Generating-function approach for bond percolations in hierarchical networks

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We study bond percolations on hierarchical scale-free networks with the open bond probability of the shortcuts \( \tilde{p} \) and that of the ordinary bonds \( p \). The system has a critical phase in which the percolating probability \( P \) takes an intermediate value \( 0 < P < 1 \). Using generating function approach, we calculate the fractal exponent \( \psi \) of the root clusters to show that \( \psi \) varies continuously with \( \tilde{p} \) in the critical phase. We confirm numerically that the distribution \( n_s \) of cluster size \( s \) in the critical phase obeys a power law \( n_s \propto s^{-\tau} \), where \( \tau \) satisfies the scaling relation \( \tau = 1 + \psi^{-1} \). In addition the critical exponent \( \beta(\tilde{p}) \) of the order parameter varies as \( \tilde{p} \), from \( \beta \simeq 0.164694 \) at \( \tilde{p} = 0 \) to infinity at \( \tilde{p} = \tilde{p}_c = 5/32 \).

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I. INTRODUCTION

Dynamics on and of complex networks have been one of the focuses of attentions since late 1990s [1,2]. Real networks, e.g., WWW, Internet, food-web, often have complex properties such as scale-free degree distribution [4], small-world property [5], etc., to demand more extensive framework of statistical physics to investigate the interplay between dynamics and such network topology. Many analytical and numerical works about the effects of network topology on processes such as percolation, interacting spin systems, epidemic processes, have been reported [4].

Among these issues, unusual phase transitions of percolations and spin systems on some networks have attracted our current interests [7–16]. For example, the percolations on some growing network models undergo an infinite order transition with a Thouless (BKT)-like singularity: (i) the relative size of the largest component vanishes in an essentially singular way at the transition point, so that the transition is of infinite order, and (ii) the mean number \( n_s \) of clusters with size \( s \) per node (or the cluster size distribution in short) decays in a power-law fashion with \( s \),

\[
    n_s \propto s^{-\tau},
\]

in a finite region below the transition point where no giant component exists [2,11]. A similar non-ordered phase with some power-law behavior, a critical phase, has also been observed in bond percolations on the enhanced binary tree [17–19], which is one of nonamenable graphs (NAGs) [20,21]. The system on a NAG takes three distinct phases according to the open bond probability \( p \) as follows: (i) the non-percolating phase \( 0 \leq p \leq p_{c1} \), in which only finite size clusters exist, (ii) the critical phase \( p_{c1} \leq p \leq p_{c2} \) in which there are infinitely many infinite clusters, and (iii) the percolating phase \( p_{c2} \leq p \leq 1 \) in which the system has a unique infinite cluster. Here infinite cluster means a cluster whose mass diverges with system size \( N \) as \( N^\phi \) with \( 0 < \phi < 1 \). To profile the critical phase it is useful to calculate the fractal exponent \( \psi_L \) defined as \( s_{\text{max}} \propto N^\psi_L \), where \( s_{\text{max}} \) is the mean size of the largest components in the system with \( N \) nodes. Note that \( \psi_L \) corresponds to \( d_f/d \) for percolating clusters having the fractal dimension \( d_f \) on \( d \)-dimensional Euclidean lattices. Recent paper [17] has shown numerically that the above phases are characterized as (i) \( \psi_L(p) = 0 \) for \( p < p_{c1} \), (ii) continuously increasing of \( \psi_L(p) < \psi_L(p) = 1 \) with \( p \), where \( n_s \) also behaves as (1) with \( p \)-dependent \( \tau \) satisfying

\[
    \tau = 1 + \psi_L^{-1},
\]

for \( p_{c1} < p < p_{c2} \), and (iii) \( \psi_L(p) = 1 \) for \( p > p_{c2} \). The scaling relation (2) indicates that \( \psi_L \) plays a role of the natural cut-off exponent of \( n_s \) as shown in the growing random tree in which \( p_{c1} = 0 \) and \( p_{c2} = 1 \) [11]. In general the growing random networks are considered to have \( p_{c1} = 0 \) with finite \( p_{c2} \) [7,10].

There exist other systems having a similar phase. They are in a special class of hierarchical scale-free networks, called (decorated) \((u,v)\)-flower introduced comprehensively in [22,23]. Berker et al. [16] have studied bond percolations on the decorated \((2,2)\)-flower by renormalization group (RG) to show the existence of a critical phase (as known as the partially ordered phase [24]), where RG flow converges onto the line of nontrivial stable fixed points. But we have little knowledge about physical properties of the critical phase, i.e., how it is critical.
In this paper, we investigate bond percolations on the decorated (2,2)-flower with two different probabilities $\tilde{p}$ and $p$, which are the open bond probability of the shortcuts and that of the ordinary bonds, respectively. Here we adopt a generating function approach to calculate the fractal exponent, the cluster size distribution, and the order parameter for an arbitrary combination of fractal exponent, the cluster size distribution, and the open bond probability $\tilde{p}$, to reveal a complete picture about the phases of this model. Our calculations show (i) the fractal exponent $\psi$ and $\beta$ of the order parameter depend on the existing probability $\tilde{p}$ of the shortcuts, and (ii) $n_\alpha$ is a power-law at all the point in the critical phase, and its exponent $\tau$ also depends on $\tilde{p}$.

The organization of this paper is as follows: In Sec. II we introduce the (2,2)-flower and the decorated (2,2)-flower, and briefly review the previous studies for the percolation on the flowers $[16, 22, 23]$. In Sec. III we introduce the generating functions, and derive those recursion relations to calculate the order parameter, fractal exponent, and cluster size distribution. The main results are presented in Sec. IV and Section V is devoted to summary.

**II. MODEL**

In this section we briefly introduce the (2,2)-flower and the decorated (2,2)-flower. The (2,2)-flower $F_n$ of the $n$-th generation is constructed recursively as illustrated in Fig 1. At $n = 0$, the flower $F_0$ consists of two nodes connected by a bond. Hereafter we call these nodes roots. For $n \geq 1$, $F_n$ is obtained from $F_{n-1}$, such that each existing bond in $F_{n-1}$ is replaced by two parallel paths consisting of two bonds each. The decorated (2,2)-flower $\tilde{F}_n$, a variant of the (2,2)-flower $F_n$, is given by adding shortcuts to $F_n$, as illustrated in Fig 1(d).

The network properties of these two flowers have been reported in $[22, 23]$. The number of nodes $N_n$ of the $n$-th generation is $N_n = 2(4^n + 2)/3$ and the degree distribution has a scale-free form $P(k) \propto k^{-\gamma}$ with $\gamma = 3$ for both flowers. One of the important differences between $F_n$ and $\tilde{F}_n$ appears in those dimensionality. Since the diameter $L_n$ of $F_n$ is $L_n = 2^n$, the dimension $d$ of the underlying network defined as $N_n \propto L_n^d$ is 2. On the other hand $\tilde{F}_n$ is known to have small-world property $L_n \sim \ln N_n$ corresponding to $d \sim \infty$. In addition $\tilde{F}_n$ has a high clustering coefficient $C \sim 0.820$, in contrast to $C = 0$ for $F_n$.

In the present work we consider the bond percolation on $\tilde{F}_n$ with the open bond probability $p$ of the bonds constituting $F_n$ (the ordinary bonds) and that of the shortcuts $\tilde{p}$ being given independently. The standard bond percolation on $F_n$ and $\tilde{F}_n$ are given by setting $\tilde{p} = 0$ and $\tilde{p} = p$, respectively. Note that the latter is also given by setting $p = 0$ because $F_n$ with $p = \tilde{p}$ and $F_{n+1}$ with $p = 0$ is exactly the same.

The phase diagram is solved exactly by RG technique $[16, 22]$. Let $P^{(n)}$ be a probability that both roots are in the same cluster of a bond configuration on $\tilde{F}_n$ with fixed $\tilde{p}$. The initial value is set to $P^{(0)} = p$. In the large size limit, the system is regarded as percolating if

**FIG. 2:** (Color online) Phase diagram of the bond percolation on the decorated (2,2)-flower $[16]$. The blue-solid and the red-dashed lines denote $p^c(\tilde{p})$ (stable fixed point) and $p_u(\tilde{p})$ (unstable fixed point), respectively. The shaded region represents the critical phase.

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![FIG. 1: (Color online) Construction of the (2,2)-flower $F_n$ and the decorated (2,2)-flower $\tilde{F}_n$. (a) Each bond is replaced by two parallel paths consisting of two bonds each at next generation. (b) The flower $F_{n+1}$ of the $n + 1$-th generation is obtained by joining four copies of $F_n$. (c) Realization of $F_n$ with $n = 0, 1, 2, 3$. (d) Realization of $\tilde{F}_n$ with $n = 0, 1, 2, 3$. $\tilde{F}_n$ is obtained by adding the shortcuts (orange-dashed line in bond replacement (a)) to $F_n$. The shortcuts are not replaced by others in each iteration.](image-url)
nonzero stable fixed points, \( P \) for \( \tilde{p} \) between the two giving the phase boundary, and one unstable fixed point corresponding to the partially ordered phase and the or-beral phase, respectively, and one unstable fixed point between the two giving the phase boundary, \( P = p_c(\tilde{p}) \).

For \( \tilde{p} > p_c \), on the other hand, there is only one stable fixed point at \( P = 1 \), so that the system is always percolating.

Two special cases, \( \tilde{p} = 0 \) and \( \tilde{p} = p \), were investigated in [22]. Here let us recall their results briefly. For the case of \( \tilde{p} = 0 \), i.e., the standard bond percolation on \( F_n \), Eq. (3) gives the critical point \( p_c(\tilde{p} = 0) = (\sqrt{5} - 1)/2 \). A simple RG argument then gives the critical exponents at \( p_c \): the exponent \( \beta \simeq 0.164694 \) of the order parameter, i.e., the fraction of the largest component \( P_\infty \sim (p-p_c)^\beta \), and \( \nu \simeq 1.63528 \) of the correlation length \( \xi \sim |p-p_c|^{-\nu} \), which are close to those of the two dimensional regular systems. On the other hand the same argument for \( \tilde{p} = p \) gives the infinite order transition, i.e., \( \beta \to \infty \).

According to the above definition of percolation the percolating cluster should include both of the root nodes. It is then convenient to consider the mean size \( (s_0)_n \) of the cluster including both roots (referred to as the root cluster) on \( F_n \) instead of \( s_{\text{max}}(N_n) \) to characterize the criticality:

\[
(s_0)_n \propto N_n^\psi,
\]

where \( \psi \) is the fractal exponent for the root cluster. Note that \( \psi \) behaves essentially the same as \( \psi_L \) for the growing random trees [11].

III. GENERATING FUNCTIONS

In this section we describe how to utilize generating functions to calculate the fractal exponent \( \psi \) and the cluster size distribution \( n_k(p) \) with \( \tilde{p} \) fixed. First let us consider the bond percolation on \( F_n \) with open bond probability \( p \) with \( \tilde{p} = 0 \). We introduce three basic quantities on \( F_n \): the probability \( t_k^{(n)}(p) \) that both roots are connected to the same cluster of size \( k \), the probability \( s_k^{(n)}(p) \) that the left (right) root is connected to a cluster of size \( k \) (\( l \)) but these clusters are not the same, and the mean number \( u_k^{(n)}(p) \) of clusters of size \( k \) to which neither of the roots is connected. For the sake of convenience the roots are not counted in the cluster size \( k \) or \( l \) for \( t_k^{(n)}(p) \) and \( s_k^{(n)}(p) \). The corresponding generating functions are defined as

\[
T_n(x) = \sum_{k=0}^{\infty} t_k^{(n)}(p)x^k, \tag{5a}
\]
\[
S_n(x,y) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} s_{k,l}^{(n)}(p)x^ky^l, \tag{5b}
\]
\[
U_n(x) = \sum_{k=0}^{\infty} u_k^{(n)}(p)x^k. \tag{5c}
\]
The self-similar structure of \( F_n \) allows us to obtain the recursion relations for the above generating functions

\[
T_{n+1}(x) = x^2T_n^4(x) + 4x^2T_n^3(x)S_n(x, x) + 2xT_n^2(x)S_n^2(x, 1),
\]

\[
S_{n+1}(x, y) = S_n^2(x, 1) + 2S_n(x, y)S_n(x, 1) + 2xyS_n^2(x, y) + \left[ xT_n(x) + yT_n(y) \right] + S_n^2(x, y) \left[ x^2T_n^2(x) + y^2T_n^2(x) \right],
\]

\[
U_{n+1}(x) = 4U_n(x) + 2xS_n^2(x, 1),
\]

as illustrated in Fig[3] and Fig[4]. The initial conditions are given as \( T_0(x) = p \), \( S_0(x, y) = q = 1 - p \) and \( U_0(x) = 0 \).

It is convenient to rewrite these recursion formulas in terms of functions of single variable \( x \) only. In order to this we introduce new functions \( V_n(x) = S_n(x, x) \) and \( R_n(x) = S_n(1, 1) \) to obtain

\[
T_{n+1}(x) = \mathcal{T} \left[ T_n(x), V_n(x), R_n(x, x) \right] \equiv x^2T_n^4(x) + 4x^2T_n^3(x)V_n(x) + 2xT_n^2(x)R_n^2(x),
\]

\[
V_{n+1}(x) = \mathcal{V} \left[ T_n(x), V_n(x), R_n(x, x) \right] \equiv R_n^4(x) + 4xT_n(x)V_n(x)R_n^2(x) + 4x^2T_n^2(x)V_n^2(x),
\]

\[
R_{n+1}(x) = \mathcal{R} \left[ T_n(x), R_n(x, x) \right] \equiv R_n^2(x) \left[ 1 + xT_n(x) \right]^2,
\]

\[
U_{n+1}(x) = \mathcal{U} \left[ R_n(x, x), U_n(x, x) \right] \equiv 4U_n(x) + 2xR_n^2(x).
\]

Indeed it is this form that enables us to obtain the solutions for large \( n \) numerically.

Now the construction of the recursion relations for \( F_n \) with \( \bar{p} \) fixed is straightforward. Let \( \bar{T}_n(x) \), \( \bar{V}_n(x) \), \( \bar{R}_n(x) \) and \( \bar{U}_n(x) \) denote the corresponding generating functions on \( F_n \). By using the above formula (\ref{8a})-(\ref{7d}) with these functions one can construct the generating functions \( \bar{T}_{n+1}(x) \), \( \bar{V}_{n+1}(x) \), \( \bar{R}_{n+1}(x) \) and \( \bar{U}_{n+1}(x) \) on the decorated \((2, 2)\)-flower of the next generation without the shortcut directly connecting the roots:

\[
\bar{T}_{n+1}(x) = \mathcal{T} \left[ \bar{T}_n(x), \bar{V}_n(x), \bar{R}_n(x, x) \right],
\]

\[
\bar{V}_{n+1}(x) = \mathcal{V} \left[ \bar{T}_n(x), \bar{V}_n(x), \bar{R}_n(x, x) \right],
\]

\[
\bar{R}_{n+1}(x) = \mathcal{R} \left[ \bar{T}_n(x), \bar{R}_n(x, x) \right],
\]

\[
\bar{U}_{n+1}(x) = \mathcal{U} \left[ \bar{R}_n(x, x), \bar{U}_n(x, x) \right].
\]

The flower \( \bar{F}_{n+1} \) is made by adding the shortcut to the intermediate one with probability \( \bar{p} \) and one thus obtains

\[
\bar{T}_{n+1}(x) = T_{n+1}(x) + \bar{p}V_{n+1}(x),
\]

\[
\bar{V}_{n+1}(x) = \bar{q}V_{n+1}(x),
\]

\[
\bar{R}_{n+1}(x) = \bar{q}R_{n+1}(x),
\]

\[
\bar{U}_{n+1}(x) = U_{n+1}(x),
\]

where \( \bar{q} = 1 - \bar{p} \). The initial conditions are given as \( \bar{T}_0(x) = p, \bar{V}_0(x) = R_0(x) = q \) and \( \bar{U}_0(x) = 0 \). One can easily check the probability conservation \( \bar{T}_n(1) + \bar{V}_n(1) = 1 \) by the iteration (\ref{11}).

Once these generating functions are obtained one can evaluate various quantities of the present interest. For example the percolation probability \( P^{(n)} \) is given as

\[
P^{(n)} = \bar{T}_n(1),
\]

\[
Q^{(n)} \equiv 1 - P^{(n)} = \bar{S}_n(1, 1) = \bar{V}_n(1) = \bar{R}_n(1).
\]

Note that (\ref{9a}) is re-obtained by putting \( x = 1 \) to (\ref{7a}) and using (\ref{13}). The mean number of the root cluster \( \langle s_0 \rangle_n \) (or the order parameter \( F_\infty(p) \)) and the cluster size distribution \( n_s(p) \) on \( \bar{F}_n \) are given as

\[
\langle s_0 \rangle_n = \bar{T}_n^\prime(1) + \bar{V}_n^\prime(1),
\]

\[
P^{(n)}(p) = \frac{\langle s_0 \rangle_n}{N_n} = \tau_n + \sigma_n,
\]

\[
n_s(p) = \frac{\bar{u}_s(p)}{N_n},
\]

where the prime denotes the first derivative with respect to \( x \), and we put \( \tau_n = T_n^\prime(1)/N_n \) and \( \sigma_n = V_n^\prime(1)/N_n \).

It is useful to consider the derivatives of recursion relations (\ref{9a})-(\ref{13}) for evaluating \( P^{(n)}(p) \). By noticing \( \bar{V}_n^\prime(1) = 2R_n^\prime(1) \) we obtain the recursion relations for \( \tau_n \) and \( \sigma_n \) as
\[
\left( \frac{\sigma_{n+1}}{\tau_{n+1}} \right) = \frac{N_n}{N_{n-1}} \left( 2(1 + P(n)) \left[ (P(n))^2 + \tilde{p}Q(n)(1 + P(n)) \right] \right) \left( \frac{4\tilde{q}(Q(n))^2(1 + P(n))}{1 - \tilde{q}(Q(n))^2(1 + P(n))} \right) \left( \frac{\sigma_n}{\tau_n} \right)
\]

where we recall \( P = \lim_{n \to \infty} P(n) \) and \( Q = \lim_{n \to \infty} Q(n) \). Note that this expression is an extension of Eq. (31) in [22].

\[ \text{IV. RESULTS} \]

To profile the critical phase we calculate the fractal exponent \( \psi \). In the ordered phase we have trivially \( \psi = 1 \). Otherwise the fixed points \( P(<1) \) of the RG equation [43] satisfy \( \tilde{q}(1 - P)(1 + P)^2 = 1 \). The recursion relation (15) is then reduced to

\[
\left( \frac{\sigma_{n+1}}{\tau_{n+1}} \right) = \left( \frac{1}{4} P \alpha \right) \left( \frac{\sigma_n}{\tau_n} \right), \tag{16}
\]

where \( \alpha = (1 - P)/(1 + P) \). By using the largest eigenvalue \( \lambda(P) \) of the above matrix,

\[
\lambda(P) = \frac{1}{4} \left[ (3 - 2\alpha) + \sqrt{1 - 4\alpha(1 - 2P) + 4\alpha^2} \right], \tag{17}
\]

we can calculate the fractal exponent \( \psi \) on the fixed points in the same way as [22] does:

\[
\psi(P) = 1 + \frac{\ln \lambda(P)}{\ln 4}. \tag{18}
\]

The \( \tilde{p} \)-dependence of \( \psi \) is shown in Fig. 5. We find that (i) for \( \tilde{p} < \tilde{p}_c = 5/32 \), \( \psi \) on the (un)stable fixed points increases (decreases) with increasing \( \tilde{p} \), and (ii) for \( \tilde{p} > \tilde{p}_c \), \( \psi \) is equal to one irrespective of both \( \tilde{p} \) and \( \tilde{p}_c \), which means that the system is always in the percolating phase. Let us consider the \( \tilde{p} \)-dependence of \( \psi \) with \( \tilde{p} < \tilde{p}_c \) fixed. In the critical phase \( p < p_c(\tilde{p}) \) the percolation probability \( P(n) \) goes to \( p^*(\tilde{p}) \) and the exponent \( \psi = \psi(p^*(\tilde{p})) \) is constant in this region. At the critical point \( p = p_c(\tilde{p}) \) the fixed point is \( P = p_c(\tilde{p}) \) itself and thus \( \psi \) discontinuously changes to \( \psi(p_c(\tilde{p})) \), and jumps again to one for the percolating phase \( p > p_c(\tilde{p}) \). This behavior can be also confirmed directly by evaluating the \( N_n \)-dependence of \( \langle s_0 \rangle_n \) numerically (not shown). This result indicates that the probability \( \tilde{p} \) of the shortcuts essentially determines how the system is critical in the partially ordered critical phase. On the other hand, for the standard bond percolation on the decorated \( (2,2) \)-flower \( (\tilde{p} = p) \), the fractal exponent \( \psi \) varies continuously with open bond probability \( p \) as observed on a NAG [17].

\[
\text{FIG. 5: (Color online) Fractal exponent} \psi \text{ on the stable fixed points} p^*(\tilde{p}) \text{ (blue-solid line) and unstable fixed points} p_s(\tilde{p}) \text{ (red-dashed line).}
\]

We also evaluate numerically the recursion relations [43 - 46] to obtain the cluster size distribution \( n_s \) on \( F_n \) given by (13). We should observe a power-law behavior of \( n_s \) in the whole region of the critical phase. To check this behavior we assume a finite size scaling form

\[
n_s(N) = s^{-\tau} f(sN^{-\psi}), \tag{19}
\]

where \( \psi \) is the fractal exponent obtained above and the scaling function \( f(x) \) behaves as

\[
f(x) \sim \begin{cases} \text{rapidly decaying func.} & \text{for } x \gg 1, \\ \text{constant} & \text{for } x \ll 1, \end{cases} \tag{20}
\]

and \( \tau \) satisfies the scaling relation [11 17]

\[
\tau = 1 + \psi^{-1}. \tag{21}
\]

As discussed in [11] a plausible argument leads us to the relation as follows: By assuming \( n_s \propto s^{-\tau} \) in the critical phase, a natural cutoff \( s_{\text{max}} \) of the cluster size distribution (a natural cutoff of the degree distribution was introduced in [22]) is given as

\[
N \int_{s_{\text{max}}}^{\infty} n_s ds \simeq 1 \rightarrow s_{\text{max}} \propto N^{1/\tau}. \tag{22}
\]

Here we emphasize that \( s_{\text{max}} \) plays just a role of characteristic cutoff of the distribution and does not need to
be strictly the mean size of the largest clusters. In this sense we can replace the fractal exponent $\psi_L$ of $s_{\text{max}}$ with $\psi$ of $\langle s_0 \rangle$, so that the relation (21) follows. Indeed the largest cluster not containing the roots is expected to the one containing the top-most (or, equivalently, bottom-most) node in the right figure of Fig. 4(c) contributing the second term of (6c). The mean size of the cluster is proportional to $\tilde{V}_c(1)$ and so to $\langle s_0 \rangle$. Therefore one can conclude that the characteristic size grows with $N$ not faster than $N^\psi$.

Our finite size scaling for $n_s$ is indeed well fitted on both stable and unstable fixed points as shown in Fig. 6. Note that for the sake of (21) no fitting parameter remains. The scaling also works at any $p$ in the critical phase (Fig. 4(c)), but the convergence is not so rapid as on the fixed points.

Finally we iterate Eq. (15) numerically to obtain the order parameter $F_\infty(p)$ on $F_\infty(p)$ with $\tilde{p} = 0, 0.1, 0.15, \text{and } 5/32 (= \tilde{p}_c)$ (from right to left). Here $n$ is taken to $10^6(\gg 1)$.

FIG. 7: (Color online) Fraction of the largest components $P_\infty(p)$ on $F_\infty(p)$ with $\tilde{p} = 0, 0.1, 0.15$, and $5/32 (= \tilde{p}_c)$ (from right to left). Here $n$ is taken to $10^6(\gg 1)$.

FIG. 8: $\tilde{p}$-dependence of the critical exponent $\beta$ on the phase boundary $p = p_c(\tilde{p})$. Inset shows that of $\beta^{-1}$.
where

$$\Lambda(P) = \left. \frac{\partial P^{(n+1)}}{\partial P^{(n)}} \right|_P = 4\tilde{q}P(1-P^2) = 2(1-\alpha). \quad (24)$$

Figure 8 shows the $\tilde{p}$-dependence of $\beta$. We find that $\beta$ increases continuously with $\tilde{p}$, from $\beta = 0.164694$ at $\tilde{p} = 0$ to $\beta = \infty$ at $\tilde{p} = \tilde{p}_c = 5/32$. At $\tilde{p} = \tilde{p}_c$ we expand Eq. (3) near $p_c(\tilde{p}_c)$ to obtain

$$\Delta P^{(n+1)} \approx \Delta P^{(n)} + \frac{9}{8}(\Delta P^{(n)})^2,$$  

which is correct as long as the second term in the r.h.s. is much less than the first one, or equivalently, $n$ is much less than $n^* \approx 1/\Delta P^{(0)} = |p - p_c|^{-1}$. For $n > n^*$, $P^{(n)}$ goes to 1 rapidly and so we obtain

$$P_\infty \sim \lambda^n \sim \exp \left( -\frac{\text{const.}}{p - p_c} \right), \quad (27)$$

where $\lambda = \lambda(p_c(\tilde{p}_c))$ given by Eq. (17). We thus find an essential singularity in the order parameter at $\tilde{p} = \tilde{p}_c$.

Note that $\beta$ is apparently related to $\psi$ through $\lambda(p_c(\tilde{p}))$ as shown in [22] (see Eqs. [18] and [23]). It is, however, not the case in the (off-boundary) critical phase where the nontrivial stable fixed points $p^*(\tilde{p})$ dominate the criticality while the order parameter vanishes there.

V. SUMMARY

We have investigated bond percolations on the decorated $(2,2)$-flower with two different probabilities $p$ and $\tilde{p}$. Our generating function approach has revealed that the system is in the critical phase for $p < p_c(\tilde{p})$ and $\tilde{p} < \tilde{p}_c = 5/32$. We have evaluated the fractal exponent $\psi$ and confirmed the power-law behavior of $n_s$ in the critical phase as well as those dependence on $p$ and $\tilde{p}$ and the validity of the scaling relation $\tau = 1 + \psi^{-1}$.

We have also examined the critical exponent $\beta$ in the percolating phase and found that $\beta$ also varies as $\tilde{p}$ from $\beta \approx 0.164694$ at $\tilde{p} = 0$, where the network is two-dimensional-like, to $\beta = \infty$ at $\tilde{p} = \tilde{p}_c$, where the dimensionality of the underlying network is infinite.

It is only at $\tilde{p} = \tilde{p}_c$ that the percolation on the decorated $(2,2)$-flower shows an infinite order transition with the BKT-like singularity as percolations on growing networks do [11,10]. The finiteness of $\beta$ for $\tilde{p} < \tilde{p}_c$ suggests that the existence of some critical phase adjacent to the normal ordered phase is not enough for the network to have such an essential singularity in the order parameter and thus an infinite order phase transition. At present we have none of the key to reveal necessary conditions for the existence of the BKT-like singularity. Further study would be required to clarify the relation between these interesting properties of the phase transition.

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