Black holes with Lagrange multiplier and potential in mimetic-like gravitational theory: multi-horizon black holes

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Abstract. In this paper, we consider the mimetic-like field equations coupled with the Lagrange multiplier and the potential to derive non-trivial spherically symmetric black hole (BH) solutions. We divided this study into three cases: in the first one, we choose the Lagrange multiplier and the potential to vanish and derive a BH solution that coincides with the BH of the Einstein general relativity despite the non-vanishing value of the mimetic-like scalar field. The first case is consistent with the previous studies in the literature where the mimetic theory coincides with GR [1]. In the second case, we derive a solution with a constant value of the potential and a dynamical value of the Lagrange multiplier. This solution has no horizon, and therefore, the obtained space-time does not correspond to the BH. In this solution, there appears a region of the Euclidian signature where the signature of the diagonal components of the metric is \((+,+,+,+\)) or the region with two times where the signature is \((+,+,−,−)\). Finally, we derive a BH solution with non-vanishing values of the Lagrange multiplier, potential, and mimetic-like scalar field. This BH shows a soft singularity compared with the Einstein BH solution. The relevant physics of the third case is discussed by showing their behavior of the metric potential at infinity, calculating their energy conditions, and studying their thermodynamical quantities. We give a brief discussion on how our third case can generate a BH with three horizons as in the de Sitter-Reissner-Nordström black hole space-time, where the largest horizon is the cosmological one and two correspond
to the outer and inner horizons of the BH. Even in the third case, the region of the Euclidian signature or the region with two times appears. We give a condition that such unphysical region(s) is hidden inside the black hole horizon and the existence of the region(s) becomes less unphysical. We also study the thermodynamics of the multi-horizon BH and consider the extremal case, where the radii of two horizons coincide with each other. We observe that the Hawking temperature and the heat capacity vanish in the extremal limit. Finally, we would like to stress the fact that in spite that the field equations we use have no cosmological constant, our BH solutions of the second and third case behave asymptotically as AdS/dS.

**Keywords:** Exact solutions, black holes and black hole thermodynamics in GR and beyond, gravity, modified gravity

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1 Introduction

The key of diffeomorphism invariance in the Einstein general relativity is to stimulate the redundancy in the presentation of the dynamical degrees of freedom in exchange for improving the formulation’s simplicity and elegance. The metric $g_{\alpha\beta}$ that has ten components are employed to describe two dynamical degrees of freedom for the graviton field. Thus, it is natural to search for an amended gravitational theory without enlarging the degrees of freedom of the gravitational system when keeping the diffeomorphism invariance. A few years ago, this amended gravitational theory was constructed in [2], using the idea of rolling the dynamical metric $g_{\alpha\beta}$ which depends on an auxiliary metric $\bar{g}_{\alpha\beta}$.

Mimetic gravitational theory is considered one of the most attractive theories of gravity. Without inserting any extra matter field, the theory represents the dark piece of the universe that is given by a geometrical effect [2]. In the mimetic theory, the conformal degree of freedom of gravitational field is isolated by inserting the relation between the physical metric $g_{\alpha\beta}$, the auxiliary metric $\bar{g}_{\alpha\beta}$ by using a mimetic field which is the scalar field, as follows,

$$g_{\alpha\beta} = \mp \left( \bar{g}^{\mu\nu} \partial_\mu \zeta \partial_\nu \zeta \right) \bar{g}_{\alpha\beta}. \quad (1.1)$$

Here $\bar{g}^{\mu\nu}$ is the inverse of $\bar{g}_{\mu\nu}$. Equation (1.1) implies that the mimetic field yields

$$g^{\alpha\beta} \partial_\alpha \zeta \partial_\beta \zeta = \mp 1. \quad (1.2)$$

Therefore, $\partial_\alpha \zeta$ is timelike and spacelike (the signature of $g_{\alpha\beta}$ is chosen in this study as $(g_{\mu\nu}) = \text{diag}(+,-,-,-)$) when we consider the positive and negative signs in (1.1) or (1.2), respectively.

The field equations of the gravitational action are equal to the one which can derive from the action written in terms of the physical metric with the prescription of the restriction (1.2), through the use of the Lagrange multiplier [2]. The conformal degree of freedom is dynamical quantity even in the absence of matter, and this imitates the situation of cold dark matter evolution of our universe in the background. Moreover, it has been explained that the scalar field can imitate the gravitational behavior of any configuration of matter [3, 4]. Thereafter, the mimetic model was expanded to the studies of inflation, theories with non-singular cosmological, dark energy, and black hole solutions [1, 3, 5–46]. Moreover, it was argued that in the four-dimensional Einstein-Maxwell theory and for asymptotically AdS space-time, there
exist BH solutions whose event horizons could have zero or negative constant curvature and therefore, their topologies are no longer the two-sphere $S^2$ \cite{47–54}. Later, many modifications of mimetic gravity have been established, $f(R)$ mimetic gravity, mimetic gravity with the Lagrange multipliers \cite{55, 56}. Moreover, the presence of potential and the Lagrange multiplier supported the possibility for recognizing different cosmologies, \cite{57}. By including the potential of the scalar field corresponding to the mimetic field, the model loses the original physical meaning as the dark matter but the potential gives effectively time-dependent vacuum energy and the potential can take the role of the dark matter and/or dark energy in a unified way.

It has been also proved that the original setting of the mimetic theory forecasts that gravitational wave (GW) propagates at the speed of light, which agrees with the results of the event GW170817 and its optical counterpart \cite{46, 58}. The mimetic gravity is also generalized to the $f(R)$ mimetic gravity \cite{35, 42, 45, 57, 59–71} and the Gauss-Bonnet mimetic gravity \cite{36, 72–74}. Especially, a unified formalism of early inflation and late-time acceleration in the frame of the mimetic $f(R)$ gravity was also constructed in \cite{43}, where the authors confirmed the inflationary era in contrast to the $f(R)$ gravity.

An exact solution in GR theory and its modifications could be considered an important tool. For example, the thermodynamics and dynamics of the gravitational model are often investigated by using exact solutions. Moreover, introducing a new technique to find the exact solutions is also an important and natural way to construct modified gravity \cite{75}. It is the aim of this study we are going to consider mimetic-like gravity theory, in which we will fix the Lagrange multiplier, the potential of the mimetic-like scalar field, and the unknown of the metric potentials. The structure of this research is as follows: in section 2, we give the cornerstone of the mimetic-like gravity coupled with potential and the Lagrange multiplier. Also in section 2, we apply the field equations of mimetic theory coupled to the Lagrange multiplier and potential to a spherically symmetric space-time that has two unknown functions and derives the non-linear differential equations which have five unknown functions. We solve these non-linear differential equations under three different constraints related to the potential and the Lagrange multiplier and derive BH solutions that are different from the BH in GR. The invariants of these BHs are calculated in section 2 and we show that the singularity of our BHs is much milder compared with GR BHs. In the BH solutions, we find, there is a case that space-time has three horizons which may be the cosmological horizon and the inner and outer BH horizons. Furthermore, in space-time, there appears the region of the Euclidian signature where the signature of the diagonal components of the metric are $(+, +, +, +)$ or the region with two times where the signature is $(+, +, -, -)$. We find a condition that the region(s) is hidden inside the black hole horizon and the existence of the region(s) becomes less unphysical. We also study the thermodynamics of the multi-horizon BH. In section 3, we study the energy conditions of these BH and show that when the potential has a constant value, the strong energy condition is not satisfied while when the potential and the Lagrange multiplier have non-trivial forms, all the energy conditions are satisfied. In section 4, we study the thermodynamics of the different BHs derived in this study and show that all the thermodynamics of the BHs are satisfied and consistent with the results presented in the literature \cite{29, 76–78}. We summarized the results of this study in the final section 5.

2 Spherically symmetric BH solutions in mimetic-like gravity coupled with potential and Lagrange multiplier

The expression of “mimetic dark matter” was presented in the literature by Mukhanov and Chamseddine \cite{3}, in spite that such theories had already been discussed in \cite{4, 11, 56, 79}.
In this study, we are going to present the mimetic-like gravity coupled with potential and the Lagrange multiplier. The gravitational action of the mimetic-like gravity coupled with the Lagrange multiplier $\lambda$ and the potential $V(\zeta)$ takes the following form [35],

$$S = \int dx^4 \sqrt{-g} \left\{ R(g_{\mu\nu}) - V(\zeta) + \lambda \left( g^{\mu\nu} \partial_\mu \zeta \partial_\nu \zeta + 1 \right) \right\} + L_{\text{matt}}, \quad (2.1)$$

with $L_{\text{matt}}$ being the Lagrangian of the matter fluid present. In this paper, we choose the gravitational coupling and the light speed to be unity. Varying the action (2.1) w.r.t. the metric tensor $g_{\mu\nu}$, we get the following field equations,

$$0 = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \frac{1}{2} g_{\mu\nu} \left\{ \lambda \left( g^{\rho\sigma} \partial_\rho \zeta \partial_\sigma \zeta + 1 \right) - V(\zeta) \right\} - \lambda \partial_\mu \zeta \partial_\nu \zeta + \frac{1}{2} T_{\mu\nu}, \quad (2.2)$$

where $T_{\mu\nu}$ is the energy momentum tensor corresponding to the matters. In this study, we put $T_{\mu\nu} = 0$. Additionally, varying the action w.r.t. the mimetic scalar field $\zeta$, we get

$$2 \nabla^\mu (\lambda \partial_\mu \zeta) + V'(\zeta) = 0, \quad (2.3)$$

where the “prime” refers to the differentiation w.r.t. the mimetic scalar $\zeta$. Finally, varying the action (2.1) w.r.t. the Lagrange multiplier $\lambda$, we get

$$g^{\mu\nu} \partial_\mu \zeta \partial_\nu \zeta = -1, \quad (2.4)$$

and as it can be seen, the above equation is identical to eq. (1.2).

Now, we are going to apply the field equations of the mimetic theory, (2.2), and (2.3) to a spherically symmetric space-time that has two unknown functions $h(r)$ and $h_1(r)$ as follows,

$$ds^2 = h(r)dt^2 - \frac{dr^2}{h_1(r)} - r^2 \left( d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right). \quad (2.5)$$

We will determine $h(r)$ and $h_1(r)$ by using the field equations. We also assume that the mimetic scalar field only depends on the radial coordinate $r$. Applying the field equations (2.2) and (2.3) to the space-time in (2.5), we obtain the $(t, t)$-component of the field equation (2.2),

$$0 = -2h - 2h_1 - 2rh' + Vr^2, \quad (2.6)$$

the $(r, r)$-component of the field equation (2.2),

$$0 = 2h - 2h_1 h'r - 2h_1 h - 2\lambda \zeta'^2 h_1 h r^2 + V hr^2, \quad (2.7)$$

both of the $(\theta, \theta)$ and $(\phi, \phi)$-components of the field equation (2.2),

$$0 = h'^2 h_1 r - 2h_1 h'h - 2h_1 h'^2 - h'h_1 hr - 2h''h_1 hr + 2Vh^2r, \quad (2.8)$$

and the field equation corresponding to (2.3),

$$0 = \lambda \zeta'^2 h_1 h'r - V' hr + 2\lambda \zeta'^2 h_1 hr + 2\lambda \zeta''h_1 \zeta'h + \lambda \zeta'^2 h_1 h + 4\lambda \zeta''h_1 h, \quad (2.9)$$

where $h \equiv h(r), h_1 \equiv h_1(r), V \equiv V(r), \lambda \equiv \lambda(r), h' = \frac{dh}{dr}, h'_1 = \frac{dh_1}{dr}, V' = \frac{dV}{dr}, \zeta' = \frac{d\zeta}{dr}$, and $\lambda' = \frac{d\lambda}{dr}$. We will solve the aforementioned nonlinear differential equations, (2.6), (2.7), (2.8), and (2.9) for the following three different cases:
Case I: $V(r) = \lambda(r) = 0$. When $V(r) = \lambda(r) = 0$, the analytic solution of the nonlinear differential equations, (2.8) and (2.9) takes the following forms

$$
\begin{align*}
    h(r) &= h_1(r) = 1 + \frac{2C}{r} = 1 - \frac{2M}{r}, \\
    \zeta(r) &= \sqrt{r^2 - 2rM + M \ln \left( r - M + \sqrt{r^2 - 2M} \right)}.
\end{align*}
$$

Equation (2.10) shows that when the potential and the Lagrange multiplier are vanishing, we return to the well-known BH of GR, i.e., the Schwarzschild solution. It might be interesting to note that the mimetic-like scalar field of eq. (2.10) satisfies the constraints (1.2). The relevant physics of the Schwarzschild space-time is well studied in the literature.

Case II: $V(r) = \text{const} = c_1$. Before discussing this case, let us remind of the result presented in [80], where the authors showed that for a mimetic theory, if the mimetic field is not trivial, i.e., it is a dynamical quantity, and if eq. (1.2) is valid, then the theory may provide a spherically symmetric solution that differs from the Einstein GR. Moreover, in [81], the authors showed that when the Einstein tensor is not vanishing, a new class of spherically symmetric solutions which are different from the Schwarzschild one can emerge from the mimetic theory. Now we are ready to see if the studies presented in [80, 81] are consistent with case II.

Using the constrains $V(r) = \text{const} = c_1$ in eqs. (2.8) and (2.9), we get

$$
\begin{align*}
    h_1(r) &= 1 + \frac{c_1}{6} r^2 - \frac{M}{r} = 1 + \Lambda r^2 - \frac{M}{r}, \\
    \lambda(r) &= \frac{1}{(c_2 - \int \frac{3\sqrt{r}}{(6r + c_1 r^3 - 6M)^{3/2}} dr)} \ln r + \frac{1}{4\Lambda^{3/2} r^2} - \frac{M}{6\Lambda^{3/2} r^3} - \frac{3}{32\Lambda^{5/2} r^4}, \\
    \zeta(r) &= \int \frac{dr}{\sqrt{1 + \Lambda r^2 - \frac{M}{r}}} \approx \ln r + \frac{1}{4\Lambda^{3/2} r^2} - \frac{M}{6\Lambda^{3/2} r^3} - \frac{3}{32\Lambda^{5/2} r^4},
\end{align*}
$$

where $\Lambda = \frac{c_2}{6}$. By the notation $\approx$, we show the asymptotic behavior when $r$ is large enough. We choose the constants of the integration to give the asymptotic behavior after $\approx$ for $h_1$.

Note that $h(r)$ can be rewritten as

$$
\begin{align*}
    h(r) &= h_1(r)e^{\int \frac{2d\rho}{\rho''(h_1(\rho))^{3/2}}} \\
    &= h_1(r)e^{\int \frac{2d\rho}{\rho''(h_1(\rho))^{3/2}}},
\end{align*}
$$

For the solution $h_1(r) = 0$

1. When $c_1 > 0$, only one real and positive solution.
2. When $c_1 < 0$,
There appears the region of Euclidian signature where the signature of the diagonal components asymptotically behaves as AdS/dS, and also the potential asymptotes a constant value as it comes from the constant of integration. Therefore \( h(r) \) does not vanish and therefore there is no horizon, which tells that the space-time given in (2.11) is not a black hole. It might be, however, interesting that there is no horizon, which tells that the space-time given in (2.11) is not a black hole. It might be, however, interesting that \( h_1(r) \) has zero(s), and therefore the signature of \( h_1(r) \) changes. There appears the region of Euclidian signature where the signature of the diagonal components of the metric is \((+, +, +, +)\) or the region with two times where the signature is \((+, +, -, -)\).

**Case III: \( \lambda(r) \neq 0 \) and \( V(r) \neq 0. \)** There are many solutions of the field equations (2.8) and (2.9). We will discuss, however, only the physical solution which has a metric potential \( h \) asymptotically behaves as AdS/dS, and also the potential asymptotes a constant value as it should be for any physical model. The solution which we will discuss has the following form,

\[
\begin{align*}
\lambda(r) &= -\frac{1}{5c-2r^3+6Mr^2}, \\
V(r) &= \frac{1}{2(5c-2r^3+6Mr^2)^2} \left[ 96c^4 r^{11} - 864c^{10}M c_4 + 2592c^{9} r^2 M^2 - 2c_5 r^9 c_4 \\
&+ 18c_4^3 r^5 M c_4 - 2592c_4^3 r^2 M^3 + 2c_5 r^7 + 2880c^2 c_4 r^5 M - 5040 M^2 r^8 c_4^2 \\
&+ 35c_4^2 c_5 r^6 c_4 + 2560 c_4^2 r^6 M - 2400 c_4^3 r^5 + 2784 c_4^2 r^5 M + 17c_5 r^4 c_4 - 5472 M^2 r^4 c_4 \\
&- 1080 c_4^3 c_2 - 28c_5 r^3 M c_4 + 4032 c_4 M^3 r^3 + 1320 c_4^2 M^2 r^2 \\
&- 10c_5 r c^4 + 1440 M c^2 c_4^2 + 900 c^3 \right] \\
&\approx 6c_4 - \frac{c_4 c_5}{8r^2}, \\
\lambda(r) &= \frac{1}{5c-2r^3+6Mr^2} \left[ 5c_4^2 c_5 r^6 c_4^2 - 5c_5 r c_4^2 + c_5 r^9 c_4 + 4c_4 r^4 c_4 - 2c_5 r^6 M + c_5 r^7 \\
&- 10c_5 r c_4 - 240 c_4^2 r^2 - 720 M^2 r^2 c_4^2 - 300 c_4^3 r^5 + 1200 c_4 r^5 M - 240 r^6 c_4 \\
&- 2160 M^2 r^2 c_4^2 + 720 c_4^2 r^5 M - 60 r^3 c_4^2 + 1440 c_4 M^3 r^3 - 120 c_4^2 M^2 r^2 \\
&+ 720 r^2 M^2 c_4^2 + 300 c_4^3 \right] \\
&\approx \frac{c_4 c_5}{8r^2} + \frac{c_4 (9M c_4 - 240 c_4)}{8r^3} + \frac{52 c_4 c_5 M^2 - 720 c_4^2 M + c_5}{8r^4},
\end{align*}
\]

(2.14)
where $c_4$, $c_5$ are constants of integration and $c_6$ is defined as

$$c_6 = -\frac{27 M^2 c_4 c_5}{16} - \frac{c_5}{16} + 45 c_4^2 M.$$  

(2.15)

Equation (2.14) shows that when $c_4 = c_5 = 0$, we return to the well-known Schwarzschild BH of GR. Using eq. (2.14), we get the mimetic-like scalar field in the form

$$\zeta(r) = \int \frac{2}{\sqrt{-\left[(c_4 r^3 - 2Mr^2 - c_4)(-16r^3 + 96Mr^2 + c_5 r - 144M^2 r - 80 c_4)\right]}} dr \\ \approx \ln \left(\frac{r}{\sqrt{c_4}}\right) - \frac{c_5}{64 \sqrt{c_4} r^2} - \frac{M}{3 c_4 \frac{1}{2} r^3} + \frac{5 \sqrt{c_4}}{3 r^3} - \frac{c_5 M}{16 \sqrt{c_4} r^3}. \quad (2.16)$$

Using eq. (2.16), we can write the radial coordinate in terms of the mimetic-like scalar field $\zeta$ as

$$r = e^{\frac{1}{2} W\left(\frac{(c_5 c_4 - 16) e^{-2 \sqrt{c_4}}}{2 c_4}\right)} + \zeta \sqrt{c_4}. \quad (2.17)$$

Here $W(x)$ is the Lambert function defined by $x = W(x)e^{W(x)}$. Using eq. (2.17), in the asymptotic form of the potential given by eq. (2.14), we get

$$V(\zeta) = \frac{1}{16 \sqrt{\text{Y}}} \{2c_5 c_4 \text{Y}^3 + 6c_4 (3M c_5 - 80 c_4) \text{Y}^2 + 2 (c_5 - 18M c_4 [80 c_4 + 3M c_5]) \text{Y} + 25 (c_5 - 576 M^2) c_4^2 + 60 (9c_3 M^3 - 8) c_4 + 14M c_3 \} \approx V_0 + V_1 \zeta + V_2 \zeta^2, \quad (2.18)$$

where $\text{Y} = e^{\frac{1}{2} W\left(\frac{(c_5 c_4 - 16) e^{-2 \sqrt{c_4}}}{2 c_4}\right)} + \zeta \sqrt{c_4}$, $V_0$, $V_1$, and $V_2$ are constants given by the constants $c_4$, $c_5$, and $M$. Using eq. (2.14) in (2.5), when $r$ is large enough, we get the line element in the following form,

$$ds^2 \approx \left(1 + c_4 r^2 - \frac{2 M}{r} - \frac{c_4}{r^3}\right) dt^2 - \frac{dr^2}{1 + c_4 r^2 - \frac{2 c_5}{16} - \frac{2 M}{r} + \frac{c_5}{r^2}} - r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (2.19)$$

As we will see soon, the parameter $c_4$ must be negative to avoid the singularity. Therefore eq. (2.19) shows that our black hole asymptotes to dS.

In the solution (2.14), there appear many constants of integration $c_4$, $c_5$, and $M$. Generally speaking, some combinations of the constants could fix the parameters specifying the model and could be included in the potential $V(\zeta)$ in eq. (2.18). However, it is a shortcoming of our formulation that it is difficult to combine the constants into the parameters specifying the model under consideration. Despite this difficulty, eq. (2.14) is indeed a solution of a specific model.
Using eq. (2.5), we get the curvature invariants as:

\[
R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{1}{4h^2r^2} \left\{ 4r^4h''^2h_1^2 + 4r^4hh_1h' \left( hh_1' - h_1h' \right) h'' + r^4h'^4h_1^2 - 2r^4h'^3h_1h'h \right. \\
+ r^2h^2 \left( h_1^2r^2 + 8h_1^2 \right) h'' + 8h^4 \left( h_1^2r^2 + 2 \left( h_1^3 - 1 \right) \right) \left. \right\}, \\
R_{\mu\nu}R^{\mu\nu} = \frac{1}{8h^2r^2} \left\{ 4r^4h''^2h_1^2 + 4h \left[ h \left( h_1^3 - 2 \right) h' - rh''h_1 + 2h_1h' \right] h_1^2h'' + r^4h'^4h_1^2 \\
+ r^2h^2 \left( 12h_1^2 + h_1^2r^2 \right) h'' - 2r^3h_1 \left( h_1^3 + 2h_1 \right) h' \\
+ 4rh^3 \left( 2h_1h_1 - 4h_1 + 4h_1^2 + h_1^2r^2 \right) h' \\
+ 4h^4 \left( 3h_1^2r^2 + 4r \left( h_1 - 1 \right) h_1 + 4 \left( h_1 - 1 \right) \right) \right\}, \\
R = \frac{2h''h_1h_1^2 - h''^2h_1r^2 + rh \left( 4h_1 + rh_1 \right) h' + 4h^4 \left( rh_1 + h_1 - 1 \right)}{2r^2h^2}. \tag{2.20}
\]

Here \( R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \), \( R_{\mu\nu}R^{\mu\nu} \), and \( R \) represent the Kretschmann scalar, the Ricci tensor square, and the Ricci scalar, respectively.

For the solution in (2.14), when \( r \) is large, we find

\[
R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \approx 24c_1^2 - \frac{3c_4^2c_5}{2r^2} + \frac{3c_4^2(160c_4 + 3c_5M)}{4r^3} + \frac{c_4M(3c_4M + 32)}{64r^4}, \\
R_{\mu\nu}R^{\mu\nu} \approx 36c_1^2 - \frac{9c_4^2c_5}{4r^2} + \frac{9c_4^2(160c_4 + 3c_5M)}{8r^3} + \frac{3c_4M(16 + c_4M)}{64r^4}, \\
R \approx 12c_1 - \frac{3c_4c_5}{8r^2} + \frac{3c_4(160c_4 + 3c_5M)}{16r^3} + \frac{c_5}{8r^4}. \tag{2.21}
\]

On the other hand, when \( r \) is small, we find

\[
R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \sim R_{\mu\nu}R^{\mu\nu} \sim R \sim O \left( r^{-3} \right). \tag{2.22}
\]

Equation (2.22) shows that the BH solution (2.14) has a softer singularity at \( r = 0 \) compared with the BH in general relativity, where \( \left( R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}, R_{\mu\nu}R^{\mu\nu}, R \right) = \left( \frac{24(\Lambda^2r^6 + 2M^2)}{r^6}, 36\Lambda^2, 12\Lambda \right) \).

The curvature invariants in (2.20) might diverge when \( b(r) = 0 \) because the expressions in (2.20) include the inverse power of \( h(r) \). We should note, however, that \( h_1(r) \) given by eq. (2.14) can be rewritten as follows,

\[
h_1(r) = \frac{r^3h(r) \left( h \left( 4r - 12M \right)^2 - c_5r + 80c_4 \right)}{4 \left( 5c_4 - 2r^3 + 6Mr^2 \right)^2}. \tag{2.23}
\]

Therefore when \( h(r) \) vanishes, \( h_1(r) \) also vanishes. As we will show, this tells that the zeros of \( h_1(r) \) cancels the zeros of \( h(r) \) in the curvature invariants and the curvature invariants (2.20) become finite and do not diverge. We now consider the general case that \( h_1 \) is written by \( h_1 = h_2h \) as in (2.23). Here \( h_2 \) does not vanish and is regular at the zeros \( h(r) = 0 \) of \( h(r) \).
By substitute the expression \( h_1 = h_2 h \) into (2.20), we find

\[
R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = h''^2 h_2^2 + h_2 h'_2 h + \frac{h'^2 h_2^2}{4} + \frac{4h_2 h'^2}{r^2} + \frac{2 \left( (h'_2 h + h_2 h')^2 r^2 + 2 (h_2 h^2 - 1)^2 \right)}{r^4}, \quad (2.24)
\]

\[
R_{\mu\nu}R^{\mu\nu} = \frac{h''^2 h_2^2}{2} + \frac{h'' h'_2 h_2^2}{2} + \frac{h'_2 h'' h_2^2}{r^2} + \frac{h'(h'_2 h + h_2 h') h_2 h}{r^2} + \frac{3h_2^2 h_2^2}{2r^2} + \frac{h'^2 h_2^2}{8} + \frac{h'' h'_2 h_2^2}{r^2} + \frac{3(h'_2 h + h_2 h') r^2 + 4r (h_2 h - 1)(h'_2 h + h_2 h') + 4 (h_2 h - 1)^2}{2r^4}, \quad (2.25)
\]

\[
R = 2h'' h_2 + \frac{2r h_2 h'}{r} + \frac{h'_2 h'}{2} + \frac{2 (r h'_2 h + r h_2 h' + h_1)}{r^2}. \quad (2.26)
\]

Therefore the invariants are surely finite at the zeros \( h(r) = 0 \).

We should also note that the denominator in the expression (2.23) can vanish in general. If the denominator vanishes, \( h_1(r) \) diverges and therefore the curvature invariants (2.20) diverge and a singularity appears. The singularity can be avoided if we choose the parameter \( c_4 \) satisfy the following condition,

\[
5c_4 + 8M^3 < 0, \quad (2.27)
\]

because the maximum of \(-2r^3 + 6Mr^2\) is \(8M^3\) when \( r = 2M \).

It might be also interesting that the factor \( r(4r - 12M)^2 - c_5 r + 80c_4 \) can vanish and it generates an extra zero(s) besides the zeros of \( h(r) \). The existence of the zero tells that there appears the region of Euclidian signature where the signature of the diagonal components of the metric are \((+,+),+(+)\) or the region with two times where the signature is \((+,+,-,-)\). Such regions might be interesting but could not be physically acceptable. Let a radius of the black hole horizon(s) be \( r_{BH} \). If we choose the parameter \( c_5 \) to satisfy

\[
-c_5 r_{BH} + 80c_4 > 0, \quad (2.28)
\]

as long as \( r > r_{BH} \), the factor \( r(4r - 12M)^2 - c_5 r + 80c_4 \) is positive and does not vanish and therefore the unphysical region(s) is hidden inside the black hole horizon and the existence of the region(s) becomes less unphysical.

Because the equation \( h(r) = 0 \) has three real positive solutions, we may write \( h(r) \) in the following form,

\[
h(r) = 1 + c_4 r^2 - \frac{2M}{r} - \frac{c_4}{r^3} = h_3(r)(r - r_1)(r - r_2)(r - r_3), \quad (2.29)
\]

where we assume \( r_1 < r_2 < r_3 \). We plot eq. (2.29) for specific values of the mass, the constants \( c_4 \), and \( c_5 \) in figure 1.

The behavior of the Lagrange multiplier, the potential, and the mimetic-like scalar field are drawn in figure 2(a), figure 2(b) and figure 2(c), where they have positive value.

### 3 Energy conditions

Energy conditions provide important tools to examine and better understand cosmological models and/or strong gravitational fields. We are interested in the study of energy conditions
Figure 1. Multi-horizon plot of eq. (2.29) against coordinate $r$ for the BH (2.14).

Figure 2. (a) Behavior of the Lagrange multiplier of eq. (2.14); (b) the potential of eq. (2.14); and (c) the mimetic-like scalar field of eq. (2.14).

in the mimetic theory coupled with the Lagrange multiplier and potential because this is the first time to derive non-trivial BH solutions in Case III given by (2.14). The energy conditions are classified into four categories: strong energy (SEC), weak energy (WEC), null energy (NEC), and dominant energy (DEC) conditions [82, 83]. To fulfill these conditions, the following inequalities should be verified:

\[
\begin{align*}
\text{SEC:} & \quad \rho + p_r \geq 0, \quad \rho + p_t \geq 0, \quad \rho - p_r - 2p_t \geq 0, \\
\text{WEC:} & \quad \rho \geq 0, \quad \rho + p_r \geq 0, \\
\text{NEC:} & \quad \rho \geq 0, \quad \rho + p_t \geq 0, \\
\text{DEC:} & \quad \rho \geq |p_r|, \quad \rho \geq |p_t|,
\end{align*}
\] (3.1)
where the energy-momentum tensor $T^\zeta_{\mu\nu}$ for the mimetic-like scalar field $\zeta$ and the Lagrange multiplier field $\lambda$ is defined as

$$T^\zeta_{\mu\nu} = \frac{1}{2} g_{\mu\nu} V(\zeta) + \lambda \partial_\mu \zeta \partial_\nu \zeta,$$  \hspace{1cm} (3.2)

with $T^\zeta_0^0 = \rho$, $T^\zeta_1^1 = -p_r$ and $T^\zeta_2^2 = T^\zeta_3^3 = -p_t$ are the density, radial pressure, and tangential pressure, respectively. Note that the constraint (1.2) is used in eq. (3.2). The form of the components of the energy-momentum tensor have the form,

$$\rho = \frac{V}{2} = \frac{c_1}{2}, \quad p_r = \frac{V}{2} - \lambda(r) h_1(r) \zeta', \quad p_t = \frac{V}{2} = \frac{c_1}{2}. \hspace{1cm} (3.3)$$

Straightforward calculations of the BH solution (2.14) give

**SEC:** \hspace{1cm} $\rho + p_r = \frac{(c_5 r^4 - 240 c_4 r^3 + 720 r^2 M c_4 + 5 c_5 r - 300 c_4^2) (c_4 - c_4 r^5 - r^3 + 2 M r^2)}{(5 c_4 - 2 r^3 + 6 M r^2)^3 r^2} > 0,$

$$\rho - p_r - 2 p_t = \frac{1}{(5 c_4 - 2 r^3 + 6 M r^2)^3 r^2} \left[ 96 r^{10} c_4 (9 M - r) + 3 (c_5 - 864 M^2) c_4 r^9 + 6 \left[ 80 c_4^2 + 3 M (144 M^2 - c_5) c_4 \right] r^8 - (2160 c_4^2 M + c_5) r^7 + 3 c_4 \left[ 10 (144 M^2 - c_5) c_4 + 112 \right] r^6 + 12 c_4 \left[ 175 c_4^2 - 132 M^2 \right] r^5 + c_4 \left( 3312 M^2 - 13 c_5 \right) r^4 + 6 \left[ 170 c_4^2 + 3 \left( c_5 M - 144 M^3 \right) c_4 \right] r^3 - 144 c_4^2 M r^2 - 5 \left( 144 M^2 - c_5 \right) c_4^2 r - 600 c_4^3 \right] > 0,$$

**WEC:** \hspace{1cm} $\rho = \frac{1}{4 (5 c_4 - 2 r^3 + 6 M r^2)^3 r^2} \left[ 96 r^{10} c_4 (9 M - 1) + (2 c_5 - 2592 M^2) c_4 r^9 + 720 c_4^2 + 18 M \left( 144 M^2 - c_5 \right) c_4 \right] r^8 - 2 \left( 1440 c_4^2 M + c_5 \right) r^7 + 35 \left( 144 M^2 - c_5 \right) c_4^2 + 576 c_4^2 + 2 c_5 M \right] r^6 + \left( 2400 c_4^3 - 2784 c_4 M \right) r^5 + c_4 \left( 5472 M^2 - 17 c_5 \right) r^4 + \left[ 1080 c_4^2 + 28 \left( c_5 M - 144 M^3 \right) c_4 \right] r^3 - 1320 c_4^2 M r^2 + 10 \left( c_5 - 144 M^2 \right) c_4^2 r - 900 c_4^3 \right] > 0,$

$$\rho + p_r = \frac{(c_5 r^4 - 240 c_4 r^3 + 720 r^2 M c_4 + 5 c_5 r - 300 c_4^2) (c_4 - c_4 r^5 - r^3 + 2 M r^2)}{(5 c_4 - 2 r^3 + 6 M r^2)^3 r^2} > 0,$$

**NEC:** \hspace{1cm} $\rho = \frac{1}{4 (5 c_4 - 2 r^3 + 6 M r^2)^3 r^2} \left[ 96 r^{10} c_4 (9 M - 1) + (2 c_5 - 2592 M^2) c_4 r^9 + 720 c_4^2 + 18 M \left( 144 M^2 - c_5 \right) c_4 \right] r^8 - 2 \left( 1440 c_4^2 M + c_5 \right) r^7 + 35 \left( 144 M^2 - c_5 \right) c_4^2 + 576 c_4^2 + 2 c_5 M \right] r^6 + \left( 2400 c_4^3 - 2784 c_4 M \right) r^5 + c_4 \left( 5472 M^2 - 17 c_5 \right) r^4 + \left[ 1080 c_4^2 + 28 \left( c_5 M - 144 M^3 \right) c_4 \right] r^3 - 1320 c_4^2 M r^2 + 10 \left( c_5 - 144 M^2 \right) c_4^2 r - 900 c_4^3 \right] > 0,$

$$\rho + p_t = 0,$$

**DEC:** \hspace{1cm} $\rho \geq |p_r| \quad \text{(satisfied)}, \quad \rho \geq |p_t| \quad \text{(satisfied)}. \hspace{1cm} (3.4)$

Figures 3(a)–3(d) shows the energy conditions of the BH solution (2.14) and we find that the solution (2.14) satisfies all the conditions in eq. (3.1).
Figure 3. Schematic plots of WEC, SEC, NEC and DEC given by eq. (2.14). All the above figures are plotted for \( M = 0.93, c_4 = -10^7 \) and \( c_5 = 100.1 \).

4 Thermodynamics and stability

We consider another approach in physics to deeply elucidate the two BHs in (2.14) by investigating their thermodynamical behaviors. Accordingly, we present the main tools of the thermodynamic quantities.

The entropy is given by [84, 85]

\[
S(r_+) = \frac{1}{4}A = \pi r_+^2,
\]

(4.1)

where \( A \) is the area of the horizon and the horizon radius \( r_+ \) satisfies the condition \( h(r_+) = 0 \).

The total mass contained within the event horizon \( (r_+) \) can be obtained by solving the equation \( h(r_+) = 0 \) with respect to \( M \),

\[
M(r_+) = \frac{c_4 r_+^5 + r_+^3 - c_4}{2r_+^2}.
\]

(4.2)

The behavior of eq. (4.2) is drawn in figure 4(a).

The Hawking temperature is generally defined as follows [86–89]

\[
T_+ = \frac{h'(r_+)}{4\pi},
\]

(4.3)

where we may assume \( h'(r_+) \neq 0 \). Using eq. (4.3), the Hawking temperature of BH (2.14) can be calculated as:

\[
T_+ = \frac{2c_4 r_+^5 + 2M(r_+) r_+^2 + 3c_4}{4\pi r_+^4} = \frac{3c_4 r_+^5 + r_+^3 + 2c_4}{4\pi r_+^4}.
\]

(4.4)

The behavior of the Hawking temperature given by eq. (4.4) is displayed in figure 4(b). There appears the critical point, which corresponds to the extremal limit where the radii of the two horizons coincide with each other.

The stability of the BH solution is an essential topic that can be studied at the dynamic and perturbative levels [90–92]. To investigate the thermodynamic stability of BHs, the
Figure 4. Schematic plots of the thermodynamic quantities of the BH solution (2.14) for a negative values of the constant $c_4$: (a), the mass-radius relation of the horizon (4.2); (b), typical behavior of the temperature of horizon (4.4) for $c_4 = -0.001$. There appears the critical point $r_+ \sim 60$, which corresponds to the extremal limit. And (c) the heat capacity, (4.6), which vanishes at the extremal limit $r_+ \sim 60$. indicates that we obtain a negative heat capacity only when $r_+$ is smaller than the critical point $r_+ \sim 60$.

The horizon heat capacity for $c_4 = -0.001$.
for the cold dark matter. Thus, it is serious to test the mimetic gravitational theory in the astrophysics frame, by investigating possible new BH solutions by including the Lagrange multiplier and potential terms.

To achieve this study, we gave the field equations of the mimetic gravitational theory coupled with the potential, \( V(r) \) and Lagrange multiplier, \( \lambda(r) \), and applied them to a four-dimensional spherically symmetric space-time having two unknown functions, \( h(r) \) and \( h_1(r) \), of the radial coordinate \( r \). We divided the study of the resulting non-linear differential equation into three cases: Case I: \( V(r) = \lambda(r) = 0 \), Case II: \( V(r) = \text{constant} \) and \( \lambda(r) \neq 0 \), and Case III: \( V(r) \) and \( \lambda(r) \) are not vanishing. We show that the first case is not different from the solution in the Einstein GR and is consistent with the studies presented in the literature [1]. The solution of the second case does not describe BH because there is no horizon. Interestingly, however, the space-time, in this case, includes the region of the Euclidian signature or the region with two times even the solution could not be physical. In the third case, we derived non-trivial forms of the potential, Lagrange multiplier, and the metric components. The BH of this case is different from GR, and the main source of this difference comes from the non-trivial forms of the potential, and the Lagrange multiplier. The asymptotic form of this BH is AdS/dS and the source of this AdS/dS comes from the potential whose asymptote behaves like a cosmological constant. In the third case, space-time has three horizons in general.

We calculated the curvature invariants, i.e., the Kretschmann \( K = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} \), the Ricci tensor squared \( R_{\alpha\beta}R^{\alpha\beta} \), and the Ricci scalar \( R \), to investigate the possible singularities of the BH solutions derived from the third case and showed that all the BH solutions had true singularities at \( r = 0 \). We explained that mimetic-like gravity coupled with potential and the Lagrange multiplier produces softer singularities compared with those of GR BHs. Moreover, we studied the energy conditions related to the third case and showed that the SEC of the second case is violated, however, all the energy conditions of the third case are verified. We also studied the thermodynamics of the third case and found that the Hawking temperature and the heat capacity vanish in the extremal limit.

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