Unified theory in the worldline approach
DCPT-14/57

James P. Edwards
Centre for Particle Theory, Department of Mathematical Sciences,
University of Durham, Durham DH1 3LE, UK

Abstract
We explore the unified theories of $SU(5)$ and $SU(6)$ using the worldline approach for chiral fermions coupled to a background field via a Wilson loop in the fundamental representation of these groups. Representing the path ordering and chiral projection operators with functional integrals has previously been shown to reproduce the chirality and representations present in the standard model. This paper shows that for $SU(5)$ the $\bar{5}$ and $10$ representations – into which the left handed fermionic content of the standard model fits – appear naturally with the familiar chirality. We carry out the same analysis for $SU(6)$ but the result resists compatibility with an embedding of the standard model.

Keywords: Quantum Electrodynamics, Standard Model, Unification, Wilson Loop

1. Introduction
A recent model of chiral fermions in the worldline approach demonstrated how the sum over gauge group representations and chiralities present in the standard model can be generated \cite{1}. This sum was constructed for a single generation of fermions and was supplemented by a sterile neutrino. The worldline formalism can greatly simplify some calculations \cite{2,3} so it is of interest to consider it in the context of the standard model and other unified theories. Furthermore the first quantised model presented in \cite{1} has an underlying string theory \cite{7} which generalises to non-Abelian interactions so it is natural to consider the consequences of using different symmetry groups in that context.

We shall demonstrate that the representations and chiralities of the standard model particles as described by the standard $SU(5)$ unified theory can also be generated in this approach. The next section briefly reviews the argument and notation in \cite{1} and in Section \ref{section} the model is applied to the unified theories of $SU(5)$ and $SU(6)$.

2. Fields and worldlines
We consider a left- or right-handed massless fermion moving in a background gauge field $A$. $A$ transforms in the adjoint representation of some symmetry group which is described by anti-Hermitian Lie algebra generators $\{T_R\}$. In the worldline approach the variation of the effective action, $\delta_A \Gamma [A]$, was given in \cite{1}

\[ \delta_A \ln \int \mathcal{D} (\bar{\xi} \xi) e^{-S[\bar{\xi} \xi]} = - \int_0^\infty \frac{dT}{T} \int \mathcal{D} (w, \psi) e^{-S[w, \psi]} \times \mathcal{P} \text{tr} \left( g (2\pi) \int_0^{2\pi} dt \, \dot{\psi} \cdot \hat{\omega} \psi \cdot \delta A \right). \] (1)

Here $g (t)$ is the super-Wilson line describing the coupling of the fermion to the gauge field

\[ g (t) = \mathcal{P} \exp \left( - \int_0^t A^{RT_R} dt \right) \] (2)
and
\[ A = \dot{\omega} \cdot A + \frac{T}{2} \psi^{\mu} F_{\mu\nu} \psi^{\nu}. \] (3)

The Grassmann variables $\psi^{\mu}$ are introduced to represent the $\gamma$-matrices and the action $S[w, \psi]$ is standard and of no importance for this paper. Their boundary conditions are interpreted depending on the chirality of the fermion. For left handed fermions the path integral with periodic boundary conditions on $\psi$ and $\dot{\psi}$ is subtracted from that with anti-periodic boundary conditions whereas for right handed fermions the two contributions are summed. These combinations insert the appropriate projection operator into the path integral.

The path ordering can be represented with the functional integrals by introducing a set of anti-commuting operators $\tilde{\phi}_r$ and $\tilde{\phi}_s$ satisfying $\{\tilde{\phi}_r, \tilde{\phi}_s\} = \delta_{rs}$ with action $S_{\phi} = \int \tilde{\phi} \cdot \phi \, dt$. It is possible to use this approach to reproduce the path ordered exponential in (1) but we follow [1], motivated by the consideration of interacting tensionless spinning strings [6, 7], to consider as the path ordered exponential in (1) but we follow [1], motivated by the consideration of interacting tensionless spinning strings [6, 7], to consider as

\[ \int_0^{\infty} \frac{dT}{T} \int e^{-S[w, \psi]} \int_0^{2\pi} dt \psi \cdot \dot{\omega} \psi \cdot \frac{\delta Z[A]}{\delta A}. \] (4)

where
\[ Z[A] = \int D(\tilde{\phi}, \phi) e^{-\frac{1}{2} \int_0^{2\pi} \tilde{\phi}(A + \delta A) \phi}. \] (5)

The evaluation of $Z[A]$ was shown in [1] to provide the correct chirality and representation assignments for the standard model. Five pairs of $\tilde{\phi}$ and $\phi$ were used to represent the Lie algebra generators of $SU(3) \times SU(2) \times U(1)$ and the result of evaluating (4) with anti-periodic boundary conditions on all Grassmann variables was added to that with periodic boundary conditions imposed.

3. Unified theory

The assignment used in [1]

\[ (T^R) = i \left( \begin{array}{ccc} \frac{1}{2} \lambda^R \otimes 1_2 & & \\
 & \frac{1}{2} \gamma^a \otimes 1_3 & \\
 & & \frac{1}{2} 1_2 \otimes 1_3 - \frac{1}{3} 1_3 \otimes 1_2 \end{array} \right) \] (6)

incorporating the standard model generators inside a five dimensional algebra is reminiscent of the Georgi-Glashow method of its embedding in $SU(5)$. This motivates the consideration of this group as the underlying symmetry without purposefully arranging for the standard model content to appear. We shall show that with the general use of $SU(5)$ as the gauge group the procedure introduced in [1] yields the familiar $5 \oplus 10 \oplus 1$ representations into which the matter content of the standard model fits. The chirality associated with these representations is also in agreement with the Georgi-Glashow model so that that these representations are favoured by that work.

Integrating over $\phi$ and $\dot{\phi}$ in (4) leads to a determinant which was evaluated in [1]:

\[ Z[A] = \det \left( i \left( \frac{d}{dt} + A \right) \right) \propto \begin{cases} \det \sqrt{g(\sqrt{2})} + 1/\sqrt{g(2\pi)} & A/P \\ \det \sqrt{g(\sqrt{2})} - 1/\sqrt{g(2\pi)} & P \end{cases} \] (7)

where $A/P$ and $P$ refer to anti-periodic and periodic boundary conditions respectively. The string model which underlies (4) requires us to take traceless generators so that $SU(N)$ is a consistent choice of symmetry group. The Lie group valued object $g(2\pi)$ can be rotated onto the Cartan subalgebra

\[ g(2\pi) = \exp(\alpha_i H_i) \quad i = 1 \ldots 4 \] (8)

whereby its eigenvalue equation can be expressed in terms of the weights of the representation under which it transforms. The calculations are essentially straightforward. In particular the Wilson loop is usually considered to be in the fundamental representation $5$, where we have shown that the determinants in (7) can be written as a sum over traces of $g(2\pi)$ in different representations:

\[ \det \left( i \left( \frac{d}{dt} + A \right) \right) \propto \begin{array}{c} \text{tr} (g_5) + \text{tr} (g_{10}) + \text{tr} (g_{\bar{1}0}) + \text{tr} (g_5) \\
+ 2 \text{tr} (g_0) \end{array} \] (9)
for anti-periodic boundary conditions and
\[
\det \left( i \left( \frac{d}{dt} + A \right) \right) \propto \\
\text{tr} \left( g_5 \right) - \text{tr} \left( g_{10} \right) + \text{tr} \left( g_{10}^* \right) - \text{tr} \left( g_5 \right)
\] (10)
when periodic boundary conditions are imposed. In the above equation the subscripts denote the representation in which the trace is to be taken.

The final step is to include with the latter the factor of $\gamma_5$ arising with periodic boundary conditions on $\psi$ and $\bar{\psi}$ and to take the sum. The result is the representations and chirality assignments which are well known in this unified theory:

\[
(\text{tr} \left( g_6 \right) + \text{tr} \left( g_{20} \right) + \text{tr} \left( g_6^* \right)) P_L \\
+ (\text{tr} \left( g_{15} \right) + 2 + \text{tr} \left( g_{15}^* \right)) P_R
\] (11)

The Georgi-Glashow model places a left-handed conjugate down quark colour triplet and a left-handed isospin doublet into the $\bar{5}$ representation. Into the $10$ representation is placed a left-handed colour triplet of conjugate up quarks, a left-handed colour triplet and isospin doublet of up and down quarks and a left-handed conjugate electron. The trivial representations that appear here may be relevant to the discussion of neutrino masses.

3.1. Other unified theories

We apply the same technique to a second unified theory. Those of interest are those into which the standard model can be embedded and recovered after spontaneous symmetry breaking at some unification scale. $SU(6)$ features in the literature and has the property that the $SU(5)$ we have considered above can be embedded into it in a natural way. We now determine the representations and chiralities which appear if $g(2\pi)$ is taken to transform in the $6$ representation.

We find the determinants as follows. For anti-periodic boundary conditions on Grassmann fields
\[
\det \left( i \left( \frac{d}{dt} + A \right) \right) \propto \\
\text{tr} \left( g_6 \right) + \text{tr} \left( g_{15} \right) + \text{tr} \left( g_{20} \right) + \text{tr} \left( g_{15}^* \right) + \text{tr} \left( g_6^* \right) \\
+ 2\text{tr} \left( g_0 \right)
\] (12)
and for periodic boundary conditions
\[
\det \left( i \left( \frac{d}{dt} + A \right) \right) \propto \\
- \text{tr} \left( g_6 \right) + \text{tr} \left( g_{15} \right) - \text{tr} \left( g_{20} \right) + \text{tr} \left( g_{15}^* \right) - \text{tr} \left( g_6^* \right) \\
+ 2\text{tr} \left( g_0 \right)
\] (13)
which are also multiplied by the $\gamma_5$ when the boundary conditions are correlated as described above. Summing the contributions from each set of boundary conditions determines the chiralities selected by this model:

\[
(\text{tr} \left( g_6 \right) + \text{tr} \left( g_{20} \right) + \text{tr} \left( g_6^* \right)) P_L \\
+ (\text{tr} \left( g_{15} \right) + 2 + \text{tr} \left( g_{15}^* \right)) P_R
\] (14)

There have been a few attempts to form a unified theory with gauge group $SU(6)$ [8, 9]. The general approach places the contents of the $\bar{5}$ representation of $SU(5)$ into the $\bar{6}$ of $SU(6)$ along with an exotic fermion, $N$. The $15$ is constructed as the anti-symmetric product $6 \otimes_A 6$ into which fall the remaining standard model particles and conjugate particles to $N$, but in this construction the particles in the $6$ and $\bar{15}$ representations have the same chirality. The result in (14) is inconsistent with this assignment and furthermore the conjugate representations share the same chirality. This presents a major barrier to any placement of the standard model particles into the representations of (14).

4. Concluding remarks

We have demonstrated that the model presented in [1] can be used when the symmetry group is $SU(5)$ and that in that case it provides the low dimensional representations which are used in order to accommodate the standard model matter content. We have chosen the Wilson loop to transform in the fundamental representation for each group – different choices here lead to the appearance of different representations and chiralities.

In combination with the simplifications to some calculations provided by the worldline formulation it would seem valuable to pursue this programme for the standard model and other unified
theories. It is certainly simpler to sum the correlated boundary conditions on the functional integrals than to sum over representations and chiralities by hand and the appearance of \([\text{I}]\) may provide a guiding principle in how the known matter content of the universe can be arranged.

It is a privilege to thank Sam Fearn, Henry Maxfield and Paul Mansfield for useful discussions. The author is also grateful to Paul Mansfield and Thomai Tsiftsi for support, patience and for critiquing the manuscript. This research is supported by STFC via a studentship and in part by the Marie Curie network GATIS (gatis.desy.eu) of the European Union’s Seventh Framework Programme FP7/2007-2013/ under REA Grant Agreement No 317089.

References

[1] P. Mansfield, “The Fermion Content of the Standard Model,” 2014.
[2] C. Schubert, “Perturbative quantum field theory in the string inspired formalism,” Phys. Rept., vol. 355, pp. 73–234, 2001.
[3] F. Bastianelli, O. Corradini, and P. A. G. Pisani, “Worldline approach to quantum field theories on flat manifolds with boundaries,” JHEP, vol. 59, p. 059, 2007.
[4] M. Mondragon, L. Nellen, M. G. Schmidt, and C. Schubert, “Axial couplings on the worldline,” Phys. Lett., vol. B366, pp. 212–219, 1996.
[5] S. Samuel Nucl. Phys., vol. B149, p. 517, 1979.
[6] J. P. Edwards and P. Mansfield, “Delta-function Interactions for the Bosonic and Spinning Strings and the Generation of Abelian Gauge Theory,” 2014.
[7] J. P. Edwards and P. Mansfield, “QED as the tensionless limit of the spinning string with contact interaction,” 2014.
[8] A. Hartanto and L. Handoko, “Grand unified theory based on the SU(6) symmetry,” Phys. Rev., vol. D71, p. 095013, 2005.
[9] M. Fukugita, T. Yanagida, and M. Yoshimura, “\(N\) anti-\(N\) oscillation without left-right symmetry,” Phys. Lett., vol. B109, p. 369, 1982.