Multi-Instanton Check of the Relation Between the Prepotential $F$ and the Modulus $u$
 in $N = 2$ SUSY Yang-Mills Theory

Nicholas Dorey

Physics Department, University College of Swansea
Swansea SA2 8PP UK  n.dorey@swansea.ac.uk

Valentin V. Khoze

Department of Physics, Centre for Particle Theory, University of Durham
Durham DH1 3LE UK  valya.khoze@durham.ac.uk

and

Michael P. Mattis

Theoretical Division T-8, Los Alamos National Laboratory
Los Alamos, NM 87545 USA  mattis@pion.lanl.gov

By examining multi-instantons in $N = 2$ supersymmetric $SU(2)$ gauge theory, we derive, on very general grounds, and to all orders in the instanton number, a relationship between the prepotential $F(\Phi)$, and the coordinate on the quantum moduli space $u = \langle \text{Tr}\Phi^2 \rangle$. This relation was previously obtained by Matone in the context of the explicit Seiberg-Witten low-energy solution of the model. Our findings can be viewed as a multi-instanton check of the proposed exact results in supersymmetric gauge theory.

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1. Introduction.

The concept of electric-magnetic duality has led to remarkable recent progress in understanding the low energy physics of certain (3+1) dimensional supersymmetric gauge theories. In this note we focus on pure $N = 2$ supersymmetric $SU(2)$ gauge theory, the exact low-energy limit of which has been given by Seiberg and Witten [1]. As sketched below, their solution consists of explicit expressions for the vevs $a(u)$ and $a_D(u)$ of the Higgs field $A$ and its dual field $A_D$, as functions of the parameter $u$

$$u \equiv \langle \frac{1}{2} A^a A^a \rangle . \quad (1)$$

Alternatively, Matone [2] has recast these as predictions for the $N = 2$ prepotential $F(a)$ and the parameter $u \equiv G(a)$ as functions of the vev $a$; explicitly, one finds [2,3]

$$G(a) = \pi i \left( F(a) - \frac{1}{2} a F'(a) \right), \quad (2a)$$

$$\left( 4\Lambda^4 - G^2 \right) G''(a) + \frac{1}{4} a \left( G'(a) \right)^3 = 0 . \quad (2b)$$

These equations are naturally Taylor expanded in the RG-invariant scale parameter $\Lambda^4$, the $n$th power of which captures the contribution of the $n$-instanton sector. Thus instantons provide an important means of checking the Seiberg-Witten solution (2) directly—without appealing to duality, nor to arguments about the number and nature of the singularities in the analytic $u$ plane. That Eq. (2b) holds at the 1-instanton and 2-instanton level (with $G$ eliminated in favor of $F$ as per Eq. (2a)) has been verified in Refs. [4] and [5], respectively. Subsequently, using the methods of Refs. [4,5], the authors of Ref. [6] have verified Eq. (2a) as well, through the 2-instanton level. In this note, we point out that Eq. (2a) is in fact built into the instanton calculus, and holds automatically at the $n$-instanton level, for all $n$.

2. Review

The particle content of pure $N = 2$ supersymmetric $SU(2)$ gauge theory consists, in $N = 1$ language, of a gauge multiplet $W^a_\alpha = (v^a_m, \lambda^a)$ coupled to a complex chiral matter multiplet $\Phi^a = (A^a, \psi^a)$ which transform in the adjoint representation of $SU(2)$, $a = 1, 2, 3$. Here $v^a_m$ is the gauge field, $A^a$ is the Higgs field, Weyl fermions $\lambda^a$ and $\psi^a$ are the gaugino and Higgsino. Classically, the adjoint scalar $A^a$ can have an arbitrary complex vev, $(0, 0, a)$, breaking $SU(2)$ down to $U(1)$. $N = 2$ supersymmetry remains unbroken, and protects the flat direction in $a$ from being lifted by quantum corrections. Different values
of \( a \) span a one complex dimensional family of theories, known as the quantum moduli space.

The isospin component of the fields that is aligned with the vev \((0, 0, a)\) remains massless, whereas the remaining two components acquire a mass \( M_W = \sqrt{\sigma} |a| \). For length scales \( x \gg 1/M_W \) the massless modes can be described by Wilsonian effective action in terms of an \( N = 1 \) photon superfield \( W_\alpha = (v_m, \lambda) \) and chiral superfield \( \Phi = (A, \psi) \). \( N = 2 \) supersymmetry restricts the terms in the effective action with not more than two derivatives or four fermions to the following form \([7,8,1]\):

\[
\mathcal{L}_{\text{eff}} = \frac{1}{4\pi} \text{Im} \left[ \int d^4 \theta F'(\Phi) \bar{\Phi} + \int d^2 \theta \frac{1}{2} F''(\Phi) W^\alpha W_\alpha \right].
\]  

Thus \( \mathcal{L}_{\text{eff}} \) is specified by a single object, the holomorphic prepotential \( F \). Its second derivative is just the running complexified coupling \( \tau(a) \) \([1]\):

\[
F''(a) = \tau(a) = \frac{4\pi i}{g^2(a)} + \frac{\vartheta(a)}{2\pi},
\]

where \( \vartheta \) is the effective theta-parameter. The first derivative \( F'(\Phi) \) defines the superfield \( \Phi_D \) dual to \( \Phi \). This is the local field of the dual magnetic description of the low-energy theory \([4]\). It has vev

\[
a_D = \langle \Phi_D \rangle = \langle A_D \rangle = F'(a).
\]

The vevs \( a \) and \( a_D \) provide alternative local parametrizations of the moduli space of the theory. However, neither \( a \) nor \( a_D \) is a good global coordinate on the quantum moduli space; instead they are traded for a gauge-invariant parameter \( u \), Eq. \([4]\), which is. By determining the behavior of \( a(u) \) and \( a_D(u) \) in the vicinity of their hypothesized singularities, Seiberg and Witten were able to reconstruct them exactly \([1]\):

\[
a = \frac{\sqrt{2}}{\pi} \int_{-2\Lambda^2}^{2\Lambda^2} \frac{dx\sqrt{x-u}}{\sqrt{x^2 - 4\Lambda^4}},
\]

\[
a_D = \frac{\sqrt{2}}{\pi} \int_{2\Lambda^2}^{u} \frac{dx\sqrt{x-u}}{\sqrt{x^2 - 4\Lambda^4}}.
\]

Here \( \Lambda \) is the dynamically generated scale of the effective \( U(1) \) theory which can be matched \([4]\) to the dynamical scale of the microscopic \( SU(2) \) theory, \( \Lambda_{\text{PV}} \), computed in the Pauli-Villars regularization scheme—the natural scheme for doing instanton calculations. These expressions are singular at \( u \to \pm 2\Lambda^2 \) and \( u \to \infty \). Physically, the singularities at \( +2\Lambda^2 \) and \( -2\Lambda^2 \) correspond, respectively, to the monopole and the dyon becoming massless,
while the singularity at $\infty$ is a perturbative one-loop effect and follows from the asymptotic freedom of the original theory.

In order to extract the prepotential from Eq. (6), one formally inverts Eq. (6), $u = G(a)$, and then uses Eqs. (5) and (6) to obtain:

$$F''(a) = \frac{a_D'(u)}{a'(u)} \bigg|_{u=G(a)}. \quad (7)$$

This procedure yields the following expansions for $u$ and $F$:

$$u \equiv G(a) = G_{\text{clas}}(a) + G_{\text{inst}}(a) = \frac{1}{2} a^2 + \sum_{n=1}^{\infty} G_n \left( \frac{\Lambda}{a} \right)^{4n} a^2, \quad (8)$$

and

$$F(a) = F_{\text{pert}}(a) + F_{\text{inst}}(a) = \frac{i}{2\pi} a^2 \log \frac{2a^2}{e^3 \Lambda^2} - \frac{i}{\pi} \sum_{n=1}^{\infty} F_n \left( \frac{\Lambda}{a} \right)^{4n} a^2. \quad (9)$$

That the expansion for $F$ has this general form had been known for some time $[7]$ with the $n$th term in the series being an $n$-instanton effect. The new information in the Seiberg-Witten solution is the numerical value $[9,2]$ of each of the coefficients $F_n$: in particular, $F_1 = 1/2$, with the higher $F_n$’s ($F_2 = 5/16$, $F_3 = 3/4$, $F_4 = 1469/29$, $F_5 = 4471/40$, ...) being determined by a recursion relation implied by Eqs. $[2]$. This constitutes a set of highly non-trivial predictions for all multi-instanton contributions to the low-energy physics in this theory.

From hereon in we focus on the relation $(2^a)$. Differentiating both sides with respect to $u$ gives the following constancy condition on the Wronskian $W(a,a_D) = a(u)a_D'(u) - a_D(u)a'(u)$:

$$1 = -\frac{i\pi}{2} W(a,a_D). \quad (10)$$

Up to two constants (an integration constant, and the multiplicative factor in Eq. $[10]$), both of which are fixed by examining the asymptotically free large-$|u|$ regime $[3]$, Eq. $(2^a)$ is therefore tantamount to the condition $dW/du = 0$. This, in turn, implies that $a$ and $a_D$ satisfy a common linear homogeneous second-order differential equation with a vanishing

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1 Numerical values of $F_n$ depend on the normalization of $a$ and the prescription for $\Lambda$. In our conventions $[3]$, the $F_n$ are those of Ref. $[9]$ times a factor of $2^{6n-2}$. 

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first-derivative term. For the specific Seiberg-Witten solution (3) this equation has in fact been found, and reads [2,9]

\[
(4\Lambda^2 - u^2) \frac{d^2}{du^2} \left( \frac{1}{4} \right) a = \left( (4\Lambda^2 - u^2) \frac{d^2}{du^2} - \frac{1}{4} \right) a_D = 0 .
\] (11)

But it is clear that Eq. (2a), unlike Eq. (2b), is more general than the specific solution (3). As reviewed in Ref. [10], this linear ODE condition is in fact the natural paradigm for understanding the monodromy conditions [1] on \( a \) and \( a_D \).

3. Multi-Instanton Check of Eq. (2a).

Expanding Eq. (2a) in the instanton number \( n \) gives the following simple rewrite [2]:

\[
G_n = 2n F_n .
\] (12)

This relation has recently been checked for \( n = 1 \) and \( n = 2 \) in Ref. [3]. In what follows we give the proof of Eq. (12) for all instanton numbers.

To understand the multi-instanton contributions to \( G \) and \( F \), we first need to review some general features of the collective coordinate measure. In bosonic pure gauge theory the general \( n \)-instanton solution of Atiyah, Drinfeld, Hitchin and Manin (ADHM) [11] contains \( 8n \) collective coordinates. In \( N = 2 \) supersymmetric gauge theory, these are augmented by \( 8n \) fermionic collective coordinates which parametrize the gaugino and higgsino adjoint fermion zero modes [12]. Denoting the \( 8n \) unconstrained bosonic and fermionic coordinates as \( X_i \) and \( \chi_i \), respectively, one may express the measure for \( n \) ADHM instantons [13] in \( N = 2 \) supersymmetric Yang-Mills theory as follows (see Sec. 7.5 of [5]):

\[
\int d\mu_n = \frac{1}{S_n} \int \left( \prod_{i=1}^{8n} dX_i d\chi_i \right) \left( J_{\text{bose}} / J_{\text{fermi}} \right)^{1/2} \exp(-S_{n-\text{inst}}) .
\] (13)

Here \( J_{\text{bose}} \) (\( J_{\text{fermi}} \)) is the Jacobian for the bosonic (fermionic) collective coordinates, while the super-multi-instanton action \( S_{n-\text{inst}} \) was calculated in Sec. 7.4 of Ref. [5]. An important simplification in a supersymmetric theory is that there are no additional small-fluctuations 't Hooft determinants to be calculated, as the non-zero eigenvalues for bosonic and fermionic excitations cancel identically. Finally, as explained in [13,5], to obtain the correct normalization of the measure, one must divide out the relevant symmetry factor \( S_n \).

In truth, the construction of \( S_n \), \( J_{\text{bose}} \) and \( J_{\text{fermi}} \) depends on one’s ability to isolate \( 8n \) independent bosonic collective coordinates from the unfortunately overcomplete set of
ADHM variables—currently an unsolved problem for \( n \geq 3 \). However, these issues are entirely irrelevant for present purposes; this is because Eq. (12) is diagonal in instanton number (unlike Matone’s recursion relation \([2]\) that follows from Eq. (2b), which connects different \( n \) sectors).

As explained in detail in Ref. \([5]\), most of the zero modes of the vevless super-multi-instanton are lifted by the vev of the adjoint Higgs field through the Yukawa terms in the action. We shall find it convenient to single out from the measure \( d\mu_n \) the integrations over the eight unlifted modes: global translations, \( d^4x_0 \), and the supersymmetric collective coordinates for gaugino and higgsino, \( d^2\xi_1 \) and \( d^2\xi_2 \),

\[
\int d\mu_n = \int d^4x_0 \ d^2\xi_1 \ d^2\xi_2 \int d\tilde{\mu}_n . \tag{14}
\]

The general \( n \)-instanton contribution to the functional integral for the quantum modulus \( u \) is simply

\[
G(a) \equiv u \equiv \langle \frac{1}{2} A^a A^a \rangle = \frac{1}{2} \int d\mu_n \ A^a_{\text{inh}} A^a_{\text{inh}} . \tag{15}
\]

Here \( A^a_{\text{inh}} \) is the solution of the inhomogeneous Euler-Lagrange equation,

\[
\mathcal{D}^2 \ A^a = \sqrt{2} i \ [ \lambda, \psi] , \tag{16}
\]

in the super-multi-instanton background. We use undertwiddling for fields in \( SU(2) \) matrix notation, \( A^a \equiv \sum_{a=1,2,3} A^a \tau^a / 2 \), where \( \tau^a \) are Pauli matrices. \( A^a_{\text{inh}} \) was found for the general case in Sec. 7.3 of Ref. \([5]\).

For the problem at hand we need not use the complete expression for \( A^a_{\text{inh}} \). Since the action \( S_{\text{n-inst}} \) does not depend on the the supersymmetric collective coordinates for gaugino and higgsino, \( \xi_1 \) and \( \xi_2 \), Grassmann integrations over these unlifted fermionic collective coordinates will produce a non-zero result only if there is an explicit dependence on \( \xi_1 \) and \( \xi_2 \) in the integrand \( A^a_{\text{inh}} A^a_{\text{inh}} \) of \([15]\). Thus it suffices to keep only the term in \( A^a_{\text{inh}} \) with explicit bilinear dependence on \( \xi_1 \) and \( \xi_2 \), and to drop the rest. This term, which we denote \( A^a_{\text{ss}} \), is a solution of Eq. \([16]\) with the right-hand side made of only supersymmetric fermion zero modes for \( \lambda \) and \( \psi \). The nice observation of Ref. \([3]\) is that this bilinear piece is proportional to the ADHM field strength,\(^2\)

\[
\tilde{A}^{\text{ss}}_a = \sqrt{2} \xi_2 \sigma^{mn} \xi_1 v^\text{ADHM}_{mn} , \tag{17}
\]

\(^2\) This observation follows immediately from some of the explicit formulae in Ref. \([3]\); see Eq. (4.3b) in the 1-instanton sector; and in the general \( n \)-instanton sector, compare the expression for \( v_{mn} \) given in Sec. 6.1, on the one hand, with the relevant bilinear piece of the Higgs extracted from Eqs. (7.23) and (7.5) on the other hand.
so that
\[ A_s^a A_s^a = -\xi_1^2 \xi_2^2 v_{mn}^a v_{mn}^a \ADHM. \] (18)

Substituting this into Eq. (15), performing the Grassmann integrals, \( \int d^2 \xi_1 \xi_1^2 \int d^2 \xi_2 \xi_2^2 = 1 \), and integrating over global translations,
\[
\frac{1}{2} \int d^4 x_0 v_{mn}^a v_{mn}^a \ADHM = 16\pi^2 n ,
\] (19)
gives the general \( n \)-instanton expression for \( G(a) \) in terms of the reduced \( n \)-instanton measure \( d\tilde{\mu}_n \) of Eq. (14):
\[
G(a) \mid_{n\text{-inst}} \equiv G_n \left( \frac{\Lambda}{a} \right)^{4n} a^2 = -16\pi^2 n \int d\tilde{\mu}_n .
\] (20)

In what follows we derive the general \( n \)-instanton expression for \( F(a) \):
\[
F(a) \mid_{n\text{-inst}} \equiv -\frac{i}{\pi} F_n \left( \frac{\Lambda}{a} \right)^{4n} a^2 = 8\pi i \int d\tilde{\mu}_n .
\] (21)
The advertised all-orders relation (12) follows immediately from a comparison of (20) and (21).

The calculation of \( F_n \) proceeds in a few short steps:
1. Write out \( L_{\text{eff}} \), Eq. (3), in components as per Ref. [5]:
\[
L_{\text{eff}} = \frac{1}{4\pi} \text{Im} \left[ -F''(A) \left( \partial_m A^\dagger \partial^m A + i\psi \phi \bar{\psi} + i\lambda \phi \bar{\lambda} + \frac{1}{2} (v_{mn}^{\text{SD}})^2 \right) \\
+ \frac{1}{\sqrt{2}} F'''(A) \lambda \sigma^{mn} \psi v_{mn} + \frac{1}{4} F''''(A) \psi^2 \lambda^2 \right] ,
\] (22)
ignoring the (sub-leading) dependence on the auxiliary component fields.
2. Concentrate on the four-fermion term in \( L_{\text{eff}} \):
\[
\frac{-i}{32\pi} F''''(a) \psi^2 \lambda^2 ,
\] (23)
which gives for the 4-point Green function:
\[
\langle \bar{\lambda}_\alpha(x_1) \bar{\lambda}_\beta(x_2) \bar{\psi}_\gamma(x_3) \bar{\psi}_\delta(x_4) \rangle = \\
\frac{-i}{8\pi} F''''(a) \int d^4 y \epsilon^{\alpha\beta} S_{\alpha\bar{\alpha}}(x_1 - y) S_{\beta\bar{\beta}}(x_2 - y) \epsilon^{\gamma\delta} S_{\gamma\bar{\gamma}}(x_3 - y) S_{\delta\bar{\delta}}(x_4 - y) ,
\] (24)
in terms of Weyl fermion propagator \( S(x)_{\alpha\bar{\alpha}} \).
3. Consider the $n$-instanton contribution to this Green function:

$$\langle \bar{\lambda}_\alpha(x_1) \bar{\lambda}_\beta(x_2) \bar{\psi}_\gamma(x_3) \bar{\psi}_\delta(x_4) \rangle = \int d\mu_n \bar{\lambda}_\alpha(x_1) \bar{\lambda}_\beta(x_2) \bar{\psi}_\gamma(x_3) \bar{\psi}_\delta(x_4),$$  \hspace{1cm} (25)

where the long-distance limit of the anti-fermion components of the multi-instanton

satisfy

$$\bar{\psi}_\alpha(x) = i\sqrt{2} \frac{\partial S_{n\text{-inst}}}{\partial a} \xi^a_1 S_{\alpha\dot{\alpha}}(x-x_0),$$  \hspace{1cm} (26)

and likewise

$$\bar{\lambda}_\alpha(x) = -i\sqrt{2} \frac{\partial S_{n\text{-inst}}}{\partial a} \xi^a_2 S_{\alpha\dot{\alpha}}(x-x_0).$$  \hspace{1cm} (27)

Here we carefully distinguish between $a$ and $\bar{a}$ and use the fact that $S_{n\text{-inst}}$ is at most linear in $a$ (bilinear in $a$ and $\bar{a}$).

4. Performing the Grassmann integrations over $d^2 \xi_1$ and $d^2 \xi_2$ on the right hand side of (25),

$$\int d^4 x_0 \varepsilon^{\alpha\beta} S_{\alpha\dot{\alpha}}(x_1-x_0) S_{\beta\dot{\beta}}(x_2-x_0) S_{\gamma\dot{\gamma}}(x_3-x_0) S_{\delta\dot{\delta}}(x_4-x_0) \int d\bar{\mu}_n \left( \frac{\partial S_{n\text{-inst}}}{\partial a} \right)^4$$  \hspace{1cm} (28)

and using the fact that the only dependence on the vev $a$ in the measure $d\bar{\mu}_n$ is through $\exp(-S_{n\text{-inst}})$ and that $S_{n\text{-inst}}$ is linear in $a$, we finally obtain

$$\frac{\partial^4}{\partial a^4} \int d\bar{\mu}_n \int d^4 x_0 \varepsilon^{\alpha\beta} S_{\alpha\dot{\alpha}}(x_1-x_0) S_{\beta\dot{\beta}}(x_2-x_0) S_{\gamma\dot{\gamma}}(x_3-x_0) S_{\delta\dot{\delta}}(x_4-x_0).$$  \hspace{1cm} (29)

5. Comparing the right hand sides of Eqs. (24) and (29) we derive the desired expression (21), and thus $G_n = 2n F_n$.

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3 In particular see Eq. (7.32) in the more recent, slightly expanded version of Ref. 3, where $S_{n\text{-inst}}$ is given explicitly in the general case of complex vev. The corresponding expression in the old version of 3 is specific to the case of real vev and does not make this distinction.
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