On extension of quantum channels and operations to the space of relatively bounded operators

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Abstract

We analyse possibility to extend a quantum operation (sub-unital normal CP linear map on the algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on a separable Hilbert space $\mathcal{H}$) to the space of all operators on $\mathcal{H}$ relatively bounded w.r.t. a given positive unbounded operator.

We show that a quantum operation $\Phi$ can be uniquely extended to a bounded linear operator on the Banach space of all $\sqrt{G}$-bounded operators on $\mathcal{H}$ provided that the operation $\Phi$ is $G$-limited: the predual operation $\Phi^*$ maps the set of positive trace class operators $\rho$ with finite $\text{Tr}\rho G$ into itself.

Assuming that $G$ has discrete spectrum of finite multiplicity we prove that for a wide class of quantum operations the existence of the above extension implies the $G$-limited property.

Applications to the theory of Bosonic Gaussian channels are considered.

1 Introduction

Unital and subunital completely positive (CP) normal linear maps between algebras of bounded operators play important role in the quantum theory. Unital CP linear maps called quantum channels describe evolution of open quantum systems (in the Heisenberg picture), subunital CP linear maps called quantum operations are also used essentially, in particular, in the theory of quantum measurements [5, 14, 16].

In the standard description of a quantum system dynamics a quantum channel (operation) acts on bounded observables – Hermitian bounded operators on a separable Hilbert space associated with the system. But for many important physical quantities (the position, momentum, etc.) the corresponding observables are unbounded Hermitian operators. So, the question arises of extending the quantum channel (operation) to unbounded observables. For a given channel (operation) $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ this question can be solved by using the Stinespring representation

$$\Phi(A) = V^*_{\Phi}(A \otimes I_{\mathcal{K}})V_{\Phi}$$

(1)
where $V_\Phi$ is an isometry (contraction) from the system space $\mathcal{H}$ into the tensor product of this space and some separable Hilbert space $\mathcal{K}$ (typically called the environment space). Indeed, for any Hermitian unbounded operator $A$ the r.h.s. of (1) defines some linear operator on $\mathcal{H}$. But we can say nothing about the domain of this operator, since in general the range of $V_\Phi$ does not coincide with $\mathcal{H} \otimes \mathcal{K}$. In particular, we can not assert that the domain of this operator is dense in $\mathcal{H}$. 

For quantum channels (operations) of special classes the existence of an adequate extension to unbounded operators are well known. For example, any Bosonic Gaussian channel is well defined on the algebra of all polynomials of canonical observables (which are unbounded operators) and transforms this algebra into itself [5, Ch.12]. In this paper we analyze the extension problem in full generality not assuming in advance any special properties of a quantum channel (operation).

Speaking about extension of quantum channels (operations) to unbounded operators we will restrict attention to operators relatively bounded w.r.t. the operator $\sqrt{G}$ (briefly $\sqrt{G}$-bounded operators), where $G$ is a given positive densely defined operator on $\mathcal{H}$, i.e. to the operators $A$ well defined on the domain $D(\sqrt{G})$ of $\sqrt{G}$ such that

$$\|A\varphi\|^2 \leq a^2\|\varphi\|^2 + b^2\|\sqrt{G}\varphi\|^2, \quad \forall \varphi \in D(\sqrt{G}),$$

for some nonnegative numbers $a$ and $b$ [6, 12]. This is explained, in particular, by the following physical reason: the quantum observables corresponding to important physical quantities are $\sqrt{G}$-bounded operators provided that either $G$ or some power of $G$ is a Hamiltonian of a quantum system.

The linear space of all $\sqrt{G}$-bounded operators can be made a Banach space by equipping it with any of the operator $E$-norms $\| \cdot \|_E^G$ induced by $G$ [10]. For given $E > 0$ the operator $E$-norm of a bounded operator $A$ is defined as

$$\|A\|_E^G = \sup_{\rho \in S(\mathcal{H}) : \text{Tr}G\rho \leq E} \sqrt{\text{Tr}A\rho A^*},$$

where the supremum is over all states $\rho$ (positive trace class operators on $\mathcal{H}$ with unit trace) such that $\text{Tr}G\rho \leq E$. This definition is naturally extended to all $\sqrt{G}$-bounded operators (see Section 2.2).

In this paper we prove that formula (1) defines a bounded linear operator on the Banach space $\mathcal{B}_G(\mathcal{H})$ of all $\sqrt{G}$-bounded operators with the norm $\| \cdot \|_E^G$ provided that

$$\text{Tr}G\Phi^*_\ast(\rho) < +\infty \quad \text{for any state } \rho \text{ such that } \text{Tr}G\rho < +\infty,$$  

(2)

where $\Phi^*_\ast$ is the predual map to the channel (operation) $\Phi$. We also prove that for a large class of quantum channels and operations condition (2) is necessary for continuity of the map $A \mapsto \Phi(A)$ on $\mathcal{B}_G(\mathcal{H})$ w.r.t. the norm $\| \cdot \|_E^G$. So, if $\Phi$ is a quantum channel

\footnote{It is easy to construct a channel $\Phi$ and a positive operator $A$ such that the domain of the r.h.s. of (1) contains only the zero vector.}

\footnote{The advantages of the norm $\| \cdot \|_E^G$ in comparison with the equivalent norm commonly used on set of relatively bounded operators are described in Section 2.2.}
(operation) from this class then formula (1) defines a bounded linear operator on the Banach space \( \mathfrak{B}_G(\mathcal{H}) \) if and only if condition (2) holds.

By noting that any Bosonic Gaussian channel \( \Phi \) satisfies condition (2) provided that \( G \) is the number operator in the multimode Bosonic system we consider some applications of the above general results to these channels.

2 Preliminaries

2.1 Basic notations

Let \( \mathcal{H} \) be a separable infinite-dimensional Hilbert space, \( \mathfrak{B}(\mathcal{H}) \) – the algebra of all bounded operators on \( \mathcal{H} \) with the operator norm \( \| \cdot \| \) and \( \mathfrak{T}(\mathcal{H}) \) – the Banach space of all trace-class operators on \( \mathcal{H} \) with the trace norm \( \| \cdot \|_1 \) (the Schatten class of order 1) \[6, 12\]. Let \( \mathfrak{T}_+(\mathcal{H}) \) be the positive cone in \( \mathfrak{T}(\mathcal{H}) \) and \( \mathfrak{S}(\mathcal{H}) \) the set of quantum states – operators in \( \mathfrak{T}_+^{(\mathcal{H})} \) with unit trace \[5\].

Denote by \( I_H \) the unit operator on a Hilbert space \( H \) and by \( \text{Id}_H \) the identity transformation of the Banach space \( \mathfrak{T}(\mathcal{H}) \).

A completely positive (CP) normal unital (corresp., subunital) linear map \( \Phi : \mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H}) \) is called quantum channel (corresp., operation) in the Heisenberg picture \[4\]. Its predual map \( \Phi^* : \mathfrak{T}(\mathcal{H}) \to \mathfrak{T}(\mathcal{H}) \) defined by the relation

\[
\text{Tr}\Phi^*(\rho)A = \text{Tr}\Phi(A)\rho, \quad A \in \mathfrak{B}(\mathcal{H}), \quad \rho \in \mathfrak{T}(\mathcal{H}),
\]

is called quantum channel (corresp., operation) in the Schrodinger picture \[4\].

For any quantum channel (corresp., operation) \( \Phi \) the Stinespring theorem (cf.\[13\]) implies existence of a separable Hilbert space \( K \) and an isometry (corresp., contraction) \( V_\Phi : \mathcal{H} \to \mathcal{H} \otimes K \) such that

\[
\Phi(A) = V_\Phi^* (A \otimes I_K) V_\Phi, \quad A \in \mathfrak{B}(\mathcal{H}),
\]

and, respectively,

\[
\Phi^*(\rho) = \text{Tr}_K V_\Phi \rho V_\Phi^*, \quad \rho \in \mathfrak{T}(\mathcal{H}),
\]

where \( \text{Tr}_K \) denotes the partial trace over the space \( K \). The Stinespring representation \[3\] is called minimal if the linear span of the set \( \{ (A \otimes I_K) V_\Phi \varphi \mid A \in \mathfrak{B}(\mathcal{H}), \varphi \in \mathcal{H} \} \) is dense in \( \mathcal{H} \otimes K \). The dimension of the space \( K \) in the minimal Stinespring representation is called the Choi rank of a channel (operation) \[5\] Ch.6).

For any quantum operation \( \Phi : \mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H}) \) the following inequality (called Kadison's inequality)

\[
\Phi(A^*)\Phi(A) \leq \| \Phi(I_H) \| \Phi(A^*A)
\]

holds for any \( A \in \mathfrak{B}(\mathcal{H}) \).

4The map \( \Phi \) is called normal if \( \Phi(\sup A_\lambda) = \sup \Phi(A_\lambda) \) for any increasing net \( A_\lambda \subset \mathfrak{B}(\mathcal{H}) \) \[2, 4\]. This property is equivalent to existence of the predual map \( \Phi^* : \mathfrak{T}(\mathcal{H}) \to \mathfrak{T}(\mathcal{H}) \). The map \( \Phi \) is called unital (corresp., subunital) if \( \Phi(I_H) = I_H \) (corresp., \( \Phi(I_H) \leq I_H \)).
We will consider unbounded densely defined positive operators on $\mathcal{H}$ having discrete spectrum of finite multiplicity. In Dirac’s notations any such operator $G$ can be represented as
\[ G = \sum_{k=0}^{+\infty} E_k |\tau_k\rangle \langle \tau_k| \] (5)
and the domain $\mathcal{D}(G) = \{ \varphi \in \mathcal{H} \mid \sum_{k=0}^{+\infty} E_k^2 |\langle \tau_k | \varphi \rangle|^2 < +\infty \}$, where \( \{ \tau_k \}_{k=0}^{+\infty} \) is the orthonormal basis of eigenvectors of $G$ corresponding to the nondecreasing sequence \( \{ E_k \}_{k=0}^{+\infty} \) of eigenvalues tending to $+\infty$. We will use the following (cf. [17])

**Definition 1.** An operator $G$ having representation (5) is called discrete.

### 2.2 Relatively bounded operators and the operator $E$-norms

In this paper we will consider operators on a separable Hilbert space $\mathcal{H}$ relatively bounded with respect to a given positive semidefinite densely defined operator. For our purposes it is convenient to denote this positive operator by $\sqrt{G}$ assuming that $G$ is a positive semidefinite operator on $\mathcal{H}$ with dense domain $\mathcal{D}(G)$ such that
\[ \inf \{ \| G\varphi \| \mid \varphi \in \mathcal{D}(G), \| \varphi \| = 1 \} = 0. \] (6)

A linear operator $A$ is called relatively bounded w.r.t. the operator $\sqrt{G}$ (briefly, $\sqrt{G}$-bounded) if $\mathcal{D}(\sqrt{G}) \subseteq \mathcal{D}(A)$ and
\[ \| A\varphi \|^2 \leq a^2 \| \varphi \|^2 + b^2 \| \sqrt{G}\varphi \|^2, \quad \forall \varphi \in \mathcal{D}(\sqrt{G}), \] (7)
for some nonnegative numbers $a$ and $b$ [6]. The $\sqrt{G}$-bound of $A$ (denoted by $b_{\sqrt{G}}(A)$ in what follows) is defined as the infimum of the values $b$ for which (7) holds with some $a$. If the $\sqrt{G}$-bound is equal to zero then $A$ is called $\sqrt{G}$-infinitesimal operator (infinitesimally bounded w.r.t. $\sqrt{G}$) [6, 8, 12].

Since $\sqrt{G}$ is a closed operator, the linear space $\mathcal{D}(\sqrt{G})$ equipped with the inner product
\[ \langle \varphi | \psi \rangle^G_E = \langle \varphi | \psi \rangle + \langle \varphi | G| \psi \rangle / E, \quad E > 0, \]
is a Hilbert space denoted by $\mathcal{H}_E^G$ in what follows [7]. A restriction of any $\sqrt{G}$-bounded operator to the set $\mathcal{D}(\sqrt{G})$ can be treated as bounded operator from $\mathcal{H}_E^G$ into $\mathcal{H}$ and, vice versa, any bounded operator from $\mathcal{H}_E^G$ into $\mathcal{H}$ induces a $\sqrt{G}$-bounded operator on $\mathcal{H}$. Thus, the linear space of all $\sqrt{G}$-bounded operators on $\mathcal{H}$ equipped with the norm
\[ \| A \|^G_E = \sup_{\varphi \in \mathcal{D}(\sqrt{G})} \frac{\| A\varphi \| \sqrt{\| \varphi \|^2 + \| \sqrt{G}\varphi \|^2 / E}}{\| \varphi \|} \] (8)
is a Banach space [3]. For our purposes it is more convenient to use the equivalent norm
\[ \| A \|^G_E \doteq \sup_{\rho \in \mathcal{B}(\mathcal{H}); \text{Tr}G\rho \leq E} \sqrt{\text{Tr}A\rho A^*} \] (9)

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5We identify operators coinciding on $\mathcal{D}(\sqrt{G})$. 
on the linear space of all $\sqrt{G}$-bounded operators. The supremum here is over all states $\rho$ in $\mathcal{S}(\mathcal{H})$ such that $\text{Tr}G\rho \leq E$. By Lemma 5 in [10] for any $\sqrt{G}$-bounded operator $A$ the function $\rho \mapsto A\rho A^*$ is well defined on the set $\mathcal{S}_G = \{\rho \in \mathcal{S}_+(\mathcal{H}) \mid \text{Tr}G\rho < +\infty\}$ by the expression

$$A\rho A^* = \sum_i |\alpha_i\rangle\langle\alpha_i|, \quad |\alpha_i\rangle = A|\varphi_i\rangle,$$  \hfill (10)

where $\rho = \sum_i |\varphi_i\rangle\langle\varphi_i|$ is any decomposition of $\rho \in \mathcal{S}_G$ into 1-rank positive operators. This function is affine and takes values in $\mathcal{S}_+(\mathcal{H})$. So, the r.h.s. of (9) is well defined for any $\sqrt{G}$-bounded operator $A$. Due to condition (6) the supremum in (9) can be taken over all operators $\rho \in \mathcal{S}_+(\mathcal{H})$ such that $\text{Tr}G\rho \leq E$ and $\text{Tr}\rho \leq 1$ [10, Prop.3].

The norm $\|\cdot\|_E$ called the operator $E$-norm in [10] can be also defined by the following equivalent expressions

$$\|A\|_E^G = \sup \left\{ \sqrt{\sum_i \|A\varphi_i\|^2} \mid \{\varphi_i\} \subset \mathcal{D}(\sqrt{G}) : \sum_i \|\varphi_i\|^2 \leq 1, \sum_i \|\sqrt{G}\varphi_i\|^2 \leq E \right\} \hfill (11)$$

and

$$\|A\|_E^G = \sup \left\{ \|A \otimes I_K\varphi\| \mid \varphi \in \mathcal{D}(\sqrt{G} \otimes I_K) : \|\varphi\| \leq 1, \|\sqrt{G} \otimes I_K\varphi\|^2 \leq E \right\}, \hfill (12)$$

where $K$ is a separable infinite-dimensional Hilbert space. If $G$ is a discrete operator (Def.1) then all the above expressions are simplified as follows

$$\|A\|_E^G = \sup \left\{ \|A\varphi\| \mid \varphi \in \mathcal{D}(\sqrt{G}) : \|\varphi\| \leq 1, \|\sqrt{G}\varphi\|^2 \leq E \right\}. \hfill (13)$$

Validity of the simplified expression (13) in the case of arbitrary positive operator $G$ is an interesting open question (see the Appendix in [10]).

For any $\sqrt{G}$-bounded operator $A$ both norms $\|A\|_E^G$ and $\|A\|_E^G$ are nondecreasing functions of $E$ tending to $\|A\| \leq +\infty$ as $E \to +\infty$. They are related by the inequalities

$$\sqrt{1/2}\|A\|_E^G \leq \|A\|_E^G \leq \|A\|_E^G, \hfill (14)$$

which show the equivalence of these norms on the set of all $\sqrt{G}$-bounded operators [10]. Moreover, for any $\sqrt{G}$-bounded operator $A$ the functions $E \mapsto \|A\|_E^G$ and $E \mapsto \|A\|_E^G$ are completely determined by each other via the following expressions ([10, Th.3A]):

$$\|A\|_E^G = \sup_{t>0} \|A\|_t^G \sqrt{1 + t}, \quad \|A\|_E^G = \inf_{t>0} \|A\|_t^G \sqrt{1 + 1/t}, \quad E > 0.$$
One of the main advantages of the norm $\|A\|_E^G$ is the concavity of the function $E \mapsto [\|A\|_E^G]^p$ for any $p \in (0, 2]$ and any $\sqrt{G}$-bounded operator $A$ which essentially simplifies quantitative analysis of functions depending on $\sqrt{G}$-bounded operators [10, Section 5]. This property implies, in particular, that

$$\|A\|_{E_1}^G \leq \|A\|_{E_2}^G \leq \sqrt{E_2/E_1}\|A\|_{E_1}^G$$

for any $E_2 > E_1 > 0$. (15)

Hence for given operator $G$ all the norms $\|\cdot\|_E^G$, $E > 0$, are equivalent on the set of all $\sqrt{G}$-bounded operators. By inequalities [14] the same is true for the norms $\|\|\cdot\|_E^G$, $E > 0$.

Another advantage of the norm $\|A\|_E^G$ essentially used in this paper is the possibility to estimates the norms $\|\Phi(A)\|_E^G$ via the norm $\|A\|_E^G$ of any bounded operator $A$, where $\Phi$ is a 2-positive linear transformation of $\mathcal{B}(\mathcal{H})$ satisfying the particular conditions [10, Proposition 5E].

Denote by $\mathcal{B}_G(\mathcal{H})$ the linear space of all $\sqrt{G}$-bounded operators equipped with any of the equivalent norms $\|A\|_E^G$, $E > 0$. The equivalence of the norms $\|A\|_E^G$ and $\|\|A\|_E^G$ mentioned before implies that $\mathcal{B}_G(\mathcal{H})$ is a (nonseparable) Banach space. The $\sqrt{G}$-bound $b_\sqrt{G}(\cdot)$ is a continuous seminorm on $\mathcal{B}_G(\mathcal{H})$, for any operator $A \in \mathcal{B}_G(\mathcal{H})$ it can be determined by the formula

$$b_\sqrt{G}(A) = \lim_{E \to +\infty} \frac{\|A\|_E^G}{\sqrt{E}},$$

where the limit can be replaced by infimum over all $E > 0$ [10, Theorem 3B].

The closed subspace $\mathcal{B}_G^0(\mathcal{H})$ of $\mathcal{B}_G(\mathcal{H})$ consisting of all $\sqrt{G}$-infinitesimal operators, i.e. operators with the $\sqrt{G}$-bound equal to 0, coincides with the completion of $\mathcal{B}(\mathcal{H})$ w.r.t. any of the norms $\|\cdot\|_E^G$, $E > 0$ [10, Theorem 3C].

We will use the following observation [10, Lemma 4, Theorem 3A].

**Lemma 1.** If $A$ is a $\sqrt{G}$-bounded operator on $\mathcal{H}$ then for any separable Hilbert space $\mathcal{K}$ the operator $A \otimes I_\mathcal{K}$ naturally defined on the set $\mathcal{D}(\sqrt{G}) \otimes \mathcal{K}$ has a unique linear $\sqrt{G} \otimes I_\mathcal{K}$-bounded extension to the set $\mathcal{D}(\sqrt{G} \otimes I_\mathcal{K})$. This extension (also denoted by $A \otimes I_\mathcal{K}$) has the following property

$$A \otimes I_\mathcal{K}\left(\sum_i |\varphi_i\rangle \otimes |\psi_i\rangle\right) = \sum_i A|\varphi_i\rangle \otimes |\psi_i\rangle$$

for any countable sets $\{|\varphi_i\rangle\} \subset \mathcal{D}(\sqrt{G})$ and $\{|\psi_i\rangle\} \subset \mathcal{K}$ such that $\sum_i \|\sqrt{G}\varphi_i\|^2 < +\infty$ and $\langle\psi_i|\psi_j\rangle = \delta_{ij}$, which implies that $\|A \otimes I_\mathcal{K}\|_{E \otimes I_\mathcal{K}}^G = \|A\|_E^G$ for any $E > 0$.

**Remark 1.** Property (17) implies that

$$(A \otimes I_\mathcal{K})(I_\mathcal{H} \otimes W)|\varphi\rangle = (I_\mathcal{H} \otimes W)(A \otimes I_\mathcal{K})|\varphi\rangle$$

for any $\varphi \in \mathcal{D}(\sqrt{G} \otimes I_\mathcal{K})$ and a partial isometry $W \in \mathcal{B}(\mathcal{K})$ s.t. $I_\mathcal{H} \otimes W^*W|\varphi\rangle = |\varphi\rangle$.

\(^*\)The function $E \mapsto [\|A\|_E^G]^p$ is not concave in general for any $p \in (0, 2]$ [10, Section 3.1].

\(^9\) $\mathcal{D}(\sqrt{G}) \otimes \mathcal{K}$ is the linear span of all the vectors $\varphi \otimes \psi$, where $\varphi \in \mathcal{D}(\sqrt{G})$ and $\psi \in \mathcal{K}$.


### 2.3 On G-limited quantum operations

Let $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a unital CP normal linear map (quantum channel in the Heisenberg picture) and $\Phi^* : \mathcal{I}(\mathcal{H}) \to \mathcal{I}(\mathcal{H})$ its predual map (quantum channel in the Schrödinger picture). Let $G$ be a positive (semidefinite) operator on $\mathcal{H}$ with dense domain $\mathcal{D}(G)$ satisfying condition (6). If $G$ is treated as a Hamiltonian of a quantum system associated with the space $\mathcal{H}$ then for any state $\rho$ in $\mathcal{S}(\mathcal{H})$ the value of $\text{Tr} G \rho$ (finite or infinite) is the (mean) energy of (the system in) the state $\rho$. The states with finite energy can be produced (in principal) in a physical experiment while the states with infinite energy are physically unrealizable. So, it is reasonable to assume that a real (physically realizable) quantum channel $\Phi^*$ maps states with finite energy into states with finite energy, i.e.

$$
\text{Tr} G \Phi^*(\rho) < +\infty \text{ for any state } \rho \text{ such that } \text{Tr} G \rho < +\infty. 
$$

(18)

A quantum channel $\Phi$ satisfying the following formally stronger condition

$$
Y_{\Phi}(E) = \sup \{ \text{Tr} G \Phi^*(\rho) \mid \rho \in \mathcal{S}(\mathcal{H}), \text{Tr} G \rho \leq E \} < +\infty
$$

for some $E > 0$ is called energy-limited in [17], where it is pointed that this condition holds for many quantum channels used in applications. Due to condition (6) the function $E \mapsto Y_{\Phi}(E)$ is concave on $\mathbb{R}_+$. So, if $Y_{\Phi}(E)$ is finite for some $E > 0$ then it is finite for all $E > 0$. The quantity $Y_{\Phi}(E)/E$ can be called the energy amplification factor of a quantum channel (operation) $\Phi$.

In fact, conditions (18) and (19) are equivalent. This follows from Lemma 3 below.

To not be limited to the case when $G$ is a Hamiltonian of a quantum system we will use the following

**Definition 2.** A quantum operation $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is called $G$-limited if equivalent conditions (18) and (19) hold.

The structure of $\mathcal{S}(\mathcal{H})$ as a convex set implies the following useful fact.

**Proposition 1.** If $G$ is a positive unbounded discrete operator (5) then

$$
Y_{\Phi}(E) = \sup \left\{ \text{Tr} G \Phi^*(|\varphi\rangle\langle\varphi|) \mid \varphi \in \mathcal{H}_*, \|\sqrt{G}\varphi\| \leq E, \|\varphi\| = 1 \right\}
$$

for any quantum operation $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$, where $\mathcal{H}_*$ is the linear span of all the eigenvectors $\tau_1, \tau_2, \ldots$ of $G$, i.e. the supremum in (19) can be taken only over pure states corresponding to the vectors in $\mathcal{H}_*$.

Proposition 1 follows from Lemma 2 below, since the functions $\rho \mapsto \text{Tr} G \rho$ and $\rho \mapsto \text{Tr} G \Phi^*(\rho)$ are affine and lower semicontinuous.

**Lemma 2.** Let $G$ be a positive unbounded discrete operator (5) on a Hilbert space $\mathcal{H}$ and $f$ a convex lower semicontinuous function on $\mathcal{S}(\mathcal{H})$. Then

$$
\sup \{ f(\rho) \mid \rho \in \mathcal{S}(\mathcal{H}), \text{Tr} G \rho \leq E \} = \sup \left\{ f(|\varphi\rangle\langle\varphi|) \mid \varphi \in \mathcal{H}_*, \|\sqrt{G}\varphi\|^2 \leq E, \|\varphi\| = 1 \right\},
$$

\text{i.e. sub-unital CP normal linear map.}
where $\mathcal{H}_*$ is the linear span of all the eigenvectors $\tau_1, \tau_2, \ldots$ of $G$.

Proof. Denote by $\mathcal{H}_n$ and $P_n$ the subspace spanned by the eigenvectors $\tau_1, \tau_2, \ldots, \tau_{n-1}$ and the projector on this subspace correspondingly.

Assume that $A = \sup \{ f(\rho) : \rho \in \mathcal{S}(\mathcal{H}), \text{Tr}G\rho \leq E \}$ is finite. For given arbitrary $\varepsilon > 0$ let $\rho_\varepsilon$ be a state in $\mathcal{S}(\mathcal{H})$ such that $\text{Tr}G\rho_\varepsilon \leq E$ and $f(\rho_\varepsilon) > A - \varepsilon$. For each $n$ let $\rho_n = [\text{Tr}P_n\rho_\varepsilon]^{-1}P_n\rho_\varepsilon P_n$. It is easy to see that $\text{Tr}G\rho_n \leq E$ provided that $E_n \geq E$ and that the sequence $\{\rho_n\}$ tends to $\rho_\varepsilon$. By the lower semicontinuity of $f$ we have

$$\liminf_{n \to +\infty} f(\rho_n) \geq f(\rho_\varepsilon) > A - \varepsilon.$$ 

It follows that there is $m$ such that $f(\rho_m) > A - 2\varepsilon$. Since $\varepsilon$ is arbitrary, this shows that

$$A = \sup_n \sup \{ f(\rho) : \rho \in \mathcal{S}(\mathcal{H}_n), \text{Tr}G\rho \leq E \}.$$ 

(20)

By using Lemma 2 in [10] and the convexity of $f$ we obtain

$$\sup \{ f(\rho) : \rho \in \mathcal{S}(\mathcal{H}_n), \text{Tr}G\rho \leq E \} = \sup \left\{ f(|\varphi\rangle\langle\varphi|) : \varphi \in \mathcal{H}_n, \|\sqrt{G}\varphi\|^2 \leq E, \|\varphi\| = 1 \right\}.$$ 

This and (20) imply the assertion of the lemma, since $\mathcal{H}_* = \bigcup_n \mathcal{H}_n$.

The case $A = +\infty$ is considered similarly. \(\square\)

Lemma 3. Any finite concave nonnegative function on a closed convex subset of a Banach space is bounded on this subset.

Proof. Let $f$ be a finite concave nonnegative function on a closed convex subset $X$ of a Banach space. Assume that for each $n \in \mathbb{N}$ there is $x_n \in X$ such that $f(x_n) > 2^n$. Let $x_* = \sum_{n=1}^{+\infty} 2^{-n}x_n$ and $x_*^m = (1 - p_m)^{-1}\sum_{n=1}^{m} 2^{-n}x_n$, where $p_m = \sum_{n>m} 2^{-n}$, $m \in \mathbb{N}$. By the concavity and nonnegativity of the function $f$ we have

$$f(x_*) \geq (1 - p_m)f(x_*^m) \geq \sum_{n=1}^{m} 2^{-n}f(x_n) > m$$

for any $m$. This implies that $f(x_*) = +\infty$. \(\square\)

3 The main results

Part A of the following theorem gives a sufficient condition for existence of a continuous linear extension of a quantum channel (operation) to the Banach space $\mathcal{B}_G(\mathcal{H})$ of $\sqrt{G}$-bounded operators (described in Section 2.2). Part B implies that for a large class of quantum channels (operations) this condition is also necessary for existence of such extension.
Theorem 1. Let $\Phi : \mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H})$ be a CP normal linear map s.t. $\Phi(I_{\mathcal{H}}) \leq I_{\mathcal{H}}$ and $G$ a positive semidefinite densely defined operator on $\mathcal{H}$ satisfying condition (A).

A) If the map $\Phi$ is $G$-limited (Def. 2) then it is continuous on $\mathfrak{B}(\mathcal{H})$ w.r.t. the norm $\|\cdot\|_E^G$ for any $E > 0$. Moreover, any Stinespring representation (3) of $\Phi$ defines a unique bounded linear operator (also denoted by $\Phi$) on the Banach space $\mathfrak{B}_G(\mathcal{H})$ of $\sqrt{G}$-bounded operators such that

$$\|\Phi(A)\|_E^G \leq \sqrt{\|\Phi(I_{\mathcal{H}})\|_E^G A\|_E^G},$$

for any $A \in \mathfrak{B}_G(\mathcal{H})$ and $E > 0$, where $Y_\Phi(E)$ is the function defined in (10). The operator $\Phi$ is bounded w.r.t. the seminorm $b_{\sqrt{G}}(\cdot)$:

$$b_{\sqrt{G}}(\Phi(A)) \leq \sqrt{k_\Phi \|\Phi(I_{\mathcal{H}})\|_E b_{\sqrt{G}}(A)} \text{ for any } A \in \mathfrak{B}_G(\mathcal{H}),$$

where $k_\Phi = \lim_{E \to +\infty} Y_\Phi(E)/E$, in particular, $\Phi$ maps the subspace $\mathfrak{B}_G^0(\mathcal{H})$ of $\sqrt{G}$-infinitesimal operators into itself.\(^\text{11}\)

B) If $G$ is a discrete operator (Def. 4) and one of following conditions holds

a) the map $\Phi$ has finite Choi rank;

b) $\Phi(I_{\mathcal{H}}) = \Phi(P)$ for some finite rank projector $P$,

then continuity of $\Phi$ on $\mathfrak{B}(\mathcal{H})$ w.r.t. the norm $\|\cdot\|_E^G$ for some $E > 0$ implies that the map $\Phi$ is $G$-limited.

Remark 2. Since the continuity of the map $\Phi$ on $\mathfrak{B}(\mathcal{H})$ w.r.t. the norm $\|\cdot\|_E^G$ is necessary for existence of the bounded linear extension of $\Phi$ to $\mathfrak{B}_G(\mathcal{H})$ mentioned in part A of Theorem 1, this theorem shows that the existence of such extension is equivalent to the $G$-limited property for CP linear maps satisfying one of the conditions in part B (provided that $G$ is a discrete operator).

Remark 3. Condition a) in part B of Theorem 1 means that the map $\Phi$ has the Kraus representation $\Phi(A) = \sum_k V_k^*AV_k$ with a finite number of summands.

Condition b) means that the image of $\mathfrak{S}(\mathcal{H})$ under the predual map $\Phi_\ast$ is contained in $\mathfrak{S}(\mathcal{H}_P)$, where $\mathcal{H}_P$ is the finite-dimensional range of $P$.

Conditions a) and b) are complementary to each other: if $\Phi$ satisfies condition a) then the complementary map $\hat{\Phi}$ satisfies condition b) and vice versa.\(^\text{12}\)

Proof of Theorem 1. A) To prove continuity of $\Phi$ on $\mathfrak{B}(\mathcal{H})$ w.r.t. the norm $\|\cdot\|_E^G$ it suffices to assume the 2-positivity of $\Phi$. Indeed, Kadison’s inequality (4) implies that

$$\text{Tr}[\Phi(A)]^*\Phi(A)\rho \leq \|\Phi(I_{\mathcal{H}})\|_E \text{Tr}[\Phi(A^*A)\rho = \|\Phi(I_{\mathcal{H}})\|_E \|A\|_E^G \|A\|_E^G \rho \leq \|\Phi(I_{\mathcal{H}})\|_E [\|A\|_E^G]^2]$$

\(^\text{11}\)The concavity of the function $E \mapsto Y_\Phi(E)$ implies that the nonnegative function $E \mapsto Y_\Phi(E)/E$ is non-increasing and hence has a finite limit as $E \to +\infty$.

\(^\text{12}\)If $\Phi$ has the Stinespring representation (3) then the complementary map $\hat{\Phi}$ is defined as $\hat{\Phi}(A) = V_\Phi^*(I_{\mathcal{H}} \otimes A)V_\Phi$, $A \in \mathfrak{B}(\mathcal{K})$. Assuming that $\mathcal{K}$ is a subspace of $\mathcal{H}$ we may consider $\hat{\Phi}$ as a map from $\mathfrak{B}(\mathcal{H})$ into itself.
for any $A \in \mathfrak{B}(\mathcal{H})$ and any $\rho \in \mathfrak{S}(\mathcal{H})$ such that $\text{Tr}G\rho \leq E$ (since the condition $\Phi(I_{\mathcal{H}}) \leq I_{\mathcal{H}}$ guarantees that $\text{Tr}\Phi_{\ast}(\rho) \leq 1$). Hence, by using inequality (15) we obtain
\[
\|\Phi(A)\|_E^G \leq \sqrt{\|\Phi(I_{\mathcal{H}})\|_E^G} \|A\|_{Y_{\Phi}(E)}^G \leq \max\left\{1, \sqrt{Y_{\Phi}(E)/E}\right\} \sqrt{\|\Phi(I_{\mathcal{H}})\|_E^G}.
\]

Let $A$ be an arbitrary operator in $\mathfrak{B}_G(\mathcal{H})$. If the map $\Phi$ has representation (3) then the predual map has the form $\Phi_{\ast}(\rho) = \text{Tr}_{K}V_{\Phi}\rho V_{\Phi}^\ast$. So, the finiteness of $Y_{\Phi}(E)$ shows that $V_{\phi}|\varphi \rangle \in D(\sqrt{G} \otimes I_{K})$ for any $\varphi \in D(\sqrt{G})$. By Lemma 1 the operator $A \otimes I_{K}$ has a unique extension to the set $D(\sqrt{G} \otimes I_{K})$ satisfying (17). So, the operator $V_{\phi}^\ast[A \otimes I_{K}]V_{\Phi}$ is well defined on $D(\sqrt{G})$. It does not depend on representation (3). Indeed, for any other Stinespring operator $V_{\phi} : \mathcal{H} \rightarrow \mathcal{H} \otimes K'$ there is a partial isometry $W : K \rightarrow K'$ such that $V_{\phi} = (I_{\mathcal{H}} \otimes W)V_{\Phi}$ and $V_{\phi} = (I_{\mathcal{H}} \otimes W^\ast)V_{\Phi}$ [5, Ch.6]. So, by Remark 1 we have $V_{\phi}^\ast[A \otimes I_{K}]V_{\phi}|\varphi \rangle = [V_{\phi}^\ast]^\ast[A \otimes I_{K}]V_{\phi}|\varphi \rangle$ for any $\varphi \in D(\sqrt{G})$.

Let $\{\varphi_k\}$ be a set of vectors in $\mathcal{H}$ such that $\sum_k \|\varphi_k\|^2 \leq 1$ and $\sum_k \|G\varphi_k\|^2 \leq E$. Since $V_{\phi}^\ast V_{\phi} = \Phi(I_{\mathcal{H}})$, we have $\|V_{\phi}\|^2 = \|\Phi(I_{\mathcal{H}})\| \leq 1$ and hence $\sum_k \|V_{\phi}\varphi_k\|^2 \leq 1$.

Let $\rho = \sum_k \langle \varphi, \varphi_k \rangle |\varphi, \varphi_k \rangle$ be an operator in the unit ball of $\mathcal{F}_{+}(\mathcal{H})$. Then $\text{Tr}G\rho = \sum_k \|\sqrt{G}\varphi_k\|^2 \leq E$ and hence $\sum_k \|\sqrt{G} \otimes I_{K}V_{\phi}\varphi_k\|^2 = \text{Tr}G\Phi_{\ast}(\rho) \leq V_{\Phi}(E)$. So, expression (11) implies that
\[
\sum_k \|V_{\phi}^\ast[A \otimes I_{K}]V_{\phi}\varphi_k\|^2 \leq \|\Phi(I_{\mathcal{H}})\| \sum_k \|A \otimes I_{K}\|_{Y_{\Phi}(E)}^G \|V_{\phi}\varphi_k\|^2 \leq \|\Phi(I_{\mathcal{H}})\| \left[\|A \otimes I_{K}\|_{Y_{\Phi}(E)}^G\right]^2.
\]

By Lemma 1 the r.h.s. of this inequality coincides with $\|\Phi(I_{\mathcal{H}})\| \left[\|A\|_{Y_{\Phi}(E)}^G\right]^2$. Thus, inequality (21) follows from expression (11) for $\|V_{\phi}^\ast[A \otimes I_{K}]V_{\phi}\|_E^G$ and inequality (15).

Inequality (22) is proved by using formula (16) and the first inequality in (21).

B) Assume that
\[
\|\Phi(A)\|_E^G \leq k_E\|A\|_E^G, \quad \forall A \in \mathfrak{B}(\mathcal{H}),
\]
for some $E > 0$ and $k_E \in (0, +\infty)$. It follows from (15) that the boundedness relation (23) with finite $k_E$ holds for all $E > 0$.

Assume that the operator $G$ has representation (5). Denote by $\mathcal{H}_{n}$ and $P_{n}$ the subspace spanned by the eigenvectors $\tau_1, \tau_2, \ldots, \tau_{n-1}$ and the projector on this subspace correspondingly. Let (3) be the minimal Stinespring representation for $\Phi$. It means that
\[
\text{lin}\{(A \otimes I_{K})V_{\phi}|\varphi \rangle | A \in \mathfrak{B}(\mathcal{H}), \varphi \in \mathcal{H}\} \text{ is dense in } \mathcal{H} \otimes K, \quad (24)
\]
where lin denotes the linear span of a subset of $\mathcal{H} \otimes K$.

Assume that $\Phi$ has finite Choi rank, i.e. $\dim K < +\infty$. Let $A$ be any operator in the unit ball $\mathfrak{B}_1(\mathcal{H})$ of $\mathfrak{B}(\mathcal{H})$. Since $AV_{\phi}P_{n} \in \mathfrak{B}(\mathcal{H})$ for all $n$ and $\|A\|_{Y_{\Phi}(E)}^G \leq \|A\| \|\sqrt{G}P_{n}\|_E^G \leq \sqrt{E}$, it follows from (23) that
\[
\|V_{\phi}^\ast(A \otimes I_{K})(\sqrt{G}P_{n} \otimes I_{K})V_{\phi}\varphi\| = \|V_{\phi}^\ast(A\sqrt{G}P_{n} \otimes I_{K})V_{\phi}\varphi\| \leq k_E\sqrt{E} \quad (25)
\]
for any unit vector $\varphi \in D(\sqrt{G})$ such that $\|\sqrt{G}\varphi\|^2 \leq E$. 

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It follows from (24) that the function \( F_{\Phi}(\psi) = \sup_{A \in \mathfrak{B}_1(\mathcal{H})} \| V_{\Phi}^*(A \otimes I_K)\psi \| \) does not take zero values on the unit sphere in \( \mathcal{H} \otimes K \). Indeed, otherwise there is a unit vector \( \psi \in \mathcal{H} \otimes K \) such that \( \langle \psi | A \otimes I_K V_{\Phi} | \varphi \rangle = 0 \) for all \( A \in \mathfrak{B}_1(\mathcal{H}) \) and \( \varphi \in \mathcal{H} \). Thus, Lemma 4 below and (25) imply that

\[
\text{Tr}GnP_{\Phi}(|\varphi\rangle\langle\varphi|) = \|(\sqrt{G}P_n \otimes I_K)V_{\Phi,\varphi}\| \leq \delta^{-2}k_E^2E
\]

for any unit vector \( \varphi \in \mathcal{D}(\sqrt{G}) \) such that \( \| \sqrt{G}\varphi \| \leq E \), all \( n \) and some \( \delta > 0 \). Hence, \( \text{Tr}G_{\Phi}(|\varphi\rangle\langle\varphi|) \leq \delta^{-2}k_E^2E \) for any such \( \varphi \). By Proposition 1 the map \( \Phi \) is \( G \)-limited.

Assume that \( \Phi(P) = \Phi(I_K) \), where \( P \) is a finite rank projector in \( \mathfrak{B}(\mathcal{H}) \). We will show first that (23) implies that the vector \( V_{\Phi} | \varphi \rangle \) does not lie in \( \mathcal{D}(\sqrt{G} \otimes I_K) \).

Thus, since \( \text{Tr}G_{\Phi}(|\varphi\rangle\langle\varphi|) \leq \delta^{-2}k_E^2E \) for any such \( \varphi \), it is easy to show that the assumption \( \| B \|_E^G = o(\sqrt{E}) \) as \( E \to +\infty \). Indeed, since

\[
\sum_{k=0}^{+\infty} c_k E_k \sum_j |\langle \tau_k \otimes \tau_j | V_{\Phi} | \varphi \rangle |^2 = +\infty,
\]

where \( \{ \tau_j \} \) is an orthonormal basic in \( K \).

It is easy to construct a sequence \( \{ c_k \} \subset (0,1) \) vanishing as \( k \to +\infty \) such that

\[
\sum_{k=0}^{+\infty} c_k E_k \sum_j |\langle \tau_k \otimes \tau_j | V_{\Phi} | \varphi \rangle |^2 = +\infty.
\]

It follows that the vector \( V_{\Phi} | \varphi \rangle \) does not belong to the domain of the operator \( B \otimes I_K \), where

\[
B = \sum_{k=0}^{+\infty} \sqrt{c_k E_k} | \tau_k \rangle \langle \tau_k |\]

is a positive densely defined operator on \( \mathcal{H} \). Note that \( \| B \|_E^G = o(\sqrt{E}) \) as \( E \to +\infty \). Indeed, since

\[
\| B \|_E^G = \sup \left\{ \sum_{k=0}^{\infty} c_k E_k x_k^2 \middle| \sum_{k=0}^{\infty} E_k x_k^2 = E, \sum_{k=0}^{\infty} x_k^2 = 1 \right\},
\]

it is easy to show that the assumption \( \| B \|_E^G \geq C\sqrt{E} \) for some \( C > 0 \) and all sufficiently large \( E \) leads to a contradiction.

Thus, \( B \in \mathfrak{B}_1^G(\mathcal{H}) \) and hence the sequence \( \{ BP_n \} \subset \mathfrak{B}(\mathcal{H}) \) tends to the operator \( B \) w.r.t. the norm \( \| \cdot \|_E^G \) [10] Remark 7]. Thus, \( \{ BP_n \} \) is a Cauchy sequence w.r.t. the norm \( \| \cdot \|_E^G \).

Let \( P_{n,m} = P_m - P_n \) for any \( m > n \) and \( E = \| \sqrt{G}\varphi \|^2 \). It follows from (23) that

\[
\| V_{\Phi}^*(ABP_{n,m} \otimes I_K) V_{\Phi} \|_E^G = \| \Phi(ABP_{n,m}) \|_E^G \leq k_E \| ABP_{n,m} \|_E^G \leq k_E \| BP_{n,m} \|_E^G,
\]

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for any \( A \in \mathfrak{B}(\mathcal{H}) \). Thus,

\[
\sup_{A \in \mathfrak{B}(\mathcal{H})} \| V^*_\psi (A \otimes I_K) (B \otimes I_K) \psi_{n,m} \| \leq k_E \| BP_{n,m} \|_E^G,
\]

where \(|\psi_{n,m}\rangle = (P_{n,m} \otimes I_K) V_\psi |\varphi\rangle\).

Since \( \Phi(P) = \Phi(I_\mathcal{H}) \), where \( P \) is a projector such that \( \text{rank} P = d < +\infty \), the vector \( V_\psi |\varphi\rangle \) lies in \( \mathcal{H}_d \otimes \mathcal{K} \), where \( \mathcal{H}_d = P(\mathcal{H}) \) is a \( d \)-dimensional subspace of \( \mathcal{H} \). It follows that \( V_\psi |\varphi\rangle = \sum_{i=1}^d |\alpha_i\rangle \otimes |\beta_i\rangle \), where \{\( \alpha_i \)\} and \{\( \beta_i \)\} are orthogonal sets of vectors in \( \mathcal{H}_d \) and \( \mathcal{K} \) correspondingly. We may assume that \( \|\beta_i\| = 1 \) for all \( i \).

For given \( n \) and \( m \) denote by \( \mathcal{H}_{n,m} \) the image of the subspace \( \mathcal{H}_d \) under the projector \( P_{n,m} \). Let \( U_{n,m} \) be any unitary operator in \( \mathfrak{B}(\mathcal{H}) \) such that \( U_{n,m} (\mathcal{H}_{n,m}) \subseteq \mathcal{H}_d \). Then all the vectors \( \|(U_{n,m} B \otimes I_K) \psi_{n,m}\|^{-1} |(U_{n,m} B \otimes I_K) \psi_{n,m}\rangle \) belong to the set

\[
\mathcal{A}_{\beta_1,...,\beta_d} = \left\{ \sum_{i=1}^d |\eta_i\rangle \otimes |\beta_i\rangle : \{\eta_i\} \subset \mathcal{H}_d, \sum_{i=1}^d \|\eta_i\|^2 = 1 \right\}.
\]

The set \( \mathcal{A}_{\beta_1,...,\beta_d} \) is compact, since it is the image of the unit sphere in the direct sum of \( d \) copies of \( \mathcal{H}_d \) under the continuous map

\[
(|\eta_1,...,\eta_d\rangle \mapsto \sum_{i=1}^d |\eta_i\rangle \otimes |\beta_i\rangle).
\]

The function \( F_{V^*_\psi}(\psi) = \sup_{A \in \mathfrak{B}(\mathcal{H})} \| V^*_\psi (A \otimes I_K) \psi \| \) is lower semicontinuous on \( \mathcal{H} \otimes \mathcal{K} \) (as the least upper bound of the family of continuous functions \( \varphi \mapsto \| V^*_\psi (A \otimes I_K) \psi \|, \quad A \in \mathfrak{B}(\mathcal{H}) \)). So, this function attains its infimum on the compact set \( \mathcal{A}_{\beta_1,...,\beta_d} \) at some vector \( \psi_0 \in \mathcal{A}_{\beta_1,...,\beta_d} \). It was mentioned before that (24) implies that the function \( F_{V^*_\psi} \) is not equal to zero on the unit sphere in \( \mathcal{H} \otimes \mathcal{K} \). Thus,

\[
\inf \{ F_{V^*_\psi}(\psi) \mid \psi \in \mathcal{A}_{\beta_1,...,\beta_d} \} = F_{V^*_\psi}(\psi_0) > 0.
\]

Denoting this infimum by \( \delta \), we obtain

\[
F_{V^*_\psi}((B \otimes I_K) \psi_{n,m}) = F_{V^*_\psi}((U_{n,m} B \otimes I_K) \psi_{n,m})
\geq \delta \|(U_{n,m} B \otimes I_K) \psi_{n,m}\| = \delta \|(B \otimes I_K) \psi_{n,m}\| \quad \forall n, m,
\]

where the first equality follows from the definition of the function \( F_{V^*_\psi} \).

Since \( \{BP_n\} \) is a Cauchy sequence in \( \mathfrak{B}(\mathcal{H}) \) w.r.t. the norm \( \| \cdot \|_E^G \), the r.h.s. of (26) tends to zero as \( n, m \to +\infty \). Thus, (26) and (27) imply that \( \{(B \otimes I_K) \psi_n\} \), where \( |\psi_n\rangle = (P_n \otimes I_K) V_\psi |\varphi\rangle \), is a Cauchy sequence of vectors in \( \mathcal{H} \otimes \mathcal{K} \). So, it has a limit in \( \mathcal{H} \otimes \mathcal{K} \). Since the sequence \( \{|\psi_n\rangle\} \) tends to the vector \( V_\psi |\varphi\rangle \) and the operator \( B \otimes I_K \) is closed, the vector \( V_\psi |\varphi\rangle \) belongs to the domain of \( B \otimes I_K \) contradicting the above assumption. Thus,

\[
V_\psi |\varphi\rangle \in \mathcal{D}(\sqrt{G} \otimes I_K) \quad \text{for any} \quad \varphi \in \mathcal{D}(\sqrt{G}).
\]
The density of $D(\sqrt{G})$ in $\mathcal{H}$ shows that the minimal subspace $\mathcal{H}_d$ containing the supports of all the states $\text{Tr}_K V_\phi \rho V_\phi^*$, $\rho \in \mathcal{S}(\mathcal{H})$, is spanned by the eigenvectors of all the states $\text{Tr}_K V_\phi |\varphi\rangle \langle \varphi| V_\phi^*$, $\varphi \in D(\sqrt{G})$, corresponding to nonzero eigenvalues. By using this it is easy to show that $\mathcal{H}_d \subset D(\sqrt{G})$. It follows that the restriction of $\sqrt{G}$ to the subspace $\mathcal{H}_d$ is a bounded operator, and hence the restriction of $\sqrt{G} \otimes I_K$ to the subspace $\mathcal{H}_d \otimes K$ is a bounded operator. Thus,

$$
\varphi \mapsto \text{Tr} \Phi_*(|\varphi\rangle \langle \varphi|) = \| \sqrt{G} \otimes I_K \Phi \varphi \|^2
$$

is a bounded function on the unit ball of $\mathcal{H}$. It follows that $\rho \mapsto \text{Tr} \Phi_*(\rho)$ is a bounded function on $\mathcal{S}(\mathcal{H})$. □

**Lemma 4.** Let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert space and $\dim \mathcal{K} < +\infty$. Let $C$ be any operator in $B(H \otimes K)$. If

$$
F_C(\varphi) = \sup_{A \in \mathcal{B}_1(\mathcal{H})} \| C(A \otimes I_K) \varphi \| > 0 \text{ for all } \varphi \neq 0 \text{ then } \inf_{\varphi \in [H \otimes K]_1} F_C(\varphi) > 0
$$

where $\mathcal{B}_1(\mathcal{H})$ is the unit ball in $\mathcal{B}(\mathcal{H})$ and $[H \otimes K]_1$ is the unit sphere in $H \otimes K$, and this infimum is attainable.

**Remark 4.** The assertion of Lemma 4 is not valid if $\dim \mathcal{K} = +\infty$. □

**Proof.** Note first that the function $F_C(\varphi)$ is lower semicontinuous on $H \otimes K$ (as the least upper bound of the family of continuous functions $\varphi \mapsto \| C(A \otimes I_K) \varphi \|$, $A \in \mathcal{B}(\mathcal{H})$).

Let $\{\tau_k\}$ be an orthonormal basis in $\mathcal{H}$, $n = \dim \mathcal{K}$ and $A$ be the set of unit vectors $H \otimes K$ having the representation

$$
|\varphi\rangle = \sum_{k=1}^n |\tau_k\rangle \otimes |\psi_k\rangle, \quad \{\psi_k\} \subset \mathcal{K}, \quad \sum_{k=1}^n \|\psi_k\|^2 = 1.
$$

Since $A$ is the image of the unit sphere in the direct sum of $n$ copies of $\mathcal{K}$ under the continuous map

$$
(\psi_1, ... \psi_n) \mapsto \sum_{k=1}^n |\tau_k\rangle \otimes |\psi_k\rangle,
$$

the set $A$ is compact. So, the lower semicontinuous function $F_C(\varphi)$ attains its infimum on $A$ at some vector $\varphi_0 \in A$ and hence this infimum is positive by the assumption.

To complete the proof of the lemma it suffices to note that

$$
\inf_{\varphi \in [H \otimes K]_1} F_C(\varphi) = F_C(\varphi_0).
$$

This follows from the definition of $F_C$, since any vector in $[H \otimes K]_1$ can be represented as $(U \otimes I_K)|\varphi\rangle$, where $U$ is a unitary operator in $\mathcal{B}(\mathcal{H})$ and $\varphi$ is a vector in $A$. □
4 Generalizations to quantum channels and operations between different systems

For simplicity in the previous sections we restrict attention to CP linear transformations of the algebra \( \mathfrak{B}(\mathcal{H}) \). All the results presented therein are directly generalized to CP normal linear maps from \( \mathfrak{B}(\mathcal{H}) \) into \( \mathfrak{B}(\mathcal{H}') \), where \( \mathcal{H} \) and \( \mathcal{H}' \) are separable Hilbert spaces. Since all such spaces are isomorphic, the only essential feature of this case is the possibility to choose different operators \( G \) and \( G' \) on these spaces. So, we have to generalize the notion of \( G \)-limited quantum operation as follows.

Definition 3. A quantum operation \( \Phi : \mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H}') \) is called \( GG' \)-limited if

\[
\Tr G \Phi_*(\rho) < +\infty \quad \text{for any} \quad \rho \in \mathcal{S}(\mathcal{H}') \quad \text{such that} \quad \Tr G' \rho < +\infty,
\]

where \( \Phi_* : \mathfrak{S}(\mathcal{H}') \to \mathfrak{S}(\mathcal{H}) \) is the predual operation.

By Lemma 3 condition (29) is equivalent to the following

\[
Y_\Phi(E) \doteq \sup \left\{ \Tr G \Phi_*(\rho) \mid \rho \in \mathcal{S}(\mathcal{H}'), \Tr G' \rho \leq E \right\} < +\infty
\]

for some \( E > 0 \). Since the function \( E \mapsto Y_\Phi(E) \) is concave, the finiteness of \( Y_\Phi(E) \) for some \( E > 0 \) implies its finiteness for all \( E > 0 \).

The following proposition generalizes Proposition 1 in Section 2.3. It is proved similarly by using Lemma 2.

Proposition 2. If \( G' \) is a positive unbounded discrete operator (3) then

\[
Y_\Phi(E) \doteq \sup \left\{ \Tr G \Phi_*(|\varphi\rangle\langle\varphi|) \mid \varphi \in \mathcal{H}_*, \|\sqrt{G'}\varphi\|^2 \leq E, \|\varphi\| = 1 \right\},
\]

for any quantum operation \( \Phi : \mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H}') \), where \( \mathcal{H}_* \) is the linear span of all the eigenvectors \( \tau_1, \tau_2, \ldots \) of \( G' \), i.e., the supremum in (30) can be taken only over pure states corresponding to the vectors in \( \mathcal{H}'_* \).

Note that any CP normal linear map \( \Phi : \mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H}') \) has the Stinespring representation (3), where \( V_\Phi \) is a linear operator from \( \mathcal{H}' \) into \( \mathcal{H} \otimes \mathcal{K} \).

Theorem 2. Let \( \Phi : \mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H}') \) be a normal CP linear map s.t. \( \Phi(I_{\mathcal{H}}) \leq I_{\mathcal{H}'} \).

Let \( G \) and \( G' \) be positive semidefinite operators densely defined, respectively, on \( \mathcal{H} \) and \( \mathcal{H}' \) satisfying condition (4).

A) If the map \( \Phi \) is \( GG' \)-limited (Def 3) then it is continuous w.r.t. the norms \( \|\cdot\|_E \) and \( \|\cdot\|_{E'} \) on \( \mathfrak{B}(\mathcal{H}) \) and \( \mathfrak{B}(\mathcal{H}') \) for any \( E > 0 \). Moreover, any Stinespring representation (3) of \( \Phi \) defines a unique bounded linear operator (also denoted by \( \Phi \)) from \( \mathfrak{B}_{G}(\mathcal{H}) \) into \( \mathfrak{B}_{G'}(\mathcal{H}') \) such that

\[
\|\Phi(A)\|_{E'} \leq \sqrt{\Tr \Phi(I_{\mathcal{H}})\|\Phi(A)\|_{Y_\Phi(E)}} \leq \sqrt{\max \{1, Y_\Phi(E)/E\} \|\Phi(I_{\mathcal{H}})\|\|A\|_E}
\]

for any \( A \in \mathfrak{B}_{G}(\mathcal{H}) \) and \( E > 0 \), where \( Y_\Phi(E) \) is the function defined in (29). The operator \( \Phi \) is bounded w.r.t. the seminorms \( b_{\sqrt{G}}(\cdot) \) and \( b_{\sqrt{G'}}(\cdot) \) on \( \mathfrak{B}_{G}(\mathcal{H}) \) and \( \mathfrak{B}_{G'}(\mathcal{H}') \):

\[
b_{\sqrt{G}}(\Phi(A)) \leq \sqrt{k_\Phi \|\Phi(I_{\mathcal{H}})\|} b_{\sqrt{G}}(A) \quad \text{for any} \quad A \in \mathfrak{B}_{G}(\mathcal{H}),
\]
where \( k_\Phi = \lim_{E \to +\infty} Y_\Phi(E)/E \), in particular, \( \Phi \) maps \( \mathfrak{B}_G^0(\mathcal{H}) \) into \( \mathfrak{B}_G'(\mathcal{H}') \).

B) If \( G' \) is a discrete operator (Def.7) and one of following conditions holds

a) the map \( \Phi \) has finite Choi rank;

b) \( \Phi(I_\mathcal{H}) = \Phi(P) \) for a finite rank projector \( P \in \mathfrak{B}(\mathcal{H}) \) and \( G \) is a discrete operator

then continuity of \( \Phi \) w.r.t. the norms \( \| \cdot \|_E \) and \( \| \cdot \|_{E'} \) on \( \mathfrak{B}(\mathcal{H}) \) and \( \mathfrak{B}(\mathcal{H}') \) for some \( E > 0 \) implies that the map \( \Phi \) is \( GG' \)-limited.

Theorem 2 is proved by obvious modifications of the arguments from the proof of Theorem 1.

Remark 5. Since continuity of a map \( \Phi : \mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H}') \) w.r.t. the norms \( \| \cdot \|_E \) and \( \| \cdot \|_{E'} \) is necessary for existence of the bounded linear extension of \( \Phi \) to \( \mathfrak{B}_G(\mathcal{H}) \) mentioned in part A of Theorem 2, part B of this theorem shows that the existence of such extension implies the \( GG' \)-limited property of a map \( \Phi \) (provided that one of the conditions a) and b) holds).

5 Bosonic Gaussian Channels

In this section we apply Theorem 2 to the class of Bosonic Gaussian channels playing central role in the modern quantum information theory [5, 15].

Let \( \mathcal{H} \) and \( \mathcal{H}' \) be the spaces of irreducible representation of the Canonical Commutation Relations (CCR)

\[
W(z_1)W(z_2) = \exp \left( -\frac{i}{2} z_1^T \Delta z_2 \right) W(z_1 + z_2), \quad z_1, z_2 \in Z,
\]

\[
W'(z_1)W'(z_2) = \exp \left( -\frac{i}{2} z_1^T \Delta' z_2 \right) W'(z_1 + z_2), \quad z_1, z_2 \in Z',
\]

where \((Z, \Delta)\) and \((Z', \Delta')\) are symplectic spaces, \(\{W(z)\}_{z \in Z}\) and \(\{W'(z)\}_{z \in Z'}\) are the families of Weyl operators in \(\mathfrak{B}(\mathcal{H})\) and \(\mathfrak{B}(\mathcal{H}')\) correspondingly [3, Ch.12]. Denote by \(s\) and \(s'\) the numbers of modes of the systems, i.e. \(2s = \text{dim } Z\) and \(2s' = \text{dim } Z'\).

A Bosonic Gaussian channel \( \Phi_{K,\alpha,\ell} : \mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H}') \) is defined via the action on the Weyl operators:

\[
\Phi_{K,\alpha,\ell}(W(z)) = W'(Kz) \exp \left[ i \ell z - \frac{1}{2} z^T \alpha z \right], \quad z \in Z,
\]

where \( K : Z \to Z' \) is a linear operator, \( \ell \) is a 2s-dimensional real row and \( \alpha \) is a real symmetric \((2s) \times (2s)\) matrix satisfying the inequality \( \alpha \geq \pm \frac{1}{2} [\Delta - K^T \Delta' K] \) [3, 5].

Any such channel can be transformed by appropriate displacement unitaries to the Bosonic Gaussian channel with \( \ell = 0 \) and the same matrix \( \alpha \) [5]. So, we will restrict attention to Gaussian channels \( \Phi_{K,\alpha,0} \) typically called centred and denoted by \( \Phi_{K,\alpha} \) in what follows.
Let \( R = [q_1, p_1, \ldots, q_s, p_s]^\top \) be the \( 2s \)-vector of the canonical observables – linear unbounded operators on the space \( \mathcal{H} \) with common dense domain satisfying the commutation relations

\[
[q_i, p_j] = i\delta_{ij}I_\mathcal{H}, \quad [q_i, q_j] = [p_i, p_j] = 0 \quad \forall i, j.
\]

Physically, the operators \( q_1, \ldots, q_s \) and \( p_1, \ldots, p_s \) can be treated, respectively, as generalized position and momentum observables of the \( s \)-mode quantum oscillator.

Let \( R' = [q'_1, p'_1, \ldots, q'_{s'}, p'_{s'}]^\top \) be the \( 2s' \)-vector of the canonical observables of the Bosonic system described by the space \( \mathcal{H}' \).

Any Gaussian channel \( \Phi_{K,\alpha} : \mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H}') \) can be correctly extended to the algebra of all polynomials in the canonical observables \( q_1, \ldots, q_s \) and \( p_1, \ldots, p_s \). Moreover, it transforms any such polynomial into a polynomial in the canonical observables \( q'_1, \ldots, q'_{s'} \) and \( p'_1, \ldots, p'_{s'} \) of the same order. This property can be proved by differentiating the relation \( (32) \) at the point \( z = 0 \) with the help of Baker-Campbell-Hausdorff formula \( [5] \). In particular, by using this way one can show that

\[
\Phi_{K,\alpha}(R^\top R) = \Phi_{K,\alpha}\left(\sum_{i=1}^{s} q_i^2 + p_i^2\right) = [R']^\top K^\top K R' + I_{\mathcal{H}'} \text{Sp}, \tag{33}
\]

where \( \text{Sp} \) denotes the spur (trace) of a \( (2s) \times (2s) \) matrix.

In many cases the Hamiltonian (energy observable) of a Bosonic system has the form \( R^\top \varepsilon R \), where \( \varepsilon \) is a real positive nondegenerate matrix \( [5, 15] \). So, we will assume that

\[
G = R^\top \varepsilon R - c_\varepsilon I_\mathcal{H} \quad \text{and} \quad G' = [R']^\top \varepsilon' R' - c_{\varepsilon'} I_{\mathcal{H}'}, \tag{34}
\]

where \( \varepsilon \) and \( \varepsilon' \) are real positive nondegenerate \( (2s) \times (2s) \) and \( (2s') \times (2s') \) matrices, \( c_\varepsilon \) and \( c_{\varepsilon'} \) are the infima of the spectrum of the positive operators \( R^\top \varepsilon R \) and \( [R']^\top \varepsilon' R' \) correspondingly. In the following lemma we use the notion of a \( GG' \)-limited channel \( (\text{Def}3) \) and the function \( Y_\Phi(E) \) defined in \( (30) \).

**Lemma 5.** Let \( G \) and \( G' \) be the operators defined in \( (34) \). Then any Gaussian channel \( \Phi_{K,\alpha} : \mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H}') \) is \( GG' \)-limited and

\[
Y_{\Phi_{K,\alpha}}(E) \leq aE + ac_{\varepsilon'} - c_\varepsilon + b, \tag{35}
\]

where \( a = \|\varepsilon\| K^2/m(\varepsilon') \), \( m(\varepsilon') \) is the minimal eigenvalue of \( \varepsilon' \), and \( b = \|\varepsilon\| \text{Sp} \).\(^{13}\)

**Proof.** By using relation \( (33) \) we obtain

\[
\Phi_{K,\alpha}(R^\top \varepsilon R) \leq \|\varepsilon\| \|\Phi_{K,\alpha}(R^\top R) \| \leq \|\varepsilon\| \|K\|^2 [R']^\top R' + \|\varepsilon\| I_{\mathcal{H}'} \text{Sp} \\
\leq \|\varepsilon\| \|K\|^2 m(\varepsilon')^{-1} [R']^\top \varepsilon' R' + \|\varepsilon\| I_{\mathcal{H}'} \text{Sp}.
\]

Hence

\[
\text{Tr}G[\Phi_{K,\alpha}](\rho) = \text{Tr} \Phi_{K,\alpha}(G) \rho \leq a\text{Tr}G'\rho + ac_{\varepsilon'} - c_\varepsilon + b
\]

\(^{13}\)Here \( \|A\| \) denotes the operator norm of a matrix \( A \).
for any state $\rho \in C(H')$. This inequality implies (35). □

Speaking about extension of Gaussian channels to unbounded operators we may (w.l.o.g.) chose on the role of $G$ and $G'$ the number operators

$$N = \frac{1}{2}(R^\top R - sI_H) \quad \text{and} \quad N' = \frac{1}{2}([R']^\top R' - s'I_{H'}) .$$

(36)

Indeed, since $m(\varepsilon)R^\top R \leq R^\top \varepsilon R \leq \|\varepsilon\|R^\top R$, where $m(\varepsilon)$ is the minimal eigenvalue of $\varepsilon$, it is easy to see that $C_G(H) = C_N(H)$ for any operator $G$ defined in (34) and that the norms $\|\cdot\|_E$ and $\|\cdot\|_{E'}$ are equivalent. The same arguments show that $C_{G'}(H') = C_{N'}(H')$ for any operator $G'$ defined in (34) and that the norms $\|\cdot\|_{E'}$ and $\|\cdot\|_{E'}$ are equivalent.

It is easy to see that the Banach space $C_N(H)$ (corresp. $C_{N'}(H')$) contains all the canonical observables $q_1, \ldots, q_s$ and $p_1, \ldots, p_s$ (corresp., $q'_1, \ldots, q'_s$ and $p'_1, \ldots, p'_s$) and their linear combinations.

If $G = N$ and $G' = N'$ then it follows from (35) that

$$Y_{\Phi_{K,\alpha}}(E) \leq F_{K,\alpha}(E) = aE + (as' - s + b)/2 ,$$

where $a = \|K\|^2$, $b = Sp\alpha$, $s = \dim Z/2$ and $s' = \dim Z'/2$. Hence Theorem 2 implies the following

**Proposition 3.** Let $N$ and $N'$ be the number operators (36) and $\Phi_{K,\alpha}$ a centered Gaussian channel from $C(H)$ to $C(H')$. Then

- the channel $\Phi_{K,\alpha}$ is continuous w.r.t. the norms $\|\cdot\|_E^N$ and $\|\cdot\|_{E'}^N$ on $C(H)$ and $C(H')$ for any $E > 0$;
- any Stinespring representation of $\Phi_{K,\alpha}$ defines a unique bounded linear operator from $C_N(H)$ to $C_{N'}(H')$ such that

$$\|\Phi_{K,\alpha}(A)\|_{E'}^N \leq \|A\|_{F_{K,\alpha}(E)}^N \leq \max\left\{ 1, \sqrt{F_{K,\alpha}(E)/E} \right\} \|A\|_{E'}^N, \quad \forall E > 0 ,$$

(37)

for any $A \in C_N(H)$, where $F_{K,\alpha}(E)$ is the above defined function.

- $b_{\sqrt{N}}(\Phi_{K,\alpha}(A)) \leq K b_{\sqrt{N}}(A)$ for any $A \in C_N(H)$, in particular, $\Phi_{K,\alpha}$ maps the subspace $C_N^0(H)$ of $\sqrt{N}$-infinitesimal operators into the similar subspace $C_{N'}^0(H')$.

**Remark 6.** It is well known that any Gaussian channel $\Phi_{K,\alpha}$ is well defined on the linear span of the canonical observables $q_1, \ldots, q_s$ and $p_1, \ldots, p_s$ contained in $C_N(H)$ and maps it into the linear span of the canonical observables $q'_1, \ldots, q'_s$ and $p'_1, \ldots, p'_s$ contained in $C_{N'}(H')$. Proposition 3 states that the channel $\Phi_{K,\alpha}$ is correctly extended.

\[14\] The operators $E$-norms of the observables $q$ and $p$ in the case $s = 1$ are estimated in [10, Sect.5].

\[15\] $C_N(H)$ and $C_{N'}(H')$ are the Banach spaces of $\sqrt{N}$-bounded operators on $H$ and $\sqrt{N'}$-bounded operators on $H'$ equipped with the norms $\|\cdot\|_E^N$ and $\|\cdot\|_{E'}^N$, correspondingly (see Section 2.2).
to the space $\mathfrak{B}_N(\mathcal{H})$ of all $\sqrt{N}$-bounded operators on $\mathcal{H}$ and that this extension maps $\mathfrak{B}_N(\mathcal{H})$ into $\mathfrak{B}_N'(\mathcal{H}')$ continuously w.r.t. the norms $\| \cdot \|_N^E$ and $\| \cdot \|_{N'}^E$.

**Example: one-mode attenuation/amplification Gaussian channel.** Assume that $\mathcal{H}' = \mathcal{H}$, $Z = Z'$, $s = s' = 1$. Consider the channel $\Phi_{K,\alpha} : \mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H})$, where

$$K = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \quad \text{and} \quad \alpha = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}, \quad \lambda = N_c + |k^2 - 1|/2.$$

This is a one-mode attenuation/amplification channel, $N_c \geq 0$ is the power of environment noise and $k > 0$ is the coefficient of attenuation/amplification [5].

In this case one can obtain explicit expression for the function $Y_{\Phi_{K,\alpha}}(E)$. Indeed, formula (33) implies that $\Phi_{K,\alpha}(N) = k^2 N + ((k^2 - 1)/2 + \lambda)I_{\mathcal{H}}$. Hence

$$Y_{\Phi_{K,\alpha}}(E) = F_{k,N_c}(E) = k^2 E + (k^2 - 1)/2 + \lambda = \begin{cases} k^2 E + N_c, & k \leq 1 \\ k^2 E + N_c + (k^2 - 1), & k > 1 \end{cases}$$

So, it follows from (21) that

$$\| \Phi_{K,\alpha}(A) \|_E^N \leq \| A \|_{F_{k,N_c}(E)}^N \leq \max \left\{ 1, \sqrt{F_{k,N_c}(E)/E} \right\} \| A \|_E^N, \quad \forall E > 0, \quad (38)$$

for any $A \in \mathfrak{B}_N(\mathcal{H})$. If $\Phi_{K,\alpha}$ is a quantum-limited attenuator ($k < 1$, $N_c = 0$) then $\| \Phi_{K,\alpha}(A) \|_E^N \leq \| A \|_{k^2 E} \leq \| A \|_E^N$ for any $A \in \mathfrak{B}_N(\mathcal{H})$, i.e. $\Phi_{K,\alpha}$ is a contraction w.r.t. the norm $\| \cdot \|_E^N$ for any $E > 0$.

Since $\lim_{E \to +\infty} F_{k,N_c}(E)/E = k^2$ for any $k > 0$ and $N_c \geq 0$, we have

$$b_{\sqrt{N}}(\Phi_{K,\alpha}(A)) \leq k b_{\sqrt{N}}(A) \quad \text{for all} \quad A \in \mathfrak{B}_N(\mathcal{H}).$$

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