T-DUALITY OF NSR SUPERSTRING: THE
WORLDSHEET PERSPECTIVE

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Abstract

We formulate target space duality symmetry of NSR superstring from the
perspectives of worldsheet. The worldsheet action is presented in the super-
space formalism in the presence of massless backgrounds. We start from a
$D$-dimensional target space worldsheet action and compactify the theory on a
d-dimensional torus, $T^d$. It is assumed that the backgrounds are independent
of compact (super)coordinates. We adopt the formalism of our earlier work
to introduce dual supercoordinates along compact directions and introduce the
corresponding dual backgrounds. It is demonstrated that combined equations
of motion of the two sets of coordinates can be expressed in a manifestly $O(d, d)$
covariant form analogous to the equations of motions for closed bosonic string
derived by us. Furthermore, we show that the vertex operators associated with
excited massive levels of NSR string can be expressed in an $O(d, d)$ invariant
form generalizing earlier result for closed bosonic string.

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1. INTRODUCTION

The rich symmetry contents of string theory have played a cardinal role in understanding diverse attributes of string theory and have provided deep incisive insights into its dynamics. The target space duality (T-duality) is a very special feature of string theory and has attracted considerable attentions over two decades [1, 2]. This symmetry owes its existence to one-dimensional nature of string and we do not encounter the analog of T-duality in field theoretic description of a point particle. Therefore, it is natural to explore the symmetry from the perspective of the worldsheet. The T-duality symmetry, associated with closed bosonic string, has been investigated from the worldsheet perspective in a general framework almost two decades ago [3]. The closed bosonic string was considered in the presence of its massless backgrounds and $d$ of its target space coordinates were compactified on a torus. The backgrounds along compact directions were allowed to depend on noncompact string coordinates and were assumed to be independent of compact coordinates. It was recognized that the worldsheet equations of motion are conserved currents along compact directions. We introduced dual coordinates and corresponding dual backgrounds to derive another set of equations of motion. The two sets of equations of motion were be suitably combined to derived $O(d, d)$ covariant equations [3] where $d$ is the number of compact coordinates.

We may recall that the first hint of the T-duality was unraveled in the context of Regge-phenomenology in the context of strong interactions in the garb of FESR duality. Therefore, intuitively, it is quite tempting to conjecture that the excited stringy levels might possess some of the attributes of T-duality. It might be useful to explore T-duality symmetry of massive states from the perspective of the worldsheet. Indeed, existence of such a symmetry for excited massive levels of closed bosonic string [4] was demonstrated in the sense that the vertex operators for these levels were cast in an $O(d, d)$ invariant form. These results were derived in in a simple scenario.

On the other hand when T-duality is analyzed in a general setting its salient features and powerful applications are exhibited from the view point of the target space in the effective action approach. To recapitulate, when we envisage evolution of a closed string in the background of its massless excitations and demand (quantum) conformal invariance the backgrounds are constrained through the $\beta$-function equations which are computed perturbatively in the worldsheet $\sigma$-model approach. These equations of motion enable us to introduce the effective action whose variation reproduces the "equations of motion". Let us toroidally compactify the effective action to lower dimensions and examine symmetries of the reduced effective action. The reduced action can be cast in a manifestly $O(d, d)$ invariant form following the Scherk-Schwarz [5] dimensional reduction scheme if the theory is compactified on a $d$-dimensional torus ($T^d$) and the massless backgrounds are independent of the compact coordinates whereas they carry spacetime dependence. The availability of a manifestly $O(d, d)$ invariant reduced effective action has been very useful to explore various aspects of
string theory in diverse directions. The target space duality has attracted a lot of attentions from different perspectives over a long period. We refer to some early papers [6] and interested reader may consult reviews for comprehensive list of papers [2]. The purpose of this investigation is primarily two fold: We investigate target space duality symmetry of NSR superstring from the perspective of the worldsheet. A comprehensive understanding of $O(d, d)$ symmetry for NSR string in this framework is lacking. We adopt the worldsheet superspace approach to study the T-duality. We demonstrate that an $O(d, d)$ covariant worldsheet evolution equation can be obtained for NSR superstring. In order to achieve our objectives we express the NSR action in the superspace (in superconformal gauge) in the presence of its massless excitations such as graviton and two-form antisymmetric tensor. When the target space is compactified on $T^d$ and the backgrounds are independent of these super coordinates, we generalize our earlier results [3] for closed string, to cast the corresponding worldsheet equations of motion in a manifestly $O(d, d)$ covariant form.

The other goal is to examine existence of T-duality symmetry for excited massive levels of NSR string when the theory is compactified on $T^d$. We achieve this objective by introducing T-duality doublet of vectors in superspace which are the basic building blocks of all vertex operators; thus substantially improving earlier results on T-duality for massive states [4]. We introduce a set of projection operators to express vertex operators of massive levels in an $O(d, d)$ invariant form. As an illustrative example we consider compactification of type IIB theory to $AdS_3 \otimes S^3 \otimes T^4$ and focus our attention to NS-NS sector. We show, in this case, how the T-duality invariant vertex operators can be constructed for the excited massive levels of superstring.

Let us very briefly recapitulate some of the essential results of T-duality from the worldsheet point of view in the case of closed bosonic string. To begin with, consider the worldsheet action in the presence of constant backgrounds, $\hat{G}_{\hat{\mu}\hat{\nu}}$ and $\hat{B}_{\hat{\mu}\hat{\nu}}$

\[
S = \frac{1}{2} \int d\sigma d\tau \left( \gamma^{ab} \hat{G}_{\hat{\mu}\hat{\nu}} \partial_a X^\mu \partial_b X^\nu + \epsilon^{ab} \hat{B}_{\hat{\mu}\hat{\nu}} \partial_a X^\mu \partial_b X^\nu \right) \tag{1}
\]

where $\gamma^{ab} = \text{diag}(1, -1)$ worldsheet metric in the orthonormal gauge, $\hat{\mu}, \hat{\nu} = 0, 1, 2, \ldots, \hat{D}-1$, $\hat{G}_{\hat{\mu}\hat{\nu}}$ is a constant metric of the target space, $\hat{B}_{\hat{\mu}\hat{\nu}}$ is the constant antisymmetric tensor; consequently, the presence of the last term in (1) does not contribute to the equations of motion. The duality symmetry is manifestly exhibited if we express the resulting canonical Hamiltonian density in the form

\[
\hat{H}_c = \frac{1}{2} \hat{V}^T \hat{M} \hat{V} \tag{2}
\]

where

\[
\hat{V} = \left( \begin{array}{c} \hat{P}_\mu \\ \hat{X}^{\hat{\mu}} \end{array} \right) \tag{3}
\]

$P_\mu$ being the conjugate momenta and prime denoting the $\sigma$-derivative. Moreover, the
$2\hat{D} \times 2\hat{D}$ symmetric matrix [7, 8]

$$\hat{M} = \begin{pmatrix} \hat{G}^{-1} & -\hat{G}^{-1} \hat{B} \\ \hat{B} \hat{G}^{-1} & \hat{G} - \hat{B} \hat{G}^{-1} \hat{B} \end{pmatrix}$$  \hspace{1cm} (4)$$

is defined in terms of the constant backgrounds appearing in the worldsheet action above. The Hamiltonian density (2) is invariant under the global $O(\hat{D}, \hat{D})$ transformation

$$\hat{M} \rightarrow \hat{\Omega} \hat{M} \hat{\Omega}^T, \quad \hat{V} \rightarrow \hat{\Omega} \hat{V}, \quad \hat{\Omega}^T \hat{\eta} \hat{\Omega} = \hat{\eta}, \quad \hat{\Omega} \in O(\hat{D}, \hat{D}), \quad \hat{\eta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$  \hspace{1cm} (5)$$

$\hat{\eta}$ is the $O(\hat{D}, \hat{D})$ metric and $\mathbf{1}$ is $\hat{D} \times \hat{D}$ unit matrix and $\hat{V}$ is the $O(\hat{D}, \hat{D})$ vector and $\hat{M} \in O(\hat{D}, \hat{D})$. This symmetry is generalization of the duality symmetry $P_{\hat{\mu}} \leftrightarrow X^{\hat{\mu}}$.

In the framework of Lagrangian formulation, the $O(\hat{D}, \hat{D})$ symmetry is exhibited as follows. In presence of constant backgrounds, the worldsheet equations of motion for string coordinates $\{X^{\hat{\mu}}(\sigma, \tau)\}$ are set of conservation laws

$$\partial_a J^a_{\hat{\mu}} = 0$$  \hspace{1cm} (6)$$

where the current is given by

$$J^a_{\hat{\mu}} = \gamma^{ab} \hat{G}_{\hat{\mu} \hat{\nu}} \partial_b X^{\hat{\nu}} + \epsilon^{ab} \hat{B}_{\hat{\mu} \hat{\nu}} \partial_b X^{\hat{\nu}}$$  \hspace{1cm} (7)$$

Thus locally, one can express the two dimensional current as:

$$J^a_{\hat{\mu}} = \epsilon^{ab} \partial^b \hat{Y}^{\hat{\mu}}$$  \hspace{1cm} (8)$$

where $\{\hat{Y}_{\hat{\mu}}\}$ are set of dual coordinates. The next step is to introduce a set of dual backgrounds $\hat{G}$ and $\hat{B}$ and finally a dual action. The equations of motion for $\hat{Y}_{\hat{\mu}}$ involving dual backgrounds is also a conservation law. The two sets of conservation laws (equations of motion consisting of $2\hat{D}$ vectors) can be combined to derive a single $O(\hat{D}, \hat{D})$ covariant equations of motion [9]. Subsequently, this result has been derived for the cosmological case i.e. when the backgrounds $G$ and $B$ are time dependent [10].

In the more general case, alluded to in the introduction, where the the massless backgrounds assume spacetime dependence the following results hold. Let us compactify the string on a d-dimensional torus $T^d$ and decompose the worldsheet string coordinates into two sets

$$X^{\hat{\mu}} = (X^\mu, Y^\alpha), \quad \mu = 0, 1, ..D - 1; \quad \alpha = 1, 2, ..d$$  \hspace{1cm} (9)$$

so that $\hat{D} = D + d$. $X^\mu$ and $Y^\alpha$ are the spacetime and compact coordinates respectively. We suppress explicit dependence of these coordinates on $\sigma$ and $\tau$ from now on. For such a toroidal compactification we can decompose the backgrounds as

$$\hat{e}_{\hat{\mu}}^{\hat{\nu}} = \begin{pmatrix} e^\mu_{\nu}(X) & A^{(1)\beta}(X) E^\alpha(X)_{\beta}(X) \\ 0 & E^\alpha_{\beta}(X) \end{pmatrix}$$  \hspace{1cm} (10)$$
The spacetime metric is $g_{\mu\nu} = e^r_{\mu} g^{(0)}_{rs} e^s_{\nu}$ and the internal metric is $G_{\alpha\beta} = E^{a}_{\alpha} \delta_{ab} E^{b}_{\beta}$; $g^{(0)}_{rs}$ is the D-dimensional flat space Lorentzian signature metric. $A^{(1)}_{\mu} \beta$ are gauge fields associated with the d isometries and it is assumes that the backgrounds depend of coordinates $X^\mu$ and are independent of $Y^\alpha$. Similarly, the antisymmetric tensor background, depending only on $X^\mu$ can be decomposed as

$$\hat{B}_{\mu\nu} = \begin{pmatrix} B_{\mu\nu}(X) & B_{\mu\alpha}(X) \\ B_{\nu\beta}(X) & B_{\alpha\beta}(X) \end{pmatrix}$$ (11)

Here we note the presence of gauge fields $B_{\mu\alpha}$ due to compactification as expected. We go through the steps of deriving equations of motion for the compact coordinates, $Y^\alpha$ from the $\sigma$-model action and note that these still correspond to conservation laws since backgrounds and gauge fields are independent of compact string coordinates. Although the set of equations are more complicated in the general setting, eventually, after long and tedious calculations, the worldsheet equations of motion can be cast in an $O(d,d)$ covariant form as was demonstrated by us [3].

There is another interesting approach to dualities in the worldsheet approach. In this formulation the number of string coordinates are doubled and this in scenario some of the nice features of conventional worldsheet approach are not maintained; however, it has been argued that such doubling might have deep significance [9, 11] in string theory. Recently, a new formulation of field theory has been introduced where $O(D,D)$ invariant action is constructed, $D$ being the number of spacetime dimensions which is doubled [12, 13]. At this stage we have not been able to establish connection of our formulation with double field theory.

We mention en passant that the case of superstring is not straightforward if we deal with worldsheet bosonic coordinates and their fermionic partners. We address this case in the next section.

2. T-DUALITY OF COMPACTIFIED NSR STRING

A free superstring (NSR string) action can be expressed as sum of actions for set of left moving and right moving bosons and fermions. Therefore, unlike the closed bosonic string case, we do not see the $P \leftrightarrow X'$ duality (which is same as $\sigma \leftrightarrow \tau$ duality) so explicitly in the resulting Hamiltonian density in the presence of fermionic coordinates. In fact the more transparent duality symmetry is to study the transformation properties of left moving and right moving fields under $\sigma \leftrightarrow \tau$ interchange. The holomorphic fields do not change sign whereas antiholomorphic ones do. If we introduce constant backgrounds as in the case of closed bosonic string the analog of noncompact $O(\hat{D}, \hat{D})$ symmetry does not emerge so neatly. The target space duality for superstrings in the NSR formulation has been studied in the past, however, we feel that this problem deserves further attention.

Let us recapitulate evolution of NSR string in the background of massless excitations. One starts with the superworldsheet action in two dimensional superspace where the components of superfield are the bosonic coordinates, (NSR) Majorana fermions and
auxiliary fields and the backgrounds are functions of the superfields. We expand the backgrounds in terms of component fields, eliminate the auxiliary fields in order to arrive at NSR superstring action in the presence of massless backgrounds with component the fermionic and boson fields only. If we envisage the case where backgrounds are independent of some of the coordinates (now backgrounds and their derivatives depend only on bosonic coordinates), then it is very hard to arrive at duality invariant/covariant equations as was achieved by Maharana and Schwarz [3]. Das and Maharana [14] considered NSR string action in superspace and adopted the technique introduced by Maharana and Schwarz [3] for the closed bosonic string to get analogous equations of motion. However, they were unable to arrive at at duality covariant equations of motion although they obtained interesting results for a special case. In this case the $Z_2$ duality conditions are recovered

$$\partial_\pm X^\mu \rightarrow \pm \partial_\pm X^\mu, \quad \psi_\pm^\mu \rightarrow \pm \psi_\pm^\mu$$

Moreover, Siegel [15] considered superstring in superspace in a Hamiltonian phase space approach to study dualities. Subsequently, there have been attempts to reveal duality symmetries on superstring [16]. Thus far worldsheet equations of motion for superstrings respecting target space duality symmetry has not been derived in a systematic manner at par with the results of closed bosonic string.

The NSR superstring action in two dimensional superspace is given by

$$S = -\frac{1}{2} \int d\sigma d\tau d^2\theta \bar{D} \Phi^\mu \left( \hat{G}_{\hat{\mu}\hat{\nu}}(\hat{\Phi}) - \gamma_5 \hat{B}_{\hat{\mu}\hat{\nu}}(\hat{\Phi}) \right) D\Phi^\nu$$

(13)

We have adopted superorthonormal gauge in arriving at this action. Here $\hat{G}_{\hat{\mu}\hat{\nu}}(\hat{\Phi})$ and $\hat{B}_{\hat{\mu}\hat{\nu}}(\hat{\Phi})$ are the graviton and 2-form backgrounds which depend on the superfield $\hat{\Phi}$. It has expansion in component fields as

$$\hat{\Phi}^\mu = X^\mu + \bar{\theta} \psi^\mu + \bar{\psi}^\mu \theta + \frac{1}{2} \bar{\theta} \theta F^\mu$$

(14)

where $X^\mu, \psi^\mu$ and $F^\mu$ are the bosonic, fermionic and auxiliary fields respectively. The covariant derivatives in superspace are defined to be

$$D_a = \frac{\partial}{\partial \theta_a} - i(\gamma^a \theta)a \partial_a, \quad \bar{D}_a = -\frac{\partial}{\partial \theta_a} + i(\bar{\theta} \gamma^a)a \partial_a$$

(15)

where $\partial_a$ stands for worldsheet derivatives ($\sigma$ and $\tau$) and the convention for $\gamma$ matrices are

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_5 = \gamma^0 \gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(16)

The resulting equations of motion from (13) are

$$\bar{D} \left( \hat{G}_{\hat{\mu}\hat{\nu}}(\hat{\Phi}) - \gamma_5 \hat{G}_{\hat{\mu}\hat{\nu}}(\hat{\Phi}) \right) D\Phi^\nu = 0$$

(17)
The equations for component fields can be obtained by expanding the backgrounds in terms of them and utilizing the definitions of superspace derivatives (15). Let us consider a compactification [17] scheme such that target space is compactified on $T^d$: $\hat{M}_D = M_D \otimes T^d$. The metric and 2-form backgrounds are decomposed as

$$\hat{G}_{\mu\nu} = \begin{pmatrix} g_{\mu\nu}(\phi) & 0 \\ 0 & G_{ij}(\phi) \end{pmatrix}, \quad \hat{B}_{\mu\nu} = \begin{pmatrix} B_{\mu\nu}(\phi) & 0 \\ 0 & B_{ij}(\phi) \end{pmatrix}$$  \hspace{1cm} (18)

Note that the backgrounds only depend on spacetime superfields, $\phi^\mu$. We decompose the superfields as

$$\hat{\Phi}^\mu = (\phi^\mu, W^i)$$  \hspace{1cm} (19)

where $\mu, \nu = 0, 1, 2, ..D-1$ and $i, j = 1, 2, ..d$ with $\hat{D} = D + d$. Note that the two superfields have the expansions

$$\phi^\mu = X^\mu + \bar{\theta}\psi^\mu + \bar{\psi}\theta^\mu + \frac{1}{2}\bar{\theta}\bar{\theta}F^\mu$$  \hspace{1cm} (20)

and

$$W^i = Y^i + \bar{\theta}\chi^i + \bar{\chi}\theta^i + \frac{1}{2}\bar{\theta}\bar{\theta}F^i$$  \hspace{1cm} (21)

$\chi^i$ are two dimensional Majorana spinors. In this compactification scheme, the equations of motion for the superfields $\phi^\mu$ is exactly analogous to (17) where we replace $\Phi$ with $\phi$ and the backgrounds with $g_{\mu\nu}(\phi)$ and $B_{\mu\nu}(\phi)$. Let us focus attention on the evolution equations and the dynamics of superfields along compact directions. The action is

$$S = -\frac{1}{2} \int d\sigma d\tau d^2\bar{\theta} \bar{\Sigma}^i \left( G_{ij}(\phi) - \gamma_5 B_{ij}(\phi) \right) \Sigma^j D\hat{W}_i$$  \hspace{1cm} (22)

The superderivatives are defined in (15) above. The equations of motion for $\{W^i\}$ are

$$\bar{\mathcal{D}}\left( \left[ G_{ij}(\phi) - \gamma_5 B_{ij}(\phi) \right] D\hat{W}_j \right) = 0$$  \hspace{1cm} (23)

In view of the fact that $G$ and $B$ depend only on $\phi^\mu$, we may introduce a dual free superfield $\tilde{W}_i$ satisfying following equation locally which is consistent with (23)

$$\left( G_{ij}(\phi) - \gamma_5 B_{ij}(\phi) \right) D\hat{W}_j = D\tilde{W}_i$$  \hspace{1cm} (24)

and the dual superfield satisfies the constraint: $\bar{\mathcal{D}}D\tilde{W}_i = 0$. Note that (24) is reminiscent of the dual coordinate introduced for closed string by us in the case of closed closed string under a similar scenario [3]. We can go further and opt for a first order formalism and consider the Lagrangian density

$$\mathcal{L} = \frac{1}{2} \Sigma^i \left( G_{ij}(\phi) - \gamma_5 B_{ij}(\phi) \right) \Sigma^j - \bar{\Sigma}^i D\hat{W}_i$$  \hspace{1cm} (25)
The $\Sigma^i$ variation leads to

$$
(G_{ij}(\phi) - \gamma_5 B_{ij}(\phi)) \Sigma^j = D\tilde{W}_i
$$

(26)

and $\tilde{W}_i$ variation implies $D\Sigma^i = 0$. Therefore, when $\Sigma^i = DW^i$ we recover (23).

We are in a position to introduce a dual Lagrangian density in terms of the dual superfields, $\tilde{W}_i$ and a set of dual backgrounds $G^{ij}(\phi)$ and $B^{ij}(\phi)$; whereas the former of the two backgrounds is symmetric in its indices the latter is antisymmetric.

$$
\mathcal{L}_{\tilde{W}} = -\frac{1}{2} D\tilde{W}_i \left( G^{ij}(\phi) - \gamma_5 B^{ij}(\phi) \right) D\tilde{W}_j
$$

(27)

where the two dual backgrounds, $(G, B)$, are related to the original background fields, $(G, B)$ through the following equations

$$
G = \left( G - BG^{-1} B \right)^{-1} \quad \text{and} \quad B = - \left( G - BG^{-1} B \right)^{-1} B G^{-1}
$$

(28)

Notice that $(G - BG^{-1} B)^{-1}$ is symmetric since $(G - BG^{-1} B)$ is symmetric and it is easy to check that $B$ is antisymmetric. The equations of motion associated with (27) is

$$
D \left[ \left( G(\phi) - \gamma_5 B(\phi) \right) D\tilde{W} \right] = 0
$$

(29)

Our next step is to write down a pair of equations relating the superfields and their duals which will lead us to T-duality covariant equations of motion. This is facilitated by inspecting the two sets of equations of motion (23) and (29) resulting from the original Lagrangian density and its dual which correspond to two conservation laws. After straightforward and a little tedious calculations we arrive at following two equations

$$
DW^i = \gamma_5 (G^{-1} B)^i_j DW^j + G^{-1ij} D\tilde{W}_j
$$

(30)

$$
D\tilde{W}_i = \gamma_5 (G^{-1} B)^i_j D\tilde{W}_j + G^{-1ij} DW^j
$$

(31)

Using (28) in (31) we get two sets of equations relating $DW^i$ and $D\tilde{W}_i$. Let us define a 2d-dimensional $O(d, d)$ vector (each one is a superfield) such that

$$
U = \begin{pmatrix}
W^i \\
\tilde{W}_i
\end{pmatrix}
$$

(32)

and a matrix

$$
\mathcal{M} = \begin{pmatrix}
G^{-1} & \gamma_5 G^{-1} B \\
-\gamma_5 B G^{-1} & 1G - 1BG^{-1} B
\end{pmatrix}
$$

(33)
where $\mathbf{1}$ is the $2 \times 2$ unit matrix and $\gamma_5$ is two dimensional diagonal matrix defined earlier. The $\mathcal{M}$ matrix has properties of the familiar $M$-matrix introduced in dimensional reduction of closed bosonic string: $\mathcal{M} \in O(d, d)$ and corresponding metric is $\eta$. The dimensions are further doubled due to the presence of two component Majorana fermions and is reflected by the appearance of $\mathbf{1}$ and $\gamma_5$ in the $\mathcal{M}$-matrix. The two equations (30) and (31) can be combined to a single matrix equation

$$DU = \mathcal{M}\eta U \quad (34)$$

It follows from the definition of the $O(d, d)$ vector $U$ that $\mathcal{D}DU = 0$. It holds by virtue of the fact that the two components of $U$ satisfy $\mathcal{D}DW^i = 0$ and $\mathcal{D}\tilde{W}_i = 0$ from our original equations (they are dual superfields of each other). Therefore, we arrive at an $O(d, d)$ covariant equations of motion for coordinates along compact directions

$$\mathcal{D}\left(\mathcal{M}\eta U\right) = 0 \quad (35)$$

Thus (35) generalizes the closed string $O(d, d)$ covariant equations of motion to NSR superstring.

Let us return to the evolution equation for the superfields corresponding to noncompact coordinates

$$\mathcal{D}\left(g_{\mu\nu}(\phi) - \gamma_5B_{\mu\nu}(\phi)\right)D\phi_\nu = 0 \quad (36)$$

Notice that due to the dependence of backgrounds on the superfield $\phi$ these are "dynamical" equations unlike the case of compact coordinates which were identified as conservation laws. More important point to note is that these equations are T-duality invariant since the background tensors and these superfields are inert under the action of the T-duality noncompact symmetry group. Therefore, we conclude that the resulting equations of motion for a superstring compactified on $T^d$ can be cast in an $O(d, d)$ covariant form. In the next section, we shall consider an illustrative example.

3. TYPE IIB COMPACTIFICATION ON $\text{AdS}_3 \otimes S^3 \otimes T^4$

We envisage a scenario where our results can be concretely realized. In the presence of NS-NS 3-form flux we can write down a worldsheet action for the case at hand. Notice that $\text{AdS}_3$ and $S^3$ correspond to target space of constant negative and positive curvatures respectively. Therefore, if we introduce appropriate NS-NS three form fluxes, we can describe the Lagrangian in these two sectors as sum of two WZW Lagrangians. The presence of WZ term renders the theory conformally invariant and has the interpretation of the background antisymmetric tensor fields. Moreover, for the present scenario the associated field strengths are constant. The radii of
these two spaces are to be such that the cosmological constants arising from constant positive and negative curvatures of $S^3$ and $AdS_3$ correspondingly sum up to zero. The worldsheet description of NSR string on $AdS_3 \otimes S^3$ can be formulated as WZW model on group manifolds $SL(2,R) \otimes SU(2)$ as is well known. Thus the full worldsheet action is decomposed into sum of three parts: one corresponds to superconformal theory on $SL(2,R)$ the other being $SU(2)$ and the third part is the one describing a supersymmetric $\sigma$-model along compact direction as we have discussed in the previous section.

Let us briefly consider bosonic WZW model for $SU(2)$ group whose action is

$$S_B = \frac{1}{4\lambda^2} \int d\sigma d\tau \text{Tr}\left( \partial_a g^{-1}(\sigma, \tau) \partial^a g(\sigma, \tau) \right) + \frac{k}{16\pi} \int_B \text{Tr}\left( g^{-1}(\sigma, \tau) dg(\sigma, \tau) \wedge g^{-1}(\sigma, \tau) dg(\sigma, \tau) \right)$$  \hspace{1cm} (37)

where $g \in SU(2)$ and satisfies the constraint $gg^\dagger = 1$, $1$ being the unit matrix. Note the following features: (i) $\lambda$ and $k$ are dimensionless coupling constants. For a compact group like $SU(2)$, $k$, the coupling constant appearing in front of the WZ term is quantized for the consistency of the quantized theory. (ii) $g$ should be smoothly extended to a 3-dimensional manifold and its boundary, $B$, is the worldsheet (actually one should define complex variables in terms of $(\sigma, \tau)$ and this action in those variables in the standard manner.) The theory is conformally invariant at the special point $\lambda^2 = \frac{4\pi}{k}$.

The case of (bosonic) string on noncompact $SL(2,R)$ manifold is similar to $SU(2)$ with some differences. (i) The matrix $\tilde{g} \in SL(2,R)$ satisfies the constraint $\tilde{g}\tilde{g}^T = \zeta$ to be contrasted with $g \in SU(2)$ group element. Here $\zeta$ is the $SL(2,R)$ metric with property $\zeta^2 = -1$ and it can be chosen to be

$$\zeta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$  \hspace{1cm} (38)

(ii) In this case the coefficient of WZ, $k$ term need not be quantized.

We shall consider the supersymmetric WZW model for $SU(2)$ from now on. The action is \cite{18, 19, 20, 21}

$$S = \frac{1}{4\lambda^2} \int d\sigma d\tau d^2\theta \bar{D}G^\dagger DG + \frac{k}{16\pi} \int d\sigma d\tau d^2\theta \int_0^1 dt \frac{dG^\dagger}{dt} \bar{D}G^\dagger \gamma_5 DG$$  \hspace{1cm} (39)

The matrix $G$ defined in the superspace satisfies constraints $GG^\dagger = 1$. In order to define the WZ term as an integral over a three dimensional space one defines extension of the superfield to 3-dimensions so that $t = 0$ corresponds to the boundary i.e. at that point the two dimensional superfield is defined on the worldsheet and two dimensional $\gamma_5$ is defined already. Several remarks are in order at this point: (i) When the $G \in SU(2)$ matrix is expressed in terms of component fields, auxiliary field is eliminated, the $d\theta$ integration is done, the resulting action contains quartic
fermionic coupling and the theory is not necessarily conformally invariant for arbitrary \( \lambda^2 \) and \( k \). (ii) At the special point \( \lambda^2 = \frac{4\pi}{k} \), theory is conformally invariant and the quartic fermionic coupling disappears. Therefore, for a superstring on a group manifold the two coupling constants are related (and \( k \) is quantized). Moreover, at the conformal point, the equations of motion of the superfields (G-matrices), decompose into holomorphic and antiholomorphic parts and they take the form of two separate current conservation equations. This feature is most elegantly displayed if we expand the super-matrix in \( G \) in light cone variables as

\[
G(\sigma, \tau, \theta) = g(\sigma, \tau) \left( 1 + i\theta^+ \psi_+(\sigma, \tau) + i\theta^- \psi_-(\sigma, \tau) + i\theta^+ \theta^- F(\sigma, \theta) \right)
\]  

(40)

Corresponding light cone superderivatives are

\[
D_\pm = \frac{\partial}{\partial \theta^\pm} - i\theta^\pm \partial_\pm \quad \text{with} \quad \partial_\pm = \partial_\tau \pm \partial_\sigma
\]  

(41)

Note that the chiral fermions \( \psi_\pm \) are matrices taking value in the Lie algebra and \( F \) is the auxiliary field. The constraint \( GG^\dagger = 1 \) results in relations between component fields, \( g, \psi_\pm \) and \( F \). At the conformal point the equation of motions become

\[
D_\pm J_\pm = 0, \quad J_\pm = -iG^{-1}D_\pm G
\]  

(42)

We therefore, note that classically we solve for NSR string on \( S^3 \).

The case of NSR string on \( AdS_3 \), proceeds similarly once we take into account the subtleties associated with the noncompact \( SL(2, R) \) group.

We have discussed in detail how to construct worldsheet action for compact coordinates in the case of NSR string on \( T^d \). We showed that the equations of motion can be cast in \( O(d, d) \) covariant form since the equations of motion are conservation laws in the superspace.

We argue that the string coordinates and backgrounds, parametrizing target space \( AdS_3 \) and \( S^3 \), transform trivially under the T-duality group associated with compact directions. Therefore, those equations of motion are \( O(d, d) \) invariant. This completes our study of T-duality symmetry for type IIB string on \( AdS_3 \otimes S^3 \otimes T^4 \).

4. VERTEX OPERATORS FOR EXCITED MASSIVE LEVELS

We study construction of duality symmetric vertex operators of NSR string in this section. There are interests in excited massive levels of strings for diverse reasons. It is argued that in Planckian energy scattering processes stringy states play a very important role in restoring good high energy behavior of the amplitudes [22, 23]. Furthermore, it is conjectured that there might be enhanced symmetry at these asymptotic energies [24]. There are some evidences that excited massive states of closed string might be endowed with higher symmetries which are not fully explored so far [25, 26, 27]. Recently, it has been shown that for bosonic closed string compactified
on $T^d$ the vertex operators associated with excited massive states can be expressed in an $O(d, d)$ invariant form. This was achieved in a simple framework. We worked in the weak field approximation when strings is considered in the background of massive excited states. These vertex operators were first expressed in terms $\sigma$-derivatives of $X^\mu$ and the canonical conjugate momenta of the compact coordinates $\sigma$ and/or $\tau$ derivatives. The vertex operators are required to fulfill following conditions. (i) All vertex operators are required to be $(1, 1)$ operators with respect to the stress energy momentum tensors, $(T_{++}, T_{--})$, of the free string [26]. This is a powerful constraint and it leads to the 'equations of motion' and 'gauge conditions' for the massive backgrounds when they are arbitrary functions of string coordinates. (ii) At each mass level one constructs 'vertex functions' from the basic building blocks such as $\partial X^\mu, \partial \bar{X}^\mu$ and $\partial$ or $\bar{\partial}$ acting on these building blocks. Note that we do not admits terms like $\partial \bar{\partial} X^\mu$ or $\bar{\partial} \partial X^\mu$ in vertex functions since these objects and derivative of such objects vanish as a virtue of free string equations of motion. (iii) The structure of vertex function of a given type, at each mass level, is constrained by the level matching conditions since there is no preferred point on a closed string loop. (iv) The vertex operator of a given mass level is sum of all such vertex functions. The vertex operator is required to be $(1, 1)$; consequently, at a given mass level, the vertex functions are related to each other (see [4] for details). At each mass level there are excitations of various angular momenta of same mass. In other words the states belong to the irreducible representations of $SO(D - 1)$. (v) When we compactify the theory to lower dimensions: $M_D \otimes T^d$, all the states of a given level are classified according to irreducible representations of $SO(D - 1)$ including the states coming from excitations in compact directions (these are all scalars) since total degrees of freedom (at each level) remains the same in both the cases.

In order to construct the desired vertex operators and study their duality properties, let us discuss some of the essential ingredients. Consider a closed string in background of constant metric

$$V_G = \frac{1}{2} \left( \hat{G}_{\hat{\mu}\hat{\nu}} X'^{\hat{\mu}} X'^{\hat{\nu}} + \hat{G}^{\hat{\mu}\hat{\nu}} P_{\hat{\mu}} P_{\hat{\nu}} \right)$$

where $P_{\hat{\mu}}$ is the conjugate momenta. Under the T-duality $P_{\hat{\mu}} \leftrightarrow X'^{\hat{\mu}}$, $\hat{G} \leftrightarrow \hat{G}^{-1}$. If we included constant $\hat{B}$ background we must have $\hat{M} \leftrightarrow \hat{M}^{-1}$ under this duality. In order to see how it works for the first excited massive level consider the vertex operator for the leading Regge trajectory

$$V^{(1)} = \hat{F}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}} \partial X^\hat{\mu} \partial X^\hat{\nu} \partial X^\hat{\rho} \partial X^\hat{\lambda}$$

where $\hat{F}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}}$ is the constant background tensor. When we express $\partial X^\hat{\mu}$ and $\partial \bar{X}^\hat{\mu}$ in terms of canonical momenta and $X'$ then (44) has sixteen terms of various types as products of $P$ and $X'$. There is a term which is product of four $P$'s and another term which is product of four $X'$s as given below.
\[ F^{(1)}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}} P\hat{\mu} P\hat{\nu} P\hat{\rho} P\hat{\lambda}, \quad F^{(2)}_{\mu\hat{\nu}\hat{\rho}\hat{\lambda}} X^{\hat{\mu}} X^{\hat{\nu}} X^{\hat{\rho}} X^{\hat{\lambda}} \]  

There are eight terms with a set of four terms which is product of three \( P \)'s and one \( X' \) and another set of four terms with product of three \( X' \)'s and a \( P \)

\[ F^{(3)}_{\hat{\lambda}} \hat{\mu} P\hat{\mu} P\hat{\nu} P\hat{\rho} X^{\hat{\lambda}}, \quad F^{(4)}_{\hat{\lambda}} \hat{\mu} P\hat{\mu} X^{\hat{\nu}} X^{\hat{\rho}} P\hat{\lambda} \]  

Finally, there are six terms of the type where we have product of two \( P \)'s and two \( X' \)'s

\[ F^{(5)}_{\hat{\mu}\hat{\nu}} X^{\hat{\mu}} X^{\hat{\nu}} P\hat{\mu} P\hat{\lambda} \]  

Now if we interchange \( P\hat{\mu} \leftrightarrow X^{\hat{\mu}} \) and look for duality symmetry we find that \( F^{(1)} \leftrightarrow F^{(2)} \), the four tensors of the type \( F^{(3)} \) interchange with precisely the four tensors of \( F^{(4)} \) type. Finally, the six tensors of \( F^{(5)} \) interchange amongst themselves with appropriate indices \[28\]. Thus, for constant backgrounds \( F^{(i)} \), we have evidence of the \( Z_2 \) duality. As a simple example let us work in the plane wave background i.e. the tensors \( F^{(i)} \) are multiplied by plane waves. Then conformal invariance leads to mass shell conditions and gauge conditions on these tensors. Moreover, there are three more "vertex functions" for the first excited massive states in addition to one which corresponds to the leading Regge trajectory. However, when we impose restrictions of conformal invariance only vertex function associated with leading Regge trajectory survives for the first excited massive level. Furthermore, as we consider higher and higher levels the number of states keep increasing and at very large mass the level degeneracy grows exponentially as is well known. Therefore, the above procedure to verify conjectured duality becomes unmanageable. A more efficient technique was introduced \[4\] to construct duality invariant vertex operators for closed string. We shall briefly recapitulate the result and then proceed to generalize the underlying idea for NSR string in what follows.

Let us focus our attentions on vertex functions associated with compact coordinates, \( Y^i(\sigma, \tau) \). (i) It is recognized that the basic building blocks are \( \partial Y^i = P^i + Y^n \), \( P^i \) being the conjugate momenta; we have suppressed \( \sigma \) and \( \tau \) dependence of \( Y^i \) and \( P_i \) here and everywhere from now on. Similarly, \( \bar{\partial} Y^i = P^i - Y^n \). These building blocks are operated upon by \( \partial \) and \( \bar{\partial} \) respectively and \( \partial \bar{\partial} Y^i = 0 \) by equations of motions. The vertex functions are products of a string of \( P^i \pm Y^n \) and operators obtained by actions of \( \partial, \bar{\partial} \). The resulting tensor is contracted with backgrounds of appropriate rank tensors which depend only on target space (string) coordinates \( \{ X^\mu \} \). (ii) The strategy is as follows: recall that \( O(d, d) \) vector is a doublet

\[ V = \left( \begin{array}{c} P_i \\ Y^n \end{array} \right) \]  

where indices are raised and lowered by \( \delta_{ij} \) and \( \delta_{ij} \). Using an appropriate projection operator the products of \( P^i \pm Y^n \) are re-expressed as products the components of the
$O(d,d)$ vectors $V_i$. Similarly the higher derivatives of these building blocks can be converted to $O(d,d)$ vectors. The product now transforms like a tensor under $O(d,d)$ transformations. This tensor can be contracted with an $X$-dependent suitable tensor. The resulting vertex function is now $O(d,d)$ invariant. As an example consider a generic vertex function for the $n^{th}$ massive level

$$\partial^p Y^{\alpha_1} \partial^q Y^{\alpha_2} \partial^{p'} Y^{\alpha_3} \ldots \partial^{p''} Y^{\alpha_i} \partial^{q'} Y^{\alpha_j} \partial^{r'} Y^{\alpha_k} \ldots, \quad p + q + r = n + 1, \quad p' + q' + r' = n + 1 \quad (49)$$

The constraints on $p, q, r, p', q', r'$ follows from the level matching condition. To convert this tensor to an $O(d,d)$ tensor we refer to the results of [4] and quote that $\partial^p Y = \partial^{p-1}(P + Y')$, $\partial^{p'} Y = \partial^{p'-1}(P - Y')$. Then using the projection operators, $\Delta_{\pm}$, we get

$$\partial^p Y = \Delta_{+}^{p-1} \left( P_+ V + \eta P_- V \right), \quad \partial^{p'} Y = \Delta_{-}^{p'-1} \left( P_+ V - \eta P_- V \right),$$

where $P_\pm$ is another projection operator and $\eta$ being the $O(d,d)$ metric. Thus the tensor in (49) can be converted into product of $O(d,d)$ vectors. The vertex function is given by

$$V_{n+1} = A_{klm..k'l'm'.} \Delta_+^{p-1} V^k_+ \Delta_-^{q-1} V^l_- \Delta_+^{r-1} V^m_+ \Delta_-^{p'-1} V^{k'}_- \Delta_-^{p'-1} V^{l'}_- \Delta_-^{p'-1} V^{m'}_- \quad (50)$$

where $V_\pm = (P_\pm V \pm \eta P_- V)$ with $p + q + r = n + 1$ and $p' + q' + r' = n + 1$. Note that superscripts $\{k, l, m; k', l', m'\}$ appearing on $V_\pm$ in eq. (50) are the indices of the components of the projected $O(d,d)$ vectors. Moreover, $A_{klm..k'l'm'.}$ is $X$-dependent $O(d,d)$ tensor. Note that (50) will be $O(d,d)$ invariant if coefficients $A$-tensors transform as

$$A_{klm..k'l'm'} \rightarrow \Omega^p_k \Omega^q_l \Omega^r_m \ldots \Omega^{p'}_{k'} \Omega^{q'}_{l'} \Omega^{r'}_{m'} A_{pqr..p'q'r'} \quad (51)$$

since each term in the product $\Delta_+^{p-1} V^k_+ \ldots \Delta_-^{p'-1} V^{k'}_-$, above, transforms like an $O(d,d)$ vector.

We mention in passing that the constraints of conformal invariance need not be imposed at this stage while we are investigating duality symmetries. Those requirements further restrict the structures of vertex operators and provide useful relations among vertex functions besides imposing mass shell conditions for a given mass level.

We intend to derive analogous results for the vertex functions of the excited massive states of NSR string. Notice that for the first excited level on the leading Regge trajectory for NSR string will have a lot more terms compared to (44) since we can construct additional terms which contract with chiral worldsheet fermions. For example, we can have generic terms like

$$G^{(1)}_{\mu\nu\lambda\delta} \partial X^\mu \partial X^\nu \psi_-^\lambda \psi_-^\delta \partial X^\delta, \quad G^{(2)}_{\mu\nu\lambda\delta} \partial X^\mu \partial X^\nu \psi_-^\lambda \psi_-^\delta \partial X^\delta$$

$$G^{(3)}_{\mu\nu\lambda\delta} \partial X^\mu \partial X^\nu \psi_-^\lambda \psi_-^\delta \partial X^\delta, \quad G^{(4)}_{\mu\nu\lambda\delta} \partial X^\mu \partial X^\nu \psi_-^\lambda \psi_-^\delta \partial X^\delta$$

(52)
and several other terms where \( \partial \) is replaced by \( \bar{\partial} \) and \( \psi_+ \) is replaced by \( \psi_- \) so long as we ensure, to start with, we have maintained same dimensionality for product of left movers and right movers with respect to the two stress energy momentum tensors. The discrete \( Z_2 \) symmetry alluded to in (12) will be maintained if we take into account all required terms for the vertex operator under considerations. It is quite obvious even keeping track of all terms for the vertex operators of some of the low lying excited massive levels is going to be not very efficient if we want to check the conjectured T-duality for superstrings in terms of the bosonic coordinates and NSR fermions. So far there is no construction of manifestly \( O(d, d) \) invariant vertex operators for NSR string along compact directions even for the massless sector i.e. massless scalars that arise from compactification of \( \hat{G} \) and \( \hat{B} \).

Therefore, we resort to the superfield approach and consider vertex functions for massive excited states constructed out of the superderivatives of superfields. There are two types of generic vertex functions

(i) \( \bar{D}W_i \bar{D}W^i_2 \ldots \bar{D}W^{i_m} \bar{D}W^{j_2} \ldots \bar{D}W^{j_m} \). These correspond to leading Regge trajectories.

(ii) \( \bar{D}'W_i \bar{D}'W^i_2 \ldots \bar{D}'W^{i_m}D^{p'}W^{j_1}D^{q'}W^{j_2} \ldots D^{r'}W^{j_m} \) and we require \( p + q + r = p' + q' + r' \).

We proceed with following step for the case (i) (I) Recall that \( U \) is an \( O(d, d) \) vector, whose upper component is \( W_i \) and the lower component is its dual \( \tilde{W}_i \). Introduce two projection operators

\[
\tilde{P}_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{P}_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
\] (53)

Note that \( \tilde{P}_+ U = W \).

(II) Introduce a doublet through the pair \((D, \bar{D})\).

\[
\mathcal{D} = \begin{pmatrix} D \\ \bar{D} \end{pmatrix}
\] (54)

Then projection operators

\[
\tilde{\Delta}_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{\Delta}_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
\] (55)

Note that \( \tilde{\Delta}_\pm \) are \( 2d \times 2d \) dimensional projectors whereas \( \tilde{\Delta}_\pm \) are \( 2 \times 2 \) projectors.

The vertex functions which assume the form given in (i) above can be cast as products of \( O(d, d) \) vectors

\[
\tilde{\Delta}_- \mathcal{D} \tilde{P}_+ U^{\alpha_1} \ldots \tilde{\Delta}_- \mathcal{D} \tilde{P}_+ U^{\alpha_m} \tilde{\Delta}_+ \mathcal{D} \tilde{P}_+ U^{\beta_1} \ldots \tilde{\Delta}_+ \mathcal{D} \tilde{P}_+ U^{\beta_m}
\] (56)

Thus we have an \( O(d, d) \) tensor of rank \( 2m \). We contract it with a tensor, which depends on spacetime superfield \( \partial^\mu \) to construct an \( O(d, d) \) invariant vertex function for the \( n^{th} \) massive level.

\[
V_{n+1} = T_{\alpha_1 \ldots \alpha_m \beta_1 \ldots \beta_m} \tilde{\Delta}_- \mathcal{D} \tilde{P}_+ U^{\alpha_1} \ldots \tilde{\Delta}_- \mathcal{D} \tilde{P}_+ U^{\alpha_m} \tilde{\Delta}_+ \mathcal{D} \tilde{P}_+ U^{\beta_1} \ldots \tilde{\Delta}_+ \mathcal{D} \tilde{P}_+ U^{\beta_m}
\] (57)
If the $O(d, d)$ vector transforms as: $U^{\alpha_1} \rightarrow \Omega^{\alpha_1}_{\alpha_1'} U^{\alpha_1'}$, then we require

$$T_{\alpha_1... \alpha_m \beta_1... \beta_m} \rightarrow \Omega^{\alpha_1'}_{\alpha_1} \cdot \cdots \cdot \Omega^{\alpha_m}_{\alpha_m} \cdot \Omega^{\beta_1'}_{\beta_1} \cdots \cdot \Omega^{\beta_m}_{\beta_m} T_{\alpha_1' ... \alpha_m' \beta_1' ... \beta_m'}$$  \hspace{1cm} (58)

so that the vertex function $V_{n+1}$ is T-duality invariant.

Now we focus attention on the second type of vertex function mentioned in (ii) above. Note that a typical term appearing in the product is like $(\hat{T})^p$ and $(\hat{D})^{\nu}$. We can use the projection operators introduced in (I) and (II) above to express products of such terms as

$$\hat{\Delta}^p \hat{P}_+ U^{\alpha_1} \Delta^q \hat{P}_+ U^{\alpha_2} \cdots \hat{\Delta}^{p'} \hat{P}_+ U^{\alpha_m} \Delta^{q'} \hat{P}_+ U^{\beta_1} \Delta^{q'} \hat{P}_+ U^{\beta_2} \cdots \Delta^{p'} \hat{P}_+ U^{\beta_m}$$  \hspace{1cm} (59)

Now this is an $O(d, d)$ tensor of rank $p + q + r$ satisfying the level matching condition.

As in the previous case, we have to just contract with a tensor (which depends on superfield $\phi$) to get an $O(d, d)$ invariant vertex function.

So far we have left out two other possibilities which we dwell upon now. There are two more types of vertex functions in a given level:

(a) We can have a situation that the vertex function has product of mixed set of operators i.e. some of the superfields correspond to spacetime coordinates and some to compact ones.

$$T_{\mu_1... \mu_k \alpha_1... \alpha_i \beta_1... \beta_s} \hat{D}^{\mu_1} \cdots \hat{D}^{\mu_k} \hat{D}^{W^{\alpha_1} \hat{D}^{W^{\alpha_2} \cdots \hat{D}^{W^{\alpha_i} \hat{D}^{W^{\beta_1} \cdots \hat{D}^{W^{\beta_s}}}}}}$$  \hspace{1cm} (60)

First notice that $\phi^\mu$ is inert under T-duality transformations. Similarly, all the spacetime indices of the tensor $T_{\mu_1... \mu_k \alpha_1... \alpha_i \beta_1... \beta_s}$ do not get transformed under T-duality. Moreover, all the spacetime indices of $T$ are contracted with spacetime superfields so that effectively we deal with a tensor with ”internal” indices which are contracted with product of building blocks consisting of superderivatives of $W$’s i.e $\hat{D}$ or $D$ acting on $W$’s. We already presented a prescription of constructing $O(d, d)$ invariant vertex functions out of such products. Therefore, any arbitrary vertex function of a given massive level can be expressed in an $O(d, d)$ invariant form.

(b) There is another class of vertex functions which are product of the superderivatives of the spacetime superfields only. However, this class of vertex functions are automatically T-duality invariant since the superderivatives $\phi^\mu$ and corresponding $\phi$-dependent tensors are not sensitive to $O(d, d)$ transformations.

In conclusion for an NSR string compactified on $T^d$, we can express all vertex functions at each massive level in T-duality invariant form.

We address another point in the context of type IIB theory. It is well known that this theory in endowed with S-duality symmetry. Its massless spectrum in NS-NS sector for critical dimension ($\hat{D} = 10$) consists of graviton, $\hat{g}_{\mu \nu}$, 2-form antisymmetric tensor, $\hat{B}_{\mu \nu}^{(1)}$ and dilaton, $\hat{\phi}$. The R-R sector is axion, $\hat{\chi}$, 2-form antisymmetric tensor, $\hat{B}_{\mu \nu}^{(2)}$ and a four form tensor $\hat{C}_{\mu \nu \rho \lambda}^{(4)}$ whose field strength is required to be self dual. The effective action for the type IIB theory may be expressed in an S-duality invariant
form. When we toroidally compactify the theory to lower dimension, the reduced effective can also written in S-duality invariant form. Therefore, starting from NS-NS backgrounds, we can generate RR backgrounds of the reduced theory; however, the reduced tensors of four form $C^{(4)}$ cannot be generated from the NS-NS backgrounds. Let us closely examine the case when type IIB theory is compactified on $T^4$ to a six dimensional theory and focus our attention on the moduli and the vector fields coming from reduction of backgrounds. We expect that the massless states coming from NS-NS sector will be classified according to representations of $O(4,4)$. In fact the moduli parametrizes the coset $\frac{O(4,4)}{O(4) \otimes O(4)}$ as was demonstrated by us [3]. The counting is quite simple: the moduli coming from compactification of the graviton and the 2-form antisymmetric tensor add up to 16 as expected. The gauge fields originating from metric and antisymmetric tensor (from NS-NS sector) belong to the vector representation of $O(4,4)$. Indeed, in the NS-NS sector the worldsheets exhibit presence of all these massless fields, if we follow prescriptions of ref [3]. Let us turn to the R-R sector. There are 9 scalar appearing due to compactification of the 2-form, $B^{(2)}$, the 4-form $C^{(4)}$ besides the axion. The number of vector fields are eight: four from $B^{(2)}$ and four from $C^{(4)}$. We should also take into account the underlying S-duality symmetry: $SL(2,\mathbb{Z})$. It is more appropriate to classify massless states of the toroidally compactified six dimensional theory combining the states from NS-NS and RR sector. The arguments are along the same line as classification of branes (hence classifying the background tensors) in the context of toroidal compactification of type IIB theory and M-theory [29, 30, 31]. They belong to representations of $O(5,5)$ from this perspective. Whereas the 25 ($=16+9$) moduli parametrize the coset $\frac{O(5,5)}{O(5) \otimes O(5)}$, the 16 ($=8+8$) massless vectors belong to the spinor representation of $O(5,5)$. The other backgrounds, in the six dimensional theory, also belong to appropriate representation of this group.

We note that one can study the T-duality attributes of the NS-NS massless backgrounds of the theory compactified on $T^4$ in the worldsheets approach presented here. The massive excited backgrounds along the compact direction, in the NS-NS sector, can be coupled to corresponding worldsheets supercoordinates. We are able to express the vertex operators for each of such levels in a manifestly T-duality invariant form. However, it is not possible to construct similar vertex operators for the RR sector in the present formulation.

5. SUMMARY AND CONCLUSIONS

In this article we focused our attention to investigate T-duality properties of NSR string in its massless backgrounds. We have shown that the worldsheets equation of motion of the NSR string compactified on $T^d$ can be expressed in an $O(d,d)$ covariant form in the presence of backgrounds. In order to achieve this objective it is most appropriate to adopt the worldsheets superfield formalism. The equations of

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In this article we focused our attention to investigate T-duality properties of NSR string in its massless backgrounds. We have shown that the worldsheets equation of motion of the NSR string compactified on $T^d$ can be expressed in an $O(d,d)$ covariant form in the presence of backgrounds. In order to achieve this objective it is most appropriate to adopt the worldsheets superfield formalism. The equations of
motion corresponding to superfields along compact directions were recognized to be conservation laws when the background components along compact directions depend only on spacetime superfield coordinates. We introduced a set of dual superfields and a set of dual backgrounds to write a corresponding dual action. The equation of motion for this case is also expressed as a conservation law. It is shown that the two sets of equations of motion can be judiciously combines and cast in an $O(d,d)$ covariant form.

We generalized an earlier prescription for closed bosonic string case to construct vertex operators for massive excited levels of NSR string in the superspace formulation. It was shown that the vertex operators for each massive level can be cast in an $O(d,d)$ invariant form. Moreover, for a special case when type IIB theory is compactified on $AdS_3 \otimes S^3 \otimes T^4$, it is possible to construct vertex operators for excited massive levels in $O(4,4)$ invariant form. This is an improvement over our result for the closed bosonic string case where the spacetime geometry was taken to be flat.

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References

1. M. B. Green, J. H. Schwarz and E. Witten, Superstring Theory, Vol I and Vol II, Cambridge University Press, 1987;
J. Polchinski, String Theory, Vol I and Vol II, Cambridge University Press, 1998;
K. Becker, M. Becker and J. H. Schwarz, String Theory and M-Theory: A Modern Introduction, Cambridge University Press, 2007;
B. Zwiebach, A First Course in String Theory, Cambridge University Press, 2004.

2. For reviews: A. Giveon, M. Porrati and E. Rabinovici, Phys. Rep. C244 1994 77;
J. E. Lidsey, D. Wands, and E. J. Copeland, Phys. Rep. C337 2000 343;
M. Gasperini and G. Veneziano, Phys. Rep. C373 2003 1.

3. J. Maharana, J. H. Schwarz, Nucl. Phys. B390 (1993) 3.

4. J. Maharana, Nucl. Phys. B843 (2011) 753; arXiv:10101434.

5. J. Scherk and J. H. Schwarz, Nucl. Phys. B194 (1979) 61.

6. K. Kikkawa and M. Yamazaki, Phys. Lett. 149B (1984) 357;
N. Sakai and I. Sanda, Prog. Theor. Phys. 75 (1986) 692;
V. P. Nair, A Shapere, A. Strominger, and F. Wilczek, Nucl. Phys. 287B (1987) 402;
B. Sathiapalan, Phys. Rev. Lett. 58 (1987) 1597;
R. Dijkgraaf, E. Verlinde, and H. Verlinde, Commun. Math. Phys. 115 (1988 649;
K. S. Narain, Phys. Lett. B169 (1986) 41;
K. S. Narain, M. H. Sarmadi, and E. Witten, Nucl. Phys. B279 (1987) 369;
P. Ginsparg, Phys. Rev. D35 (1987) 648;
P. Ginsparg and C. Vafa, Nucl. Phys. B289 (1987) 414;
S. Cecotti, S. Ferrara and L. Giraldello, Nucl. Phys. B308 (1988) 436;
R. Brandenberger and C. Vafa, Nucl. Phys. B316 (1988) 391;
M. Dine, P. Huet, and N. Seiberg, Nucl. Phys. B322 (1989) 301;
J. Molera and B. Ovrut, Phys. Rev. D40 (1989) 1146;
G. Veneziano, Phys. Lett. B265 1991 287;
A. A. Tseytlin and C. Vafa, Nucl. Phys. B372 (1992) 443;
M. Rocek and E. Verlinde, Nucl. Phys. 373 (1992) 630;
J. H. Horne, G. T. Horowitz, and A. R. Steif, Phys. Rev. Lett. 68 (1992) 568;
A. Sen, Phys. Lett. B271 (1992) 295.
7. A. Shapere and F. Wilczek, Nucl. Phys. B320 (1989) 669; A. Giveon, E. Rabinovici, and G. Veneziano, Nucl. Phys. B322 (1989) 167; A. Giveon, N. Malkin, and E. Rabinovici, Phys. Lett. B220 (1989) 551; W. Lerche, D. Lüst, and N. P. Warner, Phys. Lett. B231 (1989) 417.

8. K. Meissner and G. Veneziano, Phys. Lett. B267 (1991) 33; Mod. Phys. Lett. A6 (1991) 3397; M. Gasperini, J. Maharana, and G. Veneziano, Phys. Lett. B272 1991 277; Phys. Lett. B296 1992 51.

9. M. J. Duff, Nucl. Phys. B335 (1990) 610.

10. J. Maharana, Phys. Lett. B296 (1992) 65; hep-th/9205015.

11. E. Witten, Phys. Rev. Lett. 61 (1988) 670; A. A. Tseytlin, Phys. Lett. B242 (1990) 163; Nucl. Phys. B350 (1991) 395; Phys. Rev. Lett. 66 (1991) 545.

12. T. Kugo and B. Zwiebach, Prog. Th. Phys. 87 (1992) 801; C. Hull and B. Zwiebach, JHEP, 0909 (2009) 099, arXiv:0904.4664; C. Hull and B. Zwiebach, JHEP 0909 (2009) 090, arXiv:0908.1792; A. Dabholkar and C. Hull, JHEP 0605 (2006)009, arXiv:hep-th/0512005; O. Hohm, C. Hull and B. Zwiebach, JHEP 1008 (2010) 008, arXiv:1006.4823; O. Hohm, S. K. Kwak and B. Zwiebach, Double Field Theory of Type II Strings, arXiv:1107.0008.

13. D. S. Berman and D. C. Thompson, Phys. Lett. B662 (2008) and references therein.

14. A. Das and J. Maharana Mod. Phys. Lett. A9 (1994) 1361; hep-th/9401147.

15. W. Siegel, Phys. Rev. D48 (1993) 2826; hep-th/9308138.

16. E. Alvarez,L. Alvarez-Gaume and Y. Lozano, Phys. Lett. B336 (1994); hep-th/9406206; S.F. Hassan, Nucl. Phys. B460 (1995) 362; hep-th/9504148; T. Curtright, T. Uematsu and C. Zachos, Nucl. Phys. 469 (1996) 488; hep-th/9601096; B. Kulik and R. Roiban, JHEP 0209 (2002) 007; hep-th/0012010.

17. S. F. Hasan and A. Sen, Nucl. Phys. B375 (1992) 103.

18. E. Abdalla and M.C.B. Abadalla, Phys. Lett. B152 (1984) 50. E.Abdalla and K. Rothe, Nonperturbative Methods in Two Dimensional Quantum Field Theory, World Scientific, Singapore 1991. Also see J. Maharana, Mod. Phys. Lett. A20 (2005) 2317.
19. P. di Vecchia, V. G. Knizhnik, J. L. Peterson and P. Rossi, Nucl. Phys. B253 (1985) 701.

20. E. Braaten, T. Curtright and C. Zachos, Nucl. Phys. B260 (1984) 630.

21. For a review see O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, Phys. Rep. C323 (2000) 183.

22. D. J. Gross, P. Mende; Phys. Lett. B197 (1987) 129; Nucl. Phys. B303 (1988) 407.

23. D. Amati, M. Ciafaloni, G. Veneziano, Phys. Lett. B197 (1987) 81; Int. J. Mod. Phys. A3 (1988) 1615; Phys. Lett. B216 (1989) 41; Phys. Lett. B289 (1989) 87; Nucl. Phys. B403 (1993) 707.

24. D. J. Gross, Phys. Rev. Lett. 60 (1988).

25. J. Maharana and G. Veneziano (unpublished works, 1986, 1991 and 1993).
   J. Maharana, Novel Symmetries of String Theory, in String Theory and Fundamental Interactions, Springer Lecture Notes in Physics, Vol. 737 p525, Ed. G. Gasperini and J. Maharana Springer 2008, Berlin Heidelberg.

26. E. Evans and B. Ovrut, Phys. Rev. D39 (1989) 3016; Phys. Rev. D41 (1990) 3149; J-C. Lee and B. A. Ovrut, Nucl. Phys. B336 (1990) 222.

27. R. Akhoury and Y. Okada; Nucl. Phys. B318 (1989) 176.

28. J. Maharana, Phys. Lett. B695 (2011) 370; arXiv:10101727.

29. J. H. Schwarz, Phys. Lett. B360 (1995) 13, arXiv:hep-th/9508143.

30. J. H. Schwarz, Phys. Lett. B367 (1996) 97, arXiv:hep-th/9509148.

31. J. Maharana, Phys. Lett. B372 (1996) 53, arXiv:hep-th/9511159.