The Asymptotic of the Number of Permutations whose Cycle Lengths Are Prime Numbers

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Abstract

Let $A$ be a set of natural numbers and let $S_{n,A}$ be the set of all permutations of $[n] = \{1, 2, \ldots, n\}$ with cycle lengths belonging to $A$. Furthermore, let $|A(n)|$ denote the cardinality of the set $A(n) = A \cap [n]$. The limit $\rho = \lim_{n \to \infty} |A(n)|/n$ (if it exists) is called the density of set $A$. It turns out that, as $n \to \infty$, the cardinality $|S_{n,A}|$ of the set $S_{n,A}$ essentially depends on $\rho$. The case $\rho > 0$ was studied in detail by several authors under certain additional conditions on $A$. In 1999, Kolchin \cite{10} noticed that there is a lack of studies on classes of permutations for which $\rho = 0$. In this context, he also proposed investigations of certain particular cases. In this paper, we consider the permutations whose cycle lengths are prime numbers, that is, we assume that $A = \mathcal{P}$, where $\mathcal{P}$ denotes the set of all primes. For this class of permutations, the Prime Number Theorem implies that $\rho = 0$. In this paper, we show that, as $n \to \infty$, the ratio $|S_{n,\mathcal{P}}|/(n-1)!$ approaches a finite limit, which we determine explicitly. Our method of proof employs classical Tauberian theorems.

Key words: $A$-permutations, primes, Tauberian type theorems, Mertens theorem

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1 Introduction, Motivation and Statement of the Main Result

We start with some notation and conventions that will be used freely in the text of the paper.

The letter $p$ without subscript denotes a prime number. We write $\mathcal{P}$ for the set of all primes. If the primes in $\mathcal{P}$ are arranged in increasing order, then $p_k$ denotes the $k$th smallest prime, $k = 1, 2, \ldots$ For a function $\psi(y)$ of a real variable $y$ which assumes real or complex values, we write $\sum_p \psi(p)$ instead of $\sum_{p \in \mathcal{P}} \psi(p)$. We also use the usual notation $\pi(y)$ for the number of primes not exceeding $y > 0$.

Further on, $A$ always denotes a set of positive integers. Let $[n] = \{1, 2, \ldots, n\}$. Then, we define the set $A(n) = A \cap [n]$. We denote the cardinality of any set $M$ by $|M|$. The limit

$$\rho = \lim_{n \to \infty} \frac{|A(n)|}{n},$$

if it exists, is called the density of the set $A$. Clearly, $0 \leq \rho \leq 1$.

We write $S_{n,A}$ for the set of permutations of $n$ letters whose cycle lengths belong to $A$. We denote the cardinality of $S_{n,A}$ by $P_{n,A} = |S_{n,A}|$. For convenience, we also let denote $P_n = |S_{n,\mathcal{P}}|$ if $A = \mathcal{P}$ and $P_{n,1} = |S_{n,\mathcal{P} \cup \{1\}}|$ if $A = \mathcal{P} \cup \{1\}$.

For two sequences of real numbers $\{u_n\}_{n \geq 1}$ and $\{v_n\}_{n \geq 1}$, we write $u_n \sim v_n$ if $\lim_{n \to \infty} u_n/v_n = 1$. In a similar and traditional manner, we shall also explore the commonly used symbol $O(.)$; for its definition, we refer the reader, e.g., to [3, 14, 19].

Finally, we give the approximations of two constants that will be used further, namely the Euler’s constant

$$\gamma = 0.577215$$

and the constant

$$c = \gamma - \sum_p \left( \log \left( \frac{1}{1 - 1/p} - \frac{1}{p} \right) \right) = 0.261497,$$

which appears in an asymptotic estimate of the partial sum $\sum_{p \leq y} 1/p$ and in Mertens’ formula for the partial product $\prod_{p \leq y} (1 - 1/p)$ as $y \to \infty$. More details may be found in [19, Theorems 1.10, 1.12].
The problem on the asymptotic enumeration of permutations whose cycle lengths are constrained has attracted the interest of several authors in the middle of the last century. For more details, we refer the reader e.g., to [10, Section 4.4 and Chapter 5], [14, Section 8.2], [15], and [21, Chapter 3]. The last two chapters of Kolchin’s monograph [10] are devoted to important asymptotic results on the cardinalities $|S_{n,A}|$ for various sets $A$. He also discussed in detail the existing approaches and methods applied in this area. It seems that one of the main reasons for the interest in asymptotic enumeration problems of this type is the relationship between the cardinalities of sets of permutations $S_{n,A}$, for certain particular sets $A$, and the theory of equations containing an unknown permutation of $n$ letters.

The enumeration of permutations with cyclic structure constrained by a certain set $A$ is based on the following generating function:

$$f_A(z) := \sum_{n=0}^{\infty} \frac{P_{n,A}z^n}{n!} = \exp \left( \sum_{k \in A} \frac{z^k}{k} \right).$$

(1.3)

A proof of this identity may be found, e.g., in [10, Theorem 5.1.2]. It turns out that, for several $A$’s, the asymptotic behavior of $P_{n,A}$ depends on the density $\rho$, given by (1.1). The case $\rho > 0$ was studied under several additional conditions on $A$ by many authors. As an illustration, below we give a typical asymptotic result obtained by Yakimiv in [20] (see also [21, Theorem 3.3.1]).

**Theorem 1.1** (See Yakimiv [20],) Suppose that, for a certain $A$, its density is $\rho > 0$. Moreover, for $m \geq n$ and $m = O(n)$, we assume that

$$\left|\left\{k : k \leq m, k \in A, m - k \in A\right\}\right|/n \to \rho^2$$

(1.4)

as $n \to \infty$. Then, we have

$$P_{n,A} \sim n!n^{\rho-1}e^{L(n) - \gamma\rho}/\Gamma(\rho),$$

(1.5)

where $L(n) = \sum_{k \in A(n)} 1/k - \rho \log n$ and $\Gamma(.)$ is the Euler’s gamma function.

**Remark 1.1** Condition (1.4) shows that the set $A$ may be considered as a realization of a random set containing each integer with probability $\rho$ independently from the other integers. It is also interesting to note that, under entirely different conditions on $A$, Kolchin [9] (see also [10, Theorem 4.4.10]) obtained the same asymptotic equivalence (1.5).

3
There are many examples of permutations with constraints on their cycle structure whose basic generating function \((1.3)\) has a single dominant singularity in the complex plane at \(z = 1\). In \([14\text{ Sections } 11 \text{ and } 12]\), Odlyzko classified the methods used to extract asymptotic information about coefficients of analytic generating functions. He suggested two main classes of generating functions, depending on whether their main singularity is large or small. Functions with large singularities (i.e., ones that grow rapidly as the argument approaches the circle of convergence) are usually analyzed using the saddle point method; see \([14\text{ Sections } 12.1 \text{ and } 12.2]\). In \([10\text{ Chapters } 4 \text{ and } 5]\) Kolchin demonstrated this method in the context of permutations with constraints on their cycles. On the other hand, some generating functions of the form \((1.3)\) have small singularities on the circle of convergence and admit applications of other methods: Tauberian theory and transfer theorems (including those due to Darboux and Jungen); see \([14\text{ Sections } 8.2, \text{11.1, and } 11.2]\).

In \([10\text{ Section } 4.4]\), Kolchin surveyed asymptotic results on constrained permutations whose density \(\rho\) is positive and noticed that there is a lack of studies in the case \(\rho = 0\). Kolchin concluded his discussion with an open problem: he proposed to handle particular cases of A’s with density \(\rho = 0\). This, in fact, motivates us to study the class of permutations \(S_{n,p}\) with cardinality \(P_n\). The Prime Number Theorem in its minimal form \((\pi(n) \sim \frac{n}{\log n}, n \to \infty)\) implies that \(\rho = 0\). (For more details, historical remarks and stronger results on the remainder term in the Prime Number Theorem, we refer the reader to \([19\text{ p. } 12 \text{ and Chapter II.4}]\).) We also note that \(P_n/n!\) is the probability that a permutation of \(n\) letters, chosen uniformly at random from the set of all such permutations, has cycle lengths which are prime numbers. The sequence \((P_n)_{n \geq 1}\) is A218002 in the On-Line Encyclopedia of Integer Sequences \([17]\).

Below we state our main result.

**Theorem 1.2** We have

\[
\lim_{n \to \infty} \frac{P_n}{(n-1)!} = e^c,
\]

where \(c\) is the constant given by \((1.2)\) and \(e^c \approx 1.298873\).

**Remark 1.2** From Theorem \([1.2]\) it follows that, for large enough \(n\), the probability that a random permutation has cycle lengths, which are prime
Remark 1.3 It is well known that the set of the permutations of $[n]$, equipped with the function composition as a group operation, is the symmetric group $S_n$ of $n$ elements. The trivial permutation (bijection) $I$ that assigns each element of $[n]$ to itself serves as an identity for $S_n$. The order $O_n(s)$ of a permutation $s \in S_n$ is the smallest positive integer $m$ such that $s^m = I$, where $s^m$ denotes the $m$th iteration of $s$. It is easy to see that $O_n(s)$ equals the least common multiple of the cycle lengths of $s$. Let $T_n(s)$ be the product of the cycle lengths of the permutation $s$. Clearly, for all $s \in S_n$, we have $O_n(s) = T_n(s)$. It turns out that, for a permutation $s$ selected uniformly at random from $S_n$, $\log O_n$, suitably centered and scaled, converges weakly, as $n \to \infty$, to the standard normal distribution. There are several proofs of this remarkable theorem; the first one was given by Erdős and Turán [4]. For more details and references, we refer the reader to [1, Sections 1.1 and 5.6] and [10, Section 5.4]. Erdős and Turán’s proof is based on showing that $\log O_n$ is relatively close to $\log T_n$, while the latter quantity can be analyzed in a simpler way. Our Theorem 1.2 shows that, for large $n$, the probability that $O_n$ and $T_n$ completely coincide approaches the ratio given by (1.6).

The proof of Theorem 1.2 relies on (i) classical results from number theory (Mertens’ theorems on a sum and product of quantities depending on the primes and a weak form of the prime number theorem; see, e.g., [19, Chapter I.1]), (ii) a classical tauberian theorem due to Hardy and Littlewood [8] (see also [16, Section 9]) and (iii) a simple technical lemma dealing with formal power series (see, e.g., [2] and [14, Section 7].

Our paper is organized as follows. In the next Section 2, we give some preliminary results. First, we briefly discuss the location of the singularities of the underlying generating function in the complex domain and show why the application of Flajolet and Odlyzko’s transfer theorems [6], described also in [14, Section 11.1], fails. Our method of proof avoids the complex plane and tools from complex analysis. We use a well known real Tauberian theorem due to Hardy and Littlewood [8]. As a preparation, we first prove two lemmas establishing the behavior of the series $\sum_p z^p/p$ and $\sum_p z^p$ as $z \to 1^-$. For the sake of completeness, at the end of Section 2, we include the statements of the aforementioned Tauberian theorem and a technical lemma related to formal power series. The proof of Theorem 1.2, based on the verification of the Hardy-Littlewood’s tauberian conditions, is given in Section 3.
2 Preliminary Results

We first set in (1.3) \( A = \mathcal{P} \), \( P_{n,A} = P_n \), and \( f_A(z) = f(z) \). Hence we have

\[ f(z) := \sum_{n=0}^{\infty} \frac{P_n z^n}{n!} = \exp \left( \sum_{p} \frac{z^p}{p} \right). \]  

(2.1)

For convenience, we also let \( \varphi(z) \) denote the logarithm of the function \( f(z) \), namely,

\[ \varphi(z) := \sum_{p} \frac{z^p}{p} = \sum_{n=1}^{\infty} \frac{z^{p_n}}{p_n}, \]  

(2.2)

where in the last equality we have used the sequence of primes arranged in increasing order. (For complex values of \( z \), by \( \log z \) we mean the main branch of the logarithmic function, defined by the usual agreement: \( \log z < 0 \) if \( 0 < z < 1 \).) It is clear that \( \varphi(z) \) is analytic for \( |z| < 1 \). Moreover, Fabry gap theorem immediately implies that the circle of convergence \( |z| = 1 \) is a natural boundary of \( \varphi \) and \( f \). For the sake of completeness, we state it below and refer the reader to [11, Section 5.3] for its proof.

**Theorem 2.1** (See [11].) Suppose that the power series

\[ g(z) = \sum_{n=1}^{\infty} a_n z^{\lambda_n} \]

has radius of convergence 1, where \( 0 \leq \lambda_1 < \lambda_2 < \ldots \). If \( \lim_{n \to \infty} \lambda_n / n = \infty \), then the circle of convergence \( |z| = 1 \) is a natural boundary of \( g \).

To show how Fabry gap theorem can be applied, we recall that from the prime number theorem in its minimal form, namely \( \pi(y) \sim y / \log y \) as \( y \to \infty \), it follows that the \( n \)th smallest prime \( p_n \) satisfies

\[ p_n \sim n \log n \quad \text{as} \quad n \to \infty; \]  

(2.3)

see, e.g., [19] Part I, Chapter 1.0, Exercise 14]. Setting \( \lambda_n = p_n, n = 1, 2, \ldots \), in Theorem 2.1 from (2.3) we observe that \( |z| = 1 \) is a natural boundary of \( \varphi \) and \( f \).

It is now clear that applications of Flajolet and Odlandko’s transfer theorems [6] and Darboux and Jungen theorems [14, Sections 11.1 and 2] fail.
since both require an analytic continuation of \( f(z) \) beyond the circle of convergence \(|z| = 1\). The aforementioned methods are usually used to extract asymptotic information for the coefficients of analytic generating functions with small dominant singularities on their circle of convergence (recall that a classification of the singularities of the generating functions and a discussion on this topic may be found in [14, Chapter 11]). We will show further that the function \( f(z) \) has logarithmic order of growth at \( z = 1 \), which implies that \( z = 1 \) is a small singular point. Hence an application of the saddle point method for extracting information about the coefficients of \( f(z) \) also fails. In fact, \( f(z) \) repeats the asymptotic properties for an invalid application of this method given by Odlyzko in [14, Example 12.3]. As mentioned earlier, we prefer to base our proof on a classical Tauberian theorem due to Hardy and Littlewood [8]. We state it below in the form given by Postnikov [16, Section 9].

**Theorem 2.2** Let the coefficients of the series \( h(z) = \sum_{n=0}^{\infty} h_n z^n \) with radius of convergence 1 be real. Suppose that the following two conditions are satisfied:

a) The limit

\[
\lim_{z \to 1^-} h(z) = h,
\]

exists, and

b) there is a constant \( C > 0 \), such that

\[
h_n \leq \frac{C}{n} \quad \text{or} \quad h_n \geq -\frac{C}{n}.
\]

Then \( \sum_{n=0}^{\infty} h_n \) converges, and its sum is \( h \).

Our next task in this section is to take up the question concerning the asymptotic behavior of \( \varphi'(z) \) and \( f(z) \) as \( z \to 1^- \), where \( \varphi \) and \( f \) are defined by (2.2) and (2.1), respectively. We start with a lemma dealing with \( \varphi' \). A similar statement without a precise specification of the error term given below can be found in [12, Lemma 4].

**Lemma 2.1** If \( \varphi(z) \) is defined by (2.2), then, as \( z \to 1^- \), we have

\[
\varphi'(z) = \frac{1}{(1-z) \log \frac{1}{1-z}} + O \left( \frac{1}{(1-z) \log^2 \frac{1}{1-z}} \right).
\]
Proof. It follows from (2.2) that
\[ \varphi'(z) = \frac{1}{z} \sum_p z^p. \] (2.4)

Further on, we apply an argument similar to that given in [5] and [18]. It uses an integral representation of \( \sum_p z^p \) based on an extension of the definition of the prime number counting function \( \pi(y) \) given as follows: \( \pi(y) = 0 \) for \( 0 \leq y < p_1 = 2 \) and \( \pi(y) = n \) for \( p_n \leq y < p_{n+1}, n = 1, 2, \ldots \), where \( (p_n)_{n\geq1} \) is the sequence of the primes arranged in increasing order. For convenience, in (2.4) we change the variable \( z \) by
\[ z = e^{-t}, \quad t > 0. \] (2.5)

Hence, as \( t \to 0^+ \), we obtain
\[ \varphi'(z) \big|_{z=e^{-t}} = (1 + O(t)) \int_0^\infty e^{-yt} d\pi(y) = (1 + O(t)) \int_0^\infty te^{-yt} \pi(y) dy \]
\[ = (1 + O(t)) \int_0^\infty \pi(x/t) e^{-x} dx = (1 + O(t))(I_1(t) + I_2(t)), \] (2.6)

where in the second relation we integrated by parts and in the last one we set
\[ I_1(t) = \int_0^{1/2} \pi(x/t) e^{-x} dx, \quad I_2(t) = \int_{1/2}^\infty \pi(x/t) e^{-x} dx. \]

For \( I_1(t) \) we use the trivial bound \( \pi(x/t) \leq x/t \). Thus, for enough small \( t > 0 \),
\[ 0 \leq I_1(t) \leq \frac{1}{t} \int_0^{1/2} x e^{-x} dx \]
\[ = \frac{1}{t} \left( -xe^{-x} \big|_0^{1/2} + \int_0^{1/2} e^{-x} dx \right) = \frac{1}{t} O(t^{1/2}) = O(t^{-1/2}). \] (2.7)

The estimate of \( I_2(t) \) follows from a weak form of the prime number theorem (see, e.g., [19, p. 22]). It is enough for us to deal with the following estimate:
\[ \pi(y) = \frac{y}{\log y} + O\left( \frac{y}{\log^2 y} \right), \quad y > 1. \]
(For a more precise estimate of the error term in the Prime Number Theorem, we refer the reader to [19, Chapter II, Section 4.1]). Furthermore, for \( x \geq t^{1/2} \), we have \( \log x \geq -\frac{1}{2} \log \frac{1}{t} \). Hence we get

\[
\pi(x/t) = \frac{x}{t \log \frac{1}{t} + \log x} + O\left(\frac{x}{t \left( \log \frac{1}{t} + \log x \right)^{2}}\right)
\]

\[
= \frac{x}{t \log \frac{1}{t}} \left( 1 + O\left(\frac{|\log x|}{\log \frac{1}{t}}\right)\right) + O\left(\frac{x}{t \log^{2} \frac{1}{t}}\right)
\]

\[
= \frac{x}{t \log \frac{1}{t}} + O\left(\frac{x(1 + |\log x|)}{t \log^{2} \frac{1}{t}}\right). \tag{2.8}
\]

We recall that in (2.7) we have already shown that

\[
\int_{0}^{t^{1/2}} xe^{-x} dx = O(t^{1/2}). \tag{2.9}
\]

Combining (2.8) with (2.9), we obtain

\[
I_{2}(t) = \frac{1}{t \log \frac{1}{t}} \int_{t^{1/2}}^{\infty} xe^{-x} dx + O\left(\frac{1}{t \log^{2} \frac{1}{t}} \int_{t^{1/2}}^{\infty} x(1 + |\log x|)e^{-x} dx\right)
\]

\[
= \frac{1}{t \log \frac{1}{t}} \left( \int_{0}^{\infty} xe^{-x} dx + O(t^{1/2}) \right) + O\left(\frac{1}{t \log^{2} \frac{1}{t}}\right)
\]

\[
= \frac{1}{t \log \frac{1}{t}} + O\left(\frac{1}{t \log^{2} \frac{1}{t}}\right) + O\left(\frac{1}{t \log \frac{1}{t}}\right)
\]

\[
= \frac{1}{t \log \frac{1}{t}} + O\left(\frac{1}{t \log^{2} \frac{1}{t}}\right), \quad t \rightarrow 0^{+}. \tag{2.10}
\]

Hence, by (2.6), (2.7) and (2.10),

\[
\varphi'(z)|_{z=e^{-t}} = \frac{1}{t \log \frac{1}{t}} + O\left(\frac{1}{t \log^{2} \frac{1}{t}}\right), \quad t \rightarrow 0^{+}. \tag{2.11}
\]

To complete the proof of the lemma, it remains to recall (2.5) and notice that

\[
t = \log \frac{1}{z} = -\log z = -\log (1 - (1 - z))
\]

\[
= 1 - z + O((1 - z)^{2}), \quad z \rightarrow 1^{-}. \quad \Box \tag{2.12}
\]
Remark 2.1 We note that De Bruijn \cite[Section 3.14, Exercise 5]{3} proposed an asymptotic series in the sense of Poincaré for $\varphi'(z)$ by using a different method.

We now turn our attention to the asymptotic behavior of the generating function $f(z)$, defined by \eqref{2.1}. A weaker form of the next lemma is given in \cite{13}.

**Lemma 2.2** We have

$$f(z) = e^{c \log \frac{1}{1-z}} + O(1),$$

as $z \to 1^-$, where $c$ and $e^c$ are the constants, defined in Theorem 1.2.

**Proof.** We again make the change of variable \eqref{2.5} and consider first $\varphi(e^{-t})$, defined by \eqref{2.2}. We start our computations with the following representation:

$$\varphi(e^{-t}) = \sum_p \frac{1 - (1 - e^{-pt})}{p} = \varphi_1(e^{-t}) + \varphi_2(e^{-t}) + \varphi_3(e^{-t}),$$

where

$$\varphi_1(e^{-t}) = \sum_{p \leq \frac{1}{t}} \frac{1}{p},$$

$$\varphi_2(e^{-t}) = -\sum_{p \leq \frac{1}{t}} \frac{1 - e^{-pt}}{p},$$

$$\varphi_3(e^{-t}) = \sum_{p > \frac{1}{t}} \frac{e^{-pt}}{p}.$$

The asymptotic estimate of $\varphi_1(e^{-t})$, as $t \to 0^+$, is obtained by means of Metens’ first and second theorems from number theory; see, e.g., \cite[Theorems 1.10 and 1.12]{19}. These theorems provide an estimate for the asymptotic growth of the partial sum $\sum_{p \leq y} 1/p$ as $y \to \infty$. Thus, for $\varphi_1$, we obtain:

$$\varphi_1(e^{-t}) = \log \log \frac{1}{t} + c + O \left( \frac{1}{\log \frac{1}{t}} \right),$$

(2.14)
where the constant $c$ is defined by (1.2). The asymptotic estimate of $\varphi_2(e^{-t})$ will be based on the Prime Number Theorem in its minimal form. We apply first the inequality \((1 - e^{-pt})/p \leq t\) to each term in $\varphi_2(e^{-t})$. Then, from the Prime Number Theorem it follows that

$$|\varphi_2(e^{-t})| \leq t \sum_{p \leq \frac{1}{t}} 1 \sim \frac{1}{\log \frac{1}{t}},$$

(2.15)
as $t \to 0^+$. Finally, for $\varphi_3(e^{-t})$, we use the estimate (2.11). We have

$$\varphi_3(e^{-t}) \leq t \sum_{p > \frac{1}{t}} e^{-pt} < t \sum_{p} e^{-pt} = \frac{1}{\log \frac{1}{t}} + O \left( \frac{1}{\log^2 \frac{1}{t}} \right) = O \left( \frac{1}{\log \frac{1}{t}} \right).$$

(2.16)

Combining (2.13) - (2.16) all together, we deduce that

$$\varphi(e^{-t}) = \log \log \frac{1}{t} + c + O \left( \frac{1}{\log \frac{1}{t}} \right), \quad t \to 0^+. $$

Then, by (2.1) and (2.2)

$$f(e^{-t}) = e^c \left( \log \frac{1}{t} \right) \left( 1 + O \left( \frac{1}{\log \frac{1}{t}} \right) \right) = e^c \log \frac{1}{t} + O(1), \quad t \to 0^+, \quad (2.17)$$

and an application of (2.12) completes the proof. \(\square\)

**Remark 2.2** From Lemma 2.2 it follows that the generating function $f(z)$ satisfies the hypotheses of the well-known **Hardy-Littlewood-Karamata Tauberian theorem** (see, e.g., [7, Section 7.11] and [14, Theorem 8.7]). This theorem deals with a series $\sum_{n \geq 0} b_n z^n$ with non-negative coefficients $b_n$ and radius of convergence $1$, which, as $z \to 1^-$, satisfies the asymptotic equivalence $\sum_{n \geq 0} b_n z^n \sim (1 - z)^{-\alpha} L(1/(1 - z))$, with $\alpha \geq 0$, where the function $L(y)$ varies slowly at infinity (that is, for every $u > 0$, $L(uy) \sim L(y)$ as $y \to \infty$). For $f(z)$, we have $\alpha = 0$ and $L(y) = \log y$. Hence Hardy-Littlewood-Karamata’s theorem implies that

$$\sum_{k=0}^{n} \frac{P_k}{k!} \sim e^c \log n,$$

which is the main result of [13].
We conclude this section with an easy general result that is applicable both to convergent and purely formal power series. For its proof and more comments, we refer the reader to [2] (see also [14, Section 7]).

Lemma 2.3 Suppose that \( F(z) = \sum_{n \geq 0} F_n z^n \) and \( G(z) = \sum_{n \geq 0} G_n z^n \) are power series with radii of convergence \( r_F > r_G \geq 0 \), respectively. Suppose that \( G_n - 1/G_n \to r_G \) as \( n \to \infty \). If \( F(r_G) \neq 0 \), and

\[
\begin{align*}
H(z) &= F(z)G(z) = \sum_{n \geq 0} H_n z^n, \\
\end{align*}
\]

then, as \( n \to \infty \),

\[
H_n \sim F(r_G)G_n. 
\]

(2.18)

3 Proof of Theorem 1.2

We first note that the sequence \( (P_n/(n-1)!)_{n \geq 0} \) is not monotonic for \( n > 3 \), which makes the proof of Theorem 1.2 not straightforward. (We recall that some properties of the sequence \( (P_n)_{n \geq 0} \) are discussed in [17, A218002].) One way to overcome possible difficulties is to extend the set of permutations \( S_{n,P} \) to \( S_{n,P \cup \{1\}} \). It is clear that, in addition to cycles of lengths from \( P \), the permutations from \( S_{n,P \cup \{1\}} \) may also contain cycles of length 1 and thus \( S_{n,P} \subset S_{n,P \cup \{1\}} \). So, we first take up the question of enumerating the cardinalities \( P_{n,1} = |S_{n,P \cup \{1\}}| \). Let \( f_1(z) \) denote the exponential generating function of the sequence \( (P_{n,1})_{n \geq 0} \). Then, by (2.1) and (2.2),

\[

t_{1}(z) := \sum_{n=8}^{\infty} \frac{P_{n,1} z^n}{n!} = e^{z+\varphi(z)} = e^z f(z),
\]

(3.1)

\[
\begin{align*}
{f_1}'(z) &= \sum_{n=1}^{\infty} \frac{P_{n,1}}{(n-1)!} z^{n-1} = e^z (f(z) + f'(z)) = e^z f(z)(\varphi'(z) + 1).
\end{align*}
\]

(3.2)

Our proof consists of two parts. First, we shall prove that \( (1 - z)f_1'(z) \) satisfies the Hardy-Littlewood’s conditions of Theorem 2.2, which in turn will give the limit of the ratio \( P_{n,1}/(n-1)! \) as \( n \to \infty \). Then, using Lemma 2.3, we shall transfer this result to the limit of the sequence \( (P_n/(n-1)!)_{n \geq 0} \).

To verify condition a) of Theorem 2.2, we first apply the estimates obtained in Lemmas 2.1 and 2.2 inserting them into the right-hand side of (3.2).
We have
\[ f'_1(z) = \left( e^{c+1}(1 + o(1)) \log \frac{1}{1 - z} + O(1) \right) \]
\[ \times \left( 1 + \frac{1}{(1 - z) \log \frac{1}{1 - z}} + O \left( \frac{1}{(1 - z) \log^2 \frac{1}{1 - z}} \right) \right) \]
\[ = \frac{e^{c+1}}{1 - z} + O \left( \log \frac{1}{1 - z} \right) + O \left( \frac{1}{(1 - z) \log \frac{1}{1 - z}} \right) \]
\[ = \frac{e^{c+1}}{1 - z} + O \left( \frac{1}{(1 - z) \log \frac{1}{1 - z}} \right), \quad z \to 1^{-}. \]

Therefore
\[ \lim_{z \to 1^{-}} (1 - z)f'_1(z) = e^{c+1}, \]
and so, \((1 - z)f'_1(z)\) satisfies the Hardy-Littlewood’s condition a) with \(h = e^{c+1}\).

We next focus on the second inequality of condition b) of Theorem 2.2. The coefficient of \(z^n\) in \((1 - z)f'_1(z)\) equals
\[ \frac{P_{n+1,1}}{n!} \frac{n!}{(n-1)!} = \frac{1}{n!}(P_{n+1,1} - np_{n,1}), \quad n \geq 1. \quad (3.3) \]

This coefficient is non-negative if and only if, for every \(n \geq 1\),
\[ P_{n+1,1} - np_{n,1} \geq 0. \quad (3.4) \]
To show that this is true, we shall apply here a simple combinatorial argument. We shall construct a subset
\[ \Sigma_{n+1} \subset S_{n+1, \mathbb{P} \cup \{1\}} \quad (3.5) \]
with cardinality \(|\Sigma_{n+1}| = nP_{n,1}\) in the following way: (i) we select an element \(j \in [n]\) in \(n\) ways and compose an one-element cycle \(j \leadsto j\) of a permutation \(s \in \Sigma_{n+1}\); (ii) then, we consider the \(n\)-element set \([n+1] \setminus \{j\}\) and construct the other cycles of \(s\), whose lengths are primes, in \(P_{n,1}\) ways. So, we have established (3.5), which in turn implies the inequality (3.4). Thus, using (3.3), we conclude that, for any \(C > 0\), \((1 - z)f'_1(z)\) satisfies condition b) of Theorem 2.2. Since the \(n\)th partial sum of the series
\[ P_{1,1} + \sum_{n=1}^{\infty} \left( \frac{P_{n+1,1}}{n!} \frac{n!}{(n-1)!} - \frac{P_{n,1}}{(n-1)!} \right) \]
we get the desired result.
equals $P_{n,1}/(n-1)!$, from Theorem 2.2 it follows that
\[
\lim_{n \to \infty} \frac{P_{n,1}}{(n-1)!} = e^{c+1}.
\] (3.6)

We now proceed to the application of Lemma 2.3. The right-hand side of (3.1) suggests to set in Lemma 2.3 $H(z) = f_1(z)$, $F(z) = e^z$ and $G(z) = f(z)$. Moreover, we obviously have $r_F = \infty$. Thus we can readily get the limit given in Theorem 1.2, if we show that
\[
r_G = \lim_{n \to \infty} \frac{P_{n,1}/(n-1)!}{P_{n+1,1}/n!} = \lim_{n \to \infty} \frac{n P_{n,1}}{P_{n+1,1}} = 1.
\] (3.7)

Indeed, if (3.7) is true, then (2.18) of Lemma 2.3 becomes
\[
e^{c+1} = e \lim_{n \to \infty} \frac{P_n}{(n-1)!},
\]
which completes the proof of Theorem 1.2. To establish (3.7), we first note that, for all $n \geq 1$, (3.4) implies the following inequality:
\[
\frac{P_{n,1}/(n-1)!}{P_{n+1,1}/n!} \leq 1.
\] (3.8)

The corresponding lower bound for this ratio follows from (3.6). In fact, for any $\epsilon \in (0, e^{c+1}/2)$, there exists a number $n_0 = n_0(\epsilon)$ such that, for all $n \geq n_0$,
\[
e^{c+1} - \epsilon < \frac{P_{n,1}}{(n-1)!} < e^{c+1} + \epsilon.
\]
Hence it is not difficult to deduce the bound:
\[
\frac{P_{n,1}/(n-1)!}{P_{n+1,1}/n!} > 1 - 2e^{-c-1}\epsilon.
\] (3.9)

Combining (3.8) and (3.9), we obtain (3.7) and thus the proof of the theorem is completed. □

**Remark 3.1** It is possible to obtain a more precise result than that in Theorem 1.2 applying the quantitative form of Hardy-Littlewood’s theorem given by Postnikov [16, Section 13, Corollary 1]. Thus, after some additional computations, one can get an estimate for the error term in Theorem 1.2. We conjecture that it is of order $O(1/\log \log n)$. 

14
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