WEAK STABILITY AND LARGE TIME BEHAVIOR FOR THE CAUCHY PROBLEM OF THE VLASOV-MAXWELL-BOLTZMANN EQUATIONS

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Abstract. The Cauchy problem for the Vlasov-Maxwell-Boltzmann equations (VMB) is considered. First the renormalized solution to the Vlasov equation with the Lorentz force is discussed and the difficulty on the partial differentiability of the coefficients is overcome. Then the weak stability of the renormalized solutions to the Cauchy problem of VMB is established using the compactness of velocity averages and a renormalized formulation. Furthermore, the large time behavior of the renormalized solutions to VMB is studied and it is proved that the density of particles tends to a local Maxwellian as the time goes to infinity.

1. Introduction

Since the work of DiPerna and Lions [10] on the Cauchy problem for the Boltzmann equation twenty years ago, it has been a well-known open problem to extend their theory to the Vlasov-Maxwell-Boltzmann equations. Among the difficulties, how to define the characteristics of the Vlasov-Maxwell-Boltzmann equations is a major obstacle. In this paper, we will give the following partial results: the weak stability and large time behavior of the renormalized solutions to the Vlasov-Maxwell-Boltzmann equations, and existence of the renormalized solutions to the Vlasov equation with the Lorentz force. The fundamental model for dynamics of dilute charged particles is described by the Vlasov-Maxwell-Boltzmann equations (VMB) of the following form [5, 7, 16, 18, 23, 27]:

\[
\begin{align*}
\frac{\partial f}{\partial t} + \xi \cdot \nabla_x f + (E + \xi \times B) \cdot \nabla_\xi f &= Q(f, f), & x \in \mathbb{R}^3, \quad \xi \in \mathbb{R}^3, \quad t \geq 0, \\
\frac{\partial E}{\partial t} - \nabla \times B &= -j, \quad \text{div}B = 0, \quad \text{on} \quad \mathbb{R}^3_x \times (0, \infty), \\
\frac{\partial B}{\partial t} + \nabla \times E &= 0, \quad \text{div}E = \rho, \quad \text{on} \quad \mathbb{R}^3_x \times (0, \infty), \\
\rho &= \int_{\mathbb{R}^3} f d\xi, \quad j = \int_{\mathbb{R}^3} f \xi d\xi, \quad \text{on} \quad \mathbb{R}^3_x \times (0, \infty),
\end{align*}
\]

where \( f = f(t, x, \xi) \) is a nonnegative function for the density of particles which at time \( t \) and position \( x \) move with velocity \( \xi \) under the Lorentz force

\[
E + \xi \times B,
\]
\( E \) is the electric field, \( B \) is the magnetic field, the function \( j \) is called the current density, and the function \( \rho \) is the charge density. The collision operator \( Q(f,f) \), which acts only on the velocity dependence of \( f \) (this reflects the physical assumption that collisions are localized in space and time), is defined as

\[
Q(f,f) = \int_{\mathbb{R}^3} d\xi_s \int_{S^2} d\omega b(\xi - \xi_s, \omega)(f'f'_s - ff_s),
\]

with \( \omega \in S^2 \), the unit sphere in \( \mathbb{R}^3 \), where \( b = b(z, \omega) \) denotes the collision kernel which is a given nonnegative function defined on \( \mathbb{R}^3 \times S^2 \), and

\[
f_s = f(t, x, \xi_s), \quad f' = f(t, x, \xi'), \quad f'_s = f(t, x, \xi'_s),
\]

with

\[
\xi' = \xi - (\xi - \xi_s, \omega)\omega, \\
\xi'_s = \xi_s + (\xi - \xi_s, \omega)\omega,
\]

which yield one convenient parametrization of the set of solutions to the law of elastic collisions

\[
\xi' + \xi'_s = \xi + \xi_s, \\
|\xi'|^2 + |\xi'_s|^2 = |\xi|^2 + |\xi_s|^2.
\]

The interpretation of \( \xi, \xi_s, \xi', \xi'_s \) is the following: \( \xi, \xi_s \) are the velocities of two colliding molecules immediately before collision, while \( \xi', \xi'_s \) are the velocities immediately after the collision. Those unknown functions \( f, E, \) and \( B \) are strongly coupled, and the constraint on the divergence of \( E \) will be ensured provided that the conservation of charge holds; that is,

\[
\frac{\partial \rho}{\partial t} + \text{div}_x j = 0,
\]

since

\[
0 = \frac{\partial}{\partial t}(\text{div}_x E - \rho) = \text{div}_x E_t - \rho_t \\
= \text{div}_x(\nabla_x \times B - j) - \rho_t \\
= -\rho_t - \text{div}_x j,
\]

due to the fact \( \text{div}(\nabla \times v) = 0 \) for any vector-valued function \( v \). Similarly, the magnetic field \( B \) remains divergence free if it is so initially.

The VMB equations are integro-differential equations which provide a mathematical model for the statistical evolution of dilute charged particles. The construction of global solutions to VMB has been open for a long time until only a few years ago. In Guo [18], a unique global in time classical solution near a global Maxwellian (independent of space and time) was constructed. See also Strain [27] for the extension to the Cauchy problem. Notice that, Lions constructed in [23] a very weak solution to VMB, which is usually called a measure-valued solution, using Young’s measure to deal with the nonlinearity.

For the particles without collision (cf. [1, 9, 15, 16, 24, 26]), or when the molecules are so rare that they do not interact with each other, VMB becomes the so-called Vlasov-Maxwell
system (VM),
\[
\begin{align*}
\frac{\partial f}{\partial t} + \xi \cdot \nabla_x f + (E + \xi \times B) \cdot \nabla_\xi f &= 0, \quad x \in \mathbb{R}^3, \quad \xi \in \mathbb{R}^3, \quad t \geq 0, \\
\frac{\partial E}{\partial t} - \nabla \times B &= - j, \quad \text{div} B = 0, \quad \text{on} \quad \mathbb{R}^3 \times (0, \infty), \\
\frac{\partial B}{\partial t} + \nabla \times E &= 0, \quad \text{div} E = \rho, \quad \text{on} \quad \mathbb{R}^3 \times (0, \infty), \\
\rho &= \int_{\mathbb{R}^3} f d\xi, \quad j = \int_{\mathbb{R}^3} f\xi d\xi, \quad \text{on} \quad \mathbb{R}^3 \times (0, \infty).
\end{align*}
\]

Note that (1.2a) is a transport equation with a divergence free coefficient, that is
\[
\text{div}_{x,\xi}(\xi, E + \xi \times B) = 0.
\]

This property ensures that the solution will remain the same integrability as the initial data. With the help of this observation and velocity averaging lemma, DiPerna and Lions proved in [9] the global existence in time of weak solutions to VM with large initial data. For the smooth solutions to VM, we refer the readers to Glassey [16] and Schaeffer [26].

The main goal of this paper is to show the weak stability and the large time behavior of the renormalized solutions to VMB. To this end, we will need an existence result of the renormalized solution to the Vlasov equation (1.2a). Notice that the Vlasov equation is a transport equation with only partially $W^{1,1}_{loc}$ regularity, since usually we can not expect any differentiability on the magnetic field $B$ and the electric field $E$ from the conservation of energy. Inspired by the result in Bouchut [3] and Le Bris-Lions [20], we will first show the existence of renormalized solutions to the Vlasov equation. The presence of a non-trivial magnetic field $B(x,t)$, a natural consequence of the celebrated Maxwell theory for electromagnetism, creates severe mathematical difficulty in studying the weak stability of weak solutions and the construction of global in time solutions for VMB. Our first result on weak stability is built on our above mentioned new result about renormalized solutions to the Vlasov equation with the aid of the velocity average lemma (DiPerna-Lions [9] and DiPerna-Lions-Meyer [13]) and some techniques from Lions [22, 23]. Our second result on renormalized solutions to VMB is their large time behavior, since from the physical point of view, the density of particles is assumed to converge to an equilibrium represented by a Maxwellian function of the velocity as the time $t$ becomes large. Our results heavily depend on, apart from the weak compactness property,

- the existence of renormalized solutions to the Vlasov equation;
- a renormalized formulation, which is crucial to make sure that the quadratic term $Q(f, f)$ is meaningful in $D'$ (sense of distributions); and
- the velocity averaging lemma [9,13], which is crucial for the convergence of non-linear term $(E + \xi \times B) \cdot \nabla_\xi f$.

The stability of renormalized solutions under weak convergence yields a consequence on the propagation of smoothness for those solutions. Indeed, a sequence of renormalized solutions $\{f_n\}_{n=1}^{\infty}$ to VMB is relatively strongly compact in $L^1([0,T] \times \mathbb{R}^6)$ if and only if the sequence of the corresponding initial data $\{f_{0n}\}_{n=1}^{\infty}$ is relatively strongly compact in $L^1(\mathbb{R}^6)$. In other words, under our assumption on the collision kernel and the integrability of the electric field and the magnetic field, no oscillations develop unless they are present from the beginning.
In order to prove our results, the standard \textit{a priori} estimates derived from the conservation laws and H theorem are very useful, and in addition we need some assumptions on the integrability of the electric field \(E(t,x)\) and the magnetic field \(B(t,x)\). More precisely, besides the standard estimate of \(E\) and \(B\) in \(L^\infty(0,T; L^2(\mathbb{R}^3))\), we need to assume that \(E\) is uniformly bounded in \(L^\infty(0,T; L^5(\mathbb{R}^3))\) and \(B\) is uniformly bounded in \(L^\infty(0,T; L^s(\mathbb{R}^3))\) for some \(s > 5\). The reasons for these requirements on \(E\) and \(B\) are twofold: (I) when we define the characteristics for the Vlasov equation, we need a bound on \(E\) in \(L^\infty(0,T; L^5(\mathbb{R}^3))\); (II) the averaging lemma (cf. \cite{23}), combining with the uniform bound of \(\int_{\mathbb{R}^3} f d\xi\) in \(L^\infty(0,T; L^\frac{5}{2}(\mathbb{R}^3))\) and the uniform bound of \(\int_{\mathbb{R}^3} \xi f d\xi\) in \(L^\infty(0,T; L^\frac{1}{2}(\mathbb{R}^3))\), implies the compactness of the first two moments of \(f\) on \(L^p(0,T; L^p_{\text{loc}}(\mathbb{R}^3))\) for any \(1 \leq p < \frac{5}{4}\), which is enough to ensure the convergence of the nonlinear Lorentz force term in the sense of the distributions provided that \(E\) and \(B\) are uniformly bounded in \(L^\infty(0,T; L^5(\mathbb{R}^3))\) and \(L^\infty(0,T; L^s(\mathbb{R}^3))\) for some \(s > 5\).

We now remark that throughout this work we never claim the existence of renormalized solutions to VMB. Actually, all results in this paper are based on the assumption of such an exact existence or the existence of a sequence of approximating solutions. One possible direction to address the existence problem may be based on the construction of a sequence of exact solutions or approximating solutions with the requirement that the electric field \(E\) is uniformly bounded in \(L^\infty(0,T; L^5(\mathbb{R}^3))\). We notice that the hyperbolic property of the Maxwell equations also demonstrates some difficulties if we want to improve the integrability of the electric field and the magnetic field. How to fulfill this strategy is still an open question and will be the topic of our future research.

When the Lorentz force disappears, that is \(E + \xi \times B = 0\), VMB becomes the classical Boltzmann equation. For the Cauchy problem of the classical Boltzmann equation, in \cite{10} DiPerna and Lions proved the global existence of renormalized solutions with angular cut-off collision kernel and arbitrary initial data, see also \cite{1, 5, 8, 9, 11, 22, 23} and the references cited therein. Later, Hamdache extended this existence result to a bounded domain in \cite{19}. The method explored for the existence result was the analysis of the weak stability of solutions. The argument strongly relied on some compactness properties (see \cite{22}) which hold for sequences of renormalized solutions. In \cite{23}, Lions extended the similar weak stability and global existence result to the Vlasov-Poisson-Boltzmann equations. For the extension to the Landau equation, see Villani \cite{28}. For the long time behavior of the Boltzmann equations, see \cite{7, 11, 12, 28}.

This paper will proceed as follows. We will discuss the renormalized solution to the Vlasov equation in Section 2. Section 3 is devoted to stating \textit{a priori} estimates for VMB, main assumptions and main results on the weak stability of renormalized solutions to VMB. Then, Theorem 3.1 on weak stability and Theorem 3.2 on the propagation of smoothness will be proved in Section 4 and Section 5, respectively. In Section 6, we study the large time behavior and establish mathematically the convergence of \(f\) to a local Maxwellian satisfying the Vlasov-Maxwell equations. Finally, in Section 7, we explain an extension of our results to the relativistic Vlasov-Maxwell-Boltzmann equations.
2. Renormalized Solutions to the Vlasov Equation

In this section, we consider the Vlasov equation of the form:
\[ \partial_t f + \xi \cdot \nabla_x f + (E + \xi \times B) \cdot \nabla_\xi f = 0, \]  
with \( B(x,t) \in L^\infty(0,T;L^2(\mathbb{R}^3_x)) \) and \( E(x,t) \in L^\infty(0,T;L^2 \cap L^5(\mathbb{R}^3_x)). \)

If we set \( y = (x, \xi) \in \mathbb{R}^6, \) \( B = (\xi, E + \xi \times B) \in \mathbb{R}^6, \) then (2.1) becomes a standard transport equation
\[ \partial_t f + B \cdot \nabla y f = 0. \]  
The question of whether the Vlasov equation has renormalized solutions is not only useful when the normalized solution to VMB system is considered, but also has its own interest due to the lower regularity of the coefficients. The renormalized solutions mean that (2.2) still holds if we replace \( f \) by \( \beta(f) \) with a suitable \( \beta. \) Over past twenty years, there are many important progress about the renormalized solutions to (2.2). More precisely, DiPerna and Lions showed in [8] the existence of renormalized solutions when the coefficients \( B \) are many important progress about the renormalized solutions to (2.2). More precisely, DiPerna and Lions showed in [8] the existence of renormalized solutions when the coefficients \( B \) are bounded variations) field in [2] (for related work, see [3]). Also, in 2004 Le Bris and Lions extended in [20] the DiPerna-Lions theory tothe case that the coefficient has only partial regularity.

For the VMB or the Vlasov equation, the velocity \( B \) is no longer in \( W^{1,1}_{(x,\xi),loc}. \) Inspired by [3] [20], we claim that we still can prove the existence of a renormalized solution to (2.1) under the conditions that \( E(x,t) \in L^\infty(0,T;L^2(\mathbb{R}^3) \cap L^5(\mathbb{R}^3)), \) and \( B(x,t) \in L^\infty(0,T;L^2(\mathbb{R}^3)). \) This is a crucial step for establishing renormalized solutions to the Vlasov-Maxwell-Boltzmann equations.

**Theorem 2.1.** Assume that \( B(x,t) \in L^\infty(0,T;L^2(\mathbb{R}^3_x)) \) and \( E(x,t) \in L^\infty(0,T;L^2 \cap L^5(\mathbb{R}^3_x)). \) Let \( f_0 \in L^1 \cap L^\infty(\mathbb{R}^6) \) and \( |\xi|^2 f_0 \in L^1(\mathbb{R}^6). \) Then there exists a solution to (2.1) (and hence to (2.2)) such that
\[ f(t,x,\xi) \in L^\infty([0,T], L^1_{x,\xi} \cap L^\infty_{x,\xi}(\mathbb{R}^3_x)), \]
and \( |\xi|^2 f \in L^\infty(0,T;L^1(\mathbb{R}^6)), \) satisfying the initial condition \( f|_{t=0} = f_0(x,\xi). \) Furthermore, if \( f_0 \in L^\infty_x(L^1_{\xi}(\mathbb{R}^3)), \) then \( f \in L^\infty(0,T;L^\infty_x(L^1_{\xi}(\mathbb{R}^3))), \) and hence the solution is unique.

To begin with the proof, notice that \( B = (B_1,B_2) \) satisfies
\[ \text{div}_x B_1 = \text{div}_x B_2 = 0, \]
with
\[ B_1(x,\xi) = \xi \in W^{1,1}_{x,loc}(\mathbb{R}^3) \quad \text{(it does not depend on } x), \]
\[ B_2(x,\xi) = E + \xi \times B \in L^1_{x,loc}(\mathbb{R}^3;W^{1,1}_{x,loc}(\mathbb{R}^3)). \]
The proof of this theorem is divided into three steps. The uniqueness is a crucial issue which is the consequence of the following two lemmas, the first one dealing with regularization, and the second one stating the uniqueness. Finally, we will show the existence part.
Now we denote the mollifier $\kappa_\varepsilon$ as 
\[ \kappa_\varepsilon = \frac{1}{\varepsilon^n} \kappa \left( \frac{x}{\varepsilon} \right), \quad \kappa \in D(\mathbb{R}^3), \quad \int_{\mathbb{R}^3} \kappa = 1, \quad \kappa \geq 0, \] 
where $D(\mathbb{R}^3) = C_0^\infty(\mathbb{R}^3)$. Then, we have the following two lemmas.

**Lemma 2.1.** Let $f = f(t, x, \xi) \in L^\infty([0, T], L^1_{(x, \xi)} \cap L^\infty(\mathbb{R}^6))$ be a solution of (2.2), and $\kappa_\varepsilon$ and $\kappa_\mu$ be two regularizations with two different scalings, respectively, in the variable $x$ and $\xi$. Then, for any $\varepsilon > 0$, there exists a number $\mu(\varepsilon)$ with $0 < \mu(\varepsilon) \leq \varepsilon^2$ such that 
\[ f_{\varepsilon, \mu(\varepsilon)} = (f * \kappa_\varepsilon) * \kappa_\mu(\varepsilon) \] 
is a smooth (in $(x, \xi)$) solution of 
\[ \frac{\partial f_{\varepsilon, \mu(\varepsilon)}}{\partial t} + B \cdot \nabla f_{\varepsilon, \mu(\varepsilon)} = A_\varepsilon, \]
with 
\[ \lim_{\varepsilon \to 0} A_\varepsilon = 0, \quad \text{in} \quad L^\infty([0, T], L^1_{(x, \xi), loc} \cap L^\infty(\mathbb{R}^6)). \]

**Lemma 2.2.** Let $f = f(t, x, \xi) \in L^\infty([0, T], L^1_{(x, \xi)} \cap L^\infty(\mathbb{R}^3))$ be a nonnegative solution of (2.2) with zero initial data $f_0 = 0$. If, in addition, $|\xi|^2 f \in L^\infty([0, T], L^1(\mathbb{R}^6))$ and $f \in L^2_x(L^1_\xi)$, then $f = 0$ for all time $t > 0$.

We now prove these two lemmas, and then finally complete the proof of Theorem 2.1

**2.1. Proof of Lemma 2.1.** We will use the mollifier to regularize the function $f$ in $\xi$ and $x$, while we assume that $f$ is differentiable with respect to $t$ (the results below hold also for the general case from a standard mollification in $t$ with the help of Lebesgue’s dominated theorem.) All the functional spaces used here are local, which is clearly enough for such a regularization result.

We first regularize in the $\xi$ variable by convoluting (2.2) with $\kappa_\mu$ to get 
\[ \frac{\partial f * \kappa_\mu}{\partial t} + (\xi \cdot \nabla_x f) * \kappa_\mu + ((E + \xi \times B) \cdot \nabla_\xi f) * \kappa_\mu = 0. \]
(2.3)

Denoting by 
\[ [(E + \xi \times B) \cdot \nabla_\xi, \kappa_\mu](f) = (E + \xi \times B) \cdot \nabla_\xi (f * \kappa_\mu) - \kappa_\mu * ((E + \xi \times B) \cdot \nabla_\xi f). \]

Then, (2.3) can be rewritten as 
\[ \frac{\partial f * \kappa_\mu}{\partial t} + (\xi \cdot \nabla_x f) * \kappa_\mu + (E + \xi \times B) \cdot \nabla_\xi (f * \kappa_\mu) = [(E + \xi \times B) \cdot \nabla_\xi, \kappa_\mu](f). \]
(2.4)

It is a standard fact (see [8]) that 
\[ I_1^\mu := [(E + \xi \times B) \cdot \nabla_\xi, \kappa_\mu](f) \to 0 \quad \text{in} \quad L^1_{(x, \xi), loc} \] 
as $\mu \to 0$. Indeed, this is clear for smooth coefficients and $f$, while the general case follows as in [8] by dense property through the estimate 
\[ ||(E + \xi \times B) \cdot \nabla_\xi, \kappa_\mu)(f)||_{L^1_{x, loc}} \leq C ||E + \xi \times B||_{W^{1, 1}_{x, loc}} ||f||_{L^\infty_{x, \xi}}, \]
which then implies the following standard estimate by integrating in $x$, 
\[ ||(E + \xi \times B) \cdot \nabla_\xi, \kappa_\mu)(f)||_{L^1_{(x, \xi), loc}} \leq C ||E + \xi \times B||_{L^1_{x, loc}(W^{1, 1}_{x, loc})} ||f||_{L^\infty_{x, \xi}}. \]
Next, we regularize in the $x$ variable by convoluting (2.4) with $\kappa_\varepsilon$ for $f_\mu = f \ast \kappa_\mu$ to obtain,

\[
\frac{\partial f_\mu \ast \kappa_\varepsilon}{\partial t} + (\xi \cdot \nabla_x f) \ast \kappa_\mu \ast \kappa_\varepsilon + (E + \xi \times B) \cdot \nabla_\xi (f_\mu \ast \kappa_\varepsilon) = [(E + \xi \times B) \cdot \nabla_\xi, \kappa_\varepsilon] (f_\mu) + I_1^{\mu} \ast \kappa_\varepsilon.
\]

(2.6)

We now successively deal with each terms on the right-hand side of (2.6). First, it is easy to observe that for fixed $\mu$, we have

\[I_1^{\mu} \ast \kappa_\varepsilon \to I_1^{\mu}, \quad \text{as} \ \varepsilon \to 0,\]

in $L^1_{(x,\xi),\text{loc}}$, which together with (2.5) implies that

\[
\lim_{\mu \to 0} \lim_{\varepsilon \to 0} I_1^{\mu} \ast \kappa_\varepsilon = 0, \quad \text{in} \ L^1_{(x,\xi),\text{loc}}.
\]

(2.7)

Second, for the first term on the right-hand side of (2.6), we have

\[
[(E + \xi \times B) \cdot \nabla_\xi, \kappa_\varepsilon] (f_\mu) = (E + \xi \times B) \cdot \nabla_\xi (\kappa_\varepsilon \ast f_\mu) - \kappa_\varepsilon \ast ((E + \xi \times B) \cdot \nabla_\xi f_\mu)
\]

\[
= (E + \xi \times B) \cdot ((\nabla_\xi f_\mu) \ast \kappa_\varepsilon) - \kappa_\varepsilon \ast ((E + \xi \times B) \cdot \nabla_\xi f_\mu)
\]

\[
= [(E + \xi \times B), \kappa_\varepsilon] (\nabla_\xi f_\mu).
\]

The latter bracket can be controlled as follows:

\[
\left\| [(E + \xi \times B), \kappa_\varepsilon] (\nabla_\xi f_\mu) \right\|_{L^1_{(x,\xi),\text{loc}}} \leq C \|E + \xi \times B\|_{L^1_{(x,\xi),\text{loc}}} \|\nabla_\xi f_\mu\|_{L^\infty_{(x,\xi)}}.
\]

Hence, for fixed $\mu$, we have

\[
\lim_{\varepsilon \to 0} [(E + \xi \times B) \cdot \nabla_\xi, \kappa_\mu] (f_\mu) = 0,
\]

in $L^1_{(x,\xi),\text{loc}}$. This implies,

\[
\lim_{\mu \to 0} \lim_{\varepsilon \to 0} [(E + \xi \times B) \cdot \nabla_\xi, \kappa_\mu] (f_\mu) = 0,
\]

(2.8)

in $L^1_{(x,\xi),\text{loc}}$. By a standard diagonalization procedure, for any $\varepsilon > 0$, we can find $\mu(\varepsilon)$ with $0 < \mu(\varepsilon) \leq \varepsilon^2 \to 0$ such that

\[
\lim_{\varepsilon \to 0} I_1^{\mu(\varepsilon)} \ast \kappa_\varepsilon = 0, \quad \text{in} \ L^1_{(x,\xi),\text{loc}}.
\]

and

\[
\lim_{\varepsilon \to 0} [(E + \xi \times B) \cdot \nabla_\xi, \kappa_{\mu(\varepsilon)}] (f_{\mu(\varepsilon)}) = 0, \quad \text{in} \ L^1_{(x,\xi),\text{loc}}.
\]

To complete the proof of this lemma, it remains to show the following convergence for the above chosen $\mu(\varepsilon)$:

\[
I_2^{\mu(\varepsilon)} \varepsilon = (\xi \cdot \nabla_x f) \ast \kappa_{\mu(\varepsilon)} \ast \kappa_\varepsilon - \xi \cdot \nabla_x (f_{\mu(\varepsilon)} \ast \kappa_\varepsilon)
\]
in $L^1_{(x,\xi),loc}$. Indeed, we can control $I_2^{(\varepsilon)}$ as
\[
|I_2^{(\varepsilon)}| = \int_{\mathbb{R}^6} \left| (\xi - \eta) \cdot \nabla_x f(x - \zeta, \xi - \eta) - \xi \cdot \nabla_x f(x - \zeta, \xi - \eta) \right| \kappa_{\mu(\varepsilon)} \kappa_\varepsilon d\zeta d\eta
\]
\[
= \int_{\mathbb{R}^6} \left| \eta \cdot \nabla_x f(x - \zeta, \xi - \eta) \right| \kappa_{\mu(\varepsilon)} \kappa_\varepsilon d\zeta d\eta
\]
\[
= \int_{\mathbb{R}^6} \eta \cdot \nabla_\xi \kappa_{\varepsilon}(\xi) \kappa_{\mu(\varepsilon)}(\eta) f(x - \zeta, \xi - \eta) d\zeta d\eta
\]
\[
\leq \frac{\mu(\varepsilon)}{\varepsilon} \int_{\mathbb{R}^6} |\varepsilon \nabla_\xi \kappa_{\varepsilon}| \kappa_{\mu(\varepsilon)} f(x - \zeta, \xi - \eta) d\zeta d\eta.
\]
Thus, we deduce that, for any compact subset $K \subset \mathbb{R}^6$, by Fubini’s theorem,
\[
\|I_2^{(\varepsilon)}\|_{L^1(K)} = \int_K \left| \int_{\mathbb{R}^6} \eta \cdot \nabla_\xi \kappa_{\varepsilon}(\xi) \kappa_{\mu(\varepsilon)}(\eta) f(x - \zeta, \xi - \eta) d\zeta d\eta \right| dxd\xi
\]
\[
\leq C \frac{\mu(\varepsilon)}{\varepsilon} \left( \int_{\mathbb{R}^3} |\varepsilon \nabla_\xi \kappa_{\varepsilon}| d\xi \right) \sup_{|\xi| \leq \varepsilon, |\eta| \leq \mu(\varepsilon)} \|f(x - \zeta, \xi - \eta)\|_{L^1(K)} \quad (2.9)
\]
\[
\leq C \frac{\mu(\varepsilon)}{\varepsilon} \sup_{|\xi| \leq \varepsilon, |\eta| \leq \mu(\varepsilon)} \|f(x - \zeta, \xi - \eta)\|_{L^1(K)}.
\]
Since $f \in L^1_{(x,\xi),loc}$, one has, according to the continuity of translation in $L^1(K)$,
\[
\sup_{|\xi| \leq \varepsilon, |\eta| \leq \mu(\varepsilon)} \|f(x - \zeta, \xi - \eta) - f(x, \xi)\|_{L^1(K)} \to 0, \quad \text{as} \quad \varepsilon \to 0,
\]
and hence,
\[
\sup_{|\xi| \leq \varepsilon, |\eta| \leq \mu(\varepsilon)} \|f(x - \zeta, \xi - \eta)\|_{L^1(K)} \quad \text{is uniformly bounded for all} \quad \varepsilon \leq 1.
\]
Thus, if we let $\varepsilon \to 0$ and $0 \leq \mu(\varepsilon) \leq \varepsilon^2$, we deduce from (2.9) that
\[
I_2^{(\varepsilon)} \to 0, \quad \text{in} \quad L^1_{(x,\xi),loc} \quad \text{as} \quad \varepsilon \to 0. \quad (2.10)
\]
Therefore, the lemma follows from (2.7), (2.8), and (2.10), and we complete the proof of this lemma. \hfill \Box

Next, we turn to the proof of Lemma 2.2.

2.2. Proof of Lemma 2.2 Let $f$ be a nonnegative solution as claimed in Theorem 2.1. We introduce two cut-off functions, respectively, with respect to each variable $x$ and $\xi$. For $m, n \in \mathbb{N}$, we denote them by
\[
\psi_m(x) = \psi \left( \frac{x}{m} \right), \quad \text{and} \quad \phi_n(\xi) = \phi \left( \frac{\xi}{n} \right),
\]
where $\psi \in \mathcal{D}(\mathbb{R}^3)$, $\psi \geq 0$, $\psi = 1$ for $|x| \leq 1$ and $\psi = 0$ for $|x| \geq 2$; and the analogous properties are required on $\phi$ with respect to the variable $\xi$. We first multiply (2.1) by $\phi_n$ and integrate over $\xi$ space to obtain
\[
\frac{\partial}{\partial t} \int_{\mathbb{R}^3} f \phi_n d\xi + \text{div}_x \left( \int_{\mathbb{R}^3} \xi f \phi_n d\xi \right) + \int_{\mathbb{R}^3} (E + \xi \times B) \cdot \nabla_\xi f \phi_n d\xi = 0. \quad (2.11)
\]
For the last term in (2.11), we deduce, due to\( \text{div}_\xi (E + \xi \times B) = 0 \),

\[
\int_{\mathbb{R}^3} (E + \xi \times B) \cdot \nabla_\xi f \, d\xi = - \int_{\mathbb{R}^3} f (E + \xi \times B) \cdot \nabla_\xi \phi_n \, d\xi = - \int_{\mathbb{R}^3} \frac{1 + |\xi|}{n} \frac{E + \xi \times B}{1 + |\xi|} \cdot \nabla_\xi \phi \left( \frac{\xi}{n} \right) \, d\xi.
\]

Now we multiply (2.11) by \( \psi_m \) and integrate over \( x \) space to deduce

\[
\frac{d}{dt} \int_{\mathbb{R}^6} f \psi_m \phi_n \, dx + \int_{\mathbb{R}^3} \psi_m \text{div}_x \left( \int_{\mathbb{R}^3} \xi f \phi_n \, d\xi \right) \, dx = \int_{\mathbb{R}^3} \psi_m \int_{\mathbb{R}^3} f \frac{1 + |\xi|}{n} \frac{E + \xi \times B}{1 + |\xi|} \cdot \nabla_\xi \phi \left( \frac{\xi}{n} \right) \, d\xi \, dx.
\]  

(2.12)

Hence, using the integration by parts for the second term in (2.12), we have

\[
\frac{d}{dt} \int_{\mathbb{R}^2} f \psi_m \phi_n \, dx \int_{\mathbb{R}^3} \nabla_x \psi_m \cdot \left( \int_{\mathbb{R}^3} \xi f \phi_n \, d\xi \right) \, dx = \int_{\mathbb{R}^3} \psi_m \int_{\mathbb{R}^3} f \frac{1 + |\xi|}{n} \frac{E + \xi \times B}{1 + |\xi|} \cdot \nabla_\xi \phi \left( \frac{\xi}{n} \right) \, d\xi \, dx.
\]  

(2.13)

Next, we proceed to control the two integral terms in (2.13). Indeed, for the second term in (2.13), we have

\[
\left| \int_{\mathbb{R}^3} \nabla_x \psi_m \cdot \left( \int_{\mathbb{R}^3} \xi f \phi_n \, d\xi \right) \, dx \right| = \left| \int_{\mathbb{R}^3} \frac{1}{m} \nabla_x \psi \left( \frac{x}{m} \right) \cdot \left( \int_{\mathbb{R}^3} \xi f \phi_n \, d\xi \right) \, dx \right| 
\]

\[
\leq C \frac{1}{m} \| \xi f \chi_{\{m \leq |x| \leq 2m, |\xi| \leq 2n\}} \|_{L^1((x, \xi))} \rightarrow 0,
\]

as \( m \rightarrow \infty \) and \( n \rightarrow \infty \). Here we used \( \xi f \in L^\infty([0, T], L^1(\mathbb{R}^6)) \), because

\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |\xi| f \, dx \, d\xi \leq R \int_{\mathbb{R}^3} \int_{\mathbb{R}^3 \cap \{|\xi| \leq R\}} f \, dx \, d\xi + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3 \cap \{|\xi| > R\}} |\xi| f \, dx \, d\xi 
\]

\[
\leq R \int_{\mathbb{R}^3} \int_{\mathbb{R}^3 \cap \{|\xi| \leq R\}} f \, dx \, d\xi + \frac{1}{R} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3 \cap \{|\xi| > R\}} |\xi|^2 f \, dx \, d\xi 
\]

\[
\leq R \| f \|_{L^1(\mathbb{R}^6)} + \frac{1}{R} \| \xi^2 f \|_{L^1(\mathbb{R}^6)} 
\]

\[
\leq 2 \| f \|_{L^1(\mathbb{R}^6)}^{\frac{1}{2}} \| \xi^2 f \|_{L^1(\mathbb{R}^6)}^{\frac{1}{2}} 
\]

by optimizing the value of \( R \).

On the other hand, for \( m \) fixed, we claim that the term on the right-hand side of (2.13) goes to zero as \( n \) goes to infinity by Lebesgue’s dominated convergence theorem. Indeed,
as $\nabla \phi$ is $L^\infty$ and supported in the annular $\{1 \leq |\xi| \leq 2\}$, we have for almost all $x \in \mathbb{R}^3$,

$$
\left| \psi_m \int_{\mathbb{R}^3} f \frac{1 + |\xi| E + \xi \times B}{1 + |\xi|} \cdot \nabla_{\xi} \phi \left( \frac{\xi}{n} \right) d\xi \right|
\leq \psi_m \int_{\mathbb{R}^3} f \frac{1 + |\xi| E + \xi \times B}{1 + |\xi|} \left| \nabla_{\xi} \phi \left( \frac{\xi}{n} \right) \right| d\xi
\leq 2 \|\nabla \phi\|_{L^\infty} \psi_m \|f\chi_{\{n \leq |\xi| \leq 2n\}}\|_{L^1_{\xi}} (|E| + |B|)
\rightarrow 0,
$$
as $n \rightarrow \infty$, since for almost all $x \in \mathbb{R}^3$, $f(x, \cdot) \in L^1(\mathbb{R}^3_{\xi})$. In addition, by the Cauchy-Schwarz inequality, we have,

$$
\left| \psi_m \int_{\mathbb{R}^3} f \frac{1 + |\xi| E + \xi \times B}{1 + |\xi|} \cdot \nabla_{\xi} \phi \left( \frac{\xi}{n} \right) d\xi \right|
\leq \psi_m \int_{\mathbb{R}^3} f \frac{1 + |\xi| E + \xi \times B}{1 + |\xi|} \left| \nabla_{\xi} \phi \left( \frac{\xi}{n} \right) \right| d\xi
\leq 2 \|\nabla \phi\|_{L^\infty} \|f\|_{L^1_{\xi}} (|E| + |B|)
\leq 4 \|\nabla \phi\|_{L^\infty} (\|f\|_{L^2_{\xi}}^2 + |E|^2 + |B|^2).
$$

and the right-hand side is in $L^1_x$, since $f \in L^2_x(L^1_{\xi})$ and $E, B \in L^2_x$. Thus, Lebesgue’s theorem applies and we get the convergence of the term on the right-hand side of (2.13) to zero as $n$ goes to infinity, and $m$ being kept fixed.

Collecting the behaviors of those two terms, we obtain with (2.13), as $n$, and next $m$, go to infinity,

$$
\frac{d}{dt} \int_{\mathbb{R}^3} f dx d\xi = 0.
$$

As $f_0 = 0$, this yields $f = 0$ for all $t$ since $f \geq 0$ and this concludes the proof. \hfill $\square$

Having proved Lemma 2.1 and Lemma 2.2 we are now ready to complete the proof of Theorem 2.1 as follows.

2.3. **Proof of Theorem 2.1** Assume for the time being that we have at hand two solutions $f_1$ and $f_2$ to (2.1) satisfying the regularity stated in Theorem 2.1 and sharing the same initial value. In view of the interpolation between $L^1$ and $L^\infty$, and the fact

$$
f_i \in L^\infty([0, T], L^1_x \cap L^\infty_{x,\xi} (\mathbb{R}^3)) \cap L^\infty([0, T], L^\infty_x (\mathbb{R}^3, L^1_\xi(\mathbb{R}^3)))
$$

for $i = 1, 2$, we deduce that

$$
f_i \in L^\infty([0, T], L^2_x (\mathbb{R}^3, L^1_\xi(\mathbb{R}^3))).
$$

By virtue of Lemma 2.1 their difference $f = f_1 - f_2$ satisfies

$$
\frac{\partial f_{\mu(\varepsilon), \varepsilon}}{\partial t} + B \cdot \nabla_{(x, \xi)} f_{\mu(\varepsilon), \varepsilon} = \mathcal{A}_{\varepsilon},
$$

(2.15)
with the same notation as in Lemma 2.1. Since \( f_{\mu(\varepsilon),\varepsilon} \in C^\infty(\mathbb{R}^6) \), we multiply (2.15) by \( \beta'(f_{\mu(\varepsilon),\varepsilon}) \) for some function \( \beta \in C^1(\mathbb{R}) \) with \( \beta' \) bounded, and obtain
\[
\frac{\partial \beta(f_{\mu(\varepsilon),\varepsilon})}{\partial t} + \mathcal{B} \cdot \nabla_{(x,\xi)} \beta(f_{\mu(\varepsilon),\varepsilon}) = \mathcal{A}_\varepsilon \beta'(f_{\mu(\varepsilon),\varepsilon}).
\]

By letting \( \varepsilon \) go to zero, we obtain the equation
\[
\frac{\partial \beta(f)}{\partial t} + \mathcal{B} \cdot \nabla_{(x,\xi)} \beta(f) = 0,
\]
in \( L^\infty([0,T]; L^1(\mathbb{R}^6)) \) for such functions \( \beta \). Now, letting \( \beta \) approximate the absolute value function, we end up with
\[
\frac{\partial |f|}{\partial t} + \mathcal{B} \cdot \nabla_{(x,\xi)} |f| = 0.
\]

This implies that we have a nonnegative solution \( |f| \) to (2.1), which vanishes at initial time and belongs to the functional space stated in Lemma 2.2. Applying Lemma 2.2, we get \( |f| = 0 \), that is, \( f_1 = f_2 \). There remains now to prove the existence part.

Existence in the functional space \( L^\infty([0,T]; L^1(\mathbb{R}^6)) \) is given in a straightforward way by an application of Proposition 2.1 of [8]. For the sake of consistency, let us only mention here that it is a simple matter of regularization of the vector field \( \mathcal{B} \) appearing in (2.1). That is, one introduces the solution \( f_\alpha \) to
\[
\frac{\partial f_\alpha}{\partial t} + \mathcal{B}_\alpha \cdot \nabla f_\alpha = 0, \quad \text{in } (0, \infty) \times \mathbb{R}^6,
\]
where \( \mathcal{B}_\alpha = \kappa_\alpha * \mathcal{B} \in L^1([0,T]; C^\infty(\mathbb{R}^6)) \) converges to \( \mathcal{B} \), then shows the desired estimates on \( f_\alpha \), and finally passes to the limit.

Next, the non-standard part we have to prove here is the fact that such a solution necessarily satisfies \( |\xi|^2 f \in L^\infty([0,T], L^1(\mathbb{R}^6)) \). This is actually a consequence of the specific form of the transport equation and of the regularization process we have already done. Indeed, first, by the method of characteristics, we know if \( f_0 \geq 0 \) a.e in \( \mathbb{R}^6 \), then \( f(t) \geq 0 \) a.e in \( \mathbb{R}^6 \) for all \( t \geq 0 \). Then, formally we multiply (2.1) by \( |\xi|^2 \) to obtain
\[
\frac{\partial (|\xi|^2 f)}{\partial t} + |\xi|^2 \xi \cdot \nabla_x f + |\xi|^2 (E + \xi \times B) \cdot \nabla_\xi f = 0.
\]
Then we integrate the above identity over \( \xi \) on \( \mathbb{R}^3 \) to deduce
\[
\frac{d}{dt} \int_{\mathbb{R}^6} |\xi|^2 f \, dx \, d\xi = \int_{\mathbb{R}^6} \text{div}_\xi(|\xi|^2 (E + \xi \times B)) \, f \, dx \, d\xi,
\] (2.16)
since
\[
\int_{\mathbb{R}^6} |\xi|^2 \xi \cdot \nabla_x f \, dx \, d\xi = - \int_{\mathbb{R}^6} \text{div}_x(|\xi|^2 \xi) \, f \, dx \, d\xi = 0.
\]
For the term on the right-hand side of (2.16), we have
\[
\int_{\mathbb{R}^6} \text{div}_\xi(|\xi|^2 (E + \xi \times B)) \, f \, dx \, d\xi = 2 \int_{\mathbb{R}^6} (\xi \cdot Ef) \, dx \, d\xi,
\]
since
\[
\text{div}_\xi(|\xi|^2 \xi \times B) = 2\xi \cdot (\xi \times B) + |\xi|^2 \text{div}_\xi(\xi \times B) = 0.
\]
Also, notice that, for a.e $x \in \mathbb{R}^3$,

\[
\int_{\mathbb{R}^3} |\xi| f d\xi \leq \int_{\{|\xi| \leq R\}} R f d\xi + \int_{\{|\xi| > R\}} |\xi| f d\xi \\
\leq \omega_3 R^4 \|f\|_{L^\infty(\mathbb{R}^6)} + R^{-1} \int_{\{|\xi| > R\}} |\xi|^2 f d\xi \\
\leq C \left( \int_{\mathbb{R}^3} |\xi|^2 f d\xi \right)^{\frac{4}{5}},
\]

where $\omega_3$ is the volume of the unit ball in $\mathbb{R}^3$, and in the last inequality $R$ is taken to be

\[
R = \left( \int_{\mathbb{R}^3} |\xi|^2 f d\xi \right)^{\frac{1}{5}}.
\]

Hence, we have the following estimate, by the Hölder inequality,

\[
\left| \int_{\mathbb{R}^6} \text{div}_\xi(|\xi|^2 (E + \xi \times B)) f d\xi d\xi \right| = 2 \left| \int_{\mathbb{R}^6} (\xi \cdot Ef) d\xi d\xi \right| \\
\leq 2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |\xi| f|E| d\xi d\xi \\
\leq C \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} |\xi|^2 f d\xi \right)^{\frac{4}{5}} |E| d\xi \\
\leq C \left( \int_{\mathbb{R}^6} |\xi|^2 f d\xi d\xi \right)^{\frac{1}{5}} \|E\|_{L^\infty([0,T];L^5(\mathbb{R}^3))}.
\]

Substituting this back to (2.16), we obtain

\[
\frac{d}{dt} \int_{\mathbb{R}^6} |\xi|^2 f d\xi d\xi \leq C \left( \int_{\mathbb{R}^6} |\xi|^2 f d\xi d\xi \right)^{\frac{4}{5}} \|E\|_{L^\infty([0,T];L^5(\mathbb{R}^3))}.
\]

This implies

\[
\int_{\mathbb{R}^6} |\xi|^2 f(t) d\xi d\xi \leq \int_{\mathbb{R}^6} |\xi|^2 f_0 d\xi d\xi + CT^5,
\]

for all $t \in [0, T]$.

Finally, we show that the solution $f$ necessarily belongs to $L^\infty([0,T];L^\infty(\mathbb{R}^3,L^1(\mathbb{R}^3)))$ if \( f_0 \in L^\infty_x(\mathbb{R}^3,L^1(\mathbb{R}^3)) \). This is actually also a consequence of the specific form of the transport equation and of the regularization process as mentioned earlier. Indeed, we mollify $E + \xi \cdot B$ by $\kappa_\alpha$ to obtain,

\[
\frac{\partial f_\alpha}{\partial t} + \xi \cdot \nabla_x f_\alpha + (E + \xi \times B)_\alpha \cdot \nabla_\xi f_\alpha = 0.
\]

Integrating (2.18) over $\xi$ in $\mathbb{R}^3$, one has, thanks to the fact that $\text{div}_\xi(E + \xi \times B)_\alpha = 0$,

\[
\frac{\partial}{\partial t} \int_{\mathbb{R}^3} f_\alpha d\xi + \xi \cdot \nabla_x \int_{\mathbb{R}^3} f_\alpha d\xi = 0.
\]
That is equivalent to saying that \( \int_{\mathbb{R}^3} f_\alpha d\xi \) satisfies a conservation form, which yields the conservation over time of

\[ \left\| \int_{\mathbb{R}^3} f_\alpha d\xi \right\|_{L^\infty_t}. \]

Hence, \( f_\alpha \in L^\infty(0, T; L^\infty_x(\mathbb{R}^3, L^1(\mathbb{R}_\xi^3))) \). By letting \( \alpha \to 0 \), one obtains

\[ f \in L^\infty(0, T; L^\infty_x(\mathbb{R}^3, L^1(\mathbb{R}_\xi^3))). \]

The proof of Theorem 2.1 is complete. \( \square \)

Remark 2.1. The assumption \( E \in L^\infty(0, T; L^5(\mathbb{R}^3)) \) is only needed to show the uniform estimate

\[ |\xi|^2 f \in L^\infty(0, T; L^1(\mathbb{R}^6)). \]

We now turn to the extension of the previous result to less regular initial data through the notion of renormalized solutions in the spirit of \([8]\). As in \([8]\), we consider the set \( L^0 \) of all measurable functions \( f \) on \( \mathbb{R}^6 \) with value in \( \mathbb{R}^4 \) such that \( \text{meas}\{ |f| > \lambda \} < \infty \), for all \( \lambda > 0 \). For any \( \beta \in C_{0,0}(\mathbb{R}) \), bounded and vanishing near zero, we thus have \( \beta(f) \in L^1 \cap L^\infty(\mathbb{R}^6) \) for any \( f \in L^0 \). As in \([8]\), we shall say that a sequence \( f_n \) is bounded (respectively, converges) in \( L^0 \) whenever \( \beta(f_n) \) is bounded (respectively, converges) in \( L^1 \) for any such \( \beta \). But now we need some additional assumptions on our initial data, and that is why we consider the subset \( L^{00} \) of \( L^0 \) consisting of functions \( f \) satisfying

\[ \int_{\{|f(x,\xi)| > \delta\}} |\xi|^2 dx d\xi \leq c_\delta < \infty, \quad \forall \delta > 0. \]

This subset is equipped with the topology induced by that of \( L^0 \). For any \( f \in L^{00} \), we have \( |\xi|^2 \beta(f) \in L^1_{(x,\xi)}(\mathbb{R}^6) \). Indeed, for \( \delta \) small enough such that \( \beta \) vanishes on \([0, \delta]\), we have

\[
\int_{\mathbb{R}^6} |\xi|^2 |\beta(f(x,\xi))| dx d\xi = \int_{\{|f(x,\xi)| > \delta\}} |\xi|^2 |\beta(f(x,\xi))| dx d\xi + \int_{\{|f(x,\xi)| \leq \delta\}} |\xi|^2 |\beta(f(x,\xi))| dx d\xi \\
\leq ||\beta||_{L^\infty} c_\delta + 0 < \infty.
\]

It follows that if we choose \( f_0 \) in \( L^{00} \), then \( \beta(f_0) \) is a convenient initial condition for the transport equation considered in Theorem 2.1. We therefore say that \( f \) is a renormalized solution of (2.1) complemented by an initial condition \( f_0 \in L^{00} \) whenever \( \beta(f) \) is a solution of (2.1) in the sense of Theorem 2.1 with the initial condition \( \beta(f_0) \).

3. Stability of Vlasov-Maxwell-Boltzmann Equations: Main Results

Let us begin by recalling that the general Vlasov-Maxwell-Boltzmann equations (1.1) has the collision operator \( Q(f, f) \) which can be written as

\[ Q(f, f) = Q^+(f, f) - Q^-(f, f), \]

where

\[ Q^+(f, f) = \int_{\mathbb{R}^3} d\xi^* \int_{S^2} d\omega b(\xi - \xi^*, \omega) f' f'\].
and
\[ Q^{-}(f, f) = \int_{\mathbb{R}^3} d\xi \int_{S^2} d\omega b(\xi - \xi_s, \omega) ff^* = fL(f), \]
with
\[ L(f) = A * \xi f, \quad A(z) = \int_{S^2} b(z, \omega) d\omega, \quad z \in \mathbb{R}^3. \]

The collision kernel \( b \) in the collision operator \( Q \) is a given function on \( \mathbb{R}^3 \times S^2 \). We shall always assume the so-called angular cut-off kernel throughout the rest of the paper, that is, \( b \) satisfies
\[ b \in L^1(B_R \times S^2) \quad \text{for all} \quad R \in (0, \infty), \quad b \geq 0 \]
where \( B_R = \{ z \in \mathbb{R}^3 : |z| < R \} \), and
\[
\begin{cases}
    b(z, \omega) \quad \text{depends only on} \quad |z| \quad \text{and} \quad |(z, \omega)|, \\
    (1 + |z|^2)^{-1} \left( \int_{z+B_R} A(\xi) d\xi \right) \to 0, \quad \text{as} \quad |z| \to \infty, \quad \text{for all} \quad R \in (0, \infty).
\end{cases}
\]

A classical example of such angular cut-off collision kernels is given by the so-called hard-spheres model where we have
\[ b(z, \omega) = |(z, \omega)|. \]

The VMB system \( \text{(1.1)} \) is complemented with the initial conditions
\[
\begin{align*}
    f|_{t=0} = f_0, & \quad \text{on} \quad \mathbb{R}^6, \quad \text{with} \quad f_0 \geq 0, \\
    E|_{t=0} = E_0, & \quad B|_{t=0} = B_0 \quad \text{on} \quad \mathbb{R}_x^3,
\end{align*}
\]
with the usual compatibility condition
\[ \text{div}B_0 = 0, \quad \text{and} \quad \text{div}E_0 = \rho_0 = \int_{\mathbb{R}^3} f_0 d\xi, \quad \text{on} \quad \mathbb{R}_x^3. \]

We state below our main stability results concerning the Cauchy problem of the Vlasov-Maxwell-Boltzmann system \( \text{(1.1)} \) and \( \text{(3.1)} \). We assume that \( f_0 \) satisfies
\[
\int_{\mathbb{R}^6} f_0(1 + \nu + |\xi|^2 + |\log f_0|)dxd\xi + \int_{\mathbb{R}^3} (|E_0|^2 + |B_0|^2)dx < \infty,
\]
where \( \nu = \nu(x) \) is some function in \( \mathbb{R}^3 \) satisfying
\[ \nu \geq 0, \quad (1 + \nu)^{\frac{1}{2}} \quad \text{is Lipschitz on} \quad \mathbb{R}^3, \quad e^{-\nu} \in L^1(\mathbb{R}^3). \]

Using the classical identity (see Lemma 2.1 in [4]),
\[
\int_{\mathbb{R}^3} Q(f, f) \zeta(\xi)d\xi = \frac{1}{4} \int_{\mathbb{R}^6} f d\xi \int_{S^2} \int_{S^2} d\omega b(f' f' - ff^*) (\zeta + \zeta_s - \zeta' - \zeta'_s),
\]
we deduce the following local conservation laws of mass, momentum and kinetic energy:
\[
\frac{\partial \rho}{\partial t} + \text{div}_x j = 0,
\]
\[
\frac{\partial}{\partial t} \left( \int_{\mathbb{R}^3} f\xi d\xi + E \times B \right) + \text{div}_x \left( \int_{\mathbb{R}^3} \xi \otimes \xi f d\xi + \left( \frac{|E|^2 + |B|^2}{2} Id - E \otimes E - B \otimes B \right) \right) = 0,
\]
\[
\frac{\partial}{\partial t} \left( \int_{\mathbb{R}^3} f |\xi|^2 d\xi \right) + \text{div}_x \left( \int_{\mathbb{R}^3} \xi |\xi|^2 f d\xi \right) - 2E \cdot \int_{\mathbb{R}^3} \xi f d\xi = 0, \quad (3.6)
\]
for \((x,t) \in \mathbb{R}^3 \times (0, \infty).\) In fact, while (3.4) and (3.6) are easy to verify, we need to pay more attention to (3.5). To verify (3.5), we first multiply (1.1a) by \(\xi\) and integrate with respect to \(\xi\) to obtain
\[
\frac{\partial}{\partial t} \int_{\mathbb{R}^3} f \xi d\xi + \text{div}_x \int_{\mathbb{R}^3} \xi \otimes \xi f d\xi = - (\rho E + j \times B). \quad (3.7)
\]
Note that
\[
E \text{div} E + (\nabla \times E) \times E = \text{div}(E \otimes E) - \frac{1}{2} \nabla |E|^2.
\]
Thus it yields the following, combined with (1.1b) and (1.1c),
\[
\frac{\partial}{\partial t} \int_{\mathbb{R}^3} (E \times B) + \text{div}_x \left( \frac{|E|^2 + |B|^2}{2} Id - E \otimes E - B \otimes B \right) = -(\rho E + j \times B). \quad (3.8)
\]
Then, adding (3.7) and (3.8) together gives (3.5). Integrating (3.4)-(3.6) in \(x\) over \(\mathbb{R}^3\), we deduce the following global conservation of mass, momentum and total energy
\[
\frac{d}{dt} \int_{\mathbb{R}^6} f dx d\xi = 0, \quad \text{for} \quad t \geq 0, \quad (3.9)
\]
\[
\frac{d}{dt} \left( \int_{\mathbb{R}^6} f \xi dx d\xi + \int_{\mathbb{R}^3} E \times B dx \right) = 0, \quad (3.10)
\]
\[
\frac{d}{dt} \int_{\mathbb{R}^6} f |\xi|^2 dx d\xi - 2 \int_{\mathbb{R}^3} E \cdot \int_{\mathbb{R}^3} \xi f d\xi dx = 0, \quad \text{for} \quad t \geq 0. \quad (3.11)
\]
On the other hand, multiplying (1.1b) by \(E\), multiplying (1.1c) by \(B\), integrating them in \(x\) over \(\mathbb{R}^3\), and then summing them together, we obtain
\[
\frac{d}{dt} \int_{\mathbb{R}^3} (|E|^2 + |B|^2) dx = -2 \int_{\mathbb{R}^3} E \cdot j dx.
\]
Substituting the above identity back to (3.11), one obtains
\[
\frac{d}{dt} \left( \int_{\mathbb{R}^6} f |\xi|^2 dx d\xi + \int_{\mathbb{R}^3} (|E|^2 + |B|^2) dx \right) = 0, \quad \text{for} \quad t \geq 0. \quad (3.12)
\]
Therefore, if we assume that the initial condition \(f_0\) as (3.2), we deduce from (3.9), (3.10) and (3.12) that
\[
\sup_{t \in [0, T]} \int_{\mathbb{R}^6} f (1 + \nu + |\xi|^2) dx d\xi + \int_{\mathbb{R}^3} (|E|^2 + |B|^2) dx \leq C(T) \quad (3.13)
\]
for some nonnegative constant \(C(T)\) that depends only on \(T\) and on the initial data. Indeed, we observe that we have, multiplying (1.1a) by \(\nu(x)\) and then integrating over \(\xi\),
\[
\frac{\partial}{\partial t} \left( \int_{\mathbb{R}^3} f \nu(x) d\xi \right) + \text{div}_x \left( \int_{\mathbb{R}^3} f \nu(x) \xi d\xi \right) = \int_{\mathbb{R}^3} f \xi \cdot \nabla_x \nu(x) d\xi
\]
\[
\leq \frac{1}{2} \int_{\mathbb{R}^3} f |\xi|^2 d\xi + \frac{1}{2} \rho(t,x) |\nabla \nu|^2
\]
\[
\leq \frac{1}{2} \int_{\mathbb{R}^3} f |\xi|^2 d\xi + C \int_{\mathbb{R}^3} f d\xi + C \int_{\mathbb{R}^3} f \nu d\xi,
\]
since \((1 + \nu)^\frac{1}{2}\) is Lipschitz. In particular, we deduce
\[
\frac{d}{dt} \int_{\mathbb{R}^6} f \nu(x) dx d\xi \leq C + \frac{1}{2} \int_{\mathbb{R}^6} f(|\xi|^2 + \nu(x)) dx d\xi.
\]
Then (3.13) follows from the above inequality and Grönwall’s inequality.

The final formal bound we wish to obtain is deduced from the entropy identity. Multiplying (1.1a) by \(\log f\), using (3.3), we obtain, at least formally,
\[
\frac{d}{dt} \int_{\mathbb{R}^6} f \log f dx d\xi + \frac{1}{4} \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^6} d\xi d\xi_\ast \int_{S^2} B(f' f_\ast' - ff_\ast) \log \frac{f' f_\ast'}{ff_\ast} = 0.
\] (3.14)

Since the second term is clearly nonnegative, we deduce in particular that
\[
\sup_{t \geq 0} \int_{\mathbb{R}^6} f \log f dx d\xi \leq \int_{\mathbb{R}^6} f_0 \log f_0 dx d\xi.
\] (3.15)

This inequality together with a lemma in [22] implies
\[
\sup_{t \in [0, T]} \int_{\mathbb{R}^6} f|\log f| \leq C(T).
\]
Also, if we go back to (3.14), we deduce that,
\[
\int_0^T dt \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^6} d\xi d\xi_\ast \int_{S^2} b(f' f_\ast' - ff_\ast) \log \frac{f' f_\ast'}{ff_\ast} \leq C(T).
\]

In conclusion, we obtain the following bounds:
\[
\sup_{t \in [0, T]} \left( \int_{\mathbb{R}^6} f(1 + |\xi|^2 + \nu(x) + |\log f|) dx d\xi + \int_{\mathbb{R}^3} (|E|^2 + |B|^2) dx \right) \leq C(T);
\]
\[
\int_0^T \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^6} d\xi d\xi_\ast \int_{S^2} b(f' f_\ast' - ff_\ast) \log \frac{f' f_\ast'}{ff_\ast} \leq C(T).
\] (3.16)

Now we give the definition of Renormalized Solutions to VMB.

**Definition 3.1.** A triple \((f(t,x,\xi), E(t,x), B(t,x))\) with \(f \geq 0\) is said to be a renormalized solution to VMB (1.1) if for all \(T \in (0, \infty)\), we have
- \(f(t,x,\xi) \in C([0,T]; L^1(\mathbb{R}^6))\), \(E, B \in C([0,T]; L^2(\mathbb{R}^3))\) and (3.16) holds;
- for any \(\beta \in C^1([0,\infty))\) satisfying that \(\beta(0) = 0\) and \(\beta'(1+t)\) is bounded in \([0,\infty)\),
  \[
  \frac{\partial}{\partial t} \beta(f) + \xi \cdot \nabla_x \beta(f) + (E + \xi \times B) \cdot \nabla_\xi \beta(f) = \beta'(f)Q(f,f)
  \] (3.17)
holds in \(\mathcal{D}'\) (sense of distributions); and
- (1.1b) and (1.1c) hold in \(\mathcal{D}'\).

One of the main objectives in the rest of this paper is devoted to the stability of renormalized solutions to VMB. More precisely, we consider a sequence of initial data \(\{(f^n_0, E^n_0, B^n_0)\}_{n=1}^\infty\) satisfying (3.2) with \(f^n_0 \geq 0\), a.e. in \(\mathbb{R}^6\) and converging to \((f_0, E_0, B_0)\). Then, corresponding to those initial conditions, we suppose that there is a sequence of
Theorem 3.1 (Weak Stability). Suppose that \( \{(f^n, E^n, B^n)\}_{n=1}^{\infty} \) is a sequence of renormalized solutions to VMB \( (1.1) \) satisfying \( (3.16) \), with initial data \( \{(f_0^n, E_0^n, B_0^n)\}_{n=1}^{\infty} \) satisfying \( (3.2) \), \( f_0^n \geq 0 \), a.e. in \( \mathbb{R}^6 \) and converging weakly to \((f_0, E_0, B_0)\) in \( L^1(\mathbb{R}^6) \times (L^2(\mathbb{R}^3))^6 \); and \((f, E, B)\) is a weak-* limit of \( \{(f^n, E^n, B^n)\} \) in \( L^\infty(0, T; L^1(\mathbb{R}^6)) \times (L^\infty(0, T; (L^2(\mathbb{R}^3))^6) \). Then the sequence \( \{f_n\} \) satisfies:

1. For all \( \psi \in C(\mathbb{R}^3) \) such that \( \frac{\|\psi(\xi)\|}{1 + |\xi|^2} \to 0 \) as \( |\xi| \to \infty \), \( \int_{\mathbb{R}^3} f^n \phi d\xi \) converges to \( \int_{\mathbb{R}^3} \psi d\xi \) in \( L^p([0, T], L^1(\mathbb{R}^3)) \) for all \( 1 \leq p < \infty \).

2. \( L(f^n) \) converges to \( L(f) \) in \( L^p([0, T]; L^1(\mathbb{R}^3 \times K)) \) for all \( 1 \leq p < \infty \), \( T \in (0, \infty) \), \( K \) compact set in \( \mathbb{R}^3 \).

3. For all \( \phi \in L^\infty(\mathbb{R}^3) \) with compact support, \( \int_{\mathbb{R}^3} Q^\pm(f^n, f^n) \phi d\xi \) converges locally in measure to \( \int_{\mathbb{R}^3} Q^\pm(f, f) \phi d\xi \). And \( Q^\pm(f^n, f^n)(1 + f^n)^{-1} \) are relatively weakly compact in \( L^1(\mathbb{R}^3 \times K \times (0, T)) \) for all \( T \in (0, \infty) \), compact set \( K \) in \( \mathbb{R}^3 \).

4. \( Q^+(f^n, f^n) \) converges locally in measure to \( Q^+(f, f) \).

Moreover, if
\[
\|E^n\|_{L^\infty(0, T; L^5(\mathbb{R}^3))}
\]
is uniformly bounded, and
\[
\|B^n\|_{L^\infty(0, T; L^s(\mathbb{R}^3))}
\]
for some \( s > 5 \), then the weak limit \((f, E, B)\) is a renormalized solution of \( (1.1) \) with the initial data \((f_0, E_0, B_0)\).

Remark 3.1. Due to the convexity of \( x \ln x \) and the monotonicity of \((x - y) \ln \frac{x}{y}\) for all \( x, y > 0 \), we can show, as in \( (1.1) \),
\[
\int_{\mathbb{R}^6} f(t) \ln f(t) d\xi \leq \liminf_{n \to \infty} \int_{\mathbb{R}^6} f^n(t) \ln f^n(t) d\xi,
\]
and
\[
\int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^6} d\xi d\xi' \int_{S^2} d\omega (f' f'_s - f f_s) \ln \frac{f' f'_s}{f f_s} \leq \liminf_{n \to \infty} \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^6} d\xi d\xi' \int_{S^2} d\omega (f'' f''_s - f f'_s) \ln \frac{f'' f''_s}{f f'_s},
\]
for all \( t \geq 0 \). This entropy estimate is crucial for the long time behavior of renormalized solutions.

A consequence of the weak stability is the propagation of smoothness of renormalized solutions.

Theorem 3.2 (Propagation of Smoothness). If, in addition to the assumptions in Theorem 3.1, \( f_0^n \) converges in \( L^1(\mathbb{R}^6) \) to \( f_0 \), then \( f^n \) converges to \( f \) in \( C([0, T]; L^1(\mathbb{R}^6)) \) for all \( T \in [0, \infty) \), and \((f, E, B)\) is a renormalized solution of \( (1.1) \) if \((f^n, E^n, B^n)\) is a sequence of renormalized solutions.
Remark 3.2. The assumption that $E(x,t)$ is uniformly bounded in $L^\infty(0,T; L^5(\mathbb{R}^3))$ is crucial for Theorems 3.1 and 3.2, because of the nonlinear term associated with the Lorentz force. Notice that usually from Maxwell’s equations, we can only obtain the a priori estimates on $E$ and $B$ in $L^\infty(0,T; L^2(\mathbb{R}^3))$.

4. Proof of Theorem 3.1: Weak Stability

This section is devoted to the proof of Theorem 3.1. We divide the proof into two steps. In the first step we show why the first four statements of the theorem hold. Then we concentrate in the second step on the proof of the fact that the weak limit is indeed a renormalized solution of Vlasov-Maxwell-Boltzmann equations. We remark that the first step is essentially an adaptation of the results and methods of [10, 22, 23], while the second one requires a new result of renormalized solutions for the Vlasov-Maxwell equations.

4.1. Step One. In this subsection, we are aiming at proving the first statement of Theorem 3.1 following the spirit of [10]. Then the second and the third statements can be shown exactly as in [10]. Finally, once the first three statements hold, the fourth statement will immediately follows from the argument in [22]. Therefore, for the sake of conciseness, we only give the detailed proof of the first statement of Theorem 3.1.

In order to prove the first statement, we first recall that for all compact sets $K \subset \mathbb{R}^3$ and $T \in (0,\infty)$, we have

$$
\int_{\mathbb{R}^3 \times K} (1 + f^n)^{-1} Q^{-}(f^n, f^n) dxd\xi \leq \int_{\mathbb{R}^3 \times K} L(f^n) dxd\xi \\
= \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} f^n(x, \xi, t) \int_{K} A(\xi - \xi_s) d\xi d\xi_s \\
\leq C \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} f^n(x, \xi, t) (1 + |\xi_s|^2) d\xi < \infty,
$$

(4.1)
due to the assumption on the collision kernel $b$, hence,

$$(1 + f^n)^{-1} Q^{-}(f^n, f^n) \quad \text{is bounded in} \quad L^\infty(0,T; L^1(\mathbb{R}^3 \times K)).$$

Also, we observe that we have (see [10]),

$$Q^+(f^n, f^n) \leq 2Q^-(f^n, f^n) + \frac{1}{\ln 2} \int_{\mathbb{R}^3} d\xi_s \int_{S^2} bd\omega(f^n' - f^n) \ln \frac{f^n' f^n_{s'}}{f^n f^n_s},$$

which, combining with (3.16) and (4.1), implies

$$(1 + f^n)^{-1} Q^+(f^n, f^n) \quad \text{is bounded in} \quad L^1(0,T; L^1(\mathbb{R}^3 \times K)) \quad (4.2)$$

for all compact sets $K \subset \mathbb{R}^3$ and $T \in (0,\infty)$.

Next, we observe that since $f^n$ is a renormalized solution of VBM (1.1), we have, for $\beta = \beta_\delta = \frac{t}{1 + \delta t}$,

$$\left( \frac{\partial}{\partial t} + \xi \cdot \nabla_x \right) \beta_\delta(f^n) = \beta_\delta'(f^n)Q(f^n, f^n) - \text{div}_x((E^n + \xi \times B^n)\beta_\delta(f^n)) \quad (4.3)$$

in $\mathcal{D}'$. In order to apply the velocity averaging results in [9, 13], we remark that (4.1) and (4.2) imply that $\beta_\delta(f^n)Q(f^n, f^n)$ is bounded in $L^1(0,T; L^1(\mathbb{R}^3 \times K))$ for all compact
subsets $K$ of $\mathbb{R}_x^3$. And also we observe that $\beta_\delta(f^n)$ is bounded in $L^\infty((0, T) \times \mathbb{R}^6)$, and hence, $\text{div}_x ((E^n + \xi \times B^n) \beta_\delta(f^n))$ is bounded in $L^2((0, T) \times \mathbb{R}^3, H^{-1}_\xi(\mathbb{R}^3))$. Denoting 
$$
T_\delta(f^n) = \beta'_\delta(f^n)Q(f^n, f^n),
$$
and decomposing $\beta_\delta(f^n)$ into 
$$
g^n, \text{ and } h^n \text{ by }
$$
$$
\left(\frac{\partial}{\partial t} + \xi \cdot \nabla_x \right) u^n = T_\delta(f^n)\chi\{t, x, \xi: |T_\delta(f^n)| \leq M\} 
- \text{div}_x ((E^n + \xi \times B^n)\beta_\delta(f^n)\chi\{t, x, \xi: |T_\delta(f^n)| \leq M\}), \tag{4.4}
$$
$$
\left(\frac{\partial}{\partial t} + \xi \cdot \nabla_x \right) g^n = -\text{div}_x ((E^n + \xi \times B^n)\beta_\delta(f^n)\chi\{t, x, \xi: |T_\delta(f^n)| > M\}), \tag{4.5}
$$
$$
\left(\frac{\partial}{\partial t} + \xi \cdot \nabla_x \right) h^n = T_\delta(f^n)\chi\{t, x, \xi: |T_\delta(f^n)| > M\} \tag{4.6}
$$
for $M > 1$, where 
$$
h^n|_{t=0} = g^n|_{t=0} = 0, \quad u^n|_{t=0} = \beta_\delta(f^n_0),
$$
and $\chi$ is the characteristic function of sets. Because $\{T_\delta(f^n)\}_{n=1}^\infty$ is weakly compact in $L^1((0, T) \times \mathbb{R}^6)$ due to the facts that $\beta'(t) = \frac{1}{(1+M)t} \leq \frac{1}{1+\tau}$ and $\frac{1}{1+\tau} Q(f^n, f^n)$ is weakly compact in $L^1((0, T) \times \mathbb{R}^6)$, and because, from (4.6), 
$$
h^n(t, x + t\xi, \xi) = \int_0^T \int_{\mathbb{R}^3} T_\delta(f^n)\chi\{t, x, \xi: |T_\delta(f^n)| \geq M\}(\tau, x + \xi\tau, \xi)d\tau,
$$
it follows that, uniformly with respect to $n$, 
$$
\int_0^T \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |h^n(t, x, \xi)|d\xi dx dt \to 0, \quad \text{as} \quad M \to \infty. \tag{4.7}
$$
Similarly, from the compactness of $T_\delta(f^n)$, we deduce that 
$$
S^n := (E^n + \xi \times B^n)\beta_\delta(f^n)\chi\{t, x, \xi: |T_\delta(f^n)| \geq M\} \to 0 \tag{4.8}
$$
in $L^1_{loc}((0, T) \times \mathbb{R}^6)$ as $M \to \infty$. From (4.5), we have 
$$
g^n(t, x + t\xi, \xi) = \int_0^t -\text{div}_x ((E^n + \xi \times B^n)\beta_\delta(f^n)\chi\{t, x, \xi: |T_\delta(f^n)| > M\})(\tau, x + \xi\tau, \xi)d\tau.
$$
Thus, for any $\psi \in D_\xi(\mathbb{R}^3)$, we deduce from the above identity that 
$$
\int_{\mathbb{R}^3} g^n(t, x + t\xi, \xi)\psi(\xi)d\xi 
= \int_0^t \int_{\mathbb{R}^3} ((E^n + \xi \times B^n)\beta_\delta(f^n)\chi\{t, x, \xi: |T_\delta(f^n)| > M\})(\tau, x + \xi\tau, \xi) \cdot \nabla_\xi \psi d\xi d\tau.
$$
Therefore, from the weak compactness of $S^n$, the above identity with (4.8) implies 
$$
\int_{\mathbb{R}^3} g^n(t, x, \xi)\psi d\xi \to 0, \quad \text{as} \quad M \to \infty, \tag{4.9}
$$
in $L^1_{loc}((0, T) \times \mathbb{R}^3)$. 
On the other hand, since \( \{u^n\}_{n=1}^\infty \) and \( \{T_\delta(f^n)\chi_{\{(t,x,\xi): |T_\delta(f^n)| \leq M\}}\}_{n=1}^\infty \) are bounded sequences in \( L^2((0,T) \times \mathbb{R}^6) \), and \( \text{div}_\xi((E^n + \xi \times B^n)\beta_\delta(f^n)) \) is bounded in \( L^2((0,T) \times \mathbb{R}^3; H_\xi^{-1}(\mathbb{R}^3)) \), by the velocity averaging lemma (Theorem 3 in [9]), we deduce that

\[
\int_{\mathbb{R}^3} u^n \psi(\xi) d\xi \quad \text{is bounded in} \quad H^{\frac{1}{2}}((0,T) \times \mathbb{R}^3),
\]

for all \( \psi \in \mathcal{D}(\mathbb{R}^3) \). Thus, \( \{\int_{\mathbb{R}^3} u^n \psi(\xi) d\xi\}_{n=1}^\infty \) is compact in \( L^2((0,T) \times \mathbb{R}^3) \) and is locally compact in \( L^1((0,T) \times \mathbb{R}^3) \), which, combining with (4.7) and (4.9), implies that

\[
\int_{\mathbb{R}^3} \beta_\delta(f^n) \psi d\xi \quad \text{is relatively compact in} \quad L^p((0,T), L^1_{\text{loc}}(\mathbb{R}^3)) \quad (4.10)
\]

for all \( 1 \leq p < \infty, \psi \in \mathcal{D}(\mathbb{R}^3) \).

The first statement of the theorem for \( \psi \in \mathcal{D}(\mathbb{R}^3) \) then follows from (4.10) and (3.16), since it suffices to observe that we have for all \( R > 1 \),

\[
0 \leq f^n - \beta_\delta(f^n) \leq \delta R f^n + f^n \chi_{\{f^n > R\}} \leq \delta R f^n + C f^n \ln f^n \ln R, \quad (4.11)
\]

and then take the limit as \( R \to \infty \) and \( \delta \to 0 \). Next, for a general \( \psi \in C(\mathbb{R}^3) \) such that \( \psi(\xi)(1 + |\xi|^2)^{-1} \to 0 \) as \( |\xi| \to \infty \), we introduce

\[
\eta_M = \eta \left( \frac{\cdot}{M} \right),
\]

for \( M > 1 \), where \( \eta \in \mathcal{D}(\mathbb{R}^3) \), \( 0 \leq \eta \leq 1 \), \( \eta = 1 \) on \( B_1 \). Then the first statement holds for \( \psi \eta_M \), and the first statement will be valid for such a \( \psi \) provided that

\[
\sup_n \int_0^T \int_{K \times \mathbb{R}^3} f^n |\psi|(1 - \eta_M) dx d\xi \to 0 \quad \text{as} \quad M \to \infty, \quad (4.12)
\]

for compact subsets \( K \in \mathbb{R}^3 \). Indeed, (4.12) follows from (3.16) since

\[
\int_0^T \int_{K \times \mathbb{R}^3} f^n |\psi|(1 - \eta_M) dx d\xi \\
\leq C \sup_{|\xi| \geq M} \frac{|\psi(\xi)|}{1 + |\xi|^2} \int_0^T \int_{K \times \mathbb{R}^3} f^n (1 + |\xi|^2) \chi_{\{|\xi| \geq M\}} \\
\leq C \sup_{|\xi| \geq M} \frac{|\phi(\xi)|}{1 + |\xi|^2}
\]

for some \( C > 0 \) independent of \( n \).

**4.2. Step Two.** We now aim at proving that \((f, E, B)\) is a renormalized solution of VBM. First of all, we claim that it is enough to show that

**Lemma 4.1.** If \( f \in L^\infty((0,T); L^1(\mathbb{R}^6)) \), the equation (3.17) holds if and only if

\[
\frac{\partial}{\partial t} \ln(1 + f) + \text{div}_\xi(\xi \ln(1 + f)) + (E + \xi \times B) \cdot \nabla_\xi \ln(1 + f) = \frac{1}{1 + f} Q(f, f), \quad (4.13)
\]

in \( \mathcal{D}' \).
Proof. On one hand, if $f$ is a renormalized solution to VMB, then (4.13) automatically holds since $\beta(f) = \ln(1 + f) \in C^1([0, \infty))$ with $\beta(0) = 0$ and $\beta'(f)(1 + f) = 1$.

On the other hand, if (4.13) holds, we claim that $f$ is a renormalized solution to VMB. Indeed, denoting

$$\sigma(s) = \beta(e^s - 1)$$

for all $\beta(t) \in C^1([0, \infty))$ with $\beta(0) = 0$ and $\beta'(t)(1 + t) \leq C$. Then, we have

$$\partial_t \sigma(f) = \sigma'(f) \partial_t f;$$
$$\nabla_x \sigma(f) = \sigma'(f) \nabla_x f;$$
$$\nabla_\xi \sigma(f) = \sigma'(f) \nabla_\xi f;$$

Multiplying (4.1) by $\sigma'(\ln(1 + f))$, we obtain,

$$\frac{\partial}{\partial t} \sigma(\ln(1 + f)) + \text{div}_x(\xi \sigma(\ln(1 + f))) + (E + \xi \times B) \cdot \nabla_\xi \sigma(\ln(1 + f)) = \sigma'(\ln(1 + f)) \frac{1}{1 + f} Q(f, f),$$

in the sense of distributions. The proof of this lemma is complete.

The rest of this subsection is devoted to the proof of (4.13). Recall that we deduce, from a priori estimate (3.16) and weak passages to the limit,

$$\sup_{t \in [0, T]} \left( \int_{\mathbb{R}^6} f(1 + |\xi|^2 + \nu(x) + |\log f|) dx d\xi + \int_{\mathbb{R}^3} (|E|^2 + |B|^2) dx \right)$$
$$+ \int_0^T \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^6} d\xi d\xi' \int_{S^2} d\omega (f' f' - f f) \log \frac{f f'}{f f'} < \infty,$$

for all $T \in (0, \infty)$. Now the strategy to prove (4.13) is the following: we first consider

$$\beta_\delta(f^n) = f^n(1 + \delta f^n)^{-1}$$

for $\delta \in (0, 1]$ and weakly pass to the limit as $n$ goes to $\infty$ in the equation satisfied by $\beta_\delta(f^n)$; then for the equation satisfied by the limit of $\beta_\delta(f^n)$ as $n \to \infty$, we use $\beta$ to renormalize it and let $\delta$ go to 0 to recover (4.13). To begin with, without loss of generality, in view of (3.16), we can assume

$$f^n \to f \quad \text{weakly}^* \quad \text{in} \quad L^\infty(0, T; L^1(\mathbb{R}^6));$$
$$B^n \to B \quad \text{weakly}^* \quad \text{in} \quad L^\infty(0, T; L^2(\mathbb{R}^3));$$
$$E^n \to E \quad \text{weakly}^* \quad \text{in} \quad L^\infty(0, T; L^2(\mathbb{R}^3) \cap L^5(\mathbb{R}^3)).$$

Furthermore, without loss of generality, extracting subsequence if necessary, we may assume that for all $\delta > 0$

$$\beta_\delta(f^n) \to \beta_\delta \quad \text{weakly in} \quad L^p(\mathbb{R}^6 \times (0, T)); \quad (4.16)$$
For all $T \in (0, \infty)$, $1 \leq p \leq \infty$. Furthermore, because of the third statement and the equi-integrability, we may assume that
\[
(1 + \delta f^n)^{-\frac{p}{2}} Q^\pm(f^n, f^n) \to Q^\pm \quad \text{weakly in } L^1(\mathbb{R}^3_x \times K \times (0, T)),
\]
for all compact sets $K \subset \mathbb{R}^3_x$ and $T \in (0, \infty)$.

Notice that, since $f^n$ is a renormalized solution of VMB, (4.3) holds with $\beta(f^n)$ replaced by $\beta_\delta(f^n)$ for all $\delta > 0$ and we want to pass to the limit in these equations as $n$ goes to $\infty$.

To this end, we deduce from the first statement of Theorem 3.1 that $\rho^n$ and $j^n$ converge in $L^p(0, T; L^1(\mathbb{R}^3_x))$ to $\rho$ and $j$, respectively for all $1 \leq p < \infty$ and $T \in (0, \infty)$. We then pass to the limit in (4.3) and we obtain
\[
\frac{\partial}{\partial t} \beta_\delta + \text{div}_x (\xi \beta_\delta) + (E + \xi \times B) \cdot \nabla \beta_\delta = Q^+_\delta - Q^-_\delta, \quad x \in \mathbb{R}^3, \quad \xi \in \mathbb{R}^3, \quad t \geq 0,
\]
\[
\frac{\partial E}{\partial t} - \nabla \times B = -j, \quad \text{div} B = 0, \quad \text{on } \mathbb{R}^3_x \times (0, \infty),
\]
\[
\frac{\partial B}{\partial t} + \nabla \times E = 0, \quad \text{div} E = \rho, \quad \text{on } \mathbb{R}^3_x \times (0, \infty),
\]
\[
\rho = \int_{\mathbb{R}^3} f d\xi, \quad j = \int_{\mathbb{R}^3} f \xi d\xi, \quad \text{on } \mathbb{R}^3_x \times (0, \infty),
\]
in $\mathcal{D}'$. Here, for the convergence of the nonlinear term $(E^n + \xi \times B^n) \cdot \nabla \beta_\delta(f^n)$, we need to show, for all $\phi \in \mathcal{D}((0, \infty) \times \mathbb{R}^6)$,
\[
\int_0^t \int_{\mathbb{R}^6} \phi(E^n + \xi \times B^n) \cdot \nabla \beta_\delta(f^n) d\xi dx ds
\]
\[
= -\int_0^t \int_{\mathbb{R}^6} \nabla \xi \phi \cdot (E^n + \xi \times B^n) \beta_\delta(f^n) d\xi dx ds,
\]
since
\[
\text{div}_x (E^n + \xi \times B^n) = 0.
\]
If we take $\phi = \overline{\phi}(t, x) \Phi(\xi)$ (which is enough by dense property) for $\overline{\phi} \in \mathcal{D}((0, \infty) \times \mathbb{R}^3)$ and $\Phi \in \mathcal{D}(\mathbb{R}^3)$, we can rewrite the term on the right-hand side of (4.21) as
\[
-\int_0^t \int_{\mathbb{R}^3} \overline{\phi}(t, x) (E^n + \xi \times B^n) \cdot \left( \int_{\mathbb{R}^3} \psi(\xi) \beta_\delta(f^n) d\xi \right) d\xi dx ds,
\]
by letting $\psi = \nabla \xi \Phi$. In fact, on one hand, by (4.10) or the velocity averaging lemma in (13), $\int_{\mathbb{R}^3} \psi(\xi) \beta_\delta(f^n) d\xi$ and $\int_{\mathbb{R}^3} \xi \psi(\xi) \beta_\delta(f^n) d\xi$ strongly converge to $\int_{\mathbb{R}^3} \psi(\xi) \beta_\delta d\xi$ and $\int_{\mathbb{R}^3} \xi \psi(\xi) \beta_\delta d\xi$ in $L^p(0, T; L^1(\mathbb{R}^3))$ respectively. On the other hand, since $\psi \in C_0(\mathbb{R}^3)$ and $\beta_\delta(t) \leq t$, we have, using (2.17),
\[
\left\{ \int_{\mathbb{R}^3} \xi \psi \beta_\delta(f^n) d\xi \right\}_{n=1}^{\infty}
\]
is uniformly bounded in $L^\infty(0, T; L^1(\mathbb{R}^3))$, and
\[
\left\{ \int_{\mathbb{R}^3} \psi \beta_\delta(f^n) d\xi \right\}_{n=1}^{\infty}
\]
is uniformly bounded in $L^\infty(0, T; L^1(\mathbb{R}^3))$. 
The latter is true, because, for all $R > 1$,
\[
\int_{\mathbb{R}^3} |\psi| |\beta_\delta(f^n)| d\xi = \int_{\{\xi \leq R\}} |\psi| |\beta_\delta(f^n)| d\xi + \frac{1}{R^2} \int_{\{\xi > R\}} |\xi|^2 |\psi| |\beta_\delta(f^n)| d\xi \\
\leq \|\psi\|_{L^\infty} \left( R^3 |B(0,1)| \|\beta_\delta(f^n)\|_{L^\infty} + \frac{1}{R^2} \|\xi^2 \beta_\delta(f^n)\|_{L^1_3(\mathbb{R}^3)} \right) \tag{4.22}
\]
\[
\leq \|\psi\|_{L^\infty} \left( R^3 |B(0,1)| + \frac{1}{R^2} \|\xi^2 f^n\|_{L^1_3(\mathbb{R}^3)} \right),
\]
where $|B(0,1)|$ denotes the Lebesgue measure of the unit ball $B(0,1)$ in $\mathbb{R}^3$, and by taking
\[
R = \left( \frac{\delta \|\xi^2 f^n\|_{L^1_3(\mathbb{R}^3)}}{|B(0,1)|} \right)^{\frac{1}{3}},
\]
(4.22) becomes
\[
\int_{\mathbb{R}^3} |\psi| |\beta_\delta(f^n)| d\xi \leq 2 \|\psi\|_{L^\infty} \left( \frac{|B(0,1)|^{\frac{2}{3}} \|\xi^2 f^n\|_{L^1_3(\mathbb{R}^3)}}{\delta^\frac{2}{3}} \right).
\]
Therefore,
\[
\left\{ \int_{\mathbb{R}^3} |\psi| |\beta_\delta(f^n)| d\xi \right\}_{n=1}^\infty \text{ is uniformly bounded in } L^\infty(0,T; L^6_3(\mathbb{R}^3)),
\]
\[
\text{since } \{\|\xi^2 f^n\|_{L^1_3}\}_{n=1}^\infty \text{ is uniformly bounded in } L^\infty(0,T; L^6(\mathbb{R}^6)).
\]
Thus
\[
\int_{\mathbb{R}^3} \xi^s |\psi| |\beta_\delta(f^n)| d\xi \rightarrow \int_{\mathbb{R}^3} \xi^s |\psi| |\beta_\delta| d\xi \text{ in } L^p(0,T; L^s(\mathbb{R}^3)) \text{ for all } 1 \leq s < \frac{5}{4},
\]
and
\[
\int_{\mathbb{R}^3} \psi |\beta_\delta(f^n)| d\xi \rightarrow \int_{\mathbb{R}^3} \psi |\beta_\delta| d\xi \text{ in } L^p(0,T; L^s_{loc}(\mathbb{R}^3)) \text{ for all } 1 \leq r < \frac{5}{3},
\]
for all $1 \leq p < \infty$.

The weak convergence of $E^n$ in $L^5((0,T) \times \mathbb{R}^3)$, combined with the strong convergence of $\int_{\mathbb{R}^3} \psi(\xi) |\beta_\delta(f^n)| d\xi$, implies
\[
\int_0^t \int_{\mathbb{R}^6} \nabla \psi \cdot E^n |\beta_\delta(f^n)| d\xi dx ds \rightarrow \int_0^t \int_{\mathbb{R}^6} \nabla \psi \cdot E |\beta_\delta| d\xi dx ds.
\]
The similar argument goes to the second part of the nonlinear term
\[
\int_0^t \int_{\mathbb{R}^3} \bar{\phi}(t,x) B^n \times \left( \int_{\mathbb{R}^3} \psi(\xi) |\beta_\delta(f^n)| d\xi \right) dx ds,
\]
due to the weak convergence of $B^n$ in $L^q((0,T) \times \mathbb{R}^3)$ for $q > 5$. That is,
\[
\int_0^t \int_{\mathbb{R}^6} \nabla \psi \cdot \xi \times B^n |\beta_\delta(f^n)| d\xi dx ds \rightarrow \int_0^t \int_{\mathbb{R}^6} \nabla \psi \cdot \xi \times B |\beta_\delta| d\xi dx ds.
\]

Next, since $\beta_\delta(f^n) \in L^1(\mathbb{R}^6) \cap L^\infty(\mathbb{R}^6)$, we know that $\beta_\delta \in L^\infty(\mathbb{R}^6) \cap L^1(\mathbb{R}^6)$. Also, since $|\xi|^2 f^n \in L^\infty(0,T; L^1(\mathbb{R}^6))$, we know that $\beta_\delta(f^n)|\xi|^2 \in L^\infty(0,T; L^1(\mathbb{R}^6))$ and $\{\beta_\delta(f^n)\}_{n=1}^\infty$
is weakly compact in $L^\infty(0, T; L^1(\mathbb{R}^6))$. Hence $|\xi|^2\beta_\delta \in L^\infty(0, T; L^1(\mathbb{R}^6))$. Thus, for any $\sigma > 0$, we have
\[
\int_{\{\beta_\delta > \sigma\}} |\xi|^2 d\xi < \frac{1}{\sigma} \int_{\{\beta_\delta > \sigma\}} |\xi|^2 \beta_\delta d\xi \leq \frac{1}{\sigma} \int_{\mathbb{R}^6} |\xi|^2 \beta_\delta d\xi < \infty.
\]
Therefore, Theorem 2.1 implies that $\beta_\delta$ is a renormalized solution of (4.20).

As $\delta \to 0$, we claim that
\[
\beta_\delta \to f, \quad \text{in} \quad C([0, T]; L^1(\mathbb{R}^6)),
\]
as $\delta \to 0$.

**Proof.** We start with proving the continuity of $\beta_\delta$ with respect to $t \geq 0$ with values in $L^p(\mathbb{R}^6)$ for all $1 \leq p < \infty$. To this end, we remark that if we regularize by convolution $\beta_\delta$ into $\beta_\delta^R$ as in Lemma 2.1, we obtain
\[
\frac{\partial}{\partial t} \beta_\delta^R + \nabla \cdot \nabla \beta_\delta^R + (E + \xi \cdot B) \cdot \nabla \beta_\delta^R = Q^R - Q^\delta + r^\varepsilon
\]
where $r^\varepsilon \to 0$ in $L^1(0, T; L^1_{loc}(\mathbb{R}^6))$ as $\varepsilon$ goes to 0 for all $T \in (0, \infty)$. Hence, it is easy to see from (4.23) that, $\beta_\delta^R \in C([0, \infty); L^p(\mathbb{R}^6))$ for $1 \leq p < \infty$. Note that $\beta_\delta$ is a renormalized solution to the VM (4.20a). Subtracting (4.23) from (4.20), multiplying the result by $|\beta_\delta - \beta_\delta^R|^2 (\beta_\delta - \beta_\delta^R)$, and then integrating over $\mathbb{R}^6$, we obtain
\[
\frac{d}{dt} \int_{\mathbb{R}^6} |\beta_\delta - \beta_\delta^R|^2 p d\xi \to 0 \quad \text{in} \quad L^1(0, T), \quad \text{as} \quad \varepsilon \to 0
\]
for all $1 \leq p < \infty$, $T \in (0, \infty)$. It follows that $\beta_\delta \in C([0, T]; L^1(\mathbb{R}^6))$.

Next, we show that $f \in C([0, \infty); L^1(\mathbb{R}^6))$. Indeed, because of (4.16), we have for all $T \in (0, \infty)$, as in (4.11)
\[
\sup_{t \in [0, T]} \sup_{n \geq 1} \|f^n - \beta_\delta(f^n)\|_{L^1(\mathbb{R}^6)} \to 0 \quad \text{as} \quad \delta \to 0.
\]
Hence, by the lower semi-continuity of the weak convergence, we obtain
\[
\sup_{t \in [0, T]} \|f - \beta_\delta\|_{L^1(\mathbb{R}^6)} \leq \sup_{t \in [0, T]} \liminf_{n \to \infty} \|f^n - \beta_\delta(f^n)\|_{L^1(\mathbb{R}^6)} \to 0 \quad \text{as} \quad \delta \to 0,
\]
and this implies $\beta_\delta$ converges in $C([0, T]; L^1(\mathbb{R}^6))$ to $f$. \qed

Now we can state the equation (4.20a) more precisely. To this end, we observe that
\[
\frac{1}{1 + \frac{\delta}{t}} \leq \frac{1}{1 + \frac{\delta(t)}{t}} \leq \frac{1}{1 + \frac{\delta}{t}},
\]
are convex on $[0, \infty)$, therefore we have
\[
\beta_\delta \leq \beta_\delta(f), \quad h_\delta \geq (1 + \delta f)^{-2} \quad \text{a.e on} \quad \mathbb{R}^6 \times (0, \infty).
\]
In addition $\frac{t}{1 + \delta(t)} = \beta_\delta(t)(1 - \delta \beta_\delta(t))$, because the function $x(1 - \delta x)$ is a concave function, hence
\[
g_\delta \leq \beta_\delta(1 - \delta \beta_\delta) \quad \text{a.e on} \quad \mathbb{R}^6 \times (0, \infty).
\]
Furthermore, because of the second statement of Theorem 3.1, we deduce that
\[
Q^\delta_\delta = g_\delta L(f) \quad \text{a.e on} \quad \mathbb{R}^6 \times (0, \infty).
\]
And, using the fourth statement of Theorem 3.1, we could also deduce that
\[ Q^+_\delta = h_\delta Q^+(f, f) \quad \text{a.e on } \mathbb{R}^6 \times (0, \infty). \] (4.29)

We finally use the fact that \( \beta_\delta \) is a renormalized solution of (4.20) to write
\[
\frac{\partial}{\partial t} \beta(\beta_\delta) + \text{div}_x (\xi \beta(\beta_\delta)) + (E + \xi \times B) \cdot \nabla \xi \beta(\beta_\delta) = (1 + \beta_\delta)^{-1}Q^+_\delta - (1 + \beta_\delta)^{-1}Q^-_\delta.
\] (4.30)

And we wish to recover (4.13) by letting \( \delta \) go to 0. Recall that we already showed in Lemma 4.2 that \( \beta_\delta \) converges to \( f \) in \( C([0, T]; L^1(\mathbb{R}^6)) \) for all \( T \in (0, \infty) \). Therefore, in order to complete the proof of Theorem 3.1, it only remains to show

**Lemma 4.3.**
\[
Q^\pm_\delta(1 + \beta_\delta)^{-1} \text{ are weakly relatively compact in } L^1(\mathbb{R}_x^3 \times K \times (0, T))
\] (4.31)
for all compact sets \( K \subset \mathbb{R}_x^3 \) and \( T \in (0, \infty) \), and
\[
(1 + \beta_\delta)^{-1}Q^-_\delta \to (1 + f)^{-1}Q^-(f, f), \quad \text{a.e} \quad (4.32a)
\]
\[
(1 + \beta_\delta)^{-1}Q^+_\delta \to (1 + f)^{-1}Q^+(f, f), \quad \text{a.e} \quad (4.32b)
\]
as \( \delta \) goes to 0.

**Proof.** We will follow the lines of the argument in [23] and begin with \( Q^-_\delta \). Without loss of generality, we may assume that \( \beta_\delta \) converges a.e. to \( f \) as \( \delta \) goes to 0. Then, (4.32a) follows since
\[
(1 + \beta_\delta)^{-1}Q^-_\delta = (1 + \beta_\delta)^{-1}g_\delta L(f) \to (1 + f)^{-1}fL(f)
\]
a.e. as \( \delta \to 0 \) provided we show that \( g_\delta \) converges a.e. to \( f \).

This is easy since we have for all \( R > 1 \),
\[
0 \leq f^n - f^n(1 + \delta f^n)^{-2} \leq 3R\delta f^n + f^n\chi_{\{f^n > R\}},
\]
hence \( g_\delta \) converges to \( f \) in \( C([0, T]; L^1(\mathbb{R}^6)) \) for all \( T \in (0, \infty) \) by the uniform integrability of \( f^n \) and the lower semi-continuity of the weak convergence. We now prove (4.31) for \( Q^-_\delta \) by first observing that (4.28) yields
\[
0 \leq (1 + \beta_\delta)^{-1}Q^-_\delta = (1 + \beta_\delta)^{-1}g_\delta L(f)
\]
\[
\leq (1 - \delta \beta_\delta) \frac{\beta_\delta}{1 + \beta_\delta} L(f) \leq L(f), \quad \text{a.e.}
\]
And we conclude the proof of (4.31) for \( Q^-_\delta \) by the equi-integrability, since \( L(f) \in L^\infty(0, T; L^1(\mathbb{R}_x^3 \times K)) \) for all compact sets \( K \subset \mathbb{R}_x^3 \) and \( T \in (0, \infty) \).

Next, we turn to the proof of (4.31) for \( Q^+_\delta \) and (4.32b). We begin with (4.31). We recall the following classical inequality for all \( M > 1 \),
\[
Q^+(f^n, f^n) \leq MQ^-(f^n, f^n) + \frac{1}{\ln M} \bar{e}^n
\] (4.33)
where
\[
\bar{e}^n = \int_{\mathbb{R}^3} d\xi \int_{S^2} b d\omega (f^{n^*} f^{n^*} - f^n f^n) \ln \frac{f^n f^{n^*}}{f^n f^{n^*} f^{n^*} f^n}
\]
is positive and bounded in \( L^1(\mathbb{R}^3 \times (0, T)) \) for all \( T \in (0, \infty) \). Without loss of generality, we may assume that \( \bar{e}^n \) converges weakly in the sense of measures to some bounded
nonnegative measure $\tilde{e}$ on $\mathbb{R}^6 \times [0, \infty)$ and we denote by $\tilde{e}_0$ its regular part with respect to the usual Lebesgue measure, that is, $\tilde{e}_0 = \frac{\partial \tilde{e}}{\partial y}$. Let $(y, t) \in \mathbb{R}^6 \times (0, T)$. Dividing (4.33) by $(1 + \delta f^n)^2$ and letting $n$ go to $\infty$, we obtain
\[ Q^+_\delta \leq MQ^-_\delta + \frac{1}{\ln M} \tilde{e}, \]

hence
\[ Q^+_\delta \leq MQ^-_\delta + \frac{1}{\ln M} \tilde{e}_0 \quad \text{a.e. on} \quad \mathbb{R}^6 \times (0, \infty). \]

Then (4.31) for $Q^+_\delta$ follows since we already show it for $Q^-_\delta$ and the integrability of $\tilde{e}_0$.

We finally prove (4.32) for $Q^+_\delta$. We first remark that we have for all $R > 0$,
\[ Q^+(f^n, f^n) \geq (1 + \delta f^n)^{-2} Q^+(f^n, f^n) \]
\[ \geq (1 + \delta R)^{-2} Q^+(f^n, f^n) \chi_{\{f^n \leq R\}}. \]  

In particular, if we multiply (4.33) by $\psi \in C^\infty_0 (\mathbb{R}^3)$ with $\psi \geq 0$, we find by letting $n$ go to $\infty$ and using the third statement of Theorem 3.1
\[ \int_{\mathbb{R}^3} Q^+_\delta(f, f) \psi d\xi \geq \int_{\mathbb{R}^3} Q^+_\delta \psi d\xi \quad \text{a.e. on} \quad \mathbb{R}^3 \times (0, \infty). \]

Indeed, the integrated left-hand side converges locally in measure while the right-hand side converges weakly in $L^1$ and this is enough to pass to the limit in the inequality a.e. on $\mathbb{R}^3 \times (0, \infty)$. Therefore, we have for all $\delta \in (0, 1]$,
\[ Q^+(f, f) \geq Q^+_\delta. \]  

Next, we use the other part of the inequality (4.34) and we write for $\tau \in (0, 1]$, using (4.33),
\[ (1 + \delta R)^{-2}(1 + \tau L(f^n))^{-1} Q^+(f^n, f^n) \]
\[ \leq (1 + \delta f^n)^{-2} Q^+(f^n, f^n) + (1 + \tau L(f^n))^{-1} \chi_{\{f^n > R\}} Q^+(f^n, f^n) \]
\[ \leq (1 + \delta f^n)^{-2} Q^+(f^n, f^n) + \frac{1}{\ln M} \frac{\epsilon^n}{M} + \frac{M}{\tau} \chi_{\{f^n > R\}}. \]  

We then observe that $Q^+(f^n, f^n)(1 + \tau L(f^n))^{-1}$ is relatively weakly compact in $L^1(\mathbb{R}^6 \times (0, T))$ for all $T \in (0, \infty)$ since it is bounded above by $\frac{\epsilon^n}{M} + M \tau f^n$ for all $M > 1$. Hence, we may assume without loss of generality that it converges weakly in $L^1(\mathbb{R}^6 \times (0, T))$ for all $T \in (0, \infty)$. We claim that its weak limit is given by $(1 + \tau L(f))^{-1} Q^+(f, f)$. Indeed, if $\psi \in L^\infty(\mathbb{R}^3_x)$ with compact support, we have
\[ \int_{\mathbb{R}^3} (1 + \tau L(f^n))^{-1} Q^+(f^n, f^n) \psi d\xi = \int_{\mathbb{R}^3} Q^+(f^n, f^n) \psi^n d\xi, \]

where $\psi^n$ is uniformly bounded in $L^\infty(\mathbb{R}^3_x)$, and has a uniform compact support and $\psi^n \to \psi_\tau = (1 + \tau L(f))^{-1} \psi$ in $L^p((0, T) \times \mathbb{R}^6)$ for all $1 \leq p < \infty$. This is enough to enable us to deduce
\[ \int_{\mathbb{R}^3} Q^+(f^n, f^n) \psi^n d\xi \to \int_{\mathbb{R}^3} Q^+(f, f) \psi_\tau d\xi \]

locally in measure on $\mathbb{R}^3_x \times [0, \infty)$, which yields the claim.
Applying Theorem 3.1, we know that $f \in VBM$. In particular, we know that we have, setting $\gamma$, we can assume which, combining with (4.35), implies that finally $\tau$ go to 0, that 

$$Q^+(f, f) \leq \lim_{\delta \to 0} Q_{\delta}^+ \quad \text{a.e.}$$

which, combining with (4.35), implies that

$$Q^+(f, f) = \lim_{\delta \to 0} Q_{\delta}^+ \quad \text{a.e.}$$

The proof is complete. \hfill \Box

Putting together the conclusion of Step One, Lemmas 4.1-4.3, we finish the proof of Theorem 3.1.

5. PROOF OF THEOREM 3.2: PROPAGATION OF SMOOTHNESS

In this section, we prove Theorem 3.2. First, without loss of generality, in view of (3.16), we can assume

$$f^n \to f \quad \text{weakly* in } L^\infty(0, T; L^1(\mathbb{R}^6));$$

$$B^n \to B \quad \text{weakly* in } L^\infty(0, T; L^2(\mathbb{R}^3) \cap L^5(\mathbb{R}^3));$$

$$E^n \to E \quad \text{weakly* in } L^\infty(0, T; L^2(\mathbb{R}^3) \cap L^5(\mathbb{R}^3)).$$

Applying Theorem 3.1, we know that $f \in C([0, \infty); L^1(\mathbb{R}^6))$ is a renormalized solution of VBM. In particular, we know that we have, setting $\gamma_{\delta}(f) = \frac{1}{\delta} \ln(1 + \delta f)$,

$$\frac{\partial}{\partial t} \gamma_{\delta}(f) + \text{div}(\xi \gamma_{\delta}(f)) + (E + \xi \times B) \cdot \nabla \gamma_{\delta}(f) = \gamma_{\delta}'(f)Q^+(f, f) - f\gamma_{\delta}'(f)L(f),$$

in $\mathcal{D}'$. It is easy to deduce that, $\gamma_{\delta}(f) \in C([0, \infty); L^p(\mathbb{R}^6))$ for all $1 \leq p < \infty$ since

$$\gamma_{\delta}(f) \in C([0, \infty); L^1(\mathbb{R}^6)) \cap L^\infty(0, \infty; L^1(\mathbb{R}^6)),$$

hence

$$\gamma_{\delta}(f)|_{t=0} = \gamma_{\delta}(f_0) \quad \text{a.e. on } \mathbb{R}^6.$$

The strategy of the proof of Theorem 3.2 goes as follows. First of all, we introduce, without loss of generality, the weak limit of $\gamma_{\delta}(f^n)$ in $L^p(\mathbb{R}^6 \times (0, T))$ for all $T \in (0, \infty)$ and $1 \leq p < \infty$, and we denote it by $\gamma_{\delta}$ (note the difference from the notation $\gamma_{\delta}(f^n)$ throughout this section). The first step is to show that $\gamma_{\delta}$ is a supersolution of (5.1). In the second step, we deduce that $\gamma_{\delta} = \gamma_{\delta}(f)$ and that $f^n$ converges to $f$ a.e. or in $L^1(\mathbb{R}^6 \times (0, T))$ for all $T \in (0, \infty)$. Finally in the third step, we show that $f^n$ converges to $f$ in $C([0, T]; L^1(\mathbb{R}^6))$, thus proving Theorem 3.2.
Applying Theorem 3.1 and a similar argument in Section 4, we can show that $\gamma_\delta$ satisfies:

$$0 \leq \gamma_\delta \leq \gamma_\delta(f) \quad \text{a.e. on } \mathbb{R}^6 \times (0, \infty),$$

and

$$\frac{\partial \gamma_\delta}{\partial t} + \text{div}_x(\xi \gamma_\delta) + (E + \xi \times B) \cdot \nabla_\xi \gamma_\delta = Q^+_\delta - Q^-_\delta,$$

in $\mathcal{D}'$, where $Q^+_\delta$, $Q^-_\delta$ are respectively the weak limits in $L^1(\mathbb{R}^3 \times K \times (0, T))$ for all compact sets $K \subset \mathbb{R}^3$ of $(1+\delta f^n)^{-1}Q^+(f^n, f^n)$, $(1+\delta f^n)^{-1}Q^-(f^n, f^n)$. For the weak limit function $\gamma_\delta$, we claim

**Lemma 5.1.**

$$\gamma_\delta \in C([0, \infty); L^p(\mathbb{R}^6))$$

for all $1 \leq p < \infty$.

**Proof.** In fact, we claim that the weak limit $\gamma_\delta$ is a renormalized solution of (5.3) and then

$$\gamma_\delta \in C([0, \infty); L^p(\mathbb{R}^6))$$

for all $1 \leq p < \infty$. For this purpose, we introduce

$$\gamma^\varepsilon_\delta(f^n) = \gamma_\delta(\beta_\varepsilon(f^n))$$

for $\varepsilon \in (0, 1]$ and denote its weak limit by $\gamma^\varepsilon_\delta$. Then, the proof in Section 4 applies and shows that the weak limit $\gamma^\varepsilon_\delta \in C([0, \infty); L^p(\mathbb{R}^6))$ is a renormalized solution of

$$\frac{\partial}{\partial t} \gamma^\varepsilon_\delta + \text{div}_x(\xi \gamma^\varepsilon_\delta) + (E + \xi \times B) \cdot \nabla_\xi \gamma^\varepsilon_\delta = \gamma'\delta(\beta_\varepsilon(f))\beta'_\varepsilon(f)Q^+_\delta, \quad \text{in } \mathcal{D}'$$

where the notation $\overline{g}$ means the weak limit of the sequence $\{g_n\}_{n=1}^\infty$ in $L^1_{loc}$. Next, we claim

$$0 \leq \gamma_\delta(f^n) - \gamma^\varepsilon_\delta(f^n) \leq f^n - \beta_\varepsilon(f^n) \to 0 \quad \text{in } L^1(\mathbb{R}^6)$$

uniformly in $n \geq 1$, $t \in [0, T]$. Indeed, since the sequence $\{f_n\}_{n=1}^\infty$ is equi-integrable, for any $\eta > 0$, there exists two positive numbers $D$ and $R$ such that

$$\sup_{n \in \mathbb{N}} \int_{(0, T] \times B_R \times B_R} f^n dt dx d\xi \leq \eta,$$

and

$$\sup_{n \in \mathbb{N}} \int_{\{f^n \geq D\}} f^n dt dx d\xi \leq \eta.$$

Hence, in particular,

$$\sup_{n \in \mathbb{N}} \int_{\{(f^n \geq D) \cap [0, T] \times B_R \times B_R} f^n dt dx d\xi \leq \eta.$$
Therefore, we have
\[
\sup_{n \in \mathbb{N}} \int_{[0,T] \times \mathbb{R}^3 \times \mathbb{R}^3} (f^n - \beta_\varepsilon(f^n)) \, dt \, dx \, d\xi
\]
\[
\leq \sup_{n \in \mathbb{N}} \int_{[0,T] \times B_R \times B_R} f^n \, dt \, dx \, d\xi
\]
\[
+ \sup_{n \in \mathbb{N}} \int_{\{f^n \geq D\} \cap [0,T] \times B_R \times B_R} f^n \, dt \, dx \, d\xi
\]
\[
+ \sup_{n \in \mathbb{N}} \int_{\{f^n \leq D\} \cap [0,T] \times B_R \times B_R} (f^n - \beta_\varepsilon(f^n)) \, dt \, dx \, d\xi
\]
\[
\leq 2 \eta + D^2 R^6 T \varepsilon,
\]
(5.6)
since \(f^n - \beta_\varepsilon(f^n) \leq D^2 \varepsilon\) if \(f^n \leq D\). Thus, letting first \(\varepsilon\) go to 0 and then \(\eta\) go to 0 in (5.6), we deduce (5.5).

Similarly, we have
\[
0 \leq 1 - \beta'_\varepsilon(f^n) \rightarrow 0 \text{ in } L^1(\mathbb{R}^6)
\]
uniformly in \(n \geq 1, t \in [0,T]; \)
\[
0 \leq (\gamma'_\delta(\beta_\varepsilon(f^n)) - \gamma'_\delta(f^n)) \beta'_\varepsilon(f^n) Q^-(f^n,f^n)
\]
\[
\leq \frac{\varepsilon f^n}{1 + \delta f^n} Q^-(f^n,f^n) \rightarrow 0 \text{ in } L^1(0,T; L^1(\mathbb{R}^3_\xi \times K))
\]
uniformly in \(n \geq 1\), for all compact sets \(K \subset \mathbb{R}^3_\xi; \)
\[
0 \leq (\gamma'_\delta(\beta_\varepsilon(f^n)) - \gamma'_\delta(f^n)) \beta'_\varepsilon(f^n) Q^+(f^n,f^n)
\]
\[
\leq \frac{\varepsilon f^n}{1 + \delta f^n} Q^+(f^n,f^n) \rightarrow 0 \text{ in } L^1(0,T; L^1(\mathbb{R}^3_\xi \times K))
\]
uniformly in \(n \geq 1\), for all compact sets \(K \subset \mathbb{R}^3_\xi; \)

Thus, letting \(\varepsilon\) go to 0 in (5.4), we deduce that \(\gamma_\delta\) is a renormalized solution to (5.3).

Hence, from (5.3), we deduce that \(\frac{\partial \gamma_\delta}{\partial t} \in L^1(0,T; W^{-n,1}(\mathbb{R}^3))\) for \(n > 0\) large enough. Also, we know that, since \(\gamma_\delta(t)\) is a strictly concave function,
\[
0 \leq \gamma_\delta(t) \leq \gamma_\delta(f) \leq f \in L^\infty(0,T; L^1(\mathbb{R}^6)).
\]

Hence, by the Aubin-Lions lemma in [21], we know that
\[
\gamma_\delta \in C([0,T]; W^{-s,1}(\mathbb{R}^6)).
\]

But actually, we know
\[
\gamma_\delta \in L^\infty([0,T], L^p(\mathbb{R}^6))
\]
for all \(1 \leq p < \infty\). Thus, by the interpolation, we know that
\[
\gamma_\delta \in C([0,T]; L^p(\mathbb{R}^6)).
\]
for all $1 \leq p < \infty$. \hfill \Box

5.1. **Step One: $\gamma_\delta$ is a supersolution of (5.1).** Without loss of generality, we may assume that we have

$$
\gamma_\delta(f^n) = \frac{1}{1 + \delta f^n} \to \zeta_\delta \quad \text{weakly* in } \ L^\infty(\mathbb{R}^6 \times (0, \infty)),
$$

and

$$
f^n \gamma_\delta(f^n) = \frac{f^n}{1 + \delta f^n} \to \theta_\delta \quad \text{weakly* in } \ L^\infty(\mathbb{R}^6 \times (0, \infty)).
$$

Furthermore, since $\gamma_\delta(f), -t\gamma_\delta(f)$ are convex on $[0, \infty)$, we deduce the following inequalities:

$$
\zeta_\delta \geq \frac{1}{1 + \delta f} = \gamma_\delta(f), \quad \theta_\delta \leq \frac{f}{1 + \delta f} = f \gamma_\delta(f), \quad (5.7a)
$$

$$
\gamma_\delta \leq \frac{1}{\delta} \ln(1 + \delta f) = \gamma_\delta(f) \quad \text{a.e. in } \mathbb{R}^6 \times (0, \infty). \quad (5.7b)
$$

We claim

**Lemma 5.2.**

$$
Q^-_\delta = \theta_\delta L(f) \quad \text{a.e. on } \mathbb{R}^6 \times (0, \infty). \quad (5.8)
$$

**Proof.** In fact, it is enough to verify that (5.8) holds in $[0, T] \times B_R \times B_R$, where $B_R$ is the ball with radius $R$ and centered at the origin in $\mathbb{R}^3$. Due to the second statement of Theorem 3.1, we know that $L(f^n)$ converges a.e. to $L(f)$ in $[0, T] \times B_R \times B_R$. By Egorov’s Theorem (2.5), for any $\varepsilon > 0$, there exists a subset $E \subset [0, T] \times B_R \otimes B_R$ with $|E| \leq \varepsilon$ such that $L(f^n)$ converges uniformly to $L(f)$ on $E^c$. Thus, for all $\phi \in L^\infty(\mathbb{R}^6 \times (0, T))$,

$$
\left| \int_0^T \int_{B_R} \int_{B_R} \phi \left( \frac{f^n}{1 + \delta f^n} L(f^n) - \theta_\delta L(f) \right) \, dx \, d\xi \, dt \right|
\leq ||\phi||_{L^\infty} \sup_n \int_E |L(f^n)| + ||\theta_\delta||_{L^1} |L(f)| \, dx \, d\xi \, dt
+ ||\phi||_{L^\infty} \int_{E^c} \left| \frac{f^n}{1 + \delta f^n} - \theta_\delta \right| L(f) \, dx \, d\xi \, ds
+ ||\phi||_{L^\infty} |E^c| \sup_{E^c} |L(f^n) - L(f)|.
$$

The first term can be made arbitrarily small uniformly in $n$, due to the equi-integrability of $(L(f^n))_{n=1}^\infty$. The second term also goes to 0 since $L(f) \in L^1((0, T) \times B_R \times B_R)$. And the third term goes to 0 as $n$ goes to $\infty$ since the uniform convergence of $L(f^n)$ to $L(f)$ in $E^c$. Thus, (5.8) is verified. \hfill \Box

Similarly, we have

**Lemma 5.3.**

$$
Q^+_\delta = \zeta_\delta Q^+_\delta(f, f) \quad \text{a.e. on } \mathbb{R}^6 \times (0, \infty). \quad (5.9)
$$

**Proof.** Indeed, let $\mathcal{A}$ be an arbitrary compact subset of $\mathbb{R}^6 \times [0, \infty)$. By the Egorov’s theorem and the fourth statement of Theorem 3.1, for each $\varepsilon > 0$ there exists a measurable set $E$ with the measure of $E$ not greater than $\varepsilon$ (i.e., $|E| \leq \varepsilon$), up to a subsequence
$Q^+(f^n, f^n)$ converges uniformly to $Q^+(f, f)$ on $E^c$ and $Q^+(f, f)$ is integrable on $E^c$. Then, for all $\phi \in L^\infty(\mathbb{R}^6 \times (0, \infty))$ supported in $\mathcal{A}$, we have

\[
\left| \int_{\mathcal{A}} \phi \{ \gamma_\delta'(f^n)Q^+(f^n, f^n) - \zeta_\delta Q^+(f, f) \} dx d\xi dt \right| \\
\leq \| \phi \|_{L^\infty} \int_E \left| \gamma_\delta'(f^n)Q^+(f^n, f^n) - \zeta_\delta Q^+(f, f) \right| dx d\xi dt \\
+ \int_{E \cap \mathcal{A}} \phi \{ \gamma_\delta'(f^n) - \zeta_\delta \} Q^+(f, f) dx d\xi dt \\
+ \| \phi \|_{L^\infty} |E^c \cap \mathcal{A}| \sup_{E^c} |Q^+(f^n, f^n) - Q^+(f, f)|,
\]

where the third term goes to 0 as $n$ goes to $\infty$, for each $\varepsilon > 0$ by the uniform convergence of $Q^+(f^n, f^n)$ to $Q^+(f, f)$ on $E^c$. And so does the second term since

\[
\phi \chi_E : Q^+(f, f) \in L^1(\mathbb{R}^6 \times (0, \infty)).
\]

Finally, since $\gamma_\delta'(f^n)Q^+(f^n, f^n)$ is weakly relatively compact in $L^1(\mathbb{R}^3_x \times K \times (0, T))$ for all compact sets $K \subset \mathbb{R}^3$, the first term can be made arbitrarily small uniformly in $n$ if we let $\varepsilon$ go to 0.

Notice also that $\zeta_\delta Q^+(f, f) \in L^1(\mathbb{R}^3_x \times K \times (0, T))$ by following the similar argument as before, we can show that $\zeta_\delta Q^+(f, f) \wedge R$ is the weak limit of $\frac{1}{1+\delta f^n} (Q^+(f^n, f^n) \wedge R)$, where $a \wedge b = \min\{a, b\}$. Thus, (5.9) follows. \hfill $\square$

Now, we use (5.7) - (5.9) in (5.1) to obtain

\[
\frac{\partial \gamma_\delta}{\partial t} + \text{div}_x (\xi \gamma_\delta) + (E + \xi \times B) \cdot \nabla \gamma_\delta \geq \gamma_\delta'(f)Q(f, f) \tag{5.10}
\]

in $\mathcal{D}'$. We conclude this first step by proving that $\gamma_\delta$ satisfies the initial condition:

\[
\gamma_\delta|_{t=0} = \gamma_\delta(f_0).
\]

Indeed, in view of the equation satisfied by $\gamma_\delta(f^n)$, we know that

\[
\frac{\partial \gamma_\delta(f^n)}{\partial t} \in L^1(0, T, W^{-n,1}(\mathbb{R}^6))
\]

for $n > 0$ large enough, which, combined with the fact $\gamma_\delta(f^n) \in L^\infty(0, T; L^1(\mathbb{R}^6))$ and the Aubin-Lions lemma, implies that

\[
\gamma_\delta(f^n) \rightarrow \gamma_\delta \quad \text{in} \quad C([0, T]; W^{-s,1}(\mathbb{R}^6))
\]

for any $s > 1$. But, by the assumption, $\gamma_\delta(f^n)|_{t=0} = \gamma_\delta(f_0^n)$ converges in $L^1(\mathbb{R}^6)$ and thus in $W^{-1,1}(\mathbb{R}^6)$ to $\gamma_\delta(f_0)$. Thus, we conclude that $\gamma_\delta$ satisfies the initial condition.

5.2. Step Two: $\gamma_\delta = \gamma_\delta(f)$ and $f^n$ converges in $L^1$ to $f$. To this end, we consider

\[
\gamma_\delta(f) - \gamma_\delta = \tau_\delta \in C([0, \infty); L^p(\mathbb{R}^6))
\]

and observe that $\tau_\delta$ satisfies, in view of (5.1) and (5.10),

\[
\frac{\partial}{\partial t} \tau_\delta + \text{div}_x (\xi \tau_\delta) + (E + \xi \times B) \cdot \nabla \xi \tau_\delta \leq 0 \tag{5.11}
\]
Then, for $\tau_{\delta}$ we have

**Lemma 5.4.** 

$\tau_{\delta} = 0$.

**Proof.** Formally, we only need to integrate (5.11) over $\mathbb{R}^6$ to get

$$
\frac{d}{dt} \int_{\mathbb{R}^6} \tau_{\delta} dx \leq 0 \quad \text{in } D'(0, \infty).
$$

(5.13)

Then (5.13) with (5.12) yield: $\tau_{\delta} = 0$ on $\mathbb{R}^6 \times (0, \infty)$.

Our main objective now is to justify (5.13). In order to do so, we introduce the function

$$
\phi(z) = \begin{cases} 
1, & \text{if } |z| \leq 1; \\
0, & \text{if } |z| \geq 2.
\end{cases}
$$

Notice that $\beta_\varepsilon(\tau_{\delta}) = \frac{\tau_{\delta}}{1 + \varepsilon \tau_{\delta}}$ also satisfies (5.11) and (5.12), and $\beta_\varepsilon(\tau_{\delta}) \in C([0, \infty); L^p(\mathbb{R}^6))$ for $1 \leq p < \infty$ and $\varepsilon > 0$, since

$$
|\beta_\varepsilon(x) - \beta_\varepsilon(y)| \leq |x - y|
$$

for all $x, y \geq 0$. Then we multiply (5.11) by $\phi \left( \frac{x}{n} \right) \phi \left( \frac{\xi}{n} \right)$, and integrate the resulting inequality over $\mathbb{R}^6 \times (0, t)$ for all $t \geq 0$ to obtain

$$
\int_{\mathbb{R}^6} \beta_\varepsilon(\tau_{\delta}) \phi \left( \frac{x}{n} \right) \phi \left( \frac{\xi}{n} \right) dx d\xi \\
\leq \int_0^t ds \int_{\mathbb{R}^6} dx d\xi \beta_\varepsilon(\tau_{\delta}) \cdot \left( \frac{\xi}{n} \cdot \nabla \phi \left( \frac{x}{n} \right) \cdot \phi \left( \frac{\xi}{n} \right) \right) \\
+ \frac{1}{n} \left( E + \xi \times B \right) \cdot \nabla \phi \left( \frac{\xi}{n} \right) \phi \left( \frac{x}{n} \right).
$$

(5.14)

Recall that

$$
\sup \left\{ \int_{\mathbb{R}^6} \beta_\varepsilon(\tau_{\delta}) |\xi|^2 dx d\xi : \ t \in [0, T], \ \varepsilon \geq 0, \ \delta \geq 0 \right\}
\leq \sup_{t \in [0, T]} \left\{ \int_{\mathbb{R}^6} f |\xi|^2 dx d\xi \right\} < \infty,
$$

(5.15)

since $\beta_\varepsilon(\tau_{\delta}) \leq \tau_{\delta} \leq \gamma_{\delta}(f) \leq f$ for all $T \in (0, \infty)$. Hence, for the terms on the right hand side of (5.14), we have

$$
\int_0^t ds \int_{\mathbb{R}^6} dx d\xi \beta_\varepsilon(\tau_{\delta}) \left| \frac{\xi}{n} \cdot \nabla \phi \left( \frac{x}{n} \right) \right| \phi \left( \frac{\xi}{n} \right)
\leq \int_0^t \int_{\mathbb{R}^6} \beta_\varepsilon(\tau_{\delta}) \chi_{\{ n \leq |x| \leq 2n \}} 2 \| \phi \|_{L^\infty} \| \nabla \phi \|_{L^\infty} dx d\xi
\leq 2 \int_0^t \int_{\mathbb{R}^6} f \chi_{\{ n \leq |x| \leq 2n \}} 2 \| \phi \|_{L^\infty} \| \nabla \phi \|_{L^\infty} dx d\xi \to 0
$$
as \( n \to \infty \) and

\[
\int_0^t ds \int_{\mathbb{R}^6} dxd\xi \beta_\varepsilon(\tau_\delta) \frac{1}{n} \phi \left( \frac{x}{n} \right) \left| (E + \xi \times B) \cdot \nabla \phi \left( \frac{\xi}{n} \right) \right|
\]

\[
\leq \left( \int_0^t ds \int_{\mathbb{R}^6} dxd\xi \beta_\varepsilon(\tau_\delta) \frac{1}{n} |E + \xi \times B| \chi_{\{n \leq |\xi| \leq 2n\}} dx d\xi \right) \| \phi \|_{L^\infty} \| \nabla \phi \|_{L^\infty}.
\]

Observing that, because of (5.15),

\[
\left\| \int_{\mathbb{R}^3} d\xi \beta_\varepsilon(\tau_\delta) \frac{1}{n} \chi_{\{n \leq |\xi| \leq 2n\}} \right\|_{L^1(0,t;L^1(\mathbb{R}^3))} \leq \frac{1}{n^3} \left\| \int_{\mathbb{R}^3} d\xi f |\xi|^2 \chi_{\{n \leq |\xi| \leq 2n\}} \right\|_{L^1(0,t;L^1(\mathbb{R}^3))}
\]

\[
= \frac{1}{n^3} \varepsilon_n, \quad \text{with} \quad \varepsilon_n \to 0,
\]

while of course we have for some \( C_\varepsilon > 0, \)

\[
\left\| \int_{\mathbb{R}^3} d\xi \beta_\varepsilon(\tau_\delta) \frac{1}{n} \chi_{\{n \leq |\xi| \leq 2n\}} \right\|_{L^1(0,t;L^\infty(\mathbb{R}^3))} \leq C_\varepsilon n^2.
\]

Therefore, we deduce from the Hölder inequality that we have for all \( \varepsilon > 0, \)

\[
\int_{\mathbb{R}^3} d\xi \beta_\varepsilon(\tau_\delta) \frac{1}{n} \chi_{\{n \leq |\xi| \leq 2n\}} \to 0,
\]

in \( L^1(0,t;L^p(\mathbb{R}^3)) \) for all \( 1 \leq p \leq \frac{5}{2} \) as \( n \to \infty \), and hence in particular in \( L^1(0,t;L^2(\mathbb{R}^3)) \). This implies

\[
\int_0^t ds \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \beta_\varepsilon(\tau_\delta) \frac{1}{n} \chi_{\{n \leq |\xi| \leq 2n\}} d\xi |E| dx \to 0, \quad \text{in} \quad L^1((0,t) \times \mathbb{R}^6) \quad (5.16)
\]

as \( n \to \infty \), since \( E \in L^\infty(0,t;L^2(\mathbb{R}^3)) \).

On the other hand, using (2.17) and the fact \( \beta_\varepsilon(\tau_\delta) \leq f \), we obtain

\[
\int_0^t ds \int_{\mathbb{R}^6} dxd\xi \beta_\varepsilon(\tau_\delta) \frac{1}{n} |\xi| |B| \chi_{\{n \leq |\xi| \leq 2n\}}
\]

\[
\leq \frac{2}{n} \int_0^t ds \left( \int_{\mathbb{R}^3} f |\xi| d\xi \right) |B| dx \leq \frac{C}{n} \int_0^t ds \left( \int_{\mathbb{R}^3} f |\xi|^2 d\xi \right) \frac{4}{3} |B| dx \leq \frac{C}{n} \sup_{s \in (0,t)} \left\{ \int_{\mathbb{R}^3} f |\xi|^2 d\xi \right\} \| B \|_{L^\infty(0,t;L^5(\mathbb{R}^3))} \to 0,
\]

as \( n \to \infty \). Hence, combining (5.16) and (5.17) together, we get

\[
\int_0^t ds \int_{\mathbb{R}^6} dxd\xi \beta_\varepsilon(\tau_\delta) \frac{1}{n} |E + \xi \times B| \chi_{\{n \leq |\xi| \leq 2n\}} \to 0
\]

as \( n \to \infty \).

Finally, letting first \( n \) go to \( \infty \) and then \( \varepsilon \) go to 0 in (5.14), we deduce, by Fatou’s lemma,

\[
\int_{\mathbb{R}^6} \tau_\delta(x,\xi,t) dx d\xi \leq 0, \quad \text{for all} \quad t \geq 0,
\]
which, combined with (5.12), implies that \(\tau_\delta = 0\) on \(\mathbb{R}^6 \times (0, \infty)\) almost everywhere. \(\square\)

In other words, \(\gamma_\delta(f^n)\) weakly converges to \(\gamma_\delta(f)\). Since \(\gamma_\delta\) is strictly concave on \([0, \infty)\), we deduce from classical functional analysis arguments that \(f^n\) converges in measure to \(f\) on \(\mathbb{R}^6 \times (0, T)\) for all \(T \in (0, \infty)\), see [14]. This convergence implies that

\[
f^n \to f \quad \text{in} \quad L^p(0, T; L^1(\mathbb{R}^6)),
\]

(5.18)

for all \(1 \leq p < \infty\) and \(T \in (0, \infty)\). Indeed, by the equi-integrability of the sequence \(\{f^n\}_{n=1}^\infty\) and the integrability of \(f\) in \(L^1([0, T] \times \mathbb{R}^6)\), we know that for any \(\epsilon > 0\), there exists \(R > 0\) and \(\delta > 0\) such that

\[
\sup_{n \in \mathbb{N}} \int_{\{(0, t) \times B_R \times B_R\}^c} |f^n - f| \, dt \, dx \, \xi \leq \epsilon,
\]

(5.19)

and

\[
\sup_{n \in \mathbb{N}} \int_G |f^n - f| \, dt \, dx \, \xi \leq \epsilon,
\]

(5.20)

for all set \(G \subset [0, T] \times \mathbb{R}^6\) with \(|G| \leq \delta\).

On the other hand, on the set \([0, T] \times B_R \times B_R\), since up to a subsequence, \(f^n\) converges to \(f\) almost everywhere, by Egorov’s theorem, for the given \(\delta > 0\) as above, there exists a subset \(H \subset [0, T] \times B_R \times B_R\) with \(|H| \leq \delta\) such that \(f^n\) converges uniformly to \(f\) on \(([0, T] \times B_R \times B_R) \cap H^c\). Therefore, using (5.20),

\[
\int_{[0, T] \times B_R \times B_R} |f^n - f| \, dt \, dx \, \xi = \int_{([0, T] \times B_R \times B_R)^c \cap H^c} |f^n - f| \, dt \, dx \, \xi + \int_{H^c} |f^n - f| \, dt \, dx \, \xi \\
\leq \epsilon + \int_{([0, T] \times B_R \times B_R)^c \cap H^c} |f^n - f| \, dt \, dx \, \xi.
\]

(5.21)

Notice that the last term in (5.21) tends to 0 as \(n \to \infty\) since the uniform convergence of \(f^n\) to \(f\) in \(H^c\). Hence, combining (5.19), (5.20) and (5.21), we conclude that

\[
f^n \to f \quad \text{in} \quad L^1(0, T; L^1(\mathbb{R}^6)),
\]

which, with the uniform bound of \(f^n\) in \(L^\infty(0, T; L^1(\mathbb{R}^6))\), implies (5.18).

5.3. Step Three: The convergence in \(C([0, T]; L^1(\mathbb{R}^6))\). It only remains to show that \(f^n\) converges to \(f\) in \(C([0, T]; L^1(\mathbb{R}^6))\) using (5.18). Indeed, because of (3.16) and (4.25), it is clearly enough to show that, for each \(\delta > 0\), \(T \in (0, \infty)\), \(K\) compact set in \(\mathbb{R}^6\), we have

\[
\beta_\delta(f^n) \to \beta_\delta(f) \quad \text{in} \quad C([0, T]; L^1(K)).
\]

(5.22)

For this purpose, we take \(\phi \in C_0^\infty(\mathbb{R}^6)\) such that \(\phi = 1\) on \(K\), \(\phi \geq 0\), and we use (4.3) to deduce that for all \(t \geq 0\),

\[
\int_{\mathbb{R}^6} \beta_\delta(f^n) \phi \, dx \, \xi = \int_0^t \int_{\mathbb{R}^6} dx \, \xi \left(\frac{2\beta_\delta(f^n)}{1 + \delta f^n} \times Q(f^n, f^n) \phi \\
+ \beta_\delta(f^n)^2 (\xi \cdot \nabla x \phi + (E^n + \xi \times B^n) \cdot \nabla_x \phi)\right).
\]

(5.23)
Then, due to (5.18), \( \beta_\delta(f^n) \) converges to \( \beta_\delta(f) \) in \( L^p(\mathbb{R}^6 \times (0, T)) \) for all \( 1 \leq p < \infty \) and \( T \in (0, \infty) \), and one can check easily that the right-hand side of (5.23) converges uniformly in \( t \in [0, T] \) to the same expression with \( f^n \) replaced by \( f \). Since \( \beta_\delta(f) \) is a renormalized solution, this expression is also given by \( \int_{\mathbb{R}^6} \beta_\delta(f)^2 \phi dxd\xi \). In other words, we have

\[
\int_{\mathbb{R}^6} \beta_\delta(f^n)^2 \phi dxd\xi \rightarrow \int_{\mathbb{R}^6} \beta_\delta(f)^2 \phi dxd\xi,
\]

uniformly in \( t \in [0, T] \), for all \( T \in (0, \infty) \).

In addition, since (4.3) implies

\[
\frac{\partial \beta_\delta(f^n)}{\partial t} \in L^1(0, T; W^{-n,1}(\mathbb{R}^6))
\]

for large enough \( n > 0 \), and

\[
\beta_\delta(f^n) \in L^1(0, T; L^1(\mathbb{R}^6)),
\]

by the Aubin-Lions lemma, we know that \( \beta_\delta(f^n) \) converges to \( \beta_\delta(f) \) in \( C([0, T]; W^{-s,1}_{\text{loc}}(\mathbb{R}^6)) \) for any \( s > 1 \). Therefore, if we consider \( L^2_\beta = L^2(\text{supp}\phi, \phi dx) \), since \( \{\beta_\delta(f^n)\}_n \) is bounded in \( L^\infty(0, T; L^2_\beta) \), we deduce that \( \beta_\delta(f^n) \) converges uniformly on \([0, T]\) to \( \beta_\delta(f) \) in \( L^2_\beta \) endowed with the weak topology, which, combined with (5.24) and the fact that \( \beta_\delta(f) \in C([0, \infty); L^2_\beta) \) implies that \( \beta_\delta(f^n) \) converges to \( \beta_\delta(f) \) in \( L^2_\beta \) strongly and uniformly in \([0, T]\).

Hence, (5.22) follows.

### 6. Large Time Behavior

In this section, we are devoted to the study of the large time behavior of the renormalized solution to VMB. Indeed, let \( f(t, x, \xi) \) be a renormalized solution to VMB with finite energy and finite entropy in view of (3.10). Then, for every sequence \( \{t_n\}_{n=1}^\infty \) going to infinity, there exists a subsequence \( \{t_{n_k}\}_{k=1}^\infty \) and a local time-dependent Maxwellian \( m \) such that \( f_{n_k}(t, x, \xi) = f(t + t_{n_k}, x, \xi) \) converges weakly in \( L^1((0, T) \times \mathbb{R}^6) \) to \( m \) for every \( T > 0 \). More precisely, we have the following theorem:

**Theorem 6.1.** Let \( f(t, x, \xi) \) be a renormalized solution to VMB and assume that \( b > 0 \) almost everywhere. Then, for every sequences \( t_n \) going to infinity, there exists a subsequence \( t_{n_k} \) and a local time-dependent Maxwellian \( m(t, x, \xi) \) such that \( f_{n_k}(t, x, \xi) = f(t + t_{n_k}, x, \xi) \) converges weakly in \( L^1((0, T) \times \mathbb{R}^6) \) to \( m(t, x, \xi) \) for every \( T > 0 \). Moreover, the Maxwellian satisfies the Vlasov-Maxwell equations:

\[
\begin{align*}
\frac{\partial m}{\partial t} + \xi \cdot \nabla_x m + (E + \xi \times B) \cdot \nabla_\xi m &= 0, \\
\frac{\partial E}{\partial t} - \nabla \times B &= -\int_{\mathbb{R}^3} m\xi d\xi, \quad \text{div} B = 0, \\
\frac{\partial B}{\partial t} + \nabla \times E &= 0, \quad \text{div} E = \int_{\mathbb{R}^3} m d\xi,
\end{align*}
\]

in the sense of renormalizations.
Remark 6.1. When the spatial domain is a periodic box or a bounded domain with the reverse reflection boundary or the specular reflection boundary, we can expect, as in [7, 12], that the local Maxwellian \( m \) in Theorem 6.1 is actually global; that is, \( m \) is independent of \( t, x \).

Remark 6.2. Our large time behavior result is only sequential; that is, the Maxwellian could depend on our choice of the sequence \( \{t_n\}_{n=1}^\infty \).

Proof of Theorem 6.1. Notice that since \( f(t, x, \xi) \) is a renormalized solution to VMB, it automatically holds:

\[
\sup_{t \in [0, \infty)} \left( \int_{\mathbb{R}^6} f(1 + |\xi|^2 + \nu(x) + |\log f|) d\xi + \int_{\mathbb{R}^3} (|E|^2 + |B|^2) dx \right)
+ \int_0^\infty \int_{\mathbb{R}^3} d\xi \int_{\mathbb{R}^6} d\omega b(f' f_s - f f_s) \log \frac{f' f_s}{f f_s} < \infty.
\]

Therefore, \( f_n(t, x, \xi) = f(t + t_n, x, \xi) \) is weakly compact in \( L^1((0, T) \times \mathbb{R}^6) \) for every \( T > 0 \) and each sequence of positive numbers \( \{t_n\}_{n=1}^\infty \) going to \( \infty \). Similarly, \( E_n(t, x) = E(t + t_n, x) \), \( B_n(t, x) = B(t + t_n, x) \) are weakly compact in \( L^\infty(0, T; L^2(\mathbb{R}^3)) \). Then, the weak compactness of \( f_n(t, x, \xi) \) in \( L^1((0, T) \times \mathbb{R}^3) \) implies that there exists a subsequence \( \{t_{n_k}\}_{k=1}^\infty \) and a function \( m \in L^1((0, T) \times \mathbb{R}^6) \) such that the function \( f_{n_k} \) converges weakly to \( m \) in \( L^1((0, T) \times \mathbb{R}^6) \) while the weak compactness of \( B_{n_k}(t, x) \) and \( E_{n_k}(t, x) \) implies that we can choose \( t_{n_k} \) such that \( B_{n_k} \) and \( E_{n_k} \) converge weakly* to \( B \) and \( E \) respectively in \( L^\infty(0, T; L^2(\mathbb{R}^3)) \). Notice that, applying the velocity average lemma, we know

\[
\int_{\mathbb{R}^3} f_{n_k} d\xi \to \int_{\mathbb{R}^3} m d\xi \quad \text{in} \quad L^1(0, T; L^1(\mathbb{R}^3)),
\]

and

\[
\int_{\mathbb{R}^3} f_{n_k} \xi d\xi \to \int_{\mathbb{R}^3} m \xi d\xi \quad \text{in} \quad L^1(0, T; L^1(\mathbb{R}^3)).
\]

Hence, according to (1.1b) and (1.1c), the electric field \( E \) and the magnetic field \( B \) satisfies

\[
\frac{\partial E}{\partial t} - \nabla \times B = -\int_{\mathbb{R}^3} m \xi d\xi,
\]

\[
\frac{\partial B}{\partial t} + \nabla \times E = 0,
\]

with

\[
\text{div} B = 0, \quad \text{div} E = \int_{\mathbb{R}^3} m d\xi,
\]

in the sense of distributions.

In order to prove that \( m \) is a Maxwellian, we denote

\[
d_k := \int_0^T \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^6} d\xi d\xi_\ast \int_{S^2} d\omega b(f_{n_k}' f_{n_k}' - f_{n_k} f_{n_k}) \log \frac{f_{n_k}' f_{n_k}'}{f_{n_k} f_{n_k}}.
\]

Then, the estimate (6.2) implies that \( d_k \) converges to 0 as \( k \) goes to \( \infty \).
On the other hand, in view of the first statement of Theorem 3.1 or arguing as [11], for all smooth nonnegative functions $\psi, \phi$ with compact support, we have, up to a subsequence,

$$
\int_{\mathbb{R}^6} d\xi d\xi^* \int_{S^2} d\omega b f_{n_k'} f_{n_k} \phi(\xi) \psi(\xi^*) \\
\to \int_{\mathbb{R}^6} d\xi d\xi^* \int_{S^2} d\omega m(t, x, \xi') m(t, x, \xi^*) \phi(\xi) \psi(\xi^*),
$$

(6.3)

and

$$
\int_{\mathbb{R}^6} d\xi d\xi^* \int_{S^2} d\omega b f_{n_k} f_{n_k} \phi(\xi) \psi(\xi^*) \\
\to \int_{\mathbb{R}^6} d\xi d\xi^* \int_{S^2} d\omega b m(t, x, \xi') m(t, x, \xi^*) \phi(\xi) \psi(\xi^*),
$$

(6.4)

for almost all $(t, x) \in [0, T] \times \mathbb{R}^3$.

Furthermore, since $C(\mathbb{R}^3)$ is separable, we can also assume the convergence in (6.3) and (6.4) holds for all nonnegative function in $C(\mathbb{R}^3)$. Since $P(x, y) = (x - y) \ln(\frac{x}{y})$ is a nonnegative convex function for $x, y > 0$, we have,

$$
0 \leq \int_{\mathbb{R}^6} d\xi d\xi^* \int_{S^2} d\omega b (m'm'_* - mm_*) \log \frac{m'm'_*}{mm_*} \psi(\xi^*) \phi(\xi) \\
\leq \liminf_{k \to \infty} d_k = 0,
$$

for almost all $(t, x) \in [0, T] \times \mathbb{R}^3$. Hence,

$$
b(m'm'_* - mm_*) \log \frac{m'm'_*}{mm_*} \psi(\xi^*) \phi(\xi) = 0,
$$

almost all $(t, x) \in [0, T] \times \mathbb{R}^3$. The nonnegativity of the function $P(x, y)$ and the strict positivity of $b$ ensure that

$$m'm'_* = mm_*,
$$

for almost all $(t, x, \xi, \xi^*, \omega) \in (0, T) \times \mathbb{R}^3 \times S^2$. According to Lemma 2.2 of [4] or Section 3.2 of [5], $m$ is a Maxwellian. Thus,

$$Q(m, m) = 0.
$$

Also, in view of Theorem 3.1 $m$ is still a renormalized solution to VMB, hence

$$\frac{\partial m}{\partial t} + \xi \cdot \nabla_x m + (E + \xi \times B) \cdot \nabla_\xi m = 0,
$$

in the sense of renormalizations. The proof is complete. \qed
7. Remark on The Relativistic Vlasov-Maxwell-Boltmann Equations

An extension of our analysis is possible to the relativistic Vlasov-Maxwell-Boltzmann equations of the form (cf. \[6, 17\]):

\[
\begin{align*}
\frac{\partial f}{\partial t} + \hat{\xi} \cdot \nabla_x f + (E + \hat{\xi} \times B) \cdot \nabla_\xi f & = Q(f, f), \quad x \in \mathbb{R}^3, \quad \xi \in \mathbb{R}^3, \quad t \geq 0, \\
\frac{\partial E}{\partial t} - \nabla \times B & = -j, \quad \text{div} B = 0, \quad \text{on } \mathbb{R}^3 \times (0, \infty), \\
\frac{\partial B}{\partial t} + \nabla \times E & = 0, \quad \text{div} E = \rho, \quad \text{on } \mathbb{R}^3 \times (0, \infty), \\
\rho & = \int_{\mathbb{R}^3} fd\xi, \quad j = \int_{\mathbb{R}^3} \hat{\xi} d\xi, \quad \text{on } \mathbb{R}^3 \times (0, \infty),
\end{align*}
\]

with

\[\hat{\xi} := \frac{\xi}{\sqrt{1 + |\xi|^2}},\]

and

\[Q(f, f) = \int_{\mathbb{R}^3} d\xi_s \int_{S^2} d\omega \frac{b(\xi - \xi_s, \omega)}{\sqrt{1 + |\xi|^2} \sqrt{1 + |\xi_s|^2}} (f' f' - ff').\]

The corresponding conservation laws are given by

\[\frac{\partial \rho}{\partial t} + \text{div}_x j = 0,
\]

\[\frac{\partial}{\partial t} \left( \int_{\mathbb{R}^3} f \hat{\xi} d\xi + E \times B \right)
\]

\[+ \text{div}_x \left( \int_{\mathbb{R}^3} \hat{\xi} \otimes \hat{\xi} d\xi + \left( \frac{|E|^2 + |B|^2}{2} \mathbb{I} - E \otimes E - B \otimes B \right) \right) = 0,
\]

and

\[\frac{\partial}{\partial t} \left( \int_{\mathbb{R}^3} f \sqrt{1 + |\xi|^2} d\xi + |E(t, x)|^2 + |B(t, x)|^2 \right)
\]

\[+ \text{div}_x \left( \int_{\mathbb{R}^3} \hat{\xi} |\xi|^2 d\xi + 2E(t, x) \times B(t, x) \right) = 0.
\]

Then, we can deduce that for any \(t \in [0, T]\)

\[\int_{\mathbb{R}^6} f(t, x, \xi) (\sqrt{1 + |\xi|^2} + \sqrt{1 + |x|^2}) dx d\xi \leq C(T),
\]

which implies \(f|\xi| \in L^\infty(0, T; L^1(\mathbb{R}^6)).\) Hence, following the lines in Section 2, the existence of renormalized solution to the relativistic Vlasov equation can be verified. More importantly, we can further release the requirement on the integrability of the electric field in \(L^5\), since we no longer need the estimate on \(f|\xi|^2\) in \(L^\infty(0, T; L^1(\mathbb{R}^6)).\)

Note that for the relativistic VMB, the magnetic field has the same integrability in the variable \(x\) as the magnetic field due to equivalence between \(\xi\) and \(\sqrt{1 + |\xi|^2}\) when \(\xi\) is sufficiently large. More precisely, we have
Proposition 7.1. Assume that \( f \in L^\infty((0,T) \times \mathbb{R}^6) \). Then for any solution satisfying the above conservation laws, one has
\[
\| \rho(t,x) \|_{L^\infty(0,T;L^{\frac{4}{3}}(\mathbb{R}^3))} \leq C, \quad \| j(t,x) \|_{L^\infty(0,T;L^{\frac{4}{3}}(\mathbb{R}^3))} \leq C,
\]
where the constant \( C \) depends on the energy of the initial data and on \( \| f \|_{L^\infty((0,T) \times \mathbb{R}^6)} \).

Proof. Indeed, we have
\[
\rho(t,x) = \int_{|\xi| \leq R} f(t,x,\xi)d\xi + \int_{|\xi| > R} f(t,x,\xi)d\xi \\
\leq C \frac{1}{R^3} \| f \|_{L^\infty} + R^{-1} \int_{\mathbb{R}^3} f \sqrt{1 + |\xi|^2}d\xi \\
\leq C \left( \int_{\mathbb{R}^3} f \sqrt{1 + |\xi|^2}d\xi \right)^{\frac{1}{4}}
\]
where for the last inequality, we optimize \( R \) by taking
\[
R = \left( \int_{\mathbb{R}^3} f \sqrt{1 + |\xi|^2}d\xi \right)^{\frac{1}{4}}.
\]
The same computation also works for \( j \).

For any sequence of \( f^n \) as in Section 3, by the H Theorem and (7.2), \( f^n \) is weakly compact in \( L^1((0,T) \times (\mathbb{R}^6)) \). And then, we can follow the lines in Section 4 and Section 5 to show the corresponding weak stability for the relativistic VMB. One difference is that, due to Proposition 7.1, we need to assume the electric field \( E(t,x) \), and the magnetic field \( B(t,x) \) are uniformly bound in \( L^\infty(0,T;L^\alpha(\mathbb{R}^3)) \) for some \( \alpha > 4 \). When the weak stability and the existence of renormalized solutions to (7.1) are concerned, a different assumption on the collision kernel need to assume, that is,
\[
(1 + |z|^2)\left( \int_{z+BR} \frac{A(\xi)}{\sqrt{1 + |\xi|^2}}d\xi \right) \to 0, \quad \text{as} \quad |z| \to \infty, \quad \text{for all} \quad R \in (0,\infty).
\]

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