ITERATIVE BERNSTEIN SPLINES TECHNIQUE APPLIED TO FRACTIONAL ORDER DIFFERENTIAL EQUATIONS

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Abstract. In this work we will discuss about an approximation method for initial value problems associated to fractional order differential equations. For this method we will use Bernstein spline approximation in combination with the Banach’s Fixed Point Theorem. In order to illustrate our results, some numerical examples will be presented at the end of this article.

1. Introduction. In this work it is presented an approximation method based on the Banach’s Fixed Point Principle, combined with Bernstein splines polynomial approximation in order to solve the following fractional order initial value problem:

\[
\begin{aligned}
D^\alpha x(t) &= f(t, x(t)), \quad t \in [0, T] \\
x^{(k)}(0) &= x_0^k, \quad k = 0, 1, \ldots, \lceil \alpha \rceil - 1
\end{aligned}
\]  

(1)

with $T, \alpha > 0$, $x_0^0 \in \mathbb{R}$, $f \in C([0, T] \times \mathbb{R})$ and $\lceil \alpha \rceil$ denote the smallest integer greater than (or equal to) $\alpha$. We will split the $[0, T]$ interval in subintervals and at each iterative step of constructing the sequence of successive approximations we will replace each value with its Bernstein polynomial approximation.

This technique can be used for various other fractional order differential or integral equations:

1. Two-point boundary value problems:

\[
\begin{aligned}
D^\alpha x(t) &= f(t, x(t)), \quad t \in [0, T] \\
ax(0) + bx(T) &= c
\end{aligned}
\]

which are equivalent with the following fractional order Volterra-Fredholm integral equation (see Lemma 3.2 in [1] and Lemma 6.40 from [5]):

\[
x(t) = \frac{c}{a + b} + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} f(s, x(s)) \, ds - \frac{b}{(a + b)\Gamma(\alpha)} \int_0^T (T - s)^{\alpha - 1} f(s, x(s)) \, ds
\]

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2. Volterra integral equations:

\[
x(t) = g(t) + \frac{1}{\Gamma(\alpha)} \int_0^t H(t,s)f(s,x(s))(t-s)^{\alpha-1} \, ds, \quad t \in [0,T]
\]

3. Fredholm integral equations:

\[
x(t) = g(t) + \frac{1}{\Gamma(\alpha)} \int_0^T H(t,s)f(s,x(s))(t-s)^{\alpha-1} \, ds, \quad t \in [0,T].
\]

Fractional order differential and integral equations have many applications in different domains like mechanics, (bio-)chemistry (modelling of polymers and proteins), electrical engineering (transmission of ultrasound waves), medicine (modelling of human tissue under mechanical loads) [5], viscoelasticity [9]. For instance, fractional order differential equations models the thermoelastic phenomena in porous materials (see [2]). The well-known Bagley-Torvik fractional differential equation arise in viscoelasticity (see [3]), while the fractional Kersten-Krasil’shchik coupled KdV-nKdV system is involved in the study of multi-component plasmas (see [7]). In the hematopoietic stem cell modeling, singular fractional integro-differential equations are used (see [11]).

The "First Conference on Fractional Calculus and its Applications” was organized by Bertram Ross at the University of New Haven in June 1974 as it is stated in [6].

There are some other numerical methods like collocation method (see [12]), or collocation method combined with an operational matrix for Bernoulli wavelets (see [13]). The Laplace transform was used in [8] for solving linear fractional differential equations, while the Adams-Bashfoth technique was developed in [5] for the numerical solution of fractional order initial value problems in the Capito’s sense. The trapezoidal product integration for solving fractional order initial value problems was constructed in [4]. In this work, a Bernstein splines integration technique is developed containing as a simple particular case the trapezoidal product integration.

2. Existence and uniqueness of the solution. The initial value problem (1) is equivalent with the following Volterra integral equation:

\[
x(t) = \sum_{k=0}^{[\alpha]-1} \frac{t^k}{k!} x_0^k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,x(s)) \, ds, \quad t \in [0,T]
\]

(Lemma 6.2 from [5]).

The existence and uniqueness of the solution is proved in Theorem 6.1 and Theorem 6.5 from [5] using Schauder’s Fixed Point Theorem and Weissinger’s Fixed Point Theorem.
We will denote
\[ S(t) = \sum_{k=0}^{[\alpha]-1} \frac{t^k}{k!} x_0^k \]
and we will consider the space \( X = (C[0,T], \|\cdot\|_\tau) \) where
\[ \|x\|_\tau = \max_{t \in [0,T]} |x(t)| e^{-\tau t}, \quad x \in X \]
is the Bielecki norm and the operator \( A : X \to X \) defined by:
\[ A(x(t)) = S(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s)) \, ds. \]

**Theorem 2.1.** If \( f \in C([0,T] \times \mathbb{R}) \), is Lipschitz on its second argument with the Lipschitz constant \( L \), and \( \tau \) is chosen so that
\[ \tau \geq \frac{L}{\tau^\alpha} \]
then the initial value problem (1) has a unique solution \( x^* \in C[0,T] \).

**Proof.** First we will consider the ball \( B_R = \{ x \in X | \|x - S\|_\tau \leq R \} \) with \( R = \frac{\|f\|_\tau}{\tau} \)
and we will prove that \( A(B_R) \subseteq B_R \).

Let \( x \in B_R \) and calculate
\[ |A(x(t)) - S(t)| \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s)) \, ds \]
\[ \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s)) e^{-\tau s} e^{\tau s} \, ds \]
\[ \leq \frac{\|f\|_\tau}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{\tau s} \, ds \]
The integral will be calculated, by using the change of variable \( u = \tau(t-s) \):
\[ \int_0^t (t-s)^{\alpha-1} e^{\tau s} \, ds = \int_0^{\tau t} (\frac{u}{\tau})^{\alpha-1} e^{-u} \frac{du}{\tau} = \frac{e^{\tau t}}{\tau^\alpha} \int_0^{\tau t} u^{\alpha-1} e^{-u} \, du \leq \frac{e^{\tau t}}{\tau^\alpha} \Gamma(\alpha) \]
Now we obtain the inequality:
\[ |A(x(t)) - S(t)| \leq \frac{\|f\|_\tau}{\tau^\alpha} e^{\tau t} \]
and multiplying it by \( e^{-\tau t} \) we obtain the required relation:
\[ \|A(x) - S\|_\tau \leq \frac{\|f\|_\tau}{\tau^\alpha} = R \]
so $A(x) \in B_R$. In the next step in a similar way, we obtain

$$
|A(x)(t) - A(y)(t)| \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s,x(s)) - f(s,y(s))| \, ds
\leq \frac{L}{\Gamma(\alpha)} \|x - y\|_\tau \int_0^t (t-s)^{\alpha-1} e^{\tau s} \, ds
\leq \frac{L}{\tau^\alpha} \|x - y\|_\tau e^{\tau t}
$$

obtaining

$$
\|A(x) - A(y)\|_\tau \leq \frac{L}{\tau^\alpha} \|x - y\|_\tau, \quad \forall x, y \in X
$$

From (3) it follows that $A$ is a contraction, so it has an unique fixed point, which is, obviously the solution of the equation (2).

Another approach for obtaining an existence and uniqueness result is to impose a contraction condition to the kernel function under the norm $\|\cdot\|_\infty$. This contraction condition will be the same as the condition that ensures the convergence of the proposed iterative method. This is the reason for which we have considered the both two existence and uniqueness results.

**Theorem 2.2.** If $f \in C([0,T] \times \mathbb{R})$, is Lipschitz on its second argument with the Lipschitz constant $L$, and $LT^\alpha < \Gamma(\alpha + 1)$, then the initial value problem (1) has a unique solution $x^* \in C[0,T]$.

**Proof.**

$$
|A(x)(t) - A(y)(t)| \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s,x(s)) - f(s,y(s))| \, ds
\leq \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \|x - y\|_\infty
\leq \frac{LT^\alpha}{\Gamma(\alpha + 1)} \|x - y\|_\infty
$$

This operator is a contraction with the constant

$$
w = \frac{LT^\alpha}{\Gamma(\alpha + 1)} < 1
$$

so it has a unique fixed point, which is obviously the solution of the equation (2).

**3. Boundedness of the sequence of successive approximation.** Considering the sequence of successive approximations associated to this integral operator, $x_{m+1} = A(x_m)$, $m \in \mathbb{N}$, with any starting point $x_0$ (usually we choose the term without the integral), by the Banach’s Fixed Point Theorem, we get the following estimates:

$$
|x^*(t) - x_m(t)| \leq \frac{w^m}{1-w} |x_1(t) - x_0(t)|
$$

and

$$
|x^*(t) - x_m(t)| \leq \frac{w}{1-w} |x_m(t) - x_{m-1}(t)|, \quad \forall m \in \mathbb{N}^*, \forall t \in [0,T].
$$
We will denote $S(t) = \sum_{k=0}^{[\alpha]-1} \frac{t^k}{k!} x_0^k$. The sequence of successive approximations can be written as follows:

$$x_{m+1}(t) = S(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x_m(s)) \, ds, \quad t \in [0, T], \quad m \in \mathbb{N} \quad (5)$$

with $x_0(t) = S(t)$. Considering $M_0 \geq 0$ the maximum of $f$ (being continuous on $[0, T]$), so $|f(t, x_0(t))| \leq M_0, \forall t \in [0, T]$, we get:

$$|x_1(t) - x_0(t)| = \left| S(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x_0(s)) \, ds - S(t) \right|$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, x_0(s))| \, ds$$

$$\leq \frac{M_0 T^\alpha}{\Gamma(\alpha + 1)}$$

So the relation (4) becomes

$$|x^*(t) - x_m(t)| \leq \frac{w^m M_0 T^\alpha}{\Gamma(\alpha + 1) (1-w)} \quad (6)$$

**Proposition 1.** Under the conditions of Theorem 2.2 the sequence of successive approximations is uniformly bounded.

**Proof.** We will consider the following recursive relation between the items of the sequence:

$$|x_{m+1}(t) - x_m(t)| = \frac{1}{\Gamma(\alpha)} \left| \int_0^t (t-s)^{\alpha-1} (f(s, x_m(s)) - f(s, x_{m-1}(s))) \, ds \right|$$

$$\leq \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |x_m(s) - x_{m-1}(s)| \, ds$$

$$\leq w \|x_m - x_{m-1}\|_\infty, \quad \forall t \in [0, T]$$

obtaining by induction ($w < 1$):

$$|x_{m+1}(t) - x_m(t)| \leq w^m \|x_1 - x_0\|_\infty \leq \|x_1 - x_0\|_\infty$$

resulting the following relation:

$$|x_{m+1}(t) - x_m(t)| \leq \frac{M_0 T^\alpha}{\Gamma(\alpha + 1)}, \quad \forall t \in [0, T], \quad \forall m \in \mathbb{N}$$
Because $S$ is a polynomial function on $[0, T]$, it is continuous, so we can consider $M_S > 0$, such as $|S(t)| \leq M_S$. In this way, for $x_m$, we have:

$$
|x_m(t)| \leq |x_m(t) - x_{m-1}(t)| + \cdots + |x_1(t) - x_0(t)| + |x_0(t)| \\
\leq (w^{m-1} + w^{m-2} + \cdots + 1) \|x_1 - x_0\|_{\infty} + M_S \\
\leq \frac{1 - w^m}{1 - w} M_0 T^\alpha + M_S \\
\leq \frac{M_0 T^\alpha}{\Gamma(\alpha + 1)(1 - w)} + M_S = R, \ \forall t \in [0, T], \ \forall m \in \mathbb{N}
$$

So the uniform boundedness of the sequence is proved.

Consider the following notation:

$$
F_m(t) = f(t, x_m(t)) \ \forall t \in [0, T], \ m \in \mathbb{N}
$$

Using the Lipschitz property of $f$, we obtain:

$$
|F_m(t)| \leq |f(t, x_m(t)) - f(t, x_0(t))| + |f(t, x_0(t))| \\
\leq \frac{LM_0 T^\alpha}{\Gamma(\alpha + 1)(1 - w)} + M_0 = M, \ \forall t \in [0, T], \ m \in \mathbb{N}^*
$$

so the sequence $(F_m)_{m \in \mathbb{N}}$ is uniformly bounded, too.

4. Bernstein approximation. Let us consider a uniform partition of the interval $[0, T]$, having the knots $t_i = ih$, $i = 0, n$, where $h = \frac{T}{n}$ is the stepsize. On each subinterval $[t_{i-1}, t_i], \ i = 1, n$ and for each iterative step $m \in \mathbb{N}$, we will replace the value of the continuous function $F_m$ by its Bernstein polynomial approximation of a given degree $q \geq 1$:

$$
B_{m,i}(t) = \frac{1}{h^q} \sum_{k=0}^{q} C_q^k (t - t_{i-1})^k (t_i - t)^{q-k} F_m \left( t_{i-1} + \frac{kh}{q} \right), \ t \in [t_{i-1}, t_i]. \ (7)
$$

For the Bernstein approximation formula $F_m(t) = B_{m,i}(t) + R_{m,i}(t)$, we will use the inequality of Lorentz in order to estimate its remainder:

$$
|R_{m,i}(t)| \leq \frac{5}{4} \omega \left( F_m, \frac{h}{\sqrt{q}} \right), \ \forall t \in [t_{i-1}, t_i], \ \forall i = 1, n, \ \forall m \in \mathbb{N}.
$$

Now, based on the relation (7) the formula (5) can be written as:

$$
x_m(t_k) = S(t_k) + \frac{1}{h^q} \sum_{i=1}^{n} \sum_{j=1}^{k} \left( t_k - s \right)^{q-1} \left( B_{m-1,i}(s) + R_{m-1,i}(s) \right) \left( t_{i-1} + \frac{kh}{q} \right), \ \forall m \in \mathbb{N}.
$$

(8)

If we compute the integrals from (8) by using the Bernstein polynomial formula (7) and the change of variable $s = t_{i-1} + uh$, we obtain

$$
\int_{t_{i-1}}^{t_i} (s - t_{i-1})^j (t_i - s)^{q-j} (t_k - s)^{\alpha-1} \, ds = h^{q+\alpha} \psi_{j,k}(i)
$$

where

$$
\psi_{j,k}(i) = \int_{0}^{1} u^j (1-u)^{q-j} (k - i + 1 - u)^{\alpha-1} \, du.
$$
In this way, for \( m \in \mathbb{N}^* \), it obtains the following iterative algorithm:

\[
x_m \left( t_k + \frac{lh}{q} \right) = S \left( t_k + \frac{lh}{q} \right) + \frac{h^{\alpha}}{\Gamma(\alpha)} \sum_{i=1}^{k} \sum_{j=0}^{q} C_q^j \psi_{j,k+\frac{l}{q}}(i) F_{m-1,i} \left( t_{i-1} + \frac{jh}{q} \right) + \sum_{j=0}^{q} C_q^j \psi_{j,k+\frac{l}{q}}(k+1) F_{m-1,k} \left( t_k + \frac{jh}{q} \right) + R_{m,k+\frac{l}{q}}
\]

for \( k = 0, n - 1, l = 0, q \), where

\[
\psi_{j,k+\frac{l}{q}}(i) = \begin{cases} 
\int_0^\frac{l}{q} u^j \left( 1 - u \right)^{q-j} \left( k + \frac{l}{q} - i + 1 - u \right)^{\alpha-1} du, & i \leq k \\
\int_0^\frac{jh}{q} u^j \left( 1 - u \right)^{q-j} \left( \frac{jh}{q} - u \right)^{\alpha-1} du, & i = k + 1.
\end{cases}
\]

Here, we have denoted \( F_{m-1,i} \left( t_{i-1} + \frac{jh}{q} \right) = f \left( t_{i-1} + \frac{jh}{q}, x_{m-1} \left( t_{i-1} + \frac{jh}{q} \right) \right) \), where \( x_{m-1} \left( t_{i-1} + \frac{jh}{q} \right) \) is the calculated value of \( x_m \) at a given step, without the remainder. The expression from (8) can be written in the form

\[
x_m \left( t_k + \frac{lh}{q} \right) = x_m \left( t_k + \frac{lh}{q} \right) + R_{m,k+\frac{l}{q}}, \quad k = 0, n - 1, \ l = 0, q. \tag{9}
\]

After we obtain the values in some points of the intervals for the last iterative step, we can extend our results, constructing a Bernstein spline approximation for these results. More exactly, by using the values \( x_m \left( t_k + \frac{lh}{q} \right), \ k = 0, n - 1, \ l = 0, q \) from the last iterative step, we can use the following formula on each subinterval \([t_{i-1}, t_i]\), with \( i = 1, n\):

\[
B_{m,q}(t) = \frac{1}{h^q} \sum_{j=0}^{q} C_q^j (t - t_{i-1})^j (t_i - t)^{q-j} \cdot x_m \left( t_{i-1} + \frac{jh}{q} \right), \ t \in [t_{i-1}, t_i].
\]

5. Main result.

**Main Theorem.** If \( f \in C([0,T] \times \mathbb{R}) \), is Lipschitz on its second argument with the Lipschitz constant \( L \), and \( LT^\alpha < \Gamma(\alpha + 1) \), the difference between the exact solution of the equation (1) and the calculated values \( x_m(t_k), \ k = 0, n, \ m \in \mathbb{N}^* \) from (9) is approximated by the following relation:

\[
|x^* - x_m| \leq \frac{M_0 T^\alpha w^m}{(1 - w) \Gamma(\alpha + 1)} + \frac{5 T^\alpha \omega \left( F_{m-1}, \frac{h}{\sqrt{q}} \right)}{4(1 - w) \Gamma(\alpha + 1)}, \ \forall k = 0, n, \ m \in \mathbb{N}^*. \tag{10}
\]

For the Bernstein approximation of the solution, the error estimation is:

\[
|x^* - B_m| \leq \frac{M_0 T^\alpha w^m}{(1 - w) \Gamma(\alpha + 1)} + \frac{5 T^\alpha \omega \left( F_{m-1}, \frac{h}{\sqrt{q}} \right)}{4(1 - w) \Gamma(\alpha + 1)} + \frac{5}{4} \omega \left( V_m, \frac{h}{\sqrt{q}} \right), \ \forall t \in [0,T], \ m \in \mathbb{N}^*. \tag{11}
\]

where \( V_m : [0,T] \to \mathbb{R} \) is given by

\[
V_m(t) = x_m \left( t + \frac{t_{i+1} - t}{h} \right) - x_m \left( t_i \right) + \frac{t - t_i}{h} \left( x_m \left( t_{i+1} \right) - x_m \left( t_i \right) \right) \tag{12}
\]
and \( \overline{\omega} \left( F_m, \frac{h}{\sqrt{q}} \right) = \max_{i=0}^{m} \omega \left( F_i, \frac{h}{\sqrt{q}} \right) \).

Proof. From the relation (8), for \( k = \overline{1, n} \), we get

\[
x_1(t_k) = S(t_k) + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_k - s)^{\alpha-1} (B_{0,i}(s) + R_{0,i}(s)) \, ds
\]

with \( |R_1(t_k)| \leq \frac{5T_0}{4\Gamma(\alpha+1)} \omega \left( F_0, \frac{h}{\sqrt{q}} \right) \).

In order to obtain the relation for \( R_2 \), let us calculate:

\[
x_2(t_k) = S(t_k) + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{1}{h^q} \sum_{j=0}^{q} C^j_q (s - t_{i-1})^j (t_i - s)^{\gamma-j} F_1 \left( t_{i-1} + \frac{jq}{q} \right) (t_k - s)^{\alpha-1} \, ds
\]

\[
+ \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{1}{h^q} \sum_{j=0}^{q} C^j_q (s - t_{i-1})^j (t_i - s)^{\gamma-j} R_{F_1} \left( t_{i-1} + \frac{jq}{q} \right) (t_k - s)^{\alpha-1} \, ds
\]

\[
+ \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{1}{h^q} \sum_{j=0}^{q} C^j_q (s - t_{i-1})^j (t_i - s)^{\gamma-j} R_{1,i} \left( t_{i-1} + \frac{jq}{q} \right) (t_k - s)^{\alpha-1} \, ds
\]

where

\[
|R_{F_1}| = |F_1(t) - \overline{F_1}(t)| = |f(t, x_1(t)) - f(t, \overline{x_1}(t))| \leq L |x_1(t) - \overline{x_1}(t)| = LR_1(t) \leq L \frac{5T_0}{4\Gamma(\alpha+1)} \omega \left( F_0, \frac{h}{\sqrt{q}} \right)
\]

Now, by using the change of variable \( s - t_{i-1} = uh \), we have

\[
\sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{1}{h^q} \left[ \sum_{j=0}^{q} \left( C^j_q (s - t_{i-1})^j (t_i - s)^{\gamma-j} \right) (t_k - s)^{\alpha-1} \right] \, ds =
\]

\[
= h^{q+\alpha} \sum_{i=1}^{k} \int_{0}^{1} \frac{1}{h^q} \left[ \sum_{j=0}^{q} C^j_q u^j (1-u)^{\gamma-j} \right] (k-i+1-u)^{\alpha-1} \, du \leq \frac{T_0}{\alpha}
\]

So, we obtained:

\[
|R_2(t_k)| \leq \left[ 1 + \frac{LT_0}{\Gamma(\alpha+1)} \right] \frac{5T_0}{4\Gamma(\alpha+1)} \omega \left( F_1, \frac{h}{\sqrt{q}} \right)
\]

By induction, for \( m \geq 2 \), we get

\[
|R_m(t_k)| \leq \left[ 1 + w + \ldots + w^{m-1} \right] \frac{5T_0}{4\Gamma(\alpha+1)} \omega \left( F_{m-1}, \frac{h}{\sqrt{q}} \right), \, k = \overline{1, n}.
\]

Because \( |x^*(t_k) - \overline{x_m}(t_k)| \leq |x^*(t_k) - x_m(t_k)| + |x_m(t_k) - \overline{x_m}(t_k)| \), by (13) and (6), we obtained the requested inequality (10).

For the second part of the theorem, \( V_m \) is continuous on \([0, T]\) with \( V_m(t_i) = \overline{x_m(t_i)} \), \( \forall i = 0, m \), and thus \( B_{m,q} \) interpolates \( V_m \) on the endpoints of the intervals.
By the inequality of Lorentz we have \(|V_m(t) - B_{m,q}(t)| \leq \frac{5}{4} \omega(V_m, \frac{h}{\sqrt{q}})\) for all \(t \in [t_i, t_{i+1}]\), \(i = 0, n - 1\). Then, we get
\[
|x_m(t) - V_m(t)| = \frac{t_{i+1} - t}{h} \left( \frac{1}{m} \sum_{j=0}^{m} x_j(t_i) - x_m(t_i) \right) + \frac{t - t_i}{h} \left( \frac{1}{m} \sum_{j=0}^{m} x_j(t_{i+1}) - x_m(t_{i+1}) \right)
\]
\[
\leq \left( \frac{t_{i+1} - t + t - t_i}{h} \right) \max_{i=0, \ldots, n-1} \{|x_m(t_i) - x_m(t)|, |x_m(t_{i+1}) - x_m(t_{i+1})|\}
\]
Then, by considering
\[
|\star(t) - B_{m,q}(t)| \leq |x^*(t) - x_m(t)| + |x_m(t) - V_m(t)| + |V_m(t) - B_{m,q}(t)|
\]
we obtain the estimate (11).

We have now \(\lim_{h \to 0} \omega(F_{m-1}, \frac{h}{\sqrt{q}}) = \lim_{h \to 0} \omega(V_m, \frac{h}{\sqrt{q}}) = 0\), because the functions \(F_m\) and \(V_m\) are continuous, for \(m \in \mathbb{N}^+\) and because we considered \(w \in (0, 1)\), by (10) and (11) we obtain the convergence of this method. \(\square\)

If we include a supplementary Lipschitz condition for \(f\), more precisely the Lipschitz condition in the first argument, then we can prove that the presented method has the order of convergence at least \(O(h^\alpha)\). This fact is stated in the following result.

**Corollary 1.** Under the conditions of Theorem 5, if there exists \(\gamma \geq 0\) such that
\[
|f(t_1, u) - f(t_2, u)| \leq \gamma |t_1 - t_2|, \quad \forall t_1, t_2 \in [0, T], \quad \forall u \in [-R, R]
\]
then, by considering \(L_S = \|S'\|_\infty = \max\{|S'(t) : t \in [0, T]\}\), in the case \(\alpha \in (0, 1]\), the estimate (10) becomes
\[
|x^*(k) - x_m(k)| \leq \frac{M_0 T^\alpha w^m}{(1 - w) \Gamma(\alpha + 1)} + \frac{5 T^\alpha \cdot \left(\gamma + L S h^{\alpha - 1}\right)}{4 (1 - w) \Gamma(\alpha + 1)}, \quad \forall k = 0, \ldots, m \in \mathbb{N}^+.
\]

If \(\alpha > 1\), this estimate is
\[
|x^*(k) - x_m(k)| \leq \frac{M_0 T^\alpha w^m}{(1 - w) \Gamma(\alpha + 1)} + \frac{5 T^\alpha \cdot \left(\gamma + L S h^{\alpha - 1}\right)}{4 (1 - w) \Gamma(\alpha + 1)}, \quad \forall k = 0, \ldots, m \in \mathbb{N}^+.
\]

Consequently, for \(\alpha \in (0, 1]\) the order of convergence is \(O(h^\alpha)\), while if \(\alpha > 1\), the order of convergence is \(O(h)\).

**Proof.** Let us consider \(t_1, t_2 \in [0, T]\) and by (5), using a similar reasoning as in [5], page 90, we get
\[
|x_m(t_1) - x_m(t_2)| \leq \begin{cases} 
L_S |t_1 - t_2| + \frac{2 M}{\Gamma(\alpha + 1)} \cdot |t_1 - t_2|^\alpha, & \text{if } \alpha \in (0, 1] \\
L_S |t_1 - t_2| + \frac{M}{\Gamma(\alpha + 1)} \left(|t_1 - t_2|^\alpha + |t_1 - t_2|^\alpha + |t_2 - t_2|^\alpha\right), & \text{if } \alpha > 1.
\end{cases}
\]
Now, in the second part of this inequality, for the case $\alpha > 1$, we use the Lagrange's Mean Value Theorem obtaining $|t_1^2 - t_2^2| \leq \alpha |t_1 - t_2| \cdot \xi^{\alpha-1}$, with $t_1 < \xi < t_2$. Then, this inequality becomes

$$|x_m (t_1) - x_m (t_2)| \leq L_S |t_1 - t_2| + \frac{M}{\Gamma (\alpha + 1)} \left(|t_1 - t_2|^{\alpha} + \alpha |t_1 - t_2| \cdot T^{\alpha-1}\right)$$

for all $t_1, t_2 \in [0, T]$, in the case $\alpha > 1$. Now, since $-R \leq x_m (t) \leq R$, $\forall m \in \mathbb{N}^*$, $\forall t \in [0, T]$, by using the supplementary Lipschitz condition we get

$$|F_m (t_1) - F_m (t_2)| \leq \gamma |t_1 - t_2| + L |x_m (t_1) - x_m (t_2)| \leq$$

$$\leq \left\{ \begin{array}{ll}
(\gamma + L \cdot L_S) |t_1 - t_2| + \frac{2M}{\Gamma (\alpha + 1)} \cdot |t_1 - t_2|^{\alpha}, & \text{if } \alpha \in (0, 1] \\
(\gamma + L \cdot L_S) |t_1 - t_2| + \frac{LM}{\Gamma (\alpha + 1)} \left(|t_1 - t_2|^{\alpha} + \alpha |t_1 - t_2| \cdot T^{\alpha-1}\right), & \text{if } \alpha > 1.
\end{array} \right.$$  

Then,

$$\varpi \left( F_{m-1}, \frac{h}{\sqrt[4]{R}} \right) \leq \left\{ \begin{array}{ll}
(\gamma + L \cdot L_S) \frac{h}{\sqrt[4]{R}} + \frac{2M}{\Gamma (\alpha + 1)} \cdot \left( \frac{h}{\sqrt[4]{R}} \right)^{\alpha}, & \text{if } \alpha \in (0, 1] \\
(\gamma + L \cdot L_S) \frac{h}{\sqrt[4]{R}} + \frac{LM}{\Gamma (\alpha + 1)} \left( \frac{h}{\sqrt[4]{R}} \right) + \alpha \frac{h}{\sqrt[4]{R}} \cdot T^{\alpha-1}, & \text{if } \alpha > 1
\end{array} \right.$$  

and the estimates (14) and (15) are obtained. \qed

6. **Numerical examples.** In order to prove our results, we will apply our method on the following numerical examples:

**Example 1.** The initial value problem

$$\begin{cases}
D^\frac{4}{3} x (t) = \frac{1}{3^5} t^2 x^3 - \frac{1}{3^5} t^2 \left( t^2 + 1 \right)^{\frac{3}{2}} + \frac{\Gamma (\frac{4}{3})}{\Gamma (\frac{3}{2})} t^\frac{3}{2}, & t \in [0, 1] \\
x (0) = 1
\end{cases}$$

has the exact solution $x^* (t) = t^\frac{3}{2} + 1$. We will consider $m = 20$ iterations and $n = 10$, $n = 20$, $n = 50$, $n = 100$ partition nodes on the $[0, T]$ interval and the Bernstein polynomial approximation of order $q = 1$ and $q = 4$.

| $t_m$ | $n = 10$ | $n = 20$ | $n = 50$ | $n = 100$ |
|-------|----------|----------|----------|----------|
| 0.0   | 0.00     | 0.00     | 0.00     | 0.00     |
| 0.1   | 3.31E-02 | 1.06E-02 | 2.88E-03 | 1.11E-03 |
| 0.2   | 1.74E-02 | 6.48E-03 | 1.84E-03 | 7.17E-04 |
| 0.3   | 1.35E-02 | 5.14E-03 | 1.47E-03 | 5.77E-04 |
| 0.4   | 1.18E-02 | 4.55E-03 | 1.31E-03 | 5.14E-04 |
| 0.5   | 1.12E-02 | 4.32E-03 | 1.25E-03 | 4.91E-04 |
| 0.6   | 1.12E-02 | 4.35E-03 | 1.26E-03 | 4.96E-04 |
| 0.7   | 1.19E-02 | 4.63E-03 | 1.34E-03 | 5.28E-04 |
| 0.8   | 1.34E-02 | 5.21E-03 | 1.51E-03 | 5.96E-04 |
| 0.9   | 1.62E-02 | 6.27E-03 | 1.82E-03 | 7.18E-04 |
| 1.0   | 2.11E-02 | 8.18E-03 | 2.37E-03 | 9.37E-04 |

Table 1. The numerical results for $q = 1$
The initial value problem

\[ D^2 x(t) = \frac{1}{2} x^2 - \frac{1}{2} (t^{1.9} - 1)^2 + \frac{\Gamma(2.9)}{\Gamma(1.4)} t^{0.4}, \quad t \in [0, 1] \]

\[ x(0) = -1, \quad x'(0) = 0 \]

has the exact solution \( x^*(t) = t^{1.9} - 1 \). We will consider \( m = 20 \) iterations and \( n = 20, n = 50, n = 100 \) partition nodes on the \([0, T]\) interval and the Bernstein polynomial approximation of order \( q = 1 \) and \( q = 4 \).

| \( t_i \) | \( n = 10 \) | \( n = 20 \) | \( n = 50 \) | \( n = 100 \) |
|---------|---------|---------|---------|---------|
| 0, 0    | 0.00    | 0.00    | 0.00    | 0.00    |
| 0, 1    | 8.75E-03 | 3.24E-03 | 9.16E-04 | 3.58E-04 |
| 0, 2    | 5.35E-03 | 2.05E-03 | 5.91E-04 | 2.32E-04 |
| 0, 3    | 4.24E-03 | 1.64E-03 | 4.75E-04 | 1.87E-04 |
| 0, 4    | 3.76E-03 | 1.46E-03 | 4.24E-04 | 1.67E-04 |
| 0, 5    | 3.57E-03 | 1.39E-03 | 4.04E-04 | 1.59E-04 |
| 0, 6    | 3.60E-03 | 1.40E-03 | 4.08E-04 | 1.61E-04 |
| 0, 7    | 3.83E-03 | 1.49E-03 | 4.35E-04 | 1.72E-04 |
| 0, 8    | 4.31E-03 | 1.68E-03 | 4.91E-04 | 1.94E-04 |
| 0, 9    | 5.19E-03 | 2.03E-03 | 5.91E-04 | 2.33E-04 |
| 1, 0    | 6.78E-03 | 2.65E-03 | 7.71E-04 | 3.05E-04 |

Table 2. The numerical results for \( q = 4 \).

Example 2. The initial value problem

\[
\begin{align*}
D^2 x(t) &= \frac{1}{2} x^2 - \frac{1}{2} (t^{1.9} - 1)^2 + \frac{\Gamma(2.9)}{\Gamma(1.4)} t^{0.4}, \\
\phi(t) &= 0, \\
x(0) &= -1, \\
x'(0) &= 0
\end{align*}
\]

has the exact solution \( x^*(t) = t^{1.9} - 1 \). We will consider \( m = 20 \) iterations and \( n = 20, n = 50, n = 100 \) partition nodes on the \([0, T]\) interval and the Bernstein polynomial approximation of order \( q = 1 \) and \( q = 4 \).

| \( t_i \) | \( n = 20 \) | \( n = 50 \) | \( n = 100 \) |
|---------|---------|---------|---------|
| 0, 0    | 0.00    | 0.00    | 0.00    |
| 0, 1    | 2.16E-03 | 6.69E-04 | 2.65E-04 |
| 0, 2    | 3.28E-03 | 9.65E-04 | 3.75E-04 |
| 0, 3    | 4.03E-03 | 1.17E-03 | 4.50E-04 |
| 0, 4    | 4.58E-03 | 1.31E-03 | 5.04E-04 |
| 0, 5    | 5.00E-03 | 1.42E-03 | 5.45E-04 |
| 0, 6    | 5.32E-03 | 1.51E-03 | 5.76E-04 |
| 0, 7    | 5.57E-03 | 1.57E-03 | 6.01E-04 |
| 0, 8    | 5.79E-03 | 1.63E-03 | 6.22E-04 |
| 0, 9    | 5.99E-03 | 1.68E-03 | 6.41E-04 |
| 1, 0    | 6.20E-03 | 1.74E-03 | 6.61E-04 |

Table 3. The numerical results for \( q = 1 \).

| \( t_i \) | \( n = 20 \) | \( n = 50 \) | \( n = 100 \) |
|---------|---------|---------|---------|
| 0, 0    | 0.00    | 0.00    | 0.00    |
| 0, 1    | 7.29E-04 | 2.17E-04 | 8.48E-05 |
| 0, 2    | 1.07E-03 | 3.09E-04 | 1.19E-04 |
| 0, 3    | 1.30E-03 | 3.71E-04 | 1.43E-04 |
| 0, 4    | 1.47E-03 | 4.17E-04 | 1.60E-04 |
| 0, 5    | 1.60E-03 | 4.51E-04 | 1.72E-04 |
| 0, 6    | 1.69E-03 | 4.77E-04 | 1.82E-04 |
| 0, 7    | 1.77E-03 | 4.98E-04 | 1.90E-04 |
| 0, 8    | 1.84E-03 | 5.15E-04 | 1.96E-04 |
| 0, 9    | 1.90E-03 | 5.31E-04 | 2.02E-04 |
| 1, 0    | 1.96E-03 | 5.48E-04 | 2.09E-04 |

Table 4. The numerical results for \( q = 4 \).
Example 3. The initial value problem
\[
\begin{aligned}
D_x^\alpha x(t) &= \frac{1}{3}x^3 - \frac{1}{3}t^4 + \Gamma\left(\frac{7}{3}\right) t, & t \in [0, 1] \\
x(0) &= 1
\end{aligned}
\]

has the exact solution \( x^e(t) = t^{\frac{4}{3}} \). We will consider \( m = 20 \) iterations and \( n = 10, n = 20, n = 50, n = 100 \) partition nodes on the \([0, T]\) interval and the Bernstein polynomial approximation of order \( q = 1 \) and \( q = 4 \).

| \( t_i \) | \( n = 10 \) | \( n = 100 \) |
|---|---|---|
| 0  | 0.00 | 0.00 |
| 0.3 | 4.44E-016 | 4.16E-016 |
| 0.6 | 1.33E-015 | 1.22E-015 |
| 0.8 | 3.02E-014 | 2.33E-015 |
| 1.0 | 7.24E-009 | 9.07E-011 |

Table 5. The numerical results for \( q = 1 \).

| \( t_i \) | \( n = 10 \) | \( n = 100 \) |
|---|---|---|
| 0  | 0.00 | 0.00 |
| 0.3 | 4.16E-016 | 3.61E-016 |
| 0.6 | 1.22E-015 | 1.22E-015 |
| 0.8 | 4.66E-015 | 2.33E-015 |
| 1.0 | 1.01E-009 | 8.16E-011 |

Table 6. The numerical results for \( q = 4 \).

7. Concluding remarks. Analysing the results obtained in the Tables 1–6 we can notice that we obtain better results as we increase the number of nodes in the interval and we obtained also better results for \( q = 4 \) than for \( q = 1 \). But I consider that increasing the order of the polynom is not necessary an improvement of the obtained result in every case, it depends on the shape of the approximated function on each subinterval \([t_i, t_{i+1}]\). For a linear function, for example, it is more appropriate to use \( q = 1 \) than \( q = 4 \).

For linear equations the solution is also available in analytic form as it is specified in [5].

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