Ruled nodal surfaces of Laplace eigenfunctions and injectivity sets for the spherical mean Radon transform in $\mathbb{R}^3$.

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Abstract

It is proved that if a Paley-Wiener family of eigenfunctions of the Laplace operator in $\mathbb{R}^3$ vanishes on a real-analytically ruled two-dimensional surface $S \subset \mathbb{R}^3$ then $S$ is a union of cones, each of which is contained in a translate of the zero set of a nonzero harmonic homogeneous polynomial. If $S$ is an immersed $C^1$–manifold then $S$ is a Coxeter system of planes. Full description of common nodal sets of Laplace spectra of convexly supported distributions is given. In equivalent terms, the result describes ruled injectivity sets for the spherical mean transform and confirms, for the case of ruled surfaces in $\mathbb{R}^3$, a conjecture from [1].

1 Introduction

Nodal sets are zeros of the Laplace eigenfunctions. They play an important role in understanding of the wave propagation.

The geometry of a single nodal set can be very complicated and hardly can be well understood. On the other hand, simultaneous vanishing of large families of eigenfunctions on large sets occurs rarely and hence it is naturally to expect that common nodal sets in that case should be pretty special and have a simple geometry.
Bourgain and Rudnick [8] obtained a result of such type for two-dimensional torus $T^2$. They proved that only geodesics can serve common nodal curves for infinitely many Laplace eigenfunctions on $T^2$. For tori in high dimensions, they proved that Gauss-Kronecker curvature of the common nodal hypersurfaces must be zero. Analogous question for the sphere in the Euclidean space is still open.

In this article, we address to the similar questions for Euclidean spaces. The case of $\mathbb{R}^2$ was studied in [1], in equivalent terms of injectivity sets for the spherical mean Radon transform. Translated back to the language of nodal sets, the result of [1] says that one-dimensional parts of common nodal sets of large families of eigenfunctions are Coxeter system of straight lines in the plane. There, by large families of eigenfunctions, Laplace spectra of compactly supported functions were understood.

In the course of that result, it was conjectured in [1] that in higher dimensions, common nodal surfaces for large families of eigenfunctions (injectivity sets of the spherical mean transform) are cones - translates of the zero sets of solid harmonics (harmonic homogeneous polynomials). In this article, we confirm Conjecture from [1] for a special case of ruled surfaces in $\mathbb{R}^3$. The proof develops ideas from the article [4] of E.T. Quinto and the author.

Although ruled surfaces (unions of straight lines) are, in a sense, close to cones (union of straight lines with a common point), proving conical structure of ruled nodal surfaces in dimensions higher than two was elusive for a long time.

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3 Main results

We will formulate the main results of this article in two equivalent terms: 1) on the language of nodal surfaces and 2) on the language of injectivity sets.

We start with the nodal surfaces version.

3.1 Nodal surfaces version

Let $\varphi_\lambda, \lambda > 0$, be a family of eigenfunctions of the Laplace operator $\Delta$ in $\mathbb{R}^3$. More precisely, each function $\varphi_\lambda$ is a solution of the Helmholtz equation

$$\Delta \varphi_\lambda = -\lambda^2 \varphi_\lambda.$$ 

In particular, $\varphi_\lambda$ can be identically zero function.

**Definition 3.1** The family $\varphi_\lambda$ is a Paley-Wiener family if it can be extended in the complex plane $\lambda \in \mathbb{C}$ as an even entire function, satisfying the growth condition

$$|\varphi_\lambda(x)| \leq C(1 + |\lambda|)^N e^{(R+|x|) |\text{Im}\lambda|}.$$ 

for some positive constants $C, R$ and for some natural $N$.

By cone in $\mathbb{R}^d$, we understand union of straight lines having a common point—the vertex of the cone. We call a cone $C$ harmonic cone if there exists a nonzero harmonic homogeneous polynomial (solid harmonic) $h$ and a vector $a$ such that

$$C \subset a + h^{-1}(0).$$

**Definition 3.2** Let $S$ be a surface in $\mathbb{R}^3$. We call $S$ irreducible real analytically ruled surface if

1. There exists a closed continuous curve $\gamma \subset \mathbb{R}^3$ such that $S$ is the union of straight lines, $S = \cup_{a \in \gamma} L_a$, passing through points $a \in \gamma$.

2. Locally, $S$ is the image of the (parametrizing) mapping

$$(-1,1) \times \mathbb{R} \ni (t,\lambda) \to u(t,\lambda) = u(t) + \lambda e(t),$$

where $I \ni t \to (u(t),e(t)) \in \mathbb{R}^3 \times \mathbb{R}^3$ are real analytic maps and $|e(t)| = 1.$
The curve $\gamma$ is called the base curve, the vector $e(t)$-directional vector, the straight lines $L_t = \{u(t) + \lambda e(t), \lambda \in \mathbb{R}\}$ are called rulings, or ruling or generating lines. Real analytically ruled surface are, by definition, finite unions of irreducible those.

**Remark 3.3**

1. The parametrizing mapping $u(t, \lambda)$ is nor necessarily defines a parametrization of $S$ as a manifold, since the regularity condition is not required.

2. Real analytically ruled surfaces are not necessarily everywhere real analytic, and even differentiable. For example, a cone which is not a plane is not differentiable at its vertex.

Now we are ready to formulate the main results of this article.

**Theorem 3.4** Let $S$ be an irreducible real-analytically ruled surface with no parallel generating lines, then $S$ is the common nodal set for a Paley-Wiener family if and only if $S$ is a harmonic cone.

In the reducible case, we have

**Theorem 3.5** Let $S$ be a real-analytically ruled surface in $\mathbb{R}^3$, with no parallel generating lines. If $S$ is the common nodal for a Paley-Wiener family of eigenfunctions then $S$ is the union of finite number of harmonic cones, $S = \bigcup_{j=1}^N C_j$ such that for any $1 \leq i < j \leq N$ the intersection $C_i \cap C_j \neq \emptyset$ and one the two cases are possible:

1. $C_i \cap C_j$ is the vertex of one of the cones $C_i, C_j$,

2. $C_i \cap C_j$ is transversal and is an unbounded curve.

Conjecture from [1] (see section 4 for the details) claims that, in fact, the vertices of the cones $C_i$ coincide and therefore $S$ is a single cone. However, we are not able to prove that at the moment.

**Definition 3.6** The union $\Sigma = \bigcup_{j=1}^N \Pi_j$ of $N$ hyperplanes in $\mathbb{R}^d$ having a common point is called Coxeter system if $\Sigma$ is invariant with respect to all the reflections around the planes $\Pi_j$, $j = 1, ..., N$.

Notice that Coxeter systems are harmonic cones, i.e., are, up to translations, zero sets of solid harmonics.
**Theorem 3.7** If in Theorem 3.5 $S$ is an immersed $C^1$–surface then $S$ is a Coxeter system.

Remind that immersed $C^1$–surface is the image of a two-dimensional $C^1$–manifold under a $C^1$–mapping with non-degenerating differential.

Finally, we will formulate one more result about common nodal surfaces for special Paley-Wiener families of eigenfunctions: spectral projections of convexly supported distributions:

**Theorem 3.8** Let $f \in D'_{\text{comp}}(\mathbb{R}^3)$ be a nonzero compactly supported distribution or continuous function and

$$f = \int_0^{\infty} \varphi_\lambda d\lambda$$

be the Laplace spectral decomposition of $f$ ([27]). Assume that the boundary of the unbounded connected component of $\mathbb{R}^3 \setminus \text{supp} f$ is a real analytic strictly convex closed surface. If

$$N = \bigcap_{\lambda > 0} \varphi_{\lambda}^{-1}(0)$$

then $N = S \cup V$ where either $V = \emptyset$ or $V$ is an algebraic variety of $\dim V \leq 1$ and either $S = \emptyset$ or $S$ is one of the three surfaces:

1. $S$ is a harmonic cone.

2. $S$ is the union of two harmonic cones, $S = C_1 \cup C_2$ such that either $C_1 \cap C_2 = \{b_1\}$ or $C_1 \cap C_2 = \{b_2\}$, where $b_1, b_2$ are the vertices of the corresponding cones.

3. $S$ is the union of three harmonic cones, $S = C_1 \cup C_2 \cup C_3$, with the vertices $b_1, b_2, b_3$, correspondingly, such that either

$$C_1 \cap C_2 = \{b_1\}, \quad C_2 \cap C_3 = \{b_2\}, \quad C_3 \cap C_1 = \{b_3\}$$

or

$$C_1 \cap C_2 = \{b_2\}, \quad C_2 \cap C_3 = \{b_3\}, \quad C_3 \cap C_1 = \{b_1\}.$$

We conjecture that, in fact, $b_1 = b_2 = b_3$ which would lead to confirming Conjecture 4.2 in a more complete form.
3.2 Injectivity sets version

The spherical mean Radon transform is defined as the mean value

\[ R_f(x,t) = \int_{|\theta|=1} f(x + t\theta) dA(\theta) \]

of \( f \) over the sphere \( S(x,t) \) centered at \( x \in \mathbb{R}^d \) of radius \( t > 0 \). Here \( dA \) is the normalized area measure on the unit sphere \( \{|\theta| = 1\} \) in \( \mathbb{R}^d \).

The operator \( R \) can be extended to distributions \( f \in D'(\mathbb{R}^d) \). Namely, for each vector \( a \in \mathbb{R}^d \) define the averaging operator

\[ R_a\psi(x) := \int_{SO(n)} \psi(a + \omega(x - a)) d\omega, \]

where \( d\omega \) is the normalized Haar measure on the orthogonal group \( SO(n) \). The relation between this averaging operator and the operator \( R \) is given by

\[ (R_a\psi)(x) = R\psi(a,|x - a|). \]

Now, if \( f \in D'(\mathbb{R}^d) \) and \( a \in \mathbb{R}^d \), then we define the new distribution \( R_a f \) by the following action on test-functions \( \psi \):

\[ \langle R_a f, \psi \rangle = \langle f, R_a \psi \rangle. \]

It is easy to see that this definition is consistent with the definition of the action of the operator \( R_a \) on functions.

Denote \( R_S \) the restriction of the transform \( R \) on the set \( S \times (0,\infty) \):

\[ R_S : C_{\text{comp}}(\mathbb{R}^d) \ni f \rightarrow Rf|_{S \times (0,\infty)}. \]

**Definition 3.9** We call a set \( S \subset \mathbb{R}^d \) injectivity set if given a distribution \( f \in D'_{\text{comp}}(\mathbb{R}^d) \) such that \( R_a f = 0 \) for all \( a \in S \) then \( f = 0 \). Equivalently, \( S \) is injectivity set if the operator \( R_S \) is injective, i.e. for every function \( f \in C_{\text{comp}}(\mathbb{R}^d) \)

\[ Rf(x,t) = 0 \text{ for all } x \in S \text{ implies } f = 0. \]

The equivalence of definition for functions and distributions can be easily proved by convolving distributions with radial smooth functions.
Spherical mean Radon transform\(^1\) plays an important role in applications, namely, in thermo- and photoacoustic tomography (cf. \cite{16}), which is used in the medical imaging \cite{15}. The mathematical problem behind this is to recover \(f\) from the data \(Rf(x,t)\), \(x \in S\), \(t > 0\). The uniqueness of the recovery is equivalent to the injectivity of the operator \(R_S\) and therefore the first question to be answered is to understand for what observation surfaces \(S\) the operator \(R_S\) is injective, i.e., to understand the injectivity sets. Of course, the case \(d = 3\) is most important from the point of view of the applications.

**Definition 3.10** Let \(\{\varphi_\lambda\}_{\lambda > 0}\) be a measurable family of Laplace eigenfunction: \((\Delta + \lambda^2)\varphi_\lambda = 0\) in \(\mathbb{R}^d\). We will call the function

\[
\varphi(x) = \int_0^\infty \varphi_\lambda(x)\lambda^{d-1}d\lambda
\]

(2)

**generating function**, assuming that the integral converges (which can be achieved by a proper normalization \(\varphi_\lambda \rightarrow c(\lambda)\varphi_\lambda\)). The family \(\varphi_\lambda\) is called a Laplace spectral decomposition of \(f\).

The definition can be extended to distributions \(f \in D'(\mathbb{R}^d)\) if to understand the spectral decomposition of \(f\) in the distributional sense.

The weight factor \(\lambda^{d-1}\) is added for convenience and can be omitted by including it to \(\varphi_\lambda\).

The link between common nodal sets and injectivity sets in the question is very simple: they just coincide (see Proposition \[6.1\]).

Let us briefly explain this relation. It is proved in (\cite{21}, Theorem 3.10) that

the family \(\varphi_\lambda\) of eigenfunctions in \(\mathbb{R}^d\) is Paley-Wiener if (and if and only if, when \(d\) is odd) the integral (2) defines a compactly supported distribution \(f \in D'(\mathbb{R}^d)\).

The spectral decomposition \(\{\varphi_\lambda\}\) can be recovered from the generating distribution \(f\) by means of the convolutions

\[
\lambda^{d-1}\varphi_\lambda = j^{\frac{d-1}{2}} * f
\]

(3)

\(^1\)We refer to Radon transform because the operator \(R\) is defined on complexes of spheres with restricted centers and of arbitrary radii. Such varieties are analogous to varieties of planes with restricted set of normal vectors and arbitrary distances to the origin which are natural in the study of the plane Radon transform.
of \( f \) with the normalized Bessel function
\[
j_{\frac{d-2}{2}}(x) = (2\pi)^{-\frac{d}{2}} \frac{J_{\frac{d-2}{2}}(|\lambda x|)}{(|\lambda x|)^{\frac{d-2}{2}}}.
\]

It follows that \( S \subset \cap_{\lambda > 0} \mathcal{S}_\lambda^{-1}(0) = 0 \) if and only if \( R_f|_{S \times (0, \infty)} = 0 \).

Remind that the condition \( R_f|_{S \times (0, \infty)} = 0 \) for \( f \in D'(\mathbb{R}^d) \) means that the average distribution \( R_a f \), defined in (1), is the zero distribution: \( R_a f = 0 \) for all \( a \in S \).

Thus, we have

**Proposition 3.11** A set \( S \subset \mathbb{R}^d \) serves a common nodal set for a nontrivial family \( \{\varphi_\lambda\} \) if and only if \( R_f|_{S \times (0, \infty)} = 0 \) for some nonzero compactly supported distribution (or continuous function) \( f \), i.e., if and only if \( S \) fails to be a set of injectivity for the spherical mean Radon transform \( R \).

Using that equivalence, we can reformulate Theorems 3.4 and 3.5 in the equivalent form:

**Theorem 3.12** Let \( S \) be a real-analytically ruled surface in \( \mathbb{R}^3 \). If \( S \) fails to be an injectivity set then \( S \) is one of the surfaces enlisted in Theorem 3.5. If \( S \) is irreducible (see Definition 3.2) then \( S \) fails to be an injectivity set if and only if \( S \) is a harmonic cone.

The following theorem is a translation, on the injectivity sets language, of Theorem 3.8. It follows from Theorem 3.5 and [1], [7] and is an equivalent version of Theorem 3.8. Here the condition refers to the geometric shape of the support of the generating distribution.

**Theorem 3.13** Let \( f \in D'_{\text{comp}}(\mathbb{R}^3) \) be nonzero compactly supported distribution or continuous function. Assume that the boundary of the unbounded connected component of \( \mathbb{R}^3 \setminus \text{supp} f \) is a real analytic strictly convex closed surface. If \( R_f(x, t) = 0 \) for all \( x \in S \) and \( t > 0 \) then \( S \) is one of the surfaces enlisted in Theorem 3.5.

The proof of Theorems 3.8 and 3.13 is based on Theorem 3.5 and the results of [1],[7] (Theorem 4.6 the next section) about ruled structure of observation surfaces for convexly supported functions.
4 Background

In dimension $d = 2$, the problem of describing injectivity sets was completely solved in [AQ1]. Let us formulate the result. Denote

$$\Sigma_N = (t \cos k \frac{\pi}{N}, t \sin k \frac{\pi}{N}), \quad k = 0, 1, ..., N - 1, \quad -\infty < t < \infty,$$

the (Coxeter) system of $N$ straight lines passing through the origin and having equal angles between the adjacent lines.

**Theorem 4.1** [1] A set $S \subset \mathbb{R}^2$ is a set of injectivity if and only if $S$ is contained in no set of the form $(a + \omega(\Sigma_N)) \cup V$, where $a \in \mathbb{R}^2$, $\omega$ is a rotation in the plane and $V$ is a finite set, invariant under reflections around the lines from the Coxeter system $a + \omega(\Sigma_N)$.

Observe that the Coxeter system $\omega(\Sigma_N)$ coincides with the zero set of the polynomial $h(x, y) = \text{Im}(e^{i\varphi}(x + iy)^N)$, where $\omega$ is the rotation for the angle $\varphi$. The polynomial $h(x, y)$ represents the general form of harmonic homogeneous polynomial in the plane. That observation gives rise to the following conjecture about how injectivity sets look like in arbitrary dimension.

**Conjecture 4.2** [1] Let $S \subset \mathbb{R}^d$ fail to be an injectivity set. Then $S \subset (a + h^{-1}(0)) \cup V$, where $h$ is a harmonic homogeneous polynomial (spatial harmonic) and $V$ is an algebraic variety in $\mathbb{R}^d$ of dimension $\dim V \leq d - 2$.

Since in odd dimensions, as it was mentioned in subsection 3.2, non-injectivity sets are precisely common nodal sets of Paley-Wiener families, Conjecture 4.2 can be reformulated as following:

**Conjecture 4.3** A set $S \subset \mathbb{R}^d$, $d$ is odd, is a common nodal set for a Paley-Wiener family of Laplace eigenfunctions if and only if $S \subset (a + h^{-1}(0)) \cup V$, where the vector $a$, the variety $V$ and the polynomial $h$ are as in Conjecture 4.2.

**Remark 4.4** A partial case of non-injectivity sets in Conjecture are Coxeter systems of hyperplanes. They are arrangements of $N$ hyperplanes with a common point, invariant under reflections around each the hyperplane from the system. The Coxeter systems correspond to the case of completely reducible harmonic homogeneous polynomials $h$, i.e., those represented as products

$$h = l_1 \cdots l_N$$

of $N = \text{deg} h$ linear forms.
Here are some evidences for Conjecture 4.2 (see [5]):

- Any harmonic cone is a non-injectivity set, i.e., if \( h \) is a non-zero harmonic homogeneous polynomial, then \( S := h^{-1}(0) \) is a non-injectivity set. Namely, define \( f(x) := \alpha(r) h(\theta) \) where \( r, \theta \) are the spherical coordinates: \( x = r\theta, |\theta| = 1 \) and \( \alpha(r) \) is a non-zero smooth even compactly supported function on \( \mathbb{R} \). It is an easy exercise to prove that \( Rf(x,t) = 0 \) for all \( x \in S, t > 0 \).

- If \( V \) is an algebraic variety of \( \text{dim} V \leq d - 2 \) then there exists a nonzero \( f \in C_{\text{comp}}(\mathbb{R}^d) \) such that \( Rf(x,t) = 0 \) for all \( (x,t) \in V \times (0,\infty) \). ([5], Theorem 3.2).

So far, only partial results towards Conjecture 4.2 are obtained [4, 7, 2].

It was proved in [2] that among cones only zero sets of spatial harmonics fail to be injectivity sets. Therefore, the main difficulty in proving Conjecture 4.2 is checking that non-injectivity sets are necessarily cones.

The following two results can be considered as certain steps in that direction:

**Theorem 4.5 ([3])** Let \( f \) be a compactly supported continuous function or distribution in \( \mathbb{R}^d \). Assume that \( \text{supp}f \) is the union of disjoint balls or \( \text{supp}f \) is finite. If \( S \subset \mathbb{R}^d \) and \( R_S f = 0 \) then \( S \subset (a + h^{-1}(0)) \cup V \), where \( a \in \mathbb{R}^d \), \( h \) is a nonzero harmonic homogeneous polynomial and \( V \) is an algebraic variety of \( \text{dim} V \leq d - 2 \).

The next result deals with functions with convex compact supports and can be viewed as a motivation for Theorems 3.8 and 3.13.

**Theorem 4.6 ([7], [4]).** Let \( f \in C_{\text{comp}}(\mathbb{R}^d) \) be a compactly supported function. Suppose that the outer boundary \( \Gamma = \partial \text{supp}f \) is convex. If \( S \subset \mathbb{R}^d \) is such that \( Rf|_{S \times (0,\infty)} = 0 \) then \( S \) is ruled, i.e., \( S \) is the union of straight lines. Moreover, the ruling lines intersect \( \Gamma \) orthogonally at each point where \( S \) is differentiable.

By outer boundary \( \partial \text{supp}f \) we understand the boundary \( \partial (\mathbb{R}^d \setminus \text{supp}f)_\infty \) of the unbounded connected component of the complement.
Remark 4.7 In fact, the ruled structure of $S$ was established in [7] under much milder conditions for $\Gamma$ for example, under assumption of $C^2$ smoothness of $\Gamma$. However, in the proofs of Theorems 3.8 and 3.13 we will use the weaker version, Theorem 4.6 because some additional properties delivered by the convexity of support will be exploited.

5 The strategy of the proof of the main result

The main result of this article is Theorem 3.4. Theorem 3.5 is deduced from Theorem 3.4. Theorems 3.7, 3.8 follow from Theorems 3.4 and 3.5. All the theorems can be viewed as results towards proving Conjecture 4.2-4.3.

The proof of Theorem 3.4 falls apart into several steps:

**Step 1.** First, we prove that the common nodal surface $S$ for Paley-Wiener family is algebraic and lies in the zero set of a nontrivial harmonic polynomial. In a different setting, that fact was first observed in [18] (see also [4]).

**Step 2.** Next, we formulate local symmetry property, which is based on the results of [1], [22] about cancelation of analytic wave front sets. The corollary of that property says is that any surface $S$ having a pair of antipodal points-points of smoothness, such that the segment joining them is orthogonal to the surface, fails to be a common nodal surface for a Paley-Wiener family.

**Step 3.** Assuming that $S$ is not a cone and using compactness argument we find two generating (ruling) straight lines on $S$ with the maximal distance between them. Then we pick two closest points $a, b \in S$ on those extremal lines. If those extremal points $a, b$ are regular then the previous step implies that $S$ cannot be nodal. Otherwise, one of the extremal points is singular and we encounter the problem of characterization of singularities of algebraic real analytically ruled surfaces in $\mathbb{R}^3$.

**Step 4.** We obtain the required characterization of the singularities (Theorem 9.1), which is a key ingredient of the proof of the main result.

**Step 5.** The final arguments are as follows. Theorem 9.1 claims that singular points are conical or of cuspidal type. The corollary 9.4 is that either the irreducible ruled algebraic surface $S$ is a cone or it is a uniqueness set of harmonic polynomials. However, the latter option is ruled out (Step 1). Therefore, we conclude that $S$ is a cone (in the irreducible case) or a union of cones (in the reducible case). Finally, the proof that the cones are
harmonic easily follows by homogenization of harmonic polynomial vanishing on $S$ (obtained on Step 1). This completes the proof.

**Remark 5.1** Essentially, steps 1-3 were presented in the [4]. It was proved there that if the extremal points (Step 3) are regular then the surface is an injectivity set (not nodal). The description of singular points obtained in Theorem 9.1 allowed us to further develop the idea of [4] and push forward proving the conical structure of the nodal ruled surfaces, which is the main result of this article.

6 Preliminary observations

In this section, we briefly present auxiliary facts that we will need in the sequel. Most of them are exposed in [1]. It will be convenient to combine those facts in one proposition:

**Proposition 6.1** Let $\Phi = \{\varphi_\lambda, \lambda > 0,\}$ be a nonzero family of Laplace eigenfunctions in $\mathbb{R}^d$ with compactly supported generating distribution $f \in D'(\mathbb{R}^d)$ i.e.,

$$f = \int_0^\infty \varphi_\lambda d\lambda.$$  

We omit the factor $\lambda^{d-1}$ by including it into $\varphi_\lambda$. Clearly, this does not effect on the zero sets of $\varphi_\lambda$.

Denote

$$N_f = \{x \in \mathbb{R}^d : Rf(x,t) = 0, \forall (x,t) \in S \times (0,\infty)\}$$

and

$$N(\Phi) = \cap_{\lambda > 0} \varphi_\lambda^{-1}(0).$$

Then

1. $N_f = N(\Phi)$ and therefore common nodal sets and non-injectivity sets are the same.

2. The set $N(\Phi)$ is algebraic and has the form

$$N(\Phi) = S \cup V,$$

where $S = \emptyset$ or $S$ is a real algebraic hypersurface: $S = Q^{-1}(0)$, where $Q$ is a nonzero real polynomial, and $V$ is an algebraic variety of $\dim V \leq d - 2$ (maybe, empty as well).
3. There is a nonzero real harmonic polynomial \( H \) vanishing on \( S \), i.e. \( S \subset H^{-1}(0) \).

**Proof**

1. We have \( f = \int_0^\infty \varphi_\lambda d\lambda \), where the equality is understood in the distributional sense.

Then for any test-function \( \psi \) and for any \( a \in \mathbb{R}^d \):

\[
\langle R_a f, \psi \rangle = \langle f, R_a \psi \rangle = \int_\mathbb{R} \langle \varphi_\lambda(x), R_a \psi \rangle d\lambda.
\]

Further,

\[
\langle \varphi_\lambda, R_a \psi \rangle = \int_{\mathbb{R}^d} \int_{SO(n)} \varphi_\lambda(x) \psi(a + \omega(x - a)) dx d\omega.
\]

Change of variables \( y = a + \omega(x - a) \) yields

\[
\langle \varphi_\lambda(x), R_a \psi \rangle = \langle R_a \varphi_\lambda, \psi \rangle.
\]

The Laplace eigenfunctions are also eigenfunctions of the averaging operator \( R_a \):

\[
(R_a \varphi_\lambda)(x) = j_{\frac{d-2}{2}}(\lambda |a|) \varphi_\lambda(x).
\]

Therefore, we have

\[
\langle R_a f, \psi \rangle = \langle \int_0^\infty j_{\frac{d-2}{2}}(\lambda |x - a|) \varphi_\lambda(a) d\lambda, \psi(x) \rangle.
\]

We see that \( R_a f = 0 \) if and only if

\[
\int_0^\infty j_{\frac{d-2}{2}}(\lambda |x - a|) \varphi_\lambda(a) d\lambda = 0
\]

for all \( x \). The latter integral equation is satisfied if and only if \( \varphi_\lambda(a) = 0 \). Thus, \( N_f = N(\Phi) \) and the statement 1 is proved.

2. Decompose the (even) normalized Bessel function \( j_{\frac{d-2}{2}}(\lambda t) \) into power series:
Then we have from (3):

$$\varphi_\lambda(x) = \sum_{k=0}^{\infty} c_k \lambda^{2k} |x|^{2k} f.$$  

Therefore, $x \in N(\Phi)$ if and only if $|x|^{2k} f = 0$, $k = 0, 1, ...$

Notice that

$$Q_k(x) = c_k |x|^{2k} f = c_k < |x - y|^{2k}, f >,$$

where the right hand side stands for the action of the distribution $f$ with respect to $y$. It follows that $Q_k$ is a polynomial, of $\deg Q_k \leq 2k$.

From (3) $\varphi_\lambda(x) = 0$ is equivalent to $Q_k(0) = 0$, $k = 0, 1, ...$ and hence

$$N(\Phi) = \bigcap_{k=0}^{\infty} Q_k^{-1}(0).$$

Denote $Q$ the greatest common divisor (over $\mathbb{C}$) of $Q_k$. Then

$$N(\Phi) = (Q^{-1}(0) \cap \mathbb{R}^d) \cup V,$$

where $V$ is the intersection of $\mathbb{R}^d$ with the zero varieties of coprime polynomials and hence $\dim V < d - 1$.

To complete the proof of the statement 2, we have to show that the polynomial $Q$ has real coefficients. We will do that at the end of the proof.

3. Substitution (3) into Helmholtz equation:

$$\Delta \sum_{k=0}^{\infty} \lambda^{2k} Q_k = -\lambda^2 \sum_{k=0}^{\infty} \lambda^{2k} Q_k$$

yields

$$\Delta Q_k = -Q_{k-1}.$$
Not all polynomials $Q_k$ are identically zero. Indeed, suppose that $Q_k = c_k |x|^{2k} \ast f \equiv 0$ for all $k = 0, 1, \ldots$. Since $f$ has compact support and the linear combinations of the polynomials $|y|^{2k}$ approximate, in the $C^\infty$ topology on compact sets, any radial smooth function $\alpha(|y|^2)$, we have $\alpha \ast f \equiv 0$. Taking Fourier transform, we obtain $\hat{\alpha} \hat{f} \equiv 0$ which implies $\hat{f} = 0$ due to the arbitrariness of the radial function $\alpha$. Then $f = 0$ which is not true.

Let $k = k_0$ be the minimal $k$ such that $Q_k \neq 0$ and denote

$$H = Q_{k_0}.$$ 

Then

$$\Delta H = -Q_{k_0-1} = 0$$

and hence $H$ is harmonic. This proves the statement 3.

It remains to prove that, in fact, $Q$ is a real polynomial, i.e. has the real coefficients. To this end, we first will prove the third statement.

Let

$$H = H_1 \cdots H_q$$

be the decomposition into irreducible, over $\mathbb{C}$, polynomials. Let us prove that all polynomials $H_i$ are real.

Consider the operation of complex conjugation of coefficients:

$$H^*(z) = \overline{H(z)}, \ z \in \mathbb{C}^d.$$ 

Since $H = Q_{k_0}$ has real coefficients, we have

$$H^* = H_1^* \cdots H_q^* = H_1 \cdots H_q.$$ 

Therefore, each $H_i^*$ coincides with some $H_j$. If for some $i \neq j$ holds $H_i^* = H_j$ then $H$ is divisible by $H_iH_i^*$ and represents as

$$H = H_iH_i^*R,$$

for some polynomial $R$. Since in the real space $\mathbb{R}^d$ we have $H_i^* = \overline{H_i}$, we have in $\mathbb{R}^d$:

$$H = |H_i|^2 R.$$ 

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However, Brelot-Choquet theorem [9] says that no non-negative real polynomial can divide a real nonzero harmonic polynomial. Therefore, the only possibility is that $H_i = H_i^*$ for all $i$. That means that $H_i$ are real polynomials.

The greatest common divisor $Q$ divides $H$ and therefore is a product of some $H_i$. Since every polynomial $H_i$ has real coefficients, $Q$ does so.

If $Q$ is constant, i.e., all $Q_k$ are coprime, then $S = Q^{-1}(0) = \emptyset$. Otherwise, $S$ is a hypersurface in $\mathbb{R}^d$. Indeed, if $\dim S < n-1$ then $\mathbb{R}^d \setminus Q^{-1}(0)$ is connected and hence everywhere $Q \geq 0$ everywhere or $Q \leq 0$. However, this impossible, since Brelot-Choquet theorem states that preserving sign polynomials cannot divide harmonic polynomials. This completes the proof of Proposition.

7 Local symmetry and antipodal points

Definition 7.1 Let $S \subset \mathbb{R}^d$ and let $a, b \in S$, $a \neq b$, be two distinct points in $S$. We call $a$ and $b$ antipodal points if

1. $S$ is a $C^1$-hypersurface near the points $a, b$.
2. $a - b \perp T_a(S)$, $a - b \perp T_b(S)$, where $T_a(S), T_b(S)$ are tangent spaces to $S$ at $a$ and $b$ correspondingly.

Theorem 7.2 ([1], [3]). If $S \subset \mathbb{R}^d$ has a pair of antipodal points $a, b$ and $S$ is real analytic in neighborhoods of those points, then $S$ is an injectivity set.

Example The hyperboloid $x_1^2 + x_2^2 - x_3^2 = 1$ in $\mathbb{R}^3$ has antipodal points, for example, $(\pm 1, 0, 0)$ and hence is an injectivity set.

The proof of Theorem 7.2 is based on the following theorem about certain symmetry of the support of functions with zero spherical means on a surface:

Theorem 7.3 ([1]). Let $S$ be a real analytic hypersurface and $a \in S$. Let $f \in C_{comp}(\mathbb{R}^d)$ be a compactly supported function such that $Rf|_{S \times (0,\infty)} = 0$. Let $x \in \text{supp} f$ be a point of local extremum for the distance function $d(x) := |x - a|$ and denote

$$x^* = x - 2(x - a, \nu_a)\nu_a$$

($\nu_a$ is the unit normal vector of $S$ at $a$), the point, symmetric to $x$ with respect to the tangent plane $T_a(S)$ (mirror point). Then $x^* \in \text{supp} f$. 17
The proof of Theorem 7.3 uses microlocal analysis and results about cancellation of analytic wave front sets at mirror points ([1], [13], [12], [22]).

We are going to exploit Theorem 7.3 for algebraic surfaces $S = Q^{-1}(0)$, where $Q$ is a real nonconstant polynomial. However, Theorem 7.3 cannot be applied directly as $S$ is not necessarily everywhere real analytic and, moreover, even differentiable. Nevertheless, $S$ is real analytic everywhere outside of the critical set

$$critS := \{x \in S : \nabla Q(x) = 0\},$$

which is a nowhere dense subset of $S$. It is enough to establish a local symmetry property, though in a slightly weaker form than in Theorem 7.3.

Let us introduce some notations and definitions. Given a point $a \in S$ in a neighborhood of which $S$ is $C^1$ surface we denote

$$\sigma_a : x \rightarrow x - 2(x - a, \nu_a)\nu_a,$$

the reflection of $\mathbb{R}^d$ around the tangent plane $T_a(S)$. Here $\nu_a$, as above, is the unit normal vector to $S$.

For any $a \in S$ and $r > 0$ denote

$$K_{a,r} := \{x \in \text{suppf} : |x - a| = r\},$$

the intersection of $\text{suppf}$ with the sphere $S_r(a) = \{|x - a| = r\}$.

**Theorem 7.4** *(Local Symmetry Property)* Let $S \subset \mathbb{R}^d$ be a hypersurface, real analytic except for a nowhere dense subset. Let $f \in C_{\text{comp}}(\mathbb{R}^d)$ be such that $Rf|_{S \times (0, \infty)} = 0$. Let $a \in S$ be a $C^1$ point. Define

$$r = \max\{|x - a| : x \in \text{suppf}\}.$$

Then

$$\sigma_a(K_{a,r}) \cap \text{suppf} \neq \emptyset.$$

**Proof** is based on compactness arguments.

Denote for simplicity $K = K_{a,r}$, $K^* = \sigma_a(K_{a,r})$. If $E \subset S$ is the set where $S$ is not real analytic, the the point $a$ is a limit point of $S \setminus E$ and hence we can find a sequence $a_n \in S \setminus E$ such that

$$\lim_{n \to \infty} a_n = a.$$
The surface $S$ is real analytic at any point $a_n$ and the tangent planes

$$T_{a_n}(S) \rightarrow T_a(S), \ n \rightarrow \infty.$$ 

Denote

$$r_n = \max\{|a_n - x| : x \in \text{suppf}\}$$

and let $x_n \in \text{suppf}$ be such that

$$|a_n - x_n| = r_n.$$ 

By the construction, for all $x \in \text{suppf}$ holds

$$|a_n - x| \leq |a_n - x_n| = r_n.$$ 

By Theorem 7.3, the $T_{a_n}(S)$–symmetric point

$$x^*_n = \sigma_{a_n}(x_n) \in \text{suppf}.$$ 

Using compactness of $\text{suppf}$, choose a convergent subsequence

$$x_{n_k} \rightarrow x_0 \in \text{suppf}, \ k \rightarrow \infty.$$ 

Taking, if necessarily, a subsequence one more time, we can assume that also

$$r_{n_k} \rightarrow r_0.$$ 

Then, taking limits $a_n \rightarrow a, x_n \rightarrow x_0, r_n \rightarrow r_0$, we will have

$$|a - x_0| = r_0$$

and for any $x \in \text{suppf}$ :

$$|a - x| \leq r_0.$$ 

Those two inequalities show that

$$r_0 = r,$$

where $r$ is defined in the formulation, and

$$x_0 \in K = K_{a,r}.$$
Now,
\[ x_n^* = x_n - 2(x_n - a_n, \nu_{a_n})\nu_{a_n} \to x_0 - 2(x_0 - a, \nu_a)\nu_a = x_0^*, \]
as \( n \to \infty \). Since \( x_n^* \in \text{supp} f \) then \( x_0^* \in \text{supp} f \). Therefore \( K^* \cap \text{supp} f \neq \emptyset \). Theorem is proved.

Theorems 7.3 and 7.4 can be viewed as non-linear versions of the following global symmetry property, which follows from the uniqueness for Cauchy problem for the wave equation:

**Theorem 7.5** ([10], Ch.VI, 8.1) Let \( \Pi \) be a hyperplane in \( \mathbb{R}^d \) and \( f \in C(\mathbb{R}^d) \). Then \( Rf|_{\Pi \times (0,\infty)} = 0 \) if and only if \( f \) is odd with respect to reflections around \( \Pi \).

Obviously, \( \text{supp} f \) in Theorem 7.5 is \( \Pi \)-symmetric. Theorem 7.3 states that if the hyperplane \( \Pi \) is replaced by a hypersurface \( S \) then, still, certain symmetry of \( \text{supp} f \) holds, though in a much weaker (local) sense.

The proof of Theorem 7.2 is geometric and is given in [1]. We present it here to make the text of this article more self-sufficient.

**Proof of Theorem 7.2**

We will present an analytic exposition of the geometric proof given in [1].

We want to prove that if \( f \in C_{\text{comp}}(\mathbb{R}^d) \) and \( Rf(x,r) = 0 \) for all \( x \in S \) and \( r > 0 \) then \( f = 0 \) or, equivalently, \( \text{supp} f = \emptyset \). We assume that \( f \neq 0 \) and will arrive at a contradiction.

Since the tangent planes at \( a \) and \( b \) are parallel, the unit normal vectors \( \nu_a \) and \( \nu_b \) can be chosen equal
\[ \nu_a = \nu_b = \nu = \frac{b - a}{|b - a|}. \]

Denote as above
\[ \sigma_a(x) = x - 2(x - a, \nu)\nu = x - 2\frac{(x - a, b - a)}{|b - a|^2} (b - a). \]

the reflection around the tangent plane \( T_a(S) \) and let \( \sigma_b \) be the analogous reflection for the point \( b \).

Denote
\[ r_1 = \max\{|x - a| : x \in \text{supp} f\}. \]

Consider two cases:
1. \( r_1 < |a - b| \),

2. \( r_1 \geq |a - b| \).

In the first case, \( \text{supp} f \) lies on one side of \( T_b(S) : \)
\[
\langle x - b, \nu \rangle < 0, \ x \in \text{supp} f
\]
and therefore the entire \( T_b(S) \) - symmetric set \( \sigma_b(\text{supp} f) \) is disjoint from \( \text{supp} f \). This contradicts to Theorem 7.4.

Consider now the case \( r_1 \geq |a - b| \) and denote
\[
r_2 := \sqrt{r_1^2 - |a - b|^2}.
\]

We claim that \( \text{supp} f \subset \overline{B(b, r_2)} \), i.e. \( |x - b| \leq r_2 \) for all \( x \in \text{supp} f \). To prove that, consider
\[
r = \max\{|x - b| : x \in \text{supp} f\}.
\]
Then \( \text{supp} f \subset B(b, r) \) and it suffice to prove that \( r \leq r_2 \).

Suppose that \( r > r_2 \). Denote
\[
K = K_{b,r} = \text{supp} f \cap \{x \in \mathbb{R}^d : |x - b| = r\}.
\]

By Theorem 7.4, \( K^* = \sigma_b(K) \) meets \( \text{supp} f \). That means that there is \( x_0 \in K \) such that \( \sigma_b(x_0) \in K \), i.e.,
\[
x_0 \in \text{supp} f, \ |x_0 - b| = r \ and \ x_0^* = \sigma_b(x) \in \text{supp} f.
\]

Since \( x_0 \in \text{supp} f \) then by definition of \( r_1 : \)
\[
|x_0 - a| \leq r_1.
\]

Therefore,
\[
r_1^2 \geq |x_0 - a|^2 = \langle x_0 - b + (b - a), x_0 - b + (b - a) \rangle = |x_0 - b|^2 + |b - a|^2 + 2\langle x_0 - b, b - a \rangle.
\]

Taking into account that
\[
|x_0 - b| = r, \ |b - a|^2 = r_1^2 - r_2^2,
\]
we obtain the inequality
\[
\langle x_0 - b, b - a \rangle = r_2^2 - r^2 < 0.
\]

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But the same applies to the symmetric point $x^*_0 = \sigma_b(x_0)$ because $x^*_0$ meets the same conditions $x^*_0 \in supp f$ and $|x^*_0 - b| = |x_0 - b| = r$. Thus, also

$$\langle x^*_0 - b, b - a \rangle < 0.$$  

Substitution

$$x_0 = \sigma_b(x_0) = x_0 - 2 \frac{\langle x - b, b - a \rangle}{|b - a|^2} (b - a)$$

yields

$$-\langle x_0 - b, b - a \rangle < 0.$$  

The obtained contradictions shows that $r \leq r_2$ and hence

$$supp f \subset \overline{B(b, r)} \subset \overline{B(b, r_2)}.$$  

Then we repeat the argument, replacing $a$ by $b$ and $r_1$ by $r_2$, and obtain

$$supp f \subset \overline{B(a, r_3)},$$

where $r_3 = \sqrt{r_2^2 - |a - b|^2}$.  

Proceeding this way, we construct the sequence

$$r_{n+1} = \sqrt{r_n^2 - |a - b|^2}, \text{ i.e., } r_n = \sqrt{r_1^2 - (n - 1)|a - b|^2},$$

such that

$$supp f \subset \overline{B(a, r_{2k+1})}, \text{ } supp f \subset \overline{B(b, r_{2k})}.$$  

For large enough $n$ we will have $r_n < |b - a|$ which, as it explained above, is impossible. Therefore, the only possible conclusion is that $supp f = \emptyset$ and $f = 0$. Therefore, $S$ is an injectivity set.

### 8 Ruled surfaces

Let $S$ be a real analytically ruled surface in $\mathbb{R}^3$ (see Definition 3.2). In accordance with the definition, $S$ consists of straight lines, intersecting the fixed base curve $\gamma$.

More precisely, $S$ is locally the image of a map

$$(t, \lambda) \rightarrow u(t, \lambda) = u(t) + \lambda e(t),$$
where
\[ u(t) : I \to \mathbb{R}^3, \quad e(t) : I \to S^2, \quad I = (-1, 1), \]
are real analytic vector-functions.

We denote \( L_t \) the straight line
\[ L_t = \{ u(t) + \lambda e(t), \; \lambda \in \mathbb{R} \}. \]

**Lemma 8.1** The parametrizing mapping \( u(t) \) of the base curve \( \gamma \) can be chosen so that the tangent vector to the base curve and the directional vector are orthogonal:
\[ \langle u'(t), e(t) \rangle = 0, \; t \in (-1, 1). \] (4)

**Proof** For any function \( \lambda(t) \) we have
\[ u(t, \lambda) = u(t) + \lambda(t)e(t) + (\lambda - \lambda(t))e(t). \]

Then \( \mu = \lambda - \lambda(t) \) is a new parameter on the line \( u(t) + \mathbb{R}e(t) \) and therefore \( S \) is the image of the mapping \( \hat{u}(t, \mu) = \hat{u}(t) + \mu e(t), \) where \( \hat{u}(t) = u(t) + \lambda(t)e(t). \)

The function \( \lambda(t) \) is to be found from the condition
\[ \langle \hat{u}'(t), e(t) \rangle = \langle u'(t) + \lambda'(t)e(t) + \lambda(t)e'(t), e(t) \rangle = \langle u'(t), e(t) \rangle + \lambda'(t) = 0. \]

We have used here the that \( \langle e(t), e(t) \rangle = 1 \) and \( \langle e'(t), e(t) \rangle = 0. \) Therefore \( \lambda(t) \) can be taken
\[ \lambda(t) = - \int_{t_0}^{t} \langle u'(t), e(t) \rangle dt. \]

The condition of real analyticity preserves for \( u(t) + \lambda(t)e(t). \)

From now on, we assume that the parametrization \( u(t, \lambda) \) satisfies the orthogonality condition (4).

### 8.1 Regularity of the line foliation at smooth points

In this subsection, we will prove that the line foliation of \( S \) is regular at the points where the surface \( S \) is differentiable.

Notice that, in Definition 3.2, the parametrizing mapping \( u(t, \lambda) \) is not assumed necessarily regular, i.e. the condition nondegeneracy of the Jacobi matrix may be not fulfilled.
Definition 8.2 We call a point \( a \in S \) of a ruled surface \( S \subset \mathbb{R}^3 \) **regular with respect to a parametrization** \( I \times I : \exists (s, \sigma) \to w(s, \sigma), a = w(0, 0) \), where \( I = (-1, 1) \), if

1. The mappings \( \mathbb{R} \ni \sigma \to w(s, \sigma) \) parametrize the same line foliation of \( S \).

2. The mapping \( w(s, \sigma) \) is differentiable and regular at \( (0, 0) \), i.e., the partial derivatives \( \partial_s w(0, 0), \partial_\sigma w(0, 0) \) are linearly independent (and then span the tangent space \( T_a(S) \)).

We will call a just **regular point** of the given line foliation, if \( a \) is regular with respect to some parametrization \( w(s, \sigma) \).

Lemma 8.3 Let \( S_0 \) be a ruled surface with \( C^1 \) open base curve \( W \subset S_0 \), i.e., \( S_0 = \bigcup_{w \in W} L_w \), where \( L_w \) is a straight line passing through the point \( w \in W \). Suppose that \( L_w \perp W, w \in W \). If \( S \) is a \( C^1 \) near a point \( a \in W \) then \( a \) is a regular point of the the foliation \( \{ L_w, w \in W \} \).

**Proof** Let \( \Omega_a \) be the neighborhood of \( a \) where \( S \) is \( C^1 \).

\[
I \ni s \to w(s) \in W
\]

where \( I \) is an open interval, be a \( C^1 \) parametrization of the base curve \( W \), and \( \tau(w(s)) = w'(s) \) – the tangent vector to \( W \).

Let \( \nu(x), x \in \Omega_a \), be the unit normal \( C^1 \) vector field on \( \Omega_a \). The surface \( S \) is differentiable at \( a \), hence the normal unit vector \( \nu(a) \) is well defined, and \( \nu(x) \) is \( C^1 \) mapping on \( \Omega_a \).

Then the cross-product

\[
E(w) = \nu(w) \times \tau(w)
\]

is both orthogonal to \( W \) and tangent to \( S_0 \) and hence \( E(w) \) is the directional vector of the generating line \( L_w \). The vector field \( E(w), w \in W \) is \( C^1 \). Let

\[
I \ni s \to w(s) \in W
\]

where \( I \) is an open interval, be a \( C^1 \) parametrization of the base curve \( W \). Then the mapping

\[
I \times I \ni (s, \sigma) \to w(s, \sigma) = w(s) + \sigma E(s), \sigma \in \mathbb{R}^3,
\]

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where
\[ e(s) := e(w(s)), \]
parametrizes the given line foliation \( \{L_w\} \) and satisfies Definition 8.2 of regular point.

Indeed, \( w(s,\sigma) \) is differentiable at \((0,0)\), because \( w(s) \) and \( E(w(s)) \) are differentiable. The vectors
\[ \partial_s w(0,0) = \tau(w), \quad \partial_{\sigma} w(0,0) = E(0) \]
are nonzero and orthogonal to each other, hence the point \((0,0)\) is regular with respect to the parametrization \( w(s,\sigma) \) of the given foliation,

Lemma is proved.

9 The structure of real analytically ruled algebraic surfaces near singular points

In this section we study singular points of algebraic real-analytically ruled surfaces in \( \mathbb{R}^3 \). We did not find a relevant result in the literature, for, to our knowledge, singular points of ruled surfaces and caustics of normal fields (cf. [6]) are classified for either generic surfaces or in the case of stable singularities, while in our situation, the surface and a point are given and cannot be perturbed.

The following theorem, combined with Theorem 7.2, will be one of key points in the proof of the main result of this article:

**Theorem 9.1** Let \((-1,1) \ni t \to u(t) \in \mathbb{R}^3 \) and \((-1,1) \ni t \to e(t) \in S^2 \) be two real analytic mappings. Denote \( S \) the ruled surface \( S := \{u(t)+\lambda e(t), t \in (-1,1), \lambda \in \mathbb{R}\} \) and assume that \( S \) is algebraic. Then the following four cases are possible:

1. \( S \) is \( C^1 \)-manifold and the line foliation \( \{L_t\} \) is regular at any point \( a \in S \).
2. \( S \) is a plane.
3. \( S \) is a cone, i.e. all the lines \( L_t \) have a common point (vertex).
4. \( S \) has a cuspidal (double tangency) point \( a \in S \), which means the following: if \( H \) is a polynomial vanishing on \( S \) and \( H(x + a) = H_k(x) + H_{k+1} + \ldots + H_N(x) \), where \( H_j \) are homogeneous polynomials of degree \( j \) and \( H_k \neq 0 \), then \( H_k \) is divisible by a nonzero degenerate quadratic form \( Q(x) = (A_1x_1 + A_2x_2 + A_3x_3)^2 \).

**Remark 9.2** In both cases 1 and 2 \( S \) is a smooth manifold, but in case 2, when \( S \) is a plane, the given line foliation can be singular (have caustics). For example, all the lines \( L_t \) can pass through the same point, so that \( S \) belongs to case 3, or there can be caustics of more complicated forms. On the other hand, planes can be viewed also as a regular ruled surface (foliated into parallel lines) but this foliation can be different from the initial one.

**Example 9.3** It was proved in [11] that generic ruled surface in \( \mathbb{R}^3 \) is equivalent, near its singular point, to Whitney umbrella, which is the image \( S \) of the mapping

\[
(t, \lambda) \rightarrow (t^2, \lambda, \lambda t).
\]

Whitney umbrella is algebraic surface with the equation

\[
z^2 - yx^2 = 0.
\]

The origin \( a = (0, 0, 0) \) is the only singular point. Whitney umbrella is a typical ruled surface with cuspidal singular point, as defined in case 4 of Theorem 9.1. Indeed, any polynomial \( H \) vanishing on \( S \) is divisible by \( x_3^2 - x_1x_2^2 \). Then the minor homogeneous term \( H_k \) of \( H \) is divisible by \( x_3^2 \), i.e., the property 3 holds with \( Q(x_1, x_2, x_3) = x_3 \).

The important corollary of Theorem 9.1 is:

**Corollary 9.4** Let \( S \) be as in Theorem 9.1. Suppose that \( S \subset H^{-1}(0) \), where \( H \) is a nonzero harmonic polynomial. Then \( S \) is the surface of one of the first three cases in Theorem 9.1.

**Proof** Suppose that \( S \) is a surface of the four type, i.e, \( S \) has a cuspidal point \( a \in S \). Let \( H \) be a harmonic polynomial such that \( H|S = 0 \). Then the minor term \( H_k \) in the homogeneous decomposition

\[
H(x + a) = H_k(x) + \ldots H_M(x)
\]
is divisible by a nonzero quadratic polynomial $A^2(x)$ where $A(x) = A_1x_1 + A_2x_2 + A_3x_3$ is a nonzero linear form. Then

$$H_k(x) = 0, \nabla H_k(x) = 0,$$

whenever $A(x) = 0$.

Thus, $H_k$ satisfies on the plane $\Pi = \{A(x) = 0\}$ both the zero Dirichlet and Neumann conditions. Since $H_k$ is harmonic, this implies $H_k = 0$ identically. Therefore, the homogeneous decomposition of $H$ begins with $H_{k+1}$. The same argument yields $H_{k+1} = 0$. Proceeding this way, we obtain $H = 0$. This contradiction shows that case 4 is impossible.

### 9.1 Outline of the proof of Theorem 9.1

First of all, we will show that if $a$ is not a conical point of $S$ then by a suitable changing parameters $t$ (reparametrization) and $\lambda$ (rescaling), we can pass to a parametrization (11) of $S$ of the form

$$u(s, \sigma) = s^m v_m + \sigma s^m e_0 + D(s, \sigma) \tau,$$

where $v_m, e_0, \tau$ are nonzero pairwise orthogonal vectors and $D(s, \sigma)$ is a nonzero (if $S$ is not a plane) real analytic function.

Then we show that if $m$ is odd then $S$ is $C^1$–differentiable at $a$ and, even more, $a$ is a regular point of the line foliation on $S$ (Lemmas 9.11 and 8.3).

In the case of even $m$ we reduce the situation, by consequent descending the power $m$, to the case of even $m$ and $D$ not even function of $s$ (we assume that $D \neq 0$ identically since otherwise $S$ is a plane).

Then we prove in Lemma 9.10 that in this case the point $a$ is of cuspidal type, i.e., the fourth case of Theorem 9.1 takes place.

Thus, we conclude that if $S$ contains no cuspidal points then either $S$ is a plane or a cone, or the power $m$ associated with any point $a \in S$ is odd and therefore $S$ is everywhere $C^1$ differentiable and the line foliation is everywhere regular.

### 9.2 Preliminary constructions

Let $a$ be a singular point of the real analytically ruled surface $S$.

As it is showed in Lemma 8.1, we can choose the parametrization $u(t, \lambda) = u(t) + \lambda e(t)$ near $a$ so that $\langle u'(t), e(t) \rangle = 0$. Using translation we can always
move $a$ to the origin and assume that $a = 0$. We can also assume that the value of the parameter corresponding to the point $a$ is $t = 0$.

**Lemma 9.5** Let $a = u(0)+\lambda_0 e(0) = 0$ be a singular point of the ruled surface $S$. Then the parametrizing mapping $u(t, \lambda) = u(t) + \lambda t$ can be rewritten as $u(t, \mu) = u(t) + \mu e(t)$, where

$$v(t) = u(t) + \lambda_0 e(t), \quad \mu = \lambda - \lambda_0$$

and

1. $v'(0) = 0$.

2. If $v(t) = 0$ identically then $S$ is a cone with the vertex 0. Otherwise, $v(t)$ decomposes in a neighborhood of $t = 0$ into power series:

$$v(t) = v_m t^m + v_{m+1} t^{m+1} + \ldots, \quad v_m \neq 0,$$

where $m \geq 2$, $v_j$ are vectors in $\mathbb{R}^3$.

3. $\langle v_m, e(0) \rangle = 0$.

**Proof** Since $a$ is singular, the vectors

$$\frac{\partial u}{\partial t}(0, \lambda_0) = u'(0) + \lambda_0 e'(0), \quad \frac{\partial u}{\partial \lambda}(0, \lambda_0) = e(0)$$

are linearly dependent at $0, \lambda_0$:

$$c_1 (u'(0) + \lambda_0 e'(0)) + c_2 e(0) = 0,$$

for some $c_1, c_2 \in \mathbb{R}, c_1^2 + c_2^2 \neq 0$.

The unit vector $e(0)$ is orthogonal both to $u'(0)$ and $e'(0)$, therefore $c_2 = 0$ and

$$u'(0) + \lambda_0 e'(0)) = 0.$$

Now rewrite $u(t, \lambda)$ as

$$u(t, \lambda) = u(t) + \lambda_0 e(t) + (\lambda - \lambda_0)e(t),$$

and denote $\lambda - \lambda_0 = \mu$. Then we get the parametrization

$$u(t, \mu) = v(t) + \mu e(t), u(0, 0) = a,$$

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where \[ v(t) = u(t) + \lambda_0 e(t), \]

Then \[ v(0) = u(0) + \lambda_0 e(0) = 0, \quad v'(0) = 0. \]

The two cases are possible

1) \( v(t) \equiv 0. \)

Then \( u(t, \lambda_0) = u(t) + \lambda_0 e(t) = v(t) = 0, \) i.e., all the lines \( L_t \) pass through the origin and therefore \( S \) is a cone with the vertex 0.

2) \( v(t) \) is not identical zero.

Then by real analyticity:

\[ u(t, \mu) = v_m t^m + \ldots + \mu(e_0 + e_1 t + \ldots), \quad (6) \]

where \( v_m \neq 0. \) Since \( v'(0) = 0 \) then \( m \geq 2. \)

Also we have

\[ \langle v'(t), e(t) \rangle = \langle u'(t) + \lambda_0 e'(t), e(t) \rangle = 0. \]

Thus,

\[ \langle mv_m t^{m-1} + \ldots, e_0 + e_1 t + \ldots \rangle = 0 \]

and dividing by \( t^{m-1} \) and letting \( t \to 0 \) yields

\[ \langle v_m, e_0 \rangle = 0. \]

Lemma is proved.

On the next step, we will replace the parameters \( \mu, t \) by new parameters \( \sigma, s \) which are more convenient for further investigation. We start with re-scaling the parameter \( \mu \) on the ruling lines.

### 9.3 Re-scaling: changing the linear parameter \( \mu. \)

Thus, by Lemma 9.3, the surface \( S \) is parametrized, near \( a = 0, \) by the mapping \( u(t, \mu) = v(t) + \mu e(t), \) where

\[ v(t) = \sum_{j=m}^{\infty} v_j t^j, \quad e(t) = \sum_{j=0}^{\infty} e_j t^j \]

and \( \langle v_m, e(0) \rangle = 0. \)
Let $\tau$ be a unit vector orthogonal both to $v_m$ and $e_0$. Then the triple $v_m, e_0, \tau$

constitutes a basis in $\mathbb{R}^3$.

Decompose the vector-coefficients $v_m, v_{m+1}, ...$ and $e_0, e_1, ...$, into linear combinations of the basis vectors:

$$v_j = A_j v_m + B_j e_0 + C_j \tau, \quad j \geq m,$$

$$e_j = \hat{A}_j v_m + \hat{B}_j e_0 + \hat{C}_j \tau, \quad j \geq 0,$$

and since $v_m, e_0, \tau$ constitute the basis, one has

$$A_m = 1, B_m = 0, C_m = 0, \quad \hat{A}_0 = 0, \hat{B}_0 = 1, \hat{C}_0 = 0.$$

Substitute the expressions for $v_j, e_j$ into the power series for $v(t)$ and $e(t)$ leads to:

$$v(t) = A(t)v_m + B(t)e_0 + C(t)\tau,$$

$$e(t) = \hat{A}(t)v_m + \hat{B}(t)e_0 + \hat{C}(t)\tau,$$

where we have denoted

$$A(t) = \sum_{j=m}^{\infty} A_j t^j, \quad B(t) = \sum_{j=m+1}^{\infty} B_j t^j, \quad C(t) = \sum_{j=m+1}^{\infty} C_j t^j \quad (7)$$

and

$$\hat{A}(t) = \sum_{j=1}^{\infty} \hat{A}_j t^j, \quad \hat{B}(t) = \sum_{j=0}^{\infty} \hat{B}_j t^j, \quad \hat{C}(t) = \sum_{j=1}^{\infty} \hat{C}_j t^j. \quad (8)$$

Correspondingly, the parametrizing function $u(t, \mu) = v(t) + \mu e(t)$ takes the form

$$u(t, \mu) = (A(t) + \mu \hat{A}(t))v_m + (B(t) + \mu \hat{B}(t))e_0 + (C(t) + \mu \hat{C}(t))\tau. \quad (9)$$

Let us fix a real number $\sigma \in \mathbb{R}$ and write the functional equation

$$B(t) + \mu \hat{B}(t) = \sigma (A(t) + \mu \hat{A}(t)). \quad (10)$$

This equation defines the parameter $\mu$ as a function of $\sigma$ and $t$:

$$\mu = \mu(\sigma, t) = \frac{\sigma A - B}{B - \sigma A}.$$
Since from (7) \[ B(t) = B_{m+1} t^{m+1} + ...; \quad \hat{B}(t) = 1 + B_1 t + ... \]
and
\[ A(t) = t^m + A_{m+1} t^{m+1} + ...; \quad \hat{A}(t) = A_1 t + ... , \]
and \( m > 1 \), we obtain
\[
\mu = \sigma A - B = \sigma t^m + ... - B_{m+1} t^{m+1} + ... \\
(1 + \hat{B}_1 t + ...) - \sigma (\hat{A}_1 t + ...) 
\]
and hence
\[
\mu = \mu(t) = \sigma t^m + o(t^m). 
\]

Then the coefficient \( A(t) + \mu(t, \sigma) \hat{A}(t) \) in front of \( v_m \) in (9) is
\[
A(t) + \mu(t, \sigma) \hat{A}(t) = A_m t^m + ... + (A_m \sigma t^m + ...) (A_1 t + ...) = A_m t^m + o(t), \ t \to 0. 
\]

**Remark 9.6** The base curve \( \{ t \to v(t) \} \) of the foliation is given by the condition \( \mu = 0 \) which corresponds, due to (10), to
\[
\sigma = \frac{B(t)}{A(t)} = B_{m+1} t + o(t). 
\]

### 9.4 Re-parametrization: changing the parameter \( t \) of the base curve.

Now introduce the new parameter \( s \) by the relation
\[
s^m = A(t) + \mu \hat{A}(t) = t^m + o(t), \ t \to 0. 
\]
If \( m \) is odd, then the real parameter \( s = s(t) \) is well defined near \( t = 0 \). If \( m \) is even then \( s = s(t) \) near \( t = 0 \) is the real branch of \( (A(t) + \mu \hat{A}(t))^{1/m} \) for which
\[
s = s(t) = t + o(t). 
\]
Thus, that asymptotic holds for both odd and even \( m \).

From (9) and (10), one can rewrite, in a neighborhood of \( s = 0 \), the function \( u(t, \mu) \) as a function of the new parameters \( s, \sigma \):
\[
u(s, \sigma) = s^m v_m + \sigma s^m e_0 + D(s, \sigma) \tau, \tag{11} 
\]
where we have denoted

\[ D(s, \sigma) := C(t) + \mu \hat{C}(t). \]

Since \( s = t + o(t) \), we have from (7),(8):

\[ C(t) = C_{m+1} t^{m+1} + o(t^{m+1}) = C_{m+1} s^{m+1} + o(s^{m+1}), \]

\[ \hat{C}(t) = \hat{C}_1 t + o(t) = \hat{C}_1 s + o(s), \]

\[ \mu = \sigma s^m + o(s^m). \]

Then we have

\[ D(s, \sigma) = C(t) + \mu \hat{C}(t) = (C_{m+1} + \sigma \hat{C}_1)s^{m+1} + o(s^{m+1}). \tag{12} \]

**Lemma 9.7** If \( H \) is a polynomial vanishing on \( S \) and \( H = H_k + H_{k+1} + .. \) is its decomposition into homogeneous polynomials, then \( H_k(x) = 0 \) for all vectors \( x \in \text{span}\{v_m, e_0\} \).

**Proof** We have \( H(u(s, \sigma)) = 0 \) for all \( \sigma \in \mathbb{R} \) and \( s \) close to 0. From (12), \( D(s, \sigma) = o(s^m) \) and then formula (11) implies

\[ H(u(s, \sigma)) = H_k(s^m v_m + \sigma s^m e_0 + o(s^m)) + H_{k+1}(s^m v_m + \sigma s^m e_0 + o(s^m)) + ... = 0. \]

Since \( H_j \) are homogeneous of degree \( j \), dividing by \( s^m \) and letting \( s \to 0 \) yields:

\[ H_k(v_m + \sigma e_0) = 0. \]

Then

\[ H_k(\alpha v_m + \alpha \sigma e_0) = \alpha^k H_k(v_m + \sigma e_0) = 0 \]

for any \( \alpha \in \mathbb{R} \). Since \( \sigma \) is arbitrary, the real numbers \( \alpha, \alpha \sigma \) are arbitrary as well, and hence \( H \) vanishes on any linear combination of the vectors \( v_m \) and \( e_0 \). Lemma is proved.

**Lemma 9.8** If \( D(s, \sigma) = 0 \) identically then \( S \) locally is a plane (case 1 of Theorem 9.1).

**Proof** If \( D(s, \sigma) \equiv 0 \) then we have from (11) \( u(s, \sigma) = s^m v_m + \sigma e_0 \) and hence the image of \( u \) is contained in the plane spanned by the vectors \( v_m \) and \( e_0 \).
Lemma 9.9 Suppose that $D(s, \sigma)$ is not identically zero. Then a suitable change of the parameter $s$ leads to one of the following cases: hold for the power $m$ in (11):

1. The integer $m$ in (11) is odd.
2. $m$ is even but $D(s, \sigma)$ is not an even function with respect to $s$.

Proof We will consequently descend the power $m$ until we reach one of the above cases.

If $m$ is odd then we are done. Suppose that $m$ is even, $m = 2m'$. If $D(s, \sigma)$ is an not even with respect to $s$, then we are done.

If $D(s, \sigma)$ is still even in $s$ then $D(s) = D'(s^2)$, where $D'$ is a new function, real analytic in $s$ near 0.

Then introduce new parameter

$$s' = s^2$$

and pass to the new parameter $s'$ and the new parametrizing function

$$u(s', \sigma) = (s')^{m'}v_m + (s')^{m'}e_0 + D'(s')\tau,$$

which extends as a real analytic function to negative values of $s'$.

If, again, both functions $(s')^{m'}$ and $D'(s')$ are even, we introduce the new parameter

$$s'' = (s')^2.$$

Proceeding that way, we finally end up either with odd $m$ or with even $m$ but not even (with respect to $s$) function $D(s, \sigma)$. Lemma is proved.

9.5 The case of even $m$

The following lemma shows that the case of even power $m$ leads to the case 4 in Theorem 9.11 of double tangency at the singular point $a$ (which here is assumed to be $a = 0$):

Lemma 9.10 Let $m$ be even and let $D(s, \sigma)$ be not identically zero function (i.e. due to (11) the surface $S$ is not a plane). Then $a$ is a cuspidal point as defined in case 4 of Theorem 9.11.

Proof
9.5.1 Extracting the even part of $D(s, \sigma)$

By Lemma 9.9 we can make, by means of a suitable reparametrization, the function $D(s, \sigma)$ not even with respect to the variable $s$.

We fix an arbitrary $\sigma$ such that $D(s, \sigma)$ is not even in $s$. By the construction, the power series for $D$ contains no powers of $s$ less than $m + 1$:

$$D(s, \sigma) = \sum_{j=m+1}^{\infty} D_j(\sigma)s^j.$$

Since $D$ is not an even function with respect to $s$, there exists at least one odd exponent $j$ with $D_j(\sigma) \neq 0$ near $\sigma = 0$. Denote

$$j_0 = \min \{ j \geq m + 1 : j \text{ is odd and } D_j(\sigma) \neq 0 \}.$$

Let us split the above power series into two parts:

$$D(s, \sigma) = D_1(s, \sigma) + D_2(s, \sigma),$$

where

$$D_1(s, \sigma) = \sum_{j=m+1}^{j_0-1} D_{1,j}(\sigma)s^j,$$

$$D_2(s, \sigma) = \sum_{j=j_0}^{\infty} D_{2,j}(\sigma)s^j.$$

Then $D_1$ is even:

$$D_1(-s, \sigma) = D_1(s, \sigma)$$

because all the powers $j = m + 1, \ldots, j_0 - 1$ are even.

Now, we have:

$$D_1(s, \sigma) = D_{1,m+1}s^{m+1} + o(s^{m+1})$$

and

$$D_2(s, \sigma) = D_{2,j_0}s^{j_0} + o(s^{j_0}),$$

with $j_0$ odd. It is important that

$$D_{2,j_0} \neq 0. \tag{13}$$

Substituting the above representations for $D_1(s, \sigma)$ and $D_2(s, \sigma)$ into formula (11) for $u(s, \sigma)$ we obtain

$$u(s, \sigma) = s^mv_m + \sigma s^me_0 + (D_1(s, \sigma) + D_2(s, \sigma))\tau. \tag{14}$$
9.5.2 Taylor series for $H(u(s, \sigma))$

Now, let $H$ be a polynomial vanishing on $S$:

$$H(x) = 0 \quad \forall x \in S.$$ 

We want to prove that $S$ has a double tangency at $a$, more precisely, that the property 2) of Theorem 9.1 is satisfied for the polynomial $H$.

From the representation (14), we have

$$H(u(s, \sigma)) = H(s^m v_m + \sigma s^m e_0 + (D_1(s, \sigma) + D_2(s, \sigma) \tau)) = 0.$$

Now, let us write Taylor formula for the polynomial $H$, at the point $s^m v_m + \sigma s^m e_0 + D_1(s, \sigma) \tau$, on the vector $D_2(s, \sigma) \tau$.

It yields:

$$H(u(s, \sigma)) = \sum_{r=0}^{\text{deg} H} d^r H(s^m v_m + \sigma s^m e_0 + D_1(s, \sigma) \tau; D_2(s, \sigma) \tau) = 0, \quad (15)$$

where $d^r H(a; h)$ stands for the $r$–th differential of $H$ at a point $a$, evaluated on a vector $h$.

Replacing $s$ by $-s$, we have, taking into account that $D_1$ is even in $s$, one more relation:

$$H(u(-s, \sigma)) = \sum_{r} d^r H(s^m v_m + \sigma s^m e_0 + D_1(s, \sigma) \tau, D_2(-s, \sigma) \tau) = 0. \quad (16)$$

Now, if we subtract the second identity from the first one, then the term corresponding to $r = 0$ cancels and we will have:

$$H(u(s, \sigma)) - H(u(-s, \sigma)) = \sum_{r=1}^{\text{deg} H} T_r = 0 \quad (17)$$

where we have denoted

$$T_r = d^r H(s^m v_m + \sigma s^m e_0 + D_1(s, \sigma) \tau; D_2(s, \sigma) \tau) - d^r H(s^m v_m + \sigma s^m e_0 + D_1(s, \sigma) \tau; D_2(-s, \sigma) \tau). \quad (18)$$

Here we have used that $m$ is even and $D(-s, \sigma) = D(s, \sigma)$.
9.5.3 Contribution of the first differential

Now let us look at the first term \( T_1 \) in the expression (17) - (18), corresponding to the first differential of \( H \):

\[
T_1 = \langle \nabla H(s^m v_m + \sigma s^m e_0 + D_1(s, \sigma) \tau), (D_2(s, \sigma) - D_2(-s, \sigma)) \tau \rangle + \text{higher order differentials.}
\]

(19)

Notice that the first term in the power series for \( D_2(s, \sigma) \) is \( D_2, j_0(\sigma) s^{j_0} \), where \( j_0 \) is odd. Therefore,

\[
D_2(s, \sigma) - D_2(-s, \sigma) = 2D_2, j_0 s^{j_0} + o(s^{j_0}).
\]

(20)

Also,

\[
D_1(s, \sigma) = D_1, m+1 s^m + o(s^{m+1}).
\]

(21)

Now decompose \( H \)

\[
H = H_k + \ldots + H_{\text{deg} H}
\]

into sum of homogeneous polynomials, \( \text{deg} H_j = j \), and substitute the decomposition into (19):

\[
T_1 := dH_k(...) - dH_k(...) + dH_{k+1}(...) - dH_{k+1}(...) + \text{higher order differentials.}
\]

Here all the differentials \( dH_k \) are evaluated at the point

\[
s^m v_m + \sigma s^m e_0 + D_1(s, \sigma) \tau
\]

and on the vector

\[
D_2(\pm s, \sigma) \tau,
\]

depending whether we have \(+\) or \(-\) in front of \( dH_k \) in (19).

Now using (20), (21) and homogeneity of \( H_k \) we obtain

\[
T_1 = s^{(k-1)m+j_0} \langle \nabla H_k(v_m + \sigma e_0 + (D_1, s + o(s)) \tau, (2D_{2,j_0} + o(s)) \tau) \rangle + s^{km+j_0} \langle \nabla H_{k+1}(\ldots), \tau \rangle + \ldots,
\]

and at last

\[
T_1 = 2D_{2,j_0} s^{(k-1)m+j_0} \langle \nabla H_k(v_m + \sigma e_0), \tau \rangle + o(s^{(k-1)m+j_0}).
\]

(22)

Similarly, substituting the above asymptotic (20), (21) of \( D_1 \) and \( D_2 \) into the next homogeneous terms \( H_{k+1}, H_{k+2}, \ldots \) leads to the expressions similar to (22) were \( k \) is replaced by \( k + 1, k + 2 \) and so on. Therefore, the least power that comes from \( H_{k+1}, H_{k+2}, \ldots \) is \( s^{km+j_0} \).
9.5.4 Contribution of the higher differentials

Let us turn now to the higher differentials and consider the contribution of the terms corresponding to $d^2H_k, d^3H_k$, ... in the asymptotic near $s = 0$.

Consider now the term $T_2$ in (17), corresponding to the second differential $d^2H_k$:

$$d^2H(s^mv_m + \sigma s^me_0 + (D_1, \sigma s^{m+1} + o(s^{m+1})\tau; (2D_{2,j_0} s^{j_0} + o(s^{j_0}))(\tau)).$$

The asymptotic of (17) near $s = 0$ is determined again by the minimal degree homogeneous polynomial $H_k$, more precisely, by the difference

$$d^2H_k(s^m v_m + \sigma s^m e_0 + (D_1, \sigma s^{m+1} + o(s^{m+1})\tau, (2D_{2,j_0} s^{j_0} + o(s^{j_0}))\tau),$$

which comes from the minor homogeneous term $H_k$ in $H$.

By the homogeneity, it equals to

$$4D_{2,j_0}(\sigma)^2 s^{(k-2)m+2j_0} d^2H_k(v_m + \sigma e_0 + o(s), \tau) + o(s^{(k-2)m+2j_0}).$$

However,

$$(k - 2)m + 2j_0 = (k - 1)m + j_0 - m + j_0 > (k - 1)m + j_0,$$

because $j_0 - m > 0$.

Moreover, for the next terms, coming from the higher differentials $d^r$, we will have the following order of the asymptotic

$$(k - r)m + rj_0 = (k - 1)m + j_0 - (r - 1)m + (r - 1)j_0 > (k - 1)m + j_0.$$

Thus, we see that only the first differential $dH_k$ of the minor homogeneous term $H_k$ contributes the term $s^{(k-1)m+j_0}$ of the minimal power to the asymptotic of $H(u(s, \sigma))$ near $s = 0$.

Therefore, the main term of the asymptotic, which is determined by the minimal power of $s$, equals to

$$H(u(s, \sigma)) - H(-s, (-s, \sigma)) = 2D_{2,j_0}(\sigma)s^{(k-1)m+j_0}\langle\nabla H_k(v_m + \sigma e_0), \tau\rangle + ...$$
9.5.5 Double tangency property

Since the left hand side is identically zero

\[ H(u(s, \sigma)) - H(-s, u(-s, \sigma)) = 0, \]

the main term of the asymptotic is zero as well. It follows then from \( D_{2,j_0} \neq 0 \) that

\[ \langle \nabla H_k(v_m + \sigma e_0), \tau \rangle = 0. \]

Now recall that \( \sigma \) is an arbitrary real number. Since the polynomial \( H_k \) is homogeneous, we have

\[ \langle \nabla H_k(h), \tau \rangle = 0, \]

for all \( h \in \Pi := \text{span}\{v_m, e_0\} \). Since the vector \( \tau \) is orthogonal to the plane \( \Pi \), the normal derivative

\[ \frac{\partial H_k}{\partial \tau} = 0 \]

on \( \Pi \).

Also, we know from Lemma 9.7 that \( H_k = 0 \) on \( \Pi \). Thus \( H_k \) vanishes on \( \Pi \) at least to the second order and therefore if to define linear form

\[ A(x) = \langle (x, \tau) \rangle, \]

then \( H \) is divisible by \( Q^2 : H = A^2 R \).

Lemma is proved.

9.6 The case of odd \( m \)

Lemma 9.11 If \( m \) is odd then the surface \( S \) is differentiable at \( a = 0 \). If, moreover, \( S \) is differentiable in a neighborhood of the point \( a \) then \( S \) is a \( C^1 \) manifold there.

Proof By Lemma 9.5 the surface \( S \) is the image of the function

\[ u(t, \mu) = v(t) + \mu e(t), \]

where

\[ v(t) = v_m t^m + \ldots; \ e(t) = e_0 + e_1 t + \ldots, \ m = 2s + 1. \]
Since \( m \) is odd, the curve parametrized by
\[
 u(t, 0) = v(t), t \in I = (-\varepsilon, \varepsilon),
\]
is differentiable, which follows from the change of the parameter \( t^m = s \):
\[
 v(s) = v_m s + v_{m+1} s^{\frac{m+1}{m}} + \ldots = v_m s + o(s).
\]
We also have from definition (5) of \( v(t) \) and Lemma 8.1:
\[
 \langle v'(t), e(t) \rangle = \langle u'(t) + \lambda_0 e'(t), e(t) \rangle = 0.
\]
Therefore, the image of the function \( u(t, \mu) \) describes a ruled surface consisting of straight lines orthogonal to the differentiable curve \( v : I \to \mathbb{R}^3 \).

Apply an orthogonal transformation so that the triple \( v_m, e_0, \tau \) becomes the axis. Denote \( x_1, x_2, x_3 \) the coordinates of points in the basic \( v_m, e_0, \tau \).

Then, according to (11), the mapping \( u(s, \sigma) \) has the following representation in the new coordinates:
\[
 u(s, \sigma) = (x_1, x_2, x_3) = (s^m, \sigma s^m, D(s, \sigma)).
\]

We have
\[
 x_1 = s^m, \\
 x_2 = \sigma s^m, \\
 x_3 = D(s, \sigma),
\]
and therefore
\[
 s = \frac{x_1}{s^m}, \sigma = \frac{x_2}{x_1}.
\]

The function \( D(s, \sigma) \) is real analytic at \( s = 0, \sigma = 0 \):
\[
 D(s, \sigma) = \sum_{\alpha, \beta \in \mathbb{Z}^+} c_{\alpha, \beta} s^\alpha \sigma^\beta,
\]
in a neighborhood of \( s = 0, \sigma = 0 \).

Moreover, according to (12), \( D(s, \sigma) = o(s^m), s \to 0 \) and hence
\[
 \alpha \geq m + 1
\]
in the Taylor series for $D$.

Substituting the expressions for $s, \sigma$ through $x_1, x_2$ yields the representation of the function

$$x_3 = z(x_1, x_2) = D(x^{\frac{1}{m}}_1, \frac{x_2}{x_1})$$

as a Newton-Puiseux fractional power series:

$$z(x_1, x_2) = \sum_{\alpha=m+1, \beta=0}^{\infty} c_{\alpha,\beta} x_1^{\frac{\alpha}{m} - \beta} x_2^\beta. \quad (23)$$

### 9.6.1 Differentiability of $z(x_1, x_2)$ at $(0,0)$

We know that the line $L_0 = \{\lambda e_0, \lambda \in \mathbb{R}\}$ is one of the generating lines and belongs to $S$. In the coordinates $x_1, x_2, x_3$, the line $L_0$ has the equation $x_1 = x_3 = 0$. Since $x_3 = z(x_1, x_2)$ is the equation of $S$, we conclude that

$$\lim_{x_1 \to 0} z(x_1, x_2) = 0$$

for any fixed $x_2$. This implies that the series $(23)$ contains only positive powers of $x_1$.

Therefore, the series can be rewritten as

$$z(x, y) = \sum_{\nu>0, \beta\geq0} b_{\nu,\beta} x_1^\nu x_2^\beta, \quad (24)$$

where we have introduced the new coefficients

$$b_{\nu,\beta} = c_{\alpha,\beta}, \quad \nu = \frac{\alpha}{m} - \beta.$$

In our case $\nu$ is strictly positive because $z(0,0,0) = 0$.

Notice, that since $m$ is odd, the fractional power $x_1^\nu$ is well defined for $x_1 < 0$ as well, so the decomposition $(24)$ holds in a full neighborhood of $(0,0)$.

The general term in the Newton-Puiseux series $(24)$ is of homogeneity degree

$$\nu + \beta = (\frac{\alpha}{m} - \beta) + \beta = \frac{\alpha}{m} > 1 + \frac{1}{m}.$$

The series $(24)$ can be written in the polar coordinates

$$x_1 = r\cos \theta, \quad x_2 = r\sin \theta$$
as
\[ z(x_1, x_2) = \sum_{\nu > 0, \beta \geq 0} b_{\nu,\beta} r^{\nu + \beta} (\cos \theta)^\nu (\sin \theta)^\beta. \]

Since the exponents \( \nu, \beta \geq 0 \) then \( |\cos \theta|^\nu, |\sin \theta|^\beta \leq 1 \), the inequality
\[ \nu + \beta > 1 + \frac{1}{m} \]
implies
\[ z(x_1, x_2) = o(r), \quad r \to 0. \]
Therefore the function \( z(x_1, x_2) \) is differentiable at \((0, 0)\) with \( dz(0, 0) = 0 \). Lemma is proved.

**9.6.2 \textbf{S is differentiable in a neighborhood of} a \textbf{implies} S is C^1**

If \( S \), which is the graph of the function \( z(x_1, x_2) \) is differentiable in a neighborhood of \( a = 0 \) then the \( z(x_1, x_2) \) is differentiable at any point in a neighborhood \( U \) of \((0, 0)\). Due to (24)
\[
\frac{\partial z}{\partial x_1}(x_1, x_2) = \sum_{\nu > 0, \beta \geq 0} b_{\nu,\beta} \nu x_1^{\nu-1} x_2^\beta. \tag{25}
\]
Since \( \nu > 0 \) is fractional, the number \( \nu - 1 \) can be negative. However, this is not the case, because if series (25) contains negative powers of \( x_1 \) then for small \( x_2 \neq 0 \) we have \( \lim_{x_1 \to 0} \frac{\partial z}{\partial x_1}(x_1, x_2) = \infty \) which contradicts to the differentiability of \( z(x_1, x_2) \) at the points \((0, x_2)\) with small \( x_2 \). Then the series
\[
\frac{\partial z}{\partial x_1}(x_1^m, x_2) = \sum_{\nu > 0, \beta \geq 0} b_{\nu,\beta} \nu x_1^{m(\nu-1)} x_2^\beta
\]
is a power series, since \( m(\nu - 1) = \alpha - m\beta - m \) is integer and nonnegative. Power series are continuous in their domains of convergence, therefore \( \frac{\partial z}{\partial x_1}(x_1^m, x_2) \) is continuous in a neighborhood of \((0, 0)\). Since \( m \) is odd, the mapping \( x_1 \mapsto x_1^m \) is a homeomorphisms and hence the continuity of \( \frac{\partial z}{\partial x_1}(x_1^m, x_2) \) follows.

Same argument implies the continuity of the \( \frac{\partial z}{\partial x_2} \) since \( x_2 \) the series (24) in just a usual power series with respect to \( x_2 \). The proof of Lemma is completed.
9.7 End of the proof of Theorem 9.1

Now we are ready to finish the proof of Theorem 9.1. We start with assumption that $S$ is neither a plane nor a cone. Then we have to prove that either the surface $S$ is $C^1$ manifold and the line foliation is everywhere regular or $S$ has a cuspidal point.

Lemma ?? says that cuspidal singular points $a \in S$ correspond to the even associated powers $m$ in decomposition (11). Therefore, if $S$ is free of cuspidal points, then for any singular (with respect to the initial parametrization of our line foliation) point the associated power is odd.

But Lemma 9.11 implies that then $S$ is differentiable at any singular point $a \in S$. Surely, $S$ is also differentiable at any regular point. Therefore $S$ is differentiable everywhere. But then the second assertion of Lemma 9.11 yields that $S$ is $C^1$ - manifold and the line foliation of $S$ is everywhere regular (with respect to some parametrization of the line foliation).

Thus, we have proven that one of the fourth cases enlisted in Theorem 9.1 holds. Theorem is proved.

10 Irreducible case. Proof of Theorem 3.4

10.1 Extremal ruling lines and antipodal points

For any two ruling straight lines $L_t, L_s \subset S$ define the distance function

$$d(t, s) := \text{dist}(L_t, L_s) = \min\{|u - v| : u \in L_t, v \in L_s\}.$$

Lemma 10.1 If $d(s, t) = 0$ for all $t, s$ then $S$ is a cone.

Proof The condition implies that any two ruling lines meet. Fix two non-parallel ruling lines $L_t, L_s$. They intersect at some point $a \in L_t \cap L_s$.

Due to real analyticity of the one-dimensional connected family $\{L_t\}$ of the ruling lines, the two cases are possible:

1) all the lines $L_t$ pass through the point $a$, and then $S$ is a cone with the vertex $a$,

2) at most finite number of lines $L_{t_1}, \ldots, L_{t_N}$ contain $a$.

Suppose that case 2) takes place. Take any third ruling line $L_r$ for $r \neq t_1, \ldots, t_N$. Since any two ruling lines have a common point, the line $L_r$ must
intersect both lines $L_t$, $L_s$ at points different from $a$. This implies that $L_r$ belongs to the two-dimensional plane $\Pi$ spanned by $L_t, L_s$. Therefore all but at most finite number of ruling lines belong to $\Pi$. This implies that the union of those lines $S = \Pi$. Therefore $S$ is a 2-plane which, of course, is a cone.

Now we are interested in the case when $d(s,t)$ is not identically zero function.

**Lemma 10.2** If $S$ is not a cone then there are two maximally distant ruling lines $L_{t_0}, L_{s_0}$, i.e., the distance function $d(t,s)$ attains its maximum:

$$d(t_0, s_0) = \max_{t,s} d(t,s) > 0.$$

at some values $t_0, s_0$ of the parameters.

**Proof** The function $d(s,t)$ is defined on the compact set $[-1,1] \times [-1,1]$. It is upper semi-continuous, i.e., the upper limit

$$\limsup_{(t,s) \to (t_0,s_0)} d(t,s) \leq d(t_0,s_0).$$

Indeed, let $a = u(t_0) + \lambda_0 e(t_0) \in L_{t_0}$, $b = u(s_0) + \mu_0 e(s_0) \in L_{s_0}$, be the points on the straight lines $L_{t_0}, L_{s_0}$ such that

$$|a - b| = \text{dist}(L_{t_0}, L_{s_0}).$$

If $(t_n, s_n) \to (t_0, s_0)$ then

$$a_n = u(t_n) + \lambda_0 e(t_n) \to a, b_n = u(s_n) + \mu_0 e(s_n) \to b.$$

Then we have

$$d(s_n, t_n) \leq |a_n - b_n|$$

and hence

$$\lim_{n \to \infty} d(t_n, s_n) \leq \lim_{n \to \infty} |a_n - b_n| = |a - b| = d(t_0, s_0).$$

Due to the arbitrariness of the sequence $(t_n, s_n) \to (t_0, s_0)$, the function $d(t,s)$ is upper semi-continuous. By Weierstrass theorem it attains its maximal value $d(t_0, s_0)$. Since $d(t,s)$ is not identically zero function, we have $|a - b| = d(t_0, s_0) > 0$. We will call $a, b$ extremal points.
Lemma 10.3 Suppose that the line foliation of \( S \) contains no parallel lines. Suppose that the surface \( S \) is differentiable at the extremal points \( a \) and \( b \) and the foliation \( S = \bigcup_t L_t \) is regular at both extremal points \( a \) and \( b \). Then \( a \) and \( b \) are antipodal points (see Definition 7.1).

According to Definition 8.2, regularity means that near the points \( a \) and \( b \), the surface \( S \) is the image of the mappings

\[
w_a(t, \lambda) = w_a(t) + \lambda E_a(t), \quad w_b(s, \mu) = w_b(s) + \lambda E_b(s),
\]

correspondingly, which define the same foliation and are differentiable and regular at the points \((t_0, \lambda_0), (s_0, \mu_0)\). Here \( a = u_a(t_0, \lambda_0), \ b = u_b(s_0, \mu_0)\).

We denote the straight lines

\[
L_t = \{u_a(t) + \lambda E_a(t), \lambda \in \mathbb{R}\}, \quad L_s = \{u_b(s) + \mu E_b(s), \mu \in \mathbb{R}\}.
\]

The tangent spaces at \( a \) and \( b \) are spanned by the corresponding partial derivatives, which are linearly independent due to regularity:

\[
T_a(S) = \text{span}\{\partial_t u(t_0, \lambda_0), e(t_0)\},
\]

\[
T_b(S) = \text{span}\{\partial_t u(s_0, \mu_0), e(s_0)\}.
\]

We know that the function

\[
\lambda \to |u(t_0, \lambda) - u(s_0, \mu)|^2
\]

attains minimum at \( \lambda = \lambda_0, \mu = \mu_0 \). Therefore, the partial derivatives vanish at \((t_0, \lambda_0)\).

Differentiation in \( \lambda \) at \( t = t_0, \lambda = \lambda_0 \) yields

\[
\langle e(t_0), u(t_0, \lambda_0) - u(s_0, \mu_0) \rangle = \langle e(t_0), a - b \rangle = 0.
\]

Analogously, differentiation in \( \mu \) gives

\[
\langle e(s_0), a - b \rangle = 0. \quad (26)
\]

For any pair \( L_t, L_s \) of the constituting the surface \( S \) straight lines, denote \( a(t, s), b(t, s) \) the points

\[
a(t, s) = u(t) + \lambda(t, s)e(t), \quad b(t, s) + \mu(t, s)e(s),
\]
belonging to the lines $L_t, L_s$ correspondingly, at which the distance between the lines is attained:

$$d(t, s) = \text{dist}(L_t, L_s) = |a(t, s) - b(t, s)|.$$

The coefficients $\lambda(t, s), \mu(t, s)$ can be found from the orthogonality conditions

$$\langle a(t, s) - b(t, s), e(t) \rangle = 0, \langle a(t, s) - b(t, s), e(s) \rangle = 0.$$

The solutions of the corresponding linear system are

$$\lambda(t, s) = -\frac{\langle e(t), e(s) \rangle \langle u(t) - u(s), e(s) \rangle + \langle u(t) - u(s), e(s) \rangle}{1 - \langle e(t), e(s) \rangle^2},$$

$$\mu(t, s) = \frac{\langle e(t), e(s) \rangle \langle u(t) - u(s), e(s) \rangle - \langle u(t) - u(s), e(t) \rangle}{1 - \langle e(t), e(s) \rangle^2}.$$

The denominator is different from zero as the lines $L_t, L_s$ are not parallel by the condition and hence $1 - \langle e(t), e(s) \rangle \neq 0$.

The above formulas show that the functions $\lambda(t, s), \mu(t, s)$ are differentiable at the point $(t_0, s_0)$.

Since the distance function $d(t, s)$ attains its maximum at $t_0, s_0$ we have

$$\partial_t d(t_0, s_0) = \langle a'(t_0, s_0) - b'(t_0, s_0), a - b \rangle \geq 0,$$

$$\partial_s d(t_0, s_0) = \langle a'(t_0, s_0) - b'(t_0, s_0), a - b \rangle \geq 0,$$

or

$$\langle u'(t_0) + \lambda'(t_0, s_0)e(t_0) + \lambda_0 e'(t_0), a - b \rangle = 0,$$

$$\langle u'(s_0) + \mu'(t_0, s_0)e(s_0) + \mu_0 e'(s_0), a - b \rangle = 0.$$

Since $a - b$ is orthogonal to $e(t_0)$ and $e(s_0)$, we obtain:

$$\langle u'(t_0) + \lambda_0 e'(t_0), a - b \rangle = 0,$$

$$\langle u'(s_0) + \mu_0 e'(s_0), a - b \rangle = 0.$$

Therefore the vector $a - b$ is orthogonal to the vectors $(\partial_\lambda u)(t_0, \lambda_0)$ and to $(\partial_\lambda u)(t_0, \lambda_0)$ which span the tangent plane $T_a(S)$. Thus,

$$a - b \perp T_a(S).$$

Analogously,

$$a - b \perp T_b(S).$$

That means that the points $a$ and $b$ are antipodal.
10.2 End of the proof of Theorem 3.4

The "if" part.

Notice, that the "if" statement holds in any dimension $d$. Suppose that $S$ is a harmonic cone with a vertex $a$. This means that there exists a nonzero harmonic homogeneous polynomial (solid harmonic) $h$ such that

$$h(a + x) = 0, \ \forall x \in S.$$ 

By shifting, we can assume $a = 0$.

Define

$$\varphi_\lambda(x) = \int_{|\omega|=1} e^{i\lambda<x,\omega>} h(\omega) dA(\omega).$$

Then

$$\Delta \varphi_\lambda = -\lambda^2 \varphi_\lambda.$$ 

Now fix $x_0 \in \mathbb{R}^d \setminus 0$ such that $h(x_0) = 0$. Denote $SO_x(d)$ the group of orthogonal transformations $\rho \in SO(d)$ of $\mathbb{R}^d$ such that $\rho(x_0) = x_0$. Then

$$\varphi_\lambda(x_0) = \varphi_\lambda(\rho(x_0)) = \int_{|\omega|=1} e^{i\lambda<x_0,\rho^{-1}(\omega)>} h(\omega) dA(\omega) = \int_{|\omega|=1} e^{i\lambda<x_0,\rho^{-1}(\omega)>} h(\omega) dA(\omega).$$

Change of variables $\omega' = \rho^{-1}(\omega)$ leads to

$$\varphi_\lambda(x_0) = \int_{|\omega'|=1} e^{i\lambda<x,\omega'>} h(\omega') dA(\omega').$$

Integrating the equality in $\omega'$ against normalized Haar measure $d\rho$ on $SO(d)$ yields

$$\varphi_\lambda(x_0) = \int_{|\omega'|=1} e^{i\lambda<x_0,\omega'>} \tilde{h}(\omega') d\omega', \tag{27}$$

where $\tilde{h}(\omega')$ is the average

$$\tilde{h}(\omega') = \int_{\rho \in SO_x(d)} h(\rho \omega') d\rho.$$ 

The function $\tilde{h}(\omega')$ is a spherical harmonic, invariant under rotations $\rho \in SO(d)$, preserving $x_0$, and therefore it is proportional to the zonal harmonic $Z_{x_0}$ with the pole $x_0/|x_0|$, of the same degree as $\tilde{h}$:

$$\tilde{h} = cZ_{x_0}. \tag{28}$$
However,
\[ \frac{1}{|x|^{\deg h}} h(x) = \tilde{h}(\frac{x}{|x|}) = h(\frac{x}{|x|}) = 0, \]
because \( \rho x = x \) and \( h(x) = 0 \) and \( h \) is homogeneous. On the other hand the value of the zonal harmonic at its pole is
\[ Z_x(\frac{x}{|x|}) = \alpha \Omega_{d-1}, \]
where \( \alpha \) is the dimension of the space of spherical harmonics of degree \( \deg h \) and \( \Omega_{d-1} \) is the area of the unit sphere in \( \mathbb{R}^d \). (19, Corollary 2.9), Therefore, we have form (28):
\[ c\alpha \Omega_{d-1}^{-1} = 0 \]
and \( c = 0 \). Then (28) implies \( \tilde{h} \equiv 0 \) and then \( \varphi_\lambda(x_0) = 0 \) because of (27).

Thus, we have proven \( \varphi_\lambda(x_0) = 0 \) whenever \( h(x_0) = 0 \) and hence the harmonic cone \( h^{-1}(0) \) is a common nodal set for a nontrivial Paley-Wiener family of eigenfunctions.

**The "only if" part**

We assume that an irreducible real analytically ruled hypersurface \( S \subset \mathbb{R}^3 \), without parallel generating lines, is contained in the common zero set of a Paley-Wiener family of eigenfunctions. We need to prove that \( S \) is a cone.

We start with the case when the foliation \( \{L_t\} \) of \( S \) is everywhere regular. In particular, it is regular at the extremal points \( a, b \) at which the distance function \( d(t, s) \) attains its maximum. Then the points \( a \) and \( b \) are antipodal by Lemma 10.3 and then Theorem 7.2 implies that \( S \) is an injectivity set. By Proposition 6.1, this contradicts to the assumption that \( S \) is the common nodal set for Paley-Wiener family of eigenfunctions.

Therefore \( S \) has at least one singular point, say, \( a \). By Corollary 9.4 of Theorem 9.1 \( a \) is a conical point. This means that \( a \) belongs to an open family of lines \( \{L_t\} \). Since \( S \) is irreducible, the base curve \( \gamma \) that parametrizes the family \( L_t \) is real analytic and connected. Therefore, all lines \( L_t \) pass through \( a \) and therefore \( S \) is a cone with the vertex \( a \).

Moreover, \( S \) is a harmonic cone. Indeed, we know from Proposition 6.1 that there exists a nonzero harmonic polynomial \( H \) such that \( S \subset H^{-1}(0) \). Since \( S \) is a cone with the vertex \( a \) we have
\[ H(a + \lambda(x - a)) = 0 \]
for all \( x \in S \) and \( \lambda \in \mathbb{R} \). Therefore, if \( H(a + u) = \sum_{j=0}^{N} H_j(u) \) is the homogeneous decomposition, then \( H_j(x - a) = 0, j = 0, ..., N \) and it remains to note that all \( H_j \) are harmonic and homogeneous. Then \( a + S \subset h^{-1}(0) \), where \( h \) can be taken any nonzero polynomial \( H_j \). Theorem 3.4 is proved.

11 Reducible case. Proof of Theorem 3.5

Now we turn to the proof of more general Theorem 3.5 where we do not assume that the base curve \( \gamma \) of the ruled surface \( S \) is connected.

In general situation, \( S \) decomposes into irreducible components:

\[
S = \bigcup_{j=1}^{M} S_j,
\]

where each \( S_j \) is a real analytically ruled surface with a real analytic closed connected base curve \( \gamma_j \). So, the ruled surface \( S \) is parametrized by the base curve

\[
\gamma = \gamma_1 \cup ... \cup \gamma_M.
\]

Each surface \( S_j \) satisfies all the conditions of Theorem 3.4 and therefore is a harmonic cone with a vertex \( a_j \in S_j \). All we need now is to prove the additional properties of the decomposition of \( S \) into union of cones, claimed in Theorem 3.5.

We will start with proving that the cones pairwise meet.

**Lemma 11.1** *If there are \( i, j \) such that \( S_i \cap S_j = \emptyset \) then \( S \) is an injectivity set.*

**Proof** Assume that \( S \) fails to be an injectivity set. Since \( S_i \) and \( S_j \) do not meet, any two generating lines \( L_a, a \in \gamma_i \) and \( L_b, b \in \gamma_j \), are disjoint and \( \text{dist}(L_a, L_b) > 0 \).

Since there are no parallel generating lines, the function \( (a, b) \to \text{dist}(L_a, L_b) \) is continuous and attains its *minimum*. Let \( a_0 \in S_i, b_0 \in S_j \) are the points where the minimal distance between the generating lines is realized:

\[
|a_0 - b_0| = \min_{a \in S_i, b \in S_j} \text{dist}(L_a, L_b) > 0.
\]

The two cases are possible:

1. \( a_0 \) and \( b_0 \) are regular points of the foliation \( S = \cup_{a \in S} L_a \).
2. One of the points \( a_0, b_0 \) is a singular point.

Let \( a_0 = u(t_0, \lambda_0), \ b_0 = u(s_0, \mu_0) \).

In the case 1, the equations:

\[
\frac{\partial|u(t, \lambda) - u(s, \mu)|}{\partial t}(t_0, \lambda_0, s_0, \mu_0) = \frac{\partial|u(t, \lambda) - u(s, \mu)|}{\partial s}(t_0, \lambda_0, s_0, \mu_0) = 0,
\]

\[
\frac{\partial|u(t, \lambda) - u(s, \mu)|}{\partial \lambda}(t_0, \lambda_0, s_0, \mu_0) = \frac{\partial|u(t, \lambda) - u(s, \mu)|}{\partial s}(t_0, \lambda_0, s_0, \mu_0) = 0
\]

yield that the vector \( a_0 - b_0 \) is orthogonal to the tangent spaces \( T_{a_0}(S), T_{b_0}(S) \).

In other words, \( a_0 \) and \( b_0 \) are antipodal points. By Lemma ??, \( S \) is an injectivity set. Contradiction.

Consider now the case 2, i.e., assume that one of the extremal points, say, \( a_0 \) is singular. Since \( S \) is not an injectivity set, \( a_0 \) is a conical point, due to Theorem 9.1. The ruled surface \( S_i \) has the real analytic connected base curve \( \gamma_i \) hence \( S_i \) is a cone with the vertex \( a_0 \).

Now, the straight lines \( L_{t_0} \subset S_i \) and \( L_{s_0} \subset S_j \) are the closest generating lines belonging to \( S_i \) and \( S_j \) correspondingly. Since \( a_0 \in L_{t_0}, b_0 \in L_{s_0} \) are the closest points, we have

\[ L_{t_0}, L_{s_0} \perp [a_0, b_0]. \]

However, since \( S_i \) is the cone with the vertex \( a_0 \), all the straight lines \( L_t \) generating \( S_i \) all pass through \( a_0 \). If \( L_t \) is not orthogonal to \([a_0, b_0]\) then

\[ \text{dist}(L_t, L_{s_0}) < |a_0 - b_0| = \text{dist}(L_{t_0}, L_{s_0}) \]

which is impossible.

Therefore, for all generating lines \( L_t \subset S_i \) we have

\[ L_t \perp [a_0, b_0] \]

and hence \( L_t \subset \Pi \), where \( \Pi \) is the plane passing through \( a_0 \) and orthogonal to \([a_0, b_0]\). Then \( S_i \) coincides with the plane \( \Pi \) and \( S_i = \Pi \) can be viewed as a line foliation, regular at \( a_0 \). If the second extremal point \( b_0 \) is regular for the given foliation \( \{L_t\} \) then both points \( a_0, b_0 \) are regular antipodal points and \( S \) is an injectivity set. If \( b_0 \) is a conical point, then the same argument with closest generating lines shows that \( S_j \) is a plane. Then again \( a_0, b_0 \) are regular antipodal points and \( S \) is an injectivity sets. Lemma is proved.

Now we will prove that the cones intersect transversally.
Lemma 11.2. If some $S_i$ and $S_j$ are tangent at a point $a$ which is not a vertex of any cone $S_i, S_j$ then $S$ is an injectivity (not nodal) set.

Proof. We saw in the proof Theorem 9.1 that if $a$ is not a vertex of the cone $S_i$ then it is either the point of real analyticity or a point of differentiability, which is a singular point of the line foliation and corresponding to the case of odd $m$ in the parametrization (11). The same is true for the cone $S_j$.

After a suitable translation and rotation, we can make $a = 0$ and

$$T_a(S_i) = T_a(S_j) = \{x_3 = 0\}.$$  

The representation (24) shows that the surfaces $S_i, S_j$ are defined near $a = 0$ as the graphs:

$$S_i : x_3 = z_i(x_1, x_2),$$

$$S_j : x_3 = z_j(x_2, x_2),$$

where

$$z_i(x_1, x_2) = o(r), \quad z_j(x_1, x_2) = o(r), \quad r = \sqrt{x_1^2 + x_2^2} \to 0.$$  

Moreover, by the construction, these functions are algebraic and for some odd integers $m, n$ the functions

$$z_i(x_1^m, x_2), z_j(x_1^n, x_2)$$

are real analytic.

If $S$ is not an injectivity set, then due to Proposition 6.1 there exists the nonzero harmonic polynomial $H$ vanishing on $S$ (Proposition 6.1). Since $H = 0$ on $S_i = \{x_3 - z_i(x_1, x_2) = 0\}$ the polynomial

$$H(x_1^{m_1}, x_2, x_3) = 0$$

whenever $\rho_i(x) := x_3 - z_i(x_1^{m_1}, x_2) = 0$.

The function $\rho$ is real analytic and $\nabla \rho \neq 0$ hence the polynomial $H$ is divisible by $\rho$ which means that

$$H(x_1^{m_1}, x_2, x_3) = (x_3 - z_i(x_1^{m_1}, x_2))R(x_1, x_2, x_3),$$

where $R$ is real analytic near 0.
Since $S_i$ and $S_j$ can coincide only on a nowhere dense subset, and $H = 0$ on $S_j$, the function $R$ must vanish on the surface $\rho_j(x) := x_3 - z_j(x_1^{mn}, x_2) = 0$. Further, since both functions $H$ and $\rho_j$ are real analytic and $\nabla \rho_j \neq 0$, the function $R$ is divisible by $\rho_j$, meaning that

$$R = \rho_j G,$$

where the function $G$ is real analytic near 0.

Finally, returning to $x_1$ instead of $x_1^{mn}$ we have

$$H(x) = (x_3 - z_i(x_2, x_3))(x_3 - z_j(x_1, x_2))G(x_1^{1/2}, x_2, x_3).$$

Decompose

$$G(x_1^{1/2}, x_2, x_3) = \sum_{\alpha, \beta, \gamma \geq 0} x_1^{\alpha/2} x_2^{\beta} x_3^{\gamma},$$

and let $G_0$ be the sum of the terms with the minimal homogeneity degree

$$\frac{\alpha_0}{mn} + \beta_0 + \gamma_0.$$

If

$$H = H_k + H_{k+1} + ... + H_N, H_k \neq 0,$$

is the homogeneous decomposition for $H$, then since $z_i, z_j = o(r), r \to 0$ we have for the minimal degree homogeneous term:

$$H_k(x) = x_3^2 G_0(x).$$

Thus,

$$H(x_1, x_2, 0) = 0.$$

Notice that $G_0$ is a polynomial with respect to $x_1^{1/2}, x_2, x_3$. Therefore, differentiation in $x_3$ yields:

$$\partial_{x_3} H(x) = 2x_3 G_0(x) + x_3^2 \partial_{x_3} G_0(x)$$

and hence

$$\partial_{x_3} H(x_1, x_2, 0) = 0.$$

However, the polynomial $H_k$ is harmonic and satisfies the overdetermined Dirichlet-Neumann conditions on the plane $x_3 = 0$. This implies $H_k \equiv 0$. This contradiction completes the proof.
11.1 End of the proof of Theorem 3.5

First of all, according to Theorem 3.4, each irreducible ruled component of $S$ is a harmonic cone and therefore, $S$ is the union of harmonic cones, $S = \bigcup_{j=1}^{N} S_j$.

Moreover, the vertices are the only singular points of the cones $S_i$. The cones $S_i$ are real analytic everywhere except, maybe, for the vertex. If $S_i$ is differentiable at the vertex then $S_i$ is a plane and, of course, is real analytic everywhere.

Further, Lemma 11.1 implies that $S_i \cap S_j \neq \emptyset$ for any $i \neq j$, since otherwise $S$ is an injectivity set. In turn, Lemma 11.2 says that $S_i \neq S_j$ is transversal. The intersection $S_i \cap S_j$ is either 0-dimensional (discrete) or one-dimensional. In the latter case the intersection is a curve.

In the case when $S_i \cap S_j$ is discrete, then since $S_i$, $S_j$ are two-dimensional, any point $a \in S_i \cap S_j$, at which $S_i$ and $S_j$ are differentiable, must be a tangency point, which is not the case. Therefore, $a$ must be singular for either cone $S_i, S_j$ and hence is a vertex of one of them. Theorem 3.5 is proved.

12 Coxeter systems of planes. Proof of Theorem 3.7

Theorem 3.4 asserts that $S$ is a cone. The only cone which has no differentiable singularities is a plane. Therefore, if $S$ in Theorem 3.4 is differentiable surface then $S$ is a plane.

Then Theorem T:mainmain2 follows from

Lemma 12.1 Any finite union $S$ of hyperplanes in $\mathbb{R}^d$ is an injectivity set unless $S$ can be completed to a Coxeter system.

Proof

We will give the proof for the case $d = 3$ which is under consideration in this article.

Let

$$ S = \bigcup_{i=1}^{N} \Pi_i $$

where $\Pi_i$ are the hyperplanes. Suppose that $S$ fails to be an injectivity set. Then there exists a nonzero function $f \in C_{\text{comp}}(\mathbb{R}^3)$ such that $Rf(x,t) = 0, \ t > 0$, for all $x \in S$. It is known [10], v.II, that then $f$ is odd with respect to reflections around each plane $\Pi_i$. 

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Denote $W_{\Pi_1, \ldots, \Pi_N}$ the group generated by the reflections around the planes $\Pi_1, \ldots, \Pi_N$.

Now we are going to use the additional information about existence of nonzero harmonic polynomial vanishing on $S$ (Proposition 6.1), which rules out, due to Maximal Modulus Principle, the possibility for the action of the group $W_{\Pi_1, \ldots, \Pi_N}$ to have compact fundamental domain.

If $N = 2$ then the angle between $\Pi_1$ and $\Pi_2$ must be a rational multiple of $\pi$ since otherwise

$$\cup_{w \in W_{\Pi_1, \Pi_2}} w(\Pi_1) \cup \cup_{w \in W_{\Pi_1, \Pi_2}} w(\Pi_2)$$

is dense in $\mathbb{R}^3$ and then $f = 0$ identically because $f$ vanishes on each $\Pi_1, \Pi_2$. Therefore $S$ is a subsystem of the Coxeter system generated by the planes $\Pi_1, \Pi_2$.

Let $N \geq 3$. The following cases are possible:

1. all the planes $\Pi_i, i = 1, \ldots, N$, have a common point,
2. there are two parallel planes $\Pi_{i_1}, \Pi_{i_2}$,
3. there are three planes $\Pi_{i_1}, \Pi_{i_2}, \Pi_{i_3}$ that bound a right triangular prism,
4. $N \geq 4$ and there are four planes $\Pi_{i_1}, \Pi_{i_2}, \Pi_{i_3}, \Pi_{i_4}$ that bound a bounded simplex.

In the first case, the reflection group $W$ generated by the planes $\Pi_i$ must be finite, since otherwise $\cup_{w \in W_{\Pi_1, \ldots, \Pi_N}} w(S)$ is dense in $\mathbb{R}^3$ and then $f = 0$. Therefore, in the first case $S$ can be included in a Coxeter system of planes.

The second case is impossible, since $supp f$, being symmetric both with respect to $\Pi_{i_1}$ and $\Pi_{i_2}$, must be unbounded, which is not the case.

In the third case, the normal vectors $\nu_1, \nu_2, \nu_3$ of the corresponding planes are linearly dependent and span a plane $P$ orthogonal to all $P_{i_j}, j = 1, 2, 3$.

For any $b \in \mathbb{R}^3$ the intersection $(P + b) \cap (\Pi_{i_1} \cup \Pi_{i_2} \cup \Pi_{i_3})$ is a triangle $L_1, L_2, L_3$ in the 2-plane $P + b$, bounded a triangle.

The restriction $f|_{P + b}$ can be regarded as a compactly supported function defined in $\mathbb{R}^2$, and this function is odd-symmetric with respect to the lines $L_1, L_2, L_3$.

In particular, it has zero spherical means on the lines. As it was proven in Proposition 6.1, if $f$ is not identically zero on $P + b$ then there is a nonzero harmonic polynomial vanishing on $L_1 \cup L_2 \cup L_3$ which is impossible due to
Maximum Modulus Principle since the union contains a bounded contour. Therefore, \( f = 0 \) on \( P + b \) and then \( f = 0 \) everywhere as \( b \) is arbitrary. Thus, the third case is ruled out as well.

Also, the fourth case is impossible, since if \( f \) is not zero then we again have contradiction with existence of a nonzero harmonic polynomial vanishing on \( S \), as in the previous case. Lemma is proved.

**Proof of Theorem 3.7** Since any two-dimensional cone in \( \mathbb{R}^3 \), which is a differentiable surface, is a two-dimensional plane, Theorem 3.5 implies that the surface \( S \) in Theorem 3.7 is a finite union of 2-planes and hence is a Coxeter system of planes, due to Lemma 12.1.

### 13 Proof of Theorem 3.8 (the case of convexly supported generating function)

#### 13.0.1 Lemmas

We are given a nonzero function \( f \in C_{\text{comp}}(\mathbb{R}^3) \). Consider the set

\[
N_f = \{ x \in \mathbb{R}^3 : Rf(x,t) = 0, \ \forall t > 0 \}.
\]

By Proposition 6.1 the set \( N_f \) represents as

\[
N_f = S \cup V,
\]

where \( S \) is either empty or an algebraic hypersurface

\[
S = Q^{-1}(0),
\]

where \( Q \) is a polynomial, dividing a nonzero harmonic polynomial \( H \). We assume \( S \neq \emptyset \).

Denote \( \Gamma \) the outer boundary of \( \text{suppf} \). By the condition, \( \Gamma \) is strictly convex real analytic closed hypersurface. Theorem 4.6 yields that the observation surface \( S \) is foliated into straight lines, each of which intersects orthogonally, at two points, the strictly convex surface \( \Gamma \).

The surfaces \( \Gamma \) and \( S \) intersect orthogonally. The intersection

\[
\gamma : \Gamma \cap S
\]

is a curve, smooth at all points \( a \in \gamma \) at which \( S \) is smooth.
Lemma 13.1 The surface $S$ is a real analytically ruled surface.

Proof Denote

$$\gamma = \Gamma \cap S.$$ 

Pick a point $a \in \gamma$. Let $T_a(\Gamma)$ be the tangent plane. Applying translation and rotation, one can assume that $a = 0$ and

$$T_a(\Gamma) = \{x_3 = 0\}.$$ 

The projection

$$\pi : T_a(\Gamma) \mapsto \Gamma$$

along the normals to $\Gamma$ is well defined in a neighborhood

$$U \subset T_a(\Gamma)$$

of $a$.

Since $\Gamma$ is real analytic, the normal field to $\Gamma$ is real analytic as well and hence $\pi$ is real analytic diffeomorphism near $a = 0$. Also, $\pi(U \cap S)$ is an open neighborhood of $a$ in $\gamma$.

It is easy to understand that the polynomial $Q$ is not identically zero on $T_a(S)$ since $S = Q^{-1}(0)$ is transversal to $T_a(S)$ near $a$. Therefore, the intersection

$$C := T_a(\Gamma) \cap S$$

is an open algebraic curve in the plane $T_a(\Gamma) = \{z = 0\}$, defined by the equation $C = \{Q(x, y, 0) = 0\}$.

Then we use Puiseux theorem ([17], Ch.II, 9.6; [20], Thm. 2.1.1; [11], Ch.2,p. 3-11) which claims that each branch $C_i$ of $C$ is parametrized either by

$$I \ni t \mapsto (0, t, 0),$$

or by

$$I \ni t \mapsto (t^m, \alpha_i(t), 0)$$

where $I$ is an open interval (which can be taken $I = (-1, 1)$), $m$ is natural and $\alpha_i(t)$ is a real analytic function.

Then $\gamma$ decomposes, near $a$, into the union of the curves $\gamma_i = \pi(C_i)$ and each $\gamma_i$ is the image $\gamma_i = u(I)$ where the mapping

$$I \ni t \mapsto u_i(t) = \pi(t^m, B_i(t), 0).$$
is real analytic, because \( \pi \) is so. By Corollary 9.4 of Theorem 9.1, the ruled surface

\[
S_i = \{ u_i(t) + \lambda \nu(u_i(t)), t \in I, \lambda \in \mathbb{R} \},
\]

where \( \nu \) is unit normal vector to \( \Gamma \), is real analytically ruled surface.

**Lemma 13.2** Let \( a \) be the vertex of the cone \( C_i \). Let \( \gamma_i \) be a connected closed subarc of \( C_i \cap \Gamma \) where \( \Gamma \) is the outer boundary of \( \text{suppf} \). Then the distance \( |x - a| \) from \( a \) to an arbitrary point \( x \in \gamma_i \) is constant.

**Proof** Consider the parametrization \( u(t, \lambda) = u(t) + \lambda e(t), t \in I \), of the cone \( C_i \). The mapping \( t \mapsto u(t) \) parametrizes the curve \( \gamma_i = C_i \cap \Gamma \). Consider the distance function

\[
d(t) = |a - u(t)|^2.
\]

Then

\[
d'(t) = (a - u(t), u'(t)).
\]

Since \( a \) is the vertex of \( C_i \), it belongs to any line \( L_t \). Therefore \( a = u(t) + \lambda(t)e(t) \) and hence

\[
d'(t) = (a - u(t), u'(t)) = \lambda(t)(e(t), u'(t)) = 0,
\]

because \( u'(t) \) is tangent to \( \Gamma \), \( e(t) \) is the directional vector of the Line \( L_t \) and \( L_t \) is orthogonal to \( \Gamma \), as stated in Theorem 4.6.

**Lemma 13.3** If two cones \( C_i \), \( C_j \) meet outside of \( \text{suppf} \) then they have a common vertex and hence the union \( C_i \cup C_j \) is itself a cone.

**Proof** The cones \( C_i, C_j \) consist of straight lines orthogonal to the outer boundary \( \Gamma \) of \( \text{suppf} \). Also, \( \Gamma \) is a real analytic strictly convex surface. If \( C_i \) meet \( C_j \) in the exterior of \( \Gamma \) then \( C_i \) and \( C_j \) share a ruling straight line \( L \) passing through a common point of the two cones and orthogonal to \( \Gamma \). The vertices of both cones \( C_i \) and \( C_j \) belong to \( L \). The common line \( L \) meets the convex surface \( \Gamma \) at two points \( b^+, b^- \):

\[
\{b^+, b^- \} = L \cap \Gamma.
\]

Let \( \gamma_i \) and \( \gamma_j \) be the connected closed subarcs of the smooth curves \( C_i \cap \Gamma \) and \( C_j \cap \Gamma \), correspondingly, containing the point \( b^+ \).

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Then \( \gamma_i, \gamma_j \) are smooth closed curves on \( \Gamma \), sharing the common point \( b^+ \in \gamma_i \cap \gamma_j \).

Suppose that \( \gamma_i \) and \( \gamma_j \) are tangent at \( b^+ \) and let \( \tau \) be the common tangent vector at \( b^+ \). Since the tangent planes of the cones \( C_i \) and \( C_j \) coincide:

\[
T_{b^+}(C_i) = T_{b^+}(C_j) = \text{span}\{L, e\},
\]

the two cones are tangent. However, this is impossible due to Lemma 11.2.

Thus, the two closed curves \( \gamma_i \) and \( \gamma_j \) intersect at \( b^+ \) transversally. Then they must intersect in at least one more point, \( c \in \Gamma \). Then both cones \( C_i, C - j \) contain the straight line \( L_c \) intersecting \( \Gamma \) orthogonally at the point \( c \). The two cases are possible:

1. \( c \neq b^- \).
2. \( c = b^- \).

In the case 1, the straight lines \( L \) and \( L_c \) are different. Both of them belong to the cones \( C_i \) and \( C_j \) and hence the intersection of the two lines \( L \cap L_c \) is just a single point which is the vertex of both \( C_i \) and \( C_j \). Thus, \( C_i \) and \( C_j \) share the vertex and Lemma is proved in this case.

In the case 2 the two straight lines coincide, \( L = L_c \), as they both pass through the points \( b^+ \) and \( b^- = c \). Let \( a_i, a_j \) be the vertices of the the cones \( C_i, C_j \) correspondingly. By Lemma 13.2, the distance \( |x - a_i| \) is constant on \( \gamma_i \). Since \( b^+, b^- \in \gamma_i \), we have

\[
|b^+ - a_i| = |b^- - a_i|.
\]

The three points \( a, b^+, b^- \) belong to the same line \( L \) and therefore, \( a \) is the midpoint:

\[
a_i = \frac{1}{2}(b^+ + b^-).
\]

The same can be repeated for \( \gamma_j \) and then we obtain

\[
a_j = \frac{1}{2}(b^+ + b^-).
\]

Thus, \( a_i = a_j \) and the statement of Lemma is true in the case 2 as well.

**Lemma 13.4** Suppose that \( S_i \cap S_j \) is 0-dimensional. Then
1. \( S_i \cap S_j \subset \{ c_i, c_j \} \), where \( c_i, c_j \) are the vertices of the cones \( S_i, S_j \) correspondingly.

2. If \( S_i \cap S_j = \{ c_i, c_j \} \) then \( c_i = c_j \).

**Proof** We know that \( S_i \) and \( S_j \) are differentiable everywhere except maybe at the vertices. If \( a \in S_i \cap S_j \) and \( a \neq c_i, a \neq c_j \), then \( a \) is the point of smoothness for both \( S_i \) and \( S_j \) and hence the cones \( S_i, S_j \) cannot intersect at \( a \) transversally since in this case the intersection \( S_i \cap S_j \) must be one-dimensional. Therefore, \( S_i \) and \( S_j \) are tangent at \( a \). This possibility is ruled out by Lemma 11.2. This proves the statement 1.

If \( S_i \cap S_j = \{ c_i, c_j \} \) and \( c_i \neq c_j \) then both cones \( S_i \) and \( S_j \) contain the straight line passing through the vertices \( c_i \) and \( c_j \). This contradicts to the assumption that the intersection is 0-dimensional.

**Lemma 13.5** If \( S_i \cap S_j \) is one-dimensional then the cones \( S_i \) and \( S_j \) share the vertex so that \( S_i \cup S_j \) is a cone.

**Proof** Let \( \gamma = S_i \cap S_j \). If the curve \( \gamma \) is unbounded, then \( S_i \) and \( S_j \) intersect outside of \( \Gamma \) and by Lemma 13.3 \( S_i \) and \( S_j \) have a common vertex. Otherwise, \( \gamma \) is a bounded curve. It is also closed as it is algebraic. Then \( \gamma \) bounds two-dimensional domains \( D_i \) and \( D_j \) on the surfaces \( S_i, S_j \) correspondingly. Therefore, \( S_i \cap S_j \) contain a cycle \( D_i \cup D_j \). However, it is impossible due to Maximum Modulus Principle, since there exists a nonzero harmonic polynomial \( H \) vanishing on \( S_i \cup S_j \).

**Corollary 13.6** If \( S_i \) and \( S_j \) have different vertices, \( c_i \neq c_j \), then \( S_i \cap S_j \) consists of a single point, which is either \( c_i \) or \( c_j \).

**Proof** The intersection \( S_i \cap S_j \) is discrete (0-dimensional) since otherwise the cones \( S_i, S_j \) have equal vertices, by Lemma 13.5. Then Lemma 13.4 says the intersection coincides with one of the vertices.

**13.0.2 End of the proof of Theorem 3.8**

Let us group all the cones \( S_i \) whose vertices coincide. The union of such cones is again a cone and hence the union \( S \) can be regrouped in the union

\[
S = C_1 \cup \ldots \cup C_P
\]
of cones $C_i$ with pairwise different vertices $b_i$. Each $C_i$ is the union of the cones $S_j$ with equal vertices. Due to Lemma ??, the pairwise intersections $C_i \cap C_j$, $i \neq j$, are 0-dimensional.

First of all, all the cones $C_j$ are harmonic. Indeed, we know that there is a nonzero harmonic polynomial $H$ vanishing on $S$. By translation, we can assume the the vertex $b_i$ of the cone $C_i$ is $b_i = 0$. Since $C_i$ is a cone, we have

$$H(\lambda x) = 0$$

for all $x \in C_i$ and all $\lambda \in \mathbb{R}$. If $H = H_0 + \ldots + H_N$ is the homogeneous decomposition, then $H_0(x) + \lambda H_1(x) + \ldots + \lambda^N H_N(x) = 0$ and hence $H_k(x) = 0$ for all $k$. If $h = H_j$ is any nonzero homogeneous polynomial then $h(x) = 0$ for all $x \in C_i$ and hence $C_i$ is a harmonic cone.

Further, we know that for any $i \neq j$ the intersection $S_i \cap S_j$ is either $c_i$ or $c_j$. It follows that for the cones $C_i$, which are unions of groups of $S_j$, holds $C_i \cap C_j \subset \{b_i, b_j\}$. If $C_i \cap C_j = \{b_i, b_j\}$ then both cones $C_i$ and $C_j$ contain the points $b_j \neq b_j$ and hence, the straight line through these points, which is not the case.

Thus, $C_i \cap C_j$ is a single point, which is a vertex of $C_1$ or $C_2$:

$$C_i \cap C_j = \{b_i\} \text{ or } \{b_j\}. \quad (29)$$

**Lemma 13.7** $P \leq 3$.

**Proof** Suppose that $P \geq 4$. Consider the cones $C_1, C_2, C_3, C_4$. We have

$$C_1 \cap C_2 = \{b_1\} \text{ or } \{b_2\}.$$

Without loss of generality, we can assume that

$$C_1 \cap C_2 = \{b_1\}.$$

Then

$$C_1 \cap C_3 = \{b_3\}.$$

Indeed, if $C_1 \cap C_3 = \{b_1\}$ then $b_1 \in C_3, b_1 \in C_2$ and therefore

$$b_1 \subset \{b_2, b_3\},$$

which is impossible because $b_1, b_2, b_3$ are all different. For the same reason,

$$C_1 \cap C_4 = \{b_4\}.$$

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Now, 
\[ C_2 \cap C_3 = \{b_2\}, \]

because otherwise \( C_2 \cap C_3 = \{b_3\} \) and then \( b_3 \in C_2, b_3 \in C_1 \) and therefore
\[ b_3 \in \{b_1, b_2\} \]

which is not the case.

Now consider the intersection of \( C_2 \) and \( C_4 \):
\[ C_2 \cap C_4 = \{b_2\} \text{ or } \{b_4\}. \]

If \( C_2 \cap C_4 = \{b_2\} \) then we have
\[ b_2 \in C_4, b_2 \in C_3 \]

and therefore
\[ b_2 \in \{b_3, b_4\} \]

which is not the case. If, alternatively, \( C_2 \cap C_4 = \{b_4\} \), then we have \( b_4 \in C_2, b_4 \in C_1 \) and therefore
\[ b_4 \in \{b_1, b_2\}, \]
which is not the case. Thus, neither option is possible. Thus, \( P \leq 3 \). Lemma is proved.

Let us continue the proof of Theorem 3.8.

If \( P = 1 \) then \( S = C_1 \) is a cone and, moreover, a harmonic cone. This is the case 1) in Theorem 3.5.

Suppose \( P = 2 \) so that \( S = C_1 \cup C_2 \). Formula (29) leads to the case 2) of Theorem 3.7.

Finally, suppose that \( P = 3 \) and therefore
\[ S = C_1 \cup_2 \cup C_3. \]

**Lemma 13.8** No two cones of \( C_1, C_2, C_3 \) can have vertices belonging to the third one.

**Proof** Suppose, for example, that
\[ b_1, b_2 \in C_3. \]
We know that $C_1 \cap C_2$ is either $b_1$ or $b_2$. In the first case we have $b_1 \in C_2$ and also $b_1 \in C_3$. Hence

$$b_1 \in C_2 \cap C_3.$$ 

This implies that either $b_1 = b_2$ or $b_1 = b_3$. Neither is possible as all the vertices are different.

In the second case we have $b_2 \in C_1$ and also $b_2 \in C_3$. Then $b_2 \in C_1 \cap C_3$, which is either $b_1$ or $b_3$ and we have the same kind of contradiction. Lemma is proved.

Now we can finish the proof of Theorem 3.8 in the case $S = C_1 \cup C_2 \cup C_3$. We have $C_1 \cap C_2 =$ is either $b_1$ or $b_2$. If

$$C_1 \cap C_2 = \{b_1\},$$

then $C_2 \cap C_3$ can be only $b_2$ since otherwise $b_1, b_3 \in C_2$ which is ruled out by Lemma 13.8. Analogously, $C_3 \cap C_1$ cannot be equal to $b_1$ since then $b_1 \in C_2 \cap C_3$ and hence $b_1$ is either $b_2$ or $b_3$ which is not the case.

The case $C_1 \cap C_2 = \{b_2\}$ is treated in a similar way. Thus, finally we conclude that in the case $P = 3$ the configuration of the cones is exactly as it is pointed out in the case 3 of Theorem 3.8. Theorem is proved.

14 Concluding remarks

- Proving in full Conjecture 4.2 for ruled surfaces requires proving that the configurations of cones in Theorem 3.5 is itself a cone, i.e., the vertices of all the cones $C_i$ coincide.

- Proving Conjecture 4.2 in general case requires proving that common nodal sets for Paley-Winer families of Laplace eigenfunctions are ruled surfaces. Then one could apply Theorems 3.4 and 3.5 to pass from ruled surfaces to cones.

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