WEYL $n$-ALGEBRAS AND THE SWISS CHEESE OPERAD

NIKITA MARKARIAN

ABSTRACT. We apply Weyl $n$-algebras to prove formality theorems for higher Hochschild cohomology. We present two approaches: via propagators and via the factorization complex. It is shown that the second approach is equivalent to the first one taken with a new family of propagators we introduce.

INTRODUCTION

The present paper continues studies of Weyl $n$-algebras begun in [MT15, Mar17, Mar16]. We describe how these ideas can be applied to prove formality theorems, which are isomorphisms between higher Hochschild cohomology of polynomial algebras and Weyl $n$-algebras. The substantial part of this paper is rephrasing and generalization of the pioneer paper [Kon03], where the formality for usual Hochschild cohomology was firstly proved, in terms of Weyl $n$-algebras.

The construction from [Kon03] depends on choice of a propagator. There is another approach to formality via the factorization homology of Weyl $n$-algebras, which was implicitly stated and used in [Mar16]. We show, that for the usual Hochschild cohomology this formality is equivalent to the one introduced in [Kon03] but with a different propagator. Due to the geometric nature of this approach, all coefficients of this morphism are rational. It leads us to a surprising conjecture that a family of propagators we define gives formalities with rational coefficients.

Two approaches to the formality described in the present paper resemble two approaches to the Kontsevich integral of a knot. The first one using iterated integrals (see e.g. [CDM12, Part 3]) is similar to the approach via propagator. The second partly conjectural approach (see [Mar16] and references therein) corresponds to the one via the factorization complex.

First three section of the paper does not contain any new material. In the first and second sections, we recall basic definitions for the present series of papers of the Fulton–MacPherson operad, Weyl $n$-algebras and the factorization complex.

The third section is devoted to the notion of the Swiss Cheese operad, which was introduced in [Vor99, Kon99], and the higher Hochschild cohomological complex. A module over the Swiss Cheese operad is a triple of an $e_n$-algebra, an $e_{n-1}$-algebra and some additional data, which is referred to as an action of the $e_n$-algebra on the $e_{n-1}$-algebra. The main result of [Tho16] states that given an action of an $e_n$-algebra on an $e_{n-1}$-algebra, there is a morphism from this $e_n$-algebra to the higher Hochschild complex of $e_{n-1}$-algebra. We demonstrate, that for $n = 2$ if one takes the usual Hochschild cohomological complex as a model for higher Hochschild complex, the corresponding morphism of $L_\infty$-algebras is the one appeared in the proof of the formality theorem in [Kon03]. The Hochschild cohomological complex

The study has been funded within the framework of the HSE University Basic Research Program and the Russian Academic Excellence Project '5-100'.
introduced in in [Ger63] is a dg-Lie (not $L_\infty$!) algebra. It seems to be an important feature, that it is equipped with a pre-Lie algebra structure, which is not compatible with the differential. It would be highly interesting to find some explanation of the existence of such a model and discover some higher-dimensional generalization of it. This generalization must be a dg-Lie algebra model of the $L_\infty$-algebra of the higher Hochschild complex.

In the fourth section we construct a quasi-isomorphism between the Weyl $n$-algebra and the higher Hochschild cohomological complex of the polynomial algebra. This is a higher-dimensional generalization of the main construction of [Kon03] given in terms of Weyl $n$-algebras. As was mentioned in [Kon03] for $n = 2$, this construction works with any propagator. But [Kon03] and later papers explore merely the same propagator and its slight variations like in [RW]. The Conjecture 1 we formulate implies that there are other interesting propagators to work with.

In the fifth section following [Mar16] we use the factorization complex to build formality morphisms. These formalities turn out to be equal to the ones from the previous section for some particular propagators. The terms of these formality morphisms are given by integrals similar to the ones from [AS92].

The key point in the factorization complex approach for an $e_3$-algebra and 1-sphere (see also [Mar16]) is an isomorphism between the Hochschild cohomological complex of the polynomial algebra and the Hochschild homological complex of the $e_3$-algebra. The latter is equipped with the cyclic structure. But the quasi-isomorphism given by Proposition 13 does not respect it. Consequently, there is some interesting interaction between this structure and the formality given by the factorization complex approach. The paper [Mar16] may be considered as the first step in studying this interaction. Besides as was mentioned in [Kon03] and developed in later papers (see [RW] and references therein) the set of formalities for the usual Hochschild cohomological complex of a polynomial algebra is equipped with a rich additional structure such as the Grothendieck–Teichmüller Lie algebra action. The interaction of this structure and the structure mentioned above is a subject for future research.

Acknowledgments. I am grateful to D. Calaque, V. Dotsenko, B. Feigin, A. Khoroshkin, S. Merkulov, B. Shoikhet, D. Tamarkin and A. Voronov for fruitful discussions.

1. Weyl $n$-algebras

1.1. Fulton–MacPherson operad. Let $\mathbb{R}^n$ be an affine space. For a finite set $S$ denote by $(\mathbb{R}^n)^S$ the set of ordered $S$-tuples in $\mathbb{R}^n$. Let $\mathcal{C}^0(\mathbb{R}^n)(S) \subset (\mathbb{R}^n)^S$ be the configuration space of distinct ordered points in $\mathbb{R}^n$ labeled by $S$. In [GJ, Mar99] (see also [Sal01] and [AS92]) the Fulton–MacPherson compactification $\mathcal{C}(\mathbb{R}^n)(S)$ of $\mathcal{C}^0(\mathbb{R}^n)(S)$ is introduced. This is a manifold with corners and a boundary with interior $i: \mathcal{C}^0(\mathbb{R}^n)(S) \to \mathcal{C}(\mathbb{R}^n)(S)$. There is a projection $\pi: \mathcal{C}(\mathbb{R}^n)(S) \to (\mathbb{R}^n)^S$ such that $\pi \circ i: \mathcal{C}^0(\mathbb{R}^n)(S) \to (\mathbb{R}^n)^S$ is the natural embedding. For any $S' \subset S$ there is the projection map

$$\mathcal{C}(\mathbb{R}^n)(S) \to \mathcal{C}(\mathbb{R}^n)(S'),$$

which forgets points.

The natural action of the group of affine transformations on $\mathcal{C}^0(\mathbb{R}^n)(S)$ is lifted to $\mathcal{C}(\mathbb{R}^n)(S)$. Denote by $\text{Dil}(n)$ its subgroup consisting of dilatations with positive
coefficients and shifts. Group \( \text{Dil}(n) \) acts freely on \( \mathcal{C}(\mathbb{R}^n)(S) \) and the quotient is isomorphic to the fiber \( \pi^{-1}(\vec{0}) \), where \( \vec{0} \in (\mathbb{R}^n)^S \) is the \( S \)-tuple sitting at the origin (see e.g. [Mar17, 2.2]). Denote any of these isomorphic manifolds by \( \text{FM}_n^S \). The sequence \( \text{FM}_n^S \) may be equipped with a structure of an unital operad in the category of topological spaces, for details see [Mar17, 2.2] and other references above.

**Definition 1.** The sequence of topological spaces \( \text{FM}_n^S \) with the unital operad structure as above is called the Fulton–MacPherson operad.

Given a topological operad, one may produce a dg-operad by taking complexes of chains of its components.

**Definition 2.** Denote by \( \text{fm}_n \) the operad of \( \mathbb{R} \)-chains of \( \text{FM}_n \).

Real numbers appear here are to simplify things, all object and morphism we shall use may be defined over rationals. By chains we mean the complex of de Rham currents, that is why we need real chains. Mostly below we will consider the cooperad of de Rham cochains of \( \text{FM}_n \).

**Proposition 1.** Operad \( \text{fm}_n \) is weakly homotopy equivalent to \( e_n \), the operad of chains of the little discs operad.

**Proof.** See [Sal01, Proposition 3.9] and [Mar17, 3.3]. □

Spaces \( \text{FM}_n^S \) are acted on by the general linear group, and, in particular, by its maximal compact subgroup \( SO(n) \), we suppose that a scalar product on the space is chosen. One may consider the semi-direct product of \( SO(n) \) and the Fulton–MacPherson operad and algebras over it. But we will need only the following special case of such algebras.

**Definition 3** ([Mar17], Definition 3). We say that a dg-algebra \( A \) over \( \text{fm}_n \) is invariant, if all structure maps of complexes

\[
\text{fm}_n \otimes A \otimes \cdots \otimes A \rightarrow A
\]

are invariant under the action of group \( SO(n) \) on complexes of operations of \( \text{fm}_n \).

Note that we mean invariance on the level of complexes, not up to homotopy.

1.2. **Weyl \( n \)-algebras.** The algebras over operad \( \text{fm}_n \) we need below are Weyl \( n \)-algebras. Recall its definition, which slightly differs from the one given in [Mar17]. The difference is in the quantization parameter \( h \): in the mentioned paper we considered algebras over formal series of \( h \) since below we suppose that \( h = 1 \).

Let \( n > 1 \) be a natural number. Let \( V \) be a \( \mathbb{Z} \)-graded finite-dimensional vector space over the base field \( k \) of characteristic zero containing \( \mathbb{R} \) equipped with a non-degenerate skew-symmetric pairing \( \omega : V \otimes V \rightarrow k \) of degree \( 1 - n \). Let \( k[V] \) be the polynomial algebra generated by \( V \). Denote by

\[
\partial_\omega : k[V] \otimes k[V] \rightarrow k[V] \otimes k[V]
\]

the differential operator that is a derivation in each factor and acts on generators as \( \omega \).

Consider \( \text{FM}_n(\mathbb{2}) \), the space of 2-ary operations of the Fulton–MacPherson operad. This is homeomorphic to the \( (n - 1) \)-dimensional sphere. Denote by \( v \) the standard \( SO(n) \)-invariant \( (n - 1) \)-differential form on it. For any two-element subset \( \{i, j\} \subset S \) denote by \( p_{ij} : \text{FM}_n(S) \rightarrow \text{FM}_n(\mathbb{2}) \) the map that forgets all points
except ones marked by $i$ and by $j$. Denote by $v_{ij}$ the pullback of $v$ under projection $p_{ij}$. Let $\alpha$ be an element of endomorphisms of $k[V]^S \otimes_{\text{Aut}(S)} C^*(\text{FM}_n(S))$ (where $C^*(-)$ is the de Rham complex) given by

$$\alpha = \sum_{i,j \in S} \partial_{ij} \omega \wedge v_{ij},$$

where $\partial_{ij}$ is the operator $\partial_\omega$ applied to the $i$-th and $j$-th factors.

**Proposition 2.** The composition

$$k[V]^S \xrightarrow{\exp(\mu)} k[V]^S \otimes C^*(\text{FM}_n(S)) \rightarrow k[V] \otimes C^*(\text{FM}_n(S)),$$

where $\mu$ is the product in the polynomial algebra, defines an algebra over the operad $\text{fm}_n$ with the underlying space $k[V]$.

**Proof.** This is a simple check. \qed

The algebra defined in this way is obviously invariant under the action of $SO(n)$, thus it is invariant (see Definition 3).

**Definition 4 ([Mar17]).** For a pair $(V, \omega)$ and $n > 1$ as above the invariant $\text{fm}_n$-algebra given by Proposition 2 is called the Weyl $\text{fm}_n$-algebra or the Weyl $n$-algebra. Denote it by $W_n(V)$.

The Weyl 1-algebra is the usual Weyl algebra generated by a $\mathbb{Z}$-graded finite-dimensional vector space $V$ with relations $[x, y] = (x, y)$, where $x, y \in V$ and $(\cdot, \cdot)$ is a perfect pairing of degree 0 on $V$. Below we will use this definition only in Proposition 13.

Note that Proposition 1 provides us with a notion of Weyl $e_n$-algebras.

The natural map of of operads $\text{fm}_m \rightarrow \text{fm}_n$ for $m < n$ induces the functor from $\text{fm}_m$-algebras to $\text{fm}_n$-algebras. As in [Mar16] denote it by $\text{obl}^m_n$.

**Proposition 3.** For $m < n$ the $\text{fm}_m$-algebra $\text{obl}^m_n W_n(V)$ is isomorphic to the commutative polynomial algebra $k[V]$.

**Proof.** It follows from the very definition of the Weyl $\text{fm}_n$-algebra. \qed

1.3. **Lie algebra.** Recall the construction of a morphism from the shifted $L_\infty$ operad to $\text{fm}_n$, see e. g. [Mar17 2.3].

Spaces of operations of the Fulton–MacPherson operad are equipped with a stratification labeled by trees as follows. As an operad of sets $\text{FM}_n$ is freely generated by $\mathcal{C}_k^0(\mathbb{R}^n)(S)/\text{Dil}(n)$. Denote by $\mu$ the map from this free operad to the free operad with one generator in each arity, which sends generators to generators. Elements of the latter operad are enumerated by rooted trees. The map above sends $\mathcal{C}_k^0(\mathbb{R}^n)/\text{Dil}(n)$ to the star tree with $k$ leaves. For a tree $t \in T(S)$ denote by $[\mu^{-1}(t)] \in C_*(\text{FM}_n(S))$ the chain presented by its preimage under $\mu$. The operad $L_\infty$ is a semi-free operad with generators labeled by trees, see e. g. [GK91]. One may see that $[\mu^{-1}(\cdot)]$ commutes with differentials. It gives us the following statement.

**Proposition 4.** Map $[\mu^{-1}(\cdot)]$ as above gives a morphism

$$L_\infty[1 - n] \rightarrow \text{fm}_n$$

from shifted $L_\infty$ operad to the dg-operad $\text{fm}_n$ of chains of the Fulton–MacPherson operad.
Definition 5. For a \( \mathfrak{fm}_n \)-algebra \( A \) call its pull-back under (2) the associated \( L_\infty \)-algebra and denote it by \( L(A) \).

Consider the \( L_\infty \)-algebra \( L(\mathcal{W}^n(V)) \) associated with the Weyl \( n \)-algebra. By the very definition, all operations on it are given by integration of closed forms by chains of the Fulton–MacPherson operad. But one may see, that chains representing higher operations (that is operations, which are not compositions of Lie brackets) in \( L_\infty \) are all homologous to zero, because \( L_\infty \) is a resolution of the Lie operad. Thus \( L(\mathcal{W}^n(V)) \) is a \( \mathbb{Z} \)-graded Lie algebra, that is all higher operations vanish. This Lie algebra \( L(\mathcal{W}^n(V)) \) is a deformation of the Abelian one. The first order deformation gives the Poisson Lie algebra: the underlying space is the \( \mathbb{Z} \)-graded commutative algebra \( k[V] \), the bracket is defined by \( \omega: V \otimes V \to k \) on generators and satisfies the Leibniz rule.

Proposition 5. For \( n > 1 \) Lie algebra \( L(\mathcal{W}^n(V)) \) is isomorphic to the Poisson Lie algebra of \( (k[V], \omega) \).

Proof. Clear, because for \( n > 1 \) the square of the de Rham cochain \( v \) is zero. \( \square \)

2. Factorization complex

2.1. Factorization complex. The factorization complex of an algebra over the framed \( n \)-disks operad on a manifold is the tensor product of the right module over the framed \( n \)-disk operad corresponding to the manifold and the left one defined by the algebra, see e. g. [Gin15]. For an invariant \( \mathfrak{fm}_n \)-algebra we will use a simplified version of this definition following [Mar17].

Let \( M \) be a \( n \)-dimensional oriented manifold. In the same way, as for \( \mathbb{R}^n \), there is the Fulton–MacPherson compactification \( \mathcal{C}(M)(S) \) of the space \( \mathcal{C}^0(M)(S) \) of ordered pairwise distinct points in \( M \) labeled by \( S \). Locally it is the same thing. Inclusion \( \mathcal{C}^0(M)(S) \to \mathcal{C}(M)(S) \) is a homotopy equivalence, there is a projection \( \mathcal{C}(M)(S) \xrightarrow{\pi} M^S \).

Recall that a point in the Fulton–MacPherson compactification \( \mathcal{C}(\mathbb{R}^n)(S) \) of the configuration space of \( \mathbb{R}^n \) looks like a configuration from the configuration space \( \mathcal{C}^0(\mathbb{R}^n)(S') \) with elements of \( \mathbf{FM}_n \) sitting at each point of the configuration. It follows that spaces \( \mathcal{C}(\mathbb{R}^n)(\bullet) \) form a right \( \mathbf{FM}_n \)-module, which is freely generated by \( \mathcal{C}^0(\mathbb{R}^n)(\bullet) \) as a set. The same is nearly true for the Fulton–MacPherson compactification of any oriented manifold \( M \). But to define such an action one needs to choose coordinates at the tangent space of any point of a configuration of \( \mathcal{C}(M)(S) \). To fix it one has to consider either only framed manifolds or introduce framed configuration space. For invariant algebras these problems vanish.

Definition 6 ([Mar17, Proposition 3]). For an invariant unital \( \mathfrak{fm}_n \)-algebra \( A \) and an oriented manifold \( M \) the factorization complex \( \int_M A \) is the complex given by
the colimit of the diagram
\[
\bigoplus_{S'} C_\ast (\mathcal{E}(M)(S')) \otimes_{\text{Aut}(S')} A \otimes S'
\]
\[
\bigoplus_{i: S' \rightarrow S} C_\ast (\mathcal{E}^0(M)(S)) \otimes_{\text{Aut}(S)} (\mathfrak{fm}_n(i^{-1}s) \otimes_{\text{Aut}(i^{-1}s)} A \otimes (i^{-1}s))
\]
\[
\bigoplus_{S} C_\ast (\mathcal{E}^0(M)(S)) \otimes_{\text{Aut}(S)} A \otimes S
\]

where the summation in the middle runs over maps between finite sets, the downwards arrow is given by the left action of \(\mathfrak{fm}_n\) on \(A\) for \(\text{Im} i\) and the unit for \(S \setminus \text{Im} i\) and the upwards arrow is given by the right action of \(\mathfrak{fm}_n\) on \(C_\ast (\mathcal{E}(M)(\bullet))\).

Note that relations (3) include in particular colimits
\[
\bigoplus_{S'} C_\ast (\mathcal{E}(M)(S')) \otimes_{\text{Aut}(S')} A \otimes S'
\]
\[
\bigoplus_{i: S' \rightarrow S} C_\ast (\mathcal{E}(M)(S)) \otimes_{\text{Aut}(S)} A \otimes S
\]

where the upward arrow is the projection, which forgets points labeled by \(S \setminus S'\).

Proposition 6. For a \(\mathbb{Z}\)-graded vector space \(V\) the cohomology of the factorization complex \(\int_M k[V]\) of the polynomial algebra \(k[V]\) is isomorphic to \(k[V \otimes H_\ast (M)]\).

Proof. The factorization complex of a polynomial algebra \(k[V]\) on a manifold \(M\) is isomorphic to \(\bigoplus_i C_\ast (M^k) \otimes_{\Sigma_i} V^\otimes i\) by the very definition. It follows the statement. \(\square\)

Proposition 7. (1) The factorization complex \(\int_{M^k} A\) of an invariant \(\mathfrak{fm}_n\)-algebra on a closed compact oriented \(k\)-manifold \(M^k\) is naturally equipped with a structure of \(\mathfrak{fm}_{n-k}\)-algebra.

(2) For a fiber bundle \(E^k \xrightarrow{F_k} B^{n-k}\) with closed compact oriented base and fiber and an invariant \(\mathfrak{fm}_n\)-algebra \(A\)
\[
\int_{B^{n-k}} \left( \int_{F_k} A \right) = \int_{E^k} A,
\]
where \(\int_{F_k} A\) is a \(\mathfrak{fm}_{n-k}\)-algebra by the previous item.

Proof. See [GTZ10, Section 5] and references therein. \(\square\)

This theorem may be formulated for maps more general than projections of fiber bundles. To define push-forward in a more general situation one needs to introduce factorization sheaves, see [AFT14, Gin15] for details. The construction from Subsection 5.2 below is an example of such a push-forward.
2.2. Factorization complex of a disk. The factorization complex is homotopy invariant. In particular, it means, that the factorization complex of a disk is trivial. It is stated in two subsequent propositions.

Denote by $\mathbb{D}^n$ the open disk $\{x \in \mathbb{R}^n \mid |x| < 1\}$ and by $\mathbb{D}^n$ the closed disk $\{x \in \mathbb{R}^n \mid |x| \leq 1\}$.

**Proposition 8.** For a $\mathfrak{fm}_n$-algebra $A$ the factorization complex $\int_{\mathbb{D}^n} A$ is homotopy equivalent to $A$ and embedding of any point $p \to \mathbb{D}^n$ induces a quasi-isomorphism $A = \int_p A \xrightarrow{\sim} \int_{\mathbb{D}^n} A$.

**Proof.** See e. g. [Mar17] Prop. 5.\qed

**Proposition 9.** For a invariant $\mathfrak{fm}_n$-algebra $A$ the factorization complex $\int_{\mathbb{D}^n} A$ is homotopy equivalent to $A$ and the embedding $\mathbb{D}^n \to \mathbb{D}^n$ induces a quasi-isomorphism $\int_{\mathbb{D}^n} A \xrightarrow{\sim} \int_{\mathbb{D}^n} A$.

**Proof.** The following proof is taken from [GTZ 5.2]. Consider the projection $p: \mathbb{D}^n \to [0, 1]$, which sends point $x$ to $|x|$. The fiber over a non-zero point is the sphere $S^{n-1}$. The factorization complex of $\int_{S^{n-1}} A$ is a $e_1$-algebra and $A$ as a complex is a module over it (see [Lur] [Fra13] and also [GTZ] Proposition 5.8). As it follows from gluing property of the factorization complex, the factorization complex $\int_{\mathbb{D}^n} A$ is quasi-isomorphic to

$$A \xrightarrow{\int_{S^{n-1}}} \mathbb{D}^n \int_{S^{n-1}} A,$$

which follows the statement of the proposition. \qed

From the proof of this proposition it follows that the factorization complex $\int_{\mathbb{D}^n} A$ is equipped with a structure of $(\int_{S^{n-1}} A)$-module. As this complex is quasi-isomorphic to $A$ it follows that the underlying complex of $A$ itself is a $\int_{S^{n-1}} A$-module ([GTZ Lemma 5.12]).

3. Swiss Cheese operad

3.1. Swiss Cheese operad. Let $\mathbb{R}^n$ be an affine space. Denote by $\mathbb{R}^n_{\geq 0}$ and $\mathbb{R}^n_{> 0}$ subsets $\{x \in \mathbb{R}^n \mid x_0 \geq 0\}$ and $\{x \in \mathbb{R}^n \mid x_0 > 0\}$ correspondingly, where $x_0$ is the coordinate function. Denote by $\mathcal{E}^0(\mathbb{R}^n_{\geq 0})(S)$ the configuration space of distinct ordered points in $\mathbb{R}^n_{\geq 0}$ labeled by $S$. Points inside $\mathbb{R}^n_{> 0} \subset \mathbb{R}^n_{\geq 0}$ are called closed and points on the boundary $\mathbb{R}^n_{> 0} \subset \mathbb{R}^n_{\geq 0}$ are called open. Denote by $\mathcal{E}(\mathbb{R}^n_{> 0})(S)$ the closure of $\mathcal{E}^0(\mathbb{R}^n_{\geq 0})(S)$ in $\mathcal{E}(\mathbb{R}^n)(S)$. This is a manifold with corners and a boundary. There is a projection $\pi: \mathcal{E}(\mathbb{R}^n_{> 0})(S) \to (\mathbb{R}^n_{> 0})^S$, which restricts on $\mathcal{E}^0(\mathbb{R}^n_{\geq 0})(S)$ to the natural embedding.

Let us define a stratification of $\mathcal{E}^0(\mathbb{R}^n_{\geq 0})(S)$ that is a continuous map to a poset. The poset is $\{O < C\}^S$, where $\{O < C\}$ is the poset consisting of two elements. For $s \in S$ the $s$-component of this map is $C_s$, if the point of the configuration labeled by $s$ is closed and is $O_s$ if it is open. One may see that taking closures of strata in $\mathcal{E}(\mathbb{R}^n_{> 0})(S)$ defines a Whitney stratification of the latter space with the same indexing poset. Denote the indexing map by

$$\varpi: \mathcal{E}(\mathbb{R}^n_{> 0})(S) \to \{O < C\}^S.$$
Denote by $\text{Dil}(n-1)$ the subgroup of affine transformations of $\mathbb{R}^n$ consisting of dilatations with positive coefficients and shifts along the hyperplane $\{x_0 = 0\}$. Group $\text{Dil}(n-1)$ acts freely on $\mathscr{H}(\mathbb{R}_{>0}^n)(S)$. The quotient is isomorphic to the fiber $\pi^{-1}(\vec{0})$, where $\vec{0} \in (\mathbb{R}_{>0}^n)^S$ is the $S$-tuple sitting at the origin. It follows that $\pi^{-1}(\vec{x})$ for any $S$-tuple $\vec{x} \in (\mathbb{R}_{>0}^n)^S$ in the interior is isomorphic to $\text{FM}^S_n$.

The sequence of manifolds with corners $\text{SC}^S_n$ form a colored operad called the Swiss Cheese operad introduced in [Vor99, Kon99]. Describe it as an operad of sets. This colored operad has two colors: points may be open and closed. Note that the set of colors is a poset, that is a category, rather than a set, and there are only operations compatible with this structure. This operad of sets is free and is generated by the following operations. The set of $S$-ary generating operations from $S$ closed points to a close point equals to the quotient of $\mathscr{C}^0(\mathbb{R}^n)(S) \hookrightarrow \mathscr{C}(\mathbb{R}^n)(S)$ by $\text{Dil}(n)$, which is embedded in $\text{FM}^S_n$. The set of operations from $C$ closed and $O$ open points to an open point equals to the quotient of the configuration space of $C$ distinct points in $\mathbb{R}_{>0}^n$ and $O$ distinct points in $\mathbb{R}_{=0}^{n-1}$ factored out by the $\text{Dil}(n-1)$ group action. The action of the symmetric group is straightforward and the composition is analogous to the one of the Fulton–MacPherson operad.

Below we do not need exactly the notion of this colored operad, but the action of a $\text{fm}_n$-algebra on a $\text{fm}_{m}$-algebra we define below is essentially the action of this operad.

3.2. Action. For a space $X$ with a stratification given by $\varpi: X \to P$, where $P$ is a posetal category, we say that a constructible sheaf with values in a category $C$ is $\varpi$-combinatorial if its restriction to each stratum is constant. A combinatorial sheaf is defined by a functor $P \to C$, see e. g. [GK94, 1.5].

Consider a triple $(A, M, \varepsilon)$ consisting of a unital $\text{fm}_n$-algebra $A$, a $\text{fm}_{n-1}$-algebra $M$ and a map of unital $\text{fm}_{n-1}$-algebras $\varepsilon: \text{obl}_n A \to M$. Denote by $A^\vee$ and $M^\vee$ the linear dual complexes. The triple defines a functor from category $\{O < C\}^S$ to complexes, which sends $C$ to $A^\vee$, $O$ to $M^\vee$ and $\varepsilon^\vee$ to the only non-trivial morphism of this category. The tensor power of this functor gives a functor from $\{O < C\}^S$ to complexes. Denote by $F_{\varepsilon^\vee}$ the combinatorial sheaf of complexes over $\mathscr{H}^S_{\bullet}$ associated with this functor.

**Definition 7** ([Vor99, Kon99, Tho16]). For a triple $(A, M, \varepsilon)$ as above, an action of $A$ on $M$ is defined by maps of complexes

$$M^\vee \to C^\ast(\text{SC}^S_n, F_{\varepsilon^\vee})$$

such that

1. (compatibility) their restriction on $\mathbb{R}_{=0}^{n-1} \subset \mathbb{R}_{\geq 0}^n$ are given by the $\text{fm}_{n-1}$-algebra structure on $M$;
(2) (factorization) they factor through the limit of the diagram
\[
\begin{array}{ccc}
\bigoplus_{S'} C^\ast(S^\ast) & \xrightarrow{\nabla} & \bigoplus_{i: S' \to (C \cup O)} C^\ast(\mathcal{C}^\ast(\mathbb{R}_{\geq 0}^n)(C \cup O)) \\
\downarrow & & \downarrow \\
\bigoplus_{C \cup O} C^\ast(\mathcal{C}^\ast(\mathbb{R}_{\geq 0}^n)(C \cup O)) & \otimes & (A \otimes M) = (A \otimes M)^{\mathcal{C}}(\mathcal{C}^\ast(\mathbb{R}_{\geq 0}^n)(\bullet)) \\
\end{array}
\]

where \(\mathcal{C}^\ast(\mathbb{R}_{\geq 0}^n)(C \cup O)\) means the configuration space of \(C\) closed and \(O\) open distinct points, summation in the middle runs over maps between finite sets, which are surjective on \(O\), the upwards arrow is given by the left coaction of \(\mathfrak{m}_n\) on \(A\) for \(\text{Im} i \cap C\), action of \(A\) on \(M\) for \(\text{Im} i \cap O\) and the unit for \(S \setminus \text{Im} i\), and the downwards arrow is given by the right coaction of \(\mathfrak{m}_n\) on \(C^\ast(\mathcal{C}^\ast(\mathbb{R}_{\geq 0}^n)(\bullet))\).

This definition resembles Definition 3 of the factorization complex. It is not a coincidence, the action may be defined as a factorization sheaf of a special form, see for details \[AFT14, Gin15\].

### 3.3. Higher Hochschild cohomology

Let \(M\) be an invariant \(\mathfrak{m}_{n-1}\)-algebra. The factorization complex \(\int_{S^{n-2}} M\) is an \(e_1\)-algebra and the underlying complex of \(M\) is a module over it, see e. g. \[GTZ\] and the remark after Proposition 9.

**Definition 8** \[Fra13, GTZ\]. Define the higher Hochschild cohomological complex of an invariant \(\mathfrak{m}_{n-1}\)-algebra \(M\) by
\[
CH_{e_{n-1}}(M, M) = R\mathcal{H}om_{S^{n-2} M}(M, M).
\]

The higher Hochschild complex of an \(e_{n-1}\)-algebra is equipped with an \(e_{n-1}\)-algebra structure. It may be defined rather explicitly (see \[GTZ\]): it is given by the composition of the target of \(R\mathcal{H}om\). By \[Lur\] and \[GTZ\] the higher Hochschild cohomology is the derived centralizer of the identity map of an \(e_{n-1}\)-algebra to itself. By \[Lur\] it is equipped with a canonical \(e_{n-1}\)-algebra structure. It is shown there that the mentioned \(e_{n-1}\) structure on \(CH_{e_{n-1}}(M, M)\) may be lifted to an \(e_{n-1}\)-algebra structure.

Action in the sense of Definition 7 of an \(e_n\)-algebra \(A\) on an \(e_{n-1}\)-algebra \(M\) induces a morphism of \(e_n\)-algebras
\[
A \to CH_{e_{n}}(A, M),
\]
see \[Tho16\]. In terms of the triple \((A, M, \varepsilon)\) it may be defined as follows. Given a chain in the complex \(\int_{S^{n-1}} M\) consider the following chain in the complex dual to \(C^\ast(\mathcal{C}^\ast(S^n, C^\ast))\): its open points are given by this chain, where \(\overline{D}^{n-1}\) is the unit disc in \(\mathbb{R}^{n-1}\) and the only closed point is \((t, 0, \ldots, 0)\) labeled by an element of \(A\), where \(t \in \mathbb{R}_{>0}\). Consider the limit of this configuration as \(t\) approaches 0. Convolution with the action gives a map
\[
A \to \text{Hom}(\int_{D^{n-1}} M, M) \simeq \text{Hom}(M, M).
\]
One may see, that the resulting element of $\Hom^\bullet(M, M)$ is a homomorphism of $(\int_{\mathcal{S}_{n-2}} M)$-modules. It gives us a map

$$A \to CH^\bullet_{e_{n-1}}(M, M).$$

**Proposition 10.** The map (7) is a morphism of $e_n$-algebras. The $e_n$-algebra $CH^\bullet_{e_{n-1}}(M, M)$ is the final object in the category of $e_n$-algebras acting on $M$.

**Proof.** This is the main result of [Tho16].

Being defined as in [9], the higher Hochschild cohomological complex is not equipped with an explicit $e_n$-algebra structure (whereas the $e_{n-1}$-algebra structure can be made explicit, see [GTZ]). In particular, the $L_\infty$-structure on $L(CH^\bullet_{e_{n-1}}(M, M))$ is rather implicit.

But for $n = 2$ the higher Hochschild cohomological complex is the usual Hochschild cohomological complex and it is equipped with a Lie bracket due to Gerstenhaber [Ger63]. If a $\mathfrak{fm}_2$-algebra $A$ acts on an algebra $M$, one may build an explicit $L_\infty$-morphism from $L(A)$ to the Hochschild cohomological complex of $M$ equipped with the Gerstenhaber bracket. Note that the latter is a dg-Lie algebra, it has no higher $L_\infty$-operations. It would be interesting to generalize this construction for higher dimensions.

Let $A$ be a $\mathfrak{fm}_2$-algebra acting on a $\mathfrak{fm}_1$-algebra $M$. Define a chain in the complex dual to $C^\ast(\mathcal{SC}_n^2, \mathcal{C}_2)$ depending on $k$ elements of $A$ and $l$ elements of $M$. Let $B_2 = \{x \in \mathbb{R}^2_0 \mid |x| < 1\}$ and $B_1 = \{x \in \mathbb{R}^1_{>0} \mid |x| < 1\}$. Define chain $\tilde{c}$ by

$$\tilde{c}(a_1, \ldots, a_k; m_1, \ldots, m_l) = [e^0(B^2)(k)] \otimes_{\Sigma_k} (a_1 \otimes \cdots \otimes a_k) \cup [e^0(B^1)(1)] \otimes_{\Sigma_l} (m_1 \otimes \cdots \otimes m_l),$$

where $[e^0(B^2)(k)]$ and $[e^0(B^1)(1)]$ are cycles in $C_\ast(\mathcal{C}_2, \mathbb{R}^2_{>0})(S)$ presented by the configuration space of distinct points lying in $B_2$ and $B_1$. Consider the fiberwise closure of this chain with respect to the projection of configuration spaces, which forgets closed points. Take its intersection with the subset consisting of configurations with all closed points lying over the origin. Denote the resulting chain by $c(a_1, \ldots, a_k; m_1, \ldots, m_l)$. The convolution of this chain with action defines a map from $A^\otimes k \otimes M^\otimes l$ to $M$, which is symmetric in $A$’s. Thus it defines a map

$$c: A^\otimes k | l + 2k - 2 \to \bigoplus_l \Hom(M^\otimes l, M).$$

**Proposition 11.** For a $\mathfrak{fm}_2$-algebra $A$ acting on a $\mathfrak{fm}_1$-algebra $M$ map (8) defines a $L_\infty$-morphism from $L(A)$ to the Hochschild cohomological complex of $M$.

**Proof.** This is the Theorem from [Kon03 6.4] slightly rephrased from [Mar17].

### 4. Propagator approach

4.1. **Propagator.** Consider the stratified space $\mathcal{SC}_n^2$ defined in Subsection 3.1. One may see, that this is the higher dimensional generalization of the "eye" from [Kon03]. The stratum with two closed points is the interior of the "eye", two strata with one open and one closed point are "eyelids", which are hemispheres, the stratum with two open points is the "eye corner(s)", which is a $(n - 2)$-dimensional sphere.
The manifold with corners $\text{SC}^2_n$ has two connected components of the boundary. The first one consists of two "eyelids" glued by the "eye corner(s)". The second one is the "iris", the $(n-1)$-dimensional sphere, which magnifies collisions of two closed points.

The following definition is a straightforward high-dimensional generalization of the differential of the angle map from [Kon03, 6.2].

**Definition 9.** A $n$-propagator is a smooth closed differential $(n−1)$-form on $\text{SC}^2_n$, such that

1. its restriction on the "lower eyelid" is zero and
2. its restriction on the "iris" is the standard volume form on the sphere.

Consider some examples of propagators. We define a differential form on the interior of the "eye" and leave to the reader to check that it continues to the boundary. The interior consists of pairs of distinct points in $\mathbb{R}^n_{>0}$ modulo the $\text{Dil}(n−1)$ action, that is $\mathcal{C}^0(\mathbb{R}^n_{>0})(2)/\text{Dil}(n−1)$. Denote such a pair by $(s,t)$. When $t$ tends to the boundary $\mathbb{R}^{n−1}_{=0} \subset \mathbb{R}^n_{\geq0}$, the pair $(s,t)$ tends to the "lower eyelid".$n$

**Example 0.** The first example is a high-dimensional generalization of the propagator used in [Kon03].

For a pair $(s,t)$ as above denote by $\overline{7} \in \mathbb{R}^n_{<0}$ the image of the reflection of $t$ with respect to the boundary hyperplane. Consider two maps from $\mathcal{C}^0(\mathbb{R}^n_{>0})(2)$ to $S^{n−1}$ which send $(s,t)$ to directions given by vectors $s−t$ and $s−\overline{7}$. The difference between pullbacks of the standard volume form on the sphere under the first and the second maps is a $\text{Dil}(n−1)$-invariant closed $(n−1)$-form on $\mathcal{C}^0(\mathbb{R}^n_{>0})(2)$. Its continuous extension to the boundary satisfies conditions of the Definition 9. Denote this propagator by $\Phi^n_0$.

**Example k.** The previous example is the first in a series.

For $k \in \mathbb{N}$ consider the embedding $\mathbb{R}^n \rightarrow \mathbb{R}^{n+k}$ as the coordinate plane. For any point, $t \in \mathbb{R}^n_{>0} \subset \mathbb{R}^n$ denote by $S_t$ the only $k$-sphere in $\mathbb{R}^{n+k}$, which contains $t$, has its center on the plane $\mathbb{R}^{n−1}_{=0} \subset \mathbb{R}^n$ and lies in the plane perpendicular to this plane. Consider the space of triples $(s,t,p)$, where $s,t \in \mathbb{R}^n_{>0} \subset \mathbb{R}^n$, $s \neq t$ and $p \in S_t$. Denote by $\varpi$ the projection from this space to the configuration space of two distinct ordered points $\mathcal{C}^0(\mathbb{R}^2_{>0})(2)$, which forgets the third term of the triple.

On the configuration space $\mathcal{C}^0(\mathbb{R}^{n+k})(2)$ of two distinct points of $\mathbb{R}^{n+k}$ consider the standard differential $(n+k−1)$-form $\upsilon$, which is the pullback of the standard volume form of the $(n+k−1)$-sphere under the projection given by the direction of the vector connecting two points. Denote by the same letter the $(n+k−1)$-form on the space of triples as above, which is the pullback of $\upsilon$ under the map which forgets the middle term of the triple. Denote by $\Phi^k_n$ the differential $(n−1)$-form on $\mathcal{C}^0(\mathbb{R}^2_{>0})(2)$ given by the integration of $\upsilon$ along the projection $\varpi$:

$$\Phi^k_n(s,t) = \int_{\varpi} \upsilon(s,p).$$

(9)

One may see, that this form is closed, invariant under $\text{Dil}(n−1)$ and allows the smooth extension to $\text{SC}^2_n$, which obeys conditions of Definition 9 that it is a propagator.

In this example one may replace the sphere with any figure in the Euclidean space, which does not contain the origin and once intersects any ray from the origin.
For example, one may take an ellipsoid. If one consider a family of ellipsoids in $\mathbb{R}^{n+k}$, or any other figures, which tends to the cylinder over $S^l$, the corresponding propagator tends to $\Phi^l_n$.

4.2. **Action from a propagator.** In subsection 1.2 following [Mar17] we construct the Weyl algebra over the operad $\mathfrak{m}_n$. Given a propagator one may, in the same manner, construct an action of this $\mathfrak{m}_n$-algebra on the polynomial algebra considered as a $\mathfrak{m}_{n-1}$-algebra.

Let $U$ be a $\mathbb{Z}$-graded finite-dimensional vector space over the base field $k$ of characteristic zero containing $\mathbb{R}$ and $U^\vee$ be the dual space. Then $V = U \oplus U^\vee[1-n]$ is naturally equipped with the perfect skew-symmetric pairing of degree $1 - n$. By Definition 4 this data gives us the Weyl $\mathfrak{m}_n$-algebra $W^n(U \oplus U^\vee[1-n])$ with the underlying complex $k[U \oplus U^\vee[1-n]]$.

Consider the triple $(k[U \oplus U^\vee[1-n]], k[U], \varepsilon)$ where $\varepsilon : k[U \oplus U^\vee[1-n]] \to k[U]$ is the natural map, which sends all generators from $U^\vee[1-n]$ to zero.

As in Definition 7 denote by $\mathcal{F}_\varepsilon$ the combinatorial sheaf over $SC^S_n$ associated with $\varepsilon^\vee$. There is a natural map

$$
\mathcal{F}_\varepsilon \to (k[U \oplus U^\vee[1-n]]^\vee)^S
$$

from $\mathcal{F}_\varepsilon$ to the constant sheaf given by the augmentation.

Fix a $n$-propagator $\mathfrak{p}$.

For any two-element subset $\{i, j\} \subset S$ denote by $p_{ij} : SC_n(S) \to SC_n(2)$ the map that forgets all points except ones marked by $i$ and by $j$. Denote by $p_{ij}$ the pullback of $\mathfrak{p}$ under projection $p_{ij}$. Let $\alpha$ be an element of endomorphisms of $k[U \oplus U^\vee[1-n]]^\vee \otimes_{Aut(S)} C^*(SC_n(S))$ (where $C^*(-)$ is the de Rham complex) given by

$$
\alpha = \sum_{i,j \in S} \partial^i_j \wedge p_{ij},
$$

where $\partial^i_j$ is the operator $\partial^i$ applied to the $i$-th and $j$-th factors, where the operator $\partial^i$ is given by (11) for the standard bilinear form of degree $1 - n$ on $U \oplus U^\vee[1-n]$.

As in Proposition 2 consider the map

$$
(k[U \oplus U^\vee[1-n]]^\vee)^S \to C^*(SC^S_n) \otimes k[U \oplus U^\vee[1-n]]
$$

given by the composition of $\exp(\alpha)$ and $\mu$.

Composing the map dual to (11)

$$(k[U \oplus U^\vee[1-n]]^\vee) \to C^*(SC^S_n) \otimes (k[U \oplus U^\vee[1-n]]^\vee)^\vee = C^*(SC^S_n, (k[U \oplus U^\vee[1-n]]^\vee)^\vee)^S$$

with

$$
\varepsilon^\vee : k[U]^\vee \to (k[U \oplus U^\vee[1-n]]^\vee)^\vee
$$

we get the map

$$
k[U]^\vee \to C^*(SC^S_n, (k[U \oplus U^\vee[1-n]]^\vee)^\vee)^S.
$$

**Proposition 12.** For a $n$-propagator $\mathfrak{p}$ the map uniquely factors through the map induced by the map of sheaves (10) and the resulting map

$$
k[U]^\vee \to C^*(SC^S_n, \mathcal{F}_\varepsilon^\vee)
$$

defines an action of $W^n(U \oplus U^\vee[1-n])$ on the polynomial algebra $k[U]$ considered as a $\mathfrak{m}_{n-1}$-algebra.
Proof. The existence of the unique factorization follows from the first property of the propagator from Definition 9. The second property guarantees the second condition of Definition 7. □

Note that to get such an action one could start with a gadget more general, than the propagator: one can use a closed form on \(SC^2_n\) taking values in \(V^\vee \otimes V^\vee\), which vanishes on the "lower eyelid" and when restricted on the "iris" equals to the standard volume form multiplied by the standard bilinear form of degree \(1-n\) on \(V\).

4.3. Formality. Proposition 12 gives an action of \(W^m(U \oplus U^\vee[1-n])\) on the polynomial algebra \(k[U]\) considered as an \(fm_{n-1}\)-algebra. By Proposition 10, this action gives a morphism of \(e_{n-1}\)-algebras
\[
W^m(U \oplus U^\vee[1-n]) \to CH^*_{e_{n-1}}(k[U], k[U]).
\]

Theorem 1. Map (14) is a quasi-isomorphism.

Proof. Let us first evaluate \(CH^*_{e_{n-1}}(k[U], k[U])\). By Proposition 6 \(\int_{S_{n-1}} k[U] = k[U \oplus U][n-2]\). From (6) we get
\[
CH^*_{e_{n-1}}(k[U], k[U]) = RHom^*_{\int_{S_{n-2}} k[U]}(k[U], k[U]) = k[U \oplus U^\vee[1-n]].
\]

The commutative algebra structure on the \(k[U \oplus U^\vee[1-n]]\) comes from the \(e_{n-1}\)-algebra structure on \(CH^*_{e_{n-1}}(k[U], k[U])\) by the very definition of the latter. But by Proposition 3 \(W^m(U \oplus U^\vee[1-n])\) as an \(e_{n-1}\)-algebra is also a free commutative algebra generated by \(U \oplus U^\vee[1-n]\). Since (14) is a morphism of \(e_{n-1}\)-algebras, to prove the statement we need to show that (14) defines an isomorphism on generators. This may be easily checked by the explicit definition of the morphism given before Proposition 11. □

Combining this theorem with Proposition 5 we get the following corollary.

Corollary 1. Map (8) defines a quasi-isomorphism between \(L(CH^*_{e_{n-1}}(k[U], k[U]))\) and the Poisson Lie algebra of \((k[U \oplus U^\vee[1-n]], \omega)\) as \(L_\infty\)-algebras, where \(\omega\) is the standard bilinear form on \(U \oplus U^\vee[1-n]\).

Being combined with Proposition 11 it gives us the main result of [Kon03] about the quasi-isomorphism between polyvector fields and the Hochschild cohomological complex of polynomial algebra. In that paper only propagator \(\Phi_2^0\), which is defined in Subsection 4.1, is used, although it is mentioned there that any other propagator also does the job. It is known, that coefficients of the above formality quasi-isomorphism for this propagator are given by integrals similar to multiple zeta values (see e.g. [RW]), in particular, they are conjecturally irrational.

Results of the next section lead us to the following conjecture.

Conjecture 1. The formality morphism given by Corollary 1 with propagator \(\Phi_n^k\) for \(k > 0\) has rational coefficients.

In the next section, we shall prove this conjecture in the two-dimensional case (Corollary 3).
5. Factorization complex approach

5.1. Factorization complex of a sphere. By Proposition \[2\] for an invariant \(\mathfrak{m}_n\)-algebra \(A\) and an oriented closed manifold \(M^k\) of dimension \(k < n\) the complex \(\int_{M^k} A\) is a \(\mathfrak{m}_{n-k}\)-algebra. If \(A\) is a Weyl \(n\)-algebra the natural candidate for \(\int_{M^k} A\) is a Weyl \((n-k)\)-algebra again.

Conjecture 2. For an oriented closed \(k\)-manifold \(M^k\) and a Weyl \(n\)-algebra \(\mathcal{W}^n(V)\), where \(n > k\), the factorization complex \(\int_{M^k} \mathcal{W}^n(V)\) is quasi-isomorphic to \(\mathcal{W}^{n-k}(V \otimes H_*(M^k))\) as a \(\mathfrak{m}_{n-k}\)-algebra, where \(H_*(M^k)\) is the homology of \(M^k\) negatively graded and equipped with the Poincaré pairing.

The factorization complex on a sphere is of particular interest, see e. g. [GTZ04, Proposition 6.2].

Proposition 13. For a natural \(k < n\) the \(\mathfrak{m}_{n-k}\)-algebra \(\int_{S^k} \mathcal{W}^n(V)\) is quasi-isomorphic to \(\mathcal{W}^{n-k}(V \oplus V[k])\), where \(V \oplus V[k]\) is equipped with the natural perfect pairing of degree \(1 - n + k\).

Proof. Fix a point \(p \in S^k\). Denote by \([\mathcal{E}^0(S^k \setminus p)(S)]\) the cycle in \(C_*(\mathcal{E}(S^k))(S)\) presented by the configuration space of distinct points of \(S^k\) labeled by \(S\) distinct from \(p\). Denote by \([p]\) the cycle in \(C_*(\mathcal{E}(S^k)(S))\) presented by point \(p\). Let \([x \otimes y]\) be a monomial representing an element of \(\mathcal{W}^{n-k}(V \oplus V[k])\), where \(x \in k[V]\) and \(y \in k[V[k]]\) are monomials. Let \(y = l_1 \cdots l_i\) be a factorization of \(y\) into linear factors.

Define a map from \(\mathcal{W}^{n-k}(V \oplus V[k])\) to the factorization complex \(\int_{S^k} \mathcal{W}^n(V)\) by

\[ [x \otimes y] \mapsto [p] \otimes x \cup [\mathcal{E}^0(S^k \setminus p)[1]] \otimes \Sigma_1 l_1 \otimes \cdots \otimes l_i \]

The image is closed because a Weyl \(n\)-algebra is commutative as Weyl \(k\)-algebra by Proposition \[3\] and the cycle above is the standard cycle representing a class in the factorization complex of a polynomial algebra generalizing the standard class for the Hochschild complex [Lod98, Proposition 1.3.12] (compare with [GTZ10, Definition 2]). It is easy to show by the direct calculation, that this map respects the \(\mathfrak{m}_{n-k}\)-algebra structure. The crucial point here is using relations \[4\]. \qed

5.2. Action. As in Subsection \[4.1\] for \(k \in \mathbb{N}\) define the embedding \(\mathbb{R}^n \rightarrow \mathbb{R}^{n+k}\) as the coordinate plane and consider the projection

\[ \mathbb{R}^{n+k} \rightarrow \mathbb{R}_{>0}^n, \]

which sends a point to the only intersection with \(\mathbb{R}^n \rightarrow \mathbb{R}^{n+k}\) of the only \(k\)-sphere in \(\mathbb{R}^{n+k}\), which contains this point, has its center on the plane \(\mathbb{R}_{=0}^{n-1} \subset \mathbb{R}^n\) and lies in the plane perpendicular to this plane.

For an invariant \(\mathfrak{m}_{n+k}\)-algebra \(A\) the product of the operad gives a \(\text{Dil}(n-1)\)-invariant map

\[ A^V \rightarrow \bigoplus_S C^*(\mathcal{E}^0(\mathbb{R}^{n+k})(S)) \otimes A^{V \otimes S} \]

Denote by \(A^{V \otimes \mathcal{S}}\) the locally constant sheaf over \(\coprod S \mathcal{E}^0(\mathbb{R}^{n+k})(S)\) with the fiber equal to \(A^{V \otimes S}\). Then the right side of \[16\] is \(H^*(\coprod S \mathcal{E}^0(\mathbb{R}^{n+k})(S), A^{V \otimes \mathcal{S}})\).

Now we need to take the push-forward of the factorization sheaf with respect to the map \[15\]. The following construction, being a generalization of Proposition \[2\] could be formulated in terms of the relative factorization complex. But the relative factorization complex is a cosheaf rather than a sheaf. That is why it
is more convenient to work with the linear dual thing. Define the dual relative factorization complex $(\int_{\mathbb{R}^{n+k}/\mathbb{R}_{\geq 0}^{n+k}} A)^{\vee}$ of $A$ of the map \((15)\), which is a complex of sheaves over $\mathbb{R}_{\geq 0}^{n}$, in analogy with \((13)\) as the limit of the diagram

\[
\begin{array}{ccc}
\bigoplus_{S} C_{\mathbb{R}_{\geq 0}^{n}}(\mathcal{E}(\mathbb{R}^{n+k})(S^r)) & \otimes & A^{\vee} \otimes S^r \\
\downarrow & & \\
\bigoplus_{i: S' \to S} C_{\mathbb{R}_{\geq 0}^{n}}(\mathcal{E}^{0}(M)(S)) & \otimes & \bigotimes_{Aut(S')} (\mathfrak{fm}_k(s^{-1})) \otimes (\mathfrak{fm}_k(i^{-1}k)) \otimes A^{\otimes (i^{-1}k)} \otimes \mathfrak{Aut}(S^r) \\
\end{array}
\]

where the operad $\mathfrak{fm}_k$ coacts along fibers of \((15)\), this coaction is trivial over $\mathbb{R}_{=0}^{n-1} \subset \mathbb{R}_{\geq 0}^{n}$.

In the diagram

\[
\begin{array}{ccc}
C^{*}(\mathcal{E}(\mathbb{R}^{n+k})(S)) & \otimes & A^{\vee} \otimes S \\
\uparrow & & \leftarrow \\
A^{\vee} = (\int_{\mathbb{R}^{n+k}} A)^{\vee} \\
\end{array}
\]

the isomorphism in the bottom row is given by Proposition \ref{prop:iso} the horizontal arrow is the embedding in the top term of the diagram \((17)\) and the vertical is dual to the action of $\mathfrak{fm}_{n+k}$ operad on $A$. The dashed arrow, which makes the diagram commutative, exists because relations dual to \((17)\) are contained in the relation defining the big factorization complex $\int_{\mathbb{R}^{n+k}} A$. The dashed arrow is $\text{Dil}(n-1)$-invariant and it gives us a map

\[
A^{\vee} \to C^{*}(\mathcal{SC}_{S}^{n}\left(\int_{\mathbb{R}^{n+k}/\mathbb{R}_{\geq 0}^{n+k}} A\right)^{\vee})
\]

Recall that for a $\mathfrak{fm}_m$-algebra $A$ and $m < n$ we denote by $\text{obl}^m_{n+k}$ $A$ the $\mathfrak{fm}_m$-algebra with the same underlying complex as $A$ and the operadic structure induced by the natural map of operads $\mathfrak{fm}_m \to \mathfrak{fm}_n$. A particular case of the following proposition was crucial for the second part of \cite{Mar16}.

**Proposition 14.** For an invariant $\mathfrak{fm}_{n+k}$-algebra $A$ consider the triple

\[
\left(\int_{S^k} A, \text{obl}^n_{n+k} A, \varepsilon\right),
\]

where $\int_{S^k} A$ is an $\mathfrak{fm}_n$-algebra due to Proposition \ref{prop:inc} and $\varepsilon: \int_{S^k} A \to A$ is the natural map induced by the map from the sphere to a point. Then the complex $\mathcal{F}_{\varepsilon}^{\vee}$ is quasi-isomorphic to the complex $(\int_{\mathbb{R}^{n+k}/\mathbb{R}_{\geq 0}^{n+k}} A)^{\vee}$ defined above and map \((18)\) defines an action of $\int_{S^k} A$ on $\text{obl}^n_{n+k} A$ in the sense of Definition \ref{def:action}.

**Proof.** One may see that the complex $(\int_{\mathbb{R}^{n+k}/\mathbb{R}_{\geq 0}^{n+k}} A)^{\vee}$ is constructible with respect to the stratification \cite{3} with fibers and the restriction map equal to ones of $\mathcal{F}_{\varepsilon}^{\vee}$, what follows that they are quasi-isomorphic. Axioms of the action (Definition \ref{def:action})...
follow from the very definition of \( \mathfrak{fm}_{n+k} \)-algebra and the one of the factorization complex.

Applying Proposition \[10\] to the action constructed above we get for any \( \mathfrak{fm}_{n+k} \)-algebra \( A \) a natural map of \( \mathfrak{fm}_n \)-algebras.

\[
\int_{S^k} A \to CH^*_{c_{n-1}}(\text{obl}^{n-1}_{n+k} A, \text{obl}^{n-1}_{n+k} A),
\]

where \( CH^*_{c_{n-1}}(\cdot, \cdot) \) is the higher Hochschild cohomology, see Subsection 3.3.

5.3. Formality via the factorization complex. We are going to show that for Weyl \( (n+k) \)-algebras map \[19\] is a quasi-isomorphism.

Recall, that by Proposition \[13\] the \( \mathfrak{fm}_n \)-algebra \( \int_{S^k} W^{n+k} (V) \) is isomorphic to \( W^{n}(V \oplus V[k]) \), and \( \text{obl}^{n-1}_{n+k} W^{n+k}(V) \) is isomorphic to \( k[V] \) by Proposition \[9\].

**Proposition 15.** For a Weyl \( (n+k) \)-algebra \( W^{n+k}(V) \) the action of \( \int_{S^k} W^{n+k}(V) = W^n(V \oplus V[k]) \) on \( \text{obl}^{n-1}_{n+k} W^{n+k}(V) = k[V] \) defined by Proposition \[14\] is isomorphic to the action defined by Proposition \[12\] for propagator \( \Phi_k^n \) given by \[9\].

**Proof.** Substituting quasi-isomorphism from Proposition \[14\] to the definition of the action from Proposition \[12\] we immediately get the statement. \( \square \)

Proposition \[10\] gives a map of \( \mathfrak{fm}_n \)-algebras

\[
\int_{S^k} W^{n+k}(V) \to CH^*_{c_{n-1}}(\text{obl}^{n-1}_{n+k} W^{n+k}(V), \text{obl}^{n-1}_{n+k} W^{n+k}(V)) = CH^*_{c_{n-1}}(k[V], k[V])
\]

**Theorem 2.** The map \[20\] is a quasi-isomorphism with rational coefficients.

**Proof.** Combining Theorem \[1\] with Proposition \[15\] we get the statement. To see that coefficients are rational note that the action is given by the product in the operad \( \mathfrak{fm}_{n+k} \). Thus coefficients of the action \[18\] are given by integration of an integral cocycle by an integer cycle. \( \square \)

Combining this theorem with Proposition \[5\] and Proposition \[13\] we get the following corollary.

**Corollary 2.** Map \[20\] gives a quasi-isomorphism between two \( L_\infty \)-algebras: the higher Hochschild cohomology Lie algebra \( L(CH^*_{c_{n-1}}(k[V], k[V])) \) and the Poisson Lie algebra of \( (k[V \oplus V^*[n]], \omega) \), where \( \omega \) is the standard bilinear form on \( [V \oplus V^*[n]] \) with rational coefficients.

Let \( V \) be a vector space with a perfect symmetric pairing. Applying this corollary to Weyl 3-algebra \( W^3(V[1]) \) and combining it with Corollary \[1\] and Proposition \[15\] we get Conjecture \[1\] in dimension two.

**Corollary 3.** For a vector space \( V \) the formality morphism given by Corollary \[1\] with propagator \( \Phi_k^2 \) for \( k > 0 \) gives a quasi-isomorphism between \( L(CH^*_{c_1}(k[V], k[V])) \) and the Lie algebra of polyvector fields \( k[V \oplus V^*[1]] \) with rational coefficients.
References

[AFT14] David Ayala, John Francis, and Hiro Tanaka. Factorization homology of stratified spaces. *Selecta Mathematica*, 23:293–362, 2014.

[AS92] Scott Axelrod and I. M. Singer. Chern-Simons perturbation theory. In *Proceedings of the XXth International Conference on Differential Geometric Methods in Theoretical Physics, Vol. 1, 2 (New York, 1991)*, pages 3–45. World Sci. Publ., River Edge, NJ, 1992.

[CDM12] Sergei Chmutov, Sergei Duzhin, and Jacob Mostovoy. *Introduction to Vassiliev knot invariants*. Cambridge University Press, Cambridge, 2012.

[Fra13] John Francis. The tangent complex and hochschild cohomology of $E_n$-rings. *Compositio Mathematica*, 149(3):430480, 2013.

[Ger63] Murray Gerstenhaber. The cohomology structure of a n associative ring. *Annals of Mathematics*, 78(2):267–288, 1963.

[Gin15] Gregory Ginot. *Notes on factorization algebras, factorization homology and applications*, pages 429–552. Springer International Publishing, Cham, 2015.

[GJ] Ezra Getzler and J. D. S. Jones. Operads, homotopy algebra and iterated integrals for double loop spaces. arXiv:hep-th/9403055v1.

[GK94] Victor Goncharov and Mikhail Kapranov. Koszul duality for operads. *Duke Math. J.*, 76(1):203–272, 1994.

[GTZ] Grégoire Ginot, Thomas Tradler, and Mahmoud Zeinalian. Higher Hochschild cohomology, Brane topology and centralizers of $E_n$-algebra maps. arXiv:1205.7056 [math.AT].

[GTZ10] Grégoire Ginot, Thomas Tradler, and Mahmoud Zeinalian. Higher hochschild homology, topological chiral homology and factorization algebras. *Communications in Mathematical Physics*, 326:635–686, 2010.

[Kon99] Maxim Kontsevich. Operads and motives in deformation quantization. *Letters in Mathematical Physics*, 48:35–72, 1999.

[Kon03] Maxim Kontsevich. Deformation quantization of Poisson manifolds. *Lett. Math. Phys.*, 66:157–216, 2003.

[Lod98] Jean-Louis Loday. *Cyclic homology*, volume 301 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 1998.

[Lur] Jacob Lurie. *Higher Algebra*. [http://www.math.harvard.edu/~lurie/papers/HigherAlgebra.pdf](http://www.math.harvard.edu/~lurie/papers/HigherAlgebra.pdf)

[Mar16] Nikita Markarian. Weyl $n$-algebras and the Kontsevich integral of the unknot. *J. Knot Theory Ramifications*, 25(1):185–204, 1999.

[Mar17] Nikita Markarian. Weyl $n$-algebras. *Comm. Math. Phys.*, 350(2):421–442, 2017.

[MT15] Nikita Markarian and Hiro Lee Tanaka. *Factorization Homology in 3-Dimensional Topology*, pages 213–231. Springer International Publishing, Cham, 2015.

[RW] Carlo A. Ross and Thomas Willwacher. P. Etingof’s conjecture about Drinfeld associators. arXiv:1404.2047 [math.QA].

[Sal01] Paolo Salvatore. Configuration spaces with summable labels. In *Cohomological methods in homotopy theory (Bellaterra, 1998)*, volume 196 of *Progr. Math.*, pages 375–395. Birkhäuser, Basel, 2001.

[Tho16] Justin Thomas. Kontsevich’s swiss cheese conjecture. *Geom. Topol.*, 20(1):1–48, 2016.

[Vor99] Alexander A Voronov. The swiss-cheese operad. In *Homotopy invariant algebraic structures (Baltimore, MD, 1998)*, pages 365–373. Amer. Math. Soc., Providence, RI, 1999.

National Research University Higher School of Economics, Russian Federation, Department of Mathematics, 20 Myasnitskaya str., 101000, Moscow, Russia

E-mail address: nikita.markarian@gmail.com