Many-body delocalization dynamics in long Aubry-André quasiperiodic chains

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Abstract

We theoretically study quench dynamics in an interacting one-dimensional spin chain with a quasi-periodic on-site field, also known as the interacting Aubry-André model of many-body localization. Using the time-dependent variational principle we assess the late-time behaviour for chains up to $L = 50$. The choice of periodicity of the quasi-periodic potential influences the dynamics, which is explained in terms of properties of the non-interacting problem. Furthermore, the decay of antiferromagnetic order is faster than a power law, and finite-size effects on the value of the critical disorder are weaker than in the purely random case.

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1 Introduction

Many-body localization (MBL) describes the localization of particles in an interacting many-body system due to the presence of disorder \[1-4\], which can be viewed as a generalization of Anderson localization \[5\] to interacting systems. This phenomenon is of interest for aiding our understanding of the general mechanisms by which ergodicity is broken in quantum systems, which is crucial for technological applications such as protecting qubits used for quantum computing against decoherence.

Many of the early works studying MBL, both theoretical \[6,7\] and numerical \[8-10\], focused on the case where one considers a system that is Anderson-localized, adding interactions to such a system. Such systems are popularly modelled using one-dimensional (1D) disordered spin chains, such as the XXZ Heisenberg chain with a random on-site field, or the equivalent (through a Jordan-Wignier transformation) model of interacting hard-core bosons with a random on-site potential. However, some recent experiments \[11-13\] using ultra-cold atoms instead use a related but slightly different system, where the on-site potential is not purely random but rather quasi-periodic. The corresponding non-interacting model, known in 1D as the Aubry-André model \[14\], differs from the Anderson model. The transition from a localized eigenspectrum to a delocalized one occurs not at infinitesimal disorder strength, but at a finite value of \(W = 2\) (in units used in this work). Nevertheless, an MBL transition from a delocalized to a localized eigenspectrum is still expected to occur in the interacting case at a critical disorder \(W_c > 2\).

In purely random systems, numerical studies find the appearance of power laws in transport properties, which are attributed to Griffiths effects \[15,16\]. Such effects are due to rare events: either rare thermal regions that serve to delocalize an otherwise localized phase, or rare strongly disordered regions that act as bottlenecks for transport \[17\]. Recent theoretical work based on renormalisation group approaches suggests that the former scenario, based on “avalanches” of delocalized grains, drives the MBL transition \[18,19\]. Experimentally, power laws describing particle transport are also found in quasi-periodic disordered systems \[12\]. This is rather unexpected, since rare events should be absent in a deterministic potential, such as a quasi-periodic potential.

In a recent work, Khemani \textit{et al.} studied the difference between the two types of disorder \[20\]. They describe two distinct universality classes, and argue that previous numerical studies for purely random disorder, focusing primarily on exact diagonalization studies of small systems, are strongly affected by finite-size effects. The authors attribute the violation of Harris-Chayes bounds \[21\] in numerical studies of small systems \(L \leq 20\) to such effects. Indeed, our recent numerical results confirm that exact diagonalization studies of purely random models are tainted by strong finite-size effects and substantially underestimate the critical disorder strength \[22\].

One should wonder whether the appearance of power laws in quasi-periodic systems is merely an artefact of studying insufficiently large systems and long times. Numerically, this is difficult because on the one hand, exact-diagonalization studies can easily probe long times accurately, but the small systems thus accessible “feel” the effects of boundary conditions more strongly at late times. In one such exact diagonalization study, analysis of the return probability in quasi-periodic systems shows robust power-law tails for systems of 16 lattice sites \[23\]. On the other hand, a recent study using a self-consistent Hartree-Fock approximation suggests that the power laws seen in the quasi-periodic case are only transient, whereas
they are robust in the purely random case \cite{24} at least up to times (in units of lattice hopping) \(O(10^4)\). However, the Hartree-Fock approximation is \textit{a priori} uncontrolled, and it is not clear how well it captures the exact dynamics.

The aim of the present work is to investigate the differences between purely random and quasi-periodic disorder at system sizes inaccessible to exact diagonalization, using the newly developed numerical technique of the time-dependent variational principle as applied to matrix product states. Using this technique, we find that finite-size effects in this system are substantially smaller than for purely random disorder in the sense that the critical disorder does not appear to have a strong dependence on system size. On the other hand, we also find a breakdown of power-law behaviour in larger systems: the decay becomes faster than the power law with increasing time. This indicates that the appearance of power laws in exact-diagonalization studies or experimental studies of quasi-periodic systems was due to considering too small systems or insufficiently long times. Furthermore, we find that the choice of the specific quasi-periodic potential substantially impacts transport properties as well as the value of the critical disorder \(W_c\), which is explained by the influence of periodicity on the properties of the potential and thus of the corresponding single-particle states. Our findings are consistent with the aforementioned predictions of Khemani \textit{et al.} concerning two different universality classes for the MBL transitions with random and quasi-periodic potentials \cite{20}.

2 Model and method

2.1 Quasi-periodic Heisenberg XXZ chain

We consider the Heisenberg XXZ chain with a quasi-periodic field, also known as the interacting Aubry-André model, on a lattice of length \(L\) with open boundary conditions, as described by the Hamiltonian

\[
\mathcal{H} = \sum_{i=1}^{L} \left[ \frac{J}{2} \left( S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+ \right) + \Delta S_i^z S_{i+1}^z + \phi_i S_i^z \right],
\]

where \(\phi_i = (W/2) \cos(2\pi \Phi i + \phi_0)\) represents the quasi-periodic field with strength \(W\), \(\Phi\) is a parameter describing the periodicity of the underlying potential and \(\phi_0 \in [0, 2\pi)\) is a random phase taken from a uniform distribution. If \(\Phi\) is chosen to be irrational, the period of the potential is infinite. In this work we choose \(\Phi = (\sqrt{5} - 1)/2\), the inverse golden ratio, unless noted otherwise. The \(S\)-operators represent standard Pauli matrices, so that the case \(\Delta = 1\) corresponds to the isotropic Heisenberg chain. This problem can be mapped to a particle representation using a Jordan-Wigner transformation, wherein \(J\) is the hopping element for the single-particle transport to adjacent sites, and \(\Delta\) represents the inter-particle interaction. In the case \(\Delta = 0\), Eq. (1) maps to the Aubry-André model of non-interacting particles in a quasi-periodic field \cite{14}, which exhibits a transition from a fully ergodic (delocalized) eigenspectrum below a critical disorder \(W/J = 2\) to a fully localized spectrum above it. In the following, we set \(J \equiv 1\) as the unit of energy, and \(\hbar \equiv 1\).

A purely random disordered model differs from the quasi-periodic model \cite{11} as it has elements \(\phi_i\) which are uncorrelated from site to site. A quantitative comparison between the two cases requires a non-uniform distribution \(\phi_i = (W/2) \cos(\varphi_i)\), with a site-dependent phase \(\varphi_i \in [0, 2\pi)\), cf. Ref. \cite{20}.
We numerically simulate the dynamics of an initial anti-ferromagnetic state:

$$|\psi\rangle(t = 0) = \{\uparrow, \downarrow, \ldots, \uparrow, \downarrow\},$$

(2)

and compute the imbalance $I$ as a function of time, which quantifies how much of the initial anti-ferromagnetic order remains at a certain time $t$. It is defined as:

$$I(t) = \frac{1}{L} \sum_{i=1}^{L} (-1)^{i} \langle S_{z}^{i}(t) \rangle.$$  

(3)

It is easy to verify that the initial value of the imbalance $I(t = 0) = 1$. For an ergodic system, the long-time time-average of $I(t)$ vanishes in the thermodynamic limit. The initial state (2) is appealing for studies of thermalization, because it is typically the fastest-thermalizing initial state. Furthermore, for this choice of the initial state the imbalance (3) directly probes density-density correlations [25] while only requiring knowledge of spin densities, simplifying both numerical analysis and experimental probing of MBL. In the case of purely random disorder, the imbalance decays according to a power law: $I(t) \propto t^{-\beta}$ [16, 22], where $\beta$ is a disorder-dependent power law. A possible criterion for pinpointing the MBL transition is the vanishing of $\beta$, indicating saturation of the imbalance. If such a saturation can be extrapolated to $t \to \infty$ the system is localized. A recent work [26] used this procedure to estimate the transition (in our choice of units) at $W_{c} \approx 5$, with a somewhat lower value obtained through a finite-size scaling collapse (however, the accuracy of such finite-size scaling approaches restricted to small systems was questioned in Ref. [20]). A different approach employed in Ref. [23], using a different choice for $\Phi$, yields $W_{c} \approx 6$.

### 2.2 Time-dependent variational principle

To compute the time dynamics, we use the time-dependent variational principle (TDVP). We apply the TDVP to matrix product states (MPS) [27, 28], a type of tensor network that functions as a variational ansatz within the subspace of weakly entangled states [29]. Our implementation is identical to the one used in our previous work [22]. Implicitly, time evolution due to the TDVP is described by:

$$\frac{d|\psi\rangle}{dt} = -iP_{\text{MPS}}\mathcal{H}|\psi\rangle,$$

(4)

where $P_{\text{MPS}}$ projects the time evolution back onto the variational manifold. In our implementation, we start with a product state described by an MPS with bond dimension of 1, and rapidly increase the bond dimension in a few time steps to a desired value $\chi$, keeping the bond dimension fixed during time evolution. The number of variational parameters in the matrix product state ansatz scales polynomially in $\chi$ and in the system size $L$, as opposed to the full many-body wave function, whose complexity scales exponentially in $L$. A beneficial property of the TDVP is that time evolution with such a fixed bond dimension conserves globally conserved quantities such as the energy, as opposed to older MPS-based methods, such as time-evolving block decimation [30]. Recently, the performance of the TDVP has been studied in several works [22, 31, 32]; these findings suggest that the TDVP is particularly suited to studying dynamics of weakly entangled states close to the MBL transition.
3 Results

We focus on the case $L = 50$ and $\Delta = 1$, unless stated otherwise, and consider dynamics up to $t = 300$. Using the TDVP we numerically compute the dynamics and track the behaviour of spin density $\langle S^z_i \rangle$ at the different sites $i \in [1, L]$ for $O(500)$ different realizations of disorder, i.e. different random values of $\phi_0$. For the sake of comparison, we also consider the case $L = 16$ where we compute the dynamics exactly using the TDVP with unrestricted bond dimension. We opt to consider only the spin density here and not, e.g., the entropy of entanglement, since the former is readily measured in the experiment while the latter is difficult to probe experimentally, despite recent advances \[33\].

3.1 Imbalance

First, consider the imbalance (3). We compute the dynamics using the TDVP. In Fig. 1 we show dynamics for various choices of $W$ on both sides of the transition. For relatively strong disorder, $W > 5$, we consider a bond dimension of $\chi = 32$, whereas we consider $\chi = \{32, 64\}$ for weaker disorder. A reduction in bond dimension will typically lead to enhanced delocalization \[22\], implying that if we find saturation at a certain $\{W, \chi\}$ this saturation likely persists in the limit $\chi \to \infty$.

![Figure 1: Time evolution of the imbalance (3) for various choices of disorder strength $W$, periodicity $\Phi$, and bond dimension $\chi$, averaged over disorder realizations (i.e. different random phases $\phi_0$). Coloured solid lines indicate the results for lattice size $L = 50$ and $\Phi = (\sqrt{5} - 1)/2$, the dotted black line shows numerically exact results for $L = 16$ at the same value of $\Phi$. Dashed lines show results for $L = 50$ using different choices for the parameter $\Phi = \{\sqrt{2}, \sqrt{2}/2\}$. Results shown for different $\chi$ use independent disorder realizations. Note the log-log scales and differing $y$-axes.](image)

In the case $\Phi = (\sqrt{5} - 1)/2$, we find an apparent saturation of the imbalance at $W = 5$ and its decay at $W = 4$ (for the system size $L = 50$). We still find a weak decay for $W = 4.5$ and the same system site $L = 50$ (not shown), so we infer as a rough estimate for the critical disorder $W_c \simeq 5$. Strictly speaking, the numerical analysis cannot exclude a very slow delocalization that shows up only at times much longer than those achieved numerically. From this point of view, the numerically found values yield a lower bound for $W_c$. Note that this estimate is much lower than the estimate $W_c \simeq 11$ (in units of this paper) obtained for purely random disorder in large chains using the same method \[22\]. While the distributions of...
on-site potential are somewhat different in both models, this cannot explain a large difference in $W_c$. We speculate that a considerably larger value of $W_c$ in the random problem is related to delocalizing effect of rare “ergodic spots”, which are also responsible for stronger finite-size effects on the value of $W_c$, see a discussion in Sec. 4. In addition, in the quasi-periodic case we find an elevated probability for finding relatively large values of $|\phi_i - \phi_{i+1}|$ (see the peaks in Fig. 2), which might result in a smaller $W_c$. Clearly, the critical disorder is expected to depend on the strength of interactions $\Delta$; however, in this work we do not investigate the effect of changing $\Delta$.

For $W = 5$, the results for $L = 16$ and $L = 50$ are very similar, with only a small offset in the imbalance at late times. This is likely due to boundary effects, as the open boundary conditions act as a barrier and enhance localization. For strong disorder, $W = 8$, there is essentially no influence of the system size visible. This indicates that the localization length in this case is very small, on the order of a single lattice site. The value to which the imbalance saturates in the considered time window does depend somewhat on $\Phi$. Interestingly, on the localized side of the transition, we also find slow oscillations that do not disappear upon disorder averaging. This phenomenon is absent in the purely random case, and we attribute it to long-range correlations in the quasi-periodic potential. This is confirmed by comparing to the case where the quasi-periodic period $\Phi$ is not the inverse golden ratio, but different irrational numbers $\Phi = \{\sqrt{2}, \sqrt{2}/2\}$ (see the dashed lines in Fig. 1). We further confirm that this is not merely a finite-size effect by comparing to the case $L = 40, \Phi = (\sqrt{5} - 1)/2$ (not shown), which is quantitatively very similar to the case $L = 50$, exhibiting the same oscillations.

For weaker disorder $W = 4$ and $\Phi = (\sqrt{5} - 1)/2$, we clearly observe a decaying imbalance which is a manifestation of delocalization. In this case, we also find a substantial increase in the rate of delocalization when the system size is increased from $L = 16$ to $L = 50$. As is seen in the figure, for times $t \gtrsim 200$ we start to see a dependence on bond dimension, indicating that the numerical results are fully reliable only up to this time for the case $W = 4$. For longer times, one would need a still larger bond dimension $\chi$ to achieve the full convergence. Note that the results for different values of $\chi$ are obtained using independent realizations of the quasi-periodic fields, so that convergence is checked simultaneously with the number of realizations and the bond dimension $\chi$. We also find a clear signature of “bending” of the curve, which on the log-log scale indicates that decay is in fact faster than the power law. We analyze this feature more thoroughly in the following section.

Strikingly, the dynamics is substantially altered by changing the periodicity parameter $\Phi$. For $W = 5$, the imbalance appears to saturate for the choices $\Phi = (\sqrt{5} - 1)/2$ and $\Phi = \sqrt{2}$, but decays for $\Phi = \sqrt{2}/2$, with substantially different values of the disorder-averaged imbalance as a function of time, suggesting that the critical disorder $W_c$ varies with $\Phi$. We also observe convergence with $\chi$ over the full time window in the cases $\Phi = \sqrt{2}$ and $\Phi = (\sqrt{5} - 1)/2$, but not for $\Phi = \sqrt{2}/2$, which indicates that the growth of the entropy of entanglement also significantly depends on $\Phi$. We have verified that this is due to the interplay between interactions and disorder by comparing to the non-interacting case $\Delta = 0$ (see left panel of Fig. 2), in which case the imbalance saturates for the choices of $\Phi$ considered here. The dependence of the imbalance decay on $\Phi$ is also clearly seen for $W = 4$. Indeed, we observe no significant decay for the choice $\Phi = \sqrt{2}$, a significant decay for the inverse golden ratio $\Phi = (\sqrt{5} - 1)/2$, and a much stronger decay for $\Phi = \sqrt{2}/2$. In the latter case, numerical convergence with $\chi$ is found up to $t \approx 100$.

The dependence on $\Phi$ can be explained in the following way. Consider the consecutive
Figure 2: **Left:** evolution of the average imbalance in the non-interacting case $\Delta = 0$ for disorder $W = 5$, computed using the TDVP. The imbalance saturates at different values for different choices of the periodicity $\Phi$. **Right:** histograms of the distribution of the differences between consecutives values of the disordered on-site potential, for various choices of $\Phi$. For comparison, also the purely random case is shown. The number of realizations is $R = 5000$, and system size $L = 50$.

The values of the disordered potential at sites $\phi_i$ and the next site $\phi_{i+1}$. We plot the distribution $P$ of the absolute differences $|\phi_i - \phi_{i+1}|$ normalized by the disorder strength $W$ in the right panel of Fig. 2. For each value of $\Phi$, the support of the distribution is limited from above by a certain value $<1$ which is different for different values of $\Phi$. The corresponding peak in the distribution is similar in origin to the van Hove singularities in the density of states [34,35]. Hence, there is an effective disorder strength $\tilde{W}(\Phi) < W$ which characterizes the potential variation between the neighbouring sites. Since we consider quite strong disorder, for which the localization length of the non-interacting problem is of the order of lattice spacing, this strongly influences the spatial extension of localized states. This is supported by the data for the imbalance in non-interacting problem, see left panel of of Fig. 2. Indeed, the value at which the imbalance saturates depends on $\Phi$, consistent with the “effective disorder strength” (width of the distribution) in the right panel. A quantitative dependence of quench dynamics on the choice of $\Phi$ was also found for an extended version (including next-nearest neighbour hopping terms) of the non-interacting, extended Aubry-André model [36].

Comparing Figs. 1 and 2, we see that the dependence of the degree of localization in the single-particle problem on $\Phi$ is inherited by the many-body problem. In particular, the model with $\Phi = \sqrt{2}/2$, which has the weakest “effective disorder” and thus larger spreading of single-particle localized states out of the three considered values of $\Phi$, shows the most efficient many-body delocalization. Similarly, the case $\Phi = \sqrt{2}$, which yields the the strongest “effective disorder” and thus the smallest spreading of single-particle localized states, is the most resistant with respect to many-body delocalization.
3.2 Decay of the imbalance

To analyze the decay of the imbalance more closely, we characterize it by a (in general, time-dependent) power-law exponent $\beta$. In the following, and in the remainder of the paper, we consider $\Phi = (\sqrt{5} - 1)/2$ and bond dimensions $\chi = \{32, 48, 64\}$.

![Figure 3: Flowing power-law exponent $\beta(t)$ characterizing the decay of imbalance according to the weight (5) for various values of the disorder strength $W$. Oscillations in the imbalance shown in the middle panel of Fig. 1 lead to oscillations in the time-dependent power-law coefficient $\beta$ for $W = 5$.](image)

Since we cannot probe time scales over many decades, we use the following procedure to investigate the time dependence of $\beta$. First, we consider the window of times $\tau \in [30, 300]$. Then, we perform a power-law fit to $I(\tau; t) \propto \tau^{-\beta(t)}$, where data points are weighted according to a Gaussian (the normalization is irrelevant here as only the relative weight influences the fitting procedure):

$$\exp \left[ \frac{-(t - \tau)^2}{2\sigma^2} \right].$$

We repeat this procedure for all $t \in [30, 300]$ and compute fits to $I(\tau; t)$, yielding $\beta(t)$. The benefit of this procedure over using, e.g., shifting time windows to perform fits, is that it results in a smooth behaviour of $\beta(t)$. The standard deviation is chosen as sufficiently large, $\sigma = 50$, such that persistent, fast oscillations due to the initial condition are washed out. Such oscillations, that do not vanish upon disorder averaging, were previously found in Ref. [37] in the dynamics of the mean-square displacement; an observation reproduced in our results (see Fig. 1). In the case of power-law behaviour we expect $\beta(t)$ to be approximately constant in time, whereas an increase of $\beta$ as a function of time indicates stronger than power-law decay [24]. In a previous work on a purely random case [22], we found and essentially time-independent $\beta$, i.e., a power-law decay, for the times $t \leq 100$.

The time dependence of the flowing exponent $\beta(t)$ is shown in Fig. 3 for various choices of $W$. For $W = 4$ (left panel), i.e., on the ergodic side but fairly close to the MBL transition, we find that $\beta(t)$ for a long chain ($L = 50$) strongly increases with time, implying that the decay is faster than power law. For a small system $L = 16$, however, the power law remains a good fit over the entire time window, as evidenced by the fact that the fitted coefficient is $\sim 0.025$ to a good approximation.

For stronger disorder, $W \geq 5$, i.e., in the localized regime, $\beta(t)$ remains small and shows oscillations around zero. These oscillations originate from oscillations of the imbalance in the
localized regime, see Fig. 1 and discussion in Sec. 3.1.

3.3 Distribution of spin densities

Figure 4: Normalized distribution of $\langle S^z \rangle$ over all lattice sites and realizations of disorder at a fixed time $t = 100$, for various choices of the disorder strength $W$. The histogram uses 30 equidistant bins.

Figure 5: Interpolated filled contour plot of the distribution of $\langle S^z \rangle$ over all lattice sites and realizations of disorder as a function of time, for various choices of the disorder strength $W$. The colour indicates the value of $P(\langle S^z \rangle)$. Each “slice” in time corresponds to a histogram as in Fig. 4 with 30 equidistant bins. Values of $P(\langle S^z \rangle) > 1.5$ are indicated with the same colour.

Another quantity that can be directly probed experimentally is the distribution of spin densities $P(\langle S^z \rangle)$, where we consider the normalized distribution over all sites $i$ and disorder realizations. We focus again on the case $\Phi = (\sqrt{5} - 1)/2$. A convenient feature of $P(\langle S^z \rangle)$ is that it can be naturally defined also for random initial conditions where $\forall i: \langle S_i^z \rangle(t = 0) = \pm 1$, in addition to the initial Néel state used in this work. In the eigenstates of small, purely random systems, the study of a similar quantity reveals a bimodal distribution on the localized side of the MBL transition. A comparable picture emerges from the study of dynamics, with a bimodal distribution sharply peaked at $-1$ and 1. This is a signature of the memory of the initial state, which has a distribution consisting of two $\delta$-peaks at $-1$ and 1.
The results for the time $t = 100$ and three different values of disorder are shown in Fig. 4. On the ergodic side ($W = 4$), the distribution is broader than for stronger disorder, with lower peaks and a higher “valley.” In order to visualize the dynamics of the spin distributions, we show them as a function of time in Fig. 5 in the form of a contour plot. This representation demonstrates clearly qualitative differences between the ergodic and localized regimes. For $W = 4$ one can see the peaks close to $\langle S^z \rangle = \pm 1$ slowly fading, whereas they are persistent in the cases $W = 5$ and $W = 8$. The persistence of peaks is a signature of localization analogous to the saturation of the imbalance (3). In the strongly localized case, $W = 8$, barely any dynamics in $P(\langle S^z \rangle)$ is visible.

4 Conclusions

We have investigated many-body localization in the quench dynamics of a quasi-periodic spin chain at late times for moderately large chains up to $L = 50$, using the time-dependent variational principle applied to matrix product states. Using an initially antiferromagnetically ordered state, we have determined the time evolution of spin densities for various choices of the quasi-periodic field strength (disorder strength) $W$ and periodicity parameter $\Phi$. Our key findings can be summarized as follows.

First, quite surprisingly, our results demonstrate that the critical strength $W_c$ of the quasi-periodic field corresponding to the MBL transition at given interaction strength ($\Delta = 1$) depends on $\Phi$ in an essential way, as opposed to the non-interacting case (the Aubry-André model), where the transition is at $W = 2$ for almost any irrational $\Phi$. The dependence of $W_c$ on $\Phi$ complicates a systematic, quantitative comparison between quasi-periodic and purely random disordered systems, as one needs to exclude that differences between the two are not due to peculiarities related to the specific choice(s) of $\Phi$. The dependence on $\Phi$ is explained by considering the statistical properties of neighbouring values of the potential. It is found that the width of this distribution depends in a non-trivial way on $\Phi$. This influences properties of the corresponding localized single-particle states, and, as a result, the susceptibility of the system to the many-body-delocalizing effect of the interaction. The found dependence of $W_c$ on $\Phi$ is probably the main reason for differing estimates for the critical disorder in previous works, Refs. [23,26].

It is worth mentioning that the analysis of the imbalance decay is complicated by the appearance of slow oscillations, which are found to be related to the long-range correlations that occur in quasi-periodic potentials. This effect might be alleviated by averaging over randomized initial conditions instead of the fixed anti-ferromagnetic initial state. On the other hand, randomizing the initial state can itself be related to rare events [12], which might obscure the physics of the quasi-periodic model.

Our second main conclusion is that the decay of the imbalance (3) on the ergodic side is not described by a power law. Instead, the decay is faster than a power law, as is manifest in an increase of the “flowing power-law exponent” $\beta(t)$ with time $t$. This supports the results of Hartree-Fock simulations in Ref. [24].

Our results also demonstrate that small systems, such as $L = 16$, are insufficient to observe the acceleration of decay as compared to a power law, in consistency with the observation of apparent power laws in exact-diagonalization study of Ref. [23]. It is possible that the shift-inverse exact diagonalization technique [39], which can access systems up to $L = 26$, will...
be sufficient to detect deviations from the power-law behavior.

Finally, we find, by comparing to exact numerical results for a relatively small system with $L = 16$, that finite-size effects on the critical disorder strength are considerably weaker than in the purely random case. Indeed, contrary to the case of purely random disorder, our estimate for the critical disorder $W_c \approx 5$ for $\Phi = (\sqrt{5} - 1)/2$ does not depend strongly on system size. This agrees with predictions from Ref. [20]. A possible explanation for this difference is the appearance of extended, rare thermal regions in the purely random case, which can act as baths for global delocalization of the system in accordance with the “quantum avalanche” scenario [18,19]. In a quasi-periodic system, this mechanism of delocalization by rare regions is not operative.

On the other hand, we find that increasing system size leads to an increase in the rate of delocalization on the ergodic side, as measured by the decay of the imbalance, in agreement with Ref. [12]. This behavior is similar to the one found in the case of purely random disorder [22]. Hence, while there do not appear to be strong finite-size effects on the strength of the quasi-periodic field that separates the ergodic and non-ergodic regimes, there are still noticeable finite-size effects on how quickly the system thermalizes.

Experimentally, the system sizes probed in this work can be assessed using cold atoms [12] or ion traps [40]. The system length $L = 50$ that we used is a typical number for these experiments. Thus, an experimental setup that sufficiently suppresses noise and particle loss over times $t > 100$ ought to be able to confirm our findings. The analysis that we have performed, both for the imbalance and the spin density distribution, can be straightforwardly applied to experimental data in order to analyze and visualize the dynamics of many-body delocalization.

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