MARGULIS-RUELLE INEQUALITY FOR GENERAL MANIFOLDS

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Abstract. In this paper we investigate the Margulis-Ruelle inequality for general Riemannian manifolds (possibly noncompact and with boundary) and show that it always holds under integrable condition.

1. Introduction

In the theory of differentiable dynamical systems, Margulis-Ruelle inequality \[6\] is a fundamental formulation. This inequality establishes the bridge between the chaotic index metric entropy and the stable index Lyapunov exponent, being employed especially frequently in the study of hyperbolic behaviors in various settings (uniformly, nonuniformly or partially hyperbolic systems).

Let \( f \) be a continuous map on a metric space \( M \). For \( f \)-invariant Borel probability measure \( \mu \) and measurable partition \( \xi \) of \( M \) with partition entropy \( H_\mu(\xi) = \int \log \mu^{-1}(\xi(x)) d\mu < \infty \), the metric entropy of \( f \) with respect to \( \xi \) is given by

\[
 h_\mu(f, \xi) = \lim_{n \to +\infty} \frac{1}{n} H_\mu(\bigvee_{i=0}^{n-1} f^{-i}(\xi))
\]

and the metric entropy \( h_\mu(f) \) of \( f \) with respect to \( \mu \) is defined as the supremum of \eqref{eq1} over all \( \xi \) with finite partition entropy.

For \( f \) differentiable on a Riemannian manifold \( M \), relative to a direction \( v \in T_x M \) with \( x \in M \), Lyapunov exponent along \( v \) is given by the limit

\[
 \lambda_i(f, x) = \lim_{n \to +\infty} \frac{1}{n} \log \|D_x f^n v\|,
\]

which exists (possibly \(-\infty\)), for almost every point \( x \) of every \( f \)-invariant Borel probability measure \( \mu \) by Oseledets theorem \[3, 4\] if \( \int \log^+ \|D_x f\| d\mu < \infty \), where \( \log^+ t = \max\{\log t, 0\} \). Moreover, the finite dimension of \( M \) brings about exactly the same number of values of the limit \eqref{eq2}, which we denote as \( \lambda_i(f, x) \) (\( 1 \leq i \leq \dim M \)).

When the manifold \( M \) is compact and without boundary, the above two mechanisms could be linked together as the layout of Margulis-Ruelle inequality \[4\]: the metric entropy of \( \mu \) is not beyond the sum of positive Lyapunov exponents:

\[
 h_\mu(f) \leq \int \sum_{\lambda_i(f, x) > 0} \lambda_i(f, x) d\mu.
\]

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However, for general manifolds, the availability of (1.3) is unknown yet. There are many spaces which are expected to be taken into account, but not in the previous context, such as $\mathbb{R}^n$, Lie groups $\text{SL}(n, \mathbb{R})$, billiard tables, moduli spaces, etc, where both noncompactness and boundary lead to the actual obstructions. As the case exhibits, if $M$ is noncompact, the inequality (1.3) possibly doesn't hold [3]. With boundary permissible, (1.3) was dealt with in (2), involving systems on manifolds with finite capacity. Regarding the two aspects, in this paper, we investigate this problem in a general situation, i.e., for manifolds which are possibly both noncompact and with boundary.

Let $M$ be a Riemannian manifold, $f$ a $C^1$ map from an open subset $U \subset M$ to $M \setminus \partial M$, and Borel probability measure $\mu$ on $M \setminus \partial M$ is $f$-invariant: $\mu(f^{-1}(B)) = \mu(B)$ for any Borel set $B \subset M \setminus \partial M$. The general setting is as follows:

(A) **Distortion on $f$:** there exist $0 < \alpha < 1, C, a > 1$ such that for any $x \in U$, $y \in B(x, \min\{1, d^2(x, \partial M)\})$,

$$\frac{\|D_x f\| - \|D_y f\|}{d(x, y)^a} \leq C d_0(x, \partial M)^{-a},$$

where $d_0(x, \partial M) = \min\{d(x, \partial M), d^{-1}(x, x_0)\}$ for any preassigned $x_0 \in M$.

(B) **Integrability on $\mu$:**

$$\int_U \max\{\log^+ \|D_x f\|, |\log d_0(x, \partial M)|, |\log \rho_b(x)|, |\log N_b(x)|\} \, d\mu < \infty,$$

where $\rho_b(x)$ for some $b \geq 1$ is the regular radius given by the minimum of $\varrho_b(y)$ with $d_0(y, \partial M) \geq d_0(x, \partial M)$ and $\varrho_b(y) = \max\{0 < r \leq 1 : \|D_w \exp_{x_i}\| \leq b, \|D_{x^i} \exp_{x_i}^{-1}\| \leq b, \forall x_i \in B(y, r), i = 1, 2, w = \exp_{x_i}^{-1}(x_2)\}; N_b(x)$ is the regular tankage given by the minimal cardinality among covers of $\{y : d_0(y, \partial M) \geq d_0(x, \partial M)\}$ by balls with regular radius.

**Remark.** The properties on $d_0$ is independent of the choice of $x_0$. In fact, $d^{-1}(x, x_0)$ represents the distance of $x$ relative to the infinity: “away from $x_0$” is equivalent to “close to the infinity”.

**Remark.** (A)/(B) are naturally satisfied if $M$ is compact and without boundary. In (A), we adopt $\|D_x f\| - \|D_y f\|$ instead of $\|D_x f - D_y f\|$, which avoids the assumptions on the local trivialization of manifolds as in [2] since we just compare real numbers $\|D_x f\|$ and $\|D_y f\|$. In (B), the first one is a natural property guaranteeing the existence of Lyapunov exponents in Oseledets theorem, the next three could be verified in usual analytic settings with some nondegenerate polynomial estimates, for example, for any measure absolute to Lebesgue measure such that

- the density decreases polynomially of order bigger than $\dim M$ near infinity and increases polynomially of order smaller than $\dim M$ near boundary;

- the regular radius decreases polynomially and the regular tankage increases polynomially near infinity and boundary.

**Theorem 1.1.** Let $M$ be a Riemannian manifold and $f$ a $C^1$ map from an open subset $U \subset M$ to $M \setminus \partial M$ satisfying (A). Then for any $f$-invariant Borel probability measure satisfying (B), Margulis-Ruelle inequality holds true.

The difficulties to establish (1.3) mainly arise from four aspects: firstly, the norm of derivarives may grow to the infinity; secondly, the regular neighborhoods on
which $f$ behaves like diffeomorphisms may be destroyed by the boundary; thirdly, the distortion of manifold has no uniform bounds; and fourthly, the tankage may increase very largely when approaching the infinity and boundary. To overcome these difficulties, we take advantage of partitions which are adaptive to the above items and also we show the finiteness of such partitions’ entropy in Section 2 for the feasibility of later dynamical entropy computation. In Section 3, we carry on analysis of entropy for evolutions, with respect to the (singular) part near boundary and infinity, and the (regular) part away from boundary and infinity. In the present situation, we adopt infinite countable partition instead of finite partition as in [2], which will make the integrable condition of Theorem 1.1 more directly applicable to the proof of (1.3).

2. Construction of partitions and partition entropy

For convenience, we consider the minimum between $d_0$ and 1, which we denote by $d_*$. Notice that the properties in (A)(B) concerning $d_0$ also hold for $d_*$. Consider iterations $f^m$ with $m \in \mathbb{N}$. Associated to the influences of noncompactness and boundary, we define for any $n \in \mathbb{N}$,

$$A_n = \left\{ x \in U : \quad \Pi_{j=0}^{m-1} \|Df^j(x)f\|^* \leq 2^n, \quad \Pi_{j=0}^{m-1} d_)*(f^j(x), \partial M) \geq 2^{-n}, \quad \Pi_{j=0}^{m-1} \rho_b(f^j(x)) \geq 2^{-n}, \quad \Pi_{j=0}^{m-1} N_b(f^j(x)) \leq 2^n \right\},$$

where $t^* = \max\{t, 1\}$ for $t \in \mathbb{R}$.

It holds that $A_1 \subset A_2 \subset \cdots$ and $\lim_{n \to +\infty} \mu(A_n) = 1$ by (B). We are going to construct partitions with box-like elements whose sizes are given differently according to $A_n$.

Denote $d = \dim M$. Relative to any $x \in M \setminus \partial M$, under an orthonormal basis $\{e_1, \cdots, e_d\}$ of $T_xM$, a box centered at $y$ with sides $2a_1 > 0$ ($1 \leq i \leq d$) is understood as

$$\exp_x(\Gamma(\exp^{-1}_x(y); a_1, \cdots, a_d)) := \exp_x(\{\exp^{-1}_x(y) + \sum_{1 \leq i \leq d} t_i a_i e_i, \quad -1 \leq t_i \leq 1\}).$$

In particular, for any $a > 0$, we denote $\Gamma_x(a)$ as the cube $\exp_x(\Gamma(0; a, \cdots, a))$ and for any integer $l \geq 1$, $\mathcal{V}(x,a,l)$ is defined as a partition of $\Gamma_x(a)$ consisting of subcubes

$$\exp_x(\{\sum_{1 \leq i \leq d} \frac{k_i a}{2^l} e_i + \sum_{1 \leq i \leq d} t_i \frac{a}{2^l} e_i : \quad -1 \leq t_i \leq 1\}), \quad 1 - 2^l \leq k_i \leq 2^l - 1, \quad \frac{k_i + 1}{2} \in \mathbb{Z}.$$
and define
\[ \varepsilon_n = \frac{1}{\sqrt[d]{2}n!} , \quad \forall n \in \mathbb{N}. \]

Fix \( n \in \mathbb{N} \). In order to cover \( A_n \), we take \( F(A_n, \varepsilon_n) = \{ x_1^{(n)}, \ldots, x_k^{(n)} \} \) with \( k_n = \#F(A_n, \varepsilon_n) \). It holds that \( \bigcup_{1 \leq i \leq k_n} \Gamma_{x_i^{(n)}}(\varepsilon_n) \supset A_n \) since \( \Gamma_{x_i^{(n)}}(\varepsilon_n) \supset \bar{B}(x_i^{(n)}, \varepsilon_n) \). Moreover,
\[
\Gamma_{x_i^{(n)}}(\varepsilon_n) \subset \bar{B}(x_i^{(n)}, \sqrt[d]{n}\varepsilon_n) = \bar{B}(x_i^{(n)}, \frac{1}{2n!}), \quad \forall 1 \leq i \leq k_n.
\]

We point out that the closure of any ball \( B \) centered at \( x \in A_n \) with radius \( \leq \sqrt[d]{n}\varepsilon_n \) intersects at most \((4[b,d])^d\) such \( \Gamma_{x_i^{(n)}}(\varepsilon_n) \). Indeed, if not, then the number of \( x_i^{(n)} \) in \( \bar{B}(x, 2\sqrt[d]{n}\varepsilon_n) \subset \Gamma_x(2\sqrt[d]{n}\varepsilon_n) \subset \bar{B}(x, \frac{1}{2}\varepsilon_n) \subset B(x, \rho_b(x)) \) is more than \((4[b,d])^d\), which leads to at least two of them staying in the same element of the equi-partition of \( \Gamma_x(2\sqrt[d]{n}\varepsilon_n) \) by cubes with sides \( \frac{2\sqrt[d]{n}\varepsilon_n}{4[b,d]} \), thus their distance on \( M \) not bigger than \( \frac{2\sqrt[d]{n}\varepsilon_n}{4[b,d]^2} \sqrt[d]{4b} \leq \varepsilon_n \), that contradicts the choice of \( x_i^{(n)} \).

To obtain a partition of \( A_n \), we define
\[
B_{n;1} = A_n \cap \Gamma_{x_1^{(n)}}(\varepsilon_n),
B_{n;2} = A_n \cap \Gamma_{x_2^{(n)}}(\varepsilon_n) \setminus B_{n;1},
B_{n;3} = A_n \cap \Gamma_{x_3^{(n)}}(\varepsilon_n) \setminus \bigcup_{1 \leq i \leq 2} B_{n;i},
\]
and so on.

For \( s \geq n + 1 \), taking \( F(A_s \setminus A_{s-1}, \varepsilon_s) = \{ x_1^{(s)}, \ldots, x_k^{(s)} \} \) with \( k_s = \#F(A_s \setminus A_{s-1}, \varepsilon_s) \), define
\[
B_{s;1} = (A_s \setminus A_{s-1}) \cap \Gamma_{x_1^{(s)}}(\varepsilon_s),
B_{s;2} = (A_s \setminus A_{s-1}) \cap \Gamma_{x_2^{(s)}}(\varepsilon_s) \setminus B_{s;1},
\]
and so on.

Then we may get two measurable partitions
\[
\mathcal{P} = \left\{ B_{s;i} : 1 \leq i \leq k_s, \ s \geq n \right\},
\]
\[
\tilde{\mathcal{P}} = \left\{ E_s = \bigcup_{1 \leq i \leq k_s} B_{s;i} : s \geq n \right\}.
\]
In fact, one may see that \( E_n = A_n \) and \( E_s = A_s \setminus A_{s-1} \) for \( s \geq n + 1 \).
We continue by using \( l \) to partition each \( B_{s,i} \) into smaller sets. To be precise, denote \( \mathcal{Y}(x_i^{(s)}, \varepsilon_s, l) = \{ \Gamma_1^{(s,i,l)}, \Gamma_2^{(s,i,l)}, \ldots, \Gamma_{2^d}^{(s,i,l)} \} \) and let

\[
\begin{align*}
B_{s,i,1} &= B_{s,i} \cap \Gamma_1^{(s,i,l)}, \\
B_{s,i,2} &= B_{s,i} \cap \Gamma_2^{(s,i,l)} \setminus B_{s,i,1}, \\
& \quad \vdots \\
B_{s,i,2^d} &= B_{s,i} \cap \Gamma_{2^d}^{(s,i,l)} \setminus (\cup_{1 \leq j \leq 2^d-1} B_{s,i,j}).
\end{align*}
\]

Then, for any \( l \geq 1 \), one can obtain a partition as the following

\[
\mathcal{P}_n^{(l)} = \left\{ B_{s,i,j} : 1 \leq i \leq k_s, 1 \leq j \leq 2^l d, \ s \geq n \right\}.
\]

It is obvious that \( \bar{\mathcal{P}} \prec \mathcal{P} \prec \mathcal{P}_n^{(l)} \) mod 0 for any \( l \geq 1 \). For convenience of statement, we denote \( \mathcal{P}_n^{(0)} = \mathcal{P} \).

In the above construction, \( n \) stands for the regularity relative to the bounday and infinity; \( l \) is used to make the size of partitions arbitrarily small. The choice of \( l \) will be confirmed in the estimate of entropy with respect to evolutions.

Note that \( A_s \) is covered by at most \( 2^s \) balls with regular size (which is \( \leq 1 \)) and each of them can be further covered by at most \( C_{0,1} \left( \frac{4}{\varepsilon_s} \right)^d \) balls with radius \( \frac{\varepsilon_s}{4} \) (each such ball is contained in a ball centered at \( A_s \) with radius \( \frac{\varepsilon_s}{2} \)), thus \( k_s \leq 2^s \cdot C_{0,1} \left( \frac{4}{\varepsilon_s} \right)^d \). Moreover, each element of \( \mathcal{P}_n^{(0)} \) is partitioned by \( 2^l d \) cubes in the constructoin of \( \mathcal{P}_n^{(l)} \). Thus, for any \( s \geq n \),

\[
\# \left\{ B_{s,i,j} : 1 \leq i \leq k_s, 1 \leq j \leq 2^l d \right\} \leq 2^s \cdot C_{0,1} \left( \frac{4}{\varepsilon_s} \right)^d \cdot 2^l d
\]

\[
= C_{0,1} \left( 4 \sqrt{d} \right)^d 2^{s(1+l_1)l_1 + ld} = C_{0,2} 2^{s(1+l_1)l_1 + ld}.
\]

Before going any further, we first show the finiteness of partition entropy \( H_\mu(\mathcal{P}_n^{(l)}) \) so that the constructed partitions are feasible in the computation of dynamical entropy.

**Lemma 2.1.** \( H_\mu(\mathcal{P}_n^{(l)}) < \infty \) for any integer \( l \geq 0 \).

**Proof.** Denote

\[
p(x) = \max \left\{ \Pi_{j=0}^{m-1} \| D f^j(x) f \|_s^s, \Pi_{j=0}^{m-1} d^\ast_\varepsilon^{-1} (f^j(x), \partial M), \right. \\
\left. \Pi_{j=0}^{m-1} \rho_b^{-1} (f^j(x)), \Pi_{j=0}^{m-1} N_b (f^j(x)) \right\}.
\]

Note that if \( x \in B_{s,i,j} \) with \( s \geq n + 1 \), then \( x \notin A_{s-1} \), so \( p(x) > 2^{s-1} \). It follows that

\[
\int_{E_n} \log p(x) d\mu(x) = \sum_{s=n}^{+\infty} \int_{E_s} \log p(x) d\mu(x)
\]

\[
\geq \int_{E_n} \log p(x) d\mu(x) + \sum_{s=n+1}^{+\infty} \int_{E_s} \log 2^{s-1} d\mu(x).
\]
Together with (B), we deduce
\[ \sum_{s=n}^{+\infty} \mu(E_s) < +\infty. \]

Therefore,
\[ H_\mu(P^{(l)}_n) \leq \sum_{s=n}^{+\infty} \mu(E_s) \log(C_0 2^{s(1+l_1 d) + l_1 d}) < +\infty. \]

\[ \square \]

### 3. Estimates on Entropy for Dynamical Evolutions

This section is devoted to the estimate of metric entropy of \( g \) by using the constructed partitions \( P^{(l)}_n \). By Lemma 2.1, the entropy of \( P^{(l)}_n \) is finite, thus
\[ h_\mu(g, P^{(l)}_n) = \lim_{t \to +\infty} \frac{1}{t} \mu\left( \int_{g^{-i}(P^{(l)}_n)} \right) \]
\[ = \lim_{t \to +\infty} \frac{1}{t} \left( H_\mu(P^{(l)}_n) + \sum_{i=1}^{t-1} H_\mu\left( \bigcup_{j=0}^{i-1} g^{-j}(P^{(l)}_n) \right) \right) \]
\[ \leq H_\mu(g^{-1}(P^{(l)}_n) | P^{(l)}_n). \]

Furthermore, it holds that
\[ H_\mu(g^{-1}(P^{(l)}_n) | P^{(l)}_n) = H_\mu(g^{-1}(\tilde{P}^{(l)}_n) \cup g^{-1}(\tilde{P})) | P^{(l)}_n) \]
\[ = H_\mu(g^{-1}(\tilde{P}) \cup P^{(l)}_n) + H_\mu(g^{-1}(\tilde{P}^{(l)}_n) | P^{(l)}_n \cup g^{-1}(\tilde{P})) \]
\[ := I + II. \]

(1) **Estimate on I.** Observe that by Lemma 2.1
\[ I \leq H_\mu(g^{-1}(\tilde{P})) = H_\mu(\tilde{P}) \leq H_\mu(P^{(0)}_n) < +\infty. \]

Moreover, \( \mu(E_n) \to 1 \) as \( n \to \infty \), thus, by (2.1) and the integrability of \( \log p(x) \),
\[ \lim_{n \to \infty} \sum_{s=n+1}^{+\infty} \int_{E_s} \log 2^{s-1} d\mu(x) = 0. \]

So, taking \( n \) large enough, one can make
\[ \sum_{s=n+1}^{+\infty} \mu(E_s) \log 2^{sl_1 d} < 1. \]

Although we can obtain I is finite by (3.1), the partition entropy \( H_\mu(P^{(0)}_n) \) depending on \( g = f^m \) may be very large as \( m \) increases. For the estimate of dynamical entropy, by taking \( l \) large, we are going to deduce a controllable linear bound for I relative to the increase of \( m \).
Notice that
\[ I = \sum_{s=\infty}^{+\infty} \sum_{B \in \mathcal{P}^{(s)}_n} -\mu(g^{-1}(E_s) \cap B) \log \frac{\mu(g^{-1}(E_s) \cap B)}{\mu(B)} \]
\[ = \int \left[ \sum_{s=n}^{+\infty} -\mu(g^{-1}(E_s) \cap \mathcal{P}^{(s)}_n(x)) \log \frac{\mu(g^{-1}(E_s) \cap \mathcal{P}^{(s)}_n(x))}{\mu(\mathcal{P}^{(s)}_n(x))} \right] d\mu(x) \]
\[ = \sum_{s=n}^{+\infty} \int [-\mu(g^{-1}(E_s) \cap \mathcal{P}^{(s)}_n(x)) \log \frac{\mu(g^{-1}(E_s) \cap \mathcal{P}^{(s)}_n(x))}{\mu(\mathcal{P}^{(s)}_n(x))}] d\mu(x). \]

Since \( I \big|_{s=0} < +\infty \), there exists \( L \geq n \) such that
\[ \sum_{s=L+1}^{+\infty} \int [-\mu(g^{-1}(E_s) \cap \mathcal{P}^{(s)}_n(x)) \log \frac{\mu(g^{-1}(E_s) \cap \mathcal{P}^{(s)}_n(x))}{\mu(\mathcal{P}^{(s)}_n(x))}] d\mu(x) < 1. \]

Recalling that \( \mathcal{P}^{(s)}_n \prec \mathcal{P}^{(s)}_n \) for every \( l \geq 1 \), by the convexity of \(-x \log x\), we have
\[ \int \left[ \sum_{s=L+1}^{+\infty} -\mu(g^{-1}(E_s) \cap \mathcal{P}^{(s)}_n(x)) \log \frac{\mu(g^{-1}(E_s) \cap \mathcal{P}^{(s)}_n(x))}{\mu(\mathcal{P}^{(s)}_n(x))} \right] d\mu(x) \]
\[ = \sum_{s=L+1}^{+\infty} \int [-\mu(g^{-1}(E_s) \cap \mathcal{P}^{(s)}_n(x)) \log \frac{\mu(g^{-1}(E_s) \cap \mathcal{P}^{(s)}_n(x))}{\mu(\mathcal{P}^{(s)}_n(x))}] d\mu(x) \]
\[ \leq \sum_{s=L+1}^{+\infty} \int [-\mu(g^{-1}(E_s) \cap \mathcal{P}^{(s)}_n(x)) \log \frac{\mu(g^{-1}(E_s) \cap \mathcal{P}^{(s)}_n(x))}{\mu(\mathcal{P}^{(s)}_n(x))}] d\mu(x) \]
\[ < 1. \]

For the expression of \( I \), the term \( \mathcal{P}^{(s)}_n \) stays below in the form of conditional entropy. In order to make \( I \) small, we can let \( \mathcal{P}^{(s)}_n \) much finer, i.e., let \( l \) large. To be precise, we let \( l \) satisfy
\[ \frac{1}{2^{l-2-m(L+1)}} < \min\{1 - 2^{-s}, 2^{-1-s}\} \left( \frac{1}{2^{l-2-m(L+1)}} \right)^o C2^{aL} < 2^{\frac{1}{2}} - 1. \]

For \( B = B_{k,i,j} \) contained in some \( E_k \), denote \( s_B = \min\{s : g(B) \cap E_s \neq \emptyset\} \). We split the discussion into two cases:

(i) \( s_B > L \). Then
\[ \sum_{s=n}^{+\infty} -\mu(g^{-1}(E_s) \cap B) \log \frac{\mu(g^{-1}(E_s) \cap B)}{\mu(B)} = \sum_{s=L+1}^{+\infty} -\mu(g^{-1}(E_s) \cap B) \log \frac{\mu(g^{-1}(E_s) \cap B)}{\mu(B)}. \]
So,
\[ \sum_{s_B > L} \sum_{s=n}^{+\infty} -\mu(g^{-1}(E_s) \cap B) \log \frac{\mu(g^{-1}(E_s) \cap B)}{\mu(B)} \]
\[ \leq \int \left[ \sum_{s=L+1}^{+\infty} -\mu(g^{-1}(E_s) \cap \mathcal{P}^{(s)}_n(x)) \log \frac{\mu(g^{-1}(E_s) \cap \mathcal{P}^{(s)}_n(x))}{\mu(\mathcal{P}^{(s)}_n(x))} \right] d\mu(x) < 1. \]
(ii) \( s_B \leq L \).

**Lemma 3.1.** There exists a constant integer \( C_1 > 0 \) such that \( g(B) \subset A_{s_B + C_1 \cdot m} \).

**Proof.**

Step 1: \( \text{diam}(g(B)) \leq \frac{1}{2^{k_1}} \).

By definition, \( B \subset \Gamma_j^{(k,i,\ell)} \subset \Gamma_j^{(k,i)}(\varepsilon_k) \subset \bar{B}(x_i^{(k)}, \frac{1}{2^{k_1}}) \). For any \( y \in \bar{B}(x_i^{(k)}, \frac{1}{2^{k_1}}) \) with \( x \in A_k \),

\[
\|D_y f - D_x f\| \leq Cd(x, y)^a d_x^{-a}(x, \partial M) \leq C \left( \frac{1}{2^{k_1}} \right)^a 2^a k < 2^{\frac{1}{2}} - 1,
\]

\[
d(f(x), f(y)) \leq d(x, y) \max_{u \in B(x, \frac{1}{2^{k_1}})} \|D_u f\| \leq \frac{1}{2^{k_1}} (2^k + 1)
\]

By induction, suppose for \( 1 \leq j \leq m - 1 \),

\[
\|D_{f^{j-1}(y)} f_1 - D_{f^{j-1}(x)} f_1\| \leq C \left( \frac{1}{2^{k_1}} \right)^a 2^a k < 2^{\frac{1}{2}} - 1,
\]

\[
d(f^j(x), f^j(y)) \leq \frac{1}{2^{k_1}} (2^k + 1).
\]

Then one has

\[
\|D_{f^j(y)} f - D_{f^j(x)} f\|
\]

\[
\leq Cd(f^j(x), f^j(y))^a d_x^{-a}(f^j(x), \partial M) \leq C \left( \frac{1}{2^{k_1}} \right)^a 2^a k < 2^{\frac{1}{2}} - 1,
\]

\[
d(f^{j+1}(x), f^{j+1}(y)) \leq \frac{1}{2^{k_1}} (2^k + 1) \leq \frac{1}{2^{k_1}} (2^k + 1).
\]

Hence,

\[
\|D_y g\| \leq \Pi_{j=0}^{m-1} \left( \|D_{f^j(x)} f_1\| + 2^{\frac{1}{2}} - 1 \right) \leq \Pi_{j=0}^{m-1} \left( 2^{\frac{1}{2}} \|D_{f^j(x)} f_1\| \right) \leq 2^{k_1},
\]

which gives rise to

\[
\text{diam}(g(B)) \leq \text{diam}(B) \max_{y \in \bar{B}(x_i^{(k)}, \frac{1}{2^{k_1}})} \|D_y g\| \leq \frac{2}{2^{k_1} + 1} 2^{k_1} \leq \frac{1}{2^2}.
\]

Step 2: Choose \( y \in g(B) \cap E_{s_B} \), then \( d(f^j(u), f^j(y)) \leq \frac{1}{2^{s_B - s_B + 1}} \) for any \( u \in \bar{B}(y, \frac{1}{2^{s_B}}) \) and \( 0 \leq j \leq m - 1 \).

Note that when \( d(u, y) \leq \frac{1}{2^{s_B}} \),

\[
\|D_u f\| \leq \|D_y f\| + d(u, y)^a Cd(y, \partial M)^{-a}
\]

\[
\leq 2^{s_B} + \left( \frac{1}{2^{s_B}} \right)^a C 2^{a s_B}
\]

\[
\leq 2^{s_B} + \left( \frac{1}{2^{s_B}} \right)^a C 2^{a L}
\]

\[
\leq 2^{s_B} + 1 \leq 2^{s_B + 1} \leq 2^{L + 1}.
\]

By induction, suppose for \( 1 \leq j \leq m - 1 \), \( d(f^{j-1}(u), f^{j-1}(y)) \leq \frac{1}{2^{s_B - s_B + 1}} \) and \( \|D_{u_1} f\| \leq 2^{L + 1} \) for any \( u_1 \) with \( d(u_1, f^{j-1}(y)) \leq \frac{1}{2^{s_B - s_B + 1}} \), then
d(f^j(u), f^j(y)) \leq d(f^{j-1}(u), f^{j-1}(y))2^{L+1} \leq \frac{1}{2^{2-j(L+1)}}.

Moreover, for any \( u_1 \) with \( d(u_1, f^j(y)) \leq \frac{1}{2^{2-j(L+1)}} \),
\[
\|D_{u_1}f\| \leq \|D_{f^{j}(y)}f\| + d(u_1, f^j(y))\|C d(f^{j}(y), \partial M)^{-a} \[
\leq 2^sB + \left(\frac{1}{2^{2-j(L+1)}}\right)^a C 2^{sB} \[
\leq 2^sB + \left(\frac{1}{2^{2-j(L+1)}}\right)^a C 2^{sL} \[
\leq 2^sB + 1 \leq 2^{L+1}.
\]

Step 3: \( u \in A_{sB+C_1,m} \) for some constant integer \( C_1 > 0 \).

We need check the properties of \( d_* \), \( \|\cdot\|^* \), \( \rho_b \) and \( N_b \) for \( u \).

1. \( \Pi_{j=0}^{m-1} d_*(f^j(u), \partial M) \geq 2^{-sB - 1} \).

By the choice of \( l \), for \( 0 \leq j \leq m - 1 \),
\[
d(f^j(u), \partial M) \geq d(f^j(y), \partial M) - d(f^j(u), f^j(y)) \geq 2^{-\frac{1}{\alpha}} d(f^j(y), \partial M),
\[
d(f^j(u), x_0) \leq d(f^j(y), x_0) + d(f^j(u), f^j(y)) \leq 2^s \max\{d(f^j(y), x_0), 1\},
\]
which implies that
\[
d_*(f^j(u), \partial M) \geq 2^{-\frac{1}{\alpha}} d_*(f^j(y), \partial M).
\]

So,
\[
\Pi_{j=0}^{m-1} d_*(f^j(u), \partial M) \geq (2^{-\frac{1}{\alpha}})^m \Pi_{j=0}^{m-1} d_*(f^j(z), \partial M) \geq 2^{-sB - 1}.
\]

2. \( \Pi_{j=0}^{m-1} \|D_{f^j(u)}f\|^* \leq 2^{sB + 1} \).

\[
\left\|D_{f^j(u)}f - \|D_{f^j(y)}f\|\right\| \leq d(f^j(u), f^j(y))\|C d(f^j(y), \partial M)^{-a} \leq 2^{\frac{1}{\alpha}} - 1,
\]
which gives
\[
\|D_{f^j(u)}f\|^* \leq 2^{\frac{1}{\alpha}} \|D_{f^j(y)}f\|^*.
\]

So,
\[
\Pi_{j=0}^{m-1} \|D_{f^j(u)}f\|^* \leq (2^{\frac{1}{\alpha}})^m \Pi_{j=0}^{m-1} \|D_{f^j(z)}f\|^* \leq 2^{sB + 1}.
\]

3. \( \Pi_{j=0}^{m-1} \rho_b(f^j(u)) \geq 2^{-sB - 1} \).

For any \( z \) satisfying \( d_0(z, \partial M) \geq d_0(f^j(u), \partial M) \), there exists \( z' \in B(z, \frac{1}{2^{2-j(L+1)}}) \) satisfying \( d(z', \partial M) \geq d_0(f^j(y), \partial M) \). Since
\[
d(z, z') \leq \frac{1}{2^{2-j(L+1)}} \leq (1 - 2^{-\frac{1}{\alpha}}) \rho_b(f^j(y)),
\]
so,
\[
\rho_b(z) \geq 2^{-\frac{1}{\alpha}} \rho_b(f^j(y)).
\]

By the arbitrariness of \( z \), we have
\[
\rho_b(f^j(u)) \geq 2^{-\frac{1}{\alpha}} \rho_b(f^j(y)).
\]

So,
\[
\Pi_{j=0}^{m-1} \rho_b(f^j(u)) \geq (2^{-\frac{1}{\alpha}})^m \Pi_{j=0}^{m-1} \rho_b(f^j(z)) \geq 2^{-sB - 1}.
\]

4. \( \Pi_{j=0}^{m-1} N_b(f^j(u)) \leq C_1 2^{sB} \) for some constant integer \( C_{1,0} > 0 \).
If \( d_0(f^j(y), \partial M) \leq d_0(f^j(u), \partial M) \), it is obvious that \( N_b(f^j(u)) \leq N_b(f^j(y)) \). Assume \( d_0(f^j(y), \partial M) > d_0(f^j(u), \partial M) \). Then

\[
\rho_b(f^j(u)) \leq \rho_b(f^j(y)) \leq 2^\frac{b}{\pi} \rho_b(f^j(u)).
\]

So, there exists a constant integer \( C_{1,0} > 1 \), such that any ball \( B(v, \rho_b(f^j(y))) \) with \( d_0(v, \partial M) \geq d_0(f^j(y), \partial M) \) can be covered by \( C_{1,0} \) balls centered at \( B(v, \rho_b(f^j(y))) \) with radius \( \frac{1}{2} \rho_b(f^j(u)) \geq 2^{-\frac{b}{\pi}} \rho_b(f^j(y)) \), which we denote as

\[
B(y_{v,1}, \frac{1}{2} \rho_b(f^j(u))), \ldots, B(y_{v,c_1}, \frac{1}{2} \rho_b(f^j(u))).
\]

Moreover, for any \( z \) satisfying \( d_0(z, \partial M) \geq d_0(f^j(u), \partial M) \), there exists \( z' \in \bar{B}(z, \frac{1}{2} \rho_b(f^j(u))) \subset B(z, \frac{1}{2} \rho_b(f^j(u))) \) satisfying \( d_0(z', \partial M) \geq d_0(f^j(y), \partial M) \). Thus, if \( \{ v : d_0(v, \partial M) \geq d_0(f^j(y), \partial M) \} \) is covered by \( B(v_1, \rho_b(f^j(y))), \ldots, B(v_{N_b(f^j(y))}, \rho_b(f^j(y))) \), then \( B(y_{v_1,1}, \rho_b(f^j(u))), 1 \leq i \leq N_b(f^j(y)), 1 \leq j \leq C_{1,0} \), constitute a cover of \( \{ z : d_0(z, \partial M) \geq d_0(f^j(u), \partial M) \} \), which gives

\[
N_b(f^j(u)) \leq C_{1,0} N_b(f^j(y)).
\]

So,

\[
\Pi_{j=0}^{m-1} N_b(f^j(u)) \leq C_{1,0}^m \Pi_{j=0}^{m-1} N_b(f^j(y)) \leq C_{1,0}^m 2^{s_B}.
\]

By 1°, 2°, 3°, 4°, denoting \( C_1 = \left\lfloor \frac{\log C_{1,0}}{\log 2} \right\rfloor \), we have \( u \in A_{s_B+C_1m} \). Together with the arbitrariness of \( u \), we deduce that \( g(B) \subset A_{s_B+C_1m} \).

\( \square \)

Notice that \( g(B) \cap E_j = \emptyset \) for any \( j < s_B \), so

\[
\sum_{s_B \leq L} \sum_{s=n}^{+\infty} -\mu(g^{-1}(E_s) \cap B) \log \frac{\mu(g^{-1}(E_s) \cap B)}{\mu(B)}
= \sum_{s_B \leq L} \sum_{s=s_B}^{s_B+C_1m} -\mu(g^{-1}(E_s) \cap B) \log \frac{\mu(g^{-1}(E_s) \cap B)}{\mu(B)}
\leq \sum_{s_B \leq L} (C_1m + 1)e^{-1} \mu(B) \leq (C_1m + 1)e^{-1}.
\]

Therefore,

\[
I = \left( \sum_{s_B > L} + \sum_{s_B \leq L} \right) \sum_{s=n}^{+\infty} -\mu(g^{-1}(E_s) \cap B) \log \frac{\mu(g^{-1}(E_s) \cap B)}{\mu(B)} \leq 1 + (C_1m + 1)e^{-1}.
\]
(2) \textbf{Estimate on II.} We have

$$
\Pi = \sum_{s=n}^{+\infty} \sum_{F \in \mathcal{P}_n^{(i)}} \sum_{B \in \mathcal{P}_n^{(i)}, B \subseteq E_s} -\mu(F \cap g^{-1}(E_s) \cap g^{-1}(B)) \log \frac{\mu(F \cap g^{-1}(E_s) \cap g^{-1}(B))}{\mu(F \cap g^{-1}(E_s))}
$$

$$
= \sum_{F \in \mathcal{P}_n^{(i)}} \sum_{B \in \mathcal{P}_n^{(i)}, B \subseteq E_s} -\mu(F \cap g^{-1}(B)) \log \frac{\mu(F \cap g^{-1}(B))}{\mu(F \cap g^{-1}(E_s))}
$$

$$
+ \sum_{s=n+1}^{+\infty} \sum_{F \in \mathcal{P}_n^{(i)}} \sum_{B \in \mathcal{P}_n^{(i)}, B \subseteq E_s} -\mu(F \cap g^{-1}(B)) \log \frac{\mu(F \cap g^{-1}(B))}{\mu(F \cap g^{-1}(E_s))}
$$

$$
:= \Pi_1 + \Pi_2 .
$$

On the one hand, for any \( F \in \mathcal{P}_n^{(i)} \), if \( F \) is in some \( E_k \), then \( \text{diam}(g(F)) \leq \frac{2^{k+1}}{2^{k+1}} \leq \frac{1}{2^{k+1}} \), which implies \( \exp_{\Gamma_{x_i}(\varepsilon)}(F) \cap \Gamma_{x_i}(\varepsilon) \) is contained in a cube with sides \( \frac{2\varepsilon}{2^{k+1}} \) whenever \( g(F) \cap \Gamma_{x_i}(\varepsilon) \neq \emptyset \). Note that there exist at most \( k_s \) such \( \Gamma_{x_i}(\varepsilon_s) \), and each of them is partitioned by cubes with sides \( \frac{2\varepsilon_s}{2^{k+1}} = \frac{2}{2^{i+1} \sqrt{d}} \) in the construction of \( \mathcal{P}_n^{(i)} \). So, for any \( s \geq n + 1 \),

$$
\#\{ B \in \mathcal{P}_n^{(i)} : B \cap g(F) \neq \emptyset, B \subseteq E_s \} \leq k_s \left( \left[ \frac{2\varepsilon_s}{2^{i+1} \sqrt{d}} \right] + 2 \right)^d \leq C_2 2^{4s l_1 d},
$$

where \( C_2 = C_0([4b \sqrt{d}] + 2)^d \). Hence,

$$
\Pi_2 \leq \sum_{s=n+1}^{+\infty} \sum_{F \in \mathcal{P}_n^{(i)}} \mu(F \cap g^{-1}(E_s)) \log(C_2 2^{4s l_1 d})
$$

$$
= \sum_{s=n+1}^{+\infty} \mu(g^{-1}(E_s)) \log(C_2 2^{4s l_1 d}) = \sum_{s=n+1}^{+\infty} \mu(E_s) \log(C_2 2^{4s l_1 d}) < \log C_2 + 4 .
$$

On the other hand,

$$
\Pi_1 = \sum_{F \in \mathcal{P}_n^{(i)}, F \subseteq E_n} \sum_{B \in \mathcal{P}_n^{(i)}, B \subseteq E_n} -\mu(F \cap g^{-1}(B)) \log \frac{\mu(F \cap g^{-1}(B))}{\mu(F \cap g^{-1}(E_n))}
$$

$$
+ \sum_{F \in \mathcal{P}_n^{(i)}, F \subseteq \bigcup_{s \geq n+1} E_s} \sum_{B \in \mathcal{P}_n^{(i)}, B \subseteq E_n} -\mu(F \cap g^{-1}(B)) \log \frac{\mu(F \cap g^{-1}(B))}{\mu(F \cap g^{-1}(E_n))}
$$

$$
:= \Pi_{1,1} + \Pi_{1,2} .
$$

For any \( F \subseteq E_n, s \geq n + 1 \), it holds that

$$
\#\{ B \in \mathcal{P}_n^{(i)} : B \cap g(F) \neq \emptyset, B \subseteq E_n \} \leq k_n \left( \left[ \frac{2b}{2^{i-1} \sqrt{d}} \right] + 2 \right)^d \leq C_2 2^{4n l_1 d},
$$
which implies
\[ \Pi_{1,2} \leq \sum_{F \in \mathcal{P}_n^{(l)}, F \subseteq \cup_{i \geq n+1} E_i} \mu(F \cap g^{-1}(E_n)) \log(C_2 2^{4nl_i d}) \]
\[ \leq \mu(\cup_{i \geq n+1} E_n) \log(C_2 2^{4nl_i d}) \leq \log C_2 + 4. \]

Now, it is left to estimate \( \Pi_{1,1} \), which is exactly the term contributing Lyapunov exponent. Observe that
\[ \Pi_{1,1} \leq \sum_{F \in \mathcal{P}_n^{(l)}, F \subseteq E_n} \log \# \{ B \in \mathcal{P}_n^{(l)}, B \subseteq E_n : g(F) \cap B \neq \emptyset \} \cdot \mu(F \cap g^{-1}(E_n)) \]
\[ \leq \sum_{F \in \mathcal{P}_n^{(l)}, F \subseteq E_n} \log \# \{ B \in \mathcal{P}_n^{(l)}, B \subseteq E_n : g(F) \cap B \neq \emptyset \} \cdot \mu(F). \]

To estimate the last term in the above inequality, we consider a standard argument in advance. For any \( \beta > 0 \), denote by \( \xi \) a partition of \( \mathbb{R}^d \) into boxes as follows:
\[ \left\{ [q_1 \beta, (q_1 + 1) \beta] \times \cdots \times [q_d \beta, (q_d + 1) \beta] : q_i \in \mathbb{Z}, 1 \leq i \leq d \right\}. \]

For any box \( \Gamma(x; v_1, \cdots, v_d; a_1, \cdots, a_d) = \{ x + \sum_{1 \leq i \leq d} t_i a_i v_i, \ 0 \leq t_i \leq 1 \} \) with unit vectors \( v_i \) and sides \( a_i > 0 \) (\( 1 \leq i \leq d \)), denote by \( \phi(\Gamma(x; v_1, \cdots, v_d; a_1, \cdots, a_d)) \) the minimum number of elements in \( \xi \) whose union covers \( \Gamma(x; v_1, \cdots, v_d; a_1, \cdots, a_d) \).

**Lemma 3.2** (Lemma 12.5 of [3]). There exists a constant \( c > 0 \) such that for any box \( \Gamma(x; v_1, \cdots, v_d; a_1, \cdots, a_d) \),
\[ \phi(\Gamma(x; v_1, \cdots, v_d; a_1, \cdots, a_d)) \leq c \Pi_{i=1}^d \max\{a_i, 1\}. \]

Denote by \( (D_x g)^{\wedge \kappa} \) the linear map on the \( \kappa \)-th exterior algebra of the tangent space \( T_x M \) induced by \( D_x g \) and let \( \| (D_x g)^{\wedge \kappa} \| = \max_{1 \leq \kappa \leq d} \| (D_x g)^{\wedge \kappa} \| \).

**Lemma 3.3.** There exists \( C_3 > 0 \) such that for any \( F \in \mathcal{P}_n^{(l)} \) and \( F \subseteq E_n \), we have for any \( x \in F \),
\[ \# \{ B \in \mathcal{P}_n^{(l)}, B \subseteq E_n : g(F) \cap B \neq \emptyset \} \leq C_3 \| (D_x g)^{\wedge \kappa} \|. \]

**Proof.** Note that \( F \) is a subset of some \( \Gamma_{n,l}^{n,i,l} \). The image of \( \Gamma_{n,l}^{n,i,l} \) by \( g \) intersects at most \( (4[b d])^d \) elements \( \Gamma_{n,l}(\varepsilon_n) \). By a scaling of the size, applying Lemma 3.2, it holds that for \( \gamma_{n,l,i}^{n,j} := \exp_{x_n^{(n)}}(\Gamma(\sum_{1 \leq i \leq d} k_i \varepsilon_n 2^{-l} e_i ; e_1, \cdots, e_d ; 2 \varepsilon_n 2^{-l}, \cdots, 2 \varepsilon_n 2^{-l}) := \exp_{x_n^{(n)}}(\Gamma), \)
\[ \# \{ G \in \ell_{2} 2^{-1} : (D_{\exp_{x_n^{(n)}}^{-1}(x)}(\exp_{x_n^{(n)}}^{-1}(x) g \exp_{x_n^{(n)}}(x))) \cap G \neq \emptyset \} \]
\[ \leq c \| D_{\exp_{x_n^{(n)}}^{-1}(x)}(\exp_{x_n^{(n)}}^{-1}(x) g \exp_{x_n^{(n)}}(x)) \| \]
\[ \leq c b^2 \| (D_x g)^{\wedge \kappa} \|. \]
Denote \( \tilde{x} = \exp^{-1}_{x_u(n)}(x) \), \( \tilde{g} = \exp^{-1}_{x_u(n)} g \exp_{x_u(n)} \). Take arbitrary \( G \in \xi_{2\varepsilon_n}^{2-1} \) such that \((D_2\tilde{g})^{-1}(G) \cap \Gamma \neq \emptyset\). Then for any \( y, z \in (D_2\tilde{g})^{-1}(G) \cap \Gamma \), we denote \( \tau = \frac{y - z}{\|y - z\|} \) and \( \sigma \) to be the line segment linking \( y \) and \( z \). Then, for the chosen \( n \), taking \( l \) sufficiently large, we have

\[
d(\tilde{g}(y), \tilde{g}(z)) = \int_\sigma \| (D_2\tilde{g}) \tau \| dz \\
\leq \int_\sigma (\| (D_2\tilde{g}) \tau \| + 1) dz \\
\leq \text{diam}(G) + \text{length}(\sigma) \\
\leq \text{diam}(G) + \text{diam}(\Gamma) \\
\leq \frac{2}{2^{n_1 + l}} + \frac{2}{2^{n_1 + l}} = \frac{4}{2^{n_1 + l}}.
\]

Hence,

\[
\# \{ G \in \xi_{2\varepsilon_n}^{2-1} : G \cap \tilde{g}(\Gamma) \neq \emptyset \} \\
\leq cb^2 \|(D_2g)^\wedge\| \cdot (\left\llbracket \frac{8}{2\varepsilon_n 2^{-l}} \right\rrbracket + 2)^d \\
= cb^2 \|(D_2g)^\wedge\| \cdot (\left\llbracket 4\sqrt{d} \right\rrbracket + 2)^d.
\]

Considering \( B \) as the part of the image of \( G \) by \( \exp_{x_u(n)}(G) \) in the construction of \( P_n^{(l)} \) and letting \( C_3 = cb^2(4\sqrt{d} + 2)^d(4[b d])^d \), one gets the estimate of the lemma. \( \square \)

By Lemma 3.3 we have
\[ \Pi_{1,1} \leq \sum_{F \in \mathcal{P}^{(l)}, F \subset E} \int_{F} \log(C_3 \|(D_x g)^\hat{x}\|)d\mu \]
\[ \leq \log C_3 + \int_{M} \log \|(D_x g)^\hat{x}\|d\mu \]
\[ = \log C_3 + \int_{M} \log \|(D_x f^m)^\hat{x}\|d\mu. \]

**Proof of Theorem 1.1**  For any \( m \in \mathbb{N}, \) sufficiently large \( n \) and \( l, \) we have obtained
\[ h_\mu(f^m, \mathcal{P}^{(l)}) \leq \Pi_1 + \Pi_1 + \Pi_{1,1} \]
\[ \leq 1 + (C_1 m + 1)e^{-1} + (\log C_2 + 4) + (\log C_2 + 4) + \log C_3 + \int_{M} \log \|(D_x f^m)^\hat{x}\|d\mu. \]

Letting \( l \to +\infty, \) we get
\[ h_\mu(f^m) \]
\[ \leq 1 + (C_1 m + 1)e^{-1} + 2(\log C_2 + 4) + \log C_3 + \int_{M} \log \|(D_x f^m)^\hat{x}\|d\mu. \]

Therefore,
\[ h_\mu(f) = \lim_{m \to +\infty} \frac{1}{m} h_\mu(f^m) \leq C_1 e^{-1} + \int_{M} \sum_{\lambda_i(f, x) > 0} \lambda_i(f, x) d\mu. \]

Furthermore,
\[ h_\mu(f) = \lim_{m \to +\infty} \frac{1}{m} h_\mu(f^m) \leq \lim_{m \to +\infty} \frac{1}{m} \int_{M} \sum_{\lambda_i(f^m, x) > 0} \lambda_i(f^m, x) d\mu \]
\[ = \int_{M} \sum_{\lambda_i(f, x) > 0} \lambda_i(f, x) d\mu. \]

This completes the proof of Theorem 1.1. \( \square \)

**References**

[1] G. Froyland, S. Lloyd and A. Quas, Coherent structures and isolated spectrum for Perron-Frobenius cocycles, *Ergodic Theory Dynam. Systems.*, 30, 729-756, 2010.
[2] A. Katok, J. Strelcyn, F. Ledrappier and F. Przytycki, Invariant manifolds, entropy and billiards; smooth maps with singularities, *Lecture Notes in Math.*, 1222, Springer-Verlag, Berlin, 1986.
[3] R. Mañé, Ergodic theory and differentiable dynamics, Springer-Verlag, Berlin, 1987.
[4] V. I. Oseledets, A multiplicative ergodic theorem, *Trans. Moscow Math. Soc.*, 19, 197-231, 1968.
[5] F. Riquelme, Counterexamples to Ruelle’s inequality in the noncompact case, *Ann. Inst. Fourier (Grenoble)*, 67, 23-41, 2017.
[6] D. Ruelle, An inequality for the entropy of differentiable maps, *Bol. Soc. Bras. Mat.*, 9, 83-88, 1978.

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