Asymptotic behavior of extremal functions to an inequality involving Hardy potential and critical Sobolev exponent

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Abstract

In this paper, we study the asymptotic behavior of radial extremal functions to an inequality involving Hardy potential and critical Sobolev exponent. Based on the asymptotic behavior at the origin and the infinity, we shall deduce a strict inequality between two best constants. Finally, as an application of this strict inequality, we consider the existence of nontrivial solution of a quasilinear Brezis-Nirenberg type problem with Hardy potential and critical Sobolev exponent.

Key Words: asymptotic behavior, extremal functions, Hardy potential, critical Sobolev exponent, Brezis-Nirenberg type problem

Mathematics Subject Classifications: 35J60.

1 Introduction.

In this paper, we study the asymptotic behavior of extremal functions to the following inequality involving Hardy potential and critical Sobolev exponent:

$$C \left( \int_{\mathbb{R}^N} \frac{|u|^{p_*}}{|x|^{b p_*}} \, dx \right)^{p/p_*} \leq \int_{\mathbb{R}^N} \left( \frac{|Du|^p}{|x|^{ap}} - \frac{|u|^p}{|x|^{(a+1)p}} \right) \, dx,$$

where $1 < p < N$, $0 \leq a < \frac{N-p}{p}$, $a \leq b < (a+1)$, $p_* = \frac{Np}{N-(a+1-b)p}$, $\mu < \mu_1$, $\mu_1$ is the best constant in the Hardy equality.

*Supported by Grant 10101024 and 10371116 from the National Natural Science Foundation of China. e-mail:bjxuan@ustc.edu.cn (B. Xuan)
Asymptotic behavior of extremal functions

We shall show that for $\mu < \overline{\mu}$ the best constant of inequality (1.1) is achievable. Furthermore, the extremal functions of inequality (1.1) is radial symmetric. Then we study the asymptotic behavior of the radial extremal functions of inequality (1.1) at the origin and the infinity. At last, for any smooth bounded open domain $\Omega \subset \mathbb{R}^N$ containing 0 in its insides, we shall deduce a strict inequality between two best constants $S_{\lambda, \mu}(p, a, b; \Omega)$ and $S_{0, \mu}(p, a, b; \Omega) = S_{0, \mu}$:
\[
S_{\lambda, \mu}(p, a, b; \Omega) < S_{0, \mu},
\]
if $\lambda > 0$, where $S_{0, \mu}$ and $S_{\lambda, \mu}(p, a, b; \Omega)$ will be defined in Section 2 and 4 respectively. We believe that the strict inequality (1.2) will be useful to study the existence of quasilinear elliptic problem involving Hardy potential and critical Sobolev exponent. As an application of this strict inequality, we consider the existence of nontrivial solution of a quasilinear Brezis-Nirenberg problem with Hardy potential and critical Sobolev exponent.

In their famous paper [5], Brezis and Nirenberg studied problem:
\[
\begin{cases}
-\Delta u = \lambda u + u^{2^* - 1}, & \text{in } \Omega, \\
u > 0, & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega.
\end{cases}
\]
Since the embedding $H^1_0(\Omega) \hookrightarrow L^{2^*}(\Omega)$ is not compact where $2^* = 2N/(N-2)$, the associated energy functional does not satisfy the (PS) condition globally, which caused a serious difficulty when trying to apply standard variational methods. Brezis and Nirenberg successfully reduced the existence of solutions of problem (1.3) into the verification of a special version of the strict inequality (1.2) with $p = 2, a = b = \mu = 0$. To verify (1.2) in their case, they applied the explicit expression of the extremal functions to the Sobolev inequality, especially the asymptotic behavior of the extremal functions at the origin and the infinity. Brezis-Nirenberg type problems have been generalized to many other situations (see [8, 9, 10, 13, 16, 18, 19, 24, 26, 27] and references therein).

Recently, Jannelli [15] introduced the term $\mu \frac{u}{|x|}$ in the equation, that is,
\[
\begin{cases}
-\Delta u - \mu \frac{u}{|x|^2} = \lambda u + u^{2^* - 1}, & \text{in } \Omega, \\
u > 0, & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega.
\end{cases}
\]
He studied the relation between critical dimensions for $\lambda \in (0, \lambda_1)$ and $L^2_{\text{loc}}$ integrability of the associated Green function, where $\lambda_1$ is the first eigenvalue of operator $-\Delta - \mu \frac{1}{|x|^2}$ on $\Omega$ with zero-Dirichlet condition. Ruiz and Willem [20] also studied problem (1.4) under various assumption on the domain $\Omega$, and even for $\mu \leq 0$. Those proofs in [15] and [20] were reduced to verify the strict
inequality (1.2) with $p = 2, a = b = 0$. In 2001, Ferrero and Gazzola \[11\] considered the existence of sign-changed solution to problem (1.4) for larger $\lambda$. They distinguished two distinct cases: resonant case and non-resonant cases of the Brezis-Nirenberg type problem (1.4). For the resonant case, they only studied a special case: $\Omega$ is the unit ball and $\lambda = \lambda_1$. The general case was left as an open problem. In 2004, Cao and Han \[7\] completed the general case. In all the references cited above, the asymptotic behavior of the extremal functions at the origin and the infinity was applied to derive the local (PS) condition for the associated energy functional.

The rest of this paper is organized as follows. In section 2, we shall show that the best constant of (1.1) is achieved by some radial extremal functions. Section 3 is concerning with the asymptotic behavior of the radial extremal functions. In Section 4, we first derive various estimates on the approximation extremal functions, and then establish the strict inequality (1.2). In section 5, based on this strict inequality, we obtain the existence results of nontrivial solution of a quasilinear Brezis-Nirenberg problem.

2 Radial extremal functions

In order to obtain the extremal functions of (1.1). We consider the following extremal problem:

$$S_{0, \mu} = \inf \left\{ Q_\mu(u) : u \in \mathcal{D}^{1,p}_{a,b}(\mathbb{R}^N), \|u; L^p_b(\mathbb{R}^N)\| = 1 \right\},$$

(2.1)

where

$$Q_\mu(u) = \int_{\mathbb{R}^N} \frac{|Du|^p}{|x|^{ap}} \, dx - \mu \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{(a+1)p}} \, dx,$$

and

$$\mathcal{D}^{1,p}_{a,b}(\mathbb{R}^N) = \{ u \in L^p_b(\mathbb{R}^N) : |Du| \in L^p_a(\mathbb{R}^N) \}$$

is the closure of $C^\infty_0(\mathbb{R}^N)$ under the norm $\|u\|_{\mathcal{D}^{1,p}(\mathbb{R}^N)} = \||Du|; L^p_a(\mathbb{R}^N)\|$.

For any $\alpha, q$, the norm of weighted space $L^q_\alpha(\mathbb{R}^N)$ is defined as

$$\|u; L^q_\alpha(\mathbb{R}^N)\| = \left( \int_{\mathbb{R}^N} \frac{|u|^q}{|x|^{\alpha q}} \, dx \right)^{\frac{1}{q}}.$$

Similar to Lemma 2.1 in \[12\], one can easily obtain the following Hardy inequality with best constant $\overline{\mu} = (\frac{N-(a+1)p}{p})^p$:

$$\overline{\mu} \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{ap}} \, dx \leq \int_{\mathbb{R}^N} \frac{|Du|^p}{|x|^{ap}} \, dx.$$

(2.2)
Thus, for \( \mu < \overline{\mu} \), \( Q_\mu(u) \geq 0 \) for all \( u \in \mathcal{D}_{a,b}^{1,p}(\mathbb{R}^N) \), and the equality holds if and only if \( u \equiv 0 \). From the so-called Caffarelli-Kohn-Nirenberg inequality [13], \( S_{0,\mu} < \infty \).

**Lemma 2.1** If \( \mu \in (0, \overline{\mu}) \), \( b \in [a, a+1) \), then \( S_{0,\mu} \) is achieved at some nonnegative function \( u_0 \in \mathcal{D}_{a,b}^{1,p}(\mathbb{R}^N) \). In particular, there exists a solution to the following “limited equation”:

\[
-\operatorname{div}\left( \frac{|Du|^{p-2}Du}{|x|^{ap}} \right) - \mu \frac{|u|^{p-2}u}{|x|^{(a+1)p}} = \frac{|u|^{p^*-2}u}{|x|^{b^*}},
\]

(2.3)

**Proof.** The achievability of \( S_{0,\mu} \) at some \( u_0 \in \mathcal{D}_{a,b}^{1,p}(\mathbb{R}^N) \) with \( \|u_0; L_b^{p^*}(\mathbb{R}^N)\| = 1 \) is due to [23] for \( p = 2 \) and [22] for general \( p \). Without loss of generality, suppose that \( u_0 > 0 \), otherwise, replace it by \(|u_0|\). It is easy to see that \( u_0 \) satisfies the following Euler-Lagrange equation:

\[
-\operatorname{div}\left( \frac{|Du|^{p-2}Du}{|x|^{ap}} \right) - \mu \frac{|u|^{p-2}u}{|x|^{(a+1)p}} = \delta \frac{|u|^{p^*-2}u}{|x|^{b^*}},
\]

where \( \delta = Q_\mu(u_0)/\|u_0; L_b^{p^*}(\mathbb{R}^N)\| = Q_\mu(u_0) = S_{0,\mu} > 0 \) is the Lagrange multiplier. Set \( \overline{u} = c_0u_0, c_0 = S_{0,\mu}^{1/p} \), then \( \overline{u} \) is a solution to equation (2.3). \( \blacksquare \)

In fact, all the dilation of \( u_0 \) of the form \( \sigma^{-N-(a+1)p} u_0(\sigma \cdot) \) are also minimizers of \( S_{0,\mu} \). In order to obtain further properties of the minimizers of \( S_{0,\mu} \), let’s recall the definition of the Schwarz symmetrization (see [14]). Suppose that \( \Omega \subset \mathbb{R}^N \), and \( f \in C_0(\Omega) \) is a nonnegative continuous function with compact support, the the Schwarz symmetrization \( S(f) \) of \( f \) is defined as

\[
S(f)(x) = \sup \{t : \mu(t) > \omega_N|x|^N\}, \quad \mu(t) = \{x : f(x) > t\},
\]

where \( \omega_N \) denotes the volume of the standard \( N \)-sphere. Applying those properties of Schwarz symmetrization in [14], we have the following lemma:

**Lemma 2.2** For \( v \in \mathcal{D}_{a,b}^{1,p}(\mathbb{R}^N) \setminus \{0\}, k \geq 0 \), define

\[
R(v) = \frac{\int_{S^{N-1}} \int_0^{+\infty} \left\{ k^{1-p-\frac{a}{p}} \left( |\partial_\rho v|^2 + \frac{|\Lambda v|^2}{\rho^2} \right) \rho^{N-1} - k^{1-p-\frac{a}{p}} \mu |v|^p \rho^{N-1-p} \right\} \, d\rho \, dS}{\int_{S^{N-1}} \int_0^{+\infty} |v|^{p^*} \rho^{(\frac{N}{p^*}+1)} \, d\rho \, dS},
\]

where \( \partial_\rho \) is the directional differential operator along direction \( \rho \) and \( \Lambda \) is the tangential differential operator on \( S^{N-1} \). Then

\[
\inf\{R(v) : v \in \mathcal{D}_{a,b}^{1,p}(\mathbb{R}^N) \text{ is radial}\} = \inf\{R(v) : v \in \mathcal{D}_{a,b}^{1,p}(\mathbb{R}^N)\}.
\]
Proof. By the density argument, it suffices to prove the lemma for \(v \in C^\infty_0(\mathbb{R}^N)\). Let \(v^*\) be the Schwarz symmetrization of \(v\). Noting that \(\Lambda v^* = 0, p_* \leq \frac{Np}{N-p}\), and applying those properties of Schwarz symmetrization in [14], we have

\[
\int_{S_{N-1}} \int_0^{+\infty} |v^*|^p \rho^{(N-p)p - 1} d\rho dS \geq \int_{S_{N-1}} \int_0^{+\infty} |v|^p \rho^{(N-p)p - 1} d\rho dS = 1,
\]

\[
k^{1-p-\frac{p}{p_*}} \int_{S_{N-1}} \int_0^{+\infty} (|\partial_\rho v^*|^2 + \frac{|\Lambda v^*|^2}{\rho^2})^{p/2} \rho^{N-1} d\rho dS \leq k^{1-p-\frac{p}{p_*}} \int_{S_{N-1}} \int_0^{+\infty} (|\partial_\rho v|^2 + \frac{|\Lambda v|^2}{\rho^2})^{p/2} \rho^{N-1} d\rho dS
\]

and

\[
k^{1-p-\frac{p}{p_*}} \mu \int_{S_{N-1}} \int_0^{+\infty} |v^*|^p \rho^{N-1-p} d\rho dS \geq k^{1-p-\frac{p}{p_*}} \mu \int_{S_{N-1}} \int_0^{+\infty} |v|^p \rho^{N-1-p} d\rho dS.
\]

Thus, we have

\[
\int_{S_{N-1}} \int_0^{+\infty} \{k^{1-p-\frac{p}{p_*}} (|\partial_\rho v^*|^2 + \frac{|\Lambda v^*|^2}{\rho^2})^{p/2} \rho^{N-1} - k^{1-p-\frac{p}{p_*}} \mu |v^*|^p \rho^{N-1-p}\} d\rho dS
\]

\[
\leq \int_{S_{N-1}} \int_0^{+\infty} \{k^{1-p-\frac{p}{p_*}} (|\partial_\rho v|^2 + \frac{|\Lambda v|^2}{\rho^2})^{p/2} \rho^{N-1} - k^{1-p-\frac{p}{p_*}} \mu |v|^p \rho^{N-1-p}\} d\rho dS.
\]

That is,

\[
R(v^*) \leq R(v),
\]

thus,

\[
\inf \{R(v) : v \in \mathcal{D}_a^{1,p}(\mathbb{R}^N) \text{ is radial} \} \leq \inf \{R(v) : v \in \mathcal{D}_a^{1,p}(\mathbb{R}^N) \}.
\]

On the other hand, it is trivial that

\[
\inf \{R(v) : v \in \mathcal{D}_a^{1,p}(\mathbb{R}^N) \text{ is radial} \} \geq \inf \{R(v) : v \in \mathcal{D}_a^{1,p}(\mathbb{R}^N) \}.
\]

Lemma 2.3. If \(\mu \in (0, \frac{p}{1})\), \(b \in [a, a + 1]\), then all the minimizers of \(S_{0, \mu}\) is radial. In particular, there exists a family of radial solutions to equation (2.3).

Proof. We rewrite those integrals in \(S_{0, \mu}\) in polar coordinates. Noting that \(|Du|^2 = |\partial_r u|^2 + \frac{1}{r^2} |\Lambda u|^2|\), we have

\[
Q_\mu(u) = \int_{S_{N-1}} \int_0^{+\infty} (|\partial_r u|^2 + \frac{1}{r^2} |\Lambda u|^2)^{p/2} r^{N-1-a p} dr dS
\]

\[
- \mu \int_{S_{N-1}} \int_0^{+\infty} |u|^p r^{N-1-(a+1)p} dr dS.
\]

(2.4)
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Making the change of variables \( r = \rho^k, \quad k = \frac{N-p}{N-(a+1)p} \geq 1, \) from (2.4), we have

\[
Q_\mu(u) = k^{1-p} \int_{S^{N-1}} \int_0^{+\infty} \left( |\partial_\rho u|^2 + k^2 |\Lambda u|^2 \rho^{p/2} \right) \rho^{N-1} d\rho dS
- k\mu \int_{S^{N-1}} \int_0^{+\infty} |u|^p \rho^{N-p-1} d\rho dS. \tag{2.5}
\]

On the other hand, the restriction condition \( \|u; L^p_b(\mathbb{R}^N)| = 1 \) becomes

\[
k \int_{S^{N-1}} \int_0^{+\infty} |u|^p \rho^{(N-p)p^*-1} d\rho dS = 1. \tag{2.6}
\]

To cancel the coefficient \( k \) in (2.6), let \( v = k^{\frac{1}{p}} u \), then we have the following equivalent form of \( S_{0,\mu} \):

\[
S_{0,\mu} = \inf \left\{ \int_{S^{N-1}} \int_0^{+\infty} \left\{ k^{1-p} \frac{p}{p^*} (|\partial_\rho v|^2 + k^2 |\Lambda v|^2 \rho^{p/2}) \rho^{N-1} \right. \right.
- k^\frac{1}{p} \mu |v|^p \rho^{N-1-p} \left. \right\} d\rho dS : v \in \mathcal{D}^{1,p}_{a,b}(\mathbb{R}^N), \tag{2.7}
\]

\[
\int_{S^{N-1}} \int_0^{+\infty} |v|^p \rho^{\frac{(N-p)p^*}{p}-1} d\rho dS = 1. \tag{2.8}
\]

Since \( k \geq 1 \), we have

\[
S_{0,\mu} \geq \inf \left\{ \int_{S^{N-1}} \int_0^{+\infty} \left\{ k^{1-p} \frac{p}{p^*} (|\partial_\rho v|^2 + \frac{|\Lambda v|^2}{\rho^2}) \rho^{N-1} \right. \right.
- k^\frac{1}{p} \mu |v|^p \rho^{N-1-p} \left. \right\} d\rho dS : v \in \mathcal{D}^{1,p}_{a,b}(\mathbb{R}^N), \tag{2.8}
\]

\[
\int_{S^{N-1}} \int_0^{+\infty} |v|^p \rho^{\frac{(N-p)p^*}{p}-1} d\rho dS = 1. \tag{2.8}
\]

From Lemma [2.2], we know that the left side hand is achieved at some radial function, and the inequality in (2.8) becomes equality if and only if \( v \) is radial. Thus, all the minimizers of \( S_{0,\mu} \) is radial. \( \blacksquare \)

3 Asymptotic behavior of extremal functions

In this section, we describe the asymptotic behavior of radial extremal functions of \( S_{0,\mu} \). Our argument here is similar to that in §3.2 of [1]. Let \( u(r) \) be a nonnegative radial solution to (2.3). Rewriting in polar coordinates, we have

\[
(r^{N-1-ap}|u'|^{p-2}u')' + r^{N-1} \left( \mu \frac{|u|^{p-2}u}{r(a+1)p} + \frac{|u|^{p^*} u}{r^p b^*} \right) = 0. \tag{3.1}
\]
Asymptotic behavior of extremal functions

Set \[ t = \log r, \quad y(t) = r^\delta u(r), \quad z(t) = r^{(1+\delta)(p-1)}|u'(r)|^{p-2}u'(r), \quad (3.2) \]
where \( \delta = \frac{N-(a+1)p}{p} \). A simple calculation shows that

\[
\begin{align*}
\frac{dy}{dt} &= \delta y + |z|^\frac{2-p}{p-1}z; \\
\frac{dz}{dt} &= -\delta z - |y|^{p-2}y - \mu|y|^{p-2}y.
\end{align*}
\]

(3.3)

It follows from (3.3) that \( y \) satisfies the following equation:

\[
(p-1)|\delta y - y'|^{p-2}(\delta y' - y'') + \delta|\delta y - y'|^{p-2}(\delta y - y') - \mu y^{p-1} - y^{p-1} = 0. \quad (3.4)
\]

It is easy to see that the complete integral of the autonomous system (3.3) is

\[
V(y, z) = \frac{1}{p^*}|y|^{p^*} + \frac{\mu}{p}|y|^p + \frac{p-1}{p}|z|^\frac{p}{p-1} + \delta y z.
\]

(3.5)

Similar to Lemma 3.6-3.9 in [1], we have the following four lemmas. We will omit proofs of the first three lemmas because one only needs to replace \( \delta = \frac{N-p}{p} \) there by \( \delta = \frac{N-(a+1)p}{p} \) in our case. The interested reader can refer to [1]. The idea of the fourth Lemma is also similar to that of Lemma 3.9 in [1], with different choice of function \( \xi \). We shall write down its complete proof for completeness.

Lemma 3.1 \( y \) and \( z \) are bounded.

Lemma 3.2 For any \( t \in \mathbb{R}^N \), \( (y(t), z(t)) \in \{(y, z) \in \mathbb{R}^2 : V(y, z) = 0\} \).

Lemma 3.3 There exists \( t_0 \in \mathbb{R} \), such that \( y(t) \) is strictly increasing for \( t < t_0 \); and strictly decreasing for \( t > t_0 \). Furthermore, we have

\[
\max_{t \in \mathbb{R}} y(t) = y(t_0) = \left[ \frac{N}{N-(a+1-b)p} \delta^p - \mu \right]^{\frac{1}{p^*+1}}
\]

(3.6)

Lemma 3.4 Suppose that \( y \) is a positive solution to (3.4) such that \( y \) is increasing in \((-\infty, 0)\) and decreasing in \((0, +\infty)\), then there exist \( c_1, c_2 > 0 \), such that

\[
\lim_{t \to -\infty} e^{(l_1-\delta)t}y(t) = y(0)c_1 > 0; \quad (3.7)
\]

\[
\lim_{t \to +\infty} e^{(l_2-\delta)t}y(t) = y(0)c_2 > 0, \quad (3.8)
\]

where \( l_1, l_2 \) are zeros of function \( \xi(s) = (p-1)s^p - (N-(a+1)p)s^{p-1} + \mu \) such that \( 0 < l_1 < l_2 \).
Proof. First, it is easy to see that \( l_1 < \delta < l_2 \). Next, we prove (3.7) step by step and omit the proof of (3.8).

1. It follows from (3.3) that

\[
\frac{d}{d t} (e^{-(\delta-l_1)t} y(t)) = - (\delta - l_1) e^{-(\delta-l_1)t} y(t) + e^{-(\delta-l_1)t} (\delta y(t) + |z|^{1/p-1}) \\
= e^{-(\delta-l_1)t} y(t) (l_1 - \frac{|z(t)|^{1/p}}{y(t)}).
\]

(3.9)

Rewriting the above equation into the integral form, we have

\[
e^{-(\delta-l_1)t} y(t) = y(0) e^{-\int_{t_0}^{t} (l_1 - \frac{|z(s)|^{1/p}}{y(s)}) ds}.
\]

(3.10)

2. Let \( H(s) = \frac{|z(s)|^{1/p}}{y(s)} \).

**Claim:** \( H(s) \) is a increasing function from \((-\infty, 0]\) into \((l_1, \delta]\).

In fact, we shall prove that \( H'(s) > 0 \) for \( s < 0 \). Otherwise, we prove by contradiction, suppose that there exists \( s_0 < 0 \) such that \( H'(s_0) \leq 0 \). A direct computation shows that

\[
H'(s) = - \frac{\frac{1}{p-1} y(s) z'(s) |z(s)|^{\frac{2-p}{p-1}} - |z(s)|^{\frac{1}{p-1}} y'(s)}{y^2(s)}.
\]

Replacing formulas of \( y'(s_0) \) and \( z'(s_0) \) from (3.3), and noting that (3.5) and Lemma 3.2, it follows that

\[
H'(s_0) = (\frac{1}{p} - \frac{1}{p_*}) y^{p_*}(s_0) \leq 0,
\]

which contradicts to the fact that \( y > 0 \). Thus, \( H'(s) > 0 \), and hence \( H \) is strictly increasing on \((-\infty, 0]\).

On the other hand, from (3.3) and \( y'(0) = 0 \), we have \( H(0) = \delta \); from (3.5), it follows that \( \lim_{t \to -\infty} H(s) = l_1 \), which proves our claim.

3. (3.7) holds.

From the above claim and (3.10), it follows that \( e^{-(\delta-l_1)t} y(t) > 0 \) is decreasing on \((-\infty, 0]\), and hence the limit \( \lim_{t \to -\infty} e^{-(\delta-l_1)t} y(t) \) exists. Set

\[
\alpha \equiv \lim_{t \to -\infty} e^{-(\delta-l_1)t} y(t) = y(0) e^{\int_{t_0}^{0} (H(s)-l_1) ds}.
\]

To prove (3.7), it suffices to show that \( \alpha < -\infty \). From (3.3) and (3.5), a direct computation shows that

\[
H'(s) = - \frac{(a+1-b)p}{(p-1)(N-(a+1-b)p)} H(s)^{2-p} \xi(H(s)),
\]
where
\[ \xi(s) = (p - 1)s^p - (N - (a + 1)p)s^{p-1} + \mu. \]

From the definitions of \( l_1, l_2 \), we may suppose that
\[ H'(s) = (H(s) - l_1)(H(s) - l_2)g(H(s)), \]
where \( g \) is a continuous negative function on the interval \([l_1, \delta] \), thus satisfies \(|g(H(s))| \geq c_1 > 0 \). From (3.10), it follows that
\[ \alpha = \lim_{t \to -\infty} e^{(\delta - l_1)t} y(t) = y(0)e^{\int_{-\infty}^{\delta}(H(s) - l_1)ds} = y(0)e^{\int_{l_1}^{\delta}[(H(s) - l_2)g(H(s))]^{-1}dH(s)}. \]

Since \( l_2 > \delta \) and \(|g(H(s))| \geq c_1 \) on \([l_1, \delta] \), we know that
\[ \int_{l_1}^{\delta}[(H(s) - l_2)g(H(s))]^{-1}dH(s) < +\infty, \]
that is, \( \alpha < +\infty \), thus (3.7) follows.

In the following corollary, we rewrite these conclusions on \( y \) into those on the positive solution \( u \in \mathcal{D}^{1,p}(\mathbb{R}^N) \) of equation (3.1).

**Corollary 3.5** Let \( u \in \mathcal{D}^{1,p}(\mathbb{R}^N) \) be a positive solution of equation (3.1). Then there exists two positive constants \( C_1, C_2 > 0 \) such that
\[
\lim_{r \to 0} r^{l_1}u(r) = C_1 > 0, \quad \lim_{r \to +\infty} r^{l_2}u(r) = C_2 > 0. \tag{3.11}
\]

and
\[
\lim_{r \to 0} r^{l_1+1}|u'(r)| = C_1l_1 > 0, \quad \lim_{r \to +\infty} r^{l_2+1}|u'(r)| = C_2l_2 > 0. \tag{3.12}
\]

**Proof.** From (3.2), we know \( u(r) = r^{-\delta}y(t) \). Applying Lemma 3.4 directly, we have
\[
\lim_{r \to 0} r^{l_1}u(r) = \lim_{t \to -\infty} e^{(l_1-\delta)t} y(t) = y(0)c_1 = C_1 > 0, \]
\[
\lim_{r \to +\infty} r^{l_2}u(r) = \lim_{t \to +\infty} e^{(l_2-\delta)t} y(t) = y(0)c_2 = C_2 > 0. \]

Noting that \( \lim_{t \to -\infty} H(t) = l_1 \) and \( \lim_{t \to +\infty} H(t) = l_2 \), it follows that
\[
\lim_{r \to 0} r^{l_1}u(r) \cdot H(t) = \lim_{r \to 0} r^{l_1}u(r) \cdot \left( \frac{|z(t)|^{\frac{1}{s}}} {y(t)} \right) = \lim_{r \to 0} r^{l_1}u(r) \cdot \frac{r^{1+\delta}|u'(r)|} {r^\delta u(r)} = \lim_{r \to 0} r^{l_1+1}|u'(r)| = C_1l_1 > 0. \tag{3.13}
\]
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and

$$\lim_{r \to +\infty} r^{l_2}u(r) \cdot H(t) = \lim_{r \to +\infty} r^{l_2}u(r) \cdot \left| \frac{z(t)}{y(t)} \right|^{\frac{1}{p-1}} = \lim_{r \to +\infty} r^{l_2}u(r) \cdot \frac{r^{1+\delta}|u'(r)|}{r^\delta u(r)}$$

$$= \lim_{r \to +\infty} r^{l_2+1}|u'(r)| = C_2l_2 > 0.$$  

(3.14)

Next, we shall give a uniqueness result of positive solution of equation (3.1).

**Theorem 3.6** Suppose that $u_1(r)$ and $u_2(r)$ are two positive solutions of equation (3.1). Let $(y_1(t), z_1(t))$ and $(y_2(t), z_2(t))$ be two solutions to ODE system (3.5) corresponding to $u_1(r)$ and $u_2(r)$ respectively. If

$$\max_{t \in (\infty, +\infty)} y_1(t) = y_1(0) = \left[ \frac{N}{N-(a+1-b)p} (\delta^p - \mu) \right]^{\frac{1}{p-\rho}},$$

(3.15)

and $y_2(0) = y_1(0)$. Then $(y_1(t), z_1(t)) = (y_2(t), z_2(t))$, hence $u_1 = u_2$.

**Proof.** The proof is similar to that of Theorem 3.11 in [1].

Similar to Theorem 3.13 in [1], we resume the above results together and obtain the following theorem which describes the asymptotic behavior of all the radial solutions to equation (3.1).

**Theorem 3.7** All positive radial solutions to equation (2.3) have the form:

$$u(\cdot) = \varepsilon^{-\frac{N-(a+1)p}{p}} u_0(\frac{\cdot}{\varepsilon}),$$

(3.16)

where $u_0$ is a solution to equation (2.3) satisfying $u_0(1) = y(0) = \left[ \frac{N}{N-(a+1-b)p} (\delta^p - \mu) \right]^{\frac{1}{p-\rho}}$. Furthermore, there exist constants $C_1, C_2 > 0$ such that

$$0 < C_1 \leq \frac{u_0(x)}{(|x|^{l_1/\delta} + |x|^{l_2/\delta})^{-\delta}} \leq C_2,$$

(3.17)

where $l_1, l_2$ are the two zeros of function $\xi(s) = (p-1)s^p - (N-(a+1)p) s^{p-1} + \mu$ satisfying $0 < l_1 < l_2$. 


4  Strict inequality (1.2)

In this section, applying the asymptotic behavior of the solutions to equation (2.3) obtained in the previous section, we give some estimates on the extremal function of $S_{0, \mu}$. Let $u_0$ be an extremal function of $S_{0, \mu}$ with $\|u_0; L^p_0(\mathbb{R}^N)\| = 1$. From the discussion in Section 2 and 3, we know that $u_0$ is radial, and for all $\varepsilon > 0$,

$$U_\varepsilon(r) = \varepsilon^{-(N-(a+1)p)/p} u_0(r/\varepsilon)$$

is also an extremal function of $S_{0, \mu}$, and there exists a positive constant $C_\varepsilon$ such that $C_\varepsilon U_\varepsilon$ is a solution to equation (2.3). In fact, from the proof of Lemma 2.1, we know that

$$C_\varepsilon = 2^{-p_0} S_{p_0, \mu}^{(a+1)} = C_0,$$

which is independent of $\varepsilon$, denoted by $C_0$. Set $u_\varepsilon^* = C_0 U_\varepsilon$, then from equation (2.3) we have

$$Q_\mu(u_\varepsilon^*) = \|u_\varepsilon^*; L^p_0\|^{p_*} = S_{p_0, \mu}^{(a+1)} = S_{p_0, \mu}^{(a+1)} N.$$  (4.1)

For any $\varepsilon > 0$, and $m \in \mathbb{N}$ large enough such that $B_{1/m} \subseteq \Omega$, define

$$u_\varepsilon^m(x) = \begin{cases} u_\varepsilon^*(x) - u_\varepsilon^*(1/m), & x \in B_{1/m} \setminus \{0\}; \\ 0, & x \in \overline{\Omega \setminus B_{1/m}}. \end{cases}$$  (4.2)

**Lemma 4.1** Set $\varepsilon = m^{-h}$, $h > 1$. Then as $m \to \infty$, we have

$$Q_\mu(u_\varepsilon^m) \leq S_{0, \mu}^{(a+1)} N + O(m^{-(h-1)[(a+1)l_2p-N]}),$$  (4.3)

and

$$\|u_\varepsilon^*; L^p_0\|^{p_*} \geq S_{0, \mu}^{(a+1)} N - O(m^{-(h-1)[(b+l_2)p-N]}),$$  (4.4)

where and afterward $O(m^{-\alpha})$ denotes a positive quantity which is $O(m^{-\alpha})$, but is not $o(m^{-\alpha})$, as $m \to \infty$.

**Proof.** We shall only prove (4.3), and omit the prove of (4.4).

Since $Q_\mu(u_\varepsilon^m) = \int_{\mathbb{R}^N} |Du_\varepsilon^m|_p^p dx - \mu \int_{\mathbb{R}^N} |u_\varepsilon^m|^{p_0} dx$, we estimate each term in $Q_\mu(u_\varepsilon^m)$ as follows:

$$\int_{\Omega} |Du_\varepsilon^m|_p^p dx = \int_{B_{1/m}} |Du_\varepsilon^m|_p^p dx$$

$$= \int_{\mathbb{R}^N} |Du_\varepsilon^m|_p^p dx - \int_{\mathbb{R}^N \setminus B_{1/m}} |Du_\varepsilon^m|_p^p dx$$  (4.5)

$$\leq \int_{\mathbb{R}^N} |Du_\varepsilon^m|_p^p dx.$$
and

\[ \int_{\Omega} \left| u_\varepsilon^m \right|^p dx = \int_{B_{\frac{1}{m}}} \frac{\left( u_\varepsilon^*(x) - u_\varepsilon^*(\frac{1}{m}) \right)^p}{|x|^{(a+1)p}} dx \]

\[ \geq \int_{B_{\frac{1}{m}}} \frac{u_\varepsilon^*(x)^p - pu_\varepsilon^*(\frac{1}{m})u_\varepsilon^*(x)^{p-1}}{|x|^{(a+1)p}} dx \]

\[ = \int_{\mathbb{R}^N} \frac{u_\varepsilon^*(x)^p}{|x|^{(a+1)p}} dx - \int_{\mathbb{R}^N \setminus B_{\frac{1}{m}}} \frac{u_\varepsilon^*(x)^p}{|x|^{(a+1)p}} dx - pu_\varepsilon^*(\frac{1}{m}) \int_{B_{\frac{1}{m}}} \frac{u_\varepsilon^*(x)^{p-1}}{|x|^{(a+1)p}} dx. \]  

(4.6)

On the other hand, from the definition of \( u_\varepsilon^* \), we have

\[ \int_{\mathbb{R}^N \setminus B_{\frac{1}{m}}} \frac{u_\varepsilon^*(x)^p}{|x|^{(a+1)p}} dx = C_0^p \omega_N \int_{\frac{1}{m}}^{+\infty} \frac{\varepsilon^{-[N-(a+1)p]}u_0(\frac{\varepsilon}{\varepsilon})^p}{r^{(a+1)p}} r^{N-1} dr \]

\[ = C_0^p \omega_N \int_{m^{-h-1}}^{+\infty} u_0(t)^p t^{N-1-(a+1)p} dt \]

\[ = \mathcal{O}(m^{-(h-1)\left((a+1)l_2p-N\right)}), \]  

(4.7)

where in the second equality, we make the change of variable \( t = \varepsilon \), and in the last equality, we use the asymptotic behavior of \( u_0 \) at the infinity, since \( h > 1 \), hence \( m^{-h} \to \infty \) as \( m \to \infty \). Note that \( \xi'(l_2) = p(1-l_2)^{-1} \) and \( m^{-h-1} \to \infty \) as \( m \to \infty \). Similarly, we can estimate the last integration in (4.6) as follows:

\[ u_\varepsilon^*(\frac{1}{m}) \int_{B_{\frac{1}{m}}} \frac{u_\varepsilon^*(x)^{p-1}}{|x|^{(a+1)p}} dx = C_0^p \omega_N u_0(\frac{1}{m}) \int_{\frac{1}{m}}^{1} \frac{\varepsilon^{-[N-(a+1)p]}u_0(\frac{\varepsilon}{\varepsilon})^{p-1}}{r^{(a+1)p}} r^{N-1} dr \]

\[ = C_0^p \omega_N u_0(m^{-h-1}) \int_{0}^{m^{-h-1}} u_0(t)^{p-1} t^{N-1-(a+1)p} dt \]

\[ \leq C_0^p \omega_N C_2 m^{-(h-1)l_2p} [C + m^{-(h-1)[N-(a+1)p-1]}] \]

\[ = \mathcal{O}(m^{-(h-1)[(a+1)l_2]_p-N}), \]  

(4.8)

where the last equality is from \( \xi(l_2) = 0 \) and so \( N-(a+1)p-(p-1)l_2 = \mu/l_2^{p-1} > 0 \). Thus, (4.3) follows from (4.5)-(4.8).  

\[ \square \]

**Lemma 4.2** Set \( \varepsilon = m^{-h}, \ h > 1 \). If \( c < (a+1+l_2)p - N \), then

\[ \int_{\mathbb{R}^N} \frac{|u_\varepsilon^m(x)|^p}{|x|^{(a+1)p-c}} dx \geq \mathcal{O}(m^{-ch}). \]  

(4.9)
Proof. A direct computation shows that
\[
\int_{\mathbb{R}^N} \frac{|u^m_\varepsilon(x)|^p}{|x|^{(a+1)p-c}} \, dx = \int_{B_{\frac{1}{m}}^\perp} \frac{(u^*_\varepsilon(x) - u^*_\varepsilon(\frac{1}{m}))^p}{|x|^{(a+1)p-c}} \, dx \\
\geq \int_{B_{\frac{1}{m}}^\perp} u^*_\varepsilon(x)^p - pu^*_\varepsilon(\frac{1}{m})u^*_\varepsilon(x)^{p-1} \, dx \\
= \int_{\mathbb{R}^N} \frac{u^*_\varepsilon(x)^p}{|x|^{(a+1)p-c}} \, dx - \int_{\mathbb{R}^N \setminus B_{\frac{1}{m}}^\perp} \frac{u^*_\varepsilon(x)^p}{|x|^{(a+1)p-c}} \, dx - pu^*_\varepsilon(\frac{1}{m}) \int_{B_{\frac{1}{m}}^\perp} u^*_\varepsilon(x)^{p-1} \, dx.
\]
We estimate each of the above integrations as follows:
\[
\int_{\mathbb{R}^N} \frac{u^*_\varepsilon(x)^p}{|x|^{(a+1)p-c}} \, dx = C^p_0 \omega_N \varepsilon \int_0^\infty u_0(t)^p t^{N-1-(a+1)p+c} \, dt = \mathcal{O}(\varepsilon^{-ch}), \quad (4.10)
\]
\[
\int_{\mathbb{R}^N \setminus B_{\frac{1}{m}}^\perp} \frac{u^*_\varepsilon(x)^p}{|x|^{(a+1)p-c}} \, dx = C^p_0 \omega_N \varepsilon \int_{m^{h-1}}^\infty u_0(t)^p t^{N-1-(a+1)p+c} \, dt \\
= \mathcal{O}(m^{-(h-1)(a+1+\ell_2)p-N-c}) \quad (4.11)
\]
and
\[
u^*_\varepsilon(\frac{1}{m}) \int_{B_{\frac{1}{m}}^\perp} \frac{u^*_\varepsilon(x)^{p-1}}{|x|^{(a+1)p-c}} \, dx = C^p_0 \omega_N u_0(\frac{1}{me}) \frac{\varepsilon^{-[N-(a+1)p]} u_0(\varepsilon^{-1})^{p-1}}{r^{(a+1)p-c}} \int_0^{m^{h-1}} \int_0^{m^{h-1}} u_0(t)^p t^{N-1-(a+1)p+c} \, dt \\
\leq C^p_0 \omega_N C m^{-h-1} \ell_2 p^{-c} \left[ C + m^{h-1}(N-(a+1)p-(p-1)\ell_2) \right] \\
= \mathcal{O}(m^{-(h-1)(a+1+\ell_2)p-N-c}) \quad (4.12)
\]
Note that since \( c < (a+1+\ell_2)p-N \), we have \(-ch > -(h-1)(a+1+\ell_2)p-N-c\), that is, we prove the lemma. 

Let \( \Omega \) be a smooth bounded open domain in \( \mathbb{R}^N \) with \( 0 \in \Omega \), define \( \mathfrak{D}^{1,p}_{\alpha,b}(\Omega) \) as the closure of \( C_0^\infty(\Omega) \) under the norm \( \|u\|_{\mathfrak{D}^{1,p}_{\alpha,b}(\Omega)} = \|Du; L^p_b(\Omega)\| \) and
\[
S_{\lambda,\mu}(p, a, b; \Omega) = \inf \left\{ Q_{\lambda,\mu}(u) : u \in \mathfrak{D}^{1,p}_{\alpha,b}(\Omega), \|u; L^p_b(\Omega)\| = 1 \right\}, \quad (4.13)
\]
where
\[
Q_{\lambda,\mu}(u) = \int_{\Omega} \frac{|Du|^p}{|x|^{ap}} \, dx - \mu \int_{\Omega} \frac{|u|^p}{|x|^{(a+1)p}} \, dx - \lambda \int_{\Omega} \frac{|u|^p}{|x|^{(a+1)p-c}} \, dx.
\]
If \( \lambda = 0 \), by rescaling argument, it is easy to show that \( S_{0,\mu}(p, a, b; \Omega) = S_{0,\mu} \). But for \( \lambda > 0 \), we shall have a strict inequality between \( S_{\lambda,\mu}(p, a, b; \Omega) \) and \( S_{0,\mu} \).
Theorem 4.3. If $\mu \in (0, \frac{N}{p})$, $\lambda > 0$, $b \in [a, a + 1)$, $c \in (0, (a + 1 + l_2)p - N)$, then the strict inequality (1.2) holds.

Proof. We shall study

$$\frac{Q_{\lambda, \mu}(u^m)}{\|u^m\|_{L^p_b(\Omega)}}.$$

It follows from Lemma 4.1 and 4.2 that

$$Q_{\lambda, \mu}(u^m) = Q_\mu(u^m) - \lambda \int_\Omega \frac{|u^m(x)|^p}{|x|^{(a+1)p-c}} \, dx$$

$$\leq S_{0, \mu}^{(a+1-b)p} + O(m^{-(h-1)((a+1+l_2)p-N)}) - O(m^{-ch})$$

and

$$\|u^m\|_{L^p_b(\Omega)} \geq S_{0, \mu}^{(a+1-b)p} - O(m^{-(h-1)((b+l_2)p-N)p_*})$$

$$= S_{0, \mu}^{(a+1-b)p_*} - O(m^{-(h-1)((a+1+l_2)p-N)})$$

Thus, we have

$$\frac{Q_{\lambda, \mu}(u^m)}{\|u^m\|_{L^p_b(\Omega)}} \leq \frac{S_{0, \mu}^{(a+1-b)p} + O(m^{-(h-1)((a+1+l_2)p-N)}) - O(m^{-ch})}{S_{0, \mu}^{(a+1-b)p_*} - O(m^{-(h-1)((b+l_2)p-N)p_*}) - O(m^{-ch})}$$

$$= S_{0, \mu} + O(m^{-(h-1)((a+1+l_2)p-N)}) - O(m^{-ch}).$$

If $c \in (0, (a + 1 + l_2)p - N)$, we can choose $h$ large enough such that $c < (h-1)(a+1+l_2)p - N)/h$ and so $-ch > -((h-1)((a+1+l_2)p - N)$, thus as $m$ large enough, (1.2) holds.

5 Application

In this section, as an application of the strict inequality of (1.2), we consider the existence of nontrivial solutions to the following quasilinear Brezis-Nirenberg type problem involving Hardy potential and Sobolev critical exponent:

$$\begin{cases} -\text{div}\left(\frac{|Du|^{p-2}Du}{|x|^{ap}}\right) - \mu \frac{|u|^{p-2}u}{|x|^{(a+1)p}} = \frac{|u|^{p_*-2}u}{|x|^{(a+1)p_*}} + \lambda \frac{|u|^{p-2}u}{|x|^{(a+1)p-c}}, & \text{in } \Omega, \\ u = 0, & \text{on } \partial \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded domain with $C^1$ boundary and $0 \in \Omega$, $1 < p < N$, $p_* = \frac{Np}{N-(a+1-b)p}$, $0 \leq a < \frac{N-p}{p}$, $a \leq b < (a + 1)$, $c > 0$; $\lambda$, $\mu$ are two positive real parameters.
To obtain the existence result, let’s define the energy functional $E_{\lambda, \mu}$ on $\mathcal{D}_{a,b}^{1,p}(\Omega)$ as

$$E_{\lambda, \mu}(u) = \frac{1}{p} \int_{\Omega} \left[ \frac{|Du|^p}{|x|^{ap}} - \mu \frac{|u|^p}{|x|^{(a+1)p}} - \frac{\lambda}{|x|^{(a+1)p-c}} |u|^p \right] \, dx - \frac{1}{p^*} \int_{\Omega} |u|^{p^*} \, dx.$$

It is easy to see that $E_{\lambda, \mu}$ is well-defined in $\mathcal{D}_{a,b}^{1,p}(\Omega)$, and $E_{\lambda, \mu} \in C^1(\mathcal{D}_{a,b}^{1,p}(\Omega), \mathbb{R})$. Furthermore, all the critical points of $E_{\lambda, \mu}$ are weak solutions to (5.1). We shall apply the Mountain Pass Lemma without (PS) condition due to Ambrosetti and Rabinowitz [2] to ensure the existence of (PS) sequence of $E_{\lambda, \mu}$ at some Mountain Pass type minimax value level $\beta$. Then the strict inequality (1.2) implies that $\beta < \frac{a+1-b}{N} S_{0,\mu}^{\frac{p}{a+1-p}}$. Finally, combining the generalized concentration compactness principle and a compactness property called singular Palais-Smale condition due to Boccardo and Murat [3](cf. also [12]), we shall obtain the existence of nontrivial solutions to (5.1).

Let’s define two more functionals on $\mathcal{D}_{a,b}^{1,p}(\Omega)$ as follows:

$$I_\mu(u) = \frac{1}{p} \int_{\Omega} \frac{|Du|^p}{|x|^{ap}} \, dx - \mu \int_{\Omega} \frac{|u|^p}{|x|^{(a+1)p}} \, dx,$$

and denote $\mathcal{M} = \{ u \in \mathcal{D}_{a,b}^{1,p}(\Omega) : J(u) = 1 \}$. For $\mu \in (0, \overline{\mu})$, the Hardy inequality shows that $\frac{1}{p} \int_{\Omega} \frac{|Du|^p}{|x|^{ap}} \, dx - \frac{\mu}{p} \int_{\Omega} \frac{|u|^p}{|x|^{(a+1)p}} \, dx$ is nonnegative measure on $\Omega$. The classical results in the Calculus of Variations(cf. [21]) show that $I_\mu$ is lower semicontinuity on $\mathcal{M}$. On the other hand the compact imbedding theorem in [25] implies that $\mathcal{M}$ is weakly closed. Thus the direct method ensure that $I_\mu$ attains its minimum on $\mathcal{M}$, denote $\lambda_1 = \min\{ I_\mu(u) : u \in \mathcal{M} \} > 0$. From the homogeneity of $I_\mu$ and $J$, $\lambda_1$ is the first nonlinear eigenvalue of problem:

$$\begin{cases}
-\text{div} \left( \frac{|Du|^{p-2}Du}{|x|^{ap}} \right) - \mu \frac{|u|^{p-2}u}{|x|^{(a+1)p}} = \lambda \frac{|u|^{p-2}u}{|x|^{(a+1)p-c}}, & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega.
\end{cases} \quad (5.2)
$$

The following lemma indicates that $E_{\lambda, \mu}$ satisfies the geometric condition of Mountain Pass Lemma without (PS) condition due to Ambrosetti and Rabinowitz [2], the proof is direct and omitted.

**Lemma 5.1** If $\mu \in (0, \, \overline{\mu})$, $\lambda \in (0, \, \lambda_1)$, then

(i) $E_{\lambda, \mu}(0) = 0$;

(ii) $\exists \alpha, r > 0$, s.t. $E_{\lambda, \mu}(u) \geq \alpha$, if $\|u\| = r$;
(iii) For any \( v \in \mathcal{D}^{1,p}_{a,b}(\Omega) \), \( v \neq 0 \), there exists \( T > 0 \) such that \( E_{\lambda, \mu}(tv) \leq 0 \) if \( t > T \).

For \( v \in \mathcal{D}^{1,p}_{a,b}(\Omega) \) with \( \|v\| > r \) and \( E_{\lambda, \mu}(v) \leq 0 \), set
\[
\beta := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} E_{\lambda, \mu}(\gamma(t)),
\]
where
\[
\Gamma := \{ \gamma \in C([0,1], \mathcal{D}^{1,p}_{a,b}(\Omega)) \mid \gamma(0) = 0, \gamma(1) = v \}.
\]
It is easy to see that \( \beta \) is independent of the choice of \( v \) such that \( E_{\lambda, \mu}(v) \leq 0 \), and furthermore \( \beta \geq \alpha \). If \( \beta \) is finite, from Lemma 5.1 and Mountain Pass Lemma, there exists a (PS)\( _\beta \) sequence \( \{u_m\}_{m=1}^\infty \) of \( E_{\lambda, \mu} \) at level \( \beta \), that is, \( E_{\lambda, \mu}(u_m) \to \beta \) and \( E_{\lambda, \mu}'(u_m) \to 0 \) in the dual space \( \mathcal{D}^{1,p}_{a,b}(\Omega)' \) of \( \mathcal{D}^{1,p}_{a,b}(\Omega) \) as \( m \to \infty \).

**Lemma 5.2** If \( \mu \in (0, \bar{\mu}), \lambda \in (0, \lambda_1) \), then the strict inequality (1.2) is equivalent to
\[
\beta < \frac{a+1-b}{N} S^{\frac{N}{p+1-a-bp}}_{0, \mu}.
\]

**Proof.** 1. (1.2) \( \Rightarrow \) (5.3).

Let \( v_1 \) be a function such that \( \|v_1; L^p_b(\Omega)\| = 1 \), and \( Q_{\lambda, \mu}(v_1) < S_{0, \mu} \). We have
\[
\beta \leq \sup_{0 < t < \infty} E_{\lambda, \mu}(tv_1) = \sup_{0 < t < \infty} \left( \frac{tp}{p} Q_{\lambda, \mu}(v_1) - \frac{tp^*}{p^*} \right)
\leq \left( \frac{1}{p} - \frac{1}{p^*} \right) Q_{\lambda, \mu}(v_1)^{p^*} = \frac{a+1-b}{N} Q_{\lambda, \mu}(v_1)^{\frac{N}{p+1-a-bp}}
\leq \frac{a+1-b}{N} S^{\frac{N}{p+1-a-bp}}_{0, \mu}.
\]

2. (5.3) \( \Rightarrow \) (1.2).

Since \( \lambda < \lambda_1 \), for \( u = g(t) = tv \) with \( t \) closed to 0, we have \( (DE_{\lambda, \mu}(u), u) > 0 \); while for \( u = g(1) = v \), we have

\[
(DE_{\lambda, \mu}(v), v) < pE_{\lambda, \mu}(v) \leq 0.
\]

Consider function \( f(t) = E_{\lambda, \mu}(tv) \in C^1([0,1], \mathbb{R}) \), we have that \( f'(t) > 0 \) for \( t \) closed to 0, and \( f'(1) \leq 0 \). From the medium value theorem, there exists \( t_0 \in (0,1) \) such that \( f'(t_0) = 0 \), that is, for \( u = t_0v \), we have

\[
(DE_{\lambda, \mu}(u), u) = Q_{\lambda, \mu}(u) - \|u; L^p_b(\Omega)\|^{p^*} = 0.
\]

Thus a direct computation shows that
\[
\frac{Q_{\lambda, \mu}(u)}{\|u; L^p_b(\Omega)\|^p} = Q_{\lambda, \mu}(u)^{1-p/p^*} = \left( \frac{N}{a+1-b} E_{\lambda, \mu}(u)^{(a+1-b)p} \right)^{\frac{N}{p+1-a-bp}},
\]
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that is,

$$\beta = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} E_{\lambda, \mu}(\gamma(t)) \geq \frac{a + 1 - b}{N} S_{\lambda, \mu}(p, a, b, \Omega) \left( \frac{N}{a+1-b} \right)^p.$$ 

Hence (5.3) \implies (1.2).

**Lemma 5.3** If \( \mu \in (0, \bar{\mu}), \lambda \in (0, \lambda_1) \), then any \((PS)\_\beta\) sequence of \(E_{\lambda, \mu}\) is bounded in \(D_{a,b}^{1,p}(\Omega)\).

**Proof.** Suppose that \( \{u_m\}_{m=1}^{\infty} \) is a \((PS)\_\beta\) sequence of \(E_{\lambda, \mu}\). As \( m \to \infty \), we have

$$\beta + o(1) = E_{\lambda, \mu}(u_m) = \frac{1}{p} \int_{\Omega} \left[ \frac{|Du_m|^p}{|x|^{(a+1)p}} - \mu \frac{|u_m|^p}{|x|^{(a+1)p}} - \lambda \frac{|u_m|^p}{|x|^{(a+1)p-c}} \right] dx - \frac{1}{p^*} \int_{\Omega} |u_m|^{p^*} dx \quad (5.5)$$

and

$$o(1) \|\varphi\| = (DE_{\lambda, \mu}(u_m), \varphi) = \int_{\Omega} \left[ \frac{|Du_m|^{p-2}Du_m \cdot D\varphi}{|x|^{ap}} - \mu \frac{|u_m|^{p-2}u_m\varphi}{|x|^{(a+1)p-c}} - \lambda \frac{|u_m|^{p-2}u_m\varphi}{|x|^{(a+1)p-c}} \right] dx - \int_{\Omega} \frac{|u_m|^{p-2}u_m\varphi}{|x|^{bp^*}} dx, \quad (5.6)$$

for any \( \varphi \in D_{a,b}^{1,p}(\Omega) \). From (5.5) and (5.6), as \( m \to \infty \), it follows that

$$p_s \beta + o(1) - o(1) \|u_m\| = p_s E_{\lambda, \mu}(u_m) - (DE_{\lambda, \mu}(u_m), u_m)$$

$$= \left( \frac{p_s}{p} - 1 \right) \int_{\Omega} \left[ \frac{|Du_m|^p}{|x|^{ap}} - \mu \frac{|u_m|^p}{|x|^{(a+1)p}} - \lambda \frac{|u_m|^p}{|x|^{(a+1)p-c}} \right] dx$$

$$\geq \left( \frac{p_s}{p} - 1 \right) \left( 1 - \frac{\lambda}{\lambda_1} \right) \int_{\Omega} \left[ \frac{|Du_m|^p}{|x|^{ap}} - \mu \frac{|u_m|^p}{|x|^{(a+1)p}} \right] dx$$

$$\geq \left( \frac{p_s}{p} - 1 \right) \left( 1 - \frac{\lambda}{\lambda_1} \right) \left( 1 - \frac{\mu}{\bar{\mu}} \right) \|u_m\|^p.$$ 

Thus, \( \{u_m\}_{m=1}^{\infty} \) is bounded in \(D_{a,b}^{1,p}(\Omega)\) if \( \mu \in (0, \mu), \lambda \in (0, \lambda_1) \).

From the boundedness of \( \{u_m\}_{m=1}^{\infty} \) in \(D_{a,b}^{1,p}(\Omega)\), we have the following medium convergence:

$$u_m \rightharpoonup u \text{ in } D_{a,b}^{1,p}(\Omega), \ L_1^p(\Omega) \text{ and } L_b^{p^*}(\Omega),$$
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\[ u_m \to u \text{ in } L^r_\alpha(\Omega) \text{ if } 1 \leq r < \frac{Np}{N-p}, \frac{\alpha}{r} < (a+1) + N \left( \frac{1}{r} - \frac{1}{p} \right), \]

\[ u_m \to u \text{ a.e. in } \Omega. \]

In order to obtain the strong convergence of \( \{u_m\}_{m=1}^\infty \) in \( L^p_{1,b}(\Omega) \), we need the following generalized concentration compactness principle (cf. also [22] and [23] and references therein), the proof is similar to that in [17] and we omit it.

**Lemma 5.4 (Concentration Compactness Principle)** Suppose that \( M(\mathbb{R}^N) \) is the space of bounded measures on \( \mathbb{R}^N \), and \( \{u_m\} \subset \mathfrak{D}^{1,p}_{a,b}(\Omega) \) is a sequence such that:

\[ u_m \rightharpoonup u \quad \text{in } \mathfrak{D}^{1,p}_{a,b}(\Omega), \]

\[ \xi_m := (|x|^{-ap}|Du_m|^p - \mu|x|^{-(a+1)p}|u_m|^p) \, dx \rightharpoonup \xi \quad \text{in } M(\mathbb{R}^N), \]

\[ \nu_m := |x|^{-bp^*}|u_m|^{p^*} \, dx \rightharpoonup \nu \quad \text{in } M(\mathbb{R}^N), \]

\[ u_m \to u \quad \text{a.e. on } \mathbb{R}^N. \]

Then there are the following statements:

(1) There exists some at most countable set \( J \), a family \( \{x^{(j)} : j \in J\} \) of distinct points in \( \mathbb{R}^N \), and a family \( \{\nu^{(j)} : j \in J\} \) of positive numbers such that

\[ \nu = |x|^{-bp^*}|u|^{p^*} \, dx + \sum_{j \in J} \nu^{(j)} \delta_{x^{(j)}}, \quad (5.7) \]

where \( \delta_x \) is the Dirac-mass of mass 1 concentrated at \( x \in \mathbb{R}^N \).

(2) The following inequality holds

\[ \xi \geq (|x|^{-ap}|Du|^p - \mu|x|^{-(a+1)p}|u|^p) \, dx + \sum_{j \in J} \xi^{(j)} \delta_{x^{(j)}}, \quad (5.8) \]

for some family \( \{\xi^{(j)} > 0 : j \in J\} \) satisfying

\[ S_{0,\mu}(\nu^{(j)})^{p/p^*} \leq \xi^{(j)}, \quad \text{for all } j \in J. \quad (5.9) \]

In particular, \( \sum_{j \in J} (\nu^{(j)})^{p/p^*} < \infty. \)

**Lemma 5.5** If \( \mu \in (0, \mu), \lambda \in (0, \lambda_1), \) let \( \{u_m\}_{m=1}^\infty \) be a \((PS)_\beta\) sequence of \( E_{\lambda,\mu} \) at level \( \beta \) defined above. \( (7.3) \) implies that \( \nu^{(j)} = 0 \) for all \( j \in J \), that is, up to a subsequence, \( u_m \to u \) in \( L^p_{1,b}(\Omega) \) as \( m \to 0. \)
that for any exponent of $p \in T$ it follows that $-\mu$.

Thus, form (5.7) and Lemma 5.4, (5.11) holds.

Proof. From Lemma 5.3, $\{u_m\}_{m=1}^{\infty}$ is bounded in $D_{a,b}^{1,p}(\Omega)$, then we have that $|Du_m|^{p-2}Du_m$ is bounded in $(L^p(\Omega; |x|^{-ap}))^N$, where $p'$ is the conjugate exponent of $p$, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$. Without loss of generality, we suppose that $T \in (L^p(\Omega; |x|^{-ap}))^N$ such that

$$|Du_m|^{p-2}Du_m \rightarrow T \text{ in } (L^p(\Omega; |x|^{-ap}))^N.$$\hspace{1cm}

Also, $|u_m|^{p-2}u_m$ is bounded in $L^{p'}(\Omega; |x|^{-(a+1)p})$, $|u_m|^{p-2}u_m$ is bounded in $L^{p'}(\Omega; |x|^{-bp_\ast})$, and $u_m \rightarrow u$ almost everywhere in $\Omega$, thus it follows that $|u_m|^{p-2}u_m \rightarrow |u|^{p-2}u$ in $L^{p'}(\Omega; |x|^{-(a+1)p})$ and

$|u_m|^{p-2}u_m \rightarrow |u|^{p-2}u$ in $L^{p'}(\Omega; |x|^{-bp_\ast}).$

From the compactness imbedding theorem in [25], it follows that $|u_m|^{p-2}u_m \rightarrow |u|^{p-2}u$ in $L^{p'}(\Omega; |x|^{-(a+1)p+c}).$

Taking $m \rightarrow \infty$ in (5.6), we have

$$\int_{\Omega} \frac{T \cdot D\varphi}{|x|^{ap}} dx = \mu \int_{\Omega} \frac{|u|^{p-2}u\varphi}{|x|^{(a+1)p}} dx + \lambda \int_{\Omega} \frac{|u|^{p-2}u\varphi}{|x|^{(a+1)p-c}} dx + \int_{\Omega} \frac{|u|^{p-2}u\varphi}{|x|^{bp_\ast}} dx,$$ (5.10)

for any $\varphi \in D_{a,b}^{1,p}(\Omega)$. Let $\varphi = \psi u_m$ in (5.6), where $\psi \in C(\bar{\Omega})$, and take $m \rightarrow \infty$, it follows that

$$\int_{\Omega} \psi d\xi + \int_{\Omega} \frac{uT \cdot D\psi}{|x|^{ap}} dx = \int_{\Omega} \psi d\nu + \lambda \int_{\Omega} \frac{|u|^{p\psi}}{|x|^{(a+1)p-c}} dx.$$ (5.11)

Let $\varphi = \psi u$ in (5.10), it follows that

$$\int_{\Omega} \frac{uT \cdot D\psi}{|x|^{ap}} dx + \int_{\Omega} \frac{\psi T \cdot Du}{|x|^{ap}} dx = \mu \int_{\Omega} \frac{|u|^{p\psi}}{|x|^{(a+1)p}} dx + \lambda \int_{\Omega} \frac{|u|^{p\psi}}{|x|^{(a+1)p-c}} dx + \int_{\Omega} \frac{|u|^{p\ast \psi}}{|x|^{bp_\ast}} dx,$$ (5.12)

Thus, form (5.7) and Lemma 5.4, (5.11)–(5.12) implies that

$$\int_{\Omega} \psi d\xi = \int_{\Omega} \frac{\psi T \cdot Du}{|x|^{ap}} dx - \mu \int_{\Omega} \frac{|u|^{p\psi}}{|x|^{(a+1)p}} dx + \int_{\Omega} \psi d\nu - \int_{\Omega} \frac{|u|^{p\ast \psi}}{|x|^{bp_\ast}} dx$$

$$= \int_{\Omega} \frac{\psi T \cdot Du}{|x|^{ap}} dx - \mu \int_{\Omega} \frac{|u|^{p\psi}}{|x|^{(a+1)p}} dx + \sum_{j \in J} \nu^{(j)} \psi(x^{(j)}).$$ (5.13)
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Letting $\psi \to \delta_{x(j)}$, we have

$$\xi(j) = \nu(j).$$

Combining with (5.9), it follows that $\nu(j) \geq S_{0, \mu}(\nu(j))^{p/q}$, which means that

$$\nu(j) \geq S_{0, \mu}^{N}, \hspace{1cm} (5.14)$$

if $\nu(j) \neq 0$. On the other hand, taking $m \to \infty$ in (5.5), and using (5.13) with $\psi \equiv 1$, (5.7) and (5.10), it follows that

$$\beta = \frac{1}{p} \int_{\Omega} \frac{\partial T \cdot Du}{|x|^{(a+1)p-c}} \, dx - \frac{\lambda}{p} \int_{\Omega} \frac{|u|^p}{|x|^{(a+1)p-c}} \, dx$$

(5.15)

From (5.14), (5.15), (5.3) implies that $\nu(j) = 0$ for all $j \in J$. Hence we have

$$\int_{\Omega} \frac{|u_m|^p}{|x|^{(a+1)p-c}} \, dx \to \int_{\Omega} \frac{|u|^p}{|x|^{(a+1)p-c}} \, dx,$$

as $m \to \infty$. Thus, the Brezis-Lieb Lemma [4] implies that, up to a subsequence, $u_m \to u$ in $L^p_0(\Omega)$ as $m \to 0$.

In order to deduce the almost everywhere convergence of $Du_m$ in $\Omega$ and to obtain existence of nontrivial solution to (5.1), we shall apply the variational approach supposed in [12] and a convergence theorem due to Boccardo and Murat(cf. Theorem 2.1 in [3]), so we suppose that $a = 0$, and $D^{1,p}_{a,b}(\Omega) = W^{1,p}_0(\Omega)$.

**Theorem 5.6** If $a = 0, \mu \in (0, \bar{\mu}), \lambda \in (0, \lambda_1), b \in [0, 1), c \in (0, (1+l_2)p-N)$, then there exists a nontrivial solution to (5.1).

**Proof.** Apply the variational approach supposed in [12] and a convergence theorem in [3], there exists a subsequence of $\{u_m\}_{m=1}^{\infty}$, still denoted by $\{u_m\}_{m=1}^{\infty}$, such that

$$u_m \to u \in W^{1,q}_0(\Omega), \hspace{1cm} q < p.$$
which implies that $u$ is a solution to (5.1) in sense of distributions. Since $u \in W_0^{1,p}(\Omega)$, by density argument, $u$ is a weak solution to (5.1). Next, we shall show that $u \not\equiv 0$.

In fact, from the homogeneity and Lemma 5.5, we have

\[
0 < \alpha \leq \beta = \lim_{m \to \infty} E_{\lambda, \mu}(u_m) = \lim_{m \to \infty} \left[ E_{\lambda, \mu}(u_m) - \frac{1}{p}(DE_{\lambda, \mu}(u_m), u_m) \right]
\]

\[
= \lim_{m \to \infty} \left( \frac{1}{p} - \frac{1}{p_*} \right) \int_{\Omega} \frac{|u_m|^{p_*}}{|x|^{b p_*}} \, dx
\]

\[
= \left( \frac{1}{p} - \frac{1}{p_*} \right) \int_{\Omega} \frac{|u|^{p_*}}{|x|^{b p_*}} \, dx.
\]

Thus, $u \not\equiv 0$.  

In sight of Theorem 5.6, we conjecture that the conclusion is also true for $0 < a < \frac{N-p}{p}$.

**Conjecture 5.7** If $0 \leq a < \frac{N-p}{p}$, $\mu \in (0, \infty)$, $\lambda \in (0, \lambda_1)$, $b \in [a, a+1)$, $c \in (0, (a+1+l_2)p - N)$, then there exists a nontrivial solution to (5.1).

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