Relativistic quantum mechanics:
A Dirac’s point-form inspired approach

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Abstract

This paper describes a tentative relativistic quantum mechanics approach inspired by Dirac’s point-form, which is based on the physics description on a hyperboloid surface. It is mainly characterized by a non-standard relation of the constituent momenta of some system to its total momentum. Contrary to instant- and front-form approaches, where it takes the form of a 3-dimensional $\delta(\cdots)$ function, the relation is given here by a Lorentz-scalar constraint. Thus, in the c.m. frame, the sum of the constituent momenta, which differs from zero off-energy shell, has no fixed direction, in accordance with the absence of preferred direction on a hyperboloid surface. To some extent, this gives rise to an extra degree of freedom entering the description of the system of interest. The development of a consistent formalism within this picture is described. Comparison with other approaches is made.

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1 Introduction

Among the different forms of relativistic quantum mechanics, the point form is the less known one. Following the classification made by Dirac [1], this approach implies the description of physics on a hyperboloid surface, $x^2 = \tilde{\tau}$, which is invariant under a Lorentz transformation. This property reflects itself in the construction of the Poincaré algebra. The four components of the momentum operator, $P^\mu$, contain the interaction while rotation and boost operators, which leave this surface unchanged, are kinematical.

An implementation of the point-form approach has been considered in the literature [2, 3, 4]. It has been used later on for the calculation of form factors in different hadronic systems [5, 6, 7, 8, 9] as well as theoretical ones [10, 11]. However, as noticed by Sokolov [2], this “point-form” approach, where the generators of the Poincaré algebra evidence the same kinematical or dynamical character as above, differs from Dirac’s one. It assumes that physics is described on a hyperplane, $v \cdot x = \tau$, perpendicular to the velocity of the system [2]. The kinematical character of boosts is somewhat trivial in this case since, at the same time as the system is boosted, the frame employed for the description changes.

An attempt to justify this approach from considering the description of physics on a hyperboloid was recently made [12], using approximations however. The contribution of the interaction to the momentum operator, “$P^\mu$”, which is so derived, is proportional to the velocity of the system. Actually, this result turns out to be an exact one but obtained from describing physics on a hyperplane perpendicular to the velocity of the system. One therefore recovers the implementation of the “point-form” approach that was recognized by Sokolov as being different from Dirac’s point-form [2]. An implementation of this last approach therefore remains an open problem.

In the present work, we propose ourselves to make some steps in this direction. Each approach is, in particular, characterized by the relation of the constituent momenta to the total momentum of the system under consideration. In the instant- or front-form approaches for instance, the sum of the constituent momenta is equal to the total momentum carried by the system (instant-form) or deviates from it by an amount proportional to the orientation of the hyperplane which the physics is described on (front-form). However, for the Dirac’s point form, where no orientation in Minkowski space is a priori preferred, the relation must necessarily take the form of a Lorentz-scalar constraint. The simplest one that can thus be imagined for a two-body system, compatible with the fact that the standard 3-momentum conservation should be recovered in the non-relativistic limit (small binding), takes the form

$$\left(\vec{p}_1 + \vec{p}_2 - \vec{P}\right)^2 - (e_1 + e_2 - E_P)^2 = -(p_1 + p_2 - P)^2 = 0. \quad (1)$$

Such a constraint implies that the sum of the constituent momenta deviates from the total momentum of the system like in other approaches (front form) but, most importantly, the departure takes the form of a vector whose orientation is not determined. This introduces a somewhat new, non-trivial, degree of freedom in describing a given system. Whether this can be implemented consistently in some formalism is not straightforward. Difficulties arise as soon as one considers the equation that should be fulfilled by the wave
function describing the system. Everything concerned with the total momentum, \( \vec{P} \), or the sum of the constituent momenta, \( \vec{p}_1 + \vec{p}_2 \), should factor out so that the solution can be expressed in terms of the solution of a mass equation involving only internal variables. Thus, it has to be shown that the constraint given by Eq. (1) is part of the wave function. We mentioned that the direction of the momenta carried by the constituents is not fixed. How this direction, which enters the wave function, is affected by the interaction is another important aspect. Lastly, there is the possibility that the interaction itself depends on the total momentum but this problem is no more than the one appearing in instant- and front-form approaches. How to solve this problem was considered by Bakamjian and Thomas [13] (see also Ref. [11] for a practical application of interest here).

The plan of the paper is as follows. In the second section, we describe how we can disentangle the external and internal degrees of freedom, with the aim to derive a mass operator involving only the last ones. This is achieved by making an appropriate change of variables and showing that the orientation taken by the sum of the constituent momenta with respect to the total momentum, \( \vec{P} \), despite it is not determined, is nevertheless conserved by the interaction. The third section is devoted to a comparison with other forms. In particular, we briefly show how the present implementation of the point-form approach corrects for the systematic failure of an earlier version [4] to provide the expected asymptotic power law behavior of form factors. A conclusion and further discussion is given in the fourth section. As the approach presented here rather involves new aspects, many details are important. These ones are given in the appendix, keeping only in the main text the minimal ingredients required for its understanding.

2 Equation for the wave function and mass operator

In this section, we determine the expression that the wave function of a two-body system composed of scalar constituents could take in an approach inspired from Dirac’s point-form. The relation to a mass operator that could be used in other forms, or taken from them, is made.

- Relation to a description on a hyperboloid surface
The derivation of an interaction operator may start from the current expression of the 4-momentum, \( P^\mu \), in terms of the free-particle momenta and the Lagrangian interaction density:

\[
\text{“}P^\mu \text{”} = \text{“}p^\mu \text{”} + \text{“}P^\mu_{\text{int}} \text{”} = \text{“}p^\mu \text{”} - \int d^4x \; g^{\mu\nu} f_\nu(x) \mathcal{L}_{\text{int}}(x), \tag{2}
\]

where quotation marks have been introduced to distinguish the operator and the corresponding eigenvalue. Extra terms involving the field derivatives don’t need to be considered here. The function \( f_\nu(x) \) involves the hypersurface which the integral is performed over. It takes different expressions depending on the form under consideration. In the case of a hyperplane defined by the orientation \( \lambda_\nu \), the general function is given by \( f_\nu(x) = \lambda_\nu \delta(\lambda \cdot x - \tau) \). In the present case, the expression of interest, whose derivation is given in appendix \( \mathbb{A} \) reads:

\[
f_\nu(x) = 2 x_\nu \left( \theta(U \cdot x) \delta(\tau - x^2) - \theta(-U \cdot x) \delta(\tau + x^2) \right) + \cdots, \tag{3}
\]
where the $\delta(\tau - x^2)$ function, for instance, implies that the hypersurface to be integrated over is a hyperboloid (for $\tau > 0$). The $\theta(\cdots)$ functions indicate how the two parts of the hypersurface contribute, the relative sign, in relation with space-time symmetry, being essential to get a meaningful contribution (see appendix A). They involve a 4-vector, $U^\mu$, that will be defined later on in terms of the momenta of particles participating to the interaction. This choice is the simplest one which is possible within the point-form approach so that to ensure the Lorentz invariance of the argument of the $\theta(\cdots)$ functions. The above expression of $f_\nu(x)$ also evidences a dependence on the $\tau$ variable, which plays a role analogous to the time in the instant form. It can be used to specify the hypersurface of interest and is such that the whole space-time volume is covered when it varies between its extreme limits ($-\infty$ and $+\infty$ here). Any value of this “time” should be acceptable but corresponding mathematical developments are generally quite involved. Due to obvious “time”-symmetry reasons, some simplification is however expected with the particular choice considered by Dirac [1], $\tau = 0$, where the hyperboloid surface reduces to the light cone while the contribution represented by dots in Eq. (3) cancels. This is similar to making the current choice $t = 0$ in the instant form, where the time evolution could be accounted for by an appropriate equation involving the Hamiltonian. In order to provide a comparison, the two cases are illustrated in Fig. 4 for various values of the “time” variable. Paying attention to how the limit $\tau \to 0$ should be taken (see appendix A), the relevant function $f_\nu(x)$ that we will use reads:

$$f_\nu(x) = x_\nu \delta(x^2) \left[ \theta(U \cdot x) - \theta(-U \cdot x) \right].$$

(4)

In field theory, the Lagrangian density $\mathcal{L}(x)$ would have a well-defined meaning. In relativistic quantum mechanics, this density has to be re-interpreted for a part. On the one hand, the $x$ coordinate refers to the system as if this one could be considered as an elementary one. On the other hand, the density accounts for degrees of freedom that are integrated out, like meson exchanges, leading to a mass operator describing the internal structure of the system of interest. It also involves the plane waves describing the interacting particles. These various inputs can be considered as relevant to a minimal space-time description of relativistic quantum mechanics. They allow one to recover the usual expressions pertinent to the implementation of instant- and front-forms approaches. As an example of interest in the present work, we write the interaction contribution to “$P^\mu$”, Eq. (2), for spinless-scalar particles exchanging a spinless meson of mass $\mu$:

$$"P^\mu_{int}" = -\int d\vec{p}_1 d\vec{p}_2 d\vec{p}_1' d\vec{p}_2' \int d^4 x x^\mu \delta(x^2) \left[ \theta(U \cdot x) - \theta(-U \cdot x) \right] \epsilon^{(p-p')x} \times \frac{a^*(\vec{p}_1) a^*(\vec{p}_2)}{(2\pi)^3 (2e_1)^{1/2} (2e_2)^{1/2}} \frac{4m^2 g^2}{\mu^2 + \cdots} \frac{a(\vec{p}_1')} {a(\vec{p}_2')} \frac{a(\vec{p}_1')}{(2\pi)^3 (2e_1')^{1/2} (2e_2')^{1/2}}.$$

(5)

While standard notations are mostly employed, we stress that the quantity $p^\mu$ is a shortened notation to represent the sum of the momenta relative to particles 1 and 2, $p^\mu = p_1^\mu + p_2^\mu$. We notice that the integral over $x$ replaces the one which provides the 3-momentum conservation in the instant form. Also, the momentum transfer carried by the exchanged meson, that appears at the denominator of its propagator, has been purposely replaced by dots. As will be seen below, this term could be affected by consistency conditions pertinent to a relativistic quantum mechanics approach. For an illustration purpose, the interaction has been given the form of a single-boson exchange but it could already
Figure 1: Transverse representation of hypersurfaces at different “times” in both instan-
taneous and point-form approaches (left and right parts respectively). The last one is given for the
choice, $U^\mu = 1, 0, 0, 0$. For convenience, the variable $\bar{\tau}$ is denoted $\bar{\tau}$. The arrows indicate
how the hypersurfaces evolve when the “time” varies from $-\infty$ to $+\infty$. The counterpart
of the hyperboloid surface with a space-like character, which is rarely mentioned and
corresponds to a different sign of the “time” $\bar{\tau}$, is represented by the dashed curve.

have at this point an effective character and account for some field-theory corrections
through the choice of the coupling or that of the boson mass [14].

• Basic equation

Instead of looking for a solution of Eq. (2), we consider an equivalent problem that
consists in searching for a solution of the following equation:

$$P^2 \Phi_P(p_1, p_2) = \left( p^2 + p \cdot P_{\text{int}} + P_{\text{int}} \cdot p + P_{\text{int}}^2 \right) \Phi_P(p_1, p_2),$$

where $\Phi_P(p_1, p_2)$ is the two-body wave function. This equation offers the advantage
to be closer to a Lorentz-invariant mass equation, facilitating the derivation of a mass
operator. In the following, we ignore the last term at the r.h.s. ($P_{\text{int}}^2$). Quite generally, its
contribution could be incorporated in the middle terms (from a phenomenological view
point). In the present case however, it will turn out to vanish, very much like in front-form
approaches where $P_{\text{int}}^2$ is proportional to a 4-vector $\omega^\mu$ with the property $\omega^2 = 0$ (see for
instance Ref. [15]).

After replacing $P^2$ in Eq. (6) by the eigenvalue $M^2$, and using the expression of the
integral over $x$ appearing in Eq. (3), given by Eq. (59) of appendix A we can now cast
the above equation into the following basic form:

$$\left( M^2 - p^2 \right) \Phi_P(p_1, p_2)$$
\[ = - \int \int \frac{d\vec{p}_1'}{(2\pi)^3} \frac{d\vec{p}_2'}{(2\pi)^3} \frac{1}{(2e_1)^{1/2}} \frac{4m^2 g^2}{\mu^2 + \cdots} \frac{1}{(2e_2)^{1/2}} \times (p + p') \cdot \partial_{\mu - \nu} \left( 4\pi^2 \delta((p - p')^2) \left[ \theta(U \cdot (p - p')) - \theta(U \cdot (p' - p)) \right] \right) \Phi_P(\vec{p}_1', \vec{p}_2'). \]

An equation whose Lorentz invariance is more transparent for some factors is obtained by making the change:

\[ \Phi_P(\vec{p}_1, \vec{p}_2) = \frac{1}{(2e_1)^{1/2}} \frac{1}{(2e_2)^{1/2}} \Phi_P^{(1)}(\vec{p}_1, \vec{p}_2). \]

Together with making the derivative in the integrand, the equation reads:

\[ (M^2 - p^2) \Phi_P^{(1)}(\vec{p}_1, \vec{p}_2) = -\frac{1}{8\pi^4} \int \int \frac{d\vec{p}_1'}{2e_1'} \frac{d\vec{p}_2'}{2e_2'} \frac{4m^2 g^2}{\mu^2 + \cdots} \times \left( U \cdot (p + p') \delta((p - p')^2) \delta(U \cdot (p - p')) \right) \\
+ (p^2 - p'^2) \delta((p - p')^2) \left[ \theta(U \cdot (p - p')) - \theta(U \cdot (p' - p)) \right] \Phi_P^{(1)}(\vec{p}_1', \vec{p}_2'). \]

We now want to see whether, or under which conditions, the above equation admits a mass spectrum independent of the total momentum, \( \vec{P} \). It is stressed that it does not involve at the r.h.s. any 3-dimensional \( \delta(\cdots) \) function ensuring the conservation of some momentum like in the instant- or front-form formalisms. For the sake of comparison, we here give this equation for the general case of a hyperplane of orientation \( \lambda^\mu \):

\[ (M^2 - p^2) \Phi_P^{(1)}(\vec{p}_1, \vec{p}_2) = -\frac{1}{(2\pi)^3} \int \int \frac{d\vec{p}_1'}{2e_1'} \frac{d\vec{p}_2'}{2e_2'} \frac{4m^2 g^2}{\mu^2 + \cdots} \times \frac{\lambda \cdot (p + p')}{\lambda_0} \delta(\vec{p} - \vec{p}' - \frac{\lambda}{\lambda_0} (e_p - e_{p'})) \Phi_P^{(1)}(\vec{p}_1', \vec{p}_2'). \]

It is noticed that, in the limit \( |\vec{\lambda}/\lambda_0| \to 1 \) (front-form case), the 3-dimensional \( \delta(\cdots) \) function in this last equation implies those present in the previous one (assuming also \( \lambda^\mu \propto U^\mu \)). The converse result does not hold however.

- **Essential aspects of a solution**

To obtain a solution of the above equation, we first assume that the 4-vector, \( U^\mu \), can be written as:

\[ U^\mu = c (p - P)^\mu + c' (p' - P)^\mu, \]

where \( c \) and \( c' \) are arbitrary coefficients at this point. It is then possible to show the following relation (see appendix [B]):

\[ \left( U \cdot (p + p') \delta((p - p')^2) \delta(U \cdot (p - p')) \right) \\
+ (p^2 - p'^2) \delta((p - p')^2) \left[ \theta(U \cdot (p - p')) - \theta(U \cdot (p' - p)) \right] \delta((p' - P)^2) \\
= \delta((p - P)^2) \delta((p - p') \cdot (p' - P)) \delta((p' - P)^2) \frac{c (p^2 - M^2) + c' (p'^2 - M^2)}{c + c'}. \]
An important consequence of this result is that, if the wave function appearing on the r.h.s. of Eq. (9) is proportional to \( \delta((p' - P)^2) \), then, the wave function appearing on the l.h.s. is proportional to \( \delta((p - P)^2) \). This factor is therefore part of the solution that is looked for. Introducing the change:

\[
\Phi^{(1)}_P(\vec{p}_1, \vec{p}_2) = \delta((p - P)^2) \Phi^{(2)}_P(\vec{p}_1, \vec{p}_2),
\]

(13)
a further reduction of Eq. (9) is possible:

\[
\left(M^2 - p^2\right) \Phi^{(2)}_P(\vec{p}_1, \vec{p}_2) = -\frac{1}{8 \pi^4} \int \int \frac{d\vec{p}_1'}{2 e_1'} \frac{d\vec{p}_2'}{2 e_2'} \frac{4 m^2 g^2}{\mu^2 + \cdots} \times \delta((p - P) \cdot (p' - P)) \delta((p' - P)^2) \frac{c (p^2 - M^2) + c' (p'^2 - M^2)}{c + c'} \Phi^{(2)}_P(\vec{p}_1', \vec{p}_2').
\]

(14)

From examining this last equation, another feature comes out. It concerns the orientation of the 3-vectors, \( \vec{p} - \vec{P} \) and \( \vec{p}' - \vec{P} \). Taking into account that \( (p - P)^2 = (p' - P)^2 = 0 \), it is appropriate to introduce unit vectors defined as:

\[
\vec{u} = \frac{\vec{p} - \vec{P}}{e - E_P}, \quad \vec{u}' = \frac{\vec{p}' - \vec{P}}{e' - E_P}.
\]

(15)

These vectors determine for a part how the momentum carried by the constituents depart from the total momentum carried by the system. The presence of the function \( \delta((p - P) \cdot (p' - P)) \) in Eq. (14) then implies that the angle between the two 3-vectors, \( \vec{u} \) and \( \vec{u}' \) is zero (see appendix [14]). Therefore, the orientation of this vector is not affected by the interaction. It remains conserved through the rescattering processes that solving Eq. (14) implies, with the result:

\[
\vec{u} = \frac{\vec{p} - \vec{P}}{e - E_P} = \frac{\vec{p}' - \vec{P}}{e' - E_P} = \frac{\vec{p}'' - \vec{P}}{e'' - E_P} = \cdots.
\]

(16)

At this point, the nature of the 4-vector \( U^\mu \) introduced at the beginning of this subsection can be made more precise. As can be checked from Eq. (7) for instance, present results do not depend on its scale. Moreover, as a consequence of the three \( \delta(\cdots) \) functions appearing on the r.h.s. of Eq. (12), it has a zero norm. The 4-vectors \( U^\mu \) and (1, \( \vec{u} \)) can therefore be identified up to a factor. It is also noticed that the 4-vector \( U^\mu \) offers many similarities with the 4-vector, \( \omega^\mu \), which determines the orientation of the front in front-form approaches. In this case, an equation like Eq. (16) is obtained but with \( \vec{u} \) replaced by the orientation of the front, \( \vec{n} \) (see for instance Ref. [15]). These similarities may be useful to understand the relationship between different approaches. However, there are major differences. The orientation \( \vec{u} \) has to be integrated over while \( \vec{n} \) is a fixed orientation. On the other hand, the unit character of \( \vec{u} \) is obtained from a consistency requirement while, for \( \vec{n} \), it is given as part of the formalism. Finally, but not independently, the underlying formalisms imply physics description on quite different hypersurfaces (light cone in one case and hyperplane tangent to it in the other).

- **Change of variables**
  The next step in dealing with Eq. (7) is to show that its mass spectrum is independent
of the total momentum, \( \vec{P} \), or to determine under which conditions this is realized. This can be achieved by reducing this equation to a unique one, independent of the total momentum. In this order, we follow the approach pioneered by Bakamjian and Thomas [13], with appropriately adapting their work for the instant-form case to the present one.

We first make a change of variables which, for one-particle states, preserves the relation \( p_1^2 = m^2 \). Not surprisingly, it is taken here as a Lorentz-type transformation. For a part, it contains the standard Lorentz transformation which relates a system with a finite momentum to that one with zero momentum. For another part, it contains a Lorentz-type transformation which relates momenta of constituents in the center of mass to internal variables, \( \vec{k} \) and \( \vec{u} \). Moreover, we require that Eq. (16) be automatically satisfied. For particle 1, the change of variable thus takes the form:

\[
\vec{p}_1 = \vec{k} + \vec{w} \frac{\vec{w} \cdot \vec{k}}{\sqrt{1 + \vec{w}^2 + 1}} + \vec{w} e_k, \\
e_1 = \sqrt{1 + \vec{w}^2} e_k + \vec{w} \cdot \vec{k}. 
\] (17)

Expressions for particle 2 are obtained by making in these equations the change: \( \vec{k} \rightarrow -\vec{k} \).

From them, one easily obtains the following equalities:

\[
\vec{p} = \vec{p}_1 + \vec{p}_2 = 2 e_k \vec{w}, \\
e = e_1 + e_2 = 2 e_k \sqrt{1 + \vec{w}^2}. 
\] (18)

Equation (16) is then identically fulfilled for any \( \vec{u} \) (including \( |\vec{u}| \neq 1 \)) by taking:

\[
\vec{w} = \frac{\vec{P}}{2 e_k} + \frac{\vec{u}}{2 e_k} \frac{4 e_k^2 - M^2}{u \cdot P + \sqrt{(u \cdot P)^2 + (1 - \vec{u}^2)(4 e_k^2 - M^2)}}, \\
\sqrt{1 + \vec{w}^2} = \frac{E_P}{2 e_k} + \frac{1}{2 e_k} \frac{4 e_k^2 - M^2}{u \cdot P + \sqrt{(u \cdot P)^2 + (1 - \vec{u}^2)(4 e_k^2 - M^2)}}. 
\] (19)

In a next step, the above changes are made in Eq. (14), which then involves an integral over \( \vec{k}' \) and \( \vec{w}' \). Taking advantage of the two \( \delta(\cdot \cdot \cdot) \) functions that appear in this equation, the second of the integral can be easily performed (see some details in appendix C). The following equation for \( \Phi^{(2)}_P(\vec{p}_1, \vec{p}_2) \) is thus obtained:

\[
\left( M^2 - 4 e_k^2 \right) \Phi^{(2)}_P(\vec{p}_1, \vec{p}_2) = \int \frac{d\vec{k}'}{(2\pi)^3 e_{k'}} \left( -\frac{4 m^2 g^2}{\mu^2 + \cdots} \right) \frac{\sqrt{4 e_{k'}^2 - M^2}}{\sqrt{4 e_k^2 - M^2}} \times \left( \frac{c (4 e_{k'}^2 - M^2) + c' (4 e_k^2 - M^2)}{(c + c') \sqrt{4 e_{k'}^2 - M^2} \sqrt{4 e_k^2 - M^2}} \right) \Phi^{(2)}_P(\vec{p}_1', \vec{p}_2'). 
\] (20)

Assuming for a while that one can forget about the meson propagator, this equation would have the desired properties. The solutions do not depend on the total momentum, \( \vec{P} \), and only involve the internal variable, \( \vec{k} \). Referring to the solution of an equation that
is more symmetrical in variables $\tilde{k}$ and $\tilde{k}'$:

\[
(M^2 - 4 e_k^2) \phi_0(\tilde{k}) = \int \frac{d\tilde{k}'}{(2\pi)^3} \frac{1}{\sqrt{e_{\tilde{k}'} \mu^2 + \cdots}} \frac{-4 m^2 g^2}{\sqrt{e_{\tilde{k}'} \mu^2 + \cdots}} \phi_0(\tilde{k}')
\times \left( \frac{c (4 e_{\tilde{k}'}^2 - M^2) + c' (4 e_k^2 - M^2)}{(c + c') \sqrt{4 e_{\tilde{k}'}^2 - M^2}} \right),
\]

the solution of Eq. (20) may read:

\[
\Phi^{(2)}_P (\tilde{p}_1, \tilde{p}_2) = \frac{\sqrt{e_k} \phi_0(\tilde{k})}{\sqrt{4 e_{\tilde{k}}^2 - M^2}}.
\]

Thus, apart from the interaction term represented by a meson propagator in Eqs. (7), (9), (14), or (20), which was not considered explicitly till here, it has been possible to get rid of their dependence on the total momentum, $\mathbf{P}$.

When the change of variable made above is applied to the meson propagator, it is found that the term corresponding to the usual squared momentum transfer, represented by dots in the above equations, depends on the $\tilde{k}$ variable, but also on the total momentum, possibly through the $\tilde{u}$ variable. Removing this total momentum dependence represents the third step in getting a relevant mass operator. The problem was considered in Ref. [11] with some details. We therefore only remind the main point underlying the solution. This total momentum-dependent term also evidences an off-shell character and, thus, its contribution has the same order as higher-order terms in the interaction. It is expected that adding these terms to the interaction kernel should restore the total momentum independence, with the consequence that the whole interaction, necessarily effective, only depends on the $\tilde{k}$ variable. We thus recover the same kind of constraint as the one emphasized first in the instant-form case by Bakamjian and Thomas [13] and in other forms since then (see Ref. [16] for a review). The expression of the mass operator for the present two-body system may then read:

\[
M^2 = 4 e_k^2 + 4 m \tilde{V},
\]

where $\tilde{V}$ only depends on the internal variable $\tilde{k}$ and is normalized like the interaction appearing in a standard Schrödinger equation. For the one-boson exchange contribution considered above, this interaction in momentum space up to off-shell effects thus reads:

\[
\tilde{V}(\tilde{k}, \tilde{k}') = -\sqrt{\frac{m}{e_k}} \frac{g^2}{\mu^2 + (\tilde{k} - \tilde{k}')^2} \sqrt{\frac{m}{e_{\tilde{k}'}}}.
\]

Of course, the interaction can contain higher order terms or, if a phenomenological approach is used, the parameters, such as the coupling constant $g^2$, could be fitted to some data, accounting for higher-order effects as already mentioned.

- **Off-shellness of the interaction**

Examination of Eq. (21) shows the presence of a factor that evidences a dependence on the definition of the 4-vector $U^\mu$ through the dependence on the coefficients, $c$ and $c'$. It is noticed that this factor is equal to 1 on energy shell, independently of the values of $c$ and
\( c \propto (4\epsilon_k^2 - M^2)^{1/2} \), \( c' \propto (4\epsilon_{k'}^2 - M^2)^{1/2} \). \( \text{(25)} \)

- **Normalization and orthogonality of the solutions**

The normalization of the solutions of the mass operator, Eq. (23), labelled by an index \( \alpha \), may be given by:

\[
\int \frac{d^3k}{(2\pi)^3} \phi_0^\alpha(k) \phi_0^{\alpha'}(k) = N^\alpha \delta^\alpha,\alpha'. \quad \text{(26)}
\]

The derivation of the above expression is a standard one. It consists in sandwiching the mass operator, Eq. (23), between two solutions, possibly different, and making the operator to act respectively on the right and on the left. By taking the difference of the results so obtained, one gets zero at the r.h.s. using the hermiticity of the operator, while the l.h.s. is given by:

\[
\left(M_\alpha^2 - M_{\alpha'}^2\right) \int \frac{d^3k}{(2\pi)^3} \phi_0^\alpha(k) \phi_0^{\alpha'}(k) = 0. \quad \text{(27)}
\]

This equation implies that the integral is zero if the masses are different, or a constant otherwise, as given by Eq. (26). For our purpose, this result has to be extended to take into account that solutions we are interested in also depend on the total momentum of the system and the 4-vector \( U^\mu \) or \((1, \vec{u})\). Applying the same method for Eq. (9) and noticing that the part involving the internal variable \( \vec{k} \) factors out, one gets the following expression for the normalization:

\[
N(\vec{P}_\alpha, \vec{P}_{\alpha'}) = \frac{1}{(2\pi)^6} \int \frac{d\vec{p}_1}{2e_1} \frac{d\vec{p}_2}{2e_2} \Phi_\alpha^{(2)}(\vec{p}_1, \vec{p}_2) M_\alpha \sqrt{p^2 - M_\alpha^2} \\
\times 8\pi \delta ((p - P_\alpha)^2) \delta ((p - P_\alpha) \cdot (p - P_{\alpha'})) \delta ((p - P_{\alpha'})^2) \\
\times \Phi_{\alpha'}^{(2)}(\vec{p}_1, \vec{p}_2) M_{\alpha'} \sqrt{p^2 - M_{\alpha'}^2} \\
= E_{P_\alpha} \delta (\vec{P}_\alpha - \vec{P}_{\alpha'}) N^\alpha \delta^{\alpha,\alpha'} \int \frac{d\vec{u}}{2\pi} \delta (1 - \vec{u}^2) \frac{M_\alpha^2}{(u \cdot P_\alpha)^2}. \quad \text{(28)}
\]

Some details about dealing with the \( \delta(\cdots) \) functions appearing in the above equation are given in appendix D. We notice that the last integral over \( \vec{u} \) can be easily calculated, with the result:

\[
\int \frac{d\vec{u}}{2\pi} \delta (1 - \vec{u}^2) \frac{M_\alpha^2}{(u \cdot P_\alpha)^2} = \int \frac{d\hat{u}}{4\pi} \frac{M_\alpha^2}{(\hat{u} \cdot P_\alpha)^2} = 1,
\]

which is obviously Lorentz invariant. Actually, this property can be extended to integrals that have a similar structure. It supposes that the extra factor in the integral exhibits the form of a Lorentz scalar while its dependence on the 4-vector, \((1, \vec{u})\), is invariant.
under the change of scale, \((1, \hat{u}) \to (\lambda, \lambda \hat{u})\). This property is especially important to get Lorentz-invariant expressions for form factors. Using Eq. (29), one recovers the standard normalization for states with different momenta:

\[
N(\vec{P}_\alpha, \vec{P}_{\alpha'}) \equiv E_{P_\alpha} \delta \left( \vec{P}_\alpha - \vec{P}_{\alpha'} \right) \ N^{\alpha\alpha} \delta^{\alpha\alpha'} = E_{P_\alpha} \delta \left( \vec{P}_\alpha - \vec{P}_{\alpha'} \right) \ N^\alpha \delta^{\alpha\alpha'},
\]

implying \(N^{\alpha\alpha} = N^\alpha\) for the conventions employed here. A related expression can be obtained from integrating Eq. (28) over \(\vec{P}_\alpha\). It evidences that the normalization \(N^{\alpha\alpha}\) expressed in terms of the particle momenta does involve an integration over the \(\vec{u}\) or \((\vec{p}_1 + \vec{p}_2)\) variable, contrary to other approaches. It is given by:

\[
N^{\alpha\alpha} = \int \frac{1}{(2\pi)^6} \frac{d\vec{p}_1}{2e_1} \frac{d\vec{p}_2}{2e_2} \left( \Phi^{(2)}_{P_\alpha}(\vec{p}_1, \vec{p}_2) \right)^2 8\pi^2 \delta((p - P_\alpha)^2) M^2_\alpha
\]

\[
= \int \frac{d\vec{k}}{(2\pi)^3} \left( \phi^{(2)}_0(k) \right)^2 \int \frac{d\vec{u}}{2\pi} \delta(1 - \vec{u}^2) \frac{M^2_\alpha}{(u \cdot P_\alpha)^2}.
\]

• Poincaré algebra

The general form of the Poincaré algebra for the point-form approach is known [16]. We concentrate here on the 4-momentum operator, \(P^\mu\). Let’s remind its general expression:

\[
P^0 = M \sqrt{1 + V^2}, \quad \vec{P} = M \vec{V},
\]

where \(\vec{V}\) is the velocity of the system of interest. For a two-body one, these operators can be obtained from the free-particle contributions following the Bakamjian-Thomas construction adapted to the present case. A change of variables relative to particles 1 and 2 is first made in terms of the velocity, \(\vec{V}\), and the internal variable \(\vec{k}\):

\[
\vec{p}_{1e} = \vec{k} + \vec{V} \frac{\vec{V} \cdot \vec{k}}{\sqrt{1 + V^2 + 1}} + \vec{V} e_k, \quad e_{1e} = \sqrt{1 + V^2} e_k + \vec{V} \cdot \vec{k},
\]

\[
\vec{p}_{2e} = -\vec{k} - \vec{V} \frac{\vec{V} \cdot \vec{k}}{\sqrt{1 + V^2 + 1}} + \vec{V} e_k, \quad e_{2e} = \sqrt{1 + V^2} e_k - \vec{V} \cdot \vec{k},
\]

with the result:

\[
P^0_0 = 2e_k \sqrt{1 + V^2}, \quad \vec{P}_0 = 2e_k \vec{V}.
\]

Anticipating on what follows, a subscript e (for effective) has been introduced at the variables relative to particles 1 and 2. The second step is to replace the factor \(2e_k\) by \(M\) in Eq. (34), which can be done without modifying the algebra provided that the interaction fulfills standard constraints.

At first sight, the expression of \(\vec{V}\) in terms of the single-particle momenta can be obtained from Eq. (33), with the result \(\vec{V} = (\vec{p}_{1e} + \vec{p}_{2e})/(2e_k)\). However, it is not a priori guaranteed that the momenta appearing in this expression are the genuine ones (appearing in the Lagrangian density). From comparing this expression with that one given by Eq. (2) (3-momentum part), it can be easily guessed that the answer is negative. A more satisfying expression supposes an extra change of variables, which has some relationship with the Lorentz-type transformation given by Eq. (17):

\[
\vec{p}_{ie} = \vec{p}_i + \vec{t} \frac{\vec{t} \cdot \vec{p}_i}{\sqrt{1 + t^2 + 1}} - \vec{t} e_i, \quad e_{ie} = \sqrt{1 + t^2} e_i - \vec{t} \cdot \vec{p}_i \ (i = 1, 2),
\]
where the shortened notations, \( e = e_1 + e_2 \), \( \vec{p} = \vec{p}_1 + \vec{p}_2 \), and \( \vec{p} - \vec{P} = \vec{u} (e - E_P) \), have been used again. The sum of the 4-momenta is given by:

\[
\vec{p} e = \vec{p} - \vec{t} \left( e - \frac{\vec{t} \cdot \vec{p}}{\sqrt{1 + t^2 + 1}} \right) = 2 e_k \left( \vec{p} - \vec{u} \frac{4 e_k^2 - M^2}{2 u \cdot P} \right),
\]

\[
e_e = \sqrt{1 + t^2} e - \vec{t} \cdot \vec{p} = \frac{2 e_k}{M} \left( e - \frac{4 e_k^2 - M^2}{2 u \cdot P} \right).
\]

The above equations may look artificial as they only involve some rewriting of previous ones. Their relevance becomes clearer when a comparison of their right hand side is made with the expression of the 4-momentum \( P^\mu \), Eq. (2). Apart from a factor \( 2 e_k/M \), the similarity of these results is complete provided that the single-particle momenta, \( p_i \), are identified with the Lagrangian ones and that the factor \((4 e_k^2 - M^2)/(2 u \cdot P)\) is replaced by the interaction as expected from the mass equation, Eq. (21). This allows one to identify the velocity vector as:

\[
\vec{V} = \frac{1}{M} \left( \vec{p} + \vec{u} \frac{4 m \vec{V}}{2 u \cdot P} \right).
\]

We notice that the Lorentz-type transformation given by Eq. (17), apart from an expected sign difference, is simpler than the one considered here. The reason is that the transformation performed in the former case makes a term to appear that exactly cancels the interaction one, leaving as a net result a quantity that can be identified to the total 4-momentum, without any renormalization. Instead, the transformation given by Eqs. (35, 36), made in the context of the Bakamjian-Thomas construction, is less natural, requiring that the factor \( 2 e_k \) appearing in Eq. (34) be changed into the mass operator, \( M \), by introducing the interaction. The difference mainly involves terms of the order \((1 - M/(2 e_k))\) (apart from expected \( \vec{V} \) terms).

### 3 Comparison with other approaches

In this section, we compare different features pertinent to the implementation of various forms of relativistic quantum mechanics. They concern the change of variables that allows one to show the Lorentz invariance of the mass spectrum and the condition that the interaction has to fulfill in this order. As the work was motivated by the peculiar behavior of form factors obtained in an earlier implementation of the point form approach, we will consider more specifically some aspects of the present implementation with this respect. We in particular show how the expected asymptotic behavior is recovered.

- **Change of variables in different forms**
  When it is compared to other forms, the present implementation of the point form evidences a close relationship as for the change of the physical variables, \( \vec{p}_i \), to the total
momentum and the internal variable, \( \vec{k} \). The transformation is essential to make use of the solutions of a mass operator in order, for instance, to calculate form factors in different approaches. Interestingly, it is given in all cases by a unique expression which generalizes Eq. (19). It now involves a 3-vector \( \vec{w} \) that takes the following form:

\[
\vec{w} = \frac{\vec{P}^2}{2 e_k} + \frac{\vec{\xi}}{2 e_k} \frac{4 e_k^2 - M^2}{\sqrt{(\xi \cdot P)^2 + \xi^2 (4 e_k^2 - M^2)} + \xi \cdot P},
\]

\[
\sqrt{1 + \vec{w}^2} = \frac{E}{2 e_k} + \frac{\xi^0}{2 e_k} \frac{4 e_k^2 - M^2}{\sqrt{(\xi \cdot P)^2 + \xi^2 (4 e_k^2 - M^2)} + \xi \cdot P}.
\]

(39)

The 4-vector \( \xi^\mu \) appearing in the above expression is specific of each approach. Its choice reflects the symmetry properties of the hypersurface which the physics is described on. It is uniquely defined within each approach and independent of the physical system under consideration. As can be noticed, the 4-vector \( \xi^\mu \) multiplies a term \((4 e_k^2 - M^2)\) that can be cast into an interaction one, employing Eq. (23). This is in accordance with the expectation that changing the hypersurface implies the dynamics. Apart from these features, it can be seen that the above expression is independent of the scale of the 4-vector \( \xi^\mu \). Thus, up to an irrelevant scale, the 4-vector \( \xi^\mu \) is given as follows:

- **instant form**
  \[
  \xi^0 = 1, \quad \vec{\xi} = 0,
  \]
  (40)

- **front form**
  \[
  \xi^0 = 1, \quad \vec{\xi} = \vec{n},
  \]
  (41)

where \( \vec{n} \) is a unit vector with a fixed direction \((\xi^2 = 0)\),

- **present point form**
  \[
  \xi^0 = 1, \quad \vec{\xi} = \vec{u},
  \]
  (42)

where \( \vec{u} \) is a unit vector that can point to any direction \((\xi^2 = 0)\).

To get an invariant mass spectrum in the present approach, it has been found that conditions on the interaction have to be fulfilled. They are quite similar to those required in the instant- and front-form approaches as the consideration of a simple one-boson exchange interaction model shows. These conditions are a necessary ingredient. They have a close relationship with the fact that higher order terms in the interaction have to be included to recover the above invariance property when a field-theory approach, based on time-ordered diagrams for instance, is used.

**Comparison with an earlier implementation of the point form**
The present implementation of the point-form approach differs from an earlier one with the above two respects: constraint on the interaction and choice of \( \xi^\mu \). In this last approach, the relation of the physical variables, \( \vec{p}_i \), to the total momentum and the internal one assumes the same form as Eq. (39). However, contrary to the present case, the 4-vector \( \xi^\mu \) is determined by the properties of the system under consideration, mainly its velocity...
The corresponding relation, $\xi^\mu \propto (P^\mu/M)$, allows one to considerably simplify Eq. (39), which then reads:

\[
\vec{w} = \frac{\vec{P}}{M}, \quad \sqrt{1 + \vec{w}^2} = \frac{E_P}{M}.
\] (43)

A first consequence, already mentioned in the introduction, is that the kinematical character of the boost transformation is somewhat trivial, since both the system and the frame are affected. As a result, the constraint that the interaction has to fulfill is much weaker than in the other forms, including the implementation of the point form presented in this work. A Lorentz invariant one-boson exchange interaction is sufficient and no higher order contribution is required to get a Lorentz invariant mass spectrum. This can be checked by repeating the different steps that lead to the derivation of a mass operator, Eq. (23), or by looking at Eqs. (39, 40) of Ref. [17], where this property is explicitly used. This result suggests that this “point form” is not on the same footing as the other forms or the present one.

A second consequence concerns applications of the formalism such as calculations of form factors. Implying initial and final states with different velocities, and therefore different $\xi^\mu$, it turns out that these states are described on hypersurfaces defined differently. Nothing prevents one from proceeding that way but it cannot be considered as the most convenient one. Actually, the approach does not account for interaction effects which are required so that physics can be described on a hypersurface uniquely defined and independent of the physical system under consideration. Indeed, Eq. (43) can be recovered by replacing $2e_k$ by $M$ in Eq. (39), which amounts to discard the interaction. Including only the boost effect common to all other approaches but without the above minimal consistency requirements, this approach rather likes a pre-Dirac one.

An observation related to the above remarks concerns the expectation value of the 4-momentum operator. In the earlier implementation of the point-form approach, this quantity is given by

\[
P^\mu = M \left< \frac{P^\mu}{2e_k} \right>,
\] (44)

which is the simplest possible choice. It assumes that the constituent-momentum and interaction parts are separately proportional to the velocity of the system. A zero 3-momentum value (c.m. case) assumes that $\vec{p} = \vec{p}_1 + \vec{p}_2 = 0$. Such a result holds more generally; it however involves quasi particles rather than physical ones. When considering these last degrees of freedom, the present implementation evidences a different pattern, perhaps more complicated but more consistent with describing physics on a unique hypersurface. A zero value can be obtained by averaging the non-zero 3-vector, $\vec{p}_1 + \vec{p}_2$, over all directions but it can also be checked that this single-particle contribution, for some direction, is cancelled by an interaction part, accordingly to Eq. (38). This example provides a nice illustration of the dynamical character of the momentum operator in the point-form approach. A graphical representation for a system at rest is given in Fig. [2]. More generally, the sum of the constituent momenta fluctuates around the total momentum with an amplitude determined by the interaction\(^1\).

\(^1\)To some extent, there is a similarity with the "Zitterbewegung" effect, but in momentum rather than in configuration space. We are grateful to Ica Stancu for proposing this analogy.
Figure 2: Graphical representation of a configuration representing contributions to the total momentum for a system at rest due to momenta, $\vec{p}_1$ and $\vec{p}_2$, of particles 1 and 2, and to the interaction part, $(2m/M) \tilde{V} \tilde{u}$. It is reminded that the two contributions, which sum up here to zero, separately cancel in an earlier implementation of the point-form approach [2, 3, 4]. Contrary to the front-form approach, the interaction part and, consequently, the sum of the particle momenta, $\vec{p}_1 + \vec{p}_2$, points isotropically to all directions, which is sketched by circles for the momentum configuration drawn in the figure.

- **Asymptotic behavior of form factors**

Form factors calculated in the earlier implementation of the point-form approach show a fall off faster than expected. For the ground state of a system made of scalar constituents, the asymptotic behavior is $Q^{-8}$, instead of $Q^{-4}$ [10, 11]. The difference can be traced back to a peculiarity of the formalism that changes the dependence on a factor $Q^2$ into a dependence on $Q^2 (1 + Q^2/(4M^2))$ [6]. A fast fall off also appears in the case where the mass of the system is small compared to the sum of the constituent masses (the pion for instance) with the result that the charge radius scales like the inverse of the mass [5].

The question arises whether the present implementation of the point form solves these problems.

Detailed calculations of form factors along the present approach will be presented elsewhere. We only show here, schematically, how the approach allows one to get the expected power law $Q^{-4}$ for the scalar constituent case (ground state). At high $Q^2$, the form factor is given by the Born amplitude whose graphical representation is given in Fig. 3 for the Breit-frame configuration and in the limit of small binding. In order to get an estimate, the value taken by the momentum of the intermediate particle, $\vec{p}^*$, has to be determined. The amplitude thus depends on the relation that implies this momentum, the spectator one and the total momentum and, therefore, on the form of relativistic quantum mechanics under consideration.
Figure 3: Virtual scalar particle or photon absorption on a two-body system in the Born approximation. A similar diagram with a different time ordering of the boson exchange and the interaction with the external probe should be considered. The kinematical definitions refer to the Breit-frame and an infinitesimally small binding. The momentum of the intermediate particle, $p^*$, depends on the form of relativistic quantum mechanics.

For the present point-form approach, the relation (11) reads:

$$|\vec{p}^* - \frac{1}{4} \vec{q} - \frac{1}{2} \vec{q}| = |e^* + e_{Q/4} - 2 e_{Q/4}|.$$  \hspace{1cm} (45)

Writing $p^* = \alpha \frac{q}{4}$, one gets in the high $Q$ limit the equation $|\alpha - 3| Q = |\alpha - 1| Q$, which implies $\alpha \to \infty$ (no real interest here) and $\alpha = 2$. Solutions with a component of $p^*$ perpendicular to $q$ are also possible (for instance $\alpha = 3$ and a component of $p^*$ perpendicular to $q$ equal to 4 in units of $Q/4$, which corresponds to the front-form case with $q^+ = 0$).

In the earlier “point-form” approach, the equation to be fulfilled is given by:

$$\vec{p}^* - \frac{\vec{q}}{4} = (e^* + e_{Q/4}) \frac{\vec{q}/2}{2 e_{Q/4}}.$$  \hspace{1cm} (46)

The above equation is verified with $\alpha = 3 + Q^2/(4 m^2)$, which contrary to the above case or the instant-form one ($\alpha = 3$), tends to $\infty$ when $Q \to \infty$. To determine how the Born amplitude depends on the choice of the form, one has to insert the appropriate expression of the $k$ variable in the factor that results from the meson and the two-body propagator in Fig. 3. In the limit of large momentum transfers, one thus obtains:

$$\frac{g^2}{\mu^2 + (k - k')^2} \frac{1}{4 \epsilon_k^2 - M^2} \simeq \frac{32 g^2}{2 \alpha (Q^2 + 4 m^2)} \frac{8}{2 \alpha (Q^2 + 4 m^2)}.$$  \hspace{1cm} (47)

Replacing $\alpha$ by its expression, one gets the following asymptotic behaviors:

$$16 g^2 Q^{-4} \text{ (present point form)},$$

$$4 g^2 \frac{Q^{-4}}{(1 + Q^2/(16 m^2))^2} \propto Q^{-8} \text{ (earlier “point form”).}$$  \hspace{1cm} (48)
It can be seen that the difference between the two results at high $Q^2$ is essentially due to a factor $(1 + Q^2/(16 m^2))$ that multiplies each $Q^2$ term. As can be checked in the small binding limit, this factor is identical to the one already found in [6], reminded at the beginning of this subsection. It is noticed that the front-form case with $q^+ = 0$ gives the asymptotic behavior $4 g^2 Q^{-4}$, in agreement with the exact result, while the instant-form one gives an extra factor $16/9$, requiring in any case two-body currents to get it.

4 Conclusion

In this work devoted to the description of a two-body system within relativistic quantum mechanics, we considered a point-form approach inspired by the Dirac’s one, which implies that physics is described on a hyperboloid surface. The main difficulty resides in solving the corresponding wave equation, taking into account that the integration of plane waves over a hyperboloid surface provides expressions that are not so easy to deal with as the 3-dimensional $\delta(\cdots)$ functions obtained for a hyperplane. For the solution we were able to find, which corresponds to a particular case considered by Dirac, the relation of the constituent momenta to the total momentum of the two-body system we are interested in is given by the equation:

$$\vec{p}_1 + \vec{p}_2 - \vec{P} - \vec{u} (e_1 + e_2 - E_P) = 0,$$

where $\vec{u}$ is a unit vector whose direction is to be integrated over. Provided that the interaction has the appropriate properties pertinent to the construction of the Poincaré algebra, it turns out that this direction is conserved by the interaction. This property greatly simplifies the search for a solution. At the same time, it allows one to establish some relationship with the front-form approach where a similar relation holds but for a fixed direction (defined by the orientation of the hyperplane which physics is described on). In view of an expected unitary equivalence between different forms of relativistic quantum mechanics, this relationship is not surprising.

The most striking feature evidenced by the approach arises from the above relation of the constituent momenta to the total momentum of the system of interest. The sum of these constituent momenta at a given $\vec{P}$, contrary to other forms, is not a fixed quantity. In the c.m. frame, it differs from zero off-energy shell and, moreover, points isotropically to all spatial directions, in agreement with the absence of preferred direction on the hyperboloid surface. Despite the unfamiliar character of this approach, it nevertheless appears that a consistent scheme could be developed. In particular, the correct total momentum is recovered when the contribution to it due to the interaction is accounted for, as expected from the dynamical character of the 4-momentum in the point-form approach.

Concerning other aspects of the present implementation of the point-form approach, it appears that the relation of the physical momenta to the total momentum and an internal variable is formally quite similar to that for the instant- and front-form approaches. The only difference is in the choice of a 4-vector which characterizes the hypersurface which physics is described on and reduces to the 4-vector $(1, \vec{u})$ in the present case. The

\footnote{In spite of approximations in Eq. (47), the factor is an exact consequence of Eq. (46).}
derivation of the mass operator requires the same kind of constraints than in the other forms. The standard expression of the normalization in terms of the solution of the mass operator is recovered, with a difference in its derivation however. It implies a further integration over the orientation of the total momentum carried by the constituents, $\vec{u}$.

The present work was motivated for a part by the drawbacks evidenced by the form factors calculated in an earlier “point-form” implementation \cite{10, 11}. In comparison to an “exact” calculation, the fall off was too fast. For the ground state of a system made of scalar constituents, the asymptotic behavior was rather given by the power law $Q^{-2n}$ where $Q^{-n}$ is expected. The slope at small momentum transfers was scaling like $M^{-2}$, providing a squared charge radius that tends to $\infty$ when the mass tends to zero! As this “point-form” implementation was implying hyperplanes, moreover different for the initial and final states in the case of form factors, the question arises of what this approach is responsible for in the peculiar features evidenced by these form factors. The comparison with results obtained in the present point-form implementation, based on describing the physics on a hyperboloid, suggests that the first problem is specific of the earlier “point-form” implementation. As worked out in Ref. \cite{17}, this problem could be cured by considering specific two-body currents. The other problem, related to the dependence of form factors on the momentum transfer $Q$ through the quantity $Q/2M$ in both cases, remains complete. Up to recently, it could be thought that this second problem was a characteristic feature of the point-form approach. Examination of a few not well known results obtained in the light-front approach for a frame $q^+ \neq 0$ \cite{18, 19, 20} or in the instant-form approach for a parallel kinematics \cite{11} shows similar features when a one-body current is retained. This relationship clearly points to an important role of missing two-body currents, which have been identified in some field-theory based calculations \cite{18, 19, 20, 11} but are different from those considered in Ref. \cite{17}. Thus, the present point-form implementation solves one of the problems raised by an earlier one, namely the wrong power-law behavior of the asymptotic form factor. With this respect, it now compares to the other forms of relativistic quantum mechanics. The other problem, illustrated by the scaling of the charge radius with the inverse of the mass of the system, appears to be a more general one. It probably points to a sizeable violation of Poincaré space-time translation invariance in calculating form factors \cite{21}.

As the present and the earlier implementations of the point-form approach satisfy the main properties pertinent to the related Poincaré algebra, one may wonder why they differ in their predictions. Looking for some genuine explanation, it appears that the difference resides in how the velocity vector, $\vec{V}$, is related to the physical degrees of freedom. In the simplest case and for a two-body system, $\vec{V} \propto (\vec{p}_1 + \vec{p}_2)$, which is reminiscent of the free particle system. However, these momenta could have an effective character and have a more complicated expression in terms of the physical momenta and the interaction, quite consistently with the dynamical character of the momentum in the point-form approach. To some extent, this offers some freedom that is used to fulfill the requirement that physics be described on the same hypersurface, whatever the system under consideration. Such a property could not be obtained in the simplest case.

While considering equations with physics described on a hyperboloid, the form of the solution we found was partly guessed on the basis of minimal symmetry and consistency requirements. On the other hand, it sounds that the calculation of form factors in the
present point-form implementation amounts to make an appropriately weighted average of the front-form form factor over the orientation of the front. In some sense, this relationship is fortunate and can provide useful hints. However, one can wonder why the point-form calculation of form factors should reduce to this simple recipe. This feature being closely related to the solution we found, the question arises whether this is the more general one. Finding an answer is a task for the future. For the time being, we believe that the present solution evidences features different enough from what other forms of relativistic quantum mechanics show and, in this respect, represents a stimulating starting point for further research.

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A Dealing with a hyperboloid surface: limit $\tilde{\tau} = 0$

We here consider expressions involving the integration of plane waves over a hyperboloid surface, $x^2 - \tilde{\tau} = 0$. This corresponds to $\tilde{\tau} > 0$ but, actually, the part with $\tilde{\tau} < 0$, outside the light cone, should be considered for completeness. Moreover, the forward and backward “times” have to be included in the discussion. As explained in the text, we concentrate on the particular case $\tilde{\tau} = 0$, which is considerably simpler. For illustration, we consider a minimal but essential application, and show how the contribution of a free scalar particle to the 4-momentum is recovered. To allow for a comparison, some results for the case of a hyperplane are first reminded.

The contribution of free scalar particles to the 4-momentum may be generally written as:

$$\tag{49} \quad P^\mu_0 = \int d^4 x \; f_\nu(x) \; \partial^\nu \phi(x) \; \partial^\mu \phi(x),$$

where $f_\nu(x)$ characterizes the surface which the integral is performed over. Keeping the only term of interest here, one gets:

$$\tag{50} \quad P^\mu_0 = \int \frac{d\vec{p} \; d\vec{p}'}{(e_p \; e_{p'})^{1/2} \; (2\pi)^3} \; a^*(\vec{p}) \; a(\vec{p}') \; p^\mu \; p'^\nu \; I_\nu,$$

with

$$\tag{50} \quad I_\nu = \int d^4 x \; f_\nu(x) \; e^{i(p-p') \cdot x}.$$

Anticipating on a generalization of the hyperplane case to different hypersurfaces, the function $f_\nu(x)$ may be written as the limit for $\tilde{\tau} \to 0$ of the function:

$$\tag{51} \quad f_\nu^\tau (x) = -\partial_\nu \; \Theta(x, \tilde{\tau})$$

with

$$\tag{51} \quad \Theta(x, \tilde{\tau}) = \theta(\xi \cdot x) \; \theta(\tilde{\tau} - F(x)) + \theta(-\xi \cdot x) \; \theta(\tilde{\tau} + F(x)),$$

where $F(x)$ and $\xi^\mu$ are arbitrary at this point but should be chosen accordingly to the hypersurface of interest.
The above Θ(χ, τ) function\(^3\) determines a partition of space (0 for τ = −∞ and 1 for τ = ∞). Its introduction does not modify the derivation of the 4-momentum conservation in a simple case which therefore represents a minimal requirement whereas its derivative with respect to τ (or x) defines the hypersurface of interest in term of the τ variable. This is illustrated by the following relations:

\[
(2\pi)^4 \delta^4(p - p') = \int d^4x e^{i(p - p')x} = \int d^4x e^{i(p - p')x} \int_{-\infty}^{\infty} d\tilde{\tau} \left( \frac{d}{d\tilde{\tau}} \Theta(x, \tilde{\tau}) \right)
\]

\[
= \int_{-\infty}^{\infty} d\tilde{\tau} \int d^4x e^{i(p - p')x} \left( \theta(\xi \cdot x) \delta(\tilde{\tau} - F(x)) + \theta(-\xi \cdot x) \delta(\tilde{\tau} + F(x)) \right). \tag{52}
\]

The hypersurface defined by the last factor at the r.h.s. of this equation must be such that the whole space-time volume be visited when τ varies from −∞ to +∞.

**Hyperplane case**

For a hyperplane with orientation \(λ_μ\), the function \(f_ν^\tau(x)\) may be written as:

\[
f_ν^\tau(x) = -\partial_ν \theta(τ - λ \cdot x) = λ_ν \delta(λ \cdot x - τ), \tag{53}\]

corresponding to take in Eq. \([51]\) \(F(x) = (λ \cdot x)^2\) and \(ξ^μ = λ^μ\).

It is noticed that the expression of \(f_ν^\tau(x)\) at \(τ = 0\) is unchanged when \(x \to -x\). Using the corresponding result of the integral over the variable \(x\):

\[
I_ν = \int d^4x \lambda_ν \delta(λ \cdot x) e^{i(p - p')x} = \frac{λ_ν}{λ_0} \left( 2\pi \right)^3 \delta(p - p') \frac{λ}{λ_0} (e_ν - e_ν')
\]

\[
= \lambda_ν \left( 2\pi \right)^3 \frac{e_ν}{λ} \delta(p - p') \tag{54}
\]

one gets the desired expression:

\[
"P_0^\nu" = \int dp a^\nu(p) a(p) p^ν. \tag{55}\]

As expected from Eq. \([53]\) for τ = 0, the result is independent of the scale of the 4-vector, \(λ^μ\). It does not depend either on its direction.

**Hyperboloid case**

In dealing with a hyperboloid surface, we start from Eq. \([51]\) which, as previously reminded, is consistent with minimal requirements. Disregarding a possible constant term, the function \(F(x)\) that appears there is now taken as \(F(x) = x^2\) while the general 4-vector \(ξ^μ\), essential to distinguish between backward and forward times, is denoted \(U^μ\) with \(U^2 \geq 0\). The function \(f_ν^\tau(x)\) thus obtained reads:

\[
f_ν^\tau(x) = 2x_ν \left( \theta(U \cdot x) \delta(\tilde{\tau} - x^2) - \theta(-U \cdot x) \delta(\tilde{\tau} + x^2) \right) + \ldots, \tag{56}\]

where dots represent a contribution irrelevant for our purpose. In considering the limit of the above expression for \(τ = 0\), it is appropriate to rewrite it as follows:

\[
f_ν^\tau(x) = x_ν \left( \left( \delta(\tilde{\tau} - x^2) + \delta(\tilde{\tau} + x^2) \right) \left[ \theta(U \cdot x) - \theta(-U \cdot x) \right] \right.
\]

\[
+ \left( \delta(\tilde{\tau} - x^2) - \delta(\tilde{\tau} + x^2) \right) \left[ \theta(U \cdot x) + \theta(-U \cdot x) \right]\right) + \ldots. \tag{57}\]

\(^3\)The notations \(\tilde{\tau}\) and \(\tau\) introduced below differ in that they respectively involve constraints chosen here as quadratic and linear in the \(x\) variable.
Due to the symmetrical character of the integration on positive and negative values of $\tilde{\tau}$, the last term as well as the dots part can be ignored. Concerning the first term, we notice that the two $\delta(\cdots)$ functions exclude each other. When taking the limit $\tilde{\tau} \to 0$, where the hyperboloid defined by the equation $\tilde{\tau} - x^2 = 0$ reduces to the light-cone, only one of these terms should be therefore retained. The above expression then simplifies to read:

$$f_\nu(x) = x_\nu \delta(x^2) \left[ \theta(U \cdot x) - \theta(-U \cdot x) \right].$$

(58)

The last factor may look surprising but, together with the front factor $x_\nu$, it simply accounts for the space-time symmetry, $x \to -x$, and in particular for the symmetry between backward and forward times in the case $\nu = 0$ and $U^\mu = 1, 0, 0, 0$.

When considering the contribution to the total momentum at $\tilde{\tau} = 0$, expressions given by Eq. (50) have again to be considered. The quantity $I_\nu$, which now involves the expression of $f_\nu(x)$ given by Eq. (58), now reads:

$$I_\nu = \int d^4x \, x_\nu \delta(x^2) \left[ \theta(U \cdot x) - \theta(-U \cdot x) \right] e^{i(p-p') \cdot x}$$

$$= -i \partial_{(p-p')} \left( \int d^4x \, \delta(x^2) \left( \theta(U \cdot x) - \theta(-U \cdot x) \right) e^{i(p-p') \cdot x} \right)$$

$$= 4 \pi^2 \partial_{(p-p')} \left( \delta((p-p')^2) \theta(U \cdot (p-p')) - \delta((p-p')^2) \theta(U \cdot (p'-p)) \right)$$

$$= 8 \pi^2 U_\nu \delta((p-p')^2) \delta(U \cdot (p-p'))$$

$$+ 8 \pi^2 (p-p')_\nu \delta'(((p-p')^2) \left[ \theta(U \cdot (p-p')) - \theta(U \cdot (p'-p)) \right].$$

(59)

The above result can be extended without any change to a many-body system. Steps have been detailed as some of them should be considered with caution. It has been assumed that the 4-vector $U^\mu$ was independent of $p - p'$. This may be the case for the present application. However, for other ones considered in this work, this will not be always the case. We checked that, for those applications, the corresponding contribution contains a factor which vanishes due to the presence of a $\delta(\cdots)$ function. What matters is that the relevant dependence on $p - p'$, in the $\theta(U \cdot (p-p'))$ function for instance, is already factored out. On the other hand, for the application considered here, the very last term in Eq. (59) ($8 \pi^2 \cdots$) cannot contribute since it contains the factor $p - p'$ that should ultimately vanish (conservation of the 4-momentum). As to the other term, $\delta(U \cdot (p-p')) \delta((p-p')^2)$, one could guess that the two $\delta(\cdots)$ functions produce $\delta(\cdots)$ functions implying the components of $(p-p')$ along respectively some direction and the transverse ones (once the other one has been accounted for), then giving rise to the expected 3-dimensional $\delta(\cdots)$ function, $\delta(\vec{p} - \vec{p}')$ (see Eq. (75) of appendix D). One then gets:

$$I_\nu = U_\nu (2 \pi)^3 \frac{e p \delta(\vec{p} - \vec{p}')} {U \cdot p}.$$

(60)

This result is quite similar to Eq. (53) obtained for a hyperplane surface. It shows how various $\delta(\cdots)$ functions implying scalar arguments combine to give a 3-dimensional $\delta(\cdots)$ function. Let’s finally notice that the combination of the $\theta(\cdots)$ in Eq. (55) allows one to get rid of a term that has the wrong imaginary phase and, moreover, no expected role here.
B Getting a solution of the wave equation

In looking for a solution of a wave equation, Eq. (9) for instance, we have in mind that the effect of the interaction should leave unchanged the function, \( f_P(p_1, p_2) \), which relates the constituent momenta to the total momentum of the system under consideration. In the instant-form case, where the interaction conserves the 3-momentum, this is trivially fulfilled by taking \( f_P(p_1, p_2) = \delta(p_1 + p_2 - \vec{P}) \), as can be checked from the equality:

\[
\delta(p_1 + p_2 - \vec{P}) = \int d(p_1' + p_2') \delta(p_1 + p_2 - p_1' - p_2') \delta(p_1' + p_2' - \vec{P}).
\]  

(61)

In the present case, the \( \delta(\cdots) \) functions appearing in the interaction involve some quadratic dependence on 4-momenta, see Eq. (60). They cannot be simply factorized into terms involving the components of these 4-momenta separately. Thus, the derivation of a relation that would have the structure of the above equation is not so trivial.

- Dealing with Lorentz-scalar \( \delta(\cdots) \) functions

The key quantity that we have to deal with and replaces the 3-dimensional \( \delta(\cdots) \) function ensuring momentum conservation in the above equation is obtained from contracting \( I_\nu \), Eq. (59), with \((p + p')^\nu\) where \( p \) now represents the sum of the 4-momenta of particles 1 and 2. It is given by:

\[
J = 8 \pi^2 \left( U \cdot (p + p') \delta((p - p')^2) \delta(U \cdot (p - p')) + (p^2 - p'^2) \delta'((p - p')^2) \left[ \theta(U \cdot (p - p')) - \theta(U \cdot (p' - p)) \right] \right). 
\]

(62)

In order to find a solution to our problem, we assume that the 4-vector \( U \) can be written as a linear combination of 4-momenta, \( p - P \) and \( p' - P \), as in Eq. (60). Moreover, we assume that the above quantity, \( J \), is multiplied by the function \( \delta'( (p' - P)^2 ) \), which could be a part of the solution we look for. To deal with Eq. (62), the trick is to insert \( P \) terms so that to get an expression involving mainly combinations of \( p - P \) and \( p' - P \) and to cancel part of the arguments in the \( \delta(\cdots) \) functions taking into account the other ones. As an example, we give the various steps concerned with the first term in Eq. (62):

\[
\delta((p - p')^2) \delta(U \cdot (p - p')) \delta((p' - P)^2) \\
= \delta((p - P)^2 - 2(p - P) \cdot (p' - P) + (p' - P)^2) \delta((p' - P)^2) \\
\times \delta \left( c \left( (p - P)^2 - (p - P) \cdot (p' - P) \right) - c' \left( (p' - P)^2 - (p - P) \cdot (p' - P) \right) \right) \\
= \delta((p - P)^2 - 2(p - P) \cdot (p' - P)) \delta((p' - P)^2) \\
\times \delta \left( (p - P) \cdot (p' - P) \left( c + c' \right) \right) \\
= \delta((p - P)^2) \delta((p - P) \cdot (p' - P)) \delta((p' - P)^2) \left( c + c' \right)^{-1}. 
\]

(63)

Using the various \( \delta(\cdots) \) functions, the factor multiplying the above term in Eq. (62), \( U \cdot (p + p') \), can be transformed as follows:

\[
U \cdot (p + p') = 2 U \cdot P = c(p^2 - M^2) + c'(p'^2 - M^2).
\]

(64)
The second term in Eq. (62) can be similarly dealt with except for the derivative of the \( \delta(\cdots) \) function. This one can be transformed away by an integral by parts that changes the \( \theta(\cdots) \) function into a \( \delta(\cdots) \) function. While doing so, one has to assume that the remaining part of the integrand does not depend on \( (p - P) \cdot (p' - P) \). This condition is no more than the constraint that the interaction in the mass operator has to fulfill in any case in the various forms of relativistic quantum mechanics. Gathering the different results, one obtains:

\[
\begin{align*}
&\left( U \cdot (p + p') \, \delta((p - p')^2) \, \delta(U \cdot (p - p')) \right) \\
&\quad + (p^2 - p'^2) \, \delta'((p - p')^2) \left[ \theta(U \cdot (p - p')) - \theta(U \cdot (p' - p)) \right] \, \delta((p' - P)^2) \\
&= \delta((p - P)^2) \, \delta((p - P) \cdot (p' - P)) \, \delta((p' - P)^2) \, \frac{c \, (p'^2 - M^2) + c' \, (p'^2 - M^2)}{c + c'}.
\end{align*}
\]

(65)

**Relation** \( \bar{u} = \bar{u}' \)

It is now interesting to consider the consequences implied by the simultaneous presence of the three \( \delta(\cdots) \) functions in the last equation. We first notice that those of them with the argument \( (p - P)^2 \) or \( (p' - P)^2 \) lead to relations such as:

\[
\bar{u} = \frac{\bar{p} - \bar{P}}{e - E_P}, \quad \bar{u}' = \frac{\bar{p}' - \bar{P}}{e' - E_P},
\]

(66)

where \( \bar{u} \) and \( \bar{u}' \) are unit vectors. Considering now the second \( \delta(\cdots) \) function at the last line of Eq. (65), it is seen that it can be written in a way where the dependence on \( \bar{u} \) and \( \bar{u}' \) is factorized:

\[
\delta((p - P) \cdot (p' - P)) = \delta\left((1 - \bar{u} \cdot \bar{u}') (e - E_P) (e' - E_P)\right)
\]

\[
= \frac{\delta\left(1 - \bar{u} \cdot \bar{u}'\right)}{|e - E_P| \, |e' - E_P|}.
\]

(67)

As \( \bar{u} \) and \( \bar{u}' \) are unit vectors, this last relation implies that their relative angle is zero, hence the relation:

\[
\bar{u} = \bar{u}'.
\]

(68)

It is important to notice that such a result could not be obtained if one of \( \delta(\cdots) \) function with the argument \( (p - P)^2 \) or \( (p' - P)^2 \) was absent in Eq. (65). For practical purposes, it supposes some information on the wave function \( (1 - \bar{u}^2 = 0) \). This feature contrasts with the instant or front forms where a 3-dimensional equality also holds (conservation of the 3-momentum in the instant form for instance, see Eq. (11)). In these cases, the relation occurs at the level of the interaction alone, independently of the wave function.

Another important consequence concerns the 4-vector \( U^\mu \). Its definition in the interaction context, Eq. (11), was involving both the initial and final states. The above result makes the definition free of ambiguity, allowing one to identify the 4-vectors \( U^\mu \) and \( u^\mu \) (up to a factor). On the one hand, one has:

\[
(e - E_P, \bar{p} - \bar{P}) \propto (e' - E_P, \bar{p}' - \bar{P}) \propto (1, \bar{u}) \propto (U^0, \bar{U}).
\]

(69)
On the other hand, the proportionality factor does not matter. The 4-vector $U^u$ is defined up to a factor in any case and, moreover, it has a zero norm, which is essential to get rid of various scales.

C Change of variables

The Jacobian transformation resulting from the changes of variables, Eqs. (17) and (19), are successively given by:

\[
\int \frac{d\vec{p}_1}{2 e_1} \frac{d\vec{p}_2}{2 e_2} \ldots = \int d\vec{k}' \frac{2 e_k}{\sqrt{1 + w^2}} \ldots \\
= \int d\vec{k}' \frac{2 e_k}{\sqrt{(u \cdot P)^2 + (1 - \vec{u}'^2) (4 e_k^2 - M^2)}} \\
\times \left( \frac{4 e_k^2 - M^2}{2 e_k (u \cdot P + \sqrt{(u \cdot P)^2 + (1 - \vec{u}'^2) (4 e_k^2 - M^2)})} \right)^3 \ldots (70)
\]

where $u \cdot P = E_P - \vec{u} \cdot \vec{P}$. Two quantities that enter $\delta(\cdots)$ functions in Eq. (14) are more simply expressed in terms of the $\vec{u}$ variable, using Eqs. (18, 19):

\[
\vec{p} - \vec{P} = \vec{u} \frac{4 e_k^2 - M^2}{u \cdot P + \sqrt{(u \cdot P)^2 + (1 - \vec{u}'^2) (4 e_k^2 - M^2)}},
\]

\[
e - E_P = \frac{4 e_k^2 - M^2}{u \cdot P + \sqrt{(u \cdot P)^2 + (1 - \vec{u}'^2) (4 e_k^2 - M^2)}} \ldots \ldots (71)
\]

while the $\delta(\cdots)$ functions read:

\[
\delta((p' - P)^2) = \delta(1 - \vec{u}'^2) \left( \frac{u' \cdot P + \sqrt{(u' \cdot P)^2 + (1 - \vec{u}'^2) (4 e_{k'}^2 - M^2)}}{4 e_{k'}^2 - M^2} \right)^2 \\
\delta((p - P) \cdot (p' - P)) = \delta(1 - \vec{u} \cdot \vec{u}') \frac{u \cdot P + \sqrt{(u \cdot P)^2 + (1 - \vec{u}^2) (4 e_k^2 - M^2)}}{4 e_k^2 - M^2} \\
\times \frac{u' \cdot P + \sqrt{(u' \cdot P)^2 + (1 - \vec{u}'^2) (4 e_{k'}^2 - M^2)}}{4 e_{k'}^2 - M^2}. \ldots (72)
\]

Some simplification occurs when the Jacobian and the $\delta(\cdots)$ functions are put together:

\[
\int \frac{d\vec{p}_1'}{2 e_1'} \frac{d\vec{p}_2'}{2 e_2'} \delta((p - P) \cdot (p' - P)) \delta((p' - P)^2) \ldots \\
= \int d\vec{k}' \frac{\delta(1 - \vec{u} \cdot \vec{u}') \delta(1 - \vec{u}'^2)}{u \cdot P + \sqrt{(u \cdot P)^2 + (1 - \vec{u}'^2) (4 e_k^2 - M^2)}} \\
\times \frac{2 e_k}{ \sqrt{(u' \cdot P)^2 + (1 - \vec{u}'^2) (4 e_{k'}^2 - M^2)} \sqrt{(u' \cdot P)^2 + (1 - \vec{u}'^2) (4 e_{k'}^2 - M^2)}} \ldots \\
= \int d\vec{k}' \frac{\pi}{e_{k'} (4 e_{k'}^2 - M^2)} \ldots \ldots (73)
\]
In writing the last line, we took into account the relation, \(1 - \vec{u}'^2 = 0\), while assuming that the factor represented by dots does not depend on \(\vec{u}'\).

## D Normalization

The present appendix is devoted to the \(\delta(\cdots)\) functions appearing in the general expression of the normalization, Eq. (28) or also in Eq. (59). By using the conditions implied by the different \(\delta(\cdots)\) functions, one first successively gets:

\[
\begin{align*}
\delta((p - P)^2) & \delta((p - P) \cdot (p - P')) \delta((p - P')^2) \\
= \delta((p - P)^2) & \delta((p - P) \cdot (P - P')) \delta((p - P')^2) \\
= \delta((p - P)^2) & \delta((p - P) \cdot (P - P')) \delta((P - P')^2) \\
= \delta((p - P)^2) & \left( \frac{u \cdot P + \sqrt{(u \cdot P)^2 + (1 - \vec{u}'^2)(4 e_k^2 - M^2)}}{4 e_k^2 - M^2} \right) \\
\times & \delta(u \cdot (P - P')) \delta((P - P')^2), \\
\end{align*}
\]

(74)

where the last term of the equality is obtained by rewriting the term, \(\delta((p - P) \cdot (P - P'))\).

Assuming in a next step \(P^2 = P'^2\), the two last factors can be shown to be equivalent to a 3-dimensional-\(\delta(\cdots)\) function which implies the conservation of the total 3-momentum. In this order, it is convenient to introduce the components of \((\vec{P} - \vec{P}')\) parallel and perpendicular to \(\vec{u}\). Some detail follows:

\[
\begin{align*}
\delta(u \cdot (P - P')) & \delta((P - P')^2) \\
= \delta(E_P - E_{P'} - \vec{u} \cdot (\vec{P} - \vec{P}')) & \delta((E_P - E_{P'})^2 - (\vec{P} - \vec{P}')^2) \\
= \delta(E_P - E_{P'} - \vec{u} \cdot (\vec{P} - \vec{P}')) & \delta((\vec{P} - \vec{P}')^2) \\
= \pi & \frac{\delta((\vec{P} - \vec{P}')_||)}{1 - \vec{u} \cdot \vec{P}/E_P} \delta^2((\vec{P} - \vec{P}')_\perp) \\
= \pi & \frac{E_P}{u \cdot P} \delta(\vec{P} - \vec{P}'). \\
\end{align*}
\]

(75)

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