Binary input reconstruction for linear systems: a performance analysis

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Abstract

Recovering the digital input of a time-discrete linear system from its (noisy) output is a significant challenge in the fields of data transmission, deconvolution, channel equalization, and inverse modeling. A variety of algorithms have been developed for this purpose in the last decades, addressed to different models and performance/complexity requirements. In this paper, we implement a straightforward algorithm to reconstruct the binary input of a one-dimensional linear system with known probabilistic properties. Although suboptimal, this algorithm presents two main advantages: it works online (given the current output measurement, it decodes the current input bit) and has very low complexity. Moreover, we can theoretically analyze its performance: using results on convergence of probability measures, Markov Processes, and Iterated Random Functions we evaluate its long-time behavior in terms of mean square error.

1 Introduction

Consider the input/output linear system

\[
\begin{align*}
  x_k &= q x_{k-1} + w u_{k-1} \quad k = 1, \ldots, K \\
  y_k &= cx_k + n_k
\end{align*}
\]

with \( K \in \mathbb{N} \) (possibly tending to infinity), \( u_k \in \{0, 1\} \) for \( k = 0, \ldots, K - 1 \), \( x_k \in \mathbb{R} \) for \( k = 0, \ldots, K \), \( y_k, n_k \in \mathbb{R} \) for \( k = 1, \ldots, K \), \( q, w, c \in \mathbb{R} \), and \( q \in (0, 1) \) to preserve stability. Our aim is to recover the binary input \( u_k \), in an online fashion, given the output \( y_k \) corrupted by a noise \( n_k \). To this purpose, we retrieve a low-complexity algorithm introduced in Fagnani and Fosson (2009) and discussed in Fosson (2011a,b), and we propose a comprehensive theoretical analysis of its performance. As a result of the analysis, we will be able to evaluate the performance as a function of the system’s parameters.
The digital signal reconstruction problem is a paradigm in data transmissions, where signals arising from finite alphabets are sent over noisy continuous channels, and in hybrid frameworks, where digital and analog signals have to be merged in the same system. In Fagnani and Fosson (2009), a particular instance of model (1) was derived as time discretization of a convolution system and the input estimation described as a deconvolution problem. The same can be achieved for model (1): if we consider the system

\[
\begin{align*}
  x'(t) &= ax(t) + bu(t) \quad t \in [0, T] \\
  y(t) &= cx(t) + n(t) \quad x(0) = x_0 \\
  u(t), x(t), y(t), \quad a, b, c \in \mathbb{R}, a < 0
\end{align*}
\]  

we have

\[
x(t) = e^{ta}x_0 + b \int_0^t e^{a(t-s)}u(s)ds.
\]

(3)

Given

\[
u(t) = \sum_{k=0}^{K-1} u_k \mathbb{1}_{[k\tau,(k+1)\tau]}(t), \quad u_k \in \mathcal{U} = \{0, 1\}, \quad \tau > 0
\]

(4)

we can discretize in the following way: by defining

\[
q := e^{\tau a} \in (0, 1) \\
w := \frac{1 - e^{\tau a}}{-a} = -\frac{b}{a}(1 - q)
\]

(5)

we obtain

\[
x_k = q^k x_0 + bq^k \int_0^{k\tau} e^{-as} \sum_{h=0}^{K-1} u_h \mathbb{1}_{[h\tau,(h+1)\tau]}(s) ds
\]

\[
= q^k x_0 + bq^k \sum_{h=0}^{k-1} u_h \int_{h\tau}^{(h+1)\tau} e^{-as} ds
\]

\[
= q^k x_0 + \frac{b}{-a} \sum_{h=0}^{k-1} u_h e^{-a(h+1)\tau}(1 - e^{a\tau})
\]

\[
= q^k x_0 + w \sum_{h=0}^{k-1} u_h q^{k-1-h}
\]

(6)

from which we have the recursive formula

\[
x_k = qx_0 + wu_{k-1}.
\]

(7)
In system (2), recovering $u(t)$ basically consists in the inversion of the convolution integral $y(t) = c e^{ta} x_0 + cb \int_0^t e^{a(t-s)} u(s) d + n(t)$ (where $n(t)$ represents an additive noise), which is a long-standing mathematical ill-posed problem: small observation errors may produce defective solutions. Several estimation approaches have been studied in the last fifty years and the literature on deconvolution is widespread: we refer the reader to early papers Tikhonov (1963); Tikhonov and Arsenin (1977) and to later Arya and Holden (1978); Sparacino and Cobelli (1996); Starck et al. (2002); Fagnani and Pandolfi (2002), which show also some possible applications in geophysics, astronomy, image processing and biomedical systems. For more references, see Fosson (2011).

Most of known deconvolution methods exploit the regularity of the input function to provide good estimations. This work instead is a contribution for deconvolution in case of discontinuous input functions.

Considering a binary alphabet, which has been chosen mainly to keep the analysis straightforward, is consistent with many applications: the output of several digital devices, such as computers and detection devices Fosson (2011a), are binary. Nevertheless, the algorithm and the analysis presented in this paper could be generalized to larger alphabets with no much effort.

In Fagnani and Fosson (2009), low-complexity decoding algorithms were introduced, derived from the optimal BCJR Bahl et al. (1974) algorithm, and applied to perform the deconvolution of the system (2) with $a = 0$ and $b = c = 1$. In this work, we apply the simplest of those algorithms, the so-called One State Algorithm (OSA for short) to the system (1). We then describe the performance in terms of Mean Square Error (MSE) for long-time transmissions, through a probabilistic analysis arising from the Markovian behavior of the algorithm. The scheme of the analysis is the same proposed in Fagnani and Fosson (2009), but leads to completely different scenarios: while for $a = 0$, $b = c = 1$ standard ergodic theorems for denumerable Markov Processes were sufficient to compute the MSE, in the present case the denumerable model does not proved the expected results, and more sophisticated arguments are used, arising from Markov Processes, Iterated Random Functions (IRF for short) and sequences of probability measures.

The paper is organized as follows. In Section 2 we complete the description of the system, giving some observations and probabilistic assumptions; in Sections 3 and 4 we present our algorithm and some simulations. The core of the paper is the performance analysis provided in Section 5. Finally we propose some concluding observations. Notice that Sections 2 and 3 mainly retrieve the model presented in Fagnani and Fosson (2009) and Fosson (2011a).
1.1 Notation

We use the following notation throughout the paper: P indicates a discrete probability, while $P(\cdot, \cdot)$ is the transition probability kernel of a Markov Process; $\mathbb{E}$ is the stochastic mean. $\mathcal{B}(S)$ indicates the Borel $\sigma$-field of a space $S$. Given a bounded measurable function $v$ defined on a space $S$, $Pv(x) = \int_S v(y)P(x, dy)$. For every measure $\mu$ on $(S, \mathcal{B}(S))$ and $F \in \mathcal{B}(S)$, $\mu P(F) = \int_D P(x, F)\mu(dx)$. The complementary error function $\text{erfc}$ is defined as $\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{+\infty} e^{-t^2} dt$, $x \in \mathbb{R}$; the indicator function $1_{A}(x)$ is equal to one if $x \in A$, and zero otherwise. Moreover we often use the following acronyms: OSA for One State Algorithm, MSE for Mean Square Error, IRF for for Iterated Random Functions, MAP for Maximum a Posteriori.

2 Problem Statement

Let us develop a deeper understanding of the problem and specify some assumptions.

First notice that, given $x_k = qx_{k-1} + w_{k-1}$, we have

$$x_k = q^k x_0 + w \sum_{h=0}^{k-1} u_{k-1-h}q^{h}$$

which shows that each $x_k$ is determined by the initial state $x_0$ and by a binary polynomial of degree $k - 1$ in $q$. From now onwards, let $x_0 = 0$, so that, for any $k = 0, 1, \ldots, K$,

$$x_k \in \mathcal{X} = \left\{ \sum_{h=0}^{K} \mu_h q^h, \mu_h \in \{0, 1\} \right\}.$$  \hspace{1cm} (8)

Moreover, let us introduce some prior probabilistic information:

Assumption 1: the additive noise $n_k$ is white Gaussian, that is, $n_1, \ldots, n_K$ are realizations of independent Gaussian random variables $N_1, \ldots, N_K$, with null mean and variance $\sigma^2$.

Assumption 2: the binary inputs $u_0, \ldots, u_{K-1}$ are realizations of independent Bernoulli random variables $U_0, \ldots, U_{K-1}$ with parameter $\frac{1}{2}$.

Input and noise are also supposed to be mutually independent. Under these assumptions the system can be rewritten in probabilistic terms as follows (capital letters are used instead of small letters to indicate random
quantities): for $k = 1, \ldots, K$,

$$
\begin{aligned}
U_{k-1} &\sim \text{Ber}(1/2) \\
N_k &\sim \mathcal{N}(0, \sigma^2) \\
X_k &= qX_{k-1} + wU_{k-1} \quad (X_0 = 0) \\
Y_k &= cX_k + N_k.
\end{aligned}
$$

(9)

While Assumption 1 is realistic in physical terms, Assumption 2 is less motivated by applications, where source bits are often not independent (for example, they may be governed by a Markov Chain). We have however imposed it for simplicity of treatment, although extensions to more sophisticated prior distributions do not require much effort. Similarly, we have chosen to propose a one-dimensional problem to make the analysis more readable, while the structure would be almost the same also for multidimensional problems (see Fosson (2011a)).

Given this setting, we aim at providing a method to decode the bit $u_{k-1}$ at each time step $k = 1, \ldots, K$, based on the current measurement $y_k$ and the probabilistic properties of the system. In order to perform this online recovery, the algorithm is allowed to store a few information (just one real value) about the state $x_k$.

## 3 Binary Input Reconstruction

The One State Algorithm (OSA for short) introduced in Fagnani and Fosson (2009) fits the requirements described in the previous section.

The OSA is a suboptimal version of the Bahl, Cocke, Jelinek, and Raviv algorithm (most known as BCJR Bahl et al. (1974)), a prominent decoding algorithm used for convolutional codes. The BCJR performs a maximum a posteriori estimation (MAP) of the input bit sequence by evaluating the probabilities of the states of the encoder, through a forward and a backward recursion; it is optimal in the sense that it minimizes the Mean Square Error:

$$
\text{MSE}(U, \hat{U}) := \frac{1}{K} \sum_{k=0}^{K-1} \mathbb{E}(U_k - \hat{U}_k)^2
$$

(10)

where $U = (U_0, \ldots, U_{K-1})$ and $\hat{U} = (\hat{U}_0, \ldots, \hat{U}_{K-1})$ is the estimated input sequence. In the binary case, this is equivalent to

$$
\text{MSE}(U, \hat{U}) = \frac{1}{K} \sum_{k=0}^{K-1} P(U_k \neq \hat{U}_k).
$$
As shown in Fagnani and Fosson (2009), the BCJR can be adapted to the binary input deconvolution problem with optimal results, but with complexity drawbacks when the transmission is long. This motivated the introduction of the OSA, a BCJR-based method that consists only in a forward recursion and that stores only one state at each iteration step. More precisely, the OSA pattern is as follows:

1. Initialization of the state estimate \( \hat{x}_0 \);

2. For \( k = 1, \ldots, K \): given \( y_k \in \mathbb{R} \) and \( \hat{x}_{k-1} \),
   \[
   \hat{u}_{k-1} = \begin{cases} 
   0 & \text{if } |y_k - cq\hat{x}_{k-1}| \leq |y_k - (cq\hat{x}_{k-1} + cw)| \\
   1 & \text{otherwise.}
   \end{cases}
   \]
   \[
   \hat{x}_k = q\hat{x}_{k-1} + w\hat{u}_{k-1}.
   \] (11)

Typically, we assume to know \( x_0 \), so that we can initialize correctly by \( \hat{x}_0 = x_0 \). This point will be discussed later in Section 5. Given the probabilistic setting previously introduced, the OSA can also be written as:

\[
\begin{cases}
   \hat{U}_{k-1} = 1_{(cq\hat{x}_{k-1} + cw, +\infty)}(Y_k) \\
   \hat{X}_k = q\hat{X}_{k-1} + w\hat{U}_{k-1}.
\end{cases}
\] (12)

While the BCJR estimates the probabilities of all the possible states at each step, the OSA individuates the most likely state and assumes it to be the correct one; on the basis of this state estimate, it decides on the current bit. The decoding is performed with a MAP decision on the current bit, which in our probabilistic setting (Bernoulli input and Gaussian noise) reduces to the comparison between two Euclidean distances.

The OSA is suboptimal, but presents two main good properties: (a) it is low-complexity, both for number of computations and storage locations; (b) it is causal, that is, it uses only the past and the present information to decode the current bit. Therefore, (a) it can be applied to our case in which the number of states is (not countably) infinite and (b) it can be used online, making unnecessary the complete transmission before starting deconvolution, this feature being fundamental to study long time transmissions.

In Fagnani and Fosson (2009), we introduced other causal algorithms: the Causal BCJR, which is a version of BCJR performing only the forward recursion, and the Two States Algorithm (TSA), which works basically the same as OSA, but estimates the two best states at each step with their probabilities of being the correct ones. The TSA is then oriented to soft
decoding; in the differentiation case \( a = 0, b = c = 1 \), it was proved to have similar performance to Causal BCJR and better than the OSA. Nevertheless, neither the Causal BCJR nor the TSA are efficient for system (1). The first one, in fact, presents complexity drawbacks due to the number of states. The second one, instead, has performance too much similar to the OSA, in spite of higher complexity: owing to the structure of the state space \( \mathcal{X} \), the two best states turn out to be very close to each other, which does not enhance the information provided by the OSA.

### 3.1 Similar algorithms

We notice that our setting and decoding procedure (11) are very similar to the Decision Feedback Equalizer introduced to mitigate the effects of channels’ intersymbol inference (ISI, see Pulford (1992) for a complete review). As in our case, the model considered in channel equalization is a linear system with digital input, and the goal is the input recovery for equalization purposes. Various methods have been proposed in literature and much effort has been addressed towards complexity reduction (see, e.g., Eyuboglu and Qureshi (1988); Duel-Hallen and Heegard (1989); Williamson et al. (1992); Quevedo et al. (2007)). Typically, complexity is reduced by collecting information only from fixed time blocks, which is also our attempt; more precisely we consider only the current measurement and an estimation of the previous state, that is, “minimal” blocks of length one, but extensions to larger time blocks are possible to improve the performance.

The recovery techniques in the cited works present many analogies with ours. For example, the method introduced in Quevedo et al. (2007), if restricted to minimal blocks, differs from ours only for the introduction of a prior distribution on the state \( x_k \).

Nevertheless, an outstanding difference lies in the model: channel equalization exploits the input estimate to provide a feedback equalizer to the system, while our final aim is just the input recovery.

Given the several connections, in our future work we will study possible implementations of our low-complexity algorithms, derived from BCJR, for channel equalization and propose more detailed comparisons.

### 4 Simulations

We now show a few simulations’ results, obtained by 2000 Monte Carlo Runs of 320 bit transmissions. We consider the system derived from (2), with \( \tau = 1, q = e^a, w = -\frac{b}{a}(1 - q) \). We show the behavior of the MSE with
Figure 1: Mean Square Error in function of the Signal-To-Noise Ratio $\frac{\sigma_w^2}{\sigma^2}$ (in dB); $b = c = 1$, $a = -2, -1, -0.5, -0.3$

Figure 2: A zoom that highlights the gain obtained by decreasing $a$. 
respect to $\frac{c^2w^2}{\sigma^2}$, that can be interpreted as the Signal-To-Noise-Ratio (SNR for short) of the transmission: since for each $k$, the transmitted signal is $cx_k \in \{cq_{x_{k-1}}, cq_{x_{k-1}} + cw\}$ then $c^2w^2$ is proportional to the signal power. As expected, the MSE tends to zero when the SNR is large, while for small SNR tends to $\frac{1}{2}$.

If we fix $b = c = 1$ and let $a$ vary, we obtain a slight gain (that is, a lower MSE curve) by decreasing $a$, as shown in Figures 1-2. In other terms, more stable systems are preferable. This phenomenon will be retrieved in Section 5.

5 Analysis of the Algorithm

For simplicity, in the next we will assume $w > 0$, the analysis in the case $w < 0$ being analogous.

The goal of this section is the analytic evaluation of the Mean Square Error for the One State Algorithm, in case of long-time transmissions.

The analysis starts from the definition of the following Markov Process:

\[
\begin{align*}
D_k &= \tilde{X}_k - X_k = qD_{k-1} + w(\tilde{U}_{k-1} + U_{k-1}) \quad k = 1, 2, \ldots \\
D_0 &= \alpha
\end{align*}
\] (13)

For any $k = 1, 2, \ldots$, if $D_{k-1} = z$ then $D_k \in \{qz, qz + w, qz - w\}$, and the transition probabilities are:

\[
P(z, qz + w) = P(\tilde{U}_k = 1, U_k = 0 | D_k = z) = \frac{1}{4} \text{erfc} \left( \frac{cqz + cw/2}{\sigma \sqrt{2}} \right)
\]

\[
P(z, qz - w) = P(\tilde{U}_k = 0, U_k = 1 | D_k = z) = \frac{1}{4} \text{erfc} \left( \frac{-cqz + cw/2}{\sigma \sqrt{2}} \right)
\]

\[
P(z, qz) = 1 - P(z, qz + w) - P(z, qz - w).
\] (14)

Since for any $k \in \mathbb{N}$, $D_k \in \left\{ w \sum_{h=0}^{k-1} \mu_h q^h, \mu_h \in \{-1, 0, 1\} \right\}$, if we fix $D_0 = 0$, $(D_k)_{k \in \mathbb{N}}$ is a Markov Process on the denumerable state space

\[
\left\{ w \sum_{h=0}^{\infty} \mu_h q^h, \mu_h \in \{-1, 0, 1\} \text{ such that } (\mu_h)_{h \in \mathbb{N}} \text{ is definitely null} \right\}.
\] (15)
By definitely null, we mean that for any $D_k$ the coefficients $\mu_h$ with $h \geq k$ are null. This set is denumerable since any $D_k$ can be seen as the ternary representation of a non-negative integer. Notice that fixing $D_0 = 0$ just means that $x_0$ is known.

The key point of the analysis is that, for large $k$, the MSE of the OSA can be computed using the ergodic properties of the Markov Process $(D_k)_{k \in \mathbb{N}}$; more precisely, we require the existence of a stationary distribution. In the next, we will propose two different ways to study the stationary distribution: the first one does not depend on the initial state $D_0 \in \mathcal{D}$ (where $\mathcal{D}$ is compact set which will be defined shortly), but requires some contractive properties; the second one is valid even in the non-contractive case, but depends on the initial state.

For both methods, the presented setting is not still adequate to study the possible stationary distributions, since the states of $(D_k)_{k \in \mathbb{N}}$ are transient: when the process visits a state, then it leaves it definitely (except for a negligible set of states that have a periodic ternary representation, for example $0, \pm w/(1 - q)$); moreover, the process is not irreducible since there is no reciprocal communication between the states (see Fagnani and Fosson (2009) for more details). Thus we conclude that no hypotheses are fulfilled to apply the standard ergodic results for denumerable Markov Processes (see Fagnani and Fosson (2009)). In other terms, if a stationary distribution exists, it does not concentrate on single states.

This suggests to consider $(D_k)_{k \in \mathbb{N}}$ on a non-denumerable state space. In particular, we can extend (15) to
\[
\left\{ w \sum_{h=0}^{\infty} \mu_h q^h, \mu_h \in \{-1, 0, 1\} \right\}.
\]

(16)

It is interesting to notice that if $q \geq \frac{1}{3}$, then the set (16) coincides with the closed interval $\left[ -\frac{w}{1-q}, \frac{w}{1-q} \right]$, while if $q < \frac{1}{3}$ it is a Cantor set included in $\left[ -\frac{w}{1-q}, +\frac{w}{1-q} \right]$ (for a proof of this fact, see Fosson (2011a)). Let us then consider as state space
\[
\mathcal{D} = \left[ -\frac{w}{1-q}, +\frac{w}{1-q} \right]
\]
and study the ergodic properties of the Markov Process $(D_k)_{k \in \mathbb{N}}$ on $\mathcal{D}$.

Before continuing the analysis, let us introduce some rigorous notions that will be used in the next.
Let $\mathcal{B}(\mathcal{D})$ be the Borel $\sigma$-field of $\mathcal{D}$. We call transition probability kernel (see, e.g., Meyn and Tweedie [1993] Section 3.4.1) an application $P : \mathcal{D} \times \mathcal{B}(\mathcal{D}) \to [0,1]$ such that

(i) for each $F \in \mathcal{B}(\mathcal{D})$, $P(\cdot, F)$ is a non-negative measurable function;
(ii) for each $x \in \mathcal{D}$, $P(x, \cdot)$ is a probability measure on $(\mathcal{D}, \mathcal{B}(\mathcal{D}))$.

Given a bounded measurable function $v$ on $\mathcal{D}$, we denote by $Pv$ the bounded measurable function defined as

$$Pv(x) = \int_{\mathcal{D}} v(y)P(x, dy).$$

Furthermore, let $\mu$ be a measure on $(\mathcal{D}, \mathcal{B}(\mathcal{D}))$: we define the measure $\mu P$ by

$$\mu P(F) = \int_{\mathcal{D}} P(x,F)\mu(dx) \quad F \in \mathcal{B}(\mathcal{D}).$$

We finally define the $n$-th power of the transition kernel $P$ by $P^1(x,F) = P(x,F)$ and $P^n(x,F) = \int_{\mathcal{D}} P(x,dy)P^{n-1}(y,F)$. It is easy to see that $P^n(x,F)$ are transition kernels, too.

At this point, we can make explicit the relationship between the MSE and $(\mathcal{D}_k)_{k \in \mathbb{N}}$. In the next, we will always consider $D_0 = 0$, if not differently specified. Given the transition probability kernel $P$ of $(\mathcal{D}_k)_{k \in \mathbb{N}}$, defined by (14), we have

$$\text{MSE}(U, \hat{U}) = \frac{1}{K} \sum_{k=0}^{K-1} P(\hat{U}_k \neq U_k) =$$

$$= \frac{1}{K} \sum_{k=0}^{K-1} \int_{\mathcal{D}} P(\hat{U}_k \neq U_k|D_k = z)P^k(\alpha, dz) = \frac{1}{K} \sum_{k=0}^{K-1} P^k g(\alpha)$$

(17)

where

$$g(z) = P(\hat{U}_k \neq U_k|D_k = z)$$

(18)

and $D_0 = \alpha$ is any initial state in $\mathcal{D}$. Therefore $P^k g$ (and in particular its behavior for large $k$) will be the object of our further analysis.

In the sequel, we will distinguish two main scenarios: when $(\mathcal{D}_k)_{k \in \mathbb{N}}$ has some contractive properties and when it has not. In the first scenario, we can exploit the theory of Iterated Random Functions to prove that $P^k g$ converges, while in the second one we will use known results of convergence of probability measures.
5.1 Contractive case

Let \( l \)-Lip(\( D \)) be the set of all the Lipschitz functions with Lipschitz constant equal to \( l \) on \( D \). We define the Kantorovich (or Wasserstein) distance \( d_W \) between probability measures (see (Rachev 1991, Section 2.1, Example 3.2.2)) as

\[
d_W(\mu, \nu) = \sup_{f \in l \text{-Lip}(D)} \left| \int_D f \, d(\mu - \nu) \right|.
\]  

We can prove the following

**Theorem 1.** If \( \frac{c^2w^2}{\sigma^2} > 4 \) and \( q < \frac{1}{3+\frac{2}{c\sqrt{w}}} \) or if \( \frac{c^2w^2}{\sigma^2} \leq 4 \) and \( q \leq \frac{1}{1+\frac{2}{c\sqrt{w}}} \),

then,

\[
\lim_{K \to \infty} \text{MSE}(U, \hat{U}) = \int_D g \, d\mu \quad \text{for any } D_0 \in \mathcal{D}
\]

where \( \mu \) is the unique probability measure such that \( \sup_{x \in \mathcal{D}} d_W(P_k(x, \cdot), \mu(\cdot)) \xrightarrow{k \to \infty} 0 \), \( P \) being the kernel of \((D_k)_{k \in \mathbb{N}}\).

Notice that \( g(z) \) is time-invariant (i.e., does not depend on \( k \)) and can be analytically computed. In fact, given \( D_{k-1} = z \), \( D_k = qz \) if and only if \( \hat{U}_k = U_k \), \( D_k = qz + w \) if and only if \( \hat{U}_k = 1 \) and \( U_k = 0 \), \( D_k = qz - w \) if and only if \( \hat{U}_k = 0 \) and \( U_k = 1 \) and

\[
g(z) = P(z, qz + w) + P(z, qz - w).
\]

Furthermore, the probability measure \( \mu \) can be numerically evaluated.

Recall that, as already noticed, \( \frac{c^2w^2}{\sigma^2} \) can be interpreted as the Signal-To-Noise-Ratio. Moreover, the bounds on \( q \) can be interpreted as the necessity of a stronger stability for convergence.

In order to prove the theorem, let us introduce some elements from the Iterated Random Functions theory.

5.1.1 Iterated Random Functions

Let \((\mathcal{D}, d)\) be a complete metric space and \( S \) be a measurable space. Consider a measurable function \( w : \mathcal{D} \times S \to \mathcal{D} \) and for each fixed \( s \in S \), \( w_s(x) := w(x, s) \), \( x \in \mathcal{D} \). Let \((I_k)_{k \in \mathbb{N}}\) be a stochastic sequence in \( S \) such that \( I_0, I_1, \ldots \) are independent, identically distributed. Then, the set \( \{w_{I_k}(x), k \in \mathbb{N}\} \) is a family of random functions. The systems obtained by iterating such random functions, called Iterated Random Functions (IRF), are studied for diverse purposes: for example, IRF with contractive properties are used to
construct fractal sets, see [Hutchinson (1981); Diaconis and Freedman (1999)].

More interesting for our study is the exploitation of IRF to study Markov Processes. Given an IRF and a starting state \( x \in \mathcal{D} \), we can define the induced Markov Process \((Z_k(x))_{k \in \mathbb{N}}\) as

\[
Z_k(x) := w_{I_{k-1}} \circ w_{I_{k-2}} \circ \cdots \circ w_{I_0}(x) \quad (k \geq 1)
\] (22)

and analyze its asymptotic behavior through the properties of \( w_{I_k}(x) \), \( k \in \mathbb{N} \). It has been proved that if the \( w_{I_k}(x) \) have some contractive properties, the transition probability kernel of \( Z_n(x) \) converges to a probability measure, unique for all the initial states \( x \in \mathcal{D} \). The required contractive properties may be slightly different: [Diaconis and Freedman (1999)] studied the case of Lipschitz functions \( w_{I_k}(x) \) “contracting on average”, while similar results have been obtained by Stenflo (2001) without the continuity requirement on \( w_{I_k}(x) \), by Steinsaltz (1999) for “locally contractive” functions, and by Jarner and L.Tweedie (2001) for “non-separating on average” functions. A useful survey on the argument has been recently proposed by Iosifescu (2009).

Let us show how to exploit the IRF theory in our framework.

The evolution of \((D_k)_{k \in \mathbb{N}}\) can be modeled by IRF. We consider the complete metric space \( \mathcal{D} = \left[-\frac{w}{1-q}, \frac{w}{1-q}\right] \) naturally endowed with the Euclidean metric \( d \) of \( \mathbb{R} \), the measurable space \( \mathcal{S} = \{0, 1\} \times \mathbb{R} \) and the stochastic process

\[
I_k = (U_k, N_{k+1}), \quad k \in \mathbb{N}
\]
on \( \mathcal{S} \), and we define the random function

\[
w_{I_k}(x) = qx + w(\text{c}_{\text{iqx}+cw(\frac{1}{2} - U_k), +\infty})(N_{k+1}) - wU_k, \quad x \in \mathcal{D}
\] (23)

that describes the dynamics of \((D_k)_{k \in \mathbb{N}}\). The key result for our purpose is the following theorem (here stated for compact spaces), which does not require continuity:

**Stenflo’s Theorem** [Stenflo 2001, Theorem 1] Suppose that there exists a constant \( l < 1 \) such that

\[
\mathbb{E}[d(w_{I_0}(x), w_{I_0}(y))] \leq l \ d(x, y)
\] (24)

for all \( x, y \in \mathcal{D} \), \((\mathcal{D}, d)\) being a compact metric space. Then there exist a unique probability measure \( \mu \) and a positive constant \( \gamma_{\mathcal{D}} \) such that

\[
\sup_{x \in \mathcal{D}} d_W \left(P^n(x, \cdot), \mu(\cdot)\right) \leq \frac{\gamma_{\mathcal{D}}}{1 - l} n \quad n \geq 0
\] (25)
where \( P^n(x, \cdot) \) is the \( n \)-step transition probability kernel of the Markov Process \( Z_n(x) \).

Now, Theorem 1 can be proved by applying the Stenflo’s Theorem.

5.1.2 Proof of Theorem 1

Let us analyze the condition (24). Consider \( x, y \in D \) with \( x > y \) (recall that \( q > 0, w > 0 \)). Let \( H = H(x, y, I_0) \) and \( \mathcal{I}_u \) respectively be defined as

\[
H := 1_{(cqx + cw(\frac{1}{2} - U_0), cqy + cw(\frac{1}{2} - U_0))}(N_1) \\
\mathcal{I}_u := \frac{1}{\sqrt{2\pi}\sigma} \int_{cqx + cw(\frac{1}{2} - u)}^{cqw(\frac{1}{2} - u)} e^{-\frac{n^2}{2\sigma^2}} \, dn \\
= \frac{1}{2} \text{erfc} \left( c\frac{qy + w(\frac{1}{2} - u)}{\sigma\sqrt{2}} \right) - \frac{1}{2} \text{erfc} \left( c\frac{qx + w(\frac{1}{2} - u)}{\sigma\sqrt{2}} \right).
\]

Hence,

\[
\mathbb{E} \left[ |w(U_0, N_1)(x) - w(U_0, N_1)(y)| \right] = \mathbb{E} \left[ |q(x - y) - wH| \right] \\
= \sum_{u \in \{0, 1\}} P(U_0 = u) \frac{1}{\sqrt{2\pi}\sigma} \int_{\mathbb{R}} e^{-\frac{n^2}{2\sigma^2}} |q(x - y) - wH| \, dn \\
= \frac{1}{2} \sum_{u \in \{0, 1\}} \frac{1}{\sqrt{2\pi}\sigma} \int_{\mathbb{R}} e^{-\frac{n^2}{2\sigma^2}} |q(x - y) - wH| \, dn \\
= \frac{1}{2} \sum_{u \in \{0, 1\}} |q(x - y) - w| \mathcal{I}_u + q(x - y)(1 - \mathcal{I}_u). \tag{26}
\]

If \( q(x - y) > w \), then \( \mathbb{E} \left[ |w(U_0, N_1)(x) - w(U_0, N_1)(y)| \right] < q(x - y) \) and the contraction would be proved with \( l = q \). This is never the case when \( q < \frac{1}{3} \) and \( |x - y| \leq 2\frac{w}{1 - q} < \frac{w}{q} \).

Let us then consider \( q(x - y) < w \). We can write

\[
\mathbb{E} \left[ |w(U_0, N_1)(x) - w(U_0, N_1)(y)| \right] \\
= \frac{1}{2} \sum_{u \in \{0, 1\}} (w - q(x - y)) \mathcal{I}_u + q(x - y)(1 - \mathcal{I}_u) \tag{27} \\
\leq \frac{1}{2} \sum_{u \in \{0, 1\}} w \mathcal{I}_u + q(x - y).
\]
The last expression is obtained by neglecting \(- \sum_{u \in \{0,1\}} q \ (x - y) I_u\), which is the sum of two second degree terms in \((x - y)\), since, by the integral mean value theorem,

\[ I_u = \frac{1}{\sqrt{2\pi \sigma}} c q (x - y) e^{-\frac{n_0^2}{2\sigma^2}} \]  

(28)

for some \(n_0 \in (cqy + cw (\frac{1}{2} - u) , cqx + cw (\frac{1}{2} - u))\), \((n_0 \neq 0)\). The remaining terms are of order one. Notice that

\[ \frac{1}{2} \sum_{u \in \{0,1\}} w I_u + q(x - y) = F(x) - F(y) \]  

(29)

where

\[ F(x) = qx - \frac{w}{4} \text{erfc} \left( \frac{cq x + c^w x}{\sigma \sqrt{2}} \right) - \frac{w}{4} \text{erfc} \left( \frac{cq x - c^w x}{\sigma \sqrt{2}} \right). \]

Therefore, the thesis is achieved if \(F(x)\) is a contraction; since \(F(x)\) is differentiable and monotone increasing, its Lipschitz constant is the maximum of its first derivative:

\[ F'(x) = q + \frac{cwq}{2\sigma \sqrt{2\pi}} \left[ \exp \left( -\frac{(cq x + c^w x)^2}{2\sigma^2} \right) + \exp \left( -\frac{(cq x - c^w x)^2}{2\sigma^2} \right) \right]. \]  

(30)

In order to find the maximum of \(F'(x)\), let us compute \(F''(x)\):

\[ F''(x) = \]

\[ = \frac{cwq}{2\sigma \sqrt{2\pi}} \frac{2(cq x + c^w x)}{2\sigma^2} cq \exp \left( -\frac{(cq x + c^w x)^2}{2\sigma^2} \right) + \]

\[ - \frac{cwq}{2\sigma \sqrt{2\pi}} \frac{2(cq x - c^w x)}{2\sigma^2} cq \exp \left( -\frac{(cq x - c^w x)^2}{2\sigma^2} \right) \]

which is null for \(x\) satisfying:

\[ (qx + \frac{w}{2}) \exp \left( -\frac{c^2 q w x}{\sigma^2} \right) + (qx - \frac{w}{2}) = 0 \]  

(31)

a solution of which is \(x = 0\). Now, considering that \(F'(x)\) is determined by a mixture of two Gaussians, two cases may occur: (a) \(x = 0\) is the maximum of \(F'(x)\); (b) \(x = 0\) is a minimum for \(F'(x)\) and there are two symmetric
maxima \((F''(x)\) is an even function) at \(x_0 \in (0, \frac{w}{1-q})\) and \(-x_0\), but \(x_0\) cannot be analytically computed from the exponential equation \((31)\). Let us study the sign of \(F''(x)\) for \(x \to 0\) in order to determine the nature of the point \(x = 0\) for \(F'(x)\). Notice that \(F''(x) > 0 \Leftrightarrow -\left( qx + \frac{w}{2} \right) \exp\left( -\frac{c^2qw x}{\sigma^2} \right) - \left( qx - \frac{w}{2} \right) > 0. \) (32)

Moreover, if \(x \to 0\), \(\exp\left( -\frac{c^2qw x}{\sigma^2} \right) \sim 1 - \frac{c^2qw x}{\sigma^2}\) and

\[-\left( qx + \frac{w}{2} \right) \exp\left( -\frac{c^2qw x}{\sigma^2} \right) - \left( qx - \frac{w}{2} \right) \sim -2qx + \frac{c^2qw^2 x}{2\sigma^2}. \) (33)

Finally, if \(\frac{c^2w^2}{\sigma^2} > 4\)

\[-2qx + \frac{c^2qw^2 x}{2\sigma^2} \to 0^+ \text{ for } x \to 0^+ \] (34)

\[-2qx + \frac{c^2qw^2 x}{2\sigma^2} \to 0^- \text{ for } x \to 0^- . \] (35)

In conclusion, \(x = 0\) is a maximum point for \(F'(x)\) if and only if \(\frac{c^2w^2}{\sigma^2} < 4\), that is, only for large noise.

Let us now study \(\frac{c^2w^2}{\sigma^2} > 4\) \((x = 0\) is a minimum point\) and let us state conditions that make \(F(x)\) contractive. In particular, consider \(x > 0\) and \(\sigma^2\) close to zero: by \((31)\), \(|x - \frac{w}{2q}|\) tends to zero more quickly than \(\sigma^2\), hence \(\exp\left( -\frac{(qx - \frac{w}{2})^2}{2\sigma^2} \right)\) tends to one and the maximum of \(F'(x)\) (see \((30)\)) may assume very large values.

More in general, we observe that the points \(x = \pm \frac{w}{2q}\) are tricky as they are the unique points where the OSA fails: for these values, the error probability given by \((14)\) is \(\frac{1}{2}\), no matter which is the noise variance. This “singular” phenomenon is more evident when the noise is small; in terms of \(F(x)\), it causes large variations (then the loss of the contractivity) in a neighborhood of the point \(\pm \frac{w}{2q}\), the radius of the neighborhood being larger for smaller \(\sigma^2\).

Let us set in the case \(q < \frac{1}{3}\), so that \(\pm \frac{w}{2q} \notin D\). Under this assumption,
for any $x \in D$,

$$\exp\left(-\frac{(cq + \frac{w}{x})^2}{2\sigma^2}\right) < \exp\left(-\frac{(cq \frac{w}{1-q} + \frac{w}{x})^2}{2\sigma^2}\right)$$

$$\exp\left(-\frac{(cq - \frac{w}{x})^2}{2\sigma^2}\right) < \exp\left(-\frac{(cq \frac{w}{1-q} - \frac{w}{x})^2}{2\sigma^2}\right)$$

hence

$$F'(x) \leq q + \frac{cwq}{\sigma \sqrt{2\pi}} \exp\left(-\frac{c^2w^2 \left(\frac{1-3q}{2(1-q)}\right)^2}{2\sigma^2}\right).$$

As for $t \geq 0$, $\max te^{-t^2} = \frac{1}{\sqrt{2\pi}}$, 

$$\frac{q}{\sqrt{\pi}} \cdot \frac{2}{1-3q} \cdot \frac{1}{\sqrt{2\pi}} < 1 \implies F'(x) < 1$$

In conclusion a sufficient condition for average contractivity is

$$q < \frac{1}{3 + \sqrt{\frac{2}{\pi}}}.$$ 

Let us now study the case $\frac{c^2w^2}{\sigma^2} < 4$ ($x = 0$ is the maximum point):

$$F'(x) \leq F'(0) = q + 2 \frac{cwq}{2\sigma \sqrt{2\pi}} \exp\left(-\frac{c^2w^2}{8\sigma^2}\right)$$

$$\leq q + 2 \frac{q}{\sqrt{2e\pi}}$$

then $F'(x) < 1$ when

$$q < \frac{1}{1 + \sqrt{\frac{2}{\pi}}}.$$ 

In conclusion, we have stated that if $\frac{c^2w^2}{\sigma^2} > 4$ and $q < \frac{1}{3 + \sqrt{\frac{2}{\pi}}}$ or if $\frac{c^2w^2}{\sigma^2} \leq 4$ and $q \leq \frac{1}{1 + \sqrt{\frac{2}{\pi}}}$, then the hypotheses of Stenflo’s Theorem are fulfilled.

Now, let us prove the convergence of the Mean Square Error.
Since $g \in L_g\text{-Lip}(\mathcal{D})$ where $L_g = \max_{z \in \mathcal{D}} |g'(z)|$ and

$$|g'(z)| = \frac{cq}{2\sigma\sqrt{2\pi}} \left| e^{-\frac{(cqz+cw)^2}{2\sigma^2}} + e^{-\frac{(cqz-cw)^2}{2\sigma^2}} \right| \leq \frac{cq}{\sigma\sqrt{2\pi}} \quad (39)$$

then $L_g \leq \frac{cq}{\sigma\sqrt{2\pi}}$ is finite. Since for any $L > 0$

$$\sup_{f \in L\text{-Lip}(\mathcal{D})} \left| \int_{\mathcal{D}} f d(\mu - \nu) \right| = \sup_{f \in L\text{-Lip}(\mathcal{D})} L \left| \int_{\mathcal{D}} \frac{1}{L} f d(\mu - \nu) \right|$$

we have

$$\sup_{x \in \mathcal{D}} \left| P^k g(x) - \int_{\mathcal{D}} g d\mu \right| = \sup_{x \in \mathcal{D}} \left| \int_{\mathcal{D}} g(z) P^k(x, dz) - \int_{\mathcal{D}} g d\mu \right|$$

$$\leq \sup_{x \in \mathcal{D}} \sup_{f \in L_g\text{-Lip}} \left| \int_{\mathcal{D}} f(z) P^k(x, dz) - \int_{\mathcal{D}} f d\mu \right| \quad (40)$$

$$= \sup_{x \in \mathcal{D}} L_g d_W(P^k(x, \cdot), \mu(\cdot)) \xrightarrow{k \to \infty} 0.$$  

The convergence is then assured also for the Cesàro sum, for any initial state $D_0 = \alpha \in \mathcal{D}$:

$$\frac{1}{K} \sum_{k=0}^{K-1} P^k g(\alpha) \xrightarrow{K \to \infty} \int_{\mathcal{D}} g d\mu \quad \forall \alpha \in \mathcal{D}. \quad (41)$$

Notice that the initial state does not affect the convergence value if it is contained in $\mathcal{D}$, but $\mathcal{D}$ has been obtained by fixing $D_0 = 0$: this seems not coherent. However, even if we consider $D_0 = \alpha \notin \mathcal{D}$, given the dynamics of $D_k$ ($\alpha$ is multiplied by $q$ at each step), $\mathcal{D}$ turns out to be the “limit” state space, and with high probability $D_k$ enters $\mathcal{D}$ for some finite $k$, so it makes sense to reduce to $\mathcal{D}$. Further details are here omitted for brevity, but one must be convinced that considering the initial error lying in a compact set centered at 0 is a suitable choice.

### 5.2 Non contractive case

If the hypotheses of Theorem 1 are not fulfilled, we can prove the following
Theorem 2. For any initial state $D_0 = \alpha$, there exists a unique probability measure $\phi(\alpha, \cdot)$ such that

$$\lim_{K \to \infty} \text{MSE}(U, \hat{U}) = \phi(g(\alpha))$$

(42)

where $\phi(g(\alpha)) = \int g(z)\phi(\alpha, dz)$.

We recall that although this result holds also for the contractive case, the IRF argument is preferable in that case since the convergence value is independent from $\alpha$.

5.2.1 Proof of Theorem 2

Let $A_{x,n}$ the set of the points that $D_k$ can reach in $n$ steps starting from $x$, i.e., $A_{x,n} = \{q^n x + w \sum_{i=0}^{n-1} \alpha_i q^i, \alpha_i \in \{-1, 0, 1\}\}$

Lemma 1. For any $f \in l_f$-Lip$(D)$, there exists a positive constant $M_f$ such that $P^n f \in \frac{M_f}{1 - q}$-Lip$(D)$ for any $n \in \mathbb{N}$.

Proof. We have

$$\max_{x \in D} \max_{y \in A_{x,1}} \left| \frac{d}{dx} P(x, y) \right| \leq \frac{cq}{\sigma \sqrt{2\pi}}$$

(43)

then the three functions $P(x, qx), P(x, qx + w)$ and $P(x, qx - w)$ are Lipschitz with constant $l := \frac{cq}{\sigma \sqrt{2\pi}}$ in $D$. For any $x, x_0 \in D$ and any $(y, y_0) \in \{(qx, qx_0), (qx + w, qx_0 + w), (qx - w, qx_0 - w)\}$,

$$\begin{align*}
|P(x, y)f(y) - P(x_0, y_0)f(y_0)| &= |P(x, y)f(y) \pm P(x_0, y_0)f(y) - P(x_0, y_0)f(y_0)| \\
&\leq ||f||_\infty |P(x, y) - P(x_0, y_0)| + P(x_0, y_0)|f(y) - f(y_0)| \\
&\leq ||f||_\infty l|x - x_0| + P(x_0, y_0)l_f|y - y_0| \\
&\leq (||f||_\infty l + P(x_0, y_0)l_f)\max |x - x_0|.
\end{align*}$$

(44)

Thus,

$$\begin{align*}
|Pf(x) - Pf(x_0)| &= |P(x, qx)f(qx) - P(x_0, qx_0)f(qx_0) + P(x, qx + w)f(qx + w) \\
&\quad - P(x_0, qx_0 + w)f(qx_0 + w) + P(x, qx - w)f(qx - w) \\
&\quad - P(x_0, qx_0 - w)f(qx_0 - w)| \\
&\leq (3||f||_\infty l|x - x_0| + l_f q)\max |x - x_0|.
\end{align*}$$

(45)

In conclusion, $Pf \in L_1$-Lip$(D)$ where $L_1 = 3l||f||_\infty + l_f q$. 

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Now, given any $n \in \mathbb{N}$ and $x = x_0 + \delta$,

$$
\left| P^n f(x) - P^n f(x_0) \right| = \left| \sum_{z \in A_{x,1}} P(x, z)P^{n-1}f(z) - \sum_{z_0 \in A_{x_0,1}} P(x_0, z_0)P^{n-1}f(z_0) \right|
$$

$$
= \left| \sum_{z \in A_{x,1}} P(x, z)P^{n-1}f(z) \pm \sum_{z_0 \in A_{x_0,1}} P(x_0, z_0)P^{n-1}f(z_0) \right|
$$

$$
- \sum_{z_0 \in A_{x_0,1}} P(x_0, z_0)P^{n-1}f(z_0)
$$

$$
= \sum_{z \in A_{x,1}} \left| P(x, z) - P(x - \delta, z - q\delta) \right| P^{n-1}f(z)
$$

$$
+ \sum_{z_0 \in A_{x_0,1}} P(x_0, z_0) \left| P^{n-1}f(z_0 + q\delta) - P^{n-1}f(z_0) \right|
$$

$$
\leq \sum_{z \in A_{x,1}} \left| P(x, z) - P(x - \delta, z - q\delta) \right| ||f||_{\infty}
$$

$$
+ \sum_{z_0 \in A_{x_0,1}} P(x_0, z_0) \left| P^{n-1}f(z_0 + q\delta) - P^{n-1}f(z_0) \right|
$$

$$
\leq 3\delta ||f||_{\infty} + \sum_{z_0 \in A_{x_0,1}} P(x_0, z_0) \left| P^{n-1}f(z_0 + q\delta) - P^{n-1}f(z_0) \right|.
$$

(46)

If $n = 2$,

$$
\left| P^2 f(x) - P^2 f(x_0) \right| \leq 3\delta ||f||_{\infty} + \sum_{z_0 \in A_{x_0,1}} P(x_0, z_0)L_1q\delta \leq (3l||f||_{\infty} + L_1q) \delta
$$

that is, $P^2 f \in L_2\text{-Lip}(\mathcal{D})$ where $L_2 = 3l||f||_{\infty} + L_1q$. At this point, by iterating (46), we obtain that for any $n \in \mathbb{N}$, $P^n f \in L_n\text{-Lip}(\mathcal{D})$ where $L_n = 3l||f||_{\infty} + L_{n-1}q$. Moreover, by recursion,

$$
L_n = 3l||f||_{\infty}(1 + q + \ldots + q^{n-1}) + q^n l_f \leq \frac{M_f}{1 - q},
$$

$$
M_f := \max\{3l||f||_{\infty}, l_f\}.
$$

Let us recall that a sequence of measures $\{\mu_n\}_{n \in \mathbb{N}}$ is said to be \textit{weakly convergent} to a measure $\mu$ if $\lim_{n \to \infty} \int f(x) \, d\mu_n = \int f(x) \, d\mu$ for every continuous and bounded function $f$. In the next, we will denote weakly convergence by $\mu_n \xrightarrow{w} \mu$. 20
Lemma 2. Let $\mathcal{P}_N(x, \cdot) = \frac{1}{N} \sum_{n=0}^{N-1} P^n(x, \cdot)$, $N \in \mathbb{N}$. For any $x \in \mathcal{D}$, there exist a subsequence $\mathcal{P}_{N_j}(x, \cdot)$, $j, N_j \in \mathbb{N}$, and a probability measure $\phi(x, \cdot)$ such that $\mathcal{P}_{N_j}(x, \cdot) \xrightarrow{w} \phi(x, \cdot)$.

Proof. This is a simple consequence of Prohorov’s Theorem (see, e.g., [Billingsley, 1968, Theorem 6.1); in our context tightness is trivial since the space $\mathcal{D}$ is compact].

Lemma 3. If all the convergent subsequences of $\mathcal{P}_N(x, \cdot)$ weakly converge to the same $\phi(x, \cdot)$, then also $\mathcal{P}_N(x, \cdot)$ weakly converges to $\phi(x, \cdot)$.

Proof. Again this is a consequence of Prohorov’s Theorem (see, e.g., [Billingsley, 1968, Theorem 2.3) ]

Given Lemmas 2 and 3 to prove Theorem 2 it is sufficient to show that all the convergent subsequences of $\mathcal{P}_N(x, \cdot)$ converge to $\phi(x, \cdot)$.

Let us suppose that there exist a subsequence $\{M_i\}_{i \in \mathbb{N}} \neq \{N_j\}_{j \in \mathbb{N}}$ and a probability measure $\psi(x, \cdot) \neq \phi(x, \cdot)$ on $\mathcal{D}$ such that $\mathcal{P}_{M_i}(x, \cdot) \xrightarrow{w} \psi(x, \cdot)$.

First notice that for any $m \in \mathbb{N}$, by applying the Dominated Convergence Theorem,

$$P^m \phi f(x) = \int_{y \in \mathcal{D}} P^m(x, dy) \phi f(y)$$

$$= \int_{y \in \mathcal{D}} P^m(x, dy) \lim_{j \to +\infty} \int_{z \in \mathcal{D}} \frac{1}{N_j} \sum_{n=0}^{N_j-1} P^n(y, dz) f(z)$$

$$= \lim_{j \to +\infty} \frac{1}{N_j} \sum_{n=0}^{N_j-1} P^{n+m} f(x)$$

$$= \lim_{j \to +\infty} \frac{1}{N_j} \left( \sum_{n=0}^{N_j-1} P^n f(x) + \sum_{n=N_j}^{N_j-1+m} P^n f(x) - \sum_{n=0}^{m-1} P^n f(x) \right)$$

$$= \phi f(x).$$

Similarly, exploiting the continuity of $P^m f$, we obtain

$$\phi P^m f(x) = \phi f(x).$$

The same can be clearly said for $\psi$.

Now, for any $M_i \in \mathbb{N}$

$$\mathcal{P}_{M_i} \phi f(x) = \phi f(x)$$
and
\[ \lim_{i \to \infty} P_{M_i} \phi f(x) = \phi f(x). \]

If \( f \) is \( l_f \)-Lip(\( D \)), then \( P_{N_i} f(x) \) are equicontinuous by Lemma \[\text{[1]}\] and clearly also equibounded by \( ||f||_{\infty} \). Therefore, by Ascoli-Arzelà Theorem, \( \phi f(x) \) is continuous and \( \lim_{i \to \infty} P_{M_i} \phi f(x) = \psi \phi f(x) \) In conclusion,
\[ \phi f(x) = \psi \phi f(x). \]

Analogously, one can prove that \( \psi f(x) = \phi \psi f(x) \) and, since by Dominated Convergence \( \psi \phi f(x) = \phi \psi f(x) \), we obtain \( \phi f(x) = \psi f(x) \). To summarize, we have proved that, for any \( x \in D \), there exists a unique probability measure \( \phi(x, \cdot) \) such that
\[ \lim_{N \to \infty} P_N f(x) = \phi f(x) \]
for any \( f \in l_f \)-Lip(\( D \)).

The thesis of Theorem \[\text{[2]}\] follows by considering \( f = g \).

The arguments used in this proof partially arise from the proof of (Meyn and Tweedie, 1993, Theorem 12.4.1).

5.3 Simulations vs Theoretical Results

The convergence values \( \int_D g \, d\mu \) and \( \phi g(\alpha) \) can be numerically evaluated. In Figure 3 we show their consistency with the simulations previously presented. Notice that for simulations we have assumed to know the initial state \( x_0 \), so that \( D_0 = 0 \). Since analytical results are asymptotic, while simulations’ results are obtained by averaging transmissions of 320 bits, we intuitively conclude that the rate of convergence is fast.

6 Conclusion

In this paper, we have proposed using the One State decoding algorithm to recover the binary input of a linear system and we have analyzed its behavior. When the system has particular contractive properties, the analysis is based on Iterated Random Functions, while in the non-contractive case known results from convergence of probability measures can be exploited. The theoretical results allow to predict the Mean Square Error of the One State Algorithm for long-time transmissions, given the parameters of the system and some prior probabilistic information. Simulations and theoretical results are consistent.
Figure 3: Simulations vs Theoretical Results: MSE for $b = c = 1$, $a = -2, -1, -0.5, -0.3$. 
The One State Algorithm could be extended to multi-dimensional problems and to the recovery of digital inputs arising from larger source alphabets and with different probabilistic distributions. Moreover, its use for problems with feedback, such as channel equalization, should be further studied.

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