From data to reduced-order models via moment matching

Azka M. Burohman\textsuperscript{a,b,c} Bart Besselink\textsuperscript{a,b} Jacquelien M. A. Scherpen\textsuperscript{a,c} M. Kanat Camlibel\textsuperscript{a,b}

\textsuperscript{a}Jan C. Willems Center for Systems and Control, University of Groningen, The Netherlands.
\textsuperscript{b}Bernoulli Institute for Mathematics, Computer Science and Artificial Intelligence, University of Groningen, The Netherlands. (e-mail: \{a.m.burohman, b.besselink, m.k.camlibel\}@rug.nl)
\textsuperscript{c}Engineering and Technology Institute Groningen, University of Groningen, The Netherlands. (e-mail: j.m.a.scherpen@rug.nl)

Abstract

A new method for data-driven interpolatory model reduction is presented in this paper. Using the so-called data informativity perspective, we define a framework that enables the computation of moments at given (possibly complex) interpolation points based on time-domain input-output data only, without explicitly identifying the high-order system. Instead, by characterizing the set of all systems explaining the data, necessary and sufficient conditions are provided under which all systems in this set share the same moment at a given interpolation point. Moreover, these conditions allow for explicitly computing these moments. Reduced-order models are then derived by employing a variation of the classical rational interpolation method. The condition to enforce moment matching model reduction with prescribed poles is also discussed as a means to obtain stable reduced-order models. An example of an electrical circuit illustrates this framework.

Key words: Model reduction, data-driven model reduction, data informativity, moment matching, interpolatory model reduction.

1 Introduction

In the modeling of complex phenomena, e.g., in modern engineering systems, high-order dynamical systems naturally appear. They result from either the inherent complexity of (engineering or physical) systems or from the discretization of partial differential equations. The approximation of the input-output behavior of such a model by a model of lower order is known as model reduction, which has become a crucial tool in the analysis and control of complex systems. Several model reduction techniques for linear systems have been developed in the systems and control community, of which a thorough exposition can be found in [2]. The most popular techniques can roughly be divided into two main groups. The first group is based on energy functions characterized by Gramians and contains balancing methods [14,30,32] and optimal Hankel norm approximation [19]. The other contains interpolatory or Krylov projection-based methods and this paper belongs to this second group.

Interpolatory techniques form a popular class of model reduction approaches, since they are numerically stable and, therefore, applicable to models of very large order. These methods are aimed at constructing a reduced-order model whose transfer function interpolates that of the original high-order model at selected interpolation points, e.g., [8]. Moment matching techniques form an example of interpolatory methods and were originally developed in the field of numerical mathematics, see, e.g., [15,21]. By exploiting Krylov subspaces and projection, these methods achieve interpolation without explicitly evaluating the transfer function. Additionally, these methods are well suited for interpolating also the (higher-order) derivatives of the transfer function known as moments, see also [17,22]. A time-domain perspective on moment matching is given in [10], building on a relation between projection matrices for moment matching and Sylvester equations [16,18]. This time-domain perspective enables extensions of moment matching to more general system classes, see [36].

The majority of the existing model reduction methods rely on the availability of a state-space model or transfer function of the system to be reduced. In this paper,
however, we develop a data-driven moment matching method, i.e., based exclusively on time-domain input-output measurements on the high-order system. This is motivated by the observation that, in many cases, an explicit system model (or access to it) is not available.

A data-driven model reduction approach by moment matching has been introduced in [8, Chapter 4]. Relying on frequency-domain data, this so-called Loewner framework has strong connections to classical rational interpolation, see [1,5,9]. Rational interpolation has been studied extensively and allows for obtaining systems that, in addition to achieving interpolation of frequency-domain data, guarantee further system properties such as a minimal McMillan degree or stability [6,7,11]. Taking a model reduction framework for obtaining reduced-order systems that achieve interpolation as well as further properties, see [4,28]. Specifically, the preservation of stability [20] and passivity [3] is considered, as is optimal approximation in the $H_2$ system norm, see [12]. To enable the use of time-domain data (rather than frequency-domain data) in this framework, [31] estimates transfer function values at given interpolation points by exploiting the relation between time- and frequency-domain data via the (discrete) Fourier transform, after which standard interpolatory methods can be used.

Data-driven model reduction from given time-domain data has also become an attractive topic in recent years. Belonging to the class of moment matching methods, algorithms for computing (a least-square approximation of) moments of linear or nonlinear systems are proposed in [35], building on the framework of [10]. These (estimated) moments are then used to construct families of reduced-order models. This methods however relies on specifically chosen input data to guarantee that the resulting data (obtained from a steady-state response) is suitable for estimating a moment. In the class of Gramian-based methods, and in case the input data are assumed to be persistently exciting, a balanced realization of the high-order system generating the input-output data can be obtained using algorithms proposed in [27]. A behavioral approach is used in [34] to obtain so-called lossless or bounded-real balanced realizations. After obtaining such balanced realization, the reduced-order model can readily be computed. In another approach, if measurements on the state trajectories are available, dynamic mode decomposition (DMD) provides a way to find a linear model that best fits the given trajectories in the $L_2$ sense as in [33]. In [29], if the state trajectories are available, a lower-order model expressed in terms of data is uncovered such that its complexity matches that of the given data. A data-driven model reduction approach for a class of nonlinear systems is presented in [25].

In contrast to existing works, this paper presents a method for obtaining moments at a given interpolation point directly from time-domain input-output data, without requiring that the data are sufficiently rich to uniquely identify the high-order system (that generates the data). In particular, the contributions of this paper are threefold.

First, we introduce a new framework for computing moments from data, without necessarily identifying the full model. This perspective builds on the concept of data informativity introduced in [39], which aims to find conditions on a given data set such that specific system properties can be concluded from them. This perspective has been applied successfully in the scope of system analysis [13,39] and control design [37,38]. We stress that this framework does not require that the (high-order) system can be fully identified, as would for example be implied by the availability of impulse response data [23] or by persistently exciting data, see the so-called fundamental lemma in [41]. The importance of such persistently exciting data is well-known in the field of system identification [26,40]. Following this perspective, we define the concept of data informativity for interpolation at some interpolation point $\sigma$ as a notion that guarantees that the data is rich enough to uniquely find the value of the transfer function of the high-order system at $\sigma$. We then generalize this concept to higher-order moments.

Second, we provide necessary and sufficient conditions for given input-output data to be informative for interpolation and/or moment matching of order $k$ at a given interpolation point. We show that these conditions are weaker than those for identification, such that there is no need to fully identify the higher-order system. These conditions also lead to an approach for the computation of the moments on the basis of data. We stress that this approach yields the exact moments, instead of an approximation.

Finally, we construct the reduced-order model based on the computed moments. Our approach has a natural connection to classical rational interpolation. Therefore, the computation of the reduced-order model in this paper is similar to the rational interpolation method presented in [9]. In addition, inspired by [10], we show that our method enables to obtain a reduced-order model with prescribed poles, which enables the construction of stable reduced-order models.

The remainder of this paper is organized as follows. First, a detailed problem setting is given in Section 2. In Section 3, data informativity for interpolation is introduced, which includes an illustrative example. Next, informativity for higher-order moments and the computation of them are presented in Section 4. The computed moments are then exploited to compute a reduced-order model in Section 5, which includes the discussion on reduction with prescribed poles. Finally, conclusions are stated in Section 6.
\section{Problem formulation}

In this section, we will introduce a framework to deal with data-driven model reduction via interpolation and moment matching.

Consider the discrete-time input-output system of the form
\[ y_{t+n} + \tilde{p}_{n-1}y_{t+n-1} + \cdots + \tilde{p}_1y_{t+1} + \tilde{p}_0y_t = \tilde{q}_n u_{t+n} + \tilde{q}_{n-1}u_{t+n-1} + \cdots + \tilde{q}_1u_{t+1} + \tilde{q}_0u_t, \tag{1} \]
where \( u \) denotes the scalar input, \( y \) the scalar output and \( t \in \mathbb{N} \) the discrete time. We denote the parameters of (1) by
\[ \tilde{p} = \begin{bmatrix} \tilde{p}_0 & \tilde{p}_1 & \cdots & \tilde{p}_{n-1} \end{bmatrix} \in \mathbb{R}^{1 \times n} \]
and
\[ \tilde{q} = \begin{bmatrix} \tilde{q}_0 & \tilde{q}_1 & \cdots & \tilde{q}_n \end{bmatrix} \in \mathbb{R}^{1 \times (n+1)}. \]
Throughout the paper, we assume that the parameters \( \begin{bmatrix} \tilde{q} & -\tilde{p} \end{bmatrix} \) are unknown but \( n \geq 0 \) is known and we have access to input-output data given by
\[ U = \begin{bmatrix} u_0 & u_1 & \cdots & u_T \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} y_0 & y_1 & \cdots & y_T \end{bmatrix} \]
that are generated by the ‘true’ system (1), i.e.,
\[ y_{t+n} + p_{n-1}y_{t+n-1} + \cdots + p_1y_{t+1} + p_0y_t = q_n u_{t+n} + q_{n-1}u_{t+n-1} + \cdots + q_1u_{t+1} + q_0u_t, \tag{2} \]
for \( t = 0, 1, \ldots, T - n \) and some \( T \geq n \).

We do not assume that the data \( (U, Y) \) uniquely identify the system (1). To determine all systems that are compatible with the data \( (U, Y) \), consider an input-output system of the form
\[ y_{t+n} + p_{n-1}y_{t+n-1} + \cdots + p_1y_{t+1} + p_0y_t = q_n u_{t+n} + q_{n-1}u_{t+n-1} + \cdots + q_1u_{t+1} + q_0u_t, \tag{3} \]
The data \( (U, Y) \) can be generated by this system if and only if
\[ \tilde{y}_{t+n} + \tilde{p}_{n-1}\tilde{y}_{t+n-1} + \cdots + \tilde{p}_1\tilde{y}_{t+1} + \tilde{p}_0\tilde{y}_t = \tilde{q}_n \tilde{u}_{t+n} + \tilde{q}_{n-1}\tilde{u}_{t+n-1} + \cdots + \tilde{q}_1\tilde{u}_{t+1} + \tilde{q}_0\tilde{u}_t \]
for \( t = 0, 1, \ldots, T - n \). This system of linear equations can be written more compactly as
\[ \begin{bmatrix} q & -p \end{bmatrix} \begin{bmatrix} H_n(U) \\ H_n(Y) \end{bmatrix} = \begin{bmatrix} \tilde{y}_n & \cdots & \tilde{y}_T \end{bmatrix}, \tag{4} \]
where
\[ p = \begin{bmatrix} p_0 & p_1 & \cdots & p_{n-1} \end{bmatrix} \in \mathbb{R}^{1 \times n}, \]
\[ q = \begin{bmatrix} q_0 & q_1 & \cdots & q_n \end{bmatrix} \in \mathbb{R}^{1 \times (n+1)}, \]
and \( H_{\ell}(U) \) denotes the Hankel matrix of depth \( \ell \) obtained from \( U \), i.e.,
\[ H_{\ell}(U) = \begin{bmatrix} \tilde{u}_0 & \tilde{u}_1 & \cdots & \tilde{u}_{T-\ell} \\ \tilde{u}_1 & \tilde{u}_2 & \cdots & \tilde{u}_{T-\ell+1} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{u}_{\ell-1} & \tilde{u}_{\ell} & \cdots & \tilde{u}_T \end{bmatrix} \in \mathbb{R}^{(\ell+1) \times (T-\ell+1)}. \]
The Hankel matrix \( H_{\ell}(Y) \) is defined in a similar fashion and, finally, \( H_{\ell}(Y) \) is obtained from \( H_{\ell}(Y) \) by deleting its last row. Now, we can identify the set of all systems that can generate the data \( (U, Y) \) with the set
\[ \Sigma_{U, Y} = \left\{ q - p \in \mathbb{R}^{1 \times (2n+1)} \mid (4) \text{ holds} \right\}. \tag{5} \]
Since the data \( (U, Y) \) are generated by the ‘true’ system (1), i.e., (2) is satisfied, we clearly have that \( \begin{bmatrix} \tilde{q} & -\tilde{p} \end{bmatrix} \in \Sigma_{U, Y} \). An obvious question to ask is when the data uniquely determine the ‘true’ system.

**Definition 1** The data \( (U, Y) \) are informative for system identification if \( \Sigma_{U, Y} \) is a singleton.

Data informativity for system identification can easily be characterized as follows.

**Proposition 2** The data \( (U, Y) \) are informative for system identification if and only if
\[ \text{rank} \begin{bmatrix} H_n(U) \\ H_n(Y) \end{bmatrix} = \text{rank} \begin{bmatrix} H_n(U) \\ H_n(Y) \end{bmatrix} = 2n + 1. \tag{6} \]

**PROOF.** It is obvious that the first and the second equality in (6) are equivalent to the existence and uniqueness of the solution of the linear equation (4), respectively. \( \Box \)

**Remark 3** The condition (6) implies that
\[ \text{rank} H_n(U) = n + 1 \tag{7} \]
and hence
\[ \Pi_{\perp H_n(U)} = I - H_n(U)^T (H_n(U)H_n(U)^T)^{-1} H_n(U) \]
is well-defined. Using this, it can be shown that (6) implies
\[
\text{rank} \left( H_n(Y) \Pi_{H_n(U)} \right) = n, \tag{8}
\]
which is a well-known sufficient condition for system identification, see [40, Chapter 9]. Since the data \((U, Y)\) are generated by a system of order \(n\), Guttman’s rank additivity formula [42, page 14] implies that (8) together with (7) is equivalent to (6).

**Remark 4** In system identification, sufficient conditions for identification are typically stated in terms of the so-called persistency of excitation condition on the input data only. Here, the data \(U\) are persistently exciting of order \(\ell + 1\) if there exists an integer \(T\) such that \(H_t(U)\) has full rank \(\ell + 1\). Then, a sufficient condition for system identification (i.e., uniquely constructing a transfer function from data) is that \(U\) is persistently exciting of order \(2n + 1\), see [40, 41]. It is clear that this implies (7) and (8), which is equivalent to (6) by Remark 3. We stress however that (6) is necessary and sufficient for system identification.

### 2.1 Moments of a discrete-time linear system

Next, we will introduce moments of the system (3). To do so, we first introduce the forward shift operator \(z\) defined as \(z f_t = f_{t+1}\). Then, we rewrite (3) in the form
\[
\begin{align*}
(z^n + p_{n-1}z^{n-1} + \cdots + p_1z + p_0) y_t &= (q_n z^n + q_{n-1}z^{n-1} + \cdots + q_1z + q_0) u_t.
\end{align*}
\tag{9}
\]
We denote the polynomial on the left- and right-hand sides of (9), respectively, as
\[
\begin{align*}
P(z) &= z^n + p_{n-1}z^{n-1} + \cdots + p_1z + p_0, \\
Q(z) &= q_n z^n + q_{n-1}z^{n-1} + \cdots + q_1z + q_0.
\end{align*}
\tag{10}
\]

Now, we are in a position to define the 0-th moment for the system (3).

**Definition 5 (0-th moment)** Given an interpolation point \(\sigma \in \mathbb{C}\), a number \(M_0 \in \mathbb{C}\) is said to be a 0-th moment at \(\sigma\) of the discrete-time system (3) if
\[
P(\sigma) M_0 = Q(\sigma).
\tag{11}
\]
In this case, we also write \(M_0 = M_0(\sigma)\).

**Remark 6** For a discrete-time system (3) with transfer function
\[
G(z) = \frac{Q(z)}{P(z)},
\]
the 0-th moment at \(\sigma\) is typically defined as the complex number \(Q(\sigma)/P(\sigma)\). This, however, requires that \(P(\sigma) \neq 0\). In other words, the 0-th moment is not defined when \(\sigma\) is a pole of \(G(z)\). The notion in Definition 5 is a slight generalization as it allows to define a moment in case both \(P(\sigma) = 0\) and \(Q(\sigma) = 0\), i.e., there is a pole-zero cancellation at \(\sigma\). We stress that minimality is not assumed for (3). Therefore, any complex number \(M_0\) is regarded as a 0-th moment at \(\sigma\) by Definition 5 in case \(P(\sigma) = Q(\sigma) = 0\).

Given (10), condition (11) can be written as the linear equation
\[
\begin{bmatrix}
q - p
\end{bmatrix}
\begin{bmatrix}
\gamma_n(\sigma) \\
M_0 \gamma_n(-1)(\sigma)
\end{bmatrix} = M_0 \sigma^n,
\tag{12}
\]
where
\[
\gamma_\ell(z) = \begin{bmatrix} z & \vdots & z^\ell \end{bmatrix}.
\]
Then, the expression (12) allows for defining
\[
\Sigma_{\sigma, M_0} = \left\{ \begin{bmatrix} q - p \end{bmatrix} \in \mathbb{R}^{1 \times (2n + 1)} \mid (12) \text{ holds} \right\},
\tag{13}
\]
as the set of all (parameters of) systems of order \(n\) that have 0-th moment \(M_0\) at \(\sigma\).

Let \(f^{(j)}\) denote the \(j\)-th derivative of \(f\), i.e.,
\[
f^{(j)}(z) = \frac{d^j}{dz^j} f(z).
\]
Next, we define higher order moments in a recursive manner.

**Definition 7** \((k\text{-th moment})\) Given an interpolation point \(\sigma \in \mathbb{C}\) and \(j\)-th moments (at \(\sigma\)) \(M_j\) for \(j = 0, 1, \ldots, k - 1\), then, a number \(M_k \in \mathbb{C}\) is said to be a \(k\)-th moment at \(\sigma\) of the discrete-time system (3) if
\[
Q^{(k)}(\sigma) = \sum_{j=0}^{k} \binom{k}{j} M_j P^{(k-j)}(\sigma),
\tag{14}
\]
where
\[
\binom{k}{j} = \frac{k!}{j!(k-j)!}
\]
is the binomial coefficient. In this case, we also write \(M_k = M_k(\sigma)\).

**Remark 8** The definition of \(k\)-th moment in Definition 7 is related to the classical definition of moments as in, e.g., [2] and [10]. Classically, the \(k\)-th moment of a
transfer function $G(z)$ at $\sigma$ is defined by its $k$-th derivative with respect to $z$ evaluated at $z = \sigma$, i.e.,

$$M_k(\sigma) = \frac{d^k}{dz^k}G(z) \bigg|_{z=\sigma}, \quad k \geq 0.$$  

With $G(z) = Q(z)/P(z)$, the derivatives (up to order $k$) of $G(z)$, $P(z)$ and $Q(z)$ satisfy

$$Q^{(k)}(z) = \sum_{j=0}^{k} \binom{k}{j} G^{(j)}(z) P^{(k-j)}(z). \quad (15)$$

Evaluating (15) at $z = \sigma$ yields (14). Hence, (14) reduces to the classical definition of $k$-th moment when $P(\sigma) \neq 0$. Nonetheless, Definition 7 is a slight generalization for systems $G$.

Similarly as before, (14) can be written as a linear equation in $[q \quad p]$ as

$$\begin{bmatrix} q & p \end{bmatrix} \begin{bmatrix} \gamma^{(k)}(\sigma) \\ \sum_{j=0}^{k} \binom{k}{j} M_j \gamma^{(k-j)}(\sigma) \end{bmatrix} = \sum_{j=0}^{k} \binom{k}{j} \frac{n!}{(n-k+j)!} M_j \sigma^{n-k+j},$$

where

$$\gamma^{(j)}(\sigma) = \frac{d^j}{dz^j} \gamma(z) \bigg|_{z=\sigma}.$$  

We define the set of all (parameters of) systems having $M_k$ as the $k$-th moment at $\sigma$ by

$$\Sigma_{\sigma,M_k}^k = \left\{ \left[ q \quad p \right] \in \mathbb{R}^{1 \times (2n+1)} \bigg| (16) \text{ holds} \right\}.$$  

Note that these sets are also defined by a recursion.

### 2.2 Informativity problem for moment matching

Recall that we are interested in the moments of the true system (1), but only have the input-output data $(U, Y)$ available. Given an interpolation point $\sigma \in \mathbb{C}$, we are interested in finding (necessary and sufficient) conditions for the data $(U, Y)$ to be sufficiently rich to allow for computing the moments at $\sigma$. These conditions will allow for the construction of reduced-order models that achieve moment matching.

Starting with the 0-th moment, recall that all systems with moment $M_0$ at $\sigma$ are given by the set $\Sigma_{\sigma,M_0}^0$. However, as $[q \quad p] \in \Sigma_{U,Y}$, it is sufficient to ask for $\Sigma_{U,Y} \subseteq \Sigma_{\sigma,M_0}^0$. This motivates the following definition.

**Definition 9** The data $(U, Y)$ are informative for interpolation at $\sigma$ if there exists a unique $M_0$ such that $\Sigma_{U,Y} \subseteq \Sigma_{\sigma,M_0}^0$.

Note that the condition $\Sigma_{U,Y} \subseteq \Sigma_{\sigma,M_0}^0$ requires all systems explained by the data to have the same 0-th moment $M_0$ at $\sigma$. It is clear that, if the data $(U, Y)$ are informative for system identification (see Definition 1) and $P(\sigma) \neq 0$, then such unique $M_0$ exists. In this paper, we however look for conditions for informativity for interpolation that are strictly weaker than those for identification. This motivates the following problem statement.

**Problem 1** Find necessary and sufficient conditions such that $(U, Y)$ are informative for interpolation at $\sigma$. Furthermore, if the data satisfy these conditions, then find $M_0$.

Once we know that the data are informative for interpolation at $\sigma$ (i.e., for matching of the 0-th moment), we are also interested to match higher-order moments.

**Definition 10** The data $(U, Y)$ are informative for moment matching of order $k$ at $\sigma$ if

(1) they are informative for moment matching of order $j$ at $\sigma$ for $j = 0, 1, 2, \ldots, k-1$ (where moment matching of order 0 is understood as interpolation), and

(2) there exists a unique $M_k$ such that $\Sigma_{U,Y} \subseteq \Sigma_{\sigma,M_k}^k$.

Note that Definition 10 follows the recursive structure of the notion of $k$-th moment in Definition 7. Finally, we are interested in the following problem.

**Problem 2** Find necessary and sufficient conditions such that $(U, Y)$ are informative for moment matching of order $k$ at $\sigma$, with $k = 1, 2, \ldots$. Furthermore, if the data satisfy these conditions, then find $M_k$.

### 3 Data informativity for interpolation

In this section, we will provide necessary and sufficient conditions under which the data are informative for interpolation (i.e., moment matching of order 0) at a given interpolation point.

The following lemma will play an instrumental role in the characterization of informativity for interpolation. We note that this lemma builds upon some basic results in linear algebra given in Appendix A.

**Lemma 11** Let $\sigma$ and $M_0$ be complex numbers. Then, the inclusion $\Sigma_{U,Y} \subseteq \Sigma_{\sigma,M_0}^0$ holds if and only if there
exists \( \xi \in \mathbb{C}^{T-n+1} \) such that
\[
\begin{bmatrix}
H_n(U) & 0 \\
H_n(Y) & -\gamma_n(\sigma)
\end{bmatrix}
\begin{bmatrix}
\xi \\
M_0
\end{bmatrix} =
\begin{bmatrix}
\gamma_n(\sigma) \\
0
\end{bmatrix}.
\] (17)

**PROOF.** Note that \( \Sigma_{U,Y} \subseteq \Sigma_{\sigma,M_0}^0 \) is equivalent to saying that every solution of (4) is also a solution of (12). Note that (12) can be split into a real and imaginary part. Then, since \([q - p]\) is real, we have by Lemma A.1 in the appendix that \( \Sigma_{U,Y} \subseteq \Sigma_{\sigma,M_0}^0 \) if and only if
\[
\text{leftker } \begin{bmatrix}
H_n(U) \\
H_n(Y)
\end{bmatrix} \subseteq \text{leftker } \text{Re} \begin{bmatrix}
\gamma_n(\sigma) \\
M_0 \gamma_n(\sigma)
\end{bmatrix},
\] (18)
and
\[
\text{leftker } \begin{bmatrix}
H_n(U) \\
H_n(Y)
\end{bmatrix} \subseteq \text{leftker } \text{Im} \begin{bmatrix}
\gamma_n(\sigma) \\
M_0 \gamma_n(\sigma)
\end{bmatrix},
\] (19)
where \( \text{Re} \) and \( \text{Im} \) denote, respectively, real and imaginary parts. Note that (18) and (19) are equivalent to the existence of real vectors \( \xi_R \) and \( \xi_I \) such that
\[
\begin{bmatrix}
H_n(U) \\
H_n(Y)
\end{bmatrix}
\xi_R = \text{Re} \begin{bmatrix}
\gamma_n(\sigma) \\
M_0 \gamma_n(\sigma)
\end{bmatrix},
\] (20)
and
\[
\begin{bmatrix}
H_n(U) \\
H_n(Y)
\end{bmatrix}
\xi_I = \text{Im} \begin{bmatrix}
\gamma_n(\sigma) \\
M_0 \gamma_n(\sigma)
\end{bmatrix}.
\] (21)
Summing (20) and (21) and denoting \( \xi = \xi_R + i \xi_I \), we can conclude that \( \Sigma_{U,Y} \subseteq \Sigma_{\sigma,M_0}^0 \) if and only if there exists \( \xi \in \mathbb{C}^{T-n+1} \) such that
\[
\begin{bmatrix}
H_n(U) \\
H_n(Y)
\end{bmatrix}
\xi = \begin{bmatrix}
\gamma_n(\sigma) \\
M_0 \gamma_n(\sigma)
\end{bmatrix}.
\] \(\Box\)

**Remark 12** Note that the system parameters \([q - p]\) are restricted to be real (see the sets (5) and (13)) whereas the interpolation points \( \sigma \) might be taken complex. Hence, the main result of Lemma 11 is that the (real-valued) inclusion \( \Sigma_{U,Y} \subseteq \Sigma_{\sigma,M_0}^0 \) is equivalent to the possibly complex-valued, condition (17).

By using the above result, we provide the first main result of this paper: necessary and sufficient conditions for data informativity for interpolation.

**Theorem 13** The data \((U,Y)\) are informative for interpolation at \( \sigma \) if and only if
\[
\text{rank} \begin{bmatrix}
H_n(U) & 0 \\
H_n(Y) & \gamma_n(\sigma)
\end{bmatrix} = \text{rank} \begin{bmatrix}
H_n(U) & 0 \\
H_n(Y) & \gamma_n(\sigma)
\end{bmatrix}
\] (22)
and
\[
\text{rank} \begin{bmatrix}
H_n(U) \\
H_n(Y) \gamma_n(\sigma)
\end{bmatrix} = \text{rank} \begin{bmatrix}
H_n(U) \\
H_n(Y) \gamma_n(\sigma)
\end{bmatrix} + 1.
\] (23)

**PROOF.** By Definition 9, the data \((U,Y)\) are informative for interpolation at \( \sigma \) if there exists a unique \( M_0 \) such that \( \Sigma_{U,Y} \subseteq \Sigma_{\sigma,M_0}^0 \). On the one hand, the existence of such an \( M_0 \) is equivalent to the existence of \( \xi \) such that
\[
\begin{bmatrix}
H_n(U) \\
H_n(Y) \gamma_n(\sigma)
\end{bmatrix}
\xi = \begin{bmatrix}
\gamma_n(\sigma) \\
0
\end{bmatrix}
\] (24)
due to Lemma 11. On the other hand, the uniqueness of \( M_0 \) is equivalent to the implication
\[
\begin{bmatrix}
H_n(U) \\
H_n(Y) \gamma_n(\sigma)
\end{bmatrix}
\begin{bmatrix}
\xi_1 \\
\eta_1
\end{bmatrix} =
\begin{bmatrix}
H_n(U) \\
H_n(Y) \gamma_n(\sigma)
\end{bmatrix}
\begin{bmatrix}
\xi_2 \\
\eta_2
\end{bmatrix} \Rightarrow \eta_1 = \eta_2.
\] (25)

Therefore, the data are informative for interpolation at \( \sigma \) if and only if (24) and (25) hold. Clearly, (24) and (22) are equivalent. It follows from Lemma A.2 in the appendix that (25) is equivalent to (23). \( \Box \)

**Remark 14** The conditions (22) and (23) in Theorem 13 allow for an insightful interpretation. On the one hand, (23) implies the existence of a system \([q - p] \in \Sigma_{U,Y} \) such that \( P(\sigma) \neq 0 \). On the other hand, (22) guarantees that if there exists \([q - p] \in \Sigma_{U,Y} \) such that \( P(\sigma) = 0 \), then \( Q(\sigma) = 0 \).

An important consequence of Theorem 13 is that the data do not need to be informative for system identification in order to be for interpolation. Thus, it is possible that infinitely many systems explain the same data and they all have the same moment at a given interpolation point.

When \( \sigma \in \mathbb{C} \setminus \mathbb{R} \), it is natural to ask how data informativity for interpolation at \( \sigma \) is related to informativity for its complex conjugate. The following proposition answers this question, based on the results of Lemma 11 and Theorem 13.
In view of Theorem 13, (27) and (28) imply that the 0-th moment at σ is \( \bar{M}_0 \), then the 0-th moment at \( \bar{\sigma} \) is \( \bar{M}_0 \).

Proposition 15 The data \( (U, Y) \) are informative for interpolation at \( \sigma \) if and only if they are informative for interpolation at \( \bar{\sigma} \). Moreover, if the 0-th moment at \( \sigma \) is \( M_0 \), then the 0-th moment at \( \bar{\sigma} \) is \( \bar{M}_0 \).

**PROOF.** only if: Suppose that the data \( (U, Y) \) are informative for interpolation at \( \sigma \). In view of Theorem 13, this means that (22) and (23) hold. By (22), there exist \( \xi \in \mathbb{C}^{T-n+1} \) and \( M_0 \in \mathbb{C} \) such that (17) holds. By taking the complex conjugate of (17), we get

\[
\begin{bmatrix}
H_\gamma(U) & 0 \\
H_\gamma(Y) & -\gamma_\gamma(\bar{\sigma})
\end{bmatrix}
\begin{bmatrix}
\xi \\
\bar{M}_0
\end{bmatrix} =
\begin{bmatrix}
\gamma_\gamma(\bar{\sigma}) \\
0
\end{bmatrix},
\]

where we have used that \( \gamma_\gamma(\sigma) = \gamma_\gamma(\bar{\sigma}) \). This implies that

\[
\text{rank}
\begin{bmatrix}
H_\gamma(U) & 0 & \gamma_\gamma(\bar{\sigma}) \\
H_\gamma(Y) & \gamma_\gamma(\bar{\sigma}) & 0
\end{bmatrix}
= \text{rank}
\begin{bmatrix}
H_\gamma(U) & 0 \\
H_\gamma(Y) & \gamma_\gamma(\bar{\sigma})
\end{bmatrix}.
\]

From (23), we see that

\[
\text{rank}
\begin{bmatrix}
H_\gamma(U) & 0 \\
H_\gamma(Y) & \gamma_\gamma(\bar{\sigma})
\end{bmatrix}
= \text{rank}
\begin{bmatrix}
H_\gamma(U) \\
H_\gamma(Y)
\end{bmatrix} + 1.
\]

In view of Theorem 13, (27) and (28) imply that the data \( (U, Y) \) are informative for interpolation at \( \bar{\sigma} \). From (26), we can conclude that \( \bar{M}_0 \) is the 0-th moment at \( \bar{\sigma} \).

**if:** This part is evident since it readily follows from the fact that \( \bar{\sigma} = \sigma \) and the above proof. \( \square \)

To illustrate the first main result of this paper, we provide the following example.

Example 16 Consider the RL circuit depicted in Figure 1, which is a slight extension of [24, Example 22]. We take the currents through the inductors \( L_1, L_2, L_3 \) and \( L_4 \) as the states of the system, so \( n = 4 \). The input is the voltage \( V_d \). Finally, as the output, we take the

| t  | U   | Y   |
|----|-----|-----|
| 0  | 2   | 0   |
| 1  | 1.7071 | 0.25 |
| 2  | 0.7500 | 0.3726 |
| 3  | 0.2071 | 0.3659 |
| 4  | 0.6875 | 0.2755 |
| 5  | 0.5196 | 0.1738 |
| 6  | 0.1094 | 0.1262 |
| 7  | 0.7606 | 0.1592 |
| 8  | 1.0352 | 0.2488 |
| 9  | 0.7266 | 0.3367 |
| 10 | 0.0107 | 0.365 |

current through the first inductor \( L_1 \). This leads to the continuous-time dynamical system

\[
\dot{x} = \begin{bmatrix}
\frac{-R_1}{L_1} & \frac{R_2}{L_1} & \frac{-R_3}{L_1} & \frac{R_4}{L_1} \\
\frac{-R_2}{L_2} & \frac{R_3}{L_2} & \frac{-R_4}{L_2} & \frac{R_5}{L_2} \\
\frac{-R_3}{L_3} & \frac{R_4}{L_3} & \frac{-R_5}{L_3} & \frac{R_6}{L_3} \\
\frac{-R_4}{L_4} & \frac{R_5}{L_4} & \frac{-R_6}{L_4} & \frac{R_7}{L_4}
\end{bmatrix} x + \begin{bmatrix}
\frac{1}{L_1} \\
\frac{1}{L_2} \\
\frac{1}{L_3} \\
\frac{1}{L_4}
\end{bmatrix} u,
\]

\[
y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x.
\]

Let the inductances be given by \( L_1 = L_2 = L_3 = L_4 = 1 \text{ H} \). For the resistors, we have \( R_1 = 0.5 \Omega, R_2 = 8 \Omega, R_3 = 5 \Omega, R_4 = 1 \Omega, R_5 = 4 \Omega \). We stress that we assume that the exact model (29) is unknown and that only input-output data are available. Specifically, we assume that the continuous input \( u \) is the result of applying a zero-order hold to a discrete input \( U \) with sampling time \( \Delta t = 0.2 \text{ s} \). The continuous output \( y \) is assumed to be sampled with the same frequency and we consider input-output data for \( T = 20 \text{ samples} \) (4 seconds). This leads to the samples given in Table 1 and depicted in Figure 2.

It can be verified that condition (6) does not hold for these data. As such, the data are not informative for system identification. Instead, there are (infinitely) many systems of the form (3) with order \( n = 4 \) that explain the data. Bode plots of some systems that explain the data are given by Figure 3.

Let us check the informativity condition for interpolation at \( \sigma = \frac{-R_2}{\sqrt{2}} + \frac{R_3}{\sqrt{2}} \). A direct computation of the ranks of the matrices in Theorem 13 leads to

\[
\text{rank}
\begin{bmatrix}
H_\gamma(U) \\
H_\gamma(Y)
\end{bmatrix}
= 9,
\]

\[
\text{rank}
\begin{bmatrix}
H_\gamma(U) & 0 & \gamma_\gamma(\frac{-R_2}{\sqrt{2}} + \frac{R_3}{\sqrt{2}}) \\
H_\gamma(Y) & \gamma_\gamma(\frac{-R_2}{\sqrt{2}} + \frac{R_3}{\sqrt{2}}) & 0
\end{bmatrix}
= 9,
\]

Figure 1. RL circuit with four inductors and five resistors.
Hence, the data \( (U, Y) \) are informative for interpolation at \( \sigma = 0 \). This is also apparent from the Bode plot in Figure 3, as not all the magnitudes and phases coincide at low frequency.

4 Data informativity for higher-order moment matching

The previous section addressed data informativity for interpolation, i.e., for uniquely identifying the 0-th moment at a given interpolation point \( \sigma \). In the current section, we focus on higher-order moments.

We recall that higher-order moments are defined recursively, see Definition 7. Then, the following results provide a counterpart of Lemma 11 and Theorem 13 for higher-order moments.

**Lemma 17** Let \( \sigma \) and \( M_j \) for \( j = 0, 1, \ldots, k \) be complex numbers. Then, the inclusion \( U, Y \subseteq \Sigma^k_{\sigma,M_k} \) holds if and only if there exists \( \xi \in \mathbb{C}^{T-n+1} \) such that

\[
\begin{bmatrix} H_n(U) & 0 \\ H_n(Y) & \gamma_n(0) \end{bmatrix} \begin{bmatrix} \xi \\ M_k \end{bmatrix} = \begin{bmatrix} \sum_{j=0}^{k-1} \gamma_n^{(k)}(\sigma) \\ \sum_{j=0}^{k} M_j \gamma_n^{(k-j)}(\sigma) \end{bmatrix}.
\]

**Proof.** Note that \( U, Y \subseteq \Sigma^k_{\sigma,M_k} \) is equivalent to saying that every solution of (4) is also a solution of (16). The rest of the proof is similar to that of Lemma 11, hence it is omitted. \( \square \)

Using Lemma 17, the following result provides necessary and sufficient conditions for data informativity for higher-order moment matching.

**Theorem 18** Let \( k > 0 \) and suppose that the data \( U, Y \) are informative for moment matching of order \( j \) at \( \sigma \) for all \( j \) with \( 0 \leq j < k \). Let \( M_j \) denote the corresponding moments. Then, the data \( U, Y \) are informative for moment matching of order \( k \) at \( \sigma \) if and only if

\[
\begin{bmatrix} H_n(U) & 0 \\ H_n(Y) & \gamma_n(0) \end{bmatrix} \begin{bmatrix} \gamma_n^{(k)}(\sigma) \\ \sum_{j=0}^{k-1} \gamma_n^{(k)}(\sigma) + \sum_{j=0}^{k} M_j \gamma_n^{(k-j)}(\sigma) \end{bmatrix} = \begin{bmatrix} H_n(U) & 0 \\ H_n(Y) & \gamma_n(0) \end{bmatrix}.
\]

**Proof.** The assumption given in this theorem that the data are informative for moment matching of order \( j \) at \( \sigma \) for all \( j \) with \( 0 \leq j < k \) satisfies the first condition of Definition 10. In addition, the data are informative for moment matching of order \( k \) at \( \sigma \) if there exists a
unique $M_k$ such that $\Sigma_{U,Y} \subseteq \Sigma_{M_k}$. The existence of such an $M_k$ is equivalent to the existence of $\xi$ such that

$$
\begin{bmatrix}
H_n(U) & 0 \\
H_n(Y) & -\gamma_n(\sigma) - \sum_{j=0}^{k-1} \binom{k}{j} M_j \gamma_n^{(k-j)}(\sigma)
\end{bmatrix} \begin{bmatrix}
\xi \\
M_k
\end{bmatrix} = \begin{bmatrix}
\gamma_n(\sigma)
\end{bmatrix}
$$

(32)
due to Lemma 17. Clearly, this is equivalent to (31).

Note that the right hand side of (32) does not play a role in determining the uniqueness of $M_k$. Since the data have been assumed to be informative for interpolation at $\sigma$, condition (23) in Theorem 13 holds. This guarantees the uniqueness of $M_k$ since the matrix in the left hand side of (32) is the same to that of (24). $\Box$

Analogous to Theorem 13, this theorem provides conditions under which all systems explaining the data have the same higher-order moment at a given interpolation point.

The conjugacy relation between 0-th moments at conjugate pairs of interpolation points can be extended for higher-order moments as well.

Proposition 19 Let $k > 0$. The data $(U,Y)$ are informative for moment matching of order $k$ if and only if they are informative for moment matching of order $k$ at $\sigma$. Moreover, if the $k$-th moment at $\sigma$ is $M_k$, then the $k$-th moment at $\sigma$ is $M_k$.

PROOF. The proof is similar to that of Proposition 15. $\Box$

We illustrate the results of Theorem 18 by means of an example.

Example 20 Consider the system and input-output data studied in Example 16. Let the interpolation point be $\sigma = 0.5$. Then, we have

\[
\begin{align*}
\text{rank } \begin{bmatrix}
H_4(U) & 0 \\
H_4(Y) & \gamma_4(0.5)
\end{bmatrix} &= 9, \\
\text{rank } \begin{bmatrix}
H_4(U) & 0 \\
H_4(Y) & \gamma_4(0.5)
\end{bmatrix} &= 9.
\end{align*}
\]

Moreover, we also have

\[
\begin{align*}
\text{rank } \begin{bmatrix}
H_4(U) & 0 & \gamma_4^{(1)}(0.5) \\
H_4(Y) & \gamma_4(0.5) & M_0 \gamma_4^{(1)}(0.5)
\end{bmatrix} &= 9.
\end{align*}
\]

Hence, the data are informative for up to first-order moment matching at $\sigma = 0.5$. By solving linear equations (17) and (30) (for $k = 1$), we obtain $M_0 = 0.0827$ and $M_1 = -1.7247$, respectively.

So far, our discussion considered a single interpolation point $\sigma$ and its desired order of moment $k$. Let a collection of pairs of interpolation points and their desired order of moments

$$
P = \{ (\sigma_i, k_i) \mid i = 1, 2, \ldots, s \}$$

be given. In view of Propositions 15 and 19, we assume that $\sigma_i \neq \sigma_j$ whenever $i \neq j$. By applying Theorems 13 and 18, one can verify whether the data are informative for moment matching for each pair. If so, (17) and (30) result in the corresponding moments

$$
M_i = \{ (\sigma_i, M_j^i) \mid j = 0, 1, \ldots, k_i \},$$

where $M_j^i$ denotes the $j$-th moment at $\sigma_i$.

5 Reduced-order model by data-driven moment matching

In this section, we will investigate how reduced-order models can be computed from data that are informative for moment matching. As the results of Theorems 13 and 18 lead to the computation of moments at given interpolation points, obtaining such reduced-order model is essentially a rational interpolation problem.

Let a reduced-order model $\hat{\Sigma}$ of order $r$ be given by

\[
(z^r + \hat{p}_r z^{r-1} + \cdots + \hat{p}_1 z + \hat{p}_0) y_t = (\hat{q}_r z^r + \hat{q}_{r-1} z^{r-1} + \cdots + \hat{q}_1 z + \hat{q}_0) u_t.
\]

(35)

As before, we collect the parameters of (35) as $\hat{p} = [\hat{p}_0 \; \hat{p}_1 \; \cdots \; \hat{p}_{r-1}] \in \mathbb{R}^{1 \times r}$ and $\hat{q} = [\hat{q}_0 \; \hat{q}_1 \; \cdots \; \hat{q}_r] \in \mathbb{R}^{1 \times (r+1)}$. Denote by $P(z)$ and $Q(z)$ the polynomials on the left- and right-hand side of (35), respectively. Then, the model (35) interpolates or matches the 0-th moment at $\sigma$ if $M_0$ in (17) satisfies $Q(\sigma) = M_0 \hat{P}(\sigma)$ which is equivalent to

$$
\begin{bmatrix}
\hat{q} - \hat{p} \\
M_0 \hat{r}^{-1}(\sigma)
\end{bmatrix} = M_0 \sigma^r.
$$

More generally, for $P$ and $M_i$ as in (33) and (34), a reduced-order model parameterization by $[\hat{q} - \hat{p}]$ must satisfy the linear equations

$$
\begin{bmatrix}
\hat{q} - \hat{p} \\
\Gamma_r^{(1)}(\sigma_i) \\
\Gamma_r^{(2)}(\sigma_i)
\end{bmatrix} = \begin{bmatrix}
\Gamma_r^{(1)}(\sigma_i) \\
\Gamma_r^{(2)}(\sigma_i)
\end{bmatrix}
$$

(36)
Clearly, \( \hat{\Sigma} \) holds.

We note that in this paper, we do not restrict \( \hat{\Sigma} \) to be minimal, i.e., the polynomials \( \hat{P}(z) \) and \( \hat{Q}(z) \) might not be coprime. In addition, it is also obvious that if \( r \geq k^* - 1 \) where \( k^* = \sum_{i=1}^{s} (k_i + 1) \), then (37) always holds.

**Remark 22** The rational interpolation problem with minimality constraint has been studied using either the Loewner framework [1,5] or Prony’s method [7,9].

The following example illustrates the computation of the reduced-order model

**Example 23** From Example 16 and 20, we have \( \mathcal{P} = \{(\sigma_1, 0), (\sigma_2, 1)\} \) and \( \mathcal{M}_i \) as follows:

\[
\mathcal{M}_1 = \{(\sigma_1, M_1^1)\} = \left\{ \left( \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}, 0.0031 - 0.1417i \right) \right\},
\]

\[
\mathcal{M}_2 = \{(\sigma_2, M_2^1), (\sigma_2, M_2^2)\} = \{(0.5, 0.0827), (0.5, -1.7247)\}.
\]

Let us first verify condition (37) with \( r = 1 \). We have

\[
\text{rank} \begin{bmatrix} \Gamma^1(\sigma_1) & \Gamma^1(\sigma_2) \\ \Gamma^M_1(\sigma_1) & \Gamma^M_1(\sigma_2) \end{bmatrix} = 4.
\]

but

\[
\text{rank} \begin{bmatrix} 1.0000 & 0 & 1.0000 & 0 \\ 0.7071 & 0.7071 & 0.5000 & 1.0000 \\ 0.0031 & -0.1417 & 0.0827 & -1.7247 \\ 0.1023 & -0.0980 & 0.0413 & -0.7797 \end{bmatrix} = 3.
\]

In this case, \( \hat{\Sigma}_{1,2} = \emptyset \).

Now we let \( r = 2 \). We have

\[
\text{rank} \begin{bmatrix} \Gamma^2(\sigma_1) & \Gamma^2(\sigma_2) \\ \Gamma^M_2(\sigma_1) & \Gamma^M_2(\sigma_2) \end{bmatrix} = 4.
\]

We note that in this paper, we do not restrict \( \hat{\Sigma} \) to be minimal, i.e., the polynomials \( \hat{P}(z) \) and \( \hat{Q}(z) \) might not be coprime. In addition, it is also obvious that if \( r \geq k^* - 1 \) where \( k^* = \sum_{i=1}^{s} (k_i + 1) \), then (37) always holds.
Often reduced-order models are expected to preserve certain properties of the original models. Stability is one of the most common as well important property to be preserved. Next, we investigate conditions under which one can choose a stable reduced-order model. Note that stability of a reduced-order model $[\hat{q} - \hat{p}] \in \Sigma_{r,P}$ is purely determined by $\hat{p}$. Motivated by this observation, we provide necessary and sufficient condition such that $\hat{p}$ can be chosen arbitrarily while $[\hat{q} - \hat{p}] \in \Sigma_{r,P}$.

**Theorem 24** Given $P$ and $M$, as in (33) and (34), respectively and let $k^* = \sum_{i=1}^{s}(k_i + 1)$. If $r \geq k^* - 1$ then for every $\hat{p} \in \mathbb{R}^{1 \times r}$ there exists $\hat{q} \in \mathbb{R}^{1 \times (r+1)}$ such that $[\hat{q} - \hat{p}] \in \Sigma_{r,P}$.

**PROOF.** We know that $[\hat{q} - \hat{p}] \in \Sigma_{r,P}$ if and only if

$$
\begin{bmatrix}
\hat{q} - \hat{p} - 1 \\
\Gamma_1 \\
\Gamma_2 
\end{bmatrix} = 0,
$$

(40)

where

$$
\begin{align*}
\Gamma_1 &= \begin{bmatrix} \Gamma_r(\sigma_1) & \Gamma_r(\sigma_2) & \cdots & \Gamma_r(\sigma_s) \end{bmatrix}, \quad \text{and} \\
\Gamma_2 &= \begin{bmatrix} \Gamma_M(\sigma_1) & \Gamma_M(\sigma_2) & \cdots & \Gamma_M(\sigma_s) \end{bmatrix}.
\end{align*}
$$

It is clear that $r \geq k^* - 1$ guarantees $\Sigma_{r,P} \neq \emptyset$. Particularly, since the structure of $\Gamma_1$ is a Vandermonde-like matrix, then $\operatorname{rank}(\Gamma_1) = k^*$ which implies (37). Equation (40) also means that for every $\hat{p}$ there exists $\hat{q}$ such that $[\hat{q} - \hat{p}] \in \Sigma_{r,P}$ if and only if

$$
\text{rowspace}(\Gamma_2) \subseteq \text{rowspace}(\Gamma_1).
$$

(41)

Since we know that $\operatorname{rank}(\Gamma_1) = k^*$, i.e., rowspace$(\Gamma_1) = \mathbb{R}^{1 \times k^*}$, then indeed (41) holds. This finalizes the proof. $\Box$

**Example 25** Consider $P$ as in Example 23. Suppose we desire to choose $\hat{p} = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$, then taking $r = 3$, we obtain a reduced-order $\Sigma_{3,P}$ as

$$
\begin{align*}
(z^3 + z^2 + 1)y_t \\
= (-4.2037z^3 + 7.7895z^2 - 6.8636z + 2.1235)u_t.
\end{align*}
$$

6 Conclusion

Motivated by data-driven model reduction, the concept of data informativity for moment matching using input-output data is proposed in this paper. The main results are necessary and sufficient conditions based on the
rank of the Hankel matrix of the data and given (complex) interpolation points for computing the moments at these points. These conditions are weaker than those for system identification, which implies that the data-driven approach for moment matching can be performed on data that are not informative for system identification. Namely, the precise value of moments can be extracted even though the high-order system model is not available. The moments computed from the informativity framework are then exploited to construct a reduced-order model by solving a rational interpolation problem. Moreover, the condition to enforce prescribed poles upon the reduced-order model is discussed.

This work shows the advantages of using the informativity framework for data-driven model reduction by moment matching. We are confident that this gives the way to solve other data-driven model reduction problems, e.g., with preserving specific system properties. This study forms one of the issues of our future research.

Acknowledgements

This paper is based on research developed in the DSSC Doctoral Training Programme, co-funded through a Marie Skłodowska-Curie COFUND (DSSC 754315).

References

[1] B. D. O. Anderson and A. C. Antoulas. Rational interpolation and state-variable realizations. Linear Algebra and its Applications, 137:479–509, 1990.
[2] A. C. Antoulas. Approximation of Large-Scale Dynamical Systems. Advances in Design and Control, SIAM, Philadelphia, PA, 2005.
[3] A. C. Antoulas. A new result on passivity preserving model reduction. Systems & Control Letters, 54(4):361–374, 2005.
[4] A. C. Antoulas. The Loewner framework and transfer functions of singular/rectangular systems. Applied Mathematics Letters, 54:36–47, 2016.
[5] A. C. Antoulas and B. D. O. Anderson. On the scalar rational interpolation problem. IMA Journal of Mathematical Control and Information, 3(2-3):61–88, 1986.
[6] A. C. Antoulas and B. D. O. Anderson. On the problem of stable rational interpolation. Linear Algebra and its Applications, 122:301–329, 1989.
[7] A. C. Antoulas, J. A. Ball, J. Kang, and J. C. Willems. On the solution of the minimal rational interpolation problem. Linear Algebra and its Applications, 137:511–573, 1990.
[8] A. C. Antoulas, C. A. Beattie, and S. Gugercin. Interpolatory Methods for Model Reduction. SIAM, Philadelphia, PA, 2020.
[9] A. C. Antoulas and J. C. Willems. Minimal rational interpolation and Prony’s method. In A. Bensoussan and J. L. Lions, editors, Analysis and Optimization of Systems. Lecture Notes in Control and Information Sciences, volume 144, pages 297–306. Springer, Berlin, Heidelberg, 1990.
[10] A. Astolli. Model reduction by moment matching for linear and nonlinear systems. IEEE Transactions on Automatic Control, 55(10):2321–2336, 2010.
[11] J. A. Ball and J. Kim. Stability and McMillan degree for rational matrix interpolants. Linear Algebra and its Applications, 196:207–232, 1994.
[12] C. Beattie and S. Gugercin. Realization-independent $H_2$-approximation. In Proceedings of the 51st IEEE Conference on Decision and Control (CDC), pages 4953–4958, 2012.
[13] J. Eising, H. L. Trentelman, and M. K. Camlibel. Data informativity for observability: An invariance-based approach. In Proceedings of the European Control Conference, pages 1057–1059, 2020.
[14] D. F. Enns. Model reduction with balanced realizations: An error bound and a frequency weighted generalization. In The 23rd IEEE Conference on Decision and Control, pages 127–132, 1984.
[15] P. Feldmann and R. W. Freund. Efficient linear circuit analysis by Padé approximants via the Lanczos process. IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems, 14(5):639–649, 1995.
[16] K. Gallivan, A. Vandendorpe, and P. Van Dooren. On the generality of multipoint Padé approximations. In Proceedings of the 15th IFAC World Congress, pages 331–336, 2002.
[17] K. Gallivan, A. Vandendorpe, and P. Van Dooren. Model reduction of MIMO systems via tangential interpolation. SIAM Journal on Matrix Analysis and Applications, 26(2):328–349, 2004.
[18] K. Gallivan, A. Vandendorpe, and P. Van Dooren. Sylvester equations and projection-based model reduction. Journal of Computational and Applied Mathematics, 162(1):213–229, 2004.
[19] K. Glover. All optimal Hankel-norm approximations of linear multivariable systems and their $L_\infty$-error bounds. International Journal of Control, 39(6):1115–1193, 1984.
[20] I. V. Gosea and A. C. Antoulas. Stability preserving post-processing methods applied in the Loewner framework. In Proceedings of the 50th IEEE Workshop on Signal and Power Integrity, pages 1–4, 2016.
[21] E. Grimme. Krylov Projection Methods for Model Reduction. PhD thesis, University of Illinois at Urbana-Champaign, Urbana, IL, 1997.
[22] S. Gugercin, A. C. Antoulas, and C. Beattie. $H_2$ model reduction for large-scale linear dynamical systems. SIAM Journal on Matrix Analysis and Applications, 30(2):609–638, 2008.
[23] B. L. Ho and R. E. Kálmán. Effective construction of linear state-variable models from input/output functions. Automatisierungstechnik, 14(1-12):545–548, 1966.
[24] H. J. Jongsma, H. L. Trentelman, and M. K. Camlibel. Model reduction of networked multiagent systems by cycle removal. IEEE Transactions on Automatic Control, 63(3):657–671, 2017.
[25] Y. Kawano, B. Besselink, J. M. A Scherpen, and M. Cao. Data-driven model reduction of monotone systems by nonlinear dc gains. IEEE Transactions on Automatic Control, 65(5):2094–2106, 2019.
[26] L. Ljung. System Identification: Theory for the User. Prentice-Hall, Englewood Cliffs, NJ, 1999.
[27] I. Markovsky, J. C. Willems, P. Rapisarda, and B. L. M. De Moor. Algorithms for deterministic balanced subspace identification. Automatica, 41(5):755–766, 2005.
[28] A. J. Mayo and A. C. Antoulas. A framework for the solution of the generalized realization problem. Linear Algebra and its Applications, 425(2-3):634–662, 2007.
A Auxiliary results

For the sake of completeness, we present the following two elementary linear algebra results that are employed in proving the main results.

Lemma A.1 Let $A_i \in \mathbb{R}^{n \times m}$ and $b_i \in \mathbb{R}^{1 \times m}$, for $i = 1, 2$. Consider the sets $X_i = \{ \xi \in \mathbb{R}^{1 \times n} \mid A_i b_i = 0 \}$ and assume that $X_1$ is nonempty. Then, $X_1 \subseteq X_2$ if and only if
\[
\text{leftker} \begin{bmatrix} A_1 \\ b_1 \end{bmatrix} \subseteq \text{leftker} \begin{bmatrix} A_2 \\ b_2 \end{bmatrix}. \tag{A.1}
\]

**Proof.** If: Suppose that (A.1) holds. Then, there exists $F$ such that
\[
\begin{bmatrix} A_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} A_1 \\ b_1 \end{bmatrix} F.
\]
This readily implies that $X_1 \subseteq X_2$.

Only if: Suppose that $X_1 \subseteq X_2$. Let
\[
\begin{bmatrix} \xi \\ \eta \end{bmatrix} \in \text{leftker} \begin{bmatrix} A_1 \\ b_1 \end{bmatrix}.
\]
We distinguish two cases:

Case 1: $\eta \neq 0$. By (A.2), we have $(-\xi/\eta) A_1 = b_1$, i.e., $-\xi/\eta \in X_1$. Since $X_1 \subseteq X_2$, we also have that $(-\xi/\eta) A_2 = b_2$, which is equivalent to
\[
\begin{bmatrix} \xi \\ \eta \end{bmatrix} \in \text{leftker} \begin{bmatrix} A_2 \\ b_2 \end{bmatrix}.
\]

Case 2: $\eta = 0$. By (A.2), we have $\xi A_1 = b_1$, i.e., $\xi \in \text{leftker} A_1$. Let $\xi$ be such that $\xi A_1 = b_1$. Then, $(\xi + \alpha \xi) A_1 = b_1$ for any $\alpha \in \mathbb{R}$. Since $X_1 \subseteq X_2$, then we also have $(\xi + \alpha \xi) A_2 = b_2$. This leads to
\[
\frac{\xi}{\alpha} A_2 + \xi A_2 = \frac{b_2}{\alpha}.
\]
For $\alpha \to \infty$, this implies that $\xi A_2 = 0$. As such we have
\[
\begin{bmatrix} \xi \\ 0 \end{bmatrix} \in \text{leftker} \begin{bmatrix} A_2 \\ b_2 \end{bmatrix}.
\]
In both cases, we have that
\[
\begin{bmatrix} \xi \\ \eta \end{bmatrix} \in \text{leftker} \begin{bmatrix} A_2 \\ b_2 \end{bmatrix}.
\]
Therefore, we can conclude that
\[
\text{leftker} \begin{bmatrix} A_1 \\ b_1 \end{bmatrix} \subseteq \text{leftker} \begin{bmatrix} A_2 \\ b_2 \end{bmatrix}. \quad \square
\]

Lemma A.2 Let $a \in \mathbb{C}^n$ and $A \in \mathbb{C}^{n \times n}$. Then, the following statements are equivalent:

(i) If
\[
\begin{bmatrix} A & a \\ \xi_1 \\ \eta_1 \end{bmatrix} = \begin{bmatrix} A & a \\ \xi_2 \\ \eta_2 \end{bmatrix},
\]
then $\eta_1 = \eta_2$. 

[29] N. Monshizadeh. Amidst data-driven model reduction and control. *IEEE Control Systems Letters*, 4(4):833–838, 2020.
[30] B. Moore. Principal component analysis in linear systems: Controllability, observability, and model reduction. *IEEE Transactions on Automatic Control*, 26(1):17–32, 1981.
[31] B. Peherstorfer and K. Willcox. Dynamic data-driven model reduction: adapting reduced models from incomplete data. *Advanced Modeling and Simulation in Engineering Sciences*, 3(1):1–22, 2016.
[32] L. Pernebo and L. Silverman. Model reduction via balanced state space representations. *IEEE Transactions on Automatic Control*, 27(2):382–387, 1982.
[33] J. L. Proctor, S. L. Brunton, and J. N. Kutz. Dynamic mode decomposition with control. *SIAM Journal on Applied Dynamical Systems*, 15(1):142–161, 2016.
[34] P. Rapisarda and H. L. Trentelman. Identification and balanced state space representations. *IEEE Transactions on Automatic Control*, 27(2):382–387, 1982.
[35] G. Scarciotti and A. Astolfi. Data-driven model reduction by moment matching for linear and nonlinear systems. *Automatica*, 79:340–351, 2017.
[36] G. Scarciotti and A. Astolfi. Nonlinear model reduction by moment matching. *Foundations and Trends in Systems and Control*, 4(3–4):224–409, 2017.
[37] H. L. Trentelman, H. J. Van Waarde, and M. K. Camlibel. An informativity approach to data-driven tracking and regulation. *arXiv preprint arXiv:2009.01553*, 2020.
[38] H. J. Van Waarde, M. K. Camlibel, and M. Mesbahi. From noisy data to feedback controllers: non-conservative design via a matrix $S$-lemma. *arXiv preprint arXiv:2006.00870*, 2020.
[39] H. J. Van Waarde, J. Eising, H. L. Trentelman, and M. K. Camlibel. Data informativity: a new perspective on data-driven analysis and control. *IEEE Transactions on Automatic Control*, 2020. To appear.
[40] M. Verhaegen and V. Verdult. *Filtering and System Identification: A Least Squares Approach*. Cambridge University Press, New York, NY, 2007.
[41] J. C. Willems, P. Rapisarda, I. Markovsky, and B. L. M. De Moor. A note on persistency of excitation. *Systems & Control Letters*, 54(4):325–329, 2005.
[42] F. Zhang (Ed.). *The Schur Complement and its Applications*. Springer Science & Business Media, New York, NY, 2005.
(ii) $\text{rank } [A \ a] = \text{rank } A + 1$.

**Proof.** Note that $(i)$ holds if and only if $\ker [A \ a] = \ker A \times \{0\}$. Equivalence of this relation to $(ii)$ follows from the rank-nullity theorem. $\square$