The inertia of stress

Rodrigo Medina

Instituto Venezolano de Investigaciones Científicas,
IVIC, Apartado 21827, Caracas 1020A, Venezuela

We present a simple example in which the importance of the inertial effects of stress is evident. The system is an insulating solid narrow disc whose faces are uniformly charged with charges of equal magnitude and opposite signs. The motion of the system in two different directions is considered. It is shown how the contributions to energy and momentum of the stress that develops inside the solid to balance the electrostatic forces have to be added to the electromagnetic contributions to obtain the results predicted by the relativistic equivalence of mass and energy.

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I. INTRODUCTION

Recently, a proposal for the solution of the century old problem of the self-interaction of a charged particle was presented. One of the puzzles of this problem is that the momentum of the electromagnetic field of a particle with electrostatic energy $U_e$ moving with velocity $v$ is not $U_e \gamma v$ as required by relativity, but is $\frac{1}{2} U_e \gamma v$. It was shown that the discrepancy is due to the neglect of the inertia of the stress that is present in the particle to balance the electrostatic repulsion. Unlike the inertia of energy, which is well known, many physicists are not aware of the inertia of pressure (stress). In many cases such an effect is negligible, but for the case of the stress produced by electrostatic interactions, it is comparable to the inertial effects of the electromagnetic fields. If the inertia of stress is neglected, the calculations are inconsistent. In this paper we give an example in which these considerations are explicitly shown.

II. THE SYSTEM

We consider a solid disc of insulating material with radius $R$ and thickness $h$ such that $h \ll R$ (see Fig. 1). For simplicity we assume that the material has a unit relative dielectric constant, $\epsilon = \epsilon_0$. Both faces of the disc are uniformly charged with opposite charges $Q$ and $-Q$. The axis of the disc is parallel to the $z$ axis. The lower face is positively charged. The surface charge density of the lower face is $\sigma = Q/A$, where $A = \pi R^2$. If we neglect border effects, the electric field is $E = (\sigma/\epsilon_0) \hat{z}$ at any point inside the material and zero outside. The electrostatic energy is

$$U_e = \frac{\epsilon_0}{2} \int dV E^2 = \frac{\sigma^2}{2\epsilon_0} Ah = \frac{Q^2 h}{2\epsilon_0 A}. \quad (1)$$

The electrostatic interactions produce stresses in the solid disc. The stress tensor $\vec{P}$ is defined so that the total force that the surroundings produce on a body is the opposite of the integral of the stress tensor over the surface of the body. That is,

$$\mathbf{F} = - \int_S \mathbf{dA} \cdot \vec{P}. \quad (2)$$

FIG. 1: The electric field of a charged disc.

The opposite charges in the faces attract each other, compressing the body (see Fig. 2). A positive stress develops for surfaces parallel to the faces. In contrast, the repulsion between charged elements of the same face produces a radial stretching of the body. Hence, for surfaces
parallel to the $z$ axis there is a negative stress that balances the repulsion. That is, the stress tensor is diagonal; $P_{11}$ and $P_{22}$ are equal and negative and $P_{33}$ is positive (see Fig. 3).

\[ P_{33} A dh = -dW = dU_e. \] (3)

Then

\[ P_{33} = \frac{1}{A} \frac{dU_e}{dh} = \frac{1}{2\varepsilon_0} \frac{Q^2}{A^2} = \frac{\sigma^2}{2\varepsilon_0}. \] (4)

If the radius is increased, the work done by the stress is equal to the increase in the electrostatic energy

\[ P_{11} 2\pi Rh \, dR = dU_e, \] (5)

and

\[ P_{11} = P_{22} = \frac{1}{2\pi R h} \, dU_e = \frac{1}{h} \frac{dU_e}{h} = -\frac{\sigma^2}{2\varepsilon_0}. \] (6)

If we define $P = \sigma^2/(2\varepsilon_0)$, the stress tensor is

\[ \sigma = \begin{pmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & P \end{pmatrix}. \] (7)

Let $m_0$ be the rest mass of the disc if it were not charged and $U_e$ be the electrostatic energy. Then the equivalence between mass and energy predicts that if the disc moves with velocity $v$, its energy is $(m_0 c^2 + U_e)\gamma$ and its momentum is $(m_0 + U_e/c^2)\gamma v$, where $\gamma = [1 - (v/c)^2]^{-1/2}$.

The contributions of the electromagnetic fields to the energy and momentum are obtained by integrating the energy density $u$ and the Poynting vector $S$ over the volume:

\[ U_{em} = \int u dV = \int \left( \frac{\varepsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2 \right) dV, \] (8)

and

\[ P_{em} = \frac{1}{c^2} \int dV S = \varepsilon_0 \int dV E \times B. \] (9)

We will now evaluate the electromagnetic contributions for the disc moving in two different directions.

### III. Motion Parallel to the Axis of the Disk

Consider the disc moving with velocity $v = \hat{z}v$ (see Fig. 4). Quantities for the moving body are denoted by a prime. The circular faces remain the same, $A' = A$, so the charge density is the same $\sigma' = \sigma$, but the thickness is reduced by the Lorentz contraction $h' = h\gamma^{-1}$. The electric field is also the same $E' = E$. Inside the disc $\partial E/\partial t = 0$, so there is no magnetic field $B' = 0$. Then $u' = u$ and $S' = 0$. Finally

\[ U_{em} = u' Ah' = uAh\gamma^{-1} = U_e\gamma^{-1}, \] (10)

and

\[ P_{em} = 0. \] (11)

We see that there is no contribution to the momentum even though we might have expected $c^{-2}U_e\gamma v$ and that the energy decreases as $\gamma^{-1}$ instead of increasing as $\gamma$. Considering only the electromagnetic fields does not give the correct result.

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IV. MOTION PERPENDICULAR TO THE AXIS OF THE DISK

Now consider the disc moving with velocity \( \mathbf{v} = v \mathbf{x} \) (see Fig. 5). The thickness is the same \( h' = h \), but the faces become elliptical because of Lorentz contraction in the \( x \)-direction. The area is reduced \( A' = A \gamma^{-1} \). The charge density is increased \( \sigma' = \sigma \gamma \) and so is the electric field \( \mathbf{E}' = \gamma \mathbf{E} \). There is also a magnetic field inside the disc produced by the two sheets of opposite currents. The field can be calculated using Ampère’s law

\[
\mathbf{B}' = -\mu_0 \sigma \gamma v \mathbf{y} = \frac{1}{c^2} \mathbf{v} \times \mathbf{E}'.
\]

The Poynting vector is

\[
S' = \frac{1}{\mu_0} \mathbf{E}' \times \mathbf{B}' = \frac{1}{\varepsilon_0} (\sigma \gamma)^2 \mathbf{v},
\]

and the electromagnetic momentum is

\[
P_{\text{em}} = \frac{Ah \gamma^{-1}}{c^2} S' = 2 \frac{U_e}{c^2} \gamma \mathbf{v}.
\]

The electromagnetic energy is

\[
U_{\text{em}} = u' Ah \gamma^{-1} = \left[ \frac{1}{2\varepsilon_0} (\sigma \gamma)^2 + \frac{\mu_0}{2} (\sigma \gamma v)^2 \right] Ah \gamma^{-1}
\]

\[= U_e \gamma \left[ 1 + \left( \frac{\mathbf{v} \cdot \mathbf{v}}{c^2} \right)^2 \right].
\]

In this case the result is also not as might be expected. The energy has an extra \((\mathbf{v}/c)^2\) term and the momentum is twice the expected value. The energy and momentum of the electromagnetic field do not form a four-vector, and the effective mass is anisotropic. Something is missing and is the inertia of stress.

![Disc moving in the x-direction. The thickness is the same, but the area of the faces is reduced by Lorentz contraction. The electric field \( \mathbf{B}' = c^{-2} \mathbf{v} \times \mathbf{E}' \). Therefore there is a contribution to the momentum.](image)

V. THE INERTIA OF STRESS

We will use the following relativistic conventions, \( x^0 = -x_0 = ct, x^1 = x_1 = x, x^2 = x_2 = y, \) and \( x^3 = x_3 = z \). Greek indices take the values 0–3, and Latin indices take the values 1–3. The unit vectors in the direction of the spatial axes are \( \mathbf{e}_i \). Repeated indices indicate an implicit sum. The four-velocity \( u^\alpha \) is related to the velocity \( \mathbf{v} = v_i \mathbf{e}_i \) by, \( u^i = \gamma v_i \) and \( u^0 = c \gamma \).

The relativistic dynamics of a continuous medium is ruled by the energy and momentum conservation equation.

\[
\nabla_\alpha (\Theta^{\alpha \beta} + P^{\alpha \beta}) = f^\beta,
\]

where \( f^\beta \) is the force density four-vector, \( \Theta^{\alpha \beta} \) is the energy, and momentum density four-tensor, and \( P^{\alpha \beta} \) is the stress four-tensor. Both \( \Theta^{\alpha \beta} \) and \( P^{\alpha \beta} \) are symmetric tensors. The spatial components of \( f^\beta \) form the force density, \( \mathbf{f} = f^i \mathbf{e}_i \). The temporal component is proportional to the power density, \( f^0 = (\mathbf{f} \cdot \mathbf{v})/c \). The energy-momentum tensor is obtained from the four-velocity by

\[
\Theta^{\alpha \beta} = \tilde{\mu} u^\alpha u^\beta.
\]

The four-scalar \( \tilde{\mu} \) is the density of the rest mass with respect to the proper volume (the volume of an element at rest). The usual rest mass density is \( \mu = \rho \gamma \). The spatial part of \( \Theta^{\alpha \beta} \) is the momentum current density \( \Theta^{ij} = \gamma v_i v_j \), and \( \Theta^{00} \) is the energy density, \( \Theta^{00} = \rho c^2 \gamma \). The other elements of \( \Theta^{\alpha \beta} \) with only one temporal index are the energy current density vector \( c \Theta^{0i} \mathbf{e}_i \), and the momentum density vector \( c^{-1} \Theta^{0i} \mathbf{e}_i \).

The stress four-tensor \( P^{\alpha \beta} \) reduces to the purely spatial stress tensor when the matter element is at rest. When the element is moving, there are temporal components that contribute to the energy density \( P^{00} \) and to the momentum density \( P^{0i} / c \). That is, the stress has inertial effects. Because \( u^\alpha \) is purely temporal at rest, we have

\[
P^{\alpha \beta} u_\beta = 0.
\]

Equation (18) is valid in any reference frame and can be used to obtain the temporal components of the stress, which are

\[
P^{0i} = \frac{1}{c} P^{ij} v_j,
\]

and

\[
P^{00} = \frac{1}{c^2} P^{ij} v_i v_j,
\]

where \( P^{ij} \) is the stress tensor in that frame.

By separating the spatial and temporal components, Eq. (18) reduces to the momentum and power equations.

\[
\frac{\partial}{\partial t} (\mu \gamma \mathbf{v} + \mathbf{P} \cdot \mathbf{v}/c^2) + \nabla \cdot (\mu \gamma \mathbf{vv} + \mathbf{P}) = \mathbf{f},
\]
and
\[ \frac{\partial}{\partial t} (\mu^2 \gamma + \mathbf{v} \cdot \mathbf{P} \cdot \mathbf{v} / c^2) + \nabla \cdot (\mu^2 \gamma \mathbf{v} + \mathbf{P} \cdot \mathbf{v}) = \mathbf{f} \cdot \mathbf{v}. \] (21)

It is interesting to compare these equations with the non-relativistic ones, which are:
\[ \frac{\partial}{\partial t} (\mu \mathbf{v}) + \nabla \cdot (\mu \mathbf{v} \mathbf{v}) = \mathbf{f}, \] (22)
and
\[ \frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mu \mathbf{u} \mathbf{v}) = \mathbf{f} \cdot \mathbf{v}. \] (23)

Here \( \mu \) is the energy density of matter which includes the kinetic and internal energies. Note that the only contributions of the stress which appear in the non-relativistic equations are those in the divergence terms. Also note that the stress that multiplies the velocity in the time derivative of Eq. (22) does not vanish in the small velocity limit \((v/c \to 0)\). This case is an example in which the non-relativistic limit \((c \to \infty)\) is different from the small velocity limit. That is, the inertia of stress is a purely relativistic phenomenon, which does not have a Newtonian explanation.

VI. STRESS CONTRIBUTIONS TO ENERGY AND MOMENTUM

We now calculate the contributions of stress to the energy and momentum. The contributions are
\[ U_S = \frac{1}{c^2} \int dV \mathbf{v} \cdot \mathbf{P} \cdot \mathbf{v} \] (24)
\[ = \int dV P^{00}, \] (25)
and
\[ P_S = \frac{1}{c^2} \int dV \mathbf{P} \cdot \mathbf{v} \] (26)
\[ = \frac{1}{c} \int dV (P^{0i} \hat{e}_i). \] (27)

The easiest way to obtain the stress tensor for the moving disc is to use the Lorentz transformation of the result for the rest frame, Eq. (17):
\[ P^{\mu \nu} = L^\mu_{\alpha} L^\nu_{\beta} P^{\alpha \beta}, \] (28)
where for motion in the \( z \)-direction, the transformation matrix is
\[ (L^\mu_{\nu}) = \begin{pmatrix} \gamma & 0 & 0 & \beta \gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta \gamma & 0 & 0 & \gamma \end{pmatrix}, \] (29)
where \( \beta = v/c \), and the four-tensor of stress is
\[ (P^{\mu \nu}) = \begin{pmatrix} \beta^2 \gamma^2 P & 0 & 0 & \beta \gamma^2 P \\ 0 & -P & 0 & 0 \\ 0 & 0 & -P & 0 \\ \beta \gamma^2 P & 0 & 0 & \gamma^2 P \end{pmatrix}. \] (30)
The energy of stress is
\[ U_S = (\beta \gamma)^2 PA' = \left( \frac{v}{c} \right)^2 \gamma U_e. \] (31)
The total energy is then
\[ U = U_S + U_{em} = \left( \frac{v}{c} \right)^2 \gamma U_e + \gamma^{-1} U_e \] (32a)
\[ = \left( \frac{v}{c} \right)^2 + \gamma^{-2} \gamma U_e = \gamma U_e, \] (32b)
as expected.

The momentum of stress is
\[ P_S = \frac{1}{c} \beta \gamma^2 PA' \hat{z} = \frac{U_e}{c^2} \gamma \mathbf{v}. \] (33)
The total momentum is also the expected value, \( \mathbf{P} = P_S + P_{em} = (U_e/c^2) \gamma \mathbf{v}. \)

Now let us consider motion in the \( x \)-direction. The matrix of the Lorentz transformation is
\[ (L^\mu_{\nu}) = \begin{pmatrix} \gamma & \beta \gamma & 0 & 0 \\ \beta \gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \] (34)
and the stress is
\[ (P^{\mu \nu}) = \begin{pmatrix} -\beta^2 \gamma^2 P & -\beta \gamma^2 P & 0 & 0 \\ -\beta \gamma^2 P & -\gamma^2 P & 0 & 0 \\ 0 & 0 & -P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}. \] (35)

The energy of stress is
\[ U_S = -(\beta \gamma)^2 PA' h = -\left( \frac{v}{c} \right)^2 \gamma U_e. \] (36)
The total energy is therefore
\[ U = U_{em} + U_S = \gamma U_e \left[ 1 + \left( \frac{v}{c} \right)^2 \right] - \left( \frac{v}{c} \right)^2 \gamma U_e = \gamma U_e, \] (37)
which is the expected result.

The momentum of stress is
\[ P_S = \frac{1}{c} \beta \gamma^2 PA' \hat{x} = -\frac{U_e}{c^2} \gamma \mathbf{v}. \] (38)

In this case the correct total momentum is
\[ \mathbf{P} = P_{em} + P_S = \frac{2U_e}{c^2} \gamma \mathbf{v} - \frac{U_e}{c^2} \gamma \mathbf{v} = \frac{U_e}{c^2} \gamma \mathbf{v}. \] (39)

Everything works if we take into account the contributions of stress.

The energy and momentum of an extended system are obtained by integrating the energy density and the momentum density over the entire volume. The spaces of two different reference frames are different three-dimensional hyperplanes in the four-dimensional Minkowski space. So, in principle, the total energy and momentum of an extended system in different frames are
not the same physical quantities. If energy and momentum are conserved, they form a four-vector. When the electromagnetic field is not free, that is, when there are charges and currents, the field itself is stressed. This stress contributes to the energy and momentum of the field, which is why these quantities do not form a four-vector. The electromagnetic forces produce stresses in matter. The stress of matter also contributes to the energy and momentum, and also do not form a four-vector. The contributions of the stress of matter are exactly opposite to those of the stress of the field. Thus, if the contributions of the stress of matter are added to those of the field, the total energy and momentum transform as a four-vector.

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