SELF-IMPROVING PROPERTY OF THE FAST DIFFUSION EQUATION

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Abstract. We show that the gradient of the $m$-power of a solution to a singular parabolic equation of porous medium-type (also known as fast diffusion equation), satisfies a reverse Hölder inequality in suitable intrinsic cylinders. Relying on an intrinsic Calderón-Zygmund covering argument, we are able to prove the local higher integrability of such a gradient for $m \in \left(\frac{4n-2n+1}{n+2}, 1\right)$. Our estimates are satisfied for a general class of growth assumptions on the nonlinearity. In this way, we extend the theory for $m \geq 1$ (see \cite{GS16} in the list of references) to the singular case. In particular, an intrinsic metric that depends on the solution itself is introduced for the singular regime.

1. Introduction and main result

The aim of this paper is to study regularity properties of the gradient of non-negative solutions to nonlinear, parabolic, partial differential equations, whose prototype is the singular porous medium equation

$$u_t - \Delta u^m = u_t - \text{div} \left( mu^{m-1} Du \right) = 0 \quad 0 < m \leq 1.$$  \hspace{1cm} (PME)

When $m = 1$, the nonlinear behavior disappears and (PME) reduces to the standard heat equation. When $m \neq 1$, the equation is quasi-linear and its modulus of ellipticity is $u^{m-1}$. When $m < 1$, this quantity blows up as $u \to 0$, the diffusion process dominates over the time evolution (i.e. the diffusion speed is very large) and therefore, the equation is said to be singular. The high speed of propagation is the reason for the name of the equation, which is often referred to as fast diffusion equation. Nevertheless, in the following we will prefer and regularly use the term singular porous medium equations.

Equations of this form arise in applications both from Physics and Mathematics. For example, when modelling the anomalous diffusion of hydrogen plasma across a purely poloidal octupole magnetic field, \cite{Berry-77, Berry-78, Berry-80} have shown that the diffusion equation may be written as the one-dimensional (PME)

$$\frac{\partial}{\partial x} \left[ n^{-\frac{1}{2}} \frac{\partial n}{\partial x} \right] = F(x) \frac{\partial n}{\partial t} \quad 0 \leq x \leq 1,$$  \hspace{1cm} (1.1)

where the geometrical factor $F(x)$ is a strictly positive function determined by the octupole geometry.

A singular porous medium model was proposed by Carleman (\cite{Car}) to study the diffusive limit of kinetic equations. He considered just two types of particles in a one dimensional setting, moving with speeds $c$ and $-c$. If we denote the densities respectively with $u$ and $v$, we can write their simple dynamics as

$$\begin{align*}
\partial_t u + c \partial_x u &= k(u,v)(v-u) \\
\partial_t v - c \partial_x v &= k(u,v)(u-v)
\end{align*}$$

for some interaction kernel $k(u,v) \geq 0$. In a typical case, one assumes $k = (u + v)^\alpha c^2$ with $\alpha \in (0, 1]$. If we now write down the equations for $\rho = u + v$ and $j = c(u - v)$ and pass to the limit as $c = \frac{1}{\varepsilon} \to \infty$, we will obtain to the first order in powers of $\varepsilon$ (see \cite{LT})

$$\partial_t \rho = \frac{1}{2} \partial_x \left( \frac{1}{\rho^{1-\alpha}} \partial_x \rho \right),$$  \hspace{1cm} (1.2)

which is exactly our case with $m = 1 - \alpha \in [0, 1)$ (what exactly means $m = 0$ is beyond the purpose of this work).
As for an example coming not from physical modelling, but from pure Mathematics, the study of the Yamabe Flow in $\mathbb{R}^n$ ($n > 2$) leads to consider
\[
\partial_t v = \Delta v^m, \quad m = \frac{n-2}{n+2} \in (0,1). \tag{1.3}
\]

Obviously, our list is a very partial one, and there are plenty of other examples: the interested reader can refer, for example, to [V06].

In this paper we are interested in the order of integrability of $|Du^m|$; we will deal only with the range
\[
m \in \left(\frac{n-2}{n+2}, 1\right), \tag{1.4}
\]
(the case $m > 1$ has already been dealt with in [GS16]), and we will study a general class of equations which have the same structure as (PME). Notice that we are not considering the full interval $m \in (0,1)$: the lower bound on $m$ is quite typical, when dealing with regularity issues for the singular porous medium equation (see, for example, the discussion in [DGV, Chapter 6, Paragraph 21]).

Given a bounded, open set $E \subset \mathbb{R}^n$ with $n \geq 2$, and $T > 0$, let $E_T \equiv E \times (0,T)$. For $m$ as in (1.4), and $f \in L^\infty_{loc}(E_T)$, we will consider nonnegative solutions to
\[
u_t - \text{div} A(x,t,u,Du^m) = f \quad \text{weakly in } E_T. \tag{1.5}
\]

The vector field
\[
A: E_T \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n
\]
is only assumed to be measurable, and we suppose there exist constants $0 < \nu \leq L < \infty$ such that
\[
A(x,t,u,\xi) \cdot \xi \geq \nu |\xi|^2 \quad \text{for a.e. } (x,t) \in E_T
\]
\[
A(x,t,u,\xi) \leq L |\xi|.
\]
By [DGV, Chapter 3, § 5], the structure conditions (1.6) ensure that (1.5) is parabolic. Notice that the model problem (PME) corresponds to the case $\nu = L = 1$ and $f \equiv 0$.

1.1. Weak solutions and sub(super)-solutions. A function
\[
u \in C^0_\text{loc}(0,T;L^{m+1}_{\text{loc}}(E)) \quad \text{with} \quad u^m \in L^2_{\text{loc}}(0,T;W^{1,2}_{\text{loc}}(E)) \tag{1.7}
\]
is a local, weak sub(super)-solution to (1.5)-(1.6) if satisfies the integral identity
\[
\int\int_{E_T} -u\varphi_t + A(x,t,u,Du^m) \cdot D\varphi \, dx \, dt \leq \int\int_{E_T} f \varphi \, dx \, dt \tag{1.8}
\]
for all possible choices of nonnegative test functions $\varphi \in C^\infty_0(E_T)$. This guarantees that all the integrals in (1.8) are convergent.

A local, weak solution to (1.5)-(1.6) is both a sub- and a super-solution, i.e., it satisfies the integral identity
\[
\int\int_{E_T} -u\varphi_t + A(x,t,u,Du^m) \cdot D\varphi \, dx \, dt = \int\int_{E_T} f \varphi \, dx \, dt \tag{1.9}
\]
for all possible choices of test functions $\varphi \in C^\infty_0(E_T)$.

By a standard mollification argument, it is possible to use $u^m$ as test function (otherwise, handling the time derivative of $u$ becomes a delicate issue). Let $\zeta: \mathbb{R} \to \mathbb{R}$,
\[
\zeta(s) \equiv \begin{cases}
C \exp \frac{1}{|s|^2 - 1} & |s| < 1 \\
0 & |s| \geq 1
\end{cases}
\]
be the standard mollifier ($C$ is chosen in order to have $||\zeta||_{L^1(\mathbb{R})} = 1$) and define the family
\[
\zeta^\varepsilon(s) = \frac{1}{\varepsilon} \zeta\left(\frac{s}{\varepsilon}\right), \quad \varepsilon > 0.
\]
Since we need a time regularization, given \( \varphi \in C_0^\infty(E_T) \), we consider the family of mollifiers \( \{ \zeta^\varepsilon \} \), with

\[
\varepsilon < \text{dist (spt } \varphi, E_T),
\]

and we set

\[
\varphi_\varepsilon(x,t) = (\varphi \ast \zeta^\varepsilon)(x,t) = \int_{\mathbb{R}} \varphi(x,t-s)\zeta^\varepsilon(s) \, ds.
\]

We insert \( \varphi_\varepsilon \) as test function in (1.9), change variables and apply Fubini’s theorem to obtain

\[
\iint_{E_T} -u_\varepsilon \varphi_\varepsilon + A_\varepsilon(x,t,u,Du^m) \cdot D\varphi \, dxdt = \iint_{E_T} f_\varepsilon \varphi \, dxdt,
\]

where the subscript in \( u_\varepsilon, f_\varepsilon \), and \( A_\varepsilon \) denotes the mollification with respect to time.

We conclude this introductory section with our main result.

**Theorem 1.1** (Local higher integrability). Let \( u \geq 0 \) be a local, weak solution to (1.5)-(1.6) in \( E_T \) for \( \frac{(n-2)}{n+2} < m < 1 \), and \( f \in L^\infty_{\text{loc}}(E_T) \). Then, there exists \( \varepsilon_0 > 0 \), depending only on \( n, m, \nu \), and \( L \) of (1.6), such that

\[
u^m \in L^{2+\varepsilon}_{\text{loc}}(0,T;W^{1,2+\varepsilon}_{\text{loc}}(E)) \quad \forall \varepsilon \in (0,\varepsilon_0).
\]

Theorem 1.1 is a straightforward consequence of local quantitative estimates. We provide two different versions, a first one for standard parabolic cylinders \( B_t(x_0) \times (t_0-r^2, t_0) \) (see Theorem 7.4), and a second version on the so-called *intrinsic* cylinders (see below for their definition), which inherit the natural scaling properties of the solution (see Theorem 7.3).

The assumption that the right-hand side locally lies in \( L^\infty_{\text{loc}}(E_T) \) is not the sharpest possible one: this choice has been done in order to simplify some of the computations to follow, but more general hypotheses could be made.

### 1.2. Novelty and Significance.

For *elliptic* equations and systems, Meyers & Elcrat [ME75] showed that the gradients of solutions locally belong to a slightly higher Sobolev space than expected a priori. The main tools are a reverse Hölder inequality for \( |Du| \) and an application of *Gehring’s lemma* (see the original paper [Geh73] and also [GM79, Str80]). The method works for equations with \( p \)-growth, hence degenerate and singular elliptic equations of \( p \)-Laplacian type are allowed.

Giaquinta & Struwe [GS82] extended the elliptic, local, higher integrability result to parabolic equations. However, in their work, in order to derive the reverse Hölder inequality, the diffusion term \( A \) has a linear growth with respect to \( |Du| \), so that degenerate and singular equations are ruled out.

The main obstruction to the extension to the degenerate/singular setting is given by the lack of homogeneity in the energy estimates. This problem can be overcome by using the so-called *intrinsic parabolic geometry*, originally developed by DiBenedetto & Friedman [DBF85, DiB93] in the context of the parabolic \( p \)-Laplace equation. It is a scaling, which depends on the solution itself. Under a more physical point of view, the diffusion process evolves at a time scale which depends instant by instant on \( u \) itself; the homogeneity is recovered, once the time variable is rescaled by a factor that depends on the solution in a suitable way.

Later on, by rephrasing these ideas in the context of intrinsic Calderón-Zygmund coverings, Kinnunen & Lewis [KL00] were able to show that gradients of solutions to equations with the same structure as the parabolic \( p \)-Laplacian enjoy a higher integrability property, namely

\[ Du \in L^{p+\varepsilon}_{\text{loc}}(E_T), \quad \text{for some } \varepsilon > 0. \]

This result holds under very general structural assumptions on the operator, and minimal conditions on the right-hand side. The values of \( p \) cover the full degenerate range \( p > 2 \), but are restricted to the super-critical singular range \( \frac{2n}{n+2} < p < 2 \) (this phenomenon is analogous to the above-mentioned restriction on the value of \( m \) in (1.4)). Kinnunen & Lewis’s result was then extended to many different contexts: just to mention a few of them, see [Bög08, BP10, Par09a, Par09b].
Notwithstanding the large amount of results for solutions to $p$-Laplacian-like equations, the higher integrability result for the porous medium equations has remained open for quite some years.

With respect to the $p$-Laplacian situation, the porous medium equation presents a distinctive difficulty: if $c$ is a positive constant, and $u$ is a local solution to (PME), then in general $u - c$ is not a solution: this seemingly small obstacle makes it impossible to apply to the porous medium equation the approach known for the $p$-Laplacian. Indeed, the latter is based on (scaling invariant) Sobolev-Poincaré inequalities of Gagliardo-Nirenberg type in space-time, which are invariant by a constant (i.e. the mean value).

When dealing with the higher integrability for the gradient of solutions to the porous medium equation, in [GS16] we overcame this difficulty by splitting the problem into two cases: degenerate and non-degenerate regimes. This is a very common approach, going back to DeGiorgi. It has been used to get estimates for solutions of PDEs in many different contexts. If we just limit ourselves to solutions to (PME), without pretending to give a full list of all the relevant results, this method is a key tool to derive Harnack inequalities, as well as to prove Hölder continuity: just as an example, see [DGV]. However, as far as we can say, its use in the context of gradient estimates for the porous medium equation had not been tried before.

A second novelty of [GS16] is in Calderón-Zygmund covering, where we used cylinders which are intrinsically scaled with respect to what seems the natural quantity here, namely $u^{m-1}$, where $u$ is the solution. In other terms, we considered cylinders of the type $Q_{\theta \rho^2, \rho}$ whose space-time scaling is adapted to the solution $u$ via the coupling

$$\iint_{Q_{\theta \rho^2, \rho}} u^{m+1} \, dx \, dt \approx \theta^{-\frac{m+1}{m-1}}.$$

Finally, there is a third crucial point in the proof of the higher integrability of [GS16]: In the usual approach, as, for example, in [KL00], one constructs a covering of super-level sets of the spatial gradient with intrinsic cylinders. However, this is not possible for the cylinders which are intrinsically scaled with respect to $u$. Therefore, we weakened this property to so-called sub-intrinsic cylinders, i.e. cylinders for which the mean value integral of $u^{m+1}$ is not approximately equal to the right-hand side, but only bounded from above by it.

In [GS16] nonnegative solutions to scalar equations are considered; systems have been dealt with in [BDKS2018]: as a by-product of the vectorial case, Bögelein-Duzaar-Korte & Scheven were able to consider also signed solutions in the scalar case. Moreover, in [GS16] weak solutions are defined assuming that $Du^\frac{1}{m-1} \in L^2_{loc}(E_T)$, and as a result they end up satisfying $Du^\frac{1}{m-1} \in L^2_{loc}(E_T)$; in [BDKS2018] the authors start with solutions satisfying $Du^m \in L^2_{loc}(E_T)$, and as a result, they prove that $Du^m \in L^{2^*}(E_T)$.

Finally, coming to (1.5)–(1.6) for $0 < m < 1$, even though the regularity for solutions has been widely studied (see for example [Chen-92, DiB93, DiKV]), much less is known about the properties of the gradient, and, in particular, the higher integrability of $Du^m$ seems to be completely open. The approach we use in these notes, largely follows the same path developed in [GS16], and described above, the main difference being perhaps in the notion of solution. Indeed, here we assume that $Du^m \in L^2_{loc}(E_T)$, and this generates some technical difficulties. On the other hand, we avoided any use of the so-called expansion of positivity in dealing with the non-degenerate regime. Therefore, the method could now be employed to study vector-valued problems and signed solutions, but both for simplicity and to strengthen the emphasis on the technical novelties of this work, once more we have just focused on nonnegative solutions to scalar equations.

The construction of the intrinsic cylinders originally introduced in [Sch13] to study the parabolic $p$-Laplacian for $p \geq 2$, is here extended to the singular setting, and the proof is given with all details; indeed, we think it is of independent interest, and it might find applications in other contexts.

In order to underline the potential use of the intrinsic cylinders built in Lemma 3.1, we show that they can be used to provide bounds on the solutions in a new, clean formulation (see Proposition 5.2), even though qualitative and quantitative bounds of solutions to the singular porous medium equations are well-known (see, for example, [DGV, Appendix B]).

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2. Preliminaries

2.1. Notation. Consider a point \( z_o = (x_o, t_o) \in \mathbb{R}^{n+1} \) and two parameters \( \rho, \tau > 0 \). The open ball with radius \( \rho \) and center \( x_o \) will be denoted by

\[
B_{\rho}(x_o) \equiv \{ x \in E : |x - x_o| < \rho \},
\]

whereas the time interval with width \( \tau \) and center \( t_o \) will be denoted by

\[
A_{\tau}(t_o) \equiv \{ t \in (0, T) : t_o - \tau < t < t_o + \tau \}.
\]

Finally, we define the time-space cylinder by

\[
Q_{\tau, \rho}(z_o) \equiv (t_o - \tau, t_o + \tau) \times B_{\rho}(x_o).
\]

As we prove local estimates, the reference point is never of importance, and we often omit it by writing \( B_{\rho} \) and \( Q_{\tau, \rho} \).

The symbol \( | \cdot | \) stands for the Lebesgue measure, either in \( \mathbb{R}^n \) or \( \mathbb{R}^{n+1} \), and the dimension will be clear from the context.

Moreover, we write \( f \sim g \) if there exist constants \( c, C > 0 \), which depend only on the data, such that \( cf \leq g \leq Cf \).

2.2. An auxiliary result.

**Proposition 2.1.** Given \( u, a \geq 0 \) and \( 0 < m < 1 \), we have

\[
\frac{1}{2}(u - a)(u^m - a^m) \leq \int_a^u (y^m - a^m) \, dy \leq (u - a)(u^m - a^m) \quad \text{if } u \geq a,
\]

\[
\frac{m}{2}(a - u)(a^m - u^m) \leq \int_a^u (a^m - y^m) \, dy \leq (a - u)(a^m - u^m) \quad \text{if } u < a.
\]

**Proof.** The function \( f(y) = y^m - a^m \) is concave, so the integral in (2.1) is bounded from below by the area of the triangle, and from above by the area of the rectangle.

The integral in (2.2) is estimated from above using the area of the rectangle once more; using the triangle, the bound form below follows from the inequality

\[
\frac{1}{2} m [a^{m-1} (a - u)] (a - u) \geq \frac{1}{2} m (a^m - u^m) (a - u),
\]

since \( 0 < m < 1 \) and \( u < a \). \( \square \)

2.3. Constants and data. As usual, the letter \( c \) is reserved to positive constants, whose value may change from line to line, or even in the same formula. We say that a generic constant \( c \) depends on the data, if \( c = c(n, m, \nu, L) \), where \( \nu \) and \( L \) are the quantities introduced in (1.6).

Let \( \eta \in L^\infty(E) \) and nonnegative; we let

\[
(g)_E^\eta = \frac{1}{\| \eta \|_{L^1(E)}} \int_E g \, \eta \, dx.
\]

In the special case of \( \eta \equiv 1 \), we write

\[
(g)_E^1 =: (g)_E = \int_E g \, dx.
\]

We will frequently use what we will refer to in the following as the best constant property. For positive \( \eta \) we have, for any \( c \in \mathbb{R} \) and \( q \in [1, \infty) \)

\[
\left( \frac{1}{\| \eta \|_{L^1(E)}} \int_E | g - (g)_E^\eta \|_E^q \, \eta \, dx \right)^{\frac{1}{q}} \leq 2 \left( \frac{1}{\| \eta \|_{L^1(E)}} \int_E | g - c \|_E^\eta \, \eta \, dx \right)^{\frac{1}{q}}.
\]

Moreover, if \( 0 \leq \eta \leq 1 \), by (2.3) one obviously obtains that

\[
\left( \frac{1}{\| \eta \|_{L^1(E)}} \int_E | g - (g)_E^\eta \|_E^q \, \eta \, dx \right)^{\frac{1}{q}} \leq 2 \left( \frac{1}{\| \eta \|_{L^1(E)}} \int_E | g - (g)_E^\eta \|_E^q \, dx \right)^{\frac{1}{q}},
\]
and as a consequence, that
\[
|(g)_E^q - (g)_E| \leq \left( \frac{1}{\eta \| \cdot \|_{L^1(E)}} \int_E |g - (g)_E^q| \eta \, dx \right)^\frac{1}{q} \\
\leq 2 \left( \frac{1}{\eta \| \cdot \|_{L^1(E)}} \int_E |g - (g)_E| \eta \, dx \right)^\frac{1}{2}.
\]

(2.5)

We will also use the following estimate that was first proved in [DKS11, Lemma A.2].

**Lemma 2.2.** For \( q \in (\frac{1}{2}, \infty) \) we have
\[
\int_E |g^q - (g)_E^q|^2 \, dx \leq c_0 \int_E |g^q - (g^q)_E|^2 \, dx \leq c_1 \int_E |g - (g)_E|^q \, dx
\]
where \( c_0 \) and \( c_1 \) are constants that depend only on the data. The same estimate holds for \( q \in (0, \frac{1}{2}) \) in case
\[
\sup_E |g| \leq K(g)_E
\]
for some \( K > 0 \).

**Proof.** By the Fundamental Theorem of Calculus, and the orthogonality of the mean value to constants we have
\[
\int_E |g^q - (g)_E^q|^2 \, dx \sim \int_E (g^{2q-1} - (g)_E^{2q-1}) \cdot (g - (g)_E) \, dx
\]
\[
= \int_E (g^{2q-1} - (g^q)_E^{2q-1}) \cdot (g - (g)_E) \, dx.
\]

In case that \( q > \frac{1}{2} \), we find that
\[
\left| g^{2q-1} - (g^q)_E^{2q-1} \right| \sim (g + (g^q)^{\frac{1}{2}})^{2q-2} \left| g - (g^q)^{\frac{1}{2}} \right|,
\]
and conclude by dividing in the cases \( |u - (g)_E| \leq c \left| g - (g^q)^{\frac{1}{2}} \right| \) and its opposite.

In case that \( q \in (0, \frac{1}{2}) \), we find
\[
\int_E |g^q - (g)_E^q|^2 \, dx \sim \int_E (g + (g)_E)^{2q-2} |g - (g)_E|^2 \, dx
\]
\[
\leq (g)_E^{2q-2} \int_E |g - (g)_E|^2 \, dx
\]
\[
\leq (g)_E^{2q-2} \int_E \left| g - (g^q)^{\frac{1}{2}} \right|^2 \, dx
\]
\[
\leq c_K \int_E (g + (g)_E)^{2q-2} \left| g - (g^q)^{\frac{1}{2}} \right|^2 \, dx
\]
\[
\leq c_K \int_E (g + (g^q)^{\frac{1}{2}})^{2q-2} \left| g - (g^q)^{\frac{1}{2}} \right|^2 \, dx \sim \int_E |g^q - (g^q)_E|^2 \, dx
\]

\( \Box \)

The restriction on \( q > \frac{1}{2} \) in Lemma is rather strong. However, surprisingly enough, a different argument allows to circumvent the problems of small \( m \). As the following lemma shows, for any convex quantity a “mean change” is possible.

**Lemma 2.3.** Let \( \eta \in L^\infty(E) \) and nonnegative. For \( m \in (0, 1) \), \( \rho \geq \frac{1}{m} \), and \( u \in L^p(E) \) and nonnegative, we have
\[
\left( \frac{1}{\| \eta \|_1} \int_E \left| u^m - ((u)_E^p)^m \right|^p \eta \, dx \right)^\frac{1}{p} \leq c_o \left( \frac{1}{\| \eta \|_1} \int_E \left| u^m - (u)_E^m \right|^p \eta \, dx \right)^\frac{1}{p}
\]
\[
\leq 2c_o \left( \frac{1}{\| \eta \|_1} \int_E \left| u^m - ((u)_E^p)^m \right|^p \eta \, dx \right)^\frac{1}{p},
\]
where $c_\alpha$ is a constant that depends only on $m$ and $p$.

**Proof.** The first estimate follows by the best constant property. Concerning the second estimate, let

$$\lambda = (u)_{E}^{\eta} \quad \text{and} \quad e^m = (u^m)_{E}^{\eta}.$$  

By Jensen’s inequality $e \leq \lambda$. Now, for any $a \in \mathbb{R}_+$ we estimate

$$\left(\frac{1}{\|\eta\|_1} \int_E |u^m - \lambda^m|^p \eta \, dx\right)^{\frac{1}{p}} \leq c \left(\frac{1}{\|\eta\|_1} \int_E |u^m - a^m|^p \eta \, dx\right)^{\frac{1}{p}} + c |a^m - \lambda^m|.$$

Since for $x \in \{u > 2a\}$

$$u^m(x) \leq |u^m(x) - a^m| + a^m \leq |u^m(x) - a^m| + \frac{u^m(x)}{2^m},$$

we find

$$u^m \chi_{\{u > 2a\}} \leq \frac{2^m}{2^m - 1} |u^m - a^m| \chi_{\{u > 2a\}}.$$

Now

$$|a^m - \lambda^m| \sim (a + \lambda)^{m-1} |a - \lambda| \leq (a + \lambda)^{m-1} \frac{1}{\|\eta\|_1} \int_E |u - a| \eta \, dx$$

$$\leq (a + \lambda)^{m-1} \frac{1}{\|\eta\|_1} \int_E |u - a| \chi_{\{u \leq 2a\}} \eta \, dx + (a + \lambda)^{m-1} \frac{1}{\|\eta\|_1} \int_E |u - a| \chi_{\{u > 2a\}} \eta \, dx,$$

$$\leq \frac{a^{m-1}}{\|\eta\|_1} \int_E |u - a| \chi_{\{u \leq 2a\}} \eta \, dx + \frac{\lambda^{m-1}}{\|\eta\|_1} \int_E u \chi_{\{u > 2a\}} \eta \, dx$$

$$\leq c \frac{1}{\|\eta\|_1} \int_E (a + u)^{m-1} |u - a| \chi_{\{u \leq 2a\}} \eta \, dx$$

$$+ \left(\frac{1}{\|\eta\|_1} \int_E u \chi_{\{u > 2a\}} \eta \, dx\right)^m$$

$$\leq c \frac{1}{\|\eta\|_1} \int_E |u^m - a^m| \chi_{\{u \leq 2a\}} \eta \, dx + c \left(\frac{1}{\|\eta\|_1} \int_E |u^m - a^m|^m \chi_{\{u > 2a\}} \eta \, dx\right)^m,$$

and this implies the result by Jensen’s inequality. \qed

We close the section with the following estimate, which was originally derived in [Sch13, Lemma A.1]. See also [GS16].

**Lemma 2.4.** Let $Q_1 \subset Q$ be two cylinders and $f \in L^q(Q)$ for some $q \in [1, \infty)$. If for some $\varepsilon \in (0, 1)$ we have

$$|\langle f \rangle_{Q_1}| \leq \varepsilon |\langle |f|^q \rangle_{Q_1}|^{\frac{1}{q}},$$

then

$$|\langle f \rangle_{Q_1}| \leq \varepsilon |\langle |f|^q \rangle_{Q_1}|^{\frac{1}{q}} \leq \frac{\varepsilon}{1 - \varepsilon} \left(1 + \left(\frac{|Q|}{|Q_1|}\right)^{\frac{1}{q}} \left(\int_Q |f - \langle f \rangle_{Q_1}|^q \, dx dt\right)^{\frac{1}{q}}\right).$$
3. Constructing Proper Cylinders

Since the construction of intrinsic cylinders below has the potential to be significant for future application not only for the porous medium equation, but also for general singular systems of various kinds, we will use a notation that emphasizes the generality of the approach. Indeed, the covering can be introduced with respect to any integrable function \( f \), related to a scaling exponent \( p \) or \( m \).

Consider a degenerate or singular diffusion equation or system: by intrinsic we mean a scaling that inherits the local ellipticity coefficient (i.e. the diffusion coefficient) with respect to the mean value of the function. For the porous medium equation, the ellipticity coefficient is of order \( m u^{m-1} \). In this case we will call \( Q_{s_0}^{\sqrt{s_0}/\theta} \) a \( K \)-intrinsic cylinder, if

\[
\frac{\theta}{K} \leq (u^{m+1})^{\frac{1}{2m}} Q_{s_0}^{\sqrt{s_0}/\theta} \leq K \theta,
\]

for some \( K \geq 1 \). We will call \( Q_{s_0}^{\sqrt{s_0}/\theta} \) a \( K \)-sub-intrinsic cylinder, if only the estimate from above holds in (3.1). Observe, that (in the intrinsic case) \( \theta^{-1} \sim u^{-1} \) is somehow proportional to the scaling of ellipticity.

In the case of the p-Laplacian

\[
\begin{align*}
& u \in C^{0}_{\text{loc}}(0,T; L^{p}_{\text{loc}}(E)) \cap L^{p}_{\text{loc}}(0,T; W^{1,p}_{\text{loc}}(E)) \\
& u_t - \text{div} (|Du|^{p-2} Du) = f \quad \text{weakly in } E_T,
\end{align*}
\]

the ellipticity coefficient is of order \( |Du|^{p-2} \). Here one usually considers the size \( \lambda \sim |Du| \). The canonic analogue of \( \theta \) is therefore \( \lambda^{2-p} \), and \( p - 1 \) corresponds to \( m \).

Observe that this type of cylinders were initially introduced for the degenerate p-Laplacian (cf. [Sch13]). For this reason alone, we will stick to the exponent \( p=\gamma+1 \) and we will relate it to a general integrable function \( f \); one only needs to substitute \( f \) with \( |Du|^{p} \) or \( u^{\gamma+1} \), or any other suitable function, depending on the problem under consideration.

In terms of the function \( f \) and the exponent \( p \), we will then call \( Q_{s_0}^{\sqrt{s_0}/\theta}(z) =: Q_{s_0}^{\sqrt{s_0}/\lambda^{p-2}}(z) \) a \( K \)-intrinsic cylinder, if

\[
\frac{\theta}{K} \leq (|f|)_{Q_{s_0}^{\sqrt{s_0}/\lambda^{p-2}}} \leq K \lambda^{2-p} = K \theta,
\]

for some \( K \geq 1 \), and we will call \( Q_{s_0}^{\sqrt{s_0}/\lambda^{p-2}} \) a \( K \)-sub-intrinsic cylinder, if only the estimate from above in (3.3) holds true.

In the following, we will avoid any reference to the constant \( K \), meaning that the cylinders will be either intrinsic or sub-intrinsic for some proper \( K \). Moreover, \( Q^p_{s_0} \), \( Q^{\lambda_0}_{s_0} \), and \( Q_{s_0}(r(t,x)) \) will all denote the same object: the use of either one of the three possible notation depends on the particular geometric feature we want to emphasize.

**Lemma 3.1.** Let \( p \in (\frac{2n}{(n+2)}, 2) \). Let \( Q_{S,R}(t,x) \subset \mathbb{R}^{n+1} \), \( f \in L^1(Q_{S,R}(t,x)) \) and \( \hat{b} \in (0,1/2) \), such that \( b < (n+2)p - 2n \). For every \( 0 < s \leq S \) there exist \( r(s), \lambda_0, \text{ and } Q_{s,r(t,x)} \) with the following properties. Let \( s, \sigma \in (0,R] \) and \( s < \sigma \), then

(a) \( 0 \leq r(s) \leq R \) and \( r(s) =: \sqrt{\lambda_0^{2-p}s} =: \sqrt{s/\theta} \). In particular, \( Q_{s,r(t,x)}(t,x) =: Q_{s}^{\lambda_0}(t,x) =: Q_{s}(t,x) \subset E_T \).

(b) \( r(s) \leq \left( \frac{\lambda_0}{\lambda_0^*} \right)^{\hat{b}} r(\sigma) \), the function \( r = r(s) \) is continuous and strictly increasing on \( [0,S] \). In particular, \( Q_{s}^{\lambda_0} \subset Q_{s}^{\lambda_0^*} \).

(c) \( \iint_{Q_{s}^{\lambda_0}} |f| \ dx \ dt \leq \lambda_0^* \), i.e. \( Q_{s}^{\lambda_0^*} \) is sub-intrinsic.

(d) If \( r(s) < \left( \frac{\lambda_0^*}{\lambda_0^*} \right)^{\hat{b}} r(\sigma) \), then there exists \( s_1 \in [s, \sigma) \) such that \( Q_{s_1}^{\lambda_0} \) is intrinsic.

(e) If for all \( s \in (s_1, \sigma) \), \( Q_{s_1}^{\lambda_0} \) is strictly sub-intrinsic, then \( \lambda_0^{2-p} \leq \left( \frac{\lambda_0^*}{\lambda_0^*} \right)^{\beta} \lambda_0^* \) for all \( s \in [s_1, \sigma] \) and \( \beta = (1 - 2\hat{b}) \in [0,1) \).

(f) For \( \gamma \in (0,1] \), we have

\[
r(s) \leq c \gamma^{-\hat{b}} r(s),
\]
\[ |Q_{\gamma s}^{\lambda_2}|^{-1} \leq c_2^{-n} |Q_{s}^{\lambda_2}|^{-1}, \]
\[ \lambda_2^2 - p \leq c_2^{-\max\{2a, b\}} \lambda_s^2 - p, \]

with \( \lambda = \hat{b} + \frac{2}{2p - (2 - p)|n|} \). Observe, that \( \lambda \to \infty \), as \( p \to \frac{2n}{n + 2} \).

(g) For \( c > 1 \) we have \( Q_{cs, cr(s)} \subset Q_{c_2, cr(s)} \subset Q_{c_2, cr(s)} \) for \( c = c_2^{\frac{1}{p}} \) and \( r = \max\{c, c_2^{\frac{1}{p}}\} \).

The constant \( c \) only depends on the dimension \( n \) and on \( p \).

Proof. Let \( Q_{s, t}(t, x) \subset E_\gamma \). In the following we often omit the point \( (t, x) \). Moreover, whenever we integrate both in space and in time, we use the symbol \( dz \). We start by defining for every \( s \in (0, S] \) the quantity

\[ \tilde{r}(s) = \sup \left\{ \rho \in R \mid \left( \int_0^{t + \frac{1}{2}} \int_{B_\rho(x)} |f| \, dz \right)^{2-p} \rho^2 |B_\rho|^{\frac{2}{2-p}} \leq s^2 \right\}. \]

The function \( \tilde{r}(s) \) is well defined and strictly positive for \( r > 0 \). This is due to the fact that

\[ b_0 := (p - 2)n + 2p = (n + 2)p - 2n > 0, \]

since we assume \( p \in \left( \frac{2n}{n + 2}, 2 \right) \). We define \( \hat{\lambda}_s \) by the equation \( \hat{\lambda}_s^2 - p = \frac{s}{\tilde{r}(s)} \). By construction we find that

\[ \left( \int_{Q_{s, \tilde{r}(s)}} |f| \, dz \right)^{2-p} \tilde{r}(s)^{2p} \leq s^p. \] (3.4)

This implies that

\[ \left( \int_{Q_{s, \tilde{r}(s)}} |f| \, dz \right)^{2-p} \leq \left( \frac{s}{\tilde{r}(s)^2} \right)^p, \]

that is

\[ \int_{Q_{s, \tilde{r}(s)}} |f| \, dz \leq \hat{\lambda}_s^p. \] (3.5)

Next, we will show that \( \tilde{r}(s) \) is continuous for \( s \in (0, S] \). First of all, it is enough to show that \( \kappa(s) := \tilde{r}(s)^{\frac{2}{2-p}} |B_{\tilde{r}(s)}|^{-1} \) is continuous.

Take \( 0 < s_1 < s < S \). For \( \varepsilon > 0 \), there exists a \( \delta > 0 \), such that

\[ \int_{t-s}^{t-s_1} \int_{B_R} |f| \, dz \leq \varepsilon \quad \text{for all} \quad s - s_1 < \delta. \]

Now, in case \( \kappa(s_1) < \kappa(s) \leq R^{\frac{2}{2-p}} |B_R|^{-1} \), we find that

\[ \kappa(s) \int_{t-s}^{t} \int_{B_{\tilde{r}(s)}} |f| \, dz - \kappa(s_1) \int_{t-s}^{t} \int_{B_{\tilde{r}(s_1)}} |f| \, dz \leq s^\frac{2}{2-p} - s_1^\frac{2}{2-p} \leq c\delta^\frac{2}{2-p}, \]

and hence

\[ \left[ \int_{t-s_1}^{t} \int_{B_{\tilde{r}(s_1)}} |f| \, dz \right] \kappa(s) - \kappa(s_1) \leq c\delta^\frac{2}{2-p}, \]

which concludes this argument, since \( \int_{t-s_1}^{t} \int_{B_{\tilde{r}(s_1)}} |f| \, dz > 0 \) when \( \tilde{r}(s_1) < R \).

In case \( \kappa(s) \leq \kappa(s_1) \leq R^{\frac{2}{2-p}} |B_R|^{-1} \), we find

\[ \kappa(s) \int_{t-s_1}^{t} \int_{B_{\tilde{r}(s_1)}} |f| \, dz + \varepsilon \geq \kappa(s) \int_{t-s}^{t} \int_{B_{\tilde{r}(s)}} |f| \, dz \geq s^\frac{2}{2-p}, \]

since \( \int_{t-s_1}^{t} \int_{B_{\tilde{r}(s_1)}} |f| \, dz > 0 \). Hence

\[ \int_{t-s_1}^{t} \int_{B_{\tilde{r}(s_1)}} |f| \, dz \kappa(s_1) - \kappa(s) \leq s_1^\frac{2}{2-p} - s^\frac{2}{2-p} + \varepsilon \leq \varepsilon. \]

This concludes the proof of the continuity, as the case \( \tilde{r}(s) = \tilde{r}(s_1) \) is trivial.
Now it might happen, that $s < \sigma$ and $\tilde{r}(s) > \tilde{r}(\sigma)$. To avoid that, for $\hat{b} \in (0, b_0)$ we define

$$r(s) = \min_{s \leq a \leq S} \left( \frac{s}{a} \right)^{\hat{b}} \tilde{r}(a).$$

The minimum exists, as $\left( \frac{s}{a} \right)^{\hat{b}} \tilde{r}(a)$ is continuous in $a$. If we take $\sigma \in (s, S]$ we find that

$$r(s) = \min \left\{ \min_{s \leq a \leq \sigma} \left( \frac{s}{a} \right)^{\hat{b}} \tilde{r}(a), \left( \frac{s}{\sigma} \right)^{\hat{b}} r(\sigma) \right\}$$

we find that $r(s) < r(\sigma)$. Now we define $\lambda_s := \left( \frac{(s)^2}{s} \right)^{\hat{b}} \geq \tilde{\lambda}_s$ and $Q_s^{\lambda_s} := Q_s, r(s)$. By this definition we find (a) and (b), as $\lim_{s \to 0} r(s) \leq \lim_{s \to 0} \left( \frac{s}{S} \right)^{\hat{b}} R = 0$.

We show (c), by (3.4)-(3.5)

$$\iint_{Q_s, r(s)} |f| \, dz \leq \frac{|B_{r(s)}|}{|B_{r(s)}|} \iint_{Q_s, r(s)} |f| \, dz = \frac{\lambda_s}{\lambda_s} \iint_{Q_s, r(s)} |f| \, dz \leq \lambda_s^{2n+2p-2n} \leq \lambda_s^{\gamma_s}.$$

To prove (d) we assume that $r(s) < \left( \frac{s}{\sigma} \right)^{\hat{b}} r(\sigma)$. Then there exists $s_1 \in [s, \sigma)$, such that

$$\left( \frac{s}{s_1} \right)^{\hat{b}} \tilde{r}(s_1) = r(s) = \min_{s \leq a \leq S} \left( \frac{s}{a} \right)^{\hat{b}} \tilde{r}(a).$$

This implies that $\tilde{r}(s_1) = r(s_1)$. Since by our assumption we also have that

$$r(s_1) < \left( \frac{s_1}{\sigma} \right)^{\hat{b}} r(\sigma) \leq \left( \frac{s_1}{S} \right)^{\hat{b}} R,$$

we deduce by (3.4) that $Q_{s_1, r(s_1)} = Q_{s_1}^{\lambda_{s_1}}$ is intrinsic. This implies (d).

By (d), if $Q_s^{\lambda_s}$ is strictly sub-intrinsic for all $a \in (s, \sigma)$, then $r(a) = \left( \frac{s}{a} \right)^{\hat{b}} r(\sigma)$ for all $a \in (s, \sigma)$. Now we compute

$$\lambda_s^{2n-\gamma} = \frac{a}{r(a)^2} = \left( \frac{s}{\sigma} \right)^{2\hat{b}} r(\sigma)^2 = \left( \frac{s}{\sigma} \right)^{2\hat{b} - \gamma_s},$$

and this yields (e), with $\beta = 1 - 2\hat{b}$.

To prove (f) we take $\gamma \in (0, 1)$. First we show the estimate for $r(s)$. If $r(\gamma s) = \gamma^{\hat{b}} r(s)$ there is nothing to show. If $r(\gamma s) < \gamma^{\hat{b}} r(s)$, we find by (d) that there is a $\gamma_1 \in [\gamma, 1)$ with $r(\gamma s) = \left( \frac{s}{\gamma} \right)^{\hat{b}} r(\gamma_1 s)$ and $Q_{\gamma_1 s}^{\lambda_{\gamma_1 s}}$ is intrinsic. Next, by the sub-intrinsicity of $Q_s^{\lambda_s}$ and the intrinsicity of $Q_{\gamma_1 s}^{\lambda_{\gamma_1 s}}$ and by the calculation made in (3.5) and (3.4), we find that

$$\left( \iint_{Q_{\gamma_1 s}, r(\gamma s)} |f| \, dz \right)^{2-p} \leq s^2 = \frac{\gamma_1^2 s^2}{\gamma_1^2} = \frac{1}{\gamma_1^2} \left( \iint_{Q_{\gamma_1 s}, r(\gamma s)} |f| \, dz \right)^{2-p} \left( r(\gamma s) \right)^{2p-(2-p)n} |B_1|^{2p-(2-p)n}.$$

Since by construction $Q_{\gamma_1 s}^{\lambda_{\gamma_1 s}} \subset Q_{s}^{\lambda_s}$, this implies that

$$r(s)^{2p-(2-p)n} \leq \left( \frac{1}{\gamma_1} \right)^{2p-(2-p)n} r(\gamma s)^{2p-(2-p)n} |B_1|^{2p-(2-p)n}.$$

and so

$$r(s) \leq \frac{c}{\gamma_1^{2p-2p(n)}} r(\gamma s) = \frac{c}{\gamma_1^{2p-2p(n)}} \left( \frac{\gamma_1}{\gamma} \right)^{\hat{b}} r(\gamma s) \leq \frac{c}{\gamma^{\hat{b} + \frac{2p-2p(n)}{2-p(n)}}} r(\gamma s) = c\gamma^{-\hat{a}} r(\gamma s),$$

with $\hat{a} = \hat{b} + \frac{2p-2p(n)}{2-p(n)}$. Hence the estimate on $r(s)$ is proved.
The estimate on the size of the intrinsic cylinders is a direct consequence. In order to give an estimate for the $\lambda_s$, once more we may assume that $r(\gamma s) < \gamma^b r(s)$, since otherwise $\lambda_{\gamma s}^{2-p} = \gamma^b \lambda_s^{2-p}$ and there is nothing to show. Hence, by the above definitions and (c) we find

$$\lambda_{\gamma s}^{2-p} = \left(\frac{\gamma}{\gamma_{s1}}\right)^\beta \lambda_{s1}^{2-p} = \left(\frac{\gamma}{\gamma_{s1}}\right)^\beta \gamma_{s1}^s \frac{r(\gamma s)}{r(\gamma_{s1}s)} \leq \left(\frac{\gamma}{\gamma_{s1}}\right)^\beta \left(\frac{r(s)}{r(\gamma_{s1}s)}\right)^2 \lambda_s^{2-p} \leq \gamma^{2\hat{a}-2} \lambda_s^{2-p}$$

which closes the argument for (f). Finally, (g) follows directly by (b) and (f).

In the sequel, we will make heavy use of the fact that for $\gamma \in (0,1]$, up to a constant,

$$r(\gamma s) \leq \gamma^b r(s) \leq \gamma^b \theta^a r(\gamma s),$$

$$\theta_s \leq \gamma^{2\hat{a}-1} \theta_{s\gamma} \leq \gamma^{2(\hat{b}-\hat{a})} \theta_s.$$  \hfill (3.8)

We will also use the following version of Vitali’s covering from [GS16, Lemma 5.4]. It is inspired by [Ste93, Chapter 1, Lemma 1 and Lemma 2]. See also [EV92, Paragraph 1.5, Theorem 1].

**Lemma 3.2.** Let $\Omega \subset \mathbb{R}^M$ and $R \in \mathbb{R}$. Let there be given a two-parameter family $\mathcal{F}$ of nonempty and open sets

$$\{U(x, r) | x \in \Omega, r \in (0, R]\},$$

which satisfy

(i) They are nested, that is,

$$\text{for any } x \in \Omega, \text{ and } 0 < s < r \leq R, U(x, s) \subset U(x, r);$$  \hfill (3.9)

(ii) There exists a constant $c_1 > 1$, such that

$$U(x, r) \cap U(y, r) \neq \emptyset \Rightarrow U(x, r) \subset U(y, c_1 r).$$  \hfill (3.10)

(iii) There exists a constant $a > 1$ such that, for all $r \in (0, R],

$$0 < |U(x, 2r)| \leq a |U(x, r)| < \infty.$$  \hfill (3.11)

Then we can find a disjoint subfamily $\{U_i\}_{i \in \mathbb{N}} = \{U(x_i, r_{x_i})\}_{i \in \mathbb{N}}$, such that

$$\bigcup_{x \in \Omega} U(x, r_x) \subset \bigcup_{i \in \mathbb{N}} \tilde{U}_i,$$

with $\tilde{U}_i = U(x_i, 2c_1 r_{x_i})$, $|U_i| \sim |\tilde{U}_i|$ and

$$|\Omega| \leq c \sum_{i} |U_i|,$$

where the constant $c > 1$ depends only on $c_1$, $a$, and the dimension $M$.

Notice that the family of cylinders built in Lemma 3.1 satisfies (3.9) and (3.11). In the next lemma we show that for a family of these cylinders also (3.10) is satisfied and so the constructed family of intrinsic cylinders can actually provide a suitable Vitali-type covering; (3.10) essentially replaces the necessity to work in a metric space, which is most commonly assumed in order to apply Vitali’s covering. In some sense, the construction of Lemma 3.1 provides a sort of intrinsic metric with respect to any integrable function.

**Lemma 3.3.** Let $p \in \left(\frac{2m}{m+2}, 2\right)$, $Q_{2S,2R} \subset E_T$, $f \in L^1(Q_{2S,2R})$, and $\hat{b} \in (0,1)$. For every $z \equiv (t, x) \in Q_{S,R}$ and $0 < s \leq S$, there exist $r(s, z)$, $\lambda_s$, and a sub-intrinsic cylinder $Q_{s,r(s,z)}(z) = (t-s,t+s) \times B_{r(s,z)}(x)$, such that all properties of Lemma 3.1 hold. In particular, the cylinders $Q_{s,r(s,z)}(z)$ form a nested family of sub-intrinsic cylinders with respect to the sizes

$$\lambda_s^{2-p} = \frac{s}{(r(s,z))^2} = \theta_{s^{m+1}}, \text{ for } m+1 = p.$$  \hfill (3.12)

Moreover, if we assume that the cylinder $Q_{2S,2R}$ is sub-intrinsic, namely that for some $K \geq 1$

$$\left(\iint_{Q_{2S,2R}} |f| \, dx dt\right)_{\frac{2-p}{p}} \leq K \frac{S}{R^2} =: K \lambda_o^{2-p} (=: K \theta_o),$$

then
(1) for all \( z \equiv (t, x) \in Q_{S, R} \), we have
\[
\lambda_{\alpha}^{2-p} \leq \lambda_{S, z}^{2-p} \leq cK^{2p\alpha} \lambda_{\alpha}^{2-p} \quad \text{or} \quad \theta_{\alpha} \leq \theta_{S, z} \leq cK^{2p\alpha} \theta_{\alpha}
\]
with \( \alpha = \frac{1}{2p-n(p-2)} \).

(2) there is a constant \( c_1 > 1 \), depending only on \( n, m, \hat{b} \), such that if \( Q_{s, r(s, z)}(z) \cap Q_{s, r(s, y)}(y) \neq \emptyset \), then
\[
Q_{s, (r(s, z))}(z) \subset Q_{c_1 s, r(c_1 s, z)}(z) \quad \text{and} \quad Q_{s, r(s, y)}(y) \subset Q_{c_1 s, r(c_1 s, z)}(z),
\]
for each \( s \leq \frac{s}{c_1} \), with \( c_0 = cK^{2p\alpha} \).

Proof. For \( z = (t, x) \in Q_{S, R} \), we will construct proper sub-intrinsic cylinders.
To prove the estimate on \( \lambda_{\alpha}^{2-p} =: \theta_{S, z} \), we have to fix the initial cube \( Q_{S, R(S(z))}(z) \), required by Lemma 3.1. We do this, by defining \( R(S, z) \) as the maximum in \( (0, R) \) such that
\[
\left( \int_{t-S/2}^{t+S/2} \int_{B_{R(S,z)}(x)} |f|^p \ dx \ dt \right)^{\frac{2-p}{p}} R(S, z)^{2p} |B_{R(S,z)}|^{p-2} \leq S^2.
\]
The above construction implies
\[
\left( \int_{Q_{S, R(S(z))}(z)} |f| \ dx \ dt \right)^{\frac{2-p}{p}} \leq \frac{S}{R(S, z)^{p-2}} =: \lambda_{S, z}^{2-p}.
\]
Since
\[
\left( \int_{t-S/2}^{t+S/2} \int_{B_{R(z)}} |f| \ dx \ dt \right)^{\frac{2-p}{p}} R^{2p} |B_R|^{p-2} \leq \left( \int_{t-2S}^{t+2S} \int_{B_{2R}} |f| \ dx \ dt \right)^{\frac{2-p}{p}} R^{2p} |B_R|^{p-2} \leq cK^p S^2
\]
and \( K \geq 1 \), we find that indeed \( R(S, z) \leq R \) and that \( \frac{R^{2p} |B_R|^{p-2}}{cK^p} \leq R(S, z)^{2p} |B_{R(S,z)}|^{p-2} \), where \( c \) depends only on \( n \) and \( p \). Hence,
\[
\frac{R}{cK^{2p-n(p-2)}} = c^{-1} K^{-p\alpha} R \leq R(S, z) \leq R,
\]
which implies the last assertion
\[
\lambda_{\alpha}^{2-p} \leq \lambda_{S, z}^{2-p} \leq c^2 K^{2p\alpha} \lambda_{\alpha}^{2-p}.
\]
This concludes the proof of (1).

Let us now come to (2). We let \( \gamma = c^{-2} K^{-2\alpha} \) (notice that \( \gamma \in (0, 1) \)), and
\[
c_0 = \max \left\{ \left( \frac{3}{\gamma} \right)^{1/b}, \frac{3}{\gamma} \right\};
\]
when \( \hat{b} \leq 1 \), we have \( c_0 = \frac{3}{\gamma} \). Take \( s \leq \frac{s}{c_0} \); without loss of generality, we may assume that \( r(s, z) \geq r(s, y) \). Now (g) of Lemma 3.1 implies that
\[
Q_{s, r(s, y)}(y) \subset Q_{3s, 3r(s, z)}(z) \subset Q_{3s, r(3s, z)}(z),
\]
and the second inclusion of (2) is satisfied with \( c_1 = 3 \).

If \( r(s, y) \geq \gamma r(s, z) \), we similarly find
\[
Q_{s, r(s, z)}(z) \subset Q_{3s, 2r(s, y)}(y) \subset Q_{3s, r(3s, y)}(y),
\]
which implies the wanted inclusion with \( c_1 = \frac{3}{\gamma} \). Hence, we are left with the case, where \( r(s, y) < \gamma r(s, z) \leq \left( \frac{s}{R} \right)^{\hat{b}} R(S, y) \).

In this case, by (d) and (e) of Lemma 3.1, there exists a \( \sigma \in [s, S] \) such that \( r(s, y) = \left( \frac{s}{R} \right)^{\hat{b}} R(S, y) \) and \( Q_{\sigma, r(\sigma, y)}(y) \) is intrinsic. Moreover, since \( r(s, z) \leq \left( \frac{s}{R} \right)^{\hat{b}} r(\sigma, z) \) by (b) of Lemma 3.1, we have that \( r(\sigma, y) = \hat{r}(\sigma, y) \leq r(\sigma, z) \). Now we let \( y = (t, x) \) and \( z = (t_1, x_1) \) and estimate.
If \( \sigma \in \left[ \frac{2}{3}, S \right) \), using that \( Q_{\sigma,r}(\sigma,y)(y) \) is intrinsic implies
\[
\frac{S^2}{9} \leq \sigma^2 = \left( \int_{t-\sigma/2}^{t+\sigma/2} \int_{B_{\tilde{r}}(\sigma,y)(x)} |f| \, d\xi \, dt \right)^{2-p} |B_{\tilde{r}}(\sigma,y)|^{p-2} (\tilde{r}(\sigma,y))^{2p} \\
\leq \left( \int_{Q_{2R,2S}} |f| \, dx \, dt \right)^{2-p} |B_{\tilde{r}}(\sigma,y)|^{p-2} (\tilde{r}(\sigma,y))^{2p} \\
\leq \left( \int_{Q_{2R,2S}} |f| \, dx \, dt \right)^{2-p} |B_{r}(\sigma,z)|^{p-2} (r(\sigma,z))^{2p} \\
\leq \left( \int_{Q_{2R,2S}} |f| \, dx \, dt \right)^{2-p} |B_{R}|^{p-2} R^{2p} \\
\leq cK^p S^2.
\]

Since the first and the last member of the inequality are similar, it means that actually \( r(s,z) \sim \tilde{r}(\sigma,y) = r(\sigma,y) \). More explicitly,
\[
|B_{r}(\sigma,z)|^{p-2} (r(\sigma,z))^{2p} \leq cK^p |B_{\tilde{r}}(\sigma,y)|^{p-2} (\tilde{r}(\sigma,y))^{2p}
\]
and so \( r(\sigma,z) \leq \tilde{c}_1 r(\sigma,y) \), for \( \tilde{c}_1 = \left[ cK^p \right]^{1-p/(2-p)} \); moreover,
\[
r(s,z) \leq \left( \frac{s}{\sigma} \right)^\tilde{b} r(\sigma,z) \leq \tilde{c}_1 \left( \frac{s}{\sigma} \right)^\tilde{b} \tilde{r}(\sigma,y) = \tilde{c}_2 r(s,y).
\]

On the other hand, if \( \sigma \in (s, \frac{2}{3}) \), since the cylinders are nested and due to (g) of Lemma 3.1, we have that \( Q_{\sigma,\tilde{r}}(\sigma,y) \subset Q_{3\sigma,3\tilde{r}}(\sigma,y) \subset Q_{3\sigma,\tilde{r}(3\sigma,y)} \). Hence, we find
\[
\sigma^2 = \left( \int_{t-\sigma/2}^{t+\sigma/2} \int_{B_{r}(\sigma,y)(x)} |f| \, d\xi \, dt \right)^{2-p} |B_{r}(\sigma,y)|^{p-2} (r(\sigma,y))^{2p} \\
\leq \left( \int_{t-\sigma/2}^{t+\sigma/2} \int_{B_{3r}(\sigma,y)(x)} |f| \, d\xi \, dt \right)^{2-p} |B_{r}(\sigma,y)|^{p-2} (r(\sigma,y))^{2p} \\
\leq \left( \int_{t-\sigma/2}^{t+\sigma/2} \int_{B_{3r}(3\sigma,z)(x)} |f| \, d\xi \, dt \right)^{2-p} |B_{r}(3\sigma,z)|^{p-2} (r(3\sigma,z))^{2p} \\
\leq 9\sigma^2,
\]
where the last inequality follows by the sub-intrinsic construction of Lemma 3.1. As before,
\[
r(s,z) \leq \left( \frac{s}{\sigma} \right)^\tilde{b} r(\sigma,z) \leq \tilde{c}_2 \left( \frac{s}{\sigma} \right)^\tilde{b} \tilde{r}(\sigma,y) = \tilde{c}_2 r(s,y)
\]
for \( \tilde{c}_2 = 9^{\frac{1}{2p-1}} (2-p) \).

Therefore, for \( s \in (0, \frac{2}{3}) \), we find that
\[
Q_{s,r}(s,z)(z) \subset Q_{c_1 s, c_1 r}(s,y)(y) \subset Q_{c_1 s, \tilde{r}(c_1 r,y)}(y),
\]
with \( c_1 = \max \left\{ 3, \frac{3}{7}, \tilde{c}_1, \tilde{c}_2 \right\} \), which finishes the proof. \( \square \)

4. Energy Estimates in General Cylinders

The main result of this section is Lemma 4.1, which concerns proper energy estimates.

**Lemma 4.1.** Let \( u \geq 0 \) be a local, weak solution to (1.5)-(1.6) in \( E_T \) with \( \frac{n-2}{n+2} \frac{m}{m-1} < m < 1 \). Fix a point \( z_o \in E_T \), and, for \( \theta > 0 \), suppose
\[
Q_{\theta(2\rho),2\rho}(z_o) = Q_{\theta(2\rho),2\rho}(x_o, t_o) \subset E_T.
\]
Then there exists a constant $\gamma = \gamma(\text{data}) > 1$ such that, for every $c \geq 0$, we have the energy estimates

$$
\begin{align*}
\text{ess sup}_{t \in \Lambda_{\rho,2}(t_0)} & \int_{B_\rho(x_0)} |u^m - c^m|^{\frac{m+1}{m}} \, dx + \text{ess sup}_{t \in \Lambda_{\rho,2}(t_0)} \int_{B_\rho(x_0)} |u^m - c^m| |u - c| \, dx \\
+ & \int_{Q_{\rho,2}(x_0)} |Du^m|^2 \, dx \, dt \\
\leq & \frac{\gamma}{\theta \rho^2} \int_{Q_{\rho(2\rho)^2,2\rho}(x_0)} |u^m - c^m| |u - c| \, dx \, dt \\
& + \frac{\gamma}{\rho^2} \int_{Q_{\rho(2\rho)^2,2\rho}(x_0)} |u^m - c^m|^2 \, dx \, dt \\
& + \gamma \int_{Q_{\rho(2\rho)^2,2\rho}(x_0)} \rho^2 |f|^2 \, dx \, dt,
\end{align*}
$$

(4.1)

and

$$
\begin{align*}
\text{ess sup}_{t \in \Lambda_{\rho,2}(t_0)} & \int_{B_\rho(x_0)} |u^m - c^m|^{\frac{m+1}{m}} \, dx + \int_{Q_{\rho,2}(x_0)} |Du^m|^2 \, dx \, dt \\
\leq & \frac{\gamma}{\theta \rho^2} \int_{Q_{\rho(2\rho)^2,2\rho}(x_0)} |u - c|^{m+1} \, dx \, dt + \frac{\gamma}{\rho^2} \int_{Q_{\rho(2\rho)^2,2\rho}(x_0)} |u - c|^{2m} \, dx \, dt \\
& + \gamma \int_{Q_{\rho(2\rho)^2,2\rho}(x_0)} \rho^2 |f|^2 \, dx \, dt.
\end{align*}
$$

(4.2)

Remark 4.2. Since we are taking $f \in L^\infty_\text{loc}(E_T)$, the last term on the right-hand side both of (4.1) and (4.2) could be further estimated with the $L^\infty$-norm; at this stage this general statement suffices.

Proof. Estimate (4.2) can be easily deduced from (4.1), taking into account that for $a, b \geq 0$, and for $0 < m < 1$ we have

$$
|a^m - b^m| \leq |a - b|^m.
$$

(4.3)

After a change of coordinates we can assume $z_0 = (0,0)$. Choose a point $t_1 \in \Lambda_{\rho(2\rho)^2}$ and in the weak formulation (1.10) take the test function

$$
\varphi \equiv \pm (u^m_\varepsilon - c^m) \eta^2 \psi^h_\varepsilon,
$$

where:

- $\eta(x,t) \in C_0^\infty(\mathbb{R}^N)$ is a cut-off function such that

  $$
  \text{spt} \eta \subset Q_{\rho,2}(x_0) \cap B_\rho(x_0), \quad 0 \leq \eta \leq 1, \quad \eta = 1 \text{ on } Q_{\rho,2}, \quad \rho |D\eta| + \theta \rho^2 |\eta_t| \leq 1000.
  $$

- $\psi^h(t)$ is a piecewise linear approximation of the characteristic function, such that

  $$
  \psi^h(t) = \begin{cases} 
  1 & t \in [-\theta(2\rho)^2 + h, t_1 - h] \\
  0 & t \in (-\theta(2\rho)^2, -\theta(2\rho)^2 + \frac{h}{20}] \cup [t_1 - \frac{h}{20}, t_1]
  \end{cases}
  $$

  and

  $$
  \left| \frac{d\psi^h}{dt} \right| \leq \frac{10}{9h}.
  $$

- $u_\varepsilon$ and $\psi^h_\varepsilon$ are the mollification of $u$ and $\psi^h$ in time direction for $\varepsilon \leq \frac{h}{20}$.

Estimates on the set $[u \geq c]$: we integrate by parts the first term on the left-hand side of (1.10), and we have

$$
- \int_{-\theta(2\rho)^2}^{t_1} \int_{B_{\rho/\varepsilon} \cap [u_\varepsilon \geq c]} u_\varepsilon \varphi_t \, dx \, dt \\
= \int_{-\theta(2\rho)^2}^{t_1} \int_{B_{\rho/\varepsilon} \cap [u_\varepsilon \geq c]} \frac{\partial u_\varepsilon}{\partial t} (u^m_\varepsilon - c^m) \eta^2 \psi^h_\varepsilon \, dx \, dt
$$
Integrating by parts (4.4) and taking the limits, first as \( \varepsilon \to 0 \) and then as \( h \to 0 \), we end up with

\[
\int_{t_1}^{t_1} \int_{B_{2r} \cap [u \geq c]} \frac{\partial}{\partial t} \left( \int_c^u (y^m - c^m) \, dy \right) \eta^2 \psi^h \, dx \, dt.
\]

(4.4)

since by definition \( \eta = 0 \) at \( t = -\theta(2\rho)^2 \). Notice that the term with the time derivative of \( \psi^h \) does not appear: this follows from the bound on \( \frac{du^h}{dx} \) and the fact that we are taking the limits first as \( \varepsilon \to 0 \) and then as \( h \to 0 \).

We estimate the first term on the right-hand side of (4.5). We use Jensen’s inequality, (2.1) and (3.3) to obtain

\[
\int_{B_{2r} \cap [u \geq c]} \left( \int_c^u (y^m - c^m) \, dy \right) \eta^2(x, t_1) \, dx \geq \frac{1}{2} \int_{B_{2r} \cap [u \geq c]} (u - c)(u^m - c^m)\eta^2(x, t_1) \, dx
\]

\[
\geq \frac{1}{4} \int_{B_{2r} \cap [u \geq c]} (u - c)(u^m - c^m)\eta^2(x, t_1) \, dx + \frac{m}{4} \int_{B_{2r} \cap [u \geq c]} (u^m - c^m)\eta^{m+1}(x, t_1) \, dx
\]

\[
= \frac{1}{4} \int_{B_{2r} \cap [u \geq c]} |u - c||u^m - c^m|\eta^2(x, t_1) \, dx + \frac{m}{4} \int_{B_{2r} \cap [u \geq c]} |u^m - c^m|^{m+1} \eta^2(x, t_1) \, dx.
\]

The second term on the right-hand side of (4.5) is estimated by

\[
\int_{t_1}^{t_1} \int_{B_{2r} \cap [u \geq c]} \left( \int_c^u (y^m - c^m) \, dy \right) \eta \, dx \, dt
\]

\[\leq \int_{t_1}^{t_1} \int_{B_{2r} \cap [u \geq c]} |u - c||u^m - c^m|\eta |\eta| \, dx \, dt.
\]

Next we consider the term with the gradient in (1.10). We differentiate \( \varphi \) and take the limits, first as \( \varepsilon \to 0 \) and then as \( h \to 0 \); collecting all the estimates yields

\[
\int_{B_{2r} \cap [u \geq c]} |u - c||u^m - c^m|\eta^2(x, t_1) \, dx + \int_{B_{2r} \cap [u \geq c]} |u^m - c^m|^{m+1} \eta^2(x, t_1) \, dx
\]

\[+ \int_{t_1}^{t_1} \int_{B_{2r} \cap [u \geq c]} \eta^2 \mathbf{A}(x, t, u, Du^m) \cdot D(u^m - c^m) \, dx \, dt
\]

\[\leq \gamma \int_{t_1}^{t_1} \int_{B_{2r} \cap [u \geq c]} |u - c||u^m - c^m| \eta |\eta| \, dx \, dt
\]

\[+ \gamma \int_{t_1}^{t_1} \int_{B_{2r} \cap [u \geq c]} |u^m - c^m| \eta |\mathbf{A}(x, t, u, Du^m)| \, dx \, dt
\]

\[+ \int_{t_1}^{t_1} \int_{B_{2r} \cap [u \geq c]} |u^m - c^m|^2 \eta |f| \, dx \, dt.
\]

Using the lower bound on \( \mathbf{A} \) we obtain

\[
\int_{t_1}^{t_1} \int_{B_{2r} \cap [u \geq c]} \eta^2 \mathbf{A}(x, t, u, Du^m) \cdot D(u^m - c^m) \, dx \, dt
\]

\[= \int_{t_1}^{t_1} \int_{B_{2r} \cap [u \geq c]} \eta^2 \mathbf{A}(x, t, u, Du^m) \cdot Du^m \, dx \, dt
\]

\[\geq \nu \int_{t_1}^{t_1} \int_{B_{2r} \cap [u \geq c]} \eta^2 |Du^m|^2 \, dx \, dt.
\]
The upper bound and Young’s inequality imply
\[
\int_{-\theta(2\rho)^2}^{t_1} \int_{B_{2\rho} \cap [u \geq c]} |u^m - c^m| \eta |A(x, t, u, Du^m)| |D\eta| \, dx \, dt
\leq L \int_{-\theta(2\rho)^2}^{t_1} \int_{B_{2\rho} \cap [u \geq c]} |u^m - c^m| \eta |Du^m| |D\eta| \, dx \, dt
\leq \frac{\nu}{2} \int_{-\theta(2\rho)^2}^{t_1} \int_{B_{2\rho} \cap [u \geq c]} \eta^2 |Du^m|^2 \, dx \, dt
+ \gamma(\nu, L) \int_{-\theta(2\rho)^2}^{t_1} \int_{B_{2\rho} \cap [u \geq c]} |u^m - c^m|^2 |D\eta|^2 \, dx \, dt.
\]

In this way we end up with
\[
\int_{B_{2\rho} \cap [u \geq c]} |u - c| |u^m - c^m| \eta^2(x, t_1) \, dx
+ \int_{B_{2\rho} \cap [u \geq c]} |u^m - c^m| \frac{|\partial u}{\partial t} (c^m - u^m_c) \eta^2 \psi^h \, dx \, dt
\leq \gamma \int_{-\theta(2\rho)^2}^{t_1} \int_{B_{2\rho} \cap [u \geq c]} |u - c| |u^m - c^m| \eta |\eta_t| \, dx \, dt \tag{4.6}
+ \gamma \int_{-\theta(2\rho)^2}^{t_1} \int_{B_{2\rho} \cap [u \geq c]} |u^m - c^m|^2 |D\eta|^2 \, dx \, dt
+ \int_{-\theta(2\rho)^2}^{t_1} \int_{B_{2\rho} \cap [u \geq c]} |u^m - c^m| \eta^2 |f| \, dx \, dt.
\]

If \( c = 0 \), the proof is finished by a straightforward application of Hölder’s inequality. If \( c > 0 \), we have to consider also the set \( [u < c] \).

**Estimates on the set \([u < c]\):** integrating by parts the first term on the left-hand side of (1.10) we have
\[
- \int_{-\theta(2\rho)^2}^{t_1} \int_{B_{2\rho} \cap [u < c]} u_c \varphi_t \, dx \, dt
= \int_{-\theta(2\rho)^2}^{t_1} \int_{B_{2\rho} \cap [u < c]} -\frac{\partial u}{\partial t} (c^m - u^m_c) \eta^2 \psi^h \, dx \, dt \tag{4.7}
= \int_{-\theta(2\rho)^2}^{t_1} \int_{B_{2\rho} \cap [u < c]} -\frac{\partial}{\partial t} \left( \int_{u}^{c} (c^m - y^m) \, dy \right) \eta^2 \psi^h \, dx \, dt.
\]

After another integration by parts and taking the limits first as \( \varepsilon \to 0 \), and then as \( h \to 0 \), we conclude
\[
- \int_{-\theta(2\rho)^2}^{t_1} \int_{B_{2\rho} \cap [u < c]} u_c \varphi_t \, dx \, dt
\rightarrow \int_{B_{2\rho} \cap [u < c]} \left( \int_{u}^{c} (c^m - y^m) \, dy \right) \eta^2(x, t_1) \, dx \tag{4.8}
- 2 \int_{-\theta(2\rho)^2}^{t_1} \int_{B_{2\rho} \cap [u < c]} \left( \int_{u}^{c} (c^m - y^m) \, dy \right) \eta \eta_t \, dx \, dt,
\]

since, as before, \( \eta \) vanishes at \( t = -\theta(2\rho)^2 \) and the term with \( \frac{d\psi^h}{dt} \) goes to 0.

We control the first term using (2.2) and (4.3):
\[
\int_{B_{2\rho} \cap [u < c]} \left( \int_{u}^{c} (c^m - y^m) \, dy \right) \eta^2(x, t_1) \, dx
\geq \frac{m}{4} \int_{B_{2\rho} \cap [u < c]} (c - u)(c^m - u^m) \eta^2(x, t_1) \, dx
\geq \frac{m}{4} \int_{B_{2\rho} \cap [u < c]} (c - u)(c^m - u^m) \eta^2(x, t_1) \, dx
+ \frac{m}{4} \int_{B_{2\rho} \cap [u < c]} (c^m - u^m)^{\frac{m+1}{m}} \eta^2(x, t_1) \, dx.
\]
\[ \frac{m}{4} \int_{B_{2r} \cap [u < c]} |u - c| |u^m - c^m| \eta^2(x, t_1) \, dx + \frac{m}{4} \int_{B_{2r} \cap [u < c]} |u^m - c^m|^{m+1} m^{-1} \eta^2(x, t_1) \, dx. \]

The other calculations with the spatial gradient are similar to those on the set \( [u \geq c] \), and finally give
\[ \int_{B_{2r} \cap [u < c]} |u - c| |u^m - c^m| \eta^2(x, t_1) \, dx + \int_{B_{2r} \cap [u < c]} |u^m - c^m|^{m+1} m^{-1} \eta^2(x, t_1) \, dx \]
\[ + \int_{-\theta(2r)^2}^{t_1} \int_{B_{2r} \cap [u < c]} |Du|^2 \eta^2 \, dx \, dt \]
\[ \leq \gamma \int_{-\theta(2r)^2}^{t_1} \int_{B_{2r} \cap [u < c]} |u - c| |u^m - c^m| \eta |\eta| \, dx \, dt \]
\[ + \gamma \int_{-\theta(2r)^2}^{t_1} \int_{B_{2r} \cap [u < c]} |u^m - c^m|^2 |D\eta|^2 \, dx \, dt \]
\[ + \int_{-\theta(2r)^2}^{t_1} \int_{B_{2r} \cap [u < c]} |u^m - c^m| \eta^2 |f| \, dx \, dt. \] (4.9)

Now if we sum the contributions (4.6) and (4.9), notice that
\[ \int_{-\theta(2r)^2}^{t_1} \int_{B_{2r} \cap [u \geq c]} |u^m - c^m| \eta |f| \, dx \, dt \]
\[ \leq \frac{1}{2\rho^2} \int_{-\theta(2r)^2}^{t_1} \int_{B_{2r} \cap [u \geq c]} |u^m - c^m|^2 + \frac{\rho^2}{2} \int_{-\theta(2r)^2}^{t_1} \int_{B_{2r} \cap [u \geq c]} |f|^2 \, dx \, dt, \]
\[ \int_{-\theta(2r)^2}^{t_1} \int_{B_{2r} \cap [u \geq c]} |u^m - c^m| \eta^2 |f| \, dx \, dt \]
\[ \leq \frac{1}{2\rho^2} \int_{-\theta(2r)^2}^{t_1} \int_{B_{2r} \cap [u \geq c]} |u^m - c^m|^2 + \frac{\rho^2}{2} \int_{-\theta(2r)^2}^{t_1} \int_{B_{2r} \cap [u \geq c]} |f|^2 \, dx \, dt, \]
and use the property of \( \eta \) and the arbitrariness of \( t_1 \), we conclude the proof. \( \square \)

A direct consequence of (4.2) for \( c = 0 \) is the following corollary, that is satisfied for sub-intrinsic cylinders:

Corollary 4.3 (sub-intrinsic). Let \( u \geq 0 \) be a local, weak solution to (1.5)-(1.6) in \( E_T \) with \( \frac{(n-2)s}{n+2} < m < 1 \). Fix a point \( z_0 \in E_T \), and, for \( s, \theta > 0 \), suppose
\[ Q_{s, \sqrt{s}/\theta}(z_0) = Q_{s, \sqrt{s}/\theta}(x_0, t_0) \subset E_T, \]
and
\[ \left( \iint_{Q_{s, \sqrt{s}/\theta}(z_0)} u^{m+1} \, dx \, dt \right)^{\frac{1-m}{m+1}} \leq K \theta, \]
for some \( K \geq 1 \). Then there exists a constant \( \gamma_K = \gamma(\text{data}, K) > 1 \) such that
\[ \frac{1}{s} \underset{s \in \Lambda_{s/2}(t_0)}{\text{ess sup}} \int_{B_{\sqrt{s}/\theta}(x_0)} u^{m+1} \, dx + \iint_{Q_{s, \sqrt{s}/\theta}(z_0)} |Du|^2 \, dx \, dt \]
\[ \leq \gamma_K \left[ \frac{\theta^{m+1}}{s} + \frac{s}{\theta} \iint_{Q_{s, \sqrt{s}/\theta}(z_0)} |f|^2 \, dx \, dt \right] \]
\[ \leq \gamma_K \left[ \frac{\theta^{m+1}}{s} + \sup_{Q_{s, \sqrt{s}/\theta}(z_0)} \left[ \frac{s}{\theta} |f|^2 \right] \right]. \]

We will also need the following energy estimates for proper truncations of \( u \). They can be seen as a corollary of Lemma 4.1. They have been proved in [DGV, Section B.3], to which we refer for all the details.
Proposition 4.4. Let $u \geq 0$ be a local, weak subsolution to (1.5)–(1.6) in $E_T$ with $\frac{(n-2)+}{m+2} < m < 1$, and consider the truncations $(u^m - k^m)_+$ for $k > 0$.

There exists a positive constant $\gamma = \gamma(m, n, \nu, L)$, such that for every cylinder $Q_{\theta(2^\rho)^2} \subset E_T$, every $k > 0$, and any nonnegative, piecewise smooth cutoff function $\zeta$ vanishing on $\partial B_{2\rho}(x_o)$,

\[
\text{ess sup}_{t_o - \theta(2^\rho)^2 < t \leq t_o} \frac{1}{m+1} \int_{B_{2\rho}(x_o)} (u^m - k^m) \frac{\partial}{\partial t} \zeta^2(x,t) dx
\]

\[
- \int_{B_{2\rho}(x_o)} \int_k^u (s^m - k^m) ds \zeta^2(x,t_o - \theta(2^\rho)^2) dx
\]

\[
+ \frac{\nu}{4} \int \int_{Q_{\theta(2^\rho)^2} \subset B_{2\rho}(x_o)} |D(u^m - k^m + 1\zeta)|^2 \zeta^2 dx dt
\]

\[
\leq \gamma \int \int_{Q_{\theta(2^\rho)^2} \subset B_{2\rho}(x_o)} u^{m+1} \chi_{\{u - k > 0\}} \zeta^2 dx dt
\]

\[
+ \gamma \int \int_{Q_{\theta(2^\rho)^2} \subset B_{2\rho}(x_o)} (u^m - k^m)^2 |D\zeta|^2 dx dt
\]

\[
+ \gamma \int \int_{Q_{\theta(2^\rho)^2} \subset B_{2\rho}(x_o)} \zeta^2 (u^m - k^m + 1\zeta^2 dx dt.
\]

5. Estimate for the Solution

In the following, we refer to the cylinders built in Lemma 3.1. In particular, if $s$ denotes the height of the cylinder and $\theta(s)$ is the corresponding scaling parameter, then $r(s)$ is given by

\[
r(s) := \sqrt{\frac{s}{\theta(s)}}.
\]

(5.1)

For simplicity, whenever possible, instead of $\theta(s)$, $r(s)$, we will write $\theta_s$, $r_s$. Moreover, $Q^\theta_s$ denotes the cylinder of height $s$, scaling parameter $\theta_s$, and related radius $r_s$ given by (5.1); when there is no possibility of misunderstanding, we will simply write $Q_s$.

Lemma 5.1. Assume that the cylinder $Q^\theta_s$ is intrinsic. Then for any $a \in (1, 2]$ also the cylinder $Q_{\theta_s}^{a\gamma}$ is intrinsic.

Proof. Assume that $Q^\theta_s$ is intrinsic: this means that, up to a constant $K \geq 1$,

\[
\theta_s = \left[ \iint_{Q^\theta_s} u^{1+m} dxd\tau \right]^{\frac{1}{1+m}}.
\]

Now, for any $a \in (1, 2]$

\[
\theta_s \leq c_1 \left[ \iint_{Q^{\theta_s}_{2\gamma}} u^{1+m} dxd\tau \right]^{\frac{1}{1+m}}
\]

\[
\leq c_2 \theta_s \quad \text{by (c) of Lemma 3.1}
\]

\[
\leq c_3 \theta_s \quad \text{by (f) of Lemma 3.1}
\]

\[
\leq c_4 \left[ \iint_{Q^{\theta_s}_{2\gamma}} u^{1+m} dxd\tau \right]^{\frac{1}{1+m}}.
\]

□

Boundedness estimate are well-known in the context of the singular porous medium equation (see, for example, [DGV, Appendix B]). Here we show that they have a simple expression, when one considers intrinsic cylinders. As a consequence, we are then able to deduce a very useful reverse Hölder inequality for powers of the solution $u$, again in intrinsic cylinders.
Let \( u \geq 0 \) be a locally bounded, local, weak solution to (1.5)-(1.6) in \( E_T \) with 
\[
\frac{(n-2)}{n+2} < m < 1.
\]
Fix \( t > 0 \), assume that \( f \in L^\infty(E_T) \), the cylinder \( B_{r(t)}(x_o) \times (t_o, t_o + t) \subset E_T \) is one of the intrinsic cylinders built in Lemma 3.1 with respect to \( u \), and
\[ \sup_{Q_{2t}} [r^2(2t)|f|^2] \leq \frac{c}{t} [\theta(2t)]^{\frac{1+m}{1-m}} \]
for some \( c > 0 \), where \( Q_{2t} \) and \( \theta(2t) \) are defined below. Then there exists a positive constant \( \gamma \) depending only on the data \( \{m, n, \nu, L\} \), such that for \( Q_{2t} := B_{r(2t)}(x_o) \times (t_o - t, t_o + t) \subset E_T \) and for any \( p > 0 \), we have
\[ \sup_{B_{r(t)}(x_o) \times [t_o, t_o + t]} u \leq \gamma \frac{1}{(1 - \sigma)^q} \left( \iint_{Q_{2t}} u^p \, dx \, dt \right)^{\frac{1}{p}}, \]
where \( q \) depends only on \( p, n, m \).

**Remark 5.3.** The intrinsic cylinders of Lemma 3.1 are centered at \( (t_o, x_o) \), whereas here we consider cylinders which have a lower vertex at \( (t_o, x_o) \). However, Lemma 3.1 can be adapted to cover this case, or, alternatively, we can deal with centered cylinders, and correspondingly rephrase Proposition 5.2.

**Proof.** We follow the same strategy used for [DGV, Proposition B.4.1]. Without loss of generality, we can assume \( (x_o, t_o) \equiv (0, 0) \). Fix \( t > 0 \), consider the cylinder
\[ Q_t^\sigma = B_{r(t)} \times (0, t) \]
and assume it to be intrinsic. By Lemma 5.1, for any \( \sigma \in (0, 1] \) we have that all the cylinders
\[ Q_{(1+\sigma)t}^\sigma = B_{r((1+\sigma)t)} \times (-\sigma t, t] \]
are intrinsic. From line to line the intrinsic constants may be different, but they are stable. For any \( \sigma \in (0, 1] \) by (5.1) we have
\[ \theta((1 + \sigma)t) r^2((1 + \sigma)t) = (1 + \sigma)t. \]
For fixed \( \sigma \in (0, 1] \) and \( j = 0, 1, 2, \ldots \) set
\[
t_j = -\sigma t - \frac{1 - \sigma}{2^j} t, \qquad \rho_j = r(-t_j + t) = r((1 + \sigma)t + \frac{1-\sigma}{2^j} t),
\]
\[ B_j = B_{\rho_j}, \qquad Q_j = B_j \times (t_j, t). \]
This is a family of nested and shrinking cylinders with common “vertex” at \( (0, t) \), and by construction
\[ Q_0 = B_{r(2t)} \times (-t, t] \quad \text{and} \quad Q_\infty = B_{r((1+\sigma)t)} \times (-\sigma t, t]. \]
Having assumed that \( u \) is locally bounded in \( E_T \), set
\[ M = \text{ess sup}_{Q_\infty} u, \quad M_\sigma = \text{ess sup}_{Q_\infty} u. \]
We first find a relationship between \( M \) and \( M_\sigma \). Denote by \( \zeta \) a nonnegative, piecewise smooth cutoff function in \( Q_j \) that equals one on \( Q_{j+1} \), and has the form \( \zeta(x, t) = \zeta_1(x)\zeta_2(t) \), where
\[
\zeta_1 = \begin{cases} 
1 & \text{in } B_{j+1} \\
0 & \text{in } \mathbb{R}^n - B_j \end{cases}, \quad |D\zeta_1| \leq \frac{1}{\rho_j - \rho_{j+1}},
\]
\[
\zeta_2 = \begin{cases} 
0 & \text{for } t \leq t_j \\
1 & \text{for } t \geq t_{j+1} \end{cases}, \quad 0 \leq \zeta_2 t \leq \frac{2^{j+1}}{(1 - \sigma)t};
\]
introduce the increasing sequence of levels
\[ k_j = k - \frac{1}{2^j} k \]
where $k > 0$ is to be chosen. Estimates (4.10) with $(u^m - k_{j+1}^m)_+$ yield

$$
\sup_{t_j \leq \tau \leq t} \int_{B_j} [(u^m - k_{j+1}^m)_+ + \zeta \frac{m+1}{m}] (x, \tau) \, dx 
\leq \frac{\nu}{4} \int_{Q_j} |D[(u^m - k_{j+1}^m)_+] + \zeta|^2 \, dx d\tau 
\leq \gamma \int_{Q_j} u^{m+1} X_{[(u^m - k_{j+1}^m)_+ > 0] |\zeta|} \, dx d\tau
$$

$$
+ \gamma \int_{Q_j} (u^m - k_{j+1}^m)_+ |D\zeta|^2 \, dx d\tau 
+ \gamma \int_{Q_j} \zeta^2 (u^m - k_{j+1}^m)_+ |f| \, dx d\tau.
$$

(5.2)

In the estimations below repeated use is made of the inequality

$$
|[u > k_{j+1}] \cap Q_j| \leq \frac{2(j+1)^s}{k^{m+1}} \int_{Q_j} (u^m - k_{j+1}^m)_+ \, dx d\tau
$$

valid for all $s > 0$. Then estimate

$$
\int_{Q_j} u^{m+1} X_{[(u^m - k_{j+1}^m)_+ > 0] |\zeta|} \, dx \leq \frac{\gamma^{2(j+1)^s}}{(1 - \sigma)^t} \int_{Q_j} (u^m - k_{j+1}^m)_+ \, dx d\tau
$$

$$
\int_{Q_j} (u^m - k_{j+1}^m)_+^2 |D\zeta|^2 \, dx \leq \gamma \frac{k^{m+1}}{(1 - \sigma)^t} \int_{Q_j} (u^m - k_{j+1}^m)_+ \, dx d\tau
$$

Moreover,

$$
\int_{Q_j} \zeta^2 (u^m - k_{j+1}^m)_+ |f| \, dx \leq \frac{1}{2} \int_{Q_j} (u^m - k_{j+1}^m)_+ \, dx d\tau
$$

$$
+ \frac{1}{2} \int_{Q_j} \rho^2 \chi_{[(u^m - k_{j+1}^m)_+ > 0]} \, dxd\tau = I + II.
$$

The first term on the right-hand side $I$ can be estimated as before, namely

$$
I \leq \frac{2^{m+1}}{(1 - \sigma)^t} \frac{\theta(2t)}{k^{1-m}} \int_{Q_j} (u^m - k_{j+1}^m)_+ \, dx d\tau.
$$

As for the second term $II$, we have

$$
\int_{Q_j} \rho^2 |f|^2 X_{[(u^m - k_{j+1}^m)_+ > 0]} \, dxd\tau \leq \int_{Q_j} r^2 (2t) |f|^2 X_{[(u^m - k_{j+1}^m)_+ > 0]} \, dxd\tau
$$

$$
\leq \sup_{Q_{2t}} \left[ r^2 (2t) |f|^2 \right] \mathbb{E} \left[ u > k_{j+1} \right] \cap Q_j
$$

$$
\leq \gamma \sup_{Q_{2t}} \left[ r^2 (2t) |f|^2 \right] \frac{2^{(j+1) \frac{m+1}{m}}}{k^{m+1}} \int_{Q_j} (u^m - k_{j+1}^m)_+ \, dx d\tau
$$

$$
\leq \gamma \frac{2^{m+1}}{(1 - \sigma)^t} \frac{\theta(2t)}{k^{1-m}} \int_{Q_j} (u^m - k_{j+1}^m)_+ \, dx d\tau.
$$

Combining these estimates, (5.2) yields

$$
\sup_{t_j \leq \tau \leq t} \int_{B_j} [(u^m - k_{j+1}^m)_+ + \zeta] \frac{m+1}{m} (x, \tau) \, dx + \int_{Q_j} |D[(u^m - k_{j+1}^m)_+] + \zeta|^2 \, dx d\tau
$$

$$
\leq \gamma \frac{2^{(m+1) \frac{m}{m}}}{(1 - \sigma)^t} \left[ 1 + \frac{\theta(2t)}{k^{1-m}} + \left( \frac{\theta(2t)}{k^{1-m}} \right)^{\frac{1-m}{m}} \right] \int_{Q_j} (u^m - k_{j+1}^m)_+ \, dx d\tau.
$$
The last term in $\cdots$ is estimated by stipulating to take

$$\frac{\theta(2t)}{k^{1-m}} \leq 1 \implies k \geq (\theta(2t))^{\frac{1}{1-m}}. \quad (5.3)$$

With this stipulation, the previous inequality implies

$$\sup_{t_j \leq \tau \leq t} \int_{B_j} [(u^m - k^m_{j+1}) + \zeta]^{\frac{m+1}{m}}(x, \tau) \, dx + \iint_{Q_j} |D[(u^m - k^m_{j+1}) + \zeta]|^2 \, dx \, d\tau$$

$$\leq \frac{2^{2(m+1)}}{(1-\sigma)\tau} \iint_{Q_j} (u^m - k^m_j)^{\frac{m+1}{m}} \, dx \, d\tau. \quad (5.4)$$

By the Hölder inequality and the Parabolic Sobolev embedding

$$\iint_{Q_{n+1}} (u^m - k^m_{j+1})^{\frac{m+1}{m}} \, dx \, d\tau \leq \left[ \sup_{t_j \leq \tau \leq t} \int_{B_j} [(u^m - k^m_{j+1}) + \zeta]^{\frac{m+1}{m}}(x, \tau) \, dx \right]^{\frac{q+1}{qm}}$$

$$\times \left( \iint_{Q_j} |D(u^m - k^m_{j+1}) + \zeta|^2 \, dx \, d\tau \right)^{\frac{1}{2} \frac{m+1}{qm}} \times \iint_{Q_j} (u^m - k^m_j)^{\frac{m+1}{m}} \, dx \, d\tau \right)^{1 - \frac{m+1}{qm}}$$

where

$$q = \frac{2(nm + m + 1)}{nm}.$$

Now set

$$Y_j = \frac{1}{|Q_j|} \iint_{Q_j} (u^m - k^m_j)^{\frac{m+1}{m}} \, dx \, d\tau.$$

Taking into account (5.4), in terms of $Y_j$ the previous inequality becomes

$$Y_{j+1} \leq \gamma \frac{b^j}{(1-\sigma)^{\frac{(m+1)(n+2)}{nm}}} \left( \theta(2t) \right)^{\frac{m+1}{qm}} Y_j^{1 + \frac{2(m+1)}{qm}} \gamma^{\frac{m+1}{qm}} Y_j^{1 + \frac{2(m+1)}{qm}}.$$

where

$$b = 2^{\frac{3(m+1)}{2} \left( 1 + \frac{2(m+1)}{qm} \right)}.$$

Now $Y_j \to 0$ as $j \to +\infty$, provided $k$ is chosen such that

$$Y_\sigma = \iint_{Q_\sigma} u^{m+1} \, dx \, d\tau = (1-\sigma)^{\frac{n+2}{m}} \left( \theta(2t) \right)^{\frac{n}{2} \frac{n(m+1) + 2m + 2}{k^{n+2}}}.$$

With this choice

$$M_{\sigma} \leq \gamma \frac{1}{(1-\sigma)^{\frac{m+1}{m}}} \left( \frac{1}{\theta(2t)} \right)^{\frac{n}{n+2}} \left( \theta(2t) \right)^{\frac{2}{n(m+1) + 2m + 2}} \iint_{Q_\sigma} u^{m+1} \, dx \, d\tau \right)^{\frac{m+1}{m}}. \quad (5.5)$$

Since $Q_\sigma$ is intrinsic, up to a constant

$$\theta(2t) = \left( \iint_{Q_\sigma} u^{m+1} \, dx \, d\tau \right)^{\frac{1-m}{1+m}}.$$

Hence,

$$M_{\sigma} \leq \gamma \frac{1}{(1-\sigma)^{\frac{m+1}{m}}} \left( \iint_{Q_\sigma} u^{m+1} \, dx \, d\tau \right)^{\frac{m+1}{m}},$$

and the estimate does not change when we take into account the previous stipulation (5.3) about $k$, once more due to the intrinsic nature of $Q_\sigma$. By the interpolation Lemma 6.1 of [Giu03] we conclude.

As a straightforward consequence, we obtain the following reverse Hölder inequality.
Corollary 5.4. Let \( u \geq 0 \) be a locally bounded, local, weak solution to (1.5)–(1.6) in \( E_T \) with 
\[
\frac{(n-2)s}{n+2} < m < 1.
\]
Fix \( t > 0 \), assume that \( f \in L^\infty_{loc}(E_T) \), the cylinder \( B_{r(t)}(x_o) \times (t_o, t_o + t] \subset E_T \) is
intrinsic with respect to \( u \), and
\[
\sup_{Q_{2t}} [r^2(2t)|f|] \leq \frac{C}{T} [\theta(2t)]^{\frac{1+m}{1-m}}.
\]
for some \( c > 0 \). Then there exists a positive constant \( \gamma \) depending only on the data \( \{m, n, \nu, L\} \), such that for \( Q_{2t} := B_{r(2t)}(x_o) \times (t_o - t, t_o + t] \subset E_T \), we have
\[
\left( \iint_{Q_{2t}} u^{m+1} \, dx \, d\tau \right)^{\frac{1}{m+1}} \leq \gamma \left( \iint_{Q_{2t}} u^m \, dx \, d\tau \right)^{\frac{1}{m}}.
\]

Remark 5.5. We point out that the derivation of the reverse Hölder estimate for \( u^m \) is the only place where we make use of the \( L^\infty \) bounds of the right-hand side. There are other, more rudimentary possibilities to derive a reverse Hölder estimate like the one in Corollary 5.4, that would allow to consider vector-valued and signed solutions, weaker assumptions on \( A \) and the right-hand side \( f \). Indeed, it can be achieved in three steps, by first applying a parabolic Poincaré inequality on \( u^m \), then using the energy estimates, and finally an interpolation argument, as in Lemma 6.1 of [Giu03]. We chose to limit ourselves to the scalar framework and nonnegative solutions, in order to allow a better understanding of the new techniques introduced in the paper, which we consider to be the priority in this paper.

6. Intrinsic Reverse Hölder Inequalities

In the next two subsections we will prove reverse Hölder inequalities in intrinsic cylinders. We will have to distinguish two different conditions, the so-called Degenerate and Non-Degenerate Regimes. As a starting point we include the following lemma, which allows to switch mean values in time.

Lemma 6.1. Take any cylinder \( Q_{2s,2r} = (t_0 - 2s, t_0 + 2s) \times B_{2r} \subset E_T \), and let \( \eta \) be a constant-in-time cut off function, such that \( \eta \in C_0^\infty((B_{2r}), [0,1]) \), \( \eta(x) \equiv 1 \) for \( x \in B_{r} \), and \( \|D\eta\|_\infty \leq \frac{\mu}{2} \). Then
for \( \eta = \frac{\tilde{\eta}}{\sqrt{|B_r|}} \) and for \( t_0 - 2s \leq \sigma < \tau \leq t_0 + 2s \) we find
\[
\left| u(\tau) \right|_{B_{2r}}^2 - u(\sigma) \right|_{B_{2r}}^2 \right| \leq c s \left( \frac{1}{T} \iint_{Q_{2s,2r}} |Du^m| \, dx \, dt + \iint_{Q_{2s,2r}} |f| \, dx \, dt \right).
\]
Similarly, if we take \( \eta = \frac{\tilde{\eta}}{\sqrt{|B_r|}} \), we obtain
\[
\left| (u(\tau)) \right|_{B_{2r}}^2 - u(\sigma) \right|_{B_{2r}}^2 \right| \leq c s \left( \frac{1}{T} \iint_{Q_{2s,2r}} |Du^m| \, dx \, dt + \iint_{Q_{2s,2r}} |f| \, dx \, dt \right).
\]

Proof. By the equation, we find that
\[
\left| (u(\tau)) \right|_{B_{2r}}^2 - u(\sigma) \right|_{B_{2r}}^2 \right| = \left| \iint_{\sigma} \partial_t u(t, \eta^2) \, dt \right|
\]
\[
= \left| c \int_{\sigma} \iint_{B_{2r}} f(x, t) \eta^2 - A(x, t, u, Du^m) \cdot D\eta \, dx \, dt \right|
\]
\[
\leq c \int_{\sigma} \iint_{Q_{2s,2r}} |Du^m| \, dx \, dt + c s \iint_{Q_{2s,2r}} |f| \, dx \, dt,
\]
where \( c > 1 \) depends only on the data. \( \square \)

6.1. The Degenerate Regime. In the following we consider an intrinsic cylinder \( Q^s_\nu \equiv Q_s \sqrt{s/n} \equiv Q_s, \sqrt{s/n} \subset E_T \), and we assume that
\[
\left( \iint_{Q^s_\nu} u^m - (u^m)_{Q^s_\nu} \right)^{\frac{m+1}{m}} \, dx \, dt \leq \varepsilon \left( \iint_{Q^s_\nu} u^{m+1} \, dx \, dt \right)^{\frac{1}{m+1}} \tag{6.3}
\]
for some \( \varepsilon > 0 \). We denote this condition as the Degenerate Regime. Our aim is to obtain a reverse Hölder inequality for \( |Du^m| \). We have the following proposition.

**Proposition 6.2.** Let \( Q^e_b \) be one of the intrinsic cylinders constructed in Lemma 3.1 and assume it satisfies (6.3). Then, there exist \( \vartheta_1 \in (0,1) \) and a constant \( c_\varepsilon > 1 \) that depends only on \( \varepsilon, \vartheta_1 \), and the data, such that

\[
\iint_{Q^e_b} |Du^m|^2 \, dx dt \leq c_\varepsilon \left( \iint_{Q_{3r,3r}(s)} |Du^m|^{2b_1} \, dx dt \right)^{\frac{1}{b_1}} + c_\varepsilon \sup_{Q_{3r,3r}(s)} |r^2(s)|f|^2|.
\]  

(6.4)

**Proof.** Without loss of generality, we can assume that

\[
\sup_{Q_{3r,3r}(s)} |r^2(s)|f|^2| \leq c \frac{\theta^{m+1}_s}{s}
\]

for some positive constant \( c \); otherwise, by Corollary 4.3, we immediately have that

\[
\iint_{Q^e_b} |Du^m|^2 \, dx dt \leq \gamma \sup_{Q_{3r,3r}(s)} |r^2(s)|f|^2|
\]

and there is nothing to prove.

In order to conclude, it suffices to show that under the previous assumption on \( f \) we have

\[
\iint_{Q^e_b} |Du^m|^2 \, dx dt \leq c_\varepsilon \left( \iint_{Q_{3r,3r}(s)} |Du^m|^{2b_1} \, dx dt \right)^{\frac{1}{b_1}} + c_\varepsilon \iint_{Q_{3r,3r}(s)} r^2(s)|f|^2 \, dx dt.
\]  

(6.5)

First observe, that by Lemma 5.1, the cylinders \( Q^e_b \) are intrinsic for \( b \in [1,2] \). By (f) of Lemma 3.1 and the best constant property, we find that for all \( b \in [1,2] \)

\[
\theta^{m+1}_{bs} \leq \frac{c}{\varepsilon} \iint_{Q^e_b} |u^m - (u^m)_{Q^e_b}|^{\frac{m+1}{m}} \, dx dt.
\]

Moreover, (g)–(c) of Lemma 3.1 and the definition of \( \theta_s = \frac{s}{(r(s))^{1/m}} \) imply

\[
\iint_{Q_{2s,\sqrt{r(s)/s}}} u^{m+1} \, dx dt = \iint_{Q_{2s,2r}(s)} u^{m+1} \, dx dt \leq c \iint_{Q_{2s,r(2s)}} u^{m+1} \, dx dt
\]

\[
\leq c \left( \frac{s}{(r(s))^{1/m}} \right)^{1/m} = c \left( \frac{2s}{(r(2s))^{1/m}} \right)^{1/m} \leq c \left[ \frac{s}{(r(s))^{1/m}} \right]^{1/m} \leq c \left[ \frac{\theta_s}{2} \right]^{1/m}.
\]

Hence, the cylinder \( Q_{2s,\sqrt{4s/r(s)}} \) satisfies the assumption of Corollary 4.3, relying on it and using the definition of \( \theta_s = \frac{s}{(r(s))^{1/m}} \) again, yields

\[
\frac{1}{s} \text{ess sup} \int_{B_{\sqrt{r(s)/s}}} u^{m+1} \, dx + \iint_{Q_{2s,\sqrt{r(s)}}} |Du^m|^2 \, dx dt \leq c \theta_s^{m+1}.
\]

This proves the first part of (6.5). Now, we choose \( \eta \) as in Lemma 6.1, assuming it has support in \( B_{3r(s)} \) and it is constant in \( B_{2r(s)} \), and define the constant

\[
\lambda^m_o = \left( \int_{A_\varepsilon} (u(t))^{\eta^2}_{B_{3r(s)}} \, dt \right)^m.
\]

We can estimate \( \lambda_o \) from below using the reverse Hölder inequality of Corollary 5.4 and the intrinsic nature of \( Q^e_b \) to find

\[
\lambda^m_o \geq \left( \iint_{A_\varepsilon} \frac{1}{B_{3r(s)}} \int_{B_{2r(s)}} u \, dx dt \right)^m \geq c \left( \iint_{Q_{2s,2r(s)}} u \, dx dt \right)^m
\]

\[
\geq c \left( \iint_{Q_{s,r(s)}} u^{m+1} \, dx dt \right)^{\frac{m}{m+1}} = c \theta_s^{\frac{m}{m+1}}.
\]
We estimate $\theta_s$ by using the best constant property, (2.4) and (2.5). For some $\alpha \in (0, 1)$ to be chosen, we have

$$
\theta_s^{m+1} = c \iiint_{Q_s^{2s}} u^{m+1} \, dx \, dt \leq \frac{c}{\e} \iiint_{Q_s^{2s}} \left| u^m - (u^m)_{Q_s^{2s}} \right|^{\frac{m+1}{m}} \, dx \, dt
$$

\begin{align*}
&\leq \frac{c}{\e} \left( \sup_{t \in A_s} \left( \int_{B_{\sqrt{s}/\sigma}} u^{m+1} \, dx \right)^{\frac{1-\alpha}{\alpha}} \int_{A_s} \left( \int_{\sqrt{s}/\sigma} u^m \right)^{\frac{m+1}{m}} \, dx \right) dt \\
&\leq \frac{c}{\e} \left( \sup_{t \in A_s} \left( \int_{B_{\sqrt{s}/\sigma}} u^{m+1} \, dx \right)^{\frac{1-\alpha}{\alpha}} \int_{A_s} \left( \int_{B_{3r(s)}} u^m \right)^{\frac{m+1}{m}} \, dx \right) dt \\
&\quad + \left| \left( (u(t))_{B_{3r(s)}}^2 \right)^m - \lambda_0^{m+1} \right| \frac{m+1}{m} \, dt.
\end{align*}

By the intrinsic nature of $Q_{2s, 4s/\theta_s}$, Corollary 4.3 and Lemma 5.1, this implies that

$$
\left( \frac{\theta_s^{m+1}}{s} \right)^{\alpha} \leq c \frac{c_s}{s^\alpha} \left( \int_{B_{3r(s)}} \left| u^m - (u(t))_{B_{3r(s)}} \right|^{\frac{m+1}{m}} \, dx \right) dt \\
+ \frac{c}{s^\alpha} \left( \int_{A_s} \left( (u(t))_{B_{3r(s)}}^2 \right)^m - \lambda_0^{m+1} \right) \frac{m+1}{m} \, dt =: (I) + (II)
$$

By the Sobolev-Poincaré inequality there exists $\vartheta_1 \in \left( \frac{1}{2}, 1 \right)$ such that

$$
\int_{B_{3r(s)}} \left| u^m - (u(t))_{B_{3r(s)}} \right|^{\frac{m+1}{m}} \, dx \leq c \int_{B_{3r(s)}} \left| u^m - (u^m)_{B_{3r(s)}} \right|^\frac{m+1}{m} \, dx
$$

$$
\leq c(r(s))^{\frac{m+1}{m}} \left( \int_{B_{3r(s)}} |D \left( u^m \right)|^{2\vartheta_1} \, dx \right)^{\frac{1}{2\vartheta_1}}.
$$

Now we choose

$$
\alpha = \frac{2\vartheta_1 m}{m+1},
$$

and we conclude that

$$
(I) \leq \frac{c_s (r(s))^{2\vartheta_1}}{s^\alpha} \iint_{Q_{2s, 3r(s)}} |D \left( u^m \right)|^{2\vartheta_1} \, dx \, dt = \left( \frac{c_s (r(s))^{2\vartheta_1}}{s^\alpha} \left( \iiint_{Q_{2s, 3r(s)}} |D \left( u^m \right)|^{2\vartheta_1} \, dx \, dt \right)^{\frac{1}{2\vartheta_1}} \right)^{\frac{1}{\vartheta_1}}.
$$

On the other hand, the Fundamental Theorem of Calculus, Lemma 6.1 and the fact that $r^2(s) \theta_s = s$ yield

$$
\left| (u(t))_{B_{3r(s)}}^2 \right|^m - \lambda_0^{m+1} \sim (u(t))_{B_{3r(s)}}^2 + \lambda_0 \right|^m - \lambda_0 \right) \frac{m+1}{m} \, dt
$$

\begin{align*}
&\leq c \lambda_0^{m+1} \int_{A_2} \left| (u(t))_{B_{3r(s)}}^2 - (u(t))_{B_{3r(s)}}^2 \right| \, d\tau \\
&\leq c \frac{c_s}{s^\alpha} \frac{1}{r(s) \theta_s} \left( \int_{Q_{2s, 3r(s)}} |D \left( u^m \right)| \, dx \, dt + \int_{Q_{2s, 3r(s)}} |f| \, dx \, dt \right).
\end{align*}

This implies that

$$
\left( \frac{\theta_s^{m+1}}{s} \right)^{\frac{1}{\vartheta_1}} \leq c(I)^{\frac{1}{\vartheta_1}} + c(II)^{\frac{1}{\vartheta_1}}
$$

\begin{align*}
&\leq \frac{c_s (r(s))^{2\vartheta_1}}{s^\alpha} \left( \iiint_{Q_{2s, 3r(s)}} |D \left( u^m \right)|^{2\vartheta_1} \, dx \, dt \right)^{\frac{1}{2\vartheta_1}} \\
&+ \frac{c_s}{s^\alpha} \left( \frac{r(s) \theta_s}{s} \left( \iiint_{Q_{2s, 3r(s)}} |D \left( u^m \right)| \, dx \, dt + \iiint_{Q_{2s, 3r(s)}} r(s) |f| \, dx \, dt \right) \right)^{\frac{m+1}{m} \frac{1}{\vartheta_1}}.
\end{align*}
= \frac{c_\varepsilon}{s} \frac{m+1}{s} \left(\iint_{Q_{s,3r(s)}} |Du|^2 dxdt\right)^{\frac{1}{m+1}}
+ \frac{c_\varepsilon}{s} \frac{m+1}{s} \left(\int Q_{s,3r(s)} |Du|^2 dxdt\right)^{\frac{1}{m+1}}
+ \left(\iint_{Q_{s,3r(s)}} |Du|^2 dxdt\right)^{\frac{1}{m+1}}
+ \left(\iint_{Q_{s,3r(s)}} r(s) |f|^2 dxdt\right)^{\frac{1}{m+1}}.

By Jensen’s inequality, this implies that
\frac{\theta_{s}^{m+1}}{s} = \theta_s^{m+1} \frac{\theta_s}{s} = \left(\frac{\theta_{s}^{m+1}}{s}\right)^{\frac{1}{m+1}} \frac{\theta_s}{s} \leq c_\varepsilon \left(\iint_{Q_{s,3r(s)}} |Du|^2 dxdt\right)^{\frac{1}{m+1}}
+ c_\varepsilon \iint_{Q_{s,3r(s)}} r(s)^2 |f|^2 dxdt.

6.2. The Non-Degenerate Regime. In the following we assume that the opposite of (6.3) holds. Namely, we consider an intrinsic cylinder \(Q_{2s}^{\theta_{2s}} \equiv Q_{2s,\sqrt{2s/\theta_{2s}}} \equiv Q_{2s,r(2s)} \subset E_{r},\) and we assume that
\left(\iint_{Q_{2s}^{\theta_{2s}}} \left|u^m - \left(u^m\right)_{Q_{2s}^{\theta_{2s}}} \right| dxdt\right)^{\frac{m+1}{m+1}} \leq \varepsilon \left(\iint_{Q_{2s}^{\theta_{2s}}} u^{m+1} dxdt\right)^{\frac{1}{m+1}}.

We denote this condition the Non-Degenerate Regime; heuristically, it implies that \(u\) is close to a solution of the linear heat equation.

**Proposition 6.3.** Let \(Q_{2s}^{\theta_{2s}}\) be one of the intrinsic cylinders constructed in Lemma 3.1 and assume it satisfies (6.7). Then, there exist \(\vartheta_2 \in (0,1)\) and a constant \(c_\varepsilon\) that depends just on \(\varepsilon, \vartheta_2\) and the data, such that
\left(\iint_{Q_{2s}^{\vartheta_{2s}}} |Du|^2 dxdt\right)^{1/2} \leq c_\varepsilon \left(\iint_{Q_{2s}^{\vartheta_{2s}}} |Du|^2 dxdt\right)^{1/2} + c_\varepsilon \iint_{Q_{2s}^{\vartheta_{2s}}} r^2(2s)|f|^2 dxdt
\leq c_\varepsilon \left(\iint_{Q_{2s}^{\vartheta_{2s}}} |Du|^2 dxdt\right)^{1/2} + c_\varepsilon \sup_{Q_{2s}^{\vartheta_{2s}}} [r^2(2s)|f|^2].

**Proof.** We assume \(Q_{2s}^{\theta_{2s}}\) is an intrinsic cylinder, where the non-degenerate regime holds. In this context, we will not make use of the reverse Hölder inequality of Corollary 5.4.

We may suppose that for some \(\delta > 0\) sufficiently small
\(\left(r(2s)\right)^2 \iint_{Q_{2s}^{\theta_{2s}}} |f|^2 dxdt \leq \delta \iint_{Q_{2s}^{\theta_{2s}}} |Du|^2 dxdt;\)
indeed, otherwise, there is nothing to prove.

Fix \(s \in [s,2s]\); by Lemma 3.3 we know that \(\theta_{s} \sim \theta_{2s}\) and \(r(s) \sim r(2s)\). For the sake of simplicity, in the following we write
\(r = \frac{r(2s)}{2},\) and \(\vartheta = \theta_{2s}\).

Without further notice, we use the fact that \(\theta^2 = \frac{\theta}{2}\). Moreover, by Lemma 2.3 we find
\(\left(\iint_{Q_{2s,2r}} \left|u^m - (u^m)_{Q_{2s,2r}} \right| dxdt\right)^{\frac{1}{m+1}} \sim \left(\iint_{Q_{2s,2r}} \left|u^m - (u^m)_{Q_{2s,2r}} \right| dxdt\right)^{\frac{1}{m+1}} \leq c_\varepsilon \left(\iint_{Q_{2s,2r}} u^{m+1} dxdt\right)^{\frac{1}{m+1}}\).
\( \leq c\varepsilon \left( \iint_{Q_{2r,2r}} |u^m - (u^m)_{Q_{2r,2r}}|^\frac{m+1}{m} \, dx \right)^{\frac{1}{m+1}} + c\varepsilon \left( \int_{Q_{2r,2r}} u^m \, dx \right)^{\frac{1}{m+1}} \)

which implies

\( (u^m)^{\frac{1}{r}}_{Q_{2r,2r}} \leq \frac{c\varepsilon}{1 - c\varepsilon^2} \left( \iint_{Q_{2r,2r}} u^{m+1} \, dx \right)^{\frac{1}{m+1}} \).

and

\( \left( \iint_{Q_{2r,2r}} |u^m - (u^m)_{Q_{2r,2r}}|^\frac{m+1}{m} \, dx \right)^{\frac{1}{m+1}} \sim \left( \iint_{Q_{2r,2r}} |u^m - (u^m)_{Q_{2r,2r}}|^1 \, dx \right)^{\frac{1}{m+1}} \)

Next, let \( \tilde{\eta} \) to be a constant-in-time cut off function, such that \( \tilde{\eta} \in C_0^\infty((B_2), [0,1]), \tilde{\eta}(x) \equiv 1 \) for \( x \in B_r \), and \( |D\tilde{\eta}|_\infty \leq \frac{c}{r} \). We set \( \eta = \frac{\tilde{\eta}}{|B_r|} \), and define

\( \lambda = (u)^\eta, \lambda(t) = (u(t))^\eta \) and \( e^m = (u^m)^\eta, e^m(t) = ((u(t))^m)^\eta \).

By Lemma 5.1 we find that

\( (u^{m+1})^{\frac{1}{Q_{2r,2r}}} \sim (u^{m+1})^{\frac{1}{Q_r, r}} \) and \( (u)_{Q_{2r,2r}} \sim (u)_{Q_r, r} \),

which directly implies that

\( (u^{m+1})_{Q_{2r,2r}} \sim (u^{m+1})^\eta, \) and \( (u)_{Q_{2r,2r}} \sim (u)^\eta \).

We conclude that

\( \lambda \sim (u^{\frac{1}{Q_{2r,2r}}} \sim (u^{\frac{1}{Q_r, r}} \sim \theta_{2s}^\frac{1}{m} \)

for all \( a \in [\frac{1}{2}, 1] \). For any \( 1/2 \leq a < b \leq 1 \) and \( \theta_{2s} \sim \lambda^{1-m} > 0 \) we find by the energy estimate of Lemma 4.1 with \( c = \lambda \)

\[ I(a) + II(a) := \frac{\lambda^{m-1}}{r^2} \sup_{t \in A_2} \left( \iint_{B_{2r}} |u^m - \lambda^m|^{\frac{m+1}{m}} \, dx \right)^{\frac{1}{m+1}} + \iint_{B_{2r}} |u^m - \lambda^m| \, dx + \iint_{Q_{2r,2r}} |Du^m|^2 \, dx \]

\[ \leq \frac{c\lambda^{m-1}}{r^2(b-a)} \iint_{Q_{br,br}} |u^m - \lambda^m| \, dx \]

\[ + \frac{c}{(b-a)^2r^2} \iint_{Q_{br,br}} |u^m - \lambda^m|^2 \, dx + c \iint_{Q_{br,br}} |r^2 |f|^2 | \, dx \]

\[ \leq \frac{c\lambda^{m-1}}{r^2(b-a)} \iint_{Q_{br,br}} |u^m - \lambda^m| \, dx \]

\[ + \frac{c}{(b-a)^2r^2} \iint_{Q_{br,br}} |u^m - \lambda^m|^2 \, dx + c\delta \iint_{Q_{br,br}} |Du^m|^2 \, dx \]

Hence, by fixing \( \delta \) sufficiently small, by absorption we find that

\[ I(a) + II(a) \leq \frac{c\lambda^{m-1}}{r^2(b-a)} \iint_{Q_{br,br}} |u^m - \lambda^m| \, dx \]

\[ + \frac{c}{(b-a)^2r^2} \iint_{Q_{br,br}} |u^m - \lambda^m|^2 \, dx \]

We continue using Young’s inequality and the fact that

\( |u^m - \lambda^m|^2 \sim (u + \lambda)^{2m-2} |u - \lambda|^2 \leq \lambda^{m-1} |u - \lambda|^2 (u + \lambda)^{m-1} \sim \lambda^{m-1} |u - \lambda| |u^m - \lambda^m|, \)
and find

\[
I(a) + II(a) \leq \frac{c}{(b-a)^2} \left( \frac{\lambda^{m-1}}{r^2} \int_{B_{br}} |u^m - \lambda^m| |u - \lambda| \, dx \right)^{1-\alpha} 
\times \int_{A_{br}} \left[ \left( \frac{\lambda^{m-1}}{r^2} \int_{B_{br}} |u^m - \lambda^m| |u - \lambda| \, dx \right)^{\alpha} + \left( \frac{1}{r^2} \int_{B_{br}} |u^m - \lambda^m|^2 \, dx \right)^{\alpha} \right] \, dt 
= (I(b))^{1-\alpha} \frac{c}{(b-a)^2} \int_{A_{br}} \left( \frac{\lambda^{m-1}}{r^2} \int_{B_{br}} |u^m - \lambda^m| |u - \lambda| \, dx \right)^{\alpha} 
+ \left( \frac{1}{r^2} \int_{B_{br}} |u^m - \lambda^m|^2 \, dx \right)^{\alpha} \, dt 
\leq \frac{1}{2} (I(b)) + \frac{c}{(b-a)^2} \left( \frac{\lambda^{m-1}}{r^2} \int_{B_{br}} |u^m - \lambda^m| |u - \lambda| \, dx \right)^{\alpha} 
+ \left( \frac{1}{r^2} \int_{B_{br}} |u^m - \lambda^m|^2 \, dx \right)^{\alpha} \, dt 
= \frac{1}{2} I(b) + \frac{c}{(b-a)^2}(II). 
\]

Since we can interpolate the essential supremum to the left-hand side, we are left to estimate (III). Observe that

\[
\int_{B_{br}} |u^m - \lambda^m| |u - \lambda| \eta \, dx = \int_{B_{br}} \chi_{\{u \geq 2\lambda\}} |u^m - \lambda^m| |u - \lambda| \eta \, dx 
+ \int_{B_{br}} \chi_{\{u < 2\lambda\}} |u^m - \lambda^m| |u - \lambda| \eta \, dx 
\leq \int_{B_{br}} \chi_{\{u \geq 2\lambda\}} u^{m+1} \eta \, dx 
+ \lambda^{1-m} \int_{B_{br}} \chi_{\{u < 2\lambda\}} |u^m - \lambda^m| \lambda^{m-1} |u - \lambda| \eta \, dx 
\leq c \left( \int_{B_{br}} u^{m+1} \, dx \right)^{\frac{m+1}{m+1}} \left( \int_{B_{br}} \chi_{\{u \geq 2\lambda\}} |u^m - \lambda^m|^{\frac{m+1}{m}} \eta \, dx \right)^{\frac{2m}{m+1}} 
+ c \lambda^{1-m} \int_{B_{br}} \chi_{\{u < 2\lambda\}} |u^m - \lambda^m| (\lambda + u)^{m-1} |u - \lambda| \eta \, dx 
\leq c \left( \lambda^{1-m} + \int_{B_{br}} u^{m+1} \, dx \right)^{\frac{1-m}{m+1}} \left( \int_{B_{br}} |u^m - \lambda^m|^{\frac{m+1}{m}} \eta \, dx \right)^{\frac{2m}{m+1}}. 
\]

This allows to conclude via Hölder’s inequality (using \(\frac{1-m}{m+1} + \frac{2m}{m+1} = 1\)) and Jensen’s inequality that

\[
(III)^{\alpha} \leq c \int_{A_{br}} \left[ \left( \frac{\lambda^{m-1}}{r^2} \int_{B_{br}} |u^m - \lambda^m| |u - \lambda| \eta \, dx \right)^{\alpha} + \left( \frac{1}{r^2} \int_{B_{br}} |u^m - \lambda^m|^2 \eta \, dx \right)^{\alpha} \right] \, dt 
\leq c \left( \int_{A_{br}} \left( \frac{1}{r^m} \int_{B_{br}} |u^m - \lambda^m|^{\frac{m+1}{m}} \eta \, dx \right)^{\frac{m}{m+1}} \right)^{\frac{2m}{m+1}}. 
\]

Next we may subtract the space mean values and find by Lemma 2.3

\[
\left( \int_{B_{br}} |u^m - \lambda^m|^{\frac{m+1}{m}} \eta \, dx \right)^{\frac{m}{m+1}} \leq c \left( \int_{B_{br}} |u^m - \lambda^m(t)|^{\frac{m+1}{m}} \eta \, dx \right)^{\frac{m}{m+1}} + c |\lambda^m(t) - \lambda^m| 
\leq c \left( \int_{B_{br}} |u^m - e^m(t)|^{\frac{m+1}{m}} \eta \, dx \right)^{\frac{m}{m+1}} + c |\lambda^m(t) - \lambda^m| 
= (i) + (ii). 
\]
Using (2.5) yields
\[
\frac{(i)^2}{r^2} \leq \frac{c}{r^2} \left( \int_{B_{2r}} |u^m - e^m(t)|^{\frac{m+1}{m}} \eta \, dx \right)^{\frac{2m}{m+1}} \leq c \left( \int_{B_{2r}} |Du^m|^{2\theta_2} \, dx \right)^{\frac{1}{\theta_2}}.
\]

We further estimate using Lemma 6.1, to get that
\[
\frac{(ii)}{r} = \frac{1}{r} |\lambda^m - \lambda^m(t)| \sim \frac{1}{r} (\lambda(t) - \lambda)\]
\[
\leq \frac{c_\lambda}{r^2(s)} \int_{Q_{2r,2r}} |Du^m| \, dx \, dt \leq c \int_{Q_{2r,2r}} |Du^m| \, dx \, dt.
\]

By choosing \( \alpha = \frac{2m\theta_2}{m+1} \) we find
\[
(III) \leq c \left( \int_{Q_{2r,2r}} |Du^m| \, dx \, dt \right)^{\alpha} + \int_{B_{2r}} |Du^m|^{2\theta_2} \, dx \, dt
\]
\[
\leq c \left( \int_{Q_{2r,2r}} |Du^m|^{2\theta_2} \, dx \, dt \right)^{\frac{1}{\theta_2}}.
\]

Hence, the result follows by the interpolation result of [Giu03, Lemma 6.1].

7. Higher Integrability

7.1. Covering. We can finally come to the core of our argument; in the following we will build a proper covering with respect to \( u \) for the level sets of the function \( |Du^m|^2 \).

We assume that we have a sub-intrinsic cylinder
\[
C\theta_o \geq \left( \int_{Q_{2r,2r,2r}} u^m+1 \, dx \, dt \right)^{\frac{1-m}{m+1}}. \tag{7.1}
\]

We start by scaling everything to the cube \( Q_{2,2} \). This can be done by introducing \( \lambda_o = \frac{1}{\theta_o} \). Then we define \( \tilde{u}(y, s) = \lambda_o u(Ry, \lambda_o^{-1}R^2s) \). For this scaled solution we find
\[
C \geq \lambda_o^{m+1} \int_{Q_{2\lambda_o^{-1}R^2,2R}} u^m+1 \, dx \, dt = \int_{Q_{2,2}} \tilde{u}^m+1 \, dy \, ds. \tag{7.2}
\]

Moreover, \( \tilde{u} \) is a weak solution in \( Q_{2,2} \) to
\[
\tilde{u}_s - \text{div} \tilde{A}(y, s, \tilde{u}, \tilde{D}\tilde{u}) = \tilde{f}
\]
with right-hand side \( \tilde{f}(y, s) = \lambda_o^m R^2 f(Ry, \lambda_o^{-1}R^2s) \), and \( \tilde{A} \) which satisfies structure conditions analogous to (1.6).

By (4.2), the previous scaling and (7.1), we find that
\[
\lambda_o^{2m} R^2 \int_{Q_{\lambda_o^{-1}R^2,2R}} |Du^m|^2 \, dx \, dt = \int_{Q_{1,2}} |D\tilde{u}^m|^2 \, dy \, ds
\]
\[
\leq c \left( \int_{Q_{2,2}} \tilde{u}^m+1 \, dy \, ds \right)^{\frac{2m}{m+1}} + c \int_{Q_{2,2}} f^2 \, dy \, ds \leq C(\tilde{f}).
\]

We introduce the notation
\[
F = |D\tilde{u}^m|^2 \chi_{Q_{2,2}}. \tag{7.3}
\]

We fix
\[
\hat{b} \in \left( 0, \min \left\{ (n+2)(m+1) - 2n, \frac{1}{2} \right\} \right) \quad \text{and} \quad \beta = (1 - 2\hat{b}) \in (0,1),
\]

such that \( 1 < \frac{1}{1 - 2\hat{b}} = \frac{1}{\beta} < \frac{m+1}{1-m} \).
Let $\gamma_1$ be the constant in Corollary 4.3 for $K = 1$, and $\delta \in (0, \frac{1}{2})$ a parameter which is to be chosen later. Finally, let $c_1 > 1$ be fixed by (2) of Lemma 3.3. Accordingly, we fix $c_o, c_2 \in (2, \infty)$ depending on $\delta$ and the data alone, such that

$$
\gamma_1 \left( \frac{2}{c_o} \right)^{\frac{m+1}{m-b}-1} = \tilde{\delta} \quad \text{and} \quad c_2 = 2c_1^3 c_o, \quad \text{with} \quad \tilde{\delta} = 3^{1/b}.
$$

(7.5)

We will apply Lemma 3.1 with respect to $\hat{u}$ and this fixed choice of $\hat{b}$ on the sub-intrinsic initial cylinder $Q_{2,2}$. To simplify the notation, for $y \in Q_{1,1}$ and $s \in (0,1)$, we denote the sub-intrinsic cube $Q_{s,r(s,y)}(y)$ defined in Lemma 3.1 with $Q(s,y)$. Moreover, we denote again with $x$ and $t$ the new variables $y$ and $s$.

**Lemma 7.1.** Let $c_2$ be fixed by (7.5). For $\frac{1}{2} \leq a < b \leq 1$, there exists a parameter $\mu_{a,b}$, defined by

$$
\mu_{a,b} = \frac{C(\hat{f}, \hat{\delta})}{|b-a|^p},
$$

with $\tau$ depending only on the data, such that if for $z \in Q_{a,a}$, the intersection $Q(c_2 s, z) \cap Q_{k,b}$ is not empty, then

$$
\iint_{Q(s,z)} F \, dx \, dt \leq \mu_{a,b}.
$$

Here $C(\hat{f}, \hat{\delta}) = c \iint_{Q_{2,2}} \hat{f}^2 \, dx \, dt + c \left( \iint_{Q_{2,2}} \hat{f}^{m+1} \, dx \right)^{\frac{2m}{m+1}}$, with $c > 0$ depending only on the data, and $\delta$.

**Proof.** If $Q(c_2 s, z) \cap Q_{k,b} \neq \emptyset$, then either $c_2 s/2 > (b-a)$, or $r(c_2 s, z) > (b-a)$. In the latter case, by (b) of Lemma 3.1 (using the fact that $S = 1$ and $R(z) \sim 1$ here), we have that $(b-a) \leq c_2 s^{\frac{b}{c_2}}$ and by (f) that $(b-a) \leq c_2^{\frac{b}{c_2}} r(s, z)$.

In case $c_2 s/2 > (b-a)$, we find that $r(s, z) \geq r(\frac{b-a}{c_2}, z) \geq c |b-a|^\frac{b}{c_2} R(z) \geq c |b-a|^\frac{b}{c_2}$. Therefore, (f) of Lemma 3.1 implies for $\tau = \max \{a, 1\} n + \frac{b}{c_2}$, that

$$
\iint_{Q(s,z)} F \, dx \, dt \leq \frac{1}{|b-a|} \tau \iint_{Q_{2,2}} F \, dx \, dt \leq \frac{C(\hat{f}, \hat{\delta})}{|b-a|^p}.
$$

(7.6)

Now, for a locally integrable function $g$, we introduce the related intrinsic maximal function

$$
\mathcal{M}(g)(z) = \sup_{Q(s,y) \ni z} \iint_{Q(s,y)} |g| \, dx \, dt,
$$

and we define

$$
M^*(g)(t,x) := \sup_{t \in I \subset (-2,2), \, x \in B \subset B_x} \iint_{I \times B} |g| \, dx \, dt.
$$

(7.7)

Notice that we have

$$
|g| \leq \mathcal{M}(g) \leq M^*(g) \quad \text{a.e.};
$$

(7.8)

therefore, $\mathcal{M}$ is continuous from $L^q \to L^q$, whenever $g \in L^q$.

We define the level sets of $F$ by

$$
O_\lambda := \{ \mathcal{M}(F) > \lambda \}.
$$

(7.9)

The next proposition is the core of the proof of the higher integrability. It constructs a covering, which allows to exploit the reverse Hölder estimates of the previous section in a suitable way. It is a covering of Calderon-Zygmund type for $F$, build using cylinders scaled with respect to $\hat{u}$.

The fact that the scaling is done with respect to $\hat{u}$, and not with respect to the function whose level sets are covered, i.e. $F$, makes things quite delicate. In this context, this seems the right way to proceed, instead of relying on the by-now standard approach of parabolic intrinsic Calderon-Zygmund covering, originally introduced by Kinnunen & Lewis [KL00].
Proposition 7.2. For $\tilde{\delta}$ fixed in (7.15) below, and $\frac{1}{a} \leq b \leq 1$, let $\mu_{a,b}$ be the quantity introduced in Lemma 7.1. Let $q = \min \{q_1, q_2\} \in (0,1)$ where $q_1$, $q_2$ are the exponents defined via the reverse Hölder estimates of Propositions 6.2 and 6.3.

For every $\lambda > \mu_{a,b}$, and every $z \in O_\lambda \cap Q_{a,a}$, there exist $Q_\lambda \subset Q_\lambda^* \subset Q_\lambda^{**} \subset Q_{b,b}$, which satisfy the following properties.

(i) $|Q_\lambda|, |Q_\lambda^*|, |Q_\lambda^{**}|$ are of comparable size.

(ii) For any $y \in Q_\lambda^*$ there exist constants $c$ and $C$ just depending on the data such that

$$\lambda \leq c \iint_{Q_\lambda^*} F \, dxdt \leq C \left( \iint_{Q_\lambda^{**}} F^q \, dxdt \right)^{\frac{1}{q}} + CM^*(\tilde{f}^2)(y).$$

(iii) $\iint_{Q_\lambda^{**}} F \, dxdt \leq 2\lambda$.

Moreover, the set $O_\lambda \cap Q_{a,a}$ can be covered by a family of cylinders $Q_i^{**} := Q_{z_i}^{**} \subset Q_{b,b}$, such that the cylinders of the family $Q_{z_i}^*$ are pairwise disjoint.

Proof. We fix $\lambda > \mu_{a,b}$, and for $z \in O_\lambda \cap Q_{a,a}$ we choose $z_\lambda$ and $s_\lambda$, such that $Q(s_\lambda, y_\lambda) \ni z$ and

$$\lambda < \iint_{Q(s_\lambda, y_\lambda)} F \, dxdt \quad \text{and} \quad \iint_{Q(\sigma, \xi)} F \, dxdt \leq 2\lambda \quad (7.11)$$

for all $Q(\sigma, \xi) \ni Q(s_\lambda, y_\lambda)$; such a cylinder certainly exists by the very definition of the set $O_\lambda$. In the following notation, for $c \in \mathbb{R}^+$, we let $\tilde{c} = c^{1/b}$, with $b$ as in (7.4). For $z \in O_\lambda \cap Q_{a,a}$, we will carefully choose cylinders according to the following table. Here we rely on (7.5); notice that we are assuming $c_0 > 2$.

Case 1: There exists $\sigma \in [2s_\lambda, c_0 s_\lambda]$ such that $Q(\sigma, y_\lambda)$ is intrinsic and the degenerate alternative holds in $Q(\sigma, y_\lambda)$. Then we let

- $Q_\lambda = Q(\sigma, y_\lambda), \quad Q_\lambda^* = Q(3\sigma, y_\lambda), \quad Q_\lambda^{**} = Q(2c_1\tilde{3}\sigma, y_\lambda)$

Case 2: There exists $\sigma \in [2s_\lambda, c_0 s_\lambda]$ such that $Q(\sigma, y_\lambda)$ is intrinsic and the non-degenerate alternative holds in $Q(\sigma, y_\lambda)$. Then we let

- $Q_\lambda = \frac{1}{2}Q(\sigma, y_\lambda), \quad Q_\lambda^* = Q(3\sigma, y_\lambda), \quad Q_\lambda^{**} = Q(2c_1\sigma, y_\lambda)$

Case 3: There exists no $\sigma \in [2s_\lambda, c_0 s_\lambda]$ such that $Q(\sigma, y_\lambda)$ is intrinsic. Then we let

- $Q_\lambda = \frac{1}{2}Q(2s_\lambda, y_\lambda), \quad Q_\lambda^* = Q(2s_\lambda, y_\lambda), \quad Q_\lambda^{**} = Q(4c_1 s_\lambda, y_\lambda)$

Since $2c_1\tilde{3}\sigma \leq c_2 s_\lambda$, Lemma 7.1 implies that $Q_\lambda^{**} \subset Q_{b,b}$ in all the above cases, as otherwise there is a contradiction to $\lambda > \mu_{a,b}$ by (7.11).

On the one hand, (g) of Lemma 3.1 implies that $Q_{z_i}^{**} \ni Q \supset Q(s_\lambda, y_\lambda)$, and therefore, we conclude that

$$\iint_{Q_{z_i}^{**}} F \, dxdt \leq 2\lambda.$$

On the other hand, since (f) of Lemma 3.1 implies that $|Q_\lambda| \approx |Q_\lambda^{**}| \approx |Q(s_\lambda, y_\lambda)|$, by (7.11) we find that

$$\lambda \leq c \iint_{Q_\lambda} F \, dxdt. \quad (7.12)$$

The proof of the reverse Hölder inequality has to be split in several sub-cases.
Case 1.
In this case \( Q_z \) is intrinsic and the degenerate alternative holds. Consequently, (7.12) and Proposition 6.2 imply
\[
\lambda \leq c \iint_{Q_z} F \, dx \, dt \leq C \left( \iint_{Q_z} F^q \, dx \, dt \right)^{\frac{1}{q}} + CM^*(\tilde{f}^2)(y),
\]
for any \( y \in Q_z^* \).

Case 2.
In this case \( Q_z^* \) is intrinsic and the non-degenerate alternative holds. Proposition 6.3 and (7.12) directly imply
\[
\lambda \leq c \iint_{Q_z} F \, dx \, dt \leq C \left( \iint_{Q_z^*} F^q \, dx \, dt \right)^{\frac{1}{q}} + CM^*(\tilde{f}^2)(y),
\]
for any \( y \in Q_z^* \).

Case 3.
This is the most delicate part. Applying Corollary 4.3 on the cylinder \( Q_z = \frac{1}{2}Q(2s_z, y_z) \) yields
\[
\lambda \leq c \iint_{Q_z} F \, dx \, dt \leq \gamma_1 \frac{m+1}{2s_z} \theta_{2s_z, y_z}^\sigma + M^*(\tilde{f}^2)(y).
\]
We now show that
\[
\gamma_1\frac{m+1}{2s_z}\frac{\theta_{2s_z, y_z}}{\theta_{2s_z, y_z}^\sigma} \leq \frac{1}{2}\lambda + M^*(\tilde{f}^2)(y). \tag{7.13}
\]
Once this is proven, then by absorption, estimate (7.10) follows.

We begin by defining \( \sigma_z := \inf \{ s \in [c_\sigma s_z, 1] : Q(s, y_z) \text{ is intrinsic} \} \). Since we are in Case 3, we find that \( \sigma_z \in (c_\sigma s_z, 1] \). (e) of Lemma 3.1 and the choice of \( c_\sigma \) in (7.5) imply that
\[
\frac{\theta_{2s_z, y_z}}{\theta_{2s_z, y_z}^\sigma} \leq \gamma_1 \left( \frac{2s_z}{\sigma_z} \right)^\beta \frac{\theta_{2s_z, y_z}}{\theta_{2s_z, y_z}^\sigma}
\]
\[
\Rightarrow \gamma_1 \frac{\theta_{2s_z, y_z}}{2s_z} \leq \gamma_1 \left( \frac{2s_z}{\sigma_z} \right)^\beta \frac{m+1}{2s_z} \theta_{2s_z, y_z}^\sigma
\]
\[
\Rightarrow \gamma_1 \frac{\theta_{2s_z, y_z}}{2s_z} \leq \gamma_1 \left( \frac{2s_z}{\sigma_z} \right)^\beta \frac{m+1}{2s_z} \theta_{2s_z, y_z}^\sigma
\]
\[
\Rightarrow \gamma_1 \frac{\theta_{2s_z, y_z}}{2s_z} \leq \gamma_1 \left( \frac{2}{c_\sigma} \right)^\beta \frac{m+1}{2s_z} \theta_{2s_z, y_z}^\sigma
\]
Notice that in (7.4) we fixed \( \beta \) large enough (that is, \( \hat{b} \) small enough) such that
\[
1 < \frac{m+1}{1-m} = \frac{m+1}{1-m} (1 - \hat{b})
\]
and in (7.5) \( c_\sigma \) is such that
\[
\gamma_1 \left( \frac{2}{c_\sigma} \right)^{\frac{m+1}{1-m} - 1} = \bar{\delta}.
\]
We conclude that
\[
\gamma_1 \frac{\theta_{2s_z, y_z}}{2s_z} \leq \bar{\delta} \frac{\theta_{2s_z, y_z}}{\sigma_z}. \tag{7.14}
\]
In the simple case \( \sigma_z \in (1/3, 1] \), we further estimate by (1) of Lemma 3.3. Indeed, since \( \lambda > 1 \) and \( \theta_o \sim 1 \), by (7.1) we find that there is a \( c_3 \geq 1 \) independent of \( \bar{\delta} \), such that
\[
\frac{\theta_{2s_z, y_z}}{\sigma_z} \leq c \theta_{1, y_z} \leq c_3 \lambda;
\]
this implies (7.13) once we choose \( \bar{\delta} \leq \frac{1}{2c_3} \).
In the difficult case $\sigma_z \in (c_0 s_z, 1/\beta)$, we use the fact that $Q(\sigma_z, y_z)$ is intrinsic. Hence by Lemma 3.1, (c) and (e) we find that

$$
\iint_{Q(\sigma_z/2, y_z)} \tilde{u}^{m+1} \, dx \, dt \leq \frac{\theta_{\sigma_z/2, y_z}^{m+1}}{2} \leq \frac{1}{2^{m+1}} \iint_{Q(\sigma_z, y_z)} \tilde{u}^{m+1} \, dx \, dt.
$$

By Lemma 2.4, this implies that $Q(\sigma_z, y_z)$ is degenerate and intrinsic. Now Proposition 6.2, (g) of Lemma 3.1, and Jensen’s inequality imply

$$
\frac{\tilde{\theta}_{\sigma_z, y_z}}{\sigma_z} \leq c_0 \left( \frac{\int_{Q(\sigma_z, y_z)} |F|^q \, dx \, dt}{\int_{Q(\sigma_z, y_z)} |F|^2 \, dx \, dt} \right)^{1/2} + M^*(\hat{f}^2)(y) \leq C \delta \iint_{Q(\sigma_z, y_z)} F \, dx \, dt + CM^*(\hat{f}^2)(y).
$$

Therefore, by the construction of $Q(s, y_z)$, we find that there is a $c_4$ independent of $\tilde{\delta}$, such that

$$
\frac{\tilde{\theta}_{\sigma_z, y_z}}{\sigma_z} \leq c_4 \tilde{\delta} + M^*(\hat{f}^2)(y).
$$

Now (7.13) follows by choosing

$$
\tilde{\delta} = \min \left\{ \frac{1}{2c_3} \frac{1}{2c_4} \right\}.
$$

This finishes the construction of $Q_z, Q_z^*$ and $Q_z^{**}$. The covering now follows by Lemma 3.3 and Lemma 3.2, for $\Omega = O_\lambda$ and $U(z, s_z) := Q_z^*$.

We can finally conclude and prove the higher integrability result.

**Theorem 7.3** (Intrinsic). Let $u \geq 0$ be a local, weak solution to (1.5)-(1.6) in the space-time cylinder $E_T$ for $m \in \left( \frac{n-2}{n+2}, 1 \right)$. There exist an exponent $p > 1$, and a constant $c$ that depends only on the data, such that for any sub-intrinsic parabolic cylinder

$$
\left( \iint_{Q_s, \sqrt{\gamma} \eta} u^{m+1} \, dx \, dt \right)^{1-m} \leq C \theta_o,
$$

we have

$$
\left( \iint_{Q_s, \sqrt{\gamma} \eta} |Du|^{2p} \, dx \, dt \right)^{1/p} \leq c \left( \iint_{Q_{2s, 2\sqrt{\gamma} \eta}} |f|^{2p} \, dx \, dt \right)^{1/p} + c' \frac{\theta_o^{m+1}}{S},
$$

where $c'$ additionally depends on $C$.

**Proof.** We define the so-called bad set

$$
U_\lambda := O_\lambda \cap \left\{ M^*(\hat{f}^2 \chi_{Q_2, z}) \leq \tilde{\varepsilon}\lambda \right\},
$$

for some $\tilde{\varepsilon}$, which will be chosen later. We proceed by providing a re-distributional estimate. We take $\frac{1}{4} \leq a < b \leq 1$ and the corresponding covering constructed in Lemma 7.2. We start by

$$
|Q_i| = |Q_i \cap U_\lambda| + |Q_i \cap (U_\lambda)^c|,
$$

where $Q_i$ is one of the cylinders $Q_z$, built in Proposition 7.2. Let us first consider the case

$$
\frac{|Q_i \cap U_\lambda|}{|Q_i|} \geq \frac{1}{2}.
$$

This implies that there exists $y \in Q_i$, such that $M^*(f)(y) \leq \tilde{\varepsilon}\lambda$. We can apply the reverse Hölder estimate (7.10) of Proposition 7.2, and obtain for some $\gamma \in (0, 1)$ that

$$
\lambda^q \leq c \left( \iint_{Q_\gamma} F \, dx \, dt \right)^q \leq \frac{C}{|Q_i|} \iint_{Q_\gamma} F^q \chi_{(F > \gamma \lambda)} \, dx \, dt + C(\gamma \lambda)^q + C\tilde{\varepsilon}\lambda^q.
$$

We now choose $\gamma$ and $\tilde{\varepsilon}$ conveniently small, such that $C(\gamma \lambda)^q + C\tilde{\varepsilon}\lambda^q = \frac{1}{2} \lambda^q$ and find

$$
\lambda |Q_i| \leq c \lambda^{1-q} \iint_{Q_\gamma} F^q \chi_{(F > \gamma \lambda)} \, dx \, dt.
$$
On the other hand,

$$|Q_i| \leq 2|Q_i \cap (U_\lambda)^c| \quad \Rightarrow \quad \lambda |Q_i| \leq 2\lambda |Q_i \cap (U_\lambda)^c|.$$ 

Therefore, in any case,

$$\lambda |Q_i| \leq C\lambda^{1-q} \int_{Q_i} F^q \chi_{\{F > \gamma \lambda\}} \, dx \, dt + 2\lambda |Q_i \cap (U_\lambda)^c|.$$ 

We proceed by using the last estimates as well as the fact that $\{Q_i^*\}_i$ covers the set $O_\lambda \cap Q_{a,a}^*$.

$$\int_{Q_{a,a} \cap O_\lambda} F \, dx \, dt \leq \sum_i \int_{Q_i^*} F \, dx \, dt \leq 2\lambda \sum_i |Q_i^*| \leq C\lambda \sum_i |Q_i|$$

$$\leq C\lambda^{1-q} \int_{Q_i} F^q \chi_{\{F > \gamma \lambda\}} \, dx \, dt + 2C\lambda |Q_i \cap (U_\lambda)^c|$$

We pick $\alpha \in (0,1)$ to be chosen later, $k \in \mathbb{N}$, multiply the above estimate by $\lambda^{-\alpha}$, and integrate from $\mu_{a,b}$ to $k$ with respect to $\lambda$. This implies

$$\frac{1}{2 - q - \alpha} \leq \int_{\mu_{a,b}^* \cap O_\lambda} F \, dx \, dt \leq C \int_{\mu_{a,b}^*} \lambda^{-\alpha} \, dx \, dt$$

We estimate from below

$$(I) \geq \int_{Q_{a,a}} F \chi_{\{F > \lambda\}} \, dx \, dt \lambda \leq \frac{\mu_{a,b}^{2 - \alpha}}{1 - \alpha}$$

$$= \int_{Q_{a,a}} F \int_{\min\{F(x), k\}} \lambda^{-\alpha} \, dx \, dt$$

$$= \frac{1}{1 - \alpha} \int_{Q_{a,a}} F \min\{F, k\}^{1 - \alpha} \, dx \, dt$$

The bound from above is analogous

$$(II) \leq C \int_{\mu_{a,b}^*} \lambda^{1-q-\alpha} \int_{Q_{b,b}} F^q \chi_{\{F > \lambda\}} \, dx \, dt \lambda$$

$$\leq \frac{c}{2 - q - \alpha} \int_{Q_{b,b}} F^q \min\{F, k\}^{2-q-\alpha} \, dx \, dt.$$ 

Finally, $(III)$ is estimated by the continuity of the maximal function and the classical integral representation via level sets. We calculate and estimate

$$(III) = \int_{\mu_{a,b}^*} \lambda^{-\alpha} \left| Q_{b,b} \cap \left\{ M^*(\tilde{F}^2) > \tilde{\varepsilon} \lambda \right\} \right| \, d\lambda \leq c \int_{Q_{b,b}} \left| M^*(\tilde{F}^2_{Q_{2,2}}) \right|^{2-\alpha} \, dx \, dt$$

$$\leq c \int_{Q_{2,2}} \tilde{F}^{2(2-\alpha)} \, dx \, dt.$$ 

All together, using the definition of $\mu_{a,b}$ from (7.6), we find that

$$\frac{1}{1 - \alpha} \int_{Q_{a,a}} F \min\{F, k\}^{1-\alpha} \, dx \, dt \leq \frac{c}{2 - q - \alpha} \int_{Q_{b,b}} F^q \min\{F, k\}^{2-q-\alpha} \, dx \, dt$$

$$+ \frac{c|b-a|^{-\tau(2-\alpha)}}{1 - \alpha} + c \int_{Q_{2,2}} \tilde{F}^{2(2-\alpha)} \, dx \, dt.$$ 

Now, we fix $\alpha \in (0,1)$ in such a way, that

$$\frac{c(1-\alpha)}{2 - q - \alpha} \leq \frac{1}{2}.$$
This implies
\[
\int_{Q_{a,a}} \min \{F, k\}^{1-\alpha} \, dx \, dt \leq \frac{1}{2} \int_{Q_{b,b}} \min \{F, k\}^{1-\alpha} \, dx \, dt + c |b - a|^{-\tau(2-\alpha)} + c \int_{Q_{2,2}} \tilde{f}^{2(2-\alpha)} \, dx \, dt.
\]

Finally, the interpolation Lemma 6.1 of [Giu03] implies that for every \( k \in \mathbb{N} \)
\[
\int_{Q_{a,a}} \min \{F, k\}^{1-\alpha} \, dx \, dt \leq c \int_{Q_{2,2}} \tilde{f}^{2(2-\alpha)} \, dx \, dt.
\]

Letting \( k \to \infty \) for \( p = 2 - \alpha > 1 \) yields that
\[
\int_{Q_{a,a}} |D\tilde{u}^m|^{2p} \, dx \, dt \leq c \int_{Q_{2,2}} \tilde{f}^{2p} \, dx \, dt.
\]

This implies the desired result by scaling back to \( u \). \( \square \)

**Theorem 7.4** (parabolic). Let \( u \geq 0 \) be a local, weak solution to (1.5)-(1.6) in the space-time cylinder \( E_T \) for \( m \in \left( \frac{n-2}{n+2}, 1 \right) \). There exist an exponent \( p > 1 \) and a constant \( c \), that depends only on the data, such that for any parabolic cylinder \( Q_{R^2, R} \subset E_T \) with
\[
\left( \int_{Q_{R^2, R}} u^{m+1} \, dx \, dt \right)^{1-m} = K,
\]
we have
\[
\left( \int_{Q_{R^2, R}} |D\tilde{u}^m|^{2p} \, dx \, dt \right)^{1-m} \leq c \sqrt{K} \left( \int_{Q_{R^2, R}} R^{2p} f^{2p} \, dx \, dt \right)^{1-m} + cK^{\frac{7}{8}} + c. \tag{7.16}
\]

**Proof.** Estimate (7.16) is proved by covering \( Q_{R^2, R} \) with proper sub-intrinsic cylinders. As (1.5)-(1.6) is essentially invariant under the classical parabolic scaling, without loss of generality, we may assume \( R = 1 \).

We define \( K \) as the number for which
\[
\left( \int_{Q_{1,1}} u^{m+1} \, dx \, dt \right)^{1-m} = K.
\]

Now let \( r \in (0, 1] \) and \( s \in (0, 1] \); any \( Q_{s,r} \subset Q_{1,1} \) satisfies
\[
\left( \int_{Q_{s,r}} u^{m+1} \, dx \, dt \right)^{m-1} \leq K.
\]

If \( \frac{(m+1)|B_r|^{m-1}}{s^2} \geq K \), then the cylinder is sub-intrinsic, since
\[
\left( \int_{Q_{s,r}} u^{m+1} \, dx \, dt \right)^{m-1} \leq \frac{r^{2(m+1)} |B_r|^{m-1}}{s^2}.
\]

If \( K \leq 1 \), we can pick \( s, r = 1 \) and the result follows by Theorem 7.3. If \( K > 1 \), we choose \( r = 1 \) and \( s = \sqrt{K} \). We can then cover \( Q_{1,1} \) by \( N \) sub-cylinders of the above type, where
\[
\left\lfloor \sqrt{K} \right\rfloor \leq N \leq \left\lceil \sqrt{K} \right\rceil,
\]
and \( \lfloor \cdot \rfloor, \lceil \cdot \rceil \) are the floor and ceiling functions, respectively. This concludes the proof by Theorem 7.3. \( \square \)
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