DUALITY FOR GENERALIZED GAN-GROSS-PRASAD RELEVANT PAIRS FOR $p$-ADIC $GL_n$

KEI YUEN CHAN

Abstract. The main goal of this article is to formulate a notion, called a generalized GGP relevant pair, governing the quotient branching law for $p$-adic general linear groups. Such notion relies on a commutation relation between derivatives (from Jacquet functors) and integrals (from parabolic inductions), for which we provide both representation-theoretic and combinatorial perspectives. Our main result proves a duality on those relevant pairs, which is compatible with a dual restriction in branching law.

1. Introduction

Branching law is a classical problem in the representation theory. Even formulating a precise branching law is a subtle problem, which has already drawn a lot of attention in recent years [GGP12, GGP20]. This article aims to setup essential ingredients towards establishing the general quotient branching law in the sequel [Ch22+d].

In Section 1.1 we first define derivatives and integrals, and explain a commutation problem between derivatives and integrals. Such commutation suggests our terminology of strongly commutative triples in Section 1.2. In Section 1.3 we give an equivalent combinatorial definition for those triples. In Section 1.4 we explain how those commutative triples arise in branching laws. The main duality result is in Section 1.5.

For the interests of readers, some results on derivatives and integrals will be discussed in wider generality e.g. inner forms of general linear groups over a non-Archimedean local field and □-irreducible representations.

1.1. Derivatives and integrals. Let $F$ be a non-Archimedean local field. Let $D$ be a finite-dimensional central division $F$-algebra. Let $G_n = GL_n(D)$ be the general linear group over $D$. Let $\text{Alg}(G_n)$ be the category of complex smooth representations of $G_n$. Let $\text{Irr}(G_n)$ be the set of irreducible smooth representations of $G_n$. Let $\text{Irr} = \bigsqcup_n \text{Irr}(G_n)$.

We first define some operators on $\text{Irr}(G_n)$. Denote, by $\times$, the normalized parabolic induction. Following Lapid-Mínguez [LM19], an irreducible representation $\sigma$ of $G_n$ is said to be □-irreducible if $\sigma \times \sigma$ is still irreducible. There are some recent interests in the problem of characterizing □-irreducible representations, see [LM19]. In particular, they classify when a regular representation is □-irreducible. Let $\text{Irr}^\square(G_n)$ be the set of □-irreducible representations of $G_n$. Let $\text{Irr}^\square = \bigsqcup_n \text{Irr}^\square(G_n)$.

Let $I^L_\sigma(\pi)$ (resp. $I^R_\sigma(\pi)$) be the unique simple submodule of $\sigma \times \pi$ (resp. $\pi \times \sigma$), called the left (resp. right) $\sigma$-integral of $\pi$. The uniqueness part is shown in [LM19] adapting the proof of Kang-Kashiwara-Kim-Oh [KKKO15]. One may view $I^L_\sigma$ and $I^R_\sigma$ as operators from $\text{Irr}$ to $\text{Irr}$.

1It is called a left multiplier in [LM16] while we prefer to use integrals to emphasis the relation to derivatives.
For \( n_1 + \ldots + n_r = n \), let \( P_{n_1,\ldots,n_r} \) be the standard parabolic subgroup containing block diagonal matrices \( \text{diag}(g_1,\ldots,g_r) \) with \( g_k \in G_k \) and upper triangular matrices. Let \( N_{n_1,\ldots,n_r} \) be the unipotent radical of \( P_{n_1,\ldots,n_r} \). We shall sometimes abbreviate \( N_{n-i,i} \) by \( N_i \). For a unipotent radical \( N \) and \( \pi \in \text{Irr}(G_k) \), we write \( \pi_N \) to be the corresponding normalized Jacquet module for \( \pi \).

For \( \sigma \in \text{Irr}^\square(G_i) \) and \( \pi' \in \text{Irr}(G_n) \), define \( D^R_\sigma(\pi) \) (resp. \( D^L_\sigma(\pi) \)) to be the unique irreducible representation (if it exists) such that

\[
D^R_\sigma(\pi) \boxtimes \sigma \hookrightarrow \pi_N, \quad \text{(resp. } \sigma \boxtimes D^L_\sigma(\pi) \hookrightarrow \pi_{N'} \text{)}
\]

where \( N = N_{n-i,i} \) (resp. \( N' = N_{i,n-i} \)). We set \( D^R_\sigma(\pi) = 0 \) (resp. \( D^L_\sigma(\pi) = 0 \)) if it does not exist. This is well-defined by [LM22, Theorem 4.1D]. By [LM19, Corollary 2.4] and the second adjointness of parabolic induction, we have \( D^R_\sigma \circ I^R_\sigma(\pi) \cong \pi \) and similarly \( I^R_\sigma \circ D^R_\sigma(\pi) \cong \pi \) if \( D^R_\sigma(\pi) \neq 0 \).

When one operator is applied on the left and another operator is applied on the right, it is natural to ask whether those operators commute. For example, the commutation for \( I^L_\sigma \circ I^R_\sigma(\pi) \cong I^L_\sigma \circ I^R_\sigma(\pi) \) is related to the associativity a binary operation on nilpotent orbits studied by Aizenbud-Lapid [AL22] and Lapid-Mínguez [LM22]. We shall use geometric lemma to impose a natural condition to guarantee \( D^R_\sigma \circ I^L_\sigma(\tau) \cong I^L_\sigma \circ D^R_\sigma(\tau) \) in Definition 1.1 below.

### 1.2. Strongly commutative triples

It is shown later in Proposition 6.2 that the following conditions in Definition 1.1 guarantees a commutation of a left integral and a right derivative.

**Definition 1.1.** Let \( \pi \in \text{Irr} \). Let \( \sigma' \in \text{Irr}^\square \) and let \( \sigma \in \text{Irr}^\square(G_r) \).

- A triple \((\sigma, \sigma', \pi)\) is said to be pre-RdLi-commutative if \( D^R_\sigma(\pi) \neq 0 \) and the composition of the maps:

\[
D^R_\sigma \circ I^L_\sigma(\pi) \boxtimes \sigma \hookrightarrow I^L_\sigma(\pi)_N, \quad \hookrightarrow (\sigma' \times \pi)_N, \quad \xrightarrow{s} \sigma' \times 1\{(\pi_N)\}
\]

is non-zero, where the first injection is the unique embedding from \((1.1)\), and the second map is induced from the embedding \( I^L_\sigma(\pi) \twoheadrightarrow \sigma' \times \pi \). Here Rd refers to right derivatives and Li refers to left integrals, and \( s : (\sigma' \times \pi)_N \rightarrow \sigma' \times 1\{(\pi_N)\} \) be the natural surjection onto the top layer in the geometric lemma (see Section 4 below for the notion \( \times 1\)).

- We say that a triple \((\sigma, \sigma', \pi)\) is strongly RdLi-commutative if \((\sigma, \sigma', \pi)\) is pre-RdLi-commutative and the map \((1.2)\) above factors through the map

\[
(\sigma' \times D^R_\sigma(\pi)) \boxtimes \sigma \hookrightarrow (\sigma' \times 1\{(\pi_N)\})
\]

induced from \((1.1)\).

Typical examples of pre-RdLi-commutative triples arise when the intersection of the cuspidal supports of \( \sigma \) and \( \sigma' \) is non-empty. Another family of examples comes from when the integral preserves the level of \( \pi \), see Corollary 1.2.

The version switching the right derivative to a left derivative and switching the left integral to a right integral can be formulated analogously, and we shall call those triples to be pre-LdRi-commutative triples and strongly LdRi-commutative triples respectively. For example, in the LdRi-version, \((1.2)\) becomes of the following form:

\[
\sigma \boxtimes D^L_\sigma \circ I^R_\sigma(\pi) \hookrightarrow I^R_\sigma(\pi)_N, \quad \hookrightarrow (\pi \times \sigma')_N, \quad \xrightarrow{s} \pi_{N(n)} \times 1\sigma, \quad \xrightarrow{\pi_{N(n)} - r} \sigma.
\]
where $\pi_{N_n(\sigma)-r}$ is defined analogously as the $\check{\times}$ by switching left and right versions, and $\pi_{N_n(\sigma)-r}^L$ is a $G_{n(\sigma)} \times G_{n(\sigma)}$-representation. We shall not use the notion $\check{\times}$ anymore explicitly.

Indeed, from Section 2 one can see that the two notions of strong commutativity can be related by the Gelfand-Kazhdan involution, and are a key to the formulation of the main duality result (Theorem 1.4 below). Rather than using the Gelfand-Kazhdan involution, one may also define by using the smooth dual functor, while such formulation has a slightly subtle issue from dualizing the geometric lemma. We shall explain such issue in more details in Ch24.

We make the following conjecture:

**Conjecture 1.2.** Let $\sigma, \sigma' \in \text{Irr} G$ and let $\pi \in \text{Irr}$. If a triple $(\sigma, \sigma', \pi)$ is pre-RdLi-commutative, then $(\sigma, \sigma', \pi)$ is strongly RdLi-commutative.

It is well-known from the work of Zelevinsky [Ze80] and Tadić [Ta90] that essentially square-integrable representations are $\square$-irreducible.

**Theorem 1.3.** (=Theorem 1.10) Conjecture 1.2 holds if both $\sigma$ and $\sigma'$ are essentially square-integrable.

1.3. **Combinatorially commutative triples.** For applications on branching laws, we are interested in $\sigma$ and $\sigma'$ to be essentially square-integrable representations. We shall give a combinatorial criteria for those strongly commutative triples, and we need some combinatorial objects.

Let $|\cdot|$ be the absolute value of $F$ and let $\text{Nrd} : G_n \to F^\times$ be the reduced norm. To each cuspidal representation $\rho$ of $G_n$, we associate a unique character $\nu_\rho(g) = |\text{Nrd}(g)|^e$ of $G_n$ with $s_\rho > 0$ so that $\rho \times \nu_\rho^{s_\rho}$ is reducible. For $a, b \in \mathbb{Z}$ with $b - a \geq 0$ and a cuspidal representation $\rho$, we call $[a, b]_\rho$ to be a segment, following [Ze80]. We consider two segments $[a, b]_\rho$ and $[a', b']_\rho'$ to be equal if $\nu_\rho^a \rho \cong \nu_\rho^{a'} \rho'$ and $\nu_\rho^b \rho \cong \nu_\rho^{b'} \rho'$. For each segment $\Delta$, let $\text{St}(\Delta)$ be the corresponding essentially square-integrable representations i.e. the unique simple quotient of $\nu_\rho^a \rho \times \cdots \times \nu_\rho^b \rho$. For a segment $\Delta$, we set

$$D^R_\Delta(\pi) = D^R_{\text{St}(\Delta)}(\pi), \quad D^L_\Delta(\pi) = D^L_{\text{St}(\Delta)}(\pi), \quad I^R_\Delta(\pi) = I^R_{\text{St}(\Delta)}(\pi), \quad I^L_\Delta(\pi) = I^L_{\text{St}(\Delta)}(\pi).$$

We define more combinatorial invariants:

- Define $\epsilon_\Delta(\pi) := \epsilon^R_\Delta(\pi)$ (resp. $\epsilon^L_\Delta(\pi)$) to be the largest non-negative integer $k$ such that $D^R_\Delta(\pi) \neq 0$ (resp. $(D^L_\Delta)^k(\pi) \neq 0$). Here the power $k$ means the composition of $D_\Delta$ for $k$-times and when $k = 0$, $D^0_\Delta$ is regarded as the identity operator.

- Define $\eta_\Delta(\pi) := \eta^R_\Delta(\pi) := (\epsilon_{[a,b]_\rho}(\pi), \epsilon_{[a+1,b]_\rho}(\pi), \ldots, \epsilon_{[b,b]_\rho}(\pi))$; (resp. $\eta^L_\Delta(\pi) := (\epsilon^L_{[a,b]_\rho}(\pi), \epsilon^L_{[a,b-1]_\rho}(\pi), \ldots, \epsilon^L_{[a,a]_\rho}(\pi))$).

When $\Delta = \{\rho\}$, it is a more standard case in [Ja07, Mi09].

**Definition 1.4.** Let $\Delta, \Delta'$ be segments. Let $\pi \in \text{Irr}$. We say that $(\Delta, \Delta', \pi)$ is a combinatorially RdLi-commutative triple if $\epsilon_\Delta(\pi) \neq 0$ (equivalently $D^R_\Delta(\pi) \neq 0$) and $\eta_\Delta(I^L_{\Delta'}(\pi)) = \eta_\Delta(\pi)$.

We have the following criteria for strong commutation, which also uses Theorem 1.3 in the proof:

**Theorem 1.5.** (=Part of Theorem 1.10) A triple $(\Delta, \Delta', \pi)$ is combinatorially RdLi-commutative if and only if $(\text{St}(\Delta), \text{St}(\Delta'))$, $\pi$ is strongly RdLi-commutative.
There is a dual formulation in Definition 10.1.

1.4. Where does the commutation appear in branching laws? These integrals are a generalization of Kashiwara’s crystal operators, as suggested by Leclerc [Le03], which is related to $p$-adic groups via the quantum Schur-Weyl duality of Chari-Pressley [CP96] and the Bernstein theory. It is also known that there are tight links of the branching laws of type $A$ algebras to crystal graphs, and so one may regard our work in [Ch22+, Ch24, Ch22+e] suggest that the Bernstein-Zelevinsky derivatives are (see e.g. [Ch21, CS21] for details), but studies in [Fa06] about canonical bases [VV99, Sc00], about categorifications [CR08, KL09] as well as block structures [Fa06].

We now explain more precisely how the commutation comes into the play of branching laws. Let $D = F$. Let $\pi_1 \in \text{Irr}(G_{n+1})$ and let $\pi_2 \in \text{Irr}(G_n)$. We refer the reader to Section 12 for a notion of derivatives $D^R_\pi$ and integral $I^L_\pi$ for multisegments $m$ and $n$, which extend the segment cases $D^R_\Delta$ and $I^L_\Delta$ respectively. We shall not introduce what the Bernstein-Zelevinsky derivatives are (see e.g. [Ch21, CS21] for details), but studies in [Ch22+, Ch24, Ch22+e] suggest that $D^R_n(\pi)$ and $D^L_n(\pi)$ are weak replacements of right and left Bernstein-Zelevinsky derivatives.

Now suppose $\text{Hom}_{G_n}(\pi_1, \pi_2) \neq 0$. Then the Bernstein-Zelevinsky theory suggests that $I^L_m \circ D^R_n(\nu^{1/2}) \cong \pi'$ for some multisegments $m$ and $n$. By the left Bernstein-Zelevinsky theory [CS21, Ch21], we also have $I^L_m \circ D^L_n(\nu^{-1/2}) \cong \pi'$. This imposes a condition on those $m, m', n, n'$:

$$I^L_m \circ D^R_n(\nu^{1/2}) \cong I^L_{m'} \circ D^L_{n'}(\nu^{-1/2}).$$

This gives that

$$D^L_{n'}(\nu^{-1/2}) \bowtie \text{St}(n') \hookrightarrow I^L_m \circ D^R_n(\nu^{1/2})_{N_1} \hookrightarrow (\text{St}(m) \times (D^R_n(\nu^{1/2}))_{N_1},$$

where $l = l_{abs}(m)$. In a special case that the Zelevinsky multisegments parametrizing $\pi$ and $\pi'$ has all segments of relative length at least 2, combinatorics forces that $(\text{St}(n'), \text{St}(m), D^R_n(\nu^{1/2}))$ is a pre-RdLi-commutative triple. This is a starting point of this article.

The goal of this article is to study in details for the segment case. Extending the segment case would be better to be dealt with the concept of minimal multisegments [Ch22+, Ch22+b] and so will be postponed to [Ch22+d].

1.5. Duality. One of our main results in this article is the following duality:

**Theorem 1.6.** (=Theorem 2.10 + Corollary 9.13) Let $\sigma, \sigma'$ be essentially square-integrable representations in $\text{Irr}$. A triple $(\sigma, \sigma', \pi)$ is strongly RdLi-commutative (pre-RdLi-commutative) if and only if $(\sigma', \sigma, I^L_{\sigma} \circ D^R_{\sigma}(\pi))$ is strongly RdLi-commutative (resp. pre-LdRi-commutative).

Theorem 1.6 will be important in the study of quotient branching laws, for which we deduce a duality for the relevant pairs (see Corollary 12.5). This corresponds to the following dual restriction in branching law:

**Proposition 1.7.** (see [Ch22] Proposition 4.1) Let $\pi \in \text{Irr}(GL_{n+1}(F))$ and let $\pi' \in \text{Irr}(GL_{n}(F))$. Then there exists a cuspidal representation $\sigma$ of $GL_2(F)$ such that $\tau \times \pi'\vee$ is irreducible and

$$\text{Hom}_{GL_n(F)}(\pi, \pi') \cong \text{Hom}_{GL_{n+1}(F)}(\tau \times \pi'\vee, \pi'\vee).$$

One can utilize these two dualities with the Bernstein-Zelevinsky machinery to give some inductive proofs in [Ch22+d].
1.6. **Key ideas in the proof of Theorem 1.3** The proof relies on a special structure from the Jacquet functor, which we refer to as an irreducible pair. Roughly speaking, the structure is a direct summand in a Jacquet module so that the followings work:

- one can deduce the strong commutativity from the pre-commutativity involving an irreducible pair (Proposition 6.12(2));
- the irreducibility pair property is preserved under the integral in a strongly commutative triple (Proposition 6.11).

In order to apply the first bullet, we need to construct some irreducible pairs, that is Proposition 8.5. The proof is basically an analysis on the terms in the geometric lemma, but a key to rule out a possibility of a structure is a structure of a big derivative studied in [Ch24] (see Lemma 5.11).

These two properties are used to prove preservation of η-invariants under integrals and derivatives involved in some strongly commutative triples. Now, for a pre-RdLi-commutative triple \((\sigma_1, \sigma_2, \pi)\), one analyses the constituents on \(\sigma_1 \times \pi N_n(\sigma_1)\) and use the invariants to force the required embedding in the strong embedding has to happen.

1.7. **Key ideas in the proof of Theorem 1.6** Theorem 1.3 reduces proving Theorem 1.6 to proving the corresponding statement for pre-commutativity. The main idea of the proof is first constructing a certain irreducible pair to obtain some new pre-commutativity (Proposition 8.5), then apply the dual theory for such pre-commutativity (Proposition 7.2) and finally use a transitivity property (Proposition 6.10) to obtain the desired pre-commutativity.

1.8. **Key ideas in the proof of Theorem 1.5** For the if direction of Theorem 1.5, it is proved along the proof of Theorem 10.3 as well. The main technical ingredient is an analysis of the terms in the geometric lemma in Proposition 8.5.

For the only if direction of Theorem 1.5, the combinatorial criteria can be used to produce a strongly RdLi-commutative triple and so one applies Proposition 5.7 and the geometry in the geometric lemma to obtain the pre-commutativity. Now, one concludes the pre-commutativity of the original case by using the transitivity in Proposition 6.11 and then obtain the strong commutativity by Theorem 1.3.

1.9. **Structure of arguments.** We think it is helpful to give links on main results. Let \(\Delta, \Delta'\) be segments. Let \(\pi \in \text{Irr} \). Let \(p = m_{\Delta}(\pi)\) (see Section 5.2).

- **(PCSeg)** Pre-commutativity for \((\text{St}(\Delta), \text{St}(\Delta'), \pi)\)
- **(PCMulti)** Pre-commutativity for \((\text{St}(p), \text{St}(\Delta'), \pi)\)
- **(SCSeg)** Strong commutativity for \((\text{St}(\Delta), \text{St}(\Delta'), \pi)\)
- **(SCMulti)** Strong commutativity for \((\text{St}(p), \text{St}(\Delta'), \pi)\)
- **(CC)** Combinatorial commutativity for \((\Delta, \Delta', \pi)\)
- **(DualPCMulti)** Dual for pre-commutativity for \((\text{St}(\Delta'), \text{St}(p), I_{\Delta} \circ D_p(\pi))\)
- **(DualPCSeg)** Dual for pre-commutativity for \((\text{St}(\Delta'), \text{St}(\Delta), I_{\Delta} \circ D_\Delta(\pi))\)
- **(DualCC)** Dual for combinatorial commutativity
1.10. Applications. Other than the duality above, the commutation (Proposition 6.2) is also useful and is straightforward from Definition 1.1 (from representation-theoretic perspective). On the other hand, Definition 1.4 (from combinatorial perspective) is useful in proving properties for the generalized GGP relevance in Ch 22+a. Further applications need to combine with the study in Ch 22+b. For example, combining with results in a series of articles Ch 22+a, Ch 22+b, one can prove the uniqueness of the relevant pairs under some minimal condition in Ch 22+b.

1.11. Organizations. Section 3 studies some uniqueness of simple submodules/quotients from □-irreducible representations. Section 4 gives some preparation on the geometric lemma. Section 5 studies a structure arising from the Jacquet functor, which we refer to an irreducible pair. Section 6 studies our main notions: pre-commutativity and strong commutativity, and we show a transitivity property for strong commutativity. Section 7 studies a duality for strong commutativity. Section 8 constructs some pre-commutativity involving an irreducible pair. Section 9 proves that pre-commutativity implies strong commutativity for essentially square-integrable representations, using results in previous sections. Section 10 shows our main result on equivalent definitions in terms of duality in Section 7 as well as a combinatorial formulation. Section 11 studies some consequences from our results. Section 12 defines a notion of generalized GGP relevant pairs and gives some examples.

1.12. Acknowledgment. Part of results in this article was announced in the workshop of minimal representations and theta correspondence in the ESI at Vienna in April 2022. The author would like to thank the organizers, Wee Teck Gan, Marcela Hanzer, Alberto Mínguez, Goran Muić and Martin Weissman, for their kind invitation for giving a talk related to this work. This project is supported in part by the Research Grants Council of the Hong Kong Special Administrative Region, China (Project No: 17305223) and the National Natural Science Foundation of China (Project No. 12322120).

2. Preliminaries

For results involving derivatives and integrals, most of the time, we shall only prove results involving \(D_\sigma = D_R^\sigma\) and \(I_\sigma = I_L^\sigma\) for \(\sigma \in \text{Irr}^\square\). The analogous result that switches between \(D_R^\sigma\) and \(D_L^\sigma\), and between \(I_R^\sigma\) and \(I_L^\sigma\) can be proved similarly. From Sections 2.2 and 2.3 we give some results that one can deduce results from one version to another.
2.1. Second adjointness. Let $J$ be the matrix in $G_n$ with 1 in the anti-diagonal entries and 0 in other entries. Let $\theta = \theta_J : G_n \to G_n$, given by $\theta(g) = Jg^{-T}J$. Then $\theta$ also deduces an auto-equivalence, still denoted by $\theta$, on the category of smooth representations of $G_n$.

We shall use the following standard fact. The proof is similar to the one in [Ch22+] Proposition 2.1 and we only sketch it.

**Proposition 2.1.** Let $\pi_1$ and $\pi_2$ be smooth representations of $G_{n_1}$ and $G_{n_2}$ respectively. Let $\pi$ be a smooth representation of $G_{n_1+n_2}$. Then

$$\text{Hom}_{G_{n_1+n_2}}(\pi_1 \times \pi_2, \pi) \cong \text{Hom}_{G_{n_1} \times G_{n_2}}(\pi_2 \boxtimes \pi_1, \pi_{N_{n_1}}).$$

**Proof.** Define $\theta'(g_1, g_2) = (g_2, g_1)$. It follows from definitions that $\theta'(\pi_{n_2}) \cong (\theta(\pi))_{N_{n_2}(\pi_{n_1})}$, where $N^-_{n_2}$ is the unipotent subgroup in the parabolic subgroup opposite to $P_{n_1,n_2}$. Then

$$\text{Hom}_{G_{n_1+n_2}}(\theta(\pi_1) \times \theta(\pi_2), \theta(\pi)) \cong \text{Hom}_{G_{n_1} \times G_{n_2}}(\theta(\pi_1) \boxtimes \theta(\pi_2), \theta(\pi)_{N_{n_2}}) \cong \text{Hom}_{G_{n_2} \times G_{n_1}}(\theta(\pi_2) \boxtimes \theta(\pi_1), \theta(\pi)_{N_{n_2}(\pi_{n_1})}).$$

Since $\pi_1, \pi_2$ and $\pi$ are arbitrary and $\theta$ is a bijection, the above isomorphism implies the proposition. \qed

One consequence on the structure of Jacquet functors is as follows:

**Corollary 2.2.** Let $\sigma \in \text{Irr} \circledR$ and let $\pi \in \text{Irr}$. Suppose $D_\sigma(\pi) \neq 0$. Then $D_\sigma(\pi) \boxtimes \sigma$ appears in both submodule and quotient of $\pi_{N(\sigma)}$.

**Proof.** By definition, $\pi$ appears as a submodule in $D_\sigma(\pi) \times \sigma$. Then, by [LM19] Corollary 2.4, $\pi$ also appears as a quotient of $\sigma \times D_\sigma(\pi)$. Now the statement follows from Proposition 2.1. \qed

2.2. Switching between left and right versions (parabolic inductions). By (2.4) and the fact that $\theta$ preserves the irreducibility, $\pi$ is $\Box$-irreducible if and only if $\theta(\pi)$ is $\Box$-irreducible. Moreover, it is a simple fact that for a cuspidal representation (resp. essentially square-integrable representation) $\rho$ of $G_n$, $\theta(\rho)$ is still cuspidal (resp. essentially square-integrable).

Using $\theta(P_{n_1,n_2}) = P_{n_2,n_1}$, we deduce that

$$\theta(\pi_1 \times \pi_2) \cong \theta(\pi_2) \times \theta(\pi_1). \tag{2.4}$$

**Lemma 2.3.** Let $\sigma \in \text{Irr} \circledR$. Let $\pi \in \text{Irr}$. Then

$$\theta(I^L_{\sigma}(\pi)) \cong I^R_{\theta(\sigma)}(\theta(\pi)), \quad \theta(I^R_{\sigma}(\pi)) \cong I^L_{\theta(\sigma)}(\theta(\pi)).$$

**Proof.** By definition, we have $I^L_{\sigma}(\pi) \hookrightarrow \sigma \times \pi$. Hence, by applying $\theta$, we have:

$$\theta(I^L_{\sigma}(\pi)) \hookrightarrow \theta(\sigma) \times \theta(\pi) \cong \theta(\pi) \times \theta(\sigma).$$

Hence, by definition, we have:

$$\theta(I^L_{\sigma}(\pi)) \cong I^R_{\theta(\sigma)}(\theta(\pi)).$$

\qed

2.3. Switching between left and right versions (Jacquet functors).

**Lemma 2.4.** Let $\pi_1$ and $\pi_2$ be smooth representations of $G_{n_1}$ and $G_{n_2}$ respectively. Let $\pi$ be a smooth representation of $G_{n_1+n_2}$. Then

$$\text{Hom}_{G_{n_1} \times G_{n_2}}(\pi_1 \boxtimes \pi_2, \pi_{N_{n_2}n_1}) \cong \text{Hom}_{G_{n_2} \times G_{n_1}}(\theta(\pi_2) \boxtimes \theta(\pi_1), \theta(\pi)_{N_{n_1}n_2}).$$
Proof. Let \(w_1\) and \(w_2\) be elements in \(G_{n_1}\) and \(G_{n_2}\) with 1 in anti-diagonal entries and 0 elsewhere. Given a map \(f \in \text{Hom}_{G_{n_2} \times G_{n_3}}(\theta(\pi_2) \boxtimes \theta(\pi_1), \theta(\pi)_{N_{n_1}})\), we define a map \(f'\) in \(\text{Hom}_{G_{n_1} \times G_{n_2}}(\pi_1 \boxtimes \pi_2, \pi_{N_{n_2}})\) determined by: for \(x_1 \in \pi_1\) and \(x_2 \in \pi_2\),
\[
f'(x_1 \otimes x_2) = f((w_2, x_2) \otimes (w_1, x_1)).
\]
Then, it is straightforward to check that \(f'\) is \(G_{n_1} \times G_{n_2}\)-equivariant. The inverse map can be similarly defined. \(\square\)

Lemma 2.5. Let \(\sigma \in \text{Irr}^{\square}\). Let \(\pi \in \text{Irr}(G_n)\). Then
\[
\theta(D^{L}_{\sigma}(\pi)) \cong D^{R}_{\theta(\sigma)}(\theta(\pi)), \quad \theta(D^{R}_{\sigma}(\pi)) \cong D^{L}_{\theta(\sigma)}(\theta(\pi)).
\]
Proof. Let \(m = n(\sigma)\). By definition, we have:
\[
D^{R}_{\sigma}(\pi) \boxtimes \sigma \hookrightarrow \pi_{N_m}.
\]
Then, by Lemma 2.4, we have:
\[
\theta(\sigma) \boxtimes \theta(D^{R}_{\sigma}(\pi)) \hookrightarrow \theta(\pi)_{N_{n-m}}.
\]
Then, by definition, we have \(\theta(D^{R}_{\sigma}(\pi)) \cong D^{L}_{\theta(\sigma)}(\theta(\pi))\). \(\square\)

2.4. Switching left and right versions by using duals. We again have the standard fact that an irreducible representation \(\sigma\) of \(G_n\) is \(\square\)-irreducible if and only if \(\sigma^\vee\) is \(\square\)-irreducible. Hence, for \(\sigma \in \text{Irr}^{\square}\), \(I_{\sigma}^{L}\) and \(I_{\sigma}^{R}\) are well-defined.

Lemma 2.6. Let \(\sigma \in \text{Irr}^{\square}\). Let \(\tau \in \text{Irr}\). Then \(I_{\sigma}^{L}(\tau)^\vee \cong I_{\sigma^\vee}^{R}(\tau^\vee)\).
Proof. By definition, we have:
\[
I_{\sigma}^{L}(\pi) \hookrightarrow \sigma \times \pi.
\]
Hence, we have a surjection from \(\sigma^\vee \times \pi^\vee \cong (\sigma \times \pi)^\vee\) to \(I_{\sigma}^{L}(\pi)^\vee\). By [LM19, Corollary 2.4], \(I_{\sigma}^{L}(\pi)^\vee\) embeds to \(\pi^\vee \times \sigma^\vee\). Thus, by definition, we have \(I_{\sigma}^{L}(\pi)^\vee \cong I_{\sigma^\vee}^{R}(\tau^\vee)\). \(\square\)

Corollary 2.7. Let \(\sigma \in \text{Irr}^{\square}\). Let \(\pi \in \text{Irr}\). Suppose \(D^{L}_{\sigma}(\pi) \neq 0\). Then \(D^{L}_{\sigma}(\pi)^\vee \cong D^{R}_{\sigma^\vee}(\tau^\vee)\).
Proof. Let \(\tau = D^{L}_{\sigma}(\pi)\). Then \(I_{\sigma}^{L}(\tau) \cong \pi\). Hence, \((I_{\sigma}^{L}(\tau))^\vee \cong I_{\sigma^\vee}^{R}(\tau^\vee)\). Hence, we have \(\pi^\vee \cong I_{\sigma^\vee}^{R}(\tau^\vee)\). Applying \(D^{L}_{\sigma^\vee}\), we have \(D^{R}_{\sigma^\vee}(\tau^\vee) \cong (D^{R}_{\sigma}(\pi))^\vee\). \(\square\)

3. Some uniqueness from \(\square\)-irreducible representations

In this section, we shall first deduce some uniqueness property from [KKKO15] or [LM19]. For \(\pi \in \text{Alg}(G_n)\), let \(n(\pi) = n\).

3.1. Uniqueness. We remark that via Frobenius reciprocity, we have \(\pi \hookrightarrow D_{\sigma}(\pi) \times \sigma\) (resp. \(\pi \hookrightarrow \sigma \times D^{L}_{\sigma}(\pi)\)) when \(D_{\sigma}(\pi) \neq 0\) (resp. \(D^{L}_{\sigma}(\pi) \neq 0\)). We shall frequently use these facts.

Earlier forms of derivatives for cuspidal representations go back to the work of Jantzen [Ja07] and Mínguez [Mi09].

Lemma 3.1. Let \(\sigma_1, \sigma_2, \ldots, \sigma_r \in \text{Irr}^{\square}\) such that \(\sigma_1 \times \ldots \times \sigma_r\) is still \(\square\)-irreducible. Let \(\pi \in \text{Irr}\). Then there exists at most one \(\omega \in \text{Irr}\) such that
\[
\omega \boxtimes \sigma_r \boxtimes \ldots \boxtimes \sigma_1 \hookrightarrow \pi_N,
\]
where \(N = N_{n(\pi) - n(\sigma_1) - \ldots - n(\sigma_r), n(\sigma_r), \ldots, n(\sigma_1)}\). Moreover, if such \(\omega\) exists, the embedding is unique.
where \( G' = G_{n-n} \times G_{n(\sigma_1)} \times \ldots \times G_{n(\sigma_r)} \) and \( G'' = G_{n-n'} \times G_{n'} \). Note that the RHS term above is further isomorphic to
\[
\Hom_{G_n}((\sigma_r \times \ldots \times \sigma_1) \times \omega, \pi)
\]
via Frobenius reciprocity and so has the multiplicity at most one by [LM19, Corollary 2.4 and Lemma 2.8]. Thus we also have the LHS term above has the multiplicity at most one, as desired.

**Lemma 3.2.** Let \( \pi \in \Irr(G_n) \). Let \( \sigma_1 \in \Irr(G_k) \) and let \( \sigma_2 \in \Irr(G_l) \). Let \( \tau = I^R_{\sigma_1} \circ I^L_{\sigma_2}(\pi) \) or \( I^L_{\sigma_2} \circ I^R_{\sigma_1}(\pi) \). Then
\[
\dim \Hom_{G_{n+k+l}}(\tau, \sigma_2 \times \pi \times \sigma_1) = 1.
\]

**Proof.** We only consider \( \tau = I^L_{\sigma_2} \circ I^R_{\sigma_1}(\pi) \) and the other one is similar. Note that \( \tau \) has to be an irreducible submodule of \( \sigma_2 \times \tau' \) for a simple composition factor \( \tau' \) in \( \pi \times \sigma_1 \). But \( \tau \mapsto \sigma_2 \times \tau' \) implies \( \tau \cong I^L_{\sigma_2}(\tau') \) and so \( \tau' = I^R_{\sigma_1}(\pi) \). Since \( I^L_{\sigma_2}(\pi) \) appears with multiplicity one in \( \pi \times \sigma_1 \) (KKKO15, Theorem 3.2, LM19, Lemma 2.8), we then have the unique embedding from \( I^L_{\sigma_2} \circ I^R_{\sigma_1}(\pi) \) to \( \sigma_2 \times \pi \times \sigma_1 \) by LM19, Lemma 2.8.

**Lemma 3.3.** Let \( \sigma_1, \sigma_2 \in \Irr(G_n) \). Let \( \tau = D^L_{\sigma_1} \circ D^R_{\sigma_1}(\pi) \) or \( D^R_{\sigma_1} \circ D^L_{\sigma_2}(\pi) \), and assume it is non-zero. Then
\[
\dim \Hom_{G_{n+1} \times G_{n'}}(\sigma_2 \boxtimes \tau \boxtimes \sigma_1, \pi_N) = 1,
\]
where \( n_1 = n(\sigma_1) \), \( n_2 = n(\sigma_2) \), \( n' = n - n_1 - n_2 \) and \( N = N_{n_2,n',n_1} \).

**Proof.** By applying Bernstein's second adjointness theorem, we have:
\[
\Hom_{G_{n+1} \times G_{n'} \times G_{n_1}}(\sigma_2 \boxtimes \tau \boxtimes \sigma_1, \pi_N) \cong \Hom_{G_n}(\sigma_1 \times \tau \times \sigma_2, \pi).
\]
By taking the dual, it is equivalent to show:
\[
\dim \Hom_{G_n}(\pi^\vee, \sigma_1^\vee \times \tau^\vee \times \sigma_2^\vee) = 1
\]
Suppose we are in the case that \( \tau = D^R_{\sigma_1} \circ D^L_{\sigma_1}(\pi) \). By Lemma 2.6 and Corollary 2.7, we have that \( I^R_{\sigma_1}(\tau^\vee) \cong I^L_{\sigma_1}(\pi)^\vee \cong D^L_{\sigma_2}(\tau^\vee) \). Hence, \( I^R_{\sigma_1} \circ I^L_{\sigma_1}(\tau^\vee) \cong \pi^\vee \). Then the dimension follows from Lemma 3.2.

4. Some geometry of orbits in the geometric lemma

Note that the uniqueness statements in Section 3 are up to a scalar. This is good for our purposes since we only concern about the image of the maps (see e.g. Definition 1.1). By abuse of terminology, we say that a diagram is commutative if the images of any composition of maps starting from the same representation and ending at the same representation coincide.
4.1. General notions for supporting orbits. Let $G$ be a connected reductive group over a non-Archimedean local field. Although we mainly consider $G_n$, we need slightly more general setting such as $G_{n_1} \times G_{n_2}$ in our discussions and so we consider a general setting.

We fix a minimal parabolic subgroup of $G$ and so fix root system for $G$. For a standard parabolic subgroup $P$ in $G$, denote by $M_P N_P$ the Levi decomposition of $P$, with $M_P$ to be Levi and $N_P$ to be unipotent, and let $\pi$ be a smooth representation of $M_P$, inflated to a $P$-representation. Let $W$ be the Weyl group of $G$. Let $l : W \to \mathbb{Z}_{\geq 0}$ be the length function. Denote by $\text{Ind}_{G}^{P} \pi$ the normalized parabolic induction from $\pi$. Recall that the underlying space for $\text{Ind}_{G}^{P} \pi$, which we shall also denote by $C^{\infty}(P \setminus G, \pi)$, is the space of smooth functions from $G$ to $\pi$ satisfying

$$f(pg) = \delta_P(p)^{1/2} f(g)$$

for any $p \in P$, where $\delta_P$ is the modular character of $P$. Let $Q$ be another parabolic subgroup of $G$ with the Levi decomposition $M_Q N_Q$. Let $W(P)$ and $W(Q)$ be the associated Weyl groups for $P$ and $Q$ respectively. Let $W_{P,Q} = W_{P,Q} (G)$ be the set of minimal representatives for the double cosets in $W(P) \setminus W(Q)$.

We shall enumerate elements in $W_{P,Q}$ labelled as $w_1, \ldots, w_r$ such that $i < j$ implies $w_i \not\geq w_j$. In particular, $w_1$ is the trivial element. The geometric lemma ([BZ77, Remarks 5.5] and [BZ77 Proposition 5.11]) asserts that $C^{\infty}(P \setminus G, \pi)_{N_Q}$ admits a filtration

$$0 \subset J_r \subset \ldots \subset J_1 = C^{\infty}(P \setminus G, \pi)_{N_Q}$$

such that

$$(4.5) \quad J_i / J_{i+1} \cong \text{Ind}_{M_P \cap N_Q}^{M} ((\pi_{M_P \cap N_Q})^{w_i}),$$

where

- $P^w = \hat{w}_i^{-1} P \hat{w}_i, N_Q^w = \hat{w}_i^{-1} N_Q \hat{w}_i$ for some representative $\hat{w}_i$ of $w_i$ in $G$;
- $(\pi_{M_P \cap N_Q})^w$ is a $M_P^{w^{-1}} \cap M_Q$-representation, which $m$ acts on the space by the action $\hat{w}_i^{-1} m \hat{w}_i$ for $m \in M_P^{w^{-1}} \cap M_Q$.

The isomorphism in (4.5), denoted by $\Phi_i$, is given as follows. For $f \in J_i$ and $m \in M_Q$,

$$\Phi_i(f)(m) = \int_{N'} (pr \circ \tilde{f})(w m m) \ dN,$$

where $\tilde{f}$ is a representative of $f$ in $C^{\infty}(P \setminus G, \pi); N' = w_i (N^\circ) \cap N_Q$; $dn$ is a Haar measure on $N'$ and $pr$ is the projection from $\pi$ to $\pi_{M_P \cap N_Q^w}$.

We say that the embedding $\lambda \mapsto (\text{Ind}_{P}^{G} \pi)_{N_Q}$ has the supporting orbit of the form $P \hat{w}_i Q$ if $\lambda \cap J_{i+1} = 0, \lambda \cap J_i \neq 0$. We say that the embedding $\lambda \mapsto (\text{Ind}_{P}^{G} \pi)_{N_Q}$ has the trivial supporting orbit if $\lambda \cap J_2 = 0$.

The trivial supporting orbit case is the most interesting case for us and is independent of a choice of an enumeration. The notion depends on the term $(\text{Ind}_{P}^{G} \pi)_{N_Q}$ which the geometric lemma carries for, and it should be clear for the context. For example, for $\pi_1 \in \text{Irr}(G_{n_1})$, $\pi_2 \in \text{Irr}(G_{n_2})$, $(\sigma_1 \times \sigma_2)_{N_m}$ means to consider the orbits $P_{n_1, n_2} \setminus G_{n_1+n_2} / Q_{n_1+n_2-m,m}$.

4.2. Geometric lemma. We shall now introduce more product notations. For a $G_{n_1} \times G_{n_2}$-representation $\pi$ and a $G_m$-representation $\tau$, inflate $\tau \boxtimes \pi$ to a $P_{m,n_1} \times G_{n_2}$-representation. Define

$$(4.6) \quad \tau \boxtimes 1 \pi := \text{Ind}_{P_{m,n_1} \times G_{n_2}}^{G_{m,n_1+n_2} \times G_{n_2}} \tau \boxtimes \pi.$$
Similarly, define
\[
\tau \times^2 \pi := \text{Ind}^{G_{n_1} \times G_{m+n_2}}_{G_{n_1} \times P_{m,n_2}} \tau \boxtimes \pi,
\]
where the Levi part $G_{n_1} \times G_m \times G_{n_2} \subset G_{n_1} \times P_{m,n_2} \subset G_{n_1} \times G_{m+n_2}$ acts on $\tau \boxtimes \pi$ by
\[
(g_1, g_2, g_3). (v_1 \boxtimes v_2) = g_2.v_1 \boxtimes (g_1, g_3).v_2
\]
and the unipotent part acts trivially.

Let $\pi$ and $\pi'$ be representations of $G_{n_1}$ and $G_{n_2}$ respectively. Then the geometric lemma asserts that $(\pi \times \pi')_{N_i}$ admits a filtration with layers of the form:
\[
\text{Ind}^{G_{n_1} \times i \times G_i}_{P_{n_1-i, n_2-i} \times P_{i,1} \times G_i} (\pi_{N_i} \boxtimes \pi'_{N_i})^\phi,
\]
where $\phi$ is a natural twist sending $G_{n_1-i} \times G_i \times G_{n_2-i} \times G_{i_2}$-representations to $G_{n_1-i} \times G_{n_2-i} \times G_i \times G_{i_2}$-representations. When $i \leq n_2$, the top layer is isomorphic to $\pi \times (\pi')_{N_i}$, which will be used often when involving Definition 11.1.

5. Irreducible pairs under derivatives

5.1. $\eta$-invariants. To discuss an important class of Rd-irreducible pairs in Definition 5.2, we need more notations. Recall that $\varepsilon$ and $\eta$ are defined in Section 1.3.

Definition 5.1. Let $\Delta = [a, b]_\rho$ be a segment. Let $\pi \in \text{Irr}$.

- We write $\eta_{\Delta}(\pi) = 0$ if $\varepsilon_{[c, b], \rho}(\pi) = 0$ for all $c = a, \ldots, b$. Similarly, we write $\eta_{\Delta}(\pi) \neq 0$ if $\varepsilon_{[c, b], \rho}(\pi)$ for some $c = a, \ldots, b$.
- We write $\eta_{\Delta}(\pi)$ if $\eta_{\Delta}(\pi')$ for $? \in \{=, \leq, <, >\}$ if $\varepsilon_{[c, b], \rho}(\pi) \varepsilon_{[c, b], \rho}(\pi')$ for all $c$ satisfying $a \leq c \leq b$.
- We similarly define the left version terminologies for $\varepsilon_{\Delta}^L$ and $\eta_{\Delta}^L$ if one uses the left derivatives $D_{\Delta}^L$ (also see (5.10) below).

5.2. Multisegment counterpart of the $\eta$-invariant. Following [Ze80], a multisegment is a multiset of non-empty segments. The irreducible representations of $G_m$ are parametrized by multisegments, see work of Zelevinsky, Tadić and Mínguez-Sécherre [Ze80, Ta90, MS14].

Two segments $\Delta_1, \Delta_2$ are linked if $\Delta_1 \not\subset \Delta_2$ and $\Delta_2 \not\subset \Delta_1$ and $\Delta_1 \cup \Delta_2$ is still a segment. Otherwise, we say that $\Delta_1$ and $\Delta_2$ are unlinked. A multisegment $\mathfrak{m}$ is said to be pairwise unlinked if any two segments $\Delta_1$ and $\Delta_2$ in $\mathfrak{m}$ are unlinked. A main property for unlinked segments $\Delta_1$ and $\Delta_2$ is the following [Ze80, Ta90]:
\[
\text{St}(\Delta_1) \times \text{St}(\Delta_2) \cong \text{St}(\Delta_2) \times \text{St}(\Delta_1).
\]
For a pairwise unlinked multisegment $\mathfrak{m} = \{\Delta_1, \ldots, \Delta_r\}$, let
\[
\text{St}(\mathfrak{m}) = \text{St}(\Delta_1) \times \ldots \times \text{St}(\Delta_r),
\]
which is irreducible and $\square$-irreducible, and is independent of the ordering by [Ze80, Ta90], and we write:
\[
D_{\mathfrak{m}}(\pi) = D_{\text{St}(\mathfrak{m})}(\pi).
\]
For a segment $\Delta = [a, b]_\rho$, a segment $[a', b']_\rho$ is said to be $\Delta$-saturated if $a' \leq a$ and $b' = b$. A multisegment $\mathfrak{m}$ is said to be $\Delta$-saturated if all segments in $\mathfrak{m}$ are $\Delta$-saturated. As we shall see later, $\Delta$-saturated segments can be used to produce more strongly commutative triples (see Proposition 5.3 and Corollary 11.1).

For $\pi \in \text{Irr}$ and a segment $\Delta = [a, b]_\rho$, define
\[
m_{\mathfrak{m}}_{\Delta}(\pi) := m_{\mathfrak{m}}^\Delta(\pi) := \sum_{k=0}^{b-a} \varepsilon_{[a+k, b], \rho}(\pi) \cdot [a + k, b]_\rho,
\]
Proof.

Suppose

Hence, coincides with \( i \). Then it follows from the uniqueness that the embedding from the summand to \( I \) is the unique non-zero map and let \( i \). Proposition 5.4.

\[ \Box \]

3.1) is a direct summand in \( \pi \). Let

For the if direction, it is a simple application of the geometric lemma, which is shown in [Ch24, Proposition 11.1].

We now prove the converse direction. Suppose \( q \in \Pi \). Let \( q' = \text{St}(\pi) \). If \( (St(q), \pi) \) is still a Rd-irreducible pair, then \( D_q(\pi) \) is an indecomposable summand in \( \pi_{N_m} \). But, from the if part, \( D_q(\pi) \) is an indecomposable summand in \( \pi_{N_m} \). From the transitivity of Jacquet functors, we then have that

\[ D_q(\pi) \ast St(q') \ast St(q) \]

is a direct summand in \( \pi_{N_{m'-m',m'}} \). On the other hand, let \( \bar{\pi} = St(\text{mr}_\Delta(\pi)) \). By the if direction which is just proved,

\[ D_{\bar{\sigma}}(\pi) \ast \bar{\pi} \]

is a direct summand in \( \pi_{N_{m'}} \). Note that \( St(q') \ast St(q) \) is not a direct summand in \( \bar{\pi}_{N_{m'}} \). Hence, the transitivity of Jacquet functors implies that the unique submodule (see Lemma 3.1)

\[ D_{\bar{\sigma}}(\pi) \ast St(q') \ast St(q) \]

does not form an indecomposable summand in \( \pi_{N_{m'-m',m'}} \). Thus we obtain a contradiction.

For applications, we shall use the following reformulation:

Proposition 5.4. Let \( \sigma \in \text{Irr}(G_{n_1}) \) and let \( \pi \in \text{Irr}(G_{n_2}) \). Let \( i_1 : \sigma \ast \pi \rightarrow I_\sigma(\pi)_{N_{n_2}} \) be the unique non-zero map and let \( i_2 : I_\sigma(\pi)_{N_{n_2}} \rightarrow \sigma \ast \pi \) be the unique non-zero map. Then \( (\sigma, \pi) \) is Rd-irreducible if and only if \( i_2 \circ i_1 \neq 0 \).

Proof. Suppose \( i_2 \circ i_1 \neq 0 \). Then the map \( i_1 \) splits and so \( \sigma \ast \pi \) is a direct summand in \( I_\sigma(\pi)_{N_{n_2}} \). We now prove the converse direction. Suppose \( \sigma \ast \pi \) is a direct summand in \( I_\sigma(\pi)_{N_{n_2}} \). Then it follows from the uniqueness that the embedding from the summand to \( I_\sigma(\pi)_{N_{n_2}} \) coincides with \( i_1 \) and the surjection from \( I_\sigma(\pi)_{N_{n_2}} \) to the summand coincides with \( i_1 \). Hence, \( i_2 \circ i_1 \) is a map multiplied by a non-zero scalar and in particular is non-zero. \( \Box \)
5.4. Trivial supporting orbit for $I_{\sigma}(\pi) \hookrightarrow \sigma \times \pi$. The following Proposition 5.6 is one important property for irreducible pairs and also is one key in proving equivalent definitions in Theorem 10.3.

**Lemma 5.5.** Let $\sigma$ be a smooth representation of $G_{n_1}$ and let $\pi$ be a smooth representation of $G_{n_2}$. Let $q : \sigma \times \pi = C^\infty(P_{n_1,n_2} \setminus G_{n_1+n_2}, \sigma \boxtimes \pi) \rightarrow C^\infty(P_{n_1,n_2} \setminus P_{n_1,n_2}, \sigma \boxtimes \pi)$ be the natural projection map. Let $\omega$ be a smooth representation of $G_{n_1+n_2}$ such that there is a non-zero composition $\omega \rightarrow \sigma \times \pi$. Then $q \circ i \neq 0$.

**Proof.** Let $n = n_1 + n_2$. By applying Frobenius reciprocity on the map $\omega \rightarrow \sigma \times \pi$, we also have a non-zero map $\omega \rightarrow \sigma \boxtimes \pi$. Then $i(x)(g) = i'(g.x)$ for $x \in \omega$ and $g \in G_n$, where we regard $g.x$ as an element in $\omega$ via the natural projection. We can choose $x^* \in \omega$ such that $i'(x^*) \neq 0$. Then $i(x^*)$ (viewed as a function from $G_n$ to $\sigma \boxtimes \pi$) has a support containing 1. This implies that $q \circ i \neq 0$. \hfill \Box

**Proposition 5.6.** Let $\sigma \in \text{Irr}^{\square}$ and let $\pi \in \text{Irr}$. Then the embedding

$$\sigma \boxtimes \pi \hookrightarrow I_{\sigma}(\pi)_{N(\pi)} \hookrightarrow (\sigma \times \pi)_{N(\pi)}$$

has the trivial supporting orbit if and only if $(\sigma, I_{\sigma}(\pi))$ is a $Ld$-irreducible pair.

**Proof.** Let $P = P_{n(\sigma), n(\pi)}$ and let $N = N_{n(\sigma), n(\pi)}$. By Lemma 5.5, the unique map from $I_{\sigma}(\pi)_N$ to $\sigma \boxtimes \pi$ (arising from Frobenius reciprocity) factors through the quotient map $(\sigma \times \pi)_N \rightarrow \sigma \boxtimes \pi$ in the geometric lemma.

Suppose $(\sigma, I_{\sigma}(\pi))$ is a $Ld$-irreducible pair. By Proposition 5.4, the composition of the unique embedding $\sigma \boxtimes \pi \rightarrow I_{\sigma}(\pi)_N$ and the quotient map above is non-zero, giving the trivial supporting orbit statement.

For the converse, the definition of the trivial supporting orbit implies that we have a non-zero composition:

$$\sigma \boxtimes \pi \hookrightarrow I_{\sigma}(\pi)_N \hookrightarrow (\sigma \times \pi)_N \rightarrow \sigma \boxtimes \pi.$$

Then, the Rd-irreducibility condition follows from Proposition 5.4 again. \hfill \Box

5.5. Some variants. We shall also need the following variation:

**Proposition 5.7.** Let $\sigma \in \text{Irr}^{\square}(G_{n_1})$. Let $\omega \in \text{Irr}$ such that $(\sigma, \omega)$ is $Ld$-irreducible. Suppose $\omega \rightarrow \sigma \times \pi$ for some smooth (not necessarily irreducible) representation $\pi$. Then, the embedding

$$\sigma \boxtimes D_{\sigma}^L(\omega) \hookrightarrow \omega_N \hookrightarrow (\sigma \times \pi)_N,$$

where $N = N_{n(\sigma), n-n(\sigma)}$, has the trivial supporting orbit.

**Proof.** It follows from the argument in Proposition 5.6 that the composition

$$\omega_N \hookrightarrow (\sigma \times \pi)_N \rightarrow \sigma \boxtimes \pi$$

is non-zero (but it is not necessarily surjective since $\pi$ is not necessarily irreducible). Thus, $\omega_N$ has a quotient of the form $\sigma \boxtimes \tilde{\pi}$ for some submodule $\tilde{\pi}$ of $\pi$. By the condition of $Ld$-irreducibility, the unique simple submodule $\sigma \boxtimes D_{\sigma}^L(\omega)$ coincides with the unique simple quotient $\sigma \boxtimes D_{\sigma}^L(\omega)$. Thus, we must have that $\tilde{\pi}$ coincides with $\pi$ and the above compositions are non-zero. Thus the supporting orbit for the embedding in the lemma has the trivial supporting orbit. \hfill \Box

We prove a situation of the trivial supporting orbit that will be used later.
Corollary 5.8. Let $\Delta = [a, b]_\rho$ and let $\overline{\Delta} = [a', b]_\rho$ for some $a' > a$. Suppose we have

$$\text{St}(\Delta) \hookrightarrow \text{St}(\overline{\Delta}) \times \pi.$$ 

Then the induced embedding $\text{St}(\overline{\Delta}) \boxtimes \text{St}([a, a'-1]_\rho)$ to $(\text{St}(\overline{\Delta}) \times \pi)_N$ has the trivial supporting orbit.

Proof. The Jacquet functor on $\text{St}(\overline{\Delta})$ is semisimple and in particular this implies that $(\text{St}(\overline{\Delta}), \text{St}(\Delta))$ is $\text{Ld}$-irreducible. Now the trivial supporting orbit statement follows from Proposition 5.7. 

5.6. Converse of Proposition 5.7. We prove the converse of Proposition 5.7 for a special case in Lemma 5.12 which relies on a property shown in Lemma 5.2 below. Some arguments appear in [Ch24], Section 9.4 and we reproduce and slightly generalize for the convenience of the reader. Lemma 5.12 will be used to show certain embedding cannot have the trivial supporting orbit in Lemma 5.2 below.

For $\pi \in \text{Irr}$, there exists a unique multiset of cuspidal representations $\rho_1, \ldots, \rho_k$ such that $\pi$ is a simple composition factor in $\rho_1 \times \ldots \times \rho_k$. We denote such multiset by $\text{csupp}(\pi)$, called the cuspidal support of $\pi$.

The Krull-Schmidt theorem asserts that any smooth representation of $G_n$ of finite length can be uniquely written as a direct sum of indecomposable representations. For $\pi \in \text{Irr}$ and $\sigma \in \text{Irr}^\dagger$ with $D_\sigma(\pi) \neq 0$, since

$$\text{Hom}_{G_n(\sigma) \times G_n(\rho)}(D_\sigma(\pi) \boxtimes \sigma, \pi_{N_n(\sigma)}) \cong \mathbb{C},$$

there is a unique indecomposable summand $\kappa$ in $\pi_{N_n(\sigma)}$ containing $D_\sigma(\pi) \boxtimes \sigma$ as a submodule. We remark that we do not know whether the embedding $\kappa$ to $\pi_{N_n(\sigma)}$ is unique in general.

Lemma 5.9. Let $\omega = \text{St}(m)$ for some pairwise unlinked multisegment $m$. Let $\sigma = \text{St}(m') \in \text{Irr}$ for some pairwise unlinked multisegment $m'$ such that $D^L_\sigma(\omega) \neq 0$. Let $\kappa$ be the indecomposable component in $\omega_{N_l}(l = l_{abs}(m) - l_{abs}(m'))$ containing $\sigma \boxtimes D^L_\sigma(\omega)$ as a submodule. Then $\kappa$ has unique irreducible submodule and unique irreducible quotient and both are isomorphic to $\sigma \boxtimes D^L_\sigma(\omega)$. Furthermore, $D^L_\sigma(\omega) \cong \text{St}(n)$ for some pairwise unlinked multisegment $n$.

Proof. We consider the component $\kappa$ of $\pi_{N_n(\sigma)}$, which has cuspidal support $\text{csupp}(\sigma), \text{csupp}(\pi) - \text{csupp}(\pi)$, as shown in [Ch22], Lemma 10.14 (which is stated for $D = F$ case, but the argument works for general $D$, also see [Ch21], Corollary 2.6). Any simple submodule of $\kappa$ is isomorphic to $\text{St}(n') \boxtimes \text{St}(n'')$ for some pairwise unlinked multisegment $n', n''$ and so $\dim \text{Hom}_{G_{n-n(\sigma)\times G_n(\sigma)}}(\sigma \boxtimes D^L_\sigma(\omega), \pi_{N_n(\sigma)}) \leq 1$ implies that $\kappa$ has unique submodule. Similar argument (or by taking dual) gives that $\kappa$ has a unique simple quotient, and the simple quotient is of the form. This implies the lemma. 

For $\pi \in \text{Alg}(G_n)$ and $\sigma \in \text{Irr}(G_k)$, define:

$$\mathbb{D}_\sigma(\pi) = \text{Hom}_{G_k}(\sigma, \pi_{N_k}),$$

where $\pi_{N_k}$ is regarded as a $G_k$-module via the embedding to the second factor in $G_{n-k} \times G_k$. Moreover, $\mathbb{D}_\sigma(\pi)$ is viewed as a $G_{n-k}$-module via the map

$$(g, f)(v) = \text{diag}(g, I_k). f(v).$$

It is called a big derivative in [Ch24]. For a segment $\Delta$, we usually write $\mathbb{D}_\Delta(\pi)$ for $\mathbb{D}_{\text{St}(\Delta)}(\pi)$. The left version of the big derivative $D^L_\sigma(\pi)$ is similarly defined by using

$$\mathbb{D}^L_\sigma(\pi) = \text{Hom}_{G_k}(\sigma, \pi_{N_n-k}),$$
where $\pi_{n-k}$ is regarded as a $G_k$-module via the embedding to the first factor in $G_k \times G_{n-k}$. Again $\mathbb{D}^L_{\sigma}(\pi)$ has a natural $G_{n-k}$-module structure.

**Lemma 5.10.** Let $\sigma = \text{St}(m) \in \text{Irr}(G_k)$ for some pairwise unlinked multisegment $m$ and let $\pi = \text{St}(n)$ for some pairwise unlinked multisegment $n$. Suppose $D^L_{\sigma}(\pi) \neq 0$. Then $\mathbb{D}^L_{\sigma}(\pi)$ satisfies the SI property i.e. the socle of $\mathbb{D}^L_{\sigma}(\pi)$ is irreducible and appears with multiplicity one in the Jordan-Hölder series of $\mathbb{D}^L_{\sigma}(\pi)$.

**Proof.** In the left version of Lemma 5.10, we have shown the socle irreducible property for $\mathbb{D}^L_{\sigma}(\text{St}(m))$ (i.e. the socle of $\mathbb{D}^L_{\sigma}(\text{St}(m))$ is irreducible and appears with multiplicity one in the Jordan-Hölder series of $\mathbb{D}^L_{\sigma}(\text{St}(m))$). On the other hand, the hypothesis and Lemma 5.13 imply that $\kappa$ contains the factor $\sigma \boxtimes D^L_{\sigma}(\text{St}(m))$ with multiplicity at least 2. This implies the non-isomorphism.

**Lemma 5.11.** (c.f. Section 9.4) Let $\sigma = \text{St}(m) \in \text{Irr}$ for some pairwise unlinked multisegment $m$. Let $\omega$ be an irreducible submodule of $\sigma \times \pi$ for some $\pi \in \text{Alg}(G_n)$. Suppose $\omega \cong \text{St}(n)$ for some pairwise unlinked multisegment $n$. Suppose $(\sigma, \pi)$ is not Rd-irreducible. Then the embedding

$$\sigma \boxtimes D^L_{\sigma}(\omega) \hookrightarrow \omega_{n_n} \hookrightarrow (\sigma \times \pi)_{n_n}$$

does not have the trivial supporting orbit.

**Proof.** Let $\kappa$ be the indecomposable component of $\pi_{n_n(\sigma)}$ which contains $\sigma \boxtimes D^L_{\sigma}(\pi)$ as a submodule. Suppose $\pi \boxtimes \sigma$ has the trivial supporting orbit via the embedding, and we shall derive a contradiction. Then by Lemma 5.13 $\kappa \hookrightarrow \sigma \boxtimes \tau$. Thus, $\kappa \cong \sigma \boxtimes \tau'$ for some submodule $\tau'$ of $\tau$. This contradicts to Lemma 5.11.

6. **Strong commutation between derivatives and integrals**

Recall that the notions of pre-commutativity and strong commutativity are defined in Definition 1.1. Section 6.2 will prove the commutativity property and Section 6.3 will study compositions of strong commutativity. The composition is useful when $\sigma, \sigma'$ are $\square$-irreducible and $\sigma \times \sigma'$ is also $\square$-irreducible, and one would like to show the strong commutation involving $\sigma \times \sigma'$ from the strong commutation of $\sigma$ and $\sigma'$.

6.1. **Pre-commutativity and trivial supporting orbits.** Pre-commutativity and trivial supporting orbits are two related concepts. The pre-commutativity emphasizes on the commutativity between integrals and derivatives while the trivial supporting orbits emphasizes on the geometry in the geometric lemma.

**Lemma 6.1.** Let $\sigma_1 \in \text{Irr}^{\square}(G_{n_1})$ and let $\sigma_2 \in \text{Irr}^{\square}(G_{n_2})$. Let $\pi$ be a smooth representation of $G_n$. Then $(\sigma_1, \sigma_2, \pi)$ is a pre-RdLi-commutative triple if and only if the embedding

$$D_{\sigma_1} \circ I_{\sigma_2}(\pi) \boxtimes \sigma_1 \hookrightarrow (\sigma_2 \times \pi)_{n_1}$$

also has the trivial supporting orbit.

**Proof.** Let $P = P_{n_2-n_1}$, $Q = P_{n+n_2-n_1-n_1}$. Then the embedding can be rewritten as:

$$D_{\sigma_1} \circ I_{\sigma_2}(\pi) \boxtimes \sigma_1 \hookrightarrow C^\infty(P \setminus G_{n+n_2}, \sigma_2 \boxtimes \pi)_{n_1}$$

The surjection $(\sigma_2 \times \pi)_{n_1} \twoheadrightarrow \sigma_2 \times \pi_{n_{n_1}}$ can be rewritten as the surjection from $(\text{Ind}_G^G \sigma_2 \boxtimes \pi)_{n_1}$ to $C^\infty(P \setminus PQ, \sigma_2 \boxtimes \pi)_{n_1} \cong \pi_{n_{n_1}} \boxtimes \pi_{n_{n_1}}$, where $C^\infty(P \setminus PQ, \sigma_2 \boxtimes \pi)$ is the space of
locally constant functions from $P \setminus PQ$ to $\sigma_2 \boxtimes \pi$. Then the composition of the embedding and the surjection coincides with the one in the definition of the pre-commutativity as well as the one in the definition of the supporting orbit. This implies the lemma.

6.2. Commutative triples. The following proposition suggests the use of 'commutative' in our terminologies:

**Proposition 6.2.** Let $(\sigma', \sigma, \pi)$ be a strongly RdLi-commutative triple. Then

$$D^R_{\sigma'} \circ I^L_{\sigma'}(\pi) \cong I^L_{\sigma'} \circ D^R_{\sigma'}(\pi).$$

**Proof.** By the factoring through condition, we have that

$$D^R_{\sigma'} \circ I^L_{\sigma'}(\pi) \boxtimes \sigma' \mapsto \sigma' \times (D^R_{\sigma'}(\pi) \boxtimes \sigma') = (\sigma \times D^R_{\sigma'}(\pi)) \boxtimes \sigma'.$$

Hence, $D^R_{\sigma'} \circ I^L_{\sigma'}(\pi) \mapsto \sigma \times D^R_{\sigma'}(\pi)$ by Künneth formula. Thus, by the uniqueness of the submodule in $\sigma \times D^R_{\sigma'}(\pi)$, we have that:

$$D^R_{\sigma'} \circ I^L_{\sigma'}(\pi) \cong I^L_{\sigma'} \circ D^R_{\sigma'}(\pi).$$

The following two lemmas give examples of pre-commutativity by using the geometric lemma.

**Lemma 6.3.** Let $\Delta_1 = [a_1, b_1]_\rho$ and $\Delta_2 = [a_2, b_2]_\rho$ be two segments with $a_1 > a_2$ or $b_1 > b_2$. Then $(\text{St}(\Delta_1), \text{St}(\Delta_2), \pi)$ is pre-RdLi-commutative.

**Proof.** We consider the case that $b_1 > b_2$ and the another case is similar. Suppose not to derive a contradiction. We consider the embedding:

$$I_{\Delta_2}(\pi) \mapsto \text{St}(\Delta_2) \times \pi.$$

Let $l = l_{ab}(\Delta_1)$. Then we have:

$$D_{\Delta_1} \circ I_{\Delta_1}(\pi) \boxtimes \text{St}(\Delta_1) \mapsto (I_{\Delta_2}(\pi))_{N_l} \mapsto (\text{St}(\Delta_2) \times \pi)_{N_l}.$$

Then one applies the geometric lemma on $(\text{St}(\Delta_2) \times \pi)_{N_l}$ and it admits a filtration with subquotients taking the form:

$$(\ast) \quad \text{St}([b', b_2]_\rho) \times \tau_1 \boxtimes \text{St}([a_2, b']_\rho) \times \tau_2,$$

where $\tau_1 \boxtimes \tau_2$ is an irreducible composition factor in $\pi_{N_l}'$ for some suitable unipotent radical $N'$. Thus, $D_{\Delta_1} \circ I_{\Delta_1}(\pi) \boxtimes \text{St}(\Delta_1)$ embeds to one of such subquotients. However, since $b_2 < b_1$, applying Frobenius reciprocity on the second factor in $(\ast)$ gives that $D_{\Delta_1} \circ I_{\Delta_2}(\pi) \boxtimes \text{St}(\Delta_1)$ cannot embed to one of above layers. This gives a contradiction. Thus, $(\text{St}(\Delta_1), \text{St}(\Delta_2), \pi)$ is pre-RdLi-commutative triple.

Following application of the geometric lemma as in the previous lemma, we have more examples on pre-RdLi-commutative triples:

**Example 6.4.**

1. Let $\sigma_1, \sigma_2 \in \text{Irr}^\square$. Suppose $\text{csupp}(\sigma_1) \cap \text{csupp}(\sigma_2) = \emptyset$. Then, for any $\pi \in \text{Irr}$ with $D_{\sigma_1}(\pi) \neq 0$, $(\sigma_1, \sigma_2, \pi)$ is a pre-RdLi-commutative triple.

2. Let $\sigma \in \text{Irr}^\square_\sigma$. Let $\Delta = [a, b]_\rho$ be a segment. Suppose $\nu^\rho_{\rho, \rho} \notin \text{csupp}(\sigma)$. Then, for any $\pi \in \text{Irr}$, $(\sigma, \text{St}(\Delta), \pi)$ is a pre-RdLi-commutative triple.

3. Let $\sigma \in \text{Irr}^\square_\sigma$. Let $\Delta = [a, b]_\rho$ be a segment. Suppose $\nu^\rho_{\rho, \rho} \notin \text{csupp}(\sigma)$. Then, for any $\pi \in \text{Irr}$, $(\text{St}(\Delta), \sigma, \pi)$ is a pre-RdLi-commutative triple.

In general, commutativity does not imply strong commutativity and we have the following example:
Example 6.5. Let \( \pi = \rho \) be an irreducible cuspidal representation. We have that \( I_\rho \circ D_\rho(\pi) \cong D_\rho \circ I_\rho(\pi) \cong \rho \). However, \((\{\rho\}, \{\rho\}, \pi)\) is not a strongly RdLi-commutative triple.

6.3. Composition for strong commutations.

**Proposition 6.6.** Let \( \sigma_1', \sigma_2', \sigma \in \text{Irr} \). Let \( \pi \in \text{Irr} \). Suppose \( D_\sigma(\pi) \neq 0 \). Let \( \omega = I_{\sigma_2} \circ I_{\sigma_1}(\pi) \). Let \( l = n(\sigma) \). If \((\sigma, \sigma_1', \pi)\) and \((\sigma, \sigma_2', I_{\sigma_1}(\pi))\) are strongly RdLi-commutative, then the composition of the following natural maps:

\[
D_\sigma(\omega) \boxtimes \sigma \rightarrow \omega_{N_1} \leftarrow (\sigma_2' \times \sigma_1' \times \pi)_{N_1} \rightarrow \sigma_2' \times 1(\sigma_1' \times 1(\pi_{N_1}))
\]

are non-zero, and factors through the natural embedding:

\[
(\sigma_2' \times \sigma_1' \times D_\sigma(\pi)) \boxtimes \sigma \rightarrow \sigma_2' \times 1(\sigma_1' \times 1(\pi_{N_1}))
\]

**Proof.** We have that \( I_{\sigma_2} \circ I_{\sigma_1}(\pi) \cong I_{\sigma_2' \times \sigma_1'}(\pi) \). We consider the following commutative diagram:

\[
\begin{array}{ccccccccc}
D_\sigma(\omega) & \boxtimes \sigma' & \xrightarrow{j} & \omega_{N_1} & \xrightarrow{i_1} & (\sigma_2' \times I_{\sigma_1}(\pi))_{N_1} & \xrightarrow{i_2} & (\sigma_2' \times \sigma_1' \times \pi)_{N_1} & \\
& \downarrow f_1 & & \downarrow q_1 & & \downarrow q_2 & & \downarrow q_3 & \\
(\sigma_2' \times D_\sigma \circ I_{\sigma_1}(\pi)) & \boxtimes \sigma' & \xrightarrow{f_3} & \sigma_2' \times I_{\sigma_1}(\pi)_{N_1} & \xrightarrow{f_4} & \sigma_2' \times 1(\sigma_1' \times \pi)_{N_1} & \\
\end{array}
\]

where the top vertical maps \( q_1 \) and \( q_2 \) are the surjections onto the toppest layer in the geometric lemma; \( q_3 \) is induced from the surjection from \((\sigma_1' \times \pi)_{N_1}\) to \((\sigma_1' \times 1(\pi_{N_1}))\); \( j \) is the unique embedding from the derivative; \( i_1 \) is the unique embedding from the integral of \( I_{\sigma_2}(I_{\sigma_1}(\pi)) \) and \( i_2 \) is induced from the unique embedding from \( I_{\sigma_1}(\pi) \) to \((\sigma_1' \times \pi}\).

The existence of \( f_1 \) and \( f_2 \) follows from strong commutativity. Then the composition \( f_2 \circ f_1 \) gives the required map for factoring through the map in the proposition. \( \square \)

**Corollary 6.7.** We use the notations in Proposition 6.6. Suppose further that \( \sigma_1' \times \sigma_2' \) is irreducible. Then \((\sigma, \sigma_1' \times \sigma_2', \pi)\) is also a strongly RdLi-commutative triple.

**Proof.** Note that \( \sigma_2' \times 1(\sigma_1' \times 1(\pi_{N_1})) \cong (\sigma_2' \times \sigma_1') \times 1(\pi_{N_1}) \). Then the corollary follows from Proposition 6.6 and the definition of a strongly RdLi-commutative triple. \( \square \)

We also have the following dual version:

**Proposition 6.8.** Let \( \sigma_1, \sigma_2, \sigma' \in \text{Irr} \). Let \( \pi \in \text{Irr} \). Let \( n = n(\pi) \). Let \( l_1 = n(\sigma_1) \) and let \( l_2 = n(\sigma_2) \). Suppose \( D_\sigma(\pi) \neq 0 \). If \((\sigma_1, \sigma', \pi)\) and \((\sigma_2, \sigma', D_\sigma(\pi))\) are strongly RdLi-commutative triples, then the composition of the following natural maps:

\[
D_{\sigma_2} \circ D_\sigma \circ I_{\sigma'}(\pi) \boxtimes \sigma_2 \boxtimes \sigma_1 \leftarrow (\sigma' \times \pi)_{N_{n+n(\sigma')-1-l_2-l_2}} \rightarrow \sigma' \times 1(\pi_{N_{n-l_2-l_2-1}})
\]

is non-zero, and factors through the natural map:

\[
(\sigma' \times D_{\sigma_2} \circ D_{\sigma_1}(\pi)) \boxtimes \sigma_2 \boxtimes \sigma_1 \leftarrow \sigma' \times 1(\pi_{N_{l_2-l_2-1}})
\]

Here \( \hat{\times} \) is defined in a similar manner to \( (6.7) \) to obtain a \( G_{n+n(\sigma)-1-l_2} \times G_{l_2} \times G_{l_2} \) representation, and the natural maps are described in detail in (6.11) below.
Proof. Let \( \omega = D_{\sigma_2} \circ D_{\sigma_1} \circ I_{\sigma'} (\pi) \) and \( \omega' = D_{\sigma_2} \circ D_{\sigma_1} (\pi) \). Let \( N = N_{n+n(\sigma')-\ell_1, \ell_2, \ell_1} \) and let \( N' = N_{n-\ell_1, \ell_2, \ell_1} \). This follows from the following commutative diagram:

\[
\begin{array}{c}
\omega \otimes \sigma_2 \otimes \sigma_1 \xrightarrow{i_1} D_{\sigma_1} \circ I_{\sigma'} (\pi)_{N_{12}} \otimes \sigma_1 \xrightarrow{i_2} (\sigma' \times \pi)_N \xrightarrow{i_3} (\sigma' \times \pi)_N' \xrightarrow{j_1} \sigma_1 \times \pi_{N_1} \\
I_{\sigma'} \circ D_{\sigma_1} (\pi)_{N_{12}} \otimes \sigma_1 \xrightarrow{j_2} (\sigma' \times D_{\sigma_1} (\pi))_{N_{12}} \otimes \sigma_1 \xrightarrow{j_3} \sigma' \times \pi_{N_1} \\
\end{array}
\]

(6.11)

The embedding \( i_1 \) comes from taking a derivative, the embedding \( i_2 \) is induced from the composition of \( D_{\sigma_1} \circ I_{\sigma'} (\pi) \otimes \sigma_1 \rightarrow I_{\sigma'} (\pi)_{N_{11}} \rightarrow (\sigma' \times \pi)_{N_{11}} \) and the embedding \( i_3 \) is induced from the embedding \( I_{\sigma'} \circ D_{\sigma_1} (\pi) \rightarrow \sigma' \times D_{\sigma_1} (\pi) \). The maps \( j_1 \) and \( j_2 \) are induced from the natural embeddings \( D_{\sigma_1} (\pi) \otimes \sigma_1 \rightarrow \pi_{N_{11}} \) and \( j_3 \) is induced from the natural embedding \( \omega' \otimes \sigma_2 \rightarrow D_{\sigma_1} (\pi)_{N_{12}} \). The leftmost vertical equality follows from Proposition 6.2.

The surjections are the natural ones from the geometric lemma, with furthermore first taking the Jacquet functor \( N_{n_1} \) and then taking the Jacquet functor \( N_{n_2} \) on the first factors.

Now the existence of \( f_1 \) comes from the strong commutativity for \( (\sigma_1, \sigma', \pi) \) and the existence of \( f_2 \) comes from the strong commutativity for \( (\sigma_2, \sigma', D_{\sigma_1} (\pi)) \). The map \( f_2 \circ f_1 \circ i \) gives the required map for the factoring through condition.

Corollary 6.9. We use the notations in Proposition 6.3. Suppose further that \( \sigma_1 \times \sigma_2 \) is also \( \square \)-irreducible. Then \( (\sigma_1 \times \sigma_2, \sigma', \pi) \) is also a strongly RdLi-commutative triple.

6.4. Transitivity of pre-commutative triples. We prove a weaker converse of Proposition 6.6.

Proposition 6.10. Let \( \sigma_2, \sigma'_2 \in \text{Irr} \) such that \( \sigma_2 \times \sigma'_2 \) is still \( \square \)-irreducible. Let \( \sigma_1 \in \text{Irr} \). Let \( \pi \in \text{Irr} \). Suppose \( (\sigma_1, \sigma_2 \times \sigma'_2, \pi) \) is pre-RdLi-commutative. Then \( (\sigma_1, \sigma_2, I_{\sigma'_2} (\pi)) \) is also a pre-RdLi-commutative triple.

Proof. Let \( n_1 = n(\sigma_1) \). Let \( \omega = I_{\sigma_2 \times \sigma'_2} (\pi) \) and let \( \omega' = I_{\sigma_2} (\pi) \). We have the following diagram

\[
\begin{array}{c}
D_{\sigma_1} (\omega) \otimes \sigma_1 \xrightarrow{i_1} \omega_{N_{11}} \xrightarrow{i_2} (\sigma_2 \times \omega')_{N_{11}} \xrightarrow{i_3} (\sigma_2 \times \sigma'_2 \times \pi)_{N_{11}} \\
\sigma_2 \times \pi_{N_{11}} \xrightarrow{i_4} \sigma'_2 \times \pi_{N_{11}} \\
\sigma_2 \times (\sigma'_2 \times \pi_{N_{11}}) \\
\end{array}
\]

where the top horizontal maps are induced from natural embeddings arising from derivatives and integrals, and the right vertical maps from the geometric lemma.

The diagram is commutative by the functoriality of the geometric lemma. By uniqueness of the embedding \( \omega \rightarrow (\sigma_2 \times \sigma'_2) \times \pi \), \( i_3 \circ i_2 \) agrees with the map induced from the unique
embedding $\pi$ to $\sigma_2 \times \sigma'_2 \times \pi$. The strongly commutative triple $(\sigma_1, \sigma_2 \times \sigma'_2, \pi)$ implies that $q_3 \circ q_2 \circ i_3 \circ i_2 \circ i_1 \neq 0$. By the commutative diagram, we then have that $q_1 \circ i_2 \circ i_1 \neq 0$, giving pre-commutativity for $(\sigma_1, \sigma_2, \omega')$.

**Proposition 6.11.** Let $\sigma_1, \sigma'_1 \in \text{Irr } \square$ such that $\sigma_1 \times \sigma'_1$ is still $\square$-irreducible. Let $\sigma_2 \in \text{Irr } \square$. Suppose $(\sigma_1 \times \sigma'_1, \sigma_2, \pi)$ is pre-RdLi-commutative. Then $(\sigma_1, \sigma_2, \pi)$ is also pre-RdLi-commutative.

**Proof.** Let $\tau = D_{\sigma_1 \times \sigma'_1} \circ I_{\sigma_2}(\pi)$. Let $n_1 = n(\sigma_1)$, $n'_1 = n(\sigma'_1)$, $n_2 = n(\sigma_2)$ and $n = n(\pi)$. We first have the following diagram:

![Diagram](https://example.com/diagram.png)

where $N = N_{n_2+n-n_1-n'_1, n'_1}$ in $G_{n+n_2}$, $N' = N_{n-n_1-n'_1, n_1}$ in $G_n$, and $N'' = N_{n+n_2-n'_1-n_1}$ in $G_{n+n_2}$ regarded as a subgroup of $G_{n+n_2-n'_1} \times G_{n'_1}$ via embedding to the first factor.

The maps $q, q_1, q_2$ are the projections from the geometric lemma, while other maps in the diagram come from natural embeddings arising from derivatives and integrals.

The left triangle in the diagram is commutative by Lemma 3.3 while the right triangle is commutative by the naturality of the geometric lemma.

Now the composition of the top maps is non-zero by the pre-commutativity of $(\sigma_1 \times \sigma'_1, \sigma_2, \pi)$. This, in particular, implies that $q_1 \circ q \circ i \neq 0$. Since $q_1 \circ q \circ i$ is obtained from taking the Jacquet functor (with respect to $N''$) on the required composition for the pre-RdLi-commutativity of $(\sigma_1 \times \sigma'_1, \sigma_2, \pi)$, the required composition is also non-zero. In other words, $(\sigma'_1, \sigma_2, \pi)$ is pre-RdLi-commutative.

**6.5. Producing some strongly RdLi-commutative triples.**

**Proposition 6.12.** (1) Let $\sigma, \sigma' \in \text{Irr } \square$. Suppose $(\sigma, \pi)$ is a $\square$-irreducible pair and $(\sigma, \sigma', \pi)$ is a pre-RdLi-commutative triple. Then $(\sigma, \sigma', \pi)$ is a strongly RdLi-commutative triple.

(2) Let $\sigma, \sigma' \in \text{Irr } \square$. Suppose $D_{\sigma} \circ I_{\sigma'}(\pi)$ does not embed to $(\sigma' \times \square_{\sigma}(\pi))/(\sigma' \times D_{\sigma}(\pi))$. (Here $\sigma' \times D_{\sigma}(\pi)$ is viewed as a submodule of $\sigma' \times \square_{\sigma}(\pi)$ via inducing from the natural map from $D_{\sigma}(\pi)$ to $\square_{\sigma}(\pi)$.) If $(\sigma, \sigma', \pi)$ is a pre-RdLi-commutative triple, then $(\sigma, \sigma', \pi)$ is a strongly RdLi-commutative triple.

**Proof.** We first prove (1). Since $(\sigma, \sigma', \pi)$ is a pre-RdLi-commutative triple, there is an embedding

$$D_{\sigma}(\pi) \otimes \sigma \hookrightarrow (\sigma' \times 1)_{\pi_{N_i}},$$

where $i = n(\sigma)$. On the other hand, $(\sigma, \pi)$ is a $\square$-irreducible pair, and so

$$\pi_{N_i} = D_{\sigma}(\pi) \otimes \sigma \times \tau.$$

Moreover, by the uniqueness of submodule of Jacquet module under $\square$-irreducible representations, $\tau$ does not have a submodule of the form $\omega \otimes \sigma_2$ and hence

$$\text{Hom}_{G_0 \times G_0}(D_{\sigma}(\pi) \otimes \sigma, (\sigma' \times 1)_\pi) \cong \text{Hom}_{G_0}(D_{\sigma}(\pi), \text{Hom}_{G_0}(\sigma, (\sigma' \times 1)_\pi)) \cong \text{Hom}(D_{\sigma}(\pi), (\sigma' \times \square_{\sigma}(\pi)) = 0,$$

where $s = n(\pi) - n(\sigma)$ and $t = n(\sigma)$. Here $\text{Hom}_{G_0}(\sigma, (\sigma' \times 1)_\pi)$ is regarded as a $G_0$-representation in a similar manner to the big derivative in Section 5.6. This implies that
the above embedding factors through the map $\left(\sigma^\prime \times D_\sigma(\pi)\right) \boxtimes \sigma$ to $\sigma^\prime \times^1 \pi_{N_i}$. This implies the strong commutativity for $(\sigma, \sigma^\prime, \pi)$.

For (2), by definition, we have the following commutative diagram:

\[
\begin{array}{ccc}
D_\sigma(\pi) \circ I_{\sigma^\prime}(\pi) \boxtimes \sigma^\prime & \rightarrow & \sigma^\prime \times^1 \pi_{N_{i}(\sigma)} \\
\downarrow u & & \downarrow \uparrow \sigma \times \oplus_{\sigma}(\pi) \boxtimes \sigma \\
\sigma^\prime \times D_\sigma(\pi) \boxtimes \sigma & \rightarrow & \sigma^\prime \times D_\sigma(\pi) \boxtimes \sigma
\end{array}
\]

It is straightforward that the assumption in (2) implies that the map $u$ factors through $i$. □

**Remark 6.13.** We have not discussed an integral version of an irreducible pair, but one possible way to define is the following. For $\pi \in \text{Irr}$ and $\sigma \in \text{Irr}^\square$, we say that $(\sigma, \pi)$ is a $Li$-irreducible pair if $(\sigma, I_\sigma(\pi))$ is a $Ld$-irreducible pair. We shall not use this anyway and shall not explore its properties further.

### 7. Dual strongly commutative triples

We now show a duality for strongly commutative triples switching left and right versions. Similar to previous situations, it will work better (at least technically) if we impose some conditions of irreducible pairs.

#### 7.1. Strongly commutative triples and irreducible pairs

The following proposition is one important and interesting result that can switch left and right versions of irreducible pairs. A key idea of the proof is the commutative diagram (7.12) below and one first figures out the orbit that contributes an embedding from top maps and then deduces the orbit from the bottom maps in order to apply Proposition 5.6.

**Proposition 7.1.** Let $\sigma_1, \sigma_2 \in \text{Irr}^\square$. Let $\pi \in \text{Irr}$. Suppose $(\sigma_1, \sigma_2, \pi)$ is a strongly $RdLi$-commutative triple and $(\sigma_1, \pi)$ is a $Rd$-irreducible pair. Then $(\sigma_1, I_{\sigma_2}(\pi))$ is also a $Rd$-irreducible pair.

**Proof.** Step 1: Establish a commutative diagram. Let $\tau = D_{\sigma_1} \circ I_{\sigma_2}(\pi)$. Let $n_1 = n(\sigma_1)$ and let $n_2 = n(\sigma_2)$. By Lemma 3.12 and Proposition 6.2 the following diagram:

\[
\begin{array}{ccc}
\sigma_2 \times \pi & \rightarrow & \sigma_2 \times D_{\sigma_1}(\pi) \times \sigma_1 \\
\downarrow I_{\sigma_2}(\pi) & & \downarrow \sigma_2 \times D_{\sigma_1}(\pi) \times \sigma_1 \\
\tau \times \sigma_1
\end{array}
\]

is commutative, where the left upward and downward maps are unique embeddings; the right downward map is induced from the embedding $\pi \hookrightarrow D_{\sigma_1}(\pi) \times \sigma_1$; and the right upward map is induced from the embedding $\tau \hookrightarrow \sigma_2 \times D_{\sigma_1}(\pi)$.  


Step 2: Show the triviality of the supporting orbit for the composition of top maps in (7.12). Now we apply the Jacquet functor \( N = N_{n_1} \) on the top maps, and obtain induced maps:

\[
\tau \boxtimes \sigma_1 \xrightarrow{j} I_{\sigma_2}(\pi)_{N_{n_1}} \xrightarrow{i_2} \sigma_2 \times (\pi \times \sigma_1)_{N_{n_1}},
\]

where \( j, i_1, i_2 \) are the natural embeddings as before and \( q_1, q_2, q' \) are quotient maps from \( \tau \boxtimes \sigma_1 \) commutes from strong commutations, and the square follows from the functoriality of the geometric lemma and definitions, and the triangle follows from the Rd-irreducibility and Proposition 5.6.

By the definition of strong commutation, we have that \( q_1 \circ i_1 \circ j \neq 0 \). The irreducible pair of \( (\sigma_1, \pi) \) also implies that \( q' \circ i_2 \circ i_1' \) is an isomorphism by definition, and \( (q' \circ i_2') \circ (q_1 \circ i_1) \neq 0 \). Thus we have that the supporting orbit for the embedding \( \tau \boxtimes \sigma_1 \rightarrow (\sigma_2 \times D_{\sigma_1}(\pi) \times \sigma_1)_N \) is trivial.

Step 3: Deduce the Rd-irreducibility using bottom maps in the commutative diagram in (7.12). Now the commutative diagram in the beginning implies that in the following diagram

\[
\tau \boxtimes \sigma_1 \xrightarrow{c} I_{\sigma_2}(\pi)_{N_{n_1}} \xrightarrow{k} (\tau \times \sigma_1)_{N_{n_1}} \xrightarrow{l} (\sigma_2 \times D_{\sigma_1}(\pi) \times \sigma_1)_{N_{n_1}},
\]

the composition of the top maps coincides with the composition of the top maps in (7.13). Thus, the composition of the top horizontal maps with the rightmost vertical map is nonzero, and so \( l \circ k \neq 0 \). In other words, the supporting orbit for the embedding \( \tau \boxtimes \sigma_1 \rightarrow (\tau \times \sigma_1)_N \) is also trivial. By Proposition 5.6 \( (\sigma_1, I_{\sigma_2}(\pi)) \) is an Rd-irreducible pair.

7.2. Dual strongly commutative triples. We now prove a duality on pre-commutativity. The key idea is that if we start from a pre-RdLi-commutative triple \( (\sigma_1, \sigma_2, \pi) \) with the condition that \( (\sigma_1, \pi) \) is a Rd-irreducible pair, then we can deduce the following commutative diagram from Lemma 3.3 for \( n_1 = n(\sigma_1), n_2 = n(\sigma_2) \) and \( n = n(\pi) \), and let
\[ \tau = D_{\sigma_1} \circ I_{\sigma_2}(\pi), \]

where the maps are the natural maps from derivatives and integrals. Then one determines the supporting orbit arising from the embedding \( i \circ q_1 \) via the property of a Rd-irreducible pair and then deduce the trivial supporting orbit arising from \( i \circ q_2 \) via the commutative diagram.

**Proposition 7.2.** Let \((\sigma_1, \sigma_2, \pi)\) be a pre-RdLi-commutative triple. Suppose \((\sigma_1, \pi)\) is a Rd-irreducible pair. Then \((\sigma_2, \sigma_1, I_{\sigma_2}(\pi))\) is a pre-LdRi-commutative triple.

**Proof.** Let \( n_1 = n(\sigma_1), n_2 = n(\sigma_2) \) and \( n = n(\pi) \). Let \( \tau = I_{\sigma_2} \circ D_{\sigma_1}(\pi) \). By Proposition 6.12 \((\sigma_1, \sigma_2, \pi)\) is a strongly RdLi-commutative triple. By Proposition 7.1 \((I_{\sigma_2}(\pi), \sigma_1)\) is also a Ld-irreducible pair. By Proposition 5.6 the embedding

\[ \tau \otimes \sigma_1 \hookrightarrow I_{\sigma_1}^R(\tau)_{N_{n_1}} \hookrightarrow (\tau \times \sigma_1)_{N_{n_1}} \]

again has the trivial supporting orbit. Let \( m = n - n_1 \). Since \( P_{m+n_1, n_1}P_{n_2, m_1} = P_{m+n_2, n_2}P_{m+n_2, n_1} \), the induced embedding in (7.14), via taking the Jacquet functor \( N_m \) on the first factor of \( G_{n_1} \times G_{n_1} \),

\[ \sigma_2 \otimes D_{\sigma_2}^L(\tau) \otimes \sigma_1 \hookrightarrow \tau_{N_{n_1}} \otimes \sigma_1 \hookrightarrow I_{\sigma_2}(\pi)_{N_{n_1}} \hookrightarrow (\tau \times \sigma_1)_{N_{n_1}}, \]

still has the trivial supporting orbit. Here \( N' = N_{n_2, m_1} \).

By Proposition 6.2 \( D_{\sigma_2}^L \circ I_{\sigma_2}^R(\tau) \cong \pi \). On the other hand, we can take the Jacquet functor \( N_{n_2, m_1} \) first to have:

\[ \sigma_2 \otimes \pi \hookrightarrow I_{\sigma_2}^R(\tau)_{N_m} \hookrightarrow (\tau \times \sigma_1)_{N_m}. \]

Then, we further take the Jacquet functor \( N_{m, n_1} \) on the second factor and we have an induced embedding:

\[ \sigma_2 \otimes D_{\sigma_1}(\pi) \otimes \sigma_1 \hookrightarrow \sigma_2 \otimes \pi_{N_{n_1}} \hookrightarrow I_{\sigma_1}^R(\tau)_{N_{n_1}} \hookrightarrow (\tau \times \sigma_1)_{N_{n_1}}. \]

By the uniqueness in Lemma 3.3 this embedding agrees with the previous one. Hence, the embedding has the trivial supporting orbit, and the corresponding orbit takes the form \( P_{n_2, m_1}P_{n_2, m_1} \). Since \( P_{n_2, m_1}P_{n_2, m_1} \subset P_{n_2, m_1}P_{n_2, m_1} \), the supporting orbit for the embedding

\[ \sigma_2 \otimes \pi \hookrightarrow I_{\sigma_2}(\pi)_{N_n} \rightarrow (\tau \times \sigma_1)_{N_n} \]

takes the form \( P_{n_2, m_1}P_{n_2, m_1} \) since \( \sigma_2 \otimes D_{\sigma_1}(\pi) \otimes \sigma_1 \) is a submodule of \( \sigma_2 \otimes \pi_{N_{n_1}} \), and so also has the trivial supporting orbit. \( \square \)

8. **Constructing some pre-commutativity**

The main goal of this section is to prove Proposition 8.5. We first illustrate some basic arguments in Lemma 8.4 and then extend to the full case of Proposition 8.5. The key idea of proving Lemma 8.4 is to first use induction and some simple application on the geometric lemma to obtain control on the structure in Lemmas 8.2 and 8.3 and one then incorporates with another induction to prove Proposition 8.5.
8.1. Completing to a Rd-irreducible pair.

Lemma 8.1. We fix a cuspidal representation \( \rho \). Let \( m \) be a multisegment such that for any segment \( \Delta \) in \( m \), \( b(\Delta) = \rho \). Let \( \Delta' \) be another segment such that \( b(\Delta') = \rho \) and \( \Delta' \subset \Delta \). If \( m \) has more than one segment, then \((St(\Delta'), St(m))\) is not a Ld-irreducible pair.

Proof. Let \( n = l_{abs}(m) \). We write all the segments in \( m \) as \( \Delta_1 = [a_1, 0], \ldots, \Delta_r = [a_r, 0] \). Relabel the segments in \( m \) such that \( a_r \leq a_{r-1} \leq \ldots \leq a_1 \). Write \( \Delta' = [a', 0] \).

The representation \((St(\Delta_1) \times \ldots \times St(\Delta_r))_{\omega_1} \) is a \( G_{n-1} \times G_l \)-representation and we only consider the summand with the cuspidal support same as \( St(\Delta) \) in the \( G_l \)-factor. Then one applies the geometric lemma on

\[
(St(\Delta_1) \times \ldots \times St(\Delta_r))_{\omega_1}
\]

and those layers that contribute to the support is of the form:

\[
(St(\Delta_1) \times \ldots \times St(\Delta_{i-1}) \times St(\Delta'_i' \times St(\Delta_{i+1}) \times \ldots \times St(\Delta_r)) \boxtimes St(\Delta'),
\]

where \( \Delta'_i = [a_i, a_i' - 1] \). For simplicity, let

\[
\omega_i = (St(\Delta_1) \times \ldots \times St(\Delta_{i-1}) \times St(\Delta'_i' \times St(\Delta_{i+1}) \times \ldots \times St(\Delta_r)).
\]

Note that \( \omega_i \) is irreducible and generic.

We have that \( \omega_i \) has unique submodule which is isomorphic to \( \omega_1 \). On the other hand, there is only one indecomposable direct summand that contains \( \omega_1 \boxtimes St(\Delta') \). Hence, all those layers have to be in that indecomposable direct summand. Now by our assumption that \( m \) has more than one segment (i.e. \( i > 1 \)), we must have more than one layer and so the direct summand is not irreducible. In other words, \((St(\Delta'), St(m))\) is not a Ld-irreducible pair. \( \square \)

For a segment \( \Delta = [a, b]_\rho \), set \( b(\Delta) = \nu(\rho) \).

Lemma 8.2. Let \( \tau \) be a smooth representation of \( G_t \). Let \( \tilde{\Delta} \) be a segment and write \( \tilde{\Delta} = [\tilde{a}, \tilde{b}]_\rho \). Suppose \( m \) satisfies the following properties:

- \( D_{m}(\tau) \neq 0 \);
- any segment in \( m \) takes the form \([a, b]_\rho \) for some \( a \leq \tilde{a} \);
- \( m \) contains more than one segment.

Suppose there exists an embedding:

\[
St(m) \hookrightarrow St(\tilde{\Delta}) \times \tau.
\]

Let \( \Delta \) be a shortest segment in \( m \). Then the induced embedding

\[
St(\Delta) \boxtimes St(m - \Delta) \hookrightarrow St(m)_{N} \hookrightarrow (St(\tilde{\Delta}) \times \tau)_{N}
\]

cannot have the trivial supporting orbit. Here \( N = N_{l_1,l_2} \) for \( l_1 = l_{abs}(\Delta) \) and \( l_2 = l_{abs}(m - \Delta) \).

Proof. If the embedding has the trivial supporting orbit, then, by definition, we have:

\[
St(\Delta) \boxtimes St(m - \Delta) \hookrightarrow St(\tilde{\Delta}) \times (\tau_{N'}),
\]

where \( N' = N_{l_1,l_2} \). We further take the Jacquet functor on \( St(\Delta) \) to give

\[
St(\tilde{\Delta}) \boxtimes St(\Delta \setminus \tilde{\Delta}),
\]

where \( \Delta \setminus \tilde{\Delta} \) is the set-theoretic subtraction. Then the embedding

\[
St(\tilde{\Delta}) \boxtimes St(\Delta \setminus \tilde{\Delta}) \boxtimes St(m - \Delta) \hookrightarrow (St(\tilde{\Delta}) \times \tau_{N'})_{N''},
\]
where $N'' = N_{l_c(\Delta)\Delta}$ is the Jacquet functor taking on the first factor, also has the trivial supporting orbit by Corollary 5.8 and the supporting orbit takes the form $P_{a,b+c}P_{a,b,c}(a = l_{abs}(\Delta), b = l_{abs}(\Delta \setminus \Delta)$ and $c = l_{abs}(m - \Delta)$. Since the above embedding is obtained by taking a Jacquet functor on the following embedding
\[(8.16) \quad St(\Delta) \boxtimes St(m - \Delta + \Delta) \hookrightarrow (St(\Delta) \times \tau)_N,\]
we also have that (8.10) also has the trivial supporting orbit and the supporting orbit takes the form $P_{a,b+c}P_{a,b,c}$. However, since $m$ contains more than one segment, Lemma 8.1 implies that $(St(\Delta), St(m))$ is not a $Ld$-irreducible pair. This contradicts to Lemma 8.2. \hfill \Box

Lemma 8.3. Fix a cuspidal representation $\rho$ and an integer $b$. Let $m$ be a multisegment such that any segment in $m$ takes the form $[a, b, \rho]$ for some integer $a$. Let $\Delta = [\Delta, \rho]$ for some $\bar{b} \geq b$ and $b \geq a \geq b$. Let $l = l_{abs}(m)$. Let $\pi \in \text{Irr}(G_n)$. If $St(m)$ embeds to $St(\Delta) \times \pi$, then the supporting orbit for the embedding:
\[D_m(\pi) \boxtimes St(m) \hookrightarrow (St(\Delta) \times \pi)_{N_i}\]
can be either one of the followings:
- the supporting orbit is trivial; or
- the supporting orbit is determined by the minimal representative $w$ in $(S_s \times S_n) \setminus S_{s+n}(S_{s+n-l} \times S_l)$ given by:
\[w^{-1}(p) = p \quad \text{for } p = 1, \ldots, s - s', \]
\[w^{-1}(p) = (s + n - l) + (p - (s - s')) \quad \text{for } p = s - s' + 1, \ldots, s.\]
and, furthermore, $D_m(\pi) \boxtimes St(m) \hookrightarrow St([\bar{a}, b, \rho]) \times 2\text{St}([b + 1, \bar{b}, \rho]) \times 1\pi_{N_i}^\rho$, where $s = l_{abs}([\bar{a}, b, \rho])$ and $s' = l_{abs}([b + 1, \bar{b}, \rho])$.

Proof. We apply the geometric lemma on $(St(\Delta) \times \pi)_{N_i}$ and the layers take the form:
\[St(\Delta_i) \times 2\text{St}(\Delta_i) \times 1\pi_{N_i}^\rho,\]
where $\Delta_i = [\bar{a}, i, \rho], \Delta_i = [i + 1, \bar{b}, \rho]$ and $l_i = l - l_{abs}(\Delta_i)$. We shall denote such layer by $\lambda_i$.
Then, we must have that
\[D_m(\pi) \boxtimes St(m) \hookrightarrow \lambda_i^\ast\]
for some $i^\ast$. By comparing cuspidal support on the term $St(m)$, we must then have that $\bar{a} \leq i^\ast \leq b$. It remains to show that when $i^\ast = b$. To this end, let $\tau = St(\Delta_i) \times 1\pi_{N_i}$ and we apply Frobenius reciprocity on
\[\text{Hom}(D_m(\pi) \boxtimes St(m)_{N_i} \times St(\Delta_i) \times 1\pi_{N_i}) \cong \text{Hom}(D_m(\pi) \boxtimes St(m)_{N_i} \times St(\Delta_i) \times 1\pi_{N_i})\]
where $s_i = l_{abs}(\Delta_i) = l_i$, and $\phi$ is a natural twist that switches from $G_{s_i} \times G_{n-l} \times G_l$-representations to $G_{n-l} \times G_{s_i} \times G_l$-representations.
Now, one studies the composition factor of $St(m)_{N_i}$. A standard argument of using the geometric lemma and the Jacquet functors on generalized Steinberg modules, one has that any composition factor of $St(m)_{N_i}$ has the form $\tau_1 \boxtimes \tau_2$ with $\nu_{\rho}^\phi \in \text{supp}(\tau_1)$. This forces that $\nu_{\rho}^\phi \in \Delta_i$, and so, with $i^\ast \leq b$, we can only have $i^\ast = b$, as desired. \hfill \Box

In the following proposition, we show that the pre-commutativity for an essentially square-integrable representation gives that of a larger generic representation, and we may first note that an easier case below is when both $\Delta = \Delta$ (in the notation of Proposition 8.4) are singletons. The proof for the general case requires deeper structure in Lemma 8.2.
Lemma 8.4. Let $\Delta = [a, b], \bar{\Delta} = [\bar{a}, \bar{b}]$ be segments. Let $m$ be a multisegment such that any segment in $m$ takes the form $[a', b]$, for some $a' \geq a$. Let $\pi \in \text{Irr}$. If $(\text{St}(\Delta), \text{St}(\bar{\Delta}), \pi)$ is a pre-RdLi-commutative triple and $D_m(\pi) \neq 0$, then $(\text{St}(m), \text{St}(\bar{\Delta}), \pi)$ is a pre-RdLi-commutative triple.

Proof. We write the segments $\Delta_i = [a_i, b_i]$ in $m$. We arrange the segments in $m$ such that:

$$a_r \leq \ldots \leq a_1.$$ 

Let $m = \{\Delta_1, \ldots, \Delta_k\}$ and let $\omega = I_n(\pi)$. We shall inductively show that $(\text{St}(m_k), \text{St}(\bar{\Delta}), \pi)$ is a pre-RdLi-commutative triple. When $k = 1$, it follows from the given hypothesis. We now assume $k \geq 2$.

Suppose $(\text{St}(m_{k+1}), \text{St}(\bar{\Delta}), \pi)$ is not a pre-RdLi-commutative triple to derive a contradiction. Let $l'_1 = l_{abs}(b_1, \bar{b}_{1}), l_2 = l_{abs}(m_k)$ and $l_3 = l_{abs}(m_{k+1})$. Then, by the second bullet of Lemma 8.3, we have an embedding of the form:

$$D_{m_k+1}(\omega) \otimes \text{St}(m_{k+1}) \hookrightarrow \text{St}([a, b]_\rho)^{\times 2 \tau},$$

where $\tau = \text{St}([b, \bar{\rho}]_\rho)^{\times \frac{1}{\mu}}(\pi N)$. Let $\text{St}(\Delta_{k+1}) \otimes \text{St}(m_k) \cong (\text{St}(\bar{\Delta}) \times \pi)_{N_2}$ be a subspace of $\text{St}(\Delta_{k+1}) \otimes \text{St}(m_k)$, where $N_2$ is the unipotent radical in $G_{n(\pi)}$.

By Lemma 8.18 there is another way to obtain the embedding via:

$$D_{m_k}(\omega) \otimes \text{St}(m_k) \hookrightarrow (\text{St}(\bar{\Delta}, \pi)_{N_2}, \pi)_{N_2},$$

and $D_{m_k+1}(\omega) \otimes \text{St}(\Delta_{k+1}) \hookrightarrow D_{m_k}(\omega)_{N_2}$.

The inductive hypothesis implies 8.19 has the trivial supporting orbit. Now if we simply consider (8.17) as a map

$$\text{St}(m_{k+1}) \hookrightarrow (\text{St}(a, \bar{\rho}_b)_{\rho}^{\times \tau})_{N_2},$$

(via any natural embedding from $\text{St}(m_{k+1})$ to $\text{St}(\Delta_{k+1}) \otimes \text{St}(m_k)$), Proposition 13.1 in Appendix implies that the embedding:

$$\text{St}(\Delta_{k+1}) \otimes \text{St}(m_k) \hookrightarrow (\text{St}(a, \bar{\rho}_b)_{\rho}^{\times \tau})_{N_2},$$

where $N_2$ is regarded as a subgroup of the second factor, also has the trivial supporting orbit.

However, it follows from Lemma 8.2 that (8.20) cannot have the trivial supporting orbit and so we arrive a contradiction. $\square$

Proposition 8.5. We use the notations in Lemma 8.4. Let $m$ be a multisegment with all segments $\Delta'$ satisfying $a(\Delta') = a(\bar{\Delta})$ and $\Delta' \subset \bar{\Delta}$. Let $m$ be a $\Delta$-saturating multisegment with $D_m(\pi) \neq 0$. If $(\text{St}(\Delta), \text{St}(m), \pi)$ is a pre-RdLi-commutative triple, then $(\text{St}(m), \text{St}(n), \pi)$ is a pre-RdLi-commutative triple.

Proof. The proof is similar to Lemma 8.4. We explain necessary modifications.

Let $\omega = I_{n}(\pi)$. We write the segments in $m$ as $\Delta'_1, \ldots, \Delta'_s$. Let $n_i = \{\Delta'_1, \ldots, \Delta'_i\}$ and let $n_i = n - n_i$. We write the segments in $m$ as $\Delta_i = \Delta'_1, \Delta'_2, \ldots, \Delta'_r$. We shall prove inductively on $r$ that $(\text{St}(n_i), \text{St}(m), \pi)$ is a pre-RdLi-commutative triple. Let $\tau_i = \text{St}(n_i) \times \pi$. We shall prove by an induction on $r$ that the embedding:

$$D_m(\omega) \otimes \text{St}(m) \hookrightarrow \omega_{n_i} \hookrightarrow (\text{St}(n_i) \times \tau_i)_{N_i}$$
has the trivial supporting orbit. For $r = 1$, the inductive hypothesis implies that the supporting orbit for the embedding:

$$D_m(\omega) \boxtimes \text{St}(m) \hookrightarrow \omega_{N_i} \hookrightarrow (\text{St}(n) \times \pi)_{N_i}$$

is trivial and so the corresponding orbit (in the sense of Section 4.1) takes the form $P_{p+n}P_{p+n-l,l}$, where $p = l_{abs}(n)$, $n = n(\pi)$ and $l = l_{abs}(m)$. Since $P_{p+n}P_{p+n-l,l} = P_{p+n}P_{p+n-l,l}$ (where $p = l_{abs}(n)$ and $p = p - p = l_{abs}(\bar{n})$), the embedding

$$D_m(\omega) \boxtimes \text{St}(m) \hookrightarrow \omega_{N_i} \hookrightarrow (\text{St}(n_i) \times \pi)_{N_i}$$

has the trivial supporting orbit and the embedding takes the form $P_{p+n}P_{p+n-l,l}$. For $r = 2$, the embedding (*) also has the trivial supporting orbit. This proves the case of $r = 1$. Now we consider $r \geq 2$. We have

$$\pi \hookrightarrow \text{St}(n_i) \times \tau_i \cong \text{St}(n_i-1) \times \text{St}(\Delta_i) \times \tau_i = \text{St}(n_i-1) \times \tau_{i-1}.$$  

**Claim 1:** The embedding:

$$D_m(\omega) \boxtimes \text{St}(m) \hookrightarrow \omega_{N_i} \hookrightarrow (\text{St}(n_i-1) \times \text{St}(\Delta_i) \times \tau_i)_{N_i}$$

has the trivial supporting orbit. Here we regard the parabolic induction is from $G_a \times G_b \times G_c$ to $G_{a+b+c}$, where $a = l_{abs}(n_i-1)$, $b = l_{abs}(\Delta_i)$ and $c = n(\tau_i)$. Since $P_{a+b+c}P_{a+b+c-l,l}$ is a closed subspace in $P_{a+b+c}P_{a+b+c-l,l}$, the claim indeed implies the embedding

$$D_m(\omega) \boxtimes \text{St}(m) \hookrightarrow \omega_{N_i} \hookrightarrow (\text{St}(n_i) \times \tau_i)_{N_i}$$

has the trivial supporting orbit, in which we regard the parabolic induction is from $G_{a+b} \times G_c$ to $G_{a+b+c}$.

It remains to prove Claim 1. Before that, we first prove another useful claim:

**Claim 2:** Let $m' = \{\Delta_1, \ldots, \Delta_{r-1}\}$ and let $l' = l_{abs}(m')$. The embedding:

$$D_{m'}(\omega) \boxtimes \text{St}(m') \hookrightarrow \omega_{N_i'} \hookrightarrow (\text{St}(n_i-1) \times \text{St}(\Delta_i') \times \tau_i)_{N_i'}$$

has the trivial supporting orbit and the orbit takes the form $P_{a+b,c}P_{a+b+c-l',l'}$. Since $P_{a+b,c}P_{a+b+c-l',l'} \subset P_{a+b,c}P_{a+b+c-l',l'}$, we then obtain the claim.

**Proof Claim 2:** By induction hypothesis on $r$, we have that the embedding:

$$D_{m'}(\omega) \boxtimes \text{St}(m') \hookrightarrow \omega_{N_i'} \hookrightarrow (\text{St}(n_i) \times \tau_i)_{N_i'},$$

has the trivial supporting orbit and the orbit takes the form $P_{a+b,c}P_{a+b+c-l',l'}$. Since $P_{a+b,c}P_{a+b+c-l',l'} \subset P_{a+b,c}P_{a+b+c-l',l'}$, we then obtain the claim.

Now we go to prove Claim 1.

**Proof of Claim 1:** We shall now prove by induction on $i$. When $i = 0$, there is nothing to prove. Now assume $i \geq 1$. We shall consider the composition

$$D_m(\omega) \boxtimes \text{St}(m) \hookrightarrow (\text{St}(n_i) \times \tau_i)_{N_i} \hookrightarrow \text{St}(n_i-1) \times (\tau_i-1)_{N_i}$$

which is non-zero by induction on $i$. Here the maps again are the natural maps from derivatives, integrals and the geometric lemma.

Now we write $(\tau_i-1)_{N_i} = (\text{St}(\Delta_i') \times \tau_i)_{N_i}$. Now we have two possibilities:

**Case 1:** Suppose the embedding

$$D_m(\omega) \boxtimes \text{St}(m) \hookrightarrow \omega_{N_i} \hookrightarrow (\text{St}(n_i-1) \times \tau_i)_{N_i} \hookrightarrow \text{St}(n_i-1) \times (\tau_i)_{N_i} \hookrightarrow \text{St}(n_i-1) \times (\tau_i-1)_{N_i}$$

(8.21)
is non-zero. In other words, the embedding has the trivial supporting orbit and the orbit takes the form \( P_{a,b,c} P_{a+b+c-l,l} \). Again, using \( P_{a,b,c} P_{a+b+c-l,l} \subset P_{a+b,c} P_{a+b+c-l,l} \), we obtain Claim 1.

**Case 2:** Suppose the embedding in Claim 1 does not hold. Then, by Lemma \( 8.3 \)
\[
D_m(\omega) \boxtimes \text{St}(m) \hookrightarrow \text{St}(\iota_{n-1}) \times_1 \text{St}(\langle a, b \rangle_\rho) \times_2 \kappa,
\]
where \( \kappa = \text{St}(\Delta_1 - [\bar{a}, b]_\rho) \times_1 \tau_{N_s} \) for \( s = l - l_{ab_\rho}([\bar{a}, b]_\rho) \).

Set \( \tilde{\kappa} = \text{St}(\iota_{n-1}) \times_1 \kappa. \) Note that \( \text{St}(\langle \bar{a}, b \rangle_\rho) \times_2 \tilde{\kappa}. \) We now regard the embedding \( 8.22 \) as \( G_t \)-representations via the second factor in \( G_{a+b+c-l} \times G_l. \) Now Claim 2 (with Proposition \( 6.12 \) in the Appendix) implies that the embedding
\[
\text{St}(\Delta_1) \boxtimes \text{St}(m') \hookrightarrow \text{St}(\langle a, b \rangle_\rho) \times_2 \kappa_{N_l},
\]
where \( N_l \) is in \( G_l, \) has trivial supporting orbit. But this then contradicts to Lemma \( 8.2. \) Hence, this case is not possible. \( \square \)

9. **Pre-commutativity \( \Rightarrow \) strong commutativity in square-integrable case**

Recall that for an essentially square-integrable representation \( \sigma, \sigma \cong \text{St}(\Delta) \) for some segment \( \Delta. \) For segments \( \Delta \) and \( \Delta', \) and \( \pi \in \text{Irr}, \) we sometimes also say that \( (\Delta, \Delta', \pi) \) is pre-RdLi-commutative (resp. strongly RdLi-commutative) if \( (\text{St}(\Delta), \text{St}(\Delta'), \pi) \) is pre-RdLi-commutative (resp. strongly RdLi-commutative). Similar terminologies are also used for RdRi-versions.

The main part of this section is to study the effect of derivatives and integrals on \( \eta_\Delta \)-invariants. Such properties will be used to verify the conditions in Proposition \( 6.12 \) to prove Theorem \( 3.10. \)

9.1. **\( \eta \)-invariant under strong commutation.**

**Lemma 9.1.** Let \( \pi \in \text{Irr}. \) Let \( \sigma \in \text{Irr}. \) For any segment \( \Delta, m_\Delta(\pi) \) is a submultisegment of \( m_\Delta(I_\sigma(\pi)), \) equivalently
\[
\eta_\Delta(\pi) \leq \eta_\Delta(I_\sigma(\pi)).
\]

**Proof.** Let \( m = m_\Delta(\pi). \) We have the embedding:
\[
\pi \hookrightarrow D_m(\pi) \times \text{St}(m).
\]
Then
\[
I_\sigma(\pi) \hookrightarrow \sigma \times \pi \hookrightarrow \sigma \times D_m(\pi) \times \text{St}(m).
\]
Thus \( D_m(I_\sigma(\pi)) \neq 0. \) This implies the lemma. \( \square \)

**Lemma 9.2.** Let \( \Delta_1, \Delta_2 \) be segments. Let \( n \) be a \( \Delta_2 \)-saturated multisegment. Suppose \( (\text{St}(\Delta_1), \text{St}(n), \pi) \) is a pre-RdLi-commutative triple. Then \( \eta_{\Delta_1}(\pi) = \eta_{\Delta_1}(I_n(\pi)). \) In particular, for any \( \Delta_1 \)-saturated segment \( \Delta, \) if \( (\Delta_1, \Delta_2, \pi) \) is a pre-RdLi-commutative triple, then \( \eta_\Delta(\pi) = \eta_\Delta(I_n(\pi)). \)

**Proof.** Let \( m = m_{\Delta_1}(\pi). \) By Lemma \( 8.3 \) \( (\text{St}(m), \text{St}(n), \pi) \) is pre-RdLi-commutative and so, by Proposition \( 6.12 \), is strongly commutative. By Proposition \( 7.1 \) \( (\text{St}(m), I_n(\pi)) \) is still Rd-irreducible. Hence, by Proposition \( 8.3 \) \( m_{\Delta_1}(I_n(\pi)) = m, \) as desired. \( \square \)

We now prove a preliminary commutativity result:

**Lemma 9.3.** Let \( (\Delta_1, \Delta_2, \pi) \) be a pre-RdLi-commutative triple. Let \( m = m_{\Delta_1}(\pi). \) Then
\[
D_m \circ I_{\Delta_2}(\pi) \cong I_{\Delta_2} \circ D_m(\pi).
\]
Proof. By Lemma 8.4 (St(m), St(Δ₂), π) is pre-RdLi-commutative and so, by Proposition 6.12, is strongly RdLi-commutative. Now the required commutativity follows from Proposition 6.2.

We now prove a dual version of Lemma 9.2. Thanks to a dual theory of Proposition 7.2, the proof is simpler and we do not have to prove a dual version of Proposition 8.5.

Lemma 9.4. Let (Δ₁, Δ₂, π) be a pre-RdLi-commutative triple. Let m = mx₁(π). Then

\[ \text{η}_{Δ₁}^L(I_{Δ₁} \circ D_m(π)) = η_{Δ₂}^L(I_{Δ₂}(π)) \]

Proof. Since (Δ₁, Δ₂, π) is pre-RdLi-commutative, Proposition 8.5 implies that (St(m), St(Δ₂), π) is pre-LdRi-commutative. Then, by Proposition 7.2 (St(Δ₂), St(m), I_{Δ₂} \circ D_m(π)) is a pre-LdRi-commutative triple. Now, by using the right version of Lemma 9.3, we have

\[ \text{η}_{Δ₂}^R(I_m \circ I_{Δ₂} \circ D_m(π)) = η_{Δ₂}^R(I_{Δ₂}(π)). \]

Combining with the right version of Lemma 9.2, we obtain the desired statement.

Lemma 9.5. Let (Δ₁, Δ₂, π) be a pre-RdLi-commutative triple. Let m = mx₁(π). Then

\[ \text{η}_{Δ₂}^L(D_m \circ I_{Δ₂}(π)) = η_{Δ₂}^L(I_{Δ₂}(π)). \]

Proof. Now, let τ = D_m \circ I_{Δ₂}(π). Then

\[ D_{Δ₁} \circ I_{Δ₂}(π) \hookrightarrow τ \times St(m - Δ₁), \]

and so the right version of Lemma 9.1 gives that

\[ η_{Δ₂}(D_{Δ₁} \circ I_{Δ₂}(π)) \geq η_{Δ₂}(τ). \]

Now, using I_{Δ₂}(π) \hookrightarrow (D_{Δ₁} \circ I_{Δ₂}(π)) \times St(Δ₁) and the right version of Lemma 9.3, again, we have:

\[ η_{Δ₂}(I_{Δ₂}(π)) \geq η_{Δ₂}(D_{Δ₁} \circ I_{Δ₂}(π)). \]

By Lemmas 9.2 and 9.4

\[ η_{Δ₂}^L(D_m \circ I_{Δ₂}(π)) = η_{Δ₂}^L(I_{Δ₂}(π)). \]

and so the above inequalities are equalities as desired.

9.2. Some more properties of ηₐ.

Lemma 9.6. Let Δ be a segment and let π ∈ Irr. Let τ be a composition factor in the Jordan-Hölder series of Dₜ(Δ) with τ \not\cong Dₜ(Δ). Then mxₜ(π) ≠ mxₜ(Dₜ(π)) = mxₜ(Δ) - Δ.

Proof. Let p = mxₜ(π). Then

\[ π \hookrightarrow Dₜ(π) \times St(p). \]

Write Δ = [a, b]ₜ. Since mxₜ(Dₜ(π)) = ∅, the Gₜ-factor (for any k) of any simple composition factor in Dₜ(π)ₚₙ cannot have the cuspidal support of the form \{ν₁, ..., νₖρ\} for some \( a ≤ c ≤ b \). Since

\[ Dₜ(π) ⊗ St(Δ) \hookrightarrow πₙₚ, (Dₜ(π) \times St(p))ₚₙₚ, \]

where \( l = l_{abs}(Δ) \), the only layer in the geometric lemma of (Dₜ(π) \times St(p))ₚₙₚ that can contribute the above embedding is the toppest one i.e. \( Dₜ(π) \times (St(p))ₚₙₚ \). Hence, \( Dₜ(π) \) embeds to \( Dₜ(π) \times Dₜ(St(p)) \). We also have that \( Dₜ(St(p)) \cong St(p - Δ) \) by [Ch24, Lemma 11.6]. Thus, we have:

\[ Dₜ(π) \hookrightarrow Dₜ(π) \times St(p - Δ). \]

Let \( l' = l_{abs}(p - Δ) \). Then we also have

\[ Dₜ(π)ₚₙₚ \hookrightarrow (Dₜ(π) \times St(p - Δ))ₚₙₚ. \]

Now we consider the component with the cuspidal support same as $\text{csupp}(\text{St}(\mathfrak{p} - \Delta))$ in the second factor of $G_{n-1-n} \times G_n (n = n(\pi))$. We apply the geometric lemma and a standard computation shows that such component is of the form $D_{\pi}(\pi) \boxtimes \text{St}(\mathfrak{p} - \Delta)$ with multiplicity one. Such form must come from $D_{\pi}(\pi)_{\mathcal{N}_{\pi}}$ and so for other composition factor in $\pi$ of $\text{D}_{\pi}(\pi, \mathcal{N}_{\pi})$, cannot have the factor of the form $\pi' \boxtimes \text{St}(\mathfrak{p} - \Delta)$ for some $\tau' \in \text{Irr}$. This implies the lemma. \hfill \Box

We also need a variant.

**Lemma 9.7.** We use the notations in Lemma 9.6. Then

$$|\mathfrak{m}_{\pi}(\pi)| \leq |\mathfrak{m}_{\pi}(\pi)| - 1.$$  

*Proof.* Again write $\Delta = [a, b]_{\mathfrak{p}}$. We have shown in Lemma 9.6 that

$$\tau \mapsto D_{\pi}(\pi) \times \text{St}(\mathfrak{p} - \Delta).$$

Let $q = \mathfrak{m}_{\pi}(\pi)$. Let $\tau_1 \boxtimes \tau_2$ be any composition factor in $D_{\pi}(\pi)_{\mathcal{N}_{\pi}}$. By using $\mathfrak{m}_{\pi}(D_{\pi}(\pi)) = \emptyset$, if $\text{csupp}(\tau_2)$ contains only cuspidal representations $\nu_1^a_{\rho}, \ldots, \nu_k^b_{\rho}$ (with certain multiplicities), then $\text{csupp}(\tau_2)$ contains only cuspidal representations $\nu_1^a_{\rho}, \ldots, \nu_k^b_{\rho}$ (with certain multiplicities).

Let $\tau_1' \boxtimes \tau_2'$ be a composition factor in $\text{St}(\mathfrak{p} - \Delta)_{\mathcal{N}_{\pi}}$ for some $k'$. Then the cuspidal representation $\nu_1^a_{\rho}$ can have at most multiplicity $|\mathfrak{m}_{\pi}(\pi)| - 1$ in $\text{csupp}(\tau_2')$. Thus, if the cuspidal representations in $\text{csupp}(\tau_2 \times \tau_2)$ lie in $\nu_1^a_{\rho}, \ldots, \nu_k^b_{\rho}$, then $\nu_k^b_{\rho}$ can appear with multiplicity at most $|\mathfrak{m}_{\pi}(\pi)| - 1$ in $\text{csupp}(\tau_2 \times \tau_2')$. This proves the lemma. \hfill \Box

For convenience, for a segment $\Delta = [a, b]_{\mathfrak{p}}$, set $a(\Delta) = \nu_1^a_{\rho}$.

**Lemma 9.8.** We use the notations in Lemma 9.6. Let $\Delta$ be any segment. Then $|\mathfrak{m}_{\Delta}(\pi)| \leq |\mathfrak{m}_{\Delta}(\pi)|$.

*Proof.* Let $q = \mathfrak{m}_{\Delta}(\pi)$. Then

$$\tau \mapsto \text{St}(q) \times \text{D}^l_{\pi}(\pi).$$

Now, we have:

$$\text{D}_{\Delta}(\pi) \boxtimes \text{St}(\Delta') \mapsto \pi_{\mathcal{N}_{\pi}} \mapsto (\text{St}(q) \times \text{D}^l_{\pi}(\pi))_{\mathcal{N}_{\pi}},$$

where $l = l_{abs}(\Delta')$. Then $\tau \boxtimes \text{St}(\Delta')$ is still a simple composition factor in $\pi_{\mathcal{N}_{\pi}}$. Then, by the geometric lemma on $(\text{St}(q) \times \text{D}^l_{\pi}(\pi))_{\mathcal{N}_{\pi}}$, $\tau$ is a composition factor of $\tau_1 \times \tau_2$ for some $\tau_1 \in \text{Irr}$ and $\tau_2 \in \text{Irr}$ such that $\tau_1 \boxtimes \tau_1'$ is a simple composition factor in $\text{St}(q)_{\mathcal{N}_{\pi}}$ for some $\tau$ and some $\tau_1' \in \text{Irr}$ and $\tau_2 \boxtimes \tau_2'$ is a simple composition factor in $D^l_{\pi}(\pi)_{\mathcal{N}_{\pi}}$ for some $\tau$ and some $\tau_2' \in \text{Irr}$. The composition factors in $\text{St}(q)_{\mathcal{N}_{\pi}}$ can be computed from the geometric lemma again and so it is a straightforward computation to give $|\mathfrak{m}_{\Delta}(\pi_1)| \leq |q|$. Since $\mathfrak{m}_{\Delta}(\pi) = \emptyset$, $\mathfrak{m}_{\Delta}(\pi_1) = \emptyset$. Thus, we have that $\tau$ is a composition factor of $\text{St}(q') \times \omega$ for some $\omega \in \text{Irr}$ with $\mathfrak{m}_{\Delta}(\omega) = \emptyset$ and a left $\tilde{\Delta}$-saturated multisegment $q'$ with $|q'| \leq |q|$. (Here a left $\tilde{\Delta}$-saturated multisegment $\mathfrak{m}$ is a multisegment with all segments $\Delta$ in $\mathfrak{m}$ satisfy $\tilde{\Delta} \subset \Delta$ and $a(\tilde{\Delta}) = a(\Delta)$.) The remaining proof is precisely the left version of the arguments in the proof of Lemma 9.7. \hfill \Box

**Lemma 9.9.** Let $\pi \in \text{Irr}$. Let $\Delta$ be a segment. Let $\mathfrak{p} = \mathfrak{m}_{\Delta}(\pi)$. Let $\tau_1 \boxtimes \tau_2$ be an irreducible composition factor in $\pi_{\mathcal{N}_{\pi}}$ (for some $k$). Then

$$|\mathfrak{m}_{\Delta}(\tau_1)| \leq |\mathfrak{m}_{\Delta}(\tau)|.$$
Proof. Suppose \(|m_{\Delta}(\tau_1)| > |m_{\Delta}(\pi)|\) to derive a contradiction. Let \(q = m_{\Delta}(\tau_1)\). Then \(\text{St}(q) \boxtimes \tau'\) is a composition factor in \(\langle \tau_1 \rangle_N\), for some \(l\) and some \(\tau' \in \text{Irr} \). Hence, \(\text{St}(q) \boxtimes \tau' \boxtimes \tau_2\) appears in a Jacquet functor of \(\pi\). By comparing cuspidal support, the submodule takes the form \(\omega \boxtimes \tau'_1 \boxtimes \tau'_2\), where

\[
\text{csupp}(\omega) = \text{csupp}(\text{St}(q)), \quad \text{csupp}(\tau_1) = \text{csupp}(\tau'_1), \quad \text{csupp}(\tau_2) = \text{csupp}(\tau'_2).
\]

Recall that any irreducible representation, particularly \(\omega\), can be realized as a submodule of the dual of a standard module. This then gives that

\[
\omega \hookrightarrow \text{St}(q') \times \omega'
\]

for some left \(\Delta\)-saturated multisegment \(q'\) and some irreducible module \(\omega'\) with \(a(\Delta) \notin \text{csupp}(\omega')\). In particular, we also have \(|q'| > |q|\).

Now, by Frobenius reciprocity, we have \(\text{St}(q') \boxtimes \kappa\) is a simple submoulde of a Jacquet module of \(\pi\) for some \(\kappa \in \text{Irr}\). This implies that \(D_{q'}^\Delta(\pi) \neq 0\). However, as we have shown that \(|q'| > |q|\), this gives a contradiction to the maximality of \(m_{\Delta}(\pi)\), as desired. \(\square\)

9.3. Strong commutativity. We now prove pre-commutativity implies strong commutativity for essentially square-integrable representations.

Theorem 9.10. Let \(\sigma_1, \sigma_2 \in \text{Irr}\) be both essentially square-integrable. Let \(\pi \in \text{Irr}\). If \((\sigma_1, \sigma_2, \pi)\) is pre-RdLi-commutative, then \((\sigma_1, \sigma_2, \pi)\) is strongly RdLi-commutative.

Proof. Write \(\sigma_1 = \text{St}(\Delta_1)\) and \(\sigma_2 = \text{St}(\Delta_2)\) for some segments \(\Delta_1\) and \(\Delta_2\). Let \(p = m_{\Delta_1}(\pi)\) and let \(\tilde{\sigma} = \text{St}(p - \Delta_1)\). We also write \(\Delta_1 = [a_1, b_1]_p\) and \(\Delta_2 = [a_2, b_2]_p\). (If \(\Delta_1\) and \(\Delta_2\) are not in the same cuspidal line, the case is easy and we shall omit that.) Then \((\text{St}(p), \pi)\) is a Rd-irreducible pair (Proposition 5.3). By Proposition 5.2, \((\text{St}(p), \text{St}(\Delta_2), \pi)\) is pre-RdLi-commutative. Hence, by Proposition 6.12(1), \((\text{St}(p), \text{St}(\Delta_2), \pi)\) is a strongly RdLi-commutative triple.

We now check the conditions in Proposition 6.12. Recall that \(D_{\Delta_1}(\pi)\) has only multiplicity one in \(D_{\Delta_1}(\pi)\) [24, Proposition 11.5]. Hence, it suffices to show that \(D_{\Delta_1}(I_{\Delta_2}(\pi))\) cannot embed to \(\text{St}(\Delta_2) \times \tau\) for any simple composition factor \(\tau \neq D_{\Delta_1}(I_{\Delta_2}(\pi))\) of \(D_{\Delta_1}(\pi)\).

Suppose not to obtain a contradiction. We then have an embedding:

\[
(*) \quad D_{\Delta_1}(I_{\Delta_2}(\pi)) \hookrightarrow \text{St}(\Delta_2) \times \tau.
\]

By Lemma 6.7, \(|m_{\Delta_1}(\tau)| \leq |m_{\Delta_1}(\pi)| - 1\).

We now consider several cases:

Case 1: \(|m_{\Delta_1}(\tau)| = |m_{\Delta_1}(\pi)| - 1\). Using (*) and \(\text{St}(\Delta_2) \times \tau \hookrightarrow \text{St}(\Delta_2) \times D_{m_{\Delta_1}(\tau)}(\tau) \times \text{St}(m_{\Delta_1}(\tau))\), we have

\[
m_{\Delta_1}(\tau) \subset m_{\Delta_1}(D_{\Delta_1}(I_{\Delta_2}(\pi))).
\]

But further looking at the number of segments in those \(m_{\Delta_1}\) from Lemmas 7.2 and 8.6, it must be an equality, giving a contradiction to Lemma 9.6.

Case 2: \(|m_{\Delta_1}(\tau)| < |m_{\Delta_1}(\pi)| - 2\). Again (*) implies that

\[
|m_{\Delta_1}(D_{\Delta_1}(I_{\Delta_2}(\pi)))| \leq |m_{\Delta_1}(\tau)| + |m_{\Delta_1}(\text{St}(\Delta_2))|,
\]

but this is impossible from Lemmas 7.2 and 8.6.

Case 3: \(|m_{\Delta_1}(\tau)| = |m_{\Delta_1}(\pi)| - 2\). Let \(p = m_{\Delta_1}(D_{\Delta_1}(\pi))\) and let \(q = m_{\Delta_1}(\tau)\). Let \(l = l_{\text{abs}}(p)\).

In such case, using (*), we have:

\[
D_p \circ D_{\Delta_1}(I_{\Delta_2}(\pi)) \boxtimes \text{St}(p) \hookrightarrow (\text{St}(\Delta_2) \times \tau)_N.
\]
Using the geometric lemma and comparing the number of segments in those $\text{mf}_{\Delta_1}$, the only possible layer that can contribute to the above embedding takes the form:

$$(\text{St}(\Delta') \times \tau_1) \boxtimes (\text{St}(\Delta') \times \tau_2),$$

where $\Delta' = [b_1 + 1, b_2]_\rho$ and $\Delta'' = [a_1, b_1]_\rho$, and $\tau_1 \boxtimes \tau_2$ is a simple composition factor in $\tau_{N_p}$ ($p = l_{abs}(p)$). Then

$$(9.23) \quad D_p \circ D_{\Delta_1}(I_{\Delta_2}(\pi)) \leftrightarrow \text{St}(\Delta') \times \tau_1$$

By Lemma 9.8, we also have

$$(9.24) \quad |\text{mf}_{\Delta_1}(\tau_1)| \leq |\text{mf}_{\Delta_2}(\pi)|$$

Note that, by using the commutativity of taking Jacquet functors on left and right, one deduces that

$$(9.23) \quad |\text{mf}_{\Delta_2}(\tau_1)| \leq |\text{mf}_{\Delta_2}(\pi)|$$

and so combines with (9.24)

$$|\text{mf}_{\Delta_2}(\tau_1)| \leq |\text{mf}_{\Delta_2}(\pi)|.$$  

With $\text{mf}_{\Delta_2}(\text{St}(\Delta')) = \emptyset$, (9.23) implies that

$$|\text{mf}_{\Delta_2}(D_p \circ D_{\Delta_1}(I_{\Delta_2}(\pi)))| \leq |\text{mf}_{\Delta_2}(\pi)|,$$

and so combining with (9.24), we have

$$|\text{mf}_{\Delta_2}(D_p \circ D_{\Delta_1}(I_{\Delta_2}(\pi)))| \leq |\text{mf}_{\Delta_2}(\pi)|.$$  

However, by Lemma 9.8, 

$$|\text{mf}_{\Delta_2}(D_p \circ D_{\Delta_1}(I_{\Delta_2}(\pi)))| = |\text{mf}_{\Delta_2}(I_{\Delta_2}(\pi))| = |\text{mf}_{\Delta_2}(\pi)| + 1,$$

giving a contradiction.

We have checked the conditions in Proposition 6.12(2), and it follows from that proposition that $(\sigma_1, \sigma_2, \pi)$ is also strongly RdLi-commutative.  

**Corollary 9.11.** Let $(\text{St}(\Delta_1), \text{St}(\Delta_2), \pi)$ be a RdLi-pre-commutative triple. Let $\Delta'$ be a $\Delta_1$-saturated segment with $D_{\Delta_1}(\pi) \neq 0$. Then $(\text{St}(\Delta'), \text{St}(\Delta_2), \pi)$ is also a strongly RdLi-commutative triple.

**Proof.** Let $\sigma_1 = \text{St}(\Delta')$. Let $p = \text{mf}_{\Delta_1}(\pi)$ and let $\bar{\sigma} = \text{St}(p - \Delta')$. The remaining follows the same argument as in the proof of Theorem 9.10.  

**Remark 9.12.** Let $\Delta = [a, b]_\rho$ and $\Delta' = [a', b]_\rho$. Suppose $(\text{St}(\Delta), \sigma_2, \pi)$ is a pre-RdLi-commutative triple. If $a' \leq a$, the above lemma implies that $(\text{St}(\Delta'), \sigma_2, \pi)$ is also a pre-RdLi-commutative triple. However, it is not true in general if $a' > a$. For example, take $\pi = \text{St}([0, 1]) \times \langle [0, 2] \rangle$. Let $\sigma = \sigma' = \text{St}([2])$. Let $\bar{\sigma} = \text{St}([0, 2])$. Then $(\sigma, \sigma', \pi)$ is a pre-RdLi-commutative triple. However, $(\bar{\sigma}, \sigma', \pi)$ is not a pre-RdLi-commutative triple.

9.4 Dual formulation for essentially square-integrable representations.

**Corollary 9.13.** Let $\sigma_1, \sigma_2 \in \text{Irr}$ be essentially square-integrable. Let $\pi \in \text{Irr}$. Then $(\sigma_1, \sigma_2, \pi)$ is a strongly RdLi-commutative triple if and only if $(\sigma_2, \sigma_1, I_{\sigma_2} \circ D_{\sigma_1}(\pi))$ is a strongly LdRi-commutative triple.

**Proof.** We only prove the only if direction and a proof for the if direction is similar. By Proposition 6.3, $\sigma' \in \text{Irr}^\square$ such that $(\sigma_1 \times \sigma', \pi)$ is Rd-irreducible pair (and $\sigma_1 \times \sigma'$ is irreducible) and $(\sigma_1 \times \sigma', \sigma_2, \pi)$ is still a strongly RdLi-commutative triple by Propositions 8.3 and 6.12(1). Thus, by Proposition 7.2 (the version that switches between left and right), $(\sigma_2, \sigma_1 \times \sigma', I_{\sigma_2} \circ D_{\sigma_1 \times \sigma'}(\pi))$ is a pre-LdRi-commutative triple. Thus, by Proposition 6.10 $(\sigma_2, \sigma_1, I_{\sigma_2} \circ D_{\sigma_1 \times \sigma}(\pi))$ is a pre-LdRi-commutative triple. By Proposition 6.2 and
Theorem 10.3. Let $D_{\sigma \times \sigma'} = D_{\sigma'} \circ D_{\sigma}$, we have that $I^R_\sigma \circ I_{\sigma'} \circ D_{\sigma \times \sigma'}(\pi) \cong D_{\sigma} \circ I_{\sigma'}(\pi)$ and so $(\sigma_2, \sigma_1, D_{\sigma} \circ I_{\sigma'}(\pi))$ is a pre-LdRi-commutative triple. Hence, by the LdRi-version of Theorem 9.10 $(\sigma_2, \sigma_1, D_{\sigma} \circ I_{\sigma'}(\pi))$ is a strongly LdRi-commutative triple. By Proposition 6.2 we of course also have $D_{\sigma} \circ I_{\sigma'}(\pi) \cong I_{\sigma'} \circ D_{\sigma}(\pi)$ and this gives the corollary.

Based on the square-integrable case, we raise the following question:

**Conjecture 9.14.** Corollary 9.13 holds for any $\sigma_1, \sigma_2 \in \text{Irr}$. 

10. Combinatorially commutations

10.1. Combinatorially commutative triples. Recall that strong commutation is defined in Definition 10.1. We also have a dual formulation:

**Definition 10.1.** Let $\Delta_1, \Delta_2$ be segments. Let $\pi \in \text{Irr}$. We say that $(\Delta_1, \Delta_2, \pi)$ is a dual combinatorially RdLi-commutative triple if

$$\eta_{\Delta_2}^L(D_{\Delta_1} \circ I_{\Delta_2}(\pi)) = \eta_{\Delta_2}^L(I_{\Delta_2}(\pi)).$$

**Remark 10.2.**

1. It is not a correct formulation (c.f. Theorem 10.3) if one changes the condition in Definition 10.1 to $\eta_{\Delta_2}^L(D_{\Delta_1}(\pi)) = \eta_{\Delta_2}^L(\pi)$ (while it is not far away, see the second remark). For example, if one takes $\pi = [-1, 0]$ and $\Delta_1 = [0]$ and $\Delta_2 = [0, 1]$, then $(\Delta_1, \Delta_2, \pi)$ is not strongly RdLi-commutative, but we still have $\eta_{\Delta_2}^L(D_{\Delta_1}(\pi)) = \eta_{\Delta_2}^L(\pi) = 0$.

2. On the other hand, it follows from Theorem 10.3 below that if $(\Delta_1, \Delta_2, \pi)$ is a dual strongly RdLi-commutative triple, then $D_{\Delta_1} \circ I_{\Delta_2}(\pi) \cong I_{\Delta_2} \circ D_{\Delta_1}(\pi)$ (Proposition 6.2). Then, we also have

$$\eta_{\Delta_2}^L(D_{\Delta_1}(\pi)) = \eta_{\Delta_2}^L(\pi).$$

3. Suppose $(\Delta_1, \Delta_2, \pi)$ is strongly RdLi-commutative. Note that if $D_{\Delta_1}^L(\pi) \neq 0$, then $(\Delta_1, \Delta_2, D_{\Delta_1}(\pi))$ is also strongly RdLi-commutative. However, the analogue is not right for the induction one i.e. $(\Delta_1, \Delta_2, I_{\Delta_2}(\pi))$ is not necessarily strongly RdLi-commutative.

4. One can also define a notion of combinatorially LdRi-commutative triple as follows. A triple $(\Delta_1, \Delta_2, \pi)$ is said to be combinatorially LdRi-commutative if $\eta_{\Delta_1}^L(I_{\Delta_2}^R(\pi)) = \eta_{\Delta_1}^L(\pi)$. It follows from definitions that $(\Delta_1, \Delta_2, \pi)$ is dual RdLi-commutative if and only if $(\Delta_2, \Delta_1, D_{\Delta_1} \circ I_{\Delta_2}(\pi))$ is strongly LdRi-commutative.

10.2. Equivalent definitions.

**Theorem 10.3.** Let $\Delta_1, \Delta_2$ be segments. Let $\pi \in \text{Irr}$. Then the followings are equivalent:

1. $(\text{St}(\Delta_1), \text{St}(\Delta_2), \pi)$ is a strongly RdLi-commutative triple;
2. $(\text{St}(\Delta_2), \text{St}(\Delta_1), I_{\Delta_2} \circ D_{\Delta_1}(\pi))$ is a strongly RdLi-commutative triple;
3. $(\Delta_1, \Delta_2, \pi)$ is a combinatorially RdLi-commutative triple;
4. $(\Delta_1, \Delta_2, \pi)$ is a dual combinatorially RdLi-commutative triple.

**Proof.** (1) $\iff$ (2) is Corollary 9.13 (1) $\Rightarrow$ (3) is proved in Lemma 6.2.
Let $\tau = I_{\Delta_2}(\pi)$. Write $p = mp(\pi, \Delta_1)$. Let $l = l_{abs}(p)$. We now prove (3) $\Rightarrow$ (1). We consider the following commutative diagram:

\[
\begin{array}{c}
D_p(\tau) \times \text{St}(p) \xrightarrow{\iota} \tau_{N_1} \xrightarrow{i'} (\text{St}(\Delta_2) \times \pi)_{N_1} \xrightarrow{\iota'} (\text{St}(\Delta_2) \times D_p(\pi) \times \text{St}(p))_{N_1} \\
\downarrow s \downarrow \downarrow s \\
\text{St}(\Delta_2) \times \pi N_1 \xrightarrow{i} \text{St}(\Delta_2) \times (D_p(\pi) \times \text{St}(p))_{N_1} \\
\end{array}
\]

Here the vertical maps are from projecting to the top layers in the geometric lemma and the horizontal maps except the leftmost one come from the unique submodules of integrals. The leftmost map comes from the unique embedding of such submodule.

Suppose (3) holds. (3) implies that $(\text{St}(p), \tau)$ is a Rd-irreducible pair by Proposition 5.3. Now, we consider the composition of the following maps

\[
\tau \mapsto \text{St}(\Delta_2) \times \pi \mapsto \text{St}(\Delta_2) \times D_p(\pi) \times \text{St}(p),
\]

and by the Rd-irreducibility of $(\text{St}(p), \tau)$ and Proposition 5.3, the composition of the toppest horizontal maps and the rightmost vertical map, which gives a map from $D_p(\pi) \times \text{St}(p)$ to $(\text{St}(\Delta_2) \times D_p(\pi)) \times \text{St}(p)$, is non-zero. Thus, tracing the commutative diagram, the map $s \circ i' \circ i$ is non-zero. This implies the pre-commutativity of $(\text{St}(p), \text{St}(\Delta_2), \pi)$. Hence, we also have the pre-commutativity of $(\text{St}(\Delta_1), \text{St}(\Delta_2), \pi)$ by Proposition 6.11. This implies the strong commutativity by Theorem 9.10 and so proves (1).

A proof for (2) $\Leftrightarrow$ (4) is similar to (1) $\Leftrightarrow$ (3). \hfill $\square$

11. Applications on constructing strongly commutative triples.

11.1. A consequence on producing more strongly commutative triples.

**Corollary 11.1.** (c.f. Corollary 9.11) Let $(\Delta_1, \Delta_2, \pi)$ be a strongly RdLi-commutative triple. Let $\Delta'$ be a $\Delta_1$-saturated segment such that $D_{\Delta'}(\pi) \neq 0$.

1. $(\Delta', \Delta_2, \pi)$ is also a strongly RdLi-commutative triple;
2. Let $\Delta''$ be another $\Delta_1$-saturated segment such that $D_{\Delta''} \circ D_{\Delta'}(\pi) \neq 0$. Then $(\Delta'', \Delta_2, D_{\Delta'}(\pi))$ is a strongly RdLi-commutative triple.

**Proof.** (1) is shown in Corollary 9.11 but we use our new results here. By definitions and Theorem 10.3, $\eta_{\Delta_1}(I_{\Delta_2}(\pi)) = \eta_{\Delta_1}(\pi)$ automatically implies that $\eta_{\Delta'}(I_{\Delta_2}(\pi)) = \eta_{\Delta'}(\pi)$. By Theorem 10.3, we have (1).

We now consider (2). By (1) and Proposition 6.2, we have $I_{\Delta_2} \circ D_{\Delta'}(\pi) \cong D_{\Delta'} \circ I_{\Delta_2}(\pi)$. Then $\eta_{\Delta_1}(I_{\Delta_2} \circ D_{\Delta'}(\pi))$ is obtained from $\eta_{\Delta_1}(I_{\Delta_2}(\pi))$ by decreasing $\varepsilon_{\Delta'}(I_{\Delta_2}(\pi))$ by 1; and similarly $\eta_{\Delta_1}(D_{\Delta'}(\pi))$ is obtained from $\eta_{\Delta_1}(\pi)$ by decreasing $\varepsilon_{\Delta'}(\pi)$ by 1. Now, the combinatorial commutation for $(\Delta_1, \Delta_2, \pi)$ implies $\eta_{\Delta_1}(I_{\Delta_2} \circ D_{\Delta'}(\pi)) = \eta_{\Delta_1}(D_{\Delta'}(\pi))$. Thus $\eta_{\Delta''}(I_{\Delta_2} \circ D_{\Delta'}(\pi)) = \eta_{\Delta''}(D_{\Delta'}(\pi))$. This implies (2) again by Theorem 10.3. \hfill $\square$

11.2. A consequence on level preserving integrals. For $\pi \in \text{Irr}$, we denote by $\text{lev}(\pi)$, the level of $\pi$ in the sense of Bernstein-Zelevinsky [BZ77] i.e., the largest integer $i$ such that the $i$-th Bernstein-Zelevinsky derivative of $\pi$ is non-zero. Since this is the only part we need those notions, we refer the reader to [Ch22+] for definitions.

It is more convenient to use the machinery of the highest derivative multisegment in [Ch22+]. We briefly recall the definition. For $\pi \in \text{Irr}$ and a cuspidal representation $\rho$ and
an integer $c$, let $m^{\rho}(\pi)$ be the segment with largest $l_{abs}(m^{\rho}(\pi))$ satisfying the following two properties:

1. for any segment $\tilde{\Delta}$ in $a(\tilde{\Delta}) = \rho$;
2. $D_{m^{\rho}(\pi)}(\pi) \neq 0$.

The highest derivative multisegment of $\pi$ is defined as:

$$h\partial(\pi) := \sum_{\rho} m^{\rho}(\pi),$$

where $\rho$ runs for all isomorphism classes of cuspidal representations.

Note that the multiplicity of the segment $[\rho, \nu^c \rho]$ in $m^{\rho}(\pi)$ is equal to

$$\varepsilon_{[\rho, \nu^c \rho]}(\pi) = \varepsilon_{[\rho, \nu^c+1 \rho]}(\pi).$$

Hence, $\varepsilon_{\Delta}(\pi)$ determines $m^{\rho}(\pi)$. Indeed, by a simple induction, one can recover all $\varepsilon_{\Delta}(\pi)$ from $m^{\rho}(\pi)$.

**Corollary 11.2.** Let $\Delta$ be a segment and let $\pi \in \text{Irr}$. Suppose $\text{lev}(I_{\Delta}(\pi)) = \text{lev}(\pi)$. Then, for any segment $\Delta'$ with $D_{\Delta}(\pi) \neq 0$, $(\Delta', \Delta, \pi)$ is a strongly RdLi-commutative triple. In particular, $D_{\Delta'} \circ I_{\Delta}(\pi) \equiv I_{\Delta} \circ D_{\Delta'}(\pi)$.

**Proof.** By Lemma 6.1, $\eta_{\tilde{\Delta}}(\pi) \leq \eta_{\tilde{\Delta}}(I_{\Delta}(\pi))$ for any segment $\tilde{\Delta}$. Let $h_1$ and $h_2$ be the highest derivative multisegments for $I_{\Delta}(\pi)$ and $\pi$ respectively. The equality on the level implies that $l_{abs}(h_1) = l_{abs}(h_2)$.

Since $\eta_{\tilde{\Delta}}$ determines all $\varepsilon_{\Delta}$ as discussed above, we have that all the inequalities are equalities. Indeed, let $p_1(\rho)$ and $p_2(\rho)$ be the number of segments in $h_1$ and $h_2$ containing $\rho$. Then the above inequality forces that $p_1(\rho) = p_2(\rho)$ for all $\rho$. But since the sum of $p_i(\rho)$ is equal to $l_{abs}(h_i)$ ($i = 1, 2$), we must have that all inequalities and equalities. This then inductively recovers that all the inequalities for $\eta$ are equalities.

Now, in particular, $\eta_{\Delta}(\pi) = \eta_{\Delta}(I_{\Delta}(\pi))$. The strong commutation then follows from Theorem 11.2. The commutation then follows from Proposition 6.2. \qed

### 11.3. Producing commutative triples for sequences.

**Corollary 11.3.** Let $\Delta, \Delta_1', \Delta_2'$ be segments and let $\pi \in \text{Irr}$. Suppose $(\Delta, \Delta_1', \pi)$ and $(\Delta, \Delta_2', I_{\Delta_1'}(\pi))$ are strongly RdLi-commutative triples. If $I_{\Delta_2'} \circ I_{\Delta_1'}(\pi) \equiv I_{\Delta_1' \cup \Delta_2'} \circ I_{\Delta_1' \cap \Delta_2'}(\pi)$, then the following conditions hold:

1. $(\Delta, \Delta_1' \cup \Delta_2', \pi)$ and $(\Delta, \Delta_1' \cap \Delta_2', I_{\Delta_1' \cup \Delta_2'}(\pi))$ are strongly RdLi-commutative triples;
2. $(\Delta, \Delta_1' \cap \Delta_2', \pi)$ and $(\Delta, \Delta_1' \cup \Delta_2', I_{\Delta_1' \cap \Delta_2'}(\pi))$ are strongly RdLi-commutative triples;
3. $I_{\Delta_2'} \circ I_{\Delta_1'} \circ D_{\Delta}(\pi) \equiv I_{\Delta_1' \cup \Delta_2'} \circ I_{\Delta_1' \cap \Delta_2'} \circ D_{\Delta}(\pi)$.

**Proof.** By using Theorem 11.3 twice, we have:

$$\eta_{\Delta}(\pi) = \eta_{\Delta}(I_{\Delta_1'}(\pi)) = \eta_{\Delta}(I_{\Delta_2'} \circ I_{\Delta_1'}(\pi)).$$

On the other hand, by Lemma 6.1,

$$\eta_{\Delta}(\pi) \leq \eta_{\Delta}(I_{\Delta_1' \cup \Delta_2'}(\pi)) \leq \eta_{\Delta}(I_{\Delta_1' \cap \Delta_2'} \circ I_{\Delta_1' \cup \Delta_2'}(\pi)).$$

Now, using $I_{\Delta_2'} \circ I_{\Delta_1'}(\pi) \equiv I_{\Delta_1' \cap \Delta_2'} \circ I_{\Delta_1' \cup \Delta_2'}(\pi)$, $\eta_{\Delta}(I_{\Delta_2'} \circ I_{\Delta_1'}(\pi)) = \eta_{\Delta}(I_{\Delta_1' \cap \Delta_2'} \circ I_{\Delta_1' \cup \Delta_2'}(\pi))$. Thus, the above inequalities are equalities. Thus, by Theorem 11.3 $(\Delta, \Delta_1', \pi)$ and $(\Delta, \Delta_2', I_{\Delta_1'}(\pi))$ are strongly RdLi-commutative triples. This proves (1).

Since $I_{\Delta_1' \cup \Delta_2'} \circ I_{\Delta_1' \cap \Delta_2'} = I_{\Delta_1' \cap \Delta_2'} \circ I_{\Delta_1' \cup \Delta_2'}$, one can similarly prove (2). Now (3) follows from Proposition 6.2. \qed
11.4. A consequence on duals. For a segment \( \Delta = [a, b] \), let \( \Delta' = [-b, -a] \).

**Corollary 11.4.** Let \( \pi \in \text{Irr} \). Let \( \Delta, \Delta' \) be segments. Then \( (\Delta, \Delta', \pi) \) is a strongly RdLi-commutative triple if and only if \( (\Delta', \Delta, \pi') \) is a strongly LdRi-commutative triple.

**Proof.** We only prove the only if direction and a proof for the if direction is similar. By Corollary 12.4, \( \eta_{\Delta}(\pi) = \eta_{\Delta'}(\pi') \) and \( \eta_{\Delta}(I_{\Delta'}(\pi)) = \eta_{\Delta'}(I_{\Delta}(\pi')) = \eta_{\Delta'}(I_{\Delta}(\pi')) \). By Theorem 10.3, \( \eta_{\Delta}(\pi) = \eta_{\Delta}(I_{\Delta}(\pi)) \). Combining equations, we have \( \eta_{\Delta}(\pi') = \eta_{\Delta}(I_{\Delta}(\pi')) \). Thus, by Theorem 10.3, \( (\Delta', \Delta, \pi') \) is a strongly LdRi-commutative triple. \( \square \)

12. Commutative triples from branching laws

12.1. Generalized Gan-Gross-Prasad relevant pairs. We shall assume \( D = F \) from now on. We simply write \( \nu(g) = |\det(g)| \). Gan-Gross-Prasad [GGP20] introduces a notion of relevant pairs for Arthur type representations, governing their branching law for general linear groups [Ch22]. Such notion can be reformulated in terms of the Bernstein-Zelevinsky derivatives, see [Ch22]. This hints some connections with derivatives (also see [Ch22+]).

We first extend the notion of strongly commutative triples from segments to multisegments. For two segments \( \Delta \) and \( \Delta' \), we write \( \Delta > \Delta' \) if \( \Delta \) and \( \Delta' \) are linked and \( a(\Delta) \cong \nu^c \cdot a(\Delta') \) for some \( c > 0 \).

**Definition 12.1.** Let \( m \) be a multisegment and \( \pi \in \text{Irr} \). We impose an ordering on \( m = \{\Delta_1, \ldots, \Delta_r\} \) such that for any \( i < j \), \( \Delta_i \neq \Delta_j \). Define

\[
D^R_m(\pi) := D^R_{\Delta_1} \circ \ldots \circ D^R_{\Delta_r}(\pi),
\]

\[
I^L_m(\pi) := I^L_{\Delta_1} \circ \ldots \circ I^L_{\Delta_r}(\pi).
\]

It is shown in [Ch22+] that the \( D^R_m(\pi) \) is independent of a choice of such ordering. Using \( D^R_{\Delta} \circ I^L_{\Delta} \) is the identity map, we also have that \( I^L_m(\pi) \) is independent of a choice of an ordering.

To switch left and right versions suitably, we also need to change the ordering:

**Definition 12.2.** Let \( m, n \) be multisegments. We again impose an ordering on \( m = \{\Delta_1, \ldots, \Delta_r\} \) (resp. \( n = \{\Delta'_1, \ldots, \Delta'_s\} \)) given by (c.f. [Zo80] Theorem 6.1): for any \( i < j \),

\( \Delta_i \neq \Delta_j \) (resp. \( \Delta'_i \neq \Delta'_j \)).

Let \( m_i = \{\Delta_1, \ldots, \Delta_i\} \) and let \( n_j = \{\Delta'_1, \ldots, \Delta'_j\} \). Let \( m, n \in \text{Mult} \). Let \( \pi \in \text{Irr} \). We say that \( (m, n, \pi) \) is a strongly RdLi-commutative triple if for any \( 1 \leq i \leq r \) and \( 1 \leq j \leq s \),

\( (\text{St}(\Delta_i), \text{St}(\Delta'_j), I_{\Delta_i-1} \circ D_{m_{i-1}}(\pi)) \) is strongly RdLi-commutative in the sense of Definition 1.1.

We remark that when \( n = \emptyset \), the condition of strong commutation is automatic. We can now define a notion of generalized relevant pairs.

**Definition 12.3.** For \( \pi \in \text{Irr} \) and \( \pi' \in \text{Irr} \), we say that \( (\pi, \pi') \) is (generalized) relevant if there exist multisegments \( m, n \) such that \( (m, n, \nu^{1/2} \pi) \) is a strongly RdLi-commutative triple and

\[
D^R_m(\nu^{1/2} \cdot \pi) \cong D^L_n(\pi').
\]

Such notion is expected to govern the quotient branching law:

**Conjecture 12.4.** Let \( \pi \in \text{Irr}(G_{n+1}) \) and let \( \pi' \in \text{Irr}(G_n) \). Then \( \text{Hom}_{G_n}(\pi, \pi') \neq 0 \) if and only if \( (\pi, \pi') \) is a relevant pair.

We will prove the conjecture in [Ch22+d].
12.2. Duality. For a multisegment $m = \{\Delta_1, \ldots, \Delta_r\}$, define $m^\vee = \{\Delta_1^\vee, \ldots, \Delta_r^\vee\}$.

**Corollary 12.5.** Let $\pi, \pi' \in \text{Irr}$. Then $(\pi, \pi')$ is relevant if and only if $(\pi^\vee, \pi^\vee)$ is relevant.

**Proof.** Then there exist multisegments $m$ and $n$ such that

$$D^R_m(\nu^{1/2} \pi) \cong D^L_n(\pi')$$

and $(m, n, \nu^{1/2} \pi)$ is a strongly RdLi-commutative triple. Then, by Corollary 2.7

$$D^L_m(\nu^{-1/2} \pi^\vee) \cong D^R_n(\pi'^\vee).$$

Hence, $D^L_{m_\nu^{1/2} n}(\pi^\vee) \cong D^R_{m_\nu^{1/2} n}(\nu^{1/2} \pi^\vee)$.

It remains to check that $(\nu^{1/2} n^\vee, \nu^{1/2} m^\vee, \nu^{1/2} \pi^\vee)$ is a strongly RdLi-commutative triple. We write $m = \{\Delta_1, \ldots, \Delta_r\}$ and $n = \{\Delta'_1, \ldots, \Delta'_s\}$ with the ordering in Definition 12.2 Let $m_j = \{\Delta_1, \ldots, \Delta_j\}$, $n_j = \{\Delta'_1, \ldots, \Delta'_j\}$ and $m_j = \{\Delta_{j+1}, \ldots, \Delta_r\}$, $n_j = \{\Delta'_{j+1}, \ldots, \Delta'_s\}$.

By definition, we have that $(\Delta_j, \Delta'_j, I_{n_{j-1}} \circ O_{m_{j-1}}(\pi))$ is a strongly RdLi-commutative triple. Thus, by the duality in Corollary 11.1 with Lemma 2.6 and Corollary 2.7 (\Delta^\vee_j, \pi^\vee_j, I_{n_j}^R \circ D^L_{m_j}(\nu^{-1/2} \pi^\vee_j)) is a strongly LdRi-commutative triple. By Corollary-10 (\Delta^\vee_j, \Delta'_j, I_{n_j}^R \circ D^L_{m_j}(\nu^{-1/2} \pi^\vee_j)) is a strongly RdLi-commutative triple.

By multiple uses of Definition 12.2 and Proposition 5.2, we have:

$$I^L_{m_j} \circ O_{m_j} \circ D^R_m(\nu^{1/2} \pi) \cong I^L_{n_j} \circ O_{n_j} \circ D^R_n(\pi').$$

and so

$$I^L_{m_j} \circ O_{m_j} \circ D^R_m(\nu^{1/2} \pi) \cong I^L_{n_j} \circ O_{n_j} \circ D^R_n(\pi').$$

Now, taking the dual and using Lemma 2.6 again, we have:

$$I^L_{m_j} \circ O_{m_j} \circ D^R_m(\nu^{-1/2} \pi^\vee) \cong I^L_{n_j} \circ O_{n_j} \circ D^R_n(\pi'^\vee).$$

Thus, combining above, we have that $(\Delta^\vee_j, \Delta'_j, I_{m_j}^L \circ D_{m_j}(\nu^{-1/2} \pi^\vee_j))$ is a strongly RdLi-commutative triple. Thus $(n^\vee, m^\vee, \pi^\vee)$ is a strongly RdLi-commutative triple. Now, we obtain that $(\nu^{1/2} n^\vee, \nu^{1/2} m^\vee, \nu^{1/2} \pi^\vee)$ is a strongly RdLi-commutative triple by imposing a shift of $\nu^{1/2}$.

12.3. Examples.

**Example 12.6.** Let $\pi$ and $\pi'$ be trivial representations of $G_{n+1}$ and $G_n$ respectively. It is clear that $\text{Hom}_{G_n}(\pi, \pi') \neq 0$. In such case,

$$\pi = \text{St}\left(\left\{\frac{n}{2}, \ldots, \left\lfloor\frac{n}{2}\right\rfloor\right\}\right), \quad \pi' = \text{St}\left(\left\{\left\lfloor\frac{(n-1)}{2}\right\rfloor, \ldots, \left\lfloor\frac{(n-1)}{2}\right\rfloor\right\}\right).$$

We have that $(\nu^{1/2} \pi)_{\nu_j} = \pi' \otimes \nu^{(n+1)/2}$. Hence, $D_{[(n+1)/2]}(\nu^{1/2} \pi) \cong \pi'$ and so $\left\{\left\lfloor\frac{1}{2}\right\rfloor, 0, \pi\right\}$ defines a strongly RdLi-commutative triple.

**Example 12.7.** Let $\Delta_1, \Delta_2$ be segments such that $\text{St}(\Delta_1)$ and $\text{St}(\Delta_2)$ are representations of $G_{n+1}$ and $G_n$ respectively. It is a well-known result from the Rankin-Selberg theory of Jacquet–Piatetski-Shapiro–Shalika that $\text{Hom}_{G_n}(\text{St}(\Delta), \text{St}(\Delta)) \neq 0$. Write $\Delta = [a, b]_\rho$ and $\Delta' = [a', b']_\rho$. We now produce strong commutations according to the following cases:

1. Case 1: $\rho \neq \nu^c \rho'$. In such case, by Example 6.3.1 and Theorem 9.10 $(\nu^{1/2} \Delta, \Delta', \pi)$ is a strongly RdLi-commutative triple and we have that $I_{\Delta'} \circ O_{\nu^{1/2} \Delta}(\nu^{1/2} \text{St}(\Delta)) \cong \text{St}(\Delta')$. 


(2) Case 2: $\rho \cong \nu^c \rho'$ for some integer $c$. By rewriting the segments if necessary, we shall assume $\rho = \rho'$ from now on.

- Case 2(a): $a' < a$. By Example 6.4 (2) and Theorem 9.10 $(\nu^{1/2} \Delta, \Delta', \nu^{1/2} \pi)$ defines the desired strongly RdLi-commutative triple.
- Case 2(b): $b' < b$. By Example 6.4 (3) and Theorem 9.10 $(\nu^{1/2} \Delta, \Delta', \nu^{1/2} \pi)$ defines the desired strongly RdLi-commutative triple.
- Case 2(c): $(\nu^{1/2} \Delta) \cap \Delta' = \emptyset$. By Example 6.4 (1) and Theorem 9.10 $(\nu^{1/2} \Delta, \Delta', \nu^{1/2} \pi)$ defines the desired strongly RdLi-commutative triple.
- Case 2(d): $a \leq a' \leq b \leq b'$. Let $\Delta = (\nu^{1/2} \Delta) \cap \Delta'$. Then $((\nu^{1/2} \Delta) \setminus \Delta, \Delta' \setminus \Delta, \nu^{1/2} \pi)$ defines a strongly RdLi-commutative triple.

**Example 12.8.** Let $\pi = \text{St}([-n/2, c - 1], [c, n/2])$ and let $\pi' = \text{St}([-n/2, (n-1)/2], (n-1)/2]$ for some $-n/2 \leq c \leq n/2$. Let

$$\Delta = [c + 1/2, (n-1)/2], \quad \Delta' = [c + 1/2, (n-1)/2].$$

Note that Example 6.4 (3) and Theorem 9.10 $(\Delta, \Delta', \pi)$ is a strongly RdLi-commutative triple. This agrees with the expectation from [Qa23+].

### 13. Appendix: Compositions of supporting orbits

We use the notations in Section 13.1. In particular, we have $P, Q$ to be standard parabolic subgroups in $G$. Let $R \subset Q$ be a standard parabolic subgroup of $G$ and let $R' = R \cap M_Q$. We enumerate the elements in $W_{PQ}(G)$ as $w_1, \ldots, w_r$ such that $i < j$ implies $w_i \not\geq w_j$. For each $i$, we also enumerate the elements in $W_{P\cap R'}(M_Q)$ as $u_{i,1}, \ldots, u_{i,j_i}$. Then we obtain an enumeration on the elements in $W_{P,R}(G)$ as:

$$w_1 u_{1,1}, \ldots, w_1 u_{1,j_1}, w_2 u_{2,1}, \ldots, w_2 u_{2,j_2}, \ldots, w_r u_{r,1}, \ldots, w_r u_{r,r_j}.$$  

If we relabel as $x_1, x_2, \ldots, x_k$ then the enumeration satisfies that $k < l$ implies that $x_k \not\geq x_l$.

**Proposition 13.1.** We use the notations above. Let $\lambda$ be a submodule in $(\text{Ind}^G_P \pi)_{N_Q}$ with the supporting orbit $PwQ$. Let $\lambda'$ be a simple submodule of $\lambda_{N_{R'}}$. Suppose the embedding

$$\lambda' \hookrightarrow \lambda_{N_{R'}} \hookrightarrow \text{Ind}_{P \cap R'}^{M_Q}(\pi_{M \cap N_Q})^w$$

has the supporting orbit $Pw'w'R$. Then the embedding

$$\lambda' \hookrightarrow (\text{Ind}^G_P \pi)_{N_R}$$

has the supporting orbit $Pw'w'R$.

**Proof.** This follows from the fact that if $w_i u_{i,j} \not\geq w_i' u_{i',j'}$, then either $w_i \not\geq w_i'$ or $u_{i,j} \not\geq u_{i',j'}$ (see [De77]).

### References

[AL22] A. Aizenbud, E. Lapid, A binary operation on irreducible components of Lusztig’s nilpotent varieties II: Definition and properties, to appear in Pure and Applied Mathematics Quarterly.

[AAGS10] A. Aizenbud, D. Gourevitch, S. Rallis and G. Schiffmann, Multiplicity one theorems, Ann. of Math. (2) 172 (2010), no. 2, 1407-1434.

[Ar96] S. Ariki, On the decomposition numbers of the Hecke algebra of type $G(m, 1, n)$, J. Math. Kyoto Univ. 36 (1996), 789-808.

[Be92] J. Bernstein. Representations of p-adic groups. Harvard University, 1992. Lectures by Joseph Bernstein. Written by Karl E. Rumelhart.

[BZ76] I. N. Bernstein and A. V. Zelevinsky, Representations of the group $GL(n, F)$, where $F$ is a non-archimedean local field, Russian Math. Surveys 31:3 (1976), 1-68.

[BZ77] I. N. Bernstein and A. V. Zelevinsky, Induced representations of reductive p-adic groups, I, Ann. Sci. Ecole Norm. Sup. 10 (1977), 441-472.
A. Borel, Linear Algebraic Groups (2nd ed.), New York: Springer-Verlag, GTM 1991.

K.Y. Chan, Homological branching law for \((GL_{n+1}(F), GL_n(F))\): projectivity and indecomposability, Invent. math. (2021), doi.org/10.1007/s00222-021-01033-5

K.Y. Chan, Restriction for general linear groups: The local non-tempered Gan-Gross-Prasad conjecture (non-Archimedean case), Journal für die reine und angewandte Mathematik (Crelles Journal), vol. 2022, no. 783, doi.org/10.1515/crelle-2021-0066

K.Y. Chan, Ext-multiplicity theorem for standard representations of \((GL_{n+1}, GL_n)\), arXiv:2104.11529 Math. Z. 303, 45 (2023), https://doi.org/10.1007/s00209-022-03198-y

K.Y. Chan, Construction of simple quotients of Bernstein-Zelevinsky derivatives and highest derivative multisegments I: reduction to combinatorics, preprint (2022)

K.Y. Chan, Construction of simple quotients of Bernstein-Zelevinsky derivatives and highest derivative multisegments II: Minimality, preprint (2022).

K.Y. Chan, On the product functor on the inner forms of general linear group over a non-Archimedean local field, Transformation Groups (2024).

K.Y. Chan, Quotient branching law for \(p\)-adic \((GL_{n+1}, GL_n)\) I: generalized Gan-Gross-Prasad relevant pairs, arXiv:2212.05919 (2022).

K.Y. Chan, Quotient branching law for \(p\)-adic \((GL_{n+1}, GL_n)\) II: Minimality, preprint (2022).

K.Y. Chan, Construction of simple quotients of Bernstein-Zelevinsky derivatives and highest derivative multisegments III: Properties of minimal sequences, preprint (2022).

K.Y. Chan, Dual of the geometric lemma and the second adjointness theorem for \(p\)-adic reductive groups, preprint (2024).

K.Y. Chan and G. Savin, A vanishing Ext-branching theorem for \((GL_{n+1}(F), GL_n(F))\), Duke Math Journal, 2021, 170 (10), 2237-2261. https://doi.org/10.1215/00127094-2021-0028

K.Y. Chan and A. Pressley, Quantum affine algebras and affine Hecke algebras, Pacific J. Math. 174 (1996), no. 2, 295-326.

J. Chuang, R. Rouquier, Derived Equivalences for Symmetric Groups and \(sl_2\)-Categorification, Annals of Mathematics, 245-298 from Volume 167 (2008), 10.4007/annals.2008.167.245

Deodhar, V.V. Some characterizations of Bruhat ordering on a Coxeter group and determination of the relative Möbius function. Invent Math 39, 187-198 (1977). https://doi.org/10.1007/BF01390109

M. Fayers, Weights of multipartitions and representations of Ariki-Koike algebras, Advances in Mathematics, Volume 206, Issue 1, 2006, Pages 112-144, doi.org/10.1016/j.aim.2005.07.017.

W.T. Gan, B.H. Gross, and D. Prasad, Symplectic local root numbers, central critical L values, and restriction problems in the representation theory of classical groups. Sur les conjectures de Gross et Prasad. I, Asterisque No. 346 (2012), 1-109.

W.T. Gan, B.H. Gross, and D. Prasad, Branching laws for classical groups: the non-tempered case, Compositio Mathematica, 156(11) (2020), 2298-2367. doi:10.1112/S0010473X20007496

Christof Geiss and Jan Schröer, Extension-orthogonal components of preprojective varieties, Trans. Amer. Math. Soc. 357 (2005), no. 5, 1953-1962.

C. Jantzen, Jacquet modules of \(p\)-adic general linear groups, Represent. Theory 11 (2007), 45-83 https://doi.org/10.1090/S1088-4165-0700316-0

Seok-Jin Kang, Massaki Kashiwara, Myungho Kim and Se-jin Oh, Simplicity of heads and socles of tensor products, Compos. Math. 151 (2015), no. 2, 377-396.

M. Khovanov and A. D. Lauda; A diagrammatic approach to categorification of quantum groups, II, Represent. Theory, 13 (2009), 309-347.

E. Lapid and A. Minguez; A determinantal formula of Tadić, Amer. J. Math. 136 (2014): 111-142.

E. Lapid, A. Minguez, On parabolic induction on inner forms of the general linear group over a non-Archimedean local field, Sel. Math. New Ser. (2016) 22, 2347-2400. Doi.org/10.1007/s00029-016-0281-7

E. Lapid, A. Minguez, Geometric conditions for \(□\)-irreducibility of certain representations of the general linear group over a non-archimedean local field, Advances in Mathematics Volume 339 (2018), 113-190, 10.1016/j.aim.2018.09.027

E. Lapid, A. Minguez, A binary operation on irreducible components of Lusztig’s nilpotent varieties II: applications and conjectures for representations of GL_n over a non-archimedean local field, to appear in Pure and Applied Mathematics Quarterly.

B. Leclerc, Imaginary vectors in the dual canonical basis of \(U_q(n)\). Transformation Groups 8, 95-104 (2003). https://doi.org/10.1007/BF03326301

Lascoux, A., Leclerc, B. and Thibon, J.Y. Hecke algebras at roots of unity and crystal bases of quantum affine algebras. Commun. Math. Phys. 181, 205-263 (1996), doi.org/10.1007/BF02101678
[MW86] C. Mœglin, J.-L. Waldspurger, Sur l’involution de Zelevinski. J. Reine Angew. Math. 372, 136-177 (1986)

[Mi09] A. Minguez, Surl’irréducitibilité d’une induite parabolique, 2009, no. 629, Crelle’s Journal, https://doi.org/10.1515/CRELLE.2009.028

[MS14] Alberto Minguez, Vincent Sécherre, Représentations lisses modulo $l$ de GLm(D), Duke Mathematical Journal, Duke Math. J. 163(4), 795-887, DOI: 10.1215/00127094-2430025

[MW12] C. Mœglin and J.-L. Waldspurger, La conjecture locale de Gross-Prasad pour les groupes spéciaux orthogonaux: le cas général, Sur les conjectures de Gross et Prasad. II, Astérisque. No. 247 (2012), 167-216.

[Qa23+] M.S. Qadri, On higher multiplicity upon restriction from $GL(n)$ to $GL(n - 1)$, preprint (2023).

[Sc00] O. Schiffmann, The Hall algebra of a cyclic quiver and canonical bases of Fock spaces, Internat. Math. Res. Notices 8 (2000), 413-440.

[Ta90] M. Tadić, Induced representations of GL (n, A) for p-adic division algebras A. Journal für die reine und angewandte Mathematik (Crelles Journal), vol. 1990, no. 405, 1990, pp. 48-77. https://doi.org/10.1515/crll.1990.405.48

[VV99] M. Varagnolo, E. Vasserot, On the decomposition matrices of the quantized Schur algebra, Duke Math. J., 100(2), 267-297, doi.org/10.1215/S0012-7094-99-10010-X

[Ze80] A. Zelevinsky, Induced representations of reductive p-adic groups II, Ann. Sci. École Norm. Sup. 13 (1980), 154-210.

Department of Mathematics, the University of Hong Kong, Hong Kong, China
Email address: kychan1@hku.hk