A CONVERSE TO SCHREIER’S INDEX-RANK FORMULA

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1. Let $G$ be a group and let $\varphi$ be a real-valued function, defined on all subgroups $G_1$ with finite index in $G$. Call $\varphi$ submultiplicative if the inequality

(1) $\varphi(G_2) \leq [G_1 : G_2] \cdot \varphi(G_1)$

holds for all pairs of subgroups $G_2 \leq G_1$ of $G$ with $G_2$ a subgroup of finite index in $G$; call it multiplicative if equality holds throughout in inequality (1).

The rank $\text{rk}(G)$ of a finitely generated group $G$ is, by definition, the minimal number of elements generating it. By a Theorem of Reidemeister’s, subgroups with finite index in finitely generated groups are finitely generated; moreover, the rank function on its subgroups of finite index is submultiplicative (see, e.g., [MKS66, p. 89, Thm. 2.7]). For finitely generated free groups, a sharper result holds: Schreier’s index-rank formula ([Sch27] or [LS01, p. 16, Prop. 3.9]) asserts that the function $\text{rk} - 1$ is multiplicative. It follows that the function $\text{rk} - 1$ is submultiplicative on every finitely generated group.

2. In [LvdD81, § 2] Lubotzky and van den Dries ask whether a finitely generated group $G$ is necessarily free if its function $\text{rk} - 1$ is multiplicative and if, in addition, $G$ is residually finite. One has, of course, to require that $G$ contains sufficiently many subgroups of finite index in order to rule out counter-examples like finitely generated infinite simple groups or, more concretely, G. Higman’s 4-generator 4-relator group $\langle a_1, a_2, a_3, a_4 | a_i a_{i+1} a_i^{-1} a_{i+2}^{-1} \rangle$ for all $i \in \mathbb{Z}/4\mathbb{Z}$ (see [Hig51] or [Ser77, p. 18, Prop. 6] for a discussion of this group).

3. In this note the question raised by Lubotzky and van den Dries is shown to have an affirmative answer:

Theorem 1. A finitely generated, residually finite group for which the function $\text{rk} - 1$ is multiplicative is necessarily free.

Theorem 1 will be a consequence of

Lemma 2. Let $G \neq \{1\}$ be a finitely generated, residually finite group. Choose a presentation $\pi: F(\mathcal{X})/R \to G$ with $F(\mathcal{X})$ a free group of rank $\text{rk}(G)$. If $R \neq \{1\}$, select an element $r \in R \setminus \{1\}$ of minimal length. Then $r$ is a member of a basis of a subgroup $F_1$ of $F$ that contains $R$ and has finite index in $F$.

Proof. Let $r$ be represented by the freely reduced word $w = s_1 s_2 \cdots s_m$ with letters $s_i$ in $\mathcal{X} \cup \mathcal{X}^{-1}$. By the minimal property of $r$, no non-empty, proper subword $s_i \cdots s_{i+1} \cdots s_j$ of $w$ represents a relator in $R$ and thus the initial segments

(2) 1, s_1, s_1 s_2, \ldots , s_1 s_2 \cdots s_{m-1}

of $w$ map to distinct elements of $G$. As the group $G$ is residually finite and finitely generated, there exists therefore a normal subgroup $G_1 \triangleleft G$ of finite index such that the initial segments listed in (2) represent distinct elements of the quotient group $G/G_1$. Let $F_1$ denote the full preimage of $G_1$ under the projection $\pi: F(\mathcal{X}) \to G$. 

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The property just stated can then be rephrased by saying that the initial segments \( w \) of \( x \) represent distinct cosets of \( F_1 \) on \( F \). The list \( w \) is thus a partial Schreier transversal of \( F_1 \) in \( F \); it can be enlarged to a Schreier transversal \( T_1 \) of \( F_1 \) in \( F \).

Let \( X_1 \) be the basis of \( F_1 \) obtained from \( T_1 \) by Schreier’s method.

Two cases now arise. If \( s_m \) belongs to \( X \), the word \( w = s_1 \cdots s_{m-1} \cdot s_m \) represents an element of the basis \( X_1 \) of \( F_1 \). Otherwise, we replace the element \( x = s_1^{-1} \) of the basis \( X \) by its inverse \( x^{-1} = s_m \) and arrive at a new basis \( X' = (X \setminus \{x\}) \cup \{s_m\} \) of \( F \). Let \( T' \) be the set of words obtained from \( T \) by rewriting each element in \( T \) as a word in \( X' \). Then \( T' \) is a transversal of the subgroup \( F_1 \) in \( F \) and the basis derived from \( X' \) and \( T' \) by Schreier’s method contains the word \( r^{-1} = (s_m^{-1} \cdots s_1^{-1}) \).

But if so \( F_1 \) admits again a basis that contains \( r \).

We are left with deducing Theorem 1 from Lemma 2. Suppose \( G \) has rank \( m \); let \( F \) be a free group of the same rank and choose an epimorphism \( \pi: F \to G \).

If \( \pi \) is injective the group \( G \) is free, as claimed. Otherwise, the kernel \( R \) of \( \pi \) is not reduced to the unit element, whence Lemma 2 gives us a subgroup \( F_1 \) of \( F \) having finite index in \( F \) and comprising \( R \) and, in addition, a basis \( X_1 \) of \( F_1 \) that contains an element \( r' \in R \). Since the function \( \text{rk} \) is multiplicative on \( F \), this basis has \((m-1) \cdot \text{card}(F/F_1) + 1\) elements, one of which lies in the kernel of \( \pi \), whence \( G_1 \) is generated by no more than \((\text{rk} \cdot G - 1) \cdot \text{card}(G/G_1)\) elements. The function \( \text{rk} \) is therefore not multiplicative on \( G \), contrary to hypothesis.

4. One may ask how far the function \( \rho = \text{rk} \cdot G \) can deviate from being multiplicative. For some groups an explicit answer is available. If, e.g., \( G \) is free abelian, the function \( \text{rk} \cdot G \) is constant; if \( G \) is an orientable surface group with presentation

\[
\langle a_1, b_1, \ldots, a_g, b_g \mid \prod_{i=1}^{g} [a_i, b_i] = 1 \rangle
\]

the facts that subgroups of finite index are again of this form and that the Euler characteristic is multiplicative imply the relation

\[
\rho(G_1) = [G : G_1] \cdot \rho(G) + (1 - [G : G_1]),
\]

valid for all subgroups \( G_1 \) of finite index in an orientable surface group.

5. I found the proof of Theorem 1 at the beginning of 1980 and communicated my argument to Alex Lubotzky; he encouraged me to turn my sketch into a formal proof. This task was carried out in 1981; the typescript was finished in December of that year. I sent one of my preprints to Jean-Pierre Serre; he commented on it in a long letter at the beginning of 1982. I am very grateful to him for his critical remarks and suggestions. They have been incorporated into this new, amended and abridged, version of my original note.

**References**

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