QUANTUM TOROIDAL ALGEBRAS AND MOTIVIC HALL ALGEBRAS
I. HALL ALGEBRAS FOR SINGULAR ELLIPTIC CURVES

SHINTAROU YANAGIDA

Abstract. We consider the motivic Hall algebra of coherent sheaves over an irreducible reduced projective curve of arithmetic genus 1. We introduce the composition subalgebra in the singular curve case, and show that it is isomorphic to the composition subalgebra for a smooth elliptic curve. As in the case of smooth elliptic case studied by Burban and Schiffmann, the reduced Drinfeld double of the composition subalgebra is isomorphic to the quantum toroidal algebra for \( \mathfrak{gl}_1 \) (also called Ding-Iohara-Miki algebra), and it inherits automorphisms induced from equivalences of the associated derived category. We show that one of the non-trivial automorphisms coincide with the one constructed by Miki in a purely algebraic manner.

0. Introduction

This paper is the first part of the study on the relationship between the quantum toroidal algebras and the motivic Hall algebras of projective curves of arithmetic genus one. In this paper we prepare some generalities on motivic Hall algebras and restate known result on the Ringel-Hall algebras for curves. The new results given in this paper is relatively few.

This paper arises from the investigation of what should be called the quantum toroidal algebra for \( \mathfrak{gl}_1 \). We will denote it by \( \hat{U} \). It has two parameters \( q_1 \) and \( q_2 \). This name comes from the quantum toroidal algebra for \( \mathfrak{gl}_n \) \( (n \geq 2) \) introduced in the work [GKV95] of Ginzburg, Kapranov and Vasserot in the middle 1990s.

The algebra \( \hat{U} \) has several other names. As far as we know, it was first introduced in the work [BuS12] of Burban and Schiffmann with the name (reduced Drinfeld double of) elliptic Hall algebra. Miki [Mi07] called \( (q, \gamma) \)-analog of \( W_{1+\infty} \). In the paper [FHHSY], B. Feigin, Hashizume, Hoshino, Shiraishi and the author called it the Ding-Iohara algebra. In the paper [FT11], B. Feigin and Tsymbaliuk also used the same name. The papers [FFJMM1, FFJMM2] of B. Feigin, E. Feigin, Jimbo, Miwa and Mukhin called quantum continuous \( \mathfrak{gl}_\infty \).

The structure of \( \hat{U} \) looks complicated when it is considered as an analogue of quantum affine algebra. However, following the approach of [BuS12], we have rather clear understanding of this algebra when we consider it as the reduced Drinfeld double of the Ringel-Hall algebra for a smooth elliptic curve \( E_{\text{sm}} \) defined over a finite field \( \mathbb{F}_q \). It has a natural \( \mathbb{Z}^2 \)-grading, where \( \mathbb{Z}^2 \) appears as \( \text{Num}(\text{Coh}(E_{\text{sm}})) \), the numerical Grothendieck group of the category \( \text{Coh}(E_{\text{sm}}) \) of coherent sheaves.

Burban and Schiffmann also constructed an action of \( \text{Aut}(D^b_{\text{coh}}(E_{\text{sm}})) \) on the algebra \( \hat{U} \). They identified an action by a certain Fourier-Mukai transform with the with Miki’s automorphism \( \theta \).

Now we want to consider the Ringel-Hall algebra of coherent sheaves over a singular elliptic curve \( E \). The first main theorem is that the composition subalgebra is isomorphic to that for a smooth elliptic curve.

Theorem (Theorem 7.3). Denote by \( U(E) \) the composition subalgebra of the Ringel-Hall algebra for \( E \). Denote by \( D_{\text{red}} U(E) \) the reduced Drinfeld double of the bialgebra \( U(E) \) Then \( D_{\text{red}} U(E) \) is isomorphic to \( \hat{U} \), where the parameter \( q_1, q_2 \) in \( \hat{U} \) appears as the inverse of the...
zeros of the motivic zeta function of $E$. Namely we put

$$\zeta_{\text{mot}}(E; z) = \frac{(1 - q_1z)(1 - q_2z)}{(1 - z)(1 - qz)}.$$

As in the case for a smooth elliptic curve studied by Burban and Schiffmann, the reduced Drinfeld double of the composition subalgebra, isomorphic to the so-called Ding-Iohara-Miki algebra, inherits automorphisms induced from equivalences of the associated derived category. We will show that one of the non-trivial automorphisms coincides with the one constructed by Miki for Ding-Iohara-Miki algebra.

**Theorem** (Theorem 7.4). There is a Fourier-Mukai transform inducing the algebra automorphism $\Phi^H = \theta$ on $D_{\text{red}}U(E) = \hat{U}$.

In the course of the study, we find that the motivic formalism of Hall algebras is convenient to circumvent technical issues on Fourier-Mukai transforms. In this paper we deal with generalities on the motivic Hall algebra for projective curves, and introduce the motivic analogue of the notions around the Ringel-Hall algebras (including the motivic version of the twisted and the extended Ringel-Hall algebras). The theorems cited above will be stated in the motivic language.

Let us briefly spell out the organization of this paper.

In §1 we give the definition of the $gl_1$-quantum toroidal algebra $\hat{U}$.

§2 is the preparation of the Grothendieck group of varieties and stacks. It will be used in the next §3 for the introduction of the motivic Hall algebras. We also introduce Kapranov’s motivic zeta function, for it will be identified with the structure constant (function) of the quantum toroidal algebra.

In the section §3 we also introduce motivic analog of the notions related to the ordinary Ringel-Hall algebras.

The sections §§4 – 6 are restatement of the known results on the Ringel-Hall algebras for a smooth curve defined on a finite field. We dare to include these facts in this paper for there is little literature treating concrete examples of motivic Hall algebras.

In the final section we will give the main results of the paper. This part is relatively short since the proofs will be almost done in the previous sections.

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**Notation.** The letter $k$ denotes a field which will be a fixed one in each context.

We denote categories by sans-serif letters like $\mathsf{A}$ and $\mathsf{S}$. For example, $\mathsf{Coh}(X)$ denotes the category of coherent sheaves on a scheme $X$.

We denote stacks by Fraktur letters like $\mathfrak{M}$ and $\mathfrak{X}$. For example, $\mathfrak{M} = \mathfrak{M}(X)$ will denote the moduli stack of coherent sheaves on a smooth projective variety $X$.

A coherent sheaf will be denoted by a calligraphy letter like $\mathcal{E}$. 
For an abelian category $\mathcal{A}$, the associated Grothendieck group will be denoted by $K(\mathcal{A})$. The class of an object $a \in \mathcal{A}$ in the Grothendieck group will be denoted by $[a] \in K(\mathcal{A})$.

1. **Quantum Toroidal Algebra for $\mathfrak{gl}_1$**

We follow this definition is given in the paper [FJMM] of B. Feigin, Jimbo, Miwa and Mukhin.

1.1. **Definition.** Let $d$ and $q$ be complex numbers such that

$$q_1 := dq^{-1}, \quad q_2 := q^2, \quad q_3 := d^{-1}q^{-1}$$

satisfies

$$q_1^{n_1}q_2^{n_2}q_3^{n_3} = 1 \text{ for } n_1, n_2, n_3 \in \mathbb{Z} \iff n_1 = n_2 = n_3.$$

**Definition 1.1.** The quantum toroidal algebra for $\mathfrak{gl}_1$, denoted by $\hat{U}$, is an associative $\mathbb{C}$-algebra generated by

$$E_k, \ F_k, \ H_r, \ K^\pm, \ q^\pm c \quad (k \in \mathbb{Z}, \ r \in \mathbb{Z}/\{0\}).$$

with the following defining relation.

$$KK^{-1} = K^{-1}K = 1, \quad q^\pm c \text{ are central, } \quad q^r q^{-c} = q^{-c}q^r = 1,$$

$$K^\pm(z)K^\pm(w) = K^\pm(w)K^\pm(z),$$

$$g(q^{-c}z, w)K^-(z)K^+(w) = g(w, q^{-c}z)K^+(w)K^-(z),$$

$$g(z, w)K^\pm(q^{(1+1)c/2}z)E(w) + g(w, z)E(w)K^\pm(q^{(1+1)c/2}z) = 0,$$

$$g(w, z)K^\pm(q^{(1+1)c/2}z)F(w) + g(z, w)F(w)K^\pm(q^{(1+1)c/2}z) = 0,$$

$$[E(z), F(w)] = \frac{1}{q - q^{-1}}(\delta(q^c z/w)K^+(z) - \delta(q z/w)K^-(w)),$$

$$g(z, w)E(z)E(w) + g(w, z)E(w)E(z) = 0,$$

$$g(z, w)F(z)F(w) + g(z, w)F(w)F(z) = 0.$$

Sym$_z$ $[X(z_1), [X(z_2), X(w)]_q]_{q^{-1}} = 0 \quad (X = E \text{ or } F)$.

Here we used the current expression

$$E(z) := \sum_{k \in \mathbb{Z}} E_k z^{-k}, \quad F(z) := \sum_{k \in \mathbb{Z}} F_k z^{-k},$$

$$K^\pm(z) := K^\pm z \exp \left( \pm(q - q^{-1}) \sum_{r=1}^{\infty} H_{\pm r} z^r \right).$$

We also used the functions $g(z, w)$ given by

$$g(z, w) := (z - q_1 w)(z - q_2 w)(z - q_3 w).$$

Finally, in the last line we used the symbol Sym$_z, w$, for the symmetrizer with respect to $z, w$, and $[X, Y]_q := XY - qYX$.

$\hat{U}$ can also be considered as an associative algebra defined over $\mathbb{Q}(q_1, q_2)$ with $q_3 := q_1^{-1}q_2^{-1}$.

In §6, we will treat $\hat{U}$ in this way.

The algebra $\hat{U}$ is $\mathbb{Z}^2$-graded by the degree assignment

$$\deg(E_k) = (1, k), \quad \deg(F_k) = (-1, k), \quad \deg(H_r) = (0, r),$$

$$\deg(K) = \deg(K^{-1}) = \deg(q^c) = (0, 0),$$

The algebra $\hat{U}$ also has a formal coproduct

$$\Delta E(z) = E(z) \otimes 1 + K^-(C_1 z) \otimes E(C_1 z), \quad \Delta F(z) = F(C_2 z) \otimes K^+(C_2 z) + 1 \otimes F(z),$$

where $C_1, C_2$ are central elements of $\hat{U}$.
\[
\Delta K^+(z) = K^+(z) \otimes K^+(C_1^{-1}z), \quad \Delta K^-(z) = K^-(C_2^{-1}z) \otimes K^-(z), \quad \Delta q^c = q^c \otimes q^c
\]
with \(C_1 := q^c \otimes 1\) and \(C_2 := 1 \otimes q^c\). This coproduct gives \(\tilde{U}\) the structure of formal bialgebra.

1.2. Automorphism. In [Mi07, Theorem 2.1], Miki constructed an (algebra) automorphism \(\theta\) of \(\tilde{U}\) such that

\[
\begin{align*}
E_0 &\mapsto -q^c H_{-1}, & F_0 &\mapsto aq^c H_1, \\
H_1 &\mapsto E_0, & H_{-1} &\mapsto -a F_0, \\
q^c &\mapsto K, & K &\mapsto q^{-c}
\end{align*}
\]
for some coefficient \(a\). The automorphism \(\theta\) exchanges the Heisenberg subalgebras \(a_h\) and \(a_v\) living on the axes in the \(\mathbb{Z}^2\) (expressing the grading of the algebra).
2. The Grothendieck ring of algebraic varieties and stacks

This section is devoted to the introduction of the Grothendieck rings of varieties and stacks, which will be used as the coefficient ring of the motivic Hall algebra.

2.1. Definitions. Let \( k \) be an arbitrary field. By a “variety over \( k \)”, we mean a reduced and separated scheme of finite type over \( k \). Denote by \( \text{Var}/k \) the category of varieties over \( k \).

**Definition 2.1.** The Grothendieck group \( K(\text{Var}/k) \) of varieties over \( k \) is the quotient of the free abelian group on isomorphism classes of varieties over \( k \), by the relations of the form

\[
[X] = [Y] + [X \setminus Y],
\]

where \( Y \) is a closed subvariety of the variety \( X \) and \( X \setminus Y \) is the complementary open subvariety.

We consider the product on \( K(\text{Var}/k) \) given by

\[
[X] \cdot [Y] = [(X \times Y)_{\text{red}}],
\]

where the product is understood to be over \( \text{Spec}(k) \). It makes \( K(\text{Var}/k) \) a commutative associative ring with the unit \( 1 = [\text{Spec}(k)] \). Hereafter we call this ring by the Grothendieck ring of varieties over \( k \).

We denote by

\[
\mathbb{L} := [\mathbb{A}^1] \in K(\text{Var}/k)
\]

the class of the affine line.

Let us see some examples of computations.

**Lemma 2.2.**

1. For the \( d \)-th general linear group \( \text{GL}_d \),

\[
[\text{GL}_d] = \mathbb{L}^{d(d-1)/2} \prod_{k=1}^{d} (\mathbb{L}^k - 1) = \mathbb{L}^{d(d-1)/2}/(\mathbb{L} - 1)^d [d]_+^1,
\]

where \([n]_q^+ := 1 + q + \cdots + q^{n-1}\) and \([n]_q^+! := [1]_q^+ \cdot [2]_q^+ \cdots [n]_q^+\) for \( n \in \mathbb{Z}_{\geq 0} \) and \( q \) an indeterminate.

2. For the Grassmann variety \( \text{Gr}(d,n) \) with \( 0 \leq d \leq n \),

\[
[\text{Gr}(d,n)] = \left[ \begin{array}{c} n \end{array} \right]_L^+ := \frac{[n]_q^+!}{[d]_q^+! [n-d]_q^+!},
\]

**Proof.** (1) See [Br12, Lemma 2.6].

(2) Considering the transitive action of \( \text{GL}_n \) on \( \text{Gr}(d,n) \), we have \([\text{Gr}(d,n)] = [\text{GL}_n] \cdot [\text{GL}_{n,d}]\), where \( \text{GL}_{n,d} \) is the isotropy subgroup of the fixed \( d \)-dimensional subspace \( k^d \) in the total \( n \)-dimensional space \( k^n \). Since \( \text{GL}_{n,d} \simeq \text{GL}_d \times \text{GL}_{n-d} \times M_{d,n-d} \), where \( M_{p,q} \) is the collection of \( p \times q \) matrices, we have

\[
[\text{GL}_n] = [\text{Gr}(d,n)] \cdot [\text{GL}_d] \cdot [\text{GL}_{n-d}] \cdot \mathbb{L}^{d(n-d)}.
\]

Using (1), one can check the result immediately. \( \square \)

2.2. Kapranov’s motivic zeta function. Let us now introduce another example of the calculation in \( K(\text{Var}/k) \), namely Kapranov’s motivic zeta function \([K]\). This function is important since it will appear in the study of motivic Hall algebra of curve as the “structure function”.

Let us recall the symmetric product of a variety. Let \( X \) be a quasi-projective variety over \( k \). For every \( n \in \mathbb{Z}_{\geq 1} \), we have a natural action of the symmetric group \( \mathfrak{S}_n \) on the product \( X^n \). Since \( X^n \) is again quasi-projective, the quotient of \( X^n \) by \( \mathfrak{S}_n \) exists. This is the \( n \)-th symmetric product of \( X \), which we denote by \( \text{Sym}^n(X) \). We define \( \text{Sym}^0(X) := \text{Spec}(k) \) for convention.
If $\mathfrak{t}$ is perfect, then $X^n$ is reduced, so that $\text{Sym}^n(X)$ is also reduced.

Kapranov [K] introduced the motivic analogue of the Hasse-Weil zeta functions for varieties as the generating series of the classes of the symmetric products in the Grothendieck ring $K(\text{Var}/\mathfrak{t})$. Before writing down the definition, we follow [Mus] to introduce a description of $K(\text{Var}/\mathfrak{t})$ in terms of quasi-projective varieties.

**Definition 2.3.** Let $\widetilde{K}(\text{Var}/\mathfrak{t})$ be the quotient of $K(\text{Var}/\mathfrak{t})$ by the subgroup generated by the relations $[X] - [Y]$, where there is a a radicial surjective morphism $X \to Y$ of varieties over $\mathfrak{t}$.

Note that $\widetilde{K}(\text{Var}/\mathfrak{t})$ is a quotient ring of $K(\text{Var}/\mathfrak{t})$. This is because if $f : X \to Y$ is surjective and radicial, then for every variety $Z$, the morphism $f \times \text{Id}_Z : X \times Z \to Y \times Z$ is surjective and radicial, for $f \times \text{Id}_Z$ is the base-change of $f$ with respect to the projection $Y \times Z \to Y$.

If $\text{char}(k) = 0$, then the canonical quotient map $K(\text{Var}/\mathfrak{t}) \to \widetilde{K}(\text{Var}/\mathfrak{t})$ is an isomorphism.

**Definition 2.4.** (1) Let $K^{\text{qpr}}(\text{Var}/\mathfrak{t})$ be the quotient of the free abelian group on isomorphism classes of quasi-projective varieties over $\mathfrak{t}$, modulo the relations $[X] = [Y] + [X \setminus Y]$

where $X$ is a quasi-projective variety and $Y$ is a closed subvariety of $X$. We have a group homomorphism from $K^{\text{qpr}}(\text{Var}/\mathfrak{t})$ to $K(\text{Var}/\mathfrak{t})$. Denote it by

$$\Phi : K^{\text{qpr}}(\text{Var}/\mathfrak{t}) \to K(\text{Var}/\mathfrak{t}).$$

(2) Define $\widetilde{K}^{\text{qpr}}(\text{Var}/\mathfrak{t})$ to be the quotient of $K^{\text{qpr}}(\text{Var}/\mathfrak{t})$ by the relations $[X] - [Y]$, where we have a surjective, radicial morphism of quasi-projective varieties $f : X \to Y$. Denote the corresponding group homomorphism by

$$\widetilde{\Phi} : \widetilde{K}^{\text{qpr}}(\text{Var}/\mathfrak{t}) \to \widetilde{K}(\text{Var}/\mathfrak{t}).$$

**Fact 2.5** ([Mus, Proposition 7.27]). If $\text{char}(k) = 0$, then $\Phi$ and $\widetilde{\Phi}$ are both isomorphisms.

Now we can explain the definition of the motivic zeta function.

**Definition 2.6** ([K, Mus]). For a quasi-projective variety $X$ over a field $\mathfrak{t}$, the *motivic zeta function* of $X$ is defined to be

$$\zeta_{\text{mot}}(X; z) := \sum_{n \geq 0} [\text{Sym}^n(X)] z^n \in 1 + z \cdot \widetilde{K}(\text{Var}/\mathfrak{t})[[z]].$$

**Fact 2.7** ([Mus, Proposition 7.28]). Assume $\mathfrak{t}$ is perfect. The map $[X] \to \zeta_{\text{mot}}(X; z)$ for $X$ a quasi-projective variety defines a group homomorphism $K(\text{Var}/\mathfrak{t}) \to 1 + t \cdot \widetilde{K}(\text{Var}/\mathfrak{t})[[t]]$, which factors through $\widetilde{K}(\text{Var}/\mathfrak{t})$.

Here is the rationality result of the motivic zeta function for a smooth projective curve, originally proved by Kapranov.

**Fact 2.8** ([K, (1.1.9) Theorem], [Mus, Theorem 7.33]). Let $\mathfrak{t}$ be a perfect field. If $X$ is a smooth, geometrically connected, projective curve of genus $g$ over $\mathfrak{t}$ which has a $\mathfrak{t}$-rational point, then $\zeta_{\text{mot}}(X; z)$ is a rational function. Moreover, we have

$$\zeta_{\text{mot}}(X; z) = \frac{f(z)}{(1 - z)(1 - Lz)}$$

for a polynomial $f$ of degree $\leq 2g$ with coefficients in $\widetilde{K}(\text{Var}/\mathfrak{t})$. 
Proof. One can prove this statement similarly as in the case of the classical Hasse-Weil zeta function. Assume for simplicity that $\mathfrak{f}$ is algebraically closed, so that a line bundle of degree 1 exists on $X$. Recall that the Abel map $a: \text{Sym}^d(X) \to \text{Pic}^0(X)$ to the Picard variety of degree 0 is the projective bundle with the fiber $a^{-1}(L) \cong \mathbb{P}(H^0(X, L)^*)$ for each $L \in \text{Pic}^0(X)$, so by the Riemann-Roch formula the fiber is isomorphic to $\mathbb{P}^{d-g}$ for $d \geq 2g - 1$, where $\mathbb{P}^n_k$ denotes the projective space of dimension $n$ over $k$. Thus we have

$$
\zeta_{\text{mot}}(X; z) = \sum_{d=0}^{2g-2} [\text{Sym}^d(X)] z^d + \sum_{d \geq \min(2g-1, 0)} [\text{Pic}^0(X)] : [\mathbb{P}^{d-g}] z^d.
$$

If $g \geq 1$, then a calculation gives

$$
\zeta_{\text{mot}}(X; z) = \sum_{d=0}^{2g-2} [\text{Sym}^d(X)] z^d + \frac{[\text{Pic}^0(X)] (L^{1-g} - 1) + z(L - L^{1-g})}{(1-z)(1-Lz)},
$$

and this expression gives the consequence. The case $g = 0$ is

$$
\zeta_{\text{mot}}(\mathbb{P}^1; z) = \frac{1}{(1-z)(1-Lz)}. \quad (2.1)
$$

For the proof in the case of the general base field $k$, see [Mus].

2.3. Grothendieck ring of stacks. By the word ‘stack’, we mean an Artin stack which is locally of finite type over $k$ unless otherwise stated. We denote by $\text{St}/k$ the 2-category of Artin stacks of finite type over $k$. Given a scheme $S$ over $k$ and a stack $X$, we denote by $X(S)$ the groupoid of $S$-valued points of $X$.

Definition 2.9. (1) A stack $X$ locally of finite type over $k$ is said to have affine stabilizers if for every $k$-valued point $x \in X(k)$ the group $\text{Iso}(x, x)$ of isomorphisms is affine.

(2) A morphism $f : X \to Y$ in the category $\text{St}/k$ will be called a geometric bijection if it is representable and the induced functor on groupoids of $k$-valued points $f(k) : X(k) \to Y(k)$ is an equivalence of categories.

Definition 2.10. Let $K(\text{St}/k)$ be the free abelian group spanned by isomorphism classes of stacks of finite type over $k$ with affine stabilizers, modulo relations

(a) $[X_1 \sqcup X_2] = [X_1] + [X_2]$ for every pair of stacks $X_1$ and $X_2$,

(b) $[X] = [\emptyset]$ for every geometric bijection $f : X \to \emptyset$,

(c) $[X_1] = [X_2]$ for every pair of Zariski fibrations $f_i : X_i \to \emptyset$ with the same fibres.

Although our abelian group $K(\text{St}/k)$ looks extremely large, there is a comparison between $K(\text{St}/k)$ and the $K(\text{Var}/k)$. Fiber product of stacks over $k$ gives $K(\text{St}/k)$ the structure of a commutative ring. We have a homomorphism of commutative rings

$$
K(\text{Var}/k) \to K(\text{St}/k) \quad (2.2)
$$

by considering a variety as a representable stack. Now we cite

Fact 2.11 ([Br12, Lemma 3.9]). The morphism (2.2) induces an isomorphism of commutative rings

$$
Q : K(\text{Var}/\mathbb{C})[\text{GL}_d^{-1} \mid d \in \mathbb{Z}_{\geq 1}] \xrightarrow{\sim} K(\text{St}/\mathbb{C}).
$$

See also [T, Theorem 3.10] and [Joy07a, Theorem 4.10] for the related results.
2.4. Relative Grothendieck ring. Fix an algebraic stack $\mathcal{S}$ which is locally of finite type over $\mathfrak{f}$ and has affine stabilizers. There is a 2-category of algebraic stacks over $\mathcal{S}$. Let $\text{St}/\mathcal{S}$ denote the full subcategory consisting of objects $f : \mathcal{X} \to \mathcal{S}$ for which $\mathcal{X}$ is of finite type over $\mathfrak{f}$. Such an object will be said to have affine stabilizers if the stack $\mathcal{X}$ has.

**Definition 2.12.** Let $K(\text{St}/\mathcal{S})$ be the free abelian group spanned by isomorphism classes of objects of $\text{St}/\mathcal{S}$ with affine stabilisers, modulo relations

(a) for every pair of objects $f_1 : \mathcal{X}_1 \to \mathcal{S}$ and $f_2 : \mathcal{X}_2 \to \mathcal{S}$, a relation

$$[\mathcal{X}_1 \sqcup \mathcal{X}_2 \xrightarrow{f_1 \sqcup f_2} \mathcal{S}] = [\mathcal{X}_1 \xrightarrow{f_1} \mathcal{S}] + [\mathcal{X}_2 \xrightarrow{f_2} \mathcal{S}]$$

(b) for every commutative diagram

$$\begin{array}{ccc}
\mathcal{X}_1 & \xrightarrow{g} & \mathcal{X}_2 \\
\downarrow{f_1} & & \downarrow{f_2} \\
\mathcal{S} & & \\
\end{array}$$

with $g$ a geometric bijection, a relation

$$[\mathcal{X}_1 \xrightarrow{f_1} \mathcal{S}] = [\mathcal{X}_2 \xrightarrow{f_2} \mathcal{S}]$$

(c) for every pair of Zariski fibrations $h_1 : \mathcal{X}_1 \to Y$ and $h_2 : \mathcal{X}_2 \to Y$ with the same fibers, and every morphism $g : Y \to \mathcal{S}$, a relation

$$[\mathcal{X}_1 \xrightarrow{g \circ h_1} \mathcal{S}] = [\mathcal{X}_1 \xrightarrow{g \circ h_2} \mathcal{S}]$$.

The group $K(\text{St}/\mathcal{S})$ has the structure of a $K(\text{St}/\mathfrak{f})$-module, defined by setting

$$[\mathcal{X}] \cdot [\mathcal{Y}] \xrightarrow{f} \mathcal{S} = [\mathcal{X} \times \mathcal{Y}] \xrightarrow{f \circ p_2} \mathcal{S}$$

and extending linearly.

**Fact 2.13 ([Br12, §3.5]).** Assume that all stacks appearing have affine stabilizers.

1. A morphism of stacks $a : \mathcal{S} \to \mathfrak{T}$ induces a pushforward morphism of $K(\text{St}/\mathfrak{f})$-modules

$$a_* : K(\text{St}/\mathcal{S}) \to K(\text{St}/\mathfrak{T})$$

sending $[\mathcal{X} \xrightarrow{f} \mathcal{S}]$ to $[\mathcal{X} \xrightarrow{a \circ f} \mathfrak{T}]$.

2. A morphism of stacks $a : \mathcal{S} \to \mathfrak{T}$ of finite type induces a pullback morphism of $K(\text{St}/\mathfrak{f})$-modules

$$a^* : K(\text{St}/\mathfrak{T}) \to K(\text{St}/\mathcal{S})$$

sending $[\mathcal{Y} \xrightarrow{g} \mathfrak{T}]$ to $[\mathcal{X} \xrightarrow{f} \mathcal{S}]$, where $f$ is the morphism appearing in the following Cartesian square.

$$\begin{array}{ccc}
\mathcal{X} & \xrightarrow{f} & \mathcal{Y} \\
\downarrow{a} & & \downarrow{g} \\
\mathcal{S} & & \\
\end{array}$$

3. The pushforwards and pullbacks are functorial, i.e., $(b \circ a)_* = b_* \circ a_*$ and $(b \circ a)^* = a^* \circ b^*$ for composable morphisms $a, b$ of stacks with the required properties.

4. Given a Cartesian square

$$\begin{array}{ccc}
\mathfrak{U} & \xrightarrow{c} & \mathcal{Y} \\
\downarrow{d} & & \downarrow{b} \\
\mathcal{S} & \xrightarrow{a} & \mathfrak{T} \\
\end{array}$$

of stacks, we have

$$b^* \circ a_* = c_* \circ d^* : K(\text{St}/\mathcal{S}) \to K(\text{St}/\mathcal{Y}).$$
For every pair of stacks \((\mathcal{S}_1, \mathcal{S}_2)\), we have a morphism (Künneth morphism)
\[ K : K(\text{St}/\mathcal{S}_1) \otimes K(\text{St}/\mathcal{S}_2) \to K(\text{St}/\mathcal{S}_1 \times \mathcal{S}_2) \]
of \(K(\text{St}/k)\)-modules given by
\[ [X_1 \xrightarrow{f_1} \mathcal{S}_1] \otimes [X_2 \xrightarrow{f_2} \mathcal{S}_2] \mapsto [X_1 \times X_2 \xrightarrow{f_1 \times f_2} \mathcal{S}_1 \times \mathcal{S}_2]. \]

3. Motivic Hall algebra

In this section we recall the notion of the motivic Hall algebra originally given by Joyce [Joy07b] for the framework of the Donaldson-Thomas type curve-counting invariant [Joy06a, Joy07a, Joy06b, JS12]. We mainly follow the notations given in the paper of Bridgeland [Br12].

Fix a field \(k\), and let \(X\) be a projective variety over \(k\).
Denote by \(\text{Coh}(X)\) the category of coherent sheaves on \(X\), and by \(\mathcal{D}_{\text{coh}}(X)\) the bounded derived category of \(\mathcal{O}_X\)-modules whose cohomology sheaves are coherent and which are zero except for only finitely many degrees.

Recall that an object of \(\mathcal{D}_{\text{coh}}(X)\) is called perfect if it is isomorphic in \(\mathcal{D}_{\text{coh}}(X)\) to a bounded complex of locally free sheaves of finite rank. A sheaf is called perfect if it is perfect as an object of \(\mathcal{D}_{\text{coh}}(X)\). If \(X\) is smooth, then every object of \(\mathcal{D}_{\text{coh}}(X)\) is perfect.

3.1. Moduli of flags. We begin with the setting of moduli stacks of flags of coherent sheaves on \(X\). Since we are considering projective varieties which are not necessarily smooth, the phrase ‘perfect’ will appear to circumvent technical issues. As mentioned above, if \(X\) is smooth, then one may remove the perfectness condition.

Let \(\mathcal{M}^{(n)} = \mathcal{M}^{(n)}(X)\) denote the moduli stack of \(n\)-flags of perfect coherent sheaves on \(X\).

The objects over a scheme \(S\) are chains of monomorphisms of perfect coherent sheaves on \(S \times X\) of the form
\[ 0 = \mathcal{E}_0 \hookrightarrow \mathcal{E}_1 \hookrightarrow \cdots \hookrightarrow \mathcal{E}_n = \mathcal{E} \tag{3.1} \]
such that each factor \(\mathcal{F}_i := \mathcal{E}_i / \mathcal{E}_{i-1}\) is \(S\)-flat. If
\[ 0 = \mathcal{E}_0' \hookrightarrow \mathcal{E}_1' \hookrightarrow \cdots \hookrightarrow \mathcal{E}_n' = \mathcal{E} \]
is another such object over a scheme \(T\), then a morphism in \(\mathcal{M}^{(n)}\) lying over a morphism of schemes \(f : T \to S\) is a collection of isomorphisms of sheaves
\[ \theta_i : f^*(\mathcal{E}_i) \to \mathcal{E}_i' \]
such that each diagram
\[
\begin{array}{ccc}
\mathcal{F}_i := \mathcal{E}_i / \mathcal{E}_{i-1} & \xrightarrow{\theta_i} & \mathcal{E}_i' \\
\downarrow f^* & & \downarrow \theta_{i+1} \\
\mathcal{E}_i & \xrightarrow{\theta_{i+1}} & \mathcal{E}_{i+1}'
\end{array}
\]
commutes. We also denote \(\mathcal{M} := \mathcal{M}^{(1)}\).

As in [Br12, Lemma 4.1], one can show that \(\mathcal{M}^{(n)}\) is an algebraic stack using the relative Quot scheme and induction on \(n\).

There are morphisms of stacks
\[ a_i : \mathcal{M}^{(n)} \to \mathcal{M} \]
for \(1 \leq i \leq n\), defined by sending a flag (3.1) to its \(i\)-th factor \(\mathcal{F}_i = \mathcal{E}_i / \mathcal{E}_{i-1}\). There is another morphism
\[ b : \mathcal{M}^{(n)} \to \mathcal{M} \]
sending a flag (3.1) to the sheaf \(\mathcal{E}_n = \mathcal{E}\).

The morphisms \(a_i\)'s and \(b\) satisfy the iso-fibration property in the sense of [Br12, §A.1]. A morphism \(f : \mathcal{X} \to \mathcal{Y}\) of stacks is called an iso-fibration if the following holds. Let \(S\) be
any scheme and θ : a → b be an isomorphism in the groupoid \( \mathcal{Y}(S) \). Suppose there is an \( a' \in \mathcal{X}(S) \) such that \( f(a') = a \). Then there is an isomorphism \( \theta' : a' \to b' \) in \( \mathcal{X}(S) \) such that \( f(\theta') = \theta \). See [Br12, Lemma A.1] for the claim which yields the iso-fibration property of \( a_i \)'s and \( b \).

Using the iso-fibration property of the morphisms \( a_i \) and \( b \), we find that for \( n \in \mathbb{Z}_{>1} \) there is a Cartesian square

\[
\begin{array}{c}
\mathcal{M}(n) \xrightarrow{t} \mathcal{M}(2), \\
\downarrow s \downarrow a_i \\
\mathcal{M}(n-1) \xrightarrow{b} \mathcal{M}
\end{array}
\]

where \( s \) and \( t \) send a flag (3.1) to the flags

\( \mathcal{E}_1 \hookrightarrow \ldots \hookrightarrow \mathcal{E}_{n-1} \) and \( \mathcal{E}_{n-1} \hookrightarrow \mathcal{E}_n \)

respectively.

As noted in [Br12], the morphism \( (a_1, a_2) \) is not representable. However the following lemma holds, and it will be used for the definition of the motivic Hall algebra.

**Lemma 3.1.** The morphism \( (a_1, a_2) \) is of finite type.

**Proof.** The proof of [Br12, Lemma 4.2] using regularity of sheaves can be applied in our situation since we are considering the perfect sheaves. \( \square \)

### 3.2. The definition of the motivic Hall algebra

Recall that we denote by \( \mathcal{M} = \mathcal{M}(X) \) the moduli stack of coherent sheaves over the smooth projective variety \( X \).

**Definition 3.2.**

1. Set \( H(X) := K(\text{St}/\mathcal{M}) \).

2. Consider the diagram

\[
\begin{array}{cc}
\mathcal{M}(2) & \mathcal{M} \\
\downarrow (a_1,a_2) \downarrow \mathcal{M} \times \mathcal{M}
\end{array}
\]

where the morphisms \( a_1, a_2 \) and \( b \) send a flag \( E_1 \hookrightarrow E \) to the sheaves \( E_1, E/E_1 \) and \( E \) respectively. Now introduce a morphism \( m : H(X) \otimes H(X) \to H(X) \) of \( K(\text{St}/\mathcal{E}) \)-modules by the composition

\[
m : H(X) \times H(X) = K(\text{St}/\mathcal{M}) \times K(\text{St}/\mathcal{M}) \xrightarrow{K} K(\text{St}/\mathcal{M} \times \mathcal{M})
\]

\[
\xrightarrow{(a_1,a_2)^*} K(\text{St}/\mathcal{M}(2)) \xrightarrow{b_*} K(\text{St}/\mathcal{M}) = H(X),
\]

which we call the convolution product. We will also use the symbol \( \odot \) for the convolution product as a binary operator.

**Remark 3.3.** The convolution product can be rewritten as

\[
[X_1 \xrightarrow{f_1} \mathcal{M}] \odot [X_2 \xrightarrow{f_2} \mathcal{M}] = [\mathfrak{Z} \xrightarrow{boh} \mathcal{M}],
\]

where \( \mathfrak{Z} \) and \( h \) are defined by the following Cartesian square.

\[
\begin{array}{c}
\mathfrak{Z} \xrightarrow{h} \mathcal{M}(2) \\
\downarrow \downarrow \mathcal{M}(2)
\end{array}
\]

\[
\begin{array}{c}
\mathfrak{Z} \xrightarrow{h} \mathcal{M}(2) \\
\downarrow \downarrow \mathcal{M}(2)
\end{array}
\]

\[
\begin{array}{c}
\mathfrak{Z} \xrightarrow{h} \mathcal{M}(2) \\
\downarrow \downarrow \mathcal{M}(2)
\end{array}
\]

where \( \mathfrak{Z} \) and \( h \) are defined by the following Cartesian square.
Fact 3.4 ([Br12, Theorem 4.3]). The convolution product $m$ gives $H(X)$ the structure of an associative unital algebra over $K(\text{St}/t)$. The unit element is $1 := [M_0 \hookrightarrow M]$, where $M_0 \cong \text{Spec}(k)$ is the substack of zero objects.

3.3. Coproduct. As in the case of the ordinary Ringel-Hall algebra, the motivic Hall algebra $H(X)$ has a coproduct, and if $\dim X = 1$ then $H(X)$ is a bialgebra. Also one can introduce the extended algebra and the Hall pairing similar to the ordinary Ringel-Hall algebra. In this subsection we introduce the motivic version of these notions. We assume the readers are familiar to the ordinary Ringel-Hall case, and omit some detailed arguments. For the ordinary case, we cite [G95] as the original literature, and cite [S, Lect. 1] as a nice review.

Hereafter let $X$ be a smooth projective variety over a field $k$. The Euler form $\chi(E, F) := \sum_i \dim_k \text{Ext}^i(E, F)$ on the category $\text{Coh}(X)$ induces a bilinear form on the Grothendieck group $K(\text{Coh}(X))$. Denote the left radical with respect to this form by $K(\text{Coh}(X))^\perp$, and set

$$\text{Num}(X) := K(\text{Coh}(X))/K(\text{Coh}(X))^\perp, \quad (3.2)$$

which is usually called the numerical Grothendieck group. Let $\Gamma \subset \text{Num}(X)$ be the submonoid generated by the effective classes. Throughout this paper, an element of $K(\text{Coh}(X))$ or $\text{Num}(X)$ representing $E \in \text{Coh}(X)$ is denoted by $E \in K(\text{Coh}(X))$ or $\text{Num}(X)$.

Then the stack $\mathcal{M} = \mathcal{M}(X)$ has a decomposition

$$\mathcal{M} = \bigsqcup_{\alpha \in \Gamma} \mathcal{M}_\alpha.$$ 

Here $\mathcal{M}_\alpha$ denotes the open and closed substack of $\mathcal{M}(X)$ whose objects have the classes $\alpha \in \Gamma$ in $\text{Num}(X)$.

The injection $\mathcal{M}_\alpha \hookrightarrow \mathcal{M}$ induces a $K(\text{St}/t)$-module injective homomorphism $K(\text{St}/\mathcal{M}_\alpha) \hookrightarrow K(\text{St}/\mathcal{M})$, and the above decomposition yields the direct sum decomposition

$$H(X) = \bigoplus_{\alpha \in \Gamma} H(X)_\alpha, \quad H(X)_\alpha := K(\text{St}/\mathcal{M}_\alpha) \quad (3.3)$$

of $K(\text{St}/t)$-module. Then $H(X)$ is a $\Gamma$-graded algebra with respect to $\diamond$.

Let us fix a square root $L^{1/2} = \sqrt{L}$ of $L$, and set

$$K := K(\text{St}/t)[L^{\pm 1/2}].$$

Definition 3.5. (1) Let $H_{tw}(X)$ be the $K$-algebra structure on $H(X) = K(\text{St}/\mathcal{M}(X))$ with the product

$$x_\alpha \diamond x_\beta := L^{\chi(\alpha, \beta)/2} x_\alpha \diamond x_\beta.$$ 

Here $x_\alpha$ (resp. $x_\beta$) is an element of $H(X)_\alpha$ (resp. $H(X)_\beta$) in the decomposition (3.3).

(2) Let $H_{\text{ext}}(X)$ be the $K$-algebra which is obtained as an extension of $H_{tw}(X)$ by

$$\{k_\alpha \mid \alpha \in \text{Num}(X)\}$$

with the following relations.

$$k_\alpha \diamond k_\beta = k_{\alpha + \beta}, \quad k_\alpha \diamond x_\beta = L^{(\chi(\alpha, \beta) + \chi(\beta, \alpha))/2} x_\beta \diamond k_\alpha.$$ 

(3.4)

Here $\alpha, \beta \in \text{Num}(X)$ and $x_\beta \in H(X)_\beta$.

We immediately have
Lemma 3.6. The extended motivic Hall algebra $H_{\text{ext}}(X)$ is $\Gamma$-graded algebra with respect to $\ast$.

Next we introduce the motivic version of the Green coproduct [G95]. Denote by $\hat{\otimes}$ the completion of the tensor product with respect to the grading (3.3).

**Definition 3.7.** (1) Define $\Delta : H(X) \to H(X) \hat{\otimes} H(X)$ as

$$\Delta : H(X) = K(\text{St}/\mathcal{M}) \xrightarrow{b^*} K(\text{St}/\mathcal{M}^{(2)}) \xrightarrow{(a_1,a_2)_*} K(\text{St}/\mathcal{M} \times \mathcal{M}) \to K(\mathcal{M}) \hat{\otimes} K(\mathcal{M}) = H(X) \hat{\otimes} H(X).$$

(2) Define $\Delta : H_{\text{ext}}(X) \to H_{\text{ext}}(X) \hat{\otimes} H_{\text{ext}}(X)$ by

$$\Delta(xk_\alpha) = \delta_\alpha \circ \Delta(x).$$

Here $\delta_\alpha : H(X) \hat{\otimes} H(X) \to H_{\text{ext}}(X) \hat{\otimes} H_{\text{ext}}(X)$ is defined as a linear extension of

$$\delta_\alpha(x \otimes y) := x k_{\alpha} \otimes y k_{\alpha}$$

with $y_\beta \in H(X)_\beta$ and $x \in H(X)$.

As an immediate consequence of the ordinary Ringel-Hall algebra associated to a finitary abelian category of global dimension 1 [G95], we have

**Proposition 3.8.** If $X$ is a smooth projective curve over a field $\mathfrak{f}$, then $H_{\text{ext}}(X)$ is a $K$-bialgebra with respect to $\ast$ and $\Delta$.

Next we introduce the motivic version of Green’s scalar product.

**Definition 3.9.** Define the non-degenerate bilinear form $(\cdot, \cdot)_H$ on $H_{\text{ext}}(X)$ by

$$(\cdot, \cdot)_H : H_{\text{ext}}(X) \otimes H_{\text{ext}}(X) \to K,$$

$$(\mathbb{M}_E)_{k_\alpha}, (\mathbb{M}_F)_{k_\beta})_H := \delta_{E,F} \frac{a_E}{a_F} \frac{\langle \chi(\alpha,\beta) + \chi(\beta,\alpha) \rangle / 2}{\langle \chi(\alpha,\beta) \rangle},$$

where $E, F \in \text{Coh}(X)$ and $\mathbb{M}_E$ (resp. $\mathbb{M}_F$) denotes the moduli stack of coherent sheaves isomorphic to $E$ (resp. $F$). The symbol $[\mathbb{M}_E]$ is the abbreviation of $[\mathbb{M}_E \hookrightarrow \mathcal{M}]$. $[\mathbb{M}_F]$ is similar. Finally

$$a_E := [\text{Aut}(E)] \in K(\text{Var}/\mathfrak{f})$$

is the class in $K(\text{Var}/\mathfrak{f})$ of the algebraic group of automorphisms of $E$.

**Proposition 3.10.** If $X$ is a smooth projective curve, then $(\cdot, \cdot)_H$ is a Hopf pairing, namely $(x \ast y, z)_H = (x \otimes y, \Delta z)_H$ holds. Here the bilinear form on the tensor product is defined as $(x \otimes y, z \otimes w)_H := (x, z)_H (y, w)_H$.

3.4. Steinitz’s classical Hall algebra and the subalgebra generated by torsion sheaves. Before starting the detailed discussion of the motivic Hall algebra $H(X)$ for a smooth projective curve $X$, we summarize here the result on the classical Hall algebra of Steinitz in terms of motivic language. It will be necessary for the argument on the subalgebra of $H(X)$ generated by the elements corresponding to torsion sheaves on $X$.

Consider a projective variety $X$ defined over a fixed field $\mathfrak{f}$. The category $\text{Tor}^{\text{perf}}(X)$ of perfect torsion sheaves is abelian and closed under extension. If $X$ is smooth, then $\text{Tor}^{\text{perf}}(X) = \text{Tor}(X)$, which is the category of torsion sheaves on $X$. $\text{Tor}^{\text{perf}}(X)$ also has a decomposition

$$\text{Tor}^{\text{perf}}(X) = \prod_x \text{Tor}_x$$

(3.5)

where $x$ ranges over the set of regular points of $X$ and $\text{Tor}_x$ is the category of torsion sheaves supported at $x$. $\text{Tor}_x$ is equivalent to the category of finite-dimensional modules over the local ring at $x$. If moreover dim $X = 1$, then, by the regularity of $x$, $\text{Tor}_x$ is also equivalent
to the category of nilpotent representations of the Jordan quiver over the residue field \( \mathfrak{e}_x \) at \( x \).

Let us take a regular point \( x \) of a projective variety \( X \), and denote by \( \mathcal{M}_{\text{tor},x} \) the moduli stack of torsion sheaves supported on \( x \). It is a substack of \( \mathcal{M} = \mathcal{M}(X) \), and we can apply the definition and arguments of the motivic Hall algebra to \( \mathcal{M}_{\text{tor},x} \).

**Definition 3.11.** For a regular point \( x \in X \), let us denote by \( H(X)_{\text{tor},x} \) the motivic Hall algebra which is \( K(\text{St/} \mathcal{M}_{\text{tor},x}) \) as a \( K \)-module.

Obviously \( H(X)_{\text{tor},x} \) is a subalgebra of \( H(X) \). If \( \dim X = 1 \), then it is also a sub-bialgebra since it is closed under the coproduct \( \Delta \).

Recall that the ordinary Ringel-Hall algebra of torsion sheaves supported at a closed point is isomorphic to Steinitz’s classical Hall algebra (see [S, Lect. 2] for instance). Here we present the result in the motivic language.

**Lemma 3.12.** Let \( C_{\text{sm}} \) be a smooth projective curve.

1. The bialgebra \( H(C_{\text{sm}})_{\text{tor},x} \) is commutative and co-commutative.
2. As an algebra \( H(C_{\text{sm}})_{\text{tor},x} \) is isomorphic to polynomial algebra \( K[e_{1,x}, e_{2,x}, \ldots] \) with infinite generators \( \{ e_{d,x} \}_{d \in \mathbb{Z}_{\geq 1}} \).
3. Denote by \( \mathfrak{e}_x \) the residue field of the point \( x \). Then for any \( d \in \mathbb{Z}_{\geq 1} \) we have
   \[
   \Delta(e_{d,x}) = \sum_{r=0}^{d} L_x^{-r(d-r)} e_{r,x} \otimes e_{n-r,x},
   \]
   where \( L_x = [\mathbb{A}^1] \in K(\text{Var/} \mathfrak{e}_x) \) is the class of affine line defined over the field \( \mathfrak{e}_x \).
4. The Hall pairing \( (\cdot, \cdot)_H \) on \( H(C_{\text{sm}})_{\text{tor},x} \) is given by
   \[
   (e_{m,x}, e_{n,x})_H = \frac{\delta_{m,n}}{L_x^n(1 - L_x^{-1})}.
   \]

**Proof.** For the ordinary Ringel-Hall algebra case, see for example [S, Theorem 2.6], where the representation of Jordan quiver is utilized. The proof works in the motivic setting. \( \square \)

Now let us turn to Steinitz’s Hall algebra. We follow [Mac95] on the notations of symmetric functions.

**Definition 3.13.**

1. Denote by \( \Lambda \) the space of symmetric functions over \( \mathbb{Z} \).
2. The \( n \)-th elementary symmetric function \( \sum_{1 \leq i_1 < \cdots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n} \) is denoted by \( e_n \).
3. Denote by \( p_n := \sum x_i^n \) the \( n \)-th power-sum symmetric function.

Recall that \( e_n \)’s give rise to a basis \( \{ e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots \} \lambda \) of \( \Lambda \) parametrized by partitions \( \lambda = (\lambda_1, \lambda_2, \ldots) \), and \( p_n \)’s give rise to a \( \mathbb{Q} \)-basis \( \{ p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots \} \lambda \) of \( \Lambda \otimes \mathbb{Q} \). \( \Lambda \) is an associative commutative algebra under the (usual) multiplication. It also has a coassociative cocommutative coproduct \( \Delta \) given by
   \[
   \Delta(e_n) = \sum_{r=0}^{n} e_r \otimes e_{n-r},
   \]
   which makes \( \Lambda \) into a bialgebra. This bialgebra is also endowed with the Hopf pairing \( (\cdot, \cdot) \) give by
   \[
   (p_m, p_n) = \delta_{m,n} \frac{n}{q^n - 1},
   \]
   for an indeterminate \( q \).

**Definition 3.14.** Denote by \( \Lambda_q \) the bialgebra \( \Lambda \otimes \mathbb{Q}[q^{\pm 1}] \), where the product is given by the multiplication of symmetric functions and the coproduct is given by (3.7). We always consider \( \Lambda_q \) endowed with the Hall pairing (3.8).
It is also well-known that the Hall-Littlewood symmetric functions form the unique orthonormal basis of $\Lambda \otimes \mathbb{Q}[q^{\pm 1}]$ with respect to this pairing subject to a triangular condition in the expansion with respect to monomial symmetric functions.

Comparing the bialgebra structures on $\Lambda_q$ and $H(C_{sm})_{tor,x}$, we have the following result.

**Corollary 3.15.** If $C_{sm}$ is a smooth projective curve and $x$ is an arbitrary closed point of $C_{sm}$, then we have an isomorphism of bialgebras

$$\phi_x : H(C_{sm})_{tor,x} \to \Lambda_{L_x}, \quad e_{d,x} \mapsto L_x^{-d(d-1)/2} e_d.$$  

This map is also an isometry in terms of the Hall pairing $(\cdot, \cdot)_H$ given in (3.6) and the pairing $(\cdot, \cdot)$ in (3.8).

Finally we will introduce some notations on the subalgebra of $H(X)$ generated by torsion sheaves. Let us go back to the general situation where $X$ is a projective variety.

**Definition 3.16.** Let $X$ be a projective variety.

1. Denote by $M_{tor}$ the moduli stack of perfect torsion sheaves on $X$.
2. Denote by $H(X)_{tor} = K(\text{St}/M_{tor})$ the motivic Hall algebra generated by perfect torsion sheaves on $X$.

For a smooth curve $C_{sm}$, $H(C_{sm})_{tor}$ is a sub-bialgebra of $H(C_{sm})$ since $\text{Tor}_{perf}(C_{sm}) = \text{Tor}(C_{sm})$ is closed under extension and splitting. For any projective variety $X$, the decomposition (3.5) yields

$$H(X)_{tor} = \bigotimes_x H(X)_{tor,x}$$

as $\mathbb{K}$-algebras, where $x$ runs over the set of regular points of $X$.

We close this subsection by introduction of elements of $H(X)_{tor}$ for future use.

**Definition 3.17.** For $d \in \mathbb{Z}_{\geq 1}$ define $t_{d,x} \in H(X)_{tor,x}$ by

$$t_{d,x} := \begin{cases} [d]^{\deg(x)} \phi_x^{-1}(p_{d/\deg(x)}) & \text{if } \deg(x)|d \\ 0 & \text{otherwise} \end{cases}$$

and

$$t_d := \sum_{x \in X} t_{d,x} \in H(X)_{tor}.$$  

**Definition 3.18.** Let $M_{(0,d)}$ be the moduli stack of perfect torsion sheaves on $X$ with degree $d \in \mathbb{Z}_{\geq 1}$. It is naturally a substack of $M$. Define the element $1_{(0,d)} \in H(X)$ by

$$1_{(0,d)} := [M_{(0,d)}] \hookrightarrow M.$$  

The final remark is

**Lemma 3.19 ([S, Lemma 4.10]).** The elements $t_d \in H(X)$ satisfies

$$1 + \sum_{d \geq 1} 1_{(0,d)} z^d = \exp \left( \sum_{d \geq 1} \frac{t_d}{[d] \sqrt{\pi}} z^d \right). \quad (3.9)$$

**Proof.** We copy the proof given in [S, Lemma 4.10]. In the decomposition $H(X)_{tor} \simeq \bigotimes_x H(X)_{tor,x}$ we have

$$\exp \left( \sum_{d \geq 1} \frac{t_d}{[d]} z^d \right) = \prod_x \exp \left( \sum_{d \geq 1} \frac{t_{d,x}}{[d]} z^d \right)$$

and

$$1 + \sum_{d \geq 1} 1_{(0,d)} z^d = \prod_x (1 + \sum_{d \geq 1} 1_{(0,d),x} z^d)$$
with \(1_{(0,d),x} := [\mathcal{M}_{(0,d),x} \hookrightarrow \mathcal{M}_{\text{tor},x}]\), it is enough to show that
\[
\exp\left(\sum_{d \geq 1} \frac{t_d x}{d} z^d\right) = 1 + \sum_{d \geq 1} 1_{(0,d),x} z^d.
\]
Using Definition 3.17, we have
\[
\exp\left(\sum_{d \geq 1} \frac{t_d x}{d} z^d\right) = \phi_x^{-1}\left(\exp\left(\sum_{d \geq 1} z^{d\deg(x)} \frac{p_d}{d}\right)\right)
\]
\[
= \phi_x^{-1}\left(\exp\left(\sum_{d \geq 1} z^{d\deg(x)} \frac{p_d}{d}\right)\right) = \phi_x^{-1}(1 + \sum_{d \geq 1} z^{d\deg(x)} h_d)
\]
In the last equality we used the generating function formula of the complete symmetric functions \(h_n\)'s. Then by the definition of \(\mathcal{M}_{(0,d),x}\) we have the desired consequence. \(\square\)

3.5. The composition subalgebra and the Drinfeld double. In this subsection \(C\) is a projective curve over a field \(\mathbb{k}\). The motivic Hall algebra is in general too big to study, and we want to introduce a subalgebra which is easy to handle. We follow the work of Schiffmann and Vasserot [SV11, §6] where they consider the subalgebra generated by certain averages of line bundles and torsion sheaves.

**Definition 3.20.** Let \(\mathcal{M}_{\text{tor}} = \mathcal{M}_{\text{tor}}(C)\) be the moduli stack of perfect torsion sheaves on \(C\) and \(\mathcal{M}_{\text{lf},1} = \mathcal{M}_{\text{lf},1}(C)\) be the moduli stack of line bundles on \(C\).

1. The subalgebra of \(\text{H}_{\text{ext}}(C)\) generated by these substacks \(\mathcal{M}_{\text{tor}}\) and \(\mathcal{M}_{\text{lf},1}\) is called the **composition subalgebra** and denoted by \(U(C)\).

2. The subalgebra of \(\text{H}_{\text{ext}}(C)\) generated by these substacks and \(\{k_\alpha \mid \alpha \in \text{Num}(C)\}\) is called the **extended composition subalgebra** and denoted by \(U_{\text{ext}}(C)\).

**Remark 3.21.**
1. For \(n \in \mathbb{Z}\), denote by \(\mathcal{M}_{\text{lf},(1,n)}\) the moduli stack of line bundles of degree \(n\). Also we set
\[
1_{\text{ss}}^{(1,n)} := [\mathcal{M}_{\text{lf},(1,n)} \hookrightarrow \mathcal{M}] \in \text{H}(C).
\]
   (If denotes the word ‘locally free’, and ss denotes the word ‘semi-stable’.) Recall the elements \(t_d \in \text{H}(C)_{\text{tor}}\) given in Definition 3.17. Then \(U(C)\) is generated by
\[
1_{\text{ss}}^{(1,n)} (n \in \mathbb{Z}), \quad t_d (d \in \mathbb{Z}_{\geq 1})
\]

2. For a line bundle \(L\) on \(C\), let us denote by \(\mathcal{M}_{L}\) the moduli stack of coherent sheaves isomorphic to \(L\). Then we have \(\mathcal{M}_{\text{lf},(1,n)} = \sqcup_{L \in \text{Pic}^n(C)} \mathcal{M}_{L}\). Thus \(1_{\text{ss}}^{(1,n)}\) is a summation of the elements associated to \(L\). It corresponds to the generator \(\sum_{L \in \text{Pic}^{n}(C)} [L] \) of the composition algebra in [SV11]. The generator of the other type \(\sum_{T \in \text{Tor}(C), \deg(T) = d} [T]\) appearing in [SV11] corresponds to our \(1_{(0,d)}\).

By the definition of the coproduct \(\Delta\), we have

**Lemma 3.22.** For a smooth curve \(C_{\text{sm}}\), \(U(C_{\text{sm}})\) and \(U_{\text{ext}}(C_{\text{sm}})\) are closed under the coproduct \(\Delta\). Hence they are formal \(\mathbb{K}\)-bialgebra.

Recalling that \(\Gamma \subset \text{Num}(C)\) is the submonoid generated by the effective classes (see (3.2) and the lines below it), we also have

**Lemma 3.23.** \(U(C)\) and \(U_{\text{ext}}(C)\) are \(\Gamma\)-graded algebras.

Next we recall the notion of Drinfeld double, following [Jos95, §3.2] and [BuS12, Appendix B].

**Fact 3.24 (\cite{D86}).** Let \(H\) be a (topological) bialgebra with a Hopf pairing \((\cdot, \cdot)_H\). Let \(H^+ := H\) be the bialgebra \(H\) itself, and \(H^-\) be the bialgebra which is isomorphic to \(H\) as algebra and equipped with the opposite coproduct. Finally let \(DH\) be the associative algebra generated by \(H^\pm\) modulo the relations
(1) \( H^{\pm} \) are subalgebras.
(2) For \( a, b \in H \)
\[
\sum a^{\pm}_1 b^\pm_2 (a^{(2)}, b^{(1)})_H = \sum b^\pm_1 a^\pm_2 (a^{(1)}, b^{(2)})_H.
\]
Here we used the Sweedler notation \( \Delta(a) = \sum a^{(1)} \otimes a^{(2)} \).

Then the multiplication map \( H^{+} \otimes H^{-} \rightarrow DH \) is an isomorphism of vector spaces, and \( DH \) has a unique bialgebra structure such that the map \( H^{+} \rightarrow DH \) given by \( a \mapsto a \otimes 1 \) and the one \( H^{-} \rightarrow DH \) given by \( a \mapsto 1 \otimes a \) are both injections of bialgebras.

**Definition 3.25.** We call \( DH \) the Drinfeld double of the bialgebra \( H \) with respect to the Hopf pairing \( (\cdot, \cdot)_H \).

We also need the reduced version of the Drinfeld double. Let \( g \) be a Kac-Moody Lie algebra and \( g' \) its derived algebra. Denote by \( U_q(b') \) the quantum enveloping algebra of a Borel subalgebra \( b' \subset g' \). It is closed under the coproduct and equipped with a Hopf pairing.

**Fact 3.26** ([EdG6]). Let \( \mathbb{C}[q^{\pm 1}][K_i \mid i \in I] \) be the quantised enveloping algebra of the Cartan subalgebra \( h \subset b \). Then we have an isomorphism of bialgebras
\[
DU_q(b')/(K_i \otimes K_i^{-1} - 1 \mid i \in I) \simeq U_q(g),
\]
Xiao [X97] introduced the reduced Drinfeld double for the ordinary Ringel-Hall algebra. Here we mimic his definition in the motivic case.

**Definition 3.27.** Let \( C_{sm} \) be an irreducible smooth curve and consider the motivic Hall algebra \( H(C_{sm}) \) and the extended algebra \( H_{ext}(C_{sm}) \) equipped with the Hopf pairing \( (\cdot, \cdot)_H \) in Definition 3.9. Define the reduced Drinfeld double \( D_{red}H(C_{sm}) \) of \( H(C_{sm}) \) to be
\[
D_{red}H(C_{sm}) := DH_{ext}(C_{sm})/(k_\alpha \otimes k_\alpha^{-1} - 1 \mid \alpha \in Num(X)).
\]

We may also consider the reduced Drinfeld double of the extended composition subalgebra \( U_{ext}(X) \), since it is a bialgebra with the Hopf pairing \( (\cdot, \cdot)_H \) and contains the extension part \( \{k_\alpha \mid \alpha \in Num(X)\} \).

**Definition 3.28.** Denote by \( D_{red}U(C_{sm}) \) the reduced Drinfeld double of \( U(C_{sm}) \), which is defined to be
\[
D_{red}U(C_{sm}) := DU_{ext}(C_{sm})/(k_\alpha^+ \otimes k_{-\alpha}^+ - 1 \mid \alpha \in Num(C_{sm})).
\]

We immediately have

**Lemma 3.29.** \( D_{red}H(C_{sm}) \) and \( D_{red}U(C_{sm}) \) are \( Num(C_{sm}) \)-graded.

We also have the following triangular decomposition.

**Lemma 3.30.** The multiplication map gives an isomorphism
\[
U(C_{sm})^+ \otimes_k \mathbb{K} X \otimes_k U(C_{sm})^- \rightarrow D_{red}U(C_{sm})
\]
of modules over \( \mathbb{K} = K(\text{St/}\mathfrak{t})[L^{1/2}] \), where \( U(C_{sm})^{\pm} \) are the copies of \( U(C_{sm}) \) and \( \mathbb{K}_{C_{sm}} := \mathbb{K}[Num(C_{sm})] \).

4. The case of projective line

In this section we review the work of Kapranov [K97], which identifies the composition subalgebra of the Ringel-Hall algebra for the projective line \( \mathbb{P}^1 \) with the upper triangular part of the quantum affine algebra \( U_q(\hat{\mathfrak{sl}_2}) \). We will present this review in terms of the motivic Hall algebra. Several notations in this subsection are due to the review of Schiffmann [S] and the paper of Burban and Schiffmann [BuS13]. Let us also refer the paper of Baumann and Kassel [BaK01] for the detailed account.
In this section we sometimes use the following symbols for \( d \in \mathbb{Z} \).

\[
[d]_\mathcal{E} := \frac{L^{d/2} - L^{-d/2}}{L^{1/2} - L^{-1/2}} = L^{-(d-1)/2}(1 + L + \cdots + L^{d-1}) \in \mathbb{K} = K(\text{St}/\mathbb{C})[L^{\pm 1/2}]
\]

### 4.1. The motivic Hall algebra of projective line.

Let \( \mathbb{P}^1 \) be the projective line defined over a fixed field \( \mathfrak{f} \). In the setting of the last subsection, set \( \mathfrak{M} := \mathfrak{M}(\mathbb{P}^1) \), the moduli stack of coherent sheaves on \( \mathbb{P}^1 \). Our algebra \( H_{\text{ext}}(\mathbb{P}^1) = K(\text{St}/\mathfrak{M}) \) is defined on the ring \( \mathbb{K} := K(\text{St}/\mathfrak{f})[L^{\pm 1/2}] \).

**Fact 4.1.** The indecomposable objects of the category \( \text{Coh}(\mathbb{P}^1) \) are

1. line bundles \( \mathcal{O}_{\mathbb{P}^1}(n) \) \((n \in \mathbb{Z})\),
2. torsion sheaves \( \mathcal{O}_{l[x]} := \mathcal{O}_{\mathbb{P}^1}/\mathfrak{m}_x^l \), where \( x \) is a closed point of \( \mathbb{P}^1 \) and \( l \in \mathbb{Z}_{\geq 1} \).

Let us recall that the numerical Grothendieck group of \( \text{Coh}(\mathbb{P}^1) \) has the description \( \text{Num}(\text{Coh}(\mathbb{P}^1)) \cong \mathbb{Z}^2 \), induced by the map

\[
\text{Coh}(\mathbb{P}^1) \ni E \mapsto (\text{rk}(E), \deg(E)) \in \mathbb{Z}^2
\]

attaching the rank and degree to each coherent sheaf \( E \). The Euler pairing is calculated as

\[
\chi((r_1, d_1), (r_2, d_2)) = r_1 r_2 + r_1 d_2 - r_2 d_1.
\]

(4.1)

In particular the extended part \( \{ k_\alpha \mid \alpha \in \text{Num}(X) \} \) of \( H_{\text{ext}}(\mathbb{P}^1) \) is generated by the two elements

\[
k := k_{(1,0)}, \quad c := k_{(0,1)}.
\]

Thus, by Remark 3.21, the composition subalgebra \( U(\mathbb{P}^1) \) is generated by

\[
1_{ss}^{1}\, (n \in \mathbb{Z}), \quad t_r \, (r \in \mathbb{Z}_{\geq 1}), \quad k, \quad c,
\]

where we set

\[
1_{ss}^{1} := [\mathfrak{M}_{lf,(1,1)} \hookrightarrow \mathfrak{M}] \in H(\mathbb{P}^1)
\]

with \( \mathfrak{M}_{lf,(1,1)} \) the moduli stack of line bundles of degree \( n \).

Now we can explain the relation of these generators derived by Kapranov [K97, §5], where a general framework of the Hopf algebra of automorphic forms was used. Let us note that Baumann and Kassel [BaK01] also computed the relation by giving explicit description of extensions of line bundles and torsion sheaves.

**Fact 4.2.** The elements \( 1_{ss}^{1}, t_r, k \) and \( c \) satisfy the following relations.

1. \( c \) is central.
2. \( [k, t_r] = 0 = [t_r, t_s] \) for all \( r, s \in \mathbb{Z}_{\geq 1} \).
3. \( k \ast 1_{ss}^{1} = L 1_{ss}^{1} \ast k \) for all \( n \in \mathbb{Z} \).
4. \( [t_r, 1_{ss}^{1}]=\frac{2[r]}{r} 1_{ss}^{1}\) for all \( n \in \mathbb{Z} \) and \( r \in \mathbb{Z}_{\geq 1} \).
5. \( 1_{ss}^{1} \ast 1_{ss}^{1} \ast 1_{ss}^{1} \) for all \( n, m, n+1 \in \mathbb{Z} \).

Here \([, ,] \) denotes the commutator with respect to the product \( * \) on \( H_{\text{ext}}(\mathbb{P}^1) \).

**Proof.** We sketch the outline of the proof following [BaK01, §§2–3] and [S, §4.3].

The relation (1), \( [k, t_r] = 0 \) in (2), and (3) can be obtained by the definition (3.4) of the product \( * \) and the formula (4.1). Namely, from \( \chi((0, d_1), (r_2, d_2)) + \chi((r_2, d_2), (0, d_1)) = 0 \) we obtain (1) and \([k, t_r] = 0 \). Also from \( \chi((1, 0), (1, n)) + \chi((1, n), (1, 0)) = 2 \) we obtain (3).

The relation \( [t_r, t_s] = 0 \) in (2) holds obviously since there is no non-trivial extension between torsion sheaves.

For the relation (4), recall that an extension of the torsion sheaf \( \mathcal{O}_{l[x]} \) by the line bundle \( \mathcal{O}_{\mathbb{P}^1}(n) \) is isomorphic to \( \mathcal{O}_{\mathbb{P}^1}(n + m) \oplus \mathcal{O}_{(l-m)[x]} \):

\[
\begin{array}{c}
0 \rightarrow \mathcal{O}_{\mathbb{P}^1}(n) \rightarrow \mathcal{O}_{\mathbb{P}^1}(n + m) \oplus \mathcal{O}_{(l-m)[x]} \rightarrow \mathcal{O}_{l[x]} \rightarrow 0.
\end{array}
\]
Note that a morphism $f : \mathcal{O}_{P^1}(n) \to \mathcal{O}_{P^1}(n + m) \oplus \mathcal{O}_{(l-m)[z]}$ is injective if and only if the image of $f$ is not included in $\mathcal{O}_{(l-m)[z]}$. Then by the formulas

$$\dim_t \text{Hom}(\mathcal{O}_{P^1}(m), \mathcal{O}_{P^1}(n)) = \max(0, n - m + 1),$$
$$\dim_t \text{Hom}(\mathcal{O}_{P^1}(m), \mathcal{O}_{(l)[z]} = l$$

of the dimension of vector spaces of morphisms, we find that the stack of extensions of the above form is given by $\mathbb{P}(\mathbb{A}^{m+1} \times \mathbb{A}^{l-m}) \setminus \mathbb{P}(\mathbb{A}^{l-m})$, whose class in $K(\text{Var}/\mathfrak{t})$ is $\mathbb{L}^{l-m}[m + 1]_L$ (by Lemma 2.2 (2)). Thus we have

$$\dim_t \text{Hom}(\mathcal{O}_{P^1}(m), \mathcal{O}_{P^1}(n)) = \max(0, n - m + 1),$$
$$\dim_t \text{Hom}(\mathcal{O}_{P^1}(m), \mathcal{O}_{(l)[z]} = l$$

Thus we have

$$1_{(1,n)}*1^\text{ss}_{(1,n)} = \sqrt{L}^{-l} \sum_{0 \leq m \leq l} \mathbb{L}^{l-m}[m + 1]_L[\mathcal{M}_{\mathcal{O}_{P^1}(n+m) \oplus T} \twoheadrightarrow \mathcal{M}]$$
$$= \sum_{0 \leq m \leq l} \sqrt{L}^{-m}[m + 1]_Z[\mathcal{M}_{\mathcal{O}_{P^1}(n+m) \oplus T} \twoheadrightarrow \mathcal{M}]$$
$$= \sum_{0 \leq m \leq l} [m + 1]_Z \sum_{a \geq l} 1_{(0,l-m)}$$

(4.2)

Here $\mathcal{M}_{\mathcal{O}_{P^1}(n+m) \oplus T}$ is the moduli stack of coherent sheaves isomorphic to $\mathcal{O}_{P^1}(n+m) \oplus T$ with $T$ a torsion sheaf of degree $l - m$. In the last equality, we used the fact that the extension in the opposite way is always trivial, i.e., $\text{Ext}^1(\mathcal{O}_{P^1}(n), \mathcal{O}_{(l)[z]} = 0$. Then the equation (4.2) can be rewritten as

$$\zeta_{\text{mot}}(\mathbb{P}^1; w/\sqrt{L}z)e(z)h(w) = h(w)e(z),$$

where we used the the generating series

$$e(z) := \sum_{n \in \mathbb{Z}} 1^\text{ss}_{(1,n)}z^n, \quad h(z) := \sum_{l \geq 1} 1_{(0,l)}z^l$$

and the motivic zeta function (2.1) for $\mathbb{P}^1$. Now the relation (4) follows from the definition (3.9) of $t_r$.

Finally let us check the relation (5). An extension

$$0 \longrightarrow \mathcal{O}_{P^1}(n) \xrightarrow{f} \mathcal{F} \xrightarrow{g} \mathcal{O}_{P^1}(m) \longrightarrow 0$$

(4.3)

is a non-trivial one only if $m > n$ and $\mathcal{F} \simeq \mathcal{O}_{P^1}(n + p) \oplus \mathcal{O}_{P^1}(m - p)$, with $p \in \mathbb{Z}$ and $1 \leq p \leq (m - n)/2$.

Let us consider the non-trivial case. The space of extensions of this form with fixed $p$ is given by the space of non-trivial morphisms $f : \mathcal{O}_{P^1}(n) \to \mathcal{O}_{P^1}(n + p) \oplus \mathcal{O}_{P^1}(m - p)$ such that the cokernel is a line bundle. Writing $f = f_1 \oplus f_2$ with $f_1 : \mathcal{O}_{P^1}(n) \to \mathcal{O}_{P^1}(m + p)$ and $f_2 : \mathcal{O}_{P^1}(n) \to \mathcal{O}_{P^1}(m - p)$, the condition for $f$ is that $f_1$ and $f_2$, considered as homogeneous polynomials in $\mathfrak{t}[z, w]$ of degrees $p$ and $m - n - p$, are coprime. Let $P_{p,m-n-p}$ be the stack of such coprime polynomials $(f_1, f_2)$. These $P_{a,b}$ ($a, b \in \mathbb{Z}_{\geq 0}$) satisfy the following relation.

$$\mathfrak{t}[z, w]_a \times \mathfrak{t}[z, w]_b = \prod_{d=0}^{\min(a-d,b-d)} \mathbb{P}(\mathfrak{t}[z, w]_d) \times P_{a-d,b-d}.$$
Let us come back to the counting extensions of the form (4.3). Since the automorphism \( O(n) \) is given by \( \mathbb{A}^* \) (non-zero scalars) and \([\mathbb{A}^*] = L - 1\), we see that the non-trivial extensions give the term

\[
(L^2 - 1)L^{m-n-1}[\mathcal{M}_{O(n+p)\oplus O(m-p)}]
\]

in \( 1^s_{(1,m)} \circ 1^s_{(1,n)} \). Here \( \mathcal{M}_{O(a)\oplus O(b)} \) with some \( a, b \in \mathbb{Z} \) denotes the moduli stack of the vector bundles isomorphic to \( O(a) \oplus O(b) \).

In the case of the trivial extensions, i.e., \( \mathcal{F} \simeq O(n) \oplus O(m) \), let us write the morphisms \( f, g \) as

\[
f = f_1 \oplus f_2, \quad g = g_1 \oplus g_2
\]

with \( f_1 \in \text{End}(O(n)), f_2, g_1 \in \text{Hom}(O(n), O(m)) \) and \( g_2 \in \text{End}(O(m)) \). Then we see that \( f_1 \) and \( g_2 \) are automorphisms, \( f_2 \) is arbitrary and \( g_1 = -g_2 \circ f_2 \circ f_1 \). Thus the pair \((f, g)\) is parametrized by

\[
\text{Aut}(O(n)) \times \text{Aut}(O(m)) \times \text{Hom}(O(n), O(m)).
\]

Since the equivalent relations of the trivial extensions are given by \( \text{Aut}(O(n)) \times \text{Aut}(O(m)) \), we see that the trivial extensions give the term

\[
[\text{Hom}(O(n), O(m))] \mathcal{M}_{O(n)\oplus O(m)} = L^{m-n+1}[\mathcal{M}_{O(n)\oplus O(m)}]
\]

in \( 1^s_{(1,m)} \circ 1^s_{(1,n)} \).

Considering the twist factor \( \sqrt{L} \chi_{(1,m),(1,n)} = \sqrt{L}^{1+m-n} \) in \( 1^s_{(1,m)} \circ 1^s_{(1,n)} \), we finally have

\[
1^s_{(1,m)} \circ 1^s_{(1,n)} = \sqrt{L}^{m-n+3}[\mathcal{M}_{O(n)\oplus O(m)}] + \sqrt{L}^{m-n-1}(L^2 - 1) \sum_{p=1}^{[\frac{(m-n)}{2}]}[\mathcal{M}_{O(n+p)\oplus O(m-p)}].
\]

Now the relation (5) is an equivalent expression of this formula. \( \square \)

4.2. Relation to the quantum affine algebra. Let us introduce symbols for the quantum affine algebra of \( \mathfrak{sl}_2 \). Let

\[
Q = \mathbb{Z}\alpha \oplus \mathbb{Z}\delta
\]

denote the root lattice of the affine Lie algebra associated to \( \mathfrak{sl}_2 \). We begin with the observation due to Kapranov [K97].

**Fact 4.3.** The map

\[
K(\text{Coh}(\mathbb{P}^1)) \to Q, \quad (\mathcal{E}) \mapsto \text{rk}(\mathcal{E})\alpha + \text{deg}(\mathcal{E})\delta
\]

gives an isometry of lattices, where the pairing on \( K(\text{Coh}(\mathbb{P}^1)) \) is given by the symmetrized Euler form

\[
(\mathcal{E}_1, \mathcal{E}_2) := \chi(\mathcal{E}_1, \mathcal{E}_2) + \chi(\mathcal{E}_2, \mathcal{E}_1),
\]

and the pairing on \( Q \) is taken to be the Killing form.

By this identification, the isomorphic classes of indecomposable objects in \( \text{Coh}(\mathbb{P}^1) \) gives the set of positive roots

\[
\Phi_+ := \{\alpha + n\delta \mid n \in \mathbb{Z}\} \cup \{n\delta \mid n \in \mathbb{Z}_{\geq 1}\}.
\]

Hereafter, it will be called the **non-standard set of positive roots** (following [S, Corollary 4.4]).

Let \( \mathcal{L}\mathfrak{b}_+ \) be the Borel subalgebra of the loop algebra \( \mathcal{L}\mathfrak{sl}_2 \). We understand that it our subalgebra is associated to the non-standard set \( \Phi_+ \) of positive roots. Now we introduce the Borel subalgebra \( U_o(\mathcal{L}\mathfrak{b}_+) \) of the quantum loop algebra \( U_q(\mathcal{L}\mathfrak{sl}_2) \) of \( \mathfrak{sl}_2 \). We dare to write down the definition for clarifying our argument.
**Definition 4.4.** Let $v$ be an indeterminate, and $R$ be the subring of the field $\mathbb{Q}(v)$ consisting of the rational functions having poles only at 0 or at roots of 1. The associative algebra $U_v(Lb_+)$ over $R$ is generated by

$$E_n, \ (n \in \mathbb{Z}), \ H_l, \ (l \in \mathbb{Z} \setminus \{0\}), \ K^{\pm 1}, \ C^{\pm 1/2}$$

with the relations

$$C^{\pm 1/2} \text{ is central}, \quad C^{1/2}C^{-1/2} = 1,$$
$$[K, H_l] = [H_l, H_m] = 0,$$
$$KE_n = v^2 E_n K,$$
$$[H_l, E_n] = \frac{[2lv]}{l} C^{-|l|/2} E_{l+n},$$
$$E_{m+1}E_n - v^2 E_n E_{m+1} = v^2 E_m E_{n+1} - E_{n+1}E_m$$

with $[2lv] := \frac{v^2 l - v - 2l}{v - v^2}$.

We denote by $U_{\sqrt{v}}(Lb_+)$ the specialization of $U_v(Lb_+)$ at $v = 1/\sqrt{l}$. Precisely speaking, we set $\sqrt{Q} := \mathbb{Q}[v^{\pm 1}]/(v^{-2} - l) \cong \mathbb{Q}[\sqrt{1}]$ and $U_{\sqrt{v}}(Lb_+) := U_v(Lb_+) \otimes_R \sqrt{Q}$.

Now we can state the fundamental result. Recall that the element $c$ in $U(\mathbb{P}^1)$ is central. Let us introduce $c^{1/2} := k_{(0,1/2)}$, which is the square root of $c$, and set

$$K_{P^1} := K[c^{1/2}] = K(\text{St}/\mathfrak{t})[L^{1/2}, c^{1/2}].$$

**Fact 4.5** ([BaK01, K97]). The assignment

$$E_n \mapsto 1_{(1,n)}^{ss} \ (n \in \mathbb{Z}), \quad H_l \mapsto t_l c^{-l/2} \ (l \in \mathbb{Z} \setminus \{0\}), \quad K \mapsto k, \quad C^{\pm 1/2} \mapsto c^{\pm 1/2}$$

extends to an isomorphism of algebras

$$U_{\sqrt{v}}(Lb_+) \otimes_{\sqrt{Q}} K \xrightarrow{\sim} U(\mathbb{P}^1) \otimes_K K_{P^1} \subset H_{\text{ext}}(\mathbb{P}^1) \otimes_K K_{P^1}.$$

**Proof.** This is the consequence of Fact 4.2 and the definition of $U_q(Lb_+)$. \hfill \Box

### 4.3. The Drinfeld double.

Next we want to consider the Drinfeld double. In order to do that, we study the coproduct on $H_{\text{ext}}(\mathbb{P}^1)$.

**Fact 4.6** ([K97, Theorem 3.3]). (1) For any $d \in \mathbb{Z}_{\geq 1}$ we have

$$\Delta(t_d) = t_d \otimes 1 + c^d \otimes t_d, \quad (4.4)$$

(2) For any $n \in \mathbb{Z}$ the following equalities hold.

$$\Delta(1_{(1,n)}^{ss}) = 1_{(1,n)}^{ss} \otimes 1 + \sum_{l \geq 0} \theta_l k c^{n-l} \otimes 1_{(1,m-l)}^{ss}. \quad (4.5)$$

Here $\theta_l \in H(\mathbb{P}^1) \ (l \in \mathbb{Z}_{\geq 0})$ is defined by the following generating function.

$$\sum_{l \geq 0} \theta_l z^l = \exp \left( (L^{1/2} - L^{-1/2}) \sum_{d \geq 1} t_d z^d \right).$$

**Proof.** (1) Since the category of torsion sheaves is closed under taking subobjects and quotients, we immediately find from Definition 3.7 of $\Delta$ that

$$\Delta(1_{(0,d)}) = \sum_{m \in \mathbb{Z}, \ 0 \leq m \leq d} 1_{(0,d-m)} k_{(0,m)} \otimes 1_{(0,m)}.$$

Then using the relation (3.9) between $1_{(0,d)}$’s and $t_d$’s and the fact that $c$ is central, we have the consequence.

(2) Since censoring by line bundles preserves extensions, it is enough to study the case $n = 0$, namely the decomposition of the trivial line bundle $\mathcal{O}$. We sketch the outline following [S, Example 4.12].
It is enough to count the surjection $O \to \oplus_i O_{n_i[x_i]}$. Each morphism $O \to O_{n_i[x_i]}$ is parametrized by $\text{Hom}(O, O_{n_i[x_i]}) \setminus \text{Hom}(O, O_{(n_i-1)[x_i]})$, whose class in $K(\text{St} / \mathfrak{t})$ is equal to $\mathbb{L}^{2n_i}(1 - \mathbb{L}^{-1})$, so that we have

$$\Delta(1^{ss}_{(1,0)}) = 1^{ss}_{(1,0)} \otimes 1 + \sum_{d \geq 0} \mathbb{L}^{d/2} u_d k_{(1,-d)} 1^{ss}_{(1,-d)}$$

with

$$u_d := \sum_{(x_i, n_i)} \prod_i (1 - \mathbb{L}^{-1}) t_{n_i[x_i]},$$

where $t_{n[x]} := [\mathcal{M}_{0,n[x]} \to \mathcal{M}]$ is the element corresponding to the moduli stack $\mathcal{M}_{0,n[x]}$ of torsion sheaves of degree $n$ with support on $x$, and the summation is taken over the set of tuples of distinct points $x_i$’s and degrees $n_i$’s with $\sum_i n_i = d$. Finally a quick observation yields $\mathbb{L}^{d/2} u_d = \theta_d$.

$\square$

**Remark 4.7.** In the proof we don’t use the property of $\mathbb{P}^1$, but only use the property of a smooth curve.

Next we study the Hall pairing $(\cdot, \cdot)_H$ on $U(\mathbb{P}^1)$.

**Fact 4.8 ([BuS13, Lemma 4.4]).**

(1)

$$(t_d, t_l)_H = \delta_{d,l} \frac{[2d]\sqrt{\pi}}{d \mathbb{L}^{1/2} - \mathbb{L}^{-1/2}}.$$

(2)

$$(t_d, \theta_l)_H = \delta_{d,l} \frac{[2d]\sqrt{\pi}}{d} \quad (4.6)$$

**Proof.** (1) Decomposing into the support, we have $t_d = \sum_{x \in \mathbb{P}^1} t_{d,x}$, where $t_{d,x}$ is defined in Definition 3.17. Recalling the value (3.6) of the Hall pairing, we can calculate $(t_d, t_l)_H$ as follows.

$$(t_d, t_l)_H = \sum_{x \in \mathbb{P}^1} \deg(x) \frac{([d]\sqrt{\pi})^2}{d \mathbb{L}^{d-1}} = \frac{1}{d} \frac{([d]\sqrt{\pi})^2}{\mathbb{L}^{d-1}} \frac{[P_d]}{2} = \frac{1}{d} \frac{[2d]\sqrt{\pi}}{\mathbb{L}^{1/2} - \mathbb{L}^{-1/2}}.$$

Here $[P_d] \in K(\text{Var} / \mathfrak{t})$ is given by

$$\zeta_{\text{mot}}(\mathbb{P}^1; z) = \exp\left( \sum_{d \geq 1} [P_d] \frac{z^d}{d} \right)$$

(2) is a restatement of (1) in terms of $\theta_l$. See [BuS13, Lemma 4.4] for the detail.

$\square$

**Remark 4.9.** The proof will also work for an arbitrary smooth curve.

Consider the algebra

$$\widetilde{DU}(\mathbb{P}^1) := D_{\text{red}} U(\mathbb{P}^1) \otimes_K K_{\mathfrak{p}}$$

which has a triangular decomposition

$$DU(\mathbb{P}^1) \otimes_K K_{\mathfrak{p}} = U(\mathbb{P}^1)^+ \otimes_K K[\text{Num}(\mathbb{P}^1)][c^{\pm 1/2}] \otimes_K U(\mathbb{P}^1)^-.$$

Following [BuS13, §4], we replace the generator $t_d^\pm (d \in \mathbb{Z}_{\geq 1})$ of $U(\mathbb{P}^1)^\pm$ by

$$\tilde{t}_d := t_d^\pm c^{-d/2}.$$

Then (4.4) is rewritten as

$$\Delta(\tilde{t}_d^\pm) = \tilde{t}_d^\pm \otimes c^{-d/2} + c^{d/2} \otimes \tilde{t}_d^\pm.$$

It is also convenient to rewrite (4.5) as

$$\Delta(1^{ss, +}_{(1,n)}) = 1^{ss, +}_{(1,n)} \otimes 1 + \sum_{t \geq 0} \tilde{\theta}^+_t k c^{-n/2} \otimes 1^{ss, +}_{(1,n-d)}$$
The formula (4.6) is rewritten as

\[ \tilde{\theta}_d^+ := \theta_d^+ e^{\frac{d}{2}}. \]

The formula (4.6) is rewritten as

\[ (\tilde{\theta}_d, \theta_l)_H = \delta_{d,l} \frac{[2d]\sqrt{\tau}}{d} \]

Now by Definition 3.28 of the reduced Drinfeld double, the algebra \( \tilde{DU}(\mathbb{P}^1) \) is described in the following way.

**Proposition 4.10** ([BuS13, §4]). \( \tilde{DU}(\mathbb{P}^1) \) is the associative algebra with the generators

\[ 1_{(1,n)}^{ss, \pm} (n \in \mathbb{Z}), \quad t_d^{\pm} (d \in \mathbb{Z}_{\geq 1}), \quad k^{\pm 1}, \quad c^{\pm 1/2} \]

modulo the relation

1. \( c^{\pm 1/2} \) are central, \( c^{1/2}c^{-1/2} = 1 \) and \( kk^{-1} = 1 = k^{-1}k \).
2. \( [k, t_d^{\pm}] = 0 = [t_d^{\pm}, t_m^{\pm}] = 0 \) for any \( l, m \in \mathbb{Z}_{\geq 1} \).
3. \( k 1_{(1,n)}^{ss, \pm} = \mathbb{L}^{1/2} 1_{(1,n)}^{ss, \pm} k \) for any \( n \in \mathbb{Z} \).
4. For any \( d \in \mathbb{Z}_{\geq 1} \) and \( n \in \mathbb{Z} \) we have

\[ [t_d^{+}, 1_{(1,n)}^{ss, \pm}] = \frac{[2d]}{d} 1_{(1,n+d)}^{ss, \pm} c^{d/2}. \]

5. For any \( d \in \mathbb{Z}_{\geq 1} \) and \( n \in \mathbb{Z} \) we have

\[ [t_d^{+}, 1_{(1,n)}^{ss, \mp}] = -\frac{[2d]}{d} 1_{(1,n-d)}^{ss, \mp} c^{d/2}. \]

6. For any \( d, l \in \mathbb{Z}_{\geq 1} \) we have

\[ [t_d^{+}, t_l^{+}] = \delta_{d,l} \frac{[2d]}{d} \frac{c^d - c^{-d}}{\mathbb{L}^{1/2} - \mathbb{L}^{-1/2}}. \]

7. For \( n, m \in \mathbb{Z} \) we have

\[ [1_{(1,m)}^{ss, +}, 1_{(1,n)}^{ss, -}] = \begin{cases} \frac{1}{\mathbb{L}^{1/2}} \tilde{\theta}_e^{+} k c^{(m+n)/2} & \text{if } n > m \\ 0 & \text{if } n = m \\ \frac{1}{\mathbb{L}^{1/2}} \tilde{\theta}_e^{-} k^{-1} c^{-(m+n)/2} & \text{if } n < m \end{cases}. \]

Although we have already described the quantum loop algebra \( U_v(L\mathfrak{sl}_2) \) in Fact 3.26 as the Drinfeld double of \( U_v(L\mathfrak{sl}_2) \), we have dare to write down the definition for the clarification. Recall Definition 4.4 where we set \( R \subset \mathbb{Q}(v) \) consisting of the rational functions having poles only at 0 or at roots of 1.

**Definition 4.11.** \( U_v(L\mathfrak{sl}_2) \) is the associative algebra over \( R \) generated by

\[ E_n^{\pm} (n \in \mathbb{Z}), \quad H_r (r \in \mathbb{Z} \setminus \{0\}), \quad K^{\pm 1}, \quad C^{\pm 1/2} \]

subject to the following relations.

1. \( C^{\pm 1/2} \) are central, \( C^{1/2}C^{-1/2} = 1 \) and \( KK^{-1} = K^{-1}K = 1 \).
2. \( [K, H_r] = 0 \) for any \( r \in \mathbb{Z} \setminus \{0\} \).
3. \( [K, E_n^{\pm}] = v^{\mp 2} E_n^{\pm} K \) for any \( n \in \mathbb{Z} \).
4. For any \( r, s \in \mathbb{Z} \setminus \{0\} \) we have

\[ [H_r, H_s] = \delta_{r+s,0} \frac{[2r]}{r} \frac{C^m - C^{-m}}{v - v^{-1}}. \]

5. For any \( n \in \mathbb{Z} \) and \( r \in \mathbb{Z} \setminus \{0\} \) we have

\[ [H_r, E_n^{\pm}] = \pm \frac{[2r]}{r} E_{n+r}^{\mp} C^{\mp |r|/2}. \]
(6) For any $m, n \in \mathbb{Z}$ we have
\[ E^±_m E^±_{n+1} + E^±_n E^±_{m+1} = v^{±2}(E^±_{m+1} E^±_n + E^±_{m+1} E^±_n). \]

(7) For any $m, n \in \mathbb{Z}$ we have
\[ [E^+_m, E^-_n] = \frac{v}{v - v^{-1}}(\Psi^+_m C^{(m-n)/2} - \Psi^-_{m+n} C^{(n-m)/2})K^{sgn(m+n)}, \]
where $sgn(n)$ denotes the sign of the integer $n$, and $\Psi^±_{±d} (d \in \mathbb{Z}_{≥1})$ are given by
\[ 1 + \sum_{d≥1} \Psi^±_{±d} z^d = \exp(±(v^{-1} - v) \sum_{d≥1} H^±_{±d} z^d) \]
and $\Psi^±_{±d} = 0 (d \in \mathbb{Z}_{≥1}).$

Remark 4.12. We followed [BuS13, Definition 4.9]. The choice of generators are crucial for the construction of the isomorphism explained below.

Now we can state the main result of this section. Recall that $\widetilde{Q} := \mathbb{Q}[v^{±1}]/(v^{−2} - L)$.

**Theorem 4.13** ([BuS13, Proposition 4.11]). Let $U_{\sqrt{\tau}}(L S_{l2}) := U_v(L S_{l2}) \otimes_R \widetilde{Q}$. Then the map $U_{\sqrt{\tau}}(L S_{l2}) \to DU(\mathbb{P}^1)$ given by
\[ E^±_n \mapsto 1^{\pm}_n \ (n \in \mathbb{Z}), \quad H^±_d \mapsto ±\tilde{t}^±_{±d} \ (d \in \mathbb{Z}_{≥1}), \quad K \mapsto k, \quad C^{1/2} \mapsto c^{1/2} \]
is an isomorphism of algebras.

**Proof.** One can show that the map sends $\Psi^±_{±l}$ to $\tilde{t}^±_{±l}$ for any $l \in \mathbb{Z}_{≥1}$. Then by comparing the relations in Proposition 4.10 and Definition 4.11, one finds that the map is well-defined.

Now the map is checked to be an isomorphism with the help of the basis consisting of the elements of the form $E^±_\lambda H_{µ^±} K^{a} C^{b/2} E^±_{±} H_{µ}$, where $\lambda, µ^±, ν$ are non-decreasing finite sequences of integers, $E^±_{±} := E^±_{±1} E^±_{±2} \cdots$ and so on, and $a, b \in \mathbb{Z}$. We omit the detail and refer the proof of [BuS13, Proposition 4.11]. \[\square\]

5. **The case of an irreducible projective curve of arbitrary arithmetic genus**

In this section we denote by $C$ an irreducible reduced projective curve $C$ defined over a field $k$, and consider the motivic Hall algebra of $C$. In the smooth case our result is just a motivic restatement of the one given in [K97] and explained in [S, §4.11].

5.1. **The composition subalgebra.** As mentioned at Lemma 3.6, the Hall algebra $H(C)$ and the extended Hall algebra $H_{ext}(C)$ are $Γ$-graded, where $Γ \subset \text{Num}(C)$ is the submonoid generated by effective classes. We have
\[ \text{Num}(C) \twoheadrightarrow \mathbb{Z}, \quad \mathcal{E} \mapsto (\text{rk}(\mathcal{E}), \deg(\mathcal{E})) \]
as modules, and we denote by $k_{(r,d)}$ with $(r, d) \in \mathbb{Z}^2$ the element of the extension part of $H_{ext}(C)$. The Riemann-Roch theorem yields
\[ χ(\mathcal{E}, \mathcal{F}) = (1 - g) \text{rk}(\mathcal{E}) \text{rk}(\mathcal{F}) + \text{rk}(\mathcal{E}) \deg(\mathcal{F}) - \text{rk}(\mathcal{F}) \deg(\mathcal{E}), \]
where $g$ is the arithmetic genus of $C$.

As in the case of $\mathbb{P}^1$, the algebra $H(C)$ or $H_{ext}(C)$ is too big to treat, and we will mainly study the composition subalgebra $U(C)$ and its extended version $U_{ext}(C)$ given in Definition 3.20.

Let us denote by $M_{0}(0,d)$ the moduli stack of perfect torsion sheaves of degree $d$, and by $M_{l}(1,n)$ the moduli stack of line bundles of degree $n$. These are substacks of $M = M(C)$, the moduli stack of perfect coherent sheaves on $C$.

By definition, $U(C)$ is the subalgebra of $H(C_{sm})$ generated by elements
\[ 1_{(0,d)} := [M_{0}(0,d) \to M] \ (d \in \mathbb{Z}_{≥1}) \]
and
\[ 1^{ss}_{(1,n)} := [\mathcal{M}_{l,(1,n)} \hookrightarrow \mathcal{M}] \quad (n \in \mathbb{Z}). \]

We will replace the generators \(1_{(0,d)}\) by the following elements \(t_d\).

**Definition 5.1.** Define \(t_d\) (\(d \in \mathbb{Z}_{\geq 1}\)) and \(\theta_l\) (\(l \in \mathbb{Z}_{\geq 0}\)) by the generating functions
\[ 1 + \sum_{d \geq 1} 1_{(0,d)} z^d = \exp \left( \sum_{d \geq 1} \frac{t_d}{[d] \sqrt[L]{d}} z^d \right) \]
and
\[ \sum_{l \geq 0} \theta_l z^l = \exp \left( (L^{1/2} - L^{-1/2}) \sum_{d \geq 1} \frac{t_d}{[d] \sqrt[L]{d}} z^d \right). \]

As stated in [S, Lemma 4.51], one can calculate coproducts of these generators.

**Lemma 5.2.** For \(d \in \mathbb{Z}_{\geq 1}\) and \(n \in \mathbb{Z}\) we have
\[ \Delta(t_d) = t_d \otimes 1 + k_{(0,d)} \otimes t_d, \]
\[ \Delta(1^{ss}_{(1,n)}) = 1^{ss}_{(1,n)} \otimes 1 + \sum_{l \geq 0} \theta_l k_{(1,n-l)} \otimes 1^{ss}_{(1,n-l)}. \]

**Proof.** This is the same as Fact 4.6. As mentioned in Remark 4.7, the proof works for an arbitrary curve. \(\square\)

As for the Hopf pairing \((\cdot, \cdot)_H\), we have the following formulas.

**Lemma 5.3.** For any \(d, l \in \mathbb{Z}_{\geq 1}\) and \(m, n \in \mathbb{Z}\) we have
\[ (t_d, 1^{ss}_{(1,n)})_H = 0, \]
\[ (t_d, t_l)_H = \delta_{d,l} \frac{[d] \sqrt[L]{d}}{d} \frac{[C(d)]}{L^d - 1}, \]
\[ (1^{ss}_{(1,m)}, 1^{ss}_{(1,n)})_H = \delta_{m,n} \frac{[\text{Pic}^n(C)]}{L^d - 1}. \]

Here \([C(d)] \in K(\text{Var}/k)\) is defined by
\[ \zeta_{mot}(C; z) = \exp \left( \sum_{d \geq 1} [C(d)] \frac{z^d}{d} \right) \]

**Proof.** The smooth case is stated in [K97] and explained in [S, Lemma 4.52]. We copy the proof here.

By Definition 3.9 of the pairing \((\cdot, \cdot)_H\), the first is obvious.

As mentioned in Remark 4.9, the proof of Fact 4.8 works in this case, and we have the second formula.

For the third formula, note that \(1^{ss}_{(1,n)} = \sum_{\mathcal{L}} [\mathcal{M}_L \hookrightarrow \mathcal{M}], \) where \(\mathcal{L}\) runs over the isomorphism classes of line bundles of degree \(n\), and \(\mathcal{M}_L\) denotes the stack for the line bundles isomorphic to \(\mathcal{L}\). Since \(\text{Aut}(\mathcal{L}) \simeq k^*\) so that \(a_L = [\text{Aut}(\mathcal{L})] = L - 1 \in K(\text{Var}/k)\), we have the result. \(\square\)

We close this subsection by explaining Kapranov’s result of expressing the relation of generators of the composition subalgebra in the current form.

**Proposition 5.4.** Set
\[ x^+(z) := \sum_{n \in \mathbb{Z}} 1^\theta_{(1,n)} z^n, \quad \psi(z) := \exp \left( \sum_{d \geq 1} \frac{t_{(0,d)}}{[d] \sqrt[L]{d}} z^d \right). \]

Then in the algebra \(U(C)\) the following relations hold.
\[ \zeta_{mot}(C; w/z)x^+(z)x^+(w) = \zeta_{mot}(C; w/z)x^+(z)x^+(w), \]
5.2. The slope stability of coherent sheaves on curves. For the future use, in particular for the discussion of the Hall algebras for elliptic curves, let us recall the notion of slope stability of coherent sheaves on a projective curve. The slope stability was originally introduced by Mumford in the smooth case to construct coarse moduli spaces of semistable coherent sheaves on curves using geometric invariant theory. Here we give statements for arbitrary (not necessarily smooth) projective curves, following Simpson’s generalization [Si94]. Among a large amount of literature on this topic, we only cite [HL] for the detail.

In this subsection $C$ denotes a projective curve defined over a field $k$. $\text{Coh}(C)$ denotes the category of coherent sheaves on $C$. For $\mathcal{E} \in \text{Coh}(C)$, the rank $\text{rk}(\mathcal{E})$ and the degree $\text{deg}(\mathcal{E})$ is defined by $\text{rk}(\mathcal{E}) := \alpha_1(\mathcal{E})/\alpha_1(\mathcal{O}_C)$ and $\text{deg}(\mathcal{E}) := \alpha_0(\mathcal{E})/\alpha_1(\mathcal{O}_C)$, where $\alpha_i(-) \in \mathbb{Z}$ is given by the equation $\chi(- \otimes \mathcal{O}_C(mH)) = ma_1(-) + \alpha_0(-)$. Here $H$ denotes the fixed ample line bundle of $C$. By the Riemann-Roch theorem, this definition coincides with the classical one in the case of smooth curves.

**Definition 5.5.** (1) For $\mathcal{E} \in \text{Coh}(C)$, the slope $\mu(\mathcal{E})$ is the rational number defined by

$$\mu(\mathcal{E}) := \frac{\text{deg}(\mathcal{E})}{\text{rk}(\mathcal{E})} \in \mathbb{Q} \cup \{\infty\},$$

where if $\text{rk}(\mathcal{E}) = 0$ then we set $\mu(\mathcal{E}) := \infty$.

(2) A coherent sheaf $\mathcal{E}$ is called semistable if $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$ holds for any non-zero subsheaf $\mathcal{F} \subset \mathcal{E}$. $\mathcal{E}$ is called stable if $\mu(\mathcal{F}) < \mu(\mathcal{E})$ holds for any non-zero proper subsheaf $\mathcal{F} \subsetneq \mathcal{E}$.

(3) For $\nu \in \mathbb{Q} \cup \{\infty\}$, denote by $S_{\nu}$ the full subcategory of $\text{Coh}(C)$ consisting of semistable sheaves of slope $\nu$.

In order to distinguish this notion of stability from others, we call it slope stability. We obviously have $S_{\infty} = \text{Tor}(C)$, the category of torsion sheaves on $C$. Here are some fundamental properties of semistable sheaves.

**Fact 5.6.** (1) $\text{Hom}(S_{\nu}, S_{\nu'}) = 0$ if $\nu > \nu'$.

(2) For any $\nu \in \mathbb{Q} \cup \{\infty\}$, the category $S_{\nu}$ is abelian and closed under extension. Moreover it is a finite length category, i.e., every object has a finite composition sequence with simple factors, and the simple objects are stable sheaves.

Let us also recall the Harder-Narasimhan filtration.

**Fact 5.7.** For any $\mathcal{E} \in \text{Coh}(C)$ there exists a unique filtration

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \mathcal{E}_n = \mathcal{E}$$

such that every factor $\mathcal{F}_i := \mathcal{E}_i/\mathcal{E}_{i-1}$ is semistable and $\mu(\mathcal{F}_1) > \mu(\mathcal{F}_2) > \cdots > \mu(\mathcal{F}_n)$. It will be called the Harder-Narasimhan filtration of $\mathcal{E}$.

By the existence and uniqueness of the Harder-Narasimhan filtration, we immediately have the following statement.

**Corollary 5.8.** The motivic Hall algebra for $C$ has a decomposition

$$\text{H}(C) \sim \bigoplus_{n} \bigoplus_{\mu_1 > \mu_2 > \cdots > \mu_n} \text{H}(C)_{\mu_1} \otimes \text{H}(C)_{\mu_2} \otimes \cdots \otimes \text{H}(C)_{\mu_n}$$

as $\mathbb{K}$-module. Here for $\nu \in \mathbb{Q} \cup \{\infty\}$ $\text{H}(C)_{\nu}$ denotes $\mathbb{K}(\text{St}/\mathcal{M}_{\nu})$, the subalgebra generated by perfect semistable coherent sheaves with slope $\nu$. ($\mathcal{M}_{\nu}$ denotes the moduli stack of perfect semistable coherent sheaves with slope $\nu$.)
6. The case of a smooth elliptic curve

We now recall the work of Burban and Schiffmann [BuS12] on the Hall algebra of a smooth elliptic curve. Our review will be done in the motivic version, but the argument is essentially the same with theirs. Hereafter $E_{\text{sm}}$ denotes a smooth elliptic curve defined over a fixed field $\mathfrak{k}$.

6.1. Generalities on coherent sheaves on a smooth elliptic curve. The Euler pairing on $K(\text{Coh}(E_{\text{sm}}))$ is given by

$$\chi(\mathcal{E}, \mathcal{F}) = \text{rk}(\mathcal{E}) \text{deg}(\mathcal{F}) - \text{rk}(\mathcal{F}) \text{deg}(\mathcal{E}).$$

As for the numerical Grothendieck group $\text{Num}(E_{\text{sm}})$ (see (3.2) for the definition), we have

$$\text{Num}(E_{\text{sm}}) \xrightarrow{\sim} \mathbb{Z}^2$$

by the map

$$\mathcal{E} \mapsto (\text{rk}(\mathcal{E}), \text{deg}(\mathcal{E})).$$

Let us recall the Mumford stability explained in the previous subsection §5.2. The existence of Harder-Narasimhan filtration and Corollary 5.8 tell us that the subcategory $S_\nu$ of semistable sheaves of slope $\nu$ is the building block of the motivic Hall algebra $H(E_{\text{sm}})$. Now it is time to recall the fundamental result of Atiyah [A57] on the classification of semisimple sheaves on elliptic curves. We cite the result in the following form [S, Theorem 4.45].

**Fact 6.1** ([A57]). For any $\nu, \nu' \in \mathbb{Q} \cup \{\infty\}$, there is an equivalence of categories $\Phi_{\nu, \nu'} : S_\mu \xrightarrow{\sim} S_{\nu'}$. In particular, $S_\mu \xrightarrow{\sim} S_\infty = \text{Tor}(E_{\text{sm}})$.

**Remark** 6.2. The equivalence can be realized as a Fourier-Mukai transform [Muk81] on $D^b_{\text{coh}}(E_{\text{sm}})$, which will be explained in §6.4.

Let us recall that $H(E_{\text{sm}})_\mu = K(\text{St}/\mathfrak{M}_\nu)$ denotes the subalgebra of $H(E_{\text{sm}})$ generated by the semistable sheaves of slope $\nu$ (see Corollary 5.8). Fact 6.1 implies that the equivalence $\phi_{\nu, \infty}$ induces an isomorphism between $H(E_{\text{sm}})_\infty$ and $H(E_{\text{sm}})_\nu$.

**Definition 6.3.**

(1) For $\nu \in \mathbb{Q}$ denote by $\phi_{\infty, \nu} : H(E_{\text{sm}})_\infty \xrightarrow{\sim} H(E_{\text{sm}})_\nu$ the isomorphism of bialgebras induced by the equivalence $\Phi_{\infty, \nu}$.

(2) For $\nu \in \mathbb{Q}$ and $d \in \mathbb{Z}_{\geq 1}$, define $t_{\nu, d} := \phi_{\infty, \nu}(t_d)$, where $t_d \in H(E_{\text{sm}})_\infty = H(E_{\text{sm}})_{\text{tor}}$ is given in Definition 3.17.

(3) Denote by $U(E_{\text{sm}})$ the subalgebra of $H(E_{\text{sm}})$ generated by $\{t_{\nu, d} | \nu \in \mathbb{Q} \cup \{\infty\}, d \in \mathbb{Z}_{\geq 1}\}$. Also we define $U(E_{\text{sm}})_\nu := U(E_{\text{sm}}) \cap H(E_{\text{sm}})_\nu$ for any $\nu \in \mathbb{Q} \cup \{\infty\}$.

The last definition looks natural from the view of Atiyah’s result (Fact 6.1). Previously (Definition 3.20) we defined the composition algebra $U(E_{\text{sm}})$ as the subalgebra of $H(E_{\text{sm}})$ generated by torsion sheaves and line bundles. Here we have

**Fact 6.4** ([SV11]). $U(E_{\text{sm}})$ in Definition 6.3 coincides with the one in Definition 3.20 for a smooth elliptic curve $E_{\text{sm}}$.

**Corollary 6.5** ([BuS12]).

(1) The multiplication map induces an isomorphism

$$\bigoplus_{n} \bigoplus_{\mu_1 > \mu_2 > \cdots > \mu_n} U(E_{\text{sm}})_{\mu_1} \otimes U(E_{\text{sm}})_{\mu_2} \otimes \cdots \otimes U(E_{\text{sm}})_{\mu_n} \xrightarrow{\sim} U(E_{\text{sm}})$$

of $\mathbb{K}$-modules.

(2) $U(E_{\text{sm}})_\nu \simeq \mathfrak{k}[t_{\nu, 1}, t_{\nu, 2}, \ldots]$ for any $\nu \in \mathbb{Q} \cup \{\infty\}$.

**Proof.** (1) As in Corollary 5.8, the existence of Harder-Narasimhan filtration implies the result.

(2) By the isomorphism

$$\phi_{\infty, \nu}^{-1} : U(E_{\text{sm}})_\infty \xrightarrow{\sim} U(E_{\text{sm}})$$
and the description
\[ U(E_{sm})_\infty = \mathfrak{t}[t_1, t_2, \ldots], \]
we have the result. \( \square \)

6.2. The composition subalgebra for smooth elliptic curve. [BuS12] gave a presentation of the double of the algebra \( U(E_{sm}) \) which is \( \mathbb{Z}^2 \)-graded and \( \text{SL}(2, \mathbb{Z}) \)-invariant. Let us state it in the motivic language. Hereafter we assume \( \mathfrak{t} \) is algebraically closed, so that the formula in Fact 2.8 of Kapranov’s motivic zeta function \( \zeta_{\text{mot}}(E_{sm}; z) \) applies. For an elliptic curve \( X \) it is expressed as
\[
\zeta_{\text{mot}}(E_{sm}; z) = 1 + \frac{[\text{Pic}^0(E_{sm})]z}{(1 - z)(1 - \mathbb{L}z)} = \frac{(1 - q_1 z)(1 - q_2 z)}{(1 - z)(1 - \mathbb{L}z)}.
\]
where \( q_1 \) and \( q_2 \) are conjugate in an algebraic extension of \( K(\text{Var}/\mathfrak{t}) \), and satisfy \( q_1 q_2 = \mathbb{L} \) and \( q_1 + q_2 = \mathbb{L} + 1 - [\text{Pic}^0(E_{sm})] \).

Following [BuS12], we introduce several symbols. We will often use the subset
\[
(Z^2)^+ := (\mathbb{Z}_{\geq 1} \times \mathbb{Z}) \cup (\{0\} \times \mathbb{Z}_{\geq 1})
\]
of \( \mathbb{Z}^2 \) for the grading of \( U(E_{sm}) \).

**Definition 6.6.**

1. For \( i \in \mathbb{Z} \) set
\[
c_i := (q_1^{i/2} - q_1^{-i/2})(q_2^{i/2} - q_2^{-i/2})[i] \sqrt{\mathbb{L}}.
\]

2. Recalling \( t_{r,d} \) in Definition 6.3, set \( t_{(r,d)} := t_{d/r.gcd(r,d)} \) for \( (r, d) \in (Z^2)^+ \).

3. For \( x = (r, d) \in (Z^2)^+ \), set
\[
gcd(x) := \begin{cases} 
gcd(r, d) & \text{if } r \neq 0 \\
d & \text{if } r = 0. 
\end{cases}
\]

4. For \( x = (r, d) \in (Z^2)^+ \), set
\[
\mu(x) := \begin{cases} 
d/r & \text{if } r \neq 0 \\
\infty & \text{if } r = 0. 
\end{cases}
\]

5. For \( x, y \in (Z^2)^+ \), set \( \epsilon(x, y) := \text{sgn}(\text{det}(x, y)) \in \{\pm 1\} \).

**Remark 6.7.** \( \text{gcd}(x) \) was denoted by \( \text{deg}(x) \) in [BuS12].

Obviously \( U(E_{sm}) \) is generated by the elements \( \{t_{(r,d)} \mid (r, d) \in (Z^2)^+\} \).

**Fact 6.8** ([BuS12]). By the assignment \( t_{r,d} \mapsto T_{(r,d)} \) for \( (r, d) \in (Z^2)^+ \), the composition subalgebra \( U(E_{sm}) \) is isomorphic to the associative algebra generated by the elements
\[
\{T_{r,d} \mid (r, d) \in (Z^2)^+\}
\]
subject to the following relations.

1. If \( x, y \in (Z^2)^+ \) satisfy \( \mu(x) = \mu(y) \), then \([T_x, T_y] = 0\).

2. If \( x, y \in (Z^2)^+ \) satisfy \( \gcd(x) = 1 \) and if there is no interior lattice point in the triangle with vertices \( 0, x \) and \( x + y \) in \( \mathbb{Z}^2 \), then
\[
[t_y, t_x] = \epsilon_{x,y} \epsilon_{\gcd(y)} \frac{\theta_{x+y}}{\mathbb{L}^{1/2} - \mathbb{L}^{-1/2}},
\]
where \( \theta_x \)'s are defined by the generating series
\[
\sum_{i \geq 1} \theta_{ix_0} z^i = \exp \left( (\mathbb{L}^{1/2} - \mathbb{L}^{-1/2}) \sum_{j \geq 1} t_{jx_0} z^j \right)
\]
for \( x_0 \in (Z^2)^+ \) with \( \gcd(x_0) = 1. \)
We will not give the detail here. The proof in [BuS12] utilizes $\text{SL}(2,\mathbb{Z})$-action on the reduced Drinfeld double of $U(E_{\text{sm}})$, which is explained in the next subsection.

6.3. The Drinfeld double. The reduced Drinfeld double $D_{\text{red}}U(E_{\text{sm}})$ of the $U(E_{\text{sm}})$ for a smooth elliptic curve $E_{\text{sm}}$ is determined by [BuS12]. It has the grading with respect to the subset

$$(\mathbb{Z}^2)^* := \mathbb{Z}^2 \setminus \{(0, 0)\} = (\mathbb{Z}^2)^+ \sqcup (\mathbb{Z}^2)^-,$$

of $\mathbb{Z}^2$, where $(\mathbb{Z}^2)^- := -(\mathbb{Z}^2)^+ = \{(r, d) \in \mathbb{Z}^2 \mid r < 0 \text{ or } (r = 0 \text{ and } d < 0)\}$. We will use the notation $x = (r, d)$ for an element of $(\mathbb{Z}^2)^*$, and define $\deg(x) \in \mathbb{Z}$ and $\mu(x) \in \mathbb{Q} \sqcup \{\pm \infty\}$ as in Definition 6.6. Precisely speaking, we have

**Definition 6.9.**

1. For $x = (r, d) \in (\mathbb{Z}^2)^*$, set

$$\gcd(x) := \begin{cases} \gcd(r, d) & \text{if } r \neq 0 \\ d & \text{if } r = 0. \end{cases}$$

Thus we have $\deg(x) \in \mathbb{Z}_{>0}$ for $x \in (\mathbb{Z}^2)^+$ and $\deg(x) \in \mathbb{Z}_{<0}$ for $x \in (\mathbb{Z}^2)^-$.

2. For $x = (r, d) \in (\mathbb{Z}^2)^*$, set

$$\mu(x) := \begin{cases} \frac{d}{r} & \text{if } r \neq 0 \\ \infty & \text{if } r = 0 \text{ and } d > 0 \\ -\infty & \text{if } r = 0 \text{ and } d < 0. \end{cases}$$

3. For $x, y \in (\mathbb{Z}^2)^*$, set $\epsilon(x, y) := \text{sgn} (\det(x, y)) \in \{\pm 1\}$.

4. For $x, y \in (\mathbb{Z}^2)^*$, set $\epsilon(x) := \begin{cases} 1 & \text{if } x \in (\mathbb{Z}^2)^+ \\ -1 & \text{if } x \in (\mathbb{Z}^2)^-. \end{cases}$

**Fact 6.10 ([BuS12]).** $D_{\text{red}}U(E_{\text{sm}})$ is isomorphic to the associative algebra generated by the elements $\{t_{(r, d)} \mid (r, d) \in (\mathbb{Z}^2)^*\}$ and $\{k_{r, d} \mid r, d \in \mathbb{Z}\}$ modulo the following relations.

1. $k_{r, d}k_{r', d'} = k_{r+r', d+d'}$.
2. If $x, y \in (\mathbb{Z}^2)^*$ satisfy $\mu(x) = \mu(y)$, then

$$[t_x, t_y] = \delta_{x, -y} \epsilon_{\gcd(x)} \frac{k_x - k_x^{-1}}{L^{1/2} - L^{-1/2}}.$$

3. If $x, y \in (\mathbb{Z}^2)^*$ satisfy $\gcd(x) = 1$ and if there is no interior lattice point in the triangle with vertices $\mathbf{0}$, $x$ and $x + y$ in $\mathbb{Z}^2$, then

$$[t_y, t_x] = \epsilon_{x, y} \epsilon_{\gcd(y)} k_{\alpha(x, y)} \frac{\theta_{x+y}}{L^{1/2} - L^{-1/2}},$$

where $\alpha(x, y)$ is given by

$$\alpha(x, y) := \begin{cases} \frac{1}{2} \epsilon_x (\epsilon_x x + \epsilon_y y - \epsilon_{x+y}(x + y)) & \text{if } \epsilon_{x, y} = 1 \\ \frac{1}{2} \epsilon_y (\epsilon_x x + \epsilon_y y - \epsilon_{x+y}(x + y)) & \text{if } \epsilon_{x, y} = -1. \end{cases}$$

and $\theta_x$ is determined by

$$\sum_{i \geq 1} \theta_{ix_0} z^i = \exp \left( (L^{1/2} - L^{-1/2}) \sum_{j \geq 1} t_{jx_0} z^j \right)$$

for $x_0 \in (\mathbb{Z}^2)^+$ with $\gcd(x_0) = 1$. 


As mentioned in the previous subsection, the proof in [BuS12] utilizes the SL(2, Z)-action. Notice that the relations of the algebra in the above Fact are obviously SL(2, Z)-(quasi)invariant. The SL(2, Z)-action on $D_{\text{red}}U(E_{\text{sm}})$ comes from the autoequivalences of $D_{\text{coh}}^{b}(E_{\text{sm}})$, which will be explained in the next subsection.

Finally we relate $D_{\text{red}}U(E_{\text{sm}})$ to the quantum toroidal algebra for $\mathfrak{g}l_{1}$.

**Fact 6.11** ([BuS12, Theorem 5.4]). The algebra given in Fact 6.10 is isomorphic to the $\mathfrak{g}l_{1}$-quantum toroidal algebra $\tilde{U}$ after tensoring coefficient field. Thus we have

$$D_{\text{red}}U(E_{\text{sm}}) \sim \tilde{U} \otimes \mathbb{K}(q_{1}, q_{2}) \mathbb{K}.$$  

Here recall that $\mathbb{K} = K(\text{St}/\mathfrak{k})[L^{\pm 1/2}]$, and $\tilde{U}$ is considered as an algebra defined over $\mathbb{Q}(q_{1}, q_{2})$ with $q_{3} = q_{1}^{-1}q_{2}^{-1}$. The ring homomorphism $\mathbb{Q}(q_{1}, q_{2}) \rightarrow \mathbb{K}$ is given by mapping $q_{1}$ and $q_{2}$ to the inverse of zeros of the motivic zeta function

$$\zeta_{\text{mot}}(E_{\text{sm}}; z) = \frac{1 + (L + 1 - \text{Pic}(E_{\text{sm}}))z + Lz^{2}}{(1 - z)(1 - Lz)} = \frac{(1 - q_{1}z)(1 - q_{2}z)}{(1 - z)(1 - Lz)}.$$  

**Proof.** Let us explain the outline of the proof explained in [BuS12]. Notice that the relations in Proposition 5.4 coincides with those of $\tilde{U}$ except the Serre-like relation. Let us denote by $\tilde{U}'$ the algebra generated by the same generators in $\tilde{U}$ satisfying the relations except the Serre-like relation. Denote by $\tilde{U}$ the algebra given in Fact 6.10. Then one can find that there is a surjection $\tilde{U}' \rightarrow \tilde{U}$. By the analysis of the convex paths in $\mathbb{Z}^{2}$ (corresponding to Harder-Narasimhan filtrations of coherent sheaves on $E_{\text{sm}}$), one can show that the Serre-like relation coincides with the kernel of the above surjection. See [S12] for the precise account for this analysis. \[\square\]

### 6.4. The automorphism

The Drinfeld double $D_{\text{red}}U(E_{\text{sm}})$, or more generally $D_{\text{red}}U(C_{\text{sm}})$ for a smooth curve $C_{\text{sm}}$, may be considered as the Hall algebra of the root category $R(E_{\text{sm}}) := D_{\text{coh}}^{b}(E_{\text{sm}})/[1]^{2}$, where $[1]$ denotes the shift of complexes, and the quotient symbol means taking the orbit category. This idea is realized in the recent work of Bridgeland [Br13]. In this paper we will not treat this approach and use the definition of Drinfeld double itself.

Let us recall that the description of the group $\text{Aut}(D_{\text{coh}}^{b}(E_{\text{sm}}))$ of auto-equivalences of $D_{\text{coh}}^{b}(E_{\text{sm}})$ for a smooth elliptic curve $E_{\text{sm}}$ defined over a field $\mathfrak{k}$. We will also mention some generalities on Fourier-Mukai transforms. The basic references are [BBH] and [H].

By the fundamental result of Orlov [O97], for any smooth projective variety $X$ defined over a algebraically closed field $\mathfrak{k}$, every auto-equivalence of $D_{\text{coh}}^{b}(X)$ can be realized as a Fourier-Mukai transform [Muk81]. Let $Y$ be another smooth projective variety and $\mathcal{F}$ be an object of $D_{\text{coh}}^{b}(X \times Y)$, namely a bounded complex of coherent sheaves on the product variety $X \times Y$. Then the functor $\Phi_{\mathcal{F}} : D_{\text{coh}}^{b}(X) \rightarrow D_{\text{coh}}^{b}(Y)$ is defined to be

$$\Phi_{\mathcal{F}}(-) := R\text{p}_{Y*}(\mathcal{F} \overset{L}{\otimes} p_{X}^{*}(-)),$$

where $p_{X} : X \times Y \rightarrow X$ and $p_{Y} : X \times Y \rightarrow Y$ are natural projections and the symbols $R\text{p}_{Y*}$ and $\otimes$ denote the derived functors. We call this functor a Fourier-Mukai transform if it gives an equivalence of categories.

It is also known by [O02] that for a (smooth) abelian variety $A$, any smooth projective variety $X$ with equivalent derived category $D_{\text{coh}}^{b}(X) \simeq D_{\text{coh}}^{b}(A)$ is also a smooth abelian variety of the same dimension, and moreover if $E_{\text{sm}}$ is a smooth elliptic curve, then $X$ is isomorphic to $E_{\text{sm}}$. Thus $\text{Aut}(D_{\text{coh}}^{b}(E_{\text{sm}}))$ is generated by Fourier-Mukai transforms $\Phi_{\mathcal{F}}$ with $\mathcal{F} \in D_{\text{coh}}^{b}(E_{\text{sm}} \times E_{\text{sm}})$.

For any smooth projective variety $Y$, we have natural equivalences of $D_{\text{coh}}^{b}(Y)$ given by

- the shift $[1]$ of complexes,
- automorphism of $Y$,
- tensoring with a line bundle of degree 0.
These can be also realized by Fourier-Mukai transforms, and consist the subgroup \( \mathbb{Z} \oplus \text{Aut}(Y) \ltimes \text{Pic}^0(Y) \) of Aut(\( \mathcal{D}_{\text{coh}}^b(Y) \)). Let us define the group FM(\( Y \)) by the short exact sequence

\[
1 \longrightarrow \mathbb{Z} \oplus \text{Aut}(Y) \ltimes \text{Pic}^0(Y) \longrightarrow \text{Aut}(\mathcal{D}_{\text{coh}}^b(Y)) \longrightarrow \text{FM}(Y) \longrightarrow 1 .
\]

(6.1)

If \( A \) is an abelian variety, then a set of generators of FM(\( A \)) is given by \( \{ \Phi_F \} \), where \( F \)'s are the universal family of semi-homogeneous sheaves.

In the case of a smooth elliptic curve \( E_{\text{sm}} \), the description of FM(\( E_{\text{sm}} \)) is extremely simple, and the result is

\[
\text{FM}(E_{\text{sm}}) = \langle T_\mathcal{O}, \Phi_P \rangle \cong \text{SL}(2, \mathbb{Z}).
\]

Here we denoted by \( T_\mathcal{O} \) the spherical twist by the structure sheaf \( \mathcal{O} = \mathcal{O}_{E_{\text{sm}}} \), which is defined by

\[
T_\mathcal{O}(-) := \text{Cone}(\text{RHom}(\mathcal{O}, -) \otimes \mathcal{O} \xrightarrow{\text{ev}} -).
\]

We also denoted by \( \Phi_{\mathcal{P}_{E_{\text{sm}}}} \) the Fourier-Mukai transforms defined by the Poincare bundle \( \mathcal{P}_{E_{\text{sm}}} \) on \( E_{\text{sm}} \times \text{Pic}^0(E_{\text{sm}}) \). (\( \mathcal{P}_X \) is the universal family of the Jacobian variety Pic\( ^0(X) \)).

One can understand the isomorphism (6.2) in terms of K-theoretic Fourier-Mukai transforms \( \Phi_{\mathcal{F}_X}^K \), which is the automorphism on the Grothendieck group

\[
K(\mathcal{D}_{\text{coh}}^b(E_{\text{sm}})) = K(\text{Coh}(E_{\text{sm}}))
\]

induced by \( \Phi_F \). We can further consider the induced automorphisms on the numerical Grothendieck group Num(\( E_{\text{sm}} \)).

Denote it by \( t_\mathcal{O} \) and \( \phi_P \) the automorphisms on Num(\( E_{\text{sm}} \)) induced by \( T_\mathcal{O} \) and \( \Phi_P \). Since Num(\( E_{\text{sm}} \)) \( \cong \mathbb{Z}^2 \), they can be realized as matrices. The result is

\[
t_\mathcal{O} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad \phi_P = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

In fact, this argument gives us a homomorphism FM(\( E_{\text{sm}} \)) \( \rightarrow \text{SL}(2, \mathbb{Z}) \) of groups, and one can show that it is indeed an isomorphism.

Now note that the part Aut(\( E_{\text{sm}} \)) \( \ltimes \text{Pic}^0(E_{\text{sm}}) \) acts trivially on \( \text{D}_{\text{red}} \text{U}(E_{\text{sm}}) \). Thus it is natural to consider the action of \( \mathbb{Z} \) (generated by shifts of complexes) and FM(\( E_{\text{sm}} \)) on \( \text{D}_{\text{red}} \text{U}(E_{\text{sm}}) \). By the description (6.1) and (6.2), these parts coincide with the universal over \( \text{SL}(2, \mathbb{Z}) \) of \( \text{SL}(2, \mathbb{Z}) \), which sits in the short exact sequence

\[
1 \longrightarrow \mathbb{Z} \longrightarrow \widehat{\text{SL}}(2, \mathbb{Z}) \longrightarrow \text{SL}(2, \mathbb{Z}) \longrightarrow 1 ,
\]

acts on the algebra \( \text{D}_{\text{red}} \text{U}(E_{\text{sm}}) \).

To write down the \( \widehat{\text{SL}}(2, \mathbb{Z}) \)-action, we need to introduce the winding number. Note that we can lift the natural \( \text{SL}(2, \mathbb{Z}) \)-action on the circle \( S^1 = \mathbb{RP}^1 = (\mathbb{R}^2 \setminus \{0\})/\mathbb{R}_{>0} \) to an \( \widehat{\text{SL}}(2, \mathbb{Z}) \)-action on \( \mathbb{R} \) by the identification \( S^1 = \mathbb{R}/2\mathbb{Z} \).

**Definition 6.12.**

1. For a \( (r, d) \in \mathbb{Z}^* \), denote by \( (r : d) \) the corresponding element of \( S^1 = \mathbb{RP}^1 \), and by \( (r : d) \in \mathbb{R} \) any lift of \( (r : d) \).
2. For a slope \( d/r \in \mathbb{Q} \cup \{ \pm \infty \} \) and an element \( \gamma \in \widehat{\text{SL}}(2, \mathbb{Z}) \), define the winding number \( w(\gamma, d/r) \in \mathbb{Z} \) by

\[
w(\gamma, d/r) := \begin{cases} \#(\mathbb{Z} \cap [r : d), \gamma((r : d)] \) & \text{if } (r : d) \leq \gamma((r : d)) \\ -\#(\mathbb{Z} \cap [r : d), \gamma((r : d)] \) & \text{otherwise} \end{cases}
\]

Here \( [r : d], \gamma((r : d)) \) denotes the interval between \( (r : d) \) and \( \gamma((r : d)) \) in \( \mathbb{R} \).
Fact 6.13 ([BuS12, (6.16)]). The group $\widehat{\text{SL}(2, \mathbb{Z})}$ acts on $D_{\text{red}} U(E_{\text{sm}})$ by
\[
\gamma(k_x) = k_{\gamma(x)}, \quad \gamma(t_x) = t_{\Phi(x)}^w(\gamma, \mu(x))
\]
for $\gamma \in \text{SL}(2, \mathbb{Z})$.

Remark 6.14. For the ordinary Hall algebra $\mathcal{H}_{\text{ext}}(A)$ of a finitary abelian category $A$, the existence of the action of $\text{Aut}(D^b A)$ on the Drinfeld double of the algebra is shown by Cramer [C10].

Finally we can state the second main result of [BuS12].

Fact 6.15 ([BuS12]). The Fourier-Mukai transform $\Phi_P$ with the Poincze bundle as its kernel induces the algebra automorphism $\Phi^H = \theta$ on $D_{\text{red}} U(E_{\text{sm}}) = \hat{U} \otimes K$.

7. The case for irreducible projective curves of arithmetic genus one

In this section we will give the main result of this paper. We investigate the motivic-Hall algebras for singular irreducible projective curves of arithmetic genus one, namely, a singular elliptic curve with a node or a cusp. The final result will be quite similar to the smooth elliptic case.

In this section $k$ denotes a fixed algebraically closed field of characteristic zero. $C$ denotes an irreducible reduced projective curve over $k$, so that it may be singular.

7.1. Semistable sheaves on singular elliptic curves and Fourier-Mukai transforms.

We follow [BuK06] for the description of semistable sheaves on $C$.

The numerical Grothendieck group $\text{Num}(C)$ is isomorphic to $\mathbb{Z}^2$ as modules by the homomorphism
\[
\text{Num}(C) \xrightarrow{\sim} \mathbb{Z}^2, \quad \mathcal{E} \longmapsto (\text{rk}(\mathcal{E}), \deg(\mathcal{E})).
\]
Recalling §5.2, we denote by $\mu(\mathcal{E}) := \deg(\mathcal{E})/\text{rk}(\mathcal{E})$ the slope of $\mathcal{E} \in \text{Coh}(C)$, and consider the slope stability of coherent sheaves. Denote by $S_\nu$ the full subcategory of $\text{Coh}(C)$ consisting of semistable sheaves of slope $\nu$.

Here is the fundamental fact on the semistable sheaves on $C$.

Fact 7.1 ([BuK06, Corollary 4.3]). For any $\nu \in \mathbb{Q} \cup \{\infty\}$, the category $S_\nu$ is equivalent to the category $S_\infty = \text{Tor}(C)$ of torsion sheaves.

The equivalence is realized by Fourier-Mukai transform. Recall the exact sequence (6.1).
\[
1 \rightarrow \mathbb{Z} \oplus \text{Aut}(C) \times \text{Pic}^0(C) \rightarrow \text{Aut}(D^b_{\text{coh}}(C)) \rightarrow \text{FM}(C) \rightarrow 1.
\]
By [BuK05], the subgroup of $\text{FM}(C)$ generated by the spherical twists $T_{D_C}$ and $T_{D_{x_0}}$, where $x_0$ is a regular point of $C$, is isomorphic to $\text{SL}(2, \mathbb{Z})$.

In the category $S_\infty = \text{Tor}(C)$, the stable sheaf is precisely the structure sheaves $\mathcal{O}_x$ of closed points $x$ of $C$. If $C$ is singular with one singular point $s$, then all the structure sheaf $\mathcal{O}_s$ is the unique non-perfect stable sheaf in $S_\infty$.

As a consequence of Fact 7.1, for each $\nu \in \mathbb{Q} \cup \{\infty\}$ there is precisely one object of $S_\nu$ which is stable but not perfect.

7.2. Motivic zeta function. Here we study the motivic zeta functions for irreducible curve of arithmetic genus one. Recall that $k$ denotes an algebraically closed field of characteristic zero, and $E$ denotes an irreducible reduced projective curve of arithmetic genus 1 over $k$.

Proposition 7.2. The motivic zeta function $\zeta_{\text{mot}}(E; z)$ of the curve $E$ has the form
\[
\zeta_{\text{mot}}(E; z) = \frac{1 + az + Lz^2}{(1-z)(1-Lz)}
\]
with $a \in K(\text{Var}/k)$. 
Proof. It follows from the result of [AP96], where the result was shown for the ordinary Hasse-Weil zeta function in the case when $E$ is defined on the finite field $\mathbb{F}_q$. \qed

7.3. The composition subalgebra. Now we consider the motivic Hall algebra for a projective curve $E$ of arithmetic genus 1. Recall Definition 3.20 in §3.5 where we defined the composition subalgebra $U(E)$. On the reduced Drinfeld double of $U(E)$, we have a similar result as in the case of a smooth elliptic curve. Let $\tilde{U}$ be the $\mathfrak{gl}_1$-quantum toroidal algebra considered to be defined over $\mathbb{Q}(q_1, q_2)$.

**Theorem 7.3.** $D_{\text{red}}U(E)$ is isomorphic to $\tilde{U} \otimes_{\mathbb{Q}(q_1, q_2)} \mathbb{K}$. Here the ring homomorphism $\mathbb{Q}(q_1, q_2) \to \mathbb{K} = K(\text{St}/\mathfrak{t})$ is defined by mapping $q_1$ and $q_2$ to the inverse of the zeros of the motivic zeta function of $E$. Namely we put

$$\zeta_{\text{mot}}(E; z) = \frac{(1 - q_1 z)(1 - q_2 z)}{(1 - z)(1 - L z)}.$$  

**Proof.** As mentioned at Fact 6.10, the isomorphism $D_{\text{red}}U(E)_{\text{sm}} \simeq \tilde{U} \otimes \mathbb{K}$ comes from the $\text{SL}(2, \mathbb{Z})$-action. As we observed in §7.1, we have that action on $D_{\text{red}}U(E)$. Proposition 5.4 claims that the relation on the composition subalgebra only depends on the motivic zeta function $\zeta - a; z$. Since by Proposition 7.2 $\zeta(E; z)$ and $\zeta(E_{\text{sm}}; z)$ have the same form, we know that the generators of $U(E)$ satisfy the same relations as $U(E_{\text{sm}})$. Thus $D_{\text{red}}U(E)$ has the same description as $D_{\text{red}}U(E_{\text{sm}})$ has in Fact 6.10. \qed

By the comparison of the Fourier-Mukai transforms, we also have

**Theorem 7.4.** There is a Fourier-Mukai transform inducing the algebra automorphism $\theta_1$ on the algebra $D_{\text{red}}U(E) = \tilde{U} \otimes \mathbb{K}$.

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Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan

E-mail address: yanagida@kurims.kyoto-u.ac.jp