MONODROMY TRANSFORM APPROACH IN THE THEORY OF INTEGRABLE REDUCTIONS OF EINSTEIN’S FIELD EQUATIONS AND SOME APPLICATIONS

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A brief sketch of the formulation of the monodromy transform approach and corresponding integral equation methods as well as of various applications of this approach for solution of integrable symmetry reductions of Einstein’s field equations is presented.

1. Introduction

For various nonlinear systems integrable by the well known Inverse Scattering Method (called sometimes also the Scattering Transform), the spaces of solutions are parameterized in terms of the scattering data of the corresponding potentials in the associated Schrödinger-like equation (associated spectral problem). The scattering data consist of a set of coordinate independent functions of a spectral parameter which characterize uniquely every potential (solution) and which can serve as the "coordinates" in the space of solutions of a given completely integrable system.

In some physically important cases of the symmetry reduced Einstein equations, the spaces of local solutions also can be parameterized by a finite set of coordinate independent functions of a complex ("spectral") parameter which determine the branching (monodromy) properties of a fundamental solution of associated linear systems. These data exist for any local solution and thus, in the infinite-dimensional space of local solutions we have two systems of "coordinates" – the sets of functional parameters whose particular values characterize every local solution uniquely:

the field components: $g_{ik}(x^1, x^2), A_i(x^1, x^2), \ldots$  
the monodromy data: $u_{\pm}(w), v_{\pm}(w), \ldots$

The key difference between these "coordinates" is that the field components should satisfy the field equations, while the space of monodromy data functions is unconstraint: for arbitrarily chosen set of these functions there exists a uniquely determined local solution of the field equations. The "coordinate transformation" from the monodromy data to the field components effectively solves the field equations. That is why we call the approach using this transformation for solution of symmetry reduced Einstein equations as the "monodromy transform" approach.

The construction of the monodromy transform-provides a unified general base for solving of various integrable symmetry reductions of Einstein’s field equations.

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including the Einstein equations for vacuum, the Einstein-Maxwell and the Einstein-Maxwell-Weyl equations for gravitational, electromagnetic and classical neutrino fields as well as for the Einstein equations in higher dimensions which determine the low-energy dynamics of the bosonic sector of some string gravity models.

A large variety of physically different types of field configurations can be considered in the framework of this approach. These include the stationary axisymmetric fields of compact sources or asymptotically non-flat fields describing the interaction of these sources with various external fields, the fields of accelerated sources with boost-rotation or boost-translation symmetries, various wave fields such as colliding and nonlinearly interacting waves with smooth profiles or some discontinuities on the wavefronts and having plane, spherical, cylindrical, toroidal or some other forms of the fronts, as well as different inhomogeneous cosmological models with two commuting spatial symmetries. Below we outline some key-points of the monodromy transform approach and mention some its applications.

2. Parameterization of the solution space by monodromy data

For electrovacuum Einstein-Maxwell fields depending on two coordinates, any local solution with the complex Ernst potentials $E$ and $\Phi$, is characterized uniquely by the monodromy data which consist of the four functions of the spectral parameter $w$ holomorphic in some local regions of the spectral plane:

$$\{E(x^1, x^2), \Phi(x^1, x^2)\} \rightarrow \{u_\pm(w), v_\pm(w)\}$$  \hspace{1cm} (1)

For vacuum fields $\Phi(x^1, x^2) \equiv 0 \leftrightarrow v_\pm(w) \equiv 0$ and the space of solutions is parameterized by the monodromy data which consist of two arbitrary holomorphic functions $u_\pm(w)$. For the structure of the monodromy data for other fields see 1, 2.

To determine the monodromy data for given solution of Einstein equations, one should solve an overdetermined linear system of differential equations whose coefficients depend on the field components of a given solution and their first derivatives.

3. Constructing solutions for arbitrary monodromy data

All components and potentials of a general local solution of electrovacuum Einstein-Maxwell equations can be expressed in quadratures in terms of the monodromy data 1 and of the corresponding solution of a master system of linear singular integral equations whose kernels and rhs are expressed algebraically in terms of the monodromy data. In particular, given monodromy data, the Ernst potentials are

$$E(x^1, x^2) = \epsilon_o - \int_L [\lambda_\zeta k(\zeta)]^{u_\pm}(\zeta) d\zeta, \hspace{1cm} \Phi(x^1, x^2) = \int_L [\lambda_\zeta k(\zeta)]^{v_\pm}(\zeta) d\zeta$$  \hspace{1cm} (2)

where $\epsilon_o = \pm 1$ is the value of $E$ at some initial point; $\zeta \in L$ and the contour $L$ on the spectral plane consists of two disconnected parts $L_+$ and $L_-$ with the endpoints $(\xi_o, \xi)$ and $(\eta_o, \eta)$ depending on the coordinates $x^1$, $x^2$ and coordinates of a chosen initial point; the value $[\lambda_\zeta]'$ is a jump at the point $\zeta \in L$ of a “standard” branching function $\lambda = \sqrt{(\zeta - \xi)(\zeta - \eta)/(\zeta - \xi_o)(\zeta - \eta_o)}$ and the “weight” $(\pi/2)k(\zeta) \equiv 1 +$
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solutions with arbitrary finite number of free parameters including

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4. Applications

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lems for hyperbolic cases3,8 as well as of the boundary value problems for elliptic
cases of integrable reductions of Einstein’s field equations. It is clear, however, that

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