The convergence rate of a golden ratio algorithm for equilibrium problems

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Abstract In this paper, we establish the $R$-linear rate of convergence of a golden ratio algorithm for solving an equilibrium problem in a Hilbert space. Several experiments are performed to show the numerical behavior of the algorithm and also to compare with others.

Keywords Equilibrium problem · Strongly pseudomonotone bifunction · Lipschitz-type condition

Mathematics Subject Classification (2000) 65J15 · 47H05 · 47J25 · 47J20 · 91B50.

1 Introduction

The equilibrium problem (EP), in the sense of Blum and Oettli [3, 18], is a general model which unifies in a simple form various mathematical models as optimization problems, variational inequalities, fixed point problems and Nash equilibrium problem, see in [5, 13]. Together with studying the existence of solution, the iterative method for approximating solutions of problem (EP) is also studied and has received a lot of interest by many authors. Some notable iterative methods for solving problem (EP) can be found, for example, in [1, 6, 7, 8, 9, 12, 17, 19, 20, 21] and the references cited therein. In this paper, we concern with the rate of convergence of an algorithm, namely the golden ratio algorithm, for solving problem (EP) in a Hilbert space.

One of the most popular methods for approximating a solution of problem (EP) is the proximal point method [17, 12] which is constructed around the resolvent of cost bifunction. The resolvent of a bifunction is regarding a regularization equilibrium subproblem. The regularization solutions can converge finitely or asymptotically to a solution of original problem (EP). However, the problem of this method is that in practice the computation of a value of resolvent mapping at a point, i.e., finding a regularization solution, is often expensive and we have to use a inner loop. This is of course time-consuming. Another method for
solving problem (EP) is the proximal-like method \[6\] which includes optimization problems. Recently, the authors in \[19\] have investigated and extended further the convergence of the proximal-like method in \[6\] under other assumptions that the cost bifunction is pseudomonotone and satisfies a Lipschitz-type condition. The proximal-like methods in \[6,19\] are also called the extragradient method due to the early results of Korpelevich \[14\] on saddle point problem. We can use some known convex optimization programming to compute in the extragradient method. This is reason to explain why the latter is often easier to solve numerically than the proximal point method in \[12,17\].

In view of the extragradient method in \[6,19\], it is seen that per each iteration we have to solve two convex optimization subproblems on feasible set. This can be expansive if cost bifunction and feasible set have complex structures. Then, in recent years, many variants of the extragradient method have been developed to reduce the complexity of algorithm as well as to weaken assumptions imposed on cost bifunction, see, for example, in \[10,11,15\].

Very recently, motivated by the results of Malitsky in \[16\], Vinh in \[22\] has presented and analyzed the weak convergence of a new algorithm, namely Golden Ratio Algorithm (GRA), under the two hypotheses of pseudomonotonicity and Lipschitz-type condition imposed on cost bifunction. The main advantage of GRA is that over each iteration the algorithm only requires to compute a convex optimization program. A modification of GRA is also presented where the algorithm uses a sequence of variable stepsizes being diminishing and nonsummable. By strengthening the pseudomonotonicity by the strongly pseudomonotonicity, Vinh has established the strong convergence of this modification. A question arises naturally here as follows:

**Question:** Can we establish the convergence rate of GRA?

In this paper, we give a positive answer to the aforementioned question. Under the assumptions of strongly pseudomonotonicity and Lipschitz-type condition of cost bifunction, we prove that algorithm GRA converges \(R\)-linearly. The remainder of this paper is organized as follows: In Sect. 2 we collect some definitions and preliminary results used in the paper. Sect. 3 deals with presenting algorithm GRA and establishing its convergence rate. Finally, in Sect. 4 several numerical results are reported to show the behavior of the algorithm in comparison with others.

## 2 Preliminaries

Let \(H\) be a real Hilbert space and \(C\) be a nonempty closed convex subset of \(H\). Let \(f : C \times C \to \mathbb{R}\) be a bifunction with \(f(x,x) = 0\) for all \(x \in C\). The equilibrium problem for the bifunction \(f\) on \(C\) is stated as follows:

\[
\text{Find } x^* \in C \text{ such that } f(x^*,y) \geq 0, \ \forall y \in C. \quad (EP)
\]

Let \(g : C \to \mathbb{R}\) be a proper, lower semicontinuous, convex function. The proximal operator \(\text{prox}_{\lambda g}\) of \(g\) with some \(\lambda > 0\) is defined by

\[
\text{prox}_{\lambda g}(z) = \arg\min \left\{ \lambda g(x) + \frac{1}{2}||x-z||^2 : x \in C \right\}, \quad z \in H.
\]

The following is an important property of the proximal mapping (see \[2\] for more details)
Throughout this paper, we assume that the function $F$ or all $x$ in any Hilbert space, we have the following result, see, e.g., in [2, Corollary 2.14].

Lemma 2.2 For all $x, y \in H$ and $\alpha \in \mathbb{R}$, the following equality always holds

$$||\alpha x + (1 - \alpha)y||^2 = \alpha||x||^2 + (1 - \alpha)||y||^2 - \alpha(1 - \alpha)||x - y||^2.$$ 

Throughout this paper, we assume that the function $f(\cdot,y)$ is convex, lower semicontinuous and $f(x,\cdot)$ is hemicontinuous on $C$ for each $x, y \in C$. In order to establish the rate of convergence of algorithm GRA, we consider the following assumptions:

(SP): $f$ is strongly pseudomonotone, i.e., there exists a constant $\gamma > 0$ such that

$$f(x,y) \geq 0 \implies f(y,x) \leq -\gamma||x - y||^2 \text{ for all } x, y \in C.$$ 

(LC): $f$ satisfies the Lipschitz-type condition, i.e., there exist $c_1 > 0, c_2 > 0$ such that

$$f(x,y) + f(y,z) \geq f(x,z) - c_1||x - y||^2 - c_2||y - z||^2 \text{ for all } x, y, z \in C.$$ 

Under these assumptions, problem (EP) has the unique solution, denoted by $x^\dagger$. For solving problem (EP) with the conditions (SP) and (LC), Vinh in [22] introduced the following modified golden ratio algorithm (MGRA1): Set $\varphi = \frac{\sqrt{5} - 1}{2}$ (the golden ratio), take $x_0, y_1 \in C$ and compute

$$\begin{cases} 
  x_n = \frac{(\varphi - 1)x_n + x_{n-1}}{\varphi}, \\
  y_{n+1} = \arg \min \{ \lambda_n f(y_n, y) + \frac{1}{2}||x_n - y||^2 : y \in C \}, 
\end{cases} \quad (\text{MGRA1})$$

where $\{\lambda_n\} \subset (0, +\infty)$ is a sequence satisfying the hypotheses:

$$\begin{align*}
\text{(C1): } & \lim_{n \to \infty} \lambda_n = 0, \\
\text{(C2): } & \sum_{n=0}^{\infty} \lambda_n = +\infty.
\end{align*}$$

Without the condition (LC), Vinh introduced the second modified golden ratio algorithm (MGRA2) which is based on the early results in [20]: Take $x_0, y_1 \in C$ and compute

$$\begin{cases} 
  x_n = \frac{(\varphi - 1)x_n + x_{n-1}}{\varphi}, \\
  g_n \in \partial f(y_n, \cdot)(y_n), \\
  \lambda_n = \frac{\beta_n}{\max\{1, ||g_n||\}}, \\
  y_{n+1} = P_C(x_n - \lambda_n g_n), 
\end{cases} \quad (\text{MGRA2})$$

where $\{\beta_n\} \subset (0, +\infty)$ is a sequence satisfying the hypotheses:

$$\begin{align*}
\text{(C3): } & \sum_{n=0}^{\infty} \beta_n = +\infty, \\
\text{(C4): } & \sum_{n=0}^{\infty} \beta_n^2 < +\infty.
\end{align*}$$

In [22], Vinh proved that the sequence $\{x_n\}$ generated by MGRA1 or MGRA2 converges strongly to the unique solution $x^\dagger$ of problem (EP) without any estimate of the rate of convergence of the sequence $\{x_n\}$. We remark here that algorithm MGRA1 cannot converge
linearly. Indeed, consider our problem for $C = H = \mathbb{R}$ and $f(x, y) = x(y - x)$. The unique solution of the problem is $x^* = 0$. It is easy to see that $f$ satisfies the conditions (SP) and (LC). It follows from the definition of $y_{n+1}$ and a simple computation that

$$y_{n+1} = x_n - \lambda_n y_n.$$  

(1)

Since $x_n = \frac{(\phi - 1)x_n + x_{n-1}}{\phi}$, we obtain that $y_n = \frac{1}{\phi - 1}x_n - \frac{1}{\phi}y_{n-1}$. Thus, from the relation (1), we obtain

$$\frac{\phi}{\phi - 1}y_{n+1} - \frac{1}{\phi - 1}x_n = y_n = x_n - \lambda_n \left[ \frac{\phi}{\phi - 1}x_n - \frac{1}{\phi - 1}y_{n-1} \right].$$

or

$$x_{n+1} = (1 - \lambda_n)x_n + \frac{\lambda_n}{\phi}y_{n-1}. \tag{2}$$

Since $\lambda_n \to 0$, without loss of generality, we can assume that $\{\lambda_n\} \subset (0, 1)$. If choose $x_0, y_1 > 0$, from the definition of $x_1$, we obtain that $x_1 > 0$. Thus, from the relation (2), we get by the induction that $x_n > 0$ for all $n \geq 0$. Now, assume that $\{x_n\}$ converges linearly to the solution $x^* = 0$, i.e., there exists a number $\sigma \in (0, 1)$ such that $||x_{n+1} - x^*|| \leq \sigma ||x_n - x^*||$ or $x_{n+1} \leq \sigma x_n$ for all $n \geq 0$. This together with the relation (2) implies that

$$x_{n+1} = (1 - \lambda_n)x_n + \frac{\lambda_n}{\phi}y_{n-1} \geq (1 - \lambda_n)\frac{x_n + \lambda_n}{\phi} \geq (1 - \lambda_n)\frac{x_n}{\phi} \geq (1 - \lambda_n)\frac{x_n}{\phi} \geq \frac{\lambda_n}{\phi}x_n + \frac{\lambda_n}{\phi}y_{n-1}.$$  

Thus

$$1 \geq \frac{1 - \lambda_n}{\phi} + \frac{\lambda_n}{\phi} \sigma^2, \text{ } \forall n \geq 0. \tag{3}$$

Passing to the limit in the relation (3) as $n \to \infty$ and using the hypothesis (C1), we obtain that $1 \geq \frac{1}{\phi}$. This is contrary because $\sigma \in (0, 1)$. This says that the sequence $\{x_n\}$ does not converge linearly to the solution $x^* = 0$. Thus, algorithm MGRA1 cannot converge linearly.

In the next section, we will present in details algorithm GRA with a fixed stepsize and establish the $R$-linear rate of convergence of the algorithm.

### 3 The $R$-linear rate of convergence of GRA

In this section, we study the rate of convergence of the following golden ratio algorithm.

**Algorithm 3.1 (Golden Ratio Algorithm for Equilibrium Problem)**.

**Initialization**: Set $\varphi = \frac{\sqrt{5} - 1}{2}$. Choose $\bar{x}_0 \in H$, $x_1 \in C$ and $\lambda \in \mathbb{R}$ such that $0 < \lambda < \varphi \frac{\max\{c_1, c_2\}}{4 \max\{c_1, c_2\}}$.

**Iterative Steps**: Assume that $\bar{x}_{n-1} \in H$, $x_n \in C$ are known, calculate $x_{n+1}$ as follows:

$$\begin{cases} 
\tilde{x}_n = \frac{(\varphi - 1)x_n + x_{n-1}}{\varphi}, \\
x_{n+1} = \text{prox}_{\lambda f}(\tilde{x}_n).
\end{cases}$$

We can use the following stopping criterion for Algorithm 3.1. If $x_{n+1} = x_n = \tilde{x}_n$, then stop and $\tilde{x}_n$ is the solution of problem (EP). This follows from the definition of $x_{n+1}$ and Remark 2.4. Thus, if Algorithm 3.1 terminates then the solution of the problem can be found. Otherwise, we have the following main result.
Theorem 3.1 Under the conditions (SM) and (LC), the sequence \( \{x_n\} \) generated by Algorithm 3.1 converges R-linearly to the unique solution \( x^* \) of problem (EP).

Proof It follows from the definition of \( x_{n+1} \) and Lemma 2.1 that

\[
\langle \bar{x}_n - x_{n+1}, x - x_{n+1} \rangle \leq \lambda (f(x_n, x) - f(x_{n+1}, x_{n+1})), \quad \forall x \in C,
\]

which, with \( x = x^t \), follows that

\[
2 \langle \bar{x}_n - x_{n+1}, x^t - x_{n+1} \rangle \leq 2\lambda (f(x_n, x^t) - f(x_{n+1}, x_{n+1})).
\]

Applying the equality \( 2 (a, b) = ||a||^2 + ||b||^2 - ||a - b||^2 \) to the relation (6), we obtain

\[
||\bar{x}_n - x_{n+1}||^2 + ||x_{n+1} - x^t||^2 - ||\bar{x}_n - x^t||^2 \leq 2\lambda (f(x_n, x^t) - f(x_{n+1}, x_{n+1})).
\]

Using the relation (4) with \( n := n - 1 \), we obtain

\[
\langle \bar{x}_{n-1} - x_n, x - x_n \rangle \leq \lambda (f(x_n, x) - f(x_{n-1}, x_{n-1})), \quad \forall x \in C.
\]

Substituting \( x = x_{n+1} \) into (7), we get

\[
\langle \bar{x}_{n-1} - x_n, x_{n+1} - x_n \rangle \leq \lambda (f(x_{n+1}, x_{n+1}) - f(x_{n-1}, x_n)).
\]

Noting that from the definition of \( \bar{x}_n \), we have \( \bar{x}_{n-1} - x_n = \varphi (\bar{x}_n - x_n) \). Then, from the relation (8), we have the following estimate,

\[
2\varphi \langle \bar{x}_n - x_{n+1}, x_{n+1} - x_n \rangle \leq 2\lambda (f(x_{n+1}, x_{n+1}) - f(x_{n-1}, x_n)).
\]

Thus, as the relation (9), developing the product \( 2 \langle \bar{x}_n - x_n, x_{n+1} - x_n \rangle \), we get

\[
\varphi ||\bar{x}_n - x_n||^2 + \varphi ||x_{n+1} - x_n||^2 - \varphi ||x_{n+1} - \bar{x}_n||^2 \leq 2\lambda (f(x_{n+1}, x_{n+1}) - f(x_{n-1}, x_n)).
\]

Adding the relations (6) and (9), and using the condition (LC), we obtain

\[
||x_{n+1} - x^t||^2 - ||\bar{x}_n - x^t||^2 + (1 - \varphi) ||x_{n+1} - \bar{x}_n||^2 + \varphi ||\bar{x}_n - x_n||^2 + 2\lambda f(x_{n+1}, x^t) \leq 2\lambda f(x_{n+1}, x^t) + 2\lambda c_1 ||x_{n+1} - x_n||^2 + 2\lambda c_2 ||x_n - x_{n+1}||^2.
\]

Since \( 0 < \lambda < \frac{\varphi}{\max(x_n, x^t)} \), we find that \( \lambda < \frac{\varphi}{4c_1} \) and \( \lambda < \frac{\varphi}{4c_2} \). Thus, there exists a number \( \varepsilon \in (0, 1) \) such that

\[
\lambda < \frac{\varepsilon \varphi}{4c_1} \quad \text{and} \quad \lambda < \frac{\varepsilon \varphi}{4c_2}.
\]

Combining these inequalities with the relation (10), we get

\[
||x_{n+1} - x^t||^2 - ||\bar{x}_n - x^t||^2 + (1 - \varphi) ||x_{n+1} - \bar{x}_n||^2 + \varphi ||\bar{x}_n - x_n||^2 \leq 2\lambda f(x_{n+1}, x^t) + \frac{\varepsilon \varphi}{2} ||x_{n+1} - x_n||^2 + \frac{\varepsilon \varphi}{2} ||x_n - x_{n+1}||^2.
\]
Since $x^*$ is the solution of problem (EP) and $x_0 \in C$, $f(x_0, x^*) \geq 0$. Hence, from the strong pseudomonotonicity of $f$, we derive $f(x_n, x^*) \leq -\gamma ||x_n - x^*||^2$. This together with the last inequality implies that
\[
||x_{n+1} - x^*||^2 + \frac{\phi}{2}||x_n - x_{n+1}||^2 \leq ||x_n - x^*||^2 + \frac{\phi}{2}||x_{n-1} - x_n||^2
\]
\[- (1 - \phi)||x_{n+1} - \bar{x}_n||^2 - \phi||\bar{x}_n - x_n||^2 - 2\lambda \gamma ||x_n - x^*||^2.
\] (11)

Moreover, from the definition of $\bar{x}_n$ and Lemma 2.2, we have
\[
||x_{n+1} - x^*||^2 = \frac{\phi}{\phi - 1}||x_{n+1} - x^*||^2 - \frac{1}{\phi - 1}||\bar{x}_n - x^*||^2
\]
\[+ \frac{\phi}{(\phi - 1)^2}||\bar{x}_{n+1} - \bar{x}_n||^2
\leq \frac{\phi}{\phi - 1}||x_{n+1} - x^*||^2 - \frac{1}{\phi - 1}||\bar{x}_n - x^*||^2
\]
\[+ \frac{1}{\phi}||x_{n+1} - \bar{x}_n||^2.
\] (12)

Thus, from the relations (11), (12) and the fact $1 - \phi - \frac{1}{\phi} = 0$, we obtain
\[
\frac{\phi}{\phi - 1}||x_{n+1} - x^*||^2 + \frac{\phi}{2}||x_n - x_{n+1}||^2 \leq \frac{\phi}{\phi - 1}||\bar{x}_n - x^*||^2 + \frac{\phi}{2}||x_{n-1} - x_n||^2
\]
\[- (1 - \phi)||x_{n+1} - \bar{x}_n||^2 - \phi||\bar{x}_n - x_n||^2 - 2\lambda \gamma ||x_n - x^*||^2
\]
\[\leq \frac{\phi}{\phi - 1}||\bar{x}_n - x^*||^2 + \frac{\phi}{2}||x_{n-1} - x_n||^2 - 2\lambda \gamma ||x_n - x^*||^2.
\] (13)

Note that from the definition of $\bar{x}_n$, we obtain $x_n = \frac{\phi}{\phi - 1}\bar{x}_n - \frac{1}{\phi - 1}\bar{x}_{n-1}$. Thus, it follows from Lemma 2.2 that
\[
||x_n - x^*||^2 = \frac{\phi}{\phi - 1}||\bar{x}_n - x^*||^2 - \frac{1}{\phi - 1}||\bar{x}_{n-1} - x^*||^2 + \frac{\phi}{(\phi - 1)^2}||\bar{x}_n - \bar{x}_{n-1}||^2
\]
\[\geq \frac{\phi}{\phi - 1}||\bar{x}_n - x^*||^2 - \frac{1}{\phi - 1}||\bar{x}_{n-1} - x^*||^2.
\] (14)

From the relations (13) and (14), we see that
\[
\frac{\phi}{\phi - 1}||\bar{x}_{n+1} - x^*||^2 + \frac{\phi}{2}||x_{n+1} - x_n||^2 \leq \frac{\phi}{\phi - 1}(1 - 2\gamma \lambda)||\bar{x}_n - x^*||^2
\]
\[+ \frac{2\gamma \lambda}{\phi - 1}||\bar{x}_{n-1} - x^*||^2 + \frac{\phi}{2}||x_n - x_{n-1}||^2.
\] (15)

Set $\alpha_n = \frac{\phi}{\phi - 1}||\bar{x}_n - x^*||^2$, $b_n = \frac{\phi}{2}||x_n - x_{n-1}||^2$ and $\alpha = 2\lambda \gamma$, the inequality (15) can be rewritten as
\[
a_{n+1} + b_{n+1} \leq (1 - \alpha)a_n + \frac{\alpha}{\phi}a_{n-1} + \epsilon b_n.
\] (16)

Let $r_1 > 0$ and $r_2 > 0$. Now, the relation (16) can be rewritten as
\[
a_{n+1} + r_1a_n + b_{n+1} \leq r_2(a_n + r_1a_{n-1}) + \epsilon b_n + (1 - \alpha - r_2)a_n + \frac{\alpha}{\phi}(r_1r_2)a_{n-1}.
\] (17)
Choose \( r_1 > 0 \) and \( r_2 > 0 \) such that \( 1 - \alpha - r_2 + r_1 = 0 \) and \( \frac{r_2}{\phi} - r_1 r_2 = 0 \). Thus

\[
    r_1 = \frac{\alpha - 1 + \sqrt{(\alpha - 1)^2 + \frac{2\alpha}{\phi}}}{2} \quad \text{and} \quad r_2 = \frac{1 - \alpha + \sqrt{(\alpha - 1)^2 + \frac{2\alpha}{\phi}}}{2}.
\]

Consider the function

\[
    f(t) = \frac{1 - t + \sqrt{(t-1)^2 + \frac{2t}{\phi}}}{2}, \quad t \in [0, +\infty).
\]

We have

\[
    f'(t) = \frac{-1 + \frac{t-1+\frac{1}{\phi\sqrt{(t-1)^2 + \frac{2t}{\phi}}}}{2}}{2 \sqrt{(t-1)^2 + \frac{2t}{\phi}}} < 0
\]

because of \( \frac{1}{\phi} < 1 \). Thus, \( f(t) \) is non-increasing on \( [0, +\infty) \). Hence, \( 0 < r_2 = f(\alpha) < f(0) = 1 \). Now, set \( \theta = \max \{ \varepsilon, r_2 \} \) and note that \( \theta \in (0, 1) \) then from the relation (17), we obtain

\[
    a_{n+1} + r_1 a_n + b_{n+1} \leq \theta(a_n + r_1 a_{n-1} + b_n), \; \forall n \geq 1. \tag{18}
\]

Thus, we obtain by the induction that

\[
    a_{n+1} + r_1 a_n + b_{n+1} \leq \theta^n(a_1 + r_1 a_0 + b_1). \tag{19}
\]

We can reduce that

\[
    \frac{\theta^n}{\varphi-1}||\tilde{x}_{n+1} - x^*||^2 = a_{n+1} \leq \theta^n(a_1 + r_1 a_0 + b_1),
\]

or

\[
    ||\tilde{x}_{n+1} - x^*||^2 \leq M\theta^n,
\]

where \( M = \frac{\varphi-1}{\varphi} (a_1 + r_1 a_0 + b_1) \), or the sequence \( \{ \tilde{x}_n \} \) converges R-linearly. Since \( x_n = \frac{\varphi}{\varphi-1} \tilde{x}_n - \frac{1}{\varphi-1} \tilde{x}_{n-1} = \left( 1 + \frac{1}{\varphi-1} \right) \tilde{x}_n - \frac{1}{\varphi-1} \tilde{x}_{n-1} \), we derive

\[
    ||x_n - x^*|| \leq \left( 1 + \frac{1}{\varphi-1} \right) ||\tilde{x}_n - x^*|| + \frac{1}{\varphi-1} ||\tilde{x}_{n-1} - x^*||.
\]

Hence, the sequence \( \{ x_n \} \) also converges R-linearly. This completes the proof.
4 Numerical experiments

In this section, we present some experiments to illustrate the numerical behavior of Algorithm 3.1 (GRA) and also to compare with other algorithms. Two algorithms used here to compare are MGRA1 [22, Algorithm 4.1] and MGRA2 [22, Algorithm 4.2]. The convergence of these algorithms is established under a same assumption of strong pseudomonotonicity of bifunction, while the Lipschitz-type condition are only for Algorithm 3.1 and MGRA1. In order to show the computational performance of the algorithms, we describe the behavior of the sequence $D_n = ||x_n - \text{prox}_{f}(\alpha_n)(x_n)||^2$, $n = 0, 1, 2, \ldots$ when number of iterations (# Iterations) is performed or execution time (Elapsed Time) in the second elapses. Noting that $D_n = 0$ iff $x_n$ is the solution of the considered problem.

Considering a generalization of the Nash-Cournot oligopolistic equilibrium model in [4,5] with the affine price and fee-fax functions. Assume that there are $m$ companies that produce a commodity. Let $x$ denote the vector whose entry $x_j$ stands for the quantity of the commodity produced by company $j$. We suppose that the price $p_j(x)$ is a decreasing affine function of $s$ with $s = \sum_{j=1}^{m} x_j$, i.e., $p_j(s) = \alpha_j - \beta_j s$, where $\alpha_j > 0$, $\beta_j > 0$. Then the profit made by company $j$ is given by $f_j(x) = p_j(s)x_j - c_j(x_j)$, where $c_j(x_j)$ is the tax and fee for generating $x_j$. Suppose that $C_j = [x_{j\min}, x_{j\max}]$ is the strategy set of company $j$, then the strategy set of the model is $C := C_1 \times C_2 \times \ldots \times C_m$. Actually, each company seeks to maximize its profit by choosing the corresponding production level under the presumption that the production of the other companies is a parametric input. A commonly used approach to this model is based upon the famous Nash equilibrium concept. We recall that a point $x^* \in C = C_1 \times C_2 \times \ldots \times C_m$ is an equilibrium point of the model if

$$f_j(x^*) \geq f_j(x^*[x_j]) \quad \forall x_j \in C_j, \quad \forall j = 1, 2, \ldots, m,$$

where the vector $x^*[x_j]$ stands for the vector obtained from $x^*$ by replacing $x_j^*$ with $x_j$. By taking $f(x,y) := \psi(x,y) - \psi(x,x)$ with $\psi(x,y) := -\sum_{j=1}^{m} f_j(x|y_j)$, the problem of finding a Nash equilibrium point of the model can be formulated as:

$$\text{Find } x^* \in C \text{ such that } f(x^*,x) \geq 0 \quad \forall x \in C.$$ 

Now, assume that the tax-fee function $c_j(x_j)$ is increasing and affine for every $j$. This assumption means that both of the tax and fee for producing a unit are increasing as the quantity of the production gets larger. In that case, the bifunction $f$ can be formulated in the form

$$f(x,y) = \langle Px + Qy + q, y - x \rangle,$$

where $q \in \mathbb{R}^m$ and $P, Q$ are two matrices of order $m$ such that $Q$ is symmetric positive semidefinite and $Q - P$ is symmetric negative semidefinite. The bifunction $f$ satisfies the Lipschitz-type condition with $c_1 = c_2 = ||P - Q||/2$. In order to assure that all the algorithms can work, the data here are generated randomly such that $Q - P$ is symmetric negative definite. In this case, the bifunction $f$ is strongly pseudomonotone. We perform the numerical computations in $\mathbb{R}^m$ with $m = 100, 200, 300$; the feasible set is $C = [-2,5]^m$; the starting point $x_1 = x_0$ are generated randomly in the interval $[0,1]$. The data are generated as follows: All the entries of $q$ is generated randomly and uniformly in $(-2,2)$ and the two matrices $P, Q$ are also generated randomly such that their conditions hold. All the

1 We randomly choose $\lambda_{1k} \in (-2,0)$, $\lambda_{2k} \in (0,2)$, $k = 1, \ldots, m$. We set $\bar{Q}_1, \bar{Q}_2$ as two diagonal matrixes with eigenvalues $\{\lambda_{1k}\}_{k=1}^{m}$ and $\{\lambda_{2k}\}_{k=1}^{m}$, respectively. Then, we construct a positive semidefinite matrix $Q$ and a negative definite matrix $T$ by using random orthogonal matrixes with $\bar{Q}_2$ and $\bar{Q}_1$, respectively. Finally, we set $P = Q + T$. 

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optimization subproblems are effectively solved by the function \textit{quadprog} in Matlab 7.0.

We take $\lambda = \lambda_i := \frac{\beta_i}{p_i}$, $i = 1, 2, 3, 4$ for Algorithm 3.1 (GRA), where $p_1 = 0.9$, $p_2 = 0.7$, $p_3 = 0.5$, $p_4 = 0.3$ and $\lambda_n = \beta_n = \frac{1}{e^{100}}$ for the MGRA1 and MGRA2. The numerical results are shown in Figs. 1-6. In view of these figures, we see that Algorithm 3.1 works better than other algorithms in both execution time and number of iterations.

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Fig. 5 $D_n$ and # Iterations (in $\mathbb{R}^{300}$)

Fig. 6 $D_n$ and Time (in $\mathbb{R}^{300}$)

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