Finiteness of the basic intersection cohomology of a Killing foliation

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Abstract We prove that the basic intersection cohomology $\mathbb{H}_p^\ast(M/F)$, where $F$ is the singular foliation determined by an isometric action of a Lie group $G$ on the compact manifold $M$, is finite dimensional.

Keywords Basic intersection cohomology · Lie group actions

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1 Introduction

This paper deals with an action $\Phi : G \times M \to M$ of a Lie group on a compact manifold preserving a riemannian metric on it. The orbits of this action define a singular foliation $F$ on $M$. Putting together the orbits of the same dimension we get a stratification of $M$. This structure is still very regular. The foliation $F$ is in fact a conical foliation and we can define the basic intersection cohomology $\mathbb{H}_p^\ast(M/F)$ (cf. [10]). This invariant becomes the basic cohomology $H^\ast(M/F)$ when the action $\Phi$ is almost free, and the intersection cohomology $\mathbb{H}_p^\ast(M/G)$ when the Lie group $G$ is compact.

The aim of this work is to prove that this cohomology $\mathbb{H}_p^\ast(M/F)$ is finite dimensional. This result generalizes [3] (almost free case), [11] (abelian case) and [10] (compact case).
The paper is organized as follows. In Sect. 2 we present the foliation $\mathcal{F}$. The basic intersection cohomology $H^p_\mathcal{F}(M/\mathcal{F})$ associated to this foliation is studied in Sect. 3. Twisted products are studied in Sect. 4. The finiteness of $H^p_\mathcal{F}(M/\mathcal{F})$ is proved in Sect. 5.

In the sequel $M$ is a connected, second countable, Hausdorff, without boundary and smooth (of class $C^\infty$) manifold of dimension $m$. All the maps are considered smooth unless something else is indicated.

2 Killing foliations determined by isometric actions

We study in this work the foliations induced by isometric actions: the Killing foliations. These foliations are examples of the conical foliations for which the basic intersection cohomology has been defined (see [10,11]). We present this geometrical framework in this section.

2.1 Killing foliations

A smooth action $\Phi: G \times M \to M$ of a Lie group $G$ on a manifold $M$ is a isometric action when there exists a riemannian metric $\mu$ on $M$ preserved by $G$.

The connected components of the orbits of the action $\Phi$ determine a partition $\mathcal{F}$ on $M$. In fact, this partition is a singular riemannian foliation that we shall call Killing foliation (cf. [7]). Notice that $\mathcal{F}$ is also a conical foliation in the sense of [10,11]. Classifying the points of $M$ following the dimension of the leaves of $\mathcal{F}$ one gets the stratification $\mathcal{S}_\mathcal{F}$ of $\mathcal{F}$. It is determined by the equivalence relation $x \sim y \iff \dim G_x = \dim G_y$. The elements of $\mathcal{S}_\mathcal{F}$ are called strata.

In the particular case where the closure of $G$ in the isometry group of $(M, \mu)$ is a compact Lie group 1 we shall say that the action $\Phi$ is a tame action. In fact, a smooth action $\Phi: G \times M \to M$ is tame if and only if it extends to a smooth action $\Phi: K \times M \to M$ where $K$ is a compact Lie group containing $G$ (cf. [6]). The group $K$ is not unique, but we always can choose $K$ in such a way that $G$ is dense in $K$. We shall say that $K$ is a tamer group. Here the strata of $\mathcal{S}_\mathcal{F}$ are $K$-invariant closed submanifolds of $M$.

Since the aim of this work is the study of $\mathcal{F}$ and not the action $\Phi$ itself, we can consider that the Lie group is connected. Let us see that.

**Proposition 1** Let $\Phi: G \times M \to M$ is a tame action. Let $G_0$ be the connected component of $G$ containing the unity element. The Killing foliation defined by the restriction $\Phi: G_0 \times M \to M$ is also $\mathcal{F}$.

**Proof** The partition $\mathcal{F}$ is defined by this equivalence relation:

$x \sim y \iff \exists$ continuous path $\alpha: [0, 1] \to G(x)$ such that $\alpha(0) = x$ and $\alpha(1) = y$.

Since the map $\Delta: G \to G(x)$, defined by $\Delta(g) = \Phi(g, x) = g \cdot x$, is a submersion (see for example [2]) then

$x \sim y \iff \exists$ continuous path $\beta: [0, 1] \to G$ such that $\beta(0) = e$ and $\beta(1) \cdot x = y$.

and by definition of $G_0$

$x \sim y \iff \exists$ continuous path $\beta: [0, 1] \to G_0$ such that $\beta(0) = e$ and $\beta(1) \cdot x = y$.

This gives the result. □

1 This is always the case when the manifold $M$ is a compact.
When $G$ is connected, the tamer group $K$ has richer properties.

**Proposition 2** Let $G$ be a connected Lie subgroup of a compact Lie group $K$. If $G$ is dense in $K$ then $G \triangleleft K$ and the quotient group $K/G$ is commutative.

**Proof** The Lie algebra $\mathfrak{g}$ is $\text{Ad}_G$-invariant and hence, by density, $\text{Ad}_K$-invariant. Then $\mathfrak{g}$ is an ideal of $\mathfrak{k}$. The connectedness of $G$ gives that $G$ is a normal subgroup of $K$. Since $\text{Ad}_G$ acts trivially on $\mathfrak{t}/\mathfrak{g}$, $\mathfrak{t}/\mathfrak{g}$ is abelian (see for example [8, p. 628]).

2.2 Particular tame actions

A *trio* is a triple $(K, G, H)$, with $K$ is a compact Lie group, $G$ a normal subgroup of $K$ and $H$ a closed subgroup of $K$. We present now some tame actions associated to a trio $(K, G, H)$. They are going to be intensively used in this work. First of all we need some definitions.

- The action $\Phi_l : K \times K \to K$ is defined by $\Phi_l(g, k) = g \cdot h$. For each element $u$ of the Lie algebra $\mathfrak{t}$ of $K$, we shall write $X^u$ the associated (right invariant) vector field. It is defined by $X^u(k) = T_kR_k(u)$ where $R_k : K \to K$ is given by $R_k(\ell) = \ell \cdot k$.

- The action $\Phi_r : K \times K \to K$ is defined by $\Phi_r(g, k) = k \cdot g^{-1}$. For each element $u \in \mathfrak{t}$ of $K$, we shall write $X_u$ the associated (left invariant) vector field. It is defined by $X_u(k) = -T_kL_k(u)$ where $L_k : K \to K$ is given by $L_k(\ell) = k \cdot \ell$.

- The action $\Psi : K \times K/H \to K/H$ is defined by $\Psi(g, h) = (g \cdot h)$. For each element $u \in \mathfrak{t}$, we shall write $Y_u$ the associated vector field. Since the canonical projection $\pi : K \to K/H$ is a $K$-equivariant map, then we have $\pi_*X^u = Y_u$ for each $u \in \mathfrak{t}$.

- The action $\Gamma : H \times H \to H$ is defined by $\Gamma(g, h) = g \cdot h$. For each element $u$ of the Lie algebra $\mathfrak{h}$ of $H$ we write $Z^u$ the associated (right invariant) vector field.

The associated actions we are going to use are the following.

(a) The restriction $\Phi_l : G \times K \to K$, which induces the regular Killing foliation $\mathcal{K}$.

(b) The restriction $\Phi_r : G \times K \to K$, which induces the regular Killing foliation $\mathcal{K}$.

Since $G \triangleleft K$, the foliation $\mathcal{K}$ is determined by the family of vector fields $\{X^u / u \in \mathfrak{g}\}$, where $\mathfrak{g}$ is the Lie algebra of $G$, and also by the family $\{X_u / u \in \mathfrak{g}\}$. The orbits $G(k) = Gk = kG$ have the same dimension $\dim G$.

(c) The restriction $\Psi : G \times K/H \to K/H$, which induces the regular Killing foliation $\mathcal{D}$.

The foliation $\mathcal{D}$ is determined by the family of vector fields $\{Y_u / u \in \mathfrak{g}\}$. The orbits $G(kH)$ have the same dimension $\dim G - \dim(G \cap H)$.

(d) The restriction $\Gamma : (G \cap H) \times H \to H$, which induces the regular Killing foliation $\mathcal{C}$.

The foliation $\mathcal{C}$ is determined by the family of vector fields $\{Z^u / u \in \mathfrak{g} \cap \mathfrak{h}\}$. The orbits $(G \cap H)(k)$ have the same dimension $\dim(G \cap H)$.

(e) The restriction $\Phi_r : GH \times K \to K$, which induces the regular Killing foliation $\mathcal{E}$.

Notice that $GH$ is a Lie group since $G$ is normal in $K$. The foliation $\mathcal{E}$ is, in fact, determined by the vector fields $\{X_u / u \in \mathfrak{g} + \mathfrak{h}\}$. The orbits $(GH)(k)$ have the same dimension $\dim G + \dim H - \dim(G \cap H)$.
2.3 Twisted product

In order to prove the finiteness of the basic intersection cohomology we decompose the manifold in a finite number of simpler pieces. These are the twisted products we introduce now.

We fix a trio \((K, G, H)\) and a smooth action \(\Theta : H \times N \to N\) of \(H\) on the manifold \(N\). The \textit{twisted product} is the quotient \(K \times_H N\) of \(K \times N\) by the equivalence relation \((k, z) \sim (k \cdot h^{-1}, \Theta(h, z) = h \cdot z)\). The element of \(K \times_H N\) corresponding to \((k, z) \in K \times N\) is denoted by \(<k, z>\). This manifold is endowed with the tame action

\[\Phi : G \times (K \times_H N) \to (K \times_H N),\]

defined by \(\Phi(g, <k, z>) = <g \cdot k, z>\). We denote by \(\mathcal{W}\) the induced Killing foliation.

We also use the following tame action, namely, the restriction

\[\Theta : (G \cap H) \times N \to N\]

whose induced Killing foliation is denoted by \(\mathcal{N}\).

The canonical projection \(\Pi : K \times N \to K \times_H N\) relates the involved foliations as follows:

\[(a)\] \(\Pi_\mathcal{W}(K \times I) = \mathcal{W}\), where \(I\) is the pointwise foliation (since the map \(\Pi\) is \(G\)-equivariant).

\[(b)\] \(\mathcal{S}_{\mathcal{W}} = \{\Pi(K \times S) / S \in \mathcal{S}_{\mathcal{N}}\} = \Pi([K] \times \mathcal{S}_{\mathcal{N}})\) (since \(G_{<k,z>} = k(G \cap H)\cdot k^{-1}\)).

3 Basic intersection cohomology

In this section we recall the definition of the basic intersection\(^2\) cohomology and we present the main properties we are going to use in this work. For the rest of this section, we fix a conical foliation \(\mathcal{F}\) defined on a manifold \(M\). The associated stratification is \(\mathcal{S}_\mathcal{F}\). The regular stratum of is denoted by \(R_\mathcal{F}\). We shall write \(m = \dim M, r = \dim \mathcal{F}\) and \(s = m - r = \text{codim}_M \mathcal{F}\).

We are going to deal with differential forms on a product \((\text{manifold}) \times [0, 1[^p, they are restrictions of differential forms defined on \((\text{manifold}) \times] - 1, 1[^p\).

3.1 Perverse forms

Recall that a \textit{conical chart} is a foliated diffeomorphism \(\varphi : (\mathbb{R}^{m-n-1} \times c\mathcal{S}^n, \mathcal{H} \times c\mathcal{G}) \to (U, \mathcal{F}_U)\) where \((\mathbb{R}^{m-n-1}, \mathcal{H})\) is a simple foliation and \((\mathbb{S}^n, \mathcal{G})\) is a conical foliation without 0-dimensional leaves. We also shall denote this chart by \((U, \varphi, S)\) where \(S\) is the stratum of \(\mathcal{S}_\mathcal{F}\) verifying \(\varphi(\mathbb{R}^{m-n-1} \times \{\varnothing\}) = U \cap S\).

The differential complex \(\mathcal{I}_\mathcal{F}^\ast(M \times [0, 1[^p)\) of \textit{perverse forms} of \(M \times [0, 1[^p\) is introduced by induction on depth \(\mathcal{S}_\mathcal{F}\). When this depth is 0 then

\[\mathcal{I}_\mathcal{F}^\ast(M \times [0, 1[^p) = \Omega^\ast(M \times [0, 1[^p).\]

Consider now the generic case. A perverse form of \(M \times [0, 1[^p\) is first of all a differential form \(\omega \in \Omega^\ast(R_\mathcal{F} \times [0, 1[^p\) such that,

\[
\begin{cases}
\text{the pull-back} & \left(\varphi \times \mathbb{I}_{[0,1[^p}\right)^\ast \Omega \in \Omega^\ast(\mathbb{R}^{m-n-1} \times \mathcal{G} \times [0, 1[^p)
\end{cases}
\]

extends to \(\omega_\varphi \in \mathcal{I}_\mathcal{F}^\ast(\mathbb{R}^{m-n-1} \times \mathcal{S}^n \times [0, 1[^{p+1}\)

\(^2\) We refer the reader to [10,11] for details.
for any conical chart \((U, \varphi)\), where \(\mathbb{I}_x\) stands for the identity map. Notice that \(\Omega^\ast(M)\) is included on \(\Pi^\ast_x(M)\).3

### 3.2 Perverse degree

The amount of transversality of a perverse form \(\omega \in \Pi^\ast_x(M)\) with respect to a singular stratum \(S \in S_x\) is measured by the perverse degree \(||\omega||_S\). We recall here the definition of local perverse degree \(||\omega||_{U}\) of \(\omega\) relatively to a conical chart \((U, \varphi, S)\):

1. \(||\omega||_{U} = -\infty\) when \(\omega \equiv 0\) on \(\mathbb{R}^{m-n-1} \times R^g \times \{0\}\).
2. \(||\omega||_{U} \leq p\), with \(p \in \mathbb{N}\), when \(\omega\) verifies \(||\omega||_{U} \leq p\) where the vectors \(\{v_0, \ldots, v_p\}\) are tangent to the fibers of \(\Pi^\ast_x: \mathbb{R}^{m-n-1} \times R^g \times \{0\} \to U \cap S\).

This number does not depend on the choice of the conical chart (cf. [11, Proposition 1.3.1]). Finally, we define the perverse degree \(||\omega||_S\) by

\[
||\omega||_S = \sup \left\{ ||\omega||_{U} \middle| (U, \varphi, S) \text{ conical chart} \right\}.
\]

The perverse degree of \(\omega \in \Omega^\ast(M)\) verifies \(||\omega||_S \leq 0\) for any singular stratum \(S \in S_x\) (cf. 3.1).

### 3.3 Basic cohomology

The basic cohomology of the foliation \(\mathcal{F}\) is an important tool to study its transversal structure and plays the role of the cohomology of the orbit space \(M/\mathcal{F}\), which can be a wild topological space. A differential form \(\omega \in \Omega^\ast(M)\) is basic if \(i_X\omega = i_Xd\omega = 0\), for each vector field \(X\) on \(M\) tangent to the foliation \(\mathcal{F}\). For example, a function \(f\) is basic iff \(f\) is constant on the leaves of \(\mathcal{F}\). We shall write \(\Omega^\ast(M/\mathcal{F})\) for the complex of basic forms. Its cohomology \(H^\ast(M/\mathcal{F})\) is the basic cohomology of \((M, \mathcal{F})\). We also use the relative basic cohomology \(H^\ast((M, F)/\mathcal{F})\), that is, the cohomology computed from the complex of basic forms vanishing on the saturated set \(F \subset M\). The basic cohomology does not use the stratification \(S_x\).

### 3.4 Basic intersection cohomology

A perversity is a map \(\overline{\rho}: S_x^\sigma \to \mathbb{Z} \cup \{-\infty, \infty\}\), where \(S_x^\sigma\) is the family of singular strata. The constant perversity \(\iota\) is defined by \(\iota(S) = \iota\), where \(\iota \in \mathbb{Z} \cup \{-\infty, \infty\}\).

The basic intersection cohomology appears when one considers basic perverse forms whose perverse degree is controlled by a perversity. We shall put

\[
\Omega^\ast_{\overline{\rho}}(M/\mathcal{F}) = \left\{ \omega \in \Pi^\ast_x(M) \middle| \omega \text{ is basic and } \max (||\omega||_S, ||d\omega||_S) \leq \overline{\rho}(S) \forall S \in S_x^\sigma \right\}
\]

the complex of basic perverse forms whose perverse degree (and that of its derivative) is bounded by the perversity \(\overline{\rho}\). The cohomology \(H^\ast_{\overline{\rho}}(M/\mathcal{F})\) of this complex is the basic intersection cohomology5 of \((M, \mathcal{F})\) relatively to the perversity \(\overline{\rho}\).

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3 Through the restriction \(\omega \mapsto \omega_{R_x}\).
4 The map \(P_\varphi: \mathbb{R}^{m-n-1} \times \mathbb{R}^g \times [0, 1] \to U\) is defined by \(P_\varphi(x, y, t) = \varphi(x, [y, t])\).
5 BIC for short.
Consider a twisted product $K \times_N N$. Perversities on $K \times_N N$ and $K \times N$ are determinate by perversities on $N$ by the formula (cf. 2.3 (b)):

\[ \overline{p}(K \times S) = \overline{p}(\Pi(K \times S)) = \overline{p}(S). \]  

(1)

3.5 Mayer-Vietoris

This is the technique we use in order to decompose the manifold in nicer pieces. An open covering \([U, V]\) of \(M\) by saturated open subsets is a basic covering. It possesses a subordinated partition of the unity made up of basic functions defined on \(M\) (see [9]). For a such covering we have the Mayer-Vietoris short sequence

\[ 0 \to \Omega^*_\mathcal{P}(M/\mathcal{F}) \to \Omega^*_\mathcal{P}(U/\mathcal{F}) \oplus \Omega^*_\mathcal{P}(V/\mathcal{F}) \to \Omega^*_\mathcal{P}((U \cap V)/\mathcal{F}) \to 0, \]

where the map are defined by \( \omega \mapsto (\omega, \omega) \) and \((\alpha, \beta) \mapsto \alpha - \beta\). The third map is onto since the elements of the partition of the unity are controlled functions, i.e. elements of \(\Omega^0_\mathcal{P}(-)\) (cf. 3.2). Thus, the sequence is exact. This result is not longer true for more general coverings.

We shall use in this work the two following local calculations (see [11, Proposition 3.5.1 and Proposition 3.5.2] for the proofs).

**Proposition 3** Let \((\mathbb{R}^k, \mathcal{H})\) be a simple foliation. Consider \(\overline{p}\) a perversity on \(M\) and define the perversity \(\overline{p}\) on \(\mathbb{R}^k \times M\) by \(\overline{p}(\mathbb{R}^k \times S) = \overline{p}(S)\). The canonical projection \(\text{pr} : \mathbb{R}^k \times M \to M\) induces the isomorphism

\[ \mathbb{H}^i_\mathcal{P}(M/\mathcal{F}) \cong \mathbb{H}^i_\mathcal{P}\left(\mathbb{R}^k \times M / \mathcal{H} \times \mathcal{F}\right). \]

**Proposition 4** Let \(\mathcal{G}\) be a conical foliation without 0-dimensional leaves on the sphere \(S^n\). A perversity \(\overline{p}\) on \(cS^n\) gives the perversity \(\overline{p}\) on \(S^n\) defined by \(\overline{p}(S) = \overline{p}(S \times \{0, 1\})\). The canonical projection \(\text{pr} : S^n \times \{0, 1\} \to S^n\) induces the isomorphism

\[ \mathbb{H}^i_\mathcal{P}(cS^n / c\mathcal{G}) = \begin{cases} \mathbb{H}^i_\mathcal{P}(S^n / \mathcal{G}) & \text{if } i \leq \overline{p}(\{0\}) \\ 0 & \text{if } i > \overline{p}(\{0\}). \end{cases} \]

In the next section we shall need the following technical Lemma.

**Lemma 1** Let \(\Phi : K \times M \to M\) be a smooth action, where \(K\) is a compact Lie group, and let \(V\) be a fundamental vector field of this action. Consider a normal subgroup \(G\) of \(K\) and write \(\mathcal{F}\) the associated conical foliation on \(M\). Then, the interior operator \(i_V : \Omega^*_\mathcal{P}(M/\mathcal{F}) \to \Omega^{*-1}_\mathcal{P}(M/\mathcal{F})\) is well defined, for any perversity \(\overline{p}\).

**Proof** Since the question is a local one, then it suffices to consider the case where \(M\) is a twisted product \(K \times_N N\).\(^6\) Notice that the blow up \(\Pi : K \times N \to K \times_N N\) is a \(K\)-equivariant map relatively to the action \(\ell \cdot (k, z) = (\ell \cdot k, z)\). This gives \(\Pi_*(X^u, 0) = V\) for some \(u \in \mathfrak{k}\).

From Lemma 2 we know that it suffices to prove that the operator

\[ i_{(X^u, 0)} : \Omega^*_\mathcal{P}(K \times N / K \times N) \to \Omega^{*-1}_\mathcal{P}(K \times N / K \times N) \]

is well defined. Since \(G \triangleleft K\) then the vector field \(X^u\) preserves the foliation \(\mathcal{K}\). So, it suffices to prove that the operator

\[ i_{(X^u, 0)} : \Omega^*_\mathcal{P}(K \times N) \to \Omega^{*-1}_\mathcal{P}(K \times N) \]

\(^6\) In fact, \(N\) is an euclidean space \(\mathbb{R}^d\) et \(\Theta\) is an orthogonal action.
is well defined. This comes from the fact that $X^u$ acts on the $K$-factor while the perversity conditions are measured on the $N$-factor (cf. (1)). □

4 The BIC of a twisted product

We compute now the BIC of a twisted product $K \times_h N$ (cf. 2.3) for a perversity $\overline{p}$ (cf. (1)).

**Lemma 2** The natural projection $\Pi : K \times N \to K \times_h N$ induces the differential monomorphism

$$\Pi^* : \Omega^*_p(K \times_h N/\mathcal{W}) \longrightarrow \Omega^*_p(K \times N/K \times N).$$

Moreover, given a differential form $\omega$ on $K \times_h R\mathcal{W}$, we have:

$$\Pi^* \omega \in \Omega^*_p(K \times N/K \times N) \iff \omega \in \Omega^*_p(K \times_h N/\mathcal{W}).$$

**Proof** Notice that the injectivity of $\Pi^*$ comes from the fact that $\Pi$ is a surjection. For the rest, we proceed in several steps.

(a) A foliated atlas for $\pi : K \to K/H$.

Since $\pi : K \to K/H$ is a $H$-principal bundle then it possesses an atlas $\mathcal{A} = \{ \varphi : \pi^{-1}(U) \to U \times H \}$ made up with $H$-equivariant charts: $\varphi(k \cdot h^{-1}) = (\pi(k), h \cdot h_0)$ if $\varphi(k) = (\pi(k), h_0)$. We study the foliation $\varphi_* K$. This equivariance property gives $\varphi_* X_u = (0, Z^u)$ for each $u \in \mathfrak{g} \cap \mathfrak{h}$. Thus, the trace of the foliation $\varphi_* K$ on the fibers of the canonical projection $pr : U \times H \to U$ is $\mathcal{C}$. On the other hand, since the map $\pi$ is a $G$-equivariant map then $\pi_* K = \mathcal{D}$, which gives $\varphi_* \varphi_* K = \mathcal{D}$. We conclude that $\varphi_* \mathcal{C} \subset \mathcal{D} \times \mathcal{C}$. By dimension reasons we get $\varphi_* \mathcal{C} = \mathcal{D} \times \mathcal{C}$. The atlas $\mathcal{A}$ is an $H$-equivariant foliated atlas of $\pi$.

(b) A foliated atlas for $\Pi : K \times N \to K \times_h N$.

We claim that $\mathcal{A}_u = \{ \overline{\varphi} : \pi^{-1}(U) \times_h N \to U \times N / (U, \varphi) \in \mathcal{A} \}$ is a foliated atlas of $K \times_h N$ where the map $\overline{\varphi}$ is defined by $\overline{\varphi}(< k, z >) = (\pi(k), (\Theta((\varphi^{-1}(\pi(k), e))^{-1} \cdot k), z))$. This map is a diffeomorphism whose inverse is $\overline{\varphi}^{-1}(u, z) = (< \varphi^{-1}(u, e), z >)$. It verifies

$$\overline{\varphi}_* \mathcal{W} \xrightarrow{2.3(a)} \overline{\varphi}_* \Pi_* (\mathcal{K} \times \mathcal{I}) = \overline{\varphi}_* \Pi_* (\varphi^{-1} \times \mathcal{I}_N)_* (\mathcal{D} \times \mathcal{C} \times \mathcal{I}).$$

A straightforward calculation shows $\overline{\varphi}_* \Pi_* (\varphi^{-1} \times \mathcal{I}_N) = (\mathcal{I}_U \times \Theta)$. Since $\mathcal{C}$ is defined by the action $\Gamma$ then $\Theta_* (\mathcal{C} \times \mathcal{I}) = \mathcal{N}$. Finally we obtain $\overline{\varphi}_* \mathcal{W} = \mathcal{D} \times \mathcal{N}$.

(c) Last Step. Given $(U, \varphi) \in \mathcal{A}_u$, we have the commutative diagram

$$\begin{array}{ccc}
U \times H \times N & \xrightarrow{\varphi^{-1} \times \mathcal{I}_N} & K \times N \\
Q \downarrow & & \Pi \\
U \times N & \xrightarrow{\overline{\varphi}^{-1}} & K \times_h N
\end{array}$$

where $Q(u, h, z) = (u, h^{-1} \cdot z)$, $\Pi^{-1}(\text{Im } \overline{\varphi}^{-1}) = \text{Im } (\varphi^{-1} \times \mathcal{I}_N)$ and the rows are foliated imbeddings. Now, since (2) and (3) are local questions then it suffices to prove that...
\[ Q^* : \Omega^*_{\mathfrak{p}}(U \times N / D \times \mathcal{N}) \rightarrow \Omega^*_{\mathfrak{p}}(U \times H \times N / D \times \mathcal{C} \times \mathcal{N}) \text{ is well-defined, and} \]
\[ Q^* \omega \in \Omega^*_{\mathfrak{p}}(U \times H \times N / D \times \mathcal{C} \times \mathcal{N}) \iff \omega \in \Omega^*_{\mathfrak{p}}(U \times N / D \times \mathcal{N}), \text{ for any } \omega \in \Omega^*(U \times R_N). \]

This comes from the fact that the map
\[
\nabla : (U \times H \times N, D \times C \times \mathcal{N}) \rightarrow (U \times H \times N, D \times C \times \mathcal{N}),
\]
defined by \( \nabla(u, h, z) = (u, h, h^{-1} \cdot z) \), is a foliated diffeomorphism and \( Q = \text{pr}_0 \circ \nabla \), with \( \text{pr}_0 : U \times H \times N \rightarrow U \times N \) canonical projection (cf. Proposition 3).

4.1 The Lie algebra \( \mathfrak{k} \)

We suppose in this paragraph that \( G \) is also dense on \( K \). Choose \( \nu \) a bi-invariant riemannian metric on \( K \), which exists by compactness. Consider
\[
B = \{ u_1, \ldots, u_d, u_{a+1}, \ldots, u_b, u_{b+1}, \ldots, u_c, u_{c+1}, \ldots, u_f \}
\]
an orthonormal basis of the Lie algebra \( \mathfrak{k} \) with \( \{ u_1, \ldots, u_b \} \) basis of the Lie algebra \( \mathfrak{g} \) of \( G \) and \( \{ u_{a+1}, \ldots, u_c \} \) basis of the Lie algebra \( \mathfrak{h} \) of \( H \). For each indice \( 1 \leq i \leq f \) we shall write \( X_i \equiv X_{ui} \) and \( X^i \equiv X^{\mu_i} \) (cf. 2.2).

Let \( \gamma_l \in \Omega^1(K) \) be the dual form of \( X_l \), that is, \( \gamma_l = i_{X_l} \nu \). Notice that \( \delta_{ij} = \gamma_j(X_i) \). These forms are invariant by the left action of \( K \). Since the flow of \( X^j \) is the multiplication on the left by \( \exp(\mu_j) \) then \( L_{X^j} \gamma_l = 0 \) for each \( 1 \leq j \leq f \).

For the differential, we have the formula \( d\gamma_l = \sum_{1 \leq i < j \leq f} C_{ij}^l \gamma_i \wedge \gamma_j \), where \( [X_i, X_j] \) is \( \sum_{l=1}^f C_{ij}^l X_l \), and \( 1 \leq i, j, l \leq f \). We have several restrictions on these coefficients. Since \( G \triangleleft K \) then \( \mathfrak{g} \) is an ideal of \( \mathfrak{k} \) and therefore we have
\[
C_{ij}^l = 0 \quad \text{for} \quad i \leq b < l.
\]
Since \( K/G \) is an abelian group (cf. Proposition 2) then the induced bracket on \( \mathfrak{k}/\mathfrak{g} \) is zero and therefore we have
\[
C_{ij}^l = 0 \quad \text{for} \quad b < i, j, l \leq f.
\]
These equations imply that
\[
d\gamma_l = 0 \quad \text{for each} \quad b < l. \tag{4}
\]

The \( \mathcal{E} \)-basic differential forms in \( \bigwedge^*(\gamma_1, \ldots, \gamma_f) \) are exactly \( \bigwedge^*(\gamma_{c+1}, \ldots, \gamma_f) \) since they are cycles and the family \( \{ X_1, \ldots, X_c \} \) generates the foliation \( \mathcal{E} \). This gives
\[
H^*\left( K / \mathcal{E} \right) = \bigwedge^*(\gamma_{c+1}, \ldots, \gamma_f). \tag{5}
\]

4.2 Two actions of \( H / H_0 \)

The Lie group \( H \) preserves the foliation \( \mathcal{N} \) since the Lie group \( G \cap H \) is a normal subgroup of \( H \). Put \( H_0 \) the connected component of \( H \) containing the unity element. Since it is a connected compact Lie group then a standard argument shows that
\[
\left( H^*_{\mathfrak{p}}(N / \mathcal{N}) \right)_{H_0} = H^*\left( \left( \Omega^*_{\mathfrak{p}}(N / \mathcal{N}) \right)_{H_0} \right) = H^*_{\mathfrak{p}}(N / \mathcal{N}) \tag{6}
\]

\footnote{Since \( G \cap H \triangleleft H \).}
conclude that the finite group $H/H_0$ acts naturally on $H^i_p(N/N)$.

Since $H_0$ is a connected Lie subgroup of $GH$ then $(H^*(K/E))_0 = H^*(K/E)$. We conclude that the finite group $H/H_0$ acts naturally on $H^*(K/E)$.

**Proposition 5** Let $(K, G, H)$ be a trio with $G$ connected and dense in $K$. Then

$$H^i_p(K \times_H N/W) = \left(H^* (K/E) \otimes H^i_p(N/N)\right)_0/H_0.$$ 

**Proof** Using the blow up $\Pi: K \times N \to K \times_H N$, the computation of $H^i_p(K \times_H N/W)$ can be done by using the complex Im $\Pi^*: \Omega^*_p(K \times_H N/F) \to \Omega^*_p(K \times N/K \times N)$ (cf. Lemma 2). We study this complex in several steps. We fix $B = \{u_1, \ldots, u_f\}$ an orthonormal basis of $\mathfrak{t}$ as in 4.1.

(i) **Description of $\Omega^*_p(K \times R_N)$**.

A differential form $\omega \in \Omega^*_p(K \times R_N)$ is of the form

$$\eta + \sum_{1 \leq i_1 < \cdots < i_t \leq f} \gamma_{i_1} \wedge \cdots \wedge \gamma_{i_t} \wedge \eta_{i_1, \ldots, i_t},$$

where the forms $\eta, \eta_{i_1, \ldots, i_t} \in \Omega^*_p(K \times R_N)$ verify $i_{X_j} \eta = i_{X_j} \eta_{i_1, \ldots, i_t} = 0$ for each $1 \leq j \leq f$ and each $1 \leq i_1 < \cdots < i_t \leq f$.

(ii) **Description of $\Pi^*_p(K \times R_N)$**.

Since the foliation $K$ is regular then we always can choose a conical chart of the form $(U_1 \times U_2, \varphi_1 \times \varphi_2)$ where $(U_1, \varphi_1)$ is a foliated chart of $(K, K)$ and $(U_2, \varphi_2)$ is a conical chart of $(N, N)$. The local blow up of the chart $(U_1 \times U_2, \varphi_1 \times \varphi_2)$ is constructed from the second factor without modifying the first one. So, the differential forms $\gamma_i$ are always perverse forms and a differential form $\omega \in \Pi^*_p(K \times N)$ is of the form (7) where $\eta, \eta_{i_1, \ldots, i_t} \in \Pi^*_p(K \times N)$ verify $i_{X_j} \eta = i_{X_j} \eta_{i_1, \ldots, i_t} = 0$ for each $1 \leq j \leq f$ and each $1 \leq i_1 < \cdots < i_t \leq f$.

(iii) **Description of $\Omega^*_p(K \times R_N/K \times N)$**.

Take $\omega \in \Omega^*_p(K \times R_N/K \times N)$. Since $\mathcal{K}$ is generated by the family $\{X_j \mid 1 \leq j \leq b\}$ then $L_{X_j} \omega = 0$ for any $1 \leq j \leq b$, or equivalently, $R_X^\omega \omega = \omega$ for each $g \in G$ since $G$ is connected. By density, $R_{X_j}^\omega \omega = \omega$ for each $k \in K$ and therefore $L_{X_j} \omega = 0$ for any $1 \leq j \leq f$ since $K$ is connected. We conclude that $L_{X_j} \eta = L_{X_j} \eta_{i_1, \ldots, i_t} = 0$ for any $1 \leq j \leq f$ and each $1 \leq i_1 < \cdots < i_t \leq f$. This gives $\omega \in \bigwedge^* (\gamma_1, \ldots, \gamma_f) \otimes \Omega^*_p(R_N)$. The $N'$-basic differential forms of $\Omega^*_p(R_N)$ are exactly $\Omega^*_p(R_N/N)$. The $K$-basic differential forms of $\bigwedge^* (\gamma_1, \ldots, \gamma_f)$ are exactly $\bigwedge^* (\gamma_{b+1}, \ldots, \gamma_f)$ (cf. (4)). From these two facts, we get

$$\Omega^*_p(K \times R_N/K \times N) = \bigwedge^* \left(\gamma_{b+1}, \ldots, \gamma_f\right) \otimes \Omega^*_p(R_N/N)$$

as differential graduate commutative algebras.

(iv) **Description of $\Pi^*_p(K \times N/K \times N)$**.

From (ii) and (iii) it suffices to control the perverse degree of the forms

$$\eta + \sum_{b+1 \leq i_1 < \cdots < i_t \leq f} \gamma_{i_1} \wedge \cdots \wedge \gamma_{i_t} \wedge \eta_{i_1, \ldots, i_t} \in \bigwedge^* (\gamma_{b+1}, \ldots, \gamma_f) \otimes \Pi^*_p(N).$$
Consider $S$ a stratum of $\mathcal{S}_N$. From $||\gamma_i||_{K \times S} = 0$ and $||\eta||_{K \times S} = ||\eta||_S$, we get $||\gamma_1 \land \ldots \land \gamma_{i\ell} \land \eta_{i_1}, \ldots, i_{\ell}||_{K \times S} = ||\eta_{i_1}, \ldots, i_{\ell}||_S$. We conclude that

$$\Omega^*_p(K \times N/K \times N) \cong \bigwedge^*(\gamma_{b+1}, \ldots, \gamma_f) \otimes \Omega^*_p(N/N)$$

(cf. 2.3 (b)).

(v) Description of $\text{Im } \{\Pi^*: \Omega^*_p(K \times_h N/\mathcal{F}) \rightarrow \Omega^*_p(K \times N/K \times N)\}$.

We denote by $\{W_{a+1}, \ldots, W_c\}$ the fundamental vector fields of the action $\Theta: H \times N \rightarrow N$ associated to the basis $\{u_{a+1}, \ldots, u_c\}$. Consider now the action $\Upsilon: H \times (K \times N) \rightarrow (K \times N)$ defined by $\Upsilon(h, (k, z)) = (k \cdot h^{-1}, \Theta(h, z))$. Its fundamental vector fields associated to the basis $\{u_{a+1}, \ldots, u_c\}$ are $\{(\mathcal{X}_{a+1}, W_{a+1}), \ldots, (\mathcal{X}_c, W_c)\}$. Given $h \in H$, we take $\Upsilon_h: K \times N \rightarrow K \times N$ the map defined by $\Upsilon_h(k, z) = \Upsilon(h, (k, z))$. Then, we have

$$\text{Im } \Pi^* = \left\{ \omega \in \bigwedge^*(\gamma_{b+1}, \ldots, \gamma_f) \otimes \Omega^*_p(N/N) \left/ \begin{array}{l}
\text{(i) } i_{\mathcal{X}_i} \omega = -i_{W_i} \omega \text{ if } a < i \leq c \\
\text{(ii) } L_{\mathcal{X}_i} \omega = -L_{W_i} \omega \text{ if } a < i \leq c \\
\text{(iii) } (\Upsilon_h)^* \omega = \omega \text{ for } h \in H
\end{array} \right. \right\}$$

Let $H_0$ be the unity connected component of $H$. Recall that the subgroup $H_0$ is normal in $H$ and that the quotient $H/H_0$ is a finite group. Conditions (ii) gives that $\omega$ is $H_0$-invariant. So, condition (iii) can be replaced by: (iv) $(\Upsilon_h)^* \omega = \omega$ for $h \in H/H_0$. Therefore

$$\text{Im } \Pi^* = \left\{ \omega \in \bigwedge^*(\gamma_{b+1}, \ldots, \gamma_f) \otimes \Omega^*_p(N/N) \left/ \begin{array}{l}
\text{(i) } i_{\mathcal{X}_i} \omega = -i_{W_i} \omega \text{ if } a < i \leq c \\
\text{(ii) } L_{\mathcal{X}_i} \omega = -L_{W_i} \omega \text{ if } a < i \leq c
\end{array} \right. \right\}$$

Since the group $H/H_0$ is a finite one, we get that the cohomology $H^*(\text{Im } \Pi^*)$ is isomorphic to $\left( H^*(A^*) \right)^{H/H_0}$, where $A^*$ is the differential complex

$$\left\{ \omega \in \bigwedge^*(\gamma_{b+1}, \ldots, \gamma_f) \otimes \Omega^*_p(N/N) \left/ \begin{array}{l}
\text{(i) } i_{\mathcal{X}_i} \omega = -i_{W_i} \omega \text{ if } a < i \leq c \\
\text{(ii) } L_{\mathcal{X}_i} \omega = -L_{W_i} \omega \text{ if } a < i \leq c
\end{array} \right. \right\}.$$
We get that $A^*$ is the differential complex

$$
\begin{align*}
\left\{ \omega \in \bigwedge (\gamma_{b+1}, \ldots, \gamma_f) \otimes \Omega^*_p(N/N) \right\} & \quad \begin{array}{l}
(i) \ i_X \omega = -i_{W_i} \omega \text{ if } b < i \leq c \\
(ii) \ 0 = L_{W_i} \omega \text{ if } b < i \leq c 
\end{array} \\
= \bigwedge (\gamma_{b+1}, \ldots, \gamma_f) \otimes \Omega^*_p(N/N) & \quad \begin{array}{l}
(i) \ i_X \omega = -i_{W_i} \omega \text{ if } b < i \leq c \\
(ii) \ 0 = L_{W_i} \omega \text{ if } b < i \leq c 
\end{array}
\end{align*}
$$

A straightforward computation gives that a form $\omega \in \bigwedge (\gamma_{b+1}, \ldots, \gamma_c) \otimes \Omega^*_p(N/N)$ verifying (i) is in fact

$$
\omega = \omega_0 + \sum_{b < i_1 < \cdots < i_c \leq c} (-1)^{\ell} \gamma_{i_1} \wedge \cdots \wedge \gamma_{i_c} \wedge (i_{W_{i_1}} \cdots i_{W_{i_c}} \omega_0) \tag{8}
$$

for some $\omega_0 \in \Omega^*_p(N/N)$ (cf. Lemma 1).

Consider now $b < i, j \leq c$. Since $K \times G$ is an abelian group (cf. Proposition 2) and $H$ is a Lie group then $[W_i, W_j] = \sum_{l=b+1}^c C_{ij} W_l$. Then, $i_{[W_i, W_j]} \omega_0 = 0$ since the foliation $\mathcal{N}$ is defined by the action of $G \cap H$. So, the canonical writing of a form $\omega \in B^*$ is (8) for some $\omega_0 \in \left\{ \eta \in \Omega^*_p(N/N) / L_{W_i} \eta = 0 \text{ if } b < i \leq c \right\} = \left( \Omega^*_p(N/N) \right)^{H_0}$. Then, the operator $\Delta : B^* \rightarrow \left( \Omega^*_p(N/N) \right)^{H_0}$, defined by $\Delta(\omega) = \omega_0$, is a differential isomorphism. We conclude that the differential complex $A^*$ is isomorphic to $\bigwedge (\gamma_{b+1}, \ldots, \gamma_f) \otimes \left( \Omega^*_p(N/N) \right)^{H_0}$ and therefore $H^*(A^*) \cong H^* (K/\mathcal{E}) \otimes \mathbb{H}^*_p(N/N)$ (cf. (5) and (6)). Since the operator $\Delta$ is $(H/H_0)$-equivariant (cf. 4.2) then we get

$$
\mathbb{H}^*_p(K \times_H N/\mathcal{W}) = H^* (\text{Im } \Pi^*) = \left( H^* (A) \right)^{H/H_0} = \left( H^* (K/\mathcal{E}) \otimes \mathbb{H}^*_p(N/N) \right)^{H/H_0}.
$$

This ends the proof. \hfill \Box

4.3 Remarks

(a) When the Lie group $G$ is commutative then $K$ is also commutative. Differential forms $\gamma_\bullet$ are $K$-invariants on the left and on the right, so $\left( H^* (K/\mathcal{E}) \right)^H = H^* (K/\mathcal{E})$ and therefore

$$
\mathbb{H}^*_p(K \times_H N/\mathcal{W}) = H^* (K/\mathcal{E}) \otimes \left( \mathbb{H}^*_p(N/N) \right)^{H/H_0} = H^* (K/\mathcal{E}) \otimes \left( \mathbb{H}^*_p(N/N) \right)^H
$$

as it has been proved in [11, Proposition 3.8.4].

(b) Since the foliation $\mathcal{E}$ is a riemannian foliation defined on a compact manifold then we know that the cohomology $H^* (K/\mathcal{E})$ is finite (cf. [4]). So, the finiteness of $\mathbb{H}^*_p(K \times_H N/\mathcal{W})$ depends on the finiteness of $\mathbb{H}^*_p(N/N)$. 

\[ Springer \]
5 Finiteness of the BIC

We prove in this section that the BIC of a Killing foliation on a compact manifold is finite dimensional. First of all, we present two geometrical tools we shall use in the proof: the isotropy type stratification and the Molino’s blow up.

We fix an isometric action $Φ: G × M → M$ on the compact manifold $M$. We denote by $F$ the induced Killing foliation. For the study of $F$ we can suppose that $G$ is connected (see Lemma 1). We fix $K$ a tamer group. Notice that the group $G$ is normal in $K$ and the quotient $K / G$ is commutative (cf. Proposition 2).

5.1 Isotropy type stratification

The isotropy type stratification $S_{K, M}$ of $M$ is defined by the equivalence relation\(^8\):

$$x ∼ y ⇔ K_x \text{ is conjugated to } K_y.$$  

When depth $S_{K, M} > 0$, any closed stratum $S ∈ S_{K, M}$ is a $K$-invariant submanifold of $M$ and then it possesses a $K$-invariant tubular neighborhood $(T, τ, S, R^m)$ whose structural group is $O(m)$. Recall that there are the following smooth maps associated with this neighborhood:

+ The radius map $ρ: T → [0, 1[)$ defined fiberwise from the assignation $[x, t] → t$. Each $t ≠ 0$ is a regular value of the $ρ$. The pre-image $ρ^{-1}(0)$ is $S$. This map is $K$-invariant, that is, $ρ(k · z) = ρ(z)$.
+ The contraction $H: T × [0, 1] → T$ defined fiberwisely from $([x, t], r) → [x, rt]$. The restriction $H_t: T → T$ is an embedding for each $t ≠ 0$ and $H_0 ≡ τ$. We shall write $H(z, t) = t · z$. This map is $K$-invariant, that is, $t · (k · z) = k · (t · z)$.

The hyper-surface $D = ρ^{-1}(1/2)$ is the tube of the tubular neighborhood. It is a $K$-invariant submanifold of $T$. Notice that the map

$$∇: D × [0, 1[ → T,$$

defined by $∇(z, t) = (2t) · z$ is a $K$-equivariant smooth map, where $K$ acts trivially on the $[0, 1]$-factor. Its restriction $∇: D × ]0, 1[ → T \setminus S$ is a $K$-equivariant diffeomorphism.

Denote $S_{min}$ the union of closed (minimal) strata and choose $T_{min}$ a disjoint family of $K$-invariant tubular neighborhoods of the closed strata. The union of associated tubes is denoted by $D_{min}$. Notice that the induced map $∇_{min}: D_{min} × ]0, 1[ → T_{min} \setminus S_{min}$ is a $K$-equivariant diffeomorphism.

5.2 Molino’s blow up

The Molino’ blow up [7] of the foliation $F$ produces a new foliation $\tilde{F}$ of the same kind but of smaller depth. We suppose depth $S_{K, M} > 0$. The blow up of $M$ is the compact manifold

$$\tilde{M} = \left\{ (D_{min} × ]0, 1[) \bigrcup ((M \setminus S_{min}) × \{−1, 1\}) \right\} / ∼,$$

where $(z, t) ∼ (∇_{min}(z, |t|), t/|t|)$, and the map $L: \tilde{M} → M$ defined by

$$L(v) = \begin{cases} ∇_{min}(z, |t|) & \text{if } v = (z, t) ∈ D_{min} × ]0, 1[ \\ z & \text{if } v = (z, j) ∈ (M \setminus S_{min}) × \{−1, 1\}. \end{cases}$$

\(^8\) For notions related with compact Lie group actions, we refer the reader to [1].
Notice that $\mathcal{L}$ is a continuous map whose restriction $\mathcal{L} \colon \hat{M} \setminus \mathcal{L}^{-1}(S_{\text{min}}) \to M \setminus S_{\text{min}}$ is a $K$-equivariant smooth trivial 2-covering.

Since the map $\nabla_{\text{min}}$ is $K$-equivariant then $\Phi$ induces the action $\hat{\Phi} : K \times \hat{M} \to \hat{M}$ by saying that the blow-up $\mathcal{L}$ is $K$-equivariant. The open submanifolds $\mathcal{L}^{-1}(T_{\text{min}})$ and $\mathcal{L}^{-1}(T_{\text{min}} \setminus S_{\text{min}})$ are clearly $K$-diffeomorphic to $D_{\text{min}} \times [1, \infty)$ and $D_{\text{min}} \times (1, 0[\cup]0, 1]$ respectively.

The restriction $\hat{\Phi} : G \times \hat{M} \to \hat{M}$ is an isometric action with $K$ as a tamer group. The induced Killing foliation is $\hat{\mathcal{F}}$. Foliations $\mathcal{F}$ and $\hat{\mathcal{F}}$ are related by $\mathcal{L}$ which is a foliated map. Moreover, if $S$ is a not minimal stratum of $S_{K,M}$ then there exists an unique stratum $S' \in S_{K,\hat{M}}$ such that $\mathcal{L}^{-1}(S) \subset S'$. The family $\{S' / S \in S_{K,M}\}$ covers $\hat{M}$ and verifies the relationship: $S_1 < S_2 \iff S_1' < S_2'$. We conclude the important property
\[
\text{depth } S_{K,\hat{M}} < \text{depth } S_{K,M}. \tag{9}
\]

5.3 Finiteness of a tubular neighborhood

We suppose $\text{depth } S_{K,M} > 0$. Consider a closed stratum $S \in S_{K,M}$. Take $(T, \tau, S, \mathbb{R}^m)$ a $K$-invariant tubular neighborhood. We fix a base point $x \in S$. The isotropy subgroup $K_x$ acts orthogonally on the fiber $\mathbb{R}^m = \tau^{-1}(x)$. So, the induced action $\Lambda_x : G_x \times \mathbb{R}^m \to \mathbb{R}^m$ is an isometric action, it gives the Killing foliation $\mathcal{N}$ on $\mathbb{R}^m$.

**Proposition 6** If the BIC of $(\mathbb{R}^m, \mathcal{N})$ is finite dimensional then the BIC of $(T, \mathcal{F})$ is also finite dimensional.

**Proof** We proceed in two steps.

(a) $K_y = K_x$ for each $y \in S$.

The canonical projection $\pi : S \to S/K$ is an homogeneous bundle with fiber $K/K_x$. For any open subset $V \subset S/K$ the pull back $\tau^{-1}\pi^{-1}(V)$ is a $K$-invariant subset of $T$, then we can apply the Mayer-Vietoris technics to this kind of subsets (cf. 3.5).

Since the manifold $S/K$ is a compact one then we can find a finite good covering $\{U_i / i \in I\}$ of it (cf. [2]). An inductive argument on the cardinality of $I$ reduces the proof of the Lemma to the case where $T = \tau^{-1}\pi^{-1}(V)$, where $V$ is a contractible open subset of $S/K$.

Here, the manifold $T$ is $K$-equivalently diffeomorphic to $V \times (K \times K_x \mathbb{R}^m)$, where $K$ does not act on the first factor. So, the natural retraction of $V$ to a point gives a $K$-equivariant retraction of $T$ to the twisted product $K \times K_x \mathbb{R}^m$. Now the result comes directly from 4.3(b) since $(K, G, K_x)$ is a trio.

(b) General case.

The stratum $S$ is $K$-equivariantly diffeomorphic to the twisted product $K \times N(K_x)F$ where $N(K_y)$ is the normalizer of $K_x$ on $K$ and $F = S^{K_x}$. So, the tubular neighborhood $T$ is $K$-equivariantly diffeomorphic to the twisted product $K \times N(F)N$ where $N$ is the manifold $\tau^{-1}(F)$. The previous case gives that the BIC of $(N, \mathcal{F}_N)$ is finite dimensional. Now the result comes directly from 4.3(b) since $(K, G, N(K_x))$ is a trio.

The main result of this work is the following

**Theorem 1** The BIC of the foliation determined by an isometric action on a compact manifold is finite dimensional.
Proof} Let $\mathcal{F}$ be a Killing foliation defined on a compact manifold $M$ induced by an isometric action $\Phi: G \times M \to M$ where $G$ is a Lie group. Without loss of generality we can suppose that the Lie group $G$ is a connected one (cf. Lemma 1). We fix a tamer group $K$. We know that $G$ is normal in $K$ and the quotient group $K/G$ is commutative (cf. Proposition 2).

Let us consider the following statement

$$\mathfrak{A}(U, \mathcal{F}) = \text{"The BIC } \mathbb{H}^*_p(U/F) \text{ is finite dimensional for each perversity } p,"$$

where $U \subset M$ is a $K$-invariant submanifold. We prove $\mathfrak{A}(M, \mathcal{F})$ by induction on $\dim M$. The result is clear when $\dim M = 0$. We suppose $\mathfrak{A}(W, \mathcal{F})$ for any $K$-invariant compact submanifold $W$ of $M$ with $\dim W < \dim M$ and we prove $\mathfrak{A}(M, \mathcal{F})$. We proceed in several steps.

First step: 0-depth. Let us suppose depth $S_{K,M} = 0$. Since $G < K$ and $K_x$ is conjugated to $K_y$ then $G_{x,y}$ is conjugated to $G_y$, $\forall x, y \in M$. We get that the foliation $\mathcal{F}$ is a (regular) riemannian foliation (cf. [7]). Its BIC is just the basic cohomology (cf. 3.3). Then $\mathfrak{A}(M, \mathcal{F})$ comes from [4].

Second step: Inside $M$. Let us suppose depth $S_{K,M} > 0$. The family $\{M/S_{\min}, T_{\min}\}$ is a basic covering of $M$ and the we get the exact sequence (cf. 3.5)

$$0 \to \Omega^*_p(M/F) \to \Omega^*_p((M/S_{\min})/\mathcal{F}) \oplus \Omega^*_p(T_{\min}/\mathcal{F}) \to \Omega^*_p((T_{\min}/S_{\min})/\mathcal{F}) \to 0.$$  

The Five Lemma gives

$$\mathfrak{A}(T_{\min}/S_{\min}, \mathcal{F}), \mathfrak{A}(T_{\min}, \mathcal{F}) \text{ and } \mathfrak{A}(M/S_{\min}, \mathcal{F}) \implies \mathfrak{A}(M, \mathcal{F}).$$

Since $T_{\min}/S_{\min}$ is $K$-diffeomorphic to $D_{\min} \times ]0, 1[$ (cf. (5.1)) then $\mathfrak{A}(D_{\min}, \mathcal{F}) \implies \mathfrak{A}(T_{\min}/S_{\min}, \mathcal{F})$. The inequality $\dim D_{\min} < \dim M$ gives

$$\mathfrak{A}(T_{\min}, \mathcal{F}) \text{ and } \mathfrak{A}(M/S_{\min}, \mathcal{F}) \implies \mathfrak{A}(M, \mathcal{F}).$$

In order to prove $\mathfrak{A}(T_{\min}, \mathcal{F})$ it suffices to prove $\mathfrak{A}(T, \mathcal{F})$ where $(T, \tau, S, \mathbb{R}^m)$ is a $K$-invariant tubular neighborhood of closed stratum $S$ of $S_{K,M}$. Following Proposition 6 we have

$$\mathfrak{A}(\mathbb{R}^m, N) \implies \mathfrak{A}(T, \mathcal{F}) \implies \mathfrak{A}(T_{\min}, \mathcal{F}).$$

Consider the orthogonal decomposition $\mathbb{R}^m = \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$, where $\mathbb{R}^{m_1} = (\mathbb{R}^m)^{G_x}$. The only fixed point of the restriction $A_x: G_x \times \mathbb{R}^{m_2} \to \mathbb{R}^{m_2}$ is the origin. So, there exists a Killing foliation $G$ on the sphere $S^{m_2-1}$ with $(\mathbb{R}^{m_1} \times \mathbb{R}^{m_2}, \mathcal{F}) = (\mathbb{R}^{m_1} \times cS^{m_2-1}, \mathcal{T} \times cG)$. Propositions 3 and 4 give:

$$\mathfrak{A}(S^{m_2-1}, G) \implies \mathfrak{A}(\mathbb{R}^{m_1} \times cS^{m_2-1}, \mathcal{T} \times cG) \implies \mathfrak{A}(\mathbb{R}^m, N).$$

Finally, since $\dim S^{m_2-1} < m \leq \dim T \leq \dim M$ we have

$$\mathfrak{A}(M/S_{\min}, \mathcal{F}) \implies \mathfrak{A}(M, \mathcal{F}). \quad (10)$$

Third step: Blow-up. Let us suppose depth $S_{K,M} > 0$. The family $\{L^{-1}(M/S), L^{-1}(T_{\min})\}$ is a basic covering of $\widehat{M}$ and we get the exact sequence (cf. 3.5)

$$0 \to \Omega^*_p(\widehat{M}/\overline{\mathcal{F}}) \to \Omega^*_p(L^{-1}(M/S)/\overline{\mathcal{F}}) \oplus \Omega^*_p(L^{-1}(T_{\min})/\overline{\mathcal{F}}) \to \Omega^*_p(L^{-1}(T_{\min}/S)/\overline{\mathcal{F}}) \to 0.$$  

Following 5.2 we have that

$^{9}$ It is given by the orthogonal action $A_x: G_x \times S^{m_2-1} \to S^{m_2-1}$.

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Now, the Five Lemma gives
\[ \mathfrak{A}(D_{\min}, \hat{F}) \text{ and } \mathfrak{A}(\hat{M}, \hat{F}) \implies \mathfrak{A}(M \setminus S_{\min}, F). \]
But, the inequality \( \dim D_{\min} < \dim M \) gives
\[ \mathfrak{A}(\hat{M}, \hat{F}) \implies \mathfrak{A}(M \setminus S_{\min}, F). \tag{11} \]

Forth step: Final blow-up. When depth \( S_{K,M} = 0 \) we get \( \mathfrak{A}(M, F) \) from the First step. Let us suppose depth \( S_{K,M} > 0 \). From (10) and (11) we get
\[ \mathfrak{A}(\hat{M}, \hat{F}) \implies \mathfrak{A}(M, F), \]
with depth \( S_{K,\hat{M}} < \) depth \( S_{K,M} \) (cf. (9)). By iterating this procedure we get
\[ \mathfrak{A}(\hat{M}, \hat{F}) = \mathfrak{A}(\hat{\hat{M}}, \hat{\hat{F}}) \implies \cdots \implies \mathfrak{A}(\hat{M}, \hat{F}) \implies \mathfrak{A}(M, F), \]
with depth \( S_{K,\hat{M}} = 0 \). We finish the proof by applying again the First Step. □

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