Stratification, superfluidity and magnetar QPOs

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ABSTRACT
The violent giant flares of magnetars excite QPOs which persist for hundreds of seconds, as seen in the X-ray tail following the initial burst. Recent studies, based on single-fluid barotropic magnetar models, have suggested that the lower-frequency QPOs correspond to magneto-elastic oscillations of the star. The higher frequencies, however — in particular the strong 625 Hz peak — have proved harder to explain, except as high mode multipoles. In this work we study the time evolutions of non-axisymmetric oscillations of two-fluid Newtonian magnetars with no crust. We consider models with superfluid neutrons and normal protons, and poloidal and toroidal background field configurations. We show that multi-fluid physics (composition-gradient stratification, entrainment) tends to increase Alfvén mode frequencies significantly from their values in a single-fluid barotropic model. The higher-frequency magnetar QPOs may then be naturally interpreted as Alfvén oscillations of the multi-fluid stellar core. The lower-frequency QPOs are less easily explained within our purely fluid core model, but we discuss the possibility that these are crustal modes.

Key words: stars: neutron – stars: magnetic fields – stars: oscillations – magneto-hydrodynamics (MHD)

1 INTRODUCTION
The quasi-periodic oscillations (QPOs) of magnetars provide a tantalising possibility of probing the interior physics of neutron stars. Detected in the aftermath of giant flares, these QPOs are thought to relate directly to oscillation modes of the underlying star, perturbed by the hugely energetic flare. The theoretical challenge is to provide sufficiently good models of magnetar oscillations that the observed QPOs may be convincingly linked to particular modes; this could provide potential constraints on the stellar equation of state and magnetic field configuration.

Magnetars are thought to be slowly-rotating and magnetically-powered neutron stars (NSs), with dipole surface values reaching $10^{15}$ G — an exceptional value, even for neutron stars. Three of these objects have been seen to produce enormously energetic ‘giant flares’: SGR 0526-66 (Barat et al. 1983), SGR 1900+14 and SGR 1806-20 (Israel et al. 2004; Strohmayer & Watts 2002; Watts & Strohmayer 2006). In the latter two cases, QPOs were detected in the flare’s decaying X-ray tail, between 100 and 400 seconds after the initial burst. A number of frequencies were identified within each tail, of varying duration and intensity.

Initially these QPOs were identified with crustal shear modes, and there was hope of using these to constrain the equation of state (Duncan 1998; Israel et al. 2005). Since these optimistic early studies, the picture has become more complex, due to the effect of the magnetic field: the crust and core are coupled, with modes becoming magneto-elastic in character (Levin 2006; Glampedakis et al. 2006; Gabler et al. 2012). Attempts to model observed QPOs as global modes were dealt a blow by Levin (2007), who used a toy model to show that axisymmetric oscillations of a magnetic star could form a continuum of frequencies, with discrete crustal modes quickly damped and QPOs representing the edges of the Alfvén continuum. Numerical simulations supported this conclusion in the axial case (Sotani et al. 2008), but found that polar oscillations were discrete modes, with no evidence for a continuum (Sotani & Kokkotas 2009). These two studies considered axisymmetric oscillations on a purely poloidal background field.

There are now indications that situations with coupling between axial and polar sectors may destroy the continuum. Two studies of non-axisymmetric magnetar oscillations — for purely toroidal and purely poloidal background fields — found only discrete modes (Lander et al. 2010; Lander & Jones 2011). These seemed at odds with the results for axisymmetric studies; the key difference may be that non-axisymmetric Alfvén modes are not purely axial...
or purely polar. For axisymmetric modes, a purely poloidal background field has decoupled axial and polar modes, but a mixed poloidal-toroidal field couples the two sectors. These coupled axial-polar oscillations were also found to be discrete modes (Colaiuda & Kokkotas 2012).

Despite the great increase in sophistication of magnetar QPO modelling, a major missing ingredient in all these studies has been the multi-fluid, non-barotropic nature of neutron star matter. A large fraction of mature neutron stars (including magnetars) is composed of superfluid neutrons and superconducting protons (Baym et al. 1969; Ho et al. 2012), and the resulting oscillation spectrum will surely be different from that of a barotropic star. Furthermore, there are indications that no stable magnetic equilibria exist in barotropic stars (Reisenegger 2009; Lander & Jones 2012) — in which case it is clearly undesirable to use them as background models when studying oscillations. The reason why many magnetar QPO studies have not encountered these instabilities so far is that they specialise to axisymmetric oscillations on poloidal-field backgrounds; the strongest instabilities appear for non-axisymmetric perturbations (Markev & Tayler 1973; Wright 1973).

In this work we try to advance the modelling of magnetar QPOs, by studying the oscillations of a stratified multi-fluid star. We consider poloidal and toroidal magnetic field geometries. Two main simplifications we make are that the whole star is fluid (with no crust), and that the neutrons are superfluid but the protons normal; the validity of these assumptions is discussed in the following sections.

We begin by reviewing in Section 2 the main formalism for studying the dynamics of a magnetised two-fluid star. First we determine the background configurations and then we derive the linearised dynamical equations. In Section 3 we present our results, where we explore the differences from barotropic single-fluid magnetar models. Finally in Section 4 we discuss our results in the context of magnetar QPOs and potential future improvements.

2 EQUATION OF MOTION

The basic matter constituents of a neutron star are neutrons, protons and electrons (npe), although more exotic particle species may exist in an inner core region. In this work we assume a npe-composition in the entire volume of the star and neglect the inner core.

From a dynamical point of view, we may assume that the core’s protons and electrons can be considered as a single co-moving fluid, as the electromagnetic interaction locks them on timescales much smaller than the dynamical oscillation periods. The dynamics of this system can be therefore described by a two-fluid model comprising a gas of superfluid neutrons and a neutral mixture of protons and electrons that we call for simplicity “protons”. Neutron and proton quantities will be indicated with a subscript roman n and p, respectively.

The equations to study the dynamics of a superfluid and magnetised star are, in the ideal MHD approximation, the mass and momentum conservation equations of each fluid constituent, the induction equation for the magnetic field, and the Poisson equation for the gravitational potential. These equations read (Glampedakis et al. 2011):

\[ \partial_t \rho_n + \nabla \cdot (\rho_n \mathbf{v}_n) = 0, \]

\[ \partial_t \rho_p + \nabla \cdot (\rho_p \mathbf{v}_p) = 0, \]

\[ \partial_t \rho = \nabla \times (\mathbf{v}_p \times \mathbf{B}), \]

\[ \nabla^2 \Phi = 4\pi G \rho, \]

where the labels x and y (with x \(\neq y\)) denote the fluid component n and p. The quantities \(\rho_n, \rho_p, \mathbf{v}_n, \mathbf{v}_p\) are, respectively, the mass density, chemical potential and velocity of each fluid constituent, while the relative velocity is denoted by \(\mathbf{w}_{xy} = \mathbf{v}_x - \mathbf{v}_y\). The gravitational potential and the magnetic field are described by \(\Phi\) and \(\mathbf{B}\), respectively, and \(\rho = \rho_n + \rho_p\) is the total mass density.

The parameter \(\varepsilon_x\) accounts for the entrainment between nucleons, which is a non dissipative effect that in neutron stars is due to the strong interaction. The main effect of entrainment is to couple the nucleon motion by inducing a relative dragging between neutrons and protons. As a result the conjugate momentum of each component is not aligned with its velocity, see equation (2). The entrainment parameter can be also written in terms of the nucleon’s effective mass \(m^*\) by using the relation \(\varepsilon_x = 1 - m^* / m_n\) (Prix & Rieutord 2002).

The vector field \(\mathbf{F}_x\) in equation (2) represents the force density that acts on the x fluid component. The magnetic interaction and the mutual friction are the forces expected in a magnetised superfluid star with no crust. For normal (not superconducting) protons the interaction with the magnetic field is given by the Lorentz force:

\[ \mathbf{F}_L = \frac{1}{4\pi} (\nabla \times \mathbf{B}) \times \mathbf{B}, \]

where we have used Ampère’s law to replace the charge density current. The mutual friction is instead a dissipative force mediated by superfluid vortices and operates in rotating stars on both the fluid components. In this work we are interested in the oscillation spectrum of magnetars, which are slowly rotating objects that can be well described by non-rotating stellar models. We can therefore neglect the effects of rotation and mutual friction. More details on the impact of mutual friction on the oscillation spectrum in unmagnetised stars can be found in Passamonti & Andersson (2011).

Under these assumptions, the force density in equation (2) is given by \(\mathbf{F}_n = \mathbf{F}_L\) and \(\mathbf{F}_p = 0\).

2.1 Equation of State

The equation of state (EoS) can be described by an energy functional

\[ \mathcal{E} = \mathcal{E} (\rho_n, \rho_p, w_{np}^2), \]
that ensures Galilean invariance. The chemical potential \( \hat{\mu}_x \) and the entrainment parameter \( \varepsilon_x \) are then defined by
\[
\hat{\mu}_x \equiv \frac{\partial \mathcal{E}}{\partial \rho_x} \bigg|_{\rho_p, \omega_x} ,
\]
\[
\varepsilon_x \equiv 2\rho_x \frac{\partial \mathcal{E}}{\partial \rho_p} \bigg|_{\rho_x, \omega_y} .
\]

If the relative velocity between the two fluids is small, a typical situation in most astrophysical systems, equation (6) can be expanded in series
\[
\mathcal{E} = \mathcal{E}_0 (\rho_n, \rho_p) + a_0 (\rho_n, \rho_p) u_{np}^2 + \mathcal{O} (u_{np}^4) ,
\]
and the bulk EoS \( \mathcal{E}_0 \) and the entrainment parameter \( a_0 \) can be independently specified at \( u_{np} = 0 \). This approximation is certainly valid in our star’s model as a non zero relative velocity appears at first perturbation order. From equation (8) it follows that the entrainment parameter \( \varepsilon_x \) is given by \( \rho_x \varepsilon_x = 2a_0 \).

We consider an analytical EoS, which is a two-fluid analogue of a polytropic model (Prix & Rieutord 2002; Andersson et al. 2002; Passamonti et al. 2009; Lander et al. 2012):
\[
\mathcal{E}_0 = k_n \rho_n^n + k_p \rho_p^p ,
\]
where the coefficients \( k_n \) are constants and \( \gamma_k \) is related to the adiabatic index \( N_k \) by the standard definition \( \gamma_k = 1 + 1/N_k \). Despite its simplicity, this EoS allows us to construct neutron star models with composition gradients when \( N_n \neq N_p \). Stratification is a relevant property for the oscillation spectrum and may be crucial for the stability of the background magnetic configuration.

2.2 Background models: magnetic two-fluid stellar equilibria

We begin by modelling a neutron star as an axisymmetric body in Newtonian gravity. For this system it is natural to use cylindrical polar coordinates \( (\varpi, \phi, z) \), where the \( z \)-axis is aligned with the symmetry axis of the star’s magnetic field.

We assume the whole interior of the magnetar is multi-fluid, with the neutrons superfluid, but the protons normal rather than superconducting. Although this is done for simplicity, it may be realised in magnetars if their internal field strength reaches \( \sim 10^{16} \) G; at this ‘upper critical field’ superconductivity is broken (Glampedakis et al. 2011). We do not account for the presence of the neutron star crust in this study.

Since a full description of the equilibrium equations and their solution are presented in Lander et al. (2012), we provide only a brief summary of the important details here. We neglect the inertia of the electrons and absorb their chemical potential into that of the protons, so our equations describe a two-fluid system. From equation (2), we see that the separate Euler equations governing the two fluids are:
\[
\nabla (\hat{\mu}_n + \Phi) = 0 ,
\]
\[
\nabla (\hat{\mu}_p + \Phi) = \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{4\pi \rho_p} .
\]

As in the single-fluid case we have Poisson’s equation (1)

for the gravitational potential, from which we see that the behaviour of the two fluids is coupled, despite the neutrons being a superfluid.

The equilibrium magnetic field of an axisymmetric barotropic model (Prix & Rieutord 2002; Passamonti et al. 2009; Lander et al. 2012), we provide only a brief summary of the important details here. We neglect the inertia of the electrons and absorb their chemical potential into that of the protons, so our equations describe a two-fluid system. From equation (2), we see that the separate Euler equations governing the two fluids are:
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2.3 Perturbation Equations

The perturbation equations for studying magnetised superfluid stars can be derived by linearising equations (1)-(6) and using as dynamical variables the mass density, velocity and magnetic field perturbations. However, we prefer to define a new set of perturbation variables which both reflect the dynamical degrees of freedom of a two-fluid system and simplify the implementation of the boundary conditions. A pulsating two-fluid star behaves dynamically as a coupled harmonic oscillator in which we may discern a co- and counter-moving relative motion between the two constituents. In general, these two degrees of freedom are coupled, but in some particular cases they can be completely decoupled — for instance in a non-stratified stellar model with no magnetic field.

Following Passamonti et al. (2009), we use equations (11-12) to derive the following system of dynamical equations for non-rotating stars:
\[
\partial_t \Phi = -\nabla \delta P + \frac{\nabla P}{\rho} \delta \rho - \rho \nabla \delta \Phi + \rho_p \delta (F_L/\rho_p) ,
\]
\[
\partial_t \delta \rho = -\nabla \cdot \mathbf{f} ,
\]
\[
\partial_\phi \delta \chi_p = -\nabla \cdot \delta \mathbf{D} - \delta \mathbf{f} \cdot \nabla x_p .
\]
where we have defined \( \gamma_\epsilon = 1 - \epsilon_n - \epsilon_p \) and the following fluid perturbation variables:
\[
f = \rho_n \delta v_n + \rho_p \delta v_p ,
\]
\[
\delta \rho = \delta \rho_n + \delta \rho_p ,
\]
\[
\delta \mathbf{D} = \rho_p (1 - x_p) \delta w_{pa} ,
\]
\[
\delta \chi_p = \rho \delta x_p .
\]

The quantity \( \delta P \) is the total pressure perturbation that for
a co-rotating background reads
\[
\nabla \delta P = \delta (\rho_u \nabla \mu_u + \rho_B \nabla \mu_B) ,
\]
while \(\delta \beta = \delta \mu_u - \delta \mu_B\) describes the deviation from beta equilibrium induced by oscillations and \(\delta \Phi\) is the gravitational potential perturbation.

To simplify the boundary conditions we choose a “flux” variable also for the magnetic field perturbation, i.e. we define \(b = \rho_B \delta B\). The induction equation then reads
\[
\partial_t b = \frac{\nabla \times (f \times B)}{\rho_B} + \frac{\nabla \times (\delta D \times B)}{\rho_B} - \frac{\nabla \rho}{\rho_B} \times (f \times B) - \frac{\sum \rho}{\rho_B} \times (D \times B) ,
\]
while the perturbation of the Lorentz force in equations (11) and (15) assumes the following form:
\[
\delta F_L = \left(\nabla \times B\right) \times \frac{b}{\rho_B} + \left(\nabla \times b\right) \times \frac{B}{\rho_B} - \frac{1}{\rho_B^2} \left(\nabla \rho_B \times b\right) \times B .
\]

It is worth noticing that the Alfvén velocity in a two-fluid model is given by (Andersson, Glampedakis & Samuelsson 2009)
\[
v_A^2 = \frac{B^2}{4 \pi \rho_B} .
\]
which depends on the proton density instead of the total mass density as in the single-fluid case.

The perturbation of the gravitational potential is obtained by solving the linearised Poisson equation
\[
\nabla \delta \Phi = 4 \pi G \delta \rho .
\]

Due to the symmetry of our background configurations, any perturbation variables may be Fourier expanded with respect to the coordinate \(\phi\). More precisely, in orthonormal spherical coordinates the mass density as well as any other perturbation function can be written in the following form (Jones et al. 2002)
\[
\delta \rho = \sum_{m=0}^{\infty} \left[ \delta \rho_m^+ (t, r, \theta) \cos m \phi + \delta \rho_m^- (t, r, \theta) \sin m \phi \right] ,
\]
where \(m\) is the azimuthal harmonic index. With this transformation the perturbation equations decouple with respect to \(m\) and the problem becomes two-dimensional in \((r, \theta)\).

### 2.3.1 Boundary conditions

For non-rotating stars with purely poloidal or toroidal magnetic fields, the two-dimensional numerical domain extends over the region \(0 \leq r \leq R\) and \(0 \leq \theta \leq \pi/2\), where \(R\) is the stellar radius. To study the evolution of non-axisymmetric oscillations (\(m \neq 0\)) we must therefore specify the boundary conditions at the star’s origin, rotation axis, equator and surface. In particular, for \(m \geq 2\) the regularity of the linearised equations leads to a zero condition for all the scalar and vector perturbation fields at the origin \((r = 0)\) and rotation axis \((\theta = 0)\), e.g. \(\delta \rho = 0\). At the equator \((\theta = \pi/2)\), the reflection symmetry splits the perturbation variables in two sets of opposite parity (Jones et al. 2002). If we represent all the scalar and vector perturbation fields by, respectively, the mass density and mass flux perturbations, the polar-led perturbations satisfy the following conditions at the equator:
\[
\partial_t \delta \rho = \partial_t f^r = f^\theta = \partial_t f^\phi = 0 .
\]
The conditions for the axial-led class are
\[
\delta \rho = f^r = \partial_t f^\theta = f^\phi = 0 .
\]
The boundary conditions for the magnetic field perturbations depend on the background field configuration (Lander et al. 2010). For a poloidal field we have
\[
\partial_t b^r = \partial_t b^\theta = b^\phi = 0
\]
for polar-led perturbations and
\[
\partial_t b^r \equiv \partial_t b^\theta = b^\phi = 0
\]
for the axial-led class. For a toroidal field, by contrast, the polar-led perturbations obey condition (31) and the axial-led perturbations obey (30).

In our purely fluid models, i.e. with no crust, the two fluid components extend over the entire stellar volume. The crust is an important constituent of a neutron star which we intend to implement in a future work. In the current study the position of the star’s surface \((r = R)\) is given by the vanishing of the proton and neutron mass density, i.e. \(\rho_u = \rho_n = 0\). For the perturbation variables we require that the Lagrangian perturbation of the chemical potentials vanish at the surface, i.e.
\[
\Delta g_{\Sigma} = \delta g_{\Sigma} + \xi_{\Sigma} \nabla \tilde{g}_{\Sigma} = 0 ,
\]
where \(\xi_{\Sigma}\) is the Lagrangian displacement associated with each fluid constituent (Andersson et al. 2004). These conditions may be expressed in terms of pressure and chemical potential perturbations as \(\Delta P = \Delta \beta = 0\). For polytropic models, the surface condition may be further simplified to \(\delta P = 0\) (Passamonti et al. 2007). From the definition of the stellar surface \((\rho_u = 0)\) it follows that all the “flux” variables are zero at \(r = R\), i.e. \(f = D = b = 0\).

### 2.3.2 Numerical Code

We study the time evolution of equations (13)–(17) and (23) with a numerical code based on a MacCormack algorithm. The code is an extension of those already developed for a superfluid star by Passamonti et al. (2009) and a magnetised star by Lander et al. (2010). In these two references the reader can find all the technical details. In order to speed up the numerical simulation we neglect the perturbation of the gravitational potential and adopt the so-called Cowling approximation. In this way we do not have to solve the elliptic equation (25) which is time consuming. This approximation has anyway a tiny effect on the Alfvén mode frequencies and it is suitable for the aims of this work.

We consider for the axial- and polar-led perturbations the same initial conditions used by Lander et al. (2010) and Lander & Jones (2011).

### 3 RESULTS

We are interested in studying the effects of two-fluid physics on magnetar dynamics. A realistic magnetar model has complex physics which includes superfluid/superconducting
components, crust, realistic EoS, a strong magnetic field and magnetosphere. In this work we make a first step toward the introduction of superfluid physics in magnetised stars, and study its impact on the Alfvén mode spectrum. Here, we focus on purely fluid models, while the presence of a crust and the study of magneto-elastic oscillations will be addressed in future work.

We consider both stratified and unstratified stellar models which can be obtained with an appropriate choice of the polytropic indices in the EoS. More precisely, models with constant (non-constant) proton fraction can be determined by setting $N_p = N_n$ ($N_p \neq N_n$). For the unstratified case, we choose $N_n = N_p = 1$ and study the effects of entrainment and proton fraction on the oscillation mode frequencies. For the stratified case, we construct models with an increasing composition-gradient stratification and study its impact on the spectrum.

This work studies non-axisymmetric oscillations of stars with purely poloidal and purely toroidal fields. We are able to investigate both axial- and polar-led modes of a toroidal-field star, but the axial-led perturbations of a poloidal field are subject to the rapidly-growing Tayler instability (Wright 1973; Markey & Tayler 1973; Lander & Jones 2011), which dominates the evolutions. In this case we consider only polar-led modes. We specialise to the case of oscillations of azimuthal index $m = 2$ for brevity. Whilst it is possible to study modes of higher $m$ using our code, we anticipate that they will be similarly affected by two-fluid physics; in addition, higher-$m$ modes are probably more susceptible to damping in a real magnetar and hence less likely to survive to produce the observed long-lived QPOs.

### 3.1 Oscillations of unstratified magnetars

We begin by studying unstratified models, which are described by the EoS with $N_p = N_n = 1$.

From the plane-wave analysis of Andersson et al. (2004), we expect the Alfvén mode frequencies to scale as

$$\sigma = \sqrt{\frac{\varepsilon_n}{x_p}} \sigma_0,$$

where $\sigma_0$ is the mode frequency of a single-fluid star, and $\varepsilon_n$ is given by

$$\varepsilon_n = \frac{1 - \varepsilon_n}{1 - \varepsilon_n - \varepsilon_p}.$$

The presence of the proton fraction $x_p$ in equation (33) is due to the different definition of the Alfvén velocity in a two-fluid system, as shown in equation (25). Equation (33) will guide the analysis of our numerical results, and we also expect that in the appropriate ‘single-fluid’ limits (in terms of $x_p$, $\varepsilon_n$, $\varepsilon_n$) we should recover the mode frequencies reported by Lander & Jones (2010) for a background toroidal field and Lander & Jones (2011) for a background poloidal field.

For all the results reported here for unstratified stars, we take a fiducial average field strength of $10^{16}$ G, which corresponds to a polar-cap value of $5 \times 10^{15}$ G for a poloidal field. This is a little larger than observed magnetar fields, but the correspondingly shorter Alfvén timescale allows for faster numerical evolutions. We have, however, checked that
for all our models the mode frequencies exhibit the expected linear scaling with magnetic field strength.

In figure 1 we plot the dependence of polar-led Alfvén mode frequencies on the proton fraction \( x_p \) for a star with a purely poloidal field and zero entrainment. In this and following figures, the mode frequencies \( \nu = \sigma/(2\pi) \) are shown both in dimensionless units, \( \nu/\langle G\rho_0 \rangle^{1/2} \) (where \( \rho_0 \) is the central mass density), and in physical units, for which we have considered a star with mass \( M = 1.4M_\odot \) and radius \( R = 10 \text{ km} \). It is evident from figure 1 that the expected relationship from the plane-wave analysis, i.e. \( \sigma \propto x_p^{-1/2} \), is borne out by our full numerical analysis. This is not surprising as in a non-stratified model the relevant parameters are constant. Most notably, we are able to determine the spectrum from a small proton fraction, \( x_p = 0.05 \), up to the single-fluid case \( x_p = 1 \), where we recover the results for a single-fluid magnetised star.

Considering a model with \( x_p = 0.1 \) and a purely poloidal magnetic field, we show in figure 2 the scaling of the polar-led Alfvén modes with the entrainment \( \varepsilon_e \). We explore a wide parameter space: from the fully-entrained ‘single-fluid’ limit, to entrainment values expected in the neutron star’s core, passing by the unentrained case \( \varepsilon_e = 1 \). The agreement with the expected scaling \( \sigma \propto \varepsilon_e^{1/2} \) is again very good. The single-fluid results can be obtained in the strong entrainment limit with large effective mass \( m^* \gg m_x \), which leads to \( \varepsilon_e \propto x_p \) and thus to \( \sigma \propto \sigma_0 \), see equation (33).

The typical effective masses expected in a neutron star core without strong pinning range in the interval \( 0.93 < m_n^*/m_n \lesssim 1 \) and \( 0.4 < m_p^*/m_p \lesssim 0.95 \) (Chamel 2008). These values, which lead to \( 1 \lesssim \varepsilon_e^{1/2} \lesssim 1.7 \), may increase the mode frequencies with respect to the single-fluid case by a factor of five in a star with \( x_p = 0.1 \). It is then natural to wonder whether this strong effect on the spectrum may be influenced by the presence of an elastic crust.

The conditions in the inner crust, where we expect a lattice of ions permeated by a gas of superfluid neutrons, can be in fact very different. Recent calculations show that superfluid neutrons may be efficiently entrained by nuclei due to Bragg scattering, and that the neutron effective mass at the bottom of the crust can be large as \( m_n^* \approx 14m_n \) (Chamel 2012). This strong entrainment can limit the relative two-fluid motion and produces a 10% correction of the shear mode frequencies determined with a single-fluid model (Andersson et al. 2009; Samuelsson & Andersson 2009; Passamonti & Andersson 2012; Sotani et al. 2012). However, the crust’s entrainment may be less relevant for the Alfvén modes due to their global nature. A rough estimate can be determined by taking the effective masses of the crust and the core calculated by Carter et al. (2005) and Chamel (2003, 2006, 2008). If we insert these values in the unstratified model with \( x_p = 0.1 \) used by Passamonti & Andersson (2012) and take an average of the entrainment profile, we obtain \( \sqrt{\langle \varepsilon_e \rangle} = 1.2 \). In our purely two-fluid star this value would increase the Alfvén mode frequencies by a factor of 3.8 with respect to the single-fluid case. This estimate must be considered with caution, as the coupling of the magnetic field and the crust may lead to a different dynamical evolution and results. This issue will be addressed in a future paper.

When the star has a purely toroidal magnetic field we are able to evolve stably both axial and polar initial data and extract mode frequencies. In figure 3 we show the effects of proton fraction on the axial- and polar-led Alfvén mode frequencies for a star with a toroidal field and zero entrainment. In this case too the scaling of the mode frequencies is well approximated by equation (33) and we are able to recover the results of single-fluid magnetised models.

From figures 1 and 3 we are able to compare the effect of different magnetic field geometries on the oscillation spectrum of a magnetar. Let us concentrate on the results for \( x_p^{-1/2} \approx 3.2 \), i.e. \( x_p = 0.1 \), and just the polar-led modes (ignoring the single axial-led mode in figure 3). For a background poloidal field we find five widely-spaced Alfvén modes in the broad range 50 – 500 Hz. By contrast, for a purely toroidal field there are only three modes, in the narrower interval 180 – 320 Hz. These differences suggest the possibility of constraining the field geometry with future QPO observations.

Finally, in a star with a toroidal field we find the Alfvén modes again scale in the expected manner with entrainment \( \sigma \propto \varepsilon_e^{1/2} \), as for the poloidal-field case. Since the plot of this contains no new information, however, we omit it for brevity.

3.2 Oscillations of stratified magnetars

We now turn our attention to the oscillation spectrum of stratified models and focus on purely toroidal magnetic fields. The poloidal configurations are currently less numerically stable, and we want to return to these at a later date. We use a weaker magnetic field than in the previous subsection here, with an average strength of \( 5 \times 10^{15} \text{ G} \) — again for reasons of numerical stability.

The effects of stratification can be studied by using equation (10) with \( N_n \neq N_p \). We consider a sequence of
stars with various composition gradients by setting $N_\rho = 1$ and exploring a range of proton polytropic indices, $0.7 \leq N_\rho \leq 1.5$. For this range we are able to run the code for a sufficient evolution time to extract mode frequencies reliably (15+ Alfvén times). The central proton fraction is $x_p = 0.15$ for all these models and tends to zero (unity) for $N_\rho > 1$ ($N_\rho < 1$). The frequencies of the axial- and polar-led $m = 2$ Alfvén modes are shown in figure 4 for stars with toroidal field $B = 5 \times 10^{15}$ G and zero entrainment. The mode frequencies $\nu = \sigma/(2\pi)$ are given in physical units for a star with $M = 1.4M_\odot$ and $R = 10$ km. Note that in this stratified stellar sequence the central mass density $\rho_c$ does not scale linearly with $N_\rho$. Therefore, the dimensionless mode frequencies cannot be shown in figure 4 together with the physical values.

Our results show that composition gradients may significantly affect the Alfvén modes with respect to unstratified models. The $m = 2$ modes at $N_\rho = 1.5$ are a factor of about 1.38 larger than the unstratified case $N_\rho = 1$. As shown recently by Lander et al. (2012), a “realistic” proton gradient can be reproduced by the EoS with a proton-fluid index close to $N_\rho = 2$. Unfortunately, this value is slightly beyond our current numerical capacity. However, considering a linear fit of the mode frequencies shown in figure 4 we find that a model with $N_\rho = 2$ should oscillate with frequencies roughly 1.67 larger than the $N_\rho = 1$ case. For instance, the linear fit for the polar-led Alfvén mode which has the lowest frequencies in figure 4 is given by

$$\nu = (20.0 + 56.9N_\rho) \text{ Hz.}$$

(35)

From the unstratified model we can identify the modes already calculated by Lander et al. (2010) in the single-fluid limit. In figure 4 they are represented with a cross.

4 DISCUSSION

The main result of this work is that the multi-fluid physics of a magnetar is likely to result in considerably higher Alfvén-mode frequencies than a standard single-fluid barotropic model would indicate. Although this was predicted at a qualitative level by an earlier plane-wave analysis (Andersson et al. 2009), this paper is the first quantitative study of the oscillation spectrum of a multi-fluid magnetar.

We have studied the time evolution of non-axisymmetric oscillations of purely two-fluid stars with superfluid neutrons and normal (not superconducting) protons. The effects of an elastic crust will be included in future work. We have considered various magnetic field geometries, proton fractions, entrainment and composition stratifications and analysed their impact on the Alfvén mode spectrum.

Starting with unstratified models, we find that realistic values for the proton fraction and entrainment lead to Alfvén mode frequencies three to four times larger than the single-fluid results. If we move from an unstratified star to one with a more realistic composition gradient the mode frequencies may be increased by another factor of $\approx 1.67$. From these results we can extrapolate that a ‘typical’ multi-fluid magnetar could have QPO frequencies which are roughly a factor of five/six higher than expected from a single-fluid unstratified model. Based on our results and using equations (33) and (35), we may summarise the expected scaling of Alfvén modes in the presence of superfluid physics as:

$$\sigma \approx 6.3\sigma_0 \left[ 0.15 + 0.85 \left( \frac{N_\rho}{2} \right) \right] \left( \frac{\epsilon_p}{1.3} \right)^{1/2} \left( \frac{x_p(0)}{0.1} \right)^{-1/2},$$

(36)

where $\sigma$ is a mode of a multi-fluid magnetar, $\sigma_0$ is the corresponding single-fluid mode, and $x_p(0)$ is the central proton fraction.

Furthermore, our results suggest that there is some hope for constraining the magnetic field geometry of a magnetar from the QPO frequency distribution. For a typical unstratified stellar model with a poloidal field we find five Alfvén modes, in the wide range $50 – 500$ Hz; instead, for a toroidal field there are three modes in a far narrower range, $180 – 320$ Hz.

Let us consider a magnetar with average field strength $B = 10^{16}$ G. Although this seems rather high, it corresponds to a more reasonable polar-cap value of $5 \times 10^{15}$ G if we assume a poloidal-field geometry; if the field is more ‘buried’ (e.g. with a strong interior toroidal field) then it may have an average strength of $10^{16}$ G but a polar-cap value of only $10^{15}$ G. For this field strength, typical single-fluid lowest-order Alfvén mode frequencies are roughly $10 – 50$ Hz for $m = 0$ and $50 – 150$ Hz for $m = 2$; higher $l$-multipoles or values of $m$ produce higher frequencies (Colaiuda et al. 2009; Lander et al. 2010; Lander & Jones 2000).

Figure 4. Axial- and polar-led $m = 2$ Alfvén modes for a series of stratified stars with purely toroidal magnetic field with $B = 5 \times 10^{15}$ G, zero entrainment and $N_\rho = 1$. The horizontal axis displays the proton polytropic index $N_\rho$ which is related to the degree of stratification. Stratification is zero for $N_\rho = 1$ and increases for values of $N_\rho$ much different from $N_\rho$. The central proton fraction is $x_p = 0.15$ for all these stellar models. On the vertical axis we show the mode frequency in physical units for a star with $M = 1.4M_\odot$ and $R = 10$ km. The curves are represented with the same notation used in figure 4.

2 This refers to $m = 0$ axial modes; Satani & Kokkotas (2000) finds a fundamental polar $m = 0$ mode at 300 Hz.
Temporarily ignoring the various simplifications in our model, let us look at the repercussions of formula (35) for identification of magnetar QPOs. The strong 150 Hz QPO of SGR 1806-20 or the 155 Hz QPO of SGR 1900+14 may be interpreted as an axisymmetric Alfvén QPO of the magnetar’s core — the multi-fluid equivalent of a 25-Hz peak predicted by single-fluid models. Similarly, the long-lived 625 Hz peak of SGR 1806-20 would then be the multi-fluid equivalent of a 100-Hz single-fluid mode, which is in the range expected for an $m = 2$ Alfvén mode.

The above discussion does not provide an interpretation for the lower-frequency magnetar QPOs and neglects the effect of the crust. The effect of superfluidity on the crustal shear modes is similar to what is described in this paper for the Alfvén modes. However, a realistic entrainment (Chamel 2011, Gabler et al. 2012) may have a very different quantitative impact on these two classes of modes. In fact, the effective mass of neutrons may be quite large at the bottom of the crust leading to a total correction of the shear mode of about 10% with respect to the single-fluid models. The Alfvén modes however may be less affected by this entrainment configuration which is confined only in a limited region of the star. In conclusion, multi-fluid dynamics with a realistic entrainment may increase significantly the Alfvén modes as described in the last paragraph, but only produce a 10 percent correction to the shear modes. Therefore, we suggest that the observed magnetar QPOs may fall into two classes: below roughly 50 Hz they may be magneto-elastic crustal modes, whilst above this value they could represent Alfvén oscillations of the multi-fluid core.

The attractive feature of our interpretation is that it does not rely on the long-term excitation of high multipoles of the magnetar’s oscillation modes, but instead suggests that observed QPOs are all low-order modes originating from the core or crust. Nonetheless, our description of a magnetar is still rather approximate. We must refine our superfluid magnetised models and also include an elastic crust. One key issue is whether magnetically-modified elastic modes are really able to explain all the lower-frequency QPOs. In the axisymmetric case, two-fluid physics may move the Alfvén continuum to higher frequencies. In this way, more crustal modes could be outside the continuum and live for a longer time. Alternatively, some low-frequency QPOs could still be core modes if the magnetic field is weaker than expected, or if the two-fluid enhancement of axisymmetric oscillations is less significant than predicted by equation (36).

Our conclusions are based on a multi-fluid star where the dynamics of neutrons and protons is essentially decoupled at linear perturbation order, interacting only through the entrainment. We assume that the magnetar’s internal field strength is above the critical value at which superconductivity is destroyed and hence that the protons form a normal fluid, but this is far from certain. If instead the protons form a type-II superconductor, the oscillation spectrum will certainly be affected. The nature of the oscillations changes and their frequency is altered by a factor of $\sqrt{\mu/\mu_0}$, where $\mu \approx 10^5 \text{G}^2$ (Mendell 1998). In addition, the presence of a magnetic force on the neutrons and coupling between proton fluxtubes and neutron vortices could lower the frequency of oscillations with respect to our results (van Hoven & Levin 2008, Glampedakis et al. 2011). These important issues deserve more quantitative attention in future studies of magnetar QPOs.

ACKNOWLEDGEMENTS

We acknowledge support from the German Science Foundation (DFG) via SFB/TR7. We are also pleased to thank A. Colaiuda, K. Glampedakis and K. Kokkotas for fruitful discussions.

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