Higher Order Theories and its Relationship with Noncommutativity

Oscar Sánchez-Santos*

Departamento de Física,
Universidad Autónoma Metropolitana-Iztapalapa
San Rafael Atlixco 186, C.P. 09340, México D.F., México.

and

José David Vergara †

Instituto de Ciencias Nucleares,
Universidad Nacional Autónoma de México,
Apartado Postal 70-543, México 04510 DF, México

Abstract

We present a relationship between noncommutativity and higher order time derivative theories using a method perturbative. We introduce a generalization of the Chern-Simons Quantum Mechanics for higher order time derivatives. This model presents noncommutativity in a natural way when we project to states of low energy. Compared with the usual model, our system presents noncommutativity without the necessity of taking the limit of strong field. We quantized the theory using a Bopp’s shift of the noncommutative variables and we obtain an spectrum without negatives energies. In addition we extend the model to high order derivatives and noncommutativity with variable dependent parameter.

1 Introduction

Theories with higher order time derivatives occur naturally in several areas of physics. However a characteristic of the ordinary Hamiltonian version of these theories

*oscarsanbuzz@yahoo.com.mx
†vergara@nucleares.unam.mx
is that this Hamiltonian is linear in the momenta \[2\] and in consequence the energy is unbounded from below. However, in most cases, higher order derivative theories can be treated by approximation methods \[2, 3\]. An essential point in this construction is the elimination of the high energy degrees of freedom of the theory and that the symplectic structure is modified by the procedure. In this work we show that this fact has as consequence that naturally appears noncommutativity in the system and so there exist a relationship between noncommutativity and higher order derivative theories.

In order to show the relation between higher order time derivative theories and noncommutativity we begin by summarizing the theory of Chern-Simons quantum mechanics and we show how the noncommutativity arise in the spatial variables.

The theory of Chern-Simons with derivatives of first order \[4\] describes a point particle of mass \(m\), confined to a quadratic potential and it moves in a plane perpendicular to a magnetic field, the Lagrangian of the system is given by

\[
L = \frac{m}{2} \ddot{x}_i^2 - \frac{\kappa}{2} x_i^2 + \alpha \epsilon_{ij} x_i \dot{x}_j,
\]

where, \(\epsilon_{ij}\) is the two-dimensional Levy-Civita symbol. In the limit of zero mass the system is reduced to

\[
L_0 = \alpha \epsilon_{ij} x_i \dot{x}_j - \frac{\kappa}{2} x^2.
\]

Now, following the Dirac’s method of quantization with constraints \[5\], we obtain the momenta

\[
p_i = -\alpha \epsilon_{ij} x_j,
\]

and the constraints

\[
\chi_i = p_i + \alpha \epsilon_{ij} x_j \approx 0.
\]

The evolution of these constraints do not generate more constraints then by computing the Poisson brackets we obtain

\[
\{\chi_i, \chi_j\} = 2 \alpha \epsilon_{ij}.
\]

We can see that this matrix is invertible then according to the Dirac formalism that means that we have second-class constraints, in consequence the symplectic structure is given by the Dirac brackets

\[
\{A, B\}_D = \{A, B\} + \frac{1}{2\alpha} \{A, \chi_i\} \epsilon_{ij} \{\chi_j, B\}.
\]

Then, following Dirac procedure we promote this brackets to commutators in the quantum theory, then the Heisenberg algebra for the system (2) is

\[
[x_i, p_j] = \frac{i \delta_{ij}}{2}, \quad [x_i, x_j] = -\frac{i \epsilon_{ij}}{2\alpha}, \quad [p_i, p_j] = -i \frac{\alpha \epsilon_{ij}}{2}.
\]

where we have done \(\hbar = 1\). In consequence in the limit of zero mass we have noncommutativity in the spatial variables.
2 Lagrangian and Constraints

Now, the general idea of this paper is to see if it is possible generalize the above result to the case of theories with high order time derivatives. First, we will consider an extension of the Chern-Simons Quantum Mechanics to a second order time derivative theory \([6]\), with an additional harmonic term, the Lagrangian chosen has the form

\[
L = \frac{m}{2} \dddot{x}_i^2 - \frac{\kappa}{2} \dot{x}_i^2 + \alpha \epsilon_{ij} \dddot{x}_i \dddot{x}_j.
\]  

(8)

In order to study the canonical formalism of this theory we follow the Ostrogradski procedure in this case the generalized canonical momenta are defined by

\[
p_i = \frac{\partial L}{\partial \dddot{x}_i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dddot{x}_i} \right) \quad \text{and} \quad \pi_i = \frac{\partial L}{\partial \dddot{x}_i},
\]  

(9)

For the Lagrangian (8), one finds

\[
p_i = m \dddot{x}_i + 2 \alpha \epsilon_{ij} \dddot{x}_j \quad \text{and} \quad \pi_i = -\alpha \epsilon_{ij} \dddot{x}_j.
\]  

(10)

For this theory our phase space is defined by \((x_i, \dot{x}_i, p_i, \pi_i)\), i.e. this theory has, in principle, a higher number of degrees of freedom. However, due to the equations (10) the variables of the phase space are not independent, then we have constraints. These constrains are

\[
\phi_i = \pi_i + \alpha \epsilon_{ij} \dddot{x}_j.
\]  

(11)

On the other hand, according to the Ostrogradski formalism the canonical Hamiltonian, is given by

\[
H_c = \frac{p_i \dddot{x}_i}{2} + \frac{\kappa}{2} \dot{x}_i^2 + \frac{m}{2\alpha} \epsilon_{ij} \pi_i \dddot{x}_j - \frac{\epsilon_{ij}}{2\alpha} \pi_i p_j.
\]  

(12)

We can see that the first term of this Hamiltonian is linear in the momenta \(p_i\), this show us that the Hamiltonian is unbounded from below. In the next section we will show how to fix this problem, to finish this section, we compute the evolution of the constraints using the total Hamiltonian given by

\[
H = \frac{p_i \dddot{x}_i}{2} + \frac{\kappa}{2} \dot{x}_i^2 + \frac{m}{2\alpha} \epsilon_{ij} \pi_i \dddot{x}_j - \frac{\epsilon_{ij}}{2\alpha} \pi_i p_j + \lambda_i \phi_i.
\]  

(13)

Using this Hamiltonian, the evolution of the constraints results

\[
\dot{\phi}_i = \{\phi_i, H\} = \frac{\epsilon_{ij}}{2\alpha} \phi_j + 2\alpha \epsilon_{ik} \lambda_k \approx 0.
\]  

(14)

From the above equation we can determine the Lagrange multipliers

\[
\lambda_i \approx -\frac{\phi_i}{4\alpha^2}.
\]
and in consequence we don’t have more constraints. Furthermore, these constraints are second class, with the Poisson bracket given by

\[ \{ \phi_i, \phi_j \} = 2\alpha \epsilon_{ij}, \]  

(15)

Now, according of Dirac formalism we have to construct the Dirac brackets, these take the following form

\[ \{ A, B \}_D = \{ A, B \} - \{ A, \phi_i \} \{ \phi_i, \phi_j \}^{-1} \{ \phi_j, B \}. \]  

(16)

In particular, for our theory we have that the matrix \( \{ \phi_i, \phi_j \}^{-1} \), is given by

\[ \{ \phi_i, \phi_j \}^{-1} = -\frac{\epsilon_{ij}}{2\alpha}, \]  

(17)

Now, by promoting these brackets to commutators we obtain the following algebra between our operators

\[ [x_i, p_j] = i\delta_{ij} \mathbb{I}, \quad [x_i, x_j] = 0, \quad [\dot{x}_i, \dot{x}_j] = -\frac{i}{2\alpha} \epsilon_{ij}, \]  

(18)

From this algebra we see that the variables associated with the velocities are noncommutative. This result was obtained without the necessity of taking any class of limit in counterpart to the first order theory. However, the Hamiltonian associated with higher-order theory still contains problems with the state of minimum energy, in addition to this, the quantization can not be done directly because we have a non-canonical algebra. In the next subsection we will show how to resolve these two problems.

### 2.1 Perturbative Approximation and Quantum Spectrum

In order to obtain a theory without high order time derivatives in our model and in this way eliminate the states of negative energy. We will use the perturbative method proposed in [3]. This method will allows us to write the terms with high order derivatives in terms of first order derivatives. The following scheme will be used in the next sections, so we review this procedure. The equations of motion for the Lagrangian (8) are

\[ \ddot{x}_i = -\frac{\kappa}{m} x_i - \frac{2\alpha}{m} \epsilon_{ij} x^{(3)}_j. \]  

(19)

Now we assume that the contribution of the high order term is weaker than the other terms in the Lagrangian, consequently we make the assumption that \( \alpha << 1 \). Then, the second order time derivatives can be approached as

\[ \ddot{x}_i \approx -\left( \frac{\kappa}{m} + \frac{4\alpha^2 \kappa^2}{m^4} \right) x_i + \left( \frac{2\alpha \kappa}{m^2} + \frac{16\alpha^3 \kappa^2}{m^5} \right) \epsilon_{ij} \dot{x}_j + \mathcal{O}(\alpha^4). \]  

(20)
Higher orders in $\alpha$, are obtained by iterating the equations of motion. The next step is to built the symplectic form, by using the brackets (22) and the constraints (11) we obtain

$$\Omega = \frac{\omega_{AB}}{2} d\alpha \wedge d\beta = \delta_{ij} dp_i \wedge dx_j + \alpha \epsilon_{ij} dx_i \wedge d\dot{x}_j. \tag{21}$$

Now, with our approximations the momenta (10) are given to order $\alpha^3$ as

$$p_i = \left( m - \frac{4\alpha^2 \kappa}{m^2} \right) \dot{x}_i - \left( \frac{2\alpha \kappa}{m} + \frac{8\alpha^3 \kappa^2}{m^4} \right) \epsilon_{ij} x_j + O(\alpha^4), \quad \pi_i = -\alpha \epsilon_{ij} \dot{x}_j. \tag{22}$$

Introducing the above momenta (22) in the symplectic form (21), we obtain

$$\Omega = \left( m - \frac{4\alpha^2 \kappa}{m^2} \right) \delta_{ij} d\dot{x}_i \wedge dx_j + \left( \frac{2\alpha \kappa}{m} + \frac{8\alpha^3 \kappa^2}{m^4} \right) \epsilon_{ij} dx_i \wedge dx_j + \alpha \epsilon_{ij} d\dot{x}_i \wedge d\dot{x}_j + O(\alpha^4). \tag{23}$$

This two-form is the approximation to order $\alpha^3$ to the symplectic structure. In matrix form $\omega_{AB}$ and its inverse $\omega^{AB}$, are given by

$$\omega_{AB} = \begin{pmatrix}
-\frac{4\alpha \kappa}{m} - \frac{16\alpha^3 \kappa^2}{m^4} & 0 & -m + \frac{4\alpha^2 \kappa}{m^2} & 0 \\
0 & m - \frac{4\alpha^2 \kappa}{m^2} & 0 & 0 \\
0 & 0 & -2\alpha & 0 \\
0 & 0 & 0 & m^2
\end{pmatrix}, \tag{24}$$

$$\omega^{AB} = \begin{pmatrix}
-\frac{2\alpha}{m^2} - \frac{32\alpha^3 \kappa}{m^4} & 0 & \frac{1}{m} + \frac{12\alpha^3 \kappa}{m^4} & 0 \\
0 & \frac{1}{m} + \frac{12\alpha^3 \kappa}{m^4} & 0 & 0 \\
0 & 0 & \frac{4\alpha}{m^2} + \frac{80\alpha^3 \kappa^2}{m^8} & 0 \\
0 & 0 & 0 & \frac{4\alpha}{m^2}
\end{pmatrix}. \tag{25}$$

By using the matrix (25) we read the basic new brackets, $(\omega^{AB})_{ij} = \{z_i, z_j\}_D$ (where $z_i = \{x_1, x_2, \dot{x}_1, \dot{x}_2\}$), explicitly these parenthesis are given by

$$\{x_i, x_j\}_D = \left( \frac{2\alpha}{m^2} + \frac{32\alpha^3 \kappa}{m^5} \right) \epsilon_{ij}, \quad \{\dot{x}_i, \dot{x}_j\}_D = \left( \frac{4\alpha \kappa}{m^3} + \frac{80\alpha^3 \kappa^2}{m^6} \right) \epsilon_{ij}, \tag{26}$$

$$\{x_i, \dot{x}_j\}_D = \frac{1}{m} \left( 1 + \frac{12\alpha^2 \kappa}{m^3} \right) \delta_{ij}. \tag{27}$$

To avoid the additional extra constant factor in (27), we define

$$\rho_i = \left( 1 - \frac{12\alpha^2 \kappa}{m^3} \right) m \dot{x}_i, \tag{28}$$

in consequence the basic parenthesis in this case are

$$\{x_i, x_j\}_D = \left( \frac{2\alpha}{m^2} + \frac{32\alpha^3 \kappa}{m^5} \right) \epsilon_{ij}, \quad \{\rho_i, \rho_j\}_D = \left( \frac{4\alpha \kappa}{m} - \frac{16\alpha^3 \kappa^2}{m^4} \right) \epsilon_{ij}. \tag{29}$$
\[ \{x_i, \rho_j\}_D = \delta_{ij}. \]

On the other hand, if we introduce the momenta (22) in the Hamiltonian (12) and the definition of \( \rho_i \), we obtain the Hamiltonian in terms of the new variables. So, to third order in \( \alpha \) we get

\[
H = \frac{1}{2m} \left( 1 + \frac{16\alpha^2\kappa}{m^3} \right) \rho_i^2 + \frac{\kappa}{2} x_i^2 + \left( \frac{2\alpha\kappa}{m^2} + \frac{32\alpha^3\kappa^2}{m^5} \right) \epsilon_{ij} x_i \rho_j + O(\alpha^4). \tag{30}
\]

In this way, directly from the high order theory we get a noncommutative theory with Dirac brackets in the reduced phase space given by (29) and Hamiltonian (30). The interesting feature of the high order theory (8) is that contains the noncommutativity without taking any limit in the kinetical term in contrast with the first order Chern-Simons quantum mechanics of (1).

Now, the more simple way to quantize this noncommutative theory is to map noncommutative phase space to the ordinary phase space \([7, 8]\), which satisfy the following commutation relations

\[
\{\bar{x}_i, \bar{x}_j\} = \{\bar{\rho}_i, \bar{\rho}_j\} = 0, \quad \{\bar{x}_i, \bar{\rho}_j\} = \delta_{ij}. \tag{31}
\]

The mapping that relates the new variables to the old variables is given by

\[
x_i = A_{ij} \bar{x}_j + B_{ij} \bar{\rho}_j, \quad \rho_i = C_{ij} \bar{x}_j + D_{ij} \bar{\rho}_j, \tag{32}
\]

In the which \( A, B, C \) and \( D \) are \( 2 \times 2 \) transformation matrices. Following the procedure proposed in [9], one can easily get the conditions that the transformation matrices should satisfy, these are

\[
A_{ik} B_{jk} - B_{ik} A_{jk} = \left( \frac{2\alpha}{m^2} + \frac{32\alpha^3\kappa}{m^5} \right) \epsilon_{ij}, \quad C_{ik} D_{jk} - D_{ik} C_{jk} = \left( \frac{4\alpha\kappa}{m} - \frac{16\alpha^3\kappa^2}{m^4} \right) \epsilon_{ij}; \tag{33}
\]

\[
A_{ik} D_{jk} - B_{ik} C_{jk} = \delta_{ij}.
\]

If we choose for \( A \) and \( D \) diagonal matrices so that \( A_{ij} = a\delta_{ij}, D_{ij} = b\delta_{ij} \), and in addition we select for \( B \) and \( C \) antisymmetric matrices, then we get

\[
B_{ij} = -\frac{1}{a} \left( \frac{\alpha}{m^2} + \frac{16\alpha^3\kappa}{m^5} \right) \epsilon_{ij}, \quad C_{ij} = \frac{1}{b} \left( \frac{2\alpha\kappa}{m} - \frac{8\alpha^3\kappa^2}{m^4} \right) \epsilon_{ij}, \quad B_{ik} C_{jk} = (ab-1)\delta_{ij}; \tag{34}
\]

resolving these set of equations for \( b \) up to quadratic order in \( \alpha \), we obtain

\[
b \approx \frac{1}{a} - \frac{2\alpha^2\kappa}{am^3} + O(\alpha^4). \tag{35}
\]

Therefore, the transformations take the following form

\[
x_i = a\bar{x}_i - \frac{1}{a} \left( \frac{\alpha}{m^2} + \frac{16\alpha^3\kappa}{m^5} \right) \epsilon_{ij} \bar{\rho}_j + \ldots, \quad \rho_i = \frac{1}{a} \left( 1 - \frac{2\alpha^2\kappa}{am^3} \right) \bar{\rho}_i + a \left( \frac{2\alpha\kappa}{m} - \frac{4\alpha^3\kappa^2}{m^4} \right) \epsilon_{ij} \bar{x}_j + \ldots \tag{36}
\]
These transformations (31) will allow us to quantized our theory. Introducing the transformations (36) in the Hamiltonian (30), this takes the form

\[ H = A(\alpha, \kappa, m)\bar{\rho}_i^2 + B(\alpha, \kappa, m)x_i^2 + C(\alpha, \kappa, m)\epsilon_{ij}\bar{x}_i\bar{\rho}_j, \]  

where, the constants parameters (A, B, C), up to third order in \( \alpha \), are

\[ A(\alpha, \kappa, m) = \frac{1}{2ma^2} \left( 1 + \frac{9\alpha^2\kappa}{m^4} + O(\alpha^4) \right), \]

\[ B(\alpha, \kappa, m) = a^2 \left( \frac{\kappa}{2} - \frac{2\alpha^2\kappa^2}{m^3} + O(\alpha^4) \right), \]

\[ C(\alpha, \kappa, m) = -\frac{\alpha\kappa}{m^2} - \frac{8\alpha^3\kappa^2}{m^5} + O(\alpha^4). \]  

We recognize (37) as the Hamiltonian for the commutative, isotropic 2-dimensional harmonic oscillator, with a coupling term proportional to the \( L_z \) angular momentum.

To quantize the theory we use the coordinate representation \(|\bar{x}_i\rangle\). In this case the momenta \( \bar{\rho}_i \) are promoted directly to operators. In consequence the Hamiltonian (37) takes the form

\[ \hat{H} = -A\frac{\partial^2}{\partial \bar{x}_i^2} + B\bar{x}_i^2 - iC\epsilon_{ij}\frac{\partial}{\partial \bar{x}_j}, \]  

where we have choose the normal ordering. To solve the eigenvalue problem we write this operator in polar coordinates and it takes the form

\[ \left[ A\left( \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{\hat{L}^2}{r^2} \right) - Br^2 - C\hat{L} \right] \psi(r, \theta) = E\psi(r, \theta). \]  

It is convenient to introduce the following redefinition, given in [7],

\[ z = \sqrt{\frac{B}{A}}r^2. \]  

Using this redefinition the action of the angular momentum operator results

\[ \hat{L}\psi(r, \theta) = l\psi(r, \theta) = lZ(z)\phi(\theta), \]  

where \( l \) takes the values 0, ±1, ±2, ..., therefore, the resulting equation for \( Z(z) \) is

\[ zZ''(z) + (1 - z)Z'(z) + \left[ E - \frac{l^2}{4z^2} \right] Z(z) = 0, \]  

with \( E \) given by

\[ E = \frac{1}{4\sqrt{AB}}E - lC - \frac{1}{2}. \]  

The general solution for wave equation is given in terms of the generalized Laguerre polynomials

\[ \psi_{n, l}(z, \theta) = Nz^{l/2}L_{n, l}^{|l|}(z) \exp\left( -\frac{z}{2} + il\theta \right), \]  

where

\[ L_{n, l}^{|l|}(z) = \frac{1}{2^{|l|}n!} \left( \frac{d}{dz} \right)^{|l|} (z^n e^{-z}) \]  

and

\[ n, l = 0, 1, 2, ... \]  

are non-negative integers.
with
\[ L_n^r(z) = z^{-r} \exp(z) \frac{d^n}{dz^n} \left( z^{n+r} \exp(-z) \right). \] 

(46)

Here, \( N \) is the proper normalization constant and \( n_r \) is the radial quantum number. Therefore, in general the quantum spectrum is given by
\[ E_{n_r,l} = 2\sqrt{AB}(2n_r + |l| + 1) + lC. \] 

(47)

with quantum numbers taking values \( n_r = 0, 1, 2, ..., l = 0, \pm 1, \pm 2, ... \). Note that the spectrum only depends of the constant \( a \), the mass of the system and the parameters \( \kappa, \alpha \), so that the spectrum of the system is uniquely determined. Also, this spectrum has a well-defined minimum energy state. In order to make clear this fact, we define the following positive numbers \( (n_+, n_-) \)[7], which are determined as follows
\[ n_r = n_+ + \frac{l - |l|}{2}, \quad l = n_+ - n_. \] 

(48)

Introducing these quantum numbers in the energy (47), we obtain
\[ E_{n_+, n_-} = \sqrt{\frac{\kappa}{m}} \left[ 1 + \frac{5\alpha^2 \kappa}{2m^3} + \mathcal{O}(\alpha^4) \right] (n_+ + n_- + 1) + \left[ \frac{\alpha \kappa}{m^2} + \frac{8\alpha^3 \kappa^2}{m^5} + \mathcal{O}(\alpha^5) \right] (n_+ - n_). \] 

(49)

Therefore, for minimum energy state we get
\[ E_{0,0} = \sqrt{\frac{\kappa}{m}} \left[ 1 + \frac{5\alpha^2 \kappa}{2m^3} + \mathcal{O}(\alpha^4) \right], \] 

(50)

this energy is positive definite. We can also see that in the limit \( \alpha \to 0 \) we recover the usual case of two harmonic oscillators.

3 Model of Chern-Simons with Higher Derivatives

In the previous section was shown that noncommutativity and a high order derivative theory are closely related, we show this through the Chern-Simons quantum mechanics of second order. In this section we will introduce an additional extension of this model, we will consider now a model with \( n \)-th order derivatives. With this in mind the Lagrangian of the model is given by
\[ L = \frac{m}{2} \dot{x}_i^2 + \frac{\kappa}{2} x_i^2 + \alpha \epsilon_{ij} x_j^{(n-1)} x_i^{(n)}, \quad i = 1, 2. \] 

(51)

Here \( \alpha \) is a constant parameter that measure the high order character of the theory. According to the Ostrogradski formalism the momenta are defined as
\[ p_{m_i} = \sum_{k=m}^{n} \left( \frac{d}{dt} \right)^{k-m} \frac{\partial L}{\partial x_i^{(k)}}. \] 

(52)
In particular for the case of \( m = n \) we have

\[ p_{ni} = -\alpha \epsilon_{ij} x_j^{(n-1)}. \] (53)

Following Dirac method, the above relation is a constraint, since by Ostrogradski the \( n-1 \)-th derivative is part of the configuration space, and the full phase space of the theory is given by \( \{ x_i, p_{1i}, \dot{x}_i, p_{2i}, \ddot{x}_i, p_{3i}, \ldots, x_i^{(n-1)}, p_{ni} \} \). Consequently we get the constraints

\[ \phi_i = p_{ni} + \alpha \epsilon_{ij} x_j^{(n-1)} \approx 0. \] (54)

We observe that these constraints tell us that the \( n \)-th momenta are not independent of each other. Moreover the Poisson brackets between these constraints are given by

\[ \{ \phi_i, \phi_j \} = 2\alpha \epsilon_{ij}. \] (55)

This matrix is invertible and in consequence we are dealing with second class constraints. The corresponding Dirac brackets are

\[ \{ A, B \}_D = \{ A, B \} + \frac{1}{2\alpha} \{ A, \phi_i \} \epsilon_{ij} \{ \phi_j, B \}. \] (56)

What follows now is to identify the phase space variables and obtain the algebra. Making this identification we have the following non-zero brackets

\[ \{ x_i, p_{1j} \}_D = \delta_{ij}, \ldots, \{ x_i^{(n-2)}, p_{n-1j} \}_D = \delta_{ij}, \quad \{ x_i^{(n-1)}, x_j^{(n-1)} \}_D = -\frac{\epsilon_{ij}}{2\alpha}. \] (57)

At this point we can conclude by brief inspection that the results for \( n = 2 \), corresponds to the previous section. Now, the next step is to build the symplectic structure, using the Dirac brackets (57) and applying the second class constraints to obtain a non degenerate form, the process results in

\[ \Omega = \sum_{m=2}^{n-1} dp_{m-1i} \wedge dx_i^{(m-2)} + \alpha \epsilon_{ij} dx_i^{(n-1)} \wedge dx_j^{(n-1)}. \] (58)

To avoid the problems of a Hamiltonian not bounded from below we apply the perturbative method of [3]. The equations of motion, for this system are

\[ \ddot{x}_i = -\frac{\kappa}{m} x_i + (-1)^{n-1} 2\alpha \epsilon_{ij} x_j^{(2n-1)}. \]

To order \( \alpha \), we rewrite these equations as

\[ \ddot{x}_i \approx -\frac{\kappa}{m} x_i + \frac{2\alpha \kappa^{n-1}}{m^n} \epsilon_{ij} \dot{x}_j + \mathcal{O}(\alpha^2), \] (59)
to obtain higher orders in $\alpha$ we need to iterate the equations of motion. In general for the high order derivatives, we get

$$x_{i}^{(2k)} \approx \left(-\frac{\kappa}{m}\right)^{k} x_{i} + (-1)^{k+1} \frac{2k\alpha\kappa^{n+k-2}}{m^{n+k-1}} \epsilon_{ij} x_{j} + O(\alpha^{2}),$$

$$x_{i}^{(2k+1)} \approx \left(-\frac{\kappa}{m}\right)^{k} \dot{x}_{i} + (-1)^{k} \frac{2k\alpha\kappa^{n+k-1}}{m^{n+k}} \epsilon_{ij} x_{j} + O(\alpha^{2}), \quad \text{with} \quad k = 1, 2, 3, \ldots \quad (60)$$

On the other hand, the momenta are given by

$$p_{1i} = m\dot{x}_{i} + (-1)^{n}2\alpha\epsilon_{ij} \alpha x_{j}^{(2n-1)},$$

$$p_{ni} = (-1)^{n-m-1}2\alpha\epsilon_{ij} \alpha x_{j}^{(2n-m-1)}, \quad \text{for} \quad m = 2, 3, 4, \ldots, n-1,$$

$$p_{ni} = -\alpha\epsilon_{ij} x_{j}^{(n-1)} \quad (61)$$

where $n$ is the order of theory. As we can see the momenta are proportional to the time derivatives, and using the approximations (60), we can replace these derivatives, either by the positions or by the first time derivative. In this way the symplectic structure (58), is reduced to

$$\Omega = m\delta_{ij}d\dot{x}_{i} \wedge dx_{j} + \frac{n\alpha\kappa^{n-1}}{m^{n-1}}\epsilon_{ij} dx_{i} \wedge dx_{j} + \frac{(n-1)\alpha\kappa^{n-2}}{m^{n}}\epsilon_{ij} d\dot{x}_{i} \wedge d\dot{x}_{j}. \quad (62)$$

Defining $\rho_{i} = m\dot{x}_{i}$ in the same way that in the case of the theory of order two, the symplectic two-form is reduced to

$$\Omega = \delta_{ij}d\rho_{i} \wedge dx_{j} + \frac{n\alpha\kappa^{n-1}}{m^{n-1}}\epsilon_{ij} dx_{i} \wedge dx_{j} + \frac{(n-1)\alpha\kappa^{n-2}}{m^{n}}\epsilon_{ij} d\rho_{i} \wedge d\rho_{j} + O(\alpha^{2}). \quad (63)$$

Using this two-form we read the Dirac brackets of the reduced theory

$$\{x_{i}, x_{j}\}_{D} = \frac{2(n-1)\alpha\kappa^{n-2}}{m^{n}}\epsilon_{ij}, \quad \{p_{i}, \rho_{j}\}_{D} = \frac{2n\alpha\kappa^{n-1}}{m^{n-1}}\epsilon_{ij}, \quad \{x_{i}, \rho_{j}\}_{D} = \delta_{ij}. \quad (64)$$

these brackets are valid to first order in $\alpha$. The resulting Hamiltonian to first order in $\alpha$, is given by

$$H = \frac{\rho_{i}^{2}}{2m} + \frac{\kappa}{2} x_{i}^{2} + \frac{2(n-1)\alpha\kappa^{n-1}}{m^{n}}\epsilon_{ij} x_{i} x_{j} + O(\alpha^{2}), \quad (65)$$

As an example we consider the model with third-order derivatives, which has the following Lagrangian

$$L = \frac{m}{2} \dot{x}_{i}^{2} + \frac{\kappa}{2} x_{i}^{2} + \alpha\epsilon_{ij} x_{j}^{(2)} x_{j}^{(3)}. \quad (66)$$

The momenta associated with this theory are

$$p_{i} = \frac{\partial L}{\partial x_{i}(1)} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}_{i}^{(2)}} \right) + \frac{d^{2}}{dt^{2}} \left( \frac{\partial L}{\partial x_{i}^{(3)}} \right) = m x_{i}(1) - 2\alpha\epsilon_{ij} x_{j}^{(4)}. \quad (67)$$
\[ p_{2i} = \frac{\partial L}{\partial x_i^{(2)}} - \frac{d}{dt} \left( \frac{\partial L}{\partial x_i^{(3)}} \right) = 2\alpha \epsilon_{ij} x_j^{(3)}. \]  
\[ (68) \]

\[ p_{3i} = \frac{\partial L}{\partial x_i^{(3)}} = -\alpha \epsilon_{ij} x_j^{(2)}. \]  
\[ (69) \]

As previously anticipated, the momenta associated with the derivative of highest order, define a constraint in the theory, that results

\[ \phi_i = p_{3i} + \alpha \epsilon_{ij} x_j^{(2)}. \]  
\[ (70) \]

The resulting Dirac brackets are given by

\[ \{ x_i, p_{1j} \}_D = \{ \dot{x}_i, p_{2j} \}_D = \delta_{ij} \quad \{ \dot{x}_i, \dot{x}_j \}_D = -\frac{\epsilon_{ij}}{2\alpha}. \]  
\[ (71) \]

In consequence, we obtain for the symplectic structure the following expression

\[ \Omega = \delta_{ij} dp_{1i} \wedge dx_j + \delta_{ij} dp_{2i} \wedge \dot{x}_i + \alpha \epsilon_{ij} d\ddot{x}_i \wedge \ddot{x}_j. \]  
\[ (72) \]

On the other hand, to first-order in \( \alpha \) the momenta are reduced to

\[ p_{1i} \approx m \dot{x}_i - \frac{2\alpha \kappa^2}{m^2} \epsilon_{ij} x_j + \mathcal{O}(\alpha^2), \]

\[ p_{2i} \approx -\frac{2\alpha \kappa}{m} \epsilon_{ij} \dot{x}_j + \mathcal{O}(\alpha^2), \]

\[ p_{3i} \approx \frac{\alpha \kappa}{m} \epsilon_{ij} x_j + \mathcal{O}(\alpha^2). \]  
\[ (73) \]

With the symplectic two-form and these approximations we have the following set of Dirac brackets to first order in \( \alpha \)

\[ \{ x_i, x_j \}_D = \frac{2\alpha \kappa}{m^3} \epsilon_{ij}, \quad \{ \rho_i, \rho_j \}_D = \frac{6\alpha \kappa^2}{m^2} \epsilon_{ij}, \quad \{ x_i, \rho_j \}_D = \delta_{ij}. \]  
\[ (74) \]

Finally, the reduced Hamiltonian is given by

\[ H = \frac{p_i^2}{2m} + \frac{\kappa}{2} x_i^2 + \frac{4\alpha \kappa^2}{m^3} \epsilon_{ij} x_i \rho_j + \mathcal{O}(\alpha^2). \]  
\[ (75) \]

So, in this case we get again a noncommutative theory, with the Dirac’s brackets promoted to commutators and a Hamiltonian equivalent to (30) and the quantization of this system can be done following a similar sequence of steps as those used in Sec. 2.1.
4 Theory with Noncommutative Local Parameter

So far, starting from a high order time derivative theory we have obtained a noncommutative theory with constant noncommutative parameter. The idea of this section is to show that it is possible to generalize this result to the case of a position dependent noncommutative parameter. We begin by consider a Lagrangian similar to the introduced in Ref. [10], given by

\[
L = \frac{m_r \dot{r}^2}{2} - V(r) + \frac{m}{2} \dot{x}_i^2 - \frac{k}{2} x_i^2 + \frac{\theta f(r)}{2} \epsilon_{ij} \dot{x}_i \dot{x}_j. \tag{76}
\]

Noting now that in counterpart of previous example with constant parameter, in this new theory we aggregate the dynamics in the variable \(r\) and we have done \(\alpha = \theta f(r)/2\).

The momenta associate to this Hamiltonian are given by

\[
p_r = m_r \dot{r}, \quad p_i = m \dot{x}_i + \theta f(r) \epsilon_{ij} \dot{x}_j + \frac{\theta f'(r)}{2m_r} p_r \epsilon_{ij} \dot{x}_j, \quad \pi_i = -\frac{\theta f(r)}{2} \epsilon_{ij} \dot{x}_j. \tag{77}
\]

From the above expressions we observe that we have a constraint, that results

\[
\chi_i = \pi_i + \frac{\theta f(r)}{2} \epsilon_{ij} \dot{x}_j. \tag{78}
\]

From the evolution of this constraint we found the associated Lagrange multiplier, given by

\[
\lambda_i \approx \frac{f'(r)p_r}{2m \theta f(r)^2} \epsilon_{ij} \rho_j - \frac{m \pi_i}{\theta^2 f(r)^2} - \frac{f'(r)p_r \rho_i}{4mm_r \theta f(r)} + \frac{f'(r)^2 \pi_i \rho_j^2}{4m^2m_r f(r)^2} - \frac{\epsilon_{ij} \rho_j}{2 \theta f(r)}. \tag{79}
\]

Note that the Lagrange multiplier is well defined only if the function \(f(r)\) does not vanish in the interval of definition of the \(r\) variable. This is more clearly seen, from the Poisson bracket of the constraints, where we get

\[
\{\chi_i, \chi_j\} = \theta f(r) \epsilon_{ij}. \tag{80}
\]

So, for nonvanishing \(f(r)\) we obtain second class constraints. Therefore, we can proceed to construct the Dirac brackets

\[
\{A, B\}_D = \{A, B\} + \{A, \chi_i\} \frac{\epsilon_{ij}}{\theta f(r)} \{\chi_j, B\}. \tag{81}
\]

Our phase space is formed by \(z^A = \{r, x_i, \dot{x}_i, p_r, p_i, \pi_i\}\). Working with the constraints the algebra of the reduced phase space is

\[
\{r, p_r\}_D = 1, \quad \{x_i, p_j\}_D = \delta_{ij}, \quad \{\dot{x}_i, \dot{x}_j\}_D = -\frac{\epsilon_{ij}}{\theta f(r)}, \quad \{p_r, \dot{x}_i\}_D = \frac{f'(r)}{2f(r)} \dot{x}_i. \tag{82}
\]
By using the above algebra we obtain the symplectic two form

$$\Omega = dp_r \wedge dr + \delta_{ij} dp_i \wedge dx_j + \frac{\theta f(r)}{2} \epsilon_{ij} \dot{x}_i \wedge \dot{x}_j + \frac{\theta f'(r)}{4} \epsilon_{ij} \dot{x}_j d\dot{x}_i \wedge dr. \quad (83)$$

Now reducing the theory to first order by applying the equations of motion we obtain, to lower order approximation in $\theta$, the following

$$p_i = m \dot{x}_i - \frac{\theta k f(r)}{m} \epsilon_{ij} x_j + \frac{\theta f'(r)}{2m_r} \epsilon_{ij} \dot{x}_j + O(\theta^2). \quad (84)$$

In consequence, the Hamiltonian takes the form

$$H = \frac{p^2}{2m_r} + V(r) + \frac{\rho^2}{2m} + \frac{k}{2} x_i^2 + \frac{\theta k f(r)}{2m^2} \epsilon_{ij} x_i \rho_j + O(\theta^2). \quad (85)$$

Using these approximations in the symplectic two-form, we obtain the following algebra

$$\{r, p_r\}_D = 1, \quad \{x_i, \rho_j\}_D = \delta_{ij} + \frac{\theta f'(r)}{m m_r} \epsilon_{ij}$$

$$\{p_r, p_i\}_D = \frac{\theta f'(r)}{m m_r} \epsilon_{ij} \rho_j - \frac{2\theta k f'(r)}{m} \epsilon_{ij} x_j,$$

$$\{x_i, x_j\}_D = \frac{\theta f(r)}{m^2} \epsilon_{ij}, \quad \{p_r, x_i\}_D = \frac{\theta f'(r)}{2m} \epsilon_{ij} \rho_j,$$

$$\{\rho_i, r\}_D = \frac{\theta f'(r)}{mm_r} \epsilon_{ij} \rho_j, \quad \{\rho_i, \rho_j\}_D = \frac{2\theta k f(r)}{m} \epsilon_{ij}. \quad (86)$$

where we have used the redefinition $\rho_i = m \dot{x}_i$. To quantize this system it is possible to follow similar steps to the performed in Sec. 2.1, also can be useful to use some of the ideas reported in [11], since the noncommutativity parameter is not constant.

5 Conclusions

We have derived the relationship between the higher order theories and noncommutativity using a perturbative method. The relationship was shown for a Quantum Mechanics generalization of Chern-Simons, was proved that the noncommutativity arises naturally when we project the states to the low energy states of the high order theory. It is interesting to compare the results of this paper and Ref. [8], who studied the same model, a Hamiltonian similar to (30) and commutation relations related to (29). However, they proposed the model directly, whereas in our case this model is the result of a high order time derivative theory. Furthermore, in our model, unlike its counterpart of first order, it is not necessary to cancel the kinetic term and the noncommutativity arises
automatically from the projection to lower energy states. Thus in this example we have shown that the noncommutativity can be seen as a result of making sense, by perturbation theory, of a high order time derivative theory. Also, we have derived that this result is extended to the case of high order time derivative theories and in the case of a nonconstant noncommutative parameter.

Acknowledgments

The authors acknowledge partial support from DGAPA-UNAM grant PAPIIT -IN111210 and PROMEP/103.5/13/9043 (O.S.).

References

[1] K. S. Stelle, Classical Gravity with Higher Derivatives, General Relativity and Gravitation, 9, 353 (1978).
[2] D. A. Eliezer and R. P. Woodard, Nucl. Phys. B325, 389 (1989).
[3] Tai-Chung Cheng, Pei-Ming Ho, Mao-Chuang Yeh, Nucl. Phys. B 625 (2002) 151.
[4] G. V. Dune, R. Jackiw, and C. A. Trugenberger, Phys. Rev. D 41, 661 (1990).
[5] A. J. Hanson , T. Regge and C. Teitelboim, Constrained Hamiltonian Systems, Accademia Nazionale dei Lincei, Roma, 1976.
[6] J. Lukierki, P. C. Stichel and W. J. Zakrzewski, Ann. of Phys. 260 (1997) 224.
[7] A. Smailagic and E. Spallucci, Phys. Rev. D 65 (2002) 107701.
[8] J. Jing, F.H. Liu and J.F. Chen, Phys. Rev. D 78, 125004 (2008).
[9] K. Li, J. Wang and C. Chen, Mod. Phys. Lett. A20 (2005) 2165.
[10] M. Gomes, V. G. Kupriyanov and A. J. da Silva, J. Phys. A43, 285301 (2010).
[11] V.G. Kupriyanov, Quantum mechanics with coordinate dependent noncommutativity, [arXiv:1204.4823]