SOME BINOMIAL SUMS INVOLVING ABSOLUTE VALUES

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ABSTRACT. We consider several families of binomial sum identities whose definition involves the absolute value function. In particular, we consider centered double sums of the form

\[ S_{\alpha,\beta}(n) := \sum_{k, \ell} \left( \frac{2n}{n+k} \right) \left( \frac{2n}{n+\ell} \right) |k^\alpha - \ell^\beta|, \]

obtaining new results in the cases \( \alpha = 1, 2 \). We show that there is a close connection between these double sums in the case \( \alpha = 1 \) and the single centered binomial sums considered by Tuenter [The Fibonacci Quarterly 40 (2002), 175–180]. The double sums are of independent interest because the special case

\[ \sum_{k, \ell} \left( \frac{2n}{n+k} \right) \left( \frac{2n}{n+\ell} \right) |k^2 - \ell^2| = 2n^2 \left( \frac{2n}{n} \right)^2 \]

arises in recent proofs of lower bounds for the Hadamard maximal determinant problem. We give a closed form for an analogous triple sum

\[ \sum_{k, \ell, m} \left( \frac{2n}{n+k} \right) \left( \frac{2n}{n+\ell} \right) \left( \frac{2n}{n+m} \right) |(k^2 - \ell^2)(k^2 - m^2)(\ell^2 - m^2)|, \]

and make a conjecture regarding an analogous quadruple sum. Our results can be interpreted as giving expectations arising from certain symmetric Bernoulli random walks.

1. INTRODUCTION

The problem of finding a closed form for the binomial sum

\[ \sum_{k, \ell} \left( \frac{2n}{n+k} \right) \left( \frac{2n}{n+\ell} \right) |k^2 - \ell^2| \] (1.1)

arises in an application of the probabilistic method to the Hadamard maximal determinant problem [5]. Because of the double-summation and the absolute value occurring in (1.1), it is not obvious how to apply standard techniques [9, 13, 17]. A closed-form solution

\[ 2n^2 \left( \frac{2n}{n} \right)^2 \]

was proved by Brent and Osborn in [4], and a simpler proof was found by Prodinger [14]. In this paper we give a shorter proof and generalise the result of [4, 14] to include a wider class of binomial sums with the distinguishing feature that an absolute value occurs in the summand.

Specifically, we consider certain \( d \)-fold binomial sums of the form

\[ S(n) := \sum_{k_1, \ldots, k_d} \prod_{i=1}^d \left( \frac{2n}{n+k_i} \right) |f(k_1, \ldots, k_d)|, \] (1.2)

Research of the first author supported in part by Australian Research Council grant DP140101417.
where \( f : \mathbb{Z}^d \rightarrow \mathbb{Z} \) is a homogeneous polynomial and \( |f| \) will be called the weight function. For example, a simple case is \( d = 1 \), \( f(k) = k \). This case was considered by Best [2] in an application to Hadamard matrices and was set as a problem in the 1974 Putnam competition [1]. The closed-form solution is

\[
\sum_k \binom{2n}{n+k} |k| = n \binom{2n}{n}.
\]

A generalisation \( f(k) = k^r \) (for a fixed \( r \in \mathbb{N} \)) was considered by Tuenter [16], and shown to be expressible using Dumont-Foata polynomials [6, 7]. Tuenter gave an interpretation in terms of the moments of the distance to the origin in a symmetric Bernoulli random walk. It is easy to see that this interpretation generalises: \( 4^{-nd} S(n) \) is the expectation of \( |f(k_1, \ldots, k_d)| \) if we start at the origin and take \( 2n \) random steps \( \pm \frac{1}{2} \) in each of \( d \) dimensions, thus arriving at the point \( (k_1, \ldots, k_d) \in \mathbb{Z}^d \) with probability

\[
4^{-nd} \prod_{i=1}^d \binom{2n}{n+k_i}.
\]

A further generalisation replaces \( \binom{2n}{n+k_i} \) by \( \binom{2n_i}{n_i+k_i} \), allowing the number of random steps \( 2n_i \) in dimension \( i \) to depend on \( i \). With a suitable modification to the definition of \( S \), we could also drop the restriction to an even number of steps in each dimension.\(^2\) We briefly consider such a generalisation in §2.

Tuenter’s results for the case \( d = 1 \) were generalised in [3]. In this paper we concentrate on the case \( d = 2 \). We also give a new result for \( d = 3 \) and a conjecture for \( d = 4 \).

There are many binomial coefficient identities in the literature, e.g. 500 are given by Gould in [10]. Many such identities can be proved via generating functions [11, 17] or the Wilf-Zeilberger algorithm [13]. Nevertheless, we hope that the reader will find our results interesting, in part because of the applications mentioned above, and in part because proofs using standard techniques are not known in many of the cases where the sum involves an absolute value.

An outline of the paper follows.

In §2 we consider a special class of double sums that can be reduced to the single sums of [3, 12, 16].

In §3 we consider a generalisation of the motivating case (1.1) described above: \( f(k, \ell) = (k^\alpha - \ell^\alpha)^\beta \). In the case \( \alpha = 2 \) we give recurrence relations that allow such sums to be evaluated in closed form for any given positive integer \( \beta \). The recurrence relations naturally split into the cases where \( \beta \) is even (easy) and odd (more difficult).

In §4 we consider analogous triple and quadruple sums. A closed form for one such triple sum is given in Theorem 4.1, and Conjecture 4.1 gives a conjectured closed form for the analogous quadruple sum.

In §5 we state several double sum identities that have all been confirmed numerically for many different values of the independent variable \( n \), but in some cases have not been proved rigorously. The known proofs are given or outlined in §6.

Finally, §7 sketches some asymptotic results which provide insight into the structure of the solutions found in §3 for the weight function \( |k^\alpha - \ell^\alpha|^\beta \), and confirm that various conjectures

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1Best [2] proved a lower bound on the maximum excess of a Hadamard matrix, using the probabilistic method. This is relevant to the Hadamard maximal determinant problem [8].

2For example, in the case \( d = 1 \) we could consider \( \sum_k \binom{\ell}{k} |f(n-2k)| \).
made in \cite{3} have the correct asymptotic behaviour as $n \to \infty$. The asymptotic results explain why we typically have one form of solution, involving a factor $(\frac{2n}{n})^2$, for even $\alpha$, and a different form, involving a factor $(\frac{4n}{2n})^2$, for odd $\alpha$.

**Notation.** The set of all integers is $\mathbb{Z}$, the set of non-negative integers is $\mathbb{N}$, and the set of positive integers is $\mathbb{N}^+$. The variables $\alpha, \beta, i, \ldots, n, p, \ldots, s$ denote integers. The expectation of a random variable $X$ is denoted by $\mathbb{E}[X]$.

The binomial coefficient $\binom{n}{k}$ is defined to be zero if $k < 0$ or $k > n$ (and hence always if $n < 0$). Using this convention, we often avoid explicitly specifying upper and lower limits on $k$ or excluding cases where $n < 0$.

The notation $A = B$ means that $A = B$ is conjectured to be an identity, based on numerical evidence, but we do not claim to have a rigorous proof.

In the definition of the weight function $|f|$, we always interpret $0^0$ as 1.

2. **Some double sums reducible to single sums**

Tuenter \cite{16} considered the binomial sum

$$S_\beta(n) := \sum_k \binom{2n}{n+k} |k|^\beta, \quad (2.1)$$

and a generalisation\footnote{It is a generalisation because $S_\beta(n) = U_\beta(2n)$, but $U_\beta(n)$ is well-defined for all $n \in \mathbb{N}$. The generalisation is needed in \cite{4}.} to

$$U_\beta(n) := \sum_k \binom{n}{k} \left| \frac{n}{2} - k \right|^\beta \quad (2.2)$$

was given in \cite{3}.

Tuenter showed that

$$S_{2\beta}(n) = Q_\beta(n) 2^{2n-\beta}, \quad S_{2\beta+1}(n) = P_\beta(n)n\binom{2n}{n}, \quad (2.3)$$

where $P_\beta(n)$ and $Q_\beta(n)$ are polynomials of degree $\beta$ with integer coefficients, satisfying certain three-term recurrence relations, and expressible in terms of Dumont-Foata polynomials \cite{6, 7}.

Closed-form expressions for $S_\beta(n)$, $P_\beta(n)$, $Q_\beta(n)$ are given in \cite{3, 12}.

In this section we consider the double sum

$$T_\beta(m, n) := \sum_{k, \ell} \binom{2m}{m+k} \binom{2n}{n+\ell} |k - \ell|^\beta \quad (2.4)$$

and show that it can be expressed as a single sum of the form \eqref{2.1}.

**Theorem 2.1.** For all $\beta, m, n \in \mathbb{N}$, we have

$$T_\beta(m, n) = S_\beta(m+n),$$

where $T_\beta$ is defined by \eqref{2.4} and $S_\beta$ is defined by \eqref{2.1}.

\footnote{Care must be taken when interpreting the closed-form expressions in \cite{12}, because the definition of the binomial coefficient $\binom{n}{k}$ used in \cite{12} is different from our definition – in particular, $\binom{n}{k} := n(n-1) \cdots (n-k+1)/k!$ is nonzero for $n < 0, k \geq 0$.}
Proof. If $\beta = 0$ then $T_0(m,n) = 2^{(m+n)} = S_0(m+n)$. Hence, we may assume that $\beta > 0$ (so $0^\beta = 0$). Let $d = \ell - k$. We split the sum (2.4) defining $T_\beta(m,n)$ into three parts, corresponding to $k > \ell$, $k < \ell$, and $k = \ell$. The third part vanishes. If $k > \ell$ then $d = k - \ell$ and $k = d + \ell$; if $k < \ell$ then $d = \ell - k$ and $\ell = d + k$. Thus, we get

$$T_\beta(m,n) = \sum_{d>0} \sum_{\ell} \binom{2m}{m+d+\ell} \binom{2n}{n+\ell} d^\beta + \sum_{d>0} \sum_{k} \binom{2m}{m+k} \binom{2n}{n+k+d} d^\beta = \sum_{d>0} d^\beta \sum_{\ell} \binom{2m}{m+d+\ell} \binom{2n}{n-\ell} + \sum_{d>0} d^\beta \sum_{k} \binom{2n}{n+k+d} \binom{2m}{m-k}.$$ 

By Vandermonde’s identity, the inner sums over $k$ and $\ell$ are both equal to $\binom{2m+2n}{m+n+d}$. Thus,

$$T_\beta(m,n) = 2 \sum_{d>0} \binom{2m+2n}{m+n+d} d^\beta = \sum_d \left( \frac{2m+2n}{m+n+d} \right) |d|^\beta = S_\beta(m+n).$$

Remark 2.1. If $m = n$ then, by the shift-invariance of the weight $|k-\ell|^\beta$, we have

$$T_\beta(n,n) = \sum_{k,\ell} \binom{2n}{k} \binom{2n}{\ell} |k-\ell|^\beta = S_\beta(2n). \tag{2.5}$$

There is no need for the upper argument of the binomial coefficients to be even in (2.5). We can adapt the proof of Theorem 2.1 to show that, for all $n \in \mathbb{N}$,

$$\sum_{k,\ell} \binom{n}{k} \binom{n}{\ell} |k-\ell|^\beta = S_\beta(n).$$

3. CENTERED DOUBLE SUMS

In this section we consider the centered double binomial sums defined by

$$S_{\alpha,\beta}(n) := \sum_{k,\ell} \binom{2n}{n+k} \binom{2n}{n+\ell} |k^\alpha - \ell^\beta|^\beta. \tag{3.1}$$

Note that $S_{1,\beta}(n) = T_\beta(n,n)$, so the case $\alpha = 1$ is covered by Theorem 2.1. Thus, in the following we can assume that $\alpha \geq 2$. Since we mainly consider the case $\alpha = 2$, it is convenient to define

$$W_\beta(n) := S_{2,\beta}(n) = \sum_{k,\ell} \binom{2n}{n+k} \binom{2n}{n+\ell} |k^2 - \ell^\beta|^\beta. \tag{3.2}$$

3.1. $W_\beta$ for odd $\beta$. The analysis of $W_\beta(n)$ naturally splits into two cases, depending on the parity of $\beta$. We first consider the case that $\beta$ is odd. A simpler approach is possible when $\beta$ is even, as we show in §3.3.

As mentioned in §1, the evaluation of $W_1(n)$ was the motivation for this paper, and is given in the following theorem. It was proved in [44], but we give a new and shorter proof in [6].

\footnote{The double sum $S_{\alpha,\beta}(n)$ should not be confused with the single sum $S_\alpha(n)$ of [2].}
Theorem 3.1 (Brent and Osborn).

\[ W_1(n) = \sum_{k, \ell} \left( \binom{2n}{n+k} \binom{2n}{n+\ell} \right) |k^2 - \ell^2| = 2n^2 \left( \frac{2n}{n} \right)^2. \]

Numerical evidence suggests the following generalisation of Theorem 3.1. The conjecture has been confirmed numerically for all \( m, n \leq 100. \)

Conjecture 3.1. For all \( m, n \in \mathbb{N}, \)

\[ \sum_{k, \ell} \left( \binom{2m}{m+k} \binom{2n}{n+\ell} \right) |k^2 - \ell^2| \geq 2mn \left( \binom{2m}{m} \right) \left( \binom{2n}{n} \right), \]

with equality if and only if \( m = n. \)

3.2. Recurrence relations for the odd case. Theorem 3.1 gives \( W_1(n). \) We show how \( W_3(n), W_5(n), \ldots \) can be computed using recurrence relations. More precisely, we express the double sums \( W_{2k+1}(n) \) in terms of certain single sums \( G_k(n, m), \) and give a recurrence for the \( G_k(n, m). \) We then show that \( W_{2k+1}(n) \) is a linear combination of \( P_k(n), \ldots, P_{2k}(n), \) where the polynomials \( P_m(n) \) are as in (2.3), and the coefficients multiplying these polynomials satisfy another recurrence relation.

To avoid ambiguities of the \( \Sigma' \) notation, define

\[ f_q = \begin{cases} 1 & \text{if } q \neq 0; \\ \frac{1}{2} & \text{if } q = 0. \end{cases} \]

Using symmetry and the definition (3.2) of \( W_k(n), \) we have

\[ W_{2k+1}(n) = 8 \sum_{q=0}^{n} \sum_{p=0}^{n} \left( \binom{2n}{n+p} \binom{2n}{n+q} \right) (p^2 - q^2)^{2k+1} f_q; \] (3.3)

the factor \( f_q \) allows for terms which would otherwise be counted twice.

Let \( m = p - q. \) Since \( p^2 - q^2 = m(m+2q), \) we can write the double sum \( W_{2k+1}(n)/8 \) in (3.3) as

\[ \sum_{q=0}^{n} \sum_{p=0}^{n} \left( \binom{2n}{n+p} \binom{2n}{n+q} \right) (p^2 - q^2)^{2k+1} f_q = \sum_{m \geq 0} m^{2k+1} G_k(n, m), \] (3.4)

where

\[ G_k(n, m) := \sum_{q \geq 0} \left( \binom{2n}{n+m+q} \right) \left( \binom{2n}{n+q} \right) (m + 2q)^{2k+1} f_q. \] (3.5)

Observe that \( G_k(0, m) = 0. \) For convenience we define \( G_k(-1, m) = 0. \) We observe that \( G_k(n, m) \) satisfies a recurrence relation, as follows.

Lemma 3.2. For all \( k, m, n \in \mathbb{N}, \)

\[ G_{k+2}(n, m) = 2(4n^2 + m^2)G_{k+1}(n, m) - (4n^2 - m^2)^2 G_k(n, m) + 64n^2(2n - 1)^2 G_k(n - 1, m). \] (3.6)
Proof. If \( n = 0 \) the proof of (3.6) is trivial, since \( G_k(0, m) = G_k(-1, m) = 0 \). Hence, suppose that \( n > 0 \). We observe that

\[
[(m + 2q)^4 - 2(4n^2 + m^2)(m + 2q)^2 + (4n^2 - m^2)^2]\left(\frac{2n}{n + m + q}\right)\left(\frac{2n}{n + q}\right)
= 16(n + m + q)(n - m - q)(n + q)(n - q)\left(\frac{2n}{n + m + q}\right)\left(\frac{2n}{n + q}\right)
= 64n^2(2n - 1)^2\left(\frac{2n - 2}{n - 1 + m + q}\right)\left(\frac{2n - 2}{n - 1 + q}\right).
\]

Now multiply each side by \((m + 2q)^{2k+1}f_q\) and sum over \( q \geq 0 \).

The recurrence (3.6) may be used to compute \( G_k(n, m) \) for given \((n, m)\) and \( k = 0, 1, 2, \ldots \), using the initial values

\[
G_0(n, m) = \frac{n}{2} \left(\frac{2n}{n}ight)\left(\frac{2n}{n + m}\right)
\]

and

\[
G_1(n, m) = \frac{4n^2 + (2n - 5)m^2}{2n - 1} G_0(n, m).
\]

These initial values may be verified from the definition (3.5) by standard methods [13] – we omit the details.

Write \( g_k(n, m) = 0 \) if \( G_k(n, m) = 0 \), and otherwise define \( g_k(n, m) \) by

\[
G_k(n, m) = \left(\frac{2n}{n}\right)\left(\frac{2n}{n + m}\right) g_k(n, m).
\]

The recurrence (3.6) for \( G_k \) gives a corresponding recurrence for \( g_k \):

\[
g_{k+2}(n, m) = 2(4n^2 + m^2)g_{k+1}(n, m) - (4n^2 - m^2)^2g_k(n, m)
+ 16n^2(n^2 - m^2)g_k(n - 1, m),
\]

with initial values

\[
g_0(n, m) = \frac{n}{2}, \quad g_1(n, m) = \frac{4n^2 + (2n - 5)m^2}{2n - 1} g_0(n, m).
\]

Note that the \( g_k(n, m) \) are rational functions in \( n \) and \( m \); for computation with bivariate polynomials over \( \mathbb{Z} \), \( g_k(n, m) \) can be multiplied by \((2n - 1)(2n - 3)\cdots(2n - (2k - 1))\). If \( n \) is fixed, then \( g_k(n, m) \) is an even polynomial in \( m \) and, from the recurrence (3.7), the degree is \( 2k \). This suggests that we should define rational functions \( \gamma_{k,j}(n) \) by

\[
g_k(n, m) = \sum_{j=0}^{k} \gamma_{k,j}(n)m^{2j}.
\]

For \( j < 0 \) or \( j > k \) we define \( \gamma_{k,j}(n) = 0 \). From the recurrence (3.7), we obtain the following recurrence for the \( \gamma_{k,j}(n) \):

\[
\gamma_{k+2,j}(n) = 8n^2\gamma_{k+1,j}(n) + 2\gamma_{k+1,j-1}(n) - 16n^4\gamma_{k,j}(n) + 8n^2\gamma_{k,j-1}(n)
- \gamma_{k,j-2}(n) + 16n^4\gamma_{k,j}(n - 1) - 16n^2\gamma_{k,j-1}(n - 1).
\]

(3.8)
The $\gamma_{k,j}(n)$ can be computed from (3.8), using the initial values

$$
\begin{align*}
\gamma_{0,0}(n) &= n/2, \\
\gamma_{1,0}(n) &= 2n^3/(2n - 1), \\
\gamma_{1,1}(n) &= n(2n - 5)/(4n - 2).
\end{align*}
$$

(3.9)

Using the definition of $\gamma_{k,j}(n)$ and (3.3)-(3.5), we obtain

$$
W_{2k+1}(n) = 4 \left(\frac{2n}{n}\right) \sum_{j=0}^{k} \gamma_{k,j}(n) S_{2k+2j+1}(n).
$$

(3.10)

Since $S_{2r+1}(n) = P_r(n) n^{2n}$, we obtain the following theorem, which shows that the double sums $W_{2k+1}(n)$ may be expressed in terms of the same polynomials $P_m(n)$ that occur in expressions for the single sums of [3, 16].

**Theorem 3.2.**

$$
W_{2k+1}(n) = 4n \sum_{j=0}^{k} \gamma_{k,j}(n) P_{k+j}(n) \cdot \left(\frac{2n}{n}\right)^2,
$$

(3.10)

where the polynomials $P_{k+j}(n)$ are as in (2.3), and the $\gamma_{k,j}(n)$ may be computed from the recurrence (3.8) and the initial values given in (3.9).

The factor occurring before the binomial coefficient in (3.10) is a rational function $\omega_k(n)$ with denominator $(2n-1)(2n-3) \cdots (2n-2\lceil k/2 \rceil + 1)$. Thus, we have the following corollary of Theorem 3.2

**Corollary 3.3.** If $k \in \mathbb{N}$ and $W_k(n)$ is defined by (3.2), then

$$
W_{2k+1}(n) = \omega_k(n) \left(\frac{2n}{n}\right)^2,
$$

where

$$
\omega_k(n) \prod_{j=1}^{[k/2]} (2n - 2j + 1)
$$

is a polynomial of degree $2k + [k/2] + 2$ over $\mathbb{Z}$. The first four cases are:

$$
\begin{align*}
\omega_0(n) &= 2n^2, \\
\omega_1(n) &= 2n^3(8n^2 - 12n + 5) \frac{2n}{2n - 1}, \\
\omega_2(n) &= 2n^3(128n^4 - 512n^3 + 800n^2 - 568n + 153) \frac{2n}{2n - 1}, \\
\omega_3(n) &= \frac{2n^3 \overline{\omega}_3(n)}{(2n - 1)(2n - 3)}, \text{ where} \\
\overline{\omega}_3(n) &= 9216n^7 - 86016n^6 + 350464n^5 - 802304n^4 + 1106856n^3 - 914728n^2 + 417358n - 80847.
\end{align*}
$$
3.3. $W_{\beta}$ for even $\beta$. Now we consider $W_{\beta}(n)$ for even $\beta$. This case is easier than the case of odd $\beta$ because the absolute value in the definition (3.2) has no effect when $\beta$ is even. Theorem 3.3 shows that $W_{2r}(n)$ can be expressed in terms of the single sums $S_0(n), S_2(n), \ldots, S_{4r}(n)$ or, equivalently, in terms of the polynomials $Q_0(n), Q_1(n), \ldots, Q_{2r}(n)$. Hence $2^{2r-4n}W_{2r}(n)$ is a polynomial over $\mathbb{Z}$ of degree $2r$ in $n$.

**Theorem 3.3.** For all $n \in \mathbb{N}$,

$$W_{2r}(n) = \sum_k (-1)^k \binom{2r}{k} S_{2k}(n) S_{4r-2k}(n)$$

$$= 2^{-4n-2r} \sum_k (-1)^k \binom{2r}{k} Q_k(n) Q_{2r-k}(n),$$

where $Q_r(n)$ and $S_r(n)$ are as (2.1)–(2.3) of [2], and $W_{\beta}(n)$ is defined by (3.2).

**Proof.** From the definition of $W_{2r}(n)$ we have

$$W_{2r}(n) = \sum_i \sum_j \left( \binom{2n}{n+i} \binom{2n}{n+j} (i^2 - j^2)^{2r} \right).$$

Write

$$(i^2 - j^2)^{2r} = \sum_k (-1)^k \binom{2r}{k} i^{4r-2k} j^{2k},$$

change the order of summation in the resulting triple sum, and observe that the inner sums over $i$ and $j$ separate, giving $S_{4r-2k}(n) S_{2k}(n)$. This proves the first part of the theorem. The second part follows from (2.3).

For example, the first four cases are

- $W_0(n) = 2^{4n}$,
- $W_2(n) = 2^{4n-1} n(2n-1)$,
- $W_4(n) = 2^{4n-2} n(2n-1)(18n^2 - 33n + 17)$,
- $W_6(n) = 2^{4n-3} n(2n - 1)(900n^4 - 4500n^3 + 8895n^2 - 8055n + 2764)$.

It follows from Theorem 3.3 that the coefficients of $2^{2r-4n}W_{2r}(n)$ are in $\mathbb{Z}$, but it is not obvious how to prove the stronger result, suggested by the cases above, that the coefficients of $2^{r-4n}W_{2r}(n)$ are in $\mathbb{Z}$. We leave this as a conjecture. It has been verified numerically for $r \leq 10$.

4. **Some triple and quadruple sums**

In Theorem 4.1 we give a triple sum that is analogous to the double sum of Theorem 3.1.

**Theorem 4.1.** For all $n \in \mathbb{N}$,

$$\sum_{i,j,k} \left( \binom{2n}{n+i} \binom{2n}{n+j} \binom{2n}{n+k} \right) |(i^2 - j^2)(i^2 - k^2)(j^2 - k^2)| = 3n^3(n-1) \binom{2n}{n} 2^{2n-1}.$$

Since our proof of Theorem 4.1 is long and tedious, we relegate it to the Appendix. The proof is similar to the first proofs [4, 14] of Theorem 3.1, but necessarily more complicated because of the triple rather than double sum. We challenge the reader to find a simpler and more illuminating proof!
A conjectured quadruple-sum analogue of Theorem 4.1 verified numerically for all \( n \in \mathbb{N}^+ \) in the range \( 1 \leq n \leq 30 \), is:

**Conjecture 4.1.** For all \( n \in \mathbb{N}^+ \),

\[
\sum_{i,j,k,\ell} \binom{2n}{n+i} \binom{2n}{n+j} \binom{2n}{n+k} \binom{2n}{n+\ell} |(i^2 - j^2)(i^2 - k^2)(j^2 - k^2)(j^2 - \ell^2)(k^2 - \ell^2)| \]
\[
\quad \equiv 96n^5(n-1)^3 \left( \frac{2n-4}{n-1} \right) \left( \frac{2n}{n} \right)^3.
\]

5. Further identities and conjectures

Here we state various identities that are either proved or conjectured on the basis of numerical evidence. They have been proved in the cases indicated by the use of the equality sign “\( = \)” rather than “\( \equiv \)”. Proofs are are outlined in §6. In the cases indicated by “\( \equiv \)”, the conjectures have been confirmed numerically for all \( n \in \mathbb{N}^+ \) in the range \( 1 \leq n \leq 100 \).

**Centered double sums.** Recall that, from the definition (3.1), we have

\[
S_{\alpha,1}(n) = \sum_{i,j} \binom{2n}{n+i} \binom{2n}{n+j} |i^\alpha - j^\alpha|.
\]

Here we give known and conjectured values of \( S_{\alpha,1}(n) \) for \( 1 \leq \alpha \leq 8 \). We observe that (5.6) follows from Theorem 2.4 since \( S_{1,1}(n) = T_1(n,n) \), and (5.2) is equivalent to Theorem 3.1. It appears that, for even \( \alpha \), \( S_{\alpha,1}(n) \) is a rational function of \( n \) multiplied by \( \left( \frac{2n}{n} \right)^2 \), but for odd \( \alpha \), it is a rational function of \( n \) multiplied by \( \left( \frac{4n}{2n} \right) \).

\[
S_{2,1}(n) = 2n^2 \left( \frac{2n}{n} \right)^2,
\]
\[
S_{4,1}(n) = \frac{2n^3(4n-3)}{2n-1} \left( \frac{2n}{n} \right)^2,
\]
\[
S_{6,1}(n) = \frac{2n^3(11n^2 - 15n + 5)}{2n-1} \left( \frac{2n}{n} \right)^2,
\]
\[
S_{8,1}(n) = \frac{2n^3(80n^4 - 306n^3 + 428n^2 - 266n + 63)}{(2n-1)(2n-3)} \left( \frac{2n}{n} \right)^2,
\]
\[
S_{1,1}(n) = 2n \left( \frac{4n}{2n} \right),
\]
\[
S_{3,1}(n) \equiv \frac{4n^2(5n-2)}{4n-1} \left( \frac{4n-1}{2n-1} \right),
\]
Following are some similar [conjectured] identities. We observe that, since \( i^4 - j^4 = (i^2 + j^2)(i^2 - j^2) \), \((5.10)\) is easily seen to be equivalent to \((5.3)\). Similarly, since \( i^6 - j^6 = (i^2 + i^2 j^2 + j^3)(i^2 - j^2) \), any two of \((5.4), (5.11)\) and \((5.13)\) imply the third.

\[
\sum_{i,j} \binom{2n}{n+i} \binom{2n}{n+j} |i^2(j^2-j^2)| = \frac{n^3(4n-3)}{2n-1} \left( \frac{2n}{n} \right)^2, \tag{5.10}
\]

\[
\sum_{i,j} \binom{2n}{n+i} \binom{2n}{n+j} |i^4(j^2-j^2)| = \frac{n^3(10n^2-14n+5)}{2n-1} \left( \frac{2n}{n} \right)^2, \tag{5.11}
\]

\[
\sum_{i,j} \binom{2n}{n+i} \binom{2n}{n+j} |ij(j^2-j^2)| = \frac{2n^3(n-1)}{2n-1} \left( \frac{2n}{n} \right)^2, \tag{5.12}
\]

\[
\sum_{i,j} \binom{2n}{n+i} \binom{2n}{n+j} |i^2 j^2(j^2-j^2)| = \frac{2n^4(n-1)}{2n-1} \left( \frac{2n}{n} \right)^2, \tag{5.13}
\]

\[
\sum_{i,j} \binom{2n}{n+i} \binom{2n}{n+j} |i^3 j^3(j^2-j^2)| = \frac{2n^4(n-1)(3n^2-6n+2)}{(2n-1)(2n-3)} \left( \frac{2n}{n} \right)^2. \tag{5.14}
\]

### 6. Some proofs

The following lemma and its method of proof are useful for proving some of the identities given in \((5)\). Our proof is based on the analysis of certain random walks; a purely combinatorial proof is possible but less natural.

**Lemma 6.1.** Let \( P(x,y) \) be a bivariate polynomial such that

\[
P(p+q,p-q) = \sum_{\alpha, \beta} a_{\alpha, \beta} p^\alpha q^\beta,
\]

and let

\[
S = \sum_{i,j} \binom{2n}{n+i} \binom{2n}{n+j} P(i,j).
\]

Then

\[
S = \sum_k \binom{2n}{k} \sum_{\alpha, \beta} a_{\alpha, \beta} T_\alpha(k) T_\beta(2n-k), \tag{6.1}
\]

where\(^6\)

\[
T_\alpha(k) = \sum_\ell \binom{k}{\ell} (\ell - k/2)^\alpha.
\]

\(^6\)Here \( T_\alpha(k) \) should not be confused with the double sum \( T_\beta(m,n) \) of \((5)\).
Proof. Observe that \( 2^{-4n}S \) is the expectation \( E[P(i, j)] \) of \( P(i, j) \) where \( i \) and \( j \) are each generated by independent symmetric Bernoulli random walks, each walk having \( 2n \) steps each \( \pm \frac{1}{2} \). Thus \( i \in \{ -n, 1 - n, \ldots, n - 1, n \} \) occurs with probability \( 2^{-2n}(\binom{2n}{i+1}) \). Similarly, \( j \in \{ -n, 1 - n, \ldots, n - 1, n \} \) occurs with probability \( 2^{-2n}(\binom{2n}{j+1}) \). Since the walks are independent,

\[
E[P(i, j)] = 2^{-4n}S = 2^{-4n}\sum_{i, j} \binom{2n}{n+i} \binom{2n}{n+j} P(i, j).
\]

Consider one such pair of walks, with steps \((i_1, \ldots, i_{2n})\) and \((j_1, \ldots, j_{2n})\). Thus

\[
i = \sum_{\lambda=1}^{2n} i_{\lambda}, \quad j = \sum_{\mu=1}^{2n} j_{\mu}.
\]

Let

\[
U = \{ \lambda : 1 \leq \lambda \leq 2n \text{ and } i_{\lambda} = j_{\lambda} \}, \quad \overline{U} = \{ \lambda : 1 \leq \lambda \leq 2n \text{ and } i_{\lambda} \neq j_{\lambda} \},
\]

and let \( k = |U| \), so \( k \) is the number of times that a step in the first walk has the same sign as the corresponding step in the second walk. By independence of the two walks, \( k \in \{0, 1, \ldots, 2n\} \) occurs with probability \( 2^{-2n}(\binom{2n}{k}) \). Let \( \ell = |\{ \lambda \in U : i_{\lambda} = +1/2 \}|, m = |\{ \lambda \in \overline{U} : i_{\lambda} = +1/2 \}|.

Since the first walk has \( \ell + m \) steps of \(+1/2\) and \( 2n - \ell - m \) steps of \(-1/2\), we have \( i = \ell + m - n \). Similarly, the second walk has \( \ell + 2n - k - m \) steps of \(+1/2\) and \( k + m - \ell \) steps of \(-1/2\), so \( j = \ell - k - m + n \). Thus, if \( i = p + q, j = p - q \), we have

\[
p = \frac{i + j}{2} = \ell - \frac{k}{2}, \quad q = \frac{i - j}{2} = m - \frac{2n - k}{2}.
\]

If \( U \) is fixed, then a pair \((\ell, m)\) occurs with probability

\[
2^{-k}\binom{k}{\ell} \cdot 2^{-2n}(\binom{2n-k}{m}) = 2^{-2n}\binom{k}{\ell} \binom{2n-k}{m}.
\]

Thus, for given \( k \), the expectation of \( p^\alpha q^\beta \) is

\[
2^{-2n}\sum_{\ell} \binom{k}{\ell} \binom{\ell - k/2}{m} \sum_{m} \binom{2n-k}{m} \binom{m - 2n - k/2}{m} \beta,
\]

and by the definition of \( T_\alpha(k) \) this is just

\[
2^{-2n}T_\alpha(k)T_\beta(2n - k).
\]

Since the probability of a given \( k \in \{0, 1, \ldots, 2n\} \) is \( 2^{-2n}(\binom{2n}{k}) \), it follows that

\[
E[P(i, j)] = 2^{-4n}S = 2^{-2n}\sum_k \binom{2n}{k} \sum_{\alpha, \beta} 2^{-2n}a_{\alpha, \beta}T_\alpha(k)T_\beta(2n - k).
\]

Multiplying by \( 2^{4n} \), the result follows. \( \square \)

**Remark 6.2.** Only even values of \( \alpha \) and \( \beta \) need to be included in the sum (6.1), as \( T_\alpha(k) \) vanishes if \( \alpha \) is odd. \( T_\alpha(k) \) satisfies a recurrence

\[
4T_\alpha+2(k) = k^2T_\alpha(k) - 4k(k - 1)T_\alpha(k - 2).
\]

with initial values \( T_0(k) = 2^k, T_1(k) = 0 \). November 2014 11
Proof. If \( P(x, y) \) is replaced by \( |P(x, y)| \) and \((k/2 - \ell)\) by \([k/2 - \ell]\). In such cases odd values of \( \alpha \) and \( \beta \) need to be included. In some cases of interest we can evaluate the inner sum in \(6.1\) in closed form, thus reducing \( S \) from a double sum to a single sum. The single sum may then be amenable to standard summation techniques.

To illustrate these remarks, we prove \(5.10\). We can not use Lemma \(6.1\) directly because of the absolute value occurring in the weight, but we can follow the proof of Lemma \(6.1\) with minor modifications.

**Lemma 6.4.**
\[
\sum_{i,j} \left( \frac{2n}{n+i} \right) \left( \frac{2n}{n+j} \right) |i^2 - j^2| = \frac{n^3(4n-3)}{2n-1} \left( \frac{2n}{n} \right)^2.
\]

**Proof.** If \( i = p + q, \ j = p - q \), then \( |i^2 - j^2| = 4(p + q)^2|pq| \). Assume that \( i \) and \( j \) are generated by independent Bernoulli random walks as in the proof of Lemma \(6.1\). Thus the sum in Lemma \(6.4\) say \( S \), is
\[
S = 2^{4n} \mathbb{E}[|i^2 - j^2|] = 2^{4n+2} \mathbb{E}[p^2|pq| + 2pq|pq| + q^2|pq|],
\]
where \( \mathbb{E}[\cdots] \) denotes the expectation over the given distribution of \((i, j)\) values. Observe that the distribution of \( pq \) is symmetric about the origin, so \( \mathbb{E}[pq|pq|] = 0 \). Also, by symmetry, \( \mathbb{E}[p^2|pq|] = \mathbb{E}[q^2|pq|] \). Thus
\[
S = 2^{4n+3} \mathbb{E}[|p^3q|].
\]
Now we have to modify the definition of \(T_\alpha(k)\) in Lemma \(6.1\) to take account of the absolute values. To replace \(T_\alpha(k)\) we define\(^7\)
\[
U_\alpha(k) = \sum_{\ell} \binom{k}{\ell} |\ell - k/2|^\alpha.
\]
Now, following the proof of Lemma \(6.1\) we obtain
\[
S = 8 \sum_k \binom{2n}{k} U_3(k) U_1(2n - k). \tag{6.2}
\]
Since the closed forms for \(U_1(k)\) and \(U_3(k)\) depend on the parity of \( k \), we split \(6.2\) into even and odd parts:
\[
\frac{S}{8} = \sum_k \left( \frac{2n}{2k} \right) U_3(2k) U_1(2n - 2k) + \sum_k \left( \frac{2n}{2k - 1} \right) U_3(2k - 1) U_1(2n - 2k + 1). \tag{6.3}
\]
Now
\[
U_1(2k) = k \binom{2k}{k}, \quad U_1(2k + 1) = (2k + 1) \binom{2k}{k},
\]
\[
U_3(2k) = k^2 \binom{2k}{k}, \quad U_3(2k - 1) = \frac{k(4k - 3)}{8} \binom{2k}{k}.
\]
Substituting these values into \(6.3\) and observing that
\[
\binom{2n}{2k} \binom{2n}{k} \binom{2n - 2k}{n - k} = \binom{2n}{k} \binom{n}{k}^2,
\]

\(^7\)Here \(U_\alpha(k)\) is the same as the binomial sum \(U_r(n)\) defined in \(3\) with \((r, n) \mapsto (\alpha, k)\). Closed forms for this sum are given in \(3\) Theorem 3].
we obtain, after some simplification,
\[ S = 2(4n - 3) \binom{2n}{n} \sum_k k^2 \binom{n}{k}^2. \]

The sum over \( k \) here is easily evaluated by standard methods, which give
\[ \sum_k k^2 \binom{n}{k}^2 = \frac{n^3}{2(2n - 1)} \binom{2n}{n}. \]

Thus, the result follows. \( \square \)

Splitting into even and odd parts, we obtain the required sum
\[ \sum S_k \]
The sum over specifically, with \( n \) in Corollary 3.3, and can prove (5.3), (5.4), (5.5), (5.10), (5.11) and (5.13) above. More

Remark 6.5. We can prove Theorem 3.1 by slight modifications of the proof of Lemma 6.4. The proof given here is considerably simpler than the proofs given in 4, 14. The same method does not work for (5.7), (5.8), (5.9), (5.12) or (5.14), because of the odd powers of \( i \) and \( j \) occurring in the weight function.

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Table 1. Some proof summaries

| weight | expression in terms of \( C_{\alpha,\beta}(n) \) | application |
|--------|---------------------------------|-------------|
| \(| i^2 - j^2 | 4C_{1,1}(n) | W_1(n) |
| \(| i^2 - j^2 | 16C_{2,2}(n) | W_2(n) |
| \(| i^2 - j^2 | 64C_{3,3}(n) | W_3(n) |
| \( i^4 - j^4 | 16C_{1,3}(n) | (5.3) |
| \( i^6 - j^6 | 24C_{1,5}(n) + 40C_{3,3}(n) | (5.4) |
| \( i^8 - j^8 | 32C_{1,7}(n) + 224C_{3,5}(n) | (5.5) |
| \( i^2(i^2 - j^2) | 8C_{1,3}(n) | (5.10) |
| \( i^4(i^2 - j^2) | 8C_{1,5}(n) + 24C_{3,3}(n) | (5.11) |
| \( i^2 j^2(i^2 - j^2) | 8C_{1,5}(n) - 8C_{3,3}(n) | (5.13) |

Remark 6.5. Using the same approach, we can give an independent proof of the cases given in Corollary 3.3 and can prove (5.3), (5.4), (5.5), (5.10), (5.11) and (5.13) above. More specifically, with
\[ C_{\alpha,\beta}(n) := \sum_k \binom{2n}{k} U_\alpha(k) U_\beta(2n - k), \]
various proofs are summarised in Table 1. The same method does not work for (5.7), (5.8), (5.9), (5.12) or (5.14), because of the odd powers of \( i \) and \( j \) occurring in the weight function.
7. Asymptotics of Some Double Sums

We can verify the dominant terms in the conjectures of §5 and gain some insight into the structure of the conjectured closed forms via an asymptotic analysis. The idea is simply that the finite sums may be regarded as Riemann sums approximating a double integral. When we change to polar coordinates, the double integral splits into a product of two single integrals that are easy to evaluate. The reason why the double integral splits is that, in all the cases that we consider, the weight function $|f|$ is homogeneous.

Consider the sum $S_{\alpha,\beta}(n)$ defined by (3.1). If $|j| = O(n^{1/2+\varepsilon})$ where $\varepsilon < 1/6$, then

$$\left(\frac{2n}{n+j}\right) \sim \frac{2^{2n}}{\sqrt{\pi n}} e^{-j^2/n} \quad \text{as } n \to \infty.$$ 

Thus, using the method of tail-exchange [11, Ch. 9], we have

$$S_{\alpha,\beta}(n) \sim \frac{2^{4n}n^{\alpha\beta/2}}{\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-\left(x^2+y^2\right)} |x^\alpha - y^\beta| \, dx \, dy \quad \text{as } n \to \infty.$$ 

Changing to polar coordinates $(x,y) = (r \cos \theta, r \sin \theta)$, the double integral becomes

$$\int_{0}^{\infty} e^{-r^2} r^{\alpha+1} \, dr \int_{0}^{2\pi} |\cos^\alpha \theta - \sin^\alpha \theta|^\beta \, d\theta. \quad (7.1)$$

The first integral in (7.1) is just $\Gamma(\alpha\beta/2 + 1)/2 = \alpha \beta \Gamma(\alpha\beta/2)/4$. For the second integral, we observe that $\cos^\alpha \theta \geq \sin^\alpha \theta$ for $\theta \in [-\pi/4, \pi/4]$, and by well-known properties of $\cos \theta$ and $\sin \theta$ it is easy to see that

$$\int_{0}^{2\pi} |\cos^\alpha \theta - \sin^\alpha \theta|^\beta \, d\theta = 4 \int_{-\pi/4}^{\pi/4} (\cos^\alpha \theta - \sin^\alpha \theta)^\beta \, d\theta.$$ 

Thus, we have sketched a proof of the following theorem.

**Theorem 7.1.** If $\alpha$, $\beta$ are positive and $S_{\alpha,\beta}(n)$ is defined by (3.1), then

$$S_{\alpha,\beta}(n) \sim K_{\alpha,\beta} 2^{4n} n^{\alpha\beta/2} \quad \text{as } n \to \infty,$$

where

$$K_{\alpha,\beta} = \frac{\alpha \beta \Gamma(\alpha\beta/2)}{\pi} \int_{-\pi/4}^{\pi/4} (\cos^\alpha \theta - \sin^\alpha \theta)^\beta \, d\theta.$$ 

For example, if $\alpha = 2$, $\beta = 1$, we have $K_{2,1} = 2/\pi$, so Theorem 7.1 gives $S_{2,1}(n) \sim 2^{4n+1} n/\pi$. This is in agreement with the exact formula $S_{2,1}(n) = 2n^2 (2n)_{n}^2$, since $(2n)_{n} \sim 2n^2/\sqrt{n\pi}$.

We consider the case $\beta = 1$ in more detail since this sheds some light on the conjectures of §5. Define $c_0 := 1$ and

$$c_k := \pi K_{k,1} = k \Gamma(k/2) \int_{-\pi/4}^{\pi/4} (\cos^k \theta - \sin^k \theta) \, d\theta \quad \text{for } k \geq 1.$$ 

Using integration by parts, we obtain the recurrence

$$c_k = \left(\frac{k-1}{2}\right) c_{k-2} + \left(3 + (-1)^k\right) 2^{-k/2} \Gamma(k/2) \quad \text{for } k \geq 2. \quad (7.2)$$

Since $c_0 = 0$ and $c_1 = \sqrt{2\pi}$, the recurrence gives $c_2 = 2$, $c_3 = 5\sqrt{2\pi}/4$, $c_4 = 4$, $c_5 = 43\sqrt{2\pi}/16$, $c_6 = 11$, $c_7 = 531\sqrt{2\pi}/64$, $c_8 = 40$, etc.
Eliminating the Gamma function terms from two instances of (7.2) gives a more elegant three-term recurrence

\[ c_{k+2} = \left( \frac{3k + 2}{4} \right) c_k - \left( \frac{k(k - 1)}{8} \right) c_{k-2} \quad \text{for } k \geq 2. \]  

(7.3)

The scaled even sequence \((2^{-k/2} c_{2k})_{k \geq 1} = (1, 4, 22, 160, 1624, \ldots)\) is sequence A087547, and the scaled odd sequence \((2^{2k} c_{2k+1}/\sqrt{2\pi})_{k \geq 0} = (1, 5, 43, 531, 8601, \ldots)\) is sequence A090470, in the Online Encyclopedia of Integer Sequences [15].

In \S 5 we give formulas for \(S_{\alpha,1}(n)\), \(1 \leq \alpha \leq 8\) (conjectured for \(\alpha \in \{3, 5, 7\}\), proved in the other cases). The leading terms of these formulas agree with the asymptotic behaviour predicted by Theorem 7.1. For example, (5.5) implies that

\[ S_{8,1}(n) \sim 40n^5 \left( \frac{2n}{\pi} \right)^2 \sim \frac{40n^4}{\pi} 2^{4n}. \]

Since \(K_{8,1} = c_8/\pi = 40/\pi\), this is in agreement with Theorem 7.1.

Similarly, (5.9) implies that

\[ S_{7,1}(n) \sim \frac{531n^4}{4} \left( \frac{4n - 3}{2n - 3} \right) \sim \frac{531n^4}{32} \left( \frac{4n}{2n} \right) \sim \frac{531n^{7/2}}{32\sqrt{2\pi}} 2^{4n}, \]

which is in agreement with

\[ K_{7,1} = c_7/\pi = \frac{531}{32\sqrt{2\pi}}. \]

From the recurrence (7.3), \(c_k\) is rational if \(k\) is even, and is a rational multiple of \(\sqrt{2\pi}\) if \(k\) is odd. This is consistent with the conjecture that \(S_{\alpha,1}(n)\) is a rational function of \(n\) multiplied by \(\left( \frac{2n}{n} \right)^2\) if \(\alpha\) is even, and a rational function of \(n\) multiplied by \(\left( \frac{4n}{2n} \right)\) if \(\alpha\) is odd.

**Appendix: Outline of proof of Theorem 4.1**

**Proof.** Since the result is easily verified for \(n \in \{0, 1\}\), we can assume that \(n \geq 2\). Let \(S\) denote the triple sum in the statement of Theorem 4.1. We avoid the absolute value function by some observations using symmetries, obtaining

\[ S = 24 \sum_{k=0}^{n} \sum_{j=0}^{k} \sum_{i=-j}^{j} \left( \frac{2n}{n+i} \right) \left( \frac{2n}{n+j} \right) \left( \frac{2n}{n+k} \right) (k^2 - j^2)(k^2 - i^2)(j^2 - i^2). \]

Thus

\[ \frac{S}{24(2n)(2n-1)} = -\sum_{k=0}^{n} \sum_{j=0}^{k} \sum_{i=-j}^{j} \left( \frac{2n}{n+i} \right) \left( \frac{2n}{n+j} \right) \left( \frac{2n-2}{n-1+k} \right) (k^2 - i^2)(j^2 - i^2) \]

\[ + \sum_{k=0}^{n} \sum_{j=0}^{k} \sum_{i=-j}^{j} \left( \frac{2n}{n+i} \right) \left( \frac{2n-2}{n-1+j} \right) \left( \frac{2n}{n+k} \right) (k^2 - i^2)(j^2 - i^2), \]

and

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\[
\frac{S}{24(2n)^2(2n-1)^2} = \sum_{k=0}^{n} \sum_{j=0}^{k} \sum_{i=-j}^{j} \left( \frac{2n}{n+i} \right) \left( \frac{2n-2}{n-1+j} \right) \left( \frac{2n-2}{n-1+k} \right) (k^2 - i^2)
\]

- \[
\sum_{k=0}^{n} \sum_{j=0}^{k} \sum_{i=-j}^{j} \left( \frac{2n-2}{n-1+i} \right) \left( \frac{2n}{n+j} \right) \left( \frac{2n-2}{n-1+k} \right) (k^2 - i^2)
\]

- \[
\sum_{k=0}^{n} \sum_{j=0}^{k} \sum_{i=-j}^{j} \left( \frac{2n}{n+i} \right) \left( \frac{2n-2}{n-1+j} \right) \left( \frac{2n}{n-1+k} \right) (j^2 - i^2)
\]

+ \[
\sum_{k=0}^{n} \sum_{j=0}^{k} \sum_{i=-j}^{j} \left( \frac{2n-2}{n-1+i} \right) \left( \frac{2n}{n-1+j} \right) \left( \frac{2n}{n+k} \right) (j^2 - i^2)
\]

\[
T = -\sum_{k=0}^{n} \sum_{j=0}^{k} \sum_{i=-j}^{j} \left( \frac{2n}{n+i} \right) \left( \frac{2n-2}{n-1+j} \right) \left( \frac{2n-4}{n-2+k} \right)
\]

+ \[
\sum_{k=0}^{n} \sum_{j=0}^{k} \sum_{i=-j}^{j} \left( \frac{2n}{n+i} \right) \left( \frac{2n-4}{n-2+j} \right) \left( \frac{2n-2}{n-1+k} \right)
\]

+ \[
\sum_{k=0}^{n} \sum_{j=0}^{k} \sum_{i=-j}^{j} \left( \frac{2n-2}{n-1+i} \right) \left( \frac{2n}{n+j} \right) \left( \frac{2n-4}{n-2+k} \right)
\]

- \[
\sum_{k=0}^{n} \sum_{j=0}^{k} \sum_{i=-j}^{j} \left( \frac{2n-4}{n-2+i} \right) \left( \frac{2n}{n+j} \right) \left( \frac{2n-2}{n-1+k} \right)
\]

- \[
\sum_{k=0}^{n} \sum_{j=0}^{k} \sum_{i=-j}^{j} \left( \frac{2n-2}{n-1+i} \right) \left( \frac{2n-4}{n-2+j} \right) \left( \frac{2n}{n+k} \right)
\]

+ \[
\sum_{k=0}^{n} \sum_{j=0}^{k} \sum_{i=-j}^{j} \left( \frac{2n-4}{n-2+i} \right) \left( \frac{2n-2}{n-1+j} \right) \left( \frac{2n}{n+k} \right)
\]

Defining \( T := \frac{S}{[24(2n)^2(2n-1)^2(2n-2)(2n-3)]} \), we have

Now comes a long but elementary process that we only sketch. First, all occurrences of \( \binom{2n-2}{n+m} \) are replaced by \( \binom{2n-4}{n+m} + 2 \binom{2n-4}{n+m-1} + \binom{2n-4}{n+m-2} \).
Next, all the triple sums are simplified by changing indices and taking extra terms into account. This leaves us with only a large number of single sums. This gives

\[ T = \left( \frac{2n - 4}{n - 2} \right) \cdot \sum_{k=0}^{n} \left( \frac{2n - 4}{n - 1 + k} \right) \left( \frac{2n}{n+k} \right) + \sum_{k=0}^{n} \left( \frac{2n - 4}{n - 2 + k} \right) \left( \frac{2n}{n+k} \right) \]

\[ - \sum_{k=0}^{n} \left( \frac{2n - 4}{n - 1 + k} \right) \left( \frac{2n}{n-2+k} \right) + \sum_{k=0}^{n} \left( \frac{2n - 4}{n + k + 1} \right) \left( \frac{2n - 4}{n-2+k} \right) \]

\[ + \left( \frac{2n}{n} \right) \cdot \left[ - \sum_{k=0}^{n} \left( \frac{2n - 4}{n - 2 + k} \right)^2 + \sum_{k=0}^{n} \left( \frac{2n - 4}{n - 1 + k} \right)^2 \right] \]

\[ - \left( \frac{2n}{n} \right) \cdot \sum_{k=0}^{n} \left( \frac{2n - 4}{n - 2 + k} \right) \left( \frac{2n - 4}{n-1+k} \right) - \left( \frac{2n}{n} \right) \left( \frac{2n - 4}{n-3} \right) \sum_{k=0}^{n} \left( \frac{2n - 4}{n - 2 + k} \right) \]

\[ + \left( \frac{2n}{n} \right) \cdot \left( \frac{2n - 4}{n - 3} \right) + \left( \frac{2n}{n} \right) \left( \frac{2n - 4}{n - 2} \right) \left( \frac{2n - 4}{n - 1} \right) \]

\[ + 2 \sum_{k=0}^{n} \left( \frac{2n - 4}{n - 1 + k} \right) \left( \frac{2n}{n+k} \right) \cdot \sum_{k=0}^{n} \left( \frac{2n - 4}{n - 1 + k} \right) \]

\[ - 2 \sum_{k=0}^{n} \left( \frac{2n - 4}{n - 2 + k} \right) \left( \frac{2n}{n+k+1} \right) \cdot \sum_{k=0}^{n} \left( \frac{2n - 4}{n - 2 + k} \right). \]

At this stage, we can ask Maple to evaluate \( T \). After some human interaction, and using the assumption that \( n \geq 2 \) so the factorials are well-defined, we obtain

\[ T = 2^{2n-4} \frac{(2n-2)!(2n-4)!}{n!(n-1)!^2(n-2)!}, \]

and thus \( S = 3 \cdot 2^{2n-1} \frac{(2n)!^2}{n!(n-1)!^2(n-2)!} \), which is equivalent to the desired result. \( \square \)

References

[1] Anonymous, William Lowell Putnam Mathematical Competition, 1974, http://www.math-olympiad.com/35th-putnam-mathematical-competition-1974-problems.html
[2] M. R. Best, The excess of a Hadamard matrix, Nederl. Akad. Wetensch. Proc. Ser. A 80 = Indag. Math. 39 (1977), 357–361.
[3] R. P. Brent, Generalising Tuenter’s binomial sums, arXiv:1407.3533v3, 16 July 2014.
[4] R. P. Brent and J. H. Osborn, Note on a double binomial sum relevant to the Hadamard maximal determinant problem, arXiv:1309.2795v2, 12 Sept. 2013.
[5] R. P. Brent, J. H. Osborn and W. D. Smith, Lower bounds on maximal determinants of binary matrices via the probabilistic method, arXiv:1402.6817v2, 13 March 2014.
[6] L. Carlitz, Explicit formulas for the Dumont-Foata polynomial, Discrete Mathematics 30 (1980), 211–225.

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[7] D. Dumont and D. Foata, Une propriété de symétrie des nombres de Genocchi, *Bulletin de la Société Mathématique de France* **104** (1976), 433–451.

[8] N. Farmarkis and S. Kounias, The excess of Hadamard matrices and optimal designs, *Discrete Mathematics* **67** (1987), 165–176.

[9] R. W. Gosper, Jr., Decision procedure for indefinite hypergeometric summation, *Proc. Natl. Acad. Sci. USA* **75**, 1 (1978), 40–42.

[10] H. W. Gould, *Combinatorial Identities*, 2nd edition, Morgantown, West Virginia, USA, 1972.

[11] R. L. Graham, D. E. Knuth and O. Patashnik, *Concrete Mathematics*, Addison-Wesley, Reading, Massachusetts, 1989.

[12] V. J. W. Guo and J. Zeng, Factors of binomial sums from the Catalan triangle, *Journal of Number Theory* **130** (2010), 172–186. Also arXiv:0909.0307v2, 30 Sept. 2009.

[13] M. Petkovšek, H. S. Wilf and D. Zeilberger, *A = B*, A. K. Peters Ltd, Wellesley, Mass., 1996.

[14] H. Prodinger, A short and elementary proof for a double sum of Brent and Osburn, arXiv:1309.4351v1, 17 Sept. 2013, 2 pp.

[15] N. J. A. Sloane *et al*, The On-Line Encyclopedia of Integer Sequences, [https://oeis.org/](https://oeis.org/)

[16] H. J. H. Tuenter, Walking into an absolute sum, *The Fibonacci Quarterly* **40** (2002), 175–180. Also arXiv:math/0606080v1, 4 June 2006.

[17] H. S. Wilf, *generatingfunctionology*, 3rd edition, A. K. Peters Ltd, Wellesley, Mass., USA, 2006.

MSC2010: 05A10, 11B65 (Primary); 05A15, 05A19, 44A60, 60G50 (Secondary)

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