Study on generalized fuzzy fractional human liver model with Atangana–Baleanu–Caputo fractional derivative

Lalchand Verma, Ramakanta Meher
Department of Mathematics and Humanities, S.V. National Institute of Technology, Surat 395007, India

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Abstract This study aims to develop a novel fuzzy fractional model for the human liver that incorporates the ABC fractional differentiability, also known as ABC gH-differentiability, based on the generalized Hukuhara derivative. In addition, a novel fuzzy double parametric $q$-homotopy analysis method with a generalized transform and ABC gH-differentiability is used to deal with the fuzzy mathematical model and examine its convergence analysis. The stability of the unique equilibrium point for the fuzzy fractional human liver model and the existence of a unique solution in the proposed model are investigated using the Arzela–Ascoli theorem and Schauder’s fixed-point theory. Some numerical experiments are conducted to visualize better results and test the proposed method’s efficacy. The results of the $q$-HAShTM employing the presented approaches coincide with most of the clinical data, providing it more precise and superior to the generalized Mittag–Leffler function method.

1 Introduction

Mathematical modelling is crucial for preventing metabolic risk factors, including hypertension, diabetes, and obesity, as well as for studying the human body’s kinetics of the endocrine and metabolic systems. However, several models have been proposed to explain how the liver functions, but these models have only been applied to integer-order differential equations. Although many gains in hepatic surgery have been attributed to technological advancements, there is no doubting the importance of a detailed understanding of the interior architecture of the liver in achieving better results [1]. Based on the liver’s critical role, researchers are increasingly focusing on developing mathematical models that can characterize the liver’s performance. Some notable efforts have been made in this approach by Čelechovská [2], Calvetti et al. [3], Repetto and Tweedy [4] Friendman and Hao [5], etc. Many scientists and researchers have recently proved that the fractional models describe natural phenomena accurately and systematically better than their integer-order equivalents using conventional time-derivatives [6–10]. In recent days, fractional calculus has been used to describe various complicated biological systems [11, 12]. Although these studies produced better results than other standard models with integer order, a satisfactory level of accuracy could not be attained over the entire period due to the new definition of common fractional derivatives, which renders those operators impractical for the description of non-local dynamics [13–15].

The fuzzy set theory provides an effective option for analysing uncertain situations. There are several applications of fuzzy set theory, including fixed-point theory and topology, fractional calculus and consumer electronics. Many researchers and scientists have a keen interest in the study of fractional calculus, which also includes fractional-order integrals and derivatives. Due to its precise and accurate observations, fractional calculus has numerous applications in modern physical and biological processes. The Interval or fuzzy formulations can also be implemented using fractional models in real-world contexts. Fractional differential equations with uncertainty introduced by Agrawal et al. [16]. They solved them using Riemann-differentiability Liouville’s and Hukuhara’s differences. The Hukuhara difference, first presented by Bede et al. [17, 18], and later on, became a significant topic among academicians and researchers. Since fractional situations are common in real-world situations, so many researchers generalized it using a generalisation of the strongly generalised differentiability. The $q$-homotopy Shehu transforms technique (Sartanpara and Meher [19, 20]) with Caputo derivative ($q$-HAShTM) and fuzzy double parametric technique (Meher et al. [21–23]) are used for solving various real-world problems. Alqudah et al. [24] have addressed the fuzzy Cauchy reaction–diffusion models using the Caputo fractional derivative (CFD) and Atangana–Baleanu (AB) fractional derivative operator with generalized Hukuhara differentiability. Similarly, Smadi et al. [25] investigated fuzzy fractional differential equations with unknown constraints coefficients and initial conditions in view of the Atangana–Baleanu–Caputo differential operator whereas Alderremy et al. [26] studied the fractional...
COVID-19 model with ABC derivative in the fuzzy environment. The fuzzy-fractional variational problems were discussed by Zhang et al. [27] using the gH-Atangana–Baleanu fuzzy-fractional derivative in Caputo sense.

When a colouring substance called bromsulphthalein (BSP) is injected into the blood, presuming that no other organ in the body takes up BSP. It is the only one that absorbs and secretes it straight into the bile, the blood’s BSP concentration can be assessed at various periods. Many researchers and scientists have worked on the human liver disease through mathematical and clinical interventions in recent years. Verma et al. [28] discussed human liver failure post-liver resection. Baleanu et al. [29] studied the mathematical modelling of the human liver under the Caputo–Fabrizio fractional order. Ahamd et al. [30] discussed the human liver model in uncertainty under the Caputo fractional derivative. Ameen et al. [31] have investigated the analytical and numerical solution of time-fractional of the human liver model under Caputo sense, while Rasid et al. [32] have analysed the human liver’s oscillatory and complex behaviour with a non-singular kernel.

This model is inspired by the discussion above and the applicability of ABC gH-differentiability. Upon utilising the generalised Hukuhara difference, here we will study the application of ABC gH-differentiability, which is an explicit method based on the generalised Hukuhara derivative. This study presents a novel mathematical fuzzy liver model based on the generalised Hukuhara difference and the ABC fractional derivative. The fractional ABC derivative [33, 34] is utilise here to construct and employ an interval technique for interval modelling to find the parametric interval solutions. A novel fuzzy double parametric q-homotopy analysis method with a generalised transform and ABC derivative is used to deal with the fuzzy mathematical model and examine its solution convergence. The existence of a unique solution in the proposed model is investigated using the Arzela–Ascoli theorem and Schauder’s fixed-point theory. Some numerical experiments are conducted to visualise better results and test the proposed method’s efficacy. Finally, real-world clinical data show that the novel fractional model outperforms the existing integer-order model with ordinary temporal derivatives.

The following is the outline for this paper: Sect. 2 discusses the preliminaries, while Sect. 3 discusses the mathematical formulation of the problem with the inclusion of equilibrium and stability. The existence and uniqueness of the solution of the proposed model are discussed in Sect. 4. Section 5 discusses the applications of q-HASHSTM’s to the proposed biological model, while Sect. 6 covers the convergence analysis of the problem. The result and discussion of the study and the final remarks are presented in Sects. 7 and 8.

2 Preliminaries

This section discusses the fundamentals of fuzzy set theory, the basic definition of fractional derivatives and integrals, and their main characteristics.

Definition 1 [33, 35] Let $X$ be collection of object denoted by $s$, then a fuzzy set $\tilde{B}$ in $X$ is a set of order pairs:

$$\tilde{B} = \{(s, \mu_{\tilde{B}}(s) : s \in X)\}$$

(2.1)

where $\mu_{\tilde{B}}(s)$ is grade of membership and $\mu_{\tilde{B}}(s) \in [0, 1]$.

Definition 2 [34] A fuzzy set is said to be the fuzzy number if it is convex, piece-wise continuous membership function and normalized in real line $\mathbb{R}$.

Definition 3 [16] $\rho$-cut is a crisp set denoted and defined as

$$[\tilde{B}]^\rho = \{x \in X, \quad \mu_{\tilde{B}}(s) \geq \rho\}$$

(2.2)

Definition 4 [23] A triangular fuzzy number (TFN) is denoted by $\tilde{B} = [R_1, R_2, R_3]$ and its membership function $\tilde{B} = [R_1, R_2, R_3]$ can be defined as

$$\mu_{\tilde{B}}(s) = \begin{cases} 0 & s \leq R_1 \\ \frac{s-R_1}{R_2-R_1} & R_1 < s \leq R_2 \\ \frac{R_3-s}{R_3-R_2} & R_2 < s \leq R_3 \\ 0 & s \geq R_3 \end{cases}$$

(2.3)

Using concept of $\rho$-cut, TFN $\tilde{B} = [R_1, R_2, R_3]$ can be written in interval form as $\tilde{B} = [(R_2 - R_1)\rho + R_1, R_3 - (R_3 - R_2)\rho]$, where $\rho \in [0, 1]$.

Definition 5 [22] Let $P = [P, \overline{P}]$ be an interval. In double parametric form, it can be stated as

$$P = \delta(\overline{P} - P) + P, \quad \delta \in [0, 1].$$

(2.4)
**Definition 6** [34] A fuzzy mapping is $D : \mathbb{E} \times \mathbb{E} \to \mathbb{R}$ and $J_1, J_2 \in \mathbb{E}$ is a fuzzy number. Then, the $H$-distance is

$$D(J_1(\xi), J_2(\xi)) = \max\{\sup_{\xi \in [0, 1]}|J_1(\xi) - J_2(\xi)|, \sup_{\xi \in [0, 1]}|J_1(\xi) - J_2(\xi)|\}$$

**Definition 7** [17] A fuzzy valued function $\nu : [a, b] \to \mathbb{E}$ is said to be fuzzy continuous at point $y_0 \in [a, b]$ if for any $\epsilon > 0$, $\exists \delta > 0$ such that

$$D(\nu(y), \nu(y_0)) < \epsilon; \quad \text{whenever } |y - y_0| < \delta$$

**Definition 8** [33] A fuzzy valued function $\nu : [b, c] \to \mathbb{E}$ at point of $a_0$ with generalized Hukuhara derivative can be defined as

$$\nu'_H(a_0) = \lim_{h \to 0} \frac{\nu(a_0 + h) - \nu(a_0)}{h}$$

If $\nu'_H(a_0) \in \mathbb{E}$ exist, then $\nu$ is said to be generalized Hukuhara differentiable (gH-differentiable) at point $a_0$. Also, $D\nu'_H$ is [(i)-gH] differentiable at point $a_0$ if

(i) $[D\nu'_H(a_0)]^\rho = [D\nu(a_0; \rho), \nu(a_0; \rho)]$.

(ii) and that $D\nu'_H$ is [(ii)-gH] differentiable at point $a_0$, if

$$[D\nu'_H(a_0)]^\rho = [D\nu(a_0; \rho), D\nu(a_0; \rho)]$$

**Definition 9** [36] For a gH-differentiable function $f$, the fuzzy gH-ABC fractional derivative of order $\beta \in (0, 1)$ can be defined as

$$\left(\frac{ABC D_\beta^\beta f}{\xi H}\right)(t) = \frac{M(\beta)}{1 - \beta} \int_0^t E_\beta\left[\frac{-t}{1 - \beta} \left( t - x \right)^{\beta} \right] f'_H(x)dx$$

Therefore, $f(x) = [f(x), \overline{f}(x)]$ is a parametric form of $f(x)$, then the ABC fractional derivative in fuzzy sense can be written as

$$\left[ABC D^\beta f_{i, H}(x)\right]^\rho = \left[ABC D^\beta f(x, \rho), ABC D^\beta \overline{f}(x, \rho)\right]$$

Thus, we can write the [(i)-gH] differentiable in double parametric form as

$$ABC D^\beta f_{i, H}(x, \rho, \delta) = \delta ABC D^\beta \overline{f}(x, \rho) + ABC D^\beta f(x, \rho)$$

where $M(\beta)$ is the normalization constant with $M(0) = M(1) = 1$. Since here $ABC [(i) - g H]$ differentiability of the problem exists, there is no need to consider the $ABC [(ii) - g H]$ differentiability. The $E_\beta$ is the Mittag–Leffler operator that can be defined as

$$E_\beta(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\beta k + 1)}$$

**Definition 10** [20] The Shehu transform of ABC fractional derivative can be defined as

$$\mathbf{S} ABC D_0^\beta (f(x)) = \frac{B(\beta)}{1 - \beta + \beta \left( \frac{x}{2} \right)^{\beta}} \left( V(s, u) - \frac{u}{s} f(0) \right)$$

**Key point:** Let us consider a closed norm space $Z = \chi^2$ where $\chi = C[0, T]$ have the norm defined as follows $||M|| = ||(\tilde{z}, \tilde{w})|| = \max_{t \in [0, T]} ||\tilde{z}(t) + \tilde{w}(t)||$.

**Lemma 1** [36] For the given problem with $0 < \beta < 1$, the anti-derivative of the fractional-order can be defined as follows:

$$ABC D_{t_0}^\beta \tilde{V}(t) = \varphi(t, \tilde{V}(t), t \in [0, T], V(0) = V_0$$

is

$$\tilde{V}(t) = V_0 + \frac{1 - \beta}{M(\beta)} y(t) + \frac{\beta}{\Gamma(\beta) M(\beta)} \times \int_0^t (t - p)^{\beta - 1} y(p)dp.$$
Theorem 1  Let $\mathbb{B}$ be a subset of $\mathfrak{X}$, which is convex. Let us assume that the two operators $\Omega_1, \Omega_2$ with

(i) $\Omega_1 v + \Omega_2 v \in \mathbb{B}$ for each $v \in \mathbb{B}$.
(ii) $\Omega_1$ is contraction.
(iii) a continuous and compact set is $\Omega_2$. Then, the operator equation $\Omega_1 v + \Omega_2 v = v$ has at last one solution.

3 Mathematical formulation of the model with ABC fractional order

The study of the human liver, in some instances, is prevalent among the investigators, so here we have considered a fractional order human liver model in an uncertain case, i.e. with a fuzzy sense. Initially, Čelechovská discussed it as an integer-order model in 2004 [2], and here, the model is extended to an uncertain case. The authors employed the clinical data gathered by the Bromsulphthalein (BSP) test to investigate the parametric study of the proposed model in a crisp case. By assuming $z(t)$ and $w(t)$ as the amount of BSP in the blood and liver, respectively, at time $t$, the integer-order model is formulated by Čelechovská [2]

$$\frac{dz(t)}{dt} = -Az(t) + Bw(t)$$
$$\frac{dw(t)}{dt} = Az(t) - (B + D)w(t)$$

(3.1)

where $A, B, & D$ are the known constants and denotes the transfer rate and initial condition are $(z(0), w(0)) = (I, 0)$. In Eq. (3.1), we change the integer-order to fuzzy fractional order ones in ABC sense as shown

$$ABC\mathcal{D}_+^\beta z_H(t) = -A \odot \tilde{z}(t) + B \odot \tilde{w}(t)$$
$$ABC\mathcal{D}_+^\beta w_H(t) = A \odot \tilde{z}(t) - (B + D) \odot \tilde{w}(t)$$

(3.2)

Here $(\tilde{z}(0), \tilde{w}(0)) = (\tilde{a}, \tilde{b})$ and $\tilde{a} & \tilde{b}$ are fuzzy number. The R.H.S. of the system Eq. (3.2) has the dimension time$^{-1}$; however, when the derivative order is changed to $\beta$, the dimension of the L.H.S. becomes time$^{-\beta}$. We can change the coefficients in the following equation to ignore the dimensional mismatching [29]

$$ABC\mathcal{D}_+^\beta z_H(t) = -A^\beta \odot \tilde{z}(t) + B^\beta \odot \tilde{w}(t)$$
$$ABC\mathcal{D}_+^\beta w_H(t) = A^\beta \odot \tilde{z}(t) - (B^\beta + D^\beta) \odot \tilde{w}(t)$$

(3.3)

Put $A^\beta = a, \ B^\beta = b, \ D^\beta = d$ in Eq. (3.3), we get

$$ABC\mathcal{D}_+^\beta z_H(t) = -a \odot \tilde{z}(t) + b \odot \tilde{w}(t)$$
$$ABC\mathcal{D}_+^\beta w_H(t) = a \odot \tilde{z}(t) - (b + d) \odot \tilde{w}(t)$$

(3.4)

with fuzzy initial conditions

$$\tilde{z}(0) \geq 0, \ \tilde{w}(0) \geq 0$$

(3.5)

3.1 Equilibrium and stability

Equation (3.4) defines a homogeneous system of linear time-invariant fuzzy fractional differential equation with an equilibrium at the origin denoted as $E^* = (0, 0)$, In this regard, the matrix coefficient for the system Eq. (3.4) is given by

$$I = \begin{bmatrix} -a & b \\ a & -(b + d) \end{bmatrix}$$

(3.6)

The characteristic polynomial of the matrix (3.5) can be obtained as

$$\lambda^2 + (a + b + d)\lambda + ad = 0$$

(3.7)

Since $ad > 0$ and $(a + b + d) > 0$, the real components of the eigenvalues of $I$ are negative. As a result, the system (3.4) is asymptotically stable.
4 Existence of solution for the fractional model

This section discusses the existence and uniqueness of the solution with the fuzzy fractional approach of the proposed model (3.4). Here, we discuss the existence, uniqueness, and stability of the proposed model under ABC derivative with fractional order using Banach fixed point in a fuzzy sense. The following is the reformulation of the proposed model:

\[
\Phi_1(t, \tilde{z}, \tilde{w}) = \Phi_2(t, \tilde{z}, \tilde{w}) = a \circ \tilde{z}(t) - (b + d) \circ \tilde{w}(t)
\]  

where \(\Phi_1\) and \(\Phi_1\) are fuzzy function.

Upon using Eq. (4.1) for \(0 < \beta \leq 1\), we can write the proposed model as

\[
^\text{ABC}D^\beta \tilde{V}(t) = \Phi(t, \tilde{V}(t))
\]

\[
\tilde{V}(0) = \tilde{V}_0
\]

With help of Lemma 1, the system of Eq. (4.2) can be written as

\[
\tilde{V}(t) = \tilde{V}_0 + \left[\Phi(t, \tilde{V}(t)) - \Phi_0(t)\right] \frac{1 - \beta}{M(\beta)} + \frac{\beta}{M(\beta)\Gamma(\beta)} \int_0^t (t - p)^{\beta - 1} \Phi(p, \tilde{V}(p)) dp
\]

where

\[
\tilde{V}(t) = \begin{cases} \tilde{z}(t) \\ \tilde{w}(t) \end{cases}
\]

\[
\tilde{V}_0 = \begin{cases} \tilde{z}_0 \\ \tilde{w}_0 \end{cases}
\]

\[
\Phi(t) = \begin{cases} \Phi_1(t, \tilde{z}, \tilde{w}) \\ \Phi_2(t, \tilde{z}, \tilde{w}) \end{cases}
\]

\[
\Phi_0(t) = \begin{cases} \Phi_1(0, \tilde{z}_0, \tilde{w}_0) \\ \Phi_2(0, \tilde{z}_0, \tilde{w}_0) \end{cases}
\]

Using Eqs. (4.2) and (4.3), we can define the two operators \(\Omega_1\) and \(\Omega_2\), as

\[
\Omega_1 v = \Phi_0 + [\Phi(t, \tilde{V}(t)) - \Phi_0(t)] \frac{1 - \beta}{M(\beta)} + \frac{\beta}{M(\beta)\Gamma(\beta)} \int_0^t (t - p)^{\beta - 1} \Phi(p, \tilde{V}(p)) dp
\]

Now, upon using the fixed point theorem, here we will show the theoretical analysis for the system of equations.

\(M_1 \exists \epsilon_1 \) and \(\epsilon_2\) are constants, such that

\[
|\Phi(t, \tilde{V}(t))| \leq \epsilon_1 |\tilde{V}(t)| + \epsilon_2
\]

\(M_2 \exists \text{ constants } K_p \) for all \(v, v_1 \in \mathbb{X}\) such that

\[
|\Phi(t, \tilde{V}(t)) - \Phi(t, \tilde{V}_1(t))| \leq K_p |v - v_1|
\]

**Theorem 2** If \((M_1)\&(M_2)\) holds, then the system of equation (4.3) has at least one solution and the proposed system (3.4) also has a unique solution if

\[
\frac{(1 - \beta)K_p}{M(\beta)} < 1
\]

Proof We will start by proving that \(\Omega_1\) is a contraction by using the Banach contraction theorem, for this let \(v_1 \in \mathbb{B}\), where \(\mathbb{B} = \{v \in \mathbb{X} : r \geq |v|, 0 < r\}\) is a convex closed set. We define Eq. (4.8) by

\[
||\Omega_1 v - \Omega_2 v_1|| = \frac{(1 - \beta)}{M(\beta)} \times \max_{v \in [0, r]} |\Phi(t, \tilde{V}(t)) - \Phi(t, \tilde{V}_1(t))|
\]

\[
\leq \frac{(1 - \beta)}{M(\beta)} ||v - v_1||
\]
So $\Omega_1$ is closed and hence contraction.

Now we can show that the $\Omega_2$ is well defined throughout the entire domain and compact in comparison form, as well as that $\Omega_2$ is continuous and bounded. Since $\Phi$ is continuous and $v \in B$, it can be defined as follows:

$$||\Omega_2(v)|| = \max_{t \in [0,T]} \frac{\beta}{M(\beta)\Gamma'(\beta)} \times |\int_0^t (t - p)^{\beta - 1} \Phi(p, \tilde{V}(p))dp|$$

\[\leq \frac{\beta}{M(\beta)\Gamma'(\beta)} \int_0^t (t - p)^{\beta - 1} |\Phi(p, \tilde{V}(p))|dp\]
\[\leq \frac{T^\beta}{M(\beta)\Gamma'(\beta)} [\epsilon_1 r + \epsilon_2] \tag{4.14}\]

Hence, from Eq. (4.14), it can be seen that the operator $\Omega_2$ is bounded and for equi-continuous. Let $t_1 > t_2 \in [0, T]$, such that

$$||\Omega_2 \tilde{V}(t_1) - \Omega_2 \tilde{V}(t_2)|| = \frac{\beta}{M(\beta)\Gamma'(\beta)} \times |\int_0^{t_1} (t_1 - p)^{\beta - 1} \Phi(p, \tilde{V}(p))dp - \int_0^{t_2} (t_2 - p)^{\beta - 1} \Phi(p, \tilde{V}(p))dp|$$
\[\leq \frac{\beta}{M(\beta)\Gamma'(\beta)} [t_1^\beta - t_2^\beta] \tag{4.15}\]

As $t_1 \to t_2$, so R.H.S of Eq. (4.15) tending to zero as the operator $\Omega_2$ is continuous, so

$$||\Omega_2 \tilde{V}(t_1) - \Omega_2 \tilde{V}(t_2)|| \to 0, \quad as \quad t_1 \to t_2 \tag{4.16}\]

Hence, it is proved that $\Omega_2$ is equi-continuous; which is uniformly continuous using Arzela–Ascoli theorem, as $\Omega_2$ is bounded. It can also be added that that the system of equation has at least one solution. As a result, using Schauder’s fixed-point theorem, it can be concluded that the system (3.4) has at least one solution.

4.1 Uniqueness result

**Theorem 3** If the integral Eq. (4.3) has a unique solution under the assumption $(M_2)$, then the system of Eq. (3.4) also has the unique solution if

$$\left[\frac{(1 - \beta)K_p}{M(\beta)} + \frac{T^\beta K_p}{M(\beta)\Gamma'(\beta)}\right] < 1 \tag{4.17}\]

**Proof** Let us assume that an operator $N : \mathfrak{X} \to \mathfrak{X}$ defined by

$$T \tilde{V}(t) = \tilde{V}_0 + \left[\Phi(t, \tilde{V}(t)) - \Phi(0,t)\right] \frac{1 - \beta}{M(\beta)} + \frac{\beta}{M(\beta)\Gamma'(\beta)} \int_0^t (t - p)^{\beta - 1} \Phi(p, \tilde{V}(p))dp, \quad t \in [0, T] \tag{4.18}\]

Let $v, v_1 \in \mathfrak{X}$ then

$$||Tv - tv_1|| \leq \frac{1 - \beta}{M(\beta)} \max_{t \in [0,T]} |\Phi(t, \tilde{V}(t)) - \Phi(t, \tilde{V}_1(t))|$$
\[+ \frac{\beta}{M(\beta)\Gamma'(\beta)} \max_{t \in [0,T]} \left|\int_0^t (t - p)^{\beta - 1} \times \Phi(p, \tilde{V}(p))dp - \int_0^t (t - p)^{\beta - 1} \times \Phi(p, \tilde{V}_1(p))dp\right|\]
\[\leq \left[\frac{(1 - \beta)K_p}{M(\beta)} + \frac{T^\beta K_p}{M(\beta)\Gamma'(\beta)}\right]||v - v_1|| \tag{4.19}\]

where

$$\Lambda = \left[\frac{(1 - \beta)K_p}{M(\beta)} + \frac{T^\beta K_p}{M(\beta)\Gamma'(\beta)}\right] \tag{4.20}\]

Thus, the operator $T$ is contraction in Eq. (4.19). So, Eq. (4.3) has a unique solution. As a result, the considered system of Eq. (3.4) has a unique solution as well. \[\square\]

4.2 Stability of the problem

In this section, we find the stability of the suggested system by making a modest adjustment to $\psi \in C[0, T]$ and only satisfying $0 = \beta(0) so,

(i) $|\psi(\cdot)| \leq \xi$, for $\xi > 0$

(ii) $ABC^\beta(\tilde{V}^\beta(t)) = \gamma(t, \tilde{V}(t)) + \psi(t)$, for all $t \in [0, T]$
Lemma 2 Let the solution of the converted problem [36] be

\[ A^BCD^\beta_0 \tilde{V}_g(t) = \Phi(t, \tilde{V}(t)) + \Psi(t) \]  \hspace{1cm} (4.21)

with initial condition

\[ \tilde{V}(0) = \tilde{V}_0 \]  \hspace{1cm} (4.22)

that satisfies

\[ |\tilde{V}(t) - \left( \tilde{V}_0(t) + \left[ \Phi(t, \tilde{V}(t)) - \Phi_0(t) \right] \frac{1 - \beta}{M(\beta)} + \frac{\beta}{\Gamma(\beta)M(\beta)} \int_0^t (t - p)^{\beta - 1} \Phi(p, \tilde{V}(p)) dp \right) | \leq T^\beta, \beta \xi \]  \hspace{1cm} (4.23)

where

\[ T^\beta, \beta = \frac{T^\beta + \Gamma(\beta)(1 - \beta)}{\Gamma(\beta)M(\beta)} \]  \hspace{1cm} (4.24)

Theorem 4 With Eq. (4.23) together with the assumption \( M_2 \), the solution of Eq. (4.3) is Ulam–Hyers stable (UHS), and therefore, the analytical solution for the suggested model is Ulam–Hyers stable if \( \Lambda < 1 \).

Proof Let us assume that \( v_1 \) be a unique solution and \( v \) be any solution of Eq. (4.3), then

\[ ||\tilde{V}(t) - \tilde{V}_1(t)|| = |\tilde{V}(t) - \left( \tilde{V}_0(t) + \left[ \Phi(t, \tilde{V}(t)) - \Phi_0(t) \right] \frac{1 - \beta}{M(\beta)} + \frac{\beta}{\Gamma(\beta)M(\beta)} \int_0^t (t - p)^{\beta - 1} \Phi(p, \tilde{V}(p)) dp \right) | \]

\[ \leq |\tilde{V}(t) - \left( \tilde{V}_0(t) + \left[ \Phi(t, \tilde{V}(t)) - \Phi_0(t) \right] \frac{1 - \beta}{M(\beta)} + \frac{\beta}{\Gamma(\beta)M(\beta)} \int_0^t (t - p)^{\beta - 1} \Phi(p, \tilde{V}(p)) dp \right) | \]

\[ + |\tilde{V}_0(t) + \left[ \Phi(t, \tilde{V}(t)) - \Phi_0(t) \right] \frac{1 - \beta}{M(\beta)} + \frac{\beta}{\Gamma(\beta)M(\beta)} \int_0^t (t - p)^{\beta - 1} \Phi(p, \tilde{V}(p)) dp \right) | \]

\[ - |\tilde{V}_0(t) + \left[ \Phi(t, \tilde{V}(t)) - \Phi_0(t) \right] \frac{1 - \beta}{M(\beta)} + \frac{\beta}{\Gamma(\beta)M(\beta)} \int_0^t (t - p)^{\beta - 1} \Phi(p, \tilde{V}(p)) dp \right) | \]

\[ \leq |v - v_1| + \frac{\beta T^\beta K}{\Gamma(\beta)M(\beta)} ||v - v_1|| \]

\[ \leq \xi T^\beta, \beta + \Delta ||v - v_1|| \]  \hspace{1cm} (4.25)

From Eq. (4.25), we can write

\[ ||\tilde{V} - \tilde{V}_1|| \leq \xi T^\beta, \beta \]  \hspace{1cm} (4.26)

Hence, from Eq. (4.26), Eq. (4.3) is UH stable; therefore, the proposed model is UH stable.

Let us take a look at the following hypotheses:

(i) \( |\Psi(t)| \leq \psi(t) \xi \), and \( \xi > 0 \)

(ii) \( A^BCD^\beta(\tilde{V}_g(t)) = \Upsilon(t, \tilde{V}(t)) + \Psi(t), \forall \ t \in [0, T] \)

Lemma 3 [36] The next equation will satisfy Eq. (4.21),

\[ |\tilde{V}(t) - \left( \tilde{V}_0(t) + \left[ \Upsilon(t, \tilde{V}(t)) - \Upsilon_0(t) \right] \frac{1 - \beta}{M(\beta)} + \frac{\beta}{\Gamma(\beta)M(\beta)} \int_0^t (t - p)^{\beta - 1} \phi(p, \tilde{V}(p)) dp \right) | \]

\[ \leq \psi(t) \chi T^\beta, \beta \]  \hspace{1cm} (4.27)

Theorem 5 Upon using lemma (4.2), the solution of system of fuzzy differential equation is Ulam–Hyers–Rassias (UHR) stable and consequently generalized UHR stable.

Proof Let us assume that the system of fractional differential equation be \( v_1 \in \mathcal{I} \) and \( v \in \mathcal{I} \) be a solution of Eq. (4.3), then

\[ ||\tilde{V}(t) - \tilde{V}_1(t)|| = |\tilde{V}(t) - \left( \tilde{V}_0(t) + \left[ \Upsilon(t, \tilde{V}(t)) - \Upsilon_0(t) \right] \frac{1 - \beta}{M(\beta)} + \frac{\beta}{\Gamma(\beta)M(\beta)} \int_0^t (t - p)^{\beta - 1} \phi(p, \tilde{V}(p)) dp \right) | \]

\[ \leq |\tilde{V}(t) - \left( \tilde{V}_0(t) + \left[ \Upsilon(t, \tilde{V}(t)) - \Upsilon_0(t) \right] \frac{1 - \beta}{M(\beta)} + \frac{\beta}{\Gamma(\beta)M(\beta)} \int_0^t (t - p)^{\beta - 1} \phi(p, \tilde{V}(p)) dp \right) | \]
The fuzzy human liver problem (3.4) and initial condition are

\[ 123 \]

Upon applying the Shehu transform in Eq. (5.1), we get

\[ 123 \]

Hence, the solution of Eq. (4.3) is stable.

5 Application of \( q \)-HAShTM

This section discusses the fuzzy computation of the human liver problem under the \( q \)-HAShTM through a double-parametric approach.

5.1 Fuzzy computation of HLM

The fuzzy human liver problem (3.4) and initial condition are

\[ 123 \]

Upon applying the Shehu transform in Eq. (5.1), we get

\[ 123 \]

Hence, the solution of Eq. (4.3) is stable.

\[ 123 \]
Using Taylor series, the expansion of $P_i(0, \frac{\rho}{\delta}; q), \bar{P}_i(0, \frac{\rho}{\delta}; q) \quad i = 1, 2$ with respect to $q$ yields

\[
\begin{align*}
\left\{ & P_1(0, \frac{\rho}{\delta}; q) = z(0, \frac{\rho}{\delta}) + \sum_{m=1}^{\infty} \tilde{z}_m(t, \frac{\rho}{\delta})q^m \\
& \bar{P}_1(0, \frac{\rho}{\delta}; q) = \tilde{z}(0, \frac{\rho}{\delta}) + \sum_{m=1}^{\infty} \tilde{\tilde{z}}_m(t, \frac{\rho}{\delta})q^m
\end{align*}
\]

and

\[
\begin{align*}
\left\{ & P_2(0, \frac{\rho}{\delta}; q) = w(0, \frac{\rho}{\delta}) + \sum_{m=1}^{\infty} \tilde{w}_m(t, \frac{\rho}{\delta})q^m \\
& \bar{P}_2(0, \frac{\rho}{\delta}; q) = \tilde{w}(0, \frac{\rho}{\delta}) + \sum_{m=1}^{\infty} \tilde{\tilde{w}}_m(t, \frac{\rho}{\delta})q^m
\end{align*}
\]

where $\tilde{z}_m(t, \frac{\rho}{\delta}) = \frac{1}{m} \frac{\partial P_1(t, \rho, \delta; q)}{\partial q^m} \bigg|_{q=0}$ and $\tilde{\tilde{z}}_m(t, \frac{\rho}{\delta}) = \frac{1}{m} \frac{\partial \bar{P}_1(t, \rho, \delta; q)}{\partial q^m} \bigg|_{q=0}$ and $\tilde{w}_m(t, \frac{\rho}{\delta}) = \frac{1}{m} \frac{\partial P_2(t, \rho, \delta; q)}{\partial q^m} \bigg|_{q=0}$ and $\tilde{\tilde{w}}_m(t, \frac{\rho}{\delta}) = \frac{1}{m} \frac{\partial \bar{P}_2(t, \rho, \delta; q)}{\partial q^m} \bigg|_{q=0}$. If the $H(t, \rho, \delta), h, n$ and initial guesses are properly chosen. Thus, the series in (5.16) converges at $q = \frac{1}{n}$, we obtain

\[
\begin{align*}
\left\{ & \tilde{z}_m(t, \frac{\rho}{\delta}) = \tilde{z}(0, \frac{\rho}{\delta}) + \sum_{m=1}^{\infty} \tilde{z}_m(t, \frac{\rho}{\delta}) \left( \frac{1}{n} \right)^m \\
& \tilde{\tilde{z}}_m(t, \frac{\rho}{\delta}) = \tilde{\tilde{z}}(0, \frac{\rho}{\delta}) + \sum_{m=1}^{\infty} \tilde{\tilde{z}}_m(t, \frac{\rho}{\delta}) \left( \frac{1}{n} \right)^m
\end{align*}
\]

and

\[
\begin{align*}
\left\{ & \tilde{w}_m(t, \frac{\rho}{\delta}) = \tilde{w}(0, \frac{\rho}{\delta}) + \sum_{m=1}^{\infty} \tilde{w}_m(t, \frac{\rho}{\delta}) \left( \frac{1}{n} \right)^m \\
& \tilde{\tilde{w}}_m(t, \frac{\rho}{\delta}) = \tilde{\tilde{w}}(0, \frac{\rho}{\delta}) + \sum_{m=1}^{\infty} \tilde{\tilde{w}}_m(t, \frac{\rho}{\delta}) \left( \frac{1}{n} \right)^m
\end{align*}
\]
The deformation equation of the $m^{th}$ order can be written as

$$
\begin{cases}
S[\bar{\zeta}_m(t, \rho, \delta) - \psi_m \bar{\zeta}_{m-1}(t, \rho, \delta)] = hH(t, \rho, \delta)R_{1,m}[\bar{\zeta}_{m-1}(t, \rho, \delta), \bar{w}_{m-1}(t, \rho, \delta)] \\
S[\bar{\zeta}_m(t, \rho, \delta) - \psi_m \bar{\zeta}_{m-1}(t, \rho, \delta)] = hH(t, \rho, \delta)R_{1,m}[\bar{\zeta}_{m-1}(t, \rho, \delta), \bar{w}_{m-1}(t, \rho, \delta)]
\end{cases}
$$

(5.16)

and

$$
\begin{cases}
S[w_m(t, \rho, \delta) - \psi_m w_{m-1}(t, \rho, \delta)] = hH(t, \rho, \delta)R_{2,m}[\bar{\zeta}_{m-1}(t, \rho, \delta), \bar{w}_{m-1}(t, \rho, \delta)] \\
S[w_m(t, \rho, \delta) - \psi_m w_{m-1}(t, \rho, \delta)] = hH(t, \rho, \delta)R_{2,m}[\bar{\zeta}_{m-1}(t, \rho, \delta), \bar{w}_{m-1}(t, \rho, \delta)]
\end{cases}
$$

(5.17)

where

$$
\begin{cases}
R_{1,m}[\bar{\zeta}_{m-1}(t, \rho, \delta), \bar{w}_{m-1}(t, \rho, \delta)] = S[\bar{\zeta}_{m-1}(t, \rho, \delta)] - \left(1 - \frac{\psi}{\pi}\right) \frac{\alpha}{\pi} \chi(0, \rho, \delta) \\
\frac{1-\beta+\rho(\frac{s}{\pi})}{B(\rho)} \times S[-a\bar{\zeta}_{m-1}(t, \rho, \delta) + b\bar{w}_{m-1}(t, \rho, \delta)]
\end{cases}
$$

(5.18)

and

$$
\begin{cases}
R_{2,m}[\bar{w}_{m-1}(t, \rho, \delta), \bar{w}_{m-1}(t, \rho, \delta)] = S[\bar{w}_{m-1}(t, \rho, \delta)] - \left(1 - \frac{\psi}{\pi}\right) \frac{\alpha}{\pi} \chi(0, \rho, \delta) \\
\frac{1-\beta+\rho(\frac{s}{\pi})}{B(\rho)} \times S[-a\bar{w}_{m-1}(t, \rho, \delta) + b\bar{w}_{m-1}(t, \rho, \delta)]
\end{cases}
$$

(5.19)

and

$$\psi = \begin{cases} o, & m \leq 1 \\ n, & m > 1 \end{cases}$$

(5.20)

Upon applying the inverse Shehu transform to Eqs. (5.16) and (5.17), we get

$$\begin{align*}
\bar{\zeta}_m(t, \rho, \delta) &= \psi_m \bar{\zeta}_{m-1}(t, \rho, \delta) + S^{-1}\left[hH(t, \rho, \delta)R_{1,m}[\bar{\zeta}_{m-1}(t, \rho, \delta), \bar{w}_{m-1}(t, \rho, \delta)]\right] \\
\bar{\zeta}_m(t, \rho, \delta) &= \psi_m \bar{\zeta}_{m-1}(t, \rho, \delta) + S^{-1}\left[hH(t, \rho, \delta)R_{1,m}[\bar{\zeta}_{m-1}(t, \rho, \delta), \bar{w}_{m-1}(t, \rho, \delta)]\right]
\end{align*}
$$

(5.21)

and

$$\begin{align*}
\bar{w}_m(t, \rho, \delta) &= \psi_m \bar{w}_{m-1}(t, \rho, \delta) + S^{-1}\left[hH(t, \rho, \delta)R_{2,m}[\bar{\zeta}_{m-1}(t, \rho, \delta), \bar{w}_{m-1}(t, \rho, \delta)]\right] \\
\bar{w}_m(t, \rho, \delta) &= \psi_m \bar{w}_{m-1}(t, \rho, \delta) + S^{-1}\left[hH(t, \rho, \delta)R_{2,m}[\bar{\zeta}_{m-1}(t, \rho, \delta), \bar{w}_{m-1}(t, \rho, \delta)]\right]
\end{align*}
$$

(5.22)

Similarly, upon solving the equation, we can get the approximate solution of Eq. (5.9) as

$$\begin{align*}
\bar{z}(t, \rho, \delta) &= \bar{z}_0(t, \rho, \delta) + \frac{\bar{z}_1(t, \rho, \delta)}{n} + \frac{\bar{z}_2(t, \rho, \delta)}{n^2} + \ldots \\
\bar{z}(t, \rho, \delta) &= \bar{z}_0(t, \rho, \delta) + \frac{\bar{z}_1(t, \rho, \delta)}{n} + \frac{\bar{z}_2(t, \rho, \delta)}{n^2} + \ldots
\end{align*}
$$

(5.23)

and

$$\begin{align*}
\bar{w}(t, \rho, \delta) &= \bar{w}_0(t, \rho, \delta) + \frac{\bar{w}_1(t, \rho, \delta)}{n} + \frac{\bar{w}_2(t, \rho, \delta)}{n^2} + \ldots \\
\bar{w}(t, \rho, \delta) &= \bar{w}_0(t, \rho, \delta) + \frac{\bar{w}_1(t, \rho, \delta)}{n} + \frac{\bar{w}_2(t, \rho, \delta)}{n^2} + \ldots
\end{align*}
$$

(5.24)
5.2 Fuzzy double parametric technique HLM

This section discusses the solution of the fuzzy fractional model (3.4) using the $q$-homotopy analysis transform method ($q$-HASTM) [21]. Upon applying the $\rho$-cut in Eq. (3.4) in the interval form of [(i)-gH] derivative, we get

\[
\begin{align*}
ABC D^\beta z(t, \rho), ABC D^\beta \bar{z}(t, \rho) & = -a[\bar{z}(t, \rho), \bar{z}(t, \rho)] + b[w(t, \rho), \bar{w}(t, \rho)] \\
ABC D^\beta w(t, \rho), ABC D^\beta \bar{w}(t, \rho) & = a[\bar{z}(t, \rho), \bar{z}(t, \rho)] - (b + d)[w(t, \rho), \bar{w}(t, \rho)]
\end{align*}
\]  
(5.25)

having initial conditions

\[
\begin{align*}
[\bar{z}(0, \rho), \bar{z}(0, \rho)] & = [50\rho + 200, 300 - 50\rho] \\
[w(0, \rho), \bar{w}(0, \rho)] & = [0.1\rho, 0.2 - 0.1\rho]
\end{align*}
\]  
(5.26)

Upon applying another parameters $\delta$ in Eqs. (5.25) and (5.26), we obtain

\[
\begin{align*}
\left[\delta \left(ABC D^\beta \bar{z}(t, \rho) - ABC D^\beta \bar{z}(t, \rho)\right) + ABC D^\beta \bar{z}(t, \rho)\right] & = -a[\bar{z}(t, \rho) - \bar{z}(t, \rho)] \\
& + b[\delta (\bar{w}(t, \rho) - w(t, \rho)) + w(t, \rho)] \\
\left[\delta \left(ABC D^\beta \bar{w}(t, \rho) - ABC D^\beta \bar{w}(t, \rho)\right) + ABC D^\beta w(t, \rho)\right] & = a[\delta (\bar{z}(t, \rho) - \bar{z}(t, \rho)) \\
& + \bar{z}(t, \rho)] - (b + d)[\delta (\bar{w}(t, \rho) - w(t, \rho)) + w(t, \rho)]
\end{align*}
\]  
(5.27)

and

\[
\begin{align*}
\left[\delta (\bar{z}(0, \rho) - \bar{z}(0, \rho)) + \bar{z}(0, \rho)\right] & = \delta[(300 - 50\rho) - (50\rho + 200)] + 50\rho + 200 \\
\left[\delta (\bar{w}(0, \rho) - w(0, \rho)) + w(0, \rho)\right] & = \delta[(0.2 - 0.1\rho) - (0.1\rho)] + 0.1\rho
\end{align*}
\]  
(5.28)

Here, $\rho$ and $\delta$ are the two fuzzy parameters, and $\rho, \delta \in [0, 1]$ is the double parametric form of the fuzzy fractional model (5.27) and with initial conditions (5.28). As a result, we can write Eq. (5.27) and (5.28) using the definition 5 and 9 as

\[
\begin{align*}
\left[\delta \left(ABC D^\beta \bar{z}(t, \rho) - ABC D^\beta \bar{z}(t, \rho)\right) + ABC D^\beta \bar{z}(t, \rho)\right] & = ABC D^\beta \bar{z}_{i.gH}(t, \rho, \delta) \\
\left[\delta \left(ABC D^\beta \bar{w}(t, \rho) - ABC D^\beta \bar{w}(t, \rho)\right) + ABC D^\beta w(t, \rho)\right] & = ABC D^\beta \bar{w}_{i.gH}(t, \rho, \delta)
\end{align*}
\]  
(5.29)

\[
\begin{align*}
\delta (\bar{z}(t, \rho) - \bar{z}(t, \rho)) & = \bar{z}(t, \rho, \delta) \\
\delta (\bar{w}(t, \rho) - w(t, \rho)) & = \bar{w}(t, \rho, \delta)
\end{align*}
\]  
(5.30)

Thus, Eqs. (5.27) and (5.28) can be written with the help of Eq. (5.29) to (5.32) as

\[
\begin{align*}
ABC D^\beta \bar{z}_{i.gH}(t, \rho, \delta) & = -a\bar{z}(t, \rho, \delta) + b\bar{w}(t, \rho, \delta) \\
ABC D^\beta \bar{w}_{i.gH}(t, \rho, \delta) & = a\bar{z}(t, \rho, \delta) - (b + d)\bar{w}(t, \rho, \delta)
\end{align*}
\]  
(5.33)

and

\[
\begin{align*}
\bar{z}(t, \rho, \delta) & = \delta(100 - 100\rho) + 50\rho + 200, \\
\bar{w}(t, \rho, \delta) & = \delta(0.2 - 0.2\rho) + 0.1\rho
\end{align*}
\]  
(5.34)

Upon applying the Shehu transform in Eq. (5.33), we get

\[
\begin{align*}
\frac{B(\beta)}{1 - \beta + \beta \bar{z}} \left[SL\bar{z}(t, \rho, \delta) - \left(\frac{\mu}{s}\right)\bar{z}(0, \rho, \delta)\right] & = SL[-a\bar{z}(t, \rho, \delta) + b\bar{w}(t, \rho, \delta)] \\
\frac{B(\beta)}{1 - \beta + \beta \bar{w}} \left[SL\bar{w}(t, \rho, \delta) - \left(\frac{\mu}{s}\right)\bar{w}(0, \rho, \delta)\right] & = SL[a\bar{z}(t, \rho, \delta) - (b + d)\bar{w}(t, \rho, \delta)]
\end{align*}
\]  
(5.35)

Upon simplifying Eq. (5.35), it becomes
Clearly for where
\[
\frac{1}{S(t, \rho, \delta)} - \frac{\mu}{S} = \frac{1 - \beta + \beta(\frac{q}{\rho})}{B(\beta)} \times S[-a \tilde{z}(t, \rho, \delta) + b \tilde{w}(t, \rho, \delta)]
\]
\[
\frac{1}{S(\tilde{w}(t, \rho, \delta)) - \frac{\mu}{S}} = \frac{1 - \beta + \beta(\frac{q}{\rho})}{B(\beta)} \times S[a \tilde{z}(t, \rho, \delta) - (b + d) \tilde{w}(t, \rho, \delta)]
\]

Now we can define the two parameters using q-homotopy analysis method as
\[
N_1[P_1(t, \rho, \delta; q), P_2(t, \rho, \delta; q)] = S[P_1(t, \rho, \delta)] - \frac{1}{S} P_1(0, \rho, \delta) - \frac{1 - \beta + \beta(\frac{q}{\rho})}{B(\beta)} \times S[-a P_1(t, \rho, \delta) + b P_2(t, \rho, \delta)]
\]
\[
N_2[P_1(t, \rho, \delta; q), P_2(t, \rho, \delta; q)] = S[P_2(t, \rho, \delta)] - \frac{1}{S} P_2(0, \rho, \delta) - \frac{1 - \beta + \beta(\frac{q}{\rho})}{B(\beta)} \times S[a P_1(t, \rho, \delta) - (b + d) P_2(t, \rho, \delta)]
\]

Thus, the deformation equation is
\[
(1 - nq)S[\tilde{z}(t, \rho, \delta; q) - \tilde{z}(0, \rho, \delta)] = hq H(t, \rho, \delta) N_1[P_1(t, \rho, \delta; q), P_2(t, \rho, \delta; q)]
\]
\[
(1 - nq)S[\tilde{w}(t, \rho, \delta; q) - \tilde{w}(0, \rho, \delta)] = hq H(t, \rho, \delta) N_2[P_1(t, \rho, \delta; q), P_2(t, \rho, \delta; q)]
\]

where \( P_1(t, \rho, \delta; q) \) & \( P_2(t, \rho, \delta; q) \) are unknown functions and \( q \in [0, \frac{1}{n}] \) is an embedding parameter, \( \tilde{z}(0, \rho, \delta) \) & \( \tilde{w}(0, \rho, \delta) \) are initial guesses. \( S[.] \) is the Shehu transform, \( H(t, \rho, \delta) \neq 0 \) is an auxiliary function and \( h \neq 0 \) is a nonzero auxiliary parameters. Clearly for \( q = 0 \) and \( q = \frac{1}{n} \), we have
\[
P_1(0, \rho, \delta; q) = \tilde{z}(0, \rho, \delta); \quad P_1(t, \rho, \delta; q) = \tilde{z}(t, \rho, \delta)
\]
\[
P_2(0, \rho, \delta; q) = \tilde{w}(0, \rho, \delta); \quad P_2(t, \rho, \delta; q) = \tilde{w}(t, \rho, \delta)
\]

As the value of \( q \) increases from 0 to \( \frac{1}{n} \), the solutions \( P_1(0, \rho, \delta; q) \) and \( P_1(t, \rho, \delta; q) \) converge from the initial approach \( \tilde{z}(0, \rho, \delta) \) and \( \tilde{w}(0, \rho, \delta) \) to the solution \( \tilde{z}(t, \rho, \delta) \) and \( \tilde{w}(t, \rho, \delta) \), respectively.

Using Taylor series, the expansion of \( P_i(t, \rho, \delta; q), \ i = 1, 2 \) with respect to \( q \), yields
\[
P_1(t, \rho, \delta; q) = \tilde{z}(0, \rho, \delta) + \sum_{m=1}^{\infty} z_m(t, \rho, \delta) q^m
\]
\[
P_2(t, \rho, \delta; q) = \tilde{w}(0, \rho, \delta) + \sum_{m=1}^{\infty} w_m(t, \rho, \delta) q^m
\]

where \( \tilde{z}_m(t, \rho, \delta) = \left. \frac{\partial P_i(t, \rho, \delta; q)}{\partial q} \right|_{q=0} \) and \( \tilde{w}_m(t, \rho, \delta) = \left. \frac{\partial P_i(t, \rho, \delta; q)}{\partial q} \right|_{q=0}. \) If the \( H(t, \rho, \delta), h, n \) and initial guesses are properly chosen. Thus, the series in (5.16) converges at \( q = \frac{1}{n} \), we obtain
\[
\tilde{z}_m(t, \rho, \delta) = \tilde{z}(0, \rho, \delta) + \sum_{m=1}^{\infty} z_m(t, \rho, \delta) \left( \frac{1}{n} \right)^m
\]
\[
\tilde{w}_m(t, \rho, \delta) = \tilde{w}(0, \rho, \delta) + \sum_{m=1}^{\infty} w_m(t, \rho, \delta) \left( \frac{1}{n} \right)^m
\]

The deformation equation of the \( m^{th} \) order can be written as
\[
S[\tilde{z}_m(t, \rho, \delta) - \psi_m \tilde{z}_{m-1}(t, \rho, \delta)] = hH(t, \rho, \delta) R_{1,m}[\tilde{z}_{m-1}(t, \rho, \delta), \tilde{w}_{m-1}(t, \rho, \delta)]
\]
\[
S[\tilde{w}_m(t, \rho, \delta) - \psi_m \tilde{w}_{m-1}(t, \rho, \delta)] = hH(t, \rho, \delta) R_{2,m}[\tilde{z}_{m-1}(t, \rho, \delta), \tilde{w}_{m-1}(t, \rho, \delta)]
\]

where
\[
R_{1,m}[\tilde{z}_{m-1}(t, \rho, \delta), \tilde{w}_{m-1}(t, \rho, \delta)] = S[\tilde{z}_{m-1}(t, \rho, \delta)] - (1 - \frac{\psi}{n}) \tilde{z}(0, \rho, \delta)
\]
\[
1 - \frac{\beta + \beta(\frac{q}{\rho})}{B(\beta)} \times S[-a \tilde{z}_{m-1}(t, \rho, \delta) + b \tilde{w}_{m-1}(t, \rho, \delta)]
\]

\[
R_{2,m}[\tilde{w}_{m-1}(t, \rho, \delta), \tilde{w}_{m-1}(t, \rho, \delta)] = S[\tilde{w}_{m-1}(t, \rho, \delta)] - (1 - \frac{\psi}{n}) \tilde{w}(0, \rho, \delta)
\]
\[
1 - \frac{\beta + \beta(\frac{q}{\rho})}{B(\beta)} \times S[a \tilde{z}_{m-1}(t, \rho, \delta) - (b + d) \tilde{w}_{m-1}(t, \rho, \delta)]
\]
and
\[ \psi = \begin{cases} \alpha, & m \leq 1 \\ n, & m > 1 \end{cases} \] (5.44)

Upon applying the inverse “Shehu transform” to Eq. (5.42), we get
\[ \tilde{z}_m(t, \rho, \delta) = \psi_m \tilde{z}_{m-1}(t, \rho, \delta) + S^{-1} \left[ hH(t, \rho, \delta)R_{1,m} \tilde{z}_{m-1}(t, \rho, \delta), \tilde{w}_{m-1}(t, \rho, \delta) \right] \]
\[ \tilde{w}_m(t, \rho, \delta) = \psi_m \tilde{w}_{m-1}(t, \rho, \delta) + S^{-1} \left[ hH(t, \rho, \delta)R_{2,m} \tilde{z}_{m-1}(t, \rho, \delta), \tilde{w}_{m-1}(t, \rho, \delta) \right] \] (5.45)

Solving Eq. (5.45) for different values of \( m = 1, 2, 3, \ldots \) and \( H(t, \rho, \delta) = 1 \), we get
\[ \tilde{z}_1(t, \rho, \delta) = S^{-1} \left[ hR_{1,1} \tilde{z}_0(t, \rho, \delta), \tilde{w}_{m-1}(t, \rho, \delta) \right] \]
\[ = -h \left( -\tilde{z}(0, \rho, \delta)a + b\tilde{w}(0, \rho, \delta) \right) \left( 1 - \beta + \frac{\beta t^\beta}{\Gamma(\beta + 1)} \right) \]
\[ \tilde{z}_2(t, \rho, \delta) = - (h + n)h \left( -\tilde{z}(0, \rho, \delta)a + b\tilde{w}(0, \rho, \delta) \right) \left( 1 - \beta + \frac{\beta t^\beta}{\Gamma(\beta + 1)} \right) \]
\[ - h \left( ah(-a\tilde{z}(0, \rho, \delta) + b\tilde{z}(0, \rho, \delta)) \left( \beta - 1 \right)^2 + \frac{-2\beta^2 + 2\beta \beta^\beta}{\Gamma(\beta + 1)} + \frac{\beta^2 + 2\beta}{\Gamma(2\beta + 1)} \right) \]
\[ - hh(\tilde{z}(0, \rho, \delta) - (b + d)\tilde{w}(0, \rho, \delta)) \left( \beta - 1 \right)^2 + \frac{-2\beta^2 + 2\beta \beta^\beta}{\Gamma(\beta + 1)} + \frac{\beta^2 + 2\beta}{\Gamma(2\beta + 1)} \) \]
\[ \tilde{w}_1(t, \rho, \delta) = S^{-1} \left[ hR_{2,1} \tilde{z}_0(t, \rho, \delta), \tilde{w}_0(t, \rho, \delta) \right] \]
\[ = -h \left( \tilde{z}(0, \rho, \delta)a - (b + d)\tilde{w}(0, \rho, \delta) \right) \left( 1 - \beta + \frac{\beta t^\beta}{\Gamma(\beta + 1)} \right) \]
\[ \tilde{w}_2(t, \rho, \delta) = - (h + n)h \left( \tilde{z}(0, \rho, \delta)a - (b + d)\tilde{w}(0, \rho, \delta) \right) \left( 1 - \beta + \frac{\beta t^\beta}{\Gamma(\beta + 1)} \right) \]
\[ - h \left( -ah(-a\tilde{z}(0, \rho, \delta) + b\tilde{z}(0, \rho, \delta)) \left( \beta - 1 \right)^2 + \frac{-2\beta^2 + 2\beta \beta^\beta}{\Gamma(\beta + 1)} + \frac{\beta^2 + 2\beta}{\Gamma(2\beta + 1)} \right) \]
\[ + (b + d)h(\tilde{z}(0, \rho, \delta) - (b + d)\tilde{w}(0, \rho, \delta)) \left( \beta - 1 \right)^2 + \frac{-2\beta^2 + 2\beta \beta^\beta}{\Gamma(\beta + 1)} + \frac{\beta^2 + 2\beta}{\Gamma(2\beta + 1)} \] (5.46)

Similarly, upon solving the equation, we can get the approximate solution of Eq. (5.33) as
\[ \tilde{z}(t, \rho, \delta) = \tilde{z}_0(t, \rho, \delta) + \frac{\tilde{z}_1(t, \rho, \delta)}{n} + \frac{\tilde{z}_2(t, \rho, \delta)}{n^2} + \ldots \]
\[ \tilde{w}(t, \rho, \delta) = \tilde{w}_0(t, \rho, \delta) + \frac{\tilde{w}_1(t, \rho, \delta)}{n} + \frac{\tilde{w}_2(t, \rho, \delta)}{n^2} + \ldots \] (5.47)

6 Convergence analysis of the fuzzy fractional human liver model

This section discusses the convergence analysis of the proposed fuzzy fractional human liver model.

**Theorem 6** Let the series \( \sum_{m=0}^{\infty} \tilde{z}_m(t, \rho, \delta) \left( \frac{1}{n} \right)^m \) and \( \sum_{m=0}^{\infty} \tilde{w}_m(t, \rho, \delta) \left( \frac{1}{n} \right)^m \) be uniformly convergent to \( \tilde{z}(t, \rho, \delta) \) and \( \tilde{w}(t, \rho, \delta) \), respectively, and yield by the \( m^{th} \) order deformation Eq. (5.42). Also, we assume that both series \( \sum_{m=0}^{\infty} \tilde{z}_m(t, \rho, \delta) (\tilde{z}_i, \tilde{w}_i)_{m=0}^{\infty} \tilde{w}_m(t, \rho, \delta) \) are convergent. Then, \( \tilde{z}(t, \rho, \delta) \) and \( \tilde{w}(t, \rho, \delta) \) are the exact solution of the system of Eq. (5.33).

**Proof** Let us consider the series \( \sum_{m=0}^{\infty} \tilde{z}_m(t, \rho, \delta) \left( \frac{1}{n} \right)^m \) is uniformly convergent to \( \tilde{z}(t, \rho, \delta) \), so we can write
\[ \lim_{m \to \infty} \tilde{z}(t, \rho, \delta) \left( \frac{1}{n} \right)^m = 0, \text{ for all } t \in \mathbb{R}^+ \] (6.1)
So we have
Thus, from Eq. (6.1) and (6.2), we can write

\[
\sum_{m=1}^{\infty} S \left[ \tilde{z}_m(t, \rho, \delta) - \psi_m \tilde{z}_{m-1}(t, \rho, \delta) \right] \left( \frac{1}{n} \right)^m = \lim_{m \to \infty} S \left[ \tilde{z}_m(t, \rho, \delta) \right] \left( \frac{1}{n} \right)^m = 0
\]
\[ \tilde{z}(t) = 1, \rho = 1, \delta = 0, h = 0.91 \]

\[ \tilde{w}(t) = 1, \rho = 1, \delta = 0, h = 0.91 \]

**Fig. 2** Comparison of exact and the approximate solution of \( \tilde{z}(t) \) and \( \tilde{w}(t) \), respectively

\[ \tilde{z}(t) = 1, \rho = 1, \delta = 0, h = 0.91 \]

\[ \tilde{w}(t) = 1, \rho = 1, \delta = 0, h = 0.91 \]

**Fig. 3** 2D figure for different values of fractional order in crisp case

Hence,

\[ hH \sum_{m=1}^{\infty} R_{1,m}[ \tilde{z}_{m-1}(t, \rho, \delta), \tilde{w}_{m-1}(t, \rho, \delta)] = \sum_{m=0}^{\infty} S \left[ \tilde{z}_m(t, \rho, \delta) - \psi_m \tilde{z}_{m-1}(t, \rho, \delta) \right] \left( \frac{1}{n} \right)^m = 0 \]  \( \text{(6.4)} \)

Since \( h, H \neq 0 \), we obtain

\[ \sum_{m=1}^{\infty} R_{1,m}[ \tilde{z}_{m-1}(t, \rho, \delta), \tilde{w}_{m-1}(t, \rho, \delta)] = 0 \]  \( \text{(6.5)} \)

Similarly, we can prove

\[ \sum_{m=1}^{\infty} R_{2,m}[ \tilde{z}_{m-1}(t, \rho, \delta), \tilde{w}_{m-1}(t, \rho, \delta)] = 0 \]  \( \text{(6.6)} \)

Now, we can get Eq. (5.43) as

\[ 0 = \sum_{m=1}^{\infty} S_1 \tilde{z}_{m-1}(t, \rho, \delta) - \sum_{m=1}^{\infty} \left( 1 - \frac{\psi_m}{n} \right) \left( \frac{u}{\beta} \right) \tilde{z}(0, \rho, \delta) \]
\[ 0 = \sum_{m=1}^{\infty} S[\hat{w}_{m-1}(t/\delta, \rho, \delta)] - \sum_{m=1}^{\infty} \left( 1 - \frac{\psi_m}{n} \right) \hat{w}(0, \rho, \delta) \]
As a result, \( \tilde{z}(t, \rho, \delta) \) and \( \tilde{w}(t, \rho, \delta) \) are the exact solution of the system (5.35). Hence, the proof is complete. \( \square \)

7 Result and discussion

Here, a double parametric approach with \( q \)-HAShTM is considered in the gH-ABC sense to study the fuzzy fractional human liver model (3.4) in an uncertain environment. The values used for the proposed model parameters are as follows: \( a = 0.054736, b = 0.0152704, d = 0.0093906 \), where the initial condition is considered with an uncertain parameter in a fuzzy sense. The effect of the auxiliary parameter \( h \) is investigated to study the convergence of the series solution obtained by \( q \)-HAShTM. The \( h \)-curve has been plotted to define the value of \( h \) in finding the approximate solution of Eq. (3.4). The \( h \)-curve is displayed in Fig. 1a, b for the different values of fractional order \( \beta \). According to this diagram, the convergence zone of the approximation solution is the horizontal line parallel to the \( h \)-axis, and the convergence method is assured for the value of \( h \), i.e. \(-1.1 \leq h \leq -0.4\). In addition, the obtained results of \( q \)-HAShTM are compared with the clinical data in Tables 1 and 2 as obtained by Pro. Evzen Hrncir in 1985 [2]. From Tables 1 and 2, it can be observed that the generalized Atangana–Baleanu–Caputo model with \( \beta = 1 \) integer model approximate solution of \( q \)-HAShTM is closer to the real experimental observation. The obtained results support the efficacy of the proposed method, as the reaction of the new fractional model, i.e. the ABC model, corresponds to real-world clinical observations. Furthermore, Fig. 2a, b represents the comparison of real data and approximate solution of the amount of the BPS at time \( t \) in the blood \( \tilde{z}(t) \) and liver \( \tilde{w}(t) \), respectively. Next, we have computed the \( q \)-HAShTM solution for different value of fractional order \( \beta = 1, 0.95, 0.90, 0.85, 0.80 \) in crisp case, i.e. \( \rho = 1, \delta = 0 \) of \( \tilde{z}(t) \) and \( \tilde{w}(t) \), respectively (see Fig. 3a, b).

Figure 4a, b represents the amount of the BPS at time \( t \) in the blood \( \tilde{z}(t) \) and liver \( \tilde{w}(t) \) for different values of \( \beta \) in uncertain case, i.e. \( \rho = 0.1, \delta = 0.1 \) and similarly the 3D Figs. 5a, b and 6a, b represent the lower and upper bound solution of \( \tilde{z}(t) \) and \( \tilde{w}(t) \) in
Fig. 7 2D fuzzy figure of $\tilde{z}(t)$ for different values of $\rho$

fuzzy sense, respectively. Finally, the 2D Figs. 7a–d and 8a–d represent the fuzzy solution of the proposed problem for different values of time $t$.

8 Conclusion

In this study, a novel fuzzy fractional model of the human liver involving the generalised Atangana–Baleanu–Caputo, i.e. $gH$-ABC derivative, is considered with a novel double parametric approach. Furthermore, the study of the existence of a unique solution using Banach’s fixed point theory in a fuzzy sense demonstrated the problem’s stability. In addition, the uniqueness of the solution has been obtained using the Arzela–Ascoli theorem and Schauder’s fixed-point theory and investigated the converging of the proposed
model. The efficiency of the newly proposed method was verified by the numerical experiments as shown in Tables 1, 2 and Fig. 2. Finally, the tables’ findings revealed that the obtained solutions are closer to the real-world clinical data.

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Declarations

Conflict of interest The author declares there is no conflict of interest.

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