(1 + 1)-dimensional gauge symmetric gravity model and related exact black hole and cosmological solutions in string theory

S. Hoseinzadeh, A. Rezaei-Aghdam *

Department of Physics, Faculty of Science, Azarbaijan Shahid Madani University, 53714-161, Tabriz, Iran

A R T I C L E   I N F O

Article history:
Received 15 April 2017
Received in revised form 16 August 2017
Accepted 23 August 2017
Available online xxxx
Editor: N. Lambert

A B S T R A C T

We introduce a four-dimensional extension of the Poincaré algebra \( \mathcal{N} \) in \( (1 + 1) \)-dimensional space-time and obtain a \( (1 + 1) \)-dimensional gauge symmetric gravity model using the algebra \( \mathcal{N} \). We show that the obtained gravity model is dual (canonically transformed) to the \( (1 + 1) \)-dimensional anti de Sitter (AdS) gravity. We also obtain some black hole and Friedmann–Robertson–Walker (FRW) solutions by solving its classical equations of motion. Then, we study an \( \mathcal{N} \)-gauged Wess–Zumino–Witten (WZW) model and obtain some exact black hole and cosmological solutions in string theory. We show that some obtained black hole and cosmological solutions are exact as the metric obtained in solutions of our gauge symmetric gravity model.

© 2017 The Author(s). Published by Elsevier B.V. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/). Funded by SCOAP³.

1. Introduction

\( (1 + 1) \)-dimensional gravity has been extensively studied by proposing various models. Two of the gravitational theories of most interest are single out by their simplicity and group theoretical properties. One of them is proposed by Jackiw [1] and Teitelboim [2] (Liouville gravity) which is equivalent to the gauge theory of gravity with (anti) de Sitter group \([3–5]\). The other one is the string-inspired gravity \([6–8]\) which is equivalent to the gauge theory of the Poincaré group \( ISO(1, 1) \) [7] and its central extension \([9–13]\).

Recently, two algebras namely the Maxwell algebra \([14,15]\) and the semi-simple extension of the Poincaré algebra \([16]\) have been applied to construct some gauge invariant theories of gravity in four \([16–20]\) and three \([21–23]\)-dimensional space-times. These algebras have been also applied to string theory as an internal symmetry of the matter gauge fields \([24]\). The Maxwell algebra in \( (1 + 1) \)-dimensional space-time, is the well-known central extension of the Poincaré algebra which, as we discussed above, has been applied to construct a \( (1 + 1) \)-dimensional gauge symmetric gravity action \([9,10]\). In this paper, we introduce a new four-dimensional extension of the Poincaré algebra \( \mathcal{N} \) in \( (1 + 1) \)-dimensional space-time, which is obtained from the 16-dimensional semi-simple extension of Poincaré algebra in \( (3 + 1) \)-dimensional space-time \([16]\), by reduction of the dimensions of the space. Then, we construct a \( (1 + 1) \)-dimensional gauge symmetric gravity model, using this algebra. We obtain some black hole and cosmological solutions by solving its equations of motion.

On the other hand, in string theory, two-dimensional exact black hole has been found by Witten \([6]\). Another black hole solution to the string theory has been presented in \([25]\) both in Schwarzschild-like and target space conformal gauges. Exact three-dimensional black string and black hole solutions in string theory have also been found in \([26,27]\). Here, we study the string theory in \( (1 + 1) \)-dimensional space-time, and show that some obtained black hole and cosmological solutions of the gravity model, are exact solutions of the beta function equations (in all loops).

The outline of this paper is as follows: In section 2, we construct a \( (1 + 1) \)-dimensional gauge symmetric gravity model using a four-dimensional gauge group related to the algebra \( \mathcal{N} \). Then, by presenting a canonical map, we show that the obtained gravity model is dual (canonically transformed) to the \( (1 + 1) \)-dimensional AdS gravity model. In section 3, we solve the equations of motion and obtain some black hole and Friedmann–Robertson–Walker (FRW) cosmological solutions. Finally, in section 4, we study an \( \mathcal{N} \)-gauged Wess–Zumino–Witten (WZW) model, and show that some of the resulting string backgrounds, which are exact \( (1 + 1) \)-dimensional solutions of the string theory, are the same as the black hole and cosmological solutions obtained for our gravity model. Section five, contains some concluding remarks.

* Corresponding author.

E-mail addresses: hoseinzadeh@azaruniv.edu (S. Hoseinzadeh), rezaei-a@azaruniv.edu (A. Rezaei-Aghdam).

http://dx.doi.org/10.1016/j.physletb.2017.08.068

0370-2693/© 2017 The Author(s). Published by Elsevier B.V. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/). Funded by SCOAP³.
2. (1 + 1)-dimensional gravity from a non-semi-simple extension of the Poincaré gauge symmetric model

The Poincaré algebra $\delta \sigma (1, 1)$ in (1 + 1)-dimensional space-time has the following form:

$$[J, P_a] = \epsilon_{ab} P_b, \quad [P_a, P_b] = 0,$$

where $\epsilon_{01} = -1$, and $J$ and $P_a$ ($a = 0, 1$) are generators of the rotation and translations in space-time, and the algebra indices $a = 0, 1$ can be raised and lowered by the (1 + 1)-dimensional Minkowski metric $\gamma_{ab} = (\eta_{00} = -1, \eta_{11} = 1)$ such that $P_a = \gamma_{ab} P^b$. In $D = 1 + 1$, a four-dimensional non-semi-simple extension of the Poincaré algebra $\mathcal{N} = (P_a, J, Z)$ has the following form:

$$[J, P_a] = \epsilon_{ab} P_b, \quad [P_a, P_b] = k \epsilon_{ab} Z, \quad [Z, P_a] = -\frac{\Lambda}{k} \epsilon_{ab} P^b,$$

where $Z$ is the new generator and $k$ and $\Lambda$ are constants. For $\Lambda = 0$, which leads to $[Z, P_a] = 0$, the above algebra reduces to a soluble algebra which is called the centrally extended Poincaré algebra (or Maxwell algebra in 1 + 1 dimensions) [9–11]. We construct the $\mathcal{N}$-algebra valued one-form gauge field as follows:

$$h_i = h^a_i X^a = \epsilon^{ab} P_a + \omega_i J + A_i Z, \quad i, \ j = 0, 1$$

where the indices $i, j = 0, 1$ are the space-time indices, and the one-form fields have the following forms:

$$e^a = \epsilon^{ab} dx^b, \quad \omega_i = \omega_i dx^i, \quad A = A_i dx^i,$$

where $e^a, \omega_i, A_i$ are the vierbein, spin connection and the new gauge field, respectively. Using the following infinitesimal gauge parameter:

$$u = \rho^P P_a + \tau^J J + \lambda Z,$$

and the gauge transformation as follows:

$$h_i \rightarrow \tilde{h}^j_i = U^{-1} h_j U - U^{-1} \eta U,$$

with $U = e^{-u} \approx 1 - u$ and $U^{-1} = e^u \approx 1 + u$, we obtain the following transformations of the gauge fields:

$$\delta \epsilon^a = -\tilde{\delta} \rho^a - \epsilon^{ab} e_j \left(\tau - \Lambda k \right) + \epsilon^{ab} \rho_b \left(\omega_i - \Lambda k A_i \right),$$

$$\delta \omega_i = -\tilde{\delta} \tau,$$

$$\delta A_i = -\tilde{\delta} \lambda - k \epsilon^{ab} e_j A_b,$$

where the torsion $T_{ij}^a$, standard Riemannian curve $R_{ij}$ and the new gauge field strength $F_{ij}$ have the following forms:

$$T_{ij}^a = \partial_j e_i^a + \epsilon^{ab} \left(\omega_j - \omega_i - \frac{\Lambda}{k} A_j \right) A_i,$$

$$R_{ij} = \partial_i \omega_j - \partial_j \omega_i,$$

$$F_{ij} = \partial_i A_j + k \epsilon^{ab} e_j A_b.$$ 

Now, one can write the gauge invariant action as [10]

$$I = \frac{1}{2} \int \eta_A R^A \left(\epsilon = \frac{1}{2} \int d^2 x \epsilon^A \eta_A R^A \right),$$

$$= \frac{1}{2} \int d^2 x \epsilon^A \left(\eta_A T_{ij}^a + \eta_2 \phi_{ij} + \eta_3 F_{ij} \right),$$

$$= \frac{1}{2} \int d^2 x \epsilon^A \left(\eta_1 \left(\epsilon \eta^A + \epsilon^{ab} \phi \left(\omega_j - \frac{\Lambda}{k} A_j \right) \right) + \eta_2 \delta \omega_j + \eta_3 \left(\partial_i A_j + \frac{1}{2} k \epsilon^{ab} e_i A_j \right) \right).$$

This action is invariant under the gauge transformations (4) and the following transformations of the fields $\eta_1, \eta_2$ and $\eta_3$:

$$\delta \eta_1 = k \epsilon^{ab} \eta_3 \rho_b - \epsilon^{ab} \eta_2 \left(\omega_i - \Lambda k A_i \right),$$

$$\delta \eta_2 = -\epsilon^{ab} \eta_2 \rho_b,$$

$$\delta \eta_3 = \Lambda k \epsilon^{ab}.$$

Now, we will show that the model (9) is dual to the (1 + 1)-dimensional AdS gravity. We know that $SO(2, 1)$ gauge symmetric gravity action can be obtained by use of the following algebra (anti de Sitter algebra for $k' \neq 0$):

$$[J, P_a] = \epsilon_{ab} P^b, \quad [P_a, P_b] = k \epsilon^{ab} f,$$

as follows [10]:

$$I = \int \int d^2 x \epsilon^A \left(\eta_1 \left(\epsilon \eta^A + \epsilon^{ab} \phi \left(\omega_j - \frac{\Lambda}{k} \right) \right) + \eta_2 \delta \omega_j + \eta_3 \left(\partial_i A_j + \frac{1}{2} k' \epsilon^{ab} e_i A_j \right) \right).$$

An SO(2, 1) invariant action for two-dimensional gravity was first constructed in [3] where the aim was to reconstruct the proposed two-dimensional Einstein equation from a two-dimensional gauge theory of gravity. Although the notation adopted in [3] is different from our notation, but it can be shown that the action constructed in [3] is equivalent to the (1 + 1)-dimensional AdS gravity model (12). Now, by selecting $\eta_3 = -\frac{k'}{2} \eta_2$ in our model (9), it is dual (canonically transformed) to the AdS gravity (12); i.e. the following map:

$$\omega_i \rightarrow \omega_i - \Lambda, \quad e_i^a \rightarrow e_i^a, \quad \eta_2 \rightarrow \eta_2, \quad k' = -\Lambda,$$

transforms the AdS gravity model (12) to our model (9). In the following, we will show that this map is a canonical one. The canonical Poisson-bracket and the Hamiltonian related to the AdS gravity model (12) are as follows:

$$\{\vec{\phi} e_j \} = \frac{\Lambda}{2} A_j,$$

$$\{\vec{\phi} \vec{\omega}_j \} = \frac{\Lambda}{2} \delta (x - y),$$

where $\vec{\phi} e_i = \phi_\rho A_i$, $\vec{\phi} \vec{\omega}_j = \phi_\omega \omega_j$ and $\vec{\phi} \vec{\omega}_j = \phi_\omega \omega_j$. One can easily show that this action is equivalent to the AdS gravity action (12) with $k' = -\Lambda$. 

---

1. This algebra is isomorphic to the four-dimensional Lie algebra $A_3 \oplus A_1$. [28].

2. Note that the commutation relation $[J, Z] = 0$ can be obtained from the Jacobi identity $[J, [P_a, P_b]] \equiv$ cyclic terms $= 0$.

3. Centrally extended Poincaré algebra (or Maxwell algebra) in 1 + 1 dimensions is isomorphic to the four-dimensional Lie algebra $A_3 \oplus A_1$. [28].

4. The SO(2, 1) invariant action for two-dimensional gravity constructed in [3] is $I = \frac{1}{2} \epsilon^{abc} \epsilon_{abc} \phi$ where $\phi = \phi_\omega A_i$, $\phi_\rho A_i = \phi_\rho A_i$, $\phi_\omega \omega_i = \phi_\omega \omega_i$ and $\phi_\rho A_i = \phi_\rho A_i$ where $a, b, c = 0, 1, 2$. The field $\phi_\rho A_i$ contains both the spin connection $\omega_i$ and vierbein $e_i$ where $a, b, c = 0, 1, 2$.
\[\tilde{H} = \int d^3x \left( \tilde{\Pi}_e^a \partial_t e^a + (\tilde{\Pi}_\omega^a) \partial_t \omega^a \right)
+ (\tilde{\Pi}_{\eta}^a \partial_t \eta^a + (\tilde{\Pi}_{\eta}^\omega \partial_t \eta^\omega) - \tilde{I} = 2 \int d^2x \tilde{\eta}_a \partial_t e^a + \tilde{\eta}^\omega \partial_t \omega^a \right) - \tilde{I}. \]

(14)

where the coordinates of the space-time are \((x^0, x^i) = (t, r)\) such that \(\partial_t = \partial_0 = \partial_{x^0}\) is the inverse Minkowski metric, and the conjugate momentums corresponding to the fields are as follows:

\[\tilde{\Pi}_e^a = \frac{\partial \tilde{\mathcal{L}}}{\partial (\partial_t e^a)} = e^{\omega^a} \tilde{\eta}_a, \quad \tilde{\Pi}_\omega^a = \frac{\partial \tilde{\mathcal{L}}}{\partial (\partial_t \omega^a)} = e^{\omega^a} \tilde{\eta}_2, \]

\[\tilde{\Pi}_{\eta}^a = \frac{\partial \tilde{\mathcal{L}}}{\partial (\partial_t \eta^a)} = -e^{\omega^a} e^\omega, \quad \tilde{\Pi}_{\eta}^\omega = \frac{\partial \tilde{\mathcal{L}}}{\partial (\partial_t \eta^\omega)} = -e^{\omega^a} \omega^a. \]

(15)

The map (13) is a canonical transformation and easily it can be shown that, under this map, the canonical Poisson-brackets and the Hamiltonian (14) related to the \(\text{AdS}_5\) gravity model (12) are transformed to the following Poisson-brackets and the Hamiltonian related to our model (9):

\[\{ (\Pi_e^a) \}, \{ \omega^a(y) \} = e^{\omega^a} \tilde{\eta}_a(y), \]

\[\{ (\Pi_\omega^a) \}, \{ \omega^a(y) \} = e^{\omega^a} \tilde{\eta}_2(y), \]

\[\{ (\Pi_A^a) \}, \{ \omega^a(y) \} = e^{\omega^a} \tilde{\eta}_3(y), \。

\[\{ (\Pi_{\eta}^a) \}, \{ \eta^a(y) \} = x^a(y), \]

\[\{ (\Pi_{\eta}^\omega) \}, \{ \eta^\omega(y) \} = x^\omega(y), \]

\[H = \int d^3x \left( (\Pi_e^a \partial_t e^a + (\Pi_\omega^a \partial_t \omega^a + (\Pi_A^a \partial_t A^a) + (\Pi_{\eta}^a \partial_t \eta^a + (\Pi_{\eta}^\omega \partial_t \eta^\omega) - \tilde{I}
+ 2 \int d^2x \tilde{\eta}_a \partial_t e^a + \tilde{\eta}_2 \partial_t \omega^a + \tilde{\eta}_3 \partial_t A^a - \tilde{I}. \]

(16)

where the conjugate momentums corresponding to the fields are given as follows:

\[\Pi_e^a = \frac{\partial \mathcal{L}}{\partial (\partial_t e^a)} = e^{\omega^a} \eta_a, \quad \Pi_\omega^a = \frac{\partial \mathcal{L}}{\partial (\partial_t \omega^a)} = e^{\omega^a} \eta_2, \]

\[\Pi_A^a = \frac{\partial \mathcal{L}}{\partial (\partial_t A^a)} = e^{\omega^a} \eta_3, \quad \Pi_{\eta}^a = \frac{\partial \mathcal{L}}{\partial (\partial_t \eta^a)} = -e^{\omega^a} e^\omega, \]

\[\Pi_{\eta}^\omega = \frac{\partial \mathcal{L}}{\partial (\partial_t \eta^\omega)} = -e^{\omega^a} \omega^a. \]

(17)

Note that the conjugate momentums are transformed under the map (13) as:

\[\tilde{\Pi}_e^a \rightarrow (\Pi_e^a), \quad \tilde{\Pi}_\omega^a \rightarrow (\Pi_\omega^a), \quad \tilde{\Pi}_{\eta}^a \rightarrow (\Pi_{\eta}^a), \quad \tilde{\Pi}_{\eta}^\omega \rightarrow (\Pi_{\eta}^\omega). \]

Since the Poisson-brackets and Hamiltonian of the model are preserved under the map (13), then the \(\text{AdS}\) gauge gravity model (12) is dual to our gravity model (9), and each can be transformed to the other by the canonical transformation (13), of course with \(\eta_3 = -\frac{1}{2} \eta_2\). Finally, under the map (13), the equations of motion for the \(\text{AdS}_5\) gravity (12) also transform to the equations of motion related to our model (9).

The equations of motion with respect to the fields \(\eta_a, \eta_2, \eta_3\) have the following forms, respectively:

\[\eta_1^a \partial_t e^a + \eta_2^a \partial_t \omega^a + \eta_3^a \partial_t A^a = 0, \quad (18)\]

\[\eta_1^a \partial_t \eta_a + \frac{1}{2} k \epsilon^{ab} \omega^a \omega_b = 0, \quad (19)\]

and the equations of motion with respect to the fields \(e^a, \omega^a, A^a\) are obtained as follows, respectively:

\[\eta_1^a \partial_t \omega^a + \frac{1}{2} k \epsilon^{ab} \omega^a \omega_b = 0, \quad (19)\]

\[\eta_1^a \partial_t \omega_a + \frac{1}{2} k \epsilon^{ab} \omega^a \omega_b = 0. \]

In the next section, we will try to solve these equations and obtain different solutions of them.

3. Solutions of the equations of motion

3.1. Radial solutions for \(\Lambda \neq 0\)

Using the following Ansatz for the metric:

\[ds^2 = e^{2} e^{2} \eta_0 d\text{r}^2 + M^2 r^2 dr^2, \]

where \(\{ x^0, x^i \} = (t, r)\) are the coordinates of the space-time \((0 < r < \infty, -\infty < t < \infty)\), one can obtain the following solution for the equations of motion (18)–(19):

\[M^2 = \frac{1}{-\Lambda N^2 + C_4}, \quad \eta_0 = C_2 N(r), \quad \eta_1 = 0, \quad \eta_2 = C_2 + C_4, \quad \eta_3 = 0, \quad \omega(r) = C_3 dt + f(r) dr, \]

\[A(r) = \frac{k}{\Lambda} \left( (C_3 - \Lambda N^2 + C_4) dt + f(r) dr \right). \]

where \(C_1, C_2, C_3, \text{ and } C_4\) are arbitrary constants, and \(N(r), f(r)\) are arbitrary functions of \(r\). The solution (21) describes a space-time with a constant scalar curvature \(\mathcal{R} = 2\Lambda\).

3.1.1. \(\text{AdS} \) black hole solution

For \(N^2 = -\Lambda^2 + b\), the solution (21) reduces to the following \(\text{AdS}\) black hole solution:

\[ds^2 = -N^2(r) dt^2 + \frac{dr^2}{N^2(r)}, \]

and

\[\eta_0(r) = C_2 N(r), \quad \eta_1(r) = 0, \quad \eta_2(r) = -C_2 r + C_1, \quad \eta_3(r) = \frac{\Lambda}{k} C_2 r, \quad \omega(r) = C_3 dt + f(r) dr, \quad A(r) = (kr + \frac{k}{\Lambda} C_3) dt + \frac{k}{\Lambda} f(r) dr, \]

\[3(\text{23})\]
where $b$ is a constant. Now, we calculate the mass (energy) of solution (23) using the ADM definition of mass (energy) as discussed in [29]. Varying the action (9) produces a bulk term, which is zero using the equations of motion, plus a boundary term which can be cancelled by adding the following boundary term to the Lagrangian:

$$L_B = - \partial_t \left( \eta_\rho \phi^a \eta_\sigma \phi^b + \eta_\sigma \phi^a \eta_\rho \phi^b + \eta_\rho A_0 \right),$$

(24)
together with an appropriate boundary condition. This boundary term is identified as the mass (energy) of solution. Our boundary condition is using the obtained values for fields in the solution (23) at spatial infinity ($r \to \pm \infty$). Then, the mass of the solution is obtained as follows:

$$m = \int_{-\infty}^{+\infty} dr L_B = \left( \eta_\rho \phi^a \eta_\sigma \phi^b + \eta_\sigma \phi^a \eta_\rho \phi^b + \eta_\rho A_0 \right)\bigg|_{-\infty}^{+\infty}.$$  

(25)

which using (22) and (23) turns out to be

$$m = C_2 b - C_1 C_3.$$  

(26)
The Kretschmann scalar for this metric is

$$K = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = 4\Lambda^2.$$  

(27)
Consequently, this solution has two singularities at the following points:

$$r_\pm = \pm \sqrt{\frac{b}{-\Lambda}},$$  

(28)
which are not the curvature singularities, but the coordinate singularities, and can be removed by definition of a new coordinate system. Using the Ansatz (22), we obtain another solution for the equations of motion (18)–(19) as follows:

$$N^2(r) = -\Lambda r^2 - 2Dr + C_3, \quad \eta_0(r) = C_2 N(r).$$

$$\eta_1(r) = 0, \quad \eta_2(r) = -C_2 r + C_1, \quad \eta_3(r) = \frac{C_2}{k} (\Lambda r + D),$$

$$\omega(r) = C_4 dt + f(r) dr, \quad A(r) = (kr + C_5) dt + \frac{k}{\Lambda} f(r) dr,$$

(29)
where $C_1, C_2, C_3, C_4, C_5$ and $D = \frac{1}{2} C_5 - C_4$ are arbitrary constants, and $f(r)$ is a function of $r$. The value of the Kretschmann scalar of this solution is same as that of the previous one $K = 4\Lambda^2$, and it has two coordinate singularities at points

$$r_\pm = \pm \frac{-D \pm \sqrt{D^2 + \Lambda C_3}}{\Lambda}.$$  

(30)
Using new coordinate $\rho = r + \frac{D}{\sqrt{\Lambda}}$, the latter solution (29) transforms to the AdS black hole solution (23). This also can be achieved by choosing $D = 0$ and $C_3 = -b$.

3.1.2. Black hole solutions

For negative $\Lambda$, by assuming $N(r) = \sinh(\sqrt{-\Lambda} r - b)$ and $C_3 = -\Lambda$, the solution (21) reduces to the following black hole solution:

$$ds^2 = -\sinh^2(\sqrt{-\Lambda} r - b) dt^2 + dr^2,$$

(31)
$$\eta_0(r) = C_2 \sinh(\sqrt{-\Lambda} r - b), \quad \eta_1(r) = 0,$$

$$\eta_2(r) = -\frac{C_2}{\sqrt{-\Lambda}} \cosh(\sqrt{-\Lambda} r - b) + C_1,$$

$$\eta_3(r) = -\frac{C_2}{k} \sqrt{-\Lambda} \cosh(\sqrt{-\Lambda} r - b),$$

$$\omega(r) = \frac{k}{\Lambda} \left( (C_3 - \sqrt{-\Lambda} \cosh(\sqrt{-\Lambda} r - b)) dt + f(r) dr \right).$$

(32)
where $b$ is an arbitrary constant.

For positive $\Lambda$, by assuming $N(r) = \sin(\sqrt{\Lambda} r - b)$ and $C_3 = \Lambda$, the solution (21) reduces to the following black hole solution:

$$ds^2 = -\sin^2(\sqrt{\Lambda} r - b) dt^2 + dr^2,$$

(33)
$$\eta_0(r) = C_2 \sin(\sqrt{\Lambda} r - b), \quad \eta_1(r) = 0,$$

$$\eta_2(r) = \frac{C_2}{\sqrt{\Lambda}} \cos(\sqrt{\Lambda} r - b) + C_1,$$

$$\eta_3(r) = -\frac{C_2}{k} \sqrt{\Lambda} \cos(\sqrt{\Lambda} r - b),$$

$$\omega(r) = \frac{k}{\Lambda} \left( (C_3 - \sqrt{\Lambda} \cos(\sqrt{\Lambda} r - b)) dt + f(r) dr \right).$$

(34)
Both of the solutions (31) and (33) have coordinate singularities at

$$r = -\frac{b}{\sqrt{\Lambda}},$$  

(35)
which can be removed by suitable coordinate transformations, because of their finite Ricci and Kretschmann scalars.

3.2. Radial solutions for $\Lambda = 0$

As previously discussed, for $\Lambda = 0$, the non-semi-simple extension of the Poincaré algebra $\mathcal{N}$, reduces to the centrally extended Poincaré algebra. Then, the (1+1)-dimensional gauge symmetric gravity model (9) reduces to the following central extension of Poincaré gauge symmetric action [9–11]:

$$I = \int d^2 x \mathcal{L} \left( \eta_0 \left( \partial_t \phi^a + \phi^b \partial_\rho \omega_j \right) + \eta_2 \partial_i \omega_j + \frac{1}{2} \eta_3 \left( \partial_i \phi^a + \phi^b \partial_\rho \phi^c \right) \right).$$  

(36)
We use the Ansatz (20) to solve the equations of motions (18)–(19) by inserting $\Lambda = 0$ in them, and obtain the following solution:

$$M^2(r) = \left( \frac{1}{D_1} \frac{dN(r)}{dr} \right)^2, \quad \eta_0(r) = -\frac{kD_2}{D_1} N(r), \quad \eta_1(r) = 0,$$

$$\eta_2(r) = \frac{kD_2}{2D_1} N^2(r) + D_3, \quad \eta_3(r) = D_2,$$

$$\omega(r) = D_1 dt, \quad A(r) = \left( \frac{k}{2D_1} N^2(r) + D_4 \right) dt + g(r) dr,$$  

(37)
where $D_1, D_2, D_3$ and $D_4$ are arbitrary constants, and $N(r), g(r)$ are arbitrary functions of $r$. The solution (37) describes a Ricci-flat space-time with zero scalar curvature ($R = 0$). For $N^2(r) = 2D_1 r - D_5$, the metric in solution (37) reduces to the following Schwarzschild-type metric:

$$ds^2 = -(2D_1 r - D_5) dt^2 + \frac{dr^2}{2D_1 r - D_5},$$  

(38)
where $D_5$ is an arbitrary constant. The metric (38) has a coordinate singularity at

$$r = \frac{D_5}{2D_1}.$$  

(39)

Please cite this article in press as: S. Hoseinzadeh, A. Rezaei-Aghdam, (1+1)-dimensional gauge symmetric gravity model and related exact black hole and cosmological solutions in string theory, Phys. Lett. B (2017), http://dx.doi.org/10.1016/j.physletb.2017.08.068
For $N^2(r) = (D_1 r - D_0)^2$, the metric in solution (37) turns out to be as follows:

$$ds^2 = -(D_1 r - D_0)^2 dt^2 + dr^2,$$

(40)

and has a coordinate singularity at

$$r = \frac{D_0}{D_1},$$

(41)

where $D_0$ is an arbitrary constant. In section 4, we study the gauged Wess–Zumino–Witten model, and show that the solution (37) is an exact solution to the string theory, and specially the latter metric solution (40) describes an exact $(1 + 1)$-dimensional Ricci-flat black hole in string theory.

3.3. Friedmann–Robertson–Walker (FRW) solutions

Cosmology of the two-dimensional Jackiw–Teitelboim gravity model is studied in Ref. [30]. Moreover, some cosmological solutions of the string-inspired gravity coupled to the matter field, both for dust-filled and radiation-filled space-times have been discussed in Ref. [31]. Here, to obtain some cosmological solutions for the equations of motions (18)–(19), we use Friedmann–Robertson–Walker metric as follows:

$$ds^2 = -dt^2 + a^2(t) \frac{dr^2}{1 - \kappa r^2},$$

(42)

where $a(t)$ is the scale factor and describes the expansion of the world, and $\kappa$ is a constant which can be equal to $-1$, 0 or $+1$ only. In $1 + 1$ dimensions, one can use the following coordinate transformation:

$$r \to \frac{1}{\sqrt{-\kappa}} \sinh(\sqrt{-\kappa} r) \quad \text{for } \kappa < 0$$

(43)

$$r \to \frac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa} r) \quad \text{for } \kappa > 0$$

(44)

to write the metric (42) as follows:

$$ds^2 = -dt^2 + a^2(t) dr^2.$$  

(45)

Using the Ansatz (45) for the metric in the equations (18)–(19) and after some calculations, one can obtain three different solutions corresponding to the negative, positive or zero values of $\Lambda$, as follows:

3.3.1. FRW solution for $\Lambda < 0$

We have the following solution for the negative $\Lambda$:

$$a(t) = \frac{\dot{a}(0)}{\sqrt{-\Lambda}} \sin(\sqrt{-\Lambda} t) + a(0) \cos(\sqrt{-\Lambda} t),$$

$$\omega(t, r) = \frac{\Lambda}{k} \left( h(t, r) dt + s(t, r) dr \right),$$

$$A(t, r) = h(t, r) dt + s(t, r) + \frac{k}{\sqrt{-\Lambda}} \tilde{\xi}_1(t),$$

$$\eta_0(r) = \frac{dg(r)}{dr}, \quad \eta_1(t, r) = -\frac{1}{\sqrt{-\Lambda}} \tilde{\xi}_1(t) g(r),$$

$$\eta_2(t, r) = a(t) g(r), \quad \eta_3(t, r) = -\frac{\Lambda}{k} a(t) g(r),$$

where

$$\tilde{\xi}_1(t) = a(0) \sin(\sqrt{-\Lambda} t) - \frac{\dot{a}(0)}{\sqrt{-\Lambda}} \cos(\sqrt{-\Lambda} t),$$

$$s(t, r) = \int dt \frac{\dot{a}(t, r)}{\sqrt{-\Lambda}},$$

(47)

$$g(r) = C_1 \cosh(\sqrt{\Lambda} r) - C_2 \sinh(\sqrt{\Lambda} r),$$

$$\lambda = \dot{a}^2(t) - \Lambda a^2(t),$$

and $C_1$ and $C_2$ are constants, $h(t, r)$ is an arbitrary function of $t$ and $r$, and dot denotes derivative with respect to the timelike coordinate $t$. $a(0)$ and $\dot{a}(0)$ in solution (46) are the initial values of scale factor $a(t)$ and its time derivative $\dot{a}(t)$ at $t = 0$, respectively. Because the fields in the solution (46) are functions of the radial coordinate $r$, this solution is not a homogenous solution. In order to obtain a homogeneous solution, $h(t, r)$ must be $r$-independent $h(t, r) = h_0(t)$ and $C_1 = C_2 = 0$, where $h_0(t)$ is a function of timelike coordinate $t$ only. Then, by this choice, all of the fields will be functions of the coordinate $t$ only, and spatial homogeneity will be achieved. This solution will collapse at

$$t = \frac{1}{\sqrt{-\Lambda}} \arctan \left( -\frac{a(0)}{\dot{a}(0)} \right).$$

(48)

The Hubble parameter $H(t)$ for this solution is as follows:

$$H(t) = \frac{\dot{a}(t)}{a(t)} = \frac{\dot{a}(0) \cos(\sqrt{-\Lambda} t) - \sqrt{-\Lambda} a(0) \sin(\sqrt{-\Lambda} t)}{a(0) \sin(\sqrt{-\Lambda} t) + \sqrt{-\Lambda} a(0) \cos(\sqrt{-\Lambda} t)}.$$  

(49)

Using $\dot{a}(t) = \Lambda a(t)$, the deceleration parameter $q(t)$ can be obtained as follows:

$$q(t) = \frac{(\dot{a}(0) \sin(\sqrt{-\Lambda} t) + \sqrt{-\Lambda} a(0) \cos(\sqrt{-\Lambda} t))^2}{(\dot{a}(0) \sin(\sqrt{-\Lambda} t) - \sqrt{-\Lambda} a(0) \cos(\sqrt{-\Lambda} t))^2},$$

(50)

which is obviously positive, and shows that the expansion of the universe is decelerating.

3.3.2. FRW solution for $\Lambda > 0$

For positive $\Lambda$, one obtains the following solution:

$$a(t) = \frac{\dot{a}(0)}{\sqrt{\Lambda}} \sinh(\sqrt{\Lambda} t) + a(0) \cosh(\sqrt{\Lambda} t),$$

$$\omega(t, r) = \frac{\Lambda}{k} \left( \hat{h}(t, r) dt + \hat{s}(t, r) dr \right),$$

$$A(t, r) = \hat{h}(t, r) dt + \hat{s}(t, r) + \frac{k}{\sqrt{\Lambda}} \hat{\xi}_2(t),$$

$$\eta_0(r) = \frac{d\hat{g}(r)}{dr}, \quad \eta_1(t, r) = -\frac{1}{\sqrt{\Lambda}} \hat{\xi}_2(t) \hat{g}(r),$$

$$\eta_2(t, r) = a(t) \hat{g}(r), \quad \eta_3(t, r) = -\frac{\Lambda}{k} a(t) \hat{g}(r),$$

where

$$\hat{\xi}_2(t) = a(0) \sinh(\sqrt{-\Lambda} t) - \frac{\dot{a}(0)}{\sqrt{\Lambda}} \cosh(\sqrt{-\Lambda} t).$$

(51)
\[ \dot{h}(t, r) = \int dt \frac{\partial \tilde{h}(t, r)}{\partial r}, \]

\[ \xi_2(t) = a(0) \sinh(\sqrt{\Lambda} t) + \frac{\dot{a}(0)}{\sqrt{\Lambda}} \cosh(\sqrt{\Lambda} t), \]

\[ \ddot{g}(r) = \begin{cases} 
D_1 r + D_2 & \dot{\lambda} = 0 \\
D_1 \cosh(\sqrt{-\lambda} r) + D_2 \sinh(\sqrt{-\lambda} r) & \dot{\lambda} > 0 \\
D_1 \cosh(\sqrt{-\lambda} r) + D_2 \sin(\sqrt{-\lambda} r) & \dot{\lambda} < 0 
\end{cases}, \]

where \( D_1 \) and \( D_2 \) are constants, and \( \tilde{h}(t, r) \) is an arbitrary function. This solution is not homogenous, and as the previous solution, in order to have a homogenous solution we must put \( \tilde{h}(t, r) = \tilde{h}_0(t) \) and \( D_1 = D_2 = 0 \). This solution will collapse for \( \frac{\dot{a}(0)}{a(0)} > \frac{\sqrt{\Lambda}}{\sqrt{\Lambda}} \), at \( t = \frac{1}{\sqrt{\Lambda}} \arctan \left( -\sqrt{\Lambda} \frac{a(0)}{\dot{a}(0)} \right) \), but for \( \frac{\dot{a}(0)}{a(0)} < \sqrt{\Lambda} \), it does not collapse. The Hubble parameter \( H(t) \) for this solution is as follows:

\[ H(t) = \frac{\dot{a}(0)}{a(0)} = \sqrt{\Lambda} \left( \frac{\dot{a}(0) \cosh(\sqrt{\Lambda} t) + \sqrt{\Lambda} a(0) \sinh(\sqrt{\Lambda} t)}{\dot{a}(0) \sinh(\sqrt{\Lambda} t) + \sqrt{\Lambda} a(0) \cosh(\sqrt{\Lambda} t)} \right), \]

Using \( \dot{a}(t) = \Lambda a(t) \), the deceleration parameter \( q(t) \) can be obtained as follows:

\[ q(t) = -\frac{\dot{a}(t) \ddot{a}(t)}{\dot{a}^2(t)} = \left( \frac{\dot{a}(0) \sinh(\sqrt{\Lambda} t) + \sqrt{\Lambda} a(0) \cosh(\sqrt{\Lambda} t)}{\dot{a}(0) \cosh(\sqrt{\Lambda} t) + \sqrt{\Lambda} a(0) \sinh(\sqrt{\Lambda} t)} \right)^2, \]

which is obviously negative. Note that for \( \dot{a}(0) = \sqrt{\Lambda} a(0) \), the scale factor of the solution (51) has the following exponential form:

\[ a(t) = a(0)e^{\sqrt{\Lambda} t}, \]

and leads to a constant Hubble parameter \( H = \sqrt{\Lambda} \) and negative deceleration parameter \( q = -1 \), which means that the expansion of the universe is accelerating.

### 3.3.3. FRW solution for \( \Lambda = 0 \)

For \( \Lambda = 0 \), we obtain the following solution:

\[ a(t) = a(t) + a(0), \quad \omega(t, r) = \ddot{a}(0) \partial r, \]

\[ A(t, r) = \tilde{h}(t, r) dt + \left[ \ddot{a}(0) \right] t + a(0)(t + E_3) \, dr, \]

\[ \eta_0(t, r) = \frac{d\bar{g}(r)}{dr}, \quad \eta_1(t, r) = -\dot{a}(0) g(r), \]

\[ \eta_2(t, r) = -a(t) \bar{g}(r), \quad \eta_3(t, r) = 0, \]

where

\[ \bar{g}(r) = E_1 \cosh(\dot{a}(0) r) + E_2 \sinh(\dot{a}(0) r), \]

\[ E_1, E_2 \] and \( E_3 \) are arbitrary constants, and \( \tilde{h}(t, r) \) is an arbitrary function of \( t \) and \( r \). To obtain a homogenous solution, \( \tilde{h}(t, r) \) must be independent of coordinate \( r \) \( \tilde{h}(t, r) = \tilde{h}_0(t) \), and also we must have \( E_1 = E_2 = 0 \). This solution obviously will collapse at \( t = -\frac{a(0)}{\dot{a}(0)} \).

The Hubble parameter \( H(t) \) and the deceleration parameter \( q(t) \) for this solution can be obtained as follows:

\[ H(t) = \frac{\dot{a}(0)}{a(0)t + a(0)}, \quad q(t) = 0, \]

which show that expansion of the universe is without acceleration. In the next section, by studying the gauged Wess-Zumino-Witten model, we show that the cosmological metric solution (57) is also an exact solution to the string theory.

### 4. \( \mathcal{A}_{A_4} \) \( \mathcal{A}_{A_4} \)-gauged Wess-Zumino-Witten model

As we have explained in the introduction of this paper, it has been shown that two-dimensional string-inspired gravity model [6–8] is equivalent to the gauge theory (36) of the centrally extended Poincaré algebra (the Maxwell algebra in 1 + 1 spacetime dimensions) [9–11]. So we anticipate that the gravity model (36) which has the Maxwell symmetry, and the string theory obtained by a gauged WZW model on the Maxwell algebra \( (\cong \mathcal{A}_{A_4}) \), both have some common properties such as a common solution to them. In this section, we try to find some exact solutions to the string theory (obtained by gauged Wess-Zumino-Witten model using \( \mathcal{A}_{A_4} \)) which coincide with the solutions of our \((1 + 1)\)-dimensional gravity model. We have mentioned in the footnote 3 that the Maxwell algebra in 1 + 1 dimensions is isomorphic to the Lie algebra \( \mathcal{A}_{A_4} \) [28]:

\[ [X_2, X_3] = X_1, \quad [X_2, X_4] = X_2, \quad [X_3, X_4] = -X_3, \]

where \( \{X_i\}, i = 1 \ldots 4 \) are the bases of the Lie algebra. We display the corresponding Lie group with \( \mathcal{A}_{A_4} \). Now, we study \( \mathcal{G} / H \) gauged Wess-Zumino-Witten model to obtain an exact solution to the string theory. Let the Lie group \( \mathcal{G} \) be \( \mathcal{G} = \mathcal{A}_{A_4} \), and \( H = \mathcal{A}_{A_1} \otimes \mathcal{A}_{A_1} \) is its subgroup which is a direct product of two one-dimensional Abelian non-compact Lie groups \( \mathcal{A}_{A_1} \) and \( \mathcal{A}_{A_1} \) generated by the bases \( X_1 \) and \( X_4 \), respectively. If \( g \) is an element of the Lie group \( \mathcal{G} \), then the Wess-Zumino-Witten action can be written as follows [32]:

\[ L(g) = \frac{k}{4\pi} \int d^3z \left( g^{-1} \partial g \cdot g^{-1} \partial g \right) \]

\[ -\frac{k}{2\pi} \int d^3\theta e^{i\vec{B}\cdot\vec{r}} \left( g^{-1} \partial g \cdot \left( g^{-1} \partial g \cdot g^{-1} \partial g \right) \right), \]

where \( \vec{B} \) is a three-dimensional manifold with the coordinates \( \vec{r}^\alpha = \{z, \bar{z}, y\} \), and \( \Sigma \) is its boundary with the local complex coordinates \( \{\bar{z}, \bar{\bar{z}}\} \). Furthermore, we use the notations \( \partial = \frac{\partial}{\partial z} \) and \( \partial_\bar{z} = \frac{\partial}{\partial \bar{z}} \), and \( d\bar{z} \) and \( d\bar{\bar{z}} \) denote the measures \(|d\bar{z}|^2\) and \(|d\bar{\bar{z}}|^2\), respectively. By introducing the gauge fields \( \mathbf{A}, \mathbf{A} \) which takes their values in the Lie algebra of \( \mathcal{G} \), the gauged Wess-Zumino-Witten action having the local axial symmetry \( g \rightarrow hgh \), \( \mathbf{A} \rightarrow h(A + \partial) \) and \( \mathbf{A} \rightarrow h^{-1}(A - \partial) \) \( h \in H \) can be written as follows [33]:

\[ L(g, \mathbf{A}) = L(g) + \frac{k}{2\pi} \int d^3z \left( \frac{1}{2} \left( \mathbf{A} \cdot \partial \mathbf{A} - \mathbf{A} \cdot \partial \mathbf{A} \right) + \langle \mathbf{A}, \mathbf{A} \rangle \right), \]

Here, we consider the Lie algebra \( \mathcal{H} = \mathcal{A}_{A_1} \otimes \mathcal{A}_{A_1} \) valued gauge fields \( \mathbf{A} \) and \( \mathbf{A} \) as follows:
\[ A = A_1 X_1 + A_2 X_2, \quad \tilde{A} = \tilde{A}_1 X_1 + \tilde{A}_2 X_2. \] (63)

We parameterize \( G = A_{4,8} \) group by the following group element:

\[ g = e^{a X_1} e^{b X_2} e^{i \xi_1 X_1} e^{i \xi_2 X_2}, \] (64)

where \( a, b, \xi_1, \xi_2 \in \mathbb{R} \). Then, using the group element (64), and the following non-degenerate ad-invariant bilinear quadratic form [34] on the Lie algebra \( A_{4,8} \) (60):

\[ \langle X_1, X_4 \rangle = \alpha, \quad \langle X_2, X_3 \rangle = -\alpha, \quad \langle X_4, X_4 \rangle = \beta, \] (65)

one can rewrite the Wess–Zumino–Witten action (61) as follows:

\[ L(g) = \frac{k}{4\pi} \int d^2 z \left( \alpha(\partial \tilde{a} \nu + \tilde{a} \partial \nu) + \alpha u(\partial \tilde{b} \tilde{\nu} + \tilde{b} \partial \tilde{\nu}) \right) \] (66)

\[ -\kappa \alpha \tilde{a} \tilde{b} + \tilde{a} \tilde{b} \nu - \tilde{b} \tilde{a} \nu, \] (67)

where \( \alpha \) and \( \beta \) are arbitrary constants (\( \alpha \) should be nonzero in order that the ad-invariant bilinear quadratic form (65) be non-degenerate). We gauge \( A_{4,8}^{(1)} \otimes A_{4,8}^{(1)} \) subgroup generated infinitesimally by \( \delta g = \epsilon_1 (X_1 g + g X_1) + \epsilon_2 (X_4 g + g X_4) \) which using (64) gives the following transformations of the group parameters:

\[ \delta a = 2\epsilon_1, \quad \delta b = -2\epsilon_2, \quad \delta u = 2\epsilon_2, \quad \delta \nu = 2\epsilon_2, \] (68)

which together with the following transformations of the gauge fields:

\[ \delta A_1 = -\tilde{a} \epsilon_1, \quad \delta \tilde{A}_1 = -\tilde{a} \epsilon_1, \quad (i = 1, 2) \] (69)

where \( \epsilon_1 \) and \( \epsilon_2 \) are gauge parameters. Then, the resultant gauged WZW action is as follows:

\[ L(g, A) = L(g) + \frac{k}{2\pi} \int d^2 z \left( \partial \tilde{a} \nu A_1 + \left( \alpha(\partial \tilde{a} + b \partial \tilde{b}u - bu \partial \tilde{b}) \right) + \beta \partial \tilde{\nu} \right) A_2 + \alpha \tilde{a} \nu \tilde{A}_1 + \left( \alpha(\partial \tilde{a} + u \partial \tilde{b}) + \beta \partial \tilde{\nu} \right) \tilde{A}_2 \] (70)

\[ + 2\alpha A_1 \tilde{A}_1 + 2\alpha A_1 \tilde{A}_1 + (2\beta - \alpha \beta) A_2 \tilde{A}_2. \]

Variations of the action (69) with respect to the gauge fields \( A_1 \) and \( \tilde{A}_1 \) (i = 1, 2) gives the following equations:

\[ A_1 = -\frac{1}{2} \left( \partial \tilde{a} + u \partial \tilde{b} + \frac{1}{2} bu \partial \tilde{b} \right), \quad A_2 = -\frac{1}{2} \partial \tilde{\nu}. \]

\[ \tilde{A}_1 = -\frac{1}{2} \left( \partial \tilde{a} + b \partial \tilde{b} - \frac{1}{2} bu \partial \tilde{b} \right), \quad \tilde{A}_2 = -\frac{1}{2} \partial \tilde{\nu}. \] (71)

using which one eliminates the gauge fields in (69), and obtains the following gauged Wess–Zumino–Witten action:

\[ L(g, A) = \frac{k}{4\pi} \int d^2 z \left( \frac{1}{2} \partial \tilde{b} \partial \tilde{\nu} \tilde{A}_2 + \alpha \tilde{a} \nu (u \partial \tilde{b} - bu \partial \tilde{b}) \right) \] (72)

\[ -\kappa \alpha \partial \tilde{\nu} \tilde{A}_2 + \frac{k}{4\pi} \int d^2 z \left( \frac{1}{2} \partial \tilde{b} \partial \tilde{\nu} \tilde{A}_2 + \alpha \tilde{a} \nu (u \partial \tilde{b} - bu \partial \tilde{b}) \right) \]

\[ -\kappa \alpha \tilde{a} \tilde{b} + \tilde{a} \tilde{b} \nu - \tilde{b} \tilde{a} \nu. \] (73)

This action is independent of the parameter \( a \), and now we can fix the gauge by setting \( b = u \), such that the gauged WZW action (71) turns out to be

\[ L(g, A) = -\frac{k\alpha}{2\pi} \int d^2 z \left( \frac{1}{4} u^2 \partial \tilde{\nu} \tilde{a} + \tilde{a} \tilde{u} \tilde{u} \right) \] (74)

\[ \frac{1}{2} \int d^2 z \left( -\tilde{a} \tilde{\nu} \partial \tilde{a} + (\alpha + \beta) \tilde{a} \tilde{b} \tilde{\nu} \right) \] (75)

where \( \tilde{\nu} \) is a constant and \( \tilde{\nu}(r) \) is an arbitrary function of the spatial coordinate \( r \). Then, the obtained gauged WZW action (74) describes a string propagating in a space-time with the following metric

\[ ds^2 = -\tilde{\nu}(r) dt^2 + \left( \frac{1}{\tilde{\nu}(r)} \right)^2 dr^2. \] (76)

By assuming \( \tilde{\nu} = D_1 \) and \( \tilde{\nu}(r) = N(r) \), the metric solution (75) is precisely same as the metric in (37) obtained as a solution of our gravity model (9). As we have discussed before, by assuming \( \tilde{\nu}(r) = D_1 \) or \( \tilde{\nu}(r) = N(r) \), the metric (75) reduces to the metric (40) which describes an exact Ricci-flat black hole in string theory. Now, for obtaining the dilaton field, we consider the following one-loop beta function equations in 1 + 1 dimensions [35]:

\[ R_{\mu \nu} + 2\nu_{\mu \nu} \nu = 0, \] (77)

\[ R + \frac{8}{k} + 4\nu^2 \phi - 4(\nu \phi)^2 = 0, \] (78)

where \( R \) and \( R_{\mu \nu} \) are the scalar curvature and Ricci tensor of the target space, \( \phi \) is the dilaton field, and \( \frac{8}{k} \) is the cosmological constant term. By requiring that the metric (75) must obey the one-loop beta function equations (76) and (77), we obtain the following relation for the dilaton field using (76):

\[ \phi(t, r) = \tilde{\nu}(r)(c_1 \cosh(Dt) + c_2 \sinh(Dt)), \] (79)

where \( c_1 \) and \( c_2 \) are some real constants. By substituting \( \phi(t, r) \) in (77), one can obtain the following relation between the constants \( c_1 \) and \( c_2 \):

\[ D^2(c_1)^2 - c_2^2 = \frac{2}{k}. \] (80)

In the same way, using another field redefinition as follows:

\[ u(t) = \frac{\alpha t + \beta}{\alpha}, \quad v(r) = \alpha r, \] (81)

where \( \alpha \) and \( \beta \) are arbitrary real constants, the gauged WZW action (72) turns out to have the following form:

\[ L(g, A) = \frac{k\alpha}{2\pi} \int d^2 z \left( \alpha u \tilde{b} \tilde{a} + (\alpha + \beta) \tilde{a} \tilde{b} \tilde{\nu} \right) \] (82)

which describes a string propagating in a space-time with the following cosmological metric:

\[ ds^2 = -dt^2 + (\alpha + \beta)^2 dr^2. \] (83)

By assuming \( \alpha = \tilde{\nu}(0) \) and \( \beta = 0 \), this metric is precisely same as the FRW metric (57) which is obtained by solving the equations of motion for our gravity model discussed in the previous section. In the same way for obtaining the previous dilaton field, we use (76) and (77) to obtain the following dilaton field corresponding to the metric (82):
\[ \phi(t, r) = (\alpha \tau + \beta)(d_1 \cosh(\alpha r) + d_2 \sinh(\alpha r)), \]
\[ \alpha^2(d_2^2 - d_1^2) = \frac{2}{R}, \]  \hspace{1cm} (83)
where \( d_1 \) and \( d_2 \) are real constants. Note that the black hole metric (40) converts to the FRW metric (82), and vice versa, using the following coordinate transformation:
\[ t \to \tilde{t}, \quad r \to \tilde{r}. \]  \hspace{1cm} (84)

5. Conclusions

We have presented a four-dimensional extension of the Poincaré algebra in \((1 + 1)\)-dimensional space-time. Using this algebra, we have constructed a gauge theory of gravity, which is dual (canonically transformed) to the AdS gauge theory of gravity, under special conditions. We have also obtained black hole and Friedmann–Robertson–Walker (FRW) cosmological solutions of this model. Then, using \( A_4 \) and \( A_{3d+i} \) gauged Wess–Zumino–Witten action, we have shown that some of the black hole and cosmological solutions of our gravity model are exact \((1 + 1)\)-dimensional solutions of string theory. In this paper, we have shown that the Ricci-flat \((\mathcal{R} = 0)\) solutions of our gravity model are also exact solutions to the string theory, only. But, it remains an interesting question to be investigated if our solutions of the gravity model \((\mathcal{R} = 2\Lambda)\) are also exact solutions to the string theory? Analysis of the constraints of the gravity model (9) and its quantization are some of the interesting open problems which may lead to some desired results. Another interesting problem may be the possibility of obtaining the \((1 + 1)\)-dimensional gravity model (9) by an appropriate dimensional reduction from a gauge invariant \((2 + 1)\)-dimensional gravity model (see [12,13]).

Acknowledgements

We would like to express our heartfelt gratitude to M.M. Sheikh-Jabbari for his useful comments. This research was supported by a research fund No. 217D4310 from Azerbaijan Shahid Madani University.

References

[1] R. Jackiw, Lower dimensional gravity, Nucl. Phys. B 252 (1985) 343–356.
[2] C. Teitelboim, Gravitation and hamiltonian structure in two-space–time dimensions, Phys. Lett. B 126 (1983) 41–45.
[3] Takeshi Fukuyama, Kiyoshi Kikkawa, Gauge theory of two-dimensional gravity, Phys. Lett. B 160 (1985) 259–262.
[4] K. Isler, C.A. Trugenberger, Gauge theory of two-dimensional quantum gravity, Phys. Rev. Lett. 63 (1989) 834.
[5] A. Chamseddine, D. Wyler, Gauge theory of topological gravity in \(1 + 1\) dimensions, Phys. Lett. B 228 (1989) 75.
[6] E. Witten, On string theory and black holes, Phys. Rev. D 44 (1991) 314.
[7] H. Verlinde, Black holes and strings in two-dimensions, in: The Sixth Marcel Grossman Meeting on General Relativity, 1991, pp. 813–831.
[8] C. Callan, S. Gottsang, A. Harvey, A. Strominger, Evansen black holes, Phys. Rev. D 45 (1992) 1005, arXiv:hep-th/9111056.
[9] D. Gangevi, R. Jackiw, Gauge invariant formulations of lineal gravity, Phys. Rev. Lett. 69 (1992) 233–236, arXiv:hep-th/9203056.
[10] R. Jackiw, Gauge theories for gravity on a line, Theor. Math. Phys. 92 (1992) 979–987, arXiv:hep-th/9206093.
[11] D. Gangevi, R. Jackiw, Poincaré gauge theory for gravitational forces in \((1 + 1)\)-dimensions, Ann. Phys. 225 (1993) 229–263, arXiv:hep-th/9302026.
[12] A. Achúcarro, Lineal gravity from planar gravity, Phys. Rev. Lett. 70 (1993) 1037–1040, arXiv:hep-th/9207108.
[13] G. Grignani, G. Nardelli, Poincaré gauge theories for linear gravity, Nucl. Phys. B 412 (1994) 320–344, arXiv:gr-qc/9209013.
[14] H. Bacy, P. Combe, J.L. Richard, Group-theoretical analysis of elementary particles in an external electromagnetic field. I. The relativistic particle in a constant and uniform field, Nuovo Cimento A 67 (1970) 267–295.
[15] H. Bacy, P. Combe, J.L. Richard, Nuovo Cimento A 70 (1970) 289–312.
[16] R. Schrader, The Maxwell group and the quantum theory of particles in classical homogenous electromagnetic fields, Fortsch. Phys. 20 (1972) 701–734.
[17] D.V. Soroka, V.A. Soroka, Gauge semi-simple extension of the Poincaré group, Phys. Lett. B 707 (2012) 160–162, arXiv:1101.1592 [hep-th].
[18] A. de Azcarraga, K. Kamimura, J. Lukierski, Generalized cosmological term from Maxwell symmetries, Phys. Rev. D 83 (2011) 124036, arXiv:1012.4402 [hep-th].
[19] A. de Azcarraga, K. Kamimura, J. Lukierski, Maxwell symmetries and some applications, Int. J. Mod. Phys. Conf. Ser. 23 (2014) 01160, arXiv:1201.2850 [hep-th].
[20] O. Cebecioglu, S. Kibaroğlu, Gauge theory of the Maxwell–Weyl group, Phys. Rev. D 90 (2014) 084053, arXiv:1404.3960 [hep-th].
[21] O. Cebecioglu, S. Kibaroğlu, Maxwell-affine gauge theory of gravity, Phys. Lett. B 751 (2015) 131–134, arXiv:1503.09003 [hep-th].
[22] P. Salgado, R.J. Szabo, O. Valdivia, Topological gravity and transgression holography, Phys. Rev. D 89 (2014) 084077, arXiv:1401.3653 [hep-th].
[23] L. Diaz, O. Fierro, F. Izaurieta, N. Merino, E. Rodriguez, P. Salgado, O. Valdivia, A generalized action for \((2 + 1)\)-dimensional Chern–Simons gravity, J. Phys. A 45 (2012) 255207, arXiv:1311.2215 [hep-th].
[24] S. Hoseinzadeh, A. Rezaei-Aghdam, 2 + 1 dimensional gravity from Maxwell and semi-simple extension of the Poincaré gauge symmetric models, Phys. Rev. D 90 (2014) 084008, arXiv:1402.0230 [hep-th].
[25] S. Hoseinzadeh, A. Rezaei-Aghdam, Exact three dimensional black hole with gauge fields in string theory, Eur. Phys. J. C 75 (2015) 227, arXiv:1501.02451 [hep-th].
[26] G. Mandal, A.M. Sengupta, S.R. Wadia, Classical solutions of two-dimensional string theory, Mod. Phys. Lett. A 6 (1991) 1685–1692.
[27] J. Horne, G. Horowitz, Exact black string solutions in three dimensions, Nucl. Phys. B 368 (1992) 444, arXiv:hep-th/9108001.
[28] G.T. Horowitz, D.L. Welch, Exact three dimensional black holes in string theory, Phys. Rev. Lett. 71 (1993) 328–331, arXiv:hep-th/9302126.
[29] J. Patera, R.T. Sharp, P. Winternitz, H. Zassenhaus, Invariants of real low dimension Lie algebras, J. Math. Phys. 17 (1976) 986–994.
[30] Dongsu Bak, D. Gangevi, R. Jackiw, Energy momentum conservation in general relativity, Phys. Rev. D 49 (1994) 5173–5181, arXiv:hep-th/93110025, Erratum: Phys. Rev. D 52 (1995) 3753.
[31] M. Cadon, S. Mignemi, Cosmology of the Jackiw–Teitelboim model, Gen. Relativ. Gravit. 34 (2002) 2101–2109, arXiv:gr-qc/0202066.
[32] R.B. Mann, S.F. Ross, Gravitation and cosmology in \((1 + 1)\)-dimensional dilaton gravity, Phys. Rev. D 47 (1993) 3312–3318, arXiv:hep-th/9206022.
[33] E. Witten, Nonabelian bosonization in two dimensions, Commun. Math. Phys. 92 (1984) 435–472.
[34] K. Gawedzki, A. Kupiainen, GH conformal field theory from gauged WZW model, Phys. Lett. B 215 (1988) 119.
[35] C.R. Nappi, E. Witten, A WZW model based on a non-semi-simple group, Phys. Rev. Lett. 71 (1993) 3751–3753, arXiv:hep-th/9310112.
[36] C.G. Callan, E.J. Martinec, M.J. Perry, D. Friedan, Strings in background fields, Nucl. Phys. B 262 (1985) 593.