Phases of $\text{QCD}_3$ with three families of fundamental flavours

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Abstract

We explore the phase diagram for an $SU(N)$ gauge theory in $2 + 1$ dimensions with three families of fermions with different masses, all in the fundamental representation. The phase diagram is three dimensional and contains cuboid, planar and linear quantum regions, depending on the values of the fermionic masses. Among other checks, we consider the consistency with boson/fermion dualities and verify the reduction of the phase diagram to the one and two-family diagrams.

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In recent years, a sustained effort has been dedicated to studying the phases of 2+1 dimensional gauge theories, inspired by applications in condensed matter systems and supersymmetric dualities. One important outcome was the proposal of infrared dualities between non-supersymmetric Chern-Simons gauge theories with fundamental matter \([1,2]\) where certain fermion-boson dualities were conjectured. This avenue included theories with various gauge groups and matter representations \([2–11]\). A specific fermion-boson duality that is of interest for our work is

\[
SU(N)_k + F \text{Fermions} \leftrightarrow U(k + F/2)_{-N} + F \text{Scalars},
\]

and its time-reversal version.

However, these dualities do not describe the full phase diagram of the gauge theory in 2 + 1 dimensions for any number of flavours, colors, and level. Komargodski and Seiberg considered such a general case and presented the complete phase diagram of three dimensional \(SU(N)_k\) gauge theory with Chern-Simons level \(k\) coupled to \(F\) fundamental Dirac fermions with mass \(m\). The diagram was drawn as a function of \(m\) \([12]\). The main new feature of the phase diagram is that the infrared (IR) description has a quantum phase that is hidden semiclassically for a large number of flavours. The discussion was extended in \([13–16]\).

The next step is to consider fermions in fundamental representation with different masses. One step in this direction was to introduce two sets of fundamental fermions with different masses \(m_1 \neq m_2\), where the phase diagram acquires a two-dimensional structure and includes planes of quantum phases in the IR \([17,18]\). Our goal in this work is to consider a theory with three sets of fundamental fermions with three different masses and consider the corresponding three-dimensional diagram. We will see that there are three types of quantum regions of dimensions 1, 2, or 3 and will perform various consistency checks on the perturbations of infrared descriptions.

In the rest of this section, we review the description of the one and two-family cases to explain how the flavour symmetry breaking mechanism helps to understand the phase diagram for small masses when the semiclassical description fails. In section 2, we give the details of the IR description for the three-family case for various ranges of \(k\) and draw the full phase diagram. In section 3, we perform various checks which, on the one hand, verify our proposal and, on the other hand,
confirm the consistency of the results for one family and two families of fermions [12, 17, 18]. Section 4 contains the conclusions.

1.1 Review of the one-family case

The work [12] considered the phase diagram of $SU(N)$ gauge theory in $2 + 1$ dimensions with Chern-Simons level $k$ and $F$ fermions in the fundamental representation. The theory has a global flavour symmetry $U(F)$, which is spontaneously broken for small values of $k$. The spontaneous breaking of the flavour symmetry allows them to split the phase diagram into two cases as follows:¹

1. $k \geq F/2$: In this case, the theory in the IR is semiclassically accessible, and the phase diagram is described by the asymptotic theories obtained after integrating the fermions out when their mass $m$ is positive or negative. The two phases are then $SU(N)_{k+F/2}$ for positive $m$, which has a level-rank dual $U(k+F/2)_N$ and $SU(N)_{k-F/2}$ for negative $m$ with a level-rank dual $U(k-F/2)_N$. The two phases are separated by a second-order phase transition described by a conformal field theory (CFT); they are gapped and are described by pure topological quantum field theories (TQFT). The phase diagram is as in figure 1, where the transition between the two asymptotic phases occurs at the blue point.

$$SU(N)_k + F \psi \leftrightarrow U(k+F/2)_N + F \phi$$

$SU(N)_{k-\frac{F}{2}}$ $\downarrow \quad \downarrow m = 0 \quad \downarrow \quad SU(N)_{k+\frac{F}{2}}$

$U(k-F/2)_N$ $m \rightarrow \pm \infty$ $SU(N)_{k-F/2}$

Figure 1: Phase diagram of $SU(N)_k + F \psi$ with $k \geq F/2$.

2. $k < F/2$: In this case, the theory is semiclassically accessible only for large mass ($m \rightarrow \pm \infty$). The asymptotic theories are $SU(N)_{k+F/2} \leftrightarrow U(k+F/2)_N$ for large positive $m$ and $SU(N)_{k-F/2} \leftrightarrow U(F/2-k)_N$ for large negative $m$. However, integrating out the scalars from the dual bosonic theory $U(k+F/2)_N + F \phi$ for large negative mass squared leads to a sigma-model phase. This sigma-model can be as weakly coupled as we like, unlike the original fermionic description, which is strongly coupled with no sigma-models in its phase. In [12], the authors suggested that for the fermionic theory there is some value of the number of flavours $F^*$ at which the $U(F)$ symmetry is spontaneously broken into $U(F/2 + k) \times U(F/2 - k)$, leading to a sigma-model $\sigma$ in the IR that matches the bosonic phase and is given by the Grassmannian

$$Gr(F/2+k,F) = \frac{U(F)}{U(F/2+k) \times U(F/2-k)}.$$  \hspace{1cm} (1.2)

¹We use the notations and conventions of [12, 17, 18]
The sigma-model is a purely quantum gapless phase that does not appear semiclassically. The phase diagram now consists of the two asymptotic topological phases and a new quantum region for small $m$, which separates the two topological phases. The quantum region lies between two transition points that are described by some positive mass $m^+$ on the right and some negative mass $m^-$ on the left, as shown in figure 2. The authors also conjectured the existence of a new duality in the form $SU(N)_k + F\psi \leftrightarrow U(F/2-k)_N + F\phi$ to cover the phase diagram for negative $m$.

$SU(N)_{k+\frac{F}{2}} \quad m^- \quad \sigma \quad m^+ \quad SU(N)_{k-\frac{F}{2}}$

$U(F/2-k)_N \quad U(k+F/2)_{-N}$

Figure 2: Phase diagram of $SU(N)_k + F\psi$ with $k < F/2$.

### 1.2 Review of the two-family phase diagrams

We now move to [17, 18] where the authors considered the case when the $F$ fermions are split into two sets of fermions: $p\psi_1$ fermions with mass $m_1$ and $(F-p)\psi_2$ fermions with mass $m_2$. The flavour symmetry is now explicitly broken into $U(p) \times U(F-p)$ and the phase diagram looks different for three particular cases: $k \geq F/2$, $F/2-p \leq k < F/2$, and $0 \leq k < F/2-p$, where the range of $p$ is such that $0 \leq p < F/2$. In analogy to the one-family case, there are two dual bosonic theories $U(k+F/2)_{-N} + p\phi_1 + (F-p)\phi_2$ and $U(F/2-k)_N + p\phi_1 + (F-p)\phi_2$ for small values of $k$ to cover the full phase diagram.

1. $k \geq F/2$: There is no flavour symmetry breaking, and the four topological theories describe the phase diagram. Integrating out both $\psi_1$ and $\psi_2$ when their masses are both positive and negative, one obtains various topological phases $T_a(m_1 > 0, m_2 > 0), T_b(m_1 < 0, m_2 > 0), T_c(m_1 < 0, m_2 < 0), T_d(m_1 > 0, m_2 < 0))$:

   $T_a : SU(N)_{k+\frac{F}{2}} \leftrightarrow U(k+F/2)_{-N}$,
   $T_b : SU(N)_{k+\frac{F}{2}} \leftrightarrow U(k+F/2-p)_{-N}$,
   $T_c : SU(N)_{k-\frac{F}{2}} \leftrightarrow U(k-F/2)_{-N}$,
   $T_d : SU(N)_{k-\frac{F}{2}} \leftrightarrow U(k-F/2+p)_{-N}$.

These topological phases are separated by critical theories represented as red lines in figure 3a. The critical theories are $SU(N)_{k+\frac{F}{2}} + (F-p)\psi_2$ on the horizontal red line and $SU(N)_{k-\frac{F}{2}} + p\psi_1$ on the vertical red line. The blue point is a transition point that separates the four different topological phases. We will use the term type 1 phase diagram to label this case.
2. $F/2 - p \leq k < F/2$: The asymptotic phases can be found similarly by sending both masses to $\pm \infty$. The topological phases $T_a$, $T_b$, and $T_d$ remain as in equation (1.3) while $T_c$ becomes

$$T_c \rightarrow \tilde{T}_c : SU(N)_{k - \frac{p}{2}} \leftrightarrow U(F/2 - k)_N.$$ \hfill (1.4)

There are quantum regions in this case where the flavour symmetry is spontaneously broken. The strategy is to check the asymptotic phases where we send only one of the masses to $\pm \infty$, and the theory becomes one-family with a shifted level. The theory is strongly coupled and has sigma-model descriptions $\sigma_{bc}$ and $\sigma_{cd}$ for small and/or negative $m_1$ and $m_2$, respectively. The sigma-models have target spaces with the following Grassmannians

$$\sigma_{bc} : Gr(F/2 - k, F - p) = \frac{U(F - p)}{U(F/2 - k) \times U(k + F/2 - p)}, \hfill (1.5)$$

$$\sigma_{cd} : Gr(F/2 - k, p) = \frac{U(p)}{U(F/2 - k) \times U(k - F/2 + p)}. \hfill (1.6)$$

The theory also has a sigma-model $\sigma$ on the diagonal line $m_1 = m_2$, where it is reduced to the one-family case. In this transition, $\sigma$ acts as a phase transition between $\sigma_{bc}$ and $\sigma_{cd}$. The red line that separates $T_a$ and $T_d$ corresponds to the critical theory $SU(N)_{k + \frac{p}{2} + (F - p) \psi_2}$ while the red line separating $T_a$ and $T_b$ corresponds to the critical theory $SU(N)_{k + \frac{p}{2} + p \psi_1}$. There also exist critical theories separating the topological theories and the quantum phases, which are described via the dual bosonic theories. For example, the red line that separates $T_b$ and $\sigma_{bc}$ is given by the critical theory $U(F/2 - k)_N + (F - p) \phi_2$, and this applies similarly for the remaining red lines. The full phase diagram for this case is shown in figure 3b, and we label it as a type II phase diagram.

3. $0 \leq k < F/2 - p$: In this range, the phase diagram is similar to the previous case in the sense that there exist topological phases that can be found asymptotically plus expected quantum regions due to the symmetry breaking scenario. The topological phases $T_a$, $T_b$, and $\tilde{T}_c$ remain the same while $T_d$ becomes

$$T_d \rightarrow \tilde{T}_d : SU(N)_{k - \frac{p}{2} + p} \leftrightarrow U(F/2 - p - k)_N. \hfill (1.7)$$

In this case, the quantum phases are $\sigma$ models on the diagonal line with a Grassmannian given by equation (1.2). They are $\sigma_{bc}$ which is a plane describing the quantum region for small and negative $m_2$ with a Grassmannian given by equation (1.5) and $\sigma_{ad}$ for small and positive $m_2$ with a Grassmannian

$$\sigma_{ad} : Gr(F/2 + k, F - p) = \frac{U(F - p)}{U(F/2 + k) \times U(F/2 - p - k)}, \hfill (1.8)$$

The phase diagram is shown in figure 3c which is a type III phase diagram. As in type II, the $\sigma$ line acts as a transition between $\sigma_{bc}$ and $\sigma_{ad}$.

To summarize, the two-family theory has the following features:

- For $k \geq F/2$, the theory in the IR is described by the four topological field theories in equation (1.3).
(a) Type I: $k \geq F/2$

(b) Type II: $F/2 - p \leq k < F/2$

(c) Type III: $0 \leq k < F/2 - p$

Figure 3: Phases of $SU(N)_k + p \psi_1 + (F - p) \psi_2$.

- For small $k$, the IR description is given by four topological field theories and a line of sigma-model $\sigma$ separating two planes of sigma-models $(\sigma_{bc}, \sigma_{cd})$ or $(\sigma_{bc}, \sigma_{ad})$.

The theory passes various consistency checks such as matching the phases of the dual bosonic theory near the transition points as well as perturbing the $\sigma$ model to make sure the description is valid when both masses are small [17].

2 The theory with three families

We now consider the case when the $F$ fermions are split into three sets of fermions: $p$ fermions $\psi_1$ with mass $m_1$, $q$ fermions $\psi_2$ with mass $m_2$ and $F - p - q$ fermions $\psi_3$ with mass $m_3$. The global flavour symmetry $U(F)$ is explicitly broken to $U(p) \times U(q) \times U(F - p - q)$. In order to keep the analysis under control, we consider that the ranges of $p$ and $q$ are such that $0 < p \leq q \leq F/2$. The range of $k$ diagram is divided into five cases: $k \geq F/2$, $F/2 - q \leq k < F/2$, $F/2 - p \leq k < F/2 - q$, $F/2 - (p + q) \leq k < F/2 - p$, and $0 \leq k < F/2 - (p + q)$, each one with a different phase diagram.

Our strategy for finding the topological phases is by sending the three masses to $\pm \infty$. For small values of $k$, we look for the quantum regions in two steps: the first is to find the asymptotic limits when two of the masses are sent to $\pm \infty$ and then when we send only one of the three masses to $\pm \infty$. The theory on the three-dimensional diagonal line $m_1 = m_2 = m_3$ is reduced to the one-family case with the phase diagram described by figures 1 and 2, which means that we always expect $\sigma$ to appear as a phase for all the ranges with $k < F/2$.

Case 1: $k \geq F/2$

For this range, the theory in the IR is weakly coupled with no symmetry breaking scenario. We describe the phase diagram by eight topological field theories and their level-rank dual descriptions; six critical lines separate these phases. The topological phases are determined by finding the asymptotic limits of the three masses, and they appear in the following ranges of the masses $m_1$, $m_2$, and $m_3$ as $T_1(m_1 > 0, m_2 > 0, m_3 > 0), T_2(m_1 > 0, m_2 > 0, m_3 < 0), T_3(m_1 > 0, m_2 < 0, m_3 > 0), T_4(m_1 < 0, m_2 > 0, m_3 > 0), T_5(m_1 > 0, m_2 < 0, m_3 > 0), T_6(m_1 <$
The phase diagram is three-dimensional in the space \((m_1, m_2, m_3)\) and the six different projections are shown in figure 4. The separation between the eight topological theories occurs via planes of critical theories, each point on the critical planes belonging to one of the following critical theories

\begin{align}
T_1 : SU(N)_{k+p+q/2} & \leftrightarrow U(k+p+q-F/2)_{-N} \\
T_2 : SU(N)_{k+p+q-q/2} & \leftrightarrow U(k+p+q-F/2)_{-N} \\
T_3 : SU(N)_{k+p-q/2} & \leftrightarrow U(k+p-F/2)_{-N} \\
T_4 : SU(N)_{k+p-q-p/2} & \leftrightarrow U(k+p-F/2)_{-N} \\
T_5 : SU(N)_{k+p-q-p} & \leftrightarrow U(k+p-F/2)_{-N} \\
T_6 : SU(N)_{k+p-q-q/2} & \leftrightarrow U(k+p-F/2)_{-N} \\
T_7 : SU(N)_{k+p-q-p-q/2} & \leftrightarrow U(k+p-F/2)_{-N} \\
T_8 : SU(N)_{k+p-q-p-q} & \leftrightarrow U(k+p-F/2)_{-N} 
\end{align}

The phase diagram is three-dimensional in the space \((m_1, m_2, m_3)\) and the six different projections are shown in figure 4. The separation between the eight topological theories occurs via planes of critical theories, each point on the critical planes belonging to one of the following critical theories

\begin{align}
C^+_1 : SU(N)_{k+p+q/2} + q \psi_2 + (F-p-q) \psi_3 , \\
C^+_2 : SU(N)_{k+p/2} + p \psi_1 + (F-p-q) \psi_3 , \\
C^+_3 : SU(N)_{k+p-q-p/2} + p \psi_1 + q \psi_2 .
\end{align}
\( C_1^\pm, C_2^\pm \), and \( C_3^\pm \) describe the planes on the mass axes \( m_1, m_2, \) and \( m_3, \) respectively. The plus or minus sign indicates whether the theory is along the positive or negative axes.

**Case 2:** \( F/2 - q \leq k < F/2 \)

For this and all the remaining ranges of \( k \) we need two bosonic dual descriptions to fill in the full phase diagram, as conjectured in [12]. The bosonic theories are \( U(F/2 + k)_{-N} + p \phi_1 + q \phi_2 + (F - p - q) \phi_3 \) and \( U(F/2 - k)_{N} + p \phi_1 + q \phi_2 + (F - p - q) \phi_3, \) where \( \phi_1, \phi_2, \) and \( \phi_3 \) are scalars in the fundamental representation of \( SU(N). \) The three masses asymptotic limits yield the same topological theories as in equation (2.1) except for \( T_8, \) which becomes

\[
T_8 \rightarrow \tilde{T}_8 : SU(N)_{k-\frac{p}{2}} \leftrightarrow U(F/2 - k)_N . \tag{2.3}
\]

The two mass asymptotic limits produce a one-family theory with a shifted level. In some limits, the shifted level becomes lower than the remaining number of flavours divided by two, and the theory is strongly coupled. The three-dimensional phase diagram includes quantum regions described by sigma-models. The quantum regions appear as cuboids in the three-dimensional picture. The cuboid quantum phases are the signature of the three-family theory as the planes of sigma-models were signatures of the two-family case. The theory with \( F/2 - q \leq k < F/2 \) has a cuboid sigma-model in its phase diagram in the following cases:

- \( m_2, m_3 \rightarrow -\infty: \) the theory becomes \( SU(N)_{k-F/2+p/2 + p} \psi_1 \). The cuboid quantum region is described by a sigma-model \( \sigma_1^c \), and we will denote the cuboid Grassmanians by their corresponding sigma-models. The Grassmannian for this region is

\[
\sigma_1^c = \frac{U(p)}{U(F/2 - k) \times U(k - F/2+p)} . \tag{2.4}
\]

- \( m_1, m_3 \rightarrow -\infty: \) the theory becomes \( SU(N)_{k-F/2+q/2 + q} \psi_2 \) which has a sigma-model \( \sigma_2^c \) given by

\[
\sigma_2^c = \frac{U(q)}{U(F/2 - k) \times U(k - F/2+q)} . \tag{2.5}
\]

- \( m_1, m_2 \rightarrow -\infty: \) the theory is \( SU(N)_{k-p/2-q/2 + (F - p - q)} \psi_3 \) with a sigma-model \( \sigma_3^c \) given by

\[
\sigma_3^c = \frac{U(F - p - q)}{U(F/2 - k) \times U(F/2 - p - q+k)} . \tag{2.6}
\]

Now let us perform a consistency check by sending one mass to infinity where the theory with the remaining flavours is reduced to a two-family case with shifted level. In each limit, we check the relation between the remaining number of flavours and the shifted level, which determines whether the phase diagram is type I, II, or III. We summarize the check as follows:

(i) \( m_1 \rightarrow +\infty: \) we integrate \( \psi_1 \) out, and the theory is reduced to

\[
SU(N)_{k+\frac{q}{2} + q} \psi_2 + (F - p - q) \psi_3 \equiv SU(N)_{k_1^+} + q \psi_2 + (F_1 - q) \psi_3 , \tag{2.7}
\]

where \( k_1^+ = k + p/2 \) and \( F_1 = F - p \) with \( k_1^+ > F_1/2. \) Due to the range of \( k_1^+ \), this region of the three-dimensional phase diagram has a type I phase diagram with the topological theories \( T_1, T_2, T_3, \) and \( T_5. \)
(ii) $m_1 \to -\infty$: integrating $\psi_1$ out leads to

$$SU(N)_{k-\frac{q}{2}} + q \psi_2 + (F - p - q) \psi_3 \equiv SU(N)_{k_1^-} + q \psi_2 + (F_1 - q) \psi_3,$$  

(2.8)

where $k_1^- = k - p/2$ and $F_1/2 - q < k_1^- < F_1$. Due to the range of $k_1^-$, the phase diagram is then of type II with the topological theories $T_5$, $T_6$, $T_7$, and $\tilde{T}_8$. Alongside these topological phases, we have a sigma-model on the diagonal given by

$$\sigma^d_{23} : Gr(F_1/2 + k_1^-, F_1) = \frac{U(F - p)}{U(F/2 - k) \times U(F/2 - p + k)}.$$  

(2.9)

This diagonal sigma-model is not a line but rather a plane region in the three-dimensional picture. Only one side appears here; this will become clearer in section 4. The horizontal and vertical sigma-models are $\sigma_{78}$ which separates $T_7$ and $\tilde{T}_8$ and $\sigma_{68}$ separating $T_6$ and $\tilde{T}_8$. They are given by

$$\sigma_{78} : Gr(F_1/2 - k_1^-, F_1 - q) = \frac{U(F - p)}{U(F/2 - k) \times U(F/2 - p - q + k)} \equiv \sigma_{3}^c,$$  

(2.10)

$$\sigma_{68} : Gr(F_1/2 - k_1^-, q) = \frac{U(q)}{U(F/2 - k) \times U(k - F/2 + q)} \equiv \sigma_{2}^c.$$  

(2.11)

We notice that $\sigma_{78} \equiv \sigma_{3}^c$ which means that $\sigma_{78}$ is just one side of the three-dimensional quantum region $\sigma_{3}^c$, the same thing applies for $\sigma_{68}$ which is one side of $\sigma_{2}^c$.

(iii) $m_2 \to +\infty$: we integrate $\psi_2$ out and the theory becomes

$$SU(N)_{k+\frac{q}{2}} + p \psi_1 + (F - p - q) \psi_3 \equiv SU(N)_{k_2^+} + p \psi_1 + (F_1 - p) \psi_3,$$  

(2.12)

where $k_2^+ = k + p/2$ and $F_2 = F - q$ with $k_2^+ > F_2/2$ and the phase diagram is of type I with the topological phases $T_1$, $T_2$, $T_4$, and $T_6$.

(iv) $m_2 \to -\infty$: integrating $\psi_2$ gives

$$SU(N)_{k-\frac{q}{2}} + p \psi_1 + (F - p - q) \psi_3 \equiv SU(N)_{k_2^-} + p \psi_1 + (F_1 - p) \psi_3,$$  

(2.13)

where $k_2^- = k - p/2$ and $F_2/2 - q < k_2^- < F_2$. The phase diagram is then of type II with the topological theories $T_3$, $T_5$, $T_7$, and $\tilde{T}_8$. The diagonal sigma-model is

$$\sigma^d_{13} : Gr(F_2/2 + k_2^-, F_2) = \frac{U(F - q)}{U(F/2 - k) \times U(F/2 - q + k)}.$$  

(2.14)

The quantum phase $\sigma_{78}$ also exists as in equation (2.10), alongside with the phase $\sigma_{58}$ that separates $T_5$ and $\tilde{T}_8$ and is given by

$$\sigma_{58} : Gr(F_2/2 - k_2^-, p) = \frac{U(p)}{U(F/2 - k) \times U(k - F/2 + p)} \equiv \sigma_{5}^c.$$  

(2.15)

We should emphasize here that the equivalence between these phases and the cuboid sigma-models is itself a consistency check of our analysis.
Figure 5: Phases of $SU(N)_{k_1 + 2F_{-p - q}^2} + p \psi_1 + q \psi_2 + (F - p - q) \psi_3$ with $F/2 - q \leq k < F/2$.

(v) $m_3 \rightarrow +\infty$: after integrating $\psi_3$ out the theory is

$$SU(N)_{k_1 + 2F_{-p - q}^2} + p \psi_1 + q \psi_2 \equiv SU(N)_{k_1^+} + p \psi_1 + (F_3 - p) \psi_2,$$

(2.16)

where $k_3^+ = k + (F - p - q)/2$ and $F_3 = p + q$, the relation $k_3^+ > F_3/2$ holds, and we have a type I phase diagram with the topological phases $T_1, T_3, T_4$, and $T_7$.

(vi) $m_3 \rightarrow -\infty$: integrating $\psi_3$ gives

$$SU(N)_{k_1 - 2F_{-p - q}^2} + p \psi_1 + q \psi_2 \equiv SU(N)_{k_3^-} + p \psi_1 + (F_3 - p) \psi_3,$$

(2.17)

where $k_3^- = k - (F - p - q)/2$ and $F_3/2 - q < k_3^- < F_3$, but $p > q$ which makes this case cover the range of type II phase diagram. The phase diagram has the topological phases $T_2, T_5, T_6$, and $T_8$. The diagonal sigma-model of this side is

$$\sigma_{12}^d : Gr(F_3/2 + k_3^-, F_3) = \frac{U(p + q)}{U(F/2 - k) \times U(k - F/2 + p + q)},$$

(2.18)

while the horizontal and vertical quantum regions are $\sigma_{68}$ and $\sigma_{58}$ respectively and given by equations (2.11) and (2.15).

We summarize the phase diagram for this range of $k$ in figure 5 and the IR theory has the following phases; eight topological phases $(T_1 - T_8)$, three cuboid sigma-models $(\sigma_1^d, \sigma_2^d, \sigma_3^d)$, and...
three planes of sigma-models \((\sigma^d_{12}, \sigma^d_{13}, \sigma^d_{23})\), as well as a line of sigma-model \(\sigma\) that appears on the diagonal line \(m_1 = m_2 = m_3\). In the limiting case \(q = 0\), the phase diagram becomes equivalent to the \(k = F/2\) case where all the sigma-models, as well as the topological theories \(T_6\) and \(T_8\) trivialize. The phase diagram is then reduced to a two-dimensional phase diagram with three topological theories \(T_a, T_b,\) and \(T_d\), as well as a trivial theory \(SU(N)_0\).

**Case 3:** \(F/2 - p \leq k < F/2 - q\)

The theory has two bosonic dual descriptions. The topological phases are given by equations (2.1) and (2.3) except for \(T_6\) which becomes

\[
T_6 \rightarrow \tilde{T}_6 : SU(N)_{k-\frac{p}{2}+q} \leftrightarrow U(F/2 - q - k)_N .
\]  

(2.19)

\(T_8\) is also replaced by \(\tilde{T}_8\) as before.

The cuboid quantum regions exist in the following cases:

- \(m_2 \rightarrow +\infty\) and \(m_3 \rightarrow -\infty\): the theory is reduced to \(SU(N)_{k-\frac{F}{2}+\frac{p}{2}+q} + p \psi_1\) which has a sigma-model phase given by

\[
\tilde{\sigma}^1 = \frac{U(p)}{U(F/2 - q - k) \times U(k - F/2 + p + q)} .
\]  

(2.20)

- \(m_2, m_3 \rightarrow -\infty\): the theory has the same cuboid sigma-model as in equation (2.4).

- \(m_2 \rightarrow +\infty\) and \(m_1 \rightarrow -\infty\): the theory is reduced to \(SU(N)_{k-p/2+q/2} + (F-p-q) \psi_3\) with a sigma-model given by

\[
\tilde{\sigma}^3 = \frac{U(F-p-q)}{U(F/2-p+k) \times U(F/2-q-k)} .
\]  

(2.21)

- \(m_2, m_3 \rightarrow -\infty\): the theory has the same cuboid sigma-model \(\sigma^3\) as in equation (2.6).

The phases that appear when we take one of the masses to \(\pm\infty\) are as follows:

(i) \(m_1 \rightarrow +\infty\): this is similar to case 2, where we have a type I phase diagram with only topological phases.

(ii) \(m_1 \rightarrow -\infty\): this limit is different from the previous case, as in this range of \(k\) the value of \(k^-_1\) lies between \(F_1/2 - p\) and \(F_1/2 - q\), which gives a type IIII phase diagram. The sigma-models \(\sigma^d_{23}\) and \(\sigma_{78}\) remain the same, while a new phase \(\sigma_{46}\) appears between \(T_4, \tilde{T}_6\) and is given by

\[
\sigma_{46} : Gr(F_1/2 + k^-_1, F_1 - q) = \frac{U(F-p-q)}{U(F/2-p+k) \times U(F/2-q-k)} \equiv \tilde{\sigma}^5 .
\]  

(2.22)

(iii) \(m_2 \rightarrow +\infty\): the range of \(k^+_2\) within this range of \(k\) is \(F_2/2 - p \leq k^+_2 < F_2/2\) and the phase diagram is of type II with a new diagonal sigma-model \(\tilde{\sigma}^4_{13}\) given by

\[
\tilde{\sigma}^4_{13} : Gr(F_2/2 + k^+_2, F_2) = \frac{U(F-q)}{U(F/2+k) \times U(F/2-q-k)} .
\]  

(2.23)
The phases of case 3 are summarized in figure 6 and they are eight topological field theories \((T_{1-5}, \tilde{T}_6, T_7, \tilde{T}_8)\), five cuboid sigma-models \((\sigma_1^d, \tilde{\sigma}_1^d, \sigma_2^d, \tilde{\sigma}_2^d, \sigma_3^d)\), four planes of sigma-models \((\sigma_1^d, \sigma_2^d, \sigma_3^d, \sigma_4^d)\), as well as the one-dimensional sigma-model \(\sigma\). In the limiting case \(q = 0\), the Chern-Simons level range becomes \(F/2 - p \leq k < F/2 - q\) and the phase diagram is reduced to figure 3b.
**Case 4: \( F/2 - (p + q) \leq k < F/2 - p \)**

Following the same procedure, we found that this case has the same topological phases as in case 3 with an additional replacement of \( T_5 \) with \( \bar{T}_5 \) as

\[
T_5 \rightarrow \bar{T}_5 : SU(N)_{k-\frac{p}{2}+p} \leftrightarrow U(F/2 - p - k)_N.
\] (2.25)

The two mass asymptotic limits show that the theory has the following cuboid sigma-models

- \( m_1 \rightarrow +\infty \) and \( m_3 \rightarrow -\infty \):
  \[
  \sigma_2^c = \frac{U(q)}{U(F/2 - p - k) \times U(F/2 - 2p + q)}.
  \] (2.26)

- \( m_1 \rightarrow +\infty \) and \( m_2 \rightarrow -\infty \):
  \[
  \sigma_3^c = \frac{U(F - p - q)}{U(F/2 - p - k) \times U(F/2 - q + k)}.
  \] (2.27)

along with \( \bar{\sigma}_1^c, \bar{\sigma}_5^c, \) and \( \sigma_3^c \).

The six sides of the three-dimensional picture have the following phases:

(i) \( m_1 \rightarrow +\infty \): \( F_1/2 - q \leq k_1^+ < F_1/2 \) and the theory has a type II phase diagram with a diagonal sigma-model \( \bar{\sigma}_{23}^d \) given by

\[
\bar{\sigma}_{23}^d : Gr(F_1/2 + k_1^+, F_1) = \frac{U(F - p)}{U(F/2 + k) \times U(F/2 - p - k)}. \] (2.28)

This diagonal sigma-model separates two other sigma-models given by

\[
\sigma_{35} : Gr(F_1/2 - k_1^+, F_1 - q) = \frac{U(F - p - q)}{U(F/2 - p - k) \times U(F/2 - q + k)} \equiv \bar{\sigma}_5^c, \] (2.29)

\[
\sigma_{25} : Gr(F_1/2 - k_1^+, q) = \frac{U(q)}{U(F/2 - p - k) \times U(F/2 - q + k)} \equiv \bar{\sigma}_2^c. \] (2.30)

(ii) \( m_1 \rightarrow -\infty \): this limit is similar to case 3 with \( \sigma_{23}^d, \sigma_{78}, \) and \( \sigma_{46} \) appear as quantum phases.

(iii) \( m_2 \rightarrow +\infty \): this is also similar to case 3 with \( \sigma_{13}^d, \sigma_{46}, \) and \( \sigma_{26} \).

(iv) \( m_2 \rightarrow -\infty \): we have \( k_2^- < F_2/2 - p \) within this range of \( k \), which makes this limit to be of type III with \( \sigma_{13}^d, \sigma_{78}, \) and \( \sigma_{35} \).

(v) \( m_3 \rightarrow +\infty \): this remains of type I phase diagram just like in cases 2 and 3.

(vi) \( m_3 \rightarrow -\infty \): this limit gives a shifted level within the range \( F_3/2 - p < k_3^- < F_3/2 \), which makes this case a type II phase diagram. However, \( k_3^- \) is always negative in this range of \( k \), which requires a flip of the masses signs to get the right phase diagram. This allows sigma-models to appear for small \( m_2 \) but positive \( m_1 (\sigma_{25}) \) and small \( m_1 \) with positive \( m_2 (\sigma_{26}) \) instead of \( \sigma_{68} \) and \( \sigma_{58} \), as shown in figure 7f. \( \sigma_{25} \) and \( \sigma_{26} \) are the correct phases to appear in this limit as they are part of \( \bar{\sigma}_1^c \) and \( \bar{\sigma}_2^c \) while \( \sigma_{68} \) and \( \sigma_{58} \) are part of \( \sigma_1^c \) and \( \sigma_2^c \) which do not appear in this range of \( k \).
Figure 7: Phases of $SU(N)_{k + p \psi_1 + q \psi_2 + (F - p - q) \psi_3}$ with $F/2 - (p + q) \leq k < F/2 - p$.

Figure 7 summarizes the phases of case 4, which includes the following: the eight topological phases $(T_1, T_5, T_6, T_7, T_8)$, five cuboid sigma-models $(\sigma_{13}, \sigma_{23}^d, \sigma_{23}^d, \sigma_{13}^d, \sigma_{33}^d)$, five planes of sigma-models $(\sigma_{13}, \sigma_{23}^d, \sigma_{23}^d, \sigma_{13}^d, \sigma_{33}^d)$ together with the sigma-model line $\sigma$. In the limiting case $q = 0$, the phase diagram is equivalent to a theory with level $k = F/2 - p$ where all the sigma-models and the topological theories $T_2$ and $T_5$, trivialize. The phase diagram is then reduced to a two-dimensional phase diagram with three topological theories $T_a, T_b$, and $T_c$, as well as a trivial theory $SU(N)_0$.

Case 5: $0 \leq k < F/2 - (p + q)$

This last possible range of $k$ has a three-dimensional phase diagram with the same topological phases as in case 4 except that $T_2$ is now

$$T_2 \rightarrow \tilde{T}_2 : SU(N)_{k - \frac{F}{2} + p + q} \rightarrow U(F/2 - p - q - k)_{N}.$$  \hspace{1cm} (2.31)

The topological theories come along with $\sigma_3^c, \tilde{\sigma}_3^c, \tilde{\sigma}_5^c$, and a new cuboid region when we send $m_1, m_2 \rightarrow +\infty$ given by

$$\tilde{\sigma}_3^c = \frac{U(F - p - q)}{U(F/2 + k) \times U(F/2 - p - q - k)}.$$ \hspace{1cm} (2.32)

The one mass asymptotic limits are now
Figure 8: Phases of \(SU(N)k + p \psi_1 + q \psi_2 + (F - p - q) \psi_3\) with \(0 \leq k < F/2 - (p + q)\).

(i) \(m_1 \to +\infty: p/2 \leq k_1^+ < F_1/2 - q\) which gives a type III phase diagram with the quantum regions \(\sigma_{12}^d, \sigma_{35}\) as well as a new region for small \(m_3\) but positive \(m_2\):

\[
\sigma_{12} : \text{Gr}(F_1/2 + k_1^+, F_1 - q) = \frac{U(F - p - q)}{U(F/2 + k) \times U(F/2 - p - q - k)} \equiv \tilde{\sigma}_c^3. \quad (2.33)
\]

(ii) \(m_1 \to -\infty: |k_1^-| < F_1/2 - q\) giving a type III phase diagram as in case 4 with the phases \(\sigma_{12}^d, \sigma_{35}, \sigma_{78}\).

(iii) \(m_2 \to +\infty: q/2 < k_2^+ < F_2/2 - p\) and the phase diagram is of type III with the phases \(\sigma_{13}^d, \sigma_{46}, \text{and} \sigma_{12}\).

(iv) \(m_2 \to -\infty: \text{this is a type III phase diagram with} \sigma_{13}^d, \sigma_{78}, \text{and} \sigma_{35}\).

(v) \(m_3 \to +\infty: \text{type I phase diagram}\).

(vi) \(m_3 \to -\infty: \text{this limit has a type I phase diagram with no quantum regions as it satisfies} \ |k_3^-| > F_3/2.\)

We summarize the phases of case 5 in figure 8 and they are: eight topological field theories \((T_1, \tilde{T}_2, T_3, T_4, \tilde{T}_5, \tilde{T}_6, T_7, \tilde{T}_8)\), four cuboid sigma-models \((\sigma_{13}^d, \sigma_{23}^d, \sigma_3^5, \sigma_5^5)\), four planes of sigma-models \((\sigma_{13}^d, \sigma_{13}^d, \sigma_{23}^d, \sigma_{23}^d)\) and the one-dimensional sigma-model \(\sigma\). The limiting case \(q = 0\) reproduces the phase diagram in figure 3c.
3 Consistency checks

In this section, we zoom on the region around the blue critical points on our figures and use the boson/fermion duality adapted to our three-family case.

3.1 Planar sigma-models

Before we start looking at the bosonic phases, we give an alternative way of the reduction to the two-family case when two of the masses are equal. This reduction gives more insights into the nature of the diagonal sigma-models that appear in the three-family theory.

1. \((m_1 = m_2, m_3)\) plane:
   The theory is reduced to \(SU(N)_k + \bar{p} \psi_1 + (F - \bar{p}) \psi_3\) with \(\bar{p} = p + q\) fermions of mass \(m_1 = m_2\) and \(F - \bar{p}\) of mass \(m_3\). The phase diagram is then of type I in case 1, type II in cases 2, 3, and 4, and type III in case 5 of section 2. The phase diagrams are now reduced to the phases in figure 9. In figure 9b, the quantum phases are \(\sigma_{28}\) and \(\sigma_{28}\) where

   \[
   \sigma_{28} : \text{Gr}(F/2 - k, \bar{p}) = \frac{p + q}{U(F/2 - k) \times U(k - F/2 + p + q)} \equiv \sigma_{12}^d. \tag{3.1}
   \]

   We note that \(\sigma_{28}\) is equivalent to \(\sigma_{12}^d\), which clearly shows that \(\sigma_{12}^d\) is not a line of quantum phase but rather a plane, and it appears in cases 2, 3, and 4. The quantum regions in figure 9c are \(\sigma_{78}\) and \(\sigma_{12}\), given by equations (2.10) and (2.33).

![Figure 9: Phase diagrams in the limiting case \(m_1 = m_2\).](image)

2. \((m_1 = m_3, m_2)\) plane:
   The theory is reduced to \(SU(N)_k + q \psi_2 + (F - q) \psi_3\) with \(q\) fermions of mass \(m_1 = m_3\) and \(F - q\) fermions of mass \(m_2\). The phase diagram is now of type I in case 1, type II in case 2, and type III in cases 3, 4, and 5. The phase diagram is summarized in figure 10 with the following quantum regions: in figure 10b we have \(\sigma_{68}\) and \(\sigma_{38}\) where

   \[
   \sigma_{38} : \text{Gr}(F/2 - k, F - q) = \frac{F - q}{U(F/2 - k) \times U(F/2 - q + k)} \equiv \sigma_{13}^d. \tag{3.2}
   \]
In figure 10c the quantum phases are $\sigma_{38}$ and $\sigma_{16}$ where
\[
\sigma_{16} : Gr(F/2 + kk, F - q) = \frac{F - q}{U(F/2 + k) \times U(F/2 - q - k)} \equiv \bar{\sigma}_{13}^d.
\] (3.3)

We conclude that the diagonal sigma-model $\sigma_{13}^d$ appears in all the cases except case 1 while $\bar{\sigma}_{13}^d$ appears only in cases 3, 4, and 5 of section 2.

3. $(m_1, m_2 = m_3)$ plane:

The theory is reduced to $SU(N)_k + p \psi_1 + (F - p) \psi_3$ where $\psi_2 = \psi_3$. The phase diagram is of type I in case 1, type II only in cases 2 and 3 and type III in cases 4 and 5. The phase diagram is summarized in figure 11, wherein figure 11b, the quantum regions are $\sigma_{48}$ and $\sigma_{48}$ with
\[
\sigma_{48} : Gr(F/2 - k, F - p) = \frac{F - p}{U(F/2 - k) \times U(F/2 - p + k)} \equiv \sigma_{23}^d.
\] (3.4)

In figure 11c the quantum phases are $\sigma_{48}$ and $\sigma_{48}$ where
\[
\sigma_{48} : Gr(F/2 + k, F - p) = \frac{F - p}{U(F/2 + k) \times U(F/2 - p - k)} \equiv \bar{\sigma}_{23}^d,
\] (3.5)

which shows that the diagonal sigma-model $\sigma_{23}^d$ appears cases 2 and 3 while $\bar{\sigma}_{23}^d$ appears in cases 4 and 5.

3.2 Matching the bosonic phases

Near the critical points, the fermionic theory with three families of fermions is conjectured to have a bosonic dual description with gauge group $U(n)$ and three sets of scalar fields in the fundamental representation of the gauge group which are $p \phi_1$, $q \phi_2$ and $(F - p - q) \phi_3$. In $U(n)$, $n$ and $l$ take the value $k \pm F/2$ and $\pm N$, respectively. We split the $F$ scalars into three sets which can acquire independent mass deformations to be denoted by $M_i^2$, $i = 1, 2, 3$.

This bosonic theory has six gauge invariants operators which can be written in terms of the three scalars as
\[
X = \phi_1 \phi_1^\dagger, \quad Y = \phi_2 \phi_2^\dagger, \quad Z = \phi_3 \phi_3^\dagger, \\
U = \phi_1 \phi_1^\dagger, \quad W = \phi_1 \phi_3^\dagger, \quad T = \phi_2 \phi_3^\dagger.
\] (3.6)
where $X$, $Y$, and $Z$ are positive semidefinite diagonal Hermitian matrices of dimensions $p$, $q$, and $F - p - q$, respectively. We consider a scalar potential for the critical theory including up to quartic order in the scalar field, which is further deformed by symmetry breaking mass operators. Written in terms of the six gauge invariants operators, this is

$$V = M_1^2 \text{Tr} X + M_2^2 \text{Tr} Y + M_3^2 \text{Tr} Z + \lambda (\text{Tr}^2 X + \text{Tr}^2 Y + \text{Tr}^2 Z + 2 \text{Tr} X \text{Tr} Y + 2 \text{Tr} X \text{Tr} Z) + \mu (\text{Tr} X^2 + \text{Tr} Y^2 + \text{Tr} Z^2 + 2 \text{Tr} U U^\dagger + 2 \text{Tr} W W^\dagger + 2 \text{Tr} T T^\dagger), \quad (3.7)$$

where $\lambda$ and $\mu$ are Lagrange multipliers. The quartic couplings are chosen such that the full $U(F)$ flavour symmetry is preserved. We choose $\mu \geq 0$, which requires $\mu + \min(n, F) \lambda > 0$ for the potential to be bounded from below.

Consider that $X$, $Y$, and $Z$ have $r_1$, $r_2$, and $r_3$ degenerate eigenvalues $x$, $y$, and $z$ respectively such that

$$\text{Tr} X = r_1 x, \quad \text{Tr} Y = r_2 y, \quad \text{Tr} Z = r_3 z. \quad (3.8)$$

The gauge group $U(n)$ is never Higgsed when the squared masses of $X$, $Y$, and $Z$ are non-negative. In this case, all the six gauge invariants operators vanish on-shell, so there is no scalar condensation. All matter fields are integrated out due to being massive, and one obtains a topological $U(n)_l$ theory in the infrared.

On the other hand, if at least one of the scalars has a negative mass squared, the minimum of the potential can be found by solving the equations of motion

$$M_1^2 + 2 \lambda (\text{Tr} X + \text{Tr} Y + \text{Tr} Z) + 2 \mu X = 0, \quad (3.9)$$
$$M_2^2 + 2 \lambda (\text{Tr} X + \text{Tr} Y + \text{Tr} Z) + 2 \mu Y = 0, \quad (3.10)$$
$$M_3^2 + 2 \lambda (\text{Tr} X + \text{Tr} Y + \text{Tr} Z) + 2 \mu Z = 0. \quad (3.11)$$

It also implies that $U = W = T = 0$. Solving the equations of motion gives the following eigenvalues.
\[ x = \frac{\lambda (M_2^2 r_2 + M_3^2 r_3) - M_1^2 [\mu + \lambda (r_2 + r_3)]}{2\mu [\mu + \lambda (r_1 + r_2 + r_3)]} , \tag{3.12} \]
\[ y = \frac{\lambda (M_1^2 r_1 + M_3^2 r_3) - M_2^2 [\mu + \lambda (r_1 + r_3)]}{2\mu [\mu + \lambda (r_1 + r_2 + r_3)]} , \tag{3.13} \]
\[ z = \frac{\lambda (M_1^2 r_1 + M_2^2 r_2) - M_3^2 [\mu + \lambda (r_1 + r_2)]}{2\mu [\mu + \lambda (r_1 + r_2 + r_3)]} . \tag{3.14} \]

It can be easily seen that minimizing the potential always requires the maximization of \( r_1 + r_2 + r_3 \). The ranks \( r_1, r_2, \) and \( r_3 \) are non-negative integers satisfying the following conditions

\[
\begin{align*}
    r_1 & \leq \min(n, p), \\
    r_2 & \leq \min(n, q), \\
    r_3 & \leq \min(n, F - p - q), \\
    r_1 + r_2 + r_3 & \leq \min(n, F) .
\end{align*}
\tag{3.15}
\]

The constraints in equation (3.15) and the sign of the mass squared of each gauge invariant operator defines the phases that appear in the bosonic theory. The bosonic theory experiences Higgsing of the gauge group or Higgsing plus spontaneous symmetry breaking except when \( M_1^2 \), \( M_2^2 \), and \( M_3^2 \) are all non-negative, as discussed above.

The phase diagram of the bosonic theory can be divided into five cases:

1. \( q \leq p \leq F - p - q \leq F < n; F < n \) does not allow a spontaneous symmetry breaking for the flavour symmetry \( U(F) \). We expect to have eight different regions to describe the phase diagram in this range. Region \( A \) describes the theory when all the mass squares are non-negative with no scalar condensation. The regions \( B, C, \) and \( D \) are reached when only one scalar mass squares is negative, allowing a condensation for \( \phi_1, \phi_2, \) or \( \phi_3 \), respectively. There are also three regions \( E, F, \) and \( G \) where two of the scalars condense before integrating them out. The last region, \( H \), describes a phase when the three scalars condense simultaneously. The phases of the bosonic theory in this range are summarized in table 1.

| Region | \( r_1 \) | \( r_2 \) | \( r_3 \) | Phase |
|--------|--------|--------|--------|-------|
| A      | 0      | 0      | 0      | \( U(n) \) |
| B      | \( p \) | 0      | 0      | \( U(n - p) \) |
| C      | 0      | \( q \) | 0      | \( U(n - q) \) |
| D      | 0      | 0      | \( F - p - q \) | \( U(n - F + p + q) \) |
| E      | \( p \) | \( q \) | 0      | \( U(n - p - q) \) |
| F      | \( p \) | 0      | \( F - p - q \) | \( U(n - F + q) \) |
| G      | 0      | \( q \) | \( F - p - q \) | \( U(n - F + p) \) |
| H      | \( p \) | \( q \) | \( F - p - q \) | \( U(n - F) \) |

Table 1: Phases of the bosonic theory with \( q \leq p \leq F - p - q \leq F < n \).

These phases are only allowed when \( n = F/2 + k \) where the phases reproduce the topological theories of equation (2.1), matching the phases of case 1 in the fermionic description.

2. \( q \leq p \leq F - p - q \leq n < F \): In this range there is a possibility of spontaneous symmetry breaking which allows sigma-models to appear in the bosonic phases. The sigma-models
appear when there is a condensation of more than one scalar. The region $E$, where $\phi_1$ and $\phi_2$ condense, splits into two subregions: $E_1$ where only the constraint on $r_1$ is saturated and $E_2$ where the constraint on $r_2$ is saturated. The same scenario occurs for regions $F$ and $G$, while region $H$ splits into three subregions, each of them is described when one of the constraints on $r_1$, $r_2$, or $r_3$ is saturated.

| Region | $r_1$ | $r_2$ | $r_3$ | Phase                        |
|--------|-------|-------|-------|------------------------------|
| A      | 0     | 0     | 0     | $U(n)_t$                     |
| B      | $p$   | 0     | 0     | $U(n-p)_t$                   |
| C      | 0     | $q$   | 0     | $U(n-q)$                     |
| D      | 0     | 0     | $F-p-q$ | $U(n-F+p+q)_t$               |
| $E_1$  | $p$   | $n-p$ | 0     | $Gr(n-p,q)$                  |
| $E_2$  | $n-q$ | $q$   | 0     | $Gr(n-q,p)$                  |
| $F_1$  | $p$   | 0     | $n-p$ | $Gr(n-p,F-p-q)$              |
| $F_2$  | $n-F+p+q$ | 0 | $F-p-q$ | $Gr(n-F+p+q,p)$              |
| $G_1$  | 0     | $q$   | $n-q$ | $Gr(n-q,F-p-q)$              |
| $G_2$  | 0     | $n-F+p+q$ | $F-p-q$ | $Gr(n-F+p+q,q)$              |
| $H_1$  | $p$   | $q$   | $n-p-q$ | $Gr(n-p-q,F-p-q)$           |
| $H_2$  | $n-F+p$ | $q$ | $F-p-q$ | $Gr(n-F+q,p)$                |

Table 2: Phases of the bosonic theory with $q \leq p \leq F - p - q \leq n < F$.

The phases of the bosonic theory in this range are summarized in table 2. For $n = F/2 + k$, these phases are $(\sigma_1^t, \bar{\sigma}_1^t, \sigma_2^t, \bar{\sigma}_2^t, \sigma_3^t, \bar{\sigma}_3^t)$ and are equivalent to the phases of the fermionic theories in cases 2, 3, and 4 of section 2. For $n = F/2 - k$, the bosonic phases are $(\sigma_3^t, \bar{\sigma}_3^t, \sigma_3^t, \bar{\sigma}_3^t)$ which match the fermionic phases in case 5 of section 2, as expected.

3. $q \leq p \leq n < F - p - q \leq F$: In this range, the regions A, B, and C are similar to the previous cases. Since $n < F - p - q$, $r_1$ is saturated to $n$ and region $D$ now shrinks to a smaller region $D_1$ with a sigma-model phase. The regions $E_1$ and $E_2$ remain the same as in the previous case while only the $F_1$, $G_1$, and $H_1$ subregions appear in this case. Each of the remaining

| Region | $r_1$ | $r_2$ | $r_3$ | Phase                        |
|--------|-------|-------|-------|------------------------------|
| A      | 0     | 0     | 0     | $U(n)_t$                     |
| B      | $p$   | 0     | 0     | $U(n-p)_t$                   |
| C      | 0     | $q$   | 0     | $U(n-q)$                     |
| $D_1$  | 0     | 0     | $n$   | $Gr(n,F-p-q)$                |
| $E_1$  | $p$   | $n-p$ | 0     | $Gr(n-p,q)$                  |
| $E_2$  | $n-q$ | $q$   | 0     | $Gr(n-q,p)$                  |
| $F_1$  | $p$   | 0     | $n-p$ | $Gr(n-p,F-p-q)$              |
| $G_1$  | 0     | $q$   | $n-q$ | $Gr(n-q,F-p-q)$              |
| $H_1$  | $p$   | $q$   | $n-p-q$ | $Gr(n-p-q,F-p-q)$           |

Table 3: Phases of the bosonic theory with $q \leq p \leq n < F - p - q \leq F$. 20
subregions shares the same phase with one of the other regions (e.g., the subregion $F_2$ has the same sigma-model as in $D_1$).

The phases of this case are summarized in table 3. The only allowed substitution, in this case, is $n = F/2 - k$, giving $(\bar{\sigma}^c_1, \sigma^c_2, \bar{\sigma}^c_3, \sigma^c_5, \bar{\sigma}^c_3)$. These quantum phases are equivalent to the phases of the fermionic theory in cases 4 and 5 of section 2.

4. $q \leq n < p \leq F - p - q \leq F$: In this case, the regions $A$ and $C$ do not experience any spontaneous symmetry breaking. Since $n < (p, F - p - q)$, both $r_1$ and $r_3$ are saturated to $n$ in regions $B$ and $D$, which shrink to smaller regions $B_1$ and $D_1$ with sigma-model phases. The subregions $F_1$ and $H_1$ now join the subregion $B_1$ to form a broader region sharing the same sigma-model, and the same happens for $F_2$ and $H_2$ which join the subregion $D_1$.

| Region | $r_1$ | $r_2$ | $r_3$ | Phase          |
|--------|-------|-------|-------|---------------|
| $A$    | 0     | 0     | 0     | $U(n)_l$      |
| $B_1$  | $n$   | 0     | 0     | $Gr(n, p)$    |
| $C$    | 0     | $q$   | 0     | $U(n - q)$    |
| $D_1$  | 0     | 0     | $n$   | $Gr(n, F - p - q)$ |
| $E_2$  | $n - q$ | 0     | $q$   | $Gr(n - q, p)$ |
| $G_1$  | 0     | $q$   | $n - q$ | $Gr(n - q, F - p - q)$ |

Table 4: Phases of the bosonic theory with $q \leq n < p \leq F - p - q \leq F$.

The phases of this case are summarized in table 4. Only $n = F/2 - k$ is allowed in this case which gives $(\sigma^c_1, \bar{\sigma}^c_2, \sigma^c_3, \bar{\sigma}^c_5)$, which match the phases of the fermionic theory in case 3 of section 2.

5. $n < q \leq p \leq F - p - q \leq F$: In this range, all the single condensation cases experience spontaneous symmetry breaking scenario where each of the corresponding ranks is saturated to $n$ and produces a sigma-model. The double and triple condensation cases share the same sigma-model as in the single condensation case.

| Region | $r_1$ | $r_2$ | $r_3$ | Phase          |
|--------|-------|-------|-------|---------------|
| $A$    | 0     | 0     | 0     | $U(n)_l$      |
| $B_1$  | $n$   | 0     | 0     | $Gr(n, p)$    |
| $C_1$  | 0     | $n$   | 0     | $Gr(n, q)$    |
| $D_1$  | 0     | 0     | $n$   | $Gr(n, F - p - q)$ |

Table 5: Phases of the bosonic theory with $n < q \leq p \leq F - p - q \leq F$.

The phases are now reduced to include only regions $A$, $B_1$, $C_1$, and $D_1$, as shown in table 5. For $n = F/2 - k$, the phases are $(\sigma^c_1, \sigma^c_2, \sigma^c_3)$, matching the fermionic phases in case 2 of section 2.

An additional and straightforward consistency check is to reduce the bosonic theory to the two-family case by putting $q = 0$ where the tables 1, 2, 3, 4, and 5 reduce to the tables in [17].
3.3 Perturbing the lower dimension sigma-models

We saw in the previous subsection how to match the phases of the bosonic and the fermionic theories around the critical points by considering perturbations in the bosonic dual descriptions. We now want to perturb the diagonal sigma-models in both two-dimensional and three-dimensional pictures. We do this by adding a mass term which explicitly breaks the flavour symmetry $U(F)$, as considered in [17] for the two-family case. The target space of the sigma model $\sigma$ is

$$\sigma : Gr(n, F) = \frac{U(F)}{U(n) \times U(F - n)},$$

(3.16)

where again $n$ can be either $F/2 + k$ or $F/2 - k$. $\sigma$ appears on the diagonal line of the three different limiting cases discussed in the previous subsection, which show that there exist three different possibilities of the mass deformation corresponding to deforming the mass of each of the scalars independently.

For $(m_1, m_2 = m_3)$ plane, the theory has $p \phi_1 + (F - p) \phi_2$ scalar fields. This allows us to perturb $\sigma$ by deforming the masses of $\phi_1$ or $\phi_2$. The result is independent of the choice of the scalar set that we deform so let us consider the deformation of the mass of $\phi_1$ by adding an infinitesimal mass squared $\delta M_1^2$ to $M_1^2$. Hence we have four possibilities:

- If $\delta M_1^2 > 0$ and $F - p > n$, $\phi_3$ condenses first Higgsing the gauge group $U(n)$, then one can integrate $\phi_1$ out and the resulting sigma-model has a Grassmannian $Gr(n, F - p)$.

- If $\delta M_1^2 > 0$ but $F - p < n$ the condensation of $\phi_3$ partially Higgs the gauge group down to $U(n - F + p)$ and then can be integrated out followed by integrating out $\phi_1$. This gives a sigma-model with a Grassmannian $Gr(F - p, p)$.

- For $\delta M_1^2 < 0$ and $p > n$, $\phi_1$ condenses first with a complete Higgsing of the gauge group which leads to a sigma-model with a Grassmannian $Gr(n, p)$.

- For $\delta M_1^2 < 0$ and $p < n$, the theory has a sigma-model with a Grassmannian $Gr(F - n, F - p)$.

By substituting $n = F/2 \pm k$ one obtains Grassmannians describing the sigma-models $\sigma_{23}^d$, $\sigma_1^c$, and $\bar{\sigma}_{23}^d$, which match the theories around $\sigma$ in figure 11c.

A similar result is obtained when considering the $(m_2, m_1 = m_3)$ plane, where we deform the mass of $\phi_1$ to perturb the sigma model leading to the Grassmannians $Gr(n, F - q)$, $Gr(F - n, q)$, $Gr(n, q)$, and $Gr(F - n, F - q)$. These Grassmannians correspond to the sigma-models $\sigma_{13}^d$, $\sigma_2^c$, and $\bar{\sigma}_{13}^d$ matching the phases around $\sigma$ in figure 10c. Lastly, the mass deformation of $\phi_3$ in the $(m_1 = m_2, m_3)$ plane leads to the Grassmannians $Gr(n, F - p - q)$, $Gr(F - n, p + q)$, $Gr(n, p + q)$, and $Gr(F - n, F - p - q)$, describing the target space of the sigma-models $\sigma_5^d$, $\sigma_{12}^d$, and $\bar{\sigma}_5^d$ which match the phases around $\sigma$ in the fermionic theory, as shown in figure 9c.

We now move to perturb the other diagonal sigma-models when we simultaneously deform the mass of two scalars. We consider the perturbation of pairs of diagonal sigma-models as follows:

1. Perturbing $\sigma_{23}^d$ and $\sigma_{23}^d$: we rewrite both theories in the general form $Gr(n, F - p)$ where $\sigma_{23}^d$ can be found by substituting $n = F/2 - k$ while $\sigma_{23}^d$ is found by substituting $n = F/2 + k$. This can be obtained by deforming the mass of $\phi_1$ with $\delta M_1^2 > 0$ and $F - p > n$ on the $(m_1, m_2 = m_3)$ plane. In addition, we deform the mass of $\phi_2$ and check the four possibilities.
Now we have a perturbation of $Gr(n, F_1)$ with $\delta M^2_2 > 0$ or $\delta M^2_2 < 0$. As in the previous discussion, this gives sigma-models with Grassmannians $Gr(n, F-p-q)$, $Gr(F-p-n, q)$, $Gr(n, q)$, and $Gr(F-p-n, F-p-q)$. For $n = F/2 - k$, these Grassmannians correspond to $\sigma^2_{\delta}, \sigma^3_{\delta},$ and $\sigma^2_{\bar{\delta}}$ matching the fermionic phases around $\sigma^2_{23}$, as shown in figures 5b, 6b, 7b, and 8b. For $n = F/2 + k$, these Grassmannians correspond to $\sigma^2_{\bar{\delta}}, \sigma^3_{\bar{\delta}},$ and $\sigma^2_{\bar{\delta}}$ matching the fermionic phases around $\sigma^2_{\bar{23}}$, as shown in figures 7a and 8a.

We should clarify that not all the Grassmannians are allowed when we make the substitution $n = F/2 \pm k$ where they are subject to $k$ being non-negative and the constraint of the first deformation, which is $F - p > n$ in this case.

2. Perturbing $\sigma^d_{13}$ and $\sigma^d_{13}$: these two theories have Grassmannians written in a single form $Gr(n, F - q)$ which is obtained by deforming the mass of $\phi_2$ with $\delta M^2_2 > 0$ for $F - q > n$ on the $(m_2, m_1 = m_3$) plane. By adding a deformation to the mass of $\phi_1$, the resulting sigma-models have Grassmannians $Gr(n, F-p-q)$, $Gr(F-q-n, p)$, $Gr(n, p)$, and $Gr(F-q-n, F-p-q)$. For $n = F/2 - k$, these Grassmannians correspond to $\sigma^2_{\delta}, \sigma^3_{\delta},$ and $\sigma^2_{\bar{\delta}}$ and match the phases around $\sigma^d_{13}$. For $n = F/2 + k$, the Grassmanians correspond to $\sigma^2_{\bar{\delta}}$ and $\sigma^3_{\bar{\delta}}$, which match the phases around $\sigma^d_{13}$ in the fermionic picture.

3. Perturbing $\sigma^d_{12}$: we start from $Gr(n, p + q)$ which can be read from deforming the mass of $\phi_3$ with $\delta M^2_3 < 0$ and $p + q > n$ on the $(m_1 = m_2, m_3)$ plane. An extra deformation of the mass of $\phi_1$ gives sigma-models, the only allowed sigma-models have Grassmannians $Gr(n, q)$ and $Gr(n, p)$. These Grassmannians correspond to $\sigma^2_{\delta}$ and $\sigma^3_{\delta}$, which are the only phases that appear around $\sigma^d_{12}$.

4 Conclusion

We investigated the IR behaviour of $SU(N)$ gauge theory coupled to three-families of flavours in the fundamental representation, extending the previous work for one family [12] and two families [17, 18]. Our description covers the full phase diagram in all the possible ranges of the Chern-Simons level and leads to a three-dimensional phase diagram which is well described semiclassically by topological and gapped phases for $k \geq F/2$. In addition to the topological phases, we encounter one-dimensional, planar and cuboid sigma-models, which are all quantum phases. The cuboid sigma-models are an intrinsic feature of the three-dimensional phase diagram which appear when one of the fermion masses become small.

We also provided consistency checks such as matching the phases of the bosonic dual descriptions with the fermionic ones and perturbing the diagonal sigma-model via mass deformations to match the off-diagonal lines phases. The reduction to the two-family case by describing the various planes when two of the masses are equal reproduces the results of [17, 18].

The order of the phase transitions that appear in the phase diagrams is only known in some limits to be second-order phase transitions [19–22]. It would be interesting to study such transitions in the two and three-dimensional phase diagrams when more than one fermion is massless and derive the type of phase transition. We leave this for future work.
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