INTEGRAL MEANS INEQUALITIES, CONVOLUTION, AND UNIVALENT FUNCTIONS

DANIEL GIRELA AND CRISTÓBAL GONZÁLEZ

Dedicated to Fernando Pérez González on the occasion of his retirement

Abstract. We use the Baernstein star-function to investigate several questions about the integral means of the convolution of two analytic functions in the unit disc. The theory of univalent functions plays a basic role in our work.

1. Introduction

Let $D = \{ z \in \mathbb{C} : |z| < 1 \}$ and $T = \{ z \in \mathbb{C} : |z| = 1 \}$ denote the open unit disc and the unit circle in the complex plane $\mathbb{C}$. We let also $\mathcal{Hol}(D)$ be the space of all analytic functions in $D$ endowed with the topology of uniform convergence in compact subsets.

If $0 \leq r < 1$ and $f \in \mathcal{Hol}(D)$, we set

$$M_p(r, f) = \left( \int_{-\pi}^{\pi} |f(re^{it})|^p \frac{dt}{2\pi} \right)^{1/p}, \quad \text{if } 0 < p < \infty,$$

$$M_\infty(r, f) = \sup_{|z|=r} |f(z)|.$$

For $0 < p \leq \infty$, the Hardy space $H^p$ consists of those $f \in \mathcal{Hol}(D)$ such that

$$\|f\|_{H^p} \overset{\text{def}}{=} \sup_{0 \leq r < 1} M_p(r, f) < \infty.$$

We refer to [6] for the theory of $H^p$-spaces.

If $f, g \in \mathcal{Hol}(D)$,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad g(z) = \sum_{n=0}^{\infty} b_n z^n \quad (z \in D),$$

the (Hadamard) convolution $(f * g)$ of $f$ and $g$ is defined by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n, \quad z \in D.$$

We have the following integral representation

$$(f \ast g)(z) = \frac{1}{2\pi i} \int_{|\xi|=r} f\left(\frac{z}{\xi}\right) g(\xi) \frac{d\xi}{\xi}, \quad |z| < r < 1,$$
The convolution operation \( \star \) makes \( \text{Hol}(D) \) into a commutative complex algebra with an identity

\[
I(z) = \frac{1}{1 - z} = \sum_{n=0}^{\infty} z^n, \quad z \in \mathbb{D}.
\]

We refer to [14] for the theory of the convolution of analytic functions and its connections with geometric function theory.

Following [15], we shall say that a function \( F \in \text{Hol}(D) \) is bound preserving if for every \( f \in H^\infty \) we have that \( f \star F \in H^\infty \) and

\[
\|f \star F\|_{H^\infty} \leq \|f\|_{H^\infty}.
\]

Sheil-Small [15, Theorem 1.3] (see also [14, p. 123]) proved that a function \( F \in \text{Hol}(D) \) is bound preserving if and only if there exists a complex Borel measure \( \mu \) on \( \mathbb{T} \) with \( \|\mu\| \leq 1 \) such that

\[
F(z) = \int_{\mathbb{T}} \frac{d\mu(\xi)}{1 - z\xi}, \quad z \in \mathbb{D}.
\]

The measure \( \mu \) is a probability measure if and only if \( F \) is convexity preserving, that is, for any \( f \in \text{Hol}(D) \) the range of \( f \star F \) is contained in the closed convex hull of the range of \( f \) [14 pp. 123, 124].

It turns out that if \( F \) is bound preserving and \( 1 \leq p \leq \infty \), then for every \( f \in H^p \) we have that \( f \star F \in H^p \) and

\[
\|f \star F\|_{H^p} \leq \|f\|_{H^p}.
\]

Actually, the following stronger result holds.

**Theorem 1.** Suppose that \( f, F \in \text{Hol}(D) \) with \( F \) being bound preserving. Then

\[
M_p(r, f \star F) \leq M_p(r, f), \quad 0 < r < 1,
\]

whenever \( 1 \leq p \leq \infty \).

**Proof.** Since \( F \) is bound preserving, there exists a complex Borel measure \( \mu \) on \( \mathbb{T} \) with \( \|\mu\| \leq 1 \) such that

\[
F(z) = \int_{\mathbb{T}} \frac{d\mu(\xi)}{1 - z\xi} = \sum_{n=0}^{\infty} \left( \int_{\mathbb{T}} \xi^n d\mu(\xi) \right) z^n, \quad z \in \mathbb{D}.
\]

If \( f(z) = \sum_{n=0}^{\infty} a_n z^n \ (z \in \mathbb{D}) \), we have

\[
(f \star F)(z) = \sum_{n=0}^{\infty} a_n \left( \int_{\mathbb{T}} \xi^n d\mu(\xi) \right) z^n
\]

\[
= \int_{\mathbb{T}} \left( \sum_{n=0}^{\infty} a_n \xi^n z^n \right) d\mu(\xi) = \int_{\mathbb{T}} f(\xi z) d\mu(\xi), \quad z \in \mathbb{D}.
\]
This immediately yields (1.2) for \( p = \infty \). Now, if \( 1 \leq p < \infty \), using Minkowski’s integral inequality we obtain

\[
M_p(r, f \ast F) = \left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \int_T f(r\xi e^{i\theta})d\mu(\xi) \right|^p d\theta \right]^{1/p} \\
\leq \left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \int_T |f(r\xi e^{i\theta})|d|\mu|(\xi) \right)^p d\theta \right]^{1/p} \\
\leq \int_T \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(r\xi e^{i\theta})|^p d\theta \right)^{1/p} d|\mu|(\xi) \\
= \int_T M_p(r, f) d|\mu|(\xi) \leq M_p(r, f).
\]

□

2. Star-type inequalities

The main purpose of this article is studying the possibility of extending Theorem 1 to cover other integral means, at least for some special classes of functions. In order to do so, we shall use the method of the star-function introduced by A. Baernstein [2, 3].

If \( u \) is a subharmonic function in \( \mathbb{D} \setminus \{0\} \), the function \( u^* \) is defined by

\[
u^*(re^{i\theta}) = \sup_{|E|=2\theta} \int_E u(re^{it})dt, \quad 0 < r < 1, \quad 0 \leq \theta \leq \pi,
\]

where \( |E| \) denotes the Lebesgue measure of the set \( E \). The basic properties of the star-function which make it useful to solve extremal problems are the following [3]:

- If \( u \) is a subharmonic function in \( \mathbb{D} \setminus \{0\} \), then the function \( u^* \) is subharmonic in \( \mathbb{D}^+ = \{ z = re^{i\theta} : 0 < r < 1, 0 < \theta < \pi \} \) and continuous in \( \{ z = re^{i\theta} : 0 < r < 1, 0 \leq \theta \leq \pi \} \).

- If \( v \) is harmonic in \( \mathbb{D} \setminus \{0\} \), and it is a symmetric decreasing function on each of the circles \( \{|z| = r\} \) (\( 0 < r < 1 \)), then \( v^* \) is harmonic in \( \mathbb{D}^+ \) and, in fact, \( v^*(re^{i\theta}) = \int_0^\theta v(re^{it})dt \).

The relevance of the star-function to obtain integral means estimates comes from the following result.

**Proposition A** ([3]). Let \( u \) and \( v \) be two subharmonic functions in \( \mathbb{D} \). Then the following two conditions are equivalent:

(i) \( u^* \leq v^* \) in \( \mathbb{D}^+ \).

(ii) For every convex and increasing function \( \Phi : \mathbb{R} \rightarrow \mathbb{R} \), we have that

\[
\int_{-\pi}^{\pi} \Phi \left( u(re^{i\theta}) \right) d\theta \leq \int_{-\pi}^{\pi} \Phi \left( v(re^{i\theta}) \right) d\theta, \quad 0 < r < 1.
\]

Proposition A yields the following result about analytic functions.

**Proposition B.** Let \( f \) and \( g \) be two non-identically zero analytic functions in \( \mathbb{D} \). Then the following conditions are equivalent:

(i) \( (\log |f|)^* \leq (\log |g|)^* \) in \( \mathbb{D}^+ \).
For every convex and increasing function $\Phi : \mathbb{R} \to \mathbb{R}$, we have that

$$
\int_{-\pi}^{\pi} \Phi \left( \log |f(re^{i\theta})| \right) d\theta \leq \int_{-\pi}^{\pi} \Phi \left( \log |g(re^{i\theta})| \right) d\theta, \quad 0 < r < 1.
$$

Since for any $p > 0$ the function $\Phi$ defined by $\Phi(x) = \exp(px)$ ($x \in \mathbb{R}$) is convex and increasing we deduce that if $f$ and $g$ are as in Proposition B and $(\log |f|)^* \leq (\log |g|)^*$ in $\mathbb{D}^+$, then

$$M_p(r, f) \leq M_p(r, g), \quad 0 < r < 1,$$

for all $p > 0$.

The main achievement in the use of the star-function by A. Baernstein in [3], was the proof that the Koebe function $k(z) = \frac{z}{(1-z)^2}$ ($z \in \mathbb{D}$) is extremal for the integral means of functions in the class $S$ of univalent functions (see [6] and [13] for the notation and results regarding univalent functions). Namely, Baernstein proved that if $f \in S$ then

$$ (\pm \log |f|)^* \leq (\pm \log |k|)^* $$

and, hence,

$$\int_{-\pi}^{\pi} |f(re^{i\theta})|^p d\theta \leq \int_{-\pi}^{\pi} |k(re^{i\theta})|^p d\theta, \quad 0 < r < 1,$$

for all $p \in \mathbb{R}$. In particular, we have that if $f \in S$ and $0 < p \leq \infty$, then

$$M_p(r, f) \leq M_p(r, k), \quad 0 < r < 1.$$

Subsequently the star-function has been used in a good number of papers to obtain bounds on the integral means of distinct classes of analytic functions (see, e.g., [4, 11, 5, 8, 9, 12]).

Coming back to convolution, the following questions arise in a natural way.

**Question 1.** Let $f, g, F, G$ be analytic functions in $\mathbb{D}$ with $|F|$ and $|G|$ being symmetric decreasing on each of the circles $\{|z| = r\}$ and suppose that

$$(\log |f|)^* \leq (\log |F|)^* \quad \text{and} \quad (\log |g|)^* \leq (\log |G|)^*.$$ 

Does it follow that $(\log |f \ast g|)^* \leq (\log |F \ast G|)^*$?

**Question 2.** Let $F$ and $f$ be two analytic functions in $\mathbb{D}$ and suppose that $F$ is bound preserving. Can we assert that $(\log |f \ast F|)^* \leq (\log |f|)^*$?

We shall show that the answer to these two questions is negative. Regarding Question 1 we have the following result.

**Theorem 2.** There exist two functions $F_1, F_2 \in \text{Hol}(\mathbb{D})$ with

$$(\log |F_j|)^* \leq (\log |I|)^*, \quad \text{for } j = 1, 2,$$

and such that

$$(2.1) \quad \text{the inequality } (\log |F_1 \ast F_2|)^* \leq (\log |I \ast I|)^* \text{ does not hold.}$$

Here, $I$ is the identity element of the convolution defined in (1.1), that is, $I(z) = \frac{1}{1-z}$ ($z \in \mathbb{D}$). Hence $I \ast I = I$.

**Proof.** Let $h$ be an odd function in the class $S$ with Taylor expansion

$$h(z) = z + a_3 z^3 + a_5 z^5 + \ldots$$
with \(|a_5| > 1\). The existence of such an \(h\) was proved by Fekete and Szegö (see [7, p.104]).

Set also

\[
(2.2) \quad h_1(z) = \frac{h(z)}{z} = 1 + a_3 z^2 + a_5 z^4 + \ldots, \quad z \in \mathbb{D}.
\]

It is well known that there exists a function \(H \in S\) such that \(h(z) = \sqrt{H(z^2)}\) (see [7, p.64]).

Set \(k_2(z) = \sqrt{k(z^2)} = \frac{z}{1-z^2}\) and \(J(z) = \frac{k_2(z)}{z} = \frac{1}{1-z^2}\) (\(z \in \mathbb{D}\)). By Baernstein’s theorem we have \((\log |H|)^* \leq (\log |k|)^*\), a fact which easily implies that \((\log |h_1|)^* \leq (\log |J|)^*\).

Now, it is clear that \(J\) is subordinate to \(I\) and then, using [11, Lemma 2], we see that \((\log |J|)^* \leq (\log |I|)^*\). Thus it follows that

\[
(2.3) \quad (\log |h_1|)^* \leq (\log |I|)^*.
\]

For \(n = 1, 2, 3, \ldots\), we define \(f_n\) inductively as follows

\[
f_1 = h_1 \quad \text{and} \quad f_n = f_{n-1} * f_1, \quad \text{for } n \geq 2.
\]

In other words, \(f_n = h_1 * \cdots * h_1\). Clearly, \((2.2)\) yields

\[
f_n(z) = 1 + a_3^n z^2 + a_5^n z^4 + \ldots.
\]

Since \(|a_5| > 1\), it follows that \(|a_5^n| \to \infty\), as \(n \to \infty\). This is equivalent to saying that

\[
|f_n^{(4)}(0)| \to \infty, \quad \text{as } n \to \infty.
\]

Then it follows that the family \(\{f_n^{(4)} : n = 1, 2, 3, \ldots\}\) is not a locally bounded family of holomorphic functions in \(\mathbb{D}\). Using [11, Theorem 16, p.225] we see that the same is true for the family \(\{f_n : n = 1, 2, 3, \ldots\}\). Take \(p \in (0,1)\), then \(I \in H^p\). Since a bounded subset of \(H^p\) is a locally bounded family [6, p.36], it follows that

\[
(2.4) \quad \sup_{n \geq 1} \|f_n\|_{H^p} = \infty.
\]

Now, \((2.4)\) implies that \(\|f_n\|_{H^p} > \|I\|_{H^p}\) for some \(n\). Using Proposition [13] we see that this implies that

the inequality \((\log |f_n|)^* \leq (\log |I|)^*\) is not true for some \(n\).

Let \(N\) be the smallest of all such \(n\). Using \((2.3)\) and the fact that \(f_1 = h_1\), it follows that that \(N > 1\).

Then it is clear that \((2.1)\) holds with \(F_1 = f_1, F_2 = f_{N-1}\). \(\square\)

We have the following result regarding Question [2]

**Theorem 3.** There exist \(f, F\) analytic and univalent in \(\mathbb{D}\) such that \(F\) is convexity preserving and with the property that the inequality \((\log |f * F|)^* \leq (\log |f|)^*\) does not hold.

The following lemma will be used in the proof of Theorem [3]

**Lemma 1.** Let \(f, F \in \mathcal{H}o l((D)\) and suppose that \(F(0) = 1\), \(F\) is convexity preserving, and that \(f\) and \(f * F\) are zero-free in \(\mathbb{D}\) and satisfy the inequality \((\log |f * F|)^* \leq (\log |f|)^*\). Then we also have that

\[
(2.5) \quad \left(\log \left\lfloor \frac{1}{f * F} \right\rfloor \right)^* \leq \left(\log \left\lfloor \frac{1}{F} \right\rfloor \right)^*.
\]
Proof. Set \( u = \log |f \ast F|, \) \( v = \log |f|. \) Then \( u \) and \( v \) are harmonic in \( \mathbb{D}, \) \( u(0) = v(0), \) and \( u^* \leq v^*. \) Then it follows that, for \( 0 < r < 1 \) and \( 0 \leq \theta \leq \pi, \)

\[
(-u)^*(re^{i\theta}) = \sup_{|E|=2\theta} \int_{E} u(re^{it})dt = \sup_{|E|=2\theta} \left( -\int_{-\pi}^{\pi} u(re^{it})dt + \int_{[-\pi,\pi] \setminus E} u(re^{it})dt \right) = -2\pi u(0) + u^*(re^{i(\pi-\theta)}) \leq -2\pi v(0) + v^*(re^{i(\pi-\theta)}) = (-v)^*(re^{i\theta}).
\]

Hence, we have proved that \((-u)^* \leq (-v)^*\) which is equivalent to (2.5). \( \square \)

**Proof of Theorem 3.** Set

\[
f(z) = \frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} (n+1)z^n, \quad F(z) = 1 - \frac{1}{2}z, \quad z \in \mathbb{D}.
\]

Clearly, \( f \) and \( F \) are analytic, univalent, and zero-free in \( \mathbb{D}. \) Also

\[(f \ast F)(z) = 1 - z, \quad z \in \mathbb{D}.
\]

Hence \( f \ast F \) is also zero-free in \( \mathbb{D}. \) Notice that \( \frac{1}{f \ast F} \notin H^\infty \) and \( \frac{1}{f} \in H^\infty. \) Then it follows that

\[(2.6) \quad \text{the inequality } \left( \log \left| \frac{1}{f \ast F} \right| \right)^* \leq \left( \log \left| \frac{1}{f} \right| \right)^* \text{ does not hold.}
\]

Now, it is a simple exercise to check that

\[F(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - \cos \theta}{1 - e^{i\theta}z} d\theta
\]

and then it follows that \( F \) is convexity preserving. Then, using (2.6) and Lemma 11 it follows that the inequality \( (\log |f \ast F|)^* \leq (\log |f|)^* \) does not hold, as desired. \( \square \)

We close the paper with a positive result, determining a class of univalent functions \( \mathcal{Z} \) such that (1.2) is true for all \( p > 0, \) whenever \( f \in \mathcal{Z} \) and \( F \) is convexity preserving.

A domain \( D \) in \( \mathbb{C} \) is said to be Steiner symmetric if its intersection with each vertical line is either empty, or is the whole line, or is a segment placed symmetrically with respect to the real axis. We let \( \mathcal{Z} \) be the class of all functions \( f \) which are analytic and univalent in \( \mathbb{D} \) with \( f(0) = 0, \) \( f'(0) > 0, \) and whose image is a Steiner symmetric domain. The elements of \( \mathcal{Z} \) will be called Steiner symmetric functions. Using arguments similar to those used by Jenkins 10 for circularly symmetric functions, we see that a univalent function \( f \) with \( f(0) = 0 \) and \( f'(0) > 0 \) is Steiner symmetric if and only if it satisfies the following two conditions: (i) \( f \) is typically real and (ii) \( \text{Re} \ f \) is a symmetric decreasing function on each of the circles \( \{|z| = r\} \) \((0 < r < 1). \) Then it follows that if \( f \in \mathcal{Z} \) then for every \( r \in (0, 1), \) the domain \( f (\{|z| < r\}) \) is a Steiner symmetric domain and, hence, the function \( f_r \) defined by \( f_r(z) = f(rz) \) \((z \in \mathbb{D}) \) belongs to \( \mathcal{Z} \) and it extends to an analytic function in the closed unit disc \( \overline{\mathbb{D}}. \) Now we can state our last result.

**Theorem 4.** Suppose that \( f \in \mathcal{Z} \) and let \( F \) be an analytic function in \( \mathbb{D} \) which is convexity preserving. We have, for every \( p > 0, \)

\[(2.7) \quad M_p(r, f \ast F) \leq M_p(r, f), \quad 0 < r < 1.
\]
Proof. In view of Theorem 1 we only need to prove (2.7) for 0 < p < 1. Let \( \mu \) be the probability measure on \( \mathbb{T} \) such that \( F(z) = \int_{\mathbb{T}} \frac{d\mu(\xi)}{1-\xi z} \) (\( z \in \mathbb{D} \)). Then we have

\[
(f * F)(z) = \int_{\mathbb{T}} f(\xi z)d\mu(\xi).
\]

Since \( F \) is convexity preserving, for 0 < \( r < 1 \), we have that \( (f_r * F)(\mathbb{D}) \) is contained in the closed convex hull of \( f_r(\mathbb{D}) \). This easily yields

\[
\min_{z \in \mathbb{D}} \Re f_r(z) \leq \min_{z \in \mathbb{D}} \Re (f_r * F)(z), \quad \max_{z \in \mathbb{D}} \Re (f_r * F)(z) \leq \max_{z \in \mathbb{D}} \Re f_r(z).
\]

By the remarks in the previous paragraph, we find that, for all \( r \in (0, 1) \), \( f_r \) belongs to \( Z \) and extends to an analytic function in the closed unit disc \( \overline{\mathbb{D}} \). Finally, we claim that

\[
\Re (f_r * F) \leq (\Re f_r)^*, \quad 0 < r < 1.
\]

Once this is proved, using Proposition 6 of [5], we deduce that

\[
M_p(r, f * F) = \|f_r * F\|_{H^p} \leq \|f_r\|_{H^p} = M_p(r, f), \quad 0 < p \leq 2,
\]

finishing our proof.

So we proceed to prove (2.9). Fix \( r \in (0, 1) \) and set \( u = \Re (f_r * F), \quad v = \Re f_r \). Using (2.8), we have, for 0 < \( R < 1 \) and 0 < \( \theta < \pi \),

\[
u^*(Re^{i\theta}) = \sup_{|E|=2\theta} \int_E u(Re^{i\theta})dt = \sup_{|E|=2\theta} \int_E \int_T v(Re^{i\theta}\xi)d\mu(\xi)dt
\]

\[
= \sup_{|E|=2\theta} \int_E \int_T v(Re^{i\theta}\xi)d\mu(\xi) \leq \int_T v^*(Re^{i\theta})d\mu(\xi) = v^*(Re^{i\theta}).
\]

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ANÁLISIS MATEMÁTICO, UNIVERSIDAD DE MÁLAGA, CAMPUS DE TEATINOS, 29071 MÁLAGA, SPAIN
E-mail address: girela@uma.es

ANÁLISIS MATEMÁTICO, UNIVERSIDAD DE MÁLAGA, CAMPUS DE TEATINOS, 29071 MÁLAGA, SPAIN
E-mail address: cmge@uma.es