I. INTRODUCTION

Quantum coherence, being one of the defining features of quantum mechanics, underlies the fundamental phenomena of quantum interference and plays a significant role in physics and quantum information processing (QIP), such as quantum cryptography [1–3], quantum metrology [4, 5], nanoscale thermodynamics [6–8], and energy transport in biological systems [9]. Based on the general framework of quantum resource theories [10–12], a systematic framework of coherence have been introduced [13, 14], based on which various coherence measures have been defined [14–26]. Meanwhile, the framework of coherence has been extended from a single party to the multipartite scenario with several applications, such as quantum state merging [27], coherence localization [28] and incoherence teleportation [29]. Studies of the inter-conversion between coherence and other multipartite nonclassical correlations, such as entanglement [15, 30, 31], discord [32] and nonlocality [33], also highlight the fundamental role of quantum coherence. With the rapid development of quantum hardware in realizing large-scale multipartite systems, the ability of efficiently quantifying the coherence would thus offer an operationally meaningful benchmarking tool and benefit our understanding of QIP tasks.

Several experiments have been reported regarding the efficient detection of robustness of coherence [34, 35]. However, the experimental detection of general coherence measure, such as the relative entropy of coherence [14], is still missing. Theoretical proposals to estimate general multipartite coherence without costly state tomography have also been proposed [36–38]. While the initial proposals either need copies of the prepared multipartite state [36] or complicated post-processing [37], the spectrum estimation method was recently proposed [38], which only requires local measurements and easy-to-compute post-processing. Nevertheless, the performance of the spectrum estimation method highly depends on the choice of the measurements, and how it works for a general multipartite state still needs further study. Moreover, existing works generally focus on the lower bound of coherence. For a given quantum state, the maximal entanglement or discord it can generates are upper bounded by its coherence [15, 32], which makes detecting the upper bound of coherence important as it can indicates whether the given quantum state can generate sufficient resource for a certain QIP task.

In this work, we theoretically address these issues by proposing two methods that can respectively detect the lower and upper bound of coherence for all multi-qubit stabilizer states. The lower bound detection is based on the spectrum estimation method [38] and the stabilizer theory [39, 40], which only requires few local observable measurements for stabilizer states. The upper bound detection is based on the monogamy of coherence with a single local measurement. Experimentally, we prepare five stabilizer states of up to four qubits and demonstrate how few number of measurements could enable us to infer multipartite coherence.

II. THEORY

A. Lower bound estimation

Under the computational basis $\{ |i\rangle : i \in \{0, 1\}^\otimes n \}$ of an $n$-qubit state, we consider the relative entropy of coherence [14]

$$C_{\text{RE}}(\rho) = S_{\text{VN}}(\rho_d) - S_{\text{VN}}(\rho),$$

with $S_{\text{VN}} = -\text{tr}[\rho \log_2 \rho]$ being the von Neumann entropy and $\rho_d = \sum_{i} (\langle i | \rho | i \rangle) | i \rangle \langle i |$ being the diagonal part of $\rho$. The relative entropy of coherence characterizes the asymptotic distillable coherence under different types of incoherent operations [41, 42], quantifies the genuine randomness that can be extracted from measuring the quantum state in the computational basis [43–45], captures the deviation from thermodynamic equilibrium [46], etc. We thus focus on the estimation, in particular, the lower and upper bounds, of the relative entropy of coherence for general multipartite states.

The lower bound $l^c(\rho)$ of the coherence $C_{\text{RE}}(\rho)$ can be obtained by spectrum estimation and the majorization theory [47] as

$$C_{\text{RE}}(\rho) \geq l^c(\rho) = S_{\text{VN}}(d) - S_{\text{VN}}(d \vee (\land_{p \in X} p)),$$
where \(d = (d_1, \ldots, d_{2^n})\) are the diagonal elements of \(\rho\), 
\(p = (p_1, \ldots, p_{2^n})\) is the estimated probability distribution of the measurement on a certain entangled basis \(\{\psi_k\}\) for \(n\), 
\(\forall\) is majorization joint, and \(\land_{p \in \mathbb{X}} p\) is the majorization meet of all probability distributions in \(\mathbb{X}\) [38]. Here the majorization joint and meet are defined based on majorization. Specifically, given two probability distributions \(a = (a_1, a_2, \ldots, a_n)\) and \(b = (b_1, b_2, \ldots, b_n)\) with \(a_1 \geq a_2 \geq \cdots \geq a_n\) and \(b_1 \geq b_2 \geq \cdots \geq b_n\), \(a\) is majorized by \(b\) (written as \(a \prec b\)) if it satisfies \(\sum_{i=1}^{n} a_i \leq \sum_{i=1}^{n} b_i\) for all \(k = 1, 2, \ldots, n\). A probability distribution \(c\) is called the majorization (meet) of \(a\) and \(b\) if it satisfies: (i) \(c \succ a, b\), \(c \prec a, b\), and (ii) \(c \prec \hat{c}(a, b)\) for any \(\hat{c}\) that satisfies \(a, b \prec \hat{c}(a, b)\) [47].

Here, we consider \(p\) is selected from the set \(X\), which satisfies \(X = \{p\}|Ap \geq \alpha, Bp = \beta\). \(A\) and \(B\) are matrices and \(\alpha\) and \(\beta\) are vectors. “\(\geq\)” represents component-wise comparison.

To calculate \(l^c(\rho)\) via Eq. 2, it is crucial to set constraints \(Ap \geq \alpha\) and \(Bp = \beta\) with experimentally collected data, then find the “largest” distribution \(p\) majorized by all probability distributions in \(X\), i.e., \(\land_{p \in \mathbb{X}} p\).

According to the hermiticity of density matrix \(\rho\), we can set \(A\) as \(2^n\)-dimensional identity matrix and \(\alpha = 0\) for \(Ap \geq \alpha\), by which \(p_k \geq 0\) is guaranteed. In the following section, we introduce the procedure to construct constraint \(Bp = \beta\) via stabilizer formalism.

### B. Constructing constraint via stabilizer formalism

An observable \(S_i\) stabilizes an \(n\)-qubit state \(|\psi\rangle\) if \(|\psi\rangle\) is an eigenstate with eigenvalue +1 of \(S_i\), i.e., \(S_i |\psi\rangle = |\psi\rangle\). The set \(\mathbb{S}\) of operators \(S_i\) is the stabilizer of \(|\psi\rangle\), and \(|\psi\rangle\) is a so-called a stabilizer state [40, 48]. For an \(n\)-qubit state, there are \(n\) stabilizing operators \(\{S_1, \ldots, S_n\}\) that can uniquely determine \(|\psi\rangle\). Here \(S_1, \ldots, S_n\) are the generators of the set \(\mathbb{S}\), and we denote \(\mathbb{S} = \langle S_1, \ldots, S_n \rangle\). Note that \(|\psi\rangle\) is not only stabilized by \(\{S_1, \ldots, S_n\}\), but also their products. Thus, there could be in total \(2^n\) stabilizer operators in \(\mathbb{S}\).

Given an \(n\)-qubit stabilizer state \(|\psi\rangle\) associated with stabilizer \(\mathbb{S}\), there exists an orthonormal basis \(\{|\psi_k\rangle\}_{k=1}^{2^n}\) including \(|\psi_1\rangle = |\psi\rangle\), where \(|\psi_k\rangle\) is uniquely specified by \(\mathbb{S}\) with different eigenvalues, i.e., \(S_i|\psi_k\rangle = a_{ik}|\psi_k\rangle\) with eigenvalues \(a_{ik} = \pm 1\). For example, the Bell state \(|\Phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}\) can be specified by \(\mathbb{S} = \langle X^{(1)} X^{(2)}, Z^{(1)} Z^{(2)} \rangle\). Hereafter, \(X^{(j)}, Y^{(j)}, Z^{(j)}\) denote the Pauli matrices and \(\sigma_x, \sigma_y, \sigma_z\) acting on the \(j\)-th qubit. In the following, qubit index \(j\) may be omitted if there is no confusion. The four Bell states \(|\Phi^\pm\rangle\) and \(|\Psi^\pm\rangle\) are specified by \(\mathbb{S} = \langle XX, ZZ \rangle\) as well, with eigenvalues of \((-1, +1), (+1, 1), (-1, -1)\) and \((-1, +1)\) respectively. The basis \(\{|\psi_k\rangle\}_{k=1}^{2^n}\) associated with the same stabilizer \(\mathbb{S}\) is also called graph-diagonal basis [49, 50] as stabilizer state is equivalent to a graph state under local Clifford operations [51].

In the graph-diagonal basis, the stabilizing operator can be written as \(S_i = \sum_k a_{ik} |\psi_k\rangle\langle\psi_k|\), and its expected value on a given quantum state \(\rho\) is \(\langle S_i \rangle = \text{tr}(S_i \rho) = \sum_k p_k a_{ik}\), where the parameters \(p_k = \langle \psi_k | \rho | \psi_k \rangle\) form a probability distribution \(p = (p_1, \ldots, p_{2^n})\) with \(p_k \geq 0\) and \(\sum_k p_k = 1\).

The expected values of stabilizer \(S\) on \(\rho\) leads to \(2^n\) equations, and can be represented in matrix form

\[
\begin{bmatrix}
    a_{11} & \cdots & a_{12^n} \\
    \vdots & \ddots & \vdots \\
    a_{2^n 1} & \cdots & a_{2^n 2^n}
\end{bmatrix}
\begin{bmatrix}
    p_1 \\
    \vdots \\
    p_{2^n}
\end{bmatrix}
= \begin{bmatrix}
    \langle S_1 \rangle \\
    \vdots \\
    \langle S_{2^n} \rangle
\end{bmatrix}
\]

from which we can construct the constraint \(Bp = \beta\). However, in practice, there does not always exist solutions of \(\land_{p \in \mathbb{X}} p\) in Eq. 2 with constraint Eq. 3 (as reflected by our experimental results). The experimentally generated state always has a distance to target state due to the inevitable imperfections, which might lead to no solution of \(\land_{p \in \mathbb{X}} p\) with inputs of \(\{\langle S_i \rangle\}\). On the other hand, experimentally obtained \(\langle S_i \rangle\) is always associated with statistical errors. We address these issues by introducing the experimental standard deviation \(\sigma_i\) of \(\langle S_i \rangle\) and relax the constraint Eq. 3 to an inequality form of

\[
\begin{align*}
\langle S_1 \rangle - w\sigma_1 & \leq B \cdot p \leq \langle S_1 \rangle + w\sigma_1 \\
\vdots \\
\langle S_{2^n} \rangle - w\sigma_{2^n} & \leq \langle S_{2^n} \rangle + w\sigma_{2^n}
\end{align*}
\]

where \(w\sigma_i\) with \(w \geq 0\) is the deviation to the mean value \(\langle S_i \rangle\) represented in \(\sigma_i\). To this end, an experimentally accessible constraint is formulated as \(\beta_- \leq Bp \leq \beta_+\). In practice, instead of measuring all the stabilizers, which is impractical for a large quantum system, we can select a small subset of stabilizers so that the number of measurement does not scale exponentially to the number of qubits. Note that \(\langle I^n \rangle = 1\) must be set in Eq. 4 to ensure \(\sum_k p_k = 1\). If we apply the scheme on the graph-diagonal states \(\rho = \sum_k \lambda_k |\psi_k\rangle\langle\psi_k|\) with \(\lambda = (\lambda_1, \ldots, \lambda_{2^n})\) the spectrum of \(\rho\), we have \(\rho = \lambda I\).

Thus, \(d \land \lambda = \lambda I\) implies \(l^c(\rho) = C_{\mathbb{E}}(\rho)\), which indicates that the estimated lower bound of coherence is tight for graph-diagonal states.

We emphasize that relaxing the constraint to \(\beta_- \leq Bp \leq \beta_+\) does not increasing the risk of overestimation of \(l^c(\rho)\). Suppose that \(X_1\) and \(X_2\) are two feasible sets of probability distributions, and satisfy \(X_1 \subseteq X_2\), \(X_1\) and \(X_2\) are restricted to \(d \land \lambda_{p \in X_1} p\) and \(d \land \lambda_{p \in X_2} p\), otherwise the result of Eq. 2 is 0. According to the definition of majorization meet, \(\land_{p \in \mathbb{X}} p\) is the “largest” distribution majorized by all probability distributions in \(X\). Therefore, \(\land_{p \in \mathbb{X}} p\) becomes “smaller” when we enlarge the range of \(X\), i.e., \(\land_{p \in X_1} p \land \lambda_{p \in X_2} p\), which implies \(\langle S(d) - S(\land_{p \in X_1} p) \rangle > S(d) - S(\land_{p \in X_2} p)\).

Thus, we conclude that \(l^c(\rho)\) decreases with enlarging the range of \(X\).

Similar constraints can also be formulated for multi-qubit states that do not obviously fit the stabilizer formalism. The stabilizing operators of such kind of \(n\)-qubit states \(|\psi\rangle\) could be determined by finding its unitary dynamics \(U^\psi\) acting on \(|0\rangle^{\otimes n}\), i.e., \(|\psi\rangle = U^\psi|0\rangle^{\otimes n}\) [48]. As \(|0\rangle^{\otimes n}\) is stabilized by
in strictly incoherent operation. We define a completely positive and trace-preserving (CPTP) \( \rho, \) \( \Lambda \) for all \( i \). Let \( \{ \lambda, |\varphi_\alpha \rangle \} \) be the spectral decomposition of \( \rho \), i.e.,

\[
\rho = \sum_{\alpha} \lambda_\alpha |\varphi_\alpha \rangle \langle \varphi_\alpha |
\]

where \( |\varphi_\alpha \rangle = \sum_{i,j} c_{\alpha i} c_{\alpha j}^* |i \rangle \langle j | \) with \( \sum_{\alpha} |\varphi_\alpha \rangle \langle \varphi_\alpha | = 1 \) for all \( \alpha \). Thus, \( \rho \) form a set \( \mathcal{M}(d) \) with \( d = \sum_\alpha \lambda_\alpha |c_{\alpha} |^2 \).

We define a completely positive and trace-preserving (CPTP) map \( \Lambda_1(\cdot) = \sum_\alpha K_\alpha \cdot K_\alpha^\dagger \) associated with Kraus operators

\[
K_\alpha = \sum_{i,j,k,l} \sqrt{\lambda_{i,j} \delta_{k,l}} |i \rangle \langle j | \langle k | \langle l |
\]

for all \( \alpha \).

According to the definition of strictly incoherent operation \( [52, 53], \) \( K_\alpha \) is a strictly incoherent operator and \( \Lambda_1(\cdot) \) is a strictly incoherent operation.

For the case that there are \( m \) (\( m < 2^n \)) non-zero elements in \( d \), we first transform \( |\psi_d \rangle \langle \psi_d | \) into a block diagonal matrix via a permutation matrix \( M \), i.e.,

\[
|\psi_d \rangle \langle \psi_d | = M^{-1} |\psi_m \rangle \langle \psi_m | \otimes \mathbb{I}_{2^n-m}.
\]

To upper bound the coherence of an \( n \)-qubit state \( \rho \), we measure it in the computational basis \( \{ |i \rangle \} \), which yields a distribution \( d = (d_1, \ldots, d_{2^n}) \). Based on the monogamy of the relative entropy of coherence \([14]\), the coherence of \( \rho \) is upper bounded by the coherence \( u_c(\rho) \) of state \( |\psi_d \rangle = \sum_{i=1}^{2^n} \sqrt{d_i} |i \rangle \), i.e.,

\[
C_{RE}(\rho) \leq u_c(\rho) = C_{RE}(|\psi_d \rangle \langle \psi_d |).
\]

Note that this lower bound is tight for pure states, and is capable for various coherence measures.

### III. EXPERIMENTAL DEMONSTRATION

Next, we demonstrate the capability of our scheme by estimating the coherence of several typical multi-qubit states.

We firstly generate photon pairs by a periodically poled potassium titanyl phosphate (PPKTP) crystal in a Sagnac interferometer \([56]\), which is bidirectionally pumped by an ultraviolet (UV) laser diode with central wavelength at 405 nm (as shown in Fig. 1a). The two photons are entangled in the polarization degree of freedom (DOF), i.e., \( |\Psi_{ab}^+ \rangle = (|H_1 V_0 \rangle + |V_1 H_0 \rangle) / \sqrt{2} \) with \( H \) the horizontal polarization and \( V \) the vertical polarization.

We extend photon to its path DOF by beam displacer (BD), which transmits vertical polarization and deviate horizontal polarization, i.e., \( |H \rangle \rightarrow |H \rangle h \) and \( |V \rangle \rightarrow |V \rangle v \) with \( h \) and \( v \) the path DOF \([57–59]\). The qubit is encoded polarization DOF as \( |H(V) \rangle \rightarrow |0(1) \rangle \), and path DOF as \( |h(v) \rangle \rightarrow |0(1) \rangle \). In our experiment, we denote the qubits encoded in polarization DOF as 1 and 3, while the qubits encoded in path DOF as 2 and 4. As shown in Fig. 1b-d, with different experimental setup configurations, we can generate various 4-qubit states, including \( |\text{GHZ}_4 \rangle = (|0000 \rangle + |1110 \rangle) / \sqrt{2}, \) \( |\text{C}4 \rangle = (|0000 \rangle + |0011 \rangle + |1100 \rangle - |1111 \rangle) / 2 \) and \( |\text{W}_4 \rangle = (|0001 \rangle + |0010 \rangle + |0100 \rangle + |1000 \rangle) / 2 \). Moreover, \( |\text{GHZ}_3 \rangle = (|000 \rangle + |111 \rangle) / \sqrt{2} \) and \( |\text{W}_3 \rangle = (|000 \rangle + |010 \rangle + |100 \rangle) / \sqrt{3} \) can be obtained by extending one photon to polarization and path DOF while keeping the other in polarization DOF (See Appendix B for more details).

For each experimentally generated state \( \rho_{\text{expt}} \) of stabilizers \( \{S_i \} \) associated with the corresponding statistical errors \( \sigma_i \). We refer to Appendix A for details of stabilizing operators of \( |\psi \rangle \) and the corresponding graph-diagonal basis. The measured expected values of stabilizing operators are presented in Appendix B. With the measured \( \{S_i \} \) and \( \sigma_i \), we construct the constraint Eq. 4. Thus, we can calculate \( l_{\omega,m}(\rho_{\text{expt}}) \) via solving Eq. 2, where \( \omega \) represents the sets of deviations and \( m \) is the number of \( \{S_i \} \) we construct the constraint. We set \( \omega \) as non-negative integers from 0 to 3, and \( m \) from 1 to \( 2^n - 1 \) as \( \{S_i \} = 1 \) must be set in the constraint. In our calculation, we treat the case of no solution as \( l_{\omega,m}(\rho_{\text{expt}}) = 0 \), and the case of \( l_{\omega,m}(\rho_{\text{expt}}) > 0 \) as valid solution. For a fixed \( m \), there are \( 2^{n-1} \) subsets of \( \{S_i \} \), and the maximum \( l_{\omega,m}(\rho_{\text{expt}}) \) is shown in Fig. 2. We observe that valid \( l_{\omega,m}(\rho_{\text{expt}}) \) can not be obtained with \( m \leq 3 \) stabilizers for \( \rho_{\text{expt}} \), \( m \leq 1 \) stabilizers for \( \rho_{\text{expt}} \), and \( m \leq 2 \) stabilizers for \( \rho_{\text{expt}} \).
stabilizers for other three states. With the increasing of \( m \), the maximum \( I^c_{\omega,m}(\rho_{\text{expt}}) \) increases accordingly. When \( m \) gets close to \( 2^n - 1 \), there exists situations that we cannot obtain \( I^c_{\omega,m}(\rho_{\text{expt}}) \) for smaller \( w \) (the values are 0 at the right lower corner of each figure in Fig. 2). As aforementioned, this is caused by the experimental imperfections, such as slight misalignment of optical elements during data collection, which introduces small variation of prepared \( \rho_{\text{expt}} \). The issue is improved by extending the range of \( \langle S^\psi_1 \rangle \), i.e., increasing \( \omega \). As shown in Fig. 2b-Fig. 2e, most \( I^c_{\omega,m}(\rho_{\text{expt}}) \) has valid solutions for large \( m \) by setting \( \omega = 3 \). Moreover, the accuracy of estimated \( I^c_{\omega,m}(\rho_{\text{expt}}) \) increases along with \( m \) as well. We investigate this by calculating the normalized distance between \( I^c_{\omega,m}(\rho_{\text{expt}}) \) and \( C_{\text{RE}}(\rho_{\text{expt}}) \), i.e., \( 1 - \frac{I^c_{\omega,m}(\rho_{\text{expt}})}{C_{\text{RE}}(\rho_{\text{expt}})} \). \( C_{\text{RE}}(\rho_{\text{expt}}) \) is calculated with Eq. 1 by reconstructing \( \rho_{\text{expt}} \) via quantum state tomographic technology (See Appendix B for the reconstructed \( \rho_{\text{expt}} \)). The distances between maximal \( I^c_{3,m}(\rho_{\text{expt}}) \) and \( C_{\text{RE}}(\rho_{\text{expt}}) \) are shown in Fig. 3a, from which we observe that the distance drops down quickly with increasing of \( m \) and tends to converge at \( m = 5 \).

As aforementioned, the choice of selecting \( m \) stabilizers from \( \{S^\psi_i\} \) is not unique except \( m = 2^n - 1 \). An important property is the successful probability of obtaining valid \( I^c_{\omega,m} \) by randomly selecting \( m \) stabilizers. We show the percentage of valid \( I^c_{\omega,5}(\rho_{\text{expt}}) \) (\( I^c_{\omega,5}(\rho_{\text{expt}}) > 0 \)) for \( m = 5 \) in Fig. 3b. By increasing \( \omega \), the probability of getting valid \( I^c_{\omega,5}(\rho_{\text{expt}}) \) is enhanced, especially for \( \rho_{\text{expt}} \). However, one cannot increasing \( \omega \) arbitrarily. A larger \( \omega \) represents a smaller probability we could obtain \( I^c_{\omega,5}(\rho_{\text{expt}}) \) in range \([S_i] + (w - 1)\sigma_i, [S_i] + w\sigma_i]\) as well as \( [S_i] - w\sigma_i, [S_i] - (w - 1)\sigma_i] \), which is less than 0.3% for \( \omega = 3 \). This is also reflected by normalized distance of maximal \( I^c_{\omega,5}(\rho_{\text{expt}}) \) shown in Fig. 3c, which indicates the inaccuracy of \( I^c_{\omega,5}(\rho_{\text{expt}}) \) increases when we extend the range of \( [S_i] \). Also, the results in Fig. 3c agree
with our claim that relaxing the constraint decreases the estimated value of $l^c(\rho)$. We conclude that set $\omega = 3$ is reasonable in experiment, under which we observe the probability of getting valid $l^c_{\omega,5}(\rho_{\text{expt}})$ is 100% for $\rho_{\text{expt}}^{\text{GHZ}_3}$, $\rho_{\text{expt}}^{\text{W}_3}$, $\rho_{\text{expt}}^{C_4}$, $\rho_{\text{expt}}^{\text{GHZ}_4}$ and $\rho_{\text{expt}}^{\text{C}_4}$, and that of 86% for $\rho_{\text{expt}}^{\text{GHZ}_4}$. Note that the estimated $l^c(\rho)$ of $\rho_{\text{expt}}^{\text{GHZ}_3}$, $\rho_{\text{expt}}^{\text{W}_3}$, $\rho_{\text{expt}}^{C_4}$ and $\rho_{\text{expt}}^{\text{GHZ}_4}$ is slightly more accurate than that of $\rho_{\text{expt}}^{\text{GHZ}_4}$ as shown in Fig. 3a and Fig. 3c. The main reason is that the fidelity of prepared $\rho_{\text{expt}}^{\text{W}_3}$ is slightly lower than that of other four states as shown in Appendix B. The lower fidelity implies a larger distance between prepared state and target state, which indicates $\lambda \succ d \lor (\wedge_{p \in X} p)$ so that $C_{\text{RE}}(\rho) = S(d) - S(\lambda) > l^c(\rho) = S(d) - S(d \lor (\wedge_{p \in X} p))$. However, it does not indicate the lower fidelity always leads to the bigger gap between $C_{\text{RE}}(\rho)$ and $l^c(\rho)$. If the prepared state is still a graph-diagonal state after considering the experimental imperfections, the estimated lower bound on such state is tight as well, i.e., $l^c(\rho) = C_{\text{RE}}(\rho)$. In Appendix C, we analyze our experimental imperfections and show how it affect the prepared states.

Finally, we estimate the upper bound $u^c$ of $\rho_{\text{expt}}^\psi$ by measuring the probability distribution $d_{\text{expt}}^{\psi}$ on basis of $Z^{\otimes n}$. The probability distribution $d_{\text{expt}}^{\psi}$ are shown in Appendix B, by which we can calculate $u^c(\rho_{\text{expt}}^\psi)$ according to Eq. 8. The results of $u^c(\rho_{\text{expt}}^\psi)$ are shown with blue dash line in Fig. 3d. For all the five states, we observe that $C_{\text{RE}}(\rho_{\text{expt}}^\psi)$ lies within the range bounded by $l^c_{3,5}(\rho_{\text{expt}}^\psi)$ and $u^c(\rho_{\text{expt}}^\psi)$.

### IV. CONCLUSION

To conclude, we introduce an efficient and experimentally friendly estimation method for detecting coherence of multipartite states. We demonstrate that the coherence with high accuracy as well as high successful probability can be efficiently estimated with a few measurements for various multi-qubit states. The procedure to obtain the lower bound is based on few measurements of the stabilizing operators, similar to multipartite entanglement detection [60] and multipartite Bell inequalities [61]. It thus indicates that coherence and other resources can be inferred by the same set of measurements, which can further benefit our understanding of the connection between coherence and other resources [15, 30, 31, 33].

There are several open follow-up problems. The scheme to detect the lower bound of multipartite coherence is efficient and tight for graph-diagonal states as well as special types of quantum states that admit efficient classical representation (such as the W state). Whether our scheme will work for more general multi-qubit quantum state is an interesting future work. Besides, one may concern whether our scheme can be generalized to high-dimensional cases. For the multi-qubit stabilizer state, its stabilizing operators constructed by generalized Pauli group are generally non-Hermitian [62–64], which cannot be observed directly in experiment. Recent studies indicate that expected values of non-Hermitian operators could be measured via weak measurements [65, 66]. However, the definite answer may require rather sophisticated analysis. Finally, it is worth noting that a fidelity-based
method was recently proposed to detect the lower bound of multipartite coherence via the convex roof construction [67]. Thus, another open follow-up question is whether our scheme can be further generalized to other coherence measures while maintaining their appealing feature of efficiency and simplicity.

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Appendix A: Stabilizing operators and graph-diagonal basis

1. GHZ state

For $n$-qubit GHZ states, the generators of $S^{GHZ}$ are

$$S_{i}^{GHZ} = Z^{(i-1)}Z^{(i)} \text{ for } i = 2, 3, ..., n,$$

and denoted as $S^{GHZ} = (S_{1}^{GHZ}, S_{2}^{GHZ}, ..., S_{n}^{GHZ})$. For $|GHZ_{3}\rangle$, the generators are $S_{1}^{GHZ} = XXZ$, $S_{2}^{GHZ} = ZZX$. Then, all stabilizing operators can be obtained by multiplying them with each other, i.e.,

$$S_{4}^{GHZ} = S_{1}^{GHZ}S_{2}^{GHZ} = -YYX,$$

$$S_{5}^{GHZ} = S_{1}^{GHZ}S_{3}^{GHZ} = -XYY,$$

$$S_{6}^{GHZ} = S_{2}^{GHZ}S_{3}^{GHZ} = ZIZ,$$

$$S_{7}^{GHZ} = S_{1}^{GHZ}S_{2}^{GHZ}S_{3}^{GHZ} = -YXY,$$

$$S_{8}^{GHZ} = III.$$

The GHZ$_{3}$-diagonal basis can be obtained from the computational basis $|k_{1}k_{2}k_{3}\rangle$ by acting a 3-qubit unitary operation $U^{GHZ} = (\otimes_{i=1}^{3}H_{i}) \cdot (CZ_{12}) \cdot (CZ_{13}) \cdot (CZ_{14}) \cdot (H_{1} \otimes H_{2} \otimes H_{3} \otimes H_{4})$, where

$$CZ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (A5)$$

is the unitary matrix of controlled-Z (CZ) operation. Sixteen GHZ$_{4}$-diagonal bases are determined by $|\psi_{i}^{GHZ}_{4}\rangle = U^{GHZ}|k_{1}k_{2}k_{3}k_{4}\rangle$ and shown in Table I.

| $k_{1}k_{2}k_{3}$ | $|\psi_{1}^{GHZ}_{4}\rangle$ | $(|S_{1}^{GHZ}_{4}\rangle, |S_{2}^{GHZ}_{4}\rangle, |S_{3}^{GHZ}_{4}\rangle)$ |
|------------------|-----------------|-----------------|
| 000              | $(|000\rangle + |111\rangle)/\sqrt{2}$ | $(+,+,+,+)$ |
| 001              | $(|000\rangle - |111\rangle)/\sqrt{2}$ | $(+,+-,+,+)$ |
| 010              | $(|001\rangle + |110\rangle)/\sqrt{2}$ | $(+,+,+,+)$ |
| 011              | $(|001\rangle - |110\rangle)/\sqrt{2}$ | $(+,+,+,+)$ |
| 100              | $(|010\rangle + |101\rangle)/\sqrt{2}$ | $(+,+,+,+)$ |
| 101              | $(|010\rangle + |101\rangle)/\sqrt{2}$ | $(+,+,+,+)$ |
| 110              | $(|100\rangle - |011\rangle)/\sqrt{2}$ | $(+,-,+,+)$ |
| 111              | $(|101\rangle - |010\rangle)/\sqrt{2}$ | $(+,-,+,+)$ |

2. Cluster state

The generators of a $n$-qubit cluster state $|\tilde{C}_{n}\rangle$ are

$$S_{1}^{\tilde{C}_{n}} = X^{(1)}Z^{(2)},$$

$$S_{i}^{\tilde{C}_{n}} = Z^{(i-1)}X^{(i)}Z^{(i-1)} \text{ for } i = 2, 3, ..., n-1, \quad (A6)$$

The generators of 4-qubit linear graph are $S_{1}^{\tilde{C}_{4}} = XXZZ$, $S_{2}^{\tilde{C}_{4}} = ZXXZ$, $S_{3}^{\tilde{C}_{4}} = ZZXX$, and the corresponding state is $|C_{4}\rangle = (N^{0000} + N^{0011} + N^{1000} - N^{1100})/2$. Note that $|C_{4}\rangle$ can be transformed to the common representation $|C_{4}\rangle = (|0000\rangle + |0011\rangle + |1100\rangle - |1111\rangle)/2$ by local unitary $H_{1}H_{2}H_{3}$, i.e., $|C_{4}\rangle = H_{1}H_{2}H_{3}|C_{4}\rangle$. Accordingly, the generators of $|C_{4}\rangle$ are transformed to $ZZZZ, XXYZ$. Other stabilizing
The Cluster-diagonal basis can be obtained from the computational basis $|\psi^{GHZ}_4\rangle$ by acting a 4-qubit unitary operation $U^{GHZ} = (I \otimes CZ_{3} \otimes I_{4}) \cdot (CNOT_{12} \otimes CNOT_{34}) \cdot (H_{1} \otimes I_{2} \otimes H_{3} \otimes I_{4})$. Then, the Cluster-diagonal bases are determined by $|\psi^{C}_{4}\rangle = U^{GHZ}|k_{1}k_{2}k_{3}k_{4}\rangle$ and shown in Table III.

### 3. W state

$|W_{3}\rangle$ can be transformed from $|000\rangle$ by unitary $U^{W_{3}} = (XZI + IXX + ZZX)/\sqrt{3}$, i.e., $|W_{3}\rangle = U^{W_{3}}|000\rangle$[60]. Thus the generators of $|W_{3}\rangle$ are derived by

$$
S_{1}^{W_{3}} = U^{W_{3}}Z \Pi U^{W_{3}}^\dagger = \frac{1}{3} (ZIY + 2YYZ + 2XZX),
S_{2}^{W_{3}} = U^{W_{3}}Z \Pi U^{W_{3}}^\dagger = \frac{1}{3} (IZY + 2YZY + 2XXZ),
S_{3}^{W_{3}} = U^{W_{3}}Z \Pi U^{W_{3}}^\dagger = \frac{1}{3} (IZ + 2YZY + 2ZXX).
$$

The $W_{3}$-diagonal bases are shown in Table IV.

| $k_{1}k_{2}k_{3}k_{4}$ | $|\psi^{C}_{4}\rangle$ | ($|S_{1}^{W_{3}}\rangle, |S_{2}^{W_{3}}\rangle, |S_{3}^{W_{3}}\rangle$) |
|----------------------|----------------------|----------------------------------|
| 0000                | (0000) + (1111) / 2  | (+1, +1, +1, +1) |
| 0001                | (0001) + (1110) / 2  | (+1, +1, +1, -1) |
| 0010                | (0010) + (1101) / 2  | (+1, +1, -1, +1) |
| 0011                | (0011) + (1100) / 2  | (+1, +1, -1, -1) |
| 0100                | (0100) + (1011) / 2  | (+1, -1, +1, +1) |
| 0101                | (0101) + (1010) / 2  | (+1, -1, +1, -1) |
| 0110                | (0110) + (1001) / 2  | (+1, -1, -1, +1) |
| 0111                | (0111) + (1000) / 2  | (+1, -1, -1, -1) |

Table II. The GHZ$_4$-diagonal bases and the corresponding expected values of generators ($|S_{1}^{GHZ_{4}}\rangle, |S_{2}^{GHZ_{4}}\rangle, |S_{3}^{GHZ_{4}}\rangle, |S_{4}^{GHZ_{4}}\rangle$).

| $k_{1}k_{2}k_{3}k_{4}$ | $|\psi^{C}_{4}\rangle$ | ($|S_{1}^{W_{3}}\rangle, |S_{2}^{W_{3}}\rangle, |S_{3}^{W_{3}}\rangle$) |
|----------------------|----------------------|----------------------------------|
| 0000                | (0000) + (1111) / 2  | (+1, +1, +1, +1) |
| 0001                | (0001) + (1110) / 2  | (+1, +1, +1, -1) |
| 0010                | (0010) + (1101) / 2  | (+1, +1, -1, +1) |
| 0011                | (0011) + (1100) / 2  | (+1, +1, -1, -1) |
| 0100                | (0100) + (1011) / 2  | (+1, -1, +1, +1) |
| 0101                | (0101) + (1010) / 2  | (+1, -1, +1, -1) |
| 0110                | (0110) + (1001) / 2  | (+1, -1, -1, +1) |
| 0111                | (0111) + (1000) / 2  | (+1, -1, -1, -1) |

Table III. The Cluster-diagonal bases and the corresponding expected values of generators ($|S_{1}^{C}\rangle, |S_{2}^{C}\rangle, |S_{3}^{C}\rangle, |S_{4}^{C}\rangle$).

Other stabilizing operators are

$$
S_{5}^{C_{4}} = -YYZI, S_{5}^{C_{4}} = ZIIX, \\
S_{6}^{C_{4}} = ZZZZ, S_{6}^{C_{4}} = XXYX, \\
S_{7}^{C_{4}} = XXIZ, S_{7}^{C_{4}} = -IZYY, \\
S_{8}^{C_{4}} = YXXY, S_{8}^{C_{4}} = -ZYIY, \\
S_{9}^{C_{4}} = -IYYI, S_{9}^{C_{4}} = XYXY, \\
S_{10}^{C_{4}} = -YYXY, S_{10}^{C_{4}} = IIYY, \\
S_{11}^{C_{4}} = YXYX, S_{11}^{C_{4}} = -YYIZ, \\
S_{12}^{C_{4}} = YIYY, S_{12}^{C_{4}} = ZXYX, \\
S_{13}^{C_{4}} = -ZYYI, S_{13}^{C_{4}} = XXZX, \\
S_{14}^{C_{4}} = -IZYY, S_{14}^{C_{4}} = XYYX, \\
S_{15}^{C_{4}} = YXXY, S_{15}^{C_{4}} = -ZZZX, S_{16}^{C_{4}} = III.
$$

The W$_3$-diagonal bases are shown in Table IV.

| $k_{1}k_{2}k_{3}$ | $|\psi^{W_{3}}\rangle$ | ($|S_{1}^{W_{3}}\rangle, |S_{2}^{W_{3}}\rangle, |S_{3}^{W_{3}}\rangle$) |
|------------------|------------------|----------------------------------|
| 000              | (0001) + (0100) | (+1, +1, +1) |
| 001              | (0000) - (0101) | (+1, +1, -1) |
| 010              | (0001) + (0100) | (+1, -1, +1) |
| 011              | (0000) - (0101) | (+1, -1, -1) |
| 100              | (0100) + (1001) | (-1, +1, +1) |
| 101              | (0101) + (1000) | (-1, +1, -1) |
| 110              | (0101) + (1000) | (-1, -1, +1) |
| 111              | (0100) + (1001) | (-1, -1, -1) |

Table IV. The W$_3$-diagonal bases and the corresponding expected values of generators ($|S_{1}^{W_{3}}\rangle, |S_{2}^{W_{3}}\rangle, |S_{3}^{W_{3}}\rangle$).

Similarly, We can find a possible unitary operator $U^{W_{3}} = (ZZZX + ZZXZ + ZXXZ + XIII)/2$ to generate $|W_{3}\rangle$ from
Thus, the generators of $|W_4\rangle$ can be obtained, i.e.,

$$S^W_1 = \frac{1}{2} (YZZY + IYZY + IIY + IIY + ZIY + IIY + IIIZ + IIIZ),$$

$$S^W_2 = \frac{1}{2} (ZYYZ + IIYI + IIY + IIIZ + IIIZ),$$

$$S^W_3 = \frac{1}{2} (ZXYZ + ZXYI + XII + XII),$$

$$S^W_4 = \frac{1}{2} (ZXYZ + ZXYI + XII + XII),$$

$$S^W_5 = \frac{1}{2} (ZYYZ + ZXYI + XII + XII),$$

$$S^W_6 = \frac{1}{2} (ZYYZ + ZXYI + XII + XII),$$

$$S^W_7 = \frac{1}{2} (ZYYZ + ZXYI + XII + XII),$$

$$S^W_8 = \frac{1}{2} (ZYYZ + ZXYI + XII + XII),$$

$$S^W_9 = \frac{1}{2} (ZYYZ + ZXYI + XII + XII),$$

$$S^{W, 10} = \frac{1}{2} (ZYYZ + ZXYI + XII + XII),$$

$$S^{W, 11} = \frac{1}{2} (ZYYZ + ZXYI + XII + XII),$$

$$S^{W, 12} = \frac{1}{2} (ZYYZ + ZXYI + XII + XII),$$

$$S^{W, 13} = \frac{1}{2} (ZYYZ + ZXYI + XII + XII),$$

$$S^{W, 14} = \frac{1}{2} (ZYYZ + ZXYI + XII + XII),$$

$$S^{W, 15} = \frac{1}{2} (ZYYZ + ZXYI + XII + XII),$$

The $W_4$-diagonal bases are shown in Table V.

### Appendix B: Details of experimental realizations and results

In this Appendix, we provide further details about our experimental setup. It would be useful to bear in mind the following:

(i) A half-wave plate (HWP) @ $\theta$ performs the unitary transformation $U_{\text{HWP}} = \cos 2\theta (|H\rangle\langle H| - |V\rangle\langle V|) + \sin 2\theta (|H\rangle\langle V| - |V\rangle\langle H|)$ on a polarization state, where $\theta$ is the angle between fast axis of HWP and vertical polarization.

(ii) A beam displacer (BD) transmits a vertically polarized photon but deviates a horizontally polarized one.

(iii) A polarized beam splitter (PBS) transmits a horizontally polarized photon but reflects a vertically polarized one.

(iv) A quarter-wave plate (QWP) @ $\theta$ performs the unitary transformation $U_{\text{QWP}} = \frac{1}{\sqrt{2}} (I_z + i \sin 2\theta (|H\rangle\langle V| - |V\rangle\langle H|), \text{ where } I_z = |H\rangle\langle H| + |V\rangle\langle V| \text{ and } \theta \text{ is the angle between fast axis of QWP and vertical polarization, on a polarization state.}$

| $k_1 k_2 k_3 k_4$ | $|\psi_k^{W_4}\rangle$ | $(\langle \psi_k^{W_4}\rangle, \langle \psi_k^{W_4}\rangle, \langle \psi_k^{W_4}\rangle, \langle \psi_k^{W_4}\rangle)$ |
|-------------------|-----------------|-----------------------------------------------|
| 0000 | 0 | (+1, +1, +1, +1) |
| 0001 | 0 | (1, -1, 1, +1) |
| 0010 | 0 | (-1, +1, +1, +1) |
| 0011 | 0 | (-1, -1, 1, +1) |
| 0100 | 0 | (+1, +1, 1, +1) |
| 0101 | 0 | (1, -1, -1, +1) |
| 0110 | 0 | (-1, +1, +1, +1) |
| 0111 | 0 | (-1, -1, 1, +1) |
| 1000 | 0 | (+1, +1, 1, +1) |
| 1001 | 0 | (-1, +1, +1, +1) |
| 1010 | 0 | (+1, +1, 1, +1) |
| 1011 | 0 | (-1, -1, -1, +1) |
| 1100 | 0 | (+1, +1, 1, +1) |
| 1101 | 0 | (-1, -1, -1, +1) |
| 1110 | 0 | (+1, -1, -1, +1) |
| 1111 | 0 | (-1, -1, -1, +1) |

### 1. Polarization-entangled photon source

As shown in Fig. 4, the power of the pump light can be adjusted by a HWP and a PBS. After the PBS, horizontal polarization $|H\rangle$ is rotated to $|+\rangle = \frac{1}{\sqrt{2}} (|H\rangle + |V\rangle)$ by a HWP set at 22.5°. Pump beam is focused into PPKTP crystal with beam waist of 74μm by two lenses L1, whose focal length is 75mm and 125mm respectively. PPKTP crystal, with dimensions of 10 mm (length) x 2 mm (width) x 1 mm (thickness) and polishing length of $\Lambda = 10.025\mu m$, is held in a home-built copper oven and the temperature is controlled by a homemade temperature controller, which is set at 29°C to realize the optimum type-II phase matching at 810 nm. Then,
the pump beam is split by a dual-wavelength PBS, and coherently pumps PPKTP in the clockwise and counterclockwise direction respectively. The clockwise and counterclockwise photons are recombined at the dual-wavelength PBS to generate polarization-entangled photons with ideal form of $|\Psi_{13}^+\rangle = \frac{1}{\sqrt{2}}(|H_1V_3\rangle + |V_1H_3\rangle)$. Two photons are filtered by narrow band filter (NBF) with full width at half maximum (FWHM) of 3nm, and coupled into single-mode fiber by lenses of focal length 200 mm ($L_2$ and $L_3$) and objective lenses (not shown in Fig. 4).

2. Experimental setups to generate multi-qubit state

FIG. 5. Illustration of the experimental setups to generate multiqubit state (a) the 4-qubit GHZ state $|\Psi_{GHZ}^+\rangle$, (b) the 4-qubit Cluster state $|C_4\rangle$, (c) the 4-qubit W state $|W_4\rangle$, (d) the 3-qubit GHZ state $|\Psi_{GHZ}^+\rangle$, and (e) the 3-qubit W state $|W_3\rangle$.

We extend photon to its path degree of freedom (DOF) by BD with length of 28.3 mm and clear aperture of 10 mm×10 mm. The BD transmits vertical polarization and deviates horizontal polarization by 3 mm. Specifically, The experimental setups to generate the five multiqubit states $|\Psi_{GHZ}^+\rangle$, $|C_4\rangle$, $|W_4\rangle$, $|\Psi_{GHZ}^+\rangle$, and $|W_3\rangle$ are shown in Fig. 5(a)-(e), respectively. The step-by-step calculations are shown in Eq. (B1)-Eq. (B3).

$$|\Psi_{ab}^+\rangle = \frac{1}{\sqrt{2}}(|H_1V_3\rangle + |V_1H_3\rangle)$$

$\xrightarrow{\text{HWP@45° \ on path of photon } a} \frac{1}{\sqrt{2}}(|V_1V_3\rangle + |H_1H_3\rangle)$

$\xrightarrow{\text{BD1}} \frac{1}{\sqrt{2}}(|V_1|V_2\rangle|V_3\rangle + |H_1|H_2\rangle|H_3\rangle) = |\Psi_{GHZ}^+\rangle$

$\xrightarrow{\text{BD2}} \frac{1}{\sqrt{2}}(|V_1|V_2\rangle|V_3\rangle|V_4\rangle + |H_1|H_2\rangle|H_3\rangle|H_4\rangle) = |\Psi_{GHZ}^+\rangle$

$$|\Psi_{ab}^+\rangle = \frac{1}{\sqrt{2}}(|H_1V_3\rangle + |V_1H_3\rangle)$$

$\xrightarrow{\text{HWP@45° \ on path of photon } a} \frac{1}{\sqrt{2}}(|V_1V_3\rangle + |H_1H_3\rangle)$

$\xrightarrow{\text{HWP@22.5° \ on path of photon } b} \frac{1}{2}(|H_1H_3\rangle - |V_1V_3\rangle + |H_1V_3\rangle + |V_1H_3\rangle)$

$\xrightarrow{\text{BD1, BD2}} \frac{1}{2}(|V_1|V_2\rangle|H_3\rangle|H_4\rangle - |V_1|V_2\rangle|V_3\rangle|V_4\rangle + |H_1|H_2\rangle|H_3\rangle|H_4\rangle + |H_1|H_2\rangle|V_3\rangle|V_4\rangle) = |C_4\rangle$
\[ |\psi_{ab}^{+} \rangle = \frac{1}{\sqrt{2}} (|H_1 V_3 \rangle + |V_1 H_3 \rangle) \]
\[ \begin{align*}
\text{BD}_1 & \quad \frac{1}{\sqrt{2}} \left( (|H_1 \rangle |h_2 \rangle |v_3 \rangle + |V_1 \rangle |v_2 \rangle |H_3 \rangle) \\
\text{HWP@45°} & \quad \frac{1}{\sqrt{2}} \left( (|H_1 \rangle |h_2 \rangle |V_3 \rangle + |V_1 \rangle |v_2 \rangle |H_3 \rangle) \\
\text{BD}_2 & \quad \frac{1}{\sqrt{3}} \left( (|H_1 \rangle |h_2 \rangle |V_3 \rangle |v_4 \rangle + |H_1 \rangle |v_2 \rangle |H_3 \rangle |H_4 \rangle + |H_1 \rangle |v_2 \rangle |H_3 \rangle |h_4 \rangle) \\
\text{HWP@45°} & \quad \frac{1}{\sqrt{3}} \left( (|H_1 \rangle |h_2 \rangle |V_3 \rangle |v_4 \rangle + |V_1 \rangle |h_2 \rangle |H_3 \rangle |V_4 \rangle + |V_1 \rangle |h_2 \rangle |V_3 \rangle |h_4 \rangle + |H_1 \rangle |v_2 \rangle |V_3 \rangle |h_4 \rangle) \\
\text{BD}_3 & \quad \frac{1}{\sqrt{6}} \left( (|H_1 \rangle |h_2 \rangle |V_3 \rangle |v_4 \rangle + |V_1 \rangle |h_2 \rangle |H_3 \rangle |V_4 \rangle + |V_1 \rangle |h_2 \rangle |V_3 \rangle |h_4 \rangle + |H_1 \rangle |v_2 \rangle |V_3 \rangle |h_4 \rangle) \\
\text{HWP@45°} & \quad \frac{1}{\sqrt{6}} \left( (|H_1 \rangle |h_2 \rangle |V_3 \rangle |v_4 \rangle + |V_1 \rangle |h_2 \rangle |H_3 \rangle |V_4 \rangle + |V_1 \rangle |h_2 \rangle |V_3 \rangle |h_4 \rangle + |H_1 \rangle |v_2 \rangle |H_3 \rangle |h_4 \rangle) = |W_4 \rangle
\end{align*} \]

3. Measurement and quantum state tomography

If the photon is encoded either in polarization DOF or path DOF, we first perform measurement on polarization DOF and then the path DOF. As illustrated in Fig. 6, the measurement basis of qubit on polarization DOF \( \alpha |H \rangle + \beta |V \rangle \) is determined by HWP at \( \theta_1 \) and QWP at \( \theta_2 \). The measurement basis of qubit on path DOF \( \gamma |h \rangle + \delta |v \rangle \) is determined by HWP at \( \theta_3 \) and QWP at \( \theta_4 \). Finally, a PBS is applied before the photon arrives detector. With this setting, the measurement on basis \( \langle \alpha |H \rangle + \beta |V \rangle \otimes \langle \gamma |h \rangle + \delta |v \rangle \) is achieved. The specific calculations are shown in Eq. (B4). If the photon is only encoded in polarization DOF, the measurement on basis \( \alpha |H \rangle + \beta |V \rangle \) is implemented by a QWP, HWP and PBS.

\[ \begin{align*}
\text{HWP@45°} & \quad (|H \rangle \otimes \langle \gamma |h \rangle + \delta |v \rangle) \\
\text{QWP@45°} & \quad (|V \rangle \otimes \langle \gamma |h \rangle + \delta |v \rangle) \\
\text{BD}_1 & \quad \gamma |V \rangle |h \rangle + \delta |H \rangle |v \rangle \\
\text{HWP@45°} & \quad \gamma |V \rangle |h \rangle + \delta |H \rangle |h \rangle \\
\text{BD}_2 & \quad \gamma |H \rangle + \delta |V \rangle |h \rangle \\
\text{QWP@45°} & \quad |H \rangle |h \rangle \\
\text{PBS} & \quad |H \rangle |h \rangle
\end{align*} \]

With this experimental setting, we can perform measurement on arbitrary basis. The experimental results of \( \langle S_1^x \rangle \) and \( \alpha_{\text{expt}} \) are shown in Fig. 7a and b, respectively. Moreover, we reconstruct experimentally generated states \( \rho_{\text{expt}}^{\text{GHZ}} \), \( \rho_{\text{expt}}^{\text{C}} \), \( \rho_{\text{expt}}^{\text{W}} \), \( \rho_{\text{expt}}^{\text{F}} \), \( \rho_{\text{expt}}^{\text{W}} \) and \( \rho_{\text{expt}}^{\text{F}} \) by quantum state tomography[48]. The results are shown in Fig. 8, from which we calculate the fidelity.

![FIG. 6. Experimental setups of measurement on a photon encoded in polarization DOF and path DOF.](image)

\[ F^0 = \text{tr}(\rho_{\text{expt}}^{\text{GHZ}} |\psi\rangle \langle \psi|). \] We observe that \( F_{\text{GHZ}}^{\text{expt}} = 0.9643 \pm 0.0003, F_{\text{W}}^{\text{expt}} = 0.9589 \pm 0.0005, F_{\text{GHZ}}^{\text{expt}} = 0.9571 \pm 0.0003, F_{\text{C}}^{\text{expt}} = 0.9497 \pm 0.0002 \] and \( F_{\text{W}}^{\text{expt}} = 0.915 \pm 0.001 \). Also, the relative entropy of coherence \( C_{\text{RE}}(\rho_{\text{expt}}^{\text{GHZ}}) \) can be calculated by \( C_{\text{RE}}(\rho) = S_{\text{VN}}(\rho) - S_{\text{VN}}(\rho) \), according to which we obtain \( C_{\text{RE}}(\rho_{\text{expt}}^{\text{GHZ}}) = 0.875 \pm 0.002, C_{\text{RE}}(\rho_{\text{expt}}^{\text{C}}) = 1.479 \pm 0.004, C_{\text{RE}}(\rho_{\text{expt}}^{\text{W}}) = 0.906 \pm 0.002, C_{\text{RE}}(\rho_{\text{expt}}^{\text{F}}) = 1.806 \pm 0.002 \) and \( C_{\text{RE}}(\rho_{\text{expt}}^{\text{W}}) = 1.964 \pm 0.003 \).

**Appendix C: Experimental imperfections**

In our experiment, the main imperfection of prepared multi-qubit states comes from the imperfection of the polarization entangled-photon pair, which is caused by the mode mismatch of overlapping lights on PBS in Fig. 4. The noisy state can be
With such a noisy state and following the procedure of generating multi-qubit state in Appendix B 2, we calculate the noisy $|\psi\rangle$ from $\rho$ to $\rho^{\Phi^+} = (|HH\rangle + |VV\rangle)/\sqrt{2}$ by
\[
\rho^{\Phi^+} = \mathcal{E}(|\Phi^+\rangle\langle\Phi^+|) = (1-\mu)|\Phi^+\rangle\langle\Phi^+| + \frac{\mu}{2}(|HH\rangle\langle HH| + |VV\rangle\langleVV|).
\]

(C1)

With such a noisy state and following the procedure of generating multi-qubit state in Appendix B 2, we calculate the noisy multi-qubit states in computational basis (CB)
\[
\rho^{\text{GHZ}}_{\text{CB}} = (1-\mu)|\text{GHZ}_3\rangle\langle\text{GHZ}_3| + \frac{\mu}{2}(|000\rangle\langle000| + |111\rangle\langle111|),
\]
\[
\rho^{\text{GHZ}}_{\text{CB}} = (1-\mu)|\text{GHZ}_4\rangle\langle\text{GHZ}_4| + \frac{\mu}{2}(|0000\rangle\langle0000| + |1111\rangle\langle1111|),
\]
\[
\rho^{\text{C}_4}_{\text{CB}} = (1-\mu)|\text{C}_4\rangle\langle\text{C}_4| + \frac{\mu}{4}(|0000\rangle\langle0000| + |0101\rangle\langle0101| + |0110\rangle\langle0110| + |1101\rangle\langle1101|),
\]
\[
\rho^{\text{W}_4}_{\text{CB}} = (1-\mu)|\text{W}_4\rangle\langle\text{W}_4| + \frac{\mu}{4}(|0000\rangle\langle0000| + |0101\rangle\langle0101| + |0110\rangle\langle0110| + |1101\rangle\langle1101|).
\]

(C2) \hspace{1cm} (C3) \hspace{1cm} (C4) \hspace{1cm} (C5) \hspace{1cm} (C6)

In the graph-diagonal basis (GDB)
\[
\rho^{\text{GHZ}}_{\text{GDB}} = U^{\text{GHZ}_3}\rho^{\text{GHZ}_3}_{\text{CB}}U^{\text{GHZ}_3\dagger} = \sum_{k=1}^{8} \lambda_k |\psi_k^{\text{GHZ}_3}\rangle\langle\psi_k^{\text{GHZ}_3}| = (1-\mu)|\psi_1^{\text{GHZ}_3}\rangle\langle\psi_1^{\text{GHZ}_3}| + \frac{\mu}{2}|\psi_4^{\text{GHZ}_3}\rangle\langle\psi_4^{\text{GHZ}_3}|,
\]
\[
\rho^{\text{GHZ}_4}_{\text{GDB}} = U^{\text{GHZ}_4}\rho^{\text{GHZ}_4}_{\text{CB}}U^{\text{GHZ}_4\dagger} = \sum_{k=1}^{16} \lambda_k |\psi_k^{\text{GHZ}_4}\rangle\langle\psi_k^{\text{GHZ}_4}| = (1-\mu)|\psi_1^{\text{GHZ}_4}\rangle\langle\psi_1^{\text{GHZ}_4}| + \frac{\mu}{2}|\psi_4^{\text{GHZ}_4}\rangle\langle\psi_4^{\text{GHZ}_4}|,
\]
\[
\rho^{\text{C}_4}_{\text{GDB}} = U^{\text{C}_4}\rho^{\text{C}_4}_{\text{CB}}U^{\text{C}_4\dagger} = \sum_{k=1}^{16} \lambda_k |\psi_k^{\text{C}_4}\rangle\langle\psi_k^{\text{C}_4}| = (1-\mu)|\psi_1^{\text{C}_4}\rangle\langle\psi_1^{\text{C}_4}| + \frac{\mu}{4}|\psi_4^{\text{C}_4}\rangle\langle\psi_4^{\text{C}_4}| + |\psi_{13}^{\text{C}_4}\rangle\langle\psi_{13}^{\text{C}_4}| + \frac{\mu}{16}|\psi_{16}^{\text{C}_4}\rangle\langle\psi_{16}^{\text{C}_4}|.
\]

(C7) \hspace{1cm} (C8) \hspace{1cm} (C9)

are still graph-diagonal states under dephasing channel, which means that the estimated lower bounds on such states are tight, and it is easy to check that $\rho^{W_3}_{\text{GDB}} = U^{W_3}\rho^{W_3}_{\text{CB}}U^{W_3\dagger} \neq \sum_k \lambda_k |\psi_k^{W_3}\rangle\langle\psi_k^{W_3}|$ and $\rho^{W_4}_{\text{GDB}} = U^{W_4}\rho^{W_4}_{\text{CB}}U^{W_4\dagger} \neq \sum_k \lambda_k |\psi_k^{W_4}\rangle\langle\psi_k^{W_4}|$.

In experiment, we observe that the normalized distance of estimated $\rho^{W_4}_{\text{expt}}$ is larger than that of other states as the prepared $\rho^{W_4}_{\text{expt}}$ has lower fidelity compared to other states. More importantly, it is not a graph-diagonal state any more under dephasing.

FIG. 7. Experimental results of (a) expected values of $\langle S^y \rangle$ and (b) probabilities of $d^{\text{expt}}_{\text{GHZ}}$, $d^{\text{expt}}_{\text{C}_4}$, $d^{\text{expt}}_{\text{W}_3}$ and $d^{\text{expt}}_{\text{W}_4}$, respectively. The black grids represent the values for ideal states.
channel, which causes that there is only 5% difference in state fidelity (shown in Appendix B) but 15% difference in normalized distance (shown in Fig. 3c).

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