Recurrence for vertex-reinforced random walks on $\mathbb{Z}$ with weak reinforcements.

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Abstract

We prove that any vertex-reinforced random walk on the integer lattice with non-decreasing reinforcement sequence $w$ satisfying $w(k) = o(k^\alpha)$ for some $\alpha < 1/2$ is recurrent. This improves on previous results of Volkov [9] and Schapira [6].

Keywords. Self-interacting random walk; reinforcement; recurrence and transience

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1 Introduction

In this paper, we consider a one-dimensional vertex-reinforced random walk (VRRW) with non-decreasing weight sequence $w : \mathbb{N} \to (0, \infty)$, that is a stochastic process $X = (X_n)_{n \geq 0}$ on $\mathbb{Z}$, starting from $X_0 = 0$, with transition probabilities:

$$P\{X_{n+1} = X_n \pm 1 \mid \mathcal{F}_n\} = \frac{w(Z_n(X_n \pm 1))}{w(Z_n(X_n + 1)) + w(Z_n(X_n - 1))}$$

where $\mathcal{F}_n \overset{\text{def}}{=} \sigma(X_1, \ldots, X_n)$ is the natural filtration of the process and $Z_n(x) \overset{\text{def}}{=} \# \{0 \leq k \leq n, X_k = x\}$ is the local time of $X$ on site $x$ at time $n$. This process was first introduced by Pemantle in [3] and then studied in the linear case $w(k) = k + 1$ by Pemantle and Volkov in [5]. They proved the surprising fact that the walk visits only finitely many sites. This result was subsequently improved by Tarrès [7, 8] who showed that the walk eventually gets stuck on exactly 5 consecutive sites almost surely. When the reinforcement sequence grows faster than linearly, the walk still gets stuck on a finite set but whose cardinality may be smaller than 5, see [1, 9] for details. On the other hand, Volkov [9] proved that for sub-linearly growing weight sequences of order $n^\alpha$ with $\alpha < 1$, the VRRW is either transient or recurrent. The main result of this paper is to show that the walk is, indeed, recurrent.

Theorem 1.1. Assume that the weight sequence satisfies $w(k) = o(k^\alpha)$ for some $\alpha < 1/2$. Then $X$ is recurrent i.e. it visits every site infinitely often almost-surely.

Let us mention that, simultaneously with the writing of this paper, a similar result was independently obtained by Chen and Kozma [2] who proved recurrence for the VRRW with weights of order $n^\alpha$, $\alpha < 1/2$, using a clever martingale argument combined with previous local time estimates from Schapira [6]. The argument in this paper, while also making use of a martingale, is self-contained and does not rely upon previous results of Volkov [9] or Schapira [6]. In particular, we do not require any assumption on the regular variation of the weight function $w$.

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2 A martingale

Obviously, multiplying the weight function by a positive constant does not change the process $X$. Thus, we now assume without loss of generality that $w(0) = 1$. We define the two-sided sequence $(a_x)_{x \in \mathbb{Z}}$ by

$$a_x \overset{\text{def}}{=} \begin{cases} 
1 - \frac{\varepsilon}{(x+1)^2} & \text{for } x \geq 0 \\
\frac{\varepsilon}{2} & \text{for } x < 0 
\end{cases}$$

where $\varepsilon > 0$ will be chosen later during the proof of the theorem. Define also

$$A_k \overset{\text{def}}{=} \prod_{x=-k}^{x=0} a_x \in (0, 1).$$

We construct from $X$ two processes $(M_n)_{n \geq 0}$ and $(\Delta_n(z), z < X_n)_{n \geq 0}$ in the following way:

1. Initially set $M_0 \overset{\text{def}}{=} 0$ and $\Delta_0(z) \overset{\text{def}}{=} 1$ for all $z < 0 = X_0$.

2. By induction, $M_n$ and $(\Delta_n(z), z < X_n)$ having been constructed,

- if $X_n = x$ and $X_{n+1} = x - 1$, then
  $$M_{n+1} \overset{\text{def}}{=} M_n - a_x \Delta_n(x-1) \quad \Delta_{n+1}(z) \overset{\text{def}}{=} \Delta_n(z) \quad \text{for } z < x - 1,$$

- if $X_n = x$ and $X_{n+1} = x + 1$, then
  $$M_{n+1} \overset{\text{def}}{=} M_n + a_x \Delta_n(x-1) \frac{w(Z_n(x-1))}{w(Z_n(x+1))} \quad \Delta_{n+1}(z) \overset{\text{def}}{=} \begin{cases} 
\Delta_n(z) & \text{for } z < x, \\
a_x \Delta_n(x-1) \frac{w(Z_n(x-1))}{w(Z_n(x+1))} & \text{for } z = x.
\end{cases}$$

Note that the quantities $\Delta$ have a simple interpretation: for any $n$ and $z < X_n$, the value $\Delta_n(z)$ is positive and corresponds to the increments of $M_n$, the last time before $n$ that the walk $X$ jumped from site $z$ to site $z + 1$ (with the convention $\Delta_n(z) = 1$ for negative $z$ if no such jumps occurred yet). By extension, we also define $\Delta_n \overset{\text{def}}{=} \Delta_n(X_n)$ at the current position as the "would be" increment of $M_n$ if $X$ makes its next jumps to the right (at time $n+1$) i.e.

$$\Delta_n \overset{\text{def}}{=} a_{X_n} \Delta_n(X_n-1) \frac{w(Z_n(X_n-1))}{w(Z_n(X_n+1))}.$$

We will also use the notation $\tau_y$ to denote the hitting time of site $y$,

$$\tau_y \overset{\text{def}}{=} \inf \{ n \geq 0, X_n = y \} \in [0, \infty].$$

**Proposition 2.1.** The process $M$ is an $\mathcal{F}_n$-martingale and, for $n \geq 0$, we have

$$M_n = \sum_{i=0}^{n-1} 1_{\{X_{i+1} = X_i + 1\}} \left(1 - a_{X_i+1} 1_{\{3 \notin (i,n], X_j = X_i\}}\right) \Delta_i + \frac{1}{2} \inf_{i \leq n} X_i \quad (1)$$

In particular, for $y = 1, 2, \ldots$, the process $M_{n \wedge \tau_y}$ is bounded below by $-y/2$, hence it converges a.s.

**Proof.** Since $\Delta_n(\cdot)$ and $Z_n(\cdot)$ are $\mathcal{F}_n$-measurable, by definition of $M$,

$$\mathbb{E}[M_{n+1} \mid \mathcal{F}_n] = \mathbb{E} \left[ M_n + a_{X_n} \Delta_n(X_n - 1) \left( \frac{w(Z_n(X_n - 1))}{w(Z_n(X_n + 1))} \cdot 1_{\{X_{n+1} = X_n + 1\}} - 1_{\{X_{n+1} = X_n - 1\}} \right) \mid \mathcal{F}_n \right]$$

$$= M_n + a_{X_n} \Delta_n(X_n - 1) \left( \frac{w(Z_n(X_n - 1))}{w(Z_n(X_n + 1))} \mathbb{P}\{X_{n+1} = X_n + 1 \mid \mathcal{F}_n\} - \mathbb{P}\{X_{n+1} = X_n - 1 \mid \mathcal{F}_n\} \right)$$

$$= M_n$$
thus $M$ is indeed a martingale. Furthermore, by construction, at each time $i$ when the process $X$ crosses an edge $\{x, x+1\}$ from left to right, the process $M$ increases by $\Delta_i = \Delta_{i+1}(x) > 0$. If at some later time, say $j > i$, $X$ crosses this edge again (and thus in the other direction), the martingale decreases by $a_{x+1}\Delta_i(x) = a_{x+1}\Delta_i$. Moreover, by convention $\Delta_0(z) = 1$ and $a_z = \frac{1}{2}$ for $z < 0$ so that $M$ decreases by $\frac{1}{2}$ each time it crosses a new edge of the negative half line for the first time. Putting these facts together, we get exactly (1). Finally, since $a_z < 1$ for any $z \in \mathbb{Z}$, each term in the sum (1) is positive, hence $M_{n\wedge \tau_{\tau_y}}$ is bounded below by $\frac{1}{2} \inf_{t \leq n \wedge \tau_{\tau_y}} X_t \geq -y/2$.

\begin{proposition}
Let $y > 0$. For $n \leq \tau_{-y}$, we have

$$\Delta_n(z) \geq \frac{A_y}{w(Z_n(z))w(Z_n(z+1))} \text{ for any } -y \leq z \leq X_n. \tag{2}$$

\end{proposition}

\begin{proof}
We prove by induction on $n$ that for $n \leq \tau_{-y}$,

$$\Delta_n(z) \geq \frac{\prod_{i=-y}^z a_i}{w(Z_n(z))w(Z_n(z+1))} \text{ for any } -y \leq z \leq X_n. \tag{3}$$

Recalling that $w(k) \geq 1$ and $a_k \leq 1$ for any $k$, it is straightforward that (2) holds for $n = 0$. Now, assume the result for $n$ and consider the two cases:

- If $X_{n+1} = X_n - 1$. Then for any $-y \leq z \leq X_{n+1}$, we have $\Delta_{n+1}(z) = \Delta_n(z)$ whereas $w(Z_{n+1}(z)) \geq w(Z_n(z)).$ Thus (3) holds for $n + 1$.

- If $X_{n+1} = X_n + 1$. Again, we have $\Delta_{n+1}(z) = \Delta_n(z)$ for any $-y \leq z \leq X_n$. It remains to check that $\Delta_{n+1}(X_{n+1})$ satisfies the inequality:

$$\Delta_{n+1}(X_{n+1}) = \Delta_n + a_{X_{n+1}}\Delta_{n+1}(X_{n+1}) \geq a_{X_{n+1}} \frac{\prod_{i=-y}^z a_i}{w(Z_{n+1}(X_n))w(Z_{n+1}(X_n+1))w(Z_{n+1}(X_n + 1))w(Z_{n+1}(X_n + 1))}$$

\end{proof}

We can now recover, with our assumptions on $w$, Volkov’s result stating that the walk does not get stuck on any finite interval.

\begin{proposition}
For any $y > 0$, we have

$$\limsup_n X_n = +\infty \text{ on the event } \{\tau_{-y} = \infty\}.$$

\end{proposition}

\begin{proof}
On $\{\tau_{-y} = \infty\}$, the combination of (1) and Proposition 2\textsuperscript{22} give

$$M_n \geq \sum_{i=0}^{n-1} 1_{\{X_{i+1} = X_i + 1\}} \left(1 - a_{X_{i+1}}1_{\{3j > i, X_{i+1} = X_i\}}\right) \frac{A_y}{w(Z_i(X_i))w(Z_i(X_i + 1))} \frac{y - 1}{2} \tag{4}$$

Denoting by $e_n = (s_n, s_n + 1)$ the edge which has been most visited at time $n$, we deduce that on the event $\{\tau_{-y} = \infty\}$,

$$M_n \geq Z_n(e_n) (1 - a_{s_{n+1}}) \frac{A_y}{w(Z_n(s_n))w(Z_n(s_n + 1))} \frac{y - 1}{2},$$

where $Z_n(e_n)$ denotes the number of times the edge $e_n$ has been crossed from left to right before time $n$. Using that $\max(Z_n(s_n), Z_n(s_n + 1)) \leq 2Z_n(e_n)$ and that $w(k) = o(\sqrt{E})$ and that $M_{n\wedge \tau_{\tau_y}}$ converges, we conclude that on $\{\tau_{-y} = \infty\}$, either $Z_n(e_n)$ remains bounded or $a_{s_{n+1}}$ takes values arbitrarily close to $1$.

In any case, this means that $X$ goes arbitrarily far to the right hence $\limsup_n X_n = +\infty$.

\end{proof}
3 Proof of theorem 1.1

Fix $y > 0$ and consider the event $\mathcal{E}_y = \{\inf_n X_n = -y + 1\}$. Pick $v > 0$ and define $N_z$ to be the number of jumps of $X$ from site $z$ to site $z+1$ before time $\tau$, (according to the previous proposition $\tau$ is finite on $\mathcal{E}_y$ so all the $N_z$ are finite). From (4), grouping together the contributions to $M$ of each edge $(z, z+1)$, we get, on $\mathcal{E}_y$,

\[
M_{\tau} \geq A\sum_{z=-y+1}^{v-1} \frac{1 + (N_z - 1)(1 - a_v)}{w(Z_{\tau_z}(z))w(Z_{\tau_z}(z+1))} - \frac{y - 1}{2}
\]

\[
\geq A\sum_{z=-y+1}^{v-1} \frac{1 + (N_z - 1)(1 - a_v)}{w(N_z + N_z)w(N_z + N_z + 1)} - \frac{y - 1}{2}
\]

\[
\geq A\sum_{z=-y+1}^{v-1} \frac{\frac{1}{2} + N_z(1 - a_v)}{w(N_z + N_z + 1)} - \frac{y - 1}{2}
\]

\[
\geq CA\sum_{z=-y+1}^{v-1} \frac{\frac{1}{2} + \frac{N_z}{(v+2)\epsilon}}{(N_z + N_z)\alpha(N_z + N_z + 1)\alpha} - \frac{y - 1}{2}
\]

where $C > 0$ and $\alpha < 1/2$ only depend on the weight function $w$. Finally, lemma 4.1 below states that if we choose $\epsilon > 0$ small enough, the sum above becomes arbitrarily large as $v$ tends to infinity. On the other hand, we also know that $M$ converges on this event so necessarily $P(\mathcal{E}_y) = 0$. Since this result holds for any $y > 0$, we get inf $X_n = -\infty$ a.s. By symmetry, sup $X_n = +\infty$ a.s. which implies that the walk visits every site of the integer lattice infinitely often almost surely.

4 An analytic lemma

Lemma 4.1. For any $0 < \alpha < \frac{1}{2}$, there exists $\epsilon > 0$ such that

\[
\limsup_{K \to \infty} \inf\left\{ \sum_{i=0}^{K} \frac{\frac{1}{2} + \frac{b_i}{(K+2)\epsilon}}{(b_{i-1} + b_i)^\alpha(b_i + b_{i+1})^\alpha} \right\} = \infty
\]

(with the convention $b_{-1} = b_{K+1} = 0$).

Proof. The idea is to group the $b_i$’s into packets with respect to their value. Consider a reordering of the $b_i$’s:

\[
\tilde{b}_0 \geq \tilde{b}_2 \geq \ldots \geq \tilde{b}_K.
\]

Fix a positive integer $l$ and group these numbers into $l + 1$ packets

\[
\tilde{b}_0, \ldots, \tilde{b}_{K_1}, \tilde{b}_{K_1 + 1}, \ldots, \tilde{b}_{K_2}, \ldots, \tilde{b}_{K_{l-1}}, \ldots, \tilde{b}_{K_l}, \tilde{b}_{K_l + 1}, \ldots, \tilde{b}_K.
\]

We can choose the $K_i$’s growing geometrically such that the sizes of the packets satisfy

\[
\# P_1 \geq \frac{K}{4^l}, \quad \# P_2 \geq 3(\# P_1 + \ldots + \# P_{l-1}).
\]

We now regroup each term of the sum (6) according to which packet the central $b_i$ (the one appearing in the numerator) belongs. Assume by contradiction that the sum (6) is bounded, say by $A$.

We first consider only the terms corresponding to packet $P_1$. Since there are at least $K\frac{1}{4^l}$ terms, we obtain the inequality

\[
A \geq \frac{K}{4^l} \frac{\frac{1}{2} + \frac{b_{K_1}}{K+2}+\epsilon}{(b_{0})^\alpha(b_{0})^\alpha} \geq C\tilde{b}_{K_1} \frac{b_{K_1}}{K^\frac{1}{2}b_0^\alpha}
\]

where the constant $C$ does not depend on $K$ or on the sequence $(b_i)$. We now deal with packets $k = 2, \ldots, l$. Thanks to (6) and since every denominator in (6) involves only two $b_j$ other than the one
appearing in the numerator, there are least one third of the terms belonging to packet $\mathcal{P}_k$ that do not contain any $b_j$ from a packet with smaller index (i.e. with larger value). So, there is at least $\frac{K}{3}$ such terms for which we can get a lower bound the same way we did for packet $\mathcal{P}_1$. We deduce that, summing over the terms corresponding to packet $\mathcal{P}_k$,

$$\tilde{b}_{K_k} \leq CK^2 \tilde{b}_{K_{k-1}}^{2\alpha} \quad \text{for } i = 1, \ldots, l,$$

(7)

with the convention $K_0 = 0$ and where $C$ again does not depend on $K$ or $(b_i)$. Finally, we obtain a lower bound for the sum of the terms belonging to the last packet $\mathcal{P}_{l+1}$ by taking $\frac{1}{2}$ as the lower bound for the numerator, and considering only the terms for which no $b_i$'s from any other packet appear in the denominator (again, there are at least $\frac{K}{3}$ such terms). This give the inequality

$$K \leq C \tilde{b}_{K_l}^{2\alpha}.$$

(8)

Combining (7) and (8), we get by induction that for some constant $C$ (depending on $l$),

$$K \leq CK \epsilon^{(2\alpha+(2\alpha)^2+\ldots+(2\alpha)^i)} \tilde{b}_{0}^{(2\alpha)^{i+1}} \leq C K \epsilon^{(2\alpha)^{i+1}} \tilde{b}_{0}^{(2\alpha)^{i+1}}.$$

For $\epsilon$ small enough such that $\frac{\epsilon}{1-2\alpha} \leq \frac{1}{2}$ we obtain

$$\tilde{b}_0 \geq \frac{1}{C} K \epsilon^{-\frac{1}{(2\alpha)^{i+1}}}.$$

Recalling that the sum (5) contains the term corresponding to $\tilde{b}_0$ but is also, by assumption, bounded above by $A$, we find

$$\frac{1}{2} + \frac{\tilde{b}_0}{(K+2)^{i+1}} \leq C K \epsilon^{(2\alpha)^{i+1}}.$$

Finally, we choose $l$ large enough such that $\frac{1}{2} - 2\alpha - 1 - \epsilon > 0$ and we get a contradiction by letting $K$ tends to infinity.

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