,-j}(t) = t^d B^{-}_{d,j}(t^{-1})$$

(14)

**Proof.** There is a bijection between $B^{+}_{d,j}$ and $B^{-}_{d,-j}$ given as $\sigma \mapsto \bar{\sigma}$, where $\bar{\sigma} = (-j, -\sigma_2, \ldots, -\sigma_d)$. It is clear that $\sigma \in B^{+}_{d,j}$ if and only if $\bar{\sigma} \in B^{-}_{d,-j}$. As we know that des$_B(\sigma) + \text{des}_B(\bar{\sigma}) = d$, therefore, (13) follows. Similarly, (14) also holds. □

Let

$$T_{d,j}(t) := B^{+}_{d,j}(t^2) + \frac{1}{t} B^{-}_{d,j}(t^2).$$

Since for any $\sigma \in B^{-}_{d,j}$, des$_B(\sigma)$ is always strictly positive. Therefore, there is no constant term involved in $B^{-}_{d,j}(t^2)$. So, the right hand side of the above expression is a polynomial.

**Theorem 4.5.** For $d \geq 1$ and $1 \leq j \leq d$, we have

$$T_{d,j}(t) = [1 + t + (d - 2)t^2]T_{d-1,j}(t) + 2(1 - t^2) \frac{d}{dt}(T_{d-1,j}(t))$$

(15)

**Proof.** Using product and chain rule, we have the following

$$\frac{d}{dt}(T_{d-1,j}(t)) = 2t \left[ \frac{d}{dt}(B^{+}_{d-1,j}(t^2)) + \frac{1}{t} \frac{d}{dt}(B^{-}_{d-1,j}(t^2)) \right] - \frac{1}{t^2} B^{+}_{d-1,j}(t^2)$$

(16)
Using (11), (12) and (16), we have
\[ T_{d,j}(t) = B_{d,j}^+(t^2) + \frac{1}{t} B_{d,j}^-(t^2) \]
\[ = 2(d-2)t^2 B_{d-1,j}^+(t^2) + 2t^2(1-t^2) \frac{d}{dt}(B_{d-1,j}^+(t^2)) + B_{d-1,j}(t^2) + \]
\[ \frac{1}{t}[2(d-2)t^2 B_{d-1,j}^-(t^2) + 2t^2(1-t^2) \frac{d}{dt}(B_{d-1,j}^-(t^2)) + t^2 B_{d-1,j}(t^2)] \]
\[ = 2(d-2)t^2 T_{d-1,j}(t) + 2t^2(1-t^2) \frac{1}{2x} \frac{d}{dt}(T_{d-1,j}(t)) + \]
\[ \frac{1}{t^2} B_{d-1,j}^-(t^2)] + (1+t)B_{d-1,j}(t^2) \]
\[ = 2(d-2)t^2 T_{d-1,j}(t) + t(1-t^2) \frac{d}{dt}(T_{d-1,j}(t)) + \frac{1}{t}(1-t^2)B_{d-1,j}(t^2) + \]
\[ (1+t)[B_{d-1,j}^+(t^2) + B_{d-1,j}^-(t^2)] \]
\[ = [1 + t + (d-2)t^2] T_{d-1,j}(t) + 2(1-t^2) \frac{d}{dt}(T_{d-1,j}(t)). \]

\[ \square \]

The following is the key result for proving real rootedness of \( B_j^+(t) \) and \( B_j^-(t) \). It generalizes the relation [YZ15 Equation (11)] for \( j = 1 \).

**Proposition 4.6.** For \( d \geq 1 \) and \( 1 \leq j \leq d \),
\[ (1+t)^{d-1}A_{d,j}(t) = T_{d,j}(t) \] (17)

**Proof.** We will show that both sides of (17) satisfy the same recurrence relation. It can be easily verified that the equality hold for \( d \leq 2 \). Let
\[ S_{d,j}(t) := (1+t)^{d-1}A_{d,j}(t). \]

It is clear that
\[ \frac{d}{dt}(S_{d-1,j}(t)) = (1+t)^{d-1} \frac{d}{dt}(A_{d-1,j}(t)) + (d-1)S_{d-1,j}(t) \] (18)

Using (10) and (18), we have
\[ S_{d,j}(t) = (1+t)^{d-1}A_{d,j}(t) \]
\[ = (1+t)^{d-1}[(1+t(t-2))A_{d-1,j}(t) + t(1-t) \frac{d}{dt}(A_{d-1,j}(t))] \]
\[ = (1+t)(1+t(t-2))S_{d-1,j}(t) + t(1-t^2) \frac{d}{dt}(S_{d-1,j}(t)) - (d-1)S_{d-1,j}(t) \]
\[ = [1 + t + (d-2)t^2] S_{d-1,j}(t) + t(1-t^2) \frac{d}{dt}(S_{d-1,j}(t)). \]

which gives the same recurrence relation (15) as \( T_{d,j} \). \[ \square \]

Since \( B_{d,j}^+(t^2) \) involves only even powers in \( t \) and \( \frac{1}{t} B_{d,j}^-(t^2) \) involves only odd powers in \( t \), so we have the following corollary.
Corollary 4.7. For \( d \geq 1 \) and \( 1 \leq j \leq d \), we have

\[
B_{d,j}^+(t) = E_2((1 + t)^{d-1}A_{d,j}(t)),
\]

and

\[
B_{d,j}^-(t) = E_2(t(1 + t)^{d-1}A_{d,j}(t)),
\]

where \( E_r \) is the operator on formal series defined by

\[
E_r \left( \sum_{k \geq 0} c_k t^k \right) = \sum_{k \geq 0} c_{rk} t^k.
\]

Let's recall a result from [Bre88] which is a key tool to prove the real rootedness of the polynomials \( B_{d,j}^+(t) \) and \( B_{d,j}^-(t) \).

Theorem 4.8. ([Bre88, Theorem 3.5.4]) Let \( p(x) = \sum_{i=0}^m a_i x^i \) be a polynomial having only real non-positive zeros. Then for each \( r \in \mathbb{N} \), the polynomial \( E_r(p(x)) \) (defined in (21)) has only real non-positive zeros.

Theorem 4.9. The polynomials \( B_{d,j}^+(t) \) and \( B_{d,j}^-(t) \) are real-rooted for all \( d \geq 1 \) and \( 1 \leq j \leq d \).

Proof. It follows from Corollary 4.7, Theorem 4.8 and the fact that \( A_{d,j}(t) \) are real-rooted, see [BW08]. \( \square \)

A collection of polynomials \( f_1, f_2, \ldots, f_k \in \mathbb{R}[t] \) is said to be compatible if for all non-negative real numbers \( c_1, c_2, \ldots, c_k \), the polynomial \( \sum_{i=1}^k c_i f_i \) has only real zeros. The polynomials \( f_1, f_2, \ldots, f_k \in \mathbb{R}[t] \) are pairwise compatible if for all \( i, j \in \{1, \ldots, k\} \), \( f_i \) and \( f_j \) are compatible. By [CS07, 2.2], the polynomials \( f_1, f_2, \ldots, f_k \) with positive leading coefficients are pairwise compatible if they are compatible. In [VS13, Theorem 6.3], the authors gave some conditions under which a set of compatible polynomials are mapped to another set of compatible polynomials.

Theorem 4.10. Given a set of polynomials \( f_1, f_2, \ldots, f_k \in \mathbb{R}[t] \) with positive leading coefficients satisfying for all \( 1 \leq i < j \leq k \) that

1. \( f_i \) and \( f_j \) are compatible, and
2. \( tf_i \) and \( f_j \) are compatible.

Define another set of polynomials \( g_1, \ldots, g_k' \in \mathbb{R}[t] \) by

\[
g_l(t) = \sum_{i=0}^{n_l-1} t f_i + \sum_{i=n_l}^k f_i,
\]

for \( 1 \leq l \leq k' \), \( 0 \leq n_1 \leq n_2 \leq \cdots n_{k'} \leq k \). Then for all \( 1 \leq i < j \leq k' \), we have

a: \( g_i \) and \( g_j \) are compatible, and
b: \( tg_i \) and \( g_j \) are compatible.
**Proof of Theorem 4.1.** Let us fix the order of polynomials $B_{d,j}^+(t)$ for $j \in \{\pm 1, \pm 2, \ldots, \pm d\}$ to apply Theorem 4.10. Define

$$f_i := \begin{cases} B_{d,i}^+(t), & 1 \leq i \leq d; \\ B_{d,i-2d-1}^+(t), & d + 1 \leq i \leq 2d. \end{cases}$$

We claim that the set of polynomials $\{f_i : 1 \leq i \leq 2d\}$ is compatible. We show it by induction on $d$. For $d = 1$, it is trivial. For $d = 2$, we have $f_1 = 1$ and $f_i = t$ for $2 \leq i \leq 4$. It is clear that these polynomials are pairwise compatible. Moreover, $tf_i$ and $f_j$ for $1 \leq i < j \leq 4$ are also compatible.

By Lemma 3.5 (3) and (4), the polynomials $f_i$ satisfy the recurrence relation which has the same form required in Theorem 4.10. Therefore, by induction hypothesis, our claim is true. In particular, $\{f_j = B_{d,j}^+(t) : 1 \leq j \leq d\}$ is compatible for all $d \geq 1$. Since $h_i(\Delta)$ is non-negative for all $i$, the $h$-polynomial

$$h(\text{Int}(\Delta), t) = \sum_{i=0}^{d} h_i(\Delta)B_{d+1,i+1}^+(t)$$

is real-rooted. \hfill \Box

At this point, we are in position to relate our results to the Charney-Davis Conjecture. A $(d - 1)$-dimensional simplicial complex $\Delta$ with non-negative reciprocal $h$-vector satisfies the Charney-Davis Conjecture if $(-1)^{\lfloor \frac{d}{2} \rfloor}h(\Delta, -1) \geq 0$ holds.

**Corollary 4.11.** The Charney-Davis conjecture holds for the interval subdivision of a $(d - 1)$-dimensional simplicial complex $\Delta$ for which $h_i(\Delta) \geq 0$ and $h_i(\Delta) = h_{d-i}(\Delta)$ for $0 \leq i \leq d$.

**Proof.** Since $\Delta$ has $h_i(\Delta) \geq 0$ and $h_i(\Delta) = h_{d-i}(\Delta)$ for $0 \leq i \leq d$ so by Corollary 3.6, $\text{Int}(\Delta)$ has a reciprocal $h$-polynomial $h(\text{Int}(\Delta), t)$. By Theorem 4.1 we also know that $h(\text{Int}(\Delta), t)$ has only real zeros. Since the coefficients of $h(\text{Int}(\Delta), t)$ are non-negative and $h_0(\text{Int}(\Delta)) = 1$, it follows that the zeros of $h(\text{Int}(\Delta), t)$ are all strictly negative. Therefore, if $\beta$ is a zero of $h(\text{Int}(\Delta), t)$ then $1/\beta$ is also a zero. Thus, the zeros are either $-1$ or come in pairs $\beta < -1 < 1/\beta < 0$. If $-1$ is a zero of $h(\text{Int}(\Delta), t)$ then the assertion follows trivially. If $-1$ is not a zero then $d$ must be even and

$$h(\text{Int}(\Delta), -1) = \prod_{i=1}^{d/2} (-1 - \beta_i)(-1 - 1/\beta_i),$$

where for all $1 \leq i \leq d/2$, $\beta_i < -1 < 1/\beta_i < 0$, which shows that $h(\text{Int}(\Delta), -1)$ has sign $(-1)^{d/2}$. Thus

$$(-1)^{d/2}h(\text{Int}(\Delta), -1) \geq 0,$$

which implies the assertion. \hfill \Box
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Abdus Salam School of Mathematical Sciences, Government College University, Lahore, Pakistan

Department of Mathematics, Lahore University of Management Sciences, Lahore, Pakistan

E-mail address: imrananwar@sms.edu.pk
E-mail address: shaheen.nazir@lums.edu.pk