Percolation and lattice animals: exponent relations, and conditions for $\theta(p_c) = 0$

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Abstract We examine the percolation model on $Z^d$ by an approach involving lattice animals and their surface-area-to-volume ratio. For $\beta \in [0, 2(d - 1))$, let $f(\beta)$ be the asymptotic exponential rate in the number of edges of the number of lattice animals containing the origin which have surface-area-to-volume ratio $\beta$. The function $f$ is bounded above by a function which may be written in an explicit form. For low values of $\beta$ ($\beta \leq 1/p_c - 1$), equality holds, as originally demonstrated by F.Delyon. For higher values ($\beta > 1/p_c - 1$), the inequality is strict.

We introduce two critical exponents, one of which describes how quickly $f$ falls away from the explicit form as $\beta$ rises from $1/p_c - 1$, and the second of which describes how large clusters appear in the marginally subcritical regime of the percolation model. We demonstrate that the pair of exponents must satisfy certain inequalities, while other such inequalities yield sufficient conditions for the absence of an infinite cluster at the critical value. The first exponent is related to one of a more conventional nature in the scaling theory of percolation, that of correlation size. In deriving this relation, we find that there are two possible behaviours, depending on the value of the first exponent, for the typical surface-area-to-volume ratio of an unusually large cluster in the marginally subcritical regime.

This paper provides an account of the central aspects of the approach, including the proofs of the main results. In the longer report [5], complete proofs of all of the assertions are given.

Keywords Percolation, lattice animals, critical exponents.

1 Introduction

Percolation on the integer lattice $Z^d$ is one of the most fundamental and intensively studied models in the rigorous theory of statistical mechanics.
Many aspects of the behaviour of the model in the subcritical and supercritical regime have been determined rigorously. The problem of understanding the behaviour of the model at criticality, and interplay between this behaviour and that for parameter values nearby, has been addressed widely by physicists, but the search for proofs of many of their predictions continues. These predictions typically take the form of asserting the value of critical exponents, and thereby describe the power-law decay or explosion of characteristics of the model near criticality.

In this paper, we examine the percolation model by an approach involving lattice animals, divided according to their surface-area-to-volume ratio. Throughout, we work with the bond percolation model in $\mathbb{Z}^d$. However, the results apply to the site or bond model on any infinite transitive amenable graph with inessential changes.

For any given $p \in (0,1)$, two lattice animals with given size are equally likely to arise as the cluster $C(0)$ containing the origin provided that they have the same surface-area-to-volume ratio. For given $\beta \in (0,\infty)$, there is an exponential growth rate in the number of edges for the number of lattice animals up to translation that have surface-area-to-volume ratio very close to $\beta$. This growth rate $f(\beta)$ may be studied as a function of $\beta$. To illustrate the connection between the percolation model and the combinatorial question of the behaviour of $f$, note that the probability that the cluster containing the origin contains a large number $n$ of edges is given by

$$
\mathbb{P}_p(|C(0)| = n) = \sum_m \sigma_{n,m} p^n (1-p)^m,
$$

where $\sigma_{n,m}$ is the number of lattice animals that contain the origin, have $n$ edges and $m$ outlying edges. We rewrite the right-hand-side to highlight the role of the surface-area-to-volume ratio, $m/n$:

$$
\mathbb{P}_p(|C(0)| = n) = \sum_m (f_n(m/n)p(1-p)^{m/n})^n. \tag{1}
$$

Here $f_n(\beta) = (\sigma_n,\lfloor \beta n \rfloor)^{1/n}$ is a rescaling that anticipates the exponential growth that occurs. We examine thoroughly the link between percolation and combinatorics provided by Equation 1.

An overview of the approach is now given, in the form of a description of the organisation of the paper. In Section 2 we describe the model, and define notations, before stating the combinatorial results that we will use. The proofs are largely omitted, as are a few results in later sections. (We refer the interested reader to the report [5], in which all proofs are given in full, along with some notes on the literature.) The combinatorial results assert the existence of the function $f$ and describe aspects of its behaviour, Theorem 2.2 implying that

$$
\log f(\beta) \leq (\beta + 1) \log(\beta + 1) - \beta \log \beta \text{ for } \beta \in (0,2(d - 1)). \tag{2}
$$
F. Delyon showed that equality holds for $\beta \in (0, 1/p_c - 1)$. Theorem 2 implies that the inequality is strict for higher values of $\beta$. The marked change, as $\beta$ passes through $1/p_c - 1$, in the structure of large lattice animals of surface-area-to-volume ratio $\beta$ is a combinatorial analogue of the phase transition in percolation at criticality. The notion of a collapse transition for animals has been explored in [3].

In Section 3, two scaling hypotheses are introduced, each postulating the existence of a critical exponent. One of the exponents, $\varsigma$, describes how quickly $f(\beta)$ drops away from the explicit form given on the right-hand-side of (2) as $\beta$ rises above $1/p_c - 1$. The other, $\lambda$, describes how rapidly decaying in $n$ is the discrepancy between the critical value and that value on the subcritical interval at which the probability of observing an $n$-edged animal as the cluster to which the origin belongs is maximal. The first main result, Theorem 3.1, is then proved: the inequalities $\lambda < 1/2$ and $\varsigma \lambda < 1$ cannot both be satisfied, because they imply that the mean cluster size is uniformly bounded on the subcritical interval, contradicting known results.

In Section 4, sufficient conditions for the absence of an infinite cluster at the critical value are proved. Theorem 4.1 asserts that $\varsigma < 2$ or $\lambda > 1/2$ are two such conditions. Except for some borderline cases, the range of values remaining after Theorems 3.1 and 4.1 is specified by $\lambda < 1/2$ and $\varsigma \lambda > 1$. In Theorem 4.3, where we see that in this case, such a sufficient condition may be expressed in terms of the extent to which the asymptotic exponential rate $f(\beta)$ is underestimated by its finite approximants $f_n(\beta)$ for a certain range of values of $\beta$. The extent of underestimation is related to combinatorial exponents such as the entropic exponent (see, for example, [6]).

In Section 5, we relate the value of $\varsigma$ to an exponent of a more conventional nature in the scaling theory of percolation, that of correlation size (see Theorem 5.1). Suppose that we perform an experiment in which the surface-area-to-volume ratio of the cluster to which the origin belongs is observed, conditional on its having a very large number of edges, for a $p$-value slightly below $p_c$. How does the typical measurement, $\beta_p$, in this experiment behave as $p$ tends to $p_c$? The value $\beta_p$ tends to lie somewhere on the interval $(1/p_c - 1, 1/p - 1)$. In Theorem 5.2, we determine that there are two possible scaling behaviours. The inequality $\varsigma < 2$ again arises, distinguishing the two possibilities. If $\varsigma < 2$, then $\beta_p$ scales much closer to $1/p_c - 1$ while if $\varsigma > 2$, it is found to be closer to $1/p - 1$.

2 Notations and combinatorial results

Throughout, we work with the bond percolation model on $\mathbb{Z}^d$, for any given $d \geq 2$. This model has a parameter $p$ lying in the interval $[0, 1]$. Nearest neighbour edges of $\mathbb{Z}^d$ are declared to be open with probability $p$, these choices being made independently between distinct edges. For any vertex
$x \in \mathbb{Z}^d$, there is a cluster $C(x)$ of edges accessible from $x$, namely the collection of edges that lie in a nearest-neighbour path of open edges one of whose members contains $x$ as an endpoint. The percolation probability $\theta(p)$ as a function of $p$ may then be written $\theta(p) = \mathbb{P}(|C(0)| = \infty)$. To demonstrate the continuity of $\theta$, it suffices to show that $\theta(p_c) = 0$ (cf [4]), where $p_c$ denotes the critical value, namely the infimum of those values of $p$ for which $\theta$ is positive.

**Definition 2.1** A lattice animal is the collection of edges of a finite connected subgraph of $\mathbb{Z}^d$. An edge of $\mathbb{Z}^d$ is said to be outlying to a lattice animal if it is not a member of the animal, and if there is an edge in the animal sharing an endpoint with this edge. We adopt the notations:

- for $n, m \in \mathbb{N}$, set $\Gamma_{n,m}$ equal to the collection of lattice animals in $\mathbb{Z}^d$ one of whose edges contains the origin, having $n$ edges, and $m$ outlying edges. Define $\sigma_{n,m} = |\Gamma_{n,m}|$. The surface-area-to-volume ratio of any animal in $\Gamma_{n,m}$ is said to be $m/n$.

- for each $n \in \mathbb{N}$, define the function $f_n : [0, \infty) \to [0, \infty)$ by
  \[
  f_n(\beta) = (\sigma_{n,\lceil \beta n \rceil})^{1/n}
  \]

On another point of notation, we will sometimes write the index set of a sum in the form $nS$, with $S \subseteq (0, \infty)$, by which is meant $\{m \in \mathbb{N} : m/n \in S\}$. We require some results about the asymptotic exponential growth rate of the number of lattice animals as a function of their surface-area-to-volume ratio. The proofs of the theorems stated here are given in [3].

**Theorem 2.1**

1. For $\beta \in [0, \infty) - \{2(d-1)\}$, $f(\beta)$ exists, being defined as the limit $\lim_{n \to \infty} f_n(\beta)$.
2. for $\beta > 2(d-1)$, $f(\beta) = 0$.
3. for $\beta \in (0, 2(d-1))$, $n \in \mathbb{N}$, $f_n(\beta)$ satisfies $f_n(\beta) \leq L^{1/n}n^{1/n}f(\beta)$, where the constant $L$ may be chosen uniformly in $\beta \in (0, 2(d-1))$.

**Theorem 2.2**

1. $f$ is log-concave on the interval $(0, 2(d-1))$.
2. Introducing $g : (0, 2(d-1)) \to [0, \infty)$ by means of the formula
   \[
   f(\beta) = g(\beta)\frac{(\beta + 1)^{\beta+1}}{\beta^\beta},
   \]
   we have that
   \[
   g(\beta) \begin{cases} 
   1 & \text{on } (0, \alpha), \\
   < 1 & \text{on } (\alpha, 2(d-1)),
   \end{cases}
   \]
   where throughout $\alpha$ denotes the value $1/p_c - 1$.  

4
Remark The assertion that $g = 1$ on $(0, \alpha]$ was originally proved by Delyon. We include here the proof of the other part of the theorem.

**Proof** We must show that, for $\beta \in (\alpha, 2(d-1))$, $g(\beta)$ is strictly less than one. Let $\beta$ lie in this interval. Let $p = 1/(1 + \beta)$. Note that $p < p_c$, and that

$$
\mathbb{P}_p(|C(0)| = n) \geq \mathbb{P}_p(C(0) \in \Gamma_{n,\lfloor \beta n \rfloor})
= |\Gamma_{n,\lfloor \beta n \rfloor}| \frac{\beta^{\lfloor \beta n \rfloor}}{(1 + \beta)^{n+\lfloor \beta n \rfloor}}
= (f_n(\beta))^n \frac{\beta^{\lfloor \beta n \rfloor}}{(1 + \beta)^{n+\lfloor \beta n \rfloor}}.
$$

Taking logarithms yields

$$
\frac{\log \mathbb{P}_p(|C(0)| = n)}{n} \geq \log f_n(\beta) + \frac{\beta n \log \beta}{n} - \left(1 + \frac{\beta n}{n}\right) \log(1 + \beta),
$$

from which it follows that

$$
\liminf_{n \to \infty} \frac{\log \mathbb{P}_p(|C(0)| = n)}{n} \geq \log f(\beta) + \beta \log \beta - (1 + \beta) \log(1 + \beta). \tag{3}
$$

The right-hand-side of (3) is equal to $\log g(\beta)$, by definition. The exponential decay rate for the probability of observing a large cluster in the subcritical phase was established in [1]. Since $p < p_c$, this means the left-hand-side of (3) is negative. This implies that $g(\beta) < 1$, as required.

\[ \square \]

3 Critical exponents and inequalities

We introduce two scaling hypotheses, each of which proposes the existence of a critical exponent. We then state and prove the first main theorem, which demonstrates that a pair of inequalities involving the two exponents cannot both be satisfied.

**Hypothesis** ($\lambda$)

**Definition 3.1** For each $n \in \mathbb{N}$, let $t_n \in (0, p_c)$ denote the least value satisfying the condition

$$
\sum_m \sigma_{n,m} t_n^m (1 - t_n)^m = \sup_{p \in (0, p_c]} \sum_m \sigma_{n,m} p^n (1 - p)^m. \tag{4}
$$

That is, $t_n$ is some point at or below the critical value at which the probability of observing an $n$-edged animal as the cluster to which the origin belongs is maximal. It is reasonable to suppose that $t_n$ is slightly less than $p_c$, and that the difference decays polynomially in $n$ as $n$ tends to infinity.
Definition 3.2 Define \( \Omega^+_\lambda = \{ \beta \geq 0 : \liminf_{n \to \infty} (p_c - t_n)/n^{-\beta} = \infty \} \), and \( \Omega^-_\lambda = \{ \beta \geq 0 : \limsup_{n \to \infty} (p_c - t_n)/n^{-\beta} = 0 \} \). If \( \sup \Omega^-_\lambda = \inf \Omega^+_\lambda \), then hypothesis (\( \lambda \)) is said to hold, and \( \lambda \) is defined to be equal to the common value.

So, if hypothesis (\( \lambda \)) holds, then \( p_c - t_n \) behaves like \( n^{-\lambda} \), for large \( n \). We remark that it would be consistent with the notion of a scaling window about criticality that the probability of observing the cluster \( C(0) \) with \( n \)-edges achieves its maximum on the subcritical interval on a short plateau whose right-hand endpoint is the critical value. If this is the case, then \( t_n \) should lie at the left-hand endpoint of the plateau. To be confident that \( p_c - t_n \) is of the same order as the length of this plateau, the definition of the quantities \( t_n \) could be changed, so that a small and fixed constant multiples the right-hand-side of (4). In this paper, any proof of a statement involving the exponent \( \lambda \) is valid if it is defined in terms of this altered version of the quantities \( t_n \).

Hypothesis (\( \varsigma \))

This hypothesis is introduced to describe the behaviour of \( f \) for values of the argument just greater than \( \alpha \). Theorem 2.2 asserts that the value \( \alpha \) is the greatest for which \( \log f(\beta) = (\beta + 1) \log(\beta + 1) - \beta \log \beta \); the function \( g \) was introduced to describe how \( \log f \) falls away from this function as \( \beta \) increases from \( \alpha \). Thus, we phrase hypothesis (\( \varsigma \)) in terms of \( g \).

Definition 3.3 Define \( \Omega^-_\varsigma = \{ \beta \geq 0 : \liminf_{\delta \to 0} (g(\alpha + \delta) - g(\alpha))/\delta^\beta = 0 \} \), and \( \Omega^+_\varsigma = \{ \beta \geq 0 : \limsup_{\delta \to 0} (g(\alpha + \delta) - g(\alpha))/\delta^\beta = -\infty \} \). If \( \sup \Omega^-_\varsigma = \inf \Omega^+_\varsigma \), then hypothesis (\( \varsigma \)) is said to hold, and \( \varsigma \) is defined to be equal to the common value.

If hypothesis (\( \varsigma \)) holds, then greater values of \( \varsigma \) correspond to a smoother behaviour of \( f \) at \( \alpha \). For example, if \( \varsigma \) exceeds \( N \) for \( N \in \mathbb{N} \), then \( f \) is \( N \)-times differentiable at \( \alpha \).

Theorem 3.1 Suppose that hypotheses (\( \varsigma \)) and (\( \lambda \)) hold. If \( \lambda < 1/2 \), then \( \varsigma \lambda \geq 1 \).

Proof We prove the Theorem by contradiction, assuming that the two hypotheses hold, and that \( \lambda < 1/2 \), \( \varsigma \lambda < 1 \). We will arrive at the conclusion that the mean cluster size, given by \( \sum_n n \mathbb{P}_p(|C(0)| = n) \), is bounded above, uniformly for \( p \in (0, p_c) \). That this is not so is proved in [1]. Note that

\[
\sup_{p \in (0, p_c)} \sum_n n \mathbb{P}_p(|C(0)| = n) \leq \sum_n n \mathbb{P}_{t_n}(|C(0)| = n).
\]

We write

\[
\mathbb{P}_{t_n}(|C(0)| = n) = \sum_m \sigma_{n,m} t_n^m (1 - t_n)^m, \tag{5}
\]

and split the sum on the right-hand-side of (5). To do so, we use the following definition.
Definition 3.4 For $n \in \mathbb{N}$, let $\alpha_n$ be given by $t_n = 1/(1 + \alpha_n)$. For $G \in \mathbb{N}$, let $D_n(=D_n(G))$ denote the interval

$$D_n = (\alpha_n - G\{\log(n)/n\}^{1/2}, \alpha_n + G\{\log(n)/n\}^{1/2}).$$

Now,

$$\sum_m \sigma_n,m t_n^n(1-t_n)^m = C_1(n) + C_2(n) + C_3(n),$$

where the terms on the right-hand-side are given by

$$C_1(n) = \sum_{m \in nD_n} \sigma_n,m t_n^n(1-t_n)^m,$$

$$C_2(n) = \sum_{m \in (0,2(d-1))-D_n} \sigma_n,m t_n^n(1-t_n)^m$$

and

$$C_3(n) = \sum_{m \in \{2(d-1)n,...,2(d-1)n+2d\}} \sigma_n,m t_n^n(1-t_n)^m.$$

Definition 3.5 Let the function $\phi : (0, \infty)^2 \to \mathbb{R}$ be given by

$$\phi(\alpha, \beta) = (\beta + 1) \log(\beta + 1) - \beta \log \beta + \beta \log \alpha - (\beta + 1) \log(\alpha + 1).$$

Remark. That $\phi \leq 0$ is straightforward.

Lemma 3.2 The function $\phi$ satisfies

$$\phi(\alpha, \alpha + \gamma) = -\frac{\gamma^2}{2\alpha(\alpha + 1)} + O(\gamma^3).$$

The trivial proof is omitted.

We have that

$$\sum_n C_2(n) = \sum_n \sum_{m \in n((\alpha,2(d-1))-D_n)} \left( f_n(m/n) \frac{\alpha_n^{m/n}}{(1 + \alpha_n)^{1+m/n}} \right)^n$$

$$\leq L \sum_n \sum_{m \in n((\alpha,2(d-1))-D_n)} \exp \{n\phi_{\alpha_n,m/n}\},$$

where the inequality is valid by virtue of Theorem 2.1 and the fact that $g \leq 1$. Lemma 3.2 implies that

$$\sum_{m \in n((\alpha,2(d-1))-D_n)} \exp \{n\phi_{\alpha_n,m/n}\} \leq (2(d-1) - \alpha)n^{-K},$$

where $K$ may be chosen to be arbitrarily large by an appropriate choice of $G$. It is this consideration that determines the choice of $G$. The miscellaneous term $C_3$ is treated by the following lemma.
Lemma 3.3 There exists \( r \in (0, 1) \) such that, for \( n \) sufficiently large and for \( m \in \{2(d-1)n, \ldots, 2(d-1)n + 2d\} \), we have that

\[
\sigma_{n,m} \leq \frac{(1 + m/n)^{n+m}}{(m/n)^m} r^n.
\]

Proof See [5].

We find that the \( m \)-indexed summand in \( C_3(n) \) is at most \( r^n \exp n\phi_{\alpha_n, m/n} \): thus \( C_3(n) \leq (2d + 1)r^n \). Note that \( C_1 \) satisfies

\[
C_1(n) = \sum_{m \in nD_n} \left( f_n(m/n) \frac{\alpha^n_{m/n}}{(1 + \alpha_n)^{1+m/n}} \right)^n 
\leq Ln \sum_{m \in nD_n} g(m/n)^n \exp(n\phi_{\alpha_n, m/n}),
\]

where the inequality is a consequence of Theorems 2.1 and 2.2. The fact that the function \( \phi \) is nowhere positive implies that \( C_1(n) \leq Ln \sum_{m \in nD_n} g(m/n)^n \).

Hence the desired contradiction will be reached if we can show that

\[
\sum_n n \sum_{m \in nD_n} g(m/n)^n
\]

is finite. As such, the proof is completed by the following lemma.

Lemma 3.4 Assume hypotheses (\( \zeta \)) and (\( \lambda \)). Suppose that \( \lambda < 1/2 \) and that \( \zeta \lambda < 1 \). Then, for \( \epsilon \in (0, 1 - \zeta \lambda) \) and \( n \in \mathbb{N} \) sufficiently large,

\[
\sum_{m \in nD_n} g(m/n)^n \leq \exp -n^{1-\zeta \lambda - \epsilon}.
\]

Proof Let \( \zeta^* > \zeta \) and \( \lambda^* > \lambda \) be such that \( \lambda^* < 1/2 \) and \( \zeta^* \lambda^* < \zeta \lambda + \epsilon \). By hypothesis (\( \zeta \)), there exists \( \epsilon' > 0 \) such that

\[
d \in (0, \epsilon') \text{ implies } g(\alpha + \delta) - g(\alpha) < -\delta^\zeta^*.
\]

From Theorems 2.1 and 2.2 it follows that \( \sup_{\beta \in [\alpha + \epsilon', 2(d-1)]} g(\beta) < 1 \), which shows that the contribution to the sum in (7) from all those terms indexed by \( m \) for which \( m/n > \alpha + \epsilon' \) is exponentially decaying in \( n \). Thus, we may assume that there exists \( N_1 \) such that for \( n \geq N_1 \), if \( m \in D_n^* \) then \( m/n - \alpha < \epsilon' \). Note that, by hypothesis (\( \lambda \)), \( \alpha_n - \alpha \geq n^{-\lambda^*} \) for sufficiently large. Hence, there exists \( N_2 \) such that, for \( n \geq N_2 \),

\[
\alpha_n - G(\log(n)/n)^{1/2} \geq \alpha + n^{-\lambda^*} - G(\log(n)/n)^{1/2} \geq \alpha + (1/2)n^{-\lambda^*}.
\]
For \( n \geq \max\{N_1, N_2\} \) and \( m \in nD_n^* \),
\[
g(m/n) \leq 1 - (m/n - \alpha)^\varsigma
\leq 1 - (\alpha_n - G(\log(n)/n)^{1/2} - \alpha)^\varsigma
\leq 1 - ((1/2)n^{-\lambda^*})^\varsigma.
\]
So, for \( n \geq \max(N_1, N_2) \),
\[
\sum_{m \in nD_n^*} g(m/n)^n \leq (2G(n \log(n))^{1/2})[1 - C'n^{-\lambda^*}]^n,
\]
for some constant \( C' > 0 \). There exists \( g \in (0, 1) \), such that for large \( n \),
\[
[1 - C'n^{-\lambda^*}]^n \leq g^{n^{1-\varsigma/2}}.
\]
This implies that
\[
\sum_{m \in nD_n^*} g(m/n)^n \leq h^{n^{1-\varsigma/2}} \text{ for large } n \text{ and } h \in (g, 1).
\]
From \( \varsigma^* \lambda^* < \varsigma \lambda + \epsilon \), we find that
\[
\sum_{m \in nD_n^*} g(m/n)^n \leq \exp(-n^{1-\varsigma \lambda - \epsilon}) \text{ for large } n,
\]
as required. \( \square \)

4 Sufficient conditions for \( \theta(p_c) = 0 \)

In this section, we give two theorems, demonstrating sufficient conditions for the continuity of the percolation probability in terms of inequalities on \( \varsigma \) and \( \lambda \).

Theorem 4.1 Assume that hypotheses (\( \varsigma \)) and (\( \lambda \)) hold.

1. Suppose that \( \varsigma < 2 \). Then \( \theta(p_c) = 0 \).

2. Suppose that \( \lambda > 1/2 \). Then \( \theta(p_c) = 0 \).

The proof of Theorem 4.1 will exploit the characterisation of continuity provided by the following lemma.

Definition 4.1

- Let \( \sigma(p) = \sum_n \sum_m \sigma_{n,m}p^n(1-p)^m \).
- Let \( \sigma_N(p) = \sum_{n \leq N} \sum_m \sigma_{n,m}p^n(1-p)^m \)
Lemma 4.2 A necessary and sufficient condition for $\theta(p_c) = 0$ is that $\sigma_n$ tends uniformly to $\sigma$ on the interval $(0, p_c)$.

**Proof** See [5].

**Proof of Theorem 4.1** By Lemma 4.2, to establish that $\theta(p_c) = 0$, it suffices to show that $\sigma_n$ tends to $\sigma$ uniformly on $(0, p_c)$. We begin by verifying this condition under the hypotheses of the first part of the Theorem. We will show that

$$
\sum_n \sum_m \sigma_{n,m} \sup_{p \in (0, p_c)} p^n (1 - p)^m < \infty. \quad (8)
$$

This will do because

$$
\sup_{p \in (0, p_c)} (\sigma(p) - \sigma_N(p)) = \sup_{p \in (0, p_c)} \sum_n \sum_m \sigma_{n,m} p^n (1 - p)^m
\leq \sum_{n \geq N+1} \sum_m \sigma_{n,m} \sup_{p \in (0, p_c)} p^n (1 - p)^m.
$$

So the condition stated in (8) implies the uniform convergence of $\sigma_n$ to $\sigma$ on the subcritical interval.

Note that

$$
\sup_{p \in (0, p_c)} p^n (1 - p)^m = \begin{cases} 
\left( \frac{n}{n+m} \right)^n \left( \frac{m}{n+m} \right)^m & \text{for } n/(n+m) \leq p_c \\
\frac{n}{p_c} (1 - \frac{n}{p_c})^m & \text{for other pairs } (n, m).
\end{cases}
$$

This observation allows us to decompose the sum appearing in (8):

$$
\sum_n \sum_m \sigma_{n,m} \sup_{p \in (0, p_c)} p^n (1 - p)^m = \sum_{n=1}^{\lfloor n\alpha \rfloor} \sum_{m=1}^{n} \sigma_{n,m} p_c^n (1 - p_c)^m + \sum_n \sum_{m > \lfloor n\alpha \rfloor} \sigma_{n,m} \left( \frac{n}{n+m} \right)^n \left( \frac{m}{n+m} \right)^m. \quad (9)
$$

Now,

$$
\sum_{n=1}^{\lfloor n\alpha \rfloor} \sum_{m=1}^{n} \sigma_{n,m} p_c^n (1 - p_c)^m \leq \sum_n \sum_m \sigma_{n,m} p_c^n (1 - p_c)^m,
$$

which is less than or equal to one, being the critical probability that the origin lies in a finite cluster.

Set $A$ equal to the second sum on the right-hand-side of (10). It suffices to show that $A$ is finite. Our strategy is to split each of the summands of $n$ into two parts, each of which is a sum over $m$ in an interval which has an $n$-dependence. The first sum, $A_1$, will include those $m$-values sufficiently close to $n\alpha$ that this term can be bounded in terms of the critical probability of observing a large cluster. The second sum, $A_2$, will be shown to decay quickly, under the assumption that $\varsigma < 2$. 
Write $A = A_1 + A_2$, where

$$A_1 = \sum_n \sum_{m=[na]+1}^{[na+n^{1/2}]+1} \sigma_{n,m} \left( \frac{n}{n+m} \right)^n \left( \frac{m}{n+m} \right)^m,$$

and $A_2 = \sum_n \sum_{m>[na+n^{1/2}]+1} \sigma_{n,m} \left( \frac{n}{n+m} \right)^n \left( \frac{m}{n+m} \right)^m$.

Recalling that $\alpha = 1/p_c - 1$,

$$A_1 = \sum_n \sum_{m=[na]+1}^{[na+n^{1/2}]+1} \sigma_{n,m} p_c^n (1 - p_c)^m \exp \left( -n \phi(\alpha, m/n) \right),$$

where the function $\phi$ was specified in Definition 3.2. For each $m \in \{[na], \ldots, [na+n^{1/2}] + 1\}$, $c_m \in (0, 3/2)$, where $c_m$ is given by $m/n = \alpha + c_m n^{-1/2}$. Lemma 3.2 implies that for any sufficiently large $C'$, there exists $N_1$ such that for all $n \geq N_1$, and for $m \in \{[na]+1, \ldots, [na+n^{1/2}] + 1\}$,

$$-\phi(\alpha, m/n) \leq 9/[8n\alpha(\alpha + 1)] + C'/n^{3/2}.$$ 

From this, we deduce that for $n \geq N_1$ and $m \in \{[na]+1, \ldots, [na+n^{1/2}] + 1\}$, $\exp (-n\phi(\alpha, m/n))$ is bounded above, by $C$, say. So,

$$A_1 \leq \sum_{n<N_1} \sum_{m=[na]+1}^{[na+n^{1/2}]+1} \sigma_{n,m} \left( \frac{n}{n+m} \right)^n \left( \frac{m}{n+m} \right)^m$$

$$+ C \sum_{n \geq N_1} \sum_{m\in\{[na]+1, \ldots, [na+n^{1/2}] + 1\}} \sigma_{n,m} p_c^n (1 - p_c)^m,$$

which is finite, as desired.

We now seek to bound $A_2$:

$$A_2 = \sum_n \sum_{m>[na+n^{1/2}]+1} \left( f_n(m/n)(\frac{n}{n+m})(\frac{m}{n+m})^{m/n} \right)^n$$

$$\leq L \sum_n \sum_{m=[na+n^{1/2}]+2}^{2(d-1)n-1} n \left( f_n(m/n)(\frac{n}{n+m})(\frac{m}{n+m})^{m/n} \right)^n$$

$$+ \sum_n \sum_{m=2(d-1)n+2}^{2(d-1)n+2d} \sigma_{n,m} \left( \frac{n}{n+m} \right)^n \left( \frac{m}{n+m} \right)^m,$$

where the inequality follows from Theorem 2.1 and the fact that $g \leq 1$. By Lemma 3.3, there exists $r \in (0, 1)$ such that, for $n$ sufficiently large,

$$\sum_{m=2(d-1)n}^{2(d-1)n+2d} \sigma_{n,m} \left( \frac{n}{n+m} \right)^n \left( \frac{m}{n+m} \right)^m \leq (2d+1) r^n.$$
It follows from the definition of the function \( g \) that
\[
A_2 \leq L \sum_{n}^{2(d-1)n-1} \sum_{m=\lfloor na+n^{1/2} \rfloor +2} \ng(m/n)^n + (2d+1) \sum_{n} r^n. \tag{10}
\]

To bound the first term in the expression on the right-hand-side of (10), let \( \epsilon \in (0, 2-\varsigma) \). Let \( \delta' > 0 \), be such that, for \( \delta \in (0, \delta') \), \( g(\alpha+\delta) - g(\alpha) < -\delta^{\varsigma+\epsilon} \). Let \( \gamma \in (0, 1) \) be such that
\[
\sup_{\beta \in (\alpha+\delta', 2(d-1))} g(\beta) < \gamma.
\]

Note that
\[
\sum_{n} \left\lfloor n(\alpha+\delta') \right\rfloor \sum_{m=\lfloor na+n^{1/2} \rfloor +2} \ng(m/n)^n
\leq \sum_{n} \left\lfloor n(\alpha+\delta') \right\rfloor \sum_{m=\lfloor na+n^{1/2} \rfloor +2} n \left( 1 - (m/n - \alpha)^{\varsigma+\epsilon} \right)^n
\leq \delta' \sum_{n} n^{2 \left( 1 - n^{-\frac{\alpha+\delta'}{2}} \right)} n.
\]

Since \( \varsigma + \epsilon < 2 \), this expression is finite. Note also that
\[
\sum_{n}^{2(d-1)n-1} \sum_{m=\lfloor n(\alpha+\delta') \rfloor +1} \ng(m/n)^n \leq 2(d-1) \sum_{n} n^2 \gamma^n < \infty.
\]

We deduce that \( A_2 \) is finite and in doing so, complete the proof of the first part of Theorem 4.1.

We now prove the second part of the Theorem. A sufficient condition for continuity is
\[
\sum_{n} \sum_{m} \sigma_{n,m} t_n^m (1-t_n)^m < \infty. \tag{11}
\]

Indeed, the supremum over \( p \) in \((0, p_c)\) of \( \sigma - \sigma_N \) is bounded above by the expression in (11) with the sum in \( n \) being taken over values exceeding \( N-1 \). By Lemma 4.2 if (11) holds, then \( \theta(p_c) = 0 \).

The fact that \( t_n \leq p_c \) implies that \( t_n^m (1-t_n)^m \leq p_c^m (1-p_c)^m \) provided that \( n/(n+m) > p_c \), which holds if and only if \( m \leq \lfloor na \rfloor \). From this, we may deduce that
\[
\sum_{n} \sum_{m=1}^{\lfloor na \rfloor} \sigma_{n,m} t_n^m (1-t_n)^m \leq \sum_{n} \sum_{m=1}^{\lfloor na \rfloor} \sigma_{n,m} p_c^m (1-p_c)^m
\leq \sum_{n} \sum_{m} \sigma_{n,m} p_c^m (1-p_c)^m \leq 1
\]
To verify the condition in (11), we must bound the expression
\[
\sum_{n} \sum_{m=[n\alpha]+1}^{2(d-1)n+2d} \sigma_{n,m} t_n^m (1 - t_n)^m.
\] (12)

To do so, we make the following definition.

**Definition 4.2** For \( G \in \mathbb{N} \), let \( D_n^* (= D_n^* (G)) \) denote the interval
\[
D_n^* = (\max \{ \alpha, \alpha_n - G(\log(n)/n)^{1/2} \}, \alpha_n + G(\log(n)/n)^{1/2}),
\]
where the constants \( \{ \alpha_n : n \in \mathbb{N} \} \) were specified in Definition 3.4.

Allowing that \( G \) will be determined slightly later, we write the expression in (12) in the form
\[
\sum_{n} \sum_{m=2(d-1)n}^{2(d-1)n+2d} \sigma_{n,m} t_n^m (1 - t_n)^m
\]
\[
+ \sum_{n} \sum_{m=2(d-1)n}^{2(d-1)n+2d} \sigma_{n,m} t_n^m (1 - t_n)^m.
\] (13)

An argument identical to that by which the term \( C_2 \) was bounded in the proof of Theorem 3.1 yields
\[
\sum_{n} \sum_{m=2(d-1)n}^{2(d-1)n+2d} \sigma_{n,m} t_n^m (1 - t_n)^m \leq \sum_{n} n^{-K},
\]
where \( K \) may be chosen to be arbitrarily large by an appropriate choice of \( G \), thereby determining how \( G \) is chosen. The third term in (13) was labelled \( C_3(n) \) in the proof of Theorem 3.1 and was shown to be bounded above by \((2d+1)^r n \) for \( n \) sufficiently high. We have that
\[
\sum_{n} \sum_{m \in D_n^*} \sigma_{n,m} t_n^m (1 - t_n)^m
\]
\[
= \sum_{n} \sum_{m \in D_n^*} \sigma_{n,m} \frac{\alpha^m}{(1+\alpha)^{n+m}} \exp n \Phi(\alpha_n, \alpha, m/n),
\] (14)
where
\[
\Phi(\gamma, \alpha, \beta) = \beta \log \gamma - (\beta + 1) \log(\gamma + 1) - \beta \log \alpha + (\beta + 1) \log(\alpha + 1)
\]
\[
= \beta \log(1 + (\gamma - \alpha)/\alpha) - (\beta + 1) \log(1 + (\gamma - \alpha)/(1 + \alpha))
\]
\[
= - \frac{(\gamma - \alpha)^2}{2} \left[ \beta/\alpha^2 - (\beta + 1)/(1 + \alpha)^2 \right]
\]
\[
+ \frac{(\gamma - \alpha)(\beta - \alpha)}{\alpha(\alpha + 1)} + O[(\gamma - \alpha)^3].
\]
We are supposing that hypothesis (λ) holds, and that λ > 1/2. Let λ’ satisfy λ > λ’ > 1/2. In this context,

\[ \Phi(\alpha_n, \alpha, \beta) = \frac{(\alpha_n - \alpha)(\beta - \alpha)}{\alpha(\alpha + 1)} - \frac{(\alpha_n - \alpha)^2}{2}[\beta/\alpha^2 - (\beta + 1)/(1 + \alpha)^2] + O(n^{-3\lambda'}). \]

Now, β ∈ \( D^* \) implies that there exists \( C' > 0 \) such that \( \beta - \alpha \leq C'n^{-\lambda'} + C'(\log(n)/n)^{1/2} \); since \( \lambda' > 1/2 \), we may write \( \beta - \alpha \leq C'n^{-\lambda'} + O(n^{-3\lambda'}). \) This implies that, for all \( n \) and \( \beta \in D^* \), \( \exp n\Phi(\alpha_n, \alpha, \beta) < C' \), where once again the value of \( C' \) may have changed. Recalling that \( \alpha = 1/p_c - 1 \), we deduce from (14) that

\[ \sum \sum_{m \in nB^*} \sigma_{n,m} t_n^m (1 - t_n)^{m/n} \leq C' \sum \sum_{m \in nB^*} \sigma_{n,m} p_n^m (1 - p_c)^{m/n} \leq C', \]

proving the second part of Theorem 3.1.

We now examine the case where \( \lambda < 1/2 \) and \( \varsigma \lambda > 1 \).

**Definition 4.3** Let \( n \in \mathbb{N} \), and \( \beta \in (0, 2(d - 1)) \). Set

\[ a_n(\beta) = \left( \frac{f'_n(\beta)}{f(\beta)} \right)^n \]  

(15)

**Remark** The quantities \( a_n(\beta) \) appear in the factorisation of \( \sigma_n, \lfloor \beta n \rfloor \),

\[ \sigma_n, \lfloor \beta n \rfloor = a_n(\beta) g(\beta)^n \left( \frac{(\beta + 1)^{\beta + 1}}{\beta^\beta} \right)^n. \]

As such, they measure the extent to which the exponential growth rate \( f(\beta) \) is underestimated by \( \sigma_{n,m} \).

Performing a similar analysis to that undertaken during each part of Theorem 4.1 yields the following result. Its proof appears in [5].

**Theorem 4.3** Assume that hypotheses (ς) and (λ) hold. Suppose that \( \lambda < 1/2 \) and \( \varsigma \lambda > 1 \). Let \( K \) be large. Then there exist constants \( \epsilon > 0 \) and \( C > 0 \) such that for each \( n \in \mathbb{N} \),

\[ \epsilon \sum_{m \in nB(\alpha, n^{-1/2})} a_n(m/n) \leq \sum_m \sigma_{n,m} p_n^m (1 - p_c)^m \leq \sum_{m \in nB(\alpha, C(\log n/n)^{1/2})} a_n(m/n) + n^{-K} \]  

(16)
and
\[
\epsilon \sum_{m \in nB(\alpha, n^{-1/2})} a_n(m/n) \leq \sum_m \sigma_{n,m} t^n_n (1 - t^n_n)^m \leq \sum_{m \in nB(\alpha, C(\log n/n)^{1/2})} a_n(m/n) + n^{-K}.
\] (17)

**Remark** Here, \(B(a, b)\) denotes the interval \((a - b, a + b)\). Note also that it follows from Theorem 4.3 that the condition
\[
\sum_{m \in nB(\alpha, C(\log n/n)^{1/2})} a_n(m/n) < \infty
\]
implies that \(\theta(p_c) = 0\), without recourse to scaling hypotheses. In examining this condition, bounds on the entropic exponent are revelant (see [6]).

## 5 Scaling law

In this section, we examine the exponential decay rate in \(n\) for the probability of the event \(\{C(0) = n\}\) for \(p\) slightly less than \(p_c\) by our combinatorial approach. In doing so, we relate the quantity \(\gamma\) to the exponent for correlation size, and see how the scaling behaviour for the typical surface-area-to-volume ratio of unusually large clusters in the marginally subcritical regime depends on the value of \(\gamma\).

**Definition 5.1** Let \(q : (0, p_c) \to [0, \infty)\) be given by
\[
q(p) = \lim_{n \to \infty} -\frac{\log \mathbb{P}_p(|C(0)| = n)}{n}.
\]

Define \(\Omega^\gamma_+ = \{\gamma \geq 0 : \liminf_{p \uparrow p_c} \frac{q(p)}{(p_c - p)^\gamma} = \infty\}\) and \(\Omega^\gamma_- = \{\gamma \geq 0 : \limsup_{p \uparrow p_c} \frac{q(p)}{(p_c - p)^\gamma} = 0\}\). If \(\sup \Omega^\gamma_+ = \inf \Omega^\gamma_-\), then hypothesis \((q)\) is said to hold, and \(q\) is defined to be equal to the common value.

**Remark** The existence of \(q\) follows from a standard subadditivity argument. The quantity \(q\) might reasonably be called the exponent for ‘correlation size’.

**Theorem 5.1** There exists \(\delta' > 0\) and \(p_0 \in (0, p_c)\) such that \(p \in (p_0, p_c)\) implies that \(q(p)\) is given by
\[
\inf_{\beta \in (\alpha, \alpha + \delta')} -\log g(\beta) - \phi(1/p - 1, \beta).
\]
The facts that \( \lim \) implies that, for a small constant \( c \),

\[
\beta \quad \text{since} \quad \sigma
\]

Theorem 5.1 allows us to deduce a scaling law that relates the combinatorially defined exponent \( \varsigma \) to one which is defined directly from the percolation model.

**Theorem 5.2** Assume hypothesis (\( \varsigma \)).

- Suppose that \( \varsigma \in (1, 2) \). Then hypothesis (\( q \)) holds and \( q = 2 \).
- Suppose that \( \varsigma \in (2, \infty) \). Then hypothesis (\( q \)) holds and \( q = \varsigma \).

**Proof** Suppose that \( \varsigma \in (1, 2) \). Choose \( \epsilon > 0 \) so that \( 1 < \varsigma - \epsilon < \varsigma + \epsilon < 2 \).

There exists constants \( C_1, C_2 > 0 \) such that, for \( p \in (p_0, p_c) \) and \( \beta \in (\alpha, \alpha + \delta') \),

\[
(\beta - \alpha)^{\varsigma + \epsilon} + C_1(\beta - (1/p - 1))^2 \leq -\log(\beta) - \phi(1/p - 1, \beta) \tag{18}
\]

Applying Theorem 5.1 we find that

\[
(\beta_p - \alpha)^{\varsigma + \epsilon} + C_1(\beta_p - (1/p - 1))^2 \leq q(p), \tag{19}
\]

where \( \beta_p \in [\alpha, \alpha + \delta'] \) denotes a value at which the infimum in the interval \([\alpha, \alpha + \delta']\) of the first term in (18) is attained. Let \( y_p = 1/p - 1 - \alpha \), and let \( \sigma_p \) satisfy \( \beta_p = \alpha + y_p^{\sigma_p} \). Then \( \beta_p \) and \( \sigma_p \) satisfy

\[
(\varsigma + \epsilon)(\beta_p - \alpha)^{\varsigma + \epsilon - 1} = -2C_1(\beta_p - (1/p - 1)) \tag{20}
\]

Since \( \beta_p \leq 1/p - 1, \sigma_p \geq 1 \). From this and (20) follows \( \lim_{p \uparrow p_c} \sigma_p \geq 1/(\varsigma + \epsilon - 1) \). Applying (20) again, we deduce that \( \lim_{p \uparrow p_c} \sigma_p = 1/(\varsigma + \epsilon - 1) \).

Substituting \( \sigma_p \) in (19) yields

\[
y_p^{\sigma_p(\varsigma + \epsilon)} + C_1(y_p - y_p^{\sigma_p})^2 \leq q(p). \tag{19}
\]

The facts that \( \lim_{p \uparrow} \sigma_p > 1 \) and \( \lim_{p \uparrow} \sigma_p(\varsigma + \epsilon) = (\varsigma + \epsilon)/(\varsigma + \epsilon - 1) > 2 \) imply that, for a small constant \( c \), \( c(p_c - p)^2 \leq q(p) \) for values of \( p \) just less than \( p_c \). A similar analysis in which \( q(p) \) is bounded below by the infimum on the interval \([\alpha, \alpha + \delta']\) of the third expression in (18) implies that for large \( C, q(p) \leq C(p_c - p)^2 \), in a similar range of values of \( p \). Thus hypothesis (\( q \)) holds, and \( q = 2 \).
In the case where \( \varsigma > 2 \), let \( \epsilon > 0 \) be such that \( \varsigma > 2 + \epsilon \). Defining \( \sigma'_p \) by \( \beta_p = 1/p - 1 - y^p_\sigma' \), we find that

\[
(\varsigma + \epsilon)\left( y_p - y^p_\sigma' \right)^{\varsigma + \epsilon - 1} = 2C_1y^p_\sigma'.
\]  
(21)

Note that \( \beta_p \geq \alpha \) implies that \( \sigma'_p \geq 1 \). From (21), it follows that \( \lim \inf_{p \uparrow p_c} \sigma'_p \geq \varsigma + \epsilon - 1 \). Since \( \varsigma + \epsilon - 1 > 1 \), applying (21) again shows that the limit \( \lim \inf_{p \uparrow p_c} \sigma'_p \) exists and in fact equals \( \varsigma + \epsilon - 1 \). Substituting \( \sigma'_p \) in (18) yields

\[
(y_p - y^p_\sigma')^{\varsigma + \epsilon} + C_1y^p_\sigma' \leq q(p).
\]

The fact that \( \lim \inf_{p \uparrow p_c} \sigma'_p > 1 \) implies that \( c(p_c - p)^{\varsigma + \epsilon} \leq q(p) \) for values of \( p \) just less than \( p_c \). Making use of the inequality \( \varsigma > 2 + \epsilon \) in considering the infimum of the third term appearing in (18) yields in this case \( q(p) \leq C(p_c - p)^{\varsigma - \epsilon} \) for similar values of \( p \). Thus, since \( \epsilon \) may be chosen to be arbitrarily small, we find that, if \( \varsigma > 2 \), then hypothesis \( (g) \) holds, and that \( q = \varsigma \). □

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