Sharp Estimates for the First Eigenvalues of the Bi-drifting Laplacian

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Abstract

In the present paper we study some kinds of the problems for the bi-drifting Laplacian operator and get some sharp lower bounds for the first eigenvalue for these eigenvalue problems on compact manifolds with boundary (also called a smooth metric measure space) and weighted Ricci curvature bounded inferiorly.

keywords: Drifting Laplacian; Bakry-Emery Ricci Curvature; Eigenvalues

1 Introduction

For a given complete n-dimensional Riemannian manifold \((M, \langle \cdot, \cdot \rangle)\) with a metric \(\langle \cdot, \cdot \rangle\), the triple \((M, \langle \cdot, \cdot \rangle, e^{-\phi}dv)\) is called a smooth metric measure space, where \(\phi : M \rightarrow \mathbb{R}\) is a smooth real-valued function on \(M\) and \(dv\) is the Riemannian volume element related to \(\langle \cdot, \cdot \rangle\) (sometimes, we also call \(dv\) the volume density).

On a smooth metric measure space \((M, \langle \cdot, \cdot \rangle, e^{-\phi}dv)\), we can define the so-called drifting Laplacian (also called weighted Laplacian) \(L_\phi\) as follows

\[
    L_\phi := \Delta - \langle \nabla \phi, \nabla (\cdot) \rangle,
\]

where \(\nabla\) and \(\Delta\) are the gradient operator and the Laplace operator, respectively. Some interesting results concerning eigenvalues of the drifting Laplacian can be found, for instance, in [10, 13, 19, 27, 37], among others.

On smooth metric measure spaces, we can also define the so-called Bakry-Emery Ricci tensor \(Ric_\phi\) by

\[
    Ric_\phi = Ric + \nabla^2 \phi, \tag{1.1}
\]
which is also called the weighted Ricci curvature. Here, $Ric$ is the Ricci curvature on $M$. The equation $Ric_\phi = k\langle \cdot, \cdot \rangle$ for some constant $k$ is just the gradient Ricci soliton equation, which plays an important role in the study of Ricci flow. For $k = 0$, $k > 0$ and $k < 0$, the gradient Ricci soliton $(M, \langle \cdot, \cdot \rangle, e^{-\phi}dv, k)$ is called steady, shrinking, and expanding, respectively.

In [8], Chen-Cheng-Wang-Xia gave some lower bounds for the first eigenvalue of four kinds of eigenvalue problems of the biharmonic operator on compact manifolds with boundary and positive Ricci curvature, where two of them are in the direction of the buckling and clamped plate problems. Posteriorly in [13], Du and Bezerra extended the results of Chen-Cheng-Wang-Xia for the bi-drifting Laplacian operator. In particular, in the Theorems 1.7-1.9 they obtained lower estimates for the first eigenvalue of some eigenvalue problems for the bi-drifting Laplacian operator, defined in an smooth metric measure space $(M, \langle \cdot, \cdot \rangle, e^{-\phi}d\nu)$ with boundary $\partial M$ and a limiting condition in the Bakry-Emery Ricci curvature, if the weighted mean curvature $H_\phi$ of $\partial M$ is nonnegative. The weighted mean curvature $H_\phi$ will be defined in the next section.

In the first part of this paper, in the Theorem 1.1 and Theorem 1.2 we will improve the results of Du and Bezerra, removing the condition at the weighted mean curvature of the boundary. The results are shown below.

**Theorem 1.1** Let $(M, \langle \cdot, \cdot \rangle, e^{-\phi}d\nu)$ be an $n(\geq 2)$-dimensional compact connected smooth metric measure space with boundary $\partial M$ and denote by $\nu$ the outward unit normal vector field of $\partial M$. Denote by $\lambda_1$ the first eigenvalue with Dirichlet boundary condition of the drifting Laplacian of $M$ and let $\Gamma_1$ be the first eigenvalue of the problem:

$$\begin{cases}
L_\phi^2 u = \Gamma u & \text{in } M, \\
u u = \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial M.
\end{cases}$$

Assume that

$$Ric_\phi \geq \frac{|
abla \phi|^2}{na} + b,$$  

for some positive constants $a$ and $b$. Then we have

$$\Gamma_1 > \lambda_1 \left( \frac{\lambda_1}{n(a+1)} + b \right).$$

Changing the first equation in 1.2 and maintaining the boundary condition, we have the following result:

**Theorem 1.2** Under the assumption of Theorem 1.1 let $\eta_1$ the first eigenvalue of the problem :

$$\begin{cases}
L_\phi^2 u = -\eta L_\phi u & \text{in } M, \\
u u = \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial M.
\end{cases}$$

Then we have

$$\eta_1 > \frac{\lambda_1}{n(a+1)} + b.$$
The eigenvalue problems (1.2) and (1.5) should be compared with the clamped plate problem and the buckling problem for drifting operator, respectively. The later two ones are as follows:

\[
\begin{aligned}
\begin{cases}
L_0^2 u = \Lambda u & \text{in } M, \\
u = \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial M.
\end{cases}
\end{aligned}
\] (1.7)

\[
\begin{aligned}
\begin{cases}
L_0^2 u = -\eta L_0 u & \text{in } M, \\
u = \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial M.
\end{cases}
\end{aligned}
\] (1.8)

The Theorem 1.1 and Theorem 1.2 shows that the first eigenvalue of problems (1.2) and (1.5) are closely related to the first Dirichlet eigenvalue of the drifting Laplacian. Next we will be interested in some types Steklov eigenvalue problems of the bi-drifting Laplace operator. The eigenvalue problems we are interested in are as follows:

\[
\begin{aligned}
\begin{cases}
L_0^2 u = 0 & \text{in } M, \\
u = \frac{\partial (L_0 u)}{\partial \nu} + \xi u = 0 & \text{on } \partial M.
\end{cases}
\end{aligned}
\] (1.9)

\[
\begin{aligned}
\begin{cases}
L_0^2 u = 0 & \text{in } M, \\
u = \frac{\partial^2 u}{\partial \nu^2} - q \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial M.
\end{cases}
\end{aligned}
\] (1.10)

\[
\begin{aligned}
\begin{cases}
L_0^2 u = 0 & \text{in } M, \\
u = \frac{\partial (L_0 u)}{\partial \nu} + \xi u = 0 & \text{on } \partial M.
\end{cases}
\end{aligned}
\] (1.11)

and

\[
\begin{aligned}
\begin{cases}
L_0^2 u = 0 & \text{in } M, \\
u = \frac{\partial (L_0 u)}{\partial \nu} + \beta \Delta u + \varsigma u = 0 & \text{on } \partial M.
\end{cases}
\end{aligned}
\] (1.12)

Elliptic problems with parameters in the boundary conditions are called Steklov problems from their first appearance in [33]. The problem (1.9) was considered by Kuttler [25] and Payne [29] who studied the isoperimetric properties of the first eigenvalue \(p_1\) which is the sharp constant for \(L^2\) a priori estimates for solutions of the (second order) Laplace equation under nonhomogeneous Dirichlet boundary conditions. One can see that \(p_1\) is positive and given by

\[
p_1 = \min_{w \mid \partial M = 0, \ w \neq \text{const.}} \frac{\int_M (L_0 w)^2 e^{-\phi} dv}{\int_{\partial M} (\frac{\partial w}{\partial \nu})^2 e^{-\phi} dA}. \quad (1.13)
\]

The problem (1.10) is a natural Steklov problem for the drifting Laplacian, and is equivalent to (1.9) when the mean curvature of \(\partial M\) and \(\phi\) are constants. In Theorem 1.3 we have a sharp relation between the first eigenvalues of (1.9) and (1.10).
Theorem 1.3 Let \((M, \langle \cdot, \cdot \rangle, e^{-\phi} dv)\) be an \(n\)-dimensional compact smooth metric measure space with boundary \(\partial M\) and non-negative Ricci Bakry-Emery curvature. Denote by \(p_1\) and \(q_1\) the first eigenvalue of the problems (1.9) and (1.10), respectively. Then we have
\[
q_1 \geq \frac{p_1}{n(a+1)},
\]
with equality holding if and only if \(M\) is isometric to a ball in \(\mathbb{R}^n\).

The next result is a sharp lower bound for \(p_1\).

Theorem 1.4 Let \((M, \langle \cdot, \cdot \rangle, e^{-\phi} dv)\) be an \((n \geq 2)\)-dimensional compact smooth metric measure space. Suppose that
\[
\text{Ric}_\phi \geq \frac{\vert \nabla \phi \vert^2}{na} - b,
\]
for some positive constants \(a\) and \(b\). Denote by \(\lambda_1\) the first eigenvalue with Dirichlet boundary condition of the drifting Laplacian of \(M\) and let \(p_1\) be the first eigenvalue of the problem (1.9). If the weighted mean curvature of \(\partial M\) is bounded below by a positive constant \(c\), then we have
\[
p_1 \geq \frac{n(a+1)(n-1)c\lambda_1}{n(a+1)(\lambda_1+b) - \lambda_1},
\]
with equality holding if and only if \(M\) is isometric to a Euclidean \(n\)-ball of radius \(1/c\).

The problem (1.11) was first studied in [26] where some estimates for the first non-zero eigenvalue \(\xi\) were obtained. When \(M\) is an Euclidean ball, all the eigenvalues of the problem (1.10) have been recently obtained in [35]. Also, the authors proved an isoperimetric upper bound for \(\xi_1\) when \(M\) is a bounded domain in \(\mathbb{R}^n\). The Rayleigh-Ritz formula for \(\xi_1\) is:
\[
\xi_1 = \min \frac{\int_M (L \phi w)^2 e^{-\phi} dv}{\int_{\partial M} w^2 e^{-\phi} dA},
\]
where \(0 \neq w \in H^2(M), \int_{\partial M} w = 0 = \partial_v w|_{\partial M}\).

The problem (1.12) is a so called Wentzell problem for the bi-drifting Laplace operator which is motivated by (1.11) and the following Wentzell-Laplace problem:
\[
\begin{cases}
\Delta u = 0 \text{ in } M, \\
-\beta \Delta u + \partial_v u = \lambda u \text{ on } \partial M,
\end{cases}
\]
where \(\beta\) is a given non-negative number. The problem (1.18) has been studied recently, in [12], [35], etc. The first non-zero eigenvalue of (1.12) can be characterized as
\[
\zeta_{1,\beta} = \min \frac{\int_M (L \phi w)^2 e^{-\phi} dv + \beta \int_{\partial M} \vert \nabla w \vert^2 e^{-\phi} dA}{\int_{\partial M} w^2 e^{-\phi} dA},
\]
(1.19)
where $0 \neq w \in H^2(M), \int_{\partial M} w = 0 = \partial_{\nu} w|_{\partial M}$.

From (1.17), one can see that if $\beta > 0$, $\xi_1$ is the first non-zero eigenvalue of the Steklov problem (1.11) and $\lambda_1$ the first non-zero eigenvalue of the drifting Laplacian of $\partial M$, then we have

$$
\zeta_{1,\beta} \geq \xi_1 + \beta \lambda_1,
$$

(1.20)

with equality holding if and only if any eigenfunction $f$ corresponding to $\zeta_{1,\beta}$ is an eigenfunction corresponding to $\xi_1$ and $f|_{\partial M}$ is an eigenfunction corresponding to $\lambda_1$. Our last result is a lower bound for $\zeta_1$.

**Theorem 1.5** Let $(M, \langle \cdot, \cdot \rangle, e^{-\phi} dv)$ be an $n$-dimensional compact smooth metric measure space with boundary $\partial M$ and suppose that (1.15) is satisfied. Assume that the principal curvatures of $\partial M$ are bounded below by a positive constant $c$ and denote by $\zeta_1$ the first eigenvalue of the problem(1.12). Then we have

$$
\zeta_1 > \frac{cn(a+1)\lambda_1 \mu_1}{n(a+1)(\mu_1 + \beta) - \mu_1} + \beta \lambda_1,
$$

(1.21)

where $\mu_1$ and $\lambda_1$ are the first nonzero Neumann eigenvalue of the drifting Laplacian of $M$ and the first nonzero eigenvalue of the drifting Laplacian of $\partial M$, respectively.

**2 Proof of Theorems**

In this section, we will prove the theorems of the section 1. Before doing this, let us recall the Reilly’s formula. Let $M$ be an $n$-dimensional compact manifold $M$ with boundary $\partial M$. We will often write $\langle \cdot, \cdot \rangle$ the Riemannian metric on $M$ as well as that induced on $\partial M$. Let $\nabla$ and $\Delta$ be the connection and the Laplacian on $M$, respectively. Let $\nu$ be the unit outward normal vector of $\partial M$. The shape operator of $\partial M$ is given by $S(X) = \nabla_X \nu$ and the second fundamental form of $\partial M$ is defined as $II(X, Y) = \langle S(X), Y \rangle$, here $X, Y \in \partial M$. The eigenvalues of $S$ are called the principal curvatures of $\partial M$.

We will denote the weighted measure by $d\mu = e^{-\phi} dv$ and $d\vartheta = e^{-\phi} dA$ on $M$ and $\partial M$, respectively. The weighted mean curvature as a natural generalization of the mean curvature for Riemann manifolds with density and is defined by

$$
H_\phi = H - \frac{1}{n-1} \phi_\nu,
$$

(2.1)

where $H$ denotes the usual mean curvature of $\partial M$, given by $H = \frac{1}{n-1} tr S$, and $tr S$ denotes the trace of $S$. For a smooth function $f$ defined on an $n$-dimensional compact manifold $M$ with boundary, Ma and Du ([27]) extended the Reilly’s formula for Riemann manifolds with density and showed that the
The following identity holds if

\[
\int_M (L_\phi f)^2 - |\nabla^2 f|^2 - \text{Ric}_\phi(\nabla f, \nabla f) d\mu \\
= \int_{\partial M} 2(L_\phi f) f_\nu + (n-1)H_\phi(f_\nu)^2 + II(\nabla f, \nabla f) d\vartheta. 
\]  
(2.2)

Here \( \mathcal{L}_\phi = \Delta - \langle \nabla \phi, \nabla (\cdot) \rangle \) and \( |\nabla^2 f| \) are drifting operator and the Hessian of \( f \) on \( \partial M \), with respect to the induced metric on \( \partial M \), respectively.

**Proof of Theorem 1.1** Let \( f \) be an eigenfunction of the problem corresponding to the first eigenvalue \( \Gamma_1 \), that is,

\[
L_\phi^2 f = \Gamma_1 f \quad \text{in} \quad M, \quad f = \frac{\partial^2 f}{\partial \nu^2} = 0 \quad \text{on} \quad \partial M. 
\]  
(2.3)

Multiplying the first equality in (2.3) by \( f \) and integrating on \( M \), follows from the divergence theorem

\[
\Gamma_1 \int_M f^2 d\mu = \int_M (L_\phi f)^2 d\mu - \int_{\partial M} hL_\phi f d\vartheta, 
\]  
(2.4)

where \( h = \frac{\partial f}{\partial \nu}|_{\partial M} \). We now consider the following Lemma (5):

**Lemma 2.1** Let \( M \) an \( n \)-dimensional Riemannian manifold with boundary \( \partial M \) and let \( f \in C^\infty(M) \). Then for all \( p \in M \) we have

\[
L_\phi f = \mathcal{L}_\phi f + (n-1)H_\phi f_\nu + \nabla^2 f(\nu, \nu), 
\]  
(2.5)

where \( H_\phi \) is the weighted mean curvature of \( \partial M \) and \( L_\phi \) and \( \mathcal{L}_\phi \) are the drifting Laplacian operators defined in \( M \) and \( \partial M \) respectively.

Since \( f|_{\partial M} = \frac{\partial^2 f}{\partial \nu^2}|_{\partial M} = 0 \) we have

\[
L_\phi f|_{\partial M} = (n-1)H_\phi h. 
\]  
(2.6)

From Reilly’s formula, we infer

\[
\int_M (L_\phi f)^2 - |\nabla^2 f|^2 - \text{Ric}_\phi(\nabla f, \nabla f) d\mu = (n-1) \int_{\partial M} H_\phi h^2 d\vartheta. \]
(2.7)

Combining (2.4), (2.6) and (1.1), we get

\[
\Gamma_1 = \frac{\int_M (|\nabla^2 f|^2 + \text{Ric}_\phi(\nabla f, \nabla f)) d\mu}{\int_M f^2} \frac{\int_M f^2}{\int_M f^2} \geq \frac{\int_M (|\nabla^2 f|^2 + \frac{\nabla \phi}{na} + b) |\nabla f|^2) d\mu}{\int_M f^2}. 
\]  
(2.8)
We get easily that
\[(\triangle f)^2 = (L_\phi f + \langle \nabla \phi, \nabla f \rangle)^2 \geq \frac{(L_\phi f)^2}{a+1} - \frac{\langle \nabla \phi, \nabla f \rangle^2}{a}. \tag{2.9}\]
The Schwarz inequality implies that
\[|\nabla^2 f|^2 \geq \frac{1}{n}(\triangle f)^2, \tag{2.10}\]
with equality holding if and only if
\[\nabla^2 f = \frac{\triangle f}{n}. \tag{2.11}\]
It then follows from the Schwarz inequality that
\[|\nabla^2 f|^2 \geq \frac{1}{n}(\triangle f)^2 \geq \frac{(L_\phi f)^2}{n(a+1)} - \frac{\langle \nabla \phi, \nabla f \rangle^2}{na}. \tag{2.12}\]
Therefore, substituting (2.12) in (2.8), we have
\[
\Gamma_1 \geq \frac{\int_M \left( \frac{(L_\phi f)^2}{n(a+1)} - \frac{|\nabla \phi|^2|\nabla f|^2}{na} + \left( \frac{|\nabla \phi|^2}{na} + b \right)|\nabla f|^2 \right) d\mu}{\int_M f^2 d\mu}
\]
\[= \frac{\int_M \left( \frac{(L_\phi f)^2}{n(a+1)} + b|\nabla f|^2 \right) d\mu}{\int_M f^2 d\mu}. \tag{2.13}\]
with equality holding if and only (2.11) holds and
\[Ric_\phi = \frac{|\nabla \phi|^2}{na} + b.\]
On the other hand, since \(f\) is not a zero function which vanishes on \(\partial M\), we know that
\[\int_M (L_\phi f)^2 d\mu \geq \lambda_1 \int_M |\nabla f|^2 d\mu \geq \lambda_1^2 \int_M f^2 d\mu. \tag{2.14}\]
with equality holding if and only if \(f\) is a first eigenfunction of the Dirichlet problem for drifting Laplacian of \(M\). Thus by (2.13) and (2.14) we conclude that
\[\Gamma_1 \geq \lambda_1 \left( \frac{\lambda_1}{n(a+1)} + b \right). \tag{2.15}\]
suppose that \(\Gamma_1 = \lambda_1 \left( \frac{\lambda_1}{n(a+1)} + b \right)\) is valid. Then (2.9) becomes
\[(\triangle f)^2 = (L_\phi f + \langle \nabla \phi, \nabla f \rangle)^2 = \frac{(L_\phi f)^2}{a+1} - \frac{\langle \nabla \phi, \nabla f \rangle^2}{a}. \tag{2.16}\]
which means that \( \phi \) is not a constant and \( \Delta f - \frac{1}{a} \langle \nabla f, \nabla \phi \rangle = 0 \) holds everywhere on \( M \). Multiplying the above inequality by \( f \) and integrating on \( M \) with respect to \( e^{\frac{\phi}{2}} d\nu \) give that

\[
0 = \int_M f \left( \Delta f - \frac{1}{a} \langle \nabla f, \nabla \phi \rangle \right) e^{\frac{\phi}{2}} d\nu = - \int_M |\nabla f|^2 e^{\frac{\phi}{2}} d\nu \quad (2.17)
\]

From above equality, we know that \( f \) is a constant function on \( M \), which is a contradiction since \( f \) is the first eigenfunction of bi-drifting Laplacian and cannot be a constant. Therefore, we have \( \Gamma_1 > \lambda_1 \left( \frac{\lambda_1}{n(a+1)} + b \right) \).

**Proof of Theorem 1.2** The discussions are similar to those in the proof of Theorem 1.1. Let \( g \) be the eigenfunction of the problem (1.5) corresponding to the first eigenvalue \( \eta_1 \), that is,

\[
L_2 \phi g = -\eta_1 \Delta g \text{ in } M, \quad g = \partial^2 g / \partial \nu^2 = 0 \text{ on } \partial M. \quad (2.18)
\]

Multiplying the first equality in (2.18) by \( g \) and integrating on \( M \), follows from the divergence theorem that

\[
\eta_1 \int_M |\nabla g|^2 d\mu = \int_M (L_\phi g)^2 d\mu - \int_{\partial M} s L_\phi g d\vartheta, \quad (2.19)
\]

where \( s = \partial g / \partial \nu |_{\partial M} \). Also, we have

\[
L_\phi g |_{\partial M} = (n-1) H_\phi s. \quad (2.20)
\]

Hence

\[
\eta_1 = \frac{\int_M (L_\phi g)^2 d\mu - (n-1) \int_{\partial M} H_\phi s^2 d\mu}{\int_M |\nabla g|^2 d\mu},
\]

which, by hypothesis and by Reilly’s formula and (2.14) gives

\[
\eta_1 \geq \frac{\int_M \left( |\nabla^2 g|^2 + \left( \frac{|\nabla \phi|^2}{na} + b \right) |\nabla g|^2 \right) d\mu}{\int_M |\nabla g|^2 d\mu} \geq \frac{\int_M \left( \frac{(L_\phi g)^2}{n(a+1)} + b |\nabla g|^2 \right) d\mu}{\int_M |\nabla g|^2 d\mu}. \quad (2.21)
\]

We can see that (2.14) also holds for \( g \), and therefore

\[
\eta_1 \geq \frac{\lambda_1}{n(a+1)} + b.
\]
In a similar way to what was done in the proof of the Theorem 1.1 if we suppose that \( \eta_1 = \frac{\lambda_1}{n(a+1)} + b \), similarly to what was done,

\[
0 = \int_M g \left( \Delta g - \frac{1}{a} (\nabla g, \nabla \phi) \right) e^{\frac{1}{a}\phi} dv = - \int_M |\nabla g|^2 e^{\frac{1}{a}\phi} dv
\]

which is a contradiction since \( g \) is the first eigenfunction of (1.5) and cannot be a constant. Therefore, we have \( \eta_1 > \frac{\lambda_1}{n(a+1)} + b \).

\[\square\]

**Proof of Theorem 1.3.** Let \( w \) be an eigenfunction corresponding to the first eigenvalue \( q_1 \) of the problem (1.10), that is,

\[
\begin{align*}
& L_2^2 w = 0 \text{ in } M, \\
& w = -q_1 \frac{\partial w}{\partial \nu}, w_{\partial M} = 0 \quad \text{on } \partial M.
\end{align*}
\]

Note that \( w \) is not a constant since \( w|_{\partial M} = 0 \). Let \( \eta = \partial_{\nu} w|_{\partial M} \). Then \( \eta \neq 0 \), otherwise, we would deduce from \( w|_{\partial M} = 0 \) that

\[
\int_{\partial M} \left( L_2^2 w \right) e^{-\phi} dA = \int_M \left( L_2^2 w \right) e^{-\phi} dv = 0,
\]

and (2.5) that

\[
L_2^2 w (\partial_{\nu} w + (n-1) H_2 w + \nabla^2 w(\nu, \nu))|_{\partial M} = 0.
\]

By divergence theorem, we have

\[
0 = \int_{\partial M} (L_2^2 w \nabla w \cdot \nu) e^{-\phi} dA = \int_M \left( |\nabla L_2^2 w|^2 + L_2^2 w L_2^2 w \right) e^{-\phi} dv = 0,
\]

and so \( L_2^2 w = 0 \) on \( M \). We also have

\[
0 = \int_{\partial M} \left( w \nabla w \cdot \nu \right) e^{-\phi} dA = \int_M \left( |\nabla w|^2 + w L_2^2 w \right) e^{-\phi} dv,
\]

that implies \( w = 0 \). This is a contradiction. From \( w|_{\partial M} = 0 \), one gets again from divergence theorem that

\[
\int_M \left( \nabla w, \nabla L_2^2 w e^{-\phi} dv = - \int_M w L_2^2 w e^{-\phi} dv = 0,
\]

and therefore

\[
\int_{\partial M} \left( L_2^2 w \nabla w \cdot \nu \right) e^{-\phi} dA = \int_M \left( (\nabla L_2^2 w, \nabla w) + (L_2^2 w)^2 \right) e^{-\phi} dv
\]

\[= \int_M (L_2^2 w)^2 e^{-\phi} dv. \tag{2.28}\]
Since that \( \nabla^2 w(\nu, \nu) |_{\partial M} = w_{\nu\nu}|_{\partial M} = (\nabla w, \nabla \nu)|_{\partial M}, \) by (2.5) we get
\[
L_\phi w |_{\partial M} = (n-1)H_\phi w_\nu + q_1 w_\nu, \tag{2.29}
\]
and together (2.28) given us
\[
q_1 = \frac{\int_M (L_\phi w)^2 \, d\mu - (n-1) \int_{\partial M} H_\phi \eta^2 \, d\theta}{\int_{\partial M} \eta^2 \, d\theta}, \tag{2.30}
\]
which, combining with Reilly’s formula and (2.12), gives
\[
q_1 \geq \frac{1}{n(a+1)} \frac{\int_M (L_\phi w)^2 d\mu}{\int_{\partial M} \eta^2 d\theta}. \tag{2.31}
\]
On the other hand, we have from the variational characterization of \( p_1 \) (cf. (1.13))
\[
\frac{\int_M (L_\phi w)^2 d\mu}{\int_{\partial M} \eta^2 d\theta} \geq p_1. \tag{2.32}
\]
By (2.31) and (2.32), we get \( q_1 \geq \frac{p_1}{n(a+1)} \). From [54], Theorem 1.3, we know that equality holding if and only if \( M \) is isometric to a ball in \( \mathbb{R}^n \). This proves the Theorem 1.3. \( \square \)

**Proof of Theorem 1.4** Let \( f \) be an eigenfunction corresponding to the first eigenvalue \( p_1 \) of the problem (1.9), that is
\[
\begin{aligned}
L_\phi^2 f &= 0 \text{ in } M, \\
\frac{\partial f}{\partial \nu} &= 0 \text{ on } \partial M.
\end{aligned} \tag{2.33}
\]
Set \( \eta = \frac{\partial f}{\partial \nu}|_{\partial M} \). Then
\[
p_1 = \frac{\int_M (L_\phi f)^2 e^{-\phi} \, dv}{\int_{\partial M} \eta^2 e^{-\phi} dA}. \tag{2.34}
\]
Substituting \( f \) into Reilly’s formula and using (1.15), we have
\[
\int_M (L_\phi f)^2 - |\nabla^2 f|^2 e^{-\phi} \, dv \geq \int_M \left( \frac{|\nabla \phi|^2}{na} - b \right) |\nabla f|^2 + (n-1)c \int_{\partial M} \eta^2 e^{-\phi} dA,
\]
By (2.12) and (2.14) we have
\[
\left( 1 - \frac{1}{n(a+1)} + \frac{b}{\lambda_1} \right) \int_M (L_\phi f)^2 e^{-\phi} \, dv \geq (n-1)c \int_{\partial M} \eta^2 e^{-\phi} dA,
\]
\[
p_1 \geq \frac{n(a+1)(n-1)c\lambda_1}{n(a+1)(\lambda_1 + b) - \lambda_1}. \tag{2.35}
\]
If the equality sign holds in (2.35), using the same arguments as in the proof of 1.3 ([34], Theorem 1.3), we conclude that $M$ is isometric to a ball in $\mathbb{R}^n$ of radius $1/c$.

**Proof of Theorem 1.5** From (1.20), we only need to show that the first non-zero eigenvalue $\xi_1$ of the problem (1.11) satisfies

$$\xi_1 > \frac{cn(a + 1)\lambda_1 \mu_1}{n(a + 1)(\mu_1 + b) - \mu_1}. \quad (2.36)$$

Let $f$ be an eigenfunction corresponding $\xi_1$:

$$\begin{cases}
L_\phi^2 u = 0 \text{ in } M, \\
\frac{\partial f}{\partial \nu} = \frac{\partial (L_\phi f)}{\partial \nu} + \xi_1 f = 0 \text{ on } \partial M.
\end{cases} \quad (2.37)$$

Let $z = f|_{\partial M}$; then $z \neq 0$ and

$$\xi_1 = \frac{\int_M (L_\phi f)^2 d\mu}{\int_{\partial M} z^2 d\theta}. \quad (2.38)$$

Substituting $f$ into Reilly’s formula, we have

$$\int_M ((L_\phi f)^2 - |\nabla^2 f|^2)e^{-\phi} dv = \int_M Ric_\phi(\nabla f, \nabla f)e^{-\phi} dv + \int_{\partial M} II(\nabla z, \nabla z)e^{-\phi} dA$$

$$\geq \int_M \left(\frac{|\nabla \phi|^2}{na} - b\right)|\nabla f|^2 e^{-\phi} dv$$

$$+ c\int_{\partial M} |\nabla z|^2 e^{-\phi} dA. \quad (2.39)$$

Since $\partial_e f|_{\partial M} = 0$, we have

$$\int_M (L_\phi f)^2 e^{-\phi} dv \geq \mu_1 \int_M |\nabla f|^2 e^{-\phi} dv. \quad (2.40)$$

It follows from (2.37) that $\int_{\partial M} z d\theta = 0$. Indeed, by (2.37) we have

$$\xi_1 \int_{\partial M} f e^{-\phi} dA = \int_{\partial M} \frac{\partial}{\partial \nu} (f - L_\phi f)e^{-\phi} dA$$

$$= \int_M div(\nabla f - \nabla L_\phi f)e^{-\phi} dv$$

$$= \int_M (L_\phi f - L_\phi^2 f)e^{-\phi} dv$$

$$= \int_M (L_\phi f)e^{-\phi} dv = 0.$$

So we have from the Poincaré inequality that

$$\int_{\partial M} |\nabla z|^2 e^{-\phi} dA \geq \lambda_1 \int_{\partial M} z^2 e^{-\phi} dA. \quad (2.41)$$
Combining (2.12) and (2.38)-(2.41), we get

$$\xi_1 \geq \frac{cn(a+1)\lambda_1 \mu_1}{n(a+1)(\mu_1+b) - \mu_1}.$$  \hspace{1cm} (2.42)

Let us show by contradiction that the equality in (2.42) can’t occur. In fact, if (2.42) take equality sign, then we must have (2.11) occurring on $M$, that is,

$$\nabla^2 f = \frac{\triangle f}{n} \langle \cdot, \cdot \rangle.$$  \hspace{1cm} (2.43)

Thus for a tangent vector field $X$ of $\partial M$, we have from by expression above and $\partial_{\nu} f|_{\partial M} = 0$ that

$$0 = \nabla^2 f(\nu, X) = X\nu f - (\nabla_X \nu) f = -\langle \nabla_X \nu, \nabla z \rangle.$$  \hspace{1cm} (2.43)

In particular, we have

$$II(\nabla z, \nabla z) = 0.$$  \hspace{1cm} (2.43)

This is impossible since $II = cI$ and $z$ is not constant. This finishes the proof Theorem 1.5.

The method used for the demonstration of the results is classic and has been widely used in articles in the bibliography. The Bérard article [3] is a pioneering reference to the generalized Simons equation satisfied for the second fundamental form of an immersion in a Riemannian manifold. The Simons type inequalities used can be deduced from the Bérard article.

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