High-Confidence Policy Optimization: Reshaping Ambiguity Sets in Robust MDPs

Bahram Behzadian ∗ Reazul Hasan Russel ∗ Marek Petrik
Department of Computer Science
University of New Hampshire
Durham, NH 03824 USA
bahram, rrussel, mpetrik @ cs.unh.edu

Abstract

Robust MDPs are a promising framework for computing robust policies in reinforcement learning. Ambiguity sets, which represent the plausible errors in transition probabilities, determine the trade-off between robustness and average-case performance. The standard practice of defining ambiguity sets using the $L_1$ norm leads, unfortunately, to loose and impractical guarantees. This paper describes new methods for optimizing the shape of ambiguity sets beyond the $L_1$ norm. We derive new high-confidence sampling bounds for weighted $L_1$ and weighted $L_\infty$ ambiguity sets and describe how to compute near-optimal weights from rough value function estimates. Experimental results on a diverse set of benchmarks show that optimized ambiguity sets provide significantly tighter robustness guarantees.

1 Introduction

Markov decision processes (MDPs), the fundamental model that underlies reinforcement learning (Bertsekas and Tsitsiklis, 1996; Puterman, 2005; Sutton and Barto, 2018), assume that the exact transition probabilities and rewards are available. Reinforcement learning problems, however, require that the model is estimated from data. Transition probabilities can be particularly difficult to estimate, and even small errors can significantly degrade the quality of the optimal policy (Wiesemann, Kuhn, and Rustem, 2013). This work tackles the batch reinforcement learning setting in which transition probabilities must be estimated from a fixed and limited set of logged data.

Robust MDPs (RMDPs) are a convenient model for computing policies that are insensitive to small errors in transition probabilities (Nilim and Ghaoui, 2004; Iyengar, 2005; Wiesemann, Kuhn, and Rustem, 2013). RMDPs compute the best policy for the worst-case errors in the transition probabilities from a given ambiguity set (or an uncertainty set). The model can also be seen as a zero-sum game against an adversarial nature in which the decision-maker picks actions and the nature chooses the transition probabilities.

With ambiguity sets defined by appropriate concentration inequalities, RMDPs compute policies that maximize high-confidence return guarantees even with limited off-policy batch data (Delage and Ye, 2010; Petrik, Ghavamzadeh, and Chow, 2016; Tirinzoni et al., 2018; Russel and Petrik, 2018). The benefits of computing and optimizing return guarantees are myriad and range from ensuring the safety of deployed solutions, to better solutions with bad data, to indicating when more data collection is necessary.

Prior work on RMDPs has relied mostly on ambiguity sets defined by the $L_1$ norm for which there are both appropriate concentration inequalities and fast algorithms (Iyengar, 2005; Petrik and Subramanian, 2014; Petrik, Ghavamzadeh, and Chow, 2016; Auer, Jaksch, and Ortner, 2009; Strehl and

∗Equal contribution
Littman, 2004). These prior methods construct ambiguity sets that are too large and, regrettably, provide guarantees that are too pessimistic to be practical. Instead of changing the size of $L_1$ ambiguity sets, which is an open problem, we show that return guarantees can be improved by changing the shape of ambiguity sets by using weighted problem-specific norms.

We develop, as our main contribution, a new approach for optimizing the shape of ambiguity sets by choosing problem-specific weighted $L_1$ and $L_\infty$ norms. We also derive new concentration inequalities that extend previous results from the uniform $L_1$ norm ambiguity sets (Weissman et al., 2003) to weighted $L_1$ and $L_\infty$ sets, which can be used to provide better high-confidence guarantees on the optimized return. Recent results show that RMDPs with weighted $L_1$ norms can also be solved very efficiently (Ho, Petrik, and Wiesemann, 2018). Beyond choosing good weights for ambiguity sets, our results provide insights into which norms are appropriate for which problem. We limit our attention to tabular MDPs; the extension of the work to large MDPs is important but is beyond the scope of this paper. Our goal is broadly similar to Gupta (2019) and Petrik and Russell (2019), but our methods apply also to frequentist guarantees in addition to Bayesian ones.

The remainder of the paper is organized as follows. Section 2 describes the framework and the goals of this work. The algorithms for optimizing the shape of an ambiguity set are developed in Section 3. Then, Section 4 establishes the new finite-sample bounds for weighted $L_1$ and $L_\infty$ ambiguity sets. Section 5 describes connections to other related work, and the experimental results in Section 6 show that an appropriate choice of the ambiguity set can significantly improve the solution quality in several benchmark domains. We conclude and discuss future work in Section 7.

2 Framework

We aim to compute policies with the best possible high-confidence return guarantees for MDPs that are estimated from batch samples. This is a common problem in batch and model-based reinforcement learning. This section reviews basic properties of MDPs and RMDPs that we need to establish our results.

We consider MDP models with a finite (and relatively small) number of states $S = \{1, \ldots, S\}$ and actions $A = \{1, \ldots, A\}$. The decision-maker can take any action $a \in A$ in every state $s \in S$ and receives a reward $r_{s,a} \in \mathbb{R}$. The action also results in a transition to the next state $s'$ according to the true and unknown transition probabilities $p_{s,a}^{\star} \in \Delta^S$. We use $P^\star : S \times A \rightarrow \Delta^S$ to denote the transition kernel and $p_{s,a}$ to denote the vector of transition probabilities from state $s$ and action $a$.

The objective, when solving an MDP, is to compute a policy $\pi : S \rightarrow A$ that maximizes the infinite-horizon $\gamma$-discounted return $\rho(\pi)$ (Puterman, 2005). The return $\rho(\pi)$ for a policy $\pi$ and a given transition kernel $P$ is defined as follows: $\rho(\pi, P) = \mathbb{E} \left[ \sum_{t=0}^{\infty} \gamma^t \cdot r_{S_t, \pi(S_t)} \right]$. Ideally, the optimal policy $\pi^\star$ could be computed to maximize the true discounted return $\pi^\star \in \arg\max_{\pi \in \Pi} \rho(\pi, P^\star)$, where $\Pi$ is the set of all stationary deterministic policies. This is impossible when the true transition probabilities $P^\star$ are unknown and estimated from samples.

Robust MDPs address the challenge of unknown $P^\star$ by considering a broader set of possible transition probabilities. Instead of computing the best policy for the unknown transitions $P^\star$, they compute the best policy for the worst-case choice of transitions from a given ambiguity set $\mathcal{P} \subseteq \{ P : S \times A \rightarrow \Delta^S \}$:

$$\max_{\pi \in \Pi} \min_{P \in \mathcal{P}} \rho(\pi, P).$$ (1)

Since the optimization in (1) in its general form is NP-hard (Nilim and Ghaoui, 2004; Iyengar, 2005), much research has focused on sa-rectangular ambiguity sets $\mathcal{P}$ which allow nature to make an independent choice for each state and action (Wiesemann, Kuhn, and Rustem, 2013; Le Tallec, 2007). SA-rectangular ambiguity sets are defined as $\mathcal{P} = \times_{s \in S, a \in A} \mathcal{P}_{s,a}$, where $\mathcal{P}_{s,a} \subseteq \Delta^S$ denotes the ambiguity set for state $s$ and action $a$. The optimal robust value function $\psi^\star \in \mathbb{R}^S$ in an sa-rectangular RMDP must satisfy the robust Bellman optimality condition:

$$\hat{\psi}^\star(s) = \max_{a \in A} \min_{p \in \mathcal{P}_{s,a}} \left( r_{s,a} + \gamma p^T \psi^\star \right).$$ (2)

An optimal robust policy $\hat{\pi}^\star$ is greedy with respect to $\hat{\psi}^\star$ as is the case in MDPs. The set $\mathcal{P}_{s,a}$ is often defined as (Iyengar, 2005; Petrik, Ghavamzadeh, and Chow, 2016):

$$\mathcal{P}_{s,a} = \left\{ p \in \Delta^S : \|p - \bar{p}_{s,a}\|_1 \leq \psi_{s,a} \right\},$$
where $p_{s,a}$ is a nominal transition probability. Increasing the budget $\psi_{s,a} \in \mathbb{R}_+$ also increases the robustness of the optimal policy.

RMDPs with properly constructed ambiguity sets optimize for the highest high-confidence lower bound on the MDP return. Consider $1 - \delta$ to be the desired confidence level with $\delta \in [0, 1]$. One can readily show that as long as:

$$\mathbb{P} \left[ p_{s,a}^* \in P_{s,a}, \forall s \in S, a \in A \right] \geq 1 - \delta,$$

then $\hat{v}^*(s) \leq v^*(s)$ for probability $1 - \delta$ for all $s \in S$ simultaneously. To satisfy this requirement, when the transition probabilities are estimated from a dataset $D$, the budget can be set to:

$$\psi_{s,a} = \sqrt{\frac{2}{n_{s,a}} \log \frac{SA2^S}{\delta}}.$$

Here, $n_{s,a}$ is the number of transitions from state $s$ and action $a$ in the dataset $D$ (Petrik, Ghavamzadeh, and Chow, 2016; Petrik and Russell, 2019).

**Research Objective** As outlined above, we want to construct ambiguity sets to maximize the guaranteed return for a given confidence level $1 - \delta$. Optimizing for such an ambiguity set for every $s$ and $a$ can be stated as the following conceptual optimization problem:

$$\max_{P_{s,a}} \min_{p_{s,a} \in P_{s,a}} \left( r_{s,a} + \gamma p^T \hat{v}^* \right)$$

subject to:

$$\mathbb{P} \left[ p_{s,a}^* \in P_{s,a}, \forall s \in S, a \in A \right] \geq 1 - \delta. \quad (3)$$

Because the Bellman operator is monotone, maximizing the value of each state individually maximizes the return (Petrik and Russell, 2019). The distributionally-constrained optimization problem in (3) is intractable (Ben-Tal, Ghaoui, and Nemirovski, 2009) and depends on the optimal robust value function $\hat{v}^*$ which is unknown and depends on $P$. To mitigate these issues, we restrict our attention to optimizing the weights of $L_p$-based ambiguity sets and assume to be given a rough estimate of $\hat{v}^*$.

### 3 Optimizing Ambiguity Set Weights

In this section, we outline the general approach to tackling the desired optimization in (3). We relax the problem and use strong duality theory to get bounds that can be optimized tractably.

As noted above, maximizing the guaranteed return can be achieved by maximizing the Bellman update for every state. To this effect, assume some fixed $s \in S$ and $a \in A$ and let $z$ denote an estimate of the optimal robust value function: $z = r_{s,a} + \gamma \hat{v}$. The robust Bellman update in (2) for $s$ and $a$ then simplifies to:

$$q(z) = \min_{p_{s,a} \in \Delta} \left\{ p^T z : \|p - \bar{p}_{s,a}\| \leq \psi_{s,a} \right\}. \quad (4)$$

In the remainder of the section, we drop the $s, a$ subscripts when they are obvious from the context.

The impact of the choice of the norm in (4) on the value of $q(z)$ is not trivial, and we are not aware of a technique that could be used to optimize it directly. We instead maximize a lower bound on this value that the following theorem establishes.

**Theorem 3.1.** The estimate of expected next value can be bounded from below as:

$$q(z) \geq \bar{p}^T z - \min_{\lambda \in \mathbb{R}} \psi \|z + \lambda 1\|_*, \quad (5)$$

where $\|\cdot\|_*$ is the dual norm to the norm in (4).

Recall that the dual norm is defined as:

$$\|z\|_* = \sup \left\{ z^T x : \|x\| \leq 1 \right\}.$$

It is well known that dual norms to $L_1$, $L_2$, and $L_\infty$ are norms $L_\infty$, $L_2$, and $L_1$ respectively.
Proof. By relaxing the non-negativity constraints on \( p \), we get the following optimization problem:

\[
q(z) \geq \min_{p \in \mathbb{R}^S} \left\{ p^T z : \|p - \bar{p}\| \leq \psi, \ 1^T p = 1 \right\}
\]

Here, \( 1 \) is a vector of all ones of the appropriate size. Dualizing this optimization problem and following algebraic manipulation, detailed in Appendix A.2, we get the desired lower bound. \( \square \)

The lower bound in (5) is still hard to optimize, but, as we show next, it has a simpler form for weighted \( L_1 \) and \( L_\infty \) norms. Also, choosing any fixed \( \lambda \) also provides a lower bound which, we also show later, can be readily maximized.

We focus on ambiguity sets defined in terms of weighted \( L_1 \) and \( L_\infty \), which are defined for positive weights \( w \in \mathbb{R}^+ \) as:

\[
\|z\|_{1,w} = \sum_{i=1}^S w_i |z_i|, \quad \|z\|_{\infty,w} = \max_{i=1,\ldots,S} w_i |z_i|.
\]

The dual norms for a weighted \( L_1 \) norm is a weighted \( L_\infty \) norm as Lemma A.1 shows. Using this fact, Theorem 3.1 can be specialized to \( L_1 \) weighted ambiguity sets as follows.

**Corollary 3.1 (Weighted \( L_1 \) Ambiguity Set).** Suppose that \( q(z) \) is defined in terms of a weighted \( L_\infty \) norm for some \( w > 0 \). Then \( q(z) \) can be lower-bounded as follows:

\[
q(z) = \min_{p \in \Delta^S} \left\{ p^T z : \|p - \bar{p}\|_{1,w} \leq \psi \right\} \\
\geq \bar{p}^T z - \psi \|z - \lambda 1\|_{\infty,w}
\]

for any \( \lambda \in \mathbb{R} \). Moreover, when \( w = 1 \), the bound is tightest when \( \lambda = (\max_i z_i + \min_i z_i)/2 \) and the bound turns to \( q(z) \geq \bar{p}^T z - \psi \|z\|_s \) with \( \|\cdot\|_s \) representing the span semi-norm.

Since the dual norm of a dual norm is the original norm, we also get a similar result for weighted \( L_\infty \) ambiguity sets.

**Corollary 3.2 (Weighted \( L_\infty \) Ambiguity Set).** Suppose that \( q(z) \) is defined in terms of a weighted \( L_\infty \) norm for some \( w > 0 \). Then \( q(z) \) can be lower-bounded as follows:

\[
q(z) = \min_{p \in \Delta^S} \left\{ p^T z : \|p - \bar{p}\|_{\infty,w} \leq \psi \right\} \\
\geq \bar{p}^T z - \psi \|z - \lambda 1\|_{1,w},
\]

for any \( \lambda \). Moreover, when \( w = 1 \), the bound is tightest when \( \lambda \) is the median of \( z \).

The optimal \( \lambda \) being a median follows because maximization over \( \lambda \) values is identical to the formulation of the optimization problem for the quantile regression.

The utility of Corollaries 3.1 and 3.2 is twofold: 1) we will use them to decide whether \( L_1 \) or \( L_\infty \) ambiguity sets are more appropriate for a given problem, and 2) we will use them to improve solution quality by optimizing the weights involved.

### 3.1 Optimizing Norm Weights

In this section, we introduce tractable methods that optimize weights \( w \) in the ambiguity set in order to maximize \( q(z) \). We start with weighted \( L_1 \)-constrained sets and then describe the approach for the \( L_\infty \)-constrained sets.

The objective is to choose \( w \) that will maximize the lower bound on \( q(z) \) established in Corollary 3.1 as follows:

\[
\max_{w \in \mathbb{R}^+_1} \left\{ \bar{p}^T z - \psi \|z - \bar{\lambda} 1\|_{\infty,w} : \sum_{i=1}^S w_i^2 = 1 \right\}
\]

(6)

The value \( \bar{\lambda} \) in (6) is fixed ahead of time and does not change with \( w \). The constraint \( \sum_{i=1}^S w_i^2 = 1 \) serves to normalize \( w \) in order to preserve the desired robustness guarantees with the same \( \psi \). This
is because scaling both \( w \) and \( \psi \) simultaneously by an identical factor leaves the ambiguity set unchanged. This regularization constraint is motivated by the finite-sample guarantees in Section 4 and our empirical results.

Next, omitting terms that are constant with respect to \( w \) simplifies the optimization to:

\[
\begin{align*}
\argmin_{w \in \mathbb{R}_+^S} & \quad \| z - \bar{z} \|_\infty : \sum_{i=1}^S w_i^2 = 1 \\
\end{align*}
\]

(7)

The nonlinear optimization problem in (7) is convex and can be, surprisingly, solved analytically.

Let \( b_i = |z_i - \bar{z}| \) for \( i = 1, \ldots, S \). Introducing an auxiliary variable \( t \) further simplifies the optimization problem:

\[
\begin{align*}
\min_{t, w \in \mathbb{R}_+^S} & \quad t : t \geq b_i / w_i, \sum_{i=1}^S w_i^2 = 1 \\
\end{align*}
\]

(8)

The constraints \( w > 0 \) cannot be active (because of \( 1 / w_i \)) and may be safely ignored. Then, the convex optimization problem in Equation (8) has a linear objective, \( S + 1 \) variables (\( w \)’s and \( t \)), and \( S + 1 \) constraints. All constraints are active, therefore, in the optimal solution \( w^* \) (Bertsekas, 2003) which must satisfy:

\[
w_i^* = b_i / \sqrt{\sum_{j=1}^S b_j^2}.
\]

(9)

Since \( \sum_i w_i^2 = 1 \) implies \( \sum_i b_i^2 / t^2 = 1 \), we conclude that \( t = \sqrt{\sum_i b_i^2} \).

Following the same approach for the weighted \( L_\infty \) ambiguity set, the equivalent optimization of (8) becomes:

\[
\begin{align*}
\min_{w > 0} & \quad \sum_{i=1}^S b_i / w_i : \sum_{i=1}^S w_i^2 = 1 \\
\end{align*}
\]

(10)

Again, the inequality constraints on weights \( w > 0 \) can be relaxed. Using the necessary optimality conditions (and a Lagrange multiplier), one solution for the optimal weights \( w \) are:

\[
w_i^* = \frac{b_i^{1/3}}{\sqrt{\sum_{j=1}^S b_j^{2/3}}}
\]

(11)

Next, we establish new finite-sample bounds for these new types of ambiguity sets.

### 4 Finite-Sample Guarantees

In this section, we describe new sampling bounds that can be used to construct ambiguity sets that provide desired sampling guarantees. We describe both frequentist and Bayesian methods.

#### 4.1 Bayesian Credible Intervals (BCI)

In Bayesian statistics, credible intervals are comparable to classical confidence intervals (Murphy, 2012). An important advantage of using Bayesian techniques for robust optimization is that they can effectively leverage prior domain knowledge (Bertsekas and Tsitsiklis, 2002).

Petrik and Russell (2019) suggest an approach to construct ambiguity regions from credible intervals. The method starts with sampling from the posterior probability distribution of \( P^* \) given data \( D \) to estimate the mean transition probability \( \bar{p}_{s,a} = \mathbb{E}_{p^*}[p^*_{s,a}|D] \). Then the smallest possible ambiguity set around the mean is obtained by solving the following optimization problem for each state \( s \) and action \( a \):

\[
\psi_{s,a}^B = \min_{\psi \in \mathbb{R}^+} \left\{ \psi : P \left[ \| p_{s,a}^* - \bar{p}_{s,a} \| > \psi \mid D \right] < \frac{\delta}{SA} \right\}.
\]

Finally, the Bayesian ambiguity set can be obtained by:

\[
\mathcal{P}_{s,a}^B = \{ p \in \Delta^S : \| p - \bar{p}_{s,a} \| \leq \psi_{s,a}^B \}.
\]

This construction applies easily to any form of norm used in the construction of ambiguity sets. That is, it is easy to generalize this method for both weighted \( L_1 \) and weighted \( L_\infty \) ambiguity sets.
Algorithm 1: Weighted Bayesian Credible Intervals (WBCI)

Input: Distribution $\theta$ over $p^*_{s,a}$, confidence level $\delta$, sample count $n$, weights $w$

Output: Nominal point $\bar{p}_{s,a}$ and $\psi_{s,a}$

1. Sample $X_1, \ldots, X_n \in \Delta^S$ from $\theta$: $X_i \sim \theta$;
2. Nominal point: $\bar{p}_{s,a} \leftarrow (1/n) \sum_{i=1}^n X_i$;
3. Compute distances $d_i \leftarrow \|\bar{p}_{s,a} - X_i\|_{p,w}$ and sort in increasing order;
4. $\psi_{s,a} \leftarrow d_{\lceil (1-\delta)n \rceil}$;
5. return $\bar{p}_{s,a}$ and $\psi_{s,a}$;

that we study in this work. Algorithm 1 summarizes the simple algorithm that constructs Bayesian ambiguity sets in quasi-linear time.

The following example demonstrates how different norm weights impact the shape of the ambiguity set.

Example 4.1. Consider an MDP with 3 states $s_1, s_2, s_3$ and a single action $a_1$. True & unknown transition probability is $P^*(s_1, a_1, \cdot) = [0.3, 0.2, 0.5]$, and the value function is $v = [0, 0, 1]$. The contours of posterior probability distribution and the ambiguity sets for state $s_1$ are shown projected onto a simplex in Figure 1. The green set is constructed with unweighted $L_1$ norm and the orange set is constructed with optimized weights for the $L_1$ norm. Although both sets have the same probability measure, the weighted set yields a better return estimate for $v^*$.

4.2 Weighted Frequentist Confidence Intervals (WFCI)

We present two new finite-sample bounds that can be used to construct frequentist ambiguity sets with weighted $L_p$ norms. These bounds are necessary to guarantee high-confidence return guarantees. These results significantly extend the existing bounds which have been limited to the $L_1$ deviation (Weissman et al., 2003; Auer, Jaksch, and Ortner, 2010; Dietterich, Taleghan, and Crowley, 2013; Petrik and Russell, 2019).

Theorem 4.2 (Weighted $L_1$ Error Bound). Suppose that $\bar{p}_{s,a}$ is the empirical estimate of the transition probability obtained from $n_{s,a}$ samples for some $s \in S$ and $a \in A$. If the weights $w \in \mathbb{R}^S_+$ are sorted in a non-increasing order $w_i \geq w_{i+1}$, then:

$$
\mathbb{P} \left[ E \geq \psi_{s,a} \right] \leq 2 \sum_{i=1}^{S-1} 2^{S-i} \exp \left( -\frac{\psi_{s,a}^2 n_{s,a}}{2w_i^2} \right),
$$

where $E = \|\bar{p}_{s,a} - p^*_{s,a}\|_{1,w}$.
Importantly, replacing the sum in the theorem above by a uniform upper bound on $w_i$ would be insufficient to improve ambiguity sets. Theorem A.2 further tightens Theorem 4.2 by using Bernstein’s inequality in place of Hoeffding’s inequality.

The next theorem establishes a new finite-sample bound for weighted $L_\infty$ sets.

**Theorem 4.3 (Weighted $L_\infty$ Error Bound).** Suppose that $\hat{p}_{s,a}$ is the empirical estimate of the transition probability obtained from $n_{s,a}$ samples for some $s \in S$ and $a \in A$. Then:

$$P[E \geq \psi_{s,a}] \leq 2 \sum_{i=1}^{S} \exp \left( -\frac{2 \psi_{s,a}^2 n_{s,a} w_i^2}{w_i} \right),$$

where $E = \|\hat{p}_{s,a} - p_{s,a}^\ast\|_{\infty,w}$.

The proofs of both theorems are deferred to Appendix A.

Theorems 4.2 and 4.3 establish the error bounds that can be used to construct ambiguity sets of appropriate size. Unlike with the standard error bound, $\psi_{s,a}$ cannot be determined readily from the bounds analytically. However, since the confidence level function is monotonically increasing, $\psi_{s,a}$ can be easily determined numerically using a bisection method.

Recall that the weights in (6) are optimized under a constraint that $\sum_{i=1}^{S} w_i^2 = 1$ to preserve the confidence guarantee regardless of the weight scales. The constraint is derived from an approximation of the guarantee in Theorem 4.3 (similar for Theorem 4.2) by linearizing it from Jensen’s inequality:

$$\sum_{i=1}^{S} \exp \left( -\frac{2 \psi_{s,a}^2 n_{s,a} w_i^2}{w_i} \right) \approx S \exp \left( -\frac{2 \psi_{s,a}^2 n_{s,a}}{S} \right).$$

Then, taking the log of $\delta$ and the right-hand-side above and applying Jensen’s inequality again gives us:

$$-\frac{1}{2 \psi_{s,a}^2 n_{s,a}} \log \left( \frac{\delta}{S} \right) \leq \frac{1}{S} \sum_{i=1}^{S} w_i^2 .$$

Therefore, a constant value of $\sum_{i=1}^{S} w_i^2$ provides an upper bound on the confidence in the equation above. We emphasize that this is not a bound but rather an approximation due to the linearization step.

5 Implications and Related Work

Several methods have been proposed in the literature to construct ambiguity sets and to mitigate their sensitivity. One important factor in this regard is the underlying rectangularity assumption (Wiesemann, Kuhn, and Rustem, 2013). A rectangular ambiguity set leads to a tractable but overly pessimistic solution (Iyengar, 2005; Nilim and El Ghaoui, 2005). Most common methods for constructing rectangular ambiguity sets operate in a classical frequentist setting where the ambiguity sets are defined as a plausible region of deviation from the expectation (Ben-Tal et al., 2013; Nilim and Ghaoui, 2004). This deviation is constrained by an $L_p$-norm, KL-divergence, $\phi$-divergence, or Wasserstein metric (Abdullah et al., 2019; Petrik, Chow, and Ghavamzadeh, 2016; Lim, Xu, and Mannor, 2013; Xu and Mannor, 2012). In contrast, we consider in this paper a weighted-$L_p$-norm where the weights adapt contextually based on the problem.

Petrik and Russell (2019) and Gupta (2019) propose methods for constructing rectangular norm-bounded ambiguity sets in a Bayesian setting and show the superiority of them against their frequentist counterpart. Eliminating the rectangularity assumption helps in reducing conservativeness, but the problem becomes computationally intractable. Mannor, Mebel, and Xu (2012, 2016) and Tirinzoni et al. (2018) propose tractable approximate methods to construct coupled non-rectangular ambiguity sets with sound worst-case performance.

In addition to robust reinforcement learning, ambiguity sets play an equally important role in guiding exploration. Derman et al. (2019) and Russel, Gu, and Petrik (2019) propose methods for safe and robust exploration minimizing worst-case regret. There has been no work, to the best of our knowledge, studying the impact of using weighted norms in defining the ambiguity (or uncertainty) sets for guiding exploration in reinforcement learning.
Remark 5.1. Our results also provide new insights into which ambiguity sets better fit to which problems. Combining Corollary 3.1 with Theorem 4.2 and Corollary 3.2 with Theorem 4.3 implies that the $L_1$ norm is preferable to the $L_\infty$ norm when:
\[
\|v - \bar{v}\|_1 > \sqrt{n} \|v\|_s.
\]
Here, $v$ is the optimal value function, $\bar{v}$ is the mean value of the value function over all states, and $\|\cdot\|_s$ is the span semi-norm.

6 Empirical Evaluation

In this section, we empirically evaluate the advantage of using weighted ambiguity sets in Bayesian and frequentist settings. We assess $L_1$ and $L_\infty$-bounded ambiguity sets, both with weights and without weights. We compare Bayesian credible regions with frequentist’s Hoeffding and Bernstein style sets. We start by assuming a true underlying model that produces the simulated datasets containing 100 samples for each state and action. The frequentist methods use these datasets to construct an ambiguity set. Bayesian methods combine the data with a prior to compute a posterior distribution and then draw 10,000 samples from the posterior distribution to construct a Bayesian ambiguity set. We use an uninformative uniform prior over the reachable next states for all the experiments unless otherwise specified. This prior is somewhat informative in the sense that it contains the knowledge of non-zero transitions implied by the datasets. The performance of the methods is evaluated by the guaranteed robust returns computed for a range of different confidence levels. We strengthen the weighted $L_1$ error bound by a factor of two to match with the unweighted one.

Single Bellman Update. In this experiment, we set up a very trivial problem to meticulously examine our proposed method. We consider a transition from a single state $s_0$ and an action $a_0$ leading to 5 terminal states $s_1, \ldots, s_5$. The value functions are assumed to be fixed and known. The prior is uniform Dirichlet over the next states. Plots in Figure 2 and Figure 3 show a comparison of average guaranteed returns for 100 independent trials. The weighted methods outperform unweighted methods in all instances. Also, the weighted BCI methods are significantly better than other frequentist methods. It is also apparent from the plot that $L_\infty$-constrained methods can outperform in case of sparse value functions as shown in Figure 3. This result bears out the theoretical prediction in Remark 5.1.

RiverSwim. We consider the standard RiverSwim (Strehl and Littman, 2008) domain for evaluating our methods (see Appendix B.1 for RiverSwim MDP graph). The process follows by sampling synthetic datasets from the true model and then computing the guaranteed robust returns for different methods. We use a uniform Dirichlet distribution over the next states as prior. Table 1 summarizes the results. All the weighted methods dominate unweighted methods, and the weighted $L_1$ BCI
method provides the highest guaranteed return. The return of the optimal policy for the true model is $56,687$. At the 50% confidence level, the gap between the optimal return and guaranteed return is reduced by 34% and 13% for weighted $L_1$ BCI and weighted $L_\infty$ Hoeffding sets respectively over the standard uniform weight sets.

**Population Growth Model.** We also apply our method in an exponential population growth model (Kery and Schaub, 2012). Our model constitutes a simple state-space with exponential dynamics. At each time step, the land manager has to decide whether to apply a control measure to reduce the growth rate of the species. We refer to Tirinzoni et al. (2018) for more details of the model. The results are summarized in Table 2. Returns for all the methods are negative, which implies a high management cost. Policies computed with frequentist and unweighted methods yield a very high cost. Bayesian and weighted methods significantly outperform other methods. The return of the optimal policy for the true model is $18,448$. At the 50% confidence level, the gap between the optimal return and guaranteed return is reduced by over 75% for both weighted $L_1$ BCI and weighted $L_\infty$ Hoeffding over the standard uniform weight.

**Inventory Management Problem.** Next, we take the classic inventory management problem (Zipkin, 2000). The inventory level is discrete and limited by the number of states $S$. The purchase cost, sale price, and holding cost are $2.49, 3.99,$ and $0.03$ respectively. The demand is sampled from a normal distribution with a mean $S/4$ and a standard deviation of $S/6$. The initial state is 0 (empty stock). Table 3 summarizes the computed guaranteed returns of different methods at 0.5 and 0.95 confidence levels. The guaranteed returns computed with Bayesian and weighted methods are significantly higher than other methods in this problem domain. The return of the optimal policy for the true model is $550$. At the 50% confidence level, the gap between the optimal return and guaranteed return is reduced by 50% and 30% for weighted $L_1$ BCI and weighted $L_\infty$ Hoeffding sets respectively over the standard uniform weight.

**Cart-Pole.** We evaluate our method on Cart-Pole, a standard RL benchmark problem (Sutton and Barto, 2018; Brockman et al., 2016). We collect samples of 100 episodes from the true dynamics. We fit a linear model with that dataset to generate synthetic samples and aggregate nearby states on a resolution of 200 using K-nearest neighbor strategy. The results are summarized in Table 4. Again, in this case, all the Bayesian and weighted methods outperform other methods. The return of the optimal policy for the true model is $51$. At the 50% confidence level, the gap between the optimal return and guaranteed return is reduced by 64% and 71% for weighted $L_\infty$ BCI and weighted $L_\infty$ Hoeffding sets respectively over the standard uniform weight.
| Methods | Uniform | Weighted |
|---------|---------|----------|
| Bayesian $L_1$ BCI | 5290 | **23155** |
| Bayesian $L_\infty$ BCI | 5290 | 20673 |
| Bayesian $L_1$ Hoeffding | 490 | 634 |
| Bayesian Bernstein $L_\infty$ Hoeffding | 490 | **7976** |

Table 1: Guaranteed robust return for the RiverSwim experiment.

| Methods | Uniform | Weighted |
|---------|---------|----------|
| Bayesian $L_1$ BCI | -98659 | **-9356** |
| Bayesian $L_\infty$ BCI | -132781 | -35934 |
| Bayesian $L_1$ Hoeffding | -116167 | -106078 |
| Bayesian Bernstein $L_\infty$ Hoeffding | -132737 | **-31761** |

Table 2: Guaranteed robust return for the Population experiment.

| Methods | Uniform | Weighted |
|---------|---------|----------|
| Bayesian $L_1$ BCI | 310 | **428** |
| Bayesian $L_\infty$ BCI | 177 | 278 |
| Bayesian $L_1$ Hoeffding | 192 | 245 |
| Bayesian Bernstein $L_\infty$ Hoeffding | 132 | **255** |

Table 3: Guaranteed robust return for the Inventory experiment.

| Methods | Uniform | Weighted |
|---------|---------|----------|
| Bayesian $L_1$ BCI | 41.11 | 47.33 |
| Bayesian $L_\infty$ BCI | 39.95 | **47.48** |
| Bayesian $L_1$ Hoeffding | 9.89 | 45.11 |
| Bayesian Bernstein $L_\infty$ Hoeffding | 37.52 | **47.35** |

Table 4: Guaranteed robust return for the Cart-Pole experiment.
7 Conclusion

We proposed a new approach for optimizing the shape of the ambiguity sets that goes beyond the conventional $L_1$-constrained ambiguity sets studied in the literature. We showed that the optimal shape is problem dependent and is driven by the characteristics of the value function. We derived new sampling guarantees, and our experimental results show that the problem-dependent shapes of the ambiguity set can significantly improve return guarantees. Future work needs to focus on tightening frequentist sampling bounds or replace them with alternative techniques like bootstrapping.

7.1 Acknowledgments

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A Technical Proofs

A.1 Dual Norm of Weighted $L_1$

Lemma A.1. Let $\|\cdot\|_{1,w}$ be the weighted $L_1$ norm on $\mathbb{R}^n$. The associated dual norm $\|\cdot\|_{\infty,\frac{1}{w}}$ is defined as:

$$\|z\|_{\infty,\frac{1}{w}} = \sup\{z^T x : \|x\|_{1,w} \leq 1, w \in \mathbb{R}^{n_+}\}.$$

Proof. Assume we are given a set of positive weights $w \in \mathbb{R}^{n_+}$ for the following weighted $L_1$ optimization problem:

$$\max_{x} z^T x \quad \text{s.t.} \quad \|x\|_{1,w} \leq 1 \quad \quad \quad \text{(12)}$$

we have:

$$x^T z = \sum_{i=1}^{n} x_i z_i \leq \sum_{i=1}^{n} |x_i z_i|$$

$$(a) \leq \sum_{i=1}^{n} |x_i||z_i| = \sum_{i=1}^{n} w_i |x_i| \frac{1}{w_i} |z_i|$$

$$\leq \max_{i=1,\ldots,n} \left\{ \frac{1}{w_i} |z_i| \right\} \cdot \sum_{i=1}^{n} w_i |x_i| = \max_{i=1,\ldots,n} \left\{ \frac{1}{w_i} |z_i| \right\} \cdot \|x\|_{1,w}$$

$$(b) \leq \max_{i=1,\ldots,n} \left\{ \frac{1}{w_i} |z_i| \right\} = \|z\|_{\infty,\frac{1}{w}}.$$

Here, (a) follows from the Cauchy-Schwarz inequality and (b) follows from the constraint $\|x\|_{1,w} \leq 1$ of (12). 

A.2 Proof of Theorem 3.1

Proof. The inner optimization objective function for RMDPs for $L_p$-constrained ambiguity sets are defined as follows:

$$q(z) = \min_{p \in \Delta^n} \left\{ p^T z : \|p - \bar{p}\| \leq \psi \right\}.$$

Let $q = p - \bar{p}$. We can reformulate the optimization problem using the new variable $q$:

$$\min_{q} (q + \bar{p})^T z \quad \text{s.t.} \quad \|q\| \leq \psi \quad 1^T(q + \bar{p}) = 1 \implies 1^T q = 0 \quad \quad \quad q \geq -\bar{p}.$$  

If $\psi$ is sufficiently small and $\bar{p}$ is sufficiently large, we can relax the problem by dropping the $q \geq -\bar{p}$ constraint. Since $\bar{p}^T z$ is a fixed number, we continue with:

$$\bar{p}^T z + \min_{q} q^T z \quad \text{s.t.} \quad \|q\| \leq \psi \quad 1^T q = 0$$

We then change the minimization form to maximization:

$$\bar{p}^T z - \max_{q} -q^T z \quad \text{s.t.} \quad \|q\| \leq \psi \quad 1^T q = 0$$

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By applying the method of Lagrange multipliers, we obtain:

$$\min_{\lambda} \max_{q} \quad -q^T z - \lambda(q^T 1) = q^T (-z - \lambda 1)$$

s.t. $\|q\| \leq \psi$

Letting $x = \frac{q}{\psi}$, we get:

$$\min_{\lambda} \max_{x} \quad \psi \cdot x^T (-z - \lambda 1) .$$

s.t. $\|x\| \leq 1$

Given the definition of the dual norm, $\|z\|_* = \sup\{z^T x \mid \|x\| \leq 1\}$, we have:

$$q(z) \geq p^T z - \min_{\lambda} \psi\|z + \lambda 1\|_* .$$

A.3 Proof of Theorem 4.2 (Weighted $L_1$ Error Bound)

In this section, we describe a proof of a bound on the $L_{1,w}$ distance between the estimated transition probabilities $p$ and the true one $p^*$ over each state $s \in S = \{1, \ldots, S\}$ and action $a \in A = \{1, \ldots, A\}$. The proof is an extension to Lemma C.1 (L1 error bound) in Petrik and Russell (2019).

Proof. Let $q_{s,a} = \bar{p}_{s,a} - p^*_{s,a}$. To shorten notation in the proof, we omit the $s, a$ indexes when there is no ambiguity. We assume that all weights are non-negative. First, we will express the $L_{1,w}$ norm of $q$ in terms of an optimization problem. It is worth noting that $1^T q = 0$. Let $1_{Q_1}, 1_{Q_2} \in \mathbb{R}^S$ be the indicator vectors for some subsets $Q_1, Q_2 \subset S$ where $Q_2 = S \setminus Q_1$. According to Lemma A.1 we have:

$$\|q\|_{1,w} = \max_{z} \left\{ z^T q : \|z\|_{\infty, \frac{1}{w}} \leq 1 \right\} $$

$$= \max_{Q_1, Q_2 \in 2^S} \left\{ 1_{Q_1}^T W q + 1_{Q_2}^T W(-q) : Q_2 = S \setminus Q_1 \right\} .$$

Here weights are on the diagonal entries of $W$. Using the expression above, we can bound the probability as follows:

$$P \left[ \max_{Q_1, Q_2 \in 2^S} \left\{ 1_{Q_1}^T W q + 1_{Q_2}^T W(-q) \right\} \geq \psi \right] \leq P \left[ \max_{Q_1, Q_2 \in 2^S} \left\{ 1_{Q_1}^T W q \right\} \geq \psi \frac{1}{2} \right] + P \left[ \max_{Q_1, Q_2 \in 2^S} \left\{ 1_{Q_2}^T W(-q) \right\} \geq \psi \frac{1}{2} \right]$$

$$\leq \sum_{Q_1 \in 2^S} P \left[ 1_{Q_1}^T W q \geq \psi \frac{1}{2} \right] + \sum_{Q_2 \in 2^S} P \left[ 1_{Q_2}^T W(-q) \geq \psi \frac{1}{2} \right]$$

$$= \sum_{Q_1 \in 2^S} P \left[ 1_{Q_1}^T W (\bar{p} - p^*) \geq \psi \frac{1}{2} \right] + \sum_{Q_2 \in 2^S} P \left[ 1_{Q_2}^T W(\bar{p} - p^*) \geq \psi \frac{1}{2} \right]$$

$$\leq \sum_{Q_1 \in 2^S} \exp \left( -\frac{\psi^2 n}{2\|1_{Q_1}^T W\|^2_\infty} \right) + \sum_{Q_2 \in 2^S} \exp \left( -\frac{\psi^2 n}{2\|1_{Q_2}^T W\|^2_\infty} \right)$$

$$= \sum_{i=1}^{S-1} 2^{S-i} \exp \left( -\frac{\psi^2 n}{2w_i^2} \right).$$

(a) follows from union bound, and (b) follows from Hoeffding’s inequality. (c) follows by $Q_i = Q_2$ and sorting weights $w = \{w_1, \ldots, w_n\}$ in non-increasing order.

Theorem A.2 (weighted $L_1$ error bound using Bernstein’s inequality). Suppose that $\bar{p}_{s,a}$ is the empirical estimate of the transition probability obtained from $n_{s,a}$ samples for some $s \in S$ and
a ∈ A. If the weights \( w \in \mathbb{R}^S_+ \) are sorted in non-increasing order \( w_i \geq w_{i+1} \), then the following holds when using Bernstein’s inequality:

\[
P \left[ \| \mathbf{p}_{s,a} - \mathbf{p}_{s,a}^* \|_1 \geq \psi_{s,a} \right] \leq 2 \sum_{i=1}^{S-1} 2^{S-i} \exp \left( -\frac{3\psi^2 n}{6w_i^2 + 4\psi w_i} \right)
\]

where \( \mathbf{w} \in \mathbb{R}^S_+ \) is the vector of weights. The weights are sorted in non-increasing order.

\textbf{Proof.} The proof is similar to the proof of Theorem 4.2 until section b. The proof continues from section (b) as follows:

\[
\begin{align*}
\sum_{\mathcal{Q}_1 \in 2^S} & \exp \left( -\frac{3\psi^2 n}{24\sigma^2 + 4c\psi} \right) + \sum_{\mathcal{Q}_2 \in 2^S} \exp \left( -\frac{3\psi^2 n}{24\sigma^2 + 4c\psi} \right) \\
\sum_{\mathcal{Q}_1 \in 2^S} & \exp \left( -\frac{3\psi^2 n}{6\| \mathbf{1}_{\mathcal{Q}_1} \mathbf{W} \|^2_\infty + 4\psi \| \mathbf{1}_{\mathcal{Q}_1} \mathbf{W} \|^\infty} \right) + \sum_{\mathcal{Q}_2 \in 2^S} \exp \left( -\frac{3\psi^2 n}{6\| \mathbf{1}_{\mathcal{Q}_2} \mathbf{W} \|^2_\infty + 4\psi \| \mathbf{1}_{\mathcal{Q}_2} \mathbf{W} \|^\infty} \right) \\
= & 2 \sum_{i=1}^{S-1} 2^{S-i} \exp \left( -\frac{3\psi^2 n}{6w_i^2 + 4\psi w_i} \right).
\end{align*}
\]

Here (b) follows from Bernstein’s inequality where \( \sigma^2 \) is the mean of variance of random variables, and \( c \) is their upper bound (Devroye, Györfi, and Lugosi, 2013). In the weighted case, with conservative estimate of variance \( \sigma^2 = \| \mathbf{1}_{\mathcal{Q}_i} \mathbf{W} \|^2_\infty / 4 \), and \( c = \| \mathbf{1}_{\mathcal{Q}_i} \mathbf{W} \|^\infty \), because the random variables are drawn from Bernoulli distribution with the maximum possible variance of 1/4. (d) follows by sorting weights \( \mathbf{w} \) in non-increasing order.

\( \square \)

\textbf{Lemma A.3.} \textit{(L∞ Error Bound)} For a given \( s \in S \) and \( a \in A \), we have:

\[
P \left[ \| \mathbf{p}_{s,a} - \mathbf{p}_{s,a}^* \|_\infty \geq \psi_{s,a} \right] \leq 2S \exp(-2\psi^2 n_{s,a}).
\]

And, equivalently, in term of \( \delta \):

\[
P \left[ \| \mathbf{p}_{s,a} - \mathbf{p}_{s,a}^* \|_\infty \geq \sqrt{\frac{1}{2n_{s,a}} \log \frac{2S}{\delta}} \right] \leq \delta.
\]

\textbf{Proof.} First, we will express the \( L_\infty \) distance between two distributions \( \mathbf{p} \) and \( \mathbf{p}^* \) in terms of an optimization problem. Let \( \mathbf{1}_i \) be the indicator vector for an index \( i \in S \):

\[
\begin{align*}
\| \mathbf{p}_{s,a} - \mathbf{p}_{s,a}^* \|_\infty &= \max \left\{ \mathbf{z}^T (\mathbf{p}_{s,a} - \mathbf{p}_{s,a}^*) : \| \mathbf{z} \|_1 \leq 1 \right\} \\
&= \max_{i \in S} \{ 1_i(\mathbf{p}_{s,a} - \mathbf{p}_{s,a}^*), -1_i(\mathbf{p}_{s,a} - \mathbf{p}_{s,a}^*) \}.
\end{align*}
\]

Using the expression above, we can bound the probability in the Lemma as follows:

\[
P \left[ \| \mathbf{p}_{s,a} - \mathbf{p}_{s,a}^* \|_\infty \geq \psi \right] = \mathbb{P} \left[ \max_{i \in S} \{ 1_i(\mathbf{p}_{s,a} - \mathbf{p}_{s,a}^*), -1_i(\mathbf{p}_{s,a} - \mathbf{p}_{s,a}^*) \} \geq \psi_{s,a} \right]
\]

\[
\leq S \max_{i \in S} \mathbb{P} \left[ 1_i(\mathbf{p}_{s,a} - \mathbf{p}_{s,a}^*) \geq \psi_{s,a} \right] + S \max_{i \in S} \mathbb{P} \left[ -1_i(\mathbf{p}_{s,a} - \mathbf{p}_{s,a}^*) \geq \psi_{s,a} \right]
\]

\[
\leq 2S \exp(-2\psi^2 n_{s,a}).
\]

(a) follows from union bound and (b) follows from Hoeffding’s inequality since \( \mathbf{1}_i^T \mathbf{p} \in [0, 1] \) for any \( i \in S \) and its mean is \( \mathbf{1}_i^T \mathbf{p}^* \). \( \square \)
A.4 Proof of Theorem 4.3 (Weighted $L_\infty$ Error Bound)

Proof. First, we will express the weighted $L_\infty$ distance between two distributions $\bar{p}$ and $p^*$ in terms of an optimization problem. Let $1_i \in \mathbb{R}^S$ be the indicator vector for an index $i \in S$:

$$
\|\bar{p}_{s,a} - p^*_{s,a}\|_{\infty, w} = \max_{z} \left\{ z^T W (\bar{p}_{s,a} - p^*_{s,a}) : \|z\|_1 \leq 1 \right\} \\
= \max_{i \in S} \left\{ 1_i W (\bar{p}_{s,a} - p^*_{s,a}), -1_i W (\bar{p}_{s,a} - p^*_{s,a}) \right\}.
$$

Here, weights are on the diagonal entries of $W$. Using the expression above, we can bound the probability in the Lemma as follows:

$$
P \left[ \|\bar{p}_{s,a} - p^*_{s,a}\|_{\infty, w} \geq \psi \right] = P \left[ \max_{i \in S} \left\{ 1_i W (\bar{p}_{s,a} - p^*_{s,a}), -1_i W (\bar{p}_{s,a} - p^*_{s,a}) \right\} \geq \psi_{s,a} \right]
$$

\[ \leq \left( a \right) S \max_{i \in S} P \left[ 1_i W (\bar{p}_{s,a} - p^*_{s,a}) \geq \psi_{s,a} \right] + S \max_{i \in S} P \left[ -1_i W (\bar{p}_{s,a} - p^*_{s,a}) \geq \psi_{s,a} \right]
\]

\[ \leq \left( b \right) 2 \sum_{i=1}^{S} \exp \left( -\frac{\psi_{s,a}^2 n}{2 w_i^2} \right). \]

(a) follows from union bound and (b) follows from Hoeffding’s inequality since $1_i^T \bar{p} \in [0, 1]$ for any $i \in S$ and its mean is $1_i^T p^*$.

\[ \square \]

B Supplementary Material

B.1 RiverSwim MDP Graph

![RiverSwim Graph](image)

Figure 4: RiverSwim problem with six states and two actions (left-dashed arrow, right-solid arrow). The agent starts in either $s_1$ or $s_2$. 

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