On the moments and the distribution of aggregate discounted claims in a Markovian environment

Shuanming Li\textsuperscript{a} and Yi Lu\textsuperscript{b}
\textsuperscript{a}Centre for Actuarial Studies, Department of Economics
The University of Melbourne, Australia
\textsuperscript{b}Department of Statistics and Actuarial Science
Simon Fraser University, Canada

Abstract

This paper studies the moments and the distribution of the aggregate discounted claims (ADC) in a Markovian environment, where the claim arrivals, claim amounts and forces of interest (for discounting) are influenced by an underlying Markov process. Specifically, we assume that claims occur according to a Markovian arrival process (MAP). The paper shows that the joint Laplace transform of the ADC occurred in each state of the environment process by any specific time satisfies a first order partial differential equation through which a recursive formula is derived for the moments of the ADC occurred in certain states (a subset). We also study two types of covariances of the ADC occurred in any two subsets of the state space and with two different time lengths. The distribution of the ADC occurred in certain states by any specific time is also investigated. Numerical results are also presented for a two-state Markov-modulated model case.

Keywords: Aggregate discounted claims; Markovian arrival process; stochastic interest; partial integro-differential equation; covariance

1 Introduction

Consider a line of business or an insurance portfolio to be insured by a property and casualty insurance company. Suppose that random claims arrive in the future according to a counting process, denoted by \{N(t)\}_{t \geq 0}, i.e., \( N(t) \) is the random number of claims up to time \( t \). Assume that \( \{T_n\}_{n \geq 1} \) is a sequence of random claim occurrence times and \( \{X_n\}_{n \geq 1} \) is a sequence of corresponding random positive claim amounts (also called claim severities), and \( \delta(t) \) is the force of interest at time \( t \) which is modeled by a stochastic process. Then \( S(t) \) defined by

\[
S(t) = \sum_{n=1}^{N(t)} X_n e^{-\int_0^{T_n} \delta(s) ds}, \quad t \geq 0. \tag{1}
\]

is the aggregate discounted claims (ADC) up to certain time \( t \), or the present value of the total amounts paid out by the company up to time \( t \), which describes the random
change over time of insurer’s future liabilities at present time. Accordingly, \( \{S(t)\}_{t \geq 0} \) is the ADC process (compound discounted claims) for this business. At a fixed time \( t \), the randomness of \( S(t) \) comes from the number of claims up to time \( t \), claim occurrence times and corresponding sizes as well as the values of \( \delta(t) \), \( 0 \leq s \leq t \). It is an important quantity in the sense that at time of issue (\( t = 0 \)), this quantity would help the insurer set a premium for this particular line of business, and predict and manage their future liabilities.

A simple case of (1) is that the counting process \( \{N(t)\}_{t \geq 0} \) is a homogeneous Poisson process, independent of claim amounts, and the force of interest is deterministic. In this paper, we assume that the counting process \( \{N(t)\}_{t \geq 0} \) is a Markovian arrival process (MAP) with representation \((\gamma, D_0, D_1)\), introduced in Neuts (1979). That is, claim arrivals are influenced by an underlying continuous-time Markov process \( \{J(t)\}_{t \geq 0} \) on state space \( E = \{1,2,\ldots,m\} \) with an \( m \times m \) intensity matrix \( D \) and initial distribution \( \gamma \), where \( \mathbf{D} = \mathbf{D}_0 + \mathbf{D}_1 = (d_{0,ij}) + (d_{1,ij}) \), and is assumed to be irreducible. Precisely, \( d_{0,ij} \) represents the intensity of transitions from state \( i \) to state \( j \) without claim arrivals, while \( d_{1,ij}(\geq 0) \) represents the intensity of transitions from state \( i \) to state \( j \) with an accompanying claim, having a cumulative distribution function \( F_i \), density function \( f_i \), \( k \)-th moment \( \mu_i^{(k)} \), and Laplace transform \( \hat{f}_i(s) = \int_0^\infty e^{-sx}f_i(x)dx \). Here, the process \( \{J(t)\}_{t \geq 0} \) models the random environment, which affects the frequency and the severity of claims and thus the insurance business; for example, it is well known that the weather or climate conditions have impacts on the automobile or property and casualty insurance claims.

Moreover, we assume that the force of interest process \( \{\delta(t)\}_{t \geq 0} \) in (1) is also governed by the same Markov process \( \{J(t)\}_{t \geq 0} \) and is assumed constant while staying at certain state, that is, when \( J(t) = i \), \( \delta(t) = \delta_i(\geq 0) \), for all \( i \in E \). As the force of interest used for evaluation is mainly driven by the local or global economics conditions we would reasonably model its random fluctuations by a stochastic process which is different from \( \{J(t)\}_{t \geq 0} \). Technically, we can assume a two dimensional Markov process as the environment or background process and other mathematical treatments would be the same as we do below. Hence, we make the above assumption in this paper to simplify notations and presentations. We note that studies of the influence of economic conditions such as interest and inflation on the classical risk theory can be found in Taylor (1979), Delbaen and Haezendonk (1987), Willmot (1989), and Garrido and Léveille (2004).

The MAP has received considerable attention in recent decade due to its versatility and feasibility in modeling stochastic insurance claims dynamics. The MAP includes the Poisson process, renewal process with the inter-arrival times following phase-type distributions and Markov-modulated Poisson process as special cases, that are intensively studied in actuarial science literature. Detailed characteristics and properties of MAPs can be found in Neuts (1979) and Asmussen (2003). Below we present a brief literature review on the related work based on model (1) (including its special cases).

Most of the studies on model (1) are under the assumption that \( \{\delta(t)\}_{t \geq 0} \) is deterministic. For the ADC, Léveille and Garrido (2001a) give explicit expressions for its first two moments in the compound renewal risk process by using renewal theory arguments, while Léveille and Garrido (2001b) further derive a recursive formula for the moments calculation. Léveille et al. (2010) study the moment generating function (mgf) of the ADC by finite and infinite time under a renewal risk model or a delayed renewal risk model. Recently, Wang
et al. (2018) study the distribution of discounted compound phase-type renewal sums using the analytical results of their mgf obtained in Léveille et al. (2010). Jang (2004) obtains the Laplace transform of the distribution of the ADC using a shot noise process. Woo and Cheung (2013) derive recursive formulae for the moments of the ADC, using the techniques in Léveille and Garrido (2001b), for a renewal risk process with certain dependence between the claim arrival and the amount caused. The impact of the dependency on the ADC are illustrated numerically. Kim and Kim (2007) derive simple expressions for the first two moments of the ADC when the rates of claim arrivals and the claim sizes depend on the states of an underlying Markov process. Ren (2008) studies the Laplace transform and the first two moments of the ADC following a MAP process, and Li (2008) further derives a recursive formula for the moments of the discounted claims for the same model. Bargés et al. (2011) study the moments of the ADC in a compound Poisson model with dependence introduced by a Farlie-Gumbel-Morgenstern (FGM) copula; Jang and Ramli (2014) further derive Neumann series expression of the recursive moments by using the method of successive approximation.

There are only few papers that study model (1) with a stochastic process \( \{\delta(t)\}_{t \geq 0} \) in the literature of actuarial science. Léveille and Adékambi (2011, 2012) study the covariance and the joint moments of the discounted compound renewal sum at two different times with a stochastic interest rate where the Ho-Lee-Merton and the Vasicek interest rate models are considered. Their idea of studying the covariance and the joint moments is adopted and extended in this paper. Here, we assume that the components of the ADC process \( \{S(t)\}_{t \geq 0} \) described by (1), the number of claims, the size of the claims and the force of interest for discounting, are all influenced by a same Markovian environment process, which enhances the flexibility of the model parameter settings. It follows that \( S(t) \) depends on the trajectory of this underlying process whose states may represent different external conditions or circumstances which affect insurance claims. The main objective of this paper is to study the moments and the distribution of \( S(t) \) given in (1), occurred in certain states (e.g., certain conditions) by time \( t \).

In general, while the expectation of \( S(t) \) at any given time \( t \) can be used as a reference for insurer’s liability, the higher moments of \( S(t) \), describing further characteristics of the random variable such as the variability around the mean and how extreme outcomes could go, may be used to determine the marginals on reserves. Furthermore, the distributional results regarding \( S(t) \) would be useful for obtaining the risk measures such as the value at risk and the conditional tail expectation, which may help insurers prevent or minimize their losses from extreme cases.

Our work is basically a generalization of some aforementioned studies. We first obtain formulae for calculating mean, variance and distribution of the ADC occurred in a subset of states at a certain time. The subset may represent a collection of similar conditions that the insurer would consider them as a whole. We then derive explicit matrix-analytic expressions for covariances of the ADC occurred in two subsets of the state space at a certain time and that occurred in certain subset of states with two different time lengths. The motivation of studying these two types of covariance is that we believe they could reveal the correlation between the random discounted sums either between different underlying conditions or with different time lengths, and the information would be helpful for insurers to set their capital requirements for preventing future losses, and make strategic and contingency plans.
Moreover, we obtain a matrix form partial integro-differential equation satisfied by the
distribution function of the ADC occurred in certain subset of states. The equation can be
solved numerically to obtain the probability distribution of the ADC, which again could be
useful for measuring insurers’ risks of insolvency.

The rest of paper is organized as follows. In Section 2, we study the joint Laplace
transforms of the ADC occurred in each state by time \( t \) and pay attention to some special
cases. Recursive formulae for calculating the moments of the ADC occurred in certain
states are obtained. A formula for computing the covariance of the ADC occurred in two
subsets of the state space is derived in Section 3, while the covariance of the ADC occurred
in certain states with two different time lengths is studied in Section 4. The distribution
of the ADC occurred in certain states is investigated in Section 5. Finally, some numerical
illustrations are presented in Section 6.

2 The Laplace transforms and moments

We first decompose \( S(t) \) into \( m \) components as

\[
S(t) = \sum_{j=1}^{m} S_j(t),
\]

where

\[
S_j(t) = \sum_{n=1}^{N(t)} X_n I(J(T_n) = j) e^{-\int_0^{T_n} \delta(s) ds}
\]

is the ADC occurred in state \( j \in E \), with \( I(\cdot) \) being the indicator function. For given
\( k(1 \leq k \leq m) \), \( 1 \leq l_1 < l_2 < \ldots < l_k \leq m \), denote \( E_k = \{l_1, l_2, \ldots, l_k\} \subseteq E \), a sub-state
space of \( E \). We then define

\[
S_{E_k}(t) = \sum_{j \in E_k} S_j(t),
\]

to be the ADC occurred in the subset of state space \( E_k \). In particular, \( S_E(t) = S(t) \) and
\( S_{\{j\}}(t) = S_j(t) \). If \( \delta(t) = 0 \) and \( X_i \equiv 1 \) for all \( i \in \mathbb{N}^+ \), then \( S_{E_k}(t) = N_{E_k}(t) \), where \( N_{E_k}(t) \)
is the number of claims occurred in the sub-state space \( E_k \) by time \( t \).

Let \( \mathbb{P}_i \) and \( \mathbb{E}_i \) denote conditional probability and conditional expectation given \( J(0) = i \),
respectively. Define

\[
iL(\xi_1, \xi_2, \ldots, \xi_m; t) = \mathbb{E}_i \left[ e^{-\sum_{j=1}^{m} \xi_j S_j(t)} \right], \quad \xi_j \geq 0, t \geq 0, i \in E,
\]

to be the joint Laplace transform of \( S_1(t), S_2(t), \ldots, S_m(t) \), given that the initial state is \( i \).
In particular, we have

\[
iL(\xi; t) = \mathbb{E}_i \left[ e^{-\xi S(t)} \right] = iL(\xi; \xi, \ldots, \xi; t),
\]
\[
iL_{E_k}(\xi; t) = \mathbb{E}_i \left[ e^{-\xi S_{E_k}(t)} \right] = iL(\xi_1, \xi_2, \ldots, \xi_m; t) \bigg|_{\xi_j = \xi I(j=\{l_n\}, n=1,2,\ldots,k)}
\]
\[ iL_j(\xi_j; t) = \mathbb{E}_i \left[ e^{-\xi_j S_j(t)} \right] = iL(\xi_1, \xi_2, \ldots, \xi_m; t) \bigg|_{\xi_k = 0, k \neq j}. \]

We define, for \( n \in \mathbb{N}^+ \), the \( n \)-th moment of \( S(t), S_j(t) \), and \( S_{E_k}(t) \), respectively, as

\[ iV^{(n)}(t) = \mathbb{E}_i \left[ S^n(t) \right], \quad i \in E, \]
\[ iV_j^{(n)}(t) = \mathbb{E}_i \left[ S^n_j(t) \right], \quad i, j \in E, \]
\[ iV_{E_k}^{(n)}(t) = \mathbb{E}_i \left[ S^n_{E_k}(t) \right], \quad 1 \leq k \leq m, \]
given that the initial state is \( i \).

We write the following column vectors for the Laplace transforms

\[
\begin{align*}
\mathbf{L}(\xi_1, \xi_2, \ldots, \xi_m; t) &= \begin{pmatrix} 1L(\xi_1, \xi_2, \ldots, \xi_m; t) \\ \vdots \\ mL(\xi_1, \xi_2, \ldots, \xi_m; t) \end{pmatrix}^\top, \\
\mathbf{L}(\xi; t) &= \begin{pmatrix} 1L(\xi; t) \\ \vdots \\ mL(\xi; t) \end{pmatrix}^\top, \\
\mathbf{L}_{E_k}(\xi; t) &= \begin{pmatrix} 1L_{E_k}(\xi; t) \\ \vdots \\ mL_{E_k}(\xi; t) \end{pmatrix}^\top, \\
\mathbf{L}_j(\xi; t) &= \begin{pmatrix} 1L_j(\xi; t) \\ \vdots \\ mL_j(\xi; t) \end{pmatrix}^\top,
\end{align*}
\]

with \( \mathbf{L}(0; t) = \mathbf{L}_{E_k}(0; t) = \mathbf{L}_j(0; t) = \mathbf{1} = (1, 1, \ldots, 1)^\top. \)

In this section, we first show that \( \mathbf{L}(\xi_1, \xi_2, \ldots, \xi_m; t) \) satisfies a matrix form first order linear partial differential equation, and derive recursive formulae for calculating the moments of various ADC depending on the initial state of the underlying Markovian process. We also consider some special cases.

**Theorem 1** \( \mathbf{L}(\xi_1, \xi_2, \ldots, \xi_m; t) \) satisfies the following first order linear partial differential equation:

\[
\frac{\partial \mathbf{L}(\xi_1, \xi_2, \ldots, \xi_m; t)}{\partial t} + \delta \sum_{j=1}^{m} \xi_j \frac{\partial \mathbf{L}(\xi_1, \xi_2, \ldots, \xi_m; t)}{\partial \xi_j} = \mathbf{D}_0 \mathbf{L}(\xi_1, \xi_2, \ldots, \xi_m; t) + \mathbf{f}(\xi_1, \xi_2, \ldots, \xi_m) \mathbf{D}_1 \mathbf{L}(\xi_1, \xi_2, \ldots, \xi_m; t),
\]

where \( \delta = \text{diag}(\delta_1, \delta_2, \ldots, \delta_m) \) and \( \mathbf{f}(\xi_1, \xi_2, \ldots, \xi_m) = \text{diag}(\hat{f}_1(\xi_1), \hat{f}_2(\xi_2), \ldots, \hat{f}_m(\xi_m)) \).

**Proof:** For an infinitesimal \( h > 0 \), conditioning on three possible events which can occur in \([0, h]\): no change in the MAP phase (state), a change in the MAP phase accompanied by no claims, and a change in the MAP phase accompanied by a claim, we have

\[
iL(\xi_1, \xi_2, \ldots, \xi_m; t) = \left[ 1 + d_{0,ii}h \right] iL(\xi_1 e^{-\delta_1 h}, \xi_2 e^{-\delta_2 h}, \ldots, \xi_m e^{-\delta_m h}; t - h)
+ \sum_{k=1, k \neq i}^{m} d_{0,ik}h \ kL(\xi_1 e^{-\delta_1 h}, \xi_2 e^{-\delta_2 h}, \ldots, \xi_m e^{-\delta_m h}; t - h)
+ \sum_{k=1}^{m} d_{1,ik}h \hat{f}_k(\xi_1 e^{-\delta_1 h}, \xi_2 e^{-\delta_2 h}, \ldots, \xi_m e^{-\delta_m h}; t - h).\]

(3)
Taylor’s expansion gives
\[
i L(\xi_1 e^{-\delta h}, \xi_2 e^{-\delta h}, \ldots, \xi_m e^{-\delta h}; t - h) \\
= i L(\xi_1, \xi_2, \ldots, \xi_m; t) - h \frac{\partial}{\partial t} i L(\xi_1, \xi_2, \ldots, \xi_m; t) - \delta_i h \sum_{l=1}^{m} \xi_l \frac{\partial}{\partial \xi_l} i L(\xi_1, \xi_2, \ldots, \xi_m; t) + o(h),
\]
where \(\lim_{h \to 0} (o(h)/h) = 0\). Substituting the expression above into (3), dividing both sides by \(h\), and letting \(h \to 0\), we have
\[
\delta_i \sum_{l=1}^{m} \xi_l \frac{\partial}{\partial \xi_l} i L(\xi_1, \xi_2, \ldots, \xi_m; t) + \frac{\partial}{\partial t} i L(\xi_1, \xi_2, \ldots, \xi_m; t) \\
= \sum_{k=1}^{m} d_{0,ik} k L(\xi_1, \xi_2, \ldots, \xi_m; t) + \sum_{k=1}^{m} d_{1,ik} \hat{f}_i(\xi) k L(\xi_1, \xi_2, \ldots, \xi_m; t). \tag{4}
\]
Rewriting (4) in matrix form gives (2). \(\square\)

**Remark 1** Using the same argument, we have the follow results.

1. \(L_E_k(\xi; t)\) satisfies the following first order linear partial differential equation:
\[
\frac{\partial L_E_k(\xi; t)}{\partial t} + \delta \xi \frac{\partial L_E_k(\xi; t)}{\partial \xi} = D_0 L_E_k(\xi; t) + \hat{f}_k(\xi) D_1 L_E_k(\xi; t), \tag{5}
\]
where \(\hat{f}_k(\xi)\) is an \(m \times m\) diagonal matrix with the \(l_i\)-th entry being \(\hat{f}_i(\xi)\), for \(i = 1, 2, \ldots, k\) and all other entries being 1.

2. \(L(\xi; t)\) satisfies
\[
\frac{\partial L(\xi; t)}{\partial t} + \delta \xi \frac{\partial L(\xi; t)}{\partial \xi} = D_0 L(\xi; t) + \hat{f}(\xi) D_1 L(\xi; t),
\]
where \(\hat{f}(\xi) = \text{diag}(\hat{f}_1(\xi), \hat{f}_2(\xi), \ldots, \hat{f}_m(\xi))\).

3. \(L_j(\xi_j; t)\) satisfies
\[
\frac{\partial L_j(\xi_j; t)}{\partial t} + \delta \xi_j \frac{\partial L_j(\xi_j; t)}{\partial \xi_j} = D_0 L_j(\xi_j; t) + \hat{f}_j(\xi_j) D_1 L_j(\xi_j; t),
\]
where \(\hat{f}_j(\xi_j) = \text{diag}(1, 1, \ldots, \hat{f}_j(\xi_j), 1, \ldots, 1)\).

We now study the moments of the ADC considered in Theorem 1. Denote the vectors of the \(n\)-th moment of the corresponding ADC as
\[
V_n(t) = (1 V_n(t), 2 V_n(t), \ldots, m V_n(t))^T,
\]
\[
V_{n,E_k}(t) = (1 V_{E_k}(t), 2 V_{E_k}(t), \ldots, m V_{E_k}(t))^T,
\]
\[
V_{n,j}(t) = (1 V_{j}(t), 2 V_{j}(t), \ldots, m V_{j}(t))^T.
\]
From equation (5), we obtain in Theorem 2 a matrix form first order differential equation satisfied by the moments of $S_{E_k}(t)$, $V_{n,E_k}(t)$, and then in Theorem 3 obtain recursive formulae for calculating them.

**Theorem 2** The moments of $S_{E_k}(t)$ satisfies

$$V'_{n,E_k}(t) + (n\delta - D_0 - D_1) V_{n,E_k}(t) = \sum_{r=1}^{n} \binom{n}{r} \mu_r D_1 V_{n-r,E_k}(t), \quad n \in \mathbb{N}^+,$$  \hspace{1cm} (6)

with initial conditions $V_{n,E_k}(0) = 0$ and $V_{0,E_k}(t) = 1$, and in particular

$$V'_{1,E_k}(t) + (\delta - D_0 - D_1) V_{1,E_k}(t) = \mu_1 D_1 1, \quad t \geq 0,$$

where $\mu_r = \text{diag}(\mu_1^{(r)}, \mu_2^{(r)}, \ldots, \mu_m^{(r)})$, $I_{E_k}$ is an $m \times m$ diagonal matrix with the $i$-th entry being 1, for $i = 1, 2, \ldots, k$, and all other diagonal entries being 0.

**Proof:** By Taylor’s expansion, we have

$$\hat{f}_i(\xi) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n \xi^n}{n!} \mu_i^{(n)}.$$

In matrix notation,

$$\hat{f}_{E_k}(\xi) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n \xi^n}{n!} I_{E_k} \mu_n.$$

Substituting (7) together with

$$L_{E_k}(\xi; t) = \sum_{n=0}^{\infty} \frac{(-1)^n \xi^n}{n!} V_{n,E_k}(t)$$

into (5) and equating the coefficients of $\xi^n$ give (6). \hfill \Box

**Corollary 1** We have the following results for the moments of $S(t)$ and $S_j(t)$.

(i) $V_n(t)$ satisfies the matrix form first order linear differential equation:

$$V'_n(t) + (n\delta - D_0 - D_1) V_n(t) = \sum_{r=1}^{n} \binom{n}{r} \mu_r D_1 V_{n-r}(t), \quad n \in \mathbb{N}^+,$$

where $V_n(0) = 0$ and $V_0(t) = 1$. In particular, $V_1(t)$ satisfies

$$V'_1(t) + (\delta - D_0 - D_1) V_1(t) = \mu_1 D_1 1, \quad t \geq 0.$$

(ii) $V_{n,j}(t)$ satisfies the matrix form first order linear differential equation:

$$V'_{n,j}(t) + (n\delta - D_0 - D_1) V_{n,j}(t) = \sum_{r=1}^{n} \binom{n}{r} I_j \mu_r D_1 V_{n-r,j}(t),$$
where $I_j = I_{(j)}$ is a diagonal matrix with $j$-th entry being 1, and 0 otherwise, $V_{n,j}(0) = 0$ and $V_{0,j}(t) = 1$. In particular, $V_{1,j}(t)$ satisfies

$$V_{1,j}'(t) + (\delta - D_0 - D_1)V_{1,j}(t) = I_j \mu_1 D_1 \mathbf{1}, \quad t \geq 0.$$

Solving differential equation (6) with $V_{n,E_k}(0) = 0$, we obtain the following recursive formulae for $V_{n,E_k}(t)$.

**Theorem 3** For $t > 0$ and $n \in \mathbb{N}^+$, we have

$$V_{n,E_k}(t) = \sum_{r=1}^{n} \binom{n}{r} \int_{0}^{t} e^{-\left(n \delta - (D_0 + D_1)\right)x} I_{E_k} \mu_r D_1 V_{n-r,E_k}(t-x) dx.$$

In particular,

$$V_{1,E_k}(t) = \int_{0}^{t} e^{-\left(\delta - (D_0 + D_1)\right)x} dx I_{E_k} \mu_1 D_1 \mathbf{1}$$

$$= [\delta - (D_0 + D_1)]^{-1} \left[I - e^{-[\delta - (D_0 + D_1)]t}\right] I_{E_k} \mu_1 D_1 \mathbf{1}.$$

Clearly, we have $V_{1,E_k}(t) + V_{1,E_k^c}(t) = V_1(t)$, where $E_k^c = E \setminus E_k$.

**Corollary 2** If we set $E_k = E$ and $E_k = \{j\}$ in Theorem 3, we have the following recursive formulae for the moments of $S(t)$ and $S_j(t)$:

$$V_n(t) = \sum_{k=1}^{n} \binom{n}{k} \int_{0}^{t} e^{-\left(n \delta - (D_0 + D_1)\right)x} \mu_k D_1 V_{n-k}(t-x) dx,$$

$$V_{n,j}(t) = \sum_{k=1}^{n} \binom{n}{k} \int_{0}^{t} e^{-\left(n \delta - (D_0 + D_1)\right)x} I_{j} \mu_k D_1 V_{n-k,j}(t-x) dx.$$

In particular,

$$V_1(t) = [\delta - (D_0 + D_1)]^{-1} \left[I - e^{-[\delta - (D_0 + D_1)]t}\right] \mu_1 D_1 \mathbf{1},$$

$$V_{1,j}(t) = [\delta - (D_0 + D_1)]^{-1} \left[I - e^{-[\delta - (D_0 + D_1)]t}\right] I_{j} \mu_1 D_1 \mathbf{1}.$$

**Remark 2** When $t \to \infty$, we have the following asymptotic results for the moments of the ADC for $n \in \mathbb{N}^+$:

$$V_{n,E_k}(\infty) = [n \delta - (D_0 + D_1)]^{-1} \sum_{r=1}^{n} \binom{n}{r} I_{E_k} \mu_r D_1 V_{n-r,E_k}(\infty),$$

$$V_n(\infty) = [n \delta - (D_0 + D_1)]^{-1} \sum_{r=1}^{n} \binom{n}{r} \mu_r D_1 V_{n-r}(\infty),$$

$$V_{n,j}(\infty) = [n \delta - (D_0 + D_1)]^{-1} \sum_{r=1}^{n} \binom{n}{r} I_{j} \mu_r D_1 V_{n-r,j}(\infty).$$
where \( V_{0,E_k}(\infty) = V_{0}(\infty) = V_{0,j}(\infty) = 1 \).

3 The covariance of ADC occurred in two sub-state spaces

In this section, we first calculate the joint moment of the ADC occurred in two subsets of the state space and then the covariance between them could be calculated.

For \( 1 \leq l_1 < l_2 < \ldots < l_k \leq m \) and \( 1 \leq n_1 < n_2 < \ldots < n_j \leq m \), where \( 2 \leq k + j \leq m \). Denote \( E_k = \{l_1, l_2, \ldots, l_k\} \) and \( E_j = \{n_1, n_2, \ldots, n_j\} \) to be two disjoint subsets of \( E \), i.e., \( E_k \cap E_j = \emptyset \). The aggregate discounted claim amounts occurred in \( E_k \) and \( E_j \) are

\[
S_{E_k}(t) = \sum_{i \in E_k} S_i(t), \quad S_{E_j}(t) = \sum_{i \in E_j} S_i(t).
\]

Define

\[
iL_{E_k,E_j}(\xi_k, \xi_j; t) = \mathbb{E}_i \left[ e^{-\xi_k S_{E_k}(t)} - \xi_j S_{E_j}(t) \right]
\]

to be the joint Laplace transform of \( S_{E_k}(t) \) and \( S_{E_j}(t) \). Let \( L_{E_k,E_j}(\xi_k, \xi_j; t) \) be a column vector with the \( i \)-th entry being \( iL_{E_k,E_j}(\xi_k, \xi_j; t) \). Moreover, let

\[
iV_{E_k,E_j}(t) = \mathbb{E}_i \left[ S_{E_k}(t) S_{E_j}(t) \right]
\]

be the joint moment of \( S_{E_k}(t) \) and \( S_{E_j}(t) \). Denote \( V_{E_k,E_j}(t) \) as an \( m \times 1 \) column vector with \( i \)-th entry being \( iV_{E_k,E_j}(t) \).

It follows from (2) that

\[
\frac{\partial L_{E_k,E_j}(\xi_k, \xi_j; t)}{\partial t} + \delta \xi_k \frac{\partial L_{E_k,E_j}(\xi_k, \xi_j; t)}{\partial \xi_k} + \delta \xi_j \frac{\partial L_{E_k,E_j}(\xi_k, \xi_j; t)}{\partial \xi_j} = D_0 L_{E_k,E_j}(\xi_k, \xi_j; t) + \hat{L}_{E_k,E_j}(\xi_k, \xi_j; \mu_1 \mathbf{D}_1 L_{E_k,E_j}(\xi_k, \xi_j; t), \quad (8)
\]

where \( \hat{L}_{E_k,E_j}(\xi_k, \xi_j) \) is a diagonal matrix with the \( l_i \)-th entry being \( \hat{f}_{l_i}(\xi_k) \), for \( i = 1, 2, \ldots, k \), with the \( n_i \)-th entry being \( \hat{f}_{n_i}(\xi_j) \), for \( i = 1, 2, \ldots, j \), and all other elements being 1.

Taking partial derivatives with respect to \( \xi_k \) and \( \xi_j \) on both sides of equation (8), setting \( \xi_k = 0 \) and \( \xi_j = 0 \), and noting that

\[
\frac{\partial^2 V_{E_k,E_j}(t)}{\partial \xi_k \partial \xi_j} \bigg|_{\xi_k=0, \xi_j=0},
\]

we obtain the following matrix form first order differential equation for \( V_{E_k,E_j}(t) \):

\[
V'_{E_k,E_j}(t) + [2\delta - D_0 - D_1] V_{E_k,E_j}(t) = \mathbf{I}_{E_k} \mu_1 \mathbf{D}_1 V_{1,E_j}(t) + \mathbf{I}_{E_j} \mu_1 \mathbf{D}_1 V_{1,E_k}(t).
\]

Solving it gives

\[
V_{E_k,E_j}(t) = \int_0^t e^{-\left(2\delta - (D_0 + D_1)\right)x} \mathbf{I}_{E_k} \mu_1 \mathbf{D}_1 V_{1,E_j}(t-x) dx
\]
Using the law of iterated expectation, we have

\[ + \int_0^t e^{-\left(2\delta - (D_0 + D_1)\right)x} I_{E_j} \mu_1 D_1 V_{1,E_k}(t-x)dx. \]

When \( t \to \infty \), we have the expression below for the joint moments of \( S_{E_k}(\infty) \) and \( S_{E_j}(\infty) \):

\[ V_{E_k,E_j}(\infty) = \left[2\delta - (D_0 + D_1)\right]^{-1} \left[I_{E_k} \mu_1 D_1 V_{1,E_j}(\infty) + I_{E_j} \mu_1 D_1 V_{1,E_k}(\infty)\right]. \]

**Remark 3** If \( E_k = \{k\} \) and \( E_j = \{j\} \), and \( k \neq j \), we have

\[ V_{\{k\},\{j\}}(t) = \int_0^t e^{-\left(2\delta - (D_0 + D_1)\right)x} I_{j} \mu_1 D_1 V_{1,k}(t-x)dx + \int_0^t e^{-\left(2\delta - (D_0 + D_1)\right)x} I_{k} \mu_1 D_1 V_{1,j}(t-x)dx. \]

When \( t \to \infty \), the joint moment of \( S_k(\infty) \) and \( S_j(\infty) \) can be expressed as

\[ V_{\{k\},\{j\}}(\infty) = \left[2\delta - (D_0 + D_1)\right]^{-1} \left[I_{j} \mu_1 D_1 V_{1,k}(\infty) + I_{k} \mu_1 D_1 V_{1,j}(\infty)\right]. \]

**Remark 4** If two subsets \( E_k \) and \( E_j \) are not disjoint, i.e., \( E_k \cap E_j = E_{kj} \neq \emptyset \), then

\[ \text{Cov}_i \left(S_{E_k}(t), S_{E_j}(t)\right) = \text{Cov}_i \left(S_{E_k \setminus E_{kj}}(t) + S_{E_{kj}}(t), S_{E_j \setminus E_{kj}}(t) + S_{E_{kj}}(t)\right) \]

\[ = \text{Cov}_i \left(S_{E_k \setminus E_{kj}}(t), S_{E_{kj}}(t)\right) + \text{Cov}_i \left(S_{E_j \setminus E_{kj}}(t), S_{E_{kj}}(t)\right) + \text{Cov}_i \left(S_{E_k \setminus E_{kj}}(t), S_{E_j \setminus E_{kj}}(t)\right) + \text{Var}_i \left(S_{E_{kj}}(t)\right). \]

All the covariance terms in the expression above are for ADC occurred in two disjoint sets.

### 4 The covariance of the ADC with two different time lengths

In this section, we investigate the covariance of the ADC occurred in two (overlapped) time periods, i.e., we want to evaluate

\[ \text{Cov}_i(S_{E_k}(t), S_{E_k}(t+h)) \triangleq \text{Cov}(S_{E_k}(t), S_{E_k}(t+h) \mid J(0) = i) \]

\[ = E_i [S_{E_k}(t)S_{E_k}(t+h)] - E_i[S_{E_k}(t)]E_i[S_{E_k}(t+h)], \]

for \( t, h > 0 \) and \( E_k = \{l_1, l_2, \ldots, l_k\} \) with \( k \leq m \).

As \( S_{E_k}(t+h) = S_{E_k}(t+h) - S_{E_k}(t) + S_{E_k}(t) \), we have

\[ E_i [S_{E_k}(t)S_{E_k}(t+h)] = E_i \left[ S_{E_k}^2(t) \right] + E_i \left[ S_{E_k}(t) \left( S_{E_k}(t+h) - S_{E_k}(t) \right) \right]. \quad (9) \]

Define \( \mathcal{F}_t = \sigma(S(v); 0 \leq v \leq t) \) to be \( \sigma \)-algebra generated by the ADC process by time \( t \). Using the law of iterated expectation, we have

\[ E_i \left[ S_{E_k}(t) \left( S_{E_k}(t+h) - S_{E_k}(t) \right) \right] \]
\[
= \mathbb{E}_i \{ \mathbb{E} \left[ S_{E_k}(t)(S_{E_k}(t + h) - S_{E_k}(t)) \mid \mathcal{F}_t \right] \}
= \mathbb{E}_i \{ S_{E_k}(t) \mathbb{E} \left[ (S_{E_k}(t + h) - S_{E_k}(t)) \mid \mathcal{F}_t \right] \}
= \mathbb{E}_i \left\{ S_{E_k}(t)e^{-\int_0^t \delta(s)ds} \mathbb{E} \left[ (S_{E_k}(t + h) - S_{E_k}(t)) \mid \mathcal{F}_t \right] \right\}
= \mathbb{E}_i \{ S_{E_k}(t)e^{-\int_0^t \delta(s)ds} \mathbb{E} \left[ (S_{E_k}(t + h) - S_{E_k}(t)) \mid \mathcal{F}_t \right] \}
\]

where \( S_{E_k}(t + h) \) is the present value, at time \( t \), of the claims occurred in states within \( E_k \) over \((t, t + h)\).

Denote \( M_{E_k}(t) = (M_{i,j,E_k}(t))_{m \times m} \), where

\[
M_{i,j,E_k}(t) = \mathbb{E}_i \{ S_{E_k}(t)e^{-\int_0^t \delta(s)ds}I(J(t) = j) \}.
\]

Conditioning on the events that may occur over an infinitesimal interval \((0, \Delta t)\), we have

\[
M_{i,j,E_k}(t) = (1 + d_{0,i,E_k}(t) - 2\delta t)M_{i,j,E_k}(t - \Delta t) + \sum_{l \neq i} d_{0,i,E_k}(t)e^{-2\delta \Delta t}M_{i,l,E_k}(t - \Delta t)
+ \sum_{l = 1}^m d_{1,i,E_k}(t)e^{-2\delta \Delta t} \left[ I(i \in E_k) \mu_i^{(1)} \mathbb{E}_i (e^{-\int_0^t \delta(s)ds}I(J(t) = j)) + M_{i,j,E_k}(t - \Delta t) \right].
\]

We can obtain a matrix form differential equation for \( M_{E_k}(t) \) from (11) as follows:

\[
M_{E_k}'(t) = (D_0 + D_1 - 2\delta)M_{E_k}(t) + I_{E_k} \mu_1 D_1 \nu(t),
\]

with \( M_{E_k}(0) = 0 \), where \( \nu(t) \) is a matrix with \((i,j)\)-th element being

\[
v_{i,j}(t) = \mathbb{E}_i \left[ e^{-\int_0^t \delta(s)ds}I(J(t) = j) \right], \quad i,j \in E.
\]

It is easy to show that \( \nu(t) = e^{(D_0 + D_1 - \delta)t} \), with \( \nu(0) = I \) and \( \nu(\infty) = 0 \). Solving (12) gives

\[
M_{E_k}(t) = \int_0^t e^{-\left(2\delta -(D_0 + D_1)\right)x} I_{E_k} \mu_1 D_1 \nu(t - x)dx.
\]

Let \( q_{i,j}(t) = \mathbb{P}_i(J(t) = j) \) and \( Q(t) = (q_{i,j}(t))_{m \times m} \) is the transition matrix of the underlying Markov process \( \{J(t)\}_{t \geq 0} \) at time \( t \). It follows from Ren (2008) that \( Q(t) = e^{(D_0 + D_1)t} \).

Denote \( R_{E_k}(t, t + h) \) as a column vector with the \( i \)-th entry being \( \mathbb{E}_i [S_{E_k}(t)S_{E_k}(t + h)] \). It follows from (9) and (10) that

\[
R_{E_k}(t, t + h) = V_{2,E_k}(t) + (M_{E_k} \circ Q)(t)V_{1,E_k}(h),
\]

where \((M_{E_k} \circ Q)(t)\) is the Hadamard product of \( M_{E_k}(t) \) and \( Q(t) \), i.e., the \((i,j)\)-th element
of \((M_{E_k} \circ Q)(t)\) is \(M_{i,j,E_k}(t) \times q_{i,j}(t)\).

**Remark 5** If \(E_k = E\) or \(E_k = \{k\}\), formula (13) simplifies to the joint moment of \(S(t)\) and \(S(t + h)\), or the joint moment of \(S_k(t)\) and \(S_k(t + h)\).

## 5 The distributions of the ADC

In this section, we investigate the distributions of \(S_{E_k}(t)\) and its two special cases, \(S(t)\) and \(S_k(t)\), for \(E_k = \{l_1, l_2, \ldots, l_k\} \subseteq E\). To precede, we define for \(x \geq 0\) and \(i \in E\),

\[
G_i(x, t) = \mathbb{P}_i(S(t) \leq x),
\]

\[
G_{i,k}(x, t) = \mathbb{P}_i(S_k(t) \leq x),
\]

\[
G_{i,E}(x, t) = \mathbb{P}_i(S_{E_k}(t) \leq x),
\]

with the following conditions:

\[
G_i(x, 0) = G_{i,k}(x, 0) = G_{i,E}(x, 0) = 1, \quad x \geq 0,
\]

\[
G_i(0, t) = \mathbb{P}_i(N(t) = 0),
\]

\[
G_{i,k}(0, t) = \mathbb{P}_i(N_k(t) = 0),
\]

\[
G_{i,E}(0, t) = \mathbb{P}_i(N_{E_k}(t) = 0),
\]

where \(N_{k}(t) = \sum_{i=1}^{N(t)} I(J(T_i) = k)\) is the number of claims occurred in state \(k\) and \(N_{E_k}(t) = \sum_{j \in E_k} N_j(t)\) is the number of claims occurred in the subset \(E_k\). Denote

\[
G(x, t) = (G_1(x, t), G_2(x, t), \ldots, G_m(x, t))^\top,
\]

\[
G_k(x, t) = (G_{1,k}(x, t), G_{2,k}(x, t), \ldots, G_{m,k}(x, t))^\top,
\]

\[
G_{E_k}(x, t) = (G_{1,E_k}(x, t), G_{2,E_k}(x, t), \ldots, G_{m,E_k}(x, t))^\top.
\]

We present in the theorem below that \(G_{E_k}(x, t)\) satisfies a first order partial integro-differential equation.

**Theorem 4** \(G_{E_k}(x, t)\) satisfies the following matrix form first order partial integro-differential equation:

\[
\frac{\partial G_{E_k}(x, t)}{\partial t} - x \delta \frac{\partial G_{E_k}(x, t)}{\partial x} = (D_0 + D_1 - I_{E_k} D_1)G_{E_k}(x, t) + \int_0^x I_{E_k} f(y)D_1 G_{E_k}(x - y, t)dy,
\]

with initial conditions

\[
G_{E_k}(x, 0) = 1, \quad G_{E_k}(0, t) = e^{(D_0 + D_1 - I_{E_k} D_1)t}1,
\]

where \(G_{E_k}(0, t)\) is the solution of the differential equation obtained from (14) by setting
Proof: Using the same arguments as in Section 2, we have by conditioning on events that may occur over \((0, h]\) that

\[
G_{i,E_k}(x, t) = [1 + d_{0,ii}^h]G_{i,E_k}(xe^{\delta h}, t - h) + \sum_{j=1, j\neq i}^m d_{0,ij}^h G_{j,E_k}(xe^{\delta h}, t - h) \\
+ \sum_{j=1}^m d_{1,ij}^h G_{j,E_k}(xe^{\delta h}, t - h), \quad i \not\in E_k. \tag{16}
\]

It follows from Taylor’s expansion that

\[
G_{i,E_k}(xe^{\delta h}, t - h) = G_{i,E_k}(x, t) + \delta_i xh \frac{\partial G_{i,E_k}(x, t)}{\partial x} - h \frac{\partial G_{i,E_k}(x, t)}{\partial t} + o(h). \tag{17}
\]

Substituting (17) into (16), rearranging terms, dividing both sides by \(h\), and taking limit as \(h \to 0\), give

\[
\frac{\partial G_{i,E_k}(x, t)}{\partial t} - \delta_i x \frac{\partial G_{i,E_k}(x, t)}{\partial x} = \sum_{j=1}^m d_{0,ij}^h G_{j,E_k}(x, t) + \sum_{j=1}^m d_{1,ij}^h G_{j,E_k}(x, t), \quad i \not\in E_k.
\]

For \(i \in E_k = \{l_1, l_2, \ldots, l_k\}\), we have

\[
G_{i,E_k}(x, t) = [1 + d_{0,ii}^h]G_{i,E_k}(xe^{\delta h}, t - h) + \sum_{j=1, j\neq i}^m d_{0,ik}^h G_{j,E_k}(xe^{\delta h}, t - h) \\
+ \sum_{j=1}^m d_{1,ij}^h \int_0^{xe^{\delta h}} f_i(y) G_{j,E_k}(xe^{\delta h} - y, t - h) dy.
\]

Taylor’s expansion gives

\[
\frac{\partial G_{i,E_k}(x, t)}{\partial t} - \delta_i x \frac{\partial G_{i,E_k}(x, t)}{\partial x} = \sum_{j=1}^m d_{0,ij}^h G_{j,E_k}(x, t) + \sum_{j=1}^m d_{1,ij}^h \int_0^{xe^{\delta h}} f_i(y) G_{j,E_k}(x - y, t) dy, \quad i \in E_k.
\]

Equations for \(i \in E_k\) and \(i \not\in E_k\) can be expressed in matrix form (14). 

Remark 6 If we set \(E_k = E\) and \(E_k = \{k\}\), respectively, we have the following results:

\[
\frac{\partial G(x, t)}{\partial t} - x\delta \frac{\partial G(x, t)}{\partial x} = D_0^h G(x, t) + \int_0^x f(y)D_1^h G(x - y, t) dy, \tag{18}
\]

\[
\frac{\partial G_k(x, t)}{\partial t} - x\delta \frac{\partial G_k(x, t)}{\partial x} = (D_0 + D_1 - I_k D_1)G_k(x, t) + \int_0^x I_k f(y)D_1^h G_k(x - y, t) dy, \tag{19}
\]

\(x = 0\).
with initial conditions
\[
G(x, 0) = 1, \quad G(0, t) = e^{D_0 t} 1, \\
G_k(x, 0) = 1, \quad G_k(0, t) = e^{(D_0 + D_1 - I_k D_1) t} 1.
\]

Here \(G_k(0, t)\) is the solution of the differential equation obtained from (19) by setting \(x = 0\).

**Remark 7** The matrix form partial integro-differential equation (14) with corresponding initial condition (15) may be solved numerically as follows.

(a) For two infinitesimal \(h_1\) and \(h_2\), we set \(G_{k_1}(lh_1, 0) = 1\), for \(l = 1, 2, \ldots\), and we calculate \(G_{k_1}(0, nh_2)\) using (15) for \(n = 1, 2, \ldots\).

(b) With (14), \(G_{k_1}(lh_1, nh_2)\) can be calculated recursively, for \(n, l = 1, 2, \ldots\), by
\[
G_{k_1}(lh_1, nh_2) = \left[I - lh_2 \delta - h_2(D_0 + D_1 - I_{E_k} D_1)\right]^{-1} \\
\times \left[G_{k_1}(lh_1, (n-1)h_2) - lh_2 \delta G_{k_1}((l-1)h_1, nh_2) \right. \\
+ h_2 h_1 \sum_{j=0}^{l-1} I_{E_k} f(jh_1) D_1 G_{k_1}((l-1-j)h_1, nh_2) \bigg].
\]

**Remark 8** If \(f_i(x) = \beta_i e^{-\beta_i x}, \beta_i > 0\), then \(f(x) = \beta e^{-\beta x}\), with \(f'(x) = -\beta f(x)\), where \(\beta = \text{diag}(\beta_1, \beta_2, \ldots, \beta_m)\). Taking partial derivative with respect to \(x\) on both sides of equation (14) and performing some manipulations, we obtain the following second order partial differential equation for \(G_{k_1}(x, t)\):
\[
\frac{\partial^2 G_{k_1}(x, t)}{\partial t \partial x} - x \delta \frac{\partial^2 G_{k_1}(x, t)}{\partial x^2} + I_{E_k} \beta \frac{\partial G_{k_1}(x, t)}{\partial t} \\
+ (\delta - D_0 - D_1 + I_{E_k}(x \delta \beta + D_1)) \frac{\partial G_{k_1}(x, t)}{\partial x} - I_{E_k}(\beta D_0 + D_1) G_{k_1}(x, t) = 0.
\]

This partial differential equations can also be solved numerically by using forward finite difference methods.

**Remark 9** Li et al. (2015) show that, when \(\delta(s) = 0\), \(G_i(x, t)\) can be used to find an expression for the density of the time of ruin in a MAP risk model.

### 6 Numerical Illustrations

In this section, we consider a two-state Markov-modulated with intensity matrix
\[
A = \begin{pmatrix}
-1/4 & 1/4 \\
3/4 & -3/4
\end{pmatrix}.
\]
We also assume that \( f_1(x) = e^{-x}, f_2(x) = 0.5e^{-0.5x}, x > 0, \lambda_1 = 1, \lambda_2 = 2/3, \delta_1 = 0.03, \delta_2 = 0.05. \) Table 1 gives the first moments of \( S_1(t) \) and \( S_2(t) \) and their covariance for \( t = 1, 2, 5, 10, 20, 30, \infty \), given \( J(0) = 1 \) and \( J(0) = 2 \), respectively, in which the covariances, for \( i = 1, 2 \), are calculated by

\[
\text{Cov}_i(t) \triangleq \text{Cov}(S_1(t), S_2(t)|J(0) = i) = \mathbb{E}_i[S_1(t)S_2(t)] - \mathbb{E}_i[S_1(t)]\mathbb{E}_i[S_2(t)].
\]

It shows that, as expected, the expected values of \( S_1(t) \) and \( S_2(t) \) (and hence \( S(t) \)) are increasing in \( t \) given \( J(0) = i \) for \( i = 1, 2 \). It is not surprised to see that \( S_1(t) \) and \( S_2(t) \) are negatively correlated for any \( t \) as claims occurred in the two states compete each other. Moreover, the larger the time \( t \), the more the negative correlation between \( S_1(t) \) and \( S_2(t) \).

**Table 1:** Expected values and covariances of \( S_1(t) \) and \( S_2(t) 

| \( t \) | \( J(0) = 1 \) | \( J(0) = 2 \) |
|---|---|---|
| \( \mathbb{E}_1[S_1(t)] \) | \( \mathbb{E}_1[S_2(t)] \) | \( \text{Cov}_1(t) \) | \( \mathbb{E}_2[S_1(t)] \) | \( \mathbb{E}_2[S_2(t)] \) | \( \text{Cov}_2(t) \) |
| 1 | 0.8948 | 0.1196 | -0.0599 | 0.2690 | 0.9444 | -0.1412 |
| 2 | 1.6665 | 0.3607 | -0.2832 | 0.8117 | 1.4717 | -0.5475 |
| 5 | 3.7056 | 1.1998 | -1.3303 | 2.6996 | 2.4452 | -1.8361 |
| 10 | 6.6248 | 2.4695 | -2.9252 | 5.5563 | 3.6966 | -3.4208 |
| 20 | 11.1330 | 4.4336 | -5.0170 | 9.9757 | 5.6221 | -5.4630 |
| 30 | 14.3123 | 5.8188 | -6.1938 | 13.0922 | 6.9800 | -6.6142 |
| \( \infty \) | 21.9178 | 9.1324 | -7.9012 | 20.5479 | 10.2283 | -8.2962 |

Figure 1 plots the variances of \( S(t) \), \( S_1(t) \), and \( S_2(t) \) with initial state \( J(0) = 1 \).

Figure 1 plots the variances of \( S(t) \), \( S_1(t) \), and \( S_2(t) \), given \( J(0) = 1 \). The variances all increase with time \( t \). The variance of \( S(t) \) is bigger than those of \( S_1(t) \) and \( S_2(t) \) for a fixed \( t \). When time \( t \) goes to \( \infty \), the three variances converge.

Tables 2 and 3 display the covariances of the ADC at time \( t \) and \( t + h \), given \( J(0) = 1 \), for some selected \( t \) values and for \( h = 1 \) and \( h = 5 \). It is shown that \( S(t) \) and \( S(t + h) \),
$S_1(t)$ and $S_1(t+h)$, and $S_2(t)$ and $S_2(t+h)$, are all positively correlated. Moreover, when $t$ increases, the covariances increase, and when $h$ increases, the covariances decrease. When $t \to \infty$, the covariances of the pairs $S(t)$ and $S(t+h)$, $S_i(t)$ and $S_i(t+h)$ converge to the variances of $S(\infty)$ and $S_i(\infty)$, respectively. Similar patterns should be expected for $J(0) = 2$.

| $t$  | $J(0) = 1$ | $J(0) = 1$ | $J(0) = 1$ |
|------|------------|------------|------------|
|      | Cov$_1(S(t), S(t+1))$ | Cov$_1(S_1(t), S_1(t+1))$ | Cov$_1(S_2(t), S_2(t+1))$ |
| 1    | 1.9327     | 1.7143     | 0.5328     |
| 2    | 3.9024     | 3.2169     | 1.7021     |
| 5    | 9.5545     | 7.3372     | 5.8448     |
| 10   | 17.0771    | 12.8965    | 11.4471    |
| 20   | 26.6637    | 20.2686    | 18.0782    |
| 30   | 31.9796    | 24.5961    | 21.2571    |
| $\infty$ | 40.3073   | 32.2449    | 23.8648    |

Table 3: Covariances of discounted claims at $t$ and $t + 5$

| $t$  | $J(0) = 1$ | $J(0) = 1$ | $J(0) = 1$ |
|------|------------|------------|------------|
|      | Cov$_1(S(t), S(t+5))$ | Cov$_1(S_1(t), S_1(t+5))$ | Cov$_1(S_2(t), S_2(t+5))$ |
| 1    | 0.8651     | 1.2213     | 0.4437     |
| 2    | 1.3181     | 2.0228     | 1.4775     |
| 5    | 3.4481     | 4.5219     | 5.2676     |
| 10   | 7.5945     | 8.5121     | 10.5187    |
| 20   | 15.2651    | 14.9882    | 16.9435    |
| 30   | 21.6039    | 19.7868    | 20.2186    |
| $\infty$ | 40.3073   | 32.2449    | 23.8648    |

Finally, we display in Figure 2 the numerical values of the distribution function of $S(t)$ with initial state $i$, $G_i(x, t) = P_i(S(t) \leq x)$, for $t = 1$ and 4, $0 \leq x \leq 25$ and $i = 1, 2$. Note that $G(x, t) = (G_1(x, t), G_2(x, t))^T$ satisfies the partial differential equation (18); its solution can be obtained numerically. From the graph, it shows clearly that the probability of $S(t)$ being bigger than a fixed $x$ is smaller for small values of $t$ as expected. For most of $x$ values, $G_1(x, t)$ is bigger than $G_2(x, t)$ due to the fact that the underlying Markov process in our example tends to stay in state 1 more often than staying at state 2.

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Figure 2: (a) The distribution of $S(1)$ for $J(0) = 1, 2$. (b) The distribution of $S(4)$ for $J(0) = 1, 2$.

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Author/s:
Li, S; Lu, Y

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