Relativistic Newtonian gravitation

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Abstract

Newton + EP + local SR ⇒ all the GR results for central symmetric static gravitational fields

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1 Introduction

The aim of the present paper is to show that Newton’s [1] treatment of gravitation can be reinterpreted on the basis of the equivalence principle (EP) and combined with the local validity of Special Relativity (SR), providing all the General Relativity (GR) [2] predictions in the case of a static central symmetric field. The results are compared to other approaches and to the discussion of the Sagnac effect.

2 Newtonian time and the Equivalence principle

a) absolute time from the EP

Newtonian space-time consists of Euclidean space and “absolute” time, a notion which clearly conflicts with the basics of Special Relativity. This justifies the general consensus about the fact that the inclusion of SR into Newton’s theory be forbidden from the beginning, leaving as the only solution a complete reformulation of the entire problem of gravity, which is in fact the case of the General Relativity approach.

However, it is our aim to show that there is a simple and direct way to give a meaning to “relativistic corrections to Newton gravity”, on the basis of the Equivalence Principle
and Special Relativity. This can be done in a substantially unique way, and reproduces all the General Relativity results for static central gravitational fields.

The purpose of the EP is to use frames associated to free falling observers to “eliminate” gravity locally, to the first order in the space-time displacements from a given point. For the moment, however, we take into account the EP only to replace the Newtonian notion of time with a notion which takes into account the effects of gravity in terms of Free Falling Frames.

We assume central symmetry and the validity at large distances of the Newtonian description: space is Euclidean and time satisfies, in that limit, all the synchronization properties defining Minkowski frames in the absence of gravity. We also assume stationarity with respect to the time at infinity. Then, the EP, which asserts that gravity effects are not felt by free falling observers, suggests to define time by clocks in free fall from infinity.

We assume therefore that clocks can be arranged to fall freely from infinity, starting at all times, along radial trajectories, with zero initial velocity; they provide a unique notion of time, defined for all space-time points. We will adopt such a notion of time as the “EP absolute time” \( t \) with the same role as Newton’s absolute time.

Let us also remark that the time needed for free falling clocks to reach a given point in space from infinity is independent from the starting time and that therefore time intervals along the trajectory coincide with the corresponding ones at \( \infty \).

Notice that no velocity parameter appears in such a construction, due to the null (or very small) velocity of the clocks at infinity. Other possible constructions, with a non-zero velocity at infinity, would require the use of SR to account for the initial motion near infinity, preventing a clear separation of roles between SR and EP.

Clearly, the introduction of the above notion of time has important consequences, even on the description of space alone, since the very identification of the space variables and of space geometry concerns, by definition, space-time points at the same time. This affects in particular the notion of space distances, which will be defined as measured by sequences of small rods, with their ends at the same time.

The same construction can be performed for radial trajectories reaching infinity (at time \(+\infty\)) with zero velocity. All our results will be independent of the choice between the corresponding (alternative) notions of time. To be definite we will consider in the following the case of infalling velocities i.e. \( v(r) < 0 \).

b) Newton laws

Adopting the above reformulation for time, we now endorse the Newton principles of gravitation, for a centrally symmetric static gravitational field:

1) space is assumed to be euclidean. This amounts, due to central symmetry, to the euclidean relation between radial and angular distances

\[
\int dl = 2\pi r \quad (1)
\]

2) radial free fall is asserted to be given by the Newtonian velocity law

\[
v^2(r) = \frac{2GM}{r} = -2\Phi(r) \quad (2)
\]

As well known, since the same law applies to all bodies, eq. (2) includes, for the case of radial free fall from infinity, the basic form of the EP, i.e. the uniqueness of free fall
trajectories for given initial position and velocity.

We emphasize that all the above notions refer to measured space and time intervals; space distances are assumed to be given by the euclidean expressions in “Newton coordinates” $x$; the velocity in eq.(2) is defined by the above measurements of space distances and by time intervals given by free falling clocks.

Opposite to the ordinary GR point of view, no “coordinate independence” appears here. Put differently, we have

1) chosen to discuss the property of space and time in presence of gravity in “intrinsic coordinates”, obtained in terms of Euclidean coordinates near infinity, extended by using times and distances measured by radially free falling observers.

2) assumed Euclidean space and Newton’s free fall law in such coordinates

The velocity of light does not appear in the above considerations: only the Newton constant $G$ and the mass $M$ enter in the above description of space and time. On such a basis, no dimensionless parameter arises and Eqs.(1),(2) are forced by dimensional analysis and space flatness at infinity, which determines in particular the value $2\pi$ in eq.(1).

c) Free Falling Frames and the EP

Until now, the EP has only been used to derive a notion of time, in which Newton laws have been expressed. Let us now formulate the complete EP. To this purpose, the essential step is to introduce, around each space-time point, local variables associated to Free Falling Frames.

Observers falling along radii starting at infinity (at time minus infinity) with zero velocity employ, around a trajectory $X(t)$, time intervals measured by free falling clocks, and space distances from $X(t)$, measured by (small) rods with ends at the same time and therefore given by euclidean expressions in $x - X(t)$. Using $dX/dt = v(X(t)), v(r)$ given by Eq.(2), the corresponding differentials at a space-time point $r_0, t_0$ are

$$dt' = dt$$
$$dr' = dr - v(r_0) dt$$
$$dx_{\perp}' = dx_{\perp}$$

with $dx_{\perp}'$ the space displacements in the directions orthogonal to $r$, given by the differential $dx_{\perp}$ of local Newton cartesian coordinates orthogonal to the radius.

Even if eqs.(3) have the form of Galilei transformations, they have a very different nature, since they describe “small” (infinitesimal) displacements in Free Falling Frames, on the l.h.s., in terms of global coordinates in the r.h.s..

It is a fundamental fact that the above relations are not given by Lorentz transformations. They have the form of Galilei transformations because they arise from the use of a common, “absolute”, notion of time.

Eqs.(3) should not be interpreted therefore as a low velocity approximation of Lorentz transformations. They are exact in our approach, for all values of the free fall velocity. By the above derivation, they hold independently of eq.(2), which only fixes the value of $v$; moreover, only the third equation should be modified (by an $r$ dependent factor) in the absence of the Euclidean relation eq.(1).

In fact in the present Section we are only discussing the modifications to the inertia principle, which holds both in relativistic and non relativistic physics and has little to do with the velocity of light, which was in fact never mentioned in the above discussion.
\[ dr' - dr = -v(r)dt \] (\(r, t\))

Figure 1: The clocks of the infinitesimal Equivalence Principle Inertial Frames (EPIFs), starting at any time, carry at each point the same time rate \((t = t')\). The space-time effects of gravitation are determined by the infinitesimal invariant Minkowski interval \(ds^2 = c^2dt'^2 - dr'^2\) in the EPIFs coordinates of Eqs (3). This furnishes the metric in the global coordinates \((r, t)\) which provide all the relativistic corrections to Newtonian absolute time and space.
Second, in the relativistic case, the use of a Lorentz transformation in Eq. (3) would lead, as we shall see, to a trivial local Minkowski structure, excluding gravity effects on clocks and light deflection.

The differentials $dx'$ and $dt'$ representing the description of space and time in Free Falling Frames, to the first order in the displacements from a free fall trajectory, are the substitute of coordinates satisfying the inertia principle. They will be denoted as (infinitesimal) Equivalence Principle Inertial Frames (EPIFs) and are the object of the following form of the Einstein’s Equivalence Principle:

All the physical laws which can be written, in the absence of gravity, in inertial frames in terms of local variables and their first order variations around each point, hold true in the presence of gravity in terms of the same variables in EPIFs.

It is important to notice that the differentials defining EPIFs do not in general define coordinates, even locally. In fact the differential form

$$dr' = dr - v(r) dt$$

is not integrable, unless the velocity field $v(r)$ is constant (the trivial inertial case), since

$$\partial v/\partial r = \partial 1/\partial t = 0$$

is precisely its integrability condition.

On the contrary for the time variable the restriction to the infinitesimal interval is not essential and in fact $t = t'$ is the gravity free time measured by clocks on EPIFs, which does not suffer from the limitations produced by gravity on the space variables. Clearly, even if the formulation of the EP only uses EPIFs, its implications crucially depend on the relation between the above differentials at different points, given by Eqs. (3). In other terms, the introduction of global coordinates, in our case Newton coordinates, and the expression of the EPIF differentials in terms of them is an essential step for an effective use of the EP.

It goes without saying that the assumption of the same free fall velocity for all objects implies that feathers and lead balls will experience the same gravitational effects: their inertial and gravitational masses are equal.

3 Relativistic Physics

a) Newton’s mechanics from the principle of least action in EPIFs

Let us first show how by use of the EP one can reproduce classical non-relativistic mechanics for a particle in the gravitational field of a mass $M$.

Classical Mechanics can be indeed formulated in terms of the principle of least action in inertial frames. The implementation of the EP in the non relativistic free Lagrangian $L = m/2 \dot{x}^2$ is immediate. Following Eqs. (3), it is enough to express the velocity in the local EPIF

$$\dot{r} \rightarrow \dot{r} - v(r)$$

The principle of least action in EPIFs for a free particle in the presence of gravity is thus given by
\[ \delta \int dt \left( \frac{m}{2} \left[ (\dot{r} - v(r))^2 + r^2 (\dot{\theta})^2 \right] \right) = 0. \quad (4) \]

The Lagrange equations yield for the radial coordinate

\[
\frac{d}{dt}(\dot{r} - v(r)) + (\dot{r} - v(r))\frac{dv}{dr} - r(\dot{\theta})^2 = \dot{r} - \frac{d}{dr}(1/2v^2(r)) - r(\dot{\theta})^2 = 0. \quad (5)
\]

Since \(1/2 v^2(r) = GM/r\), the Newton’s radial equation of motion is obtained, and the same holds also for the angular variables. A constant \(v\) would only amount to a change of inertial frame.

b) The invariant Minkowski interval

Relativistic physics is governed by the “infinitesimal” invariant interval. By the EP, the invariant interval has the standard Minkowski form in EPIFs, in which the ordinary inertial frame laws hold and light propagates isotropically and always with velocity \(c\), to first order in space-time displacements:

\[
\begin{align*}
ds^2 &= c^2 dt'^2 - dr'^2 - dx'^2_{\perp}.
\end{align*}
\]

(6)

The local (first order) validity of the principles of SR implies that \(ds^2\) is still given by the same expression, Eq. (6), in the coordinates employed by any observer around the given space-time point, to the first order, independently of his motion. All the SR results hold therefore locally, for all observers (on arbitrary trajectories) to first order in the coordinates defined by their clocks and rods, with the Minkowski interval given by the ordinary expression.

The above expression of the EPIF differentials in terms of globally defined variables allows to write the Minkowski intervals, all of the same form in their EPIF variables around different points, in global coordinates:

\[
\begin{align*}
ds^2 &= c^2 dt^2 - (dr - v(r) dt)^2 - dx^2_{\perp} = \\
&= c^2(1 - v^2(r)/c^2) dt^2 + 2v(r) dt dr - dr^2 - dx^2_{\perp}.
\end{align*}
\]

(7)

Eq. (7) gives nothing else than the Painlevé - Gullstrand metric (P-G), a solution of the GR equations in a central field, obtained here (as a solution of no whatsoever equation other than Newton’s law) on the pure basis of euclidean space and absolute “free fall” time. Even if built on Euclidean space and absolute time, it represents nevertheless a non-Minkowskian space-time, due to the crossed term. The P-G metric is equivalent to the Schwarzschild metric (Ss), the relation being given by a change of the time variable (see following sections).

Clearly, in our approach, SR enters at a different stage with respect to the description of radial free fall. The latter is given by the Newton free fall velocity in global coordinates; the velocity of light enters in the local relativistic structure of space time, which is trivial in EPIFs and globally determined by Eqs. (6,3).

All SR effects depend therefore on the gravitational “weak field” parameter

\[
\epsilon(r) \equiv 2GM/rc^2 = v^2(r)/c^2 \equiv -2\Phi(r)/c^2
\]

(8)

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c) Clocks ticking and red shift

How time flows in a gravitational field for observers at rest, in the above (P-G) coordinates, is immediately got from the P-G metric. Indeed, by setting $dr = 0$,

$$d\tau^2 = (1 - v^2(r)/c^2)\, dt^2 = (1 - \epsilon(r))\, dt^2$$

relates the (P-G) Newtonian free fall absolute time with the relativistic invariant interval $d\tau$ measured by observers at rest in the P-G coordinates, thus defining his proper time.

Since, as observed above, time intervals at fixed space points coincide with time intervals at infinity, the frequency of light propagating from infinity remains constant in (the above, P-G) time, and therefore frequencies observed by observers at rest are given by (the inverse of) the above relation.

Because the velocity of light remains the same for all observers, this can also expressed in terms of wavelengths, relating the one at $\infty$ $\lambda_{\infty}$ to the one at $r$ $\lambda_r$,

$$\lambda_r = (1 - \epsilon(r))^{1/2} \lambda_{\infty} \simeq (1 - \epsilon(r)/2) \lambda_{\infty}$$

For small radial distances $h$,

$$\Delta \omega/\omega \simeq \epsilon(r)/2 \, (h/r)$$

i.e. the well known red shift. For moving sources, one has to add the usual Doppler effect.

d) Light cones

Light velocity is obtained by setting to zero the invariant interval Eq.(7). The velocity in directions orthogonal to the radius takes the value

$$c_\perp = r d\theta/dt = c (1 - v^2/c^2)^{1/2}. \tag{10}$$

Along the radius, the velocity is

$$c_r = dr/dt = \pm c - v(r), \tag{11}$$

$v = v(r)$ given by eq.(2). Both equations directly follow from eqs. by ordinary (Galilean) vector composition of the (isotropic) velocity $c$ in EPIFs with the EPIF velocity $v$,

$$c_{PG} = c + v(r). \tag{12}$$

For completeness, we recall the results for the Schwarzschild metric

$$ds^2 = (1 - v^2/c^2)c^2\, dt^2_S - dr^2_S/(1 - v^2/c^2) - r^2\, d\Omega^2. \tag{13}$$

There the radial velocity is

$$dr/dt = \pm c \,(1 - v^2/c^2) \tag{14}$$
Figure 2: Light cones as a function of $r$ in the Painlevé-Gullstrand and Schwarzschild metric respectively. A singularity at $R_S$ only arises for the second metric; in the P-G coordinates simply, because of the distorted light cone, nothing can get out of the hypothetical black hole.
whereas the tangential one is the same as in the P-G coordinates.

Thus the Schwarzschild light cones shrink for decreasing distances down to a pseudo
singularity at $R_S = 2MG/c^2$. In the P-G coordinates, they rotate, without any sin-
gularity. If one believes in the validity of the extrapolation of Newtonian dynamics to
such extreme (to be discussed later), a physical effect emerges from the SR constraints
in EPIFs.

Indeed, since light can take at most the vertical values $\pm c$, when $|v(r)| > c$ light
cannot propagate upwards for positive times.

This happens at $R_S$ where we are in the presence of a black hole. Reversing the sign
of time is equivalent to reversing the free fall velocity in the above construction. In this
case, light cannot propagate inwards (since this time $v(r) - c > 0$), for positive times.
With this choice of the sign of time we are therefore in the presence of a "white hole".

Notice that the interior dynamics is irrelevant in the sense that for any reasonable
assumption about matter distribution (shell, constant density etc.), consistently with
our essentially classical treatment of free fall, $|v_{int}| \geq |v_{R_S}|$ within $R_S$.

c) **Relativistic Mechanics**

The formulation of relativistic mechanics is immediate via the EP, which only amounts
to substitute stationarity of the Minkowski invariant interval in inertial frames with the
same for invariant intervals in EPIFs. The corresponding action is therefore obtained,
as in the non-relativistic case, by the substitution $\dot{r} \rightarrow \dot{r} - v(r)$

$$\delta A = \delta \int L \, dt = \delta \int ds = \delta \int dt \left( m c^2 \sqrt{1 - 1/c^2[(\dot{r} - v(r))^2 + r^2(\dot{\theta})^2]} \right) = 0 \quad (15)$$

The equation of motion are given by the corresponding Euler Lagrange equations.
They are equivalent to the geodesic equations in the metric defined by the above invari-
ant interval, i.e. in the P-G metric.

Both solve in fact the same variational problem, the ordinary geodesic equations
being obtained from a parametrization of trajectories with the proper time $\tau$ and the
substitution of the Lagrangian with its square (which is allowed since the Lagrangian
associated to the proper time parametrization takes the value 1 on the solution of the
stationarity problem).

4 **Light deflection**

Since our invariant interval coincides with the one of the Painlevé-Gullstrand solution
of the Einstein equations and the principle of stationary action amounts to geodesic
motion in the corresponding metric, all the results of GR for the dynamics of particles
and light follow.

We show below that these results can also be derived directly in our approach in a
rather elementary way.

\footnote{Let us also mention that the use of proper times is inappropriate in the many body case.}
The problem of light bending has been paramount in assessing the view of space distortion associated to the Schwarzschild solution. Indeed, classically, (see e.g. [6]) it is well known that Newtonian mechanics can account for light deflection, at variance however by a factor of 2 from the GR result and from experimental data. This is explained in Schwarzschild coordinates by saying that Newton just reproduces the time part \( g_{00} \) of the metric tensor and that the space part \( g_{ii} \) is the new fundamental contribution of GR.

Let us first recall the classical treatment and then discuss the contributions arising from the EP. We restrict to first order in the (relativistic) parameter \( \epsilon(R) \), which completely covers the experimental situation.

Take a luminous ray grazing the sun, coming from \( \infty \) and calculate the light deflection as measured by a distant observer (we on the earth), practically at \( \infty \).

a) Newtonian light deflection

To first order, the bending angle of light associated to the unperturbed trajectory \( (x = ct, y = R) \), is given by \( \theta(x) = -c_y/c, \ c_y \) the \( y \) component of the light velocity, \( c \) the unperturbed velocity. If light is assumed to accelerate according to Newton’s law

\[
\theta(x) \simeq -c_y(x)/c = \int_{-\infty}^{t} \frac{\partial}{\partial R} \Phi(ct', R) \, dt'/c = \int_{-\infty}^{x} \frac{\partial}{\partial R} \Phi(x', R) \, dx'/c^2 \tag{16}
\]

The deflection angle is then obtained by integration over the whole \( x \) axis in terms of the relativistic weak field parameter \( 2GM/c^2 R = \epsilon(R) \) as

\[
\theta = \theta(\infty) = \epsilon(R) \tag{17}
\]

b) Wave fronts and light velocity

In order to discuss light deflection as a refraction effect produced by position dependent velocities, let us see how it can be obtained in general, for small angles, in terms of wave fronts. The Newtonian result will be shown to follow from the light velocity given by the pure time component of the Schwarzschild metric, while the full GR result will follow from the above (very elementary) P-G light velocity.

We will consider the propagation of light in a first approximation along straight lines (see Fig.2) at different heights , with velocity \( dx/dt = c(x, y) \) and calculate the orientation of wave fronts, at \( y \simeq R \).

Since, as motivated before, light frequency \( \omega \) remains constant for the P-G time (and also for the Schwarzschild time, see below), the phase \( \varphi \) of the wave changes with the time it takes a wavefront to travel in the \( x \) direction

\[
d\varphi = dx/\lambda(x) = \omega \, dx/c_x(x, y),
\]

\( c_x \) the velocity of propagation along the \( x \) axis. Thus, to first order, the difference of the wavefronts at different heights

\[
\frac{\partial}{\partial y} \varphi(x, y) \simeq - \int_{-\infty}^{x} \frac{\partial}{\partial y} c_x(x', y)^{-1} \, \omega \, dx' \tag{18}
\]
determines the bending. With the same sign convention as above the bending angle $\theta$ of the wavefront at $y \simeq R$ is given by

$$\theta(x) = -\frac{\partial}{\partial R} \varphi(x, R) \lambda(x) \simeq -\int_{-\infty}^{x} \left( \frac{\partial}{\partial R} c_x(x', R)^{-1} \right) c_x(x, R) \, dx'$$

and the deflection angle is $\theta = \theta(\infty)$.

c) Schwarzschild

The modifications of the light velocity given by the “pure time component” of the Schwarzschild metric, $ds^2 = c^2 \left( 1 - \frac{v^2(r)}{c^2} \right) dt^2 - dx^2$, are independent of the direction and given by

$$c(x, y) = c \left( 1 - \frac{v^2(r)}{c^2} \right)^{1/2} \simeq c(1 - \epsilon)$$

and therefore Eq. (19) gives the deflection angle

$$\theta \simeq \int_{-\infty}^{\infty} \frac{\partial}{\partial R} \Phi(x, R)/c^2 \, dx,$$

which coincides with the Newtonian expression, eqs. (16)-(17). The Newtonian equation of motion is in fact equivalent to the stationarity principle for the optical length in such a metric.

For the complete Schwarzschild metric, the velocity of propagation of light along the $x$ axis, with $y = R = r \cos \alpha$, is given by

$$dx^2 ((1 - v^2/c^2)^{-1} \sin^2 \alpha + \cos^2 \alpha) = (1 - v^2/c^2) dt^2$$

which gives

$$c_x(x, R)^{-1} \simeq c^{-1} \left( 1 + \epsilon(r) \frac{1 - R^2/(x^2 + R^2)}{2} \right).$$

The integral of the last term is finite and independent of $R$, so that the result changes by the well known factor of 2.

d) deflection from EPIF Galilean light composition

In our approach light velocity is given by the Galilei formula, eq. (12).

By imposing to first order its propagation along the $x$ axis, the $y$ components cancel in eq. (12) and therefore the $x$ component is given by

$$c_x = (c^2 - v_y^2(r))^{1/2} + v_x(r)$$

Since $v(r)/c$ is if of order $\epsilon(R)^{1/2}$, to first order in $\epsilon$

$$c_x^{-1} = c^{-1} ((1 - v_y^2(r)/c^2)^{1/2} + v_x(r)/c)^{-1} \simeq$$

$$c^{-1} \left( 1 + v_y^2/2c^2 - v_x(r)/c + v_y^2(r)/c^2 \right) = c^{-1} (1 - v_y^2/2c^2 - v_x(r)/c + \epsilon(r))$$

The second term is proportional to $\Phi(r) R^2/(x^2 + R^2)$ and its integral is independent of $R$, as above. The third is antisymmetric in $x$ and its integral vanishes; the last term gives the GR result,

$$\theta = \theta(\infty) = 2 \epsilon(R)$$

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Figure 3: Spherically symmetric light propagation in the EPIFs is vectorially composed with the free fall velocity $v(r)$ to yield a resultant $c_x$ along the unperturbed trajectory. The phase variation and hence the bending of the wave front as a mirage effect comes from the dependence of the $x$ velocity $c_x$ on the height $y$. 
The factor 2 has emerged from the second order expansion of the inverse of the velocity in the parameter $\sqrt{\epsilon(r)}$.

Notice that in the P-G coordinates light velocity is different on its way towards and away from the source of gravitation, due to the linear dependence on the free fall velocity. As we have seen, light deflection arises as second order effect in the free fall velocity. Linear terms are present for wave fronts at finite distances from the source, but their effect depends crucially on the relation between wave front and simultaneity, which is of course different in different coordinate systems.

5 Perihelion precession

Let us show how the perihelion precession can be calculated directly, for motion close to a circular orbit and to first order in $\epsilon$, from the above equations of motion.

The relativistic Lagrangian is given by eq. (15). The angular equation of motion is

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = -\frac{d}{dt}(L^{-1} \dot{r}^2 \dot{\theta}) = 0,$$

i.e.,

$$L \equiv r^2 \frac{d\theta}{dt} L^{-1} = r^2 \frac{d\theta}{ds} = \text{const}$$

The radial motion is given by:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0.$$

By using the proper time $s$, $ds/dt = L$, we obtain

$$\frac{d^2 r}{ds^2} = v(r)(d/ds L^{-1}) - L^{-2} d\Phi/dr + L^2/r^3 = 0.$$

This equation is equivalent to the Schwarzschild equation

$$\frac{d^2 r}{ds^2} = -d\Phi/dr + L^2/r^3(1 - 3/2\epsilon)$$

since they solve the same variational problem in the same variables, and therefore implies the GR result for the perihelion. Even if the P-G the radial equation is more involved (a dependence on $r$ and $dr/ds$ being hidden in the terms involving the Lagrangian), a calculation of the precession effect in the above approximations is straightforward.

Let us derive it directly from the equations of motion in our (P-G) time variable. The radial equation (26) reads

$$-\frac{d}{dt} (L^{-1} \dot{r}) + (d/dt L^{-1}) v(r) - L^{-1} d\Phi/dr + L L^2/r^3 = 0.$$

Multiplying by $L$,

$$\frac{d^2 r}{dt^2} = -(\dot{r} - v) L \frac{d}{dt} L^{-1} - d\Phi/dr + L^2 L^2/r^3$$

Circular orbits are given by

$$- d\Phi/dr + L^2 L^2/r^3 = 0.$$
and their frequency is
\[ \omega_0^2 \equiv \dot{\theta}^2 = L^2 L^2 / r^4 = GM/r^3, \]  \quad (30)
the same as in the Newton case.

Eq. (29) differs from Newton’s equation by the term
\[ L^2 = 1 + 3\epsilon \]
For small oscillations around a circular orbit the term \( \dot{r} L \dot{L}^{-1} \), quadratic in \( \dot{r} \), can be dropped; using \( \dot{v}^2 = -2\Phi \),
\[ L \sim 1 + (\Phi + \dot{v} \dot{r} - L^2/2r^2)/c^2 \]
and the circular orbit constraint, the terms linear in \( \dot{r} \) of Eq. (28) are readily seen to cancel, corresponding to the absence of damping. As a result, the only contribution of the first term in the r.h.s of Eq. (28) is
\[ -v^2/c^2 d^2 r/dt^2 = -\epsilon d^2 r/dt^2 . \]
Eq. (28) therefore reduces to
\[ (1 + \epsilon) d^2 r/dt^2 = -d\Phi/dr + L^2 L^2/r^3 = -d\Phi/dr + (1 - \epsilon - L^2/r^2 c^2) L^2/r^3 . \]  \quad (31)
The frequency for circular orbits is thus given by
\[ \omega_r^2(1 + \epsilon) = d^2\Phi/dr^2 - L^2 d/dr L^2/r^3 + L^2/r^3 d/dr (\epsilon(r) + L^2/r^2 c^2) \sim \omega_0^2 + d\Phi/dr 2d\epsilon/dr \]
so that
\[ \omega_r^2 \sim \omega_0^2(1 - 3\epsilon) = GM/r^3 (1 - 3\epsilon) \]
i.e.
\[ \omega_r/\omega_0 \simeq (1 - 3/2 \epsilon) \]
and the precession angle is therefore
\[ \frac{\Delta \phi}{2\pi} = \frac{3}{2} \epsilon \simeq \frac{3GM}{rc^2} \]  \quad (32)

Notice, in connection to MOND [7], that the relativistic corrections appearing in the above equations for nearly circular orbits are \( O(GM/rc^2) \) and that no effects \( O(v^2/r) \) appear. In other words comparable velocities (e.g. Earth and orbiting HI lines), even at very different radii, have the same sort of relativistic corrections (with negligible effect in the second case).

In conclusion, in static central gravity, GR effects are all given by relativistic effects on Newton gravity, corrected for the EP notion of time. This gives an alternative interpretation to the widespread statement (e.g. Schiff [8] ) that the perihelion precession provides the test of GR. The point is that time ticking and light bending only depend on the validity of the “Newtonian” assumptions 1) and 2) to the lowest order in \( \epsilon \); on the contrary, the perihelion result would be affected by \( \epsilon \) corrections to 1) and 2).
The Newtonian fall velocity and the mass

The previous discussion shows that static centrally symmetric gravity can be accounted for by Euclidean space and Newton free fall law, exactly as Coulomb law describes electrostatic fields, independently of the other Maxwell equations.

As repeatedly stressed, all the results depend on the form of \(v(r)\). Let us now try to justify Newton’s formula for free fall discussing possible “relativistic” effects.

To start with, assuming that energy is the source of gravitation, the mass in the potential term in Newton formula should be corrected both by the self energy and by the kinetic term.

Energy conservation for our free falling particle would thus read

\[
m_0 v^2 / 2 = \frac{GMm_0}{r} \left( 1 - \frac{GM}{c^2 r} + \frac{v^2}{2c^2} \right) \quad (33)
\]

or

\[
v^2 / 2 - \frac{GM}{r} = \frac{GM}{c^2 r} \left( \frac{v^2}{2} - \frac{GM}{r} \right) \quad (34)
\]

whence

\[
v^2 = 2GM/r \quad (35)
\]

The result only uses conservation of energy and is independent of the use of the non relativistic formula. This puts Newton’s law on a somewhat safer ground in the sense that the above energy corrections cancel out.

It is paramount to underline that the self energy correction to the mass, which embodies the fact that the graviton is itself a source of gravity, a relativistic effect, is cancelled in our approach by the relativistic kinetic corrections of the mass.

This has to be contrasted with what happens in the PPN parametrization (see e.g. [9]), where the non linearity of Einstein equations appears in the form (destitute of any measurement prescription)

\[
h_{00} = 2GM/c^2 x \left( 1 - \frac{GM/c^2 x}{1} \right)
\]

Thus one concludes, remarkably, that the non linearity of gravitation depends on the formulation.

We also notice that the Newtonian \(1/r^2\) form of the force is crucial in canceling possible contributions from external masses, within a reasonable schematization of the outer world as a homogeneous sphere.

Moreover, in a possible relativistic extension of Eq. (33),

\[
\frac{m_0}{\sqrt{1 - v^2/c^2}} - \frac{GMm_0}{c^2 r} \left( \frac{1}{\sqrt{1 - v^2/c^2}} - \frac{GM}{c^2 r} \right) = m_0 \quad (36)
\]

additional higher order terms in \(v^2/c^2\) i.e. in \(\epsilon\) in the l.h.s. of Eq. (35) would result in first order corrections to Mercury’s precession which are ruled out. In that respect the LLR experiments [10] of the free falling Earth-Moon system in the gravitational field of the Sun should also exclude such relativistic corrections with higher accuracy. In addition, apart from the cancellation of energy corrections in the r.h.s. as above, the
inertial mass in the l.h.s. cannot be gravitationally corrected by an additional factor \(-GM/c^2r\) since one would get in this case for the escape velocity \(v^2 = 4GM/r\).

One is therefore led to conclude that free fall is determined only by \(GM\) without kinetic and self energy corrections and that gravity does not contribute to the inertial mass \(m_o\).

This can be summarized as

\[
m_i = m_0 = m_0(1 - GM/c^2r + v^2/2c^2) = m_0 = m_G
\]

i.e. that one mass is enough, and cancels out in the motion in a given field.

Gravitational quantum interference experiments \[11\] are very far from providing possible additional information.

In conclusion our treatment based on Newton’s law in the free falling frame gets validated beyond expectations and all the geometrical relations of GR are direct consequences. Of course the possible distinction between different substances (WEP), ruled out by the terrestrial experiments of Eötvös-Wash \[12\], is automatically implemented here.

Let us finally comment on the role of moving frames which play such a fundamental role in SR and GR, although sometimes with a misleading interpretation.

In the former case they represent a physical entity: e.g., in the flying muon frame the atmosphere thickness is shorter than the one measured on earth and the time needed to reach it is correspondingly shorter.

In the latter the accelerating falling frame is the basis for a construction which describes gravity in terms of local “inertial” frames. When such local frames are expressed in terms of global coordinates and the latter are interpreted as inertial coordinates in a hypothetical world without gravity (in a sense reminiscent of how one switches on e.m.), gravity effects appear in terms of a distortion of space-time distances, i.e., of a metric.

Coordinates play therefore a role comparable to that of different gauges in e.m., also in connection with the discussion of “perturbative” effects on the “free” theory (here, the Minkowski space-time). In particular, the common statement that Newton’s law violates causality is as wrong as it is for Coulomb’s law in QED (where causality is explicit in the Lorentz gauge). In general, the use of different coordinates does not represent different solutions of the same physical problem, but just two different ”languages”.

Thus the popular expression that GR dilates spacial distance has no relation with the physical fact that in SR distances for moving particles are shorter. That statement should be supplemented by the phrase: in the Ss metric, whereas in the P-G it does not happen. In that sense the P-G metric, with its clear and physically founded combination of local SR with a global ”Galilean” free fall law (also preserving at infinity the gravity free absolute time), has an interpretation which is ”closer to reality”.

7 From the P-G to the Ss metric

In paragraph 3, c) and d), the Ss metric has been mentioned. In this paragraph its relation to the P-G metric will be elucidated by explicitly reconsidering how the combination of Galilean transformations for the free fall with the local space time Minkowski structure yield the physical i.e. invariant proper time and length measured on earth.
Disregarding earth motion, this corresponds to a frame at rest in the P-G coordinates. Summarizing:

\[(dr, dt)_\infty^{\text{Newton}} \implies_{\text{Gal.}} (dr', dt')_{\text{EPIF}} \implies_{\text{L.T.}} (d\rho, d\tau)_{\text{terrestrial}}\]

Local time and space coordinates \(d\tau\) and \(d\rho\) are thus obtained as

\[
d\tau = \gamma(v)(dt' + vdr') = \gamma(v)(dt + v(dr - vdt)) = dt/\gamma(v) + \gamma(v)vdr \\
\]

\[
d\rho = \gamma(dr' + vdt') = \gamma(dr - vdt + vdt) = \gamma dr
\]

where \(\gamma(v) = \gamma(v[r]) = \frac{1}{\sqrt{1-v^2/c^2}} = \frac{1}{\sqrt{1-2GM/c^2r}}\).

The last equation shows that the "switching on" of gravitation can be interpreted as a dilation of proper lengths which is sort of astonishing since it goes the opposite way with respect to SR.

According to the second to last, times are indeed shortened, as in SR, were it not for a space dependent term. Since that term does not alter the time rate at a given space point, it can be dropped through a redefinition of the global time:

\[dt \rightarrow dt - v/c^2/(1-v^2/c^2)dr_S \equiv d\tau_S\]

Then the metric takes the Schwarzschild form

\[ds^2 = c^2d\tau^2 - d\rho^2 = (1 - v^2/c^2)c^2dt_S^2 - dr_S^2/(1 - v^2/c^2) - r^2 d\Omega^2\]

The difference between the two notions of time is particularly evident in the treatment of light deflection, where in the P-G metric a linear factor in the velocity shows up, with a presumed huge effect when observed half way (e.g. on earth in the measurement of parallaxes). The point is that one must not confuse the P-G notion of simultaneity with the one defined by clocks at rest e.g. on earth (apart from an easy relativistic correction for its motion).

It should also be noticed that the Ss metric has spurious singularities not only at the S radius \(r = 2GM/c^2\), as well known, but also at \(r = \infty\) where the Newton time \(t\) and \(t_S\) differ by \(\approx \sqrt{r}\).

In conclusion the P-G coordinates have the advantage of the underlined physical foundation, the lack of singularities, no necessity of an equation of motion beyond Newton's law and can be directly and simply applied to all processes, apart from the discussion of equal time geometrical effects, as the parallax, where the Ss coordinates give a notion of simultaneity which applies more directly to terrestrial measurements.

Let us recall that the requirement to eliminate the off diagonal term of the P-G metric is generally accomplished just by adding in an ad hoc way the previous velocity dependent term in the relation among times

\[
\begin{pmatrix}
  dt \\
  dr
\end{pmatrix}
= \begin{pmatrix}
  1 & -v/c^2/(1-v^2/c^2) \\
  0 & 1
\end{pmatrix}
\times
\begin{pmatrix}
  dt_S \\
  dr_S
\end{pmatrix}
\]

without any discussion about its physical meaning, nor about its effects in the interpretation of experiments.
Finally, let us underline an inherent "paradox" of GR. The pretension that coordinate independence of the formulation is fundamental backfires, in the sense that Newton’s absolute time not only has the right of citizenship, but gives rise to an independent description based on fundamental physical motivations.

8 On alternative derivations

Ever since the appearance of GR, the endeavor to find other solutions than Schwarzschild’s, to "derive" it from SR and to eventually propose alternative theories has been paramount.

To start with, let us recall that Einstein’s rebuttal of the Painlevé-Gullstrand solution has led to an ostracism (their metric is not even mentioned in most textbooks) which has lasted till almost the end of the last century. Only recently the P-G metric has been reevaluated as a singularity-free solution. In addition, it has been realized that it could be obtained directly, without recursion to GR.

This possibility has provoked a heap of warnings: that it could be only heuristic, that it only may apply to the weak field case, accompanied as well by the (trivial) argument that it cannot reproduce all of GR results. In connection with the first points we want to comment on some of the most relevant and cited articles.

Schiff’s [8] work had already been criticized by Schild [13]. The usual result for time had been obtained by using again a SR argument, comparing local time with that of gravity free infinity via a flying time-shortened clock. However his (incomplete) argument about space is incorrect since in the end, contrary to SR, the velocity of light is not constant nor isotropic. His statement about Mercury’s perihelion being the crucial test of GR has already been commented upon above.

Kasner’s [14] work is relevant in the present context because of his discussion of the necessity of supplementary assumptions in order to derive the Ss metric on the basis of pre-general-relativistic physics alone, i.e., SR, the Einstein EP and the "Newtonian limit".

This is not contradictory with our findings. In reality, the Newton law is used here globally, not only to first order at infinity, implemented by the two (almost unescapable because of our motivations) subsidiary conditions on space (length of the circumference) and absolute time. They can be seen as substitutes of Kasner’s two additional “postulates”, which serve the same aim but which are, in our opinion, less transparent and motivated.

Czerniawsky’s point of view [15] is the closest to ours. Our assumption on the Euclidean properties of space, at equal “free fall” times, is somewhat hidden in his considerations about the EP. As a result, one does not see, in his treatment, the dependence of any approach based on free fall and EP on two functions of the radius (eqs. (1), (2)), a general fact already recognized in [16]. On the other hand we agree with Czerniawsky’s considerations on the difference in the notion of simultaneity between the Ss and PG metrics and on the physical significance of the time-reversed PG metric.

Finally Visser [17] and Padmanabhan [18] have strived to maintain the inadequacy of the free fall approach for the following reasons: to be only a weak field approximation of a more general theory and to be heuristic since it does not reproduce the Kerr
metric. The first point has already been commented upon. The second is irrelevant in the present context. For rotating masses results have been reproduced successfully via gravitomagnetism in a parameter free way just from SR without the need to invoke GR. [19].

Concerning the rest of the literature, let us also underline again the difference between time and space coordinates in the sense that time contraction can be simply obtained via SR arguments, and problems seem to arise only in the treatment of space. As regards the first point, the gravitational effects on time flow had been obtained by Einstein using the mass energy equivalence.

Consider an atom at $B = r' = r + h$ and an identical one at $A = r$. Then the photon emitted by B reaches A, because of the energy mass equivalence, with a greater energy due to the effect of the gravitational field. The photon frequencies at the two places are related by

$$\hbar \omega (1 - GM/c^2 r) = \hbar \omega' (1 - GM/c^2 r')$$

from which it trivially follows

$$\omega = \omega' \frac{1 - GM/c^2 r'}{1 - GM/c^2 r} \simeq \omega' (1 + gh/c^2)$$

This implies the reverse relation for times

$$t' = t \frac{1 - GM/c^2 r'}{1 - GM/c^2 r}$$

i.e. that time runs quicker in regions of smaller gravitational field. When the comparison is made with respect to $\infty$, where gravity is absent, one gets for the proper time at $r$ denoted by $\tau$

$$t' = t_\infty = \tau/(1 - GM/c^2 r)$$

and this agrees with the above result from the invariant interval.

Notice that a basic form of principle of equivalence has been tacitly assumed: atoms are the same (as locally measured) in different points of a gravitational field. Otherwise a correction factor would arise.

This goes along with the parallel argument about atomic energy levels. The mass $m$ at rest in a gravitational field of $M$ at the height $RT$ has an energy

$$E_0 = m_0 c^2 (1 - GM/c^2 RT)$$

and at $RT + h$

$$E' = m_0 c^2 (1 - GM/c^2 (RT + h))$$

It follows that at the earth surface

$$E' - E_0 \simeq m_0 c^2 GM/c^2 RT^2 h = m_0 gh$$

This energy difference exactly corresponds in classical terms to the gravitational potential energy difference or, in other words, to the work done against the standard Newtonian force $F = mg$. From that it trivially follows the well known local energy conservation of the atom-photon system.
Thus the gravitational interaction energy plays a fundamental role. This has to be contrasted with what happens in free fall motion, which provides the basis of our dynamical treatment, (where it is cancelled by the kinetic energy, thanks to energy conservation). This basic difference between "falling" and "at rest" situations questions the interpretation of GR usually associated to the Schwarzschild solution, as already mentioned in connection with non linearity.

Concerning the above arguments, the essential point is, in our opinion, that the simplicity of the treatment of time relies on a first order approximation, which is always implicit in use of the notion of fixed points in space. As remarked above, such a notion is coordinate dependent, even if the corresponding notions coincide in the Ss and PG coordinates (also implying the same notion of time intervals, at a fixed space point).

9 Rotating frames and the Sagnac effect

We pass now to a subject, not directly related to gravitation, whose treatment may help in shedding some more light on the use of metrics and of synchronization.

The Sagnac effect has a long history, remarkable practical applications and has caused a considerable amount of discussions concerning its connection with SR and GR.

In its standard form two counter propagating photon beams in a circular waveguide mounted on a disk are made to interfere after having traveled one circumference. When the disk is put in rotation with angular velocity $\Omega$, the interference figure is seen to shift by an amount proportional to $\Omega$.

Let us consider the problem from the point of view of the external observer (inertial frame). For him, light propagates of course with velocity $c$; however when the disk rotates the interference of the two light waves is observed at a moving angle, $\theta = \Omega t$.

The lengths $l_{1,2}$ of the two paths satisfy

$$l_1 - l_2 = 2\theta r,$$

where $\theta$ is the shift of the angle in the traveling time and $r$ the radius of the disk (with time and distances measured in the fixed frame). This corresponds, for light of frequency $\omega$, to a shift in phase

$$\Delta \varphi = 2\theta r \omega/c$$

To first order in $\Omega$, the traveling time of the two rays is $t \simeq 2\pi r/c$ and therefore

$$\Delta \varphi \simeq \frac{4\pi r^2 \omega \Omega}{c^2} = \frac{4\omega \Omega S}{c^2},$$

$S$ standing for the area perpendicular to the rotation axis enclosed by the given contour.

This is all, since the effect is frame independent. However for the sake of the argument and in order to make contact with the previous treatment of gravity, let us consider it from another point of view.

On the invariant interval in rotating frames

The above kinematical constraint about the meeting of two rays at a moving point can of course be written as the condition of meeting at the same point in a rotating
coordinate system and can be therefore discussed in terms of light propagation in such coordinates. This does not mean that quantities measured “on a rotating body” enter the discussion and in fact the introduction of such “physical frames”, in particular of local frames associated to observers at each point on the disk, is not necessary.

Consider then a uniformly rotating reference system, whose local cylindrical coordinates are denoted by \((t, r, z, \phi_R)\) connected to those of the fixed inertial one \((t, r, z, \phi)\) by

\[
\phi_R = \phi - \Omega t;
\]

the invariant interval in the rotating system reads

\[
ds^2 = c^2 dt^2 - (rd\phi_R + \Omega r dt)^2 - dz^2 - dr^2,
\]

in complete analogy with gravity in our Newtonian coordinates. This immediately yields that light propagates tangentially with velocity

\[
c_\perp = \pm c - \Omega r,
\]
as in the composition of light velocity with that of the free fall frame in gravity.

The previous equation can be rewritten in terms of \(v = \Omega r\) and \(dy = rd\phi\)

\[
ds^2 = c^2 (1 - v^2/c^2) dt^2 - 2v dt dy - dy^2 - dr^2 - dz^2 \tag{51}
\]

The similarity with the P-G formula is once more apparent. The essential difference is that now \(v\) is independent of \(y\), and in fact the differential \(rd\phi = dy + v dt\), from Eq.(49), corresponding to the EPIF differential \(dr' = dr - v(r) dt\), is now exact.

The above difference between light velocities in the two directions easily leads to the same result as before. We emphasize that the analysis applies to first order in \(\Omega\), that no relativistic effect appear to that order and that the above discussion of the relativistic interval has nothing to do with Lorentz transformations, rather expressing the interval in the inertial frame in terms of different coordinates (intervals in such coordinates coinciding with measured intervals “on the moving disk” only in the non-relativistic limit).

It is also of some interest to write the above relativistic interval in “Schwarzschild coordinates”: the off-diagonal term can be disposed of along the previous lines via the transformation

\[
dt_S = dt - \frac{v}{(1 - \frac{v^2}{c^2})} dy \tag{52}
\]

and for the relevant part (i.e. apart from \(dr^2\) and \(dz^2\) terms) the invariant interval takes the “Schwarzschild form”

\[
ds^2 = c^2 (1 - \frac{v^2}{c^2}) dt^2_S - \frac{dy^2}{1 - \frac{v^2}{c^2}} \tag{53}
\]

In both forms, the invariant interval is not Minkovskian, and in fact light velocity is different from \(c\) both in the “P-G” and in the “Schwarzschild” coordinates, where the tangential velocity is direction independent:

\[
c_\perp = c (1 - \frac{v^2}{c^2})
\]
This is compatible with the Sagnac effect because Eq. (52) only gives rise to a local notion of time and global, topological, effects are hidden in the angular nature of the $y$ variable. In $S_s$ coordinates the Sagnac effect comes in fact from the difference in time coordinates obtained after following closed paths. The time difference between two path enclosing the circle in opposite directions is

$$ \Delta t_S = \frac{2}{c^2} \oint \frac{\Omega r^2}{(1 - \frac{\Omega r^2}{c^2})^2} d\phi $$

which for low angular velocities yields

$$ \Delta T = \frac{4\Omega S}{c^2} $$

corresponding to the result obtained via elementary considerations at the beginning of the paragraph.

10 Conclusions

In the present work Einstein’s equations have been shown to be unnecessary in the static symmetric case, where all GR results have been obtained via EP from relativistic corrections to Newton’s law.

The only calculational ingredient has been a variational principle (like the Fermat one for light) using the Euler-Lagrange equations for the infinitesimal invariant non Minkovskian interval, thus providing a simple formulation for what has been considered so far a rather complicated and necessarily formal description of gravitation.

The connection with the $S_s$ GR solution, which is superfluous given the explicit calculations of the present approach (and which had been considered as a necessary endorsemen for the Painlevé-Gullstrand solution) only helps in clarifying the arbitrariness of attaching physical significance to the metric. In particular, space-time curvature is perfectly compatible with Newtonian absolute time and Euclidean space.

The treatment of rotating frames, which played a role in the genesis of GR, has been shown to be unrelated to gravitational effects. It just leads at each point to a metric of the same form, which nevertheless simply follows from a local coordinate transformation.

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References

[1] I. Newton, Philosophiae Naturalis Principia Mathematica (Mathematical Principles of Natural Philosophy), London, 1687; Cambridge, 1713; London, 1726.
[2] A. Einstein, Die Grundlage der allgemeinen Relativitätstheorie, Annalen der Physik. 49, 1916, S. 769-822.

[3] P. Painlevé, La mécanique classique et la théorie de la relativité, C. R. Acad. Sci. (Paris) 173 677-680 (1921).

[4] A. Gullstrand, Allgemeine Lösung des statischen Einkörper-Problems in der Einsteinschen Gravitationstheorie, Arkiv. Mat. Astron. Fys. 16(8) 115 (1922).

[5] Schwarzschild, K. (1916). "Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie". Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften. 7: 189-196.

[6] Berkeley Physics Course, Mechanics, McGraw-Hill (1962)

[7] Milgrom, M. (1983), Astrophysical Journal. 270: 365-370.

[8] L. I. Schiff, Am. J. Phys. 28 (1960) 340.

[9] T. Damour, La Matematica, vol. IV, G. Einaudi Editore, 453 (2010)

[10] J. G. Williams, S. G. Turyshev and D. H. Boggs, arXiv: gr-qc/0411113v2 (2004), Int. J. Mod. Phys. D 18, 1129 (2009). https://doi.org/10.1142/S021827180901500X

[11] R. Colella, A. W. Overhauser and S. A. Werner. Phys. Rev. Lett., 34 (1975), p. 1472.

[12] Eöt-Wash Group, Class. Quantum Grav. 29, 184002 (2012)

[13] A. Schild, Am. J. Phys. 28, 778, (1960)

[14] K. Kassner, Eur. J. Phys. 36 (2015) 065031, arXiv:1502.00149 [pdf, ps, other]

[15] Jan Czerniawski, arxiv.org/abs/gr-qc/0201037 (2004)

[16] R. P. Gruber et al., Am. J. Phys. 56, 265 (1988)

[17] M. Visser arXiv: gr-qc/0309072v3 2004

[18] T. Padmanabhan, link.springer.com/article/10.1007/s12045-009-0101-x

[19] P. Christillin and L. Barattini, Gravitomagnetic forces and quadrupole gravitational radiation from special relativity, arXiv:1206.4593 (2013)