A Note on Arithmetic Cohomologies for Number Fields

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In this short article, we develop a new cohomology theory for number fields as a part of our Program for Geometric Arithmetic.

1. Arithmetic Cohomology Groups

Let $F$ be a number field with discriminant $\Delta_F$. Denote its (normalized) absolute values by $S_F$, and write $S_F = S_{\text{fin}} \cup S_{\infty}$, where $S_{\infty}$ denotes the collection of all archimedean valuations. For simplicity, we use $v$ (resp. $\sigma$) to denote elements in $S_{\text{fin}}$ (resp. $S_{\infty}$).

Denote by $A = \mathcal{A}_F$ the ring of adeles of $F$, and $GL_r(A)$ the rank $r$ general linear group over $A$, and write $A := A_{\text{fin}} \oplus A_{\infty}$ and $GL_r(A) := GL_r(A)_{\text{fin}} \times GL_r(A)_{\infty}$ according to their finite and infinite parts.

For any $g = (g_{\text{fin}}; g_{\infty}) = (g_v; g_\sigma) \in GL_r(A)$, define the injective morphism $i(g) := i(g_{\infty}) : F^r \to A^r_{\infty}$ by $(f) \mapsto (g_{\sigma} \cdot f)$, $F^r(g) := \text{Im}(i(g))$ and

$$A^r(g) := \{(a_v; a_\sigma) \in A^r : \exists f \in F^r \text{ s.t. } g_v^{-1}(a_v) \in \mathcal{O}_v, \ g_\sigma^{-1}(f) \in \mathcal{O}_\sigma, \ \forall v \text{ and } (a_\sigma) = i(g_{\infty})(f)\}.$$ Then we have the following 9-diagram with all columns and roots exact:

\[
\begin{array}{ccc}
0 & \to & A^r(g) \cap F^r(g) \\
\downarrow & & \downarrow \\
0 & \to & A^r(g) \\
\downarrow & & \downarrow \\
0 & \to & F^r(g) \\
\downarrow & & \downarrow \\
0 & \to & A^r/\mathcal{A}^r(g) \cap F^r(g) \\
\downarrow & & \downarrow \\
0 & \to & A^r/\mathcal{A}^r(g) \\
\downarrow & & \downarrow \\
0 & \to & A^r/\mathcal{A}^r(g) + F^r(g) \\
\downarrow & & \downarrow \\
0 & \to & 0 \\
\end{array}
\]

Motivated by this and Weil’s adelic cohomology theory for divisors over algebraic curves, (see e.g., [Serre] and [Weil]), we introduce the following

Definition. For any $g \in GL_r(A)$, define its 0-th and 1-st arithmetic cohomology groups by

$$H^0(F, g) := A^r(g) \cap F^r(g), \quad \text{and} \quad H^1(F, g) := A^r/\mathcal{A}^r(g) + F^r(g).$$

Theorem. (Serre Duality= Pontrjagin Duality) As locally compact groups,

$$H^1(F, g) \simeq H^0(F, \hat{k}_F \otimes g^{-1}).$$

Here $k_F$ denotes an idelic canonical element of $F$.

Remark. For many $v \in S_{\text{fin}}$, denote $\partial_v$ the local differential of $f_v$, the $v$-completion of $F$ at $v$. Denote by $\mathcal{O}_v$ the valuation ring and $\pi_v$ any local parameter. Then $\partial_v = \pi_v^{\text{ord}_v(\partial_v)} \cdot \mathcal{O}_v$ and we call $k_F := (\partial_v^{\text{ord}_v(\partial_v)}; 1)$ an idelic canonical element of $F$.

Proof. As usual, introduce a basic character $\chi$ on $A$ by $\chi = (\chi_v^{(r)}; \chi_\sigma^{(r)})$ where $\chi_v := \lambda_v \circ \text{Tr}_{Q_v}$ with $\lambda_v : Q_v \to Q_v/\mathbb{Z}_v \hookrightarrow \mathbb{Q}/\mathbb{Z} \hookleftarrow \mathbb{R}/\mathbb{Z}$, and $\chi_\sigma := \lambda_\infty \circ \text{Tr}_{\mathbb{R}^r}$ with $\lambda_\infty : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$. Then the pairing $(x, y) \mapsto \int_F x(y) \chi(y) \, d\sigma$ gives
$e^{2\pi i(x\cdot y)}$ induces a natural isomorphism $\hat{A}^r \simeq A^r$. Moreover, with respect to this pairing, $(\mathcal{O}^r_v)^\perp = \partial_v^r$ and $F^r(g)^\perp \simeq F^r(g^{-1}) = F^r(k_F \otimes g^{-1})$. Thus, we could complete the proof by the fact that

\[
\left( A^r(g) \cap F^r(g) \right)^\perp = \left( A^r(g) \right)^\perp + \left( F^r(g) \right)^\perp.
\]

2. Arithmetic Counts

Motivated by the fact that the dimension of a vector space is equal to the dimension of its dual space, one of the basic principles we adopt in counting locally compact groups is the following:

**Counting Axiom.** If $\#_{ga}$ is a count for a certain class of locally compact groups $G$, then $\#_{ga}(G) = \#_{ga}(\hat{G})$.

Clearly, this is compatible with the Pontrjagin duality. Thus, our counts of arithmetic cohomology groups should be based on Fourier analysis over these groups, or better, the Fourier inverse formula for Fourier transforms. In this way, practically, it is natural to use the Plancherel formula to do counts.

While we may use any reasonable test functions on $A^r$ to do the counts, for simplicity and also as a continuation of a more classical mathematics, we set $f := \prod_v f_v \cdot \prod_\sigma f_\sigma$. Here $f_v$ is the characteristic function of $O^r_v$; $f_\sigma(x_\sigma) := e^{-\pi |x_\sigma|^2/2}$ if $\sigma$ is real; and $f_\sigma(x_\sigma) := e^{-\pi |x_\sigma|^2}$ if $\sigma$ is complex.

**Definition.** (1) The arithmetic counts of the 0-th and the 1-st arithmetic cohomology groups for $g \in \text{Gl}_r(A)$ are defined to be

\[
\#_{ga}(H^0(F, g); f, dx) := \int_{H^0(F, g)} |f(x)|^2 dx;
\]

\[
\#_{ga}(H^1(F, g); \hat{f}, d\xi) := \int_{H^1(F, g)} |\hat{f}(\xi)|^2 d\xi.
\]

Here $dx$ denotes (the restriction of) the standard Haar measure on $A$, $d\xi$ denotes (the induced quotient measure from) the dual measure, and $\hat{f}$ denotes the Fourier transform of $f$.

(2) The 0-th and the 1-st arithmetic cohomologies of $g \in \text{Gl}_r(A)$ is defined to be

\[
h^0(F, g) := \log \left( \#_{ga}(H^0(F, g)) \right) \quad \text{and} \quad h^1(F, g) := \log \left( \#_{ga}(H^1(F, g)) \right).
\]

3. Serre Duality and Riemann-Roch

For the arithmetic cohomologies just introduced, we have the following

**Theorem.** (1) (Serre Duality) $h^1(F, g) = h^0(F, k_F \otimes g)$;

(2) (Riemann-Roch Theorem)

\[
h^0(F, g) - h^1(F, g) = \deg(g) - \frac{r}{2} \cdot \log |\Delta_F|.
\]

**Proof.** (1) is a direct consequence of the topological version of Serre duality and the Plancherel Formula.

(2) is a direct consequence of the Serre duality just proved and the Poisson summation formula by noticing that $\left( H^0(F, g) \right)^\perp = H^0(F, k_F \otimes g^{-1})$. This then completes the proof.
4. Comments

Our work here is motivated by the works of Weil, Tate, van der Geer and Schoof, and Li, as well as the works of Lang, Arakelov, Szpiro, Neukirch, Connes, Deninger, Borisov, and Moreno. For details, please see the references below, in particular [Weng2]. As an application, we could introduce new yet natural non-abelian zeta functions for number fields, based on a discussion about intersection stability ([Weng1]).

As it stands, one may start from only $A_\infty$ to introduce arithmetic cohomogy groups and do corresponding counts. This has the advantage of being more concrete, and applies well say in Arakelov theory. On the other hand, a more general theory indeed works well for a much wider class of characters and test functions. We leave this to the reader. (See e.g., [Tate] and [Moreno].)

Finally, I would like to thank Deninger for his discussion and encouragement.

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