Characteristic polynomials of real symmetric random matrices

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Abstract

It is shown that the correlation functions of the random variables $\det(\lambda - X)$, in which $X$ is a real symmetric $N \times N$ random matrix, exhibit universal local statistics in the large $N$ limit. The derivation relies on an exact dual representation of the problem: the $k$-point functions are expressed in terms of finite integrals over (quaternionic) $k \times k$ matrices. However the control of the Dyson limit, in which the distance of the various parameters $\lambda$'s is of the order of the mean spacing, requires an integration over the symplectic group. It is shown that a generalization of the Itzykson-Zuber method holds for this problem, but contrary to the unitary case, the semi-classical result requires a finite number of corrections to be exact. We have also considered the problem of an external matrix source coupled to the random matrix, and obtain explicit integral formulae, which are useful for the analysis of the large $N$ limit.

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1 Introduction

The spectrum of eigenvalues of complex Hamiltonians are often modelled by a random matrix theory, in which the random matrices belong to various ensembles according to the symmetries of the physical problem. The most common space-time symmetries of the Hamiltonian lead to the consideration of ensembles of real, complex or quaternionic random matrices. In the simplest case one considers Gaussian probability distributions. This simple choice is in many cases sufficient since it is now understood that the local statistics of the eigenvalues are universal, i.e. largely independent of the probability distribution. The most commonly studied Gaussian ensembles, called GOE, GUE and GSE, are invariant under the orthogonal, unitary or symplectic groups, respectively, and they all have important applications [1, 2, 3].

In this article we follow our previous study for the GUE case of the characteristic polynomials of random matrices [4]. If $X$ is an $N \times N$ random matrix, whose characteristic polynomial is $\det(\lambda - X)$, we consider the average of products of such characteristic polynomials defined by

$$F_k(\lambda_1, \ldots, \lambda_k) = \langle \prod_{l=1}^{k} \det(\lambda_l - X) \rangle. \quad (1)$$

In the GUE case we have derived in a previous article explicit formulae for those correlation functions, found then their asymptotic behavior for large $N$, and proved their universality in the short distance limit in which the differences $\lambda_i - \lambda_j$ is the order of the mean spacing of the eigenvalues of $X$. As usual the orthogonal and symplectic cases are more difficult to handle. It turns out that there is a hidden duality in these problems between $N$ the size of the matrices, and $k$ the number of points in the correlation functions: we may turn the integrals over $N \times N$ matrices into integrals over $k \times k$ matrices and, since we are interested in large $N$ -finite $k$ limit, this is the required tool for obtaining the large $N$ limit by the saddle-point method. We return below to the GUE case and exhibit its self-duality. However the GOE case turns out to be dual to the GSE. This duality, in the simpler case of all $\lambda_i$'s
equal, has been discussed recently within the orthogonal polynomial method [7, 8]. In both cases one may use geometry to reduce further the number of integrations. In the GUE case it relies on the Harish-Chandra-Itzykson-Zuber formula (HIZ) [14, 15]. For the GOE case it turns out that the $N - k$ duality maps the problem into the GSE case, and the use of the geometry of the symmetric space $U(2k)/Sp(k)$ leads to considerable simplifications. At the orders that we have considered one finds that there is a generalization of (HIZ) ; it is well-known that in that case the WKB approximation happens to be exact. For the GSE problem that one finds here, it is WKB, plus a finite number of corrections, which happens to be exact. Therefore we shall begin be re-exposing the unitary case at the light of this duality and of the HIZ formula. We shall then proceed to the GOE ensemble.

2 Survey of the unitary ensemble

For the Gaussian unitary ensemble (GUE), the random matrix $X$ is a complex $N \times N$ Hermitian matrix, with a probability distribution function

$$P(X) = \frac{1}{Z} \exp(-\frac{N}{2} \operatorname{tr} X^2).$$

The average $< ... >$ means integration with the normalized weight $P(X)$, with the Euclidean measure $\prod_i dX_{ii} \prod_{i<j} d\Re X_{ij} d\Im X_{ij}$. It is easily shown that the $F_2(\lambda_1, \lambda_2)$ reduces, up to a trivial factor, to the the kernel [3] $K_N(\lambda_1, \lambda_2)$ which characterizes the correlation functions of the eigenvalues of $X$. When all the $\lambda_j$’s are nearby, i.e. in the short distance scaling region in which $N$ is large and the products $N(\lambda_i - \lambda_j)$ are finite, $F_{2k}(\lambda_1, ..., \lambda_{2k})$ becomes, within an appropriate scaling, a universal function, i.e. independent of the specific distribution $P(X)$. When all the $\lambda_j$ are equal,

$$F_{2k}(\lambda) = F_{2k}(\lambda, ..., \lambda) = < [\det(\lambda - X)]^{2k} >$$

(3)
the 2k-th moment of the characteristic polynomial. In the large N limit, we have derived earlier [4, 5]

\[ F_{2k}(\lambda) = \gamma_k [2\pi N \rho(\lambda)]^k, \] (4)

in which \( \rho(\lambda) \) is the density of eigenvalues, and \( \gamma_k \) is a universal factor. This number had been first computed for the circular unitary ensemble by Keating and Snaith who used the Selberg integral formula [9].

There are several different derivations for those results. Let us here expose the duality which was mentioned in the introduction. We introduce Grassmann variables \( c \) and \( \bar{c} \), normalized to

\[ \int dcd\bar{c}e^{iN\bar{c}c} = 1. \] (5)

Then the characteristic polynomial may be written as

\[ \det(\lambda - X) = \int \prod_{a=1}^{N} d\bar{c}_a dc_a e^{iN\bar{c}_a(\lambda \delta_{ab} - X_{ab})c_b}. \] (6)

Repeating this \( k \)-times

\[ \prod_{a=1}^{k} \det(\lambda_a - X) = \int \prod_{a=1}^{N} \prod_{\alpha=1}^{k} d\bar{c}_{a\alpha} dc_{a\alpha} e^{iN\sum_{\alpha=1}^{k} \bar{c}_{a\alpha}(\lambda_a \delta_{ab} - X_{ab})c_{b\alpha}}. \] (7)

The (normalized) integration over \( X \), in presence of a matrix source \( Y \), yields

\[ \int dX e^{-\frac{N}{2} \text{Tr}X^2 + iN\text{tr}XY} = e^{-\frac{N}{4} \text{Tr}Y^2}. \] (8)

We may now apply this to the matrix \( Y_{ab} = -\sum_{\alpha=1}^{k} \bar{c}_{a\alpha}c_{b\alpha} \), generated by (7). Then one finds easily that

\[ \text{Tr}(Y^2) = -\sum_{\alpha\beta=1}^{k} \sum_{a=1}^{N} \bar{c}_{a\alpha}c_{a\beta} \sum_{b=1}^{N} \bar{c}_{b\beta}c_{b\alpha} = -\text{tr}(\hat{\gamma}^2) \] (9)

with

\[ \hat{\gamma}_{\alpha\beta} = \sum_{a=1}^{N} \bar{c}_{a\alpha}c_{a\beta}. \] (10)

Our notation are as follows : ”Tr” refers here to N-dimensional space, whereas ”tr” refers to matrices acting in the k-dimensional space. We introduce next an auxiliary
matrix $k \times k$ hermitian matrix $B$, such that
\[ e^{\frac{N}{2} \text{tr} \gamma^2} = \int dB \exp \left( -\frac{N}{2} \text{tr} B^2 + N \text{tr} \gamma B \right), \tag{11} \]
integrate over the Grassmann variables (which are now decoupled in the original $N$-dimensional space) and end up with
\[ F_k(\lambda_1 \cdots \lambda_k) = \int dB \det(\Lambda - iB)^N \exp \left( -\frac{N}{2} \text{tr} B^2 \right) \tag{12} \]
\[ = e^{\frac{N}{2} \text{tr} \Lambda^2} \int dB (\det B)^N \exp \left( -\frac{N}{2} \text{tr} B^2 + iN \text{tr} \Lambda B \right) \]
in which $\Lambda$ is the diagonal matrix $(\lambda_1, \ldots, \lambda_k)$. The problem is thus mapped into Gaussian integrals over $k \times k$ hermitian matrices as announced. This dual representation is of course well adapted to the $k$-fixed, $N$-large, limit that we are considering, since (12) contains $k^2$ variables instead of $N^2$ in our starting point. It is not difficult to proceed from (12) and derive the scaling results which were given in [4].

However it turns out that it is simpler, and necessary in view of what comes next for the GOE problem, to integrate out the unitary degrees of freedom in (12). This is done through the HIZ formula [14, 15] which gives the integral over the unitary group $U(k)$:
\[ \int dU e^{iN \text{tr} UXU^\dagger Y} = C_N \frac{\det_{1 \leq i, j \leq k} e^{iN x_i y_j}}{\Delta(x_1, \cdots, x_k) \Delta(y_1, \cdots, y_k)} \tag{13} \]
in which the $x_i$’s and $y_i$’s are the eigenvalues of the Hermitian $X$ and $Y$ respectively; $\Delta(x_1, \cdots, x_k)$ is the Van der Monde determinant
\[ \Delta(x_1, \cdots, x_k) = \prod_{i<j} (x_i - x_j). \tag{14} \]
It is well-known that the formula (13) happens to be exact semi-classically, i.e. if one retains only the sum of the $k!$ stationary points in the space of unitary matrices, weighted by the Gaussian fluctuations around each of them. Higher corrections happen to cancel exactly. This leads immediately to an integral over the $k$ eigenvalues $b_l$ of $B$, rather than over the $k^2$ matrix elements:
\[ F_k(\lambda_1, \cdots, \lambda_k) = Ce^{\frac{N}{2} \text{tr} \Lambda^2} \int (\prod_{l=1}^k db_l)^N e^{-\frac{N}{2} \sum_{l=1}^k (b_l^2 - 2b_l \lambda_l)} \prod_{l<p} \frac{(b_l - b_p)}{\lambda_l - \lambda_p} \tag{15} \]
When we consider simply the $k$-th moment of the characteristic polynomials, namely the case in which $(\lambda = \lambda_1 = \cdots = \lambda_k)$, the previous formula reduces to

$$F_k(\lambda) = C \exp \frac{Nk}{2} \lambda^2 \int \left( \prod_{l=1}^{k} db_l b_l^N \right) e^{-\frac{N}{2} \sum_{l=1}^{k} (b_l^2 - 2ib_l \lambda)} \prod_{l<l'} (b_l - b_l')^2 \quad (16)$$

Note that the above representations of the correlation functions of the characteristic polynomials, in terms of integrals over $k$ variables, are exact for any size $N \times N$ of the random matrices. It is then simple to find the large-$N$ limit of those functions by saddle-point integration. If we focus to even values of $k$, and substitute $2k$ to $k$ (the odd case is doable of course, but it leads to an oscillatory behavior) the saddle point equation for each $b_l$ is

$$b_l^2 - i \lambda b_l - 1 = 0 \quad (17)$$

whose roots are $b_l^\pm = (i \lambda_l \pm \sqrt{4 - \lambda_l^2})/2$. In the scaling limit, in which the $\lambda_l - \lambda_l'$ are of order $1/N$, the eigenvalues $b_j$ are very close and one must pay attention to the Vandermonde determinant in the integration measure. Finally the leading saddle-points correspond to equal numbers of $b_l$ close to either $b^+$ or $b^-$, with $b^\pm = (i \lambda \pm \sqrt{4 - \lambda^2})/2$. There are thus $\binom{2k}{k} = 2k!/k!k!$ saddle-points of equal weight. The combinatorial factor $\gamma_k$ of (4) is then simply

$$\gamma_k = \binom{2k}{k} \frac{(h_k)^2}{h_{2k}} = \frac{k-1}{l} \prod_{l=0}^{k-1} \frac{l!}{(k+l)!} \quad (18)$$

where we have used

$$h_k = \frac{1}{(2\pi)^{k/2}} \int_{-\infty}^{\infty} \prod_{1}^{k} dx_j e^{-\frac{1}{2} \sum_{i=1}^{k} x_i^2} \prod_{l<l'} (x_l - x_l')^2$$

$$= \prod_{l=0}^{k} l! \quad (19)$$

(This formula is used when we consider the gaussian fluctuations near the saddle-point in which $k$ of the $b_j$’s are near $b^+$ and the other half are close to the saddle point $b^-$. )
Finally this representation through an integral over a finite matrix matrix $B$, may be generalized to the case of an external matrix source $A$ coupled to the random matrix $X$ \cite{3, 11, 12}.

$$
\int dX \prod_{l=1}^{2k} \det(\lambda_l - X) e^{-\frac{N}{2} \text{tr}X^2 + N \text{tr}AX}
= \frac{1}{\Delta(\lambda)} \int db_1 \cdots b_N \prod_{i=1}^{2k} (a_i - b_i) \prod_{l < l'} (b_l - b_{l'}) e^{-\frac{N}{2} \sum b_l^2 + iN \sum \lambda_b b_l}
$$

(20)

We shall now transpose these techniques to real symmetric random matrices.

### 3 Real symmetric matrices and characteristic polynomials

Again we consider

$$F_k(\lambda_1, \ldots, \lambda_k) = \int dX e^{-\frac{N}{2} \text{tr}X^2} \prod_{i=1}^{k} \det(\lambda_i - X),$$

(21)

in which $X$ is a real symmetric $N \times N$ matrix, $X = X^T$. It is worth remembering that real symmetric matrices form the Lie algebra of the symmetric space $U(N)/O(N)$ (the Lie algebra of $U(N)$ consists of $N^2$ complex hermitian matrices; the imaginary part of those matrices are the $N(N-1)/2$ antisymmetric generators of $O(N)$; the real parts are the $N(N+1)/2$ real symmetric generators of the coset).

Using again Grassmann variables, and the representation \cite{3} of the characteristic determinants we are led again to an integration over real symmetric matrices in the presence of the matrix source $Y = -\sum_{\alpha=1}^{k} \bar{c}_{\alpha} c_{\alpha}$. This gives

$$
\int dX e^{-\frac{N}{2} \text{tr}X^2 + iN \text{tr}XY} = e^{-\frac{N}{4} \text{Tr}(Y^2 + YY^T)}.\]$$

(22)

We have dealt earlier with

$$\text{Tr}(Y^2) = -\text{tr}(\gamma^2)$$

(23)

which led to the integral \cite{11} over an hermitian $k \times k$ matrix. In addition we have here

$$\text{Tr}(YY^T) = \text{tr}(UV)$$

(24)
with the matrices $U$ and $V$ defined by

$$U_{\alpha\beta} = \sum_{a=1}^{N} \bar{c}_{a\alpha} c_{a\beta}$$  \hspace{1cm} (25)$$
$$V_{\alpha\beta} = \sum_{a=1}^{N} c_{a\alpha} \bar{c}_{a\beta}$$  \hspace{1cm} (26)

Defining the complex conjugation of Grassmann variables as $(\bar{c}_1 c_2)^* = \bar{c}_2 c_1$, we have $\gamma = \gamma^\dagger$, $V^\dagger = U$. Therefore, we may again decompose the remaining quartic terms in the $c$'s and $\bar{c}$'s as

$$e^{-\frac{N}{4} \text{tr}(UV)} = \int dD e^{-N \text{tr}(D^\dagger D + \frac{1}{2} V^\dagger D + \frac{1}{2} D^\dagger V)}$$  \hspace{1cm} (27)$$

where $D$ is a complex $k \times k$ antisymmetric matrix, $D = -D^T$. Then we have

$$F_k(\lambda_1, \ldots, \lambda_k) = \int \prod d\bar{c}_{a\alpha} dc_{a\alpha} e^{iN \sum_{a=1}^{k} \sum_{a=1}^{N} \lambda_{a} \bar{c}_{a\alpha} c_{a\alpha}}$$
$$\times \int dB dD e^{-N \text{tr}(B^2 + D^\dagger D) + N \sum c_{a\alpha} c_{a\beta} B_{\alpha\beta}}$$
$$\times e^{-\frac{1}{2} D^\dagger_{\alpha\beta} c_{a\beta} \bar{c}_{a\alpha} - \frac{1}{2} D_{\alpha\beta} c_{a\alpha} \bar{c}_{a\beta}}.$$  \hspace{1cm} (28)$$

Those auxiliary matrices $B$ and $D$ allow us to integrate over each pair $\bar{c}_a, c_a$ independently of the other pairs. It is convenient to define

$$\psi_a = \left( \begin{array}{c} \bar{c}_a \\ c_a \end{array} \right).$$  \hspace{1cm} (29)$$

For a given antisymmetric matrix $M$, $(M = -M^T)$, we then have the following formula :

$$\int \prod d\bar{c}_{a\alpha} dc_{a\alpha} e^{2 \psi_{a\alpha} M_{\alpha\beta} \psi_{\beta\alpha}}$$
$$= \left[ \int d\bar{c}_a dc_a e^{iN \psi_{a\alpha} M_{\alpha\beta} \psi_{\beta\alpha}} \right]^N$$
$$= [-\text{Pf} M]^N = [\text{det} M]^{N/2}$$  \hspace{1cm} (30)$$

where Pf is the pfaffian of the antisymmetric matrix $M$. Applying this to our problem, we deal here with

$$M = \left( \begin{array}{ccc} D & \Lambda - iB^T \\ -\left( \Lambda - iB \right) & D^\dagger \end{array} \right)$$  \hspace{1cm} (31)$$
in which \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_k) \). Thus we finally obtain

\[
F_k(\lambda_1, \ldots, \lambda_k) = \int dBdDe^{-N \text{Tr}(B^2 + D^\dagger D)}[-\text{Pf}M]^N
\]  

(32)

This representation (32) of \( F_k \) in terms of a finite number of integrals, here \( 2k^2 - k \) integrals (one \( k \times k \) hermitian matrix \( B \), one complex antisymmetric \( k \times k \) matrix \( D \)), again exact for any \( N \), is a solution to the problem. However, contrary to the GUE case, it turns out that a direct use of the saddle-point equations fail in the scaling limit. In other words every term of the perturbative expansion around the saddle-point turns out to be relevant in the regime in which the products \( N(\lambda_i - \lambda_j) \) are finite.

One may use the remaining invariances of this representation, to reduce further the number of integrations. A unitary transformation \( U \), among the \( c_a \), which diagonalizes the Hermitian matrix \( B \), one of the block matrices of \( M \), transforms \( D \) into \( D \rightarrow U^* D U^T \); in other words one can diagonalize \( B \) and keep for \( D \) an antisymmetric matrix. Therefore, applying the Harish-Chandra-Itzykson-Zuber formula [14, 15] for the integration over the unitary group, i.e. over the relative unitary transformation between the diagonal matrix \( \Lambda \) and the eigenbasis of \( B \), we obtain

\[
F_k(\lambda_1, \ldots, \lambda_k) = \frac{e^{N \text{Tr}A^2}}{\Delta(\Lambda)} \int \prod_{l=1}^k db_l \Delta(b) \int dDe^{-N \sum b_l^2 + 2iN \sum \lambda_a b_a - N \text{Tr}D^\dagger D} \times (-\text{Pf}M)^N
\]  

(33)

where now the matrix \( B \) in \( M \) is diagonal ; we have reduced the integrations to \( k^2 \) variables, instead of \( 2k^2 - k \). However it turns out that this is still unsufficient : the inapplicability of the saddle-point method in the scaling limit is still a problem if we proceed from (33). It is thus necessary to return to the underlying geometry of the space of matrices \( M \) in the representation (31). In order to make the quaternionic structure more apparent we return to (28) and define the spinor

\[
\psi_{aa} = \begin{pmatrix} \bar{c}_{aa} \\ c_{aa} \end{pmatrix}
\]  

(34)
and the adjoint

$$\bar{\psi}_{a\alpha} = \left( -\bar{c}_{a\alpha} \bar{c}_{a\alpha} \right).$$  \hfill (35)

Then the quadratic form in the Grassmann variables of (28) takes the form (repeated

$$N$$ times for each index $$a$$ that we drop)

$$\sum_{\alpha,\beta=1}^{k} \left( \bar{c}_{\alpha} c_{\beta} B_{\beta\alpha} - \frac{i}{2} D_{\gamma,\alpha}^{\dagger} c_{\gamma} c_{\alpha} - \frac{i}{2} D_{\beta,\alpha} \bar{c}_{\alpha} \bar{c}_{\beta} + i \lambda_{\alpha} \bar{c}_{\alpha} c_{\alpha} \right) = \frac{1}{2} \sum_{\alpha,\beta=1}^{k} \bar{\psi}_{a\alpha} \left( q_{a,\beta} + i \lambda_{a} \delta_{a\beta} \right) \psi_{\beta}$$

in which the $$q_{a,\beta}$$ are quaternionic matrix elements, i.e. linear combination of the
Pauli matrices. The identification in terms of $$2 \times 2$$ matrices is thus

$$q_{a,\beta} = \left( \begin{array}{cc} B_{a,\beta} & -i D_{a,\beta}^{*} \\ -i D_{a,\beta} & B_{a,\beta}^{*} \end{array} \right).$$ \hfill (37)

This defines a self-dual quaternion matrix $$\mathbb{1}, \mathbb{3}$$, i.e.

$$q_{a,\beta}^{\dagger} = q_{\beta a}$$ \hfill (38)

and $$q_{a,\beta} q_{\beta a}$$ is a multiple of identity. Let $$\tilde{M}_0$$ be the quaternionic matrix whose elements are the quaternion $$q_{a,\beta}$$ and $$\tilde{M}$$ the quaternionic matrix with elements.

$$\tilde{M}_{a,\beta} = q_{a,\beta} + i \lambda_{a} \delta_{a\beta}. $$ \hfill (39)

The Grassmannian integration leads to

$$\int \prod dc_{\alpha} dc_{\alpha} \exp - \frac{N}{2} \sum_{\alpha,\beta=1}^{k} \bar{\psi}_{\alpha} \left( q_{a,\beta} + i \lambda_{a} \delta_{a\beta} \right) \psi_{\beta} = \det \tilde{M} = -PfM,$$ \hfill (40)

in which $$\det$$ denotes the quaternionic determinant $$\mathbb{1}, \mathbb{3}$$. In addition

$$tr(B^2 + DD^{\dagger}) = tr(\tilde{M}_0^{\dagger}) = \sum_{a,\beta} q_{a,\beta} q_{\beta a}. $$ \hfill (41)

It may be clarifying to show this quaternionic construction explicitly for the

$$k = 2$$ case. There one has

$$M = \left( \begin{array}{cccc} 0 & d & \lambda_{1} - i B_{11} & -i B_{21} \\ -d & 0 & -i B_{12} & \lambda_{2} - i B_{22} \\ -\lambda_{1} + i B_{11} & i B_{12} & 0 & -d^{*} \\ -i B_{21} & -i B_{22} & d^{*} & 0 \end{array} \right),$$ \hfill (42)
\[ - \text{PfM} = |d|^2 + (\lambda_1 - iB_{11})(\lambda_2 - iB_{22}) + |B_{12}|^2. \]  
(43)

The equivalent quaternionic construction is

\[ \tilde{M} = \begin{pmatrix} q_{11} + i\lambda_1 & q_{12} \\ q_{21} & q_{22} + i\lambda_2 \end{pmatrix} \]  
(44)

with

\[ q_{11} = B_{11} \mathbf{1}, \quad q_{12} = (\Re B_{12}) \mathbf{1} + i(\Im B_{12}) \sigma_3 + i(\Re d) \sigma_1 + i(\Im d) \sigma_2 \]

\[ q_{21} = q_{12}^\dagger, \quad q_{22} = B_{22} \mathbf{1} \]  
(45)

and

\[ Q \det(\tilde{M}) = (q_{11} + i\lambda_1)(q_{22} + i\lambda_2) - q_{12}q_{12}^\dagger \]

\[ = (B_{11} + i\lambda_1)(B_{22} + i\lambda_2) + |B_{12}|^2 + |d|^2 = -\text{PfM}. \]  
(46)

Therefore we end up for the correlation functions with the following duality:

\[ F_k(\lambda_1, \ldots, \lambda_k) = \int d\tilde{M} (Q \det(\tilde{M}))^N e^{-N\text{tr}(\tilde{M}^2)}. \]  
(47)

The original integral over real symmetric \( N \times N \) matrices is replaced by integrals over quaternionic matrices which depend upon \( 2k^2 - k \) degrees of freedom. Those matrices are the generators for the symmetric space \( U(2k)/Sp(k) \).

This representation (47) in terms of a finite number of integration variables is a priori well adapted to the large \( N \)-limit. However it turns out that in the scaling limit of interest, the contributions of the non-gaussian fluctuations around the saddle-points are all relevant. Therefore it is necessary to eliminate first the "angular" degrees of freedom. When all the \( \lambda_i \) are equal, namely if we consider the moments of the characteristic polynomials, one can simply diagonalize the symplectic matrices in terms of \( k \) eigenvalues and then proceed to the large \( N \) limit. This is done in the next section. However if the \( \lambda_i \)'s are unequal we need some equivalent of the HIZ formalism, which will be described afterwards.
4 Moments of the characteristic polynomials

We first note the trivial \( k = 1 \) case: \( F_1(\lambda) = \langle \det(\lambda - X) \rangle \) is simply

\[
F_1(\lambda) = \int_{-\infty}^{\infty} db e^{-Nb^2} (\lambda - ib)^N ,
\]

which, up to a trivial factor, is the Hermite polynomial \( H_N(\sqrt{N}\lambda) \) which has an oscillatory behavior for large \( N \) when \( \lambda \) belongs to the support of Wigner’s semicircle. Therefore we consider from now on the more interesting even correlation functions.

When all the \( \lambda_i \)'s are equal, the matrix \( \Lambda \) is a multiple of identity, and one can diagonalize the quaternionic matrix \( \tilde{M} \) through a transformation belonging to the symplectic group \( Sp(2k) \). The transformation of \( \tilde{M} \) into the diagonal matrix \( T = \text{diag}(t_1, ... t_k) \) yields the Jacobian \( J = \Delta(t)^N \), \( (\Delta(t) \) is the Vandermonde determinant \( \prod_{i<j} (t_i - t_j) \)).

This gives simply

\[
F_{2k}(\lambda) = \langle \det(\lambda - X)^{2k} \rangle = C \int \prod_{l=1}^{2k} dt_l (\prod_{l=1}^{2k} t_l)^N \prod_{l<p} (t_l - t_p)^4 e^{-N \sum t_l^2 + i2N\lambda \sum t_l}
\]

The integral representation \( (49) \) is well suited to the study of the large \( N \) limit. Exponentiating \( t_l^N \) term as \( e^{N\log t_l} \), the integrand is of the form \( \exp(-N \sum \frac{1}{2} f(t_l)) \) with

\[
f(t) = t^2 - 2i\lambda t - \log t
\]

The saddle points for every \( t_l \) are solutions of \( f'(t_c) = 0 \), i.e.

\[
2t^2 - 2i\lambda t - 1 = 0
\]

The two solutions are given by

\[
t^\pm = \frac{1}{2}(i\lambda \pm \sqrt{2 - \lambda^2})
\]
The difference \((t^+ - t^-)\) is proportional to the semi-circular density of eigenvalues of the GOE ensemble:

\[
t^+ - t^- = \pi \rho(\lambda)
\]

(53)

where \(\rho(\lambda) = \sqrt{2 - \lambda^2}/\pi\). Expanding \(t_i\) around either \(t^+\) or \(t^-\), we find that the leading saddle-points are those in which half of the \(t_i(l = 1, ..., 2k)\) are near \(t^+\) and the remaining half near \(t^-\). (Other choices give oscillatory contributions in \(\exp(-N \sum_1^{2k} f(t_i))\), which damp the large \(N\)-limit). Therefore we have to add the \((2k)!/(k!k!)\) leading saddle-points corresponding to the distribution of half of the \(t_i\)'s near \(t^+\), and the other half near \(t^-\). The measure term given by the 4-th power of the Vandermonde determinant, yields a factor \((\pi \rho(\lambda))^4\) from the \(k\) variables near \(t^+\) and the \(k\) near \(t^-\). The exponent \(f\) is then expanded around \(t^+\) or \(t^-\), and the remaining integral factorizes into an integration around \(t^+\) and another one around \(t^-\). The integration around \(t^+\) is

\[
\int_{-\infty}^{\infty} \prod_{l=1}^{k} dt_i e^{-\frac{N}{2} f''(t^+)(t-t^+)^2} \prod_{i<j}^{k} (t_i - t_j)^4 = \left(\frac{1}{\sqrt{N f''(t^+)}}\right)^{2k^2-k} \prod_{l=1}^{k} (2l)!.
\]

(54)

Noting \(t^+ t^- = -1\), and \(t^+ - t^- = \pi \rho(\lambda)\), we find \(f''(t^+) f''(t^-) = (\pi \rho(\lambda))^2\). We need to fix the normalization constant \(C\) in (49). It is obtained from the integral,

\[
\int \prod dt_i e^{-\frac{N}{2} \sum t_i^2} \prod_{i<j}^{2k} (t_i - t_j)^4 = \left(\frac{1}{N}\right)^{4k^2-k} \prod_{l=1}^{2k} (2l)!
\]

(55)

The constant \(C\) in (49) is thus the inverse of this number. This constant appears as a normalization for the n-point correlation function of the Gaussian symplectic ensemble [3]. Thus including this normalization constant, \(F_{2k}(\lambda)\) becomes in the large \(N\) limit as

\[
F_{2k}^{(GOE)} = \gamma_k N^{2k^2} (2\pi \rho(\lambda))^{2k^2+k}
\]

(56)

\[
\gamma_k = \frac{(2k)!}{k!k!} \frac{[\prod_{l=1}^{k} (2l)!]^2}{\prod_{l=1}^{2k} (2l)!}
\]

\[
= \prod_{l=1}^{k} \frac{(2l-1)!}{(2k+2l-1)!}
\]

(57)
For example, in the case of $F_2(\lambda) = \langle (\det(\lambda - X))^2 \rangle$, it gives

$$F_2(\lambda)^{\text{GOE}} = \frac{1}{6} N^2 (2\pi \rho(\lambda))^3$$  \hspace{1cm} (58)

This result agrees with the result which one would deduce from $\lim \rho(\lambda_1, \lambda_2)/(\lambda_1 - \lambda_2)$, where the limit means $\lambda_1 \to \lambda$ and $\lambda_2 \to \lambda$, (details are given in appendix A).

The value of $\gamma_k$ agrees with the result of COE (circular orthogonal ensemble) found by Keating and Snaith \cite{9} through the Selberg integral formula. In the COE case, however, the density of state is a constant, and the factor $\rho(\lambda)$ is absent. This result is to be compared with the earlier result for the GUE,

$$F_{2k}^{\text{GUE}} = \gamma_k (2N \pi \rho(\lambda))^{k^2}$$  \hspace{1cm} (59)

where $\gamma_k$ is given by (18). This universal constant $\gamma_k$ appears also in the average of the moments of the Riemann $\zeta$-function, and it has a number theoretical meaning \cite{4, 9, 10}.

5 Correlations of characteristic polynomials

The integral representation (12) of the correlations functions $F_k(\lambda_1, \ldots, \lambda_k)$ is not unitary invariant unless the $\lambda_i$’s are all equal. Therefore if we parametrize the matrix $B$ as $B = U^\dagger b U$, in which $b$ is a diagonal matrix, we have to consider the HIZ integral

$$\int dU \exp i \text{tr} U^\dagger b U = \frac{\det \exp i \lambda_\rho b_j}{\Delta(b) \Delta(\lambda)}$$  \hspace{1cm} (60)

which is well-known to be WKB exact. This explains why, in the GUE case, it is equally possible, to apply the saddle-point method with or without integrating out the unitary group.

In the symplectic case we are not aware of any similar explicit result; however it will be shown now, at least for the lowest values of $k$, that the integral over the symplectic group can be performed exactly. The result is remarkably that, in this case, WKB plus a finite number of corrections is exact.
In the $k = 2$ case, we have

$$F_2(\lambda_1, \lambda_2) = \langle \det(\lambda_1 - X) \det(\lambda_2 - X) \rangle \quad (61)$$

As shown in the previous section, it is given by the integral (49). We first evaluate explicitly the angular integral. We first diagonalize $B$ by a unitary transformation, and then write the eigenvalues $b_1$ and $b_2$ in terms of new parameters $t$ and $c$,

$$b_1 = (1 - c)t_1 + ct_2$$
$$b_2 = ct_1 + (1 - c)t_2 \quad (62)$$

Then we have $b_1 + b_2 = t_1 + t_2$, and $b_1 - b_2 = (1 - 2c)(t_1 - t_2)$. Since the integrand is a function of $|d|^2$, we change variable to $|d|^2 = c(1 - c)(t_1 - t_2)^2$; then we have $b_1^2 + b_2^2 + 2|d|^2 = t_1^2 + t_2^2$, and $b_1b_2 - |d|^2 = t_1t_2$. Finally since the parameter $c$ is restricted to the interval $0 < c < 1$, we replace it by $c = \sin^2 \theta$.

This leads to

$$F_2(\lambda_1, \lambda_2) = \int_0^1 dc \int_{-\infty}^{+\infty} dt_1 dt_2 (1 - 2c) (t_1 - t_2)^3 (t_1t_2)^N e^{-N(t_1^2 + t_2^2) + 2iN(t_1\lambda_1 + t_2\lambda_2) - 2iNc(t_1 - t_2)(\lambda_1 - \lambda_2)} \quad (63)$$

The integration over $c$ yields

$$F_2(\lambda_1, \lambda_2) = \int_{-\infty}^{+\infty} dt_1 dt_2 (t_1t_2)^N e^{-N(t_1^2 + t_2^2) + 2iN(t_1\lambda_1 + t_2\lambda_2)} \times \left[ \frac{1}{N} \left( \frac{t_1 - t_2}{\lambda_1 - \lambda_2} \right)^2 + \frac{i}{N^2} \frac{t_1 - t_2}{(\lambda_1 - \lambda_2)^3} \right] \quad (64)$$

When $\lambda_1 = \lambda_2 = \lambda$, it reduces as expected to (49). This formula may be easily checked for finite values of $N$, since it reduces to Gaussian integrals over $t_1$ and $t_2$; for instance in the simplest case $N = 1$, it gives $F_2(\lambda_1, \lambda_2) \sim \lambda_1\lambda_2 + 1$, which agrees with the direct calculation (in this case the trivial integral over the real axis $\int dx(\lambda_1 - x)(\lambda_2 - x)e^{-\frac{1}{2}x^2}$).

However the representation (64), which is exact for any $N$, makes it clear

i) that the large $N$-limit may be found through a saddle-point integration over $t_1$ and
$t_2$; this will be done below.

ii) That in the universal local limit of interest, in which $N$ goes to infinity, $\lambda_1 - \lambda_2$ goes to zero and $N(\lambda_1 - \lambda_2)$ remains finite, the large $N$-limit could not have been taken earlier. If, for instance, we had used the saddle-point method at the level of (33), we would have missed the second term in the bracket of (64). If, at the early level of (33), we had recognized that the regime of interest requires to expand beyond the Gaussian approximation to the saddle-point, it would have appeared unexpectedly that the expansion stops after the first correction. Therefore (64) could have been obtained by a semi-classical approximation with a finite number of corrections, here just one. This is analogous, although not as simple, to the Harish-Chandra-Itzykson-Zuber formula for the GUE case, which is semi-classically exact, without any correction term [18]. The above integration over $c$ is therefore, for $k = 2$, the corresponding HIZ formula for the symplectic group.

For higher values of $k$ we need a more elaborate strategy. The HIZ formula may be easily derived by considering the Laplacian operator [10],

$$L = -\frac{\partial^2}{\partial X_{ij}^2}.$$

(65)

Its eigenfunctions are plane waves

$$Le^{iN\text{tr}\Lambda X} = (N^2\text{tr}\Lambda^2)e^{iN\text{tr}\Lambda X}.$$  

(66)

One can construct a unitary invariant eigenfunction of $L$, for the same energy $N^2\text{tr}\Lambda^2$, by the superposition

$$I = \int dU e^{iN\text{tr}\LambdaUXU^\dagger},$$

(67)

which is nothing but the HIZ integral. The integral being unitary invariant, it is a function of the $k$ eigenvalues $t_i$ of $X$. The same considerations hold for the three ensembles $\beta = 1, 2$ and 4, corresponding to the orthogonal, unitary and symplectic ensemble, with

$$I = \int e^{N\text{tr}\Lambda gXg^{-1}} dg.$$  

(68)
The Laplacian, expressed in terms of a differential operator on the eigenvalues $t_i$ reads

$$\left[ \sum_{i=1}^{k} \frac{\partial^2}{\partial t_i^2} + \beta \sum_{i=1, (i \neq j)}^{k} \frac{1}{t_i - t_j} \frac{\partial}{\partial t_i} \right] I = -\epsilon I, \quad (69)$$

with the eigenvalue $\epsilon$

$$\epsilon = N^2 \sum_{i=1}^{k} \lambda_i^2 \quad (70)$$

The $t$-dependent eigenfunctions of this Schrödinger operator have a scalar product given by the measure

$$\langle \varphi_1 | \varphi_2 \rangle = \int dt_1 \cdots dt_k |\Delta(t_1 \cdots t_k)|^\beta \varphi_1^*(t_1 \cdots t_k) \varphi_2(t_1 \cdots t_k) \quad (71)$$

The measure becomes trivial if one multiplies the wave function by $|\Delta|^{\beta/2}$. Thus if one changes $I(t)$ to

$$\psi(t_1 \cdots t_k) = |\Delta(t_1 \cdots t_k)|^{\beta/2} I(t_1 \cdots t_k), \quad (72)$$

one obtains the Hamiltonian,

$$\left[ \sum_{i=1}^{k} \frac{\partial^2}{\partial t_i^2} - \beta(\frac{\beta}{2} - 1) \sum_{i<j}^{k} \frac{1}{(t_i - t_j)^2} \right] \psi = -\epsilon \psi. \quad (73)$$

For $\beta = 2$, the solution is again given by plane waves in the $t_i$ and (taking into account the symmetry under permutations of $I$), one obtains the HIZ formula.

In the $\beta = 4$ case, the problem is less trivial, but simple for finite values of $k$. For $k = 2$, a solution of this equation is

$$\psi_0 = e^{iN(\lambda_1 t_1 + \lambda_2 t_2)} \left( 1 + \frac{2i}{N(t_1 - t_2)(\lambda_1 - \lambda_2)} \right) \quad (74)$$

The symmetry of $I$ under permutation of the $t_i$'s leads then to the solution

$$\psi = e^{iN(\lambda_1 t_1 + \lambda_2 t_2)} \left( 1 + \frac{2i}{N(t_1 - t_2)(\lambda_1 - \lambda_2)} \right) + e^{iN(\lambda_1 t_2 + \lambda_2 t_1)} \left( 1 + \frac{2i}{N(t_2 - t_1)(\lambda_1 - \lambda_2)} \right) \quad (75)$$

Then, after multiplication by the Vandermonde factor, we obtain the required symplectic HIZ formula (for $k = 2$),

$$I = \frac{1}{(t_1 - t_2)^2(\lambda_1 - \lambda_2)^2} \psi \quad (76)$$
For general $k$ ($\beta = 4$), the solution of (73) is of the form

$$\psi_0 = e^{iN(\lambda_1t_1+\cdots+\lambda_k t_k)}\chi$$

(77)

where $\chi$ satisfies

$$\left[\sum_{i=1}^k \frac{\partial^2}{\partial t_i^2} + 2iN\sum_{i=1}^k \lambda_i \frac{\partial}{\partial t_i} - \sum_{i<j} \frac{4}{(t_i - t_j)^2}\right]\chi = 0$$

(78)

The operator $\sum_{i=1}^k \frac{\partial^2}{\partial t_i^2} - \sum_{i<j} \frac{4}{(t_i - t_j)^2}$ annihilates the function $\Delta^{-1}(t_1, \cdots, t_k)$. Consequently the solution of (78) may be written

$$\chi(t_1 \cdots t_k) = \frac{f(t_1 \cdots t_k)}{\Delta(t_1 \cdots t_k)}$$

(79)

in which $f(t_1 \cdots t_k)$ is a polynomial of degree $k(k-1)/2$ in the $t_i$'s. Defining

$$\tau_{ij} = N(\lambda_i - \lambda_j)(t_i - t_j)$$

(80)

one finds for $k = 3$

$$\chi = \left[1 - \frac{2i}{L} \left(\frac{1}{\tau_{12}} + \frac{1}{\tau_{23}} + \frac{1}{\tau_{31}}\right) - 4 \left(\frac{1}{\tau_{12}\tau_{23}} + \frac{1}{\tau_{23}\tau_{31}} + \frac{1}{\tau_{31}\tau_{12}}\right) - 12i \frac{1}{\tau_{12}\tau_{23}\tau_{31}}\right].$$

(81)

Again, as for $k = 2$, one sees that the successive terms in the r.h.s. of (81) are of same order in the limit of interest, and again they could have been obtained through a finite number of corrections to a semi-classical calculation. It is remarkable that the series of $\chi$ stops at the order the inverse of the Vandermonde ; thus the symplectic HIZ integral is expressed as the sum of a finite number of terms. The successive coefficients of each term are determined by the equation (73).

Using this modified HIZ formula for the symplectic case, we obtain for the $k=3$ case, $F_3(\lambda_1, \lambda_2, \lambda_3)$ which is expressed by

$$F_3(\lambda_1, \lambda_2, \lambda_3) = \int dt_1 dt_2 dt_3 e^{-N(t_1^2 + t_2^2 + t_3^2) + 2iN(\lambda_1 t_1 + \lambda_2 t_2 + \lambda_3 t_3)}(t_1 t_2 t_3) N \left(\frac{\Delta(t)}{\Delta(\lambda)}\right)^2$$
\[
\times \left[ 1 + i \left( \frac{1}{N(\lambda_1 - \lambda_2)(t_1 - t_2)} + \frac{1}{N(\lambda_2 - \lambda_3)(t_2 - t_3)} + \frac{1}{N(\lambda_3 - \lambda_1)(t_3 - t_1)} \right) - \frac{1}{N^2(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(t_2 - t_3)} - \frac{1}{N^2(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1)(t_2 - t_3)(t_3 - t_1)} - \frac{1}{N^2(\lambda_3 - \lambda_1)(\lambda_1 - \lambda_2)(t_3 - t_1)(t_1 - t_2)} - \frac{3}{2} \frac{1}{N^3 \Delta(t) \Delta(t)} \right]
\] (82)

where \( \Delta(t) = (t_1 - t_2)(t_2 - t_3)(t_3 - t_1) \). (Note that the symmetries of the integrand allowed us to keep only the single solution (81), without adding permutations). For low values of \( N \) (i.e. \( N=1 \) or 2, one verifies easily this result by a direct integration over \( 1 \times 1 \) or \( 2 \times 2 \) matrices.

Higher values of \( k \) may be handled in a similar way, but the combinatorics become quite heavy. For instance in an appendix the solution of the case \( k = 4 \) is given explicitly and, although again it consists of a finite number of terms, it is quite cumbersome.

As is now clear, those integral representations make it easy to find the scaling limit (large \( N \), finite \( N(\lambda_i - \lambda_j) \)). For instance for \( k = 2 \) one finds the saddle point values of \( t_1 \) and \( t_2 \) from (64) in the large \( N \) limit,

\[ t_i = \frac{i}{2} \lambda_i + \frac{\epsilon}{2} \sqrt{2 - \lambda_i^2} \] (83)

where \( i = 1, 2 \) and \( \epsilon = \pm 1 \). We use the parametrization

\[ \lambda_i = \sqrt{2} \cos \theta_i. \]

There are a priori four saddle-points given by (83), but the two dominant ones are \( t_1 = \frac{i}{\sqrt{2}} e^{-i\theta_1}, t_2 = \frac{1}{\sqrt{2}} e^{i\theta_2} \) for \( \epsilon = \pm 1 \). In the short distance limit, \( N \) large and \( N(\theta_1 - \theta_2) = N\theta_{12} \) finite, we obtain

\[ F_2(\lambda_1, \lambda_2) = \sum_{\epsilon = \pm 1} e^{-2 i \epsilon N \theta_{12} \sin^2 \theta} \left[ \frac{1}{(N\theta_{12})^2} + \frac{i \epsilon}{2(N\theta_{12})^3 \sin^2 \theta} \right] \] (84)

where \( \theta = (\theta_1 + \theta_2)/2 \). The semi-circle density of states is given by \( \rho(\lambda) = \frac{\sqrt{2}}{\pi} \sin \theta \), and \( \theta_{12} = -\frac{1}{\sqrt{2}} \frac{\lambda_1 - \lambda_2}{\sin \theta} \). Thus we obtain in the scaling short distant limit,

\[ F_2(\lambda_1, \lambda_2) = C \left[ \frac{\cos x}{x^2} - \frac{\sin x}{x^3} \right] \] (85)

where \( x = \pi N(\lambda_1 - \lambda_2) \rho(\lambda) \). It is interesting to note that this function may be expressed as a half-integer Bessel function, since \( \left( \frac{\cos x}{x^2} - \frac{\sin x}{x^3} \right) = -\sqrt{\frac{\pi}{2x^3}} J_{3/2}(x). \)
In the unitary case, the sine kernel is similarly a half integer Bessel function since
\[
\frac{\sin x}{x} = \sqrt{\frac{\pi}{2x}} J_{1/2}(x).
\]

6 Extension to an external matrix source

In the GUE case, when an external matrix source \( A \) is coupled to an Hermitian random matrix \( X \), as we have discussed earlier, \( F_{2k}(\lambda_1, ..., \lambda_{2k}) \) is given by (20). The degrees of freedom provided by the eigenvalues of \( A \) are useful to study a number of new universality classes [6, 17]. For instance by tuning the eigenvalues \( a_i \) of the external source matrix \( A \), we can study the problem of a closing gap in the spectrum of random hermitian matrices [6]. Thus it is interesting to consider this external source problem for real symmetric matrices as well.

One can always assume that the external source matrix \( A \) is diagonal. In the method of integration over Grassmann variables used in section three, it is simple to include the external matrix \( A \):

\[
\int e^{-\frac{N}{2}\text{tr}X^2 + N\text{tr}AX + iN\text{tr}XY} dX = e^{-\frac{N}{2}\text{tr}[(Y-iA)^2 + (Y-iA)(Y^T-iA)]}
\]

(86)

where \( Y = -\sum_{\alpha} \bar{c}_a a_{\alpha} \). Since \( A \) is diagonal, the term \( \text{tr}AX \) gives simply the extra term \( \exp[iN\sum_{\alpha} a_{\alpha} c_{\alpha a}] \) in the integrand of \( F_k \) in (28). Therefore, repeating the calculations of section 2, we find that eq.(30) is modified as follows:

\[
\int \prod d\bar{c}_j d_{c_{j\alpha}} e^{iN_{a}^{\alpha}\psi_{\alpha j}M^{(j)}_{a\beta}\psi_{\beta}}
\]

\[
= \prod_{j=1}^{N} \int d\bar{c}_j d_{c_{j\alpha}} e^{iN_{a}^{\alpha}\psi_{\alpha j}M^{(j)}_{a\beta}\psi_{\beta}}
\]

\[
= \prod_{j} [-\text{Pf}M^{(j)}] = \prod_{j} [\text{det} M^{(j)}]/2
\]

(87)

where \( \text{Pf}M^{(a)} \) is the pfaffian of the antisymmetrix matrix \( M^{(a)} \) given by

\[
M^{(j)} = \begin{pmatrix}
D & \Lambda + a_j \mathbf{1} - iB \\
-(\Lambda + a_j \mathbf{1} - iB) & \Lambda + a_j \mathbf{1} - iB^T
\end{pmatrix}.
\]

(88)
in which \( \Lambda = \text{diag}(\lambda_1, ..., \lambda_k) \) and \( a_j 1 = \text{diag}(a_j, ..., a_j) \). Thus we finally obtain

\[
F_k(\lambda_1, ..., \lambda_k) = \int dB dDe^{-N \text{tr}(B^2 + D^\dagger D)} \prod_{j=1}^N \{-\text{Pf}(M^{(j)})\}
\] (89)

This integral can be expressed in terms of a quaternion matrix \( Q \), which can be diagonalized by the symplectic group \( Sp(k) \). When all the \( \lambda_i \)'s are equal to a single \( \lambda \), we get

\[
F_k(\lambda, ..., \lambda) = \int \prod_{l=1}^k \prod_{j=1}^N (t_l - ia_j) \prod_{l < l'} (t_l - t_{l'})^4 e^{-N \sum t_l^2 + 2iN \lambda \sum t_l} \prod_{l=1}^k dt_l .
\] (90)

For the case of different \( \lambda_i \)'s, this formula is modified by an extra factor as in the previous section.

As an example of the usefulness of the above representation, we choose an external source with only two opposite eigenvalues \( \pm c \), with half of the eigenvalues equal to \( +c \) and the other half to \( -c \). This gives a factor \((t_l^2 + c^2)^{N/2} = \exp \frac{N}{2} \log (t_l^2 + c^2)\) in the integrand. Expanding it in powers of \( t_l^2 \), the total coefficient of \( t_l^2 \) in the exponent vanishes for \( c = 1/\sqrt{2} \). Therefore in the large \( N \) limit, we obtain at this new critical point

\[
< [\text{det}(X)]^k > = \int e^{-N \text{tr} Q^4} dQ
\] (91)

where \( Q \) is a \( k \times k \) symmetric quaternionic matrix, and the the average \(< ... >\) is evaluated with the distribution in the presence of the external source whose eigenvalues are \( \pm c = \pm 1/\sqrt{2} \).

One could make other choices for the eigenvalues of the external source matrix \( A \), and obtain thereby higher multicritical points with terms such as \( \text{tr} Q^n \) in the exponent, in analogy with the GUE case in an external matrix source [12].

7 Summary

In this article, an exact representation of the k-point functions \( \langle \prod_{a=1}^k \text{det}(\lambda_a - X) \rangle \), averaged over \( N \times N \) real symmetric random matrices, has been derived in terms of
an integral over quaternionic $k \times k$ matrices, invariant under the unitary symplectic group. This representation leads to an easy calculation of the moments of the characteristic polynomials ($\lambda_1 = \cdots = \lambda_k$). In the large $N$-limit one finds

$$F_{2k}^{(GOE)} = \prod_{l=1}^{k} \frac{(2l - 1)!}{(2k + 2l - 1)!} N^{2k^2} (2\pi \rho(\lambda))^{2k^2+k},$$

(92)

to be compared to the earlier result for the GUE,

$$F_{2k}^{(GUE)} = \prod_{l=0}^{k-1} \frac{l!}{(k+l)!} (2N\pi \rho(\lambda))^{k^2}.$$ 

(93)

For unequal $\lambda_a$'s, in spite of the fact that the integral representation involves a finite number of variables, in the large $N$-limit the corrections to the saddle-point, in the scaling regime $N(\lambda_i - \lambda_j)$ finite, are not negligible. A generalization of the HarishChandra-Itzyson-Zuber formula is shown to solve the problem. Remarkably this formula is "nearly" semi-classical, in the sense that it happens that the semi-classical expansion terminates after a few terms, a number of terms which increases with $k$ but not with $N$. Then the saddle-point method may easily be applied for large $N$, and this leads to explicit asymptotic formulae for the correlation functions of the characteristic polynomials. Finally this may be generalized to include an external matrix source in the probability measure. Real symmetric random matrices appear as models of numerous physical time-reversal invariant Hamiltonians. For instance the orthogonal matrix model with an external source has been investigated as a model of glassy behavior [13]. The results of the present work for the moments and for the correlation functions in an external source may be of interest for such problems.
Appendix A: The solution for $k = 4$

From (78) and (79), the polynomial $f$ satisfies

$$
\sum_{i=1}^{k} \frac{\partial^2 f}{\partial t_i^2} + 2 \sum_{i=1}^{k} i N \lambda_i f \left( \Delta \frac{\partial}{\partial t_i} \Delta \right) + 2i N \sum_{i=1}^{k} \lambda_i \frac{\partial f}{\partial t_i} = 0 \quad (A.1)
$$

The solution of this equation is obtained by a perturbation expansion in powers of the $\lambda_i$'s, but it ends at the level of the Vandermonde $\Delta(\lambda_1 \cdots \lambda_4)$. Using the notation of (80), $\tau_{ij} = N(t_i - t_j)(\lambda_i - \lambda_j)$, we obtain

$$
f = C [1 - \frac{i}{4} \left( \tau_{12} + \tau_{13} + \tau_{14} + \tau_{23} + \tau_{24} + \tau_{34} \right) \\
- \frac{1}{12} \left( \tau_{12} \tau_{13} + \tau_{12} \tau_{14} + \tau_{13} \tau_{14} + \tau_{12} \tau_{23} + \tau_{23} \tau_{24} + \tau_{12} \tau_{24} \right) \\
+ \tau_{14} \tau_{34} + \tau_{14} \tau_{24} + \tau_{24} \tau_{34} + \tau_{23} \tau_{34} + \tau_{13} \tau_{34} + \tau_{13} \tau_{23} \right) \\
- \frac{1}{18} \left( \tau_{12} \tau_{34} + \tau_{13} \tau_{24} + \tau_{14} \tau_{23} \right) \\
\frac{i}{24} \left( \tau_{12} \tau_{13} \tau_{14} + \tau_{12} \tau_{23} \tau_{24} + \tau_{13} \tau_{23} \tau_{34} + \tau_{14} \tau_{24} \tau_{34} \right) \\
\frac{i}{36} \left( \tau_{12} \tau_{13} \tau_{23} + \tau_{12} \tau_{14} \tau_{24} + \tau_{13} \tau_{14} \tau_{34} + \tau_{23} \tau_{24} \tau_{34} \right) \\
+ \tau_{14} \tau_{34} \tau_{23} + \tau_{14} \tau_{24} \tau_{23} + \tau_{12} \tau_{24} \tau_{34} + \tau_{12} \tau_{23} \tau_{34} \\
+ \tau_{12} \tau_{14} \tau_{23} + \tau_{13} \tau_{14} \tau_{23} + \tau_{12} \tau_{13} \tau_{34} + \tau_{12} \tau_{14} \tau_{34} \\
+ \tau_{13} \tau_{34} \tau_{24} + \tau_{13} \tau_{24} \tau_{23} + \tau_{14} \tau_{24} \tau_{13} + \tau_{13} \tau_{12} \tau_{24} \right) \\
+ \frac{1}{72} \left( \tau_{12} \tau_{23} \tau_{34} \tau_{14} + \tau_{12} \tau_{13} \tau_{24} \tau_{34} + \tau_{13} \tau_{14} \tau_{24} \tau_{23} + \tau_{12} \tau_{14} \tau_{24} \tau_{34} + \tau_{12} \tau_{14} \tau_{23} \tau_{24} \right) \\
+ \tau_{12} \tau_{14} \tau_{24} \tau_{34} + \tau_{12} \tau_{13} \tau_{23} \tau_{34} + \tau_{12} \tau_{13} \tau_{24} \tau_{34} + \tau_{12} \tau_{13} \tau_{23} \tau_{24} + \tau_{12} \tau_{14} \tau_{23} \tau_{34} \\
+ \tau_{14} \tau_{24} \tau_{23} \tau_{34} + \tau_{13} \tau_{24} \tau_{23} \tau_{34} + \tau_{12} \tau_{14} \tau_{13} \tau_{34} + \tau_{14} \tau_{13} \tau_{24} \tau_{34} \\
- \frac{i}{144} \left( \tau_{12} \tau_{13} \tau_{24} \tau_{34} + \tau_{12} \tau_{14} \tau_{24} \tau_{13} \tau_{34} + \tau_{12} \tau_{14} \tau_{24} \tau_{13} \tau_{34} \right) \\
+ \frac{1}{288} \tau_{12} \tau_{13} \tau_{14} \tau_{24} \tau_{23} \tau_{34} \right]. \quad (A.2)
$$

The HIZ integral is obtained by requiring the symmetry under permutation of the $t_i$'s in the final expression for $I$, thus from (83) (and a replacement of $\lambda$ by $2\lambda$),

$$
I = C \left[ \frac{1}{\Delta(t)\Delta(\lambda)^3} \right] (f(t_1, ..., t_k) + \text{perm.of } f) \quad (A.3)
$$

22
where the last term means that one adds the terms in which one permutes the $t_i$’s for fixed $\lambda$’s.

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