Bias of reconstructing the dark energy equation of state from the Padé cosmography

Yang Liu · Zhengxiang Li · Hongwei Yu · Puxun Wu

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Abstract
Padé cosmography has been widely used to probe the cosmic evolution and to investigate the nature of dark energy. In this paper, we find that the Padé approximant can describe the cosmic evolution better than the standard cosmography, and if the luminosity distance \(d_L(z)\) described by the Padé approximant is used to reconstruct the dark energy equation of state \(w(z)\), then the reconstructed \(w(z)\) will approach a constant, i.e. \(1/3\) or 0, when the redshift is very high. This result is general since it is independent of the coefficients in the Padé approximant and the value of the present dimensionless matter density parameter. This intrinsic character will bias the \(w(z)\) reconstruction and lead to misconception of the property of dark energy. Therefore, one must exercise caution in investigating the property of dark energy from Padé cosmography when the high redshift data, i.e. \(z > 2\), are included.

Keywords Accelerated cosmic expansion · Cosmography · Padé approximant

1 Introduction

Many observations including the type Ia supernovae (SNIa) (Riess et al. 1998; Perlmutter et al. 1999), the cosmic microwave background radiation (CMB) (Spergel et al. 2003, 2007), the baryon acoustic oscillation (BAO) (Eisenstein et al. 2005), and so on, have indicated that our universe is undergoing an accelerating expansion. To explain this observed mystery, usually, an exotic dark energy component is assumed to exist in our universe. The simplest candidate of dark energy is the cosmological constant \(\Lambda\), whose equation of state parameter \(w\) (\(w \equiv \frac{P_{DE}}{\rho_{DE}}\)) equals to \(-1\), where \(P_{DE}\) and \(\rho_{DE}\) are the pressure and energy density of dark energy, respectively. Although the cosmological constant plus cold dark matter (\(\Lambda CDM\)) model is well consistent with most of the observational data, it however suffers from both the fine tuning problem and the coincidence problem. Furthermore, the value of the Hubble constant \((H_0)\) of the \(\Lambda CDM\) model determined from the CMB data is in severe tension with that from the nearby SNIa (Riess et al. 2018a,b; Aghanim et al. 2019; Wu et al. 2017). Alternatively, many other dark energy models, the quintessence scalar field dark energy model (Ratra and Peebles 1988), for instance, have been proposed to explain the current accelerated cosmic expansion. When these dark energy models are used to study the property of dark energy, the result obtained is unavoidably model-dependent. To circumvent this dependence, one may directly parametrize the equation of state \(w(z)\) of dark energy in the investigation of the property of dark energy. One such popular model is the CPL parametrization (Chevallier and Polarski 2001; Linder 2003). Apparently, the results depend on the parametrization forms.

A better way to study the property of dark energy is to directly reconstruct it from cosmological observations. In this regard, the cosmography is a popular method used to probe the cosmic expansion history and the property of dark energy (Visser 2015; Dunsby and Luongo 2016; Capozziello et al. 2019). The standard cosmography (SC) is to Taylor expand the luminosity distance-redshift relation \(D_L(z)\) at the present time \(t_0\) or the present redshift \(z_0 = 0\). Taking a truncation on the Taylor series, one can express \(D_L(z)\) as

H. Yu
hwyu@hunnu.edu.cn

P. Wu
pxwu@hunnu.edu.cn

1 Department of Physics and Synergistic Innovation Center for Quantum Effects and Applications, Hunan Normal University, Changsha, Hunan 410081, China
2 Department of Astronomy, Beijing Normal University, Beijing 100875, China
a function of redshift by using some constants

\[ D_L(z) = \frac{c}{H_0} d_L(z) = \frac{c}{H_0} z^{1 + \alpha_1 z + \alpha_2 z^2 + \alpha_3 z^3 + \alpha_4 z^4 + \cdots} \tag{1} \]

where \( d_L(z) \) is called the \( H_0 \) free luminosity distance which is dimensionless, \( c \) is the speed of light, and

\[ \alpha_1 = \frac{1}{2} (1 - q_0), \]

\[ \alpha_2 = -\frac{1}{6} \left(1 - q_0 - 3q_0^2 + j_0\right), \]

\[ \alpha_3 = \frac{1}{24} \left(2 - 2q_0 - 15q_0^2 - 15q_0^3 + 5j_0 + 10q_0j_0 + s_0\right), \]

\[ \alpha_4 = -\frac{1}{20} - \frac{9}{40} j_0 + \frac{1}{12} j_0^2 - \frac{1}{120} j_0 - \frac{1}{20} q_0 - \frac{11}{12} j_0 q_0 + \frac{27}{40} q_0^2 + \frac{7}{8} j_0 q_0^2 + \frac{11}{8} q_0^3 + \frac{7}{8} q_0^4 - \frac{11}{12} j_0 q_0^3 - \frac{1}{8} q_0^4. \tag{2} \]

Here, \( q_0, j_0, s_0 \) and \( l_0 \) are the present deceleration, jerk, snap and lerk parameters, respectively. Using observational data to constrain these parameters, one can obtain \( d_L(z) \), and then derive the cosmic evolution and the property of dark energy. Nevertheless, the convergence region of the Taylor series of the luminosity distance \( d_L \) is only around \( z = 0 \), and thus the results from the SC will be unreliable when the observational data with \( z > 1 \) are used. This is the convergence problem of the cosmographic approach (Cattoën and Visser 2007), and it becomes more and more severe as the observational distance increases.

To alleviate this convergence problem, two different methods are proposed: the \( y \)-variable method and the Padé one. The former is to enlarge the convergence radius of the cosmographic approach by using a relationship (Cattoën and Visser 2007; Aviles et al. 2012), e.g. \( y = z/(1 + z) \), to replace the redshift variable \( z \), so that \( y \rightarrow constant \) when \( z \rightarrow \infty \). The Padé method is based on the rational function approximation (Padé 1892), which can alleviate the convergence problem by exploiting the difference in the order of the numerator and denominator. The cosmography based on the Padé polynomials has been used widely to discuss the cosmic kinematics and investigate the property of dark energy (Adachi and Kasai 2012; Wei et al. 2014; Aviles et al. 2014; Gruber and Luongo 2014; Dunsby and Luongo 2016; Zaninetti 2016; Zhou et al. 2016; Capozziello et al. 2019b, 2020). Recent studies have indicated that the Padé approximation has a better fit to high redshift data than the \( y \)-variable (Capozziello et al. 2020). On the other hand, using the Padé approximant for the luminosity distance, the dark energy equation of state can be directly reconstructed (Saini et al. 2000; Huterer and Turner 2001; Sahni and Starobinsky 2006). Direct reconstruction of the dark energy equation of state would involve a second-order derivative of the distance with respect to redshift. This means that small deviations in fitting the luminosity distance \( d_L(z) \) will be magnified in the reconstruction of \( w(z) \), and will cause unstable results (Huterer and Turner 2001; Maor et al. 2001; Gerke and Efstathiou 2002). In this paper, we find that there is indeed an intrinsic defect in deriving the dark energy equation of state from the Padé approximation of the luminosity distance. When \( z \) is very large, \( w(z) \) given from the luminosity distance \( d_L(z) \) described by the Padé polynomial approaches a constant, i.e. 0 or 1/3, which may result in a bias for the property of dark energy reconstructed from observational data with the Padé approximant. Therefore, although the Padé cosmography is a viable way to investigate the cosmic evolution, it may lead to some unreliable results on the property of dark energy.

This paper is arranged as follows. In Sect. 2, we briefly review how to approximate the luminosity distance using the Padé approximation, and diagnose the Padé cosmography by using the one-parameter method. We analyze the evolution of the reconstructed \( w(z) \) from the Padé approximant at high redshifts in Sect. 3. Finally, we conclude in Sect. 4.

### 2 The Padé cosmography

#### 2.1 Padé approximant

The Padé approximant is obtained by expanding a function as a ratio of numerator and denominator power series. Its radii of convergence are usually broader than that of a Taylor series since the Taylor expansion converges only near the expansion point. For a generic function \( f(x) \), its Taylor series expansion has the form \( f(x) = \sum_{i=0}^{\infty} c_i x^i \), which converges in some neighborhood of the origin. The Padé approximant of order \((n,m)\) to \( f(x) \) is defined to be a rational function \( P_{n,m}(x) \) expressed in a fractional form:

\[ P_{n,m}(x) = \frac{\sum_{i=0}^{n} a_i x^i}{1 + \sum_{i=1}^{m} b_i x^i}, \tag{3} \]

where \( n \) and \( m \) denote the highest order of the numerator and denominator, respectively. The total order of the Padé polynomial is \( n + m \). Approximating this function at \( x = 0 \), the coefficients \((a_0, \ldots, a_n, b_1, \ldots, b_m)\) can be obtained in the following way:

\[ P_{n,m}(0) = f(0), \]

\[ P'_{n,m}(0) = f'(0), \]

\[ \ldots \]
where $P_{n,m}^0(0) = f^{(n+m)}(0)$,
\begin{equation}
P_{n,m}^{(n+m)}(0) = f^{(n+m)}(0),
\end{equation}

where $P_{n,m}^0(0)$ is the first-order derivative of $P_{n,m}(x)$ at $x = 0$ and $P_{n,m}^{(n+m)}(0)$ is the $n + m$ order derivative of $P_{n,m}(x)$ at $x = 0$.

The coefficients of the Padé approximant can also be obtained as follows. If the Taylor series of $f(x)$ is truncated at order $k$, $f(x) = \sum_{i=0}^{k} c_i x^i$, one can assume:
\begin{equation}
c_0 + c_1 x + c_2 x^2 + \cdots + c_k x^k = \frac{a_0 + a_1 x + \cdots + a_n x^n}{1 + b_1 x + \cdots + b_m x^m},
\end{equation}

where $k$ must satisfy the relation $k \geq n + m$, otherwise the whole set of linear equations will not be obtained. Multiplying both sides of this equation by the denominator $(1 + b_1 x + \cdots + b_m x^m)$, and setting coefficients of the same order to be equal, we have
\begin{align}
c_0 &= a_0, \\
c_1 + c_0 b_1 &= a_1, \\
\vdots \\
c_n + c_{n-1} b_1 + \cdots + c_0 b_n &= a_n, \\
c_{n+1} + c_n b_1 + \cdots + c_0 b_{n+1} &= 0,
\end{align}

By solving this system of linear equations, the value of coefficients $(a_i, b_i)$ can be obtained. It is easy to see that a function, which can be expanded in a Taylor series, can always be approximated by a Padé approximant.

### 2.2 One-parameter diagnostic on Padé cosmography

The Padé approximation has been widely used in cosmology (Wei et al. 2014; Gruber and Luongo 2014; Mehrabi and Basilakos 2018; Capozziello et al. 2019b; Rezaei 2019; Rezaei and Malekjani 2021). Usually, we can use the Padé approximant to express the luminosity distance $d_L(z)$:
\begin{equation}
d_L(z) = \frac{z + \sum_{i=2}^{n} a_i z^i}{1 + \sum_{i=1}^{m} b_i z^i},
\end{equation}

where $a_0 = 0$ and $a_1 = 1$ have been used, which arise from the requirements that $d_L(z) = 0$ and $d_L'(z) = 1$ must be satisfied at $z = 0$. Since $d_L(z)$ is an increasing function of redshift, it is required that $n \geq m$ in Eq. (7).

The discussion in the above subsection shows that one can use the Padé approximant to approximate the SC. In this case, the coefficients $(a_i, b_i)$ in the Padé approximants can be expressed with the cosmological parameters $(q_0, j_0, \ldots)$ by solving Eq. (6). In the framework of the flat $\Lambda$CDM model, the Hubble parameter has the form
\begin{equation}
H_{\Lambda\text{CDM}}(z) = H_0 \sqrt{\Omega_{m0}(1 + z)^3 + 1 - \Omega_{m0}},
\end{equation}

after neglecting the radiation energy density, where $\Omega_{m0}$ is the present density parameter of pressureless matter. Then, the cosmological parameters $q_0, j_0, s_0$ and $l_0$ can be, respectively, expressed as
\begin{align}
q_0 &= -1 + \frac{3}{2} \Omega_{m0}, \\
j_0 &= 1, \\
s_0 &= 1 - \frac{9}{2} \Omega_{m0}, \\
l_0 &= 1 + 3 \Omega_{m0} - \frac{27}{2} \Omega_{m0}^2.
\end{align}

The values of these parameters depend only on $\Omega_{m0}$, which also determines the coefficients $(a_i, b_i)$ when using the Padé approximant to approximate the SC. Using data to constrain $\Omega_{m0}$, we can discuss the deviation of the cosmography from the $\Lambda$CDM. This method is called the one-parameter diagnostic, which has been used to investigate the SC in Aviles et al. (2017).

To diagnose the Padé cosmography with the one-parameter method, we use the Pantheon SNIa sample (Scolnic et al. 2018), which consists of 1024 data points. We consider four popular Padé approximants to express the luminosity distance $d_L(z)$ (Aviles et al. 2014; Capozziello et al. 2020):
\begin{align}
P_{2,1} &= \frac{z + a_2 z^2}{1 + b_1 z}, \\
P_{2,2} &= \frac{z + a_2 z^2}{1 + b_1 z + b_2 z^2}, \\
P_{3,1} &= \frac{z + a_2 z^2 + a_3 z^3}{1 + b_1 z}, \\
P_{3,2} &= \frac{z + a_2 z^2 + a_3 z^3}{1 + b_1 z + b_2 z^2}.
\end{align}

It has been found that these Padé approximants have small deviations from the $\Lambda$CDM model (Aviles et al. 2014; Capozziello et al. 2020) by using different observational data including SNIa (Suzuki et al. 2012; Scolnic et al. 2018), BAO (Percival et al. 2010), the Observational Hubble Data (Jimenez and Loeb 2002) and the CMB shift parameter (Ade et al. 2016) to constrain the model parameters.

As a comparison, we also diagnose the SC with the one-parameter method. We use $SC_n$ to denote the $n$-th order luminosity distance. For example, $SC_3$ represents the 3rd order luminosity distance. Constraints on $\Omega_{m0}$ can be obtained
by minimizing the $\chi^2$-square:

$$\chi^2 = \sum_{i=1}^{N} \frac{[\mu_{i}^{\text{obs}}(z_i) - \mu_{i}^{\text{th}}(H_0, \Omega_{m0}, z_i)]^2}{\sigma_{\mu_{i}}^{\text{obs}}},$$

where $N = 1048$, $\mu_{i}^{\text{obs}}$ and $\sigma_{\mu_{i}}^{\text{obs}}$ are the distance modulus and the corresponding error of the Pantheon SNIa, respectively, and $\mu_{i}^{\text{th}} = 25 + 5 \log[D_L(z_i)/\text{Mpc}]$ is the distance modulus from the SC or the Padé approximant at $z_i$. In Eq. (11), $H_0$ is a noise parameter, which is marginalized by using the method given in Nesseris and Perivolaropoulos (2004). In our analysis, the CosmoMC code is used.\(^1\)

The results are shown in Fig. 1 and the best fit values of $\Omega_{m0}$ at the 1σ confidence level (CL), respectively, are:

- $\Omega_{m0}|_{P_{2,1}} = 0.325 \pm 0.021$,
- $\Omega_{m0}|_{P_{3,1}} = 0.278 \pm 0.022$,
- $\Omega_{m0}|_{P_{2,2}} = 0.283 \pm 0.021$,
- $\Omega_{m0}|_{P_{3,2}} = 0.300 \pm 0.022$,
- $\Omega_{m0}|_{SC_3} = 0.262 \pm 0.014$,
- $\Omega_{m0}|_{SC_4} = 0.278 \pm 0.025$,
- $\Omega_{m0}|_{SC_5} = 0.227 \pm 0.017$,
- $\Omega_{m0}|_{\Lambda CDM} = 0.298 \pm 0.022$. \(^{(12)}\)

It is easy to see that when the number of model parameters is the same, the one-parameter diagnostic shows that the Padé cosmography is apparently better than the SC. Although $P_{2,1}$, $P_{3,1}$ and $P_{2,2}$ are consistent with the $\Lambda CDM$ only at the margin of 1σ, $P_{3,2}$ gives almost the same result as the $\Lambda CDM$. Thus, the Padé cosmography can provide an unbiased estimation of the background cosmology.

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3 Reconstructing the equation of state of dark energy from Padé cosmography

3.1 $w(z)$ reconstructed from Padé approximant

Using the Padé approximant to approximate the luminosity distance, one can obtain the expression of the Hubble parameter

$$H(z) = \left[ (z + 1)^2 \sum_{i=1}^{m} (b_iz_i + 1) \right]^{-1} / \left[ (z + 1) \times \sum_{i=0}^{n} ia_iz_i^{-1} \sum_{i=1}^{m} (b_iz_i + 1) - \sum_{i=0}^{n} a_iz_i^{-1} \sum_{i=1}^{m} (b_iz_i + 1) \right]$$

after assuming a spatially flat FLRW universe, where Eq. (7) and the following relation

$$H(z) = \left[ \frac{d}{dz} \left( \frac{dL(z)}{dz} \right) \right]^{-1}$$

have been used.

Assuming that the universe is filled with pressureless matter and dark energy, we can obtain the equation of state of dark energy from the Hubble parameter

$$w(z) = \frac{(2(1+z)/3)(\ln H') - 1}{1 - (H_0/H)^2} - \frac{\Omega_{m0}(1+z)^3}{\Omega_{m0}(1+z)^3}. \quad (15)$$

Substituting $H(z)$ (Eq. (13)) and its first derivative into Eq. (15), one can reconstruct the dark energy equation of state from the Padé approximant. Apparently, $w(z)$ depends on the second derivative of $d_L(z)$ with respect to

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\(^1\)The CosmoMC code is available at https://cosmologist.info/cosmomc.
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Fig. 2 The evolution of the reconstructed dark energy equation of state from the Padé cosmography. $w_{2,1}$, $w_{3,1}$, $w_{2,2}$ and $w_{3,2}$ represent the results from $P_{2,1}$, $P_{3,1}$, $P_{2,2}$ and $P_{3,2}$, respectively. The coefficients $a_2 = 1.7$, $a_3 = 1$, $b_1 = 1$, and $b_2 = 0.5$, and $\Omega_{m0}$ is set to be 0.3. The black dotted and orange dashed lines denote $w = 0$ and $w = 1/3$, respectively.

$z$, which results in that the reconstructed result may be unreliable. Here, we find that the reconstructed $w(z)$ approaches a constant when $z$ is very large. This will render the reconstructed result biased. Since the value of a rational function as $z \rightarrow \infty$ is only dependent on the highest-order term in the numerator and denominator, we replace the summation terms, i.e. $\sum_{i=0}^n a_i z^i$ and $\sum_{j=1}^m b_j z^j$, in $w(z)$ with the highest order term $a_n z^n$ and $b_m z^m$, and then obtain

$$w(z)|_{z \rightarrow \infty} = \frac{A_0 + A_{-1} z^{-1}}{B_0 + B_{-1} z^{-1} + B_{2n-2m-1} z^{2n-2m-1}}, \quad (16)$$

where the coefficients are defined as:

$A_0 = b_m^5 \left(2m^2 + m(3 - 4n) + 2n^2 - 3n + 1\right)$,

$A_{-1} = b_m^5 \left(6m^2 + m(8 - 12n) + 6n^2 - 8n + 1\right)$,

$B_0 = 3b_m^5 (m - n + 1)$,

$B_{-1} = 3b_m^5 (2m - 2n + 1)$,

$B_{2n-2m-1} = -3\Omega_{m0} a_n^2 b_m^3 (m - n + 1)^3$. \quad (17)

It is worth noting that when $n = m + 1$, the coefficients become $A_0 = B_0 = B_{2n-2m-1} = 0$ and $A_{-1} = \frac{1}{3} B_{-1} = 1$, which implies that $w$ approaches $1/3$. When $n = m$, we have $A_0 = \frac{1}{3} B_0 = -1$ and thus $w \rightarrow 1/3$. The $n > m + 1$ case is different from the previous two cases since the term containing $B_{2n-2m-1}$ in Eq. (16) must be taken into account. Due to $2n - 2m - 1 > 0$, it is easy to obtain that $w(z) \rightarrow 0$ when $z$ is very large. Therefore, we find that in the high redshift regions the equation of state of dark energy reconstructed from the Padé cosmography will approach a constant, i.e. 0 or 1/3. This character will bias our understanding of the property of dark energy. To show this character clearly, we plot in Fig. 2 the evolutionary curves of the reconstructed $w(z)$ from four popular Padé approxi-
mants given in Eq. (10). In this figure, \( w_{2,1}, w_{3,1}, w_{2,2} \) and \( w_{3,2} \) represent the dark energy equation of state reconstructed from \( P_{2,1}, P_{3,1}, P_{2,2} \) and \( P_{3,2} \), respectively. It is easy to see that when the redshift \( z \) is about 10, then \( w_{2,1}, w_{2,2} \) and \( w_{3,2} \) converge to 1/3 and \( w_{3,1} \) approaches 0. There exists a singularity in the reconstructed \( w_{2,1} \) and \( w_{2,2} \), and the redshift where the singularity appears depends on the values of the coefficients of the Padé approximants.

To further discuss the bias in reconstructing the property of dark energy from the Padé cosmography, in the following we will use the mock SNIa data based on the ΛCDM model and the real data from the Pantheon SNIa sample (Scolnic et al. 2018) to constrain the free parameters in the Padé approximants, respectively, and then study the evolution of \( w(z) \) from four popular approximants given in Eq. (10).

### 3.2 Mock data

Since the maximum redshift of the Pantheon sample is 2.26 and the total data number is only 1048, the observational data may not give tight constraints on high redshift regions of \( w(z) \), and thus are not effective in analyzing the bias discussed in the above subsection. Therefore, here we mock SNIa distances on the basis of the upcoming Wide Field InfraRed Survey Telescope (WFIRST), which was the highest-ranked large space-based mission of the 2010 decadal surveys. A primary objective of this mission is to precisely constrain the nature of dark energy with multiple probes, including SNIa. Optimistically, the Imaging: Allz strategy of WFIRST could collect ∼13500 SNIa in the redshift range 0 < \( z \) < 3 (Hounsell et al. 2018). The number of SNIa that can be discovered by WFIRST is expected to follow the volumetric rates (Rodney et al. 2014; Graur et al. 2014),

\[
R_{\text{SNIa}}(z) = \begin{cases} 
2.5 \times (1 + z)^{1.5} (10^{-5} \text{ yr}^{-1} \text{ Mpc}^{-3}), \\
9.7 \times (1 + z)^{0.5} (10^{-5} \text{ yr}^{-1} \text{ Mpc}^{-3}), \\
& 1 < z < 3.
\end{cases}
\]

(18)

As the expected detection rate is low for \( z > 3 \) SNIa, we do not simulate events at those redshifts. For a sample of SNIa, the total uncertainty of the distances \( d_L \) consists of two main components, the statistical uncertainty \( \sigma_{\mu, \text{stat}} \) and the systematic uncertainty \( \sigma_{\mu, \text{sys}} \), i.e. \( \sigma_{\mu, \text{tot}}^2 = \sigma_{\mu, \text{stat}}^2 + \sigma_{\mu, \text{sys}}^2 \). With the fractional statistical uncertainty (panel h of Fig. 7 in Hounsell et al. (2018)) and all ingredients of the systematic uncertainty (Figs. 9 and 10 in Hounsell et al. (2018), including wavelength dependent calibration, nonlinearity, contamination, host-mass evolution, intrinsic scatter, population drift, and zero-point uncertainties) taken into consideration, we can generate a simulated distance modulus sample of SNIa in a given fiducial model. The fiducial value \( \mu^\text{fid}(z) \) is obtained from the spatially flat ΛCDM model with the model parameters being \( H_0 = 70 \text{ km s}^{-1} \text{ Mpc}^{-1} \) and

![Image](https://via.placeholder.com/150)

Table 1 The constraints on Padé approximants from the mock SNIa data

| \( P_{2,1} \) | Mean(\( \sigma \)) | 0.68 CL | Mean(\( \sigma \)) | 0.68 CL | Mean(\( \sigma \)) | 0.68 CL | Mean(\( \sigma \)) | 0.68 CL |
|---|---|---|---|---|---|---|---|---|
| \( a_2 \) | 1.343(0.045) | +0.045 | 1.348(0.147) | +0.127 | 1.331(0.127) | +0.118 | 8.911(1.716) | +2.205 |
| \( b_1 \) | 0.509(0.020) | +0.020 | 0.511(0.100) | +0.087 | 0.501(0.080) | +0.080 | 8.221(1.697) | +2.232 |

Table 2 The constraints on Padé approximants from the Pantheon SNIa data

| \( P_{2,1} \) | Mean(\( \sigma \)) | 0.68 CL | Mean(\( \sigma \)) | 0.68 CL | Mean(\( \sigma \)) | 0.68 CL | Mean(\( \sigma \)) | 0.68 CL |
|---|---|---|---|---|---|---|---|---|
| \( a_2 \) | 1.381(0.218) | +0.181 | 4.078(2.688) | +1.498 | 1.370(0.543) | +0.341 | 8.454(2.154) | +3.113 |
| \( b_1 \) | 0.544(0.148) | +0.121 | 3.015(2.463) | +1.313 | 0.532(0.443) | +0.274 | 7.566(2.053) | +2.947 |

Note—The mean of different coefficients \( (a_i, b_i) \) with the standard deviation \( \sigma \) and the 68% CL. The results are obtained from the mock SNIa data.
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Fig. 3 Contour plots for the coefficients of four Padé approximants. Top left, top right, bottom left and bottom right represent $P_{2,1}$, $P_{3,1}$, $P_{2,2}$ and $P_{3,2}$, respectively. The red and blue colors show the constraints from the Pantheon SNIa sample and the mock data, respectively.

Fig. 4 The evolution of the relative deviation $\delta(z)$ obtained from 13570 mocked SNIa in the range $0 < z < 3$. For a sample containing $N_{\text{SN}} = 13570$ mocked SNIa, we draw a random number from the Gaussian distribution $N(\sigma_{\mu,\text{tot}}, \sigma_{\mu,\text{tot}}/\sqrt{N_{\text{SN}}})$ as the uncertainty $\sigma_{\mu}(z)$. For simplicity and approximate predictions, we only take the diagonal covariance matrix into account in our analysis. For the distance modulus of the mocked data, we use $\mu_{\text{mock}}(z) = \mu_{\text{fid}}(z)$ at each data point in order to compare with the exact $-1$ line. From this mocked SNIa sample, the coefficients in the Padé approximants can be constrained by minimizing the $\chi$-square given in Eq. (11).

3.3 Results

The constraints on coefficients are shown in Fig. 3 and summarized in Tables 1 and 2. Figure 3 gives the contours of the model parameters of the Padé approximants, in which red and blue colors represent the Pantheon and the mock data, respectively. It is easy to see that the results from the Pantheon sample are consistent with those from the mock data, but the latter gives very strong constraints.

To discuss the deviations of the Padé cosmography from the fiducial model used to mock data, we investigate the evolution of $\delta(z)$, which is defined as:

$$\delta(z) = \frac{|d^\text{Padé}_L(z) - d^\Lambda_{\text{CDM}}(z)|}{d^\Lambda_{\text{CDM}}(z)}.$$ (19)
Fig. 5 The evolution of the dark energy equation of state with 1σ error reconstructed from Padé cosmography. \( w_{2,1}, w_{3,1}, w_{2,2} \) and \( w_{3,2} \) represent the results from \( P_{2,1}, P_{3,1}, P_{2,2} \) and \( P_{3,2} \), respectively.

In Fig. 4, we show our results. One can see that four Padé polynomials can describe very well the cosmic evolution since the relative deviation \( \delta \) is less than 0.8% in the redshift region between 0 to 3, and \( P_{3,2} \) gives the best fit result due to that \( \delta \) is < 0.4% in this case.

After the free parameters in the Padé approximants having been determined by the mock data, one can use Eq. (15) with \( \Omega_{m0} = 0.3 \) to obtain the evolution of the dark energy equation of state, which is shown in Fig. 5. One can see that \( w_{2,1} \) deviates from the \(-1\) line in the redshift region around 1 and in the very low redshift regions at the 1σ confidence level, while \( w_{3,1}, w_{2,2} \) and \( w_{3,2} \) are consistent with \(-1\) at the 1σ CL when \( z \) is less than about 2.2. In the high redshift region, i.e. \( z > 2.8 \), it is easy to see that all of the reconstructed \( w(z) \) apparently deviate from \(-1\) line and are larger than \(-1\). This is because when \( z \) is very large \( w_{2,1}, w_{2,2} \) and \( w_{3,2} \) approach \( 1/3 \), and \( w_{3,1} \) approaches 0, while the dark energy model used to mock data is the cosmological constant.

### 4 Conclusion

In this paper, we investigate the reconstruction of the dark energy equation of state from the Padé cosmography. We first diagnose the Padé cosmography by using the one-parameter method and find that it can give a description of the background cosmology better than the standard cosmography. Then, we obtain that the dark energy equation of state approaches a constant, i.e. \( 1/3 \) or 0, when the redshift is very large. This result is general since it is independent of the coefficients in the Padé approximant and the value of \( \Omega_{m0} \). This intrinsic character will bias the \( w(z) \) reconstruction and lead to misconception of the property of dark energy. By using the mock data based on the \( \Lambda \)CDM model, we demonstrate that the Padé approximant can describe the cosmic evolution very well since the relative deviation from the fiducial model is less than 0.008. However, when reconstructing the equation of state of dark energy, the reconstructed results in the high redshift regions apparently devi-
ate from the $-1$ line, which is used to mock data. This arises as a result of the fact that the reconstructed dark energy equation of state from the Padé cosmography approaches a constant at very high redshifts, while the equation of state of dark energy used to mock data is $-1$. Our results indicate that one must exercise caution in reconstructing the property of dark energy from the Padé cosmography when the high redshift data, i.e. $z > 2$, are used to constrain model parameters. Thus, the viable way to probe the property of dark energy from the Padé approximant is to use it directly to express the dark energy equation of state (Rezaei et al. 2017, 2020) rather than the luminosity distance.

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Data Availability Data available on request from the authors.

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