RIGIDITY AND SYMMETRY: ONLY ONE KIND OF SYMMETRY ALLOW NON-ZERO REAL SYMMETRIC SOLUTION

QIXIANG YANG

Abstract. Leray guessed that, a blow-up solution should have similar structure as its initial data and proposed to consider self-similar solution. But Necas-Ruzicka-Sverak proved in 1996 that such solution should be zero. That is to say, Navier-Stokes equations have rigidity for self-similar structure. Recently, Yang-Yang-Wu found that the symmetry property plays an important role in the proof of ill-posedness result. Further, Yang applied Fourier transformation to consider symmetric solutions. He has shown that a party of symmetric solution should be zero and there exists some symmetric property can result in symmetric solution.

In this paper, we consider the symmetry related to the independent variables of initial data and we analyze the symmetric structure of non-linear term. (i) We have found out what kinds of symmetric properties can generate symmetric solutions and we have also proved that the rest symmetric properties allow only zero solutions in some sense. For real initial data, we prove there exists only one kind of symmetry can generate non-zero symmetric solution. (ii) Further, to understand the structure of $B(u,v)$, we show it is sufficient to consider all the symmetric cases. (iii) Thirdly, we establish the well-posedness for some big initial values. (iv) Lastly, we apply such symmetric result to the Navier-Stokes equations on the domain and we prove the existence of smooth solution with energy conservation.

1. Motivation, Introduction to rigidity and symmetry

We consider rigidity and symmetry of the following incompressible Navier-Stokes equations on the half-space $\mathbb{R}_+ \times \mathbb{R}^3$:

\[
\begin{align*}
&\partial_t u - \Delta u + u \cdot \nabla u - \nabla p = 0, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\
&\nabla \cdot u = 0, \\
&u(0, x) = u_0(x).
\end{align*}
\]

By the way, we consider its applications to the relative equations on the domain. Denote

\[
\begin{align*}
A(u) &= u \cdot \nabla u = \sum_i \partial_i (u_i u) ; \\
A(u, u) &= \nabla A(u) = \sum_{i, j} \partial_i \partial_j u_i u_j ; \\
C(u) &= \mathbb{P} \nabla (u \otimes u) = A(u) - (-\Delta)^{-1} \nabla A(u, u) ; \\
B(u, u)(t, x) &= \int_0^t e^{(t-s)\Delta} \mathbb{P} \nabla (u \otimes u) ds ,
\end{align*}
\]

A solution of the above Cauchy problem (1.1) is then obtained via the integral equation

\[
u(t, x) = e^{t\Delta} u_0(x) - B(u, u)(t, x).
\]
Which can be solved by a fixed-point method whenever the convergence is suitably defined in some function space. Denote
\begin{align}
\label{1.4}
u^0(t,x) &= e^{\Delta t}u_0;
\bar{u}^{r+1}(t,x) &= u^0(t,x) - B(u^r, u^r)(t,x), \quad \forall r = 0, 1, 2, \cdots.
\end{align}
For $u_0 \in X^1_0$, if there exists $X^3$ such that $e^{\Delta t}u_0 \in X^3$ and $u^r$ converge to some function $u(t,x) \in X^3$, then $u(t,x)$ is the solution of (1.3) and $u(t,x)$ is called to be the mild solutions of (1.1). The notion of mild solution was pioneered by Kato-Fujita [8] in 1960s.

During the latest decades, many important results about the mild solutions of (1.1) have been established; see for example, Cannone [2, 3], Germain-Pavlovic-Staffilani [4], Giga-Inui-Mahalov-Saal[5], Giga-Miyakawa [6], Kato [7], Koch-Tataru [10], Lei-Lin [11], Wu [20, 21, 22, 23], Xiao [24, 25] and their references including Kato-Ponce [9] and Taylor [19] (see also the book [12]). Further, applying the wavelets, mild solutions have extended to many function spaces: Trieble-Lizorkin spaces, Besov Morrey spaces and Trieble-Lizorkin Morrey spaces (see [13, 14, 15, 16, 27, 28]). Among all these results, Koch-Tataru’s $BMO^1$ is the biggest initial data spaces for non-phase spaces cases. Giga-Inui-Mahalov-Saal and Lei-Lin studied $P_{1,1}^{-1}$, which is the biggest initial data spaces for phase spaces cases, see also [26]. There are many well-posedness results for many kinds of initial data spaces. Hence an interesting idea is to consider the structure of the bilinear operator $B(u,v)$.

The rigidity has been studied extensively for partial differential equations. For incompressible Navier-Stokes equations, Leray speculated that, a blow-up solution should have similar structure as its initial data and proposed to consider self-similar solution. Necas-Ruzicka-Sverak [18] proved in 1996 that the only possible self-similar solution is zero. That is to say, there exists rigidity phenomenon for self similar solution. In this paper, we consider mainly the relation between symmetry and rigidity. There are two category of symmetry for velocity field. One is the symmetry on the component of velocity field which has been studied by many people. See Abidi-Zhang [1] and Yang [26]. Another is the symmetry for the independent variables of velocity field. By applying the Fourier transform and other skills, Yang [26] has found a party of symmetric solution should be zero and proved there exists one kind of symmetry property which allow symmetric solution. Further, Yang-Yang-Wu [29] found that symmetric structure plays an important role on the ill-posedness of (1.1).

To understand better the structure of non-liner term $B(u,u)(t,x)$, we consider the symmetry of independent variables of velocity $u(t,x)$ in this paper. We show also, to understand the structure of $B(u,v)$, we show it is sufficient to consider all the symmetric cases. Further, for symmetric initial data, we can see $u^r(t,x)$ in the above (1.4) converges always for some special points $x$ even the norm of the initial data is big. In this paper, we consider the following four terms:

(i) We have found out what kinds of symmetric properties can generate symmetric solutions and we have also proved that the rest symmetric properties allow only zero solutions in some sense. For \textbf{real initial data, we prove there exists only one kind of symmetry can generate non-zero symmetric solution}. By the way, we get also the well-posedness of some big initial value.

(ii) For two arbitrary symmetric solenoidal vector fields $u(t,x)$ and $v(t,x)$, usually, $B(u,v)$ may have no symmetric property. But, we prove that any solution $u(t,x)$ can be decomposed as the sum of 8 matched symmetric solenoidal vector field $[u^\alpha(t,x)]_{\alpha \in [0,1]^3}$ and for $\alpha, \beta \in [0,1]^3$, each $B(u^\alpha, u^\beta)(t,x)$ has symmetric property. That is to say, to understand
the structure of the non-linear term $B(u, u)$, we need only to consider the symmetric situation in the harmonic analysis point.

(iii) Lastly, we apply such symmetric result to consider the Navier-Stokes equations on the domain and we prove the existence of smooth solution with energy conservation.

At the end of this section, we introduce some notations which will be used throughout this paper. $\forall \alpha \in \mathbb{N}$, denote $m(\alpha) = \alpha (\text{mod} 2) \in \{0, 1\}$. Further, $\forall \alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$, denote $m(\alpha) = (\alpha_1 (\text{mod} 2), \alpha_2 (\text{mod} 2), \alpha_3 (\text{mod} 2)) \in \{0, 1\}^3$.

2. Rigidity, transitivity and domain

2.1. Symmetry solution and rigidity. Let $u^\text{re}$ and $u^\text{im}$ be the real part and imaginary part of a function $u(x)$. A function has symmetry property means

**Definition 2.1.** (i) We say that a real function $u(x)$ has symmetry property, if there exists function $f : \{0, 1\}^3 \to \{0, 1\}$ such that $(u,f)$ satisfies,

\begin{align}
\forall x = (x_1, x_2, x_3) \in \mathbb{R}^3,
\end{align}

\begin{align}
&u((-1)^{\alpha_1}x_1, (-1)^{\alpha_2}x_2, (-1)^{\alpha_3}x_3) = (-1)^{f(\alpha)}u^\text{re}(x_1, x_2, x_3),
\end{align}

(ii) We say that a complex function $u(x) = (u^\text{re} + iu^\text{im})$ has symmetry property, if its real part and imaginary part also satisfy symmetry property. Hence, there exists two functions $f, g : \{0, 1\}^3 \to \{0, 1\}$ such that $(u^\text{re}, f)$ and $(u^\text{im}, g)$ satisfy (2.1). That is to say, $\forall \alpha = (\alpha_1, \alpha_2, \alpha_3), \beta = (\beta_1, \beta_2, \beta_3) \in \{0, 1\}^3, \forall (x_1, x_2, x_3) \in \mathbb{R}^3$,

\begin{align}
&u^\text{re}((-1)^{\alpha_1}x_1, (-1)^{\alpha_2}x_2, (-1)^{\alpha_3}x_3) = (-1)^{f(\alpha)}u^\text{re}(x_1, x_2, x_3),
\end{align}

\begin{align}
&u^\text{im}((-1)^{\beta_1}x_1, (-1)^{\beta_2}x_2, (-1)^{\beta_3}x_3) = (-1)^{g(\beta)}u^\text{im}(x_1, x_2, x_3).
\end{align}

There are 8 kinds of symmetry property for a real valued function, there are $8 \times 8 = 64$ kinds of symmetry property for a complex valued function. For a general vector field $u_0$,

**Definition 2.2.** We say that $u_0 = (u_1^\text{re} + iu_1^\text{im}, u_2^\text{re} + iu_2^\text{im}, u_3^\text{re} + iu_3^\text{im})$ has symmetry property, if for $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2, \beta_3) \in \{0, 1\}^3$, there exist $f_\alpha(\alpha)$ and $g_\beta(\beta) \in \{0, 1\}$ such that the real part $u_0^\text{re}$ satisfies (2.2) and the imaginary part $u_0^\text{im}$ of $u_0$ satisfies (2.3). That is to say, $\forall (x_1, x_2, x_3) \in \mathbb{R}^3$,

\begin{align}
&u_0^\text{re}((-1)^{\alpha_1}x_1, (-1)^{\alpha_2}x_2, (-1)^{\alpha_3}x_3) = (-1)^{f_\alpha(\alpha)}u_0^\text{re}(x_1, x_2, x_3),
\end{align}

\begin{align}
&u_0^\text{im}((-1)^{\beta_1}x_1, (-1)^{\beta_2}x_2, (-1)^{\beta_3}x_3) = (-1)^{g_\beta(\beta)}u_0^\text{im}(x_1, x_2, x_3).
\end{align}

For each $\alpha$ fixed, there are 8 possibility for $f_\alpha(\alpha)$ and $g_\beta(\beta)$. Hence there exist $(8 \times 8)^3 = 262144$ kinds of possibility for $\{f_\alpha(\alpha), g_\beta(\beta)\}$. That is to say, there exist 262144 kinds of symmetry property for general complex valued vector function.

If $u$ is complex, then the non-linear term $B(u, u)$ have $3 \times (11+12+9) = 96$ terms which include products of differentiable functions and integration. If $u$ is real, the non-linear term $B(u, u)$ have $3 \times (3+6) = 27$ terms which include products of differentiable functions and integration. There are many factors which can change the symmetry property of $B(u, u)(t, x)$, hence it is not so easy to visualize that one can control the symmetry property of solution. In this paper, we find out all the possible cases which can generate symmetric solutions. For real valued initial data, we have:
Theorem 2.3. Let $u(t, x)$ be the real valued solution of the equations (1.1), $u(t, x)$ has the same symmetric property for all $t \geq 0$, if and only if one of the following conditions holds:

(i) the real valued solenoidal vector field $u_0(x)$ satisfies the following condition (2.6)

$$f_\tau(x) = \alpha_\tau^r, \forall \tau = 1, 2, 3, \alpha \in \{0, 1\}^3,$$

(ii) the real valued solenoidal vector field $u_0(x)$ satisfies the following condition (2.7)

$$\mathcal{P}\nabla(e^{\Lambda t}u_0 \otimes e^{\Lambda t}u_0) = 0 \text{ in } (\mathcal{S}'(\mathbb{R}^3))^3, \forall t \geq 0.$$

The above theorem tells us, if $u_0$ does not satisfy the symmetric property (2.6), then $u_0$ must be a solution of equation (2.7). That is to say, the solution $u(t, x)$ has almost rigidity.

Let $u(t, x) = (u_1^r e + iu_1^m, u_2^r e + iu_2^m, u_3^r e + iu_3^m)^t$ be the strong solution of (1.1) satisfying

$$\forall l = 1, 2, 3, u_1^r e \text{ and } u_l^m \text{ are not constant.}$$

For complex valued initial data, to save the length of the paper, we assume that $u(t, x)$ satisfies (2.8). Under this constraint, 8 kinds of symmetric initial values can result in symmetric solutions. Denote $e_1 = (1, 0, 0), e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$. We have

Theorem 2.4. Let $u(t, x)$ be the strong solution of (1.1) satisfying (2.8). $u(t, x)$ has the same symmetric property for all $t \geq 0$ if and only if one of the following conditions is satisfied

(i) $\{f_\tau\}$ satisfies (2.6) and $\{g_\tau\}$ satisfies (2.9)

$$g_\tau(\beta) = g_1(m(e_1 + e_r + \beta)), \forall \tau = 2, 3, \beta \in \{0, 1\}^3.$$

(ii) $u_0(x)$ satisfies the condition (2.7).

2.2. Big initial data. The equation (2.7) is very complex. So we do not get the complete rigidity and we get only almost rigidity. We did not know how to deal the equation (2.7) with zero divergence

$$\begin{cases}
\mathcal{P}\nabla(e^{\Lambda t}u_0 \otimes e^{\Lambda t}u_0) = 0 \text{ in } (\mathcal{S}'(\mathbb{R}^3))^3, \forall t \geq 0, \\
\nabla \cdot u_0 = 0 \text{ in } \mathcal{S}'(\mathbb{R}^3).
\end{cases}$$

The above equations (2.10) allow non-zero solution, for example $u_0 = (c_1, c_2, c_3)^t$ where $c_1, c_2$ and $c_3$ are constants. But we did not know whether they allow non-constant solution. Even we limit to more particular cases, we consider the below equation (2.12), we did not know how to solve it. For $f(x_1, x_2)$ and $g(x_3)$, denote $f(x_1, x_2) = (4\pi t)^{-1} \int \int f(x_1 - y_1, x_2 - y_2) \exp(-\frac{(y_1 + y_2)^2}{4t}) dy_1 dy_2$ and $g(x_3) = (4\pi t)^{-\frac{3}{2}} \int g(x_1 - y_3) \exp(-\frac{y_3^2}{4t}) dy_3$. Let $f(x_1, x_2)$ and $g(x_3)$ be functions satisfying

$$f((-1)^\alpha x_1, (-1)^\beta x_2) = f(x_1, x_2), \forall \alpha, \beta = 0, 1,$$

$$\partial_1 [\partial_1 \partial_2^2 f] = \partial_2 [\partial_1^2 \partial_2 f],$$

$$g(-x_3) = g(x_3).$$

Denote $u_1 = g(x_3)\partial_1 \partial_2^2 f$ and $u_2 = -g(x_3)\partial_1^2 \partial_2 f$ and $u_3 = 0$. We have
Theorem 2.5. (i) If \( f \) and \( g \) satisfies the conditions (2.11), (2.12) and (2.13), then \( u_0 = (u_1, u_2, 0)' \) is a solution of (2.10).

(ii) If \( u_0 \in (S'(\mathbb{R}^3))^3 \) satisfying the equation (2.7), then there exists strong solution \( u(t, x) = e^{it}u_0 \) for (1.1).

(iii) For any symmetric solenoidal vector field \( u_0 \) satisfies (2.7), the equations (1.1) allow symmetric solution.

Remark 2.6. If the equations (2.10) allow non-constant solution or the equation (2.12) has solution satisfying (2.11), then \( u_0 \) can be chosen as big initial data and the above theorem tells us we have wellposedness for big initial data:

2.3. Weyl-Helmholtz project operator and transitivity of symmetry. Most of the time, the Weyl-Helmholtz project operator \( \mathbb{P}V(u \otimes v) \) does not map symmetric solenoidal vector field \( u \) and \( v \) to symmetric solenoidal vector field. But in this subsection, we can decompose arbitrary solenoidal vector field \( u(t, x) \) into eight kinds of matched symmetric solenoidal vector field \( u_0(t, x) \) such that \( \mathbb{P}V(u_0(t, x)) \otimes u_0(t, x) \) are always symmetric solenoidal vector field. We introduce first the definition of matched symmetric property:

Definition 2.7. Given two solenoidal vector fields \( u = (u_1^r + iu_1^m, u_2^r + iu_2^m, u_3^r + iu_3^m)' \) and \( v = (v_1^r + iv_1^m, v_2^r + iv_2^m, v_3^r + iv_3^m)' \). Say \( u \) and \( v \) have matched symmetric properties, if \( \forall \alpha \in [0, 1]^3 \) and \( \tau = 1, 2, 3 \), there exist functions \( f_1^\tau(\alpha), f_2^\tau(\alpha), g_1^\tau(\alpha) \) and \( g_2^\tau(\alpha) \) belong to \( [0, 1] \) such that all \( (u_1^\tau, f_1^\tau), (u_2^\tau, f_2^\tau), (v_1^\tau, g_1^\tau) \) and \( (v_2^\tau, g_2^\tau) \) satisfy (2.1) and the following matched condition

\[
m(f_1^\tau + g_2^\tau) = m(f_2^\tau + g_1^\tau), \forall \tau = 1, 2, 3.
\]

For arbitrary solenoidal vector field \( u(t, x) \), we can divide it to be the sum of 8 matched symmetric solenoidal vector field.

Theorem 2.8. Let \( u(t, x) \) be any solenoidal vector field. let \( v_1^\alpha(t, x) \) be the real part of \( v_\alpha(t, x) \) and let \( v_2^\alpha(t, x) \) be the imaginary part of \( v_\alpha(t, x) \). There exists 8 symmetric solenoidal vector fields \( v_\alpha(t, x) \) such that \( \sum_{\alpha \in [0, 1]^3} v_\alpha(t, x) \) and \( v_\alpha^\tau(t, x) \) has the same symmetry as \( \mathbb{P}V v_\alpha^\tau(t, x) \).

For two matched symmetric solenoidal vector field, we prove the bilinear operator \( B(u, v) \) maps always symmetric solenoidal vector fields to symmetric solenoidal vector field. Let \( B(u, v) = (B_1^\alpha(u, v) + iB_2^\alpha(u, v), B_1^\alpha(u, v) + iB_2^\alpha(u, v)) \). We have

Theorem 2.9. Let \( u = (u_1^r + iu_1^m, u_2^r + iu_2^m, u_3^r + iu_3^m)' \) and \( v = (v_1^r + iv_1^m, v_2^r + iv_2^m, v_3^r + iv_3^m)' \) be solenoidal vector field with matched symmetric properties. That is to say, \( \forall \alpha \in [0, 1]^3 \) and \( \tau = 1, 2, 3 \), there exist functions \( f_1^\tau(\alpha), f_2^\tau(\alpha), g_1^\tau(\alpha) \) and \( g_2^\tau(\alpha) \) belong to \( [0, 1] \) such that all \( (u_1^\tau, f_1^\tau), (u_2^\tau, f_2^\tau), (v_1^\tau, g_1^\tau) \) and \( (v_2^\tau, g_2^\tau) \) satisfy (2.14). Then \( B(u, v) \) has symmetry property and all \( (B_1^\tau(u, v), f_2^\tau + g_2^\tau), (B_2^\tau(u, v), f_1^\tau + g_1^\tau) \) satisfy (2.1).

Remark 2.10. (i) If \( u(t, x) \) in (1.4) have the same symmetric property for all \( \tau \geq 0 \) and \( t \geq 0 \), the limitation of \( u(t, x) \) have no blow-up phenomenon on certain coordinate axis.

(ii) The above Theorem 2.8 and 2.9 told us, to understand the structure of the non-linear terms, we need only to know the structure for symmetric initial data in the sense of harmonic analysis.
2.4. Solution on the domain. By consequence, we study Navier-Stokes equations on the domain $\Omega$. We consider the following incompressible Navier-Stokes equations on the space $\mathbb{R}_+ \times \Omega$,

$$
\begin{align*}
\begin{cases}
\partial_t u - \Delta u + u \cdot \nabla u - \nabla p = 0, & (t, x) \in \mathbb{R}_+ \times \Omega, \\
\nabla \cdot u = 0, \\
u(0, x) = u_0(x),
\end{cases}
\end{align*}
$$

(2.15)

where initial data $u_0(x)$ is real valued.

When considering Navier-Stokes equations on the domain, usually, one makes first zero extension of initial data, then considers well-posedness on the whole spaces. The solution of the domain is the restriction of the solution on the whole space to the domain. There exists loss of energy for such extension. Given $\Omega = \mathbb{R}^2 \times \mathbb{R}_+$. If one uses symmetric extension, by the above symmetric theorem, there exists not certainly symmetric solution. That is to say, the energy inside $\Omega$ and outside $\Omega$ of the solution for the extended data are not certainly equal. Hence we assume the real initial data $u_0(x)$ satisfies the following symmetry property.

$$
\begin{align*}
-u_{0,1}(-x_1, x_2, x_3) &= u_{0,1}(x_1, -x_2, x_3) &= u_{0,1}(x_1, x_2, x_3) \\
u_{0,2}(-x_1, x_2, x_3) &= -u_{0,2}(x_1, -x_2, x_3) &= u_{0,2}(x_1, x_2, x_3) \\
u_{0,3}(-x_1, x_2, x_3) &= u_{0,3}(x_1, -x_2, x_3) &= u_{0,3}(x_1, x_2, x_3),
\end{align*}
$$

(2.16)

In this paper, we search energy conservation smooth solution of (2.15) with domain $\Omega = \mathbb{R}^2 \times \mathbb{R}_+$. We will see such symmetry property is sufficient and necessary for smooth solution with energy conservation. Denote

$$
P_s u_0(x) = \begin{cases}
u_0(x), \\
u_{0,1}(x), \nu_{0,2}(x), \nu_{0,3}(x) & x_3 \geq 0; \\
u_{0,1}(x), \nu_{0,2}(x), -\nu_{0,3}(x) & x_3 < 0.
\end{cases}
$$

(2.17)

Definition 2.11. We say $u(x) \in (\dot{H}^1(\mathbb{R}^2 \times \mathbb{R}_+))^3$, if $P_s u(x) \in (\dot{H}^1(\mathbb{R}^2))^3$.

As a byproduct of the study of symmetry property, we find out smooth solution with energy conservation.

Theorem 2.12. Given $m > 2$. If the real initial data $u_0$ satisfies $\text{div} u_0 = 0$, symmetry property (2.16) and $\|u_0\|_{(\dot{H}^1(\mathbb{R}^2 \times \mathbb{R}_+))^3}$ being small, then the Navier-Stokes equations (2.15) have a $C^{m-1}$ smooth solution $u(t, x)$ with energy conservation and satisfying symmetry property (2.16).

The above theorem 2.12 can be extend to Besov spaces, Triebel-Lizorkin spaces, Besov-Morrey spaces and Triebel-Lizorkin-Morrey spaces. Further, our method can be applied also to the following domain $\Omega = \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+$ or $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$.

The rest of the sections is organized as follows. In section 3, we recall some preliminaries on mild solution and prove Theorem 2.5. In section 4, we prove some preliminaries on symmetry properties. In section 5, we find out all the symmetrical species for all the solenoidal vector fields. In section 6, we prove theorem 2.4. In section 6, we prove theorem 2.3. In section 8, we prove theorems 2.8 and 2.9. In section 9, we prove theorem 2.12.

3. Mild solution

The well-posedness of (1.1) has been studied by many people in different initial data spaces. We have no intention to consider more initial data spaces. Since wavelet method is easy to see
the role of high frequency part and low frequency part, we recall a smooth solution result for Sobolev space which is studied in [13] by wavelets. In this paper, we use Meyer wavelets, see [17]. Let $Φ^0(x)$ be the father wavelet and $∀ε \in (0,1)^3\backslash\{0\}$, let $Φ^ε(x)$ be the mother wavelet. Denote

$$\Lambda = \{(ε, j, k), ε \in [0,1]^3, j \in \mathbb{Z}, k \in \mathbb{Z}^3\}.$$  

Further, $∀(ε, j, k) \in \Lambda$, denote $Φ_{jk}^ε(x) = 2^j Φ(2^j x - k)$. For $(ε, j, k) \in Λ$, let $a_{jk}^ε = \langle f, Φ_{jk}^ε \rangle$. Then

$$f(x) = \sum_{(ε, j, k) \in Λ} a_{jk}^ε Φ_{jk}(x).$$

The following proposition can be found in [13, 16, 17, 28]:

**Proposition 3.1.**

(i) $\{Φ_{jk}^ε(x)\}_{(ε, j, k) \in Λ}$ is an orthogonal basis in $L^2(\mathbb{R}^3)$.

(ii) $f(x) \in H^\frac{j}{2}(\mathbb{R}^3)$ if and only if

$$\sum_{(ε, j, k) \in Λ} 2^j |a_{jk}^ε|^2 < \infty.$$  

For $t > 0$, denote $j_0$ the smallest integer such that $2^{j_0} > 1$. For $(ε, j, k) \in Λ$, let $a_{jk}^ε(t) = \langle f(t, \cdot), Φ_{jk}^ε \rangle$. Then

$$f(t, x) = \sum_{(ε, j, k) \in Λ} a_{jk}^ε(t)Φ_{jk}^ε(x).$$

The following definition is a particular case in [13]:

**Definition 3.2.** $m, m' > 0$ if $(t, x) \in S_{m,m'}$, if the high frequency part of $f(t, x)$ satisfies

$$w_{m,\infty}^h(f) = \sup_{T > 0} \left\{ \sum_{(ε, j, k) \in Λ, j \geq j_0} 2^{2(m + \frac{1}{2})j |a_{jk}^ε(t)|^2} \right\}^{\frac{1}{2}} < \infty.$$  

$$w_{m',2}^h(f) = \left\{ \int_0^T \sum_{(ε, j, k) \in Λ, j \geq j_0} 2^{2(m + \frac{1}{2})j |a_{jk}^ε(t)|^2} dt \right\}^{\frac{1}{2}} < \infty.$$  

and the low frequency part of $f(t, x)$ satisfies

$$w_{m,\infty}^l(f) = \left\{ \sup_{T > 0} \sum_{(ε, j, k) \in Λ, j < j_0} 2^{j |a_{jk}^ε(t)|^2} \right\}^{\frac{1}{2}} < \infty.$$  

$$w_{m',2}^l(f) = \left\{ \int_0^T \sum_{(ε, j, k) \in Λ, j < j_0} 2^{2(m' + \frac{1}{2})j |a_{jk}^ε(t)|^2} dt \right\}^{\frac{1}{2}} < \infty.$$  

For $m > 1$, the equation (3.1) means the high frequency part of $f(t, x)$ belongs to $C^{m-1}(\mathbb{R}^3)$. Hence $S_{m,m'}(\mathbb{R}^3) \subset C^{m-1}(\mathbb{R}^3)$. The following theorem is a restate of the particular case of Theorem 1.1 in Li-Xiao-Yang [13].

**Theorem 3.3.** Given $m > 2, 0 < m' < 1$ and given $u_0 \in (H^\frac{j}{2})^3$ satisfying $\text{div} \ u_0 = 0$. If $\|u_0\|_{(H^\frac{j}{2})^3}$ is small, then $u^t(t, x)$ defined in (1.4) belongs to $S_{m,m'}^3$ and $u^t(t, x)$ converge to the solution $u(t, x)$ of Navier-Stokes equations (1.1) and $u(t, x) \in S_{m,m'}^3$. satisfying $\text{div} \ u(t, x) = 0$.

At the end of this section, we apply the classic mild solution skills to prove Theorem 2.5.
Proof. (i) It is easy to see $e^{i\Delta}u_1 = \partial_1^2\partial_2^2f_1(x_1, x_2)g_1(x_3)$ and $e^{i\Delta}u_2 = -\partial_1^2\partial_2^2f_1(x_1, x_2)g_1(x_3)$. Since $f_i$ satisfies (2.12), we get

$$A(e^{i\Delta}u_0) = 0, A_1(e^{i\Delta}u_0) = 0 \text{ and } A_2(e^{i\Delta}u_0) = 0.$$ 

Further $u_3 = 0$ implies $A_3(e^{i\Delta}u_0) = 0$. That is to say $u_0$ satisfies equation (2.7).

By construction, we have $\nabla u_0 = 0$. Hence $u_0$ satisfies equations (2.10).

(ii) By (1.4), we have $u(\tau, t, x) = u^0(t, x) - f_1(t, x)$, $\forall \tau \geq 1$. Hence $u(t, x) = u^0(t, x) + e^{i\Delta}u_0$ is the strong solution of the equations (1.1).

(iii) $e^{i\Delta}u_0$ has the same symmetry as $u_0$, hence $u(t, x)$ has the same symmetry as $u_0$.

$\square$

4. Preliminaries on symmetry of complex functions

In this section, we consider some preliminaries on symmetry of independent variables for complex functions $f(x) = f_1(x) + i f_2(x)$ where the real functions $f_1(x)$ and $f_2(x)$ all have symmetry or anti-symmetry properties with respect to all their independent variables.

**Definition 4.1.** Given $\alpha, \beta \in [0, 1]^3$. If $f(x)$ have the same symmetry property as functions $x^\alpha + i x^\beta$, we denote

$$T f(x) = \alpha + i \beta.$$ 

If $f(x)$ is a real function, we denote also $T f(x) = \alpha$. Further, we point out that, if $\beta = 0$, the notation $i 0$ does not mean the imaginary part is zero, $i 0$ means the imaginary part has symmetry like constant $I$.

The addition of the value of $T f$ is taken under the sense of modulation 2. Complex scale function has 64 kinds of different symmetry. Real scale function has 8 kinds of different symmetry.

**Lemma 4.2.** Given two real symmetric functions $f$ and $g$. If $f + g$ is also a symmetric function, then one of the following three conditions must be true:

1. $T f = T g;
2. f = 0;
3. g = 0.$

The product and convolution of real functions have the following basic properties:

**Lemma 4.3.** Given $\alpha, \beta \in [0, 1]^3$. If $T f(x) = \alpha, T g(x) = \beta$ and functions $f g$ and $f \ast g$ are well defined, then

$$T (f g) = T (f \ast g) = m(\alpha + \beta).$$

In particular, $\forall t > 0$ and $\phi_t(x) = t^{-\frac{3}{2}}(\frac{\alpha}{t})^3$ satisfying $T \phi(x) = 0$, we have

$$T (f + \phi_t) = T f.$$ 

(4.1)

Since the kernels of $(-\Delta)^{-1}$ and $e^{i\Delta}$ are radial function, we have

**Corollary 4.1.** Given $t > 0, \alpha, \beta \in [0, 1]^3$. If $T f = \alpha + i \beta$, then

$$T ((-\Delta)^{-1}f) = \alpha + i \beta.$$ 

$$T (e^{i\Delta}f) = \alpha + i \beta.$$
The derivatives of functions have the following properties:

**Lemma 4.4.** Given \( \alpha \in \mathbb{N}, \beta, \gamma \in \{0, 1\}^3 \). If \( T f(x) = \beta + i\gamma \), then

\[
T(\partial^\alpha f) = m(\alpha + \beta) + i m(\alpha + \gamma).
\]

**Proof.** By similarity of proof, we consider only the case \( T f = (0, 0, 0) + i(0, 0, 0) \) and \( \alpha = (1, 0, 0) \).

By the equation (4.1), we suppose that \( f(x) \) is a real smooth function. Denote \( x = (x_1, x') \), we have

\[
\partial_1 f(-x_1, x') = \lim_{h \to 0} \frac{f(-x_1 + h, x') - f(-x_1, x')}{h} = \lim_{h \to 0} \frac{f(x_1, x') - f(x_1, x')}{h} = -\partial_1 f(x_1, x').
\]

\( \square \)

5. The kinds of symmetric properties for solenoidal vector fields

In this section, we find out all the possible symmetric solenoidal vector field. For a complex vector field \( u = (u_1, u_2, u_3)' \), we can describe the symmetry by two matrices. That is to say, there exist matrices \( \alpha = (\alpha_1, \alpha_2, \alpha_3)' \) and \( \beta = (\beta_1, \beta_2, \beta_3)' \) where \( \alpha_1, \beta_1 \in \{0, 1\}^3 \), \( \forall l = 1, 2, 3 \) such that

\[
Tu_l = \alpha_l + i\beta_l, \forall l = 1, 2, 3.
\]

Complex vector field can have \( 64^3 = 262144 \) kinds of different symmetry. For two vector fields \( u \) and \( v \), the addition of the value of \( Tu + Tv \) is taken under the sense of modulation. The divergence zero property greatly limits the possible types of symmetries. In fact, we have

**Theorem 5.1.** Given \( u \) satisfies (5.1) and \( \text{div} u = 0 \). If \( u \) satisfies (2.8), then there exist \( \alpha_0, \beta_0 \in \{0, 1\}^3 \) such that

\[
\forall l = 1, 2, 3, Tu_l = m(e_l + \alpha_0) + im(e_l + \beta_0).
\]

**Proof.** The zero divergence property \( \text{div} u = \partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3 = 0 \) implies \( \partial_1 u_1 + \partial_2 u_2 \) and \( \partial_2 u_2 + \partial_3 u_3 \) have still symmetry. The fact \( u \) satisfies (2.8) implies

\[
\partial_1 u_1 = m(e_1 + \alpha_1) = m(e_2 + \alpha_2) = m(e_3 + \alpha_3)
\]

(5.3)

\[
m(e_1 + \beta_1) = m(e_2 + \beta_2) = m(e_3 + \beta_3).
\]

(5.4)

Denote

\[
\alpha_0 = m(e_1 + \alpha_1) = m(e_2 + \alpha_2) = m(e_3 + \alpha_3)
\]

(5.5)

\[
\beta_0 = m(e_1 + \beta_1) = m(e_2 + \beta_2) = m(e_3 + \beta_3).
\]

(5.6)

That is to say, the symmetry vector field with divergence zero is determined by the symmetry of the first component of vector field. Hence \( u \) satisfies (5.2). \( \square \)

If \( u \) satisfies (5.2), then there exists and only exists 64 kinds of possibilities. If there exists constant functions for real or imaginary part, similar to the proof of the above Theorem 5.1, we can prove the following theorem:
Theorem 5.2. Given $u$ satisfies (5.1) and $\text{div } u = 0$.

(i) If there are only one component $\tau$ which is constant function for all the real part function and there is no component which is constant function for imaginary function of $u$, then there exist $\alpha_0, \beta_0 \in \{0, 1\}^3$ such that

$$
\begin{align*}
Tu_l &= 0 + im(e_l + \beta_0), \quad l = \tau; \\
Tu_l &= m(e_l + \alpha_0) + im(e_l + \beta_0), \quad l \neq \tau.
\end{align*}
$$

(ii) If there are only one component $\tau$ which is constant function for all the imaginary part function and there is no component which constant function for real part function of $u$, then there exist $\alpha_0, \beta_0 \in \{0, 1\}^3$ such that

$$
\begin{align*}
Tu_l &= m(e_l + \alpha_0) + i0, \quad l = \tau; \\
Tu_l &= m(e_l + \alpha_0) + im(e_l + \beta_0), \quad l \neq \tau.
\end{align*}
$$

(iii) If both the real part and the imaginary part have $\tau$--th component which is constant. For real part, the $\tau$ component; the imaginary part, the $\tau'$ component. There exist $\alpha_0, \beta_0 \in \{0, 1\}^3$ such that

$$
\begin{align*}
Tu_l &= 0 + i0, \quad l = \tau; \\
Tu_l &= m(e_l + \alpha_0) + im(e_l + \beta_0), \quad l \neq \tau.
\end{align*}
$$

(iv) If both the real part and the imaginary part have one component which is constant. For real part, the $\tau$ component; the imaginary part, the $\tau'$ component and $\tau \neq \tau'$. There exist $\alpha_0, \beta_0 \in \{0, 1\}^3$ such that

$$
\begin{align*}
Tu_l &= 0 + im(e_l + \beta_0), \quad l = \tau; \\
Tu_l &= m(e_l + \alpha_0) + i0, \quad l = \tau'; \\
Tu_l &= m(e_l + \alpha_0) + im(e_l + \beta_0), \quad l \neq \tau, \tau'.
\end{align*}
$$

(v) If there are at least two components which are constant function for all the real part function and there is no component which is constant function for imaginary function of $u$, then there exist $\beta_0 \in \{0, 1\}^3$ such that

$$
\begin{align*}
Tu_l &= 0 + im(e_l + \beta_0), \quad l = \tau; \\
Tu_l &= 0 + im(e_l + \beta_0), \quad l \neq \tau.
\end{align*}
$$

(vi) If there are at least two components which are constant function for all the imaginary part function and there is no component which constant function for real part function of $u$, then there exist $\alpha_0 \in \{0, 1\}^3$ such that

$$
\begin{align*}
Tu_l &= m(e_l + \alpha_0) + i0, \quad l = \tau; \\
Tu_l &= m(e_l + \alpha_0) + i0, \quad l \neq \tau.
\end{align*}
$$

(vii) If there are at least two components which are constant function for both the real part and the imaginary part of $u$, then such vector fields must be constant such that

$$
Tu_l = 0 + i0, \quad \forall l = 1, 2, 3.
$$

There are 262144 kinds of different symmetric complex vector fields, but if they have zero divergence property, then

Theorem 5.3. There are 984 kinds of different symmetric complex vector fields with zero divergence property.
Proof. We note

(i) There are 64 kinds of symmetric complex vector fields in the equations \((5.2)\).

(ii) In the equations \((5.7)\), if we fix \(\tau\) and \(\beta_0\), there exists 8 kinds of symmetry property. Further, for each \(\tau\) fixed, if \(\alpha_0 = e_\tau\), then it has the similar symmetry property as one of the symmetry property in the equations \((5.2)\). There are 168 = 3 \times 8 \times (8 - 1) kinds of symmetric complex vector fields which are different to which in the equations \((5.2)\).

(iii) The same reason as above, there are 168 kinds of symmetric complex vector fields in the equations \((5.8)\) which are different to which satisfying the equations \((5.2)\).

(iv) There exist 192 = 3 \times 8 \times 8 kinds of symmetric property in the equations \((5.9)\). But there exists \(\alpha_0 = \beta_0 = e_\tau\) which has the same symmetric property as which in the equations \((5.2)\). Hence there are 3 kinds of symmetric properties which have been existed in the equations \((5.2)\).

(v) There exist 384 = 3 \times 2 \times 8 \times 8 \times 8 kinds of symmetric property in the equations \((5.10)\). But for \(\alpha_0 = e_\tau\) and \(\beta_0 = e_\tau'\) which has the same symmetric property as which in the equations \((5.2)\). Hence there are 6 kinds of symmetric properties which have been existed in the equations \((5.2)\).

(vi) There are 8 kinds of symmetric complex vector fields in the equations \((5.11)\) and \((5.12)\).

(vii) There are one kind of symmetric complex vector fields in the equations \((5.13)\).

In total, for complex solenoidal vector fields, there are 984 = 64 + 168 \times 2 + (192 - 3) + (384 - 6) + 8 \times 2 + 1 kinds of different symmetric properties.

□

For real valued solenoidal vector field, similar to the above theorems 5.1 and 5.2, there exist the following possibilities of symmetric properties:

**Theorem 5.4.** If \(u(t, x)\) is a real symmetric solenoidal vector field, then one of the following three conditions will be satisfied:

(i) \(u(t, x)\) has no constant component, then there exists \(\alpha_0 \in \{0, 1\}^3\) such that

\[
Tu_l = m(e_l + \alpha_0), \forall l = 1, 2, 3.
\]

(ii) There exists one \(\tau = 1, 2, 3\) such that \(u_{\tau}(t, x)\) is a constant function, then there exists \(\alpha_0 \in \{0, 1\}^3\) such that

\[
Tu_\tau(t, x) = 0, \forall \neq \tau, Tu_l = m(e_l + \alpha_0).
\]

(iii) If there exists more than one constant component, then \(u\) is a constant vector field and

\[
Tu_l = 0, \forall l = 1, 2, 3.
\]

There are 512 = 8 \times 8 \times 8 kinds of symmetry properties for real vector fields. Similar to the above theorem 5.3, we have

**Theorem 5.5.** There exists only 30 kinds of real symmetric solenoidal vector fields.

Proof. By counting the possibility of symmetric cases in Theorem 5.4, we have

\[
30 = 8 + 3 \times (8 - 1) + 1.
\]

□
6. Rigidity for symmetric complex solenoidal vector field

In this section, we consider the rigidity for symmetric complex solenoidal vector field satisfying (2.8). We consider first the symmetry of $A(u, u)$.

**Lemma 6.1.** If $u(x)$ satisfies (5.2), then

\[(6.1) \quad T A(u, u) = 0 + i m(\alpha_0 + \beta_0).\]

**Proof.** For $l = 1, 2, 3$, denote the real part and the imaginary part of $u_l$ to be $u_l^r$ and $u_l^i$ respectively. Hence, for $l, l' = 1, 2, 3$, the product $u_l u_{l'}$ can be written as

\[u_l u_{l'} = (u_l^r + i u_l^i)(u_{l'}^r + i u_{l'}^i) = u_l^r u_{l'}^r - u_l^i u_{l'}^i + i(u_l^r u_{l'}^i + u_l^i u_{l'}^r).\]

Since $u(x)$ satisfies (5.2), we have

\[T(u_l^r u_{l'}^r - u_l^i u_{l'}^i) = m(e_l + e_{l'}).\]

and

\[T(u_l^i u_{l'}^i) = m(e_l + \beta_0 + e_{l'} + \alpha_0) = m(e_l + e_{l'} + \alpha_0 + \beta_0),\]

\[T(u_l^r u_{l'}^i) = m(e_l + \alpha_0 + e_{l'} + \beta_0) = m(e_l + e_{l'} + \alpha_0 + \beta_0).\]

Hence

\[(6.2) \quad Tu_l u_{l'} = m(e_l + e_{l'} + i m(e_l + e_{l'} + \alpha_0 + \beta_0)).\]

The above (6.2) implies,

\[T(\sum_{l, l'} \partial_l \partial_{l'}(u_l u_{l'})) = 0 + i m(\alpha_0 + \beta_0).\]

Then we consider the symmetry of $B(u, u)$.

**Lemma 6.2.** If $u(x)$ satisfies (5.2), then

\[(6.3) \quad T(B(u, u)) = \begin{pmatrix} e_1 + i m(e_1 + \alpha_0 + \beta_0) \\ e_2 + i m(e_2 + \alpha_0 + \beta_0) \\ e_3 + i m(e_3 + \alpha_0 + \beta_0) \end{pmatrix}.\]

**Proof.** For $l' = 1, 2, 3$, according to (6.1), we have

\[T(\partial_{l'} \sum_{l, l'} \partial_l \partial_{l'}(u_l u_{l'})) = e_{l'} + i m(e_{l'} + \alpha_0 + \beta_0).\]

According to (6.2), we have

\[T(\sum_{l} \partial_l (u_l u_{l'})) = e_{l'} + i m(e_{l'} + \alpha_0 + \beta_0).\]

Hence

\[T((\mathbb{P} \nabla (u \otimes u))) = \begin{pmatrix} e_1 + i m(e_1 + \alpha_0 + \beta_0) \\ e_2 + i m(e_2 + \alpha_0 + \beta_0) \\ e_3 + i m(e_3 + \alpha_0 + \beta_0) \end{pmatrix}.\]

By corollary 4.1, $T(B(u, u))$ satisfies the equation (6.3).
The fact \( u_0 \) satisfies (2.6) and (2.9) is equivalent to the following condition: there exists \( \beta_0 \in \{0, 1\}^3 \) such that

\[
T(u_0) = \begin{pmatrix} e_1 + i m(e_1 + \beta_0) \\ e_2 + i m(e_2 + \beta_0) \\ e_3 + i m(e_3 + \beta_0) \end{pmatrix}.
\]

For complex valued initial data \( u_0 \), there are 8 kinds and only 8 kinds of symmetry property which can generate symmetric solution. Now we come to prove theorem 2.4.

**Proof.** \( u(t, x) \) has the same symmetric property for all \( t \geq 0 \),

\[
Tu(t, x) = Tu_0 = Te^{\Delta t}u_0.
\]

Further, if \( u(t, x) \) has symmetric property and satisfies (2.8), then there exists \( \alpha_0, \beta_0 \in \{0, 1\}^3 \) such that \( u(t, x) \) satisfies (5.2). According to lemma 6.2, we have

\[
T(B(u, u)) = \begin{pmatrix} e_1 + i m(e_1 + \alpha_0 + \beta_0) \\ e_2 + i m(e_2 + \alpha_0 + \beta_0) \\ e_3 + i m(e_3 + \alpha_0 + \beta_0) \end{pmatrix}.
\]

According to equation (1.3), we have

\[
e^{\Delta t}u_0(x) - u(t, x) = B(u, u)(t, x).
\]

Combine (2.8), (6.5) and (6.6), we get

\[
\alpha_0 = 0 \text{ or } B(u, u) = 0.
\]

(i) \( \alpha_0 = 0 \) implies \( u_0 \) satisfies equation (6.4).

(ii) \( B(u, u) = 0 \) implies \( u = e^{\Delta t}u_0 \) and \( \partial_t u - \Delta u = 0 \). That is to say, (2.7) is right.

If \( u_0 \) satisfies the above (i) or (ii), then \( u^*(t, x) \) in equation (1.4) are symmetric functions. Hence \( u(t, x) \) is a symmetric solution.

\( \square \)

7. **Rigidity for real initial values**

At the begin of this section, we prove first the condition (5.15) in the above theorem 5.4 will imply the condition (2.7). We will prove this fact in the following two theorems:

**Theorem 7.1.** Given \( u \) satisfies (5.15) and \( \alpha_0 \neq \varepsilon_1 \) and \( \alpha_0 \neq 0 \). Then \( u \) has the same symmetry property for all \( t \geq 0 \) if and only if (2.7) is true.

**Proof.** Given \( u \) satisfies (5.15), then \( A_r(u) = 0 \) and

\[
A(u, u) = \sum_{|r| = 0} \partial_r \partial_r (u(u u_0)).
\]

Hence, \( TA(u, u) = 0 \).

Then we consider two cases. (i) If \( u_0 \neq 0 \), then \( A_r(u) = 0 \) and \( u_{0,x} \) is a constant, and

\[
A(u, u) = \sum_{|r| = 0} \partial_r \partial_r (u(u u_0)).
\]
Hence, \( TA(u, u) = 0 \) and \( T \partial_t A(u, u) = \alpha_0 \). If \( u_\tau(t, x) \) satisfies (1.3), then
\[
C_\tau(u) = 0.
\]

Let \( l \neq \tau, l' \neq l, \tau \). We have
\[
A_I(u) = (\partial_t(u))^2 + \partial_x(u_tu_t) + \partial_x(u_\tau u_\tau) \equiv I + II.
\]

It is easy to see \( TI = e_l \) and \( TII = \alpha_0 \). If \( u_l(t, x) \) satisfies (1.3), then \( l = 0, II = 0 \). But \( II = 0 \) implies \( u_\tau u_\tau \) is a constant. That is to say, all \( u_l, l = 1, 2, 3 \) are constant and (2.7) is true.

(ii) If \( u_\tau = 0 \), then \( A_\tau(u) = 0 \), \( u_{0, \tau} = 0 \) and further
\[
A(u, u) = \sum_{|t-|t'|-|t| \neq 0} \partial_t \partial_x(u_tu_t).
\]

Hence, \( TA(u, u) = 0 \) and \( T \partial_t A(u, u) = e_l \), \( \forall l = 1, 2, 3 \). Therefore \( C_\tau(u) \) has symmetry property and \( TC_\tau(u) = e_{\tau} \). If \( Tu_\tau = TB_\tau(u, u) \), by (1.3), we have \( C_\tau(u) = 0 \). Which implies \( \partial_t A(u, u) = 0 \), hence \( A(u, u) \) is a constant. So for \( l \neq \tau \), we have \( C_\tau(u) = A_I(u) \).

Let \( l \neq \tau, l' \neq l, \tau \). We have
\[
A_I(u) = (\partial_t(u))^2 + \partial_x(u_tu_t).
\]

Then we have \( TA_I(u) = e_l \). Since \( \alpha_0 \neq 0 \), we have must \( A_I(u) = 0 \). That is to say, (2.7) is true.

\[\square\]

**Theorem 7.2.** Given \( u \) satisfies (5.15), \( \alpha_0 = 0 \) and \( u_\tau \neq 0 \). Then \( u \) has the same symmetry property for all \( t \geq 0 \) if and only if (2.7) is true.

**Proof.** Given \( u \) satisfies (5.15), then \( A_\tau(u) = 0 \) and
\[
A(u, u) = \sum_{|t-|t'|-|t| \neq 0} \partial_t \partial_x(u_tu_t).
\]

Hence, \( TA(u, u) = 0 \).

If \( u_\tau \neq 0 \), then \( A_\tau(u) = 0 \) and \( u_{0, \tau} \) is a constant, and
\[
A(u, u) = \sum_{|t-|t'|-|t| \neq 0} \partial_t \partial_x(u_tu_t).
\]

Hence, \( TA(u, u) = 0 \) and \( T \partial_t A(u, u) = \alpha_0 \). If \( u_\tau(t, x) \) satisfies (1.3), then
\[
C_\tau(u) = 0.
\]

Let \( l \neq \tau, l' \neq l, \tau \). We have
\[
A_I(u) = (\partial_t(u))^2 + \partial_x(u_tu_t) + \partial_x(u_\tau u_\tau) \equiv I + II.
\]

It is easy to see \( TI = e_l \) and \( TII = e_\tau + e_l \).

If \( u_\tau(t, x) \) satisfies (1.3), then \( II = 0 \). But \( II = 0 \) implies \( u_\tau u_\tau \) is a constant. That is to say, all \( u_l, l = 1, 2, 3 \) are constant and (2.7) is true.

\[\square\]
At the end of this section, we come to prove theorem 2.3. Note that (2.6) can be rewritten as

\begin{equation}
T(u_0) = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.
\end{equation}

For real valued initial data, there exists only one kind of symmetric property for (2.6).

**Proof.** We can classify the particular case in (5.15) where \( \alpha_0 = 0 \) and \( u_\tau = 0 \) to the case (5.14) with \( \alpha_0 = 0 \). By Theorems 7.1 and 7.2, we need only to consider the case where \( u \) satisfies (5.14). Similar to lemma 6.2, we have

\begin{equation}
T(B(u, u)) = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.
\end{equation}

\( u \) has the same symmetric property for all \( t \geq 0 \), we have

\[ Tu = Tu_0 = Te^{j\lambda}u_0. \]

According to lemma 1.3, we have

\[ B(u, u) = e^{j\lambda}u_0 - u. \]

Combine with (7.2), we get \( \alpha_0 = 0 \) in the equation (5.14) or \( B(u, u) = 0 \).

(i) \( \alpha_0 = 0 \) in the equation (5.14) implies \( u_0 \) satisfies equation (7.1)

(ii) \( B(u, u) = 0 \) implies that \( u_0 \) satisfies (2.7).

If \( u_0 \) is a real initial data and satisfies (i) or (ii), then \( u(t, x) \) in equation (1.4) are real symmetric functions. Hence \( u(t, x) \) is a real symmetric solution.

\[ \square \]

8. **Symmetric transitivity and matched symmetric solenoidal vector fields**

Theorem 2.8 can be restated as following:

**Theorem 8.1.** Given any solenoidal vector field \( u(t, x) = (u_1(t, x), u_2(t, x), u_3(t, x))^T \) and \( \beta \in [0, 1]^3 \). Then there exist 8 symmetric solenoidal vector field \( u_\alpha = u^{\alpha, re} + iu^{\alpha, im} \) where \( u^{\alpha, re} \) and \( u^{\alpha, im} \) are respectively the real part and imaginary part of \( u_\alpha \) such that

\[ u = \sum_{\alpha \in [0, 1]^3} u_\alpha \text{ and } T(u^{\alpha, re}_l u^{\alpha, im}_l) = \beta, \forall l = 1, 2, 3. \]

**Proof.** We decompose first the real part \( u^{re} \). For \( l = 1, 2, 3 \), each \( u^{re}_l \) can be decomposed as 8 symmetric functions \( \{u^{\alpha, re}_l \}_{\alpha \in [0, 1]^3} \). \( \forall \alpha \in [0, 1]^3 \) and \( l = 1, 2, 3 \), we take \( u^{\alpha, re}_l \) such that \( T u^{\alpha, re}_l = e_l + \alpha. \) It is easy to see that

\begin{equation}
T \sum_{l=1,2,3} \partial_l u^{\alpha, re}_l = \alpha, \forall \alpha \in [0, 1]^3.
\end{equation}

Now we prove that

\[ \sum_{l=1,2,3} \partial_l u^{\alpha, re}_l = 0. \]
Denote \( \tilde{u}_l^{α,\text{re}} = u_l^{α,\text{re}} - \sum_{i\neq α} u_i^{α,\text{re}} \). Since \( \sum_{l=1,2,3} \partial_l u_l^{α,\text{re}} = 0 \), by (8.1), we have \[
α = T \sum_{l=1,2,3} \partial_l u_l^{α,\text{re}} = T \sum_{l=1,2,3} \partial_l \tilde{u}_l^{α,\text{re}}.
\]

But if \( \sum_{l=1,2,3} \partial_l u_l^{α,\text{re}} \neq 0 \), then \( T \sum_{l=1,2,3} \partial_l u_l^{α,\text{re}} \neq α \). Hence we must have \[
\sum_{l=1,2,3} \partial_l u_l^{α,\text{re}} = \sum_{l=1,2,3} \partial_l \tilde{u}_l^{α,\text{re}} = 0.
\]

For the imaginary part, we repeat almost the same process with little modifications. We decompose the imaginary part \( u_l^{α,\text{im}} \). For \( l = 1, 2, 3 \), each \( u_l^{α,\text{im}} \) can be decomposed as 8 symmetric functions \( u_l^{α,i,m} \). \( \forallα \in [0, 1]^3 \) and \( l = 1, 2, 3 \), we take \( u_l^{α,i,m} \) such that \( T u_l^{α,i,m} = e_l + α + β \). It is easy to see that \[
(8.2)
T \sum_{l=1,2,3} \partial_l u_l^{α,i,m} = α + β, \forallα \in [0, 1]^3.
\]

The same reason as above for real part, we have \[
\sum_{l=1,2,3} \partial_l u_l^{α,i,m} = 0.
\]

By applying (8.1) and (8.2), we get \[
T(u_l^{α,\text{re}} u_l^{α,\text{im}}) = β, \forall l = 1, 2, 3.
\]

We get the relative decomposition. \( \square \)

We consider then the symmetry of \( A(u, v) \) for matched symmetric property.

**Lemma 8.2.** Let \( u(t, x) = (u_1(t, x), u_2(t, x), u_3(t, x))^f \) and \( v(t, x) = (v_1(t, x), v_2(t, x), v_3(t, x))^f \) be two solenoidal vector fields. There exists \( α_0, β_0, α_0', β_0' \in [0, 1]^3 \) satisfying \[
(8.3)
m(α_0 + α_0') = m(β_0 + β_0')
\]
such that \( \forall l = 1, 2, 3 \), we have \[
(8.4)
Tu_l = m(e_l + α_0) + i m(e_l + β_0);
Tv_l = m(e_l + α_0') + i m(e_l + β_0').
\]

Hence we have \[
(8.5)
TA(u, v) = 0 + i m(α_0 + β_0).
\]

**Proof.** For \( l = 1, 2, 3 \), denote the real part and the imaginary part of \( u_l \) to be \( u_l^{α,\text{re}} \) and \( u_l^{α,\text{im}} \) respectively; denote the real part and the imaginary part of \( v_l \) to be \( v_l^{α,\text{re}} \) and \( v_l^{α,\text{im}} \) respectively. Hence, for \( l, l' = 1, 2, 3 \), the product \( u_l v_{l'} \) can be written as \[
u_l v_{l'} = (u_l^{α,\text{re}} + i u_l^{α,\text{im}})(v_{l'}^{α,\text{re}} + i v_{l'}^{α,\text{im}}) = u_l^{α,\text{re}} v_{l'}^{α,\text{re}} + i u_l^{α,\text{im}} v_{l'}^{α,\text{im}} + i(u_l^{α,\text{re}} v_{l'}^{α,\text{im}} + u_l^{α,\text{im}} v_{l'}^{α,\text{re}}).
\]

Since \( u(t, x) \) and \( v(t, x) \) satisfies (8.4), we have
\[
T(u_l^{α,\text{re}} v_{l'}^{α,\text{re}}) = m(e_l + α_0 + e_{l'} + α_0') = m(e_l + e_{l'} + α_0 + α_0'),
T(u_l^{α,\text{re}} v_{l'}^{α,\text{im}}) = m(e_l + β_0 + e_{l'} + β_0') = m(e_l + e_{l'} + β_0 + β_0'),
T(u_l^{α,\text{im}} v_{l'}^{α,\text{re}}) = m(e_l + β_0 + e_{l'} + α_0') = m(e_l + e_{l'} + α_0' + β_0),
T(u_l^{α,\text{im}} v_{l'}^{α,\text{im}}) = m(e_l + α_0 + e_{l'} + β_0') = m(e_l + e_{l'} + α_0 + β_0').
\]
Hence (8.3) implies
\[
8.6 \quad Tu'\n = m(e_1 + e_r + \alpha_0 + \alpha'_0) + i m(e_1 + e_r + \alpha'_0 + \beta_0).
\]
The above (8.6) implies,
\[
8.7 \quad T\{\sum_{l'} \partial_l \partial_T(u_l'v_f)\} = m(\alpha_0 + \alpha'_0) + i m(\alpha'_0 + \beta_0).
\]

Theorem 2.9 can be restated as following:

**Lemma 8.3.** If \( u(t, x) \) and \( v(t, x) \) satisfies (8.3) and (8.4), then
\[
8.8 \quad T(B(u, v)) = \left\{ \begin{array}{l}
\{ m(e_1 + \alpha_0 + \alpha'_0) + i m(e_1 + \alpha'_0 + \beta_0) \\
m(e_2 + \alpha_0 + \alpha'_0) + i m(e_2 + \alpha'_0 + \beta_0) \\
m(e_3 + \alpha_0 + \alpha'_0) + i m(e_3 + \alpha'_0 + \beta_0)
\end{array} \right\}.
\]

**Proof.** For \( l' = 1, 2, 3 \), according to (8.5), we have
\[
8.9 \quad T[\partial_T \sum_{l'} \partial_l \partial_T(u_l'v_f)] = m(e_1 + \alpha_0 + \alpha'_0) + i m(e_1 + \alpha'_0 + \beta_0).
\]

According to (8.6), we have
\[
8.10 \quad T[\sum_{l'} \partial_l (u_l'v_f)] = m(e_1 + \alpha_0 + \alpha'_0) + i m(e_1 + \alpha'_0 + \beta_0).
\]

Hence
\[
8.11 \quad T(\nabla (u \otimes v)) = \left\{ \begin{array}{l}
m(e_1 + \alpha_0 + \alpha'_0) + i m(e_1 + \alpha'_0 + \beta_0) \\
m(e_2 + \alpha_0 + \alpha'_0) + i m(e_2 + \alpha'_0 + \beta_0) \\
m(e_3 + \alpha_0 + \alpha'_0) + i m(e_3 + \alpha'_0 + \beta_0)
\end{array} \right\}.
\]

By corollary 4.1, \( T(B(u, v)) \) satisfies the equation (8.7). \( \square \)

The following Theorem 8.4 is a combination of Theorems 2.8 and 2.9. Theorem 8.4 tells us that, if we know the structure of all the symmetric solenoidal vector field for operator \( B \), then we know the structure of all the solenoidal vector field for operator \( B \).

**Theorem 8.4.** Given any two solenoidal vector fields \( u(t, x) \) and \( v(t, x) \). Then there exists symmetric solenoidal vector fields \( u^{\alpha} \) and \( v^{\beta} \) satisfying \( u = \sum_{\alpha \in [0, 1]^3} u^{\alpha} \) and \( v = \sum_{\alpha' \in [0, 1]^3} v^{\alpha'} \) such that all the \( B(u^{\alpha}, v^{\alpha'}) \) have symmetric properties satisfying
\[
8.12 \quad B(u, v) = \sum_{\alpha, \alpha' \in [0, 1]^3} B(u^{\alpha}, v^{\alpha'}).
\]
9. Smooth solution with energy conservation

In real world, when considering Navier-Stokes equations (2.15) on the domain $\Omega$, the initial data must be real valued and the solution should keep energy conservation. For general domain $\Omega$, to solve (2.15) with initial data $u_0$, we can try to use integration equation (1.3). For example, we can restrict the operators $e^{t\Delta}, e^{(r-t)\Delta}$ and $e^{(r-t)\Delta}(-\Delta)^{-1}$ in equations (1.4) on the domain $\Omega \times \Omega$. But, even $u'(t,x)$ in (1.4) converges to some function $u(t,x)$, we did not know whether $u(t,x)$ satisfies the first equation of equations (2.15).

Another way to solve (2.15) is to extend the initial data $u_0$ to a function in $\mathbb{R}^3$. Let $P\cdot u_0(x)$ be the zero extension of $u_0$.

$$P\cdot u_0(x) = \begin{cases} u_0(x), & x \in \Omega; \\ 0, & x \notin \Omega. \end{cases}$$

We solve the equations (1.1), we get a solution $P\cdot u(t,x)$. We say the restriction of $P\cdot u(t,x)$ on the domain $\Omega$ is the relative solution of (2.15). But we know $P\cdot u(t,x)$ can not be zero outside $\Omega$. Hence there exists loss of energy. It is hard to say, $u(t,x)$ is the real solution of (2.15).

Further, let $v(x)$ be a function with $\text{supp}v(x) \subset \Omega^\circ$. We can consider other extension $P_{z,z}u_0(x)$ which is the zero extension of $u_0$.

$$P_{z,z}u_0(x) = \begin{cases} u_0(x), & x \in \Omega; \\ v(x), & x \notin \Omega. \end{cases}$$

We can consider also the restriction of solution $P_{z,z}u(t,x)$ of (1.1) with initial data $P_{z,z}u_0(x)$. There are many choice for $v(x)$. Generally speaking, we have no sufficient reason to say that $P_{z,z}u(t,x)$ is the real solution of (2.15).

Hence we consider particular domain $\Omega = \mathbb{R}^2 \times \mathbb{R}_+$. We choose $v(x)$ to be the symmetric extension of $u_0(x)$. That is to say, we extend $u_0(x)$ to $P_{z,z}u(t,x)$ in a symmetric way. To ensure the zero divergence property $\text{div} P_{z,z}u(t,x) = 0$, we can not extend all the three $u_{0,1}(x)$ with the same symmetry method. If we extend $u_{0,1}(x)$ and $u_{0,2}(x)$ with the anti-symmetric way and we have to extend $u_{0,3}(x)$ with symmetric way. If we extend $u_{0,1}(x)$ and $u_{0,2}(x)$ with the symmetric way and we have to extend $u_{0,3}(x)$ with anti-symmetric way. There are two ways to extend the initial value symmetrically. Denote $u_0(x) = (u_{0,1}(x),u_{0,2}(x),u_{0,3}(x))'$ and $\bar{x} = (x_1,x_2,-x_3)$. The first way is to extend $u_0(x)$ in the following way:

$$P_{st}u_0(x) = \begin{cases} u_0(x), \\ (-u_{0,1}(\bar{x}),-u_{0,2}(\bar{x}),u_{0,3}(\bar{x}))', \quad x_3 \geq 0; \\ u_0(x), \quad x_3 < 0. \end{cases}$$  \hspace{1cm} (9.1)

The second way is to extend $u_0(x)$ as we did in the equation (2.17). Then we can consider the solutions $P_{st}u(t,x)$ and $P_{st}u(t,x)$ of (1.1) with respectively initial data of $P_{st}u_0(x)$ and $P_{st}u_0(x)$.

For the first way, the iteration process (1.4) will change the symmetric property. Given initial data $P_{st}u_0(x)$ defined in (9.1). If $P_{st}u(t,x)$ exists, according to Theorem 2.3, $P_{st}u(t,x)$ can not be a symmetric solution of (1.1). We did not know whether the energy in the domain $\mathbb{R}^2 \times \mathbb{R}_+$ equals to the energy in the domain $\mathbb{R}^2 \times \mathbb{R}_-$.

In this paper, we search energy conservation smooth solution of (2.15) with domain $\Omega = \mathbb{R}^2 \times \mathbb{R}_+$. To get smooth solution, the real initial data $u_0(x)$ should satisfy the symmetry property (2.16). We will see such symmetry property is sufficient and necessary for smooth solution with energy conservation. In fact, there exists and only exists one kind of symmetry property generate smooth solution with energy conservation.
Theorem 9.1. Given $m > 2$. If the real initial data $u_0$ satisfies $\text{div} \, u_0 = 0$, symmetry property (2.16) and $\|u_0\|_{\dot{H}^2(\mathbb{R}^3 \times \mathbb{R}_+)}$ being small, then the Navier-Stokes equations (2.15) have a $C^{m-1}$ smooth solution $u(t, x)$ with energy conservation and satisfying symmetry property (2.16).

Proof. If $u_0$ satisfies the symmetry property (2.16), then

$$
T(P_s u_0) = \begin{pmatrix}
\epsilon_1 \\
\epsilon_2 \\
\epsilon_3 
\end{pmatrix}
$$

$u^s(t, x)$ in the iteration process (1.4) will keep the symmetric property

$$
T(u^s(t, x)) = \begin{pmatrix}
\epsilon_1 \\
\epsilon_2 \\
\epsilon_3 
\end{pmatrix}
$$

(i) $u_0 \in (\dot{H}^2(\mathbb{R}^3 \times \mathbb{R}_+))$ means $P_s u_0(x) \in \dot{H}^2(\mathbb{R}^3)$. For $\|u_0\|_{\dot{H}^2(\Omega)}$, being small, $\|P_s u_0\|_{\dot{H}^2(\mathbb{R}^3)}$ being small. Hence $u^s(t, x)$ converges to some $u(t, x) \in (S_{m,m}(\mathbb{R}^3))^3$ which is the solution of the equations (1.1). Since $S_{m,m}(\mathbb{R}^3) \subset C^{m-1}(\mathbb{R}^3)$, by Theorems 2.3 and 3.3, we get smooth symmetric solution on the whole $\mathbb{R}^3$. By restriction, we get also smooth solution on the domain $\mathbb{R}^2 \times \mathbb{R}_+$.

(ii) Further, according to Theorem 2.3, we know the energy in the domain $\mathbb{R}^2 \times \mathbb{R}_+$ equals to the energy in the domain $\mathbb{R}^2 \times \mathbb{R}_-$.

By the above two points (i) and (ii), we get smooth solution with energy conservation. \(\square\)

Remark 9.2. (i) For real valued initial data with symmetric solution, there exists only one possibility. If $u_0$ does not satisfies (2.16), then the iteration process (1.4) will not keep the symmetry property and will not keep energy conservation.

(ii) General speaking, the fluid on the domain has ripple effect near the boundary. We have proved that, if the initial data satisfies the symmetry property (2.16), then the Navier-Stokes equations (2.15) have smooth solution even at the boundary.

(iii) Our method can be applied also to the following domain $\Omega = \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+$ or $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+$.

Acknowledgement The author is financially supported by the National Natural Science Foundation of China (No.11571261).

References

[1] H. Abidi, P. Zhang Global smooth axisymmetric solutions of 3-D inhomogeneous incompressible Navier-Stokes system, Calc. Var. DOI 10.1007/s00526-015-0902-6 54 (2015), pp. 3251-3276

[2] M. Cannone, A generalization of a theorem by Kato on Navier-Stokes equation, Rev. Mat. Iberoamericana, 13(1997), pp. 673-697

[3] M. Cannone, Harmonic analysis tools for solving the incompressible Navier-Stokes equations, Handbook of Mathematical Fluid Dynamics, 3(2005), pp. 161-244

[4] P. Germain, N. Pavlović and G. Staffilani, Regularity of solutions to the Navier-Stokes equations evolving from small data in $BMO^{-1}$, Int. Math. Res. Not., 21(2007), pp.227-239

[5] Y. Giga, K. Inui, A. Mahalov and J. Saal, Uniform global solvability of the rotating Navier-Stokes equations for nondecaying initial data, Indiana University Mathematical Journal, 57 (2008), 2775-2791
[6] Y. Giga and T. Miyakawa, Navier-Stokes flow in $\mathbb{R}^3$ with measures as initial vorticity and Morrey spaces, *Comm. Partial Differential Equations*, 14(1989), pp. 577-618.

[7] T. Kato, Strong $L^p$ solutions of the Navier-Stokes equation in $\mathbb{R}^n$ with applications to weak solutions, *Math. Z.*, 187(1984), pp. 471-480.

[8] T. Kato and H. Fujita, On the non-stationary Navier-Stokes system, *Rend. Semin. Mat. Univ. Padova*, 30(1962), pp. 243-260.

[9] T. Kato and G. Ponce, Commutator estimates and the Euler and Navier-Stokes equations, *Comm. Pure Appl. Math.*, XLI(1988), pp. 891-907.

[10] H. Koch and D. Tataru, Well-posedness for the Navier-Stokes equations, *Adv. Math.*, 157(2001), pp. 22-35.

[11] Z. Lei and Fanghua Lin, Global mild solutions of Navier-Stokes equations, *Comm. Pure Appl. Math.*, 64(2011), pp. 1297-1304.

[12] P. G. Lemarié-Rieusset, *Recent development in the Navier-Stokes problems*, Boca Raton, London, New York, Washington, D. C., CRC Press, 2002.

[13] P. Li, J. Xiao and Q. Yang, Global mild solutions of fractional Navier-Stokes equations with small initial data in critical Besov-$Q$ spaces, *Electron. J. Differential Equ.*, 185(2014), pp. 1-37.

[14] P. Li and Q. Yang, Well-posedness of Quasi-Geostrophic Equations with data in Besov-$Q$ spaces, *Nonlinear Analysis*, 94(2014), pp. 243-258.

[15] P. Li and Q. Yang, Wavelets and the well-posedness of incompressible magneto-hydrodynamic equations in Besov type $Q$-space, *Journal of Mathematical Analysis and Applications*, 405(2013), pp. 661-686.

[16] C. Lin and Q. Yang, Semigroup characterization of Besov type Morrey spaces and well-posedness of generalized Navier-Stokes equations, *J. Differential Equations*, pp. 804-846.

[17] Y. Meyer, *Wavelets and operators*, Cambridge Studies in Advanced Mathematics, 37, Cambridge: Cambridge University Press, 1992.

[18] J. Necas, M. Ruzicka and V. Sverak, On Leray's self-similar solutions of the Navier-Stokes equations, *Acta. Math.*, 176 (1996), 283-294.

[19] M. E. Taylor, Analysis on Morrey spaces and applications to Navier-Stokes and other evolution equations, *Comm. Partial Differential Equations*, 17(1992), pp. 1407-1456.

[20] J. Wu, Generalized MHD equations, *J. Differential Equations*, 195(2003), 284-312.

[21] J. Wu, The generalized incompressible Navier-Stokes equations in Besov spaces, *Dyn. Partial Differ. Equ.*, 1(2004), 381-400.

[22] J. Wu, Lower bounds for an integral involving fractional Laplacians and the generalized Navier-Stokes equations in Besov spaces, *Comm. Math. Phys.*, 263(2005), pp. 803-831.

[23] J. Wu, Regularity criteria for the generalized MHD equations, *Comm. Partial Differential Equations*, 33(2008), pp. 285-306.

[24] J. Xiao, Homothetic variant of fractional Sobolev space with application to Navier-Stokes system, *Dyn. Partial Differ. Equ.*, 4(2007), pp. 227-245.

[25] J. Xiao, Homothetic variant of fractional Sobolev space with application to Navier-Stokes system revisited, *Dyn. Partial Differ. Equ.*, 11(2014), pp. 167-181.

[26] Q. Yang, Navier-Stokes equations, symmetric and uniform analytic solutions in phase space, *Dyn. Partial Differ. Equ.*, Vol. 17, No. 1 (2020), pp. 75-95.

[27] Q. Yang and P. Li, Regular wavelets, heat semigroup and application to the magneto-hydrodynamic equations with data in critical Triebel-Lizorkin type oscillation spaces, *Taiwanese J. Math.*, 20(2016), pp. 1335-1376.

[28] Q. Yang and H. Yang, $Y$ spaces and global smooth solution of fractional Navier-Stokes equations with initial value in the critical oscillation spaces, *J. Differential Equations*, 264 (7) 2018, pp. 4402-4424.

[29] Q. Yang, H. Yang and H. Wu, Ill-posedness of Navier-Stokes equations and critical Besov-Morrey spaces, 30 September, 2018, submit/2415346, 1706.08120

QIXIANG YANG, SCHOOL OF MATHEMATICS AND STATISTICS, WUHAN UNIVERSITY, WUHAN, 430072, P. R. CHINA

E-mail address: qxyang@whu.edu.cn