Minimax Theorem for Latent Games or:
How I Learned to Stop Worrying about Mixed-Nash and Love Neural Nets

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Abstract
Adversarial training, a special case of multi-objective optimization, is an increasingly useful tool in machine learning. For example, two-player zero-sum games are important for generative modeling (GANs) and for mastering games like Go or Poker via self-play. A classic result in Game Theory states that one must mix strategies, as pure equilibria may not exist. Surprisingly, machine learning practitioners typically train a single pair of agents – instead of a pair of mixtures – going against Nash’s principle. Our main contribution is a notion of limited-capacity-equilibrium for which, as capacity grows, optimal agents – not mixtures – can learn increasingly expressive and realistic behaviors. We define latent games, a new class of game where agents are mappings that transform latent distributions. Examples include generators in GANs, which transform Gaussian noise into distributions on images, and StarCraft II agents, which transform sampled build orders into policies. We show that minimax equilibria in latent games can be approximated by a single pair of dense neural networks. Finally, we apply our latent game approach to solve differentiable Blotto, a game with an infinite strategy space.

“As far as I can see, there could be no theory of games [without] the Minimax Theorem” – von Neumann (1953)

1. Introduction
Externally defined games have been used as benchmarks in artificial intelligence for decades (Samuel, 1959; Tesauro, 1995), with recent progress on increasingly complex domains such as poker (Brown and Sandholm, 2017; 2019), chess, Go (Silver et al., 2017), and StarCraft II (Vinyals et al., 2019). At the same time, remarkable progress in generative modeling of images (Wu et al., 2019) and speech synthesis (Bùiñkowski et al., 2020) has resulted from zero-sum games designed specifically to facilitate via carefully constructed arms races (Goodfellow et al., 2014). Zero-sum games also play a fundamental role in building classifiers that are robust to adversarial attacks (Madry et al., 2017).

The goal of the paper is to put learning – by neural nets – in two-player zero-sum games on a firm theoretical foundation to answer the question: What does it mean to solve a game?

In single agent learning, performance is well-defined via a fixed objective. However, it is not obvious what counts as optimal in a two-player zero-sum setting. Since each player’s goal is to maximize its payoff, it is natural to ask whether a player can maximize its utility independently of how the other player behaves. von Neumann et al. (1944) laid the foundation of game theory with the Minimax theorem, which provides a meaningful notion of optimal behavior against an unknown adversary.

In short, a meaningful solution for a two-player zero-sum game must incorporate two notions: (i) a number $V$, called the value of the game, (ii) a strategy for each player such that their average gain is at least $V$ (resp. $-V$) no matter what the other does. The optimal course of action is typically to choose a mixed strategy (a distribution over pure strategies). The existence of a value and optimal strategies is guaranteed by the Minimax theorem (von Neumann et al., 1944).

From a game theoretic perspective it is obvious that pure-equilibria need not exist and so we should work with mixed strategies. Nevertheless, machine learning practitioners typically train a single pair of agents (instead of a pair of mixtures), where each of them is represented by a neural net.

In this work, we reconcile machine learning (ML) practice with game theory. We make four main contributions. (i) We formulate latent games, a new class of games, with many application in machine learning, where the agents leverage function approximation to directly encode mixed strategies. (ii) We prove the existence of a new notion of limited-capacity-equilibrium in latent games that differs from the standard Nash equilibrium of the game. (iii) We show that in some cases, e.g. GANs, the first player can exploit the concavity of its payoff to only play pure strategies. We call such games semi-latent games and extend our

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results on latent games to semi-latent games. (iv) Finally, we show that, for agents parametrized by ReLU neural networks, such a limited-capacity-equilibrium can be achieved by a pair of pure-agents.

To sum-up, our main contribution is the proof of the existence of an approximate global minimax achieved by a single pair of ReLU neural networks for a large class of two player games that includes GANs. This result contrasts with the negative result of Jin et al. (2019) who construct a nonconvex-nonconcave game where pure global minimax does not exist.

1.1. Related work

Minimax theorems in GANs. The notion of a mixed Nash equilibrium for GANs was already originally mentioned by Goodfellow et al. (2014) but without taking into account the limited capacity of the models. Many papers have adopted a game theoretic perspective to understand GANs and motivate the computation of mixed equilibria (Arora et al., 2017; Oliehoek et al., 2018; Hsieh et al., 2019; Grnarova et al., 2018). However, these papers fail to explain why, in practice, it suffices to train only a single generator and discriminator.

We argue that generators directly encode mixed strategies using a latent structure and discriminators correspond to pure strategies. The discriminator player is not required to play mixed strategies because of the concavity of the payoff function. We show that instead the discriminator player can reach an equilibrium by playing a point-wise average of classifiers, giving a principled motivation for the practical method proposed by Durugkar et al. (2017).

Overall, even if our motivations are similar to the theoretical related work mentioned above (providing principled results), our results and conclusion are fundamentally different: we are able to explain why using a single generator and discriminator makes sense by proving that a notion of pure minimax equilibrium is achieved for GANs parametrized with neural networks.

Stackelberg games. Other notions of equilibrium have been considered in the literature. Recently, Fiez et al. (2019) and Jin et al. (2019) proved results on games where the goal is to find a Stackelberg equilibrium. Such a notion of equilibrium may be meaningful in some contexts such as adversarial training but we argue in §1.2 that the minimax theorem is fundamental for defining a valid notion of solution for a large class of machine learning applications.

Latent policies in Multi-agent RL. Our notion of latent games can be related to the latent policies used in some multi-agent reinforcement learning (RL) applications. For instance the agent trained by Vinyals et al. (2019) to play the game of StarCraft II had policies of form \( \pi(a|s, z) \) where \( z \) belongs to structured space that corresponds to a particular way to start the game or to actions it should complete during the game (for instance building a particular unit or building). The correspondence between a given structured \( z \) and specific action \( a \) to complete during the game was imposed by a loss penalty \( L(\pi(a|z)|z) \) and \( z \) was sampled according to what was played by human players: \( z \sim p_{\text{human}}(z) \).

In our work, we do not impose any semantically meaningful structure for the latent space a-priori and do not ground the latent space to any extra loss in the game. We instead propose a general theory to understand, study and play games using latent mappings.

Finding a Nash of Colonel Blotto. After its introduction by Borel (1921), finding the Nash of the Colonel Blotto game has been an open question for 85 years. Roberson (2006) found an equilibrium solution for the continuous version of the game, later extended to the discrete symmetric case by Hart (2008). The equilibrium computation in the general case remains open. Recently, Blotto has been used as a challenging use-case for equilibrium computation (Ahmadinejad et al., 2019). In the same vein, we consider a variant of Blotto to validate the efficiency of our method.

1.2. Relevance of the Minimax theorem for ML

A notorious ML application which has a minimax formulation is adversarial training where a classifier is trained to be robust against adversarial attack. From a game theoretic perspective, the adversarial attack is picked after the classifier \( f \) is set and thus corresponds to a best response. From a learning perspective, the goal is to learn to be robust to adversarial attacks specifically designed against the current classifier. Such an equilibrium is called a Stackelberg Equilibrium (Conitzer and Sandholm, 2006).

In games with imperfect information such as Colonel Blotto, Poker, or StarCraft II the players must commit to a strategy without the knowledge of the strategy picked by their opponent. In that case, the agents cannot design attacks specific to their opponent, because such attacks may be exploitable strategies. It is thus strictly equivalent to consider that the players simultaneously pick their respective strategies and then reveal them. Under these considerations a meaningful notion of playing the game must have a value and an equilibrium.

In machine learning applications, each player is trained using local information (though gradient or RL based methods). Because the behavior of the players changes slowly, they cannot have access to the best response against their opponent. In order to illustrate that point, let us consider the example of Generative Adversarial Networks. The two agents (the generator and the discriminator) are usually se-
Minimax Theorem for Latent Games

quently updated using a gradient method with similar step-sizes. During training, one cannot expect an agent to find a best response in a single (or few) gradient steps. To sum-up, since local updates are performed one must expect to reach a point that is locally stable, i.e. a local equilibrium.

1.3. Structure of the Paper

In §2, starting from the definition of a two-player zero-sum game, we define latent games, a way to play games using latent mappings, and show that they induce a notion of limited-capacity equilibrium that differs from the Nash equilibria of the original game. Then, in §3, we show that approximate limited-capacity-equilibria can be achieved with small mixtures. Finally, in §4, we prove that mixture of ReLU neural networks can be expressed with a single larger ReLU network and thus show that ReLU networks can achieve a notion of pure \( \varepsilon \)-equilibrium.

All the proof of the propositions and theorems of §2,3, and 4 can be respectively found in §C,D, and E.

2. Using Latent Agents to Solve Games

Recall the definition of a two-player zero-sum game:

**Definition 1.** A two-player zero-sum game is given by a payoff function that evaluate pairs of strategies

\[ \varphi : A \times B \rightarrow \mathbb{R} \]

Players pick strategies in \( A \) and \( B \) respectively. If the players choose \( a \in A \) and \( b \in B \), the first player receives reward \( \varphi(a, b) \) and the second receives \( -\varphi(a, b) \).

The goal for each player is to find a non-exploitable strategy or Nash equilibrium: a pair \((a^*, b^*)\) satisfying

\[ \varphi(a^*, b^*) \leq \varphi(a^*, b) \leq \varphi(a, b^*) \quad \forall a \in A, b \in B \]

Any fixed action may be exploitable and so pure Nash equilibria need not exist. von Neumann et al. (1944) introduced mixed strategies \( p \in \mathcal{P}(A) \) where \( \mathcal{P}(A) \) is the set of probability distributions over \( A \), which guarantees the existence of an equilibrium. If we consider two mixed strategies \( p \in \mathcal{P}(A) \) and \( q \in \mathcal{P}(B) \), the associated payoff is:

\[ \varphi(p, q) = \mathbb{E}_{a \sim p, b \sim q}[\varphi(a, b)] \]

For mixed strategies von Neumann et al. (1944) proved the Minimax theorem:

\[ V := \min_{q \in \mathcal{P}(B)} \max_{p \in \mathcal{P}(A)} \varphi(p, q) = \max_{p \in \mathcal{P}(A)} \min_{q \in \mathcal{P}(B)} \varphi(p, q). \] (1)

**Example 1** (Rock-Paper-Scissors). Consider the game with three actions \( \{r, p, s\} \) such that

\[ r \text{ beats } s, \quad s \text{ beats } p \quad \text{and} \quad p \text{ beats } r. \] (tps)

Playing a mixed strategy is playing a probability vector \( p \geq 0, \text{ s.t., } p_r + p_p + p_s = 1 \). The payoff of this game is

\[ \varphi(p, q) := p^\top M q \quad \text{where} \quad M \text{ correspond to } (\text{tps}). \]

The Nash equilibrium \((p^*, q^*)\) of this game is simply \( p^* = q^* \), \( p^*_r = p^*_p = p^*_s = \frac{1}{3} \).

The Nash equilibrium of a two-player zero-sum game can be cast as the solution to a linear program (Chvatal, 1983) and can thus be solved in polynomial time. However, for some games the number of strategies available may be exponential or infinite making the Nash intractable.

**Example 2** (Colonel Blotto Game). Consider two players who control armies of \( S_1 \) and \( S_2 \) soldiers respectively. Each colonel allocates their soldiers on \( K \) battlefields. A strategy for player-\( i \) is an allocation \( a_i \in A_i \) defined as

\[ A_i := \{a \in \mathbb{N}^K : \sum_{k=1}^K \|a\|_k \leq S_i , 1 \leq k \leq K \}. \] (2)

The payoff of the first player is the number of battlefields won (because more soldiers were allocated):

\[ \varphi(a_1, a_2) := \frac{1}{K} \sum_{k=1}^K \mathbb{I}\{[a_1]_k > [a_2]_k\}. \]

We present experiments with Blotto in §5. For now, note that the number of strategies grows exponentially in \( K \).

An important example is generative adversarial networks (Goodfellow et al., 2014). It is usually described as a game between a generator that create samples that aim to look similar to ‘real’ samples and a discriminator that classify samples as ‘real’ or ‘fake’. Thus, a naïve way to cast GANs according to Definition 1 is the following:

**Example 3** (Naive GAN). The first player, the discriminator, chooses \( \psi \in \mathcal{F} : \mathbb{R}^d \rightarrow \mathbb{R} \). The second player, the generator, proposes an image \( x \in [0,1]^d \). The payoff \( \varphi \) is then the ability of the first player to discriminate a real data distribution \( p_{\text{data}} \) from the generated \( x \):

\[ \varphi(\psi, x) := \mathbb{E}_{x' \sim p_{\text{data}}} \left[ \ln(\sigma(\psi(x'))) + \ln(1 - \sigma(\psi(x)) \right]. \]

The number of strategies is infinite for both players, so the space of mixed strategies is infinite dimensional and difficult to work with. Remarkably, the machine learning community has found approximate equilibria of GANs using single pairs of networks – rather than mixtures.

**Hypothesis:** Example 3 suggests that GANs use function approximation to construct mixtures of strategies.

We generalize this, showing how one can approximate the mixed Nash using generative functions in any game.

‘We decided to consider logits prediction for the discriminator because doing so the function \( \varphi(\cdot, x) \) is 1-Lipschitz
Latent Game Meta-algorithm

**Input:** Payoff $\varphi$, sets $\mathcal{F}$ and $\mathcal{G}$, and batch size $B$.

**repeat**

1. Sample $z_1, \ldots, z_B \sim \pi$ and $z'_1, \ldots, z'_B \sim \pi'$
2. $\tilde{\varphi} \leftarrow \sum_{j=1}^{B} \varphi(f(z_j), g(z_j))$ (Compute the payoff)
3. $f \leftarrow \text{oracle}(f, \mathcal{F}, \tilde{\varphi})$ (Improve the 1st player)
4. $g \leftarrow \text{oracle}(g, \mathcal{G}, -\tilde{\varphi})$ (Improve the 2nd player)

**until** An approximate equilibrium is reached

2.1. Latent Games

The games arising in machine learning are not classical normal- or extensive-form games. Rather, they often use neural nets to approximate complex functions and high dimensional distributions (Brock et al., 2019). Motivated by Examples 2 and 3, we propose to use function approximation to play games in a way that is better-suited to common use cases in machine learning.

**Definition 2** (Latent Game). Given latent space $\mathcal{Z}$, latent distribution $\pi$ on $\mathcal{Z}$ and mapping $f : \mathcal{Z} \rightarrow A$, we can define the distribution $p_f \in \mathcal{P}(A)$ as

$$a \sim p_f : a = f(z), \ z \sim \pi. \quad (3)$$

A latent game $(\varphi, \mathcal{F}, \mathcal{G})$ is a two-player zero-sum game where the players pick mappings $f \in \mathcal{F}$ and $g \in \mathcal{G}$ and obtain payoffs according to

$$\varphi(f, g) := \varphi(p_f, p_g) = \mathbb{E}_{z \sim \pi, z' \sim \pi'} \left[ \varphi(f(z), g(z')) \right]$$

where $\pi$ and $\pi'$ are two fixed latent distributions.

Algorithm 1 is a simple meta-algorithm to train agents in latent games. The function \text{oracle}(f, \mathcal{F}, \tilde{\varphi}) is a black box oracle similar to the one described in Balduzzi et al. (2019) that gives an improvement for the mappings $f$ within the class $\mathcal{F}$ using the approximate payoff $\tilde{\varphi}$. Crucially, the equilibria of latent games depend on the function classes $\mathcal{F}$ and $\mathcal{G}$, which in practice are specified by the choice of neural net architecture, see §2.2.

In games where the number of strategies grows exponentially, such as Colonel Blotto (Example 2), we cannot compute the Nash using standard methods for two-player zero-sum games. We thus reformulate Colonel Blotto as a latent game:

**Example 2** (Latent Blotto). For $i \in \{1, 2\}$, consider the function $f_i : \mathbb{R}^p \rightarrow A_i$ where $A_i$ is defined in (2). Then

$$\varphi(f_1, f_2) := \frac{1}{K} \sum_{k=1}^{K} \mathbb{P}(\left| [f_1(Z_1)]_k \right| > \left| [f_2(Z_2)]_k \right|) \quad (4)$$

where $Z_1, Z_2 \sim \mathcal{N}(0, I_p)$ are independent Gaussians.

Latent games are closely related to the RL policies used to play StarCraft II (Vinyals et al., 2019). The agent, called AlphaStar, has a latent-conditioned policy $\pi(a | s, z)$ where $z$ belongs to a structured space that represents information about how to start constructing units and buildings sampled from expert human players: $z \sim p_{human}(z)$. Given two agents $\pi_1(a | s, z)$ and $\pi_2(a | s, z)$, the payoff in the latent game is

$$\varphi(\pi_1, \pi_2) = \mathbb{P}(\pi_1 \text{ beats } \pi_2).$$

The classes $\mathcal{F}$ and $\mathcal{G}$ correspond to the neural architecture used to parameterize the policies; the priors $\pi_f$ and $\pi_g$ are the human expert distribution $p_{human}$. Latent games thus provide a useful framework for training agents on challenging games, whilst incorporating real-world constraints and additional information encoded in highly structured priors.

2.2. Minimax Theorem for Latent Games

Classical game theory introduced mixed strategies to find equilibria in games. Analogously, we need to define latent mixtures of mappings.

**Latent mixture of mappings.** Consider latent mappings $g_1$ and $g_2$ that generate probability distributions $p_{g_1}$ and $p_{g_2}$, as in (3). The latent mixture $p_{\lambda}$ of $g_1$ and $g_2$ with $\lambda \in [0, 1]$ is the mixture of these two probability distributions:

$$p_{\lambda} := \lambda p_{g_1} + (1 - \lambda)p_{g_2}. \quad (5)$$

To sample from $p_{\lambda}$, flip a biased coin with $\mathbb{P}(\text{heads}) = \lambda$. If the result is heads then sample a strategy from $p_{g_1}$ and if the result is tails then sample from $p_{g_2}$. We used the denomination latent mixture because the mixture is performed before sampling the latent variable. We can generalize latent mixtures to an arbitrary number of mappings and define

$$\Delta(\mathcal{G}) := \{\text{Latent mixtures of mappings from } \mathcal{G}\}. \quad (5)$$

More formally, $\Delta(\mathcal{G})$ is the ‘strong’ convex hull of the distributions $\{p_g, g \in \mathcal{G}\}$. Thus $\Delta(\mathcal{G})$ is a subset of $\mathcal{P}(A)$, the set of all distributions on $A$. This set is different from the set of distributions supported on $\mathcal{G}$ considered by Arora et al. (2017); Hsieh et al. (2019). The set $\Delta(\mathcal{G})$ contains ‘smaller’ mixtures because there may be many distributions supported on $\mathcal{G}$ that correspond to the same $p \in \Delta(\mathcal{G})$. For instance in Example 4, any point in the hole of the annulus is in $\Delta(\mathcal{G})$ and is at most a mixture of two elements while the set of distributions on $\mathcal{G}$ contains infinite mixtures.

Our first main result is a Minimax theorem for latent games:

**Theorem 1.** Let $(\varphi, \mathcal{F}, \mathcal{G})$ be a latent game. We have,

$$\max_{p \in \Delta(\mathcal{F})} \min_{q \in \Delta(\mathcal{G})} \varphi(p, q) = \min_{q \in \Delta(\mathcal{G})} \max_{p \in \Delta(\mathcal{F})} \varphi(p, q) \quad (6)$$

where $\Delta(\mathcal{G})$ is defined in (5). We call this value $V_L$.

\footnote{See Appendix B for a formal definition.}
Theorem 1 implies that latent games have a value and an equilibrium, the two necessary notions to define meaningful solutions for two-player zero-sum games.

It is crucial to notice that the space of latent mixtures \( \Delta(G) \) may not cover the whole space of mixed strategies \( P(A) \). Thus, the latent game \((\varphi, F, G)\) may not have the same equilibrium and value as \((\varphi, A, B)\), the original zero-sum two player game (1):

\[
V_L \neq V := \min_{p \in P(A)} \max_{q \in P(B)} \varphi(p, q).
\]

This difference, in terms of value and equilibrium with standard normal form games is not a drawback but a feature of latent games. Such latent game equilibria have a meaningful interpretation in terms of games played by agents with restricted capacity which we develop in §6 and Appendix A.

The next example builds intuition about the geometry of \( \Delta(G) \) and illustrates why the minimax theorem is essential.

**Example 4 (Annular Rock-Paper-Scissors).** Recall Rock-Paper-Scissors from Example 1. Directly represent a mixed strategy as a point in the probability simplex \( \Delta_3 \subset \mathbb{R}^3 \). Introduce a latent game \((\varphi, F, G)\) with mixed strategies expressed in the subset of the simplex (denoting 1 := \([1, 1, 1]\)):

\[
P_G = \{ p \in \Delta_3 : 0 < r \leq \|3p - 1\| \leq R \leq 1 \}.
\]

The geometry of Example 4 is represented in Fig. 1. The equilibrium for each player is the point \( O \not\in P_G \). The payoff \( \varphi \) can be geometrically interpreted as \( \varphi(p, q) = \|OP\| \cdot \|OQ\| \cdot \sin(\angle POQ) \). We have that \( \min_{p \in P_G} \varphi(p, q) = -\|OP\| \cdot R \) and \( \max_{q \in T} \varphi(p, q) = \|OQ\| \cdot R \). Thus, even though \( \varphi \) is bilinear we have,

\[
\max \min_{p \in P_G} \varphi(p, q) = -rR < rR = \min \max_{q \in P_G} \varphi(p, q).
\]

The Minimax theorem thus fails over \( G \). However, the Minimax theorem holds over latent mixtures as \( O \in \Delta(G) \).

### 2.3. Exploiting Latent Games’ Structure

Recasting Example 3 as a latent game yields a latent GAN that closely resembles the standard GAN formulation proposed by Goodfellow et al. (2014), with the difference that the second player controls a latent discriminator (instead of a pure discriminator).

**Example 3 (Latent GANs).** Let \( G : \mathbb{R}^p \rightarrow \mathbb{R}^d \) a generator and a latent discriminator \( \psi : \mathbb{R}^d \times \mathbb{R}^q \rightarrow \mathbb{R} \), the payoff \( \varphi(\psi, G) \) of the latent GAN is defined as,

\[
\sum_{x \sim \pi_{data}} \ln(\sigma(\psi(x, z))) + \sum_{z' \sim \pi'} \ln \left(1 - \sigma(\psi(G(z'), z))\right),
\]

where \( \pi \) and \( \pi' \) are latent distributions.

However, the structure of discriminators in GANs implies that there is a “reduction” from mixed to pure discriminators:

**Proposition 1.** For any latent discriminator \( \psi \) there exists a non-latent discriminator \( D \), independent of \( G \), such that,

\[
\varphi(\psi, G) \leq \varphi(D, G), \quad \forall G \in \mathcal{G},
\]

where \( D(x) := \mathbb{E}_{z \sim \pi_{data}} \psi(x, z) \) is the mean of \( \psi \) over \( z \).

The result follows from the fact that \( \varphi(\cdot, G) \) is concave for any \( G \in \mathcal{G} \). Note that this is not the case for the generator, i.e., for most of \( \psi \), the payoff \( -\varphi(\psi, \cdot) \) is not concave with respect to the mapping \( G \). For such latent games with concave payoff, we can define a new notion of latent game where the first player does not need to use a latent mapping (since any latent mapping is dominated by a pure strategy).

**Definition 3 (Semi-latent Game).** A semi-latent game \((\varphi, D, G)\) is a two-player zero-sum game for which

\[
\varphi(\cdot, g) \text{ is concave for any } g \in \mathcal{G}.
\]

The first player plays “pure”-strategies \( D \in \mathcal{D} \subset A \) and the second plays latent mappings \( g \in \mathcal{G} \). The payoff is

\[
\varphi(D, g) := \mathbb{E}_{z \sim \pi} \varphi(D, g(z))
\]

where \( \pi \) is the latent distribution for \( g \).

Notice that the set of pure strategies \( \mathcal{D} \) is only a subset of the whole strategy space \( A \). For instance, as in Example 3, when the strategy space is a function space it might not be reasonable to assume that the discriminator can pick any function. In practice, the first player often has access to a set of parametrized functions. For simplicity, we assume that, in semi-latent games, \( \mathcal{D} \) is a space of functions.

It is the case for instance for GANs, which are arguably the most prominent example in machine learning of a semi-latent game: the set \( \mathcal{D} \) corresponds to a class of discriminators and \( G \) to a class of generators (often neural networks with a given architecture).
Similarly to how latent games were denoted by a triplet\((\varphi, \mathcal{F}, \mathcal{G})\), to emphasize the importance of the limited-capacity classes of mappings\(\mathcal{F}\) and\(\mathcal{G}\), a semi-latent game is denoted\((\varphi, \mathcal{D}, \mathcal{G})\). The subtle difference is that\(\mathcal{D}\) corresponds to subset of the whole strategy space\(A\) (assumed to be a function space) while\(\mathcal{G}\) corresponds to a class of latent mappings from the latent space\(Z\) to the strategy space\(B\).

By concavity of\(\varphi(\cdot, g)\), any mixed strategy\(p\) of element of\(\mathcal{D}\) is dominated by its average strategy\(\text{Avg}(\mathcal{D})(p)\) over the mixture. Define the strong convex hull of\(\mathcal{D}\) as

\[
\text{Avg}(\mathcal{D}) := \{\text{Any average of strategies from }\mathcal{D}\}. \tag{7}
\]

Even if latent mixtures\(\mathcal{F}\) and averages\(\mathcal{G}\) are both defined using a class of mappings/functions, they significantly differ. While\(\mathcal{G}\) can be understood as a set of point-wise average of functions, the set of latent mixtures\(\mathcal{F}\) can be seen as a set of convex combinations of probability distributions induced by latent mappings.

In semi-latent games the first player leverages the concavity of its payoff to only play pure strategies. Because any mixed strategy is dominated by its average strategy, the Minimax theorem also holds for semi-latent games, only considering averages of strategies for the first player:

**Theorem 2.** Let\((\varphi, \mathcal{D}, \mathcal{G})\) be a semi-latent game. The Minimax theorem holds when considering averages of strategies for the first player and latent mixtures of mappings for the second one, i.e.,

\[
\max_{D \in \text{Avg}(\mathcal{D})} \min_{q \in \Delta(\mathcal{G})} \varphi(D, q) = \min_{q \in \Delta(\mathcal{G})} \max_{D \in \text{Avg}(\mathcal{D})} \varphi(D, q)
\]

where\(\Delta(\mathcal{G})\) and\(\text{Avg}(\mathcal{D})\) are respectively defined in\(\mathcal{F}\) and\(\mathcal{G}\). We call this value\(\mathcal{V}_{\text{SL}}\).

A meaningful notion of equilibrium can be defined in latent and semi-latent games via a minimax theorem considering an analogous (but different) notion of convex combination. In the following, we will focus on latent game but our results can be extended to semi-latent games.

### 3. Latent Games in Machine Learning

In latent games\((\varphi, \mathcal{F}, \mathcal{G})\) the players use mappings to play the game. In an ML context, these mappings are often parametrized by finite dimensional variables,

\[
\mathcal{F} := \{f_w : w \in \mathbb{R}^{p_1}\} \quad \text{and} \quad \mathcal{G} := \{g_\theta : \theta \in \mathbb{R}^{p_2}\}.
\]

Thus, the payoff can be seen as a function of\(w\) and\(\theta\),\(\varphi(w, \theta) := \varphi(f_w, g_\theta)\), and consequently there is an equivalence between the minimax theorem in the parameter space and that in the mapping space,

\[
\max_{\theta \in \mathbb{R}^{p_2}} \min_{w \in \mathbb{R}^{p_1}} \varphi(w, \theta) = \max_{f \in \mathcal{F}} \min_{g \in \mathcal{G}} \varphi(f, g)
\]

and

\[
\min_{w \in \mathbb{R}^{p_1}} \max_{\theta \in \mathbb{R}^{p_2}} \varphi(w, \theta) = \min_{g \in \mathcal{G}} \max_{f \in \mathcal{F}} \varphi(f, g).
\]

Even if one can design parametric non-convex-non-concave games where pure global minimax do not exist\((\text{Jin et al., 2019})\), latent games provide a framework where there is hope to show (approximate) minimax theorems even when the function\(\varphi\) is highly non-convex-non-concave with respect to the parameters\(w\) and\(\theta\), e.g.,\(f_w\) and\(g_\theta\) correspond to (deep) neural networks. By working in the mapping space, we prove in\textsection 4.3 that ReLU neural networks can achieve approximate global minimax equilibria.

**Approximate minimax for latent games.** A notion of an approximate solution for\(\mathcal{E}\) can be defined using\(\epsilon\)-safety sets. Such sets contain the agents that are\(\epsilon\)-close to achieving the value of the game.

**Definition 4 (\(\epsilon\)-Latent Equilibrium).** Let\((\varphi, \mathcal{F}, \mathcal{G})\) be a latent game. The\(\epsilon\)-safety sets\(\mathcal{F}_\epsilon^*\) and\(\mathcal{G}_\epsilon^*\) are the sets of\(\epsilon\) approximate solutions of\(\mathcal{E}\),

\[
\Delta(\mathcal{F})_\epsilon^* := \{p^* \in \Delta(\mathcal{F}) : \min_{q \in \Delta(\mathcal{G})} \varphi(p^*, q) \geq V_L - \epsilon\}
\]

and

\[
\Delta(\mathcal{G})_\epsilon^* := \{q^* \in \Delta(\mathcal{G}) : \max_{p \in \Delta(\mathcal{F})} \varphi(p, q^*) \leq V_L + \epsilon\}.
\]

An\(\epsilon\)-equilibrium is a pair\((p_\epsilon^*, q_\epsilon^*) \in \Delta(\mathcal{F})_\epsilon^* \times \Delta(\mathcal{G})_\epsilon^*\).

It can be shown that such approximate solutions of\(\mathcal{E}\) can be achieved with finite latent mixture of\(K\) mappings. Considering\(f_k \in \mathcal{F}\) that generates the distribution\(p_{f_k}\),\(1 \leq k \leq K\) (see Eq.3), we aim at finding the smallest mixture that is an approximate solution of the minimax

\[
K_\epsilon^\mathcal{F} := \text{Smallest } K \in \mathbb{N} \text{ s.t. } \sum_{k=1}^K \lambda_k p_{f_k} \in \Delta(\mathcal{F})_\epsilon^*.
\]

Our goal is to provide a precise bound that depends on\(\epsilon\) and on some properties of the classes\(\mathcal{F}\) and\(\mathcal{G}\).

**Theorem 3.** Let\((\varphi, \mathcal{F}, \mathcal{G})\) be a latent game. If\(\mathcal{F}\) and\(\mathcal{G}\) are compact and\(\varphi\) is\(L\)-Lipschitz then,

\[
K_\epsilon^\mathcal{F} \leq \frac{4L^2 \text{diam}(\mathcal{F})^2}{\epsilon^2} \ln(N(\mathcal{G}, \frac{\epsilon}{2L}))
\]

where\(N(\mathcal{G}, \epsilon)\) is the number of\(\epsilon\)-balls necessary to cover\(\mathcal{G}\). We have a similar bound for\(K_\epsilon^\mathcal{G}\).

The quantity\(N(\mathcal{G}, \epsilon)\) is called covering number of\(\mathcal{G}\) in the literature. When\(\mathcal{G}\) is compact, it is a finite complexity measure of the class\(\mathcal{G}\) that has been extensively studied in the context of generalization bounds\((\text{Mohri et al., 2012; Shalev-Shwartz and Ben-David, 2014})\).

Note that in practice we expect\(K_\epsilon^\mathcal{F}\) to be small. For instance in the context of\(\text{GANs}\), if the class of discriminators contains the constant function\(D(\cdot) = 0\), then this function belongs to a Nash since\(\varphi(D, \cdot) = 0\) and thus\(K_\epsilon^\mathcal{F} = 1\).

Roughly, the number\(K_\epsilon^\mathcal{F}\) expresses to what extent the set of distributions induced by the mappings in\(\mathcal{F}\) has to be ‘convexified’ to achieve an approximate equilibrium.
4. Achieving Pure-Nash with Neural Nets

We showed above that approximate equilibria in latent games can be achieved with finite latent mixtures. In this section, we investigate how it is possible to achieve pure-Nash with neural networks.

Let $\text{NN}_p$ be the set of two-layer neural networks with $p$ hidden neurons and ReLU non-linearities. Formally, function $g : \mathbb{R}^d \rightarrow \mathbb{R}^{d_{\text{out}}}$ in $\text{NN}_p$ can be written as,

$$g(x) = \sum_{i=1}^{p} a_i \text{ReLU}(c^T_i x + d_i) + b_i$$

where $a_i, b_i \in \mathbb{R}^{d_{\text{out}}}$, $c_i \in \mathbb{R}^d$, $d_i \in \mathbb{R}$, and $\text{ReLU}(x) = \max(x, 0)$. We present two results on the representative power of neural networks. The first concerns mixtures of latent mappings (5) for $\mathcal{G} = \text{NN}_p$ and the second concerns averages of strategies (7) for $\mathcal{D} = \text{NN}_p$.

4.1. Neural Nets represent mixtures of smaller Nets

When working with a latent mapping $g$, we are interested in the probability distribution $p_g$ induced by $g$. In this section, we consider $g \in \text{NN}_p : [0, 1] \rightarrow \mathbb{R}$ and the uniform prior distribution $U([0, 1])$,

$$a \sim p_g : a = g(z) \text{ where } z \sim U([0, 1]) \text{ and } g \in \text{NN}_p.$$  

We show that a latent mixture of two neural networks can be represented by a single, wider neural network.

**Proposition 2.** Any latent mixture (5) of two latent mappings $g_1, g_2 \in \text{NN}_p$ can be approximated to arbitrary precision with a single latent mapping $g_3 \in \text{NN}_{2p+1}$. Moreover, if there exist $x, x' \text{ s.t. } g_1(x) = g_2(x')$ then the latent mixture between $g_1$ and $g_2$ can be represented by a latent mapping $g_3 \in \text{NN}_{2p}$.

The proposition naturally extends to any finite latent mixture of mappings. Fig. 3b (in §E) illustrates how $g_3$ is constructed.

Unlike the universal approximation theorem, Prop. 2 shows that latent mixtures can be exactly represented by a single neural network. On the one hand, when one wants to approximate an arbitrary continuous function the number of required hidden units may be prohibitively large (Lu et al., 2017) as the error $\epsilon$ vanishes. On the other hand, the size of $g_3$ in Prop. 2 does not depend on any vanishing quantity.

The high level insight is that mixtures of ReLU neural nets can be represented by a larger ReLU neural network with a width that grows linearly with the size of the mixture or the width of the smaller ReLU networks. In §4.3 we leverage this representation property to prove that ReLU neural nets can achieve approximate pure equilibria in latent games.

4.2. Neural Nets represent average of smaller Nets

If we consider averages of functions as described in (7), we can show that point-wise averages of functions $f \in \text{NN}_p : \mathbb{R}^d \rightarrow \mathbb{R}^{d_{\text{out}}}$ can be represented by a wider neural network.

**Proposition 3.** Any point-wise average of two functions in $\text{NN}_p$ can be represented by a single function of $\text{NN}_{2p}$.

This property is illustrated in Figure 3a and extends naturally to any finite point-wise average of functions. Similarly as the Prop. 2, Prop. 3 is a representation theorem that we will leverage to show that ReLU neural nets can achieve approximate pure equilibria in semi-latent games.

4.3. Minimax Theorem for Neural Networks

Prop. 2 and 3 give interesting insights about the representative power of ReLU neural nets: as their width grows, ReLU nets can express larger latent mixtures/averages of sub-nets.

Combining these properties with Theorem 3, we show that approximate pure equilibria can be achieved with a single ReLU neural net.

**Theorem 4.** Let $(\varphi, \text{NN}_p, \text{NN}_p)$ be a latent game. For any $\epsilon > 0$ there exists a pair of networks in $\text{NN}_p$ that achieve a pure $\epsilon$-equilibrium of a lower-capacity sub-game $(\varphi, \text{NN}_{p_s}, \text{NN}_{p_u})$. The capacity of the sub-game is at least

$$p_c \geq C \epsilon \sqrt{p}.$$  

where $C > 0$ contains log-dependencies in $p$ and $\epsilon$.

An explicit formula for $C$ is provided in §E. Theorem 4 shows that latent games played with a single-pair of agents using a standard fully connected architecture can achieve a notion of weaker-capacity-pure-equilibrium. This result differs from Arora et al. (2017, Theorem 4.3) who, only in the context of GANs, design a specific architecture to achieve an approximate equilibrium.

On the one hand, if $\epsilon \sqrt{p} < 1$, then the bound is vacuous (since $p_c$ is an integer). On the other, the width only needs to grow quadratically with $\epsilon$ to achieve a non-vacuous bound. Hence, a consequence of Theorem 4 is that highly overparameterized networks can provably represent an approximate high-capacity-pure-equilibrium.

5. Application: Solving Differentiable Blotto

We apply our latent game approach to solve a differentiable version of Colonel Blotto game (Example 2). We consider a continuous relaxation of the strategy space where $S_1 = S_2$.

After renormalization we have that $A_1 = A_2 = \Delta_K$, where $\Delta_K$ is the $K$-dimensional simplex. It is important to notice that in that case a pure strategy corresponds to a point on
Figure 2. Training of latent agents to play differentiable Blotto (8) with \(K = 3\). Right: The suboptimality corresponds to the payoff of the agent against a best response. We averaged our results over 40 random seeds, the transparent shades are the standard deviations.

(a) 5000 samples using the latent mapping \(f\) and \(g\) after 0, 400, and 800 training steps. Their respective suboptimality along training has a value of 1.5, 1.2, and .5.

(b) Performance and convergence of the agents.

the simplex and a mixture of strategies corresponds to a distribution over the simplex. We replace the payoff (4) of Latent Botto by a differentiable one,

\[
\varphi(f, g) := \mathbb{E}_{z \sim \pi, z' \sim \pi'} \left[ \frac{1}{K} \sum_{k=1}^{K} \sigma([f(z) - g(z')]_k) \right] \tag{8}
\]

where \(\sigma\) is a sigmoid minus 1/2 and \(f, g : \mathbb{R}^p \rightarrow \Delta_K\). This game is studied by Ferdowsi et al. (2018) when \(S_1 > S_2\).

For the latent mappings, \(f\) and \(g\) we considered dense ReLU networks with 4 hidden layers, 16 hidden units per layer and a \(K\)-dimensional softmax output. We use a 16-dimensional Gaussian prior for the latent variable. We trained our agents using Alg. 1 with the Adam optimizer (Kingma and Ba, 2015) with \(\beta_1 = .5\) and \(\beta_2 = .99\) as oracle.

Finally, it is instructive to discuss the relative merits of the ‘true’ minimax equilibrium of game and its limited-capacity version that occurs in latent games. On the one hand, a Nash equilibrium always exists for two-player zero-sum games but this quantity may not be realistically tractable, e.g., in GANs, the optimal infinite-capacity generator is supposed to represent the distribution of ‘real world images’. In a similar vein, simple statements such as ‘are there two Nashes?’ or ‘is there a Nash that contains the strategy s?’ are NP-hard problems for two-player games (Gilboa and Zemel, 1989). If such concepts are hard to compute it seems unrealistic to expect limited capacity agents, such as humans or computers, to find equilibria (Papadimitriou, 2007).

On the other hand, our work shows that some latent equilibria can be efficiently approximated when working with neural networks. Latent equilibria capture the notion that agents–and humans–that play complex games have a limited capacity. It seems more realistic to play complex games such as Poker of StarCraft II that are multi-step with imperfect information. Thus, latent equilibria are a compelling solution concept that seem more practical than infinite capacity Nash equilibria.

6. Discussion

Solution concepts like von Neumann’s minmax theorem and Nash equilibria are fundamental to game theory because they provide meaningful targets against which performance can be measured. Loosely, the minmax theorem provides a notion of optimal behavior in two-player zero-sum games. However, games differ radically from optimization problems: equilibria can require mixed strategies whereas optima are always pure. Concretely, it is obvious that all three pure strategies in rock-paper-scissors are brutally exploitable. How, then, can single neural nets so regularly find meaningful solutions to zero-sum games like GANs?

In this work, we partially explain why practitioners have successfully trained pure neural nets in games like GANs by showing that a single pair of ReLU neural networks can achieve an notion of limited-capacity-equilibrium.

The intuition underlying our theorems is as follows. Neural nets have a particular structure that interleaves matrix multiplications and simple nonlinearities (often based on the max operator like ReLUs). The matrix multiplications in one layer of a neural net compute linear combinations of functions encoded by the other layers. In other words, neural nets are (non)linear mixtures of their sub-networks.
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APPENDIX

A. Interpretation of Equilibria in Latent Games

In latent games, players embed in mapping spaces in order to solve the game. When we consider a standard normal form game $\varphi$ that we try to solve using mappings to approximate mixtures of strategies, we are actually playing a limited capacity version of the game that heavily depends on the expressivity of the mappings in the classes $F$ and $G$.

Such a limitation may be interpreted as limitations on the skills of the players. It intuitively makes sense that such limitations would change the optimal way to play the game: the optimal way to play StarCraft II is different for players that can perform 10 versus 100 actions per second. Thus, if the goal is to train agents to compete with humans, one needs to set a class $G$ that (roughly) corresponds to human skills.

Similarly, in the context of Generative Adversarial Networks, it has been argued that setting a restricted function class for the discriminator provides a more meaningful loss and describes an achievable learning task for the generator $\text{Arora et al. (2017); Huang et al. (2017)}$. The final task is to generate pictures that are realistic according to the human metric. Such task is way looser – and thus easier to achieve – than for instance minimizing the KL divergence or the Wasserstein distance between the real data distribution and the generated distribution.

To sum-up, the equilibrium of a latent game provides a notion of limited-capacity-equilibrium that can define a target that correspond to agents with expressive and realistic behavior. In many tasks, our goal is to train agents that outperform human using human realistic limitations: it is important to constrain the agent in order to prevent it to play $10^5$ actions per minute but it is also important to constrain its opponent because we would like opponent to try to exploit the main agent in a semantically meaningful way and not by designing very specific ‘adversarial example’ strategies – e.g., very precise positions of units that breaks the vision system of the main agent – that a human player could not perform.

B. Latent Mixture of Mapping

Let us first recall that given a latent space $Z$, a distribution $\pi$ over this latent space and a latent mapping $f : Z \to A$ we can generate

$$a \sim pf : a = f(z), z \sim \pi. \quad (9)$$

Let us denote $\mathcal{P}(A)$ the set of Borel measures on $A$. For any set $X$, we can define a notion of convex combination of elements of $X$ using a measure $\mu \in \mathcal{P}(X)$,

$$\bar{x} := \int_X x d\mu(x) \quad (10)$$

For a given class of mappings $g \in G$, we define the set of latent mixture of mappings as the set of ‘convex combination’ of the distribution belonging to $\{p_g : g \in G\}$,

$$\Delta(G) := \left\{ \int_G p_g d\mu(g) : \mu \in \mathcal{P}(G) \right\} \quad (11)$$

Note that very importantly, this construction is different from considering the set of distribution with support $G$ because for any $p \in \Delta(G)$ there might exist many $\mu \in \mathcal{P}(G)$ such that $p = \int_G p_g d\mu(g)$.

**Proposition 4.** If $G$ is compact then $\Delta(G)$ is compact.

**Proof.** From (Aliprantis and Border, 2006, Chap. 15) if $G$ is compact then the set $\mathcal{P}(G)$ is compact. Finally $\Delta(G)$ is the image of $\mathcal{P}(G)$ by a continuous application and thus is compact. \hfill $\Box$

Similarly, if $D$ is compact, then $\text{Avg}(D)$ is compact.

C. Proof of results from Section 2

**Proposition 1.** For any latent discriminator $\psi$ there exists a non-latent discriminator $D$, independent of $G$, such that,

$$\varphi(\psi, G) \leq \varphi(D, G), \quad \forall G \in G,$$

where $D(\cdot) := \mathbb{E}_{z \sim \pi_X[i]}[\psi(z, z)]$ is the mean of $\psi$ over $z$.

**Proof.** This proposition results from the concavity of $\ln(\sigma(\cdot))$ and $\ln(1 - \sigma(\cdot))$. For the proof sketch let us consider a mixture $p_D$ of two feature maps such that

$$\mathbb{E}_{\psi \sim p_D}[\psi] = \bar{\psi}$$

Then, $\varphi$ is defined is (3) has the following property

$$\varphi(x, p) := \mathbb{E}_{\psi \sim p_D}[\varphi(x, \psi)] \quad (11)$$

$$= \frac{1}{n} \sum_{i=1}^n \mathbb{E}_{\psi \sim p_D}[\ln(\sigma(\psi(x_i)))] \quad (12)$$

$$= \frac{1}{n} \sum_{i=1}^n \ln(\sigma(\bar{\psi}(x_i))) + \ln(1 - \sigma(\bar{\psi}(x_i)))$$

$$\leq \frac{1}{n} \sum_{i=1}^n \ln(\sigma(\bar{\psi}(x_i))) + \ln(1 - \sigma(\bar{\psi}(x_i)))$$

$$\varphi(x, \bar{\psi}). \quad (13)$$
where the inequality comes from the concavity of $\ln(\cdot)$ and $\ln(1 - \sigma(\cdot))$. \qedhere

In its general form Minimax Sion’s theorem can be formulated as follows,

**Theorem 5 (Minimax theorem (Sion et al., 1958)).** If $U$ and $V$ are convex and compact sets and if the sublevel sets of $\varphi(\cdot, v)$ and $-\varphi(u, \cdot)$ are convex then,

$$\max_{u \in U} \min_{v \in V} \varphi(u, v) = \min_{u \in U} \max_{v \in V} \varphi(u, v)$$ \hspace{1cm} (14)

This theorem combined with Proposition 4 proves Theorems 1 and 2.

**Proof of Theorem 1 and 2.** We just need to verify the hypothesis of Theorem 5.

In the case of Theorem 1, the function $\varphi$ is convex-concave (actually bilinear by construction) with respect to the the distributions belonging to $\Delta(G)$ and $\Delta(F)$. By Proposition 4 these sets are compact and convex by construction.

In the case of Theorem 2, by assumption $\varphi(\cdot, g)$ is concave and $\text{Avg}(D)$ is compact and convex. The rest is similar as in the latent case. \qed

**D. Proof of Theorems from Section 3**

Let us first prove a lemma

**Lemma 1.** $\text{diam}(\Delta(F)) = \text{diam}(F)$.

**Proof.** Since $F \subset \Delta(F)$, we have that $\text{diam}(\Delta(F)) \leq \text{diam}(F)$. Let us show the other inequality. Let us consider $f_1, f_2 \in \Delta(F)$, then we have

$$\|u - v\| = \left\| \int_F p_f d\mu_1(f) - \int_F p_f d\mu_2(f') \right\|$$ \hspace{1cm} (15)

$$\leq \int_F \left\| p_f - p_f' \right\| d\mu_1(f) d\mu_2(f')$$ \hspace{1cm} (16)

$$\leq \int_F \int_F D d\mu_1(f) d\mu_2(f')$$ \hspace{1cm} (17)

$$= D$$ \hspace{1cm} (18)

where $D := \text{diam}(U)$. The first inequality comes from the triangle inequality, the second one comes from the definition of the diameter and the third one from Fubini and the fact that $\mu_1$ and $\mu_2$ are probability measures. \qed

**Theorem 3.** Let $(\varphi, F, G)$ be a latent game. If $F$ and $G$ are compact and $\varphi$ is $L$-Lipschitz then,

$$K^F_{\varphi} \leq \frac{4L^2 \text{diam}(F)^2}{\epsilon^2} \ln(N(G, \frac{\epsilon}{2L}))$$

where $N(G, \epsilon)$ is the number of $\epsilon$-balls necessary to cover $G$. We have a similar bound for $K^G_{\varphi}$.

**Proof.** This proof is inspired from the proof of (Lipton and Young, 1994, Theorem 2) and (Arora et al., 2017, Theorem B.3).

Since $G$ is a compact it has a finite covering number $N(G, \epsilon)$. It is the smallest number of $\epsilon$ balls needed to cover $G$ (see (Mohri et al., 2012) for details about covering number in the context of machine learning).

Let us consider $q_i, 1 \leq i \leq N(G, \frac{\epsilon}{2L})$ the center of these balls.

Let us consider $(p^*, q^*)$ an equilibrium of $(\varphi, F, G)$ (does exists because of Theorem 1).

The mixture $p^*$ can be written as,

$$p^* = \int_F p_f d\mu(f).$$ \hspace{1cm} (19)

where $\mu \in \mathcal{P}(F)$. Now let us consider the mixture

$$p^*_n := \frac{1}{n} \sum_{k=1}^{n} p_{f_k}$$ \hspace{1cm} (20)

where $f_k, 1 \leq k \leq n$ are independently drawn from $\mu$.

Using Hoeffding’s inequality, for any $q_i, 1 \leq i \leq N(G, \epsilon)$ we have that,

$$\mathbb{P}(\varphi(p^*_n, q_i) - \varphi(p^*, q_i) < \epsilon/2) \leq e^{-\frac{n\epsilon^2}{2L^2}}$$ \hspace{1cm} (21)

where $D = \text{diam}(F)$. Thus using standard union bounds,

$$\mathbb{P}\left(\varphi(p^*_n, q_i) - \varphi(p^*, q_i) < \epsilon/2, \forall 1 \leq i \leq N(G, \frac{\epsilon}{2L})\right) \leq N(G, \frac{\epsilon}{2L})e^{-\frac{n\epsilon^2}{2L^2}}$$ \hspace{1cm} (22)

Let us now consider

$$\hat{g}_n \in \arg \min_{g \in G} \varphi(p^*_n, g).$$ \hspace{1cm} (23)

Note that $\min_{g \in G} \varphi(p^*_n, g) = \min_{g \in \Delta(G)} \varphi(p^*_n, g)$ since a pure strategy always achieve that minimum. We have that $q_i$, is an $\epsilon/(2L)$ covering there exist an index $i$ such that

$$\varphi(p^*_n, q_i) \leq \varphi(p^*_n, \hat{g}_n) + \epsilon/2.$$ \hspace{1cm} (24)

and we know that

$$\varphi(p^*, q) \geq \varphi(p^*, q^*) = V_L, \forall q \in \Delta(G).$$ \hspace{1cm} (25)

Thus for $n > \frac{2L^2 \epsilon^2}{\epsilon^2} \ln(N(G, \frac{\epsilon}{2L}))$ we have,

$$\mathbb{P}(\varphi(p^*_n, q) - \varphi(p^*, q^*) < \epsilon, \forall q \in \Delta(G)) < 1$$ \hspace{1cm} (26)
Minimax Theorem for Latent Games

(a) ReLU networks of width $p$ can represent any $p$-piecewise linear function. The point-wise average $f_3$ of two ReLU neural networks $f_1$ and $f_2$ is a piecewise linear function that can be represented by a wider ReLU neural network.

Thus, Proposition 3. Any point-wise average of two functions in $\mathbb{R}^p$ can be represented by a single function of $\mathbb{R}_{2p}$.

Proof. We will prove this result for a pointwise average of $K$ functions of $\mathbb{R}^p$. Let us recall that a two-layers ReLU network of width $p$ can be written as

$$g(x) = \sum_{i=1}^{p} a_i \text{ReLU}(c_i^T x + d_i) + b_i$$

where $a_i, b_i \in \mathbb{R}^{d_{out}}$, $c_i \in \mathbb{R}^d$ and $d_i \in \mathbb{R}$. Then, let us consider $K$ such elements of $\mathbb{R}^p$, then any convex combination of these $K$ function can be written as,

$$f(x) = \sum_{k=1}^{K} \sum_{i=1}^{p} \lambda_k (a_{i,k} \text{ReLU}(c_{i,k}^T x + d_{i,k}) + b_{i,k})$$

where $\lambda_k \geq 0$, $1 \leq k \leq K$ and $\sum_{k=1}^{K} \lambda_k = 1$.

Setting $\tilde{a}_{i,k} := \lambda_k a_{i,k}$ and $\tilde{b}_{i,k} := \lambda_k b_{i,k}$, we have that

$$f(x) = \sum_{(i,k) \in [p] \times [K]} \tilde{a}_{i,k} \text{ReLU}(c_{i,k}^T x + d_{i,k}) + \tilde{b}_{i,k}$$

which belongs to $\mathbb{R}^{2p}$.

(b) Latent mixture of $g_1$ and $g_2$: For $k \in \{1, 2, 3\}$ the transformation $g_k$ maps the uniform distribution on $[0, 1]$ into a distribution on the output space: $x \sim p_y$, if $x = g_k(z), z \sim U([0, 1])$. In that case, $p_y$ and $p_y$ are respectively the uniform distribution over $I_1$ and $I_2$. The function $g_3$ represents a distribution that puts half of its mass uniformly on $I_1$ and the other half on $I_2$.

\[ \varphi(p^*_n, q) \geq \epsilon + V_L \]  \hspace{1cm} (27)

Thus,

\[ K_{\epsilon}^F \leq 4L^2D^2 \epsilon^2 \ln(N(G, \frac{\epsilon}{2\epsilon})) \]  \hspace{1cm} (28)

\[ \square \]

E. Proof of Theorem from Section 4

Proposition 3. Any point-wise average of two functions in $\mathbb{R}^p$ can be represented by a single function of $\mathbb{R}_{2p}$.

Proof. This proposition is related to the study of ReLU neural networks. See for instance (Hanin, 2019) and (Arora et al., 2018)

Definition 5. A continuous real $p$-piecewise linear function $f : [0, 1] \to \mathbb{R}$ is a piecewise linear function with $p$ pieces, i.e., there exists $0 =: x_0 < x_1 < \ldots < x_{p+1} := 1$ and $a_i, b_i \in \mathbb{R}$ such that

\[ f(x) = \sum_{i=0}^{p} 1\{x_i \leq x < x_{i+1}\}(a_i x + b_i). \]  \hspace{1cm} (32)

In this definition we assume that $f$ is continuous and thus that $a_i x_{i+1} + b_i = a_{i+1} x_{i+1} + b_{i+1}$, $0 \leq i \leq p$.

Lemma 2. For all $x_i < x_{i+1}$ and $a, b \in \mathbb{R}$ such that $a x_i + b = 0$, there exists $a_i, b_i, a, b_i, d_i, d_i$ such that

\[ a_i [c_i x + d_i]_+ + b_i = \begin{cases} 0 & \text{if } x < x_i \\ ax + b & \text{if } x_i \leq x \leq x_{i+1} \end{cases} \]  \hspace{1cm} (33)

Proof. We set $d_i = -x_i, c_i = 1, b_i = b$ and $a_i = a$. \hspace{1cm} \square

Theorem 6. (Arora et al., 2018) A two-layer ReLU network of width $p$ is a $p+1$-piecewise linear function $f : [0, 1] \to \mathbb{R}$. Conversely, Any $p+1$-piecewise linear function can be written as a two-layer ReLU network of width $p+1$.

Proof. The function encoded by a two-layers ReLu network...
of width $p$ is
\begin{equation}
g(x) = \sum_{i=1}^{p} a_i [c_i x + d_i]_+ + b_i ,
\end{equation}
where $[\cdot]_+$ stands for $\max(0, \cdot)$. A maximum of a $p$-piecewise linear function and $0$ is at most a $p+1$-piecewise linear function and a sum of a $p$-piecewise linear function and a $q$-piecewise linear function is at most $p+q-1$-piecewise linear. Thus $a_i [c_i x + d_i]_+ + b_i$ is a 2-piecewise linear function and thus $g(x)$ is a $p+1$-piecewise linear function.

Let $f$ be a $p+1$-piecewise-linear function. We have that there exists $0 =: x_0 < x_1 < \ldots < x_{p+1} := 1$ such that
\begin{equation}
f(x) = \sum_{i=0}^{p} 1\{x_i \leq x < x_{i+1}\}(a_i x + b_i).
\end{equation}
In order to prove this result we will proceed by induction. Let us assume we have built a two-layers ReLU network of width $j$ such that $g_j = \sum_{i=1}^{j} a_i [c_i x + d_i]_+ + b_i$ and,
\begin{equation}
g_j(x) = f(x) , \quad \forall x \leq x_j .
\end{equation}
Using the convention that $\sum_{i=0}^{0} a_i [c_i x + d_i]_+ + b_i = 0$ this result is true for $j = 0$.

Let us assume that this result is true for $j \geq 0$. Then if we consider the function $f - g_j$ we have that, by induction hypothesis $f(x_j) - g_j(x_j) = 0$ and by construction of $g_j$, this function is linear on $[x_j, x_{j+1}]$. Thus,
\begin{equation}
(f - g_j)(x) = ax + b , \quad \forall x_j \leq x < x_{j+1}
\end{equation}
Thus we can apply Lemma 2 to show that there exists $a_j, b_j, c_j, d_j$ such that,
\begin{equation}
a_j [c_j x + d_j]_+ + b_j = \begin{cases} 
0 & \text{if } x < x_j \\
ax + b & \text{if } x_j \leq x < x_{j+1}
\end{cases}
\end{equation}
Setting $g_{j+1}(x) = g_j(x) + a_j [c_j x + d_j]_+ + b_j$, we have that,
\begin{equation}
g_{j+1}(x) = f(x) , \quad \forall x < x_{j+1},
\end{equation}
and using the fact that $a_j x_{j+1} + b_j = a_{j+1} x_{j+1} + b_{j+1}$ we have that the induction hypothesis is verified for $j+1$:
\begin{equation}
g_{j+1}(x) = f(x) , \quad \forall x \leq x_{j+1}.
\end{equation}

\begin{proposition}
Any latent mixture (5) of two latent mappings $g_1, g_2 \in \mathbb{NN}_p$, can be approximated to arbitrary precision with a single latent mapping $g_3 \in \mathbb{NN}_{p+1}$. Moreover, if there exist $x, x'$ s.t. $g_1(x) = g_2(x')$ then the latent mixture between $g_1$ and $g_2$ can be represented by a latent mapping $g_3 \in \mathbb{NN}_2$.
\end{proposition}
\begin{proof}
We will prove the first part of this theorem for an arbitrary number $K$ of mappings. Let $g$ be a two-layers ReLU network of width $p$, the probability distribution $\pi_g$ induced by $g$ verifies,
\begin{equation}
\pi_g(S) = \ell(g^{-1}(S)) , \quad \forall S \text{ measurable in } [0, 1]^d .
\end{equation}
where $\ell$ is the Lebesgue measure on $[0, 1]^d$. By Theorem 6 any two-layers ReLU network of width $p$ is $p$ piecewise linear and thus
\begin{equation}
g(x) = \sum_{i=0}^{p} 1\{x_i \leq x < x_{i+1}\}(a_i x + b_i).
\end{equation}
For any measurable $S \subset [0, 1]$ we have,
\begin{align}
\pi_g(S) &= \ell(g^{-1}(S)) \\
&= \ell(\cup_{i=0}^{p} \{x_i \leq x < x_{i+1} , a_i x + b_i \in S\}) \\
&= \sum_{i=0}^{p} \ell(\{x_i \leq x < x_{i+1} , a_i x + b_i \in S\})
\end{align}
where the third equality is due to the fact that the intersection of $[x_i, x_{i+1}]$ are empty. Thus the convex combination $\pi$ of $\pi_{g_1}, \ldots, \pi_{g_K}$ verifies,
\begin{equation}
\pi(S) := \left( \sum_{k=1}^{K} \lambda_k \pi_{g_k} \right)(S) = \sum_{1 \leq k \leq K} \lambda_k \ell(\{x_i, k \leq x < x_{i+1,k} , a_i x + b_i, k \in S\})
\end{equation}
Using the change of variable formula
\begin{equation}
\lambda(U) = \int_{U} \lambda dl(x) = \int_{\lambda(U)} d\ell(x) = \ell(\lambda U).
\end{equation}
We can notice that
\begin{align}
&\lambda(\{x_i \leq x < x_{i+1} , a_i x + b_i \in S\}) \\
&= \ell(\{x_i \leq x < \lambda x_{i+1} , \lambda^{-1} a_i x + b_i \in S\}) \\
&= \lambda(\{x_i + b \leq x < \lambda x_{i+1} + b , \lambda^{-1} a_i (x + b) + b_i \in S\})
\end{align}
Then, setting $c_k := \sum_{i=0}^{k-1} \lambda_k, \text{ and } \tilde{x}_{i,k} := \lambda_k x_i + c_k$ we get by construction that
\begin{equation}
\tilde{x}_{i,k} \leq \tilde{x}_{i+1,k}, 1 \leq i \leq p \quad \text{and} \quad \tilde{x}_{p+1,k} = c_k + 1 = \tilde{x}_{0,k+1}
\end{equation}
Thus, the function $f$ defined as
\begin{equation}
f(x) = \sum_{0 \leq i < P} 1\{\tilde{x}_{i,k} \leq x < \tilde{x}_{i+1,k}\}(\tilde{a}_{i,k} x + \tilde{b}_{i,k}) .
\end{equation}
where $\tilde{a}_{i,k} = \lambda_k^{-1} a_i$ and $\tilde{b}_{i,k} := \lambda_k^{-1} c_k + b_i$ exactly encodes the distribution $\pi$. However, this function may not be continuous at $\tilde{x}_{p+1,k-1} = \tilde{x}_{0,k}, 1 \leq k \leq K - 1$. 
Theorem 6 only applies to continuous piecewise linear function.

In order to construct a piecewise linear function that is close to $f$ we set $(K - 1) \min_{i,j,k,l} \frac{\epsilon}{\max_{i,j,k,l} |\tilde{x}_{i,j,k,l} - \tilde{x}_{j,k,l}|} > \epsilon > 0$ and we change the definition of $\tilde{x}_{p+1,k-1}$ and $\tilde{x}_{0,k}$ for $1 \leq k \leq K - 1$:

$$\tilde{x}_{p+1,k-1} := \epsilon k - \frac{\epsilon}{2(K - 1)}$$ and $$\tilde{x}_{0,k} := \epsilon k + \frac{\epsilon}{2(K - 1)}$$

We then just link the pieces of $f$, with

$$1\{\tilde{x}_{p+1,k} \leq x < \tilde{x}_{0,k+1}\} (\tilde{a}_{p+1,k} x + \tilde{b}_{p+1,k})$$

where $\tilde{a}_{p+1,k} = \frac{f(\tilde{x}_{0,k+1}) - f(\tilde{x}_{p+1,k})}{\epsilon}$ and $\tilde{b}_{p+1,k} = f(\tilde{x}_{p+1,k}) - \tilde{x}_{p+1,k} \frac{f(\tilde{x}_{0,k+1}) - f(\tilde{x}_{p+1,k})}{\epsilon}$.

Let us finally define the $(p+1)K - 1$-piecewise linear and continuous function $\tilde{f}$,

$$\tilde{f}(x) := f(x) + \sum_{k=1}^{K-1} 1\{\tilde{x}_{p+1,k} \leq x < \tilde{x}_{0,k+1}\} (\tilde{a}_{p+1,k} x + \tilde{b}_{p+1,k}).$$

Using the total variation distance between the probability measure $\pi$ and the measure induced by $\tilde{f}$ we have that

$$TV(\pi, \pi_{f}) = \sup_{S} |\pi(S) - \pi_{f}(S)|$$

$$\leq \sum_{k=1}^{K-1} f(\tilde{x}_{p+1,k} \leq x < \tilde{x}_{0,k+1}) = \epsilon$$

For the second part of this theorem, let us first show that any $g \in NN_{p}$ there exists an increasing $g' \in NN_{p}$ that represent the same distribution. Since we have

$$g(x) = \sum_{i=0}^{p} 1\{x_i \leq x < x_{i+1}\} (a_ix_i + b_i).$$

Let us call $I_i := [x_i, x_{i+1}]$ the function $g$ maps $I_i$ into the interval $J_i := g(I_i) = [y_i, y_{i+1}]$. Since $g$ is linear on $[x_i, x_{i+1}]$, the distribution induced by $g$ is the mixture of uniform distribution on $J_i$ with weight $x_{i+1} - x_i$. By ordering the $y_i$ in an increasing order that we call $y_i$ we can see that $g$ represents a mixture of uniform distribution on $[y_i, y_{i+1}]$ (let us call $\lambda_i > 0$ the weights of the mixture).

Consequently, if we define $\tilde{x}_0 = 0, \tilde{x}_{i+1} = \tilde{x}_i + \lambda_i$, the function that goes from $y_i$ to $y_{i+1}$ between $\tilde{x}_i$ and $\tilde{x}_{i+1}$ induces the same distribution as $g$. Formally,

$$\tilde{g}(x) := \sum_{i=0}^{p} 1\{\tilde{x}_i \leq x < \tilde{x}_{i+1}\} (\tilde{a}_i x_i + \tilde{b}_i).$$

where $\tilde{x}_0 = 0, \tilde{x}_{i+1} = \tilde{x}_i + \lambda_i, \tilde{a}_i = \frac{y_{i+1} - y_i}{\tilde{x}_{i+1} - \tilde{x}_i}$ and $\tilde{b}_i = -\tilde{x}_i \frac{y_{i+1} - y_i}{\tilde{x}_{i+1} - \tilde{x}_i}$. This function is a $p$-piecewise linear function and thus can be represented by a ReLU neural network of width $p$.

Let us now consider $g_1$ and $g_2$ two mappings such that there exist $x, x' \in \mathbb{R}$ such that $g_1(x) = g_2(x)$. Then we have

$$g_1([0, 1]) \cap g_2([0, 1]) \neq \emptyset$$

Since $g_1$ induce a mixture of uniform distribution on $p$-intervals (similarly as above) that form a partition of $g_1([0, 1])$ and $g_2$ induce a mixture of uniform distribution on $p$-intervals (similarly as above) that form a partition of $g_2([0, 1])$, the latent mixture of $g_1$ and $g_2$ induce a mixture of uniform distribution on $2p$-intervals (similarly as above) that form a partition of $g_1([0, 1]) \cup g_2([0, 1])$ that is an interval since $g_1([0, 1]) \cap g_2([0, 1]) \neq \emptyset$. By the same construction as (53) there exists a ReLU neural network of width $p$ that represents the mixture of $g_1$ and $g_2$.

\[ \square \]

**Theorem 4.** Let $(\varphi, NN_{p}, NN_{p})$ be a latent game. For any $\epsilon > 0$ there exists a pair of networks in $NN_{p}$ that achieve a pure $\epsilon$-equilibrium of a lower-capacity sub-game $(\varphi, NN_{p}, NN_{p})$. The capacity of the sub-game is at least

$$p_{c} \geq C \epsilon \sqrt{p}.$$