Central extensions of groups and adjoint groups of quandles

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Abstract

This paper develops an approach for describing centrally extended groups, as determining the adjoint groups associated with quandles. Furthermore, we explicitly describe such groups of some quandles. As a corollary, we determine some second quandle homologies.

Keywords Central extension of groups, group homology, quandle, $K_2$-group.

1 Introduction

Many mathematicians have been interested in objects with symmetries $X$, e.g., reflections and the sphere; Classically, as seen in Klein’s Erlangen program, such symmetry is interpreted from subgroups in the permutation group $\text{Bij}(X,X)$, e.g., Coxeter groups and $SO(n)$. However, this interpretation is quite classical and often overlooks central information of groups. Actually, as in physical explanation of spin structure on $SO(n)$, directly formal observation of symmetric objects recover central extensions in some cases.

Quandle theory [Joy] is a study of symmetric objects. Strictly speaking, a quandle is a set with a binary operation $\triangleright : X^2 \to X$, the axioms of which are partially motivated by knot theory and braidings. The operation is roughly the conjugation of a group; Conversely, given a quandle $X$, we can define the adjoint group as the following group presentation:

$$\text{Adj}(X) = \langle e_x \ (x \in X) \mid e_{x \triangleright y}^{-1} \cdot e_{y}^{-1} \cdot e_x \cdot e_{y} \ (x, y \in X) \rangle.$$

The correspondence $X \mapsto \text{Adj}(X)$ yields a functor from the category of quandles to that of groups with left-adjointness; This adjointness is considerably suggestive in some areas (see Ger [N3] for the quantum representations, or see AG, CJKLS, FRS, Kab, Joy, N1 and references therein for knot-invariants and (pointed) Hopf algebras). Moreover, as seen in [3], $\text{Adj}(X)$ is a central extension of a subgroup $G \subset \text{Bij}(X,X)$. Furthermore, as a result of Eisermann [Eis] (see Theorem 4.1), the second quandle homology $H_2^Q(X)$ can be computed from concrete expressions of $\text{Adj}(X)$. However, as seen in the definition of $\text{Adj}(X)$ or some explicit computation [Cla2], it has been considered to be hard to dealt with $\text{Adj}(X)$ concretely.

In this paper, we develop a method for formulating practically $\text{Adj}(X)$ in a purely algebraic way. This method is roughly summarised to be ‘universal central extensions of groups modulo type-torsion’ (see [2] [3]); the main theorem 2.1 emphasizes importance of the concept of types. Furthermore, Section 4 demonstrates practical applications of the method; Actually, we succeed in determining $\text{Adj}(X)$ and the associated homologies $H_2^Q(X)$ of some quandles $X$ (up to torsion). As a special case, Subsection 4.5 compares the theorem with Howlett’s theorem [How] concerning the Schur multipliers of Coxeter groups. Furthermore, in Section 5 we will see that the method is applicable to coverings in quandle theory.
**Notation and convention.** For a group $G$, we denote by $H^n_G(G)$ the usual group homology in trivial integral coefficients. Moreover, a homomorphism $f : A \to B$ between abelian groups is said to be a $[1/N]$-isomorphism and is denoted by $f : A \cong_{[1/N]} B$, if the localization of $f$ at $\ell$ is an isomorphism for any prime $\ell$ that does not divide $N$. This paper does not need any basic knowledge in quandle theory, but assumes basic facts of group cohomology as in [Bro], Sections I, II and VII.

2 Preliminaries and the main theorem

This section aims to state Theorem 2.1. We start by reviewing quandles and their properties. A quandle [Joy] is a set, $X$, with a binary operation $\triangleleft : X \times X \to X$ such that

(i) The identity $a \triangleleft a = a$ holds for any $a \in X$.

(ii) The map $(\bullet \triangleleft a) : X \to X$ defined by $x \mapsto x \triangleleft a$ is bijective for any $a \in X$.

(iii) The identity $(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$ holds for any $a, b, c \in X$.

For example, any group is a quandle by the operation $a \triangleleft b := b^{-1}ab$; see [IV] for other examples. A quandle $X$ is said to be of type $t_X$, if $t_X > 0$ is the minimal $N$ such that $x = x \triangleleft^N y$ for any $x, y \in X$, where we denote by $\bullet \triangleleft^N y$ the $N$-times on the right operation with $y$. Note that, if $X$ is of finite order, it is of type $t_X$ for some $t_X \in \mathbb{Z}$.

Next, let us study the adjoint group $\text{Adj}(X)$ in some details. Define a right action $\text{Adj}(X)$ on $X$ by $x \cdot e_y := x \triangleleft y$ for $x, y \in X$. Note the equality

$$e_x g = g^{-1} e_x g \in \text{Adj}(X) \quad (x \in X, \ g \in \text{Adj}(X)),$$

by definitions. The orbits of this action of $\text{Adj}(X)$ on $X$ are called connected components of $X$, denoted by $O(X)$. If the action is transitive (i.e., $O(X)$ is single), $X$ is said to be connected. Furthermore, with respect to $i \in O(X)$, define a homomorphism

$$\varepsilon_i : \text{Adj}(X) \longrightarrow \mathbb{Z} \quad \text{by} \quad \left\{ \begin{array}{ll} \varepsilon_i(e_x) = 1 \in \mathbb{Z}, & \text{if } x \in X_i, \\ \varepsilon_i(e_x) = 0 \in \mathbb{Z}, & \text{if } x \in X \setminus X_i. \end{array} \right.$$ (2)

Note that the direct sum $\oplus_{i \in O(X)} \varepsilon_i$ yields the abelianization $\text{Adj}(X)_{\text{ab}} \cong \mathbb{Z}^{\oplus O(X)}$ by (1), which means that the group $\text{Adj}(X)$ is of infinite order. Furthermore, if $O(X)$ is single, we often omit writing the index $i$.

Further, we briefly review the inner automorphism group, $\text{Inn}(X)$, of a quandle $X$. Regard the action of $\text{Adj}(X)$ as a group homomorphism $\psi_X$ from $\text{Adj}(X)$ to the symmetric group $\mathfrak{S}_X$. The group $\text{Inn}(X)$ is defined as the image $\text{Im}(\psi_X) \subset \mathfrak{S}_X$. Hence we have a group extension

$$0 \longrightarrow \text{Ker}(\psi_X) \longrightarrow \text{Adj}(X) \xrightarrow{\psi_X} \text{Inn}(X) \longrightarrow 0 \quad \text{(exact)}. \quad (3)$$

By the equality (1), this kernel $\text{Ker}(\psi_X)$ is contained in the center. Therefore, it is natural to focus on their second group homology; this paper provides a general appraisal as follows:
Theorem 2.1. For any connected quandle $X$ of type $t_X$ (possibly, $X$ could be of infinite order), the second group homology $H_2^{gr}(\text{Adj}(X))$ is annihilated by $t_X$. Furthermore, the abelian kernel $\text{Ker}(\psi_X)$ in (3) is $[1/t_X]$-isomorphic to $\mathbb{Z} \oplus H_2^{gr}(\text{Inn}(X))$.

The proof will appear in §6. In conclusion, metaphorically speaking, $\text{Adj}(X)$ turns out to be the ‘universal central extension’ of $\text{Inn}(X)$ up to $t_X$-torsion; hence, this theorem emphasizes importance of the concept of types, and so as to investigate $\text{Adj}(X)$, we shall study $\text{Inn}(X)$ and $H_2^{gr}(\text{Inn}(X))$.

3 Methods on inner automorphism groups.

Following the previous theorem to study the $\text{Adj}(X)$, we shall develop a method for describing the inner automorphism group $\text{Inn}(X)$:

Theorem 3.1. Let a group $G$ act on a quandle $X$. Let a map $\kappa : X \to G$ satisfy the followings:

(I) The identity $x \cdot \kappa(y) \in X$ holds for any $x,y \in X$.

(II) The image $\kappa(X) \subset G$ generates the group $G$, and the action $X \curvearrowright G$ is effective.

Then, there is an isomorphism $\text{Inn}(X) \cong G$, and the action $X \curvearrowright G$ agrees with the natural action of $\text{Inn}(X)$.

Proof. Identifying the action $X \curvearrowright G$ with a group homomorphism $F : G \to \mathfrak{S}_X$, this $F$ factors through $\text{Inn}(X)$ by (I). Notice, for any $x,y,z \in X$, the identities

$$z \cdot \kappa(x) \kappa(y) = (z \cdot \kappa(x)) \kappa(y) = (z \kappa(y)) \kappa(x) \kappa(y) = \kappa(x) \kappa(y) \kappa(x \cdot \kappa(y)) \in X,$$

which imply $\kappa(x) \kappa(y) = \kappa(y) \kappa(x \cdot \kappa(y)) \in G$ by the effectivity in (II). Hence the epimorphism $\psi_X$ in (3) is decomposed as $\text{Adj}(X) \to G \xrightarrow{F} \text{Inn}(X)$. Moreover (II) concludes the bijectivity of $F : G \cong \text{Inn}(X)$; Hence, the agreement of the two actions follows from construction.

This theorem is applicable to many quandles, in practice. Actually, as seen in Section 4, we can determine $\text{Inn}(X)$ of many quandles $X$. However, we here explain that this lemma is inspired by the Cartan embeddings in symmetric space theory as follows:

Example 3.2. Let $X$ be a symmetric space in differential geometry. By definition, this connected $C^\infty$-manifold $X$ is equipped with a Riemannian metric such that each point $x \in X$ admits an isometry $s_y : X \to X$ that reverses every geodesic line $\gamma : (\mathbb{R},0) \to (X,y)$, meaning that $s_y \circ \gamma(t) = \gamma(-t)$. Then, we have a quandle structure on $X$ defined by $x \cdot y := s_y(x)$; see, e.g., [Joy, Eis]. Further, consider the group $G \subset \text{Diff}(X)$ generated by the symmetries $s_y$ with compact-open topology. As is well known, this $G$ is a Lie group. Then, denoting by $\kappa$ the Cartan embedding $\mathfrak{g} \to X \to G$ that sends $x$ to $\bullet \cdot x$, we can easily see that the action and $\kappa$ satisfy the conditions in Theorem 3.1. Consequently, we conclude $G \cong \text{Inn}(X)$.

Furthermore, we suggest another computation when $\text{Inn}(X)$ is perfect.

\footnotesize\begin{itemize}
   \item[\footnotesize1] In contrast to the common notation, the map is not always embedding and injective, but is an immersion.
\end{itemize}
Proposition 3.3. Let $X$ be a quandle, and $O(X)$ be the set of orbits of the action $X \acts Adj(X)$. Set the epimorphisms $\varepsilon_i : Adj(X) \to \mathbb{Z}$ associated with $i \in O(X)$ defined in [2]. If the group $Inn(X)$ is perfect, i.e., $H^1_{gr}(Inn(X)) = 0$, then we have an isomorphism

$$Adj(X) \cong \text{Ker}(\bigoplus_{i \in O(X)} \varepsilon_i) \times \mathbb{Z}^{\oplus O(X)},$$

and this $\text{Ker}(\bigoplus_{i \in O(X)} \varepsilon_i)$ is a central extension of $Inn(X)$ and is perfect. In particular, if $X$ is connected and the group homology $H^2_{gr}(Inn(X))$ vanishes, then $Adj(X) \cong Inn(X) \times \mathbb{Z}$.

Proof. We will show the isomorphism (1). Since $H^1_{gr}(Inn(X)) = 0$, we obtain an epimorphism $\text{Ker}(\psi_X) \hookrightarrow Adj(X) \overset{\text{proj}}{\longrightarrow} H^1_{gr}(Adj(X)) = \mathbb{Z}^{\oplus O(X)}$ from (3). Since $\mathbb{Z}^{\oplus O(X)}$ is free, we can choose a section $s : \mathbb{Z}^{\oplus O(X)} \to Adj(X)$ which factors through the $\text{Ker}(\psi_X)$. Hence, by the equality (1), the semi-product $Adj(X) \cong \text{Ker}(\bigoplus_{i \in O(X)} \varepsilon_i) \times \mathbb{Z}^{\oplus O(X)}$ is trivial, leading to (1) as desired. Furthermore the kernel $\text{Ker}(\bigoplus_{i \in O(X)} \varepsilon_i)$ is a central extension of $Inn(X)$ by construction, and is perfect by the Kunneth theorem and $Adj(X)_{ab} \cong \mathbb{Z}^{\oplus O(X)}$, which completes the proof. 

Remark 3.4. In general, the kernel $\text{Ker}(\bigoplus_{i \in O(X)} \varepsilon_i)$ is not always the universal central extension of the perfect group $Inn(X)$; see [N3, Theorem 4] with $g = 3$ as a counterexample.

In general, it is hard to calculate the associated groups $Adj(X)$ concretely; For example, the center of $Adj(X)$ is not so trivial as seen in the following two lemmas:

Lemma 3.5. Let $X$ be a connected quandle of type $t < \infty$. Then, for any $x, y \in X$, we have the identity $(e_x)^t = (e_y)^t$ in the center of $Adj(X)$.

Proof. For any $b \in X$, note the equalities $(e_x)^{-1}e_b e_x^t = e_{(\ldots(b_{\ldots x})\ldots)x} = e_b$ in $Adj(X)$. Namely $(e_x)^t$ lies in the center. Furthermore the connectivity admits $g \in Adj(X)$ such that $x \cdot g = y$. Hence, it follows from (1) that $(e_x)^t = g^{-1}(e_x)^t g = (e_{x\cdot g})^t = (e_y)^t$ as desired.

Lemma 3.6. Let $X$ be a connected quandle of finite order. Then its type $t_X$ is a divisor of $|\text{Inn}(X)|/|X|$.

Proof. For $x, y \in X$, we define $m_{x,y}$ as the minimal $n$ satisfying $x \triangleleft^n y = x$. Note that $(\cdot \triangleleft m_{x,y} y)$ lies in the stabilizer $\text{Stab}(x)$. Since $|\text{Stab}(x)| = |\text{Inn}(X)|/|X|$ by connectivity, any $m_{x,y}$ divides $|\text{Inn}(X)|/|X|$; hence so does the type $t_X$.

Furthermore, in some cases, we can calculate some torsion parts of their group homologies:

Lemma 3.7. Let $X$ be a connected quandle of type $t_X$. If $H^2_{gr}(Inn(X))$ is annihilated by $t_X < \infty$, then there is a $[1/t_X]$-isomorphism $H^2_{gr}(Adj(X)) \cong H^2_{gr}(Inn(X))$.

Proof. Notice the $[1/t_X]$-isomorphism $\text{Ker}(\psi_X) \cong [1/t_X] \mathbb{Z}$ from Theorem [2]. Hence, the Lyndon-Hochschild spectral sequence of (3) readily leads to the required $[1/t_X]$-isomorphism.
4 Six examples of Adj($X$) and second quandle homology

Based on the previous results on Adj($X$), this section calculates Inn($X$) and Adj($X$) for six kinds of connected quandles $X$: Alexander, symplectic, spherical, Dehn, Coxeter and core quandles. These quandles are dealt with in six subsections in turn.

Furthermore, to determine the second quandle homologies $H_2^Q(X)$ in trivial $\mathbb{Z}$-coefficients (see §6 for the definition), we will employ the following computation of Eisermann:

**Theorem 4.1** (Eis, Theorem 1.15). Let $X$ be a connected quandle. Fix an element $x_0 \in X$. Let $\text{Stab}(x_0) \subset \text{Adj}(X)$ be the stabilizer of $x_0$, and $\varepsilon : \text{Adj}(X) \to \mathbb{Z}$ be the abelianization mentioned in (2). Then, $H_2^Q(X)$ is isomorphic to the abelianization of $\text{Stab}(x_0) \cap \text{Ker}(\varepsilon)$.

4.1 Alexander quandles

We start by discussing the class of Alexander quandles. Every $\mathbb{Z}[T^{\pm 1}]$-module $X$ has a quandle structure with the operation $x \triangleright y = y + T(x - y)$ for $x, y \in X$, and is called the Alexander quandle. This operation $\cdot \triangleright y$ is roughly a $T$-multiple centered at $y$. The type is the minimal $N$ such that $T^N = \text{id}_X$ since $x \triangleleft^n y = y + T^nx - y)$. Furthermore, it can be easily verified that an Alexander quandle $X$ is connected if and only if $(1 - T)X = X$.

Let us review the concrete presentation of Adj($X$), which is due to Clauwens [Cla2]. When $X$ is connected, set up the homomorphism $\mu_X : X \otimes X \to X \otimes X$ defined by $\mu_X(x \otimes y) = x \otimes y - Ty \otimes x$. Further, he defined a group operation on $\mathbb{Z} \times X \times \text{Coker}(\mu_X)$ by setting

$$(n, x, \alpha) \cdot (m, y, \beta) = (n + m, T^nx + y, \alpha + \beta + [T^mx \otimes y]),$$

and showed that the homomorphism Adj($X$) $\to \mathbb{Z} \times X \times \text{Coker}(\mu_X)$ defined by sending $e_x$ to $(1, x, 0)$ is a group isomorphism. The lower central series of Adj($X$) is then described as

$$\text{Adj}(X) \supset X \times \text{Coker}(\mu_X) \supset \text{Coker}(\mu_X) \supset 0. \quad (5)$$

As a result, we see that the kernel of $\psi_X : \text{Adj}(X) \to \text{Inn}(X)$ equals $t_X \mathbb{Z} \times \text{Coker}(\mu_X)$.

Thanks to his presentation of Adj($X$), we can easily show a result of Clauwens that determines the homology $H_2^Q(X)$ of a connected Alexander quandle $X$. To be precise,

**Proposition 4.2** (Clauwens [Cla2]). Let $X$ be a connected Alexander quandle. The homology $H_2^Q(X)$ is isomorphic to the quotient module $\text{Coker}(\mu_X) = X \otimes X / (x \otimes y - Ty \otimes x)_{x,y \in X}$.

**Proof.** By definition we can see that the $\text{Ker}(\varepsilon_X) \cap \text{Stab}(0)$ is the cokernel $\text{Coker}(\mu_X)$. \hfill $\Box$

Further, the following proposition is immediately shown by the spectral sequence:

**Proposition 4.3.** Let $X$ be a connected Alexander quandle of finite order. If the type $t_X$ of $X$ is relatively prime to the order $|X|$, then the $t_X$-torsion of $H_3(\text{Adj}(X))$ is zero.
4.2 Symplectic quandles

Let $K$ be a commutative field, and let $\Sigma_g$ be the closed surface of genus $g$. Define $X$ to be the first homology with $K$-coefficients outside 0, that is, $X = H^1(\Sigma_g; K) \setminus \{0\} = K^{2g} \setminus \{0\}$. Using the standard symplectic 2-form $\langle \cdot, \cdot \rangle : H^1(\Sigma_g; K) \times H^1(\Sigma_g; K) \to K$, the set $X$ is made into a quandle by the operation $x \triangleleft y := \langle x, y \rangle y + x \in X$ for any $x, y \in X$, and is called a symplectic quandle (over $K$). The operation $\bullet \triangleleft y : X \to X$ is commonly called the transvection of $y$. Note that the quandle $X$ is of type $p = \text{Char}(K)$ since $x \triangleleft^N y = N(x, y)y + x$.

We will determine $\text{Inn}(X)$ and $\text{Adj}(X)$ with the symplectic quandle $X$ over $K$.

**Lemma 4.4.** Then, $\text{Inn}(X)$ is isomorphic to the symplectic group $\text{Sp}(2g; K)$.

**Proof.** As is called the Cartan-Dieudonné theorem classically, the classical group $\text{Sp}(2g; K)$ is generated by transvections $(\bullet \triangleleft y)$.

We will show the desired isomorphism. For any $y \in X$, the map $(\bullet \triangleleft y) : X \to X$ is a restriction of a linear map $K^{2g} \to K^{2g}$. It thus yields a map $\kappa : X \to GL(2g; K)$, which factors through $\text{Sp}(2g; K)$ and satisfies the conditions in Theorem 3.1. Indeed, the condition (II) follows from the previous classical theorem and the effectiveness of the standard action $K^{2g} \curvearrowright \text{Sp}(2g; K)$. Therefore $\text{Inn}(X) \cong \text{Sp}(2g; K)$ as desired. $\square$

**Proposition 4.5.** Take a field $K$ of positive characteristic and with $|K| > 10$. Let $X$ be the symplectic quandle over $K$, and $\widetilde{\text{Sp}}(2g; K)$ be the universal central extension of $\text{Sp}(2g; K)$. Then $\text{Adj}(X) \cong \mathbb{Z} \times \widetilde{\text{Sp}}(2g; K)$.

**Proof.** Since $X$ is connected and $\text{Inn}(X) \cong \text{Sp}(2g; K)$ is perfect (Lemma 4.4), Proposition 3.3 implies $\text{Adj}(X) \cong \ker(\varepsilon) \times \mathbb{Z}$. Further, $H^\text{gr}_2(\text{Adj}(X))$ is annihilated by $p$ by Theorem 2.1. Hence, following the fact [Sus] that $H^\text{gr}_2(\text{Sp}(2g; K))$ has no $p$-torsion, the kernel $\ker(\varepsilon)$ must be the universal central extension of $\text{Sp}(2g; K)$, which completes the proof. $\square$

**Remark 4.6.** This proposition holds even if $\text{Char}(K) = 0$; see [N2] for the proof. Furthermore, the paper [N2] also determines $H^Q_2(X)$ in the case where $K$ is of infinite order.

Accordingly, hereafter, we will focus on finite fields $K = \mathbb{F}_q$ with $q > 10$.

**Proposition 4.7.** Let $X$ be the symplectic quandle over $\mathbb{F}_q$. If $q > 10$, then $\text{Adj}(X) \cong \mathbb{Z} \times \text{Sp}(2g; \mathbb{F}_q)$. Furthermore, $H^\text{gr}_3(\text{Adj}(X)) \cong \mathbb{Z}/(q^2 - 1)$.

**Proof.** Since the first and second homologies of $\text{Inn}(X) \cong \text{Sp}(2g; \mathbb{F}_q)$ vanish (see [FP, Fri]), we have $\widetilde{\text{Sp}}(2g; \mathbb{F}_q) = \text{Sp}(2g; \mathbb{F}_q)$, leading to $\text{Adj}(X) \cong \mathbb{Z} \times \text{Sp}(2g; \mathbb{F}_q)$ as stated. Furthermore, the latter part follows from the result $H^\text{gr}_3(\text{Sp}(2g; \mathbb{F}_q)) \cong \mathbb{Z}/(q^2 - 1)$ in [FP, Fri]. $\square$

As a result, we will determine the second homology $H^Q_2(X)$.

**Proposition 4.8.** Let $q > 10$. If $g \geq 2$, the homology $H^Q_2(X)$ vanishes. If $g = 1$, then $H^Q_2(X) \cong (\mathbb{Z}/p)^d$, where $q = p^d$. 

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Proof. Recall Adj($X$) $\cong \mathbb{Z} \times Sp(2g; \mathbb{F}_q)$. Considering the standard action $X \vartriangleleft Sp(2g; \mathbb{F}_q)$, denote by $G_X$ the stabilizer of $(1,0,\ldots,0) \in (\mathbb{F}_q)^{2g}$. Since Theorem 4.1 immediately means $H_2^Q(X) \cong H_1^{gr}(G_X)$, we will calculate $H_1^{gr}(G_X)$ as follows. First, for $g = 1$, it can be verified that the stabilizer $G_X$ is exactly the product ($\mathbb{Z}/p)^d$ as an abelian group; hence $H_2^Q(X) \cong (\mathbb{Z}/p)^d$ in the sequel. Next, for $g \geq 2$, the vanishing $H_2^Q(X) = H_1^{gr}(G_X) = 0$ immediately follows from Lemma 4.9 below.

Lemma 4.9. Let $g \geq 2$ and $q > 10$. Let $G_X$ denote the stabilizer of the action $X \vartriangleleft Sp(2g; \mathbb{F}_q)$ mentioned above. Then the homologies $H_1^{gr}(G_X)$ and $H_2^{gr}(G_X)$ vanish.

Proof. Recall from [FP, II. §6.3] the order of $Sp(2g; \mathbb{F}_q)$ as

$$|Sp(2g; \mathbb{F}_q)| = q^{2g^2} - 1(q^{2g^2 - 1} - 1)\ldots(q^2 - 1).$$

Since $|X| = q^{2g} - 1$, the order of $G_X$ is equal to $q^{2g-1}\cdot |Sp(2g - 2; \mathbb{F}_q)|$. Thereby $H_1^{gr}(G_X)$ and $H_2^{gr}(G_X)$ are zero without $p$-torsion, because of the inclusion $Sp(2g - 2; \mathbb{F}_q) \subset G$ by definitions and the vanishing $H_1^{gr} \oplus H_2^{gr}(Sp(2g - 2; \mathbb{F}_q)) \cong 0$ without $p$ torsion.

Finally, we may focus on the $p$-torsion of $H_1^{gr} \oplus H_2^{gr}(G_X)$. Following the proof of [BTR, Proposition 4.4], there is a certain subgroup “$\Delta(Sp(2g; \mathbb{F}_q))$” of $G_X$ which contains a $p$-sylow group of $Sp(2g; \mathbb{F}_q)$ and this $\mathbb{Z}/p$-homology vanishes. Hence, $H_1^{gr} \oplus H_2^{gr}(G_X) = 0$ as required.

4.3 Spherical quandles

Let $K$ be a field of characteristic not equal to 2, and fix $n \geq 2$ in this subsection. Take the standard symmetric bilinear form $\langle , \rangle : K^{n+1} \otimes K^{n+1} \to K$. Consider a set of the form

$$S_K^n := \{ x \in K^{n+1} \mid \langle x, x \rangle = 1 \}.$$

We define the operation $x \triangleleft y$ to be $2\langle x, y \rangle y - x \in S_K^n$. The pair $(S_K^n, \triangleleft)$ is a quandle of type 2, and is referred to as a spherical quandle (over $K$). This operation $\bullet \triangleleft y$ can be interpreted as the rotation through 180-degrees with the rotating axis $y$.

Then, similar to the proof of Lemma 4.4, one can readily determine $\text{Inn}(X)$ as follows:

Lemma 4.10. If $n \geq 2$, then $\text{Inn}(S_K^n)$ is isomorphic to the orthogonal group $O(n + 1; K)$.

Next, we will focus on second homologies and $H_2^{gr}(\text{Adj}(X))$ of spherical quandles over $\mathbb{F}_q$. However, the results are up to 2-torsion, whereas the 2-torsion part is the future problem.

Proposition 4.11. Let $X$ be a spherical quandle over $\mathbb{F}_q$. Let $q > 10$. For $n \geq 3$, the second homology $H_2^Q(X)$ is annihilated by 2. If $n = 2$, then the homology $H_2^Q(X)$ is $[1/2]$-isomorphic to the cyclic group $\mathbb{Z}/(q - \delta_q)$, where $\delta_q = \pm 1$ is according to $q \equiv \pm 1$ (mod 4).

Proof. Under the standard action $X \vartriangleleft O(n + 1; \mathbb{F}_q)$, the stabilizer of $(1,0,\ldots,0) \in X$ is $O(n; \mathbb{F}_q)$. By a similar discussion to the proof of Proposition 3.7, $H_2^Q(X) \cong H_1^{gr}(O(n; \mathbb{F}_q))$ without 2-torsion. For $n \geq 3$, the abelianization of $O(n; \mathbb{F}_q)$ is $(\mathbb{Z}/2)^2$; see [FP, II. §3]; hence the $H_2^Q(X)$ is annihilated by 2 as required. Finally, when $n = 2$, the group $O(2; \mathbb{F}_q)$ is cyclic and of order $q - \delta_q$. Hence $H_2^Q(X) \cong H_1^{gr}(O(2; \mathbb{F}_q)) \cong [1/2] \mathbb{Z}/(q - \delta_q)$. 

□
Proposition 4.12. Let $q > 10$. Then $H^*_3(\text{Adj}(X)) \cong [1/2] H^*_3(O(n+1; \mathbb{F}_q))$ without 2-torsion.

Proof. As in [FR, FP], $H^*_1 \oplus H^*_2(O(n+1; \mathbb{F}_q))$ is known to be annihilated by 2. Hence, the conclusion readily results from Lemma 3.7.

4.4 Dehn quandle

Changing the subject, we now review Dehn quandles [Y]. Denote by $\mathcal{M}_g$ the mapping class group of $\Sigma_g$, and consider the set, $\mathcal{D}_g$, defined by

$$\mathcal{D}_g := \{ \text{ isotopy classes of (unoriented) non-separating simple closed curves } \gamma \text{ in } \Sigma_g \}.$$ 

For $\alpha, \beta \in \mathcal{D}_g$, we define $\alpha \triangleleft \beta \in \mathcal{D}_g$ by $\tau_\beta(\alpha)$, where $\tau_\beta \in \mathcal{M}_g$ is the positive Dehn twist along $\beta$. The pair $(\mathcal{D}_g, \triangleleft)$ is a quandle, and called (non-separating) Dehn quandle. As is well-known, any two non-separating simple closed curves are conjugate by some Dehn twists. Hence, the quandle $\mathcal{D}_g$ is connected, and is not of any type $t$. The Dehn quandle $\mathcal{D}_g$ is applicable to study 4-dimensional Lefschetz fibrations (see, e.g., [Y, Zab, N3]). The natural inclusion $\kappa : \mathcal{D}_g \to \mathcal{M}_g$ implies $\text{Inn}(\mathcal{D}_g) \cong \mathcal{M}_g$ by Lemma 3.1. Furthermore, if $g \geq 4$, there is an isomorphism $\text{Adj}(\mathcal{D}_g) \cong \mathbb{Z} \times T_g$ shown by [Ger], where $T_g$ is the universal central extension of $\mathcal{M}_g$ associated with $H^*_2(\mathcal{M}_g) \cong \mathbb{Z}$.

The result of this subsection is the following:

Proposition 4.13. If $g \geq 5$, then $H^*_2(\mathcal{D}_g) \cong \mathbb{Z}/2$.

Proof. We will use the fact that an epimorphism $G \to H$ between groups induces an epimorphism $G_{ab} \to H_{ab}$, and that $\mathcal{M}_g$, is perfect.

Fixing $\alpha \in \mathcal{D}_g$, we begin by observing the stabilizer $\text{Stab}(\alpha) \subset \text{Adj}(\mathcal{D}_g)$. Note that the map $\mathcal{D}_g \to \mathcal{M}_g$ sending $\beta$ to $\tau_\beta$ yields a group epimorphism $\pi : \text{Adj}(\mathcal{D}_g) \to \mathcal{M}_g$. Furthermore, by Proposition 3.3, the restriction of $\pi$ to $\text{Ker}(\varepsilon) \cong T_g$ coincides with the projection $T_g \to \mathcal{M}_g$. In particular, we thus have $\pi(\text{Stab}(\alpha)) = \pi(\text{Stab}(\alpha) \cap \text{Ker}(\varepsilon)) \subset \mathcal{M}_g$.

We will construct a surjection $H^*_2(\mathcal{D}_g) \to \mathbb{Z}/2$. By the virtue of Theorem 4.1, it is enough to construct a surjection from the previous $\pi(\text{Stab}(\alpha) \cap \text{Ker}(\varepsilon))$ to $\mathbb{Z}/2$ for $g \geq 2$. As is shown [PR, Proposition 7.4], we have the following exact sequence:

$$0 \to \mathbb{Z} \to \mathcal{M}_{g-1,2} \overset{\xi}{\to} \pi(\text{Stab}(\alpha)) \overset{\lambda}{\to} \mathbb{Z}/2 \quad (\text{exact}).$$

(6)

Here $\xi$ is the homomorphism induced from the gluing $(\Sigma_{g-1,2}, \partial(\Sigma_{g-1,2})) \to (\Sigma_g, \alpha)$, and $\lambda$ is defined by the transposition of the connected components of boundaries of $\Sigma_g \setminus \alpha$. By considering a hyper-elliptic involution preserving the above $\alpha$, the map $\lambda$ is surjective. Hence $\pi(\text{Stab}(\alpha) \cap \text{Ker}(\varepsilon))$ surjects onto $\mathbb{Z}/2$ as desired.

Finally, we will complete the proof. By Theorem 4.1, again, recall that $(\text{Stab}(\alpha) \cap \text{Ker}(\varepsilon))^*_ab \cong H^*_2(\mathcal{D}_g)$. To compute this, put the inclusion $\iota : \pi(\text{Stab}(\alpha)) \to \mathcal{M}_g$. By the Harer-Ivanov stability theorem (see [Iva]), the composition $\iota \circ \xi : \mathcal{M}_{g-1,2} \to \mathcal{M}_g$ induces an epimorphism

$$(\iota \circ \xi)_* : H^*_2(\mathcal{M}_{g-1,2}; \mathbb{Z}) \longrightarrow H^*_2(\mathcal{M}_g; \mathbb{Z}) \quad \text{for } g \geq 5.$$
Since $H_2(\mathcal{M}_{g-1,2};\mathbb{Z}) \cong H_2(\mathcal{M}_g;\mathbb{Z}) \cong \mathbb{Z}$ is known (see, e.g., [FM]), the epimorphism (7) is isomorphic. Let $(\iota \circ \xi)^*(T_g)$ denote the central extension of $\mathcal{M}_{g-1,2}$ obtained by $\iota \circ \xi$. Since $\mathcal{M}_g$ and $\mathcal{M}_{g-1,2}$ are perfect, the group $(\iota \circ \xi)^*(T_g)$ is also perfect by the isomorphism (7). Note that the group Stab$(\alpha) \cap \text{Ker}(\varepsilon)$ is isomorphic to $\iota^*(T_g)$. Hence the abelianization $(\text{Stab}(\alpha) \cap \text{Ker}(\varepsilon))_{\text{ab}}$ never be bigger than $\mathbb{Z}/2$. In conclusion, we arrive at the conclusion.

4.5 Coxeter quandles

We will focus on Coxeter quandles, and study the associated groups, and show Theorem 4.14.

This subsection assumes basic knowledge of Coxeter groups, as explained in [Aki, How].

Given a Coxeter graph $\Gamma$, we can set the Coxeter group $W$. Let $X_\Gamma$ be the set of the reflections in $W$, that is, the set of elements conjugate to the generators of $W$. Equipping $X_\Gamma$ with conjugacy operation, $X_\Gamma$ is made into a quandle of type 2. Denote the inclusion $X_\Gamma \hookrightarrow W$ by $\kappa$. Since $W$ subject to the center $Z_W$ effectivity acts on $X_\Gamma$, we have Inn$(X_\Gamma) \cong W/Z_W$.

Moreover, $W$ is, by definition, isomorphic to the quotient of Adj$(X_\Gamma)$ subject to the squared relations $(e_x)^2 = 1$ for any $x \in X_\Gamma$.

In this situation, we now give another easy proof of a part of the theorem shown by Howlett:

**Theorem 4.14** (A connected result in [How, §2–4]). Assume that the Coxeter quandle $X_\Gamma$ is connected. Then, the second group homology $H^\text{gr}_2(W)$ is annihilated by 2.

**Proof.** Since $X_\Gamma$ is connected, the graph $\Gamma$ must be path-connected. Recall from Theorem 2.1 that $H^\text{gr}_1(\text{Adj}(X_\Gamma)) \cong \mathbb{Z}$ and $H^\text{gr}_2(\text{Adj}(X_\Gamma))$ is annihilated by 2. Therefore, the inflation-restriction exact sequence from the central extension $\text{Adj}(X_\Gamma) \to W$ implies the desired 2-vanishing of $H^\text{gr}_2(W)$.

Finally, we will end this subsection by briefly commenting on Adj$(X_\Gamma)$. Although the computation of the 2-torsion subgroup of the homology of $W$ is widely known to be a hard problem, Howlett [How] further gave an explicit algorithm from the graph $\Gamma$ to compute the Schur multipliers and the rank of the elementary abelian 2-group $H^\text{gr}_2(W)$. Thus, in order to determine Adj$(X_\Gamma)$, we should analyse the Schur multipliers, though this paper does not. Furthermore, concerning the third homology $H_3(\text{Adj}(X_\Gamma))$ in the case $X_\Gamma$ is connected, we obtain $H_3(\text{Adj}(X_\Gamma)) \cong H_3(W)$ without 2-torsion from Lemma 3.7. However, the odd torsion of $H^\text{gr}_3(W)$ in a certain stable range are studied by Akita [Aki].

4.6 Core quandles

Given a group $G$, we let $X = G$ equipped with a quandle operation $g \triangleleft h := hg^{-1}h$, which is called core quandle and is of type 2. This last subsection will deal with core quandles [Joy], and show Proposition 4.15.

Let us give some terminologies to state the proposition. Consider the wreath product $(G \times G) \rtimes \mathbb{Z}/2$, and the commutator subgroup $[G, G]$. Set up the subgroup

$$G_1 := \{ (g, h, \sigma) \in (G \times G) \rtimes \mathbb{Z}/2 \mid gh \in [G, G] \}.$$
Further, with respect to \( x \in X \) and \((g, h, \sigma) \in \mathcal{G}_1\), we define \( x \cdot (g, h, \sigma) := (hxg^{-1})^{2\sigma-1} \), which ensures an action of \( \mathcal{G}_1 \) on \( X \). Further, consider a subgroup of the form
\[
\mathcal{G}_2 := \{(z, z, \sigma) \in (G \times G) \times \mathbb{Z}/2 \mid z^2 \in [G, G], \quad k^{-1}zk = z^{2\sigma} \text{ for any } k \in G. \},
\]
which is contained in the center of \( \mathcal{G}_1 \). Then, the quotient action subject to \( \mathcal{G}_2 \) is effective. Moreover, consider the map \( \kappa : X \to \mathcal{G}_1/\mathcal{G}_2 \) which sends \( g \) to \([(g, g^{-1}, 1)]\). Here, let us notice that this \( \mathcal{G}_1/\mathcal{G}_2 \) is generated by the image \( \text{Im}(\kappa) \). Actually, we easily verify that any element \((g, h, \sigma) \) in \( \mathcal{G}_1 \) with \( g_i, h_i \in G \) and \( gh = g_1h_1^{-1}h_1^{-1} \cdots g_mh_mg_m^{-1}h_m^{-1} \) is decomposed as
\[
\kappa(1_G)^{\frac{a+1}{2}} \cdot \kappa(gh^{-1}) \cdot \left( (\kappa(g_1h_1) \cdot \kappa(1_G) \cdot \kappa(g_1^{-1}) \cdot \kappa(h_1)) \cdots (\kappa(g_mh_m) \cdot \kappa(1_G) \cdot \kappa(g_m^{-1}) \cdot \kappa(h_m)) \right).
\]

Then, the routine discussion from Lemma 3.1 deduces the following:

**Proposition 4.15.** There is a group isomorphism \( \text{Inn}(X) \cong \mathcal{G}_1/\mathcal{G}_2 \).

This proposition implies the difficulty to determine \( \text{Inn}(X) \), in general. Thus, it also seems hard to determine \( \text{Adj}(X) \). Actually, even if \( X \) is a connected core quandle, Proposition 4.15 implies that the kernel \( \text{Ker}(\psi) \) has the complexity characterized by the second homology \( H_2^{\text{gr}}(G) \) and \( H_2^{\text{gr}}(\text{Inn}(X)) \). For example, if \( X \) the product \( h \)-copies of the cyclic group \( \mathbb{Z}/m \), i.e., \( X \) is the Alexander quandle of the form \((\mathbb{Z}/m)^h[T]/(T + 1)\), then Proposition 4.2 implies such a complexity.

### 5 On quandle coverings

This section suggests that the results in section 2 are applicable to quandle coverings.

Let us review notation of coverings in the sense of Eisermann [Eis]. A map \( f : Y \to Z \) between quandles is a (quandle) homomorphism, if \( f(a \triangleleft b) = f(a) \triangleleft f(b) \) for any \( a, b \in Y \). Furthermore, a quandle epimorphism \( p : Y \to Z \) is a (quandle) covering, if the equality \( p(\bar{x}) = p(\bar{y}) \in Z \) implies \( \bar{a} \triangleleft \bar{x} = \bar{a} \triangleleft \bar{y} \in Y \) for any \( \bar{a}, \bar{x}, \bar{y} \in Y \).

Let us mention a typical example. Given a connected quandle \( X \) with \( a \in X \), recall the abelianization \( \varepsilon_0 : \text{Adj}(X) \to \mathbb{Z} \) in (2). Then, the kernel \( \text{Ker}(\varepsilon_0) \) has a quandle operation defined by setting
\[
g \triangleleft h := e_a^{-1}gh^{-1}e_a \quad \text{for } g, h \in \text{Ker}(\varepsilon_0).
\]

We can easily see the independence of the choice of \( a \in X \) up to quandle isomorphisms. Ones write \( \hat{X} \) for the quandle \((\text{Ker}(\varepsilon_0), \triangleleft)\), which is considered in [Joy, §7]. When \( X \) is of type \( t_X \), so is the extended one \( \hat{X} \) by using Lemma 3.1. Furthermore, using the restricted action \( X \rhd \text{Ker}(\varepsilon_0) \subset \text{Adj}(X) \), we see that the map \( p : \hat{X} \to X \) sending \( g \) to \( a \cdot g \) is a covering. This \( p \) is called the universal (quandle) covering of \( X \), according to [Eis, §5].

As a preliminary, we will explore some properties of quandle coverings.

**Proposition 5.1.** For any quandle covering \( p : Y \to Z \), the induced group surjection \( p_* : \text{Adj}(Y) \to \text{Adj}(Z) \) is a central extension. Furthermore, if \( Y \) and \( Z \) are connected and \( Z \) is of type \( t_Z \), then the abelian kernel \( \text{Ker}(p_*) \) is annihilated by \( t_Z \).
Proof. Fix a section $s: Y \to Z$. For any $y \in Z$, put arbitrary $y_i \in p^{-1}(y)$. Then,

$$e_{s(y)}^{-1} e_b e_{s(y)} = e_{b \circ s(y)} = e_{b \circ y_i} = e_{y_i}^{-1} e_b e_{y_i} \in \text{Adj}(Y)$$

for any $b \in Y$. Here the second equality is because of the covering $p$. Denoting $e_{s(y)} e_{y_i}^{-1}$ by $z_i$, the equalities imply that $z_i$ is central in $\text{Adj}(Y)$. Since $e_{s(y)} = z_i e_{y_i}, \text{Adj}(Y)$ is generated by $e_{s(y)}$ with $y \in Y$ and the central elements $z_i$ associated with $y_i \in p^{-1}(y)$; Consequently, the surjection $p_*$ is a central extension.

We will show the latter part. Take the inflation-restriction exact sequence, i.e.,

$$H^g_2(\text{Adj}(Z)) \longrightarrow \text{Ker}(p_*) \longrightarrow H^g_1(\text{Adj}(Y)) \longrightarrow H^g_1(\text{Adj}(Z)) \longrightarrow 0 \quad (\text{exact}).$$

By connectivities the third map from $H^g_1(\text{Adj}(Y)) = Z$ is an isomorphism. Since Theorem 2.1 says that $H^g_2(\text{Adj}(Z))$ is annihilated by $t_Z$, so is the kernel $\text{Ker}(p_*)$ as desired.

Next, we will compute the second homology of $\widetilde{X}$ (Theorem 5.4) by showing propositions:

**Proposition 5.2.** For any connected quandle $X$, the extended one $\widetilde{X}$ above is also connected.

Proof. It is enough to show that the identity $1_{\widetilde{X}} \in \widetilde{X} = \text{Ker}(\varepsilon_0)$ is transitive to any element $h$ in $\widetilde{X}$. Expand $h \in \widetilde{X} \subset \text{Adj}(X)$ as $h = e_{x_1}^{\varepsilon_1} \cdots e_{x_n}^{\varepsilon_n}$ for some $x_i \in X$ and $\varepsilon_i \in \mathbb{Z}$. Since $h \in \text{Ker}(\varepsilon_0)$, note $\sum \varepsilon_i = 0$. The connectivity of $X$ ensures some $g_i \in \text{Adj}(X)$ so that $a \cdot g_i^{\varepsilon_i} = x_i$. Therefore $g_i^{-\varepsilon_i} e_a g_i^{\varepsilon_i} = e_a g_i^{\varepsilon_i} = e_{x_i}$ by (1). In the sequel, we have

$$(\cdots (1_{\widetilde{X}} \cdot g_1^{\varepsilon_1}) \cdots \cdot g_n^{-\varepsilon_n}) = e_{x_1}^{\sum \varepsilon_1} 1_{\widetilde{X}} (g_1^{-\varepsilon_1} e_a g_1^{\varepsilon_1}) \cdots (g_n^{-\varepsilon_n} e_a g_n^{\varepsilon_n}) = e_{x_1}^{\varepsilon_1} \cdots e_{x_n}^{\varepsilon_n} = h.$$ 

These equalities in $\widetilde{X}$ imply the transitivity of $\widetilde{X}$. \hfill \Box

**Proposition 5.3.** Let $p_* : \text{Adj}(\widetilde{X}) \to \text{Adj}(X)$ be the epimorphism induced from the covering $p : \widetilde{X} \to X$. Then, under the canonical action of $\text{Adj}(\widetilde{X})$ on $\widetilde{X}$, the stabilizer $\text{Stab}(1_{\widetilde{X}})$ of $1_{\widetilde{X}}$ is equal to $Z \times \text{Ker}(p_*)$ in $\text{Adj}(\widetilde{X})$. Furthermore, the summand $Z$ is generated by $1_{\widetilde{X}}$.

Proof. We can easily see that the stabilizer of $1_{\widetilde{X}}$ via the previous action $\text{Ker}(\varepsilon_0) = \widetilde{X} \cap \text{Adj}(X)$ is $\text{Stab}(1_{\widetilde{X}}) = \{ e_a^n \}_{a \in \mathbb{Z}} \subset \text{Adj}(X)$ exactly. Notice that any central extension of $Z$ is trivial; Since the $p_*$ is a central extension (Proposition 5.1), the restriction $p_* : \text{Stab}(1_{\widetilde{X}}) \to \text{Stab}(1_{\widetilde{X}}) = Z$ implies the required identity $\text{Stab}(1_{\widetilde{X}}) = Z \times \text{Ker}(p_*)$. \hfill \Box

**Theorem 5.4.** The second quandle homology of the extended quandle $\widetilde{X}$ is isomorphic to the kernel of the $p_* : \text{Adj}(\widetilde{X}) \to \text{Adj}(X)$. Namely, $H^g_2(\widetilde{X}) \cong \text{Ker}(p_*)$. In particular, if $t_X < \infty$, then $H^g_2(\widetilde{X})$ is annihilated by the type $t_X$, according to Proposition 5.1.

Proof. Note that $\widetilde{X}$ is connected (Proposition 5.2) and the kernel $\text{Ker}(p_*)$ is abelian (Proposition 5.3). Accordingly, the desired isomorphism $H^g_2(\widetilde{X}) \cong (\text{Ker}(\varepsilon_{\widetilde{X}}) \cap \text{Stab}(1_{\widetilde{X}}))_{ab} = \text{Ker}(p_*)$ follows immediately from Proposition 5.3 and Theorem 4.1. \hfill \Box

Finally, we now discuss the third group homology.

**Proposition 5.5.** The universal covering $p : \widetilde{X} \to X$ induces a $[1/t_X]$-isomorphism $p_* : H^g_3(\text{Adj}(\widetilde{X})) \cong H^g_3(\text{Adj}(X))$. 




Proof. By connectivity of $\tilde{X}$ and Theorem 2.1, $H_2^{gr}(\text{Adj}(\tilde{X}))$ and $H_2^{gr}(\text{Adj}(X))$ are annihilated by $t_X$. Furthermore, since the epimorphism $p : \text{Adj}(\tilde{X}) \to \text{Adj}(X)$ is a central extension whose kernel is annihilated by $t_X$ (Proposition 5.1), we readily obtain the $[1/t_X]$-isomorphism $p_* : H_3^{gr}(\text{Adj}(\tilde{X})) \cong H_3^{gr}(\text{Adj}(X))$ from the Lyndon-Hochschild sequence of $p_*$. \hfill \Box

These properties will play a key role to prove the main theorem in \cite{N1}.

6 Proof of Theorem 2.1

The purpose of this section is to prove Theorem 2.1. Let us begin by reviewing the rack space introduced by Fenn-Rourke-Sanderson \cite[FRS]. Let $X$ be a quandle with discrete topology. We set up a disjoint union $\bigcup_{n \geq 0}([0, 1] \times X)^n$, and consider the relations given by

$$(t_1, x_1, \ldots, x_{j-1}, 1, x_j, t_{j+1}, \ldots, t_n, x_n) \sim (t_1, x_1 \triangleleft x_j, \ldots, t_{j-1}, x_{j-1} \triangleleft x_j, t_{j+1}, x_{j+1}, \ldots, t_n, x_n),$$

$$(t_1, x_1, \ldots, x_{j-1}, 0, x_j, t_{j+1}, \ldots, t_n, x_n) \sim (t_1, x_1, \ldots, t_{j-1}, x_{j-1}, t_j, x_{j+1}, \ldots, t_n, x_n).$$

Then, the \textit{rack space} $BX$ is defined to be the quotient space. By construction, we have a cell decomposition of $BX$ by regarding the projection $\bigcup_{n \geq 0}([0, 1] \times X)^n \to BX$ as characteristic maps. From the 2-skeleton of $BX$, we have $\pi_1(BX) \cong \text{Adj}(X)$. Furthermore, let $c : BX \hookrightarrow K(\pi_1(BX), 1)$ be an inclusion obtained by killing the higher homotopy groups of $BX$.

**Theorem 6.1.** Let $X$ be a connected quandle of type $t$, and let $t < \infty$. For $n = 2$ and $3$, the induced map $c_* : H_n(BX) \to H_n^{gr}(\text{Adj}(X))$ is annihilated by $t$.

**Remark 6.2.** This is still more powerful and general than a result of Clauwens \cite[Proposition 4.4]{Cla}, which stated that, if a finite quandle $X$ satisfies a certain condition, then the composite $(\psi_X)_* \circ c_* : H_n(BX) \to H_n^{gr}(\text{Adj}(X)) \to H_n^{gr}(\text{Inn}(X))$ is annihilated by $|\text{Inn}(X)|/|X|$ for any $n \in \mathbb{N}$. Here note from Lemma 3.6 that $t$ is a divisor of the order $|\text{Inn}(X)|/|X|$.

Since the induced map $c_* : H_2(BX) \to H_2^{gr}(\text{Adj}(X))$ is known to be surjective (cf. Hopf’s theorem \cite[II.5]{Bro}), Theorem 2.1 is immediately obtained from Theorem 6.1 and the inflation-restriction exact sequence of \cite{F}. Hence, we may turn into proving Theorem 6.1.

To this end, we give a brief review of the rack and quandle (co)homologies. Let $C_n^R(X)$ be the free right $\mathbb{Z}$-module generated by $X^n$. Define a boundary $\partial_n^R : C_n^R(X) \to C_{n-1}^R(X)$ by

$$\partial_n^R(x_1, \ldots, x_n) = \sum_{1 \leq i \leq n} (-1)^i (\langle x_1 \triangleleft x_i, \ldots, x_{i-1} \triangleleft x_i, x_{i+1}, \ldots, x_n \rangle - \langle x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n \rangle).$$

The composite $\partial_{n-1}^R \circ \partial_n^R$ is known to be zero. The homology is denoted by $H_n^R(X)$ and is called the \textit{rack homology}. As is known, the cellular complex of the rack space $BX$ is isomorphic to the complex $(C_n^R(X), \partial_n^R)$. Furthermore, following \cite{CJKLS}, let $C_n^D(X)$ be a submodule of $C_n^R(X)$ generated by $n$-tuples $(x_1, \ldots, x_n)$ with $x_i = x_{i+1}$ for some $i \in \{1, \ldots, n-1\}$. It can be easily seen that the submodule $C_n^D(X)$ is a subcomplex of $C_n^R(X)$. Then, the \textit{quandle homology}, $H_n^Q(X)$, is defined to be the homology of the quotient complex $C_n^R(X)/C_n^D(X)$.\hfill \Box
Furthermore, we now observe concretely the map $c_n : H_n(BX) \to H^*_{gr}(\text{Adj}(X))$ for $n \leq 3$. Let us recall the (non-homogenous) standard complex $C^*_{gr}(\text{Adj}(X))$ of Adj(X); see e.g. [Bro] §I.5. The map $c_n$ can be described in terms of their complexes. In fact, Kabaya [Kab, §8.4] considered homomorphisms $c_n : C^R_n(X) \to C^*_{gr}(\text{Adj}(X))$ for $n \leq 3$ defined by setting

$$c_1(x) = e_x,$$

$$c_2(x, y) = (e_x, e_y) - (e_y, e_{xy^<}),$$

$$c_3(x, y, z) = (e_x, e_y, e_z) - (e_x, e_z, e_{yx^<}) + (e_y, e_z, e_A) - (e_y, e_{xy^<}, e_z) + (e_z, e_{xy^<}, e_{yz^<}) - (e_z, e_{xy^<}, e_A),$$

where we denote $(x<y)<z \in X$ by $A$ for short. As is shown (see [Kab §8.4]), the induced map on homology coincides with the map above $c_n$ up to homotopy.

We will construct a chain homotopy between $t \cdot c_n$ and zero, when $X$ is connected and of type $t$. Define a homomorphism $h_i : C^R_i(X) \to C^*_{i+1}(\text{Adj}(X))$ by setting

$$h_1(x) = \sum_{1 \leq j \leq t-1} (e_x, e_j^i),$$

$$h_2(x, y) = \sum_{1 \leq j \leq t-1} (e_x, e_y, e_j^i) - (e_x, e^i, e_y) - (e_y, e_{xy^<}, e_j^i) + (e_y, e_j^i, e_y),$$

$$h_3(x, y, z) = \sum_{1 \leq j \leq t-1} ((e_x, e_y, e_z, e_A) - (e_x, e_z, e_{yx^<}, e_A^i) - (e_x, e_y, e_{xy^<}, e_z) - (e_y, e_{xy^<}, e_z, e_A)$$

$$+(e_x, e_z, e_{yx^<}, e_{yz^<}) + (e_z, e_{yx^<}, e_{yz^<}, e_A) + (e_x, e_{yx^<}, e_{yz^<}) - (e_x, e_{yx^<}, e_{yz^<}),$$

$$+(e_y, e_{xy^<}, e_A, e^i_A) - (e_y, e_{xy^<}, e_{yx^<}, e_A) + (e_y, e_{xy^<}, e_{yx^<}, e_{yz^<}) + (e_y, e_{xy^<}, e_{yz^<}, e_z)).$$

**Lemma 6.3.** Let $X$ be as above. Then we have the equality $h_1 \circ \partial^R_2 - \partial^*_{gr} \circ h_2 = t \cdot c_2$.

**Proof.** Compute the both terms $h_1 \circ \partial^R_2$ and $\partial^*_{gr} \circ h_2$ in the left hand side as

$$h_1 \circ \partial^R_2(x, y) = \sum (e_x, e_j^i) - (e_{xy^<}, e_{dy<}).$$

$$\partial^*_{gr} \circ h_2(x, y) = \partial^R_{gr} (\sum (e_x, e_j^i, e_y) - (e_x, e^i, e_y) - (e_y, e_{xy^<}, e_j^i) + (e_y, e_j^i, e_y))$$

$$= \left( \sum (e_y, e_{xy^<}) - (e_x, e_j^i, e_y) + (e_x, e_j^i, e_y) - (e_y, e_{xy^<}, e_j^i) + (e_y, e_j^i, e_y) + (e_x, e_j^i, e_y) - (e_y, e_{xy^<}, e_j^i) + (e_y, e_j^i, e_y)ight)$$

$$= t((e_y, e_{xy^<}) - (e_x, e_y)) + (e_j^i, e_y) - (e_y, e_j^i, e_y) + (e_j^i, e_y) + h_1 \circ \partial^R_2(x, y),$$

$$= -t \cdot c_2(x, y) + h_1 \circ \partial^R_2(x, y).$$

Here we use Lemma 6.3 for the last equality. Hence, the equalities complete the proof. \qed

**Lemma 6.4.** Let $X$ be as above. The difference $h_2 \circ \partial^R_2 - \partial^*_{gr} \circ h_3$ is chain homotopic to $t \cdot c_3$. 


Proof. This is similarly proved by a direct calculation. To this end, recalling the notation $A = (x < y) < z$, we remark two identities

$$e_x e_A = e_x e_y e_z, \quad e_{y < z} e_A = e_{x < z} e_{y < z} \in \text{Adj}(X).$$

Using them, a tedious calculation can show that the difference $(t \cdot c_3 - h_2 \circ \partial_3^R - \partial_3^R \circ h_3)(x, y, z)$ is equal to

$$(e_y, e_z, e_A^t) - (e_y^t, e_y, e_z) + (e_y^t, e_z, e_{x < y}) - (e_y, e_x < y) + (e_z, e_{x < z}, e_y < z) - (e_z, e_y < z, e_A^t) + \sum_{1 \leq j \leq t-1} (e_y, e_y^j, e_y) - (e_y < z, e_y^j < z, e_y < z).$$

Note that this formula is independent of any $x \in X$ since the identity $(e_a)^t = (e_b)^t$ holds for any $a, b \in X$ by Lemma 3.5. However, the map $c_3(x, y, z)$ with $x = y$ is zero by definition. Hence, the map $t \cdot c_3$ is null-homotopic as desired.

Proof of Theorem 6.1. The map $t \cdot c_*$ are obviously null-homotopic by Lemmas 6.3 and 6.4.

The proof was an ad hoc computation in an algebraic way; however the theorem should be easily shown by a topological method:

Problem 6.5. Does the $t$-vanishing of the map $c_* : H_n(BX) \to H_n^R(\text{Adj}(X))$ hold for any $n \in \mathbb{N}$? Provide its topological proof. Further, how about the non-connected quandles?

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