Precision study of large-$N$ Yang-Mills theory in 2 + 1 dimensions

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Abstract

The boundstate problem in 2+1-dimensional large-$N$ Yang-Mills theory is accurately solved using the light-front Hamiltonian of transverse lattice gauge theory. We conduct a thorough investigation of the space of couplings on coarse lattices, finding a single renormalised trajectory on which Poincaré symmetries are enhanced in boundstate solutions. Augmented by existing data from finite-$N$ Euclidean lattice simulations, we obtain accurate estimates of the low-lying glueball spectrum at $N = \infty$. 

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I. INTRODUCTION

Yang-Mills theories are theoretically interesting in $2 + 1$ dimensions because their properties are very similar to the corresponding theory in $3 + 1$ dimensions, yet they can be handled much more accurately; see Ref. [1] for a review of properties and extensive references. They appear to exhibit linear confinement of heavy sources, a discrete spectrum of (glueball) boundstates, and a finite-temperature transition. Teper has recently performed a comprehensive analysis, using the standard tools of Euclidean $SU(N)$ lattice gauge theory, of $2+1$-dimensional Yang-Mills theories at $N = 2, 3, 4,$ and $5$. Hamiltonian lattice calculations have also recently been performed for finite $N$ [2,3] and, though less comprehensive, the results are mainly consistent. With data at enough values of $N$, one can contemplate an extrapolation to $N = \infty$. This is a limit of special interest for ‘analytic’ approaches to gauge theory, which often take advantage of large-$N$ simplifications. In the absence of any other criteria for the errors involved, the only way to know how well ‘analytic’ approaches are doing, for example those of Refs. [4,5], is to compare with lattice data and their extrapolation to large $N$.

A related question is: how close is $N = \infty$ to small $N$? This question can only be faithfully answered once there are accurate results in both limits. The $1/N$ expansion [3] is typically an asymptotic one and, a priori, observables in the two limits need not be close in value. The existing finite-$N$ data suggest strong suppression of corrections to the large-$N$ limit [4], a conclusion that was speculated about much earlier [6], on the basis of less reliable lattice data. If true, this fact deserves a deeper understanding.

The main objective of this paper is to address these issues for $2 + 1$ dimensional Yang-Mills theory with explicit calculations at $N = \infty$. In Refs. [10,11] (see also [14]) we used the large-$N$ limit of $2 + 1$-dimensional Yang-Mills as a test for developing the transverse lattice method of solving non-abelian gauge theories [12]. Based on an improved understanding of the sources of error in that calculation, we perform here a calculation at the next level of approximation. We obtain a renormalised light-front Hamiltonian on the transverse lattice
for both the pure-glue and heavy-source sector. From this, we obtain the glueball spectrum and the heavy-source potential. Existing finite-$N$ data, combined with our explicit large-$N$ results, are used to determine the first few coefficients of the $1/N^2$ expansion of glueball masses in string tension units.

In the next section we briefly review the transverse lattice method, the details of which have been covered elsewhere [10,11,13]. Section 3 describes the numerical search for the renormalised Hamiltonian via tests of Poincaré invariance. Our thorough investigation yields a single, well-defined candidate for the renormalised trajectory in coupling space. Results for the low-lying glueball eigenstates on this trajectory and the first few coefficients of the $1/N^2$ expansion of their masses are given. This improves upon current estimates of the large-$N$ limit, allowing us to accurately verify that corrections to it are highly suppressed. In the conclusions we discuss possible reasons for the success of analytic approaches, given their approximations.

II. TRANSVERSE LATTICE IN 2 + 1 DIMENSIONS

Adapted to 2 + 1 Yang-Mills theory, the Bardeen-Pearson transverse lattice gauge theory consists of continuum gauge potentials $\{A_0, A_2\}$ and space-time co-ordinates $\{x^0, x^2\}$, together with gauge-covariant transverse link variables $M(x^1)$ running between sites at $x^1$ and $x^1 + a$ on a transverse lattice of spacing $a$. We also use the light-front combinations $x^\pm = (x^0 \pm x^2)/\sqrt{2}$, $A^\pm = (A^0 \pm A^2)/\sqrt{2}$, et cetera. DLCQ [15] and Tamm-Dancoff cut-offs on the number of partons are used as intermediate regulators. These are extrapolated following the analysis of ref. [11]. DLCQ means that we impose anti-periodic boundary conditions $x^- \sim x^- + 2\pi K/P^+$, where $P^+$ is the total light-front momentum, and $K$ is an integer cut-off. $x^+$ remains continuous and infinite, and is used as a canonical time variable to derive a light-front Hamiltonian $P^-$. The most general action, from which $P^-$ will be derived canonically, must allow all gauge invariant operators that respect the Poincaré symmetries unviolated by the (gauge invariant) cut-offs. Since we will explicitly extrapolate
the DLCQ and Tamm-Dancoff cut-offs, only local dimension 2 operators with respect to \(\{x^+, x^-\}\) co-ordinates will be included at the outset. In the discussion hereafter we assume this limit has be taken.

Near the transverse continuum limit \(a \to 0\) corresponding to \(SU(N)\) Yang-Mills, one expects \(M\) to take values in \(SU(N)\). However, away from this limit one can allow \(M\) to be a general \(N\times N\) complex matrix, provided it still gauge transforms covariantly. One must then search this larger class of lattice theories for the renormalised trajectory that leads one to the continuum limit \(SU(N)\) Yang-Mills theory. Physical results are invariant along this trajectory and equal to the values in the full continuum limit. The trajectory may be found by renormalisation group transformations in the neighborhood of a fixed point (continuum limit). However, this is difficult for the present formulation. There are (roughly) two possibilities for the behaviour of \(M\) at a given point in the space of couplings constants: \(M\) is a massive degree of freedom (\(M = 0\) is the classical minimum); or, the ‘radial’ part of \(M\) condenses. We expect the latter to be the case near the \(a = 0\) limit of Yang-Mills, where the action should be minimized near values of \(M\) in \(SU(N)\) rather than \(M = 0\). Dealing with the condensation of the radial part, or using unitary matrices for \(M\) from the outset \([9]\), is tricky in light-front quantisation. This is what makes an analysis near \(a = 0\) difficult. On the other hand, it is straightforward to perform canonical light-front quantisation about \(M = 0\), when this is a stable minimum. If the renormalised trajectory passes into such a region, we can then study it.

An alternative way to find the renormalised trajectory is to use symmetry \([16]\). Generally speaking, we can define a quantum field theory by symmetry — in our case gauge and Poincaré invariance — and a particular continuum limit (there may be more than one). There is actually no reason why we cannot take a partial continuum limit, a limit in some space-time directions but not in others, since Poincaré invariance should relate them. Thus, in Ref. \([11]\) we proposed to take the continuum limit of Yang-Mills theory in the \(\{x^0, x^2\}\) directions, and tune couplings to impose full Poincaré invariance at finite transverse cut-off \(a\). This procedure can be carried out using light-front quantisation about \(M = 0\). Although
this regime apparently cannot contain the Poincaré-invariant theory at \( a = 0 \), numerical evidence for the existence of a renormalised trajectory was given, and has been extended to 3 + 1 Yang-Mills \([13]\). In this paper, we present conclusive numerical evidence for the case of 2 + 1 Yang-Mills.

For practical calculations, the remaining allowed operators in the action must be pared down to a finite number of independent parameters, and one must find some reasonable criteria to test Poincaré invariance. We now develop these necessary approximations, following closely our previous work.

### A. Pure-glue sector

To reduce the space of couplings to a finite dimension, we use various approximations:

1. quadratic canonical momentum operator \( P^+ \),

2. light-front momentum-independent couplings,

3. transverse locality, and

4. expansion in gauge-invariant powers of \( M \).

The reasoning behind them is described in more detail in Ref. \([13]\). We only note here, that a poor choice of approximation will simply mean that we cannot get close to the renormalised trajectory, if it exists, and accuracy will suffer accordingly. The principle physical approximation, Item 4, is the ‘colour-dielectric’ expansion about \( M = 0 \), which is applied to the light-cone gauge-fixed Hamiltonian rather than the action.

We have studied the light-cone Hamiltonian derived from the following \( SU(N) \) gauge-invariant action in the large-\( N \) limit

\[
A = \int dx^0 dx^2 \sum_{x^1} \overline{D}_a M(x^1) (\overline{D}^{\dagger} M(x^1))^{\dagger} - V_{x^1} - \frac{1}{2G^2} \text{Tr} \{ F^{\alpha\beta} F_{\alpha\beta} \}
\]  

(2.1)

where \( \alpha \in \{0, 2\} \) and
\[ \bar{D}_\alpha M(x^1) = \left( \partial_\alpha + iA_\alpha(x^1) \right) M(x^1) - iM(x^1)A_\alpha(x^1 + a) , \] (2.2)

is the transverse lattice covariant derivative. The ‘potential’ term is
\[ V_{x^1} = \mu^2 \text{Tr}\left\{ M(x^1)M^\dagger(x^1) \right\} + \frac{\lambda_1}{aN} \text{Tr}\left\{ M(x^1)M^\dagger(x^1)M(x^1)M^\dagger(x^1) \right\} \]
\[ + \frac{\lambda_2}{aN} \text{Tr}\left\{ M(x^1)M(x^1 + a)M^\dagger(x^1 + a)M^\dagger(x^1) \right\} + \frac{\lambda_3}{aN^2} \left( \text{Tr}\left\{ M(x^1)M^\dagger(x^1) \right\} \right)^2 . \] (2.3)

In light-cone gauge \( A_- = 0 \) and after eliminating \( A_+ \) by its (constraint) equation of motion, the corresponding light-front Hamiltonian is
\[ P^- = \sum_{x^1} \int dx^- - \frac{G^2}{4} \text{Tr}\left\{ J^+(x^1) \frac{1}{\partial^2} J^+(x^1) \right\} + \frac{G^2}{4N} \text{Tr} J^+(x^1) \frac{1}{\partial^2} \text{Tr} J^+(x^1) , + V_{x^1} \] (2.4)
\[ J^+(x^1) = i \left( M(x^1) \frac{\partial^+}{\partial^+} M^\dagger(x^1) + M^\dagger(x^1 - a) \frac{\partial^-}{\partial^-} M(x^1 - a) \right) . \] (2.5)

This is the most general Hamiltonian to order \( M^4 \) that obeys the other stated approximations. It can be light-front quantised and studied in a suitable Fock space at general momenta \( P^+ \) and \( P^1 \), as detailed in Ref. [11]. The eigenvalues of the exact Yang-Mills Hamiltonian yield the glueball masses \( M \) through the relativistic dispersion relation
\[ P^- = (M^2 + (P^1)^2)/2P^+ . \]

**B. Heavy Sources**

We introduce heavy sources \( \phi(x^+, x^-, x^1) \) on transverse sites. They are in the fundamental representation and of mass \( \rho \). We apply the same approximations that were made in the pure-glue sector, but here we expand to order \( M^2 \) all operators containing heavy-source fields, and work at leading non-trivial order in \( 1/\rho \). The heavy-source action is \( A_\phi = A + A_\phi \) where
\[ A_\phi = \int dx^+dx^- \sum_{x^1} (D_\alpha \phi)^\dagger D^\alpha \phi - \rho^2 \phi^\dagger \phi - \frac{\tau_1}{NG^2} \text{Tr}\left\{ F^{\alpha\beta}(x^1)F_{\alpha\beta}(x^1)W(x^1) \right\} \]
\[ - \frac{\tau_2}{NG^2} \text{Tr}\left\{ M^\dagger(x^1)F^{\alpha\beta}(x^1)M(x^1)F_{\alpha\beta}(x^1 + a) \right\} . \] (2.6)
and
\[
W(x^1) = \left( M^\dagger(x^1)M(x^1) + M(x^1)M^\dagger(x^1) \right)
\]  
(2.7)

\[D_\alpha = \partial_\alpha + iA_\alpha(x^1)\] is the usual covariant derivative for the plane \{x^0, x^2\}. After gauge fixing \(A_- = 0\), eliminating \(A_+\) in powers of \(M\) from its constraint equation, and discarding the higher orders in \(M\), the Hamiltonian resulting from \(A_+\) which satisfies the approximations is

\[
P_s^- = \int dx^1 \sum_{x^1} \frac{G^2}{4} \text{Tr} \left\{ \frac{J_\text{tot}^+ J_\text{tot}^+}{\partial_- / \partial_-} \right\} - \frac{G^2}{4N} \text{Tr} \left\{ \frac{J_\text{tot}^+}{\partial_- / \partial_-} \right\} \text{Tr} \left\{ \frac{J_\text{tot}^+}{\partial_- / \partial_-} \right\} + V_{x^1}
\]

\[+ \rho^2 \phi^\dagger \phi + \frac{\rho \tau}{aN} \phi^\dagger W \phi + \frac{2\tau_1}{N} \text{Tr} \left\{ \frac{J^+ J^+}{\partial_- / \partial_-} W \right\} + \frac{2\tau_2}{N} \text{Tr} \left\{ \frac{J^+ (x^1)}{\partial_- / \partial_-} M(x^1) \frac{J^+ (x^1 + a)}{\partial_- / \partial_-} M^\dagger(x^1) \right\}
\]  
(2.8)

with

\[
J_\text{tot}^+ = J^+ + i\phi \frac{\partial \phi^\dagger}{\partial_-} \phi^\dagger
\]  
(2.9)

Like \(P^-\), \(P_s^-\) can be studied in a suitable Fock space. The eigenvalues of \(v^+ P_s^-\), for co-moving heavy sources of velocity \(v^+\), are the usual excitation energies associated with the heavy-source potential \[14\]. If two sources are separated by \(na\) in the transverse direction \(x^1\) and by \(L\) in the longitudinal direction \(x^2\), then a rotationally invariant string tension would imply that, for large separations,

\[v^+ P_s^- \rightarrow \sigma R, \quad R = \sqrt{a^2 n^2 + L^2}
\]  
(2.10)

for the lowest eigenvalue.Demanding this rotational invariance, then comparing results at \(n = 0\) with \(L = 0\), allows one to determine \(a\) in a dimensionful unit (we use \(G^2 N\)) independent of \(\sigma\). This fixes the relative scale between \(x^1\) and \(x^2\), which will be needed for testing covariance. In practice it is relatively difficult to calculate the heavy source potential in the purely transverse direction. Consequently, we measure the string tension in this direction by compactifying space and calculating the winding mode spectrum.

\[\text{C. Poincaré Invariance}\]

We test Poincaré invariance of the theory by making measurements on eigenstates of \(P^-\) and \(P_s^-\). It turns out that a rather simple set of tests suffices to obtain an accurate
estimate of the renormalised trajectory. One of the approximations made in arriving at $P^-$ (2.4) is transverse locality. Therefore, it make sense to expand eigenvalues at fixed momenta $(P^+, P^1)$ in powers of transverse momentum thus

$$2P^+P^- = G^2 N \left( M_0^2 + M_1^2 a^2 (P^1)^2 + M_2^2 a^4 (P^1)^4 + \cdots \right). \quad (2.11)$$

Note that $G$ has dimensions of energy, and $G^2 N$ is held finite in the $N \to \infty$ limit.

$M_0, M_1, M_2, \cdots$ are dimensionless numbers which we calculate when diagonalising $P^-$. The simplest requirement of covariance is that

$$M_1^2 a^2 G^2 N - 1 = 0 \quad (2.12)$$

This ensures isotropy of the speed of light. The dimensionless quantity $a^2 G^2 N$ has already been determined above from the scale setting procedure via the string tension. Further conditions come from higher order corrections in $P^1$ in (2.11). In this work we will use only the condition (2.12) for the lowest-mass glueballs, together with conditions of rotational invariance in the heavy-source potential, to test the space of couplings of $P^-$. If our reasoning is correct and our approximations valid, we should find a well-defined trajectory on which the conditions (2.12) are accurately satisfied — in practice we introduce a $\chi^2$ test to quantify this. Moving along this trajectory should correspond to changing the spacing $a$. Eventually this would take us to the transverse continuum limit, but we will be prevented from reaching $a = 0$ by the restriction $\mu^2 > 0$, a necessary condition for quantisation about $M = 0$.

### III. RESULTS

#### A. $\chi^2$ charts

It is convenient to form dimensionless versions of the other couplings $1aG^2 \to g^2$ as $a \to 0$, but since we do not approach $a = 0$ we cannot use the continuum gauge coupling $g$. 

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\begin{equation}
m^2 = \frac{\mu^2}{G^2 N}, \quad l_i = \frac{\lambda_i}{aG^2 N}, \quad t_i = \frac{\tau_i}{\sqrt{G^2 N}}.
\end{equation}

The basic technique we follow is to search this space, with the $\chi^2$ test, for an approximation to a renormalised trajectory on which observables show enhancement of space-time symmetries violated by the cut-off $a$. The $\chi^2$ test is made up of variables to test isotropy of the speed of light in dispersion of low-lying glueballs, rotational invariance of the string tension, and rotational invariance of the potential at intermediate source separations. Since we can expect to do better with some variables than others, the weights are adjusted until we produce a sharp trajectory in coupling space where $\chi^2$ is minimized to roughly one per effective degree of freedom. In fact, altering the weights typically changes the sharpness of the trajectory and not its location. The optimum trajectory is tabulated in Table I. Full details on our computations are available at [17].

Figs. 1 and 2 show $\chi^2$ charts for a range of values of $m$ vs. $l_1$ and $m$ vs. $l_2$ near the renormalised trajectory. In each case the renormalised trajectory appears at the bottom of a well-defined and unique $\chi^2$-valley, running from large to small $m$. The behaviour of the lattice spacing as one moves along the renormalised trajectory is shown in Fig. 3. As expected, the lattice spacing gradually decreases with $m^2$ but never becomes zero for $m^2 > 0$. The fluctuations are due mainly to the difficulty in establishing the scale $\sqrt{\sigma}$. Since the $\chi^2$ is stable and small over a range of lattice spacings, we will use the point of smallest lattice spacing ($m = 0.044$) to extract physical quantities.

B. Rotational invariance

The heavy-source potential is displayed in Fig. 4. It shows better restoration of spatial symmetry than previously obtained [11]. The potential in the continuum spatial direction $x^2$ is a fit to the lowest eigenvalue as a function of $L$ of the form

\footnote{The behaviour of $l_3$, which is always very large, is clarified in Ref. [11].}
\[ v^+ P_s^- = 0.154 LG^2 N + 0.183 \sqrt{G^2 N} - \frac{0.178}{L}. \] (3.2)

One must be careful when interpreting (3.2) since the Coulomb potential in 2+1 dimensions is logarithmic. The form (3.2) should be appropriate except at the very smallest \( L \), where Coulomb corrections are expected. The \( 1/L \) term is a universal correction expected on the grounds of models of flux-string oscillations \[18\]. Universality implies that its coefficient should be invariant along the renormalised trajectory. In fact, we find that it varies slowly, a symptom that our approximation to the renormalised trajectory is not an exact scaling trajectory for this quantity and/or the form (3.2) is not sufficient to fit the potential.

C. Glueballs

The spectrum of glueballs in 2 + 1 dimensions can be classified by \(|J|^{PC}\), where \( J \) is \( SO(2) \) spin, \( C \) is charge conjugation, and the parity \( P \) is spatial reflection \( x^1 \rightarrow -x^1 \). Combinations of \( \pm J \) form parity doublets if states are Lorentz covariant. On the transverse lattice, there is enough symmetry to determine \( C, P \) and \(|J| \mod 2\). Additionally we can examine the shape of wavefunctions to help distinguish the spin of states.

The lightest glueball is a \( 0^{++} \); its mass along the renormalised trajectory is shown in Fig. 6. The anisotropy of the speed of light in the \( 0^{++} \) dispersion is less than 3\% for all the low \( \chi^2 \) points. For the point of smallest \( a \), \( \mathcal{M}_{0^{++}} = 4.10(13)\sqrt{\sigma} \). Here, we have estimated a 2–3\% error from extrapolations in DLCQ and Tamm-Dancoff cut-offs based on known analytic behaviour \[11\], and another 2–3\% from systematic finite-\( a \) errors. The fractional finite-\( a \) error estimate is based upon deviations from the relativistic dispersion condition (2.12). Figure 6 also shows the result of Teper, \( \mathcal{M}_{0^{++}}(N \rightarrow \infty) = 4.065(55)\sqrt{\sigma} \), who fit his finite-\( N \) data to a form \( A + B/N^2 \) in order to estimate the large-\( N \) limit. Teper’s large-\( N \) extrapolation and our independent direct calculation are in agreement.\[3\]

\[3\] We note that there are other finite-\( N \) lattice results which do not agree with Teper’s. Recent
Fitting Teper’s finite-$N$ data together with our large-$N$ result we find

\[
\frac{M_{0^{++}}}{\sqrt{\sigma}} = 4.118(13) + \frac{1.55(22)}{N^2} + \frac{3.38(73)}{N^4} \quad (3.3)
\]

See Fig. 6. This gives a better estimate of the large-$N$ limit than using one or the other data set alone.

Although we have made improvements to the calculation of $\sigma$, the dominant error in Fig. 5 still comes from the fluctuation of this quantity; in particular, the determination of $\sigma$ from the longitudinal direction $x^2$ is a big source of error in determining the relative scales. This error becomes so severe for most heavier glueball states, which exhibit poor covariance, that an alternative method must be used for accurate results. To remove most of the error when dealing with heavier glueballs, we set $M_{0^{++}}/\sqrt{\sigma}$ to the large-$N$ value estimated in Eqn. (3.3), then recalculated the renormalised trajectory with this constraint, id est we calculate mass ratios. To improve covariance in the lighter glueballs, at the expense of heavier states, we also restricted the $\chi^2$ to test only the dispersion of the lowest states in each charge-conjugation sector. The resulting mass ratios, at the point of lowest $\chi^2$ on the new renormalised trajectory, are shown in Table II. We also show the fit to the form

\[
\frac{M}{M_{0^{++}}} = A + \frac{B}{N^2} + \frac{C}{N^4} \quad (3.4)
\]

including Teper’s data. The convergence in $1/N^2$ is illustrated in Fig. 7.

**IV. CONCLUSIONS**

We have found that our improved transverse lattice calculations for 2 + 1 Yang-Mills in the large-$N$ limit are consistent with existing finite-$N$ data from an independent lattice method. Although both make use of lattice regulators, the methods use different quantisation procedures, elementary degrees of freedom, regulators, gauge fixing, and renormalisation Hamiltonian lattice calculations yield $M_{0^{++}} = 3.88(11)\sqrt{\sigma}$ for SU(3) compared to $M_{0^{++}} = 4.329(41)\sqrt{\sigma}$ in Ref. [1].
techniques. By combining the finite-$N$ and large-$N$ results, we have obtained accurate estimates of the lightest glueball masses and mass ratios for any $N$ in $2 + 1$ dimensions. These should provide a useful benchmark for analytic studies. We are able to confirm that $O(1/N^2)$ corrections to the large-$N$ limit are typically small. The plots of mass versus $1/N^2$ could have had any shape, but they turn out to be almost straight and almost flat. The same result also seems to be true in $3 + 1$ dimensions \cite{13}, though the data is less precise there.

Recalling how the quark model explains the (OZI) suppression of $1/N$ corrections in most channels \cite{19}, it would be interesting to know if constituent gluon models could provide an intuitive explanation of our finding.

The lightest glueball mass in units of the string tension has recently been calculated analytically in $2 + 1$ Yang-Mills for any $N$ \cite{4}. Various attempts have also been made to obtain excited glueball mass ratios via extensions the ADS/CFT correspondence at large-$N$ \cite{5}. These both use strong-coupling approximations and give qualitatively reasonable results. The transverse lattice method is also a strong-coupling one in a sense; it is a coarse lattice method. How can these methods give good results when the large-$N$ approximation almost certainly introduces a phase transition in the strong-coupling/coarse-lattice regime \cite{20}? The answer, we believe, lies in the fact that an exact renormalised trajectory, existing in an infinite-dimensional space of couplings, can avoid such transitions. Thus, it is only necessary to approximate the renormalised trajectory with a finite number of couplings to obtain results relevant to the continuum limit. One cannot extrapolate to weak coupling or fine lattice limits, but if one chooses the right variables and couplings, results can be obtained directly from the approximation to the renormalised trajectory far from these limits.

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### TABLE I. The trajectory in coupling-constant space which minimises the $\chi^2$ test of covariance.

| $m$  | $l_1$ | $l_2$ | $l_3$ | $t_1$ | $t_2$ | $\chi^2$ |
|------|-------|-------|-------|-------|-------|---------|
| 0.044 | -0.052 | -0.112 | 680.2 | -0.661 | -0.691 | 7.02    |
| 0.089 | -0.087 | -0.109 | 396.8 | -0.780 | -0.811 | 7.85    |
| 0.134 | -0.108 | -0.091 | 3.221 | -0.896 | -0.876 | 7.56    |
| 0.180 | -0.147 | -0.107 | 4.401 | -0.943 | -0.927 | 6.55    |
| 0.226 | -0.204 | -0.134 | 178.5 | -1.098 | -1.167 | 8.02    |
| 0.2765| -0.240 | -0.153 | 5.48  | -0.989 | -1.138 | 7.31    |
| 0.3275| -0.308 | -0.157 | 6.01  | -1.181 | -1.340 | 8.64    |

### TABLE II. Mass ratios for lightest glueball excited states, showing our $N = \infty$ measurement and fit coefficients including finite-$N$ data from Ref. [1]. The $2^{++}$ and $0_s^{++}$ states were not covariant-enough for reliable error estimates, and only Teper’s finite-$N$ extrapolation is shown.

| $|\mathcal{J}|^{PC}$ | $\mathcal{M}/\mathcal{M}_{0^{++}}$ | Fit coefficients |
|-------------------|-------------------------------|------------------|
|                   |                               | $C$              |
|                   |                               | $B$              |
|                   |                               | $A$              |
| $0^{--}$          | 1.35(5)                       | -14.58(1.47)     |
|                   |                               | 2.983(191)       |
|                   |                               | 1.349(6)         |
| $2^{++}$          | 1.60(17)                      | 3.233(2.724)     |
|                   |                               | -1.144(856)      |
|                   |                               | 1.743(51)        |
| $0_s^{--}$        | 1.82(6)                       | -5.839(7.488)    |
|                   |                               | 1.136(941)       |
|                   |                               | 1.824(25)        |
| $2^{-+}$          | 1.77(?)                       | -                 |
|                   |                               | 0.659(246)       |
|                   |                               | 1.697(57)        |
| $0_s^{++}$        | 1.28(?)                       | -                 |
|                   |                               | 0.770(399)       |
|                   |                               | 1.520(38)        |
FIG. 1. Minimum $\chi^2$ for a given $m$ and $l_1$. In the blank region to the right, tachyons appear in the spectrum.

FIG. 2. Minimum $\chi^2$ for a given $m$ and $l_2$. 
FIG. 3. Variation of the transverse lattice spacing along the Lorentz trajectory. The fit is $1.275m + 1.23$.

FIG. 4. The heavy-source potential. Solid line is fit to potential for sources with $x^2$-separation only; data points are values at one-link transverse separation and $x^2$-separation $L \sqrt{G^2N} = 0, 2.5, 5$. 
FIG. 5. The variation of the lightest glueball mass along the renormalised trajectory (together with the associated variation of the \( \chi^2 \)). Also shown is Teper’s extrapolation to \( N = \infty \) (ELMC).

FIG. 6. Fit to lowest glueball mass as a function of \( N \).
FIG. 7. Variation of excited glueball mass ratios with $N$. 

$\frac{M}{M_{0^{++}}}$

$1/N^2$