HOMOLOGICAL INVARIANTS OF GENERALIZED BOUND PATH ALGEBRAS

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ABSTRACT. We study some homological invariants of a given generalized bound path algebra in terms of those of the algebras used in its construction. We discuss the particular case where the algebra is a generalized path algebra and give conditions for those algebras to be shod or quasitilted.

1. INTRODUCTION

An important result in the representation theory of algebras states that a finite dimensional basic $k$-algebra $A$, where $k$ is an algebraically closed field, is isomorphic to a quotient of a path algebra $kQ_A/I_A$ where $Q_A$ is a finite quiver and $I_A$ is an admissible ideal (see below for details). This allows us to describe the finitely generated $A$-modules in terms of the representations of the corresponding quiver $Q_A$, a relation which proves to be of essential help in the theory.

In order to generalize such a construction, Coelho-Liu introduced in [6] the notion of generalized path algebras (gp-algebras for short). Instead of assigning the field $k$ to each vertex $i$ of the quiver $Q$ as in the traditional construction of path algebras $kQ$, it is assigned a finite dimensional $k$-algebra. This was further generalized by us in the article [3] where we consider also some quotients of the gp-algebras. Specifically, let $\Gamma$ denote a quiver and $\mathcal{A} = \{A_i : i \in \Gamma_0\}$ denote a family of basic finite dimensional $k$-algebras indexed by the set $\Gamma_0$ of the vertices of $\Gamma$. Consider also a set of relations $I$ on the paths of $\Gamma$ which generates an admissible ideal of $k\Gamma$. To such a data we have considered (see [3]) the generalized bound path algebra $\Lambda = k(\Gamma, \mathcal{A}, I)$ (gbp-algebra for short) with a natural multiplication given not only by the concatenation of paths of the quiver but also by the multiplication of the algebras associated with the vertices of $\Gamma$, modulo the relations in $I$ (see below for details).

The idea behind such a construction is to compare properties of the algebras in $\mathcal{A}$ and those for the algebra $\Lambda$. In the seminal work [6], the focus was more ring theoretical, and, as mentioned, the authors only considered the case where $I = 0$. We mention, for instance, [7, 9, 10], where such a particular case was also studied.

In [3, 4], we have studied the case where $I$ is not necessarily zero, thus extending the description of the representations of the algebra $\Lambda$ given in [9]. Clearly, a path algebra $A$ can be realized as a generalized one in two trivial ways: the usual description as path...
algebra through its ordinary quiver $Q_A$ but also by considering a quiver with a single vertex and no arrows and assigning to it the whole algebra $A$. In [3], we discuss when there are, besides the above two, other possibilities of realizing a path algebra as a generalized one. This is important because we can relate properties of a gp-algebra with those of the smaller algebras which appear in its definition. In [4], we studied the correspondence between modules over a gbp-algebra and representations of the corresponding quiver.

Here, our focus will be, using the description of the projective and injective modules from [4], to study some homological invariants of a given gbp-algebra in terms of those of the algebras used in its construction. This is done in Sections 3 and 4 after devoting Section 2 to preliminary concepts needed along the paper. The particular case of gp-algebras is discussed in Section 5 where we prove, for instance, that the global dimension of a gp-algebra is the maximum between one and the global dimension of the algebras assigned to each vertex (Theorem 5.1). Also, we provide a sufficient condition for a gp-algebra to belong to classes of algebras which can be defined using these invariants, such as $shod$ or quasitilted algebras (see [5, 8]).

2. Preliminaries

Along this paper, $k$ will denote an algebraically closed field. For an algebra, we mean an associative and unitary basic finite dimensional $k$-algebra. Also, given an algebra $A$, an $A$-module (or just a module) will be a finitely generated right module over $A$. We refer to [1, 2] for unexplained details on Representation Theory.

2.1. Path algebras. A quiver $Q$ is given by a tuple $(Q_0, Q_1, s, e)$, where $Q_0$ is the set of vertices, $Q_1$ is the set of arrows and $s, e : Q_1 \to Q_0$ are maps which indicate, for each arrow $\alpha \in Q_1$, the starting vertex $s(\alpha) \in Q_0$ of $\alpha$ and the ending vertex $e(\alpha) \in Q_0$ of $\alpha$. A vertex $i \in Q_0$ is called a source (respectively a sink) provided there are no arrows ending (or starting, respectively) at $i$. A path in $Q$ of length $n \geq 1$ is given by $\alpha_1 \cdots \alpha_n$, where for each $i = 1, \cdots, n - 1$, $e(\alpha_i) = s(\alpha_{i+1})$. There are also paths of length zero which are in one-to-one correspondence to the vertices of $Q$.

We shall assume that all quivers are finite, that is, both sets $Q_0$ and $Q_1$ are finite.

Naturally, given a quiver $Q$ one can assign a path algebra $kQ$ with a $k$-basis given by all paths of $Q$ and multiplication on that basis defined by concatenation. Even when $Q$ is finite, the corresponding algebra could not be finite dimensional. However, a well-known result established by Gabriel states that given an algebra $A$, there exists a finite quiver $Q$ and a set of relations on the paths of $Q$ which generates an admissible ideal $I$ such that $A \cong kQ/I$ (see [1] for details).

2.2. Generalized bound path algebras (gbp-algebras). Let $\Gamma = (\Gamma_0, \Gamma_1, s, e)$ be a quiver and $\mathcal{A} = (A_i)_{i \in \Gamma_0}$ be a family of algebras indexed by $\Gamma_0$. An $\mathcal{A}$-path of length $n$ over $\Gamma$ is defined as follows: for $n = 0$, such a path is an element of $\bigcup_{i \in \Gamma_0} A_i$, and for
\( n > 0 \), it is a sequence of the form

\[ a_1 \beta_1 a_2 \ldots a_n \beta_n a_{n+1} \]

where \( \beta_1 \ldots \beta_n \) is an ordinary path in the quiver \( \Gamma \), \( a_i \in A_{s(\beta_i)} \) if \( i \leq n \), and \( a_{n+1} \in A_{e(\beta_n)} \).

Denote by \( k[\Gamma, \mathcal{A}] \) the \( k \)-vector space spanned by all \( \mathcal{A} \)-paths over \( \Gamma \).

Then we consider the quotient vector space \( k(\Gamma, \mathcal{A}) = k[\Gamma, \mathcal{A}] / V \), where \( V \) is the subspace generated by all elements of the form

\[ (a_1 \beta_1 \ldots \beta_{j-1} (a_j^1 + \ldots + a_j^m) \beta_j a_{j+1} \ldots \beta_n a_{n+1}) - \sum_{l=1}^{m} (a_1 \beta_1 \ldots \beta_{j-1} a_j^l \beta_j \ldots \beta_n a_{n+1}) \]

or, for \( \lambda \in k \),

\[ (a_1 \beta_1 \ldots \beta_{j-1} (\lambda a_j) \beta_j a_{j+1} \ldots \beta_n a_{n+1}) - \lambda \cdot (a_1 \beta_1 \ldots \beta_{j-1} a_j \beta_j a_{j+1} \ldots \beta_n a_{n+1}) \]

The space \( k(\Gamma, \mathcal{A}) \) has a naturally defined multiplication, induced by the multiplications of the algebras \( A_i \)'s and the composition of the \( \mathcal{A} \)-paths. More explicitly, it is defined by linearity and the following rule:

\[ (a_1 \beta_1 \ldots \beta_n a_{n+1}) (b_1 \gamma_1 \ldots \gamma_m b_{m+1}) = a_1 \beta_1 \ldots \beta_n (a_{n+1} b_1) \gamma_1 \ldots \gamma_m b_{m+1} \]

if \( e(\beta_n) = s(\gamma_1) \), and

\[ (a_1 \beta_1 \ldots \beta_n a_{n+1}) (b_1 \gamma_1 \ldots \gamma_m b_{m+1}) = 0 \]

otherwise.

With this multiplication, \( k(\Gamma, \mathcal{A}) \) is an associative algebra, and since we are assuming the quivers to be finite, it has also an identity element, which is equal to \( \sum_{i \in \Gamma_0} 1_{A_i} \).

Finally, it is easy to observe that \( k(\Gamma, \mathcal{A}) \) is finite-dimensional over \( k \) if and only if so are the algebras \( A_i \) and if \( \Gamma \) is acyclic. We call \( k(\Gamma, \mathcal{A}) \) the \textbf{generalized path algebra} \( (\text{gp-algebra}) \) over \( \Gamma \) and \( \mathcal{A} \) (see [6]). In case \( A_i = k \) for every \( i \in \Gamma_0 \), this construction gives the usual path algebra \( k\Gamma \).

Using the result mentioned above in [2, 11], for each \( i \in \Gamma_0 \), we fix a quiver \( \Sigma_i \) such that \( A_i \cong k\Sigma_i / \Omega_i \), with \( \Omega_i \) an admissible ideal of \( k\Sigma_i \).

Following [3], we shall now consider quotients of generalized path algebras by an ideal generated by relations. Namely, let \( I \) be a finite set of relations over \( \Gamma \) which generates an admissible ideal in \( k\Gamma \). Consider the ideal \( (\mathcal{A}(I)) \) generated by the following subset of \( k(\Gamma, \mathcal{A}) \):

\[ \mathcal{A}(I) = \left\{ \sum_{i=1}^{t} \lambda_i \beta_{i_1} \gamma_{i_1} \beta_{i_2} \ldots \gamma_{i(m_i-1)} \beta_{i m_i} : \right. \]

\[ \sum_{i=1}^{t} \lambda_i \beta_{i_1} \ldots \beta_{i m_i} \text{ is a relation in } I \text{ and } \gamma_{ij} \text{ is a path in } \Sigma_{e(\beta_{ij})} \left\} \right. \]

The quotient \( k(\Gamma, \mathcal{A}) / (\mathcal{A}(I)) \) is said to be a \textbf{generalized bound path algebra} \( (\text{gbp-algebra}) \). We may also write \( \frac{k(\Gamma, \mathcal{A})}{(\mathcal{A}(I))} = k(\Gamma, \mathcal{A}, I) \). When the context is clear, we simply denote the set \( \mathcal{A}(I) \) by \( I \).
We use the following notation in the sequel: $\Gamma$ is an acyclic quiver, $\mathcal{A} = \{A_i : i \in \Gamma_0\}$ denotes a family of basic finite dimensional algebras over an algebraically closed field $k$, and $I$ is a set of relations in $\Gamma$ generating an admissible ideal in the path algebra $k\Gamma$. By $\Lambda = k(\Gamma, \mathcal{A}, I)$, we denote the gbp-algebra obtained from these data. Also, $A_\Lambda$ will denote the product algebra $\prod_{i \in \Gamma_0} A_i$. We denote the identity element of the algebras $A_i$ by $1_i$ instead of $1_{A_i}$. Also, for an algebra $A$, we shall denote by $\text{mod}A$ the category of finitely generated right $A$-modules.

### 2.3. Representations

In [4], we have described the representations of a gbp-algebra, including those associated to projective and injective modules. We shall now recall the results needed in the sequel.

**Definition 2.1.** Let $\Lambda = k(\Gamma, \mathcal{A}, I)$ be a gbp-algebra.

(a) A **representation** of $\Lambda$ is given by $((M_i)_{i \in \Gamma_0}, (M_\alpha)_{\alpha \in \Gamma_1})$ where

(i) $M_i$ is an $A_i$-module, for each $i \in \Gamma_0$;

(ii) $M_\alpha : M_{s(\alpha)} \to M_{t(\alpha)}$ is a $k$-linear transformation, for each arrow $\alpha \in \Gamma_1$; and

(iii) whenever $\gamma = \sum_{i=1}^t \lambda_i \alpha_{i1} \alpha_{i2} \ldots \alpha_{in_\gamma}$ is a relation in $I$ where $\lambda_i \in k$ and $\alpha_{ij} \in \Gamma_1$, then

\[
\sum_{i=1}^t \lambda_i M_{\alpha_{i1}} \circ \gamma_{i1} \circ \ldots \circ M_{\alpha_{i2}} \circ \gamma_{i2} \circ M_{\alpha_{i1}} = 0
\]

for every choice of paths $\gamma_{ij}$ over $\Sigma_{s(\alpha_{ij})}$, with $1 \leq i \leq t$, $2 \leq j \leq n_i$.

(b) We say that a representation $((M_i)_{i \in \Gamma_0}, (M_\alpha)_{\alpha \in \Gamma_1})$ of $\Lambda$ is **finitely generated** if each of the $A_i$-modules $M_i$ is finitely generated.

(c) Let $M = ((M_i)_{i \in \Gamma_0}, (M_\alpha)_{\alpha \in \Gamma_1})$ and $N = ((N_i)_{i \in \Gamma_0}, (N_\alpha)_{\alpha \in \Gamma_1})$ be representations of $\Lambda$. A **morphism of representations** $f : M \to N$ is given by a tuple $f = (f_i)_{i \in \Gamma_0}$, such that, for every $i \in \Gamma_0$, $f_i : M_i \to N_i$ is a morphism of $A_i$-modules; and such that, for every arrow $\alpha : i \to j \in \Gamma_1$, it holds that $f_\alpha M_\alpha = N_\alpha f_\alpha$, that is, the following diagram comutes:

\[
\begin{array}{ccc}
M_i & \xrightarrow{M_\alpha} & M_j \\
f_i \downarrow & & \downarrow f_j \\
N_i & \xrightarrow{N_\alpha} & N_j
\end{array}
\]

We shall denote by $\text{Rep}_k(\Gamma, \mathcal{A}, I)$ (or by $\text{rep}_k(\Gamma, \mathcal{A}, I)$) the category of the representations (or finitely generated representations, respectively) of the algebra $k(\Gamma, \mathcal{A}, I)$.

**Theorem 2.2** ([4], see also [3]). There is a $k$-linear equivalence

\[
F : \text{Rep}_k(\Gamma, \mathcal{A}, I) \to \text{Mod} k(\Gamma, \mathcal{A}, I)
\]

which restricts to an equivalence

\[
F : \text{rep}_k(\Gamma, \mathcal{A}, I) \to \text{mod} k(\Gamma, \mathcal{A}, I)
\]
2.4. **Realizing an \( A_i \)-module as a \( \Lambda \)-module.** Let \( i \in \Gamma_0 \) and let \( M \) be an \( A_i \)-module. We consider three ways of realizing \( M \) as a \( \Lambda \)-module:

**A- Natural inclusion.** \( \mathcal{I}(M) = ((M_j)_{j \in \Gamma_0}, (\phi_\alpha)_{\alpha \in \Gamma_1}) \) is the representation given by

\[
M_j = \begin{cases} 
M & \text{if } j = i \\
0 & \text{if } j \neq i
\end{cases}
\quad \text{and} \quad \phi_\alpha = 0 \text{ for all } \alpha \in \Gamma_1.
\]

By abuse of notation, we shall identify \( \mathcal{I}(M) = M \), since these two have the same underlying space.

**B-Cones.** As observed in [6], if \( k(\Gamma, \mathcal{A}) \) is a gp-algebra, then it is a tensor algebra: if \( \mathcal{A} = \prod_{i \in \Gamma_0} A_i \) is the product of the algebras in \( \mathcal{A} \), then there is an \( \mathcal{A}_A - \mathcal{A}_A \)-bimodule \( M \) such that \( k(\Gamma, \mathcal{A}) \cong T(\mathcal{A}_A, M) \).

Since \( M \) is naturally an \( \mathcal{A}_A \)-module and there is a canonical map \( \mathcal{A}_A \to \Lambda = k(\Gamma, \mathcal{A})/I \), then, by extension of scalars, \( M \) originates a \( \Lambda \)-module \( \mathcal{C}_i(M) \), which is called the **cone** over \( M \).

We now recall the following results from [4]:

**Proposition 2.3.** Given \( i \in \Gamma_0 \), we have:

1. The cone functor \( \mathcal{C}_i : \text{mod } A_i \to \text{mod } \Lambda \) is exact.
2. If \( P \) is a projective \( A_i \)-module, then \( \mathcal{C}_i(P) \) is a projective \( \Lambda \)-module.

**C-Dual cones.** The **dual cone** over \( M \) is given by \( \mathcal{C}_i^*(M) \cong D\mathcal{C}_iD(M) \), where \( D = \text{Hom}_k(-, k) \) is the usual duality functor. A dual result of Proposition 2.3 for injective modules holds true (see [4]).

We refer to [4] for further details of the above constructions.

3. **Homological dimensions**

Using the notations established above, we shall concentrate now in the comparison of some homological dimensions of \( \Lambda \) with those in the algebras \( A_i, \ i \in \Gamma_0 \). Given an algebra \( A \) and an \( A \)-module \( M \), we denote by \( \text{pd}_A M \) and by \( \text{id}_A M \) the projective and the injective dimensions of \( M \), respectively. Also, the global dimension of \( A \) is denoted by \( \text{gl.dim} A \).

3.1. **First case.** We analyse the natural inclusion of \( A_i \)-modules in \( \text{mod } \Lambda \).

**Lemma 3.1.** Let \( i \in \Gamma_0 \) and let \( M \) be an \( A_i \)-module. Then

(a) \( \text{pd}_A M \geq \text{pd}_A A_i M \).

(b) if \( i \) is a sink, then \( \text{pd}_A M = \text{pd}_A A_i M \).

(c) \( \text{id}_A M \geq \text{id}_A A_i M \).

(d) if \( i \) is a source, then \( \text{id}_A M = \text{id}_A A_i M \).

**Proof.** We shall prove only (a) and (b) since the proofs of (c) and (d) are dual.

(a) There is nothing to show if \( \text{pd}_A M = \infty \). So, assume \( M \) has finite projective dimension
be a minimal projective $\Lambda$-resolution of $M$. Since a projective resolution is in particular an exact sequence, it yields an exact sequence of $A_i$-modules at the $i$-th component:

\[
0 \longrightarrow (P_m)_i \longrightarrow \cdots \longrightarrow (P_1)_i \longrightarrow (P_0)_i \longrightarrow M \longrightarrow 0
\]

It follows from the description of the projective modules over $\Lambda$ (see [4], Subsection 5.1) that every component of a projective representation is projective (indeed, the $i$-th component is a direct sum of indecomposable projective modules over $A_i$, copies of $A_i$, or zero modules). Thus the exact sequence above is a projective resolution of $M$ over $A_i$. This implies that $\text{pd}_{A_i} M \leq m = \text{pd}_{\Lambda} M$.

(b) Because $i$ is a sink, every projective resolution of $M$ over $A_i$ is easily seen to yield a projective resolution of $M$ over $\Lambda$ with the same length. □

The next result follow now easily.

**Corollary 3.2.** $\text{gl.dim} \Lambda \geq \max\{\text{gl.dim} A_1, \ldots, \text{gl.dim} A_n\}$.

We shall see below examples of when equality in the above statement holds and when it does not.

3.2. **Cones and duals.** The next result, which relates the projective and the injective dimensions of a module over $A_i$ with the corresponding dimension of its cone or its dual cone, is a direct consequence of Proposition 2.3 and its dual.

**Lemma 3.3.** Given $i \in \Gamma_0$ and $M$ an $A_i$-module, Then

(a) $\text{pd}_{A_i} M = \text{pd}_{\Lambda} \mathcal{C}_i(M)$.
(b) $\text{id}_{A_i} M = \text{id}_{\Lambda} \mathcal{C}_i^*(M)$.

**Proof.** We shall prove only (a) since the proof of (b) is dual. Let

\[
0 \longrightarrow P_m \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0
\]

be a minimal projective resolution of $M$ in $\text{mod} A_i$. Thus $m = \text{pd}_{A_i} M$. Applying the functor $\mathcal{C}_i$, we have

\[
0 \longrightarrow \mathcal{C}_i(P_m) \longrightarrow \cdots \longrightarrow \mathcal{C}_i(P_1) \longrightarrow \mathcal{C}_i(P_0) \longrightarrow \mathcal{C}_i(M) \longrightarrow 0
\]

Because of Proposition 2.3 this sequence is exact. Moreover, also by Proposition 2.3 every term except possibly for $\mathcal{C}_i(M)$ is known to be projective. So this is a projective resolution in $\text{mod} \Lambda$, proving that $\text{pd}_{\Lambda} \mathcal{C}_i(M) \leq \text{pd}_{A_i} M$. Since the $i$-th component of $\mathcal{C}_i(M)$ is $M$, we know from Proposition 3.1 that the inverse inequality also holds. □
3.3. General case. Having studied the projective and injective dimensions of modules which are inclusion or cones of $A_i$-modules, we turn our attention to general $\Lambda$-representations.

**Definition 3.4.** Let $M = ((M_i)_{i \in \Gamma_0}, (\phi_\alpha)_{\alpha \in \Gamma_1})$ be a representation over $k(\Gamma, A, I)$. The support of $M$ is defined as the set of vertices $\text{supp} M = \{ i \in \Gamma_0 : M_i \neq 0 \}$.

**Proposition 3.5.** For a $\Lambda$-module $M$,

(a) $\text{pd}_\Lambda M \leq \max_{j \in \text{supp} M} \{ \text{pd}_\Lambda M_j \}$

(b) $\text{id}_\Lambda M \leq \max_{j \in \text{supp} M} \{ \text{pd}_\Lambda M_j \}$

**Proof.** We shall prove only (a) since the proof of (b) is dual.

We proceed by induction on $|\text{supp} M|$. If $|\text{supp} M| = 1$ then $M$ has only one non-zero component, say the $i$-th component, and it is clear that $\text{pd}_\Lambda M = \text{pd}_\Lambda M_i$.

Suppose $|\text{supp} M| > 1$ and that the statement holds for representations whose support is smaller than that of $M$. Then, since $\Gamma$ is acyclic, there is at least one vertex $i \in \Gamma_0$ which is a source in the full subquiver determined by $\text{supp} M$. We consider the following representations:

- $N = ((N_j)_{j \in \Gamma_0}, (\phi_\alpha)_{\alpha \in \Gamma_1})$ given by $N_i = M_i$, $N_j = 0$ if $j \neq i$, $\psi_\alpha = 0$; and $T = ((T_j)_{j \in \Gamma_0}, (\rho_\alpha)_{\alpha \in \Gamma_1})$ given by $T_i = 0$, $T_j = M_j$ if $j \neq i$, $\rho_\alpha = \phi_\alpha|_{T_i(\alpha)}$

Observe that, since the support of $N$ has size 1, it satisfies the relations in $I$. Also, it is easy to see that $T$ also satisfies these relations, because $M$ does.

We also consider two morphisms of representations $f = (f_j)_{j \in \Gamma_0} : T \rightarrow M$, and $g = (g_j)_{j \in \Gamma_0} : M \rightarrow N$, given by:

- $f_j : T_j \rightarrow M_j$, $f_i = 0$, $f_j = \text{id}_M$ if $j \neq i$; and
- $g_j : M_j \rightarrow N_j$, $g_i = \text{id}_M$, $g_j = 0$ if $j \neq i$

It is directly verified that these are in fact morphisms of representations. Thus we have a short exact sequence of representations:

$$0 \rightarrow T \overset{f}{\rightarrow} M \overset{g}{\rightarrow} N \rightarrow 0$$

It is indeed exact because the $i$-th component is

$$0 \rightarrow 0 \rightarrow M_i \overset{i_{M_i}}{\rightarrow} M_i \rightarrow 0$$

and for $j \neq i$, the $j$-component is

$$0 \rightarrow M_j \overset{i_{M_j}}{\rightarrow} M_j \rightarrow 0 \rightarrow 0$$

and these are clearly exact sequences.

We obtain that $\text{pd}_\Lambda M \leq \max\{ \text{pd}_\Lambda T, \text{pd}_\Lambda N \}$. Note that $|\text{supp} T| = n - 1$ and $|\text{supp} N| = 1$. Therefore, by the induction hypothesis,

$$\text{pd}_\Lambda N = \text{pd}_\Lambda N_i = \text{pd}_\Lambda M_i$$
\[
\text{pd}_A T \leq \max_{j \in \text{supp} T} \{ \text{pd}_A T_j \} = \max_{j \neq i} \{ \text{pd}_A T_j \} = \max_{j \neq i} \{ \text{pd}_A M_j \}
\]

Assembling the pieces together, we conclude that

\[
\text{pd}_A M \leq \max \{ \text{pd}_A N, \text{pd}_A T \} \leq \max_{j \in \text{supp} M} \{ \text{pd}_A M_j \},
\]

as we wanted to prove. \qed

4. Homological dimensions for gbp-algebras

We shall prove in this section some general results involving gbp-algebras, leaving the particular case of gp-algebras for the next section. We will adopt the following notation from here on: if \( i \) is a source vertex of \( \Gamma \), then \( \Gamma \setminus \{i\} \) shall denote the quiver obtained from \( \Gamma \) by deleting the vertex \( i \) and the arrows starting at \( i \). Moreover, if \( \Gamma \) is equipped with a set of relations \( I \), \( I \setminus \{i\} \) will be the set obtained from \( I \) by excluding the relations starting at \( i \). Also, since \( \Gamma \) is acyclic, we can iterate this process and enumerate \( \Gamma_0 = \{1, \ldots, n\} \) in such a way that \( i \) is a source vertex of \( \Gamma \setminus \{1, \ldots, i-1\} \) for every \( i \).

**Lemma 4.1.** Let \( i \in \Gamma_0 \), \( M \) be an \( A_i \)-module, and let \( (P,g) \) be its projective cover in \( \text{mod} \ A_i \). Then there is an exact sequence of \( \Lambda \)-modules:

\[
0 \rightarrow C_i(\text{Ker} \ g) \oplus L \rightarrow C_i(P) \rightarrow M \rightarrow 0
\]

where \( L \) is a \( \Lambda \)-module such that

\[
\text{supp} \ L \subseteq \{ j \in \Gamma_0 : j \neq i \text{ and there is a path } i \rightarrow j \}
\]

Moreover,

(a) \( L_j \) is free for every vertex \( j \), and

(b) If \( i \in \Gamma_0 \) is such that \( I \setminus \{1, \ldots, i\} = I \setminus \{1, \ldots, i-1\} \), then \( L \) is projective over \( \Lambda \).

**Proof.** (a) It follows from [4] (Proposition 5 and Remark 5) that \( (C_i(P))_i = P \). So, we can define a morphism of representations \( g' : C_i(P) \rightarrow M \) by establishing that \( g'_i = g \) and that \( g'_j = 0 \) for \( j \neq i \). We want to show that \( \text{Ker} \ g' = C_i(\text{Ker} \ g) \oplus L \), where \( L \) satisfies the conditions in the statement.

Let \( \{p_1, \ldots, p_r\} \) be a \( k \)-basis of \( \text{Ker} \ g \) and complete it to a \( k \)-basis \( \{p_1, \ldots, p_r, \ldots, p_s\} \) of \( P \). Also let, for every \( j \in \Gamma_0 \), \( \{a^j_1, \ldots, a^j_{i_j}\} \) be a \( k \)-basis of \( A_j \). For a path \( \gamma : i = l_0 \rightarrow l_1 \rightarrow \cdots \rightarrow l_t = j \) from \( i \) to \( j \) in \( \Gamma \) denote

\[
\theta_{\gamma,h,i_1,\ldots,i_t} = p_h \otimes \gamma_1 a^{e(\gamma_1)}_{i_1} \gamma_2 a^{e(\gamma_2)}_{i_2} \cdots \gamma_t a^{e(\gamma_t)}_{i_t} \in \text{Ker} \ g'
\]

Remember that since \( g' \) was defined as a morphism of representations, it corresponds to a morphism of \( \Lambda \)-modules, because of Theorem 2.2. Therefore,

\[
g' (\theta_{\gamma,h,i_1,\ldots,i_t}) = g' (p_h \otimes \gamma_1 a^{e(\gamma_1)}_{i_1} \gamma_2 a^{e(\gamma_2)}_{i_2} \cdots \gamma_t a^{e(\gamma_t)}_{i_t}) = g(p_h)
\]
So \( \theta_{\gamma,h,i_1,\ldots,i_t} \notin \text{Ker} \, g' \) if and only if \( \gamma \) is the zero-length path \( \varepsilon_i \) and \( r < h \leq s \). Thus we can write

\[
\text{Ker} \, g' = (\theta_{\varepsilon_i,h} : 1 \leq h \leq r) + (\theta_{\gamma,h,i_1,\ldots,i_t} : l(\gamma) > 0)
\]

\[
= (\theta_{\gamma,h,i_1,\ldots,i_t} : 1 \leq h \leq r) \oplus (\theta_{\gamma,h,i_1,\ldots,i_t} : l(\gamma) > 0 \text{ and } r < h \leq s)
\]

\[
= C_i(\text{Ker} \, g) \oplus L
\]

where \( L = (\theta_{\gamma,h,i_1,\ldots,i_t} : l(\gamma) > 0 \text{ and } r < h \leq s) \). Since the generators of \( L \) involve only paths of length strictly greater than zero, the only components of \( L \) that are non-zero are the ones over the successors of \( i \), except for \( i \) itself. Therefore the condition about the support of \( L \) in the statement is satisfied. It remains to prove the other two assertions in the statement.

To prove (a), fix \( j \in \Gamma_0 \). If \( j = i \) or if \( j \) is not a successor of \( i \), then \( L_j = 0 \), so we may suppose this is not the case. Again using the equivalence given by Theorem 2.2,

\[
L_j = L \cdot 1_j = (\theta_{\gamma,h,i_1,\ldots,i_t} : \gamma : i \leadsto j \text{ and } r < h \leq s)
\]

\[
= (p_h \otimes \gamma_1 a_{i_1}^{e(\gamma_1)} \gamma_2 a_{i_2}^{e(\gamma_2)} \ldots \gamma_t a_{i_t}^{e(\gamma_t)} : \gamma : i \leadsto j \text{ and } r < h \leq s)
\]

So \( L_j \) is isomorphic to the free \( A_j \)-module whose basis is the set of all possible elements \( p_h \otimes \gamma_1 a_{i_1}^{e(\gamma_1)} \gamma_2 a_{i_2}^{e(\gamma_2)} \ldots \gamma_t a_{i_t}^{e(\gamma_t)} \). In particular, \( L_j \) is free over \( A_j \), and this proves (a).

(b) Assume that \( I \setminus \{1,\ldots,i\} = I \setminus \{1,\ldots,i-1\} \) and let \( i^+ \) denote the set of immediate successors of \( i \). Since, by hypothesis, there are no relations starting at \( i \), we can write:

\[
L = (\theta_{\gamma,h,i_1,\ldots,i_t} : l(\gamma) > 0 \text{ and } r < h \leq s)
\]

\[
= (p_h \otimes \gamma_1 a_{i_1}^{e(\gamma_1)} \gamma_2 a_{i_2}^{e(\gamma_2)} \ldots \gamma_t a_{i_t}^{e(\gamma_t)} : l(\gamma) > 0 \text{ and } r < h \leq s)
\]

\[
= (p_h \otimes \gamma_1 a_{i_1}^{e(\gamma_1)} \otimes \gamma_2 a_{i_2}^{e(\gamma_2)} \ldots \gamma_t a_{i_t}^{e(\gamma_t)} : l(\gamma) > 0 \text{ and } r < h \leq s)
\]

\[
\cong (a_{i_1}^{e(\gamma_1)} \otimes \gamma_2 a_{i_2}^{e(\gamma_2)} \ldots \gamma_t a_{i_t}^{e(\gamma_t)} : l(\gamma) > 0)^{s-r}
\]

\[
\cong \bigoplus_{i' \in i^+} C_i(A_{i'})^{s-r}
\]

Since \( A_{i'} \) is projective over \( A_i \), then \( C_i(A_{i'}) \) is projective over \( \Lambda \) by Proposition 2.3.

We have thus shown that \( L \) is isomorphic to a direct sum of projective \( \Lambda \)-modules, and therefore it is also projective, concluding the proof.

4.1. **A special kind of gbp-algebras.** Before our next result, we need a further definition. For a vertex \( j \) of \( \Gamma \), denote by \( S_j \) the simple \( k\Gamma/I \)-module over \( j \).

**Definition 4.2.** A gbp-algebra \( \Lambda \) is called **terraced** provided for every \( i \in \Gamma_0 \) such that \( I \setminus \{1,\ldots,i\} \neq I \setminus \{1,\ldots,i-1\} \) (i.e., every time there are relations starting at \( i \)), one has

\[
\text{pd}_{k\Gamma/I} S_i \geq \max\{\text{pd}_{k\Gamma/I} S_j : j \text{ is a successor of } i\} + 1.
\]

Observe that any gp-algebra (that is, when \( I = 0 \), which makes \( k\Gamma \) hereditary) is terraced.
Theorem 4.3. Let $\Lambda = k(\Gamma, A, I)$ be a terraced gbp-algebra. Then, for every representation $M$ over $\Lambda$,

$$\text{pd}_\Lambda M \leq \max_{i \in \text{supp } M} \{ \text{pd}_{A_i} M_i, \text{pd}_{k\Gamma/I} S_i \}$$

where $S_i$ denotes the simple $k\Gamma/I$-module associated with the vertex $i$.

Proof. The proof is done by induction. First, suppose $\text{supp } M = \{ n \}$. By the assumption on the numbering of the vertices, we know that $n$ is a sink vertex of $\Gamma_0$. It follows from Lemma 3.1(b) that $\text{pd}_\Lambda M = \text{pd}_{A_n} M_n$. Since $n$ is a sink vertex, the simple $k\Gamma/I$-module $S_n$ is projective, and thus it holds that $\text{pd}_\Lambda M = \max \{ \text{pd}_{A_n} M_n, \text{pd}_{k\Gamma/I} S_n \}$. This proves the initial step of induction.

Now suppose that $\text{supp } M \subseteq \{ i, \ldots, n \}$ and that the statement is valid for representations whose support is contained in $\{ i + 1, \ldots, n \}$. Initially we are going to study the projective dimension of $M_i$ over $\Lambda$. If $i$ is a sink vertex, then, similarly to above, we have that $\text{pd}_\Lambda M_i = \max \{ \text{pd}_{A_i} M_i, \text{pd}_{k\Gamma/I} S_i \}$, so suppose $i$ is not a sink vertex. Let $(P, g)$ be a projective cover of $M_i$ over $A_i$. Then, because of Lemma 4.1, there is an exact sequence in $\text{mod } \Lambda$:

$$0 \longrightarrow C_i(\text{Ker } g) \oplus L \longrightarrow C_i(P) \longrightarrow M_i \longrightarrow 0$$

where $L$ satisfies the conditions given in the statement of the cited lemma. From this exact sequence, we deduce that

$$\text{pd}_\Lambda M_i \leq \max \{ \text{pd}_\Lambda C_i(P), \text{pd}_\Lambda (C_i(\text{Ker } g) \oplus L) + 1 \}$$

Since $P$ is projective over $A_i$, Proposition 2.3 implies that $\text{pd}_\Lambda C_i(P) = 0$. Thus

$$\text{pd}_\Lambda M_i \leq \text{pd}_\Lambda (C_i(\text{Ker } g) \oplus L) + 1 \leq \max \{ \text{pd}_\Lambda C_i(\text{Ker } g), \text{pd}_\Lambda L \} + 1$$

Using Corollary 3.3, we have

$$(4.1) \quad \text{pd}_\Lambda M_i \leq \max \{ \text{pd}_{A_i} \text{Ker } g, \text{pd}_\Lambda L \} + 1$$

Now we divide our analysis in cases:

Case 1: $\text{pd}_{A_i} \text{Ker } g \geq \text{pd}_\Lambda L$.

In this case, Equation 4.1 implies that $\text{pd}_\Lambda M_i \leq \text{pd}_{A_i} \text{Ker } g + 1 = \text{pd}_{A_i} M_i$, because $(P, g)$ is the projective cover of $M_i$.

Case 2: $\text{pd}_{A_i} \text{Ker } g \leq \text{pd}_\Lambda L$.

Now, from Equation 4.1, $\text{pd}_\Lambda M_i \leq \text{pd}_\Lambda L + 1$. In case $I \setminus \{ 1, \ldots, i \} = I \setminus \{ 1, \ldots, i - 1 \}$, from Lemma 4.1, we get that $\text{pd}_\Lambda L = 0$. Since we have already supposed in this case that $\text{pd}_{A_i} \text{Ker } g \leq \text{pd}_\Lambda L$, then $\text{pd}_{A_i} \text{Ker } g = 0$. Again from Equation 4.1, $\text{pd}_\Lambda M_i \leq 1$. Since $i$ is not a sink, we know that $S_i$ is not projective over $k\Lambda/I$ and so $\text{pd}_{k\Lambda/I} S_i \geq 1$. Thus $\text{pd}_\Lambda M_i \leq \text{pd}_{k\Lambda/I} S_i$.

Assume now $I \setminus \{ 1, \ldots, i \} \neq I \setminus \{ 1, \ldots, i - 1 \}$. By Lemma 4.1, $\text{pd}_{A_j} L_j = 0$ for every $j$, and since the support of $L$ is contained in $\{ i + 1, \ldots, n \}$, by the induction hypothesis and because $\Lambda$ is terraced:

$$\text{pd}_\Lambda L \leq \max_{j \in \text{supp } L} \{ \text{pd}_{k\Gamma/I} S_j \} \leq \text{pd}_{k\Gamma/I} S_i - 1$$
Then $\text{pd}_A M_i \leq \text{pd}_A L + 1 \leq \text{pd}_{k\Gamma/I} S_i - 1 + 1 = \text{pd}_{k\Gamma/I} S_i$.

Putting together all cases discussed above, we conclude that

$$\text{pd}_A M_i \leq \max \{ \text{pd}_A M_i, \text{pd}_{k\Gamma/I} S_i \}$$

Now, using Proposition 3.5, we have that

$$\text{pd}_A M \leq \max_{j \in \text{supp} M} \text{pd}_A M_j \leq \max_{j \in \text{supp} M} \{ \text{pd}_A M_j, \text{pd}_{k\Gamma/I} S_j \}$$

which proves the theorem.

\[ \square \]

**Corollary 4.4.** Let $\Lambda = k(\Gamma, \mathcal{A}, I)$ be a terraced gbp-algebra. Then, for every $j \in \Gamma_0$, $\text{gl.dim} A_j \leq \text{gl.dim} \Lambda$, and the following inequality holds:

$$\text{gl.dim} \Lambda \leq \max_{j \in \Gamma_0} \left\{ \text{gl.dim} \frac{k\Gamma}{I}, \text{gl.dim} A_j \right\}$$

### 4.2. Opposite algebras.

Before our next corollary, let us recall some facts concerning the opposite algebra of $\Lambda = k(\Gamma, \mathcal{A}, I)$. Denote by $\Gamma^{\text{op}}$ the quiver with the same vertices of $\Gamma$ and with reversed arrows. Also, $I^{\text{op}}$ will denote the set of relations in $\Gamma^{\text{op}}$ obtained through inversion of the arrows in $I$. Finally, $\mathcal{A}^{\text{op}} = \{ A_i^{\text{op}} : i \in \Gamma_0 \}$ is the set where $A_i^{\text{op}}$ is the opposite algebra of $A_i$. With this notation, we have that $\Lambda^{\text{op}} \cong k(\Gamma^{\text{op}}, \mathcal{A}^{\text{op}}, I^{\text{op}})$. (See [1], Proposition 2. We shall refer to this in the next proof as Fact I).

Also, in a natural way, one can describe the representations of the opposite algebra in terms of the representations of the original one using the duality functor $D = \text{Hom}_k(-, k)$. Namely, if $((M_i)_{i \in \Gamma_0}, (\phi_\alpha)_{\alpha \in \Gamma_1})$ is the representation of the $\Lambda$-module $M$, then the representation of the $\Lambda^{\text{op}}$-module $DM$ is isomorphic to $(D(M_i)_{i \in \Gamma_0}, D(\phi_\alpha)_{\alpha \in \Gamma_1})$. (See [1], Proposition 3. We shall refer to this in the next proof as Fact II). As a consequence, we will have:

**Corollary 4.5.** Let $\Lambda = k(\Gamma, \mathcal{A}, I)$ be a terraced gbp-algebra, and let $M$ be a representation over $\Lambda$. Then

$$\text{id}_\Lambda M = \max_{i \in \text{supp} M} \{ \text{id}_{A_i} M_i, \text{id}_{k\Gamma/I} S_i \}$$

where $S_i$ denotes the simple $k\Gamma/I$-module over the vertex $i$.

**Proof.** The idea is to use Theorem 4.3 and the fact that the duality functor anti-preserves homological properties. Again, let $D = \text{Hom}_k(-, k)$ denote the duality functor. Let $S'_i$ denote the simple $k\Gamma^{\text{op}}/I^{\text{op}}$-module over the vertex $i$. Thus:

$$\text{id}_\Lambda M = \text{pd}_{\Lambda^{\text{op}}} DM$$

$$= \max_{i \in \text{supp} M} \{ \text{pd}_{\Lambda_i^{\text{op}}} (DM)_i, \text{pd}_{k\Gamma^{\text{op}}/I^{\text{op}}} S'_i \}$$

(by Theorem 4.3 and Fact I)

$$= \max_{i \in \text{supp} M} \{ \text{pd}_{A_i^{\text{op}}} D(M_i), \text{pd}_{k\Gamma^{\text{op}}/I^{\text{op}}} S'_i \}$$

(by Fact II)

$$= \max_{i \in \text{supp} M} \{ \text{id}_{A_i} M_i, \text{id}_{k\Gamma/I} S_i \}$$

(by D is a duality)

\[ \square \]
4.3. **Finitistic dimension.** Given an algebra $A$, its **finitistic dimension** is given by:

$$\text{fin.dim } A = \sup \{ \text{pd}_A M : M \text{ is an } A\text{-module of finite projective dimension} \}$$

A still open conjecture, called the **Finitistic Dimension Conjecture**, states that every algebra has finite finitistic dimension.

**Proposition 4.6.** Let $\Lambda = k(\Gamma, \mathcal{A}, I)$ be a terraced gbp-algebra. Then

$$\text{fin.dim } \Lambda \leq \max_{i \in \Gamma_0} \left\{ \text{gl.dim } \frac{k\Gamma}{I}, \text{fin.dim } A_i \right\}$$

In particular, if the bound path algebra $k\Gamma/I$ has finite global dimension and $\text{fin.dim } A_i < \infty$ for each $i$, then also $\text{fin.dim } \Lambda < \infty$.

**Proof.** Let $M = ((M_i)_{i \in \Gamma_0}, (\phi_\alpha)_{\alpha \in \Gamma_1})$ be a representation of finite projective dimension over $\Lambda$. From Lemma 3.1 for every $i \in \Gamma_0$, $\text{pd}_{A_i} M_i \leq \text{pd}_A M$, so $M_i$ has finite projective dimension over $A_i$, and thus $\text{pd}_{A_i} M_i \leq \text{fin.dim } A_i$. Using Theorem 4.3

$$\text{pd}_A M \leq \max_{i \in \Gamma_0} \{ \text{pd}_{k\Gamma/I} S_i, \text{pd}_{A_i} M_i \} \leq \max_{i \in \Gamma_0} \{ \text{gl.dim } k\Gamma/I, \text{fin.dim } A_i \}.$$

Since $M$ is arbitrary, the statement follows. \qed

5. **Homological dimensions for gp-algebras**

We shall now concentrate in gp-algebras which are, as observed above, terraced gbp-algebras. We start with the following result which is a direct consequence of the above considerations.

**Theorem 5.1.** Let $\Lambda = k(\Gamma, \mathcal{A})$ be a gp-algebra, with $\Gamma$ having at least one arrow. Then

$$\text{gl.dim } \Lambda = \max_{j \in \Gamma_0} \{ 1, \text{gl.dim } A_j \}.$$ 

**Proof.** Observe that $\text{gl.dim } k\Gamma = 1$ in this case and so, by Corollary 4.4, $\text{gl.dim } \Lambda \leq \max_{j \in \Gamma_0} \{ 1, \text{gl.dim } A_j \}$. The equality now follows using Corollary 3.2 and the fact that $\Lambda$ is not semisimple (since $k\Gamma$ is not). \qed

5.1. **Shod and quasitilted algebras.** The next result is an application to the study of shod and quasitilted algebras. Quasitilted algebras were introduced in [8] as a generalization of tilted algebras, by considering tilting objects in abelian categories. We shall, however, use a characterization of quasitilted algebras, also proven in [8], which suits better our purpose here. The shod algebras were then introduced in [5] in order to generalize the concept of quasitilted. The acronym shod stands for small homological dimension, as it is clear from the definition below. We refer to [5, 8] for more details.

**Definition 5.2.** Let $A$ be an algebra. We say that $A$ is a **shod** algebra if, for every indecomposable $A$-module $M$, either $\text{pd}_A M \leq 1$ or $\text{id}_A M \leq 1$. If, besides from being shod, $A$ has global dimension of at most two, we say that $A$ is **quasitilted**.
Our next result allows us to produce a quasitilted or shod gp-algebra from other algebras. It is worth mentioning that it is not intended as a complete description of which generalized (bound) path algebras are quasitilted or shod. Before stating it, please note that every hereditary algebra is quasitilted, and thus also shod.

**Proposition 5.3.** Let $\Lambda = k(\Gamma, A)$ be a gp-algebra, with $\Gamma$ acyclic. Suppose that $A_j$ is hereditary for every $j \in \Gamma_0$, except possibly for a single vertex $i \in \Gamma_0$. Then:

(a) If $A_i$ is shod, then $\Lambda$ is shod.

(b) If $A_i$ is quasitilted, then $\Lambda$ is quasitilted.

**Proof.** (a) Let $M = ((M_j)_{j \in \Gamma_0}, (\phi_\alpha)_{\alpha \in \Gamma_1})$ be an indecomposable representation over $\Lambda$. Since $\Gamma$ is acyclic, we infer that the algebra $k\Gamma$ is hereditary and so every simple module over it will have projective and injective dimension of at most one. Observe also that, since $A_j$ is hereditary for $j \neq i$, we also have $\text{pd}_{A_j} M_j \leq 1$ and $\text{id}_{A_j} M_j \leq 1$ if $j \neq i$.

Now, since $A_i$ is shod, either $\text{pd}_{A_i} M_i \leq 1$ or $\text{id}_{A_i} M_i \leq 1$. In the former case, from Theorem 4.3, we have that $\text{pd}_{A} M \leq \max_{j \in \Gamma_0}\{\text{pd}_{A_j} M_j, \text{pd}_{k\Gamma} S_j\} \leq 1$, and in the latter, using Corollary 4.5 in an analogous manner, one obtains that $\text{id}_{A} M \leq 1$. Thus $\Lambda$ is shod.

(b) Since $A_i$ is quasitilted, it is shod and from the previous item we get that $\Lambda$ is shod. It remains to prove that $\text{gl.dim } \Lambda \leq 2$. Applying Corollary 4.4,

$$\text{gl.dim } \Lambda \leq \max_{j \in \Gamma_0}\{k\Gamma, \text{gl.dim } A_j\} \leq 2,$$

using that $A_i$ is quasitilted and that the other algebras are hereditary. \qed

**Example 5.4.** This example will show that the converse of proposition above could not hold. Let $A$ be the bound path algebra over the quiver

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

bound by $\alpha\beta = 0$, and let $\Lambda$ be the generalized path algebra given by

$$A \xrightarrow{k} A$$

We have that, with this setting, $\Lambda$ does not satisfy the hypothesis from the last proposition: there is more than one vertex upon which the algebra is quasitilted and non-hereditary. However, using [9], Theorem 3.3 or [3], Theorem 3.9, we see that $\Lambda$ is isomorphic to the bound path algebra over the quiver

$$\begin{align*}
1 & \xrightarrow{\alpha} 2 \\
2 & \xrightarrow{\beta} 4 \xrightarrow{\gamma} 5 \\
3 & \xrightarrow{\delta} 7 \\
4 & \xrightarrow{\delta} 6
\end{align*}$$

bound by $\alpha\beta = \gamma\delta = 0$. Then it is easy to see that $\Lambda$ is a quasitilted algebra. The same example shows that the converse of the above proposition also does not hold for shod algebras.
We finish our considerations with a result which is a direct consequence of Proposition 4.6.

**Proposition 5.5.** Let \( \Lambda = k(\Gamma, A) \) be a gp-algebra, with \( \Gamma \) having at least one arrow. Then

\[
\text{fin.dim} \Lambda = \max_{i \in \Gamma_0} \{1, \text{fin.dim} A_i\}
\]

In particular, if \( \text{fin.dim} A_i < \infty \) for each \( i \), then also \( \text{fin.dim} \Lambda < \infty \).

**Proof.** Just observe that \( \text{gl.dim} k\Gamma = 1 \), and use Proposition 4.6. \( \square \)

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