Introduction to critical phenomena through the fiber bundle model of fracture

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Abstract
We discuss the failure dynamics of the fiber bundle model, especially in the equal-load-sharing scheme. We also highlight the ‘critical’ aspects of their dynamics in comparison with those in standard thermodynamic systems undergoing phase transitions.

Keywords: fiber bundle model, critical behavior, recursive dynamics, fixed points, Ising model

(Some figures may appear in colour only in the online journal)

1. Introduction

Critical phenomena nicely demonstrate one important aspect of the complexity of nature, a wide-ranging field with huge developments on all fronts: theoretical, numerical and experimental. Thousands of papers, reviews and articles have been published so far on this topic and several hundred are still being published each year. However, introducing critical phenomena to college or university students, or to beginners, is not always easy. Certainly, asking them to go through the vast literature does not make sense.

Here the role of models comes into play. We plan to introduce a simple model which is intuitively appealing and has clear-cut dynamical rules. This allows us to perform some analytic calculations and find solutions, and it is also possible to check the results numerically through a few quick ‘runs’.

The fiber bundle model (FBM), introduced by Peirce in 1926\cite{1}, is such a simple model. Although it is quite old, and was designed as a model for the fracture or failure of a set of...
parallel elements (fibers), each having a breaking threshold different from the others, with the collective sharing of each failed fiber’s load, the failure dynamics in the model clearly show all the attributes of critical phenomena and the associated phase transition. Indeed, the FBM is the precise equivalent of the Ising model of magnetism, introduced by Ising [2] in 1925. The models are both identically potential, deep and versatile in their respective fields. A recently published book [3] tried to gather and explain work on the FBM from a statistical physics point of view. In this article, we want to concentrate only on the FBM’s dynamical aspects related to critical phenomena. We hope that this will aid the easy introduction of the models and critical phenomena before students undertake research on these topics. Thus, students are the target group for this article.

We arrange the article as follows: we briefly introduce the notion of critical phenomena in section 2, then we discuss a brief history of the FBM and its evolution as a fracture model in section 3. Section 4 deals with the simplest version of the FBM, the equal-load-sharing (ELS) FBM. In several sub-sections of section 4 we demonstrate the critical behavior in FBM with construction of the evolution dynamics and their solutions. We also compare the analytic results with numerical simulations in this section. We dedicate section 5 to discussion of some related works which aid understanding of the critical behavior of the FBM. Finally, some discussion and conclusions will be made in section 6.

2. Critical phenomena

Let us consider the case of a ferromagnet. Even in the absence of any external field, at temperatures \( T \) below the Curie temperature \( T_c \), one gets a finite average magnetization. This spontaneous magnetization disappears as one increases the temperature of the magnet beyond \( T_c \). This phase transition (from the ferromagnetic phase with spontaneous magnetization to the paramagnetic phase with vanishing magnetization) was seen to have some ‘critical’ aspects in the sense that the temperature variation of the magnetization \( m \) or the susceptibility \( \chi \) near the Curie point can be expressed as power laws with respect to the temperature interval from the Curie (critical) point: \( m(T) \sim |T - T_c|^\beta, \chi(T) \sim |T - T_c|^{-\gamma} \).

Additionally, the values of these powers (exponents) were found to be irrational numbers in general (indicating singular behavior of the free energy of the magnets near the ferromagnetic–paramagnetic transition point). When interpreted in terms of the elemental spin-magnetic moments, which interact through the exchange interactions, one finds that the correlation of the spin-state fluctuations at any arbitrary crystal point with that of another spin at a distance \( r \) decays as \( \exp(-r/\xi(T)) \), where the correlation length \( \xi(T) \sim |T - T_c|^{-\nu} \) diverges at \( T = T_c \) with correlation length exponent \( \nu \). This correlation length sets the scale which essentially determines the critical aspects of the thermodynamic behavior near the critical (Curie) point of the magnet. One also finds that the values of these exponents \((\beta, \gamma, \nu, \text{ etc})\) are universal in the sense that they depend only on some subtle features of the systems such as the spatial dimensionality of the system and the number of components (dimensionality) of the order parameter (magnetization) vector. They do not depend on details like the strength of the (exchange) interaction, lattice structures, etc.

In the standard models of cooperatively interacting systems in classical statistical physics, as in the Ising model, simple two-state Ising spins (representing the constituent magnetic moments) on the lattice sites interact with themselves through neighboring (exchange) interactions. In the absence of any thermal noise, or even at finite but low temperatures, the effects of the (spin–spin) interactions win over thermal noise, and induce spontaneous order without any magnetic field. This order, say ferromagnetic when the spin–spin interactions

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favor similar orientations of the spins, gets destroyed when the thermal noise, corresponding
to a temperature beyond the phase transition or Curie point, dominates and the paramagnetic
phase sets in. After intensive study over about three decades, starting in the middle of the last
century, it was established that while the thermodynamic behavior away from the phase
transition point of such systems has the usual scale-dependent variations (with the finite scale
determined by the competition between the interaction energies and temperature), behavior
becomes scale-free as the phase transition point is approached (see e.g. [4]) and is expressed
by power laws (with the powers given by some effective fractal dimensionality [5] of volume
determined by the correlation length scale which diverges at the phase transition point). This
scale invariance, often with singularities in the growth of the correlation length scale as one
approaches the transition point, had been exploited by the renormalization group theory (see
e.g. [4–6]).

3. The FBM

The FBM [1, 3, 7, 8] captures the fracture dynamics in composite materials. In this model, a
large number of parallel Hookean springs or fibers are clamped between two horizontal
platforms; the bundle hangs from the upper one (rigid) while the load hangs from the lower
one (figure 1). The springs or fibers are assumed to have identical spring constants though
their breaking strengths are different. Once the load per fiber exceeds a fiber’s own threshold,
it fails and cannot carry the load any more. The load it carried is now shared by the surviving
fibers. If the lower platform deforms under loading, fibers closer to the just-failed fiber will
absorb more of the load compared to those further away and this is called the local-load-
sharing (LLS) scheme. On the other hand, if the lower platform is rigid, the load is equally

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distributed amongst all the surviving fibers. This is called the equal-load-sharing (ELS) scheme. Obviously, for low values of the initial load (per fiber), successive failures of the fibers due to extra load-sharing remain localized, and although the strain of the bundle (given by the identical strain of the surviving fibers) grows with increasing load, the bundle as a whole does not fail. Beyond a critical value of the initial load, determined by the fiber strength distribution and the load-sharing mechanism (after each failure), the successive failures become a global one and the bundle fails. Here, the ‘order’ could be measured by the fraction of eventually surviving fibers (after a ‘relaxation time’ required for stabilization of the bundle), which decreases as the load on the bundle increases. Beyond the critical load, mentioned above, the eventual damage size becomes global and order disappears (the bundle fails). We will see in the following sections that, as we approach the critical load (either from higher or lower load), the relaxation time (time or steps required once the load is applied until no further failure occurs or until the whole bundle collapses) diverges with ‘universal’ values of the exponents (power) similar to the power laws in thermodynamic phase transitions discussed in the previous section.

Long ago, in 1926, F T Peirce introduced the FBM [1] in order to study the strength of cotton yarns in connection with textile engineering. Some static behavior of such a bundle (with ELS by all the surviving fibers, following a failure) was discussed by Daniels in 1945 [9] and the model was brought to the attention of physicists in 1989 by Sornette [10].

4. The ELS FBM

Let us consider an FBM having $N$ parallel fibers placed between two stiff clamps. Each fiber responds linearly with force $f$ to an extension or stretch $\Delta$,

$$f = \kappa \Delta,$$

where $\kappa$ is the spring constant. We consider $\kappa = 1$ for all fibers. Each fiber has a load threshold $x$ assigned to it. If the stretch $\Delta$ exceeds this threshold, the fiber fails irreversibly. In the ELS mode, the clamps are stiff and there is no non-uniform redistribution of loads among the surviving fibers, i.e. the applied load is shared equally by the remaining intact fibers.

4.1. Fiber strength distributions and the cumulative distributions

The fiber strength thresholds are drawn from a probability density $p(x)$. The corresponding cumulative probability is given by

$$P(y) = \int_0^y p(x) \, dx.$$  

The most studied threshold distributions are power-law-type distributions and Weibull distributions (see figures 2, 3).

We consider a general power-law-type fiber threshold distribution within the range $(0, 1)$,

$$p(x) \propto x^\alpha; \quad \alpha \geq 0.$$  

For normalization, we need to fulfill

$$\int_0^1 p(x) \, dx = 1.$$
Therefore we get, from equations (3), (4), the prefactor is \((1 + \alpha)\):
\[
p(x) = (1 + \alpha)x^{\alpha}.
\] (5)

The cumulative distribution takes the form
\[
P(x) = \int_0^x p(y)dy = x^{1+\alpha}.
\] (6)

When the power-law index \(\alpha = 0\), the distribution reduces to a uniform distribution with
\[
p(x) = 1; P(x) = x.
\] (7)

In figure 2 we present the probability distributions and corresponding cumulative distributions for power-law-type threshold distributions.

On the other hand the cumulative Weibull distribution has the form
\[
P(x) = 1 - \exp(-x^k),
\] (8)
where $k$ is the shape parameter, sometimes called the Weibull index. Therefore the probability distribution takes the form 

$$p(x) = kx^{k-1}\exp(-x^k).$$

(9)

In figure 3 we present the probability distributions and corresponding cumulative distributions for Weibull threshold distributions.

4.2. The load curve and the critical values

When the fiber bundle is loaded, the fibers fail according to their thresholds, the weaker before the stronger. We suppose that $N_f$ fibers have failed at stretch or load $\Delta$. Then the fiber bundle supports a force 

$$F = \kappa(N - N_f)\Delta = (N - N_f)\Delta,$$

(10)

as the spring constant has been set equal to unity. This is a discrete picture and the above equation is valid for any value of $N$, small or large. When $N$ is very large, the force on the bundle at stretch value $\Delta$ can be written as 

$$F = (N - N_f)\Delta = N(1 - P(\Delta))\Delta.$$

(11)

If we plot the normalized force ($F/N$) versus stretch value $\Delta$, we normally get a parabola like the shape shown in figure 4.

The force has a maximum at a particular $\Delta$ value ($\Delta_c$) and this is the maximum strength of the whole bundle. This point is often called the failure point or critical point of the system, beyond which the bundle collapses. Therefore we can say that there are two distinct phases of the system: a stable phase for $0 < \Delta \leq \Delta_c$ and an unstable phase for $\Delta > \Delta_c$. Now, at the critical point, setting $dF(\Delta)/d\Delta = 0$ we get 

$$1 - \Delta_c p(\Delta_c) - P(\Delta_c) = 0.$$

(12)

4.2.1. General threshold distribution. At the critical stretch ($\Delta_c$), we recall equation (12) and, using the $p(\Delta_c)$ and $P(\Delta_c)$ values for a general power-law-type distribution, we get
What is the critical strength of the bundle? If we put the $\Delta_c$ value in the force expression (equation (11)), we get

$$F_N = (1 + \alpha) \left( \frac{1}{2 + \alpha} \right)^{2+\alpha}. \quad (14)$$

Inserting $\alpha = 0$, we get

$$\Delta_c = \frac{1}{2}; \quad \frac{F_c}{N} = \frac{1}{4}, \quad (15)$$

which are the critical stretch and force values for a uniform distribution (figures 4, 5).

4.2.2. The Weibull threshold distribution. Let us move to a more general distribution of fiber thresholds, the Weibull distribution, which has been used widely in materials science. As the force has a maximum at the failure point $\Delta_c$, recalling the expression (equation (12)) and putting the Weibull $P(x)$, $p(x)$ values into it, we get

$$\exp(-\Delta^k_c) - (\Delta_c k \Delta_c^{-1} \exp(-\Delta^k_c)) = 0. \quad (16)$$

From the above equation we can easily calculate the critical stretch value

$$\Delta_c = k^{-\frac{1}{k}}; \quad (17)$$

and the critical force value

$$\frac{F_c}{N} = k^{-\frac{1}{k}} \exp\left(-\frac{1}{k}\right). \quad (18)$$

For $k = 1$, $\Delta_c = 1.0$ and $\frac{F_c}{N} = \exp(-1)$ (figures 4, 5).
4.3. Quasi-static loading versus loading by discrete steps

So far we have described the ELS model and derived the equilibrium force—elongation or stress—strain relation. We have not said anything about the means of loading, i.e. how the load/force has been applied to the bundle.

Going back to the history of the FBM we find that people discussed first the ‘weakest-link-failure’ mode of loading. This is a very slow loading process that ensures the breaking of only the weakest element (among the intact fibers). This is clearly a 'quasi-static' approach, and noise or fluctuation in the threshold distribution plays a major role (figure 6) in this type of loading process.

However, a fiber bundle can be loaded in a different way. If a finite external force or load is applied, all fibers that cannot withstand the applied stress fail. The stress on the surviving fibers then increases, which drives further fibers to break, and so on. This iterative breaking process will go on until an equilibrium with some intact fibers (those that can support the load) is reached or the whole bundle collapses. We are now going to study the average behavior of such breaking processes for a bundle of a large number of fibers following the formulations in [3, 7, 11–13].

4.4. Loading by discrete steps: the recursive dynamics

Let us assume that an external force $F$ is applied to the fiber bundle, with the applied stress denoted by

\[ \sigma(x) = \frac{F}{N} \]

Figure 6. Two realizations (continuous line and dashed line) for the force per fiber $F(x)/N = \sigma(x)$ as a function of the stretch $x$ for a bundle with $N = 50$ having a uniform distribution of fiber thresholds. For comparison, a realization with $N = 5000$ (the dotted line in the middle) is shown. Clearly, for large number of fibers, the fluctuations are tiny and the resulting force—stretch curve almost follows the parabolic average force expression $x(1 - x)$. 
the external load per fiber. We let $N_t$ be the average number of fibers that survive after $t$ steps in the stress redistribution process, with $N_0 = N$. We want to determine how $N_t$ decreases until the degradation process stops.

At a stage during the breaking process when $N_t$ intact fibers remain, the effective stress becomes

$$x_t = \frac{N\sigma}{N_t},$$

Thus

$$NP(N\sigma/N_t)$$

fibers will have thresholds that cannot withstand the load. In the next step, therefore, the number of intact fibers will be

$$N_{t+1} = N - NP(N\sigma/N_t).$$

Now we define the relative number of intact fibers as

$$n_t = \frac{N_t}{N},$$

and, therefore, equation (22) takes the form of a nonlinear recursion relation:

$$n_{t+1} = 1 - P(\sigma/n_t).$$

with $\sigma$ as the control parameter and with $n_0 = 1$ as the start value.

We can also set up a recursion for $x_t$, the effective stress $\sigma/n_t$ after $t$ iterations:

$$x_{t+1} = \frac{\sigma}{1 - P(x_t)},$$

with $x_0 = \sigma$ as the initial value. Since by (20)

$$x_t = \frac{\sigma}{n_t},$$

the two recursion relations (24) and (25) may be mapped onto each other.

In general it is not possible to solve nonlinear iterations like (24) or (25) analytically. The models with uniform ($\alpha = 0$) and linearly increasing ($\alpha = 1$) threshold distributions are however, exceptions.

In nonlinear dynamics the character of an iteration is primarily determined by its fixed points (denoted by $^*$). We are therefore interested in possible fixed points $n^*$ of (24), which satisfy

$$n^* = 1 - P(\sigma/n^*).$$

Correspondingly, fixed points $x^*$ of the iteration (25) must satisfy

$$x^* = \frac{\sigma}{1 - P(x^*)},$$

which may be written as

$$F = N\sigma = N\sigma^*(1 - P(x^*)).$$

This is precisely the relation (11) between stress and strain. Therefore the equilibrium value of $x$, for a given external stress $\sigma$, is a fixed point.

4.5. Solution of the recursive dynamics and the critical exponents

Let us illustrate these general results by an example. We consider first a power-law-type distribution (5)

$$\sigma = F/N,$$
The fixed-point equation (27) takes the form
\[(n^c)^{\alpha+2} - (n^c)^{\alpha+1} + \sigma^{\alpha+1} = 0.\] (31)
If we set \(\alpha = 0\), the threshold distribution reduces to a uniform threshold distribution and the recursion relation becomes
\[n_{t+1} = 1 - \sigma/n_t.\] (32)
Consequently the fixed-point equation assumes a harmless form:
\[(n^*)^2 - n^* + \sigma = 0,\] (33)
with solution
\[n^* = \frac{1}{2} \pm (\sigma_c - \sigma)^{1/2}.\] (34)
Here \(\sigma_c = 1/4\), the critical value of the applied stress beyond which the bundle fails completely. In (34) the upper signs give \(n^* > n_c\) which corresponds to stable fixed points. The negative sign in (34) is unphysical as for \(\sigma = 0\) it gives \(n^* = 0\). From this solution we can easily derive order parameter, susceptibility and relaxation time behavior and their exponents (see figures 7 and 8).

4.5.1. Order parameter. From the fixed-point solution, at the critical point \((\sigma = \sigma_c)\) we get
\[n_c^* = \frac{1}{2}.\] (35)
Therefore we can present the fixed-point solution as
\[n^*(\sigma) - n_c^* \propto (\sigma_c - \sigma)^\beta, \quad \beta = \frac{1}{2}.\] (36)
where \( n^\star(\sigma) - n^\star \) behaves like an order parameter, i.e. it shows a transition from non-zero to zero value at the critical point \( \sigma_c \).

4.5.2. Susceptibility. We can define the breakdown susceptibility as \( \chi = -dn^\star/d\sigma \). From the fixed-point solution we can write directly

\[
\chi \propto (\sigma_c - \sigma)^{-\gamma}, \quad \gamma = \frac{1}{2},
\]

which diverges at the critical point following a well-defined power law. This is another robust signature of a critical phenomenon.

4.5.3. Relaxation time. To track down the approach very near a fixed point, we note that close to a stable fixed point the iterated quantity changes by tiny amounts, so that one may expand in the differences \( n - n^\star \). For the model with uniform distribution of the thresholds, the recursion relation (24),

\[
n_{t+1} = 1 - \sigma/n_t,
\]

gives linear order

\[
n_{t+1} - n^\star = \frac{\sigma}{n^\star} - \frac{\sigma}{n_t} = \frac{\sigma}{n_t n^\star} (n_t - n^\star) \approx \frac{\sigma}{n^\star^2} (n_t - n^\star).
\]

Thus the fixed point is approached monotonically with exponentially decreasing steps:

\[
n_t - n^\star \propto \exp(-t/\tau),
\]

with a relaxation parameter

\[
\tau = 1/\ln(n^\star^2/\sigma) = 1/\ln\left[\left(\frac{1}{2} + \frac{1}{\sqrt{4 - \sigma}}\right)^2/\sigma\right].
\]
For the critical load, $\sigma = \sigma_c = \frac{1}{4}$, the argument of the logarithm is 1, so that apparently $\tau$ is infinite. More precisely, for $\sigma \to \sigma_c$:

$$\tau \simeq \frac{1}{4} (\sigma_c - \sigma)^{-\theta} \text{ with } \theta = \frac{1}{2}. \quad (42)$$

The divergence is a clear indication that the character of the breaking dynamics changes when the bundle becomes critical.

4.5.4. Critical slowing. How does the system behave at the critical point? If we put the critical $\sigma$ value in the recursion relation, we get

$$n_t - n^*_c \sim t^{-\phi}, \phi = 1, \quad (43)$$

which implies the relaxation dynamics is critically slow exactly at the critical stress value. Critical slowing is another known characteristic of a critical phenomenon.

4.6. Universality

So far we have obtained the dynamic critical behavior for the uniform distribution of the breaking thresholds, and the natural question is how general the results are. We can do a spot check on universality by considering a different distribution of fiber strengths. By setting $\alpha = 1$, the power-law-type distribution reduces to a linearly increasing distribution in the interval $(0,1)$,

$$p(x) = \begin{cases} 2x, & 0 \leq x \leq 1 \\ 0, & x > 1. \end{cases} \quad (44)$$

For simplicity non-dimensional variables are used. By the force–elongation relationship, the average total force per fiber,

$$F(x)/N = \begin{cases} x(1-x^2), & 0 \leq x \leq 1 \\ 0, & x > 1. \end{cases} \quad (45)$$

shows that the critical point is

$$x_c = \frac{1}{\sqrt{3}}, \quad \sigma_c = \frac{2}{3\sqrt{3}}. \quad (46)$$

In this case the recursion relation takes the form

$$n_{t+1} = 1 - (\sigma/n_t)^2, \quad (47)$$

and consequently the fixed-point equation reduces to

$$(n^*)^3 - (n^*)^2 + \sigma^2 = 0, \quad (48)$$

a cubic equation in $n^*$. Therefore there exist three solutions of $n^*$ for each value of $\sigma$. For the critical load, $\sigma_c = 2/3\sqrt{3}$, the only real and positive solution of (48) is

$$n^*_c = \frac{2}{3}. \quad (49)$$

One can show that for $\sigma < \sigma_c$, there will be an unstable fixed point with $n^* < n_c^*$, and a stable one with $n^* > n_c^*$.

To find the number of intact fibers in the vicinity of the critical point, we insert $n = \frac{2}{3} + (n - n_c)$ into (47), with the result
\[
\frac{4}{27} - (n - n_c)^2 - (n - n_c)^3 = \sigma^2 = \left(\frac{2}{3}\sigma + \sigma - \sigma_c\right)^2 = \frac{4}{27} + \frac{4}{3\sqrt{3}}(\sigma - \sigma_c) + (\sigma - \sigma_c)^2.
\]

(50)

To leading order we have
\[
(n - n_c)^2 = \frac{4}{3\sqrt{3}}(\sigma_c - \sigma).
\]

(51)

4.6.1. Order parameter. Hence for \(\sigma \leq \sigma_c\) the order parameter behaves as
\[
n(\sigma) - n_c \propto (\sigma_c - \sigma)^{\beta}, \quad \beta = \frac{1}{2},
\]

(52)
in accordance with (36).

4.6.2. Susceptibility. The breakdown susceptibility \(\chi = -dn/d\sigma\) will therefore have the same critical behavior,
\[
\chi \propto (\sigma_c - \sigma)^{-\gamma}, \quad \gamma = \frac{1}{2}
\]

(53)
as for the model with a uniform distribution of fiber strengths.

4.6.3. Relaxation time. Let us also investigate how the stable fixed point is approached. From (47) we find
\[
n_{t+1} - n^* = \frac{\sigma^2}{n^*} - \frac{\sigma^2}{n_t^2} = \frac{\sigma^2}{n^* n_t^2} (n_t^2 - n^*^2) \approx (n_t - n^*) \frac{2\sigma^2}{n^*^3}
\]

(54)
near the fixed point. Hence the approach is exponential,
\[
n_t - n^* \propto \exp(-t/\tau) \text{ with } \tau = \frac{1}{\ln(n^*^3/2\sigma^2)}.
\]

(55)
At the critical point, where \(n_c^* = 2/3\) and \(\sigma_c = 2/3\sqrt{3}\), the argument of the logarithm equals 1, so that \(\tau\) diverges when the critical state is approached. The divergence is easily seen to be of the same form,
\[
\tau \propto (\sigma_c - \sigma)^{-\theta}, \quad \theta = \frac{1}{2},
\]

(56)
as for the model with a uniform threshold distribution, equation (42).

4.6.4. Critical slowing. To find the correct behavior of the distance to the critical point, \(\Delta n_t = n_t - n_c = n_t - 2/3\), at criticality, we use the iteration (47) with \(\sigma = \sigma_c\),
\[
n_{t+1} = 1 - \frac{4/27}{n_t^2}, \quad \text{or} \quad \Delta n_{t+1} = \frac{1}{3} \Delta n_t - \frac{4/27}{\left(\frac{2}{3} + \Delta n_t\right)^2}.
\]

(57)
Near the fixed point, the deviation \(\Delta n_t = n_t - n^*\) is small. An expansion to second order in \(\Delta n_t\) yields
\[
\Delta n_{t+1} = \Delta n_t - \frac{9}{4} \Delta n_t^2,
\]

(58)
which is satisfied by

$$\Delta n_t = \frac{4}{9t} + \mathcal{O}(t^{-2}).$$

(59)

Here $\mathcal{O}$ means order of. The slow critical relaxation towards the fixed point,

$$n_t - n_c \propto t^{-\phi}, \quad \phi = 1,$$

(60)

for large $t$, is the same as for the uniform threshold distribution, formula (43).

In conclusion, we have found that the model with a linearly increasing distribution of fiber strengths possesses the same critical power laws as the model with a uniform distribution. This suggests that the critical properties are universal.

4.7. Graphical solutions of the recursive dynamics

Even if the recursive dynamics cannot be solved for each and every fiber threshold distribution, through a graphical solution scheme one can always reach the critical points. The trick is to plot $n_{t+1}$ versus $n_t$ and check where this plot touches the fixed-point line $n_{t+1} = n_t$.

For example, let us consider a Weibull distribution (with $k = 1$) of fiber thresholds (8), having cumulative distribution

$$P(x) = 1 - \exp(-x).$$

(61)

When an external stress $\sigma$ is applied, the recursion relation can be written as

$$n_{t+1} = 1 - [1 - \exp(-\sigma/n_t)] = \exp(-\sigma/n_t).$$

(62)

Now we plot $n_{t+1}$ versus $n_t$ following equation (62) for several $\sigma$ values (see figure 9). It is clear that at a particular $\sigma$ value, the $n_{t+1}$ versus $n_t$ curve touches the $n_{t+1} = n_t$ straight line at a single point and this particular $\sigma$ value is the critical stress value $\sigma_c$ for this model.
4.8. Approach to the critical point

It is important to find out how the system approaches the critical point (failure point) from below (pre-critical stress values) and above (post-critical stress values).

For uniform fiber strength distribution when the external load approaches the critical load $\sigma_c = 1/4$ from a higher value, i.e. in the post-critical region, the number of necessary iterations increases as one approaches the critical point. Near criticality, the number of iterations has a square-root divergence [13]:

$$ t_f \approx \frac{1}{2} \pi (\sigma - \sigma_c)^{-1/2}. $$

Similarly, in the pre-critical region, when the external load approaches the critical load $\sigma_c = 1/4$ from a lower value, near the critical point the number of iterations again has a square-root divergence [13] (for uniform distribution):

$$ t_f = \frac{1}{4} \ln(N)(\sigma_c - \sigma)^{-1/2}, $$

with a system-size-dependent amplitude.

We therefore find that in the FBM, there exists a two-sided critical divergence behavior (figure 10) in terms of the number of iteration steps needed to reach the critical point from below (pre-critical) and above (post-critical). A detailed derivation of the above divergences in $t_f$ are given in the appendix.

4.9. Percolation in the ELS FBM

We will now discuss the connection between the ELS FBM and the standard percolation model.

In the ELS mode, when the fiber bundle is loaded the fibers fail according to their thresholds, the weaker before the stronger. At a load or stretch $\Delta$, the fiber bundle supports a
where spring constant ($\kappa$) has been set to unity. Clearly, $P(\Delta)$ is the fraction of failed fibers at a load (extension) $\Delta$. We know that there is a certain $\Delta = \Delta_c$ value beyond which catastrophic failure occurs and the system collapses completely. We are particularly interested to find out whether or not the cluster of broken fibers percolates in the stable phase (before the failure point is reached). To answer this question we need to calculate $P(\Delta_c)$. From our analysis in section 6.2, we recall that the force has a parabolic maximum at the failure point $\Delta_c$; where the following relation is valid:

$$1 - \Delta_c p(\Delta_c) - P(\Delta_c) = 0. \tag{66}$$

Therefore

$$P(\Delta_c) = 1 - \Delta_c p(\Delta_c). \tag{67}$$

4.9.1. General threshold distribution. We consider a general power-law-type fiber threshold distribution within the range (0, 1),

$$p(x) = (1 + \alpha)x^\alpha; \quad P(x) = x^{1+\alpha}. \tag{68}$$

Putting the $p(\Delta_c)$ and $P(\Delta_c)$ values into equation (66) we get

$$\Delta_c = \left(\frac{1}{2 + \alpha}\right)^{1+\alpha}. \tag{69}$$

From the above relations, we can easily calculate the fraction of failed fibers at the failure point

$$P(\Delta_c) = \frac{1}{2 + \alpha} \leq \frac{1}{2}. \tag{70}$$

It is obvious that if $P(\Delta_c) < p_c$, ($p_c$ is the percolation threshold) the largest cluster of broken fibers does not percolate until the failure point itself is reached. Therefore in the case of a power-law-type distribution of thresholds, we do not see percolation of broken fibers in 2D until the system enters into the unstable phase. In a 3D site percolation problem the situation is different—as long as $\alpha < 1.25$, the cluster of broken fibers percolates in the stable phase (figure 11).

4.9.2. Analysis for a Weibull threshold distribution. Let us move to a more general distribution of fiber thresholds, the Weibull distribution. The cumulative Weibull distribution has the form

$$P(x) = 1 - \exp(-x^k), \tag{71}$$

where $k$ is the shape parameter. Therefore the probability distribution takes the form

$$p(x) = kx^{k-1}\exp(-x^k). \tag{72}$$

As the force has a maximum at the failure point $\Delta_c$, recalling the expression (equation (12)) and putting the $P(x)$, $p(x)$ values into it, we get
From the above equation we can easily calculate the critical extension value as
\[ \Delta_c = \frac{k}{\kappa}. \] (74)

The fraction of broken fibers at the failure point is
\[ P(\Delta_c) = 1 - \exp\left(-\frac{1}{k}\right). \] (75)

In the case of a Weibull distribution of fiber thresholds, the clusters of broken fibers can percolate before the failure point is reached (for 2D and 3D site and bond percolation scenarios) when the value of \( k \) remains within a certain window (figure 12).

5. Some related works on the FBM

In this section we draw attention to some related works on the FBM, which we believe may be regarded as essential reading in this field. If we follow quasi-static loading, the ELS model produces avalanches (successive failures of fibers at a fixed external load) of different sizes during the entire fiber-failure process. The statistics of such avalanches were analyzed by Hemmer and Hansen in their seminal paper in 1992 [14]. They found that the avalanches follow a universal power law with exponent \(-5/2\) for a mild restriction on the threshold distribution such that the load curve (force versus elongation) has a single maximum. Later, Pradhan, Hansen and Hemmer showed that the exponent of the avalanche distribution crosses over from \(-5/2\) to \(-3/2\) if we collect only the avalanches near the critical (failure) point [15]. Also, Divakaran and Dutta [16] studied the effect of discontinuity in the threshold distribution and obtained similar crossover behavior of avalanche distributions in ELS models.
In 1991, Harlow and Phoenix first introduced the LLS model \cite{17} with a simple breaking rule: when a fiber fails, the load it carried is shared by the nearest surviving fibers. The localized load-redistribution mechanism in the LLS scheme makes the model very different from the ELS scheme. In 1D, the critical strength of the LLS bundle shows a typical system-size dependence \cite{7,18}, $\sigma_c(N) \sim 1/\ln(N)$, where $N$ is the total number of fibers in the chain. However, recent studies by the Norwegian University of Science and Technology (NTNU) group have established that, in higher dimensions, the memory-independent LLS model shows non-zero critical strength \cite{19}. Biswas and Chakrabarti have studied the self-organized dynamics \cite{20} in LLS models by slightly modifying the load-redistribution rule, where the steadily increasing external load is applied at a central point of the system. The redistributed load always remains localized along the steadily growing boundary of the broken patch, and dynamic self-organization sets in.

There have been some attempts to bridge the gap between the ELS and LLS models by introducing some intermediate load-sharing rules. Hidalgo, Kun and Herrmann proposed a model \cite{21} where the load that was carried by a broken fiber is redistributed to the surviving fibers following a decaying power law in the distance from the broken fiber. Along the same lines, Pradhan, Chakrabarti and Hansen introduced a mixed-mode model \cite{22} where the ELS and LLS schemes are mixed together: when a fiber fails, a fraction $g$ of the load it carried is distributed according to the LLS rule (to the fibers at the edge of the hole containing the broken fiber) and the remaining $1 - g$ fraction of the load is distributed to all the surviving fibers. Clearly, for $g = 0$, the model reduces to a pure ELS model and for $g = 1$, it is nothing but the LLS model. They found an interesting result that the mixed-mode model crosses over from ELS to LLS behavior at $g \approx 0.8$.

Roy and Ray tried to observe another type of critical behavior \cite{23} in the ELS model in terms of a brittle to quasi-brittle transition as a function of the width of the threshold distribution. Their claim is that at (and below) a critical width value ($\delta_c$), breaking of the weakest

![Figure 12. Fraction of broken fibers versus Weibull index $k$. The curved line is $P(\Delta_c)$, which separates the stable and unstable phases of the FBM. Straight lines are the percolation threshold values.](image-url)
fiber leads to complete failure of the bundle, and at $\delta_c$ relaxation time diverges obeying a finite-size scaling law: $\tau \sim N^{\beta}(|\delta - \delta_c|^{\alpha})$ with $\alpha = \beta = 1/3$.

In section 4 we presented a mean-field treatment of the relaxation behavior of ELS models when the model is loaded by a discrete step. However, one can expect finite-size dependence of the relaxation behavior at or around the critical stress values when the system size is not large enough. Roy, Kundu and Manna have done extensive numerical studies [24] on the finite-size scaling forms of the relaxation time as a function of the deviation of stress values from the critical stress for the ELS model. In [25] Biswas and Sen investigated, mostly analytically, the maximum strength and corresponding redistribution schemes for sudden and quasi-static loading on the FBM. The universality class associated with the phase transition from partial to total failure (by increasing the load) was found to be dependent on the redistribution mechanism.

The FBM has also been applied to traffic-jam modeling [26], power-grid failure modeling [27], earthquake modeling, [28] etc. For a recent discussion on self-organized criticalities in the FBM in order to study the propagation of a crack front in heterogeneous solids, see [29]. An elegant renormalization group procedure in ELS FBM analysis has been introduced in a very recent article [30].

6. Discussion

The FBM is an extremely elegant model for studying collective dynamical failure in inhomogeneous materials. The ELS scheme (ensured by the absolute rigidity of the platforms) of equally sharing the extra load amongst the surviving fibers, after an individual fiber failure, often allows some precise analysis of the dynamics. With a simple (uniform) fiber-breaking threshold distribution, we have demonstrated here the ‘critical behavior’ and its related features like universality. The model here fits simple common sense, yet the collective failure dynamics in the bundle, and its critical behavior, are extremely intriguing. It is hoped that the unininitiated readers can appreciate the excitement.

The same model with realistic fiber threshold distribution (like the Weibull distribution or the Gumbel distribution) may not always allow such analytic studies, but their numerical analysis (often with a realistic LLS scheme for load-sharing due to local deformations of the platforms with finite rigidity) can take one to the frontiers of civil engineering applications, as practiced by professional engineers and architects.

It may be noted that there are hardly any other models in science and technology where some simplifications allow intriguing progress analytically in the basic science, yet by adding some realistic ingredients to the same model, one is taken to the forefront of engineering applications.

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Appendix. Exact solutions for pre- and post-critical relaxation

The iterative breaking process considered in section 4 ends with one of two possible end results. Either the whole bundle breaks down, or an equilibrium situation with a finite number of intact fibers is reached. The final fate depends on whether the external stress $\sigma$ on the bundle is post-critical ($\sigma > \sigma_c$), pre-critical ($\sigma < \sigma_c$), or critical $\sigma = \sigma_c$. We now investigate the total number $t_f$ of iterative steps $t$ necessary to reach the final state, and start with the post-critical situation following the formulations in [3, 13].

A.1. Post-critical relaxation

For uniform threshold distribution (7) we can explicitly and exactly follow the path of iteration. We introduce a measure $\epsilon$ of the deviation from critical value by

$$\epsilon = \sigma - \sigma_c = \sigma - \frac{1}{4},$$

where $\epsilon$ is positive. The basic iteration formula (24) is in this case

$$n_{t+1} = 1 - \frac{\sigma}{n_t}.$$  \hspace{1cm} \text{(A2)}

The fraction $n_t$ of intact fibers will decrease under the iteration, and we see from (A2) that if $n_t$ reaches the value $\sigma$, or smaller, the next iteration yields $n_{t+1} = 0$ or a negative value, i.e. complete bundle breakdown. We wish to find how many iterations, $t_f$, are needed to reach this stage.

For that purpose we solve the nonlinear iteration (A2) by converting it into a linear iteration by means of two transformations. From (A2), we can write

$$n_{t+1}n_t = n_t - \sigma = n_t - \frac{1}{4} - \epsilon.$$ \hspace{1cm} \text{(A3)}

We introduce first

$$n_t = \frac{1}{2} - y_t \sqrt{\epsilon},$$ \hspace{1cm} \text{(A4)}

with the result

$$2 \sqrt{\epsilon} = \frac{y_{t+1} - y_t}{1 + y_{t+1}y_t}.$$ \hspace{1cm} \text{(A5)}

As a second transformation we put

$$y_t = \tan v_t,$$ \hspace{1cm} \text{(A6)}

with the result

$$2 \sqrt{\epsilon} = \frac{\tan v_{t+1} - \tan v_t}{1 + \tan v_{t+1} \tan v_t} = \tan(v_{t+1} - v_t).$$ \hspace{1cm} \text{(A7)}

Hence we have now obtained the linear iteration

$$v_{t+1} - v_t = \tan^{-1}(2 \sqrt{\epsilon}),$$ \hspace{1cm} \text{(A8)}

with solution

$$v_t = v_0 + t \tan^{-1}(2 \sqrt{\epsilon}).$$ \hspace{1cm} \text{(A9)}

The iteration starts with all fibers intact, i.e. $n_0 = 1$, which by (A4) and (A6) corresponds to $y_0 = -1/\sqrt{\epsilon}$ and $v_0 = -\tan^{-1}(1/2 \sqrt{\epsilon})$. With the constant in (A9) now determined, we can express the solution in terms of the original variable:
We saw above that when \( n_t \) obtains a value in the interval \((0, \sigma)\), the next iteration gives \( n_{t+1} \leq 0 \), complete bundle failure. \( n_t \) reaching the smallest value 0 gives an upper bound \( t^u_t \) for the number of iterations. We find

\[
t^u_t(\sigma) = 1 + \frac{2\tan^{-1}(1/2\sqrt{\sigma})}{\tan^{-1}(2\sqrt{\epsilon})}.
\]

Furthermore, \( n_t \) reaching the value \( \sigma \equiv \frac{1}{2} + \epsilon \) gives a lower bound:

\[
t^l_t(\sigma) = 1 + \frac{\tan^{-1}\left(\frac{1}{2} - \epsilon\right)/\sqrt{\sigma}}{\tan^{-1}(2\sqrt{\epsilon})}.
\]

The upper and lower bounds (A11) and (A12) nicely embrace the simulation results (figure A1). When the external load is large, just a few iterations suffice to induce complete bundle breakdown. On the other hand, when the external load approaches the critical load \( \sigma_c = 1/4 \), the number of necessary iterations becomes very large. Near criticality (\( \epsilon \to 0 \)) both the upper and lower bounds, (A11) and (A12), have a square-root divergence:

\[
t_t \simeq \frac{1}{2} \pi (\sigma - \sigma_c)^{-1/2},
\]

dominating order for small \( \epsilon \).
A.2. Pre-critical relaxation

When the external stress is less than the critical value, $\sigma < \sigma_c$ we use the positive parameter

$$\epsilon = \sigma_c - \sigma$$

as a measure of the deviation from criticality. In this case the bundle is expected to relax to an equilibrium situation with a finite number of fibers intact. We are going to calculate how many iteration, $t$, are needed to reach equilibrium.

Here we consider uniform threshold distribution (7), and again we transform the non-linear iteration (38) to a linear one by means of two transformations. Introducing $\sigma = \frac{1}{4} - \epsilon$ and

$$n_t = \frac{1}{2} + \sqrt{\epsilon} \frac{z_t}{z_t}$$

into (38), we have

$$2\sqrt{\epsilon} = \frac{z_{t+1} - z_t}{1 - z_t + \epsilon t}.$$  \hspace{1cm} (A15)

A second transformation,

$$z_t = \tanh w_t,$$  \hspace{1cm} (A16)

gives

$$2\sqrt{\epsilon} = \frac{\tanh w_{t+1} - \tanh w_t}{1 - \tanh w_{t+1} \tanh w_t} \equiv \tanh(w_{t+1} - w_t).$$  \hspace{1cm} (A17)

Hence we have the linear iteration $w_{t+1} - w_t = \tanh^{-1}(2\sqrt{\epsilon})$, which gives

$$w_t = w_0 + t \tanh^{-1}(2\sqrt{\epsilon}).$$  \hspace{1cm} (A18)

As $\tanh^{-1}x = \frac{1}{2} \ln[(1 + x)/(1 - x)]$, via (A15) and (A17) the initial situation with no broken fibers, $n_0 = 1$, corresponds to $w_0 = \tanh^{-1}(2\sqrt{\epsilon})$, so that (A19) becomes

$$w_t = (1 + t) \tanh^{-1}(2\sqrt{\epsilon}).$$  \hspace{1cm} (A19)

For the original variable this corresponds to

$$n_t = \frac{1}{2} + \sqrt{\epsilon} \frac{1}{\tanh[(1 + t)^{-1}(2\sqrt{\epsilon})]}.$$  \hspace{1cm} (A20)

After an infinite number of iterations ($t \to \infty$ in (A21)) $n_t$ apparently approaches the fixed point

$$n^* = \frac{1}{2} + \sqrt{\epsilon},$$  \hspace{1cm} (A21)

which is the fixed point (34) for the uniform distribution. However, the bundle contains merely a finite number of fibers, so equilibrium should be reached after a finite number of steps. Since an equilibrium value corresponds to a fixed point, we seek fixed points $N^*$ for finite $N$.

Taking into account that the variables $N_t$ are integers, one can obtain the final result (see [3, 13])

$$t_1(\sigma) = -1 + \frac{\ln(N)}{2 \tanh^{-1}(2\sqrt{\epsilon})}.$$  \hspace{1cm} (A22)

The simulation data in figure A2 for the uniform threshold distribution are well approximated by this analytic formula. Equation (A23) shows that near the critical point the number of
iterations again has a square-root divergence:

\[ t_i = \frac{1}{4} \ln(N)(\sigma_\ell - \sigma)^{-1/2}, \tag{A24} \]

with a system-size-dependent amplitude.

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**References**

[1] Peirce F T 1926 *J. Textile Inst. Trans.* 17 355
[2] Ising E 1925 *Z. Phys.* 31 253
[3] Hansen A, Hemmer P C and Pradhan S 2015 *The Fiber Bundle Model* (Berlin: Wiley-VCH)
[4] Fisher M E 2017 *Excursions in the Land of Statistical Physics* (Singapore: World Scientific)
[5] Mandelbrot B 1982 *Fractal Geometry of Nature* (New York: W H Freeman)
[6] Wilson K G and Kogut J 1974 *Phys. Rep.* 12 75
[7] Pradhan S, Hansen A and Chakrabarti B K 2010 *Rev. Mod. Phys.* 82 499
[8] Biswas S, Ray P and Chakrabarti B K 2015 *Statistical Physics of Fracture, Breakdown, and Earthquake* (Berlin: Wiley-VCH)
[9] Daniels H E 1945 *Proc. R. Soc.* A 183 243
[10] Sornette D 1989 *J. Phys. A: Math. Gen.* 22 L243
[11] Pradhan S and Chakrabarti B K 2001 *Phys. Rev.* E 65 016113
[12] Bhattacharyya P, Pradhan S and Chakrabarti B K 2003 *Phys. Rev.* E 67 046122
[13] Pradhan S and Hemmer P C 2007 *Phys. Rev.* E 75 056112
[14] Hemmer P C and Hansen A 1992 *J. Appl. Mech.* 59 909
[15] Pradhan S, Hansen A and Hemmer P C 2005 *Phys. Rev. Lett.* 95 125501
[16] Divakaran U and Dutta A 2007 *Phys. Rev.* E 75 011117
[17] Harlow D G and Phoenix S L 1991 *J. Mech. Phys. Solids* 39 173
[18] Gomez J, Iniguez D and Pacheco A F 1993 Phys. Rev. Lett. 71 380
[19] Sinha S, Kjellstadli J T and Hansen A 2015 Phys. Rev. E 92 020401(R)
[20] Biswas S and Chakrabarti B K 2013 Phys. Rev. E 88 042112
[21] Hidalgo R C, Kun F and Herrmann H J 2002 Phys. Rev. E 65 046148
[22] Pradhan S, Chakrabarti B K and Hansen A 2005 Phys. Rev. E 71 036149
[23] Roy S and Ray P 2015 Eur. Phys. Lett. 112 26004
[24] Roy C, Kundu S and Manna S S 2013 Phys. Rev. E 87 062137
[25] Biswas S and Sen P 2015 Phys. Rev. Lett. 115 155501
[26] Chakrabarti B K 2006 Physica A 377 162–6
[27] Pahwa S, Scoglio C and Scala A 2014 Sci. Rep. 4 3694
[28] Sornette D 1992 J. Phys. I France 2 2089–96
[29] Petri A and Pontuale G 2018 J. Stat. Mech. 2018 063201
[30] Pradhan S, Hansen A and Ray P Frontiers Phys. 6 65