Some Characterizations of Slant Helices in the Euclidean Space $E^n$

Ahmad T. Ali
Mathematics Department
Faculty of Science, Al-Azhar University
Nasr City, 11448, Cairo, Egypt
E-mail: atali71@yahoo.com

Melih Turgut*
Department of Mathematics,
Buca Educational Faculty, Dokuz Eylül University,
35160 Buca, Izmir, Turkey
E-mail: melih.turgut@gmail.com, melih.turgut@ogr.deu.edu.tr

Abstract

In this work, notion of a slant helix is extended to space $E^n$. Necessary and sufficient conditions to be a slant helix in the Euclidean $n-$space are presented. Moreover, we express some integral characterizations of such curves in terms of curvature functions.

M.S.C. 2000: 53A04
Keywords: Euclidean n-space; Frenet equations; Slant helices.

1 Introduction and Statement of Results

Inclined curves or so-called general helices are well-known curves in the classical differential geometry of space curves [9] and we refer to the reader for recent works on this type of curves [6] [12]. Recently, Izumiya and Takeuchi have introduced the concept of slant helix in Euclidean 3-space $E^3$ saying that the normal lines makes a constant angle with a fixed direction [7]. They characterize a slant helix if and only if the function

$$
\frac{\kappa^2}{(\kappa^2 + \tau^2)^{3/2}} \left( \frac{\tau}{\kappa} \right)'
$$

*[Corresponding author.]
is constant. In the same space, spherical images, the tangent and the binormal indicatrix and some characterizations of such curves are presented by [8]. With the notion of a slant helix, similar works are treated by the researchers, see [1, 3, 5, 11, 13].

In this work, we consider the generalization of the concept of a slant helix in the Euclidean n-space $E^n$.

Let $\alpha : I \subset \mathbb{R} \rightarrow E^n$ be an arbitrary curve in $E^n$. Recall that the curve $\alpha$ is said to be of unit speed (or parameterized by arc-length function $s$) if $\langle \alpha'(s), \alpha'(s) \rangle = 1$, where $\langle \cdot, \cdot \rangle$ is the standard scalar product in Euclidean space $E^n$ given by

$$\langle X, Y \rangle = \sum_{i=1}^{n} x_i y_i,$$

for each $X = (x_1, \ldots, x_n), \ Y = (y_1, \ldots, y_n) \in E^n$.

Let $\{V_1(s), \ldots, V_n(s)\}$ be the moving frame along $\alpha$, where the vectors $V_i$ are mutually orthogonal vectors satisfying $\langle V_i, V_i \rangle = 1$. The Frenet equations for $\alpha$ are given by (10)

$$\begin{bmatrix}
V'_1 \\
V'_2 \\
V'_3 \\
\vdots \\
V'_{n-1} \\
V'_n
\end{bmatrix} = \begin{bmatrix}
0 & \kappa_1 & 0 & 0 & \cdots & 0 & 0 \\
-\kappa_1 & 0 & \kappa_2 & 0 & \cdots & 0 & 0 \\
0 & -\kappa_2 & 0 & \kappa_3 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & \kappa_{n-1} \\
0 & 0 & 0 & 0 & \cdots & -\kappa_{n-1} & 0
\end{bmatrix} \begin{bmatrix}
V_1 \\
V_2 \\
V_3 \\
\vdots \\
V_{n-1} \\
V_n
\end{bmatrix}.$$

Recall the functions $\kappa_i(s)$ are called the i-th curvatures of $\alpha$. If $\kappa_{n-1}(s) = 0$ for any $s \in I$, then $V_n(s)$ is a constant vector $V$ and the curve $\alpha$ lies in a $(n-1)$-dimensional affine subspace orthogonal to $V$, which is isometric to the Euclidean $(n-1)$-space $E^{n-1}$. We will assume throughout this work that all the curvatures satisfy $\kappa_i(s) \neq 0$ for any $s \in I$, $1 \leq i \leq n-1$. Here, recall that a regular curve with constant Frenet curvatures is called a W-curve [10].

**Definition 1.1.** A unit speed curve $\alpha : I \rightarrow E^n$ is called slant helix if its unit principal normal $V_2$ makes a constant angle with a fixed direction $U$.

Our main result in this work is the following characterization of slant helices in Euclidean n-space $E^n$.

**Theorem 1.2.** Let $\alpha : I \rightarrow E^n$ be a unit speed curve in $E^n$. Define the functions

$$G_1 = \int \kappa_1(s)ds, \ G_2 = 1, \ G_3 = \frac{\kappa_1}{\kappa_2}G_1, \ G_i = \frac{1}{\kappa_{i-1}} \left[ \kappa_{i-2}G_{i-2} + G'_{i-1} \right],$$

where $4 \leq i \leq n$. Then $\alpha$ is a slant helix if and only if the function

$$\sum_{i=1}^{n} G_i^2 = C$$

is constant and non-zero. Moreover, the constant $C = \sec^2 \theta$, being $\theta$ the angle that makes $V_2$ with the fixed direction $U$ that determines $\alpha$. 

2
This theorem generalizes in arbitrary dimensions what happens for \( n = 3 \), namely: if \( n = 3 \), (1.3) writes
\[
1 + \left(1 + \frac{\kappa_1^2}{\kappa_2^2}\right)G_1^2 = C.
\]
It is easy to prove that: this equation is equivalent to
\[
\frac{\kappa_1^2}{(\kappa_1^2 + \kappa_2^2)^{3/2}} \left(\frac{\kappa_2}{\kappa_1}\right)' = \frac{1}{\sqrt{C - 1}}
\]
equation (1.1) where \( C \neq 1 \).

2 Proof of Theorem 1.2

Let \( \alpha \) be a unit speed curve in \( E^n \). Assume that \( \alpha \) is a slant helix curve. Let \( U \) be the direction with which \( V_2 \) makes a constant angle \( \theta \) and, without loss of generality, we suppose that \( \langle U, U \rangle = 1 \). Consider the differentiable functions \( a_i, 1 \leq i \leq n \),
\[
(2.1) \quad U = \sum_{i=1}^{n} a_i(s) V_i(s), \quad s \in I,
\]
that is,
\[
a_i = \langle V_i, U \rangle, \quad 1 \leq i \leq n.
\]
Then the function \( a_2(s) = \langle V_2(s), U \rangle \) is constant, and it agrees with \( \cos \theta \) as follows:
\[
(2.2) \quad a_2(s) = \langle V_2, U \rangle = \cos \theta
\]
for any \( s \). Because the vector field \( U \) is constant, a differentiation in (2.1) together gives the following system of ordinary differential equation:
\[
(2.3) \quad \begin{cases}
    a'_1 - \kappa_1 a_2 &= 0 \\
    \kappa_1 a_1 - \kappa_2 a_3 &= 0 \\
    a'_i + \kappa_{i-1} a_{i-1} - \kappa_i a_{i+1} &= 0, \quad 3 \leq i \leq n - 1 \\
    a'_n + \kappa_{n-1} a_{n-1} &= 0.
\end{cases}
\]
Let us define the functions \( G_i = G_i(s) \) as follows
\[
(2.4) \quad a_i(s) = G_i(s) a_2, \quad 1 \leq i \leq n.
\]
We point out that \( a_2 \neq 0 \): on the contrary, (2.4) gives \( a_i = 0 \), for \( 1 \leq i \leq n \) and so, \( U = 0 \), which is a contradiction. Since, the first \( n \)-equations in (2.3) lead to
\[
(2.5) \quad \begin{cases}
    G_1 = \int \kappa_1(s) ds \\
    G_2 = 1 \\
    G_3 = \frac{\kappa_1}{\kappa_2} G_1 \\
    G_i = \frac{1}{\kappa_i} \left[ \kappa_{i-2} G_{i-2} + G'_{i-1} \right], \quad 4 \leq i \leq n.
\end{cases}
\]
The last equation of (2.3) leads to the following condition:

\[ G'_{n} + \kappa_{n-1} G_{n-1} = 0. \]  

(2.6)

We do the change of variables:

\[ t(s) = \int^{s} \kappa_{n-1}(u) du, \quad \frac{dt}{ds} = \kappa_{n-1}(s). \]

In particular, and from the last equation of (2.5), we have

\[ G'_{n-1}(t) = G_{n}(t) - \left( \frac{\kappa_{n-2}(t)}{\kappa_{n-1}(t)} \right) G_{n-2}(t). \]

As a consequence, if \( \alpha \) is a slant helix, substituting the equation (2.6) to the last equation, we express

\[ G''_{n}(t) + G_{n}(t) = \frac{\kappa_{n-2}(t) G_{n-2}(t)}{\kappa_{n-1}(t)}. \]

By the method of variation of parameters, the general solution of this equation is obtained

\[ G_{n}(t) = A - \int \frac{\kappa_{n-2}(t) G_{n-2}(t)}{\kappa_{n-1}(t)} \sin t \, dt \]

\[ + \left( B + \int \frac{\kappa_{n-2}(t) G_{n-2}(t)}{\kappa_{n-1}(t)} \cos t \, dt \right) \sin t, \]

where \( A \) and \( B \) are arbitrary constants. Then (2.7) takes the following form

\[ G_{n}(s) = \left( A - \int \kappa_{n-2}(s) G_{n-2}(s) \sin \kappa_{n-1}(s) ds \right) \cos \kappa_{n-1}(s) ds \]

\[ + \left( B + \int \kappa_{n-2}(s) G_{n-2}(s) \cos \kappa_{n-1}(s) ds \right) \sin \kappa_{n-1}(s) ds. \]

From (2.6), the function \( G_{n-1} \) is given by

\[ G_{n-1}(s) = \left( A - \int \kappa_{n-2}(s) G_{n-2}(s) \sin \kappa_{n-1}(s) ds \right) \sin \kappa_{n-1}(s) ds \]

\[ - \left( B + \int \kappa_{n-2}(s) G_{n-2}(s) \cos \kappa_{n-1}(s) ds \right) \cos \kappa_{n-1}(s) ds. \]

From Equation (2.7), we have

\[ \sum_{i=1}^{n-2} G_{i} G'_{i} = G_{1} G'_{1} + G_{2} G'_{2} + \sum_{i=3}^{n-2} G_{i} G'_{i} \]

\[ = \kappa_{1} G_{1} + \sum_{i=3}^{n-2} G_{i} \left[ \kappa_{i} G_{i+1} - \kappa_{i-1} G_{i-1} \right] \]

\[ = \kappa_{1} G_{1} + \sum_{i=3}^{n-2} \left[ \kappa_{i} G_{i+1} - \kappa_{i-1} G_{i-1} G_{i} \right] \]

\[ = \kappa_{1} G_{1} + \kappa_{n-2} G_{n-2} G_{n-1} - \kappa_{2} G_{2} G_{3} \]

\[ = \kappa_{n-2} G_{n-2} G_{n-1}. \]
Substituting (2.9) to the above equation and integrating it, we have:

\[ \sum_{i=1}^{n-2} G_i^n = C - \left( A - \int \left[ \kappa_{n-2}(s)G_{n-2}(s) \sin \int \kappa_{n-1} ds \right] ds \right)^2 \]

\[ - \left( B + \int \left[ \kappa_{n-2}(s)G_{n-2}(s) \cos \int \kappa_{n-1} ds \right] ds \right)^2, \tag{2.10} \]

where \( C \) is a constant of integration. Using equations (2.8) and (2.9), we have

\[ G^n_2 + G^{n-1}_2 = \left( A - \int \left[ \kappa_{n-2}(s)G_{n-2}(s) \sin \int \kappa_{n-1} ds \right] ds \right)^2 \]

\[ + \left( B + \int \left[ \kappa_{n-2}(s)G_{n-2}(s) \cos \int \kappa_{n-1} ds \right] ds \right)^2, \tag{2.11} \]

It follows from (2.10) and (2.11) that

\[ \sum_{i=1}^{n} G_i^n = C. \]

Moreover this constant \( C \) can be calculated as follows. From (2.4), together the \((n-2)\)-equations (2.5), we have

\[ C = \sum_{i=1}^{n} G_i^n = \frac{1}{a_2^n} \sum_{i=1}^{n} a_i^2 = \frac{1}{a_2^n} = \sec^2 \theta, \]

where we have used (1.3) and the fact that \( U \) is a unit vector field.

We do the converse of Theorem. Assume that the condition (2.5) is satisfied for a curve \( \alpha \). Let \( \theta \in \mathbb{R} \) be so that \( C = \sec^2 \theta \). Define the unit vector \( U \) by

\[ U = \cos \theta \left[ \sum_{i=1}^{n} G_i V_i \right]. \]

By taking account (2.6), a differentiation of \( U \) gives that \( \frac{dU}{ds} = 0 \), which it means that \( U \) is a constant vector field. On the other hand, the scalar product between the unit tangent vector field \( V_2 \) with \( U \) is

\[ (V_2(s), U) = \cos \theta. \]

Thus, \( \alpha \) is a slant helix in the space E\(^n\).

As a direct consequence of the proof, we generalize theorem 1.2 in Minkowski space for timelike curves and give an another theorem which characterizes slant helices with constant curvatures.

**Theorem 2.1.** Let \( E^n_1 \) be the Minkowski \( n \)-dimensional space and let \( \alpha : I \to E^n_1 \) be a unit speed timelike curve. Then \( \alpha \) is a slant helix if and only if the function \( \sum_{i=1}^{n} G_i^2 \) is constant, where the functions \( G_i \) are defined as in (1.2).

**Proof.** The proof carries the same steps as above and we omit the details. We only point out that the fact that \( \alpha \) is timelike means that \( V_1(s) = \alpha'(s) \) is a timelike vector field. The other \( V_i \) in the Frenet frame, \( 2 \leq i \leq n \), are unit spacelike vectors and so, the second equation in Frenet equations changes to \( V_2^2 = \kappa_1 V_1 + \kappa_2 V_3 \) (for details of Frenet equations see [4]).
Theorem 2.2. There are no slant helices with constant and non-zero curvatures (W−slant helices, i.e.) in the space $E^n$.

Proof. Let us suppose a slant helix with constant and non-zero curvatures. Then the equations in (2.3) and (2.5) hold. Since, we easily have for odd $i$, $G_i = \delta_i$, where $\delta_i \in \mathbb{R}$ and for even $i$, $G_i = \delta_i$. Then, we form

$$\sum_{i=1}^{n} G_i^2 = (\delta_1 s)^2 + (\delta_2 s)^2 + (\delta_3 s)^2 + ...$$

and it is easy to say that $\sum_{i=1}^{n} G_i^2$ is nowhere constant. By the theorem 1.2, we arrive at that there does not exist a slant helix with constant and non-zero curvatures in the space $E^n$.

3 Further Characterizations of Slant Helices in $E^n$

In this section we present new characterizations of slant helix in $E^n$. The first one is a consequence of Theorem 1.2.

Theorem 3.1. Let $\alpha : I \subset \mathbb{R} \rightarrow E^n$ be a unit speed curve in Euclidean space $E^n$. Then $\alpha$ is a slant helix if and only if there exists a $C^2$-function $G_n(s)$ such that

$$G_n = \frac{1}{\kappa_{n-1}} \left[ \kappa_{n-2} G_{n-2} + G'_{n-1} \right], \quad \frac{dG_n}{ds} = -\kappa_n(s) G_{n-1}(s),$$

where

$$G_1 = \int \kappa_1(s) ds, G_2 = 1, G_3 = \frac{\kappa_1}{\kappa_2} G_1, G_i = \frac{1}{\kappa_{i-1}} \left[ \kappa_{i-2} G_{i-2} + G'_{i-1} \right], \quad 4 \leq i \leq n-1.$$

Proof. Let now assume that $\alpha$ is a slant helix. By using Theorem 1.2 and by differentiation the (constant) function given in (1.3), we obtain

$$0 = \sum_{i=1}^{n} G_i G'_i$$

$$= G_1 \kappa_1 + G_3 \left( \kappa_3 G_4 - \kappa_2 G_2 \right) + G_4 \left( \kappa_4 G_5 - \kappa_3 G_3 \right) + ...$$

$$+ G_{n-1} \left( \kappa_{n-1} G_n - \kappa_{n-2} G_{n-2} \right) + G_n G'_n$$

$$= G_n \left( G'_n + \kappa_{n-1} G_{n-1} \right).$$

This shows (3.1). Conversely, if (3.1) holds, we define a vector field $U$ by

$$U = \cos \theta \left[ \sum_{i=1}^{n} G_i V_i \right].$$

By the Frenet equations, $\frac{dU}{ds} = 0$, and so, $U$ is constant. On the other hand, $\langle V_2(s), U \rangle = \cos \theta$ is constant, and this means that $\alpha$ is a slant helix. \qed
We end giving an integral characterization of a slant helix.

**Theorem 3.2.** Let \( \alpha : I \subset \mathbb{R} \rightarrow E^n \) be a unit speed curve in Euclidean space \( E^n \). Then \( \alpha \) is a slant helix if and only if the following condition is satisfied

\[
G_{n-1}(s) = \left( A - \int \left[ \kappa_{n-2} G_{n-2} \sin \int \kappa_{n-1} du \right] ds \right) \sin \int s \kappa_{n-1} du - \left( B + \int \kappa_{n-2} G_{n-2} \cos \int \kappa_{n-1} du \right) \cos \int s \kappa_{n-1} du.
\]

for some constants \( A \) and \( B \).

**Proof.** Suppose that \( \alpha \) is a slant helix. By using Theorem 3.1, let define \( m(s) \) and \( n(s) \) by

\[
\phi(s) = \int s \kappa_{n-1} du,
\]

\[
m(s) = G_n(s) \cos \phi + G_{n-1}(s) \sin \phi + \int \kappa_{n-2} G_{n-2} \sin \phi ds,
\]

\[
n(s) = G_n(s) \sin \phi - G_{n-1}(s) \cos \phi - \int \kappa_{n-2} G_{n-2} \cos \phi ds.
\]

If we differentiate equations (3.3) with respect to \( s \) and taking into account of (3.2) and (3.1), we obtain \( \frac{dm}{ds} = 0 \) and \( \frac{dn}{ds} = 0 \). Therefore, there exist constants \( A \) and \( B \) such that \( m(s) = A \) and \( n(s) = B \). By substituting into (3.3) and solving the resulting equations for \( G_{n-1}(s) \), we get

\[
G_{n-1}(s) = \left( A - \int \kappa_{n-2} G_{n-2} \sin \phi ds \right) \sin \phi - \left( B + \int \kappa_{n-2} G_{n-2} \cos \phi ds \right) \cos \phi.
\]

Conversely, suppose that (3.2) holds. In order to apply Theorem 3.1 we define \( G_n(s) \) by

\[
G_n(s) = \left( A - \int \kappa_{n-2} G_{n-2} \sin \phi ds \right) \cos \phi + \left( B + \int \kappa_{n-2} G_{n-2} \cos \phi ds \right) \sin \phi.
\]

with \( \phi(s) = \int s \kappa_{n-1}(u) du \). A direct differentiation of (3.2) gives

\[
G'_{n-1} = \kappa_{n-1} G_n - \kappa_{n-2} G_{n-2}.
\]

This shows the left condition in (3.1). Moreover, a straightforward computation leads to \( G'_{n}(s) = -\kappa_{n-1} G_{n-1} \), which finishes the proof. \( \square \)

**ACKNOWLEDGEMENTS:** The second author would like to thank Tübitak-Bideb for their financial supports during his Ph.D. studies.

**References**

[1] Ali, A. *Inclined curves in the Euclidean 5-space \( E^5 \)*, J. Advanced Research in Pure Math., 1 (1), 15–22, 2009.

[2] Ali, A. and López, R. *Slant helices in Minkowski space \( E^3_1 \)*, preprint 2008: [arXiv:0810.1464v1 [math.DG]].
[3] Ali, A. and López, R. *Timelike $B_2$-slant helices in Minkowski space $E^4_1$*, preprint 2008:arXiv:0810.1460v1 [math.DG].

[4] Ekmekci, N., Hacisalihoglu, H.H. and Ilarslan, K. *Harmonic Curvatures in Lorentzian Space*, Bull. Malaysian Math. Soc. (Second Series), 23 (2), 173-179, 2000.

[5] Erdogan, M. and Yilmaz, G. *Null generalized and slant helices in 4-dimensional Lorentz-Minkowski space*, Int. J. Contemp. Math. Sci., 3 (23), 1113-1120, 2008.

[6] Gluck, H. *Higher curvatures of curves in Euclidean space*, Amer. Math. Monthly, 73, 699–704, 1966.

[7] Izumiya, S. and Takeuchi, N. *New special curves and developable surfaces*, Turk J. Math., 28 (2), 531–537, 2004.

[8] Kula, L. and Yayli, Y. *On slant helix and its spherical indicatrix*, Appl. Math. Comput. 169 (1), 600607, 2005.

[9] Milman, R.S. and Parker, G.D. Elements of Differential Geometry, Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1977.

[10] Petrovic-Torgasev, M. and Sucurovic, E. *W-curves in Minkowski spacetime*, Novi. Sad. J. Math. 32 (2), 55–65, 2002.

[11] Onder, M., Kazaz, M., Kocayigit, H. and Kilic, O. *$B_2$-slant helix in Euclidean 4-space $E^4_1$*, Int. J. Contemp. Math. Sci., 2008, 3(29): 1433-1440.

[12] Scofield, P.D. *Curves of constant precession*, Amer. Math. Monthly, 102, 531–537, 1995.

[13] Turgut, M. and Yilmaz, S. *Some characterizations of type-3 slant helices in Minkowski space-time*, Involve J. Math., 2 (1), 115-120, 2009