Caratheodory’s approximation for a type of Caputo fractional stochastic differential equations

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Abstract

The Caratheodory approximation for a type of Caputo fractional stochastic differential equations is considered. As is well known, under the Lipschitz and linear growth conditions, the existence and uniqueness of solutions for some type of differential equations can be established. However, this approach does not give an explicit expression for solutions; it is not applicable in practice sometimes. Therefore, it is important to seek the approximate solution. As an extending work for stochastic differential equations, in this paper, we consider Caratheodory’s approximate solution for a type of Caputo fractional stochastic differential equations.

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1 Introduction

Recently, stochastic fractional differential equations and stochastic fractional partial differential equations have attracted more and more attention. It turns out that differential equations involving derivatives of non-integer orders have memory properties, which are called non-local properties. Because of the non-local property of the Caputo fractional derivatives in time, Caputo fractional differential equations are important to model and describe problems in many disciplines, such as engineering, physics, and chemistry. For more details, see [1–7].

Compared with the work on deterministic fractional differential equations, the study of stochastic fractional differential equations is still in its infancy. However, the majority of work is concerned about the existence and uniqueness of solutions; see [8–12]. Until quite recently, there were some authors who considered some types of Caputo fractional stochastic differential equations and Caputo fractional stochastic partial differential equations by different approaching. For example, in Ref. [13], the authors considered the existence of stable manifolds for a type of stochastic differential equations. The authors of paper [14] considered the averaging principle of a type of stochastic fractional differential under some conditions consistent with the stochastic differential equations. In [15], the
existence of global forward attracting set for stochastic lattice systems with a Caputo fractional time derivative in the weak mean-square topology is established. In [16], the asymptotic distance between two distinct solutions is considered under a temporally weighted norm. Its worth mentioning that the Euler–Maruyama type approximate results for Caputo fractional stochastic differential equations have been established by [17]. For more related work, see [12, 18–22].

The Caratheodory approximation scheme was first considered by Carhaedory for ordinary differential equations, then Bell, Mohammad and Mao extended it to the stochastic differential equations case; see [23]. To the best of our knowledge, there is no work paying attention to the Caratheodory approximation for the Caputo fractional stochastic differential equation. In this paper, we will consider the Caratheodory approximation for the following type of Caputo fractional stochastic differential equation:

\[
\begin{aligned}
\frac{D^\alpha}{dt} X_t &= f(t, X_t) \, dt + g(t, X_t) \, dB_t, \quad t \geq 0, \\
X_0 &= x_0 \in L^2(\Omega, H),
\end{aligned}
\]  

(1.1)

where \( \alpha \in \left( \frac{1}{2}, 1 \right) \). For more details see Sect. 2. The aim of this paper is to extend the Caratheodory approximate results for Eq. (1.1).

This article is organized as follows. In Sect. 2 we will give some assumptions and basic results that we need. The existence and uniqueness of solution will be discussed in Sect. 3. In the last section, we will consider the Caratheodory approximation for the Caputo fractional stochastic differential equations.

Throughout this paper, the letter \( C \) will denote positive constants whose value may change in different occasions. We will write the dependence of a constant on parameters explicitly if it is essential.

2 Preliminaries

We impose the following assumptions to guarantee the existence and uniqueness of solution, \( H \) denote a Hilbert space, its norm is denoted by \( | \cdot | \).

**H1**: Lipschitz condition: Let \( t \geq 0 \) and constant \( k > 0 \), such that, for all \( x, y \in H \),

\[
|f(t,x) - f(t,y)|^2 + |g(t,x) - g(t,y)|^2 \leq k|x - y|^2.
\]

**H2**: Growth condition: Let \( t \geq 0 \) and constant \( k > 0 \), such that, for all \( x \in H \),

\[
|f(t,x)|^2 + |g(t,x)|^2 \leq k(1 + |x|^2).
\]

The following generalization of Gronwall’s lemma for singular kernels is needed for us to establish our results; see [15, 24].

**Lemma 2.1** Suppose \( b \geq 0, \beta > 0 \) and \( a(t) \) is a nonnegative function locally integrable on \( 0 \leq t < T \) (some \( T \leq +\infty \)), and suppose \( u(t) \) is nonnegative and locally integrable on \( 0 \leq t < T \) with

\[
u(t) \leq a(t) + b \int_0^t (t-s)^{\beta-1} u(s) \, ds.
\]
Then
\[ u(t) \leq a(t) + \int_0^t \left[ \sum_{n=1}^{\infty} \frac{(b \Gamma(\beta))^n}{\Gamma(n \beta)} (t-s)^{n\beta-1} a(s) \right] ds, \quad 0 \leq t < T, \]
where \( \Gamma(\cdot) \) is the Gamma function.

3 Well-posedness

In this section, we consider the existence and uniqueness of solution for the following equation under conditions H1 and H2:

\[
\begin{aligned}
\frac{d^\alpha}{dt^\alpha} X_t &= f(t, X_t) \, dt + g(t, X_t) \, dB_t, \quad t \geq 0, \frac{1}{2} < \alpha < 1, \\
X_0 &= x_0 \in L^2(\Omega, H),
\end{aligned}
\tag{3.1}
\]

where \( B_t \) is a scalar Brownian motion, \( f \) and \( g \) are \( H \)-value functions.

Definition 3.1 An \( H \)-value \( \mathcal{F}_t \)-adapted stochastic process \( X_t, t \in [0, T] \), is called a solution of the initial value problem (3.1), if \( X_t \in C([0, T]; L^2(\Omega, H)) \) and satisfies the following integral equation:

\[
X_t = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, X_s) \, ds \\
+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s, X_s) \, dB_s.
\tag{3.2}
\]

The existence and uniqueness of solutions for Eq. (3.1) have been considered by our previous work [25]. Similar problem also considered by [16] under different framework. To make this paper self-contained, we just give the main part of the proof for the following theorem.

Theorem 3.1 ([25]) Under conditions H1 and H2, for every \( x_0 \in L^2(\Omega, H) \), Eq. (3.1) has a unique mild solution \( X_t \in C([0, T]; L^2(\Omega, H)) \).

Proof. We prove the theorem by the contraction mapping principle. Using conditions H1 and H2, Lemma 2.1, we can derive that \( X_t \in C([0, T]; L^2(\Omega, H)) \).

Let
\[
S = \{ X_t | X_t \in C([0, T]; L^2(\Omega, H)) \}
\]
equipped with the norm
\[
|f(t)|_S = \sup_{0 \leq t \leq T} E|f(t)|^2
\]
be the Banach space of all \( \mathcal{F}_t \)-adapted processes.

For any \( t \in [0, T] \) and \( X_t \in S \), define a mapping as follows:

\[
(\Phi X)(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, X_s) \, ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s, X_s) \, dB_s.
\]
It is easy to verify that

\[ \Phi(\cdot) : C([0, T], L^2(\Omega; H)) \to C([0, T], L^2(\Omega; H)). \]

Let \( X_t, Y_t \in S \), then

\[
E|\Phi X(t) - (\Phi Y)(t)|^2 \leq 2E \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [f(s,X_s) - f(s,Y_s)] \, ds \right|^2 \\
+ 2E \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [g(s,X_s) - g(s,Y_s)] \, dB_s \right|^2.
\]

Denote \( \beta = 2\alpha - 1 > 0 \), by the Cauchy–Schwartz inequality, Itô’s isometry formula and condition \( H_1 \), we have

\[
E|\Phi X(t) - (\Phi Y)(t)|^2 \leq \frac{2k(T + 1)}{\Gamma(\alpha)^2} \int_0^t (t-s)^{\beta-1} E|X_s - Y_s|^2 \, ds.
\]

Using mathematical induction methods, we can deduce the following fact:

\[
E|\Phi^n X(t) - (\Phi^n Y)(t)|^2 \leq \frac{1}{\beta} \left( \frac{2k(T + 1)}{\Gamma(\alpha)^2} \right)^n \Gamma(\beta)^n \frac{t^n}{\Gamma(n\beta)} |X_t - Y_t|_x^n.
\] (3.3)

For \( n = 1 \), by simple calculation we get

\[
E|\Phi X(t) - (\Phi Y)(t)|^2 \leq \frac{2k(T + 1)}{\Gamma(\alpha)^2} |X_t - Y_t|^2 \frac{t^\beta}{\beta},
\]

which satisfies Eq. (3.3) with \( n = 1 \).

Now, assuming that Eq. (3.3) is satisfied for \( n = j \), we claim that it is also correct for \( n = j + 1 \). We have

\[
E|\Phi^{j+1} X(t) - (\Phi^{j+1} Y)(t)|^2 \\
\leq \frac{2k(T + 1)}{\Gamma(\alpha)^2} \int_0^t (t-s)^{\beta-1} E|\Phi^j X(s) - (\Phi^j Y)(s)|^2 \, ds \\
\leq \frac{2k(T + 1)}{\Gamma(\alpha)^2} \int_0^t (t-s)^{\beta-1} \left( \frac{2k(T + 1)}{\Gamma(\alpha)^2} \right)^j \frac{1}{\beta} \Gamma(\beta)^j \frac{t^j}{\Gamma(j\beta)} |X_s - Y_s|_x^j \, ds \\
\leq \left( \frac{2k(T + 1)}{\Gamma(\alpha)^2} \right)^{j+1} \frac{1}{\beta} \Gamma(\beta)^j \frac{t^{j+1}}{\Gamma(j+1)} |X_t - Y_t|^2 \int_0^t (t-s)^{\beta-1} s^j \, ds.
\] (3.4)

To get the estimate for \( n = j + 1 \), we only need to consider the following integral:

\[
\int_0^t (t-s)^{\beta-1} s^j \, ds.
\]

Take \( s = tz \), then

\[
\int_0^t (t-s)^{\beta-1} s^j \, ds = \int_0^1 (1-z)^{\beta-1} t^{\beta-1} z^j t \, dz \\
= t^{j+1} \int_0^1 (1-z)^{\beta-1} z^j \, dz
\]
\[= t^{(j+1)\beta} B(j\beta + 1, \beta)\]
\[= t^{(j+1)\beta} \frac{\Gamma(\beta) \Gamma(j\beta + 1)}{\Gamma((j + 1)\beta + 1)},\]

where \(B(\cdot, \cdot)\) is the Beta function. Combining this result with Eq. (3.4) we have

\[
E\left| (\Phi^{j+1}X)(t) - (\Phi^{j+1}Y)(t) \right|^2 \leq \left( \frac{2k(T + 1)}{\Gamma(\alpha)^2} \right)^{j+1} \frac{1}{\beta} \frac{\Gamma(\beta) \Gamma(j\beta + 1)}{\Gamma((j + 1)\beta + 1)} X_t - Y_{t+1} \\
= \left( \frac{2k(T + 1)}{\Gamma(\alpha)^2} \right)^{j+1} \frac{1}{\beta} \frac{\Gamma(\beta) \Gamma(j\beta + 1)}{\Gamma((j + 1)\beta + 1)} \frac{t^{(j+1)\beta}}{\Gamma((j + 1)\beta + 1)} |X_t - Y_{t+1}| \\
= \left( \frac{2k(T + 1)}{\Gamma(\alpha)^2} \right)^{j+1} \frac{1}{\beta} \frac{\Gamma(\beta) \Gamma(j\beta + 1)}{\Gamma((j + 1)\beta + 1)} \frac{t^{(j+1)\beta}}{\Gamma((j + 1)\beta + 1)} |X_t - Y_{t+1}|. \tag{3.5}
\]

Then we arrive at the following estimate for all \(n\):

\[
\left| (\Phi^nX)(t) - (\Phi^nY)(t) \right|_s \leq \left( \frac{2k(T + 1)}{\Gamma(\alpha)^2} \right)^{n} \frac{1}{\beta} \frac{\Gamma(\beta) \Gamma(\beta)}{\Gamma(n\beta)} \frac{T^{n\beta}}{\Gamma(n\beta)} |X_t - Y_{t+1}|. \tag{3.6}
\]

If we can prove

\[
\left( \frac{2k(T + 1)}{\Gamma(\alpha)^2} \right)^{n} \frac{1}{\beta} \frac{\Gamma(\beta) \Gamma(n\beta)}{\Gamma(n\beta)} < 1, \tag{3.7}
\]

for sufficient large \(n\), then the theorem holds.

Consider the following series of positive terms:

\[
\sum_{n=1}^{\infty} \left( \frac{2k(T + 1)}{\Gamma(\alpha)^2} \right)^{n} \frac{1}{\beta} \frac{\Gamma(\beta) \Gamma(n\beta)}{\Gamma(n\beta)}.
\]

We will show that

\[
\left( \frac{2k(T + 1)}{\Gamma(\alpha)^2} \right)^{n} \frac{1}{\beta} \frac{\Gamma(\beta) \Gamma(n\beta)}{\Gamma(n\beta)} \rightarrow 0,
\]

as \(n \rightarrow +\infty\), which guarantees that Eq. (3.7) holds. Thanks to the d'Alembert discriminant method, we only need to justify

\[
\lim_{n \rightarrow +\infty} \frac{(2k(T + 1))^{n} \Gamma(\beta) T^{n\beta} \Gamma(n\beta)}{\Gamma((n + 1)\beta)} < 1.
\]

Use the relationship of Gamma function and the Stirling formula, represented as follows:

\[
\Gamma(x) \sim \sqrt{2\pi} e^{-x} x^{x-\frac{1}{2}}, x \rightarrow \infty.
\]
Then
\[
\lim_{n \to \infty} \left( \frac{2k(T + 1)}{\Gamma(\alpha)^2} \Gamma(\beta) T^\beta \Gamma(n\beta)}{\Gamma((n + 1)\beta)} \right) = \lim_{n \to \infty} \left( \frac{2k(T + 1)}{\Gamma(\alpha)^2} \Gamma(\beta) T^\beta e^{\beta} \sqrt{n + 1} \left( \frac{n}{n + 1} \right)^{n\beta} \frac{1}{(n\beta + \beta)^\beta} \right) = 0,
\]
which shows that \( \Phi(\cdot) \) is a contraction mapping on \( C([0, T], L^2(\Omega; H)) \) for all \( T < \infty \). This completes the proof.

4 Caratheodory's approximate solutions

In this section, we consider the Caratheodory approximation for stochastic fractional differential equations. Similar to the stochastic differential equations approach, we try to give the definition of Caratheodory's approximate solutions for stochastic fractional differential equations as follows.

For every integer \( n \geq 1 \), define \( x_n(t) = x_0 \) for \( -1 \leq t \leq 0 \) and
\[
x_n(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-s)^{\alpha-1} f(s, x_n(s - 1/n)) \, ds + \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-s)^{\alpha-1} g(s, x_n(s - 1/n)) \, dB_s
\]
for \( 0 < t \leq T \).

Note that, for \( 0 \leq t \leq \frac{1}{n} \), \( x_n(t) \) can be computed by
\[
x_n(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-s)^{\alpha-1} f(s, x_0) \, ds + \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-s)^{\alpha-1} g(s, x_0) \, dB_s,
\]
then, for \( \frac{1}{n} < t \leq \frac{2}{n} \),
\[
x_n(t) = x_n(\frac{1}{n}) + \frac{1}{\Gamma(\alpha)} \int_{\frac{1}{n}}^{t} (t-s)^{\alpha-1} f(s, x_n(s - 1/n)) \, ds + \frac{1}{\Gamma(\alpha)} \int_{\frac{1}{n}}^{t} (t-s)^{\alpha-1} g(s, x_n(s - 1/n)) \, dB_s
\]
and so on. By this approach, we can compute \( x_n(t) \) step by step on the intervals \([0, \frac{1}{n}],[\frac{1}{n}, \frac{2}{n}],\ldots\)

Lemma 4.1 Under the condition \( H2 \), for all \( n \leq 1 \), we have
\[
\sup_{0 \leq t \leq T} E|x_n(t)|^2 \leq \Omega =: r_1 (1 + E_{2\alpha-1,1}(r_2 \Gamma(2\alpha - 1) T^{2\alpha-1})) < \infty,
\]
where \( r_1 = 3E|x_0|^2 + 3 \left( \Gamma(2\alpha-1) T^{\alpha-1} \right) \Gamma(\alpha)^2 (2\alpha-1) \), \( r_2 = 3 \left( \Gamma(T+1) \right) \Gamma(\alpha)^2 (2\alpha-1) \) and \( E_{2\alpha-1,1}(\cdot) \) is a two-parameter function of the Mittag-Leffler type (see [15]).

Proof From the simple arithmetic inequality
\[
|a + b + c|^2 \leq 3(|a|^2 + |b|^2 + |c|^2),
\]
we have

\[
E|\sigma(t)|^2 \leq 3E|x_0|^2 + 3E \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{-\alpha-1} f\left(s, \frac{1}{n}\right) \right|^2 ds + 3E \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{-\alpha-1} g\left(s, \frac{1}{n}\right) dB_s \right|^2
\]

\[:= 3I_1 + 3I_2 + 3I_3.\]

By the Cauchy–Schwarz inequality and condition \(H_2\), we can estimate the term \(I_2\) as follows:

\[
I_2 \leq \frac{Tk}{\Gamma(\alpha)^2} \int_0^t (t-s)^{-2(\alpha-1)} \left( 1 + E \left| \sigma\left(s, \frac{1}{n}\right) \right|^2 \right) ds
\]

\[\leq \frac{Tk}{\Gamma(\alpha)^2} \left[ \frac{t^{2\alpha-1}}{2\alpha - 1} + \int_0^t (t-s)^{2(\alpha-1)} E \left| \sigma\left(s, \frac{1}{n}\right) \right|^2 ds \right]
\]

\[\leq \frac{kT^{2\alpha}}{\Gamma(\alpha)^2 (2\alpha - 1)} + \frac{Tk}{\Gamma(\alpha)^2} \int_0^t (t-s)^{2(\alpha-1)} \sup_{0 \leq r \leq s} E|\sigma(r)|^2 ds.
\]

Similarly, with Itô's isometry formula and condition \(H_2\), we have an estimate for the stochastic integral term:

\[
I_3 \leq \frac{k}{\Gamma(\alpha)^2} \int_0^t (t-s)^{-2(\alpha-1)} \left( 1 + E \left| \sigma\left(s, \frac{1}{n}\right) \right|^2 \right) ds
\]

\[\leq \frac{kT^{2\alpha-1}}{\Gamma(\alpha)^2 (2\alpha - 1)} + \frac{k}{\Gamma(\alpha)^2} \int_0^t (t-s)^{2(\alpha-1)} \sup_{0 \leq r \leq s} E|\sigma(r)|^2 ds.
\]

Combining the estimate for \(I_1, I_2, I_3\), we arrive at

\[
E|\sigma(t)|^2 \leq r_1 + r_2 \int_0^t (t-s)^{(2\alpha-1)-1} \sup_{0 \leq r \leq s} E|\sigma(r)|^2 ds,
\]

where we denote

\[r_1 = 3E|x_0|^2 + 3 \left( \frac{kT^{2\alpha-1}(T+1)}{\Gamma(\alpha)^2 (2\alpha - 1)} \right)
\]

and

\[r_2 = 3 \frac{k(T+1)}{\Gamma(\alpha)^2}.
\]

Note that, for \(t_1 \leq t_2\), we have

\[
\int_0^{t_1} (t_1-s)^{(2\alpha-1)-1} \sup_{0 \leq r \leq s} E|\sigma(r)|^2 ds \leq \int_0^{t_2} (t_2-s)^{(2\alpha-1)-1} \sup_{0 \leq r \leq s} E|\sigma(r)|^2 ds.
\]

Then

\[
\sup_{0 \leq r \leq t} E|\sigma(r)|^2 \leq r_1 + r_2 \int_0^t (t-s)^{(2\alpha-1)-1} \sup_{0 \leq r \leq s} E|\sigma(r)|^2 ds.
\]
Applying Lemma 2.1, we can directly obtain

\[
\sup_{0 \leq r \leq t} E|\|x_n(r)\|_t|^2 \leq r_1 \left( 1 + \int_0^t \sum_{n=1}^{\infty} \frac{(r_2 \Gamma(2\alpha - 1))^n}{\Gamma(2n\alpha - n)} (t-s)^n(2\alpha - 1) \right) ds \\
\leq r_1 \left( 1 + \sum_{n=1}^{\infty} \frac{(r_2 \Gamma(2\alpha - 1) )^n T^{2\alpha - 1}}{\Gamma(2n\alpha - n + 1)} \right) \\
= r_1 \left( 1 + E_{2\alpha-1,1}(r_2 \Gamma(2\alpha - 1) T^{2\alpha - 1}) \right) < \infty,
\]

for all \( t \in [0, T] \), where \( E_{2\alpha-1,1}(\cdot) \) is a two-parameter function of the Mittag-Leffler type (see [15]). \( \square \)

**Lemma 4.2** Under the condition H2, for all \( n \geq 1 \) and \( 0 \leq t_0 < t \leq T \) with \( t - t_0 \leq 1 \), then

\[
E|\|x_n(t) - x_n(t_0)\|_t|^2 \leq C(t - t_0)^{2\alpha - 1}.
\]

**Proof** Taking \( 0 \leq t_0 < t \leq T \), we note that

\[
E|\|x_n(t) - x_n(t_0)\|_t|^2 \\
\leq 2E \frac{1}{\Gamma(\alpha)^2} \left( \int_0^t (t-s)^{\alpha - 1} f\left(s, x_n\left(s - \frac{1}{n}\right)\right) ds - \int_0^{t_0} (t_0-s)^{\alpha - 1} f\left(s, x_n\left(s - \frac{1}{n}\right)\right) ds \right)^2 \\
+ 2E \frac{1}{\Gamma(\alpha)^2} \left| \int_0^t (t-s)^{\alpha - 1} g\left(s, x_n\left(s - \frac{1}{n}\right)\right) dB_s \right|^2 \\
- \int_0^{t_0} (t_0-s)^{\alpha - 1} g\left(s, x_n\left(s - \frac{1}{n}\right)\right) dB_s \right|^2 \\
=: 2(J_1 + J_2).
\]

For \( J_1 \), we have

\[
J_1 \leq 2E \frac{1}{\Gamma(\alpha)^2} \left( \int_0^t (t-s)^{\alpha - 1} f\left(s, x_n\left(s - \frac{1}{n}\right)\right) ds \right)^2 \\
+ 2E \frac{1}{\Gamma(\alpha)^2} \left| \int_0^{t_0} [(t-s)^{\alpha - 1} - (t_0-s)^{\alpha - 1}] f\left(s, x_n\left(s - \frac{1}{n}\right)\right) ds \right|^2 \\
=: 2J_{11} + 2J_{12}.
\]

Using the Cauchy–Schwartz inequality, \( t - t_0 \leq 1 \), we give an estimate for \( J_{11} \) as follows:

\[
J_{11} \leq \frac{1}{\Gamma(\alpha)^2} \int_0^t (t-s)^{2\alpha - 2} ds \int_0^t E \left| f\left(s, x_n\left(s - \frac{1}{n}\right)\right) \right|^2 ds \\
\leq \frac{k}{\Gamma(\alpha)^2(2\alpha - 1)} (t-t_0)^{2\alpha - 1} \int_0^t \left[ 1 + E \left| x_n\left(s - \frac{1}{n}\right) \right|^2 \right] ds \\
\leq \frac{(k + 1)k}{\Gamma(\alpha)^2(2\alpha - 1)} (t-t_0)^{2\alpha - 1},
\]
where
\[ \Omega = r_1 \left( 1 + E_{2\alpha - 1, 1} \left( r_2 \Gamma(2\alpha - 1) T^{2\alpha - 1} \right) \right) \]
has been defined in Lemma 4.1.

For \( J_{12} \), we have the following result:
\[
J_{12} = E \frac{1}{\Gamma(\alpha)^2} \left| \int_0^t \left[ (t-s)^{\alpha-1} - (t_0 - s)^{\alpha-1} \right] f \left( s, x_n \left( s - \frac{1}{n} \right) \right) ds \right|^2
\]
\[
\leq \frac{k}{\Gamma(\alpha)^2} \int_0^t \left[ (t-s)^{\alpha-1} - (t_0 - s)^{\alpha-1} \right] ds \int_0^t \left[ 1 + E \left| x_n \left( s - \frac{1}{n} \right) \right|^2 \right] ds
\]
\[
\leq C T k \frac{1}{\Gamma(\alpha)^2} \left[ \frac{(t-t_0)^{2\alpha-2} - (t-s)^{2\alpha-2}}{2\alpha - 1 \cdot 2\alpha - 1} \right]
\]
\[
\leq C T k \frac{(t-t_0)^{2\alpha-1}}{2\alpha - 1}.
\]

For \( J_2 \), taking the Itô isometry formula and condition \( H_2 \) into account, using similar estimate methods to \( J_1 \), it can be shown that
\[ J_2 \leq \frac{C k \cdot (t-t_0)^{2\alpha-1}}{\Gamma(\alpha)^2 \cdot 2\alpha - 1}. \]

Combining all the deduced estimates, we have
\[
E \left| x_n(t) - x_n(t_0) \right|^2 \leq C (t-t_0)^{2\alpha-1}.
\]

This completes the proof. \( \square \)

**Theorem 4.1** Under the conditions \( H_1 \) and \( H_2 \), let \( x(t) \) be the unique solution of equations (3.1). Then for \( n \geq 1 \)
\[
\sup_{0 \leq t \leq T} E \left| x(t) - x_n(t) \right|^2 \leq \frac{C}{n^{2\alpha-1}} \tag{4.2}
\]

**Proof** Note that
\[
x(t) - x_n(t) = \frac{1}{\Gamma(\alpha)} \left[ \int_0^t (t-s)^{\alpha-1} f(s,x(s)) ds - \int_0^t (t-s)^{\alpha-1} f \left( s, x_n \left( s - \frac{1}{n} \right) \right) ds \right]
\]
\[
+ \frac{1}{\Gamma(\alpha)} \left[ \int_0^t (t-s)^{\alpha-1} g(s,x(s)) dB_s - \int_0^t (t-s)^{\alpha-1} g \left( s, x_n \left( s - \frac{1}{n} \right) \right) dB_s \right].
\]

Hence, employing a simple arithmetic inequality, we have
\[
E \left| x(t) - x_n(t) \right|^2 \leq 2E \frac{1}{\Gamma(\alpha)^2} \left| \int_0^t (t-s)^{\alpha-1} f(s,x(s)) ds - \int_0^t (t-s)^{\alpha-1} f \left( s, x_n \left( s - \frac{1}{n} \right) \right) ds \right|^2
\]
\[ + 2E \frac{1}{\Gamma(\alpha)^2} \left| \int_0^t (t-s)^{\alpha-1}g(s,x(s)) \, dB_s - \int_0^t (t-s)^{\alpha-1}g(s,x_n(s)) \, dB_s \right|^2 \]
\[ = 2(I_1 + I_2). \]

For \( I_1 \), we have
\[ I_1 \leq 2E \frac{1}{\Gamma(\alpha)^2} \left| \int_0^t (t-s)^{2\alpha-2}E|x(s) - x_n(s)|^2 \, ds \right| \]
\[ + 2E \frac{1}{\Gamma(\alpha)^2} \left| \int_0^t (t-s)^{2\alpha-1}g(s,x(s)) \, dB_s - \int_0^t (t-s)^{2\alpha-1}g(s,x_n(s)) \, dB_s \right|^2 \]
\[ = 2(I_{11} + I_{12}). \]

Using the Cauchy–Schwarz inequality and the condition \( \textbf{H1} \), we have the following estimate for \( I_{11} \):
\[ I_{11} \leq \frac{kT}{\Gamma(\alpha)^2} \left| \int_0^t (t-s)^{2\alpha-2}E|x(s) - x_n(s)|^2 \, ds \right|. \]

Similarly, for \( I_{12} \), we have
\[ I_{12} \leq \frac{kT^{2\alpha-1}}{(2\alpha - 1)\Gamma(\alpha)^2} \left| \int_0^t E|x_n(s) - x_n(s) - \frac{1}{n}|^2 \, ds \right|. \]

Also, we can divide \( I_2 \) into two parts as follows:
\[ I_2 \leq 2E \frac{1}{\Gamma(\alpha)^2} \left| \int_0^t (t-s)^{2\alpha-1}g(s,x(s)) \, dB_s - \int_0^t (t-s)^{2\alpha-1}g(s,x_n(s)) \, dB_s \right|^2 \]
\[ + 2E \frac{1}{\Gamma(\alpha)^2} \left| \int_0^t (t-s)^{2\alpha-1}g(s,x(s)) \, dB_s - \int_0^t (t-s)^{2\alpha-1}g(s,x_n(s)) \, dB_s \right|^2 \]
\[ = 2(I_{21} + I_{22}). \]

By the Itô isometry formula, we get
\[ I_{21} \leq \frac{k}{\Gamma(\alpha)^2} \int_0^t (t-s)^{2\alpha-2}E|x(s) - x_n(s)|^2 \, ds \]
and
\[ I_{22} \leq \frac{k}{\Gamma(\alpha)^2} \int_0^t (t-s)^{2\alpha-2}E|x_n(s) - x_n(s) - \frac{1}{n}|^2 \, ds. \]

Combining with the estimate for \( I_1 \) and \( I_2 \), it is derived that
\[ E|x(t) - x_n(t)| \leq \frac{k(T+1)}{\Gamma(\alpha)^2} \int_0^t (t-s)^{2\alpha-2}E|x(s) - x_n(s)|^2 \, ds \]
\[ + \frac{kT^{2\alpha-1}}{(2\alpha - 1)\Gamma(\alpha)^2} \int_0^t E|x_n(s) - x_n(s) - \frac{1}{n}|^2 \, ds \]
\[ + \frac{k}{\Gamma(\alpha)^2} \int_0^t (t-s)^{2\alpha-2}E|x_n(s) - x_n(s) - \frac{1}{n}|^2 \, ds, \quad (4.3) \]
by Lemma 4.2, if \( s \geq \frac{1}{n} \), then
\[
E \left| x_n(s) - x_n \left( s - \frac{1}{n} \right) \right|^2 \leq \frac{C}{n^{2\alpha - 1}}.
\]
otherwise if \( 0 \leq s < \frac{1}{n} \),
\[
E \left| x_n(s) - x_n \left( s - \frac{1}{n} \right) \right|^2 = E \left| x_n(s) - x_n(0) \right|^2 \leq C s^{2\alpha - 1} \leq \frac{C}{n^{2\alpha - 1}}.
\]
Following Eq. (4.3), we have
\[
E \left| x(t) - x_n(t) \right|^2 \leq \frac{k(T + 1)}{\Gamma(\alpha)^2} \int_0^t (t - s)^{2\alpha - 2} E \left| x(s) - x_n(s) \right|^2 \, ds
\]
\[
+ \frac{k T^{2\alpha - 1}}{(2\alpha - 1) \Gamma(\alpha)^2} \int_0^t E \left| x_n(s) - x_n \left( s - \frac{1}{n} \right) \right|^2 \, ds
\]
\[
+ \frac{T}{\Gamma(\alpha)^2} \int_0^t (t - s)^{2\alpha - 2} E \left| x_n(s) - x_n \left( s - \frac{1}{n} \right) \right|^2 \, ds
\]
\[
\leq \frac{k(T + 1)}{\Gamma(\alpha)^2} \int_0^t (t - s)^{2\alpha - 2} E \left| x(s) - x_n(s) \right|^2 \, ds
\]
\[
+ \frac{k T^{2\alpha - 1}}{(2\alpha - 1) \Gamma(\alpha)^2} \frac{1}{n^{2\alpha - 1}}
\]
\[
=: q_1 \int_0^t (t - s)^{2\alpha - 2} E \left| x(s) - x_n(s) \right|^2 \, ds + q_2.
\]
Applying Lemma 2.1, we obtain
\[
E \left| x(t) - x_n(t) \right|^2 \leq q_2 \left( 1 + E_{2\alpha - 1, 1} \left( q_1 \Gamma(2\alpha - 1) T^{2\alpha - 1} \right) \right) =: \frac{C}{n^{2\alpha - 1}}.
\]
This completes the proof. \( \square \)

**Remark 4.1** When \( \alpha = 1 \), i.e. Eq. (1.1) becomes a stochastic differential equation, the convergent rate of the scheme in Theorem 4.1 coincides with the well-known convergent rate of the classical Caratheodory results; see [23].

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**Availability of data and materials**
Please contact the authors for data requests.
Competing interests
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Authors' contributions
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