Mass, zero mass and . . . nophysics

R. Saar and S. Groote

1 Loodus- ja Täppisteaduste valdkond, Füüsika Instituut,
Tartu Ülikool, W. Ostwaldi 1, 50411 Tartu, Estonia

2 PRISMA Cluster of Excellence, Institut für Physik,
Johannes-Gutenberg-Universität, Staudinger Weg 7, 55099 Mainz, Germany

Abstract

In this paper we demonstrate that massless particles cannot be considered as the limiting case of massive particles. Instead, the usual symmetry structure based on semisimple groups like $U(1)$, $SU(2)$ and $SU(3)$ has to be replaced by less usual solvable groups like the minimal nonabelian group $sol_2$. Starting from the proper orthochronous Lorentz group $Lor_{1,3}$ we extend Wigner’s little group by an additional generator, obtaining the maximal solvable or Borel subgroup $Bor_{1,3}$ which is equivalent to the Kronecker sum of two copies of $sol_2$, telling something about the helicity of particle and antiparticle states.
1 Introduction

In his paper “Sur la dynamique de l’electron” from July 1905, Henri Poincaré formulates the Principle of Relativity, introduces the concepts of Lorentz transformation and Lorentz group, and postulates the covariance of laws of Nature under Lorentz transformations [1].

In 1939, Eugène Wigner analysed the unitary representations of the inhomogeneous Lorentz group [2,3]. However, there are also nonunitary representations. As Wigner pointed out, the irreducible representations of the Lorentz group are important because they define the types of particles. The Lorentz symmetry is expressed by introducing the commutative diagram for the covariant wave functions $\psi^C$,

$$
\begin{align*}
\psi^C : & \mathbb{E}_{1,3} \ni p & \rightarrow & \psi^C(p) \\
U(\Lambda) \downarrow & \downarrow \Lambda & \downarrow & T(\Lambda)
\end{align*}
$$

(1)

implying $(U(\Lambda)\psi^C)(\Lambda p) = T(\Lambda)\psi^C(p)$. Here $\text{Lor}_{1,3} \ni \Lambda \rightarrow T(\Lambda)$ is the finite dimensional nonunitary representation.

Let the vectors $|p, \lambda\rangle$ with four-momentum $p$ and independent parameter $\lambda$ form a complete orthogonal basis of the irreducible representation of the Poincaré group. In this basis, $U(\Lambda)$ is always expressed by [4,5]

$$
U(\Lambda) = Q(W(\Lambda, p)) P(\Lambda),
$$

where $Q$ is diagonal with respect to $\lambda$, $P$ is diagonal with respect to $p$, and $W(\Lambda, p)$ is the Thomas–Wigner rotation.

The transformation law for Wigner’s wave function $\psi^W$ reads

$$
(U(\Lambda)\psi^W)(p, \lambda) = \sum_{\lambda'} Q_{\lambda\lambda'} \left( W(\Lambda, \Lambda^{-1} p)\psi^W(\Lambda^{-1} p, \lambda') \right).
$$

(2)

Here $Q(W(\Lambda, p))$ is a representation of Wigner’s little group as a subrepresentation of some Lorentz group representation $T(\Lambda)$. The explicit relation between $\psi^C$ and $\psi^W$ is given in Refs. [6, 7, 8, 9, 10, 11, 12, 13, 14].
The great importance of Wigner’s theory is that the classification of particles according to their Lorentz transformation properties is entirely determined by the representation of the little group as the subrepresentation of the representation of Lor$_{1,3} = SO^0_{1,3}$ [15] [16].

The most important cases are

\[ \hat{p} = (1, 0, 0, 0), (1, 0, 0, 1) \]

\[ \text{Lg}(\hat{p}) = \begin{array}{c} \text{SO}_3 \\ \text{E}(2) \end{array} \]

Bor$_{1,3}$

In the title we use the word “nophysics”. This word can be taken as an abbreviation, namely

nophysics = new old physics

where “old” stands for Pauli’s massless neutrino hypothesis, “new” for the solvability as symmetry applicable to massless particles. Another more detailed explanation is found in the Conclusions of this paper.

2 The little group

The little group of the four-momentum $\hat{p}$ or the stabiliser of $\hat{p}$ is the maximal closed subgroup of Lor$_{1,3}$ defined as

\[ \text{Lg}(\hat{p}) = \{ \hat{\Lambda} \in \text{Lor}_{1,3} : \hat{\Lambda} \hat{p} = \hat{p} \} \]  

The orbit of $\hat{p}$ is a subspace in $E_{1,3}$,

\[ \text{Orb}(\hat{p}) = \{ \Lambda \hat{p} : \Lambda \in \text{Lor}_{1,3} \} = \text{Lor}_{1,3} \hat{p}, \]

given as the bijection $\text{Orb}(\hat{p}) = \text{Lor}_{1,3} / \text{Lg}(\hat{p})$. For all $p \in \text{Orb}(\hat{p})$ the Thomas–Wigner rotation is given by

\[ W(\Lambda, p) = L_{\Lambda p}^{-1} \Lambda L_p, \]

where $L_p \in \text{Lor}_{1,3}$ is the representative of $p \in \text{Orb}(\hat{p})$. 

3
The characteristic feature of the massive case is that the fixed vector \( \hat{p} \) can be chosen to be \( \hat{p} = (m, 0) \), for which

\[
\text{Lg} \; \hat{p} = \text{SO}_3 \text{ locally} \equiv \text{SU}_2
\]

(\( \text{SU}_2 \) is the universal covering group of \( \text{SO}_3 \)). Since \( \text{SU}_2 \) is compact and simply connected, all its finite-dimensional irreducible representations are single-valued, unitary and parametrised by the eigenvalue \( s \) of the Casimir operator which can take on non-negative half-integer values

\[
s = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots
\]

From the local equivalence, the equivalence of the Lie algebras can be derived, i.e.

\[
\text{so}_3 = \text{ad} \; \text{su}_2 = \text{lg} \; \hat{p}.
\]

It is important that the little group \( \text{SO}_3 \) is the maximal compact simple subgroup of \( \text{Lor}_{1,3} \) while \( \text{su}_2 \) (i.e. \( \text{lg} \; \hat{p} \)) is the simplest semisimple Lie algebra, \( \text{dim}_\mathbb{R} \text{su}_2 = 3 \).

For every irreducible unitary representation of the little group \( \text{Lg} \; \hat{p} = \text{SO}_3 \) one can derive a corresponding induced representation of the Poincaré group, \( \mathcal{P}_{1,3} = \mathcal{T}_{1,3} \rtimes \text{Lor}_{1,3} \).

The irreducible representations of \( \mathcal{P}_{1,3} \) are characterised by the pairs \( (m, s) \), where the mass \( m \) is real and positive and the spin takes on the values \( s = 0, \frac{1}{2}, 1, \ldots \). The states within each irreducible representation are labelled by \( \xi = -s, -s+1, \ldots, s \) which means that massive particles of spin \( s \) have \( 2s + 1 \) degrees of freedom.

In the massless case \( m = 0 \) the representative vector \( \hat{p} \) may be taken to be

\[
\hat{p} = (\omega_p, 0, 0, \omega_p), \quad \omega = |\vec{p}|.
\]

To construct the little group, one has to solve the defining equation

\[
\hat{\Lambda} \hat{p} = \hat{p}.
\]

The result is the Euclidean group \( \mathbb{R}^2 = \text{SO}(2) \text{ locally} \equiv \mathcal{T}_2 \rtimes \text{SO}(2) \)

\[
\text{Lg} \; \hat{p} = \mathcal{E}(2) = \text{ISO}(2) = \mathcal{T}_2 \rtimes \text{SO}(2)
\]
for which the double covering group is given by

$$\bar{E}(2) = \mathcal{T}_2 \rtimes U(1)$$

where $\mathcal{T}_2$ is the Abelian two-dimensional group of translations. Thus, the little group is solvable and non-compact, and the restrictions of the finite-dimensional representations of $\text{Lor}_{1,3}$ to $E(2)$ are in general non-unitary. In fact, the only unitary, irreducible representation of $E(2)$ are one-dimensional, i.e. degenerate representations, since the subgroup $\mathcal{T}_2$ of translations has to be realised trivially [4, 5],

$$E(2) \to E(2)/\mathcal{T}_2 = \text{SO}(2). \quad (7)$$

The requirement that the representations are at most double-valued implies that only the representations $\text{SO}(2) \to U^j(\text{SO}(2)), j = 0, \pm \frac{1}{2}, \pm 1, \ldots$ are allowed. This one-dimensional (internal) freedom of massless particles is usually called the helicity. Since all the unitary representations on the orbits $p^2 = 0$ are induced by the non-faithful one-dimensional representation of the little group $E(2)$, massless particles are characterised by a discrete helicity

$$\lambda = 0, \pm \frac{1}{2}, \pm 1, \ldots \quad (8)$$

Notice that if the parity is included, the helicity takes on two values, $\lambda$ and $-\lambda$. For example, the two states $\lambda = \pm 1$ are then referred to as left-handed ($\lambda = -1$) and right-handed ($\lambda = +1$) photons.

### 2.1 The Borel subgroup

Due to the unitarity of representations of the little group $\text{Lg} \hat{\rho} = E(2)$, zero-mass particles have only a single value for the helicity (if the parity is not taken into account). Suppose, the most general determined, relativistically invariant, first order single particle equation is of the form

$$(\beta^\mu \partial_\mu + \rho) \psi(x) = 0. \quad (9)$$
Then there is a simple criterion by D. Kwoh under which this equation will have zero-mass solutions \cite{17},

**Kwoh’s lemma:** A necessary condition that Eq. (9) has a zero-mass solution is that

\[
\det(-i\beta^\mu p_\mu + \lambda p) = 0 \tag{10}
\]

for all real \(\lambda\) and all light-like \(p\), i.e. all \(p\) such that \(p^2 = 0\).

If Eq. (9) is a defining equation for a single massless particle, Kwoh’s lemma states the gauge invariance \(p \rightarrow \lambda p\) as a very special property of the theory. Therefore, it seems to be reasonable (at least mathematically) to include this gauge transformation into the little group, i.e. instead of Eq. (6) take

\[
\hat{\Lambda} \hat{p} = \lambda \hat{p} \tag{11}
\]

as defining equation for the little group, where \(\hat{p} = (\varepsilon, 0, 0, 1)\) with \(\varepsilon = \pm 1\), \(\lambda > 0\) and \(\hat{\Lambda} \in \text{Lor}_{1,3}\). If \(\Lambda = \exp(-\frac{1}{2} \omega^\mu \nu e_{\mu \nu})\), Eq. (11) yields \(\omega^\mu \nu \hat{\nu}^\mu = \delta \lambda \hat{p}^\mu\) or more explicitly

\[
\omega_{03} = \delta \lambda \varepsilon, \quad \varepsilon \omega_{01} - \omega_{13} = 0, \quad \varepsilon \omega_{02} - \omega_{23} = 0
\]

(cf. Sec. A.2). Notice that in case of \(E(2)\) as little group one has \(\delta \lambda = 0\). The solution of Eq. (11) reads

\[
\hat{\Lambda} = B^{(\varepsilon)} (\tilde{\xi}; \lambda, \omega) = \begin{pmatrix}
\frac{1}{2}(\lambda + \frac{1}{\lambda}) & \frac{1}{2}(\lambda - \frac{1}{\lambda}) & \frac{1}{\lambda} (\tilde{\xi} T \text{ rot } \omega) & \frac{1}{\lambda} (\tilde{\xi} T \text{ rot } \omega) \\
\frac{1}{\lambda} (\tilde{\xi} T \text{ rot } \omega) & \frac{1}{\lambda} (\tilde{\xi} T \text{ rot } \omega) & \frac{1}{\lambda} (\tilde{\xi} T \text{ rot } \omega) & \frac{1}{\lambda} (\tilde{\xi} T \text{ rot } \omega)
\end{pmatrix}, \tag{12}
\]

where

\[
\tilde{\xi} = \begin{pmatrix}
\xi_1 \\
\xi_2
\end{pmatrix}, \quad \text{rot } \omega = \begin{pmatrix}
\cos \omega & -\sin \omega \\
\sin \omega & \cos \omega
\end{pmatrix}, \quad \xi^2 = \xi_1^2 + \xi_2^2.
\]

If one now defines

\[
B^{(\varepsilon)}_\lambda = B^{(\varepsilon)} (0; \lambda, 0) = \begin{pmatrix}
\frac{1}{2} (\lambda + \frac{1}{\lambda}) & \frac{1}{2} (\lambda - \frac{1}{\lambda}) & \frac{1}{2} (\tilde{\xi} (\lambda - \frac{1}{\lambda})) \\
\frac{1}{2} (\lambda - \frac{1}{\lambda}) & \frac{1}{2} (\lambda + \frac{1}{\lambda}) & \frac{1}{2} (\tilde{\xi} (\lambda - \frac{1}{\lambda}))
\end{pmatrix},
\]

6
\[
R_\omega = B^{(e)}(0; 1, \omega) = \begin{pmatrix} 1 & 0^T & 0 \\
0 & \text{rot } \omega & 0 \\
0 & 0^T & 1 \end{pmatrix},
\]

\[
T^{(e)}_\xi = B^{(e)}(\xi; 1, 0) = \begin{pmatrix} 1 + \frac{1}{2} \xi^2 & \varepsilon \xi^T & -\frac{\varepsilon}{2} \xi^2 \\
\varepsilon \xi & 1 & -\xi \\
\frac{\varepsilon}{2} \xi^2 & \xi^T & 1 - \frac{1}{2} \xi^2 \end{pmatrix},
\]

(13)

the general transformation (12) can be written as

\[
B^{(e)}(\xi; \lambda, \omega) = T^{(e)}_\xi B^{(e)}_\lambda R_\omega.
\]

One easily obtains the multiplication table

\[
B^{(e)}_\mu B^{(e)}_\lambda = B^{(e)}_{\mu \lambda} = B^{(e)}_\lambda B^{(e)}_\mu,
\]

\[
R_\phi R_\omega = R_{\phi + \omega} = R_\omega R_\phi,
\]

\[
T^{(e)}_\eta T^{(e)}_\xi = T^{(e)}_{\eta + \xi} = T^{(e)}_\xi T^{(e)}_\eta,
\]

\[
B^{(e)}_\lambda T^{(e)}_\xi = T^{(e)}_{\lambda \xi} B^{(e)}_\lambda,
\]

\[
R_\omega T^{(e)}_\xi = T^{(e)}_{\text{rot } \omega \xi} R_\omega,
\]

\[
B^{(e)}(\xi; \lambda, \omega)B^{(e)}(\eta; \mu, \varphi) = B^{(e)}(\xi + \lambda \text{rot } \omega \eta; \lambda \mu, \omega + \varphi),
\]

\[
(B^{(e)}(\xi; \lambda, \omega))^{-1} = B^{(e)}(-\frac{1}{\lambda} \text{rot } (-\omega) \xi; \frac{1}{\lambda}, -\omega).
\]

(14)

From the multiplication table (14) it follows that the transformations \(B^{(e)}(\xi; \lambda, \omega)\) form a group \(\text{Bor}_{1,3}^{(e)} \subset \text{Lor}_{1,3}\) with non-compact parameter space

\[
\{\xi \in \mathbb{R}_2, \ 0 \leq \omega \leq \pi, \ \lambda > 0\}.
\]

It can easily be shown that the derived series of commutators \(\mathcal{D}\) for \(\text{Bor}_{1,3}^{(e)}\) ends in the identity id. In fact,

\[
\mathcal{D}^2(\text{Bor}_{1,3}^{(e)}) = \{\text{id}\}.
\]
Actually, $\text{Bor}_{1,3}^{(e)}$ is a maximal, solvable and non-compact subgroup of $\text{Lor}_{1,3}$, i.e. the non-compact Borel subgroup of $\text{Lor}_{1,3}$. Moreover, one obtains the Borel decomposition as the semidirect product

$$B^{(e)} \equiv \text{Bor}_{1,3}^{(e)} = T_2^{(e)} \rtimes \text{Tor}_{1,3}^{(e)}.$$  \hspace{1cm} (15)

The set $T_2^{(e)} = \text{Gen}\{T_\xi^{(e)} : \xi \in \mathbb{R}_2\}$ of unipotent elements of $\text{Bor}_{1,3}^{(e)}$ is a closed nilpotent subgroup of $\text{Bor}_{1,3}^{(e)}$. It contains the subgroup $\mathcal{D}(\text{Bor}_{1,3}^{(e)}) = (\text{Bor}_{1,3}^{(e)}, \text{Bor}_{1,3}^{(e)})$ generated by the commutators and is normal in $\text{Bor}_{1,3}^{(e)}$. $\text{Tor}_{1,3}^{(e)} = \text{Bor}_{1,3}^{(e)}/T_2^{(e)} = \text{Gen}\{B_\Lambda^{(e)}, R_\omega : \Lambda > 0; 0 \leq \omega \leq \pi\}$ is the maximal torus in $\text{Bor}_{1,3}^{(e)}$ (and in $\text{Lor}_{1,3}$) with dimension $\dim(\text{Bor}_{1,3}^{(e)}/T_2^{(e)})$, generated by the semisimple elements $B_\Lambda^{(e)}$ and $R_\omega$. By the Lie–Kolchin theorem as it is written down later, $\text{Bor}_{1,3}^{(e)}$ is upper triangular $[18, 19, 20, 21]$. At this point a remark of S. Weinberg is in order $[23]$: “For the case of zero mass there are interesting complications. The little group as Wigner pointed out is a non-semisimple group, and one must make special remarks about its invariant Abelian subalgebra.” Indeed,

$$\text{Lg} \circ \hat{p} \sim \text{Abelian}_2 \rtimes \text{Abelian}_2.$$  

is the semidirect product (15) of two two-parametric Abelian groups.

### 2.2 Jordan factorisation

The Jordan factorisation of $M \in \text{GL}_4(\mathbb{R})$ into a semisimple and an unipotent component is given by

$$M = M_u M_s.$$
Since Bor$_{1,3}$ is solvable, according to the Lie–Kolchin theorem a basis can be chosen with respect to which $B \in \text{Bor}_{1,3}$ can be put into a triangular form

$$B = \begin{pmatrix}
* & * & * & * \\
0 & * & * & * \\
0 & 0 & * & * \\
0 & 0 & 0 & *
\end{pmatrix} = \begin{pmatrix}
1 & * & * & * \\
0 & 1 & * & * \\
0 & 0 & 1 & * \\
0 & 0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
\lambda_0 & 0 & 0 & 0 \\
0 & \lambda_1 & 0 & 0 \\
0 & 0 & \lambda_2 & 0 \\
0 & 0 & 0 & \lambda_3
\end{pmatrix} = B_u B_s$$

where $B_u$ is unipotent (i.e. all eigenvalues of $B_u$ are 1) and $B_s = \text{diag} (\lambda_0, \lambda_1, \lambda_2, \lambda_3)$ is semisimple. The eigenvalues of $B$ and $B_s$ are identical. In this form, the Jordan decomposition is given by

$$\text{Bor}_{1,3} \subset U_4(\mathbb{R}) \rtimes D_4(\mathbb{R}) \equiv T_4(\mathbb{R}),$$

where $U_4(\mathbb{R})$ is the group of upper triangular unipotent matrices and $D_4(\mathbb{R})$ is the group of invertible diagonal matrices. If $B$ is solvable, then [24]

$$B = \begin{pmatrix}
\lambda_0 & * & * & * \\
0 & \lambda_1 & * & * \\
0 & 0 & \lambda_2 & * \\
0 & 0 & 0 & \lambda_3
\end{pmatrix} \Rightarrow \quad B_u = B \begin{pmatrix}
\lambda_0 & 0 & 0 & 0 \\
0 & \lambda_1 & 0 & 0 \\
0 & 0 & \lambda_2 & 0 \\
0 & 0 & 0 & \lambda_3
\end{pmatrix}^{-1}$$

and $\det B = 1$. From Refs. [18, 19, 20, 21, 22] we take the following set of well-known theorems, lemmas and propositions.

**Theorem:** Let $G$ be a connected linear algebraic group. Then $G$ contains a Borel subgroup $B$, and all other Borel subgroups of $G$ are conjugate to $B$. The homogeneous space $G/B$ is a projective variety ([20], p. 524).

**Lie–Kolchin theorem:** If $\pi : G \to \text{GL}(V)$ is a linear representation of a connected solvable group, then $\pi(G)$ leaves a flag in $V$ invariant, i.e. $\pi(G)$ can be put in triangular form (see e.g. [19], p. 406).

**Borel Fixpoint Theorem:** Let $S$ be a connected, solvable group that acts algebraically
on a projective variety $X$. Then there exists a point $x \in X$ such that (18, p. 137)
\[
\Lambda x = x \quad \text{for all } \Lambda \in S.
\]

Therefore, Eq. (11) is reasonable, as the semi-invariant of $\text{Bor}_{1,3}$ in $\mathbb{E}_{1,3}$ is the non-zero vector $\hat{p}$ spanning the $B$-stable line in $\mathbb{E}_{1,3}$ ($x = \mathbb{R}\hat{p}$).

**Lemma:** Let $V$ be a $\mathbb{C}$-vector space of dimension $n > 0$ and $S$ a connected, solvable subgroup of $\text{GL}(V)$. Then there exists a vector $v \in V \setminus \{0\}$ such that (19, p. 407)
\[
Sv = \mathbb{C}v.
\]

Let $B_s$ denote the set of semisimple elements of $B$ and $B_u$ the set of unipotent elements.

**Proposition:** If $B$ is connected and solvable, then the set $B_u$ is a closed, connected and nilpotent subgroup of $B$, containing $D(G)$ and, therefore, is normal in $B$ (19, p. 407).

From this it follows that the set $B_s$ is not a closed subgroup of $B$ because if it would be a subgroup, $B$ would be nilpotent.

**Proposition:** Let $B$ be connected and solvable and let $b \equiv \mathcal{L}(B)$ be its Lie algebra. Then the set $\mathcal{L}(B_u)$ is the set of nilpotent elements of $g$ (i.e. $\text{ad}_g u$ is nilpotent for $u \in \mathcal{L}(B_u)$) (19, p. 409).

It is important that for solvable $B$ the set $B_u$ of unipotent elements of $B$ is a connected, closed, normal and nilpotent subgroup of $B$, $B/B_u$ is a torus, and $D(B) \subset B_u$.

**Theorem (Borel):** Let $B$ be a connected, solvable, linear algebraic group. If Tor is a maximal torus of $B$, then
\[
B = B_u \rtimes \text{Tor}.
\]

Otherwise, there exists such a torus $\text{Tor} \subset B$ such that
\[
B = \text{Rad}_u(B) \rtimes \text{Tor},
\]
where \( \text{Rad}_u(B) \) is the unipotent radical of \( B \), leading to the factorisation \( [15] \) (this theorem can be found in the original work Ref. [18], p. 137 as well as in Ref. [19], p. 410).

The generators of the Borel subgroup in the representation \( [13] \) are defined by

\[
b^{(e)}_{\mu} = \left. \frac{\partial B^{(e)}(\omega_\mu)}{\partial \omega_\mu} \right|_{\hat{\omega},} \tag{16}
\]

where \( \omega_0 = \lambda, \omega_1 = \xi_1, \omega_2 = \xi_2, \omega_3 = \omega \) and \( \hat{\omega} = (1, 0, 0, 0) \). This yields the Lie algebra \( \text{bor}_{1,3}^{(e)} \) as basis for the underlying vector space to be generated by

\[
\begin{align*}
b^{(e)}_0 & = \left. \frac{\partial \lambda^{(e)}}{\partial \lambda} \right|_{\lambda = 1} = \varepsilon e_{03}, \\
b^{(e)}_1 & = \left. \frac{\partial \xi^{(e)}}{\partial \xi_1} \right|_{\xi = \hat{\xi}} = \varepsilon e_{01} + e_{31}, \\
b^{(e)}_2 & = \left. \frac{\partial \xi^{(e)}}{\partial \xi_2} \right|_{\xi = \hat{\xi}} = \varepsilon e_{02} + e_{32}, \\
b^{(e)}_3 & = \left. \frac{\partial R^\omega}{\partial \omega} \right|_{\omega = 0} = e_{21}.
\end{align*}
\tag{16}
\]

The commutator relations are \((a, b \in \{1, 2\}) \) \( [25, 26, 27, 28, 29] \)

\[
\begin{align*}
[b^{(e)}_0, b^{(e)}_a] & = b^{(e)}_a, & [b^{(e)}_0, b^{(e)}_3] & = 0, \\
[b^{(e)}_3, b^{(e)}_a] & = -\varepsilon_3 a b^{(e)}_b, & [b^{(e)}_a, b^{(e)}_b] & = 0. 
\end{align*}
\tag{17}
\]

The algebra \( \text{bor}_{1,3}^{(e)} \) is solvable because

\[
[D(\text{bor}_{1,3}^{(e)}), D(\text{bor}_{1,3}^{(e)})] = D^2(\text{bor}_{1,3}^{(e)}) = \{0\}
\]

and maximal in \( \text{Lor}_{1,3} \), i.e. the Borel algebra of \( \text{lor}_{1,3} \). Moreover,

\[
\text{bor}_{1,3}^{(e)} = \mathfrak{t}_{2}^{(e)} \times \text{tor}_{1,3}^{(e)}, \tag{18}
\]

where the vector space underlying \( \mathfrak{t}_{2}^{(e)} \) is \( \mathfrak{t}_{2}^{(e)} = \text{span}_\mathbb{R}\{b^{(e)}_1, b^{(e)}_2\} \) and that of \( \text{tor}_{1,3} = \text{car}_{1,3} \) is \( \text{tor}_{1,3} = \text{span}_\mathbb{R}\{b^{(e)}_0, b^{(e)}_3\} \) (\( \text{car}_{1,3} \) is the Cartan subalgebra of \( \text{lor}_{1,3} \)). Therefore, one can conclude that in the massless case the (enlarged) little algebra \( \text{bor}_{1,3}^{(e)} \) is a maximal, non-compact and solvable Lie subalgebra of \( \text{lor}_{1,3} \), i.e. the Borel subalgebra, and is the
semidirect sum of two abelian algebras $t_2^{(e)}$ and $tor_{1,3}^{(e)}$. Notice that there exists no Casimir operator.

In the general case $[18, 19, 20, 21]$, if $B$ is solvable, its Lie algebra $\mathfrak{b} = \mathcal{L}(B)$ is solvable. Since $B_u$ is normal in $B$, its Lie algebra $\mathfrak{n} = \mathcal{L}(B_u)$ is an ideal of $\mathfrak{b}$ and $\mathfrak{n}$ is the set of nilpotent elements of $\mathfrak{b}$. Moreover, since $\mathcal{D}(B) \subseteq B_u$ one has $[\mathfrak{b}, \mathfrak{b}] \subseteq \mathfrak{n}$.

**Theorem:** There exists a Lie subalgebra $\mathfrak{a}$ of $\mathfrak{b}$ obeying the conditions ($[19]$, p. 410)

1. $\mathfrak{a}$ is abelian and all its elements are semisimple, i.e. $\mathfrak{a} \subseteq \mathfrak{b}$;
2. as a vector space one has $\mathfrak{b} = \mathfrak{n} \oplus \mathfrak{a}$.

**Theorem:** Let $\mathfrak{b}$ be algebraic and solvable in $\mathfrak{gl}(V)$ and let $\mathfrak{n}$ be the set of nilpotent endomorphisms of $V$ in $\mathfrak{b}$. If $\mathfrak{h}$ is the maximal commutative Lie subalgebra of $\mathfrak{b}$ consisting of semisimple elements, $\mathfrak{b}$ is the semidirect product of $\mathfrak{h}$ with $\mathfrak{n}$ ($[19]$, p. 455),

$$ \mathfrak{b} = \mathfrak{n} \rtimes \mathfrak{h}. $$

The existence of Borel algebras follows from the triangular decomposition of the semisimple Lie algebra $\mathfrak{g}$ $[18, 19, 20, 21, 30, 31, 32]$, 

$$ \mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_- $$

where

$$ \mathfrak{n}_+ = \sum_{\alpha \in \Phi^+} \mathfrak{g}_\alpha, \quad \mathfrak{n}_- = \sum_{\alpha \in \Phi^+} \mathfrak{g}_{-\alpha} $$

and $\mathfrak{h}$ being the Cartan subalgebra of $\mathfrak{g}$. Then $\mathfrak{b}_\pm = \mathfrak{h} \oplus \mathfrak{n}_\pm$ are maximal solvable Lie subalgebras of $\mathfrak{g}$, called the Borel subalgebra of $\mathfrak{g}$ relative to $\mathfrak{h}$. Moreover,

$$ \mathcal{N}_\mathfrak{g}(\mathfrak{b}_\pm) = \mathfrak{b}_\pm \quad \text{and} \quad \mathcal{Z}_\mathfrak{g}(\mathfrak{b}_\pm) = \{0\} $$

where $\mathcal{N}_\mathfrak{g}$ is the normaliser and $\mathcal{Z}_\mathfrak{g}$ is the centraliser of $\mathfrak{b}_\pm$ in $\mathfrak{g}$.
2.3 Generators of the Borel subgroup

It can be easily seen that the exponential operation provides the parametrisation of the generic element $B^{(e)}(\xi; \lambda, \omega) \in \text{Bor}_{1,3},$

$$B^{(e)}_{\xi \lambda} = \exp(\tilde{\lambda} b_0^{(e)}) = 1 + \sinh \tilde{\lambda} y_0 + (\cosh \tilde{\lambda} - 1) (b_0^{(e)})^2 \quad (\lambda = e^{\tilde{\lambda}}),$$

$$T^{(e)}_\xi = \exp(\tilde{\xi} b_3^{(e)}) = 1 + \tilde{\xi} y_3 + \frac{1}{2} (\tilde{\xi} y_3)^2 = 1 + \tilde{\xi}_1 y_1 + \tilde{\xi}_2 y_2 + \frac{1}{2} \tilde{\xi}_1^2 (y_1^2) + \frac{1}{2} \tilde{\xi}_2^2 (y_2^2),$$

$$R_\omega = \exp(\omega b_3^{(e)}) = 1 + \sin \omega y_3 + (1 - \cos \theta)(y_3^2), \quad (19)$$

The general element of $\text{bor}^{(e)}_{1,3}$ can be written as

$$Y = y^\mu b_\mu = \begin{pmatrix} 0 & \varepsilon y_1 & \varepsilon y_2 & \varepsilon y_0 \\ \varepsilon y_1 & 0 & -y_3 & -y_1 \\ \varepsilon y_2 & y_3 & 0 & -y_2 \\ \varepsilon y_0 & y_1 & y_2 & 0 \end{pmatrix}. \quad (20)$$

The adjoint representation $\text{ad} Y$ in the basis $\{b_1, b_2, b_0, b_3\}$ of the semidirect sum $\rtimes$ is calculated to be

$$\text{ad} Y(= \text{Reg} Y) = \begin{pmatrix} y_0 & -y_3 & -y_1 & y_2 \\ y_3 & y_0 & -y_2 & -y_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (21)$$

From the secular equation $\det(\text{ad} Y - \lambda) = (-\lambda)^2 ((y_0 - \lambda)^2 + (y_3^2)^2)$ it follows that

$$\text{spec}(\text{ad} Y) = \{0, 0, \lambda_3 = y_0 - iy_3^3, \lambda_4 = y_0 + iy_3^3\}. \quad (22)$$

The eigenvalue problem yields two eigenfunctions $Z_3$ and $Z_4,$

$$\text{ad} Y(Z_3) = \lambda_3 Z_3 \Rightarrow Z_3 = \frac{i}{2} (b_1^{(e)} + ib_2^{(e)}),$$

$$\text{ad} Y(Z_4) = \lambda_4 Z_4 \Rightarrow Z_4 = \frac{i}{2} (b_1^{(e)} - ib_2^{(e)}), \quad (23)$$
i.e. $Z_3, Z_4 \in t_2^{(e)}$. In the special case $Y = y^0 b_0^{(e)} + y^3 b_3^{(e)}$ one obtains

$$[y^0 b_0 + y^3 b_3, Z_3] = (y^0 - iy^3)Z_3,$$
$$[y^0 b_0 + y^3 b_3, Z_4] = -(y^0 + iy^3)Z_4. \quad (24)$$

Two cases for $Y$ are important, namely $t_0^{(e)} = \frac{1}{2}(b_0^{(e)} + i b_3^{(e)})$ and $u_0^{(e)} = \frac{1}{2}(b_0^{(e)} - i b_3^{(e)})$.

Combining in pairs with $t_+ = Z_3$ and $u_+ = Z_4$,

$$t_0^{(e)} = \frac{1}{2}(b_0^{(e)} + i b_3^{(e)}), \quad t_+^{(e)} = \frac{i}{2}(b_1^{(e)} + i b_2^{(e)}),$$
$$u_0^{(e)} = \frac{1}{2}(b_0^{(e)} - i b_3^{(e)}), \quad u_+^{(e)} = -\frac{i}{2}(b_1^{(e)} - i b_2^{(e)}), \quad (25)$$

one obtains the commutation relations

$$[t_0^{(e)}, t_+] = t_+^{(e)}, \quad [u_0^{(e)}, u_+] = u_+^{(e)}, \quad [t_0^{(e)}, u_+] = 0. \quad (26)$$

The elements

$$t_0^{(e)} = \frac{1}{2}(b_0^{(e)} + i b_3^{(e)}) = \frac{1}{2}(\varepsilon e_{03} + ie_{21}) = -iJ_3^{(e)}, \quad (27)$$
$$t_+^{(e)} = \frac{i}{2}(b_1^{(e)} + i b_2^{(e)}) = \frac{1}{2}(i\varepsilon e_{01} - \varepsilon e_{02} + ie_{31} - e_{32}) = J_3^{(e)}$$

generate the minimal solvable algebra $\text{sol}_2^{(e)}(t)$ with underlying vector space given by $\text{span}_R\{t_0^{(e)}, t_+^{(e)}\}$. Similarly,

$$u_0^{(e)} = \frac{1}{2}(b_0^{(e)} - i b_3^{(e)}) = \frac{1}{2}(\varepsilon e_{03} - ie_{21}) = iK_3^{(e)}, \quad (28)$$
$$u_+^{(e)} = -\frac{i}{2}(b_1^{(e)} - i b_2^{(e)}) = -\frac{1}{2}(i\varepsilon e_{01} + \varepsilon e_{02} + ie_{31} + e_{32}) = K_3^{(e)}$$

generate the algebra $\text{sol}_2^{(e)}(u)$, and since $[\text{sol}_2^{(e)}(t), \text{sol}_2^{(e)}(u)] = 0$, one obtains the decomposition

$$\text{bor}_{1,3}^{(e)*} = \text{sol}_2^{(e)}(t) \boxplus \text{sol}_2^{(e)}(u), \quad (29)$$

where $\boxplus$ is the Kronecker sum, $A \boxplus B = A \otimes 1 + 1 \otimes B$. 

14
3 Representations

Every representation of lor\(_{1,3}\) defines a particular representation of the subalgebra bor\(_{1,3}\).
Of course, not all the representations are of that kind but those defined by lor\(_{1,3}\) are of
great importance because the classification of particles is determined by their Lorentz
transformation properties according to Eqs. (1) and (2). More precisely, the common
eigenvectors of the representation space of the solvable algebra bor\(_{1,3}\) are the possible helicity states of the particle.

**Theorem:** Let \( g \) be a solvable algebra and \( g \to \Gamma(g) \) be a representation on a finite-
dimensional vector space \( V \). Then ([22], p. 200)

1. there exists a vector \( v \in V \) which is a simultaneous eigenvector for all of \( \Gamma(g) \),

2. there exists a basis of \( V \) with respect to which all elements of \( \Gamma(g) \) are represented
by upper triangular matrices.

Notice that the common eigenvector is determined by all the elements of \( \Gamma(g) \), i.e. in our
case \( g = bor_{1,3} \) there is no need to assume \( \Gamma(t^{(e)}_2) = 0 \). In the complex spaces

\[
\text{span}_\mathbb{C}\{e_\mu : e_\mu^\rho = \eta^\rho_\mu = \delta_{\rho\mu}\}_{0}^3,
\]

the eigenvectors of the solvable algebra sol\(_2^{(e)}(t)\) are

\[
\ell_0^{(e)} = \varepsilon e_{(0)} + e_{(3)} = (\varepsilon, 0, 0, 1)^T, \quad \ell_1 = e_{(1)} + i e_{(2)} = (0, 1, i, 0)^T.
\]

Indeed,

\[
l_0^{(e)} c_0^{(e)} = \frac{1}{2} l_0^{(e)}, \quad l_0^{(e)} c_0^{(e)} = 0,
\]

\[
l_0^{(e)} c_1 = \frac{1}{2} l_1, \quad l_0^{(e)} c_1 = 0.
\]

Accordingly, the eigenvectors of sol\(_2(t)\) are \( \ell_0^{(e)} \) and \( \ell_2 = e_{(1)} - i e_{(2)} = (0, 1, -i, 0)^T \), where

\[
u_0^{(e)} c_0^{(e)} = \frac{1}{2} v_0^{(e)}, \quad v_0^{(e)} c_0^{(e)} = 0,
\]

\[
u_0^{(e)} c_2 = \frac{1}{2} v_2, \quad u_0^{(e)} c_2 = 0.
\]
Since

\[ u_0^{(e)} \ell_1 = -\frac{1}{2} \ell_1, \quad u_+^{(e)} \ell_1 = -i \ell_0^{(e)} , \]
\[ t_0^{(e)} \ell_2 = -\frac{1}{2} \ell_2, \quad t_+^{(e)} \ell_2 = i \ell_0^{(e)} \]

and \( \text{bor}^{*}_{1,3} = \text{sol}_2(t) \oplus \text{sol}_2(u) \), the subspace \( \text{span}_\mathbb{C}\{\ell_0^{(e)}, \ell_1, \ell_2\} \) is invariant under the action of \( \text{bor}_{1,3} \). However, the vector \( \ell_0^{(e)} \) is already the defining vector for \( \text{bor}_{1,3} \) (cf. Eq. (11)). Therefore, there are two helicity states \( \ell_1 \) and \( \ell_2 \) relative to \( \ell_0^{(e)} \). More precisely, using the two components \( \vec{D} \) and \( \vec{B} \) of the Lorentz group defined in Appendix A, the defining Eq. (11) yields the conditions

\[ D_3 \ell_0^{(e)} = 0, \quad B_3 \ell_0^{(e)} = 1 \ell_0^{(e)}, \quad (30iii) \]

\( \ell_1 \) is called right-handed with respect to \( \ell_0^{(e)} \), and \( \ell_2 \) is called left-handed with respect to \( \ell_0^{(e)} \). The value 1 in Eq. (30iii) may be considered as helicity 1 (not spin because there is no rotation \( \text{SO}_3 \)). The conditions (30iii) are equivalent to

\[ t_0^{(e)} \ell_0^{(e)} = u_0^{(e)} \ell_0^{(e)} = \frac{1}{2} \ell_0^{(e)}, \quad t_+^{(e)} \ell_0^{(e)} = u_+^{(e)} \ell_0^{(e)} = 0. \]

3.1 The Chevalley basis

Depending to the sign of \( \varepsilon = \pm 1 \), the elements of \( \text{sol}_2(t) \) and \( \text{sol}_2(u) \) can be represented by elements of the fundamental \( \text{sl}_2 \) (or Chevalley) basis

\[ h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \]

According to the commutator relations, for \( \varepsilon = +1 \) one can represent \( t_0^{(e)} \) and \( u_0^{(e)} \) by \( \mp \varepsilon i h / 2 \), \( t_+^{(e)} \) and \( u_+^{(e)} \) by \( i e \) and \( i f \) – or vice versa for \( \varepsilon = -1 \) (cf. Eqs. (A37, A38)). The two solvable algebras \( \text{sol}_2(e) = \text{span}_\mathbb{R}\{h, ie\} \) and \( \text{sol}_2(f) = \text{span}_\mathbb{R}\{h, if\} \) are representations of the algebras \( \text{sol}_2(t) \) and \( \text{sol}_2(u) \), respectively,

\[ t_0^{(e)} = \left( \frac{1 + \varepsilon}{4} h \right) \oplus \left( \frac{1 - \varepsilon}{2} h \right) = -i J_3^{(e)}, \quad t_+^{(e)} = \left( \frac{1 + \varepsilon}{2} ie \right) \oplus \left( \frac{1 - \varepsilon}{2} ie \right) = J_+^{(e)}. \]
\[ u_0^{(\varepsilon)} = - \left( \left( \frac{1 - \varepsilon}{4} h \right) \uplus \left( \frac{1 + \varepsilon}{4} h \right) \right) = iK_3^{(\varepsilon)}, \quad u_+^{(\varepsilon)} = \left( \frac{1 - \varepsilon}{2} i f \right) \uplus \left( \frac{1 + \varepsilon}{2} i f \right) = K_3^{(\varepsilon)}. \] (31)

This can be generalised to arbitrary representations as for instance the subrepresentations of the representation \((k, l)\) of \(lor_1, 3\). The general representation \((k, l)\) is given by the rule

\[ \pi^{(k,l)} : bor_{1,3}^{(\varepsilon)*} = sol_2^{(\varepsilon)}(e) \uplus sol_2^{(\varepsilon)}(f) \rightarrow \pi^{(k,l)}(bor_{1,3}^{(\varepsilon)*}) = \pi^{(k)}(sol_2^{(\varepsilon)}(e)) \uplus \pi^{(l)}(sol_2^{(\varepsilon)}(f)), \] (32)

where

\[ sol_2^{(\varepsilon)}(e) = \frac{1 + \varepsilon}{2} sol_2(e) \uplus \frac{1 - \varepsilon}{2} sol_2(e), \quad sol_2^{(\varepsilon)}(f) = \frac{1 - \varepsilon}{2} sol_2(f) \uplus \frac{1 + \varepsilon}{2} sol_2(f). \] (33)

By virtue of this construction one obtains

\[ \pi^{(k,l)}(t_0^{(\varepsilon)}) = - \left( \frac{i(1 + \varepsilon)}{4} \pi^{(k)}(h) \uplus \frac{i(1 - \varepsilon)}{4} \pi^{(l)}(h) \right) = \]

\[ = \left\{ \begin{array}{ll}
- \frac{i}{2} \pi^{(k)}(h) \otimes 1_l & \text{for } \varepsilon = +1, \\
- 1_k \otimes \frac{i}{2} \pi^{(l)}(h) & \text{for } \varepsilon = -1,
\end{array} \right. \]

\[ \pi^{(k,l)}(t_+^{(\varepsilon)}) = \frac{i(1 + \varepsilon)}{2} \pi^{(k)}(e) \uplus \frac{i(1 - \varepsilon)}{2} \pi^{(l)}(e) = \]

\[ = \left\{ \begin{array}{ll}
i \pi^{(k)}(e) \otimes 1_l & \text{for } \varepsilon = +1, \\
1_k \otimes i \pi^{(l)}(e) & \text{for } \varepsilon = -1,
\end{array} \right. \]

\[ \pi^{(k,l)}(u_0^{(\varepsilon)}) = \frac{i(1 - \varepsilon)}{4} \pi^{(k)}(h) \uplus \frac{i(1 + \varepsilon)}{4} \pi^{(l)}(h) = \]

\[ = \left\{ \begin{array}{ll}
1_k \otimes \frac{i}{2} \pi^{(l)}(h) & \text{for } \varepsilon = +1, \\
\frac{i}{2} \pi^{(k)}(h) \otimes 1_l & \text{for } \varepsilon = -1,
\end{array} \right. \]

\[ \pi^{(k,l)}(u_+^{(\varepsilon)}) = \frac{i(1 - \varepsilon)}{2} \pi^{(k)}(f) \uplus \frac{i(1 + \varepsilon)}{2} \pi^{(l)}(f) = \]

\[ = \left\{ \begin{array}{ll}
1_k \otimes i \pi^{(l)}(f) & \text{for } \varepsilon = +1, \\
i \pi^{(k)}(f) \otimes 1_l & \text{for } \varepsilon = -1,
\end{array} \right. \] (34)

where

\[ \pi^{(k)}(h)|k, m\rangle = 2m|k, m\rangle, \]
\[
\pi^{(k)}(e)|k, m\rangle = \rho^{(k)}_{(m)}|k, m + 1\rangle,
\]
\[
\pi^{(k)}(f)|k, m\rangle = \rho^{(k)}_{(-m)}|k, m - 1\rangle,
\]
(35)

where \(\rho^{(k)}_{(m)} = \sqrt{(k - m)(k + m + 1)}\). In the representation of \(\pi^{(k,l)}\) by direct products,
\[
\{|k, l; m_k, m_l\rangle = |k, m_k\rangle \otimes |l, m_l\rangle : -k \leq m_k \leq k; -l \leq m_l \leq l\}
\]

one obtains common eigenvectors for sol\(_2(t)\) given by
\[
\pi^{(k,l)}(t_0^{(+)})|k, l; k, m_l\rangle = k|k, l; k, m_l\rangle, \quad \pi^{(k,l)}(t_+^{(+)})|k, l; k, m_l\rangle = 0
\]
(36)

with \(m_l = -l, -l + 1, \ldots, l\), and common eigenvectors for sol\(_2(u)\) given by
\[
\pi^{(k,l)}(u_0^{(+)})|k, l; m_k, -l\rangle = l|k, l; m_k, -l\rangle, \quad \pi^{(k,l)}(u_+^{(+)})|k, l; m_k, -l\rangle = 0
\]
(37)

with \(m_k = -k, -k + 1, \ldots, k\).

### 3.2 Resolution of the solvable group

sol\(_2(e)\) is a Lie algebra of orientation-conserving affine translations Aff\(_1\). The underlying
topological space \(\mathbb{R} \times \mathbb{R}_+\) for the corresponding Lie group Sol\(_2(e)\) is simply connected and
open in the plane \(\mathbb{R}^2\). As a general element of this group can be represented by
\[
S(\beta, \alpha) = \begin{pmatrix} e^\alpha & \beta \\ 0 & 1 \end{pmatrix}
\]
\((\alpha, \beta \in \mathbb{R})\), the geometrical space on which this group acts is the real line,
\[
\text{Sol}_2(e) \ni S(\beta, \alpha) : \mathbb{R}^1 \ni x \to e^\alpha x + \beta \in \mathbb{R}^1.
\]

Because of
\[
S(\beta, \alpha)S(\beta', \alpha') = \begin{pmatrix} e^{\alpha + \alpha'} & e^\alpha \beta' + \beta \\ 0 & 1 \end{pmatrix} = S(e^\alpha \beta' + \beta, \alpha + \alpha'),
\]
the solvable group is a semidirect product of two abelian groups,
\[
\text{Sol}_2(e) = \mathbb{R} \rtimes \mathbb{R}_+.
\]

Therefore, Sol\(_2\) is
1. locally compact
2. simply connected
3. minimal non-abelian
4. non-compact
5. non-semisimple, i.e. solvable,
6. non-unimodular.

Via the exponential mapping, these two parts are generated by $e$ and $h_+ = \frac{1}{2}(1 + h)$. Therefore, one can write $\text{sol}_2(e) = \mathbb{R}e \rtimes \mathbb{R}h_+$. A similar consideration leads to $\text{sol}_2(f) = \mathbb{R}f \rtimes \mathbb{R}h_-$ where $h_- = \frac{1}{2}(1 - h)$. Moreover, $\text{sol}_2(e)$ and $\text{sol}_2(f)$ are related to the Cartan involution. Therefore, we end up with the amusing two-fold decomposition

$$\text{bor}_{1,3}^* = (\mathbb{R}e \rtimes \mathbb{R}h_+) \oplus (\mathbb{R}f \rtimes \mathbb{R}h_-)$$

where

$$h_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

It is important to note that the Kronecker sum structure is imposed by semisimplicity because $\text{bor}_{1,3}$ is a real form of $\text{so}_4(\mathbb{C})$, while the semidirect product structure is caused by solvability.

### 3.3 Weinberg’s ansatz

From Steven Weinberg we adopt the following statements which we subsume under the keyword of “Weinberg’s Ansatz” \[4, 5\]:

1. If a massless particle is equal to it’s antiparticle, it is described by the irreducible representation $(k, k)$ of the proper Lorentz group.
2. If a massless particle is not equal to its antiparticle, the particle is described by the irreducible representation \((k, 0)\) of the proper Lorentz group, the antiparticle is described by the irreducible representation \((0, k)\) of the proper Lorentz group. Note that the massless particle is defined via the Borel subgroup by the irreducible representation of the proper Lorentz group without necessity to introduce parity separately.

For the representation \((k, k)\), from Eq. (36) one obtains the helicity states associated with \(\text{sol}_2^{(+)}(e)\),

\[
|k, k; k, -k + p\rangle, \quad p = 0, 1, 2, \ldots, 2k,
\]

and from Eq. (37) one obtains the helicity states associated with \(\text{sol}_2^{(+)}(f)\),

\[
|k, k; k - p, -k\rangle, \quad p = 0, 1, 2, \ldots, 2k.
\]

However, the condition (30iii) excludes the state \(|k, k; k, -k\rangle\), since

\[
D_3^{(k,k)}|k, k; k, -k\rangle = i(k - k)|k, k; k, -k\rangle = 0,
\]

\[
B_3^{(k,k)}|k, k; k, -k\rangle = 2k|k, k; k, -k\rangle.
\]

Therefore, for the particle \((\varepsilon = +1)\) with zero mass and helicity \(\lambda = 2k\) one obtains the 4k helicity states

\[
|k, k; k, -k + p\rangle, \quad |k, k; k - p, -k\rangle, \quad p = 1, 2, \ldots, 2k
\]

relative to the central state \(|k, k; k, -k\rangle\) which is determined by the conditions

\[
t_0^{(+)}|k, k; k, -k\rangle = u_0^{(+)}|k, k; k, -k\rangle = k|k, k; k, -k\rangle,
\]

\[
t_+^{(+)}|k, k; k, -k\rangle = u_+^{(+)}|k, k; k, -k\rangle = 0.
\]

On setting \(\varepsilon = -1\) in Eq. (34), the helicity states become

\[
|k, k; -k + p, k\rangle, \quad |k, k, -k; k - p\rangle, \quad p = 1, 2, \ldots, 2k
\]
The central state $|k, k; -k, k\rangle$ is defined by the conditions

$$
t_0(-)|k, k; -k, k\rangle = u_0(-)|k, k; -k, k\rangle = k|k, k; -k, k\rangle,
$$

$$
t_+(-)|k, k; -k, k\rangle = u_+(-)|k, k; -k, k\rangle = 0.
$$

(40)

Notice that for the vector case $(\frac{1}{2}, \frac{1}{2})$ and $\varepsilon = +1$ the helicity states are given by

$$
|\frac{1}{2}, \frac{1}{2}; \frac{1}{2}, \frac{1}{2}\rangle, \quad |\frac{1}{2}, \frac{1}{2}; -\frac{1}{2}, -\frac{1}{2}\rangle
$$

relative to the central state $|\frac{1}{2}, \frac{1}{2}; \frac{1}{2}, \frac{1}{2}\rangle$. For $\varepsilon = -1$ the helicity states are the same, but now relative to the central state $|\frac{1}{2}, \frac{1}{2}; -\frac{1}{2}, \frac{1}{2}\rangle$. Therefore, the two polarisations of the photon as basic quantities in physics are determined by the proper Lorentz group $\text{Lor}_{1,3}$ without taking refuge to the parity.

As a further example, the massless particle of helicity $\lambda = 2$ is associated with the representation $(1, 1)$ and has 4 helicity states relative to the state $|1, 1; 1, -1\rangle$: two right-handed states $|1, 1; 1, 1\rangle, |1, 1; 1, 0\rangle$ and two left-handed states $|1, 1; 0, -1\rangle, |1, 1; -1, -1\rangle$.

### 3.4 Weyl equations

The Weyl equations are of the type $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$. The representation $(\frac{1}{2}, 0)$ is defined by the commutative diagram \[33, 34, 35\]

$$
\text{Lor}_{1,3} \ni \Lambda : \quad \mathbb{R}_{1,3} \ni p^\mu \quad \rightarrow \quad (\Lambda p)^\mu = \Lambda^{\mu\nu}p^\nu
$$

\[\downarrow \quad \downarrow \sigma\]

$$
\text{SL}_2(\mathbb{C}) \ni \pm A_\Lambda : \quad \mathbb{H}_2 \in \sigma(p) = \sigma_\mu p^\mu \quad \rightarrow \quad A_\Lambda \sigma(p) A_\Lambda^\dagger
$$

where the commutativity of the diagram results in

$$
A_\Lambda \sigma(p) A_\Lambda^\dagger = \sigma(\Lambda p) = \sigma_\mu \Lambda^\mu\nu p^\nu.
$$

Using $\tilde{\sigma} = (\mathbb{1}; -\bar{\sigma})$, one obtains

$$
A_\Lambda = \frac{\Lambda^\mu\nu \sigma_\mu \tilde{\sigma}_\nu}{2 \text{tr}(A_\Lambda^\dagger)}, \quad \Lambda^\mu_{\Lambda\nu} = \frac{1}{2} \text{tr}(\sigma_\mu A_\Lambda \sigma_\nu A_\Lambda^\dagger).
$$

(41)
Using the exponential representations \( \Lambda = \exp\left(-\frac{1}{2} \omega^{\mu\nu} e_{\mu\nu}\right) \) and \( A_\Lambda = \exp\left(-\frac{1}{2} \omega^{\mu\nu} m_{\mu\nu}\right) \), one obtains a relation between the generators,

\[
m_{\mu\nu} = \frac{1}{4} (e_{\mu\nu})^{\alpha\beta} \sigma_\alpha \tilde{\sigma}_\beta = \begin{cases} 
m_{kl} = \frac{i}{2} \epsilon_{kl} \delta \sigma_j \\
m_{0j} = \frac{1}{2} \sigma_j
\end{cases}
\]

One can obtain the transformation rule of the Weyl spinor by looking at the commutative diagram

\[
\psi_R : \mathbb{E}_{1,3} \ni x \longrightarrow \psi_R(x)
\]

\[
U(\Lambda) \downarrow \Lambda \quad \downarrow A_\Lambda
\]

\[
U(\Lambda) \psi_R : \mathbb{E}_{1,3} \ni \Lambda x \longrightarrow (U(\Lambda) \psi_R)(\Lambda x)
\]

implying \((U(\Lambda) \psi_R)(\Lambda x) = T^{(1/2,0)} \psi_R(x) = A_\Lambda \psi_R(x)\). The Weyl equation

\[
\tilde{\sigma}_\mu \partial^\mu \psi_R(x) = 0
\]

is given in momentum space by \( \tilde{\sigma}_\mu p^\mu \psi_R(p) = 0 \). For the standard vector \( \hat{p} = (\varepsilon, 0, 0, 1)^T \) this equation reduces to

\[
\sigma_3 \ell_R^{(e)} = \varepsilon \ell_R^{(e)},
\]

having the solutions \( \ell_R^{(e)} = (1 + \varepsilon) a \ell_1 + (1 - \varepsilon) b \ell_2 \) with \( \ell_1 = (1, 0)^T \) and \( \ell_2 = (0, 1)^T \). The Borel algebra \( \text{bor}^{(e)}_{1,3}(1/2, 0) \) can be expressed as

\[
\begin{align*}
b_0^{(e)}(1/2, 0) &= \frac{\varepsilon}{2} h, \quad b_1^{(e)}(1/2, 0) = \frac{1 + \varepsilon}{2} e - \frac{1 - \varepsilon}{2} f, \\
b_3^{(e)}(1/2, 0) &= -\frac{i}{2} h, \quad b_2^{(e)}(1/2, 0) = -\frac{i(1 + \varepsilon)}{2} e - \frac{i(1 - \varepsilon)}{2} f.
\end{align*}
\]

The algebras \( \text{sol}^{(e)}_{2}(e) \) and \( \text{sol}^{(e)}_{2}(f) \) have the form

\[
\begin{align*}
\text{sol}^{(e)}_{2}(e) &= \left\{ t_0^{(e)}(1/2, 0) = \frac{1 + \varepsilon}{4} h, \quad t_+^{(e)}(1/2, 0) = \frac{i(1 + \varepsilon)}{2} e \right\}, \\
\text{sol}^{(e)}_{2}(f) &= \left\{ u_0^{(e)}(1/2, 0) = \frac{1 - \varepsilon}{4} h, \quad u_+^{(e)}(1/2, 0) = \frac{i(1 - \varepsilon)}{2} f \right\}.
\end{align*}
\]

The common eigenvector for \( \text{sol}^{(e)}_{2}(e) \) is \( \ell_1 = (1, 0)^T \),

\[
t_0^{(e)}(1/2, 0) \ell_1 = \frac{1 + \varepsilon}{4} \ell_1, \quad t_+^{(e)}(1/2, 0) \ell_1 = 0,
\]

\[22\]
and for sol$_2^{(c)}(f)$ one obtains $\ell_2 = (0, 1)^T$,

$$u_0^{(\varepsilon)}(\frac{1}{2}, 0)\ell_2 = -\frac{1 - \varepsilon}{4} \ell_2,$$

$$u_+^{(\varepsilon)}(\frac{1}{2}, 0)\ell_2 = 0.$$ 

Therefore, the eigenvector of $\text{bor}^{(c)}_{1,3}(\frac{1}{2}, 0)$ is exactly equal to the solution of Weyl’s equation, where $\ell_1$ is right handed and $\ell_2$ is left handed. Notice that in case of the irreducible representation $(\frac{1}{2}, 0)$ of the proper Lorentz group there exists only one single solution, i.e. one helicity state $\lambda = \frac{1}{2}$.

More generally, the representation space of the Lorentz representation $(k, 0)$ is given by

$$V^{(+)}(k, 0) = \text{span}_\mathbb{C}\{|k, 0; m, 0\} : m = -k, -k + 1, \ldots, k\}.$$ 

The action of $\text{bor}^{(+)}_{1,3}(k, 0)$ on $V^{(+)}(k, 0)$ can be written as

$$t_0^{(+)}|k, 0; m, 0\rangle = m|k, 0; m, 0\rangle, \quad u_0^{(+)}|k, 0; m, 0\rangle = 0,$$

$$t_+^{(+)}|k, 0; m, 0\rangle = i\rho_{(m)}^{(k)}|k, 0; m + 1, 0\rangle, \quad u_+^{(+)}|k, 0; m, 0\rangle = 0.$$ 

Therefore, there exists only a single eigenvector $|k, 0; k, 0\rangle \in V^{(+)}(k, 0)$ of the Borel algebra $\text{bor}^{(+)}_{1,3}(k, 0)$, i.e. one a single helicity state with

$$t_0^{(+)}|k, 0; k, 0\rangle = k|k, 0; k, 0\rangle,$$

$$t_+^{(+)}|k, 0; k, 0\rangle = 0,$$

$$u_0^{(+)}|k, 0; k, 0\rangle = u_+^{(+)}|k, 0; k, 0\rangle = 0.$$ 

For $\varepsilon = -1$ the single helicity state can be written as $|k, 0; -k, 0\rangle$ with

$$t_0^{(-)}|k, 0; -k, 0\rangle = t_+^{(-)}|k, 0; -k, 0\rangle = 0,$$

$$u_0^{(-)}|k, 0; -k, 0\rangle = k|k, 0; -k, 0\rangle,$$

$$u_+^{(-)}|k, 0; -k, 0\rangle = 0.$$ 

In the irreducible case $(0, k)$ (and $\varepsilon = +1$) the representation space reads

$$V^{(+)}(0, k) = \text{span}_\mathbb{C}\{|0, k; 0, m\} : m = -k, -k + 1, \ldots, k\},$$

23
and the action of \( \text{bor}_{1,3}^{(+)}(0, k) \) is given by

\[
\begin{align*}
\text{t}_0^{(+)}|0, k; 0, m\rangle &= 0, & u_0^{(+)}|0, k; 0, m\rangle &= -m|0, k; 0, m\rangle, \\
\text{t}_+^{(+)}|0, k; 0, m\rangle &= 0, & u_+^{(+)}|0, k; 0, m\rangle &= i\rho^{(k)}_{(-m)}|0, k; 0, m - 1\rangle.
\end{align*}
\]

The only eigenvector of \( \text{bor}_{1,3}^{(+)}(0, k) \) is \(|0, k; 0, -k\rangle\) with

\[
\begin{align*}
\text{t}_0^{(+)}|0, k; 0, -k\rangle &= \text{t}_+^{(+)}|0, k; 0, -k\rangle = 0, \\
u_0^{(+)}|0, k; 0, -k\rangle &= k|0, k; 0, -k\rangle, \\
u_+^{(+)}|0, k; 0, -k\rangle &= 0.
\end{align*}
\]

For \( \varepsilon = -1 \) one obtains the action

\[
\begin{align*}
\text{t}_0^{(-)}|0, k; 0, m\rangle &= m|0, k; 0, m\rangle, & u_0^{(-)}|0, k; 0, m\rangle &= 0, \\
\text{t}_+^{(-)}|0, k; 0, m\rangle &= i\rho^{(k)}_{(m)}|0, k; 0, m + 1\rangle, & u_+^{(-)}|0, k; 0, m\rangle &= 0
\end{align*}
\]

and the only eigenvector \(|0, k; 0, k\rangle\).

### 4 Conclusions

Turning back to the title of our paper, “no” in “nophysics” can also stand for

- no semisimple (instead, solvable) solution,
- no abelian (instead, the minimal non-abelian) solution,
- no compact (but locally compact) solution,
- no unimodular (instead, non-unimodular) solution,
- no Killing form (because the Cartan–Killing metric tensor is identically zero on the derived algebra), and
- no Casimir invariant found.
If semisimplicity, as we are being used to it, stands for “yes”, solvability represents “no”. However, in our opinion, physics as it should be treated is semisimple as well as solvable, providing a good perspective for our view at the phenomenon of mass.

Taking solvability as the internal symmetry of massless particles, the photon of helicity $\lambda = 1$ is represented by $(\frac{1}{2}, \frac{1}{2})$, and Pauli’s neutrino and antineutrino of helicity $\lambda = \frac{1}{2}$ by $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$, respectively. Finally, we might ask about the graviton. If the graviton is identical to its own antiparticle, according to Weinberg’s ansatz it is represented by $(1,1)$ with helicity $\lambda = 2$. $(1,0)$ and $(0,1)$ can stand for an hypothetical massless vector boson and its antiboson, both with helicity $\lambda = 1$ but with e.g. opposite charge (e.g. massless charges $W$ bosons). However, since open charges for massless particles have not been seen in any of the accelerator experiments, the validity of Weinberg’s ansatz challenges the graviton as independent particle, i.e. the quantisation of gravity as a whole.

Finally, accepting the Borel algebra as symmetry algebra for massless particles, it is reasonable to develop a Yang–Mills theory for solvable groups, treating the elementary algebras $\text{su}_2$ and $\text{sol}_2$ on the same footing. Indeed, while the algebra $\text{su}_2$ generates semisimple algebras via the Cartan matrix, the solvable algebras are constructed gradually as semidirect sums of abelian algebras. Moreover, the solvable gauge will more immediately generate the abelian gauge of the field theory of electromagnetism, according to ideas presented by Helfer, Nuyts and others [52, 53, 54].

Acknowledgements

This work was supported by the Estonian Research Council under Grant No. IUT2-27. The authors are grateful to Z. Oziewicz for interesting discussions on the Lorentz group.
A The Lorentz group

The Lorentz group is usually given by its action on the Minkowski vector space \( \mathbb{E}_{1,3} \) with the metric \( \eta = \text{diag}(1; -1, -1, -1) \). By definition, the Lorentz group \( O_{1,3} \) preserves the invariant \( x \cdot y = x^\mu \eta_{\mu\nu} y^\nu = x^T \eta y, \ x, y \in \mathbb{E}_{1,3} \) (cf. e.g. Refs. [36, 37, 38, 39, 40, 41, 42]), i.e.

\[
\Lambda x \cdot \Lambda y = x^T \Lambda^T \eta \Lambda y = x^T \eta y = x \cdot y \Rightarrow \Lambda^T \eta \Lambda = \eta, \quad (A1)
\]

where \( O_{1,3} \ni \Lambda : \mathbb{E}_{1,3} \ni x \rightarrow \Lambda x \in \mathbb{E}_{1,3} \). In the matrix form one obtains

\[
(\Lambda x)^\mu = \Lambda^\mu_\nu x^\nu \quad \text{and} \quad \Lambda^\mu_\nu \eta_{\mu\rho} \Lambda^\rho_\sigma = \eta_{\nu\sigma}.
\]

The group \( O_{1,3} \) is topologically homeomorphic to \( O_3 \times \mathbb{R}^3 \), and the number of connected components is four. Henceforth we will consider mainly the component connected to unity, \( \text{Lor}_{1,3} = \text{SO}^0_{1,3} \), called the proper orthochronous Lorentz group,

\[
\text{Lor}_{1,3} = \{ \Lambda \in \text{M}_4(\mathbb{R}) : \Lambda^T \eta \Lambda = \eta, \ \det \Lambda = 1, \ \Lambda^0_0 \geq 1 \}. \quad (A2)
\]

\( \text{Lor}_{1,3} \) is a normal subgroup of \( O_{1,3} \). The Lorentz group of all proper orthochronous Lorentz transformations of coordinates on the Minkowski space is a six-parameter matrix Lie group. The domain of the six parameters is given by

\[
D = \{ \eta_1, \eta_2, \eta_3, \omega_1, \omega_2, \omega_3 : \eta_i \in \mathbb{R}, \ -\pi < \omega_1 \leq \pi, \ 0 < \omega_2 \leq \pi, \ -\pi < \omega_3 \leq \pi \},
\]

where the boundary points are topologically identified. The resulting region is homeomorphic to \( \mathbb{R}^3 \times \mathbb{P}_3 \) where \( \mathbb{P}_3 \) is the three-dimensional projective space, covered twice by the simply connected three-dimensional unit sphere. Therefore, \( \text{Lor}_{1,3} \) is locally compact and doubly connected, path connected, simple and reductive. The universal covering group is \( \text{SL}_2(\mathbb{C}) \), i.e.

\[
\text{Lor}_{1,3} = \text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C})/\mathbb{Z}_2 = \text{SO}_3(\mathbb{C}) \quad (A3)
\]

where realification of complex groups is understood.
A.1 Matrix representation

Assuming the Minkowski metric, it is convenient to write the defining representation for $\Lambda$ blockwise,

$$\Lambda = \begin{pmatrix} A & \vec{B}^T \\ \vec{C} & D \end{pmatrix} \quad \text{(A4)}$$

where $A = \Lambda^0_0 \geq 1$, $B_k = \Lambda^0_k$, $C_k = \Lambda^k_0$, and

$$D = (D^j_i) \in \text{GL}(3, \mathbb{R}).$$

For $\Lambda \in \text{Lor}_{1,3}$, one has

$$\Lambda^T \eta \Lambda = \eta \Rightarrow \begin{cases} A^2 = 1 + \vec{C}^T \vec{C} = 1 + |\vec{C}|^2 \\
D^T D = 1_3 + \vec{B} \vec{B}^T \\
A \vec{B} - D^T \vec{C} = 0 \end{cases} \quad \text{(A5.1)}$$

$$\Lambda \eta \Lambda^T = \eta \Rightarrow \begin{cases} A^2 = 1 + \vec{B}^T \vec{B} = 1 + |\vec{B}|^2 \\
D D^T = 1_3 + \vec{C} \vec{C}^T \\
A \vec{C} - D \vec{B} = 0 \end{cases} \quad \text{(A5.2)}$$

Using these equations, it is easy to see that

$$\Lambda^{-1} = \begin{pmatrix} A & -\vec{C}^T \\ -\vec{B} & D^T \end{pmatrix}.$$ 

Because of $\det \Lambda = 1$, one has $\det D = A \geq 1$. One can use the equations to rewrite

$$\vec{C} = \frac{1}{A} D \vec{B} \quad \text{or} \quad \vec{B} = \frac{1}{A} D^T \vec{C}$$

to obtain

$$\Lambda = \begin{pmatrix} A & \vec{B}^T \\ \frac{1}{A} D \vec{B} & D \end{pmatrix}^{1/2} = \begin{pmatrix} A & \frac{1}{A} \vec{C}^T D \\ \vec{C} & D \end{pmatrix}. \quad \text{(A6)}$$

As a consequence of this, one can apply a polar decomposition [43, 44, 45] to the elements of the Lorentz group $\text{Lor}_{1,3}$, $\Lambda = Q P = P' Q$ where $Q$ is orthogonal and $P, P'$ are real symmetric positive definite. Using the ansatz

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}, \quad P = \begin{pmatrix} A & \vec{B}^T \\ \vec{B} & D_P \end{pmatrix}, \quad P' = \begin{pmatrix} A & \vec{C}^T \\ \vec{C} & D'_{P'} \end{pmatrix} \quad \text{(A7)}$$

27
with \( R \in SO_3 \) and \( D_P, D'_P \) symmetric, one obtains

\[
R = \frac{1}{1 + A} (D + AD^{-1}T), \quad D_P = \frac{1}{1 + A} (A + D^T D), \quad D'_P = \frac{1}{1 + A} (A + DD^T).
\] (A8)

According to Tolhoek’s theorem [24], \( P = (\Lambda^T \Lambda)^{1/2} \) and \( P' = (\Lambda \Lambda^T)^{1/2} \) describe pure Lorentz transformations or boosts, where

\[
A = \frac{1}{\sqrt{1 - v^2/c^2}}, \quad \vec{C} = \frac{1}{c} A\vec{v}.
\]

Vice versa, \( \Lambda = QP(\vec{B}) = P(R\vec{B})Q \) can be written as

\[
\Lambda = \begin{pmatrix} A & \vec{B}^T \\ R\vec{B} & R + \frac{1}{1 + A} R\vec{B}\vec{B}^T \end{pmatrix} = \begin{pmatrix} A & \vec{C}^T R \\ \vec{C} & R + \frac{1}{1 + A} \vec{C}\vec{C}^T R \end{pmatrix},
\] (A9)

where \( A = \Lambda^0_0 \geq 1, \vec{B}, \vec{C} \in \mathbb{R}^3 \) and \( R \in SO_3 \). The Principal axis theorem for the group \( Lor_{1,3} \), finally, tells us that every \( \Lambda \in Lor_{1,3} \) has one of the shapes

\[
SA_s S^{-1} = S \begin{pmatrix} \text{lor} t & 0 \\ 0 & \text{rot} \omega \end{pmatrix} S^{-1} \quad \text{or}
\]

\[
SA_u S^{-1} = S \begin{pmatrix} 1 & 0 \\ 0 & N \end{pmatrix} S^{-1},
\] (A10)

where

\[
\text{lor} t = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}, \quad t \in \mathbb{R},
\]

\[
\text{rot} \omega = \begin{pmatrix} \cos \omega & -\sin \omega \\ \sin \omega & \cos \omega \end{pmatrix} \in SO_2, \quad -\pi < \omega \leq \pi.
\] (A11)

\( S \in Lor_{1,3} \) is a similarity transformation. Since \( \text{spec} \Lambda_s = \{e^t, e^{-t}, e^{i\omega}, e^{-i\omega}\} \), \( \Lambda_s \) is semisimple. On the contrary \( \Lambda_u \) us unipotent. Therefore, the Jordan form of \( N \) is

\[
\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}
\]

and \( \text{spec} \Lambda_u = \{1\} \). Finally, we note that for all \( \Lambda \in Lor_{1,3} \), \( \Lambda \) satisfies the minimal equation [46]

\[
\Lambda^4 - (\text{tr} \Lambda)\Lambda^3 + \frac{1}{2} (\text{tr} \Lambda)^2 - \text{tr} \Lambda^2 \Lambda^2 - (\text{tr} \Lambda) \Lambda + 1_4 = 0.
\]
Though it is a pure algebraic reason, one concludes from polar decomposition that the maximal compact subgroup of Lor$_{1,3}$ is isomorphic to SO$_3$. Indeed, Lor$_{1,3}$ is isomorphic to SO$_3(\mathbb{C})$, and SO$_3$ is the compact real form of the latter. Moreover, since Lor$_{1,3}$ is stable under the transposition (i.e. \( \Lambda \in \text{Lor}_{1,3} \Rightarrow \Lambda^T \in \text{Lor}_{1,3} \)), Lor$_{1,3}$ is a linear reductive group, for which the maximal compact subgroup \( K \) is determined by the Cartan involution

\[
\theta : \text{Lor}_{1,3} \ni \Lambda \to \theta(\Lambda) = \Lambda^{-1T} \in \text{Lor}_{1,3}
\]
as \( K = \{ \Lambda \in \text{Lor}_{1,3} : \theta(\Lambda) = \Lambda^{-1T} = \Lambda \} = \text{SO}_3 \). In this setting, Lor$_{1,3}$/SO$_3$ is a symmetric space.

It is important that the maximal simple compact subgroup SO$_3 \subset$ Lor$_{1,3}$ determines the internal symmetry of massive particles, i.e. the spin. On the other hand, the maximal solvable noncompact subgroup called Borel subgroup Bor$_{1,3} \subset$ Lor$_{1,3}$ determines the helicity of massless particles. Notice that Bor$_{1,3}$ is a semidirect product of the abelian subgroups \( T_2 \) and Tor$_{1,3}$, Bor$_{1,3} = T_2 \rtimes \text{Tor}_{1,3} \), where Tor$_{1,3}$ is the maximal Torus of Lor$_{1,3}$.

### A.2 Generators of Lor$_{1,3}$

In order to linearise the group Lor$_{1,3}$, one can simply differentiate it and evaluate the derivative at the identity element of the group. The tangent space at the identity element is the Lie algebra

\[
\text{lor}_{1,3} = \{ X \in M_4(\mathbb{R}) : e^{tX} \in \text{Lor}_{1,3} \forall t \in \mathbb{R} \}.
\]

(A12)

According to Lie’s theorem, the exponential map \( \exp : \text{lor}_{1,3} \to \text{Lor}_{1,3} \) is surjective. Therefore, any element \( \Lambda \in \text{Lor}_{1,3} \) that is close to unity can be written as the exponential of an element \( X \in \text{lor}_{1,3} \). From Eq. (A11) one concludes that the defining equation for an element \( X \in \text{lor}_{1,3} \) is given by

\[
X^T \eta + \eta X = 0.
\]

(A13)
Using the infinitesimal transformation

\[ \Lambda_{\mu
u} = \eta_{\mu\nu} + \omega_{\mu\nu} \equiv \eta_{\mu\nu} - \frac{1}{2}(\omega_{\rho\sigma}e^{\rho\sigma})_{\mu\nu}, \tag{A14} \]

the defining equation (A13) gives \( \omega_{\mu\nu} = -\omega_{\nu\mu} \), and one obtains six independent parameters \( \omega_{\mu\nu} \) and generators \( e^{\mu\nu} = -e^{\nu\mu} \). A generic element \( \Lambda \in \text{Lor}_{1,3} \) is written as

\[ \Lambda(\omega) = \exp\left( -\frac{1}{2}\omega_{\mu\nu}e_{\mu\nu} \right), \quad e^{\mu\nu} = -\frac{\partial}{\partial\omega_{\mu\nu}}\Lambda(\omega) \big|_{\omega=0}. \tag{A15} \]

The six independent generators \( e^{\mu\nu} \) have the form

\[ (e^{\mu\nu})_{\rho\sigma} = -\eta^{\mu\rho}\eta^{\nu\sigma} + \eta_{\mu\sigma}e^{\nu\rho} - \eta_{\nu\rho}e^{\mu\sigma}. \tag{A16} \]

and obey the commutation relation

\[ [e^{\mu\nu}, e^{\rho\sigma}] = \eta^{\mu\rho}e^{\nu\sigma} + \eta^{\nu\sigma}e^{\mu\rho} - \eta^{\mu\sigma}e^{\nu\rho} - \eta^{\nu\rho}e^{\mu\sigma}. \tag{A17} \]

The defining equation (A13) applies to the generators in the form

\[ e^{\mu\nu}T_{\eta} + \eta e^{\mu\nu} = 0. \tag{A18} \]

The minimal equation for \( X \in \text{lor}_{1,3} (\equiv \text{so}_{1,3}) \) is given by

\[ X^4 - \frac{1}{2}(\text{tr} \ X^2)X^2 + (\det \ X)1_4 = 0, \]

and \( \det X \leq 0, \text{tr} X = 0. \)

**A.3 Cartan decomposition**

Following Ref. [47], let \( \vec{e}_{(i)} \) \( (i = 1, 2, 3) \) be an orthogonal triad for \( \mathbb{R}^3 \) defined by

\[ (\vec{e}^T_{(i)})_j = (\vec{e}_{(i)})^j = \delta^j_i, \quad \vec{e}^T_{(i)}\vec{e}_{(j)} = \delta_{ij}, \quad (\vec{e}_{(i)} \times \vec{e}_{(j)})^k = \epsilon_{ijk} \]

\( (1 = \epsilon^{0123} = -\epsilon_{0123} \equiv -\epsilon_{123}) \). In terms of this triad the generators \( e^{\mu\nu} \) are given by

\[ e_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & f_{ij} \end{pmatrix} \equiv \epsilon_{ijk}D^k \]

\(^1\text{The notation } (X^T)_{\mu} = X^\mu \text{ is used in the following.}\)
or, vice versa,

\[ D_i = -\frac{1}{2} \epsilon_{0ijk} \epsilon^{jk} = \begin{pmatrix} 0 & 0 \\ 0 & -\epsilon_{ijk} \epsilon^{j(k)} \end{pmatrix} = -D^i, \]  

(A19.1)

where \((D_i)^\mu_\nu = \epsilon_{0j}^\mu_\nu, f_{ij} = \epsilon^{(i)}_j \epsilon^{(j)}_k = \epsilon^{(j)}_j \epsilon^{(i)}_k\), and

\[ \epsilon_{0i} = \begin{pmatrix} 0 & \epsilon^{(i)}_j \\ \epsilon^{(i)}_j & 0 \end{pmatrix} \equiv B_i, \]  

(A19.2)

where \((B_i)^\mu_\nu = -\eta_0^\mu_\nu + \eta_0^\nu_\mu\). A general element \(X \in \text{lor}_{1,3}\) has the form

\[ X = \tilde{\omega} \tilde{D} - \eta \tilde{B} = \begin{pmatrix} 0 & \eta^1 & \eta^2 & \eta^3 \\ \eta^1 & 0 & -\omega^3 & \omega^2 \\ \eta^2 & -\omega^3 & 0 & -\omega^1 \\ \eta^3 & -\omega^2 & \omega^1 & 0 \end{pmatrix}, \]  

(A20)

The corresponding finite Lorentz transformation is given by

\[ \Lambda = \exp \left( -\frac{1}{2} \omega^{\mu\nu} \epsilon_{\mu\nu} \right) = \exp(\tilde{\omega} \tilde{D} - \eta \tilde{B}), \]  

(A21)

where \(\omega^j = \frac{1}{2} \epsilon^{ijk} \omega_{jk}, \eta^j = \omega_{0i} = -\eta_i\). The commutation relations can be expressed as

\[ [D_i, D_j] = \epsilon_{ijk} D_k, \]
\[ [D_i, B_j] = \epsilon_{ijk} B_k, \]
\[ [B_i, B_j] = -\epsilon_{ijk} D_k. \]  

(A22)

The compact generators \(D_i\) are antisymmetric \((D_i^T = -D_i)\) while the noncompact generators \(B_i\) are symmetric \((B_i^T = B_i)\). As a consequence, the Lorentz algebra \(\text{lor}_{1,3}\) (if considered as vector space) is a symmetric Lie algebra with symmetric decomposition

\[ \tilde{\text{lor}}_{1,3} = \tilde{\text{so}}_3 \oplus \tilde{\text{p}}, \]  

(A23)

where \(\tilde{\text{p}} = \text{span}_R \{B_i\}_{1}^{3}\). Indeed, \(\text{so}_3\) is a subalgebra, \([\text{so}_3, \text{so}_3] = \text{so}_3\), but \([\text{so}_3, \text{p}] \subset \text{p}\) and \([\text{p}, \text{p}] \subset \text{so}_3\). Given the structure (A20), the generic element \(X \in \text{lor}_{1,3}\) can be split up into two parts,

\[ X = \begin{pmatrix} 0 & X^T_3 \\ X_3 & X_{(3)} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & X_{(3)} \end{pmatrix} + \begin{pmatrix} 0 & X^T_3 \\ X_3 & 0 \end{pmatrix} \]  

(A24)
where the first part is compact, $X^{T}_{(3)} = -X^{(3)}$, and contained in $so_{3}$, the second part is noncompact and contained in $p$. Notice that $so_{3}$ and $p$ are orthogonal with respect to the Killing form,

$$ (so_{3}, p) = 0. \quad \text{(A25)} $$

Since $lor_{1,3}$ is simple, the Cartan–Killing form is nonsingular on $so_{3}$ and $p$. The symmetric decomposition is determined by the Cartan involution

$$ \theta : lor_{1,3} \ni X \rightarrow \theta(X) = -X^{T} = \eta X \eta \in lor_{1,3} \quad \text{(A26)} $$

which is the only external involutive automorphism for $lor_{1,3}$. Indeed, one obtains $\theta(D_{i}) = -D_{i}^{T} = D_{i}$ and, therefore,

$$ so_{3} = \{ X \in lor_{1,3} : \theta(X) = +X \} = \text{span}_{\mathbb{R}} \{ D_{i} \}^{3}_{1} \quad \text{(A27.1)} $$

is the maximal compact subalgebra of $lor_{1,3}$. Similarly,

$$ p = \{ X \in lor_{1,3} : \theta(X) = -X \} = \text{span}_{\mathbb{R}} \{ B_{i} \}^{3}_{1} \quad \text{(A27.2)} $$

consists of the noncompact elements of $lor_{1,3}$. Therefore, one ends up with the Cartan decomposition

$$ lor_{1,3} = so_{3} + p. \quad \text{(A28)} $$

The map $SO_{3} \times p \rightarrow Lor_{1,3}$ given by

$$ SO_{3} \times p \ni (R, X) \rightarrow R \exp X \in Lor_{1,3} $$

is a diffeomorphism onto $Lor_{1,3}$. Therefore, the Cartan decomposition [(A28)] on the level of the Lie algebra $lor_{1,3}$ induces the polar decomposition [(A7)] on the level of the Lie group $Lor_{1,3}$. The exponential map is a diffeomorphism from the vector space $\bar{p}$ of symmetric matrices to the set of positive definite matrices,

$$ \exp : \bar{p} \ni X = X^{T} \rightarrow \exp X = (\exp X)^{T}. $$
A.4 Weyl’s unitary trick

It is an algebraic fact that the symmetric spaces appear in pairs. If the Cartan involution induces \( g = \mathfrak{k} + \mathfrak{p} \), it’s companion is

\[
g^{(W)} = \mathfrak{k} + i\mathfrak{p} \equiv g_{\mathbb{C}} = \mathbb{C} \otimes g. \tag{A29}
\]

If \( g \) is the Lie algebra of a noncompact connected semisimple Lie group \( G \), \( g^{(W)} \) is the Lie algebra of a second Lie group \( G^{(W)} \) which is compact. In this way the noncompact algebras appearing in the Cartan decomposition can be analytically continued to compact algebras by analytic extension,

\[
\text{lor}_{1,3} = \text{so}_3 + \mathfrak{p} \rightarrow \text{so}_3 + i\mathfrak{p} \equiv \text{lor}^{(W)}_{1,3}. \tag{A30}
\]

This analytical continuation known as Weyl’s unitary trick can be accomplished by using the matrix

\[
\Gamma = \begin{pmatrix}
i & 0 \\
0 & 1_3
\end{pmatrix}
\]

in the way

\[
\text{lor}_{1,3} \ni X \xrightarrow{\Gamma} X^{(W)} = \Gamma X \Gamma
\]

(note that Weyl’s unitary trick is not a similarity transformation). One obtains

\[
X^{(W)} = \Gamma X \Gamma = \begin{pmatrix} 0 & i\vec{X}^T \\
i\vec{X} & X_{(3)} \end{pmatrix} = \Gamma^{-1}(-\eta X)\Gamma. \tag{A31}
\]

and for the basis

\[
D_i^{(W)} = \Gamma D_i \Gamma = D_i, \quad B_i^{(W)} = \Gamma B_i \Gamma = iB_i. \tag{A32}
\]

Accordingly, the commutation relations (A22) change to their compact form

\[
[D_i^{(W)}, D_j^{(W)}] = \epsilon_{ijk} D_k^{(W)}, \\
[D_i^{(W)}, B_j^{(W)}] = \epsilon_{ijk} B_k^{(W)}, \\
[B_i^{(W)}, B_j^{(W)}] = \epsilon_{ijk} D_k^{(W)}. \tag{A33}
\]
Because of \((\eta X)^T = -\eta X\), one recovers the algebra \(so_4(\mathbb{R})\). The last algebra in turn is isomorphic to \(su_2 \oplus su_2\) where \(\oplus\) denotes the Kronecker sum of algebras, i.e. for \(a \in \mathfrak{g}\) and \(b \in \mathfrak{h}\) one has

\[
a \oplus b \equiv a \otimes 1_{\mathfrak{h}} + 1_{\mathfrak{g}} \otimes b \in \mathfrak{g} \oplus \mathfrak{h}.
\]  

(A34)

To conclude, the pair of symmetric algebras \(lor_{1,3}\) and \(lor^{(W)}_{1,3}\) is connected by Weyl’s unitary trick,

\[
lor_{1,3} \xrightarrow{\text{Weyl}} lor^{(W)}_{1,3} = S(su_2 \oplus su_2)S^\dagger.
\]  

(A35)

The splitting map

\[
S = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ -i & 0 & 0 & -i \\ 0 & 1 & 1 & 0 \end{pmatrix}
\]  

(A36)

with \(S^\dagger = S^{-1}\) gives the decomposition in the \(su_2\) basis,

\[
D_i^{(W)} \rightarrow S^\dagger D_i^{(W)} S = m_i \oplus m_i \Rightarrow D_i = m_i \oplus m_i,
\]

\[
B_i^{(W)} \rightarrow S^\dagger B_i^{(W)} S = m_i \oplus (-m_i) \Rightarrow B_i = (-im_i) \oplus (im_i),
\]  

(A37)

where \(m_j = \frac{i}{2}\sigma_j\), and the decomposition in the \(sl_2\) basis,

\[
D_1 = \frac{i}{2}(e \oplus e) + \frac{i}{2}(f \oplus f), \quad B_1 = \frac{1}{2}(e \oplus (-e)) + \frac{1}{2}(f \oplus (-f)),
\]

\[
D_2 = \frac{1}{2}(e \oplus e) - \frac{1}{2}(f \oplus f), \quad B_2 = -\frac{i}{2}(e \oplus (-e)) + \frac{i}{2}(f \oplus (-f)),
\]

\[
D_3 = \frac{i}{2}(h \oplus h), \quad B_3 = \frac{1}{2}(h \oplus (-h)).
\]  

(A38)

The meaning of Weyl’s unitary trick is that the representations of the noncompact group \(Lor_{1,3}\) may be viewed as representations of the compact group \(SO_4 = SU_2 \times SU_2/\mathbb{Z}_2\), where \(\mathbb{Z}_2 = \{(1_2, 1_2), (-1_2, -1_2)\}\) is the discrete subgroup. From the unitary nature of the representations of the compact group \(SO(4)\) one concludes the full reducibility of the finite-dimensional representations of \(Lor_{1,3}\). Note that \(SO_4\) is the real compact form of
SO\(_4(\mathbb{C}) = \text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C})/\mathbb{Z}_2\), and Lor\(_{1,3}\) is a real noncompact form. Since the simply connected group SU(2) \times SU(2) is a universal covering group of the doubly connected group SO\(_4\), their Lie algebras are isomorphic. Moreover, since su\(_2\) is a compact real form of sl\(_2(\mathbb{C})\), the construction of the representations of the algebra lor\(_{1,3}\) may be realised by using the representations of sl\(_2(\mathbb{C})\). Since SL\(_2(\mathbb{C})\) as the topological product \(\mathbb{R}^3 \times \text{SU}(2)\) is simply connected, all its representations are single-valued.

A.5 Higher dimensional representations

In the standard basis

\[
\begin{align*}
    h &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
    e &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\
    f &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
\end{align*}
\] (A39)

of sl\(_2(\mathbb{C})\) with \([h, e] = 2e\), \([h, f] = -2f\) and \([e, f] = h\), the \((2k + 1)\)-dimensional, real representations applied to states \(|k, m\rangle\) are given by

\[
\begin{align*}
    \pi^{(k)}(h)|k, m\rangle &= 2m|k, m\rangle, \\
    \pi^{(k)}(e)|k, m\rangle &= \rho^{(k)}_{(m)}|k, m + 1\rangle, \\
    \pi^{(k)}(f)|k, m\rangle &= \rho^{(k)}_{(-m)}|k, m - 1\rangle,
\end{align*}
\] (A40)

where \(k = 0, \frac{1}{2}, 1, \ldots, m = -k, -k + 1, \ldots, k\) and \(\rho^{(k)}_{(m)} = \sqrt{(k - m + 1)(k + m)}\).

**Theorem:** Let \(2k \in \mathbb{N}\) and let \((V, \pi)\) be a simple representation of sl\(_2(\mathbb{C})\) of dimension \(2k + 1\). Then ([19], p. 281)

1. \(\pi\) is equivalent to \(\pi^{(k)}\),
2. the eigenvalues of \(\pi^{(k)}(h)/2\) are \(\{-k, -k + 1, \ldots, k\} = \text{spec} \frac{1}{2}\pi^{(k)}(h)\),
3. if \(0 \neq v \in V\) verifies \(\pi^{(k)}(e)v = 0\), then \(\pi^{(k)}(h)v = 2kv\),

i.e. \(\pi^{(k)}(h)\) and \(\pi^{(k)}(e)\) have the common eigenvector \(|k, k\rangle\),
4. if \( 0 \neq v \in V \) verifies \( \pi^{(k)}(f)v = 0 \), then \( \pi^{(k)}(h)v = -2kv \),
i.e. \( \pi^{(k)}(h) \) and \( \pi^{(k)}(f) \) have the common eigenvector \( |k, -k\rangle \)

Since \( \text{su}(2) \) is the compact real form of \( \text{sl}_2(C) \) and the generators of \( \text{su}(2) \) are given by
\[
m_1 = \frac{i}{2}(e + f), \quad m_2 = \frac{1}{2}(e - f), \quad m_3 = \frac{i}{2}h, \tag{A41}
\]
one can accordingly define irreducible representations of \( \text{su}(2) \) given by
\[
\pi^{(k)}(m_1), \quad \pi^{(k)}(m_2) \quad \text{and} \quad \pi^{(k)}(m_3). \tag{A42}
\]

Following the procedure given before, the real Lie algebra \( \text{lor}_{1,3} \) may be identified via Weyl’s unitary trick with the algebra \( \text{lor}^{(W)}_{1,3} \) which splits into a Kronecker sum of two algebras \( \text{su}(2), \text{lor}^{(W)}_{1,3} \sim \text{su}(2) \bigoplus \text{su}(2) \). If \( \pi^{(k)} \) and \( \pi^{(l)} \) are representations of \( \text{su}(2) \) on the vector spaces \( V^{(k)} \) and \( V^{(l)} \), \( \pi^{(k)} \otimes \pi^{(l)} \) is a representation of the Lie algebra \( \text{lor}^{(W)}_{1,3} \) on \( V^{(k)} \otimes V^{(l)} \), defined by
\[
m_i \bigoplus m_j \rightarrow \pi^{(k,l)}(m_i \bigoplus m_j) = \pi^{(k)}(m_i) \bigoplus \pi^{(l)}(m_j). \tag{A43}
\]
The representation \( \pi^{(k,l)} \) of the Kronecker sum \( \text{su}_2 \bigoplus \text{su}_2 \) on the tensor product basis
\[
\{|k, l; m_k, m_l\} \equiv |k, m_k\rangle \otimes |l, m_l\rangle, \quad -k \leq m_k \leq k, \quad -l \leq m_l \leq l
\]
is given by
\[
\pi^{(k,l)}(m_i \bigoplus m_j)|k, l; m_k, m_l\rangle = \\
= \sum_{m=-k}^{k} (\pi^{(k)}(m_i))_{mm_k} |k, l; m, m_i\rangle + \sum_{m=-l}^{l} (\pi^{(l)}(m_j))_{mm_l} |k, l; m_k, m_l\rangle. \tag{A44}
\]

Using the representations of \( D_i, B_i \in \text{lor}_{1,3} \) in the \( \text{su}_2 \) basis, one obtains
\[
\pi^{(k,l)}(D_i) = \pi^{(k)}(m_i) \bigoplus \pi^{(l)}(m_i),
\]
\[
\pi^{(k,l)}(B_i) = (-i\pi^{(k)}(m_i)) \bigoplus (i\pi^{(l)}(m_i)) \tag{A45}
\]
with \( \pi^{(k)}(m_i) \) given by Eq. \( \text{[A42]} \).
A.6 Splitting algebra

Theorem: Any finite-dimensional representation of lor\(_{1,3}\) is isomorphic to \(\pi^{(k,l)}\) for some \(k, l = 0, \frac{1}{2}, 1, \ldots\) and is non-antihermitean. The corresponding representation of Lor\(_{1,3}\) is non-unitary [19].

The isomorphism between so\(_4\) and so\(_3\) \(\oplus\) so\(_3\) is easily realised by the choice of the basis

\[
J_i^{(\epsilon)} = \frac{1}{2}(D_i + i\epsilon B_i) = \frac{1 + \epsilon}{2}m_i \oplus \frac{1 - \epsilon}{2}m_i,
\]

\[
K_i^{(\epsilon)} = \frac{1}{2}(D_i - i\epsilon B_i) = \frac{1 - \epsilon}{2}m_i \oplus \frac{1 + \epsilon}{2}m_i
\]

with \(J_i^{(\epsilon)\dagger} = -J_i^{(\epsilon)}, K_i^{(\epsilon)\dagger} = -K_i^{(\epsilon)}\) \((\epsilon = \pm 1)\). The commutator relations are given by

\[
[J_i^{(\epsilon)}, J_j^{(\epsilon)}] = \epsilon_{ijk}J_k^{(\epsilon)},
\]

\[
[J_i^{(\epsilon)}, K_j^{(\epsilon)}] = 0,
\]

\[
[K_i^{(\epsilon)}, K_j^{(\epsilon)}] = \epsilon_{ijk}K_j^{(\epsilon)}.
\]

Note that the fact that the Lorentz algebra lor\(_{1,3}\) can be written as a Kronecker sum su(2) \(\oplus\) su(2) of two algebras does not mean that lor\(_{1,3}\) is the same as su(2) \(\oplus\) su(2) or lor\(_{1,3}^{(W)}\). Rather, they are the anti-hermitean complex representations of lor\(_{1,3}\).

A.7 Spinor representations

There are two fundamental spinor representations, from which all other may be obtained by tensor product reduction [48, 49, 50, 51]. The Lorentz covariant description needs two sets of relativistic Pauli matrices,

\[(\sigma_\mu) = (\mathbf{1}_2, \bar{\sigma}) \quad \text{and} \quad (\bar{\sigma}_\mu) = (\mathbf{1}_2, -\bar{\sigma}).\]

The relation between the real Minkowski space \(\mathbb{E}_{1,3}\) and the set of all complex hermitean 2 \(\times\) 2 matrices \(\mathbb{H}_2\) is given by

\[
\mathbb{E}_{1,3} \ni p \to \sigma(p) = \sigma_\mu p^\mu \in \mathbb{H}_2.
\]

(A48)
The correspondence is a linear isomorphism,

$$\det \sigma(p) = p^2 = p^\mu p_\mu,$$

and the characteristic polynomial

$$\det(\sigma(p) - \lambda) = \begin{cases} (p^0 + |\vec{p}| - \lambda)(p^0 - |\vec{p}| - \lambda) & \text{for } p^2 > 0, \\ \lambda(2p^2 - \lambda) & \text{for } p^2 = 0. \end{cases}$$

Therefore, if $p^2 > 0$, $\sigma(p)$ is positive semidefinite.

**Theorem:**

1. Let $\sigma(p) \in M_2(\mathbb{C})$ be positive definite.

   If $A \in M_n(\mathbb{C})$ and $\det A \neq 0$, then $A \sigma(p) A^\dagger$ is positive definite [19].

2. If $\sigma(p) \in M_2(\mathbb{C})$ is not positive definite but positive semidefinite and if $A \in M_n(\mathbb{C})$,

   then $A \sigma(p) A^\dagger$ is always positive semidefinite and not positive definite [19].

The fundamental representation $(\frac{1}{2}, 0)$ can be expressed as the commutative diagram

$$\begin{array}{ccc}
\text{Lor}_{1,3} & \ni & \Lambda \\
\downarrow & \downarrow \sigma & \downarrow \sigma \\
\text{SL}_2(\mathbb{C}) & \ni & \pm A_\Lambda \\
\end{array}$$

The continuous homomorphism relates an element $\Lambda \in \text{Lor}_{1,3}$ to two elements $\pm A_\Lambda \in \text{SL}_2(\mathbb{C})$. The group $\text{SL}_2(\mathbb{C})$ constitutes the universal covering group of $\text{Lor}_{1,3}$, i.e. $\text{Lor}_{1,3} = \text{SL}_2(\mathbb{C})/\mathbb{Z}$ (on the right hand side the realification of $\text{SL}_2(\mathbb{C})$ is understood). Using Pauli’s spin matrices one obtains

$$A_\Lambda \sigma_\nu A_\Lambda^\dagger = \sigma_\mu \Lambda^\mu_\nu,$$

$$\Lambda^\mu_\nu = \frac{1}{2} \text{tr}(\sigma_\mu A_\Lambda \sigma_\nu A_\Lambda^\dagger),$$

$$A_\Lambda = \frac{\Lambda^{\mu\nu} \sigma_\mu \sigma_\nu}{\left(\frac{1}{2} \Lambda^{\alpha\beta} \Lambda^{\gamma\delta} \text{tr}(\sigma_\alpha \sigma_\beta \sigma_\gamma \sigma_\delta)\right)^{1/2}}.$$
where

\[
4(tr A_L^\dagger)^2 = \frac{1}{2} \Lambda^{\mu \nu} \Lambda^{\rho \sigma} \text{tr} (\sigma_\mu \tilde{\sigma}_\nu \sigma_\sigma \tilde{\sigma}_\rho) = (\text{tr} \Lambda)^2 - \text{tr} \Lambda^2 + 4 + i \Lambda^{\mu \nu} \Lambda^{\rho \sigma} \epsilon_{\mu \nu \rho \sigma}
\]

(note the different order of indices). Suppose that $\Lambda$ and $A_\Lambda$ are given by $A_\Lambda \sigma_\nu A_\Lambda^\dagger = \sigma_\mu \Lambda^{\mu \nu}$, one can write

\[
A_\Lambda^\dagger \sigma_\nu A_\Lambda = \sigma_\mu (\Lambda^T)^\mu_\nu,
\]

\[
A_\Lambda^\dagger \tilde{\sigma}_\nu A_\Lambda = \tilde{\sigma}_\mu (\Lambda^{-1})^\mu_\nu,
\]

\[
A_\Lambda \tilde{\sigma}_\nu A_\Lambda^\dagger = \tilde{\sigma}_\mu (\Lambda^{-1T})^\mu_\nu.
\]

Defining $\Lambda = \exp(-\frac{1}{2} \omega^{\mu \nu} e_{\mu \nu})$ and $A_\Lambda = \exp(-\frac{1}{2} \omega^{\mu \nu} e_{\mu \nu})$, for the representation $(\frac{1}{2}, 0)$ one obtains

\[
m_{\mu \nu} = \frac{1}{4} \left( e_{\mu \nu} \right)^{\alpha \beta} \sigma_\alpha \tilde{\sigma}_\beta = \frac{1}{4} (\sigma_\mu \tilde{\sigma}_\nu - \sigma_\nu \tilde{\sigma}_\mu),
\]

\[
m_i = -\frac{1}{2} \epsilon_{ijk} m^{jk} = \frac{i}{2} \sigma_i,
\]

\[
m_{0j} = \frac{1}{2} \sigma_j = -im_j.
\]

The second fundamental representation $(0, \frac{1}{2})$ is defined by the commutative diagram

\[
\begin{array}{c}
\text{Lor}_{1,3} \ni \Lambda : \ E_{1,3} \ni p \quad \longrightarrow \quad \Lambda p \\
\downarrow \downarrow \quad \downarrow \tilde{\sigma} \\
\text{SL}_2(\mathbb{C}) \ni \pm B_\Lambda : \ H_2 \ni \bar{\sigma}(p) \quad \longrightarrow \quad B_\Lambda \bar{\sigma}(p) B_\Lambda^\dagger = \bar{\sigma}(\Lambda p)
\end{array}
\]

where $\bar{\sigma} = C^{-1} \sigma^T C = C^{-1} \sigma^* C$,

\[
C = -i \sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\]

and $\bar{\sigma}(p) = 2p_0 - \sigma(p)$. The properties of the Pauli matrices yield

\[
B_\Lambda \tilde{\sigma}_\nu B_\Lambda^\dagger = \tilde{\sigma}_\mu \Lambda^{\mu \nu},
\]

\[
\Lambda^{\mu \nu} = \frac{1}{2} \text{tr}(\tilde{\sigma}_\mu B_\Lambda \tilde{\sigma}_\nu B_\Lambda^\dagger),
\]

\[
B_\Lambda = \frac{\Lambda^{\mu \nu} \tilde{\sigma}_\mu \sigma_\nu}{\left(\frac{1}{2} \Lambda^{\alpha \beta} \Lambda^{\gamma \delta} \text{tr}(\tilde{\sigma}_\alpha \sigma_\beta \tilde{\sigma}_\gamma \sigma_\delta)\right)^{1/2}}
\]

39
and

\[ 4(\text{tr} B_\Lambda^1)^2 = \frac{1}{2} \Lambda^\mu_\nu \Lambda^\rho_\sigma \text{tr}(\bar{\sigma}_\mu \sigma_\nu \bar{\sigma}_\sigma \sigma_\rho) = (\text{tr} \Lambda)^2 - \text{tr} \Lambda^2 + 4 - i \Lambda^\mu_\nu \Lambda^\rho_\sigma \epsilon_\mu_\nu_\rho_\sigma. \]

Defining the generators of the representation \( (0, \frac{1}{2}) \) by \( B_\Lambda = \exp \left( -\frac{1}{2} \omega^\mu_\nu \tilde{m}_{\mu \nu} \right) \), one obtains

\[ \tilde{m}_{\mu \nu} = \frac{1}{4} (\epsilon^\mu_\nu)^{\alpha \beta} \tilde{\sigma}_\alpha \sigma_\beta = -\frac{1}{4} (\tilde{\sigma}_\mu \sigma_\nu - \tilde{\sigma}_\nu \sigma_\mu), \]
\[ \tilde{m}_i = -\frac{1}{2} \epsilon_{ijk} \tilde{m}^{jk} = -\frac{i}{2} \tilde{\sigma}_i = \frac{i}{2} \sigma_i, \]
\[ \tilde{m}_j = \frac{1}{2} \tilde{\sigma}_j = -\frac{1}{2} \sigma_j = i\tilde{m}_j. \]

The two nonequivalent fundamental representations \( A_\Lambda \) and \( B_\Lambda \) are related by

\[ B_\Lambda = C^{-1} A_\Lambda^\dagger C = (A_\Lambda)^{-1\dagger}. \]

References

[1] H. Poincaré, “Sur la dynamique de l’électron”, Rendiconti del Circolo matematico di Palermo 21 (1906) 129–176 (sent to the editor on July 23rd, 1905)

[2] E.P. Wigner, “On Unitary Representations of the Inhomogeneous Lorentz Group”, Annals Math. 40 (1939) 149 [Nucl. Phys. Proc. Suppl. 6 (1989) 9]

[3] E.P. Wigner, “Unitary Representations of the inhomogeneous Lorentz Group including Reflections”, Lecture at the Istanbul Summer School of Theoretical Physics (ed. by F. Gürsey), Gordan and Breach, New York and London, 1962, pp. 37–80

[4] Y. Ohnuki, “Unitary Representations of the Poincaré Group and Relativistic Wave Equations”, World Scientific, Singapore, 1976

[5] S. Weinberg, “The Quantum Theory of Fields”, Cambridge Univ. Press, 1995

[6] R. Shaw, “Unitary representations of the inhomogeneous Lorentz group”, Nuovo Cim. 33 (1964) 1074

40
[7] U.H. Niederer and L. O’ Raifeartaigh, “Realizations of the unitary representations of the inhomogeneous space-time groups. I+II”, Fortsch. Phys. 22 (1974) 111

[8] C. Fronsdal, “Unitary Irreducible Representations of the Lorentz Group”, Phys. Rev. 113 (1959) 1367

[9] F.R. Halpern and E. Branscomb, “Wigner’s Analysis Of The Unitary Representations Of The Poincare Group”, Preprint No. UCRL-12359, University of California, 1965

[10] G.W. Mackey, “Induced representations of groups and quantum mechanics”, Benjamin, New York, 1968

[11] D.J. Candlin, “Physical operators and the representations of the inhomogeneous Lorentz group”, Nuovo Cim. 37 (1965) 1396

[12] D.L. Pursey, “General theory of covariant particle equations”, Annals Phys. 32 (1965) 157

[13] G. Feldman and P.T. Mathews, “Poincaré invariance, particle fields, and internal symmetry”, Annals Phys. 40 (1966) 19

[14] N. Dragon, “Currents for Arbitrary Helicity”, arXiv:1601.07825 [hep-th]

[15] S. Sternberg, “Group theory and physics”, Cambridge Univ. Press, 1994

[16] Y.S. Kim and M.E. Noz, “Theory And Applications Of The Poincare Group”, D. Reidel Publishing Company, Dordrecht, Netherlands, 1986

[17] D. Kwoh, Senior Thesis, Princeton University, Princeton, N.J., 1970 (unpublished), cited on page 457 in: A.S. Wightman, “Relativistic wave equations as singular hyperbolic systems”, Proc. Symp. Pure Math. 23 (1973) 441

[18] Armand Borel, “Linear Algebraic Groups”, Springer, New York, 1991
[19] P. Tauvel, R.W.T. Yu, “Lie Algebras and Algebraic Groups”,  
                 Springer, New York, 2005
[20] R. Goodman, N.R. Wallach, “Symmetry, Representations, and Invariants”,  
                 Springer, New York, 2009
[21] J.P. Serre, “Lie Algebras and Lie Groups”,  
                 Lectures given at Harvard University, Benjamin, New York, 1965
[22] Daniel Bump, “Lie Groups”, Springer, New York, 2004
[23] S. Weinberg, Nucl. Phys. Proc. Suppl. 6 (1989) 67
[24] J.M. Jauch, C. Frønsdal, R. Hagedorn and H.A. Tolhoek, “The representations of the Lorentz group in quantum mechanics”, J. Math. Phys. 3 (1958) 1116
[25] D. Finkelstein, “Internal Structure of Spinning Particles”, Phys. Rev. 100 (1955) 924
[26] P. Winternitz and I. Frisch, “Invariant expansions of relativistic amplitudes and subgroups of the proper Lorentz group”, Sov. J. Nucl. Phys. 1 (1965) 889
[27] J. Patera, P. Winternitz and H. Zassenhaus, “Continuous subgroups of the fundamental groups of physics”, J. Math. Phys. 16 (1975) 1597
[28] F.G. Lastaria, “Lie subalgebras of real and complex orthogonal groups in dimension four”, J. Math. Phys. 40 (1999) 449
[29] L. Zhang and X. Xue, “The deformation of Poincare subgroups concerning very special relativity”, Sci. China Phys. Mech. Astron. 57 (2014) 859
[30] Nathan Jacobson, “Lie Algebras”, Dover Publ. Inc., 1979
[31] G.G.A. Bäuerle, E.A. de Kerf, “Lie algebras – Finite and Infinite Dimensional Lie Algebras and Applications in Physics”,  
                 North-Holland, Elsevier Science Publishers B.V., 1990
[32] A.O. Barut and R. Raczk, “Theory Of Group Representations And Applications”,
    World Scientific, Singapore, 1986

[33] Michele Maggiore, “A Modern Introduction to Quantum Field Theory”,
    Oxford Univ. Press, 2005

[34] George Sterman, “An Introduction to Quantum Field Theory”,
    Cambridge Univ. Press, 1994

[35] F.J. Yndurain, “Relativistic Quantum Mechanics and Introduction to Field Theory”,
    Springer, New York, 1996

[36] I. Gel’fand, R. Miklos and Z. Shapiro, “Representations of the Rotation and Lorentz
    Groups and Their Applications” (engl. transl.), Macmillan, New York, 1963

[37] G. Lyubarskii, “The Application of Group Theory in Physics” (engl. transl.),
    Pergamon Press, Oxford, 1960

[38] M. Naimark, “Linear Representations of the Lorentz Group” (engl. transl.),
    Macmillan, New York, 1964

[39] R.U. Sexl, H.K. Urbantke, “Relativity, Groups, Particles – Special Relativity and
    Relativistic Symmetry in Field and Particle Physics”, Springer, New York, 2001

[40] Wu-Ki Tung, “Group Theory in Physics”, World Scientific, Singapore, 1999

[41] A. Das, “The Special Theory of Relativity – A Mathematical Exposition”,
    Springer, New York, 1993

[42] A. Aste, “Weyl, Majorana and Dirac fields from a unified perspective”,
    Symmetry 8 (2016) 87

[43] D. Serre, “Matrices – Theory and Applications”, Springer, New York, 2002
[44] Abraham A. Ungar, “Thomas precession and its associated grouplike structure”, Am. J. Phys. 59 (1991) no.9, 824

[45] Abraham A. Ungar, “Parametric Realization of the Lorentz Transformation Group in Pseudo-Euclidean Spaces”, J. Geom. Symm. Phys. 38 (2015) 39

[46] Jason Hanson, “Orthogonal decomposition of Lorentz transformations”, Gen. Rel. Grav. 45 (2013) 599

[47] Robert Gilmore, “Lie Groups, Lie Algebras, and Some of Their Applications”, Dover Publ. Inc., 2002

[48] H. Joos, “On the Representation theory of inhomogeneous Lorentz groups as the foundation of quantum mechanical kinematics”, Fortsch. Phys. 10 (1962) 65

[49] E.M. Corson, “Introduction to Tensors, Spinors, and Relativistic Wave Equations”, Amer. Math. Soc., London and Glasgow, 1953

[50] A.J. MacFarlane, “On the Restricted Lorentz Group and Groups Homomorphically Related to It”, J. Math. Phys. 3 (1962) 1116

[51] R. Ticciati, “Quantum Field Theory for Mathematicians”, Cambridge Univ. Press, 1999

[52] A. Helfer, M. Hickman, C. Kozameh, C. Lucey and E.T. Newman, “Yang–Mills Equations And Solvable Groups”, Phys. Rev. D36 (1987) 1740

[53] J. Nuyts and T.T. Wu, “Yang–Mills theory for nonsemisimple groups”, Phys. Rev. D67 (2003) 025014

[54] S.C. Anco, “Symmetry properties of conservation laws”, Int. J. Mod. Phys. B30 (2016) 1640004