Geometric Interpretation of Thiemann’s Generalized Wick Transform

Guillermo A. Mena Marugán
Instituto de Matemáticas y Física Fundamental, C.S.I.C., Serrano 121, 28006 Madrid, Spain.

In the Ashtekar and geometrodynamic formulations of vacuum general relativity, the Euclidean and Lorentzian sectors can be related by means of the generalized Wick transform discovered by Thiemann. For some vacuum gravitational systems in which there exists an intrinsic time variable which is not invariant under constant rescalings of the metric, we show that, after such a choice of time gauge and with a certain identification of parameters, the generalized Wick transform can be understood as an analytic continuation in the explicit time dependence. This result is rigorously proved for the Gowdy model with the topology of a three-torus and for a whole class of cosmological models that describe expanding universes. In these gravitational systems, the analytic continuation that reproduces the generalized Wick transform after gauge fixing turns out to map the Euclidean line element to the Lorentzian one multiplied by an imaginary factor; this transformation rule differs from that expected for an inverse Wick rotation in a complex rescaling of the four-metric. We then prove that this transformation rule for the line element continues to be valid in the most general case of vacuum gravity with no model reduction nor gauge fixing. In this general case, it is further shown that the action of the generalized Wick transform on any function of the gravitational phase space variables, the shift vector, and the lapse function can in fact be interpreted as the result of an inverse Wick rotation and a constant, imaginary conformal transformation.

I. INTRODUCTION

The application of the canonical quantization program to the Ashtekar formulation of general relativity [1,2] is one of the most promising approaches to constructing a quantum theory of gravity. The Ashtekar variables are a densitized triad and a canonically conjugate connection. In these terms, the gravitational first-class constraints acquire a low-order polynomial form suitable for quantization. Owing to the fact that the Ashtekar connection is genuinely complex for Lorentzian metrics [1], one seems nonetheless bound to impose complicated reality conditions [1,3] in the quantization process in order to arrive at the physical, Lorentzian theory. This problem does not appear in the Euclidean sector of general relativity because the Ashtekar variables can be defined as real ones in this case.

Considerable progress has recently been made on the canonical quantization of gravity assuming Euclidean reality conditions for the Ashtekar variables [4]. Remarkably, the results achieved for Euclidean gravity turn out to be of significance for Lorentzian general relativity because there exists a well-established relation between both theories. This relation has been found by Thiemann [5] who constructed a generalized Wick transform (GWT) which maps the Euclidean constraint functionals to the Lorentzian ones without preserving the reality conditions. In particular, by means of this transform one can obtain Lorentzian physical states starting with the Euclidean quantum theory [5,6]. Procedures determining the physical, Lorentzian inner product have also been suggested [5]. In this sense, the transform discovered by Thiemann allows one to work only with real Ashtekar variables when quantizing general relativity.

The role of the GWT is in fact analogous to that played by the Wick rotation (WR) in other approaches to quantum gravity since they both provide a relation between the Euclidean and Lorentzian theories. In the path integral approach, e.g., it is assumed that gravitational quantum states and correlation functions can be defined by means of a sum over Euclidean histories, each of them weighed by the exponential of minus the Euclidean action [7]. This action, $I_{(E)}$, is obtained from the Lorentzian one, $S_{(L)}$, by means of a WR. There are various ways of implementing this rotation. For instance, one can carry out an analytic continuation from the Lorentzian time coordinate $t^{(L)}$ to $-it^{(E)}$, $t^{(E)}$ being the Euclidean time [7]. At least formally, this procedure generalizes to gravity the WR of the time parameter that is usually employed in ordinary quantum field theory. An alternative prescription consists in performing an analytic continuation of the lapse function $N$ to negative imaginary values [8] (i.e., $N \to -iN$, with $N$ positive), so that the diagonal time component of the metric is continued from the negative to the positive real axis [9]. These two prescriptions can be considered equivalent inasmuch as they both formally map the set of all Lorentzian geometries to its Euclidean counterpart and lead to the same expression for the gravitational Euclidean action $I_{(E)}$.

Explicitly, this action is given by $I_{(E)} = -iR \circ S_{(L)}$ where $R$ denotes the WR. For pure gravity, the standard Hilbert-Einstein action for Lorentzian metrics is [7,10]

$$S_{(L)} = \frac{1}{2} \int_{\mathcal{M}} d^{4}x \sqrt{-g} R - \int_{\partial \mathcal{M}} d^{3}x \sqrt{h} k$$

$$= \frac{1}{2} \int_{\mathcal{M}} d^{4}x N \sqrt{h} \left((3)R + k_{a}^{b}k_{b}^{a} - k^{2}\right). \quad (1.1)$$

Here we have set $8\pi G = 1$, $G$ being the gravitational constant, $\partial \mathcal{M}$ (supposed to consist of sections of constant time) is the boundary of the four-manifold $\mathcal{M}$, $k$ is the
trace of the extrinsic curvature $k_b^a$, and $g$, $R$, $h$, and $(3)\bar{R}$ denote the determinants and curvature scalars of the four-metric and the induced three-metric. A WR leads then to

$$I_{(E)} = -\frac{1}{2} \int_{\mathcal{M}} d^4x \sqrt{g} R - \int_{\partial\mathcal{M}} \sqrt{h} k^b \partial_a h_b^a - k^b \partial_a h_b^a,$$

$$= \frac{1}{2} \int_{\mathcal{M}} d^4x N \sqrt{h} \left( -R + k^a_k b_a - k^2 \right). \quad (1.2)$$

Unlike the situation found in ordinary quantum field theory in flat spacetime, where the Euclidean action is positive, the above action is not even bounded from below [7,11]. Hence, Euclidean path integrals for gravity are in principle ill-defined. It has been argued that, in order to make these integrals converge (at least in the one-loop approximation), the conformal factor must be integrated over an appropriate complex contour other than the real axis [11,12]. If it were possible to obtain meaningful path integrals in this way, their analytic continuation back to the Lorentzian section would supply physical results for the Lorentzian theory [7].

The WR for gravity must not be considered to be a rigorously defined transformation which sends each particular Lorentzian metric to a real, Euclidean one. Indeed, it is known that the analytic continuation of a Lorentzian metric does not generally admit a section on the complexified spacetime on which the metric has Euclidean signature and is real [13]. One must rather understand the WR as a series of transformation rules for the lapse function, the shift vector, and the gravitational phase variables (obtained, e.g., by any of the prescriptions commented above) such that they formally map the Lorentzian abstract line element to the Euclidean one and the Lorentzian action for gravity to the Euclidean action multiplied by a factor of $i$. It is in this sense that we will refer to the gravitational WR from now on.

On the other hand, note that the WR is used to transform a sum over Lorentzian configurations into a sum over Euclidean histories, the latter expected to be more manageable. As we have said, it is not assumed that each Lorentzian configuration has a real Euclidean counterpart which can be reached by means of a WR; only the path integrals over all configurations, either Lorentzian or Euclidean, are supposed to be related [13]. A similar philosophy underlies the introduction of the GWT. Instead of dealing with Lorentzian gravity, whose reality or complexification of the time structure, the former is a map on functions on the phase space of general relativity whose definition does not seem to rest on the availability of a time coordinate and which results in complexifying the Ashtekar variables [5]. Given the analogies between the GWT and the WR, one might nonetheless expect that there exists a close relation between the actions of these two transformations. In this work, we find this relation and show that it allows one to reach a geometric interpretation of the GWT. In fact, there has been some confusion in the literature about the existence of such a relation. When Thiemann introduced his transformation, he claimed that its action was a phase space WR [5], even though the GWT preserves Poisson brackets, while the WR does not. In addition, it was suggested in Ref. [6] that a simple spacetime interpretation of the GWT in terms of a WR could not exist; in particular, it was argued that the lapse function and the shift vector transform under the GWT in a way that is different from what an interpretation of this kind would suggest. One of the purposes of our analysis is to clarify this point. We will see that the effect of the GWT in vacuum gravity can really be interpreted as the result of an inverse WR composed with a complex conformal transformation.

With the aim at gaining insight into the relation between the GWT and the WR, we will first study some particular gravitational systems. In doing this, we also want to discuss another issue, namely, whether, under appropriate circumstances, the GWT can further admit the spacetime interpretation of an analytic continuation in the explicit time dependence that could be related to the usual prescription for carrying out the WR. In fact, one would expect this to be the case in the following situation.

Given a vacuum gravitational system, one can always find a canonical set of variables such that the configuration variables are homogeneous functions of (only) the triad. The degree of homogeneity can always be chosen to be zero, except for one of the configuration variables, $x$. Taking into account that the GWT preserves Poisson brackets and rescales the triad by a constant complex factor [5], it is then possible to prove that there exists a choice of canonical momenta such that the only phase space variables affected by the GWT are the canonical pair $(x, p_x)$, and that $x$ gets multiplied by a complex constant under the GWT, while $xp_x$ remains invariant. It hence turns out that the effect of the GWT on the three-metric (and, subsequently, on the line element) amounts to a rescaling of $x$. Suppose then that, for our particular gravitational system, $x$ is a positive variable in the sector of non-degenerate metrics and that $x = \tau \in \mathbb{R}^+$ is an acceptable choice of intrinsic time ($\tau$ being the time coordinate). This would occur, e.g., when $x$ describes a volume, area, or length element expanding with the evolution. When the above conditions are satisfied, one would expect that, after gauge fixing, the action of the GWT on the line element could be viewed as a constant complex rescaling in the explicit $\tau$-dependence. This rescaling can be interpreted as an analytic continuation in the complex $\tau$-plane. If one wants to deal only with variables defined
over the whole real axis, one can replace \((x, p_x)\) with 
\((z = \ln x, p_z = x p_x)\), and fix \(z = t = \ln r \in \mathbb{R}\). In terms
of the time coordinate \(t\), the action of the GWT would be
interpretable as a constant complex translation in the
explicit \(t\)-dependence.

The situation described above occurs in fact in the
two kinds of vacuum gravitational systems that we are
going to analyze. We will first consider the family of
Gowdy cosmologies with the spatial topology of a three-
torus \([14]\), a model with two commuting spacelike Killing
vector fields. In this model, the surfaces that contain
the orbits of the Killing vector fields are in continuous
expansion. Hence, their area provides a natural choice
for the intrinsic time variable \(x\). The other case that we
will study is that of expanding universes. By these we
understand spacetimes such that the trace of the extrinsic
curvature is nowhere vanishing. The sections of constant
time expand then with the evolution, and we can use
their volume to define the intrinsic time variable \(x\).

The rest of the paper is organized as follows. In
Sec. 2 we summarize some basic results for Euclidean and
Lorentzian gravity, both in the triad and Ashtekar formu-
lations, and review the definition of the GWT. Sec. 3 is
devoted to the Gowdy cosmologies. The case of expand-
ing universes is studied in Sec. 4. For these two types
of gravitational systems we prove that, after gauge fixing
and with a certain identification of parameters, the GWT
can indeed be interpreted as an analytic continuation per-
ceptually. The latter are raised and lowered with the identity
matrix. The fundamental Poisson brackets are

\[
\left\{ \tilde{E}_i^a(x), K^b_a(y) \right\} = \delta^a_b \delta^{(3)}_i(x-y)
\]  

(2.1)

where \(\delta^a_b\) and \(\delta^{(3)}_i\) are Kronecker deltas, \(x\) and \(y\) two points
of \(\Sigma\), and \(\delta^{(3)}\) the delta function on \(\Sigma\).

Assuming that the induced metric \(h_{ab}\) is not degener-
ate, the variables \(\tilde{E}_i^a\) and \(K^b_a\) admit the following expres-
sion in terms of the triad \(e_i^a\) and the extrinsic curve:

\[
\tilde{E}_i^a = \sqrt{h} e_i^a, \quad K^b_a = k_{ab} e_i^b.
\]  

(2.2)

Here \(h\) is the determinant of \(h_{ab}\), with \(h^{ab} = e_i^a e_i^b\).

The Gauss and vector gravitational constraints \([6]\)
read, respectively,

\[
\mathcal{G}_i \equiv \epsilon_{ij} k^j_a \tilde{E}_i^a = 0, \quad \mathcal{V}_a \equiv 2 \tilde{E}_i^a D_a K^b_i = 0.
\]  

(2.3)

where \(\epsilon_{ijk}\) is the antisymmetrical symbol, the brackets
denote antisymmetrization, and \(D_a\) is the derivative
operator defined by \(D_a \tilde{E}_i^a = 0\). In particular, we have \([15]\)

\[
D_a K^i_b = \partial_a K^i_b - \Gamma^i_{ca} K^a_c + \epsilon^{ijk} \Gamma^k_i a K^a_c.
\]  

(2.4)

In this formula, \(\Gamma^c_{ab}\) are the Christoffel symbols \([16]\), and
\(\Gamma^i_a\) is the spin connection compatible with the triad \([15]\),
namely,

\[
\Gamma^i_a = -\frac{1}{2} \epsilon^{ijk} \tilde{E}_j^k (\partial_a \tilde{E}_k^i + \Gamma^i_{ca} K^a_c).
\]  

(2.5)

\(\tilde{E}_i^a\), being the inverse of the densitized triad.

\sim

The only remaining first-class constraint is the scalar
one, which is different for Lorentzian and Euclidean met-
rics \([6]\). We can nevertheless consider both cases simul-
taneously by introducing a parameter \(\epsilon\) such that

\[
\epsilon = \begin{cases} 
-1 & \text{for Lorentzian metrics} \\
1 & \text{for Euclidean metrics}
\end{cases}
\]  

(2.6)

The scalar constraint can then be written in the form

\[
S \equiv 2 \tilde{E}_i^a \tilde{E}_j^b K^i_a K^j_b - \frac{h}{\epsilon^2} (^{(3)} R) = 0.
\]  

(2.7)

The Ashtekar variables can be obtained from \(\tilde{E}_i^a\) and
\(K^i_a\) by means of a canonical transformation which results
in replacing the momenta \(K^i_a\) with the connection \([1, 2]\)

\[
A^i_a = \Gamma^i_a + \epsilon K^i_a.
\]  

(2.8)

Note that the Ashtekar connection \(A^i_a\) is real for Eu-
clidean gravity but complex in the Lorentzian case. The
Poisson-bracket structure is given by

\[
\left\{ \frac{1}{\epsilon} \tilde{E}_i^a(x), A^i_a(y) \right\} = \delta^a_b \delta^{(3)}_i(x-y),
\]  

(2.9)

and the gravitational constraints can be expressed as \([15]\)

\[
\mathcal{G}_i = \frac{1}{\epsilon} D_a \tilde{E}_i^a = \frac{1}{\epsilon} (\partial_a \tilde{E}_i^a + \epsilon_{ijk} A^j_a \tilde{E}_k^a) = 0,
\]  

(2.10)

\[
\mathcal{V}_a \equiv \frac{1}{\epsilon} \tilde{E}_i^a F_i^{ab} = 0
\]  

(2.11)

\[
S \approx \frac{1}{\epsilon^2} \epsilon^{ijk} \tilde{E}_i^a \tilde{E}_j^b F_k^{ab} = 0,
\]  

(2.12)

II. THE GENERALIZED WICK TRANSFORM

In the triad formulation of general relativity, one can take
as elementary phase space variables the densitized triad
\(\tilde{E}_i^a\) and the contraction of the extrinsic curvature
with the triad, \(K^i_a\). These variables are defined on a
three-manifold \(\Sigma\) and can be restricted to be real. Lower-
case Latin indices from the beginning and the middle of
the alphabet stand for spatial and SU(2) indices, repre-
sently. The latter are raised and lowered with the identity
metric. The fundamental Poisson brackets are
where the symbol \( \approx \) denotes equality modulo the Gauss constraint, \( D_a \) is the gauge-covariant derivative defined by the Ashtekar connection, and \( F_{ab}^a \) is its curvature:

\[
F_{ab}^a = \partial_a A_b^a - \partial_b A_a^a + c^{\epsilon} j_{jk} A_a^j A_b^k. \quad (2.13)
\]

In order to introduce the GWT, let us first suppose that \( C(t, \rho) \) is a given function on the gravitational phase space which may explicitly depend on a certain parameter \( t, \rho \) denoting a complete set of phase space variables. Using \( C \) as an infinitesimal generator, we can construct the following family of maps on functions \( f(\rho) \) on phase space:

\[
W(t, t_0) \circ f = \sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{P} \left[ f, \int_{t_0}^t C \right] (n). \quad (2.14)
\]

Here, \( \{ \cdot, \cdot \} (n) \) is the multiple Poisson bracket [5], and the symbols \( \mathcal{P}[\int_{t_0}^t \cdot] \) denote path-ordered integration over the explicit dependence on \( t \), the contour of integration being the segment that joins \( t_0 \) with \( t \) (in the complex \( t \)-plane) and \( t_0 \) being a fixed initial value of our parameter. Explicitly we can write the \( n \)-th term of the above expression as [17]

\[
\left\{ \cdots \left\{ f(\rho), \int_{t_0}^t dt_1 \cdots \int_{t_0}^t dt_3 C(t_1, \rho) \right\} \cdots , C(t_n, \rho) \right\}.
\]

Assuming that the series in Eq. (2.14) converges, each map \( W(t, t_0) \) is an automorphism on the algebra of functions on phase space that preserves the Poisson-bracket structure. However, \( W(t, t_0) \) generally modifies the reality conditions if the contour of integration for \( t \) or the function \( C \) along this contour are not real [5,6].

In the case that \( C \) is explicitly independent of \( t \), Eq. (2.14) reduces to

\[
W(t-t_0) \circ f = \sum_{n=0}^{\infty} \frac{(t-t_0)^n}{n!} \{ f, C \} (n). \quad (2.15)
\]

The GWT is defined as the map \( W \equiv W(t-t_0 = i\pi/2) \) obtained with the infinitesimal generator

\[
C = \int_{\Sigma} d^3x K^a_i E^a_i. \quad (2.16)
\]

From Eqs. (2.1) and (2.15) it is straightforward to check that

\[
W \circ \tilde{E}^a_i = i \tilde{E}^a_i, \quad W \circ K^a_i = -i K^a_i. \quad (2.17)
\]

Since the spin connection \( \Gamma^a_i \) is a function of only the densitized triad that is invariant under constant rescalings of \( E^a_i \), we then obtain that the GWT maps the Euclidean Ashtekar connection to the Lorentzian one. Moreover, one can now readily see that the GWT maps the Euclidean constraints (2.10)-(2.12) (with \( \epsilon = 1 \)) to those of Lorentzian gravity (i.e., those with \( \epsilon = -i \)):

\[
W \circ \mathcal{G}^{(E)}_i = \mathcal{G}^{(L)}_i, \quad W \circ \mathcal{V}^{(E)}_a = \mathcal{V}^{(L)}_a, \quad W \circ \mathcal{S}^{(E)} = \mathcal{S}^{(L)}. \quad (2.18)
\]

The superscripts \((E)\) and \((L)\) are used in this formula to refer to the Euclidean and the Lorentzian theory, respectively. The above conclusion about the transformation of the constraints can also be reached from the alternative expressions (2.3) and (2.7) by using the fact that the first equation in (2.17) implies the constant conformal transformation \( W \circ \rho_{ab} = i \rho_{ab} \) and by realizing that

\[
(3)R(\rho_{ab}) = -i (3)R(\rho_{ab}),
\]

\[
\det(i\rho_{ab}) = i^3 h. \quad (2.19)
\]

To pass to quantum theory, let us assume that the infinitesimal generator \( C \) has a well-defined quantum analogue \( \hat{C} \), and (setting \( \hbar = 1 \)) introduce the operator

\[
\hat{W} = \mathcal{P} \left[ \exp \left( i \int_{t_0}^{t_0+\pi/2} \hat{C} \right) \right] = \exp \left( -\frac{\pi}{2} \hat{C} \right). \quad (2.20)
\]

The classical map \( W \) has then the quantum mechanical counterpart [5]

\[
\hat{W} \hat{f} \hat{W}^{-1} = \sum_{n=0}^{\infty} \left( \frac{\pi}{2} \right)^n \frac{[\hat{f}, \hat{C}] (n)}{n!}. \quad (2.21)
\]

where \( \hat{f} \) is a generic operator. From Eq. (2.18) it follows that \( \hat{W} \) transforms Euclidean quantum constraints into Lorentzian ones (with an appropriate choice of factor ordering). This implies that a quantum state \( |\Psi\rangle \) is annihilated by the constraints of Euclidean gravity if and only if \( \hat{W} |\Psi\rangle \) is (at least formally) a solution to the Lorentzian constraints [5,6]. In this way the physical states of Lorentzian general relativity can actually be obtained from the Euclidean quantum theory.

### III. THE GOWDY MODEL

To gain insight into the kind of relation that can exist between the GWT and the WR, we will first consider the action of the GWT in a particular gravitational system, namely, the family of Gowdy cosmologies whose sections of constant time have the topology of a three-torus. This gravitational model has been studied in Ref. [18], both classically and quantum-mechanically, although the analysis carried out there was restricted to the sector of non-degenerate Lorentzian metrics. We first review that analysis and extend it to the case of Euclidean metrics. The GWT for the model is studied at the end of this section. We will show that there exists a choice of intrinsic time such that the GWT can be interpreted as analytic continuation made in the explicit time dependence. This continuation turns out to map the Euclidean line element of the model to its Lorentzian counterpart multiplied by a factor of \( i \). We will also comment on the validity of
these results for other choices of time. In particular, we will argue that the conclusion that the GWT maps the Euclidean line element to the Lorentzian one rescaled by a complex conformal factor is gauge-independent.

### A. Lorentzian and Euclidean models

The Gowdy universes are vacuum spacetimes which possess two commuting spacelike Killing vector fields and whose sections of constant time are compact [14]. When these sections have the topology of a three-torus, one can choose a global set of coordinates \((t, \theta, \omega, \nu)\) such that \(\partial_\omega\) and \(\partial_\nu\) are the two Killing fields and \(2\pi \omega, 2\pi \nu, \theta \in S^1\) (\(S^1\) being the unit circle).

It is possible to remove almost all of the non-physical degrees of freedom of the model by a gauge-fixing procedure. One starts by imposing the gauge-fixing conditions \([18, 19]\)

\[
\bar{E}_M^a = E_3^a = \arctan \left( \frac{E_1^a}{E_2^a} \right) = 0.
\]  
(3.1)

Capital Latin letters from the middle of the alphabet denote \(SU(2)\) indices equal to 1 or 2, whereas Greek letters refer to the spatial variables \(\omega\) and \(\nu\). The Gauss and vector constraints \(G_M, V_\alpha\) are then solved by

\[
A_\theta^M = A_3^\beta = 0,
\]  
(3.2)

while the remaining Gauss constraint \(G_3\) implies

\[
\epsilon^{MN3} A_\theta^M \bar{E}_N^\alpha + \partial_\theta \bar{E}_3^\alpha = 0.
\]  
(3.3)

Using Eqs. (2.2), (2.5), and (2.8), the conditions (3.1)–(3.3) can be seen equivalent to the demand that the extrinsic curvature be symmetric, and that all the non-diagonal elements of the induced metric and the extrinsic curvature vanish, except \(h_{\omega\omega}\) and \(k_{\omega\nu}\). After the above partial gauge fixing, one can set the \(\omega\) and \(\nu\) components of the shift vector equal to zero in the four-dimensional metric [18].

Let us next rename \(E = \bar{E}^0\) and define, for both the Lorentzian and Euclidean theories,

\[
\mathcal{K} = \frac{1}{\epsilon} (A_\theta^3 - \Gamma_\theta^3), \quad \bar{K}_\alpha^M = \frac{1}{\epsilon} (A_\alpha^M - \Gamma_\alpha^M),
\]  
(3.4)

\[
\bar{K}_\alpha^\beta = \bar{K}_{\alpha M} \bar{E}_M^\beta, \quad q^{\alpha\beta} = \bar{E}_M^\alpha \bar{E}_M^\beta,
\]  
(3.5)

with \(\epsilon\) being the parameter given by Eq. (2.6). In the sector of non-degenerate induced metrics, a canonical set of real phase space variables for the reduced Gowdy model obtained with our previous gauge fixing is [18]

\[
u = \frac{\sqrt{\det(q^{\alpha\beta})}}{q^\nu}, \quad v = \frac{q^\nu}{q^\nu},
\]  
(3.6)

\[
\frac{1}{2} \ln E,
\]  
(3.7)

\[
p_\alpha = \frac{1}{2} (\bar{K}_\omega^\alpha - \bar{K}_\omega^\nu), \quad p_\nu = \bar{K}_\omega^\nu,
\]  
(3.8)

\[
H_r = -\bar{K} E - 2p_\omega, \quad \bar{K} = \bar{K}_\alpha^\alpha.
\]  
(3.9)

These variables depend only on \(\theta \in S^1\) (and on the time coordinate \(t\)). Their non-vanishing Poisson brackets are

\[
\{u(\theta), p_\alpha(\theta')\} = \{v(\theta), p_\nu(\theta')\} = \{H_r(\theta), z(\theta')\} = \{w(\theta), \bar{K}(\theta')\} = \delta(\theta - \theta')
\]  
(3.10)

where \(\delta(\theta)\) is the Dirac delta on \(S^1\). This reduced model still possesses two first-class constraints, namely, the scalar one and the vector constraint \(V_\theta\).

It is worth commenting that, from our definitions, the phase space variables \(u, v, w, \text{ and } z\) are the same for the Lorentzian and Euclidean sectors, whereas their canonically conjugate momenta \((p_\alpha, p_\nu, \bar{K}, \text{ and } -H_r)\) differ, in the Euclidean theory, from their Lorentzian counterparts by a factor of \(-i\). Except for what concerns the Poisson-bracket structure and the reality conditions, this relation between the two sets of phase space variables allows one to translate the results for the Lorentzian model to the Euclidean case. This prescription can be employed in what follows to check the consistency of our analysis.

On the other hand, using Eq. (2.2) and recalling that \(h_{\theta\alpha} = k_{\alpha\alpha} = 0\), it is not difficult to show that

\[
\bar{K} = h k_{\alpha\beta} h^{\alpha\beta}, \quad E = \sqrt{\det(h_{\alpha\beta})}.
\]  
(3.11)

Thus the variables \(\bar{K}\) and \(E\) are proportional to and have the same sign as the trace of the extrinsic curvature and the area element of the surfaces of constant \(\theta\)-coordinate and time, respectively. In addition, it is known that

\[
\bar{K}_0 = \frac{1}{\sqrt{2\pi}} \oint \bar{K}
\]  
(3.12)

(where the symbol \(\oint\) denotes integration over \(\theta \in S^1\)) is a constant of motion of the system, that is, its value is preserved both by the dynamical evolution and by all gauge transformations [20]. Thanks to this fact, one can consistently restrict all considerations to the family of geometries with non-vanishing \(\bar{K}_0\) [18]. Moreover, taking into account the invariance of the four-geometries under time reversal, one can further restrict \(\bar{K}_0\) to be positive without loss of generality [18]. One can then use the \(\theta\)-diffeomorphism gauge freedom to make \(\bar{K}\) coincide with its average on \(S^1\), namely,

\[
\bar{K} = \frac{\bar{K}_0}{\sqrt{2\pi}}
\]  
(3.13)

The corresponding classical solutions describe universes with sections of constant time and \(\theta\)-coordinate that expand forever from an initial singularity. Actually, these solutions are those analyzed in the Lorentzian theory by Gowdy for the topology of a three-torus [14, 18]. The area of the surfaces with constant \(t\) and \(\theta\) provides hence a natural candidate for defining an intrinsic time. We can then set, e.g.,

\[
z = t.
\]  
(3.14)
This choice of intrinsic time is in fact equivalent to that made by Gowdy [14], which can be expressed as \( \tau = E \). From Eqs. (3.7) and (3.14) we then get \( \tau = e^\ell > 0 \). On the other hand, it is worth noticing that, since \( E \) scales as the densitized triad and is positive for non-degenerate metrics, our choice of time satisfies the conditions that, according to our discussion in the Introduction, should guarantee that the GWT can be interpreted after gauge fixing as an analytic continuation in the explicit time dependence. We will see later in this section that such a spacetime interpretation is actually feasible.

Eqs. (3.13) and (3.14) can be imposed as gauge-fixing conditions to eliminate almost all of the non-physical degrees of freedom of our reduced model [18]. With this gauge fixing, the scalar constraint can be solved to obtain an expression of \( H_r \) in terms of the time coordinate and the variables \( K_0, u, p_u, v, \) and \( p_v \). Besides, the homogeneous part of the vector constraint \( V_0 \) implies

\[
w = \frac{w_0}{\sqrt{2\pi}} - \sum_{n=-\infty, \neq 0}^{\infty} \Pi_n e^{i n \theta},
\]

whereas its homogeneous part, which reads \( \Pi_0 = 0 \), remains as the only constraint of the system [18]. Here \( \Pi_n = \oint \Pi e^{-i n \theta} / \sqrt{2\pi} \) are the Fourier coefficients of

\[
\Pi = \partial_\theta u \ p_u + \partial_\theta v \ p_v.
\]

In Eq. (3.15) the zero-mode \( w_0 \) is left undetermined. This mode is a homogeneous degree of freedom of the reduced model attained with our gauge fixing. Its canonically conjugate momentum is \( \hat{K}_0 \). Since \( \hat{K}_0 \in \mathbb{R}^+ \), we can replace the canonical pair \( (w_0, \hat{K}_0) \) by

\[
(0, \ c_0 = \ln \hat{K}_0).
\]

The domain of definition of these variables is the entire real axis, and they satisfy \( \{b_0, c_0\} = 1 \). Finally, the conditions (3.13) and (3.14) are compatible with the dynamical evolution provided that the densitized lapse function \( N \) is given by \( \sqrt{2\pi} / \hat{K}_0 \), and the \( \theta \)-component of the shift vector \( N^\theta \) depends only on time [18]. A suitable redefinition of the \( \theta \)-coordinate sets then \( N^\theta \) equal to zero [18].

The reduced model that results from our gauge fixing admits, as a canonical set of real variables, the set formed by \( b_0, c_0 \) and the fields on the unit circle \( u, p_u, v, \) and \( p_v \). There exists a homogeneous constraint: \( \Pi_0 = 0 \). The time evolution is generated in the model by the reduced Hamiltonian density \( H_r \) (i.e., minus the momentum of the time variable) obtained by solving the scalar constraint. On the other hand, to check that the Hamiltonian density \( H_r = \pm \hat{K}_0 \) coincides in the Lorentzian case with that found in Ref. [18], it suffices to use the definition of \( K \) and the fact that, once the conditions (3.1) are imposed,

\[
\Gamma^3_0 E = \frac{1}{2} e^{-u^\prime/2} \partial_\theta v.
\]

The canonical quantization of this reduced model has been discussed in Ref. [18] by choosing as elementary variables \( b_0, c_0 \) and the Fourier coefficients \( (u_n, p_u^n, v_n, p_v^n) \) of the fields \( (u, p_u, v, p_v) \). The quantum states were represented by analytic functionals \( \Psi \) of the variables \( (c_0, u_n, v_n) \) \([n = 0, \pm 1, \ldots]\). These states can also depend on time. For the analysis of the GWT, however, it will prove convenient to make the following change of elementary variables:

\[
u'_0 = u_0 + 2\sqrt{2\pi} t, \quad b'_0 = b_0 - \sqrt{2\pi} e^{c_0} t.
\]

which is just a canonical transformation generated by the function

\[
F = p_{u_0}^0 (2\sqrt{2\pi} t - u'_0) - c_0 b'_0 - \sqrt{2\pi} e^{c_0} t.
\]

The momenta of \( u'_0 \) and \( b'_0 \) are then \( p_{u_0}^0 \) and \( c_0 \), respectively. Notice that the constraint \( \Pi_0 = 0 \) is invariant under the transformation (3.19) because \( \Pi, \) given by Eq. (3.16), does not depend on \( u_0 \) or \( b_0 \). With this change of variables, the line element of our gauge-fixed model can be written in the form [18]

\[
d\tau^2 = e^{2u + v'/2} \left[ \frac{2\pi}{e^2} e^{2u} e^{-2c_0} dt^2 + d\theta^2 \right] + e^{u - v'/2} \left[ dv^2 - 2v du dv + (v'' + v^2) dv^2 \right]
\]

where

\[
u' = \frac{u'_0}{\sqrt{2\pi}} + \sum_{n \neq 0} \frac{u_n}{\sqrt{2\pi}} e^{i n \theta} = u + 2t,
\]

\[
w' = e^{-c_0} \left( \frac{b'_0}{\sqrt{2\pi}} - \sum_{n \neq 0} \frac{\Pi_n e^{i n \theta}}{in} \right) = w - t,
\]

the last equality coming from Eqs. (3.15) and (3.17).

Since the function (3.20) depends on time, the dynamical evolution of our new set of variables is generated by the reduced Hamiltonian

\[
H^r = \oint H^r_r = \oint H_r + \partial_t F
\]

\[
= \oint \left( H_r + 2p_u - \frac{e^{c_0}}{\sqrt{2\pi}} \right)
\]

where we have used the definition of the Fourier coefficient \( p_{u_0}^0 \). Therefore we get that, with our gauge fixing,

\[
H^r = - \left( KE + \frac{\hat{K}_0}{\sqrt{2\pi}} \right).
\]

On the other hand, substituting the solution obtained for \( H_r \) from the scalar constraint [18] into Eq. (3.24), we arrive at the explicit expression

\[
H^r = - \sqrt{2\pi} e^{-c_0} \left[ 4 \left( p_u - \frac{e^{c_0}}{4\sqrt{2\pi}} \right)^2 + e^{u'} p_v^0 \right] - \frac{3e^{c_0}}{4\sqrt{2\pi}} + \frac{\sqrt{2\pi}}{16\pi^2} e^{2t} (\partial_\theta v')^2 + 4e^{-u'} (\partial_\theta v)^2.
\]
To conclude this subsection, let us comment that, starting from the quantum representation discussed for our model in Ref. [18], it is possible to construct a representation in which the quantum states are analytic functionals of the set of variables $(c_0, u_0', u_m, v_n)$ where $m = \pm 1, \pm 2, ..., n = 0, \pm 1, ...$. For each quantum state $\Phi$ in the former representation, define

$$\Phi(c_0, u_0', u_m, v_n, t) = \Psi(c_0, u_0 = u_0' - 2\sqrt{2\pi t}, u_m, v_n, t) \times \exp \left( i e^{c_0} \sqrt{2\pi t} \right).$$

(3.27)

Under this correspondence, the states $u_0' \Phi \equiv u_0' \Phi$ and $b_0' \Phi \equiv i\partial_{c_0} \Phi$ are mapped in the original representation to

$$u_0' \Phi \rightarrow (u_0 + 2\sqrt{2\pi t}) \Psi,$$

$$i\partial_{c_0} \Phi \rightarrow (i\partial_{c_0} - \sqrt{2\pi e^{c_0} t}) \Psi.$$  

(3.28)

These transformations are the quantum counterpart of the relations (3.19). Moreover, since the only constraint of this system has been represented in Ref. [18] by an operator $\Pi_0$ which involves neither multiplication by $u_0$ nor differentiation with respect to $c_0$, it follows that $\Phi$ is annihilated by $\Pi_0$ if the same happens to $\Psi$. In addition, it is straightforward to check that $\Psi$ is a solution to the Schrödinger equation $i\partial_t \Psi = H \Psi$, with $H$ a quantum operator representing the reduced Hamiltonian $\frac{1}{2} \bar{C} \bar{H} \bar{C}$, if and only if $\Phi$ satisfies the relation $i\partial_t \Phi = \bar{H}^T \Phi$ where

$$\bar{H}^T = \bar{H} - i2\sqrt{2\pi} e^{c_0} - i2\sqrt{2\pi} \partial_{u_0}'$$

(3.29)

is the quantum analogue of the Hamiltonian $H^T$ given by Eq. (3.24). Therefore Eq. (3.27) provides a well-defined transformation between the physical states of the two considered representations. Finally, it is not difficult to see that, for each constant and real value of $t$, the states $\Psi$ and $\Phi(c_0, u_0 = u_0 + 2\sqrt{2\pi t}, u_m, v_n)$ have the same norm with respect to the inner product constructed in Ref. [18], so that Eq. (3.27) can in fact be interpreted as a unitary transformation.

B. The transform

We will now consider the GWT for the Gowdy model with the topology of a three-torus. After the reduction (3.1)–(3.3) the infinitesimal generator $C$ of this transform [given by Eq. (2.16)] is expressed as

$$C = \oint (KE + \bar{K})$$

(3.30)

where we have used that $k_{0\omega} = k_{0\nu} = 0$ and that $2\pi \omega, 2\pi \nu \in S^1$. Taking then into account Eq. (3.25), the gauge fixing (3.13) and (3.14) leads to

$$C = -\oint H^T_E(E) = -H^T_E$$

(3.31)

where the label $(E)$ refers to the Euclidean model. Notice in this sense that, since the GWT acts in principle on functions of the Euclidean phase space variables, mapping the Euclidean constraints to the Lorentzian ones, the Hamiltonian density appearing in Eq. (3.31) must correspond to the Euclidean theory [i.e., $\epsilon = 1$ in Eq. (3.26)]. Hence we conclude that the infinitesimal generator of the GWT for the model coincides, after gauge fixing, with minus the generator of the dynamical evolution in Euclidean time $H^T_E$.

In the system reduction by the gauge-fixing procedure, Poisson brackets transform into Dirac ones. As a consequence, for any function $f$ that depends only on the reduced phase space variables $(c_0, b_0', u', v, \rho_1)$ (and maybe on time), the Poisson bracket $\{f, C\}$ becomes equal to $\{f, -H^T_E\}$. Therefore, when acting on functions on the reduced phase space, the action of the family of maps $W(t, t_0)$, introduced in Sec. 2, turns out to be given by Eq. (2.14) with $-H^T_E$ substituted for $C$. So far nothing has been specified about the nature of the parameter $t$ of this family of maps. Our proposal is to set this parameter equal to the time coordinate of our gauge fixing. The motivation for this proposal comes from the following considerations.

It is not difficult to check that, before imposing the gauge conditions (3.13) and (3.14), the phase space of the Gowdy model admits a canonical set of variables such that $z$ is the only element that does not commute under Poisson brackets with the infinitesimal generator of the GWT [given by Eq. (3.30)]. For instance, one can choose the variables $(u' = u + 2z, p_u, v, p_v, w = w - z, \bar{K}, \bar{H}' = -KE \bar{K}, z)$ [see Eqs. (3.6)–(3.9) and (3.22)–(3.25)]. Taking then into account that $e^{z}$ scales as the densitized triad, we conclude that, before gauge fixing, the effect of the GWT on any explicitly time-independent function on phase space is a shift in the $z$-dependence that replaces $z$ with $z + i\pi/2$. Once our time gauge is introduced, the dependence on $z$ becomes an explicit dependence on the time coordinate. Therefore, if the action of the GWT commutes with the gauge-fixing procedure, the GWT for the gauge-fixed model must be equivalent to a complex translation in the explicit time dependence by a factor of $i\pi/2$ in which the implicit time dependence is not modified. We will see that this is a result obtained by identifying the time coordinate with the parameter $t$ of the family of maps $W(t, t_0)$. Furthermore, we will show in Sec. 5 that in general relativity without symmetry reductions or gauge fixing the action of the GWT amounts to that of an inverse WR followed by a constant, complex conformal transformation. With the proposed identification of parameters, this is exactly the action that the GWT for our reduced model will have on the Euclidean line element. Thus our proposal guarantees that, on the line element, the process of model reduction and gauge fixing commutes with the GWT. On the other hand, we note that the proposed identification of parameters is devoid of significance before gauge fixing,
inasmuch as it leads to the same formulae for the GWT as one obtains by assuming that \( t \) is just an abstract parameter on which the generator \( C \) does not depend explicitly. With our gauge fixing, however, \( C \) is replaced in Eq. (2.14) by minus the reduced Euclidean Hamiltonian that depends explicitly on time. Owing to this dependence, Eq. (2.15) [with \(-H^T_{(E)} \) substituted for \( C \)] is no longer valid for our reduced system. Employing instead Eq. (2.14), we arrive at the following action of the GWT on functions on the reduced phase space:

\[
W_{t_0} \circ f = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \mathcal{P} \left[ f, \int_{t_0}^{t_0+i\frac{\pi}{2}} H^T_{(E)}(n) \right] \quad (3.32)
\]

where the path-ordered integration is over the explicit \( t \)-dependence of \( H^T_{(E)} \).

In particular, we see that the GWT for the reduced model depends on the choice of the initial Euclidean time \( t_0 \). More importantly, since \( H^T_{(E)} \) is the generator of the dynamical evolution, the map \( W_{t_0} \) can be understood as an analytic continuation in the complex plane of the time coordinate. In fact, by defining a Lorentzian time through \( t^{(L)} = t - i\pi/2 \) (where \( t \) is the Euclidean time), employing the relation \( H^T_{(E)}(t = t^{(L)} + i\pi/2) = H^T_{(L)}(t^{(L)}) \) (which follows from Eq. (3.26), with \( H^T_{(L)} \) the reduced Lorentzian Hamiltonian), and calling \( \rho \) to the complete set of (implicitly time-dependent) variables \((c_0, b_0', u', p_a, v, p_c)\), one can check that the action of \( W_{t_0} \) on any function \( f(t_0, \rho) \) on the reduced Euclidean phase space leads to the function on the Lorentzian phase space obtained from \( f \) by performing the analytic continuation \( t_0 \to t_0 + i\pi/2 \) just in the explicit time dependence, while leaving unaltered the implicit dependence on the time coordinate.

In quantum theory we expect the physical states \( \Phi(t) \) of the Euclidean \((E)\) and Lorentzian \((L)\) models to be related by the transformation \( \Phi^{(L)}(t_0) = W_{t_0} \Phi^{(E)}(t_0) \). According to Eq. (2.20) and taking into account our identification of parameters and the time dependence of the Hamiltonian, the operator \( W_{t_0} \) should be

\[
W_{t_0} = \mathcal{P} \left[ \exp \left( -i \int_{t_0}^{t_0+i\frac{\pi}{2}} H^T_{(E)} \right) \right] \quad (3.33)
\]

It is not difficult to prove [18] that the reduced Hamiltonian \( H^T_{(E)} \) commutes under Poisson brackets with the only constraint of the model \( \Pi_0 \), which is the same for the Euclidean and Lorentzian systems. Let us then assume that in quantum theory \( H^T_{(E)} \) is represented by an operator which commutes with the quantum constraint \( \Pi_0 \). Under this assumption, one can check that the Lorentzian state \( \Phi^{(L)}(t_0) = W_{t_0} \Phi^{(E)}(t_0) \) is indeed annihilated by \( \Pi_0 \) if \( \Phi^{(E)} \) is in the kernel of the constraint. Therefore the operator \( W_{t_0} \) provides a map between physical states.

Furthermore, if the Euclidean Hamiltonian \( H^T_{(E)} \) is self-adjoint on the Hilbert space of physical states, we can introduce a unitary evolution operator \( \hat{U}^{(E)} \) in the Euclidean quantum theory such that \( \Phi^{(E)}(t_1) = \hat{U}^{(E)}(t_1, t_0) \Phi^{(E)}(t_0) \). This operator has the form

\[
\hat{U}^{(E)}(t_1, t_0) = \mathcal{P} \left[ \exp \left( -i \int_{t_0}^{t_1} H^T_{(E)} \right) \right] \quad (3.34)
\]

Comparison of Eqs. (3.33) and (3.34) allows one to interpret \( W_{t_0} \) as the operator \( \hat{U}^{(E)}(t_0 + i\pi/2, t_0) \) corresponding to a continuation of the Euclidean time coordinate from \( t_0 \) to \( t_0 + i\pi/2 \).

Actually, the continuation \( t_0 \to t_0 + i\pi/2 \) transforms the Euclidean model into the Lorentzian one. To see this, notice that the two models would be completely equivalent if their Hamiltonian densities \( H'_{\tau} \) coincided. The only existing constraint \( \Pi_0 \) is the same in both cases, and the Euclidean and Lorentzian phase spaces are formally identical since they both admit a set of elementary variables with the same Poisson-bracket structure and domains of definition. On the other hand, \( H'_{\tau} \) given by Eq. (3.26) is analytic in time, while the constraint \( \Pi_0 \) and the elementary phase space variables do not display any explicit time dependence. It is then straightforward to check that the analytic continuation \( t_0 \to t_0 + i\pi/2 \) made in the explicit time dependence maps the reduced phase space, constraint and Hamiltonian of the Euclidean theory to those of the Lorentzian case.

Substitution of \( t_0 \) by \( t_0 + i\pi/2 \) in the explicit time dependence turns out to relate as well the Euclidean and Lorentzian line elements of the model. In Gowdy’s time coordinate \( \tau = e^{i\phi} \), this substitution translates into

\[
\tau \to i\tau, \quad \text{a transformation that resembles an inverse WR} \quad [7].
\]

Given Eq. (3.21), the above substitution sends the abstract Euclidean line element \( ds^2_{(E)} \) to the Lorentzian one multiplied by a constant conformal factor, namely

\[
ds^2_{(L)} = ids^2_{(E)}/\tau \quad \text{Hence, from the point of view of the set of four-geometries, the GWT can in fact be interpreted as a composition of a complex rescaling of the metric by a factor of \( i \) and an inverse WR which maps the Euclidean line element to its Lorentzian counterpart. This inverse WR can be identified with the analytic continuation \( \tau \to i\tau \) only if it is performed not in \( ds^2_{(E)} \), but in the explicit time dependence of the metric \( ds^2_{(E)}/\tau \) which belongs to the same conformal equivalence class.}

From the above discussion about the coincidence of the phase spaces and constraints of the reduced Euclidean and Lorentzian models, we expect that the corresponding quantum theories will basically differ just in possessing distinct Hamiltonian and evolution operators. The GWT, together with the Euclidean unitary evolution, implies that the Lorentzian evolution operator must be given by

\[
\hat{U}^{(L)}(t_1, t_0) = \hat{W}_t \hat{U}^{(E)}(t_1, t_0) \hat{W}_t^{-1} \quad (3.35)
\]

Substituting Eqs. (3.33) and (3.34), we then obtain that, at least formally,
\[ 
\dot{U}(t_1, t_0) = \mathcal{P} \left[ \exp \left( -i \int_{t_0}^{t_1} \dot{H}(E) \right) \right] 
\] (3.36)

where \( \Gamma \) is the contour in the complex \( t \)-plane formed by the three segments that join \( t_0 + i\pi/2 \) with \( t_0 \) with \( t_1 \), and \( t_1 \) with \( t_1 + i\pi/2 \). Assuming that the Euclidean Hamiltonian \( \dot{H}(E) \) depends analytically on \( t \), as it happens for its classical analogue, we can distort the contour \( \Gamma \) to the segment that goes from \( t_0 + i\pi/2 \) to \( t_1 + i\pi/2 \). Defining then the Lorentzian Hamiltonian by \( \dot{H}(L)(t) = \dot{H}(E)(t = t + i\pi/2) \), we finally get

\[ 
\dot{U}(t_1, t_0) = \mathcal{P} \left[ \exp \left( -i \int_{t_0}^{t_1} \dot{H}(L) \right) \right]. 
\] (3.37)

Note that the definition adopted for \( \dot{H}(L) \) is just a quantum version of the relation between the classical Hamiltonians of the Euclidean and Lorentzian models.

In conclusion, we have shown that it is possible to find an intrinsic time variable for the Gowdy model such that the action of the GWT for the reduced system amounts to an analytic continuation in the explicit time dependence. This continuation maps the Euclidean line element to the Lorentzian one rescaled by a factor of \( i \). For other equivalent choices of intrinsic time, of the form \( t' = F(t) \), the GWT would clearly be interpretable as the analytic continuation \( t' = F(t_0 + i\pi/2) \) performed in the explicit \( t' \)-dependence. Although Gowdy’s time (or any equivalent intrinsic time) provides a natural and physically relevant choice of the time coordinate for our model, there might also exist other non-equivalent choices of time. One could then ask whether our results would apply to these other possible choices of gauge as well. It is not difficult to convince oneself that the purely spacetime interpretation of the GWT as an analytic continuation in the explicit time dependence will not be reachable in a generic gauge. One can actually think of situations in which, after a choice of intrinsic time which does not satisfy the conditions that we have discussed in the Introduction, the action of the GWT would imply an analytic continuation both in the explicit and implicit time dependences. However, if the action of the GWT on the abstract line element can be defined before gauge fixing and the reduction of the system is consistently performed in the gauge-fixing procedure, in the sense that the GWT and the reduction commute (at least as far as the line element is concerned), the conclusion that the effect of the GWT on the Euclidean line element coincides with that of an inverse WR and a complex conformal transformation should be gauge-independent and therefore valid for the unreduced system. We will see that this is indeed the case in Sec. 5 where it will be proved that for general relativity with no gauge fixing the action of the GWT on the abstract line element is precisely that mentioned above.

### IV. Expanding Universes

In order to show that the results obtained in the previous section are not just an artifact of the particular gravitational system studied, we will now discuss the relation between the GWT and the WR for the case of expanding universes. By these, we will understand spacetimes whose sections of constant time possess an everywhere positive trace of the extrinsic curvature. Clearly, this hypothesis about the expansion of the universe will not be generally satisfied unless we restrict our attention to some specific class of gravitational models. We will nevertheless carry out our analysis without referring to any particular model reduction of general relativity. It is worth remarking that the analysis of expanding universes is of special interest for its applications in cosmology. The feasibility and implications of our hypothesis of expansion will be commented later in this section.

Let us start with the triad formulation of general relativity and restrict our considerations to the sector of non-degenerate three-metrics. We can first remove the gauge freedom associated with the Gauss constraint by requiring that the densitized triad be upper triangular. It is not difficult to check that this gauge fixing is well-posed and consistent provided that the induced metric is positive-definite. The Gauss constraint implies then that the extrinsic curvature must be symmetric. The gravitational system obtained with this gauge fixing can be described by the variables

\[ q^{ab} = \hat{E}^a_i \hat{E}^b_i = h k^{ab}, \]
\[ K_{ab} = K^a_i \hat{E}_{ib} = h^{-1} k_{ab}, \] (4.1)

for which the only non-vanishing Poisson brackets are

\[ \{q^{ab}(x), K_{cd}(y)\} = 2 \delta^{(3)}(x-y) \delta^a_c \delta^b_d. \] (4.2)

Here the parentheses in the lower indices denote symmetrization. Note that \( q^{ab} \) and \( K_{ab} \) are symmetric tensor densities of weights 2 and \(-1\), respectively. A canonical set of real phase space variables is then

\[ q^{11}, K_{11}, q^{22}, K_{22}, q^{33}, K_{33}, q^{12}, K_{12}, q^{13}, K_{13}, q^{23}, K_{23}. \] (4.3)

The reality conditions \( q^{ab} \in \mathcal{R} \) are not sufficient to guarantee the positivity of \( h^{ab} \); one must also require

\[ \det(q^{ab}) = h^2 > 0, \quad q^{22} q^{33} - (q^{23})^2 > 0, \quad q^{33} > 0. \] (4.4)

It is therefore most convenient to replace \( q^{ab} \) by a new set of configuration variables such that the sector of positive-definite three-metrics corresponds to unrestricted real domains of definition. This can be achieved with the change
\[ u = \ln q^{33} - \frac{2}{3} \ln h, \quad z = \frac{1}{3} \ln h, \quad (4.5) \]
\[ v = \ln \left[ q^{22} q^{33} - (q^{23})^2 \right] - \frac{4}{3} \ln h, \quad A = q^{12} h^{-\frac{2}{3}}, \quad (4.6) \]
\[ B = q^{13} h^{-\frac{2}{3}}, \quad C = q^{23} h^{-\frac{2}{3}}. \quad (4.7) \]

These equations can be viewed as part of a canonical transformation in phase space. We will call the new canonically conjugate momenta \( (p_a, p_v, p_z, p_A, p_B, p_C) \). An explicit form of these momenta can be found in the Appendix where we also display the inverse of our canonical transformation. In particular, it turns out that
\[ p_z = K_{ab} q^{ab} = \sqrt{h} k_{ab} h^{ab}, \quad (4.8) \]
that is, \( p_z \) is the densitized trace of the extrinsic curvature.

The restriction to the sector of positive-definite three-metrics is now easily implemented by requiring that the new phase space variables run over the whole real axis. On the other hand, defining
\[ \sigma = p_a + 2 p_v + A p_A + B p_B + C p_C, \quad (4.9) \]
\[ U_a^b = K_{ac} q^{cb} + \frac{\delta^b}{3} (2 \sigma - p_z), \quad (4.10) \]
\[ \bar{q}^{ab} = e^{-2z} q^{ab}, \quad (4.11) \]
we can write the vector constraint as
\[ \mathcal{V}_a = -p_z \partial_a z + \frac{2}{3} \partial_a (p_z + \sigma) - \partial_b U^b_a + \bar{\Gamma}^c_{ab} U^b_c \quad (4.12) \]
with \( \bar{\Gamma}^c_{ab} \) being the Christoffel symbols of the metric \( \bar{q}_{ab} \) (the inverse of \( q^{ab} \)). The expressions for \( U_a^b \) and \( \bar{q}^{ab} \) in terms of our new variables are given in the Appendix. A remarkable point is that they are independent of the pair \( (z, p_z) \). So all the dependence of \( \mathcal{V}_a \) on these two variables has been made explicit in Eq. (4.12).

It is straightforward to see that \( \det \bar{q}^{ab} = 1 \). As a particular consequence, we have \( \bar{\Gamma}^b_{ab} = 0 \) (where summation over \( b \) is understood). Besides, one can check that \( U_a^b \) is a tensor density of weight 1. Therefore the last two terms in Eq. (4.12) can in fact be rewritten in the form \( -\bar{D}_a U^b_a \), with \( \bar{D}_a \) the covariant derivative of the metric \( \bar{q}^{ab} \).

In addition, the scalar constraint (2.7) can be written as
\[ S = \frac{2}{3} \left( p_z^2 - \Delta \right) - \frac{1}{\epsilon^2} e^{3z} (3)R, \quad (4.13) \]
where
\[ \Delta = \frac{1}{2} \left( 3 K_{ab} K_{cd} q^{ac} q^{bd} - p_z^2 \right). \quad (4.14) \]

An expression for \( \Delta \) in terms of the variables (4.5)–(4.7) and their momenta is obtained in the Appendix. It turns out that \( \Delta \) is a positive function on phase space which does not depend on \( z \) or \( p_z \). On the other hand, we get from our definitions that \( h_{ab} = e^z \bar{q}_{ab} \). Employing then the familiar formula for the change of the curvature scalar under a conformal transformation [16] and recalling that \( \bar{\Gamma}^b_{ba} = 0 \), we obtain
\[ e^z (3)R = (3) \bar{R} - 2 \partial_b (\bar{q}^{ab} \partial_a z) - \frac{1}{2} \bar{q}^{ab} \partial_a z \partial_b z, \quad (4.15) \]
\( (3) \bar{R} \) being the curvature scalar of the metric \( \bar{q}_{ab} \). Substituting this relation into Eq. (4.13), we arrive at a formula for the scalar constraint in which all the dependence on \( z \) and \( p_z \) is explicitly displayed.

Suppose now that we restrict our attention to a sector of general relativity in which the trace of the extrinsic curvature is everywhere nonzero. From Eq. (4.8), this implies that the momentum \( p_z \) does not vanish at any point of the spacetime. Using then the invariance of the theory under a time reversal, we can limit \( p_z \) to be positive without disregarding any allowed geometry. In general, the classical solutions with \( p_z > 0 \) describe universes whose sections of constant time expand with the evolution. An appealing choice of time gauge, motivated by this expanding behaviour, is to set the metric variable \( z \) (which is an increasing function of the determinant of the induced metric) equal to the time coordinate:
\[ z = t. \quad (4.16) \]
Actually, it is easy to see that this condition fixes the gauge freedom associated with the scalar constraint provided that \( p_z \neq 0 \). Given the definition of \( z \), our gauge fixing is equivalent to setting \( h^{1/3} = \tau = e^t > 0 \). One can then easily check that our choice of intrinsic time satisfies the conditions which have been argued to guarantee that the GWT can be interpreted as an analytic continuation in the explicit time dependence after gauge fixing. On the other hand, since the volume of the sections of constant time is proportional to \( h^{1/2} = e^{3t/2} \), the choice of this volume as a cosmological, intrinsic time for expanding universes is in fact equivalent to our choice (4.16).

After gauge fixing, the dynamical evolution of the reduced system is generated, modulo the vector constraint, by the Hamiltonian \( H^T = \int \Sigma d^3x H_r \), where the reduced Hamiltonian density \( H_r \) is equal to minus the momentum \( p_z \) obtained by solving the scalar constraint. From Eqs. (4.13) and (4.15), we get
\[ H_r = -p_z = -\sqrt{\Delta + \frac{3}{2 \epsilon^2} e^{2z} (3)\bar{R}}, \quad (4.17) \]
The square root in this formula is defined so that \( \sqrt{\tau} = 1 \), and we choose its cut to lie along the negative real axis. On the other hand, with our gauge fixing, the vector constraint (4.12) becomes
\[ \mathcal{V}_a = \frac{2}{3} \partial_a (\sigma - H_r) - \bar{D}_b U^b_a. \quad (4.18) \]
Since \( \Delta \geq 0 \) on the phase space and \( t \in \mathbb{R} \), we see from Eq. (4.17) that the requirement \( p_z > 0 \) is automatically
satisfied in the Euclidean (Lorentzian) theory for models in which \((3)\dot{R}\) is strictly positive (negative). Moreover, in this case the condition \(p_z \neq 0\), which allows the gauge fixing (4.16), is still verified under an analytic continuation of the parameter \(\epsilon\) as far as its real (imaginary) part is kept positive (negative). In this sense, for gravitational models with \((3)\dot{R} > 0\), the Lorentzian Hamiltonian density \(H^L_r\) could be defined as the limit of the right-hand side of Eq. (4.17) when \(\epsilon\) tends to \(-i\) from the right half of the complex plane. This limit can be identified with that of the Euclidean Hamiltonian density \(H^E_r(\epsilon = 1)\) when \(t\) is replaced by \(t + i\eta\) and \(\eta \to (\pi/2)^-\):

\[
H^L_r(t^{(L)}) \equiv \lim_{\eta \to (\pi/2)^-} H^E_r(t = t^{(L)} + i\eta). \tag{4.19}
\]

Here \(t^{(L)}\) denotes the Lorentzian time. Similarly, for models with \((3)\dot{R} < 0\), we could define

\[
H^L_r(t) \equiv \lim_{\eta \to (\pi/2)^-} H^L_r(t^{(L)} = t - i\eta). \tag{4.20}
\]

Finally, the condition \(p_z > 0\) is also satisfied in models with \((3)\dot{R} = 0\), both in the Euclidean and Lorentzian cases, except if \(p_u = p_v = p_A = p_B = p_C = 0\), since it is only then that \(\Delta\) [given by Eq. (A21)] vanishes. Our analysis would therefore also apply to this class of gravitational systems if the points with zero momenta could be suitably removed from the reduced phase space.

For the expanding universes considered, we find from Eqs. (2.16) and (4.8) that, with our choice of gauge, the infinitesimal generator of the GWT takes the form

\[
C = -\int_{\Sigma} d^3x \, H^E_r = -H^T_r, \tag{4.21}
\]

\(H^E_r\) being the Euclidean Hamiltonian density (4.17). Thus, modulo the vector constraint, \(C\) coincides with minus the generator of the Euclidean dynamical evolution in the reduced system. This relation is an analogue of Eq. (3.31) for the Gowdy model. Therefore, Eqs. (3.32) and (3.33) are still valid for the case of expanding universes if we follow again the proposal of identifying our time coordinate with the parameter \(t\) of the family of maps (2.14). Moreover, using Eq. (3.32) and applying a line of reasoning similar to that discussed for the Gowdy model, it is possible to show that, on functions \(f\) on the reduced phase space that are invariant under diffeomorphisms (so that \(\{f, \Sigma_a\} = 0\)), the action of the GWT can be understood as an analytic continuation in the explicit \(t\)-dependence. On the other hand, assuming that our gauge fixing is well-posed in the Euclidean sector and that the Hamiltonian \(H^T_r\) can be represented by a self-adjoint operator on the Hilbert space of Euclidean physical states (that are annihilated by the vector constraint), it is possible to introduce a unitary evolution operator \(\hat{U}(E)\) in the Euclidean quantum theory through Eq. (3.34). When acting on Euclidean physical states, the operator \(\hat{W}_{t_0}\) of the GWT can then be interpreted as the evolution operator \(\hat{U}(E)(t_0 + i\pi/2, t_0)\) that provides a continuation of the Euclidean time from \(t_0\) to \(t_0 + i\pi/2\).

In addition, let us suppose that the definition (4.19) is meaningful, e.g., for models with \((3)\dot{R} > 0\) (or with \((3)\dot{R} = 0\) if the points of the reduced phase space with vanishing momenta can be consistently removed). Since the only difference between the Lorentzian and Euclidean reduced systems is found in the Hamiltonian density \(H_r\) [see Eqs. (4.17) and (4.18)], we conclude that the classical Euclidean theory can actually be mapped to the Lorentzian one by means of the analytic continuation \(t \to t + i\pi/2\), performed in the explicit time dependence. This continuation must be understood as a limit of the continuation \(t \to t + i\eta\) when \(\eta \in R\) tends to \(\pi/2\) from the left-hand side.

In quantum theory, the Lorentzian evolution operator would be given again by Eq. (3.35), or, equivalently, by (3.36). For models with \((3)\dot{R} > 0\), e.g., the Euclidean Hamiltonian \(H^T_r\) is analytic with respect to its time dependence in the whole region of the complex \(t\)-plane

\[
\left\{ t + i\eta : t \in \mathbb{R}, 0 \leq \eta < \frac{\pi}{2} \right\} \tag{4.22}
\]

and possesses a well-defined limit when \(\eta\) tends to \(\pi/2\). Provided that the quantum Hamiltonian \(\hat{H}^T_r\) has also this analytic behaviour, we can deform the contour of integration \(\Gamma\) in Eq. (3.36) in such a way that it can be taken as the limit of the segment that joins \(t_0 + i\eta\) with \(t_1 + i\eta\) when \(\eta \to (\pi/2)^-\), \(t_0\) and \(t_1\) being the initial and final times appearing in the Lorentzian evolution operator \(\hat{U}(L)\). In this way we arrive at

\[
\hat{U}(L)(t_1, t_0) = \lim_{\eta \to (\pi/2)^-} \mathcal{P} \left[ \exp \left( -i \int_{t_0 + i\eta}^{t_1 + i\eta} \hat{H}^T_r \right) \right], \tag{4.23}
\]

which coincides formally with the operator

\[
\mathcal{P} \left[ \exp \left( -i \int_{t_0}^{t_1} \hat{H}^T_r \right) \right] \tag{4.24}
\]

if we define the quantum Lorentzian Hamiltonian \(\hat{H}^T_r\) by analogy with Eq. (4.19):

\[
\hat{H}^T_r(t^{(L)}) = \lim_{\eta \to (\pi/2)^-} \hat{H}^T_r(t = t^{(L)} + i\eta). \tag{4.25}
\]

Let us finally analyze the transformation of the line element under the substitution of the Euclidean time \(t\) by \(t + i\pi/2\) in the explicit time dependence. We have seen that, before fixing the time gauge, the induced metric is given by \(h_{ab} = e^2 \vec{q}_{ab}\), \(\vec{q}_{ab}\) being the inverse of the metric (A1)-(A2), which is independent of \(z\) and \(p_z\). On the other hand, the compatibility of our gauge-fixing condition with the dynamical evolution requires that

\[
1 = \frac{dz}{dt} = \left\{ z, \int_{\Sigma} d^3x \left( \frac{N}{2} S - N^a \Sigma_a \right) \right\} \approx -\frac{2}{3} N H_r + \frac{2}{3} \partial_a N^a, \tag{4.26}
\]
where $\mathcal{V}_a$ and $S$ are the constraints (4.12) and (4.13), the symbol $\approx$ denotes equality modulo these constraints and Eq. (4.16), and $H_r$ is given by Eq. (4.17). Since the lapse function is $\sqrt{h}N$, with $h = e^{3t}$, we find that, for $z = t$, the line element can be written as

$$
\begin{align*}
\text{ds}^2 &= \frac{e^{3t}}{(e^H)^2} \left( \frac{3}{2} - \partial_a N^a \right)^2 \text{dt}^2 \\
&\quad + e^t q_{ab} (dx^a + N^a \text{dt}) (dx^b + N^b \text{dt}).
\end{align*}
$$

(4.27)

Here we have assumed that our gauge fixing is acceptable, at least in the sense of a limit (as discussed above for gravitational models with definite sign of the curvature scalar $(\delta R)$, both in the Euclidean and Lorentzian cases.

From Eqs. (4.17) and (4.27) we conclude that the substitution of $t$ by $t + i\pi/2$ (in the explicit time dependence) sends the Euclidean line element to the Lorentzian one multiplied by a factor of $i$, as it happened in the Gowdy model. Therefore we again arrive at the conclusion that the action of the GWT on the set of Euclidean four-geometries can be interpreted as a composite of a conformal transformation by a constant complex factor and an inverse WR. This WR can be performed by substituting the Euclidean time $\tau = e^t$ with $i\tau$ in the explicit time dependence of the metric $ds_\Lambda^2 (E_j) / \tau$, which is a representative of the conformal equivalence class to which the Euclidean metric $ds_\Lambda^2 (E_j)$ belongs. As to the validity of our results for other possible choices of gauge, comments similar to those presented at the end of Sec. 3 apply to the present case as well.

V. THE TRANSFORM IN VACUUM GENERAL RELATIVITY

We will now study the action of the GWT and its relation with the WR in vacuum general relativity without imposing any gauge fixing nor symmetry or model reduction. In this general case, we do not have at our disposal any natural choice of time as a function of the gravitational phase space variables. This prevents us from identifying the infinitesimal generator of the GWT with the generator of the dynamical evolution in a particular time variable. Therefore one cannot achieve for the GWT the purely spacetime interpretation of just an analytic continuation in the explicit dependence on a suitable time coordinate. However, we will see that it is still possible to relate the GWT with an inverse WR. As we commented in the Introduction, it was initially far from clear that a relation between these two transformations could actually exist [5,6]. Our purpose in this section is to clarify this point and to show that the action of the GWT can in fact be interpreted as the result of performing an inverse WR and a constant, complex conformal transformation. Recall that we regard the Lorentzian phase space variables as a series of transformation rules for the lapse, shift and gravitational phase space variables that map the Lorentzian to the Euclidean abstract line element, and the Lorentzian action to the Euclidean one rescaled by a factor of $i$.

Let us start our discussion by analyzing the behaviour of the line element under the transform introduced by Thiemann. From Eqs. (2.2) and (2.17) we obtain

$$
W \circ h_{ab} = W \circ \left( \frac{\tilde{E}_{ai} \tilde{E}_{bi}}{\det(\tilde{E}_{ij})} \right) = -ih^{ab}.
$$

(5.1)

So the action of the GWT on the induced metric is simply a constant, complex rescaling: $W \circ h_{ab} = ih_{ab}$. On the other hand, we recall that the GWT is in principle defined as a map on functions on phase space, specially designed to send the Euclidean constraints (2.3) and (2.7) (with $\epsilon = 1$) to their Lorentzian counterparts. For functions which do not depend on the phase space variables, it is nevertheless trivial to extend the definition of the GWT by using Eq. (2.14). On this kind of functions, the GWT acts then as the identity operator. It therefore seems natural to assume that the densitized lapse function and the shift vector are invariant under the GWT,

$$
W \circ N = N, \quad W \circ N^a = N^a
$$

(5.2)

because $N$ and $N^a$ are not phase space variables, but just the Lagrange multipliers of the scalar and vector constraints. A different way to arrive at these transformation rules is to demand that the action of the GWT is consistent with the expression of the extrinsic curvature [16]

$$
k_{ab} = \frac{h^{bc}}{2N} \left[ \partial_t h_{ac} - D_a (N^d h_{cd}) - D_c (N^d h_{ad}) \right].
$$

(5.3)

Recalling that the lapse function is given by $N = \sqrt{h}N$ and using Eq. (5.1), we conclude that the GWT sends the right-hand side of Eq. (5.3) to

$$
\frac{1}{i\sqrt{h}N} \frac{h^{bc}}{W \circ N} \left[ \partial_t h_{ac} - D_a (W \circ N^d h_{cd}) - D_c (W \circ N^d h_{ad}) \right].
$$

(5.4)

In addition, Eqs. (2.2) and (2.17) imply

$$
W \circ k_{ab} = W \circ (h^{-\frac{\delta}{2}} K_{ai} \tilde{E}_{bi}) = \frac{1}{i\sqrt{h}} k_{ab}.
$$

(5.5)

One can then readily see that the expression (5.4) reproduces the GWT of the extrinsic curvature for any choice of densitized lapse function and shift vector if and only if Eq. (5.2) is satisfied.

Except for a sign in the transform of the densitized lapse, Eq. (5.2) reproduces the transformation rules deduced in Ref. [6] by imposing that the GWT maps the Euclidean action to the Lorentzian one. This difference of a sign is due to the unusual convention adopted in Ref. [6] for the sign of the Euclidean gravitational action,
which is the opposite of the standard one. There has also been some confusion [6] about a possible interpretation of the transformation rules (5.2). We will show at the end of this section that such rules are precisely those that one would obtain with the continuation $N \to i N$ (which corresponds to an inverse WR) when this is composed with the conformal transformation $ds^2 \to i ds^2$.

On the other hand, making use of Eqs. (5.1) and (5.2), we get

$$W \circ N = i \sqrt{\Omega} N. \quad (5.6)$$

It is now straightforward to check that the abstract Euclidean line element,

$$ds^2_{(E)} = N^2 dt^2 + h_{ab}(dx^a + N^a dt)(dx^b + N^b dt), \quad (5.7)$$

is mapped by the GWT to

$$W \circ ds^2_{(E)} = i ds^2_{(L)} \quad (5.8)$$

where $ds^2_{(L)}$ is the Lorentzian metric. Therefore, from the point of view of the set of four-geometries, the GWT can be considered as equivalent to an inverse WR that sends the Euclidean line element to the Lorentzian one, followed by a conformal transformation of the metric,

$$ds^2_{(L)} \to \Omega^2 ds^2_{(L)},$$

in which the squared conformal factor is constant and imaginary, namely, $\Omega^2 = i$.

The geometric interpretation that has been attained for the GWT can also be achieved by analyzing the relation between the Euclidean and Lorentzian actions of the geometrodynamics formulation. From Eq. (5.1) and the formula for the curvature scalar of a conformally transformed metric [16] it follows that $W \circ (3) R(h_{ab}) = -i (3) R(h_{ab})$. Employing Eq. (5.5), it is then a simple exercise to check that the GWT maps the Euclidean action (1.2) to its Lorentzian counterpart (1.1):

$$W \circ I_{(E)} = S_{(L)}. \quad (5.9)$$

In agreement with our discussion above, this correspondence between the geometrodynamical actions can be interpreted as that obtained by composing an inverse WR with a conformal transformation of the Lorentzian metric when $\Omega^2 = i$. A conformal transformation with the factor $\Omega$ has the following action on the metric functions:

$$h_{ab} \to \Omega^2 h_{ab}, \quad N \to \Omega N, \quad N^a \to N^a. \quad (5.10)$$

The corresponding change in the extrinsic curvature can be computed from Eq. (5.3). Provided that $\Omega$ is constant, since the Christoffel symbols are invariant under a constant rescaling of the metric, we arrive at

$$k^b_a \to \Omega^{-1} k^b_a. \quad (5.11)$$

Finally, for constant $\Omega$, the change in the curvature scalar of the induced metric is the familiar one: $(3) R \to \Omega^{-2} (3) R$. One can then see that a constant conformal transformation results in multiplying the Lorentzian action (1.1) by a factor of $\Omega^2$. Since an inverse WR maps the Euclidean action $I_{(E)}$ to $-i S_{(L)}$, the composite of this rotation and a constant conformal transformation turns out to send the action $I_{(E)}$ to $-i \Omega^2 S_{(L)}$. Thus, for $\Omega^2 = i$, we recover the transformation rule (5.9).

One may wonder whether the relation found between the GWT and the inverse WR in vacuum gravity continues to be valid when one compares the effects of these two transformations not just on the abstract line element and the gravitational action, but on any function of the shift vector, densitized lapse function, and gravitational phase space variables. Actually, the answer to this question turns out to be in the affirmative provided that the inverse WR is implemented by making the analytic continuation $N \to i N$.

We will prove this statement in the triad formulation of vacuum general relativity, although our discussion can be easily extended to the Ashtekar and geometrodynamical formulations. Let us first consider the action of an inverse WR obtained by performing an analytic continuation of the (Euclidean) lapse function from positive real to positive imaginary values. Denoting this action by $R^{-1}$, we then have $R^{-1} \circ N = i N$ (with $N \in \mathbb{R}^+$). Since an inverse WR maps Euclidean metrics to Lorentzian ones, we also get

$$R^{-1} \circ N^a = N^a, \quad R^{-1} \circ \tilde{E}^a_i = \tilde{E}^a_i. \quad (5.12)$$

In particular, this equation implies $R^{-1} \circ h_{ab} = h_{ab}$. Using now the definition of the densitized lapse function and Eqs. (2.2) and (5.3), one arrives at

$$R^{-1} \circ N = i N, \quad R^{-1} \circ K^i_a = -i K^i_a. \quad (5.13)$$

On the other hand, the action of a conformal transformation on the metric functions and the extrinsic curvature is given by Eqs. (5.10) and (5.11). For $\Omega^2 = i$, the corresponding constant rescaling of the shift vector, densitized lapse function, and phase space variables of the triad formulation is

$$N^a \to N^a, \quad N \to -i N, \quad (5.14)$$

$$\tilde{E}^a_i \to i \tilde{E}^a_i, \quad K^i_a \to K^i_a. \quad (5.15)$$

It then follows that the composite of an inverse WR (obtained with the prescription $N \to i N$) and a conformal transformation with factor $\Omega^2 = i$ preserves the Poisson brackets of vacuum gravity and leaves $N^a$ and $N$ invariant. Moreover, this composite transformation has in fact the same effect as the GWT on the gravitational variables, shift, and densitized lapse, as can be seen by comparing Eqs. (5.12)–(5.13) with Eqs. (2.17) and (5.2). We hence conclude that the action of the GWT on functions on phase space that depend on the shift vector and the densitized lapse function can indeed be interpreted in vacuum general relativity as the result of an inverse WR and a constant, complex conformal transformation.
VI. CONCLUSIONS AND FURTHER COMMENTS

We have analyzed whether there exists a relation between the GWT constructed by Thiemann and the WR employed in the geometrodynamics (or triad) formulation of vacuum gravity, an issue which remained obscure in the literature [5,6]. It has been seen that the answer is in the affirmative. More precisely, the transformation rules for the lapse, shift and gravitational phase space variables are the same under the GWT as under an inverse WR composed with a conformal transformation of a (squared) factor equal to $i$, provided that the WR is implemented by means of a complex rotation of the lapse function. We have also discussed whether, in special circumstances, the GWT admits the simple spacetime interpretation of an analytic continuation in time. It has been proved that, in certain systems which possess an intrinsic time variable which behaves as a logarithm of the densitized triad under constant scale transformations, the GWT reduces, after fixing the time gauge, to an analytic continuation that is performed only in the explicit dependence on the time coordinate.

Let us describe in more detail the results obtained. We have first considered two different types of gravitational models, namely, the Gowdy model with the topology of a three-torus and a class of gravitational systems which represent universes with expanding sections of constant time. In these models, the dynamical evolution allows one to choose, as a time coordinate, a function of the induced metric which is not invariant under constant scale transformations. After a partial gauge fixing which incorporates this particular choice of time, the infinitesimal generator of the GWT turns out to coincide (modulo first-class constraints) with minus the reduced Hamiltonian of the system. Given this result, we have proposed to identify the parameter $t$ employed to define the GWT with the introduced time coordinate. This proposal has been seen to guarantee that (at least on the abstract line element and functions on phase space which are explicitly time-indepedent before fixing the time gauge) the action of the GWT and the gauge-fixing procedure commute. Furthermore, it then follows that, when acting on functions on the partially reduced phase space that commute with the existing first-class constraints, the GWT for the reduced system can be interpreted as an analytic continuation made only in the explicit time dependence. In addition, the operator that implements this GWT in the quantum theory is then formally identical to the evolution operator which, on physical states, would result in a continuation of the Euclidean time $t$ until it acquires a constant, imaginary part equal to $\pi/2$. For the models analyzed, we have also shown that this continuation in the explicit time dependence maps in fact the Euclidean theory to the Lorentzian one, because it sends the reduced phase space, first-class constraints and the reduced Hamiltonian of the Euclidean systems to their Lorentzian counterparts.

In these models, the substitution of $t$ by $t + i\pi/2$ in the explicit time dependence has been seen to transform the Euclidean line element into the Lorentzian one, though multiplied by a factor of $i$. This constant factor is irrelevant in what concerns the Einstein equations. However, its appearance in the Lorentzian line element reveals that the GWT implies a complex conformal transformation of the metric. On the other hand, we have argued that the transformation rule of the Euclidean line element under the GWT, namely $ds^2_{(E)} \rightarrow ids^2_{(L)}$, must in fact be gauge-independent. In this way, we have arrived at a geometric interpretation of the GWT as the composite of an inverse WR (which maps the set of Euclidean four-geometries to its Lorentzian counterpart) and a complex conformal transformation with $\Omega^2 = i$.

In general relativity one cannot introduce a well-posed and globally defined time gauge. Hence, one loses a purely spacetime picture for the GWT as an analytic continuation in the explicit dependence on the time coordinate. Nevertheless, we know that this transform sends the Euclidean first-class constraints to those of the Lorentzian sector. In addition, we have seen that, in vacuum general relativity with neither gauge fixing nor symmetry reduction, the transformation rules for the Euclidean action and the abstract line element under the GWT are formally the same as one obtains by composing an inverse WR and a constant conformal transformation, provided that $\Omega^2 = i$. Furthermore, a similar result has been obtained when discussing the transformation rules for the gravitational phase space variables, the shift vector, and the densitized lapse function, once we have adhered to the prescription of implementing the WR by means of an analytic continuation of the lapse function. Therefore, we conclude that, on functions on phase space that depend as well on the shift and the densitized lapse, the action of the GWT for vacuum gravity can actually be interpreted as the result of performing an inverse WR and a suitable conformal transformation.

The fact that the GWT can be related to the WR in vacuum gravity allows one to regard the transform introduced by Thiemann as a rigorous procedure to carry out the WR in general relativity, a rotation that would otherwise remain a rather formal technique for passing from the Lorentzian gravitational theory to the Euclidean one. For all possible applications of the GWT, one must nonetheless keep in mind that the action of this transform cannot be identified just with that of a pure (inverse) WR because it involves in addition a complex rotation of the conformal factor.

At first sight, it might seem striking that the GWT relates the Euclidean and Lorentzian theories while mapping the Euclidean action $I_{(E)}$ to the Lorentzian one, $S_{(L)}$, since in the path-integral approach to geometrodynamics one usually assumes that the Lorentzian weight $e^{iS_{(L)}}$ must be replaced by $e^{-I_{(E)}}$ when dealing with Euclidean histories. However, if one really considers pure
Euclidean gravity rather than a Euclidean version of Lorentzian general relativity, the weight for a Euclidean geometry in the path integral should indeed be given by $e^{I(E)}$. In this sense, a correct substitution should be the one supplied by the GWT, i.e., $I(E) \rightarrow S(L)$. On the other hand, if it were possible to arrive at a convergent Euclidean path integral for gravity, the conformal factor $\Omega$ ought to be integrated over a complex contour $[7,11]$. But once the conformal factor is taken as a complex variable, the sums over conformally transformed Euclidean metrics obtained with the weights $e^{I(E)}$ and $e^{-I(E)}$ become equivalent because the Euclidean action $I(E)$ gets multiplied by a factor of $i$ under the conformal transformation $ds^2(E) \rightarrow ids^2(E)$. So, the two considered weights differ only by a constant rotation of $\pi/4$ in the complex $\Omega$-plane. We thus see that, modulo complex rotations of the conformal factor, the GWT and the WR lead to equivalent prescriptions for the path integral over Euclidean geometries. The use of the GWT has nonetheless clear advantages since it is formulated on a much more rigorous basis than the WR for gravity, and its application in order to extract the Lorentzian physics from Euclidean general relativity is in principle independent of the quantization approach followed and does not require the availability of a classical notion of spacetime.

In this paper we have restricted our analysis of the GWT to the case of vacuum general relativity. Some of the results obtained here might be modified by the existence of a matter content. The physical implications that the existence of a Wick transform may have for gravity with matter fields and a cosmological constant [6] will be the subject of future research.

APPENDIX

In this appendix we explicitly construct the canonical transformation introduced in Sec. 4, and obtain the expressions for $\bar{q}^{ab}$, $U^b_a$ and $\Delta$ in terms of the new phase space variables.

The inverse of Eqs. (4.5)–(4.7) is given by $q^{ab} = e^{2z} \bar{q}^{ab}$, where

\begin{align}
\bar{q}^{11} &= (1 + B^2 e^{-u+v} + Y^2) e^{-v}, &\bar{q}^{22} &= X, \\
\bar{q}^{31} &= e^{u}, &\bar{q}^{12} &= A, &\bar{q}^{13} &= B, &\bar{q}^{23} &= C. 
\end{align}

(A1)

Here we have adopted the notations

\begin{align}
X &= (e^v + C^2) e^{-u}, &Y &= A e^{-v} - B C e^{-v}. 
\end{align}

(A3)

In particular, we see that $\bar{q}^{ab}$ is independent of $z$.

The above relations are part of a canonical transformation generated by the functional

\[ F = -\frac{1}{2} \int_S K_{ab} \bar{q}^{ab}(u, v, z, A, B, C) \]

(A4)

where $q^{ab}(u, v, z, A, B, C)$ denote the functions (A1)-(A2) of the new configuration variables. The new momenta can then be straightforwardly computed from the generating functional $F$. Using Eqs. (A1)-(A4), we get

\begin{align}
p_a &= e^{2z} \left( A^2 e^v - X B^2 \right) e^{-v} K_{11} \\
&\quad + e^{2z} \left[ -X K_{22} + e^u K_{33} \right], \\
p_v &= e^{2z} \left[ -\left( 1 + Y^2 \right) e^{-v} K_{11} + e^{-u+v} K_{22} \right], \\
p_A &= e^{2z} \left[ Ye^{\hat{\Theta} - v} K_{11} + K_{12} \right], \\
p_B &= e^{2z} \left[ (XB - AC)e^{-v} K_{11} + K_{13} \right], \\
p_C &= e^{2z} \left[ Y Be^{-\hat{\Theta} - v} K_{11} + Ce^{-u} K_{22} + K_{23} \right], \\
p_z &= K_{ab} q^{ab}(u, v, z, A, B, C). 
\end{align}

(A5)

(A6)

(A7)

(A8)

(A9)

After some calculations one finds that the inverse of these formulae is

\begin{align}
K_{11} &= e^{-2z+v} \Theta, \\
K_{22} &= e^{-2z+u-v} \left[ (1 + Y^2) \Theta + 2p_u \right], \\
K_{33} &= e^{-2z} \left[ (1 + Y^2 + B^2 e^{-u+v}) X e^{-v} - A^2 \right] \Theta \\
&\quad + e^{-2z} \left[ 2e^{-u} p_u + 2X e^{-v} p_v \right], \\
K_{12} &= e^{-2z} \left[ -Ye^{\hat{\Theta} + p_A} \right], \\
K_{13} &= e^{-2z} \left[ (AC - XB) \Theta + p_B \right], \\
K_{23} &= e^{-2z} \left[ Ye^{\hat{\Theta} - v} - \left( 1 + Y^2 \right) e^{-v} \right] \Theta \\
&\quad + e^{-2z} \left[ p_C - 2Ce^{-u} p_v \right], 
\end{align}

(A10)

(A11)

(A12)

(A13)

(A14)

(A15)

(A16)

where

\[ \Theta = \frac{1}{3} (p_z - 2\sigma) \]

(A17)

and $\sigma$ is the function defined in Eq. (4.9).

Substituting Eqs. (A11)–(A16) into the right-hand side of Eq. (4.10), it is possible to show that

\begin{align}
U^b_1 &= \bar{q}^{2b} p_A + \bar{q}^{3b} p_B, \\
U^b_2 &= \bar{q}^{1b} p_A + \bar{q}^{3b} p_C + 2(\bar{q}^{2b} e^u - \bar{q}^{3b} C) e^{-v} p_v, \\
U^b_3 &= \bar{q}^{1b} p_B + \bar{q}^{2b} p_C + 2(\bar{q}^{3b} e^{-u} p_v \\
&\quad + 2 (\bar{q}^{3b} X - \bar{q}^{2b} C) e^{-v} p_v). 
\end{align}

(A18)

(A19)

(A20)

All elements of the matrix $U^b_a$ are therefore independent of the canonical pair of variables $(z, p_z)$.

Finally, an expression for $\Delta$ in terms of our new phase space variables can be obtained from Eqs. (4.14), (A1)–(A3) and (A11)–(A17) after a lengthy computation. The result can be written in the form

\begin{align}
\Delta &= \sigma^2 + 3 \left[ p_u + B p_B + C p_C + (2B C e^{-u} - A) p_A \right]^2 \\
&\quad + 3e^v \left[ p_C + Ye^{\hat{\Theta} - v} p_B + (YC e^{-v} + B) e^{-u} p_A \right]^2 \\
&\quad + 3e^{u-v} \left[ p_B + Ce^{-u} p_A \right]^2 + 3e^{-u} p_A^2. 
\end{align}

(A21)
ACKNOWLEDGMENTS

The author is very grateful to P. F. González Díaz for valuable comments and discussions. He is also thankful to A. Ashtekar for helpful conversations.

[1] A. Ashtekar, “Lectures on Non-Perturbative Canonical Gravity”, eds. L.Z. Fang and R. Ruffini, World Scientific, Singapore, 1991.
[2] A. Ashtekar, Phys. Rev. Lett. 57, 2244 (1986); A. Ashtekar, Phys. Rev. D 36, 1587 (1987).
[3] A. Ashtekar, J. D. Romano, and R. S. Tate, Phys. Rev. D 40, 2572 (1989).
[4] A. Ashtekar, J. Lewandowski, D. Marolf, J. Mourão, and T. Thiemann, J. Math. Phys. 36, 6456 (1995); A. Ashtekar, Int. J. Mod. Phys. D 5, 629 (1996); A. Ashtekar and J. Lewandowski, Class. Quantum Grav. 14, A55 (1997).
[5] T. Thiemann, Class. Quantum Grav. 13, 1383 (1996).
[6] A. Ashtekar, Phys. Rev. D 53, 2865 (1996).
[7] S. W. Hawking, in: “General Relativity. An Einstein Centenary Survey”, eds. S. W. Hawking and W. Israel, Cambridge Univ. Press, Cambridge, England, 1979.
[8] See, for instance, J. J. Halliwell and S. W. Hawking, Phys. Rev. D 31, 1777 (1985).
[9] This prescription for the Wick rotation amounts to an analytic continuation of the signature parameter. Some papers that consider this continuation are: J. Greensite, Phys. Lett. B 300, 34 (1993); A. Carlini and J. Greensite, Phys. Rev. D 49, 866 (1994); V. D. Ivashchuk, Grav. Cosmol. 3, 8 (1997), and references therein.
[10] See, for instance, J. B. Hartle and S. W. Hawking, Phys. Rev. D 28, 2960 (1983).
[11] G. W. Gibbons, S. W. Hawking, and M. J. Perry, Nucl. Phys. B138, 141 (1978).
[12] P. O. Mazur and E. Mottola, Nucl. Phys. B341, 187 (1990).
[13] S. W. Hawking, Nucl. Phys. B239, 257 (1984).
[14] R. H. Gowdy, Ann. Phys. (N.Y.) 83, 203 (1974).
[15] G. A. Mena Marugán, Int. J. Mod. Phys. D 3, 513 (1994).
[16] R. M. Wald, “General Relativity”, University of Chicago Press, Chicago, 1984.
[17] All integrals in this equation are well defined if the function $C$ depends analytically on the integration parameter in a complex region which contains the segment joining $t_0$ and $t$.
[18] G. A. Mena Marugán, Phys. Rev. D 56, 908 (1997).
[19] V. Husain and L. Smolin, Nucl. Phys. B327, 205 (1989).
[20] S. Mizoguchi, Phys. Rev. D 51, 6788 (1995).
[21] This change of geometrodynamic variables is similar to others considered in the literature to arrive at a Klein-Gordon equation for general relativity. See, for instance: B. S. DeWitt, Phys. Rev. 160, 1113 (1967); K. Kuchař, in. “Quantum Gravity 2: A Second Oxford Symposium”, eds. C. J. Isham, R. Penrose, and D. W. Sciama, Clarendon Press, Oxford, 1981.