QUASICONFORMAL, LIPSCHITZ, AND BV MAPPINGS
IN METRIC SPACES

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Abstract. Consider a mapping \( f: X \to Y \) between two metric measure spaces. We study generalized versions of the local Lipschitz number \( \operatorname{Lip} f \), as well as of the distortion number \( H_f \) that is used to define quasiconformal mappings. Using these, we give sufficient conditions for \( f \) being a BV mapping \( f \in \BV_{\text{loc}}(X; Y) \) or a Newton-Sobolev mapping \( f \in N_{1,p}_{\text{loc}}(X; Y) \), with \( 1 \leq p < \infty \).

1. Introduction

Consider two metric measure spaces \((X, d, \mu)\) and \((Y, d_Y, \nu)\), and a mapping \( f: X \to Y \). For every \( x \in X \) and \( r > 0 \), one defines
\[
L_f(x, r) := \sup \{ d_Y(f(y), f(x)) : d(y, x) \leq r \}
\]
and
\[
l_f(x, r) := \inf \{ d_Y(f(y), f(x)) : d(y, x) \geq r \},
\]
and then
\[
H_f(x, r) := \frac{L_f(x, r)}{l_f(x, r)}.
\]
A homeomorphism \( f: X \to Y \) is (metric) quasiconformal if there is a number \( 1 \leq H < \infty \) such that
\[
H_f(x) := \limsup_{r \to 0} H_f(x, r) \leq H
\]
for all \( x \in X \). One also defines \( h_f \) by replacing “\( \limsup \)” with “\( \liminf \)”.

Assuming that the spaces \( X, Y \) are \( Q \)-dimensional, with \( Q > 1 \), and satisfy suitable regularity assumptions, it is known that quasiconformal mappings belong to the Newton-Sobolev space \( N_{1,Q}^{1,Q}_{\text{loc}}(X; Y) \). See Section 2 for definitions. Moreover, starting from Gehring, in the literature there are many results saying that if \( f \) satisfies a suitable relaxed version of the metric definition of quasiconformality, then \( f \) is at least in the class \( N_{1,Q}^{1,1}_{\text{loc}}(X; Y) \). A typical version of such a result is the following: assume that \( X, Y \) are locally compact and Ahlfors \( Q \)-regular, \( f: X \to Y \) is a homeomorphism, and \( 1 \leq p \leq Q \); then if

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• \( h_f < \infty \) outside a set \( E \) with \( \sigma \)-finite \( Q - p \)-dimensional Hausdorff measure, and
• \( h_f \in L^p_{\text{loc}}(X) \),
it follows that \( f \in N^{1,p}_{\text{loc}}(X;Y) \). For this, see e.g. Williams [27, Corollary 1.3].

Similarly to \( H_f \), we define the pointwise asymptotic Lipschitz number

\[
\operatorname{Lip}_f(x) := \limsup_{r \to 0} \frac{L_f(x,r)}{r},
\]

and also \( \operatorname{lip}_f \) with “\( \limsup \)” replaced by “\( \liminf \)”. If \( 1 \leq p \leq \infty \) and a continuous function \( f: \mathbb{R}^n \to \mathbb{R} \) satisfies the following:

• \( \operatorname{lip}_f < \infty \) outside a set \( E \) with \( \sigma \)-finite \( n - 1 \)-dimensional Hausdorff measure, and
• \( \operatorname{lip}_f \in L^p_{\text{loc}}(\mathbb{R}^n) \),
then \( f \in W^{1,p}_{\text{loc}}(\mathbb{R}^n) \); see Balogh–Csörnyei [2, Theorem 1.2]. Overall, the literature studying the above two types of results is extensive, see e.g. (in order of publication) Gehring [9, 10], Margulis–Mostow [21], Fang [8], Balogh–Koskela [3], Kallunki–Koskela [15], Heinonen–Koskela–Shanmugalingam–Tyson [13], Kallunki–Martio [16], Koskela–Rogovin [17], Balogh–Koskela–Rogovin [4], Wildrick–Zürcher [26, 28], Hanson [12], Williams [27], and Lahti–Zhou [19, 20]. Note that in these results, somewhat curiously there are two different exceptional sets. One exceptional set is the at most \( Q - p \)-dimensional set \( E \), whereas the condition that \( h_f \) (resp. \( \operatorname{lip}_f \)) is in \( L^p_{\text{loc}}(X) \) (resp. \( L^p_{\text{loc}}(X) \)) fails to give control in a set of zero \( \mu \)-measure, amounting to an at most \( Q \)-dimensional exceptional set.

In many contexts, it is natural to replace the Newton-Sobolev class \( N^{1,1}_{\text{loc}}(X;Y) \) with the larger class of mappings of bounded variation \( \text{BV}_{\text{loc}}(X;Y) \). Thus one can ask, is there a BV version of the above type of results? We observe that just like the exceptional set \( E \) in the case \( p = 1 \), the approximate jump set of a BV function \( f \in \text{BV}(\mathbb{R}^n) \) is also of \( \sigma \)-finite \( \mathcal{H}^{n-1} \)-measure, suggesting a connection between the two. In order to obtain a natural BV result, one needs to drop the continuity assumption on \( f \), and it is also expected that one should allow a larger exceptional set \( E \). However, without continuity, already when assuming that \( \operatorname{Lip}_f, H_f \) are in \( L^\infty(\mathbb{R}^n) \) and finite outside a set of \( \sigma \)-finite \( \mathcal{H}^{n-1} \)-measure, it can easily happen that \( f \) is not a BV mapping; see Example 3.1.

For these reasons, for a mapping \( f: X \to Y \), we define generalized versions of \( \operatorname{Lip}_f \) and \( H_f \) as follows:

\[
\operatorname{Lip}_f^{\kappa,M}(x) := \limsup_{r \to 0} \frac{L_f(x,r)}{r} \frac{\mu(B(x,Mr))}{\kappa(B(x,Mr))}, \quad x \in X,
\]

and

\[
H_f^{\kappa,M}(x) := \limsup_{r \to 0} \frac{L_f(x,r)}{L_f(x,r)} \left( \frac{\mu(B(x,Mr))}{\kappa(B(x,Mr))} \right)^{(Q-1)/Q}, \quad x \in X,
\]
where \( M \geq 1 \) and \( \kappa \) is a positive Radon measure. Note that \( \text{Lip}_f^{\mu, 1} = \text{Lip}_f \) and \( H_f^{\mu, 1} = H_f \). This type of generalized Lipschitz and distortion numbers were previously found in [18] to be useful in various contexts.

Our main results will be given in Theorems 4.1, 4.17, and 4.30. These theorems strive for generality, leading to rather long and complicated formulations, but we give a simple version in the following. The symbol \( \widetilde{H}^p \) denotes the codimension \( p \) Hausdorff measure.

**Theorem 1.1.** Suppose \((X, d, \mu)\) and \((Y, d_Y, \nu)\) are Ahlfors \( Q \)-regular, with \( Q > 1 \) and \( 1 \leq p \leq Q \). Let \( f: X \to Y \) be injective and bounded, and \( M \geq 1 \). Then:

1. If there exists a Radon measure \( \kappa \geq \mu \) such that \( \text{Lip}_f^{\kappa, M}(x) \leq 1 \) for \( \widetilde{H}^1 \)-a.e. \( x \in X \), then \( f \in \text{BV}_{\text{loc}}(X; Y) \);

2. If there exists a function \( a: X \to [1, \infty) \) belonging to \( L_{\text{loc}}^{p(Q−1)/Q}(X) \) such that \( H_f^{a, M}(x) \leq 1 \) for \( \widetilde{H}^p \)-a.e. \( x \in X \), then \( f \in \text{N}_{\text{loc}}^{1,p}(X; Y) \).

Besides being able to cover the BV case, in Example 5.6 we will see that the generalized Lipschitz and distortion numbers are sometimes also more effective at detecting Newton-Sobolev mappings \( f \in \text{N}_{\text{loc}}^{1,p}(X; Y) \), compared with the existing results in the literature. Moreover, in Remark 2.15 and Examples 3.1 and 5.8 we will see that at least under suitable circumstances, the exceptional set \( E \) can indeed be seen to correspond to the approximate jump set of a BV function, giving an interpretation of “why” one usually needs the two different exceptional sets.

2. Notation and definitions

Throughout the paper, we consider two metric measure spaces \((X, d, \mu)\) and \((Y, d_Y, \nu)\), where \( \mu \) and \( \nu \) are Borel regular outer measures. In this paper, we only consider positive outer measures. A ball is defined by \( B(x, r) := \{ y \in X : d(y, x) < r \} \), for \( x \in X \) and \( r > 0 \). For a ball \( B = B(x, r) \) and \( M > 0 \), we sometimes denote \( MB := B(x, Mr) \). In a metric space, a ball (as a set) does not necessarily have a unique center and radius, but when using this abbreviation we will work with balls for which these have been specified.

We will always assume that \( \mu \) is doubling, meaning that there is a constant \( C_d \geq 1 \) such that

\[
0 < \mu(B(x, 2r)) \leq C_d \mu(B(x, r)) < \infty
\]

for all \( x \in X \) and \( r > 0 \). Given \( Q \geq 1 \), we say that \( \mu \) (or \( X \)) is Ahlfors \( Q \)-regular if there is a constant \( C_A \geq 1 \) such that

\[
C_A^{-1}r^Q \leq \mu(B(x, r)) \leq C_A r^Q \tag{2.1}
\]

for all \( x \in X \) and \( r > 0 \).

We denote the \( s \)-dimensional Hausdorff measure by \( H^s \), \( s \geq 0 \), and it is obtained as a limit of the Hausdorff pre-measures \( H^s_R \) as \( R \to 0 \).
We always consider $1 \leq p < \infty$. For any set $A \subset X$ and $0 < R < \infty$, the restricted Hausdorff content of codimension $p$ is defined by
\[
\widetilde{\mathcal{H}}^p_R(A) := \inf \left\{ \sum_j \frac{\mu(B(x_j, r_j))}{r_j^p} : A \subset \bigcup_j B(x_j, r_j), \ r_j \leq R \right\},
\]
where we consider finite and countable coverings. The codimension $p$ Hausdorff measure of $A \subset X$ is then defined by
\[
\widetilde{\mathcal{H}}^p(A) := \lim_{R \to 0} \widetilde{\mathcal{H}}^p_R(A).
\]
In the literature, $\widetilde{\mathcal{H}}^1$ is often denoted by $\mathcal{H}$.

A continuous mapping from a compact interval into $X$ is said to be a rectifiable curve if it has finite length. A rectifiable curve $\gamma$ can always be parametrized by arc-length, so that we get a curve $\gamma: [0, \ell_\gamma] \to X$; see e.g. [11, Theorem 3.2]. We will only consider curves that are rectifiable and arc-length parametrized. If $\gamma: [0, \ell_\gamma] \to X$ is a curve and $g: X \to [0, \infty]$ is a Borel function, we define
\[
\int_\gamma g \, ds := \int_0^{\ell_\gamma} g(\gamma(s)) \, ds.
\]
The $p$-modulus of a family of curves $\Gamma$ is defined by
\[
\text{Mod}_p(\Gamma) := \inf \int_X \rho^p \, d\mu,
\]
where the infimum is taken over all nonnegative Borel functions $\rho: X \to [0, \infty]$ such that $\int_\gamma \rho \, ds \geq 1$ for every curve $\gamma \in \Gamma$. If a property holds apart from a curve family with zero $p$-modulus, we say that it holds for $p$-a.e. curve. Given any set $A \subset X$, we denote by $\Gamma_A$ the family of all nonconstant curves in $X$ that intersect $A$.

We will always denote by $\Omega$ an open subset of $X$.

**Definition 2.2.** A sequence of nonnegative functions $\{g_i\}_{i=1}^\infty$ in $L^1(\Omega)$ is equi-integrable if the following two conditions hold:

- for every $\varepsilon > 0$ there exists a $\mu$-measurable set $D \subset \Omega$ such that $\mu(D) < \infty$ and
  \[
  \int_{\Omega \setminus D} g_i \, d\mu < \varepsilon \quad \text{for all } i \in \mathbb{N};
  \]
- for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $A \subset \Omega$ is $\mu$-measurable with $\mu(A) < \delta$, then
  \[
  \int_A g_i \, d\mu < \varepsilon \quad \text{for all } i \in \mathbb{N}.
  \]

Next we give (special cases of) the Dunford–Pettis theorem, Mazur’s lemma, and Fuglede’s lemma, see e.g. [1, Theorem 1.38], [24, Theorem 3.12], and [5, Lemma 2.1], respectively.
Theorem 2.3. Let \( \{g_i\}_{i=1}^{\infty} \) be an equi-integrable sequence of nonnegative functions that is bounded in \( L^1(\Omega) \). Then there exists a subsequence such that \( g_i \to g \) weakly in \( L^1(\Omega) \).

Theorem 2.4. Let \( \{g_i\}_{i=1}^{\infty} \) be a sequence with \( g_i \to g \) weakly in \( L^1(\Omega) \). Then there exist convex combinations \( \hat{g}_i := \sum_{j=1}^{N_i} a_{i,j} g_j \), for some \( N_i \in \mathbb{N} \), such that \( \hat{g}_i \to g \) in \( L^1(\Omega) \).

By convex combinations we mean that the numbers \( a_{i,j} \) are nonnegative and that \( \sum_{j=1}^{N_i} a_{i,j} = 1 \) for every \( i \in \mathbb{N} \).

Lemma 2.5. Let \( \{g_i\}_{i=1}^{\infty} \) be a sequence of functions with \( g_i \to g \) in \( L^p(\Omega) \). Then for \( p \)-a.e. curve \( \gamma \) in \( \Omega \), we have
\[
\int_{\gamma} g_i \, ds \to \int_{\gamma} g \, ds \quad \text{as } i \to \infty.
\]

By a Radon measure \( \kappa \) we mean a locally finite Borel regular outer measure; locally finite means that for every \( x \in X \) there is \( r > 0 \) such that \( \kappa(B(x,r)) < \infty \). We will also need the following Vitali-Carathéodory theorem; for a proof see e.g. [14, p. 108]. Recall that \( 1 \leq p < \infty \).

Theorem 2.6. Let \( \kappa \) be a Radon measure on \( \Omega \) and let \( \rho \in L^p(\Omega, \kappa) \) be nonnegative. Then there exists a sequence \( \{\rho_i\}_{i \in \mathbb{N}} \) of lower semicontinuous functions on \( \Omega \) such that \( \rho \leq \rho_i+1 \leq \rho_i \) for all \( i \in \mathbb{N} \), and \( \rho_i \to \rho \) in \( L^p(\Omega, \kappa) \).

Definition 2.7. Let \( f : \Omega \to Y \). We say that a Borel function \( g : \Omega \to [0, \infty] \) is an upper gradient of \( f \) in \( \Omega \) if
\[
d_Y(f(\gamma(0)), f(\gamma(\ell_\gamma))) \leq \int_{\gamma} g \, ds \quad (2.8)
\]
for every curve \( \gamma \) in \( \Omega \). If \( g : \Omega \to [0, \infty] \) is a \( \mu \)-measurable function and (2.8) holds for \( p \)-a.e. curve in \( \Omega \), we say that \( g \) is a \( p \)-weak upper gradient of \( f \) in \( \Omega \).

We write \( f \in L^1(\Omega; Y) \) if \( d_Y(f(\cdot), f(x)) \in L^1(\Omega) \) for some \( x \in \Omega \).

Definition 2.9. The Newton-Sobolev class \( N^{1,p}(\Omega; Y) \) consists of those mappings \( f \in L^p(\Omega; Y) \) for which there exists a \( p \)-weak upper gradient \( g \in L^p(\Omega) \) of \( f \) in \( \Omega \).

The Dirichlet class \( D^p(\Omega; Y) \) consists of those mappings \( f : \Omega \to Y \) for which there exists a \( p \)-weak upper gradient \( g \in L^p(\Omega) \) of \( f \) in \( \Omega \). We define
\[
\|f\|_{D^p(\Omega; Y)} := \inf \|g\|_{L^p(\Omega)},
\]
where the infimum is taken over all \( p \)-weak upper gradients \( g \) of \( f \) in \( \Omega \).

One can define these classes also by using upper gradients instead of \( p \)-weak upper gradients, but this leads to the same result, see [5, Lemma 1.46]. In the classical setting of \( X = Y = \mathbb{R}^n \), both spaces equipped with the Lebesgue measure, we have that functions in the class \( N^{1,p}(X; Y) \) are exactly suitable pointwise representatives of
functions in the classical Sobolev class $W^{1,p}(\mathbb{R}^n;\mathbb{R}^n)$, see e.g. [5, Theorem A.2]. The Newton-Sobolev class in metric spaces was first introduced by Shanmugalingam [25].

To define the class of BV mappings, we consider the following definitions by Martio, who studied BV functions on metric spaces [22]. Given a family of curves $\Gamma$, we say that a sequence of nonnegative Borel functions $\{\rho_i\}_{i=1}^\infty$ is AM-admissible for $\Gamma$ if

$$\liminf_{i \to \infty} \int_\gamma \rho_i \, ds \geq 1 \text{ for all } \gamma \in \Gamma.$$ 

Then we let

$$AM(\Gamma) := \inf \left\{ \liminf_{i \to \infty} \int_X \rho_i \, d\mu \right\},$$

where the infimum is taken over all AM-admissible sequences $\{\rho_i\}_{i=1}^\infty$. Note that always $AM(\Gamma) \leq \text{Mod}_1(\Gamma)$. Then, BV mappings are defined as follows.

**Definition 2.10.** Given a mapping $f : \Omega \to Y$, we say that $f$ is in the Dirichlet class $D^{BV}(\Omega; Y)$ if there exists a sequence of nonnegative functions $\{g_i\}_{i=1}^\infty$ that is bounded in $L^1(\Omega)$, such that for AM-a.e. curve $\gamma$ in $\Omega$, we have

$$d_Y(f(\gamma(t_1)), f(\gamma(t_2))) \leq \liminf_{i \to \infty} \int_{\gamma([t_1,t_2])} g_i \, ds \quad (2.11)$$

for almost every $t_1, t_2 \in [0, \ell_\gamma]$ with $t_1 < t_2$. We also define

$$\|Df\| (\Omega) := \inf \left\{ \liminf_{i \to \infty} \int_\Omega g_i \, d\mu \right\},$$

where the infimum is taken over sequences $\{g_i\}_{i=1}^\infty$ as above. If also $f \in L^1(\Omega; Y)$, then we say that $f \in \text{BV}(\Omega; Y)$.

Durand-Cartagena–Eriksson-Bique–Korte–Shanmugalingam [6] show that in the case where $f$ is real-valued, $X$ is complete, and $\mu$ is a doubling measure that supports a Poincaré inequality, Definition 2.10 agrees with Miranda’s definition of BV functions in metric spaces given in [23], which in turn agrees in Euclidean spaces with the classical definition. Thus Definition 2.10 is natural for us to use.

Given a $\mu$-measurable set $A \subset X$ with $0 < \mu(A) < \infty$ and a function $u \in L^1(A)$, we denote the integral average by

$$\frac{1}{\mu(A)} \int_A u \, d\mu := \frac{1}{\mu(A)} \int_A u \, d\mu.$$

In the Euclidean space $\mathbb{R}^n$, with $n \geq 1$, BV functions $f \in \text{BV}(\mathbb{R}^n;\mathbb{R}^n)$ and their variation measures have the following structure; see [1, Section 3]. Given $x \in \mathbb{R}^n$, $r > 0$, and a unit vector $\nu \in \mathbb{R}^n$, we define the half-balls

$$B^+_\nu(x,r) := \{y \in B(x,r) : \langle y-x, \nu \rangle > 0\},$$

$$B^-_\nu(x,r) := \{y \in B(x,r) : \langle y-x, \nu \rangle < 0\},$$
where \( \langle \cdot, \cdot \rangle \) denotes the inner product. We consider the Euclidean space equipped with the Euclidean metric and \( \mu = \mathcal{L}^n \), that is, the \( n \)-dimensional Lebesgue measure. We say that \( x \in \mathbb{R}^n \) is an approximate jump point of \( f \) if there exist a unit vector \( \nu \in \mathbb{R}^n \) and distinct vectors \( f^+(x), f^-(x) \in \mathbb{R}^n \) such that

\[
\lim_{r \to 0} \int_{B_r^+(x,r)} \| f(y) - f^+(x) \| d\mathcal{L}^n(y) = 0
\]

and

\[
\lim_{r \to 0} \int_{B_r^-(x,r)} \| f(y) - f^-(x) \| d\mathcal{L}^n(y) = 0,
\]

where \( \| \cdot \| \) is the Euclidean norm in \( \mathbb{R}^n \). The set of all approximate jump points is denoted by \( J_f \). We write the Radon-Nikodym decomposition of the variation measure of \( f \) into the absolutely continuous and singular parts with respect to \( \mathcal{L}^n \) as

\[
\|Df\| = \|Df\|^a + \|Df\|^s.
\]

Furthermore, we define the Cantor and jump parts of \( Df \) by

\[
\|Df\|^c := \|Df\|^s(J_f^c), \quad \|Df\|^j := \|Df\|^s(J_f^j).
\]

Here

\[
\|Df\|^s[J_f(A)] := \|Df\|^s(J_f \cap A), \quad \text{for } \|Df\|^s\text{-measurable } A \subset \mathbb{R}^n.
\]

We get the decomposition

\[
\|Df\| = \|Df\|^a + \|Df\|^c + \|Df\|^j.
\]

For the jump part, we know that (see \([1, \text{Section 3.9}]\))

\[
d\|Df\|^j = |f^+ - f^-| d\mathcal{H}^{n-1}[J_f].
\]

Thus the jump set \( J_f \) is \( \sigma \)-finite with respect to \( \mathcal{H}^{n-1} \), that is, it can be written as an at most countable union of sets of finite \( \mathcal{H}^{n-1} \)-measure.

A rather similar decomposition holds also in more general metric measure spaces, but we will not use this.

Consider \( f : \Omega \to Y \). For every \( x \in \Omega \) and \( r > 0 \), we define

\[
L_f(x, r) := \sup\{d_Y(f(y), f(x)) : d(y, x) \leq r, y \in \Omega\}
\]

and

\[
l_f(x, r) := \inf\{d_Y(f(y), f(x)) : d(y, x) \geq r, y \in \Omega\}.
\]

Next, we define the generalized Lipschitz and distortion numbers.

**Definition 2.14.** Consider a mapping \( f : \Omega \to Y \), a Radon measure \( \kappa \) on \( \Omega \), and \( M \geq 1 \). For every \( x \in \Omega \), we define

\[
\text{Lip}_f^{\kappa, M}(x) := \limsup_{r \to 0} \frac{L_f(x, r)}{r} \frac{\mu(B(x, Mr))}{\kappa(B(x, Mr))}.
\]
We also define
\[ H_{f}^{\kappa,M}(x) := \limsup_{r \to 0} \frac{L_{f}(x,r)}{l_{f}(x,r)} \left( \frac{\mu(B(x,Mr))}{\kappa(B(x,Mr))} \right)^{(Q-1)/Q}. \]

Whenever a denominator is zero, we consider the corresponding quantity to be \( \infty \).
However, we will only consider Radon measures \( \kappa \) for which \( \kappa(B(x,r)) > 0 \) for every ball \( B(x,r) \subset \Omega \).

Remark 2.15. We will show that the generalized Lipschitz and distortion numbers can be used to give sufficient conditions for \( f \) to be a Newton-Sobolev or BV mapping. The definitions are also motivated by the fact that in the Euclidean setting, a variant of \( \text{Lip}_{f}^{a,M} \) can in fact be used to characterize Sobolev functions \( f \in W^{1,p}(\mathbb{R}^{n};\mathbb{R}^{n}) \) (see [18, Theorem 1.3]), and a different generalized distortion number can be used to detect the rank of \( \frac{dDf}{d|Df|} \) when \( f \in BV(\mathbb{R}^{n};\mathbb{R}^{n}) \) (see [18, Theorem 6.3]). In particular, it follows from the proof of [18, Theorem 6.3] that by choosing a natural pointwise representative of \( f \), we have \( H_{f} = \infty \mathcal{H}^{n-1}\) a.e. in the approximate jump set \( J_{f} \).

We will not always assume Ahlfors regularity (2.1), but nonetheless we understand \( Q > 1 \) to be a fixed parameter throughout the paper. Given any \( 1 \leq p \leq Q \), we denote the Sobolev conjugate by \( p^{*} = Qp/(Q-p) \) when \( p < Q \), and \( p^{*} = \infty \) when \( p = Q \). Our standing assumptions will be the following.

Throughout this paper we assume that \((X,d,\mu)\) and \((Y,d_{Y},\nu)\) are metric spaces equipped with Borel regular outer measures \( \mu \) and \( \nu \), such that \( \mu \) is doubling, every ball in \( Y \) has nonzero and finite measure, \( X \) is connected, and \( 1 < Q < \infty \) is fixed.

3. Preliminaries

In this section we consider some preliminary examples and results. Recall that \( \Omega \) always denotes an open subset of \( X \).

We start with the following simple example.

Example 3.1. Let \((X,d,\mu)\) be \( \mathbb{R}^{2} \) equipped with the Euclidean metric and Lebesgue measure. Let \( \Omega := (-1,1) \times (0,1) \) and
\[
f(x_1, x_2) := \begin{cases} (x_1, x_2) & \text{when } -1 < x_1 \leq 0 \text{ or } 1/(j+1) \leq x_1 < 1/j, \ j \in \mathbb{N} \text{ is odd}, \\ (2 - x_1, x_2) & \text{when } 1/(j+1) \leq x_1 < 1/j, \ j \in \mathbb{N} \text{ is even}. \end{cases}
\]
Now $f: \Omega \to \mathbb{R}^2$ is injective and bounded, and the set
\[ E := \{\text{Lip}_f = \infty\} = \{H_f = \infty\} \]
\[ = \{(x_1, x_2) \in \mathbb{R}^2: x_1 = 0, 0 < x_2 < 1\} \cup \bigcup_{j=2}^{\infty} \{(x_1, x_2) \in \mathbb{R}^2: x_1 = 1/j, 0 < x_2 < 1\} \]
has $\sigma$-finite 1-dimensional Hausdorff measure. Note that the codimension 1 Hausdorff measure $\tilde{H}^1$ is now comparable to the 1-dimensional Hausdorff measure $H^1$. Moreover, in the set $\Omega \setminus E$ we have $\text{Lip}_f, H_f = 1$, so that $\text{Lip}_f, H_f \in L^p(\Omega)$ for all $1 \leq p \leq \infty$.

But $f$ is not even in $\text{BV}_{\text{loc}}(\Omega, \mathbb{R}^2)$; this can be seen e.g. from the representation (2.13).

This example indicates that it is difficult to formulate a reasonable sufficient condition for $f$ belonging to the BV class by only using $\text{Lip}_f$ or $H_f$. Hence we are motivated to define the generalized versions of these quantities.

The following lemma will be used for upper gradient estimates on curves. Note that despite the simplicity of the lemma, one has to be careful with the fact that $h$ is not assumed to be continuous, and in fact the lemma would fail if we did not assume the sets $U_{j,k}$ to be open.

**Lemma 3.2.** Let $h: [a, b] \to Y$ for some finite interval $[a, b] \subset \mathbb{R}$. Suppose that there is a sequence of at most countable unions of sets $W_j = \bigcup_k U_{j,k}, j \in \mathbb{N}$, where each $U_{j,k} \subset \mathbb{R}$ is open and bounded, and such that $\chi_{W_j}(t) \to 1$ as $j \to \infty$ for all $t \in [a, b]$. Then
\[
d_Y(h(a), h(b)) \leq \liminf_{j \to \infty} \sum_k \text{diam } h(U_{j,k} \cap [a, b]).
\]

Here $\chi_{W_j}$ is the characteristic function of $W_j$, that is, $\chi_{W_j} = 1$ in $W_j$ and $\chi_{W_j} = 0$ outside $W_j$. Moreover, $\text{diam } h(U_{j,k} \cap [a, b]) := \sup_{s,t \in U_{j,k} \cap [a, b]} d_Y(h(s), h(t))$.

**Proof.** In [18, Lemma 3.9] this was shown for real-valued $h$. By the Kuratowski embedding theorem, every metric space, in particular $Y$, can be isometrically embedded into a Banach space, for example $V := L^\infty(Y)$; see e.g. [14, p. 100]. Denote the norm in $V$ by $\| \cdot \|_V$, and the dual of $V$ by $V^*$. We find $v^* \in V^*$ with $\|v^*\|_{V^*} = 1$ and $v^*(h(a) - h(b)) = \|h(a) - h(b)\|_V$. Now we can consider the real-valued function
\( d_Y(h(a), h(b)) = \| h(a) - h(b) \|_V \\
= v^*(h(a) - h(b)) \\
= v^*(h(a)) - v^*(h(b)) \\
\leq \lim \inf_{j \to \infty} \sum_k \text{diam } v^*(h(U_{j,k} \cap [a, b])) \) by [18, Lemma 3.9]

\[ = \lim \inf_{j \to \infty} \sum_k \text{diam } h(U_{j,k} \cap [a, b]). \]

We will also need the following measurability result.

**Lemma 3.3.** Suppose \( f : \Omega \to Y \) is continuous at every point \( x \in H \subset \Omega \), with \( \mu(\Omega \setminus H) = 0 \). Then \( \text{lip}_f \) is a \( \mu \)-measurable function on \( \Omega \).

**Proof.** For a fixed \( r > 0 \), the mapping
\[
x \mapsto \sup_{r} \{ d_Y(f(y), f(x)) : d(y, x) < r, y \in \Omega \}
\]
is lower semicontinuous at every point \( x \in H \). On the other hand, for every \( x \in \Omega \),
\[
\text{lip}_f(x) = \lim \inf_{r \to 0} \frac{L_f(x, r)}{r} = \lim \inf_{r \to 0} \frac{\sup \{ d_Y(f(y), f(x)) : d(y, x) \leq r, y \in \Omega \}}{r}
\]
\[
= \lim \inf_{r \to 0} \frac{\sup \{ d_Y(f(y), f(x)) : d(y, x) < r, y \in \Omega \}}{r}
\]
\[
= \lim_{j \to \infty} \inf_{r \in (0, 1/j) \cap \mathbb{Q}} \frac{\sup \{ d_Y(f(y), f(x)) : d(y, x) < r, y \in \Omega \}}{r}.
\]
Together these establish that \( \text{lip}_f \) is a Borel function in \( H \). Thus it is \( \mu \)-measurable in \( \Omega \). \( \square \)

The next lemma is standard; see e.g. [20, Lemma 4.8].

**Lemma 3.4.** Let \(-\infty < a < b < \infty\). Let \( h : [a, b] \to Y \) be a continuous mapping such that \( \text{lip}_h(t) < \infty \) for all \( t \in T \) for some \( L^1 \)-measurable set \( T \subset [a, b] \). Then
\[
\mathcal{H}^1(h(T)) \leq \int_T \text{lip}_h(t) \, dt.
\]

The following lemma says essentially that \( \text{lip}_f \) is a weak upper gradient of \( f \). Recall that \( 1 \leq p < \infty \).

**Lemma 3.5.** Suppose \( f : \Omega \to Y \) is continuous at \( \mu \)-a.e. point in \( \Omega \). Then for \( p \)-a.e. curve \( \gamma : [0, \ell_\gamma] \to \Omega \) we have that if \( f \circ \gamma \) is continuous and \( \text{lip}_f(\gamma(t)) < \infty \) for all \( t \in T \subset [0, \ell_\gamma] \) where \( T \) is \( L^1 \)-measurable, then we get
\[
\mathcal{H}^1(f(\gamma(T))) \leq \int_T \text{lip}_f(\gamma(t)) \, dt.
\]
Consider the set of points in $\Omega$ for which no such sequence exists; call this set $A$. Let $y \in A$. Then there exists $\delta > 0$ such that $d_Y(f(y), f(x)) \geq \delta$ for all $y \in \Omega \cap B(x, \delta) \setminus \{x\}$. Thus $A = \bigcup_{j=1}^{\infty} A_j$ with

$$A_j := \{x \in \Omega: d_Y(f(y), f(x)) \geq 1/j \text{ for all } y \in \Omega \cap B(x, 1/j) \setminus \{x\}\}.$$ 

It is enough to show that every $A_j$ is countable; suppose instead that $A_j$ is uncountable for a fixed $j$. Note that both $X$ and $Y$ are separable; see [5, Proposition 1.6]. Since $X$ is separable, there exists a collection of points $\{z_k\}_{k=1}^{\infty} \subset \Omega$ that is dense in $\Omega$. Thus $\Omega$ is covered by the countable collection of balls $\{B(z_k, (2j)^{-1})\}_{k=1}^{\infty}$. Then for some $k \in \mathbb{N}$, also the set $D := B(z_k, (2j)^{-1}) \cap A_j$ is uncountable. If $x_1, x_2 \in D$ are two distinct points, then $d(x_1, x_2) < 1/j$, and so $d_Y(f(x_1), f(x_2)) \geq 1/j$. Then the balls $B(f(x), (2j)^{-1}), x \in D$, are disjoint. But this contradicts the separability of $Y$. \hfill \square

Denote by $D_f$ the set of points in $\Omega$ where $f$ is not continuous.

**Lemma 3.7.** Let $f: \Omega \to Y$. Then $D_f \cap \{H_f < \infty\}$ is an at most countable set.

**Proof.** Let $x \in \Omega$ be a point that is not in the exceptional set of Lemma 3.6. Then we find a sequence $y_k \to x$, $y_k \neq x$, with $d_Y(f(y_k), f(x)) \to 0$. Suppose $f$ is discontinuous at $x$; it is enough to show that $H_f(x) = \infty$. Now we also find a sequence $z_k \to x$ with $d_Y(f(z_k), f(x)) \geq \delta > 0$. Fix an arbitrary $M > 0$. For all sufficiently small $r > 0$, there exists $y_k \in \Omega$ with $d(y_k, x) > r$ and $d_Y(f(y_k), f(x)) < \delta/M$, and $z_k \in \Omega$ with $d(z_k, x) < r$ and $d_Y(f(z_k), f(x)) \geq \delta$. Thus

$$\frac{L_f(x, r)}{I_f(x, r)} \geq \frac{\delta}{\delta/M} = M.$$
It follows that
\[ \limsup_{r \to 0} H_f(x, r) \geq M \quad \text{and thus} \quad \limsup_{r \to 0} H_f(x, r) = \infty, \]
since \( M > 0 \) was arbitrary. Thus \( H_f(x) = \infty \).
\[ \square \]

Finally, we give the following lemma. Recall that we denote by \( \Gamma_A \) the family of all nonconstant curves in \( X \) that intersect \( A \), and that \( 1 \leq p < \infty \).

**Lemma 3.8.** Let \( A \subset X \) with \( \mathcal{H}^p(A) = 0 \). Then \( \text{Mod}_p(\Gamma_A) = 0 \).

**Proof.** Fix \( \varepsilon > 0 \). For every \( j \in \mathbb{N} \), we find an (at most countable) collection of balls \( B_{j,k} = B(x_{j,k}, r_{j,k}) \) covering \( A \), such that \( r_{j,k} < 1/j \) and
\[ \sum_k \frac{\mu(B_{j,k})}{r_{j,k}^p} < 2^{-j} \varepsilon. \]
Define
\[ \rho := \sup_{j,k} \frac{\chi_{2B_{j,k}}}{r_{j,k}}. \]
For every \( \gamma \in \Gamma_A \), we can choose \( j > 1/\ell_\gamma \) and \( k \) such that \( \gamma \cap B_{j,k} \neq \emptyset \), and then
\[ \int_\gamma \rho \, ds \geq \int_\gamma \frac{\chi_{2B_{j,k}}}{r_{j,k}} \, ds \geq 1. \]
Now
\[ \text{Mod}_p(\Gamma_A) \leq \|\rho\|_{L^p(X)}^p = \int_X \left( \sup_{j,k} \frac{\chi_{2B_{j,k}}}{r_{j,k}} \right)^p \, d\mu \leq \sum_{j,k} \frac{\mu(2B_{j,k})}{r_{j,k}^p} \leq C_\varepsilon. \]
Letting \( \varepsilon \to 0 \), we get the result. \( \square \)

### 4. Main theorems

In this section we state and prove our main theorems. The theorems all take a rather long and complicated form, but in Section 5 we will consider more streamlined corollaries. In [18, Theorem 5.10], a somewhat similar theorem was proved in the Euclidean setting, but there (as in all of the literature, as far as we know) the mapping \( f \) was assumed to be continuous, which is of course not a natural assumption in the BV setting.

Recall that \( 1 < Q < \infty \) is a fixed parameter, and that we denote by \( \Gamma_N \) the family of all nonconstant curves in \( X \) that intersect \( N \).

**Theorem 4.1.** Suppose \( \Omega \) is nonempty, open, and bounded, \( f : \Omega \to Y \), \( M \geq 1 \), \( \kappa \) is a finite Radon measure on \( \Omega \) whose support contains \( \Omega \), and suppose \( \Omega \) admits a partition into disjoint sets \( A \), \( D \), and \( N \) such that \( \text{Lip}_f^{\kappa,M} < \infty \) in \( A \), \( H_f^{\kappa,M} < \infty \) in \( D \),
Mod$(\Gamma_X) = 0$, and
\[
\int_{\Omega} h \, d\kappa < \infty \quad \text{for a function } h \geq \text{Lip}^{\kappa,M}_{f} \chi_A + (H^{\kappa,M}_{f})^{Q/(Q-1)} \chi_D. \tag{4.2}
\]

Furthermore, suppose that for some $0 < C_\Omega < \infty$ and for all $r > 0$,
\[
\frac{\mu(B(x,r))}{r^q} \leq C_\Omega \quad \text{for all } x \in \Omega \quad \text{and} \quad \frac{\nu(B(f(x),r))}{r^q} \geq C_\Omega^{-1} \quad \text{for all } x \in D, \tag{4.3}
\]
that there is an open set $W$ such that $D \subset W \subset \Omega$ and $f$ is injective in $W$, and that there is $\beta > 0$ such that
\[
\nu(V) < \infty \quad \text{for } V := \bigcup_{y \in D} B(f(y), l_f(y, \beta)). \tag{4.4}
\]

Also let $0 < \varepsilon \leq 1$. Then $f \in D_{BV}(\Omega; Y)$ with
\[
\|Df\|_{\Omega} \leq C_1 \varepsilon^{-1/(Q-1)} \int_{\Omega} h \, d\kappa + C_1 \varepsilon (\kappa(\Omega) + \nu(V)), \tag{4.5}
\]
where $C_1$ only depends on $Q$, $M$, $\kappa$, and $C_\Omega$.

Here $\lceil b \rceil$ denotes the smallest integer at least $b \in \mathbb{R}$.

Proof. By the Vitali-Carathéodory theorem (Theorem 2.6), we can assume that the function $h$ is lower semicontinuous. Denote
\[
\Omega(\delta) := \{ x \in \Omega : \text{dist}(x, X \setminus \Omega) > \delta \}, \quad \delta > 0. \tag{4.6}
\]
Here $\text{dist}(x, X \setminus \Omega) := \inf \{ d(x,y) : y \in X \setminus \Omega \}$. We have $A = \bigcup_{j=1}^{\infty} A_j$, where $A_j$ consists of those points $x \in A \cap \Omega(j^{-1}(M+1))$ for which
\[
\sup_{0 < r \leq 1/j} \frac{L_j(x,r)}{r} \frac{\mu(B(x,Mr))}{\kappa(B(x,Mr))} \leq \text{Lip}_{f}^{\kappa,M}(x) + \varepsilon^{Q/(Q-1)} \tag{4.7}
\]
and
\[
\text{Lip}_{f}^{\kappa,M}(x) \leq h(z) + \varepsilon^{Q/(Q-1)} \quad \text{for all } z \in B(x, M/j). \tag{4.8}
\]
Using the fact that $\mu$ is doubling, for any fixed $j \in \mathbb{N}$ we find an (at most countable) collection $\{B_{j,k} = B(x_{j,k},1/j)\}_k$ of balls covering $A_j$, with $x_{j,k} \in A_j$, and such that the balls $(M+1)B_{j,k} \subset \Omega$ can be divided into at most
\[
C_M := C_d^{\lceil \log_2(18M) \rceil}
\]
collections of pairwise disjoint balls.

Analogously, we have $D = \bigcup_{j=1}^{\infty} D_j$, where $D_j$ consists of those points $y \in D \cap W(j^{-1}(M+1))$ for which
\[
\sup_{0 < s \leq 1/j} \frac{L_j(y,s)}{I_j(y,s)} \left( \frac{\mu(B(y,Ms))}{\kappa(B(y,Ms))} \right)^{(Q-1)/Q} \leq H_{f}^{\kappa,M}(y) + \varepsilon \tag{4.9}
\]
and
\[(H_f^{\kappa,M}(y))^{Q/(Q-1)} \leq h(z) + \varepsilon^{Q/(Q-1)} \quad \text{for all } z \in B(y,M/j).\]  

We find a collection \(\{\tilde{B}_{j,k} = B(y_{j,k},1/j)\}_k\) of balls covering \(D_j\), with \(y_{j,k} \in D_j\), and such that the balls \((M+1)\tilde{B}_{j,k} \subset W\) can be divided into at most \(C_M\) collections of pairwise disjoint balls.

Denote \(L_{j,k} := L_f(x_{j,k},1/j)\), \(\tilde{L}_{j,k} := L_f(y_{j,k},1/j)\), and \(\hat{l}_{j,k} := l_f(y_{j,k},1/j)\). For each \(j \in \mathbb{N}\), define
\[g_j := 2j \sum_k L_{j,k} \chi_{2B_{j,k}} + 2j \sum_k \tilde{L}_{j,k} \chi_{2\tilde{B}_{j,k}}.\]

By assumption, 1-a.e. nonconstant curve in \(\Omega\) has empty intersection with \(N\). Take such a curve \(\gamma: [0,\ell_\gamma] \to \Omega\), with \(\ell_\gamma > 0\). By a slight abuse of notation, denote also the image of a curve \(\gamma\) by the same symbol. Now we have
\[\gamma \subset A \cup D = \bigcup_{j=1}^{\infty} A_j \cup \bigcup_{j=1}^{\infty} D_j,\]
where from the definitions we can see that \(A_j \subset A_{j+1}\) and \(D_j \subset D_{j+1}\) for every \(j \in \mathbb{N}\).

Thus
\[\lim_{j \to \infty} \chi_{\bigcup_k B_{j,k}}(x) = 1 \quad \text{for every } x \in A \quad \text{and} \quad \lim_{j \to \infty} \chi_{\bigcup_k \tilde{B}_{j,k}}(y) = 1 \quad \text{for every } y \in D.\]

If \(\ell_\gamma \geq 1/j\) and \(\gamma \cap B_{j,k} \neq \emptyset\), then
\[2j \int_{\gamma} L_{j,k} \chi_{2B_{j,k}} ds \geq 2L_{j,k} \geq \text{diam } f(B_{j,k}).\]

The analogous fact holds for the balls \(\tilde{B}_{j,k}\). Thus we get
\[\int_{\gamma} g_j ds \geq 2j \sum_{k: \gamma \cap B_{j,k} \neq \emptyset} \int_{\gamma} L_{j,k} \chi_{2B_{j,k}} ds + 2j \sum_{k: \gamma \cap \tilde{B}_{j,k} \neq \emptyset} \int_{\gamma} \tilde{L}_{j,k} \chi_{2\tilde{B}_{j,k}} ds \geq \sum_{k: \gamma \cap B_{j,k} \neq \emptyset} \text{diam } f(B_{j,k}) + \sum_{k: \gamma \cap \tilde{B}_{j,k} \neq \emptyset} \text{diam } f(\tilde{B}_{j,k}).\]
Applying Lemma 3.2 with \( h \equiv f \circ \gamma : [0, \ell_\gamma] \to Y \) (assuming only \( \ell_\gamma > 0 \)), we get
\[
\begin{align*}
d_Y(f(\gamma(0)), f(\gamma(\ell_\gamma))) &\leq \liminf_{j \to \infty} \left( \sum_{k: \gamma \cap B_{j,k} \neq \emptyset} \text{diam } f(\gamma^{-1}(B_{j,k})) + \sum_{k: \gamma \cap \hat{B}_{j,k} \neq \emptyset} \text{diam } f(\gamma^{-1}(\hat{B}_{j,k})) \right) \\
&\leq \liminf_{j \to \infty} \left( \sum_{k: \gamma \cap B_{j,k} \neq \emptyset} \text{diam } f(B_{j,k}) + \sum_{k: \gamma \cap \hat{B}_{j,k} \neq \emptyset} \text{diam } f(\hat{B}_{j,k}) \right) \\
&\leq \liminf_{j \to \infty} \int_{\gamma} g_j \, ds.
\end{align*}
\] (4.11)

On the other hand, we estimate
\[
2j \sum_k L_{j,k} \mu(2B_{j,k}) \leq 2Cd \sum_k \kappa(MB_{j,k})(\text{Lip}_j^{\kappa,M}(x_{j,k}) + \varepsilon^{Q/(Q-1)}) \quad \text{by (4.7)}
\]
\[
\leq 2Cd \sum_k \int_{MB_{j,k}} (h + 2\varepsilon^{Q/(Q-1)}) \, d\kappa \quad \text{by (4.8)}
\]
\[
\leq 2CdCM \int_{\Omega} (h + 2\varepsilon^{Q/(Q-1)}) \, d\kappa.
\] (4.12)

In general, note that if balls \( B_1 = B(z_1, r_1) \) and \( B_2 = B(z_2, r_2) \) are disjoint and contained in \( W \), then from the definition of \( l_f(\cdot, \cdot) \) and from the injectivity of \( f \) it follows that
\[
B(f(z_1), l_f(z_1, r_1)) \cap B(f(z_2), l_f(z_2, r_2)) = \emptyset.
\] (4.13)

Note also that the collection \( \{B(y_{j,k}, 1/j)\}_{k=1}^\infty \) can be divided into at most \( CM \) collections of pairwise disjoint balls. In total, when \( 1/j < \beta \) we get
\[
\sum_k \nu(B(f(y_{j,k}, \hat{I}_{j,k}))) \leq CM \nu(V) < \infty
\] (4.14)

by (4.4). With \( 1/Q + (Q - 1)/Q = 1 \), by Young’s inequality we have for any \( b_1, b_2 \geq 0 \) that
\[
b_1b_2 = \varepsilon^{1/Q}b_1\varepsilon^{1/Q}b_2 \leq \frac{1}{Q} \varepsilon b_1^Q + \frac{Q - 1}{Q} \varepsilon^{1/(Q-1)}b_2^{Q/(Q-1)} \leq \varepsilon b_1^Q + \varepsilon^{-1/(Q-1)}b_2^{Q/(Q-1)}.
\] (4.15)
Consider \( j \in \mathbb{N} \) large enough that \( 1/j < \beta \). We estimate

\[
2j \sum_k \hat{L}_{j,k} \mu(2\hat{B}_{j,k})
\]

\[
\leq 2j \sum_k \hat{\gamma}_{j,k} \mu(2\hat{B}_{j,k}) \left( \frac{\kappa(M\hat{B}_{j,k})}{\mu(M\hat{B}_{j,k})} \right)^{(Q-1)/Q} (H_{f}^{\kappa,M}(y_{j,k}) + \varepsilon) \quad \text{by (4.9)}
\]

\[
\leq 2C_dC_{\Omega}^{1/Q} \sum_k \hat{\gamma}_{j,k} \kappa(M\hat{B}_{j,k})^{(Q-1)/Q} (H_{f}^{\kappa,M}(y_{j,k}) + \varepsilon) \quad \text{by (4.3)}
\]

\[
\leq (2C_d)^Q C_{\Omega} \varepsilon \sum_k \hat{\gamma}_{j,k} + \varepsilon^{-1/(Q-1)} \sum_k (H_{f}^{\kappa,M}(y_{j,k}) + \varepsilon)^{Q/(Q-1)} \kappa(M\hat{B}_{j,k}) \quad \text{by (4.15)}
\]

\[
\leq (2C_d)^Q C_{\Omega}^2 \varepsilon \sum_k \nu(B(f(y_{j,k}), \hat{t}_{j,k})) \quad \text{by (4.3)}
\]

\[
\quad + 2^{Q/(Q-1)} \varepsilon^{-1/(Q-1)} \sum_k (H_{f}^{\kappa,M}(y_{j,k})^{Q/(Q-1)} + \varepsilon^{Q/(Q-1)}) \kappa(M\hat{B}_{j,k})
\]

\[
\leq (2C_d)^Q C_{\Omega}^2 C_{M} \varepsilon \nu(V) + C_{M} 2^{Q/(Q-1)} \varepsilon^{-1/(Q-1)} \int_{\Omega} (h + 2\varepsilon^{Q/(Q-1)}) \, dk
\]

by (4.14) and (4.10). By combining this with (4.12), we get

\[
\int_{\Omega} g_j \, d\mu = 2j \sum_k \hat{L}_{j,k} \mu(2B_{j,k}) + 2j \sum_k \hat{L}_{j,k} \mu(2\hat{B}_{j,k})
\]

\[
\leq 2^{2+Q/(Q-1)} C_d C_{M} \varepsilon^{-1/(Q-1)} \int_{\Omega} (h + 2\varepsilon^{Q/(Q-1)}) \, dk + (2C_d)^Q C_{\Omega}^2 C_{M} \varepsilon \nu(V)
\]

\[
\leq C_1 \varepsilon^{-1/(Q-1)} \int_{\Omega} h \, dk + C_1 \varepsilon \kappa(\Omega) + C_1 \varepsilon \nu(V)
\]

\[
< \infty
\]

(4.16)

by the assumption (4.2), with \( C_1 := 2^{2+Q+Q/(Q-1)} C_{\Omega}^Q C_d^{Q + [\log_2(18M)]} \). By (4.11), for 1-a.e. curve \( \gamma: [0, \ell_\gamma] \to \Omega \) we have

\[
d_Y(f(\gamma(0)), f(\gamma(\ell_\gamma))) \leq \liminf_{j \to \infty} \int_{\gamma} g_j \, ds.
\]

By the properties of modulus, see e.g. [5, Lemma 1.34], it follows that for 1-a.e. curve \( \gamma: [0, \ell_\gamma] \to \Omega \), we have in fact

\[
d_Y(f(\gamma(t_1)), f(\gamma(t_2))) \leq \liminf_{j \to \infty} \int_{\gamma(t_1, t_2)} g_j \, ds
\]

for every \( 0 \leq t_1 < t_2 \leq \ell_\gamma \). Thus \( f \in D^{BV}(\Omega; Y) \) by Definition 2.10, and by (4.16) we have (4.5).
In the remaining two theorems, we show that if \( dk = a \, d\mu \) and either \( \text{Lip}_a^M \) or \( H_a^M \) is not too large, then we can get Newton-Sobolev instead of just BV regularity.

**Theorem 4.17.** Suppose \( \Omega \) is nonempty, open, and bounded, \( f: \Omega \to Y, \ M \geq 1, \ 1 \leq p < \infty, \ a \in L^p(\Omega) \) is a nonnegative function whose integral over every nonempty open subset of \( \Omega \) is nonzero, and suppose that \( \Omega \) admits a partition into disjoint sets \( A \) and \( N \) such that \( \text{Lip}_a^M < \infty \) in \( A \), \( \text{Mod}_p(\Gamma_N) = 0 \), and

\[
\int_{\Omega} (ha)^p \, d\mu < \infty \quad \text{for a function } h \geq \text{Lip}_a^M \chi_A.
\] (4.18)

Furthermore, suppose that for some \( 0 < C_\Omega < \infty \) and for all \( r > 0 \), we have

\[
\frac{\mu(B(x, r))}{r^Q} \leq C_\Omega \quad \text{for all } x \in \Omega.
\] (4.19)

Then \( f \in D^p(\Omega; Y) \) with

\[
\|f\|_{D^p(\Omega; Y)} \leq C_2\|ha\|_{L^p(\Omega)},
\] (4.20)

where \( C_2 \) only depends on \( C_d, p, \) and \( M \).

**Proof.** Fix \( 0 < \varepsilon \leq 1 \). By the Vitali-Carathéodory theorem (Theorem 2.6), we can assume that \( h \) is lower semicontinuous. We have \( A = \bigcup_{j=1}^\infty A_j \), where \( A_j \) consists of those points \( x \in A \cap \Omega(M/j) \) for which

\[
\sup_{0 < r \leq 1/j} \frac{L_f(x, r)}{r} \left( \frac{\int_{B(x, Mr)} a \, d\mu}{B(x, Mr)} \right)^{-1} \leq \text{Lip}_a^M(x) + \varepsilon
\] (4.21)

and

\[
\text{Lip}_a^M(x) \leq h(z) + \varepsilon \quad \text{for all } z \in B(x, M/j).
\] (4.22)

For any fixed \( j \in \mathbb{N} \), we find a collection \( \{B_{j,k} = B(x_{j,k}, 1/j)\}_k \) covering \( A_j \) with \( x_{j,k} \in A_j \), and such that the balls \( (M+1)B_{j,k} \) can be divided into at most \( C_M \) collections of pairwise disjoint balls.

Denote \( L_{j,k} := L_f(x_{j,k}, 1/j) \). For each \( j \in \mathbb{N} \), define

\[
g_j := 2j \sum_k L_{j,k} \chi_{2B_{j,k}}.
\]

By assumption, \( p \)-a.e nonconstant curve in \( \Omega \) avoids \( N \). Hence, just as in the estimates leading to (4.11), for \( p \)-a.e. curve \( \gamma \) in \( \Omega \) we get

\[
d_Y(f(\gamma(0)), f(\gamma(\ell_\gamma))) \leq \liminf_{j \to \infty} \int_{\gamma} g_j \, ds.
\] (4.23)
Then we estimate
\[ j^p \sum_k L_{j,k}^p \mu(2B_{j,k}) \leq \sum_k \mu(2B_{j,k}) \left( \int_{2B_{j,k}} a \, d\mu \right)^p \left( \text{Lip}_f^a (x_{j,k}) + \varepsilon \right)^p \text{ by (4.21)} \]
\[ \leq \sum_k \mu(2B_{j,k}) \left( \int_{2B_{j,k}} (h + 2\varepsilon) a \, d\mu \right)^p \text{ by (4.22)} \]
\[ \leq \sum_k \mu(2B_{j,k}) \int_{2B_{j,k}} (h + 2\varepsilon)^p a^p \, d\mu \text{ by Hölder} \]
\[ \leq C_d \sum_k \int_{2B_{j,k}} (h + 2\varepsilon)^p a^p \, d\mu \]
\[ \leq C_d C_M \int_\Omega (h + 2\varepsilon)^p a^p \, d\mu. \]  

(4.24)

Thus for all \( j \in \mathbb{N} \), we get
\[ \int_\Omega g_j^p \, d\mu \leq \int_\Omega \left( 2j \sum_k L_{j,k} \chi_{2B_{j,k}} \right)^p \, d\mu \]
\[ \leq 2^p C_d^p \sum_k \mu(2B_{j,k}) \]
\[ \leq 2^p C_d^p (h + 2\varepsilon)^p a^p \, d\mu \text{ by (4.24)} \]
\[ \leq 2^p C_d^p (h + 2\varepsilon)^p a^p \, d\mu + (2\varepsilon)^p \int_\Omega a^p \, d\mu \]
\[ < \infty \text{ by (4.18).} \]

Hence \( \{g_j\}_{j=1}^\infty \) is a bounded sequence in \( L^p(\Omega) \).

**Case \( p = 1 \):**

Recall that we defined
\[ g_j = 2j \sum_k L_{j,k} \chi_{2B_{j,k}}, \quad j \in \mathbb{N}. \]

By (4.25), this is a bounded sequence in \( L^1(\Omega) \). We will show that it is equi-integrable. The first condition of Definition 2.2 holds automatically since \( \Omega \) as a bounded set has finite \( \mu \)-measure. We check the second condition.

Suppose by contradiction that we find \( 0 < \alpha < 1 \), an infinite set \( J \subset \mathbb{N} \), and a sequence of \( \mu \)-measurable sets \( H_j \subset \Omega \) such that \( \mu(H_j) \to 0 \) and
\[ j \int_{H_j} \left( \sum_k L_{j,k} \chi_{2B_{j,k}} \right) d\mu \geq \alpha \text{ for all } j \in J. \]  

(4.26)
Choose $L$ to be the following (very large) number:

$$L := \frac{2C_d^{1+\lfloor \log_2 M \rfloor} C_M}{\alpha} \int_{\Omega} (h + 2)a \, d\mu. \quad (4.27)$$

For every $j \in \mathbb{N}$, define two sets of indices $I_1^j$ and $I_2^j$ as follows: for $k \in I_1^j$, we have

$$\frac{\mu((M+1)B_{j,k} \cap H_j)}{\mu((M+1)B_{j,k})} \leq \frac{1}{L},$$

and $I_2^j$ consists of the remaining indices. We estimate

$$j \sum_{k \in I_1^j} L_{j,k} \mu(2B_{j,k} \cap H_j) \leq j \sum_{k \in I_1^j} L_{j,k} \mu((M+1)B_{j,k})$$

$$\leq \frac{j}{L} \sum_{k \in I_1^j} L_{j,k} \mu((M+1)B_{j,k})$$

$$\leq \frac{C_d^{1+\lfloor \log_2 M \rfloor}}{L} \sum_{k \in I_1^j} L_{j,k} \mu(2B_{j,k}) \quad (4.28)$$

$$\leq \frac{C_d^{1+\lfloor \log_2 M \rfloor} C_M}{L} \int_{\Omega} (h + 2)a \, d\mu \quad \text{by (4.24)}$$

$$= \frac{\alpha}{2} \quad \text{by (4.27)}.$$

Next, we estimate

$$\mu \left( \bigcup_{k \in I_2^j} MB_{j,k} \right) \leq \sum_{k \in I_2^j} \mu((M+1)B_{j,k})$$

$$\leq L \sum_{k \in I_2^j} \mu((M+1)B_{j,k} \cap H_j) \leq LC_M \mu(H_j) \to 0 \quad \text{as } j \to \infty. \quad (4.29)$$

By writing the first four lines of (4.24) with the sums over the indices $k \in I_2^j$, we get

$$j \sum_{k \in I_2^j} L_{j,k} \mu(2B_{j,k}) \leq C_d \sum_{k \in I_2^j} \int_{MB_{j,k}} (h + 2)a \, d\mu$$

$$\leq C_d C_M \int \bigcup_{k \in I_2^j} MB_{j,k} (h + 2)a \, d\mu$$

$$\to 0 \quad \text{as } j \to \infty$$

by (4.29) and by the absolute continuity of integrals. Combining this with (4.28), we get

$$\limsup_{j \to \infty} \int_{H_j} j \sum_{k} L_{j,k} \chi_{2B_{j,k}} \, d\mu \leq \frac{\alpha}{2}.$$
This contradicts (4.26) and proves the equi-integrability of \( \{g_j\}_{j=1}^\infty \). Now by the Dunford–Pettis theorem (Theorem 2.3), we find \( g \in L^1(\Omega) \) such that by passing to a subsequence (not relabeled), we have \( g_j \to g \) weakly in \( L^1(\Omega) \). By Mazur’s lemma (Theorem 2.4), for suitable convex combinations we get the strong convergence \( \sum_{i=j}^{N_j} a_{j,l}g_l \to g \) in \( L^1(\Omega) \).

From (4.23) and Fuglede’s lemma (Lemma 2.5), we get

\[
d_Y(f(\gamma(0)), f(\gamma(\ell_{\gamma}))) \leq \liminf_{j \to \infty} \int_{\gamma} \sum_{l=j}^{N_j} a_{j,l}g_l ds = \int_{\gamma} g ds
\]

for 1-a.e. curve \( \gamma \) in \( \Omega \). Thus \( g \in L^1(\Omega) \) is a 1-weak upper gradient of \( f \) in \( \Omega \), and so \( f \in D^1(\Omega; Y) \).

**Case 1 < p < \infty:**

By (4.25), \( \{g_j\}_{j=1}^\infty \) is a bounded sequence in \( L^p(\Omega) \). By reflexivity of the space \( L^p(\Omega) \), we find a subsequence of \( \{g_j\}_{j=1}^\infty \) (not relabeled) and \( g \in L^p(\Omega) \) such that \( g_j \to g \) weakly in \( L^p(\Omega) \) (see e.g. [14, Section 2]). By Mazur’s lemma (Theorem 2.4), for suitable convex combinations we get the strong convergence \( \sum_{i=j}^{N_j} a_{j,l}g_l \to g \) in \( L^p(\Omega) \). By (4.23) and Fuglede’s lemma (Lemma 2.5), we obtain for \( p \)-a.e. curve \( \gamma \) in \( \Omega \) that

\[
d_Y(f(\gamma(0)), f(\gamma(\ell_{\gamma}))) \leq \liminf_{j \to \infty} \int_{\gamma} \sum_{l=j}^{N_j} a_{j,l}g_l ds = \int_{\gamma} g ds.
\]

Hence \( f \in D^p(\Omega; Y) \).

For all \( 1 \leq p < \infty \), by (4.25) we get

\[
\|f\|_{D^p(\Omega; Y)} \leq \|g\|_{L^p(\Omega)} \leq \limsup_{j \to \infty} \|g_j\|_{L^p(\Omega)} \leq C_2(\|ha\|_{L^p(\Omega)}^p + (2\varepsilon)^2\|a\|_{L^p(\Omega)}^p)^{1/p}
\]

with \( C_2 = 4C_d^{1/p+(1+1/p)[\log d(18M)]} \). Letting \( \varepsilon \to 0 \), this proves (4.20).

Recall that \( 1 < Q < \infty \) is a fixed parameter, and that for \( 1 \leq p \leq Q \), we denote the Sobolev conjugate by \( p^* = Qp/(Q-p) \) when \( p < Q \), and \( p^* = \infty \) when \( p = Q \).

**Theorem 4.30.** Suppose \( \Omega \) is nonempty, open, and bounded, \( f: \Omega \to Y \) is injective, \( M \geq 1 \), \( 1 \leq p \leq Q \), \( a \in L^{p^*(Q-1)/Q}(\Omega) \) is a nonnegative function whose integral over every nonempty open subset of \( \Omega \) is nonzero, and suppose \( \Omega \) admits a partition into disjoint sets \( D \) and \( N \) such that \( H^{a,M}_f < \infty \) in \( D \), \( \mu(N) = \text{Mod}_p(\Gamma_N) = 0 \), and

\[
\|ha^{p^*(Q-1)/Q}\|_{L^1(\Omega)} < \infty \quad \text{for a function } h \geq (H^{a,M}_f)^{p^*} \chi_D \tag{4.31}
\]

in the case \( 1 \leq p < Q \), and \( \|H^{a,M}_f\|_{L^\infty(\Omega)} < \infty \) in the case \( p = Q \).

Furthermore, suppose that for some \( 0 < C_{\Omega} < \infty \) and for all \( r > 0 \),

\[
\frac{\mu(B(x,r))}{r^Q} \leq C_{\Omega} \quad \text{for all } x \in \Omega \quad \text{and} \quad \frac{\nu(B(f(x),r))}{r^Q} \geq C_{\Omega}^{1} \quad \text{for all } x \in D \tag{4.32}
\]
and that there is $\beta > 0$ such that

$$\nu(V) < \infty \quad \text{for} \quad V := \bigcup_{y \in \mathcal{D}} B(f(y), l_f(y, \beta)). \quad (4.33)$$

Then $f \in D^p(\Omega; Y)$ with

$$\|f\|_{D^p(\Omega; Y)}^p \leq C_3 \nu(V) + C_3 \|ha^{p(Q-1)/(Q-p)}\|_{L^1(\Omega)} \quad (4.34)$$
in the case $1 \leq p < Q$, and

$$\|f\|_{D^Q(\Omega; Y)}^Q \leq C_4 \nu(V) \|a\|_{L^{Q-1}(\Omega)} \|H_{f,M} a\|_{L^\infty(\Omega)} \quad (4.35)$$
in the case $p = Q$. The constants $C_3, C_4$ only depend on $C_d, C_{\Omega}, p, Q,$ and $M$.

**Proof.** Part 1: Case $1 \leq p < Q$.

Fix $0 < \varepsilon \leq 1$. By the Vitali-Carathéodory theorem (Theorem 2.6), we can assume that $h$ is lower semicontinuous. We have $D = \bigcup_{j=1}^{\infty} D_j$, where $D_j$ consists of those points $y \in D \cap \Omega((M+1)/j)$ for which

$$\sup_{0 < s \leq 1/j} \frac{L_f(y, s)}{l_f(y, s)} \left( \int_{B(y, Ms)} a \, d\mu \right)^{-(Q-1)/Q} \leq H_{f,M}^a(y) + \varepsilon \quad (4.36)$$

and

$$(H_{f,M}^a(y))^{Qp/(Q-p)} \leq h(z) + \varepsilon \quad \text{for all} \quad z \in B(y, M/j). \quad (4.37)$$

As before, we find a collection $\{B_{j,k} = B(y_{j,k}, 1/j)\}_k$ of balls covering $D_j$, with $y_{j,k} \in D_j$, and such that the balls $(M+1)B_{j,k}$ can be divided into at most $C_M$ collections of pairwise disjoint balls.

Denote $L_{j,k} := L_f(y_{j,k}, 1/j)$ and $l_{j,k} := l_f(y_{j,k}, 1/j)$. For each $j \in \mathbb{N}$, define

$$g_j := 2j \sum_k L_{j,k} \chi_{2B_{j,k}}.$$

Just as in the estimates leading to (4.11), for $p$-a.e. curve $\gamma$ in $\Omega$ we get

$$d_Y(f(\gamma(0)), f(\gamma(\ell_{\gamma}))) \leq \liminf_{j \to \infty} \int_{\gamma} g_j \, ds. \quad (4.38)$$
Then we estimate
\[ \sum_{k} I_{j,k}^{p} \mu(2B_{j,k}) \]
\[ \leq \sum_{k} I_{j,k}^{p} \mu(2B_{j,k}) \left( \int_{MB_{j,k}} a \, d\mu \right)^{p(Q-1)/Q} (H_{f}^{a,M}(y_{j,k}) + \varepsilon)^{p} \quad \text{by (4.36)} \]
\[ \leq \sum_{k} I_{j,k}^{p} \mu(2B_{j,k}) \left( \int_{MB_{j,k}} a^{p(Q-1)/(Q-p)} \, d\mu \right)^{(Q-p)/Q} (H_{f}^{a,M}(y_{j,k}) + \varepsilon)^{p} \quad \text{by Hölder} \]
\[ \leq C_d C_{\Omega}^{p/Q} \sum_{k} I_{j,k}^{p} \left( \int_{MB_{j,k}} a^{p(Q-1)/(Q-p)} \, d\mu \right)^{(Q-p)/Q} (H_{f}^{a,M}(y_{j,k}) + \varepsilon)^{p} \]
\[ \text{(4.39)} \]

by the doubling property of \( \mu \) and (4.32). Using this and Young's inequality, we get
\[ \sum_{k} I_{j,k}^{p} \mu(2B_{j,k}) \]
\[ \leq (C_d C_{\Omega}^{p/Q}) Q/p \sum_{k} I_{j,k}^{p} \mu(2B_{j,k}) \]
\[ \leq C_d^{Q/p} \sum_{k} \nu(B(f(y_{j,k}), l_{j,k})) \quad \text{by (4.32)} \]
\[ + 2Qp/(Q-p) \sum_{k} \left( \int_{MB_{j,k}} a^{p(Q-1)/(Q-p)} \, d\mu \right) \left( (H_{f}^{a,M}(y_{j,k}))^{Qp/(Q-p)} + \varepsilon Qp/(Q-p) \right) \]
\[ \leq C_d^{Q/p} C_{\Omega}^{Q/p} \nu(V) + 2Qp/(Q-p) C_{M} \int_{\Omega} a^{p(Q-1)/(Q-p)} (h + \varepsilon + \varepsilon Qp/(Q-p)) \, d\mu \quad \text{by (4.37)} \]
\[ \leq C_d^{Q/p} C_{\Omega}^{Q/p} C_{M} \nu(V) + 2Qp/(Q-p) C_{M} \int_{\Omega} a^{p(Q-1)/(Q-p)} (h + 2\varepsilon) \, d\mu \]
\[ \text{(4.40)} \]

for all \( j \in \mathbb{N} \) large enough that \( 1/j < \beta \), by (4.14); note that the exact same estimate applies here. We get
\[ \int_{\Omega} g_{j}^{p} \, d\mu = \left( \int_{\Omega} \left( 2j \sum_{k} L_{j,k} \chi_{2B_{j,k}} \right)^{p} \, d\mu \right) \]
\[ \leq (2j)^{p} C_{M} \sum_{k} L_{j,k}^{p} \mu(2B_{j,k}) \]
\[ \leq C_3 \nu(V) + C_3 \int_{\Omega} a^{p(Q-1)/(Q-p)} (h + 2\varepsilon) \, d\mu \quad \text{by (4.40)} \]
\[ < \infty \quad \text{by (4.31)} \]
with $C_3 := 2^{p+Qp/(Q-p)}C_d^{Q/p+(p+1)\lceil\log_2(18M)\rceil}C_{\Omega}^2$.

**Part 1(a): Case $p = 1$.**

Recall that we are considering the sequence 
\[ g_j = 2^j \sum_{k} L_{j,k} \chi_{2B_{j,k}}. \]

By (4.41), this is a bounded sequence in $L^1(\Omega)$. We will show that it is equi-integrable. The first condition of Definition 2.2 holds automatically since $\Omega$ as a bounded set has finite $\mu$-measure. We check the second condition.

Suppose by contradiction that we find $0 < \alpha < 1$, an infinite set $J \subset \mathbb{N}$, and a sequence of $\mu$-measurable sets $H_j \subset \Omega$ such that $\mu(H_j) \to 0$ and 
\[ j \int_{H_j} \sum_{k} L_{j,k} \chi_{2B_{j,k}} \, d\mu \geq \alpha \quad \text{for all } j \in J. \] (4.42)

Choose $L$ to be the following (very large) number:
\[ L := \frac{4C_d^{\lceil\log_2 M\rceil}}{\alpha} \left[ C_d^2 C_{\Omega}^2 C_M \nu(V) + 2^{Q/(Q-1)} C_M \int_{\Omega} a(h + 2) \, d\mu \right]. \] (4.43)

For every $j \in \mathbb{N}$, define two sets of indices $I_1^j$ and $I_2^j$ as follows: for $k \in I_1^j$, we have 
\[ \frac{\mu((M+1)B_{j,k} \cap H_j)}{\mu((M+1)B_{j,k})} \leq \frac{1}{L}, \]

and $I_2^j$ consists of the remaining indices. Now
\[ j \sum_{k \in I_1^j} L_{j,k} \mu(2B_{j,k} \cap H_j) \]

\[ \leq j \sum_{k \in I_1^j} L_{j,k} \mu((M+1)B_{j,k} \cap H_j) \]

\[ \leq \frac{j}{L} \sum_{k \in I_1^j} L_{j,k} \mu((M+1)B_{j,k}) \]

\[ \leq \frac{j}{L} C_d^{\lceil\log_2 M\rceil} \sum_{k \in I_1^j} L_{j,k} \mu(2B_{j,k}) \]

\[ \leq \frac{C_d^{\lceil\log_2 M\rceil}}{L} \left[ C_d^2 C_{\Omega}^2 C_M \nu(V) + 2^{Q/(Q-1)} C_M \int_{\Omega} a(h + 2) \, d\mu \right] \quad \text{by (4.40)} \]

\[ = \frac{\alpha}{4} \quad \text{by (4.43)}. \]
Next, we estimate
\[
\mu \left( \bigcup_{k \in I_j^2} MB_{j,k} \right) \leq \sum_{k \in I_j^2} \mu((M + 1)B_{j,k}) \\
\leq L \sum_{k \in I_j^2} \mu((M + 1)B_{j,k} \cap H_j) \leq LC_M \mu(H_j) \to 0 \quad \text{as } j \to \infty.
\] (4.45)

Just as in (4.45), for any \( 0 \leq \delta < 1 \) and any \( b_1, b_2 \geq 0 \) we have
\[
b_1 b_2 \leq \delta b_1^Q + \delta^{-1/(Q-1)} b_2^{Q/(Q-1)}.
\] (4.46)

We choose
\[
\delta := \frac{1}{4} \frac{\alpha}{C_Q^2 C_{\Omega}^2 C_M \nu(V) + 1}.
\] (4.47)

By (4.36), for \( j \in \mathbb{N} \) large enough that \( 1/j < \beta \) we get
\[
j \sum_{k \in I_j^2} L_{j,k} \mu((2B_{j,k})
\leq j \sum_{k \in I_j^2} l_{j,k} \mu((2B_{j,k}) \left( \int_{MB_{j,k}} a \, d\mu \right)^{(Q-1)/Q} (H_f^{a,M}(y_{j,k}) + 1)
\leq C_d C_{\Omega}^{1/Q} \sum_{k \in I_j^2} l_{j,k} \left( \int_{MB_{j,k}} a \, d\mu \right)^{(Q-1)/Q} (H_f^{a,M}(y_{j,k}) + 1) \quad \text{by (4.32)}
\leq C_d C_{\Omega}^Q C_Q \delta \sum_{k \in I_j^2} l_{j,k}^Q + \delta^{-1/(Q-1)} \sum_{k \in I_j^2} (H_f^{a,M}(y_{j,k}) + 1)^{Q/(Q-1)} \int_{MB_{j,k}} a \, d\mu \quad \text{by (4.46)}
\leq C_d C_{\Omega}^Q C_Q \delta \sum_{k \in I_j^2} \nu(B(f(y_{j,k}, l_{j,k})) \quad \text{by (4.32)}
\quad + 2^{Q/(Q-1)} \delta^{-1/(Q-1)} \sum_{k \in I_j^2} (H_f^{a,M}(y_{j,k})^{Q/(Q-1)} + 1) \int_{MB_{j,k}} a \, d\mu
\leq C_d C_{\Omega}^Q C_Q \delta \nu(V) \quad \text{by (4.14) (the same estimate applies)}
\quad + 2^{Q/(Q-1)} \delta^{-1/(Q-1)} \sum_{k \in I_j^2} (h + 2) a \, d\mu \quad \text{by (4.37)}
\leq \frac{\alpha}{4} + 2^{Q/(Q-1)} C_M \delta^{-1/(Q-1)} \int_{\bigcup_{k \in I_j^2} MB_{j,k}} (h + 2) a \, d\mu \quad \text{by (4.47)}.
\]

Here the latter term goes to zero as \( j \to \infty \) by (4.45). Combining this with (4.44), we get
\[
\limsup_{j \to \infty} j \sum_{k} L_{j,k} \mu((2B_{j,k} \cap H_j) \leq \frac{\alpha}{2}.
\]
This contradicts (4.42) and proves the equi-integrability of \( \{g_j\}^\infty_{j=1} \).

Now by the Dunford–Pettis theorem (Theorem 2.3), we find \( g \in L^1(\Omega) \) such that by passing to a subsequence (not relabeled), we have \( g_j \to g \) weakly in \( L^1(\Omega) \). By Mazur’s lemma (Theorem 2.4), for suitable convex combinations we get the strong convergence \( \sum_{l=j}^{N_j} a_{j,l}g_l \to g \) in \( L^1(\Omega) \). From (4.38) and Fuglede’s lemma (Lemma 2.5) we get

\[
d_Y(f(\gamma(0)), f(\gamma(\ell_\gamma))) \leq \liminf_{j \to \infty} \int_\gamma \sum_{l=j}^{N_j} a_{j,l}g_l \, ds = \int_\gamma g \, ds
\]

for 1-a.e. curve \( \gamma \) in \( \Omega \). Thus \( g \in L^1(\Omega) \) is a 1-weak upper gradient of \( f \) in \( \Omega \), and so \( f \in D^1(\Omega; \gamma) \).

**Part 1(b): Case 1 < p < Q.**

By (4.41), \( \{g_j\}^\infty_{j=1} \) is a bounded sequence in \( L^p(\Omega) \). By reflexivity of the space \( L^p(\Omega) \), we find a subsequence of \( \{g_j\}^\infty_{j=1} \) (not relabeled) and \( g \in L^p(\Omega) \) such that \( g_j \to g \) weakly in \( L^p(\Omega) \). By Mazur’s lemma (Theorem 2.4), for suitable convex combinations we get the strong convergence \( \sum_{l=j}^{N_j} a_{j,l}g_l \to g \) in \( L^p(\Omega) \). By (4.38) and Fuglede’s lemma (Lemma 2.5), we obtain

\[
d_Y(f(\gamma(0)), f(\gamma(\ell_\gamma))) \leq \liminf_{j \to \infty} \int_\gamma \sum_{l=j}^{N_j} a_{j,l}g_l \, ds = \int_\gamma g \, ds
\]

for \( p \)-a.e. curve \( \gamma \) in \( \Omega \). Hence \( f \in D^p(\Omega; \gamma) \). For all \( 1 \leq p < Q \), by (4.41) we get

\[
\|f\|_{D^p(\Omega; \gamma)}^p \leq \|g\|_{L^p(\Omega)}^p \leq \limsup_{j \to \infty} \|g_j\|_{L^p(\Omega)}^p \leq C_3 \nu(V) + C_3 \|h + 2\varepsilon\|a^{(Q-1)/(Q-p)}\|_{L^1(\Omega)}.
\]

Letting \( \varepsilon \to 0 \), we get (4.34).

**Part 2: Case \( p = Q \).**

Now \( \Omega \) is the disjoint union of \( N \) and \( D \), where \( \mu(N) = \text{Mod}_Q(\Gamma_N) = 0 \), and \( H_f^{a,M} < \infty \) in \( D \) and \( \|H_f^{a,M}\|_{L^\infty(\Omega)} < \infty \), where \( a \in L^\infty(\Omega) \). For every \( y \in \Omega \), we have

\[
H_f(y) = \limsup_{r \to 0} \frac{L_f(y,r)}{l_f(y,r)} \leq \|a\|_{L^\infty(\Omega)} \limsup_{r \to 0} \frac{L_f(y,r)}{l_f(y,r)} \left( \int_{B(y,Mr)} a \, d\mu \right)^{-1/(Q-1)} = \|a\|_{L^\infty(\Omega)} H_f^{a,M}(y).
\]

In particular,

\[
\|H_f\|_{L^\infty(\Omega)} \leq \|a\|_{L^\infty(\Omega)} \|H_f^{a,M}\|_{L^\infty(\Omega)} < \infty.
\]

Thus in fact \( H_f < \infty \) in \( D \), and \( H_f \in L^\infty(\Omega) \). (Hence from now on, we can work with the ordinary distortion number \( H_f \).) We have \( D = \bigcup_{j=1}^\infty D^j \), where \( D^j \) consists of those
points \( y \in D \cap \Omega(2/j) \) for which
\[
\sup_{0 < s \leq 1/j} \frac{L_f(y, s)}{l_f(y, s)} \leq j. \tag{4.50}
\]

Fix \( j \in \mathbb{N} \). For each \( k \geq j \), we find a collection \( \{B_{k,l} = B(y_{k,l}, 1/k)\}_l \) covering \( D^j \), with \( y_{k,l} \in D^j \), and such that the balls \( 2B_{k,l} \) can be divided into at most \( C_M \) collections of pairwise disjoint balls. Define
\[
g_k := 2k \sum_l l_f(y_{k,l}, 1/k) \chi_{2B_{k,l}}, \quad k \in \mathbb{N}.
\]

Consider a curve \( \gamma \) in \( \Omega \) with \( \ell_\gamma \geq 1/k \). If \( \gamma \) intersects \( B_{k,l} \), then
\[
\mathcal{H}_1^\infty(\gamma \cap 2B_{k,l}) \geq \frac{1}{k}.
\]
Thus we have
\[
\hat{\gamma} g_k ds \geq 2k \sum_l l_f(y_{k,l}, 1/k) \mu(2B_{k,l}) \geq \sum_{l : \gamma \cap B_{k,l} \neq \emptyset} \mu(B(y_{j,l}, l_{k,l})) \quad \text{by (4.51)}
\]
where the last inequality holds since the balls \( B_{k,l} \) satisfying \( \gamma \cap B_{k,l} \neq \emptyset \) cover \( \gamma \cap D^j \) and so the sets \( f(B_{k,l}) \) with \( \gamma \cap B_{k,l} \neq \emptyset \) cover \( f(\gamma \cap D^j) \).

Denote \( L_{k,l} := L_f(y_{k,l}, 1/k) \) and \( t_{k,l} := l_f(y_{k,l}, 1/k) \). For every \( k \in \mathbb{N} \) such that \( 1/k < \beta \), we get
\[
\int_\Omega g_k^Q d\mu = \int_\Omega \left( 2k \sum_l L_{k,l} \chi_{2B_{k,l}} \right)^Q d\mu
\leq C_M^Q(2k)^Q \sum_l L_{k,l}^Q \mu(2B_{k,l})
\leq 2^{Q} C_M^Q \sum_l L_{k,l}^Q \mu(2B_{k,l}) \quad \text{by (4.32)}
\leq 2^{Q} C_M^Q \sum_l l_{k,l}^Q \mu(2B_{k,l}) \quad \text{by (4.32)}
\leq 2^{Q} C_M^Q \sum_l \nu(B(y_{j,l}, l_{k,l})) \quad \text{by (4.32)}
\leq 2^{Q} C_M^Q + 1 \sum_l \nu(V) \quad \text{by (4.14) (the same estimate applies)}
< \infty \quad \text{by (4.33)};
\]
recall that we are keeping \( j \) fixed. By reflexivity of the space \( L^Q(\Omega) \), we find a subsequence (not relabeled) and \( g \in L^Q(\Omega) \) such that \( g_k \to g \) weakly in \( L^Q(\Omega) \). By Mazur’s and Fuglede’s lemmas (Theorem 2.4 and Lemma 2.5), we find convex combinations \( \sum_{l=k}^{N_k} a_{k,l} g_l \) converging to \( g \) in \( L^Q(\Omega) \), and such that for \( Q \)-a.e. curve \( \gamma' \) in \( \Omega \) we have
\[
\int_{\gamma'} g ds = \liminf_{k \to \infty} \int_{\gamma'} \sum_{l=k}^{N_k} a_{k,l} g_l ds \geq \mathcal{H}_1^\infty(g(\gamma' \cap D^j)) \quad \text{by (4.51)}.
\tag{4.52}
\]
By the properties of modulus, see e.g. [5, Lemma 1.34], for $Q$-a.e. curve $\gamma$ in $\Omega$ we have that (4.52) holds for every subcurve $\gamma'$ of $\gamma$. For $Q$-a.e. curve $\gamma$ in $\Omega$, we also have that $\int_{\gamma} g \, ds < \infty$ (see e.g. [5, Proposition 1.37]). Fix a curve $\gamma$ satisfying these two conditions. We can write any open $U \subset (0, \ell_\gamma)$ as an at most countable union of pairwise disjoint intervals $U = \bigcup_{j}(a_j, b_j)$, and then

$$\int_{U} g(\gamma(s)) \, ds = \sum_{j} \int_{(a_j, b_j)} g(\gamma(s)) \, ds \geq \sum_{j} \mathcal{H}^1_{\infty}(f(\gamma((a_j, b_j)) \cap D^j)) \quad \text{by (4.52)}$$

by the subadditivity of $\mathcal{H}^1_{\infty}$. Then for any Borel set $S \subset (0, \ell_\gamma)$, we can take a small $\delta > 0$ and find an open set $U$ such that $S \subset U \subset (0, \ell_\gamma)$ and

$$\int_{S} g(\gamma(s)) \, ds \geq \int_{U} g(\gamma(s)) \, ds - \delta \geq \mathcal{H}^1_{\infty}(f(\gamma(U) \cap D^j)) - \delta \quad \text{by (4.53)}$$

Letting $\delta \to 0$, we get

$$\int_{S} g(\gamma(s)) \, ds \geq \mathcal{H}^1_{\infty}(f(\gamma(S) \cap D^j)).$$

Note that $\mathcal{H}^1_{\infty}$ and $\mathcal{H}^1$ have the same null sets (see e.g. [7, Lemma 2.1]). Thus, using also the Borel regularity of $\mathcal{L}^1$, we conclude that whenever $S \subset [0, \ell_\gamma]$ with $\mathcal{L}^1(S) = 0$, we have $\mathcal{H}^1(f(\gamma(S) \cap D^j)) = 0$. Recall that $\Omega = \bigcup_{j=1}^{\infty} D^j \cup N$, and that $Q$-a.e. curve avoids $N$. Thus for $Q$-a.e. curve $\gamma$ in $\Omega$, we have that if $S \subset [0, \ell_\gamma]$ with $\mathcal{L}^1(S) = 0$, then

$$\mathcal{H}^1(f(\gamma(S))) \leq \sum_{j=1}^{\infty} \mathcal{H}^1(f(\gamma(S) \cap D^j)) = 0.$$

By Lemma 3.7, there is an at most countable set $H \subset D$ such that $f$ is continuous at every point $y \in D \setminus H$. Note that $Q$-a.e. curve avoids $H$, see [14, Corollary 5.3.11]. In total, for $Q$-a.e. curve $\gamma$ in $\Omega$, we know that

$$f \circ \gamma \text{ is continuous and if } S \subset [0, \ell_\gamma] \text{ with } \mathcal{L}^1(S) = 0, \text{ then } \mathcal{H}^1(f(\gamma(S))) = 0.$$  

(4.54)

Fix a new $\delta > 0$. Recall that we define

$$H_f(y) = \limsup_{s \to 0} H_f(y, s) = \limsup_{s \to 0} \frac{L_f(y, s)}{l_f(y, s)}.$$

Note that $H_f(y) \geq 1$ for every $y \in \Omega$, by the fact that the space $X$ is connected. Thus for every $y \in \Omega$, for all sufficiently small $s > 0$ we have

$$\frac{L_f(y, s)}{l_f(y, s)} = H_f(y, s) \leq (1 + \delta)H_f(y).$$
It follows that

$$\liminf_{s \to 0} \left( \frac{L_f(y, s)}{s} \right)^Q = (1 + \delta)^Q \liminf_{s \to 0} \left( \frac{L_f(y, s)}{s} \right)^Q \leq C_f^2(1 + \delta)^Q |H_f|^Q \liminf_{s \to 0} \frac{\nu(B(f(y), l_f(y, s)))}{\mu(B(y, s))} \text{ by (4.32)}$$

$$\leq C_f^2(1 + \delta)^Q |H_f|^Q \liminf_{s \to 0} \frac{\nu(B(f(y), l_f(y, s)))}{\mu(B(y, s))}$$

for \( \mu \)-a.e. \( y \in \Omega \). For each \( j \in \mathbb{N} \), consider any \( \mu \)-measurable function \( h_j \) on \( \Omega \) that is at least

$$\Omega \ni y \mapsto \frac{\nu(B(f(y), l_f(y, 1/j)))}{\mu(B(y, 1/j))}.$$ 

By Lemma 3.3, \( \text{lip}_f \) is \( \mu \)-measurable on \( \Omega \). We get

$$\int_{\Omega} \text{lip}_f^Q \, d\mu \leq C_f^2(1 + \delta)^Q |H_f|^Q \liminf_{j \to \infty} h_j(y) \, d\mu(y) \text{ by (4.55)}$$

$$\leq C_f^2(1 + \delta)^Q |H_f|^Q \liminf_{j \to \infty} \int_{\Omega} h_j(y) \, d\mu(y) \text{ by Fatou.}$$

Consider \( j \in \mathbb{N} \) with \( 1/j < \beta \) (recall (4.33)), and take balls \( \{B(y_l, 1/j)\}_l \), with \( y_l \in \Omega \), which cover \( \Omega \), and such that the balls \( B(y_l, 2/j) \) can be divided into at most \( L \leq C_f^2 \) collections of pairwise disjoint balls \( \{B(y_l, 2/j)\}_{l \in L} \). Note that

$$\frac{\nu(B(f(y), l_f(y, 1/j)))}{\mu(B(y, 1/j))} \leq \sum_l \chi_{B(y_l, 1/j)}(y) \frac{\nu(\bigcup_{z \in B(y_l, 1/j) \cap \Omega} B(f(z), l_f(z, 1/j)))}{\mu(B(y, 1/j))}$$

$$\leq C_f^2 \sum_l \chi_{B(y_l, 1/j)}(y) \frac{\nu(\bigcup_{z \in B(y_l, 1/j) \cap \Omega} B(f(z), l_f(z, 1/j)))}{\mu(B(y, 1/j))},$$

which is clearly \( \mu \)-measurable, and so \( h_j \) can be taken to be the right-hand side. Now, if \( z \in B(y_l, 1/j) \cap \Omega \), and \( z' \in B(y_{l'}, 1/j) \cap \Omega \), where \( B(y_l, 2/j) \cap B(y_{l'}, 2/j) = \emptyset \), then from the definition of \( l_f(\cdot, \cdot) \) and from the injectivity of \( f \) it follows (just as in (4.13)) that

$$B(f(z), l_f(z, 1/j)) \cap B(f(z'), l_f(z', 1/j)) = \emptyset. \quad (4.57)$$
We get
\[
\int_\Omega h_j(y) \, d\mu(y) = C_d^2 \int_\Omega \sum_i \chi_{B(y_i,1/j)}(y) \frac{\nu \left( \bigcup_{z \in B(y_i,1/j) \cap \Omega} B(f(z), l_f(z, 1/j)) \right)}{\mu(B(y_i, 1/j))} \, d\mu(y)
\]
\[
\leq C_d^2 \sum_i \nu \left( \bigcup_{z \in B(y_i, 1/j) \cap \Omega} B(f(z), l_f(z, 1/j)) \right)
\]
\[
= C_d^2 \sum_{m=1}^L \sum_{l \in I_m} \nu \left( \bigcup_{x \in \Omega} B(f(x), l_f(x, 1/j)) \right)
\]
\[
\leq C_d^2 \sum_{m=1}^L \nu \left( \bigcup_{x \in \Omega} B(f(x), l_f(x, 1/j)) \right) \quad \text{by (4.57)}
\]
\[
\leq C_d^6 \nu \left( \bigcup_{y \in \Omega} B(f(y), l_f(y, 1/j)) \right)
\]
\[
= C_d^6 \nu(V) < \infty \quad \text{by (4.33)}.
\]
Combining this with (4.56), we get
\[
\|\text{lip}_f\|_{L^Q(\Omega)} \leq C_d^2 (1 + \delta)^Q C_d^6 \|H_f\|_{L^\infty(\Omega)} \nu(V)
\]
\[
\leq C_d^2 (1 + \delta)^Q C_d^6 \nu(V) \|a\|_{L^\infty(\Omega)}^{-1} \|H_f^{a,M}\|_{L^\infty(\Omega)} \quad \text{by (4.49)}.
\]
Letting \(\delta \to 0\), we conclude
\[
\|\text{lip}_f\|_{L^Q(\Omega)} \leq C_d^2 C_d^6 \nu(V) \|a\|_{L^\infty(\Omega)}^{-1} \|H_f^{a,M}\|_{L^\infty(\Omega)}.
\] (4.58)
Thus \(\text{lip}_f < \infty\) \(\mu\)-a.e. in \(\Omega\). Using the function \(\infty \chi_F\) where \(F \supset \{\text{lip}_f = \infty\}\) is a \(\mu\)-negligible Borel set, we see that for \(Q\)-a.e. curve \(\gamma\) in \(\Omega\) we have, with \(T := [0, \ell_\gamma] \cap \gamma^{-1}\{\text{lip}_f < \infty\}\), that \(\mathcal{L}^1([0, \ell_\gamma] \setminus T) = 0\). Moreover, \(f \circ \gamma\) is continuous by (4.54), and thus
\[
d_T(f(\gamma(0)), f(\gamma(\ell_\gamma))) \leq \mathcal{H}^1(f(\gamma([0, \ell_\gamma])))
\]
\[
\leq \mathcal{H}^1(f(\gamma(T))) \quad \text{by (4.54) (second part)}
\]
\[
\leq \int_T \text{lip}_f(\gamma(t)) \, dt \quad \text{by Lemma 3.5}
\]
\[
\leq \int_\gamma \text{lip}_f \, dt.
\]
Hence \(\text{lip}_f \in L^Q(\Omega)\) is a \(Q\)-weak upper gradient of \(f\) in \(\Omega\). In conclusion, \(f \in D^Q(\Omega; Y)\). Moreover, by (4.58) we have
\[
\|f\|_{D^Q(\Omega; Y)}^Q \leq \|\text{lip}_f\|_{L^Q(\Omega)}^Q \leq C_d^2 C_d^6 \nu(V) \|a\|_{L^\infty(\Omega)}^{-1} \|H_f^{a,M}\|_{L^\infty(\Omega)} \nu(V),
\]
proving (4.35) with $C_4 = C_d^6C^2_\Omega$. \hfill \Box

In Part 2 of the proof, the idea of first proving absolute continuity on curves, and then examining $\text{lip}_f$, comes from Williams [27]. This part of the proof was moreover quite similar to [20, Section 5], but as usual, extra difficulties arose from the fact that $f$ is not assumed to be a homeomorphism, merely injective.

5. Corollaries and examples

Since the formulations of Theorems 4.1, 4.17, and 4.30 are rather long and technical, we will consider three corollaries, as well as a few examples.

Corollary 5.1. Suppose $\Omega \subset X$ is open and bounded, $f : \Omega \to Y$ is bounded, $M \geq 1$, $\kappa \geq \mu$ is a finite Radon measure on $\Omega$, $\text{Lip}^{\kappa,M}_f(x) \leq 1$ for $\tilde{H}^1$-a.e. $x \in \Omega$, and there is $0 < C_\Omega < \infty$ such that

$$\frac{\mu(B(x,r))}{r^Q} \leq C_\Omega \quad \text{for all } x \in \Omega \text{ and } r > 0.$$ 

Then $f \in \text{BV}(\Omega; Y)$.

Proof. This is obtained by applying Theorem 4.1 with the choices: $N$ is the subset of $\Omega$ where $\text{Lip}^{\kappa,M}_f > 1$, $A = \Omega \setminus N$, $D = W = \emptyset$, $h = 1$, and $\varepsilon = 1$. Note that $\text{Mod}_1(\Gamma_N) = 0$ by Lemma 3.8, and that $f \in L^\infty(\Omega; Y) \subset L^1(\Omega; Y)$. \hfill \Box

Many BV mappings, which may be highly discontinuous, can be seen as “generalized Lipschitz” in the sense of $\text{Lip}^{\kappa,M}_f$ being bounded. Consider the following example.

Example 5.2. Let the spaces $(X,d,\mu)$ and $(Y,d_Y,\nu)$ be $\mathbb{R}^2$ and $\mathbb{R}^1$, respectively, equipped with the Euclidean metrics and the Lebesgue measures $\mathcal{L}^2$ and $\mathcal{L}^1$. Let $\Omega = (0,1) \times (0,1)$, and for $(x_1,x_2) \in \Omega$ let

$$f(x_1,x_2) := \lambda((0,x_2])$$

for a finite Radon measure $\lambda \geq \mathcal{L}^1$. If $\lambda$ is not absolutely continuous with respect to $\mathcal{L}^1$, then $f$ is obviously not Lipschitz, and if $\lambda$ has Dirac masses, $f$ is not even continuous. Thus the existing literature results written for continuous functions, as described in the Introduction, are not applicable. On the other hand, define the Radon measure

$$\kappa := \mathcal{L}^1 \times \lambda$$

on $\Omega$. For every $x = (x_1,x_2) \in \Omega$, we get

$$\text{Lip}^{\kappa,2}_f(x) = \limsup_{r \to 0} \frac{L_f(x,r)}{r} \frac{\mathcal{L}^2(B(x,2r))}{\kappa(B(x,2r))} \leq \limsup_{r \to 0} \frac{\lambda([x_2-r, x_2+r])}{r} \frac{\mathcal{L}^2(B(x,2r))}{\kappa(B(x,2r))} \leq \limsup_{r \to 0} \frac{\lambda([x_2-r, x_2+r])}{r} \frac{4\pi r^2}{2r \lambda([x_2-r, x_2+r])} \leq 2\pi.$$ (5.3)
Now Corollary 5.1 gives $f \in \text{BV}(\Omega; \mathbb{R})$.

Note that in (5.3), the factor “2” in $\kappa(B(x, 2r))$ is crucial for obtaining the quantity $\lambda([x_2 - r, x_2 + r])$ in the denominator on the third line. This demonstrates that the number $M$ in the definition of $\text{Lip}^{\kappa,M}_f$ provides crucial flexibility.

In the next corollary, the idea is that we choose $a = 1$ and then the generalized Lipschitz number reduces to the ordinary one.

**Corollary 5.4.** Suppose $\Omega \subset X$ is open and bounded, $1 \leq p < \infty, f: \Omega \to Y$ is bounded, $\text{Lip}_f(x) < \infty$ for $\tilde{H}^p$-a.e. $x \in \Omega$, $\|\text{Lip}_f\|_{L^p(\Omega)} < \infty$, and there is $0 < C_{\Omega} < \infty$ such that

$$\frac{\mu(B(x, r))}{r^q} \leq C_{\Omega} \quad \text{for all } x \in \Omega \text{ and } r > 0.$$

Then $f \in N^{1,p}(\Omega; Y)$.

**Proof.** This is obtained by applying Theorem 4.17 with the choices: $M = 1, a = 1, N$ is the subset of $\Omega$ where $\text{Lip}_f = \infty, A = \Omega \setminus N, D = W = \emptyset$, and $h = \text{Lip}_f$. Note that $\text{Mod}_{p}(\Gamma_N) = 0$ by Lemma 3.8, and that $f \in L^\infty(\Omega; Y) \subset L^p(\Omega; Y)$.

In this way, we recovered a result very similar to the known results in the literature, see e.g. [2, Theorem 1.5] and recall the results described in the Introduction. However, the main advantage of the generalized Lipschitz and distortion numbers becomes apparent when choosing suitably large measures $\kappa$ and functions $a$, which force these numbers to be bounded. This was the idea in Corollary 5.1, and also in the next corollary.

**Corollary 5.5.** Suppose $\Omega \subset X$ is open and bounded, $f: \Omega \to Y$ is injective and bounded, $M \geq 1, 1 \leq p \leq Q$, there is $a: \Omega \to [1, \infty)$ in $L^p_{\text{loc}}(\Omega)$ such that $H_f^{a,M} \leq 1$ for $\tilde{H}^p$-a.e. $x \in \Omega$, and there is $0 < C_{\Omega} < \infty$ such that

$$\frac{\mu(B(x, r))}{r^q} \leq C_{\Omega} \quad \text{and} \quad \frac{\nu(B(f(x), r))}{r^q} \geq C_{\Omega}^{-1} \quad \text{for all } x \in \Omega.$$

Then $f \in N^{1,p}(\Omega; Y)$.

**Proof.** This is obtained by applying Theorem 4.30 with the choices: $N$ is the subset of $\Omega$ where $H_f^{a,M} > 1, D = \Omega \setminus N$, and $h = 1$. Note that $\text{Mod}_{p}(\Gamma_N) = 0$ by Lemma 3.8. Moreover, as long as $\Omega$ consists of at least two points, we can choose $x_0 \in \Omega$ and a number $0 < \beta < \text{diam } \Omega$, and then

$$\nu\left(\bigcup_{y \in D} B(f(y), l_f(y, \beta))\right) \leq \nu\left(\bigcup_{y \in \Omega} B(f(y), \text{diam } f(\Omega))\right) \leq \nu(B(f(x_0), 2 \text{diam } f(\Omega))),$$

which is finite by our standing assumption that balls have finite measure. Thus (4.33) is also fulfilled.
Recall that Theorem 1.1 from the Introduction states that:

- Suppose \((X,d,\mu)\) and \((Y,d_Y,\nu)\) are Ahlfors \(Q\)-regular, and \(1 \leq p \leq Q\). Let \(f: X \to Y\) be injective and bounded, and \(M \geq 1\). Then:
  1. If there exists a Radon measure \(\kappa \geq \mu\) such that \(\text{Lip}_f^{\kappa,M}(x) \leq 1\) for \(\tilde{H}^1\)-a.e. \(x \in X\), then \(f \in \text{BV}_{\text{loc}}(X;Y)\);
  2. If there exists a function \(a: X \to [1,\infty)\) belonging to \(L^p_\text{loc}(Q-1)/Q)\) such that \(H_{f,a}^{M}(x) \leq 1\) for \(\tilde{H}^p\)-a.e. \(x \in X\), then \(f \in N^{1,p}_{\text{loc}}(X;Y)\).

\textit{Proof of Theorem 1.1.} (1): For every \(x \in X\) we find \(r > 0\) such that \(\kappa(B(x,r)) < \infty\), and then from Corollary 5.1 we get \(f \in \text{BV}(B(x,r);Y)\). In conclusion, \(f \in \text{BV}_{\text{loc}}(X;Y)\).

(2): For every \(x \in X\) we find \(r > 0\) such that \(\|a\|_{L^p_\text{loc}(Q-1)/Q}(B(x,r)) < \infty\), and then from Corollary 5.5 we get \(f \in N^{1,p}_{\text{loc}}(B(x,r);Y)\). In conclusion, \(f \in N^{1,p}_{\text{loc}}(X;Y)\).

\(\Box\)

\textbf{Example 5.6.} Let \((X,d,\mu)\) and \((Y,d_Y,\nu)\) both be \(\mathbb{R}^2\) equipped with the Euclidean metric and the Lebesgue measure \(L^2\). Let \(\Omega = (0,1) \times (0,1)\), and for \((x_1,x_2) \in \Omega\) let

\[ f((x_1,x_2)) := (f_1(x_1), f_2(x_2)), \]

where

\[ f_1(x_1) := x_1 \quad \text{and} \quad f_2(x_2) := \int_0^{x_2} g(s) \, ds \]

for a function \(g \in L^2((0,1)), g \geq 1\). For every \(x = (x_1,x_2) \in \Omega\), we have

\[ \limsup_{r \to 0} \int_{B(x_2,r)} g(s) \, ds \leq H_f(x) \leq 2 \limsup_{r \to 0} \int_{B(x_2,r)} g(s) \, ds. \quad (5.7) \]

Thus \(H_f = \infty\) in a set \((0,1) \times D\), where \(D \subset (0,1)\) may clearly be of dimension strictly greater than 0, and then \(E := (0,1) \times D\) is of dimension strictly greater than 1. Thus \(E\) is not of \(\sigma\)-finite \(H^1\)-measure, and the results in the literature, as described in the Introduction, are not applicable. (Note that \(H^1\) and \(\tilde{H}^1\) are now comparable.) On the other hand, let \(\hat{g}(x_1,x_2) := g(x_2)\) and \(a := \hat{g}^2\), so that \(a \in L^1(\Omega)\). For every
$x = (x_1, x_2) \in \Omega$, we get

$$H_f^{a,2}(x) = \limsup_{r \to 0} \frac{L_f(x,r)}{l_f(x,r)} \left( \int_{B(x,2r)} a \, d\mathcal{L}^2 \right)^{(1-2)/2} \leq 2 \limsup_{r \to 0} \int_{(x_2-r,x_2+r)} g \, ds \left( \int_{B(x,2r)} a \, d\mathcal{L}^2 \right)^{-1/2} \leq 2 \limsup_{r \to 0} \int_{(x_2-r,x_2+r)} g \, ds \left( \int_{B(x,2r)} \tilde{g} \, d\mathcal{L}^2 \right)^{-1} \text{ by Hölder} \leq 2 \limsup_{r \to 0} \int_{(x_2-r,x_2+r)} g \, ds \left( \frac{1}{4\pi r^2} \int_{(x_2-r,x_2+r)} \int_{(x_1-r,x_1+r)} g(s) \, dt \, ds \right)^{-1} = 2\pi.$$

Now Corollary 5.5 gives $f \in N^{1,1}(\Omega; \mathbb{R}^2)$.

In the literature, it is known that $E = \{ H_f = \infty \}$ cannot in general be allowed to be larger than $\sigma$-finite with respect to $\mathcal{H}^1$, see e.g. [17, Remark 1.2(b)] and [27, Remark 1.9]. However, Example 5.6 demonstrates that we can sometimes get around this limitation by using the generalized distortion number $H_f^{a,M}$. A similar example was already given previously in [18, Example 5.27], but there we were limited to being able to handle functions $g$ with a suitable structure, whereas above, $1 \leq g \in L^2((0,1))$ can be arbitrary. This flexibility is again achieved by means of the factor $M$ in $H_f^{a,M}$; in this case $M = 2$ sufficed.

On the other hand, note that clearly the mapping $f$ in Example 5.6 is actually in $N^{1,2}(\Omega; \mathbb{R}^2)$ and not only in $N^{1,1}(\Omega; \mathbb{R}^2)$. This raises the question of whether there could be a sharper version of Corollary 5.5.

Finally we return to the question of the connection between the exceptional set $E$ and the approximate jump set of a BV function. Our reasoning will be similar to [18, Corollary 5.22], but in the theory developed in that paper, $f$ was always assumed to be continuous, which is of course not natural from the viewpoint of BV theory.

Example 5.8. Suppose $f: \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism, with $n \geq 2$, and there exists a set $E$ of $\sigma$-finite $\mathcal{H}^{n-1}$-measure such that $H_f(x) < \infty$ for every $x \in \mathbb{R}^n \setminus E$. Assume also that $H_f \in L_n^{n/(n-1)}(\mathbb{R}^n)$.

Additionally, assume that $E$ can be presented as a union $E = \bigcup_{j=0}^{\infty} E_j$, where each $E_j$ is a Borel set with $\mathcal{H}^{n-1}(E_0) = 0$, $0 < \mathcal{H}^{n-1}(E_j) < \infty$ for all $j \in \mathbb{N}$, and

$$\liminf_{r \to 0} \frac{\mathcal{H}^{n-1}(B(x,r) \cap E_j)}{r^{n-1}} > 0 \quad (5.9)$$

for $\mathcal{H}^{n-1}$-a.e. $x \in E_j$, for each $j \in \mathbb{N}$. This is a regularity assumption that is satisfied for example if $E$ is an $n-1$-rectifiable set (see e.g. [1, Theorem 2.83]).
Define $\kappa$ by
\[
\kappa := \mathcal{L}^n + \sum_{j=1}^{\infty} 2^{-j} \mathcal{H}^{n-1}(E_j)^{-1} \mathcal{H}^{n-1} \lfloor_{E_j}.
\] (5.10)

At points $x \in E$ where (5.9) is satisfied, by the continuity of $f$ we get
\[
\text{Lip}_{\kappa}^{\kappa,1}(x) = \limsup_{r \to 0} \frac{L_f(x, r) \mathcal{L}^n(B(x, r))}{\kappa(B(x, r))} = 0.
\]
Thus we obtain $\text{Lip}_{\kappa}^{\kappa,1}(x) = 0$ for $\mathcal{H}^{n-1}$-a.e. $x \in E$; this gives an exceptional (Borel) $N \subset E$ with $\mathcal{H}^{n-1}(N) = 0$. Meanwhile, $H_f^{\kappa,1}(x) \leq H_f(x) < \infty$ for every $x \in \mathbb{R}^n \setminus E$.

Now, for a bounded open set $\Omega \subset \mathbb{R}^n$, we apply the first part of Theorem 4.1 with the choices $A = \Omega \cap E \setminus N$, $D = \Omega \setminus E$, and
\[
h = \text{Lip}_{\kappa}^{\kappa,1} \chi_A + (H_f^{\kappa,1})^{n/(n-1)} \chi_D;
\]
measurability is now straightforward to show, since $f$ is continuous. Note also that since $f$ is a homeomorphism, it is locally bounded and then (4.4) is satisfied with every choice of $\beta > 0$. For any $0 < \varepsilon \leq 1$, we obtain
\[
\|Df\|(\Omega) \leq C_1 \varepsilon^{-1/(n-1)} \left( \int_A \text{Lip}_{\kappa}^{\kappa,1} \, d\kappa + \int_D (H_f^{\kappa,1})^{n/(n-1)} \, d\kappa \right) + C_1 \varepsilon (\kappa(\Omega) + \mathcal{L}^n(V))
\leq C_1 \left( \varepsilon^{-1/(n-1)} \int_V H_f^{\kappa,1} \, d\mathcal{L}^n + \varepsilon \kappa(\Omega) + \varepsilon \mathcal{L}^n(V) \right).
\] (5.11)
Thus $f \in \text{BV}_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^n)$; for this, we can simply choose $\varepsilon = 1$. On the other hand, using the fact that we can choose arbitrarily small $\varepsilon > 0$, we also see that $\|Df\|$ is absolutely continuous with respect to $\mathcal{L}^n$. In the Euclidean BV theory (see e.g. [1, Section 3]), this implies that in fact $f \in W_{\text{loc}}^{1,1}(\mathbb{R}^n; \mathbb{R}^n)$.

The measure $\kappa$ given in (5.10) charges the set $E$, which is $\sigma$-finite with respect to $\mathcal{H}^{n-1}$, just like the approximate jump set $J_f$ of a BV function; recall (2.12). Thus the first line of (5.11) appears to give an upper bound for the jump part of the variation measure, $\|Df\| = |f^+ - f^-| \, d\mathcal{H}^{n-1} \lfloor_{J_f}$ (recall (2.13)). In conclusion, the exceptional set $E$ can be interpreted to correspond to the approximate jump set $J_f$ of the BV function $f$. Since $f$ is now continuous by assumption, however, the approximate jump set is empty and so we actually obtained $f \in W_{\text{loc}}^{1,1}(\mathbb{R}^n; \mathbb{R}^n)$.

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