SOME ELEMENTS OF CONNES’ NON-COMMUTATIVE GEOMETRY, AND SPACE-TIME GEOMETRY

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1. Introduction

Physics – as we practice it – rests on two pillars:

(i) The analysis of (causal sequences of events in) classical space-time, viewed as a four-dimensional, smooth Lorentzian manifold (with certain good properties).

(ii) Quantum theory.

These two pillars appear to be somewhat incompatible, in the sense that it is found to be difficult to join them into a unified theoretical framework, or, in other words, to derive them as two different (limiting) aspects of a consistent, unified theory. Fortunately, space-time can be taken to be classical and, for purposes of laboratory physics, Minkowskian, down to distance scales comparable to the Planck length (corresponding to $\sim 10^{19}$ GeV). Thus, the unification of space-time geometry with quantum theory is not an urgent issue from a pragmatic point of view. However, for the logical consistency of the building of theoretical physics, joining space-time geometry with quantum theory would appear to be a fundamental task.

For space-time to directly reveal its “quantum nature”, it would have to be explored at distance scales close to the Planck length. Since this is impossible, our thinking about the problem of unifying space-time geometry with quantum theory is necessarily speculative and must be guided by considerations of mathematical consistency, elegance and aesthetic appeal.

One standard idea in the search for a theoretical framework unifying space-time- and quantum dynamics is to attempt to formulate a fundamental quantum theory without any reference to specific classical space-time models (“background independence”) and to try to view space-time as a derived (rather than as a fundamental) structure which manifests itself in a certain limiting regime of the fundamental quantum theory (e.g. as the geometry connected with an algebra of “functions on constant loops” or of “zero modes”).

Among a variety of theoretical ideas in this direction the following two approaches have been pursued most forcefully:
(A) Superstring theory [1],[2].

(B) The study of non-commutative spaces and their non-commutative geometry, as initiated by Connes [3].

The successes and problems of approach (A) are relatively well known among theoretical physicists. One success of approach (B) consists in a new perspective in the study of gauge theories [4], in particular of the standard model [5] (see also [6]) and of grand unified models [7]. However, since approach (B) has not been worked out in much detail for examples of infinite-dimensional non-commutative geometries, yet, problems connected with quantization have remained essentially untouched.

An idea that might be promising is to look for a manifestly background-independent, or “invariant” formulation of superstring theory and then use the methods of non-commutative geometry of Connes to study its properties. Since we still do not know a completely precise form of string field theory, we might settle for a more modest goal: Since string vacua correspond to superconformal field theories, we might first try to formulate superconformal field theories in a purely algebraic way, i.e., in a form independent of a choice of a target space and a target space geometry (see e.g. [8],[9]) and then to reconstruct geometrical data from a superconformal field theory by using methods of non-commutative geometry.

For simplicity, let us consider an $N=1$ unitary superconformal field theory. [Of course, in the construction of string vacua, one must study $N=2$ superconformal field theories. While $N=1$ theories turn out to generalize real Riemannian geometry, $N=2$ theories generalize complex Kähler geometry which is more difficult, and the necessary tools have not been fully developed, yet.] We choose the Ramond sector of an $N=1$ theory. Abstractly, it can be coded into the following data; (see [9] for background material):

(i) A *algebra, $A$, of operators acting on a separable Hilbert space $H$; $A$ contains the identity operator.

(ii) A “*Dirac operator” $D$ (the Ramond generator $G_0$ of the theory) which is a self-adjoint operator on $H$. [Technically, it is useful to assume that $A$ consists of bounded
operators with the property that $[D, a]$ is a bounded operator, for arbitrary $a \in A$."

A Laplacian $\triangle$ is defined by setting $-\triangle = D^2 \geq 0$.

(iii) Z$_2$ grading: There is a unitary involution, $\Gamma$, on $H$ such that $\Gamma a = a \Gamma$, for all $a \in A$, ($A$ is "even"), but $\Gamma D = -D \Gamma$, on the domain of $D$, ($D$ is "odd"). Physically, $\Gamma = (-1)^F$, where $F$ is fermion number.

(iv) Conformal invariance: $H$ carries a projective, unitary representation of the group $\text{PSU}(1,1)$ of Möbius transformations preserving the unit circle, with generators $L_0 = L_0^*$ and $L_1, L_{-1}$, with $L_1^* = L_{-1}$. We assume that

$$D^2 \equiv -\triangle = L_0 - \frac{c}{24} \mathbb{I},$$

where $c$ is the central charge of the theory, and that the representation of $\text{PSU}(1,1)$ on $H$ defines a $*$-automorphism group of the algebra $A$.

Formally, one can now inductively define all the generators of the Ramond algebra [9]. It is an interesting problem to isolate the precise hypotheses needed to prove that the formal Ramond generators obtained from (ii), (iii), (iv) are well defined and satisfy the Ramond algebra. [One approach towards solving this problem consists in generalizing the Lüscher-Mack theorem [10].]

Our goal in this paper is to show how from data (i) - (iii) one can reconstruct a generalized (non-commutative version of) Riemannian geometry, (Sect. 2). Rather than exemplifying non-commutative Riemannian geometry in the context of superconformal field theory – which would be a highly desirable goal that is, however, still somewhat elusive, so far – we shall, in Sect. 3, discuss the easier example of Riemannian geometry on finite-dimensional, generalized commutative and non-commutative spaces, e.g. on a two-sheeted, four-dimensional manifold, as in [5], and consider analogues of the Einstein-Hilbert and the Chern-Simons action functionals. Sect. 4 contains some conclusions and an outlook.

This paper is rather mathematical in its structure, and the applicability of the mathematical formalism to physical problems remains, at least to a fairly large extent, a
matter of speculation – really convincing examples are still missing. Nevertheless, we feel that Prof. C.N. Yang might follow attempts such as the present one with some benevolence. It is a pleasure to dedicate this paper to him.

2. Some Elements of Non-Commutative Geometry, [3].

2.1. Non-commutative spaces

Let \( M \) be a smooth, compact manifold without boundary. All properties of \( M \) can be retrieved from the study of the commutative \(*\) algebra \( C^\infty(M) \) of complex-valued, smooth functions on \( M \). Since we assume that \( M \) is compact, \( C^\infty(M) \) contains the function identically equal to 1 on \( M \). Connes’ idea is to study “non-commutative spaces” in terms of non-commutative \(*\) algebras with identity, 1. It remains to be seen what a good notion of “non-commutative manifold” would consist of, [3]. Here we say that any non-commutative, unital \(*\) algebra \( A \) of bounded operators (“bounded” with respect to some \( C^* \) norm on \( A \)) defines a “non-commutative space”, also denoted by \( A \).

2.2. Non-commutative differential forms on a non-commutative space.

According to Connes [3], non-commutative differential forms on a non-commutative space \( A \) are elements of the graded, differential algebra \( \Omega(A) \) of “universal forms” over \( A \):

\[
\Omega(A) = \bigoplus_{n=0}^{\infty} \Omega^n(A)
\]

is a \( \mathbb{Z} \)-graded complex vector space such that each \( \Omega^n(A) \) is an \( A \) bimodule;

\[
\Omega^n(A) = A^\otimes(n+1)/\text{Relations} = \langle a_0 da_1 \cdots da_n : a_0, a_1, \cdots, a_n \text{ in } A \rangle/\text{Relations} ,
\]

with \( \Omega^0(A) = A \).

The “Relations” are:

\[
da = da’ \iff a - a’ = \lambda 1, \lambda \in \mathbb{C},
\]
(in particular, $d1 = 0$). Clearly, $\Omega^n(A)$ defined by (2) and (3) is a left $A$ module. For $\Omega^n(A)$ to be a right $A$ module, we must impose the Leibniz rule:

$$d(a \cdot b) = (da) \cdot b + a \cdot (db).$$  \hspace{1cm} (4)

Relation (4) could be deformed to read

$$d(a \cdot b) = (da) \cdot \theta(b) + \psi(a) \cdot (db),$$  \hspace{1cm} (5)

where $\theta$ and $\psi$ are *automorphisms of $A$. We shall, however, not pursue this generalization here. Rather, we shall view $d$ as an analogue of exterior differentiation and define an $A$-linear map $d : \Omega^n(A) \to \Omega^{n+1}(A)$, $n = 0, 1, 2, \ldots$, by setting

$$d(a_0 da_1 \cdots da_n) := 1 da_0 da_1 \cdots da_n.$$  

Since $d1 = 0$, see (3), it follows that

$$d^2 = 0,$$  \hspace{1cm} (6)

so that $\Omega(A)$ is a graded complex of vector spaces.

Thanks to (4) or (5), $\Omega(A)$ is equipped with a multiplication,

$$m : \Omega^n(A) \otimes_A \Omega^l(A) \to \Omega^{n+l}(A),$$  \hspace{1cm} (7)

$$(a_0 da_1 \cdots da_n) \cdot (b_0 db_0 \cdots db_l)$$

$$= a_0 da_1 \cdots d(a_n \cdot b_0) db_1 \cdots db_l$$
$$= a_0 da_1 \cdots d(a_{n-1} a_n) db_0 db_1 \cdots db_l$$
$$+ a_0 da_1 \cdots d(a_{n-2} a_{n-1}) da_n db_0 \cdots db_l$$
$$- \cdots$$

which belongs to $\Omega^{n+l}(A)$.

Thus $\Omega(A)$ is an algebra under $m$. Since it contains $\Omega^0(A) = A$ as a unital subalgebra, it contains an identity $1 \in A$. Furthermore, it becomes a *algebra by defining

$$(da)^* = -da^*, \quad \text{for all} \quad a \in A,$$
and hence
\[(d\alpha)^* = (-1)^{\deg\alpha+1} d\alpha^*,\]
\[\deg\alpha = \deg\alpha^* = n,\] (8)
for all \(\alpha \in \Omega^n(A),\) for all \(n.\)

The graded, differential algebra \(\Omega(A)\) plays an important role in analyzing the “topology” of a non-commutative space \(A\) using Connes’ cyclic cohomology, [3]. The classical theory emerges as a special case.

2.3. Vector bundles, connections, hermitian structures.

Classically, the space of sections of a vector bundle over a manifold \(M\) can be described as a finitely generated, projective left module for the ring \(C^\infty(M)\) of smooth functions on \(M,\) [12]. A left module \(E\) over a ring \(A\) is finitely generated iff there is a finite number of elements, \(s_1, \ldots, s_n,\) in \(E\) such that every \(s \in E\) can be written as \(s = \sum_{j=1}^n a_j s_j,\)
for some \(a_1, \ldots, a_n\) in \(A.\) The elements \(s_1, \ldots, s_n\) form a basis of \(E\) iff \(0 = \sum_{j=1}^n a_j s_j\)
implies \(a_j = 0,\) for all \(j = 1, \ldots, n.\) The left module \(E\) is called free iff it has a basis;
\(E\) is called projective iff it is a submodule of a free module \(F,\) i.e., there exists a free module \(F\) and a submodule \(G\) such that \(F = E \oplus G.\)

The theorem of Swan [12] quoted above suggests to interpret the space of sections \(E\)
of a vector bundle over a non-commutative space described by a unital *algebra \(A\) as a finitely generated, projective left \(A\) module; see [3]. [This notion of vector bundles is adequate in the context of “real geometry” but may not be useful in the holomorphic setting.]

Adapting the classical notion of connection (gauge potential) to the more general setting of vector bundles over non-commutative spaces, Connes [3] has proposed to define a connection on \(E\) as a linear map, \(\nabla,\) from \(E\) to \(\Omega^1(A) \otimes_A E\)
\[\nabla : E \rightarrow \Omega^1(A) \otimes_A E,\] (9)
satisfying the Leibniz rule

\[ \nabla(as) = da \otimes_A s + a \nabla s, \]

for all \( a \in A \) and all \( s \in E \). Defining

\[ \Omega^r(E) = \bigoplus_{n=0}^{\infty} \Omega^n(E), \]

with \( \Omega^n(E) = \Omega^n(A) \otimes_A E \), it is easy to verify that \( \nabla \) extends to \( \Omega^r(E) \), with

\[ \nabla : \Omega^n(E) \longrightarrow \Omega^{n+1}(E) \]

satisfying

\[ \nabla(\alpha \phi) = d\alpha \phi + (-1)^{\deg \alpha} \alpha \nabla \phi, \]

for all homogeneous forms \( \alpha \in \Omega^r(A) \) and all \( \phi \in \Omega^r(E) \).

This permits us to define the curvature, \( R(\nabla) \), of a connection \( \nabla \) on \( E \) by

\[ R(\nabla) := -\nabla^2 : E \longrightarrow \Omega^2(A) \otimes_A E; \]

\( R(\nabla) \) is an \( A \)-linear map satisfying

\[ R(\nabla)(as) = a R(\nabla) s, \]

for all \( a \in A \) and all \( s \in E \). By (13) and (14), the definition of \( R(\nabla) \) extends to \( \Omega^r(E) \), and one has (using (13) and \( d^2 = 0 \)) that

\[ R(\nabla)(\alpha \phi) = \alpha R(\nabla) \phi, \]

for arbitrary \( a \in \Omega^r(A) \) and arbitrary \( \phi \in \Omega^r(E) \).

The following result from module theory [13] permits us to rewrite \( R(\nabla) \) in a more concrete form: Let \( E \) and \( F \) be two left modules over a ring \( A \). Define \( \theta_{EF} : E^* \otimes_A F \rightarrow \text{Hom}_A(E,F) \) by setting \( \theta_{EF}(\sigma \otimes_A t)(s) = \sigma(s)t, \) for arbitrary \( \sigma \in E^* \) (the space of \( A \)-linear functionals on \( E \)) and arbitrary \( s \in E \) and \( t \in F \). Then \( \theta_{EF} \) is an isomorphism.
from $E^* \otimes_A F$ to $Hom_A(E, F)$ iff $E$ is finitely generated and projective. Applying this result to $R(\nabla)$, we set $E := E$, $F := \Omega^2(A) \otimes_A E$ and note that $R(\nabla) \in Hom_A(E, F)$. Since $E$ is finitely generated and projective, it follows that $R(\nabla)$ can be written as

$$R(\nabla) = \sum_{\alpha, \beta} \varepsilon_\alpha \otimes_A F^\alpha \otimes_A e^\beta.$$  \hspace{1cm} (17)

for some elements $\varepsilon_\alpha \in E^*$, $R^\alpha \beta \in \Omega^2(A)$ and $e^\beta \in E$, with

$$R(\nabla)s = \sum_{\alpha, \beta} \varepsilon_\alpha(s) F^\alpha \otimes_A e^\beta.$$  \hspace{1cm} (18)

In spite of the fact that representation (17) is not unique, it turns out to be useful. [For example, it permits one to define traces: $\text{trace}_E R(\nabla) = \sum_{\alpha, \beta} \varepsilon_\alpha(e^\beta) F^\alpha \beta \in \Omega^2(A)$, $\text{trace}_E R(\nabla)^2 = \sum_{\alpha, \beta, \gamma, \delta} \varepsilon_\gamma(e^\delta) F^\gamma \delta \varepsilon_\alpha(e^\delta) F^\alpha \beta \in \Omega^4(A), \ldots$]

Next, we recall the notion of hermitian vector bundles. An element $a \in A$ is said to be positive iff $a = \sum_i b_i^* b_i$, $b_i \in A$, (where the series is assumed to converge in the $C^*$ norm on $A$ to an element of $A$). We say that a vector bundle $E$ over $A$ is hermitian iff there is a map $\langle \cdot, \cdot \rangle : E \times E \to A$, called a hermitian inner product on $E$, with the properties:

(i) $\langle as, bt \rangle = a \langle s, t \rangle b^*$, for all $a, b$ in $A$ and all $s, t$ in $E$; $\langle \cdot, \cdot \rangle$ is linear in the first and anti-linear in the second argument.

(ii) $\langle s, s \rangle \geq 0$, for all $s \in E(= 0 \leftrightarrow s = 0)$.

(iii) The anti-linear map $s \mapsto \langle \cdot, s \rangle$ defines an isomorphism from $E$ to $E^*$.

One now shows without difficulty that $\langle \cdot, \cdot \rangle$ extends uniquely to a hermitian inner product on $\Omega(E)$ with values in the algebra $\Omega(A)$, with

$$\langle \alpha \phi, \beta \psi \rangle = \alpha \langle \phi, \psi \rangle \beta^*.$$  \hspace{1cm} (19)

for all $\alpha, \beta$ in $\Omega(A)$ and all $\phi, \psi$ in $\Omega(E)$.

Note that it follows easily from (ii) that

$$\langle \phi, \psi \rangle^* = \langle \psi, \phi \rangle.$$  \hspace{1cm} (20)
One says that a connection $\nabla$ on a hermitian vector bundle $E$ over $A$ is unitary iff

$$d\langle s, t \rangle = \langle \nabla s, t \rangle - \langle s, \nabla t \rangle.$$  (21)

[The minus sign on the R.S. of (21) is forced upon us by the convention that $(da)^* = -da^*$, for $a \in A$.] One then effortlessly shows that

$$d\langle \phi, \psi \rangle = \langle \nabla \phi, \psi \rangle - (-1)^{\deg \phi + \deg \psi} \langle \phi, \nabla \psi \rangle,$$  (22)

for arbitrary homogeneous $\phi, \psi$ in $\Omega(E)$.

Using Connes’ cyclic cohomology [3] one can now go on to define Chern characters of vector bundles $E$ over a non-commutative space $A$ which are pairings between the $K$-theory of $A$ and even cyclic cocycles. We shall not pursue this theme here but refer the interested reader to the literature, in particular to Connes’ book [3] and to [14].

What is more important for our theme is to introduce a notion of differentiable structure on a non-commutative space.

2.4. Differentiable structure on a non-commutative space

We recall that, among the basic data specifying a conformal field theory was not only a non-commutative space described by a unital *algebra $A$, but also a Dirac operator $D$ such that $[D, (\cdot)]$ acts as a derivation on $A$. This is what is required to define a differentiable structure on $A$.

Thus, consider a non-commutative space corresponding to a unital *algebra $A$. We define an even $K$-cycle for $A$ to consist of the following data:

(i) A *representation, $\pi$, of $A$ on a separable Hilbert space $H$. [Usually, we may assume that $\pi$ is a faithful representation, and we shall therefore write $a$ for both, the element $a \in A$ and the bounded operator $\pi(a)$ on $H$.]

(ii) A (possibly unbounded) selfadjoint operator $D$ on $H$ such that

$$[D, a] \text{ is bounded, for all } a \in A;$$  (23)

$$(D^2 + \mathbb{I})^{-1} \text{ is a compact operator.}$$  (24)
This condition expresses the idea that the non-commutative space under consideration is compact.

(iii) A unitary involution $\Gamma$ on $H$ (i.e., $\Gamma = \Gamma^* = \Gamma^{-1}$) such that

$$A \text{ is even under } \Gamma \text{ and } D \text{ is odd under } \Gamma,$$

(i.e., $\Gamma a = a \Gamma$, for all $a \in A$, and $\Gamma D + D \Gamma = 0$, on the domain of $D$).

If (iii) is omitted one speaks of odd $K$-cycles. An odd $K$-cycle $(\pi, H, D)$ determines an even $K$-cycle, $(\tilde{\pi}, \tilde{H}, \tilde{D}, \Gamma)$, by setting $\tilde{\pi} = \pi \oplus \pi$, $\tilde{H} = H \oplus H$, $\tilde{D} = \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix}$, and $\Gamma = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$.

Given a $K$-cycle $(\pi, H, D)$ for $A$, we can replace the somewhat monstrous graded, differential algebra $\Omega(A)$ of universal forms by a more manageable one, the differential algebra $\Omega_D(A)$ defined as follows: We define a *representation $\pi'$ of $\Omega(A)$ on $H$ by setting

$$\pi'(a_0 da_1 \cdots da_n) = a_0 [D, a_1] \cdots [D, a_n],$$

for arbitrary $a_0, a_1, \cdots, a_n$ in $A$. As shown by Connes and Lott in [5],[3], the subalgebra $\ker \pi' + d \ker \pi'$ of $\Omega(A)$ is a two-sided ideal in $\Omega(A)$; (the proof is an easy application of the Leibniz rule for $d$). This enables them to define a graded, differential algebra $\Omega_D(A)$ by setting

$$\Omega_D(A) = \Omega(A)/_{(\ker \pi' + d \ker \pi')}.$$  

Note that

$$\Omega^0_D(A) = \Omega^0(A)/_{\ker \pi'} = A/_{\ker \pi} = A,$$

since $\ker \pi'/\Omega^0(A) = \ker \pi = \{0\} - \pi$ has been assumed to be faithful – and hence

$$\Omega^1_D(A) = \Omega^1(A)/_{\ker \pi'}/\Omega^1(A) \\ \simeq \pi'(\Omega^1(A)) = \left\{ \sum_i a_0^i [D, a_1^i] : a_0^i, a_1^i \in A \right\}.$$  


and
\[
\Omega^2_D(A) = \Omega^2(A) / (\ker \pi / \Omega^2(A) + d \ker \pi / \Omega^1(A))
\approx \pi'(\Omega^2(A)) / \pi'(d \ker \pi / \Omega^1(A)).
\] (30)

The space \( Aux := \pi'(d \ker \pi / \Omega^1(A)) \) is called the “space of auxiliary fields” [5]. We note that

\[
Aux = \left\{ \sum_i [D, a^i_0] [D, a^i_1] : \sum_i a^i_0 [D, a^i_1] = 0 \right\}
= \left\{ -\sum_i a^i_0 [D, [D, a^i_1]] : \sum_i a^i_0 [D, a^i_1] = 0 \right\}.
\] (31)

In the classical situation, where \( A = C^\infty(M) \), for some even-dimensional, smooth Riemannian spin manifold \( M \), and \( D = \partial_M \), the Dirac operator on \( M \),

\[ Aux = A. \] (32)

Since \([D, (\cdot)]\) satisfies the Leibniz rule, \( \Omega^1_D(A) \) is a left and right \( A \) module. Assuming that, for a given \( K \)-cycle \((\pi, H, D)\), \( \Omega^1_D(A) \) is finitely generated and projective, the following definition becomes meaningful.

**Definition.** \( \Omega^1_D(A) \), viewed as a finitely generated, projective left \( A \) module, is called the (space of sections of the) **cotangent bundle** associated with the non-commutative space \( A \) with differentiable structure given by \((\pi, H, D)\).

Thus if \( \Omega^1_D(A) \) is finitely generated and projective it is, according to Sect. 2.3, a vector bundle over \( A \), and we can study connections on \( \Omega^1_D(A) \). Let \( \nabla \) be a connection on \( \Omega^1_D(A) \), i.e.,

\[
\nabla : \Omega^1_D(A) \rightarrow \Omega^1_D(A) \otimes_A \Omega^1_D(A)
\]
is a linear map satisfying

\[
\nabla(a\omega) = da \otimes_A \omega + a \nabla \omega,
\] (33)
for all $a \in A \simeq \pi(A)$ and all $\omega \in \Omega^1_D(A) \simeq \pi^*(\Omega^1(A))$. One defines the torsion, $T(\nabla)$, of a connection $\nabla$ on $\Omega^1_D(A)$ by the formula

$$T(\nabla) = d - m \circ \nabla,$$

(34)

see [11]. One easily verifies that $T(\nabla)$ is an $A$-linear map from $\Omega^1_D(A)$ to $\Omega^2_D(A)$; in particular,

$$T(\nabla)(a\omega) = aT(\nabla)\omega,$$

(35)

for all $a \in A$ and all $\omega \in \Omega^1_D(A)$. Since $\Omega^2_D(A)$ is generated by products of pairs of elements in $\Omega^1_D(A)$, one can always construct, from a given connection $\nabla$ on $\Omega^1_D(A)$, a new connection $\nabla'$ whose torsion, $T(\nabla')$, vanishes. [15]. [This follows from arguments similar to those leading to eq. (17).]

2.5. Integration theory and Hilbert spaces of forms.

Following Connes [3], one says that a $K$-cycle $(\pi, H, D)$ for a non-commutative space described by a unital $*$ algebra $A$ is $(d, \infty)$-summable iff

$$\text{trace}_H \left( D^2 + \mathbb{I} \right)^{-p} < \infty, \quad \text{for all } p > \frac{d}{2}. \quad (36)$$

Let $Tr_\omega$ denote the Dixmier trace [3]. The Dixmier trace is a positive, cyclic trace on the algebra $B(H)$ of all bounded operators on $H$ which vanishes on trace-class operators.

We define the integral of a form $\alpha \in \Omega^*(A)$ over a non-commutative space $A$ by setting

$$\int \alpha := \lim_{\varepsilon \searrow 0} Tr_\omega \left( \pi^*(\alpha)(D^2 + \varepsilon \mathbb{I})^{-d/2} \right)$$

$$= Tr_\omega \left( \pi^*(\alpha) \mid D \mid^{-d} \right). \quad (37)$$

[The limit $\varepsilon \searrow 0$ exists trivially, since $Tr_\omega \left( \pi^*(\alpha)(D^2 + \varepsilon \mathbb{I})^{-d/2} \right)$ is actually independent of $\varepsilon$.]

Unfortunately, the $K$-cycles encountered in supersymmetric quantum field theory are not $(d, \infty)$-summable, for any finite $d$, but there are plenty of so-called $\theta$-summable $K$-cycles [3],[14], meaning that

$$\text{trace}_H e^{-\beta D^2} < \infty, \quad (38)$$
for any $\beta > 0$. In this case, one may attempt to define the integral of an element $\alpha \in \Omega'(A)$ by the formula

$$\int \alpha := \lim_{\beta \searrow 0} \omega \frac{\text{trace}_H \left( \pi'(\alpha) e^{-\beta D^2} \right)}{\text{trace}_H \left( e^{-\beta D^2} \right)},$$

(39)

where the notation $\lim_{\beta \searrow 0} \omega$ indicates that the “limit” is defined in terms of a suitable mean on the space of uniformly bounded functions of $\beta \in (0, 1]$; see [3]. [The definition (39) is useful, e.g., if $\text{trace}_H e^{-\beta D^2}$ is bounded by $\exp \text{ const } \beta^{-s}$, as $\beta \searrow 0$, for some $s < 1$.] In the examples of Sect. 3 (which are $(d, \infty)$-summable, for some $d < \infty$), the two definitions (37) and (39) agree, but (39) has the advantage that it may still be meaningful for $\theta$-summable $K$-cycles with $d = \infty$.

Connes has shown that if $(D^2 + \varepsilon I)^{-s}$ is trace class for $s > 1$ and $\lim_{s \searrow 1} (s - 1)$ exists then

$$\int \alpha = \text{const } \lim_{s \searrow 1} (s - 1) \text{trace}_H \left( \pi'(\alpha)(D^2 + \varepsilon I)^{-s} \right),$$

and the result is independent of the choice of the mean $\omega$.

When $d < \infty$ and $\int (\cdot)$ is defined by (37), or $\int (\cdot)$ is defined by (39) and the behaviour of $\text{trace}_H e^{-\beta D^2}$ is suitably constrained, as $\beta \searrow 0$, then

$$\int a\alpha = \int a \alpha, \text{ for all } a \in A \text{ and all } \alpha \in \Omega'(A).$$

(40)

The integral permits us to define a scalar product on the space $\Omega'(A)$: For $\alpha$ and $\beta$ in $\Omega'(A)$, we set

$$(\alpha, \beta) := \int \alpha \beta^*.$$ 

(41)

This is linear in the first argument and anti-linear in the second argument and is positive-semidefinite. Let $\tilde{H}$ denote the completion of $\pi'(\Omega'(A))$, modulo the kernel of $(\cdot, \cdot)$, in the norm determined by $(\cdot, \cdot)$. Clearly $\tilde{H}$ is a Hilbert space. It carries a $^*$representation
of $A$ by bounded operators on $\tilde{H}$, determined by the equation
\[
(\tilde{\pi}(a)\tilde{\alpha}, \tilde{\beta}) := \int a\alpha\beta^* = \int \alpha\beta^* a = \int \alpha(a^*\beta)^* = (\tilde{\alpha}, \tilde{\pi}(a^*)\tilde{\beta}),
\]
where $\tilde{\alpha}$ and $\tilde{\beta}$ are the vectors in $\tilde{H}$ corresponding to the elements $\alpha, \beta$ in $\Omega(A)$. We let $\tilde{A}$ denote the von Neumann algebra obtained from $\tilde{\pi}(A)$ by taking the weak closure in $B(\tilde{H})$.

The Hilbert space $\tilde{H}$ has a filtration into subspaces
\[
\tilde{H}^{(0)} \subset \tilde{H}^{(1)} \subset \cdots \subset \tilde{H}^{(n-1)} \subset \tilde{H}^{(n)} \subset \cdots \subset \tilde{H},
\]
where $\tilde{H}^{(n)}$ is defined to be the closed subspace of $\tilde{H}$ obtained by taking the closure of $\sum_{k=0}^{n} \pi^*(\Omega^k(A))$, modulo the kernel of $(\cdot, \cdot)$, in the norm determined by $(\cdot, \cdot)$. Let $P_D^{(n)}$ denote the orthogonal projection onto $\tilde{H}^{(n)}$. It is reasonable to define the space $\hat{\Omega}^n(A)$ of “square-integrable $n$-forms” by setting
\[
\hat{\Omega}^n(A) := (I - P_D^{(n-1)}) \tilde{H}^{(n)} \equiv \tilde{H}^{(n)} \ominus \tilde{H}^{(n-1)}.
\]
In the classical case $(A = C^\infty(M), \cdots)$ $\hat{\Omega}^n(A)$ is precisely the space of square-integrable de Rham $n$-forms.

The scalar product $(\cdot, \cdot)$ permits us to choose canonical representatives in the equivalence classes in
\[
\Omega(A)/(\ker \pi^* + d\ker \pi^*) \simeq \pi^*(\Omega(A))/(\pi^*(d\ker \pi))
\]
which are identified with the elements of $\Omega_D^*(A)$: With an equivalence class $[\alpha], \alpha \in \Omega^n(A)$, defining an element of $\Omega^n_D(A)$ we associate the vector $\alpha^\perp$ in $\tilde{H}$ defined by
\[
\alpha^\perp := (I - P_{d\ker^{n-1}}) \tilde{\alpha},
\]
where $P_{d\ker^{n-1}}$ is the orthogonal projection onto the subspace of $\tilde{H}$ spanned by $d\ker \pi^* / \Omega^{n-1}(A)$. We define $\Omega_D^+(A)$ to be the linear space spanned by the forms $\{\alpha^\perp : \alpha$
a homogeneous element of $\Omega'(A)$. Since $\int a\alpha = \int \alpha a$, for all $a \in A$, $\alpha \in \Omega'(A)$, and since $d \ker \pi / \Omega^n(A)$ is closed under left and right multiplication by elements of $A$, for all $n$, $\Omega_D^\perp(A)$ is a left and right $A$ module.

We define $H^\perp$ to be the Hilbert space of differential forms obtained by taking the closure of $\Omega_D^\perp(A)$ in the norm determined by the scalar product $(\cdot, \cdot)$ introduced in (41). Clearly $H^\perp \subseteq \tilde{H}$, with equality in the classical case ($A = C^\infty(M), D = \emptyset_M, \cdots$). Since $\Omega_D^\perp(A)$ is a left and right $A$ module, for all $n$, $H^\perp$ carries a $*$-representation, $\pi^\perp$, of $A$, and it has a filtration into subspaces $H^\perp(0) \subset H^\perp(1) \subset \cdots \subset H^\perp(n) \subset \cdots \subset H^\perp$ which are invariant subspaces for $\pi^\perp(A)$. By (28) and (29),

$$H^\perp(0) = H(0), \quad H^\perp(1) = H(1),$$

and

$$\alpha^\perp = \hat{\alpha},$$

for all $\alpha \in \Omega^0(A) = \Omega_D^0(A) = A$ and all $\alpha \in \Omega_D^1(A) = \Omega_D^\perp(1) = \pi(\Omega^1(A))$.

In the classical case, we have that $H^\perp = \tilde{H}$, as

$$\alpha^\perp = \hat{\alpha}, \quad \text{for all } \alpha \in \hat{\Omega}^n(A) \text{ and all } n.$$

Moreover, for $\alpha \in \hat{\Omega}^n(A)$, the operator

$$\hat{d}\alpha := P_D^{(n+1)} \pi(d\alpha),$$

satisfies

$$\hat{d}^2 = 0.$$  \hspace{1cm} (49)

It defines standard exterior differentiation.

Another interesting special case is the following one: Suppose that $F$ is an operator on $H$ with $F^2 = \lambda I$, for some $\lambda \geq 0$, and such that $[F, a]$ is a compact operator on $H$ with the property that $| [F, a] |^l$ is trace class, for $l > d$. Suppose, moreover, that $F\Gamma + \Gamma F = 0$, where $\Gamma$ defines the $\mathbb{Z}_2$-grading on $H$. [An example is $F = \text{sign } D$, under
suitable assumptions on $A$ and $(\pi, H, D, \Gamma)$; see [3].] For $\alpha = a_0 \, da_1 \cdots da_n \in \Omega^n(A)$, $n = 0, 1, 2, \cdots$, we define

$$\pi'(\alpha) = a_0[F,a_1] \cdots [F,a_n].$$

For $x \in B(H)$ we define

$$[F,x]_{\Gamma} = \begin{cases} Fx - xF & \text{if } x\Gamma = \Gamma x \quad (x \text{ even}) \\ Fx + xF & \text{if } x\Gamma = -\Gamma x \quad (x \text{ odd}) \end{cases}$$

Then

$$[F,\pi'(\alpha)]_{\Gamma} = \pi'(d\alpha) \quad (50)$$

and hence

$$\Omega'_F(A) \simeq \pi'(\Omega(A)), \quad (51)$$

since, for $\alpha \in \ker \pi'$, $0 = [F,\pi'(\alpha)]_{\Gamma} = \pi'(d\alpha)$, i.e., $d\alpha \in \ker \pi'$.

We define integration by setting

$$\int (\cdot) = Tr_{\omega}(\cdot).$$

This enables us to define an analogue of $\tilde{H}$, the Hilbert space of differential forms, and, in this case, $H^\perp = \tilde{H}$, because of (51). We may now define an operator $\hat{d}$ on $\tilde{H}$ by setting

$$\hat{d}\hat{\alpha} := \hat{d}\alpha, \quad \text{with } \hat{d}^2 = 0. \quad (52)$$

By (50), $\hat{d}$ is well defined, and $\hat{d}^2 = 0$ follows from $d^2 = 0$. [The situation described here may be important in the study of “BRST geometry”, conformal geometry [3] and complex geometry.]

In the situation just described and in the classical case considered in equs. (47) through (49), the Hilbert space of “differential forms” is $\tilde{H}$ which is a $\mathbb{Z}_2$-graded ($\mathbb{Z}$-graded, resp.) complex for the operator $\hat{d}$ defined in (52) ((48), resp.). On the Hilbert space $\tilde{H}$, one defines a “Dirac operator on differential forms”, $\tilde{D}$, by setting

$$\tilde{D} = \hat{d} + \hat{d}^*.$$  

(53)
For topics like the definition of $C^n$-differentable structures on non-commutative spaces and cyclic homology and cohomology, we refer the reader to the literature, in particular [3],[14],[16]. [One key idea is to define integration of “top-dimensional forms” by $\int \Gamma \alpha$, where $\Gamma$ is the $\mathbb{Z}_2$-grading on $H$; see [3].]

2.6. A hermitian structure on differential forms.

The purpose of this section is to equip $\Omega^1_D(A)$ with a canonical hermitian structure. This will permit us to introduce a natural notion of unitary connections on $\Omega^1_D(A)$.

We start with some general considerations. Suppose that $\hat{v}$ is a vector in $\tilde{\pi}(A)$ in $B(\tilde{H})$. Since $\tilde{\pi}(A)$ is dense in $\tilde{H}(0)$, there exists a sequence $\{b_\kappa\} \subset \tilde{\pi}(A)$ such that

$$\hat{v} = s - \lim_{\kappa} \hat{b}_\kappa.$$

We define $v^{op}$ by setting

$$\langle v^{op} \hat{a}', \hat{a}'' \rangle = \lim_{\kappa} \int b_\kappa a' (a'')^*$$

$$= \lim_{\kappa} \langle \hat{b}_\kappa, a''(a')^* \rangle$$

$$= \langle \hat{v}, a''(a')^* \rangle.$$  (54)

The domain of $v^{op}$ contains $\tilde{\pi}(A)$. For, if $a \in \tilde{\pi}(A)$ then

$$0 \leq \langle v^{op} \hat{a}, v^{op} \hat{a} \rangle = \lim_{\kappa} \int b_\kappa a a^* b_\kappa^*$$

$$= \lim_{\kappa} \int a a^* \hat{b}_\kappa, \hat{b}_\kappa^* = \lim_{\kappa} \langle a a^* \hat{b}_\kappa^*, \hat{b}_\kappa^* \rangle$$

$$\leq ||a a^*|| \lim_{\kappa} \langle \hat{b}_\kappa^*, \hat{b}_\kappa^* \rangle = ||a a^*|| \lim_{\kappa} \int b_\kappa^* b_\kappa$$

$$= ||a a^*|| \int b_\kappa b_\kappa^*$$

$$= ||a a^*|| (\hat{v}, \hat{v}).$$

If $v^{op}$ is a bounded operator then $\{b_\kappa\}$ can be chosen such that $||b_\kappa||$ is uniformly bounded, and it follows from (54) that

$$v^{op} = w - \lim_{\kappa} b_\kappa \in \tilde{A}.$$  (55)
Next, let $\alpha$ and $\beta$ be in $\pi'(\Omega(A))$. We define
\[
\langle \alpha, \beta \rangle_D := P_D^{(0)}(\alpha\beta^*) \equiv P_D^{(0)}(\alpha\beta^*)^{op}, \tag{56}
\]
where $P_D^{(0)}$ is the orthogonal projection onto the subspace $\tilde{H}^{(0)}$ of $\tilde{H}$. By (56),
\[
\langle \langle \alpha, \beta \rangle_D, \tilde{a} \rangle \equiv \int \langle \alpha, \beta \rangle_D a^* = \int \alpha\beta^* a^* = \int a^* \alpha\beta^* = (\tilde{a}^*, \langle \beta, \alpha \rangle_D). \tag{57}
\]

From what we have shown above and definition (56) it follows that $\langle \alpha, \beta \rangle_D$ defines an operator affiliated with $\tilde{A}$. As shown in [11], it is actually a bounded operator and hence belongs to $\tilde{A}$. By using (57), it has been shown in [11] that:

(i)
\[
\langle a\alpha, b\beta \rangle_D = a\langle \alpha, \beta \rangle_D b^*, \tag{58}
\]

for arbitrary $\alpha, \beta$ in $\pi'(\Omega(A))$ and arbitrary $a$ and $b$ in $\tilde{A}$.

(ii)
\[
\langle \alpha, \alpha \rangle_D \geq 0, \quad \text{for arbitrary } \alpha \in \pi'(\Omega(A)); \tag{59}
\]

(iii) the anti-linear map $\alpha \mapsto \langle \cdot, \alpha \rangle_D$ defines an isomorphism from $\pi'(\Omega(A))$ to the space of linear functionals on $\pi'(\Omega(A))$ extending continuously to linear functionals on $\tilde{H}$ with values in $\tilde{A}$.

We conclude from (i) – (iii) that, since $\Omega_D^1(A) \simeq \pi'(\Omega^1(A))$, $\langle \cdot, \cdot \rangle_D$ defines a generalized hermitian structure on $\Omega_D^1(A)$ with values in $\tilde{A}$.

Since we interpret $\Omega_D^1(A)$ as the (space of sections of the) cotangent bundle over the non-commutative space described by $A$ we can view $\langle \cdot, \cdot \rangle_D$ as the non-commutative analogue of a Riemannian metric. Apparently, it is uniquely determined by the $K$-cycle $(\pi, H, D)$ on $A$ and the choice of integration, $\int (\cdot)$. Since $\Omega_D^1(A)$ is a left and right $\tilde{A}$ module, the metric $\langle \cdot, \cdot \rangle_D$ on $\Omega_D^1(A)$ is unitary invariant: If $U(\tilde{A})$ denotes the group of unitary elements of $\tilde{A}$ then
\[
\langle \alpha u, \beta u \rangle_D = \langle \alpha, \beta \rangle_D, \tag{60}
\]

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for arbitrary $\alpha, \beta \in \Omega^1_D(A)$ and arbitrary $u \in U(\bar{A})$.

2.7. Riemann-, Ricci- and scalar curvature; “Levi-Civita” connections on $\Omega^1_D(A)$.

In this section, we shall assume that $\Omega^1_D(A)$ is a finitely generated, projective left $A$ module. Thus, by the results of the last subsection, $\Omega^1_D(A)$ is then a hermitian vector bundle over $A$, the cotangent bundle over $A$. Let $\nabla$ be a connection on $\Omega^1_D(A)$. Thus $\nabla: \Omega^1_D(A) \to \Omega^1_D(A) \otimes_A \Omega^1_D(A)$ is a linear map satisfying the Leibniz rule (33). By (14), (15) the Riemann curvature of $\nabla$ is defined by

$$R(\nabla) := -\nabla^2$$

and is an $A$-linear map from $\Omega^1_D(A)$ to $\Omega^2_D(A) \otimes_A \Omega^1_D(A)$. Since $\Omega^1_D(A)$ is finitely generated and projective and $\Omega^2_D(A) \otimes_A \Omega^1_D(A)$ is a left $A$ module, we may apply eq. (17) and write

$$R(\nabla) = \sum_{\alpha, \beta} \varepsilon_\alpha \otimes_A R^\alpha_\beta \otimes_A e^\beta,$$

where $\varepsilon_\alpha \in \Omega^1_D(A)^*$ (which, thanks to the hermitian structure defined on $\Omega^1_D(A)$, is actually isomorphic to $\overline{\Omega^1_D(A)}$), $R^\alpha_\beta \in \Omega^2_D(A)$, and $e^\beta \in \Omega^1_D(A)$, for all $\alpha, \beta = 1, 2, 3, \cdots$. Now, by (30), $\Omega^2_D(A)$ is defined as a space of equivalence classes:

$$\Omega^2_D(A) \simeq \pi'(\Omega^2(A))/\pi'(d\ker\pi/\Omega^1(A)).$$

If we want to identify $R^\alpha_\beta$ with an element of $\pi'(\Omega^2(A))$, (i.e., with a well defined operator on the Hilbert space $H$), we shall choose the representative $R^\alpha_\beta \in \Omega^2_D(A)$ defined by (45), for $n = 2$. No such choices have to be made for $\varepsilon^\alpha$ and $e^\beta$, since $\Omega^1_D(A) \simeq \pi'(\Omega^1(A))$. In the classical case, the choice $R^\alpha_\beta$ for $R^\alpha_\beta$ identifies $R^\alpha_\beta$ with an ordinary (de Rham) differential 2-form, by (47), and (61), (62) reduce to the standard definition of the Riemann curvature tensor.

Representation (62) enables us to define the Ricci- and scalar curvature of $\nabla$ as follows:

$$Ric(\nabla) := \sum_\alpha \varepsilon_\alpha \otimes_A P_1 \left( \sum_\beta R^\alpha_\perp_\beta \cdot e^\beta \right),$$
where $P_1 := P_D^{(1)} - P_D^{(0)}$ is the orthogonal projection onto the closed subspace $\tilde{H}^{(1)} \ominus \tilde{H}^{(0)}$ of “1-forms” in $\tilde{H}$. Furthermore, we define the scalar curvature, $r(\nabla)$, of $\nabla$ by setting

$$r(\nabla) := \sum_\alpha \varepsilon_\alpha \left( P_1 \left( \sum_\beta R^{\alpha,1}_\beta \cdot e^\beta \right) \right) . \tag{65}$$

Since $\otimes_A$ and $\cdot$ are $A$-distributive and associative, $Ric(\nabla)$ and $r(\nabla)$ are defined invariantly by eqs. (64) and (65). These equations show that

$$Ric(\nabla) \in \Omega^1_D(A)^* \otimes_A \overline{\Omega^1_D(A)}, \quad r(\nabla) \in \overline{A}, \tag{66}$$

where $\overline{\Omega^1_D(A)}$ denotes the closure of $\Omega^1_D(A) \simeq \pi'(\Omega^1(A))$ in the norm determined by the scalar product $(\cdot, \cdot)$ defined in (41).

The Einstein-Hilbert action in non-commutative geometry is now defined by

$$I(\nabla) := \kappa \int r(\nabla) + \Lambda \int \Pi , \tag{67}$$

where $\kappa$ is related to Newton’s constant and $\Lambda$ is the cosmological constant; see [11].

A connection $\nabla$ on $\Omega^1_D(A)$ is said to be unitary if, for all $\alpha$ and $\beta$ in $\Omega^1_D(A)$,

$$d\langle \alpha, \beta \rangle = \langle \nabla\alpha, \beta \rangle - \langle \alpha, \nabla\beta \rangle , \tag{68}$$

see eq. (21), Sect. 2.3. As in eq. (34), Sect. 2.4., the torsion of $\nabla$ is defined by

$$T(\nabla) = d - m \circ \nabla .$$

It is tempting to define a Levi-Civita connection to be a unitary connection, $\nabla_{LC}$, on $\Omega^1_D(A)$, whose torsion, $T(\nabla_{LC})$, vanishes.

It is straightforward to show that, in the classical case, $I(\nabla_{LC})$, as given by (67), reduces to the usual Einstein-Hilbert action (with cosmological constant $\Lambda$), [11].

Remarks.

(1) In general, $\Omega^1_D(A)$ is not a free left $A$ module, i.e., the cotangent bundle over a non-commutative space is usually not a trivial bundle; as one would expect.
(2) The cotangent bundle of a non-commutative space need not admit any Levi-
Civita connection, and – if it admits such connections – the Levi-Civita connection may
not be unique.

(3) If $\Omega^1_D(A)$ is a free left $A$ module then, by definition of free modules, it has a
basis. It is natural to choose a basis $\{e^\beta\}_{\beta=1}^N$ which is orthonormal with respect to the
canonical hermitian structure $\langle \cdot , \cdot \rangle_D$ on $\Omega^1_D(A)$, i.e.,

$$\langle e^\alpha , e^\beta \rangle_D = \delta^{\alpha \beta} 1.$$ (69)

The basis elements $e^\alpha$ are analogues of the “vielbein” used in Cartan’s formalism of
Riemannian geometry. The automorphisms of $\Omega^1_D(A)$ are then generated by unitary
$N \times N$ matrices, $M = (M^\alpha_\beta)$, with matrix elements in $A$.

One may now define the Cartan structure equations in non-commutative geometry,
(see [11]): Let $\omega^\alpha_\beta \in \Omega^1_D(A)$ be defined by

$$\nabla e^\alpha = - \omega^\alpha_\beta \otimes_A e^\beta$$ (70)

and let $T^\alpha \in \Omega^2_D(A)$ be given by

$$T^\alpha = T(\nabla) e^\alpha.$$ (71)

Finally, we define $R^\alpha_\beta \in \Omega^2_D(A)$ by setting

$$R(\nabla) e^\alpha = R^\alpha_\beta \otimes_A e^\beta.$$ (72)

Then the Cartan equations are

$$T^\alpha = de^\alpha + \omega^\alpha_\beta e^\beta,$$ (73)

$$R^\alpha_\beta = d\omega^\alpha_\beta + \omega^\alpha_\gamma \omega^\gamma_\beta.$$ (74)

If $\{\varepsilon_\alpha\}$ is the basis of $\Omega^1_D(A)^*$ dual to the basis $\{e^\alpha\}$ of $\Omega^1_D(A)$ then formula (62) gives

$$R(\nabla) = \varepsilon_\alpha \otimes_A R^\alpha_\beta \otimes_A e^\beta,$$ (75)

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and if $\varepsilon_\alpha$ is identified with the element $e_\alpha \equiv e^\alpha$ of $\Omega^1_D(A)$, using the hermitian structure $\langle \cdot, \cdot \rangle_D$ on $\Omega^1_D(A)$, then

$$I(\nabla) = \kappa \int R^{\alpha, \beta}_{\alpha} e^\beta e^*_\alpha + \Lambda \int 1$$

$$= \kappa (R^{\alpha, \beta}_{\alpha} e^\beta, e_\alpha) + \Lambda \int 1 . \quad (76)$$

See [11].

(4) An alternative approach to defining a generalized Einstein-Hilbert action goes as follows: In the classical case, it is easy to show that $\int r(\nabla)$ is proportional to the constant term in the Laurent series expansion of

$$- \frac{d}{d\beta} \frac{\text{trace}_H (e^{-\beta D^2})}{\text{trace}_H (e^{-\beta D^2})} = \frac{\text{trace}_H (D^2 e^{-\beta D^2})}{\text{trace}_H (e^{-\beta D^2})}$$

around $\beta = 0$ which we denote by $\int D^2$.

Hence if $(\pi, H, D)$ is a $K$-cycle for a unital *algebra $A$ one may define a generalized Einstein-Hilbert action by setting

$$I(D) = \kappa \int D^2 + \Lambda \int 1 . \quad (77)$$

However, in general, this definition of $I$ is not equivalent to the one given in eq. (67), as we have checked for the examples discussed in Sect. 3.

The approach sketched here suffers from an ambiguity: In general the algebra, $\pi(A)'$, of all bounded operators on $H$ commuting with the operators of $\pi(A) \simeq A$ contains a non-trivial subspace, $B_{odd}$, of odd operators,

$$B_{odd} = \{ b \in \pi(A)' : b\Gamma = -\Gamma b \} .$$

Then two “Dirac operators”, $D$ and $D'$, on $H$ for which $[D, a] = [D', a]$, for all $a \in A$, may differ by an operator $b$ affiliated with $B_{odd}$, i.e., $D' = D + b$. If $b$ is a compact perturbation of $D$ then our definition of integration in eq. (37) is independent of $b$. Thus, perturbations $b$ affiliated with $B_{odd}$ and relatively compact with respect to $D$ describe
the ambiguities in the definition of the “Dirac operator” on \( H \) which propagate into the definition of \( I(D) \), as given in (77). In attempting to eliminate them one must presumably return to the tools developed in Sects. 2.5 – 2.7 and derive expressions for a Levi-Civita spin connection.

(5) To incorporate gauge fields in this formalism, one is led, according to Connes [3], to study hermitian vector bundles \( E \) over \( A \), as in Sect. 2.3, but with \( \Omega(D) \) replaced by \( \Omega_D(A) \), with connections \( \nabla : E \to \Omega_D^1(A) \otimes_A E \), whose curvature, \( R(\nabla) \), is given by the formula in eq. (17), i.e.,

\[
R(\nabla) = -\nabla^2 = \sum_{\alpha,\beta} e_\alpha \otimes_A F_{\alpha,\perp}^{\beta} \otimes e_\beta,
\]

with \( e_\alpha \in E^*, e_\beta \in E \) and \( F_{\alpha,\perp}^{\beta} \in \Omega_D^{1,2}(A) \). The Yang-Mills action functional [3] is then defined by

\[
YM(\nabla) = \sum_{\alpha,\beta} \int F_{\alpha,\perp}^{\beta} F_{\beta,\perp}^{\alpha}.
\]

3. EXAMPLES: EINSTEIN-HILBERT AND CHERN-SIMONS ACTION FOR TWO-SHEETED SPACE-TIMES.

In this section we illustrate Connes’ formalism sketched in Sect. 2 in the context of some simple examples. As in [5], we choose (Euclidean) space-time, \( X \), to consist of two copies of a four-dimensional spin manifold \( M \):

\[
X = M \times \mathbb{Z}_2.
\] (78)

We consider a non-commutative space described by an algebra \( A \) given by

\[
A = \mathcal{A}_1 \otimes C^\infty(M) \oplus \mathcal{A}_2 \otimes C^\infty(M),
\] (79)

where \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) are finite-dimensional, unital *algebras over the real or complex numbers. It is convenient to think of elements of \( A \) as operators of the form

\[
\begin{pmatrix}
\mathbb{1} \otimes a_1 & 0 \\
0 & \mathbb{1} \otimes a_2
\end{pmatrix}
\] (80)
where $a_i$ is a smooth function on $M$ with values in $A_i$, $i = 1, 2$, and $\mathbb{I}$ is the identity in the Clifford algebra, $\text{Cliff}(T^* M)$, of Dirac matrices over $M$. To define a differentiable structure on $A$, we consider even $K$-cycles $(\pi, H, D, \Gamma)$ for $A$, with:

(a) $\pi = \pi_1 \oplus \pi_2$.

(b) $\pi_i$ is a representation of $A_i \otimes C^\infty(M)$ on a Hilbert space $L^2(S_i, \tau_i, dv)$, where $S_i$ is a bundle of spinors on $M$ with values in a finitely generated, projective, hermitian left $A_i$ module $E_i$; the scalar product on $L^2(S_i, \tau_i, dv)$ is given by

$$
(\psi_1, \psi_2) = \int_M dv x \tau_i \langle \psi_1(x), \psi_2(x) \rangle_i, \tag{81}
$$

where $\tau_i$ is a normalized trace on $A_i$, $\langle \cdot, \cdot \rangle_i$ denotes the hermitian structure on $E_i$, $i = 1, 2$, and $dv_x$ is the volume element on $M$. Then we define $H$ by

$$
H = L^2(S_1, \tau_1, dv) \oplus L^2(S_2, \tau_2, dv). \tag{82}
$$

(c) The Dirac operator is given, for example, by

$$
D = \left( \begin{array}{cc}
\nabla_M \otimes \mathbb{I}_1 & \gamma^5 \otimes \phi \\
\gamma^5 \otimes \phi^* & \nabla_M \otimes \mathbb{I}_2
\end{array} \right), \tag{83}
$$

where $\nabla_M$ is the standard covariant Dirac operator on $M$, and $\mathbb{I}_i$ is the identity operator in $A_i$, $i = 1, 2$; $\phi$ is a homomorphism from $E_2$ to $E_1$, and $\phi^*$ is the adjoint homomorphism from $E_1$ to $E_2$. Finally, $\gamma^5 = \gamma^1 \gamma^2 \gamma^3 \gamma^4$, $\{\gamma^a, \gamma^b\} = -2\delta^{ab}$, $(\gamma^a)^* = -\gamma^a$, $\gamma^\mu = e^\mu_a \gamma^a$, where $e^\mu_a(x) \partial_\mu$ is a basis of the tangent space, $T_xM$, of $M$ at $x$, with $e^\mu_a e^\nu_b \delta^{ab} = g^{\mu\nu}$, and $g_{\mu\nu}$ is a Riemannian metric on $M$. We choose $dv_x$ to be the volume element corresponding to the metric $g_{\mu\nu}(x)$.

(d) The $\mathbb{Z}_2$ grading on $M$ is given by

$$
\Gamma = \left( \begin{array}{cc}
\gamma^5 & 0 \\
0 & -\gamma^5
\end{array} \right). \tag{84}
$$

A second interesting example is obtained as follows: We choose $A$ to have the form

$$
A = A \otimes C^\infty(M),
$$
where $A$ is a finite-dimensional, unital $\ast$-algebra.

An odd $K$-cycle $(\pi_0, H_0, D)$ is given by

(a') a representation $\pi_0$ of $A$ on a

(b') a Hilbert space $H_0 = L^2(S, \tau, dv)$, where $S, \tau$ and $dv$ are as above; and

(c') $D \equiv \hat{\nabla}_M \otimes \mathbb{I}$.

From this $K$-cycle we obtain an even $K$-cycle by setting

(a") $\pi = \pi_0 \oplus \pi_0$;

(b") $H = H_0 \oplus H_0$;

(c") $D = \left( \begin{array}{cc} \hat{\nabla}_M \otimes \mathbb{I} & i\gamma^5 \otimes \phi \\ -i\gamma^5\phi & -\hat{\nabla}_M \otimes \mathbb{I} \end{array} \right)$, (85)

where $\phi = \phi^* \in \text{End}(E)$; and

(d") $\Gamma = \left( \begin{array}{cc} 0 & \mathbb{I} \otimes \mathbb{I} \\ \mathbb{I} \otimes \mathbb{I} & 0 \end{array} \right)$.

Now

$\pi(a) = \left( \begin{array}{cc} \mathbb{I} \otimes \pi_0(a) & 0 \\ 0 & \mathbb{I} \otimes \pi_0(a) \end{array} \right)$, \quad a \in A,

clearly commutes with $\Gamma$, and one easily checks that $D$, as given in (85), anticommutes with $\Gamma$.

To these examples we shall now apply the methods developed in Sect. 2.

3.1. Generalized Einstein-Hilbert actions for two-sheeted space-time geometries.

In this section, we briefly review the example of general relativity on a two-sheeted space-time proposed in [11]. Let Euclidean space-time, $X$, be as in eq. (78) and the algebra $A$ as in (79), with $A_1 = A_2 = \mathbb{C}$. We choose an even $K$-cycle $(\pi, H, D, \Gamma)$ for $A$ as in (a) - (d); see eqs. (81) through (84). Then the "cotangent bundle" $\Omega^1_D(A)$ is a free left and right $A$ module, with a basis $\{e^N_a\}_{N=1}^5$ given by

$e^a = \left( \begin{array}{cc} \gamma^a & 0 \\ 0 & \gamma^a \end{array} \right) = \left( \begin{array}{cc} \gamma^\mu e^a_\mu & 0 \\ 0 & \gamma^\mu e^a_\mu \end{array} \right)$, \quad a = 1, 2, 3, 4, (87)
\[ e^5 = \begin{pmatrix} 0 & \gamma^5 \\ -\gamma^5 & 0 \end{pmatrix}. \]  
(88)

The hermitian structure on \( \Omega^1_D(A) \) is given by the trace on \( 8 \times 8 \) matrices, normalized such that \( \operatorname{tr} \mathbb{I} = 1 \). Hence

\[ \langle e^N, e^M \rangle_M = \operatorname{tr} (e^N (e^M)^*) = \delta^{NM}. \]  
(89)

Using the generalized Cartan formalism, eqs. (69) - (74), we find that, in this example, the components of a connection \( \nabla \) on \( \Omega^1_D(A) \) in the basis \( \{ e^N \}_{N=1}^5 \) are given by

\[ \omega^N_M = \begin{pmatrix} \gamma^\mu \omega^N_{1\mu M} & \gamma^5 \phi l^N_M \\ -\gamma^5 \phi l^N_M & \gamma^\mu \omega^N_{2\mu M} \end{pmatrix}. \]  
(90)

The unitarity of \( \nabla \) then implies that

\[ \omega^N_{i\mu M} = -\omega^M_{i\mu N}, \quad i = 1, 2, \quad \text{and} \quad l^N_M = -l^M_N. \]  
(91)

The expressions for the components \( R^N_{\perp M} \) of the curvature, \( R(\nabla) = -\nabla^2 \), of \( \nabla \) are found to be given by

\[ R^N_{\perp M} = \begin{pmatrix} \gamma^{\mu\nu} R^N_{1\mu\nu M} & \gamma^5 \phi Q^N_{\mu M} \\ -\gamma^5 \phi Q^N_{\mu M} & \gamma^{\mu\nu} R^N_{2\mu\nu M} \end{pmatrix}, \]  
(92)

with

\[ R^N_{i\mu\nu M} = \partial_\mu \omega^N_{i\nu M} - \partial_\nu \omega^N_{i\mu M} + \omega^N_{i\mu L} \omega^L_{i\nu M} - \omega^N_{i\nu L} \omega^L_{i\mu M}, \quad i = 1, 2, \]

\[ \gamma^{\mu\nu} = \frac{1}{2} (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu), \]

\[ Q^N_{\mu M} = \partial_\mu l^N_M + \omega^N_{1\mu M} - \omega^N_{2\mu M} + \omega^N_{1\mu L} l^L_M - l^N_L \omega^L_{2\mu M}, \]

\[ \tilde{Q}^N_{\mu M} = -\partial_\mu l^M_N + \omega^N_{1\mu M} - \omega^N_{2\mu M} + l^N_L \omega^L_{1\mu M} - \omega^N_{2\mu L} l^M_L. \]

Imposing the condition that the torsion, \( T(\nabla) \), of \( \nabla \) vanishes one deduces that

\[ \omega^a_{1\mu b} = \omega^a_{2\mu b} = \omega^a_{\mu b} \]
is the classical Levi-Civita connection derived from the metric \( g_{\mu\nu} = e_\mu^a \delta_{ab} e_\nu^b \) on \( M \), for \( a, b = 1, \ldots, 4 \);

\[
\begin{align*}
\ell^a_b &= l^b_a, \ a, b = 1, \ldots, 4, \text{ and } \ell^5_a = -\ell^a_5, \\
\omega^a_{1\mu5} &= -\omega^b_{2\mu5} = \phi l^a_b e^b_\mu; \\
\ell_5^a e^a_\mu &= -\partial_\mu \phi^{-1}.
\end{align*}
\]

(93)

The Einstein-Hilbert action defined in (76) is then calculated to be

\[
I(\nabla) = \kappa \int_M \left[ 2r - 4\phi \nabla_\mu \partial^\mu \phi^{-1} + 4\phi^2 \ell^a_5 \ell^a_5 \\
+ \phi^2 \left((\ell^a_5)^2 - \ell^a_b \ell^b_5\right) \right] \sqrt{g} \, d^4x + \Lambda \, \text{vol.}(M),
\]

(94)

where \( r \) is the scalar curvature of the classical Levi-Civita connection. The fields \( \ell^a_b \) and \( \ell^5_5 \) turn out to decouple. Setting \( \phi = e^{-\sigma} \) and eliminating \( \ell^a_b, \ell^5_5 \), one finds \([11]\).

\[
I(\nabla) = 2\kappa \int_M \left[ r - 2\partial_\mu \sigma \partial^\mu \sigma \right] \sqrt{g} \, d^4x + \Lambda \, \text{vol.}(M).
\]

(95)

Thus, in this approach, the theory of gravity on \( X = M \times \mathbb{Z}_2 \) is equivalent to general relativity on \( M \), with an additional, massless scalar field \( \sigma \) that couples to gravity via the metric on \( M \). [Some results concerning this model have been reported in \([18]\).]

Geometrically, \( e^{-\sigma(x)} \) is a measure for the distance between the two copies of \( M \) at a point \( x \in M \). [For generalizations see \([19]\).]

It is worthwhile to compare expression (95) to the one obtained from definition (17) of the Einstein-Hilbert action. The total “Dirac operator” \( D' \), with a spin connection determined from \( \omega^N_M \), see (90), and with \( \ell^a_b = \ell^5_5 = 0 \), is given by \( D' = D - \frac{1}{2} \delta \phi \delta \sigma \otimes \left( \begin{array}{cc} \gamma^5 & 0 \\ 0 & \gamma^5 \end{array} \right) \). The action \( I(D') \), defined as in (77), comes out to be

\[
I(D') = \kappa \int_M \left[ ar + be^{-2\sigma} \right] \sqrt{g} \, d^4x + \Lambda \, \text{vol.}(M),
\]

for some constants \( a \) and \( b \). Apparently, it does not coincide with (95). However, had we chosen the \( \mathbb{Z}_2 \)-grading \( \Gamma = \left( \begin{array}{cc} \gamma^5 & 0 \\ 0 & \gamma^5 \end{array} \right) \) and set

\[
D' := \left( \begin{array}{cc} \nabla_M & \delta \gamma^5 \phi \sigma \\ \delta \gamma^5 \phi \sigma & \nabla_M \end{array} \right), \ \delta \text{ some constant},
\]

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we would have obtained
\[ I(D') = \kappa \int \left[ ar - \delta^2 \partial_\mu \sigma \partial^\mu \sigma \right] + \Lambda vol.(M), \]
which essentially agrees with (95). [Details of these calculations have been carried out in [17].]

3.2. The standard model coupled to gravity.

Connes and Lott [5], [3] have shown how to construct the tree-level Lagrangian of the standard model from the formalism of non-commutative geometry. In order to construct the electroweak sector, they use the formalism sketched in Sect. 2.3 and remark (5) of Sect. 2.7. They use an algebra \( A \) as in (79), with \( A_1 = \mathbb{H} \) (the real algebra of quaternions), \( A_2 = \mathbb{C} \).

The Dirac operator is chosen as in (83), with
\[ \phi = e^{-\sigma} \phi_0, \]
for some constant homomorphism \( \phi_0 \); see [3], [5]. They set \( \sigma = 0 \). However, since we have identified \( \sigma \) as a dynamical field coupled to gravity, we choose \( \sigma \) to be an arbitrary function of \( x \in M \).

In order to couple the quarks to colour - SU(3), Connes and Lott choose the corresponding spinors to take values in an \( A-B \) bimodule, where \( A \) is as above, and \( B = (\mathbb{C} \oplus \mathbb{M}_3(\mathbb{C})) \otimes C^\infty(M) \). For three generations of fermions and a suitable choice of the Kobayashi-Maskawa mixing matrix, they obtain precisely the Lagrangian of the standard model including the Higgs field. Their construction has the following interesting features:

- The Higgs field is identified with a component of the generalized electro-weak gauge connection and thus acquires a geometrical significance.
- For the Higgs potential \textbf{not} to vanish one must require \textbf{more than one generation of fermions}. 

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At the tree level, the cosmological constant of the Connes-Lott Lagrangian vanishes naturally.

One may now proceed to calculate the one-loop effective potential of the theory, as in [20]. Making the heuristic ansatz that, to order $\hbar$, the cosmological constant of the theory retains the form imposed by the formalism of non-commutative geometry at the tree level, one finds an explicit expression for the effective potential, $V^{(1)}(H,\sigma;g)$, to order $\hbar$, where $H$ denotes the vacuum expectation value of the Higgs field, and $g_{\mu\nu}$ is the metric on $M$. Choosing $g_{\mu\nu} \equiv \eta_{\mu\nu}$ to be the flat metric (a delicate point in the argument!), one may proceed to minimize $V^{(1)}(H,\sigma;\eta)$ in $H$ and in $\sigma$. The result of the calculation [21] is quite surprising: The minimization in $\sigma$ yields one new relation between the parameters of the standard model Lagrangian which, together with the requirement that $V^{(1)}(H,\sigma;\eta)$ be stable in $H$ and $\sigma$, fixes the mass, $m_t$, of the top quark to satisfy the bounds $146$ GeV $< m_t < 147.5$ GeV and yields a relation between $m_t$ and the mass, $m_H$, of the Higgs that constrains $m_H$ to lie between $\sim 110$ GeV and $\sim 150$ GeV; [21].

Of course, these predictions have, at best, heuristic value, since the problem of fixing the form of the cosmological constant to order $\hbar$ and higher by imposing natural, geometrical constraints is not understood. However, they do suggest that gravitational effects may play a role in understanding masses of fermions and Higgses and that methods of non-commutative geometry may be useful in understanding these problems.

3.3. Chern-Simons actions and gauge theories of gravitation.

The purpose of this section is to briefly review some recent results [22] on the Chern-Simons action in non-commutative geometry. We consider a non-commutative space described by an algebra $A = \mathcal{A} \otimes C^\infty(M)$, where $\mathcal{A}$ is a finite-dimensional, unital $^*$algebra. The differentiable structure of $A$ is given by an odd $K$-cycle $(\pi_0, H_0, D)$ for $A$ with properties $(a'), (b')$ and $(c')$ specified at the beginning of Sect. 3. Let us first consider the case where the dimension, $d$, of $M$ is odd, and $M$ is a homology sphere. We
consider a vector bundle $E$ over $A$ given by $A$ itself. The components of a connection on $E$ are then given by one-forms $\pi_0(\alpha) \in \Omega^1_D(A)$, and the corresponding curvature is obtained from $\pi_0(d\alpha + \alpha^2) \in \Omega^2_D(A)$. The Chern-Simons forms are given by

$$\vartheta^{(3)} := \pi_0(\alpha) \pi_0(d\alpha) + \frac{2}{3} \pi_0(\alpha^3) \in \pi_0(\Omega^3(A)),$$

$$\vartheta^{(5)} := \pi_0(\alpha) \pi_0((d\alpha)^2) + 3 \pi_0(\alpha^3) \pi_0(d\alpha) + \frac{3}{5} \pi_0(\alpha^5) \in \pi_0(\Omega^5(A)),$$

where $\pi_0(d\alpha)$ is chosen to belong to $\Omega^2_D(A)$; etc. Chern-Simons actions are now defined by

$$I_{CS}^{(d,d)}(\alpha) := i \int \vartheta^{(d)} \equiv \int \epsilon_G(\vartheta^{(d)} | D |^{-d});$$

(96)

see eq. (37), Sect. 2.5. They turn out to agree with the classical Chern-Simons actions.

The formalism of non-commutative geometry allows us to consider Chern-Simons actions in the case where $M$ is even-dimensional, $(d = 2, 4, \cdots)$: We choose the algebra $A$ as above, but consider an even $K$-cycle $(\pi, H, D, \Gamma)$ defined as in $(a'') - (d'')$; see eqs. (85), (86). Then we define the Chern-Simons action by

$$I_{CS}^{(d+1,d)} := \int \Gamma \vartheta^{(d+1)} \equiv \int \Gamma \vartheta^{(d+1)} | D |^{-d}.$$  

(97)

What kind of actions do we obtain? For $d = 2$ and $A = M_n(\mathbb{C})$, for example, we obtain a two-dimensional topological gauge theory with action

$$I_{CS}^{(3,2)} = i c \int_{M^2} \text{tr} (\phi F),$$

(98)

where $F = F_{ij} \ dx^i \wedge dx^j$, $F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]$, with $A \in u(n)$, and $c$ is a constant. This is the theory first considered in [23]. We could also have considered the action

$$I_{CS}^{(5,2)} := \int \vartheta^{(5)} | D |^{-2}$$

and would have obtained

$$I_{CS}^{(5,2)} = i c \int_{M^2} \text{tr} [-\phi \nabla \phi \wedge \nabla \phi + \phi^3 F],$$

(99)
where $\nabla$ denotes covariant differentiation in the adjoint representation of $\mathcal{A}$. Similarly,
\[
I_{CS}^{(5,4)} = i c \int_{M^4} tr (\phi F \wedge F),
\]
an action of interest in connection with Donaldson theory, [25].

A particularly interesting example is obtained when one chooses $\mathcal{A} = C_l i f f_{\mathbb{R}}(SO(4))$. As usual one requires that the connection $\alpha$ is hermitian. After somewhat lengthy calculations [22] one finds that $I_{CS}^{(5,4)}$ and $I_{CS}^{(7,4)}$ determine Lagrangians for topological gravity theories formulated as metric-independent gauge theories of a vierbein and a spin connection coupled to a $C_l i f f_{\mathbb{R}}(SO(4))$-valued scalar field $\phi$. Details concerning these theories go beyond the scope of this review; but see [22].

4. Conclusions and outlook.

In this survey we have discussed some elements of Connes' non-commutative geometry and indicated some applications of the formalism; mostly in the context of classical field theory and for spaces which are "close" to classical commutative spaces but which are not manifolds in the classical sense. We have found that when general relativity is formulated on generalized spaces, fields such as $\sigma$ and $B_{\mu \nu}$ appear in the theory which also appear in supergravity and superstring theory and receive a geometrical interpretation: They describe the geometrical structure of discrete internal spaces. It is tempting to imagine that what we have found is the "classical regime", the geometry of a "space of zero modes", of a putative quantum theory of space-time structure which one may hope can be formulated within the formalism of infinite-dimensional non-commutative geometry.

We have also seen that many familiar topological field theories can be derived from Chern-Simons theories on generalized commutative and non-commutative spaces, typically products of a classical manifold with a discrete commutative or non-commutative "internal space". We have studied finite-dimensional examples. According to the program described in the introduction, one should extend these attempts to infinite-dimensional examples. This might shed new light on string field theory which, at least
for open, bosonic strings, has the form of a Chern-Simons theory [25]. An attempt to fit
Witten’s open string field theory into Connes’ formalism of non-commutative geometry
has been described in [26], but further work in this direction appears to be necessary
before these problems will be understood more fully. As suggested by the work in [26],
it is tempting to think that Connes’ theory of foliations (see [3]) will be useful in under-
standing gauge fixing and BRST cohomology in a deeper way which play a vital role in
the quantization of all theories with infinite-dimensional symmetries.

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