Towards a classification of vacuum near-horizons geometries

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Abstract

We prove uniqueness of the near-horizon geometries arising from degenerate Kerr black holes within the collection of nearby vacuum near-horizon geometries.

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(Some figures may appear in colour only in the online journal)

1. Introduction

A lot of effort in general relativity has been put towards classifying all suitably regular stationary solutions of the vacuum equations—see e.g. [1] and references therein. In spite of an impressive body of results, a complete description is still lacking. Indeed, there are unresolved questions concerning analyticity of the solutions, or existence of multi-component solutions, or configurations involving degenerate horizons.

Recall that degenerate Killing horizons are, by definition, those Killing horizons on which the surface gravity vanishes. Such horizons arise in an important class of stationary black holes, whose properties are rather distinct from their non-degenerate counterparts. In particular, in vacuum the metric induced by the space-time metric on the sections of degenerate Killing horizons $H$ satisfies the set of equations

$$R_{AB} - (\mathcal{D}_A \omega_B + \mathcal{D}_B \omega_A + 2 \omega_A \omega_B) = 0,$$

known as the near-horizon-geometry equations, where $\omega_A$ is a suitable field of one-forms on the horizon. The fact that [4, 8] (compare [7, 10])
all axisymmetric solutions \((g_{AB}, \omega_A)\) of (1.1) on a two-dimensional sphere arise from degenerate Kerr metrics (1.2) plays a key role in the proof that all connected degenerate stationary axisymmetric vacuum black holes are Kerr [2] (see also [3, 9]). It is expected that equation (1.2) holds without the axisymmetry assumption. The goal of this work is to establish this in a neighborhood of the Kerr near-horizon geometries. Indeed, we prove the following:

**Theorem 1.1.** Let \( S \) denote the set of pairs \((g_{AB}, \omega_A)\) on \( S^2 \) satisfying (1.1), let \( S_{\text{Kerr}} \subset S \) denote the set of such pairs arising from some Kerr solution. There exists a neighborhood \( \mathcal{U} \) of \( S_{\text{Kerr}} \) in the set of all pairs \((g_{AB}, \omega_A)\) such that

\[ S \cap \mathcal{U} = S_{\text{Kerr}}. \]

The neighborhood \( \mathcal{U} \) can be taken, e.g. in a \( C^2 \) topology, or in a \( L^2 \) topology.

The proof of theorem 1.1 requires controlling the set of zeros of \( \omega \) near the Kerr solution. For this we prove quite generally that, on \( S^2 \), \( \omega \) has exactly two zeros, each of index one. We expect this to be useful for a future complete solution of the problem at hand.

Our result suggests the validity of the following: an analytic, vacuum asymptotically flat, suitably regular space-time with a connected degenerate Killing horizon and with a near-horizon geometry close to Kerr is Kerr. Indeed, one expects that the axi-symmetry of the near-horizon geometry extends to the domain of outer communications by analyticity, so that the usual uniqueness results for connected vacuum black holes [1] apply. However, a proof along these lines would require a careful analysis of isometry-extensions off degenerate horizons, which does not seem to have been carried out so far.

### 2. The Jezierski–Kamiński variables

Let us denote by \((\hat{g}, \hat{\omega})\) the fields describing the near-horizon geometry of an extreme Kerr metric with mass \( m \). Using a coordinate \( x := \cos \theta \), one has [5]:

\[
\hat{g} = \hat{g}_{AB} \, dx^A \, dx^B = 2m^2 \left( a^{-2} \, dx^2 + a^2 \, d\varphi^2 \right),
\]

\[
\hat{\omega}_x = \frac{x}{1 + x^2}, \quad \hat{\omega}_\varphi = \frac{a^2}{1 + x^2},
\]

where \( a^2 := 2 \frac{1 - x^2}{1 + x^2} \).

Invoking the uniformization theorem, metrics on \( S^2 \) will be described by a conformal factor:

\[
g_{AB} = \exp(2U)\hat{g}_{AB}.
\]

There is a ‘gauge freedom’ in \( U \) related to the conformal transformations of \((S^2, \hat{g})\), which can be reduced as follows:

Consider any vacuum near-horizon geometry \((S^2, g, \omega)\). By a constant rescaling of \( g \) we can, and will, require that the total volume of \( g \) equals one. Next, as shown in section 3 below, there exist precisely two points, say \( p \) and \( q \), on which \( \omega \) vanishes. We can, and will, use a conformal transformation of \((S^2, \hat{g})\) to map \( p \) to the north pole and \( q \) to the south pole. This leaves the freedom of rotating \( S^2 \) around the \( z \)-axis, as well as performing a conformal transformation generated by the conformal Killing vector

\[
X := a^2 \partial_x.
\]
Let $E(g, \omega)$ denote the map which assigns the left-hand side of equation (1.1) to a metric $g$ and a one-form field $\omega$. Let $(\delta g, \delta \omega)|P(w)$ denote the linearisation of $E$ at a Kerr solution $(\tilde{g}, \tilde{\omega})$. We note that the kernel of $P$ is non-trivial, as it contains all deformations of the Kerr near-horizon geometry arising from conformal transformations of $S^2$ [6]. We will be particularly interested in

$$\Delta_{\tilde{g}} \omega = a^2 \partial_x (\omega_A dx^A) + 2a \partial_x \omega_A dx + a^2 \partial_\phi \omega_A d\phi.$$

Let us denote by $w = w_A dx^A$ the linearisation of $\omega$ at the Kerr metric, and by $u$ the linearisation of $U$. The Jezierski–Kamiński variables $(\alpha, \beta)$ are defined as [5]

$$\alpha := w_x = \frac{\partial_x}{a^2} w_\phi, \quad \beta := x w_x + \frac{1}{a^2} w_\phi,$$

and it holds that

$$\partial_x u = \frac{1}{2} \left[ w_A + \varepsilon_A^B \nabla_B \alpha + \nabla_A \beta \right],$$

$$w_x = \frac{\alpha + x \beta}{1 + x^2}, \quad w_\phi = a^2 \frac{\beta - x \alpha}{1 + x^2}.$$

Suppose that $\omega$ vanishes at both the south and north poles, and let $w_x = \partial_x (a^2 \omega_x)$ and $w_\phi = a^2 \partial_\phi \omega_\phi$ be the variations of $\omega$ associated with the conformal Killing vector $X$, as in (2.5). At the axes of rotation $x = \pm 1$ the associated functions $\alpha$ and $\beta$ read

$$\alpha = -x \partial_x \omega_\phi, \quad \beta = \partial_\phi \omega_\phi.$$

It follows from (2.2) that $\alpha$ and $\beta$ do not vanish for the Kerr metric on the axes of rotation. This will therefore remain true for all nearby geometries. We conclude that

**Lemma 2.1.** For all near horizon geometries which are $C^1$-close to Kerr, one can find a conformal transformation generated by the conformal Killing vector (2.4) such that $w_x$ vanishes at the north pole.

This fixes the conformal gauge-freedom up to rotations around the $z$-axis. This remaining freedom turns out to be irrelevant for our purposes.

Let

$$\Delta_{\tilde{g}} := \partial_x a^2 \partial_x + \partial_\phi a^{-2} \partial_\phi$$

be the Laplace operator of the metric $\tilde{g}$. Set $v := \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, B := \begin{bmatrix} x & -1 \\ 3 & 3x \end{bmatrix}, C := \begin{bmatrix} 1 & x \\ -3x & 3 \end{bmatrix},$ then the linearised near-horizon geometry equations consist of (2.7) together with

$$\Delta_{\tilde{g}} v + \frac{4a^2}{1 + x^2} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \partial_x \left( a^2 B v \right) + \partial_\phi \left( C v \right) = 0.$$

We are ready now to pass to the

**Proof of theorem 1.1.** Using the above formulation of the problem, we will show in section 4 that the linearised near-horizon geometry operator $P$ has no kernel, once the gauge-freedom inherent in the Jezierski–Kamiński formulation has been factored out.
Suppose, for contradiction, that there exists a sequence of pairwise distinct near-horizon geometries \((g_i, \omega_i)\) on \(S^2\) converging to \((\hat{g}, \hat{\omega})\) as \(i \to \infty\). In the conformal gauge just described all metrics have volume one, the zeros of each \(\omega_i\) are located at the north and south poles, and the associated functions \(\alpha_i\) and \(\beta_i\) vanish at the north pole.

Writing \(\delta g_i = g_i - \hat{g}, \delta \omega_i = \omega_i - \hat{\omega}\), we have

\[
0 = E(g, \omega) - E(\hat{g}, \hat{\omega}) = P(\delta g_i, \delta \omega_i) + O(\|\delta g_i, \delta \omega_i\|^2).
\]

Dividing by \(\|\delta g_i, \delta \omega_i\|\), and passing to a subsequence if necessary (here one can invoke elliptic regularity and the Arzela–Ascoli theorem), one finds that \(P\) has a non-trivial kernel—a contradiction. Theorem 1.1 follows.

### 3. Zeros of \(\omega\)

In this section we show the following:

**Theorem 3.1.** Consider a smooth solution of (1.1) on a two-dimensional sphere \(S^2\). Then \(\omega = \omega_A dx^A\) has exactly two zeros, each of index one.

Here the index of \(\omega\) is understood as that of the associated vector field \(\omega^A \partial_A\) obtained by raising indices with the metric.

**Proof.** The equation to be solved is

\[
\mathcal{D}(\omega_B) + \omega_A \omega_B = \frac{1}{2} R_{AB} = \frac{1}{4} R g_{AB}.
\]

Prolong by introducing the antisymmetric part of \(\mathcal{D}_A \omega_B\):

\[
\mathcal{D}_A \omega_B := F_{AB} = \frac{1}{2} \phi \epsilon_{AB},
\]

where \(\epsilon_{AB}\) is antisymmetric with \(\epsilon_{AB} \epsilon^{AC} = \delta^C_B\) and \(\phi\) is real function global on the sphere. Now the equation can be written as

\[
\mathcal{D}_A \omega_B + \omega_A \omega_B = \frac{1}{2} K g_{AB} + \frac{1}{2} \phi \epsilon_{AB},
\]

where \(K = R/2\) is the Gauss curvature.

Let \(x^A\) be normal coordinates centered at a point \(p\) where \(\omega\) vanishes, oriented consistently with the orientation of \(M\). We have

\[
|\omega|^2 = (-\frac{1}{2} \phi \epsilon_{AB} + \frac{K}{2} g_{AB})(-\frac{1}{2} \phi \epsilon_{AC} + \frac{K}{2} g_{AC}) x^B x^C + O(|x|^3)
\]

\[
= \frac{1}{4} (\phi(p)^2 + K(p)^2) |x|^2 + O(|x|^3).
\]

If \(f(p)^2 + K(p)^2 \neq 0\), we see that the zero is isolated. Next, the determinant of \(\mathcal{D}_A \omega_B(p)\), which we will denote by det, equals \(\frac{1}{4} (\phi(p)^2 + K(p)^2)\). When det does not vanish, the index of \(\omega\) at \(p\) equals the sign of det, hence one.

Commute derivatives on (3.2) to obtain...
\begin{equation}
\mathcal{D}_A \phi + 3 \omega_A \phi = e_B^A (\nabla_B K + 3 \omega_B K). \tag{3.4}
\end{equation}

By the uniformisation theorem there is a globally defined, smooth function \( u \) (not to be confused with the function \( u \) of equation (2.7)) and complex coordinate \( \zeta \) with
\begin{equation}
g_{AB} dx^A dx^B = 4 e^{2u} \frac{d\zeta d\bar{\zeta}}{P_0^2}, \tag{3.5}
\end{equation}

with \( P_0 = 1 + \zeta \bar{\zeta} \) and \( \zeta = \tan \frac{\theta}{2} e^{i\varphi} \).

For later convenience introduce \( h_{AB}^0 \) for the unit sphere metric, so that
\begin{equation}
h_{AB}^0 dx^A dx^B = 4 \frac{d\zeta d\bar{\zeta}}{P_0^2}, \tag{3.6}
\end{equation}

which is \( d\theta^2 + \sin^2 \theta d\varphi^2 \), and then \( g_{AB} = e^{2u} h_{AB}^0 \).

Introduce the null vector \( m^A \) and operator \( \delta \) by
\begin{equation}
\delta = m^A \partial_A = \frac{1}{\sqrt{2}} P_0 e^{-u} \frac{\partial}{\partial \zeta}, \tag{3.7}
\end{equation}

so that also
\begin{equation}
m_A dx^A = \frac{\sqrt{2} \rho^A}{P_0} d\bar{\zeta} \quad \text{and} \quad g_{AB} = m_A \bar{m}_B + m_B \bar{m}_A,
\end{equation}

so also
\begin{equation}
\epsilon_{AB} = i (m_A \bar{m}_B - m_B \bar{m}_A).
\end{equation}

We need the Christoffel symbols; we will get them indirectly. Write \( \delta := m^A \mathcal{D}_A \) for covariant derivative. Since \( m^A \) is null, there must be complex \( \alpha, \beta \) (not to be confused with the \( \alpha, \beta \) variables of Jezierski and Kamiński) with
\begin{equation}
\delta m^A = \alpha m^A, \quad \bar{\delta} m^A = \beta m^A,
\end{equation}

whence, by complex conjugation, also
\begin{equation}
\bar{\delta} \bar{m}^A = \bar{\alpha} \bar{m}^A, \quad \delta \bar{m}^A = \bar{\beta} \bar{m}^A.
\end{equation}

Then, since \( m_A m^A = 1 \), we deduce \( \alpha + \bar{\beta} = 0 \) (just calculate \( \delta (m_A \bar{m}^A) \)). Finally we can calculate the commutator
\begin{equation}
[\delta, \bar{\delta}] = \bar{\beta} \bar{\delta} - \beta \delta
\end{equation}

and substitute from (3.7) to deduce that
\begin{equation}
\bar{\beta} = -\alpha = \frac{\partial}{\partial \zeta} \left( \frac{1}{\sqrt{2}} P_0 e^{-u} \right). \tag{3.8}
\end{equation}

Now we have the connection coefficients explicitly.

We proceed by expanding \( \omega_A \) in the basis:
for complex function $\chi = \omega_A m^A$, and then projecting (3.2) along the basis. Contracting with $m^A m^B$ we obtain

$$ (\delta - \alpha) \chi + \chi^2 = 0. $$

(3.10)

Then with $\bar{m} A m B$ to obtain

$$ (\bar{\delta} - \beta) \chi + \bar{\chi} \chi = \frac{1}{2} (\bar{K} + i \phi). $$

(3.11)

The one-form $\omega_A$ can be written as a sum of exact and co-exact terms:

$$ \omega := \omega_A dx^A = dv - * dw, $$

where $v, w$ are real-valued, smooth functions on the sphere, unique up to additive constants, and we are using the convention that $(* dw)_A = \epsilon_A^B w_B$. Then contraction with $m^A$ gives

$$ \chi = \delta (v + iw) = \delta \psi = \frac{1}{\sqrt{2}} P_0 e^{-u} \frac{\partial \psi}{\partial \zeta}, $$

(3.12)

where we have introduced the smooth complex function $\psi = v + iw$, which is unique up to additive complex constant. Using (3.8) rewrite (3.10) as

$$ \frac{\partial}{\partial \zeta} \left( \frac{1}{\sqrt{2}} P_0 e^{-u} \chi \right) + \chi^2 = 0, $$

and substitute for $\chi$ in terms of $\psi$ to obtain

$$ \frac{\partial}{\partial \zeta} \left( \frac{1}{2} P_0^2 e^{-2u+\psi} \frac{\partial \psi}{\partial \zeta} \right) = 0. $$

This can be integrated in terms of an (at present) arbitrary antiholomorphic function $f(\bar{\zeta})$ as

$$ \frac{1}{2} P_0^2 e^{-2u+\psi} \frac{\partial \psi}{\partial \zeta} = f(\bar{\zeta}), $$

whence

$$ \chi = \frac{1}{\sqrt{2}} P_0 e^{-u} \frac{\partial \psi}{\partial \zeta} = \frac{\sqrt{\gamma}}{P_0} e^{u-\psi} f(\bar{\zeta}). $$

(3.13)

We need to constrain $f$. Note that

$$ \omega_A \omega^A = 2 \chi \bar{\chi} = \frac{4}{P_0} e^{2u-\psi} f f. $$

Since the one-form $\omega_A$ is smooth, it follows that $f$ cannot have singularities in the complex plane of $\zeta$ and must therefore be entire. To see what happens at $\zeta = \infty$ (so to speak) introduce $\eta = -\zeta^{-1}$ and consider

$$ \omega = \chi m + \bar{\chi} m = 2 e^{3u} \left( \frac{e^{-\psi} f(\bar{\zeta}) d\zeta}{(1+\zeta \bar{\zeta})^2} + \text{c.c.} \right). $$


We have
\[
\frac{f(\zeta)d\zeta}{(1 + \zeta \bar{\zeta})^2} = \frac{\eta^2 f(-\bar{\eta}^{-1})d\eta}{(1 + \eta \bar{\eta})^2},
\]
and for boundedness at \(\zeta = \infty\), which is \(\eta = 0\), we need \(\eta^2 f(-\bar{\eta}^{-1})\) bounded there. By an application of Liouville’s Theorem this forces \(f\) to be a quadratic polynomial. The roots of the quadratic may be distinct or repeated, and they can be moved about by Möbius transformation, so that w.l.o.g. there are just two cases to consider:

1. \(f = C\zeta\);
2. \(f = C\zeta^2\);

for complex constant \(C\).

The next move is to rule out the second case for \(f\).

Go back to (3.4) and contract with \(m^A\) to obtain
\[
(\delta + 3\chi)(\phi - iK) = 0.
\]
Substitute for \(\chi\) from (3.12) and multiply by \(i\) to obtain
\[
\delta((K + i\phi)e^{3\psi}) = 0.
\]
Integrate recalling (3.7) to obtain
\[
(K + i\phi)e^{3\psi} = g(\zeta),
\]
for \(g\) holomorphic in \(\zeta\). This time, the left-hand side is globally defined on the sphere so that \(g\) is a bounded holomorphic function and is therefore a constant, say \(C_1\). Thus
\[
K + i\phi = C_1 e^{-3\psi}.
\]
(3.14)

Since \(\psi\) is globally defined, (3.14) shows that \(K + i\phi\) is everywhere nonzero (if it had a zero then \(C_1 = 0\) so \(K = 0\) everywhere, and we could not be on the sphere), which will give a contradiction to case 2, as we see next.

Go back to (3.11) and substitute for \(\chi\) from (3.13). It is clear that, in case 2, all terms on the left vanish at \(\zeta = 0\), therefore so does \(K + i\phi\): contradiction! Thus \(f = C\zeta\)

and
\[
\omega_A \nu_A = \frac{4}{P_0} e^{2u - \psi - \bar{\psi}} |C|^2 \zeta \bar{\zeta}.
\]
This vanishes only at \(\zeta = 0\) in the finite \(\zeta\)-plane and, as the substitution \(\eta = -\zeta^{-1}\) shows, at \(\eta = 0\) (equivalently, \(\zeta = \infty\)—two isolated simple zeroes which Möbius transformation places at north and south poles.

We end this section by noting that it follows from equation (1.1), together with standard facts about systems of elliptic equations, that all the fields are real analytic in harmonic...
coordinates, regardless of the topology of the underlying manifold. This is already clear in any case on $S^2$ from the analysis above.

4. The kernel of $P$

It is shown in [5] that elements of the kernel of $P$ are in one-to-one correspondence with solutions of the following system of ODEs on $[-1, 1]$ for a sequence of complex functions $(\alpha_k, \beta_k)$, $k \in \mathbb{N}$:

$$\partial_x (a^2 \partial_x v_k) - \frac{k^2}{a^2} v_k + \frac{4a^2}{1 + x^2} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} v_k + \partial_x (a^2 B v_k) + ik(C v_k) = 0,$$

where

$$v_k = \begin{bmatrix} \alpha_k \\ \beta_k \end{bmatrix}.$$

The parameter $k$ denotes the $k$th coefficient of $(\alpha, \beta)$ in a Fourier series decomposition with respect to the azimuthal angle $\varphi$ on $S^2$. It has been proved in [5] that for $k \geq 8$ the equation (4.1) does not have solutions other than $\alpha_k(x) \equiv 0 \equiv \beta_k(x)$, once the relevant boundary conditions, as discussed in the next section, have been imposed. To complete the proof it remains to prove non-existence of non-zero solutions for the Fourier modes $0 < k < 8$, and to analyze the solutions with $k = 0$.

4.1. Boundary conditions

Near the north pole $\cos \theta = 1$ introduce coordinates $(x^A) = (x^1, x^2)$ defined as

$$x^1 = \rho \cos \varphi, \quad x^2 = \rho \sin \varphi,$$

with $d\rho^2 = a^{-2} dx^2$, $\rho \approx \sqrt{2(1-x)}$ for small $\rho$. (4.2)

The Kerr near-horizon metric is analytic in these coordinates. Analyticity of solutions of systems of elliptic equations implies that $U$ and $w$ are analytic in these coordinates. A similar construction applies near the south pole.

We have

$$w_A d x^A = w_A d (\rho \cos \varphi) + w_B d (\rho \sin \varphi) = \frac{w_A x^A}{\rho} d \rho + \epsilon_{AB} x^A w_B d \varphi,$$

where $\epsilon_{AB} \in \{0, \pm 1\}$ is totally antisymmetric, and where a sum over $B$ is understood. This gives

$$\alpha = \frac{w_A x^A}{a \rho} - \frac{x \epsilon_{AB} x^A w_B}{a^2}, \quad \beta = \frac{w_A x^A}{a \rho} + \frac{x \epsilon_{AB} x^A w_B}{a^2}.$$ (4.4)

Since $a^2$ behaves as $2(1-x)$ i.e. as $\rho^2$ near $\rho = 0$, a rough estimate gives $\alpha, \beta = O(\rho^{-1})$ there. However, more can be said if we use a gauge in which $w = 0$ at the north and south poles, as can always be done, and which we will assume from now on. It follows from equation (3.2) that

$$w_B = \frac{1}{2} (\delta \varphi(0) \epsilon_{AB} + \delta K(0) g_{AB}) x^A + O(|x|^3),$$ (4.5)
where $\delta \phi$ and $\delta K$ are the linearised changes of $\phi$ and $M$ associated with the linearised solution. This gives, for small $\rho$,

$$
\alpha = \frac{\delta K(0)\rho + O(\rho^2)}{2a} - \frac{\delta \phi(0)\rho^2 + O(\rho^3)}{2a^2}, \quad \beta = \frac{\delta K(0)\rho + O(\rho^2)}{2a} + \frac{\delta \phi(0)\rho^2 + O(\rho^3)}{2a^2}.
$$

(4.6)

Let $\alpha_k$ and $\beta_k$ denote the $k$th Fourier component of $\alpha$ and $\beta$ in a Fourier series with respect to $\phi$. It follows from (4.6) that

$$
\alpha_0(0) = \delta K(0) - \delta \phi(0), \quad \beta_0(0) = \frac{\delta K(0) + \delta \phi(0)}{2}.
$$

(4.7)

However, Lemma 2.1 shows that we can find a conformal gauge in which $\alpha_0$ and $\beta_0$ vanish at the north pole.

It further follows from what has been said that for $k \geq 1$ we have

$$
\alpha_k(0) = 0 = \beta_k(0).
$$

(4.8)

4.2. $k = 0$

All solutions of (4.1) with $k = 0$ are found by MAPLE without need of any manipulations of the equations. One obtains

$$
\alpha_0(x)(x^2 + 1)^2 = C_1 (x - 1)(x + 1) + C_2 (x - 1)(x + 1)[x + \ln(x - 1) - \ln(x + 1)]
$$

$$
+ C_3 \left\{ [\ln(x - 1) + \ln(x + 1)](x^2 - 1) - 2 \right\}
$$

$$
+ C_4 \left[ \text{Li}_2 \left( \frac{x + 1}{2} \right) (x^2 - 1) + \hat{\alpha}(x) \right],
$$

where $\text{Li}_2(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^2}$ is the dilogarithm, and where $\hat{\alpha}(x)$ is a lengthy explicit polynomial in $\ln(x - 1), \ln(x + 1)$ and $x$. Analyticity of $w_A dx^4$ implies that no logarithms or dilogarithms can occur in the solution, hence $C_2 = C_3 = C_4 = 0$. Alternatively, a careful analysis of the behaviour of $\alpha$ at $x = \pm 1$ together with the requirement of boundedness leads to the same conclusion. This further results in

$$
\beta_0(x) = -C_1 \frac{x \left( x^2 - 5 \right)}{(x^2 + 1)^2}.
$$

Translating into $w_A dx^4$, one obtains

$$
w_x = \frac{C_1 (x^3 - 6x^2 + 1)}{(x^2 + 1)^3}, \quad w_{\phi} = -\frac{a^2 C_1 x (x^5 + x^2 - 6)}{(x^2 + 1)^3}.
$$

Imposing the conformal gauge of Lemma 2.1 we find $C_1 = 0$, hence $\alpha_0 \equiv 0 \equiv \beta_0$.

4.3. $1 \leq k \leq 7$

When $k \geq 1$, MAPLE finds two explicit linearly independent solutions of (4.1), the sum of which we denote by $\hat{\alpha}_k$: 

\[
\dot{\alpha}_k(x)(x + i)^2(x - i)^2 = [(x^2 + 1)^2 - 2i(x^2 - 1)/k] \\
\times \left[ C_1 \left( \frac{x + 1}{x - 1} e^{-x} \right)^{\frac{1}{2}} + C_2 \left( \frac{x + 1}{x - 1} e^{-x} \right)^{-\frac{1}{2}} \right], \tag{4.9}
\]

where \( C_1 \) and \( C_2 \) are arbitrary complex constants.

Now, there is a standard way of obtaining from (4.1) two fourth order decoupled ODEs for \( \alpha_k \) and \( \beta_k \). Next, there is a standard way of obtaining a lower-order ODE when a solution is known. All this allows one to obtain

\[
\alpha_k(x) = \dot{\alpha}_k(x) + W_k(x),
\]

where \( W_k \) solves the following second-order equation

\[
(x - 1)^2 (x + 1)^2 p_2(x, k) W_k''(x) + (x + 1)(x - 1) p_1(x, k) W_k'(x) + p_0(x, k) W_k(x) = 0, \tag{4.10}
\]

and where the polynomials \( p_i(x, k) \) read

\[
p_2(x, k) = 8 \left( x^2 + 1 \right) \left[ 1/2 k^2 x^8 + k (i + 2 k) x^6 + (i k + 3 k^2 - 8) x^4 + (-i k - 2 k^2 + 16) x^2 - i k + 1/2 k^2 - 8 \right] ,
\]

\[
p_1(x, k) = 16 \left[ k (i + k) x^8 + (4 k^2 - 16) x^6 + (-2 i k + 6 k^2 + 64) x^4 + (4 k^2 - 80) x^2 + i k + k^2 + 32 \right] x ,
\]

\[
p_0(x, k) = -2 \left[ 1/2 k^4 x^{14} + k^3 (i + 7/2 k) x^{12} + 4 k \left( i k^2 + 21 k^3 / 8 - 6 i - 11 k \right) x^{10} + \left( -68 k^2 + 64 + 5 i k^3 + 35 k^4 / 2 - 40 i k \right) x^8 + \left( 40 k^2 - 128 + 35 k^4 / 2 + 240 i k \right) x^6 + \left( 88 k^2 + 256 - 5 i k^3 + 21/2 k^4 + 16 i k \right) x^4 + (4 k^2 - 384 - 4 i k^3 + 7/2 k^4 - 216 i k) x^2 - (k^2 - 24) (i k - 1/2 k^2 + 8) \right] .
\]

Equation (4.10) is a Fuchsian equation with indicial exponents \( \pm i k / 2 \) both at \( x = 1 \) and at \( x = -1 \). Near \( x = \varepsilon \in \{ \pm 1 \} \) the solutions \( W_k(x) \) have therefore expansions of the form

\[
W_k(x) = c_{1,\varepsilon,k}(x - \varepsilon)^{-i k / 2} (P_{k,\varepsilon}(x) + (x - \varepsilon)^{i k / 2} D_k \ln(x - \varepsilon)) + c_{2,\varepsilon,k}(x - \varepsilon)^{i k / 2} + o(x - 1)^{i k / 2} ,
\]

where \( c_{1,\varepsilon,k} \) and \( c_{2,\varepsilon,k} \) are complex constants, \( P_{k,\varepsilon} \) are polynomials, and the \( D_k \) values can be calculated using Maple:

\[
\{ D_k \}_{k=1}^8 = \left\{ -\frac{3}{2} (-1 + i) , -3 , \frac{27}{4} (3 + i) , -36 (7 + 5 i) , \frac{225}{8} (149 + 207 i) , -243 (225 + 952 i) , \frac{1233}{16} (-336 23 + 127 151 i) , -1728 (-302 743 + 270 849 i) \right\}.
\]

Since logarithmic terms in \( \alpha_k \) are forbidden by the analyticity properties of \( w_A dx^A \), we conclude that \( c_{1,\varepsilon,k} \) has to vanish for admissible solutions. Thus
This, together with the requirement that $\alpha_k$ vanish at $x = \pm 1$ implies that the constants $C_1$ and $C_2$ in equation (4.9) vanish, leading to

$$\alpha_k(x) = W_k(x),$$

with $W_k$ behaving near $x = \varepsilon$ as in (4.11).

To finish the proof, it remains to show that the only solution $W_k$ which is regular at both the north and south poles is zero. Assume that this is not the case. Since the problem is linear, there exists a solution which is regular at $x = -1$ (thus $c_{1,-1,k} = 0$) with $c_{2,-1,k} = 1$. We have solved (4.10) numerically, using Maple$^4$, under these conditions,

$$W_k(x) = (x + 1)^{k/2} + o((x + 1)^{k/2}),$$

where $o$ is meant for $x$ near $-1$. Because the equation is singular at $x = -1$, the numerical solutions have been found by calculating from the equation the first terms in a power series for $W_k(x)$ (up to order eight, we return to this below), and starting the numerical solution at $x \geq -1$. We then estimated the limit

$$\lim_{x \to 1} 2^k(1-x)^{k/2}W_k(x)$$

by stopping the calculation close to $x = 1$. The solutions are plotted in figure 1.
It is clear from the figure that all solutions satisfying (4.12) blow up at \( x = 1 \) as \((1 - x)^{-k/2}\), and thus do not satisfy (4.11): Indeed, for these solutions we find the following numerical estimates

\[
\{2^k c_{1,1,k}\}^k_{k=1} = \{0.3293 - 1.4994i, 0.6587 - 1.1239i, 1.3136 - 1.0047i, 2.3401 - 0.9939i, 3.9057 - 1.0507i, 6.2775 - 1.1644i, 9.8575 - 1.3368i, 14.1138 - 1.4603i\}.
\]

Hence \( \alpha_k \equiv 0 \) for \( k \in \{1, \ldots, 7\} \) as well, and thus for all \( k \in \mathbb{N} \).

The equality \( \beta_k \equiv 0 \) follows directly from this, using

\[
\beta_k(x) = 2i \left[ (-x^4k + 2(4i - k)x^2 - 8i - k)(x^2 - 1)(x^2 + 1)W_k(x) + (kx^6 + (8i + 9k)x^4 - 16i + 15k)x^2 + 8i + 7k)xW_k(x) \right] \\
\times \left[ -16x^4 + (x^2 + 1)^4k^2 + 32x^2 + 2i(x + 1)(x^2 + 1)^2(x - 1) - 16 \right]^{-1}.
\]

In order to check the convergence of \((1 - x)^{k/2}W_k(x)\) as \( x \) tends to one, we calculated the values of \( 2^k 10^{-m/2} |W_k(1 - 10^{-m})| \), for \( m = 1, \ldots, 15 \). The results for \( k = 1, \ldots, 7 \) are shown in figure 2, as calculated using an expansion near \( x = -1 \) to order eight.

Yet another test of the reliability of the results is provided by comparing the values obtained after varying the order of the expansion near \( x = -1 \). The numerical estimates of \( |2^k c_{1,1,k}| \) with \( k = 1, \ldots, 7 \), calculated using a starting value obtained from an expansion of the solution at \( x = -1 + 10^{-m} \) for \( m = 1, \ldots, 15 \) and truncated at order \( l = 2, 3, 4, 5 \), are compared to \( |2^k c_{1,1,k}| \) truncated at order \( l = 8 \) in figure 3, where \( \Delta = ||2^k c_{1,1,k}| - |2^k c_{1,1,k}|_{l=8}|| \). The points where \( \Delta = 0 \) are omitted.
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