Approximating Values of Generalized-Reachability Stochastic Games

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Abstract—Simple stochastic games are turn-based 2½-player games with a reachability objective. The basic question asks whether player Max can win a given game with at least a given probability. A natural extension are games with a conjunction of such conditions as objective. Despite a plethora of recent results on analysis of systems with multiple objectives, decidability of this basic problem remains open. In this paper, we show the Pareto frontier of the achievable values can be approximated to a given precision. In particular, our algorithm is not limited to stopping games and can be run as an anytime algorithm, always returning the current approximation and its error bound.

I. INTRODUCTION

Simple stochastic games [Con92] are zero-sum turn-based stochastic games (SG) where the objective of player Maximizer is to maximize the probability of reaching a given target set of states, while player Minimizer aims at the opposite. The basic decision problem is to determine whether there is a strategy for Max achieving at least a given probability threshold. These games are interesting both theoretically: the problem is known to be in $\text{NP} \cap \text{co-NP}$, but not known to be in $\text{P}$, and several other important game problems reduce to it [CF11] such as parity games; as well as practically: they can serve as a tool for synthesis with safety/co-safety objectives in environments with stochastic uncertainty.

Multi-objective stochastic systems have attracted a lot of attention recently, see below, both SG and the special case with one player only (Markov decision processes, MDP [Put14]). They model and enable optimization with respect to conflicting goals, where a desired trade-off is to be considered. A natural multi-dimensional generalization of the reachability threshold constraint $\mathbb{P}[\exists t \in T_i] \geq t_i$ giving rise to generalized-reachability (or multiple-reachability) stochastic games, similarly to e.g. generalized mean-payoff SG [BKTW15], [CD16]. The problem is then to decide whether a given vector of thresholds can be achieved by Maximizer. Note that since these games are not determined [CFK+13b] this corresponds to the lower-value problem.

While in the single-dimensional case reachability is comparably simple objective, e.g. a special case of mean payoff since the target states can be made absorbing, it is not the case for the multi-dimensional case. Firstly, while generalized mean-payoff MDP can be solved in $\text{P}$ [Cha07], [BBC+14], generalized reachability MDP are $\text{PSPACE}$-hard [RRS15]. Secondly, while generalized mean-payoff SG have been solved [BKTW15], [CD16], generalized reachability SG are still open. Decidability is shown for the (more general) total-reward objective, but only for the subclass of stopping SG with a 2-dimensional objective [BF16]. (An SG is stopping if under any strategies a designated set of sinks is reached almost surely.)

In this paper, we show the following:

Theorem: $\varepsilon$-approximation of the set of all achievable vectors in any SG (not necessarily stopping) with generalized-reachability objective of any dimension is effectively constructible for any given $\varepsilon > 0$.

Our algorithm provides anytime bounds on the Pareto frontier and the bounds converge in the limit to the frontier. Thus we strengthen the result of [CFK+13b], which provides a converging sequence of lower bounds on the frontier. Our complementary upper bounds allow us to approximate the frontier with any given precision, i.e., report the current precision in the case of the anytime approximation.

Convergent upper bounds on the value are notoriously known to be difficult to achieve. Until recently, the default engine for analysis in the most used probabilistic model checker Prism [KNP11] and Prism-Games [CFK+13a] used value iteration, e.g. [Put14], which converges to the value from below and the stopping criteria used could render arbitrarily wrong results [HM17]. For a solution with a given precision, one could use linear programing instead, which however, does not scale well for MDP, e.g. [KM17], and does not work at all for SG [Con92]. Therefore, [BCC+14], [HM17] extended value iteration so that it provides not only the under-approximating convergent sequence, but also an over-approximating one, calling the technique “bounded value iteration” (due to [MLG05]) or “interval iteration”, respectively.

Its essence is to collapse maximal end components (MECs) of the MDP; on an MDP without MECs the over-approximating sequence also converges to the actual value of the (modified as well as original) MDP. This technique was further extended to MDP with mean-payoff objective [ACD+17]. In contrast, for SG one cannot collapse MECs since they account for non-trivial alternating structure, as opposed to MDP, where any desired action exiting the MEC can be
taken almost surely. Therefore, a more complex procedure has been proposed for SG [KKKW18]: Depending on the current under-approximation, problematic parts of MECs are dynamically identified and their over-approximation is lowered to over-approximations of certain actions exiting the MEC, as exemplified and explained in Section III.

This paper utilizes several techniques from literature to obtain the corresponding result for the multi-dimensional case:

- Firstly, we use the Bellman operator extended to downward-closed sets (instead of just real values) [CFK+13b], allowing for value iteration in the multi-dimensional setting.
- Secondly, we exploit the technique of [KKKW18], which in the single-dimensional setting repetitively identifies the problematic parts of MECs preventing convergence.
- Thirdly, in order to apply this technique, we view our problem as a continuum of single-dimensional problems due to [FKP12].
- Fourthly, we group the single-dimensional problems into finitely many regions, in spirit similar to regions of timed automata [AD94] since they are essentially given by orderings of the approximate values of certain actions. Nevertheless, due to the projective geometry of the problem, we need to work slightly more generally with simplicial complexes, e.g. [Hat02].

The main technical difficulty is to identify (i) the parts of MECs with an unjustified too high upper bound and (ii) the value to which it should be decreased in each step, both relative to different real-valued weights put on the objectives, which are, moreover, uncountably many.

**Related work.** Already for a decade, MDP have been extensively studied in the setting of multiple objectives. Multiple objectives have been considered both qualitative, such as reachability and LTL [EKVY08], as well as quantitative, such as mean payoff [Cha07], [BBC+14], discounted sum [CFW13], or total reward [FKN+11]. The expectation has been combined with variance in [BCFK13]. Beside expectation queries, conjunctions of percentile (threshold) queries have been considered for various objectives [FKRK95], [BBC+14], [RRS17], [CKK17]. Further, for general Boolean combinations for Markov chains with total reward, [HKL17] approximates the value, while computability is still open. In contrast, [Ve15] shows that Boolean combinations over mean payoff games become quickly undecidable. For the specifics of the two-dimensional case and their interplay, see [BDD+14]. The usage of the multi-dimensional setting is discussed in [BDK+14b], [BDK14a], comparing multiple rewards and quantiles and reporting how they have practically been applied and found useful by domain experts.

More recently, SG have been also analyzed with multiple objectives, but the results are more limited [SK16]. Multiple mean-payoff objective was first examined in [BKTW15] and both the qualitative and the quantitative problems are coNP-complete [CD16]. Although Boolean combinations of mean-payoff are undecidable in general [Ve15], in certain subclasses of SG they can be approximated [BKW18]. Boolean combinations of total-reward objectives was approximated in the case of stopping games [CFK+13b] and applied to autonomous driving [CKSW13], where LTL is reduced to total reward in the case of stopping games. For dimension two, the problem is shown decidable [BF16].

While Prism-Games [KPW16] provides tool support for several multi-player multi-objective settings [KPKW18], the single-dimensional-focused tools GAVS+ [CKLB11] and GIST [CHJR10] are not maintained anymore. MultiGain [BCFK15] is limited to generalized mean-payoff SG.

Pareto frontiers can be in many settings $\varepsilon$-approximated in polynomial time [PY00]. Pareto frontiers are constructed for generalized mean-payoff objective for 2-player (non-stochastic) games in [BR15], MDP in [BBC+14], [CKK17], and SG in [BKW18]. For the generalized-reachability, the Pareto frontier is approximated for MDP in [EKVY08], but for SG the Pareto frontier is not even known to be given by finitely many points, except for dimension two [BF16]. In contrast, in the single-dimensional case, the value is known to be a multiple of a denominator that can be easily calculated from the syntactic description of the game [CH08].

**Structure of the paper** After recalling the basic notions and the gist of the technique of [KKKW18] in Section II, we illustrate the problem, the difficulties and our solution on examples in Section III. The algorithm is described and the correctness intuitively explained in Section IV. The high-level proof follows in Section V, relaying all technical proofs of claims, for the sake of readability, to Section VI. We conclude in Section VII.

## II. Preliminaries

### A. Stochastic Games

A probability distribution on a finite set $X$ is a mapping $\delta : X \to [0,1]$, such that $\sum_{x \in X} \delta(x) = 1$. The set of all probability distributions on $X$ is denoted by $D(X)$. Given a dimension $n \in \mathbb{N}$, often implicitly clear from context, and $c \in \mathbb{R}$, we let $\vec{c}$ denote the $n$-dimensional vector with all components equal to $c$. For a vector $\vec{v}$, its $i$th component is denoted $\vec{v}_i$. We compare vectors component-wise, i.e. $\vec{u} \leq \vec{v}$ if $\vec{u}_i \leq \vec{v}_i$ for all $i$.

Now we define turn-based two-player stochastic games. As opposed to the notation of e.g. [Con92], we do not have special stochastic nodes, but rather a probabilistic transition function.

**Definition 1 (SG):** A stochastic game (SG) is a tuple $(S, S_0, S_\infty, S_0, A, \Av, \delta)$, where $S$ is a finite set of states partitioned into the sets $S_0$ and $S_\infty$ of states of the player Maximizer and Minimizer, respectively, $s_0 \in S$ is the initial state, $A$ is a finite set of actions, $\Av : S \to 2^A$ assigns to every state a set of available actions, and $\delta : S \times A \to D(S)$ is a transition function that given a state $s$ and an action $a \in \Av(s)$ yields a probability distribution over successor states.

A Markov decision process (MDP) is then a special case of SG where $S_\infty = \emptyset$. Note that a Markov chain (MC) can be also seen as a special case of an MDP, where for all $s \in S$: 

\[
\delta(s) = \sum_{a \in A} \delta(s, a) = 1
\]
\(|\text{Av}(s)| = 1\). We assume that SG are non-blocking, so for all states \(s\) we have \(\text{Av}(s) \neq \emptyset\).

For a state \(s\) and an available action \(a \in \text{Av}(s)\), we denote the set of successors by \(\text{Post}(s, a) := \{s' \mid \delta(s, a, s') > 0\}\). We say a state-action pair \((s, a)\) is an \(\text{EC}\) if \(s \in \text{Av}(s)\), \((s, a)\) exits \(T\), if \(\exists t \in \text{Post}(s, a) : t \notin T\), i.e., with some probability a successor outside of \(T\) could be chosen. Further, we use \(\text{Ex}(T) = \{(s, a) \mid s \in T, a \in \text{Av}(s), (s, a) \text{ exits } T\}\) to denote all exits of a state \(T\) set \(S\). Finally, for any set of states \(T \subseteq S\), we use \(T\) and \(\bigcup T\) to denote the states of \(T\) that belong to Maximizer and Minimizer, whose states are drawn in the figures as \(\square\) and \(\bigcirc\), respectively.

The semantics of SG is given in the usual way by means of strategies and the induced Markov chain [BK08] and its respective probability space, as follows. An infinite path \(\rho\) is an infinite sequence \(\rho = s_0a_0s_1 \cdots \in (S \times A)^\omega\), such that for every \(i \in \mathbb{N}\), \(a_i \in \text{Av}(s_i)\) and \(s_{i+1} \in \text{Post}(s_i, a_i)\). Finite paths are defined analogously as elements of \((S \times A)^* \times S\). A strategy of Maximizer or Minimizer is a function \(\sigma : (S \times A)^* \times S_{\text{T}} \rightarrow D(A)\) or \((S \times A)^* \times S_{\text{O}} \rightarrow D(A)\), respectively, such that \(\sigma(s) \in D(\text{Av}(s))\) for all \(s\). We call a strategy deterministic if it maps to Dirac distributions only; otherwise, it is randomizing. A pair \((\sigma, \tau)\) of strategies of Maximizer and Minimizer induces a Markov chain \(G^{\sigma, \tau}\) with finite paths as states, \(s_0\) being initial, and the transition function \(\delta(ws, ws') = \sum_{a \in \text{Av}(s)} \sigma(ws)a \cdot \delta(s, a, s')\) for states of Maximizer and analogously for states of Minimizer, with \(\sigma\) replaced by \(\tau\). The Markov chain induces a unique probability distribution \(P^{\sigma, \tau}\) over measurable sets of infinite paths [BK08, Ch. 10].

B. End Components

Now we recall a fundamental tool for analysis of MDP called end components. An end component of a SG is then defined as the end component of the underlying MDP with both players unified.

**Definition 2 (EC):** A non-empty set \(T \subseteq S\) of states is an end component (EC) if there is a non-empty set \(B \subseteq \bigcup e \in T \text{Av}(s)\) of actions such that

1. for each \(s \in T\), \(a \in B \cap \text{Av}(s)\), we have \((s, a) \notin \text{Ex}(T)\),
2. for each \(s, s' \in T\) there is a finite path \(w = s_0 \ldots a_n s' \in (T \times B)^* \times S\), i.e. the path stays inside \(T\) and only uses actions in \(B\).

Intuitively, ECs correspond to bottom strongly connected components of the Markov chains induced by possible strategies. Hence for some pair of strategies all possible paths starting in an EC remain there. An EC \(T\) is a maximal end component (MEC) if there is no other end component \(T'\) such that \(T \subseteq T'\). Given an SG \(G\), the set of its MECs is denoted by \(\text{MEC}(G)\) and can be computed in polynomial time [CY95].

C. Generalized Reachability

For a set \(T \subseteq S\), we write \(\Diamond T := \{\text{inf. path } s_0 a_0 s_1 a_1 \cdots \mid \exists i \in \mathbb{N} : s_i \in T\}\) to denote the (measurable) set of all paths which eventually reach \(T\). A generalized-reachability objective (of dimension \(n\)) is an \(n\)-tuple \(T = (T_1, \ldots, T_n)\) of state sets \(T_i \subseteq S\). A vector \(\vec{v}\) (of dimension \(n\)) is achievable if there is a strategy \(\sigma\) of Maximizer such that for all strategies \(\tau\) of Minimizer

\[\forall i \in \{1, \ldots, n\} \quad \mathbb{P}^{\sigma, \tau}(\Diamond T_i) \geq v_i\]

Note that this corresponds to the lower value only since these games are not determined [CFK13b].

For a given state \(s\), the set of points achievable from \(s\), meaning in a game where the initial state is set to \(s\), is denoted \(\mathbb{A}(s)\) or just \(\mathbb{A}(s)\) when \(T\) is clear from context. We abbreviate \(\mathbb{A}(s_i)\) as \(\mathbb{A}\).

D. Basic Geometry Notation and Pareto Frontiers

In order to consider convex combinations of sets, we define scaling of a set \(X\) by a constant \(c \in [0, 1]\) as \(c \cdot X = \{c \cdot x \mid x \in X\}\), and the Minkowski sum of sets \(X\) and \(Y\) as \(X + Y = \{x + y \mid x \in X, y \in Y\}\). The convex hull of a set \(X\) is denoted by \(\text{conv}(X) = \{\sum_{i=1}^{k} a_i x_i \mid k \in \mathbb{N}, \forall i : x_i \in X, a_i \geq 0, \sum_{i=1}^{k} a_i = 1\}\).

A downward closure of a set \(X\) of vectors is \(\text{dwc}(X) := \{y \mid \exists x \in X : y \leq x\}\). A set \(X\) is downward closed if \(X = \text{dwc}(X)\). The set \(\mathbb{A}\) of achievable points is clearly downward closed.

It will be convenient to use a few basic notions of projective geometry, which we now recall. A direction is a line through origin \(0\); we may represent it with any vector \(\vec{v}\) on that line; all vectors \(\lambda \vec{v}\) for any \(\lambda \in \mathbb{R} \setminus \{0\}\) are equivalent and represent the same direction. For instance, direction \(\vec{d} = [(1, 0, 0)]\) denotes the \(x\)-axis and it holds \(\vec{d} = [(\lambda, 0, 0)]\) for any \(\lambda \neq 0\).

The natural basis of \(\mathbb{R}^n\) consists of unit vectors \(e_i^\perp\) pointing in the direction of axes, i.e. each \(e_i^\perp\) is given by Kronecker delta as \(e_i^\perp = 1\) for \(i = j\) and \(0\) otherwise. We define the set \(D\) of main directions as convex combinations of the directions \(\{e_i^\perp\}\) given by natural basis vectors. Intuitively, these are all directions in the first quadrant (generalized to higher dimensions).

Given a set \(X\) of vectors and a direction \(\vec{d}\), \(X\) evaluated in direction \(\vec{d}\) is the (Euclidean) length of the vector from the origin to the farthermost intersection of \(X\) and \(\vec{d}\), denoted \(X[\vec{d}] := \sup\{||\vec{x}|| \mid \vec{x} \in X, \vec{x} = [\vec{x}]\}\). Fig. 1 illustrates an evaluation of a direction on an achievable set. Intuitively, it describes what is achievable if we prefer the dimensions in the “ratio” given by \(\vec{d}\). Another example is the whole blue-red set of Fig. 4a: evaluated in [(1, 1)] it yields \(\sqrt{2}/2\).

Given a downward closed set \(X\), its Pareto frontier is the set of farthest points in each direction:

\[\mathbb{P}(X) = \{\vec{x} \mid \vec{d} \in D, \vec{d} = [\vec{x}], X[\vec{d}] = ||\vec{x}||\}\]

The Pareto frontier of a state \(s\) is the Pareto front of the set achievable in \(s\), i.e. \(\mathbb{P}(s) := \mathbb{P}(\mathbb{A}(s))\). The Pareto set of the game is \(\mathbb{P} := \mathbb{P}(s_0)\). Clearly, \(\mathbb{P} = \mathbb{P}(\mathbb{A})\) and \(\mathbb{A} = \text{dwc}(\mathbb{P})\).\(^1\)

\(^1\)Our notion of Pareto frontier captures the whole surface in the first quadrant. Other definitions such as \(\mathbb{P}^\perp = \{\vec{v} \mid \vec{v}\text{ is achievable }\land \forall \vec{v}' \text{ achievable } \vec{v}' \neq \vec{v} \text{ only capture the Pareto optimal points. For example, if the set of achievable points in the three-dimensional space is the whole unit cube then our definition returns its three sides, while the definition above returns only the singleton with the Pareto optimal point (1, 1, 1).\)
Given an SG, generalized-reachability objective \( T \), and precision \( \varepsilon > 0 \), the task is to construct sets \( L, U \subseteq \mathbb{R}^n \) such that for each direction \( \mathbf{d} \in \mathcal{D} \), \( L[\mathbf{d}] \) and \( U[\mathbf{d}] \) are effectively computable and we have
\[
L[\mathbf{d}] \leq \mathcal{P}[\mathbf{d}] \leq U[\mathbf{d}] \quad \text{and} \quad U[\mathbf{d}] - L[\mathbf{d}] < \varepsilon.
\]

**F. Multi-dimensional and Bounded Value Iteration**

In this section we recall two extensions of the standard value iteration: a generalization for multi-dimensional objectives and a “bounded” one with an over-approximating sequence. Firstly, the multi-dimensional Bellman operator for reachability, e.g. [CFK13b],
\[
\mathcal{B} : (S \cup S \times A \rightarrow 2^{\mathbb{R}^n}) \rightarrow (S \cup S \times A \rightarrow 2^{\mathbb{R}^n})
\]
works with sets \( X(s) \) and \( X(s,a) \) of points (achievable in \( s \), or in \( s \) using a \( a \), respectively) rather than single points:
\[
\mathcal{B}(X)(s) = \begin{cases} 
\bigcap_{a \in A(s)} X(s,a) & \text{if } s \in S_{\circ} \\
conv(\bigcup_{a \in A(s)} X(s,a)) & \text{if } s \in S_{\Box}
\end{cases}
\]
\[
\mathcal{B}(X)(s,a) = \left( dwc((1_{\tau}(s))) + \sum_{s' \in S} \delta(s,a,s') \cdot X(s') \right) \cap 1
\]
where \( 1_{\tau} \) is the indicator vector function of target sets, i.e. \( 1_{\tau}(s) \) equals 1 if \( s \in T \), and 0 otherwise, and \( 1 = \{ \vec{v} | \forall i: \vec{v}_i \in [0,1] \} \) is the unit box.

Intuitively, the operator works as follows. Given what can be achieved from \( s \) using now an action \( a \), we can compute the value for the minimizing state as the intersection over all actions since these points are achievable no matter what Minimizer does. For maximizing states, if there exists an action achieving a point then Maximizer can achieve it from here; moreover, we compute the convex hull since Maximizer can also randomize and, as opposed to the minimizing case with intersection, union of convex sets need not be convex. Once we have dealt with decision making on the first line, it remains to determine what can be achieved by each decision, on the second line. The achievable values are given by the weighted average of the successors’ values, but additionally, the base case of targets must be handled. Namely, whenever a state is in a target set, all values up to 1 in that dimension are achievable (but not greater than 1).

This also gives rise to an algorithm approximating \( \mathcal{P} \), which is the least fixpoint of \( \mathcal{B} \) [CFK13b]. We initialize \( L : S \cup S \times A \rightarrow 2^{\mathbb{R}^n} \) to return \( \{0\} \) everywhere, iteratively apply the Bellman operator, and then \( \lim_{k \rightarrow \infty} \mathcal{B}^k(L) = \mathcal{A} = dwc(\mathcal{P}) \) [CFK13b]. Moreover, the set is effectively presented at each step as a finite set \( \mathcal{P} \) of points on the Pareto frontier that generate it, the set is computable as \( dwc(\text{conv}(\mathcal{P})) \).

However, it is not known how to bound the difference of the actual achievable set \( \mathcal{A} \) and the approximation after \( k \) iterations. For that reason, [KKKW18] introduced for the single-dimensional case the bounded value iteration (named along the tradition of [MLG05]), a way to compute also an over-approximating sequence. If we initialize \( U \) to return \( I \) everywhere, then \( \lim_{k \rightarrow \infty} \mathcal{B}^k(U) \) is the greatest fixpoint, which is generally different from the least one. Hence [KKKW18] modifies \( \mathcal{B} \) so that it has a single fixpoint equal to the least one of the original \( \mathcal{B} \). Then both the sequence of lower bounds and of upper bounds converge to \( \mathcal{P} \), the value of the game. The modification is demonstrated in the next section, where we also illustrate the main ideas how to cope with the multi-dimensional case.

### III. Example

In this section, we illustrate the issues preventing convergence of the upper bounds and our solution on examples. All the problems are rooted in end components. Consider the EC in Fig. 2 with states \( s_1, s_2, s_3 \) and actions \( a, \ldots, g \). We start with considering a single reachability objective. Suppose the lower and upper bounds functions have already converged for the states outside of this EC to their true value, as depicted in the picture, e.g. for \( (s_2, e) \) it is \( \gamma \).

Since we are considering the single reachability objective, the standard Bellman update procedure [Put14] reduces to the following equations, where intersections become minima and unions become maxima. We write \( \mathcal{B}^k(U) \) as \( U_k \) for short.
\[
U_{i+1}(s_i) = \min \{ U_i(s_2), U_i(s_3), \gamma \}
\]
\[
U_{i+1}(s_2) = \max \{ U_i(s_1), \alpha \}
\]
\[
U_{i+1}(s_3) = \max \{ U_i(s_1), \beta \}
\]
Recall that we initialize \( L_0 \) to return 0 everywhere and \( U_0 \) to return 1 everywhere.
A. MDP

Firstly, let us briefly mention the solution of [BCC+14], [HM17] for MDP. In a maximizing MDP the first \( \min \) would also be \( \max \) and the initialization \( U_0 = \mathbf{1} \) is actually already a fixpoint, although the actual value is \( \max[\alpha, \beta, \gamma] \). Intuitively, the reason for this is that the equations set the dependencies of the values in a circular way, the process of finding the value by “asking neighbours” is not well-founded, and all states live in an illusion about a higher value (1). Since the illusion is generally shared it is a consistent model of the constraints. The solution is to detect this is an EC, collapse it, eliminating the circularity. We only keep the outgoing actions \( \alpha, \beta, \gamma \) and in the next iteration, the Bellman operator sets the value correctly to \( \max[\alpha, \beta, \gamma] \), converging to the true value.

B. Single-reachability SG

Secondly, for single-reachability SG, the EC cannot in general be collapsed since the values of the states differ, as can be seen in the following case distinction:

**Case 1:** If \( \gamma < \min(\alpha, \beta) \), then after the first iteration we have \( U_1(s_1) = \gamma \), \( U_1(s_2) = 1 \) and \( U_1(s_3) = 1 \). After the next iteration, \( U_2(s_1) = \gamma \), \( U_2(s_2) = \alpha \) and \( U_2(s_3) = \beta \), which are the actual values. In this case thus \( U_k \) converges to the value. However, note that the values of the states in the same EC are different.

**Case 2:** If \( \gamma \geq \min(\alpha, \beta) \), and say \( \alpha > \beta \), so the values of \( s_1 \) and \( s_3 \) are \( \beta \) and that of \( s_2 \) is \( \alpha \). However, \( U_k \) does not converge to this. In the first iteration, \( U_1(s_1) = \gamma \), \( U_1(s_2) = 1 \) and \( U_1(s_3) = 1 \). After the next iteration, \( U_2(s_1) = U_2(s_3) = \gamma \). After this, the upper bounds do not change, although the values of \( s_1 \) and \( s_3 \) are \( \beta \), which is smaller than \( \gamma \).

In this case thus \( U_k \) does not converge to the actual value. The EC is “bloated” [KKKW18], having unjustified large (bloated) value, which needs to be explicitly “deflated”. If we fix the strategy of the Minimizer to \( c \) as it is its best choice, only \( s_1 \) and \( s_3 \) form an EC. If we collapsed this EC, we would correctly update the value to \( \beta \). Hence the “deflating” subprocedure of the new Bellman operator imagines what if collapse happened on this substEC\(^2\) and updates the values correspondingly. To obtain an algorithm, this collapsing must be only imaginary since we cannot say for sure what to collapse. Indeed, in the case with \( \alpha < \beta \), a different EC \( \{s_1, s_2\} \) should be collapsed and if \( \alpha = \beta \) then all three states should be collapsed. Since during the approximation process we only have approximations \( (T_i) \) of the actual values, we do not know which of the cases it is and which EC is finally to be collapsed. Our guesses may change over time and we might learn the truth only in the limit. Hence we only pretend the collapse for the one-step computation and deflate only to what is for sure a current safe upper bound.

C. Generalized-reachability SG

Here we intuitively describe and illustrate the main elements of our solution.

\(^2\)A substEC of an EC \( E \) is an EC contained in \( E \).

- Figure 3: Pareto sets of \( \alpha \) (left), \( \beta \) (center) and \( \gamma \) (right) in a 2-objective setting. X-axis represents the value along the first objective and Y-axis represents the value along the second objective.
- Figure 4: (a) Visualizing the regions; and (b) the result of deflating the regions in direction \( d_2 \).

**Regions.** In the multi-dimensional case, instead of \( \alpha, \beta \) and \( \gamma \) being reals, assume they are achievable sets as given in Fig. 3. Here \( \gamma \) gives the highest values, so it is the best one for Maximizer, so Minimizer will not play it. Depending on the trade-off that Maximizer wants to achieve, \( \alpha \) or \( \beta \) might be better than the other. To this end, let \( d \) be the direction in which Maximizer wants to maximize. Depending on \( d \), Minimizer’s behaviour changes. If the objective along the x-axis is more important, then Minimizer chooses action \( c \). If on the other hand, the objective along y-axis is more important, then the Minimizer chooses action \( a \). Our algorithm identifies finitely many regions where the Minimizer has the same preferences for actions and then we deflate each region separately. In our example, we can identify three regions, as shown in Fig. 4a. Between the directions \( d_1 \) and \( d_2 \) (region), Minimizer’s best choice is action \( a \); between \( d_2 \) and \( d_3 \) (region), Minimizer’s best choice is action \( c \); and along \( d_2 \) (line), both have the same preference.

Once the region fixes the preferences of Minimizer’s actions, we can (virtually) drop some of Minimizer’s actions, identify areas which Minimizer does not want to leave and consider what happens within this simpler area.

**Deflating substECs.** In order to improve the upper bounds, we introduce the set of “cooperatively” achievable vectors for a state set \( T \) as

\[
\Psi_{\text{coop}}(T) := \left( \sum_{s \in T} d\text{wc}(\{1_{T}(s)\}) + \text{conv}(\bigcup_{(s, a) \in \text{Ex}(T_i)} \Psi(s, a)) \right) \cap 1
\]

The first summand contains the vectors that are possibly achievable by staying only in \( T \). The second summand con-
tains the vectors that are achievable by taking any desired combination of exits of Maximizer.\textsuperscript{3}

Intuitively, $\mathcal{A}_{coop}(T)$ contains all vectors that are achievable if Minimizer decides to let Maximizer roam around in $T$ freely as he likes, visit all targets in $T$ and pick any randomization over all of Maximizer’s exits. In other words, it is as if we collapsed the whole $T$ in to one Maximizer’s state. Of course, such cooperative behaviour yields an unrealistic over-approximation. However, it is not only the best approximation we can give without in-depth graph analysis, but also a key to the whole solution. Indeed, for each region and each corresponding MEC, we identify subECs, called simple (SEC), where the Minimizer does not have any choice, but cooperate unless it wants to exit the area at the cost of increasing the value. In a SEC $T$, $U^{coop}(T)$ thus provides a tight upper bound, given the current knowledge (here $U^{coop}(T)$ replaces application of $\mathcal{A}$ by its current over-approximation $U$ in $\mathcal{A}_{coop}(T)$). This is illustrated in Fig. 4b. Once these SECs are handled, Minimizer’s states outside of SECs can choose in which SECs to steer the game (based on the updated value) without circular dependencies (for that particular region). Thus deflating SECs provides enough information to proceed with the value iteration with the standard Bellman operator in the next step.

**Computing and representing regions.** As explained above, a region is given by the order of preference of actions. We can depict the achievable set of each action as in Fig. 4a, one given by $dwc(\{(0.5, 0.9)\})$, the other by $dwc(\{(0.9, 0.5)\})$. Points where their boundaries intersect represent the turning points of the preference. In this picture it is the point $(0.5, 0.5)$. When projected to the projective plane, the projections of the intersections yield a partitioning of the projective plane into different regions. In this case, it is the two lines and the point in between, see Fig. 4a. Another example is depicted in Fig. 5 with three achievable sets: two rectangles and one line segment. The fronts of the sets generate only one intersection (namely, of the two 2-dimensional sets\textsuperscript{4}), which is the point $(0.5, 0.0, 0.5)$, see Fig. 5a. This results in distinguishing 4 different regions, as shown in Fig. 5b. Firstly, there is the region that contains everything but the right side (marked as thick) of the triangle. This side of the triangle is divided into the other 3 regions like in Fig. 4a.

In order to keep the representation of regions effective, we triangulate regions into finer ones, which are convex and generated by finitely many points, see Fig. 5c. Since a region is given by the ordering of actions, it bears some resemblance to regions of timed automata [AD94]. In particular, the boundaries of regions are also separate regions, corresponding to equal preferences, see the the thin region (just $d\lambda_2$) in Fig. 4a or the single direction in Fig. 5. Instead of each simplex we consider only its interior (considering its self as a topology), hence we have open triangles, open line segments and points. To this end, we represent them as simplicial complexes [Hat02], see next section.

Further examples are illustrated in Fig. 6. In each of them one set is the tetrahedron (generated by Maximizer’s free, but exclusive choice between target sets). The other one is a box of different sizes (generated by the possibility to reach with a given probability a state in all target sets). As

\textsuperscript{3}We assume the empty union equals $\{\emptyset\}$ since that is the neutral element for achievable sets.

\textsuperscript{4}The neutral element $\{\emptyset\}$ is not considered a non-empty intersection.
Algorithm 1 Bounded Value Iteration

Input: SG G, generalized-reachability objective T, precision ε
Output: L, U: ∀d ∈ D: L[d] ≤ δε[d] ≤ U[d], U[d] − L[d] < ε

1: procedure BVI
2: for each s ∈ S do ▷ Initialization
3: L(s) ← {0} ▷ to the least and
4: U(s) ← duc({I}) ▷ the greatest values
5: repeat ▷ The new Bellman update \( \overline{\beta} \)
6: L ← \( \beta \)
7: U ← \( \beta \)
8: \( \text{U} \leftarrow \text{DEF}LATE\_\text{SECs}(G, L, U) \) ▷ New treatment
9: until max \( d \in D \), U \( s_0 \)[d] − L \( s_0 \)[d] < ε ▷ ε-approximate
10: return \( \text{frontier}(L), \text{frontier}(U) \)

Algorithm 2 DEFLATE\_SECs

1: procedure DEFLATE\_SECs(L, U) ▷ In each MEC, we compute relevant regions, find all respective SECs and decrease their upper bounds
2: \( M \leftarrow \text{MEC}(G) \) ▷ MEC decomposition of the game
3: for each \( T \in M \) do
4: \( R \leftarrow \text{GET\_REGIONS}(T, L) \)
5: for each \( R \in R \) do
6: \( S \leftarrow \text{FIND\_SECs}(T, L, R) \)
7: for each \( C \in S \) do ▷ Deflate s on \( R \)
8: redefine \( U(s) \) on \( R \) to be \( U(s) \cap U^{coop}(C) \)
9: return U

this probability varies, the box “rises” above the tetrahedron (like a floating object above the water surface), producing different intersections. While some of those, e.g. the third one, may be convex, others are not and must be triangulated. A triangulation of the second one can be found in Fig. 7.

Such finitely-generated regions enable effective groupings of “equivalent” single-dimensional optimization queries.

IV. ALGORITHM

We have seen in the example that it is important to split the set of all possible directions \( D \) into regions. Moreover, for effectiveness reasons, we restrict ourselves to finitely-generated convex ones.

Definition 3 (Region): A region \( R \subseteq D \) is a set of directions such that there are \( k \in \mathbb{N}, d_1, \ldots, d_k \in D \) so that \( R = \text{conv}(d_1, \ldots, d_k) \).

A region thus corresponds to a finitely generated cone, i.e., origin connected to a polygon (prolonged to infinity). In the following we also view the region as the set of points it contains.

We use this concept to generalize the notion of simple end component (SEC) of [KKKW18]. Intuitively, an EC is simple, if Minimizer’s best choice is to let Maximizer roam around freely in the EC and pick any combination of Maximizer’s exits. Minimizer cannot thus influence the value of the states in this EC, hence all states have the same value, namely \( \mathcal{A}^{coop}(T) \). Hence, intuitively, we can find SECs by removing all but the best (i.e. least-valued) choices of Minimizer. If in the remaining game there still exists an EC, then Minimizer cannot change the value of any state in that EC, because it is its best action to play like this, and all states have the same value.

Definition 4 (SEC): An EC \( T \) is simple, written SEC, for some region \( R \), if for every direction \( d \in R \) and all states \( s, t \in T, \mathcal{A}(s)[d] = \mathcal{A}(t)[d] \).

As we saw in the examples, the presence of SECs the over-approximation must be additionally decreased. So the idea of Algorithm 1 is to not only iteratively apply the standard Bellman updates, but to additionally deflate the (bloated) SECs.\(^5\) Note that Algorithm 1 now has a convergence criterion. This is effective since \( L \) and \( U \) are at any moment given as values for finitely many regions.

Algorithm 2 shows how SECs are treated. It first computes the MEC decomposition, because every SEC is an EC and hence part of a MEC. Then, for every MEC it computes the relevant regions as in the examples, see Section III. The invariant for each of the computed regions is, that for all directions inside this region Minimizer has the same preferences over the possible Maximizer exits. Thus, SECs are well-defined for these regions (and can be computed) since a SEC for one direction in a region is also a SEC for all directions in this region.

Since all states in a SEC \( T \) have the same value, namely \( \mathcal{A}^{coop}(T) \), we can set the upper bound of all states in \( T \) safely to \( \mathcal{U}^{coop}(T) \), which is the smallest over-approximation possible at the moment. Algorithm 2 deflate the upper bound region by region. It only updates the estimate of single regions and only for states in SEC for this region, see line 9. It also makes sure not to increase the value, so that the sequence of upper bounds stays monotonic.

In order to identify all the regions, we use the simplicial complexes, e.g. [Hat02]. We recall the formal definitions: A \((k-1)\text{-simplex}\) is a \((k\text{-dimensional})\) polytope given as the convex hull of \(k+1\) affinely independent vertices. A simplicial complex (SC) is a set of simplices closed under taking faces and such that the intersection of any its two simplices is a face of both.

\(^5\)Actually, we deflate what we currently believe are SECs based on the current approximations, which is proven safe in the subsequent sections.
Algorithm 3 GET_REGIONS

1: procedure GET_REGIONS(T ⊆ S, L)
2:     P ← SC generated by D on the projective hyperplane
3:     for s₁, s₂ ∈ Ex(T₀) do
4:         I ← frontier(L(s₁)) ∩ frontier(L(s₂))
5:         P_I ← SC generated by the projected I
6:         P ← common refinement of P and P_I
7:     return \{self-interior(s) | s ∈ P\}

Algorithm 4 FIND_SECs

1: procedure FIND_SECs(T ⊆ S, L, region R)
2:     d ← arbitrary element of R
3:     B ← \{(s, \{a ∈ Av(s) | L(s, a)[d] > L(s)[d]\}) | s ∈ T₀\}
4:     Av' ← Av \ B \> Keep optimal actions only
5:     return MEC(T|Av') \> MEC decomposition on T

Intuitively, a simplex is a point, line segment, triangle, tetrahedron etc. A simplicial complex is like a drawing consisting of these elements that ensures all that lower dimensional parts of a drawing (e.g. line as a part of a triangle) is also in the collection.

Given SCs C₁, C₂, we can create an SC C such that any s ∈ C₁∪C₂ is a finite union of elements of C [Hat02]. We call C a common refinement. This essentially involves triangulation in higher dimensions.

Algorithm 3 makes use of this as it gradually draws the picture on the projective hyperplane, as illustrated in Section III. It starts with the “generalized first quadrant”, e.g. in 3D the big triangles depicted in Figures 5, 6, 7. Then we draw all the intersections of all the frontiers. In each step we make sure by the refinement that the current areas are convex. Finally, as in timed automata, we need to return the interiors of the areas since the boundaries indicate equality of preference of actions. This is formally done by taking what we call self-interior(s), the interior of s in the topology defined by s; e.g. the interior of a line segment in 3D is empty, but within the line-segment its interior is itself without the end-points.

Finally, Algorithm 4 FIND_SECs is exactly as in [KKKW18], except that we first need to fix a direction to eliminate trade-offs and get a clear notion of an action being better than another one.

V. Correctness Proof

Our Bellman operator \( \mathfrak{B} \) is a higher order operator transforming pairs of the estimate functions: the two estimate functions \( L, U \in (S ∪ S × A) \to 2^{[0,1]^n} \) for the under-/over-approximation are transformed into a pair with the modified under- and over-approximation. It can thus be seen as

\[
\mathfrak{B} : \left((S ∪ (S × A)) \to 2^{[0,1]^n} \times 2^{[0,1]^n}\right) \to \left((S ∪ (S × A)) \to 2^{[0,1]^n} \times 2^{[0,1]^n}\right)
\]

For the next two sections, we fix an SG \( G = (S, S∪S⊙, S₀, A, Av, δ) \) and a generalized-reachability objective \( T \) and implicitly use them as parameters of \( \mathfrak{B} \). Note that for all states \( s ∈ S \), \( U₀(s) = I \) respectively \( L₀(s) = 0 \) are set by the initialization, while \( U₀(s,a) \) and \( L₀(s,a) \) are undefined. The latter is not a problem, because for every positive number \( i, Lᵢ(s,a) \) (similarly for \( U ) is implicitly computed from \( Lᵢ₋₁(s,a) \) and hence \( U₀(s,a) \) and \( L₀(s,a) \) are not needed.

We are interested in the properties of the following sequence: For all \( i ∈ \mathbb{N} \), let \( (Lᵢ, Uᵢ) = \mathfrak{B}^i(L₀, U₀) \). We use the abbreviations \( L_∞ := \limᵢ→∞ Lᵢ \) and \( U_∞ := \limᵢ→∞ Uᵢ \).

**Proposition 1: Soundness**
Algorithm 1 computes for each state \( s ∈ S \) monotonic under- and under-approximations of \( \mathfrak{A}(s) \), i.e. \( ∀i ∈ \mathbb{N} : Lᵢ(s) ≤ \mathfrak{A}(s) ≤ Uᵢ(s) \) and for \( i < j, Lᵢ(s) ≤ Lⱼ(s) \) as well as \( Uᵢ(s) ≥ Uⱼ(s) \).

**Proposition 2: Convergence from below**
For all states \( s ∈ S : L_∞(s) = \mathfrak{A}(s) \).

**Proposition 3: Convergence from above**
For all states \( s ∈ S : U_∞(s) = \mathfrak{A}(s) \).

Note that for all directions \( d \) and for all \( s ∈ S \) by definition \( \mathfrak{A}(s)[d] = \mathfrak{B}(s)[d] \). Using this and the three propositions, we can prove the main theorem.

**Theorem 1:** Algorithm 1 computes convergent monotonic under- and under-approximations of \( \mathfrak{B}(s) \) for each \( s ∈ S \). Since it is convergent, for every \( ε > 0 \) there exists an \( i \), such that for every \( s ∈ S \) and direction \( d ∈ \mathbb{R}^n : Uᵢ(s)[d] − Lᵢ(s)[d] < ε \).

By instantiating \( s \) with \( s₀ \), we solve the problem posed in Section VI-E.

**Proof of Propositions 1 and 2:** Note that for all \( i ∈ \mathbb{N} \) it holds that \( Lᵢ = \mathfrak{B}^i(L₀) \), since DEFLATE_SECs does not change the under-approximation. [BKW18, Proposition 8] proves that \( \mathfrak{B} \) is order-preserving, i.e. monotonic, and that it converges to the unique least fixpoint \( \mathfrak{A} \) when repeatedly applied to the bottom element of a complete partial order. The least possible lower bound assigns \( 0 \) to all \( S \), since there is no smaller vector that can be assigned to a state. This is exactly the definition of \( L₀ \), which implies that for all \( s ∈ S \), \( L_∞ = \mathfrak{A}(s) \). This proves Proposition 2.

For the soundness of the over-approximation we require that the additional operation performed by \( \mathfrak{B} \), namely DEFLATE_SECs, is sound and monotonic. This is proven in Section VI-A in Lemmata 5 and 6, respectively. From this and the fact that \( \mathfrak{B} \) is order-preserving, we can deduce Proposition 1.

**Proof of Proposition 3:** Note that \( U_∞ = \limᵢ→∞ \mathfrak{B}^i(U₀) \). It is proven in Lemma 7 in Section VI-B that \( \mathfrak{B} \) is a continuous operator. From this we get that \( \mathfrak{B}(U_∞) = U_∞ \). We will use this statement to derive a contradiction, i.e. we will now assume that there is a state \( s ∈ S \) such that \( U_∞(s) ≠ \mathfrak{A}(s) \) and derive from this that \( \mathfrak{B}(U_∞) < U_∞ \). In other words, applying the loop once more decreases the over-approximation.

1) Assume for contradiction, that \( ∃t ∈ S : U_∞(t) ≠ \mathfrak{A}(t) \).
Hence from Proposition 1 it follows that $U_\infty(t) > \mathfrak{A}(t)$. Thus, we can fix a state $t$ and a direction $d$, s.t. $U_\infty(t)[d] > \mathfrak{A}(t)[d]$.

2) Let $X := \{ s \mid s \in S \land \Delta(s) = \max_{s \in S} \Delta(s) \}$, where $\Delta(s) := U_\infty(s)[d] - \mathfrak{A}(s)[d]$ is the difference between our over-approximation and the true achievable set in direction $d$.

3) We also define $\Delta(s,a) := U_\infty(s,a)[d] - \mathfrak{A}(s,a)[d]$ for a state-action pair.

4) There exists some state $s \in X$ with $\Delta(s) \geq \Delta(t)$. Hence from Proposition 1 it follows that $U_\infty(s)[d] - \mathfrak{A}(s)[d] < 0$, and thus for all states $s \in X$, $\mathfrak{A}(s)[d] < U_\infty(s)[d]$, which, together with the continuity of $\mathfrak{B}$, is a contradiction.

5) If $X$ contains an EC, then we arrive at a contradiction.

Reason: If $X$ contains an EC, then $X$ only contains transient states, and there must be a bottom state $s \in X$, such that for all $a \in \mathfrak{M}(s) : (s,a)$ exits $X$. Since all actions from this state are exits, it must depend on an action leaving $X$, and thus by Fact 1 we arrive at a contradiction.

VI. Technical Details of the Proofs

A. Soundness

Lemma 1 (\$^{coop}_s$ for a Maximizer state is correct): If $s$ belongs to the Maximizer, then $\mathfrak{A}^{coop}(s) = \mathfrak{A}(s)$.

Proof: If we show that for all Maximizer states $s$, $U^{coop}(s) = \mathfrak{A}(s)$, then the lemma holds because $\mathfrak{A}$ is a fixed point of $\mathfrak{B}$.

- If $s$ is a non-target Maximizer state, then $\mathfrak{A}^{coop}(s) = conv(\cup_{a \in \mathfrak{M}(s)} \mathfrak{A}(s,a)) = conv(\cup_{a \in \mathfrak{M}(s)} \mathfrak{A}(s,a)) = \mathfrak{B}(\mathfrak{A}(s))$. While $\mathfrak{A}(s)$ may contain a self-loop action which is not contained in $\mathfrak{A}(s)$, this does not matter as the Maximizer cannot improve its value by choosing a self-loop action unless $s$ is a target. Hence, adding a $\mathfrak{A}(s,a')$ term, where $a'$ is a self-loop, to the inner union operation does not change the result.

- If $s$ is a target and a Maximizer state, then $\mathfrak{A}^{coop}(s) = \sum_{a \in T^t} r(s) + conv(\cup_{a \in \mathfrak{M}(s)} \mathfrak{A}(s,a))$. The $\sum_{a \in T^t} r(s)$ term contributes what the self-loop would have contributed in the Bellman equation, a value of 1 in the direction of the target $s$. Following a similar reasoning as above, we can easily show that $\mathfrak{A}^{coop}(s) = \mathfrak{B}(\mathfrak{A}(s))$.

Lemma 2 (\$^{coop}_s$ for a set of states is an over-approximation): Given an EC $T$, and a correct upper bound $U$ with $U(s) \geq \mathfrak{A}(s)$ for all $s \in S$, we get that $\forall s \in T : U^{coop}(s) \geq \mathfrak{A}(s)$.

Proof: Let us introduce a new Maximizer state $t$ representing $T$. Let $t$ be a target if $T$ contains at least one target. Let $\mathfrak{A}(t) = \{ a \mid (s,a) \text{ exits } T \}$, and since $t$ can randomize between any set of actions that any of the states in $T$ can choose, $\forall s \in T : (s,a)$ exits $T$. Moreover, $\forall s \in T$, $\exists s' \in T : \mathfrak{A}(s) \leq \mathfrak{A}(s')$. If this was not the case, it means that there exists some Minimizer state $s'$ that has a value greater than all Maximizer states. Since $T$ is an
EC, \( s_\circ \) has an action \( a_\circ \) whose successors are all in the EC. This implies that \( \mathfrak{A}(s_\circ, a_\circ) \) cannot be greater than \( \mathfrak{A}(s') \) for all \( s' \in T \).

Using Lemma 1, we get that \( \mathfrak{A}^{\text{coop}}(T) = \mathfrak{A}^{\text{coop}}(\{t\}) = \mathfrak{A}(t) \). Combining this with the previous argument yields \( \forall s \in T : \mathfrak{A}^{\text{coop}}(T) \geq \mathfrak{A}(s) \). Since \( U \) is correct by assumption, it follows that \( \forall s \in T : U^{\text{coop}}(T) \geq \mathfrak{A}(s) \).

**Lemma 3 (GET_REGIONS is sound and correct):** For any set of states \( T \) and bound function \( L \), the set of regions \( R \) returned by \text{getRegions}(T, L) \) has the following properties:

1. \( \bigcup_{R \in R} R = D \)
2. For each \( R \in R \), for all directions \( d \in D \) the relative order of exits is the same. More formally: \( \forall d_1, d_2 \in R, s_1, s_2 \in T, i \in \{1, 2\}, a_i \in \text{Av}(s_i), (s_i, a_i) \) exits \( T \):
   \( L(s_1, a_1)[d_1] \geq L(s_2, a_2)[d_1] \iff L(s_1, a_1)[d_2] \geq L(s_2, a_2)[d_2] \).

**Proof:** That \( \bigcup R = D \) follows easily from the initialization to the whole set and staying within this cone. Disjointness follows from the definition of \( S \), which ensures that self-interiors of any its two elements either do not intersect or equal one of them. For the second claim, consider directions \( d_1, d_2 \) such that \( L(s_1, a_1)[d_1] \geq L(s_2, a_2)[d_1] \) but \( L(s_1, a_1)[d_2] < L(s_2, a_2)[d_2] \). If the former inequality is strict, then the two directions are split by an intersection of the frontiers of \( L(s_1, a_1) \) and \( L(s_2, a_2) \). The intersection then splits also the projection and thus also the produced \( S \) and hence also the regions. If instead equality holds, then one of the directions is on the boundary. However, the boundary is then projected and turned into a face of the \( S \), hence into a simplex, whose parts are turned into regions. In either case, \( d_1 \) and \( d_2 \) lie in different regions. \( \square \)

**Lemma 4 (FIND_SECs is sound and correct, given well formed regions):** For \( T \subseteq S \), a lower bound function \( L \) and a region \( R \), where the relative ordering of exits is the same for all exits in \( R \), it holds that \( X \in \text{FIND_SECs}(T, A, \mathcal{V}, R) \iff X \) is an inclusion maximal \( S \) with respect to \( L \).

**Proof:** If a direction \( d \) is fixed, \( L(s)[d] \in \mathbb{R} \). The proof of [KKKW18, Lemma 2] works directly. We now argue that this is the case for all \( d \in R \). Given that the relative ordering of exits is same for all \( d \in R \), for any two actions \( a_1 \) and \( a_2 \), \( L(s, a_1)[d_1] \geq L(s, a_2)[d_1] \iff L(s, a_1)[d_2] \geq L(s, a_2)[d_2] \). This implies that \( \{ a \in \text{Av}(s) \mid L(s, a)[d] \geq L(s)[d] \} \) is the same for all \( d \in R \). Consequently, the set \( B \) computed on line 3 of the FIND_SECs procedure is the same for all \( d \in R \). Hence, [KKKW18, Lemma 2] immediately proves this lemma once any random \( d \in R \) is chosen. \( \square \)

**Lemma 5 (DEFLATE_SECs is monotonic):** For any set of states \( T \subseteq S \) and upper and lower bound functions \( U \) and \( L \), it holds that \( U' = \text{DEFLATE_SECs}(L, U, T) \) is pointwise smaller or equal than \( U \).

**Proof:** For a state not part of any \( M \), the upper bound is not changed. For a state part of a \( M \), \( U'(s) \) is constructed only using intersections with \( U(s) \). Since this only restricts the current upper bound, it can clearly be seen that \( U' \) will be pointwise smaller than or equal to \( U \). \( \square \)

**Lemma 6 (DEFLATE_SECs is sound):** For correct upper and lower bound functions \( U \) and \( L \) with \( L(s) \leq \mathfrak{A}(s) \leq U(s) \), for each \( s \in S \), it holds that \( U' = \text{DEFLATE_SECs}(L, U, T) \) is still correct, i.e. \( \mathfrak{A}(s) \leq U'(s) \) for all \( s \in S \).

**Proof:** From lines 3 - 9 of the DEFLATE_SECs procedure, it can be seen that the \( U \) value of a state is updated only if it is in an inclusion maximal (see Lemma 4) \( S \subseteq T \) for some region \( R \) with respect to \( L \). Now let \( s \in X \) be a state for which the upper bound is updated. Lemma 2 together with 5 allows us to conclude that at the end of procedure \( \text{DEFLATE_SECs} \), \( \mathfrak{A}(s) \leq U'(s) \) for all \( s \in S \). \( \square \)

**B. Convergence**

**Lemma 7 (Continuity):** \( \mathfrak{B} \) is Scott-continuous.

**Proof:** \( \mathfrak{B} \) operates on the domain \( D = (\mathbb{R}^n)^{S \cup \mathcal{V} \times A} \). We write \( b = (L, U) \in D \) to denote an element in this domain. We define a partial order on \( D \) and write \( b_1 \leq b_2 \) if \( b_1 \) is component-wise “worse” than \( b_2 \), i.e. \( L_1 \leq L_2 \) and \( U_1 \geq U_2 \). A directed chain \( D \in D \) is denoted as \( \{ (L_1, U_1), (L_2, U_2), \ldots \} \). In order to show Scott-continuity, we need to show that for every directed chain \( D, \sqcup (\mathfrak{B}(D)) = \mathfrak{B}(\sqcup (D)) \) (where \( \sqcup \) is the meet). [BK18, Proposition 8] shows that \( \mathfrak{B} \) is Scott-continuous. Lemma 6 and 5 show that \( \text{DEFLATE} \) is monotonic and order-preserving. Hence, \( \mathfrak{B} \) is monotonic and order-preserving and consequently it follows that \( \sqcup (\mathfrak{B}(D)) = \mathfrak{B}(\sqcup (D)) \). \( \square \)

**Proof of Fact 2:** We have the context of the proof of Proposition 3, in particular we know that \( X \subseteq S \) contains an EC and that for all states \( s \in X : \Delta(s) = \max_{s \in S} \Delta(s) = c \).

1. Let \( X' \subseteq X \) be a bottom MEC in \( X \).

**Justification:** We compute the MEC decomposition of \( X \) and pick a MEC at the end of a chain. \( X' \) exists, since there is an EC in \( X \), so there also is at least one MEC in \( X \).

2. Let \( m = \max_{s \in X'} U(s)[d] \) be the maximal upper bound in \( X' \).

3. Let \( Y := \{ s \mid s \in X' \land U(s) = m \} \) be the states with maximal upper bound in \( X' \).

4. \( \forall s \in Y, \exists a \in \text{Av}(s) : a \in X' \) exists \( Y \), i.e. all states in \( Y \) have actions that stay in \( Y \).

**Reasoning:**

- a) If for some state \( s \in Y \) all available actions left \( X \), it would have to depend on the outside of \( X \), and by Fact 1 this is a contradiction. Thus \( s \) has actions that stay in \( X \).
- b) No action can exit from \( Y \) to \( X \setminus X' \), because \( X' \) is a bottom MEC in \( X \). If an action left to some state \( t \in X \setminus X' \), then, since \( X' \) is a bottom MEC in \( X \), from \( t \) there would be no reachable EC in \( X \). Starting from \( t \), this is the same situation as when \( X \) does not contain an EC, and thus Fact 1 yields a contradiction.
- c) Not all actions can exit to \( X' \setminus Y \). \( U(s) = m \) and for every \( s' \in X' \setminus Y : U(s') < m \). So if the action exits \( Y \) to \( X' \), the upper bound of the action is smaller than
Since part of the successors have a smaller upper bound. It cannot have another successor with a higher upper bound, because it has to stay in \( X' \), and \( m \) is chosen to be the highest upper bound in \( X' \). Therefore, there has to be some action with upper bound \( m \), because \( U(s) = m \). Thus, not all actions can leave to \( X' \).

d) Aggregating the previous points: Not all actions can exit \( X \), no action can exit to \( X \setminus X' \), not all actions can exit to \( X' \setminus Y \). So some action has to remain in \( Y \).

5) Let \( Z \) be a bottom MEC in \( Y \).

\textbf{Justification:} We compute the MEC decomposition of \( X \) and pick a MEC at the end of a chain. \( Z \) exists, since by the previous step there must exists an EC in \( Y \), since all states have staying actions.

6) For all states \( s \in Z : \mathfrak{A}(s)[d] = m - c \).

\textbf{Justification:} Since \( Z \subseteq Y \), \( U_{\infty}(s)[d] = m \). Since \( Y \subseteq X \), \( \Delta(s) = c \). We get the following chain of equations: \( c = \Delta(s) = U_{\infty}(s)[d] - \mathfrak{A}(s)[d] = m - \mathfrak{A}(s)[d] \).

Reordering yields the statement.

7) Thus, \( Z \) is an SEC for region \( \{d\} \).

8) When applying \( \hat{\mathfrak{A}} \) once more, \( Z \in S \) in Line 6 of Algorithm 2.

\textbf{Reasoning:} \( X' \in MEC(G) \) by definition of \( X' \). So \( R \leftarrow \text{GET\_REGIONS}(X', L_{\infty}) \) is executed. Since \( \bigcup_{R \in R} R = \mathcal{D} \) by Lemma 3, there is some \( R \in \mathcal{R} \) with \( d \in \hat{R} \). Also by that Lemma we have that the relative order of exits for all directions in \( R \) is the same, and since it was called with \( L_{\infty} \), it is correct. Thus, we can apply Lemma 4, which proves the statement.

9) \( U_{\text{coop}}(Z)[d] < m \)

\textbf{Reasoning:} \( U_{\text{coop}}(Z)[d] \) must put positive weight on some exit of \( Z \). If it puts positive weight on some state-action-pair that exits \( X \), then by Fact 1 \( U_{\text{coop}}(Z)[d] < m \). The only other possible exit is to \( X' \setminus Y \). This is because of the fact the \( Z \) is a bottom MEC in \( Y \) and the argumentation in Step 4. For all states \( s' \in X' \setminus Y \), it holds that \( U(s')[d] < m \). If \( U_{\text{coop}}(Z)[d] \) is constructed from a convex combination of exits only to \( X' \setminus Y \), then also \( U_{\text{coop}}(Z)[d] < m \).

10) \( \forall s \in Z : \hat{\mathfrak{A}}(U_{\infty}(s)[d] = U_{\text{coop}}(Z)[d] \).

\textbf{Reasoning:} Let \( s \in Z \). The upper bound is modified by only Line 9. Since \( d \in R \) by how the algorithm found \( R \) (Step 8) and since \( U_{\text{coop}}(Z)[d] < m = U(s)[d] \), the new upper bound is exactly \( U_{\text{coop}}(Z)[d] \) for each \( s \in Z \).

11) Thus, by combining the previous two steps, we finally arrive at a contradiction, since \( \forall s \in Z : \hat{\mathfrak{A}}(U_{\infty}(s)[d] = U_{\text{coop}}(Z)[d] < U_{\infty}(s)[d] \).

\[ \Box \]

\section{VII. Conclusion}

For a given \( \varepsilon > 0 \) and a generalized-reachability stochastic game, we compute an \( \varepsilon \)-approximation of its Pareto frontier. Our algorithm can be run as an anytime algorithm, reporting the current lower and upper bounds on the frontier, due to an extended version of value iteration. We conjecture that this technique can be generalized to other models, such as concurrent games, and more complex objectives, such as total reward.

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