EXTENDED ELECTRODYNAMICS:
III. Free Photons and (3+1)-Soliton-like Vacuum Solutions

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Abstract
This paper aims to give explicitly all non-linear vacuum solutions to our non-linear field equations [1], and to define in a coordinate free manner the important subclass of non-linear solutions, which we call almost photon-like. By means of a correct introduction of the local and integral intrinsic angular momentums of these solutions, we separate the photon-like solutions through the requirement their integral intrinsic angular momentums to be equal to the Planck’s constant \( h \). Finally, we consider such solutions, moving radially to or from a given center, using standard spherical coordinates.
1 Explicit non-linear vacuum solutions

As it was shown in [1] with every nonlinear solution $F$ of our nonlinear equations (we use all notations from [1])

$$\delta F \wedge \ast F = 0, \delta \ast F \wedge \ast \ast F = 0, \delta \ast F \wedge \ast F - \delta F \wedge F = 0. \tag{1}$$

a class of $F$-adapted coordinate systems is associated, such that $F$ and $\ast F$ acquire the form respectively

$$F = \varepsilon u dx \wedge dz + u dx \wedge d\xi + \varepsilon p dy \wedge dz + p dy \wedge d\xi$$

$$\ast F = -p dx \wedge dz - \varepsilon p dx \wedge d\xi + u dy \wedge dz + \varepsilon u dy \wedge d\xi.$$

Since we look for non-linear solutions of (1), we substitute these $F$ and $\ast F$ in (1) and after some elementary calculations we obtain

$$\delta F = (u_\xi - \varepsilon u_z) dx + (p_\xi - \varepsilon p_z) dy + \varepsilon (u_x + p_y) dz + (u_x + p_y) d\xi,$$

$$\delta \ast F = -\varepsilon (p_\xi - \varepsilon p_z) dx + \varepsilon (u_\xi - \varepsilon p_z) dy - (p_x - u_y) dz - (p_x - u_y) d\xi,$$

$$F_{\mu\nu}(\delta F)^\nu dx^{\nu} = (\ast F)_{\mu\nu}(\delta \ast F)^\nu dx^{\nu} =$$

$$= \varepsilon [p(p_\xi - \varepsilon p_z) + u(u_\xi - \varepsilon u_z)] dz + [p(p_\xi - \varepsilon p_z) + u(u_\xi - \varepsilon u_z)] d\xi,$$

$$(\delta F)^2 = (\delta \ast F)^2 = -(u_\xi - \varepsilon u_z)^2 - (p_\xi - \varepsilon p_z)^2$$

A simple direct calculation shows, that the equation

$$\delta \ast F \wedge \ast F - \delta F \wedge F = 0$$

is identically fulfilled for any such $F$ and $\ast F$ with arbitrary $u$ and $p$. We obtain that our equations reduce to only 1 equation, namely

$$p(p_\xi - \varepsilon p_z) + u(u_\xi - \varepsilon u_z) = \frac{1}{2} [(u^2 + p^2)_\xi - \varepsilon (u^2 + p^2)_z] = 0. \tag{2}$$

The obvious solution to this equation is

$$u^2 + p^2 = \phi^2(x, y, \xi + \varepsilon z). \tag{3}$$

The solution obtained shows that the equations impose some limitations only on the amplitude function $\phi$ and the phase function $\varphi$ is arbitrary except that it is bounded: $|\varphi| \leq 1$. The amplitude $\phi$ is a running wave along the
specially chosen coordinate \( z \), which is common for all \( F \)-adapted coordinate systems. Considered as a function of the spatial coordinates, the amplitude \( \phi \) is arbitrary, so it can be chosen spatially finite. The time-evolution does not affect the initial form of \( \phi \), so it will stay the same in time. This shows, that among the nonlinear solutions of our equations there are \((3+1)\) soliton-like solutions. The spatial structure is determined by the initial condition, while the phase function \( \varphi \) can be used to define internal dynamics of the solution.

Recalling from [1] the substitutions
\[
u = \phi \varphi, \ p = \phi \sqrt{1 - \varphi^2},
\]
and the equality \(|A| = \phi\), we get
\[
|\delta F| = |\delta \ast F| = \frac{|\phi| |\varphi_\xi - \varepsilon \varphi_z|}{\sqrt{1 - \varphi^2}}, \ L = \frac{|A|}{|\delta F|} = \frac{\sqrt{1 - \varphi^2}}{|\varphi_\xi - \varepsilon \varphi_z|}.
\]

For the induced pseudoorthonormal bases (1-forms and vector fields) we find
\[
A = \varphi dx + \sqrt{1 - \varphi^2} dy, \ \varepsilon A^* = -\sqrt{1 - \varphi^2} dx + \varphi dy, \ R = -dz, \ S = d\xi,
\]
\[
A = -\varphi \frac{\partial}{\partial x} - \sqrt{1 - \varphi^2} \frac{\partial}{\partial y}, \ \varepsilon A^* = \sqrt{1 - \varphi^2} \frac{\partial}{\partial x} - \varphi \frac{\partial}{\partial y}, \ R = \frac{\partial}{\partial z}, \ S = \frac{\partial}{\partial \xi}.
\]

Hence, the nonlinear solutions in canonical coordinates are parametrized by one function \( \phi \) of 3 parameters and one bounded function of 4 parameters. Therefore, the separation of various subclasses of nonlinear solutions is made by imposing additional conditions on these two functions. Further we are going to separate a subclass of solutions, the integral properties of which reflect well enough the well known from the experiment integral properties and characteristics of the free photons. These solutions will be called photon-like and will be separated through imposing additional requirements on \( \phi \) and \( \varphi \) in a coordinate-free manner.

\section{Almost photon-like solutions}

We note first, that we have three invariant quantities at hand: \( \phi \), \( \varphi \), and \( L \). The amplitude function \( \phi \) is to be determined by the initial conditions, which have to be finite. So, we may impose additional conditions on \( L \) and \( \varphi \). These conditions have to express some intra-consistency among the various
characteristics of the solution. The idea, what kind of intra-consistency to use, comes from the observation that the amplitude function \( \phi \) is a first integral of the vector field \( \mathbf{V} \), i.e.

\[
\mathbf{V}(\phi) = \left( -\varepsilon \frac{\partial}{\partial z} + \frac{\partial}{\partial \xi} \right) \phi = -\varepsilon \frac{\partial}{\partial z} \phi(x, y, \xi + \varepsilon z) + \frac{\partial}{\partial \xi} \phi(x, y, \xi + \varepsilon z) = 0.
\]

We want to extend this available consistency between \( \mathbf{V} \) and \( \phi \), so we shall require the two functions \( \varphi \) and \( L \) to be first integrals of some of the available \( F \)-generated vector fields. Explicitly, we require the following:

1°. The phase function \( \varphi \) is a first integral of the three vector fields \( \mathbf{A}, \mathbf{A}^* \) and \( \mathbf{R} \): \( \mathbf{A}(\varphi) = 0, \mathbf{A}^*(\varphi) = 0, \mathbf{R}(\varphi) = 0. \)

2°. The scale factor \( L \) is a non-zero finite first integral of the vector field \( \mathbf{S} \): \( \mathbf{S}(L) = 0. \)

The requirement \( \mathbf{R}(\varphi) = 0 \) just means that in these coordinates \( \varphi \) does not depend on the coordinate \( z \). The two other equations of 1° define the following system of differential equations for \( \varphi \):

\[
\begin{align*}
-\varphi \frac{\partial \varphi}{\partial x} - \sqrt{1 - \varphi^2} \frac{\partial \varphi}{\partial y} &= 0, \\
\sqrt{1 - \varphi^2} \frac{\partial \varphi}{\partial x} - \varphi \frac{\partial \varphi}{\partial y} &= 0.
\end{align*}
\]

Noticing that the matrix

\[
\begin{pmatrix}
-\varphi & -\sqrt{1 - \varphi^2} \\
\sqrt{1 - \varphi^2} & -\varphi
\end{pmatrix}
\]

has non-zero determinant, we conclude that the only solution of the above system is the zero-solution:

\[
\frac{\partial \varphi}{\partial x} = \frac{\partial \varphi}{\partial y} = 0.
\]

We obtain that in the coordinates used the phase function \( \varphi \) depends only on \( \xi \). Therefore, in view of (4), for the scale factor \( L \) we get

\[
L = \frac{\sqrt{1 - \varphi^2}}{|\varphi_{\xi}|}.
\]
Now, the requirement 20, which in these coordinates reads
\[ S(L) = \frac{\partial L}{\partial \xi} = \frac{\partial}{\partial \xi} \frac{\sqrt{1 - \varphi^2}}{|\varphi|} = 0, \]
just means that the scale factor \( L \) is a pure constant: \( L = \text{const} \). In this way we obtain the differential equation
\[ \frac{\partial \varphi}{\partial \xi} = \pm \frac{1}{L} \sqrt{1 - \varphi^2}. \]  
(5)
The obvious solution to this equation reads
\[ \varphi(\xi) = \cos \left( \frac{\kappa \xi}{L} + \text{const} \right), \]  
(6)
where \( \kappa = \pm 1 \). It worths to note that the naturally arising characteristic frequency according to the equation
\[ \nu = \frac{c}{L}, \]  
(7)
has nothing to do with the concept of frequency in CED. In fact, the quantity \( L \) can not be defined in Maxwell’s theory.

Finally (recalling [1]) we note, that the so obtained phase function \( \varphi(\xi) \) leads to the following. The 2-form \( \text{tr}(\mathcal{R}^0) \), where \( \mathcal{R}^0 \) is the matrix of 2-forms, formed similarly to the matrix \( \mathcal{R} \), but using the basis \((A, \varepsilon A^*, R, S)\) instead of the basis \((A, \varepsilon A^*, R, S)\), is closed. In fact,
\[ \text{tr}(\mathcal{R}^0) = \varphi dx \wedge d\xi + \varphi dy \wedge d\xi - dy \wedge dz + dz \wedge d\xi \]
and since \( \varphi = \varphi(\xi) \), we get \( d\text{tr}(\mathcal{R}^0) = 0 \). Note also that the above explicit form of \( \text{tr}(\mathcal{R}^0) \) allows to define the phase function by
\[ \varphi = \sqrt{\frac{|\text{tr}(\mathcal{R}^0)|^2}{2}}. \]
This class of solutions we call almost photon-like.

Remark. If one of the two functions \( u \) and \( p \), for example \( p \), is equal to zero: \( p = 0 \), then, formally, we again have a solution, which may be called linearly polarized by obvious reasons. Clearly, the phase function of such solutions will be constant: \( \varphi = \text{const} \), so, the corresponding scale factor becomes infinitely large: \( L \rightarrow \infty \), therefore, condition 20 is not satisfied. The reason for this is, that at \( p = 0 \) the function \( u \) becomes a running wave and we get \( |\delta F| = |\delta * F| = 0 \), so the scale factor can not be defined by the relation \( L = |A|/|\delta F| \).
3 Intrinsic angular momentum (helicity) and photon-like solutions

The problem for describing the intrinsic angular momentum (IAM), or in short helicity, spin of the photon is of fundamental importance in modern physics, therefore, we shall pay a special attention to it. In particular, we are going to consider two approaches for its mathematical description. But first, some preceding comments.

First of all, there is no any doubt that every free photon carries such an intrinsic angular momentum. Since the angular momentum is a conserved quantity, the existence of the photon’s intrinsic angular momentum can be easily established and, in fact, its presence has been experimentally proved by an immediate observation of its mechanical action and its value has been numerically measured. Assuming this is so, we have to understand its origin, nature and its entire meaning for the existence and outer relations of those natural entities, called shortly photons somewhere in the first quarter of this century.

So, we begin with the assumption: every free photon carries an intrinsic angular momentum with integral value equal to the Planck’s constant $h$. According to our understanding, the photon’s IAM comes from an intrinsic periodic process. This point of view undoubtedly leads to the notion, that photons are not point-like structureless objects, they have a structure, i.e. they are extended objects. In fact, according to one of the basic principles of physics all free objects move as a whole uniformly. So, if the photon is a point-like object any characteristic of a periodic process, e.g. frequency, should come from an outside force field, i.e. it can not be free: a free point-like (structureless) object can not have the characteristic frequency.

This simple, but true, conclusion sets the theoretical physics of the first quarter of this century faced with a serious dilemma: to keep the notion of structurelessness and to associate in a formal way the characteristic frequency to the microobjects, or to leave off the notion of structurelessness, to assume the notion of extendedness and availability of intrinsically occurring periodic process and to build corresponding integral characteristics, determined by this periodic process. A look back in time shows that the majority of those days physicists had adopted the first approach, which has brought up to life quantum mechanics as a computing method, and the dualistic-probabilistic interpretation as a philosophical conception. If we set aside the widespread
and intrinsically controversial idea that all microobjects are at the same time (point-like) particles and (infinite) waves, and look impartially, in a fair-minded way, at the quantum mechanical wave function for a free particle, we see that the only positive consequence of its introduction is the legalization of frequency, as an inherent characteristic of the microobject. In fact, the probabilistic interpretation of the quantum mechanical wave function for a free object, obtained as a solution of the free Schroedinger equation, is impossible since its square is not an integrable quantity (the integral is infinite). The frequency is really needed not because of the dualistic-probabilistic nature of microobjects, it is needed because the Planck’s relation \( E = h \nu \) turns out to be universally true in microphysics, so there is no way to avoid the introduction of frequency. The question is, if the introduction of frequency necessarily requires some (linear) ”wave equation” and the simple complex exponentials of the kind \( \text{const.exp}^i(k \cdot r - \nu t) \), i.e. running waves, as ”free solutions”. Our answer to this question is ”no”. The classical wave is something much richer and a much more engaging concept, so it hardly worths to use it just because of the attribute of frequency. In our opinion, it suffices to have a periodic process at hand.

These considerations made us turn to the soliton-like objects, they present the two features of the microobjects (localized spatial extendedness and time-periodicity), simultaneously, and, therefore, seem to be more adequate theoretical models for those microobjects, obeying the Planck’s relation \( E = h \nu \). Of course, if we are interested only in the behaviour of the microobject as a whole, we can use the point-like notion, but any attempt to give a meaning of its integral characteristics without looking for their origin in the consistent intrinsic dynamics and structure, in our opinion, is not a perspective theoretical idea. And the ”stumbling point” of such an approach is just the availability of an intrinsic mechanical angular momentum, which can not be understood as an attribute of a free structureless object.

Having in view the above considerations, we are going to consider two ways to introduce and define the intrinsic angular momentum as a local quantity and to obtain, by integration, its integral value. So, these two approaches will be of use only for the spatially finite nonlinear solutions of our equations. The both approaches introduce in different ways 3-tensors (2-covariant and 1-contravariant). Although these two 3-tensors are built of quantities, connected in a definite way with the field \( F \), their nature is quite different. The first approach is based on an appropriate tensor generaliza-
tion of the classical Poynting vector. The second approach makes use of the concept of torsion, connected with the field $F$, considered as 1-covariant and 1-contravariant tensor. The first approach is pure algebraic, while the second one uses derivatives of $F_{\mu\nu}$. The spatially finite nature of the solutions $F$ allows to build corresponding integral conserved quantities, naturally interpreted as angular momentum. The scale factor $L$ appears as a multiple, so these quantities go to infinity for all linear (i.e. for Maxwell’s) solutions.

In the first approach we make use of the scale factor $L$, the isotropic vector field $V$ and the two 1-forms $A$ and $A^*$. By these four quantities we build the following 3-tensor $H$:

$$ H = \kappa \frac{L}{c} V \otimes (A \wedge A^*). \quad (8) $$

The connection with the classical vector of Poynting comes through the exterior product of $A$ and $A^*$, the 3-dimensional sense of which is just the Pointing’s vector. In components we have

$$ H_{\nu\sigma}^\mu = \kappa \frac{L}{c} V^\mu (A_\nu A^*_{\sigma} - A_\sigma A^*_{\nu}). $$

In our system of coordinates we get

$$ H = \kappa \frac{L}{c} \left( -\varepsilon \frac{\partial}{\partial z} + \frac{\partial}{\partial \xi} \right) \otimes (\varepsilon \phi^2 dx \wedge dy), $$

so, the only non-zero components are

$$ H_{12}^3 = -H_{21}^3 = -\kappa \frac{L}{c} \phi^2, \quad H_{12}^4 = -H_{21}^4 = \kappa \varepsilon \frac{L}{c} \phi^2. $$

It is easily seen, that the divergence $\nabla_\mu H_{\nu\sigma}^\mu \rightarrow \nabla_\mu H_{12}^\mu$ is equal to 0. In fact,

$$ \nabla_\mu H_{12}^\mu = \frac{\partial}{\partial z} H_{12}^3 + \frac{\partial}{\partial \xi} H_{12}^4 = \kappa \frac{L}{c} \left[ -\phi^2 + (\varepsilon \phi^2) \xi \right] = 0 $$

because $\phi^2$ is a running wave along the coordinate $z$. Since the tangent bundle is trivial we may construct the antisymmetric 2-tensor

$$ H_{\nu\sigma} = \int_{R^3} H_{4,\nu\sigma} \, dx dy dz, $$
the constant components of which are conserved quantities.

\[
H_{12} = -H_{21} = \int_{R^3} H_{4,12} dxdydz = \kappa \varepsilon \frac{L}{c} W = \kappa \varepsilon WT = \kappa \varepsilon \frac{W}{\nu}.
\]

The non-zero eigen values of \( H_{\nu\sigma} \) are pure imaginary and are equal to \( \pm iWT \). This tensor has unique non-zero invariant \( P(F) \),

\[
P(F) = \sqrt{\frac{1}{2} H_{\nu\sigma} H^{\nu\sigma}} = WT. \tag{9}
\]

The quantity \( P(F) \) will be called Planck’s invariant for the finite nonlinear solution \( F \). All finite nonlinear solutions \( F_1, F_2, ..., \) satisfying the condition

\[
P(F_1) = P(F_2) = ... = h,
\]

where \( h \) is the Planck’s constant, will be called further photon-like. The tensor field \( H \) will be called intrinsic angular momentum tensor and the tensor \( H \) will be called spin tensor or helicity tensor. The Planck’s invariant \( P(F) = WT \), having the physical dimension of action, will be called integral angular momentum, or just spin or helicity.

The reasons to use this terminology are quite clear: the time evolution of the two mutually orthogonal vector fields \( A \) and \( A^* \) is a rotational-advancing motion around and along the \( z \)-coordinate (admissible are the right and the left rotations: \( \kappa = \pm 1 \)) with the advancing velocity of \( c \) and the frequency of circulation \( \nu = c/L \). We see the basic role of the two features of the solutions: their soliton-like character, giving finite value of all integral quantities, and their nonlinear character, allowing to define the scale factor \( L \) correctly.

From this point of view the intrinsic angular momentum \( h \) of a free photon is far from being incomprehensible quantity, connected with the even more incomprehensible duality "wave-particle", and it looks as a quite normal integral characteristic of a solution, presenting a model of our knowledge of the free photon.

We proceed to the second approach to introduce IAM by recalling the definition of torsion of two (1,1) tensors. If \( G \) and \( K \) are 2 such tensors

\[
G = G^\nu_\mu dx^\mu \otimes \frac{\partial}{\partial x^\nu}, \quad K = K^\nu_\mu dx^\mu \otimes \frac{\partial}{\partial x^\nu};
\]
their torsion is defined as a 3-tensor $S_{\mu}^{\sigma} = -S_{\sigma}^{\mu}$ by the equation

$$S(G, K)(X, Y) = [GX, KY] + [KX, GY] + GK[X, Y] + KG[X, Y] - G[X, KY] - G[KX, Y] - K[X, GY] - K[GX, Y],$$

where $[,]$ is the Lie-bracket of vector fields,

$$GX = G_{\mu}^{\nu}X_{\mu} \frac{\partial}{\partial x^{\nu}}, \quad GK = G_{\mu}^{\nu}K_{\sigma}dx^{\sigma} \otimes \frac{\partial}{\partial x^{\nu}},$$

and $X, Y$ are 2 arbitrary vector fields. If $G = K$, in general $S(G, G) \neq 0$ and

$$S(G, G)(X, Y) = 2\left\{[GX, GY] + GG[X, Y] - G[X, GY] - G[GX, Y]\right\}.$$

This last expression defines at every point $x \in M$ the torsion $S(G, G)$ of $G$ with respect to the 2-dimensional plain, defined by the two vector fields $X(x)$ and $Y(x)$. Now we are going to compute the torsion $S_F$ of the nonlinear solution $F$ with respect to the intrinsically defined by the two unit vectors $A$ and $\varepsilon A^*$ 2-plain. In components we have

$$(S_F)_{\mu}^{\sigma} = 2 \left[ F_{\mu}^{\alpha} \frac{\partial F_{\sigma}^{\alpha}}{\partial x^{\alpha}} - F_{\nu}^{\alpha} \frac{\partial F_{\mu}^{\alpha}}{\partial x^{\alpha}} - F_{\alpha}^{\sigma} \frac{\partial F_{\mu}^{\alpha}}{\partial x^{\mu}} + F_{\alpha}^{\sigma} \frac{\partial F_{\nu}^{\alpha}}{\partial x^{\nu}} \right].$$

In our coordinate system

$$A = -\varphi \frac{\partial}{\partial x} - \sqrt{1 - \varphi^2} \frac{\partial}{\partial y}, \quad \varepsilon A^* = \sqrt{1 - \varphi^2} \frac{\partial}{\partial x} - \varphi \frac{\partial}{\partial y},$$

so,

$$(S_F)_{\mu}^{\sigma} A^{\mu} A^{*\nu} = (S_F)_{12}^{\sigma} (A^1 \varepsilon A^* + A^2 \varepsilon A^*).$$

For $(S_F)_{12}^{\sigma}$ we get

$$(S_F)_{12}^{1} = (S_F)_{12}^{2} = 0, \quad (S_F)_{12}^{3} = -\varepsilon (S_F)_{12}^{4} = 2\varepsilon \{p(u_{\xi} - \varepsilon u_{\varepsilon}) - u(p_{\xi} - \varepsilon p_{\varepsilon})\}.$$  

**Remark.** In our case $(S_F)_{12}^{3} = (S_{*F})_{12}^{3}$, so further we shall work with $S_F$ only.

It is easily seen that the following relation holds: $A^1 \varepsilon A^* + A^2 \varepsilon A^* = 1$. Now, for the almost photon-like solutions

$$u = \phi(x, y, \xi + \varepsilon z) \cos \left(\frac{\kappa \xi}{L} + \text{const}\right), \quad p = \phi(x, y, \xi + \varepsilon z) \sin \left(\frac{\kappa \xi}{L} + \text{const}\right)$$

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we obtain

\[(S_F)^3_{12} = -\varepsilon (S_F)^4_{12} = -2\varepsilon \frac{\kappa}{L} \phi^2,\]

\[
(S_F)^\sigma_{\mu\nu} A^\mu \varepsilon A^{*\nu} = \left[ 0, 0, -2\varepsilon \frac{\kappa}{L} \phi^2, 2\frac{\kappa}{L} \phi^2 \right].
\]

Since \(\phi^2\) is a running wave along the \(z\)-coordinate, the vector field \(S_F(A, \varepsilon A^*)\) has zero divergence: \(\nabla_\nu [S_F(A, \varepsilon A^*)]^{\nu} = 0\). Now we define the \textit{helicity vector} for the solution \(F\) by

\[\Sigma_F = \frac{L^2}{2c} S_F(A, \varepsilon A^*).\]

Since \(L = \text{const}\), then \(\Sigma_F\) has also zero divergence, so the integral quantity

\[\int (\Sigma_F)_4 dxdydz\]

does not depend on time and is equal to \(\kappa WT\). The photon-like solutions are separated in the same way by the condition \(WT = h\). Here are three more integral expressions for the quantity \(WT\). We form the 4-form

\[-\frac{1}{L} S \wedge *\Sigma_F = \frac{\kappa}{c} \phi^2 \omega_o\]

and integrate it over the 4-volume \(\mathcal{R}^3 \times L\), the result is \(\kappa WT\). Besides, we verify easily the relations

\[\frac{1}{c} \int_{\mathcal{R}^3 \times L} |A \wedge A^*| \omega_o = \frac{L^2}{c} \int_{\mathcal{R}^3 \times L} |\delta F \wedge \delta * F| \omega_o = WT.\]

Since we separate the photon-like solutions by the relation \(WT = h\), the last expressions suggest the following interpretation of the Planck’s constant \(h\). Since \(|A \wedge A^*|\) is proportional to the area of the square, defined by the two mutually orthogonal vectors \(A\) and \(\varepsilon A^*\), the above integral sums up all these areas over the whole 4-volume, occupied by the solution \(F\) during the intrinsically determined time period \(T\), in which the couple \((A, \varepsilon A^*)\) completes a full rotation. The same can be said for the couple \((\delta F, \delta * F)\) with some different factor in front of the integral. This shows quite clearly the ”helical” origin of the full energy \(W = h\nu\) of the single photon.
4 Solutions in spherical coordinates

The so far obtained soliton-like solutions describe objects, "coming from infinity" and "going to infinity". Of interest are also soliton like solutions "radiated" from, or "absorbed", by some central "source" and propagating radially from or to the center of this source. We are going to show, that our equations [1] admit such solutions too. We assume this central source to be a small ball $R_0$ with radius $r_0$, and put the origin of the coordinate system at the center of the source-ball. The standard spherical coordinates $(r, \theta, \varphi, \xi)$ will be used and all considerations will be carried out in the region outside of the ball $R_0$. In these coordinates we have

$$ds^2 = -dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\varphi^2 + d\xi^2, \quad \sqrt{|\eta|} = r^2\sin\theta.$$ 

The $*$-operator acts in these coordinates as follows:

$$\begin{align*}
*dr &= r^2\sin\theta d\theta \wedge d\varphi \wedge d\xi * (dr \wedge d\theta \wedge d\varphi) = (r^2\sin\theta)^{-1}d\xi \\
*d\theta &= -\sin\theta dr \wedge d\varphi \wedge d\xi * (dr \wedge d\theta \wedge d\xi) = \sin\theta d\varphi \\
*d\varphi &= (\sin\theta)^{-1} dr \wedge d\theta d\xi * (dr \wedge d\varphi \wedge d\xi) = -(\sin\theta)^{-1}d\theta \\
*d\xi &= r^2\sin\theta dr \wedge d\theta d\varphi * (d\theta \wedge d\varphi \wedge d\xi) = (r^2\sin\theta)^{-1}dr
\end{align*}$$

$$\begin{align*}
*(dr \wedge d\theta) &= -\sin\theta d\varphi \wedge d\xi * (d\theta \wedge d\varphi) = -(r^2\sin\theta)^{-1}dr \wedge d\xi \\
*(dr \wedge d\varphi) &= (\sin\theta)^{-1} dr \wedge d\theta d\xi * (d\theta \wedge d\varphi) = -\sin\theta dr \wedge d\varphi \\
*(dr \wedge d\xi) &= r^2\sin\theta dr \wedge d\varphi * (d\varphi \wedge d\xi) = (\sin\theta)^{-1}dr \wedge d\theta.
\end{align*}$$

We look for solutions of the following kind:

$$F = \varepsilon u dr \wedge d\theta + ud\theta \wedge d\xi + \varepsilon pdr \wedge d\varphi + pd\varphi \wedge d\xi, \quad (10)$$

where $u$ and $p$ are spatially finite functions. We get

$$\begin{align*}
*F &= \frac{p}{\sin\theta} dr \wedge d\theta + \varepsilon \frac{p}{\sin\theta} d\theta \wedge d\xi - u\sin\theta dr \wedge d\varphi - \varepsilon \sin\theta d\varphi \wedge d\xi.
\end{align*}$$

The following relations hold:

$$\begin{align*}
F \wedge F &= 2\varepsilon(up - up)dr \wedge d\theta \wedge d\varphi \wedge d\xi = 0, \\
F \wedge *F &= \left(-u^2\sin\theta + u^2\sin\theta - \frac{p^2}{\sin\theta} + \frac{p^2}{\sin\theta}\right) dr \wedge d\theta \wedge d\varphi \wedge d\xi = 0,
\end{align*}$$

i.e. the two invariants are equal to zero: $(*F)_{\mu\nu}F^{\mu\nu} = 0$, $F_{\mu\nu}F^{\mu\nu} = 0$. 

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After some elementary computation we obtain

\[ \delta F \wedge F = \delta \star F \wedge \star F = \varepsilon \left[ u (\varepsilon p_r + p_\xi) - p (\varepsilon u_r + u_\xi) \right] dr \wedge d\theta \wedge d\varphi + \]

\[ + \left[ u (\varepsilon u_r + u_\xi) - u (\varepsilon p_r + p_\xi) \right] d\theta \wedge d\varphi \wedge d\xi, \]

\[ F \wedge \star dF = \varepsilon \left[ u (\varepsilon u_r + u_\xi) \sin \theta + \frac{p (\varepsilon p_r + p_\xi)}{\sin \theta} \right] dr \wedge d\theta \wedge d\varphi - \]

\[ - \varepsilon \left[ u (\varepsilon u_r + u_\xi) \sin \theta + \frac{p (\varepsilon p_r + p_\xi)}{\sin \theta} \right] d\theta \wedge d\varphi \wedge d\xi, \]

\[ \star F \wedge \star d \star F = \left[ u (\varepsilon u_r + u_\xi) \sin \theta + \frac{p (\varepsilon p_r + p_\xi)}{\sin \theta} \right] dr \wedge d\theta \wedge d\varphi - \]

\[ - \left[ u (\varepsilon u_r + u_\xi) \sin \theta + \frac{p (\varepsilon p_r + p_\xi)}{\sin \theta} \right] d\theta \wedge d\varphi \wedge d\xi. \]

So, the two functions \( u \) and \( p \) have to satisfy the equation

\[ u (\varepsilon u_r + u_\xi) \sin \theta + \frac{p (\varepsilon p_r + p_\xi)}{\sin \theta} = 0, \quad (11) \]

which is equivalent to the equation

\[ \left( u^2 \sin \theta + \frac{p^2}{\sin \theta} \right)_{\xi} + \varepsilon \left( u^2 \sin \theta + \frac{p^2}{\sin \theta} \right)_r = 0. \quad (12) \]

The general solution of this equation is

\[ u^2 \sin \theta + \frac{p^2}{\sin \theta} = \phi^2 (\xi - \varepsilon r, \theta, \phi). \quad (13) \]

For the non-zero components of the energy-momentum tensor we obtain

\[ - Q_1^1 = - Q_4^1 = Q_4^4 = \frac{1}{4\pi r^2 \sin \theta} \left( u^2 \sin \theta + \frac{p^2}{\sin \theta} \right). \quad (14) \]

It is seen that the energy density is not exactly a running wave but when we integrate to get the integral energy, the integrand is exactly a running wave:

\[ W = \frac{1}{4\pi} \int_{R^3 - R^0} \left( Q_4^4 \mu \right) d\xi = \frac{1}{4\pi} \int_{R^3 - R^0} \left( u^2 \sin \theta + \frac{p^2}{\sin \theta} \right) dr \wedge d\theta \wedge d\phi. \]

Since the functions \( u \) and \( p \) are spatially finite, the integral energy \( W \) is finite, and from the explicit form of the energy-momentum tensor it follows the well-known relation between the integral energy and momentum: \( W^2 - c^2 p^2 = 0. \)
References:

1. Donev, S., Tashkova, M., Extended Electrodynamics: II. Properties and Invariant Characteristics of the Non-linear Vacuum Solutions, submitted for publication.