CESÀRO AVERAGE IN SHORT INTERVALS FOR GOLDBACH NUMBERS

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Abstract. Let \( \Lambda \) be the von Mangoldt function and

\[
R(n) = \sum_{h+k=n} \Lambda(h)\Lambda(k).
\]

Let further \( N, H \) be two integers, \( N \geq 2, 1 \leq H \leq N \), and assume that the Riemann Hypothesis holds. Then

\[
\sum_{n=N-H}^{N+H} R(n) \left(1 - \frac{|n-N|}{H}\right) = HN - 2\sum_{\rho} \frac{(N+H)^{\rho+2} - 2N^{\rho+2} + (N-H)^{\rho+2}}{\rho(\rho+1)(\rho+2)}
\]

\[
+ \mathcal{O}\left(N\left(\log\frac{2N}{H}\right)^2 + H(\log N)^2\log(2H)\right),
\]

where \( \rho = 1/2 + i\gamma \) runs over the non-trivial zeros of the Riemann zeta function \( \zeta(s) \).

1. Introduction

Let \( \Lambda \) be the von Mangoldt function and

\[
R(n) = \sum_{h_1+h_2=n} \Lambda(h_1)\Lambda(h_2)
\]

be the counting function for the Goldbach numbers. In this paper we are looking for an explicit formula for a Cesàro average of \( R(n) \) in short intervals. Concerning long intervals, we should mention our result in [5]: assuming the Riemann Hypothesis (RH) we have

\[
\sum_{n \leq N} R(n) = \frac{N^2}{2} - 2\sum_{\rho} \frac{N^{\rho+1}}{\rho(\rho+1)} + \mathcal{O}\left(N(\log N)^3\right),
\]

where \( N \) is a large integer and \( \rho = 1/2 + i\gamma \) runs over the non-trivial zeros of the Riemann zeta function \( \zeta(s) \). We also mention its extension to the Cesàro average case by Goldston-Yang [1] again under the assumption of RH:

\[
\sum_{n \leq N} R(n) \left(1 - \frac{n}{N}\right) = \frac{N^2}{6} - 2\sum_{\rho} \frac{N^{\rho+1}}{\rho(\rho+1)(\rho+2)} + \mathcal{O}\left(N\right).
\]
We also recall our unconditional result in \cite{6}; see also \cite{3}: let \( k > 1 \) be a real number; we have
\[
\sum_{n \leq N} R(n) \frac{(1 - n/N)^k}{\Gamma(k + 1)} = \frac{N^2}{\Gamma(k + 3)} - 2 \sum_{\rho} \frac{\Gamma(\rho)}{\Gamma(\rho + k + 2)} N^{\rho + 1} + \sum_{\rho_1} \sum_{\rho_2} \frac{\Gamma(\rho_1)\Gamma(\rho_2)}{\Gamma(\rho_1 + \rho_2 + k + 1)} N^{\rho_1 + \rho_2 + \mathcal{O}(N)},
\]
where \( \rho_1, \rho_2 \) run over the non-trivial zeros of the Riemann zeta function \( \zeta(s) \) and \( \Gamma(s) \) is Euler’s function. Our result here is

**Theorem 1.** Let \( N, H \) be two integers, \( N \geq 2, 1 \leq H \leq N \). Assume that the Riemann Hypothesis (RH) holds. Then
\[
\sum_{n = N - H}^{N + H} R(n) \left(1 - \frac{|n - N|}{H}\right) = HN - \frac{2}{H} \sum_{\rho} \frac{(N + H)^{\rho + 2} - 2N^{\rho + 2} + (N - H)^{\rho + 2}}{\rho(\rho + 1)(\rho + 2)} + \mathcal{O}\left(N\left(\log \frac{2N}{H}\right)^2 + H(\log N)^2 \log(2H)\right),
\]
where \( \rho = 1/2 + i\gamma \) runs over the non-trivial zeros of the Riemann zeta function \( \zeta(s) \).

The second difference involved in the zero-depending term is natural since it depends on the symmetric nature of the short interval Cesàro weight used in Theorem \cite{1}. Its unconditional order of magnitude is \( \ll HN \exp(-c_1(\log N)^3/5(\log \log n)^{-1/5}) + N \), where \( c_1 > 0 \) is an absolute constant, while, under the assumption of RH, it is \( \ll HN^{1/2}(\log N)^2 + N \); see Section \cite{5}.

In fact we will obtain Theorem \cite{1} as a consequence of a weighted result. Letting \( \psi(x) = \sum_{m \leq x} \Lambda(m) \), we have

**Theorem 2.** Let \( N, H \) be two integers, \( N \geq 2, 1 \leq H \leq N \) and \( y \in [-H, H] \). Assume that the Riemann Hypothesis (RH) holds. Then
\[
\max_{y \in [-H, H]} \left| \sum_{n = N - H}^{N + y} e^{-n/N} \left( R(n) - (2\psi(n) - n) \right) \left(1 - \frac{|n - N|}{H}\right) \right| \ll N(\log N)^2 \log(2H)
\]
and
\[
\left| \sum_{n = N - H}^{N + H} e^{-n/N} \left( R(n) - (2\psi(n) - n) \right) \left(1 - \frac{|n - N|}{H}\right) \right| \ll N\left(\log \frac{2N}{H}\right)^2.
\]

The better estimate for the case \( y = H \) depends on the second point of Lemma \cite{5} below in which we have a more efficient estimate for the exponential sum \( T_H(H, H; \alpha) \), defined in \cite{7}, attached to the Cesàro weight.

For \( H = N \) we can compare Theorem \cite{1} with \cite{1} and it is clear that the previously mentioned weakness of the available estimates for \( T_H(H, y; \alpha) \), again defined in \cite{7}, when \( y \neq H \) leads us to a weaker final estimate by a factor \((\log N)^3\). Unfortunately it seems that Lemma \cite{5} is optimal, see the remark after its proof, and hence this is a serious limitation for our method.

After being shown this paper, Goldston and Yang told us that it should be possible to combine their technique in \cite{1} with our Lemmas \cite{7} and \cite{8} below to remove the second error term in the statement of Theorem \cite{1}.
In order to match the case \( H = N \) with our method, we should have a more efficient way of removing the \( e^{-n/N} \) weight (which naturally arises from the use of infinite series; see [4]); unfortunately the partial summation strategy we used to achieve this goal needs a uniform result on \( y \). This leads to the first estimate in Lemma 5 and hence our global method is efficient essentially only for \( H \ll N(\log \log N)/(\log N)^3 \).

As we did in [5], we will use the original Hardy and Littlewood [2] circle method setting, i.e., the weighted exponential sum

\[
\tilde{S}(\alpha) = \sum_{n=1}^{\infty} \Lambda(n)e^{-n/N}(e(n\alpha),
\]

where \( e(x) = \exp(2\pi ix) \). Such a function was also used by Linnik [7,8].

2. Setting of the circle method

For brevity, throughout the paper we write

\[
z = \frac{1}{N} - 2\pi i\alpha,
\]

where \( N \) is a large integer and \( \alpha \in [-1/2, 1/2] \). The first lemma is an \( L^2 \)-estimate for the difference \( \tilde{S}(\alpha) - 1/z \).

**Lemma 1.** Assume RH. Let \( N \) be a sufficiently large integer and \( z \) be as in (5). For \( 0 \leq \xi \leq 1/2 \), we have

\[
\int_{-\xi}^{\xi} \left| \tilde{S}(\alpha) - \frac{1}{z} \right|^2 d\alpha \ll N\xi(1+\log(2N\xi))^2.
\]

**Proof.** This follows immediately from the proof of Theorem 1 of [4]. We just have to pay attention to the final estimate of eq. (22) on page 315 there. A slightly more careful estimate immediately gives that (22) can be replaced by

\[
\ll \sum_{\gamma_1 > 0} \exp\left(-\frac{c}{2N\eta}\right) \sum_{\gamma_2 > 0} \frac{1}{1 + |\gamma_1 - \gamma_2|^2} \ll N\eta(\log(2N\eta))^2.
\]

The final estimate follows at once. \( \square \)

The next four lemmas do not depend on RH. By the residue theorem one can obtain

**Lemma 2** (Eq. (29) of [4]). Let \( N \geq 2 \) and \( 1 \leq n \leq 2N \) be integers; let further \( z \) be as in (5). We have

\[
\int_{-\frac{1}{2}}^{\frac{1}{2}} e(-n\alpha) \frac{1}{z^2} d\alpha = ne^{-n/N} + O(1)
\]

uniformly for every \( n \leq 2N \).

**Lemma 3** (Lemma 2.3 of [5]). Let \( N \) be a sufficiently large integer and \( z \) be as in (5). We have

\[
\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \tilde{S}(\alpha) - \frac{1}{z} \right|^2 d\alpha = \frac{N}{2} \log N + O\left(N(\log N)^{1/2}\right).
\]
Let
\begin{equation}
V(\alpha) = \sum_{m=1}^{\infty} e^{-m/N} e(m\alpha) = \sum_{m=1}^{\infty} e^{-mz} = \frac{1}{e^z - 1}.
\end{equation}

**Lemma 4** (Lemma 2.4 of [5]). If \( z \) satisfies (5), then \( V(\alpha) = z^{-1} + O(1) \).

Let now
\begin{equation}
h_H(m) = H - |m| \quad \text{and} \quad T_H(N, y; \alpha) = \sum_{n=N-H}^{N+y} t_H(n-N)e(n\alpha).
\end{equation}

**Lemma 5.** Let \( N, H \) be two integers, \( N \geq 2 \), \( 1 \leq H \leq N \). For every \( y \in [-H, H) \) and \( \alpha \in [-1/2, 1/2] \), we have
\[ T_H(N, y; \alpha) \ll H \min(H; \frac{1}{\|\alpha\|}). \]
Moreover, for every \( \alpha \in [-1/2, 1/2] \), we also have
\[ T_H(N, H; \alpha) \ll \min(H^2; \frac{1}{\|\alpha\|^2}). \]

**Proof.** First of all we recall the well-known estimate
\begin{equation}
\sum_{m=1}^{u} e(m\alpha) \ll \min(u; \frac{1}{\|\alpha\|}).
\end{equation}

Let now \( y \in [-H, H) \). Then
\begin{equation}
|T_H(N, y; \alpha)| \leq \sum_{n=N-H}^{N+y} t_H(n-N) \ll H(H + y + 1) \ll H^2.
\end{equation}
Moreover if \( y \geq 0 \) we get
\begin{equation}
T_H(N, y; \alpha) = H \sum_{n=N-H}^{N+y} e(n\alpha) - \sum_{m=0}^{y} me((N+m)\alpha) - \sum_{m=1}^{H} me((N-m)\alpha) = A - B - C,
\end{equation}
say. By partial summation and (8) we get
\begin{equation}
B = y \sum_{m=0}^{y} e((N+m)\alpha) - \int_{0}^{y} \sum_{m=0}^{w} e((N+m)\alpha) \, dw \ll \frac{y}{\|\alpha\|} + \int_{0}^{y} \frac{dw}{\|\alpha\|} \ll \frac{y}{\|\alpha\|} \ll \frac{H}{\|\alpha\|}.
\end{equation}
Arguing analogously we have
\begin{equation}
C \ll \frac{H}{\|\alpha\|},
\end{equation}
while the inequality \( A \ll H/\|\alpha\| \) follows from (8). If \( y < 0 \), then we can write that
\[ T_H(N, y; \alpha) = H \sum_{n=N-H}^{N+y} e(n\alpha) - \sum_{m=-y}^{H} me((N-m)\alpha) = A - D,
\]
say, where \( A \) is defined in (10). Arguing as we did for \( B \) we get
\begin{equation}
D \ll \frac{H}{\|\alpha\|}.
\end{equation}
Combining (9)-(13) the first part of the lemma follows for every \( y \in [-H, H] \). The second part of the lemma follows by (8) and the fact that in this case we can write

\[
T_H(N, H; \alpha) = \sum_{m=-H}^{H} t_H(m)e(m\alpha)e(N\alpha) = \left| \sum_{m=1}^{H} e(m\alpha) \right|^2 e(N\alpha).
\]

\( \square \)

**Remark.** We remark that the estimate for \( T_H(N, y; \alpha), y \neq H, \) is essentially optimal. For brevity, we only deal with \( T_H(N, y; \alpha) \) for \( y \in [0, H] \). It is not hard to prove by induction that

\[
T_H(N; y, \alpha) = e(N\alpha) \sum_{n=-H}^{y} (H - |n|)e(n\alpha)
\]

\[
= \frac{e((N + y + 1)\alpha)}{1 - e(\alpha)} \cdot (y - H) + \frac{e((N + 1)\alpha)}{(1 - e(\alpha))^2} \cdot (e(y\alpha) - 2 + e(-H\alpha)).
\]

For brevity, we only deal with \( T_H(N; y, \alpha) \) for \( y \in [0, H] \). It is not hard to prove by induction that

\[
T_H(N; y, \alpha) = e(N\alpha) \sum_{n=-H}^{y} (H - |n|)e(n\alpha)
\]

\[
= \frac{e((N + y + 1)\alpha)}{1 - e(\alpha)} \cdot (y - H) + \frac{e((N + 1)\alpha)}{(1 - e(\alpha))^2} \cdot (e(y\alpha) - 2 + e(-H\alpha)).
\]

In the critical range \( \alpha \in [H^{-1}, 1/2] \) the last summand has a smaller order of magnitude than \( H\|\alpha\|^{-1} \), and this implies that the bound \( T_H(N; y, \alpha) \ll H\|\alpha\|^{-1} \) is sharp, at least when \( y \leq H/2, \) say.

We build now the zero-depending term we have in Theorem 1. The first step is the following

**Lemma 6.** Let \( N, H \) be two integers, \( N \geq 2, 1 \leq H \leq N \) and \( z \) be as in (5). For every \( y \in [-H, H] \) we have

\[
\int_{-1/2}^{1/2} T_H(N, y; -\alpha) \left( \frac{S(\alpha) - 1/z}{z} \right) d\alpha = \sum_{n=-H}^{N+y} e^{-n/N} t_H(n - N)(\psi(n) - n)
\]

\[
+ O\left( H^{3/2} N^{1/2} (\log N)^{1/2} \right).
\]

We remark that Lemma 6, which is a modified version of Lemma 2.5 of [5], is unconditional and hence it implies, using also Lemmas 7,8, that the ability of detecting the zero-depending term of the Riemann zeta function \( \zeta(s) \) in Theorem 1 does not depend on RH.

**Proof.** Writing \( \tilde{R}(\alpha) = \tilde{S}(\alpha) - 1/z \), by Lemma 4 we have

\[
\int_{-1/2}^{1/2} T_H(N, y; -\alpha) \frac{\tilde{R}(\alpha)}{z} d\alpha
\]

\[
= \int_{-1/2}^{1/2} T_H(N, y; -\alpha) \tilde{R}(\alpha) V(\alpha) d\alpha + O\left( \int_{-1/2}^{1/2} |T_H(N, y; -\alpha)| |\tilde{R}(\alpha)| d\alpha \right)
\]

\[
= \int_{-1/2}^{1/2} T_H(N, y; -\alpha) \tilde{R}(\alpha) V(\alpha) d\alpha + O\left( (H^3 N \log N)^{1/2} \right),
\]

since, by Lemmas 3 and 5 the error term above is

\[
\ll \left( \int_{-1/2}^{1/2} |T_H(N, y; -\alpha)|^2 d\alpha \right)^{1/2} \left( \int_{-1/2}^{1/2} |\tilde{R}(\alpha)|^2 d\alpha \right)^{1/2} \ll (H^3 N \log N)^{1/2}.
\]

Again by Lemma 4 we have

\[
\tilde{R}(\alpha) = \tilde{S}(\alpha) - 1/z = \tilde{S}(\alpha) - V(\alpha) + O(1)
\]
and hence (15) implies
\[
\int_{-1/2}^{1/2} T_H(N, y; -\alpha) \frac{\tilde{R}(\alpha)}{z} \, d\alpha = \int_{-1/2}^{1/2} T_H(N, y; -\alpha) \left( \tilde{S}(\alpha) - V(\alpha) \right) V(\alpha) \, d\alpha
\]
\[
+ O\left( \int_{-1/2}^{1/2} |T_H(N, y; -\alpha)| |V(\alpha)| \, d\alpha \right) + O\left( (H^3 N \log N)^{1/2} \right).
\]
\[\text{(16)}\]

The Cauchy-Schwarz inequality, Lemma 5 and the Parseval theorem imply that
\[
\int_{-1/2}^{1/2} |T_H(N, y; -\alpha)||V(\alpha)| \, d\alpha \leq \left( \int_{-1/2}^{1/2} |T_H(N, y; -\alpha)|^2 \, d\alpha \right)^{1/2} \left( \int_{-1/2}^{1/2} |V(\alpha)|^2 \, d\alpha \right)^{1/2}
\]
\[
\ll \left( H^3 \sum_{m=1}^{\infty} e^{-2m/N} \right)^{1/2} \ll (H^3 N)^{1/2}.
\]
\[\text{(17)}\]

By (16)–(17), we have
\[
\int_{-1/2}^{1/2} T_H(N, y; -\alpha) \frac{\tilde{R}(\alpha)}{z} \, d\alpha = \int_{-1/2}^{1/2} T_H(N, y; -\alpha) \left( \tilde{S}(\alpha) - V(\alpha) \right) V(\alpha) \, d\alpha
\]
\[
+ O\left( (H^3 N \log N)^{1/2} \right).
\]
\[\text{(18)}\]

Now, by (4) and (6), we can write
\[
\tilde{S}(\alpha) - V(\alpha) = \sum_{m=1}^{\infty} (\Lambda(m) - 1)e^{-m/N} e(m\alpha)
\]
so that
\[
\int_{-1/2}^{1/2} T_H(N, y; -\alpha) \left( \tilde{S}(\alpha) - V(\alpha) \right) V(\alpha) \, d\alpha
\]
\[
= \sum_{m=-H}^{y} t_H(m) \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} e^{-m_2/N} \left( \Lambda(m_1) - 1 \right) e^{-m_1/N} \int_{-1/2}^{1/2} e((m_1 + m_2 - m - N)\alpha) \, d\alpha
\]
\[
= \sum_{m=-H}^{y} t_H(m) \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} e^{-m_2/N} \begin{cases} 1, & \text{if } m_1 + m_2 = m + N, \\ 0, & \text{otherwise}, \end{cases}
\]
\[
\times \sum_{m_2=1}^{\infty} e^{-m_2/N} \left( \sum_{m_1=1}^{\infty} \left( \Lambda(m_1) - 1 \right) \right)
\]
\[
= \sum_{m=-H}^{y} t_H(m)e^{-(m+N)/N} \sum_{m_1=1}^{m+N-1} (\Lambda(m_1) - 1)
\]
\[
\times e^{-n/N} t_H(n - N)(\psi(n - 1) - (n - 1))
\]
\[\text{(19)}\]
since the condition $m_1 + m_2 = m + N$ implies that both variables are $< m + N$. Now $\psi(n) = \psi(n-1) + \Lambda(n)$, so that

$$
\sum_{n=N-H}^{N+y} e^{-n/N} t_H(n - N)(\psi(n-1) - (n-1))
= \sum_{n=N-H}^{N+y} e^{-n/N} t_H(n - N)(\psi(n) - n) + O(E),
$$

say, where, by the Brun-Titchmarsh theorem, we have $E \ll H^2 \log N$ if $0 \leq H + y \leq N^\varepsilon$ and $E \ll H^2$ otherwise. By (18)-(19) and the previous equation, we have

$$
\int_{-1/2}^{1/2} T_H(N, y; -\alpha) \frac{\tilde{R}(\alpha)}{z} \, d\alpha
= \sum_{n=N-H}^{N+y} e^{-n/N} t_H(n - N)(\psi(n) - n) + O\left( H^{3/2} N^{1/2} (\log N)^{1/2} \right).
$$

Hence (14) is proved. \qed

We need now the following lemma which is an extension of Lemma 2.6 of [5].

**Lemma 7.** Let $M > 1$ be a real number. We have that

$$
\sum_{n=1}^{M} (\psi(n) - n) = -\sum_{\rho} \frac{M^{\rho+1}}{\rho(\rho + 1)} + O(M),
$$

where $\rho$ runs over the non-trivial zeros of the Riemann zeta function $\zeta(s)$.

**Proof.** The case when $M > 1$ is an integer was proved in Lemma 2.6 of [5]. Let $M > 1$ be a non-integral real number. Hence

$$
\sum_{n=1}^{M} (\psi(n) - n) = \sum_{n=1}^{\lfloor M \rfloor} (\psi(n) - n) = -\sum_{\rho} \frac{|M|^{\rho+1}}{\rho(\rho + 1)} + O(M),
$$

by Lemma 2.6 of [5]. Writing $\rho = \beta + i\gamma$, we have

$$
\frac{|M|^{\rho+1} - M^{\rho+1}}{\rho + 1} \ll M^{\beta+1} \min\left( \frac{1}{M}; \frac{1}{|\rho + 1|} \right)
$$

and hence, by the zero-free region and the Riemann-von Mangoldt estimate, we obtain

$$
\sum_{\rho} \frac{|M|^{\rho+1} - M^{\rho+1}}{\rho(\rho + 1)} \ll \sum_{|\rho| \leq M} \frac{M^\beta}{|\rho|} + \sum_{|\rho| > M} \frac{M^\beta+1}{|\rho|^2} = o(M).
$$

By (20)-(21), Lemma 7 follows. \qed
Lemma 8. Let $N$ be a large integer and $2 \leq H \leq N$. We have that

$$
\sum_{n=N-H}^{N+H} t_H(n-N)(\psi(n) - n) = -\sum_{\rho} \frac{(N + H)^{\rho+2} - 2N^{\rho+2} + (N - H)^{\rho+2}}{\rho(\rho + 1)(\rho + 2)} + O(HN),
$$

where $\rho$ runs over the non-trivial zeros of the Riemann zeta function $\zeta(s)$.

Proof. A direct computation shows

$$
\sum_{n=N-H}^{N+H} t_H(n-N)(\psi(n) - n) = \sum_{m=0}^{H} t_H(m)(\psi(N + m) - (N + m)) + \sum_{m=0}^{H} t_H(m)(\psi(N - m) - (N - m)) - H(\psi(N) - N)
$$

(22)

$$
= \sum_{m=0}^{H} t_H(m)a(m) - H(\psi(N) - N),
$$

where we have implicitly defined $a(m) := \psi(N + m) + \psi(N - m) - 2N$. By partial summation we have

$$
\sum_{m=0}^{H} t_H(m)a(m) = -Ha(0) + \int_{0}^{H} \sum_{m=0}^{t} a(m) \, dt.
$$

It is easy to see that $a(0) = 2(\psi(N) - N)$ and that

$$
\sum_{m=0}^{t} a(m) = \sum_{n=N-t}^{N+t} (\psi(n) - n) + (\psi(N) - N)
$$

$$
= \sum_{n=1}^{N+t} (\psi(n) - n) - \sum_{n=1}^{N-t} (\psi(n) - n) + (\psi(N) - N) + ^{'}(\psi(N - t) - (N - t))
$$

$$
= -\sum_{\rho} \frac{(N + t)^{\rho+1} - (N - t)^{\rho+1}}{\rho(\rho + 1)} + O(N),
$$

where $^{'}$ indicates that the term is present only when $t$ is an integer and, in the last equality, we used Lemma 7 and the Prime Number Theorem. Summing up,
exploiting the absolute convergence of the series over the zeros of the Riemann zeta function \( \zeta(s) \), we obtain that

\[
\sum_{m=0}^{H} t_H(m)a(m) = -\sum_{\rho} \frac{1}{\rho(\rho+1)} \int_0^H \left( (N + t)^{\rho+1} - (N - t)^{\rho+1} \right) dt + O(HN)
\]

(23)

\[
= -\sum_{\rho} \frac{(N + H)^{\rho+2} - 2N^{\rho+2} + (N - H)^{\rho+2}}{\rho(\rho+1)(\rho+2)} + O(HN).
\]

Inserting (23) in (22) and using the Prime Number Theorem, Lemma 8 follows. \( \square \)

3. PROOF OF THEOREM \( \text{(1)} \)

We will get Theorem \( \text{(1)} \) as a consequence of Theorem \( \text{(2)} \). By partial summation we have

\[
\sum_{n=N-H}^{N+H} t_H(n-N) \left( R(n) - (2\psi(n) - n) \right)
\]

\[
= \sum_{n=N-H}^{N+H} e^{n/H} t_H(n-N) \left\{ R(n) - (2\psi(n) - n) \right\} e^{-n/H}
\]

\[
= e^{(N+H)/H} \sum_{n=N-H}^{N+H} e^{-n/H} t_H(n-N) \left( R(n) - (2\psi(n) - n) \right)
\]

(24)

\[
- \frac{1}{N} \int_{N-H}^{N+H} \left\{ \sum_{n=N-H}^{N+H} e^{-n/H} t_H(n-N) \left( R(n) - (2\psi(n) - n) \right) \right\} e^{w/H} dw + O(1).
\]

Inserting (2)-(3) in (24) we get

\[
\sum_{n=N-H}^{N+H} t_H(n-N) \left( R(n) - (2\psi(n) - n) \right) \ll HN \left( \log \frac{2N}{H} \right)^2 + H^2 (\log N)^2 \log(2H)
\]

and hence

\[
\sum_{n=N-H}^{N+H} t_H(n-N) R(n) = \sum_{n=N-H}^{N+H} t_H(n-N)n + 2 \sum_{n=N-H}^{N+H} t_H(n-N)(\psi(n) - n)
\]

(25)

\[+ O \left( HN \left( \log \frac{2N}{H} \right)^2 + H^2 (\log N)^2 \log(2H) \right).\]

A direct calculation proves that

\[
\sum_{n=N-H}^{N+H} t_H(n-N)n = H^2 N
\]

and hence Theorem \( \text{(1)} \) now follows inserting such an identity and Lemma \( \text{8} \) in (25) and dividing by \( H \).
4. Proof of Theorem 2

Assume $N \geq 2$, $1 \leq H \leq N$, $y \in [-H, H]$ and let $\alpha \in [-1/2, 1/2]$. Writing $\tilde{R}(\alpha) = \tilde{S}(\alpha) - 1/\alpha$, recalling (7) we have

$$\sum_{n=N-H}^{N+y} e^{-n/N} t_H(n - N) R(n) \int_{-\frac{1}{2}}^{\frac{1}{2}} \tilde{S}(\alpha)^2 T_H(N, y; -\alpha) \, d\alpha$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{T_H(N, y; -\alpha)}{z^2} \, d\alpha + 2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{T_H(N, y; -\alpha) \tilde{R}(\alpha)}{z} \, d\alpha$$

(26)

$$+ \int_{-\frac{1}{2}}^{\frac{1}{2}} T_H(N, y; -\alpha) \tilde{R}(\alpha)^2 \, d\alpha = I_1(y) + I_2(y) + I_3(y),$$

say.

Evaluation of $I_1(y)$. By Lemma 2 we obtain

$$I_1(y) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{T_H(N, y; -\alpha)}{z^2} \, d\alpha = \sum_{n=N-H}^{N+y} \frac{t_H(n - N)}{z^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-n\alpha} \, d\alpha$$

$$= \sum_{n=N-H}^{N+y} t_H(n - N) \left(ne^{-n/N} + O(1)\right)$$

(27)

Estimation of $I_2(y)$. By (14) of Lemma 6 we obtain

$$I_2(y) = 2 \sum_{n=N-H}^{N+y} e^{-n/N} t_H(n - N)(\psi(n) - n) + O\left(H^{3/2} N^{1/2} (\log N)^{1/2}\right).$$

Estimation of $I_3(y); y \in [-H, H]$. Using Lemmas 5 and 1 we have that

$$I_3(y) \ll \int_{-\frac{1}{2}}^{\frac{1}{2}} |T_H(N, y; -\alpha)||\tilde{R}(\alpha)||^2 \, d\alpha$$

$$\ll H^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} |\tilde{R}(\alpha)|^2 \, d\alpha + H \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{|\tilde{R}(\alpha)|^2}{\alpha} \, d\alpha + H \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{|\tilde{R}(\alpha)|^2}{|\alpha|} \, d\alpha$$

$$\ll H N \left(\frac{2N}{H}\right)^2 + H \sum_{k=0}^{\Theta(\log 2H)} \frac{H}{2k} \int_{\frac{1}{2}}^{\frac{1}{H}} |\tilde{R}(\alpha)|^2 \, d\alpha$$

$$\ll H N \left(\frac{2N}{H}\right)^2 + H \sum_{k=0}^{\Theta(\log 2H)} \frac{H}{2k} N^{\frac{k+1}{H}} \left(\log \frac{2k+2N}{H}\right)^2$$

(29)

$$\ll H N \left(\frac{2N}{H}\right)^2 + H N (\log N)^2 \sum_{k=0}^{\Theta(\log 2H)} 1.$$
Estimation of $I_3(H)$. Using Lemmas 5 and 1 we have that

$$I_3(H) \ll \int_{-\frac{1}{2}}^{\frac{1}{2}} |T_H(N, H; -\alpha)| \tilde{R}(\alpha)|^2 \, d\alpha \ll H^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( \tilde{R}(\alpha) \right)^2 \, d\alpha + \int_{\frac{1}{2}}^{\frac{1}{2}} \frac{\left( \tilde{R}(\alpha) \right)^2}{\alpha^2} \, d\alpha + \int_{-\frac{1}{2}}^{-\frac{1}{2}} \frac{\left( \tilde{R}(\alpha) \right)^2}{\alpha^2} \, d\alpha$$

$$\ll HN \left( \log \frac{2N}{H} \right)^2 + \sum_{k=0}^{\mathcal{O}(\log 2H)} \frac{H^2}{4^k} \int_{\frac{1}{2}}^{\frac{1}{2}} \left( \tilde{R}(\alpha) \right)^2 \, d\alpha$$

$$\ll HN \left( \log \frac{2N}{H} \right)^2 + \sum_{k=0}^{\mathcal{O}(\log 2H)} \frac{H^2}{4^k} N \frac{2^{k+1}}{H} \left( \log \frac{2^{k+2}N}{H} \right)^2$$

$$\ll HN \left( \log \frac{2N}{H} \right)^2 + HN \sum_{k=0}^{\mathcal{O}(\log 2H)} \frac{1}{2^k} \left( k + 1 + \log \frac{2N}{H} \right)^2$$

(30) $$\ll HN \left( \log \frac{2N}{H} \right)^2.$$

End of the proof. Inserting (27)-(29) into (26), for every $y \in [-H, H)$, we immediately have

$$\sum_{n=N-H}^{N+y} e^{-n/N} t_H(n - N) R(n) = \sum_{n=N-H}^{N+y} e^{-n/N} t_H(n - N)n$$

$$+ 2 \sum_{n=N-H}^{N+y} e^{-n/N} t_H(n - N)(\psi(n) - n) + \mathcal{O}\left( HN(\log N)^2 \log(2H) \right).$$

Hence

$$\sum_{n=N-H}^{N+y} e^{-n/N} t_H(n - N) \left( R(n) - (2\psi(n) - n) \right) \ll HN(\log N)^2 \log(2H)$$

for every $y \in [-H, H)$. Thus we can write

$$\max_{y \in [-H, H)} \left| \sum_{n=N-H}^{N+y} e^{-n/N} t_H(n - N) \left( R(n) - (2\psi(n) - n) \right) \right| \ll HN(\log N)^2 \log(2H).$$

The $y = H$ case follows analogously using (30) instead of (29). Dividing by $H$, Theorem 2 is proved.

5. About the order of magnitude of the zero-depending term

Let us define

$$S := \sum_{\rho} \frac{(N + H)^{\rho+2} - 2N^{\rho+2} + (N - H)^{\rho+2}}{\rho(\rho + 1)(\rho + 2)}.$$

By Lemma 8 we have

$$S = - \sum_{n=N-H}^{N+H} t_H(n - N)(\psi(n) - n) + \mathcal{O}(HN).$$
Assuming RH, we have $\psi(n) - n \ll n^{1/2}\log(n)^2$ and hence

$$S \ll H(\log N)^2 \sum_{n=N-H}^{N+H} n^{1/2} + HN \ll H(\log N)^2 \left((N+H)^{3/2} - (N-H)^{3/2}\right) + HN$$

$$\ll H^2 N^{1/2}(\log N)^2 + HN.$$ 

Dividing by $H$, the expected order of magnitude of the second difference term in Theorem 1 is, under the assumption of RH, $\ll H N^{1/2}(\log N)^2 + N$. The same strategy works in the unconditional case too. The Prime Number Theorem in the form $\psi(n) - n \ll n \exp(-c(\log n)^{3/5}(\log\log n)^{-1/5})$, where $c > 0$ is an absolute constant, leads to the final estimate

$$S \ll H \exp(-c_1(\log N)^{3/5}(\log\log n)^{-1/5}) \sum_{n=N-H}^{N+H} n + HN$$

$$\ll H^2 N \exp(-c_1(\log N)^{3/5}(\log\log n)^{-1/5}) + HN,$$

where $c_1 > 0$ is an absolute constant. Dividing by $H$, the expected order of magnitude of the second difference term in Theorem 1 is $\ll H N \exp(-c_1(\log N)^{3/5}(\log\log n)^{-1/5}) + N$.

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