Diophantine approximation with one prime of the form
\[ p = x^2 + y^2 + 1 \]

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Abstract. Let \( \varepsilon > 0 \) be a small constant. We prove that whenever \( \eta \) is real and constants \( \lambda_i \) satisfy some necessary conditions, then there exist infinitely many prime triples \( p_1, p_2, p_3 \) satisfying the inequality \( |\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta| < \varepsilon \) and such that \( p_3 = x^2 + y^2 + 1 \).

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1 Notations

The letter \( p \) with or without subscript will always denote prime numbers. We denote by \((m, n)\) the greatest common divisor of \( m \) and \( n \). Moreover, \( e(t) = \exp(2\pi it) \). As usual, \( \varphi(d) \) is Euler’s function, \( r(d) \) is the number of solutions of the equation \( d = m_1^2 + m_2^2 \) in integers \( m_j \), \( \chi(d) \) is the nonprincipal character modulo 4, and \( L(s, \chi) \) is the corresponding Dirichlet \( L \)-function. We write a congruence \( m \equiv n \pmod{d} \) as \( m \equiv n \pmod{d} \).

We denote by \( \lfloor t \rfloor \), \( \lceil t \rceil \), and \( \{ t \} \) the floor function, the ceiling function, and the fractional part function of \( t \), respectively. Let \( \lambda_1, \lambda_2, \lambda_3 \) be nonzero real numbers, not all of the same sign, such that \( \lambda_1/\lambda_2 \) is irrational. Then there are infinitely many different convergents \( a_0/q_0 \) to its continued fraction with

\[ \left| \frac{\lambda_1}{\lambda_2} - \frac{a_0}{q_0} \right| < \frac{1}{q_0^2}, \quad (a_0, q_0) = 1, \ a_0 \neq 0, \quad (1.1) \]

and \( q_0 \) arbitrarily large. Denote

\[ q_0^2 = \frac{X}{(\log X)^{22}}, \]

\[ D = \frac{X^{1/2}}{(\log X)^{52}}, \quad (1.2) \]

\[ \Delta = \frac{(\log X)^{23}}{X}, \quad (1.3) \]
\[ \theta_0 = \frac{1}{2} - \frac{1}{4} e \log 2 = 0.0289 \ldots, \] (1.4)

\[ \varepsilon = \frac{\log \log X}{\log X} \theta_0, \] (1.5)

\[ H = \frac{\log^2 X}{\varepsilon}, \] (1.6)

\[ S_{l,d,r}(\alpha, X) = \sum_{p \in J, \ p \equiv d \mod{(d)}} e(\alpha p) \log p, \quad J \subset (\lambda_0, X], \ 0 < \lambda_0 < 1, \] (1.7)

\[ S(\alpha, X) = S_{1,1;\lambda, X}(\alpha, X), \] (1.8)

\[ I_J(\alpha, X) = \int_J e(\alpha y) \, dy; \] (1.9)

\[ I(\alpha, X) = I_{\lambda, X}(\alpha, X), \] (1.10)

\[ E(x, q, a) = \sum_{p \leq x, \ p \equiv a \mod{(q)}} \log p - \frac{x}{\varphi(q)}. \] (1.11)

## 2 Introduction and statement of the result

In 1960, Linnik [10] has proved that there exist infinitely many prime numbers of the form \( p = x^2 + y^2 + 1 \), where \( x \) and \( y \) are integers. More precisely, he has proved the asymptotic formula

\[ \sum_{p \leq X} r(p - 1) = \pi \prod_{p > 2} \left( 1 + \frac{\chi(p)}{p(p - 1)} \right) \frac{X}{\log X} + O\left( \frac{X \log \log X}{\log X} \right), \] where \( \theta_0 \) is defined by (1.4).

Seven years later, Baker [1] showed that whenever \( \lambda_1, \lambda_2, \lambda_3 \) are nonzero real numbers, not all of the same sign, \( \lambda_1/\lambda_2 \) is irrational, and \( \eta \) is real, there are infinitely many prime triples \( p_1, p_2, p_3 \) such that

\[ |\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta| < \xi, \] (2.1)

where \( \xi = (\log \max p_j)^{-A} \) with arbitrary large constant \( A > 0 \).

Latter, the right-hand side of (2.1) was sharpened several times, and the best result up to now belongs to Matomäki [11] with \( \xi = (\max p_j)^{-2/9 + \delta} \) and \( \delta > 0 \). After Matomäki, inequality (2.1) was solved with prime numbers of a special form.

Let \( P_1 \) is a number with at most \( l \) prime factors. The author and Todorova [3] and the author [4] proved that (2.1) has a solution in primes \( p_i \) such that \( p_i + 2 = P_i, \ i = 1, 2, 3 \).

Very recently, the author [6] showed that (2.1) has a solution in Piatetski–Shapiro primes \( p_1, p_2, p_3 \) of type \( \gamma \in (37/38, 1) \).

In this paper, we continue to solve inequality (2.1) with prime numbers of a special type. More precisely, we prove the solvability of (2.1) with Linnik primes. Thus we establish the following theorem.

**Theorem 1.** Suppose that \( \lambda_1, \lambda_2, \lambda_3 \) are nonzero real numbers, not all of the same sign, \( \lambda_1/\lambda_2 \) is irrational, and \( \eta \) is real. Then there exist infinitely many triples of primes \( p_1, p_2, p_3 \) such that

\[ |\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta| < \frac{(\log \max p_j)^7}{(\log \max p_j)^{\theta_0}} \]

and \( p_3 = x^2 + y^2 + 1 \), where \( \theta_0 \) is defined by (1.4).
In addition, we have the following challenge.

**Conjecture 1.** Let $\varepsilon > 0$ be a small constant. Suppose that $\lambda_1, \lambda_2, \lambda_3$ are nonzero real numbers, not all of the same sign, $\lambda_1/\lambda_2$ is irrational, and $\eta$ is real. Then there exist infinitely many triples of primes $p_1, p_2, p_3$ such that $|\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta| < \varepsilon$ and $p_1 = x_1^2 + y_1^2 + 1$, $p_2 = x_2^2 + y_2^2 + 1$, and $p_3 = x_3^2 + y_3^2 + 1$.

The author wishes success to all young researchers in attacking this hard hypothesis.

## 3 Preliminary lemmas

**Lemma 1.** Let $\varepsilon > 0$ and $k \in \mathbb{N}$. There exists a $k$ times continuously differentiable function $\theta$ such that

$$\theta(y) = 1 \text{ for } |y| \leq \frac{3\varepsilon}{4},$$

$$0 < \theta(y) < 1 \text{ for } \frac{3\varepsilon}{4} < |y| < \varepsilon,$$

$$\theta(y) = 0 \text{ for } |y| \geq \varepsilon,$$

and its Fourier transform

$$\Theta(x) = \int_{-\infty}^{\infty} \theta(y)e(-xy)\,dy$$

satisfies the inequality

$$|\Theta(x)| \leq \min\left(\frac{7\varepsilon}{4}, \frac{1}{\pi|x|}, \frac{1}{2\pi|x|}\left(\frac{k}{8\varepsilon}\right)^k\right).$$

**Proof.** See [12]. \(\square\)

**Lemma 2.** Let $|\alpha| \leq \Delta$. Then for the sum (1.8) and integral (1.10), we have the asymptotic formula

$$S(\alpha, X) = I(\alpha, X) + O\left(\frac{X}{e^{(\log X)^{1/5}}}\right).$$

**Proof.** Arguing as in [14, Lemma 14], we establish the lemma. \(\square\)

**Lemma 3 [Bombieri–Vinogradov].** For any $C > 0$, we have the inequality

$$\sum_{q \leq X^{1/2}/(\log X)^C} \max_{y \leq X} \max_{(a,q) = 1} |E(y, q, a)| \ll \frac{X}{(\log X)^C}.$$  

**Proof.** See [2, Chap. 28]. \(\square\)

**Lemma 4.** Let $\alpha \in \mathbb{R}$, $a \in \mathbb{Z}$, and $q \in \mathbb{N}$ be such that $|\alpha - a/q| \leq 1/q^2$ and $(a, q) = 1$. Let

$$\Sigma(\alpha, X) = \sum_{p \leq X} e(\alpha p) \log p.$$  

Then

$$\Sigma(\alpha, X) \ll (Xq^{-1/2} + X^{4/5} + X^{1/2}q^{1/2}) \log^4 X.$$  

**Proof.** See [9, Thm. 13.6]. \(\square\)

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Lemma 5. Let \( k \in \mathbb{N} \), \( l, a, b \in \mathbb{Z} \), and \( ab \neq 0 \). Let \( x \) and \( y \) be real numbers satisfying \( k < y \leq x \). Then

\[
\# \{ p: x - y < p \leq x, \, p \equiv l \pmod{k}, \, ap + b = p' \} \ll \prod_{p|kab} \left( 1 - \frac{1}{p} \right)^{-1} \frac{y}{\varphi(k) \log^{2}(y/k)}.
\]

**Proof.** See [7, Chap. 2, Cor. 2.4.1]. □

The next two lemmas are due to C. Hooley.

Lemma 6. For any constant \( \omega > 0 \), we have

\[
\sum_{p \leq X} \left| \sum_{d|p-1} \chi(d) \right|^2 \ll \frac{X(\log \log X)^{7}}{\log X},
\]

where the constant in the Vinogradov symbol depends on \( \omega > 0 \).

Lemma 7. Let \( \omega > 0 \), and let \( F_{\omega}(X) \) be the number of primes \( p \leq X \) such that \( p - 1 \) has a divisor in the interval \( (\sqrt{X}(\log X)^{-\omega}, \sqrt{X}(\log X)^{\omega}) \). Then

\[
F_{\omega}(X) \ll \frac{X(\log \log X)^{3}}{(\log X)^{1+2\theta_{0}}},
\]

where \( \theta_{0} \) is defined by (1.4), and the constant in the Vinogradov symbol depends only on \( \omega > 0 \).

The proofs of very similar results are available in [8, Chap. 5].

4 Outline of the proof

Consider the sum

\[
\Gamma(X) = \sum_{\lambda_{0}X < p_{1}, p_{2}, p_{3} \leq X, \, \vert \lambda_{1}p_{1} + \lambda_{2}p_{2} + \lambda_{3}p_{3} + \eta \vert < \varepsilon} r(p_{3} - 1) \log p_{1} \log p_{2} \log p_{3}.
\]  

Any nontrivial lower bound of \( \Gamma(X) \) implies the solvability of \( \vert \lambda_{1}p_{1} + \lambda_{2}p_{2} + \lambda_{3}p_{3} + \eta \vert < \varepsilon \) in primes such that \( p_{3} = x^{2} + y^{2} + 1 \).

We have

\[
\Gamma(X) \geq \Gamma_{0}(X),
\]

where

\[
\Gamma_{0}(X) = \sum_{\lambda_{0}X < p_{1}, p_{2}, p_{3} \leq X} r(p_{3} - 1)\theta(\lambda_{1}p_{1} + \lambda_{2}p_{2} + \lambda_{3}p_{3} + \eta) \log p_{1} \log p_{2} \log p_{3}.
\]

Using (4.3) and the well-known identity \( r(n) = 4 \sum_{d|n} \chi(d) \), we write

\[
\Gamma_{0}(X) = 4(\Gamma_{1}(X) + \Gamma_{2}(X) + \Gamma_{3}(X)),
\]
We will estimate $I_1(X)$ and $I_3(X)$, we have to consider the sum

$$I_{l,d,j}(X) = \sum_{\lambda_0 X < p_1, p_2, p_\lambda \leq X} \theta(\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_\lambda + \eta) \log p_1 \log p_2 \log p_3,$$

where $d$ and $l$ are coprime natural numbers, and $J \subset (\lambda_0 X, X]$ is an interval. If $J = (\lambda_0 X, X]$, then we write $I_{l,d}(X)$ for simplicity.

Using the inverse Fourier transform for the function $\theta$, we get

$$I_{l,d,j}(X) = \sum_{\lambda_0 X < p_1, p_2, p_\lambda \leq X} \int_{-\infty}^{\infty} \Theta(t)e((\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_\lambda + \eta)t) \, dt$$

$$= \int_{-\infty}^{\infty} \Theta(t)S(\lambda_1 t, X)S(\lambda_2 t, X)S_{l,d,j}(\lambda_3 t, X)e(\eta t) \, dt.$$

We decompose $I_{l,d,j}(X)$ as follows

$$I_{l,d,j}(X) = I_{l,d,j}^{(1)}(X) + I_{l,d,j}^{(2)}(X) + I_{l,d,j}^{(3)}(X),$$

where

$$I_{l,d,j}^{(1)}(X) = \int_{|t| < \Delta} \Theta(t)S(\lambda_1 t, X)S(\lambda_2 t, X)S_{l,d,j}(\lambda_3 t, X)e(\eta t) \, dt,$$

$$I_{l,d,j}^{(2)}(X) = \int_{\Delta \leq |t| \leq H} \Theta(t)S(\lambda_1 t, X)S(\lambda_2 t, X)S_{l,d,j}(\lambda_3 t, X)e(\eta t) \, dt,$$

$$I_{l,d,j}^{(3)}(X) = \int_{|t| > H} \Theta(t)S(\lambda_1 t, X)S(\lambda_2 t, X)S_{l,d,j}(\lambda_3 t, X)e(\eta t) \, dt.$$

We will estimate $I_{l,d,j}^{(1)}(X)$, $I_{l,d,j}^{(3)}(X)$, $\Gamma_3(X)$, $\Gamma_2(X)$, and $\Gamma_1(X)$ in Sections 5, 6, 7, 8, and 9, respectively. In Section 10, we will complete the proof of Theorem 1.
5 Asymptotic formula for $I_{t,d;J}^{(1)}(X)$

Replace

$$S_1 = S(\lambda_1 t, X), \quad S_2 = S(\lambda_2 t, X), \quad (5.1)$$
$$S_3 = S_{t,d;J}(\lambda_3 t, X), \quad (5.2)$$
$$I_1 = I(\lambda_1 t, X), \quad I_2 = I(\lambda_2 t, X), \quad (5.3)$$
$$I_3 = \frac{1}{\varphi(d)} I_J(\lambda_3 t, X). \quad (5.4)$$

We use the identity

$$S_1 S_2 S_3 = I_1 I_2 I_3 + (S_1 - I_1) I_2 I_3 + S_1 (S_2 - I_2) I_3 + S_1 S_2 (S_3 - I_3). \quad (5.5)$$

From (1.3), (1.7), (1.9), (1.11), (5.2), (5.4), and Abel’s summation formula it follows that

$$S_3 = I_3 + O\left( \Delta X \max_{y \in (\lambda_0 X, X]} |E(y, d, l)| \right). \quad (5.6)$$

Now using (1.8)–(1.10), (5.1)–(5.6), Lemma 2, and the trivial estimations

$$S_1, S_2, I_2 \ll X, \quad I_3 \ll \frac{X}{\varphi(d)},$$

we get

$$S_1 S_2 S_3 - I_1 I_2 I_3 \ll X^3 \left( \frac{1}{\varphi(d) e^{(\log X)^{1/5}}} + \Delta \max_{y \in (\lambda_0 X, X]} |E(y, d, l)| \right). \quad (5.7)$$

Put

$$\Phi(X) = \frac{1}{\varphi(d)} \int_{|t| < \Delta} \Theta(t) I(\lambda_1 t, X) I(\lambda_2 t, X) I_J(\lambda_3 t, X) e(\eta t) \, dt. \quad (5.8)$$

Taking into account (4.10), (5.7), (5.8), and Lemma 1, we find

$$I_{t,d;J}^{(1)}(X) - \Phi(X) \ll \varepsilon \Delta X^3 \left( \frac{1}{\varphi(d) e^{(\log X)^{1/5}}} + \Delta \max_{y \in (\lambda_0 X, X]} |E(y, d, l)| \right). \quad (5.9)$$

On the other hand, for the integral (5.8), we write

$$\Phi(X) = \frac{1}{\varphi(d)} B_J(X) + \Omega, \quad (5.10)$$

where

$$B_J(X) = \int_{\lambda_0 X}^{X} \int_{\lambda_0 X}^{X} \int_{\lambda_0 X}^{X} \theta(\lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3 + \eta) \, dy_1 \, dy_2 \, dy_3$$

and

$$\Omega \ll \frac{1}{\varphi(d)} \int_{\Delta} \left| \Theta(t) \right| \left| I(\lambda_1 t, X) I(\lambda_2 t, X) I_J(\lambda_3 t, X) \right| \, dt. \quad (5.11)$$
By (1.9) and (1.10) we get
\[ I_J(\alpha, X) \ll \frac{1}{|\alpha|}, \quad I(\alpha, X) \ll \frac{1}{|\alpha|}. \] (5.12)

Using (5.11), (5.12), and Lemma 1, we deduce
\[ \Omega \ll \frac{\varepsilon}{\varphi(d)\Delta^2}. \] (5.13)

Bearing in mind (1.3), (5.9), (5.10), and (5.13), we find
\[ I^{(1)}_{l,d;J}(X) = \frac{1}{\varphi(d)} B_J(X) + \mathcal{O}\left(\varepsilon \Delta^2 X^3 \max_{y \in (\lambda_0 X, X]} |E(y, d, l)| \right) + \mathcal{O}\left(\frac{\varepsilon}{\varphi(d)\Delta^2}\right). \] (5.14)

\section{Upper bound of $I^{(3)}_{l,d;J}(X)$}

By (1.7), (1.8), (4.12), and Lemma 1 it follows that
\[ I^{(3)}_{l,d;J}(X) \ll \frac{X^3 \log X}{d} \int_{H}^{\infty} \frac{1}{t} \left( \frac{k}{2\pi t\varepsilon / 8} \right)^k \, dt = \frac{X^3 \log X}{d k} \left( \frac{4k}{\pi \varepsilon H} \right)^k. \] (6.1)

Choosing $k = \lfloor \log X \rfloor$, from (1.6) and (6.1) we obtain
\[ I^{(3)}_{l,d;J}(X) \ll \frac{1}{d}. \] (6.2)

\section{Upper bound of $\Gamma_3(X)$}

Since
\[ \sum_{d|p_3-1 \quad d \geq X/D} \chi(d) = \sum_{m|p_3-1 \quad m \leq (p_3-1)D/X} \chi\left(\frac{p_3-1}{m}\right) = \sum_{j=\pm 1} \chi(j) \sum_{m|p_3-1 \quad m \leq (p_3-1)D/X \quad (p_3-1)/m \equiv j \pmod{4}} 1, \]
from (4.7) and (4.8) it follows that
\[ \Gamma_3(X) = \sum_{m < D} \sum_{j=\pm 1} \chi(j) I_{1+jm,4m;J_m}(X), \]
where $J_m = \{ \max\{1 + mX/D, \lambda_0 X\}, X]\}$. The last formula and (4.9) yield
\[ \Gamma_3(X) = \Gamma^{(1)}_3(X) + \Gamma^{(2)}_3(X) + \Gamma^{(3)}_3(X), \] (7.1)
where
\[ \Gamma^{(i)}_3(X) = \sum_{m < D} \sum_{j=\pm 1 \quad 2|m} \chi(j) I^{(i)}_{1+jm,4m;J_m}(X), \quad i = 1, 2, 3. \] (7.2)
7.1 Estimation of $\Gamma_3^{(1)}(X)$

From (5.14) and (7.2) we deduce

$$
\Gamma_3^{(1)}(X) = \Gamma^* + O(\varepsilon \Delta^2 X^3 \Sigma_1) + O\left(\frac{\varepsilon}{\Delta^2} \Sigma_2\right),
$$

where

$$
\Gamma^* = B_{j}(X) \sum_{m<D} \frac{\varphi(4m)}{2|m} \sum_{j=\pm 1} \chi(j),
$$

$$
\Sigma_1 = \sum_{m<D, 2|m} \max_{y \in (\lambda_0 X, X]} \left| E(y, 4m, 1 + jm) \right|, \quad \Sigma_2 = \sum_{m<D} \frac{1}{\varphi(4m)}. \tag{7.4}
$$

From the properties of $\chi(k)$ we have that

$$
\Gamma^* = 0. \tag{7.5}
$$

By (1.2), (7.4)1, and Lemma 3 we get

$$
\Sigma_1 \ll \frac{X}{(\log X)^{47}}. \tag{7.6}
$$

It is well known that

$$
\Sigma_2 \ll \log X. \tag{7.7}
$$

Bearing in mind (1.3), (7.3), (7.5), (7.6), and (7.7), we obtain

$$
\Gamma_3^{(1)}(X) \ll \frac{\varepsilon X^2}{\log X}. \tag{7.8}
$$

7.2 Estimation of $\Gamma_3^{(2)}(X)$

From (4.11) and (7.2) we have

$$
\Gamma_3^{(2)}(X) = \int_{\Delta \leq |t| \leq H} \Theta(t) S(\lambda_1 t, X) S(\lambda_2 t, X) K(\lambda_3 t, X) e(\eta t) \, dt, \tag{7.9}
$$

where

$$
K(\lambda_3 t, X) = \sum_{m<D} \sum_{j=\pm 1} \chi(j) S_{1+jm, 4m; t_m}(\lambda_3 t). \tag{7.10}
$$

Suppose that

$$
\left| \alpha - \frac{a}{q} \right| \leq \frac{1}{q^2}, \quad (a, q) = 1, \quad q \in \left(\log X\right)^{22}, \frac{X}{(\log X)^{22}}. \tag{7.11}
$$

Then (1.8), (7.11), and Lemma 4 give us

$$
S(\alpha, X) \ll \frac{X}{(\log X)^{7}}. \tag{7.12}
$$
Let
\[ \mathcal{S}(t, X) = \min\{|S(\lambda_1 t, X)|, |S(\lambda_2 t, X)|\}. \] (7.13)

Using (1.1), (7.12), and (7.13) and working similarly to [3, Lemma 6], we establish that there exists a sequence of real numbers \(X_1, X_2, \ldots \to \infty\) such that
\[ \mathcal{S}(t, X_j) \ll \frac{X_j}{(\log X_j)^t}, \quad j = 1, 2, \ldots. \] (7.14)

Using (7.9), (7.13), (7.14), and Lemma 1, we obtain
\[
\Gamma_3^{(2)}(X_j) \ll \varepsilon \int_{\Delta \leq |t| \leq H} \mathcal{S}(t, X_j)\left(|S(\lambda_1 t, X_j)K(\lambda_3 t, X_j)| + |S(\lambda_2 t, X_j)K(\lambda_3 t, X_j)|\right) dt
\ll \varepsilon \int_{\Delta \leq |t| \leq H} \mathcal{S}(t, X_j)(|S(\lambda_1 t, X_j)|^2 + |S(\lambda_2 t, X_j)|^2 + |K(\lambda_3 t, X_j)|^2) dt
\ll \varepsilon \frac{X_j}{(\log X_j)^t} (T_1 + T_2 + T_3),
\] (7.15)

where
\[
T_k = \int_{\Delta} |S(\lambda_k t, X_j)|^2 dt, \quad k = 1, 2, \quad T_3 = \int_{\Delta} |K(\lambda_3 t, X_j)|^2 dt.
\]

From (1.3), (1.6), (1.8), and (7.15) after straightforward computations, we get
\[ T_k \ll HX_j \log X_j, \quad k = 1, 2. \] (7.16)

Taking into account (1.3), (1.6), (7.10), and (7.15) and proceeding as in [5, p. 14], we find
\[ T_3 \ll HX_j \log^3 X_j. \] (7.17)

By (1.5), (1.6), (7.16), and (7.17) we deduce
\[ \Gamma_3^{(2)}(X_j) \ll \frac{X_j}{(\log X_j)^t} X_j \log^5 X_j = \frac{X_j^2}{(\log X_j)^2} \ll \varepsilon \frac{X_j^2}{\log X_j}. \] (7.18)

### 7.3 Estimation of \( \Gamma_3^{(3)}(X) \)

From (6.2) and (7.2) we have
\[ \Gamma_3^{(3)}(X) \ll \sum_{d < D} \frac{1}{d} \ll \log X. \] (7.19)

### 7.4 Estimation of \( \Gamma_3(X) \)

Summarizing (7.1), (7.8), (7.18), and (7.19), we get
\[ \Gamma_3(X_j) \ll \frac{\varepsilon X_j^2}{\log X_j}. \] (7.20)
8 Upper bound of $\Gamma_2(X)$

We denote by $\mathcal{F}(X)$ the set of all primes $\lambda_0 X < p \leq X$ such that $p - 1$ has a divisor belonging to the interval $(D, X/D)$. The inequality $xy \leq x^2 + y^2$ and (4.6) yield

$$\Gamma_2(X)^2 \ll (\log X)^6 \left| \sum_{\lambda_0 X < p_1, \ldots, p_6 \leq X} \left| \sum_{d | p_3 - 1 \atop \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \leq X} \chi(d) \right| \sum_{t | p_6 - 1 \atop D < t < X/D} \chi(t) \right| \sum_{\lambda_0 X < p_1, p_2, p_4, p_5 \leq X} \left| \sum_{\lambda_1, \lambda_2, \lambda_3, \lambda_4 \leq X} \chi(d) \right|^2 \sum_{D < d < X/D} \chi(d) \right| \sum_{\lambda_0 X < p_1, p_2, p_4, p_5 \leq X} 1. \right.$$ 

The summands in the last sum for which $p_3 = p_6$ can be estimated with $O(X^{3+\varepsilon})$. Therefore

$$\Gamma_2(X)^2 \ll (\log X)^6 \Sigma_0 + X^{3+\varepsilon}, \quad (8.1)$$

where

$$\Sigma_0 = \sum_{\lambda_0 X < p_3 \leq X} \left| \sum_{d | p_3 - 1 \atop D < d < X/D} \chi(d) \right|^2 \sum_{\lambda_0 X < p_3 \leq X} \sum_{\lambda_1, \lambda_2, \lambda_3, \lambda_4 \leq X} \chi(d) \sum_{\lambda_0 X < p_3 \leq X} 1. \quad (8.2)$$

Since $\lambda_1, \lambda_2, \lambda_3$ are not all of the same sign, without loss of generality, we can assume that $\lambda_1 > 0$, $\lambda_2 > 0$, and $\lambda_3 < 0$. Now let us consider the set

$$\Psi(X) = \{ (p_1, p_2) : |\lambda_1 p_1 + \lambda_2 p_2 + D| < \varepsilon, \lambda_0 X < p_1, p_2 \leq X, D \neq X \} \quad (8.3)$$

We will find the upper bound of the cardinality of $\Psi(X)$.

Using

$$|\lambda_1 p_1 + \lambda_2 p_2 + D| < \varepsilon, \quad (8.4)$$

we write

$$\left| \frac{\lambda_1}{\lambda_2} p_1 + p_2 + \frac{D}{\lambda_2} \right| < \frac{\varepsilon}{\lambda_2}, \quad (8.5)$$

Since $\lambda_2$ is fixed, for sufficiently large $X$, we have that $\varepsilon / \lambda_2$ is sufficiently small. Therefore (8.5) implies

Case 1. $[(\lambda_1 / \lambda_2) p_1 + p_2] = [-D / \lambda_2]$ or
Case 2. $[(\lambda_1 / \lambda_2) p_1 + p_2] = [D / \lambda_2]$ or
Case 3. $[(\lambda_1 / \lambda_2) p_1 + p_2] = [-D / \lambda_2]$ or
Case 4. $[(\lambda_1 / \lambda_2) p_1 + p_2] = [D / \lambda_2]$.

We will consider only Case 1. Cases 2, 3, and 4 are treated similarly. In Case 1, we have

$$\left| \frac{\lambda_1}{\lambda_2} p_1 + p_2 = \left| -\frac{D}{\lambda_2} \right|, \quad (8.5)$$
thus
\[
\left\lfloor \left( \frac{\lambda_1}{\lambda_2} \right) + \left\{ \frac{\lambda_1}{\lambda_2} \right\} \right\rfloor p_1 + p_2 = \left\lfloor \frac{D}{\lambda_2} \right\rfloor,
\]
and therefore
\[
\left\lfloor \frac{\lambda_1}{\lambda_2} \right\rfloor p_1 + p_2 = \left\lfloor \frac{D}{\lambda_2} \right\rfloor - \left\{ \frac{\lambda_1}{\lambda_2} p_1 \right\}. \tag{8.6}
\]

Bearing in mind definition (8.3), we deduce that there exist constants \( C_1 > 0 \) and \( C_2 > 0 \) such that
\[
C_1 X \leq \left\lfloor \frac{D}{\lambda_2} \right\rfloor - \left\{ \frac{\lambda_1}{\lambda_2} p_1 \right\} \leq C_2 X.
\]
Consequently, there exists a constant \( C \in [C_1, C_2] \) such that
\[
\left\lfloor \frac{\lambda_1}{\lambda_2} \right\rfloor p_1 + p_2 = C X. \tag{8.7}
\]
Equalities (8.6) and (8.7) give us
\[
\left\lfloor \frac{\lambda_1}{\lambda_2} \right\rfloor p_1 + p_2 = C X \tag{8.8}
\]
for some constant \( C \in [C_1, C_2] \).

We established that the number of solutions of inequality (8.4) is less than the number of all solutions of all equations denoted by (8.8). According to Lemma 5, for any fixed \( C \in [C_1, C_2] \) participating in (8.8), we have
\[
\# \left\{ (p_1, p_2) : \left\lfloor \frac{\lambda_1}{\lambda_2} \right\rfloor p_1 + p_2 = C X, \lambda_0 X < p_1, p_2 \leq X \right\} \ll \frac{X \log \log X}{\log^2 X}. \tag{8.9}
\]
Taking into account that \( C \leq C_2 \), from (8.3), (8.4), and (8.9) we find
\[
\# \Psi(X) \ll \frac{X \log \log X}{\log^2 X}. \tag{8.10}
\]
Formulas (8.2), (8.3), and inequality (8.10) yield
\[
\Sigma_0 \ll \frac{X^2}{\log^4 X} (\log \log X)^2 \Sigma' \Sigma'', \quad \Sigma' = \sum_{\lambda_0 X < p \leq X} \left| \sum_{D < d < X/D} \chi(d) \right|^2, \quad \Sigma'' = \sum_{\lambda_0 X < p \leq X} \sum_{p \in F} 1. \tag{8.11}
\]
Applying Lemma 6, we obtain
\[
\Sigma' \ll \frac{X (\log \log X)^7}{\log X}. \tag{8.12}
\]
Using Lemma 7, we get
\[
\Sigma'' \ll \frac{X (\log \log X)^3}{(\log X)^{1+2\theta_0}}, \tag{8.13}
\]
where \( \theta_0 \) is denoted by (1.4).
We are now in a good position to estimate the sum $\Gamma_2(X)$. From (8.1) and (8.11)–(8.13) it follows that

$$\Gamma_2(X) \ll \frac{X^2(\log \log X)^6}{(\log X)^\theta_0} = \varepsilon X^2 \log \log X. \quad (8.14)$$

### 9 Lower bound for $\Gamma_1(X)$

From (4.5), (4.8), and (4.9) we deduce

$$\Gamma_1(X) = \Gamma_1^{(1)}(X) + \Gamma_1^{(2)}(X) + \Gamma_1^{(3)}(X), \quad (9.1)$$

where

$$\Gamma_1^{(i)}(X) = \sum_{d \leq D} \chi(d)I_1^{(i)}(X), \quad i = 1, 2, 3. \quad (9.2)$$

#### 9.1 Estimation of $\Gamma_1^{(1)}(X)$

Using formula (5.14) for $J = (\lambda_0 X, X]$ and (9.2) and treating the reminder term in the same way as for $\Gamma_3^{(1)}(X)$, we find

$$\Gamma_1^{(1)}(X) = B(X) \sum_{d \leq D} \frac{\chi(d)}{\varphi(d)} + O\left(\frac{\varepsilon X^2}{\log X}\right), \quad (9.3)$$

where

$$B(X) = \int_{\lambda_0 X}^{X} \int_{\lambda_0 X}^{X} \int_{\lambda_0 X}^{X} \theta(\lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3 + \eta) \, dy_1 \, dy_2 \, dy_3. \quad (9.4)$$

According to [3, Lemma 4], we have

$$B(X) \gg \varepsilon X^2. \quad (9.5)$$

Denote

$$\Sigma = \sum_{d \leq D} f(d), \quad f(d) = \frac{\chi(d)}{\varphi(d)}. \quad (9.6)$$

We have

$$f(d) \ll d^{-1} \log \log(10d) \quad (9.7)$$

with absolute constant in the Vinogradov symbol. Hence the corresponding Dirichlet series $F(s) = \sum_{d=1}^{\infty} f(d)/d^s$ is absolutely convergent in $\text{Re}(s) > 0$. On the other hand, $f(d)$ is a multiplicative with respect to $d$, and applying Euler’s identity, we obtain

$$F(s) = \prod_p T(p, s), \quad T(p, s) = 1 + \sum_{l=1}^{\infty} \frac{\chi(p)}{p^{ls}}. \quad (9.8)$$

By (9.6) and (9.8) we establish that

$$T(p, s) = \left(1 - \frac{\chi(p)}{p^{s+1}}\right)^{-1} \left(1 + \frac{\chi(p)}{p^{s+1}(p-1)}\right).$$
Hence we find

\[ F(s) = L(s + 1, \chi) \mathcal{N}(s), \quad (9.9) \]

where \( L(s + 1, \chi) \) is the Dirichlet series corresponding to the character \( \chi \), and

\[ \mathcal{N}(s) = \prod_p \left( 1 + \frac{\chi(p)}{p^s + 1(p - 1)} \right). \quad (9.10) \]

From the properties of the \( L \)-functions it follows that \( F(s) \) has an analytic continuation to \( \text{Re}(s) > -1 \). It is well known that

\[ L(s + 1, \chi) \ll 1 + \left| \text{Im}(s) \right|^{1/6} \quad \text{for} \quad \text{Re}(s) \geq -\frac{1}{2}. \quad (9.11) \]

Moreover,

\[ \mathcal{N}(s) \ll 1. \quad (9.12) \]

By (9.9), (9.11), and (9.12) we deduce

\[ F(s) \ll X^{1/6} \quad \text{for} \quad \text{Re}(s) \geq -\frac{1}{2}, \left| \text{Im}(s) \right| \leq X. \quad (9.13) \]

Using (9.6), (9.7), and Perron’s formula given by Tenenbaum [13, Chap. II.2], we obtain

\[ \Sigma = \frac{1}{2\pi i} \int_{\kappa - iX}^{\kappa + iX} F(s) \frac{D^s}{s} \, ds + \mathcal{O} \left( \sum_{t=1}^{\infty} \frac{D^s \log \log(10t)}{t^{1+\kappa}(1 + X \log 2X)} \right), \quad (9.14) \]

where \( \kappa = 1/10 \). It is easy to see that the error term in (9.14) is \( \mathcal{O}(X^{-1/20}) \). Applying the residue theorem we see that the main term in (9.14) is equal to

\[ F(0) + \frac{1}{2\pi i} \left( \int_{1/10-iX}^{1/2-iX} + \int_{1/2-iX}^{1/2+iX} + \int_{1/2+iX}^{1/10+iX} \right) F(s) \frac{D^s}{s} \, ds. \]

From (9.13) it follows that the contribution from the above integrals is \( \mathcal{O}(X^{-1/20}) \).

Hence

\[ \Sigma = F(0) + \mathcal{O}(X^{-1/20}). \quad (9.15) \]

Using (9.9), we get

\[ F(0) = \frac{\pi}{4} \mathcal{N}(0). \quad (9.16) \]

Bearing in mind (9.3), (9.6), (9.10), (9.15), and (9.16), we find a new expression for \( \Gamma_1^{(1)}(X) \):

\[ \Gamma_1^{(1)}(X) = \frac{\pi}{4} \prod_p \left( 1 + \frac{\chi(p)}{p(p - 1)} \right) B(X) + \mathcal{O} \left( \frac{\varepsilon X^2}{\log X} \right) + \mathcal{O}(B(X)X^{-1/20}). \quad (9.17) \]

Now (9.5) and (9.17) yield

\[ \Gamma_1^{(1)}(X) \gg \varepsilon X^2. \quad (9.18) \]
9.2 Estimation of $\Gamma_1^{(2)}(X)$

Arguing as in the estimation of $\Gamma_3^{(2)}(X)$, we get

$$\Gamma_1^{(2)}(X) \ll \frac{\varepsilon X^2}{\log X}. \quad (9.19)$$

9.3 Estimation of $\Gamma_1^{(3)}(X)$

From (6.2) and (9.2) we have

$$\Gamma_1^{(3)}(X) \ll \sum_{m<D} \frac{1}{d} \ll \log X. \quad (9.20)$$

9.4 Estimation of $\Gamma_1(X)$

Summarizing (9.1), (9.18), (9.19), and (9.20), we deduce

$$\Gamma_1(X) \gg \varepsilon X^2. \quad (9.21)$$

10 Proof of Theorem 1

Taking into account (1.5), (4.2), (4.4), (7.20), (8.14), and (9.21), we obtain

$$\Gamma(X_j) \gg \varepsilon X_j^2 = \frac{X_j^2(\log \log X_j)^7}{(\log X_j)^\theta}. \quad (10.1)$$

The last lower bound implies

$$\Gamma(X_j) \rightarrow \infty \quad \text{as} \quad X_j \rightarrow \infty. \quad (10.1)$$

Bearing in mind (4.1) and (10.1), we establish the theorem.

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