Analytic Four-Point Lightlike Form Factors and OPE of Null-Wrapped Polygons

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We obtain for the first time the analytic two-loop four-point MHV lightlike form factor of the stress-tensor supermultiplet in planar $\mathcal{N}=4$ SYM where the momentum $q$ carried by the operator is taken to be massless. Remarkably, we find that the two-loop result can be constrained uniquely by the infrared divergences and the collinear limits using the master-bootstrap method. Moreover, the remainder function depends only on three dual conformal invariant variables, which can be understood from a hidden dual conformal symmetry of the form factor arising in the lightlike limit of $q$. The symbol alphabet of the remainder contains only nine letters, which are closed under the action of the dihedral group $D_4$. Based on the dual description in terms of periodic Wilson lines (null-wrapped polygons), we also consider a new OPE picture for the lightlike form factors and introduce a new form factor transition that corresponds to the three-point lightlike form factor. With the form factor results up to two loops, we make some all-loop predictions using the OPE picture. A preliminary study at three loops using symbol bootstrap is discussed.

I. INTRODUCTION

Notable progress has been made recently in the study of form factors (FFs) of the stress-tensor supermultiplet in planar $\mathcal{N}=4$ SYM. At non-perturbative level, the form factor operator product expansion (FFOPE) was developed in [1–3], which is similar to the OPE for amplitudes [4–14]. With the FFOPE data, the three-point FF has been bootstrapped to the remarkable eight loops [15, 16], using the symbol bootstrap that has been extensively used for amplitudes [17–30]. Intriguingly, these new results reveal an antipodal duality between the three-point FF and six-point amplitude [31]; a similar connection between the two quantities was previously observed at two loops in [17]. While these studies mostly focus on the explicit three-point case, it is important to explore if they can be extended to more general FFs.

In this paper we initiate the study of lightlike form factors in which the momentum of the local operator $q$ is taken to be lightlike. In planar $\mathcal{N}=4$ SYM, such FFs are dual to polygonal Wilson lines [32–34] with a lightlike period $x_{i+n} - x_i = q$, which will also be referred to as null-wrapped polygons. This duality implies an exact directional dual conformal symmetry (DDCS) of the lightlike FFs along the direction, which was shown at the integrand level for the three-point FF of the stress-tensor multiplet up to four loops in [35] (see also [36–39]). For the integrated form factors, the DDCS implies that the finite remainder function of an $n$-point lightlike FF depends on $3n - 9$ independent dual conformal invariant ratios (similar to the counting for amplitudes [40]). Thus the first non-trivial case is the four-point FF which depends on three independent variables.

Below we present a first analytic computation of the two-loop four-point MHV lightlike FF of the stress-tensor supermultiplet in planar $\mathcal{N}=4$ SYM, which is defined as

$$ F_{O4}^{LL} = \int d^D x e^{-iq \cdot x} \langle p_1, \ldots, p_4 | O(x) | 0 \rangle \bigg|_{q^2=0}, \quad (1) $$

where $p_i(p_i^2 = 0)$ are on-shell momenta, and $q = \sum_{i=1}^4 p_i$. We use the superscript “LL” to indicate the lightlike condition of $q$. This form factor may be thought of as an $\mathcal{N}=4$ version of a Higgs-plus-four-parton amplitude in the limit where the top-quark mass is taken to be infinity while the Higgs mass is taken to be zero.

We apply the master bootstrap method [41, 42], which starts with an ansatz expanded in terms of a finite set of basis integrals [43, 44]. Remarkably, we find that the constraints of infrared (IR) divergences and collinear limits, together with the dihedral symmetry, can uniquely determine the two-loop finite remainder of the four-point lightlike FF. The remainder function only depends on three independent variables, providing an explicit check of the DDCS. Moreover, the symbol of the remainder depends only on nine letters, which are dual conformal invariant and closed under the action of the dihedral group $D_4$.

The massless limit of the operator momentum also suggests a new OPE decomposition based on the dual null-wrapped polygons. Compared with the OPE in [1–3], a new lightlike FF transition is involved, which originates from the three-point lightlike FF. We define the regularized ratio functions and also give the explicit parametrization of the OPE parameters in this new picture. Using the four-point FF results up to two loops, we are able to make certain all-order predictions. The new lightlike FF transition is expected to be determined at finite coupling using integrability methods.
The simple structure of the two-loop result as well as the OPE suggest the possibility of bootstrapping the four-point lightlike FF to higher loops. We present a preliminary study at three loops. Several future directions are mentioned in the outlook section.

II. TWO-LOOP ANSÄTZ

To construct the two-loop ansatz of $F_{D,4}^{LL}$, we first review the tree and one-loop results. Without loss of generality, we choose the operator as the chiral Lagrangian $L$ in the stress-tensor multiplet. The tree-level MHV FF takes the simple form as

$$F_{4}^{LL,(0)} = \frac{\delta^{(8)}(\sum_{i} \lambda_i \eta_i)}{(12)\langle 23 \rangle \langle 34 \rangle \langle 41 \rangle}.$$  

(2)

The one-loop FF up to finite part (for general $q^2 \neq 0$) was obtained in [34]. To bootstrap to two loops, we note that it is important to have also the higher $O(\epsilon)$ result at one loop. We perform $D$ dimensional unitarity cuts to obtain the full one-loop results to all orders in $\epsilon$ expansion. The result can be reorganized in the following form

$$F_{4}^{LL,(1)} = F_{4}^{LL,(0)} I_{4}^{LL,(1)} = F_{4}^{LL,(0)} \left( G_{1}^{(1)} + B G_{2}^{(1)} \right),$$  

(3)

where $B$ is a parity-odd factor

$$B = \frac{s_{12}s_{34} + s_{23}s_{14} - s_{13}s_{24}}{tr_5}, \quad tr_5 = 4i\epsilon(1234),$$  

(4)

and $G_{a}^{(1)}$ can be given in terms of the bubble, box and pentagon master integrals; see Appendix A for details. We point out that the parity-odd part $G_{1}^{(1)}$ contains only the pentagon master integrals and is at $O(\epsilon^3)$.

Given the one-loop structure, we propose the following ansatz of two-loop planar FF:

$$F_{4}^{LL,(2)} = F_{4}^{LL,(0)} I_{4}^{LL,(2)} = F_{4}^{LL,(0)} \left( G_{1}^{(2)} + B G_{2}^{(2)} \right),$$  

(5)

where the loop function $G_{a}^{(2)}$ can be expanded in terms of a set of the two-loop master integrals. Topologies with maximal numbers of propagators are shown in Fig. 1. We emphasize that since the operator leg $q$ is a color singlet, one needs to consider non-planar topologies. We choose the master integrals to be uniformly transcendentals (UT) integrals, which have been constructed in [49–53] based on the canonical differential equations method [54]. The most general ansatz contains 590 master integrals for each $G_{a}^{(2)}$:

$$G_{a}^{(2)} = \sum_{i=1}^{590} c_{a,i} I_{i}^{(2),UT},$$  

(6)

where $c_{a,i}$ are the coefficients to be solved. Since both the FF (which is BPS) and integral basis have uniform transcendentality degree 4, the coefficients $c_{a,i}$ are expected to be pure rational numbers independent of the regularization parameter $\epsilon$.

Practically, it is convenient to first consider masters in terms of their “symbol”, which is introduced in the study of the two-loop six-gluon amplitude [55]. The symbol simplifies transcendentals functions into tensor products of function arguments, for simple examples: $S(\log(x)) = x, S(Li_2(x)) = -(1 - x) \otimes x$. Substituting the master integral symbol results into our ansatz [55], we obtain an $\epsilon$-expansion form of the FF:

$$S(I_{4}^{LL,(2)}) = \sum_{k \geq 0} \epsilon^{k-4} \sum_{I} \alpha_{I}(\epsilon) \otimes \sum_{i=1}^{k} w_{I_{i}},$$  

(7)

where $w_{I}$ are rational functions of kinematic variables and are called symbol letters, and $\alpha_{I}(\epsilon)$ are linear combinations of $c_{a,i}$ in [55]. For the master integrals we consider, there are 31 independent letters and we review them in Appendix B. Since the FF is uniformly transcendentals, the tensor degree at a given order in $\epsilon$-expansion is fixed, e.g. the finite order has degree $k = 4$.

We determine the ansatz coefficients by various physical constraints in the next section.

III. PHYSICAL CONSTRAINTS

We first impose the symmetry property that the four-point FF has cyclic and flip symmetries which in total form a dihedral group $D_4$:

$$I_{4}^{(l)} = I_{4}^{(l)}|_{p_i \rightarrow p_{i+1}} = I_{4}^{(l)}|_{p_i \rightarrow p_{i+4}}.$$  

(8)

Note that the tree-level FF satisfies this symmetry. The symmetry property allows us to reduce the number of free parameters to 168.

Two further important constraints are the IR divergences [56, 57] and collinear factorization [58–60]. For the planar amplitudes or FFs in $N = 4$ SYM, a convenient representation to capture both the IR and collinear behavior is the BDS expansion [61, 62], which at two-loop gives:

$$I_{4}^{(2)} = \frac{1}{2} (I_{4}^{(1)}(\epsilon))^2 + f^{(2)}(\epsilon) I_{4}^{(1)}(2\epsilon) + R_{4}^{(2)} + O(\epsilon),$$  

(9)
where \( f^{(2)}(\epsilon) = -2\zeta_2 - 2\zeta_3\epsilon - 2\zeta_4\epsilon^2 \). The finite remainder function \( R \) has the nice collinear behavior

\[
R^{LL,(2)}_4 \frac{p_i \parallel p_{i+1}}{p_i} R^{LL,(2)}_3 = -6\zeta_4, \tag{10}
\]

where as mentioned in the introduction, the remainder of the three-point FF with \( q^2 = 0 \) is a pure number.

We first apply the IR and collinear constraints at the symbol level as in (7), and we are able to reduce the number of free parameters to 43. Then we consider the master integrals at the function level. It is sufficient to compute the integrals numerically with high precision, using \[59\], as well as the packages DiffExp \[60\] or AM-Flow \[61\]. We find the IR and collinear constraints at the function level can fix further 22 parameters.

Remarkably, it turns out that the remaining 21 degrees of freedom all contribute to \( \mathcal{O}(\epsilon) \) order. These terms can be safely ignored if one is only interested in up to the finite order of the two-loop FF. In other words, the two-loop finite remainder is uniquely fixed by the IR and collinear constraints.

As important cross-checks, we have also applied a spanning set of \( D \)-dimensional unitarity cuts \[59\], \[62\], \[63\] and find full consistency with the bootstrap result. This not only verifies the ansatz we made but also fixes the coefficients of the remaining 21 degrees of freedom at \( \mathcal{O}(\epsilon) \) order.

We summarize the parameters after each constraint in Table I. All master coefficients are small rational numbers (up to the factor \( B \)).

\begin{table}[h]
\centering
\begin{tabular}{ |c|c| }
\hline
Constraints & Parameters left \\
\hline
Starting ansatz & \( 590 \times 2 \) \\
Symmetries & 168 \\
IR (Symbol) & 109 \\
Collinear limit (Symbol) & 43 \\
IR (Function) & 39 \\
Collinear limit (Function) & 21 \\
Keeping up to \( \epsilon^0 \) order or via unitarity & 0 \\
\hline
\end{tabular}
\caption{Solving for parameters via master bootstrap.}
\end{table}

**IV. TWO-LOOP REMAINDER**

The two-loop finite remainder \( R^{LL,(2)}_4 \) presents several nice properties. First, all the terms depending on \( B \) cancel in the two-loop finite remainder function, namely,

\[
\mathcal{G}^{(2)}_2 = \mathcal{G}^{(1)}_2 (\epsilon) + \mathcal{O}(\epsilon). \tag{11}
\]

This is consistent with the dual Wilson line picture, which has no \( B \) factor contribution. See a similar cancellation for the six-gluon amplitude in \[64\].

Furthermore, although the four-point FF depends on five Mandelstam variables, we find the remainder function only depends on three variables

\[
u_1 = \frac{s_{12}}{s_{34}}, \quad u_2 = \frac{s_{23}}{s_{14}}, \quad u_3 = \frac{s_{123}s_{134}}{s_{234}s_{124}}, \tag{12}
\]

which are dual conformal invariant in the dual momentum space \( (x_{i+1} - x_i = p_i) \) \[65\] as shown in Figure 2. Here the dual conformal symmetry is the special conformal transformation along the lightlike \( q \) direction in the dual \( x \) space:

\[
\delta_q x_i^\mu = \frac{1}{2} x_i^2 q^\mu - (x_i \cdot q) x_i^\mu, \tag{13}
\]

see \[66\] for detailed discussion. Our result verifies the DDCS of the integrated lightlike form factor mentioned in the introduction. Explicit numerical checks are also given in Appendix C.

The two-loop remainder has transcendentality-degree four and its symbol can be expressed as:

\[
S(R^{LL,(2)}_4) = \sum_{i=1}^{1283} c_i w_{i1} \otimes w_{i2} \otimes w_{i3} \otimes w_{i4}, \tag{14}
\]

where \( w_i \) are symbol letters, and \( c_i \) are pure numbers. It turns out that the symbol alphabet of the two-loop remainder can be chosen as

\[
\{u_1, u_2, u_3, 1 - u_1, 1 - u_2, 1 - u_3, x_{1234}, x_{234q}, x_{123q}\}, \tag{15}
\]

where

\[
x_{ijkl} = \frac{\langle i|j|k|l\rangle}{\langle i|j|k|l\rangle}. \tag{16}
\]

It is easy to check that they are closed under the action of the dihedral group \( D_4 \). Note that \( x' \)’s are also functions of \( u_i \) in (12) (see Appendix B).

If we transform the \( u_i \) to \( s_{ij} \), there are 13 symbol letters appearing in the remainder, in contrast to 31 letters in the masters as reviewed in Appendix B. The DDCS imposes strong constraints on the possible letters; for example, the two letters \( x_{234q} \) and \( x_{123q} \) are not directional dual conformal invariant. Let us also comment on the properties associated to each entry: i) The first entry contains only \( s_{1i, i+1} \) and \( s_{i1, i+1} \) for \( i = 1, \ldots, 4 \); ii) The second entry contains all symbol letters; iii) The third entry is free from the letters \( s_{12} - s_{34} \) and \( s_{23} - s_{14} \); iv) The last entry is free from \( s_{12} - s_{34}, s_{23} - s_{14}, x_{1234} \), and \( x_{234q} \). The full two-loop symbol is provided in the ancillary file.

We also consider “BDS-like remainder” as in \[12\] \[10\]:

\[
\mathcal{E}_4 = \exp \left[ \frac{\Gamma_{\text{cusp}}}{4} c^{(1)}_4 + R^{LL}_4 \right]. \tag{17}
\]

\[\text{FIG. 2: Dual periodic Wilson line.}\]
where $\Gamma_{\text{cusp}} = 4g^2 + ..$ is the cusp anomalous dimension [69, 71] and the one-loop BDS-like function $E_4^{(1)}(u_1, u_2, u_3)$ can be chosen as the $D_4$-symmetrization of $W_4^{(1)}$ given later in [21] (see Appendix [1] for explicit expressions). Nicely, we find that the 1283 terms of $S(R_4^{(1)})$ in [14] cancel significantly by $S(E_4^{(1)})^2/2$, and the new $S(E_4^{(1)})$ contains only 456 terms. This implies that the BDS-like remainder has simpler properties.

The simplicity of the symbol alphabet makes it promising to explore high-loop remainders using the symbol bootstrap method. The knowledge of the OPE will be essential to provide new constraints. In the next section, we consider such a picture for the lightlike FFs.

\section{Lightlike FF OPE}

An OPE was proposed for the FF with general $q^2 \neq 0$ in [13, 14]. In principle, one may apply this program first for general $q$ and then take the lightlike limit to obtain the lightlike FFs. It is however much better if there is a straightforward picture for lightlike FFs. Indeed, we find a new OPE construction that takes into account the on-shell property of $q$ from the beginning.

Since the three-point lightlike FF is “trivial” in the sense that its finite remainder has no kinematic dependence, it should be used as a building block in the OPE construction, similar to the two-point FF for the general $q$ case. Indeed, one can decompose an $n$-sided null-wrapped polygon into a three-sided null-wrapped polygon and $n - 3$ pentagons:

$$W_n^{(n)} = \sum_{\psi_1, ..., \psi_{n-2}} \epsilon_{\psi_1, ..., \psi_{n-2}}(-E_j r_j + ip_j \sigma_j + im_j \phi_j)$$
$$\times P(0\vert \psi_1) ... P(\psi_{n-4} \vert \psi_{n-3}) F_3^{(1)}(\psi_{n-3}),$$

where $F_3^{(1)}(\psi)$ is the lightlike factor transition. Here it is instructive to make an analogy with the six-point amplitudes and three-point FF with $q^2 \neq 0$ (see Figure 3):

$$W_4^{(1)} = \sum_{\psi} e^{-E_1 r_1 + i p_1 \sigma_1 + i m_1 \phi_1} P(0 \vert \psi) F_3^{(1)}(\psi),$$
$$W_6^{(1)} = \sum_{\psi} e^{-E_1 r_1 + i p_1 \sigma_1 + i m_1 \phi_1} P(0 \vert \psi) F_3^{(1)}(\psi),$$

which makes it clear that $F_3^{(1)}(\psi)$ is a new type of transition. The similarity of the OPE decompositions in (19) implies that there may be certain connections between $F_4^{(1)}$ and $A_6, F_3$. Since the decomposition with pentagon transition parts is the same as in [1], below we only need to focus on the four-point case which is sufficient to capture all new properties.

The UV divergences of the Wilson line can be regularized by introducing the following finite conformally invariant ratio:

$$W_n^{(n)} = \frac{F_4^{(1)} \times W_{\text{eq}}}{F_3^{(1)} \times W_{\text{pent}}},$$

which is shown explicitly in Figure 3. The generalization to higher points is straightforward. The one-loop finite ratio can be obtained as [22]

$$W_4^{(1)} = -\frac{1}{2} \log \left( \frac{(1 - u_1)(1 - u_2)}{1 - u_3} \right) \log \left( \frac{(1 - u_1)(1 - u_2)u_3}{(1 - u_3)u_1} \right),$$

which indeed depends only on the three ratio variables in (12).

With the one-loop result, one can obtain $W_n^{(n)}$ from the FF remainder $R_n^{(1)}$ as (see also [1, 13])

$$W_n^{(n)} = \exp \left[ \frac{\Gamma_{\text{cusp}}}{4} W_n^{(n)} + R_n^{(1)} \right].$$

In particular, from the previous $R_4^{(2)}$ we also have the two-loop Wilson-line result $W_4^{(2)}$.

To take OPE limit, one parametrizes three symmetries of the null square by three parameters $\{\sigma, \tau, \phi\}$ [4]. They are related to the ratios $u_i$ in [21] in a nice form as shown in Figure 5 (see Appendix [13] for detail). The OPE limit is obtained by taking $\tau \to \infty$, corresponding to $s_{21} \to 0$. The expansion of the one-loop result [21] in this limit

\begin{align*}
\frac{u_1}{s_{21}} &= e^{-2\sigma}, & u_2 &= \frac{x^2}{x_{42}} = e^{-2\tau} \\
\frac{u_3}{s_{21}} &= \frac{x^2}{x_{42}} s_{21^2} = \frac{\cosh(\sigma - \tau) + \cos(\phi)}{\cosh(\sigma + \tau) + \cos(\phi)}
\end{align*}
The expansion of the two-loop result is
\[ W_4^{\text{LL,(2)}} = e^{-\tau} \cos(\phi)f^{(2)}(\phi) + O(e^{-2\tau}), \]
where \( f^{(2)}(\sigma) \) is a function of \( \sigma \) (see Appendix \[F\]).

Similar to the six-gluon amplitude \([3]\), the OPE analysis shows that the leading contribution of order \( e^{-\tau} \) should come from the lightest state: gluon excitation states \( F \) and \( \bar{F} \). Interestingly, we can see that the one-loop OPE expansion starts at \( e^{-2\tau} \), and the two-loop starts at \( e^{-4\tau} \). These suggest that the transition \( \mathcal{F}_{\Lambda}^{\psi}(\psi) \) for the single gluon excitation starts at order \( g^2 \).

Simple observation already provides a non-trivial prediction to all orders, namely,
\[ W_4^{\text{LL,(f)}}|_{e^{-1}e^{-1}} = 0, \quad \text{for all } \ell. \]

Based on \([15]\), one can also make another all-order prediction:
\[ W_4^{\text{LL,(f)}}|_{e^{-1}e^{-1}} = \cos(\phi)f^{(f)}(\phi), \]

where \( f^{(f)}(\sigma) \) can be determined by \( f^{(2)}(\sigma) \) and one-loop anomalous dimension \( \dot{\lambda}(1) \) \([72]\). See Appendix \[F\] for further detail and we leave a more systematic study to another work.

\section{VI. TOWARDS HIGH-LOOP SYMBOL BOOTSTRAP}

Based on the simple symbol alphabet \([15]\), and combined with the OPE prediction, it is natural to bootstrap the four-point FF to higher loops like the three-point case \([15,16]\). As a preliminary study, we consider the construction of the remainder using the symbol bootstrap method up to three loops.

We start with an ansatz with only the letters in \([15]\) and no \( x_{ijkl} \) in the first entry. Next, we impose the \( D_4 \) symmetry and the integrability condition for the symbol. Furthermore, we require the symbol to be zero in the collinear limit. We then apply two “minimal” entry conditions: 1) the first-entry must be either \( s_{i,i+1} \) or \( s_{i,i+1,i+2} \) (the branch cut condition), and 2) the first \( L \) entries can not all be the same \( s_{i,i+1} \) (the \( L^\text{th} \) discontinuity condition) \([15,24]\). Finally, we impose the OPE constraints at \( e^{-\tau} \) order. We summarize the constraints and the change of free parameters in Table \[II\].



| Constraints          | Parameters left |
|----------------------|-----------------|
| Starting ansatz      | 4374 354294     |
| \( D_4 \) symmetry   | 561 44409       |
| Integrability        | 72 1056         |
| Collinear limit      | 56 992          |
| Branch cut condition | 21 295          |
| \( L^\text{th} \) discontinuity condition | 8 206 |
| \( \mathcal{F}_{\Lambda}^{\psi}(\psi) \) (leading \( e^{-\tau}\gamma^L \)) | 5 193 |
| \( \mathcal{F}_{\Lambda}^{\psi}(\psi) \) (\( e^{-\tau}\gamma^L \)) \( \ell < L - 1 \) | 1 126 |

\begin{table}[h]
\centering
\begin{tabular}{|c|c|}
\hline
Loops & 2-loop 3-loop \\
\hline
\end{tabular}
\end{table}

\section{VII. OUTLOOK}

We mention several future directions based on this work.

I) One immediate problem is to determine the lightlike FF transition \( \mathcal{F}_{\Lambda}^{\psi}(\psi) \) non-perturbatively via integrability \([14,15]\). This will help to ultimately determine the high-loop four-point FF remainder via symbol bootstrap.

II) Similar to the three-point case \([17,31]\), the \( D_4 \) symmetry, as well as the connection of the alphabet (see \([72]\)), suggest possible connections between the four-point FF and the eight-point amplitude. The antipodal symmetry of the latter was studied recently in \([76]\). With the known two-loop remainder of eight-point MHV amplitude \([72,78]\), our four-point FF result provides timely concrete data to study such connections.

III) As mentioned in the introduction, the four-point FF we consider is an analogy of Higgs-plus-four-gluon scattering. It is interesting to check the maximally transcendental principle \([17,79,82]\) in this case, where the master bootstrap method can be used as in \([42]\).

IV) It would be interesting to consider the lightlike FFs at strong coupling using Y-system \([33,83-85]\) and also in the OPE limit.

V) It would be nice to extend our study to non-MHV cases or super-Wilson lines \([80,88]\), as well as local operators other than the stress-tensor supermultiplet \([89-98]\).

VI) Finally, although the focus of this paper is the lightlike FF with \( q^2 = 0 \), the master bootstrap method can be used to compute the four-point FF with general \( q^2 \neq 0 \). Although the master integrals are not fully known analytically, one can still apply the master-bootstrap method numerically with high precision, using \( e.g. \) AMFlow package \([64]\). In particular, the \( q^2 = 0 \) result provides essential input to bootstrap the general \( q^2 \neq 0 \) cases.

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**Appendix A: One-loop result**

The one-loop result can be given as the cycling summation of the following density functions

\[
\mathcal{I}_4^{(1)}(i, i+1, i+2, i+3) + \mathcal{I}_4^{(1)}(i, i+1, i+2, i+3),
\]

A1

where

\[
\mathcal{I}_4^{(1)}(1, 2, 3, 4) = -\frac{1}{2} f_{\text{Box}}^{(1)}(1, 2, 3) - \frac{1}{2} f_{\text{Box}}^{(1)}(1, 2, 3, 4) + I_{\text{Bub}}^{(1)}(1, 2, 3),
\]

A2

\[
\mathcal{I}_4^{(1)}(1, 2, 3, 4) = \frac{B - 1}{2} f_{\text{Pen}}^{(1)}(1, 2, 3, 4).
\]

Here \(B\) is the parity-odd factor defined in (3), and the one-loop UT master integrals are

\[
I_{\text{Bub}}^{(1)}(1, \ldots, n) = \frac{1 - 2\epsilon}{\epsilon} \times \begin{array}{c}
\bullet
\!
\begin{array}{c}
\downarrow
\end{array}
\end{array}_n,
\]

A3

\[
I_{\text{Box}}^{(1)}(i, j, k) = (s_{ij}s_{jk} - p_j^2 q^2) \times \begin{array}{c}
\bullet
\!
\begin{array}{c}
\downarrow
\end{array}
\end{array}_k,
\]

A4

\[
I_{\text{Pen}}^{(1)}(i, j, k, l) = \text{tr}_5 \times \mu \times \begin{array}{c}
\bullet
\!
\begin{array}{c}
\downarrow
\end{array}
\end{array}_l,
\]

A5

in which \(\mu = l^{-2\epsilon} \cdot l^{-2\epsilon}\), and \(p_j\) in \(I_{\text{Box}}^{(1)}(i, j, k)\) can be a massive momentum.

**FIG. 6: D-dimensional cut for the one-loop FF.**

The above result can be obtained via unitarity cuts. In particular, to determine the pentagon contribution, a \(D\) dimensional cut is necessary as in Figure 6. Note that the parity-odd part of the one-loop result is contained in \(\mathcal{I}_4^{(1)}(1, 2, 3, 4)\) by the pentagon master and contributes to \(\mathcal{O}(\epsilon)\) only.

**Appendix B: Symbol letters**

There are 31 independent symbol letters for all two-loop five-point master integrals we use in this paper [38]:

\[
\{s_{12}, s_{23}, s_{34}, s_{4q}, s_{q1},\}
\]

(B1)

\[
\{s_{13}, s_{24}, s_{3q}, s_{14}, s_{2q},\}
\]

\[
\{s_{12} + s_{23}, s_{23} + s_{34}, s_{34} + s_{4q}, s_{4q} + s_{q1}, s_{q1} + s_{12}, s_{12} - s_{34}, s_{23} - s_{4q}, s_{34} - s_{q1}, s_{4q} - s_{12}, s_{q1} - s_{12}, s_{12} + s_{24}, s_{23} + s_{3q}, s_{34} + s_{14}, s_{4q} + s_{2q}, s_{q1} + s_{13}, x_{1234}, x_{123q}, x_{124q}, x_{134q}, x_{234q}, \text{tr}_5(1234),\}
\]

(B2)

On the other hand, in the final remainder of the two-loop four-point FF, only nice letters in [15] are needed to express the remainder symbol as discussed in the main text. If we expand ratio variables \(u_i\) in terms of \(s_{ij}\), there are only 13 symbol letters appearing in the remainder:

\[
\{s_{12}, s_{23}, s_{34}, s_{41}, s_{1q}, s_{2q}, s_{3q}, s_{4q},\}
\]

(B3)

\[
\{s_{12} - s_{34}, s_{23} - s_{41}, x_{1234}, x_{123q}, x_{234q}\}\}
\]

(B4)

One can write \(x\) variables in other forms as

\[
x_{1234} = \frac{B + 1}{B - 1}, \quad x_{234q} = \frac{B' + 1}{B' - 1}, \quad x_{123q} = \frac{B'' + 1}{B'' - 1},
\]

where the parity-odd ratio \(B\) appears in the one-loop FF in the coefficients of UT master integrals, and \(B'\) and \(B''\) have a similar structure. They can be given in terms of three cross ratios \(u_i\) as

\[
B^2 = \frac{[u_2 + u_3 + (u_2 u_3 - 1)]^2}{Y}, \quad B'^2 = \frac{[-u_1 + u_2 + (u_1 - 2) u_2 u_3 + u_3]^2}{Y}, \quad B''^2 = \frac{[u_2 (u_3 - 2)] u_1 + u_1 + u_2 - u_3]^2}{Y},
\]

\[
Y = \frac{(u_2 - u_3)^2 + u_3^2 (u_2 u_3 - 1)^2 - 2 u_1 [u_3 (u_2 + u_3 - 4) u_2 + u_2 + u_3]}{u_1 - u_3}.
\]

The nine letters in [15] in the final remainder symbol are closed under the action of the dihedral group \(D_4\):

\[
\{u_1, u_2, u_3, 1 - u_1, 1 - u_2, 1 - u_3\} \xrightarrow{p_1 \rightarrow p_{1+1}} \{u_2, \frac{1}{u_1}, \frac{1}{u_3}, 1 - u_2, \frac{1 - u_1}{u_1}, \frac{1 - u_3}{u_3}\},
\]

(B5)

\[
\{x_{1234}, x_{234q}, x_{123q}\} \xrightarrow{p_{1 \rightarrow p_{1+1}}} \{\frac{1}{x_{1234}}, \frac{x_{1234}}{x_{123q}}, \frac{x_{234q}}{x_{123q}}\},
\]

\[
\{u_1, u_2, u_3, 1 - u_1, 1 - u_2, 1 - u_3\} \xrightarrow{p_{3 \rightarrow p_{3+1}}} \{\frac{1}{u_1}, \frac{1}{u_3}, \frac{1}{u_3}, \frac{1 - u_1}{u_1}, \frac{1 - u_2}{u_2}, \frac{1 - u_3}{u_3}\},
\]

\[
\{x_{1234}, x_{234q}, x_{123q}\} \xrightarrow{p_{3 \rightarrow p_{3+1}}} \{\frac{x_{1234}}{x_{123q}}, \frac{1}{x_{123q}}, \frac{1}{x_{234q}}\}.
\]
TABLE III: Numerical results of the two-loop four-point lightlike FF: the kinematics for the first column are chosen as \{s_{12} = -3, s_{13} = 5, s_{14} = 6, s_{23} = -7, s_{24} = -9, s_{34} = 8, tr_{5} = 3\sqrt{399}i\}, and the second column chosen as \{s_{12} = -200/63, s_{13} = 52700/10647, s_{14} = 1000/169, s_{23} = -3500/507, s_{24} = -295600/31941, s_{34} = 1600/189, tr_{5} = \sqrt{19}/21 \times 10^{3}/1521i\}. 

We consider two different set of Mandelstam variables which give the same three cross ratios as \(u_{1} = -3/8, u_{2} = -7/6, u_{3} = -95/48\). (Here the kinematics are related by a small directional dual conformal transform to avoid the interference of the analytic continuation.) The numerical results of the two-loop four-point FF and the remainder are shown in the Table III. One can see that although the form factor results are different, the remainder is the same.

Appendix C: Numerical check of DDCS

We provide some details on the parametrization of Appendix D: One-loop BDS-like functions

We provide two choices of one-loop BDS-like functions used to define the modified remainder in \([17]\). One is obtained from the \(D_{4}\)-symmetrization of \(\mathcal{W}_{4}^{LL(1)}\) in \([21]\):

\[
\mathcal{E}_{4}^{(1)} = \left[ \log \left( \frac{(1 - u_{1})(1 - u_{2})}{1 - u_{3}} \right) - \frac{1}{2} \log(u_{1}u_{2}) \right] \times \left[ \log \left( \frac{(1 - u_{1})(1 - u_{2})u_{3}}{1 - u_{3}} \right) - \frac{1}{2} \log(u_{1}u_{2}) \right] (D1)
\]

Another choice is a modification of \(\mathcal{W}_{4}^{LL(1)}\):

\[
\mathcal{E}_{4}^{\prime(1)} = \log \left( \frac{(1 - u_{1})(1 - u_{2})}{1 - u_{3}} \right) \log \left( \frac{(1 - u_{1})(1 - u_{2})u_{3}}{1 - u_{3}} \right) \frac{\log(u_{1}) \log(u_{2}) - \log(u_{1}u_{2}) \log(u_{3})}{2} . (D2)
\]

Appendix E: FFOPE parametrization

We provide some details on the parametrization of the Wilson line used in the OPE picture. We define the twistor variables as in Fig. 7. The OPE parameters \(\{\tau, \sigma, \phi\}\) are related to the three ratios \(u_{i}\) in \([12]\) as follows:

\[
u_{1} = \frac{x_{12}^{2}}{x_{31}^{2}} = \frac{(4, \bar{1})(4, 1, 2, 3)}{(4, 1)(2, 3, 4, 1)} = e^{-2\sigma}, (E1)
\]

\[
u_{2} = \frac{x_{23}^{2}}{x_{42}^{2}} = \frac{(1, \bar{2})(1, 2, 3, 4)}{(1, 2)(3, 4, 1, 2)} = e^{-2\tau}, (E2)
\]

\[
u_{3} = \frac{x_{14}^{2}x_{32}^{2}}{x_{21}^{2}x_{43}^{2}} = \frac{(1, \bar{2})(4, \bar{1})(1, 2, 3, 4)(2, 3, 1, 2)}{(4, 1)(2, 3)(1, 2)(1, 2, 4, 1)(3, 4, 2, 3)} = \cosh(\sigma - \tau) + \cos(\phi) \cosh(\sigma + \tau) + \cos(\phi) . (E3)
\]

The momentum twistors \(Z_{i}\) are related spinor variables in the standard way as

\[
Z_{i}^{A} = (\lambda_{i}^{\alpha}, \mu_{i}^{\dot{\alpha}}), \quad \mu_{i}^{\dot{\alpha}} = x_{i}^{\alpha \dot{\alpha}} \cdot \lambda_{i\alpha} = x_{i}^{\alpha \dot{\alpha}} \cdot \lambda_{i\alpha}. \quad (E4)
\]

An explicit choice of the twistor variables can be chosen as follows. Similar to the paper \([6]\), we set the following twistor constants:

\[
Z_{q} = \{1, 0, 0, 0\}, \quad Z_{4} = \{-1, 1, -1, 1\}, \quad (E5)
\]

\[
Z_{A} = \{1, -1, 0, 0\}, \quad Z_{Z} = \{-1, 1, -2, 2\}, \quad (E6)
\]

\[
Z_{T} = \{1, 1, -1, 1\}, \quad Z_{T} = \{1, 1, 0, -2\}, \quad (E7)
\]

\[
Z_{1} = \{0, 1, 0, 0\}, \quad Z_{1} = \{0, 1, 1, -1\}, \quad (E8)
\]

and those depending on the \(\{\tau, \sigma, \phi\}\) are defined via transformation

\[
Z_{2} = \{2, 2, 1, -3\}M, \quad Z_{3} = \{0, 2, 0, -2\}M, \quad (E9)
\]

\[
Z_{2} = \{2, 2, 1, -3\}\bar{M}, \quad Z_{3} = \{0, 2, 2, -4\}\bar{M}, \quad (E10)
\]

where the matrices \(M\) and \(\bar{M}\) are
One can check the self-consistence condition
\[
P(Z_i M) = P(Z_i) P(M), \tag{E13}
\]
where \( P \) denotes the periodic translation
\[
P(Z_i) \equiv Z_i = Z_i + \{0, 0, q^{\alpha \dot{\alpha}} \lambda_{\alpha \dot{\alpha}} \}, \tag{E14}
\]
and \( P(M) \equiv \bar{M} \).

Following the above parametrization of the momentum twistors, we perform the calculation (such as the one-loop result \([21]\)) within the kinematics region
\[
\{ s_{12} > 0, s_{23} < 0, s_{34} > 0, s_{14} < 0, \\
s_{123} < 0, s_{234} < 0, s_{234} > 0, s_{134} > 0 \}. \tag{E15}
\]

**Appendix F: Some details on FFOPE**

The four-point finite ratios in \([20]\) can be given as
\[
\mathcal{W}_4^{\text{LL}} = \sum_a \int d\mathbf{u} e^{-E(u)\tau + ip(u)\sigma + im(u)\phi} F^a_n(0|u) F^{\text{LL}}_{\bar{a}}(u), \tag{F1}
\]
where the sum is for the GKP eigenstates represented by \( a \) and their Bethe rapidities \( \mathbf{u} \), the integral measure is
\[
d\mathbf{u} = \prod_{i=1}^N \frac{d\mathbf{u}_i}{2\pi} \mu_{\alpha i}(u_i). \tag{F2}
\]
The leading contribution of the OPE comes from the lightest state: gluon excitation states \( F \) and \( \bar{F} \), similar to the six-gluon case (rather than the FF with \( q^2 \neq 0 \) case), and this leads to
\[
\mathcal{W}_4^{\text{LL}} = 1 + 2 \cos(\phi) \hat{f}(\tau, \sigma) + \ldots, \tag{F3}
\]
where \( \hat{f}(\tau, \sigma) = \int \frac{d\mathbf{u}}{2\pi} \mu(u) e^{-E(u)\tau + ip(u)\sigma} \mathcal{P}(0|\mathbf{u}) F^{\text{LL}}_{\bar{a}}(u) \). \tag{F4}

One can normalize the pentagon transition \( \mathcal{P}(0|\mathbf{u}) \) to be one \( \mathbb{1} \), and
\[
\mu(u) = \frac{\pi g^2}{\cosh(\pi u)} (1 + \mathcal{O}(g^2)). \tag{F5}
\]

To consider the dependence of \( \tau \), one uses \( E(u) = 1 + \gamma(u) \) and obtains
\[
\hat{f}(\tau, \sigma) = g^2 e^{-\tau} \int \frac{du}{2\pi} \cosh(\pi u) e^{-\gamma(u)\tau + ip(u)\sigma} F^{\text{LL}}_{\bar{a}}(u). \tag{F6}
\]
Since \( \gamma(u) \) is at order \( g^2 \) (and \( p(u) = 2u + \mathcal{O}(g^2) \)), one finds that at the \( \ell \)-th loop order (i.e. \( g^{2\ell} \) order), the function dressing \( e^{-\tau} \) is a polynomial of degree \( \ell - 1 \) in \( \tau \).

It is clear that, if \( F^{\text{LL}}_{\bar{a}}(u) \) to start at \( g^{2\ell} \) order:
\[
F^{\text{LL}}_{\bar{a}}(u) = g^{2\ell} y_{\bar{a}}(u) + \mathcal{O}(g^4), \tag{F7}
\]
the leading terms \( e^{-\tau} \tau^{\ell-1} \) are zero to all order. As discussed in the main text, this is what one would expect from the one- and two-loop results, and this simple observation leads to all-order predictions as \([23]\):
\[
\mathcal{W}_4^{\text{LL,}(\ell)\tau \rightarrow e^{-\tau}} = 0, \quad \text{for } \ell. \tag{F8}
\]

At two-loop order, one has the leading order expansion at \( e^{-\tau} \) as in \([24]\):
\[
\mathcal{W}_4^{\text{LL,}(2)} = e^{-\tau} \cos(\phi) f^{(2)}(\sigma) + \mathcal{O}(e^{-2\tau}), \tag{F9}
\]
with
\[
f^{(2)}(\sigma) = \int \frac{du}{2\pi} \pi y_{\bar{a}}(u) e^{2u\sigma}. \tag{F10}
\]
Using the two-loop FF result, one can extract \( f^{(2)}(\sigma) \) as
\[
f^{(2)}(\sigma) = 16 \left\{ -\cosh(\sigma) \left[ L_{31} (e^{-2\sigma}) + \sigma L_{2} (e^{-2\sigma}) \right] \\
+ \sinh(\sigma)L_{2} (1 - e^{-2\sigma}) - e^{-\sigma} \left( \frac{1}{3} \sigma^3 + \sigma^2 \right) \right\} \tag{F11}
\]
+ terms beyond symbol.
With this result, one can make another all-order prediction:

\[ \mathcal{W}_{4}^{LL}(\ell) = \tau^\ell e^{-\tau} \cos(\phi) f^{(\ell)}(\sigma) + \mathcal{O}(\tau^{\ell-3} e^{-\tau}), \]  

(F12)

where

\[ f^{(\ell)}(\sigma) = \int \frac{du}{2\pi} e^{iu\omega} \frac{(-1)^{\ell-2}}{(\ell - 2)!} \left[ E^{(1)}(u) \right] \tau^{\ell-2} \pi y_F(u) \cos(\pi u), \]  

(F13)

and \( E^{(1)}(u) \) is the one-loop anomalous dimension in \( \gamma(u) = g^2 E^{(1)}(u) + \mathcal{O}(g^4) \).

We point out that for multiple excitations, since the one-loop OPE has non-zero terms of \( e^{-m\tau} \) with \( m \geq 2 \), the transition should start at \( \ell \sim (g^0)^d \) order.

Finally, we mention that in principle the lightlike FFOPE is related to the FFOPE in [4]. For example, the regularized ratio function \( \mathcal{W}_{4}^{LL} \) can be formally obtained as a limit:

\[ \mathcal{W}_{4}^{LL} = \frac{\mathcal{W}_4^F}{\mathcal{W}_4^F_{q^2 \to 0}}, \]  

(F14)

which implies a schematical relation for the transition function as

\[ \mathbb{F}_{3}^{LL}(\psi) \sim \left[ \sum_{\psi'} e^{-E' + ip'S'} \mathbb{F}(\psi|\psi') \mathbb{F}_2(\psi') \right]_{\text{lightlike limit}}. \]

It would be interesting to explore the limit from this point of view.

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