A GENERALIZATION OF THE BLASCHKE-LEBESGUE PROBLEM TO A KIND OF CONVEX DOMAINS

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Abstract. In this paper we will introduce for a convex domain $K$ in the Euclidean plane a function $\Omega_n(K, \theta)$ which is called by us the biwidth of $K$, and then try to find out the least area convex domain with constant biwidth $\Lambda$ among all convex domains with the same constant biwidth. When $n$ is an odd integer, it is proved that our problem is just that of Blaschke-Lebesgue, and when $n$ is an even number, we give a lower bound of the area of such constant biwidth domains.

1. Introduction. If the distance between parallel supporting planes of a convex body in $\mathbb{R}^d$ is always the same, then the convex body is said to be of constant width. For $d = 2$ such convex bodies are often called orbiforms, and for $d = 3$ they are called spheriforms. A well-known example of these bodies is the Reuleaux triangle (see Figure 2(A) below). It has long been known that among all two-dimensional convex bodies of constant width $w$, the Reuleaux triangle with width equal to $w$ has the least area. This fact, to the best of our knowledge, was first proved by W. Blaschke ([5]) and H. Lebesgue ([18, 19]), and since then there have been several proofs of this result (see, for example, [7, 9, 11, 13, 14, 16, 20]). However, the Blaschke-Lebesgue problem in dimension $d \geq 3$ appears to be very difficult to solve and remains open. For partial results, see Anciaux and Georgiou [1], Anciaux and Guilloyle [2] and Bayen, Lachand-Robert and Oudet [4] and the literature therein. There are many researches on various variations of the Blaschke-Lebesgue problem (see [3, 8, 11, 12, 23]). In [21], in order to get a generalization of the Chernoff inequality ([10]), K. Ou and the first author of the present paper introduced for convex domains in the plane a function

$$w_k(\theta) = p(\theta) + p \left( \theta + \frac{2\pi}{k} \right) + \cdots + p \left( \theta + \frac{2(k-1)\pi}{k} \right)$$

to generalize the usual width function, where $p(\theta)$ is the Minkowski support function. They also posed a Blaschke-Lebesgue-style problem which motivates us to perform the research of the present paper.

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In this paper we will first introduce a function (which is called by us the biwidth function of $K$) in association with a convex domain $K$ in the Euclidean plane

$$\Omega_n(K, \theta) = w(K, \theta) + w\left(K, \theta + \frac{\pi}{n}\right),$$

where $w(K, \theta)$ is the usual width function of $K$, and then consider the similar Blaschke-Lebesgue problem for convex domains with $\Omega_n(K, \theta)$ being constant. In section 2, we will give some basic preliminaries for convex domains in the Euclidean plane. In section 3, we will introduce the function $\Omega_n(K, \theta)$, give the Fourier series expansions of the support functions of $K$ and $DK$, and construct the regular Reuleaux type polygon with even sides, denote it by $R_{2m}$ $\left(m \in \mathbb{N}\right)$. In section 4, by the properties of the mixed area, we will prove that when $n$ is an odd integer, the Reuleaux triangle has the least area among $K$, when $n$ is an even number, the $R_{2m}$ has the least area among $DK$ and give a new lower bound of the area of $K$.

2. Preliminaries.

2.1. Support function and width function of convex domains. A convex body in the Euclidean plane is usual called a convex domain. Let $K$ be a bounded convex domain, its support function is a certain real continuous function and defined by

$$h(K, u) = \max_{k \in K} \{k \cdot u \mid u \in S^1\},$$

and the width function of $K$ in direction $u$ is defined by

$$w(K, u) = h(K, u) + h(K, -u).$$

For $u = e^{i\theta}$, we can write $h(K, \theta)$ instead of $h(K, u)$ and $w(K, \theta)$ instead of $w(K, u)$. The supporting line of $K$ in direction $u$, $H(K, u)$, is defined by $H(K, u) = \{x \in \mathbb{R}^2 \mid x \cdot u = h(K, u)\}$.

![Figure 1. The support and width function of a convex domain](image-url)
direction $u$, of $K$ (see Figure 1(B)). $h(K, \theta)$ and $w(K, \theta)$ are both periodic functions with periods $2\pi$ and $\pi$, respectively. It is also easy to see that

$$w(K, \theta) = h(K, \theta) + h(K, \theta + \pi).$$

Denote by $L(\cdot)$ the perimeter of a bounded convex domain, then

$$L(K) = \int_0^{2\pi} h(K, \theta) \, d\theta,$$

which is known as Cauchy’s formula.

In order to establish most of the results contained in this paper, we need some definitions and known facts about mixed area and some other related notions associated to convex domains. Denote $A(K)$ the area of $K$, then one can define the mixed area of two convex domains.

Given two bounded convex domains $K_1$, $K_2$, their Minkowski sum is

$$K_1 + K_2 = \{k_1 + k_2 \mid k_1 \in K_1, k_2 \in K_2\}.$$

More generally, for any $t_1, t_2 \in \mathbb{R}$, the Minkowski combination is defined as

$$t_1 K_1 + t_2 K_2 = \{t_1 k_1 + t_2 k_2 \mid k_1 \in K_1, k_2 \in K_2\}.$$

**Definition 2.1.** If $K_1$, $K_2$ are bounded convex domains, then their mixed area is defined by

$$A(K_1, K_2) = \frac{1}{2} [A(K_1 + K_2) - A(K_1) - A(K_2)].$$

Since the mixed area of convex domains has been well-studied (see, for instance, [6, 15, 22, 24]), we state some propositions without proofs for later use.

**Proposition 2.2.** If $K_1, K_2, K_3$ and $K$ are convex domains, $g = e^{i\varphi}$ is a rotation, $t_1, t_2$ and $t$ are real numbers, then one gets

1. $A(K_1, K_2) = A(K_2, K_1)$,
2. $A(K, K) = A(K)$,
3. $A(gK) = A(K)$,
4. $A(gK_1, gK_2) = A(K_1, K_2)$,
5. $A(tK_1, K_2) = tA(K_1, K_2) = A(K_1, tK_2)$ for all $t \geq 0$,
6. If $K_1 \subseteq K_2$, then $A(K_1, K_3) \leq A(K_2, K_3)$ and the same with the second variable,
7. $A(K_1 + K_2, K_3) = A(K_1, K_3) + A(K_2, K_3)$,
8. $A(t_1 K_1 + t_2 K_2) = t_1^2 A(K_1) + 2t_1 t_2 A(K_1, K_2) + t_2^2 A(K_2), \quad t_1 \cdot t_2 \geq 0$.

**Proposition 2.3.** If $K$ is a convex domain, $g = e^{i\varphi}$ is a rotation, then

$$h(gK, u) = h(K, g^{-1} u), \quad \text{and}$$

$$w(gK, u) = w(K, g^{-1} u). \quad (2)$$

**Definition 2.4.** If $K$ is a convex domain, then the difference domain of $K$ is defined by

$$DK = K + (-K).$$

**Lemma 2.5.** If $K$ is a convex domain, $DK$ is the difference domain of $K$, then

$$8A(K) = 2(A(DK) + A(K + gK) + A(K - gK)) - A(DK + gDK). \quad (3)$$
Proof. From Proposition 2.2 and Proposition 2.3, one can easily obtain
\[
2(A(DK) + A(K + gK) + A(K - gK)) - A(DK + gDK)
\]
\[
= 2(2A(K) + 2A(K, -K) + 2A(K, gK) + 2A(K) + 2A(K, -gK))
- 2A(DK) - 2A(K - K, gK - gK)
\]
\[
= 2(6A(K) + 2A(K, -K) + 2A(K, gK) + 2A(K) + 2A(K) + 2A(K, -gK))
- 4A(K) - 4A(K, -K) - 4A(K, gK) - 4A(K, -gK)
\]
\[
= 8A(K).
\]

3. Convex domains with constant biwidth.

3.1. Construction of convex domain with constant biwidth. In this section, we will construct a class of convex domains with constant biwidth.

Definition 3.1. Let \( K \) be a convex domain in the plane. We call the following \( \Omega_n(K, \theta) \) the biwidth function of \( K \):
\[
\Omega_n(K, \theta) = w(K, \theta) + w(K, \theta + \frac{\pi}{n}), \quad n \in \mathbb{N},
\]
where \( w(K, \theta) \) is the width function of \( K \).

It is easy to see that \( \Omega_n(K, \theta) \) is a periodic function in \( \theta \) with period \( \pi \). We are more interested in the case when \( \Omega_n(K, \theta) \) is a constant. We call a convex domain \( K \) satisfying \( \Omega_n(K, \theta) = \Lambda \) (a constant) a convex domain with constant biwidth, and denote
\[
\mathcal{C}_n(\Lambda) = \{ K | \Omega_n(K, \theta) = \Lambda \}.
\]

Theorem 3.2. If \( K \in \mathcal{C}_n(\Lambda) \), then \( DK \in \mathcal{C}_n(2\Lambda) \).

Proof. By Definition 3.1, we obtain
\[
\Omega_n(DK, \theta) = w(DK, \theta) + w(DK, \theta + \frac{\pi}{n})
\]
\[
= w(K, \theta) + w(-K, \theta) + w(K, \theta + \frac{\pi}{n}) + w(-K, \theta + \frac{\pi}{n})
\]
\[
= w(K, \theta) + w(K, \theta + \pi) + w(K, \theta + \frac{\pi}{n}) + w(K, \theta + \frac{\pi}{n})
\]
\[
= w(K, \theta) + w(K, \theta) + w(K, \theta + \frac{\pi}{n}) + w(K, \theta + \frac{\pi}{n}),
\]
which, together with the fact that \( K \in \mathcal{C}_n(\Lambda) \), gives us \( \Omega_n(DK, \theta) = 2\Lambda \), i.e., \( DK \in \mathcal{C}_n(2\Lambda) \). 

Theorem 3.3. Let \( K \in \mathcal{C}_n(\Lambda) \), then \( L(K) = \frac{\pi\Lambda}{2} \) and \( L(DK) = \pi\Lambda \).

Proof. From Cauchy’s formula it follows that
\[
L(DK) = \int_0^{2\pi} h(DK, \theta) d\theta = \int_0^{2\pi} w(K, \theta) d\theta
\]
\[
= \frac{1}{2} \int_0^{2\pi} (w(K, \theta) + w(K, \theta + \pi/n)) d\theta = \pi\Lambda,
\]
and then \( L(K) = \frac{\pi\Lambda}{2} \). 

Theorem 3.3 is a Barbier-style result which tells us that all convex domains with constant biwidth have the same length. Among all domains with the same length $L$, the classical isoperimetric inequality implies that the circular disc with circumference $L$ has the greatest area. For convex domains of constant width $w_0$, the elegant Blaschke-Lebesgue theorem claims that the Reuleaux triangle with diameter $w_0$ has the least area. Now, we consider the Blaschke-Lebesgue-Style problem for convex domains with constant biwidth, that is, which member of the set $C_n(\Lambda) = \{K|\Omega_n(K,\theta) = \Lambda\}$ has the least area?

3.2. Fourier series expansion. Since the support function of a given convex domain is always continuous, bounded and $2\pi$-periodic, it can be expressed in the Fourier series form

$$h(K,\theta) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} (a_k \cos k\theta + b_k \sin k\theta). \quad (4)$$

where

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} h(K,\theta) \, d\theta$$

and

$$a_k = \frac{1}{\pi} \int_0^{2\pi} h(K,\theta) \cos k\theta \, d\theta, \quad b_k = \frac{1}{\pi} \int_0^{2\pi} h(K,\theta) \sin k\theta \, d\theta, \quad k = 1, 2, \cdots.$$

The width function of $K$ can be written as

$$w(K,\theta) = h(K,\theta) + h(K,\theta + \pi)$$

$$= \frac{1}{2}a_0 + \sum_{k=1}^{\infty} (a_k \cos k\theta + b_k \sin k\theta) + \frac{1}{2}a_0 + \sum_{k=1}^{\infty} [a_k \cos (\theta + \pi) + b_k \sin (\theta + \pi)]$$

$$= a_0 + 2 \sum_{k=1}^{\infty} (a_{2k} \cos 2k\theta + b_{2k} \sin 2k\theta),$$

and thus

$$w\left(K,\theta + \frac{\pi}{n}\right) = a_0 + 2 \sum_{k=1}^{\infty} \left[a_{2k} \cos \frac{2k\theta}{n} + b_{2k} \sin \frac{2k\theta}{n}\right].$$

Hence

$$\Lambda = \Omega_n(\Lambda,\theta) = w(K,\theta) + w\left(K,\theta + \frac{\pi}{n}\right)$$

$$= 2a_0 + 2 \sum_{k=1}^{\infty} \left[a_{2k} \cos \frac{2k\theta}{n} + b_{2k} \sin \frac{2k\theta}{n}\right].$$

From Theorem 3.3, we get $a_0 = \frac{L}{2}$. Expanding $\cos 2k(\theta + \frac{\pi}{n})$ and $\sin 2k(\theta + \frac{\pi}{n})$ we obtain

$$\sum_{k=1}^{\infty} \left[\left(a_{2k} + a_{2k} \cos \frac{2k\theta}{n} + b_{2k} \sin \frac{2k\theta}{n}\right) \cos 2k\theta$$

$$+ \left(b_{2k} + b_{2k} \cos \frac{2k\theta}{n} - a_{2k} \sin \frac{2k\theta}{n}\right) \sin 2k\theta\right] = 0,$$

then

$$\left\{\begin{array}{l}
a_{2k} + a_{2k} \cos \frac{2k\pi}{n} + b_{2k} \sin \frac{2k\pi}{n} = 0, \\
b_{2k} + b_{2k} \cos \frac{2k\pi}{n} - a_{2k} \sin \frac{2k\pi}{n} = 0,
\end{array}\right.$$
which implies that
\[(a_{2k}^2 + b_{2k}^2) \left(1 + \cos \frac{2k\pi}{n}\right) = 0, \quad (a_{2k}^2 + b_{2k}^2) \sin \frac{2k\pi}{n} = 0,\]
and thus
\[a_{2k} = \begin{cases} \neq 0, & \text{if } \frac{2k}{n} \text{ is odd integer;} \\ \equiv 0, & \text{other.} \end{cases} \quad (6)\]
\[b_{2k} = \begin{cases} \neq 0, & \text{if } \frac{2k}{n} \text{ is odd integer;} \\ \equiv 0, & \text{other.} \end{cases} \quad (7)\]

Now, using (4)-(7), we can obtain

**Theorem 3.4.** If \(K \in \mathbb{C}_n(\Lambda)\), then the Fourier series expansion of the support function of \(DK\) can be written as

\[h(DK, \theta) = w(K, \theta) = a_0 + 2 \sum_{k \in \mathbb{N}} (a_{2k} \cos 2k\theta + b_{2k} \sin 2k\theta)\]

and that of \(K\) can be given by

\[h(K, \theta) = \frac{1}{2} a_0 + \sum_{k \in \mathbb{N}} (a_{2k} \cos 2k\theta + b_{2k} \sin 2k\theta)\]

\[+ \sum_{k=1}^{\infty} \left[a_{2k-1} \cos (2k-1)\theta + b_{2k-1} \sin (2k-1)\theta\right].\]

### 3.3. Construction of the regular Reuleaux type polygon with even sides.

A polygon built by circular arcs is called a curvilinear polygon. A Reuleaux polygon is a curvilinear polygon where the center of each circular arc is its opposite vertex. That is why a Reuleaux polygon necessarily has odd number of sides. If all sides of a Reuleaux polygon are of equal length, then it is called regular. In this paper, we use the symbol \(R_{2m+1}\), \(m \in \mathbb{N}\), to denote a regular Reuleaux polygon of \(2m+1\) sides. Figure 2 below shows some examples of regular Reuleaux polygons.

In the following, we will describe how to construct a regular Reuleaux type polygon with even number of sides. We denote it by \(R_{2m}\). For the sake of simplicity, we assume that the radius of curvature of \(R_{2m}\) is 4.

\[\text{(a) } R_3 \quad \text{(b) } R_5 \quad \text{(c) } R_7\]

**Figure 2.** regular Reuleaux polygons with width 4
Set 
\[ \alpha = \frac{\pi}{2m}, \quad r_{2m} = \frac{2}{\cos(\alpha/2)}. \]

Draw a circle with radius \( r_{2m} \) and centered at the origin \( O \) (see Figure 3). We will build a \( 2m \)-gon inscribed in the circle. Let \( AB \) be one side of the \( 2m \)-gon, draw a circular arc connecting \( A \) and \( B \) with radius equal to 4 and centered at \( O' \) which is the midpoint of the circular arc \( \hat{CD} \), where \( C \) and \( D \) are chosen such that \( AB \parallel CD \). Similarly, other sides of the \( 2m \)-gon can be built and we can obtain the regular Reuleaux type polygon \( R_{2m} \). Obviously, \( R_{2m} \) is centrosymmetric. Some examples are given in Figure 5.

**Figure 3.** Construction of \( R_{2m} \)

**Lemma 3.5.** \( R_{2m} + gR_{2m} = 4B \), where \( g = e^{i\frac{\pi}{2m}} \). Furthermore, \( R_{2m} \in \mathbb{C}_{2m}(8) \).

**Proof.** Let \( \partial K \) denote the boundary curve of the convex domain \( K \). Recall that the form of the sum of two domains (or curves) does not depend upon the choice of the origin and is not changed by parallel displacement of the summands, and under these circumstances the sum undergoes only a parallel displacement. So we can choose the center of \( R_{2m} \) as the origin in Figure 4. We rotate \( R_{2m} \) with a rotation of angle \( \frac{\pi}{2m} \) and translate it to the regular Reuleaux \( 2m \)-gon on the right side in Figure 4, for the sake of simplicity, it is still denoted by \( gR_{2m} \).

It is well known that the sum \( R_{2m} + gR_{2m} \) is equivalent to a convex domain which is enclosed by the sum curve \( \partial R_{2m} + \partial (gR_{2m}) \). However, the sum curve \( \partial R_{2m} + \partial (gR_{2m}) \) is the locus of all points which are the sum of corresponding points of \( \partial R_{2m} \) and \( \partial (gR_{2m}) \) (Let \( l_1 \) and \( l_2 \) be parallel and similarly oriented supporting lines of the curves \( \partial R_{2m} \) and \( \partial (gR_{2m}) \) respectively. Let \( P \) and \( Q \) be the points of \( l_1 \) and \( l_2 \) with \( \partial R_{2m} \) and \( \partial (gR_{2m}) \) respectively. We say that the points \( P \) and \( Q \) are corresponding points of the curves \( \partial R_{2m} \) and \( \partial (gR_{2m}) \)).

In Figure 4, it is easy to see that \( B \) on \( \partial R_{2m} \) and arbitrary point on \( \hat{CD} \) in \( \partial (gR_{2m}) \) are corresponding points, and the sum \( B + \hat{CD} \) is \( \hat{FG} \) according to the parallelogram law. Arbitrary point on \( \hat{AB} \) in \( \partial R_{2m} \) and \( C \) on \( \partial (gR_{2m}) \) are corresponding points, and the sum \( \hat{AB} + C \) is \( \hat{EF} \). Continuing this process, the closed convex curve \( \partial R_{2m} + \partial (gR_{2m}) \) is consisted of \( 4m \) circular arcs.

On the other hand, \( \hat{CD} \) and \( I \) (the center of \( \hat{CD} \)) can be translated to \( \hat{FG} \) and \( O' \) along \( \hat{CF} \) respectively, so \( O' \) is the center of \( \hat{FG} \). From Figure 4, it is easy to
calculate $O'H = 4 - r_{2m}$, then $\widehat{FG}$ is a circular arc with center $O'$ and radius 4. Similarly, we can obtain the rest of the other $4m - 1$ arcs which are circular arcs with center $O'$ and radius 4 as well, i.e., $R_{2m} + gR_{2m} = 4B$.

Regular Reuleaux type polygons with even number of sides have been constructed, they are strictly convex domains, and their support functions are $C^1$, (see [17]), so the radius of curvature of the boundary $\partial R_{2m}$ exists a.e., and

$$\rho(R_{2m}, \theta) = h(R_{2m}, \theta) + h''(R_{2m}, \theta) \geq 0.$$  

Since $R_{2m}$ in Figure 3 is centrosymmetric, we get immediately

Lemma 3.6. For any regular Reuleaux type polygon $R_{2m}$ in Figure 3, there is a unique regular Reuleaux-type $2m$-polygon $\tilde{R}_{2m}$ such that $D\tilde{R}_{2m} = R_{2m}$.

From Lemmata 3.5 and 3.6 it follows obviously that

$$R_{2m} \in \{DK \mid K \in C_{2m}(4)\}.$$  

Lemma 3.7. If $K$ is a convex domain and $DK = R_{2m}$, then $K \in C_{2m}(4)$.

Proof. One can check that, (see Figure 6 for the case of $m = 1, 2, 3, 4$)

$$\rho(R_{2m}, \theta) + \rho\left(R_{2m}, \theta + \frac{\pi}{2m}\right) \equiv 4.$$

From $h(R_{2m}, \theta) = h(DK, \theta) = w(K, \theta)$, it can be seen that

$$\Omega_{2m}(K, \theta) + \Omega_{2m}''(K, \theta)$$
$$= w(K, \theta) + w\left(K, \theta + \frac{\pi}{2m}\right) + w''(K, \theta) + w''\left(K, \theta + \frac{\pi}{2m}\right)$$
$$= h(R_{2m}, \theta) + h\left(R_{2m}, \theta + \frac{\pi}{2m}\right) + h''(R_{2m}, \theta) + h''\left(R_{2m}, \theta + \frac{\pi}{2m}\right)$$
$$= \rho(R_{2m}, \theta) + \rho\left(R_{2m}, \theta + \frac{\pi}{2m}\right) = 4,$$
which can be considered as an ODE of $\Omega_{2m}(K, \theta)$ and its solution can be given by

$$\Omega_{2m}(K, \theta) = a \cos \theta + b \sin \theta + 4,$$

where $a$ and $b$ are constants. Since $\Omega_{2m}(K, \theta) = \Omega_{2m}(K, \theta + \pi)$, we get $a = 0$, $b = 0$ and $\Omega_{2m}(K, \theta) \equiv 4$, which means that is $K \in \mathbb{C}_{2m}(4)$.

Next we will calculate the area of $R_{2m}$.

![Figure 5. regular Reuleaux type polygons](image)

![Figure 6. Radius of curvature of the regular Reuleaux type polygons](image)
Lemma 3.8. The area of the regular Reuleaux type polygons $R_{2m}$ is
\[ A(R_{2m}) = 8\pi \left[ 1 - \frac{\tan(\alpha/2)}{\alpha} \right]. \] (8)

Proof. In Figure 3,
\[ A(O\hat{A}E'B) = A(O'\hat{A}E'B) - 2A(\triangle OO'A) = 8\alpha - 8\tan(\alpha/2) = 8(\alpha - \tan(\alpha/2)) = 8\alpha \left[ 1 - \frac{\tan(\alpha/2)}{\alpha} \right]. \]

Note that $\alpha = \frac{\pi}{2m}$, we get
\[ A(R_{2m}) = 2mA(O\hat{A}E'B) = 16m\alpha \left[ 1 - \frac{\tan(\alpha/2)}{\alpha} \right] = 8\pi \left[ 1 - \frac{\tan(\alpha/2)}{\alpha} \right]. \quad \blacksquare \]

4. The main results. In this section, without loss of generality, let $\Lambda = 4$, and denote $P_{4m}$ the regular $4m$–gon circumscribing a circle of radius 2. If $K \in \mathbb{C}_n(4)$, $B$ is a unit disc, set
\[ q_n(K) = \min_{K \in \mathbb{C}_n(4)} \frac{A(K)}{A(B)} , \quad q_n(DK) = \min_{K \in \mathbb{C}_n(4)} \frac{A(DK)}{A(2B)}. \]

They are hypothetically invariant, like the isoperimetric ratio, they can measure the area and the shape of convex domains in $\mathbb{C}_n(4)$.

Theorem 4.1. If $K \in \mathbb{C}_{2m-1}(4)$, then $w(K, \theta) \equiv 2$.

Proof. Observe that $w(K, \theta)$ is continuous. Suppose $w(K, \theta) \neq 2$, that is to say that there exists a $\theta_0$ such that $w(K, \theta_0) \neq 2$. Without loss of generality, we may assume that
\[ w(K, \theta_0) > 2. \]

By Definition 3.1, we get
\[ w\left(K, \theta_0 + \frac{\pi}{2m-1}\right) < 2, \]
\[ w\left(K, \theta_0 + 2 \cdot \frac{\pi}{2m-1}\right) > 2, \]
\[ w\left(K, \theta_0 + 3 \cdot \frac{\pi}{2m-1}\right) < 2, \]
\[ w\left(K, \theta_0 + 4 \cdot \frac{\pi}{2m-1}\right) > 2, \]
\[ \vdots \]
\[ w\left(K, \theta_0 + (2m-1) \cdot \frac{\pi}{2m-1}\right) < 2. \]

Hence
\[ w(K, \theta_0) = w(K, \theta_0 + \pi) = w\left(K, \theta_0 + (2m-1) \cdot \frac{\pi}{2m-1}\right) < 2. \]
This is a contradiction. Therefore \( w(K, \theta) \equiv 2 \).

Theorem 4.1 tells us that \( C_{2m-1}(4) \) are consists of convex domains with constant width 2, and thus one can obtain the area of the difference domains.

**Corollary 4.2.** If \( K \in C_{2m-1}(4) \), then \( DK \equiv 2B \) and \( A(DK) \equiv 4\pi \).

In this case, our problem is just that of Blaschke-Lebesgue. According to the Blaschke-Lebesgue theorem, the Reuleaux triangle has the least area, that is \( A(K) \geq A(R_3) = 2(\pi - \sqrt{3}) \), where \( R_3 \) is a Reuleaux triangle with width 2, and by Corollary 4.2, we can obtain

**Corollary 4.3.** If \( K \in C_{2m-1}(4) \), then \( q_{2m-1}(K) \equiv \frac{2(\pi - \sqrt{3})}{\pi} \approx 0.8973 \), \( q_{2m-1}(DK) \equiv 1 \).

We have considered the case when \( n \) is an odd integer. Next, we will deal with the case when \( n \) is an even integer.

**Theorem 4.4.** If \( K \in C_{2m}(4) \), then
\[
DK + gDK = 4B,
\]
where the rotation \( g = e^{i\frac{\pi}{2m}} \).

**Proof.**
\[
4 = w(K, \theta) + w(K, \theta + \frac{\pi}{2m}) = w(K, u) + w(K, gu) = w(K, u) + w(g^{-1}K, u) = h(DK, u) + h^{-1}(DK, u) = h(DK + g^{-1}DK, u).
\]
It is known that a convex domain is uniquely determined by its support function, and \( 4 = h(4B, u) \), we have
\[
DK + g^{-1}DK = 4B,
\]
and hence
\[
DK + gDK = g(DK + g^{-1}DK) = g(4B) = 4B.
\]

**Theorem 4.5.** If \( K \in C_{2m}(4) \), then \( DK \subseteq P_{4m} \), where \( P_{4m} \) represent the regular \( 4m \)-gon circumscribing a circle with radius 2.

**Proof.** If \( K \in C_{2m}(4) \), then \( w(K, \theta) + w(K, \theta + \frac{\pi}{2m}) = 4 \). Without loss of generality, we may assume that there exists a \( \theta_0 \) such that \( w(K, \theta_0) \geq 2 \), then \( w(K, \theta_0 + \frac{\pi}{2m}) \leq 2 \). Because of the continuity of \( w(K, \theta) \), the intermediate value theorem tells us that there exists a \( \theta^* \in [\theta_0, \theta_0 + \frac{\pi}{2m}] \) such that
\[
w(K, \theta^*) = 2.
\]
From this it follows that
\[
w(K, \theta^* + \frac{\pi}{2m}) = 2,
\]
\[
w(K, \theta^* + 2 \cdot \frac{\pi}{2m}) = 2,
\]
\[
w(K, \theta^* + 3 \cdot \frac{\pi}{2m}) = 2,
\]
\[
\vdots
\]
\[ w(K, \theta^* + (2m - 1) \frac{\pi}{2m}) = 2. \]

2m pairs of parallel supporting lines corresponding to \( w(K, \theta^*), w(K, \theta^* + \frac{\pi}{2m}), \ldots, w(K, \theta^* + (2m - 1) \frac{\pi}{2m}) \) can form a 4m-polygon \( L_{4m} \).

Obviously, \( L_{4m} \) is equiangular and each pair of its opposite sides are parallel, the distance between opposite sides is 2. By properties of supporting line, one can get that \( K \) is inscribed in \( L_{4m} \). If one show that the polygon \( DL_{4m} \) is a regular 4m-polygon, then one can obtain \( DK \subseteq P_{4m} \).

![Admiting a square](image)

![Admiting a regular polygon 4m-gon](image)

**Figure 7.** Admiting a regular polygon

It follows at once from the definition of addition of two polygon that \( DL_{4m} \) is equiangular, centrosymmetric and each pair of its opposite sides are parallel. Then, in Figure 7(B),

\[ \angle PBQ = \angle QCR = \angle RDS = \cdots, \quad OP = OQ = OR = OS = \cdots = 2. \]

Therefore, we can get \( QC = RC = BQ = DR = \cdots = 2\tan(\alpha/2) \). Then, we have \( BC = DC = \cdots \). It is obvious that \( DL_{4m} \) is a regular 4m-gon. On the other hand, \( K \in L_{4m} \) and \( -K \in -L_{4m} \), so \( DK \in DL_{4m} \). Thus completes the proof. \( \square \)

**Remark 1.** Here, it is worth mentioning that \( L_{4m} \) in the proof above is not certainly regular, see for instance the octagon in Figure 8, formed by two squares with the same edges.

**Corollary 4.6.** If \( K \in C_{2m}(4) \), then among \( DK \), \( R_{2m} \) has the least area, that is

\[ A(DK) \geq A(R_{2m}) = 8\pi \left[ 1 - \frac{\tan(\alpha/2)}{\alpha} \right], \quad (10) \]

and

\[ A(K) \geq 4\pi \left[ 1 - \frac{\tan(\alpha/2)}{2\alpha} - \frac{\sqrt{3}}{\pi} \right]. \quad (11) \]
Proof. If $K \in \mathbb{C}_2m(4)$, then from Theorems 4.4 and 4.5 it follows that $DK + gDK = 4B$ and thus

$$16\pi = A(4B) = A(DK + gDK) = 2A(DK) + 2A(DK, gDK),$$

$$A(DK) = \frac{A(4B)}{2} - A(DK, gDK) \geq \frac{A(4B)}{2} - A(P_{4m}, gP_{4m}) = \frac{A(4B)}{2} - A(P_{4m}, P_{4m})$$

$$= \frac{A(4B)}{2} - A(P_{4m}) = 8\pi \left[ 1 - \frac{\tan(\alpha/2)}{\alpha} \right] = A(R_{2m}),$$

where $A(P_{4m}) = \frac{8\pi \tan(\alpha/2)}{\alpha}$. Hence, this is the best lower bound.

Note that $K + gK$ and $K - gK$ are both constant width domains with width equal to 4 when $g = e^{i\frac{\pi}{6}}$ and $K \in \mathbb{C}_2m(4)$. By (3), (9) and (10), we can easily obtain

$$A(K) = \frac{1}{4} \left( A(DK) + A(K + gK) + A(K - gK) \right) - 2\pi$$

$$\geq \frac{1}{4} \left( 8\pi \left[ 1 - \frac{\tan(\alpha/2)}{\alpha} \right] \right) + \frac{1}{2} \left[ 8(\pi - \sqrt{3}) \right] - 2\pi$$

$$= 4\pi \left[ 1 - \frac{\tan(\alpha/2)}{2\alpha} - \frac{\sqrt{3}}{\pi} \right].$$

By Corollary 4.6, we have

**Corollary 4.7.** $K \in \mathbb{C}_2m(4)$, then

$$q_{2m}(DK) = 2 \left[ 1 - \frac{\tan(\alpha/2)}{\alpha} \right].$$

$q_{2m}(DK)$ is monotonically increasing as $m$ increases, and as $m \to \infty$, $q_{2m}(DK) \to 1$, and

$$q_{2m}(K) \geq 4 \left[ 1 - \frac{\tan(\alpha/2)}{2\alpha} - \frac{\sqrt{3}}{\pi} \right].$$
when $m \to \infty$,

$$q_{2m}(K) \geq 4 \left[ 1 - \frac{\tan(\alpha/2)}{2\alpha} - \frac{\sqrt{3}}{\pi} \right] \to 3 - \frac{4\sqrt{3}}{\pi} \approx 0.7947.$$

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