Poisson structures for difference equations

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Abstract
We study the existence of log-canonical Poisson structures that are preserved by difference equations of a special form. We also study the inverse problem, given a log-canonical Poisson structure to find a difference equation preserving this structure. We give examples of quadratic Poisson structures that arise for the Kadomtsev–Petviashvili (KP) type maps which follow from a travelling-wave reduction of the corresponding integrable partial difference equation.

Keywords: Poisson structures, difference equations, Hirota–Miwa equation

1. Introduction

Hamiltonian dynamical systems form a major area of study in dynamical systems, both for their mathematical structure and because of their widespread applications \[1, 10\]. The form of the paradigmatic Hamiltonian system is

\[
\dot{x} = \Omega \nabla H(x)
\]

(1)

where \(x \in \mathbb{R}^n\) with \(n\) even. The \(n \times n\) constant matrix \(\Omega\) is skew-symmetric and is the \textit{symplectic structure} of the system. Typically,

\[
\Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}
\]

(2)

where 0 and \(I\) denote, respectively, the zero and identity matrix of dimension \(\frac{n}{2}\). More generally, \(\Omega\) in (1) can be any constant skew-symmetric matrix, or more generally again, a non-constant skew symmetric matrix \(\Omega(x)\) which satisfies the Jacobi identity, in which case it is called a \textit{Poisson structure}. The Poisson structure is called non-degenerate when \(\det(\Omega(x)) \neq 0\).
These possibilities for $\Omega(x)$ can also be taken for (1) in the case that the dimension is odd in which case $\Omega(x)$ is degenerate because it is skew-symmetric and odd dimensional.

Equation (1) can be written in terms of the Poisson bracket $\{\cdot,\cdot\}$ defined by $\{x_i, x_j\}(x):=\Omega_{ij}(x)$ or, for any functions $f,g$, by

$$\{f(x), g(x)\}(x):=\sum_{1\leq i,j\leq n} \frac{\partial f}{\partial x_i}(x) \frac{\partial g}{\partial x_j}(x) \{x_i, x_j\}(x) = \nabla f(x)\Omega(x) \nabla g(x).$$

Then equation (1) becomes

$$\dot{x}_i = \{x_i, H\}(x).$$

The existence of a symplectic structure or more generally of a Poisson structure plays a key role in the geometry of (1). Darboux’s theorem says that any system (1) with arbitrary non-degenerate Poisson matrix $\Omega(x)$ can be transformed locally to the Hamiltonian form with the canonical $\Omega$ of (2). However, many systems arise naturally with a non-canonical $\Omega(x)$ and are analyzed in that given coordinate system (e.g. for geometric numerical integration, where the system is numerically approximated with the symplectic structure not converted to canonical form).

In discrete time, a map

$$M : x \mapsto x' := M(x)$$

preserves a Poisson structure $\Omega(x)$ if its Jacobian $dM(x)$, satisfies

$$dM'(x)\Omega(x) dM(x) = \Omega(x').$$

Equivalently if for any two functions $f,g$ on $\mathbb{R}^n$,

$$\{f \circ M, g \circ M\}(x) = \{f, g\}(M(x))$$

which, by using the notation $G \circ M(x) = G'$ for any function $G$, can be shortened into $\{f', g'\} = \{f, g\}'. Using (3) this is equivalent to $\{x_i', x_j'\} = \{x_i, x_j\}'$ for all $1 \leq i < j \leq n$. As in the continuous case, the existence of the Poisson structure plays a key role in the geometry of (4) (see [27]). Recall that the flow or map in $2m$ degrees of freedom satisfies Liouville–Arnol’d integrability if there exist $m$ functionally independent integrals of motion $\{I_1, I_2, \ldots, I_m\}$, in involution with respect to the Poisson structure, i.e. satisfying $\{I_i, I_j\} = 0$. Clearly, establishing this type of integrability requires knowing the Poisson structure to begin with.

In this paper we study the problem of finding a Poisson structure $\{\cdot,\cdot\}$ that is preserved by a difference equation of order $n$ of the form

$$x_n = F(x) := F(x_0, x_1, \ldots, x_{n-1}).$$

By saying that the Poisson structure $\{\cdot,\cdot\}$ is preserved by the difference equation (6) we mean that the map

$$M(x_0, x_1, \ldots, x_{n-1}) = (x_0', x_1', \ldots, x_{n-1}') := x'$$

where

$$x_i' = x_{i+1} \quad \text{for} \quad i = 0, 1, \ldots, n-1, \quad \text{and} \quad x_n = F(x)$$

is a Poisson map. By now, many authors have studied similar problems from the point of view of cluster algebras [9, 14], r-matrix approach [19], using three leg forms for $(p,p)$ reductions of maps in the ABS list [2], by considering symplectic structures [16] and many other [3, 4, 7, 8, 15, 20, 23, 25, 26].

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We will consider two families of difference equations of the form (6) with
\[ F(x) = \phi(y_1, y_2, \ldots, y_k) \]  
(9)
where
\[ y_i = r_i \cdot x = \sum_{j=0}^{n-1} r_{ij} x_j \]
is the dot product of \( x \) and \( r_i = (r_{i0}, r_{i1}, \ldots, r_{i,n-1}) \in \mathbb{R}^n \)
and
\[ F(x) = \psi(z_1, z_2, \ldots, z_k) \]
(10)
where
\[ z_i = \mathbf{x}^{r_i} = x_0^{r_{i0}} x_1^{r_{i1}} \cdots x_{n-1}^{r_{in-1}}, \]
for functions \( \phi, \psi \in C^1(\mathbb{R}^k) \). Some particular choices of the functions \( \phi \) and \( \psi \) give rise to several well known maps such as the sine-Gordon (SG), Korteweg–de Vries (KdV), modified KdV (mKdV), potential KdV (pKdV), AKP and BKP reductions [13–15, 23, 25, 26] as table 1 shows. At the end of section 2 we study the maps presented in table 1 and we provide Poisson structures that are preserved by them. For simplicity we adopt the following notation.

**Notation 1.** The bar over a sequence of numbers means that the sequence is repeated and the number of times is repeated will follow from the order of the corresponding map. For the KdV map in table 1 below the vector \( r_1 \) contains only \(-1\)'s and the vector \( r_2 \) has zero in its first two and the last element and the rest are \(1\)'s.

We will consider two families of Poisson structures that are each defined by a constant skew-symmetric matrix \( T \). Constant Poisson structures defined by
\[ \{x_i, x_j\} = \Omega_{ij}(x) = T_{ij}, \quad i, j \in \{0, 1, \ldots, n-1\} \]
(11)
and quadratic Poisson structures which are known as log-canonical (or diagonal, or Lotka–Volterra) Poisson structures [6]. These are defined by the brackets
\[ \{x_i, x_j\} = \Omega_{ij}(x) = T_{ij} x_i x_j, \quad i, j \in \{0, 1, \ldots, n-1\}. \]
(12)
It is well-known that, because of the skew-symmetry of \( T \), such brackets always satisfy the Jacobi identity, hence they are indeed Poisson brackets [21]. The rank of these Poisson structures, at a generic point, equals the rank of the constant matrix \( T \) and their Casimirs are in correspondence with the null-vectors of \( T \). If \( k = (k_0, k_1, \ldots, k_{n-1}) \) is a null-vector of \( T \) then \( k \cdot x \) is a Casimir of the constant Poisson structure while \( x^k \) is a Casimir of the quadratic Poisson structure. The log-canonical term comes from the fact that these quadratic structures are related to the constant Poisson structures by exponentiation of the coordinates. If we define new variables \( \eta_i = e^{x_i} \), where the \( x_i \) variables satisfy (11), then the bracket of \( \eta_i \) and \( \eta_j \) is
\[ \{\eta_i, \eta_j\} = T_{ij}/\eta_i \eta_j. \]

Because of this relation and because the examples we give in section 4 are of the form (10), in what follows we will focus on difference equations of the form (10) that preserve quadratic Poisson structures of the form (12). The explicit relation between maps of the form (9) and (10) is given in the following lemma.

**Lemma 1.** Suppose that \( M : \mathbb{R}^n \to \mathbb{R}^n, x \mapsto x' \) is a map of the form (7) and (8) where \( F(x) = \phi(y_1, y_2, \ldots, y_k) \) as in (9). Then \( M \) preserves the constant Poisson structure \( \{x_i, x_j\} = T_{ij} \) if and only if the map \( L : \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0} \) defined by
The map $L$ is of the form $z = \frac{v}{x}$, we only need to verify (5) for $z + e_0 = p_0 = (0, 1)$

The previous lemma allows us to present our results only for maps of the form $z = \frac{v}{x}$ or $z = \frac{v}{x^2}$. By definition, $G$ is indeed of the form (10) and is defined by the function $\psi(v_1, v_2, \ldots, v_{n-1}) = e^{G(x)}$ and $v_j = e^{g_j}$ preserves the quadratic Poisson structure $\{v_i, v_j\} = T_i j v_j$. The map $L$ is of the form (7) and (8) with $G$ as in (10).

**Proof.** By definition, $G$ is indeed of the form (10) and is defined by the function $\psi(z_1, z_2, \ldots, z_l) = e^{\phi(z_1, z_2, \ldots, z_l)}$. To verify that $L$ preserves the Poisson structure $\{v_i, v_j\} = T_i j v_j$ we only need to verify (5) for $f = v_i, i = 0, 1, \ldots, n - 2$ and $g = v_n$. We have, for any $i = 0, 1, \ldots, n - 2$,

$$\{v_i, v_{n-1}\} = T_i {n-1} v_{n-1} v_{n-1} G(v) = \sum_{j=0}^{n-1} T_i {j} v_{n-1} G(v) \frac{\partial F(x)}{\partial x_j}$$

$$= \sum_{j=0}^{n-1} T_i {j} v_{n-1} G(v) \frac{\partial F(x)}{\partial x_j}$$

$$= \sum_{j=0}^{n-1} T_i {j} v_{n-1} G(v) \frac{\partial G(v)}{\partial v_j} = \{v_i', v_{n-1}'\},$$

where in the second equality we have used our assumption that the map $M$ preserves the constant Poisson structure $(T_i j)$. The proof of the other direction is done similarly.

**Remark 1.** The previous lemma allows us to present our results only for maps of the form (10) and for quadratic Poisson structures. Then the same results will hold true for maps of the form (9) and constant Poisson structures. One can prove a more general result to cover a larger class of mappings and Poisson structures. Namely, under the assumptions of the previous lemma, if $h : \mathbb{R} \to X \subseteq \mathbb{R}$ is any differentiable function with $h'(x) \neq 0$ for all $x \in \mathbb{R}$, then the brackets $\{v_i, v_j\} = T_i j h'(h^{-1}(v_i)) h'(h^{-1}(v_j))$ define a Poisson structure on $X^p$ that is preserved by the map $L : X^p \to X^q, v = (v_0, v_1, \ldots, v_{n-1}) \mapsto v' = (v_1, v_2, \ldots, v_{n-1}, G(v))$ with $G(v_0, v_1, \ldots, v_{n-1}) = h(F(x))$ and $v_1 = h(x_1)$.

In section 2 we show that under some assumptions on the function $\psi$ we can always find a quadratic Poisson structure of the form (12) that is preserved by the map (7) and (8) (theorem 4). In section 3 we study the inverse problem; given a (log-canonical) quadratic Poisson structure to find a map of the form (7) and (8) that preserves this structure (theorems 8 and...
9). Last, in section 4 we apply our theory to maps which are obtained as reductions of known partial difference equations.

2. Finding the Poisson structure given the difference equation

We begin this section by showing that if a Poisson structure is preserved by a map of the form (7) and (8), then it must be of a specific form and the function $F$ must satisfy certain PDE’s that depend on the Poisson structure. We show in the next lemma that if $F$ is of the form (10) and the Poisson structure is of the form (12) with Toeplitz matrix $T$ then the PDE’s are transformed into a linear system of equations. Recall that a square $n \times n$ matrix $T$ is called Toeplitz if its entries $T_{i,j}$ depend only on the differences $j - i$. This means that if $T$ is Toeplitz there exist $2n - 1$ numbers $T_j$ for $j = -n + 1, -n + 1, \ldots, n - 1$ such that for any $i,j \in \{1, 2, \ldots, n\}$, $T_{i,j} = T_{j-i}$. Notice that in the case of a skew symmetric Toeplitz matrix $T$, which is the case we consider, the numbers $T_j$ have the property $T_{-j} = -T_j$, in particular $T_0 = 0$.

**Lemma 2.** Let $M$ be the map (7) and (8) and $\{\cdot, \cdot\}$ a Poisson structure with $\Omega(x)$ the corresponding Poisson matrix.

(1) The map $M$ preserves the Poisson structure $\{\cdot, \cdot\}$ if and only if the following two relations are satisfied

$$\Omega_{i+1,j+1}(x) = \Omega_{i,j}(M(x)),$$

for all $0 \leq i < j < n - 1$

and

$$\Omega_{i,n-1}(x') = \{x_{i+1}, x_n\} := \sum_{j=0}^{n-1} \frac{\partial F}{\partial x_j} \Omega_{i+1,j}(x), \quad \text{for } i = 0, 1, \ldots, n - 2.$$

(2) If the function $F$ of (10) is of the form $F(x) = z_1 \tilde{\psi}(z_2, \ldots, z_k)$ with $\tilde{\psi}$ any function in $C^1(\mathbb{R}^{k-1})$ and if $\{\cdot, \cdot\}$ is a quadratic Poisson structure of the form (12) with Toeplitz matrix $T$, then $M$ preserves $\{\cdot, \cdot\}$ if and only if

$$\sum_{j=0}^{n-1} r_{1,j} T_{j-i} = T_{n-i}, \quad \text{for } i = 1, \ldots, n - 1$$

and

$$\sum_{j=0}^{n-1} r_{\ell,j} T_{j-i} = 0, \quad \text{for } i = 1, \ldots, n - 1, \quad \ell = 2, 3, \ldots, k. \quad (13)$$

**Proof.** Item (1) is easily proved by direct computation using formula (5). For the proof of item (2) notice that, because $T$ is Toeplitz the first system of equations of item (1) is automatically satisfied while the second one becomes

$$\sum_{j=0}^{n-1} \frac{\partial F}{\partial x_j} x_{i+1} x_{j-i-1} = x_{i+1} x_n T_{n-i-1}$$

$$\iff \sum_{\ell=1}^{k} \sum_{j=0}^{n-1} \frac{\partial F}{\partial z_\ell} \frac{\partial z_\ell}{\partial x_j} x_{j-i-1} = F(x) T_{n-i-1}, \quad i = 0, 1, \ldots, n - 2. \quad (14)$$
Using that $F(x) = z_1 \hat{v} (z_2, \ldots, z_k)$ and that $x_j \frac{\partial z_j}{\partial x_j} = r_{i,j} z_l$, system (14) is transformed into the second part (13).

The linear system (13) has $k \cdot (n - 1)$ equations and $n - 1$ variables, therefore is unlikely to have a solution. Imposing some restrictions on the vectors $r_i$ we are able to reduce the size of the system and in some cases obtain general (non)existence results. We do that in the next lemma after introducing some notation.

**Notation 2.** If $r$ is any row vector we write $r^*$ for the vector obtained from $r$ by deleting its first element and we write $\hat{r}$ for the vector obtained by reversing the order of the entries of $r$. We say that the vector $r$ is symmetric if $r = \hat{r}$ and that is skew-symmetric if $r = -\hat{r}$. We will also write $T^*$ for the $(n - 1) \times (n - 1)$ minor of $T$ obtained by deleting its first row and column and $Q$ for the $(n - 1) \times n$ minor of $T$ obtained by deleting its first row. The $n \times n$ Hankel matrix $J$, defined by $J_{in+1-i} = 1$ for all $i = 1, 2, \ldots, n$, and all other entries zero, will be useful.

Using the above notation a Toeplitz $n \times n$ matrix $T$ is skew-symmetric (symmetric) if and only if $JTJ = -T$ ($JTJ = T$). For example, it is easy to see that $J^2 = I$ and for the skew-symmetric matrix $\Omega$ in (2), $J\Omega J = -\Omega$. Similarly, the vector $r$ is skew-symmetric (symmetric) if and only if $JR = -r$ ($JR = r$).

With the above notation the linear system (13) is written, in an equivalent matrix form, as

$$-r_{1,0} \mathbf{t} + T^* \mathbf{r}_1^* = \mathbf{i}, \quad -r_{\ell,0} \mathbf{t} + T^* \mathbf{r}_\ell^* = 0, \quad \ell = 2, 3, \ldots, k,$$

or equivalently again, as

$$Q \mathbf{r}_1^* = \mathbf{i}, \quad Q \mathbf{r}_\ell^* = 0, \quad \ell = 2, 3, \ldots, k,$$

where $\mathbf{t}$ is the vector

$$\mathbf{t}^* = (T_1 \quad T_2 \quad \cdots \quad T_{n-1}).$$

**Lemma 3.**

(1) If $r_{1,0} = -1$ (resp. $r_{1,0} = 1$) and $r_{\ell,0} = 0$ for $\ell = 2, 3, \ldots, n$ and if the vectors $\mathbf{r}_\ell^*$ are symmetric (resp. skew-symmetric) for all $\ell = 1, 2, \ldots, k$, then the system (15) becomes half in size. More explicitly, for any $\ell \in \{1, 2, \ldots, k\}$, the vector $T^* \mathbf{r}_\ell^*$ is skew-symmetric (resp. symmetric).

(2) If $n$ is even, $k = 2$, $r_{1,0} = -1$, $r_{2,0} = 0$ and if the vector $\mathbf{r}_1^*$ is symmetric for $\ell = 1, 2$ then the linear system (15) has a non-trivial solution.

(3) If $\mathbf{r}_\ell = (r_{\ell_1}, r_{\ell_2}, \ldots, r_{\ell_k})$ for some $r_{\ell_k} \in \mathbb{R}$ then the $\ell$th equations of (15) simplify to

$$T_i (r_{\ell_i} + 1) + \sum_{j=i+1}^{n-1} r_{\ell_j} T_j - T_{n-i} = 0, \quad i = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor.$$

**Proof.** For $r_{1,0} = -1$ the vector $\mathbf{i} + r_{1,0} \mathbf{t}$ is skew-symmetric while for $r_{1,0} = 1$ is symmetric. In order to prove item (1) it is enough to show that the vector $T^* \mathbf{r}_\ell^*$ is skew-symmetric (resp. symmetric). Let $J$ be the $(n - 1) \times (n - 1)$ Hankel matrix defined in notation 2. Then

$$JT^* \mathbf{r}_\ell^* = -J^2 T^* J \mathbf{r}_\ell^* = -T^* \mathbf{r}_\ell^*.$$
where we have used the skew-symmetry of $T^\ell$ and the symmetry of $r_i$. The symmetric case is done similarly. For item (2) notice that because $n$ is even and because the $n-1$ dimensional vector $T^\ast r_i$ is skew-symmetric, its middle element is zero. Thus the (homogeneous) linear system (15) has $2(\frac{n}{2} - 1) = n-2$ equations with $n-1$ variables $(T_1, T_2, \ldots, T_{n-1})$ and therefore no non-trivial solution. Item (3) follows by direct computation.

The next theorem about maps of the form (7) and (8) is a corollary of lemma 3.

**Theorem 4.** Let $n$ be even and $M$ the map (7) and (8) with the function $F$ of (10) being of the form

$$F(x) = x^\top \tilde{\psi}(x^\top)$$

for some real function $\tilde{\psi} \in C^1(\mathbb{R})$. If $r_{1,0} = -1, r_{2,0} = 0$ and $r_i$ symmetric for $\ell = 1, 2$ then there is a quadratic Poisson structure $\{x_i, x_j\} = T_{ij} x_i x_j$ that is preserved by the map $M$. The matrix $T$ is a skew-symmetric Toeplitz matrix with first row $\ell = (0, T_1 T_2 \ldots T_{n-1})$, where the $T_i$ are determined by the non-trivial solution of (15).

In the rest of this section we present Poisson structures that are preserved by the maps of table 1. First, notice that the vectors $r_i$ which define the mKdV and pKdV maps are identical. Therefore, according to lemma 1, a constant Poisson structure $T$ is preserved by the pKdV map if and only if the quadratic log-canonical Poisson structure defined by the matrix $T$ is preserved by the mKdV map. This is in accordance with the results of [25]. An easy calculation, using (13) or (15), shows that in even dimensions the SG map preserves the non-degenerate log-canonical Poisson structure defined by the Toeplitz matrix with first line $(0, \overline{T})$. The KdV map preserves a constant Poisson structure with Toeplitz matrix $T$ where, for $n \equiv 2 \bmod 4$ the first line of $T$ is $(0, 1, 0, 1)$ and for $n \equiv 0 \bmod 4$ is $(0, 0, 1, -1)$. These Poisson structures are non-degenerate. For $n \equiv 3 \bmod 4$ the first line of $T$ is $(0, 1, 0, -1, 0, 1)$ and $T$ is degenerate with rank 2. For all other remaining cases the linear system (13) does not have a solution and therefore there are no log-canonical (respectively constant, for the odd-dimensional SG map) Poisson structures that are preserved. In the next proposition we show that reductions of these remaining cases give rise to maps that preserve Poisson structures of our form (see also proposition 14).

**Proposition 5.**

1. For $n \equiv 1 \bmod 4$ the reduction $w_i = x_{n+1}$ of the KdV map gives rise to a map of the same form (7) and (8) with function $F$ as in (10) that preserves the log-canonical Poisson structure defined by the non-degenerate Toeplitz matrix with first line $(0, 0, 1, 0, -1, 0, 1, 0)$.

2. For the odd dimensional SG map the reduction $w_i = x_{n+1}$ gives rise to a map of the same form (7) and (8) with function $F$ as in (10) that preserves the log-canonical Poisson structure defined by the non-degenerate Toeplitz matrix with first line $(0, \overline{T})$.

3. For the even (resp. odd) dimensional mKdV map the reduction $w_i = \frac{x_i}{x_0}$ (resp. $w_i = \frac{x_i}{x_1}$) gives rise to a map of the form (7) and (8) with function $F$ as in (10) that preserves the log-canonical Poisson structure defined by the non-degenerate Toeplitz matrix with first line $(0, 1, \overline{6})$ (resp. $(0, 1, -1, T)$).

**Proof.** We only give the vectors $r_i$ obtained after the reductions since the rest are straightforward computations. For the KdV reductions the vectors $r_1$ and $r_2$ are $r_1 = (-1, 0)$ and $r_2 = (0, 0, 1, 0)$ and for the SG reductions they are $r_1 = (-1, 1, -1, T)$ and $r_2 = (0, 1, -1, T)$. For the even dimensional mKdV map they are given by $r_1 = (-1, 0)$ and $r_2 = (0, 1)$ and for the odd dimensional mKdV map by $r_1 = (-1)$ and $r_2 = (0, 1)$. □
3. Finding the difference equation given the Poisson structure

We consider now the inverse problem of finding a difference equation of the form (6) that preserves a given Poisson structure. In what follows we assume that the matrix $T$ is skew-symmetric and Toeplitz. We first show that if the map $M$ defined by (7) and (8) preserves the quadratic log-canonical Poisson structure with matrix $T$ then the function $F$ is necessarily of the form (10).

**Proposition 6.** Let $M$ be the map (7) and (8) which preserves a quadratic log-canonical Poisson structure with matrix $T$. Then the function $F$ defining $M$ is of the form (10).

**Proof.** According to lemma 2, if the map $M$ preserves the quadratic Poisson structure with matrix $T$ then

$$
\sum_{j=0}^{n-1} \frac{\partial F}{\partial x_j} T_{j-i} x_j = T_{n-i} F(x), \quad i = 1, 2, \ldots, n-1. \quad (17)
$$

If $r_1$ is a solution of the non-homogeneous linear system (13) then we can verify that $F_1 = x^{r_1}$ is a solution of (17). Since the ratio of two solutions of (17) is a solution of the corresponding homogeneous system it follows that its general solution is $F = x^{r_1} \tilde{F}(x)$ where $\tilde{F}$ is the solution of the homogeneous one. The system

$$
\sum_{j=0}^{n-1} \frac{\partial F}{\partial x_j} T_{j-i} x_j = 0, \quad i = 1, 2, \ldots, n-1, \quad (18)
$$

is linear and can be solved using the method of characteristics (see [24]). It can be verified directly that if $r_\ell$ is a solution of the homogeneous part of (13) then a solution $\tilde{F}$ of (18) will remain constant along the surface defined by $x^{r_\ell} = C$. This shows that the solution of (17) is $F(x) = x^{r_1} \tilde{\psi}(x^{r_2}, x^{r_3}, \ldots, x^{r_k})$ where $\tilde{\psi}$ is any real function of $k-1$ variables and $r_\ell$, for $\ell = 2, 3, \ldots, k$, are solutions of the homogeneous part of (13). Therefore $F$ is indeed of the form (10). □

The proof of the previous proposition shows that the existence of a map of the form (7) and (8) preserving a given log-canonical Poisson structure amounts to a solution of a linear system. Assuming that the matrix $T$ has sufficiently large rank then we can derive existence and non-existence results about maps that preserve the corresponding Poisson structure.

**Proposition 7.** Let $Q$ be the matrix obtained from $T$ by deleting its first row, as in notation 2, and $\{\cdot, \cdot\}$ the quadratic Poisson structure of the form (12) with matrix $T$.

1. If $Q$ is of maximal rank, then there exists a map $M$ of the form (7) and (8) with function $F$ of the form (10) which preserves the Poisson structure $\{\cdot, \cdot\}$.
2. If $Q$ is of rank $m \leq n-1$ and $M$ is a map of the form (7) and (8) which preserves the Poisson structure $\{\cdot, \cdot\}$, then $F$ is of the form (10) and the vectors $r_\ell$, for $\ell = 2, 3, \ldots, k$, form a linear space of dimension smaller or equal than $n-m$.
3. Assuming that the vectors $r_\ell$ for $\ell = 2, 3, \ldots, k$ are linearly independent, $Q$ is of rank $m \leq n-1$ and $M$ the map (7) and (8) with $F$ of the form (10) with $k > n - m + 1$, then $M$ does not preserve the Poisson structure $\{\cdot, \cdot\}$. 

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Proof. For item (1) we first note that from proposition 6, if the map M preserves the Poisson structure \{\cdot, \cdot\} then the function F defining M is necessarily of the form (10). The existence of such M is guaranteed by the existence of solutions of the linear system (16) which consist of k linear systems (in the r_\ell's) each one having n – 1 equations and n variables. Because Q is of maximal rank its non-homogeneous part has a solution, and due to their dimension, the rest homogeneous k – 1 linear systems have a non-trivial solution. The proof of items (2) and (3) is a consequence of the dimension of the homogeneous part of the linear system (16). □

We now show that for the non-degenerate Poisson structures, the maps that preserve them given in item (1) of the previous proposition, have r_{1,0} = –1, r_{\ell,0} = 0 for \ell = 2, . . . , k and the vectors r^*_\ell are symmetric for \ell = 1, 2, . . . , k (see [11, 17]). This serves as a partial inverse of theorem 4.

Theorem 8. Let n be even and T an n \times n matrix of full rank. Also let \{\cdot, \cdot\} be the Poisson structure \{x_i, x_j\} = T_{ij} x_i x_j and M a map of the form (7) and (8) which preserves \{\cdot, \cdot\}. Then the function F defining M is of the form (10) with

\[
F(x) = \frac{(x^*)^T r\ast}{x_0} \psi((x^*)^T r\ast),
\]

where the vectors r\ast_1, r\ast_2 are symmetric. They are explicitly given by the formulas

\[
r_{1,j} = \frac{\det(T^{(j+1,1)})}{\det(T)}, \quad j = 1, . . . , n - 1.
\]

The matrix T^{(\ell,1)} is obtained from T by replacing its jth column by the vector (a T_{n-1} T_{n-2} \cdots T_1)^T where a ∈ R and

\[
r_{2,j} = \text{cofactor}(T, 1, j + 1), \quad j = 0, 1, . . . , n - 1.
\]

The cofactor(T, 1, j + 1) is the signed determinant of the minor of T obtained by deleting its first row and j + 1 column.

Proof. From the previous proposition it follows that k \leq 2 and, because of the rank of T, it is sufficient to show that the linear systems (in r\ast_1, r\ast_2)

\[
T^* r\ast_1 = \begin{pmatrix} T_{n-1} - T_1 \\ T_{n-2} - T_2 \\ \vdots \\ T_1 - T_{n-1} \end{pmatrix}, \quad T^* r\ast_2 = 0
\]

have symmetric solutions. That, will be a consequence of the following more general result: For an m \times m skew-symmetric Toeplitz matrix R of rank m – 1 (hence m is odd) and b ∈ R^m skew-symmetric, the solutions of the linear system Rq = b are symmetric. We recall from notation 2, that J is the m \times m matrix with entries J_{m+i, m+i} = 1 for all i = 1, 2, . . . , m and all other entries zero. Then RJ = –R and Jb = –b. This shows that RJq = –JRJ^2 q = –JRq = b and therefore the vector q – Jq is a (skew-symmetric) null vector of R. Showing that R has a non-zero symmetric null vector it will imply (because of the rank of R) that q – Jq = 0 and therefore q is symmetric.
Let \( v \) be a null vector of \( R \). The previous proof (with \( q = v \) and \( b = 0 \)) shows that \( Jv \) is also a null vector of \( R \) and therefore \( v + Jv \) is a null vector of \( R \) as well. So, we can assume that \( v \) is symmetric and the proof of lemma 3 (item 2), shows that the homogeneous linear system \( Rv = 0 \) has \( \frac{n+1}{2} \) equations with \( \frac{n+1}{2} + 1 \) unknowns, therefore a non-trivial solution.

Having established that if \( T \) is an \( n \times n \) skew-symmetric Toeplitz matrix of full rank the solution of the linear system (16) has \( r_{1,0} = -1 \) and \( r_{2,0} = 0 \) we can now use Cramer’s rule to give explicit formulas for the \( r_\ell \) for \( \ell = 1, 2 \). The linear system (16) is equivalently written as

\[
Tr_1 = \begin{pmatrix} a \\ T_{n-1} \\ T_{n-2} \\ \vdots \\ T_1 \end{pmatrix}, \quad Tr_2 = \begin{pmatrix} b \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}
\]

for arbitrary \( a, b \in \mathbb{R} \). The one degree of freedom of the linear system (16) is imposed into the parameters \( a, b \). Cramer’s rule gives that

\[
r_{1,j} = \frac{\det(T(j+1,1))}{\det(T)}, \quad j = 0, 1, \ldots, n-1
\]

and similarly,

\[
r_{2,j} = \frac{\det(T(j+1,2))}{\det(T)}, \quad j = 0, 1, \ldots, n-1.
\]

Expanding the determinant \( \det(T(j+1,2)) \) with respect to its \( j + 1 \) column we get

\[
r_{2,j} = \frac{b \cofactor(T, 1, j + 1)}{\det(T)}, \quad j = 0, 1, \ldots, n-1.
\]

The factor \( \frac{b}{\det(T)} \) of the vector \( r_2 \) can be absorbed into the arbitrary function \( \tilde{\psi} \) and we get

\[
r_{2,j} = \cofactor(T, 1, j + 1), \quad j = 0, 1, \ldots, n-1.
\]

Also, because of the dimension of \( T^* \), \( r_{2,0} = \cofactor(T, 1, 1) = \det(T^*) = 0 \).

\[ \square \]

**Remark 2.** Because \( r_{2,0} = 0 \) and \( T^* r_2^* = 0 \), the vector \( r_2 \) does not depend on the entry \( T_{n-1} \) of \( T \). This is consistent with the next example and with the results of table 2. Also from the explicit form of the map \( M \) given in the previous proof it follows that the map \( M \) is invertible (it can be solved for \( x_0 \)), and also reversible, i.e. \( L^{-1}ML = M^{-1} \) for a suitable map \( L \). In our case the map \( L \) is the involution \( x \mapsto \bar{x} \).

**Example 1.** Suppose \( n \) is even and \( T_i = T_{i+1} \) for all \( i < n - 1 \), i.e. the first line of the matrix \( T \) is

\[
(0, T_1, T_{n-1}) = (0, T_1, \ldots, T_1, T_{n-1}).
\]

For generic values of \( T_1, T_{n-1} \) the matrix \( T \) is non-degenerate with determinant \( \det(T) = T_{n-1}^2T_1^{n-2} \) and the solution of (16) for \( r_2 \) is \( r_2 = (0, T_1, -T_1, 1) \). Similarly for
Table 2. Non-degenerate Poisson structures and the function $F$ which defines the map that preserves the Poisson structure.

| $(T_1, T_2, \ldots, T_{n-1})$ | Determinant | Form of the function $F$ |
|---------------------------------|-------------|--------------------------|
| $(\bar{T}, t)$                  | $t^2$       | $F = \frac{1}{4} \psi(x_{t-1} \prod_{j=1}^{n-1} x_j)$ |
| $(\bar{T}, -T, t)$              | $t^2$       | $F = \frac{1}{4} \psi(\prod_{j=1}^{n-1} x_j)$ |
| $(\bar{T}, 0, t)$               | $2^{n-4}(t + 1)^2$ | $F = \frac{1}{4} \psi(x_1 x_{t-1})$ |
| $(1, \bar{0}, t)$               | $(t + 1)^2$ | $F = \frac{1}{4} \psi(\prod_{j=1}^{n-1} x_{t-1})$ |
| $(0, 1, \bar{0}, 1, t), \frac{2}{2}$ odd | $t^2$       | $F = \frac{1}{4} \psi(\prod_{j=1}^{n-1} x_{t-1})$ |
| $(0, 1, \bar{0}, 1, t), \frac{2}{2}$ even | $16$        | $F = \frac{1}{4} \psi(\prod_{j=1}^{n-1} x_{t-1})$ |

$T_i = -T_{i+1}$ for all $i < n - 1$, the first line of $T$ becomes

$$(0, T_1, -T_1, T_1, -T_1, \ldots, -T_1, T_1)$$

and the solution of (16) is $r_2 = (0, \bar{T})$.

We now consider the odd dimensional case.

**Theorem 9.** Let $n$ be odd and $T$ an $n \times n$ matrix with $T'$ of rank $n - 1$. We assume that $F$ is a function of the form (10) which defines the map (7) and (8) which preserves the Poisson structure $\{x_i, x_j\} = T_{ij} x_i x_j$. Then if $k = 2$ the vector $r_2$ is symmetric and can be chosen such that $r_{2,0} = 1$, $r_{1,1} = -r_2^* \psi(x_1)$ and $r_{1,0} = 0$. Therefore the function $F$ is of the form

$$F(x) = \langle x^* \rangle - r_2^* \psi(x^*) \psi(x^*).$$

**Proof.** According to proposition 6 the function $F$ defining $M$ is of the form $F(x) = x^* \psi(x^*)$. From the proof of the previous theorem, the vector $r_2$ being a null vector of $T$, is symmetric. The defining relations of the vectors $r_1$ and $r_2$ are the linear systems (15) which, from our assumption that $T'$ has full rank, they have a unique solution in $r_1^*, r_2^*$. The arbitrary function $\psi$ absorbs the parameters $r_{1,0}$ and $r_{2,0}$ and can be chosen to be equal to 0 and 1 respectively. The skew-symmetry and Toeplitz form of $T$ gives that $r_1^* = -r_2^*$.

**Remark 3.** If the vector $t = (T_1, T_2, \ldots, T_{n-1})$ is symmetric then the vector $r_1^*$ can be absorbed into the arbitrary function $\psi$ and we can choose $r_1 = (-1, 0)$ while if $t$ is skew-symmetric we can choose $r_1 = (1, 0)$.

**Remark 4.** The explicit form of the map given in the previous proposition shows that the map is invertible if and only if the function $\psi$ is invertible and it is reversible, with reversing symmetry the same map $L$ as in the even dimensional case, if and only if $\psi^{-1} = \psi$.

**Example 2.** Continuing example 1 for odd $n$, if $T_i = T_{i+1} \neq 0$ for all $i < n - 1$, i.e. if the first line of $T$ is $(0, T_1, T_1, \ldots, T_1, T_{n-1})$ we get

$$r_2 = (1, \frac{T_{n-1}}{T_1}, -\frac{T_{n-1}}{T_1}, \frac{T_{n-1}}{T_1}, \ldots, -\frac{T_{n-1}}{T_1}, 1).$$
Table 3. Degenerate Poisson structures of even dimension and the function $F$ which defines the map that preserves the Poisson structure.

| $(T_1, T_2, \ldots, T_{n-1})$ | Form of the function $F$ |
|-------------------------------|-----------------------------|
| $(T, 0)$                      | $F = \frac{1}{4} \psi(z_0x_{n-1}, x_0 \prod_{j=1}^{n-1} \frac{z_j}{z_{j+1}})$ |
| $(1, -T, 0)$                  | $F = \frac{1}{4} \psi(z_0, \prod_{j=1}^{n-1} x_j)$ |
| $(1, 0, -1)$                  | $F = \frac{1}{4} \psi(x_1x_0, x_0x_{n-2})$ |
| $(1, \tilde{b}, -1)$          | $F = \frac{1}{4} \psi(z_0, \prod_{j=1}^{n-1} x_{2j-2}^{\otimes 2} \prod_{j=1}^{n-1} x_{2j-1})$ |
| $(0, 1, \tilde{b}, 1, 0); \tilde{b}$ odd | $F = \frac{1}{4} \psi(z_0, \prod_{j=1}^{n-1} \frac{z_{j+1}}{z_j})$ |

Similarly for $T_i = -T_{i+1} \neq 0$ for all $i < n - 1$, we get

$$r_2 = (1, -\frac{T_{n-1}}{T_1}, -\frac{T_{n-2}}{T_1}, -\frac{T_{n-3}}{T_1}, \ldots, -\frac{T_2}{T_1}, 1).$$

In tables 2–4 we give the vector $t$ and the form of the function $F$ which defines the map $M$ that preserves the quadratic Poisson structure $\{x_i, x_j\} = T_i x_i x_j$ where $t = (T_1, \ldots, T_{n-1})$. In table 2 we present non-degenerate Poisson structures which depend on a parameter $t \in \mathbb{R}$ and in table 3 we present the same structures with $t$ chosen such that the matrix $T$ is degenerate. In table 4 we present Poisson structures of odd dimension $n$. The results of table 2 verify remark 2, that the vector $r_2$ is not affected from the last entry of the matrix $T$ which is taken arbitrary so that the matrix is non-degenerate. In table 3 the rank of the Poisson structures is $n - 2$ and therefore the function $F$ can be a two variable function. These examples illustrate the results of theorems 8 and 9.

4. Poisson structures for known maps

We now apply our results to several families of maps and we find Poisson structures that they preserve. We also find maps that preserve Poisson structures of specific form. For simplicity we write LVPS(t) (Lotka–Volterra Poisson structure) for the quadratic Poisson structure $\{x_i, x_j\} = T_i x_i x_j$ where $T$ is a skew-symmetric Toeplitz matrix and $t = (T_1, \ldots, T_{n-1})$ with $T_{n-i} = T_i$ for all $0 \leq i < j \leq n - 1$.

First we consider maps which arise as reductions of the AKP partial difference equation [12, 22]. These maps are defined by an equation of the form

$$A x^n + B x^m + C x^2 = 0$$

which is obtained from a $(z_1, z_2, z_3)$-travelling wave reduction of the AKP equation

$$A T_{l+1,j,m} T_{l,j+1,m+1} + B T_{l,j+1,m} T_{l+1,j,m} + C T_{l,j,m+1} T_{l+1,j,m+1} = 0.$$  \hspace{1cm} (23)

We consider the $(z_1, z_2, z_3)$-travelling wave reduction $T_{l,j,m} = T_{z_1, z_2, z_3}$ where $z_1, z_2, z_3 \in \mathbb{N}$. Because of the symmetry of equation (23), the order of $(z_1, z_2, z_3)$ is irrelevant and therefore we may use the constraint $0 < z_1 < z_2 < z_3$. By writing $\tau_n$ for $T_{z_1, z_2, z_3}^{m+n}$ then the pullback of (23) under the transformation $x_0 = \frac{z_1 + z_2}{z_3}$ is the map (22) of order $n = M(z_1, z_2, z_3) = z_2 + z_3 - z_1 - 2$. The vectors $u_f^*$ in (22) are symmetric and of dimension $n-1$. For $z_3 \geq z_1 + z_2$ they are given by
Table 4. Poisson structures of odd dimension and the function $F$ which defines the map that preserves the Poisson structure.

| $(T_1, T_2, \ldots, T_{n-1})$ | Rank | Form of the function $F$ |
|-----------------------------|------|--------------------------|
| $(T)$                      | $n-1$ | $F = \frac{1}{n} \tilde{\psi}(x_0 \prod_{j=1}^{n-1} \tilde{\psi}^j)$ |
| $(1, -T)$                  | $n-1$ | $F = x_0 \tilde{\psi}(\prod_{j=0}^{n-1} s_j)$ |
| $(T, 0)$                   | $n-1$ | $F = \frac{1}{n} \tilde{\psi}(x_0 s_{n-1})$ |
| $(1, 0)$                   | $n-1$ | $F = \prod_{j=1}^{n-1} \tilde{\psi}(\prod_{j=0}^{n-1} x_j)$ |

$u_0^* = (2, 3, \ldots, z_2 - 1, \overline{z_2}, z_2 - 1, \ldots, 3, 2)$

$u_1^* = (\overline{0}, 1, 2, \ldots, z_1 - 1, \overline{z_1}, z_1 - 1, \ldots, 1, \overline{0})$.

$u_2^* = (0, \ldots, 0)$.

where the total number of zeros in $u_1^*$ is $2(z_2 - z_1 - 1)$. If $z_3 < z_1 + z_2$ the exponents $u_1^*$ coincide with the exponents of the $(z_3 - z_2, z_3 - z_1, z_3)$ reduction. Their first elements are respectively $u_{0,0} = 1$ and $u_{\ell,0} = 0$ for $\ell = 1, 2$.

Equation (23) is a special case of a more general partial difference equation, the BKP equation [22]

$$AT_{k+1,l,m} + BT_{k,l+1,m} + CT_{k,l,m+1} + DT_{k+1,l+1,m+1} = 0.$$  (24)

The same ($z_1, z_2, z_3$)-travelling wave reduction as before gives rise to the $n$th order map with $n = N(z_1, z_2, z_3) = z_1 + z_2 + z_3 - 2$, given by

$$Dx^{u_0} x_0 + Ax^{u_1} + Bx^{u_2} + Cx^{u_3} = 0.$$  (25)

For $z_3 \geq z_1 + z_2$ the vectors $u_1^*$ are

$u_0^* = (2, 3, \ldots, z_1 + z_2 - 1, \overline{z_1 + z_2}, z_1 + z_2 - 1, \ldots, 3, 2)$,

$u_1^* = (\overline{0}, 1, 2, \ldots, z_1 - 1, \overline{z_1}, z_1 - 1, \ldots, 2, 1, \overline{0})$,

$u_2^* = (\overline{0}, 1, 2, \ldots, z_1 - 1, \overline{z_1}, z_1 - 1, \ldots, 2, 1, \overline{0})$,

$u_3^* = (0, 0, \ldots, 0)$.

and for $z_3 < z_1 + z_2$ they are

$u_0^* = (2, 3, \ldots, z_3 - 1, z_3, z_3, \ldots)$,

$u_1^* = (\overline{0}, 1, 2, \ldots, z_3 - z_1 - 1, \overline{z_3 - z_1}, z_3 - z_1 - 1, \ldots, 2, 1, \overline{0})$,

$u_2^* = (\overline{0}, 1, 2, \ldots, z_3 - z_2 - 1, \overline{z_3 - z_2}, z_3 - z_2 - 1, \ldots, 2, 1, \overline{0})$,

$u_3^* = (0, 0, \ldots, 0)$.

where, in both cases, the total number of zeros in $u_1^*$ is $2(z_1 - 1)$ and in $u_2^*$ is $2(z_2 - 1)$. Their first elements are respectively $u_{1,0} = 1$ and $u_{\ell,0} = 0$ for $\ell = 1, 2$.

The above equations (22) and (25) are of the form (10) with $\psi = z_1 \tilde{\psi}(z_2, z_3, \ldots, z_k)$ and $k = 2, 3$, respectively. More explicitly the equation (22) is written as

$$x_n = x_M = x_{u_0 - u_0} \left(- \frac{B}{A} + \frac{C}{A} x^{u_2 - u_2} \right)$$  (26)

and the equation (25) as
\[ x_n = x_N = x_{\ell} = 0, \quad (t_1, t_2, \ldots, t_n) = (1, 1, \ldots, 1). \]  

The vectors \( r_t \) are related to the vectors \( u_t \) by \( r_1 = u_1 - u_0, \quad r_t = u_t - u_1. \)

Applying theorem 4 we get the following result about the AKP reductions.

**Proposition 10.** If \( z_2 + z_3 - z_1 - 2 \) is even then there is a quadratic Poisson structure of the form (12) that is preserved by the map (22).

We now look at some specific choices of \( z_1, z_2 \) and \( z_3 \).

**Proposition 11.** For each \( n \in \mathbb{N} \) even with \( n \geq 2 \), the \( n \)th order map (22) with \( z_1 = 1, z_2 = 2 \) and \( z_3 = n + 1 \) preserves the LVPS(\( t \)) with \( t = (\bar{1}, -\bar{1}, 1) \).

**Proof.** For these choices of \( z_1, z_2, z_3 \) the map (22) becomes

\[ A_{x_0} x_{\ell} x_n + B x_{\ell} + C = 0 \]

with \( u_0 = (2) \) and \( u_{\ell} = (1) \). The vectors \( r_1, r_2 \) are respectively \((-\bar{1})\) and \((0, -\bar{1})\) and the solution of the linear system (15) is \( t = (T_1, T_2, \ldots, T_{n-1}) = (1, -\bar{1}, \bar{1}). \)

Note that the LVPS(\( t \)) Poisson structure with \( t = (1, -\bar{1}, 1) \) is non-degenerate and the vector \( t \) is symmetric for any even \( n \). We show in the next proposition that this is the only family of the AKP reductions that preserves a non-degenerate Poisson structure of the form (12) with symmetric vector \( t \). For the BKP reductions we show that they cannot preserve a non-degenerate Poisson structure of the form (12).

**Proposition 12.**

1. The only AKP reductions (22) that preserve a non-degenerate Poisson structure of the form (12) with \( t = (T_1, T_2, \ldots, T_{n-1}) \) symmetric, are those corresponding to \( z_1 = 1, z_2 = 2 \) and \( z_3 = n + 1 \) for \( n \) even given in proposition 11.

2. For any choice of \( z_1 < z_2 < z_3 \), the BKP reduction (25) does not preserve a non-degenerate Poisson structure of the form (12).

**Proof.** The proof of item (2) is a consequence of item (3) of proposition 7 by noticing that the vectors \( r_2, r_3 \) (or equivalently the vectors \( u_1, u_2 \)) are linearly independent.

For the proof of item (1) notice that because \( r_{1,1} = -1 \) the solution of the linear system (15) (using the assumption that \( t \) is symmetric) would imply that the vectors \( r_1, r_2 \) (or equivalently the vectors \( u_0, u_1 \)) are null vectors of the matrix \( T^* \). Assuming that \( T \) is of full rank, the matrix \( T^* \) is of co-rank 1 and therefore \( u_0 \) and \( u_1 \) are linearly dependent. We can see from the explicit formulas of \( u_0 \) and \( u_1 \) that they are linearly depended if and only if \( z_1 = 1, z_2 = 2 \) and \( z_3 = n + 1 \) with \( n \) even.

For the AKP reduction with \( z_1 = 1, z_2 = 2 \) and \( z_3 = n + 1 \) with \( n \) odd we have a map of odd order for which the associated linear system (13) does not have a non-trivial solution. This is because (13) now has \( n - 1 \) equations with the same number of variables and from lemma 3 (item 3) we see that, for each \( j = 1, 2, \ldots, n - 1 \), there is an equation with exactly \( j \) zeros. Therefore it can be transformed into a triangular homogeneous system with non-zero diagonal elements. As it turns out there is a further reduction which gives rise to Poisson maps. These reductions are similar to the reductions given in proposition 5. We first prove a more general result.

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**Proposition 13.** Let $M$ be the map (7) and (8) with $F = z_1 \tilde{\psi}(z_2, z_3, \ldots , z_k), n$ odd, $r_{1,0} = -1, r_{0,\ell} = 0$ for $\ell = 2, 3, \ldots, k$ and $r_{\ell}^\tau$ symmetric for all $\ell = 1, 2, 3, \ldots, k$. Then the reduction $w_j = x_j x_{j+1}$, $j = 0, 1, \ldots , n - 1$ of the map $M$ gives rise to a map of order $n - 1$, which is of the form (10).

**Proof.** It is enough to show that under our hypotheses the equation $x_{n-1} x_n = \prod z_i^{-1} \psi(x_2, x_3, \ldots , x_k)$ can be written in terms of the new variables $w_i$. For this, it is enough to show that, when $n$ is even and the vector $r = (r_1, r_2, \ldots , r_n)$ is symmetric, then $x^t = \prod x^t_i$ can be written in terms of the $w_i$'s. This is possible if and only if

$$\prod_{i=1}^{n} x^t_i = w_1^t w_2^t r_1 x^t_2 - r_2 + r_1 \ldots w_{n-1}^t - r_{n-2} + \ldots - r_2 + r_1$$

which is equivalent to $r_n = r_{n-1} - r_{n-2} + \ldots - r_2 + r_1$. This follows from the symmetry of the vector $r$. □

Notice that the new exponents in the previous proposition remain symmetric. Using the previous result and theorem 4 we get the following.

**Proposition 14.** For $z_1, z_2, z_3$ such that $n = z_2 + z_3 - z_1 - 2$, is odd, the reduction $w_j = x_j x_{j+1}$, $j = 0, 1, \ldots , n - 1$, of the nth order map (22) gives rise to an $n - 1$st order map which preserves a quadratic Poisson structure.

As a special case of the previous proposition we get the following.

**Corollary 15.** For each $n \in \mathbb{N}$ odd with $n \geq 3$ the reduction $w_j = x_j x_{j+1}$, $j = 0, 1, \ldots , n - 1$, of the nth order map (22) with $z_1 = 1, z_2 = 2$ and $z_3 = n + 1$ is an $n - 1$th order mapping which preserves the LVPS($t$) with $t = (1, 0)$.

**Proof.** For the proof we only have to solve the associated linear system (13). The new map in the $w$ variables is given by

$$A w_0 w_{n-1} + B w^n + C = 0$$

with $v_0 = (1)$ and $v_1 = (1, 0, 1)$ and the system (13) becomes

$$\sum_{j=i+1}^{n-i-2} (r_{i+j-1} - 1) T_j - T_{n-i-1} = 0, \quad i = 1, 2, \ldots , \frac{n-1}{2} - 1,$$

$$\sum_{j=i+1}^{n-1} T_j = 0, \quad i = 1, 2, \ldots , \frac{n-1}{2} - 1.$$

This is a linear system with $n - 3$ equations and $n - 2$ variables. Its first column is zero and its rest $n - 3 \times n - 3$ minor is invertible since there is, for each $j = 1, 2, \ldots , n - 3$, a row with exactly $n - 2 - j$ zeros. It is not difficult to show that its solution is indeed $T_1 = 1$ and $T_i = 0$ for $i = 2, 3, \ldots , n - 2$. □

The LVPS which is preserved by the previous reduction is the Kac–van Moerbeke Poisson structure (see [18]) which is of maximal rank for any $n$.

We now give some examples for the inverse problem: to find the maps given the Poisson structures. We consider a family of Poisson structures that appeared in [5]. Let us denote
with \( v_n^{(k)} = (v_1, \ldots, v_{n-1}) \) the vector with \( v_i = 1 \) for \( i = 1, 2, \ldots, n - k - 1 \) and \( v_i = -1 \) for \( i = n - k, \ldots, n - 1 \). It was shown in [5] that for any \( n \geq 3 \) and \( k \in \mathbb{N} \) with \( 2k + 1 \leq n \) the LVPS\( (v_n^{(k)}) \) is of full rank when \( n \) is even and of co-rank 1 when \( n \) is odd. We give the form of the maps that preserve the LVPS\( (v_n^{(k)}) \) for the extreme cases \( k = 0 \) and \( 2k + 1 = n \).

**Proposition 16.** For \( n \geq 3 \) the Poisson structure LVPS\( (v_n^{(k)}) \) is preserved by maps of the form (7) and (8) with

\[
F(x) = x^r \tilde{\psi}(x^s),
\]

where \( \tilde{\psi} \) is any function in \( C^1(\mathbb{R}) \) and the \( r_1, r_2 \) are given as follows.

1. For \( k = 0 \)
   \[
   r_1 = (-1, 0), \quad r_2 = (1, -1, 1), \quad \text{if } n \text{ is odd},
   \]
   \[
   r_1 = (-1, 0), \quad r_2 = (0, 1, -1, 1), \quad \text{if } n \text{ is even}.
   \]
2. For \( 2k + 1 = n \)
   \[
   r_1 = (1, 0), \quad r_2 = (1).
   \]

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