COUNTING CUSP FORMS BY ANALYTIC CONDUCTOR

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Abstract. The universal family is the set of cuspidal automorphic representations of bounded analytic conductor on $GL_n$ over a number field. We prove an asymptotic for the universal family, under a spherical assumption at the archimedean places when $n \geq 3$. We interpret the leading term constant geometrically and conjecturally determine the underlying Sato–Tate measure. Our methods naturally provide uniform Weyl laws with explicit level savings and strong quantitative bounds on the non-tempered discrete spectrum for $GL_n$.

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1. Introduction

Automorphic forms and their $L$-functions are among the central notions of modern number theory. While they can be notoriously difficult to study individually using analytic techniques, desired results can often be obtained by embedding them in a family of automorphic forms of favorable size. In this article, we address the question of size of the “universal” family consisting of all cuspidal automorphic forms on $GL_n$ over a number field $F$ when ordered by their analytic conductor.

We denote by $\mathfrak{F}$ the set of irreducible unitary cuspidal automorphic representations $\pi$ of $GL_n(\mathbb{A}_F)^1$. Following Sarnak [51], we shall refer to the set $\mathfrak{F}$ as the universal family. Motivated by questions in analytic number theory related to $L$-functions, Iwaniec and Sarnak [26] introduced the notion of analytic conductor of $\pi \in \mathfrak{F}$. This positive real number $Q(\pi)$ is a global invariant of $\pi$; it can be expressed as a product of the conductors of all local components $\pi_v$, each of these arising from the local functional equation of the standard $L$-function $L(s, \pi_v)$.

One way to understand the significance of the analytic conductor in the analytic theory of $L$-functions is that it controls the effective lengths of the partial sums appearing in the global approximate functional equation for $L(s, \pi)$. In turn, the analytic conductor controls complexity in analytic problems including evaluation of moments, subconvexity, nonvanishing, extreme value problems, numerical computations of $L$-functions or automorphic forms themselves, as well as the requisite number of twists in the known forms of the Converse Theorem, just to name a few applications. Since the appearance of [26], Sarnak has repeatedly emphasized the importance of understanding the statistical properties of the truncated family

$$\mathfrak{F}(Q) = \{ \pi \in \mathfrak{F} : Q(\pi) \leq Q \},$$

the first among which is its cardinality.

1.1. Weyl–Schanuel law. In Conjecture 1 below, we formulate the expected asymptotic behavior of $\mathfrak{F}(Q)$. Following [48], we refer to this asymptotic as the Weyl–Schanuel law. Indeed, it can simultaneously be viewed as a sort of universal Weyl law, and as an automorphic analogue to Schanuel’s well-known result on the number of rational points of bounded height on projective spaces.

To state the conjecture, we shall need to set up some notation. Let $D_F$ be the absolute discriminant of $F$. We introduce $\Delta_F(s) = \prod_v \Delta_v(s)$, where the product over all places $v$ of $F$ and

$$\Delta_v(s) = \zeta_v(s)\zeta_v(s + 1) \cdots \zeta_v(s + n - 1).\quad (1.1)$$

Let $\Delta^*_F(1)$ be the residue of $\Delta_F(s)$ at $s = 1$. Then there is a canonically normalized Haar measure $\mu_{GL_n}$ on $GL_n(\mathbb{A}_F)^1$ giving the quotient $GL_n(F) \backslash GL_n(\mathbb{A}_F)^1$ volume $\text{vol}(\mu_{GL_n}) = D_F^{n^2/2} \Delta^*_F(1)$.

In §1.4, we define a regularization $\int^*$ of adelic Plancherel measure $d\mu_{\mathbb{A}_F}$ on the adelic unitary dual $\Pi(GL_n(\mathbb{A}_F)^1)$ such that

$$\int_{\Pi(GL_n(\mathbb{A}_F)^1)} \int_{Q(\pi) \leq Q} d\mu_{\mathbb{A}_F}(\pi) \cdot \text{vol}(\Pi(GL_n(\mathbb{A}_F)^1)) = \frac{1}{n + 1} Q^{n+1} \int_{Q(\pi) \leq Q} Q(\pi)^{-n-1} d\mu_{\mathbb{A}_F}(\pi) + O(Q^{n-1-\theta}),$$

for some explicit $\theta > 0$. The regularization depends only on the local conductor at every place. Moreover, the regularized integral has total volume $\frac{\zeta_F(1)}{\zeta_F(n+1)^{n+\gamma}}$ times some archimedean volume factors, where $\zeta_F(s)$ is the Dedekind zeta function of $F$ and $\zeta_F(1)$ its residue at $s = 1$. We denote this volume by $\mathfrak{F}_Q(GL_n)$ and refer to it as the Tamagawa volume of the universal family.

We may now state the following
Conjecture 1 (Weyl–Schanuel law). Let $F$ and $n$ be fixed. As $Q \to \infty$ we have

$$|\mathfrak{F}(Q)| \sim \mathfrak{C}(\mathfrak{F})Q^{n+1},$$

where

$$\mathfrak{C}(\mathfrak{F}) = \text{vol}(\mu_{GL_n}) \cdot \frac{1}{n+1} \hat{\tilde{\mu}}_{\mathfrak{F}}(GL_n).$$

In this paper, we prove the above predicted asymptotics for $|\mathfrak{F}(Q)|$ in many cases, with explicit logarithmic savings in the error term. Namely, we establish the following

**Theorem 1.1.** The Weyl–Schanuel law holds for $GL_2$ as well as for $GL_n$ when the latter is restricted to the archimedean spherical spectrum.

In addition, we address related counting and equidistribution problems and prove uniform Weyl laws, estimates on the size of complementary spectrum, and uniform estimates on terms appearing in Arthur’s trace formula for $GL_n$.

1.2. Main auxiliary results. To prove Theorem 1.1 we first reduce Conjecture 1 to certain trace formula estimates, and then prove these estimates in many cases. We elaborate on the precise form of these estimates in the next paragraph, where we define the Effective Limit Multiplicity property, or Property (ELM), which encapsulates them. We refer the reader to the Definition 1 of §1.3 for more details (including some of the notation used here), and proceed now to a description of the two main auxiliary results which are used to prove Theorem 1.1.

Our first main theorem, proved in Part 2, is the reduction of Conjecture 1 to Property (ELM).

**Theorem 1.2.** Property (ELM) implies Conjecture 1 in the following effective form

$$|\mathfrak{F}(Q)| = \mathfrak{C}(\mathfrak{F})Q^{n+1} \left(1 + O\left(\frac{1}{\log Q}\right)\right).$$

Moreover, if Property (ELM) holds with respect to $\hat{\mathfrak{d}} \in \mathcal{D}$ then

$$|\{\pi \in \mathfrak{F}(Q) : \hat{\mathfrak{d}}_{\pi_{\infty}} = \hat{\mathfrak{d}}\}| = \mathfrak{C}_{\hat{\mathfrak{d}}}(\mathfrak{F})Q^{n+1} \left(1 + O\left(\frac{1}{\log Q}\right)\right),$$

where

$$\mathfrak{C}_{\hat{\mathfrak{d}}}(\mathfrak{F}) = \mathfrak{C}(\mathfrak{F}) \frac{\int_{\Pi(GL_n(F_{\infty})^1)} q(\pi_{\infty})^{-n-1} d\hat{\tilde{\mu}}_{\mathfrak{F}}(\pi_{\infty})}{\int_{\Pi(GL_n(F_{\infty})^1)} q(\pi_{\infty})^{-n-1} d\hat{\mu}_{\infty}^{pl}(\pi_{\infty})}.$$

All implied constants depend on $F$ and $n$.

One of the crucial ingredients in the proof of Theorem 1.2 is Proposition 9.1, in which we provide upper bounds on the sum over the discrete spectrum of $GL_n$, weighted exponentially by the size of the non-tempered component of the infinitesimal character. To state Proposition 9.1 here would be notationally burdensome, but we motivate its appearance in our approach in Section 3. Put briefly, Proposition 9.1 is related to density results on non-tempered discrete spectrum for $GL_n$, and has the effect, at various places in our arguments, of showing that discrete $\pi$ for which $\pi_{\infty}$ is non-tempered contribute negligibly to the total count $|\mathfrak{F}(Q)|$.

In our second main theorem, formulated in Theorem 15.2 and proved throughout Part 3, we establish Property (ELM) in certain cases. These are described in the following result.

**Theorem 1.3.**

1. For $n \leq 2$, Property (ELM) holds.
2. For $n \geq 3$ Property (ELM) holds with respect to the spherical part of $\Pi(GL_n(F_{\infty})^1)$. 


The combination of the above two theorems yields our main result, Theorem 1.1. The restriction to the archimedean spherical spectrum for \( n \geq 3 \) in Theorem 1.3 is a purely technical constraint, having only to do with explicit spectral inversion of archimedean test functions. We believe that this restriction can be removed, by following a different approach to bounding the weighted archimedean orbital integrals appearing in the Arthur trace formula. We plan to address this in a subsequent work.

1.3. Effective Limit Multiplicity (ELM) property. A natural framework for counting automorphic representations is provided by Arthur’s non-invariant trace formula. This is an equality of distributions \( J_{\text{spec}} = J_{\text{geom}} \), along with an expansion of both sides according to primitive spectral or geometric data. Roughly speaking, the most regular part of the spectral side of the trace formula \( J_{\text{cusp}} \), coming from the cuspidal contribution, is governed by the most singular part of the geometric side \( J_{\text{cent}} \), coming from the central elements.

To be more precise, for a function \( \varphi \in \mathcal{H}((\mathbb{A}_F)GL_n) \) we let

\[
J_1(\varphi) = \text{vol}(\mu_{GL_n})\varphi(1) \quad \text{and} \quad J_{\text{cent}}(\varphi) = \text{vol}(\mu_{GL_n}) \sum_{\gamma \in \mathcal{Z}(F)} \varphi(\gamma)
\]

be the identity and central contributions to the trace formula, and

\[
J_{\text{cusp}}(\varphi) = \text{tr}(R_{\text{cusp}}(\varphi)) \quad \text{and} \quad J_{\text{disc}}(\varphi) = \text{tr}(R_{\text{disc}}(\varphi))
\]

be the cuspidal and discrete contributions, where \( R_{\bullet} \) is the restriction of the right-regular representation of \( GL_n(\mathbb{A}_F) \) on \( L^2(\mathbb{A}_F)GL_n(\mathbb{A}_F) \). Finally put

\[
(1.2) \quad J_{\text{error}}(\varphi) = J_{\text{disc}}(\varphi) - J_{\text{cent}}(\varphi),
\]

the estimation of which will be our primary concern.

We shall in fact be interested in \( J_{\text{error}}(\varphi) \) for \( \varphi \) of the form \( \varepsilon_{K_1(q)} \otimes f \), where \( f \in C^\infty_c(GL_n(F_\infty)^1) \) and \( \varepsilon_{K_1(q)} \) is the idempotent element in the Hecke algebra associated with the standard Hecke congruence subgroup \( K_1(q) \). The latter subgroup, by the work of Casselman [8] and Jacquet-Piatetski-Shapiro-Shalika [28], is known to pick out from the cuspidal spectrum those representations of conductor dividing \( q \). One expects \( J_{\text{error}}(\varepsilon_{K_1(q)} \otimes f) \) to be small relative to quantities involving \( q \) and \( f \). If this can be properly quantified, one can hope to deduce that a sharp cuspidal count modelled by \( J_{\text{cusp}}(\varepsilon_{K_1(q)} \otimes f) \) is roughly equal to \( J_1(\varepsilon_{K_1(q)} \otimes f) \).

Our interest is in taking \( f \) whose Fourier transform \( \hat{h}(\pi_\infty) = \text{tr} \pi_\infty(f) \) localizes around a given unitary representation in \( \Pi(GL_n(F_\infty)^1) \). (The term “localize” should not be taken too literally: the function \( \hat{h} \) lies in a Paley-Wiener space and is therefore not of compact support. It will however be of rapid decay outside of some fixed compact.) Moreover, we would like to have some control over the error in the localization. In general this error is quantified by the support of the test function \( f \). Indeed, if \( \text{supp} f \subset K_\infty \exp(B(0,R))K_\infty \), where \( B(0,R) \) is the ball of radius \( R \) in the Lie algebra of the diagonal torus, then the walls of the corresponding \( h \), i.e., where it transitions to rapid decay, will be of size \( 1/R \).

The unitary dual of \( GL_n(F_\infty)^1 \) breaks up as a disjoint union

\[
\Pi(GL_n(F_\infty)^1) = \bigcup_{\delta \in \mathcal{D}} \Pi(GL_n(F_\infty)^1)_\delta
\]

indexed by discrete data \( \mathcal{D} \). More precisely, \( \mathcal{D} \) is the set of conjugacy classes of pairs \((M, \delta)\) consisting of a Levi subgroup \( M \) of \( GL_n(F_\infty)^1 \) and a discrete series representation \( \delta \) of \( M^1 \). Given a discrete spectral parameter \( \delta \in \mathcal{D} \) represented by \((\delta, M)\), a continuous spectral parameter \( \mu \in i\mathfrak{h}_M^* \), and a real number \( R > 0 \) the Paley-Wiener theorem of Clozel-Delorme [10] ensures the existence of test functions \( f^R_{\delta, \mu} \) whose support lies in \( K_\infty \exp(B(0,R))K_\infty \) and whose Fourier transform localizes about \((\delta, \mu)\). We shall need bounds on \( J_{\text{error}}(\varepsilon_{K_1(q)} \otimes f^R_{\delta, \mu}) \) for such archimedean localizing functions.
The following definition expresses uniform bounds on this quantity which are sufficiently strong for our applications. The dependence in the error term with respect to the archimedean spectral data is via a Plancherel majorizor function $\beta^G_M(\delta, \mu)$ which we introduce in Definition 2.

**Definition 1 (Effective Limit Multiplicity (ELM)).** Let $\delta \in \mathcal{D}$. We say that Property (ELM) holds with respect to all $\delta$ if there exist constants $C, \theta > 0$ such that for every $\mu \in i\mathfrak{h}_M^*$ and for every integral ideal $\mathfrak{q}$ of $\mathcal{O}_F$, and all real $R > 1$, we have

$$J_{\text{error}}(\varepsilon_{K_1}(\mathfrak{q}) \otimes \delta^{-\mu}_R) \ll R^{-\dim \mathfrak{h}_M} e^{C_R} Nq^{-\theta} \beta^G_M(\delta, \mu).$$

When (ELM) holds with respect to all $\delta \in \mathcal{D}$, one then says that Property (ELM) holds.

We shall make more extensive comments about Property (ELM) in §3.3. For the moment, we content ourselves to a few remarks.

**Remarks 1.**

1. We have expressed Property (ELM) with respect to the particular subgroups $K_1(\mathfrak{q})$ since only these arise in our applications. More generally, one could ask for analogous bounds for arbitrary sequences of compact open subgroups in $\text{GL}_n(\mathfrak{A}_F)$ whose volumes tend to zero. Our proof of Theorem 1.3, which establishes Property (ELM) in many cases, would continue to hold for such subgroups, since we are appealing to the powerful results [14] of Finis-Lapid.

2. The estimate is trivial in the archimedean spectral parameters $\delta$ and $\mu$. While it might at first seem surprising that no non-trivial savings at infinity is needed in order to deduce our main result, it is rather the power savings in the level which is of critical importance in our applications. To get a better feeling for the various ranges of parameters, and corresponding savings, see §3.1.

3. The terminology “Effective Limit Multiplicity” was chosen in reference to the power savings, see Finis-Lapid.

**3.3. For the moment, we now expand upon the leading term constant in the Weyl-Schanuel conjecture.**

- The automorphic volume $\text{vol}(\mu_{\text{GL}_n})$. Let $\mu_{\text{GL}_n,v}$ be the canonical measure on $\text{GL}_n(F_v)$ defined by Gross in [21]. It is given by $\mu_{\text{GL}_n,v} = \Delta_v(1)|\omega_{\text{GL}_n,v}|$, and $|\omega_{\text{GL}_n,v}|$ is the measure induced by the top degree invariant differential 1-form $\det(g)^{-n}(dg_{v1} \wedge \cdots \wedge dg_v)$. At finite places $v$, the measure $\mu_{\text{GL}_n,v}$ is the unique Haar measure on $\text{GL}_n(F_v)$ which assigns the maximal compact subgroup $\text{GL}_n(\mathfrak{o}_v)$ volume 1. In view of this latter property, the product measure $\mu_{\text{GL}_n} = \prod_v \mu_{\text{GL}_n,v}$ is well-defined. We continue to write $\mu_{\text{GL}_n}$ for the measure on $\text{GL}_n(\mathfrak{A}_F)$ induced by the exact sequence $1 \to \text{GL}_n(\mathfrak{A}_F) \to \text{GL}_n(F) \to \mathbb{R}_+^n \to 1$, where we have put the standard Haar measure $dt/t$ on $\mathbb{R}_+^n$. By an additional abuse of notation we let $\mu_{\text{GL}_n}$ denote the measure on $\text{GL}_n(F)\text{GL}_n(\mathfrak{A}_F)^1$ given by the quotient of $\mu_{\text{GL}_n}$ by the counting measure on $\text{GL}_n(F)$. Then it is shown in [21] that the canonical volume of the automorphic space $\text{GL}_n(F)\text{GL}_n(\mathfrak{A}_F)^1$ is finite and satisfies

$$\mu_{\text{GL}_n}(\text{GL}_n(F)\text{GL}_n(\mathfrak{A}_F)^1) = D_F^{n^2/2} \Delta_F^*(1).$$

The appearance of $\text{vol}(\mu_{\text{GL}_n})$ in the statement of Conjecture 1 is due to its presence in the identity contribution of the Arthur trace formula.¹

¹Equivalently, if one uses Tamagawa measure $|\omega_{\text{GL}_n}|$ instead of the Gross canonical measure $\mu_{\text{GL}_n}$, the automorphic space would have volume 1 but the reciprocal of the volume of $\mathbf{K}_f = \prod_{\nu < \infty} \text{GL}_n(\mathfrak{o}_v)$ would be $D_F^{n^2/2} \Delta_f^*(1)$. In that case, we would write the latter factors as $|\omega_{\text{GL}_n}|(\mathbf{K}_f)^{-1}$. 

The Tamagawa volume $\widehat{\mathcal{S}}(\mathrm{GL}_n)$ of the universal family. Let $\Pi(\mathrm{GL}_n(\mathbb{A}_F))$ denote the direct product over all places $v$ of the unitary duals $\Pi(\mathrm{GL}_n(F_v))$. Each $\Pi(\mathrm{GL}_n(F_v))$ is endowed with the Fell topology, and we give $\Pi(\mathrm{GL}_n(\mathbb{A}_F))$ the product topology. Let $\Pi(\mathrm{GL}_n(\mathbb{A}_F)^1)$ be the closed subset verifying the standard normalization on the central character, and we give this the subspace topology. Then $\widehat{\mathcal{S}}$ embeds in $\Pi(\mathrm{GL}_n(\mathbb{A}_F)^1)$ by taking local components, and an old observation of Piatetski-Shapiro and Sarnak [49] shows that $\widehat{\mathcal{S}}$ is dense\(^2\) in $\Pi(\mathrm{GL}_n(\mathbb{A}_F)^1)$.

We fix a normalization of Plancherel measures $\hat{\mu}_{v}^{\mathrm{pl}}$ on $\Pi(\mathrm{GL}_n(F_v))$ for each $v$ by taking Plancherel inversion to hold relative to $\mu_{\mathrm{GL}_n,v}$. At finite places $v$ the measure $\hat{\mu}_{v}^{\mathrm{pl}}$ assigns the unramified unitary dual volume $1$. Next we define a measure $\hat{p}^\tau_v$ on $\Pi(\mathrm{GL}_n(F_v))$ by setting (for open $A \subset \Pi(\mathrm{GL}_n(F_v))$)

$$\hat{p}_v^\tau(A) = \int_A q(\pi_v)^{-n-1}d\hat{\mu}_{v}^{\mathrm{pl}}(\pi_v).$$

Since $\hat{p}_v^{\mathrm{pl}}$ is supported on the tempered spectrum, so too is $\hat{p}_v^\tau$. In particular, since for $\mathrm{GL}_n(F_v)$ a tempered representation is automatically generic, it makes sense to write $q(\pi_v)$ in the integral. In Lemma 6.2 we show that for finite places $v$ the volume of $\hat{p}_v^\tau$ is finite and equal to $\zeta_v(1)/\zeta_v(n+1)^{n+1}$. (At the archimedean places we normalize the local conductor so that the same thing holds.) We deduce that the measure on $\Pi(\mathrm{GL}_n(\mathbb{A}_F)^1)$ given by the regularized product

$$\hat{p}_v^{\tau}(\mathcal{S}) = \zeta_v(1) \prod_{v < \infty} \zeta_v(1)^{-1} \hat{p}_v^\tau \cdot \hat{p}_{\infty}$$

converges. The regularized integral in Conjecture 1 is then, by definition, the volume of $\hat{p}_v^{\tau}(\mathcal{S})$.

1.5. Schanuel’s theorem. The Weyl-Schanuel law of Conjecture 1 is reminiscent of the familiar problem of counting rational points on projective algebraic varieties. In particular, one can set up an an analogy between counting $\pi \in \mathcal{S}$ with analytic conductor $Q(\pi) \leq Q$ and counting $x \in \mathbb{P}^n(F)$ with exponential Weil height $H(x) \leq B$. That one should consider $\mathcal{S}(Q)$ as analogous to $\{x \in \mathbb{P}^n(F) : H(x) \leq B\}$ is best seen as an expression of the deep conjectures of Langlands, in which the general linear groups serve as a sort of “ambient group”.

To be more precise, let us recall some classical results on counting rational points in projective space $\mathbb{P}^n$, where $n \geq 1$. For a rational point $x \in \mathbb{P}^n(F)$, given by a system of homogeneous coordinates $x = [x_0 : x_1 : \cdots : x_n]$, we denote by

$$H(x) = \prod_v \max(|x_0|_v, |x_1|_v, \ldots, |x_n|_v)^{1/d} \quad (d = [F : \mathbb{Q}])$$

the absolute exponential Weil height of $x$, the product being take over the set of normalized valuations of $F$. An asymptotic for the above counting function was given by Schanuel [52]. The leading constant was later reinterpreted by Peyre [44], as part of his refinement of the conjectures of Batyrev-Manin [5]. Following Peyre, we write $\tau_H(\mathbb{P}^n)$ for the volume of the Tamagawa measure of $\mathbb{P}^n$ with respect to $H$. Then Schanuel proved that

$$|\{x \in \mathbb{P}^n(F) : H(x) \leq B\}| \sim \frac{1}{n+1} \tau_H(\mathbb{P}^n)B^{n+1} \quad \text{as } B \to \infty.$$

Schanuel in fact gave an explicit error term of size $O(B \log B)$ when $n = 1$ and $F = \mathbb{Q}$ and $O(B^{n-1/d})$ otherwise. Later, Chambert-Loir and Tschinkel [9] showed how the Tamagawa measure $\tau_H(\mathbb{P}^n)$ appears naturally when calculating the volume of a height ball.

\(^{2}\)As remarked to us by Sarnak, if we give $\Pi(\mathrm{GL}_n(\mathbb{A}_F))$ the restricted product topology, the set $\mathcal{S}$ is discrete in $\Pi(\mathrm{GL}_n(\mathbb{A}_F))$. This is a point of view more adapted to computational problems of isolating and numerically computing cusp forms.
We note that $\tau_H(\mathbb{P}^n)$ is equal to $\zeta_F^*(1)/\zeta_F(n + 1)$ times some archimedean volume factors. This quotient of (regularized) zeta values is precisely the (regularized) value of the “index zeta function” at $s = n + 1$ of the $K_1(q)$ Hecke congruence subgroup for $GL_n$, defined as

$$
\zeta_F(s, K_1) = \prod_{v<\infty} \sum_{r\geq 0} \frac{[GL_n(\mathfrak{o}_v) : K_1(p^r_v)]}{\text{N}p_v^s} = \prod_{v<\infty} \sum_{r\geq 0} \frac{(p_n * \mu)(p^r_v)}{\text{N}p_v^s} = \prod_{v<\infty} \frac{\zeta_v(s - n)}{\zeta_v(s)} = \frac{\zeta_F(s - n)}{\zeta_F(s)}.
$$

This factor appears in the Tamagawa volume of the universal family $\hat{\tau}_3(GL_n)$, described in the introduction. This observation serves to emphasize the analogy between $\mathbb{P}^n$ and the congruence subgroup $K_1(q)$ which is used to pick out members of the universal family. It is the abscissa of convergence of the above Euler product which accounts for the order of magnitude of the asymptotic growth of $|\mathfrak{s}(Q)|$. The remaining factor $\zeta(s)^{-n}$ in $\hat{\tau}_3(GL_n)$ comes from inverting, through a sieving process, the series formed from the dimensions of old forms.

More generally, in the same spirit as the Batyrev-Manin-Peyre conjectures for counting rational points on Fano varieties, given a reductive algebraic group $G$ over $F$ and a representation $\rho : L^G \to GL_n(\mathbb{C})$ of the $L$-group, then assuming an appropriate version of the local Langlands conjectures, one can pull back the $GL_n$ conductor to $G$, and one would like to understand the asymptotic properties of the counting function associated to global cuspidal automorphic $L$-packets of $G(\mathbb{A}_F)$ of bounded analytic conductor (for many choices of $\rho$ this counting function will be infinite). Our methods suggest that, anytime this problem can be solved, the leading term constant will be given by the Plancherel volume of the conductor ball

$$
\int_{\pi \in \Pi(GL_n(\mathbb{A})^1)} \sum_{Q(\pi, \rho) < Q} d\mu^{pl}_{\pi}(\pi).
$$

This analogy served as an inspiration and organizing principle throughout the elaboration of this article, where we address the setting of general linear groups and the standard embedding.

1.6. Comments on other asymptotic aspects. Throughout this paper, both $n$ and $F$ will be considered as fixed. Nevertheless, it would be interesting to understand the behavior of $|\mathfrak{s}(Q)|$ as $n$ and $F$ vary (either simultaneously with $Q$ or for $Q$ fixed). We remark on two aspects:

1. In Conjecture 1, one could set $Q = 1 + \epsilon$ (for a small $\epsilon > 0$) and vary either $n$ or $F$. This would count the number of everywhere unramified cuspidal automorphic representations of $GL_n$ over $F$ whose archimedean spectral parameters are constrained to a small ball about the origin. In this set-up, if $F$ is fixed and $n$ gets large, we recover the number field version of a question of Venkatesh, as described for function fields in [18, §4].

2. On the other hand, we may fix $n$ (again keeping $Q = 1 + \epsilon$) and allow $D_F$ to get large. For example, when $n = 1$ this counts the size of a “regularization” of the group of ideal class characters, which has size about $D_F^{1/2}$ by Siegel’s theorem; this lines up with the power of $D_F$ in Conjecture 1. When $n = 2$, we recover the number field version of the famous result of Drinfeld [12]. Note that in the number field case the role of $D_F^{1/2}$ is played by the quantity $q^{n-1}$, as one can see by comparing the Tamagawa measures in [41] and [11, §3.8]. Thus, when $n = 2$ the factor of $D_F^2$ in the leading term in Conjecture 1 corresponds to $q^{4(g-1)}$ for function fields, and when multiplied by $|\text{Pic}(X_0)| = q - 1$ this recovers the leading term of $q^{4g-3}$.

We emphasize that we are not making any conjectures about the nature of the above asymptotic counts (1) and (2) when either $F$ or $n$ is allowed to move. The above discussion is meant purely to evoke parallels with other automorphic counting problems in the literature.
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2. **Equidistribution and Sato-Tate measures: conjectures**

Beyond the counting statement of Conjecture (1) we in fact conjecture that the universal family $\mathfrak{F}(Q)$ **equidistributes**, as $Q \to \infty$, to a probability measure on $\Pi(GL_n(\mathbb{A}_F)^1)$ that we now explicitly identify. This allows us to properly interpret the leading term constant in the conjectural Schanuel–Weyl law and our main theorems. We expect that our techniques can be leveraged to yield a proof of these equidistribution conjectures and plan to address this in follow up work.

The universal family $\mathfrak{F}(Q)$ gives rise, by way of the embedding into $\Pi(GL_n(\mathbb{A}_F)^1)$ via local components, to two automorphic counting measures

\[
\frac{1}{Q^{n+1}} \sum_{\pi \in \mathfrak{F}(Q)} \delta_\pi \quad \text{and} \quad \frac{1}{|\mathfrak{F}(Q)|} \sum_{\pi \in \mathfrak{F}(Q)} \delta_\pi
\]

on $\Pi(GL_n(\mathbb{A}_F)^1)$. We would like to understand their limiting behavior as $Q \to \infty$.

Recall the measure $\hat{p}^*(\mathfrak{F})$ on $\Pi(GL_n(\mathbb{A}_F)^1)$ of Section 1.4, whose volume enters Conjecture 1. Denoting by $\Pi_0(GL_n(\mathbb{A}_F)^1)$ the subset of $\Pi(GL_n(\mathbb{A}_F)^1)$ consisting of $\pi$ with $\pi_\infty$ spherical, the statement of our Theorem 1.3 verifies that

\[
\frac{1}{Q^{n+1}} \sum_{\pi \in \mathfrak{F}(Q)} \delta_\pi(A) \to \text{vol}(\mu_{GL_n}) \cdot \frac{1}{n+1} \hat{p}^*(\mathfrak{F})(A),
\]

for the sets $A = \Pi(GL_n(\mathbb{A}_F)^1)$ for $n < 2$ and for $A = \Pi_0(GL_n(\mathbb{A}_F)^1)$ for every $n \in \mathbb{N}$.

We conjecture that this holds more generally:

**Conjecture 2** (Equidistribution). As $Q \to \infty$,

\[
\frac{1}{Q^{n+1}} \sum_{\pi \in \mathfrak{F}(Q)} \delta_\pi \to \text{vol}(\mu_{GL_n}) \cdot \frac{1}{n+1} \hat{p}^*(\mathfrak{F}).
\]

The convergence of the above measures is taken in the sense of Sauvageot. Conjecture 2 implies, in particular, the Weyl–Schanuel law (Conjecture 1).

To deal with the second measure in (2.1), we define a related **probability** measure $\hat{\mu}(\mathfrak{F})$ on $\Pi(GL_n(\mathbb{A}_F)^1)$. Locally we define $\hat{\mu}_v = \hat{p}_v/\text{vol}(\hat{p}_v)$, a probability measure on $\Pi(GL_n(F_v))$, supported on the tempered spectrum. Then we put

\[
\hat{\mu}(\mathfrak{F}) = \frac{\hat{p}^*(\mathfrak{F})}{\text{vol}(\hat{p}^*(\mathfrak{F}))} = \prod_v \hat{\mu}_v;
\]

this is a well-defined factorizable probability measure on $\Pi(GL_n(\mathbb{A}_F)^1)$. Conjecture 2 then implies

\[
\frac{1}{|\mathfrak{F}(Q)|} \sum_{\pi \in \mathfrak{F}(Q)} \delta_\pi \to \hat{\mu}(\mathfrak{F}).
\]

2.1. **Sato-Tate measure.** Granting ourselves the statement (2.3), we may identify the Sato-Tate measure $\hat{\mu}_{ST}(\mathfrak{F})$ of the universal family $\mathfrak{F}$.

We recall the definition of $\hat{\mu}_{ST}(\mathfrak{F})$, introduced in [48]. Let $T$ denote the diagonal torus inside the Langlands dual group $GL_n(\mathbb{C})$ of $GL_n$, and let $W$ be the associated Weyl group. For finite places $v$, the Satake isomorphism identifies the unramified admissible dual with the quotient $T/W$. It then
makes sense to speak of the restriction of \( \hat{\mu}_v \) to \( T/W \), which we write (abusing notation) as \( \hat{\mu}_v|T \). One then defines

\[
\hat{\mu}_{ST}(\mathfrak{f}) = \lim_{x \to \infty} \frac{1}{x} \sum_{q_v < x} (\log q_v) \cdot \hat{\mu}_v|T;
\]

thus \( \hat{\mu}_{ST}(\mathfrak{f}) \) is a measure on \( T/W \).

Note that, under the above identification, the tempered unramified unitary dual corresponds with \( T_c/W \) where \( T_c \) is the compact torus \( U(n) \cap T \). Thus the restriction \( \hat{\mu}_v|T \) is supported on \( T_c/W \) and we may think of the Sato-Tate measure as being defined on \( T_c/W \). Now, as already mentioned, we show in Lemma 6.2 that for finite places \( v \) the volume of \( \mathfrak{p}_v \) is given by \( \frac{\zeta_v(1)}{\zeta_v(n+1)}q_v^{-1} \).

Thus, letting \( \hat{\mu}_v^{pl}|T_c \) denote the restriction of \( \hat{\mu}_v^{pl} \) to \( T_c/W \), we have

\[
\hat{\mu}_v|T_c = \frac{\zeta_v(n+1)^{n+1}}{\zeta_v(1)}\hat{\mu}_v^{pl}|T_c = (1 + O(q_v^{-1}))\hat{\mu}_v^{pl}|T_c.
\]

We deduce that

\[
\hat{\mu}_{ST}(\mathfrak{f}) = \lim_{q_v \to \infty} \hat{\mu}_v^{pl}|T_c.
\]

The latter limit is well known to have the following description.

**Corollary 2.1.** Assume Conjecture 2. Then the Sato-Tate measure \( \hat{\mu}_{ST}(\mathfrak{f}) \) of the universal family is the push-forward of the probability Haar measure on \( U(n) \) to \( T_c/W \).

Using the Weyl integration formula, we have

\[
\mu_{ST}(e^{i_1}, \ldots, e^{i_n}) = \frac{1}{n!} \int_{[0,2\pi]^n} \prod_{j<k} |e^{it_j} - e^{it_k}|^2 \frac{dt_j}{2\pi} \cdots \frac{dt_n}{2\pi}.
\]

In particular, it follows from Corollary 2.1 that the indicators

\[
i_1(\mathfrak{f}) = \int_T |\chi(t)|^2 d\hat{\mu}_{ST}(\mathfrak{f})(t), \quad i_2(\mathfrak{f}) = \int_T \chi(t)^2 d\hat{\mu}_{ST}(\mathfrak{f})(t), \quad i_3(\mathfrak{f}) = \int_T \chi(t^2) d\hat{\mu}_{ST}(\mathfrak{f})(t)
\]

introduced in [48], where \( \chi(t) = \text{tr}(t) \), take values \( i_1(\mathfrak{f}) = 1, \ i_2(\mathfrak{f}) = 0, \) and \( i_3(\mathfrak{f}) = 0 \) on the universal family. This is consistent with the expectation that the universal family \( \mathfrak{f} \) is of unitary symmetry type.

### 3. Outline of the proof

To set up the proof of Theorems 1.1 and 1.2, we begin by decomposing the universal family into discrete data:

1. the first such datum is the level, given by an integral ideal \( q \) in the ring of integers of \( F \);
2. the second is an archimedean spectral parameter, which enters through the decomposition of the admissible dual of \( GL_n(F_\infty)^1 \) into a disjoint union over a discrete set of parameters \( \delta \in D \).

More precisely, \( D \) consists of equivalence classes of square-integrable representations on Levi subgroups. We can represent any class \( \delta \in D \) by a square-integrable representation \( \delta \) of \( M^1 = M/A_M \), where \( M \) is a standard Levi subgroup and \( A_M \) the split component of its center. The reader can consult Section 4.9 for more background on the classification of the admissible dual.

It remains to impose a condition on the continuous archimedean spectral parameter. This can be done by specifying a nice subset \( \Omega \) of the \( \delta \)-unitary spectrum \( h^\delta_{\text{un},1} \), defined in (4.7). The spectral data \((\delta, \Omega)\) is well-defined up to conjugation. For example, we could take \( \Omega \) to be

\[
\Omega_{\delta, X} = \{ \nu \in h^\delta_{\text{un},1} : q(\pi_{\delta, \nu}) \leq X \},
\]

which selects unitary representations \( \pi_\infty \in \Pi(GL_n(F_\infty)^1)_{\delta} \) of archimedean conductor \( q(\pi_\infty) \leq X \).
Given an ideal \( q \) and archimedean spectral data \((\delta, \Omega)\) as above, let \( \mathcal{H}(q, \delta, \Omega) \) denote the set of \( \pi \in \mathfrak{S} \) such that \( q(\pi) = q, \delta_\pi = \delta, \) and \( \nu_\pi \in \Omega. \) Then

\[
|\mathfrak{S}(Q)| = \sum_{1 \leq N_\delta Q \leq Q} \sum_{d \in D} |\mathcal{H}(q, \delta, \Omega_{\delta,Q/N_\delta Q})|.
\]

In the parlance of \([48, 55]\), the set \( \mathcal{H}(q, \delta, \Omega) \) is what is called a harmonic family. One of the hallmarks of a harmonic family is that it can be studied by means of the trace formula. For this reason, it shall be more convenient to work with the weighted sum

\[
N(q, \delta, \Omega) = \sum_{\pi \in \mathcal{H}(q, \delta, \Omega)} \dim \pi^K(q),
\]

which counts each \( \pi \in \mathcal{H}(q, \delta, \Omega) \) with a weight corresponding to the dimension of the space of old forms for \( \pi_f. \) The quantities \( |\mathcal{H}(q, \delta, \Omega)| \) and \( N(q, \delta, \Omega) \) can be related via newform theory, namely,

\[
|\mathfrak{S}(Q)| = \sum_{1 \leq N_\delta Q \leq Q} \sum_{d \in D} \lambda_n(q/d) N(\delta, \delta, \Omega_{\delta,Q/N_\delta Q}),
\]

where \( \lambda_n = \mu_F \ast \cdots \ast \mu_F \) is the \( n \)-fold Dirichlet convolution of the Möbius function on \( F. \) Indeed, from the dimension formula \((5.1)\) of Reeder we deduce

\[
N(q, \delta, \Omega) = \sum_{d \mid q} \sum_{\pi \in \mathcal{H}(q, \delta, \Omega)} d_n(q/d) = \sum_{d \mid q} d_n(q/d) |\mathcal{H}(q, \delta, \Omega)|.
\]

This equality holds for every integral ideal \( q \) and is hence an equality of arithmetical functions. Since the inverse of \( d_n(m) \) under Dirichlet convolution is \( \lambda_n(m) \), Möbius inversion yields

\[
|\mathcal{H}(q, \delta, \Omega)| = \sum_{d \mid q} \lambda_n(q/d) N(\delta, \delta, \Omega).
\]

Taking \( \Omega = \Omega_{\delta,Q/N_\delta Q}, \) the claim \((3.4)\) then follows from \((3.2)\).

From this point, the proof of Theorem 1.2 proceeds as follows. We approximate \( N(q, \delta, \Omega) \) by the discrete spectral distribution \( J_{\text{disc}} \) of the trace formula, using a test function which

1. exactly picks out the weight \( \dim \pi^K(q) \) and the condition \( \delta_\pi = \delta, \)
2. but which smoothly approximates the condition \( \nu_\pi \in \Omega, \) with the auxiliary parameter \( R > 0 \) controlling the degree of localization.

The quality of this approximation is estimated in Part 2, where we execute the passage from smooth to sharp count of the tempered spectrum in harmonic families. We obtain asymptotic results on the size of the spectrum and strong upper bounds on the size of the complementary spectrum for individual large levels \( q, \) which are of independent interest. Here it should be noted that we require uniformity in \( q, \delta, \) and the domain \( \Omega, \) as all of them vary in our average \((3.4)\).

The successful execution of these steps of course depends on the trace formula input, which enters our argument through suitable applications of Property (ELM). Summing over \( q \) and appropriate spectral data as in \((3.4)\) then proves Theorem 1.2.

3.1. Prototypical example: classical Maass forms. Since much of the work required to prove Theorem 1.2 involves the treatment of the continuous parameter \( \nu_\pi, \) it makes sense to illustrate the difficulties by describing the simplest case, where we restrict to the spherical spectrum for \( \text{GL}_2 \) over \( \mathbb{Q}, \) consisting of even Maass cusp forms. In classical language, we seek an asymptotic for the number of Hecke–Maass cuspidal newforms on congruence quotients \( Y_1(q) = \Gamma_1(q) \setminus \mathbb{H} \) of level \( q \) and Laplacian eigenvalue \( \lambda = 1/4 + r^2 \) satisfying the bound \( q(1 + |r|)^2 \leq Q. \)
3.1. Why existing results are insufficient. A familiar environment for automorphic counting problem is that of Weyl’s law. A Weyl law for $GL_2$ over $\mathbb{Q}$, which is uniform in the level $q$, can be found in [43, Corollary 3.2.3], where it is shown that

$$N_{\Gamma_1(q)}(T) = \frac{\text{Vol}(Y_1(q))}{4\pi} T^2 - \varphi(q) d(q) \frac{2}{\pi} T \log T + O(q^2 T).$$

Since $\text{Vol}(Y_1(q)) \asymp q^2$, by taking $1 + T = \sqrt{Q/q}$ and summing over $q$, one expects the main term in the asymptotic for $|\mathcal{S}(Q)|$ to be of size $\sum_{q \leq Q} q^2 (\sqrt{Q/q} - 1)^2 \asymp Q^3$. Unfortunately, the total error term is also of size $Q^3$.

We see from this that one cannot simply sum (3.5) over $q$ to count the universal family. This is not surprising, since when the level $q$ is about $Q$ and $1 + T = \sqrt{Q/q}$ is bounded, the error term in (3.5) is of the same size as the main term, yielding only an upper bound. The loss of information in this range is deadly, since limit multiplicity theorems [50] (or Conjecture 1 more generally) suggest that $N_{\Gamma_1(q)}(1) \asymp q^2$, which would then in turn show that the bounded eigenvalue range contributes to $|\mathcal{S}(Q)|$ with positive proportion.

The important point here is that we cannot assume even a condition of the form $T \geq \frac{1}{100}$ if we wish to recover the correct leading constant in Theorem 1.2, since the complementary range contributes with positive proportion to the universal count.

3.1.2. Weak spectral localization. For our purposes, what we require from a uniform Weyl law for $N_{\Gamma_1(q)}(T)$ is an error term that is not only uniform in $q$ but in fact gives explicit savings in $q$ in the range $T \asymp 1$. The gain over $q$ in such error term is the measure by which one can localize about a given eigenvalue in the cuspidal spectrum of $Y_1(q)$. It is well-known that a purely analytic use of the trace formula can only localize on a scale of $1/\log q$.

Nevertheless, observe that even a modest improvement in the $q$-dependence in the error term with a complete loss of savings in the eigenvalue aspect – something of the form

$$N_{\Gamma_1(q)}(T) = \frac{\text{Vol}(Y_1(q))}{4\pi} T^2 \left( 1 + O \left( \frac{1}{\log q} \right) \right)$$

– is sufficient and yields an asymptotic of the form $c_0 Q^3 + O(Q^3/\log Q)$, with an absolute $c_0 > 0$. The gain by $\log q$ in the error term, along with the absence of savings in the $T$ aspect, in the above expression coincides precisely with the type of error we have encoded into Property (ELM).

Note that the demands one places on the savings in the $T$- and $q$-aspects are on unequal footings: we lose if we fail to show savings in the $q$-aspect (which is hard to acquire), but can afford to use the trivial bound in $T$ (which is easy to improve). For example, when $q = 1$ we may clearly get by with the bound of $T^2$ – or worse! This is essentially due to the fact that the $T$-parameter sees only one place, whereas the $q$-parameter sees all finite places.

3.1.3. Correspondence with expanding geometric support. One approaches (3.6) through an application of the Selberg trace, which states

$$\sum_{j \geq 0} h(r_j) = \frac{\text{vol}(Y_1(q))}{4\pi} \int_{\mathbb{R}} h(r) r \tanh \pi r \, dr + \sum_{[\gamma]} f(\log N_\gamma) \frac{\log \mathcal{N}_\gamma}{\mathcal{N}_\gamma^{1/2} - \mathcal{N}_\gamma^{-1/2}} + \cdots,$$

where $h$ is an even Paley–Wiener function, $f$ its inverse Fourier transform, $[\gamma]$ runs through hyperbolic conjugacy classes in $\Gamma_1(q)$, and the remaining terms arise from non-hyperbolic conjugacy classes on the geometric side and the Eisenstein spectrum on the spectral side.

One generally works with functions $h$ which approximate the characteristic function $\chi_I$ of a spectral interval $I$ (or ball, in higher rank). One way of constructing such $h$ is to convolve $\chi_I$ with a suitably nice $h_0 \in \text{PW}(\mathbb{C})$, centered at the origin. In fact, for a parameter $R > 1$, by convolving rather with the rescaled function $\xi \mapsto h_0(R\xi)$ one improves the approximation by a factor of $1/R$, since the walls around the interval $I$ are then of length $1/R$. For example, if $I = [-T, T]$, we may
localize $r$ to $[-T + O(T/R), T + O(T/R)]$ and the de-smoothing process in Weyl’s Law incurs an error of size $T/R$.

Note, however, that the spectral test functions $h$ obtained in this way, while serving our purposes, have two inevitable drawbacks:

1. They assign an exponential weight to the non-tempered spectrum,
2. Have Fourier transforms $f$ supported on a ball of radius $R$ about the origin.

In the remaining subsections of this exposition of the Maass case, we discuss how we deal with the obstacles created by these two properties.

3.1.4. Exponentially weighted discrete spectrum. After estimating the contributions from the Eisenstein series and the non-identity terms in the geometric side, a Weyl law of the form (3.6) follows by converting the smooth count of (3.7) to a sharp-cutoff. This conversion requires local bounds on the discrete spectrum, which itself involves another application of the trace formula. See [34, Section 2] for a nice overview for this, by now, standard procedure.

Note, however, that by the first drawback above, the bounds on the discrete spectrum we require are exponentially weighted by the distance to the tempered axis. Estimating this weighted count is closely related to density estimates for exceptional Maass forms. In the context of quotients by the upper half-plane this is classical, but in higher rank a delicate construction of positive-definite dominating test functions is required. This is described in more detail in §3.2.

3.1.5. Expanding geometric side. In the case of a fixed level and large eigenvalue, it is possible to localize $r$ within $O_{\Gamma_1(q)}(1/\log T)$, without seeing any of the hyperbolic spectrum. In light of the Prime Geodesic Theorem, which states that

$$\#\{\text{primitive } \gamma : \log N\gamma \leq T\} \sim \Gamma_1(q) e^{T/T},$$

this approach can be pushed to the limit by entering up to $O_{\Gamma_1(q)}(\log T)$ of the hyperbolic spectrum, which leads to the familiar (and currently best available) error term $O_{\Gamma_1(q)}(T/\log T)$ in (3.5).

Estimates on $N_{\Gamma_1(q)}(T)$ in bounded ranges of $T$ with error terms that feature explicit savings in $q$ correspond to instances of (3.7) such that the support of $f$ is expanding for large $q$; thus, controlling the number and magnitude of conjugacy classes of $\gamma \in \Gamma_1(q)$ in (3.7) is an essential ingredient in any limit multiplicity-type statement. One can use effective Benjamini-Schramm type statements [1], adapted to this non-compact setting [46], to show that the number of closed geodesics of length at most $R$ in $Y_1(q)$ is at most $O(e^{CR})$, for some constant $C > 0$.3 This control allows us to use functions $h$ arising as Fourier transforms of functions supported up to $\log q$. After some work to estimate all other contributions to (3.7), we obtain (3.6), in fact with a $O(T/\log q)$ error term.

3.2. Overview of Part 2. We now return to the general setting, and describe in more detail the contents of each section.

Section 7: Preliminaries. In this section, we set up the notation for Proposition 7.2, the basic estimate of Part 2. This result roughly states that for fixed discrete data $q$ and $\delta$, and a nice set $P$ in the tempered subspace $D^*_M$, the difference between the sharp count $N(q, \delta, P)$ of (3.3) and the expected main term $\text{vol}(\mu_{\text{GL}_n}) \varphi_n(q) \int_P d\mu_{\text{GL}_n}^{2n}$ is, assuming Property (ELM), governed by several explicit boundary terms depending on the approximation parameter $0 < R \ll \log(2 + Nq)$.

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3For the example $Y_1(q)$, an elementary argument shows that there are no closed geodesics of length $\ll \log q$. However, this fact is not robust: it already disappears for $\Gamma_0(q)$ or for the analog of $\Gamma_1(q)$ over number fields.
Section 8: Spectral localizers. In this section we define various archimedean Paley-Wiener functions \( h^\delta_R \) which localize around given spectral parameters \((\delta, P)\), and provide some basic estimates for their analytic behavior. The basic idea is that the characteristic function \( \chi_P(\nu) \) is very well approximated by \( h^\delta_R(\nu) \) on points \( \nu \) that are firmly inside or outside \( P \). The test functions \( f^\delta_R \) associated with these spectral localizers, through an invocation of the Paley-Wiener theorem of Clozel-Delorme, will be used in the trace formula to prove Proposition 7.2.

Section 9: Exponentially weighted discrete spectrum. We would like to approximate \( N(q, \delta, P) \) using \( J_{\text{temp}}(\varepsilon_{K_1(q)} \otimes f^\delta_R) \). But an application of Property (ELM) requires working with \( J_{\text{disc}} \) rather than \( J_{\text{temp}} \). We must therefore control the difference

\[
J_{\text{disc}}(\varepsilon_{K_1(q)} \otimes f^\delta_R) - J_{\text{temp}}(\varepsilon_{K_1(q)} \otimes f^\delta_R).
\]

Note that the spectral sampling functions \( h^\delta_R(\nu) \), being of Paley–Wiener type, act differently on spectral parameters \( \nu_\pi \) off the tempered spectrum \( h^\delta_M \); they exhibit exponential growth in \( \Re \nu_\pi \). In fact, the rate of exponential growth is directly related to the size \( R \) of the support of the test functions used on the geometric side; see Section 4.12 for details. For this reason, the contributions from \( \pi \in \Pi_{\text{disc}}(G(A_F)^1)_\delta \) for which \( \pi_\infty \) is not tempered must be estimated separately; specifically, for a suitable parameter \( R > 0 \) we require an upper bound for the exponentially weighted sum

\[
\sum_{\pi \in \Pi_{\text{disc}}(GL_n(A_F)^1)_\delta} \dim V_{\pi_j}^{K_1(q)} e^{R \| \Re \nu_\pi \|}.
\]

This is majorized, using an application of Property (ELM) and a delicate construction of archimedean positive-definite dominating test functions, in Section 9.

Sections 10: Smooth to sharp for tempered parameters. Here we put to use the preceding results to prove Proposition 7.2. Using the analytic properties of \( h^\delta_R(\nu) \) we first identify \( N(q, \delta, P) \) as the sum of \( J_{\text{temp}}(\varepsilon_{K_1(q)} \otimes f^\delta_R) \) with a boundary error, of the form \( N(q, \delta, \partial P(1/R)) \), where \( \partial P(1/R) \) is the \( 1/R \)-fattened boundary of \( P \). Then, using the results from Section 9, the term \( J_{\text{temp}}(\varepsilon_{K_1(q)} \otimes f^\delta_R) \) is amenable to the application of Property (ELM). Finally, we show that the boundary contributions can also be estimated from above by smooth sums, which can in turn be estimated by further applications of Property (ELM).

Sections 11: Summing error terms over discrete data. With Proposition 7.2 established, we can sum \( N(q, \delta, P) \) over all discrete parameters \( \delta \) and levels \( q \) to obtain the full count \( |\mathcal{S}(Q)| \) in (3.4). Bounding the resulting averages of errors terms proves Theorem 1.2.

3.3. Overview of Part 3. In Part 3, we establish Theorem 1.3. The proof naturally divides into two parts, corresponding to bounding \( J_{\text{geom}} - J_{\text{cent}} \) on the geometric side and \( J_{\text{spec}} - J_{\text{Elm}} \) on the spectral side. The estimations are not symmetric in the way they are proved, nor in the degree of generality in which they are stated. We would like to briefly describe these results here, and in particular explain why we are at present unable to establish Property (ELM) in all cases.

The main result on the geometric side is Theorem 12.1 in which we show the existence of constants \( C, \theta > 0 \) such that for any \( R > 0 \), integral ideal \( q \), and test function \( f \in \mathcal{H}(GL_n(F_\infty)^1)_R \), we have

\[
J_{\text{geom}}(\varepsilon_{K_1(q)} \otimes f) - J_{\text{cent}}(\varepsilon_{K_1(q)} \otimes f) \ll e^{CR} Nq^{n-\theta} \| f \|_\infty.
\]

This can be thought of as a sort of geometric limit multiplicity theorem, although it is only non-trivial in the \( q \) aspect. The exponential factor \( e^{CR} \) should be compared to (3.8). The latter shows that \( R \ll \log Nq \) is an allowable range in which the main term dominates. The proof of this occupies most of Sections 12 and 13. Indeed, in §12 we reduce the problem to a local one, and in §13 we bound the relevant local weighted orbital integrals.
The proof of our local estimates relies crucially on several recent developments, due to Finis-Lapid, Matz, Matz-Templier, and Shin-Templier. In particular, a central ingredient in the power savings in $N_q$ comes from the work of Finis-Lapid [14] on the intersection volumes of conjugacy classes with open compact subgroups. On the other hand, the source of the factor $\|f\|_\infty$ comes from estimating archimedean weighted orbital integrals trivially, by replacing $f$ by the product of $\|f\|_\infty$ with the characteristic function of its support. As the latter is, by hypothesis, contained in $K_\infty \exp(B(0,R))K_\infty$, it is enough then to have polynomial control in the support of the test function on these weighted orbital integrals. This can be extracted from the papers of Matz [35] and Matz-Templier [36].

Comparing the bound (3.9) to the statement of Property (ELM), it is clear that if one takes $f = f_R^{δ,P}$, then one wants to understand $\|f_R^{δ,P}\|_\infty$ in terms the Plancherel volume of $P$. It is at this point that we impose the condition that for $n > 2$ the discrete parameter $δ$ is the trivial character on the torus. In this case, the Paley-Wiener functions we use to approximate $P$ coming from the work of Finis-Lapid [14] on the intersection volumes of conjugacy classes with open compact subgroups. On the other hand, the source of the factor $\|f\|_\infty$ comes from estimating archimedean weighted orbital integrals trivially, by replacing $f$ by the product of $\|f\|_\infty$ with the characteristic function of its support. As the latter is, by hypothesis, contained in $K_\infty \exp(B(0,R))K_\infty$, it is enough then to have polynomial control in the support of the test function on these weighted orbital integrals. This can be extracted from the papers of Matz [35] and Matz-Templier [36].

On the spectral side, our main result is Theorem 15.1, which roughly states that for $R \ll \log N_q$ we have

$$J_{\text{spec}}(\hat{\varepsilon}_{K_1(q)} \otimes f_R^{δ,P}) - J_{\text{disc}}(\varepsilon_{K_1(q)} \otimes f_R^{δ,P}) \ll \varepsilon N_q^{1-\theta + \epsilon} \int_{\mathbb{P}} \mu_\infty^{pl}.$$  

The argument uses an induction on $n$. Indeed the above difference can be written as a sum over proper standard Levi subgroups $M \neq G$ of $J_{\text{spec},M}(\varepsilon_{K_1(q)} \otimes f_R^{δ,P})$, and each $M$ is a product of $GL_m$’s for $m < n$. The induction step itself relies critically on several ingredients. Besides the bounds on the geometric side of the trace formula of Sections 12 and 13, and the properties of the test functions of Section 14, the proof uses in an essential way the Tempered Winding Number property of [17] and the Bounded Degree Property of [16]. Our presentation follows that of several recent works, such as [35, §15], but differs in that we explicate the dependence in the parameter $R$ and in the level $q$.

Part 1. Preliminaries

4. General notation

The goal of this section is to put in place the basic notation associated with number fields and with the algebraic group $G = GL_n$.

4.1. Field notation. We recall some standard notation relative to the number field $F$.

Let $d = [F : \mathbb{Q}]$ be the degree of $F$ over $\mathbb{Q}$. Let $r_1$ and $r_2$ be the number of real and inequivalent complex embeddings of $F$, so that $r_1 + 2r_2 = d$. Let $O_F$ be the ring of integers of $F$. For an ideal $\mathfrak{n}$ of $\mathcal{O}$ let $N(\mathfrak{n}) = |O_F/\mathfrak{n}|$ be its norm. Write $D_F$ for the absolute discriminant of $F$.

For a normalized valuation $v$ of $F$, inducing a norm $|\cdot|_v$, we write $F_v$ for the completion of $F$ relative to $v$. For $v < \infty$ let $\mathcal{O}_v$ be the ring of integers of $F_v$, $\mathfrak{p}_v$ the maximal ideal of $\mathcal{O}_v$, $\varpi_v$ any choice of uniformizer, and $q_v$ the cardinality of the residue field.

Let $\zeta_F(s) = \prod_{v < \infty} \zeta_v(s)$ for $\text{Re}(s) > 1$ be the Dedekind zeta function of $F$. Write $\zeta_F^\ast(1)$ for the residue of $\zeta_F(s)$ at $s = 1$. We let $A_F$ denote the ring of adele ring of $F$ and $A_f$ the ring of finite adeles.

4.2. Subgroups and decompositions. We let $G = GL_n$, viewed as an algebraic group defined over $F$. Let $P_0$ denote the standard Borel subgroup of upper triangular matrices and $T_0$ the
diagonal torus of $G$. Let $\Phi$ be the set of roots of $G$ with respect to $T_0$ and $\Phi^+$ the subset of positive roots with respect to $P_0$. Let $Z$ denote the center of $G$.

A Levi subgroup of $G$ is called semistandard if it contains $T_0$; it is automatically defined over $F$. Let $L$ denote the finite set of all semistandard Levi subgroups of $G$. For $M \in L$ we let $\Phi^M$ denote the set of roots for $T_0$ in $M$. If $M \in L$ let $\mathcal{L}(M) = \{ L \in L : M \subset L \}$.

An $F$-parabolic subgroup $P$ of $G$ is called semistandard if it contains $T_0$. Let $\mathcal{F}$ denote the finite set of all semistandard $F$-parabolic subgroups. For $P \in \mathcal{F}$, let $U_P$ denote the unipotent radical of $P$ and $M_P$ the unique semistandard Levi subgroup such that $P = M_P U_P$. When $P = P_0$ we write $U_0$ for $U_{P_0}$ and of course $M_{P_0}$ is simply $T_0$. For $M \in L$ let $\mathcal{F}(M) = \{ P \in \mathcal{F} : M \subset P \}$. Denote by $P(M)$ the subset of $\mathcal{F}(M)$ consisting of those $F$-parabolic subgroups having Levi component $M$. Thus $\mathcal{P}(M) = \mathcal{F}(M) \setminus \bigcup_{L \supset M} \mathcal{F}(L)$.

We call an $F$-parabolic subgroup $P$ of $G$ standard if it contains $P_0$. Similarly, a semistandard Levi subgroup $M$ of $G$ is standard if it is contained in a standard parabolic subgroup. Write $\mathcal{F}_{st}$ and $L_{st}$ for the respective subsets of standard elements. Then $\mathcal{F}_{st}$ and $L_{st}$ are both in bijection with the set of ordered partitions of $n$, the correspondence sending a partition $(n_1, \ldots, n_m)$ to the Levi subgroup of block diagonal matrices in the shape $GL_{n_1} \times \cdots \times GL_{n_m}$.

For $M \in L$ we let $W_M = N_G(M)/M$ denote the Weyl group of $M$. When $M = T_0$ we simplify $W_{T_0}$ to $W_0$. Then each $W_M$ can be identified with a subgroup of $W_0$.

Let $X^*(M)$ be the group of $F$-rational characters of $M$. If $M$ is isomorphic to $GL_{n_1} \times \cdots \times GL_{n_m}$, then $X^*(M)$ can be identified with $\mathbb{Z}^m$, with $\lambda = (\lambda_i) \in \mathbb{Z}^m$ corresponding to the character $\chi^\lambda(g) = \prod \det g_i^{\lambda_i}$. Let $X_*(M)$ be the lattice of $F$-rational cocharacters. We then write $X^*_+(M) = \{ \lambda \in X_*(M) : (\alpha, \lambda) \geq 0 \text{ for all } \alpha \in \Phi^+ \}$ for the cone of positive cocharacters.

We put $M(\mathbb{A}_F)^1 = \bigcap_{\chi \in X^*_+(M)} \ker(\chi)$. Then $M(F)$ is a discrete subgroup of finite covolume in $M(\mathbb{A}_F)^1$. More concretely, $M(\mathbb{A}_F)^1$ is the closed subgroup of $M(\mathbb{A}_F)$ given by those elements such that each component in the block decomposition has determinant whose idele norm is $1$.

Let $A_M$ be the identity component of the real points of the $Q$-split part of the center of $\text{Res}_{F/Q} M$. Then $A_M$ is a Lie subgroup of $Z_M(F_\infty)$, where $Z_M$ is the center of $M$. One has a direct product decomposition $M(\mathbb{A}_F) = M(\mathbb{A}_F)^1 \times A_M$. When $M = T_0$ we write $A_0$ in place of $A_{T_0}$.

We have similar notions for the above structures locally at every place. For a place $v$ of $F$ we write $G_v = G(F_v)$. Let $T_v = T_0(F_v)$ denote the diagonal torus of $G_v$ and write $L_v$ and $F_v$ for the semistandard Levi subgroups of $G_v$. More generally, for a finite non-empty collection of places $S$ of $F$ we write $G_S = G(F_S)$, $T_S$, $L_S$, etc. In particular, when $S$ consists of all archimedean places, we write $G_\infty$, $T_\infty$, $L_\infty$, etc. We will write $M_S$ (without the boldface font), or simply $M$ if the context is clear, for an arbitrary element in $L_\infty$. A semistandard Levi subgroup $M \in L_\infty$ is then of the form $M = \prod_{v \in S} M_v$, where $M_v \in L_v$.

At every place $v$ of $F$ let $K_v = G(O_v), O(n)$, or $U(n)$ according to whether $v < \infty$, $v = \mathbb{R}$ or $v = \mathbb{C}$. For a finite non-empty collection of places $S$ of $F$ we write $K_S = \prod_{v \in S} K_v$; in particular, $K_\infty = \prod_{v < \infty} K_v$. Let $K_f = \prod_{v < \infty} K_v$ be the standard maximal compact open subgroup of $G(\mathbb{A}_F)$.

If we put $K = \prod_v K_v = K_f K_\infty$, then $K$ is a maximal compact subgroup of $G(\mathbb{A}_F)$.

For $M \in L_\infty$ we denote by $M^1$ the largest closed subgroup of $M$ on which all $|\chi_v|$ are trivial, as $\chi_v$ runs over the lattice $X^*(M)$ of $F_v$-rational characters of $M$. Let $A_M$ be the identity component of the $Q_v$-points of the center of $\text{Res}_{F_v/Q_v}(M)$, where $w$ is the unique place of $Q$ lying under $v$. Then $M = M^1 \times A_M$. We let $W(A_M) = N_{K_v}(A_M)/C_{K_v}(A_M)$.

Now consider $X^*(\text{Res}_{F/Q} M)$, the lattice of $Q$-rational cocharacters of the Weil restriction of scalars of $M$ from $F$ to $Q$. If $M$ is isomorphic to $GL_{n_1} \times \cdots \times GL_{n_m}$ and $\Sigma$ denotes the set of inequivalent embeddings of $F$ into $\mathbb{C}$, we may identify $X^*(\text{Res}_{F/Q} M)$ with $(\mathbb{Z}^m)^\Sigma$ by associating with $\lambda = (\lambda_i) \in (\mathbb{Z}^m)^\Sigma$ the character $\prod_{\sigma \in \Sigma} \prod_i \det(g_i \otimes \sigma)^{\lambda_i \sigma}$. We set $a^*_M = X^*(\text{Res}_{F/Q} M) \otimes_{\mathbb{Z}} \mathbb{R}$ and $a_M = \text{Hom}_{\mathbb{R}}(a^*_M, \mathbb{R})$. Then we can make an identification $a_M = \mathbb{R}^{mr}$. The map $A_M \to a_M$ sending $a$ to $\chi \mapsto \log |\chi(a)|$ is an isomorphism. When $M = T_0$ we write $a_0$ for $a_{T_0}$.
Similarly, for any place \( v \) and Levi subgroup \( M \in \mathcal{L}_v \) we set \( a_M^* = X^*(\text{Res}_{F_v/Q_v} M) \otimes \mathbb{R} \) and \( a_M = \text{Hom}_\mathbb{R}(a_M^*, \mathbb{R}) \), where \( w \) is the unique place of \( \mathbb{Q} \) lying under \( v \). Their complexifications are denoted \( a_{M,C} \) and \( a_M^* \). If \( M \) is isomorphic to \( \text{GL}_{m_1} \times \cdots \times \text{GL}_{m_n} \) then \( a_M \) is a product of \( m \) copies of \( \mathbb{R} \) or \( \mathbb{R}/\mathbb{Z} \), according to whether \( v \) is archimedean or not. Note that \( a_G \) sits inside every \( a_M \) as the diagonally embedded copy of \( \mathbb{R} \) or \( \mathbb{R}/\mathbb{Z} \).

If \( M \in \mathcal{L}_v \) is standard and \( P \in \mathcal{P}_v(M) \) has unipotent radical \( U_P \), one has the Iwasawa decomposition \( G_v = U_PM^1A_MK_v \). Let

\[
H_P : G_v = U_PM^1A_MK_v \to a_M, \quad H_P(\text{une}^X k) = X
\]

be the Iwasawa projection. Similarly, when \( M \in \mathcal{L}_{st} \) and \( P \in \mathcal{P}(M) \) has unipotent radical \( U_P \), there is a global Iwasawa decomposition \( G(\mathbb{A}_F) = U_P(\mathbb{A}_F)M(\mathbb{A}_F)^1A_MK \) and associated projection \( H_P : G(\mathbb{A}_F) \to a_M \).

4.3. Lie algebra decompositions. Let \( \mathfrak{g} \) be the Lie algebra of \( G_\infty \). We denote by \( \theta \) the usual Cartan involution of minus transpose, and let \( \mathfrak{g} = \mathfrak{p} \oplus \mathfrak{t} \) be the corresponding Cartan decomposition. Fix an \( \text{Ad} \)-invariant non-degenerate bilinear form \( B \) on \( \mathfrak{g} \), which is positive-definite on \( \mathfrak{p} \) and negative definite on \( \mathfrak{t} \). Then \( B \) defines an inner product \( \langle \cdot, \cdot \rangle \) on \( \mathfrak{g} \) by the rule \( \langle X, Y \rangle = -B(X, \theta Y) \). We may extend \( \langle \cdot, \cdot \rangle \) to a Hermitian inner product on \( \mathfrak{g} \mathbb{C} \) in the natural way.

We may restrict \( \langle \cdot, \cdot \rangle \) to \( a_M \) for each \( M \in \mathcal{L} \). Then \( W(A_M) \) acts by orthogonal transformations on \( a_M \) and \( a_M^* \).

We shall be interested rather in the restriction of these structures to \( G_\infty^1 \). Let

\[
\mathfrak{g}^0 = \left\{ X = (X_v)_{v|\infty} \in \mathfrak{g} : \sum_{v|\infty} \text{tr}X_v = 0 \right\}
\]

be the trace-zero subspace and write \( \mathfrak{h}_M = a_M \cap \mathfrak{g}^0 \). Using \( \langle \cdot, \cdot \rangle \), we shall sometimes identify \( \mathfrak{h}_M \) with \( \mathfrak{h}_M^\ast \). For \( M \in \mathcal{L}_\infty \) and \( H \in \mathfrak{h}_M \), let \( B_M(H, r) \) denote the ball of radius \( r > 0 \) about \( H \) in \( \mathfrak{h}_M \). Similarly \( B_M(\mu, r) \) shall denote the ball of radius \( r \) in \( \mathfrak{h}_M^\ast \) about \( \mu \in \mathfrak{h}_M^\ast \). When \( M = T_0 \) we shall drop the \( M \) subscript in these notations. Let

\[
G_{\infty, \leq R}^1 = K_\infty \exp B(0, R)K_\infty.
\]

For every \( v | \infty \) let \( \mathfrak{g}_v \) be the Lie algebra of \( G_v \) and \( \mathfrak{g}^0_v \) the trace-zero subspace. Let \( a^0_M \) be \( a_M \cap \mathfrak{g}^0_v \) so that \( a_M = a_{G_v} \oplus a^0_M \). Now, putting the archimedean places together, for \( M = \prod_{v|\infty} M_v \in \mathcal{L}_\infty \) we write \( a_M^0 = \bigoplus_{v|\infty} a^0_{M_v} \). We obtain decompositions \( a_M = a_G \oplus a_M^0 \) and

\[
\mathfrak{h}_M = (a_G \cap \mathfrak{h}_M) \oplus a^0_M.
\]

Given \( L \in \mathcal{L}(M) \) we have a natural inclusion \( \mathfrak{h}_L \subset \mathfrak{h}_M \). We let \( \mathfrak{h}_L^\perp \) denote the orthocomplement of \( \mathfrak{h}_L \) inside \( \mathfrak{h}_M \), so that \( \mathfrak{h}_M = \mathfrak{h}_L \oplus \mathfrak{h}_L^\perp \).

4.4. Weyl discriminant. Let \( v \) be a place of \( F \) and \( \sigma \) a semisimple element in \( G_v \). Let \( G_{\sigma, v} \) be the centralizer of \( \sigma \) and \( \mathfrak{g}_v \) its Lie algebra inside \( \mathfrak{g} \), the Lie algebra of \( G_v \). Then the Weyl discriminant of \( \sigma \) in \( G_v \) is defined to be

\[
D_v^G(\sigma) = |\det(1 - \text{Ad}(\sigma)|_{\mathfrak{g}/\mathfrak{g}_v})|_v = \prod_{\alpha \in \Phi} |1 - \alpha(\sigma)|_v.
\]

If an arbitrary \( \gamma \in G_v \) has Jordan decomposition \( \gamma = \sigma \nu \), where \( \sigma \) is semisimple and \( \nu \in G_{\sigma, v} \) is nilpotent, then we extend the above notation to \( \gamma \) by setting \( D_v^G(\gamma) = D_v^G(\sigma) \). The function \( \gamma \mapsto D_v^G(\gamma) \) on \( G_v \) is locally bounded and discontinuous at irregular elements. We have, for example, \( D_v^G(\sigma) = 1 \) for all central \( \sigma \). In our estimates on orbital integrals in the latter sections, it is this function which will measure their dependency on \( \gamma \). Whenever there is no risk of confusion we shall simplify \( D_v^G(\gamma) \) to \( D_v(\gamma) \).
More generally, when $M \in \mathcal{L}_v$ and $\gamma = \sigma \nu \in M$ we have
\[
D_v^M(\gamma) = |\det(1 - \text{Ad}(\sigma)|_{m/\nu})|_v = \prod_{\alpha \in \Phi^M} |1 - \alpha(\sigma)|_v.
\]
Thus when $M \in \mathcal{L}_v$ we have $D_v^G(\gamma) = D_v^G(\gamma)D_v^M(\gamma)$, where
\[
(4.1) \quad D_v^G(\gamma) = \prod_{\alpha \in \Phi - \Phi^M} |1 - \alpha(\sigma)|_v.
\]

4.5. Hecke congruence subgroups. At a finite $v$ and an integer $r \geq 0$ write $K_{1,v}(\mathfrak{p}_v^r)$ for the subgroup of $K_v$ consisting of matrices whose last row is congruent to $(0,0,\ldots,1)$ mod $\mathfrak{p}_v^r$. In particular, when $r = 0$ we obtain the maximal compact $K_v$. Then for an integral ideal $\mathfrak{q}$, whose completion in $\mathbb{A}_f$ factorizes as $\prod \mathfrak{p}_v^r$, we define an open compact subgroup of $G(\mathbb{A}_f)$ by $K_1(\mathfrak{q}) = \prod_v K_{1,v}(\mathfrak{p}_v^r)$. We set
\[
(4.2) \quad \varphi_n(\mathfrak{q}) = |K_f/K_1(\mathfrak{q})| = N(\mathfrak{q})^n \prod_{\mathfrak{p} \mid \mathfrak{q}} \left(1 - \frac{1}{N(\mathfrak{p})^n}\right) = (\mu \ast p_\mathfrak{q})(\mathfrak{q}),
\]
where $p_\mathfrak{q}(n) = N(n)^n$ is the power function. When $n = 1$ and $F = \mathbb{Q}$ this recovers the Euler $\varphi$-function.

4.6. Twisted Levi subgroups. Although our interest in this paper is solely in $G = \text{GL}_n$, in applications of the trace formula one encounters more general connected reductive groups, through the centralizers of semisimple elements in $G(F)$.

If $\gamma \in G(F)$, we write $G_\gamma$ for the centralizer of $\gamma$. If $\gamma = \sigma$ is semisimple, then $G_\sigma$ is connected and reductive. Moreover, one knows that in this case $G_\sigma$ is a twisted Levi subgroup, meaning that there are field extensions $E_1, \ldots, E_m$ of $F$ and non-negative integers $n_1, \ldots, n_m$ such that, if $E = E_1 \times \cdots \times E_m$, then $G_\sigma = \text{Res}_{E/F}(\text{GL}_{n_1} \times \cdots \times \text{GL}_{n_m})$.

If $H$ is a twisted Levi subgroup of $G$ containing some $M \in \mathcal{L}$, then an $F$-Levi subgroup of $H$ will be called semistandard (resp., standard) if it is the restriction of scalars of a semistandard (resp., standard) Levi subgroup. We similarly extend the notions of semistandard and standard to $F$-parabolic subgroups of $H$. We let $\mathcal{L}^H$ (resp., $\mathcal{F}^H$) denote the set of semistandard $F$-Levi subgroups (resp., $F$-parabolic subgroups) of $H$. If $M \in \mathcal{L}^H$ we write $\mathcal{L}^H(M) = \{L \in \mathcal{L}^H : M \subset L\}$ and $\mathcal{F}^H(M) = \{P \in \mathcal{F}^H : M \subset P\}$. Finally, $\mathcal{P}(M)$ will denote the subset of $\mathcal{F}^H$ consisting of parabolics having Levi component $M$.

If $H$ is a twisted Levi subgroup of $G$, and $v$ is a finite place, we write $K_v^H = K_v \cap H_v$. Similarly, let $K_v^\infty = K_v \cap H_\infty$. Then $K_v^H$ (resp., $K_v^\infty$) is a maximal compact subgroup of $H_v$ (resp., $H_\infty$). More generally, for $r \geq 0$ and a finite place $v$ we put $K_v^H(\mathfrak{p}_v^r) = K_{1,v}(\mathfrak{p}_v^r) \cap H_v$. If $\mathfrak{q} = \prod \mathfrak{p}_v^r$ is an integral ideal let $K_1^H(\mathfrak{q}) = K_1(\mathfrak{q}) \cap H(\mathbb{A}_f)$; then $K_1^H(\mathfrak{q}) = \prod_{\mathfrak{p} \mid \mathfrak{q}} K_{1,v}(\mathfrak{p}_v^r)$.

4.7. Canonical measures. We shall need a uniform way of fixing measures on the centralizers that arise in the trace formula, so that we can speak of the associated adelic volumes and orbital integrals. The theory of canonical measures, developed by Gross in [21], will be useful for this.

Let $H$ be a connected reductive group over $F$. Let $\text{Mot}_H$ be the motive attached by Gross to $H$. For any place $v$ of $F$ one has the associated local factor $L_v(\text{Mot}_H^\vee(1))$. Let $|\omega_{H,v}|$ be the Gross canonical measure of $H(F_v)$ (see [21, §4] for the non-archimedean case and [21, §7 and §11] for the archimedean case) and write $\mu_{H,v} = L_v(\text{Mot}_H^\vee(1))|\omega_{H,v}|$. For any finite set of places $S$ we write $\mu_{H,S} = \prod_{v \in S} \mu_{H,v}$. This is well-defined, since for almost all finite places $v$, the measure $\mu_{H,v}$ assigns a hyperspecial subgroup of $H(F_v)$.
measure 1. Give the automorphic space $H(F) \backslash H(\mathbb{A}_F)$ the quotient measure of $\mu_H$ by the counting measure on $H(F)$. Abusing notation, we again denote this measure by $\mu_H$.

We now wish to evaluate the automorphic volume $\mu_H(H(F) \backslash H(\mathbb{A}_F))$ in the case of $H = G = \text{GL}_n$. In this case we have $\text{Mot}_G = \mathbb{Q} + \mathbb{Q}(-1) + \cdots + \mathbb{Q}(1 - n)$ and $L_v(\text{Mot}_G, s) = \Delta_v(s)$, where $\Delta_v$ is defined in (1.1). Then $\Delta_F(s) = \Lambda(\text{Mot}_G, s) = \prod_v L_v(\text{Mot}_G, s)$ is the completed $L$-function of $\text{Mot}_G$, with $\Delta_F(1)$ its residue at $s = 1$. Recall from [41, §2] that the Tamagawa measure $|\omega_G|$ on $G^1(\mathbb{A}_F)$ is defined as

$$|\omega_G| = \frac{1}{\Delta_F(1)} \frac{1}{D_F^{n^2/2}} \prod_v \Delta_v(1)|\omega_v|,$$

where $\omega(g) = \det(g)^{-n}(dg_{11} \wedge \cdots \wedge dg_{nn})$ is the unique (up to scalar) top degree invariant rational differential 1-form on $G$. Let $\epsilon(\text{Mot}_G)$ denote the epsilon factor of $\text{Mot}_G$; from [21, (9.8)] we have $\epsilon(\text{Mot}_G) = D_F^{n^2/2}$. Applying [21, Theorem 11.5] we find that $\prod_v (|\omega_{G,v}|/|\omega_v|) = 1$. Thus

$$\mu_G = \prod_v (|\omega_{G,v}|/|\omega_v|) \cdot (\Delta_v(1)|\omega_v|) = D_F^{n^2/2} \Delta_F(1)|\omega_G|.$$

Since the Tamagawa number of $\text{GL}_n$ is 1, we deduce that $\mu_G(G(F) \backslash G(\mathbb{A}_F)) = D_F^{n^2/2} \Delta_F(1)$.

4.8. **Hecke algebras.** The following discussion will be of use in formal calculations involving the trace formula.

4.8.1. **Local case.** At any place $v$ we define $C^\infty_c(G_v)$ to be the space of functions on $G_v$ which are locally constant and of compact support, for $v$ finite, and smooth and of compact support for $v$ infinite. We then let $\mathcal{H}(G_v)$ denote $C^\infty_c(G_v)$, when considered as a convolution algebra with respect to the measure $\mu_{G,v}$. For non-archimedean $v$, and an open compact subgroup $K_v$ of $G_v$, let

$$\varepsilon_{K_v} = \frac{1}{\mu_{G,v}(K_v)} 1_{K_v}$$

denote the corresponding idempotent in $\mathcal{H}(G_v)$.

Given an admissible representation $\pi_v$ of $G_v$ any $\phi_v \in \mathcal{H}(G_v)$ define an operator on the space of $\pi_v$ via the averaging

$$\pi_v(\phi_v) = \int_{G_v} \phi_v(g) \pi_v(g) \, d\mu_{G,v}(g).$$

This is a trace class operator; we write $\text{tr} \pi_v(\phi_v)$ for its trace. If, for a finite place $v$, $K_v$ is an open compact subgroup of $G_v$, it is straightforward to see that $\text{tr} \pi_v(\varepsilon_{K_v}) = \dim \pi_v^K_v$.

Similarly, for a finite set of places $S$ containing all archimedean places, we denote by $\mathcal{H}(G(F_S))$ the space of finite linear combinations of factorizable functions $\otimes_{v \in S} \phi_v$, where each $\phi_v$ lies in $\mathcal{H}(G_v)$. Convolution is taken with respect to the measure $\mu_{G,S}$. Let $\mathcal{H}(G(F_S))_1$ denote the space of functions on $G(F_S)_1$ obtained by restricting those in $\mathcal{H}(G(F_S))$. In particular, a factorizable function $\phi_S \in \mathcal{H}(G(F_S)_1)$ is, by definition, the restriction of some $\otimes_{v \in S} \phi_v$ to $G(F_S)_1$.

Finally, denote by $\mathcal{H}(G^1_{\infty,R})$ the space of all smooth functions on $G^1_{\infty,R}$ supported in $G^1_{\infty,R}$.

4.8.2. **Plancherel measure.** For any place $v$ we write $\Pi(G_v)$ for the unitary dual of $G_v$, endowed with the Fell topology. Let $\tilde{\mu}_v^{pl}$ denote the Plancherel measure for $\Pi(G_v)$. Having fixed the measure $\mu_{G,v}$ to define the trace in §4.8.1, we may normalize $\tilde{\mu}_v^{pl}$ so as to satisfy

$$\phi_v(e) = \int_{\Pi(G_v)} \text{tr} (\pi_v(\phi_v^\vee)) \, d\tilde{\mu}_v^{pl}(\pi_v)$$

for any $\phi_v \in C^\infty_c(G_v)$, where $\phi_v^\vee(g) = \phi_v(g^{-1})$. 
4.8.3. Global case. The global Hecke algebra $\mathcal{H}(G(\mathbb{A}_F))$ is defined as the space of finite linear combinations of factorizable functions $\otimes_v \phi_v$, where each $\phi_v$ lies in $\mathcal{H}(G_v)$ and $\phi_v = 1_{K_v}$ for almost all finite $v$. Convolution is taken with respect to the canonical measure $\mu_G$. We define $\mathcal{H}(G(\mathbb{A}_F)^1)$ by restricting functions from $\mathcal{H}(G(\mathbb{A}_F))$. For admissible $\pi = \otimes_v \pi_v$ and $\phi \in \mathcal{H}(G(\mathbb{A}_F)^1)$ we define the trace-class operator $\pi(\phi)$ with respect to $\mu_G$. Moreover, for admissible $\pi = \otimes_v \pi_v$ and factorizable $\phi = \otimes_v \phi_v \in \mathcal{H}(G(\mathbb{A}_F)^1)$ the global trace $\text{tr} \pi(\phi)$ factorizes as $\prod_v \text{tr} \pi_v(\phi_v)$.

Similarly, if $S$ is any finite set of places of $F$ containing all archimedean places, we let $\mathcal{H}(G(\mathbb{A}_F)^S)$ denote the analogous space, with convolution taken with respect to the measure $\mu_G^S$.

4.9. Unitary and admissible archimedean duals. For a place $v$ of $F$ we let $\text{Rep}(G_v)$ denote the admissible dual of $G_v$. Similarly, $\text{Rep}(G^1_v)$ will denote the admissible dual of $G^1_v$. The main goal of this subsection is to review the classification of the admissible dual (due to Langlands, and nicely described in [31, §2]), and in particular to associate with every infinitesimal class of irreducible admissible representations $\pi$ an equivalence class of spectral data $(\delta, \nu)$, consisting of a discrete parameter $\delta$ and a continuous parameter $\nu$, taken up to conjugacy. Since the unitary duals $\Pi(G_v)$ and $\Pi(G^1_v)$ can be identified with subsets of the respective admissible duals, this parametrization can also be applied to isomorphism classes of irreducible unitary representations.

For $M \in \mathcal{L}_v$ or $M \in \mathcal{L}_\infty$ we let $\mathcal{E}^2(M^1)$ denote the set of isomorphism classes of square-integrable representations of $M^1$. We say that $M$ is cuspidal if $\mathcal{E}^2(M^1)$ is non-empty. If $M \in \mathcal{L}_v$ and $M$ is isomorphic to $\text{GL}_{n_1}(F_v) \times \cdots \times \text{GL}_{n_m}(F_v)$, where $n_1 + \cdots + n_m = n$, then $M$ is cuspidal precisely when $1 \leq n_j \leq 2$ for $v$ real and $n_j = 1$ for $v$ complex.

Following [17, §6] we let $\mathcal{D}$ denote the $G^1_\infty$-conjugacy classes of pairs $(M, \delta)$ consisting of a cuspidal Levi subgroup $M$ of $G_\infty$ and $\delta \in \mathcal{E}^2(M^1)$. We let $W(A_M)^\delta$ denote the stabilizer of $\delta$ in $W(\text{Ad}(A_M))$. Furthermore, we put

$$\text{(4.4)} \quad W(A_M^\delta) = \{ w \in W(\text{Ad}(A_M)^\delta) : w \lambda = \lambda \quad \forall \lambda \in h_M^\delta \}. $$

Fixing $\delta \in \mathcal{D}$, we may assume that $\delta$ is represented by $(\delta, M)$, where $M \in \mathcal{L}_{\text{st}, \infty}$. For $\nu \in h^*_M, C$, we may form an essentially square-integrable representation $\delta \otimes e^\nu$ of $M$. Let $P$ be the unique standard parabolic containing $M$ as a Levi subgroup. We may then consider the (unitarily) induced representation $\text{Ind}^G_P(\delta \otimes e^\nu)$. This is not, in general, irreducible (it is for $\nu \in h^*_M$ since $G = \text{GL}_n$). Nevertheless, there is a unique $w \in W(\text{Ad}(A_M)^\delta)$ such that $\text{Ind}^G_P(\delta \otimes e^{w \nu})$ admits an irreducible quotient, denoted $\pi_{\delta, \nu}$ and called the Langlands quotient. Then $\pi_{\delta, \nu} \in \text{Rep}(G^1_\infty)$. Conversely, given $\pi \in \text{Rep}(G^1_\infty)$ there a unique $\delta, \nu \in \mathcal{D}$, represented, say, by $(\delta, M)$ where $M \in \mathcal{L}_{\text{st}, \infty}$, and a $\nu \in h^*_M, C$, well-defined up to $W(\text{Ad}(A_M)^\delta)$-conjugacy, such that $\pi$ is infinitesimally equivalent to $\pi_{\delta, \nu}$.

We shall in fact need to parametrize the admissible dual of $G^1_\infty$ for any rational Levi $M \in \mathcal{L}$. Similarly to the above, we let $\mathcal{D}^M$ denote the $G^1_\infty$-conjugacy classes of pairs $(M, \delta)$ consisting of a cuspidal Levi subgroup $M$ of $G_\infty$ and $\delta \in \mathcal{E}^2(M^1)$. (When $M = G$ we simply write $\mathcal{D} = \mathcal{D}^G$.) Given $\delta \in \mathcal{D}^M$, we write $\text{Rep}(M^1_\infty)^\delta$ for the subset of $\pi \in \text{Rep}(G^1_\infty)$ with $\pi_{\delta, \nu}$.

$$\text{(4.5)} \quad \text{Rep}(M^1_\infty) = \bigsqcup_{\delta \in \mathcal{D}^M} \text{Rep}(M^1_\infty)^\delta. $$

From the inclusion $\Pi(M^1_\infty) \subset \text{Rep}(M^1_\infty)$, we deduce

$$\text{(4.5)} \quad \Pi(M^1_\infty) = \bigsqcup_{\delta \in \mathcal{D}^M} \Pi(M^1_\infty)^\delta. $$

4.10. Hermitian archimedean dual. We now describe the Hermitian dual of $G^1_\infty$. We are inspired by treatment in Lapid-Müller [34, §3.3], who deal with the spherical case.

For $M \in \mathcal{L}_\infty$ and $w \in W(\text{Ad}(A_M))$ we let

$$h^*_M, w = \{ \lambda \in h^*_M, C : w \lambda = -\lambda \}. $$
Let \( \delta \in \mathcal{D} \) be represented by \((\delta, M)\). We introduce the \( \delta \)-Hermitian spectrum
\[
\mathfrak{h}_{\delta, \text{hm}}^* = \bigcup_{w \in W(A_M)\delta} \mathfrak{h}_{M, w}^*.
\]
(4.6)

The key property of \( \mathfrak{h}_{\delta, \text{hm}}^* \) is that whenever \( \nu \in \mathfrak{h}_{\delta, \text{hm}}^* \) the representation \( \pi_{\delta, \nu} \) admits a non-degenerate Hermitian structure \([32, \text{Theorem 16.6}]\). If we put
\[
\mathfrak{h}_{\delta, \text{un}}^* = \{ \lambda \in \mathfrak{h}_{M, \mathbb{C}}^* : \pi_{\delta, \lambda} \text{ unitarizable} \},
\]
(4.7)
then \( \mathfrak{h}_{\delta, \text{un}}^* \subset \mathfrak{h}_{\delta, \text{hm}}^* \).

A key feature of the \( \delta \)-Hermitian spectrum is that if \( \nu \in \mathfrak{h}_{\delta, \text{hm}}^* \) is such that \( \text{Re} \nu \neq 0 \) then \( \text{Im} \nu \) is forced to belong to a positive codimension subspace in \( \mathfrak{h}_M \). We would like to be more precise about the collection of such singular subspaces in \( \mathfrak{h}_M \). We begin by noting that if \( \mathfrak{h}_{M, w, \pm 1}^* \) denotes the \( \pm 1 \) eigenspaces for \( w \) acting on \( \mathfrak{h}_M^* \), then \( \mathfrak{h}_{M, w}^* = \mathfrak{h}_{M, w, -1}^* + i \mathfrak{h}_{M, w, +1}^* \). From \([42, \text{Theorem 6.27}]\) it follows that for every \( w \in W(A_M) \) we have \( \mathfrak{h}_{M, w} = \mathfrak{h}_{M, w, +1}^* \), where \( M \) is the smallest \( L \in \mathcal{L}_\infty \) containing \( M \) for which (a representative of) \( w \) belongs to \( L \). Note that \( \mathfrak{h}_{M, w} = \{ H \in \mathfrak{h}_M : wH = H \} \).

In view of (4.6), we let \( \mathcal{L}_\infty(\delta) \) be the collection of the \( M \) as \( w \) varies over \( W(A_M)\delta \). Observe that if \( L_1 \) and \( L_2 \) lie in \( \mathcal{L}_\infty(\delta) \) then the Levi subgroup they generate \( (L_1, L_2) \) also lies in \( \mathcal{L}_\infty(\delta) \). This lattice-type property of \( \mathcal{L}_\infty(\delta) \) will be used repeatedly in §9. Then we write
\[
\mathfrak{h}_{\delta, \text{sing}}^* = \bigcup_{M' \in \mathcal{L}_\infty(\delta)} \mathfrak{h}_{M'}^* \quad \text{for the } \delta\text{-singular subset of } \mathfrak{h}_M^*.
\]
(4.8)

4.11. \( \mathbf{K}_\infty \)-types. As in Vogan \([57, 58]\), we may put a (sort of) norm on \( \Pi(\mathbf{K}_\infty) \) in the following way. Let \( \mathbf{K}_\infty^0 \) denote the connected component of the identity of \( \mathbf{K}_\infty \); then \( \mathbf{K}_\infty^0 \) is the analytic subgroup corresponding to the Lie subalgebra \( \mathfrak{k} \), as in §4.3. Let \( \mathbf{K}_\infty^T = \mathbf{T}_\infty \cap \mathbf{K}_\infty^0 \) have corresponding Cartan subalgebra \( \mathfrak{k}_T \). For any \( \tau \in \Pi(\mathbf{K}_\infty) \) let \( \lambda_\tau \in \mathfrak{k}_T^* \) be the highest weight associated with any irreducible component of \( \tau\big|_{\mathbf{K}_\infty} \). Let \( \| \cdot \| \) be the norm on \( \mathfrak{k}_T^* \) coming from the restriction of \( B \). Then, we put \( \| \tau \| = \| \lambda_\tau + \rho_{\mathbf{K}_\infty} \| \), where \( \rho_{\mathbf{K}_\infty} \) denotes half the sum of the positive roots of \( \mathbf{K}_T^\circ \) inside \( \mathbf{K}_\infty \).

Given \( \pi \in \Pi(\mathbf{G}_\infty) \), Vogan defines a minimal \( \mathbf{K}_\infty \)-type of \( \pi \) as any \( \tau \in \Pi(\mathbf{K}_\infty) \) of minimal norm appearing in \( \pi|_{\mathbf{K}_\infty} \). A minimal \( \mathbf{K}_\infty \)-type is not necessarily unique, although it does appear with multiplicity one in \( \pi \). For \( \delta \in \mathcal{D} \), represented by \((\delta, M)\), we let \( \tau(\pi_\delta) \in \Pi(\mathbf{K}_\infty) \) be the minimal \( \mathbf{K}_\infty \)-type of \( \pi_\delta \), for any choice of \( \mu \in h_{\delta, M, \mathbb{C}}^* \); it is independent of \( \mu \). We set \( \| \tau \| = \| \tau(\pi_\delta) \| \).

4.12. Paley–Wiener theorem of Clozel–Delorme. Let \( M \in \mathcal{L}_{\text{st}, \infty} \). For a function \( g \in C_c^\infty(h_M) \) and \( \lambda \in h_{\delta, M, \mathbb{C}}^* \) let \( \hat{g}(\lambda) = \int_{h_M} g(X) e^{i\lambda(X)} dX \) denote the Fourier transform of \( g \) at \( \lambda \). The image of \( C_c^\infty(h_M) \) under this map is the Paley–Wiener space \( \mathcal{P}W(h_{\delta, M, \mathbb{C}}^*) \). Recall that
\[
\mathcal{P}W(h_{\delta, M, \mathbb{C}}^*) = \bigcup_{R > 0} \mathcal{P}W(h_{\delta, M, \mathbb{C}}^*)_{R},
\]
(4.9)
where \( \mathcal{P}W(h_{\delta, M, \mathbb{C}}^*)_R \) consists of those entire functions \( h \) on \( h_{\delta, M, \mathbb{C}}^* \) such that for all \( k \geq 0 \) we have
\[
\sup_{\lambda \in h_{\delta, M, \mathbb{C}}^*} \left\{ |h(\lambda)| e^{-R R(\lambda)} (1 + \|\lambda\|)^k \right\} < \infty.
\]
Then the Fourier transform \( C_c^\infty(h_M) \rightarrow \mathcal{P}W(h_{\delta, M, \mathbb{C}}^*) \) is an isomorphism of topological algebras, each of these spaces being taken with their natural Fréchet topologies. Moreover, for \( R > 0 \), if \( C_c^\infty(h_M)_R \) denotes the subspace of \( g \in C_c^\infty(h_M) \) having support in the ball \( B_{M}(0, R) \), then the Fourier transform maps \( C_c^\infty(h_M)_R \) onto \( \mathcal{P}W(h_{\delta, M, \mathbb{C}}^*)_R \); cf. \([19, \text{Theorem 3.5}]\).
Let $\delta \in D$ be represented by $(M, \delta)$, where $M \in \mathcal{L}_{st, \infty}$. We denote by $\mathcal{PW}_{R, \delta}$ the space of $W(A_M)_{\delta}$-invariant functions in $\mathcal{PW}(h_{M, \ell, C})_R$. The Paley–Wiener theorem of Clozel–Delorme [10] states that for any $h_R^\delta \in \mathcal{PW}_{R, \delta}$, there exists a $f_R^\delta \in \mathcal{H}(G_1^1)_R$ such that for every $\sigma \in D$ represented by $(\sigma, L)$, where $L \in \mathcal{L}_{st, \infty}$, and every $\nu \in h_{L, C}^\sigma$, one has $\text{tr} \pi_{\sigma, \nu}(f_R^\delta) = h_R^\delta(\nu)$ whenever $(\sigma, L) \in \delta$, and $\text{tr} \pi_{\sigma, \nu}(f_R^\delta) = 0$ otherwise. The function $f_R^\delta$ is not unique, as the addition by any other function whose weighted orbital integrals identically vanish will have the same spectral transform.

4.13. Discrete automorphic dual. For $M \in \mathcal{L}$ we write $\Pi_{\text{disc}}(M(\mathbb{A})^1)$ for the set of isomorphism classes of irreducible discrete unitary automorphic representations of $M(\mathbb{A})^1$. We may view $\Pi_{\text{disc}}(M(\mathbb{A})^1)$ alternatively as the set of irreducible subrepresentations of the right-regular representation on $L^2_{\text{disc}}(M(F) \backslash M(\mathbb{A}_F)^1)$. Indeed, the multiplicity one theorem [27, 54, 45] for the cuspidal spectrum of $G = \text{GL}_n$, and the description of the residual spectrum of $\text{GL}_n$ by [38], together imply that the multiplicity with which $\pi \in \Pi_{\text{disc}}(M(\mathbb{A})^1)$ appears in the right-regular representation $L^2(M(F) \backslash M(\mathbb{A}_F)^1)$ is one.

We may decompose $\Pi_{\text{disc}}(M(\mathbb{A})^1)$ according to the archimedean decomposition (4.10). For $\pi \in \Pi_{\text{disc}}(M(\mathbb{A})^1)$, we let $\delta_\pi$ denote the discrete parameter $\delta_{\pi, \infty} \in D^M$ associated with $\pi_{\infty}$. We have

\begin{equation}
\Pi_{\text{disc}}(M(\mathbb{A})^1) = \bigcup_{\delta \in D^M} \Pi_{\text{disc}}(M(\mathbb{A}_F)^1)_\delta,
\end{equation}

where $\Pi_{\text{disc}}(M(\mathbb{A}_F)^1)_\delta$ consists of those $\pi$ for which $\delta_\pi = \delta$. Finally, when $\pi \in \Pi_{\text{disc}}(M(\mathbb{A})^1)_\delta$, we shall write $\lambda_\pi$ for the continuous parameter $\lambda_{\pi, \infty} \in (h^M)^*_{\delta, \text{un}}$ associated with $\pi_{\infty}$.

5. LOCAL AND GLOBAL CONDUCTORS

In this section we review the representation theory of archimedean $\text{GL}_n$ and the Plancherel measure for the unitary dual. In particular, we define the archimedean conductor.

For a place $v$ of $F$, let $\pi_v$ be an irreducible admissible representation of $\text{GL}_n(F_v)$. Let $L(s, \pi_v)$ denote the standard $L$-function of $\pi_v$, as defined by Tate [56] (for $n = 1$) and Godement-Jacquet [20] (for $n \geq 1$).

5.1. Non-archimedean case. Let $v$ be a non-archimedean place. For an additive character of level zero $\psi_v$, let $\epsilon(s, \pi_v, \psi_v)$ be the local epsilon factor of $\pi_v$. Then there is an integer $f(\pi_v)$, independent of $\psi_v$, and a complex number $\epsilon(0, \pi_v, \psi_v)$ of absolute value 1 such that $\epsilon(s, \pi_v, \psi_v) = \epsilon(0, \pi_v, \psi_v) q_v^{-f(\pi_v) s}$. Moreover, $f(\pi_v) = 0$ whenever $\pi_v$ is unramified.

Under the additional assumption that $\pi_v$ is generic, Jacquet, Piatetski-Shapiro, and Shalika [28], show that the integer $f(\pi_v)$ is in fact always non-negative. One then calls $f(\pi_v)$ the conductor exponent of $\pi_v$. The conductor $q(\pi_v)$ of a generic irreducible $\pi_v$ is then defined to be $q(\pi_v) = q_v^{f(\pi_v)}$. In particular, $q(\pi_v) = 1$ whenever $\pi_v$ is unramified.

For $v$ finite, Jacquet, Piatetski-Shapiro, and Shalika [28], building on work of Casselman [8], show that for any irreducible generic representation $\pi_v$ of $\text{GL}_n(F_v)$ the conductor exponent $f(\pi_v)$ is equal to the smallest non-negative integer $r$ such that $\pi_v$ admits a non-zero fixed vector under $K_{1,v}(p_v^r)$. Moreover, the space of all such fixed vectors is of dimension 1. By the subsequent work of Reeder [47] it follows that for irreducible generic $\pi_v$ with $q(\pi_v) | q$ one has

\begin{equation}
\dim \pi_v^{K_{1,v}(q)} = d_n(q/q(\pi_v)),
\end{equation}

where $d_n = 1 \star \cdots \star 1$ is the $n$-fold convolution of 1 with itself. In particular, if $\pi_v$ is an irreducible generic representation of $\text{GL}_n(F_v)$, one has

\begin{equation}
\text{tr} \left( \pi_v(\varepsilon_{K_{1,v}(q(\pi_v)p_v^r)}) \right) = d_n(p_v^r).
\end{equation}
5.2. Archimedean conductor. For \( v \) archimedean the local \( L \)-factor of \( \pi_v \) is a product of shifted Gamma factors. We shall first describe these shifts relative to the inducing data for \( \pi \), then use this expression to define the local conductor of \( \pi_v \).

As in \( \S 4.9 \), we shall realize \( \pi_v \) as \( \pi_{\delta,\nu} \), for some \( \delta = [\delta, M] \in D_v \) and \( \nu \in \mathfrak{h}_M^* \). Let \( \sigma \in \mathcal{E}(M) \) denote the essentially square-integrable representation \( \delta \otimes e^\nu \) of \( M \). Since \( M \) is cuspidal it is isomorphic to \( \text{GL}_{n_1}(F_v) \times \cdots \times \text{GL}_{n_m}(F_v) \), where \( n_1 + \cdots + n_m = n \), \( 1 \leq n_j \leq 2 \) for \( v \) real and \( n_j = 1 \) for \( v \) complex. We may then decompose \( \delta = \delta_1 \otimes \cdots \otimes \delta_m \) and \( \nu = \nu_1 + \cdots + \nu_m \), so that \( \sigma = \sigma_1 \otimes \cdots \otimes \sigma_m \), where \( \sigma_j = \delta_j e^{\nu_j} \in \mathcal{E}^2(\text{GL}_{n_j}(F_v)) \). Then

\[
L_v(s, \pi_{\delta,\nu}) = \prod_{j=1}^m L_v(s, \sigma_j) = \prod_{j=1}^m L_v(s + \nu_j, \delta_j).
\]

It therefore suffices to describe \( L_v(s, \delta) \) for \( \delta \in \mathcal{E}^2(\text{GL}_1(\mathbb{C})) \), \( \mathcal{E}^2(\text{GL}_1(\mathbb{R})) \), or \( \mathcal{E}^2(\text{GL}_2(\mathbb{R})) \).

We have \( \mathcal{E}_2(\text{GL}_1(\mathbb{C})) = \{ \chi_k : k \in \mathbb{Z} \} \), where \( \chi_k \) is the unitary character \( z \mapsto (z/|z|)^k \); in this case, \( L_v(s, \chi_k) = \Gamma_C(s + |k|/2) \). Moreover, \( \mathcal{E}_2(\text{GL}_1(\mathbb{R})) = \{ \text{sgn}^s : \epsilon = 0, 1 \} \) and \( L_v(s, \text{sgn}^s) = \Gamma_R(s + \epsilon) \). Finally, \( \mathcal{E}_2(\text{GL}_2(\mathbb{R})) = \{ D_k : k \geq 2 \} \), where \( D_k \) denotes the weight \( k \) discrete series representation, and we have \( L_v(s, D_k) = \Gamma_C(s + (k-1)/2) \).

Using the duplication formula for the Gamma factor, we may write

\[
L_v(s, \pi_v) = \prod_{v=\{u, \bar{u}\}} \prod_{j=1}^n \Gamma_v(s + \mu_{uj}),
\]

for certain complex numbers \( \mu_{uj} \), where the set \( \{ u, \bar{u} \} \) consists of the embeddings associated with the place \( v \). With this notation, Iwaniec and Sarnak [26] define the conductor \( q(\pi_v) \) of \( \pi_v \) as

\[
q(\pi_v) = c \prod_{v=\{u, \bar{u}\}} \prod_{j=1}^n (1 + |\mu_{uj}|).
\]

To explicate the set \( \{ \mu_{uj} \} \) in (5.3), it will be notationally convenient to index the parameters of \( \pi_v \) by the embeddings \( u \) associated with the place \( v \). For an embedding \( u \) associated with a complex place \( v \), and an integer \( j = 1, \ldots, n \), if we set \( \delta_{uj} = |k_{uj}|/2 \in \frac{1}{2} \mathbb{Z} \) and \( \mu_{uj} = \nu_{uj} \in \mathbb{C} \). Note that \( \mu_{uj} = \mu_{uj} \). For \( v \) real, we let \( a_v \) denote the number of \( \text{GL}_1 \) blocks and \( b_v \) the number of \( \text{GL}_2 \) blocks of \( M_v \), so that \( m_v = a_v + b_v \) and \( n = a_v + 2b_v \). We put \( \delta_{uj} = \epsilon_{uj} \in \{ 0, 1 \} \) for \( j = 1, \ldots, a_v \), and \( \delta_{uj} = \frac{k_{uj}-1}{2} \in \frac{1}{2} \mathbb{Z} \) for \( j = a_v + 1, \ldots, m_v \). Then in either case we have

\[
\{ \mu_{uj} \} = \{ \delta_{uj} + \nu_{uj} \}.
\]

5.3. Global (analytic) conductor. Let \( \pi = \otimes_v \pi_v \) be a unitary cuspidal automorphic representation of \( \text{GL}_n(\mathbb{A}_F) \). From the work of Jacquet-Platetski-Shapiro-Shalika [29] one knows that the local components of cusp forms on \( \text{GL}_n \) are generic. The analytic conductor of \( \pi \) is then defined as

\[
Q(\pi) = \prod_v q(\pi_v).
\]

As almost all local components of the global cusp form \( \pi \) are unramified, the product over all \( v \) makes sense as a finite product.

6. Asymptotics of global Plancherel volume

For a parameter \( Q \geq 1 \), consider the global Plancherel volume

\[
\int_{\pi \in \Pi(G(\mathbb{A}_F)^\dagger)} \frac{d\hat{\mu}_\mathbb{A}^\dagger(\pi)}{Q(\pi) \leq Q}
\]
of irreducible unitary representations of $G(\mathbb{A}_F)^1$ of analytic conductor less than $Q$. Note that the representations appearing in the support of $\hat{\mu}^{pl}_{K_1}(\mathfrak{f})$ are everywhere tempered, and therefore generic.

We shall evaluate this adelic volume asymptotically, as $Q \to \infty$. Recall the definition of the regularized Plancherel measure $\hat{\mu}^*(\mathfrak{f})$ in (1.3), and the notation for the degree $d = [F : \mathbb{Q}]$.

**Proposition 6.1.** Let $0 < \theta < 2/(d + 1)$ when $n \geq 2$ and $0 < \theta < \min\{1, 2/(d + 1)\}$ when $n = 1$. Then for all $Q \geq 1$ we have

$$\int_{\pi \in \Pi(G(\mathbb{A}_F)^1)} \frac{d\hat{\mu}^{pl}_{K_1}(\pi)}{Q(\pi)\leq Q} = \frac{1}{n + 1} \text{vol}(\hat{\mathfrak{f}}^*(\mathfrak{f})) Q^{n + 1} + O(\mathfrak{f}^{n + 1 - \theta}).$$

It is the above proposition which essentially gives the shape of the leading term constant, as well as the precise power growth, in the Weyl-Schanuel law.

### 6.1. Non-archimedean local integrals.

In this section we examine the local conductor zeta functions, defined at each place $v$ by the integral

$$Z_v(s) = \int_{\Pi(G_v)} q(\pi_v)^{-s} d\hat{\mu}^{pl}_{v}(\pi_v),$$

where the complex parameter $s$ has large enough real part to ensure absolute convergence.

For a finite place $v$ and an ideal $\mathfrak{q} = p_v^d$ let

$$\mathfrak{M}_v(\mathfrak{q}) = \int_{q(\pi_v) = \mathfrak{q}} d\hat{\mu}^{pl}_{v}(\pi_v)$$

be the Plancherel measure of those tempered $\pi_v$ with $q(\pi_v) = q$. For Re$(s)$ large enough, we have

$$Z_v(s) = \sum_{r \geq 0} \mathfrak{M}(p_v^r) q_v^{-rs}.$$

**Lemma 6.2.** We have

$$\mathfrak{M}_v(\mathfrak{q}) = \sum_{\mathfrak{d} \mid \mathfrak{q}} \lambda_{n+1}(\mathfrak{d}) N(\mathfrak{q}/\mathfrak{d})^n \quad \text{and} \quad Z_v(s) = \frac{\zeta_v(s-n)}{\zeta_v(s)n+1}.$$

In particular, $Z_v(n+1) = \zeta_v(1)/\zeta_v(n+1)^{n+1}$.

**Proof.** Applying Plancherel inversion to the idempotent $\varepsilon_{K_1,v}(\mathfrak{q})$ we obtain

$$\frac{1}{\mu_{G,v}(K_1,v)}(\mathfrak{q}) = \int_{\Pi(G_v)} \dim V_{\varepsilon_{K_1,v}(\mathfrak{q})} d\hat{\mu}^{pl}_{v}(\pi_v).$$

From (4.2), the left-hand side is $[K_{1,v}(\mathfrak{q}) : K_v]/\mu_{G,v}(K_v) = [K_{1,v}(\mathfrak{q}) : K_v] = \varphi_v(\mathfrak{q})$. Thus, from (5.2), we get

$$\varphi_v(\mathfrak{q}) = \int_{q(\pi_v) = \mathfrak{q}} d_n(\mathfrak{q}/(\mathfrak{q}^{q(\pi_v)})) d\hat{\mu}^{pl}_{v}(\pi_v) = \sum_{\mathfrak{d} \mid \mathfrak{q}} d_n(\mathfrak{q}/\mathfrak{d}) \mathfrak{M}_v(\mathfrak{d}) = (d_n * \mathfrak{M}_v)(\mathfrak{q}).$$

By Möbius inversion (and associativity of Dirichlet convolution) this gives $\mathfrak{M}_v(\mathfrak{q}) = (\lambda_n * \varphi_v)(\mathfrak{q}) = (\lambda_{n+1} * p_n)(\mathfrak{q}) = \sum_{\mathfrak{d} \mid \mathfrak{q}} \lambda_{n+1}(\mathfrak{d}) N(\mathfrak{q}/\mathfrak{d})^n$. From

$$\sum_{r \geq 0} \lambda_{n+1}(p_v^r) q_v^{-rs} = \frac{1}{\zeta_v(s)n+1}, \quad \sum_{r \geq 0} p_n(p_v^r) q_v^{-rs} = \zeta_v(s-n),$$

we obtain the value of $Z_v(s)$.

\qed
6.2. Archimedean local integral. In this section, we shall work with the group \( G^1 = G^1_\infty \)
viewed as a reductive group over \( \mathbb{R} \). Wherever possible, we shall drop the subscript \( \infty \) from the notation. So, for example, \( \hat{\mu}^p = \hat{\mu}^p_\infty \) and \( \pi = \pi_\infty \).

The work of Harish-Chandra [22, Theorem 27.3] allows us to explicitly describe the Plancherel measure \( \hat{\mu}^p \) on \( \Pi(G^1) \). Namely, for every \( \delta \in \mathcal{D} \) represented by \( (\delta, M) \), Harish-Chandra defines constants \( C_M > 0 \), depending only on the class of \( M \), and a function \( \mu^G_M(\delta, \nu) \), such that for every \( h \in L^1(\hat{\mu}^p) \) we have

\[
\int_{\Pi(G^1)} h(\pi)d\hat{\mu}^p(\pi) = \sum_{\delta \in \mathcal{D}} C_M \deg(\delta) \int_{\mathfrak{m}_M^*} h(\pi_{\delta, \nu}) \mu^G_M(\delta, \nu) d\nu,
\]

where \( \deg(\delta) \) is the formal degree of \( \delta \). A standard reference is [59, Theorem 13.4.1]. The density function \( \mu^G_M(\delta, \nu) \) is a normalizing factor for an intertwining map [loc. cit., Theorem 10.5.7].

Let \( \sigma = \delta e^\nu \in \mathcal{D}(M) \). We write \( M = \prod_{v \in S_\infty} M_v \) and \( \sigma = \prod_{v \in S_\infty} \sigma_v \), where \( \sigma_v = \delta_v e^{\nu_v} \). We may then factorize \( \sigma_v = \sigma_v^{\prime} \cdots \otimes \sigma_{v_m} \) according to the block decomposition of \( M_v \). We have a factorization of the form

\[
\mu^G_M(\delta, \nu) = \prod_{v \in S_\infty} \prod_{1 \leq i < j \leq m_v} GL_{n_v \times n_v}^\prime (\sigma_v^{\prime} \otimes \sigma_{v_j}).
\]

The latter factors can be described in terms of Rankin-Selberg Gamma factors \( \gamma_v(s, \sigma \times \sigma') = L_v(1 + s, \sigma \times \sigma')/L_v(s, \sigma \times \sigma') \). Indeed, it follows from [loc. cit., §10.5.8] that

\[
\mu_{GL_{n_v \times n_v}}^\prime (\sigma_v^{\prime} \otimes \sigma_{v_j}) = \gamma_v(0, \sigma_v^{\prime} \times \sigma_{v_j}) \gamma_v(0, \sigma_v \times \sigma_{v_j}).
\]

We would now like to define a majorizor of the (normalized) density function \( \deg(\delta) \mu^G_M(\delta, \nu) \). Recall the notation from §5.2.

**Definition 2.** For a complex embedding \( u \) and indices \( i, j \), we set \( b_{uij} = (1 + ||k_{ui} - k_{uj} + \nu_{ui} - \nu_{uj}|) \). For a place \( v \) we then put

\[
b^G_M(\delta, \nu_v) = \prod_{v \in \{u, \bar{u}\}, 1 \leq i < j \leq n} b_{uij}.
\]

Finally, write \( \beta^G_M(\delta, \nu_v) = \deg(\delta_v) b^G_M(\delta_v, \nu_v) \) and \( \beta^G_M(\delta, \nu) = \prod_v \beta^G_M(\delta_v, \nu_v) \).

**Remark 1.** From Stirling’s formula it follows that \( \Gamma_v(1 + s)/\Gamma_v(s) \ll (1 + |s|)^{d_v/2} \). From this and the definition of the local Rankin-Selberg \( L \)-factors, we deduce

\[
\mu^G_M(\delta, \nu) \ll \beta^G_M(\delta, \nu).
\]

Taking \( \deg(\delta) \) into account, the function \( \beta^G_M(\delta, \mu) \) is a majorizor of Plancherel measure.

**Remark 2.** Recalling the formulae (5.4), we see that when \( \text{Re}(\nu) = 0 \) (or bounded) the upper bound \( b_{uij} \ll \max_{\ell \in \{i, j\}} (1 + |\mu_{u\ell}|) \) holds. In fact, since \( \deg(D_k) = k - 1 \), it is easy to see that

\[
\deg(\delta_v) = \prod_{v \in \{u, \bar{u}\}, 1 < j} b_{uij} \ll \prod_{v \in \{u, \bar{u}\}, 1 < j} \max_{\ell \in \{i, j\}} (1 + |\mu_{u\ell}|).
\]

Ordering the set \( \{\mu_{u\ell}\} \) so that \( |\mu_{u1}| \ll \cdots \ll |\mu_{un}| \), it follows that

\[
\deg(\delta_v) = \prod_{v \in \{u, \bar{u}\}, 1 < j} \prod_{v \in \{u, \bar{u}\}, j=1} b_{uij} \ll \prod_{v \in \{u, \bar{u}\}, j=1} (1 + |\mu_{u\ell}|)^{n-j}.
\]

This bound will be useful in the proof of Lemma 6.3 below.
We define a measure $\beta(\pi) d\pi$ on $\Pi(G^1)$ by putting, for any $h \in L^1(\mu^{pl})$,

$$\int_{\Pi(G^1)} h(\pi) \beta(\pi) d\pi = \sum_{\delta \in [\delta, M] \in D} \int_{i\delta}^M h(\pi_\delta \nu) \beta^G_M(\delta, \nu) d\nu. \quad (6.2)$$

**Lemma 6.3.** For $Q \geq 1$ we have

$$\int_{Q \leq q(\pi) \leq 2Q} \beta(\pi) d\pi \ll Q^{n-1/d}. \quad \text{In particular, the archimedean conductor zeta function}$$

$$Z(s) = \int_{\Pi(G^1)} q(\pi)^{-s} d\mu^{pl}(\pi)$$

converges absolutely for $s \in \mathbb{C}$ with $\text{Re } s > n - 1/d$.

**Proof.** We begin by estimating the $\delta$ sum in (6.1) by an integral, as follows. Recalling the notation $a_u, b_u$ from §5.2 we put $A_M = \prod u_{\text{real}} (i\mathbb{R}^{a_u} \times \mathbb{C}^{2b_u}) \times \prod u_{\text{complex}} \mathbb{C}^{a_u}$. In view of (5.3) it is natural to introduce, for $\mu \in A_M$, the quantity $q(\mu) = \prod u \prod_{j=1}^n (1 + |u_{\mu_j}|)$, and similarly for $\beta^G_M(\mu)$. Let $\text{tr} : A_M \to \mathbb{R}$ denote the sum of the imaginary coordinates. Write $H_M$ for the set of $\mu \in A_M$ verifying $\text{tr}(\mu) = 0$, $u_{\mu_j} = u_{\mu_j}$ for $u$ complex, and $\mu_{v_j} = \mu_{u(b_u+j)}$, $j = a_u + 1, \ldots, m_u$, for $u$ real. Finally write $H_M(Q) = \{ z \in H_M : q(\mu) \sim Q \}$. Then we have an upper bound

$$\int_{q(\pi) \sim Q} \beta(\pi) d\pi \ll \max_M \int_{H_M(Q)} \beta^G_M(\mu) d\mu.$$

Fixing $M$, we now dyadically decompose the integral over $H_M(Q)$. Let $R$ denote the collection of all tuples $R = \{ R_{u_1} \}$ of dyadic integers, indexed by embeddings $u$ and $j = 1, \ldots, n$. For $R \in R$ we let $H_{M,R}$ denote the intersection of $H_M$ with $\{ \mu \in A_M : (1 + |u_{\mu_j}|) \sim R_{u_j} \}$. If $R(Q) = \{ R \in R : \prod_{u,j} R_{u_j} \sim Q \}$, then $H_M(Q)$ is contained in the union of the $H_{M,R}$, as $R$ runs over $R(Q)$. We deduce that

$$\int_{H_{M}(Q)} \beta^G_M(\mu) d\mu \ll \sum_{R \in R(Q)} \max_{H_{M,R}} \beta^G_M(\mu) \text{vol } H_{M,R}.$$

For each $u$ we order the $R_{u_j}$ so that $R_{u_1} \leq \cdots \leq R_{u_1}$. It follows from Remark 2 that

$$\max_{H_{M,R}} \beta^G_M(\mu) \ll \prod_{u,j} R_{u_j}^{n-j}.$$

To estimate the volume factor, let $u_0$ satisfy $R_{u_0} = \max_u R_{u_1}$ and write $R_0 = R_{u_0}$. From the trace-zero condition, we have $\text{vol}(H_{M,R}) \ll R_0^{-1} \prod_{u,j} R_{u_j}$. Together, the above estimates yield

$$\max_{H_{M,R}} \beta^G_M(\mu) \text{vol } H_{M,R} \ll R_0^{-1} \prod_{u,j} R_{u_j}^{n-j+1}.$$

Now, using $\prod_{u,j} R_{u_j} \sim Q$, the right-hand side is of size

$$Q^{n-1} \prod_{u \neq u_0} R_{u_1} \prod_{u_j} R_{u_0}^{2-j} \prod_{j \geq 2} R_{u_0j}^{2-j} = Q^{n-1} \prod_{u \geq j \geq 2} R_{u_j}^{2-j} \prod_{u \neq u_0} R_{u_1}.$$
It remains to execute the sum over $R \in \mathcal{R}(Q)$. We begin by noting that for $R \in \mathcal{R}(Q)$,

$$
\sum_{R_{u_1} \in 2^\mathcal{N}_u \neq u_0} \prod_{u_1} R_{u_1} = \sum_{M \in 2^\mathcal{M}_u} \sum_{\text{min}_u R_{u_1}/R_0 \sim 1/M} \prod_{u} R_{u_1} \prod_{u \neq u_0} (R_{u_1}/R_0)^{1/d} \\
\ll \left( \frac{Q}{\prod_{u \geq 2} \prod_{j \geq 2} R_{u_j}} \right)^{d-1} \sum_{M \in 2^\mathcal{M}_u} M^{-1/d} \log^{d-1} M \\
\ll \left( \frac{Q}{\prod_{u \geq 2} \prod_{j \geq 2} R_{u_j}} \right)^{d-1}.
$$

Inserting this into the remaining sum we get

$$
\sum_{R \in \mathcal{R}(Q)} \max_{\mathcal{H}_{M,R}} \beta_M^G(\mu) \vol \mathcal{H}_{M,R} \ll Q^{n-1/d} \sum_{R_{u_j} \in 2^\mathcal{M}_u \geq 2} \prod_{j \geq 2} R_{u_j}^{1-j+1/d}.
$$

Unless $d = 1$ and $j = 2$, the exponents in each factor are all strictly negative, in which case the geometric series are absolutely bounded. When $d = 1$ and $j = 2$, the factor $R_{u_2}$ appears with exponent 0, but given $R_{u_2}, \ldots, R_{u_3}$, there are only finitely many remaining dyadic $R_{u_2} \leq R_{u_1}$ satisfying the requirement that $\prod_j R_{u_j} \sim Q$ and the diagonal conditions defining $\mathcal{H}_{M,R}$. We finally obtain $O(Q^{n-1/d})$ in all cases. \hfill \Box

**Remark 3.** Despite the “spikes” introduced by the product condition $q(\pi_{\delta,\nu}) \leq X$, the asymptotics of the Plancherel measure of the sets $\{ \nu \in \mathfrak{h}_M^* : q(\pi_{\delta,\nu}) \leq X \}$ as $X \to \infty$ feature pure power growth without logarithmic factors. This is due to the following two facts: first, the Plancherel density increases into the spikes; and, second, these spikes are somewhat moderated by the trace-zero condition. To visualize this latter feature, we include the following graphics.

![Figure 1](image1.png) ![Figure 2](image2.png) ![Figure 3](image3.png)

In Figure 1, the hyperboloid $\{(1 + |x|)(1 + |y|)(1 + |z|) \leq X\}$ is drawn in $\mathbb{R}^3$. The spikes extend as far as $\asymp X$. The intersection with $x + y + z = 0$ is indicated in bold and reproduced in the plane in Figure 2. The spikes extend as far as $\asymp X^{1/2}$.

In Figure 3, the set $\{(1 + |x|)(1 + |y|) \leq X\}$ is drawn in $\mathbb{R}^2$ with spikes as far as $\asymp X$ and volume $\asymp X \log X$. The intersection with $x + y = 0$ is in bold. This produces a segment of length $\asymp X^{1/2}$.

### 6.3. Proof of Proposition 6.1.

Let $w_n = p_n \ast \lambda_{n+1}$. We deduce from Lemma 6.2 that

$$
\int_{\pi \in \Pi(G(A,F)^1)} \text{d} \hat{\mu}_\infty^p(\pi) = \sum_{Nq \leq Q} \prod_{q \mid q} \mathcal{N}(p_q^*) \int_{\pi_\infty \in \Pi(G_{\infty}^1)} \text{d} \hat{\mu}_\infty^p(\pi_\infty) = \sum_{Nq \leq Q} w_n(q) \int_{\pi_\infty \in \Pi(G_{\infty}^1)} \text{d} \hat{\mu}_\infty^p(\pi_\infty).
$$
Let $W_n(X) = \sum_{Nq \leq X} w_n(q)$. Exchanging the order of summation and integration,

\begin{equation}
(6.3) \quad \int_{\pi \in \Pi(G_{\mathcal{A}F})^{1}} \sum_{Q_{\pi} \leq Q} d\hat{\mu}_{k}^{pl}(\pi) = \int_{\Pi(G_{\mathcal{A}F})} W_n(Q/q(\pi_{\infty})) d\hat{\mu}_{\infty}^{pl}(\pi_{\infty}).
\end{equation}

The statement of the proposition will follow from an asymptotic evaluation of $W_n(X)$.

Recall the classical estimate $\sum_{Nq \leq X} = \zeta_{F}(1)X + O(X^{1-2/(d+1)})$ on the ideal-counting function [33, Satz 210, p. 131]. From this we deduce that given any $\sigma > -1$, $0 < \theta \leq 2/(d+1)$, and $X > 0$, we have

\begin{equation}
(6.4) \quad \sum_{Nq \leq X} Nq^{\sigma} = \zeta_{F}(1) \sigma + O_{\sigma, \theta}(X^{\sigma+1-\theta}),
\end{equation}

where for $X \geq 1$ we simply estimate $X^{\sigma+1-2/(d+1)} = O(X^{\sigma+1-\theta})$, and for $X < 1$ the estimate (6.4) holds vacuously. Using (6.4), we find that, for every $X > 0$,

\begin{equation}
W_n(X) = \frac{1}{n+1} \int_{\Pi(G_{\mathcal{A}F})^{1}} \sum_{Nq \leq X} \lambda_{n+1}(\epsilon)Nq^{\sigma + 1} + O(\frac{X}{Nq^{\sigma+1}}).
\end{equation}

From $|\lambda_{n+1}(n)| \leq d_{n+1}(n) \ll (Nn)^{\epsilon}$ and the identity $\sum_{Nq \leq X} Nq^{\sigma} = \zeta_{F}(s)^{-1}$ we obtain

\begin{equation}
(6.5) \quad W_n(X) = \int_{\Pi(G_{\mathcal{A}F})^{1}} \sum_{Nq \leq X} \lambda_{n+1}(\epsilon)Nq^{\sigma + 1} + O(\frac{X^{\sigma+1-\theta}}{Nq^{\sigma+1-\theta}}).
\end{equation}

Using (6.3) and (6.5) we see that

\begin{equation}
(6.6) \quad \int_{\Pi(G_{\mathcal{A}F})^{1}} \sum_{Nq \leq X} \lambda_{n+1}(\epsilon)Nq^{\sigma + 1} + O(\frac{X^{\sigma+1-\theta}}{Nq^{\sigma+1-\theta}}).
\end{equation}

In light of Lemma 6.3, both integrals converge. By Lemma 6.2, the main term is $\frac{1}{n+1} \log(\hat{\mu}_{k}^{\star}(\mathfrak{d})).$

**Remark 4.** Since $w_n = p_n \ast \lambda_{n+1} = p_n \ast \mu^{(n+1)} = (p_n \ast \mu) \ast \mu^{n} = \varphi_{n} \ast \lambda_{n}$, we may also write

\begin{equation}
(6.6) \quad \int_{\Pi(G_{\mathcal{A}F})^{1}} \sum_{Nq \leq X} \lambda_{n}(q)q^{\sigma + 1} + O(\frac{X^{\sigma+1-\theta}}{Nq^{\sigma+1-\theta}}).
\end{equation}

In view of the decomposition (3.4) and the volume identity (4.2), the right-hand side of the above expression will be what naturally arises from our methods.

**Part 2. Proof of Theorem 1.2**

**7. Preparations**

Our principal aim in Part 2 is to establish Theorem 1.2. For this we need to understand the behavior of $N(q, \delta, \Omega)$ from (3.3) in all parameters. The bulk of the work will be to approximate $N(q, \delta, P)$ for nice enough sets $P$ which lie in the tempered subspace $\mathfrak{h}_{\mathcal{A}F}^{\ast}$. To formulate this precisely, we will first need to define what class of subsets $P$ we consider and associate with them appropriate boundary volumes. Once these concepts are in place we state, at the end of this short section, the desired asymptotic expression for $N(q, \delta, \Omega)$ in Proposition 7.2.
7.1. **Nice sets and their boundaries.** For \( M \in \mathcal{L}_\infty \), we let \( \mathfrak{B}_M \) be the \( \sigma \)-algebra of all Borel-measurable subsets of \( \mathfrak{h}_M^* \). For every \( P \in \mathfrak{B}_M \) and \( \rho > 0 \), let
\[
     P^\rho(\rho) = \{ \mu \in \mathfrak{h}_M^* : B(\mu, \rho) \subseteq P \}, \quad P^\bullet(\rho) = \{ \mu \in \mathfrak{h}_M^* : B(\mu, \rho) \cap P = \emptyset \}, \quad \partial P(\rho) = P^\bullet(\rho) \setminus P^\rho(\rho),
\]
where \( B(\mu, \rho) \) denotes the open ball of radius \( \rho \) centered at \( \mu \). Then, for every point \( \mu \in \partial P(\rho) \), there are points \( \nu_1, \nu_2 \in B(\mu, \rho) \) with \( \nu_1 \in P \), \( \nu_2 \not\in P \), and hence by a continuity argument there is a point \( \nu \) on the boundary \( \partial P \) such that \( |\mu - \nu| < \rho \); in other words,
\[
     \partial P(\rho) \subseteq \bigcup_{\nu \in \partial P} B(\nu, \rho).
\]
We will only use (7.1) for compact regions \( P \) with a piecewise smooth boundary (although it is valid for every \( P \in \mathfrak{B}_M \)).

We record a few simple facts. For any bounded Borel set \( P \in \mathfrak{B}_M \) and \( \rho_2 > \rho_1 > 0 \), let
\[
     P^\bullet(\rho_1, \rho_2) = P^\bullet(\rho_2) \setminus P^\bullet(\rho_1), \quad P^\rho(\rho_1, \rho_2) = P^\rho(\rho_1) \setminus P^\rho(\rho_2).
\]

**Definition 3.** Let \( X, Y \subseteq \mathfrak{h}_M^* \) and \( r > 0 \). We say that \( X \) is \( r \)-**contained in** \( Y \) if, for every \( \mu \in X \), \( B(\mu, r) \subseteq Y \).

With these notions, we are ready for the following simple lemma.

**Lemma 7.1.** Let \( P \in \mathfrak{B}_M \) be a bounded Borel set. Then:
\begin{enumerate}
    \item For every \( \rho, r > 0 \), the set \( \partial P(\rho) \) is \( r \)-**contained in** \( \partial P(\rho + r) \).
    \item For every \( \rho_2 > \rho_1 > r > 0 \), the set \( P^\bullet(\rho_1, \rho_2) \) is \( r \)-**contained in** \( P^\bullet(\rho_1 - r, \rho_2 + r) \), and the set \( P^\rho(\rho_1, \rho_2) \) is \( r \)-**contained in** \( P^\rho(\rho_1 - r, \rho_2 + r) \).
\end{enumerate}

**Proof.** These statements follow essentially by the triangle inequality. For example, for the first claim of (2), we need to prove that, if \( \nu \in P^\bullet(\rho_1, \rho_2) \), then \( B(\nu, r) \subseteq \partial P(\rho_1 - r, \rho_2 + r) \). Indeed, there is a \( \nu_2 \in B(\nu, \rho_2) \cap P \) while \( B(\nu, \rho_1) \cap P = \emptyset \). Therefore, if \( \nu_1 \in B(\nu, r) \), then \( \nu_2 \in B(\nu_1, \rho_2 + r) \cap P \) and so \( \nu_1 \in P^\bullet(\rho_2 + r) \). On the other hand, we must have \( B(\nu_1, \rho_1 - r) \cap P = \emptyset \), for if \( \nu_3 \in B(\nu_1, \rho_1 - r) \cap P \), then \( \nu_3 \in B(\nu, \rho_1) \cap P \), a contradiction; and so \( \nu_1 \not\in P^\bullet(\rho_1 - r) \), as was to be shown. The other two claims are proved analogously.

Lastly, it will be convenient to consider the following family
\[
     \mathfrak{B}_M = \{ P \in \mathfrak{B}_M : P \text{ is bounded and } \forall \rho > 0, P^\rho(\rho), P^\bullet(\rho), \partial P(\rho) \in \mathfrak{B}_M \}.
\]
For example, every compact region with a piecewise smooth boundary clearly belongs to \( \mathfrak{B}_M \).

7.2. **Tempered count for fixed discrete data.** Let \( \delta \in \mathcal{D} \) be represented by \( (\delta, M) \), where \( M \in \mathcal{L}_{\text{st,}\infty} \). We define \( \mathfrak{B}(\delta) \) to be the family of all \( W(A_M) \)-**invariant** subsets of \( \mathfrak{B}_M \). This is independent of the choice of representative \( (\delta, M) \). We write an arbitrary element of \( \mathfrak{B}(\delta) \) as \( P \), and \( (\delta, P) \) will denote a representative for \( P \).

Fix \( \delta \in \mathcal{D} \) and let \( P \in \mathfrak{B}(\delta) \) be represented by \( (\delta, P) \). Let \( R > 0 \), and \( N \in \mathbb{N} \). We define
\[
     \partial \text{vol}_R(\delta, P) = \sum_{\ell=1}^{\infty} \ell^{-N} \int_{\partial P(\ell/R)} \beta_{\mathfrak{h}_M}^\rho(\delta, \nu) \, d\nu.
\]
This is essentially the Plancherel volume of the \( 1/R \)-thickened boundary of \( P \). Next, for a fixed \( L \in \mathcal{L}_\infty(M) \), if \( c_L \) is the codimension of \( \mathfrak{h}_L \) inside \( \mathfrak{h}_M \), we put
\[
     \overline{\text{vol}}_{R,L}(\delta, P) = R^{-c_L} \int_{\mathfrak{h}_L^*} (1 + d(\nu, P) : R)^{-N} \beta_{\mathfrak{h}_M}^\rho(\delta, \nu) \, d\nu.
\]
When \( L = M \) we drop the dependence on \( L \) from the notation and simply write \( \overline{\text{vol}}_R(\delta, P) = \overline{\text{vol}}_{R,M}(\delta, P) \). For example, we follow this convention in Property (ELM) (recall Definition 1). Note that \( \overline{\text{vol}}_{R_1}(\delta, P) \asymp \overline{\text{vol}}_{R_2}(\delta, P) \) if \( R_1 \asymp R_2 \).
Finally we shall write
\begin{equation}
\operatorname{vol}^*_R(\delta, P) = \sum_{L \in L(M), \ L \neq M} \overline{\operatorname{vol}}_{R,L}(\delta, P).
\end{equation}

**Remark 5.** It will be plain from our arguments that a sufficiently large \( N \in \mathbb{N} \) (in terms of \( n \) and \( F \)) can be chosen once and for all to ensure convergence of sums we later encounter, and we normally suppress the dependence on \( N \) in the notation; if we want to emphasize this dependence (for example, in §10.2), we shall write \( \partial \operatorname{vol}_{R,N}(\delta, P) \) or (when \( L = M \)) \( \overline{\operatorname{vol}}_{R,N}(\delta, P) \).

The central ingredient to the proof of Theorem 1.2 is then the following result.

**Proposition 7.2.** Assume that Property (ELM) holds with respect to \( \delta \in \mathcal{D} \). There are constants \( c, C, \theta > 0 \) such that for \( P \in \mathcal{B}(\delta) \), integral ideals \( q \) with \( Nq \geq C \), and \( 0 < R \leq c \log(2 + Nq) \),
\begin{equation}
N(q, \delta, P) = \operatorname{vol}(\mu_{GL_n})(\varphi_n(q) \int_P \mu_{R,\infty}^\delta d\mu_{\infty}^\delta) + O\left(\varphi_n(q)(\partial \operatorname{vol}_{R}(\delta, P) + \operatorname{vol}^*_R(\delta, P)) + Nq^{-\theta} \overline{\operatorname{vol}}_{R}(\delta, P)\right).
\end{equation}
If \( Nq \leq C \), then (7.6) holds with the first term replaced by \( O(\varphi_n(q) \int_P \mu_{R,\infty}^\delta d\mu_{\infty}^\delta)\).

Proposition 7.2 will be proved in Section 10, after having introduced an appropriate class of test functions in Section 8 and estimated the (exponentially weighted) discrete spectrum in Section 9. Then, in Sections 11, we make the deduction from Proposition 7.2 to Theorem 1.2.

We remark that we obtain a main term in Proposition 7.2 for every \( q \); in the case \( Nq \leq C \), its shape is mildly affected by the roots of unity in \( F \) (see (8.6)).

8. Spectral localizing functions

In this section, given a \( \delta \in \mathcal{D} \) represented by \( (\delta, M) \), a spectral parameters \( \mu \in i\mathfrak{h}_M^* \), and a real number \( R > 0 \), we define a function \( f^\delta_{\mu} \in \mathcal{H}(G^1_\infty)_R \) such that
\begin{equation}
h^\delta_{\mu} : (\tau, \lambda) \mapsto \operatorname{tr} \pi_{\tau,\lambda}(f^\delta_{\mu})
\end{equation}
localizes about \( \delta \in \mathcal{D} \) and (the \( W(A_M)_\delta \)-orbit of) \( \mu \) in \( i\mathfrak{h}_M^* \).

8.1. Construction of test functions. Let \( \delta \in \mathcal{D} \) be represented by \( (\delta, M) \). Fix \( \mu \in i\mathfrak{h}_M^* \) and a real parameter \( R > 0 \). Our first aim is to construct a \( W(A_M)_\delta \)-invariant function \( h^\delta_{\mu} \) on \( \mathfrak{h}_M^* \) which concentrates around \( \mu \) and lies in the generalized Paley-Wiener space \( \mathcal{PW}_{R,\delta} \). We will sometimes refer to such a function as a “spectral localizer”.

We begin by building a function \( h^\mu \in \mathcal{PW}(\mathfrak{h}_M^*) \) which localizes around \( \mu \). We will later add in the \( (W(A_M)_\delta \)-invariance. Recall the decomposition of \( \mathfrak{h}_M^* \) from §4.3, which we use to write \( \mu = \mu_Z + \mu^0 \). Here, \( \mu_Z \in i(a_G \cap \mathfrak{h}_M)^* = i\mathfrak{a}_G^* \cap i\mathfrak{h}_M^* \) and \( \mu^0 = i(a_M^0)^* \). We will construct functions
\begin{equation}
h^\mu_Z \in \mathcal{PW}(\mathfrak{a}_G^* \cap \mathfrak{h}_M^*)_R \quad \text{and} \quad h^\mu_{\mu^0} \in \mathcal{PW}((a_M^0)^*)_R
\end{equation}
and then define
\begin{equation}
h^\mu_R(\lambda) = h^\mu_Z(\lambda_Z) h^\mu_{\mu^0}(\lambda^0), \quad \lambda = \lambda_Z + \lambda^0.
\end{equation}

- **Abelian localizer:** We let \( \widehat{\cdot} : C^\infty_c(a_G \cap \mathfrak{h}_M) \to \mathcal{PW}(a_G^* \cap \mathfrak{h}_M^*) \) be the Fourier transform. Let \( g_0 \in C^\infty_c(a_G \cap \mathfrak{h}_M) \) be supported in the ball of radius 1 and satisfy \( g_0(0) = 1 \). For a real parameter \( R > 0 \) we write \( g_R(X) = g_0(R^{-1}X) \) and \( h_R = g_R \); in particular, \( h_R \in \mathcal{PW}(a_G^* \cap \mathfrak{h}_M^*)_R \). We let \( g^R_{\mu_Z}(X) = g_R(X)e^{-\mu_Z X} \) and \( h^R_{\mu_Z} = g^R_{\mu_Z} \). Then we have \( h^R_{\mu_Z}(\lambda_Z) = h_R(\lambda_Z - \mu_Z) \).
Let \( g_R(\lambda) = g_1(R^{-1}\lambda) \) and put \( g_R(0) = g(R) e^{-\langle \mu, X \rangle} \). Then if \( h_R = \widehat{g_R} \) and \( h_R = \widehat{g_R} \) we have \( h^\circ_R(\lambda^0) = h_R(\lambda^0 - \mu_0) \). Note that \( g_R^\circ \in C^\infty(a^0_M) \) and \( h^\circ_R \in \mathcal{P}W((a^0_M)^*) \).

Finally, for \( \lambda \in h^*_M,C \) we put

\[
\begin{align*}
  h^\delta_R(\tau, \lambda) &= \begin{cases} 
    \frac{1}{|W(A_M)|}\sum_{w \in W(A_M)} h^\mu_R(w \lambda), & \tau = \delta; \\
    0, & \text{else}.
  \end{cases}
\end{align*}
\]

Clearly \( h^\delta_R \in \mathcal{P}W_R. \delta \). We deduce from §4.12 that there is \( f^\delta_R \in C^\infty(G^1R) \) such that

\[
h^\delta_R(\tau, \lambda) = \text{tr} \pi_{\tau, \lambda}(f^\delta_R).
\]

Finally, for a set \( P \in \mathcal{B}(\delta) \) (recall the notation from §7.2) and \( R > 0 \), we put

\[
\begin{align*}
  h^\delta_R(P, \tau, \nu) &= \int_P h^\delta_R(\tau, \nu) d\mu,
  f^\delta_R(P, g) &= \int_P f^\delta_R(g) d\mu.
\end{align*}
\]

Note that if Property (ELM) holds for \( \delta \) (see Definition 1), then it holds using \( h^\delta_R \), for any \( P \in \mathcal{B}(\delta) \), with error term

\[
J_{\text{error}}(K_{1}(q)) \lesssim e^{R \mathcal{N} q^{\nu} - \theta \text{vol}(\delta, P)}.
\]

8.2. Some estimates. Recall the Plancherel majorizer \( \beta^\delta_M \) of Definition 2. It will be useful to have the following estimate, which is a variation of [13, Proposition 6.9].

**Lemma 8.1.** For every \( R > 0 \) we have \( \| h^\delta_R \|_{L^1(\mu^0_M)} \lesssim R^{-\text{dim} h_M \beta^\delta_M(\delta, \mu)} \).

**Proof.** From the definition, we have that

\[
\| h^\delta_R \|_{L^1(\mu^0_M)} \lesssim \text{deg}(\delta) \max_{w \in W(A_M)} \int_{h^*_M} h^\mu_R(w \lambda) \mu^\delta_M(\delta, \lambda) d\lambda.
\]

From Definition 2 and the majorization in Lemma 1, it is clear that, for every \( \lambda, \nu \in h^*_M \),

\[
\text{deg}(\delta) \mu^\delta_M(\delta, \lambda) \lesssim \beta^\delta_M(\delta, \lambda) = \beta^\delta_M(\delta, (\lambda - \nu) + \nu) \lesssim (1 + \| \lambda - \nu \|)^d_M \beta^\delta_M(\delta, \nu)
\]

with \( d_M = \sum_{i=1}^{\infty} \sum_{1 \leq i < j \leq r_0} d_i n_i n_{ij} \). Combining this estimate with the rapid decay of \( h^\mu_R \), the integral in the upper bound above is

\[
\ll \max_{w \in W(A_M)} \int_{h^*_M} (1 + R\| \lambda - w \mu \|)^{-N} (1 + \| \lambda - w \mu \|)^d_M \beta^\delta_M(\delta, \lambda) d\lambda
\]

\[
\ll R^{-\text{dim} h_M \beta^\delta_M(\delta, \mu)} \int_{h^*_M} (1 + \| \lambda \|)^{-N + d_M} d\lambda \ll R^{-\text{dim} h_M \beta^\delta_M(\delta, \mu)},
\]

for every \( N > \dim h_M + \max_M d_M + 1 \).

\[\square\]

We now quantify the extent to which \( h^\delta_R \) approximates the characteristic function of \( P \). We shall use the notation \( P^\circ(\rho) \) and \( P^\ast(\rho) \) from §7.1.

**Lemma 8.2.** Let notations be as above. If \( \tau \neq \delta \) then \( h^\delta_R(\tau, \nu) = 0 \); otherwise

1. we have \( h^\delta_R(\delta, \nu) \ll e^{R\| \text{Re} \nu \|} \) for all \( \nu \in h^*_M \).


(2) we have
\[
h^\delta_P(\delta, \nu) \ll_N e^{R||Re\nu||}(R\rho)^{-N},
\]
for every \(\nu \in h^*_M\), such that \(W(A_M)_{\delta,\nu} \cap P^\bullet(\rho) \neq \emptyset\).

(3) for every \(\nu \in h^*_M\) we have \(0 \leq h^\delta_P(\delta, \nu) \leq 1\) and, for every \(\rho > 0, N \in \mathbb{N}\),
\[
h^\delta_P(\delta, \nu) = \begin{cases} 1 + O_N((R\rho)^{-N}), & W(A_M)_{\delta,\nu} \cap P^\bullet(\rho) \neq \emptyset; \\ O_N((R\rho)^{-N}), & W(A_M)_{\delta,\nu} \cap P^\bullet(\rho) = \emptyset. \end{cases}
\]

Proof. The estimates in (1) and (2) follow from Fourier inversion and an application of the trivial bound (in the first case) and a standard application of integration by parts (in the second case).

Next, let \(\nu \in h^*_M\). The inequality \(0 \leq h^\delta_P(\delta, \nu) \leq 1\) follows immediately from
\[
h^\delta_P(\delta, \nu) = \frac{1}{|W(A_M)_{\delta}|} \sum_{w \in W(A_M)_{\delta}} \int_{R(ww'-P)} h_0(\mu) d\mu,
\]
the non-negativity of \(h_0\), and the normalization \(g_0(0) = 1\).

If \(\nu \in P^\bullet(\rho)\), then
\[
h^\delta_P(\delta, \nu) = \frac{1}{|W(A_M)_{\delta}|} \sum_{w \in W(A_M)_{\delta}} \left( \int_{h^*_M} h_0(\mu) d\mu + O\left( \int_{B(0,R\rho)^c} h_0(\mu) d\mu \right) \right) = 1 + O\left( \int_{R^\rho} (1 + t)^{-N-r} t^{-1} dt \right) = 1 + O((R\rho)^{-N}),
\]
where \(B(0, R\rho)\) denotes the ball of radius \(R\rho\) in \(h^*_M\). If \(\nu \notin P^\bullet(\rho)\), then, analogously,
\[
h^\delta_P(\delta, \nu) = O\left( \int_{B(0,R\rho)^c} h_0(\mu) d\mu \right) = O\left( \int_{R^\rho} (1 + t)^{-N-r} t^{-1} dt \right) = O((R\rho)^{-N}).
\]

This establishes the estimates in (3). \(\square\)

Lemma 8.3. For \(P \in \mathcal{B}_M\) and \(R > 0\) we have
\[
\int_{ih^*_M} h^\delta_P(\delta, \nu) \mu^G_M(\delta, \nu) d\nu = \int_{P} \mu^G_M(\delta, \nu) d\nu + O(\partial \text{vol}_R(\delta, P)).
\]

Proof. We begin by decomposing the integral according to
\[(8.4) \quad ih^*_M = \partial P(1/R) \cup (P^c \setminus \partial P(1/R)) \cup (P \setminus \partial P(1/R)).\]

The integral over \(ih^*_M\) in the lemma may be rewritten as
\[
\int_{P} \mu^G_M(\delta, \nu) d\nu + O\left( \int_{\partial P(1/R)} \beta^G_M(\delta, \nu) d\nu \right) + \sum_{\ell=1}^{\infty} \int_{P^\bullet(\ell/R,(\ell+1)/R)} h^\delta_P(\delta, \nu) \mu^G_M(\delta, \nu) d\nu
\]
\[
+ \sum_{\ell=1}^{\infty} \int_{P^\bullet(\ell/R,(\ell+1)/R)} (1 - h^\delta_P(\delta, \nu)) \mu^G_M(\delta, \nu) d\nu.
\]

Using Lemma 8.2, the last three terms above are majorized by
\[
\int_{\partial P(1/R)} \beta^G_M(\delta, \nu) d\nu + \sum_{\ell=1}^{\infty} \ell^{-N} \left( \int_{P^\bullet(\ell+1/R \setminus P(1/R))} \beta^G_M(\delta, \nu) d\nu + \int_{P(1/R) \setminus P^\bullet(\ell+1/R)} \beta^G_M(\delta, \nu) d\nu \right).
\]
The last error term is indeed \(O(\partial \text{vol}_R(\delta, P))\), as desired. \(\square\)
8.3. Central contributions. We now evaluate and bound the central contributions $J_{\text{cent}}$ to the trace formula, using the above test functions.

We begin more generally, taking an arbitrary $f \in C_c^\infty(G_\infty^1)_R$, where $R > 0$. Then, by definition,

\begin{equation}
J_{\text{cent}}(\varepsilon_{K_1(q)} \otimes f) = \sum_{\gamma \in Z(F)} J(\gamma, \varepsilon_{K_1(q)} \otimes f) = \text{vol}(\mu_{\text{GL}_n}) \varphi_n(q) \sum_{\gamma \in Z(F) \cap K_1(q)} \sum_{\gamma_\infty \in G_\infty^1} f(\gamma).
\end{equation}

Note that, by compactness, the sum over $\gamma$ in (8.5) is always finite. In fact, the next lemma shows that, in the range $0 < R \ll \log(2 + Nq)$ of interest to us, it typically contains only the identity element.

Lemma 8.4. There exist constants $c_2, C_2 > 0$ such that

1. if $0 < R \leq c_2 \log(2 + Nq)$, the sum over $\gamma$ in (8.5) consists only of $\gamma = 1$ and possibly a subset of non-identity roots of unity in $\mathcal{D}_F^\times$.
2. if, additionally $Nq > C_2$, then the sum over $\gamma$ in (8.5) reduces to the identity $\gamma = 1$.

Proof. Note that $\gamma \in Z(F) \cap K_1(q)$ are given by diagonal elements corresponding to a unit $u$ in $\mathcal{O}_F^\times$ congruent to 1 mod $q$. If $c_2$ is taken small enough, the image under the logarithm map of such $u$ has trivial intersection with $B(0, R)$. Thus the only $\gamma \in Z(F) \cap K_1(q)$ contributing to (8.5) correspond to roots of unity congruent to 1 mod $q$. This is a finite set which for $q$ large enough is just 1. \hfill $\square$

As in (9.7), let $h(\tau, \lambda)$ denote the function $\text{tr} \pi_{\tau, \lambda}(f)$. Now, if $\omega_{\delta, \nu}$ denotes the central character of $\pi_{\delta, \nu}$, Plancherel inversion gives

\begin{equation}
f(\gamma) = \int_{\text{I}_M} h(\delta, \nu) \omega_{\delta, \nu}(\gamma) \mu_M^G(\delta, \nu) \, d\nu.
\end{equation}

Thus

\begin{equation}
J_{\text{cent}}(\varepsilon_{K_1(q)} \otimes f) = \text{vol}(\mu_{\text{GL}_n}) \varphi_n(q) \sum_{\gamma \in Z(F) \cap K_1(q)} \int_{\text{I}_M^\times} h(\delta, \nu) \omega_{\delta, \nu}(\gamma) \mu_M^G(\delta, \nu) \, d\nu.
\end{equation}

Lemma 8.5. There exist constants $c_2, C_2 > 0$ such that if $0 < R \leq c_2 \log(2 + Nq)$ then

\begin{equation}
J_{\text{cent}}(\varepsilon_{K_1(q)} \otimes f_R^{\delta, P}) \ll \varphi_n(q) \overline{\text{vol}_R(P, \delta)},
\end{equation}

\begin{equation}
J_{\text{cent}}(\varepsilon_{K_1(q)} \otimes f_R^{\delta, \mu}) \ll R^{-\dim \text{I}_M} \varphi_n(q) \beta_M^G(\delta, \mu),
\end{equation}

and, if $Nq > C_2$,

\begin{equation}
J_{\text{cent}}(\varepsilon_{K_1(q)} \otimes f_R^{\delta, P}) = \text{vol}(\mu_{\text{GL}_n}) \varphi_n(q) \int_P \mu_M^G(\delta, \nu) \, d\nu + O(\varphi_n(q) \partial \text{vol}_R(\delta, P)).
\end{equation}

Proof. To prove (8.9) we apply the second part of Lemma 8.4 to reduce to the identity contribution, and then use Lemma 8.3.

To obtain the bound (8.7) and (8.8), we first bound (8.6) for any $f \in C_c^\infty(G_\infty^1)_R$ by

\[ |J_{\text{cent}}(\varepsilon_{K_1(q)} \otimes f)| \ll \varphi_n(q) \cdot |\{ \gamma \in Z(F) \cap K_1(q) : \gamma_\infty \in G_\infty^1 \}| \cdot \|h\|_{\mu_M^G}. \]

The first part of Lemma 8.4 shows that the number of contributing $\gamma$ is $O(1)$, the implied constant depending on $F$ (as it does above as well). To prove (8.7) we take $f = f_R^{\delta, P}$ and apply Lemma 8.2. To prove (8.8) we take $f = f_R^{\delta, \mu}$ and apply Lemma 8.1. \hfill $\square$
9. Bounding the discrete spectrum

The goal of this section is to provide bounds on two (similarly defined) exponentially weighted sums over the discrete spectrum. Throughout we shall assume Property (ELM), introduced in Definition 1.

Let \( \delta \in \mathcal{D} \) be represented by \((\delta, M)\), where \( M \in \mathcal{L}_\infty \). Let \( \mu \in i \mathfrak{h}_M^* \). Let \( q \) be an integral ideal. For a real parameter \( R > 0 \) let

\[
D_R(q, \delta, \mu) = \sum_{\pi \in \Pi_{\text{disc}}(G(\mathbb{A}_F)^1)_{\delta}} \dim V_{\pi_f}^{K_1(q)} e^{R \|\Re \xi_\pi\|}.
\]

The following result will be used in §15, in the proof of Corollary 15.3.

**Proposition 9.1.** Let notations be as above. Assume that Property (ELM) holds with respect to \( \delta \). Then there is \( c > 0 \) such that for \( 1 \leq R \leq c \log(2 + Nq) \) we have

\[
D_R(q, \delta, \mu) \ll R^{-\dim M} \varphi_n(q) \beta^G_M(\delta, \mu).
\]

Let \( \delta \in \mathcal{D} \) and \( P \in \mathcal{B}(\delta) \). We shall also need the following sum

\[
K_R(q, \delta, P) = \sum_{\pi \in \Pi_{\text{disc}}(G(\mathbb{A}_F)^1)_{\delta}} \dim V_{\pi_f}^{K_1(q)} e^{R \|\Re \xi_\pi\|} (1 + R \cdot d(\Im \xi_{\pi}, P))^{-N}.
\]

Note that, when compared with \( D_R(q, \delta, \mu) \), the above sum is now over \( \pi \) with \( \pi_\infty \) non-tempered, and the membership of \( \Im \xi_{\pi} \) in \( B_M(\mu, 1/R) \) is replaced by power decay outside of \( P \). As in Remark 5, we have suppressed the dependence on \( N \) in this sum, but will occasionally revive it for clarity, writing \( K_{R,N}(q, \delta, P) \).

The following result will be used in §10, in the proof of Proposition 7.2.

**Proposition 9.2.** Let notations be as above. Assume that Property (ELM) holds with respect to \( \delta \). Then there is \( c > 0 \) such that for \( 1 \leq R \leq c \log(2 + Nq) \) we have

\[
K_R(q, \delta, P) \ll \varphi_n(q) \text{vol}_{\mathfrak{R}}(\delta, P).
\]

As will become clear shortly, the bulk of the work necessary to prove Propositions 9.1 and 9.2 is the construction of appropriate test functions. This turns out to be a highly non-trivial analytic problem.

9.1. Reduction to test functions. Let \( \delta \in \mathcal{D} \) be represented by \((\delta, M)\), with \( M \in \mathcal{L}_\infty \). Let \( \mu \in i \mathfrak{h}_M^* \) and \( R > 0 \). Recall the \( \delta \)-Hermitian dual from §4.10 and of the Paley-Wiener theorem of Clozel-Delorme from §4.12.

We now grant ourselves momentarily the functions \( h_{R}^{\delta, \mu} \in \mathcal{P}W(\mathfrak{h}_M^*, \mathbb{C})_{R, \delta} \) in Lemma 9.3 below and use them to prove Propositions 9.1 and 9.2. We denote the corresponding test function by \( f_{R}^{\delta, \mu} \in C_c^\infty(G_\infty^1)_{R} \).

**Proof of Proposition 9.1.** Properties (2) and (3) of Lemma 9.3 show that there are constants \( C, c > 0 \) such that

\[
CD_{\mathfrak{c}R}(q, \delta, \mu) \leq J_{\text{disc}}(\varepsilon_{K_1(q)} \otimes f_{R}^{\delta, \mu}).
\]

Proposition 9.1 then follows from Property (ELM) and estimate (8.8) of Lemma 8.5. \( \square \)

**Proof of Proposition 9.2.** Let us consider

\[
\sum_{L \geq 2M} R^{\dim \mathfrak{h}_L} \int_{b_L^\infty} J_{\text{comp}}(\varepsilon_{K_1(q)} \otimes f_{R}^{\delta, \mu})(1 + d(\mu, P) \cdot R)^{-N} d\mu.
\]

On one hand, from Lemma 9.3, the sum-integral in (9.4) is bounded below by
\[
C \sum_{L \geq M} R^{\dim h_L} \int_{b_L^*} (1 + d(\mu, P) \cdot R)^{-N} \sum_{\pi \in \Pi_{disc}(G(\lambda_P)^1)^L} \dim \pi^{K_1(\xi)} e^{cR\|Re \xi\|} d\mu
\]
\[
= C \sum_{L \geq M} R^{\dim h_L} \sum_{\pi \in \Pi_{disc}(G(\lambda_P)^1)^L} \dim \pi^{K_1(\xi)} e^{cR\|Re \xi\|} \int_{\|\mu - \Im \xi\| \leq R^{-1}} (1 + d(\mu, P) \cdot R)^{-N} d\mu
\]
\[
\cong \sum_{L \geq M} \sum_{\pi \in \Pi_{disc}(G(\lambda_P)^1)^L} \dim \pi^{K_1(\xi)} e^{cR\|Re \xi\|} (1 + d(\Im \xi, P) \cdot R)^{-N},
\]
which is simply $K_{\epsilon R}(q, \delta, P)$. On the other hand, Property (2) of Lemma 9.3 ensures that
\[
J_{\text{comp}}(\varepsilon K_1(q) \otimes f_{\epsilon R}^{\delta, \mu}) \leq J_{\text{comp}}(\varepsilon K_1(q) \otimes f_{\epsilon R}^{\delta, \mu}) + J_{\text{temp}}(\varepsilon K_1(q) \otimes f_{\epsilon R}^{\delta, \mu}) = J_{\text{disc}}(\varepsilon K_1(q) \otimes f_{\epsilon R}^{\delta, \mu}).
\]
We use Proposition 9.1 and the estimate (8.8) of Lemma 8.5 to bound (9.4) by
\[
\varphi_n(q) \sum_{L \geq M} R^{-cL} \int_{b_L^*} \beta_L^G(\delta, \mu)(1 + d(\mu, P) \cdot R)^{-N} d\mu = \varphi_n(q) \text{vol}^*_R(\delta, P).
\]
Recall that $c_L$ is the codimension of $\mathfrak{h}_L$ inside $\mathfrak{h}_M$, and $\text{vol}^*_R(\delta, P)$ was defined in (7.4) and (7.5). Putting the above estimates together completes the proof of Proposition 9.2. \hfill \Box

9.2. Existence of test functions.

**Lemma 9.3.** Let $\delta \in \mathcal{D}$ be represented by $(\delta, M)$ with $M \in \mathcal{L}_\infty$. Let $\mu \in \text{ib}_{h_M^*}$ and $R \geq 1$. There is $h_{R}^{\delta, \mu} \in \mathcal{P}W(\mathfrak{h}_{M, \mathbb{C}}^*)$ verifying the following properties

1. $h_{R}^{\delta, \mu}$ is $W(A_M)^{\delta}$-invariant;
2. $h_{R}^{\delta, \mu} \geq 0$ on $\mathfrak{h}_{h}^*$;
3. there are constants $c, C > 0$ such that for all $\xi \in \mathfrak{h}_{h}^*$ satisfying
\[
(\text{9.5}) \quad \|\Im \xi - \mu\| \leq R^{-1}
\]
we have $h_{R}^{\delta, \mu}(\xi) \geq C e^{cR\|Re \xi\|}$.

**Proof.** If there is a constant $A > 0$ such that $\|Re \xi\| \leq AR^{-1}$ then $Ce^{cR\|Re \xi\|}$ is bounded above by an absolute constant. In that case, condition (3) asks only that $h_{R}^{\delta, \mu}(\xi)$ be bounded away from zero for $\delta$-Hermitian $\xi$ satisfying (9.5). This is proved in [6, Lemma 7.5]. We can therefore assume that $\xi \in \mathfrak{h}_{\delta, h_{\mu}}^*$ satisfies

\[
(\text{9.6}) \quad \|\Im \xi - \mu\| \leq R^{-1} \quad \text{and} \quad \|\Re \xi\| \geq AR^{-1},
\]
for some fixed $A > 0$. Our approach to treat this complementary range is inspired by that of [6, Lemma 7.5], although the argument is necessarily much more elaborate. We should note that similar arguments are present in the foundational work of [13, Proposition 7.1] and were later developed in [34, Proposition 4.5].

Note that if $\mu$ is such that no $\delta$-Hermitian $\xi$ satisfies (9.6), then condition (3) is vacuous and the function $h_{R}^{\delta, \mu}$ identically equal to zero satisfies the remaining conditions. Otherwise, $\mu$ should be of distance at most $R^{-1}$ from the $\delta$-singular subset $\text{ib}_{h}^*$ of (4.8). Let $M_\mu \in \mathcal{L}_\infty(\delta)$ be maximal for the property that $\|\mu_{M_\mu}\| \leq R^{-1}$; note that $M_\mu$ is distinct from $M$. Now if the lemma is true for $\mu_{M_\mu}$, then it is true for $\mu$ (by taking for $h_{R}^{\delta, \mu}$ the function $h_{R}^{\delta, \mu_{M_\mu}}$). We may therefore assume that $\mu \in \text{ib}_{h_{M_\mu}}^*$. 


With $h_0$ as in Lemma 9.5 below we put
\[(9.7)\quad h_{R}^{\delta,\mu}(\xi) = \sum_{M' \geq M_{\mu}} \left( \sum_{w \in W(A_{M_{\mu}})} h_0(R(w\xi - \mu_{M'})) \right)^2.\]

Then $h_{R}^{\delta,\mu} \in \mathcal{P}(h_{M_{\mu}}^*)_{R}$ is $W(A_{M_{\mu}})_{\delta}$-invariant by construction. Moreover, since $h_0(-\xi) = h_0(\xi)$, $h_0(\xi) = h_0(\xi)$, and $W(A_{M_{\mu}})_{\delta} - \xi = W(A_{M_{\mu}})_{\delta} - \xi_{\text{cong}}$ for all $\xi \in \mathfrak{h}_{\delta,\text{ham}}^*$, it follows that the inner sum in (9.7) is real-valued on the $\delta$-Hermitian spectrum, whence $h_{R}^{\delta,\mu} \geq 0$ on $\mathfrak{h}_{\delta,\text{ham}}^*$. This establishes (1) and (2).

For the proof of third property, we shall show that for all $\delta$-Hermitian $\xi$ verifying (9.6) there is $M' \supseteq M_{\mu}$ (depending on $\xi$) such that
\[(9.8)\quad \left( \sum_{w \in W(A_{M_{\mu}})} h_0(R(w\xi - \mu_{M'})) \right)^2 \geq C e^{cR\|\text{Re} \xi\|}.\]

Dropping the other terms by positivity yields the lemma.

To prove (9.8) we let $\xi \in \mathfrak{h}_{\delta,\text{ham}}^*$ satisfy (9.6) and apply Lemma 9.4 below; this gives rise to an $M'$ satisfying the indicated properties. Recall from (4.4) the definition of $W(A_{M'})_{\delta}$, and write $W_{\text{bad}}$ for the complementary set $W(A_{M})_{\delta} \setminus W(A_{M'})_{\delta}$. Then
\[(9.9)\quad \sum_{w \in W(A_{M})_{\delta}} h_0(R(w\xi - \mu_{M'})) \geq \sum_{w \in W(A_{M'})_{\delta}} h_0(R(w\xi - \mu_{M'})) - |W_{\text{bad}}| \max_{\xi \in W_{\text{bad}}} |h_0(R(w\xi - \mu_{M'}))|.
\]

Note that $W(A_{M})_{\delta} \subset O(\mathfrak{h}_{M}, \langle \cdot, \cdot \rangle)$. By property (1) of Lemma 9.5, the fact that $M' \supseteq M_{\mu}$, and the definition of $W(A_{M'})_{\delta}$, we have
\[(9.10)\quad \sum_{w \in W(A_{M'})_{\delta}} h_0(R(w\xi - \mu_{M'})) = |W(A_{M'})_{\delta}||h_0(R(\xi - \mu_{M'}))| \geq |h_0(R(\xi - \mu_{M'}))|.
\]

It now suffices to establish an upper bound for the second term in (9.9). Note that (9.13) implies that $w\xi - \mu_{M'}$, for $w \in W_{\text{bad}}$, satisfies the inequalities on the left-hand side of (9.20), with $\kappa = \kappa_{r_{M'}}$. Similarly, (9.12) implies that $\xi - \mu_{M'}$ satisfies the inequality on the right-hand side of (9.20), with $\eta = \eta_{r_{M'}}$. Recalling the value of $\epsilon$, we deduce that
\[(9.11)\quad |W_{\text{bad}}| \max_{\xi \in W_{\text{bad}}} |h_0(R(w\xi - \mu_{M'}))| \leq \frac{1}{2} |h_0(R(\xi - \mu_{M'}))|.
\]

Inserting (9.10) and (9.11) into (9.9) yields
\[\sum_{w \in W(A_{M})_{\delta}} h_0(R(w\xi - \mu_{M'})) \geq \frac{1}{2} |h_0(R(\xi - \mu_{M'}))|.
\]

From this and property (2) of Lemma 9.5, with $\eta = \eta_{r_{M'}}$, the lower bound (9.8) follows. \hfill \Box

We now prove the following geometric lemma which was a crucial ingredient in the proof of Lemma 9.3.

**Lemma 9.4.** Let $\delta \in \mathcal{D}$ be represented by $(\delta, M)$ with $M \in \mathcal{L}_{\infty}$. Let $\mu \in i\mathfrak{h}_{M}^*$ be contained in a $\delta$-singular subspace $i\mathfrak{h}_{M}^*$ for some $M_{\mu} \in \mathcal{L}_{\infty}(\delta)$ strictly containing $M$.

There is a constant $A > 0$ and a finite system $(\kappa_{i}, \eta_{i})_{i=r_{M_{\mu}}, \ldots, r_{G}}$ of pairs of positive constants, depending only on $G$, satisfying the following properties:
(1) for every $i = r_{M_1}, \ldots, r_G$ the constants $(\eta_i, A)$ verify Property (2) of Lemma 9.5;
(2) for every $i = r_{M_1}, \ldots, r_G - 1$ the constants $(\kappa_i, \eta_i, A)$ verify Property (3) of Lemma 9.5 with
\[ \epsilon = \frac{1}{2} |W(A_{M_i})\delta|^{-1}; \]
(3) for all $\xi \in \mathfrak{h}_{\delta,\text{hm}}$ verifying (9.6) there is $M' \in \mathcal{L}_{\delta}(\delta)$ containing $M_\mu$ (depending on $\mu$ and $\xi$) such that
\begin{align*}
(9.12) & \quad \| \text{Im} \xi - \mu M' \| \leq \eta_{r_{M'}} \| \text{Re} \xi \| \\
(9.13) & \quad \| w \text{Im} \xi - \mu M' \| > \kappa_{r_{M'}} \| \text{Re} \xi \| \quad (w \in W(A_{M'}) \setminus W(A_{M_{\delta}})).
\end{align*}

Here we have used $r_{M'}$ to denote the $\mathbb{R}$-rank of $M'$ and $W(A_{M_{\delta}})$ is as in (4.4).

**Proof.** The values of $A > 0$ and the system $(\kappa_i, \eta_i)$ depend on fixed choices of constants depending only on $G$ which we now specify:

- For $L \in \mathcal{L}_{\infty}(\delta)$ and $\eta > 0$ let $\mathcal{T}_L(\eta)$ denote the tube of radius $\eta$ about $h^*_{\mu}$ inside $h^*_M$. Let $0 < \tilde{\eta}_1 < \tilde{\eta}_2 < \cdots < \tilde{\eta}_G$ be a fixed system of radii such that for any $L_1, L_2 \in \mathcal{L}_{\infty}(\delta)$ one has
\begin{equation}
\mathcal{T}_{L_1}(\tilde{\eta}_{L_1}) \cap \mathcal{T}_{L_2}(\tilde{\eta}_{L_2}) \subset \mathcal{T}_L(\tilde{\eta}_{L}),
\end{equation}
where $L = \langle L_1, L_2 \rangle \in \mathcal{L}_{\infty}(\delta)$ is generated by $L_1$ and $L_2$. We remark that the property (9.14) is conserved under simultaneous rescaling of all $\tilde{\eta}_r$.

- From [34, p. 136] there exists a constant $C \geq 1$ such that for all $L, L' \in \mathcal{L}_{\infty}(\delta)$ and all $\mu \in \mathfrak{h}_{L'}$ one has
\begin{equation}
\| \mu^{(L, L')} \| \leq C \| \mu^{L} \|.
\end{equation}

We first set $\eta_{r_{G}}$ and $A_{r_{G}}$ to be values of $\eta$ and $A$ for which property (2) of Lemma 9.5 hold. The remaining constants indexed by $i = r_{M_1}, \ldots, r_{G} - 1$ will be defined by downward induction on $i$.

We set $c_i = \min_{j > i} \eta_j / \tilde{\eta}_j$ and then take any $\kappa_i$ satisfying $\kappa_i < \frac{c_i \tilde{\eta}_i}{8 \| W(A_{M_i})\delta \| C}$. Applying Property (3) of Lemma 9.5 with $\kappa = \kappa_i$ and $\epsilon = \frac{1}{2} |W(A_{M_i})\delta|^{-1}$ yields constants $\eta'_i$ and $A_i$. We then set $\eta_i = \min \{ \eta'_i, \frac{1}{4 |W(A_{M_i})\delta| C} \}$. It is not hard to see, by invoking the rescaling property and the inequality $\eta_i / \tilde{\eta}_i < \eta_j / \tilde{\eta}_j$ for $i < j$, that (9.14) holds with the system of $\eta_i$’s in place of $\tilde{\eta}_i$. Finally we put $A = \max_i \{ A_i, 2 \eta_i^{-1} \}$.

For $\xi \in \mathfrak{h}_{\delta,\text{hm}}$ satisfying (9.6), let $M' \in \mathcal{L}_{\infty}(\delta)$ containing $M_\mu$ be maximal for the property
\begin{equation}
\| \mu^{M'} \| \leq \frac{1}{2} \eta_{r_{M'}} \| \text{Re} \xi \|.
\end{equation}

This is well-defined, since if $M_1$ and $M_2$ satisfy this bound, then so does $\langle M_1, M_2 \rangle \in \mathcal{L}_{\infty}(\delta)$.

Using (9.6) we have that $\| \text{Im} \xi - \mu \| \leq R^{-1} \leq A^{-1} \| \text{Re} \xi \|$, and so
\begin{equation}
\| \text{Im} \xi - \mu^{M'} \| \leq \| \text{Im} \xi - \mu \| + \| \mu^{M'} \| \leq \left( \frac{1}{2} \eta_{r_{M'}} + A^{-1} \right) \| \text{Re} \xi \| \leq \eta_{r_{M'}} \| \text{Re} \xi \|.
\end{equation}

This proves the upper bound (9.12).

We proceed to some preliminary estimates toward (9.13). We claim that for all $L \in \mathcal{L}_{\infty}(\delta)$, not contained in $M'$, we have
\begin{equation}
\| \mu^{L} \| > \frac{c_{r_{M'}} \tilde{\eta}_i}{2C} \| \text{Re} \xi \|.
\end{equation}
Assuming otherwise, we apply the inequality (9.15) with $L' = M_\mu$ to obtain the upper bound
\[ \| \mu^{(L, M_\mu)} \| \leq \frac{1}{2} c_{r_{M'}} \| \text{Re} \xi \|. \]
Now from (9.16) and using $\eta_{r_{M'}} \leq c_{r_{M'}} \eta_{r_{M'}}$, we also have
\[ \| \mu^{M'} \| \leq \frac{1}{2} c_{r_{M'}} \| \text{Re} \xi \| \]
Setting $M'' = \langle M', \langle L, M_\mu \rangle \rangle = \langle L, M' \rangle$, the compatibility of the constants $\tilde{\eta}_i$ then shows that $\| \mu^{M''} \| \leq \frac{1}{2} c_{r_{M'}} \| \text{Re} \xi \|$. Now since $L \not\subset M'$, the subgroup $M''$ is
strictly larger than $M'$. Thus $c_{r, M'} \leq \eta_{r, M''} / \tilde{h}_{r, M''}$ and $\|\mu'''\| \leq \frac{1}{2} \eta_{r, M''} \|\text{Re} \xi\|$. But this contradicts the maximality of $M'$, establishing (9.18). Moreover, we may bootstrap (9.18) to show that

$$\|((\mu_{M'})^L)\| \geq \frac{3c_{r, M'} \tilde{h}}{8C} \|\text{Re} \xi\|$$

for all $L \in \mathcal{L}_\infty(\delta)$ not contained in $M'$. This can be seen from

$$\|\mu^L - (\mu_{M'})^L\| = \|(\mu - \mu_{M'})^L\| = \|(\mu^L)^L\| \leq \|\mu^L\| \leq \frac{1}{2} \eta_{r, M'} \|\text{Re} \xi\| \leq \frac{c_{r, M'} \tilde{h}}{8C} \|\text{Re} \xi\|$$

along with the triangle inequality.

We can now prove (9.13). Arguing by contradiction, we let $w \in W(A_M) \setminus W(A_{M'})$ and suppose that $\|w \text{Im} \xi - \mu_{M'}\| \leq \kappa_{r, M'} \|\text{Re} \xi\|$. Then, using this and (9.17), we get

$$\|\mu_{M'} - w \mu_{M'}\| \leq \|\mu_{M'} - w \text{Im} \xi\| + \|w (\text{Im} \xi - \mu_{M'})\| \leq (\kappa_{r, M'} + \eta_{r, M'}) \|\text{Re} \xi\|.$$ 

From this it follows by induction on $k$ that $\|\mu_{M'} - w^k \mu_{M'}\| \leq k (\kappa_{r, M'} + \eta_{r, M'}) \|\text{Re} \xi\|$. From this and the expression $(\mu_{M'})_{M_w} = |W(A_M)|^{-1} \sum_{k=1}^{|W(A_M)|} w^k \mu_{M'}$, we conclude that

$$\|((\mu_{M'})_{M_w})\| = \|\mu_{M'} - (\mu_{M'})_{M_w}\| \leq |W(A_{M})| (\kappa_{r, M'} + \eta_{r, M'}) \|\text{Re} \xi\| < \left(\frac{1}{8} + \frac{1}{4} \frac{c_{r, M'} \tilde{h}}{C}\right) \|\text{Re} \xi\|.$$ 

Since $w \not\in W(A_{M'})$, $M_w \not\in M'$, we may now apply (9.19) with $L = M_w$ to get a contradiction. □

9.3. Stationary phase estimates. We now prove the establish the following technical result, used in the proof of Lemma 9.3. The proof is based on the principle of stationary phase, and in particular derives inspiration from standard treatments of the Fourier transform of the uniform measure on the round sphere. What makes our setting non-standard (relative to the existing literature) is the presence of a complex phase, which requires an application of a multidimensional saddle point method. Statements (2) and (3) below are shown to follow from precise asymptotic estimates obtained in this way.

**Lemma 9.5.** There is a real-valued $f_0 \in C^\infty_c(\mathfrak{h})$ whose Fourier transform $h_0(\xi) = \int_\mathfrak{h} f_0(H) e^{\langle \xi, H \rangle} dH$ satisfies:

1. $h_0(k\xi) = h_0(\xi)$ for all $k \in O(\mathfrak{h}, \langle , \rangle)$;
2. There are constants $A, B, \eta, c > 0$ such that for all $R \geq 1$ and $\sigma \geq AR^{-1}$ we have
   $$\min_{\|\text{Im} \xi\| \leq \eta \sigma, \|\text{Re} \xi\| = \sigma} |h_0(R\xi)| \geq Be^{c\sigma R}.$$ 
3. For every $\epsilon, \kappa > 0$ there is $0 < \eta \leq 1$ (depending only on $\kappa$) and $A > 1$ (depending on $\epsilon, \kappa$) such that for all $R \geq 1$ and $\sigma \geq AR^{-1}$ we have
   $$\max_{\|\text{Im} \xi\| > \kappa \sigma, \|\text{Re} \xi\| = \sigma} |h_0(R\xi)| \leq \epsilon \min_{\|\text{Im} \xi\| \leq \eta \sigma, \|\text{Re} \xi\| = \sigma} |h_0(R\xi)|.$$

**Proof.** Let $b \in C^\infty_c(\mathbb{R})$ be the bump function equal to $e^{-1/(1-x^2)}$ in $[-1, 1]$ and vanishing outside of this interval. Define $f_0(H) = b(\|H\|)$. We shall show that $f_0$ satisfies all properties of the lemma.

Let $\omega$ denote the surface measure of the unit sphere $S^{d-1}$ in $(\mathfrak{h}, \langle , \rangle)$, so that

$$h_0(\xi) = \int_0^1 e^{-1/(1-r^2)} \hat{\omega}(r\xi) r^{d-1} dr.$$ 

Then property (1) follows from the $O(\mathfrak{h}, \langle , \rangle)$-invariance of $\omega$. Properties (2) and (3) will follow from an asymptotic estimates on $h_0$. 

---

**Note:** The above text contains mathematical content that requires a high level of expertise in mathematics to fully comprehend and understand. It is important to carefully study each section and its proofs to grasp the underlying concepts. The text is a detailed explanation of a specific mathematical topic, focusing on asymptotic estimates and their implications in the context of saddle point methods. The use of symbols and notation is consistent with standard mathematical practices, and the text provides necessary definitions and theorems to build a comprehensive understanding of the subject matter.
The asymptotic behavior of $h_0(\xi)$ is markedly different depending on whether the real or imaginary part of $\xi$ plays a dominant role. For this it will be useful to introduce the parameter

$$s = \sqrt{||\text{Re} \xi||^2 - ||\text{Im} \xi||^2},$$

which is either real or purely imaginary. In this notation we shall show:

(I): If $||\text{Re} \xi|| > ||\text{Im} \xi||$ and $s \gg 1$, then

$$h_0(\xi) = \left(C + O\left(\frac{1}{\sqrt{s}}\right)\right) \frac{e^s}{s^{d/2+1}},$$

where $C = e^{-1/2\sqrt{2}d/2-3/4\pi d/2}$ is a dimensional constant;

(II): If $|s| \ll 1$, then $h_0(\xi) \ll ||\text{Re} \xi||$;

(III): If $||\text{Re} \xi|| < ||\text{Im} \xi||$ and $|s| \gg 1$, then $h_0(\xi) \ll ||\text{Im} \xi||/|s|$.

It is immediate that this implies (2) and (3). The bound in range (III) is far from optimal, but sufficient for our purposes.

Since $\langle \text{Re} \xi, \text{Im} \xi \rangle = 0$ (by the Hermitian property), there exists a $k \in SO(\mathfrak{h}, \langle , \rangle)$ such that $k \cdot \xi = ae_1^* + ibe_2^*$, with $a = ||\text{Re} \xi|| \geq 0$ and $b = ||\text{Im} \xi|| \geq 0$. Let $u : U^{d-1} \rightarrow S^{d-1}$ (with $U^{d-1}$ the $(d-1)$-unit ball), $u : X \mapsto (\sqrt{1-||X||^2}, X)$ be the standard coordinate chart on $S^{d-1}$ covering $0 \mapsto 1$. Then $\omega(\xi) + O(1)$ is

$$\int_{U^{d-1}} e^{(ae_1^* + ibe_2^*)(u(X))} \frac{dX}{\sqrt{1-||X||^2}} = \frac{2\pi^{d/2-1}}{\Gamma(d/2 - 1)} \int_0^1 e^{a\sqrt{1-t^2}} \int_{-r}^r e^{ibx \cdot r(r^2 - x^2)^{d/2-2}} dx dr$$

$$= \frac{2\pi^{d/2-1}}{\Gamma(d/2 - 1)} \int_0^1 \frac{t^{d-3}}{\sqrt{1-t^2}} \int_{-\sqrt{1-t^2}}^{\sqrt{1-t^2}} e^{a\sqrt{1-t^2-x^2} + ibx} \frac{dx}{\sqrt{1-t^2-x^2}} dt$$

$$= \frac{2\pi^{d/2-1}}{\Gamma(d/2 - 1)} \int_0^1 t^{d-3} I(a\sqrt{1-t^2}, b\sqrt{1-t^2}) dt,$$

where

$$I(a, b) = \int_{-1}^1 e^{\phi(x)} \frac{dx}{\sqrt{1-x^2}}$$

and

$$\phi(x) = a\sqrt{1-x^2} + ibx.$$

Denoting by $\mathbb{C}^+$ the upper half-plane, the phase extends to an analytic function in $\mathbb{C}^+ \cup (-1, 1)$ and extends to a continuous function $\phi(z) = \phi(x+iy)$ on $\mathbb{C}^+ \cup \mathbb{R}$ given explicitly by

$$\text{Re} \phi(x+iy) = a\sqrt{\frac{1-x^2+y^2+\sqrt{(1-x^2+y^2)^2+4x^2y^2}}{2}} - by,$$

$$\text{Im} \phi(x+iy) = -\text{sgn}(x) \cdot a\sqrt{\frac{-(1-x^2+y^2)+\sqrt{(1-x^2+y^2)^2+4x^2y^2}}{2}} + bx.$$

The integral (9.23) exhibits sharply different asymptotic behavior according to the relative sizes of $a$ and $b$, as given in the ranges (I), (II), and (III).

Range (I): In this range, the phase has a unique complex stationary point at $i\alpha = ib/s$. It is non-degenerate, with $\phi''(i\alpha) = -a/(1 - (i\alpha)^2)^{3/2} = -s^3/a^2$. We may write

$$I(a, b) = \int_{\mathcal{C}} + \int_{-1}^{-\beta} + \int_{\beta}^1,$$

where $\beta = a/s > 1$, and $\mathcal{C}$ is the arc from $-\beta$ to $\beta$ in $\mathbb{C}^+$ of the ellipse given by

$$\frac{x^2}{\beta^2} + \frac{y^2}{a^2} = 1.$$
The curve \( \mathcal{C} \) is the curve of steepest descent passing through \( i\alpha \); along it we compute

\[
1 - x^2 + y^2 = \frac{a^2}{s^2} - \frac{a^2 + b^2}{a^2}x^2, \quad \sqrt{(1 - x^2 + y^2)^2 + 4x^2y^2} = \frac{a^2}{s^2} - \frac{s^2}{a^2}x^2, 
\]

so that \( \text{Im } \phi(x + iy) = 0 \) and \( \text{Re } \phi(x + iy) = s \sqrt{1 - s^2x^2/a^2} \). The line element along \( \mathcal{C} \) is explicated as \( ds/\sqrt{1 - z^2} = \beta^{-1}dx/\sqrt{1 - x^2/\beta^2} \). We estimate crudely in the range \( |x| \geq \gamma \), where \( \gamma = \beta/\sqrt{3} \), to find that \( \int_{\mathcal{C}} \) is

\[
\frac{1}{\beta} \int_{-\beta}^{\beta} e^{s \sqrt{1 - x^2/\beta^2}} \frac{dx}{\sqrt{1 - x^2/\beta^2}} = \frac{1}{\beta} \int_{-\gamma}^{\gamma} e^{s(1 - \frac{1}{2}(x^2/\beta^2))} \left( 1 + O(x^2/\beta^2 + sx^4/\beta^4) \right) dx + O \left( e^{s \sqrt{1 - \gamma^2/\beta^2}} \right),
\]

the Taylor expansion of the phase being valid in light of \( sx \). The integrals appearing in the error terms, we find that \( \int_{\mathcal{C}} \) is

\[
\int_{1} \left( 1 - w^2 \right)^{d/2 - 2}e^{sw} \left( 1 + O \left( \frac{1}{sw} \right) \right) w^{3/2} dw = \frac{e^s}{s^{d/2 - 1}} \int_{0}^{s^{1/2}} (2u)^{d/2 - 2}e^{-u} \left( 1 + O \left( \frac{u}{s} \right) \right) du + O(e^{s/2})
\]

After some simplifications, we shall convert the above integral into a Gamma function, as follows. Changing variables \( w = \sqrt{1 - t^2} \) and then \( u = (1 - w)s \), we obtain

\[
\int_{0}^{1} \left( 1 - w^2 \right)^{d/2 - 2}e^{sw} \left( 1 + O \left( \frac{1}{sw} \right) \right) w^{3/2} dw = \frac{e^s}{s^{d/2 - 1}} \int_{0}^{s^{1/2}} (2u)^{d/2 - 2}e^{-u} \left( 1 + O \left( \frac{u}{s} \right) \right) du + O(e^{s/2})
\]

Altogether, this gives

\[
\tilde{\omega}(\xi) = \frac{e^s}{s^{(d-1)/2}} \left( (2\pi)^{(d-1)/2} + O(1/s) \right) + O(a/s^2).
\]

Inserting this into (9.21) (as well as the bound from (III) for \( r \ll 1/s \)) we find

\[
h_0(\xi) = \left( \frac{2\pi}{s} \right)^{(d-1)/2} \int_{0}^{1} e^{f(r)r^{d-1}/2} \left( 1 + O(1/rs) \right) dr + O(a/s^2),
\]

where \( f(r) = -\frac{1}{1-r^2} + rs \). Computing

\[
f'(r) = s - 2r/(1 - r^2), \quad f''(r) = (-2 - 6r^2)/(1 - r^2)^2, \quad f'''(r) \propto 1/(1 - r)^4 \quad (1 \ll r \ll 1),
\]

we find that the phase \( f(r) \) achieves the global maximum at a point \( \beta_s \) that satisfies

\[
\beta_s = 1 - \frac{1}{\sqrt{2}s} + O \left( \frac{1}{s^3/2} \right), \quad f''(\beta_s) = -(2s)^{3/2} + O(s^2), \quad \|f'''\|_{[0,(1+\beta_s)/2]} \approx s^2.
\]

Furthermore, \( f(\beta_s) = s\beta_s - \sqrt{s^2/2}\beta_s = s - \frac{1}{2\sqrt{2}} + O(1/\sqrt{s}) \). Thus, up to an error term of size

\[
O \left( a/s^2 + e^{-(c^2\sqrt{2}+O(c^3))\sqrt{s}/s^{(d-1)/2}} \right),
\]

where \( c > 0 \) is taken suitably small, we have that \( h_0(\xi) \) is

\[
\left( \frac{2\pi}{s} \right)^{(d-1)/2} \int_{0}^{\beta_s + c/\sqrt{s}} e^{f(\beta_s) + f''(\beta_s)/2(r-\beta_s)^2 + O(s^2(r-\beta_s)^2)} \left( \beta_s \right)^{d-1} + O(r - \beta_s)) dr.
\]
Completing the Gaussian, we have
\[ h_0(\xi) = \left( \frac{2\pi}{s} \right)^{(d-1)/2} \left( \frac{\sqrt{\pi}}{2^{1/4}s^{3/4}} + O \left( \frac{1}{s^2} + \frac{e^{-(c^2\sqrt{2}+O(c^3))\sqrt{s}}}{s^n} \right) \right) e^{s - \frac{1}{12s^2}}. \]

Combining everything proves (9.22).

Range (II): The transition range is \( b^2 = a^2 + O(1) \), or, equivalently, \( b = a + O(1/a) \). Let \( \mathcal{C}_a \) be the semicircle from \(-a\) to \(a\) in the upper half-plane and write \( \int_{-1}^{1} = \int_{-a}^{a} + \int_{\mathcal{C}_a}^{1} \). It is easy to see that \( \Re \phi(x + iy) = O(1) \) along \( \mathcal{C}_a \) whenever \( a^2 - x^2 = O(1) \). Otherwise, we use
\[ \sqrt{(1 - x^2 + y^2)^2 + 4x^2y^2} = \sqrt{(1 + a^2)^2 - 4x^2} \leq 1 + a^2 - \frac{2x^2}{1 + a^2}, \]
so that, whenever \( |a^2 - x^2| \gg 1 \) along \( \mathcal{C}_a \), we have
\[ \Re \phi(x + iy) \lesssim a \sqrt{1 + a^2 - x^2 - \frac{x^2}{1 + a^2} - a\sqrt{a^2 - x^2} + O(1)} \lesssim a \sqrt{a^2 - x^2 \left( \sqrt{1 + O(1/a^2)} - 1 \right)} + O(1) = O(1) \]
Keeping in mind that \( \lim_{y \to 0+} \Re \phi(x + iy) = 0 \) for \( |x| \geq 1 \), we estimate trivially in terms of the length to obtain \( I(a,b) \ll a \) and thus \( \tilde{\omega}(\xi) \ll a \) and \( h_0(\xi) \ll a \).

Range (III): Denoting \( \beta = b/|s| \) and \( ia = i|a/|s|| \), and letting \( \mathcal{C} \) be the arc as in (9.26), we find by a computation similar to (9.27) that \( \Re \phi(x + iy) = O(1) \) along \( \mathcal{C} \); furthermore, \( \lim_{y \to 0+} \Re \phi(x + iy) = 0 \) for \( |x| \gg 1 \) by direct inspection of (9.24). Shifting contours as in (9.25) and estimating trivially in terms of the length, we thus find that \( I(a,b) \ll b/|s| \) and thus \( \tilde{\omega}(\xi) \ll b/|s| \) and \( h_0(\xi) \ll b/|s| \). \( \square \)

10. Proof of Proposition 7.2

Similarly to the existing literature on uniform Weyl laws, our basic strategy in the proof of Proposition 7.2 is to relate the sharp count \( N(q, \delta, P) \) to a corresponding smooth count; we can control the smooth count via trace formula input, as represented by Property (ELM), applied to the test function \( f_R^{\delta, P} \) constructed in Section 8.

We need to pass from a smooth to a sharp test function in two terms: the central contribution \( J_{\text{cent}}(\varepsilon_{K_1(q)} \otimes f_R^{\delta, P}) \) and the smooth count over the discrete spectrum \( J_{\text{disc}}(\varepsilon_{K_1(q)} \otimes f_R^{\delta, P}) \). The central contributions were already addressed in §8.3. In the following paragraphs we treat the discrete spectrum.

10.1. Upper bounds. We must first bound the total deviations of \( h_R^{\delta, P} \) from the sharp-cutoff condition of belonging to \( P \) over the “transition zones” (where the smooth test function is transitioning between 0 and 1). This is done in Proposition 10.1 below. For the statement, recall the notion of \( 1/R \)-containment from Definition 3 and the sum \( K_R(q, \delta, P) \) from (9.2).

**Proposition 10.1.** Let \( 0 < R < c \log(2 + 4Nq) \). For any set \( S \subseteq \mathcal{R}_M \) which is \( 1/R \)-contained in some \( B \in \mathcal{R}_M \),
\[ N(q, \delta, S) \ll \varphi_n(q) \left| \text{vol}_R(B) + |J_{\text{error}}(\varepsilon_{K_1(q)} \otimes f_R^{\delta, B})| \right| + K_R(q, \delta, B). \]
In particular, this holds with \( S = \partial P(\rho) \) and \( B = \partial P(\rho + 1/R) \) where \( P \in \mathcal{R}_M \) and \( \rho > 0 \).

**Proof.** For \( B \in \mathcal{R}_M \) let us decompose
\[ J_{\text{disc}}(\varepsilon_{K_1(q)} \otimes f_R^{\delta, B}) = J_{\text{temp}}(\varepsilon_{K_1(q)} \otimes f_R^{\delta, B}) + J_{\text{comp}}(\varepsilon_{K_1(q)} \otimes f_R^{\delta, B}), \]

according to whether or not the archimedean component of the $\pi$ contributing to $J_{\text{disc}}(\varepsilon_{K_1(q)} \otimes f^\delta_R)$ is tempered. Using the non-negativity of $f$, and that $S$ is $(1/R)$-contained in $B$, we obtain that, for every $\nu \in S$,

$$h^\delta_R(\nu) = R^\delta \int_B f(R(\nu - \tau))
\quad d\tau \geq R^\delta \int_{B(\nu, 1/R)} f(R(\nu - \tau))
\quad d\tau = \int_{B(0, 1)} f(\tau)
\quad d\tau \geq \frac{1}{2}.$$  

Thus, by the non-negativity of $h^\delta_R$ on $\mathfrak{h}^*_M$, we have

$$\frac{1}{2} N(q, \delta, S) \leq J_{\text{temp}}(\varepsilon_{K_1(q)} \otimes f^\delta_R).$$

On the other hand, by definition, we have

$$J_{\text{temp}}(\varepsilon_{K_1(q)} \otimes f^\delta_R) = -J_{\text{comp}}(\varepsilon_{K_1(q)} \otimes f^\delta_R) + J_{\text{cent}}(\varepsilon_{K_1(q)} \otimes f^\delta_R) + J_{\text{error}}(\varepsilon_{K_1(q)} \otimes f^\delta_R).$$

It suffices to bound the first two terms on the right-hand side. For this, we use the estimate (8.7) of Proposition 8.5 to bound $J_{\text{cent}}(\varepsilon_{K_1(q)} \otimes f^\delta_R)$. Then, by Lemma 8.2, we have that

$$h^\delta_R(\delta, \nu) \ll N \cdot e^{R\|\Re \nu\|}(1 + R \cdot d(\Im \nu, B))^{-N}$$

for every $\nu \in \mathfrak{h}^*_M$, so that

$$J_{\text{comp}}(\varepsilon_{K_1(q)} \otimes f^\delta_R) \ll K_R(q, \delta, B).$$

The last statement of the proposition follows from Lemma 7.1, where it is shown that $\partial P(\rho)$ is $(1/R)$-contained in $\partial P(\rho + 1/R)$.

10.2. Proof of Proposition 7.2. Let $c = \min(c_1, c_2)$, where $c_1$ is defined in Property (ELM) and $c_2$ is given in Corollary 8.5. Assume that $0 < R < c \log(2 + Nq)$.

We begin by applying the decomposition (10.2). We use the bound (10.3) and then Proposition 9.2 to bound the complementary term by $\varphi_n(q) \text{vol}^*_R(\delta, P)$. We use Property (ELM) to bound $J_{\text{error}}$ by $Nq^{a-\theta} \text{vol}_R(\delta, P)$. Finally, when $Nq > C_2$, where $C_2$ is as in Corollary 8.5, then (8.9) allows us to conclude that

$$J_{\text{temp}}(\varepsilon_{K_1(q)} \otimes f^\delta_R) = \text{vol}(\mu_{\text{GL}_n}) \varphi_n(q) \int_P \mu^G_M(\delta, \nu)
\quad d\nu
\quad + O\left(\varphi_n(q)(\text{vol}_R(\delta, P) + \text{vol}_R(\delta, P)) + Nq^{a-\theta} \text{vol}_R(\delta, P)\right).$$

The case $Nq \leq C_2$ can be treated similarly, yielding an upper bound, using (8.8) of Corollary 8.5.

We now decompose the sum in $J_{\text{temp}}(\varepsilon_{K_1(q)} \otimes f^\delta_R)$ according to (8.4). Note that

$$N(q, \delta, P) = \sum_{\pi \in \Pi_{\text{disc}}(G(\mathbb{A}_F)^1)^\perp} \dim V_{\pi_f}^{K_1(q)}.$$  

Indeed, this follows from the description of the discrete spectrum by Moeglin-Waldspurger [38], where it is shown that any $\pi \in \Pi_{\text{disc}}(G(\mathbb{A}_F)^1)$ such that $\pi_\infty$ is tempered is necessarily cuspidal. Using this and Lemma 8.2, we find that $J_{\text{temp}}(\varepsilon_{K_1(q)} \otimes f^\delta_R)$ equals

$$N(q, \delta, P) + O(N(q, \delta, \partial P(1/R)))
\quad + O\left(\sum_{\ell=1}^\infty \ell^{-N} \sum_{\psi/\psi} N(q, \delta, P^{\bullet/\circ}(\ell/R, (\ell + 1)/R))\right)
\quad = N(q, \delta, P) + O\left(\sum_{\ell=1}^\infty \ell^{-N-1} N(q, \delta, \partial P(\ell/R))\right).$$
It follows from this and Proposition 10.1 that $J_{\text{temp}}(\varepsilon_{K_1(q)} \otimes f_R^\delta P) - N(q, \delta, P)$ is majorized by

\[
\sum_{\ell=1}^\infty \ell^{-N-1} \left( \varphi_n(q) \text{vol}_{R,N+1}(\delta, \partial P((\ell + 1)/R)) + K_{R,N+1}(q, \delta, \partial P((\ell + 1)/R)) + |J_{\text{error}}(\varepsilon_{K_1(q)} \otimes f_R^\delta P((\ell + 1)/R))| \right) .
\]

(10.5)

The first of the error terms in (10.5) needs only elementary manipulation to be put into the required form. Indeed, expanding into a double sum, we have

\[
\sum_{\ell=1}^\infty \ell^{-N-1} \text{vol}_{R,N+1}(\delta, \partial P((\ell + 1)/R)) \leq \sum_{\ell=1}^\infty \sum_{m=0}^\infty \ell^{-N-1}(m + 1)^{-N-1} \int_{\partial P((\ell + m + 1)/R)} \beta_M^G(\delta, \nu) \, d\nu
\]

(10.6)

= \sum_{\ell=2}^\infty \int_{\partial P(\ell/R)} \beta_M^G(\delta, \nu) \, d\nu \sum_{k_1 + k_2} (k_1 k_2)^{-N-1}

\leq \sum_{\ell=1}^\infty \ell^{-N} \int_{\partial P(\ell/R)} \beta_M^G(\delta, \nu) \, d\nu .

By definition, this last sum is just $\partial \text{vol}_R(\delta, P)$.

To treat the second term in (10.5), an elementary argument similar to the one above shows

\[
\sum_{\ell=1}^\infty \ell^{-N-1} K_{R,N+1}(q, \delta, \partial P((\ell + 1)/R)) \ll K_{R,N}(q, \delta, \partial P(1/R)),
\]

which is certainly less than $K_{R,N}(q, \delta, P)$. Proposition 9.2 then bounds this by $\varphi_n(q) \text{vol}_R^*(\delta, P)$.

It remains to treat the contribution

\[
|J_{\text{error}}(\varepsilon_{K_1(q)} \otimes f_R^G) | + \sum_{\ell=1}^\infty \ell^{-N} |J_{\text{error}}(\varepsilon_{K_1(q)} \otimes f_R^\delta P((\ell + 1)/R))| ,
\]

coming from (10.4) and (10.5), which we do by invoking Property (ELM). Indeed, from (8.3), the above sum is bounded by

\[
Nq^{n-\theta} \left( \text{vol}_R(\delta, P) + \sum_{\ell=1}^\infty \ell^{-N} \text{vol}_R(\delta, \partial P(\ell/R)) \right)
\]

where we have used the assumption that $0 < R < c \log(2 + Nq)$. Using the same arguments as in (10.6) for the second term then shows that this is $O(Nq^{n-\theta} \text{vol}_R(\delta, P))$, completing the proof. \( \square \)

11. Deducing Theorem 1.2

To pass from Proposition 7.2 to Theorem 1.2, we need to sum over the various discrete data coming from the decomposition of $|\mathcal{F}(Q)|$ in (3.4). In this section, we package together the terms arising in Proposition 7.2 after executing this summation, evaluate the main term, and bound the boundary errors. This will complete the proof of Theorem 1.2.

11.1. Summing over discrete data. Recalling the set $\Omega_{\delta,X}$ of (3.1), we put

\[
P_{\delta,X} = \Omega_{\delta,X} \cap \mathcal{D}_M^* \quad \text{and} \quad P_X = \bigcup_{\delta = [\delta,M] \in \mathcal{D}} P_{\delta,X} .
\]

Then $P_X$ can be identified with and $\Omega_X \cap \Pi_{\text{temp}}(G_{1q}^\delta)$, where $\Omega_X = \{ \pi \in \Pi(G_{1q}^\delta) : q(\pi) \leq X \}$. We use this notation to extend the definition of the boundary terms of (7.3) and (7.4) by writing

\[
\text{vol}_R(P_X) = \sum_{\delta = [\delta,M] \in \mathcal{D}} \text{vol}_R(\delta, P_{\delta,X}) ,
\]

(11.1)
and similarly for $\partial \text{vol}_R(P_X)$ and $\text{vol}^*_R(P_X)$. We shall also need to introduce the notation

$$\partial P_X(r) = \bigcup_{\delta \in \mathcal{D}} \{\pi_{\delta,\nu} : \nu \in \partial P_{\delta, X}(r)\}.$$ 

For a sequence $\mathcal{R} = (R(n))_{n \in \mathcal{O}_F}$ of real numbers indexed by integral ideals $n$, and $\theta > 0$, let

$$\overline{\text{vol}}^\theta_{\mathcal{R}}(Q) = \sum_{Nq \in Q} \sum_{q \mid q} |\lambda_n(q/\mathfrak{d})|Nq^{n-\theta}\text{vol}_{R(q)}(P_{Q/Nq}).$$

Analogously notation holds for $\partial \text{vol}_{\mathcal{R}}(Q)$ and $\text{vol}^*_R(Q)$ with $Nq^{n-\theta}$ replaced by $\varphi_n(\mathfrak{d})$.

**Definition 4.** We shall say that $\mathcal{R} = (R(n))_{n \in \mathcal{O}_F}$ is admissible if $0 < R(n) < c \log(2 + Nn)$, where $c$ is the constant in Proposition 7.2.

The following result reduces the remaining work to an estimation of error terms.

**Lemma 11.1.** Assume Property (ELM). There is $\theta > 0$ such that for any admissible sequence $\mathcal{R} = (R(n))$ we have

$$|\mathfrak{S}(Q)| = \mathcal{C}(Q)Q^{n+1} + O(Q^{n+1-\theta} + \partial \text{vol}_{\mathcal{R}}(Q) + \text{vol}^*_R(Q) + \overline{\text{vol}}^\theta_{\mathcal{R}}(Q)).$$

**Proof.** We begin by decomposing $\mathfrak{S}(Q) = \mathfrak{S}_{\text{temp}}(Q) \cup \mathfrak{S}_{\text{comp}}(Q)$, according to whether the archimedean component $\pi_{\infty}$ of the cusp form $\pi \in \mathfrak{S}(Q)$ is tempered or not.

We first dispatch with the contribution from $|\mathfrak{S}_{\text{comp}}(Q)|$. Using (3.4) along with the inequality

$$\sum_{\pi \in \Pi_{\text{cusp}}(G(\mathcal{A}_F)^1)_{\mathfrak{d}}} \sum_{\xi \in \Omega_{\delta, X} \setminus P_{\delta, X}} \dim V_{\pi}^{K_1(\delta)} \leq \sum_{\pi \in \Pi_{\text{disc}}(G(\mathcal{A}_F)^1)_{\mathfrak{d}}} \sum_{\xi \in \Omega_{\delta, X} \setminus P_{\delta, X}} \dim V_{\pi}^{K_1(\delta)} \leq K_R(\delta, P_{\delta, X}),$$

valid for any $R(\mathfrak{d}) > 0$, we find that

$$|\mathfrak{S}_{\text{comp}}(Q)| \leq \sum_{1 \leq Nq \leq Q} \sum_{q \mid q} |\lambda_n(q/\mathfrak{d})|K_R(\delta, P_{Q/Nq}).$$

Proposition 9.2 then bounds the latter sum by $\text{vol}^*_R(Q)$.

For the tempered contribution, we again apply (3.4) to get

$$|\mathfrak{S}_{\text{temp}}(Q)| = \sum_{1 \leq Nq \leq Q} \sum_{q \mid q} \lambda_n(q/\mathfrak{d})N(\mathfrak{d}, P_{Q/Nq}).$$

Proposition 7.2 show that $N(q, P_X)$ is

$$\text{vol}(\mu_{GL_n})\varphi_n(q)\int_{P_X} d\mu_{\infty} + O(\varphi_n(q)(\partial \text{vol}_R(P_X) + \text{vol}^*_R(P_X)) + Nq^{n-\theta}\text{vol}_R(P_X)),$$

for $0 < R < c \log(2 + Nq)$ and $Nq \geq C$. Summing the error terms in (11.2) over all $q$ and $q \mid q$, we recover the three boundary error terms in the lemma.

The second part of Proposition 7.2 furthermore shows that $N(q, P_X) \ll \varphi_n(q)\int_{P_X} d\mu_{\infty}^* \ll q^{n-1/d+\epsilon}$, for $Nq \leq C$. From the trivial estimate

$$\sum_{q \mid q \atop Nq \leq X} |\lambda_n(q/\mathfrak{d})|\varphi_n(\mathfrak{d}) \ll \epsilon X^nNq^{\epsilon}$$

and Lemma 6.3, we easily deduce

$$\sum_{Nq \leq Q} \sum_{q \mid q \atop Nq \leq C} |\lambda_n(q/\mathfrak{d})|\varphi_n(\mathfrak{d}) \int_{P_{Q/Nq}} d\mu_{\infty}^* \ll \epsilon Q^{n-1/d+\epsilon},$$

which is clearly admissible as an error term.
To obtain the contribution of $\mathcal{E}(\mathfrak{A})Q^{n+1}$, first note that the summation on the main term in \text{(11.2)} can be extended back over all $\mathfrak{d}$ since the contribution from divisors with $\mathbf{N}\mathfrak{d} \leq C$ is estimated as above. Then \text{(6.6)} shows that the main term in \text{(11.2)}, once summed over all $\mathfrak{q}$ and $\mathfrak{d} | \mathfrak{q}$, is equal to the adelic Plancherel volume of a conductor ball, which is then evaluated asymptotically (with a power savings error) in Proposition 6.1.

We must now prove satisfactory bounds on the error terms appearing in Lemma 11.1. The remaining sections will be dedicated to proving the following

**Proposition 11.2.** There is $\theta > 0$ and a choice of admissible sequence $\mathcal{R}$ such that

$$\partial \text{vol}_{\mathfrak{A}}(Q), \text{vol}_{\mathfrak{A}}^*(Q) \ll Q^{n+1} / \log Q, \quad \overline{\text{vol}}_{\mathfrak{A}}(Q) \ll Q^{n+1-\theta}.$$  

Putting Lemma 11.1 together with Proposition 11.2, we deduce Theorem 1.2.

11.2. A preparatory lemma. The following lemma will go a long way towards proving Proposition 11.2. It bounds very similar quantities to $\partial \text{vol}_{\mathfrak{A}}(Q), \text{vol}_{\mathfrak{A}}^*(Q),$ and $\overline{\text{vol}}_{\mathfrak{A}}(Q)$, but with the arithmetic weights in the average replaced by powers of the norm, and with the admissible sequence $\mathcal{R}$ taken to be constantly equal to the real number $R$. Dealing with these arithmetic weights and choosing an appropriate admissible sequence to prove Proposition 11.2 will be done in §11.3.

**Lemma 11.3.** Let $0 < \theta \leq 2/(d+1)$ and $\sigma > n - 1/d - 1 + \theta$. For $R \gg 1$, $Q \geq 1$, we have

$$\sum_{1 \leq Nq \leq Q} Nq^\sigma \partial \text{vol}_R(P_{Q/Nq}) = O_{\sigma,\theta}(R^{-1}Q^{\sigma+1} + Q^{\sigma+1-\theta}),$$  

$$\sum_{1 \leq Nq \leq Q} Nq^\sigma \overline{\text{vol}}_R(P_{Q/Nq}) = O_\sigma(Q^{\sigma+1}),$$  

$$\sum_{1 \leq Nq \leq Q} Nq^\sigma \text{vol}_R^*(P_{Q/Nq}) = O_\sigma(R^{-1}Q^{\sigma+1}).$$

We remark that this lemma is one of the places which put requirements on the integer $N$ implicit in the volume factors; for example, for purposes of this lemma, $N \geq 3 + d(\sigma + 1)$ suffices.

**Proof.** Applying the definitions, we get

$$\sum_{1 \leq Nq \leq Q} Nq^\sigma \partial \text{vol}_R(P_{Q/Nq}) = \sum_{\ell=1}^\infty \ell^{-N} \sum_{1 \leq Nq \leq Q} Nq^\sigma \int_{\partial P_{Q/Nq}(\ell/R)} \beta(\pi) \, d\pi.$$  

Now for any $r > 0$ we have

$$\sum_{1 \leq Nq \leq Q} Nq^\sigma \int_{\partial P_{Q/Nq}(r)} \beta(\pi) \, d\pi = \int_{\Pi(G_\infty)} \left( \sum_{\pi \in \partial P_{Q/Nq}(r)} Nq^\sigma \right) \beta(\pi) \, d\pi,$$  

upon exchanging the order of the sum and integral. From Lemma 11.4 below we deduce that the right-hand side is majorized by

$$r\left(1 + r^{d(\sigma+1)}\right)Q^{\sigma+1} \int_{\Pi(G_\infty)} q(\pi)^{-\sigma-1} \beta(\pi) \, d\pi + (1 + r)^{d(\sigma+1-\theta)}Q^{\sigma+1-\theta} \int_{\Pi(G_\infty)} q(\pi)^{-\sigma-1+\theta} \beta(\pi) \, d\pi.$$  

In view of Lemma 6.3, both integrals converge, yielding

$$\sum_{1 \leq Nq \leq Q} Nq^\sigma \int_{\partial P_{Q/Nq}(r)} \beta(\pi) \, d\pi \ll_{\sigma,\theta} r\left(1 + r^{d(\sigma+1)}\right)Q^{\sigma+1} + (1 + r)^{d(\sigma+1-\theta)}Q^{\sigma+1-\theta}.$$
Applying the above estimate with \( r = \ell/R > 0 \) and executing the sum over \( \ell \geq 1 \), we get
\[
\sum_{1 \leq Nq \leq Q} Nq^\sigma \|\text{vol}_R(P_{Q/Nq}) \| \ll \sum_{\ell=1}^\infty \ell^{-N} \left( \frac{\ell}{R} \left( 1 + \left( \frac{\ell}{R} \right)^{r(\sigma+1)} \right) Q^{\sigma+1} + \left( 1 + \frac{\ell}{R} \right)^{d(\sigma+1-\theta)} Q^{\sigma+1-\theta} \right)
\ll R^{-1} \left( 1 + R^{-d(\sigma+1)} \right) Q^{\sigma+1} + \left( 1 + R^{-d(\sigma+1-\theta)} \right) Q^{\sigma+1-\theta}.
\]
The last expression is majorized by \( R^{-1}Q^{\sigma+1} + Q^{\sigma+1-\theta} \).

For the remaining two estimates, recalling the definitions in §7.2, it is enough to show that for any standard Levi \( L \in \mathcal{L}_\infty(M) \) we have
\[
\sum_{1 \leq Nq \leq Q} Nq^\sigma \|\text{vol}_{RL}(P_{Q/Nq}) \| = O_c(R^{-cL}Q^{\sigma+1}),
\]
where \( c_L \) is the codimension of \( \mathfrak{h}_L \) inside \( \mathfrak{h}_M \). When \( L = M \) (corresponding to the second sum of the lemma, over volume terms \( \text{vol}_R(P_{Q/Nq}) \)), the above estimate follows from
\[
\sum_{1 \leq Nq \leq Q} Nq^\sigma \int_{P_{Q/Nq}} \beta(\pi) \, d\pi = \int_{\Pi(G_{\infty}) \sigma} \left( \sum_{1 \leq Nq \leq Q/q(\pi)} Nq^\sigma \right) \beta(\pi) \, d\pi
\ll \sigma Q^{\sigma+1} \int_{\Pi(G_{\infty}) \sigma} q(\pi)^{-\sigma-1} \beta(\pi) \, d\pi,
\]
the last integral converging in view of Lemma 6.3 and our assumptions on \( \sigma \). For \( L \) strictly larger than \( M \) (corresponding to the third sum of the lemma, over volume terms \( \text{vol}_R(P_{Q/Nq}) \)), similar manipulations apply, and one has only to verify the convergence of the remaining integral, but with integration of the Plancherel majorizer \( \beta_M^G(\delta,\nu) \) taking place over the planar sections \( \mathfrak{h}_L \cap P_{\delta,X} \). This can be established through elementary modifications to the proof of Lemma 6.3. \( \square \)

We now establish the following result, which was used in the proof of Lemma 11.3.

**Lemma 11.4.** Let \( \pi \in \Pi_{\text{temp}}(G_{\infty}) \). Let \( r > 0 \), \( 0 < \theta \leq 2/(d+1) \), and \( \sigma \geq -1 + \theta \). Then
\[
\sum_{\pi \in \partial P_{Q/Nq}(r)} Nq^\sigma \ll_{\sigma,\theta} r\left(1 + r^{d(\sigma+1)}\right) \left(\frac{Q}{q(\pi)}\right)^{\sigma+1+\theta} + \left(1 + r^{d(\sigma+1-\theta)}\right) \left(\frac{Q}{q(\pi)}\right)^{\sigma+1-\theta}.
\]

**Proof.** We first convert the condition \( \pi \in \partial P_{Q/Nq}(r) \) on the ideal \( q \) to a more amenable condition on the norm of \( q \). We may assume that \( \pi = \pi_{\delta,\nu} \) for a class \([M,\delta]\) and \( \nu \in i\mathfrak{h}_M^* \).

For parameters \( r, X > 0 \) we have \( \pi \in \partial P_X(r) \) precisely when there is \( \mu \in i\mathfrak{h}_M^* \) with \( \|\mu - \nu\| < r \) and \( q(\pi_{\delta,\nu}) = X \). Letting \( M_r(\pi) \) (resp., \( m_r(\pi) \)) denote the maximum (resp. minimum) value of \( q(\pi_{\delta,\nu}) \) as \( \mu \) varies over \( i\mathfrak{h}_M^* \) with \( \|\mu - \nu\| < r \), we see that \( \pi \in \partial P_{Q/Nq}(r) \) implies \( m_r(\pi) \leq Q/Nq \leq M_r(\pi) \), so that
\[
\sum_{\pi \in \partial P_{Q/Nq}(r)} Nq^\sigma \ll \sum_{Q/M_r(\pi) \leq Q/Nq \leq m_r(\pi)} Nq^\sigma.
\]

Using the asymptotic (6.4) we see that this is bounded by
\[
(11.4) \quad \frac{C_r(1)}{\sigma+1} Q^{\sigma+1} \left(\frac{1}{m_r(\pi)^{\sigma+1}} - \frac{1}{M_r(\pi)^{\sigma+1}}\right) + O\left(\frac{Q^{\sigma+1-\theta}}{m_r(\pi)^{\sigma+1-\theta}}\right).
\]

To conclude the proof of the lemma we shall need to relate \( m_r(\pi) \) and \( M_r(\pi) \) to expressions involving \( (1 + r) \) and \( q(\pi) \). This will require some basic analytic properties of the archimedean conductor \( q(\pi_{\delta,\nu}) \) as \( \nu \) varies.

For \( \nu \in i\mathfrak{h}_M^* \) and \( r > 0 \), let \( \nu_0 \in i\mathfrak{h}_M^* \) denote the translation \( \nu + r\nu_0 \), for some fixed \( \nu_0 \in i\mathfrak{h}_M^* \) in the positive chamber, and write \( \pi(r) = \pi_{\delta,\nu_0(r)} \). Since \( q(\pi_{\delta,\nu}) \) is monotonically increasing in \( \nu \),
it follows that for \( \nu_0 \) large enough we have \( M_r(\pi) \leq q(\pi(r)) \). Similarly, \( q(\pi(-r)) \leq m_r(\pi) \), for \( r \leq \frac{1}{2}\|\nu\| \), say. In this interval we have \( q(\pi(-r)) \asymp q(\pi) \), while, if \( r > \frac{1}{2}\|\nu\| \), we have
\[
q(\pi(-r)) \geq q(\pi_{a,0}) \geq (1 + \|\nu\|)^{-[F_{r,\text{r}}]} q(\pi) \geq (1 + r)^{-d} q(\pi).
\]
Thus, in either case, \( q(\pi(-r)) \geq (1 + r)^{-d} q(\pi) \), proving \( m_r(\pi) \geq (1 + r)^{-d} q(\pi) \). When inserted into the second term in (11.4) we obtain the second term of the lemma.

Now let \( s \mapsto \pi(s) \) be a unit length parametrization of the line between \( \nu(-r) \) and \( \nu(r) \) in \( i\mathbb{H}^*_M \). Since \( s \mapsto q(\pi(s)) \) is a real-valued differentiable map on the interval \([0,1]\), we have
\[
q(\pi(-r))^\sigma - q(\pi(r))^\sigma = \int_0^1 \frac{d}{ds} q(\pi(s)) = \int_0^1 \frac{d}{ds} q(\pi(s)) \frac{dq(\pi(s))}{d\pi(s)} ds.
\]
Since \( \frac{dq(\pi(s))}{d\pi(s)} \ll q(\pi(s)) \) and \( \frac{d\pi(s)}{ds} \ll r \), the latter integral is bounded by
\[
\ll r|\sigma| \int_0^1 q(\pi(s))^\sigma ds \ll q(\pi(-r))^\sigma \ll r(1 + r)^{-d} q(\pi)^\sigma.
\]
Since \( \sigma \leq 0 \) we have \((1 + r)^{-d} q(\pi(-r))^\sigma \asymp (1 + r^{-d}) \), proving \( m_r(\pi)^\sigma = m_r(\pi)^\sigma \ll r(1 + r)^{-d} q(\pi)^\sigma \).

Inserting this into the first term of (11.4) then completes the proof of the lemma. \( \square \)

11.3. End of proof. We now return to the proof of Proposition 11.2.

We first choose the sequence \( R = (R(n))_{n \in \mathbb{Q}_+} \) of the form
\[
(1.5) \quad R(n) = \begin{cases} R_1, & \text{if } Q^{1/2} < Nn \leq Q, \\ R_2, & \text{if } Nn \leq Q^{1/2}, \end{cases}
\]
where \( R_1, R_2 > 0 \) will be chosen shortly (and \( Q^\sigma \) works as a cutoff for any \( \sigma \in [0,1] \)). With the above choice of \( R \) the term \( \partial \vol_\mathcal{R}(Q) \) is equal to
\[
\sum_{Nq \leq Q} \sum_{Q^{1/2} < Nq \leq Q} |\lambda_n(q/\delta)\varphi_n(\delta)\partial \vol R_1(PQ/Qq) + \sum_{Nq \leq Q} \sum_{Nq \leq Q} |\lambda_n(q/\delta)\varphi_n(\delta)\partial \vol R_2(PQ/Qq)).
\]

Bounding the first term using
\[
\sum_{Nq \leq Q} \sum_{Nq \leq Q} |\lambda_n(q/\delta)\varphi_n(\delta) \leq Nq^n \prod_{p|q} \left(1 - Np^{-n}\right) (1 + nNp^{-n} + \binom{n}{2}Np^{-2n} + \cdots) \asymp Nq^n
\]
and second term using (11.3), the above expression can be majorized by
\[
\sum_{Nq \leq Q} Nq^n \partial \vol R_1(PQ/Qq) + Q^{n/2} \sum_{Nq \leq Q} Nq^n \partial \vol R_2(PQ/Qq).
\]
Combining this with Lemma 11.3 shows that \( \partial \vol_\mathcal{R}(Q) \ll \epsilon R_1^{-1} Q^{n+1} + R_2^{-1} Q^{n+1+\epsilon} + Q^{n+1-\theta} \).

Similarly one obtains \( \vol_\mathcal{R}(Q) \ll R_1^{-1} Q^{n+1} + R_2^{-1} Q^{2n+1+\epsilon} + \vol_\mathcal{R}(Q) \ll Q^{n+1-\theta} \).

Now let \( c > 0 \) be as in Proposition 7.2. Then taking \( R_1 = \frac{c}{2} \log Q \) and \( R_2 = c \) in the definition of \( R \) in (11.5) yields an admissible sequence according to the definition preceding Corollary 11.1. Inserting these values establishes the stated bounds of Proposition 11.2. \( \square \)

Part 3. Proof of Theorem 1.3

12. Bounding the non-central geometric contributions

Arthur defines a distribution \( J_{\text{geom}} \) on \( \mathcal{H}(G/\mathcal{K}_F)^1 \) related to geometric invariants of \( G \). This distribution \( J_{\text{geom}} \) admits an expansion along semisimple conjugacy classes of \( G(F) \), and our task in this section is to bound all but the most singular terms (the central contributions) appearing
in this expansion. We must do so uniformly with respect to the level and the support of the test functions at infinity.

**Theorem 12.1.** Let \( n \geq 1 \). There exists \( \theta > 0 \) and \( c > 0 \) satisfying the following property. For any integral ideal \( q, R > 0 \), and \( f \in \mathcal{H}(G_{\infty}^1)_R \) we have

\[
J_{\text{geom}}(\varepsilon_{K_1(q)} \otimes f) - \varphi_n(q) \sum_{\gamma \in \mathbb{Z}(F) \cap K_1(q)} f(\gamma) \ll e^{cn} q^{n-\theta \|f\|_{\infty}}.
\]

The implied constant depends on \( F \) and \( n \).

Our presentation is by and large based on the papers [14], [35], and [36]. Many aspects of our argument are simplified by the absence of Hecke operators in our context. On the other hand, we have to explicate (in various places) the dependence in \( R \).

12.1. **Fine geometric expansion.** Let \( \mathcal{O} \) denote the set of semisimple conjugacy classes of \( G(F) \). For \( G \), a semisimple conjugacy class consists of all those \( \gamma \in G(F) \) sharing the same characteristic polynomial. Then Arthur defines distributions \( J_\sigma \) associated with each \( \sigma \in \mathcal{O} \) so that

\[
J_{\text{geom}}(\phi) = \sum_{\sigma \in \mathcal{O}} J_\sigma(\phi).
\]

The fine geometric expansion expresses each \( J_\sigma(\phi) \) as a linear combination of weighted orbital integrals \( J_M(\gamma, \phi) \), where \( M \in \mathcal{L} \) and \( \gamma \in M(F) \).

More precisely, Arthur shows that for every equivalence class \( \sigma \in \mathcal{O} \), there is a finite set of places \( S_{\text{adm}}(\sigma) \) (containing \( S_{\infty} \)) which is admissible in the following sense. For any finite set of places \( S \) containing \( S_{\text{adm}}(\sigma) \), there are real numbers \( a^M(\gamma, S) \), indexed by \( M \in \mathcal{L} \) and \( M(F) \)-conjugacy classes of elements \( \gamma \in M(F) \) (and, in general, depending on \( S \)), such that

\[
J_\sigma(1_{K_S} \otimes \phi_S) = \sum_{M \in \mathcal{L}} \frac{|W_M|}{|W|} \sum_{\gamma} a^M(\gamma, S) J_M(\gamma, 1_{K_S} \otimes \phi_S)
\]

for any function \( \phi_S \in C_c^\infty(G(F_S)^1) \). In the inner sum, \( \gamma \) runs over those \( M(F) \)-conjugacy classes of elements in \( M(F) \) meeting \( \sigma \). We shall describe the integrals \( J_M(\gamma, 1_{K_S} \otimes \phi_S) \) later, in Section 12.4. For the moment, we simply record the fact that \( J_M(\gamma, 1_{K_S} \otimes \phi_S) = 0 \) for any \( \gamma \notin K_S \cap \sigma \) and \( J_M(\gamma, \phi) = J_M(\gamma, \phi_S) \) otherwise, the latter being an \( S \)-adic integral.

Following [35, §6] and [36, (10.4)], for \( \sigma \in \mathcal{O} \) and \( \gamma = \sigma \nu \in \sigma U_{G_\sigma}(F) \cap \sigma \) we let

\[
S_\sigma = S_{\text{wild}} \cup \{v < \infty : D^G_v(\sigma) \neq 1\},
\]

where \( S_{\text{wild}} \) is a certain finite set of finite places depending only on \( n \). Then [35, Lemma 6.2] or [36, Proposition 10.8] shows that one can take \( S_{\text{adm}}(\sigma) = S_\sigma \cup \infty \) in the fine geometric expansion.

We will apply the fine geometric geometric expansion in the case where

\[
(12.1) \quad S = S_\sigma \cup S_q \cup S_{\infty} \quad \text{and} \quad \phi_S = \prod_{v \in S_\sigma, v \notin S_q} 1_{K_v} \otimes \prod_{v \in S_q} \varepsilon_{K_{1,v}^1}(q) \otimes f,
\]

for certain \( f \in \mathcal{H}(G_{\infty}^1)_R \) (those appearing in the statement of Theorem 12.1).

12.2. **Contributing classes.** We now wish to bound the number of equivalence classes \( \sigma \in \mathcal{O} \) contributing to the course geometric expansion of \( J_{\text{geom}}(1_{K_1(q)} \otimes f) \), for \( f \in \mathcal{H}(G_{\infty}^1)_R \). It clearly suffices to take \( q \) trivial.

**Definition 5.** For \( R > 0 \) let \( \mathcal{O}_R \) denote the set of \( \sigma \in \mathcal{O} \) for which there is \( f \in \mathcal{H}(G_{\infty}^1)_R \) with \( J_\sigma(1_{K_\text{fin}} \otimes f) \neq 0 \).

Our main result in this subsection is the following estimate. The argument is based largely on [35, Lemma 6.10].
Proposition 12.2. There is $c > 0$, depending on $n$ and $[F : \mathbb{Q}]$, such that $|\mathcal{D}_R| \ll e^{cR}$.

Proof. Let $\mathfrak{o}$ be a semisimple $G(F)$-conjugacy class represented by some semisimple element $\sigma \in G(F)$. Let $\chi_\sigma$ denote the characteristic polynomial of $\sigma$. It is a monic polynomial of degree $n$ with coefficients in $F$, independent of the choice of $\sigma$, and the map $\mathfrak{o} \mapsto \chi_\sigma$ is a bijection onto such polynomials. We shall count the $\mathfrak{o}$ appearing in $\mathcal{D}_R$ by counting the corresponding $\chi_\sigma$.

Let $\mathcal{U}_{G_\sigma}$ denote the algebraic variety of unipotent elements in the centralizer $G_\sigma$. The condition $\mathfrak{o} \in \mathcal{D}_R$ is equivalent to the following collection of local conditions at every place $v$:

1. for every $v \nmid \infty$ there is $\nu_v \in \mathcal{U}_{G_\sigma}(F_v)$ such that the $G_v$-conjugacy class of $\sigma \nu_v$ meets $K_v$;

2. there is $\nu_v \in \mathcal{U}_{G_\sigma}(F_\infty)$ such that the $G_\infty$-conjugacy class of $\sigma \nu_\infty$ meets $G_\infty^{1, \infty, \leq R}$.

Note that any element $\tilde{\gamma} \in G(\mathbb{A}_F)$ lying in the $G(\mathbb{A}_F)$-conjugacy class of $\mathfrak{o}$ has characteristic polynomial equal to $\chi_\sigma$. From the above local conditions we deduce that the coefficients of $\chi_\sigma$ (for $\mathfrak{o} \in \mathcal{D}_R$) are $v$-integral for all finite $v$, and so lie in $O_F$. Moreover, their archimedean absolute value is bounded by $e^{cR}$ for some constant $c > 0$. Each coefficient of $\chi_\sigma$ for contributing classes $\mathfrak{o}$ then lies in the intersection of $O_F \subset F$ with $\prod_{v \mid \infty} [-X, X] \subset F_\infty$. As there are at most $O(X^{[F : \mathbb{Q}]})$ such lattice points, the proposition follows.

12.3. Bounding global coefficients. Next we bound the coefficients $a^M(\gamma, \mathfrak{q})$, for $\gamma$ lying in a contributing class $\mathfrak{q}$. Once again, we are free to assume that $\mathfrak{q}$ is trivial, so that $\mathfrak{o} \in \mathcal{D}_R$.

For any finite set of finite places $T$ we put

$$q_T = \prod_{v \in T} q_v.$$ 

We begin with the following useful result.

Lemma 12.3. For $\mathfrak{o} \in \mathcal{D}_R$ we have $q_{S_\mathfrak{o}} \ll e^{cR}$. In particular,

$$(12.2) \quad |S_\mathfrak{o}| \ll R$$

for $\mathfrak{o} \in \mathcal{D}_R$.

Proof. If $\mathfrak{o} \in \mathcal{D}_R$ and $\sigma \in G(F)$ is a semisimple element representing $\mathfrak{o}$, then there is $y \in G(F)$ such that $y^{-1}\sigma \mathcal{U}_{G_\sigma}(F) y \cap K_f \neq \emptyset$. In other words there are $y \in G(F)$ and $u \in \mathcal{U}_{G_\sigma}(F)$ such $y^{-1} \sigma uy \in K_f$. Thus, for every finite place $v$ we have $D_v(\sigma) = D_v(y^{-1} \sigma uy) \leq 1$. Moreover, it follows from [36, Lemma 3.4] that (under the same assumptions on $\mathfrak{o}$ and $\sigma$)

$$(12.3) \quad D_\infty(\sigma) \ll e^{cR}$$

Thus for $\mathfrak{o} \in \mathcal{D}_R$ we in fact have

$$S_\mathfrak{o} = S_{\text{wild}} \cup \{v < \infty : D_v(\gamma) < 1\} = S_{\text{wild}} \cup \{v < \infty : D_v(\gamma) \leq q_v^{-1}\}.$$

By the product formula, we deduce that

$$1 = \prod_{v \notin S_\mathfrak{o} \cup S_\infty} D_v(\gamma) \prod_{v \in S_\mathfrak{o}} D_v(\gamma) \prod_{v \in S_\infty} D_v(\gamma) \ll q_{S_{\text{wild}}} q_{S_\mathfrak{o}}^{-1} e^{cR} \ll e^{cR},$$

as desired.

To deduce (12.2) from this we note $\sum_{v \in S_\mathfrak{o}} 1 \leq \sum_{v \in S_\mathfrak{o}} \log q_v = \log q_{S_\mathfrak{o}} \ll R$. 

For the next estimate, we invoke the main result of [37], a corollary of which is the following. Let $\sigma$ be elliptic semisimple in $M(F)$. Then $\sigma$ is conjugate in $M(\mathbb{C})$ to a diagonal matrix $\text{diag}(\zeta_1, \ldots, \zeta_n)$. Let

$$\Delta^M(\sigma) = N \left( \prod_{i<j : \zeta_i \neq \zeta_j} (\zeta_i - \zeta_j)^2 \right),$$

where $N$ is the norm from $F$ to $Q$, and the product is taken over indices $i < j$ such that the string $\alpha_i + \cdots + \alpha_j$ lies in the set of positive roots $\Phi_+^M = \Phi_+ \cap \Phi_1^M$ for $M$. For an equivalent expression for $\Delta^M(\sigma)$ using based root data, see [36, Section 9]. Then it follows from [37] (see also [35, (22)]) that there is $\kappa > 0$ such that for any finite set of places $S$ containing $S_\infty$, if $\gamma = \sigma \nu$ is the Jordan decomposition of $\gamma$, we have

$$a^M(\gamma, S) \ll |S|^n \Delta^M(\sigma)^{\kappa} \left( \prod_{v \in S} \log q_v \right)^n$$

for $\sigma$ elliptic in $M(F)$ and $a^M(\gamma, S) = 0$ otherwise.

**Proposition 12.4.** There is $c > 0$ such that for any $M \in \mathcal{L}$, any $\sigma \in \mathfrak{O}_R$ meeting $M$, and any $\gamma \in \mathfrak{o}$, we have $a^M(\gamma, S_0) = O(c^{|R^n|})$.

**Proof.** It follows from (12.2) that the factor of $|S_0|^n$ in the upper bound (12.4) is at most $O(R^n)$. To bound the factor $\Delta^M(\sigma)$ we follow the argument of [35, Lemma 6.10, (iv)]. The eigenvalues $\zeta_1, \ldots, \zeta_n$ are the roots of the characteristic polynomial of $\sigma$. As in the proof of Proposition 12.2, for $\sigma \in \mathfrak{O}_R$, these coefficients have $v$-adic (for archimedean $v$) absolute value bounded by $O(c^R)$. An application of Rouché’s theorem shows that each $\zeta_i$ has complex absolute value bound by $O(c^R)$, from which it follows that $\Delta^M(\sigma) \ll c^R$. To bound the last factor in (12.4) we apply the first part of Lemma 12.3.

### 12.4. The constant term map and weighted orbital integrals.

In this section we review the definitions of the constant term map and the weighted orbital integrals. We fix a finite non-empty set place of places $S$ of $F$. Where possible, we will drop the subscript $S$. So, for example, $G = G_S$, $G_\sigma = G_\sigma(F_S)$, $K = K_S$, $U_{G_\sigma} = U_{G_\sigma}(F_S)$, and $D(\sigma) = D_S(\sigma)$.

Let $M \in \mathcal{L}$ and $P \in \mathcal{P}_S(M)$ with Levi decomposition $P = MU$. Let $\phi \in C_c^\infty(G)$. Then the constant term of $\phi$ along $P$ is defined by

$$\phi^{(P)}(m) = \delta_P(m)^{1/2} \int_U \int_K \phi(k^{-1} m u k) \, d\mu(k) \, du \quad (m \in M).$$

Then $\phi^{(P)} \in C_c^\infty(M)$.

We come now to the definition of the weighted orbital integral. Throughout, we let $\gamma = \sigma \nu \in G$ be the Jordan decomposition of $\gamma$.

Before stating the full formula, we begin with two special cases.

1. Assume that $G_\gamma \subset M$. Then

$$J_M(\gamma, \phi) = D(\gamma)^{1/2} \int_{G_\gamma \setminus G} \phi_u(y^{-1} \gamma y) v_M'(y) \, dy.$$ 

Here, the weight function $v_{\gamma}'$ is the volume of a certain complex hull; as a function on $G$, it is left invariant under $M$ (so the above integral is well-defined) and constant equal to 1 when $M = G$.

2. Assume $\gamma = \nu \in M$ is unipotent. Let $V_0$ be the $M$-conjugacy class of $\nu$ and denote by $V_1$ the $G$-conjugacy class given by inducing $V_0$ from $M$ to $G$ along any parabolic of $G$ containing $M$ as a Levi subgroup. (The induced class $V_1$ is independent of the choice of such a parabolic.) Then, since unipotent conjugacy classes in $G$ are of Richardson type, there is a unique standard parabolic subgroup $P_1 \subset F_S$, say with Levi decomposition $P_1 = L_1 U_1$, such that $V_1$ has dense intersection with $U_1$. (The Levi factor of $P_1$ is given by the dual partition of the Jordan form of $V_1$. See, for example, [25, §5.5].) Then

$$J_M(\nu, \phi) = \int_K \int_{U_1} \phi(k^{-1} u k) w_{M, U_1}(u) \, du \, dk.$$
Here, the weight function \( w_{M,U_1} \) is a complex-valued function defined on \( U_1 \); it is invariant under conjugacy by \( K^{L_1} \) (so that the above integral is well-defined) and constantly equal to 1 when \( M = G \). See [34, p. 143] for more details.

In fact, case (2) (indeed the general case) is obtained as a limit of linear combinations of case (1). Note that for \( \nu \) unipotent \( G_\nu \not\subset M \), unless \( M = G \). When \( \gamma = \nu \) is unipotent and \( M = G \) then the two formulae coincide, giving the invariant unipotent orbital integral. For example, the Richardson parabolic of the trivial class in \( G \) is of course \( G \) itself, so that both formulae collapse in this case to \( J_G(1, \phi) = \phi(1) \).

The general formula is considerably more complicated. It will be expressed in terms of weighted orbital integrals for the Levi components of parabolic subgroups in \( F_S^{G_\sigma}(M_\sigma) \). Indeed, for any \( R \in F_S^{G_\sigma}(M_\sigma) \), we will evaluate the unipotent orbital integrals \( J_{M,R}^{M_\sigma}(\nu, \cdot) \) on certain descent functions \( \Phi \in \mathcal{C}_c^\infty(M_R) \). More precisely, if \( \gamma = \sigma \nu \in \sigma U_{G_\sigma} \cap M \), then [4, Corollary 8.7] states that

\[
(12.8) \quad J_M(\gamma, \phi) = D(\sigma)^{1/2} \int_{G_\sigma \backslash G} \left( \sum_{R \in F_S^{G_\sigma}(M_\sigma)} J_{M,R}^{M_\sigma}(\nu, \Phi_R \phi) \right) dy,
\]

where, for \( m \in M_R \) and \( y \in G \), we have put

\[
\Phi_R \phi(m) = \delta_R(m)^{1/2} \int_{K^{G_\sigma}} \int_{N_R} \phi(\gamma^{-1} \sigma k^{-1} m n k y) v'_R(ky) \, dk \, dn.
\]

The complex-valued weight function \( v'_R \) on \( G \) is set to be

\[
(12.9) \quad v'_R(z) = \sum_{Q \in F_S(M); Q_\sigma = R, a_Q = a_R} v'_Q(z),
\]

where \( v'_Q \), defined in [2, §2], generalizes the weight \( v'_M \) to arbitrary parabolics \( Q \in F_S(M) \).

Using the expression (12.7) for the unipotent weighted orbital integral, we may write \( J_{M,R}^{M_\sigma}(\nu, \Phi_R \phi) \) more conveniently. To see this, first let \( V_0 \) denote the \( M_\sigma \)-conjugacy class of the unipotent element \( \nu \in M_\sigma \). Next we write \( V_1 \) for the induced unipotent class of \( V_0 \) to \( M_R \) along any parabolic in \( M_R \) containing \( M_\sigma \) as a Levi subgroup. (The induced class \( V_1 \) is independent of the choice of such a parabolic.) Let \( P_1 = L_1 U_1 \subset M_R \) be a Richardson parabolic for \( V_1 \). Finally, let \( V \) be the induced unipotent class of \( V_1 \) to \( G_\sigma \) along \( R \). Then the Richardson parabolic \( P = LV \subset G_\sigma \) of \( V \) satisfies \( U = U_1 N_R \). We deduce that

\[
(12.10) \quad J_{M,R}^{M_\sigma}(\nu, \Phi_R \phi) = \int_{K^{G_\sigma}} \int_{U_1} \phi(\gamma^{-1} \sigma k^{-1} my) w_{M,R}^{M_\sigma}(u) v'_R(ky) \, du \, dk,
\]

where \( w_{M,R}^{M_\sigma} \) is the trivial extension of \( w_{M_\sigma,U_1}^{M_\sigma} \) to all of \( U \). For more details, see [35, §10.4].

We make the following observations:

1. Assume \( G_\gamma \subset M \). This condition is clearly equivalent to \( G_\gamma = M_\sigma \) and the uniqueness of the Jordan decomposition then implies \( M_\sigma = G_\sigma \). In this case \( F_S^{G_\sigma}(M_\sigma) = \{ G_\sigma \} \) and the sum over \( R \) in (12.8) reduces to the single term \( R = G_\sigma \); clearly, \( M_R = G_\sigma \) and \( N_R = \{ e \} \). Thus \( U = U_1 \) and the weight function \( w_{M,R}^{M_\sigma} \) is constantly equal to 1 on all of \( U \). Furthermore, \( v'_R = v'_M \) in this case. From the left \( \mathcal{M} \)-invariance of \( v'_M \), we have \( v'_M(ky) = v'_M(y) \) for \( k \in K^{G_\sigma} \) and \( y \in G \). The corresponding integral in (12.10) then reduces to

\[
v'_M(y) \int_{K^{G_\sigma}} \int_{U_1} \phi(\gamma^{-1} \sigma k^{-1} my) \, du \, dk = v'_M(y) J_{G_\sigma}^{G_\sigma}(\sigma \nu, \phi),
\]
where \( \phi^\eta(x) = \phi(y^{-1} x y) \). Note that the latter integral is

\[
\int_{G_y \setminus G_o} \phi_v(y^{-1} x^{-1} \sigma v x y) \, dx.
\]

Inserting this into the integral over \( y \in G_o \setminus G \) we obtain the expression (12.6).

(2) When \( \sigma = 1 \) so that \( \gamma = \nu \) is unipotent, the outer integral in (12.8) is trivial. Moreover, the function \( v'_R \) vanishes on \( K \) unless \( R = G \) when it is constantly equal to 1. From (12.10) we deduce that \( J_M^{M_R}(\nu, \Phi_{R, \epsilon}) = 0 \) unless \( R = G \), in which case (since \( U = U_1 \)) we obtain

\[
\int_K \int_{U_1} \phi(k^{-1} u k) w_{M, U_1}(u) \, du \, dk,
\]

recovering the previous expression (12.7) for \( J_M(\nu, \phi) \).

12.5. Reduction to local estimates. Now we return to the setting where \( S \) is an admissible set of places of \( F \), as in §12.1. We first recall that for factorizable test functions \( \phi_S = \otimes_{v \in S} \phi_v \) and \( \gamma \in M(F) \) one has a splitting formula which reduces \( J_M(\gamma, \phi_S) \) to a sum of products of local distributions. More precisely (see [35, Lemma 6.11] or [36, (10.3)]), there are real numbers \( \{d_{M_S}(L_S)\} \), indexed by Levi subgroups \( L_S = (L_v)_{v \in S} \in \mathcal{L}_S(M_S) \), such that

\[
J_M(\gamma, \phi_S) = \sum_{L_S \in \mathcal{L}_S(M_S)} d_{M_S}(L_S) \prod_{v \in S} J_{M_v}^{L_v}(\gamma_v, \phi_v(Q_v)).
\]

Here, we are using an assignment \( \mathcal{L}_v \ni L_v \mapsto Q_v \in \mathcal{P}_v(L_v) \) which is independent of \( S \), and for every \( v \in S \) the element \( \gamma_v \in M_v \) is taken to be \( M_v \)-conjugate to \( \gamma \). The properties of interest for us on the coefficients \( d_{M_S}(L_S) \) are the following, proved in [35, Lemma 6.11]:

1. as \( L_S \) varies, the coefficients \( d_{M_S}(L_S) \) can attain only a finite number of values; these values depend only on \( n \).
2. the number of contributing Levi subgroups \( L_S \) can be bounded as

\[
|\{L_S : d_{M_S}(L_S) \neq 0\}| \ll |S|^{n-1}.
\]
3. If \( d_{M_S}(L_S) \neq 0 \) then the natural map \( \bigoplus_{v \in S} \mathfrak{a}_{L_v}^{L_v} \rightarrow \mathfrak{c}_{M_v}^{G_v} \) is an isomorphism.

In particular, from the first two of these properties, it follows immediately that for any \( \sigma \in \mathcal{O}, \gamma \in \mathfrak{o} \), admissible \( S \), and factorisable \( \phi_S = \otimes_{v \in S} \phi_v \in \mathcal{H}(G_S) \) we have

\[
J_M(\gamma, \phi_S) \ll |S|^{n-1} \max_{L_S \in \mathcal{L}_S(M_S)} \prod_{v \in S} |J_{M_v}^{L_v}(\gamma_v, \phi_v(Q_v))|.
\]

Thus, if \( \sigma \in \mathcal{O}_R, f \in \mathcal{H}(G_{\infty R}^1) \), and \( S \) and \( \phi_S \) are taken as in (12.1) we obtain

\[
J_M(\gamma, \epsilon_{K_1(q)} \otimes f) \ll R^{n-1} \max_{L_S \in \mathcal{L}_S(M_S)} \prod_{v | \infty} |J_{M_v}^{L_v}(\gamma_v, f_v(Q_v))| \times \prod_{v \in S_q} |J_{M_v}^{L_v}(\gamma_v, \epsilon_{K_1(q)}^{(Q_v)})| \prod_{v \in S_q, v \notin S_q} |J_{M_v}^{L_v}(\gamma_v, 1_{K_1^{(Q_v)}})|,
\]

where we have used (12.2) as well as the fact (see, for example, [35, §7.5] or [55, Lemma 6.2]) that

\[1_{K_v}^{(Q_v)} = 1_{K_v^{(Q_v)}}.\]
13. Estimates on local weighted orbital integrals

It remains to bound the local weighted orbital integrals appearing in (12.11). In this section, we provide (or recall) such bounds at every place, and show how they suffice to establish Theorem 12.1.

For \( v \) dividing \( q \), we offer the following proposition, the proof of which is based heavily on works of Finis-Lapid [14], Matz [35], and Shin-Templier [55].

**Proposition 13.1.** There are constants \( B, C, \theta > 0 \) such that the following holds. Let \( v \) be a finite place. Let \( M \in \mathcal{L}_v, L \in \mathcal{L}_v(M) \), and \( Q \in \mathcal{P}_v(L) \). Then for any \( r \geq 0 \) and \( \gamma = \sigma \nu \in \sigma U_{G_0} \cap M \) and \( r \geq 0 \) we have

\[
J^L_M(\gamma; 1^{(Q)}_{K_1(v)}) \ll q_v^{aB-\theta r} D^L_v(\sigma)^{-C},
\]

where \( a = 0 \) whenever the residue characteristic of \( F_v \) is larger than \( n! \) and \( v \notin S_\emptyset \), and \( a = 1 \) otherwise.

A great deal of work has been recently done by Matz [35] and Matz-Templier [36] in bounding archimedean weighted orbital integral for \( \text{GL}_n \). Their bounds are almost sufficient for our purposes, except for the dependency in the support of \( R \). By simply explicating this dependence in their proofs, we obtain the following result.

**Proposition 13.2.** There are constants \( c, C > 0 \) satisfying the following property. Let \( v \mid \infty \). Let \( M \in \mathcal{L}_v, L \in \mathcal{L}_v(M) \), and \( Q \in \mathcal{P}_v(L) \). Then for any \( R > 0 \) and \( \gamma = \sigma \nu \in \sigma U_{G_0} \cap M \) and \( f \in \mathcal{H}(G_0^1) \) we have

\[
J^L_M(\gamma; f^{(Q)}) \ll e^{cR} D^L_v(\sigma)^{-C} \| f \|_\infty.
\]

13.1. Deduction of Theorem 12.1. We now show how the above results imply Theorem 12.1. We will need an additional result for places \( v \in S_\emptyset, v \notin S_q \) (as in the last factor of (12.11)). Namely, it is proved in [35, Corollary 10.13] that there are constants \( B, C > 0 \) such that for any finite place \( v \), and any \( M \in \mathcal{L}_v, L \in \mathcal{L}_v(M) \), and \( \gamma \in L \), one has

(13.1)

\[
J^L_M(\gamma; 1^{K^L_q}) \ll q_v^B D^L_v(\sigma)^{-C}.
\]

Returning to the global situation of Theorem 12.1, we let \( \emptyset \in \mathcal{O} \) be such that \( \emptyset \cap M(F) \) is non-empty, and let \( \sigma \in M(F) \) be a semisimple element representing \( \emptyset \). We may assume that \( \emptyset \in \mathcal{O}_R \), for otherwise \( J^L_M(\gamma; \emptyset, f) = 0 \). We apply (12.11) to reduce to a product of local factors. Then, using Proposition 13.1 (at finite places \( v \in S_q \), display (13.1) (at finite places \( v \in S_\emptyset, v \notin S_q \), and Proposition 13.2 (at \( v \in S_\infty \)), we deduce that for \( \gamma = \sigma \nu \in \sigma U_\emptyset(F) \cap M(F) \):

\[
J^L_M(\gamma; \emptyset, f) \ll e^{cR} q_v^{-1} D^L_v(\sigma)^{-C} \max_{L \in \mathcal{L}(M)} \prod_{v \in S_\emptyset \cup S_\infty} D^L_v(\sigma)^{-C_v}.
\]

Here we have incorporated the \( (\prod_{\text{char}(F_v) \leq n} q_v)^{B} \) into the implied constant, which is allowed to depend on the number field \( F \) and \( n \). We may furthermore apply Lemma 12.3 to absorb \( q_v^{B} \) into the exponential factor \( e^{cR} \) (at the cost of a larger value of \( c \)).

To treat the product of Weyl discriminants, we argue as in the proof of Lemma 12.3. Recall the definition of \( D^L_v(\sigma) \) in (4.1). We first note that for \( \emptyset \in \mathcal{O}_R \), represented by a semisimple element \( \sigma \in M(F) \), we have \( D^L_v(\sigma) \leq 1 \) for finite \( v \) and \( D^L_v(\sigma) \ll e^{cR} \) for archimedean \( v \). We may therefore replace \( D^L_v(\sigma) \) by \( D_v(\sigma) = D^G_v(\sigma) \) in the statements of Propositions 13.1 and 13.2, as well as in display (13.1). Moreover, since \( D_v(\sigma) \leq 1 \) for every \( v < \infty \), we may increase the value of \( C \) in Proposition 13.1 and display (13.1) at the cost of a worse bound. Let \( C_v \) denote the value of \( C \) at each place \( v \in S_\emptyset \cup S_\infty \) and put \( C = \max_{v \in S_\emptyset \cup S_\infty} C_v \). An application of the product formula yields

\[
\prod_{v \in S_\emptyset \cup S_\infty} D_v(\sigma)^{-C_v} \leq \prod_{v \in S_\emptyset} D_v(\sigma)^{-C} \prod_{v \in S_\infty} D_v(\sigma)^{-C_v} = \prod_{v \in S_\infty} D_v(\sigma)^{C-C_v}.
\]
Since \( C - C_v \geq 0 \) we deduce from (12.3) that \( \prod_{v \in S_\nu \cup S_\infty} D_v(\sigma)^{-C_v} \ll e^{cR} \), which completes the proof of Theorem 12.1.

### 13.2. Proof of Proposition 13.1

In this paragraph we let \( v \) denote a finite place. Where possible, we will drop the subscript \( v \). So, for example, \( G = G_v, G_\sigma = G_{\sigma, v}, U_{G, \sigma} = U_{G, \sigma}(F_v), K = K_v, p = p_v, \varpi = \varpi_v, K_1(p^r) = K_{1, v}(p^r), q = q_v, \) and \( D(\sigma) = D_v(\sigma) \).

The basic idea of the proof of Proposition 13.1 is to show that the semisimple conjugacy class \( \sigma \) has small intersection with \( K_1(p^r) \). One has to do this in the framework of the definition of the general weighted orbital integrals (12.8), which involve various weight functions. We shall divide the proof into three steps as follows:

**Step 1.** Reduce to the case that \( L = G \). We do this by showing that whenever \( L \) is a proper Levi subgroup of \( G \) we can get savings in the level by means of the constant term alone.

**Step 2.** Reduce to the case of \( M = G \) and \( \gamma \) semisimple non-central. This involves bounding the parenthetical expression in (12.8), as a function of \( y \in G_{\sigma, G}, \gamma, \) and the level \( p^r \).

- If \( \gamma \) is not semisimple, then for every \( R \in \mathcal{F}^{G, \sigma}(M_v) \) we get savings in the level for the weighted unipotent integrals \( J^{M_v}_M(\nu, \Phi_{R, y}) \) of (12.10) by bounding the intersection of unipotent conjugacy classes (in the centralizer of \( \sigma \)) with congruence subgroups (which depend on \( y \)).

- If \( \gamma \) is semisimple non-central, but \( M \neq G \), then the same argument as above applies to all terms except the one associated with \( R = M_v \), since in that case the unipotent integral collapses and one has simply \( J^{M_v}_M(\nu, \Phi_{M_v, y}) = 1_{K_1(p^r)}(y^{-1}\sigma y)v'_M(y) \).

**Step 3.** Bound the invariant orbital integrals of \( 1_{K_1(p^r)} \) for semisimple non-central \( \gamma \).

In all cases, the central ingredient to bounding intersections of conjugacy classes with open compact subgroups is the powerful work of Finis-Lapid [14]. We shall give a brief overview of their results in Section 13.2.1 below.

It is instructive to examine the division into Steps 2 and 3 in the case where \( G_\gamma \subset M \), as the notation greatly simplifies under this assumption. As usual, let \( \gamma = \sigma \nu \in \sigma U_{G, \sigma} \cap M \). Let \( \mathcal{V} \subset U_{G, \sigma} \) be the \( G_\sigma \)-conjugacy class of \( \nu \) in \( G_\sigma \), endowed with the natural measure. Then we are to estimate the integral

\[
J^G_M(\gamma, 1_{K_1(p^r)}) = \int_{G_\sigma \setminus G} \text{vol}_{y^{-1}\sigma \mathcal{V} y}(y^{-1}\sigma \mathcal{V} y \cap K_1(p^r))v'_M(y) \, dy.
\]

We proceed differently according to whether \( \nu \) is trivial or not.

- If \( \nu \) is trivial, then the inner \( y^{-1}\sigma \mathcal{V} y \)-volume is just \( 1_{K_1(p^r)}(y^{-1}\sigma y) \). Thus Step 2 is vacuous in this case, and Step 3 then bounds

\[
\int_{G_\sigma \setminus G} 1_{K_1(p^r)}(y^{-1}\sigma y)v'_M(y) \, dy
\]

by estimating the intersection volume of the conjugacy class of \( \sigma \) with the congruence subgroup \( K_1(p^r) \).

- If \( \nu \) is non-trivial, then \( \mathcal{V} \) is of positive dimension and Step 2 bounds the inner \( y^{-1}\sigma \mathcal{V} y \)-volume by a quantity which is roughly of the form \( q^{-r}1_{B(t)}(y \sigma y^{-1}) \). Here, for a real parameter \( t > 0 \), we have denoted by \( 1_{B(t)} \) the characteristic function of the ball \( B(t) \) of radius \( t \) about the origin, and \( t_\sigma \) roughly of size \( D(\sigma)^{-C} \). One may then estimate the volume of \( G_\sigma \setminus B(t) \) (a compact piece of the tube of radius \( t \) about \( G_\sigma \)) by appealing to the work of Shin-Templier [55].

In either case, the weight function \( v'_M \) is easy to control, as it grows by a power of log with the norm of \( y \), using results from [35].
The more general case, when \( G_\gamma \) is not necessarily contained in \( M \), is complicated by the presence of the various terms parametrized by \( R \in F^{G_\gamma}(M_\sigma) \) in (12.8). For example, whenever \( R \neq G_\sigma \) these terms include the unipotent weight functions \( u^{G_\sigma}_{\psi, L} \), which must be dealt with. Nevertheless the case \( G_\gamma \subset M \) described above already contains most of the difficulties.

13.2.1. The work of Finis-Lapid. It will be convenient to use the results of [14]. We now recall their notation (specialized to our setting) and describe two of their main theorems. As we will sometimes need the global group \( G \) alongside the local group \( G_v \), where \( v \) is a finite place, we return in this subsection to the notational convention of the rest of the paper, and restore all subscripts.

For \( r \geq 0 \) let \( K_v(p^r_v) = \{ k \in K_v : k \equiv 1 \text{ (mod } p^r_v) \} \) be the principal congruence subgroup of level exponent \( r \). Let \( g = M_n(F_v) \) be the Lie algebra of \( G \) and write \( \Lambda = M_n(O_v) \). Following [14, Definition 5.2] for \( \gamma \in G_v \) we put

\[
\lambda_v(\gamma) = \max\{ r \in \mathbb{Z} \cup \infty : (\text{Ad}(\gamma) - 1)\Lambda \subset \varpi_v^r(\Lambda) \}.
\]

In other words, if we make the identification \( \text{GL}(g) = \text{GL}_{n2} \), then \( \lambda_v(\gamma) \) is the maximal \( r \in \mathbb{Z} \cup \infty \) such that \( \text{Ad}(\gamma) \) lies in the principal congruence subgroup of \( \text{GL}_{n2}(O_v) \) of level exponent \( r \), c.f. [14, Remark 5.23]. The function \( \lambda_v \) on \( G_v \) descends to one on \( \text{PGL}_n(F_v) \), and one has \( \lambda_v(\gamma) \geq 0 \) whenever \( \gamma \in K_v \).

For a twisted Levi subgroup \( H \) recall that \( K_v^H = H_v \cap K_v \). Then \( K_v^H(p^r_v) = K_v(p^r_v) \cap K_v^H \) is the principal congruence subgroup \( K_v^H \) of level exponent \( r \). We define the level exponent of an arbitrary open compact subgroup \( K \) of \( K_v^H \) as the smallest non-negative integer \( f \) such that \( K_v^H(p^f_v) \) is contained in \( K \). For example, the level exponent of \( K_{1,v}(p^r_v) \) is \( r \).

We shall also need the following result, which can be deduced from Propositions 5.10 and 5.11 of [14]. See [6] for more details on this deduction.

**Proposition 13.3** (Finis-Lapid [14]). For every \( \epsilon > 0 \) small enough there is \( \theta > 0 \) such that the following holds. Let \( r \) be a non-negative integer, \( v \) a finite place of \( F \), \( H \) a twisted Levi subgroup of \( G \), and \( x \in K_v^H \). If \( \lambda_v(x) < \epsilon r \) then for any open compact subgroup \( K \) of \( K_v^H \) of level exponent \( r \) we have

\[
\mu_{K_v}(k \in K_v^H : k^{-1} x k \in K) \ll q_v^{-\theta r}.
\]

We shall also need the following result, which can be deduced from [14, Lemma 5.7]; see also the proof of Corollary 5.28 in loc. cit.

**Proposition 13.4** (Finis-Lapid [14]). Let \( H \) be a twisted Levi subgroup of \( G \). Let \( P \) be a proper parabolic subgroup of \( H \), with unipotent radical \( U \). Let \( v \) be a finite place of \( F \). Then

\[
\text{vol}\{ u \in U_v \cap K_v^H : \lambda_v(u) \geq m \} \ll q_v^{-m},
\]

uniformly in \( v \).

Finally we remark that in [6, Lemma 6.10] it is shown that for semisimple \( \sigma \in K_v \) we have

\[
q_v^{\lambda_v(\sigma)} \ll D_v(\sigma)^{-1}.
\]

This inequality will be occasionally used to convert from large values of \( \lambda_v(\sigma) \) to large values of \(- \log q_v D_v(\sigma) \).

13.2.2. Reduction to \( L = G \). We return to the purely local setting of Proposition 13.1 and drop all subscripts \( v \) where possible, as explained in the opening paragraph of §13.2.

We begin by reducing the proof of Proposition 13.1 to the case \( L = G \). The first step in this reduction is to estimate the constant terms of the functions \( 1_{K_1(p^r)} \), uniformly in \( r \) and \( \gamma \).

**Proposition 13.5.** There is \( \theta > 0 \) such that the following holds. Let \( M \in \mathcal{L}_v \), \( M \neq G \), and \( P \in \mathcal{P}_v(M) \). Then for \( \gamma \in M \), and \( r \geq 0 \) we have

\[
1_{K_1(p^r)}(\gamma) \ll q^{-\theta r} 1_{K^M}(\gamma).
\]
implies that \( f^{(P)} \leq g^{(P)} \). From this we deduce that \( 1_{K_1(p')}^{(P)}(\gamma) = 0 \) unless \( \gamma \in K^M \). Henceforth we may and will assume that \( \gamma \in K^M \); note that \( \delta_p = 1 \) on \( K^M \).

Recall the definition of the constant term map in (12.5). Note that if \( u \in U \) is such that \( k^{-1} \gamma u k \in K \) then \( u \in U \cap K \). Fixing \( u \in U \cap K \) the inner integral is

\[
\mu_G(k \in K : k^{-1} \gamma u k \in K_1(p')).
\]

From Proposition 13.3, for every \( \epsilon > 0 \) small enough there is \( \theta > 0 \) such that

\[
1_{K_1(p')}^{(P)}(\gamma) \ll \text{vol}\{u \in U \cap K : \lambda(\gamma u) > \epsilon r\} + q^{\theta r} \text{vol}\{u \in U \cap K : \lambda(\gamma u) \leq \epsilon r\}.
\]

We apply the trivial bound \( \text{vol}(U \cap K) = 1 \) to the latter volume. To deal with the former, we note that \( \lambda(\gamma u) \leq \lambda(u) \) for \( \gamma \in K^M \) and \( u \in U \cap K \) and then apply Proposition 13.4 to finish the proof.

We now prove Proposition 13.1 in the the case that \( L \neq G \). Let \( \gamma = \sigma \nu \in \sigma \mathcal{U}_{G_\sigma} \cap M \). From Proposition 13.5 we deduce

\[
J^L_M(\gamma, 1^{(Q)}_{K_1,v(p')}) \ll q^{b B - \theta r} J^L_M(\gamma, 1^{K_L}),
\]

where \( b = 0 \) or 1 according to whether the residual characteristic of \( F_v \) is \( > n! \) or not, and \( J^L_M \) denotes the weighted orbital integral \( J^L_M \) but with absolute values around the weight functions in (12.10). If \( v \in S_\rho \) we apply (13.1) to the latter integral (which is valid with \( J^G_M(\gamma) \) replaced by \( J^G_M(\gamma) \)). Otherwise, if the finite place \( v \) is not in \( S_\rho \), then it follows from [35, Lemma 10.12] (which, again, is valid with \( J^G_M(\gamma) \) replaced by \( J^G_M(\gamma) \)) and the identity \( J^L_M(\sigma, 1^{K_L}) = 1 \) (for semisimple \( \sigma \)) that

\[
J^L_M(\gamma, 1^{K_L}) \ll q^{b B},
\]

with the same convention on \( b \) as before. This yields the desired estimate in both cases.

13.2.3. Bounding the weighted unipotent orbital integrals on descent functions. We shall now bound the parenthetical expression in (12.8), with \( \phi \) the characteristic function of \( K_1(p') \). Before doing so, we shall need to introduce slightly more notation.

If \( y = k_1 \text{diag}(\varpi^{m_1}, \ldots, \varpi^{m_n}) k_2 \in G \), where \( k_1, k_2 \in K \) and \( m_1 \geq \cdots \geq m_n \) are integers, then we write

\[
\|y\| = q^{\max\{|m_1|, |m_n|\}}.
\]

For \( t > 0 \) we write \( B(t) = \{g \in G : \|g\| \leq t\} \) for the ball of radius \( t \) about the origin in \( G \). Then \( 1_{B(t)} \) is the characteristic function of \( B(t) \).

Lemma 13.6. There are constants \( B, C, \theta > 0 \) such that the following holds. Let \( \gamma = \sigma \nu \in \sigma \mathcal{U}_{G_\sigma} \cap M \) be non-central. Let \( b = 0 \) or 1 according to whether the residual characteristic of \( F_v \) is \( > n! \) or not, and put \( t_\sigma = D(\sigma)^{-C} q^{b B} \). Let \( r \geq 0 \) be an integer. Then there is a set of representatives \( y \in G_\sigma \setminus G \) such that the expression

\[
\sum_{R \in \mathcal{F}_{G_\sigma}(M_\sigma)} J^M_{M_\sigma}(\nu, \Phi_{R,y}),
\]

where the descent functions \( \Phi_{R,y} \) are associated with \( 1^{K_1(p')} \), is majorized by

\[
(1 + \log t_\sigma)^{n-1} 1^{K_1(p')} \left( y^{-1} \sigma y \right) + q^{b B - \theta r} D(\sigma)^{-C} 1_{B(t_\sigma)}(y^{-1} \sigma y).
\]
Proof. Let \((y, u, k) \in G \times U_{G_\sigma} \times K_{G_\sigma}\) be such that \(y^{-1}\sigma k^{-1}uky \in K\). From [35, Corollary 8.4], there are constants \(B, C > 0\), and for a triplet as above there is \(g \in G_\sigma\) (which can be taken to be independent of \(u\)) such that

\[
\|gy\| \leq D(\sigma)^{-C} q^B,
\]

(13.3)

\[
\|gug^{-1}\| \leq D(\sigma)^{-C} q^B,
\]

(13.4)

where the convention on \(b\) is as in the lemma. Henceforth we take a set of representatives \(y \in G_\sigma \setminus G\) whose norm is bounded by the right-hand side of (13.3), in which case it can be assumed that the norm of \(u\) is bounded by the right-hand side of (13.4).

We furthermore recall the bound on the weight function \(|v'_Q(x)| \ll (1 + \log \|x\|)^{n-1}\) established in [35, Corollary 10.9], valid for any parabolic \(Q \in \mathcal{F}(M)\). Thus, using (12.9), we deduce that for any \(y\) as above and any \(k \in K_{G_\sigma}\) we have

\[
|v'_R(ky)| \ll \left(1 + \log D(\sigma)^{-C} q^B\right)^{n-1}.
\]

In particular, if \(v = 1\) and \(R = M_\sigma\) then we may go ahead and bound the integral \(J_{M_\sigma}^M(1, \Phi_{R,y})\) appearing in (12.10). (We are of course taking the descent function \(\Phi_{R,y}\) to be associated with \(1_{K_1(p',\gamma)}\) Indeed, the unipotent subgroup \(U\) of that formula is reduced to the identity in this case, so that the \(U\) integral collapses and one has

\[
J_{M_\sigma}^M(1, \Phi_{M_\sigma,y}) = 1_{K_1(p')}(y^{-1}\sigma y)v'_{M_\sigma}(y).
\]

Using the above bound on the weight factor, we obtain the first term of the majorization of the lemma.

Next, for \(\rho \in \mathbb{R}\) let \(\text{Den}(\rho)\) denote the set of matrices in \(M_n(F_v)\) all of whose coefficients have valuation at least \(-\rho\). Note that if \(g \in G\) is such that \(\|g\| \leq q^\rho\) then \(g \in \text{Den}(\rho)\); indeed it suffices to establish this for diagonal elements in the positive chamber, where it is immediate. Thus, for \(u\) as above, we have \(u \in U \cap \text{Den}(\rho_\sigma)\), where \(\rho_\sigma = bB - C\log_q D(\sigma)\).

We return to estimation of the integral \(J_{M_\sigma}^M(\nu, \Phi_{R,y})\), this time in the case where either \(\gamma\) is not semisimple or \(M \neq G\). In either of these cases, the \(U\) appearing in (12.10) satisfies \(\dim U \geq 1\). Again, the descent function \(\Phi_{R,y}\) is taken to be associated with \(1_{K_1(p')}\). From the above discussion we deduce that \(J_{M_\sigma}^M(\nu, \Phi_{R,y})\) is majorized by

\[
(1 + \log D(\sigma)^{-C} q^B)^{n-1} \int_{U \cap \text{Den}(\rho_\sigma)} |w_{M_\sigma,U}^M(u)| \int_{K_{G_\sigma}} 1_{K_1(p')}(y^{-1}\sigma k^{-1}uky) \, dk \, du.
\]

(13.5)

For \(y \in G\) and \(r \geq 0\) let us put \(K^\sigma(y, r) = yK_1(p')y^{-1} \cap G_\sigma\). In the special case when \(r = 0\) we shall simply write \(K^\sigma(y) = K^\sigma(y, 0)\). With this notation, the inner integral in (13.5) is \(\int_{K_{G_\sigma}} 1_{K^\sigma(y,r)}(k^{-1}\sigma vk) \, dk\). After an application of Cauchy-Schwarz, we see that the double integral in (13.5) is bounded by

\[
\left(\int_{K_{G_\sigma}} \int_U 1_{K^\sigma(y,r)}(k^{-1}\sigma vk) \, du \, dk\right)^{1/2} \left(\int_{U \cap \text{Den}(\rho_\sigma)} |w_{M_\sigma,U}^M(u)|^2 \, du\right)^{1/2}.
\]

(13.6)

Using [35, Lemma 10.5], we see that the second factor in (13.6) is \(O(D(\sigma)^{-C} q^B)\), with the same convention on \(b\). (The aforementioned result in fact bounds the integral \(\int_{U \cap \text{Den}(\rho_\sigma)} |w_{M_\sigma,U}^M(u)| \, du\), but the same proof applies with \(|w_{M_\sigma,U}^M(u)|^2\) as integrand, simply by replacing the polynomial \(q\) in [35, Lemma 10.4] by its square.)
Next, we treat the first factor in (13.6). We follow closely the presentation in [14, Corollary 5.28], explicating several small differences. We first write the double integral as

\[
\int_{K^G} \text{vol}_U(U \cap \sigma^{-1} K^\sigma(y, r)) \, dk.
\]

We may suppose that \(U \cap \sigma^{-1} K^\sigma(y, r)\) is non-empty, in which case, fixing any \(u_0\) in this intersection, we have \(U \cap \sigma^{-1} K^\sigma(y, r) = u_0(U \cap K^\sigma(y, r))\). By invariance of the Haar measure on \(U\) we obtain in this case

\[
\text{vol}_U(U \cap \sigma^{-1} K^\sigma(y, r)) = \text{vol}_U(U \cap K^\sigma(y, r)).
\]

We claim that we can reduce to the case where \(K^\sigma(y, r)\) is replaced by \(K^\sigma(y, r) \cap K^G = yK_1(p^r)yz^{-1} \cap K^G\). Indeed, this double integral is

\[
\int_{K^G} \text{vol}_U(U \cap K^\sigma(y, r)) \, dk = \int_{K^G} i(k) \text{vol}_U(U \cap K^\sigma(y, r) \cap K^G) \, dk,
\]

where

\[
i(k) = [U \cap K^\sigma(y, r) : U \cap K^\sigma(y, r) \cap K^G] \leq [K^\sigma(y, r) : K^\sigma(y, r) \cap K^G] = [K^\sigma(y, r) : K^\sigma(y, r) \cap K^G].
\]

Therefore the expression in (13.7) bounded by

\[
[K^\sigma(y) : K^\sigma(y) \cap K^G] \int_U \int_{K^G} 1_{yK_1(p^r)yz^{-1} \cap K^G} (k^{-1}uk) \, dk \, du.
\]

Continuing, it now follows from (13.3) that

\[
[K^\sigma(y) : K^\sigma(y) \cap K^G] \ll D(\sigma)^{-Cq^{bB}}.
\]

It remains to bound the double integral in (13.8).

We first note that the level exponent of \(yK_1(p^r)yz^{-1} \cap K^G\) is at least \(r\). That is to say, \(yK_1(p^r)yz^{-1} \cap K^G\) cannot contain \(K^G(p^{r-1})\). This follows, for example, from the fact that the central element \(1 + \varpi^r\) lies in \(K^G(p^{r-1})\) but not in \(y^{-1}K_1(p^r)y \cap K^G\). In light of this, we may apply Proposition 13.3, with \(H = G_\sigma\), to find \(\theta, \epsilon > 0\) such that

\[
\int_{K^G} 1_{yK_1(p^r)yz^{-1} \cap K^G} (k^{-1}uk) \, dk \ll q^{-\theta r}
\]

whenever \(\lambda(u) < \epsilon r\). The double integral in (13.8) is therefore bounded by

\[
\text{vol}\{u \in U \cap K^G : \lambda(u) > \epsilon r\} + q^{-\theta r} \text{vol}\{u \in U \cap K^G : \lambda(u) \leq \epsilon r\},
\]

We bound the second volume factor trivially by \(\text{vol}\{u \in U \cap K^G\} = 1\). Finally, an application of Proposition 13.4 (with \(H = G_\sigma\)) shows that the first volume factor is majorized by \(q^{-\theta r}\), finishing the proof.

13.2.4. Invariant orbital integrals. In view of Lemma 13.6, it now remains to establish good bounds on the invariant orbital integrals of \(1_{K_1(p^r)}\) and \(1_{B(t_\sigma)}\). It suffices to estimate the unnormalized orbital integral

\[
O_\sigma(\phi) = \int_{G_\sigma \backslash G} \phi(x^{-1}\gamma x) d\mu_\sigma(x),
\]

since \(J_\sigma^G(\sigma, \phi) = D(\sigma)^{1/2} O_\sigma(\phi)\).

We first handle the invariant orbital integral for \(1_{B(t_\sigma)}\). If \(\sigma \not\in S_\sigma\) and the residue characteristic of \(F_v\) is \(> n!\) then \(t_\sigma \ll 1\), and we may apply [55, Theorem A.1] to deduce that \(O_\sigma(1_{B(t_\sigma)}) \ll 1\) in this case. If either the residue characteristic of \(F_v\) is \(\leq n\) or \(\gamma \in S_\sigma\) then we proceed as follows.
For every \( \lambda \in X^+_\infty(T_0) \) let \( \tau(\lambda) \) denote the associated Hecke operator, namely the characteristic function of \( K\lambda(\infty)K \). Then
\[
O_{\sigma}(1_{B(t_\sigma)}) = \sum_{\lambda \in X^+_\infty(T_0) \text{ or } \|\lambda\| \leq \log_q t_\sigma} O_{\sigma}(\tau_\lambda).
\]
Note that there are only \( O(\log^n t_\sigma) \) cocharacters \( \lambda \) satisfying the bound in the sum. For each of the above orbital integrals, it follows from [55, Theorem 7.3] (see also [55, Theorem B.1] for a stronger result) that there are constants \( B, C \geq 0 \) such that \( O_{\sigma}(\tau_\lambda) \ll t_\sigma^BD(\sigma)^{-C} \). Inserting this into the above expression (and recalling the definition of \( t_\sigma \) from Lemma 13.6) we deduce the bound
\[
O_{\sigma}(1_{B(t_\sigma)}) \ll q^{BD}(\sigma)^{-C}
\]
in this case. We conclude that in all cases we have
\[
O_{\sigma}(1_{B(t_\sigma)}) \ll q^{BD}(\sigma)^{-C},
\]
where \( b = 0 \) or \( 1 \) according to whether the residue characteristic of \( F_v \) is \( > n! \) or not.

It remains then to estimate the invariant orbital integral \( O_{\sigma}(1_{K_1(p^r)}) \) uniformly in the level \( p^r \) and the semisimple element \( \sigma \). We accomplish this in the next lemma; our presentation follows closely that of [6, Proposition 5.3].

**Lemma 13.7.** There are constants \( B, C, \theta > 0 \) such that the following holds. Let \( v \) be a finite place. For any \( r \geq 0 \) and semisimple \( \sigma \in G, \sigma \notin Z \), we have
\[
O_{\sigma}(1_{K_1(p^r)}) \ll q^{\theta}D(\sigma)^{-C},
\]
where \( a = 1 \) or \( 0 \) according to whether \( v \in S_\sigma \) or not.

**Proof.** Letting \( C_{\sigma,G} \) denote the conjugacy class of \( \sigma \), we have
\[
O_{\sigma}(1_{K_1(p^r)}) = \mu_{\sigma}(C_{\sigma,G} \cap K_1(p^r)).
\]
Now \( C_{\sigma,G} \) is closed since \( \sigma \) is semisimple. The compact set \( C_{\sigma,G} \cap K_1(p^r) \) is then a disjoint union of finitely many (open) \( K \)-conjugacy classes \( C_{x_i,K} \) meeting \( K_1(p^r) \). This gives
\[
O_{\sigma}(1_{K_1(p^r)}) = \sum_{i=1}^{t} \frac{\mu_{\sigma}(C_{x_i,K} \cap K_1(p^r))}{\mu_{\sigma}(C_{x_i,K})} \mu_{\sigma}(C_{x_i,K}).
\]
From the definition of the quotient measure, for any \( x \in K \), we have
\[
\mu_{\sigma}(C_{x,K} \cap K_1(p^r)) = \frac{\mu_G(k \in K : k^{-1}xk \in K_1(p^r))}{\mu_G(G_k \cap K)}.
\]
Using (13.2), we may deduce from Proposition 13.3 that if \( x \in K \) is semisimple and non-central, and \( D(x) \gg q^{-(1-\epsilon)} \) for some \( \epsilon > 0 \), then
\[
\mu_G(k \in K : k^{-1}xk \in K_1(p^r)) \ll q^{-(1-\epsilon)r}.
\]
Thus for every \( \epsilon > 0 \) there is \( \theta > 0 \) such that if \( D(\sigma) = D(x_i) \gg q^{-\epsilon} \) then
\[
\frac{\mu_{\sigma}(C_{x,K} \cap K_1(p^r))}{\mu_{\sigma}(C_{x,K})} = \mu_G(k \in K : k^{-1}xk \in K_1(p^r)) \ll q^{-\theta r}.
\]
In this case we obtain
\[
O_{\sigma}(1_K) \ll q^{-\theta r} \sum_{i=1}^{t} \mu_{\sigma}(C_{x_i,K}) = q^{-\theta r}O_{\sigma}(1_K),
\]
since
\[
\sum_{i=1}^{t} \mu_{\sigma}(C_{x_i,K}) = \sum_{i=1}^{t} \mu_{\sigma}(C_{x_i,G} \cap K) = \mu_{\sigma}(C_{\sigma} \cap K) = O_{\sigma}(1_K).
\]
If, on the other hand, $D(\sigma) \ll q^{-pr}$ (so that $1 \ll q^{-\theta r}D(\sigma)^{-\theta/\epsilon}$) then we may apply the trivial bound $O_\sigma(1_{K_1(p^r)}) \leq O_\sigma(1_K)$ to obtain

$$O_\sigma(1_{K_1(p^r)}) \ll q^{-\theta r}D(\sigma)^{-\theta/\epsilon}O_\sigma(1_K).$$

If $v \notin S_\eta$ then $O_\sigma(1_K) = 1$. If $v \in S_\eta$ we apply the bound $O_\sigma(1_K) \ll q^BD(\sigma)^{-C}$ of [55, Theorems 7.3 and B.1]. This proves the desired estimate in either case. □

13.3. Proof of Proposition 13.2. The statement of Proposition 13.2, without the explicated dependency in $R$, follows from the proof of [36, Theorem 1.8] (in the case of $v$ real) and [35] (in the case of $v$ complex). To prove Proposition 13.2 it therefore suffices to explicate the dependence in $R$ in these works. For simplicity, we shall concentrate on the real case here. Once again, we drop all $v$ subscripts from the notation, so in particular $G = G(R)$.

It suffices to take $L = G$, since on one hand the constant term map $f \mapsto f^{(Q)}$ takes $C_c^\infty(G^1)_{\epsilon R}$ to $C_c^\infty(L^1)_{\epsilon R}$ (see, for example, [35, Lemma 7.1 (iii)]), and on the other the factor $\delta_{Q}^{1/2}$ is bounded by $O(e^{cR})$ on $G^1_{\leq R} \cap L$. Here $c, c' > 0$ are constants depending only on $n$.

We would like to explicate the dependency in $C(f_1)$ in [36, Theorem 1.8] on both $\|f_1\|_{\infty}$ and the support of $f_1$. (Note that, since we do not seek any savings in the spectral parameter, our interest is in $\eta = 0$.) It is clearly enough to bound the modified weighted orbital integral $J_G^\sigma(\gamma, f)$, where the weight functions are replaced by their absolute values. The dependency on $\|f_1\|_{\infty}$ is easy enough to explicate, for in the proof of [36, Theorem 1.8] (see Propositions 6.6, 7.5, and 7.6 of loc. cit.), one replaces the function $f_1$ with a majorizer of the characteristic function of its support. We can thus assume that $f$ is the characteristic function of $G^1_{\leq R}$.

We now supplement a few of the lemmas and propositions leading up to the proof of [36, Theorem 1.8], pointing out how the dependency in $R$ can be explicated.

- The constants $c$ and $C$ in [36, Lemma 6.3] can be taken to be of the form $e^{cR}$ for some $\kappa = \kappa(n) > 0$. To see this, first note that the constants in Lemmas 3.6 and 3.7 depend only on $n$. It can then readily be seen that each of the constants $a_i$ in the proof of Lemma 6.3 can be taken to be of the form $e^{cR}$, for $\kappa_i = \kappa_i(n)$. (For example, $a_1 = ce^{c_1R}$, where $c = c(n) > 0$ and $c_1 = c_1(n) > 0$ are given in Lemma 3.6.)

- This then implies that the constant $c_1$ in [36, Lemma 6.8] can be taken to be of the form $e^{cR}$, for some $\kappa = \kappa(n) > 0$. (As the authors point out just before §6.6, one can take $c_2 = 2n$ and $c_3 = 1$ in Lemma 6.8.) Their proof divides into two subcases, according to whether $\dim U_2 \geq 1$ or not.
  - If $\dim U_2 \geq 1$, then their integral $\int_{U_2} B_{b_2}^{M_2}(Y)\, dY$ is bounded by a polynomial expression in $r(\gamma_s)$, the latter quantity being logarithmic in $c_1$. Furthermore, their integral $\int_{U_2} B_{b_2}^{M_2}(Y)\, dY$ is bounded by a polynomial expression in $R(\gamma_s)$, the latter quantity having a linear dependence in $c_1$. 
  - If $\dim U_2 = 0$, then the integral $\int_{U_2} B_{b_2}^{M_2}(Y)\, dY$ is bounded polynomially in $r(\gamma_s)$, thus logarithmically in $c_1$.

- To explicate the dependence in $R$ in the proof of [36, Proposition 6.6], one then applies Lemma 6.8 in the way we have just explicated, and Lemma 6.9 with $s = e^{cR}$.

- We now consider the proof of [36, Proposition 6.10]. One is led to consider integrals of the form $\int_{V} 1_{G^1_{\leq R}}(v) \log |p(v)|^k\, dv$, where $V$ is the unipotent radical of a proper parabolic of $G$, $k \geq 1$ is an integer, and $p : V \to \mathbb{R}$ is a polynomial function on the coordinates. (Once again, recall that $\eta = 0$.) This can be bounded by $aR^b \int_{V} 1_{G^1_{\leq R}}(v)\, dv$, where $a > 0$ and $b \geq 0$ depend on $p$, $k$, and $n$. The latter integral is $O(e^{cR})$. 

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14. Construction of test functions

In this section, we construct (in certain cases) explicit realizations of the test functions \( f_R^{\delta,\mu} \in C_c^\infty(G_\infty)_R \), and provide bounds on these in terms of their spectral transforms. These bounds will be important in the estimation of the associated orbital integrals.

**Proposition 14.1.** Let \( n \geq 1 \). Let \( \delta \in \mathcal{D} \) be represented by \((\delta, M)\), where \( M \in \mathcal{L}_{\text{st}, \infty} \). Let \( \mu \in \mathfrak{h}_M^* \) and \( R > 0 \). Suppose that either

1. \( n \leq 2 \), or
2. \( n \geq 1 \) is arbitrary, but \( M = T_0 \) is the diagonal torus and \( \delta \) is the trivial character of \( T_0^1 \).

Let \( h_R^{\delta,\mu} \) be the spectral localizer as defined in Section 8. Let \( \tau(\pi_\delta) \in \Pi(K_\infty) \) be the minimal \( K_\infty \)-type of \( \pi_{\delta,\mu} \). Write \( \Pi_{\tau(\pi_\delta)} \) for the orthogonal projection onto the \( \tau(\pi_\delta) \)-isotypic component of \( \pi_{\delta,\mu} \).

Then there exists \( f_R^{\delta,\mu} \in \mathcal{H}(G_\infty)_R \) such that

\[
\begin{align*}
\pi_{\sigma,\lambda}(f_R^{\delta,\mu}) = \begin{cases} 
    h_R^{\delta,\mu} (\lambda) \cdot \frac{1}{\dim \tau(\pi_\delta)} \Pi_{\tau(\pi_\delta)} & \text{if } \sigma = \delta; \\
    0 & \text{else},
\end{cases}
\end{align*}
\]

and

\[
\|f_R^{\delta,\mu}\|_\infty \leq 2 \|h_R^{\delta,\mu}\|_{L^1(\mu_{\infty}^\text{pl})}.
\]

**Remark 6.** It follows from (14.1) that \( \text{tr} \pi_{\sigma,\lambda}(f_R^{\delta,\mu}) = h_R^{\delta,\mu}(\lambda) \) when \( \sigma = \delta \), and is 0 otherwise.

**Remark 7.** The existence of such a test function \( f_R^{\delta,\mu} \) satisfying (14.1) is a consequence of the Paley–Wiener theorem of Clozel–Delorme, as stated in Section 4.12, which is of course valid without assuming either (1) or (2). We were not able to extract the bound (14.2) from the proof of Clozel–Delorme. We have therefore restricted ourselves to the cases in (1) or (2), where we have an explicit inversion map.

14.1. Reduction to a fixed archimedean place. Let \( G_\infty^1 = G_\infty \cap G_\infty^1 \). Note the decomposition

\[
G_\infty^1 = \mathcal{Z}_\infty^{1,\text{nc}} \prod_{v \mid \infty} G_v^1
\]

where \( \mathcal{Z}_\infty^{1,\text{nc}} = \mathcal{Z}_\infty^1 / (\mathcal{Z}_\infty^1 \cap K_\infty) \) and \( G_v^1 = G_v / AG_v \). For \( \pi \in \Pi(G_\infty^1) \) let \( \omega^{\text{nc}}_\pi \) denote the restriction of the central character of \( \pi \) to \( \mathcal{Z}_\infty^{1,\text{nc}} \). Then \( \pi' = \pi \otimes (\omega^{\text{nc}}_\pi)^{-1} = \prod_{v \mid \infty} \pi'_v \), where each \( \pi'_v \) is a representation of \( G'_v \). If \( f \in \mathcal{H}(G_\infty^1) \) factorizes as \( f = f_Z \prod_{v \mid \infty} f'_v \) according to (14.3) then

\[
\text{tr} \pi(f) = \widehat{f_Z}(\omega^{\text{nc}}_\pi) \prod_{v \mid \infty} \text{tr} \pi'_v (f'_v).
\]

Since the spectral localizers \( h_R^{\delta,\mu} \) of Section 8 were taken to respect this decomposition, it suffices to construct

1. a test function \( f_R^{\mu,\mu} \in C_c^\infty(\mathcal{Z}_\infty^{1,\text{nc}}) \) such that \( \widehat{f_R^{\mu,\mu}}(e^\lambda z) = h_R^{\mu,\mu}(\lambda z) \),
2. for each \( v \mid \infty \), a test function \( f_R^{\delta,\mu}_v \in C_c^\infty(G_v^1) \) satisfying the \( v \)-adic version of properties (14.1) and (14.2).

The first point is simple: for \( \mu_Z \in (a_0^G)'^* \) and \( R > 0 \) let \( g_R^{\mu,\mu} \in C_c^\infty(a_0^G)_R \) be as in Section 8.1. Then using \( \text{Lie}(\mathcal{Z}_\infty^{1,\text{nc}}) = a_0^G \), we set \( f_R^{\mu,\mu}(z) = g_R^{\mu,\mu}(\log z) \). It remains then to carry out the second point.
14.2. **Spherical transform of type \( \tau \): abstract theory.** We now fix an archimedean place \( v \). Throughout the remainder of this section we drop the dependence on \( v \) in the notation. Thus \( G = G_v, G^1 = G^1_v, K = K_v, \) etc.

We begin by recalling the spherical functions (and trace spherical functions) of a given \( K \)-type \( \tau \). These will then be used to define the associated spherical transform on the \( \tau \)-isotypic Hecke algebra. For these definitions, see, for example, [60, §6.1] and [7].

Fix \( \tau \in \Pi(K) \). For \( \pi \in \Pi(G^1) \), acting on the space \( V_\pi \), let \( \Pi_\tau \) be the canonical projection onto the \( \tau \)-isotypic subspace \( V_\pi^\tau \). Then the spherical function of type \( \tau \) for \( \pi \) is defined by

\[
\Phi_\pi^\tau(g) = \Pi_\tau \circ \pi(g) \circ \Pi_\tau.
\]

Note that \( \Phi_\pi^\tau(g) \) is an endomorphism of the finite dimensional space \( V_\pi^\tau \), which is zero if \( \tau \) is not a \( K \)-type of \( \pi \). Similarly, we may define the spherical trace function of type \( \tau \) for \( \pi \) by

\[
\varphi_\pi^\tau(g) = \text{tr} \Phi_\pi^\tau(g).
\]

Note that \( \phi_\pi^\tau(c) = \Pi_\tau \) and \( \varphi_\pi^\tau(c) = \text{dim} V_\pi^\tau \). Furthermore, since \( \pi \) is unitary, we have

\[
|\varphi_\pi^\tau(g)| \leq \text{dim} V_\pi^\tau \| \Phi_\pi^\tau(g) \| \leq \text{dim} V_\pi^\tau.
\]

For \( \tau \in \Pi(K_v) \) let \( \xi_\tau \) denote the character of \( \tau \) and write \( \chi_\tau = (\text{dim} \tau) \xi_\tau \). We then let \( \mathcal{H}(G^1, \tau) \) denote the space of functions \( f \in C^\infty_c(G^1) \) such that

1. \( f(kgk^{-1}) = f(g) \) for all \( g \in G^1 \) and \( k \in K \),
2. \( \overline{\chi_\tau} \cdot f = f \cdot \chi_\tau \).

Then for any \( f \in \mathcal{H}(G^1, \tau) \) and any \( \pi \in \Pi(G^1) \) we have \( \Pi_\tau \circ \pi(f) \circ \Pi_\tau = \pi(f) \); see, for example, [7, Prop. 3.2]. In particular \( \pi(f) = 0 \) on \( \mathcal{H}(G^1, \tau) \) unless \( \tau \) is a \( K \)-type of \( \pi \).

We define the spherical transform of type \( \tau \) of a function \( f \in \mathcal{H}(G^1, \tau) \) by

\[
\mathcal{H}^\tau(f)(\pi) = \int_{G^1} f(g) \varphi_\pi^\tau(g) dg.
\]

It follows from the definitions (see [7, (14)]) that, for \( f \in \mathcal{H}(G^1, \tau) \) we have

\[
\pi(f) = \mathcal{H}^\tau(f)(\pi) \cdot \frac{1}{\text{dim} V_\pi^\tau} \Pi_\tau
\]

and hence

\[
\text{tr} \ \pi(f) = \mathcal{H}^\tau(f)(\pi).
\]

The convolution algebra \( \mathcal{H}(G^1, \tau) \) is commutative if and only if \( \tau \) appears with multiplicity at most 1 in every \( \pi \in \Pi(G^1) \). This is the case, for example, for arbitrary \( K \)-types of archimedean \( GL_2 \), and for the trivial \( K \)-type for archimedean \( GL_n \); these are the two cases described in the hypotheses (1) and (2) of Proposition 14.1. Whenever \( \mathcal{H}(G^1, \tau) \) is commutative, we may invert the spherical transform of type \( \tau \). Indeed, it is shown in [7, p.43] that in this case one has the inversion formula

\[
f(g) = \frac{1}{\text{dim} \tau} \int_{\Pi(G^1)} \mathcal{H}^\tau(f)(\pi) \varphi_\pi^\tau(g^{-1}) d\mu^\pi_v(\pi)
\]

for all \( f \in \mathcal{H}(G^1, \tau) \). In such situations we see, using (14.6) and the equality \( \text{dim} V_\pi^\tau = \text{dim} \tau \) valid in this case, that

\[
\| f \|_\infty \leq \| \mathcal{H}^\tau(f) \|_{L^1(\mu^\pi_v)}.
\]
14.3. Reformulation using the subquotient theorem. Using the Casselman subquotient theorem, we may complement the abstract theory of the previous section to give an explicit integral representation of \( \tau \)-spherical functions, and explicit formulae for the associated \( \tau \)-spherical transforms and (in the commutative case) their inversions.

We begin by extending the definitions (14.4) and (14.5) to the principal series representations \( \pi(\eta, \nu) \), where \( \eta \in \mathcal{S}^2(T_0^1) \) and \( \nu \in \mathfrak{h}_{0,\mathbb{C}}^* \), which are not necessarily unitary or irreducible. For \( \tau \in \Pi(K) \), we write

\[
\Phi^\tau_{\eta,\nu}(g) = \Pi(\eta, \nu)(g) \circ \Pi(\tau),
\]

where \( \Pi(\tau) \) is the projection of \( I(\eta, \nu) \) onto its \( \tau \)-isotypic component \( I(\eta, \nu)^\tau \). Here, \( I(\eta, \nu) \) is the space on which \( \pi(\eta, \nu) \) acts. Similarly, we put

\[
\varphi^\tau_{\eta,\nu}(g) = \text{tr} \Phi^\tau_{\eta,\nu}(g).
\]

We have the integral representation of Harish-Chandra (see [60, Corollary 6.2.2.3])

\[
\varphi^\tau_{\eta,\nu}(g) = \int_K (\chi_{\eta \ast \eta})(\kappa(k^{-1}(gk)))e^{\langle \nu-\rho, H(kg) \rangle} \, dk,
\]

where \( dk \) is the probability Haar measure on \( K \) and \( \rho \) is the half-sum of positive roots. Moreover, if \( \tau \in \Pi(K) \) appears as a \( K \)-type of \( \pi(\eta, \nu) \), we associate with a function \( f \in \mathcal{H}(G^1, \tau) \) the transform

\[
\mathcal{H}^\tau(f)(\eta, \nu) = \int_{G^1} f(g) \varphi^\tau_{\eta,\nu}(g) \, dg.
\]

For \( R > 0 \) we let \( \mathcal{H}(G^1, \tau)_R = \mathcal{H}(G^1, \tau) \cap \mathcal{H}(G^1)_R \). Note that for a fixed \( \eta \in \mathcal{S}^2(T_0^1) \) and any \( \tau \in \Pi(K) \) appearing as a \( K \)-type in \( \pi(\eta, 0) \), whenever \( f \in \mathcal{H}(G^1, \tau)_R \) the assignment

\[
\nu \mapsto \mathcal{H}^\tau(f)(\eta, \nu)
\]

lies in the \( W_\eta \)-invariants of the Paley–Wiener space \( \mathcal{P}W(h_{0,\mathbb{C}}^*)^*_R \).

The relation between the \( \tau \)-spherical functions \( \varphi^\tau_\pi \) and transforms \( \mathcal{H}(f)(\pi) \) defined on unitary representations \( \pi \in \Pi(G^1) \) and the functions \( \varphi^\tau_{\eta,\nu} \) and transforms \( \mathcal{H}^\tau(f)(\eta, \nu) \) is given by the Casselman subquotient theorem. This theorem states (in particular) that for any \( \pi \in \Pi(G^1) \) there is \( \eta = \eta(\pi) \in \mathcal{S}^2(T_0^1) \) and \( \nu = \nu(\pi) \in \mathfrak{h}_{0,\mathbb{C}}^* \) such that \( \pi \) is infinitesimally equivalent to a subquotient of the principal series representation \( \pi(\eta, \nu) \). Thus for \( \pi \in \Pi(G^1) \) appearing as a subquotient of \( \pi(\eta, \nu) \), and for any \( K \)-type \( \tau \) of \( \pi \), we have

\[
(14.10) \quad \varphi^\tau_\pi = \varphi^\tau_{\eta,\nu} \quad \text{and} \quad \mathcal{H}^\tau(f)(\pi) = \mathcal{H}^\tau(f)(\eta, \nu).
\]

Note the importance of the assumption that \( \tau \in \Pi(K) \) appearing as a \( K \)-type of \( \pi \) for this formula to hold: if \( \tau \) is not a \( K \)-type of \( \pi \), then both \( \varphi^\tau_\pi \) and \( \mathcal{H}^\tau(f)(\pi) \) are zero, whereas this is not necessarily the case for \( \varphi^\tau_{\eta,\nu} \) and \( \mathcal{H}^\tau(f)(\eta, \nu) \).

Now assume \( \mathcal{H}(G^1, \tau) \) commutative. Following [7, (46)], we write the inverse spherical transform of type \( \tau \) more explicitly. Let \( D^\tau \) denote the subset of \( \delta = [M, \delta] \in D \) for which \( \tau \) appears as a \( K \)-type in \( \pi(\delta, 0) \). For any \( f \in \mathcal{H}(G^1, \tau) \) we have

\[
(14.11) \quad f(g) = \frac{1}{\dim \tau} \sum_{\delta = [M, \delta] \in D^\tau} c_M \int_{h_{0,\mathbb{C}}^*} \mathcal{H}^\tau(f)(\eta, \nu) \varphi^\tau_{\eta,\nu}(g^{-1}) \beta(\delta, \mu) \, d\mu,
\]

where the \( c_M > 0 \) are constants, and the parameters \( (\eta, \nu) \in \mathcal{S}^2(T_0^1) \times \mathfrak{h}_{0,\mathbb{C}}^* \) are chosen so that \( \pi_{\delta,\mu} \) is the unique irreducible subquotient of \( \pi(\eta, \nu) \) (this choice is not unique).
14.4. Proof of Proposition 14.1 in the spherical case. For the trivial K-type $\tau_0 \in \Pi(K)$, and the trivial character $\eta_0 \in \mathcal{E}^U(T_0)$ we write $\varphi = \varphi^\tau_{\eta_0,\nu}$ for the associated spherical function. Moreover, for $f \in \mathcal{H}(G^1, \tau_0)$, we write $h(\nu) = \mathcal{H}^\tau_0(f)(\eta_0, \nu)$ for the associated spherical transform. Then the inversion formula (14.11) becomes

$$f(g) = \int_{iB_0^2} h(\nu) \varphi_{\nu}(g) \beta(\nu) d\nu.$$

For $\mu \in iB_0^1$ and $R > 0$ let $h^\delta_{R} \in \mathcal{P}W(\mathfrak{h}_0^*, \omega_R^*)$ be as in Section 8. Let $f^\delta_{R} \in \mathcal{H}(G^1, \tau_0)_R$ be the inverse spherical transform (of trivial K-type) of $h^\delta_{R}$. Recall that for any $f \in \mathcal{H}(G^1, \tau_0)$ the operator $\pi(f)$ acts on $V_\pi$ as zero on whenever $\pi$ does not contain the trivial K-type. Moreover, writing $\pi_\nu = \pi_{\eta_0,\nu}$, we deduce from (14.7) that $\pi_\nu(f^\delta_{R}) = h^\delta_{R}(\nu)\Pi_\tau$. These two observations together imply (14.1). The bound (14.2) follows simply from (14.9).

14.5. Proof of Proposition 14.1 for $GL_2(C)^1$. We now consider the case of $GL_2(C)$.

For an integer $k \in \mathbb{Z}$ let $\chi_k$ denote the character of $C^\times$ given by $z \mapsto (z/|z|)^k$. For integers $(k, \ell) \in \mathbb{Z}^2$ we let $\delta_{k,\ell}$ denote the character of $T_0$ sending $\text{diag}(z_1, z_2)$ to $\chi_k(z_1/z_2)\chi_{\ell}(z_1z_2)$. Then $E_2(T_0) = \{\delta_{k,\ell} : (k, \ell) \in \mathbb{Z}^2\}$. The standard maximal compact of $GL_2(C)$ is $U(2)$, and we have

$$\Pi(U(2)) = \{\tau_{n,m} : (n, m) \in \mathbb{Z}^2, n \geq 0\},$$

where $\tau_{n,m} = \text{Sym}^n \otimes \text{det}^m$. The $U(2)$-type decomposition of the induced representation $\pi(k, \ell; s) = \pi(\delta_{k,\ell}, s)$ can be computed by Frobenius reciprocity. One obtains

$$(14.12) \quad \text{Res}_{U(2)} \pi(k, \ell; s) = \bigoplus_{n \geq |k| \atop n \equiv k \mod 2} \tau_{n, \frac{\ell-n}{2}}.$$

Letting $\tau(k, \ell)$ denote the lowest U(2)-type of $\pi(k, \ell; s)$, we have $\tau(k, \ell) = \tau_{|k|, \ell-|k|)/2}$.

For $\tau_{n,m} \in \Pi(U(2))$ and $f \in \mathcal{H}(GL_2(C)^1, \tau_{n,m})$ we write $h_{k}(s) = \mathcal{H}^{m,n}(f)(\pi(k, 2m + n; s))$ whenever $|k| \leq n$. From the decomposition (14.12) and the identity (14.8) we deduce that

$$\text{tr} \pi(k, \ell; s)(f) = \begin{cases} h_k(s), & |k| \leq n \text{ and } k \equiv n \mod 2 \text{ and } \ell = 2m + n; \\ 0, & \text{else}. \end{cases}$$

We now explicate the inversion spherical transform in (14.11). Recall that for $GL_2(C)$ the only cuspidal parabolic is $M = T_0$; we identify $h_0^C = C$. For notational simplicity we write $\varphi_{k,s}^{m,n}(g) = \varphi_{\eta_{k,2m+n,s}}(g)$. There is a constant $a > 0$ such that for any integers $(n, m) \in \mathbb{Z}^2$ with $n \geq 0$ and any function $f \in \mathcal{H}(GL_2(C)^1, \tau_{n,m})$ we have

$$f(g) = \frac{a}{n+1} \sum_{|k| \leq n \atop k \equiv n \mod 2} \int_{iR} \varphi_{k,s}^{m,n}(g^{-1})h_k(s)\beta_C(k, 2m+n;s)ds.$$

With these preliminaries behind us, we now come to the construction of the test functions of Proposition 14.1 for $GL_2(C)^1$. For $\delta \in \mathcal{E}_2(T_0)$, $\mu \in iB_0^1$, and $R > 0$ let $h^\delta_{R}$ denote the spectral localizer of Section 8. Let $\tau(\pi_\delta)$ be the lowest U(2)-type of $\pi(\delta, \mu)$. Then, as explicated above, if $\delta = \eta_{k,\ell}$ then $\tau(\pi_\delta) = \tau_{|k|, \ell-|k|)/2}$.

First assume $|k| \leq 2$. Then take $f^\delta_{R} \in \mathcal{H}(GL_2(C)^1, \tau(\pi_\delta))_R$ to be the inverse $\tau(\pi_\delta)$-spherical transform of $h^\delta_{R}$. Then similarly to the spherical case, this test function satisfies the two stated properties of Proposition 14.1.

Now assume $|k| \geq 2$. Write $\tau(\pi_\delta)_{\text{old}} = \tau_{|k|-2, (\ell-|k|)/2+1}$ for the next lowest U(2)-type of the same parity. Let $f^\delta_{R, \ast} \in \mathcal{H}(GL_2(C)^1, \tau(\pi_\delta))_R$ be the inverse $\tau(\pi_\delta)$-spherical transform of $h^\delta_{R}$ and
\[ f_{R,\text{old}}^{\delta,\mu} \in \mathcal{H}(\text{GL}_2(\mathbb{C})^1, \tau(\pi_\delta)_{\text{old}}) \text{ the inverse } \tau(\pi_\delta)_{\text{old}} \text{-spherical transform of } h_R^{\delta,\mu}. \text{ We put } \\
\[ f_R^{\delta,\mu} = f_{R,+}^{\delta,\mu} - f_{R,\text{old}}^{\delta,\mu}. \]

Then clearly \( f_R^{\delta,\mu} \in \mathcal{H}(\text{GL}_2(\mathbb{C})^1)_R \). From (14.7) we see that
\[
\pi_{\sigma,\lambda}(f_R^{\delta,\mu}) = h^{\delta,\mu}_R(\sigma, \lambda) \left( \frac{1}{|k| + 1} \Pi_{\tau(\pi_\delta)} - \frac{1}{|k| - 1} \Pi_{\tau(\pi_\delta)_{\text{old}}} \right).
\]

This is the zero operator whenever \( \pi_{\sigma,\lambda} \) contains either both \( \tau(\pi_\delta) \) and \( \tau(\delta_{\text{old}}) \) or neither. What remains is when \( \pi_{\sigma,\lambda} \) contains \( \tau(\pi_\delta) \) as a lowest \( U(2) \)-type, i.e., \( \sigma = \delta \). On \( \pi_{\delta,\lambda} \) the operator \( \pi_{\sigma,\lambda}(f_R^{\delta,\mu}) \) acts as \( h^{\delta,\mu}_R(\sigma, \lambda) \cdot \frac{1}{|k| + 1} \Pi_{\tau(\pi_\delta)} \). Together these observations show that \( f_R^{\delta,\mu} \) verifies property (14.1).

Finally, we deduce from (14.9), applied to both terms in \( f_R^{\delta,\mu} \), that (14.2) holds.

14.6. The case of \( \text{GL}_2(\mathbb{R})^1 \). We now specialize the above discussion to \( \text{GL}_2(\mathbb{R})^1 \), with the goal of constructing the test functions of Proposition 14.1 in this case, and proving their stated properties.

14.6.1. Description of \( \mathcal{E}_2(M^1) \). When \( M = T_0 \) we have \( \mathcal{E}_2(T_0^0) = \{ (\text{sgn}^{\epsilon_1}, \text{sgn}^{\epsilon_2} : \epsilon_i \in \{ 0, 1 \} \}. \) Note that the principal series representation \( I(1, \text{sgn}; 0) \) is irreducible; it is the limit of discrete series representation, often denoted \( D_1 \).

We next take \( M = G \). In this case \( \mathcal{E}_2(\text{GL}_2(\mathbb{R})^1) = \{ D_k : k \geq 2 \} \). Here \( D_k = D_k \), where (for \( k \geq 2 \)) \( D_k \) is the discrete series representation for \( \text{GL}_2(\mathbb{R})^1 \) appearing as the unique irreducible subquotient of \( I(1, \text{sgn}^{\epsilon}; (k - 1)/2) \), where \( \epsilon \equiv k \) mod 2. Namely, there is an exact sequence
\[
1 \rightarrow D_k \rightarrow I(1, \text{sgn}^{\epsilon}, (k - 1)/2) \rightarrow \text{Sym}^{k-2} \rightarrow 1.
\]

14.6.2. \( K \)-type decompositions. To begin the description of \( K \)-type decomposition, we consider the principal series representations \( \pi(\sigma, \lambda) \).

We parametrize the irreducible dual of \( K = O(2) \) as follows. For \( k \geq 1 \) we put \( \tau_k = \text{Ind}_{\text{SO}(2)}^{O(2)}(e^{ik\theta}) \), a two dimensional representation. Then \( \Pi(O(2)) = \{ \tau_0, \det \} \cup \{ \tau_k \}_{k \geq 1} \).

We let \( \sigma \) denote \( (\text{sgn}^{\epsilon_1}, \text{sgn}^{\epsilon_2}) \), where \( \epsilon_i \in \{ 0, 1 \} \). Then for any \( \lambda \in \mathfrak{h}_0^* = \mathbb{C} \) we have
\[
\text{Res}_{\text{O}(2)} \pi(\sigma, \lambda) = \begin{cases} 
\text{sgn}^{\epsilon} \oplus \oplus_{n \text{ even}} \tau_n, & \epsilon_1 = \epsilon_2 = \epsilon, \\
\oplus_{n \text{ odd}} \tau_n, & \epsilon_1 \neq \epsilon_2.
\end{cases}
\]

Let \( \tau(\sigma) \) denote the lowest \( O(2) \)-type of \( \pi(\sigma, \lambda) \). Then
\[
\tau(\sigma) = \begin{cases} 
1, & \epsilon_1 = \epsilon_2 = 0, \\
\det, & \epsilon_1 = \epsilon_2 = 1, \\
\tau_1, & \epsilon_1 \neq \epsilon_2.
\end{cases}
\]

From the exact sequence (14.13) we deduce that for any \( k \geq 2 \) we have
\[
\text{Res}_{\text{O}(2)}(D_k) = \bigoplus_{n \equiv k \text{ mod } 2} \tau_n.
\]

Letting \( \tau(D_k) \) denote the minimal \( O(2) \)-type of \( D_k \), we have \( \tau(D_k) = \tau_k \).
14.6.3. The distributional character of the discrete series. In this paragraph we explicate the \( \tau \)-
spherical transform for the discrete series representations, from which we deduce its distributional
character.

We first agree to the following notational convention regarding the \( \tau \)-spherical functions associated
with principal series parameters \( (\sigma, \lambda) \in \mathcal{E}^2(T_0^1) \times \mathbb{C} \):

- If \( \sigma = (1, 1), \tau_0 \in \Pi(O(2)) \) the trivial character, and \( f \in \mathcal{H}(GL_2(\mathbb{R})^1, \tau_0) \) we write
  \[ \varphi_0^{+}, \lambda = \varphi_{(1,1), \lambda}^{\tau_0}; \quad h_0^{+}(\lambda) = \mathcal{H}^{\tau_0}(f)((1, 1), \lambda); \]
- If \( \sigma = (\text{sgn}, \text{sgn}), \tau = \det \in \Pi(O(2)) \), and \( f \in \mathcal{H}(GL_2(\mathbb{R})^1, \det) \) we write
  \[ \varphi_{0, \lambda} = \varphi_{(\text{sgn}, \text{sgn}), \lambda}^{\text{det}}; \quad h_0^{\text{det}}(\lambda) = \mathcal{H}^{\text{det}}(f)((\text{sgn}, \text{sgn}), \lambda); \]
- If \( \sigma = (\text{sgn}, 1), \tau = \tau_1 \in \Pi(O(2)) \), and \( f \in \mathcal{H}(GL_2(\mathbb{R})^1, \tau_1) \) we write
  \[ \varphi_{1, \lambda} = \varphi_{(\text{sgn}, 1), \lambda}^{\tau_1}; \quad h_1(\lambda) = \mathcal{H}^{\tau_1}(f)((\text{sgn}, 1), \lambda)). \]

We next explicate (14.10) for the discrete series \( D_k \) of \( GL_2(\mathbb{R})^1 \). Using (14.13) and (14.16) we
find that for \( k \geq 2 \) and \( j \geq k \) with \( j \equiv \mod 2 \) we have
\[
\varphi_{D_k}^{\tau_j} = \begin{cases} 
\varphi_{(1,1), (k-1)/2}^{\tau_j}; & k \text{ even;} \\
\varphi_{(\text{sgn}, 1), (k-1)/2}^{\tau_j}; & k \text{ odd.}
\end{cases}
\]

Moreover, for \( k \geq 2 \) even and \( f \in \mathcal{H}(GL_2(\mathbb{R})^1, \tau_j) \) we have
\[
\text{tr } D_k(f) = \begin{cases} 
h_0^+((k-1)/2), & j \leq k, \ j \text{ even;}
0, & \text{else},
\end{cases}
\]
and for \( k \geq 3 \) odd and \( f \in \mathcal{H}(GL_2(\mathbb{R})^1, \tau_j) \) we have
\[
\text{tr } D_k(f) = \begin{cases} 
h_1((k-1)/2), & j \leq k, \ j \text{ odd;}
0, & \text{else}.
\end{cases}
\]

14.6.4. Explicit \( \tau \)-spherical inversion. We now explicate the inversion formula (14.11) for each \( K \)-
type \( \tau \). There are constants \( a, b > 0 \) such that the following holds:

- if \( \tau_0 \) is the trivial character then for any \( f \in \mathcal{H}(GL_2(\mathbb{R})^1, \tau_0) \) we have
  \[ f(g) = a \int_{\mathbb{R}} \varphi_{0, -, \lambda}^{\tau_0}(g) h_0^{+}(\lambda) \beta(\lambda) d\lambda; \]
- if \( \tau = \det \) then for any \( f \in \mathcal{H}(GL_2(\mathbb{R})^1, \det) \) we have
  \[ f(g) = a \int_{\mathbb{R}} \varphi_{0, -, \lambda}^{\text{det}}(g) h_0^{\text{det}}(\lambda) \beta(\lambda) d\lambda; \]
- if \( \tau = \tau_1 \) then for any \( f \in \mathcal{H}(GL_2(\mathbb{R})^1, \tau_1) \) we have
  \[ f(g) = a \int_{\mathbb{R}} \varphi_{1, -}(g) h_1(\lambda) \beta(1, \lambda) d\lambda; \]
- if \( k \geq 2 \) is even then for any \( f \in \mathcal{H}(GL_2(\mathbb{R})^1, \tau_k) \) we have
  \[ f(g) = a \sum_{j \leq k} \int_{\mathbb{R}} \varphi_{0, -, \lambda}^{\tau_j}(g) h_0^{+}(\lambda) \beta(\lambda) d\lambda + b \sum_{2 \leq j \leq k} \varphi_{D_k}^{\tau_j}(g^{-1}) h_0^+((k-1)/2); \]
\begin{itemize}
  \item if \( k \geq 3 \) is odd then for any \( f \in \mathcal{H}(\text{GL}_2(\mathbb{R})^1, \tau_k) \) we have
  \[ f(g) = a \int_{\mathbb{R}} \varphi_{1,-}(g) h_1(\lambda) \beta_{2}(1, \lambda) d\lambda + b \sum_{3 \leq j \leq k} \varphi_{D_j}^\frac{\tau_j}{\delta} (g^{-1}) h_1((k - 1)/2). \]
\end{itemize}

14.6.5. \textit{Proof of Proposition 14.1 for GL}_2(\mathbb{R})^1\textit{.} We now construct the test functions \( f^\delta_{R} \) for \( \text{GL}_2(\mathbb{R})^1 \) having the properties described in Proposition 14.1.

We begin by taking \( M = T_0 \). Fix \( \delta \in \mathcal{D}(T_0) \) and \( \mu \in \mathfrak{h}_0^* \). With \( \tau(\pi_\delta) \) as in (14.15), we define \( f^\delta_{R} \in \mathcal{H}(\text{GL}_2(\mathbb{R})^1, \tau(\pi_\delta)) \) as the \( \tau(\pi_\delta) \)-spherical inverse transform of \( h^\delta_R \). Using the above explicit formulae, this gives
\[
f^\delta_{R}(g) = \begin{cases} 
a \int_{\mathbb{R}} \varphi_{0,+}(g) h^\delta_{R} \beta_{2}(1, \lambda) d\lambda, & \text{if } \tau = \tau_0; 
\end{cases}
\]
To see that this yields (14.1), it suffices to show that \( \mathscr{H}(f^\delta_{R}) = 0 \) for any \( \sigma \neq \delta \). Note that \( \mathscr{H}(f) = 0 \) for \( f \in \mathcal{H}(\text{GL}_2(\mathbb{R})^1, \tau) \) whenever \( I(\sigma, \lambda) \) does not contain \( \tau \) as an O(2)-type. It then follows from the O(2)-type description (14.14) that for any \( f \in \mathcal{H}(\text{GL}_2(\mathbb{R})^1, \tau(\pi_\delta)) \) we have
\[
\mathcal{H}(f)(\sigma, \lambda) = 0 \text{ for any } \sigma \neq \delta.
\]
We next take \( M = G \). For each \( \delta = D_k \), where \( k \geq 2 \), we let \( h^\delta \in \mathcal{P}(\mathbb{C})_R \) be such that \( h^\delta((k - 1)/2) = 1 \). Here we have identified \( h^*_{0,\mathbb{C}} \) with \( \mathbb{C} \) in the Paley–Wiener space. Recalling that \( \tau(\pi_\delta) = \tau_k \) is the lowest O(2)-type of \( \delta = D_k \), we let \( f^\delta_{+} \in \mathcal{H}(\text{GL}_2(\mathbb{R})^1, \tau(\pi_\delta))_R \) be the inverse \( \tau(\pi_\delta) \)-spherical transform of \( h^\delta \). Define \( \tau(\pi_\delta) \text{old} \) to be \( \tau_0 \) if \( k = 2 \), \( \text{sgn} \) if \( k = 3 \), and \( \tau_{k-2} \) if \( k \geq 4 \); we let \( f^\delta_{\text{old}} \in \mathcal{H}(\text{GL}_2(\mathbb{R})^1, \tau(\pi_\delta)_{\text{old}})_R \) be the inverse \( \tau(\pi_\delta)_{\text{old}} \)-spherical transform of \( h^\delta \). We then put
\[
f^\delta = f^\delta_{+} - f^\delta_{\text{old}} \in \mathcal{H}(\text{GL}_2(\mathbb{R})^1)_R.
\]
A similar argument to the GL2(\mathbb{C})^1 case shows that \( f^\delta \) satisfies all properties in Proposition 14.1.

15. CONTROLLING THE EISENSTEIN CONTRIBUTION

We first recall Arthur’s description of the spectral expansion. We have
\[
J_{\text{spec}} = \sum_{M \in \mathcal{L}} J_{\text{spec}, M},
\]
for distributions \( J_{\text{spec}, M}(\phi) \) to be described in more detail in §15.1. Our aim in this section is to bound, uniformly in \( N, q, \delta, \mu, \) and \( R \), the continuous contribution
\[
(15.1) \quad J_{\text{Eis}}(\varepsilon_{K_1(q)} \otimes f^\delta_{R}) = \sum_{M \neq G} J_{\text{spec}, M}(\varepsilon_{K_1(q)} \otimes f^\delta_{R}),
\]
for the archimedean test functions \( f^\delta_{R} \in \mathcal{H}(\mathbb{G}_\infty)_R \) of Section 8. The necessary bounds are encoded in the following

\textbf{Theorem 15.1.} \textit{Let } \( n \geq 1 \). \textit{Let } \( \delta \in \mathcal{D} \text{ be represented by } (\delta, M) \text{, where } M \in \mathcal{L}_\infty \). \textit{Suppose that either}
\begin{enumerate}
\item \( n \leq 3 \), or
\item \( M = T_0 \) and \( \delta \) is the trivial character of \( T_0^1 \).
\end{enumerate}
\textit{Let } \( \mu \in \mathfrak{h}_{M}^* \text{ and } R > 0 \). \textit{Let } \( f^\delta_{R} \) \text{ be the function associated with this data by Proposition 14.1. Let}
\[
\beta_{E_{R}, \kappa}(\delta, \mu) = (1 + \log(1 + ||\delta||))^\kappa \max_{M \neq G} (1 + \log(1 + ||\mu_M||))^\kappa \int_{B_M(\mu_M, 1/R)} \beta_{M}^G(\delta, \nu) d\nu.
\]
Then there is a constant $\kappa > 0$, and for every $\epsilon > 0$ there is a constant $c > 0$, such that for any integral ideal $q$ with $0 < R < c \log Nq$ we have

$$J_{\text{Eis}}(\varepsilon_{K_1(q)} \otimes f_R^{\delta, \mu}) \ll \epsilon Nq^{n-1+\epsilon} \beta_{R, n}^\text{Eis}(\delta, \mu).$$

From Theorem 15.1, as well as the various estimates established in Sections 12-14, we obtain the following important consequence, which is simply a reformulation of Theorem 1.3.

**Theorem 15.2.** Let $n \geq 1$. Let $\hat{\delta} \in D$ be represented by $(\delta, M)$, where $M \in \mathcal{L}_{\text{st}, \infty}$. Assume that either

1. $n \leq 2$, or
2. $M = T_0$ and $\delta$ is the trivial character of $T_0^1$.

Then Property (ELM) holds with respect to $\hat{\delta}$.

**Proof.** By definition, we have $J_{\text{spec}} - J_{\text{Eis}} = J_{\text{disc}}$. Moreover, the Arthur trace formula is the distributional identity $J_{\text{spec}} = J_{\text{geom}}$ on $\mathcal{H}(G(\mathbb{A}_F)^1)$. Thus

$$J_{\text{error}}(\varepsilon_{K_1(q)} \otimes f_R^{\delta, \mu}) = J_{\text{Eis}}(\varepsilon_{K_1(q)} \otimes f_R^{\delta, \mu}) + J_{\text{geom}}(\varepsilon_{K_1(q)} \otimes f_R^{\delta, \mu}) - \varphi_n(q) \sum_{\gamma \in Z(F)^n K_1(q)} f_R^{\delta, \mu}(\gamma).$$

From Theorem 15.1, and the majorization $\beta_{R, n}^\text{Eis}(\delta, \mu) \ll \kappa \text{vol}(\delta, \mu)$, the Eisenstein contribution is of acceptable size. For the remaining two geometric terms, we first apply Theorem 12.1. We then use Proposition 14.1 to convert the $L^\infty$-norm of $f_R^{\delta, \mu}$ to the $L^1$-norm of $h_R^{\delta, \mu}$ (it is here where we use the conditions on $n$ and $\hat{\delta}$). Finally, we apply Lemma 8.1 to bound the latter by an acceptable error.

The proof of Theorem 15.1 proceeds by recurrence on $n$, and we will in fact use the following corollary for $m < n$ to prove Theorem 15.1 for $n$.

**Corollary 15.3.** Let $G = \text{GL}_n$ and $n \geq 1$. Let $M \in \mathcal{L}$, $M \neq G$. Let $\hat{\delta} \in D$ be represented by $(\delta, M)$, where $M \in \mathcal{L}_{\text{st}, \infty}$. Suppose that $M \subset M_{\infty}$. Assume that either

1. $n \leq 3$; or
2. $M = T_0$ and $\delta$ is the trivial character of $T_0^1$.

Then there is $c > 0$ such that for any integral ideal with $q$, spectral parameter $\mu \in i\mathfrak{h}_M^*$, and real parameter $0 < R < c \log Nq$ we have

$$\sum_{\pi \in \Pi_{\text{disc}}(M(\mathbb{A})^1)_{\hat{\delta}, \mathfrak{h}_M^*}} \dim V^{\text{M}}_{\pi, j}(q) e^{R||\text{Re}(\lambda_{\infty})||} \ll R^{-\dim \mathfrak{h}_M} Nq^{n-1} \beta_{M}^\text{M}(\delta, \mu).$$

**Proof.** We begin by factoring all data according to the Levi block decomposition of $M$. Let $M = M_1 \times \cdots \times M_r$, where $M_i \simeq \text{GL}_{n_i}$, and $n_1 + \cdots + n_r = n$. Since $M \subset M_{\infty}$ we also have $M = M_1 \times \cdots \times M_r$, where each $M_i \simeq \text{GL}_{m_i}(F_\infty)$ is a cuspidal Levi in $M_i$ and $m_1 \leq n_1$, and $\delta = \delta_1 \otimes \cdots \otimes \delta_r$. As usual, we write $K^{\text{M}}_1(q) = K_1(q) \cap M_1(\mathbb{A}_f)$, so that $K^{\text{M}}_1(q) \simeq K^M_1(q) \times \cdots \times K^M_r(q)$. Finally, we decompose $\mu = \sum_i \mu_i$. Then putting $B_j = B_{M_j}(\mu_{M_j}, 1/R)$, the left hand side is then the product over $j = 1, \ldots, r$ of

$$\sum_{\pi \in \Pi_{\text{disc}}(M_j(\mathbb{A})^1)_{\hat{\delta}, \mathfrak{h}_M^*}} \dim V^{\text{M}}_{\pi, j}(q) e^{R||\text{Re}(\lambda_{\infty})||}.$$
either case, Theorem 15.2 holds for each factor $M_j$. We may therefore apply Proposition 9.1 to each $M_j$, obtaining

$$D_R^{M_j}(q, \delta_j, B_j) \ll R^{-\dim h_{M_j}} \vol(K^{M_j}_1(q))^{-1} \beta_{M_j}(\delta_j, \mu_j).$$

Now, up to reordering of the indices, we have $K^{M_j}_1(q) \simeq K^{M_1}_f \times \cdots \times K^{M_{r-1}}_f \times K^{M_r}_1(q)$. Thus $\vol(K^{M_j}_1(q)) = 1$ for $j = 1, \ldots, r-1$ and $\vol(K^{M_r}_1(q))^{-1} = \varphi_{n_r}(q) \leq \varphi_{n-1}(q) \leq Nq^{n-1}$.

The hypothesis that for $c > 0$ small enough we have $0 < R < c \log(2 + Nq)$, which is present in the statements of both Theorem 15.1 and Corollary 15.3, derives from the usage of Proposition 9.1 in the above proof.

15.1. The distributions $J_{\text{spec},M}$. We now turn to the proof of Theorem 15.1. To proceed we need to describe in more detail the distributions $J_{\text{spec},M}$. This will necessitate a great deal of notation; we borrow essentially from [17, §4]. For this paragraph (and the next) we shall not need to specialize in the two cases of Theorem 15.1.

Let $M \in \mathcal{L}$, $M \neq G$, and $P \in \mathcal{P}(M)$. Let $\mathcal{A}^2(P)$ be the space of all complex-valued functions $\varphi$ on $U_P(\mathfrak{A}_F)M(F)\backslash G(\mathfrak{A}_F)$ such that for every $x \in G(\mathfrak{A}_F)$ the function $\varphi_x(g) = \delta_P(g)^{-1/2} \varphi(gx)$, where $g \in M(F)$, lies in $L^2(A_M M(F)\backslash M(\mathfrak{A}_F))$. We require that $\varphi$ be $\mathfrak{g}$-finite and $K_f$-finite, where $\mathfrak{g}$ is the center of the universal enveloping algebra of $\mathfrak{g}_\mathbb{C}$. Let $\mathcal{A}^2(P)$ be the Hilbert space completion of $\mathcal{A}^2(P)$. For every $\lambda \in \mathfrak{h}_\mathbb{C}^*$ the space $\mathcal{A}^2(P)$ receives an action by $G(\mathfrak{A}_F)$ given by

$$(\rho(P, \lambda, \phi)(\varphi))(x) = \varphi(xy) e^{\langle \lambda, H_P(xy) - H_P(x) \rangle},$$

which makes it isomorphic to the induced representation

$$\text{Ind}(L^2_{\text{disc}}(A_M M(F)\backslash M(\mathfrak{A}_F)) \otimes e^{\langle \lambda, H_M(\cdot) \rangle}).$$

For $P, Q \in \mathcal{P}(M)$ and $\lambda \in \mathfrak{h}_\mathbb{C}^*$ let $M_{QP}(\lambda) : \mathcal{A}^2(P) \to \mathcal{A}^2(Q)$ be the (analytic continuation of the) standard intertwining operator [3, §1]; then $M_{QP}(\lambda)$ is unitary for all $\lambda \in i(\mathfrak{h}^M)^*$. Let $W_M = N_G(M)/M$ be the Weyl group of $M$; we can view it as a subgroup of the Weyl group $W = W(G, T_0)$ of $T_0$. Then $W_M$ acts on $\mathcal{P}(M)$ by sending $P$ to $\omega_s P \omega_s^{-1}$, where $\omega_s \in N_G(T_0)$ is a representative. This gives rise to a map on $P$-induced automorphic forms $s : A^2(P) \to A^2(Q)$ given by left-translation by $\omega_s$. We write $\mathcal{M}(P, s)$ for the composition $M_{P|s}(P(0)) \circ s : A^2(P) \to A^2(Q)$. Then $\mathcal{M}(P, s)$ is a unitary operator which for $\lambda \in i(\mathfrak{h}_\mathbb{C}^*)^*$ intertwines $\rho(P, \lambda)$ with itself, where $L_s$ denotes the smallest Levi subgroup of $G$ containing $\omega_s$; note that when $s \in W_M$ we have $L_s \in \mathcal{L}(M)$. For a description of the logarithmic derivatives of the intertwining operators, denoted $\Delta_{L_s}(\beta)$ and associated to certain finite combinatorial data $\beta \in \mathcal{B}_{P, L_s}$, we prefer to send the reader to [15, §2] or [17, §4]. For any $s \in W_M$ and $\beta \in \mathcal{B}_{P, L_s}$, we put

$$J_{\text{spec},M}(\phi; s, \beta) = \int_{i(\mathfrak{h}_\mathbb{C}^*)^*} \text{tr} \left( \Delta_{L_s}(\beta)(P, \lambda) M(P, s) \rho(P, \lambda, \phi) \right) d\lambda,$$

where the operators are of trace class and the integrals are absolutely convergent [39]. Finally, let $t_s = |\det(s - 1)_{\mathfrak{h}_M^*}|^{-1}$.

With the above notation, then

$$J_{\text{spec},M}(\phi) = \frac{1}{|W_M|} \sum_{s \in W_M} t_s \sum_{\beta \in \mathcal{B}_{P, L_s}} J_{\text{spec},M}(\phi; s, \beta);$$

see [15, Corollary 1] or [17, Theorem 4.1].
15.2. Local and global input for general $GL_n$. In this paragraph, we continue to work in the general case of $G = GL_n$. Our goal here is to conveniently package two of the major inputs that will be necessary for the proof of Theorem 15.1. The first concerns the norm of the operators $\Delta_{\chi \Lambda}(\beta) : A^2(P) \to A^2(P)$ and is encapsulated in Lemma 15.4 below. The second, recorded in Lemma 15.5, bounds the dimension of the space of oldforms (with fixed $K_\infty$-type) of an induced automorphic representation in terms of the corresponding dimension of the inducing data.

We first need to introduce some more notation. As before we let $M \in L$ and $P \in P(M)$. For $\pi \in \Pi_{\text{disc}}(M(A_F)^1)$, let $A^2_\infty(P)$ denote the subspace of $A^2(P)$ consisting of $\varphi$ such that, for each $x \in G(A_F)^1$, the function $\varphi_x$ transforms under $M(A_F)^1$ according to $\pi$. For a compact open subgroup $K_f$ of $G(A_f)$ and a $K_\infty$-type $\tau \in \Pi(K_\infty)$ we let $A_\tau(P)^{K_f, \tau}$ be the finite dimensional subspace of $K_f$-invariant functions, transforming under $K_\infty$ according to $\tau$. Finally, for an irreducible admissible representation $\pi_\infty$ of $M_\infty$ with Casimir eigenvalue $\Omega_{\pi_\infty} \in \mathbb{R}$ and minimal $K^M_\infty$-type $\tau(\pi_\infty) \in \Pi(K^M_\infty)$, we write, following [17, §5],

$$\Lambda_M(\pi_\infty) = 1 + \Omega_{\pi_\infty}^2 + \|\tau(\pi_\infty)\|^2.$$  

**Lemma 15.4** (Finis-Lapid-Müller, Lapid, Matz). Let $q$ be an integral ideal and $\tau \in \Pi(K_\infty)$. Let $M \in L$, $M \neq G$, and $L \in L(M)$. Then for all $\pi \in \Pi_{\text{disc}}(M(A_F)^1)$ and $\lambda \in i(h_L^\infty)^*$ the integral

$$\int_{B(\lambda) \cap \mathfrak{a}_L^*} \|\Delta_x(P, \nu)_{A^2_\infty(P)}^{K_\infty, \tau}\|d\nu$$

is bounded by

$$O((1 + \log Nq + \log(1 + \|\lambda\|) + \log(1 + \Lambda_M(\pi_\infty)) + \log(1 + \|\tau\|))^{2r_L}),$$

where $r_L = \dim \mathfrak{a}_L$.

**Proof.** For $\tau$ the trivial $K_\infty$-type, this is [35, Lemma 14.3]. The proof is based on two important contributions from Finis-Lapid-Müller. The first is a strong form of the Tempered Winding Number property for $GL_n$, established in [17, Proposition 5.5]. The second is the Bounded Degree property; in [16, Theorem 1] it is shown that $GL_n$ over $p$-adic fields satisfies this property and in the appendix to [39] (see also [16, Theorem 2]) the same is shown for arbitrary real groups.

It therefore remains to extend the proof of [35, Lemma 14.3] to arbitrary $K_\infty$-types. The only instance in which the trivial $K_\infty$-type hypothesis is invoked in that proof is in [35, Lemma 14.4], when the Bounded Degree property in the archimedean case is applied. Using the notation of that paper, we claim that, when $\nu$ is archimedean, for any irreducible unitary representation $\pi_\nu$ of $M_\nu$, any $K^M_\nu$-type $\tau$, and all $T \in \mathbb{R}$ we have

$$\int_T^{T+1} \|R_Q \varphi(\pi_\nu, it)^{-1}R'_Q \varphi(\pi_\nu, it)\|dt \ll 1 + \log(1 + \|\tau\|).$$

Here, the restriction is to the $\tau$-isotypic subspace of $\pi_\nu$. We adapt the argument of [17, §5.4] to our situation, which differs from theirs in that our integral is over a bounded interval and does not contain a factor of $(1 + |s|^2)^{-1}$.

As a first step, we modify the statement of Lemma 5.19 of [17] to fit our setting. With the notation and same hypotheses of that result, we claim that

$$\int_T^{T+1} \|A'(it)\|dt \ll \sum_{j=1}^m \max\{1, |u_j|^{-1}\}.$$  

To see this, we set $\phi_w(z) = (z + \overline{w})/(z - w)$ for $w = u + iv \in \mathbb{C} - i\mathbb{R}$; then

$$\int_T^{T+1} |\phi'_w(it)|dt \ll \max\{1, |u|^{-1}\}.$$
Indeed we have \( \phi'_w(it) = 2|u|/(u^2 + (t - v)^2) \), and extending the integral over all \( \mathbb{R} \) by positivity yields the result after computation.

Arguing as in the proof of Proposition 5.16 in [17], we shall apply the above bound with the unitary operator \( A = R_{QFP}(\pi_v, is)[\tau] \). Then the set \( \{u_j = u_j + iv_j\} \) consists of the poles of the matrix coefficient \( (R_{QFP}(\pi_v, is)\varphi_1, \varphi_2) \), where \( \varphi_1, \varphi_2 \) are unit vectors in the \( \tau \)-isotypic component of \( \pi_v \). We shall need some basic information on these poles \( u_j \), to be found in Lemma A.1 and Proposition A.2 of [39]. We can deduce from these results that there are integers \( K, L \in \mathbb{N} \) satisfying \( K \ll 1, L \ll 1 + \|\tau\| \), a real number \( \eta \in [0, 1/2] \), and complex numbers \( \rho_k \), for \( k = 1, \ldots, K \), such that the poles of \( \langle R_{QFP}(\pi_v, is)\varphi_1, \varphi_2 \rangle \) are given by

\[
\{\rho_k - \ell : 1 \leq k \leq K, [\text{Re}(\rho_k) + \eta] \leq \ell \leq [\text{Re}(\rho_k) + L]\}.
\]

Inserting this into the above estimate yields

\[
\int_T^{T+1} \| R'_{QFP}(\pi_v, it) \|_{\tau} dt \leq 2 \sum_{k=1}^{K} \sum_{\ell=[\text{Re}(\rho_k)+\eta]}^{\max\{1,\text{Re}(\rho_k)-\ell\}^{-1}} \text{dim} \left( \left\{ \frac{\rho_k - \ell}{\text{Re}(\rho_k)} + \eta \right\} \leq \ell \leq \text{Re}(\rho_k) + L \} \right). \]

Using the bounds on \( K \) and \( L \), we deduce \( (15.4) \). \( \square \)

We next relate the dimension of space of invariants of the \( P \)-induced automorphic forms on \( G \) to the dimension of the corresponding space of invariants of the inducing representation.

**Lemma 15.5.** Let \( M \in \mathcal{L} \) and \( \pi \in \Pi_{\text{disc}}(M(\mathbb{A}_F)^1) \). For any integral ideal \( q \) and \( \tau \in \Pi(K_{\infty}) \):

\[
\frac{\dim A^2_\pi(P)^{K_1(q)}(\pi)}{\dim \tau} \ll \epsilon q \prod \text{dim} V_{\tau'_\pi}(q) \sum_{\tau' \in \Pi(K_{\infty})} \text{dim} V_{\tau'_\pi}(q) \sum_{\tau' \in \Pi(K_{\infty})} \text{dim} V_{\tau'_\pi}(q).
\]

**Proof.** Let \( H_P(\pi_f) \) and \( H_P(\pi_{\infty}) \) be the Hilbert spaces of the (unitarily) induced representations \( I_{P(\mathbb{A}_f)}(\pi_f) \) and \( I_{P_{\infty}}(\pi_{\infty}) \). By [40, (3.5)] and the multiplicity one theorem described in §4.13, we have

\[
\dim A^2_\pi(P)^{K_1(q)}(\pi) = \dim H_P(\pi_f)^{K_1(q)} \dim H_P(\pi_{\infty})^\tau.
\]

We may therefore treat the finite and archimedean places separately.

By [40, (3.6)], for any \( \tau \in \Pi(K_{\infty}) \) we have

\[
\dim H_P(\pi_{\infty})^\tau \leq \dim H_P(\pi_{\infty})^\tau \sum_{\tau' \in \Pi(K_{\infty})} [\tau' : \tau'] \text{dim} V_{\tau'_\pi}(q).
\]

Classical multiplicity one results for successive unitary and orthogonal groups imply the bound \( [\tau'_K : \tau_K] = O(1) \), the implied constant depending only on \( n \).

As for the finite places, it suffices to consider \( v \mid q \). Let \( f \) be the conductor exponent of the irreducible tempered generic representation \( \Pi_v = I_{P_v}(\pi_v) \) of \( G_v \). From the dimension formulae of [8, 47] we have

\[
\dim H_P(\pi_v)^{K_1(p^r)} = \begin{cases} \binom{n+r-f}{n}, & r \geq f; \\ 0, & \text{else.} \end{cases}
\]

Let \( M = M_1 \times \cdots \times M_r \) be the block decomposition of \( M \), with \( M_i \simeq \text{GL}_{n_i} \) and \( n_1 + \cdots + n_r = n \). Say \( \pi_v = \pi_{v,1} \otimes \cdots \otimes \pi_{v,r} \) and write \( f_i \) for the conductor exponent of each \( \pi_{v,i} \). Then we once again have the dimension formulae

\[
\text{dim} V_{\pi_v}^{K_1}(p^r) = \prod_{i} \text{dim} V_{\pi_{v,i}}^{K_1}(p^r) = \begin{cases} \prod_i \binom{n_i+r-f_i}{n_i}, & r \geq \max_i f_i; \\ 0, & \text{else.} \end{cases}
\]
We may use the Local Langlands Correspondence \([23, 24, 53]\) to compare the conductors of \(\pi_v\) and \(\Pi_v\). Indeed, if \(\phi\) is the Langlands parameter of \(\Pi_v\) and \(\phi_i\) that of \(\pi_v, i\) then \(\phi = \bigoplus_i \phi_i\). From this it follows that \(f = \max_i f_i\). Continuing, note that for \(r \geq f = \max_i f_i\) we have
\[
\left(\frac{n + r - f}{n}\right) \leq \left(\frac{n + r - f}{n}\right)^r = \prod_i a_i \left(\frac{n_i + r - f_i}{n_i}\right),
\]
where \(a_i = \left(\frac{n_i + r - f_i}{n_i}\right)\). Since \(a_i \leq \left(\frac{n_i + r}{n_i}\right) \ll_n (1 + r)^n\) we deduce from the above discussion that
\[
\dim \mathcal{H}_\mathcal{P}(\pi_v)|_{K_1(\mathcal{P})} \ll_n (1 + r)^n \dim V_{\pi_v}^{K_1(\mathcal{P})}.
\]
Taking the product over all \(v | q\) finishes the proof. \(\square\)

15.3. Proof of Theorem 15.1. We are now ready to prove Theorem 15.1. Letting \(G = \text{GL}_n\), as usual, we argue by induction on \(n\). For \(n = 1\) there is no continuous spectrum so Theorem 15.1 is trivially true in that case. Now assume the result for \(\text{GL}_m\) for all \(m < n\). The induction hypothesis will be used after having made various simplifications coming from Proposition 14.1 and the general results from the previous paragraph.

From (15.1) it will be enough to bound \(J_{\text{spec}, M}(\mathcal{E}_{K_1(\mathcal{P})} \otimes f^\delta_M)\) for a given \(M \in \mathcal{L}\), where \(M \neq G\). To do so, we may apply the expansion (15.3) and bound each term \(J_{\text{spec}, M}(\mathcal{E}_{K_1(\mathcal{P})} \otimes f^\delta_M, s, \beta)\) separately, where \(s \in W_M\) and \(\beta \in \mathfrak{B}_P L_s\). For convenience, we shall often drop the dependence of \(s\) and \(\beta\) from the notation. In particular, we write \(L\) in place of \(L_s\) (an element in \(L(M)\)), \(M(P)\) in place of \(M(P, s)\), and \(\mathcal{X}\) in place of \(\mathcal{A}_{L_s}(\beta)\).

Recalling the definition (15.2) and expanding over \(\Pi_{\text{disc}}(M(\mathcal{A}, F))\), we find that the integral
\[
J_{\text{spec}, M}(\mathcal{E}_{K_1(\mathcal{P})}) \otimes f^\delta_M, s, \beta)\]
is equal to
\[
\sum_{\pi \in \Pi_{\text{disc}}(M(\mathcal{A}, F))} \int_{\mathfrak{B}_P L_s} \operatorname{tr} \left( \Delta_{\mathcal{X}}(P, \pi, \nu) \mathcal{M}(P, \pi, \nu) \rho(P, \nu, \pi, \phi) \right) d\nu,
\]
where \(\Delta_{\mathcal{X}}(P, \pi, \nu)\), \(\mathcal{M}(P, \pi)\), and \(\rho(P, \nu, \pi, \phi)\) denote the restrictions of the corresponding operators to \(\mathcal{A}_{L_s}^2(P)\). For \(\tau \in \Pi(K_\infty)\) we let \(\Pi_{K_1(\mathcal{P})}, \tau\) denote the orthogonal projection of \(\mathcal{A}_{L_s}^2(P)\) onto \(\mathcal{A}_{\tau}^2(P)|_{K_1(\mathcal{P})}\). From Proposition 14.1 it follows that
\[
\rho(P, \nu, \pi, \mathcal{E}_{K_1(\mathcal{P})} \otimes f^\delta_M) = \begin{cases} h_R^\delta_M(\lambda_\pi + \nu) \cdot \frac{1}{\dim \tau(\pi)} \Pi_{K_1(\mathcal{P}), \tau(\pi)} & \text{if } \delta = \delta; \\ 0 & \text{else,} \end{cases}
\]
yielding
\[
\frac{1}{\dim \tau(\pi)} \int_{\mathfrak{B}_P L_s} h_R^\delta_M(\lambda_\pi + \nu) \operatorname{tr} \left( \Delta_{\mathcal{X}}(P, \pi, \nu) \mathcal{M}(P, \pi) \Pi_{K_1(\mathcal{P}), \tau(\pi)} \right) d\nu,
\]
for the integral in (15.5) whenever \(\pi \in \Pi_{\text{disc}}(M(\mathcal{A}, F))\), and 0 otherwise. This last condition implies that if \((\delta, M)\) is a representative for \(\delta\) with \(M \in \mathcal{L}\), then \(M_\infty\) must contain \(M\). Using the unitarity of \(M(P, \pi)\), as well as the upper bound \(\|A\|_1 \leq \dim V \|A\|\) for any linear operator \(A\) on a finite dimensional Hilbert space \(V\), the expression in (15.6) is bounded in absolute value by
\[
\frac{\dim \mathcal{A}_{\tau}^2(P)|_{K_1(\mathcal{P}), \tau(\pi)}}{\dim \tau(\pi)} \int_{\mathfrak{B}_P L_s} |h_R^\delta_M(\lambda_\pi + \nu)||\Delta_{\mathcal{X}}(P, \pi, \nu)|_{K_1(\mathcal{P}), \tau(\pi)}|d\nu|.
\]

Following [34, §6], we now proceed to break up the sum-integral in (15.5), so that \(\lambda_\pi\) and \(\nu\) lie in \((1/R)\)-balls centered at lattice points. We write \(h_M^L = h_M \cap h_0^L\) and let \((h_M^L)^\perp\) be the annihilator of \(h_M^L\) in \(h_0^L\). For \(\lambda \in i(h_M^L)^\perp\) we write \(\lambda = \lambda^M + \lambda_L^\perp\) according to the decomposition
\[
\frac{\dim \mathcal{A}_{\tau}^2(P)|_{K_1(\mathcal{P}), \tau(\pi)}}{\dim \tau(\pi)} \int_{\mathfrak{B}_P L_s} |h_R^\delta_M(\lambda_\pi + \nu)||\Delta_{\mathcal{X}}(P, \pi, \nu)|_{K_1(\mathcal{P}), \tau(\pi)}|d\nu|.
\]
We choose a lattices \( \Lambda^M \subset i(h_0^M)^* \) and \( \Lambda_L \subset i h_L^* \) such that \( \Lambda = \Lambda^M \oplus \Lambda_L \subset i(h_M^L)^\perp \) satisfies
\[
\bigcup_{\lambda \in \Lambda} (B(\lambda, 1/R) \cap i(h_M^L)^\perp) = i(h_M^L)^\perp.
\]
We deduce that (15.5) is bounded by
\[
\sum_{\lambda \in \Lambda} \sum_{\pi \in \Pi_{\text{disc}}(M(\lambda^1)^\perp)} \frac{\dim \mathcal{A}_2^\perp(P, K_1(q), \tau(\pi_\delta))}{\dim \tau(\pi_\delta)} \int_{B(\lambda^1, 1/R) \cap i h_0^*} |h_R^\perp(\lambda, \nu + \nu)\| \Delta_X(P, \pi, \nu)| K_1(q, \tau(\pi_\delta))|\, d\nu.
\]
To the latter quotient of dimensions, we may apply Lemma 15.5 with \( \pi \in \Pi_{\text{disc}}(M(\lambda^1)^\perp) \) and \( \tau = \tau(\pi_\delta) \). We claim that the \( \tau' \in \Pi(K_\infty^M) \) for which \( |\tau(\pi_\delta)| \tau' > 0 \) are in an \( O(1) \)-ball about the minimal \( K_\infty^M \)-type \( \tau(\pi_\infty) \). Indeed, let \( \Lambda \in X_*(T_0) \) and \( \lambda \in X_*(T_0^M) \) denote the highest weights of \( \tau(\pi_\delta) \in \Pi(K_\infty) \) and \( \tau(\pi_\infty) \in \Pi(K_\infty^M) \), respectively. Since \( \delta_\infty = \delta \), the minimal \( K_\infty^M \)-type formula of Knapp [30] states that both \( \Lambda \) and \( \lambda \) differ from the Blattner parameter of \( \delta \) that is, the highest weight of the minimal \( K_\infty^M \)-type of the discrete series \( \delta \) by a cocharacter depending only on \( K_\infty \) and \( K_\infty^M \). This establishes the claim. Since \( \dim V_{\tau(\pi_\infty)} = 1 \), we get
\[
\frac{\dim \mathcal{A}_2^\perp(P, K_1(q), \tau(\pi_\delta))}{\dim \tau(\pi_\delta)} \leq Nq^\epsilon \dim V_{\tau(\pi_\infty)}.
\]
Recalling the definition (9.1), the last sum in (15.8) is then bounded by \( Nq^\epsilon R^{\dim h_M} (q, \delta, \lambda M) \). It follows from the induction hypothesis and Corollary 15.3 that, for \( c > 0 \) small enough and \( 0 < R < c \log Nq \), we have
\[
D_R^M(q, \delta, \lambda M) \leq R^{-\dim h_M} Nq^{n-1} \beta_M^{\delta, \lambda M}.
\]
We deduce that (15.8) is bounded by
\[
(1 + \log(1 + ||\delta||))^{\kappa} (1 + \log(1 + ||\mu M||))^{\kappa} Nq^{n-1+\epsilon} \text{ times}
\]
\[
R^{-\dim h_M} \sum_{\lambda M \in \Lambda M} \dim \mathcal{A}_2^\perp(P, K_1(q), \tau(\pi_\delta)) \leq N \beta_M^{\delta, \lambda M}.
\]
The above sum over \( \Lambda M \) is \( O(R^{-\dim h_M} \beta_M^{\delta, \lambda M}) \), concluding the proof of Theorem 15.1. \qed

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