The Natural Banach Space for Version Independent Risk Measures

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Abstract

Risk measures, or coherent measures of risk are often considered on the space $L^\infty$, and important theorems on risk measures build on that space. Other risk measures, among them the most important risk measure—the Average Value-at-Risk—are well defined on the larger space $L^1$ and this seems to be the natural domain space for this risk measure. Spectral risk measures constitute a further class of risk measures of central importance, and they are often considered on some $L^p$ space. But in many situations this is possibly unnatural, because any $L^p$ with $p > p_0$, say, is suitable to define the spectral risk measure as well. In addition to that risk measures have also been considered on Orlicz and Zygmund spaces. So it remains for discussion and clarification, what the natural domain to consider a risk measure is?

This paper introduces a norm, which is built from the risk measure, and a Banach space, which carries the risk measure in a natural way. It is often strictly larger than its original domain, and obeys the key property that the risk measure is finite valued and continuous on that space in an elementary and natural way.

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Classification: 90C15, 60B05, 62P05

1 Introduction

This paper addresses coherent measures of risk (risk measures, for short) and the natural domain (the natural space), where they can be considered. Coherent measures of risk have been introduced in the seminal paper [4] in an axiomatic way and have been investigated in a series of subsequent papers in mathematical finance since then. In the actuarial literature, however, risk measures and axiomatitc treatments have been considered already earlier, for example in Denneberg ([10]) and in this journal by Wang et al. ([27]).

We state the axioms (cf. [5]) for a convex risk measure $\rho$, mapping $\mathbb{R}$-valued random variables into the real numbers $\mathbb{R}$ or to $+\infty$. Here, the initial axioms have been adapted to follow the interpretation of loss instead of profit—the common modification in insurance—in the usual and appropriate way.

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Monotonicity: \( \rho(Y_1) \leq \rho(Y_2) \) whenever \( Y_1 \leq Y_2 \) almost surely;

Positive homogeneity: \( \rho(\lambda Y) = \lambda \rho(Y) \) whenever \( \lambda > 0 \);

Convexity: \( \rho((1 - \lambda) Y_0 + \lambda Y_1) \leq (1 - \lambda) \rho(Y_0) + \lambda \rho(Y_1) \) for \( 0 \leq \lambda \leq 1 \);

Translation equivariance\(^1\): \( \rho(Y + c) = \rho(Y) + c \) if \( c \in \mathbb{R} \).

The main observation in this paper starts with the fact that the risk measure \( \rho \) can be associated in a natural way with a seminorm, which is a norm in important cases. It is an elementary property that the risk measure is continuous with respect to the norm introduced.

We investigate this new norm for specific risk measures, starting with spectral risk measures. It turns out that the domain, where the spectral risk measure can be defined in a meaningful way, is always strictly larger than \( L^\infty \). The respective space is a Banach space, and we study its topology, which can be compared with \( L^p \) spaces. However, the topology always differs from the topology of an \( L^p \) space (cf. [13]).

A risk measure \( \rho \)—being a convex function—has a convex conjugate function, and the Fenchel–Moreau theorem allows recovering the initial function, the initial risk measure \( \rho \) in our situation. The convex conjugate function involves the dual of the initial space, for this reason it is essential to understand the dual of the Banach space associated with the risk measure. The norm on the dual space measures the growth of the random variable by involving second order stochastic dominance relations.

It is elaborated moreover in this paper that a risk measure cannot be defined in a meaningful way on a space larger than \( L^1 \).

The domain and the co-domain of spectral risk measures

The axioms characterizing risk measures have been stated above without giving the domain and the co-domain precisely. Indeed, important results are well known when considering \( \rho \) as a function on \( L^\infty, \rho : L^\infty \to \mathbb{R} \): the results include Kusuoka’s representation (cf. [18] and (3) below) and results on continuity. We state the following example.

**Proposition 1.** Every \( \mathbb{R} \)-valued risk measure \( \rho \) on \( L^\infty \) is Lipschitz-continuous with constant 1, it satisfies \( |\rho(Y_2) - \rho(Y_1)| \leq \|Y_2 - Y_1\|_{\infty} \).

**Proof.** See, e.g., [14, Lemma 4.3] for a proof.

In many situations, for example when considering the trivial risk measure \( \rho(\cdot) := \mathbb{E}(\cdot) \) or the Average Value-at-Risk, the domain \( L^\infty \) is not satisfactory large enough, the domain \( L^1 \) is perhaps more natural and convenient to consider in this situation.

Depending on the domain chosen for a risk measure, the co-domain is often specified to be \( \mathbb{R} \), or the extended reals \( \mathbb{R} \cup \{\infty\} \), in some publications even \( \mathbb{R} \cup \{\infty, -\infty\} \). In this context it should be emphasized that there is an intimate relationship between the properties continuity of a risk measure and its range, the following important result clarifies the connections:

**Proposition 2.** Consider a \( \mathbb{R} \cup \{\infty\} \)-valued, lsc. risk measure \( \rho \) defined on \( L^p \), \( 1 \leq p < \infty \), satisfying \( (M), (C) \) and \( (T) \). Suppose further that \( \{\rho < \infty\} \) has a nonempty interior. Then \( \rho \) is finite valued and continuous on the entire \( L^p \).

\(^1\)In an economic or monetary environment this is often called Cash invariance instead.
The proof is contained in [23] and in [24], Proposition 6.7. The preceding discussion of the latter reference also contains the following reformulation of the statement, which is more striking: A risk measure satisfying (M), (C) and (T) is either finite valued and continuous on the entire $L^p$, or it takes the value $+\infty$ on a dense subset.

Both results suggest to consider $\mathbb{R}$ (i.e. $\mathbb{R} \setminus \{\pm \infty\}$) valued risk measures solely, because these are precisely the finite valued and continuous risk measures.

Outline of the paper: The following Section 2 introduces the associated norm and elaborates its elementary property. The subsequent section, Section 3, addresses an elementary risk measure, the spectral risk measure. This risk measure is elementary, as every version independent risk measure can be built from spectral risk measures.

A space is introduced, which we call the space of natural domain, which is as large as possible to carry a spectral risk measure. It is verified that the associated space is a Banach space. The new norm can be used in a natural way to extend the domain of elementary risk measures, and it is elaborated which $L^p$ spaces the space of natural domain comprises.

This section contains moreover the remarkable result, that there is no finite valued risk measure on a space larger than $L^1$.

We study further the topological dual of the Banach space introduced (Section 5). It turns out the dual norm can be characterized by use of the Average Value-at-Risk, the simplest risk measure, and by second order stochastic dominance. The investigations are pushed further to more general risk measures, and an even more general Banach space to carry a general risk measure is highlighted in Section 6.

2 The norm associated with a risk measure

The results presented in this paper start along with the observation that a risk measure $\rho$ induces a (semi-)norm in the following elementary way.

**Definition 3.** Let $L$ be a vector space of $\mathbb{R}$-valued random variables on $(\Omega, \mathcal{F}, P)$ and $\rho : L \to \mathbb{R} \cup \{-\infty, \infty\}$ be a risk measure. Then

$$\|\cdot\|_\rho := \rho(|\cdot|)$$

is called associated norm, associated with the risk measure $\rho$.

**Remark 4.** If no confusion may occur we shall simply write $\|\cdot\|$ to refer to $\|\cdot\|_\rho$.

The following proposition verifies that $\|\cdot\|_\rho$ is indeed a seminorm on the appropriate vector space.

**Proposition 5** (Finiteness, and the seminorm property). Let $\rho$ be a risk measure on a vector space of $\mathbb{R}$-valued random variables. Then $\|\cdot\| = \rho(|\cdot|)$ is a seminorm on $L := \{Y : \rho(|Y|) < \infty\}$ and $\rho$ is finite valued on $L$.

**Proof.** We show first that that $\rho$ is $\mathbb{R}$-valued on $L = \{Y : \rho(|Y|) < \infty\}$. For this observe that $Y \leq |Y|$, and by monotonicity thus $\rho(Y) \leq \rho(|Y|) = \|Y\|$. Moreover it holds that $\rho(0) = 0$ \footnote{Otherwise, $\rho(0) = \rho(2 \cdot 0) = 2 \cdot \rho(0)$ would imply $1 = 2$, a contradiction.} and thus

$$0 = 2 \cdot \rho\left(\frac{1}{2} Y + \frac{1}{2} (-Y)\right) \leq 2 \cdot \left(\frac{1}{2} \rho(Y) + \frac{1}{2} \rho(-Y)\right) = \rho(Y) + \rho(-Y),$$

...
such that \(-\rho(Y) \leq \rho(-Y)\). Now \(-Y \leq |Y|\) and, again by monotonicity, \(-\rho(Y) \leq \rho(-Y) \leq \rho(|Y|) = \|Y\|\). Summarizing thus \(|\rho(Y)| \leq \|Y\|\), such that \(\rho\) is finite valued on \(L\).

Note that
\[
||\lambda \cdot Y|| = \rho(|\lambda \cdot Y|) = \rho(|\lambda| \cdot |Y|) = |\lambda| \cdot \rho(|Y|) = |\lambda| \cdot \|Y\|,
\]
and \(||\cdot||\) thus is positively homogeneous.

Next it follows from monotonicity, positive homogeneity and convexity that
\[
\|Y_1 + Y_2\| = \rho(|Y_1 + Y_2|) = 2 \cdot \rho\left(\frac{1}{2} |Y_1| + \frac{1}{2} |Y_2|\right)
\leq 2 \cdot \left(\frac{1}{2} \rho(|Y_1|) + \frac{1}{2} \rho(|Y_2|)\right) = \rho(|Y_1|) + \rho(|Y_2|)
= \|Y_1\| + \|Y_2\|,
\]
and this is the triangle inequality. \(\square\)

The next proposition elaborates, that the risk measure is continuous with respect to its associated norm. This consistency result on continuity generalizes Proposition 1.

**Proposition 6** (Continuity). Let \(\rho\) be a risk measure, defined on a vector space of \(\mathbb{R}\)-valued random variables. Then \(\rho\) is Lipschitz continuous with constant \(1\) with respect to the seminorm \(||\cdot|| = \rho(|\cdot|)\).

**Proof.** As for continuity note that
\[
\rho(Y_2) = 2 \cdot \rho\left(\frac{1}{2} Y_1 + \frac{1}{2} (Y_2 - Y_1)\right)
\leq 2 \left(\frac{1}{2} \rho(Y_1) + \frac{1}{2} \rho(Y_2 - Y_1)\right) = \rho(Y_1) + \rho(|Y_2 - Y_1|)
\]
by convexity and monotonicity. It follows that \(\rho(Y_2) - \rho(Y_1) \leq ||Y_2 - Y_1||\). Interchanging the roles of \(Y_1\) and \(Y_2\) reveals that
\[
|\rho(Y_2) - \rho(Y_1)| \leq ||Y_2 - Y_1||,
\]
the assertion. To accept that the Lipschitz constant \(1\) cannot be improved consider the particular choices \(Y_1 := 0\) and \(Y_2 := 1\) in view of translation equivariance \((T)\).

\(\square\)

## 3 Spectral risk measures

Among the initial attempts to introduce premium principles to price insurance contracts are distorted probabilities, a concept which can be summarized nowadays by distorted acceptability functionals (cf. [20]) or spectral risk measures. Spectral risk measures—or the weighted Value-at-Risk (cf. [8]), which is a more suggestive term—have been considered for example in [2, 1]. This risk measure involves the Value-at-Risk at level \(p\),
\[
\text{V@R}_p(Y) := F^{-1}_Y(p) := \inf \{y : P(Y \leq y) \geq p\},
\]
which is the left-continuous, lower semi-continuous (lsc.) quantile; the spectral risk measure (or weighted \(\text{V@R}\)) then is the functional
\[
\rho_\sigma(Y) := \int_0^1 \sigma(u) \text{V@R}_u(Y) \, du, \tag{1}
\]
mapping a random variable \( Y \) to a real number, if the integral exists.

The function \( \sigma : [0, 1] \rightarrow \mathbb{R}_0^+ \), called the spectrum or spectral function, is a weight function. To build a reasonable premium principle the function \( \sigma \) should obey some properties to be consistent with the axioms imposed on risk measures: first, associating \( Y \) with loss, \( \sigma \) should evaluate to nonnegative reals, \( \mathbb{R}_0^+ \). Higher losses should be weighted higher, thus \( \sigma \) should be nondecreasing. And finally, as \( \sigma \) represents a weight function, it is natural to request \( \int_0^1 \sigma(u) \, du = 1 \).

An important, elementary spectral risk measure satisfying all axioms above is the Average Value-at-Risk, which is specified by the spectral function

\[
\sigma_\alpha(u) := \begin{cases} 
0 & \text{if } u < \alpha \\
\frac{1}{1-\alpha} & \text{else,}
\end{cases}
\]

that is

\[
\text{AV@R}_\alpha(Y) := \frac{1}{1 - \alpha} \int_\alpha^1 V@R_\alpha(Y) \, du \quad (\alpha < 1),
\]

and for \( \alpha = 1 \) the Average Value-at-Risk per definition is

\[
\text{AV@R}_1(Y) := \lim_{\alpha \uparrow 1} \text{AV@R}_\alpha(Y) = \text{ess sup} Y \quad (\alpha = 1).
\]

The domain of spectral risk measures

It is obvious that the Average Value-at-Risk \( (\alpha < 1) \) may be well defined on \( L^1 \), with the result that

\[
|\text{AV@R}_\alpha(Y)| \leq \frac{1}{1 - \alpha} \mathbb{E}|Y| = \frac{1}{1 - \alpha} \|Y\|_1 < \infty \quad (Y \in L^1),
\]

that means that \( \text{AV@R}_\alpha \) is finite valued whenever \( Y \in L^1 \). This is not the case, however, for \( \alpha = 1 \); a restriction to the smaller space \( L^\infty \subset L^1 \) is necessary in order to ensure that \( \text{AV@R}_1 \) is finite valued,

\[
|\text{AV@R}_1(Y)| \leq \|Y\|_\infty < \infty \quad (Y \in L^\infty).
\]

Even more peculiarities appear when considering the spectral function \( \sigma(u) := \frac{1}{2\sqrt{1-u}} \). Clearly, \( \sigma \in L^q \) whenever \( q < 2 \), but \( \sigma \notin L^2 \). Hölder’s inequality can be employed to insure that \( \rho_\sigma \) is finite valued on \( L^p \) \( (p > 2, \frac{1}{q} + \frac{1}{p} = 1) \), because

\[
|\rho_\sigma(Y)| \leq \|\sigma\|_q \cdot \left( \int_0^1 F_{Y^{-1}}(u)^p \right)^{\frac{1}{p}} = \frac{1}{2} \left( \frac{2}{2 - q} \right)^{\frac{1}{q}} \cdot \|Y\|_p,
\]

and the constant \( \frac{1}{2} \left( \frac{2}{2 - q} \right)^{\frac{1}{q}} \) again exceeds every finite bound whenever \( q \) approaches 2 from below.

So what is a good space to consider \( \rho_\sigma \)? Any \( L^p \) \( (p > 2) \) guarantees that \( \rho_\sigma \) is finite valued and continuous, but \( L^2 \) is obviously too large. The naïve choice \( \bigcup_{p>2} L^p \) does not have a satisfying norm, or topology neither. (See, for different configurations, \([6, 7]\).)
Further properties and importance of spectral risk measures

A well known and essential representation of risk measures was elaborated by Kusuoka in [18] (see [17] for the statement presented below). Kusuoka’s result considers risk measures on $L^\infty$ which are version independent (also: law invariant), i.e. which satisfy $\rho(Y) = \rho(Y')$ whenever $Y$ and $Y'$ share the same law, that is if $P(Y \leq y) = P(Y' \leq y)$ for every $y \in \mathbb{R}$.

**Theorem 7** (Kusuoka’s representation). A version independent risk measure $\rho$ on $L^\infty$ of an atomless probability space $(\Omega, \mathcal{F}, P)$ has the representation

$$\rho(Y) = \sup_{\mu \in \mathcal{M}} \int_0^1 AV@R_\alpha(Y) \mu(d\alpha),$$

(3)

where $\mathcal{M}$ is a collection of probability measures on $[0, 1]$.

**Kusuoka representation of a spectral risk measure.** The Kusuoka representation of a spectral risk measure $\rho_\sigma$ is provided by the probability measure $\mu_\sigma([a, b]) := \int_a^b d\mu_\sigma(\alpha)$ on $[0, 1]$, where $\mu_\sigma$ is the nondecreasing function

$$\mu_\sigma(\alpha) := (1 - p) \sigma(p) + \int_0^p \sigma(u) du \quad (0 \leq p \leq 1), \quad \mu_\sigma(p) := 0 \quad (p < 0),$$

(4)

which satisfies $\mu_\sigma(1) = 1$ and $d\mu_\sigma(p) = (1 - p) d\sigma(p)$. It holds that

$$\rho_\sigma(Y) = \int_0^1 AV@R_\alpha(Y) \mu_\sigma(d\alpha),$$

(5)

which exposes the Kusuoka representation of a spectral risk measure (cf. [25]).

**Kusuoka representation by spectral risk measures.** Conversely, any measure $\mu$ (provided that $\mu(\{1\}) = 0$) of the representation (3) can be related to the function

$$\sigma_\mu(\alpha) = \int_0^\alpha \frac{1}{1 - u} \mu(du),$$

(6)

and it holds that

$$\int_0^1 AV@R_\alpha(Y) \mu(d\alpha) = \int_0^1 \sigma_\mu(\alpha) V@R_\alpha(Y) d\alpha = \rho_{\sigma_\mu}(Y),$$

which is a spectral risk measure.

But even the requirement $\mu(\{1\}) = 0$ can be dropped: indeed, there is a set $\mathcal{S}$ of continuous (and thus bounded) spectral functions on $[0, 1]$, such that the relation

$$\rho(Y) = \sup_{\mu \in \mathcal{M}} \int_0^1 AV@R_\alpha(Y) \mu(d\alpha) = \sup_{\sigma \in \mathcal{S}} \int_0^1 V@R_\alpha(Y) \sigma(\alpha) d\alpha = \sup_{\sigma \in \mathcal{S}} \rho_{\sigma}(Y)$$

(7)

holds (cf. [19]). This again exposes the importance of spectral risk measures, as every version independent risk measure $\rho$ can be built from spectral risk measures by (7).

Recall that Kusuoka’s representation builds on the space $L^\infty$. But again it is not clear, if, and to which larger space this risk measure can be extended, because every $\sigma$ might allow a different domain.
The space of natural domain, $L_\sigma$

Let $\sigma$ be a nonnegative, nondecreasing, integrable function with $\int_0^1 \sigma(u)\,du = 1$. For $Y$ a random variable we consider the function

$$\rho_\sigma(Y) = \int_0^1 \sigma(u) \, F_Y^{-1}(u) \, du$$

already defined in (1). For $\sigma \in L^1$ (which is a minimal requirement to insure that $\int_0^1 \sigma(u)\,du = 1$), $\rho_\sigma$ is certainly well defined for $Y \in L^\infty$, but for other random variables the integral possibly diverges. And it might diverge to $+\infty$, to $-\infty$, or be even of the indefinite form $\infty - \infty$. The following definition respects the finiteness of the spectral risk measure in view of Proposition 5.

**Definition 8.** The **natural domain** corresponding to a spectral risk measure $\rho_\sigma$ induced by a spectral function $\sigma$ is

$$L_\sigma := \{ Y \in L^0 : \|Y\|_\sigma < \infty \},$$

where

$$\|Y\|_\sigma := \rho_\sigma(|Y|).$$

Note that $|Y| \geq 0$ is positive, such that $F_{|Y|}^{-1}(\cdot) \geq 0$ is positive as well and the condition $\rho_\sigma(|Y|) < \infty$ makes perfect sense for any measurable random variable $Y \in L^0$.

**Remark 9.** The seminorm $\|\cdot\|_\sigma$ has the representation

$$\|Y\|_\sigma = \int_0^\infty \tau_\sigma(F_{|Y|}(y)) \, dy$$

in terms of the cdf $F_{|Y|}$ directly, without involving the inverse $F_{|Y|}^{-1}(\tau_\sigma(\alpha)) := \int_0^1 \sigma(u)\,du$.

**Proposition 10.** $\|\cdot\|_\sigma = \rho_\sigma(|\cdot|)$ is a norm on $L_\sigma$.

**Proof.** It was already shown in Proposition 5 that $\|\cdot\|_\sigma$ is a seminorm. What remains to be shown is that $\|\cdot\|_\sigma$ separates points. For this recall that $\sigma$ is positive, nondecreasing, and satisfies $\int_0^1 \sigma(p) \, dp = 1$, and $F_{|Y|}(\cdot)$ is a nondecreasing and positive function as well. Hence if $\int_0^1 \sigma(p) \, F_{|Y|}^{-1}(p) \, dp = 0$, then $F_{|Y|}^{-1}(\cdot) \equiv 0$, that is $Y = 0$ almost everywhere. The function $\|\cdot\|_\sigma$, thus separates points in $L_\sigma$ and $\|\cdot\|_\sigma$ hence is a norm.

The next theorem already elaborates that the set $L_\sigma$ is large enough and at least contains $L^p$, whenever $\sigma \in L^q$ (and the exponents are conjugate, $\frac{1}{p} + \frac{1}{q} = 1$).

**Theorem 11 (Comparison with $L^p$).** Let $\sigma$ be fixed.

(i) If $\sigma \in L^q$ for some $q \in [1, \infty]$ with conjugate exponent $p$, then

$$L^\infty \subset L^p \subset L_\sigma \subset L^1$$

and

$$\|Y\|_1 \leq \|Y\|_\sigma \leq \|\sigma\|_q \cdot \|Y\|_p$$

whenever $Y \in L^p$.
(ii) For \( \sigma \) bounded (i.e. \( \sigma \in L^\infty \)) it holds moreover that \( L_\sigma = L^1 \), the norms are equivalent and satisfy
\[
\|Y\|_1 \leq \|Y\|_\sigma \leq \|\sigma\|_\infty \cdot \|Y\|_1.
\]

It follows in particular from (ii) that \( P(A) \leq \|1_A\|_\sigma \leq 1 \) for measurable sets \( A \), and \( \|Y\|_\sigma = \|Y\|_1 \) for the function being constantly \( 1 \) (\( \sigma = 1 \)).

Proof. Note that \( \int_0^1 \sigma(u) \, du = 1 \) and \( \sigma(\cdot) \) is nondecreasing, hence there is a \( \bar{u} \in (0, 1) \) such that \( \sigma(u) \leq 1 \) for \( u < \bar{u} \) and \( \sigma(u) \geq 1 \) for \( u > \bar{u} \). Note as well that \( \int_0^\bar{u} 1 - \sigma(u) \, du = \int_0^1 \sigma(u) - 1 \, du \).

Then it follows that
\[
\int_0^\bar{u} (1 - \sigma(u)) \frac{1}{|Y|}(u) \, du \leq \int_0^\bar{u} (1 - \sigma(u)) \frac{1}{|Y|}(\bar{u}) \, du
\]
\[
= \int_\bar{u}^1 (\sigma(u) - 1) \frac{1}{|Y|}(\bar{u}) \, du \leq \int_\bar{u}^1 (\sigma(u) - 1) \frac{1}{|Y|}(u) \, du,
\]
because \( \frac{1}{|Y|}(\cdot) \) is increasing. After rearranging thus
\[
\|Y\|_1 = \mathbb{E}|Y| = \int_0^1 F_{|Y|}^{-1}(u) \, du \leq \int_0^1 F_{|Y|}^{-1}(u) \sigma(u) \, du = \rho_\sigma(|Y|) = \|Y\|_\sigma,
\]
which is the first assertion. The inclusion \( L_\sigma \subset L^1 \) is immediate as well, as \( \|Y\|_\sigma < \infty \) implies that \( \|Y\|_1 \leq \infty \).

The remaining inequality
\[
\|Y\|_\sigma = \int_0^1 F_{|Y|}^{-1}(u) \sigma(u) \, du \leq \left( \int_0^1 \sigma(u)^\frac{1}{p} \right)^\frac{1}{q} \cdot \left( \int_0^1 \frac{1}{|Y|}(u)^p \right)^\frac{1}{p} = \|\sigma\|_q \cdot (\mathbb{E}|Y|^p)^\frac{1}{p}
\]
is Hölder’s inequality. \( \square \)

Remark 12. The inequality \( \|Y\|_1 \leq \|Y\|_\sigma \) is also a direct consequence of Chebyshev’s sum inequality in its continuous form, which states that \( \int_0^1 f(u) \, du \cdot \int_0^1 g(u) \, du \leq \int_0^1 f(u) g(u) \, du \) whenever \( f \) and \( g \) are both nondecreasing (choose \( f = \sigma \) and \( g = F_{|Y|}^{-1} \); cf. [15]).

Theorem 13 (Comparability of \( L_\sigma \)-spaces). Suppose that
\[
c := \sup_{0 \leq \alpha < 1} \frac{\int_0^1 \sigma_2(u) \, du}{\int_0^1 \sigma_1(u) \, du}
\]
is finite \( (c < \infty) \), then
\[
\|Y\|_{\sigma_2} \leq c \cdot \|Y\|_{\sigma_1} \quad (Y \in L_{\sigma_1})
\]
and \( L_{\sigma_1} \subset L_{\sigma_2} \); \( c \) is moreover the smallest constant satisfying (10), the identity
\[
\text{id} : (L_{\sigma_1}, \|\cdot\|_{\sigma_1}) \to (L_{\sigma_2}, \|\cdot\|_{\sigma_2})
\]
thus is continuous with norm \( \|\text{id}\| = c \).
Proof. To accept (10) define the functions \( S_i(\alpha) := \int_0^1 \sigma_i(u) \, du \) \((i = 1, 2)\), then by Riemann–Stieltjes integration by parts and as \( u \mapsto F^{-1}_{|Y|}(u) \) is nondecreasing,

\[
\|Y\|_{\sigma_2} = \int_0^1 F^{-1}_{|Y|}(u) \sigma_2(u) \, du = -\int_0^1 F^{-1}_{|Y|}(u) \, dS_2(u)
\]

\[
= -F^{-1}_{|Y|}(u) S_2(u) \bigg|_0^1 + \int_0^1 S_2(u) \, dF^{-1}_{|Y|}(u) = F_{|Y|}^{-1}(0) + \int_0^1 S_2(u) \, dF_{|Y|}^{-1}(u)
\]

\[
\leq F_{|Y|}^{-1}(0) + c \cdot \int_0^1 S_1(u) \, dF_{|Y|}^{-1}(u)
\]

\[
= F_{|Y|}^{-1}(0) + c \cdot F_{|Y|}^{-1}(u) S_1(u) \bigg|_0^1 - c \cdot \int_0^1 F_{|Y|}^{-1}(u) \, dS_1(u)
\]

\[
= -F_{|Y|}^{-1}(0) (c - 1) + c \cdot \int_0^1 F_{|Y|}^{-1}(u) \sigma_1(u) \, du \leq c \cdot \|Y\|_{\sigma_1},
\]

because \( F_{|Y|}^{-1}(0) \geq 0 \) and \( c \geq 1 \) (choose \( \alpha = 0 \) in (9)).

To accept that \( c \) is the smallest constant satisfying (10) just consider the random variable \( Y = 1_{A^c} \), for which \( \|Y\|_{\sigma} = \rho_\sigma(1_{A^c}) = \int F_{|A|} \sigma(u) \, du \). The assertion follows, as the measurable set \( A \) may be chosen arbitrarily. \( \square \)

It is a particular consequence of (10) that

\[
AV@R_{\alpha_1}(\|Y\|) \leq AV@R_{\alpha_2}(\|Y\|) \leq \frac{1 - \alpha_1}{1 - \alpha_2} AV@R_{\alpha_1}(\|Y\|),
\]

which holds whenever \( \alpha_1 \leq \alpha_2 < 1 \). It should be noted, however, that \( AV@R_{\alpha_1}(Y) \leq AV@R_{\alpha_2}(Y) \leq \frac{1 - \alpha_1}{1 - \alpha_2} AV@R_{\alpha_1}(Y) \) in general.

The following representation result for spectral risk measures is well known for \( \sigma \) in an appropriate space. We extend it to \( L_\sigma \), the result will be used in the sequel.

**Proposition 14** (Representation of the spectral risk measure). \( \rho_\sigma \) has the equivalent representation \(^3\)

\[
\rho_\sigma(Y) = \sup \{ EY \cdot \sigma(U) : U \text{ is uniformly distributed} \}
\]

(11) on \( L_\sigma \).

**Remark 15.** For the Average Value-at-Risk it holds in particular that

\[
AV@R_\alpha(Y) = \sup \left\{ EY \cdot Z : EZ = 1, 0 \leq Z \leq \frac{1}{1 - \alpha} \right\}
\]

(12)

in view of the spectral function (2).

**Proof.** Consider the random variable \( Z = \sigma(U) \) for a uniformly distributed random variable \( U \), then \( P(Z \leq \sigma(\alpha)) = P(\sigma(U) \leq \sigma(\alpha)) \geq P(U \leq \alpha) = \alpha \), that is \( V@R_\alpha(Z) \geq \sigma(\alpha) \). But as

\[
1 = \int_0^1 \sigma(\alpha) \, d\alpha \leq \int_0^1 V@R_\alpha(\sigma(U)) \, d\alpha = E\sigma(U) = \int_0^1 \sigma(p) \, dp = 1
\]

it follows that

\[
V@R_\alpha(Z) = \sigma(\alpha).
\]

\(^3\)A random variable \( U \) is uniformly distributed if \( P(U \leq u) = u \) whenever \( u \in [0, 1] \).
Now $F^{-1}_Y(\cdot)$ is an increasing function, and so is $\sigma(\cdot)$. By the Hardy–Littlewood rearrangement inequality (cf. [16] and [20, Proposition 1.8] for the respective rearrangement inequality, sometimes also referred to as Hardy–Littlewood–Pólya inequality, cf. [9]) it follows thus that

$$\mathbb{E} Y \cdot \sigma(U) \leq \int_0^1 F^{-1}_Y(\alpha) \sigma(\alpha) \, d\alpha.$$ 

However, if $Y$ and $U$ are coupled in a co-monotone way, then equality is attained, that is $\mathbb{E} Y \cdot \sigma(U) = \int_0^1 F^{-1}_Y(\alpha) \sigma(\alpha) \, d\alpha$. This proves the statement in view of the definition of the spectral risk measure, (1).

The next theorem demonstrates that the spaces $L_\sigma$ really add something to $L^p$ spaces, the space $L_\sigma$ is strictly larger than $L^p$.

**Theorem 16 (L_\sigma is larger than L^p).** The following hold true:

(i) Suppose that $\sigma \in L^q$ for some $1 \leq q < \infty$. Then the space of natural domain $L_\sigma$ is strictly larger than $L^p$, $L^p \subsetneq L_\sigma \left( \frac{1}{p} + \frac{1}{q} = 1 \right)$.

(ii) In particular the space of natural domain $L_\sigma$ is (always) strictly larger than $L^\infty$, $L^\infty \subsetneq L_\sigma \left( q = 1 \right)$.

**Remark 17.** It should be noted that the statement of the latter theorem does not hold for $\sigma \in L^\infty$: In this situation $\rho_\sigma$ is well defined on $L^1$, and $L_\sigma = L^1$ by the preceding Theorem 11, (i).

**Proof.** To prove the first assertion assume that $\sigma \in L^q$ for $1 < q < \infty$. Consider the uniquely determined numbers $t_0 := 0 < t_1 < t_2 < \cdots < 1$ for which $\int_{t_{n-1}}^{t_n} \sigma(u)^q \, du = \frac{\|\sigma\|_q^q}{\zeta(p+1) \sum_{j=1}^{\infty} \frac{1}{j^p}}$ and observe that $\int_{t_{n-1}}^{t_n} \sigma(u)^q \, du = \mathbb{E} Y \cdot \sigma(U)^q \cdot \tau(U) \cdot \zeta(p+1) \sum_{j=1}^{\infty} \frac{1}{j^p}$. Define the function

$$\tau(u) := \begin{cases} n & \text{if } t_{n-1} \leq u < t_n, \end{cases}$$

let $U$ be uniformly distributed and consider the random variable

$$Y := \sigma(U)^q \cdot \tau(U). \quad (14)$$

Note, by (11), that

$$\rho_\sigma(Y) = \mathbb{E} \sigma(U) Y = \mathbb{E} \sigma(U) \sigma(U)^q \cdot \tau(U) = \mathbb{E} \sigma(U)^q \cdot \tau(U).$$

$$= \int_0^1 \sigma(u)^q \tau(u) \, du = \sum_{n=1}^{\infty} \int_{t_{n-1}}^{t_n} \sigma(u)^q \cdot n \, du$$

$$= \frac{\|\sigma\|_q^q}{\zeta(p+1)} \sum_{n=1}^{\infty} \frac{n}{n^{p+1}} = \frac{\|\sigma\|_q^q}{\zeta(p+1)} \sum_{n=1}^{\infty} \frac{1}{n^p} = \|\sigma\|_q^q \cdot \zeta(p+1) < \infty,$$

where $\zeta(p) := \sum_{n=1}^{\infty} \frac{1}{n^p}$ is Riemann’s Zeta function, the series converges whenever $p > 1$. 

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Then there does not exist a version independent, finite valued risk measure on $L$ that can be considered on a space larger than $\sigma$. This was communicated to the author by Prof. Alexander Shapiro (Georgia Tech). In brief: it does not make sense to consider risk measures on a space larger than $L$.

**Theorem 19.** Let $L \subset L^0$ be a vector space collecting $\mathbb{R}$-valued random variables on $([0,1], \mathcal{B}, \lambda)$ (the standard probability space equipped with its Borel sets) such that $L \supset L^1$ and $|Y| \in L$, if $Y \in L$. Then there does not exist a version independent, finite valued risk measure on $L$. 

**Remark 18.** Notably, the preceding proof applies for the random variable $Y = \sigma(U)^{\alpha-1} \cdot \tau(U)$ in (14) equally well whenever $1 \leq \alpha < p$, such that $L_\sigma$ is larger than $L^p$ by an entire infinite dimensional manifold.

It was demonstrated above that the space $L_\sigma$ is contained in $L^1$. The above inequality (8), $\| \cdot \|_1 \leq \| \cdot \|_\sigma$, allows to prove an even much stronger result: a finite valued risk measure cannot be considered on a space larger than $L^1$. This is the content of the following theorem, which was communicated to the author by Prof. Alexander Shapiro (Georgia Tech). In brief: it does not make sense to consider risk measures on a space larger than $L^1$.

**Theorem 19.** Let $L \subset L^0$ be a vector space collecting $\mathbb{R}$-valued random variables on $([0,1], \mathcal{B}, \lambda)$ (the standard probability space equipped with its Borel sets) such that $L \supset L^1$ and $|Y| \in L$, if $Y \in L$. Then there does not exist a version independent, finite valued risk measure on $L$. 

Next,

$$
\|Y\|^p_p = \mathbb{E}[|Y|^p] = \int_0^1 \sigma(u)^{(q-1)p} \tau(u)^p du \\
= \int_0^1 \sigma(u)^q \tau(u)^p du = \sum_{n=1}^\infty \int_{t_{n-1}}^{t_n} \sigma(u)^q \cdot n^p du \\
= \frac{\|\sigma\|^q_q}{\zeta(p+1)} \sum_{n=1}^\infty \frac{n^p}{n^{p+1} - 1} = \frac{\|\sigma\|^q_q}{\zeta(p+1)} \sum_{n=1}^\infty \frac{1}{n} = \infty.
$$

Hence, $Y \in L_\sigma$, but $Y \notin L^p$.

The second statement of the theorem is actually the first statement with $q = 1$, but the above proof needs a modification: To accept it define, as above, an increasing sequence of values by $t_0 := 0 < t_1 < t_2 < \cdots < 1$ satisfying $\int_0^{t_n} \sigma(t)dt \geq 1 - 2^{-n}$. Note, that

$$
\int_{t_{n-1}}^{t_n} \sigma(u)du \leq \int_{t_{n-1}}^{t_1} \sigma(u)du = 1 - \int_0^{t_{n-1}} \sigma(u)du \leq 2^{1-n}.
$$

Define moreover the increasing function

$$
\tau(\cdot) := \sum_{n=0}^{\infty} \mathbb{1}_{[t_n, 1]}(\cdot)
$$

(i.e. $\tau(t) = n$ if $t_{n-1} \leq t < t_n$) and observe that $\tau \rightarrow \infty$ whenever $t \rightarrow 1$.

Now let $U$ be a uniformly distributed random variable and set $Y := \tau(U)$. Then

$$
\rho_\sigma(Y) = \int_0^1 \sigma(u)\tau(u)du = \sum_{n=1}^{t_n} \int_{t_{n-1}}^{t_n} \sigma(u)\tau(u)du \\
= \sum_{n=1}^{t_n} n \cdot \int_{t_{n-1}}^{t_n} \sigma(u)du \leq \sum_{n=1}^{t_n} n \cdot 2^{1-n} = 4 < \infty,
$$

so $Y \in L_\sigma$. But $Y \notin L^\infty$, because $P(Y \geq n) \geq 1 - t_{n-1} > 0$ by definition of $\tau$. 

It was demonstrated above that the space $L_\sigma$ is contained in $L^1$. The above inequality (8), $\| \cdot \|_1 \leq \| \cdot \|_\sigma$, allows to prove an even much stronger result: a finite valued risk measure cannot be considered on a space larger than $L^1$. This is the content of the following theorem, which was communicated to the author by Prof. Alexander Shapiro (Georgia Tech). In brief: it does not make sense to consider risk measures on a space larger than $L^1$.
Proof. Suppose that \( \rho : L \rightarrow \mathbb{R} \) is a version independent, and finite valued risk measure on \( L \). Restricted to \( L^\infty \), Kusuoka’s theorem (Theorem 7) applies and \( \rho \) takes the form \( \rho (\cdot) = \sup_{\sigma \in \mathcal{S}} \rho_\sigma (\cdot) \). Choose \( Y \in L \backslash L^1 \), that is \( \mathbb{E}[|Y|] = \infty \), or \( \int_0^p F_Y^{-1} (u) \, du \rightarrow \infty \) whenever \( p \rightarrow 1 \).

Next, pick any \( \sigma \in \mathcal{S} \). Define \( Y_n := \min \{ n, \, |Y| \} \) and observe that \( \rho (Y_n) \leq \rho (|Y|) \) by monotonicity. Note that \( Y_n \in L^\infty \) and hence, by Kusuoka’s representation, (8) and the particular choice of \( Y \),
\[
\rho (|Y|) \geq \rho (Y_n) \geq \rho_\sigma (Y_n) = \|Y_n\|_\sigma \geq \|Y_n\|_1 \geq \int_0^{\mathbb{P}(|Y| \leq n)} F_Y^{-1} (u) \, du \rightarrow \infty,
\]
as \( n \rightarrow \infty \). Hence, \( \rho \) is not finite valued on \( L \).

\[\Box\]

**Theorem 20.** \( (L_\sigma, \|\cdot\|_\sigma) \) is a Banach space over \( \mathbb{R} \).

**Proof.** It remains to be shown that \( (L_\sigma, \|\cdot\|_\sigma) \) is complete. For this let \( (Y_k)_k \) be a Cauchy sequence for \( \|\cdot\|_\sigma \). By (8) the sequence \( (Y_k)_k \) is a Cauchy sequence for \( \|\cdot\|_1 \) as well, and from completeness of \( L^1 \) it follows that there exists a limit \( Y \in L^1 \). We shall show that \( Y \in L_\sigma \).

It follows from convergence in \( L^1 \) that \( (Y_k)_k \) converges in distribution, that is \( F_{Y_k} (y) \rightarrow F_Y (y) \) for every point \( y \) where \( F_Y \) is continuous and moreover \( F_{Y_k}^{-1} (\cdot) \rightarrow F_Y^{-1} (\cdot) \) (cf. [26, Chapter 21]). Now
\[
\|Y\|_\sigma = \rho_\sigma (|Y|) = \int_0^1 \sigma (t) F_Y^{-1} (t) \, dt = \int_0^1 \sigma (t) \lim_{k \rightarrow \infty} F_{Y_k}^{-1} (t) \, dt = \lim_{k \rightarrow \infty} \int_0^1 \sigma (t) F_{Y_k}^{-1} (t) \, dt \leq \lim_{k \rightarrow \infty} \int_0^1 \sigma (t) F_{Y_k}^{-1} (t) \, dt = \lim_{k \rightarrow \infty} \|Y_k\|_\sigma,
\]
by Fatou’s Lemma, which is applicable because \( F_{Y_k}^{-1} (\cdot) \geq 0 \).

As \( (Y_k)_k \) is a Cauchy sequence one may pick \( k^* \in \mathbb{N} \) such that \( \|Y_k - Y_{k^*}\|_\sigma < 1 \) for all \( k > k^* \), and hence \( \|Y_k\|_\sigma \leq \|Y_{k^*}\|_\sigma + \|Y_k - Y_{k^*}\|_\sigma < \|Y_{k^*}\|_\sigma + 1 < \infty \) by the triangle inequality. The sequence \( (Y_k)_k \) thus is uniformly bounded in its norm. Hence,
\[
\|Y\|_\sigma \leq \lim_{k \rightarrow \infty} \|Y_k\|_\sigma \leq \|Y_{k^*}\|_\sigma + 1 < \infty,
\]
that is \( Y \in L_\sigma \) and \( L_\sigma \) thus is complete. \[\Box\]

**Example 21.** Consider the spectrum \( \sigma (\alpha) = \frac{1}{\sqrt{1-\alpha}} \). It should be noted that \( L_\sigma \supset \bigcup_{p>2} L^p \), and \( \|\cdot\|_\sigma \) provides a reasonable norm on that set.

Restricted to \( L^p \), for some \( p > 2 \), the open mapping theorem (cf. [22] or [3]) insures that the norms are equivalent, that is there are constants \( c_1 \) and \( c_2 \) such that
\[
c_1 \cdot \|Y\|_p \leq \|Y\|_\sigma \leq c_2 \cdot \|Y\|_p \quad (Y \in L^p \subset L_\sigma).
\]

The latter inequalities hold just for \( Y \in L^p \), but not for \( Y \in L_\sigma \).

**Proposition 22.** Measurable, simple (step) functions are dense in \( L_\sigma \), and in particular \( L^\infty \) is dense in \( L_\sigma \).

**Proof.** Given \( Y \in L_\sigma \) and \( \varepsilon > 0 \), find \( t_0 \in (0, 1) \) such that \( \int_0^{t_0} F_Y^{-1} (u) \sigma (u) \, du < \frac{\varepsilon}{3} \) and set \( s (t) := F_Y^{-1} (t_0) \) whenever \( t \leq t_0 \). Moreover, find \( t_1 \in (0, 1) \) such that \( \int_{t_1}^{t} F_Y^{-1} (u) \sigma (u) \, du < \frac{\varepsilon}{3} \) and set
\[ s(t) := F^{-1}_Y(t_1) \text{ whenever } t \geq t_1. \] In between, as \( F^{-1}_Y(t) \) is nondecreasing on the compact \([t_0, t_1]\), there is an increasing step function \( s(t) \) such that \( |s(t) - F^{-1}_Y(t)| \sigma(t) < \frac{\varepsilon}{2} \). Let \( U \) be uniformly distributed and co-monotone with \( Y \). Then it holds that \( \|Y - s(U)\|_\sigma < \varepsilon \) by construction of the step function \( s \).

\section{The Dual of the natural domain \( L_\sigma \)}

Risk measures are convex and lower semi-continuous (cf. [17]) functions, hence they have a dual representation by involving the Fenchel–Moreau Theorem (also Legendre transformation, see below). This representation involves the dual space in a natural way, and hence it is of interest to understand the dual of the Banach space \((L_\sigma, \|\cdot\|_\sigma)\). We describe the norm of the dual and identify the dual with a subspace of \( L^1 \). The respective results are proven in this section, moreover essential properties of the dual are highlighted.

\textbf{Theorem 23 (Fenchel–Moreau).} Let \( \mathcal{Y} \) be a Banach space and \( f : \mathcal{Y} \to \mathbb{R} \cup \{\infty\} \) be convex and lower semi-continuous with \( f(Y_0) < \infty \) for an \( Y_0 \in \mathcal{Y} \). Then

\[ f^{**} = f, \]

where

\[ f^*(Z^*) := \sup_{Y \in \mathcal{Y}} Z^*(Y) - f(Y) \quad \text{and} \quad f^{**}(Y) := \sup_{Z^* \in \mathcal{Y}^*} Z^*(Y) - f^*(Z^*). \]

\textbf{Proof.} cf. [21].

Note, that a risk measure \( \rho_\sigma \) is not only lower semicontinuous, by Proposition 6 it is continuous with respect to the norm \( \|\cdot\|_\sigma \) on the Banach space \( \mathcal{Y} = (L_\sigma, \|\cdot\|_\sigma) \). By the Fenchel–Moreau theorem thus \( \rho_\sigma^{**} = \rho_\sigma \). To involve it on its natural domain \( \mathcal{Y} = (L_\sigma, \|\cdot\|_\sigma) \) its dual \( \mathcal{Y}^* = (L_\sigma, \|\cdot\|_\sigma)^* \) has to be available, and this is elaborated in the sequel.

\textbf{Definition 24.} For a spectral function \( \sigma \) and a random variable \( Z \in L^1 \) define the binary relation

\[ Z \preceq \sigma \text{ iff } \text{AV}_@R_\alpha (|Z|) \leq \frac{1}{1-\alpha} \int_0^1 \sigma(u)du \text{ for all } 0 \leq \alpha < 1, \tag{15} \]

the gauge function (Minkowski functional)

\[ \|Z\|_{\sigma^*} := \inf \left\{ \eta \geq 0 : \text{AV}_@R_\alpha (|Z|) \leq \frac{\eta}{1-\alpha} \int_0^1 \sigma(u)du \text{ for all } 0 \leq \alpha < 1 \right\} \tag{16} \]

\[ = \inf \{ \eta \geq 0 : |Z| \preceq \eta \cdot \sigma \} \]

and the set \( L_{\sigma^*}^1 := \{ Z \in L^1 : \|Z\|_{\sigma^*} < \infty \} \).

It should be noted that the relation (15), which is a kind of second order stochastic dominance relation (cf. [12, 11]), can be interpreted as a growth condition for \(|Z|\), which is a condition on \( Z \)'s tails: \( Z \preceq \eta \cdot \sigma \) can only hold true if \(|Z|\) does not grow (in quantiles) faster towards \( \infty \) than \( \eta \cdot \sigma \).

Notice as well that

\[ \|Z\|_{\sigma^*} \leq \eta \text{ if and only if } \text{AV}_@R_\alpha (|Z|) \leq \frac{\eta}{1-\alpha} \int_0^1 \sigma(u)du \text{ for all } 0 \leq \alpha < 1. \tag{17} \]
Moreover the functions $\alpha \mapsto \int_0^1 \sigma(u) du$ and $\alpha \mapsto (1 - \alpha) \text{AV@R}_\alpha (|Z|)$ are both continuous functions on $[0, 1]$, so the maximum of their difference is attained in $[0, 1]$. Hence, the infimum in (16) will be attained as well at some $\eta \geq 0$.

Example 25. For $U$ a uniformly distributed random variable it follows readily from the definition and (13) that

$$\|\sigma (U)\|_\sigma^* = 1.$$  \hfill (18)

The norm of the indicator function has the explicit form

$$\|I_A\|_\sigma^* = \frac{1}{\mathbb{P}(A)} \int_0^1 \sigma(u) du,$$  \hfill (19)

which derives from $\text{AV@R}_\alpha (I_A) = \min \left\{ 1, \frac{\mathbb{P}(A)}{1 - \alpha} \right\}$ and the particular choice $\alpha = 1 - \mathbb{P}(A)$ in (15). Immediate consequences of (19) are further the bounds $\mathbb{P}(A) \leq \|I_A\|_\sigma^* \leq 1$.

Remark 26. Given Kusuoka’s representation one may employ the measure $\mu$ directly instead of the spectral density $\sigma$ by involving (6). It holds that

$$\frac{1}{1 - \alpha} \int_0^1 \sigma(u) du = \int_0^1 \min \left\{ \frac{1}{1 - u}, \frac{1}{1 - \alpha} \right\} d\mu(u),$$

the condition $Z \preceq \sigma_\mu$ thus reads directly

$$Z \preceq \sigma_\mu \text{ iff } \text{AV@R}_\alpha (|Z|) \leq \int_0^1 \min \left\{ \frac{1}{1 - u}, \frac{1}{1 - \alpha} \right\} d\mu(u) \text{ for all } 0 \leq \alpha < 1.$$  

Notice as well that $\int_0^1 \min \left\{ \frac{1}{1 - u}, \frac{1}{1 - \alpha} \right\} d\mu(u)$ represents an expectation of a (bounded) function with respect to the measure $\mu$.

Lemma 27. The unit ball of the norm $\|\cdot\|_\sigma^*$ is

$$B_\sigma = \left\{ Z \in \mathcal{L}^1 : \text{AV@R}_\alpha (|Z|) \leq \frac{1}{1 - \alpha} \int_0^1 \sigma(u) du \text{ for all } 0 \leq \alpha < 1 \right\},$$

which is an absolutely convex set.

Proof. Just observe that

$$\text{AV@R}_\alpha (|\lambda_1 Z_1 + \lambda_2 Z_2|) \leq \text{AV@R}_\alpha (|\lambda_1 Z_1| + |\lambda_2 Z_2|)$$

$$= 2 \cdot \text{AV@R}_\alpha \left( \frac{1}{2} |\lambda_1 Z_1| + \frac{1}{2} |\lambda_2 Z_2| \right)$$

$$\leq |\lambda_1| \text{AV@R}_\alpha (|Z_1|) + |\lambda_2| \text{AV@R}_\alpha (|Z_2|)$$

by monotonicity, convexity and positive homogeneity (sub-additivity). For $Z_1, Z_2 \in B_\sigma$ and $|\lambda_1| + |\lambda_2| \leq 1$ it follows thus that $\lambda_1 Z_1 + \lambda_2 Z_2 \in B_\sigma$ and $B_\sigma$ is absolutely convex.

Monotonicity. It follows from monotonicity of the Average Value-at-Risk that

$$\|Y_1\|_\sigma^* \leq \|Y_2\|_\sigma^*, \text{ if } |Y_1| \leq |Y_2|.$$  \hfill (20)
Comparison with $L^1$. For $Z \in L^\sigma$, $\|Z\|^*_\sigma \leq \eta$ implies that $\mathbb{E}|Z| \leq \eta$ (by the choice $\alpha = 0$ in (17)), hence

$$\|Z\|_1 \leq \|Z\|^*_\sigma$$

and $L^*_\sigma \subset L^1$.

Comparison with $L^\infty$. Suppose that $\sigma$ is bounded and $Z \in L^\infty$. Then $AV@R_\alpha (|Z|) \rightarrow \|Z\|_\infty$ and $\frac{1}{\alpha} \int_0^1 \sigma(u)du \rightarrow \|\sigma\|_\infty^*$, as $\alpha \rightarrow 1$, and consequently $\|Z\|_\infty \leq \eta \cdot \|\sigma\|_\infty$ has to hold by (17) for $\eta$ to be feasible. That is,

$$\|Z\|_\infty \leq \|Z\|^*_\sigma \cdot \|\sigma\|_\infty.$$  \hspace{1cm} (22)

Upper bound. An upper bound for the norm $\|\cdot\|^*_\sigma$ is given by

$$\|Z\|^*_\sigma \leq \sup_{0 \leq u < 1} \frac{F_{|Z|}^{-1}(u)}{\sigma(u)},$$

where the conventions $\frac{0}{0} = 0$ and $\frac{1}{\infty} = \infty$ have to be employed. Indeed, if $\frac{F_{|Z|}^{-1}(u)}{\sigma(u)} \leq \eta$, then integrating gives $(1 - \alpha)AV@R_\alpha (|Z|) = \int_0^1 \frac{F_{|Z|}^{-1}(u)}{\sigma(u)}du \leq \eta \cdot \int_0^1 \sigma(u)du$, which in turn means that $\|Z\|^*_\sigma \leq \eta$. Notice, however, that $Z \mapsto \sup_{0 \leq u < 1} \frac{F_{|Z|}^{-1}(u)}{\sigma(u)}$ is not a norm, it does not satisfy the triangle inequality.

Simple functions. For $Z = \sum_{j=1}^n a_j I_{A_j}$ a simple (step) function, $\alpha \mapsto (1 - \alpha)AV@R_\alpha (|Z|) = \int_0^1 F_{|Z|}^{-1}(u)du$ is piecewise linear. As $\alpha \mapsto \int_0^1 \sigma(u)du$ is concave (this is, because $\sigma$ is increasing), the defining condition (17) has to be verified on finite many points only, such that simple functions are contained in $L^*_\sigma$.

Proposition 28. The pair $\left( L^*_\sigma, \|\cdot\|^*_\sigma \right)$ is a Banach space.

Proof. Notice first that $\|Z\|^*_\sigma = 0$ implies that $AV@R_\alpha (|Z|) = 0$ for all $\alpha < 1$, so

$$0 = \lim_{\alpha \nearrow 1} AV@R_\alpha (|Z|) = \text{ess sup} |Z|,$$

that is $Z = 0$ almost everywhere, such that $\|\cdot\|^*_\sigma$ separates points in $L^*_\sigma$.

Positive homogeneity is immediate and inherited from the Average Value-at-Risk.

As for the triangle inequality let $\eta_1$ and $\eta_2$, resp. satisfy (16) for $Z_1$ and $Z_2$, resp.. Then, by monotonicity and sub-additivity of the Average Value-at-Risk,

$$AV@R_\alpha (|Z_1 + Z_2|) \leq AV@R_\alpha (|Z_1| + |Z_2|) \leq AV@R_\alpha (|Z_1|) + AV@R_\alpha (|Z_2|)$$

such that

$$AV@R_\alpha (|Z_1 + Z_2|) \leq \frac{\eta_1 + \eta_2}{1 - \alpha} \int_0^1 \sigma(u)du,$$

that is finally $\|Z_1 + Z_2\|^*_\sigma \leq \|Z_1\|^*_\sigma + \|Z_2\|^*_\sigma$, the triangle inequality.
Finally completeness remains to be shown. For this let $Z_k$ be a Cauchy sequence. Hence there is a natural number $k^*$, such that $\|Z_k\|_\sigma^* \leq \|Z_{k^*}\|_\sigma^* + \|Z_k - Z_{k^*}\|_\sigma^* \leq \|Z_{k^*}\|_\sigma^* + 1$, that is there is $\eta \geq 0$ ($\eta$ satisfies $\eta \leq \|Z_{k^*}\|_\sigma^* + 1$) such that

$$\mathcal{AV}@\mathcal{R}_{\alpha} (|Z_k|) \leq \frac{\eta}{1 - \alpha} \int_\alpha^1 \sigma(u) \, du$$

for all $k > k^*$ and $\alpha \in (0, 1)$. Next, by (21) $Z_k$ is a Cauchy sequence for $L^1$ as well, hence there is a limit $Z \in L^1$, and $Z_k$ converges in distribution and in quantiles. By Fatou’s inequality,

$$\mathcal{AV}@\mathcal{R}_{\alpha} (|Z|) = \frac{1}{1 - \alpha} \int_\alpha^1 F_{|Z|}^{-1}(u) \, du = \frac{1}{1 - \alpha} \int_\alpha^1 \liminf_{k \to \infty} F_{|Z_k|}^{-1}(u) \, du$$

$$\leq \frac{1}{1 - \alpha} \liminf_{k \to \infty} \int_\alpha^1 F_{|Z_k|}^{-1}(u) \, du = \liminf_{k \to \infty} \mathcal{AV}@\mathcal{R}_{\alpha} (|Z_k|)$$

$$\leq \frac{\eta}{1 - \alpha} \cdot \int_\alpha^1 \sigma(u) \, du.$$

The limit $Z \in L^1$ thus satisfies the defining conditions to qualify for $L^*_\sigma$ and $\|Z\|_\sigma^* \leq \eta$. It follows that $Z \in L^*_\sigma$ and $(L^*_\sigma, \|\|_\sigma^*)$ thus is a Banach space. 

**Theorem 29.** The space $(L^*_\sigma, \|\|_\sigma^*)$ is the dual of $(L_\sigma, \|\|_\sigma)$.

**Proof.** Let $Y \in L_\sigma$ and $Z \in L^*_\sigma$ with $\|Z\|_\sigma^* =: \eta$ be chosen. Then note that

$$|EYZ| \leq |Y| \cdot |Z| \leq \int_0^1 F_{|Y|}^{-1}(u) F_{|Z|}^{-1}(u) \, du$$

by the Hardy–Littlewood–Pólya inequality. To abbreviate the notation we introduce the functions $S(u) := \int_u^1 \sigma(p) \, dp$ and $G(u) := \int_u^1 F_{|Z|}^{-1}(p) \, dp$ (the functions are well defined, because $\sigma \in L^1$ and $Z \in L^1$). Then, by Riemann–Stieltjes integration by parts,

$$\int_0^1 F_{|Y|}^{-1}(u) F_{|Z|}^{-1}(u) \, du = - \int_0^1 F_{|Y|}^{-1}(u) \, dG(u)$$

$$= - F_{|Y|}^{-1}(u) G(u) \Big|_{u=0}^{u=1} + \int_0^1 G(u) \, dF_{|Y|}^{-1}(u)$$

$$= F_{|Y|}^{-1}(0) \cdot E|Z| + \int_0^1 G(u) \, dF_{|Y|}^{-1}(u).$$

Now note that $F_{|Y|}^{-1}(\cdot)$ is an increasing function, and $G(u) = \int_u^1 F_{|Z|}^{-1}(p) \, dp \leq \eta \cdot \int_u^1 \sigma(p) \, dp = \eta \cdot S(u)$
because \( \|Z\|_\sigma^* \leq \eta \). Thus, and employing again Riemann–Stieltjes integration by parts,

\[
\begin{align*}
|\mathbb{E}YZ| & \leq F_{|Y|}^{-1}(0) \cdot \|Z\|_1 + \eta \cdot \int_0^1 S(u) \, dF_{|Y|}^{-1}(u) \\
& = F_{|Y|}^{-1}(0) \cdot \|Z\|_1 + \eta \cdot S(u) \, dF_{|Y|}^{-1}(u) |_{u=0}^1 - \eta \cdot \int_0^1 F_{|Y|}^{-1}(u) \, dS(u) \\
& = F_{|Y|}^{-1}(0) \cdot \|Z\|_1 - \eta \cdot F_{|Y|}^{-1}(0) + \eta \cdot \int_0^1 F_{|Y|}^{-1}(u) \, dS(u) \\
& = F_{|Y|}^{-1}(0) \cdot \|Z\|_1 - \eta \cdot \int_0^1 F_{|Y|}^{-1}(u) \, dS(u) \\
& = F_{|Y|}^{-1}(0) \cdot \|Z\|_1 - \eta \cdot \int_0^1 F_{|Y|}^{-1}(u) \, dS(u).
\end{align*}
\]

Finally observe that \( F_{|Y|}^{-1}(0) = \text{ess inf} |Y| \geq 0 \) and \( \|Z\|_1 - \|Z\|_\sigma^* \leq 0 \) by (21), hence

\[
|\mathbb{E}YZ| \leq \|Z\|_\sigma^* \cdot \int_0^1 F_{|Y|}^{-1}(u) \, dS(u) = \rho_\sigma(|Y|) \cdot \|Z\|_\sigma^* = \|Y\|_\sigma \cdot \|Z\|_\sigma^*.
\]

This proves that for every \( Z \in L_\sigma^* \) the linear mapping \( Y \mapsto \mathbb{E}YZ \) is continuous with respect to the norm \( \|\|_\sigma^* \).

It remains to be shown that every linear, continuous mapping \( \zeta \) in the dual of \( L_\sigma \) \( \zeta \in (L_\sigma, \|\|_\sigma^*) \) takes the form \( \zeta(Y) = \mathbb{E}YZ \) for some \( Z \in L_\sigma^* \). For this consider the (signed) measure \( \mu(A) := \zeta(1_A) \). If \( A = \bigcup_{i=1}^{\infty} A_i \) is a disjoint union of countably measurable sets, then \( 1_A = \sum_{i=1}^{\infty} 1_{A_i} \). Clearly,

\[
\left\| 1_A - \sum_{i=1}^{n} 1_{A_i} \right\|_\sigma = \int_{1-\sum_{i=1}^{\infty} P(A_i)} \sigma(u) \, du \xrightarrow{n \to \infty} 0,
\]

as \( P \) is sigma-finite and \( \sigma \in L^1 \). It follows by continuity of \( \zeta \) with respect to \( \|\|_\sigma \) that

\[
\mu(A) = \zeta(1_A) = \zeta\left( \sum_{i=1}^{\infty} 1_{A_i} \right) = \sum_{i=1}^{\infty} \zeta(1_{A_i}) = \sum_{i=1}^{\infty} \mu(A_i),
\]

hence \( \mu \) is a sigma-finite measure. If \( P(A) = 0 \), then

\[
|\mu(A)| = |\zeta(1_A)| \leq \|\zeta\| \cdot \|1_A\|_\sigma = \|\zeta\| \cdot \int_0^1 \sigma(u) \, F_{1_A}^{-1}(u) \, du = 0,
\]

because \( F_{1_A}^{-1}(u) = 0 \) for every \( u < 1 \). It follows that \( \mu(A) = 0 \), such that \( \mu \) is moreover absolutely continuous with respect to \( P \).

Let \( Z \in L^0 \) be the Radon–Nikodým derivative, \( d\mu = ZdP \). Then \( \zeta(1_A) = \mu(A) = \int_A ZdP = \int 1_A ZdP = \mathbb{E} Z \, 1_A \) and hence \( \zeta(\phi) = \mathbb{E} Z \phi \) for all simple functions \( \phi \) by linearity and \( |\mathbb{E} Z \phi| = |\zeta(\phi)| \leq \|\zeta\| \cdot \|\phi\|_\sigma \) by continuity of \( \zeta \).

Choose the function \( \phi := \text{sign} Z \) (a simple function) to see that \( \mathbb{E} |Z| \leq \|\zeta\| \), that is \( Z \in L^1 \).
Note as well that $\mathbb{E}|Z| \phi = \mathbb{E}Z \cdot \text{sign}(Z) \phi \leq \|\zeta\| \cdot \|\text{sign}(Z) \phi\|_\sigma \leq \|\zeta\| \cdot \|\phi\|_\sigma$, because $\rho_\sigma$ is monotone and $|\text{sign}(Z) \cdot \phi| \leq |\phi|$. For any measurable set $A$ (with complement denoted $A^c$) thus

$$\mathbb{E}|Z| 1_{A^c} \leq \|\zeta\| \cdot \|1_{A^c}\|_\sigma = \|\zeta\| \cdot \rho_\sigma(1_{A^c}) = \|\zeta\| \cdot \int_{P(A)}^1 \sigma(u)du,$$

and hence $\mathbb{E}|Z| \frac{1}{P(A)} \leq \|\zeta\| \cdot \int_{1-P(A)}^1 \sigma(u)du$. Taking the supremum over all sets $A$ with $P(A) \leq \alpha$ gives

$$\mathbb{A} \mathbb{V} @ R_\alpha(|Z|) = \sup_{P(A^c) \geq 1-\alpha} \mathbb{E}|Z| \frac{1}{P(A^c)} \leq \|\zeta\| \cdot \sup_{P(A) \leq \alpha} \frac{1}{1-P(A)} \int_{1-P(A)}^1 \sigma(u)du$$

by (12) and because $\sigma$ is increasing. It follows that $\|Z\|_\sigma^* \leq \|\zeta\|$ and thus $Z \in L_\sigma^*$. This completes the proof. □

**The Hahn-Banach functional.**  Let $Y \in L_\sigma$ be fixed, and let $U$ be coupled in a co-monotone way with $|Y|$. Define $Z_Y := \sigma(U) \cdot \text{sign} Y$ and observe that $F_{\sigma(U)}^{-1}(\alpha) = \sigma(\alpha)$ by (13). Hence $\mathbb{A} \mathbb{V} @ R_\alpha(\sigma(U)) = \frac{1}{1-\sigma} \int_{1-\sigma}^1 \sigma(u)du$, and it follows that $\|Z_Y\|_\sigma^* = 1$. On the other side $\mathbb{E} Y \cdot Z_Y = \mathbb{E}|Y| \cdot \sigma(U) = \int_0^1 F_{|Y|}^{-1} (u) \sigma(u) du = \|Y\|_\sigma$, $Z_Y$ thus is a maximizer of the problem

$$\|Y\|_\sigma = \max \{\mathbb{E} Y \cdot Z : \|Z\|_\sigma^* \leq 1\}.$$

**Theorem 30.** The Banach space $(L_\sigma, \|\cdot\|_\sigma)$ is reflexive iff the spectrum function $\sigma$ is unbounded, $\sigma \notin L^\infty$.

**Proof.** If $\sigma$ is bounded, then $L_\sigma = L^1$, the norms being equivalent by Theorem 11 (ii). But $L^1$ is not a reflexive space and thus $(L_\sigma, \|\cdot\|_\sigma)$ is not reflexive.

Secondly, assume that $\sigma$ is unbounded, and let $\xi$ be a continuous functional in the bi-dual, $\xi \in (L_\sigma^*, \|\cdot\|_\sigma^*)^*$, with norm $\|\xi\| < \infty$. Define the measure $\nu(A) := \xi(1_A)$. Let $A_i$ be a sequence of mutually disjoint, measurable sets and set $A := \bigcup_{i=1}^\infty A_i$. Note, that

$$\left\|1_A - \sum_{i=1}^N 1_{A_i}\right\|_\sigma^* \leq \sum_{i=N+1}^\infty 1_{A_i}\right\|_\sigma^* = \frac{1}{p_N} \int_{1-p_N}^1 \sigma(u) du \leq \frac{1}{\sigma(1-p_N)} \rightarrow 0 \quad N \rightarrow \infty$$

(where $p_N := \sum_{i=1}^N P(A_i)$) by (19), and because $\sigma$ is unbounded. From continuity of $\xi$ it follows thus that $\nu$ is sigma additive. Further, if $P(A) = 0$, then $\|1_A\|_\sigma^* = 0$ and

$$|\nu(A)| \leq \|\xi\| \cdot \|1_A\|_\sigma^* = 0,$$

$\nu$ thus is absolutely continuous with respect to $P$.

Let $Y \in L^0$ be the Radon–Nikodým density, $d\nu = Y dP$, for which

$$\xi(1_A) = \nu(A) = \int_A Y dP = \int Y 1_A dP = \mathbb{E} Y 1_A,$$
and from linearity thus $\xi(\phi) = \mathbb{E} Y \phi$ and $|\mathbb{E} Y \phi| = |\xi(\phi)| \leq \|\xi\| \cdot \|\phi\|_{\sigma}$ for a simple function $\phi$.

Finally let $U$ be coupled in a co-monotone way with $|Y|$ and let $\sigma_n$ be simple, nondecreasing step functions with $0 \leq \sigma_n \leq \sigma_{n+1} \rightarrow \sigma$ pointwise, then

$$
\|Y\|_{\sigma} = \rho_{\sigma}(|Y|) = \mathbb{E} |Y| \sigma(U) = \mathbb{E} |Y| \cdot \lim_{n \rightarrow \infty} \sigma_n(U)
\leq \liminf_{n \rightarrow \infty} \mathbb{E} |Y| \sigma_n(U) = \liminf_{n \rightarrow \infty} \mathbb{E} Y \cdot \text{sign}(Y) \sigma_n(U)
\leq \liminf_{n \rightarrow \infty} \|\xi\| \cdot \|\text{sign}(Y) \sigma_n(U)\|_{\sigma} = \|\xi\| \cdot \liminf_{n \rightarrow \infty} \|\sigma_n(U)\|_{\sigma}
\leq \|\xi\| \cdot \|\text{sign}(U)\|_{\sigma} = \|\xi\| < \infty
$$

by Fatou’s Lemma, monotonicity, (20) and (18). This proves that $Y \in L_{\sigma}$, and $L_{\sigma}$ thus is reflexive.

The following statement compares $L^*_\sigma$ spaces with spaces $L^q$, and it generalizes the relations (21) and (22) for general $L^q$ spaces. It is the dual statement to Theorem 11.

**Theorem 31 (Comparison with $L^q$).** For $\sigma \in L^q (1 \leq q \leq \infty)$ it holds that

$$
\|Z\|_q \leq \|Z\|_{\sigma} \cdot \|\sigma\|_q
$$

whenever $Z \in L^*_\sigma$, and thus $L^*_\sigma \subset L^q$.

Moreover,

$$
\frac{\|Z\|_\infty}{\|\sigma\|_\infty} \leq \|Z\|_{\sigma} \leq \|Z\|_c
$$

such that the norms $\|\cdot\|_\infty$ and $\|\cdot\|_{\sigma}$ are equivalent whenever $\sigma \in L^\infty$, and in this case $L^*_\sigma = L^\infty$.

**Proof.** Employing $L^p - L^q$ duality and $L_\sigma - L^*_\sigma$ duality it holds that

$$
\|Z\|_q = \sup_{Y \neq 0} \frac{\mathbb{E} Y |Z|}{\|Y\|_p} \leq \sup_{Y \neq 0} \frac{\|Y\|_{\sigma} \|Z\|_{\sigma}}{\|Y\|_p} \leq \sup_{Y \neq 0} \frac{\|\sigma\|_q \|Y\|_{\sigma} \|Z\|_{\sigma}}{\|Y\|_p} = \|\sigma\|_q \cdot \|Z\|_{\sigma}
$$

by (8).

The inequality, which is missing, is given by

$$
\|Z\|_{\sigma} = \sup_{Y \neq 0} \frac{\mathbb{E} Y Z}{\|Y\|_{\sigma}} \leq \sup_{Y \neq 0} \frac{\|Y\|_1 \|Z\|_{\infty}}{\|Y\|_{\sigma}} \leq \sup_{Y \neq 0} \frac{\|Y\|_{\sigma} \|Z\|_{\infty}}{\|Y\|_{\sigma}} = \|Z\|_c,
$$

again by (8). 

6 The general natural domain space $L_\mathcal{F}$

Kusuoka’s theorem (Theorem 7) and (7) suggest to consider risk measures of the form

$$
\rho_{\mathcal{F}}(\cdot) := \sup_{\sigma \in \mathcal{F}} \rho_{\sigma}(\cdot).
$$

To investigate this general type of risk measure we define the according norm and space first.
**Definition 32.** The natural domain of $\rho_\mathcal{Y}$, where $\mathcal{Y}$ is a collection of spectral functions, is

$$L_\mathcal{Y} := \{ Y \in L^1 : \|Y\|_\mathcal{Y} < \infty \},$$

where

$$\|\cdot\|_\mathcal{Y} := \rho_\mathcal{Y} (|\cdot|) = \sup_{\sigma \in \mathcal{Y}} \rho_\sigma (|\cdot|) = \sup_{\sigma \in \mathcal{Y}} \|\cdot\|_\sigma.$$  

Obviously, $L_\mathcal{Y} \subset \bigcap_{\sigma \in \mathcal{Y}} L_\sigma$. In view of Theorem 11 (ii) it is obvious as well that

$$L^\infty \subset L_\mathcal{Y} \subset L^1,$$

even more, it holds that $\|Y\|_\mathcal{Y} \leq \|Y\|_\infty$ whenever $Y \in L^\infty$, and $\|Y\|_1 \leq \|Y\|_\mathcal{Y}$, whenever $Y \in L_\mathcal{Y}$. Further, if $\sup_{\sigma \in \mathcal{Y}} \|\sigma\|_q < \infty$ is finite as well, then

$$\|Y\|_\mathcal{Y} \leq \sup_{\sigma \in \mathcal{Y}} \|\sigma\|_q \cdot \|Y\|_p$$

by Theorem 11, (i).

**Theorem 33.** The pair $(L_\mathcal{Y}, \|\cdot\|_\mathcal{Y})$ is a Banach space.

**Proof.** First of all it is clear that $\|\cdot\|_\mathcal{Y}$ is a norm on $L_\mathcal{Y}$, as it separates points, is positively homogeneous and satisfies the triangle inequality: these properties are inherited from the spaces $(L_\sigma, \|\cdot\|_\sigma)_{\sigma \in \mathcal{Y}}$.

It remains to be shown that $(L_\mathcal{Y}, \|\cdot\|_\mathcal{Y})$ is complete. So if $(Y_k)_k$ is a Cauchy sequence in $L_\mathcal{Y}$, then because of $\|\cdot\|_\sigma \leq \|\cdot\|_\mathcal{Y}$ it is a Cauchy sequence in any of the spaces $(L_\sigma, \|\cdot\|_\sigma)$ and it has a limit $Y$ there. The limit is the same for all $L_\sigma$, so $Y \in \bigcap_{\sigma \in \mathcal{Y}} L_\sigma$. Following (15) it holds that

$$\|Y\|_\mathcal{Y} = \sup_{\sigma \in \mathcal{Y}} \|Y\|_\sigma \leq \sup_{\sigma \in \mathcal{Y}} \liminf_{k \to \infty} \|Y_k\|_\sigma \leq \liminf_{k \to \infty} \sup_{\sigma \in \mathcal{Y}} \|Y_k\|_\sigma = \liminf_{k \to \infty} \|Y_k\|_\mathcal{Y}$$

by the max-min inequality. Now choose $k^* \in \mathbb{N}$ such that $\|Y_k - Y_{k^*}\|_\mathcal{Y} < 1$ for all $k > k^*$, which is possible because the sequence is Cauchy. It follows that

$$\|Y\|_\mathcal{Y} \leq \liminf_{k \to \infty} \|Y_k\|_\mathcal{Y} \leq \|Y_{k^*}\|_\mathcal{Y} + 1 < \infty,$$

and hence $Y \in L_\mathcal{Y}$, that is $L_\mathcal{Y}$ is complete. \qed

**Theorem 34.** The risk measure $\rho_\mathcal{Y}$ is finite valued on $L_\mathcal{Y}$, it is moreover continuous with respect to the norm $\|\cdot\|_\mathcal{Y}$ with Lipschitz constant 1.

**Proof.** The assertion follows from the more general Proposition 6. \qed

**Comparison of different $L_\mathcal{Y}$ spaces.** The norm of the identity

$$\text{id} : (L_\mathcal{Y}_1, \|\cdot\|_{\mathcal{Y}_1}) \to (L_\mathcal{Y}_2, \|\cdot\|_{\mathcal{Y}_2})$$

is

$$\|\text{id}\| = \sup_{\sigma_2 \in \mathcal{Y}_2} \inf_{\sigma_1 \in \mathcal{Y}_1} \sup_{0 \leq \alpha \leq 1} \frac{\int_0^1 \sigma_2 (u) \, du}{\int_0^1 \sigma_1 (u) \, du},$$

and $L_\mathcal{Y}_1 \subset L_\mathcal{Y}_2$ iff $\|\text{id}\| < \infty$. This is immediate from (9), (10) and

$$\|\text{id}\| = \inf \left\{ c > 0 : \forall \sigma_2 \in \mathcal{Y}_2 \exists \sigma_1 \in \mathcal{Y}_1 : \int_0^1 \sigma_2 (u) \, du \leq c \cdot \int_0^1 \sigma_1 (u) \, du \text{ for all } \alpha \in (0, 1) \right\}.$$
Examples

We give finally two examples for which the norm $\| \cdot \|_S$ induced by a set of spectral functions $\mathcal{S}$ coincides with the norm $\| \cdot \|_p$ on $L^p$. Note, that this is contrast to the space $L_\sigma$, as Theorem 16 insures that $L_\sigma$ is strictly larger than $L^p$.

Example 35 (Higher order semideviation). The $p-$semideviation risk measure for $0 < \lambda \leq 1$ is

$$\rho (Y) := \mathbb{E} Y + \lambda \cdot \| (Y - \mathbb{E} Y)_+ \|_p.$$ 

Then $L_\mathcal{S} = L^p$, where $\mathcal{S}$ is an appropriate spectrum to generate $\rho = \rho_\mathcal{S}$, and the norms $\| \cdot \|_\mathcal{S}$ and $\| \cdot \|_p$ are equivalent.

Proof. The generating set $\mathcal{S}$ is provided in [24] and in [25], the higher order semideviation risk measure takes the alternative form

$$\rho_\mathcal{S} (Y) := \sup_{\sigma \in L^q} \left( 1 - \frac{\lambda}{\| \sigma \|_q} \right) \mathbb{E} Y + \frac{\lambda}{\| \sigma \|_q} \rho_\sigma (Y).$$

It is evident that $\rho_\mathcal{S} (|Y|) \leq \left( 1 - \frac{\lambda}{\| \sigma \|_q} \right) \| Y \|_1 + \lambda \| Y \|_p \leq (1 + \lambda) \| Y \|_p$, such that $\rho_\mathcal{S}$ is finite valued for $Y \in L^p$. We claim that the natural domain is $L_\mathcal{S} = L^p$. For this suppose that $Y \in L_\mathcal{S} \setminus L^p$, i.e. $\| Y \|_1 < \infty$, but $\| Y \|_p = \infty$. So it holds that

$$\rho_\mathcal{S} (Y) \geq \lambda \cdot \sup_{\sigma \in L^q} \frac{\rho_\sigma (Y)}{\| \sigma \|_q} = \lambda \cdot \sup_{Z \in L^q} \mathbb{E} Y \frac{Z}{\| Z \|_q} = \lambda \cdot \| Y \|_p = \infty$$

by $L^p - L^q$ duality, hence $Y \notin L_\mathcal{S}$ and thus $L_\mathcal{S} = L^p$.

It follows by the open mapping theorem that the norms are equivalent. \( \square \)

Example 36. Theorem 16 states that $L_\sigma \supsetneq L^\infty$, that is to say $L_\sigma$ is strictly larger than $L^\infty$. This is not the case any more for the space $L_\mathcal{S}$: for this consider just the risk measure

$$\rho (Y) := \sup_{\alpha < 1} \mathbb{AV}_{\mathcal{R}_\alpha} (Y) \quad (= \text{ess sup} \ Y).$$

Then $\rho (Y) < \infty$ if and only if $\text{ess sup} \ Y < \infty$, that is $L_\mathcal{S} = L^\infty$.

7 Summary

In this paper we associate a norm with a risk measure in a natural way. The risk measure is continuous with respect to the associated norm. This point of view allows considering spectral risk measures on its natural domain, which is a Banach space and as large as possible. The space of natural domain is considerably larger than an accordant $L^p$ space for spectral risk measures.

As important representation theorems, as the Fenchel–Moreau theorem, involve the dual space, we study the dual space as well. Its norm can be described by a gauge functional, and the underlying set is characterized by second order stochastic dominance constraints, which measure the pace of growth of the random variable considered.

A consequence of the results of this paper is given by the fact that finite valued risk measures cannot be defined on a space larger than $L^1$ in a meaningful way.
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