THE TORSION GENERATING SET OF THE EXTENDED MAPPING CLASS GROUPS IN LOW GENUS CASES

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Abstract. We prove that for genus \( g = 3, 4 \), the extended mapping class group \( \text{Mod}^\pm(S_g) \) can be generated by two elements of finite orders. But for \( g = 1 \), \( \text{Mod}^\pm(S_1) \) cannot be generated by two elements of finite orders.

1. Introduction

Korkmaz has proved that the mapping class group \( \text{Mod}(S_g) \) can be generated by two elements of finite orders in [3]. Using the notation that \( \langle m, n \rangle \) (\( m, n \) are integers) to mean a group can be generated by two elements whose orders are \( m \) and \( n \) respectively, Korkmaz’s result says:

\[
\begin{array}{|c|c|}
\hline
\text{Mod}(S_g) & \text{torsion generating set consisting of two elements} \\
\hline
g = 1 & \langle 4, 6 \rangle \\
\hline
g = 2 & \langle 6, 10 \rangle \\
\hline
g \geq 3 & \langle 4g + 2, 4g + 2 \rangle \\
\hline
\end{array}
\]

It is an open problem listed in [4] that whether the extended mapping class group \( \text{Mod}^\pm(S_g) \) can be generated by two torsion elements. In [1], the author partially solved such a problem when the genus \( g \geq 5 \). In this paper, we deal with \( g = 1, 3, 4 \).

When \( g = 3, 4 \), the method and idea in the process of calculation in this paper are mostly the same as those in [1] and [3]. The reason for \( g = 3 \) and \( g = 4 \) should be treated separately is as the follow. When the genus is high, there will be plenty of space to find a simple closed curve satisfying two conditions: (1) it lies in the periodic orbit; (2) it does not intersect with some given curves. When the genus is less than 5, we cannot do this. So we use other treatment carefully. When \( g = 1 \), we use the presentation of \( GL(2, \mathbb{Z}) \) to prove it cannot be generated by two elements of finite orders. So we can summarize the result as follow:

2010 Mathematics Subject Classification. 57N05, 57M20, 20F38.

Key words and phrases. mapping class group, generator, torsion.
2. Preliminary

Notations.

(a) We use the convention of functional notation, namely, elements of the mapping class group are applied right to left, i.e. the composition $FG$ means that $G$ is applied first.

(b) A Dehn twist means a right-hand Dehn twist.

(c) We denote the curves by lower case letters $a, b, c, d$ (possibly with subscripts) and the Dehn twists about them by the corresponding capital letters $A, B, C, D$. Notationally we do not distinguish a diffeomorphism/curve and its isotopy class.

Humphries generators and the $(4g + 2)$-gon.

Humphries have proved the following theorem ([2]).

**Theorem 2.1.** Let $a_1, a_2, \ldots, a_{2g}, b_0$ be the curves as on the left-hand side of figure 1. Then the mapping class group $\text{Mod}(S_g)$ is generated by $A_i$'s and $B_0$.

The genus $g$ surface can be looked as a $(4g + 2)$-gon, whose opposite edges are glued together in pairs. $(4g + 2)$ vertices of the $(4g + 2)$-gon are glued to be two vertices.

We can also draw the curves $a_1, a_2, \ldots, a_{2g}, b_0$ on the $(4g + 2)$-gon as the right-hand side of figure 1. There is a natural rotation $\sigma$ of the $(4g + 2)$-gon that sends $a_i$ to $a_{i+1}$. In this paper, we will use the curve $c_0$ as figure 1 shows. Denote $b_i = \sigma^i(b_0), c_i = \sigma^i(c_0)$. They are also used in this paper.

We need the intersection numbers between the curves $a_j, b_k, c_l$. Consider the index $i, j, k$ in modulo $4g + 2$ classes. When viewing these curves in the

| $\text{Mod}\pm(S_g)$ | torsion generating set consisting of two elements |
|----------------------|-----------------------------------------------|
| $g = 1$              | impossible                                    |
| $g = 2$              | still unknown                                 |
| $g \geq 3$           | $\langle 2, 4g + 2 \rangle$                  |
(4g + 2)-gon, we need to be careful. Sometimes though two such curves meet at the vertex of the (4g + 2)-gon, they do not really intersect. We can perturb them a little to cancel the intersection point. The intersection numbers between \( a_j, b_k, c_l \) are listed as follow:

1. \( i(a_j, a_k) = 0 \) if and only if \( |j - k| \neq 1 \).
2. \( i(a_j, a_k) = 1 \) if and only if \( |j - k| = 1 \).
3. \( i(b_j, b_k) = 0 \) if and only if \( |j - k| \notin \{1, 2, 3, 2g - 2, 2g\} \).
4. \( i(b_j, b_k) = 1 \) if and only if \( |j - k| \in \{1, 3, 2g - 2, 2g\} \).
5. \( i(b_j, b_k) = 2 \) if and only if \( |j - k| = 2 \).
6. \( i(c_j, c_k) = 0 \) if and only if \( j = k \).
7. \( i(c_j, c_k) = 1 \) if and only if \( j \neq k \).
8. \( i(a_j, b_k) = 0 \) if and only if \( j - k \notin \{0, 4\} \).
9. \( i(a_j, b_k) = 1 \) if and only if \( j - k \in \{0, 4\} \).
10. \( i(a_j, c_k) = 0 \) if and only if \( k - j \notin \{-1, 0\} \).
11. \( i(a_j, c_k) = 1 \) if and only if \( k - j \in \{-1, 0\} \).
12. \( i(b_j, c_k) = 0 \) if and only if \( k - j \notin \{0, 1, 2, 3\} \).
13. \( i(b_j, c_k) = 1 \) if and only if \( k - j \in \{0, 1, 2, 3\} \).

Some torsion elements

Obviously we have \( \sigma^{4g+2} = 1 \). Take the reflection \( \tau \) of the regular \((4g+2)\)-gon satisfying \( \tau(b_0) = b_0 \). We can check \( (\tau B_0)^2 = 1 \). See figure 2.

![Figure 2.](image)

In [1] we know \( \text{Mod}^\pm(S_g) = \langle \sigma, \tau B_0 \rangle \) for \( g \geq 5 \). We will see it is also true for \( g = 3, 4 \).

3. The main result and the proof

**Theorem 3.1.** Let \( \tau, \sigma, B_0 \) as before. For \( g = 3, 4 \), \( \text{Mod}^\pm(S_g) = \langle \sigma, \tau B_0 \rangle \).

**Proof.** Denote the subgroup generated by \( \tau B_0 \) and \( \sigma \) as \( G \). We will prove that \( G \) includes all the elements in \( \text{Mod}^\pm(S_g) \). Similar to [1], The proof of the lemma has 4 steps.

Step 1. For every \( i, k \), we prove \( B_i B_k^{-1} \) is in \( G \).

Step 2. For every \( i, k \), we prove \( B_i A_k^{-1} \) is in \( G \).

Step 3. Using lantern relation, we prove that for every \( i \), \( A_i \) is in \( G \).
Step 4. $G = \text{Mod}^\pm(S_g)$.

The motivation of step 2 and step 3 is as follow. There is a lantern on the surface where the curves in the lantern relation appear as $a_1$, $a_3$, $a_5$, $b_0$, $b_2$, $e$, $f$ showed on the upper side of figure 3. The lantern relation $B_0B_2E = A_1A_3A_5F$ can be also written as $A_1 = (B_0A_3^{-1})(B_2A_5^{-1})(EF^{-1})$. So one Dehn twist can be decomposed into the product of pairs of Dehn twists. Draw the lantern in the $(4g + 2)$-gon as on the lower side of figure 3. We will find some of the pairs of Dehn twists we use can be expressed as the form $B_kA_i^{-1}$. When the $g \leq 2$, we cannot find a lantern on the surface.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{Figure 3.}
\end{figure}

The proof of Step 1:

The reason for $B_0B_k^{-1} \in G$ is $\sigma^k(\tau B_0)\sigma^k(\tau B_0) = B_0B_k^{-1}$. After conjugating with $\sigma^i$, we have for every $i, k$, $B_iB_{i+k}^{-1}$ is in $G$.

The proof of step 2:

Suppose the genus $g = 4$.

We already know $b_{11}$ does not intersect with $b_0$ or $b_6$. So $B_{11}B_6^{-1}$ maps the pair of curves $(b_{11}, b_0)$ to the pair of curves $(b_{11}, B_6^{-1}(b_0))$. Since $B_{11}B_6^{-1}$ is in $G$, $B_{11}(B_6^{-1}B_0^{-1}B_6)$ is in $G$. We also have for every $k$, $B_k(B_6^{-1}B_0^{-1}B_6) = (B_kB_1^{-1}) (B_{11}(B_6^{-1}B_0^{-1}B_6))$ is in $G$. See figure 4.

We know $b_1$ does not intersect with $b_5$. We can check $B_1B_5^{-1}B_6^{-1}(b_0) = a_5$. So $B_5^{-1}$ maps the pair of curves $(b_5, B_6^{-1}(b_0))$ to the pair of curves $(b_5, B_5^{-1}B_6^{-1}(b_0))$, $B_1$ maps the pair of curves $(b_0, B_5^{-1}B_6^{-1}(b_0))$ to the pair
of curves \((b_5, a_5)\). This means \(B_1B_5^{-1}\) maps the pair of curves \((b_5, B_6^{-1}(b_0))\) to the pair of curves \((b_5, a_5)\). See figure 5.

Hence \(B_5A_5^{-1}\) is in \(G\). After conjugating some power of \(\sigma\) and multiplying some \(B_iB_j^{-1}\), we have for every \(i, j\), \(B_iA_j^{-1}\) is in \(G\).

Suppose the genus \(g = 3\).

We know that \(b_0\) does not intersect with \(b_0\) or \(b_4\). So \(B_0B_4^{-1}\) maps the pair of curves \((b_0, b_0)\) to the pair of curves \((b_0, B_4^{-1}(c_0))\). We can also check when the genus is 3, \(c_0 = B_4^{-1}(b_0)\). So \(B_0C_0^{-1}\) is in \(G\). See figure 6.

After conjugating with some power of \(\sigma\) and multiplying some \(B_iB_j^{-1}\), we have for every \(i, j\), \(B_iC_j^{-1}\) and \(C_iB_j^{-1}\) are in \(G\). We also have for every \(i, j\), \(C_iC_j^{-1}\) is in \(G\).
We know $c_0$ does not intersect with $b_1$ or $b_2$. So $B_2C_0^{-1}$ maps the pair of curves $(c_0, b_1)$ to the pair of curves $(c_0, B_2(b_1))$. Then $C_0(B_2B_1B_2^{-1})$ is in $G$. For every $i$, $C_i(B_2B_1B_2^{-1})$ is also in $G$. See figure 7.

![Figure 7.](image)

We know $c_4$ does not intersect with $b_6$ or $B_2(b_1)$. So $C_4B_6^{-1}$ maps the pair of curves $(c_4, B_2(b_1))$ to the pair of curves $(c_4, B_6^{-1}B_2(b_1))$. Then $C_0(B_6^{-1}B_2B_1B_2^{-1}B_6) = a_2$. So $C_4A_2^{-1} = C_4(B_5^{-1}B_6^{-1}B_2B_1B_2^{-1}B_6) = a_2$. Conjugating with some power of $\sigma$ and multiplying $C_jC_k^{-1}$, we have for every $j, k$, $C_jA_k^{-1}$ is in $G$. Multiplying it by $B_iC_j^{-1}$, we have for every $i, k$, $B_iA_k^{-1}$ is in $G$. See figure 9.

![Figure 8.](image)

![Figure 9.](image)
The proof of step 3:
We want to show for every $i$, $A_i$ is in $G$.

Recall lantern relation, we have $B_0B_2E = A_1A_3A_5F$, or $A_1 = (B_0A_3^{-1}) (B_2A_5^{-1}) (EF^{-1})$, where $e$ and $f$ are the curves showed in figure 3. By the result of step 2, $B_0A_3^{-1}$ and $B_2A_5^{-1}$ are in $G$. What we need is to prove $EF^{-1}$ is also in $G$. Notice $EF^{-1} = (EB_i^{-1})(B_jF^{-1})$. We only need to show there exist some $i, j$ such that $EB_i^{-1}$ and $B_jF^{-1}$ are in $G$.

Suppose $g = 4$.

We can check that $f = B_3^{-1}A_6A_5A_4(b_0)$. We also know $b_7$ does not intersect with $a_4, a_5, a_6, b_3$. So $(B_7B_3^{-1})(A_6B_7^{-1})(A_5B_7^{-1})(A_4B_7^{-1})$ maps $(b_7, b_0)$ to $(b_7, f)$. Hence $B_7F^{-1}$ is in $G$. See figure 10.

Suppose $g = 3$.

The fact $f = B_3^{-1}A_6A_5A_4(b_0)$ still holds. When $g = 3$ we cannot find some $b_i$ that does not intersect with $a_4, a_5, a_6, b_3$ simultaneously. We use some curves $c_i$ instead.

At first we find $c_6$ does not intersect with $a_4, a_5, b_0$. So $(A_5C_6^{-1})(A_4C_6^{-1})$ maps $(c_6, b_0)$ to $(c_6, A_5A_4(b_0))$, $C_6(A_5A_4B_0A_4^{-1}A_5^{-1})^{-1}$ is in $G$. See figure 12.
$B_8(A_5A_4B_0A_4^{-1}A_5^{-1})^{-1} = (B_8C_6^{-1})(C_6(A_5A_4B_0A_4^{-1}A_5^{-1})^{-1})$ is also in $G$. Then we find $b_8$ does not intersect with $a_6, b_3$ or $A_5A_4(b_0)$. So $(B_8B_3^{-1})(B_8^{-1}A_6)$ maps $(b_8, A_5A_4(b_0))$ to $(b_8, B_3^{-1}A_6A_5A_4(b_0)) = (b_8, f)$. Hence $B_8F^{-1}$ is in $G$. See figure 13.

Similarly, The fact $e = A_2A_1A_4^{-1}B_1(a_5)$ still holds. When $g = 3$, we can find $c_i$ does not intersect with $a_1, a_2, a_4, a_5, b_1$. So $(A_2C_6^{-1}) (A_1C_6^{-1}) (C_6A_4^{-1}) (B_1C_6^{-1})$ maps $(a_5, c_6)$ to $(e, c_6)$. Hence $EC_6^{-1}$ is in $G$. And then multiply $C_6B_i^{-1}$, we have $EB_i^{-1}$ in $G$. See figure 14.

The proof of step 4:

Since both $B_iA_j^{-1}$ and $A_j$ are in $G$, by Humphries’s result, $G$ contains the mapping class group $\text{Mod}(S_g)$. Now $\tau B_0 \in G$ is an orientation reversing element. $\text{Mod}(S_g)$ is an index 2 subgroup of $\text{Mod}^\pm(S_g)$. So $G = \text{Mod}^\pm(S_g)$ for $g = 3, 4$. □
Theorem 3.2. For $g = 1$, $\text{Mod}^+(S_1)$ is $\text{GL}(2, \mathbb{Z})$. It cannot be generated by two elements of finite orders.

Proof. We only need to prove that $\text{PGL}(2, \mathbb{Z})$ cannot be generated by two elements of finite orders.

It is well known that $\text{PGL}(2, \mathbb{Z}) \cong D_6 \times \mathbb{Z}_2 D_4$, where $D_6$ and $D_4$ are the dihedral group of order 6 and order 4 respectively (see, for example, [5] or [6]). It has a presentation as

$$\text{PGL}(2, \mathbb{Z}) = \langle a, b, t \mid a^3 = t^2 = b^2 = 1, at = ta^2, bt = tb \rangle.$$

Every element $\alpha$ in $\text{PGL}(2, \mathbb{Z})$ can be written as a reduced form in one of the following 3 types:

1. $a^{i_1}b^{j_1} \ldots a^{i_k}b^{j_k}t$,
2. $b^{j_0}a^{i_1}b^{j_1} \ldots a^{i_k}b^{j_k}t$, or
3. $a^{i_1}b^{j_1} \ldots a^{i_k}b^{j_k}a^{j_{k+1}}t$.

Here each $i_n \in \{1, 2\}$ and each $j_n = 1$.

For an element in type (2), $b^{j_0}a^{i_1}b^{j_1} \ldots a^{i_k}b^{j_k}t$ can be conjugated to $a^{i_1}b^{j_1} \ldots a^{i_k}b^{j_k+j_0}t = a^{i_1}b^{j_1} \ldots a^{i_k}t$. So its conjugacy class is the same as an element in type (3) with a shorter word length.

For an element in type (3), $a^{i_1}b^{j_1} \ldots a^{i_k}b^{j_k}a^{j_{k+1}}t = a^{i_1}b^{j_1} \ldots a^{i_k}b^{j_k}ta^{2j_{k+1}}$ can be conjugated to $a^{i_1+2j_{k+1}}b^{j_1} \ldots a^{i_k}b^{j_k}t$. So its conjugacy class is the same as an element in type (1) or (2) with a shorter word length.

For an element in type (1), since the word length its power will be larger, it must not be of finite order. So an element of finite order in $\text{PGL}(2, \mathbb{Z})$ must be conjugated to one of the following 8 elements: $1, a, a^2, t, at, a^2t, b, bt$.

By adding in a new relation $ab = ba$, we get a quotient group homomorphic to the Cartesian product $D_6 \times \mathbb{Z}_2$, which is a finite group with the presentation

$$\langle a_1, t_1, b_1 \mid a_1^3 = t_1^2 = b_1^2 = 1, a_1t_1 = t_1a_1^2, b_1t_1 = t_1b_1, a_1b_1 = b_1a_1 \rangle.$$

For the convenience of calculation, we can think of $D_6 \times \mathbb{Z}_2$ as a permutation subgroup of the symmetric group $S_5$:

| element in $D_6 \times \mathbb{Z}_2$ | permutation | element in $D_6 \times \mathbb{Z}_2$ | permutation |
|-----------------------------------|-------------|-----------------------------------|-------------|
| 1                                 | (1)         | $b_1$                             | (45)        |
| $a_1$                             | (123)       | $a_1b_1$                          | (123)(45)   |
| $a_1^2$                           | (132)       | $a_1^2b_1$                        | (132)(45)   |
| $t_1$                             | (13)        | $a_1t_1$                          | (13)(45)    |
| $a_1t_1$                          | (23)        | $a_1^2t_1$                        | (23)(45)    |
We can check all the possible images in $D_6 \times \mathbb{Z}_2$ of the conjugacy classes in $PGL(2, \mathbb{Z})$ as follow:

| Conjugacy classes in $PGL(2, \mathbb{Z})$ | elements in $D_6 \times \mathbb{Z}_2$ |
|-----------------------------------------|----------------------------------|
| $a, a^2$                                | $a_1, a_1^2$                     |
| $t, at, a^2t$                           | $t_1, a_1t_1, a_1^2t_1$          |
| $b, bt$                                 | $b_1, b_1t_1$                    |

We have the following fact: if two elements in \{a_1, a_1^2t_1, a_1t_1, a_1^2t_1, b_1, b_1t_1\} can generate $D_6 \times \mathbb{Z}_2$, the only possible cases are $\langle a_1t_1, b_1t_1 \rangle$ and $\langle a_1^2t_1, b_1t_1 \rangle$.

Each one of $a_1t_1, a_1^2t_1, b_1t_1$ has order 2. If an element in $PGL(2, \mathbb{Z})$ is mapped to $a_1t_1, a_1^2t_1$, or $b_1t_1$, it must also have order 2. But two elements of order 2 can only generate a dihedral group, not $PGL(2, \mathbb{Z})$. □

**Remark 3.3.** Though $GL(2, \mathbb{Z})$ cannot be generated by two torsion elements, it can be generated by two elements. In fact, since

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

we have

$$GL(2, \mathbb{Z}) = \langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle.$$  

So the extended mapping class group $GL(2, \mathbb{Z})$ for $g = 1$ case: (1) can be generated by two elements; (2) cannot be generated by two elements of finite orders. This is different from the case $g \geq 3$.

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