Indications for Gluon Condensation
From Nonperturbative Flow Equations

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Abstract:
We employ nonperturbative flow equations for the description of the effective action in Yang-Mills theories. We find that the perturbative vacuum with vanishing gauge field strength does not correspond to the minimum of the Euclidean effective action. The true ground state is characterized by a nonvanishing gluon condensate.
1 Introduction

A perturbative calculation of the effective action for constant colour-magnetic fields in Yang-Mills theories indicates that the configuration of lowest Euclidean action does not correspond to vanishing fields \([1],[2]\). This has led to many interesting speculations about the nature of the QCD vacuum. Unfortunately, perturbation theory is clearly invalid in the interesting region in field space. It breaks down both for vanishing magnetic fields and for fields \(B\) corresponding to the minimum of the perturbatively calculated effective action \(\Gamma[B]\). Despite many nonperturbative indications for a nontrivial QCD vacuum from lattice studies and phenomenology, an analytical establishment of the phenomenon of gluon condensation is still lacking so far.

In this paper we present new analytical evidence for gluon condensation based on the non-perturbative method of the average action \([3]\). The average action \(\Gamma_k\) is the effective Euclidean action for averages of fields which obtains by integrating out all quantum fluctuations with (generalized) momenta \(q^2 > k^2\). It can be viewed as the standard effective action \(\Gamma\) computed with an additional infrared cutoff \(\sim k\) for all fluctuations. In the limit \(k \to 0\) the average action equals the usual effective action, \(\Gamma_0 = \Gamma\). For \(k > 0\) no infrared divergences can appear in the computations and a lowering of \(k\) allows to explore the long-distance physics step by step. The \(k\)-dependence of the average action is described by an exact nonperturbative evolution equation \([4]\). The structure of this equation is close to a perturbative one-loop equation but it involves the full propagator and vertices instead of the classical ones. For gauge theories it can be formulated in a way such that \(\Gamma_k[A]\) is a gauge-invariant functional of the gauge field \(A\) \([7]\).

The flow equation for the pure non-abelian Yang-Mills theory has been solved \([7]\) with a very simple approximation - the average action \(\Gamma_k\) has been truncated with a minimal kinetic term \(\sim Z_k F_{\mu\nu} F^{\mu\nu}\). From the \(k\)-dependence of \(Z_k\) the running of the renormalized gauge coupling \(g(k)\) has been derived for arbitrary dimension \(d\). Most strikingly, this lowest-order estimate suggests that the wave function renormalization \(Z_k\) reaches zero for \(k = k_{\infty}\) and turns negative for \(k < k_{\infty}\). Here the scale \(k_{\infty}\) can be identified with the confinement scale, i.e. the scale where the renormalized gauge coupling \(g^2(k) \sim Z_k^{-1}\) diverges. This was a first nonperturbative analytical indication for the instability of the perturbative vacuum with \(F_{\mu\nu} = 0\). Indeed, negative \(Z_k\) implies gluon condensation - the minimum of \(\Gamma_k\) must occur for a non-vanishing gauge field for all \(k < k_{\infty}\) \([8]\). It is obvious, however, that an effective action \(\sim Z_k B^2\) is insufficient to describe gluon condensation. One expects positive \(\Gamma_k\) for large \(B\) and this cannot be accomodated with a negative \(Z_k\) in the “lowest order truncation”. In this paper we enlarge the “space of actions” by considering for \(\Gamma_k\) an arbitrary function of constant magnetic fields, i.e. \(\Gamma_k \sim W_k(\frac{1}{2} B^2)\). The flow equation describes how the function \(W_k\) changes its shape, starting from a linear

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\(^1\)For the relation to earlier versions of exact renormalization group equations \([5]\) see ref. \([6]\).

\(^2\)See ref. \([8]\) for alternative formulations.
dependence \( W_k = \frac{1}{2} Z_k B^2 \) for large \( k \). We will see that within our truncation \( W_k \) develops its absolute minimum for \( B \neq 0 \) once \( k \) becomes smaller than a critical scale \( k_c \), and that the linear term indeed vanishes for \( k_{\infty} < k_c \). Despite some shortcomings of the present truncation for \( k \lesssim k_{\infty} \) (which are discussed in the conclusions) we interpret these findings as a strong analytical indication for gluon condensation!

In the following we shall consider the pure \( SU(N) \) Yang-Mills theory. For this case the exact evolution equation of ref. [7] reads \((t \equiv \ln k)\)

\[
\frac{\partial}{\partial t} \Gamma_k[A, \bar{\cal A}] = \frac{1}{2} \text{Tr}_{xcL} \left[ \left( \Gamma_k^{(2)}[A, \bar{A}] + R_k(\Gamma_k^{(2)}[\bar{A}, \bar{A}]) \right)^{-1} \frac{\partial}{\partial t} R_k(\Gamma_k^{(2)}[\bar{A}, \bar{A}]) \right] \\
- \text{Tr}_{xc} \left[ (-D^\mu[A] D_\mu[A] + R_k(-D^2[A]))^{-1} \frac{\partial}{\partial t} R_k(-D^2[A]) \right].
\]

(1.1)

Here the first trace on the r.h.s. arises from the fluctuations of the gauge field. It involves on integration over space-time (“\( x \)”) as well as a summation over color (“\( c \)”) and Lorentz (“\( L \)”) indices. The “color” trace is in the adjoint representation. The second trace is due to the Faddeev-Popov ghosts and is over space-time and color indices only. The precise form of the infrared cutoff is described by the function \( R_k \). It is convenient to choose

\[
R_k(u) = u[\exp(Z_k^{-1}u/k^2) - 1]^{-1}.
\]

(1.2)

In general the wave function renormalization \( Z_k \) could be a matrix in the space of fields which may even depend on \( \bar{A} \). (In ref. [7] we used \( Z_k \equiv 1 \) for the ghosts and a \( k \)-dependent constant \( Z_k \equiv Z_{F,k} \) for all modes of the gauge field.) Furthermore, \( \Gamma_k^{(2)}[A, \bar{A}] \) denotes the matrix of second functional derivatives of \( \Gamma_k \) with respect to \( A \), with the background field \( \bar{A} \) kept fixed. The modes of the gauge field are declared “high-frequency modes” or “low-frequency modes” depending on whether their eigenvalues with respect to \( \bar{A} \), with the background field \( \bar{A} \) kept fixed. The modes of the gauge field are declared “high-frequency modes” or “low-frequency modes” depending on whether their eigenvalues with respect to the operator \( \Gamma_k^{(2)}[\bar{A}, \bar{A}] \equiv \Gamma_k^{(2)}[A, \bar{A}] \mid_{|A|=\bar{A}} \) are larger or smaller than \( k^2 \), respectively. Therefore it is this operator which appears in the argument of \( R_k \) and in this sense \( R_k \) acts as an effective infrared cutoff by suppressing the low frequency modes. The flow equation (1.1) can be rewritten in close analogy to a one-loop formula

\[
\frac{\partial}{\partial t} \Gamma_k[A, \bar{A}] = \frac{1}{2} \frac{D}{Dt} \text{Tr}_{xcL} \ln \left[ \Gamma_k^{(2)}[A, \bar{A}] + R_k(\Gamma_k^{(2)}[\bar{A}, \bar{A}]) \right] \\
- \frac{D}{Dt} \text{Tr}_{xc} \ln \left[ -D_\mu[A] D_\mu[A] + R_k(-D^2[A]) \right].
\]

(1.3)

\(^3\)In [7] we used the classical \( S^{(2)}[\bar{A}, \bar{A}] \) rather than \( \Gamma_k^{(2)}[\bar{A}, \bar{A}] \) for this purpose. While preserving all the general properties of \( \Gamma_k \), the new flow equation is much easier to handle from a technical point of view. This change also entails the different positioning of \( Z_k \) in (1.3) relative to the one in [7]: for the simple truncation used there one has \( \Gamma_k^{(2)} = Z_{F,k} S^{(2)} \) for part of the modes. We only will use the definition (1.2) if for high momenta \( q^2 \to \infty \) one has \( u \to \infty \). We should mention, however, that for negative eigenvalues of \( \Gamma_k^{(2)}[A, \bar{A}] \) the vanishing of \( R_k \) for \( k \to 0 \) is guaranteed only if the ratio \( u/k^2 \) in (1.2) remains finite. This problem concerns mainly the approach to convexity of the effective action for \( k \to 0 \) and is of no relevance for the present work.
The derivative $\frac{D}{Dt}$ acts only on the explicit $k$-dependence of the function $R_k$, but not on $\Gamma^{(2)}_k[A, \bar{A}]$. It is now easy to describe the relation between the effective average action $\Gamma_k$ and the conventional effective action. Let us first make the approximation $\frac{D}{Dt} \to \frac{\partial}{\partial t}$ in eq. (1.3). This amounts to neglecting the running of $\Gamma_k$ on the r.h.s. of the evolution equation. It is then trivial to solve it explicitly:

$$
\Gamma_k[A, \bar{A}] = \Gamma_{\Lambda}[A, \bar{A}] + \frac{1}{2}\text{Tr}_{xcL} \left\{ \ln \left[ \Gamma^{(2)}_k[A, \bar{A}] + R_k(\Gamma^{(2)}_k[A, \bar{A}]) \right] 
- \ln \left[ \Gamma^{(2)}_\Lambda[A, \bar{A}] + R_{\Lambda}(\Gamma^{(2)}_\Lambda[A, \bar{A}]) \right] \right\}
- \text{Tr}_{xc} \left\{ \ln \left[ -D^\mu[A]D_\mu[\bar{A}] + R_k(-D^2[A]) \right] \right\}
- \ln \left[ -D^\mu[A]D_\mu[\bar{A}] + R_{\Lambda}(-D^2[\bar{A}]) \right] \right\}
+ O \left( \frac{\partial}{\partial t} \Gamma^{(2)}_k \right)
$$

(1.4)

with $\Lambda$ some appropriate high momentum scale (ultraviolet cutoff) where we may identify $\Gamma_{\Lambda}$ with the classical action $S$ including a gauge fixing term. This formula has a similar structure as a regularized expression for the conventional one-loop effective action in the background gauge [10, 2]. There are two important differences, however: (i) The second variation of the classical action, $S^{(2)}$, is replaced by $\Gamma^{(2)}_k$. This implements a kind of “renormalization group improvement” and transforms (1.4) into a sort of “gap equation”. (ii) The effective average action contains an explicit infrared cutoff $R_k$. Because

$$\lim_{u \to \infty} R_k(u) = 0, \quad \lim_{u \to 0} R_k(u) = Z_k k^2,$$

(1.5)

a $k$-dependent mass-type term is added to the inverse propagator $\Gamma^{(2)}_k$ for low frequency modes ($u \to 0$), but it is absent for high frequency modes ($u \to \infty$). For the purpose of comparison we will also consider in this paper a second choice for the cutoff, namely simply a constant

$$R_k = Z_k k^2$$

(1.6)

Despite the similarity of (1.4) with a one-loop expression, we stress that, for $k \to 0$, the solution of the original renormalization group equation (1.1) or (1.3) equals the exact effective action which includes contributions from all orders of the loop expansion. For a detailed discussion of the approximation (1.4) in the case of the abelian Higgs model we refer to [11], and to ref. [12] for the corresponding nonperturbative evolution equations.

2 Truncating the space of actions

In order to find nonperturbative approximative solutions of (1.1) we employ the following ansatz for $\Gamma_k$:

$$\Gamma_k[A, \bar{A}] = \int d^d x \, \bar{W}_k \left( \frac{1}{4} F_{\mu\nu}(x) F_{\mu\nu}(x) \right) + \frac{1}{2\alpha_k} \int d^d x \, \sum_z (D_\mu[\bar{A}](A^\mu - \bar{A}^\mu))^2$$

(2.1)
Here $W_k$ is an arbitrary function of the invariant $1/4 F^2$ with $F_{\mu\nu}^z$ the field strength of the gauge field $A$. The $k$-dependence of $W_k$ will be determined by inserting (2.1) into the evolution equation (1.1). If we think of $W_k(\theta)$, $\theta \equiv 1/4 F_{\mu\nu}^z F_{\mu\nu}^z$, as a power series in $\theta$ our ansatz contains invariants of arbitrarily high canonical dimension. Since all invariants which occur are of the form $\theta^l$, only the dimensions $4l, l = 1, 2, \ldots$, actually occur. Also for a fixed dimension $4l$ the truncation (2.1) does not contain a complete basis of operators. Nevertheless one may hope that (2.1) gives a qualitatively correct picture of the physics in the regime where the renormalization group evolution has already drastically modified the classical Lagrangian $1/4 F^2$. We remark that effective actions which depend on $\theta$ only play also a central role in the leading-log models [13] of QCD. The second term on the r.h.s. of (2.1) is a standard background gauge-fixing term [10], [7] with a $k$-dependent gauge-fixing parameter $\alpha_k$. In addition to the $k$-dependence of the function $W_k(\theta)$ we should, in principle, also compute the $k$-dependence of $\alpha_k$. We will omit this here since the general identities of ref. [7] imply that $\alpha$ is independent of $k$ in a first approximation.

We should mention at this place that a truncation is actually not defined by the terms retained but rather by specifying which invariants in the most general form of $\Gamma_k$ are omitted. One may parametrize a general $\Gamma_k$ by infinitely many couplings multiplying the infinitely many possible invariants which can be formed from the gauge fields consistent with the symmetries. In the corresponding infinite dimensional space a truncation is a projection on a subspace which is defined by setting all but the specified couplings to zero. (In our case the subspace remains infinite dimensional.) In practice, we will choose a particular test configuration $A_\mu$ corresponding to a constant magnetic field. Its detailed definition will be given below. The truncation should then be understood in the sense that we use a basis for the invariants where all invariants except those used in (2.1) vanish for the test configuration. By putting the coefficients of all invariants which vanish for the test configuration to zero the truncation is uniquely defined. A computation of $W_k$ therefore amounts to a computation of the $k$-dependent effective action for a (particular) constant magnetic field.

Upon performing the second variation of the ansatz (2.1),

$$\delta^2 \Gamma_k[A, \bar{A}] = \int d^d x \delta A_\mu^w \Gamma_k^{(2)}[A, \bar{A}]_{\mu\nu}^{yz} \delta A_\nu^z$$

we arrive at the following operator $\Gamma_k^{(2)}$:

$$\Gamma_k^{(2)}[A, \bar{A}]_{\mu\nu}^{yz} = W_k'(\theta)(D_T[A] - D_L[A])_{\mu\nu}^{yz} + W_k''(\theta)S_{\mu\nu}^{yz}[A] + \frac{1}{\alpha_k}(D_L[\bar{A}])_{\mu\nu}^{yz}$$

with

$$\theta = \frac{1}{4} F_{\mu\nu}^z F_{\mu\nu}^z$$

Here we used the notation ($w, y, z$ are adjoint group indices and $\bar{g}$ is the (bare) gauge coupling)

$$(D_T)^{yz}_{\mu\nu} = (-D^2 \delta_{\mu\nu} + 2i\bar{g} F_{\mu\nu})^{yz}$$
\[(D_L)_{\mu\nu}^{yz} = -(D \otimes D)_{\mu\nu}^{yz} = -D_{\mu}^{yw} D_{\nu}^{wz} \]
\[S_{\mu\nu}^{yz} = F_{\mu\nu}^{yw} (D^\rho D^\sigma)^{wz} \]

(2.5)

with the covariant derivative \((D_\mu [A])^{yw} = \partial_\mu \delta^{yw} - i \bar{g} A_\mu^z (T_z)^{yw}\) in the adjoint representation and \(F_{\mu\nu}^{yw} = F_{\mu\nu}^z (T_z)^{yw}\). Moreover, \(W_k^\prime\) and \(W_k^{\prime\prime}\) denote the first and the second derivative of \(W_k\) with respect to \(\theta\). In writing down eq. (2.4) we made the additional assumption that the field strength \(F_{\mu\nu} [A]\) is covariantly constant \(F_{\mu\nu;\rho} = 0\) or

\[ [D_\rho [A], F_{\mu\nu} [A]] = 0 \]

(2.6)

In fact, we may choose as a particularly convenient test field a covariantly constant color-magnetic field \([2]\) with a vector potential of the form

\[ A_\mu^z (x) = n^z A_\mu (x) \]

(2.7)

Here \(n^z\) is a constant unit vector in color space \((n^z n_z = 1)\), and \(A_\mu (x)\) is any “abelian” gauge field whose field strength

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = B \varepsilon_{\mu\nu} = \text{const} \]

(2.8)

corresponds to a constant magnetic field \(B\) along the 3-direction, say. (We define \(\varepsilon_{z1}^1 = -\varepsilon_{21}^2 = 1\), with all other components vanishing.) Hence we have \(\theta = \frac{1}{4} F_{\mu\nu} F_{\mu\nu} - \frac{1}{8} B^2\) such that \(\theta\) and \(W_k\) etc. commute with all operators. It is easy to see that (2.7) and (2.8) obey the condition (2.3). The choice (2.7) has the advantage that it allows for an explicit diagonalization of the operator \(\Gamma_{k}^{(2)}\). We finally note that \(W_k\) can be extracted from \(\Gamma_k [A, A]\) which is a gauge-invariant functional of \(A\) obtained by putting \(\bar{A} = A\). It is therefore sufficient to know \(\Gamma_k^{(2)} [A, A]\).

Before returning to the evolution equation, we list a few special properties of the covariantly constant fields, which will prove helpful later on. From (2.3) it follows that \(A_\mu\) satisfies the classical Yang-Mills equations \(D_\mu F_{\mu\nu} = 0\). This in turn is sufficient to prove that the operators \(D_T\) and \(D_L\) commute. As a consequence, one may define generalized projection operators [4]

\[ P_L = D_T^{-1} D_L \]
\[ P_T = 1 - P_L \]

(2.9)

which satisfy \(P_{T,L}^2 = P_{T,L}, P_T + P_L = 1\) and \(P_T P_L = 0 = P_L P_T\). For \(A_\mu = 0\) they reduce to the standard projectors on transverse and longitudinal modes:

\[(P_T^{(0)})_{\mu\nu} = \delta_{\mu\nu} - \partial_\mu \partial_\nu / \partial^2 \]
\[(P_L^{(0)})_{\mu\nu} = \partial_\mu \partial_\nu / \partial^2 \]

(2.10)

Furthermore, if \(A_\mu\) is of the form (2.7), it is natural to define another pair of orthogonal projectors,

\[ P_{\perp}^{yz} = \delta^{yz} - n^y n^z, \quad P_{\parallel}^{yz} = n^y n^z \]

(2.11)
which project on the spaces perpendicular and parallel to \( n^z \), respectively. For the vector potential \((2.7)\) the matrix \( A_\mu^w T_w \) reads in the adjoint representation

\[
A^w_\mu (x) \equiv (A_\mu^w T_w)^w = i f^{\nu \rho z} n_\nu A_\mu (x) \tag{2.12}
\]

The antisymmetry of the structure constants \( f^{\nu \rho z} \) implies that \( P_\parallel \) and \( P_\perp \) commute with \( D_\mu, D^2, D_T, D_L \) and \( F_{\mu \nu} \), and that

\[
P_\parallel A_\mu = 0, \quad P_\parallel D_\mu = P_\parallel \partial_\mu, \quad P_\parallel (D_T)_{\mu \nu} = -\partial^2 \delta_{\mu \nu} P_\parallel \tag{2.13}
\]

The operator \( S \) from (2.3) factorizes according to

\[
S^w_\mu = P_\parallel s_{\mu \nu}, \quad s_{\mu \nu} = F_{\mu \rho} \sigma_{\nu \rho} \partial^\sigma \tag{2.14}
\]

Hence \( S \) commutes with \( D^2, D_L \) and \( D_T \) because it annihilates the gauge-field contained in these operators:

\[
SD^2 = S \partial^2, \quad SD_T = -S \partial^2, \quad SD \otimes D = S \partial \otimes \partial \tag{2.15}
\]

In physical terms this means that those components of the gauge fluctuations \( \delta A_\mu^z \equiv a_\mu \) which are parallel to \( n^z \) decouple from \( A_\mu \) to some extent. In fact, in terms of the projections \( a_{\mu \parallel} \) the quadratic action \((2.2)\) with \((2.6)\) reads

\[
\delta^2 \Gamma_k [A, A] = \int d^4 x \left\{ a_{\mu \parallel}^2 \left[ -\partial^2 W_k' \delta_{\mu \nu} + (W_k' - \frac{1}{\alpha_k}) \partial_\mu \partial_\nu + W''_k s_{\mu \nu} \right] a_{\parallel \nu}^z \right. \\
+ \left. a_{\nu \parallel}^z \left[ W_k' D_T + \left( \frac{1}{\alpha_k} - W_k' D_L \right) a_{\parallel \nu}^z \right] a_{\mu \parallel}^z \right\} \tag{2.16}
\]

We observe that the \( a_{\parallel \mu} \)-modes couple to the external field only via the derivatives of \( W_k \equiv W_k' (\frac{1}{2} B^2) \). In a conventional one-loop calculation one uses the classical Yang-Mills Lagrangian \( \frac{1}{4} F_{\mu \nu}^2 \) rather than \( W_k (\frac{1}{4} F_{\mu \nu}^2) \). In that case the quadratic action for the small fluctuations is given by \((2.13)\) with \( W_k' = 1 \) and \( W_k'' = 0 \). Hence the one-loop determinant resulting from the integration over \( a_{\parallel \mu} \) is field-independent and may be ignored. In the present case, the \( a_{\parallel \mu} \)-modes are important for the “renormalization group improvement”, however. The quadratic form \((2.16)\) can be diagonalized even further by introducing the longitudinal and transversal projections

\[
a_{\parallel \mu} = P_L a_{\parallel \mu} = P_{L(0)} a_{\parallel \mu}, \quad a_{\parallel \nu} = P_T a_{\parallel \nu} = P_{T(0)} a_{\parallel \nu} \]

\[
a_{\perp \mu} = P_L a_{\perp \mu}, \quad a_{\perp \nu} = P_T a_{\perp \nu} \tag{2.17}
\]

By virtue of \( P_{L(0)} s = s P_{L(0)} = 0, P_{T(0)} s = s P_{T(0)} = s \) and \([P_{L(T)}, \theta] = 0\) one obtains

\[
\delta^2 \Gamma_k [A, A] = \int d^4 x \left\{ a_{\parallel \mu}^z \left[ -\partial^2 W_k' \delta_{\mu \nu} + W''_k s_{\mu \nu} \right] a_{\parallel \nu}^z \right. \\
+ \left. \frac{1}{\alpha_k} a_{\parallel \mu}^z \left[ -\partial^2 \right] a_{\parallel \mu}^z \right. \\
+ \left. a_{\perp \nu}^z \left[ W_k' D_T + \left( \frac{1}{\alpha_k} - W_k' D_L \right) a_{\perp \nu}^z \right] a_{\parallel \nu}^z \right. \\
+ \left. \frac{1}{\alpha_k} a_{\perp \nu}^z \left[ D_T a_{\perp \nu}^z \right] a_{\parallel \mu}^z \right\} \tag{2.18}
\]
This block-diagonal form of $\Gamma^{(2)}_k$ will facilitate the evolution of the traces occurring in the evaluation equation. For example, $a^{||,L}$ gives no $A$-dependent contribution and, except for an irrelevant constant, the only dependence of $\Gamma_k$ on $\alpha_k$ arises from $a^{\perp,L}$. Writing

$$\Gamma^{(2)}[A, A] = \Gamma^{(2)}_1 + \Gamma^{(2)}_2 + \Gamma^{(2)}_3 + \Gamma^{(2)}_4$$

where, in an obvious notation

$$\Gamma^{(2)}_{1,2} = \mathcal{P}^{||,L}_T \mathcal{P}^{||,L}_T \mathcal{P}^{||,L}_T \mathcal{P}^{||,L}_T, \quad \Gamma^{(2)}_{3,4} = \mathcal{P}^{\perp,\perp}_L \mathcal{P}^{\perp,\perp}_L \mathcal{P}^{\perp,\perp}_L \mathcal{P}^{\perp,\perp}_L$$

with $[\mathcal{P}^{||,L}_T, \mathcal{P}^{\perp,\perp}_L] = 0$.

We have

$$\Gamma^{(2)}_A \Gamma^{(2)}_B = 0 \quad \text{for} \quad A \neq B, \quad \text{and} \quad [\Gamma^{(2)}_A, \Gamma^{(2)}_B] = 0$$

with $[\mathcal{P}^{||,L}_T, \mathcal{P}^{\perp,\perp}_L] = 0$ etc.

3 Evolution equation for $W_k$

In this section we derive an evolution equation which governs the scale-dependence of the function $W_k$. We will choose the matrix $Z_k$ in the definition of $R_k$ (1.2) as $Z_k = 1$ for the ghosts and

$$Z_k = Z_k \mathcal{P}^{\perp}_L [\bar{A}] + \bar{Z}_k \mathcal{P}^{\perp}_L [\bar{A}]$$

for the gauge boson degrees of freedom. Here $Z_k, \bar{Z}_k$ are $k$-dependent constants and we observe that the choice (3.1) is compatible with (2.22). If we insert the truncation (2.1) into (1.1) with $\bar{A} = A$, we obtain ($\theta = \frac{1}{2}B^2$)

$$\Omega \frac{\partial}{\partial t} W_k(\theta) = \frac{1}{2} \text{Tr}_{xcL} \left[ H \left( \Gamma^{(2)}_k \mathcal{P}^{||,L}_T \mathcal{P}^{\perp,\perp}_L \mathcal{P}^{\perp,\perp}_L \mathcal{P}^{\perp,\perp}_L \right) \right]$$

where $\Gamma^{(2)}_k$ is given by (2.7) and $\Omega \equiv \int d^d x$. In (3.2) we introduced the convenient abbreviation (for $Z_k = Z_k$)

$$H(u) \equiv (u + R_k(u))^{-1} \frac{\partial}{\partial t} R_k(u)$$

3 Evolution equation for $W_k$
The integrals implied by the trace is guaranteed for the choice (1.2) only if over simple operators. We note that the ultraviolet finiteness of the momentum $D^2$ may take equivalently 1-2 plane, and a discrete quantum number $n$ exponentially for large $q$. They are exploited in appendix B to express the evolution equation for $Z$ and different factor $Z$ with $d$ by $\alpha$ 2)

In appendix A we display various trace identities for an evaluation of (3.2). They are parametrized explicitly [2]. They are parametrized $\nu$ which “lives” in the space orthogonal to the 1-2 plane, and a discrete quantum number $n = 0, 1, 2, \ldots$ which labels the Landau levels. The spectral sum for the function $\hat{H}(x) \equiv \hat{H}(W_k^L(x))$ reads

\[
\Omega^{-1} \text{Tr}_{xL}[\hat{H}(D_T)] = \frac{N^2 - 1}{2\pi} \frac{\tilde{g} |\nu_l| B}{\sum_{n=0}^{\infty}} \int \frac{d^{d-2}q}{(2\pi)^{d-2}} \cdot \{(d - 2)\hat{H}(q^2 + (2n + 1)\tilde{g} |\nu_l| B) + \hat{H}(q^2 + (2n + 3)\tilde{g} |\nu_l| B) + \hat{H}(q^2 + (2n - 1)\tilde{g} |\nu_l| B)\}
\]

(The momentum integration is absent for $d = 2$.) Here $\nu_l, l = 1, \ldots, N^2 - 1$ are the eigenvalues of the matrix $n^2 T_z$ in the adjoint representation. We note that for $n = 0$
and $q^2$ sufficiently small the eigenvalue $q^2 - \bar{g}|\nu_l|B$ in the third term on the r.h.s. of (3.6) can become negative. This is the origin of the instability [14] of the Savvidy vacuum [1], which causes severe problems if one tries to compute the standard one-loop effective action in the background of a covariantly constant magnetic field [15]. In our approach this problem is cured by the presence of an IR regulator. For $d > 2$ eq. (3.6) can be rewritten as

$$\Omega^{-1}\text{Tr}_{xL}[\hat{H}(D_T)] = \frac{\nu_d-2}{\pi} \sum_{l=1}^{N^2-1} \bar{g}|\nu_l|B \int_0^\infty dxx^{d-2} \cdot \left\{ \sum_{n=0}^\infty \hat{H}(x + (2n+1)\bar{g}|\nu_l|B) + \hat{H}(x - \bar{g}|\nu_l|B) - \hat{H}(x + \bar{g}|\nu_l|B) \right\} \quad (3.7)$$

but it cannot be simplified any further in closed form. The other traces in (3.4) are given by

$$\Omega^{-1}\text{Tr}_{x}[\hat{H}(-D^2)] = \frac{\nu_d-2}{\pi} \sum_{l=1}^{N^2-1} \bar{g}|\nu_l|B \int_0^\infty dxx^{d-2} \sum_{n=0}^\infty \hat{H}(x + (2n+1)\bar{g}|\nu_l|B) \quad (3.8)$$

and similar for $\tilde{H}$ and $H_G$.

In this paper we use two different methods in order to (approximately) compute the spectral sums (3.7) and (3.8). In appendix E we shall represent them as Schwinger proper-time integrals [16]. This method leads to compact integral representations which are valid for all values of $B$, but it has the disadvantage that it works only for a slightly simplified form of the cutoff function $R_k(x)$ defined by (1.6) which leads, as we shall see, to ultraviolet problems. The second method consists of expanding the r.h.s. of (3.7) in powers of $B$. It is applicable if $\bar{g}B \ll k^2$, and it works for any function $R_k(x)$. The condition $\bar{g}B \ll k^2$ guarantees that we may express the sum over $n$ by an Euler-McLaurin series, and that the first terms converge rapidly. In this manner (3.7) turns into

$$\Omega^{-1}\text{Tr}_{xL}[H(W_k'(\frac{1}{2}B^2)D_T)] = (N^2 - 1)\frac{\nu_d-2}{2\pi} [W_k'(\frac{1}{2}B^2)]^{-\frac{d}{2}} \int_0^\infty dx \int_0^\infty dy x^{\frac{d}{2}-2} H(x+y) \quad (3.9)$$

$$+ \frac{\nu_d-2}{\pi} \sum_{m=1}^\infty C_m \left( \sum_{l=1}^{N^2-1} \nu_l^{2m} \right) (\bar{g}B)^{2m}[W_k'(\frac{1}{2}B^2)]^{2m-\frac{d}{2}} \int_0^\infty dx \ x^{\frac{d}{2}-2} H^{(2m-1)}(x)$$

with

$$C_m = \frac{d}{(2m)!} (2^{2m-1} - 1) B_{2m} - \frac{2}{(2m-1)!} \quad (3.10)$$

Here $B_{2m}$ are the Bernoulli numbers. In a second step one has to expand the $B^2$-dependence of $W_k'$. Note that only even powers of $B$ occur in this expansion. The group-theoretical factors $\sum_{l=1}^{N^2-1} \nu_l^{2m}$ are discussed in appendix C. In particular for $SU(2)$ these factors equal 2 for all values of $m$. 

9
If we use the Euler-McLaurin series (3.9) and a similar expansion for the trace (3.8) in eq. (3.4), we find for $SU(N)$:

$$\frac{\partial}{\partial t} W_k(\theta) = \frac{v_d - 2}{2\pi} (2 - \eta) k^d \left\{ \frac{d - 1}{2} (N^2 - 1) r^d_2 \left( \frac{W_k'}{Z_k} \right)^{-\frac{d}{2}} \right\}$$

$$- \sum_{m=1}^{\infty} \tau_m (C_m - E_m) r^{d,m}_0 \left( \frac{2g^2 \theta}{k^4} \right)^m \left( \frac{W_k'}{Z_k} \right)^{2m-\frac{d}{2}}$$

$$+ \frac{v_d - 2}{2\pi} (2 - \tilde{\eta})(\tilde{Z}_k \alpha_k) \frac{d}{k^d} \left\{ \frac{1}{2} (N^2 - 1) r^d_2 - \sum_{m=1}^{\infty} \tau_m E_m r^{d,m}_0 \left( \frac{2g^2 \theta}{k^4} \right)^m (\tilde{Z}_k \alpha_k)^{-2m} \right\}$$

$$- \frac{v_d - 2}{\pi} k^d \left\{ (N^2 - 1) r^d_2 - 2 \sum_{m=1}^{\infty} \tau_m E_m r^{d,m}_0 \left( \frac{2g^2 \theta}{k^4} \right)^m \right\}$$

$$+ v_d (2 - \eta) r^d_1 k^d \left( \frac{Z_k}{W_k + 2\theta W_k'} - \frac{Z_k}{W_k'} \right) \left( \frac{W_k'}{Z_k} \right)^{1-\frac{d}{2}} + \text{const.} \quad (3.11)$$

with $\tau_m$ defined in appendix C and the constants $E_m$ given by

$$E_m = \frac{1}{(2m)!} (2^{2m-1} - 1) B_{2m}. \quad (3.12)$$

The dimensionless integrals

$$r^{d,m}_0 = - \frac{1}{2 - \eta} (Z_k k^2)^{2m-\frac{d}{2}} \int_0^\infty dx x^{\frac{d}{2}-2} H(2m-1)(x)$$

$$r^d_1 = \frac{1}{2 - \eta} (Z_k k^2)^{-\frac{d}{2}} \int_0^\infty dx x^{\frac{d}{2}-1} H(x) = \int_0^\infty dx \frac{x^{\frac{d}{2}}}{e^x - 1}$$

$$r^d_2 = \frac{1}{2 - \eta} (Z_k k^2)^{-\frac{d}{2}} \int_0^\infty dx \int_0^\infty dy y^{\frac{d}{2}-2} H(x + y)$$

$$= \int_0^\infty dx \int_0^\infty dy \frac{x^{\frac{d}{2}-2}(x + y)}{\exp(x + y) - 1} \quad (3.13)$$

occur as a consequence of (3.3) and the second equality uses (1.2). For this choice we note for later use that in 4 dimensions

$$r^{4,m}_0 = B_{2m-2}, \quad r^4_1 = r^4_2 = 2\zeta(3) \quad (3.14)$$

where $\zeta$ denotes the Riemann zeta function. In contrast, the quantities $r^4_1$ and $r^4_2$ are not well defined for the choice (1.6). The ultraviolet divergence indicates an incomplete “thinning out” of the high momentum degrees of freedom for a simple mass like infrared cutoff. Even though eq. (3.11) was derived by choosing a specific
background field, this evolution equation does not depend on the background we used for the calculation. By employing derivative expansion techniques [17], [18] it should also be possible to derive \((3.11)\) without ever specifying a background. In the case at hand the method presented here is by far simpler, however.

In eq. \((3.11)\) we have introduced the anomalous dimensions

\[
\eta = -\frac{d}{dt}\ln Z_k, \quad \tilde{\eta} = -\frac{d}{dt}\ln \tilde{Z}_k
\]  

\((3.15)\)

A convenient choice for the wave function renormalization constants used in the infrared cutoff \(R_k\) is \((k_{np} \text{ denotes the transition to the nonperturbative regime})\)

\[
Z_k = \begin{cases} 
W'_k(0) & \text{for } k > k_{np} \\
W'_{k_{np}}(0) & \text{for } k < k_{np} 
\end{cases}
\]

\(
\tilde{Z}_k = \frac{1}{\alpha_k}
\)  

\((3.16)\)

Here we account for the possibility that \(W'_k(0)\) may turn negative for \(k\) smaller than a “confinement scale” \(k_\infty\), whereas \(Z_k\) must always be strictly positive. More precisely, if \(|d\ln W'_k(0)/dt|\) grows larger than one for small scales \(k\), we choose \(k_{np}\) to be the scale where \(\eta(k_{np}) = 1\). For \(k < k_{np}\) all couplings run fast anyhow, and an improvement of the scaling properties of \(R_k\) by the introduction of a \(k\)-dependent wave function renormalization seems not necessary. The choice \(\tilde{Z}_k = \alpha_k^{-1}\) guarantees that the infrared cutoff acts on the longitudinal modes in the same way as on the transversal modes. It implies that the flow equation for \(\frac{\partial}{\partial t} W_k\) becomes independent of \(\alpha_k\) in our truncation (except for an irrelevant constant)! In this paper we can neglect the running of \(\alpha_k\). This can be inferred from a first-order approximation to the solution of the general identities which govern the dependence of \(\Gamma_k[A, \bar{A}]\) on the background field \(A[1]\). On therefore has \(\tilde{\eta} = 0\).

It is also convenient to express the flow equation in terms of renormalized dimensionless quantities

\[
\begin{align*}
g^2 &= k^{d-4}Z_k^{-1} \tilde{g}^2 \\
\vartheta &= g^2 k^{-d} Z_k \theta \\
w_k(\vartheta) &= g^2 k^{-d} W_k(\theta)
\end{align*}
\]  

\((3.17)\)

Switching to a notation where dots denote derivatives with respect to \(\vartheta\) instead of \(\theta\) and \(\partial/\partial t\) is now taken at fixed \(\vartheta\) one obtains

\[
\frac{\partial}{\partial t} w_k(\vartheta) = -(4 - \eta) w_k(\vartheta) + 4 \vartheta \dot{w}_k(\vartheta)
\]

\[
+(2 - \eta) \nu_d g^2 (\ddot{w}_k(\vartheta)) - \frac{1}{4} \left( \frac{(d-1)(d-2)}{2} (N^2 - 1) \nu_d^d \right)
\]
the Euler-McLaurin series in our case. For large
$k \eta < k \omega (\theta)$ in our truncation. If necessary, one has to replace for
$k \rightarrow 0$ the form of (3.18). Solving for
$\Lambda \eta \omega (\theta)$ the function $w_k(\theta)$ depending on
$k$ and $\omega$. One needs in addition the running of the renormalized gauge coupling $g$
and $\eta$, which are related by
$$\beta_g^2 = \frac{\partial g^2}{\partial \tau} = (d - 4 + \eta)g^2$$
For $k > k_{np}$ we have by definition $\dot{w}_k(0) = 1$ and $\eta$ can be determined by
$$\frac{\partial}{\partial \tau} \dot{w}_k(0) = 0 = \eta - 2(d - 2)v_d r_0^{d,1} \tau_1 g^2 \left((2 - \eta)C_1^d - (4 - \eta)E_1\right)$$
$$- (2 - \eta) v_d g^2 \left(2r_1^d + \frac{d(d - 1)(d - 2)}{4}(N^2 - 1)r_2^d \right) \dot{w}_k(0)$$
With $\tau_1 = N, C_1^d = \frac{d}{12} - 2, E_1 = \frac{1}{12}$ this yields
$$\eta = - \left(\frac{N}{3} v_d(d - 2)(26 - d)r_0^{d,1} g^2 - 2h_d g^2 w_2\right)$$
$$\left(1 - \frac{N}{6} v_d(d - 2)(25 - d)r_0^{d,1} g^2 + h_d g^2 w_2\right)^{-1}$$
with
$$h_d = v_d \left(2r_1^d + \frac{d(d - 1)(d - 2)}{4}(N^2 - 1)r_2^d \right)$$
$$w_2 = \dot{w}_k(0)$$
For general $d$ the constants $r_0^{d,1}, r_1^d$ and $r_2^d$ depend on the precise choice of the infrared
cutoff except for $d = 4$ where $r_0^{d,1} = 1$ is cutoff independent. The running of the
renormalized gauge coupling $g$ is now determined (cf. (3.19)). It depends on the
additional coupling $w_2$ (3.22) which will be discussed in more detail in the next
section. Specifying the initial value $g^2(\Lambda)$ and the function $w_\Lambda(\theta)$ at some high
momentum scale $\Lambda$ the form of $w_k(\theta)$ and $g^2(k)$ are completely determined by the
flow equation (3.18). Solving for $k \rightarrow 0$ the function $w_0(\theta)$ specifies the effective
action in our truncation. If necessary, one has to replace for $k < k_{np}$ eq. (3.21) by
$\eta = 0, g^2(k < k_{np}) = g^2(k_{np})$.
Before closing this section, we briefly comment on the range of convergence of
the Euler-McLaurin series in our case. For large $m$ one has
$$\lim_{m \rightarrow \infty} C_m = d \lim_{m \rightarrow \infty} E_m \sim \pi^{-2m}$$
\footnote{For $w_2 = 0$ we recover the result of ref. [4] except for the factor $(25 - d)$ in the denominator
in (3.21) which was $(24 - d)$ previously. This difference is due to a slightly different choice of the
$Z$-factors in the infrared cutoff.}
For $N = 2$ and $d = 4$ we find (cf. (3.14)) that the coefficients of $(2\vartheta)^m$ in eq. (3.18) diverge $\sim \pi^{-2m}B_{2m-2} \sim \pi^{-4m}2^{-2m}(2m - 4)!$. For small nonvanishing $\vartheta$ the first terms of the series have a very rapid apparent convergence, but the series finally diverges due to the factorial growth $\sim (2m - 4)!$. We can therefore safely use this series only for the derivatives $w^{(n)}(\vartheta = 0)$ with finite $n$ where convergence problems are absent since only a finite number of terms in the sum contributes. The situation is probably similar for $N > 2$ and/or $d \neq 4$ as well as for many other choices of the infrared cutoff. In contrast, the original sums over $n$ in eqs. (3.7), (3.8) always converge since for $B > 0$ the contributions from sufficiently high values of $n$ are exponentially suppressed. An explicit evaluation of these sums is possible for the simplified masslike IR-cutoff (1.6). This is described in appendix E where we will also see the reason for the ultraviolet divergence of $r_4^4$ and $r_2^4$ for this particular cutoff.

4 The $F^4$ approximation

In order to get a first idea of the physical contents of the evolution equation (3.11) we further simplify the problem in this section: We include in $W_k(\vartheta)$ only terms which are at most quadratic in $\vartheta$:

$$W_k(\vartheta) = W_0(k) + W_1(k)\vartheta + \frac{1}{2} W_2(k)\vartheta^2,$$

$$W_j(k) \equiv \left( \frac{d}{d\vartheta} \right)^j W_k(0) \quad (4.1)$$

Thus the truncation for $\Gamma_k$ is parametrized by three couplings:

$$\Gamma_k[A, A] = \int d^d x \left\{ W_0(k) + \frac{1}{4} W_1(k) F_{\mu\nu}^z F_{\mu\nu}^z + \frac{1}{32} W_2(k) \left( F_{\mu\nu}^z F_{\mu\nu}^z \right)^2 \right\} \quad (4.2)$$

Apart from the constant $W_0$, we keep here only the “marginal” coupling $W_1$ and the lowest “irrelevant” coupling consistent with the truncation, $W_2$. We concentrate on $d = 4$ and employ the infrared cutoff (1.2) with (3.13). The short distance or “classical” theory is specified for $k = \Lambda$ by $(Z_\Lambda = 1)$

$$W_1(\Lambda) = 1, \quad W_0(\Lambda) = W_2(\Lambda) = 0 \quad (4.3)$$

In terms of the couplings $w_0 = w(\vartheta = 0), g^2$ and $w_2$ (3.17), (3.23), one has for $k > k_{np}$

$$W_0(k) = \frac{k^4 w_0(k)}{g^2(k)}$$

$$W_1(k) = \frac{\tilde{g}^2}{g^2(k)}$$

$$W_2(k) = \frac{k^4 w_2(k)}{g^2(k)k^4} \quad (4.4)$$
or
\[ g^2(\Lambda) = g^2, \quad w_0(\Lambda) = w_2(\Lambda) = 0 \] (4.5)
with \( g^2(k) \) determined by eqs. (3.19), (3.21). For \( k < k_{np} \) we use \( g^2(k) = g^2(k_{np}) \) and

\[ W_1(k) = \frac{\bar{g}^2}{g^2(k_{np})} w_1(k) \]
\[ w_1(k) = \dot{w}_k(0) \] (4.6)

The flow equations for \( w_j(k) = w_k^{(j)}(\vartheta = 0) \) follow from appropriate partial differentiation of eq. (3.18) with respect to \( \vartheta \), as, for example

\[ \frac{\partial}{\partial t} \dot{w} = \eta \dot{w} + 4\vartheta \ddot{w} \]
\[ - (2 - \eta) v_d g^2 w^{-\frac{d}{2}} \left\{ (d - 2) \sum_{m=1}^{\infty} \tau_m (C^{d}_m - E_m) \tau_0^{d,m} (2\vartheta \dot{w})^{m-1} (2mw^2 + (4m - d)\vartheta \dot{w} \ddot{w}) \right\} \]
\[ + r_d^d \left( \frac{2\dot{w} + 2\vartheta \dot{w}^{(3)}}{\dot{w} + 2\vartheta \dot{w}} - \frac{2\vartheta \dot{w}(3\dot{w} + 2\vartheta \dot{w}^{(3)})}{(\dot{w} + 2\vartheta \dot{w})^2} - \frac{d - 2}{\dot{w}(\dot{w} + 2\vartheta \dot{w})} \right) \]
\[ + \frac{1}{4} d(d - 1)(d - 2)(N^2 - 1) r_d^d \frac{\dot{w}}{\dot{w}} \sum_{m=1}^{\infty} m \tau_m E_m r_0^{d,m} (2\vartheta)^{m-1} \] (4.7)

We observe that \( w_0 \) does not appear on the r.h.s. of the evolution equations for \( w_j, j \geq 1 \), and we omit in the following this irrelevant constant. For \( k > k_{np} \) the evolution equation for the running gauge coupling reads \((d = 4)\)

\[ \frac{\partial g^2}{\partial t} = -\frac{g^4}{24\pi^2} (11N - 3H_4w_2) \left[ 1 - \frac{g^2}{32\pi^2} (7N - 2H_4w_2) \right]^{-1} \] (4.8)

whereas for \( k < k_{np} \) one uses the flow equation for \( w_1 \)

\[ \frac{\partial}{\partial t} w_1 = \frac{g^2(k_{np})}{8\pi^2} \left( \frac{11N}{3} - H_4 \frac{w_2}{w_1^3} \right) \] (4.9)

where \( H_4 = 16\pi^2 h_4 = r_1^4 + 3(N^2 - 1)r_3^2 = 2(3N^2 - 2)\zeta(3) \) for the choice \((1,2)\). It is obvious that for \( w_2 < 0 \) the gauge coupling \( g^2(k) \) always increases \((4.8)\) until at \( k = k_{np} \) the anomalous dimension \( \eta \) reaches one. If \( w_2 \) remains negative for \( k < k_{np} \), the coupling \( w_1 \) decreases until it reaches zero at the confinement scale \( k_\infty > 0 \). (If we define \( g^2(k) = g^2(k_{np})/w_1 \), this coupling diverges at the confinement scale.) The issue is different for \( w_2 > 0 \): The coupling \( w_1 \) would not reach zero since for small enough \( w_1 \) the second term in eq. \((4.9)\) would cancel the first term. One therefore needs an estimate of \( w_2(k) \).
Let us first consider $k > k_{np}$ where the evolution equation for $w_2$ reads (with $E_2 = -\frac{7}{720}$, $C_2^4 = -\frac{67}{180}$)

$$
\begin{align*}
\frac{\partial}{\partial t} w_2 &= (4 + \eta)w_2 + \frac{g^2}{8\pi^2} \left\{ \tau_2 \partial_0^{4,2} \left( \frac{127}{45} - \frac{29}{20} \eta \right) \right. \\
&\quad + (2 - \eta) \left( 5r_1^{4} + \frac{9}{2}(N^2 - 1)r_2^{4} \right) w_2 - (2 - \eta) \left( r_1^{4} + \frac{3}{2}(N^2 - 1)r_2^{4} \right) w_3 \right\} \\
&= \frac{\partial}{\partial t} \left( \frac{w_2}{g^2} \right) = 4\frac{w_2}{g^2} + \frac{127}{360\pi^2} \tau_2 \partial_0^{4,2} \tag{4.10}
\end{align*}
$$

We observe the appearance of the coupling $w_3$. This is where the truncation of this section becomes relevant. We simply put $w_3 = 0$ here. We then observe that all remaining terms in the curly bracket in (4.10) are positive. Starting from $w_2(\Lambda) = 0$ we conclude that $w_2(k)$ is negative for all $k$ between $k_{np}$ and $\Lambda$. In the perturbative region, where $Ng^2/16\pi^2$ is small, it is easy to see that $w_2$ is of the order $g^2$. In fact, we may neglect the terms $\sim w_2^2$ and $\sim \eta$ in the curly bracket in (4.10). One finds for the ratio $\frac{w_2}{g^2}$ an infrared stable fixpoint

$$
\frac{\partial}{\partial \tau} \left( \frac{w_2}{g^2} \right) = 4\frac{w_2}{g^2} + \frac{127}{360\pi^2} \tau_2 \partial_0^{4,2} \tag{4.11}
$$

$$
w_{2\tau}(k) = -\frac{127}{1440\pi^2} \tau_2 \partial_0^{4,2} g^2(k) = -\frac{127}{270} \frac{g^2}{16\pi^2} \tag{4.12}
$$

which is approached very rapidly. (The last equation in (4.12) is for the choice (1.2) and $N = 2$.) It is interesting to insert the value (4.12) into the $\beta$-function for $g^2$ (1.8). Expanding in powers of $g^2$ one has

$$
\frac{\partial g^2}{\partial \tau} = -\frac{22N}{3} \frac{g^4}{16\pi^2} - \frac{77N^2}{3} \frac{g^6}{(16\pi^2)^2} + (3N^2 - 2)\zeta(3) \frac{g^4}{4\pi^2} w_2 \\
= -\frac{22}{3} \frac{g^4N^2}{16\pi^2} - \frac{77}{3} \left( \frac{127}{45} \zeta(3) \tau_2 \left( 1 - \frac{2}{3N^2} \right) \right) \frac{g^6N^2}{(16\pi^2)^2} \tag{4.13}
$$

We note that without the term $\sim w_2$ the coefficient $\sim g^6$ exceeds the perturbative two-loop coefficient $-\frac{204}{9}(\frac{N^2}{16\pi^2})^2$ by a little more than 10%. It is comforting to find for the choice (1.2) ($r_0^{4,2} = \frac{1}{6}$) the contribution from $w_2$ in the same order of magnitude as this difference. This gives the hope that the dominant nonperturbative physics is already contained in the $F^2$ approximation, whereas additional invariants (like $F^4$ in this section) give only moderate modifications for $k > k_{np}$. We emphasize that a full computation of $\beta_{g^2}$ in order $g^6$ should take additional invariants into account, as for example $(F\bar{F})^2$ or $(D_\mu F^{\mu\nu})^2$. We also observe that for $k = k_{np}$ one has approximately $Ng^2/16\pi^2 = \frac{2}{21}$ such that the validity of perturbation theory extends roughly to all $k > k_{np}$.

The infrared fixpoint in $w_2/g^2$ implies that the coefficient $W_2$ in (4.1) diverges $\sim k^{-4}$

$$
W_2 \sim -\frac{g^4}{k^4} \tag{4.14}
$$
Because of the infrared divergence for $k \to 0$, a result of this type could never have been found in standard perturbation theory. It could, however, be derived using the \( "\frac{D}{Dt} \approx \frac{\partial}{\partial t}\) approximation of the effective average action which we displayed in eq. \((1.4)\). For this purpose one can neglect the $W''_k$ terms on the r.h.s. of \((1.4)\) and approximate $W'_k = w_1$. Then, with \((2.6)\) inserted into \((1.4)\), one obtains for the $k$-dependent terms in $\Gamma_k$

$$
\Gamma_k[A,A] = \frac{1}{2} \text{Tr}_{xeL} \ln \left[ w_1 D_T + (w_1 - \frac{1}{\alpha_k})D \otimes D + R_k(w_1 D_T + (w_1 - \frac{1}{\alpha_k})D \otimes D) \right] \\
- \text{Tr}_{xe} \ln \left[ -D^2 + R_k(-D^2) \right]
$$

(4.15)

If one extracts the $F^{\mu\nu}_{\mu\nu}$-term from these traces, one finds a (renormalized) coefficient which equals exactly $W^2$, as extracted from the fixpoint \((4.12)\). Clearly \((4.10)\) goes beyond the \( "\frac{D}{Dt} \approx \frac{\partial}{\partial t}\) approximation. The terms proportional to $\eta$ and to $w^2_2$ could not have been obtained in this approximation.

Let us finally consider $k < k_{np}$. The evolution equation for $w_2$

$$
\frac{\partial}{\partial t} w_2 = 4w_2 + \frac{g^2(k_{np})}{4\pi^2} \left\{ 2r_0^{4.2} \left( \frac{29}{20} \frac{w_2^2}{w_1^2} - \frac{7}{180} \right) \\
+ \left( 5r_0^4 + \frac{9}{2} (N^2 - 1) r_0^4 \right) \frac{w_2^2}{w_1^2} - \left( r_0^4 + \frac{3}{2} (N^2 - 1) r_0^4 \right) \frac{w_3}{w_1^4} \right\}
$$

(4.16)

again contains only positive terms in the curly bracket if $w_3$ is neglected and $w_1^2$ is larger than $\frac{7}{261}$. (For $w_1^2 < \frac{7}{261}$ the positive term $\sim w_2^2/w_1^4$ dominates largely.) The coupling $w_2$, which is negative at the scale $k_{np}$, will therefore remain negative for $k < k_{np}$.

The investigation of this section strongly indicates a negative value of the coupling $w_2$ for all scales. We conclude that the vanishing of $W_1$ at a nonzero confinement scale $k_{\infty}$ is unavoidable within our truncation. This is a clear indication that the $F^2$ term in the effective average action changes its sign at the scale $k_{\infty}$!

5 Gluon condensation

If the coefficient $W_1$ in front of the $F_{\mu\nu}F^{\mu\nu}$ term changes sign for small enough $k$, the solution $A_\mu = 0$ cannot correspond anymore to the minimum of $\Gamma_k$. Since $\Gamma_k$ is bounded from below there must be a new solution with $F_{\mu\nu}F^{\mu\nu} > 0$ corresponding to the true minimum of $\Gamma_k$. One therefore expects a phenomenon of “gluon condensation”, i.e. the minimum of $\Gamma_0$ should occur for nonvanishing $F_{\mu\nu}F^{\mu\nu}$. In our truncation this suggests that the minimum of $W_k(\theta)$ should shift to a nonvanishing $\theta_0$ for small enough $k$. On the other hand, a negative value of $W_2$ excludes a scenario where the minimum of $W_k$ remains at $\theta = 0$ for all $k > k_{\infty}$, and then $\theta_0$ continuously moves away from zero for $k < k_{\infty}$. A second local minimum of $\Gamma_k$ must already appear for $k_i > k_{\infty}$ and become the absolute minimum for $k < k_c$, 

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with $k_\infty < k_c < k$. At the scale $k_\infty$ the local minimum at $\theta = 0$ finally disappears. This scenario is closely analogous to a first order phase transition as a function of temperature. (In fact, the jump of the minimum of $\Gamma_k$ as a function of $k$ corresponds most probably to a jump of $\Gamma_0$ as a function of temperature and therefore to a first order phase transition of the high temperature Yang Mills theory.) In order to describe such a “first order scenario” one has to keep at least the term $\sim W_3 \theta^3$ in a polynomial expansion of $W_k$. We will show in this section that a polynomial approximation

$$w_k(\vartheta) = w_1 \vartheta + \frac{1}{2} w_2 \vartheta^2 + \frac{1}{6} w_3 \vartheta^3$$

(5.1)

is indeed sufficient for a qualitative description of gluon condensation.

The first necessary ingredient is a positive coefficient $w_3(k)$. With (1.2) we find in four dimensions the following $\beta$-function for $k > k_{np}$ (for details see appendix F)

$$\frac{\partial}{\partial \tau} w_3 = (8 + \eta) w_3 + g^2 \left\{ \frac{1}{16 \pi^2} \left( -\frac{442}{315} - \frac{137}{210} \eta \right) \tau_3 + \frac{87}{30} (2 - \eta) \tau_2 w_2 - 6 \zeta(3) (2 - \eta) \left[ (12 N^2 + 23) w_3^3 - (9 N^2 + 8) w_2 w_3 + N^2 w_4 \right] \right\}$$

(5.2)

We neglect the term $\sim w_4$ consistent with our approximation. For small values of $g^2$ there is an infrared stable fixpoint of the ratio $w_3/g^2$ corresponding to positive $w_3$

$$w_{3*}(k) = \frac{221 \tau_3 g^2(k)}{37800 16 \pi^2}$$

(5.3)

Actually, all ratios $w_n/g^2$ reach perturbative fixpoints for $n \geq 3$ (cf. appendix F). This justifies the approximation (5.1) at least for small enough $g^2(k)$. (We observe $W_n \sim g^{2(n-1)} k^{-4(n-1)}$. A negative value of $w_3(k)$ would require that the positive term $\sim w_3^3$ dominates the $w_3$-independent part of (5.2). This needs $w_2 > \frac{1}{12 N^2 + 19 60 \zeta(3)}$, or, for $N = 2$, $w_2 \gtrsim \frac{1}{8}$. Such a large value of $w_2$ does not occur for $k > k_{np}$ and we conclude that $w_3$ is positive for this range of scales.

For positive $w_3$ the polynomial (5.1) is bounded from below for $\vartheta \geq 0$. For $w_1 > 0$ there is always a local minimum at $\vartheta = 0$. Two additional extrema are present if

$$w_2 > 2 w_1 w_3$$

(5.4)

There is a minimum for positive $\vartheta_0$

$$\vartheta_0 = -w_2 + \sqrt{w_2^2 - 2w_1 w_3}$$

(5.5)

and, for $w_1 > 0$, a maximum at

$$\vartheta_{\text{max}} = -w_2 - \sqrt{w_2^2 - 2w_1 w_3}$$

(5.6)
For $w_1 < 0$ the origin $\vartheta = 0$ turns to a local maximum. The critical set of couplings where the two minima are of equal height ($w_k(\vartheta_0) = 0$) corresponds to

$$w_1 = \frac{3}{8} \frac{w_2^2}{w_3} \quad (5.7)$$

For $w_3$ smaller than the critical value, the absolute minimum occurs at $\vartheta_0$. This corresponds to the phenomenon of gluon condensation, with Euclidean action in the ground state given by $w_k(\vartheta_0)$.

For $k > k_{np}$ where $w_1 = 1$ a quick inspection of the flow equations (cf. (4.12), (5.3)) shows that $2w_3$ remains larger than $w_2^2$. There is therefore only one minimum at $\vartheta = 0$. For $k < k_{np}$, however, $w_1$ decreases towards zero. In the region where $w_3$ remains positive there is necessarily a scale $k_i$ where (5.4) is met and subsequently a scale $k_c$ where $w_1(k_c)$ obeys (5.7). For $k < k_c$ gluon condensation sets in. Since $k_c > k_\infty$ the establishment of the phenomenon of gluon condensation does not need the validity of the flow equations down to the scale $k_\infty$. It is sufficient if the flow equations are a reasonable approximation for $k \geq k_c$.

What remains to be discussed in our truncation is the positivity of $w_3$. The relevant flow equation for $k < k_{np}$ reads

$$\frac{\partial}{\partial t} w_3 = 8w_3 + \frac{g^2(k_{np})}{16\pi^2w_1} \left\{ -\frac{\tau_3}{30} \left( \frac{137}{105} w_1^6 + \frac{31}{315} w_1^2 \right) + \frac{87}{15} \tau_3 w_3 w_2 \\ -12\zeta(3) \left( \frac{N^2 w_4}{w_1} - (9N^2 + 8) \frac{w_2 w_3}{w_1^2} + (12N^2 + 23) \frac{w_3^2}{w_1 w_3} \right) \right\} \quad (5.8)$$

Neglecting again $w_4$ and for $w_2 < 0$ the only positive term in the curly bracket in (5.8) is the one $\sim w_3^3/w_1^3$. For $k \to k_\infty$ this term will finally dominate the flow equation (5.8), drive $w_3$ towards zero, and invalidate the truncation (5.1). We emphasize, however, that the critical condition (5.7) is always met at a scale $k_c$ where $w_3(k_c)$ is positive. Also, from

$$\frac{\partial}{\partial t} w_2 = 4w_2 + \frac{g^2(k_{np})}{16\pi^2w_1^2} \left\{ \tau_2 \left( \frac{29}{30} w_1^2 - \frac{7}{270} w_1^4 \right) \\ + \frac{2\zeta(3)}{3} \left[ (9N^2 + 1) \frac{w_2^2}{w_1^2} - (3N^2 - 1) \frac{w_3^2}{w_1^4} \right] \right\} \quad (5.9)$$

we learn that the term $\sim w_3$, which was neglected in the previous section, is actually substantially smaller than the term $\sim w_2^2$ for $k$ between $k_i$ and $k_c$. We therefore see a clear indication for gluon condensation within the $F^6$ truncation employed in this section!

The system of the three flow equations (1.9), (5.9), and (5.8) (with $w_4 = 0$) could be solved numerically with initial conditions given at the scale $k_{np}$. It would be interesting to see if the expectation value of $\theta$, i.e.

$$\theta_0(k) = Z^{-1}_{k_{np}} k^4 g^{-2}(k_{np}) \theta_0(k) \quad (5.10)$$
has a tendency to stabilize at a fixed value before \( w_1(k) \) or \( w_3(k) \) reach zero. The corresponding value could then be taken as a rough estimate for the size of the gluon condensate. The breakdown of the approximations of the present paper for \( k \to k_\infty \) is, however, unavoidable for very general reasons that will be discussed in the final section. This excludes in the present approach a quantitative determination of the gluon condensate.

6 Conclusions

In this paper we have approximated the exact nonperturbative evolution equation for Yang-Mills theories \([7]\) by a truncation where \( \Gamma_k \) is only given as a function \( W_k(\theta) \) of \( \theta = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \). This ansatz is general enough to allow for a description of gluon condensation. A nonvanishing gluon condensate corresponds to an absolute minimum of \( W_k(\theta) \) for \( \theta_0 > 0 \) in the limit where the infrared cutoff \( k \) vanishes. We have not solved explicitly for \( k \to 0 \), but we have found strong indications for gluon condensation appearing already for nonvanishing \( k \): First, the term linear in \( \theta \) in a expansion of \( W_k(\theta) \) around \( \theta = 0 \) turns negative at \( k = k_\infty \) (sect. 4). This implies that for \( k < k_\infty \) the “trivial” configuration \( F_{\mu\nu} = 0 \) cannot correspond to the minimum of the Euclidean action \( \Gamma_k \). Second, we have found (sect. 5) that a new minimum of \( W_k(\theta) \) appears for \( k = k_i \), and that this minimum becomes the absolute minimum of \( W_k \) at the scale \( k_c \), with \( k_i > k_c > k_\infty \). The truncated set of evolution equations is qualitatively reasonable for all \( k \) down to \( k_c \) and we interpret our results as a strong indication for gluon condensation.

On the other hand, we note that for \( k \) in the vicinity of the confinement scale \( k_\infty \) our flow equations become very problematic. These problems are not originating in the general method but they occur rather as a particularity of our truncation. There are three different sorts of problems, and it is instructive to discuss them in detail:

i) The flow equations are not valid for negative \( W'_k(\theta) = \partial W_k/\partial \theta \). For \( u = W'_k(\theta)x \) the momentum integrals implied by the traces of sect. 3, i.e. \( \int dxxH(W'_k x) \), are ultraviolet finite only for \( W'_k > 0 \). In consequence, various terms in the flow equations (4.9), (5.9), (5.8) diverge for \( w_1 \to 0 \). The reason for this disease is the negative inverse propagator \( \Gamma_k^{(2)} \sim W'_k q^2 \) for \( W'_k < 0 \). This implies in particular that the high momentum modes with \( q^2 \gg k^2 \) are all unstable. Even our optimized infrared cutoff (1.2) cannot remove this problem: The function \( H(u) \) defined in (5.3) grows \( \sim -u/k^2 \) for large \( (-u/k^2) \).

The problem described here should, however, be viewed as a pure artefact of an insufficient truncation. The physics of the high momentum modes in four-dimensional Yang-Mills theory should be governed by asymptotic freedom, and \( \Gamma_k \) should be described by a term \( \sim \frac{1}{4} Z_k F_{\mu\nu} F^{\mu\nu} \) with positive \( Z_k \) for all \( k \) as far as the high momentum modes are concerned. In our truncation we have neglected the expected momentum dependence of \( Z_k \). A more reasonable approximation for the
term quadratic in the gauge field $A_{\mu}$ would be of the sort

$$\frac{1}{4} \int d^4x F^2_{\mu\nu} Z_k(-D^2[A]) F^\mu_{\nu}$$

(6.1)

with $Z_k$ depending on the covariant Laplacian in the adjoint representation. For the functional form of $Z_k(q^2)$ we expect for large $q^2$ a $k$-independent positive function $Z(q^2)$. (In lowest order the logarithmic dependence of $Z$ on $q^2$ should be determined by the one-loop $\beta$-function for the running gauge coupling. Indeed, external momenta of the gluons act as an independent infrared cutoff. For $k^2 \ll q^2$ the running of $Z_k(q^2)$ with $k$ should essentially stop whereas it is given by the one-loop $\beta$-function for $k^2 \approx q^2$.) In contrast, our truncation identifies $Z_k(q^2)$ with a constant $Z_k(0)$ may well turn negative without affecting the high momentum behaviour of $Z_k(q^2)$. We conclude that with a better truncation the high momentum modes are always stable if the momentum dependence of the propagators is properly taken into account. This implies positive $u$ for $x \to \infty$ and no ultraviolet problem can appear for the exponential cutoff (1.2). The present truncation can nevertheless be employed for all scales $k$ where the inverse propagator for $q^2 \approx k^2$ can be approximated by $Z_k(0)q^2$. One may hope that $Z_k(q^2=k^2) \approx W'_k(\theta=0)$ remains valid in the vicinity of $k_c$.

ii) For $k < k_c$ the absolute minimum of $W_k$ occurs for $\theta_0 > 0$. For this range of scales it would seem reasonable to expand $W'_k$ around $\theta_0$ rather than around 0. A polynomial expansion around the true minimum has turned out to be an important improvement for all models investigated so far with the help to the average action. The reason is that the relevant mass terms in the “ground state” appear then directly in the flow equations. Instead, an expansion around $\theta = 0$ involves the “unphysical mass terms” corresponding to a metastable configuration. At the minimum of $W_k$ one always has

$$W''_k(\theta_0(k)) = 0$$

(6.2)

$$W''_k(\theta_0(k)) = \bar{\lambda} > 0$$

(6.3)

Differentiating (6.2) with respect to $k$ yields the running of $\theta_0(k)$, i.e.

$$\frac{d\theta_0}{dt} = -\bar{\lambda}^{-1} \frac{\partial}{\partial t} W'_k(\theta_0)$$

(6.4)

This equation, together with flow equations for the couplings $W^{(n)}(\theta_0)$, could replace the flow equations of the last section for $k < k_c$.

Unfortunately, a quick inspection of eq. (3.7) reveals that the r.h.s. of such flow equations would diverge in the present truncation. It is instructive to investigate more in detail the contributions from the different modes to the evolution equation (6.4): First we have modes with a positive “transverse kinetic term” $(W'_k(\theta_0) + B^2 W''_k(\theta_0)) x = \bar{\lambda} B^2 x$ that we may call of type I. (Here $x$ denotes the square of the transverse momentum.) The contributions from type I modes as well as from
ghosts are unproblematic for $\vartheta = \vartheta_0$. The type II modes have an inverse propagator $W'_k(\vartheta)x + c$ with $c \geq 0$. Contributions from these modes suffer from the “high momentum disease” since $W'_k(\vartheta_0) = 0$. The corresponding ultraviolet divergences in the flow equations could be cured by employing $Z_k(q^2)$ as discussed above. The infrared behaviour remains without problems.

Finally, there could be modes of type III with inverse propagator $W'_k(\vartheta)x - c, c > 0$. This would lead to an infrared instability for the modes with $x \to 0$ due to a negative masslike term $-c$. Such an infrared problem cannot be removed by a modified truncation for the high momentum modes. It would be an indication that even at $\vartheta_0$ the particular configuration with constant magnetic field introduced in sect. 2 does not correspond to the minimum of $\Gamma_k$. This is obviously not possible within our truncation and we find correspondingly that no mode of type III exists in the spectrum for $\vartheta = \vartheta_0$. An expansion around $\vartheta_0$ becomes therefore possible for an enlarged truncation of the type (6.1).

We should mention that $\Gamma_k$ is, of course, not only defined for the ground state configuration. Negative eigenvalues of $\Gamma_k^{(2)}$ for a configuration like (2.7) are harmless as long as $k^2$ is much larger than the absolute value of a negative eigenvalue. Only once $k^2$ becomes of equal size strong renormalization effects set in. They lead to a “flattening” of $\Gamma_k$. This is related to the general property that $\Gamma_k$ becomes a convex quantity for $k \to 0$. We are not dealing with this “approach to convexity” [[1]] in the present investigation.

iii) We observe that the spectrum of small fluctuations around the constant magnetic field configuration (2.7) lacks Euclidean $SO(4)$ rotation symmetry. This is not surprising since $F_{\mu\nu}$ singles out two of the space directions. In addition, the spectrum is partially discrete - continuity exists only with respect to the transversal momentum. The lack of full rotation symmetry and the partial discreteness of the spectrum are actually related: For an $SO(4)$ symmetric spectrum the continuity in two momentum directions must extend to all momentum directions. One expects then a spectrum with a few separate particles, each of them having $\Gamma_k^{(2)}$ depending continuously on a generalized $SO(4)$ invariant of the type $q_\mu q^\mu$. The true ground state should be invariant under a generalized version of Poincaré transformations $[4]$ and therefore have an $SO(4)$ symmetric spectrum. In ref. $[3]$ we have proposed a candidate ground state for $d = 4$ and gauge group $SU(N), N \geq 4$, but no realistic candidate for the gauge group $SU(3)$ has been found up to now. It would be very interesting to perform an analysis similar to the one presented here for this groundstate candidate instead of the configuration (2.7).

In summary, our first attempt to investigate the phenomenon of gluon condensation with the help of nonperturbative flow equations is encouraging. Our simple configuration (2.7) and the simple truncation give qualitatively the expected behaviour of $\Gamma_k$: The minimum of $\Gamma_k$ does not occur for $F_{\mu\nu} = 0$ for sufficiently small $k$, thus indicating the phenomenon of gluon condensation. The present truncation is, however, insufficient to describe the true ground state. In consequence, a detailed quantitative analysis of the size of the gluon condensate seems premature and should
wait for an investigation involving a configuration which is more similar to the true
ground state.

Appendix A

In this appendix we collect a few trace identities which are needed in order to derive
the evolution equation for \( W_k \). We choose the covariantly constant background (2.7),
(2.8). Then eq. (2.13) implies for any function \( f \)

\[
\text{Tr}_{xL}[P_{\parallel} f (D_{\mu}, P_{\parallel}, P_{\perp})] = \text{Tr}_{xL}[f (\partial_{\mu}, 1, 0)]
\]

(A.1)

because \( \text{Tr}_{x}[P_{\parallel}] = n^z n_z = 1 \). Writing \( P_{\perp} = 1 - P_{\parallel} \) and exploiting \( S \propto P_{\parallel} \) it is also
easy to see that

\[
\text{Tr}_{xL}[P_{\perp} f (D_{\mu}, S)] = \text{Tr}_{xL}[f (D_{\mu}, 0)] - \text{Tr}_{xL}[f (\partial_{\mu}, 0)].
\]

(A.2)

Since \( F_{\mu\nu} \) is a constant matrix, the operator \( s_{\mu\nu} \) of (2.14) commutes with \( P_{L}^{(0)} \) and
\( P_{T}^{(0)} \) and satisfies \( \partial^\mu s_{\mu\nu} = 0 \). This fact can be used to show that

\[
\text{Tr}_{xL}[f (P_{L}^{(0)}, P_{T}^{(0)}; s)] = \text{Tr}_{xL}[f (0, 1; s)] + \text{Tr}_{x}[f (1, 0; 0)] - \text{Tr}_{x}[f (0, 1; 0)]
\]

(A.3)

If one subtracts the same expression with \( s = 0 \) one obtains

\[
\text{Tr}_{xL}[f (P_{L}^{(0)}, P_{T}^{(0)}; s) - f (P_{L}^{(0)}, P_{T}^{(0)}; 0)] = \text{Tr}_{xL}[f (0, 1; s)] - d\text{Tr}_{x}[f (0, 1; 0)]
\]

(A.4)

In the last step we used that \( \text{Tr}_{xL} = d\text{Tr}_{x} \) for an operator \( \sim \delta_{\mu\nu} \). In the above
identities the function \( f \) may also depend on further operators provided they commute with those displayed explicitly and do not introduce any additional colour or Lorentz index structures.

For the evaluation of \( U_1 \) in eq. (B.5) we need another important relation:

\[
\text{Tr}_{xL}[P_{L} f (D_{T})] = \text{Tr}_{x}[f (-D^2)]
\]

(A.5)

It follows from the fact that the operator \( (D_{T})_{\mu\nu} = -D^2 \delta_{\mu\nu} + 2i \bar{g} F_{\mu\nu} \), when restricted
to the space of longitudinal modes \( (a_{\mu} = (P_{L})_{\nu}^{\mu} a_{\nu}) \), has the same spectrum as \( -D^2 \)
acting on Lorentz scalars. The proof makes essential use of the identity

\[
D_{T} (D \otimes D) = (D \otimes D) D_{T} = -(D \otimes D) (D \otimes D)
\]

(A.6)

which holds true whenever the gauge field contained in the covariant derivatives obeys \( D^\mu F_{\mu\nu} = 0 \).
Appendix B

In this appendix we use the trace identities derived in appendix A in order to simplify the flow equation (3.2). The first trace on the r.h.s. of eq. (3.1) can be simplified by using the identities of appendix A. Inserting a factor of $\frac{1}{P_\parallel + P_\perp}$ leads to the decomposition

$$\text{Tr}_{xL} \left[ H(\Gamma^{(2)}_{k} [A, A]) \right] = T_1^\parallel + T_1^\perp$$

with

$$T_1^\parallel = \text{Tr}_{xL} \left[ P_\parallel H(W_k' D_T + [W_k' - \frac{1}{\alpha_k}]D \otimes D + W_k'' s) \right]$$

and

$$T_1^\perp = \text{Tr}_{xL} \left[ H(-W_k' \partial^2 + [W_k' - \frac{1}{\alpha_k}]\partial \otimes \partial + W_k'' s) \right]$$

where (A.1) was used, and

$$U_1 = \text{Tr}_{xL} \left[ H(W_k' D_T + [W_k' - \frac{1}{\alpha_k}]D \otimes D) \right]$$

and $U_2$ the sum of $T_1^\parallel$ and the second term of (B.3):

$$U_2 = \text{Tr}_{xL} \left[ H(-\partial^2[W_k'P_T^{(0)} + \frac{1}{\alpha_k}P_L^{(0)}] + W_k'' s) \right]$$

It is quite remarkable that if we now apply the identity (A.4) to $U_2$, the longitudinal contribution drops out completely and the result becomes independent of the gauge fixing parameter $\alpha_L$:

$$U_2 = \text{Tr}_{xL} [H(-\partial^2 W_k' + W_k'' s)] - d\text{Tr}_x [H(-\partial^2 W_k')]$$
The operators entering (B.7) are easily diagonalized in a plane-wave basis. A standard calculation yields, for $W'_k > 0, W''_k + B^2W''_k > 0$,

$$\Omega^{-1}U_2 = 2v_d \left(\frac{1}{W'_k + B^2W''_k} - \frac{1}{W'_k}\right) \left(\frac{1}{W'_k}\right)^{\frac{d}{2}-1} \int_0^\infty dx \, x^{\frac{d}{2}-1}H(x)$$  \hfill (B.8)

with $v_d = [2^{d+1} \pi^{d/2} \Gamma(d/2)]^{-1}$. (As always, the argument of $W_k$ and its derivatives is understood to be $\frac{1}{2}B^2$.)

Next let us simplify the trace $U_1$ by inserting a pair of projectors:

$$U_1 = \text{Tr}_{xcL} \left[ P_T H(D_T[W'_kP_T + \frac{1}{\alpha_k}P_L]) \right] + \text{Tr}_{xcL} \left[ P_L H(D_T[W'_kP_T + \frac{1}{\alpha_k}P_L]) \right]$$

$$= \text{Tr}_{xcL} \left[ P_T H(W'_kD_T) \right] + \text{Tr}_{xcL} \left[ P_L H(\frac{1}{\alpha_k}D_T) \right]$$

$$= \text{Tr}_{xcL} \left[ H(W'_kD_T) \right] + \Delta U_1(\alpha_k).$$  \hfill (B.9)

The $\alpha$-dependence of $U_1$ is contained in

$$\Delta U_1(\alpha_k) = \text{Tr}_{xcL} \left[ P_L \{ H(\frac{1}{\alpha_k}D_T) - H(W'_kD_T) \} \right]$$

$$= \text{Tr}_{xc} \left[ H(-\frac{1}{\alpha_k}D^2) - H(-W'_kD^2) \right].$$  \hfill (B.10)

In the last line of (B.10) we made use of the identity (A.5).

By a similar combination of the trace identities we can also evaluate the last term on the r.h.s. of (3.2)

$$\text{Tr}_{xcL} \left\{ P_\perp P_L \left( \tilde{H} \left( \frac{D_T}{\alpha_k} \right) - H \left( \frac{D_T}{\alpha_k} \right) \right) \right\}$$

$$= \text{Tr}_{xc} \left\{ \tilde{H} \left( -\frac{D^2}{\alpha_k} \right) - H \left( -\frac{D^2}{\alpha_k} \right) \right\} - \text{Tr}_x \left\{ \tilde{H} \left( -\frac{\partial^2}{\alpha_k} \right) - H \left( -\frac{\partial^2}{\alpha_k} \right) \right\}. \hfill (B.11)$$

Evaluating the second term in a plane wave basis yields

$$\frac{1}{2\Omega} \text{Tr}_x \left\{ \tilde{H} \left( -\frac{\partial^2}{\alpha_k} \right) - H \left( -\frac{\partial^2}{\alpha_k} \right) \right\} = v_d \int_0^\infty dx \, x^{\frac{d}{2}-1} \left( \tilde{H} \left( \frac{x}{\alpha_k} \right) - H \left( \frac{x}{\alpha_k} \right) \right)$$  \hfill (B.12)

At this point we have exploited the various trace identities as much as possible. Combining these results yields the flow equation (3.4).

**Appendix C**

In this appendix we discuss the group theoretical factors $\sum_\ell \nu_\ell^{2m}$. The LHS of the evolution equation is $\partial_t W_k$. The argument of $W_k$ is $\frac{1}{2}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}B^2$ which is manifestly independent of the unit vector $n^z$ which specifies the direction of the field in
“color space”. The r.h.s. of the evolution equation consists of expansions such as (3.9) which involve the factors \( \sum \nu^2_m \). As \( \{\nu_\ell\} \) are the eigenvalues of \( n^z T_z \), we can rewrite them as

\[
\sum \nu^2_\ell = n^{z_1} n^{z_2} \cdots n^{z_{2m}} \text{Tr}_c[T_{z_1} T_{z_2} \cdots T_{z_{2m}}] \quad (C.1)
\]

where the trace is in the adjoint representation. The question is whether the invariants \( (C.1) \) are all independent of the direction of \( n^z \). In appendix D we explain in detail that generically this is not the case. If the orbit space of the gauge group in the adjoint representation is nontrivial, different \( n^z \)'s can lead to different sums \( \sum \nu^2_\ell \). The resolution to this puzzle is as follows. For \( m = 1 \) we can use the standard orthogonality relation

\[
\text{Tr}_c[T_y T_z] = N \delta_{yz} \quad (C.2)
\]

to prove that \( \sum \nu^2_\ell = N n^z n_z = N \) is independent of the direction of \( n \). Likewise, if the symmetric invariant tensor \( \text{Tr}_c \left[ T_{(z_1 \cdots T_{z_{2m}})} \right] \) is proportional to the trivial one, \( \delta_{(z_1 z_2 \cdots z_{2m-1} z_{2m})} \), we can again use the normalization condition \( n^z n_z = 1 \) to show that the r.h.s. of \( (C.1) \) is independent of \( n \). The situation changes if there exists a totally symmetric invariant tensor \( T_{z_1 z_2 \cdots z_{2m}} \) which is different from the trivial one. Then we might have

\[
\text{Tr}_c \left[ T_{(z_1 \cdots T_{z_{2m}})} \right] = \tau_m \delta_{(z_1 z_2 \cdots z_{2m-1} z_{2m})} + T_{z_1 z_2 \cdots z_{2m}} \quad (C.3)
\]

with some coefficient \( \tau_m \). (If there exists more than one \( T \) an appropriate sum is implied.) In general \( n^{z_1} n^{z_2} \cdots T_{z_1 z_2} \cdots \) will be direction dependent \( [20] \). If some invariant tensor \( T \) exists, the correct way of deriving the evolution equation for \( W_k \) is to compare coefficients of a fixed tensor structure on both sides of the equation. Clearly the l.h.s., \( \partial_t W_k(\frac{1}{4} F^2_{\mu\nu}) \), gives rise to the trivial tensor structure only. Therefore only the \( \tau_m \)-piece of \( (C.3) \) should be kept in \( (C.1) \) and the \( (n\text{-dependent}) \) part coming from \( T \) has to be discarded. Thus \( (3.9) \) may be used on the r.h.s. of the equation for \( W_k \) provided we interpret \( \sum \nu^2_\ell \) as the coefficient \( \tau_m \). On the other hand, nontrivial \( T \) permits to construct additional invariants from an even number of covariantly conserved \( F_{\mu\nu} \). The truncation \( W(\theta) \) is not sufficient anymore to parametrize the most general effective action for constant magnetic fields of the type introduced in sect. 2 (with covariantly constant \( F_{\mu\nu} \)). The evolution equation for the new invariants can now be extracted by projecting the r.h.s. on the appropriate tensor structure. We will not pursue this generalization in the present paper.

In appendix D we show that for \( SU(2) \) this complication is absent. There exists no additional invariant tensor \( T \), and one finds the \( n\text{-independent} \) result

\[
\sum_{l=1}^{3} \nu^2_l = 2 \quad (C.4)
\]

for all \( m = 1, 2, \ldots \).
Appendix D

In this appendix we investigate in more detail the group-theoretical quantities $\sum \nu_l^2m_l$ which occur in many calculations involving covariantly constant backgrounds of the type $A_\mu^z = n^z A_\mu$. Here we consider an arbitrary (semi-simple, compact) gauge group $G$ with structure constants $f^{wyz}$. For a fixed unit vector $n^z$ we consider the matrix

$$\hat{n}^{yz} = n^w (T^w)^{yz} = if^{yzw} n^w$$  \hspace{1cm} (D.1)$$

The numbers $\nu_l, l = 1, \ldots, \dim G$ are the eigenvalues of $\hat{n}$: $\hat{n}^{yz} \psi_l^z = \nu_l \psi_l^y$. This equation can be rewritten in a more suggestive form. Let $t^z$ denote the generators of $G$ in an arbitrary representation: $[t^w, t^y] = if^{wyz} t^z$. If we define

$$\tilde{n} = n^z t^z, \quad \tilde{\psi}_l = \psi_l^z t^z$$  \hspace{1cm} (D.2)$$

the eigenvalue equation becomes

$$[\tilde{n}, \tilde{\psi}_l] = \nu_l \tilde{\psi}_l$$  \hspace{1cm} (D.3)$$

Clearly the $\nu_l$’s do not depend on the representation chosen. We would like to know how the spectrum $\{\nu_l\}$ depends on the vector $n$. First of all, it is clear that if $V$ is any group element in the $t$-representation, the matrices $\hat{n}$ and $\hat{n}' = V \hat{n} V^{-1}$ have the same spectrum, i.e. the spectrum is constant along the orbit of $G$ in the adjoint representation. If two directions $n$ and $n'$ are not related by a group transformation, then the spectra $\{\nu_l(n)\}$ and $\{\nu_l(n')\}$ can be different. Typically, for $G$ large enough [20], the orbit space is indeed nontrivial, and the spectrum “feels” the direction of $n^z$.

Let us go over from the basis $\{T^z\}$ to the Cartan-Weyl basis $\{H_i, E_{\bar{\alpha}}\}$ of the abstract Lie algebra. Here $\bar{\alpha} \in \mathbb{R}^r$ are the root vectors and $i = 1, \ldots, r \equiv \text{rank } G$. For definiteness we assume that the $t^z$’s are in the fundamental representation where we write $\{h_i, e_{\bar{\alpha}}\}$ for the Cartan-Weyl basis. Thus

$$[H_i, E_{\bar{\alpha}}] = \alpha_i E_{\bar{\alpha}} \quad \text{and} \quad [h_i, e_{\bar{\alpha}}] = \alpha_i e_{\bar{\alpha}}$$  \hspace{1cm} (D.4)$$

We assume that the generators $h_i$ of the Cartan subalgebra are given by diagonal matrices. By an appropriate transformation $\tilde{n} \rightarrow V \tilde{n} V^{-1}$ any $\tilde{n}$ can be brought to diagonal form. Therefore, in order to investigate the $n$-dependence of $\{\nu_l(n)\}$, it is sufficient to consider $\tilde{n}$’s which are in the Cartan subalgebra: $\tilde{n} = \sum_{i=1}^r n_i h_i$. For this choice

$$[\tilde{n}, e_{\bar{\alpha}}] = \left( \sum_{i=1}^r n_i \alpha_i \right) e_{\bar{\alpha}}, \quad [\tilde{n}, h_i] = 0$$  \hspace{1cm} (D.5)$$

and the nonvanishing eigenvalues $\nu_l = \nu_{\bar{\alpha}}$ are given by

$$\nu_{\bar{\alpha}} = \sum_{i=1}^r n_i \alpha_i$$  \hspace{1cm} (D.6)$$
Therefore the quantities $\sum_l \nu_l^{2m}$ can be computed explicitly from the root system:

$$\sum_{l=1}^{\text{dim}G} \nu_l^{2m} = \sum_{\text{roots} (\vec{a})} \left( \sum_{i=1}^{\text{rank} G} n_i \alpha_i \right)^{2m}$$  \hspace{1cm} (D.7)

Let us consider a few simple examples. For $G = SU(2)$ we have $\alpha = \pm 1$. Thus

$$\sum_{l=1}^{3} \nu_l^{2m} = 2, \quad m = 1, 2, 3, ...$$  \hspace{1cm} (D.8)

depends neither on the direction $n^z$ nor on the power $m$. This degeneracy can be understood by noting that for $SU(2)$ the square of the matrix (D.1) is the projector $P_{\perp}$: $\hat{n} \hat{n} = P_{\perp}$. This means that $\sum_l \nu_l^{2m} = \text{Tr}(P_{\perp}^m) = \text{Tr}(P_{\perp}) = 2$, as it should be.

Contracting eq. (C.3) with $n^z$ and comparing the result to (D.8) we see that there exists no nontrivial invariant tensor $T$ and that $\tau_m = 2$ for all $m$.

For $G = SU(3)$ we have $r = 2$, and a 2-component unit vector $(n_1, n_2)$ specifies the direction of the field in the Cartan subalgebra. Using the explicit form of the roots it is straightforward to derive that

$$\sum_{l=1}^{8} \nu_l^{2m} = 2^{1-2m} \left[ (n_1 + \sqrt{3} n_2)^{2m} + (n_1 - \sqrt{3} n_2)^{2m} + (2n_1)^{2m} \right]$$  \hspace{1cm} (D.9)

For $m = 1$ and $m = 2$ it turns out that this expression depends on $n_1$ and $n_2$ only via $n_1^2 + n_2^2 = 1$, and one obtains the direction-independent results

$$\sum_{l=1}^{8} \nu_l^2 = 3, \quad \sum_{l=1}^{8} \nu_l^4 = \frac{9}{4}$$  \hspace{1cm} (D.10)

Starting from $m = 3$, the invariants are explicitly $n$-dependent. Writing $n_1 = \cos \theta, n_2 = \sin \theta$ we find for $m = 3$

$$\sum_{l=1}^{8} \nu_l^6 = \frac{3}{16} \left[ 11 \cos^6 \theta + 15 \cos^4 \theta \sin^2 \theta + 45 \cos^2 \theta \sin^4 \theta + 9 \sin^6 \theta \right]$$  \hspace{1cm} (D.11)

As discussed in section 3, the $n$-dependence is related to the existence of a nontrivial invariant tensor $T_{z_1...z_6}$. However, we are not going to calculate the corresponding coefficient $\tau_3$ here.

**Appendix E**

In the regime where $\bar{g}B/k^2 \gtrsim 1$ the use of the Euler-McLaurin expansion (3.9) becomes questionable and we should look for an alternative representation of the spectral sums (3.7) and (3.8). In this section we use the Schwinger proper-time representation $[16]$. It can be easily applied only for the simplified cutoff function

$$R_k(x) = Z_k k^2.$$  \hspace{1cm} (E.1)
In this case one may write \((x \equiv D_T)\)

\[
H(x) \equiv \frac{\partial R_k(x)}{x + R_k(x)} = \frac{\partial}{\partial t}(Z_k k^2) \int_0^\infty ds e^{-sz_k^2} e^{-sx}
\]  \hspace{1cm} (E.2)

Inserting this representation into (3.7) we may employ

\[
\sum_{n=0}^\infty \exp[-sW'_k g|\nu_l B(2n + 1)] = \frac{1}{2 \sinh[sW'_k g|\nu_l B]}
\]

and

\[
\int_0^\infty dx x^{\frac{d}{2}} e^{-sW'_{kx}} = v_d^{-1}2^{1-d} \pi^{1-\frac{d}{2}} (sW'_{k})^{1-\frac{d}{2}}
\]

as long as \(W'_{k} > 0\). One finds

\[
\Omega^{-1} \text{Tr}_{\text{cl}}[H(W'_k D_T)] = 2(4\pi)^{-d/2} (W'_k)^{1-\frac{d}{2}}
\]

\[
\frac{\partial}{\partial t}(Z_k k^2) \sum_{l=1}^{N^2-1} \tilde{g}|\nu_l B \int_0^\infty ds \frac{s}{s^{(d-4)/2}} e^{-sz_k^2}
\]

\[
\cdot \left[ \frac{d}{2 \sinh[(sW'_k g|\nu_l B)]} - \exp(-sW'_k g|\nu_l B) + \exp(+sW'_k g|\nu_l B) \right]
\]  \hspace{1cm} (E.3)

where the last exponential is due to the unstable mode. For \(Z_k k^2 < W'_k g|\nu_l B\) it makes the \(s\)-integration divergent at the upper (i.e. IR) limit. In conventional calculations of the one-loop effective action this creates a problem from the outset, because one attempts to work at \(k^2 \rightarrow 0\) there. In the present formulation everything is well defined for \(k^2\) sufficiently large, and one interesting question is how the renormalization group flow behaves as one approaches \(k^2 \approx \theta\) from above.

In the UV limit \(s \rightarrow 0\) the terms inside the square bracket in (E.3) behave as

\[
[...] = \frac{d}{2\tilde{g}|\nu_l B W'_k(B^2/2)} \cdot \frac{1}{s} + O(s)
\]  \hspace{1cm} (E.4)

While the \(O(s)\)-terms do not lead to UV-divergences for \(d < 6\), the term \(\sim 1/s\) leads to a divergent contribution to the proper-time integral. Though the factor \(\tilde{g}|\nu_l B\) cancels against a similar one coming from the density of states, this divergent piece is still field-dependent because of the \(B\)-dependence of \(W'_k\). This UV divergence shows a failure of the truncation for the mass-type cutoff function \(R_k = Z_k k^2\). The latter may be used only together with the approximation \(W''_k = 0\) on the r.h.s. of the flow equation. The divergent piece in the proper-time integral is an irrelevant constant then.

Using (E.3) and a similar formula for the scalar traces in eq. (B.4), we obtain (up to an irrelevant constant and for \(\tilde{Z}_k = 1/\alpha_k, \tilde{\eta} = 0\))

\[
\frac{\partial}{\partial t} W_k(\frac{1}{2} B^2) = (4\pi)^{-d/2} \sum_{l=1}^{N^2-1} \tilde{g}|\nu_l B k^2 \int_0^\infty ds \frac{s^{1-d/2}}{s^{(d-4)/2}}.
\]
\[
(2 - \eta)Z_k(W'_k)^{1 - \frac{d}{2}} \left[ \frac{d - 1}{2\sinh(sW'_k\bar{g}|B|)} + 2\sinh(sW'_k\bar{g}|B|) \right] e^{-sZ_kk^2} \left\{ \frac{\exp(-sk^2)}{\sinh(sq|\nu_l|B)} \right\}
\]

(E.5)

This evolution equation is the analogue of (3.11) with the additional assumption \(W''_k = 0\). Contrary to the Euler-McLaurin series it is valid even for strong fields \(\bar{g}B \approx k^2\).

For \(\bar{g}B \ll k^2\) the r.h.s. of (E.5) can be expanded in powers of \(B\). Apart from the different form of \(R_k\), this reproduces the Euler-McLaurin expansion. Expanding up to order \(B^4\) we find, for instance,

\[
\frac{\partial}{\partial t} \left( \frac{w_2}{g^2} \right) = 4 \frac{w_2}{g^2} + \frac{127}{360\pi^2} r_2 r_0^{4,2}
\]

(E.6)

with \(r_0^{4,2} = 2\). This result is the counterpart of eq. (4.11) which had been obtained with the exponential cutoff for which \(r_0^{4,2} = 1/6\). In accordance with the (3.13) we find an additional factor of 12 in the second term on the r.h.s. of (E.6). Hence also the value of the fixpoint \(w_2^*(k)\) is 12 times larger than the result (4.12). In view of the discussion following (4.13) this means that for the mass-type cutoff \(R_k = Z_kk^2\) this higher-order correction is much larger than for the exponentially decreasing \(R_k\) of (1.2). This is probably closely related to the ultraviolet problems and indicates that, though computationally more difficult to handle, the cutoff (1.2) should be used for reliable estimates.

**Appendix F**

In this appendix we derive the flow equation in the \(F^6\) truncation. We start from the evolution equation for \(\dot{w}(\vartheta)\) which follows from differentiating (4.7) with respect to \(\vartheta\)

\[
\frac{\partial}{\partial t} \dot{w} = (4 + \eta)\dot{w} + 4\vartheta w^{(3)}
\]

\[-(2 - \eta)v_d g^2 \dot{w}^\frac{3}{4} \left\{ (d - 2) \sum_{m=1}^{\infty} \tau_m (C^d_m - E_m) r_0^{d,m} \right\}
\]

\[
\left[ (2\vartheta \dot{w}^2)^m - 1 \left( (8 - d)m - d + \frac{d^2}{2} \right) \dot{w} \dot{w}^2 + (4m - d)\dot{w} w^{(3)} \right]
\]

\[+ 2(m - 1)(2\vartheta \dot{w}^2)^{m-2} \dot{w}^2 (\dot{w} + 2\vartheta \ddot{w}) (2m\dot{w} + (4m - d)\ddot{w}) \]

\[+ r_1^d \left[ \frac{4w^{(3)} + 2\vartheta w^{(4)}}{\dot{w} + 2\vartheta \ddot{w}} - \frac{4(\dot{w} + \vartheta w^{(3)})(3\dot{w} + 2\vartheta w^{(3)}) + 2\vartheta \ddot{w}(5w^{(3)} + 2\vartheta w^{(4)})}{(\dot{w} + 2\vartheta \ddot{w})^2} \right]
\]

\[+ \frac{4\vartheta \ddot{w}(3\dot{w} + 2\vartheta w^{(3)})^2}{(\dot{w} + 2\vartheta \ddot{w})^3} \]

with \(r_0^{4,2} = 2\). This result is the counterpart of eq. (1.11) which had been obtained with the exponential cutoff for which \(r_0^{4,2} = 1/6\). In accordance with the (3.13) we find an additional factor of 12 in the second term on the r.h.s. of (E.6). Hence also the value of the fixpoint \(w_2^*(k)\) is 12 times larger than the result (4.12). In view of the discussion following (4.13) this means that for the mass-type cutoff \(R_k = Z_kk^2\) this higher-order correction is much larger than for the exponentially decreasing \(R_k\) of (1.2). This is probably closely related to the ultraviolet problems and indicates that, though computationally more difficult to handle, the cutoff (1.2) should be used for reliable estimates.
\[-d^2 \ddot{w}^2 + 3 \dddot{w} \dddot{w} + 2d \frac{\partial \dddot{w}}{\partial \dot{w}}(3 \dot{w} + 2 \partial \ddot{w}) + 3d \left( \frac{\partial \dddot{w}}{\partial \dot{w}} \right) + 2d \frac{\partial \dddot{w}}{\partial \dot{w}}(3 \dot{w} + 2 \partial \ddot{w}) \left( \frac{\dot{w}}{2} \right) \left( \frac{\partial \dddot{w}}{\partial \dot{w}} \right)\]

\[+ \frac{1}{4} d(d-1)(d-2) N \dot{r}_d \left[ \frac{w(3)}{\dot{w}} - \frac{d + 2 \ddot{w}}{2} \right] \]

\[+ 8(d - 2) \nu_d g^2 \sum_{m=1}^{\infty} m(m-1) \tau_m E_m r_0^d m(2 \vartheta)^{m-2} \]

F.1

Taking one further \( \vartheta \)-derivative at \( \vartheta = 0 \) we obtain for \( d = 4 \) and \( k > k_{np} \) the flow equation for \( w_3 \) (with \( C_2^4 E_2 = -\frac{29}{80}, C_3^4 E_3 = -\frac{137}{10080} \) and \( E_3 = \frac{31}{30240} \))

\[ \frac{\partial}{\partial t} w_3 = (8 + \eta) w_3 \]

\[ + \frac{g^2}{16 \pi^2} \left\{ \left( \frac{442}{315} - \frac{137}{210} \right) \tau_3 r_0^{4,3} + \frac{87}{5} (2 - \eta) \tau_2 r_0^{4,2} \dot{w}_2 \right. \]

\[ - (2 - \eta) r_1^4 (3 w_4 - 51 w_2 w_3 + 105 w_3^2) \]

\[-3(2 - \eta) (N^2 - 1) r_2^3 (w_4 - 9 w_2 w_3 + 12 w_3^2) \}

F.2

For the infrared cutoff (1.2) one has \( r_0^{4,2} = \frac{1}{6}, r_0^{4,3} = -\frac{1}{30} \) and we observe that the perturbatively leading term \( \sim g^2 \) is negative. This implies a positive perturbative fixpoint value for the ratio \( w_3 / g^2 \).

This discussion can easily be generalized for arbitrary \( w_n \). In lowest order in \( g^2 \) the flow equation (F.1) simplifies considerably \((k > k_{np})\)

\[ \frac{\partial}{\partial t} \dot{w} = 4 \dot{w} + 4 \partial w^{(3)} \]

\[-8(d - 2) \nu_d g^2 \sum_{m=1}^{\infty} m(m-1) \tau_m E_m r_0^d (2 \vartheta)^{m-2} \]

F.3

This implies for \( n \geq 2 \) the flow equations

\[ \frac{\partial}{\partial t} w_n = 4^{n-1} w_n - (d - 2) \nu_d g^2 2^{n+1} n! \tau_n r_0^{d,n} (C_n - 2 E_n) \]

F.4

and the infrared fixpoint values

\[ \left( \frac{w_n}{(d - 2) \nu_d g^2} \right)_* = 2^{3-n！} n! \tau_n r_0^{d,n} (C_n - 2 E_n) \]

F.5

For \( d = 4 \) and with (F.1)2 this yields

\[ w_n = \frac{d_n \nu_d^2}{16 \pi^2 } \]

\[ d_n = \frac{2^{4-n！} (2n - 1)}{(2n - 1)! \tau_n B_{2n-2} \left( \frac{2^{2n-1} - 1}{2n} B_{2n} - 1 \right) \] 

F.6
where for $SU(2)$

$$\frac{2d_2}{\tau_2} = -\frac{127}{270}, \quad \frac{2d_3}{\tau_3} = \frac{221}{37800}$$  \quad (F.7)

The series of $d_n$ is alternating as long as the bracket is dominated by -1.

Finally, for $k < k_{np}$ the flow equation for $w_3$ follows from (F.1) as

$$\frac{\partial}{\partial t} w_3 = 8w_3 + \frac{g^2}{16\pi^2 w_1^4} \left\{ \tau_2 r_0^{4,3} \left( \frac{137}{105} w_1^6 + \frac{31}{315} w_1^2 \right) + \frac{174}{5} \tau_2 r_0^{4,2} w_1^3 w_2 
- 6r_1^4 \left( \frac{w_4}{w_1} - 17 \frac{w_2 w_3}{w_1^2} + 35 \frac{w_3^2}{w_1^3} \right) 
- 6(N^2 - 1) r_2^4 \left( \frac{w_4}{w_1} - 9 \frac{w_2 w_3}{w_1^2} + 12 \frac{w_3^2}{w_1^3} \right) \right\}$$  \quad (F.8)

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