Critical temperature of inhomogeneous magnetic superconductor: effective tensor field approach

Yu N Proshin and V A Tumanov
Institute of Physics, Kazan Federal University, Kazan, Russian Federation
E-mail: yurii.proshin@kpfu.ru

Abstract. Superconducting state with the inhomogeneous effective exchange field background is studied. We calculate the critical temperature of magnetic superconductor on the basis of the Hamiltonian that takes into account the interaction of electrons with the effective exchange field in the direction of inhomogeneity. We use the local unitary rotation in spinor space to rewrite the Hamiltonian in the new basis, where this interaction is diagonal. In this case the exchange field becomes homogeneous but the effective tensor field appears. This method allows us to simplify the Gor’kov equations in many symmetric cases and to find the Green’s functions and the critical temperature. We test our approach on the known case of magnetic superconductor with helical magnetization and focus on the critical temperature and the Fulde–Ferrell–Larkin–Ovchinnikov (FFLO) states.

1. Introduction
Interplay of superconductivity and magnetism leads to numerous significant effects in general, but many of them occur only with an inhomogeneous background magnetization [1-3]. The origin of these inhomogeneities can be various: domain walls, magnetic helical structures, artificial multilayers with different magnetization directions, etc. The effective exchange field with non-uniform orientation produces the triplet superconducting correlations in a layered ferromagnet-superconductor (FS) structure [1]. The triplet pairs penetrate much deeper into the ferromagnet compared with the singlet ones. The domain ordering affects also the critical temperature of the superconducting transition [2]. The nature of this effect can be explained by the following qualitative argument: the exchange field acting on the Cooper pairs is averaged over the region of the order of superconducting correlation length [2]. Thus, the exchange is less than the average field near a domain wall, and this leads to a more intense Cooper pairing and local increase of the critical temperature ($T_c$) of the FS contacts in these regions. In the theoretical calculations the derivatives of the average exchange field with respect to the relative coordinates can be neglected [4]. The size of natural magnetic inhomogeneities is much larger compared with the superconducting coherence length but there are many interesting and practically significant effects in artificial systems out of this case. For example, the existence of $\pi$-phase magnetism in the FS superlattices [5,6] leads to increasing $T_c$. In the superconducting multilayer comprising several monolayers of iron and chromium, the critical temperature varies with large amplitude when varying the thickness of Cr layer [7]. The deposition of PdFe on the Nb film increases the critical current density [8]. The latter systems and materials are favorable for applications in spin switches like FFS, FSF and FSFS systems [9-12].

2. Effective tensor field
We use a model of collectivized electrons with Bardeen-Cooper-Schrieffer (BCS) type electron-electron interaction and ferromagnetic interaction with localized spins. Below we assume that the
Planck constant, the Boltzmann constant, and the Bohr magneton are equal to 1. Here we use the effective exchange field approximation and the Hamiltonian should be averaged over the spin operators of the localized electrons:

\[
H = \int \sum_{\sigma, \sigma'} \psi_{\sigma'}^\dagger \left( \frac{\hat{p}^2}{2m} - \mu \right) \hat{I} - (\mathbf{I}, \mathbf{\hat{\sigma}}) \psi_{\sigma} \, d\mathbf{r} + \lambda \int \psi_{\sigma}^\dagger \psi_{\sigma} \, d\mathbf{r},
\]  

(1)

where \(\psi^\dagger\) and \(\psi\) are electron creation and annihilation operators; \(m^*\) is the effective mass of the electron, \(\mu\) is the chemical potential, \(\lambda\) is the electron-electron interaction constant, \(\sigma\) is the electron spin projection, \(\hat{p}\) is the electron momentum operator. The matrix, which is responsible for the interaction between the electron spin and the mean-field localized spins, has the following form:

\[
(\mathbf{I}, \mathbf{\hat{\sigma}}) = \begin{pmatrix}
\cos(\theta) & \sin(\theta) e^{-i\phi} \\
\sin(\theta) e^{i\phi} & -\cos(\theta)
\end{pmatrix},
\]  

(2)

where \(\phi(\mathbf{r})\) and \(\theta(\mathbf{r})\) are the angles defining the magnetization direction in spherical coordinate system. We diagonalize the matrix \((\mathbf{I}, \mathbf{\hat{\sigma}})\) by rotating in the spinor space using unitary matrices. The BCS term is invariant under such transformations. The Hamiltonian \(H\) can be written in the form

\[
H = \int \sum_{\sigma, \sigma'} \psi_{\sigma'}^\dagger \left( -\frac{\hat{D}^2}{2m} - \mu \hat{I} - \mathbf{I} \sigma_3 \right) \psi_{\sigma} \, d\mathbf{r} + \int \left( \Delta(\mathbf{r}) \psi_{\sigma}^\dagger \psi_{\sigma} + \Delta^*(\mathbf{r}) \psi_{\sigma} \psi_{\sigma}^\dagger \right) \, d\mathbf{r},
\]  

(3)

where \(\hat{D} = \nabla + \hat{A}, \quad \hat{A} = \hat{U} \nabla \hat{U}^{-1} \); \(\sigma_i\) are the Pauli matrices, \(\Delta\) is the self-consistent order parameter. Here we use the standard self-consistent approximation and introduce the order parameter in the BCS term. The matrix \(\hat{U}\) is determined from the requirement \(\hat{U}(\mathbf{I}, \hat{\sigma})\hat{U}^{-1} = \mathbf{I} \sigma_3\) and has the form:

\[
\hat{U} = \begin{pmatrix}
\cos(\frac{\theta}{2}) e^{i\xi} & \sin(\frac{\theta}{2}) e^{i(\xi - \phi)} \\
-sin(\frac{\theta}{2}) e^{-i(\xi - \phi)} & \cos(\frac{\theta}{2}) e^{-i\xi}
\end{pmatrix},
\]  

(4)

where \(\xi(\mathbf{r})\) is arbitrary real function. Substituting \(\hat{U}\) into the expression for the field we obtain the following result

\[
\hat{A} = \frac{i}{2} \left( \sin(2\xi - \phi) \hat{\sigma}_1 - \cos(2\xi - \phi) \hat{\sigma}_2 \right) \nabla \theta + i \hat{\sigma}_3 \left( \sin(\frac{\theta}{2})^2 \nabla \phi - \nabla \xi \right) + \\
+ \frac{i \sin \theta}{2} \left( \cos(2\xi - \phi) \hat{\sigma}_1 - \sin(2\xi - \phi) \hat{\sigma}_2 \right) \nabla \phi.
\]  

(5)

The tensor field has freedom, because the real function \(\xi(\mathbf{r})\) is chosen arbitrarily and the observable parameters should be independent of \(\xi(\mathbf{r})\). Thus, the problem of the origin of superconducting correlations on the background of inhomogeneously directed effective exchange field is reduced to the homogeneous problem in the presence of effective tensor field. The effective tensor field has the freedom that allows us to require further invariance theory from the function \(\xi(\mathbf{r})\). The critical
temperature can be found by solving the equations for the Matsubara Green's functions. The matrix Green’s functions are selected as follows:

\[
\hat{G} = \begin{pmatrix}
G_{↑↑}(r_1,r_2,\omega) & G_{↑↓}(r_1,r_2,\omega) \\
G_{↑↑}(r_1,r_2,\omega) & G_{↑↑}(r_1,r_2,\omega)
\end{pmatrix}, \quad \hat{F} = i\hat{\sigma}_2 \begin{pmatrix}
F_{↑↑}(r_1,r_2,\omega) & F_{↑↓}(r_1,r_2,\omega) \\
F_{↑↑}(r_1,r_2,\omega) & F_{↑↑}(r_1,r_2,\omega)
\end{pmatrix}.
\] (6)

Here \( \hat{G}_{\alpha\alpha}(r_1,r_2,\omega) \) and \( \hat{F}_{\alpha\alpha}(r_1,r_2,\omega) \) are the Fourier components of the normal and anomalous Matsubara Green's functions, \( \omega \) is the Matsubara frequency. In the matrix Green's functions notation the Gorkov’s equations are rewritten as

\[
i\omega \hat{G}(r_1,r_2,\omega) - \hat{H}_0(I) \hat{G}(r_1,r_2,\omega) + \Delta^*(r_i) \hat{F}^\dagger(r_1,r_2,\omega) = \delta(r_i - r_2) \\
i\omega \hat{F}^\dagger(r_1,r_2,\omega) + \hat{H}_0(-I) \hat{F}^\dagger(r_1,r_2,\omega) + \Delta^*(r_i) \hat{G}(r_1,r_2,\omega) = 0.
\] (7)

\[
\Delta^*(r) = \frac{\lambda T}{2} \sum_\omega \text{Sp} \hat{F}^\dagger(\omega, r, r), \quad \hat{H}_0 = -\frac{\hat{D}_r^2}{2m} - \mu \hat{I} - I\hat{\sigma}_3.
\]

The prime at the summation sign means a cutoff at the Debye frequency. To calculate the critical temperature the Gor’kov equation is expanded in the linear approximation for the order parameter \([13]\).

After that we assume the conventional self-consistency equation \([13]\) with kernel \( \hat{K}^*(\omega, r, s) \)

\[
\hat{K}^*(\omega, r, s) = \hat{G}(-\omega, -I, r, s) \hat{G}(\omega, I, s, r),
\]

\[
i\omega \hat{G}(r_1,r_2,\omega) - \hat{H}_0(\theta, \varphi, \xi, I) \hat{G}(r_1,r_2,\omega) = \delta(r_1 - r_2).
\] (8)

To test our approach, we examine the case of a magnetic superconductor with helical magnetization. This case was firstly considered by Bulaevskii, Rusinov and Kulic \([14]\) for theoretical study of superconducting substances like ErRh₄B₄ where helicoidal magnetization occurs. They calculated the Green's function without diagonalization of the Hamiltonian, found the expression for free energy and analysed the phase transitions in the system. Here we focus on the critical temperature of the superconducting transition and the Fulde–Ferrell–Larkin–Ovchinnikov (FFLO) states that add new information to this well-known case. Note that we use another notations than authors \([14]\): \( \theta \) is a spatial variable only, and it is a linear function of the coordinate, and \( 2\zeta - \varphi = 0 \). The effective tensor field is coordinate independent, and the Green's function depends only on the difference \( r_1 - r_2 \). In this case the Fourier transform of the Green's function has the form

\[
\hat{G}(p,\omega) = \frac{1}{\zeta^2 - p^2 - (p,\nabla\theta)^2 / 4m^2}
\begin{pmatrix}
\zeta - I & -i\frac{(p,\nabla\theta)}{2m} \\
i\frac{(p,\nabla\theta)}{2m} & \zeta + I
\end{pmatrix},
\] (12a)

\[
\zeta = i\omega - \frac{p^2}{2m} + \mu - \frac{(\nabla\theta)^2}{8m},
\] (12b)
After inverse Fourier transform we can obtain the well-known expression for the magnetic superconductor if \( \nabla \theta = 0 \). This kernel is an oscillating function. As was shown Fulde, Ferrell [15] and Larkin-Ovchinnikov [16], such behaviour of the kernel leads to spatial oscillations of the order parameter in a certain temperature range. In the simplest case, the order parameter can be assumed as an oscillating function \( \Delta(r) \sim \exp(i kr) \). We assume that the wave vector \( k \) is along the helical magnetic structure because it is the most interesting case, in our opinion. As shown in [2], under certain strict requirements for the parameters of the superconductor-ferromagnet bilayers, the problem of finding the critical temperature can be reduced to the determination of the critical temperature of the magnetic superconductor with a renormalized effective exchange field.

As usual, the equation for the critical temperature can be obtained from the self-consistent equation. The wave vector \( k \) of the FFLO state modulation is chosen from the condition of the critical temperature maximum [3]. In Figure 1 the calculated critical temperature of magnetic superconductor is shown for the various values of the rotation parameter \( b = \frac{\nu \nabla \theta}{T_{cs}} \). Here \( T_{cs} \) is the critical temperature of bulk superconductor. In the case \( b >> 1 \) the influence of the effective exchange field on the critical temperature becomes much weaker. The same conclusion was drawn by Volkov [17] who considered superconductivity on the background of rotating magnetization using the Eilenberger equations. The FFLO state decays rapidly with decreasing the spatial period of magnetization rotation. Comparing the dependences like shown in figure 1 with different parameters of the magnetization rotation, we conclude that the influence of the magnetization rotation on the critical temperature can be reduced to suppression of the effective exchange field. For \( b >> 1 \) the suppression can be approximately represented as power law (See inset of figure 1).

Note, that in the framework of suggested approach the Usadel and Eilenberger equations can be also rewritten. Relative the known quasiclassical approach [4] we do not need to assume that the effective exchange field together with the superconducting order parameter change slowly compared

![Figure 1](image-url)

**Figure 1.** (Color online) The dependencies of critical effective exchange field \( I_e \) versus temperature \( T \) are presented at few values of the rotation parameter \( b = \frac{\nu \nabla \theta}{T_{cs}} \). The dashed lines correspond to the FFLO states. In the inset the dependence \( I_{c/2}(b) \) is shown in a double logarithmic scale. \( I_{c/2} \) is determined from \( T_c(I_c/b) = T_{cs}^{1/2} \) condition.
with the superconducting coherence length. In this sense our approach is more consistent, and the effect of magnetization inhomogeneity scale of correlation length is responsible to the appearance of the triplet component of the superconducting condensate.

3. Conclusions

The problem of superconducting correlations with the background of inhomogeneously directed effective exchange field can be reduced to the homogeneous problem in the presence of effective tensor field. The effective tensor field has the freedom that allows us to simplify Green’s functions and quasiclassical equations. Proposed approach is confirmed in known case of superconductor with helical magnetization. The critical temperature is calculated via Gor’kov equations taken into account the FFLO states. Rotation of magnetization leads to reduction of effective exchange field and increasing the critical temperature. The approach can be expanded on the other cases of magnetic inhomogeneities.

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