Extraction of inherent polarization mode from one single light beam

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Abstract:
Superposition of two independent orthogonally polarized beams is a conventional principle of creating a new light beam. Here, we intend to achieve the inverse process, namely taking inherent polarization modes out from one single light beam. However, inherent polarization modes within a light beam are always entangled together so that a stable polarization is maintained during propagating in free space. To overcome this limitation, we report an approach that breaks down the modulation symmetry of light beam, thereby disentangling the inherent polarization modes. Using polarization mode competition along with the optical pen, polarization modes are extracted at will in the focal region of an objective lens. This work demonstrates the polarization mode extraction of light beam, which will not only provide an entirely new principle of polarization modulation, but also pave ways for multi-dimensional manipulation of light field, thereby facilitating extensive developments in optics.
Introduction

As a natural property of light beam, polarization represents the inherent oscillation of electric field. Light beam with special polarization possesses unique properties [1-5] that can not only deepen our understanding about light beam, but also play a vital role in many applications of optics, including optical imaging [6], optical trapping [7, 8], optical communication [9, 10] and optical lithography [11-13]. Generally, every light beam can be considered as the combination of two independent orthogonally polarized beams. Throughout the development of optics, superposing two orthogonally polarized beams is a conventional principle of creating a light beam with arbitrary polarization [14-16], which has already written in textbook of optics. For example, the combination of a x and y linear polarized beam can give rise to a circularly polarized beam, while a radially polarized beam can be realized by overlapping a left and right circularly polarized beam together [9]. However, we wonder whether the inverse process of above conventional principle can be realized in free space. More intuitively, is it possible to take inherent polarization modes out from one single light beam?

As we know, a pair of two particular orthogonal polarization modes can create a light beam with arbitrary polarization. However, this light beam contains not just the above pair of polarization modes, but other pairs of polarization modes as well. Taking a radially polarized beam as an example. Radially polarized beam can be divided into a pair of x and y linear polarization mode or a pair of left and right circular polarization mode, respectively. Adding both pairs of polarization modes together does not affect the whole polarization state of radially polarized beam. In other words, there is not only a pair of x and y linear polarization mode within a radially polarized beam, but also other pairs of polarization modes simultaneously, such as a pair of left and right polarization mode. Supposed that there are multiple pairs of inherent polarization modes within a light beam. All inherent polarization modes are intertwined with each other so that light beam satisfied wave function can maintain a stable polarization during propagation in free space. From the viewpoint of physics optics, the entanglement of inherent polarization modes within light beam makes polarization mode extraction hopeless. However, the solve of this physics problem no doubt stands for an entirely new principle of polarization modulation like that of overlapping two independent light beams.

Here, we report an approach that overcome the entanglement of inherent polarization modes by breaking down modulation symmetry to extract arbitrary polarization mode from one single light beam in the focal region of an objective lens. As a generalized property of all light beams satisfied wave function, modulation symmetry is broken down by adjusting the weight factor of polarization modes
with the phase of light beam, thereby disentangling the inherent polarization modes. Using the polarization mode competition along with the optical pen [17], multiple polarization modes are extracted at will in the focal region of an objective lens. Polarization modes extraction of one single light beam provides an entirely new principle of polarization modulation, which may fundamentally influence the application of optics and deepen our understanding about light beam.
Results

Modulation symmetry of light beam

In classical optics, light beam satisfied wave function always keeps its polarization unchanged when propagating in free space. The stable polarization therefore requires that the inherent polarization modes are intertwined together. Otherwise, the polarization of light beam cannot maintain in free space. It is quite questionable whether the inherent polarization modes of light beam can be extracted in free space. The key to solve this problem requires a new understanding regarding the entanglement of polarization modes.

According to Supplementary Equations (2) and (3), a light beam can be composed with a left and right circular polarization modes in rectangular coordinate system, or a radial and azimuthal polarization modes in cylindrical coordinate system. These orthogonal polarization modes are modulated by different factors. For example, the right and left circular polarization modes $|R\rangle$, $|L\rangle$ are modulated by $\exp(\pm i mf(\varphi, \theta))$, while the linear polarization modes $|RVB_{\sigma}\rangle$ and $|LVB_{\sigma}\rangle$ are modulated by the factors $\cos nf(\varphi, \theta)$ and $\sin nf(\varphi, \theta)$. However, these inherent polarization modes, no matter in what coordinate system, all possess identical order $mf(\varphi, \theta)$ and $nf(\varphi, \theta)$, respectively. Here, we call this property of inherent polarization modes as modulation symmetry of light beam. Because of modulation symmetry, the inherent polarization modes are always entangled together, thereby maintaining an identical polarization of entire light beam during propagating in free space, as shown in Figure 1. That is, the entanglement of polarization modes is caused by the modulation symmetry of light beam.

Breaking down modulation symmetry

As a generalized property of light beam, modulation symmetry is the physical root of the entanglement of polarization modes, which makes it impossible to extract one polarization mode from the others. For this reason, the key to extract polarization mode from light beam lays on how to break down the modulation symmetry without affecting the target inherent polarization mode.

However, not all light beam is suitable for breaking down the modulation symmetry. Modulation asymmetry between polarization modes can only occur in the condition that the modulation factors $mf(\varphi, \theta)$ and $nf(\varphi, \theta)$ are variates (See Supplementary Note 2). That is, the light beam must be inhomogeneously polarized. Otherwise, when $mf(\varphi, \theta)$ and $nf(\varphi, \theta) = \text{const}$, the light beams in Supplementary Equation (1) stand for a homogeneously polarized beam. All polarization modes within
the light beam are always modulated simultaneously, thereby making it impossible to break down the modulation symmetry.

To break down the modulation symmetry of light beam, we take a familiar vector beam, namely \( m \)-order VVB, as an example. Mathematically, an \( m \)-order VVB can be expressed as [9, 21]

\[
E_{mc} = \exp(im\varphi)|R\rangle + \exp(-im\varphi)|L\rangle, \\
E_{wv} = \cos n\varphi|RVVB_{\sigma}\rangle + \sin n\varphi|LVVB_{\sigma}\rangle,
\]

where \(|L\rangle = [1 \quad i]\) and \(|R\rangle = [1 \quad -i]\) denote the left and right circularly polarized mode, respectively, and \(\varphi\) is the azimuthal angle. \(|RVVB_{\sigma}\rangle = [\cos\sigma\varphi \quad \sin\sigma\varphi]^T\) and \(|LVVB_{\sigma}\rangle = [-\sin\sigma\varphi \quad \cos\sigma\varphi]^T\) are two \(\sigma\)-order VVBs with orthogonal polarization. \(T\) denotes the matrix transposition operator. Here, \(m = n + \sigma\).

After modulating by \(T = \exp(i(n\varphi - \beta))\) and \(T = \cos(n\varphi - \beta)\), the \(m\)-order VVBs in Equations (1) and (2) turn into

\[
E_{mc} = \exp[i((m + n)\varphi + \beta)]|R\rangle + \exp[i((-m + n)\varphi + \beta)]|L\rangle, \\
E_{wv} = |RVVB_{2n+\sigma,-\beta}\rangle + |RVVB_{\sigma,\beta}\rangle,
\]

respectively. Here, \(|RVVB_{2n+\sigma,-\beta}\rangle\) and \(|RVVB_{\sigma,\beta}\rangle\) are \(2n + \sigma\), \(\sigma\)-order VVBs with the polarization direction of \(-\beta\) and \(\beta\), respectively, which can be written as

\[
|RVVB_{2n+\sigma,-\beta}\rangle = [\cos((2n + \sigma)\varphi - \beta) \quad \sin((2n + \sigma)\varphi - \beta)]^T, \\
|RVVB_{\sigma,\beta}\rangle = [\cos(\sigma\varphi + \beta) \quad \sin(\sigma\varphi + \beta)]^T,
\]

Equations (1) and (2) represent two classical forms of \(m\)-order VVB: one is a pair of circular polarization modes; another is a pair of linear polarization modes. For the latter one in Equation (2), \(m\)-order VVB can be realized by two orthogonally polarized VVBs with identical order \(\sigma\), namely \(|LVVB_{\sigma}\rangle\) and \(|RVVB_{\sigma}\rangle\). In case \(\sigma = 0\), \(|LVVB_{\sigma}\rangle\) and \(|RVVB_{\sigma}\rangle\) are simplified into a \(x\) and \(y\) linear polarization mode, respectively. That is, the entire \(m\)-order VVB turns into the superposition of \(x\) and \(y\) linear polarization mode with the modulation of \(\cos m\varphi\) and \(\sin m\varphi\), respectively. Because of the identical modulating factors \(m, n\) in Equations (1) and (2), both pairs of polarization modes are intertwined with each other, thereby leading to a stable polarization of \(m\)-order VVB in free space. However, when the \(m\)-order VVBs are modulated by \(T = \exp(i(n\varphi - \beta))\) and \(T = \cos(n\varphi - \beta)\),
respectively, modulation symmetries in Equations (1) and (2) are broken down. Specifically, two circular polarization modes with different topological charges $m+n$, $m-n$ are obtained in Equation (3), while the $m$-order VVB in Equation (4) is split into $2n+\sigma$ , $\sigma$-order VVBs. The break of modulation symmetry brings the hope of disentangling the inherent polarization modes of light beam, thereby providing a possibility of extracting polarization modes from one single light beam.

**Critical condition for polarization mode extraction**

Actually, the modulation symmetry can simply be broken down by a radially or azimuthally polarized beam modulated by many phases, such as vortex phase [2, 18], $\pi$-phase [19]. Nevertheless, extracting one polarization mode from light beam cannot be realized yet. The reason is that the inherent polarization modes within radially or azimuthal polarized beam are overlapped with each other even in the focal region of an objective lens. To extract polarization modes from a light beam, there are still two critical conditions that need to be satisfied. One is that the polarization modes must be spatially separated in free space; another is that only the target polarization mode retains while its counterpart can be neglected.

For the first condition, once the modulation symmetries are broken down (See Equations (3) and (4)), $m$-order VVB cannot maintain its original polarization state any longer. Theoretically, both orthogonal polarization modes not only come with pairs, but also propagate coaxially in free space. As the light beam propagates in free space, the polarization modes with larger modulation factor or higher order, namely $\exp[i((m+n)\varphi + \beta)]|L\rangle$, $\exp[i((-m+n)\varphi + \beta)]|R\rangle$ and $|RVVB_{2m+n+\sigma} \rangle$, locate at the outer ring while the polarization modes $|L\rangle$, $|R\rangle$ and $|RVVB_{\sigma} \rangle$ are at the inner ring (See Supplementary Figure 3). The longer the propagation distance of light beam, the larger the spatial separation of polarization modes. Generally, one can simply obtain the largest spatial separation by focusing $m$-order VVB with an objective lens.

For the second condition, $m$-order VVB divides into two polarization modes when modulating by $T = \exp(i(n\varphi - \beta))$ and $T = \cos(n\varphi + \beta)$. Thus, both polarization modes are of equal light intensity. Although we cannot adjust the total light intensity of polarization modes, their corresponding energy densities can be tuned at will by the orders $m$ and $n$, $\delta$. Taking the polarization modes $|L\rangle$, $|R\rangle$ in Equation (3) as an example. Supposed $m,n > 0$, $|R\rangle$ is modulated by a vortex phase with topological charge of $m+n$, while that of $|L\rangle$ is $m-n$. Because of this topological charge difference, the divergent
degree of \( |\text{L}\rangle , |\text{R}\rangle \) are not the same. Generally, the larger the topological charge, the higher the divergent degree. Higher divergent degree gives rise to lower energy density of polarization mode. For this reason, the energy density difference between both polarization modes \( |\text{L}\rangle , |\text{R}\rangle \) can be enlarged with the increment of \( \eta = \|m + n\| - \|m - n\| \). Once \( \eta \) increases to a certain value, the energy density of target polarization modes \( |\text{L}\rangle , |\text{R}\rangle \) are much larger than that of their counterparts. In this way, one can obtain one polarization mode by simply neglecting the other one. Here, we call this phenomenon as the polarization mode competition within VVB. Likewise, \( |\text{LVVB}_{\sigma}\rangle \) and \( |\text{RVVB}_{\sigma}\rangle \) can also be extracted from \( m \)-order VVB with an appropriate \( \eta \).

**Polarization mode extraction of VVB**

Taking the above two critical conditions into consideration. We verify polarization mode extraction using \( m=30 \)-order VVB in a focusing system. To achieve multiple polarization modes, the modulation factors \( T = \exp(i(n\varphi - \beta)) \) and \( T = \cos(n\varphi + \beta) \) are combined with the optical pen developed in our previous work in Ref [17]. Therefore, the overall phase of \( m=30 \)-order VVB can be expressed as [17]

\[
\psi = \text{Phase} \left\{ \sum_{j=1}^{N} [T_j \cdot \text{PF}(s_j, x_j, y_j, z_j, \delta_j)] \right\}
\]

(7)

where \( N \) indicates the number of foci; \( x_j, y_j, \) and \( z_j \) denote the position of the \( j \)-th focus in the focal region; and \( s_j \) and \( \delta_j \) are the parameters that can be used to adjust the amplitude and phase of the \( j \)-th focus, respectively. \( T_j \) represents the modulation factor of polarization modes, where \( T = \exp(i(n\varphi - \beta)) \) for \( |\text{L}\rangle , |\text{R}\rangle \) in rectangular cylindrical coordinate system and \( T = \cos(n\varphi + \beta) \) for \( |\text{RVVB}_{\sigma,\beta}\rangle \) in cylindrical coordinate system, respectively. Note that the amplitude factor of \( T = \cos(n\varphi + \beta) \) is neglected because of its little impact on the polarization modes (See Supplementary Note 3).

**Experiment**

Following the general focusing theory in Supplementary Equation 12, inherent polarization modes of \( m \)-order VVB can be extracted in the focal region by modulating the light beam with \( T = \exp(i\psi) \). It should be emphasized that modulation symmetry is broken down in free space once \( m \)-order VVB is modulated by \( T = \exp(i\psi) \). If only the light beam propagates sufficiently, one can always achieve polarization mode extraction in free space, not just in a focusing system. Here, we only take \( NA=0.01 \) as example in this paper. Figure 2 presents the schematic of the experimental setup, which is formed by two optical systems. One is a VVB creation system indicated by the green dotted box; another is a focusing system.
which is formed by an objective lens indicated by the purple dotted box. A collimated incident linearly polarized beam with a wavelength of 633 nm passes through a phase-only SLM and two lenses (L3, L4) and being converted into an m=30-order VVB by a vortex polarizer (VP). m=30-order VVB outputs from the first system is focused by an objective lens (OL) with NA=0.01 in the focusing system and the light intensity of polarization mode is recorded using a CCD. Here, the VP is realized by the Q-plate technique [20-22] and conjugate with the phase-only SLM (Spatial light modulator) by the 4f system with L3 (f3=150 mm) and L4 (f4=150 mm). Therefore, the phase of m=30-order VVB in Equation 7 can be adjusted at will by the phase-only SLM.

Figure 3 presents the experimental result of one single polarization mode extraction from m=30-order VVB in the focal region of OL. There are two kinds of polarization modes extracted from VVB for different coordinate systems. Polarization modes |L⟩, |R⟩ in Figures 3 (b, g) are created by the vortex phase $T = \exp(i(n\varphi - \beta))$ with $n = \pm 30$, as shown in Figures 3 (a, f), which are the eigenmode of the Cartesian coordinate system. In the cylindrical coordinate system, polarization modes $|\text{RVVB}_{\sigma,\beta}\rangle$ in Figures 3 (l, q, v) are realized by the phases $T = \exp(i\psi)$ with $\psi = \text{Phase}[\cos(n\varphi + \beta)]$, where $\beta = 0$, $n = 1, 2, 10$ for Figures 3 (k, p, u), respectively. Note that the polarization state of $|\text{RVVB}_{\sigma,\beta}\rangle$ can convert into that of $|\text{LVVB}_{\sigma,\beta}\rangle$ by adjusting the parameter $\beta$. All of these modes are distinguished from each other using a quarter-wave plate (indicated by the blue arrow) and a polarizer (indicated by the purple arrows), as shown in Figure 3(c-e, h-i, m-o, r-t, and w-y), which are consistent with their theoretical results in Supplementary Figure 4.

To better understand on the polarization extraction of light beam, Figure 4 present the entire principle of polarization mode extraction from VVB. There are two key factors for polarization mode extraction. One is the break of modulation symmetry; another is the polarization mode competition. Here, we take the polarization modes |L⟩ in Figure 3 (b) as example to explain the break of modulation symmetry and the polarization modes competition within VVB. When modulating by the vortex phase $T = \exp(i(n\varphi - \beta))$ with $n = m=30$ in Figure 3 (a), the modulation symmetry of VVB is broken down, thereby dividing the light beam into a left and a right polarization mode simultaneously. Although both polarization modes are equal parts of m=30-order VVB, their energy densities are far different because both modes possess a different topological charge. Generally, larger energy density of one polarization mode leads to smaller energy density of the other one.
Due to the above polarization mode competition, polarization mode with higher topological charge has a lower energy density, thereby leading to a smaller maximum light intensity at the outer ring. On the contrary, polarization mode at the inner ring possesses a lower topological charge, thereby leading to a larger maximum light intensity (See Supplementary Figure 3). For this reason, one can adjust the maximum light intensities of both polarization modes by controlling the difference of topological charge, namely $\eta = \|m + n\| - |m - n|$. Here, we define a parameter $\gamma = I_{m,\text{outer}} / I_{m,\text{inner}}$ to characterize the difference between the maximum light intensity of inner $I_{m,\text{inner}}$ and outer ring $I_{m,\text{outer}}$. In the case of Figure 3 (b), $\eta = 60$ and $\gamma = 0.0018$. When being normalized, the right circular polarization mode at the outer ring can be neglected, and only the left circular polarization mode is retained in the focal region. Likewise, one can obtain arbitrary polarization modes $\{RVVB_{\sigma,\beta}\}$ with $n = 1, 2, 10$ in Figures 3 (l, q, v) by the phases in Figures 3 (k, p, u), respectively.

Figure 3 not only demonstrates the polarization mode extraction from $m$-order VVB, but also, more importantly, verifies that polarization can be controlled by merely the phase of light beam. Thus, with the aid of optical pen, one can simply extend the single polarization mode extraction in Figure 3 into a multiple one in Figure 5. After all, there is not just only one pair of polarization modes within a light beam, but many other pairs as well. Figures 5(a, e, i) present the experimental results of extracting $3 \times 3$ polarization mode arrays in the focal plane, where (a) polarization modes $\{RVVB_{\sigma,\beta}\}$ with positive order $\sigma = 0, 1, \ldots, 7$ and a right circular polarization (RCP) mode in the position of geometry focus; (e) polarization modes $\{RVVB_{\sigma,\beta}\}$ with negative order $\sigma = 0, -1, \ldots, -7$ and a RCP mode in the position of geometry focus; (i) polarization modes $\{RVVB_{\sigma,\beta}\}$ with positive and negative order $\sigma = 0, \pm 1, \ldots, \pm 4$, respectively. Their corresponding phases are shown in Figures 5(m, n, o), respectively, the parameters of which can be found in Supplementary Note 4. As shown in Figures 5 (b-d, f-h and j-l), petals with $2|\sigma|$ number are rotated along with the polarizer indicated by the purple arrow, which are coincident with their corresponding theoretical results in Supplementary Figure 5. Besides, we also verify the function of parameter $\beta$ on the polarization mode transformation in Supplementary Figures 6, 7.

**Discussion**

Unlike the previous works, we are not attempt to combine two orthogonally polarized beams together to form a new light beam, but on the contrary, extract inherent polarization modes from one single light
beam. In principle, two orthogonally polarized beams, such as left and right circularly polarized beams, can form a new light beam. However, this light beam contains not just both above polarization modes, but many other polarization mode pairs as well. Therefore, the inverse process of overlapping two light beams in the conventional principle can provide many more possibilities in term of polarization modulation. Specifically, different coordinate system possesses different inherent polarization modes within a light beam. We have already extracted the eigenmodes $|L\rangle$, $|R\rangle$ and $|RVVB_{\sigma,\beta}\rangle$ for Cartesian and cylindrical coordinate system in the focal region. One can predict that other eigenmodes can also be obtained in the focal region for other coordinate systems.

In the followings, we would like to discuss a technical factor of polarization mode extraction. Polarization mode extraction involves one important conceptual change: modulation symmetry is the physical root of polarization mode entanglement within light beam satisfied wave function. However, the break of modulation symmetry does not assure polarization mode extraction in the focal region [2,18,19]. For example, supposed $\eta=1$, one can also achieve the modulation asymmetry of light beam, however, the inner and outer polarization modes cannot be spatially separated, thereby making it impossible to extract one polarization mode from the others. The key parameter $\eta$ not only plays a vital role in adjusting $\gamma$, but also the distance between the outer and inner polarization modes. Generally, $\eta$ should not be too small. Small $\eta$ leads to the short distance between the outer and inner polarization modes, which not only gives rise to a large $\gamma$, but also causes the interference between both modes. In case $m < |n|$, $\eta = |m + n| - |m - n|$ increases from 0 to $2m$ with the increment of $|n|$, while one can obtain an identical $\eta = 60$ in case $m < |n|$. Owing to the identical $\eta = 60$, polarization mode within $m$-order VVB can be extracted with order even higher than the order $m$ in free space.

In conclusion, we have theoretically and experimentally demonstrated extraction of inherent polarization mode from one single light beam in the focal region of an objective lens by taking $m$-order VVB as an example. The entanglement of inherent polarization modes is overcome by breaking down the modulation symmetry of VVB along with the polarization mode competition. Polarization modes $|L\rangle$, $|R\rangle$ and $|RVVB_{\sigma,\beta}\rangle$ for Cartesian and cylindrical coordinate system are therefore extracted in arbitrary positions of focal region with the aid of optical pen. This work not only conveys a physical idea that the generalized property of all light beams satisfied wave function, namely modulation symmetry, is the physical root of the entanglement of inherent polarization modes during propagating in free space,
but also presents an entirely new principle for polarization modulation, which will pave ways for polarization control based on phase modulation of one single light beam.

**Data Availability**

All data supporting the findings of this study are available from the corresponding author on request.
References

1. R. Dorn, S. Quabis, G. Leuchs, Sharper Focus for a Radially Polarized Light Beam. *Phys. Rev. Lett.* 91, 233901 (2003).
2. X. Hao, C. Kuang, T. Wang, X. Liu, Phase encoding for sharper focus of the azimuthally polarized beam. *Opt. Lett.* 35, 3928-3930 (2010).
3. Q. Zhan, Cylindrical vector beams: from mathematical concepts to applications. *Adv. Opt. Photon.* 1, 1-57 (2009).
4. Wang, H., Shi, L., Lukyanukh, B., Sheppard, C., Chong, C. T. Creation of a needle of longitudinally polarized light in vacuum using binary optics. *Nat. Photonics* 2, 501-505 (2008).
5. J. Luo, H. Zhang, S. Wang, L. Shi, Z. Zhu, B. Gu, X. Wang, X. Li, Three-dimensional magnetization needle arrays with controllable orientation. *Opt. Lett.* 44, 727-730 (2019).
6. Y. Kozawa, T. Hibi, A. Sato, H. Horanai, M. Kurihara, N. Hashimoto, H. Yokoyama, T. Nemoto, S. Sato, Lateral resolution enhancement of laser scanning microscopy by a higher-order radially polarized mode beam. *Opt. Express* 19, 15947-15954 (2011).
7. G. Rui, Q. Zhan, Trapping of resonant metallic nanoparticles with engineered vectorial optical field. *Nanophotonics-Berlin* 3, 351-361 (2014).
8. T. A. Nieminen, N. R. Heckenberg, H. Rubinsztain-Dunlop, Forces in optical tweezers with radially and azimuthally polarized trapping beams. Opt. Lett. 33, 122-124 (2008).
9. G. Milione, M. P. J. Lavery, H. Huang, Y. Ren, G. Xie, T. A. Nguyen, E. Karimi, L. Marrucci, D. A. Nolan, R. R. Alfano, A. E. Willner, 4 x 20Gbit/s mode division multiplexing over free space using vector modes and a q-plate mode (de)multiplexer. *Opt. Lett.* 40, 1980-1983 (2015).
10. B. Ndagano, I. Nape, M. A. Cox, C. Rosales-Guzman, A. Forbes, Creation and Detection of Vector Vortex Modes for Classical and Quantum Communication. *J. Light. Technol.* 36, 292-301 (2018).
11. J. Fischer, M. Wegener, Three-dimensional optical laser lithography beyond the diffraction limit. *Laser Photonics Rev.* 7, 22-44 (2013).
12. S. G. Hansen, Source mask polarization optimization. *Journal of Micro/Nanolithography, MEMS, and MOEMS* 10, 1-10 (2011).
13. L. Wei, Y. Li, K. Liu, Design of freeform illumination sources with arbitrary polarization for immersion lithography. *Proc. of SPIE* 9272, 927221 (2014).
14. D. Naidoo, F. S. Roux, A. Dudley, I. Litvin, B. Piccirillo, L. Marrucci, A. Forbes, Controlled generation of higher-order Poincaré sphere beams from a laser. *Nature Photonics,* (2015).
15. X. L. Wang, J. P. Ding, W. J. Ni, C. S. Guo, H. T. Wang, Generation of arbitrary vector beams with a spatial light modulator and a common path interferometric arrangement. *Opt. Lett.* 32, 3549-3551 (2007).
16. X. L. Wang, J. Chen, Y. N. Li, J. P. Ding, C. S. Guo, H. T. Wang, Optical orbital angular momentum from the curl of polarization. *Phys. Rev. Lett.* 105, 253602 (2010).
17. X. Weng, Q. Song, X. Li, X. Gao, H. Guo, J. Qu, S. Zhuang, Free-space creation of ultralong anti-diffracting beam with multiple energy oscillations adjusted using optical pen. *Nat. Commun.* 9, 5035 (2018).
18. Y. Zhang, X. Guo, L. Han, P. Li, S. Liu, H. Cheng, J. Zhao, “Gouy phase induced polarization transition of focused vector vortex beams,” Opt. Express 25, 25725-25733 (2017).

19. H. Ren, X. Li, M. Gu, Polarization-multiplexed multifocal arrays by a \( \pi \)-phase-step-modulated azimuthally polarized beam. Opt. Lett. 39, 6771-6774 (2014).

20. L. Marrucci, C. Manzo, D. Paparo, Optical spin-to-orbital angular momentum conversion in inhomogeneous anisotropic media. Phys. Rev. Lett. 96, 163905 (2006).

21. M. M. Sánchez-López, J. A. Davis, N. Hashimoto, I. Moreno, E. Hurtado, K. Badham, A. Tanabe, S. W. Delaney, Performance of a q-plate tunable retarder in reflection for the switchable generation of both first- and second-order vector beams. Opt. Lett. 41, 13-16 (2016).

22. P. Chen, W. Ji, B.-Y. Wei, W. Hu, V. Chigrinov, Y.-Q. Lu, Generation of arbitrary vector beams with liquid crystal polarization converters and vector-photoaligned q-plates. Appl. Phys. Lett. 107, 241102 (2015).
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Author contributions

X. Weng conceived of the research and designed the experiments. Yu Miao and Q. Zhang performed the experiments. Guanxue Wang, Yue Li and X. Weng analyzed all of the data. X. Weng and X. Gao supervised the experiments. X. Weng wrote the paper. X. Gao and S. Zhuang offered advice regarding its development. S. Zhuang directed the entire project.

Competing interests statement

The authors declare that they have no competing financial and non-financial interests to disclose.
Figures

**Figure 1.** Schematic of the entanglement of inherent polarization modes within light beam.

**Figure 2.** Schematic of experimental setup for polarization mode extraction. The entire optical system is formed by the VVB creation system (the green dotted box) and a focusing system which is formed by an objective lens (the purple dotted box). A collimated incident $x$ linearly polarized beam with a wavelength of 633 nm can convert into a $m=30$-order VVB by the VP, which is conjugated with the phase-only SLM (Spatial light modulator) by the 4f-system with $L_3$ ($f_3=150$ mm) and $L_4$ ($f_4=150$ mm). The VP is a vortex polarizer realized by the Q-plate technique. The modulated VVB output from the first system is focused by an objective lens with NA=0.01 in the focusing system and the light intensity of polarization mode is recoded using a CCD. Here, Subfigures (a) and (b) are the light intensities of $m$-order VVB without and with a polarizer (purple arrow), respectively.
Figure 3. Experimental results of single polarization mode extraction. Here, left and right circular polarization modes (b, g) are realized by the vortex phase (a, f), respectively. VVBs with the order of 1, 2, 10 are extracted by the phase \( \phi = \text{Phase}[\cos(n\phi - \beta)] \), where \( \beta = 0 \), \( n=29 \) (k), \( n=28 \) (p), \( n=20 \) (u), respectively. Subfigures (c-e, h-j, m-o, r-t, and w-y) show the light intensities passing through the polarizer indicated by the purple arrow. The intensities of all polarization modes are normalized to a unit value, and the phase scale is 0-2\( \pi \).

Figure 4. Schematic of inherent polarization mode extraction from VVB. Here, A and B present two arbitrary orthogonal polarization modes of VVB. Polarization mode A or B is extracted, even selected in the focal region of OL by breaking down modulation symmetry as well as polarization mode competition.
Figure 5. Experimental results of multiple polarization mode extraction. $3 \times 3$ multiple polarization arrays (a, e, i) are extracted simultaneously from $m=30$-order VVB in the focal region by the phases (m-o), respectively. Here, RCP indicate a circular polarization mode; the parameter $\sigma$ is the order of polarization modes $\{RVVB_{2\sigma-1,\beta}\}$ with $\beta=0$. Subfigures (b-d, f-h, and j-l) show the light intensities passing through the polarizer indicated by the purple arrow. The intensities of all polarization modes are normalized to a unit value, and the phase scale is $0-2\pi$. 
Supplementary Information

*Extraction of inherent polarization mode from one single light beam*

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**Supplementary Figure 1** Schematic of the focusing system in Figure 2. Ω is the focal sphere, with its center at O and a radius $f$, namely, the focal length of the objective lens (OL). The transmittance of the pupil filter P is expressed as Equation (7). $\theta$ is the convergent angle. $I_r(\theta)$ denotes the electric amplitude of incident $m$ order VVB.

**Supplementary Figure 2.** Polarization modes generated by accurate and approximate modulation. The polarization modes $|RVVB_{m+1,\beta=\alpha}\rangle$ (b and f) are extracted in the focal region of OL by the accurate and approximate wavefronts of $m$=30-order VVB in (a and e), respectively. Subfigures (c, d, g, and h) show the light intensities passing through the polarizer indicated by the purple arrow. The intensities of all polarization modes are normalized to a unit value, and the phase scale is $0$-$2\pi$. 
Supplementary Figure 3. Polarization modes with different order. Polarization modes $\text{RVVB}_{\sigma, \beta=0}$ with the order (e) $\sigma = 0$; (f) $\sigma = 3$; (g) $\sigma = 6$; (h) $\sigma = 9$ are extracted by the phases in (a-d), respectively. $\gamma = \frac{I_{\text{m,outer}}}{I_{\text{m,inner}}}$, where $I_{\text{m,inner}}$ and $I_{\text{m,outer}}$ are the maximum light intensity of polarization modes in the inner (IR) and outer ring (OR), respectively. The intensities of all polarization modes are normalized to a unit value, and the phase scale is 0-2$\pi$.

Supplementary Figure 4. Theoretical results of Figure 3. Here, left and right circular polarization modes (b, g) are realized by the vortex phases (a, f), respectively. VVBs with the order of 1, 2, 10 are extracted by the phase $\phi = \text{Phase} [\cos(n \varphi - \beta)]$, where $\beta = 0$, $n=29$ (k), $n=28$ (p), $n=20$ (u), respectively. Subfigures (c-e, h-j, m-o, r-t, and w-y) show the light intensities passing through the polarizer indicated by the purple arrow. The intensities of all polarization modes are normalized to a unit value, and the phase scale is 0-2$\pi$. 
Supplementary Figure 5. Theoretical results of Figure 5. 3×3 polarization mode arrays (a, e, i) are extracted from $m=30$-order VVB in the focal region by the phases (m-o), respectively. Here, RCP indicates a right circular polarization mode; the parameter $\sigma$ is the order of polarization modes $|RVVB_{\sigma, \beta, \omega}|$. Subfigures (b-d, f-h, and j-l) show the light intensities passing through the polarizer indicated by the purple arrow. The light intensities are normalized to a unit value, and the phase scale is 0-2$\pi$. 
Supplementary Figure 6. Theoretical results of multiple polarization mode extraction with different $\beta$. Polarization modes $\mathbf{RVVB}_{\sigma,\beta}$ with the order (b) $\sigma = 0$; (g) $\sigma = 1$; (l) $\sigma = -1$; (q) $\sigma = 2$; (v) $\sigma = -2$ are extracted by the phases in (a, f, k, p, u), respectively. Subfigures (c-e, h-j, m-o, r-t, and w-y) show the light intensities passing through the polarizer indicated by the purple arrow. Here, $\beta$ control the polarization direction of each polarization mode and all parameters of the phases in (a, f, k, p, u) can be found in Supplementary Note 4. The light intensities are normalized to a unit value, and the phase scale is 0-2$\pi$. 
Supplementary Figure 7. Experimental results of Supplementary Figure 6. The light intensities of all polarization modes are normalized to a unit value, and the phase scales is 0-2π.
Supplementary Note 1: Modulation symmetry of light beam

In principle, a light beam can be composed with two orthogonal polarization modes, which is also the basis of conventional interference scheme. During propagating in free space, both polarization modes are intertwined with each other so that a stable polarization state can be maintained. Generally, light beam with homogeneous and inhomogeneous polarization can be expressed as

\[
E = \begin{bmatrix}
\cos mf(\phi, \theta) & \sin mf(\phi, \theta)
\end{bmatrix}^T,
\]

where \(f(\phi, \theta)\) represents a generalized phase function, and \(\phi\) and \(\theta\) are the convergence and azimuthal angle of the focusing system in Supplementary Figure 1; \(T\) denotes the matrix transposition operator; \(m\) is the order of light beam. In case \(mf(\phi, \theta)\)=const, the light beam in Supplementary Equation (1) is a homogeneously polarized beam. For example, a x linearly polarized light beam can be obtained with \(m=0\). In case \(mf(\phi, \theta)\)=variate, the light beam possesses a inhomogeneous polarization state.

Without loss of generality, Supplementary Equation (1) can be written into two forms:

\[
E_{\text{m}+i\text{n}} = \exp\left(\text{im}f(\phi, \theta)\right)\left|\text{R}\right\rangle + \exp\left(-\text{im}f(\phi, \theta)\right)\left|\text{L}\right\rangle,
\]

\[
E_{\text{m}} = \cos nf(\phi, \theta)\left|\text{RVB}_\sigma\right\rangle + \sin nf(\phi, \theta)\left|\text{LVB}_\sigma\right\rangle,
\]

where \(|\text{L}\rangle=[1 \ 0]\) and \(|\text{R}\rangle=[1 \ -i]\) denote the left and right circular polarization mode, respectively, and \(\phi\) is the azimuthal angle. \(|\text{RVB}_\sigma\rangle=[\cos \sigma f(\phi, \theta) \ \sin \sigma f(\phi, \theta)]\) and \(|\text{LVB}_\sigma\rangle=[-\sin \sigma f(\phi, \theta) \ \cos \sigma f(\phi, \theta)]\) are the \(\sigma\)-order light beam with orthogonal polarization. \(m = n + \sigma\). In case \(\sigma = 0\), \(|\text{RVB}_\sigma\rangle\) and \(|\text{LVB}_\sigma\rangle\) are simplified into x and y linear polarization mode, respectively. That is, the \(m\)-order light beam in Supplementary Equation (1) can be formed by combining x and y linear polarization mode with the modulation of \(\cos mf(\phi, \theta)\) and \(\sin mf(\phi, \theta)\), respectively.

From Supplementary Equations (2) and (3), inherent polarization modes within light beam are modulated by different weight factors. Taking Supplementary Equation (2) as an example. Polarization modes \(|\text{R}\rangle\) and \(|\text{L}\rangle\) are modulated by \(\exp(\pm mf(\phi, \theta))\), respectively. Although the signs of both phases are inverse, their absolute values are equal, namely \(mf(\phi, \theta)\). Likewise, polarization modes \(|\text{RVB}_\sigma\rangle\) and \(|\text{LVB}_\sigma\rangle\) modulated by \(\cos nf(\phi, \theta)\) and \(\sin nf(\phi, \theta)\) also possess an identical \(nf(\phi, \theta)\) in Supplementary Equation (3). Here, we call the above modulation of inherent polarization mode as modulation symmetry of light beam. Due to modulation symmetry, all inherent polarization modes
within light beam are intertwined with each other so that the polarization in Supplementary Equation (1) remains the same during propagating in free space. This is the physics root that one cannot extract polarization modes $|L\rangle, |R\rangle, |L_{VB}\rangle$ and $|R_{VB}\rangle$ from one single light beam in free space.
**Supplementary Note 2: Breaking down the modulation symmetry of light beam**

As discussed in Supplementary Note 1, light beam that satisfied wave function possesses the modulation symmetry between inherent polarized modes, thereby maintaining a stable polarization during propagating in free space. To break down the modulation symmetry, one must destroy the identical modulation factors $mf(\phi, \theta)$ and $nf(\phi, \theta)$. Here, we impose two particular modulation factors on circular and linear polarization modes in Supplementary Equations (2) and (3), respectively.

For the case of Supplementary Equation (2), circular polarization modes with different modulation factor can be obtained by modulating the light beam with $\exp(\pm imf(\phi, \theta))$, which can be expressed as

$$E_{mcL} = \exp(i2mf(\phi, \theta)) |R\rangle + |L\rangle,$$

$$E_{mcR} = \exp(-i2mf(\phi, \theta)) |L\rangle + |R\rangle,$$

For the case of Supplementary Equation (3), the light beam can be simplified into

$$E_{mV} = |RVB_{2n+\sigma,-\beta}\rangle + |RVB_{\sigma,\beta}\rangle,$$

when modulating by $\cos(nf(\phi, \theta) - \beta)$. Here, $|RVB_{\eta,\omega}\rangle = [\cos[\eta f(\phi, \theta) + \omega] \ \sin[\eta f(\phi, \theta) + \omega]]^T$ is a $\eta$-order light beam with polarization direction $\omega$. Thus, $|RVB_{2n+\sigma,-\beta}\rangle$ and $|RVB_{\sigma,\beta}\rangle$ can be obtained with $\eta = 2n + \sigma$, $\omega = -\beta$ and $\eta = \sigma$, $\omega = \beta$, respectively. For $\beta = 0, \pi / 2$, the light beam in Supplementary Equation (6) can be simplified into

$$E_{mV\ell} = |RVB_{2n+\sigma}\rangle + |RVB_{\sigma}\rangle,$$

$$E_{mVL} = -|LVB_{2n+\sigma}\rangle + |LVB_{\sigma}\rangle,$$

Although polarization modes $|L\rangle$, $|R\rangle$, $|RVB_{2n+\sigma,-\beta}\rangle$ and $|RVB_{\sigma,\beta}\rangle$ are of different modulation factor or different order in Supplementary Equations (4), (5) and (6), we still cannot assert the modulation symmetry of light beam is broken down. For example, if $mf(\phi, \theta) = const$ or $nf(\phi, \theta) = const$, the light beam in Supplementary Equation (1) is a homogeneously polarized beam, the polarization modes of which always have an identical modulation weight factor in Supplementary Equations (4), (5) and (6). For this reason, inherent polarization modes are still intertwined together and the modulation symmetry of homogeneously polarized beam cannot be broken down.

When $mf(\phi, \theta)$ and $nf(\phi, \theta)$ are variates, Supplementary Equation (1) turns into an
inhomogeneously polarized beam. Taking a radially polarized beam as example. In case $mf(\varphi, \theta) = \varphi$, Supplementary Equation (1) represents a radially polarized beam. When modulating by $\exp(\pm i\varphi)$ and $\cos(\varphi - \beta)$, the light beam can be simplified into

\[
E_{mL} = \exp(i2\varphi)|R\rangle + |L\rangle, \tag{9}
\]

\[
E_{mR} = \exp(-i2\varphi)|L\rangle + |R\rangle, \tag{10}
\]

\[
E_{mv} = |RVB_{2-\beta}\rangle + |RVB_{o,\beta}\rangle, \tag{11}
\]

Here, we take $\sigma = 0$ in Supplementary Equation (6). According to Supplementary Equations (9-11), different polarization modes have different modulation factor or different order, which demonstrates the modulation asymmetry of light beam. The realization of modulation asymmetry by inhomogeneously polarized beam provides a possibility of disentangling the inherent polarization modes in free space.
Supplementary Note 3: Principle of polarization mode extraction

In this note, we present the entire theoretical principle of polarization mode extraction. As shown in Supplementary Figure 1, the experimental setup in Figure 2 can be simplified into a focusing system, where \( m \)-order VVB are modulated by a phase-only SLM. The numerical aperture (NA) of objective lens is 0.01.

Based on the Debye vectorial diffractive theory, the electric fields of inherent polarization mode extracted in the focal region can be expressed as [1]

\[
E = -\frac{A}{\pi} \int_0^{2\pi} \int_0^{\alpha} \sin\theta \cos^{1/2}\theta T_l_\theta(\theta) V \exp(-iks \cdot p) d\theta d\phi,
\]

where \( \theta \) and \( \phi \) are the convergent angle and azimuthal angle, respectively; and \( A \) is a normalized constant. \( \alpha = \arcsin(NA / n) \), where NA is the numerical aperture of the objective lens, and \( n \) is the refractive index in the focusing space. The wavenumber is \( k = 2\pi / \lambda \), where \( \lambda \) is the wavelength of the incident beam, and \( p = (rcos\phi, rsin\phi, z) \) denotes the position vector of an arbitrary field point. The unit vector along a ray is expressed as \( s = (-sin\theta cos\phi, -sin\theta sin\phi, cos\theta) \). \( T = \exp(i\psi) \) is the transmittance of the pupil filter \( P \), where \( \psi \) denotes the phase of \( m \) order VVB in Equation (7). \( l_\theta(\theta) \) is the electric amplitude of the incident beam, which can be expressed as [2]

\[
l_\theta(\theta) = J_1(2\beta_0 \frac{sin\theta}{sin\alpha}) \exp[-(\beta_0 \frac{sin\theta}{sin\alpha})^2],
\]

where \( \beta_0 \) is the ratio of the pupil radius to the incident beam waist. \( J_1(\bullet) \) is the Bessel function of the first kind with order 1.

In Supplementary Equation 12, \( V \) represents the propagation unit vector of the incident beam right after having passed through the lens. According to Equation 2, the incident \( m \) order VVB can be considered as the combination of two orthogonal polarization modes \( |RVVB_\sigma> \) and \( |LVVB_\sigma> \). Both polarization modes can be simplified into a \( x \) and \( y \) linear polarization mode with \( \sigma = 0 \), respectively. That is, the electric field of \( m \) order VVB can be written as [3, 4]

\[
E_{VVB} = \cos(m\phi)|x> + \sin(m\phi)|y>,
\]

where \( |x> \) and \( |y> \) denote the \( x \) and \( y \) linear polarization mode, respectively. Therefore, the unit vector \( V = \cos m\phi V_x + \sin m\phi V_y \), where \( V_x \) and \( V_y \) are the electric vector of \( |x> \) and \( |y> \), respectively, and can be written as [1]
\[
V_x = \begin{bmatrix}
\cos \theta + (1 - \cos \theta) \sin^2 \varphi \\
-1 \cos \theta \sin \varphi \cos \varphi \\
\sin \theta \cos \varphi
\end{bmatrix}; V_y = \begin{bmatrix}
-(1 - \cos \theta) \sin \varphi \cos \varphi \\
1 - (1 - \cos \theta) \sin^2 \varphi \\
\sin \theta \sin \varphi
\end{bmatrix}
\] (15)

Eventually, the focal light intensity of the polarization modes extracted from \( m \) order VVB can be obtained using \( I = |E|^2 \).

**Theoretical result**

In the following simulations, \( NA=0.01, n=1, \) and \( \beta_0 = 1 \). The unit of length in all figures is the wavelength \( \lambda \), and the light intensity is normalized to the unit value.

**Difference between accurate and approximate wavefront**

According to Equation (4), polarization modes \( \{RVVB_{2n+\sigma,-\beta}\} \) and \( \{RVVB_{\sigma,\beta}\} \) are obtained by modulating \( m \)-order VVB with \( T = \cos(n\varphi - \beta) \), which is related to not only the phase of VVB, but also the amplitude. Thus, we rewrite this wavefront of \( m \)-order VVB into \( T = Amp \cdot \exp(i\phi) \), where \( Amp = \text{Amplitude}(\cos(n\varphi - \beta)) \) and \( \phi = \text{Phase}(\cos(n\varphi - \beta)) \), respectively. Supplementary Figure 2 shows the polarization modes created via the accurate wavefront of \( T = \cos(n\varphi - \beta) \) in Supplementary Figure 2 (a) and approximate wavefront of \( T = \exp(i\phi) \) in Supplementary Figure 2 (f), respectively. Here, \( n = 29, \beta = 0 \). Polarization modes \( \{RVVB_{\sigma,\beta=0}\} \) with the order of \( \sigma = 1 \) are therefore extracted from \( m=30 \)-order VVB in the focal region, see Supplementary Figure 2 (b, g). By comparing the light intensities in Supplementary Figure 2 (c, d) with that in Supplementary Figure 2 (h, i), polarization modes created by Supplementary Figure 2 (a, f) are almost the same, with only a deviation of 0.26%. That is, the amplitude of \( T = \cos(n\varphi - \beta) \) has little impact on the extraction of polarization mode in the focal region. For this reason, we only take the phase of \( T = \cos(n\varphi - \beta) \) into account in the paper by neglecting the amplitude of \( m \)-order VVB.

**Polarization mode competition**

Supplementary Figure 3 presents the break of modulation system by modulating \( m \)-order VVB using \( T = \exp(i\phi) \) with \( \phi = \text{Phase}(\cos(n\varphi)) \). Taking Supplementary Figure 3 (h) as an example, \( m=30 \)-order VVB modulated by the phase \( T = \exp(i\phi) \) with \( n = 21 \) in Supplementary Figure 3 (d) is divided into two polarization modes \( \{RVVB_{\sigma,\beta}\} \) with the order \( \sigma = 9,51 \) and \( \beta = 0 \) in the focal region, respectively.
Both modes not only possess an identical total light intensity, but also propagate coaxially in free space. As the propagation distance increases, polarization mode with different order exhibits different degree of energy divergence, thereby leading to different position in the focal region. Generally, polarization mode with lower order locates at the inner ring while that of larger order is at the outer ring, as shown in Supplementary Figure 3 (h). This spatial separation implies the break of modulation symmetry, which further disentangles the inherent polarization modes in free space.

As discussed above, both inherent orthogonal polarization modes are two equal parts of VVB. Although the total light intensities of both modes cannot be adjusted in free space, their energy densities are merely depended on the order of polarization modes. Generally, the lower order the polarization mode, the higher the energy density. Higher energy density always leads to larger maximum light intensity of polarization mode. That is, the polarization mode at the inner ring is always brighter than that at the outer ring, as shown in Supplementary Figures 3(e-h). Here, we define a parameter $\gamma = \frac{I_{m,\text{outer}}}{I_{m,\text{inner}}}$ to characterize the difference between the maximum light intensity of inner $I_{m,\text{inner}}$ and outer ring $I_{m,\text{outer}}$. As shown in Supplementary Figure 3, different $\gamma$ can be obtained by different order of polarization modes, which can further be adjusted at will by the parameter $\eta = |m+n| - |m-n|$. Here, $|m+n|$ and $|m-n|$ are the order of polarization modes at the outer and inner ring, respectively. If only $\eta$ is sufficiently large, a small $\gamma$ can always obtained. When being normalized, polarization mode at the inner ring is retained by neglecting the other one, see Supplementary Figures 3(e-h). Here, we call this phenomenon as polarization mode competition. Note that the circular polarization mode $|L\rangle, |R\rangle$ in Equation (3) can also be analysed in the same way.

**Theoretical result of Polarization mode extraction**

Supplementary Figures 4 and 5 present the theoretical results of single and multiple polarization mode extraction in the focal region using $m=30$-order VVB. By comparing with Figures 3 and 5, the experiment results are consistent with that of theoretical predictions in Supplementary Figures 4 and 5, respectively. In addition, we also verify the polarization conversion between $|RVVB_{\sigma,\beta}\rangle$ and $|LVVB_{\sigma,\beta}\rangle$ by adjusting the parameter $\beta$. Supplementary Figures 6 and 7 present the theoretical and experimental results of extracting multiple polarization modes $|RVVB_{\sigma,\beta}\rangle$, where $\beta = -N\pi / 8$ with $N=0, 1, \ldots, 8$ and (b) $\sigma = 0$; (g) $\sigma = 1$; (l) $\sigma = -1$; (q) $\sigma = 2$; (v) $\sigma = -2$, respectively. When passing
through the polarizers indicated the purple arrow, petals with $2\sigma$ numbers are rotated along with the parameter $\beta$, which demonstrates that $|RVVB_{\sigma,\beta}\rangle$ with $\sigma \neq 0$ is converted into its orthogonal counterpart $|LVVB_{\sigma,\beta}\rangle$, see Supplementary Figures 6 (h-j, m-o, r-t, w-y). In case of $\sigma=0$ in Supplementary Figure 6 (b), $|RVVB_{\sigma,\beta}\rangle$ is simplified into a linear polarization mode with the direction of $\beta$. For example, a x, y linear polarization mode can be obtained with $\beta = 0, \pi/2\rangle$, respectively. According the malus’ law, different polarization direction of linear polarization mode manifests different light intensity. Thus, polarization modes with different brightness are obtained by passing through the polarizer indicated by the purple arrow in Supplementary Figures 6(c-e). According to the experimental and theoretical results in Supplementary Figures 6 and 7, polarization mode with order of $\sigma$ can transform into its orthogonal counterpart by merely adjusting the parameter $\beta$. Note that the phases for the extraction of above polarization modes are shown in in Supplementary Figures 6 (a, f, k, p, u), respectively, the parameters of which can be found in the Supplementary Note 4.
Supplementary Note 4: Parameters of all phases

Figure 5 and Supplementary Figure 5

The phase of $m$ order VVB:

$$\psi = \text{Phase}\left\{\sum_{j=1}^{N=9} \left[T_j \cdot \text{PF}(s_j, x_j, y_j, 0, \delta_j)\right]\right\},$$

where

- $l_1 = T_1 \cdot \text{PF}(s_1, 0, 0, 0, \delta_1)$;
- $l_2 = T_2 \cdot \text{PF}(s_2, d, 0, 0, \delta_2)$;
- $l_3 = T_3 \cdot \text{PF}(s_3, d, d, 0, \delta_3)$;
- $l_4 = T_4 \cdot \text{PF}(s_4, 0, d, 0, \delta_4)$;
- $l_5 = T_5 \cdot \text{PF}(s_5, -d, d, 0, \delta_5)$;
- $l_6 = T_6 \cdot \text{PF}(s_6, -d, 0, 0, \delta_6)$;
- $l_7 = T_7 \cdot \text{PF}(s_7, -d, -d, 0, \delta_7)$;
- $l_8 = T_8 \cdot \text{PF}(s_8, 0, -d, 0, \delta_8)$;
- $l_9 = T_9 \cdot \text{PF}(s_9, d, -d, 0, \delta_9)$;

$d = 800$;

The parameters of the phase in Figure 5 and Supplementary Figure 5(m)

$$\delta_j = 0 ; \quad T_i = \exp(-im\phi) ; \quad T_j = \cos(m_2\phi + \beta_2) \quad \text{with} \quad \beta_j = 0 \quad \text{and} \quad m_j = m - j + 2, j = 2, 3...N ; \quad m = 30 ;$$

$$s_1 = 0.45 ; \quad s_2 = 0.76 ; \quad s_3 = 1.2 ; \quad s_4 = 1.4 ; \quad s_5 = 1.6 ; \quad s_7 = 1.8 ; \quad s_8 = 2 ; \quad s_9 = 2.2 .$$

The parameters of the phase in Figure 5 and Supplementary Figure 5(n)

$$\delta_j = 0 ; \quad T_i = \exp(-im\phi) \quad \text{;} \quad T_j = \cos(m_3\phi + \beta_3) \quad \text{with} \quad \beta_j = 0 \quad \text{and} \quad m_j = m - j + 2, j = 2, 3...N ; \quad m = 30 ;$$

$$s_1 = s_2 = 0.5 ; \quad s_3 = 1.2 ; \quad s_4 = 1.4 ; \quad s_5 = s_6 = 1.6 ; \quad s_7 = 1.8 ; \quad s_8 = 2 ; \quad s_9 = 2.2 .$$

The parameters of the phase in Figure 5 and Supplementary Figure 5(o)

$$\delta_j = 0 ;$$

For $j = 1, 3, 5, 7, 9$:

$$T_j = \cos(m_4\phi + \beta_4) \quad \text{with} \quad \beta_j = 0 \quad \text{and} \quad m_j = m + (j - 1) / 2 .$$

For $j = 2, 4, 6, 8$:

$$T_j = \cos(m_5\phi + \beta_5) \quad \text{with} \quad \beta_j = 0 \quad \text{and} \quad m_j = m - j / 2 .$$

$$s_1 = 0.55 ; \quad s_2 = s_3 = 1 ; \quad s_4 = s_5 = 1.25 ; \quad s_6 = s_7 = 1.35 ; \quad s_8 = s_9 = 1.5 .$$

Final transmittance of the pupil filter $P$ in Supplementary Equation 12:

$$T = \exp(i\psi) .$$

Supplementary Figures 6 and 7

The phase of $m$ order VVB:

$$\psi = \text{Phase}\left\{\sum_{j=1}^{N=8} \left[T_j \cdot \text{PF}(s_j, x_j, y_j, 0, \delta_j)\right]\right\},$$
where \( x_j = d \cos \phi_j \), \( y_j = d \sin \phi_j \) and \( d = 800 \); \( \phi_j = 2\pi (j - 1) / N, j = 1,2,3...N \); \( \delta_j = 0 \); \( s_j = 1 \);

\[ T_j = \cos(m_1 \varphi + \beta_j) \] with \( \beta_j = -\pi (j - 1) / N, j = 1,2,3...N \).

The parameters of the phases in Supplementary Figures 6(a) and 7(a)

\[ m_1 = m = 30 \; ; \]

The parameters of the phases in Supplementary Figures 6(f) and 7(f)

\[ m_1 = m - 1 \; ; \]

The parameters of the phases in Supplementary Figures 6(k) and 7(k)

\[ m_1 = m + 1 \; ; \]

The parameters of the phases in Supplementary Figures 6(p) and 7(p)

\[ m_1 = m - 2 \; ; \]

The parameters of the phases in Supplementary Figures 6(u) and 7(u)

\[ m_1 = m + 2 \; ; \]

Final transmittance of the pupil filter \( P \) in Supplementary Equation 12:

\[ T = \exp(i\psi) \; . \]
Supplementary References

1. Richards, B., Wolf, E. Electromagnetic Diffraction in Optical Systems. II. Structure of the Image Field in an Aplanatic System. Proc. Royal Soc. A 253, 358-379 (1959).

2. Youngworth, K. S., Brown, T. G. Focusing of high numerical aperture cylindrical-vector beams. Opt. Express 7, 77-87 (2000).

3. G. Milione, M. P. J. Lavery, H. Huang, Y. Ren, G. Xie, T. A. Nguyen, E. Karimi, L. Marrucci, D. A. Nolan, R. R. Alfano, A. E. Willner, 4 x 20Gbit/s mode division multiplexing over free space using vector modes and a q-plate mode (de)multiplexer. Opt. Lett. 40, 1980-1983 (2015).

4. M. M. Sánchez-López, J. A. Davis, N. Hashimoto, I. Moreno, E. Hurtado, K. Badham, A. Tanabe, S. W. Delaney, Performance of a q-plate tunable retarder in reflection for the switchable generation of both first- and second-order vector beams. Opt. Lett. 41, 13-16 (2016).