KRUSKAL-KATONA THEOREM FOR T-SPREAD STRONGLY STABLE IDEALS

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ABSTRACT. We prove that any t-spread strongly stable ideal has a unique t-spread lex ideal with the same f-vector. We also characterize the possible f-vectors of t-spread strongly stable ideals in the "t-spread" analogue of Kruskal-Katona theorem.

INTRODUCTION

Kruskal-Katona Theorem solves the problem of characterizing the possible f-vectors of the simplicial complexes on a given vertex set. With a special importance in Combinatorial Algebra, this result has been proved by Joseph Kruskal [5] and Gyula Katona [4]. In other words, Kruskal-Katona Theorem gives an elegant answer for the following question: Given the number of faces of dimension $d-1$ of a simplicial complex $\Delta$, how many faces of dimension $d$ could the complex have?

Let $\Delta$ be a $(d-1)$-dimensional simplicial complex on the vertex set $[n]$. The f-vector of $\Delta$ is defined as a sequence of positive integers $f = (f_{-1}, f_0, f_1, \ldots, f_{d-1})$ with the property that $f_{-1} = 1$ and $f_i$ counts the faces of $\Delta$ of dimension equal to $i$ for $i \in \{0, 1, \ldots, d-1\}$. Then Kruskal-Katona Theorem claims that a sequence $f = (f_{-1}, f_0, f_1, \ldots, f_{d-1}) \in \mathbb{Z}^{d+1}$ is the f-vector of some $(d-1)$-dimensional simplicial complex if and only if $f_{-1} = 1$ and $0 < f_{i+1} \leq f_i^{(i+1)}$ for all $i$, where $f_i^{(i+1)}$ is determined by the so-called binomial or Macaulay expansion. More precisely, given two integers $a, d > 0$, let

$$a = \binom{a_d}{d} + \binom{a_{d-1}}{d-1} + \cdots + \binom{a_r}{r}$$

with $a_d > a_{d-1} > \ldots > a_r \geq r \geq 1$ be the unique binomial expansion of $a$ with respect to $d$. Then

$$a^{(d)} = \binom{a_d}{d+1} + \binom{a_{d-1}}{d} + \cdots + \binom{a_r}{r+1}.$$
An algebraic proof of Kruskal-Katona Theorem can be found in [3]. Let $\Delta$ be a $(d-1)$-dimensional simplicial complex on the vertex set $[n]$, $K$ be a field and $K\{\Delta\}$ be the exterior face ring of $\Delta$. Then $K\{\Delta\} = E/J_{\Delta}$, where $E$ is the exterior algebra of the $K$-vector space $V = \bigoplus_{i=1}^{n}Ke_i$ and $J_{\Delta} \subset E$ is the graded ideal generated by all exterior monomials $e_F = e_{i_1} \wedge \ldots \wedge e_{i_k}$ for which $F = \{i_1, \ldots, i_k\} \notin \Delta$. Since the Hilbert series of $K\{\Delta\}$ is $H_{K\{\Delta\}}(t) = \sum_{i=0}^{d} f_{i-1} t^i$, where $(f_{-1}, f_0, f_1, \ldots, f_{d-1})$ is the $f$-vector of $\Delta$, in order to get the conditions on the $f_i$’s, we need to obtain the possible Hilbert functions of graded algebras of the form $E/J$. The main step of the proof consists of showing that for each graded ideal $J \subset E$, there exists a unique square-free lexsegment ideal $J_{\text{sqlex}} \subset E$ such that $H_{E/J}(t) = H_{E/J_{\text{sqlex}}}(t)$. Therefore, it remains to understand the Hilbert series of a square-free lexsegment ideal.

In this paper, the key role is played by $t$-spread strongly stable ideals with $t \geq 1$. They have been recently introduced in [2] and they represent a special class of square-free monomial ideals. Let $t \geq 1$ be a positive integer, $K$ be a field and $S = K[x_1, \ldots, x_n]$ be the polynomial ring in $n$ variables over $K$.

A monomial $x_{i_1}x_{i_2} \cdots x_{i_d} \in S$ with $i_1 \leq i_2 \leq \cdots \leq i_d$ is called $t$-spread, if $i_j - i_{j-1} \geq t$ for $2 \leq j \leq n$ and a monomial ideal in $S$ is called a $t$-spread monomial ideal, if it is generated by $t$-spread monomials. Note that every square-free monomial is a 1-spread monomial and thus, every square-free monomial ideal is a 1-spread ideal.

Let $I$ be a $t$-spread ideal in $S$ and let $\Delta$ be its associated simplicial complex. It is natural to introduce the $f_t$-vector of $I$,

$$f_t(I) = (f_{t, -1}(I), f_{t, 0}(I), \ldots, f_{t, j}(I), \ldots),$$

where $f_{t, j}(I)$ is the cardinality of the set

$$\{ F : F \text{ is a } j \text{- dimensional face in } \Delta \text{ and } x_F = \prod_{i \in F} x_i \text{ is a } t \text{- spread monomial} \}.$$

Since $f_1(I)$ is the classical $f$-vector of $\Delta$, a natural question is the following:

**Are there any results for $f_t(I)$ which generalize the classification of the classical $f$-vectors for $t$-spread strongly stable ideals with $t \geq 1$?**

A complete answer for $t$-spread strongly stable ideals is given in the main result of this paper.

**Theorem 3.9.** Let $f = (f(0), f(1), \ldots, f(d), \ldots)$ be a sequence of positive integers and $t \geq 1$ be an integer. The following conditions are equivalent:

1. there exists an integer $n \geq 0$ and a $t$-spread strongly stable ideal $I \subset K[x_1, \ldots, x_n]$ such that $f(d) = f_{t, d-1}(I)$ for all $d$.
2. $f(0) = 1$ and $f(d+1) \leq f(d)[d]^t$ for all $d \geq 1$.

The $t$-operator $f(d) \rightarrow f(d)[d]^t$ is determined analogously to the operator which is involved in the proof of Kruskal-Katona Theorem; see Definition 3.5 and Remark 3.10.

The proof of this theorem follows the steps in the proof of Kruskal-Katona Theorem given in [3]. Thus, in Section 1, we define the $t$-spread lex ideal and we recall from [2] what a $t$-spread strongly stable ideal means. In Section 2, Theorem 2.1 shows that for any $t$-spread strongly stable ideal there exists a unique $t$-spread lex
ideals with the same \( f_t \)-vector. By Remark 2.3, a \( t \)-spread ideal may not have an associated \( t \)-spread lex ideal with the same \( f_t \)-vector. That is why we will restrict to the case when the ideal is \( t \)-spread strongly stable. Classifying all \( t \)-spread ideals which have an associated \( t \)-spread lex ideal with the same \( f_t \)-vector remains still open. In Section 3, we present a complete classification of the sequences of positive integers which are the \( f_t \)-vectors of some \( t \)-spread strongly stable ideals; see Theorem 3.9.

1. Preliminaries

Fix a field \( K \) and a polynomial ring \( S = K[x_1, \ldots, x_n] \). A monomial \( x_1^{i_1} x_2^{i_2} \cdots x_d^{i_d} \in S \) with \( i_1 \leq i_2 \leq \cdots \leq i_d \) is called \( t \)-spread, if \( i_j - i_{j-1} \geq t \) for \( 2 \leq j \leq n \). Note that any monomial is 0-spread, while the square-free monomials are 1-spread.

A monomial ideal in \( S \) is called a \( t \)-spread monomial ideal, if it is generated by \( t \)-spread monomials. For example, \( I = (x_1x_4x_8, x_2x_5x_8, x_1x_5x_9, x_2x_6x_9, x_4x_9) \subset K[x_1, \ldots, x_9] \) is a 3-spread monomial ideal, but not 4-spread, because \( x_2x_5x_8 \) is not a 4-spread monomial.

For an arbitrary monomial ideal \( I \), we denote by \( I_j \), the \( j \)-th graded component of \( I \) and call the set of \( t \)-spread monomials in \( I_j \), the \( t \)-spread part of \( I_j \) and denote it by \([I_j]_t\). Furthermore, we set

\[
f_{t,j-1}(I) = \|[S_t]_t| - [I_j]_t|.
\]

Then the vector \( f_t(I) = (f_{t,-1}(I), f_{t,0}(I), \ldots, f_{t,j}(I), \ldots) \) is called the \( f_t \)-vector of the \( t \)-spread monomial ideal \( I \). By convention, we set \( f_{t,-1} = 1 \). Note that if \( t = 1 \) then \( I \) is the Stanley-Reisner ideal of a uniquely determined simplicial complex \( \Delta \) and \( f_1(I) \) is the classical \( f \)-vector of \( \Delta \). According to [1, Theorem 5.1.7], the \( f_1 \)-vector of 1-spread monomial ideal \( I \subset S \) determines the Hilbert function of \( S/I \). This is not the case for \( t \geq 2 \). For example, \( I_1 = (x_1x_3, x_2x_4) \) and \( I_2 = (x_1x_3, x_1x_4) \) are 2-spread monomial ideals in \( K[x_1, \ldots, x_4] \) with \( f_2(I_1) = f_2(I_2) = (1, 4, 1, 0, 0, \ldots) \) and \( H(S/I_1, 3) = 12 < 13 = H(S/I_2, 3) \).

We denote by \( M_{n,d,t} \) the set of the \( t \)-spread monomials of degree \( d \) in the polynomial ring \( S \). For a monomial \( u \in S \), we set

\[
supp(u) = \{ i : x_i \mid u \} \text{ and } m(u) = \max\{ i : i \in supp(u) \}.
\]

**Definition 1.1.**
(a) A subset \( L \subset M_{n,d,t} \) is called a \( t \)-spread strongly stable set, if for all \( t \)-spread monomials \( u \in L \), all \( j \in supp(u) \) and all \( 1 \leq i < j \) such that \( x_i(u/x_j) \) is a \( t \)-spread monomial, it follows that \( x_i(u/x_j) \in L \).
(b) Let \( I \) be a \( t \)-spread monomial ideal. Then \( I \) is called a \( t \)-spread strongly stable ideal, if \([I_j]_t \) is a \( t \)-spread strongly stable set for all \( j \).

A special class of \( t \)-spread strongly stable ideals consists of \( t \)-spread lex ideals, which are defined as follows.

**Definition 1.2.**
(a) A subset \( L \subset M_{n,d,t} \) is called a \( t \)-spread lex set, if for all \( u \in L \) and for all \( v \in M_{n,d,t} \) with \( v \geq_{\text{Lex}} u \), it follows that \( v \in L \).
(b) Let \( I \) be a \( t \)-spread monomial ideal. Then \( I \) is called a \( t \)-spread lex ideal, if \([I_j]_t \) is a \( t \)-spread lex set for all \( j \).
Let $L \subset M_{n,d,t}$ be a $t$-spread lex set. Note that $L$ need not to be a $t$-spread lex set in $M_{m,d,t}$ for $m > n$. For example, $L = \{x_1x_2, x_1x_3, x_2x_3\}$ is a 1-lex set in $M_{3,2,1}$, but not in $M_{4,2,1}$. However, if $L$ is a $t$-spread strongly stable set in $M_{n,d,t}$, then $L$ remains a $t$-spread strongly stable set in $M_{m,d,t}$ for all $m > n$.

We notice that $\{I \subset S : I \text{ is a } t - \text{spread lex ideal}\} \subset \{I \subset S : I \text{ is a } t - \text{spread strongly stable ideal}\}$ and the inclusion is strict according to the following example.

**Example 1.3.** Let $I = (x_1x_3, x_1x_4, x_2x_4x_6, x_2x_4x_7) \subset \mathbb{K}[x_1, \ldots, x_7]$. Then $I$ is a 2-spread strongly stable ideal. Since $x_1x_5x_7 >_{\text{lex}} x_2x_4x_6$ and $x_1x_5x_7 \notin I$, the ideal $I$ is not a 2-spread lex ideal.

For every $L \subset M_{n,d,t}$ and for every $0 \leq \tau \leq t$, we define the $\tau$-shadow of $L$

$$\text{Shad}_\tau(L) = \{x_iv : v \in L, 1 \leq i \leq n \text{ and } x_iv \text{ is a } \tau\text{-spread monomial}\}$$

**Lemma 1.4.**

(a) Let $L \subset M_{n,d,t}$ be a $t$-spread strongly stable set.

Then $\text{Shad}_1(L) \subset M_{n,d+1,t}$ is also a $t$-spread strongly stable set.

(b) Let $L \subset M_{n,d,t}$ be a $t$-spread lex set.

Then $\text{Shad}_1(L) \subset M_{n,d+1,t}$ is also a $t$-spread lex set.

**Proof.** Let $u \in \text{Shad}_1(L)$. Then $u = wx_j$ for some $w \in L$. We may assume that $m(w) \leq j$. Otherwise, we consider $u = w'x_{m(w)}$, where $w' = x_j(w/x_{m(u)}) \in L$.

(a) Let $v = x_i(u/x_k) = x_i(wx_j)/x_k$ be a $t$-spread monomial such that $k \in \text{supp}(u)$ and $i < k$. Then we need to show that $v \in \text{Shad}_1(L)$. If $k = j$, then $v = wx_i \in \text{Shad}_1(L)$, by definition of $\text{Shad}_1(L)$. If $k \neq j$, then $x_iw/x_j \in L$ and again $v \in \text{Shad}_1(L)$.

(b) Let $v \in M_{n,d+1,t}$ with $v >_{\text{lex}} u$. Then we need to show that $v \in \text{Shad}_1(L)$.

Let $v = x_i1\cdots x_{id+1}$ with $i_1 \leq \ldots \leq i_{d+1}$ and $u = x_{k_1}\cdots x_{k_d}x_j$ with $k_1 \leq \ldots \leq k_d \leq j$. Since $v >_{\text{lex}} u$, $v' = x_i1\cdots x_{id} >_{\text{lex}} w = x_{k_1}\cdots x_{k_d}$. This shows that $v' \in L$ and therefore $v \in \text{Shad}_1(L)$ because $v = v'x_{id+1}$.

**Remark 1.5.** The assertions of the previous lemma do not remain true for every $0 \leq \tau < t$. Indeed, let $L = \{x_1x_3, x_1x_4, x_1x_5, x_2x_4\} \subset M_{5,2,2}$ be a 2-spread lex set.

Then $\text{Shad}_1(L) = \{x_1x_2x_3, x_1x_2x_4, x_1x_2x_5, x_1x_3x_4, x_1x_3x_5, x_1x_4x_5, x_2x_3x_4, x_2x_4x_5\}$ is not a 1-spread strongly stable set because $x_2x_3x_5 \notin \text{Shad}_1(L)$.

2. THE EXISTENCE OF $I^{t\text{-lex}}$

Let $I \subset S$ be a $t$-spread strongly stable monomial ideal. Then a $t$-spread lex ideal $J \subset S$ with $f_t(I) = f_t(J)$, if exists, is uniquely determined. We then denote this ideal $J$ by $I^{t\text{-lex}}$. The main purpose of this section is to prove

**Theorem 2.1.** For any $t$-spread strongly stable ideal $I$, the $t$-spread lex ideal $I^{t\text{-lex}}$ exists.

For the proof, we proceed in a similar way to the proof of the existence of $I^{\text{lex}}$ and $I^{s\text{lex}}$; see [3, Chapter 6].
For each graded component $I_j$ of $I$, let $I_j^{t\text{-llex}}$ be the $\mathbb{K}$-vector space spanned by $L_j \cup \text{Shad}_0(B_{j-1})$, where $L_j$ is the unique $t$-spread lex set with $|L_j| = ||I_j||_t$ and where $B_{j-1}$ is the set of the monomials of $I_j^{t\text{-llex}}$. If $j = 0$, then we consider $B_{j-1} = \emptyset$. We define $I^{t\text{-llex}} = \bigoplus_j I_j^{t\text{-llex}}$. This is the only possible candidate meeting the requirements of the theorem if and only if $\text{Shad}_t(L_j) \subset L_{j+1}$ for all $j$. Indeed, $I^{t\text{-llex}}$ is an ideal in $S$ because $B_j$ is a $\mathbb{K}$-basis for $I_j^{t\text{-llex}}$ and $\text{Shad}_0(B_{j-1}) \subset B_j = L_j \cup \text{Shad}_0(B_{j-1})$ for every $j \geq 0$.

Let $d$ be the smallest degree which appears in the set of the generators of $I$. We notice that $B_j = \emptyset = L_j$ for all $j < d$ and $B_d = L_d$. Then

$$\text{Shad}_t(B_j) \subset L_{j+1}$$

if and only if

$$\text{Shad}_t(L_j) \subset L_{j+1}$$

for all $j \leq d$. By using induction on $j$,

$$\text{Shad}_t(B_j) = \text{Shad}_t(L_j \cup \text{Shad}_0(B_{j-1}))$$

$$= \text{Shad}_t(L_j) \cup \text{Shad}_t(\text{Shad}_0(B_{j-1})) \subset L_{j+1} \cup \text{Shad}_t(\text{Shad}_t(B_{j-1})) = L_{j+1}$$

and

$$||I_{j+1}^{t\text{-llex}}||_t = |L_{j+1}| = ||I_{j+1}||_t$$

if and only if

$$\text{Shad}_t(L_j) \subset L_{j+1}.$$  

The proof of Theorem 2.1 is completed if we show that $|\text{Shad}_t(L_j)| \leq ||I_{j+1}||_t = |L_{j+1}|$, since $L_j$ is a $t$-spread lex set for every $j \geq 0$.

Let $L \subset M_{n,d,t}$ be a set of monomials. For every $i \in \{1+t(d-1), 2+t(d-1) \ldots, n\}$, we denote by $m_i(L)$ the number of elements $u \in L$ with $m(u) = i$ and we set $m_{\leq i}(L) = \sum_{j=1}^i m_j(L)$. Then we have the following result.

**Lemma 2.2.** Let $L \subset M_{n,d,t}$ be a $t$-spread strongly stable set. Then

(a) $m_i(\text{Shad}_t(L)) = m_{\leq i-t}(L)$ for all $i$ and

(b) $|\text{Shad}_t(L)| = \sum_{i=1}^{n-t} m_{\leq i}(L)$.

**Proof.** (b) is a consequence of (a). For the proof of (a), we note that the map

$$\varphi : \{u \in L : m(u) \leq i - t\} \rightarrow \{u \in \text{Shad}_t(L) : m(u) = i\}, u \mapsto ux_i$$

is a bijection. In fact, $\varphi$ is clearly injective. To see that $\varphi$ is surjective, let $v \in \text{Shad}_t(L)$ with $m(v) = i$. Since $v \in \text{Shad}_t(L)$, there exists $u \in L$ with $v = ux_j$ for some $j \leq i$. If $j = i$, then $m(u) < m(v)$ and $i - m(u) = m(v) - m(u) \geq t$ because $v$ is a $t$-spread monomial. In other words, $u \in \{u \in L : m(u) \leq i - t\}$ and $\varphi(u) = v$. Otherwise, $j < i$ and $i \in \text{supp}(u)$. Since $L$ is a strongly stable set and $v$ is a $t$-spread monomial, $w = x_j(u/x_i) \in L$. Thus, $v = wx_i$ and $\varphi(w) = v$. □

Now Theorem 2.1 will be a consequence of the following theorem.

**Theorem 2.3.** Let $L \subset M_{n,d,t}$ be a $t$-spread lex set and let $N \subset M_{n,d,t}$ be a $t$-spread strongly stable set with $|L| \leq |N|$. Then $m_{\leq i}(L) \leq m_{\leq i}(N)$. 

Proof. We first observe that $N = N_0 \cup N_1 x_n$ where $N_0$ is a $t$-spread strongly stable set of monomials of degree $d$ in the variables $x_1, x_2, \ldots, x_{n-1}$ and $N_1$ is a $t$-spread strongly stable set of monomials of degree $d-1$ in the variables $x_1, \ldots, x_{n-t}$. Similarly, one has the decomposition $L = L_0 \cup L_1 x_n$ where $L_0$ and $L_1$ are $t$-spread lex sets.

We prove the theorem by induction on the number of variables. For $n = 1$, the assertion is trivial. Now let $n > 1$. Since $|L| = m_{\leq n}(L)$ and $|N| = m_{\leq n}(N)$, we obtain $m_{\leq n}(L) \leq m_{\leq n}(N)$.

Note that $|L_0| = m_{\leq n-1}(L)$ and $|N_0| = m_{\leq n-1}(N)$. Thus in order to prove that $m_{\leq n-1}(L) \leq m_{\leq n-1}(N)$, we need to show that $|L_0| \leq |N_0|$. Moreover, if the inequality $|L_0| \leq |N_0|$ holds, then by applying the induction hypothesis we obtain

$$m_{\leq i}(L) = m_{\leq i}(L_0) \leq m_{\leq i}(N_0) = m_{\leq i}(N)$$

for every $i \in \{1, \ldots, n-1\}$.

Let $N_0^* \subset M_{n-1,d,t}$ be a $t$-spread lex set with $|N_0^*| = |N_0|$ and $N_1^* \subset M_{n-t,d-1,t}$ be a $t$-spread lex set with $|N_1^*| = |N_1|$. We claim that $N^* = N_0^* \cup N_1^* x_n$ is again a $t$-spread strongly stable set of monomials. Indeed, we need to show that $\text{Shad}_{t}(N_1^*) \subset N_0^*$. By Lemma 1.4, it is clear that $\text{Shad}_{t}(N_1^*)$ is a $t$-spread lex set. Thus, it suffices to prove that $|\text{Shad}_{t}(N_1^*)| \leq |N_0^*|$. Since $N$ is $t$-spread strongly stable set, $\text{Shad}_{t}(N_1) \subset N_0$. We apply Lemma 2.2 and our induction hypothesis and we obtain

$$|\text{Shad}_{t}(N_1^*)| = \sum_{i=1+(d-1)t}^{n-t} m_{\leq i}(N_1^*) \leq \sum_{i=1+(d-1)t}^{n-t} m_{\leq i}(N_1) = |\text{Shad}_{t}(N_1)| \leq |N_0| = |N_0^*|$$

Thus, we completed the proof of the fact $N^*$ is a $t$-spread strongly stable set of monomials.

Since $|N| = |N^*|$, we may replace $N$ by $N^*$ and then we can assume that $N_0$ is a $t$-spread lex set. We suppose that $n \neq (d-1)t + 1$. Otherwise, $M_{n,d,t} = \{x_1 x_{1+t} \cdots x_{1+(d-1)t}\}$ and the assertion is trivial.

Let $m = x_{j_1} \cdots x_{j_d}$ be a $t$-spread monomial and $\alpha : M_{n,d,t} \to M_{n,d,t}$ be a map defined as follows:

1. if $j_d \neq n$, then $\alpha(m) = m$.
2. if $j_d = n$ and there exists $r \in \{2, \ldots, d\}$ such that $j_r > j_{r-1} + t$, then we choose $r$ to be the largest integer with this property and define

$$\alpha(m) = x_{j_1} \cdots x_{j_{r-1}} x_{j_{r-1}+1} \cdots x_{j_{d-1}+1} x_n.$$

3. if $m = x_{n-(d-1)t} x_{n-(d-2)t} \cdots x_n$, then $\alpha(m) = x_{n-1-(d-1)t} x_{n-1-(d-2)t} \cdots x_{n-1}$.

Then $\alpha$ is a lexicographic order preserving map. Indeed, if we take $m_1 = x_{j_1} x_{j_2} \cdots x_{j_d}$ and $m_2 = x_{k_1} x_{k_2} \cdots x_{k_d}$ with $m_1 \leq_{\text{lex}} m_2$, then $\alpha(m_1) \leq_{\text{lex}} \alpha(m_2)$ in every case. For example, if $m_1 \in M_{n-1,d,t}$ and $m_2$ is as in case (2), then there exists $l \in \{1, \ldots, d\}$ such that $j_1 = k_1, \ldots, j_{l-1} = k_{l-1}$ and $j_l > k_l$ and for
(a) $r > l$, we have $\alpha(m_1) <_{lex} \alpha(m_2)$, since $j_1 = k_1, \ldots, j_{l-1} = k_{l-1}$ and $j_l > k_l$.
(b) $r < l$, we have $\alpha(m_1) <_{lex} \alpha(m_2)$, since $j_1 = k_1, \ldots, j_{r-1} = k_{r-1}$ and $j_r = k_r > k_r - 1$.
(c) $r = l$, we have $\alpha(m_1) <_{lex} \alpha(m_2)$, since $j_1 = k_1, \ldots, j_{l-1} = k_{l-1}$ and $j_l > k_l = k_l - 1$.

All the other cases can be treated in the same way.

For a set of monomials $S$ we denote by $\min S$ the lexicographically smallest element in $S$. Since both $L_0$ and $N_0$ are $t$-spread lex sets, the inequality $|L_0| \leq |N_0|$ will follow once we have shown that $\min L_0 \geq \min N_0$. Let $u = x_{k_1} \cdots x_{k_d} = \min L$ and $v = x_{j_1} \cdots x_{j_d} = \min N$. Then $\alpha(u) = \min L_0$ and $\alpha(v) = \min N_0$. Indeed, according to the three cases which define the map $\alpha$, we have:

1. $v \in N_0$ and $\alpha(v) = v \in N_0$.
2. $v \in N_1 x_n$ and $\alpha(v) = x_{j_1} \cdots x_{j_{k-1}} x_{n-1}$ with $k = \max \{r : j_r > j_r - 1\}$.
   - If $k = d$, then $\alpha(v) = x_{j_1} \cdots x_{j_{d-1}} x_{n-1} = (v/x_n) x_{n-1} \in N_0$ because $N$ is a $t$-spread strongly stable set.
   - If $k < d$, then we set $v_1 = x_{j_{k-1}} (v/x_{j_k}) \in N$, $v_2 = x_{j_{k+1}} (v_1/x_{j_{k+1}}) \in N$,
     $\ldots, v_{d-k} = x_{j_{k-1}} (v_{d-k-1}/x_{j_{d-1}}) \in N$ and $\alpha(v) = x_{n-1} (v_{d-k}/x_n) \in N_0$, since $N$ is a $t$-spread strongly stable set.
3. $v \in N_1 x_n$ and $\alpha(v) = x_{j_1} \cdots x_{j_{d-1}} x_{n-1}$ with $j_r = j_r - 1 + t$ for all $2 \leq r \leq d$.
   - Since $N$ is a $t$-spread strongly stable set, $v_1 = x_{j_{n-1}} (v/x_{j_{n-1}}) \in N$,
     $v_2 = x_{j_{n-1}} (v_{n-2}/x_{j_{n-1}}) \in N$,
     $\ldots, v_{d-1} = x_{j_{d-1}} (v_{d-1}/x_{j_{d-1}}) \in N$ and $\alpha(v) = x_{n-1} (v_{d-1}/x_n) \in N_0$.

Then $\min N_0 \leq \alpha(v)$ and $\min N_0 \geq v = \min N$. Thus, $\alpha(\min N_0) = \min N_0 \geq \alpha(v)$ and $\min N_0 = \alpha(v)$. Similarly, we obtain $\min L_0 = \alpha(u)$.

Finally we observe that $u \geq v$, since $L$ is a $t$-spread lex set and $|L| \leq |N|$. Hence we conclude that $\min L_0 = \alpha(u) \geq \alpha(v) = \min N_0$, as desired.

**Example 2.4.** Let

$I = (x_1 x_3 x_5, x_1 x_3 x_6, x_1 x_3 x_7, x_1 x_3 x_8, x_1 x_4 x_6, x_1 x_4 x_7, x_1 x_4 x_8, x_2 x_4 x_6, x_2 x_4 x_7, x_2 x_4 x_8) \subset \mathbb{K}[x_1, \ldots, x_8].$

Then $I$ is a 2-spread strongly stable ideal in $\mathbb{K}[x_1, \ldots, x_8]$ and

$I^{\text{lex}} = (x_1 x_3 x_5, x_1 x_3 x_6, x_1 x_3 x_7, x_1 x_3 x_8, x_1 x_4 x_6, x_1 x_4 x_7, x_1 x_4 x_8, x_1 x_5 x_7, x_1 x_5 x_8, x_1 x_6 x_8, x_2 x_4 x_6 x_8).$

Indeed, we have

$B_3 = L_3 = \{x_1 x_3 x_5, x_1 x_3 x_6, x_1 x_3 x_7, x_1 x_3 x_8, x_1 x_4 x_6, x_1 x_4 x_7, x_1 x_4 x_8, x_1 x_5 x_7, x_1 x_5 x_8, x_1 x_6 x_8\}$

$B_4 = L_4 \cup \operatorname{Shad}_0(B_3) = \{x_1 x_3 x_5 x_7, x_1 x_3 x_5 x_8, x_1 x_3 x_6 x_8, x_1 x_4 x_6 x_8, x_2 x_4 x_6 x_8\}$ and

$B_j = \operatorname{Shad}_0(B_{j-1})$ for all $j \geq 5,$
Theorem 3.2. [3, Theorem 6.4.5 (Kruskal-Katona)]

Let us recall the so-called binomial or Macaulay expansion of a positive integer.

Lemma 3.1. [3, Lemma 6.3.4]

Let us recall the so-called binomial or Macaulay expansion of a positive integer.

3. Possible $f$-vectors of a $t$-spread strongly stable ideal

In this section, we will give a complete answer to the following question: When is a given sequence of positive integers $f_t = (f_{t,-1}, f_{t,0}, f_{t,1}, \ldots, f_{t,d}, \ldots)$ the $f_t$-vector of a $t$-spread strongly stable ideal?

To answer this question, we would like to proceed like in the proof of Kruskal-Katona Theorem given in [3]. To this aim, we need to define a "$t$-operator" analog to the operator $a \rightarrow a^{(d)}$ which is involved in the proof of Kruskal-Katona Theorem. Let us recall the so-called binomial or Macaulay expansion of a positive integer.

Lemma 3.1. [3 Lemma 6.3.4] Let $d$ be a positive integer. Then each positive integer $a$ has a unique expansion

$$a = \binom{a_d}{d} + \binom{a_{d-1}}{d-1} + \cdots + \binom{a_r}{r}$$

with $a_d > a_{d-1} > \ldots > a_r \geq r \geq 1$. This expansion is called the binomial or Macaulay expansion of $a$ with respect to $d$.

Let $a = \binom{a_d}{d} + \binom{a_{d-1}}{d-1} + \cdots + \binom{a_r}{r}$ be the binomial expansion of $a$ with respect to $d$. Then one defines the binomial operator $a \rightarrow a^{(d)}$ by

$$a^{(d)} = \binom{a_d}{d+1} + \binom{a_{d-1}}{d} + \cdots + \binom{a_r}{r+1}.$$ 

For convenience, $0^{(d)} = 0$ for all positive integers $d$.

Theorem 3.2. [3, Theorem 6.4.5 (Kruskal-Katona)] Let $f = (f_{t,-1}, f_{t,0}, f_{t,1}, \ldots, f_{t,d-1})$ be a sequence of positive integers. Then the following conditions are equivalent:

1. There exists a simplicial complex $\Delta$ with the $f$-vector of $\Delta$, $f(\Delta) = f$;
2. $f_{t-1} = 1$ and $f_{j+1} \leq f_j^{(j+1)}$ for $0 \leq j \leq d - 2$. 

Lemma 3.3. Let the $t$-spread lex set with $(n,d,t)$-spread lex ideal may not have an associated $t$-spread lex ideal with the same $f_t$-vector. Therefore, we will restrict to $t$-spread strongly stable ideals.

As in the ordinary square-free case, the binomial expansions naturally appear in the context of the $t$-spread lex sets. Let $u = x_{i_1} \cdots x_{i_d} \in M_{n,d,t}$. We denote by $L_u$ the $t$-spread lex set $\{v \in M_{n,d,t} : v \geq_{\text{lex}} u\}$.

**Lemma 3.3.** Let $u = x_{i_1} \cdots x_{i_d} \in M_{n,d,t}$. Then

$$M_{n,d,t} \setminus L_u = \bigcup_{k=1}^{d} \{v = x_{j_k}x_{j_{k+1}} \cdots x_{j_d} : v \in M_{n,d-k+1,t}, j_k > i_k \} \prod_{l=1}^{k-1} x_{i_l}.$$ 

This union is disjoint and in particular, we obtain

$$|M_{n,d,t} \setminus L_u| = \sum_{j=1}^{d} \binom{a_j}{j}$$

with $a_j = n - i_{d-j+1} - (j - 1)(t - 1)$ for every $j \in \{1, \ldots, d\}$.

**Proof.** We will prove the equality between sets by induction on $d$. If $d = 1$, then $M_{n,1,t} = \{x_{i_1+1}, x_{i_1+2}, \ldots, x_n\}$, as desired.

If $d > 1$, then

$$M_{n,d,t} \setminus L_u = \{v = x_{j_1} \cdots x_{j_d} : v \in M_{n,d,t} \text{ and } j_1 > i_1\} \cup (M_{n,d-1,t} \setminus L_{ux_{i_1}})x_{i_1}$$

and we may apply the induction hypothesis for $M_{n,d-1,t} \setminus L_{ux_{i_1}}^{-1}$.

In order to compute the cardinality of $M_{n,d,t} \setminus L_u$, we recall that $|M_{n,d,t}| = \binom{n-(d-1)(t-1)}{d}$ by [2, Theorem 2.3]. Thus,

$$|M_{n,d,t} \setminus L_u| = \sum_{k=1}^{d} |M_{n-i_k,d-k+1,t}| = \sum_{k=1}^{d} \left( n - i_k - (d - k)(t - 1) \right) = \sum_{j=1}^{d} \left( n - i_{d-j+1} - (j - 1)(t - 1) \right) = \sum_{j=1}^{d} \binom{a_j}{j}.$$ 

\[\square\]

**Remark 3.4.** In the previous lemma, let $r \geq 1$ be the smallest integer for which $a_r > r - 1$. Then the expansion $|M_{n,d,t} \setminus L_u| = \sum_{j=r}^{d} \binom{a_j}{j}$ is the binomial expansion of $|M_{n,d,t} \setminus L_u|$ with respect to $d$. Indeed, we have

$$a_j - a_{j-1} = n - i_{d-j+1} - (j - 1)(t - 1) - n + i_{d-j+1} + (j - 2)(t - 1) = i_{d-j+2} - i_{d-j+1} - t + 1 \geq t - t + 1 = 1$$

and

$$a_j = n - i_{d-j+1} - (j - 1)(t - 1) \geq (j - 1)(t - 1) \geq i_d - i_{d-j+1} - (j - 1)t + j - 1 \geq j - 1$$

for every $j$. Moreover, $\binom{a_j}{j} = 0$ for $j < r$. 

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Definition 3.5. Let \( n, d, t \) and \( a \) be positive integers with \( a \leq \binom{n-(d-1)(t-1)}{d} \). If \( a = \binom{a}{d} + \binom{a_{d-1}}{d} + \cdots + \binom{a}{r} \) is the binomial expansion of \( a \) with respect to \( d \), then we set \( a_{r-1} = r - 2, a_{d+1} = n - (d-1)(t-1) \) and \( a_{d+2} = a_{d+1} + (t+1) \) and we define
\[
a^{[d]t} := a^{[d]t},
\]
where \( k \) is the largest integer of the interval \([-1, d - r + 1] \) with the property that \( a_{d-k+1} - a_{d-k} \geq t + 1 \) and
\[
a^{[d]t}_k := \sum_{j=d+1-k}^{d} \binom{a_j - (t-1)}{j+1} + \binom{a_{d-k} - (2t-1)}{d-k+1} + \sum_{j=r}^{d-k} \binom{a_j}{j}
\]
for all \( k \geq 0 \) and
\[
a^{[d]t-1} := \binom{n-d(t-1)}{d+1}.
\]

Again for convenience, we set \( 0^{[d]t} = 0 \) for positive integers \( d \) and \( t \).

Example 3.6. We consider \( n = 28, d = 8, t = 3 \) and \( a = 2018 \) and we notice that \( a = 2018 \leq 3003 = \binom{14}{8} = \binom{n-(d-1)(t-1)}{d} \). Then the binomial expansion of \( a \) with respect to \( d \) is
\[
2018 = \binom{13}{8} + \binom{11}{7} + \binom{10}{6} + \binom{9}{5} + \binom{7}{4} + \binom{6}{3} + \binom{5}{2}.
\]
So, we obtain \( a_1 = 0, a_2 = 5, a_3 = 6, a_4 = 7, a_5 = 9, a_6 = 10, a_7 = 11, a_8 = 13, a_9 = 14 \) and \( a_{10} = 18 \). Since \( a_2 - a_1 = 5 \geq t + 1 = 4 \), the largest integer of the interval \([-1, 7] = [-1, 8 - 2 + 1] \), with the property that \( a_{d-k+1} - a_{d-k} \geq t + 1 \), is \( k = 7 \). Thus, we have
\[
a^{[d]t} = 2018^{[8]3} = \sum_{j=2}^{8} \binom{a_j - 2}{j+1} + \binom{a_1 - 5}{2} = \binom{11}{9} + \binom{9}{8} + \binom{8}{7} + \binom{7}{6} + \binom{6}{5} + \binom{5}{4} + \binom{3}{3} = 82.
\]

The definition of the binomial operator \( a \to a^{[d]t} \) is justified by the following result.

Proposition 3.7. Let \( \emptyset \neq L \subset M_{n,d,t} \) be a \( t \)-spread lex set with \( a = |M_{n,d,t} \setminus L| \). Then
\[
|M_{n,d+1,t} \setminus 
\text{Shad}_t(L)| = a^{[d]t}.
\]

Proof. To begin with, we notice that \( a < \binom{n-(d-1)(t-1)}{d} \), since \( L \neq \emptyset \) and \( |M_{n,d,t}| = \binom{n-(d-1)(t-1)}{d} \) by [2] Theorem 2.3.

Let \( a = \binom{a}{d} + \binom{a_{d-1}}{d} + \cdots + \binom{a}{r} \) be the binomial expansion of \( a \) with respect to \( d \). Since \( a < \binom{n-(d-1)(t-1)}{d} \), we have \( a_d < n - (d-1)(t-1) \). Thus, we obtain \( n - a_d - (d-1)(t-1) \geq 1 \). Let us consider the \( t \)-spread monomial \( u = x_{i_1} \cdots x_{i_d} \) where
(a) \( i_1 := n - a_d - (d - 1)(t - 1) \), \( i_2 := n - a_{d-1} - (d - 2)(t - 1) \), \ldots , \( i_k := n - a_{d-k+1} - (d - k)(t - 1) \), \ldots , \( i_{d-1} := n - a_2 - (t - 1) \), \( i_d := n - a_1 \), if \( r = 1 \);
(b) \( i_1 := n - a_d - (d - 1)(t - 1) \), \( i_2 := n - a_{d-1} - (d - 2)(t - 1) \), \ldots , \( i_{d-r+1} := n - a_r - (r - 1)(t - 1) \), \( i_{d-r+2} := n - (r - 2)t \), \ldots , \( i_d = n \), if \( r > 1 \).

By Lemma 3.3, \([M_{n,d,t} \setminus L_u] = a = [M_{n,d,t} \setminus L] \) and \( L = L_u \), since \( L \) and \( L_u \) are \( t \)-spread lex sets.

According to Lemma 1.4, \( \text{Shad}_t(L) \) is a \( t \)-spread lex set and with Lemma 3.3 we obtain the following four cases:

1. If \( r = 1 \) and \( a_1 \geq t \), then \( n - i_d \geq t \). This follows that \( \text{Shad}_t(L) = L_{ux_n} \) and
   \[
   |M_{n,d+1,t} \setminus \text{Shad}_t(L)| = |M_{n,d+1,t} \setminus L_{ux_n}| =
   \sum_{j=2}^{d+1} \left( n - i_{d+1-j+1} - (j - 1)(t - 1) \right) = \sum_{j=2}^{d+1} \left( a_{j-1} - (t - 1) \right)
   = \sum_{j=1}^{d} \left( a_j - (t - 1) \right) = n - \left( \sum_{j=1}^{d} a_j \right) = a_{[d]}^d.
   \]
   Since \( a_1 - a_0 = a_1 + 1 \geq t + 1 \) and \( k = d \) is the largest integer of the interval \([-1, d] \) with the property that \( a_{d-k+1} - a_{d-k} \geq t + 1 \), we have \( |M_{n,d+1,t} \setminus \text{Shad}_t(L)| = a_{[d]}^d \), as desired.

2. If \( r \neq 1 \) or \( r = 1 \) and \( a_1 < t \)) and there exists \( k \in \{1, \ldots , d - r + 1\} \) such that \( a_{d-k+1} - a_{d-k} \geq t + 1 \), then we choose the largest \( k \in \{-1, d - r + 1\} \) with this property and we have \( n - i_d < t \) and \( i_{k+1} - i_k \geq 2t \). Thus, we obtain \( \text{Shad}_t(L) = L_{ux_{ik+1-t}} \) and
   \[
   |M_{n,d+1,t} \setminus \text{Shad}_t(L)| = |M_{n,d+1,t} \setminus L_{ux_{ik+1-t}}| =
   \sum_{j=1}^{d-k} \left( n - i_{d-j} - (t - 1)(j - 1) \right) + \left( n - i_{k+1} - t - (t - 1)(d - k) \right)
   + \sum_{j=d+2-k}^{d+1} \left( n - i_{d+1-j+1} - (j - 1)(t - 1) \right) = \sum_{j=r}^{d-k} \left( \sum_{j=d+2-k}^{d+1} \left( a_j \right) \right)
   = \left( a_{d-k} - (2t - 1) \right) + \sum_{j=d-k+1}^{d} \left( a_j - (t - 1) \right) = a_{[d]}^d = a_{[d]}^d.
   \]

3. If \( r \neq 1 \) or \( r = 1 \) and \( a_1 < t \)), \( a_{d-k+1} - a_{d-k} < t + 1 \) for all \( k \in \{1, \ldots , d - r + 1\} \) and \( a_d \leq n - d(t - 1) - 2 \), then \( n - i_d < t \), \( i_{k+1} - i_k < 2t \) for all \( k \in \{1, \ldots , d - 1\} \) and \( i_1 - t \geq 1 \). This implies that \( \text{Shad}_t(L) = L_{ux_{i_1-t}} \) and
   \[
   |M_{n,d+1,t} \setminus \text{Shad}_t(L)| = |M_{n,d+1,t} \setminus L_{ux_{i_1-t}}| =
   \sum_{j=1}^{d} \left( n - i_{d-j+1} - (j - 1)(t - 1) \right) + \left( n - i_1 - t - (t - 1)d \right)
   = n - \left( \sum_{j=1}^{d} a_j \right) = n.
   \]
= \sum_{j=r}^{d} \left( \frac{a_j}{j} \right) + \left( \frac{a_d - (2t - 1)}{d + 1} \right) = a_t^{[d]_t}.

Since \( a_{d-k+1} - a_{d-k} < t + 1 \) for all \( k \in \{1, \ldots, d - r + 1\} \) and \( a_{d+1} - a_d = n - (d - 1)(t - 1) - a_d \geq n - (d - 1)(t - 1) - n - d(t - 1) - 2 = t + 1 \), we obtain \(|M_{n,d+1,t} \setminus \text{Shad}_t(I)| = a_t^{[d]_t} = a_t^{[d]_t} \), as desired.

(4) If \((r \neq 1 \text{ or } r = 1 \text{ and } a_1 < t)\), \( a_{d-k+1} - a_{d-k} < t + 1 \) for all \( k \in \{1, \ldots, d - r + 1\} \) and \( a_d > n - d(t - 1) - 2 \), then \( n - i_d < t \), \( i_k + 1 - i_k < 2t \) for all \( k \in \{1, \ldots, d - 1\} \) and \( i_1 - t < 1 \). Therefore, \( \text{Shad}_t(I) = \emptyset \) and

\(|M_{n,d+1,t} \setminus \text{Shad}_t(I)| = |M_{n,d+1,t}| = a_t^{[d]_t} = a_t^{[d]_t}

because \( a_{d-k+1} - a_{d-k} < t + 1 \) for all \( k \in \{1, \ldots, d - r + 1\} \), \( a_{d+1} - a_d \leq t \) and \( a_{d+2} = a_{d+1} = t + 1 \).

\[ \square \]

**Remark 3.8.** If \( L = \emptyset \subset M_{n,d,t} \), then \( a = |M_{n,d,t} \setminus L| = |M_{n,d,t}| = \binom{n-(d-1)(t-1)}{d} \) and \( \text{Shad}_t(L) = \emptyset \). Moreover, we have \( a_{d-1} = d - 2 \), \( a_d = a_{d+1} = n - (d - 1)(t - 1) \), \( a_{d+2} = a_{d+1} + (t + 1) \) and \( a_t^{[d]_t} = a_t^{[d]_t} \), where \( k \) is the largest integer of the set \( \{-1, 0, 1\} \) with the property that \( a_{d-k+1} - a_{d-k} \geq t + 1 \). Since \( a_{d+1} - a_d = 0 \), \( k \in \{-1, 1\} \) and

1. \( a_t^{[d]_t} = \binom{n-(d-1)}{d} \) if \( k = -1 \);
2. \( a_t^{[d]_t} = \binom{n-(d+1)}{d+1} + \binom{a_{d-1}-(2t-1)}{d} = \binom{n-(d-1)}{d} \) if \( k = 1 \).

We notice that we get the same number in both cases and thus, \( a_t^{[d]_t} = \binom{n-(d-1)}{d+1} = |M_{n,d+1,t}| = |M_{n,d+1,t} \setminus \text{Shad}_t(L)| \).

We now give the main result of this paper.

**Theorem 3.9.** Let \( f = (f(0), f(1), \ldots, f(d), \ldots) \) be a sequence of positive integers and \( t \geq 1 \) be an integer. The following conditions are equivalent:

1. there exists an integer \( n \geq 0 \) and a \( t \)-spread strongly stable ideal
   \[ I \subset \mathbb{K}[x_1, \ldots, x_n] \]
   such that \( f(d) = f_{t,d-1}(I) \) for all \( d \).
2. \( f(0) = 1 \) and \( f(d+1) \leq f(d)^{[d]_t} \) for all \( d \geq 1 \).

**Proof.** According to Theorem 2.1 we may reduce to \( t \)-spread lex ideals instead of \( t \)-spread strongly stable ideals.

For (1) \( \Rightarrow \) (2), let \( I \subset \mathbb{K}[x_1, \ldots, x_n] \) be a \( t \)-spread lex ideal with \( f_{t,d-1}(I) = |M_{n,d,t} \setminus [I_d]_t| = f(d) \leq |M_{n,d,t}| = \binom{n-(d-1)(t-1)}{d} \), where \([I_d]_t\) is the \( t \)-spread part of the \( d \)-th graded component of \( I \). Since \([I_d]_t\) is a \( t \)-spread lex set in \( M_{n,d,t} \) and \( \text{Shad}_t([I_d]_t) \subset [I_{d+1}]_t \), it follows from Proposition 3.7 that

\( f(d+1) = |M_{n,d+1,t} \setminus [I_{d+1}]_t| \leq |M_{n,d+1,t} \setminus \text{Shad}_t([I_d]_t)| = f(d)^{[d]_t} \)

and of course we have \( f(0) = f_{t,-1}(I) = 1 \).
For (2) ⇒ (1), let \( n = f(1) \) and set \( S = \mathbb{K}[x_1, \ldots, x_n] \). We first show by induction on \( d \) that

\[
f(d) \leq \binom{n - (d - 1)(t - 1)}{d}.
\]

The assertion is trivial for \( d = 1 \). Now assume that \( f(d) \leq \binom{n - (d - 1)(t - 1)}{d} \) for some \( d \geq 1 \). Since \( f(d + 1) \leq f(d)\frac{d!}{|d|!} \), it remains to prove that \( f(d)\frac{d!}{|d|!} \leq \binom{n - d(t - 1)}{d} \).

Let \( f(d) = \binom{a_d}{d} + \binom{a_{d-1}}{d-1} + \cdots + \binom{a_0}{1} \) be the binomial expansion of \( f(d) \) with respect to \( d \). Then \( a_d \leq n - (d - 1)(t - 1) \). By definition, \( f(d)\frac{d!}{|d|!} = f(d)\frac{d!}{d!} \), where \( k \) is the largest integer of the interval \([-1, d + r - 1]\) with the property that \( a_{d-k+1} - a_{d-k} \geq t + 1 \).

If \( k = -1 \), then \( f(d)\frac{d!}{|d|!} = \binom{n - d(t - 1)}{d+1} \), as desired.

If \( k = 0 \), then \( a_{d+1} - a_d \geq t + 1 \) and \( a_d \leq n - d(t - 1) - 2 \). Thus,

\[
f(d)\frac{d!}{|d|!} = f(d)\frac{d!}{d!} = d \sum_{r=0}^{d} \binom{a_j}{j} + \binom{a_d - (2t - 1)}{d + 1} =
\]

\[
= f(d) + \binom{a_d - (2t - 1)}{d + 1} < \binom{a_d + 1}{d} + \binom{a_d + 1 - 2t}{d + 1} <
\]

\[
< \binom{a_d + 1}{d} + \binom{a_d + 1}{d + 1} = \binom{a_d + 2}{d + 1} \leq \binom{n - d(t - 1)}{d + 1}.
\]

If \( k \geq 1 \), then \( a_{d-k+1} - a_{d-k} \geq t + 1 \) and \( a_{d-k} + 2 \leq a_{d-k+1} - (t - 1) \). Thus,

\[
f(d)\frac{d!}{|d|!} = f(d)\frac{d!}{d!} = \sum_{j=d+1-k}^{d} \binom{a_j - (t - 1)}{j + 1} + \binom{a_{d-k} - (2t - 1)}{d - k + 1} + \sum_{j=k}^{d} \binom{a_j}{j} =
\]

\[
= \sum_{j=d+1-k}^{d} \binom{a_j - (t - 1)}{j + 1} + \binom{a_{d-k} - (2t - 1)}{d - k + 1} + a - \sum_{j=d-k+1}^{d} \binom{a_j}{j} <
\]

\[
< \sum_{j=d+1-k}^{d} \binom{a_j - (t - 1)}{j + 1} + \binom{a_{d-k} + 1}{d - k + 1} + \binom{a_{d-k} + 1}{d - k} =
\]

\[
= \sum_{j=d+1-k}^{d} \binom{a_j - (t - 1)}{j + 1} + \binom{a_{d-k} + 2}{d - k + 1} \leq
\]

\[
\leq \sum_{j=d+2-k}^{d} \binom{a_j - (t - 1)}{j + 1} + \binom{a_{d-k+1} - (t - 1)}{d - k + 2} + \binom{a_{d-k+1} - (t - 1) + 1}{d - k + 1} =
\]

\[
= \sum_{j=d+3-k}^{d} \binom{a_j - (t - 1)}{j + 1} + \binom{a_{d-k+2} - (t - 1)}{d - k + 3} + \binom{a_{d-k+1} - (t - 1) + 1}{d - k + 2} \leq \cdots \leq
\]

\[
\leq \binom{a_d - (t - 1)}{d + 1} + \binom{a_{d-1} - (t - 1) + 1}{d} \leq \binom{a_d - (t - 1) + 1}{d + 1}.
\]
In the case that \( f(d) < (n-(d-1)(t-1)) \), it follows by induction hypothesis that
\[
f(d)^{[d]}_t \leq \binom{a_d - (t - 1) + 1}{d + 1} \leq \binom{n - (d - 1)(t - 1) - (t - 1)}{d + 1} = \binom{n - d(t - 1)}{d + 1}.
\]

If \( f(d) = (n-(d-1)(t-1)) \), then
\[
f(d)^{[d]}_t = \binom{a_d - (t - 1)}{d + 1} + \binom{a_{d-1} - (2t - 1)}{d}
= \binom{n - (t - 1)(d - 1) - (t - 1)}{d + 1} + \binom{d - 2 - (2t - 1)}{d} = \binom{n - d(t - 1)}{d + 1}.
\]

Hence we have \( f(d) \leq (n-(d-1)(t-1)) = |M_{n,d,t}| \) for all \( d \). In other words, \( |M_{n,d,t}| - f(d) \geq 0 \). Let \( L_d \) be the unique \( t \)-spread lex set with \( |L_d| = |M_{n,d,t}| - f(d) \).

The existence of the \( t \)-spread lex ideal \( I \) with \( f_{t,d-1}(I) = f(d) = |M_{n,d,t}| - |L_d| \) is acquired in a similar way to the proof of the existence of \( I_{t-\text{lex}} \) in Theorem 2.1.

Thus, it remains to show that \( \text{Shad}_t(L_d) \subseteq L_{d+1} \) for all \( d \).

Since \( f(d + 1) \leq f(d)^{[d]}_t \),
\[
|M_{n,d+1,t} \setminus L_{d+1}| = f(d + 1) \leq f(d)^{[d]}_t = |M_{n,d,t} \setminus L_d|^{[d]}_t = |M_{n,d+1,t} \setminus \text{Shad}_t(L_d)|.
\]

It follows that \( M_{n,d+1,t} \setminus L_{d+1} \subseteq M_{n,d+1,t} \setminus \text{Shad}_t(L_d) \) because \( L_d \) is a \( t \)-spread lex set for every \( d \). Thus, \( \text{Shad}_t(L_d) \subseteq L_{d+1} \) for every \( d \), as desired. \( \square \)

**Remark 3.10.** For \( t = 1 \) in Theorem [3.9] we obtain Theorem 3.2. Indeed, let \( a \) and \( d \in \mathbb{Z} \). If \( a = \binom{a_d}{d} + \binom{a_{d-1}}{d-1} + \cdots + \binom{a_r}{r} \) is the binomial expansion of \( a \) with respect to \( d \), then \( a_d > a_{d-1} > \ldots > a_r \geq r \geq 1 \). We set \( a_{r-1} = r - 2 \). Since \( a_{d-(d-r+1)} + a_{d-(d-r+1)} = a_r - a_{r-1} = a_r - r + 2 \geq 2 = t + 1 \),
\[
a^{[d]}_1 = a^{[d]}_1^{d-r+1} = \sum_{j=r}^{d} \binom{a_j}{j+1} + \binom{a_{r-1} - 1}{r} = \sum_{j=r}^{d} \binom{a_j}{j+1} = a^{(d)}.
\]

**Example 3.11.** Let \( f = (1, 2, 50, 20, 15, 0, 0, \ldots) \). If \( t = 1 \), then \( f \) satisfies the second condition of Theorem 3.9, because we have
\[
f(1)^{[1]}_1 = 12^{(1)} = \binom{12}{1+1} = 66 \geq f(2) = 50,
\]
\[
f(2)^{[2]}_1 = 50^{(2)} = \binom{10}{2+1} + \binom{5}{1+1} = 130 \geq f(3) = 20,
\]
\[
f(3)^{[3]}_1 = 20^{(3)} = \binom{6}{3+1} = 15 \geq f(4) = 15 \text{ and}
\]
\[
f(4)^{[4]}_1 = 15^{(4)} = \binom{6}{4+1} = 6 \geq 0 = f(5).
\]
We notice that \( f(2) = 50 \leq f(1)^{[1]}_t = \binom{f(1) - (t-1)}{2} = \binom{13-t}{2} \) if and only if \( t = 2 \). Thus, \( f \) does not satisfy the second condition of the previous theorem while \( t > 2 \).

If \( t = 2 \), then \( f(3) = 20 \leq \binom{12-(3-1)}{3} = 120, f(3)^{[3]}_2 = f(3)^{[3]}_1 = \binom{6-(2-1)}{3+1} < \binom{6}{3} = 15 = f(4) \) and again we can not obtain a 2-spread strongly stable ideal with the \( f_2 \)-vector given by \( f \).

In conclusion, \( f = (1, 12, 50, 20, 15, 0, 0, \ldots) \) is the \( f_t \)-vector of a \( t \)-spread strongly stable ideal if and only if \( t = 1 \).

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