Pathwise asymptotics for Volterra type stochastic volatility models

Miriana Cellupica  Barbara Pacchiarotti*

Abstract. We study stochastic volatility models in which the volatility process is a positive continuous function of a continuous Volterra stochastic process. We state some pathwise large deviation principles for the scaled log-price.

Keywords: large deviations, Volterra type Gaussian processes, conditional processes.

2000 MSC: 60F10, 60G15, 60G22.

Corresponding Author: Barbara Pacchiarotti, Dipartimento di Matematica, Università di Roma Tor Vergata, Via della Ricerca Scientifica, I-00133 Roma, Italy. E-mail address: pacchiar@mat.uniroma2.it

1 Introduction

The last few years have seen renewed interest in stochastic volatility models driven by fractional Brownian motion or other self-similar Gaussian processes (see [14], [19], [20]), i.e. fractional stochastic volatility models. Here we study stochastic volatility models in which the volatility process is a positive continuous function $\sigma$ of a continuous stochastic process $\hat{B}$, that we assume to be a Volterra type Gaussian process. The main result obtained in the present paper is a generalization, to the infinite-dimensional case, of a large deviation principle for the log-price process, due (in the real case) to Forde and Zhang [13] and Gulisashvili [17]. Our result is similar to that obtained in Section 2 in a recent paper of Gulisashvili [18] where also moderate deviations are considered.

An important aspect of this paper is that the techniques used here are different from those generally used in this framework (Freidlin-Wentzell theory). The principal result we use is Chaganty Theorem (see Theorem 2.3 in [6]), where a large deviation principle for joint and marginal distributions is stated. In this way the same results contained in [13], [17] and [18] can be obtained in a more general context, see Section 7.

In the stochastic volatility models of interest, the dynamic of the asset price process $(S_t)_{t \in [0,T]}$ is modeled by the following equation

$$
\begin{cases}
    dS_t = S_t \mu(\hat{B}_t)dt + S_t \sigma(\hat{B}_t) d(\tilde{\rho}W_t + \rho B_t), \\
    S_0 = s_0 > 0,
\end{cases}
$$

where $s_0$ is the initial price, $T > 0$ is the time horizon and $\hat{B}$ is a non-degenerate continuous Volterra type Gaussian process of the form

$$
\hat{B}_t = \int_0^t K(t,s) dB_s, \quad 0 \leq t \leq T,
$$

for some kernel $K$, the processes $W$ and $B$ are two independent standard Brownian motions, $\rho \in (-1,1)$ is the correlation coefficient and $\tilde{\rho} = \sqrt{1-\rho^2}$. Remark that $\tilde{\rho}W + \rho B$ is another standard Brownian motion which has correlation coefficient $\rho$ with $B$. If $\rho \neq 0$ the model is called a correlated stochastic volatility

*Dept. of Mathematics, University of Rome “Tor Vergata”, E-mail address: pacchiar@mat.uniroma2.it
model, otherwise it is called an uncorrelated model. It is assumed that $\mu: \mathbb{R} \to \mathbb{R}$ and $\sigma: \mathbb{R} \to (0, +\infty)$ are continuous functions satisfying suitable hypotheses. The process $\tilde{\sigma}(B) = (\tilde{\sigma}(B_t))_{0 \leq t \leq T}$ describes the stochastic evolution of the volatility in the model and $\mu(B) = (\mu(B_t))_{0 \leq t \leq T}$ is an adapted return process. Note that the model here considered contains a drift term which is not present in [18].

The unique solution to the previous equation is the Doléans-Dade expression

$$S_t = s_0 \exp \left\{ \int_0^t \mu(\hat{B}_s) ds - \frac{1}{2} \int_0^t (\hat{\sigma}(\hat{B}_s))^2 ds + \tilde{\rho} \int_0^t \sigma(\hat{B}_s) dW_s + \rho \int_0^t \hat{\sigma}(\hat{B}_s) dB_s \right\}$$

for $0 \leq t \leq T$. Therefore, the log-price process $Z_t = \log S_t$, $0 \leq t \leq T$, with $Z_0 = x_0 = \log s_0$ is defined by

$$Z_t = x_0 + \int_0^t \mu(\hat{B}_s) ds - \frac{1}{2} \int_0^t (\hat{\sigma}(\hat{B}_s))^2 ds + \tilde{\rho} \int_0^t \sigma(\hat{B}_s) dW_s + \rho \int_0^t \hat{\sigma}(\hat{B}_s) dB_s.$$

Now let $\varepsilon: \mathbb{N} \to \mathbb{R}_+$ be an infinitesimal function, i.e. $\varepsilon_n \to 0$, as $n \to +\infty$. For every $n \in \mathbb{N}$, we consider the following scaled version of the stochastic differential equation

$$\begin{cases} dS^n_t = S^n_t \mu(\hat{B}^n_t) dt + \varepsilon_n S^n_t \sigma(\hat{B}^n_t) dB^n_t + \varepsilon_n \rho \int_0^t \hat{\sigma}(\hat{B}_s) dB_s, & 0 \leq t \leq T, \allowbreak \\
S^n_0 = s_0. \end{cases}$$

Here the Brownian motion $\check{\sigma} W + \rho B$ is multiplied by a small-noise parameter $\varepsilon_n$ and also the Volterra process $\check{\sigma}$ is multiplied by a small-noise parameter, i.e.

$$\check{\sigma}^n_t = \varepsilon_n \check{\sigma}_t, \quad t \in [0, T].$$

The log-price process $Z^n_t = \log S^n_t$, $0 \leq t \leq T$, in the scaled model is

$$Z^n_t = x_0 + \int_0^t \mu(\hat{B}^n_s) ds - \frac{1}{2} \varepsilon_n^2 \int_0^t (\hat{\sigma}(\hat{B}^n_s))^2 ds + \varepsilon_n \rho \int_0^t \hat{\sigma}(\hat{B}^n_s) dB^n_s + \varepsilon_n \rho \int_0^t \hat{\sigma}(\hat{B}^n_s) dB_s.$$

We will obtain a sample path large deviation principle (which is called a small-noise large deviation principle) for the family of processes $(\langle Z^n_t - x_0 \rangle_{t \in [0, T]} \rangle )_{n \in \mathbb{N}}$. A large deviation principle for $(\langle Z^n_T - x_0 \rangle_{n \in \mathbb{N}}$ can be obtained with the same techniques. But it can be also obtained by contraction and this is the approach we follow here. The paper is organized as follows.

In Section 2 we recall some basic facts about large deviations for continuous Gaussian processes (also for Gaussian diffusions) and we give some examples of Volterra type stochastic volatility models to which the large deviation principles obtained here could be applied. In particular we discuss fractional models. In Section 6, we first prove a large deviation principle for the log-price process in the uncorrelated model under mild hypotheses on the coefficients $\mu$ and $\sigma$. We assume only $\mu$ continuous, $\sigma$ continuous and positive. In [18] the uncorrelated model is a particular case of the correlated one and it is obtained under the further hypothesis that $\sigma$ is locally $\omega$-continuous (see Section 6 for the exact fact). In Section 6, we first prove a large deviation principle for a certain family $(\langle Z^{m,n}_t - x_0 \rangle_{t \in [0, T]} )_{n \in \mathbb{N}}$ with a certain good rate function $I^m$ (Section 6.1). Then, showing that the family $(I^{m,n})_{n \in \mathbb{N}}$ is an exponentially good approximation of $(I^n)_{n \in \mathbb{N}}$, we prove a large deviation principle for $(\langle Z^{m,n}_t - x_0 \rangle_{t \in [0, T]}$ with the good rate function obtained in terms of the $I^m$‘s (Section 6.2). To prove this large deviation principle we have the same hypotheses on $\sigma$ as in [18]. Finally, in Section 6.3 we give an explicit expression for the rate function (not in terms of the $I^m$’s). In Section 6.4, for the identification of the rate function, we have a more restrictive hypothesis on $\sigma$. Here we need a power growth not required in [18]. In Section 6.5, we give an application to the asymptotic estimate of the crossing probability. In Section 7, we extend the results of Sections 5 and 6 to a more general context. We get the same results for more general families of Volterra processes $(\langle \check{\sigma}^n_t \rangle_{t \in [0, T]} )_{n \in \mathbb{N}}$ (not only $(\langle \varepsilon_n \check{\sigma}_t \rangle_{t \in [0, T]} )_{n \in \mathbb{N}}$) that obey a large
deviation principle (some examples of such processes can be found in [5], [15] and [24]). For example we can consider \((\{B^n_{t}\}_{t \in [0,T]}\}_{n \in \mathbb{N}} = (\{B_{t}\}_{t \in [0,T]}\}_{n \in \mathbb{N}}\) (Example 7.6). In this case we have a small-time large deviation principle for the Volterra processes. If the Volterra process is self-similar we can pass from small-noise to small-time regime (see the discussion at the end of Section 3 in [18]), while, if the process is not self-similar, it is not generally true.

2 Large deviations for continuous Gaussian processes

We briefly recall some main facts on large deviation principles and Volterra processes we are going to use. We refer, for example, to the following classical references: Chapitre II in Azencott [1], Section 3.4 in Deuschel and Strook [12], Chapter 4 (in particular Sections 4.1, 4.2 and 4.5) in Dembo and Zeitouni [11], for large deviation principles; [10] and [22] for Volterra processes.

2.1 Large deviations

Definition 2.1. Let \(E\) be a topological space, \(\mathcal{B}(E)\) the Borel \(\sigma\)-algebra and \((\mu_n)_{n \in \mathbb{N}}\) a family of probability measures on \(\mathcal{B}(E)\); let \(\gamma : \mathbb{N} \to \mathbb{R}^+\) be a speed function, i.e. \(\gamma_n \to +\infty\) as \(n \to +\infty\). We say that the family of probability measures \((\mu_n)_{n \in \mathbb{N}}\) satisfies a large deviation principle (LDP) on \(E\) with the rate function \(I\) and the speed \(\gamma_n\) if, for any open set \(\Theta\),

\[- \inf_{x \in \Theta} I(x) \leq \liminf_{n \to +\infty} \frac{1}{\gamma_n} \log \mu_n(\Theta)\]

and for any closed set \(\Gamma\)

\[\limsup_{n \to +\infty} \frac{1}{\gamma_n} \log \mu_n(\Gamma) \leq - \inf_{x \in \Gamma} I(x).\]  

A rate function is a lower semicontinuous mapping \(I : E \to [0, +\infty]\). A rate function \(I\) is said good if \(\{I \leq a\}\) is a compact set for every \(a \geq 0\).

Definition 2.2. Let \(E\) be a topological space, \(\mathcal{B}(E)\) the Borel \(\sigma\)-algebra and \((\mu_n)_{n \in \mathbb{N}}\) a family of probability measures on \(\mathcal{B}(E)\); let \(\gamma : \mathbb{N} \to \mathbb{R}^+\) be a speed function. We say that the family of probability measures \((\mu_n)_{n \in \mathbb{N}}\) satisfies a weak large deviation principle (WLDP) on \(E\) with the rate function \(I\) and the speed \(\gamma_n\) if the upper bound (1) holds for compact sets.

Let \(U = (U_t)_{t \in [0,T]}\) be a continuous, centered, Gaussian process on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). From now on, we will denote by \(C[0,T]\) (respectively \(C_0[0,T]\)) the set of continuous functions on \([0,T]\) (respectively the set of continuous functions on \([0,T]\) starting from 0) endowed with the topology induced by the sup-norm. Moreover, we will denote by \(\mathcal{M}[0,T]\) its dual, that is, the set of signed Borel measures on \([0,T]\). The action of \(\mathcal{M}[0,T]\) on \(C[0,T]\) is given by

\[\langle \lambda, h \rangle = \int_0^T h(t) d\lambda(t), \quad \lambda \in \mathcal{M}[0,T], h \in C[0,T].\]

Remark 2.3. We say that a family of continuous processes \((\{U^n_t\}_{t \in [0,T]}\) is a LDP if the family of their laws satisfies a LDP on \(C_0[0,T]\).

The following remarkable theorem (Proposition 1.5 in [1]) gives an explicit expression of the Cramér transform \(\Lambda^*\) of a continuous centered Gaussian process \((U_t)_{t \in [0,T]}\) with covariance function \(k\). Let us recall that

\[\Lambda(\lambda) = \log \mathbb{E}[\exp(\langle U, \lambda \rangle)] = \frac{1}{2} \int_0^T \int_0^T k(t, s) d\lambda(t) d\lambda(s)\]

for \(\lambda \in \mathcal{M}[0,T]\).
Theorem 2.4. Let \((U_t)_{t \in [0,T]}\) be a continuous, centered Gaussian process, with covariance function \(k\). Let \(\Lambda^*\) denote the Cramér transform of \(\Lambda\), that is

\[
\Lambda^*(x) = \sup_{\lambda \in \mathcal{M}[0,T]} (\langle \lambda, x \rangle - \Lambda(\lambda)) = \sup_{\lambda \in \mathcal{M}[0,T]} \left( \langle \lambda, x \rangle - \frac{1}{2} \int_0^T \int_0^T k(t, s) d\lambda(t) d\lambda(s) \right).
\]

Then,

\[
\Lambda^*(x) = \begin{cases} 
\frac{1}{2} \| x \|_{\mathcal{H}}^2 & x \in \mathcal{H} \\
+\infty & x \notin \mathcal{H},
\end{cases}
\]

where \(\mathcal{H}\) and \(\| . \|_{\mathcal{H}}\) denote, respectively, the reproducing kernel Hilbert space and the related norm associated to the covariance function \(k\).

Reproducing kernel Hilbert spaces (RKHS) are an important tool to handle Gaussian processes. For a detailed development of this wide theory we can refer, for example, to Chapter 4 in [21] (in particular Section 4.3) and to Chapter 2 (in particular Sections 2.2 and 2.3) in [3]. In order to state a large deviation principle for a family of Gaussian processes, we need the following definition.

Definition 2.5. A family of continuous processes \((X^n_t)_{t \in [0,T]}\) is exponentially tight at the speed \(\gamma_n\) if, for every \(R > 0\) there exists a compact set \(K_R\) such that

\[
\limsup_{n \to +\infty} \gamma_n^{-1} \log P(X^n \notin K_R) \leq -R.
\]

If the means and the covariance functions of an exponentially tight family of Gaussian processes have a good limit behavior, then the family satisfies a large deviation principle, as stated in the following theorem which is a consequence of the classic abstract Gärtner-Ellis Theorem (Baldi Theorem 4.5.20 and Corollary 4.6.14 in [11]) and Theorem 2.4.

Theorem 2.6. Let \((X^n_t)_{t \in [0,T]}\) be an exponentially tight family of continuous Gaussian processes with respect to the speed function \(\gamma_n\). Suppose that, for any \(\lambda \in \mathcal{M}[0,T]\),

\[
\lim_{n \to +\infty} E[(\langle \lambda, X^n \rangle)] = 0
\]

and the limit

\[
\Lambda(\lambda) = \lim_{n \to +\infty} \gamma_n \text{Var}(\langle \lambda, X^n \rangle) = \int_0^T \int_0^T k(t, s) d\lambda(t) d\lambda(s)
\]

exists, for some continuous, symmetric, positive definite function \(k\), that is the covariance function of a continuous Gaussian process, then \((X^n_t)_{t \in [0,T]}\) satisfies a large deviation principle on \(C[0,T]\), with the speed \(\gamma_n\) and the good rate function

\[
I(h) = \begin{cases} 
\frac{1}{2} \| h \|_{\mathcal{H}}^2 & h \in \mathcal{H} \\
+\infty & h \notin \mathcal{H},
\end{cases}
\]

where \(\mathcal{H}\) and \(\| . \|_{\mathcal{H}}\) denote, respectively, the reproducing kernel Hilbert space and the related norm associated to the covariance function \(k\).

Remark 2.7. Suppose \((Y^n_t)_{t \in [0,T]}\) is a family of centered Gaussian processes that satisfies a large deviation principle on \(C[0,T]\) with the speed \(\gamma_n\) and the good rate function \(I\). Let \((m^n)_{n \in \mathbb{N}} \subseteq C[0,T]\), \(m \in C[0,T]\) be functions such that \(m^n \xrightarrow{C[0,T]} m\), as \(n \to +\infty\). Then, the family of processes \((X^n_t)_{t \in [0,T]}\), where \(X^n = m^n + U^n\), satisfies a large deviation principle on \(C[0,T]\) with the same speed \(\gamma_n\) and the good rate function

\[
I_X(h) = I(h - m) = \begin{cases} 
\frac{1}{2} \| h - m \|_{\mathcal{H}}^2 & h - m \in \mathcal{H} \\
+\infty & h - m \notin \mathcal{H},
\end{cases}
\]
A useful result which can help in investigating the exponential tightness of a family of continuous Gaussian processes is Proposition 2.1 in [23] where the required property follows from Hölder continuity of the mean and the covariance function.

### 2.2 Volterra type Gaussian processes

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $B = (B_t)_{t \in [0,T]}$ a standard Brownian motion. Suppose $\hat{B} = (\hat{B}_t)_{t \in [0,T]}$ is a centered Gaussian process having the following Fredholm representation,

$$
\hat{B}_t = \int_0^T K(t, s) dB_s, \quad 0 \leq t \leq T,
$$

where $T > 0$ and $K$ is a measurable square integrable kernel on $[0,T]^2$ such that

$$
\sup_{t \in [0,T]} \int_0^T K(t, s)^2 ds < \infty.
$$

For such a kernel, the linear operator $\mathcal{K} : L^2[0,T] \rightarrow L^2[0,T]$ defined by

$$
\mathcal{K}h(t) = \int_0^T K(t, s)h(s) ds,
$$

is compact. The operator $\mathcal{K}$ is called a Hilbert-Schmidt integral operator. The modulus of continuity of the kernel $K$ is defined as follows

$$
M(\delta) = \sup_{\{t_1, t_2 \in [0,T] : |t_1 - t_2| \leq \delta\}} \int_0^T |K(t_1, s) - K(t_2, s)|^2 ds, \quad 0 \leq \delta \leq T.
$$

The covariance function of the process $\hat{B}$ is given by

$$
k(t, s) = \int_0^T K(t, u)K(s, u) du, \quad t, s \in [0,T].
$$

Let us define a Volterra process.

**Definition 2.8.** The process in (3) is called a Volterra type Gaussian process if the following conditions hold for the kernel $K$:

1. $K(0, s) = 0$ for all $0 \leq s \leq T$, and $K(t, s) = 0$ for all $0 \leq t < s \leq T$;
2. There exist constants $c > 0$ and $\alpha > 0$ such that $M(\delta) \leq c\delta^\alpha$ for all $\delta \in [0,T]$.

**Remark 2.9.** Condition (a) is a typical Volterra type condition for the kernel $K$ and the integral representation in (3) becomes $\hat{B}_t = \int_0^t K(t, s) dB_s$, for $0 \leq t \leq T$. So $\hat{B}$ is adapted to the natural filtration generated by $B$. Condition (b) guarantees the existence of a Hölder continuous modification of the process $\hat{B}$. Note that other definitions for Volterra processes are allowed. For example, Definition 5 in [22] also contains the following condition

3. $K$ is injective as a transformation of functions in $L^2[0,T]$.

Thanks to Condition (c) an explicit expression for the RKHS holds, see the next Remark. We will not use condition (c) in this paper.
Remark 2.10. If \((B_t)_{t \in [0,T]}\) is a Volterra type Gaussian process with kernel \(K\), satisfying condition (c), the reproducing kernel Hilbert space \(\mathcal{H}_B\) can be represented as the image of \(L^2[0,T]\) under the integral transform \(\mathcal{K}\), i.e. \(\mathcal{H}_B = \mathcal{K}(L^2[0,T])\), equipped with the inner product
\[
\langle \varphi, \psi \rangle_{\mathcal{H}_B} = \langle K^{-1} \varphi, K^{-1} \psi \rangle_{L^2[0,T]}, \quad \varphi, \psi \in \mathcal{H}_B,
\]
(for further details, see e.g. Subsection 2.2 in [22] and [27]). Any \(\varphi \in \mathcal{H}_B\) can be represented as
\[
\varphi(t) = Kf(t) = \int_0^t K(t,s)f(s) \, ds, \quad t \in [0,T]
\]
where \(f\) belongs to \(L^2[0,T]\). If condition (c) is verified we have an identification between \(\varphi \in \mathcal{H}_B\) and \(f \in L^2[0,T]\) \((\mathcal{K}\) is a bijection from \(L^2[0,T]\) into \(\mathcal{H}_B\)).

We now discuss some Volterra processes which satisfy conditions (a) and (b) in Definition 2.8.

**Fractional Brownian motion.** The fractional Brownian motion \(Z\) with Hurst parameter \(H \in (0,1)\) is the centered Gaussian process with covariance function
\[
k_H(t,s) = \frac{1}{2} \left( t^{2H} + s^{2H} - |t-s|^{2H} \right).
\]
It is well-known that fractional Brownian motion can be represented as a Volterra process with kernel
\[
K_H(t,s) = c_H \left( \frac{1}{2} \right)^{H-1/2} (t-s)^{H-1/2} - (H-\frac{1}{2}) s^{1/2-H} \int_s^t u^{H-3/2} (u-s)^{H-1/2} du,
\]
where
\[
c_H = \left( \frac{2H \Gamma(3/2-H)}{\Gamma(2-H)} \right)^{1/2}.
\]
Notice that when \(H = 1/2\) the fractional Brownian motion reduces to the Wiener process. Condition (b) for this process, with \(\alpha = \min\{2H,1\}\), was established in [28] and Lemma 8 in [17].

**Fractional Ornstein-Uhlenbeck process.** For \(H \in (0,1)\) and \(a > 0\), the fractional Ornstein-Uhlenbeck process is given by
\[
U^H_t = \int_0^t e^{-a(t-u)} dB^H_u, \quad t \geq 0
\]
where \(B^H\) is a fractional Brownian motion and the stochastic integral appearing above can be defined using the integration by parts formula and the stochastic Fubini theorem. This gives the following equality,
\[
U^H_t = B^H_t - a \int_0^t e^{-a(t-u)} B^H_u du,
\]
and therefore the Volterra representation,
\[
U^H_t = \int_0^t K_H(t,s) dB_s,
\]
where, for \(0 \leq s \leq t \leq T\),
\[
K_H(t,s) = K_H(t,s) - a \int_s^t e^{-a(t-u)} K_H(t,u) du,
\]
(see, e.g., Proposition A.1 in [8]). Condition (b) for this process, with \(\alpha = \min\{2H,1\}\), was established in Lemma 10 in [17]. Note that this is not a self similar process, therefore large deviations for small-time cannot be deduced from large deviations for small-noise. See Example 7.6.
**Riemann-Liouville fractional Brownian motion.** For $H \in (0,1)$, the Riemann-Liouville fractional Brownian motion is defined by

$$R^H_t = \frac{1}{\Gamma(H + 1/2)} \int_0^t (t-u)^{H-1/2} dB_u, \quad t \geq 0.$$  

This process is simpler than fractional Brownian motion. However, the increments of the Riemann-Liouville fractional Brownian motion lack the stationarity property. Condition (b) for this process, with $\alpha = 2H$, was established in Lemma 7 in \[17\].

**a-th fold integrated Brownian motion.** For $a \in \mathbb{N}$, consider the Volterra process

$$Z_t = \int_0^t \frac{(t-s)^a}{a!} dB_s, \quad t \geq 0.$$  

The covariance function is

$$k(s,t) = \int_0^{s \wedge t} \frac{(t-u)^a(s-u)^a}{(a!)^2} du.$$  

This is the covariance function of the a-th fold integrated Brownian motion. For details, see \[7\]. Notice that for this process the kernel is very similar to the kernel of the Riemann-Liouville fractional Brownian motion. Condition (b) for this process, with $\alpha = 2a + 1$, trivially holds.

**Conditioned Volterra processes.** For $T > 0$ consider the centered Volterra process $\hat{Z}^T$ defined by

$$\hat{Z}^T_t = \int_0^T K(T+t,T+u) dB_u, \quad t \geq 0.$$  

The covariance function is

$$\hat{K}^T(t,s) = \int_0^{s \wedge t} K(T+t,T+u)K(T+s,T+u) du.$$  

This process can be obtained by conditioning a Volterra process with kernel $K$ to the past up to time $T$. The new kernel is $\hat{K}^T(t,s) = K(T+t,T+s)$. For major details see \[15\]. If the original kernel satisfies condition (b) in $[0,2T]$ then the new one satisfies condition (b) in $[0,T]$ and therefore the large deviation principles obtained in this paper can be applied.

Now we recall a small noise large deviation principle for the couple $(\varepsilon_n B, \varepsilon_n \hat{B})_{n \in \mathbb{N}}$ (see, for example, \[17\]). First observe that $(B, \hat{B})$ is a Gaussian process (for details see, for example, \[13\]) and therefore the following theorem is an application of Theorem 3.4.5 in \[12\]. From now on we denote by $H^1_{\alpha}[0,T]$ the Cameron-Martin space, i.e. the set of absolutely continuous functions $f$ such that $f(0) = 0$ and $f \in L^2[0,T]$.

**Remark 2.11.** It is known that reproducing kernel Hilbert space of the couple $(B, \hat{B})$ is the Hilbert space

$$\mathcal{H}_{(B,\hat{B})} = \{(f,g) \in C_0[0,T]^2 : f \in H^1_{\alpha}[0,T], g(t) = \int_0^t K(t,u)f(u) du, \quad 0 \leq t \leq T\}. \quad (5)$$

equipped with the norm

$$\| (f,g) \|_{\mathcal{H}_{(B,\hat{B})}} = \frac{1}{2} \int_0^T \dot{f}(s)^2 ds,$$

see, for example, the discussion in Section 6 in \[17\].

**Theorem 2.12.** $(\varepsilon_n B, \varepsilon_n \hat{B})_{n \in \mathbb{N}}$ satisfies a large deviation principle on $C_0[0,T]^2$ with the speed $\varepsilon_n^{-2}$ and the good rate function

$$I_{(B,\hat{B})}(f,g) = \begin{cases} \frac{1}{2} \int_0^T \dot{f}(s)^2 ds & (f,g) \in \mathcal{H}_{(B,\hat{B})} \\ +\infty & (f,g) \in C_0[0,T]^2 \setminus \mathcal{H}_{(B,\hat{B})} \end{cases} \quad (6)$$

where $H_{(B,\hat{B})}$ is defined in \[9\].
Remark 2.13. For \( f \in H^1_0[0,T] \), define
\[
\hat{f}(t) = \int_0^t K(t,u) \hat{f}(u) \, du \quad t \in [0,T].
\] (7)

Then, from Theorem 2.12 and the contraction principle, the family \((\varepsilon_n \hat{B})_{n \in \mathbb{N}}\) satisfies a large deviation principle on \( C_0[0,T] \) with the speed \( \varepsilon_n^{-2} \) and the good rate function
\[
I_B(g) = \inf \left\{ \frac{1}{2} \int_0^T \hat{f}(s)^2 \, ds : \hat{f} = g, \ f \in H^1_0[0,T] \right\},
\] (8)

with the understanding \( I_B(g) = +\infty \) if the set is empty.

2.3 Gaussian diffusion processes

Let \( X^n \) be the solution of the following stochastic differential equation
\[
\begin{aligned}
\begin{cases}
\, dX^n_t = b_n(t) \, dt + \varepsilon_n c_n(t) \, dW_t & 0 \leq t \leq T \\
\, X^n_0 = x \in \mathbb{R}.
\end{cases}
\end{aligned}
\] (9)

This is a Gaussian diffusion process. As a simple application of Theorem 3.1 in \[9\] we have the following result for Gaussian diffusion processes.

Theorem 2.14. Suppose that \( b_n \to b \) and \( c_n \to c \) in \( C[0,T] \) then the family \((X^n)_{n \in \mathbb{N}}\) of solutions to the SDE (4) satisfies a LDP with the speed \( \varepsilon_n^{-2} \) and the good rate function
\[
I(f) = \inf \left\{ \frac{1}{2} \int_0^T \hat{g}(t)^2 \, dt : x + \int_0^t b(s) \, ds + \int_0^t c(s) \hat{g}(s) \, ds = f(t), \ g \in H^1_0[0,T] \right\}
\] (10)

with the understanding \( I(f) = +\infty \) if the set is empty.

Remark 2.15. In the non-degenerate case, that is, if \( c \geq \underline{c} > 0 \) then the rate function (10) simplifies to
\[
I(f) = \begin{cases}
\frac{1}{2} \int_0^T \left( \frac{f(s) - b(s)}{c(s)} \right)^2 \, ds & f \in H^1_0[0,T] \\
+\infty & f \notin H^1_0[0,T].
\end{cases}
\]

3 Large deviations for joint and marginal distributions

In this section we introduce the Chaganty Theorem in which a large deviation principle for a sequence of probability measures on a product space \( E_1 \times E_2 \) (and then for both marginals) is obtained starting from the large deviation principle of the sequences of marginal and conditional distributions. The main reference for this topic is \[3\]. We recall, for the sake of completeness, some results about conditional distributions in Polish spaces. Let \( Y \) and \( Z \) be two random variables defined on the same probability space \((\Omega, \mathcal{F}, \mathbb{P})\), with values, respectively, in the measurable spaces \((E_1, \mathcal{E}_1)\) and \((E_2, \mathcal{E}_2)\). Let us denote by \( \mu_1 \) the (marginal) laws of \( Y \), by \( \mu_2 \) the marginal of \( Z \) and by \( \mu \) the joint distribution of \((Y,Z)\) on \((E, \mathcal{E}) = (E_1 \times E_2, \mathcal{E}_1 \times \mathcal{E}_2)\). A family of probabilities \((\mu_2(\cdot|y))_{y \in E_1}\) on \((E_2, \mathcal{E}_2)\) is a regular version of the conditional law of \( Z \) given \( Y \) if

1. For every \( B \in \mathcal{E}_2 \), the map \((E_1, \mathcal{E}_1) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})), y \mapsto \mu_2(B|y)\) is \( \mathcal{E}_1 \)-measurable.
2. For every \( B \in \mathcal{E}_2 \) and \( A \in \mathcal{E}_1 \), \( \mathbb{P}(Y \in A, Z \in B) = \int_A \mu_2(B|y)\mu_1(dy) \).
In this case we have
\[ \mu(dy,dz) = \mu_2(dz|y)\mu_1(dy). \]

In this section we will use the notation \((E, \mathcal{B})\) to indicate a Polish space (i.e. a separable, completely metrizable space) with the Borel \(\sigma\)-field, and we say that a sequence \((x_n)_{n\in\mathbb{N}} \subset E\) converges to \(x \in E\), \(x_n \to x\), if \(d_E(x_n,x) \to 0\), as \(n \to \infty\), where \(d_E\) denotes the metric on \(E\). Regular conditional probabilities do not always exist, but they exist in many cases. The following result, that immediately follows from Corollary 3.1.2 in [4], shows that in Polish spaces the regular version of the conditional probability is well defined.

**Proposition 3.1.** Let \((E_1, \mathcal{B}_1)\) and \((E_2, \mathcal{B}_2)\) be two Polish spaces endowed with their Borel \(\sigma\)-fields, \(\mu\) be a probability measure on \((E, \mathcal{B}) = (E_1 \times E_2, \mathcal{B}_1 \times \mathcal{B}_2)\). Let \(\mu_i, i = 1,2\), be the marginal probability measures on \((E_i, \mathcal{B}_i)\). Then there exists \(\mu_1\)-almost sure a unique regular version of the conditional law of \(\mu_2\) given \(\mu_1\), i.e.
\[ \mu(dy,dz) = \mu_2(dz|y)\mu_1(dy). \]

In what follows we always suppose random variables taking values in a Polish space.

Let \((E_1, \mathcal{B}_1)\) and \((E_2, \mathcal{B}_2)\) be two Polish spaces. We denote by \((\mu_n)_{n\in\mathbb{N}}\) a sequence of probability measures on the product space \((E_1 \times E_2, \mathcal{B}_1 \times \mathcal{B}_2)\) (the sequence of the joint distributions), by \((\mu_n)_{n\in\mathbb{N}}\), for \(i = 1,2\), the sequence of the marginal distributions on \((E_i, \mathcal{B}_i)\) and by \((\mu_{2n}(\cdot|x_1))_{n\in\mathbb{N}}\) the sequence of the conditional distributions on \((E_2, \mathcal{B}_2)\). Let \(\mu_1 = \mu(\cdot|x_1)\) given by Proposition 3.1, i.e.
\[ \mu_n(B_1 \times B_2) = \int_{B_1} \mu_{2n}(B_2|x_1)\mu_1(dx_1) \tag{11} \]
for every \(B_1 \times B_2\), with \(B_1 \in \mathcal{B}_1\) and \(B_2 \in \mathcal{B}_2\).

**Definition 3.2.** Let \((E_1, \mathcal{B}_1)\), \((E_2, \mathcal{B}_2)\) be two Polish spaces and \(x_1 \in E_1\). We say that the sequence of conditional laws \((\mu_{2n}(\cdot|x_1))_{n\in\mathbb{N}}\) on \((E_2, \mathcal{B}_2)\) satisfies the **LDP continuously** in \(x_1\) with the rate function \(J(\cdot|x_1)\) and the speed \(\gamma_n\), or simply, the **LDP continuity condition** holds, if
\[ (a) \text{ For each } x_1 \in E_1, J(\cdot|x_1) \text{ is a good rate function on } E_2. \]
\[ (b) \text{ For any sequence } (x_{1n})_{n\in\mathbb{N}} \text{ in } E_1 \text{ such that } x_{1n} \to x_1, \text{ the sequence of measures } (\mu_{2n}(\cdot|x_{1n}))_{n\in\mathbb{N}} \text{ satisfies a LDP on } E_2 \text{ with the rate function } J(\cdot|x_1) \text{ and the speed } \gamma_n. \]
\[ (c) J(\cdot|x) \text{ is lower semicontinuous as a function of } (x_1, x_2) \text{ in } E_1 \times E_2. \]

**Theorem 3.3.** [Theorem 2.3 in [6]] Let \((E_1, \mathcal{B}_1)\), \((E_2, \mathcal{B}_2)\) be two Polish spaces. Let \((\mu_{1n})_{n\in\mathbb{N}}\) be a sequence of probability measures on \((E_1, \mathcal{B}_1)\). For \(x_1 \in E_1\) let \((\mu_{2n}(\cdot|x_1))_{n\in\mathbb{N}}\) be the sequence of the conditional laws on \((E_2, \mathcal{B}_2)\). Suppose that the following two conditions are satisfied:
\[ (i) (\mu_{1n})_{n\in\mathbb{N}} \text{ satisfies a LDP on } E_1 \text{ with the good rate function } I_1(\cdot) \text{ and the speed } \gamma_n. \]
\[ (ii) \text{ for every } x_1 \in E_1, \text{ the sequence } (\mu_{2n}(\cdot|x_1))_{n\in\mathbb{N}} \text{ obeys the LDP continuity condition with the rate function } J(\cdot|x_1) \text{ and the speed } \gamma_n. \]

Then the sequence of joint distributions \((\mu_n)_{n\in\mathbb{N}}\), given by (11), satisfies a **WLDL** on \(E = E_1 \times E_2\) with the speed \(\gamma_n\) and the rate function
\[ I(x_1,x_2) = I_1(x_1) + J(x_2|x_1), \]
for \(x_1 \in E_1\) and \(x_2 \in E_2\). Furthermore the sequence of the marginal distributions \((\mu_{2n})_{n\in\mathbb{N}}\) defined on \((E_2, \mathcal{B}_2)\), satisfies a LDP with the speed \(\gamma_n\), and the rate function
\[ I_2(x_2) = \inf_{x_1 \in E_1} I(x_1,x_2). \]

Moreover, \((\mu_n)_{n\in\mathbb{N}}\) satisfies a LDP if \(I(\cdot,\cdot)\) is a good rate function, and in this case, also \(I_2(\cdot)\) is a good rate function.
We shall give a sufficient condition on the rate functions \( I_1(\cdot) \) and \( J(\cdot|\cdot) \) which guarantees that \( I(\cdot, \cdot) \) is a good rate function. See Lemma 2.6 in [3].

**Lemma 3.4.** In the same hypotheses of Theorem 3.3 if the set

\[
\bigcup_{x_1 \in K_1} \{ x_2 : J(x_2|x_1) \leq L \}
\]

is a compact subset of \( E_2 \) for any \( L \geq 0 \) and for any compact set \( K_1 \subset E_1 \), then \( I(\cdot, \cdot) \) is a good rate function (and therefore also \( I_2(\cdot) \) is a good rate function).

## 4 Volterra type stochastic volatility models

In the stochastic volatility models of interest the dynamic of the asset price process \((S_t)_{t \in [0,T]}\) is modeled by the following equation

\[
\begin{align*}
    dS_t &= S_t \mu(\hat{B}_t) dt + S_t \sigma(\hat{B}_t) d(\rho W_t + \rho B_t) \quad 0 \leq t \leq T, \\
    S_0 &= s_0 > 0,
\end{align*}
\]

where \( s_0 \) is the initial price, \( T > 0 \) is the time horizon, \( \hat{B} \) is a non-degenerate continuous Volterra type process as in [3] for some kernel \( K \) which satisfies the conditions in Definition 2.2 the processes \( W \) and \( B \) are two independent standard Brownian motions, \( \rho \in (-1, 1) \) is the correlation coefficient and \( \tilde{\rho} = \sqrt{1 - \rho^2} \). Remark that \( \rho W + \rho B \) is another standard Brownian motion which has correlation coefficient \( \rho \) with \( B \). If \( \rho \neq 0 \) the model is called a correlated stochastic volatility model, otherwise it is called an uncorrelated model. The equation in (12) is considered on a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})\); \((\mathcal{F}_t)_{0 \leq t \leq T}\) is the filtration generated by \( W \) and \( B \), completed by the null sets, and made right-continuous. The filtration \((\mathcal{F}_t)_{0 \leq t \leq T}\) represents the information given by the two Brownian motions. It is assumed in (12) that \( \mu : \mathbb{R} \to \mathbb{R} \) and \( \sigma : \mathbb{R} \to (0, +\infty) \) are continuous functions. It follows from (12) that the process \( \sigma(\hat{B}) = (\sigma(\hat{B}_t))_{0 \leq t \leq T} \) describes the stochastic evolution of volatility in the model and the process \( \mu(\hat{B}) = (\mu(\hat{B}_t))_{0 \leq t \leq T} \) is an adapted return process.

Equation (12) has a unique solution that can be represented as an exponential functional. The unique solution to the equation in (12) is the Dörzs-Dade exponential

\[
S_t = s_0 \exp \left\{ \int_0^t \mu(\hat{B}_s) ds - \frac{1}{2} \int_0^t \sigma(\hat{B}_s)^2 ds + \tilde{\rho} \int_0^t \sigma(\hat{B}_s) dW_s + \rho \int_0^t \sigma(\hat{B}_s) dB_s \right\}
\]

for \( 0 \leq t \leq T \) (for further details, see Section IX-2 in [25]). Therefore, the log-price process \( Z_t = \log S_t \), \( 0 \leq t \leq T \), with \( Z_0 = x_0 = \log s_0 \) is

\[
Z_t = x_0 + \int_0^t \mu(\hat{B}_s) ds - \frac{1}{2} \int_0^t \sigma(\hat{B}_s)^2 ds + \tilde{\rho} \int_0^t \sigma(\hat{B}_s) dW_s + \rho \int_0^t \sigma(\hat{B}_s) dB_s.
\]

Let \( \varepsilon_n : \mathbb{N} \to \mathbb{R}_+ \) be an infinitesimal function. For every \( n \in \mathbb{N} \), we will consider the following scaled version of the stochastic differential equation in (12)

\[
\begin{align*}
    dS^n_t &= S^n_t \mu(\hat{B}^n_t) dt + \varepsilon_n S^n_t \sigma(\hat{B}^n_t) d(\rho W_t + \rho B_t) \\
    S^n_0 &= s_0,
\end{align*}
\]

where, for every \( n \in \mathbb{N} \),

\[
\hat{B}^n_t = \varepsilon_n \hat{B}_t, \quad t \in [0, T].
\]

In the next sections we will obtain a sample path large deviation principle for the family of log-price processes \((Z^n_t - x_0)_{t \in [0,T]}\)\( n \in \mathbb{N} \). By contraction we will deduce a large deviation principle for the family \((Z^n_T - x_0)_{n \in \mathbb{N}}\) obtaining the same result contained in [17].

We will start by proving a large deviation principle for the log-price in the uncorrelated stochastic volatility model and then we extend the results to the class of correlated models.
5 LDP for the uncorrelated stochastic volatility model

We first consider, for $\rho \neq 0$, the model described by

\[
\begin{cases}
    dS_t = S_t\mu(\hat{B}_t)dt + \rho S_t\sigma(\hat{B}_t)dW_t, & 0 \leq t \leq T, \\
    S_0 = s_0 > 0,
\end{cases}
\]

(15)

where the processes $W$ and $B$ driving, respectively, the stock price and the volatility equations are two independent standard Brownian motions, so the model in (15) is an uncorrelated stochastic volatility model. The corresponding scaled model is given by

\[
\begin{cases}
    dS^n_t = S^n_t\mu(\hat{B}^n_t)dt + \epsilon_n S^n_t\hat{\rho}\sigma(\hat{B}^n_t)dW_t, & 0 \leq t \leq T, \\
    S^n_0 = s_0 > 0,
\end{cases}
\]

where, for every $n \in \mathbb{N}$, $\hat{B}^n_t$ is the Volterra process defined in equation (14). Moreover, the process $X^n_t = \log S^n_t$, $0 \leq t \leq T$, with $X^n_0 = x_0 = \log s_0$ is

\[
X^n_t = x_0 + \int_0^t \left( \mu(\hat{B}^n_s) - \frac{1}{2}\epsilon_n^2\sigma(\hat{B}^n_s)^2 \right) ds + \epsilon_n\hat{\rho}\int_0^t \sigma(\hat{B}^n_s)dW_s.
\]

(16)

We will prove that hypotheses of Chaganty's Theorem [33] hold for the family of processes

\[
(\hat{B}^n, X^n - x_0)_{n \in \mathbb{N}}.
\]

In order to guarantee the hypotheses of Chaganty’s Theorem, we will need to impose some conditions on the coefficients. First, let us recall, for future references, some well known facts on continuous functions.

**Remark 5.1.** (i) Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function and let $\varphi_n, \varphi \in C[0,T]$ be functions such that $\varphi_n \overset{C[0,T]}{\rightarrow} \varphi$, as $n \rightarrow +\infty$, then $f \circ \varphi_n \overset{C[0,T]}{\rightarrow} f \circ \varphi$, as $n \rightarrow +\infty$.

(ii) Suppose $f : \mathbb{R} \rightarrow (0, +\infty)$ is a continuous function and let $(\varphi_n)_n \subset C[0,T]$ be a sequence of equi-bounded functions, i.e., there exist $M > 0$ such that for every $n \in \mathbb{N}$, $\sup_{t \in [0,T]} |\varphi_n(t)| \leq M$, then there exist constants $\underline{f}_M, \overline{f}_M > 0$ such that, for every $n \in \mathbb{N}$ and for every $t \in [0,T],

\[
0 < \underline{f}_M \leq f(\varphi_n(t)) \leq \overline{f}_M.
\]

**Assumption 5.2.** $\sigma : \mathbb{R} \rightarrow (0, +\infty)$ and $\mu : \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions.

**Remark 5.3.** For $L > 0$, denote by $D_L$ the level sets in the Cameron Martin space, i.e.

\[
D_L = \{ f \in H^1_0[0,T] : \| f \|_{H^1_0[0,T]} \leq L \}.
\]

(17)

Then for $f \in D_L$, from the Cauchy-Schwarz inequality,

\[
|\hat{f}(t)| = \left| \int_0^t K(t,s)\hat{f}(s)ds \right| \leq \| f \|_{H^1_0[0,T]} \left( \int_0^T K^2(t,s)ds \right)^{1/2}.
\]

Therefore (thanks to conditions (a) and (b) in Definition 2.8), there exists a constant $\hat{L} > 0$ such that

\[
\sup_{f \in D_L} \sup_{t \in [0,T]} |\hat{f}(t)| \leq \hat{L}.
\]

11
Let $\mu_{1n}$ denote the law induced by $\tilde{B}^n$ on the Polish space $(E_1, B_1) = (C_0[0,T], B(C_0[0,T]))$ and for $n \in \mathbb{N}$, let $\mu_{2n}$ be the law induced by $X^n - x_0$ on $(E_2, B_2) = (C_0[0,T], B(C_0[0,T]))$. Moreover, for (almost) every $\varphi \in C_0[0,T]$, $n \in \mathbb{N}$, let $\mu_{2n}(\cdot|\varphi)$ be the conditional law of the process,

$$X^{n,\varphi} = X^n(\tilde{B}^n = \varphi(t) \quad 0 \leq t \leq T),$$

i.e. for $\varphi \in C_0[0,T]$, $\mu_{2n}(\cdot|\varphi)$ is the law of the process

$$X^{n,\varphi}_t = x_0 + \int_0^t \left( \mu(\varphi(s)) - \frac{1}{2} \varepsilon_n^2 \sigma(\varphi(s))^2 \right) ds + \varepsilon_n \bar{\rho} \int_0^t \sigma(\varphi(s)) dw_s, \quad 0 \leq t \leq T. \quad (18)$$

Let’s now check that the hypotheses of Theorem 2.14 are fulfilled. The sequence $(\mu_{1n})_{n \in \mathbb{N}}$ satisfies a LDP on $C_0[0,T]$ with the speed $\varepsilon_n^{-2}$ and the good rate function $I_{\bar{\rho}}(\cdot)$ given by (3) (condition (i) of Theorem 2.14), therefore it is enough to show that the conditions (a), (b) and (c) of LDP continuity condition are satisfied (condition (ii) of Theorem 2.14).

**Proposition 5.4.** The sequence of the conditional laws $(\mu_{2n}(\cdot|\varphi))_{n \in \mathbb{N}}$ satisfies, on $C_0[0,T]$, the LDP continuity condition with the rate function $J(\cdot|\varphi)$ and the inverse speed $\varepsilon_n^2$.

**Remark 5.5.** Notice that, for every $\varphi \in C_0[0,T]$ and $n \in \mathbb{N}$, $X^{n,\varphi}$ is a Gaussian diffusion process. We will prove that the sequence of conditional distributions obeys the conditions (a) and (b) of the LDP continuity condition by using the generalized Freidlin-Wentzell’s Theorem 2.14. The same result can be obtained by using theory of Gaussian processes. For major details see, for example, Section 5 in [24].

**Proof of Proposition 5.4.**

(a) For $\varphi \in C_0[0,T]$ we check that $(\mu_{2n}(\cdot|\varphi))_{n \in \mathbb{N}}$ obeys a LDP on $C_0[0,T]$ with the good rate function $J(\cdot|\varphi)$. With the same notation of Theorem 2.14 we have

- $b_n(t) = \mu(\varphi(t)) - \frac{1}{2} \varepsilon_n^2 \sigma(\varphi(t))^2$, then $b_n(t) \to \mu(\varphi(t))$, as $n \to +\infty$, uniformly for $t \in [0,T]$;
- $c_n(t) = \sigma(\varphi(t))$, not depending on $n$.

Then the family $(\mu_{2n}(\cdot|\varphi))_{n \in \mathbb{N}}$ satisfies a LDP on $C_0[0,T]$ with the inverse speed $\varepsilon_n^2$ and the good rate function

$$J(x|\varphi) = \inf \left\{ \frac{1}{2} \int_0^T \dot{y}(t)^2 dt : \int_0^T \mu(\varphi(s)) ds + \bar{\rho} \int_0^T \sigma(\varphi(s)) \dot{y}(s) ds = x(t), y \in H^1_0[0,T] \right\}$$

with the usual understanding $J(x|\varphi) = +\infty$ if the set is empty. If $y \in H^1_0[0,T]$ then

$$\dot{x}(t) = \mu(\varphi(t)) + \bar{\rho} \sigma(\varphi(t)) \dot{y}(t) \quad \text{a.e., with } x(0) = 0.$$

Thanks to Remark 5.1 (ii), $\sigma \circ \varphi > 0$ and the rate above simplifies to

$$J(x|\varphi) = \begin{cases} \frac{1}{2} \int_0^T \left( \frac{\dot{x}(t) - \mu(\varphi(t))}{\sigma(\varphi(t))} \right)^2 dt & x \in H^1_0[0,T] \\ +\infty & \text{otherwise} \end{cases} \quad (19)$$

(b) Let $(\varphi_n)_{n \in \mathbb{N}} \subset C_0[0,T]$ and $\varphi \in C_0[0,T]$ be functions such that $\varphi_n \xrightarrow{C_0[0,T]} \varphi$, as $n \to +\infty$. We check that the sequence $(\mu_{2n}(\cdot|\varphi_n))_{n \in \mathbb{N}}$ obeys a LDP on $C_0[0,T]$, with the (same) rate function $J(\cdot|\varphi)$. For every $n \in \mathbb{N}$, denote by $X^{n,\varphi_n}$ the process

$$X^{n,\varphi_n}_t = x_0 + \int_0^t \mu(\varphi_n(s)) ds - \frac{1}{2} \varepsilon_n^2 \int_0^t \sigma(\varphi_n(s))^2 ds + \varepsilon_n \bar{\rho} \int_0^t \sigma(\varphi_n(s)) dw_s, \quad 0 \leq t \leq T. \quad (12)$$

With the same notation of Theorem 2.14 thanks to Remark 5.1 (i), we have

\[ 12 \]
Now we show that 

\[ b_n(t) = \mu(\varphi_n(t)) - \frac{1}{2} \varepsilon_n^2 \sigma(\varphi_n(t))^2, \] 
then \( b_n(t) \to \mu(\varphi(t)) \), as \( n \to +\infty \), uniformly for \( t \in [0, T] \);

\[ c_n(t) = \bar{\rho} \sigma(\varphi_n(t)), \] 
then \( c_n(t) \to \bar{\rho} \sigma(\varphi(t)) \), as \( n \to +\infty \), uniformly for \( t \in [0, T] \).

Therefore \((\mu_n, |\varphi_n|)_{n \in \mathbb{N}}\) obeys a LDP with the inverse speed \( \varepsilon_n^2 \) and the good rate function \( J(\cdot | \varphi) \).

(c) We check that \( J(\cdot | \varphi) \) is lower semicontinuous as a function of the couple \((\varphi, x) \in C_0[0, T]^2\).
Suppose that 

\[ (\varphi_n, x_n) \xrightarrow[n \to +\infty]{} (\varphi, x). \]

If \( \lim_{n \to +\infty} J(x_n | \varphi_n) = \lim_{n \to +\infty} J(x_n | \varphi_n) = +\infty \), there is nothing to prove. Therefore we can suppose that \((x_n)_{n \in \mathbb{N}} \subset H^1_0[0, T] \) and then

\[
J(x_n | \varphi_n) = \frac{1}{2} \int_0^T \left( \frac{\dot{x}_n(t) - \mu(\varphi_n(t))}{\bar{\rho} \sigma(\varphi_n(t))} \right)^2 dt = \\
= \frac{1}{2} \int_0^T \left( \frac{\dot{x}_n(t) - \mu(\varphi_n(t))}{\bar{\rho} \sigma(\varphi(t))} \right)^2 \left( \frac{\sigma(\varphi(t))}{\sigma(\varphi_n(t))} \right)^2 dt \\
\geq \inf_{\varepsilon \in [0, T]} \left( \frac{\sigma(\varphi(t))}{\sigma(\varphi_n(t))} \right)^2 \int_0^T \left( \frac{\dot{x}_n(t) - \mu(\varphi_n(t))}{\bar{\rho} \sigma(\varphi(t))} \right)^2 dt \\
= \inf_{\varepsilon \in [0, T]} \left( \frac{\sigma(\varphi(t))}{\sigma(\varphi_n(t))} \right)^2 J(x_n + \int_0^T (\mu(\varphi(s)) - \mu(\varphi_n(s)))ds | \varphi)
\]

Now \( x_n + \int_0^T (\mu(\varphi(s)) - \mu(\varphi_n(s)))ds \to x \) in \( C_0[0, T] \), as \( n \to +\infty \). Therefore from the semicontinuity of \( J(\cdot | \varphi) \) (it is a rate function) and Remark 5.1, the claim follows.

The hypotheses of Theorem 5.3 are fulfilled, so we have the following result.

**Proposition 5.6.** The family 

\( (\hat{B}^n, X^n - x_0)_{n \in \mathbb{N}} \)

satisfies a WLDL with the speed \( \varepsilon_n^{-2} \) and the rate function

\[ I(\varphi, x) = I_{\hat{B}}(\varphi) + J(x | \varphi). \]

Furthermore \((X^n - x_0)_{n \in \mathbb{N}}\) satisfies a LDP with the speed function \( \varepsilon_n^{-2} \) and the rate function

\[ I_X(x) = \begin{cases} 
\inf_{f \in H^1_0[0, T]} \left[ \frac{1}{2} \|f\|_{H^2_0[0, T]}^2 + \frac{1}{2} \int_0^T \left( \frac{\dot{x}(t) - \mu(\bar{f}(t))}{\sigma(\bar{f}(t))} \right)^2 dt \right] & x \in H^1_0[0, T] \\
+\infty & x \notin H^1_0[0, T],
\end{cases} \]

where \( \bar{f} \) is defined in (7).

**Proof.** Thanks to Remark 2.13 and Proposition 5.4, the family \((\hat{B}^n, X^n - x_0)_{n \in \mathbb{N}}\) satisfies the hypotheses of Chaganty’s Theorem 3.3 and therefore satisfies a WLDL with the speed \( \varepsilon_n^{-2} \) and the rate function

\[ I(\varphi, x) = I_{\hat{B}}(\varphi) + J(x | \varphi), \]

for \( x \in C_0[0, T] \) and \( \varphi \in C_0[0, T] \). Furthermore \((X^n - x_0)_{n \in \mathbb{N}}\) satisfies a LDP on \( C_0[0, T] \) with the speed function \( \varepsilon_n^{-2} \) and the rate function

\[ I_X(x) = \inf_{\varphi \in C_0[0, T]} I(\varphi, x). \]

From the expressions of the rate functions \( I_{\hat{B}}(\cdot) \) in (5) and \( J(\cdot | \cdot) \) in (19), the claim follows.

Now we show that \( I_X(\cdot) \) is a good rate function and this follows from Lemma 3.4.
Lemma 5.7. Let $J : C_0[0,T] \times C_0[0,T] \rightarrow [0, +\infty]$ be defined in (19), Then, for any $L \geq 0$ and for any compact set $K_1 \subset C_0[0,T]$,

$$\bigcup_{\varphi \in K_1} \{ x \in C_0[0,T] : J(x|\varphi) \leq L \}$$

is a compact subset of $C_0[0,T]$, therefore $I_X(\cdot)$ is a good rate function.

Proof.
Let $K_1$ be a compact set of $C_0[0,T]$. For $L > 0$ let us define

$$A^L_{\varphi} = \{ x \in C_0[0,T] : J(x|\varphi) \leq L \} = \{ x \in H^1_0[0,T] : J(x|\varphi) \leq L \}.$$

For every $\varphi \in K_1$, $A^L_{\varphi}$ is a compact subset of $C_0[0,T]$ (since $J(\cdot|\varphi)$ is a good rate function). If $(\varphi_n)_{n \in \mathbb{N}} \subset \bigcup_{\varphi \in K_1} A^L_{\varphi}$, then, for every $n \in \mathbb{N}$, there exists $\varphi_n \in K_1$ such that $x_n \in A^L_{\varphi_n}$ (i.e. $J(x_n|\varphi_n) \leq L$). The sequence $(\varphi_n)_{n \in \mathbb{N}}$ is contained in $K_1$, therefore we can suppose that $\varphi_n \xrightarrow{C_0[0,T]} \tilde{\varphi} \in K_1$, as $n \to +\infty$. Straightforward computations show that exists $M > 0$ such that, for every $n \in \mathbb{N}$,

$$J(x_n|\tilde{\varphi}) \leq M.$$

Therefore for every $n \in \mathbb{N}$, $x_n \in A^M_{\tilde{\varphi}}$, which is a compact set. Then, up to a subsequence, we can suppose that $x_n \rightarrow x \in A^M_{\tilde{\varphi}}$, as $n \to +\infty$. Since $J(\cdot)|\cdot)$ is semicontinuous, then

$$J(x|\tilde{\varphi}) \leq \liminf_{n \to +\infty} J(x_n|\varphi_n) \leq L,$$

i.e. $x \in A^L_{\tilde{\varphi}} \subset \bigcup_{\varphi \in K_1} A^L_{\varphi}$ and therefore $\bigcup_{\varphi \in K_1} A^L_{\varphi}$ is compact. We are in the hypotheses of Lemma 3.4, then $I(\cdot,\cdot)$ is a good rate function, and also $I_X(\cdot)$ is a good rate function. $\square$

We summarize the sample path large deviation principle for the process $(X^n - x_0)_{n \in \mathbb{N}}$ in the following theorem.

Theorem 5.8. Under Assumptions 3.2 a large deviation principle with the speed $\varepsilon_n^{-2}$ and the good rate function

$$I_X(x) = \begin{cases} \inf_{f \in H^1_0[0,T]} & \left( \frac{1}{2} \| f \|_{H^1_0[0,T]}^2 + \frac{1}{2} \int_0^T \left( \frac{\dot{x}(t) - \mu(\overline{f}(t))}{\sigma(\overline{f}(t))} \right)^2 dt \right) & x \in H^1_0[0,T] \\ +\infty & x \notin H^1_0[0,T] \end{cases}$$

holds for the family $(X^n - x_0)_{n \in \mathbb{N}}$, where for every $n \in \mathbb{N}$, $(X^n_t)_{t \in [0,T]}$ is defined by (17).

6 LDP for the correlated Stochastic Volatility Model

We now consider a correlated Volterra type stochastic volatility model. The asset price process $(S_t)_{t \in [0,T]}$ is modeled by the following stochastic differential equation

$$\begin{cases} dS_t = S_t \mu(\tilde{B}_t) dt + S_t \sigma(\tilde{B}_t) d\rho W_t + \rho B_t, & 0 \leq t \leq T, \\ S_0 = s_0 > 0, \end{cases}$$

where $\rho \in (-1,1)$ (for $\rho = 0$ we have the uncorrelated model). Like before, let $Z_t = \log S_t$, $0 \leq t \leq T$, be the log-price process defined by (13). We are going to consider the following process

$$Z^n_t = x_0 + \int_0^t \left( \mu(\tilde{B}^n_s) - \frac{1}{2} \varepsilon_n^2 \sigma(\tilde{B}^n_s)^2 \right) ds + \varepsilon_n \rho \int_0^t \sigma(\tilde{B}^n_s) dW_s + \varepsilon_n \rho \int_0^t \sigma(\tilde{B}^n_s) dB_s, \quad (20)$$
where $Z^n_t = x_0 = \log s_0$. In this section we want to prove a sample path large deviation principle for the family $((Z^n_t - x_0)_{t \in [0,T]})_{n \in \mathbb{N}}$. Notice that

$$Z^n_t = X^n_t + \varepsilon_n \rho \int_0^t \sigma(\varepsilon_n \hat{B}_s) dB_s,$$

where $(X^n_t)_{t \in [0,T]}$ is defined in (19). The study of the correlated model is more complicated than the previous one. In fact, in this case, we should also study the behavior of the process $(V^n_t)_{t \in [0,T]}$ where

$$V^n_t = \varepsilon_n \rho \int_0^t \sigma(\varepsilon_n \hat{B}_s) dB_s, \quad 0 \leq t \leq T.$$  \hspace{1cm} (21)

Notice that this process depends on the couple $(\varepsilon_n B, \varepsilon_n \hat{B})$, but we can’t directly apply Chaganty’s Theorem to the family

$$((\varepsilon_n B, \varepsilon_n \hat{B}), Z^n - x_0)_{n \in \mathbb{N}}$$

since $V^n$ cannot be written as a continuous function of $(\varepsilon_n B, \varepsilon_n \hat{B})$ and so the LDP continuity condition is not fulfilled. To overcome this problem, we introduce a new family of processes $(Z^{n,m})_{n \in \mathbb{N}}$, where for every $m \geq 1$, $V^n$ is replaced by a suitable continuous function of $(\varepsilon_n B, \varepsilon_n \hat{B})$. Thanks to the results obtained in the previous section, we prove that the hypotheses of Chaganty’s Theorem are fulfilled for the family $((\varepsilon_n B, \varepsilon_n \hat{B}), Z^{n,m} - x_0)_{n \in \mathbb{N}}$. Then, for every $m \geq 1$, $(Z^{n,m} - x_0)_{n \in \mathbb{N}}$ satisfies a LDP with a certain good rate function $I^m$ (Section 6.1). Then, proving that the family $(Z^{n,m})_{n \in \mathbb{N}}$ is an exponentially good approximation of $(Z^n)_{n \in \mathbb{N}}$, we obtain a large deviation principle for the family $(Z^n - x_0)_{n \in \mathbb{N}}$ with the good rate function obtained in terms of the $I^m$’s (Section 6.2). Finally (in Section 6.3) we give an explicit expression for the rate function (not in terms of the $I^m$’s). In Section 6.4 we give an application of the previous results.

In this section we need some more hypotheses on coefficients $\mu$ and $\sigma$.

**Definition 6.1.** A modulus of continuity is an increasing function $\omega : [0, +\infty) \to [0, +\infty)$ such that $\omega(0) = 0$ and $\lim_{x \to 0} \omega(x) = 0$. A function $f$ defined on $\mathbb{R}$ is called locally $\omega$-continuous, if for every $\delta > 0$ there exists a constant $L(\delta) > 0$ such that for all $x, y \in [-\delta, \delta]$, the following inequality holds: $|f(x) - f(y)| \leq L(\delta)\omega(|x-y|)$.

**Remark 6.2.** For instance, if $\omega(x) = x^\alpha$, $\alpha \in (0, 1)$, the function $f$ is locally $\alpha$-Hölder continuous. If $\omega(x) = x$, the function $f$ is locally Lipschitz continuous.

Consider the following assumptions.

**Assumption 6.3.** $\sigma : \mathbb{R} \to (0, +\infty)$ is a locally $\omega$-continuous function.

**Assumption 6.4.** There exist constants $\alpha, M_1, M_2 > 0$, such that

$$\sigma(x) + |\mu(x)| \leq M_1 + M_2 |x|^\alpha, \quad x \in \mathbb{R}.$$  \hspace{1cm} (22)

### 6.1 LDP for the approximating families

In this section we suppose that Assumptions 6.2 are fulfilled.

For every $m \geq 1$, let us define the functions $\Psi_m : C_0[0,T]^2 \to C_0[0,T]$,

$$\Psi_m(f,g)(t) = \sum_{k=0}^{[\frac{mt}{T}]} \sigma(g\left(\frac{k+1}{m}T\right)) \left[f\left(\frac{k+1}{m}T\right) - f\left(\frac{k}{m}T\right)\right] + \sigma\left(g\left(\frac{mt}{T}\right)\right) \left[f(t) - f\left(\frac{mt}{T}\right)\right], \quad t \in [0,T].$$  \hspace{1cm} (22)

$t \in [0,T]$. We note that, for every $m \geq 1$, $\Psi_m$ is a continuous function on $C_0[0,T]^2$ (where we are using the sup-norm topology for both arguments of $\Psi_m$).
Combining Proposition 5.4 and the contraction principle the proof is complete. Note that
Proof. If $f$ is finite when $t \in [0, T]$, then for every $m \geq 1$, we introduce a new family of processes
\[
(Z^n_{t} - x_0)_{t \in [0, T]} \text{ for } m \geq 1, \text{ defined by}
\]
\[
Z^n_{t} = x_0 + \int_0^t \left( \mu(\varepsilon_n B_s) ds - \frac{1}{2} \varepsilon_n^2 \sigma(\varepsilon_n B_s)^2 \right) ds + \varepsilon_n \hat{\rho} \int_0^t \sigma(\varepsilon_n B_s) dW_s + \rho \Psi_m(\varepsilon_n B, \varepsilon_n \hat{B})(t).
\]
We prove a large deviation principle for the family $((Z^n_{t} - x_0)_{t \in [0, T]})_{n \in \mathbb{N}}$ (for $m \geq 1$), as $n \to +\infty$. For this purpose we check that hypotheses of Theorem 3.3 hold for the family of processes
\[
((\varepsilon_n B, \varepsilon_n \hat{B}), Z^n_{t} - x_0)_{n \in \mathbb{N}}.
\]
From Theorem 2.12 we already know that the couple $((\varepsilon_n B, \varepsilon_n \hat{B}))_{n \in \mathbb{N}}$ satisfies a large deviation principle on $C_0[0, T]^2$ with the inverse speed $\varepsilon_n^2$ and the good rate function $I_{(B, \hat{B})}(\cdot, \cdot)$ given by (18). For fixed $m \geq 1$ and $(f, g) \in C_0[0, T]^2$ our next goal is to prove that the family of conditional processes
\[
Z^{n,m,f,g} = Z^{n,m}_{t} | (\varepsilon_n B_t = f(t), \varepsilon_n \hat{B}_t = g(t)) \quad 0 \leq t \leq T
\]
satisfies a large deviation principle. For every $(f, g) \in C_0[0, T]^2$ and $t \in [0, T]$ we have
\[
Z^{n,m,f,g}_{t} = x_0 + \int_0^t \left( \mu(g(s)) ds - \frac{1}{2} \varepsilon_n^2 \sigma(g(s))^2 \right) ds + \varepsilon_n \hat{\rho} \int_0^t \sigma(g(s)) dW_s + \rho \Psi_m(f, g)(t),
\]
i.e.
\[
Z^{n,m,f,g}_{t} = X^{n,g}_{t} + \rho \Psi_m(f, g)(t),
\]
where $(X^{n,g}_{t})_{t \in [0, T]}$ is defined in (18) and the equalities are to be intended in law.

**Proposition 6.6.** If $(f, g) \in C_0[0, T]^2$, then for every $m \geq 1$, $((Z^{n,m,f,g}_{t} - x_0)_{t \in [0, T]})_{n \in \mathbb{N}}$ satisfies a large deviation principle on $C_0[0, T]$ with the speed $\varepsilon_n^2$ and the good rate function
\[
\mathcal{J}^{m}(x | (f, g)) = J(x - \rho \Psi_m(f, g)|g),
\]
where $J(\cdot | g)$ is given by (17).

**Proof.** Combining Proposition 5.4 and the contraction principle the proof is complete. Note that $\mathcal{J}^{m}(x | (f, g))$ is finite when $x - \rho \Psi_m(f, g) \in H^1_0[0, T]$. \qed

**Remark 6.5.** If $(f, g) \in \mathcal{H}_{(B, \hat{B})}$, then, we have already seen in Remark 6.5 that the function $\Psi_m(f, g)$ can be written as
\[
\Psi_m(f, g)(t) = \int_0^t \sigma \left( \hat{f} \left( \left[ \frac{m s}{T} \right] \frac{T}{m} \right) \right) f(s) ds
\]
for $t \in [0, T]$. Clearly $\Psi_m(f, g)$ is differentiable with a square integrable derivative, i.e. $\Psi_m(f, g) \in H^1_0[0, T]$. Therefore, the rate function $\mathcal{J}^{m}(\cdot | (f, g))$ is
\[
\mathcal{J}^{m}(x | (f, g)) = \begin{cases} \frac{1}{2} \int_0^T \left( \frac{\dot{x}(t) - \mu(\hat{f}(t)) - \rho \Psi_m(f, \hat{f})(t)}{\rho \sigma(\hat{f}(t))} \right)^2 dt & x \in H^1_0[0, T] \\ + \infty & x \notin H^1_0[0, T]. \end{cases}
\]
Proposition 6.8. Let \((f_n, g_n)_{n \in \mathbb{N}} \subset C_0[0, T]^2\), \((f, g) \in C_0[0, T]^2\) be functions such that \((f_n, g_n) \xrightarrow{C_0[0, T]^2} (f, g)\). Then, for every \(m \geq 1\), the family of processes \((Z_t^{n,m}(f_n, g_n) - x_0)_{t \in [0, T]}\)\(_{n \in \mathbb{N}}\), where

\[
Z_t^{n,m}(f_n, g_n) = \int_0^t \left(\mu(g_n(s))ds - \frac{1}{2} \epsilon_n^2 \sigma(g_n(s))^2\right)ds + \epsilon_n \dot{\rho} \int_0^t \sigma(g_n(s))dW_s + \rho \Psi_m(f_n, g_n)(t)
\]

satisfies a large deviation principle on \(C_0[0, T]\) with the speed \(\epsilon_n^{-2}\) and the good rate function \(\mathcal{J}^m(\cdot|\cdot(f, g))\) defined in \((22)\).

**Proof.** Since \(g_n \xrightarrow{C_0[0, T]} g\), as \(n \to +\infty\), from Proposition \ref{prop_cont_rate}(condition (b) of LDP continuity condition), we already know that the family \(((X_t^{n,g_n})_{t \in [0, T]}\)\(_{n \in \mathbb{N}}\) satisfies a LDP with the speed \(\epsilon_n^{-2}\) and the good rate function

\[
J(x|g) = \left\{ \begin{array} {cc} \frac{1}{2} \int_0^T \left(\frac{2(t) - \mu(g(t))}{\rho \sigma(g(t))}\right)^2 dt & x \in H_0^1[0, T] \\ +\infty & x \notin H_0^1[0, T] \end{array} \right.
\]

Combining this with the contraction principle, we have that the family

\(((X_t^{n,g_n} + \rho \Psi_m(f, g)(t))_{t \in [0, T]}\)\(_{n \in \mathbb{N}}\)

satisfies a large deviation principle with the speed \(\epsilon_n^{-2}\) and the good rate function \(\mathcal{J}^m(\cdot|\cdot(f, g))\) for \(x \in C_0[0, T]\). Furthermore, for every \(m \geq 1\)

\[
\Psi_m(f_n, g_n) \xrightarrow{n \to +\infty} \Psi_m(f, g)
\]

in \(C_0[0, T]\), since \(\Psi_m\) is a continuous function. Therefore the families \(((X_t^{n,g_n} + \rho \Psi_m(f_n, g_n)(t))_{t \in [0, T]}\)\(_{n \in \mathbb{N}}\) and \(((X_t^{n,g_n} + \rho \Psi_m(f, g)(t))_{t \in [0, T]}\)\(_{n \in \mathbb{N}}\) are exponentially equivalent (see Remark \ref{rem_exp_equiv}) and the proof is complete. \(\Box\)

We now want to prove the lower semicontinuity of \(\mathcal{J}^m(\cdot|\cdot, \cdot)\).

Proposition 6.9. If \(((f_n, g_n), x_n) \xrightarrow{n \to +\infty} ((f, g), x)\) in \(C_0[0, T]^2 \times C_0[0, T]\), then for every \(m \geq 1\)

\[
\liminf_{n \to +\infty} \mathcal{J}^m(x_n|(f_n, g_n)) \geq \mathcal{J}^m(x|(f, g)).
\]

**Proof.** From Proposition \ref{prop_lsc}, we have

\[
\mathcal{J}^m(x_n|(f_n, g_n)) = J(x_n - \rho \Psi_m(f_n, g_n)|g_n).
\]

Recall that for every \(m \geq 1\), \(\Psi_m\) is continuous on \(C_0[0, T]^2\). Therefore, if \(((f_n, g_n), x_n) \to ((f, g), x)\), as \(n \to +\infty\), in \(C_0[0, T]^2 \times C_0[0, T]\), then

\[
x_n - \rho \Psi_m(f_n, g_n) \xrightarrow{n \to +\infty} x - \rho \Psi_m(f, g)
\]

in \(C_0[0, T]\). Then, from the lower semicontinuity of \(J(\cdot|\cdot)\) (see Proposition \ref{prop_cont_rate}(condition (c) of LDP continuity condition))

\[
\liminf_{n \to +\infty} \mathcal{J}^m(x_n|(f_n, g_n)) = \liminf_{n \to +\infty} J(x_n - \rho \Psi_m(f_n, g_n)|g_n) \geq J(x - \rho \Psi_m(f, g)|g) = \mathcal{J}^m(x|(f, g)),
\]

which concludes the proof. \(\Box\)

17
Proposition 6.10. For $m \geq 1$, the family $((\varepsilon_n B, \varepsilon_n B), Z^{n,m} - x_0)_{n \in \mathbb{N}}$, where for every $n \in \mathbb{N}$, $Z^{n,m}$ is the process defined by $(23)$ satisfies a WLD with the speed $\varepsilon_n^{-2}$ and the rate function

$$
\mathcal{H}^m((f,g), x) = I_{(B,B)}(f,g) + \mathcal{J}^m(x|(f,g))
$$

(27)

for $x \in C_0[0,T]$ and $(f,g) \in C_0[0,T]^2$, and $(Z^{n,m} - x_0)_{n \in \mathbb{N}}$ satisfies a LDP with the speed $\varepsilon_n^{-2}$ and the rate function

$$
I^m_Z(x) = \left\{ \inf_{f \in H^0_1[0,T]} \left\{ \frac{1}{2} \mathcal{J}^m(x|(f,g)) + \frac{1}{2} \int_0^T \left( \frac{\hat{x}(t) - \mu(\hat{f}(t)) - \rho \hat{\psi}_m(f,\hat{f})(t)}{\rho \sigma(f(t))} \right)^2 dt \right\} \right\} x \in H^0_1[0,T],
$$

(28)

Therefore $(Z^{n,m} - x_0)_{n \in \mathbb{N}}$ satisfies a LDP with the speed $\varepsilon_n^{-2}$ and the rate function

$$
I^m_Z(x) = \inf_{(f,g) \in C_0[0,T]^2} \left\{ I_{(B,B)}(f,g) + \mathcal{J}^m(x|(f,g)) \right\}
$$

(29)

for $x \in C_0[0,T]$. From (6) and Remark 6.7 the claim follows. □

We conclude this section showing that, for every $m \geq 1$, $I^m$ is a good rate function.

Proposition 6.11. For every $m \geq 1$, the rate function $I^m_Z(x)$ defined in (28) is a good rate function.

For the proof of this proposition, we need the following lemma.

Lemma 6.12. Let $\mathcal{J}^m : C_0[0,T]^2 \times C_0[0,T] \rightarrow [0, +\infty]$ be the rate function defined in (20). Then, the set

$$
\bigcup_{(f,g) \in K_1} \{ x \in C_0[0,T] : \mathcal{J}^m(x|(f,g)) \leq L \}
$$

is a compact subset of $C_0[0,T]$ for any $L \geq 0$ and for any (compact) level set $K_1$ of the (good) rate function $I_{(B,B)}(\cdot, \cdot)$ defined in (6).

**Proof.** Let $K_1$ be a level set of $I_{(B,B)}(\cdot, \cdot)$. For $L \geq 0$ and $(f,g) \in K_1$ define

$$
A_{L(f,g)} = \{ x \in C_0[0,T] : \mathcal{J}^m(x|(f,g)) \leq L \}.
$$

(30)

For every $(f,g) \in K_1$, $A_{L(f,g)}$ is a compact subset of $C_0[0,T]$, since $\mathcal{J}^m(\cdot|(f,g))$ is a good rate function. From the expression of the rate function $I_{(B,B)}(\cdot, \cdot)$, we can deduce that $K_1 \subset \mathcal{H}_{(B,B)}$, where $\mathcal{H}_{(B,B)}$ is defined in (5). Therefore, for every $(f,g) \in K_1$, we have that $g = \hat{f}$ where $\hat{f}$ is defined in (7) and $\mathcal{J}^m(\cdot|(f,\hat{f}))$ is given by (28). Consider a sequence $(x_n)_{n \in \mathbb{N}} \subset \bigcup_{(f,g) \in K_1} A_{L(f,g)}$. Then, for every $n \in \mathbb{N}$, there exists $(f_n, g_n) \in K_1$ such that $x_n \in A_{L(f_n,g_n)}$ (i.e. $\mathcal{J}^m(x_n|(f_n,g_n)) \leq L$). Then, $((f_n, g_n))_{n \in \mathbb{N}} \subset K_1$ and therefore, up to a subsequence, we can suppose that $(f_n, g_n) \xrightarrow{C_0[0,T]^2} (f,g) \in K_1$, as $n \rightarrow +\infty$. Straightforward computations show that there exists a constant $M > 0$ such that

$$
\mathcal{J}^m(x_n|(f,g)) \leq M \quad \text{for every } n \in \mathbb{N}.
$$

Therefore $(x_n)_{n \in \mathbb{N}} \subset A_{L(f,g)}$, where $A_{L(f,g)}$ is the compact set defined in (30). Then, up to a subsequence, we can suppose that $x_n \xrightarrow{C_0[0,T]^2} x \in A_{L(f,g)}$, as $n \rightarrow +\infty$. Furthermore $x \in A_{L(f,g)}$ since, from Proposition 6.8

$$
\mathcal{J}^m(x|(f,g)) \leq \liminf_{n \rightarrow +\infty} \mathcal{J}^m(x_n|(f_n, g_n)) \leq L.
$$

18
Therefore \( \bigcup_{(f,g) \in K_i} A_{(f,g)}^L \) is a compact subset of \( C_0[0,T] \), for any \( L \geq 0 \) and for any level set \( K_i \) of \( I_{(B,B)}(\cdot, \cdot) \).

\[ \square \]

**Proof of Proposition 6.11** From the contraction principle, Proposition 6.11 will be stated if we show that the rate function \( \mathcal{H}^m((\cdot, \cdot), \cdot) \), defined in (27), is a good rate function. For \( L \geq 0 \) we prove that

\[
M_L = \{ ((f,g), x) \in C_0[0,T]^2 \times C_0[0,T] : \mathcal{H}^m((f,g), x) \leq L \}
\]

\[
= \{ ((f,g), x) \in \mathcal{H}(B,B) \times H^0_0[0,T] : I_{(B,B)}(f,g) + J^m(x)(f,g) \leq L \}
\]

is a compact subset of \( C_0[0,T]^2 \times C_0[0,T] \). Note that \( M_L \) is a closed subset of \( C_0[0,T] \times C_0[0,T] \) since \( \mathcal{H}^m((\cdot, \cdot), \cdot) \) is lower semicontinuous. Set \( K_i = \{ (f,g) \in \mathcal{H}(B,B) : I_{(B,B)}(f,g) \leq L \} \). It is easy to verify that

\[
M_L \subset K_i \times \bigcup_{(f,g) \in K_i} \{ x \in C_0[0,T] : J^m(x)(f,g) \leq L \}.
\]

\( K_i \) is compact since it is a level set of \( I_{(B,B)}(\cdot, \cdot) \). Then the set on the right hand side is compact from Lemma 6.12. Thus \( M_L \) is a compact set being a closed subset of a compact set. This completes the proof.

\[ \square \]

We summarize the results we have proved for the family \( ((Z_t^{n,m} - x_0)_{t \in [0,T]} \) \( )_{n \in \mathbb{N}} \) (for \( m \geq 1 \)) in the following theorem.

**Theorem 6.13.** Suppose \( \sigma \) and \( \mu \) satisfy Assumption 6.2. For every \( m \geq 1 \), a large deviation principle with the speed \( \varepsilon_n^{-2} \) and the good rate function \( I_n^m(\cdot) \) given by (28) holds for the family \( ((Z_t^{n,m} - x_0)_{t \in [0,T]} \) \( )_{n \in \mathbb{N}} \).

### 6.2 LDP for the log-price processes

In this section, we suppose that Assumptions 5.2 and 6.3 are fulfilled.

Theorem 6.13 provides a LDP for the families \( (Z^{n,m} - x_0)_{n \in \mathbb{N}} \) for every \( m \geq 1 \), but our goal is to get a LDP for the family \( (Z^n - x_0)_{n \in \mathbb{N}} \). Then, we prove that the sequence of processes \( (Z^{n,m} - x_0)_{n \in \mathbb{N}} \) is an exponentially good approximation of \( (Z^n - x_0)_{n \in \mathbb{N}} \). Let us give the definition of exponentially good approximations. The main reference for this section is [2].

**Definition 6.14.** Let \( (E, d_E) \) be a metric space and for \( \delta > 0 \), define \( \Gamma_\delta = \{ (\hat{x}, x) : d_E(\hat{x}, x) > \delta \} \subset E \times E \). For each \( n \in \mathbb{N} \) and \( m \in \mathbb{N} \), let \( (\Omega, \mathcal{F}_n, \mathbb{P}^n,m) \) be a probability space, and let \( E \)-valued random variables \( Z^n \) and \( Z^{n,m} \) be distributed according to the joint law \( \mathbb{P}^n,m \), with marginals \( \mu^n \) and \( \mu^{n,m} \) respectively. The families \( (Z^{n,m})_{n \in \mathbb{N}} \) for \( m \geq 1 \) are called **exponentially good approximations** of \( (Z^n)_{n \in \mathbb{N}} \) at the speed \( \gamma_n \) if, for every \( \delta > 0 \), the set \( \{ \omega : (Z^n, Z^{n,m}) \in \Gamma_\delta \} \) is \( \mathcal{F}_n \)-measurable and

\[
\lim_{m \to +\infty} \limsup_{n \to +\infty} \gamma_n^{-1} \log \mathbb{P}^n,m(\Gamma_\delta) = -\infty.
\]

Similarly, the measures \( (\mu^{n,m})_{n \in \mathbb{N}} \) for \( m \geq 1 \) are exponentially good approximations of \( (\mu^n)_{n \in \mathbb{N}} \) if one can construct probability spaces \( (\Omega, \mathcal{F}_n, \mathbb{P}^n,m) \) as above.

Next theorem, Theorem 3.11 in [2], states that under a suitable condition if for each \( m \geq 1 \) the sequence \( (\mu^{n,m})_{n \in \mathbb{N}} \) satisfies a large deviation principle with the rate function \( I^n \), then also \( (\mu^n)_{n \in \mathbb{N}} \) satisfies a large deviation principle with the rate function \( I^m \), obtained in terms of the \( I^m \)s.

**Theorem 6.15.** [Theorem 3.11 in 2] Assume that \( (E, \mathcal{B}(E)) \) is a Polish space and that for each \( m \geq 1 \), \( (\mu^{n,m})_{n \in \mathbb{N}} \) satisfies a LDP with the speed \( \gamma_n \) and the good rate function \( I^n \). Let \( (\mu^n)_{n \in \mathbb{N}} \) be a family of probability measures. For every \( \delta > 0 \) define

\[
\rho_n(\mu^{n,m}, \mu^n) = \inf_{\varepsilon > 0} \left\{ \mu^{n,m}(A) \leq \mu^n(A^\delta) + \varepsilon, \quad A \in \mathcal{B}(E) \right\},
\]
where
\[ A^\delta = \bigcup_{x \in A} B_\delta(x), \quad \text{and} \quad B_\delta(x) = \{ y \in E : d_E(x, y) < \delta \}, \] (31)
with \( d_E \) the metric on \( E \). If for every \( \delta > 0 \)
\[ \lim_{m \to +\infty} \limsup_{n \to +\infty} \gamma_n^{-1} \log \rho_\delta(\mu^{n,m}, \mu^n) = -\infty, \] (32)
then \( (\mu^n)_{n \in \mathbb{N}} \) satisfies a LDP with the speed \( \gamma_n \) and the good rate function \( I \) given by
\[ I(x) = \mathcal{I}(x) = \overline{I}(x), \]
where
\[ \mathcal{I}(x) = \lim_{\delta \to 0} \inf_{m \to +\infty} \inf_{y \in B_\delta(x)} I^m(y), \quad \overline{I}(x) = \lim_{\delta \to 0} \sup_{m \to +\infty} \inf_{y \in B_\delta(x)} I^m(y). \]

Proposition 6.16. [Proposition 3.16 in \[2\]] In the same hypotheses of Theorem 6.15, if

- \( I^m(x) \xrightarrow{m \to +\infty} J(x) \), for \( x \in E \);
- \( x_m \xrightarrow{m \to +\infty} x \) implies \( \liminf_{m \to +\infty} I^m(x_m) \geq J(x) \),

for some functional \( J(\cdot) \), then \( I(\cdot) = J(\cdot) \).

Remark 6.17. Let \((E, \mathcal{B}(E))\) be a Polish space and \( d_E \) the metric on \( E \). If the random variables \((Z^{n,m})_{n \in \mathbb{N}}\) for \( m \geq 1 \) are exponentially good approximations of \((Z^n)_{n \in \mathbb{N}}\) at the speed \( \gamma_n \), then (32) holds. Consider \( A \in \mathcal{B}(E) \), then
\[ \mu^{n,m}(A) = \mathbb{P}(Z^{n,m} \in A) = \mathbb{P}(Z^{n,m} \in A, d_E(Z^{n,m}, Z^n) \leq \delta) + \mathbb{P}(Z^{n,m} \in A, d_E(Z^{n,m}, Z^n) > \delta) \leq \mathbb{P}(Z^n \in A^\delta) + \mathbb{P}(d_E(Z^{n,m}, Z^n) > \delta) = \mu^n(A^\delta) + \mathbb{P}(d_E(Z^{n,m}, Z^n) > \delta), \]
where \( A^\delta \) is defined in (31). It follows that
\[ \rho_\delta(\mu^{n,m}, \mu^n) \leq \mathbb{P}(d_E(Z^{n,m}, Z^n) > \delta) = \mathbb{P}((Z^{n,m}, Z^n) \in \Gamma_\delta). \]
Therefore, for every \( \delta > 0 \)
\[ \lim_{m \to +\infty} \limsup_{n \to 0} \gamma_n^{-1} \log \rho_\delta(\mu^{n,m}, \mu^n) = -\infty \]

Proposition 6.18. The families \((Z^{n,m})_{n \in \mathbb{N}}, m \geq 1\), defined in (28) are exponentially good approximations, at the speed \( \varepsilon_n^{-2} \), of \((Z^n)_{n \in \mathbb{N}}\) defined in (27).

Proof. For every \( \delta > 0 \) we have to prove that
\[ \lim_{m \to +\infty} \limsup_{n \to +\infty} \varepsilon_n^2 \log \mathbb{P}(\|Z^{n,m} - Z^n\|_\infty > \delta) = -\infty, \] (33)
that is
\[ \lim_{m \to +\infty} \limsup_{n \to +\infty} \varepsilon_n^2 \log \mathbb{P}(\|V^n - \rho \Psi_m(\varepsilon_n B, \varepsilon_n \hat{B})\|_\infty > \delta) = -\infty, \]
where \( V^n \) and \( \Psi_m \) are defined, respectively in (21) and (22). One can easily verify that in order to prove equality (33), it is enough to show that
\[ \lim_{m \to +\infty} \limsup_{n \to +\infty} \varepsilon_n^2 \log \mathbb{P}\left(\varepsilon_n \rho \sup_{t \in [0, T]} \left| \int_0^T \sigma(\varepsilon_n B_s) - \sigma(\varepsilon_n \hat{B}_s) \right| dB_s \right) > \delta = -\infty \] (34)
Formula (34) was established, under Assumptions 5.2 and 6.3 in Lemmas 23 and 24 in [17]. This completes the proof. \( \square \)
Then, we are ready to establish a large deviation principle for the family \((Z^n - x_0)_{n \in \mathbb{N}}\).

**Theorem 6.19.** Suppose \(\sigma\) and \(\mu\) satisfy Assumptions 5.2 and 6.3. Then \((Z^n - x_0)_{n \in \mathbb{N}}\), satisfies a LDP on \(C_0[0, T]\), with the speed \(\varepsilon_n^{-2}\) and the good rate function \(I_Z\) given by

\[
I_Z(x) = I_Z^1(x) = T_Z(x),
\]

where

\[
I_Z(x) = \lim_{\delta \to 0} \inf_{m \to +\infty} \inf_{y \in B_\delta(x)} I^m_Z(y), \quad \text{and} \quad T_Z(x) = \lim_{\delta \to 0} \sup_{m \to +\infty} \inf_{y \in B_\delta(x)} I^m_Z(y)
\]

with \(B_\delta(x) = \{ y \in C_0[0, T] : \|x - y\|_\infty < \delta \}\) and \(I^m_Z\)'s are defined in (28).

**Proof.** For every \(m \geq 1\), \((Z^n,m - x_0)_{n \in \mathbb{N}}\), where \(Z^n,m\) is defined by (23), satisfies a large deviation principle with the speed \(\varepsilon_n^{-2}\) and the good rate function \(I^m_Z\). From Proposition 6.18, \(((Z^n,m - x_0)_{n \in \mathbb{N}})_{m \geq 1}\) is an exponentially good approximation (at the same speed) of \((Z^n - x_0)_{n \in \mathbb{N}}\). This completes the proof. \(\square\)

### 6.3 Identification of the rate function

In this section we suppose Assumptions 5.2, 6.3 and 6.4 are fulfilled. Theorem 6.19 provides a LDP for the family \((Z^n - x_0)_{n \in \mathbb{N}}\) with a rate function \(I_Z(\cdot)\) which is obtained in terms of the \(I^m_Z(\cdot)'s\), but our goal is to write explicitly the rate function. Let us define a measurable function \(\Psi : C_0[0, T]^2 \to C_0[0, T]\) by

\[
\Psi(f, g)(\cdot) = \left\{ \begin{array}{ll}
\int_0^1 \sigma(\hat{f}(s))\hat{f}(s) \, ds & \text{if } (f, g) \in H^1_0[0, T]/\mathcal{H}(B, B) \\
0 & \text{if } (f, g) \in C_0[0, T]^2 / \mathcal{H}(B, B)\end{array}\right. \quad (35)
\]

The function \(\Psi\) is finite on \(C_0[0, T]^2\) and, for \(f \in H^1_0[0, T]\), \(\Psi(f, \hat{f})\) is differentiable with a square integrable derivative, i.e. \(\Psi(f, \hat{f}) \in H^1_0[0, T]\).

**Remark 6.20.** Let \(D_L\) be defined as in (17). Then thanks to Remarks 5.1 and 5.3 for \(f \in D_L\) there exist constants \(\overline{\sigma}_L\), \(\underline{\sigma}_L\) and \(\overline{\mu}_L\) (depending on \(L\)) such that, for \(f \in D_L\) and \(t \in [0, T]\), we have

\[
|\mu(\hat{f}(t))| \leq \overline{\mu}_L, \quad 0 < \underline{\sigma}_L \leq \sigma(\hat{f}(t)) \leq \overline{\sigma}_L.
\]

**Remark 6.21.** Thanks to Assumptions 5.2 and 6.4 and Remark 5.3 there exist a constant \(M > 0\) such that, for \(f \in H^1_0[0, T]\) and \(t \in [0, T]\), we have

\[
\|\mu(\hat{f}(t))\| + \sigma(\hat{f}(t)) \leq M\|f\|_{H^1_0[0, T]}.
\]

Next lemma is a particular case of Lemma 2.13 in [13]. We give some details of the proof for the sake of completeness.

**Lemma 6.22.** For every \(L > 0\), if \(D_L\) is the set defined in (17), then one has,

\[
\lim_{m \to +\infty} \sup_{f \in D_L} \|\Psi(f, \hat{f}) - \Psi_m(f, \hat{f})\|_\infty = 0.
\]

**Proof.** From Lemma 22 in [17], we have

\[
\lim_{m \to +\infty} \sup_{f \in D_L} \sup_{t \in [0, T]} \left| \sigma(\hat{f}(t)) - \sigma\left(\hat{f}\left(\frac{mt}{T}\right)\right) \right| = 0.
\]
In Remark 6.23 we showed that, if \( f \in H_0^1[0, T] \), for \( t \in [0, T] \)

\[
\Psi_m(f, \dot{f})(t) = \int_0^t \sigma \left( \dot{f} \left( \frac{ms}{T} \right) \right) \dot{f}(s) ds
\]

Then, it is enough to show that for every \( L > 0 \),

\[
\lim_{m \to +\infty} \sup_{f \in D_L} \sup_{t \in [0, T]} \left| \int_0^t \left[ \sigma(\dot{f}(s)) - \sigma \left( \dot{f} \left( \frac{ms}{T} \right) \right) \right] \dot{f}(s) ds \right| = 0.
\]

For \( f \in H_0^1[0, T] \) and \( m \geq 1 \) we have

\[
\sup_{f \in D_L} \sup_{t \in [0, T]} \left| \int_0^t \left[ \sigma(\dot{f}(s)) - \sigma \left( \dot{f} \left( \frac{ms}{T} \right) \right) \right] \dot{f}(s) ds \right| \leq \sup_{f \in D_L} \int_0^T \left| \sigma(\dot{f}(s)) - \sigma \left( \dot{f} \left( \frac{ms}{T} \right) \right) \right| \dot{f}(s) ds
\]

\[
\leq \sqrt{LT} \sup_{f \in D_L} \sup_{t \in [0, T]} \left| \sigma(\dot{f}(s)) - \sigma \left( \dot{f} \left( \frac{ms}{T} \right) \right) \right|.
\]

Therefore the claim follows from (36). \( \square \)

Now let us introduce the following functional

\[
I_Z(x) = \begin{cases} 
\inf_{f \in H_0^1[0, T]} \mathcal{H}((f, \dot{f}), x) & x \in H_0^1[0, T] \\
+ \infty & x \notin H_0^1[0, T],
\end{cases}
\]

where for every \( f \in H_0^1[0, T] \),

\[
\mathcal{H}((f, \dot{f}), x) = \frac{1}{2} \|f\|_{H_0^1[0, T]}^2 + \frac{1}{2} \int_0^T \left( \dot{x}(t) - \mu(\dot{f}(t)) - \rho \dot{\Psi}(f, \dot{f})(t) \right)^2 dt
\]

and \( \Psi \) is defined in (35). We shall prove that \( I_Z(\cdot) = I_Z(\cdot) \).

**Remark 6.23.** For \( x \in H_0^1[0, T] \) we have,

\[
I_Z(x) = \inf_{f \in H_0^1[0, T]} \mathcal{H}((f, \dot{f}), x) \leq \mathcal{H}((0, 0), x) = \frac{1}{2\rho^2 \sigma^2(0)} \int_0^T (\dot{x}(t) - \mu(0))^2 dt,
\]

therefore

\[
I_Z(x) = \inf_{f \in D_{C_x}} \mathcal{H}((f, \dot{f}), x)
\]

where \( C_x = \frac{1}{2\rho^2} \int_0^T (\dot{x}(t) - \mu(0))^2 dt \) and \( D_{C_x} = \{ f \in H_0^1[0, T] : \|f\|_{H_0^1[0, T]}^2 \leq C_x \} \). Similarly, for \( x \in H_0^1[0, T] \), for every \( m \geq 1 \), we have

\[
I_{Z_m}^m(x) = \inf_{f \in D_{C_x}} \mathcal{H}_m((f, \dot{f}), x)
\]

where, we recall, \( I_{Z_m}^m(\cdot) \) is the rate function defined in (29) and

\[
\mathcal{H}_m((f, \dot{f}), x) = \frac{1}{2} \|f\|_{H_0^1[0, T]}^2 + \frac{1}{2} \int_0^T \left( \dot{x}(t) - \mu(\dot{f}(t)) - \rho \dot{\Psi}_m(f, \dot{f})(t) \right)^2 dt.
\]

In order to prove that \( I_Z(\cdot) = I_Z(\cdot) \), we have to verify that the hypotheses of Proposition 6.16 are fulfilled. We start by proving the convergence to \( I_Z(\cdot) \) of the rate functions \( I_{Z_m}^m(\cdot) \)’s.
Lemma 6.24. For every $x \in C_0[0, T]$ one has
\[
\lim_{m \to +\infty} I^n_Z(x) = \mathcal{I}_Z(x),
\]
where $I^n_Z(\cdot)$ and $\mathcal{I}_Z(\cdot)$ are defined in (28) and (37), respectively.

Proof. If $x \notin H^1_0[0, T]$, one has $I^n_Z(x) = \mathcal{I}_Z(x) = +\infty$. If $x \in H^1_0[0, T]$ we have,
\[
|I^n_Z(x) - \mathcal{I}_Z(x)| = \left| \inf_{f \in D_{C_x}} \mathcal{H}_m((f, \dot{f}), x) - \inf_{f \in D_{C_x}} \mathcal{H}((f, \dot{f}), x) \right| \leq \sup_{f \in D_{C_x}} \left| \mathcal{H}_m((f, \dot{f}), x) - \mathcal{H}((f, \dot{f}), x) \right|.
\]
Taking into account Remark 6.20 we have
\[
\sup_{f \in D_{C_x}} \left| \mathcal{H}_m((f, \dot{f}), x) - \mathcal{H}((f, \dot{f}), x) \right| \leq \frac{\rho^2}{2 \rho^2 \bar{C}_x} \sup_{f \in D_{C_x}} \left[ \int_0^T \|\hat{\psi}_m^2(f, \dot{f})(t) - \hat{\psi}^2(f, \dot{f})(t)\| dt + \frac{2}{\rho} \int_0^T |\dot{x}(t)| \|\hat{\psi}(f, \dot{f})(t) - \hat{\psi}_m(f, \dot{f})(t)\| dt \right]
\]
\[
+ \frac{2}{\rho} \int_0^T |\mu(\dot{f})(t)| \|\hat{\psi}_m(f, \dot{f})(t) - \hat{\psi}(f, \dot{f})(t)\| dt
\]

Now we study the three addends. For the first one we have,
\[
\sup_{f \in D_{C_x}} \int_0^T \|\hat{\psi}_m^2(f, \dot{f})(t) - \hat{\psi}^2(f, \dot{f})(t)\| dt \leq 2 \sigma C_x \sup_{f \in D_{C_x}, t \in [0, T]} \left| \hat{f} \left( \left[ \frac{mT}{m} \right] \right) - \sigma(\hat{f}(t)) \right|
\]
From the Cauchy-Schwarz inequality and Remark 6.20 we have,
\[
\sup_{f \in D_{C_x}} \int_0^T |\dot{x}(t)| \|\hat{\psi}(f, \dot{f})(t) - \hat{\psi}_m(f, \dot{f})(t)\| dt \leq \sqrt{C_x} \|x\|_{H^1_0[0, T]} \sup_{f \in D_{C_x}, t \in [0, T]} \left| \hat{f} \left( \left[ \frac{mT}{m} \right] \right) - \sigma(\hat{f}(t)) \right|
\]
Finally,
\[
\sup_{f \in D_{C_x}} \int_0^T |\mu(\dot{f})(t)| \|\hat{\psi}_m(f, \dot{f})(t) - \hat{\psi}(f, \dot{f})(t)\| dt \leq \sqrt{T C_x} \mu C_x \sup_{f \in D_{C_x}, t \in [0, T]} \left| \hat{f} \left( \left[ \frac{mT}{m} \right] \right) - \sigma(\hat{f}(t)) \right|
\]
The claim then follows from equation (38). \qed

It remains to show that $x_m \xrightarrow{m \to +\infty} x$ implies $\lim \inf_{m \to +\infty} I^n_Z(x_m) \geq \mathcal{I}_Z(x)$. For this purpose we need to prove that $\mathcal{I}_Z(\cdot)$ is lower semicontinuous.

Remark 6.25. For every $f \in H^1_0[0, T]$, we consider the functional
\[
\mathcal{F}(x(f, \dot{f})) = \begin{cases} \frac{1}{2} \int_0^T \left( \frac{\dot{x}(t) - \mu(\dot{f}(t)) - \rho \hat{\psi}(f, \dot{f})(t)}{\rho \sigma(\dot{f}(t))} \right)^2 dt & x \in H^1_0[0, T] \\ \|x\|_{H^1_0[0, T]} & x \notin H^1_0[0, T]. \end{cases}
\] (38)

It is easy to verify that it is the good rate function of the family $((Z^n_t(f, g))_{t \in [0, T]})_{n \in \mathbb{N}}$, where
\[
Z^n_t(f, g) = X^n_t + \rho \hat{\psi}(f, g)(t) \quad 0 \leq t \leq T,
\] (39)

therefore it is lower semicontinuous.
Remark 6.26. For $L > 0$, denote by $B_L$ the level sets in the space $H(B, \hat{B})$, i.e.

$$B_L = \{(f, g) \in H(B, \hat{B}) : \|f\|^2_{H^1_{\text{Lip}}[0, T]} \leq L\}. \tag{40}$$

$B_L$ is a compact set in $C_0[0, T]^2$ since $I_{(B, \hat{B})}(\cdot, \cdot)$ defined in (33) is a good rate function.

Lemma 6.27. The function $\Psi : C_0[0, T]^2 \to C_0[0, T]$ defined in (55) is continuous on the set $B_L$ defined in (40), for every $L > 0$.

Proof. Easily follows from Lemma 6.22 and the continuity of $\Psi_m$ (for every $m \geq 1$). □

In the next lemma we will prove that $J(\cdot|\cdot, \cdot)$ is lower semicontinuous as a function of $((f, g, x) \in B_L \times C_0[0, T]$.

Lemma 6.28. Let $((f_n, \hat{f}_n), x_n) \in B_L \times C_0[0, T]$ be a sequence of functions such that $((f_n, \hat{f}_n), x_n) \to ((f, g), x)$ in $C_0[0, T]^2 \times C_0[0, T]$. Then one has,

$$\liminf_{n \to +\infty} J(x_n|f_n, \hat{f}_n) \geq J(x|f, g)$$

where $J(\cdot|\cdot, \cdot)$ is defined in (35).

Proof. If $\liminf_{n \to +\infty} J(x_n|(f_n, \hat{f}_n)) = +\infty$, there is nothing to prove. Therefore we can suppose that $(x_n)_{n \in \mathbb{N}} \subset H^1_0[0, T]$. Since $(f_n, \hat{f}_n) \to (f, \hat{f})$ in $C_0[0, T]^2$ we have that $(f, \hat{f}) \in B_L$, i.e. $g = f$. Furthermore it is easy to show that

$$J(x_n|f_n, \hat{f}_n) \geq \inf_{t \in [0, T]} \left( \frac{\sigma(f(t))}{\sigma(f_n(t))} \right)^2 J(x_n + \rho(\Psi(f, \hat{f})(\cdot) - \Psi(f_n, \hat{f}_n)(\cdot)) + \int_0^T \mu(f(s)) - \mu(f_n(s)))ds \right) \bigg|_{f_n = f, \hat{f}_n = \hat{f}}.$$

Now

$$x_n + \rho(\Psi(f, \hat{f})(\cdot) - \Psi(f_n, \hat{f}_n)(\cdot)) + \int_0^T \mu(f(s)) - \mu(f_n(s)))ds \stackrel{C_0[0, T]}{\to} x,$$

since $x_n \to x$, $f_n \to f$ in $C_0[0, T]^2$ from Lemma 6.27 and $\mu \circ \hat{f} \to \mu \circ \hat{f}$ from Remark 5.1 (which implies $\int_0^T \mu(f_n(s))ds \to \int_0^T \mu(f(s))ds$). The claim follows from semicontinuity of $J(\cdot|f, \hat{f})$ (see Remark 5.21) and from the uniform convergence of $\sigma \circ f_n \to \sigma \circ f$. □

Lemma 6.29. The function $I_Z(\cdot)$, defined in (37) is lower semicontinuous.

Proof. It is enough to show that, for $L > 0$, the set

$$M = \{x \in C_0[0, T] : I_Z(x) \leq L\}$$

is closed. Let $(x_n)_{n \in \mathbb{N}} \subset M$ be a converging sequence of functions, $x_n \to x$ in $C_0[0, T]$. Thanks to the definition of $I_Z$ we can choose a sequence $(f_n)_{n \in \mathbb{N}} \subset H^1_0[0, T]$ such that, for every $n \in \mathbb{N}$,

$$\frac{1}{2}\|f_n\|^2_{H^1_{\text{Lip}}[0, T]} + J(x_n|f_n, \hat{f}_n) \leq I_Z(x_n) + \frac{1}{n} \leq L + \frac{1}{n} \leq (L + 1). \tag{41}$$

Therefore $(f_n, \hat{f}_n)_{n \in \mathbb{N}} \subset B_{2(L+1)}$, where $B_{2(L+1)}$ is the compact set of $C_0[0, T]^2$ defined in (40) and, up to a subsequence, we can suppose that

$$(f_n, \hat{f}_n) \to (f, \hat{f}) \in B_{2(L+1)}.$$
Now, $\mathcal{J}((\cdot, \cdot))$ is lower semicontinuous on $B_{2(L+1)} \times C_0[0, T]$ from Lemma 6.28. Then from inequality (31) and the lower semicontinuity of the norm, we have

\[ \mathcal{I}_Z(x) \leq \frac{1}{2} \|f\|_{H^2_0[0,T]}^2 + \mathcal{J}(x((f, \hat{f})) \leq \liminf_{n \to +\infty} \left( \frac{1}{2} \|f_n\|_{H^2_0[0,T]}^2 + \mathcal{J}(x_n((f_n, \hat{f}_n))) \right) \leq L. \]

Thus $x \in M$, and $M$ is a closed subset of $C_0[0,T]$. \hfill \Box

Now, we are ready prove the final lemma of this section.

**Lemma 6.30.** If $x_m \to x$ in $C_0[0,T]$, then

\[ \liminf_{n \to +\infty} I^m_Z(x_m) \geq \mathcal{I}_Z(x) \]

where $I^m_Z(\cdot)$ and $\mathcal{I}_Z(\cdot)$ are defined, respectively, in (29) and (37).

**Proof.** Suppose $x_m \to x$ in $C_0[0,T]$. If there exist $m_0 > 0$ such that $(x_m)_{m \geq m_0} \subset C_0[0,T] \setminus H^1_0[0,T]$, then $\lim_{m \to +\infty} I^m_Z(x_m) = +\infty$ and there is nothing to prove.

Otherwise, we can suppose that $(x_m)_{m \in \mathbb{N}} \subset H^1_0[0,T]$. Now, there are two possibilities:

(i) $\sup_{m \geq 1} \|x_m\|_{H^1_0[0,T]}^2 < +\infty$;

(ii) $\sup_{m \geq 1} \|x_m\|_{H^1_0[0,T]}^2 = +\infty$.

(i) The sequence $(x_m)_{m \in \mathbb{N}}$ is bounded in $H^1_0[0,T]$, therefore there exist a constant $C > 0$ (depending on $\sup_{m \geq 1} \|x_m\|_{H^1_0[0,T]}$), such that

\[ I^m_Z(x_m) = \inf_{f \in D_C} H_m((f, \hat{f}), x_m), \quad \mathcal{I}_Z(x_m) = \inf_{f \in D_C} H((f, \hat{f}), x_m). \]

Then we have,

\[ |I^m_Z(x_m) - \mathcal{I}_Z(x_m)| = \begin{cases} \inf_{f \in D_C} H_m((f, \hat{f}), x_m) - \inf_{f \in D_C} H((f, \hat{f}), x_m) \\ \sup_{f \in D_C} |H_m((f, \hat{f}), x_m) - H((f, \hat{f}), x_m)| \end{cases} \leq \lim_{m \to +\infty} [I^m_Z(x_m) - \mathcal{I}_Z(x_m)] = 0. \]

Using similar computations as in Lemma 6.29, it follows that

\[ \lim_{m \to +\infty} [I^m_Z(x_m) - \mathcal{I}_Z(x_m)] = 0. \]

Moreover, since $\mathcal{I}_Z(\cdot)$ is lower semicontinuous from Lemma 6.29, we have

\[ \liminf_{m \to +\infty} I^m_Z(x_m) = \liminf_{m \to +\infty} [I^m_Z(x_m) - \mathcal{I}_Z(x_m)] + \mathcal{I}_Z(x_m) \geq \liminf_{m \to +\infty} \mathcal{I}_Z(x_m) \geq \mathcal{I}_Z(x). \]

(ii) The sequence $(x_m)_{m \in \mathbb{N}}$ is not bounded in $H^1_0[0,T]$, therefore we can suppose that $\lim_{m \to +\infty} \|x_m\|_{H^1_0[0,T]} = +\infty$. We will prove, in this case, that

\[ \lim_{m \to +\infty} I^m_Z(x_m) = +\infty. \] (42)

For every $u > 0$ we have,

\[ I^m_Z(x_m) = \min \left\{ \inf_{\|f\|_{H^1_0[0,T]}^2 \leq u} H_m((f, \hat{f}), x_m), \inf_{\|f\|_{H^1_0[0,T]}^2 > u} H_m((f, \hat{f}), x_m) \right\} \geq \min \left\{ \inf_{\|f\|_{H^1_0[0,T]}^2 \leq u} \mathcal{J}^m((f, \hat{f})), \inf_{\|f\|_{H^1_0[0,T]}^2 > u} I((B, B)(f, \hat{f})) \right\} \]

(43)

25
where \( I(B, \hat{\theta}) \) and \( J^m(\cdot|(f, \hat{f})) \) are defined, respectively, in (6) and (26). Now we consider the two infima in (43). For the second one we have,

\[
I(B, \hat{\theta})(f, \hat{f}) = \inf_{\|f\|^2_{H^2_0[0,T]} > \|x_m\|^2_{H^2_0[0,T]}} \inf_{\|f\|^2_{H^2_0[0,T]} > \|x_m\|^2_{H^2_0[0,T]}} \frac{1}{2} \|f\|^2_{H^2_0[0,T]} \geq \frac{1}{2} \|x_m\|^2_{H^2_0[0,T]}. \tag{44}
\]

From Assumption 6.3 and the Cauchy-Schwarz inequality, we have

\[
J^m(x_m|(f, \hat{f})) \geq \frac{1}{2 \rho^2 M^2 \|f\|_{H^2_0[0,T]}^2} \left( \|x_m\|^2_{H^2_0[0,T]} - 2 \int_0^T \dot{x}(t) + \rho \hat{\Psi}_{m}(f, \hat{f})(t) \right) dt
\]

\[
\geq \frac{1}{2 \rho^2 M^2 \|f\|_{H^2_0[0,T]}^2} \left( \|x_m\|^2_{H^2_0[0,T]} - 2 M \sqrt{T} \|f\|_{H^2_0[0,T]} \|x_m\|_{H^2_0[0,T]} - 2 \rho M \|f\|_{H^2_0[0,T]} \|x_m\|_{H^2_0[0,T]} \right)
\]

Now we can choose \( u > 0 \) such that \((\alpha + 1)u < 1\), therefore, since \( \|f\|_{H^2_0[0,T]} \leq \|x_m\|_{H^2_0[0,T]} \), for large \( m \) and a suitable \( c > 0 \), one has

\[
J^m(x_m|(f, \hat{f})) \geq \frac{1}{2 \rho^2 M^2 \|f\|_{H^2_0[0,T]}^2} \left( \|x_m\|^2_{H^2_0[0,T]} - 2 M \sqrt{T} \|x_m\|_{H^2_0[0,T]}^{\alpha + 1} - 2 \rho M \|x_m\|_{H^2_0[0,T]}^{\alpha + 1} \right)
\]

\[
\geq c \|x_m\|_{H^2_0[0,T]}^{2(1-\alpha u)}.
\tag{45}
\]

So (42) follows from (43), (44), (45) and then the proof is complete. \( \square \)

We are ready to identify the rate function \( I_Z(\cdot) \) with \( I_Z(\cdot) \).

**Theorem 6.31.** Let \( I_Z(\cdot) \) be the good rate function given by Theorem 6.13. Then,

\[
I_Z(x) = I_Z(x)
\]

for every \( x \in C_0[0,T] \), where \( I_Z(\cdot) \) is given by (37).

**Proof.** From Lemma 6.24 and Lemma 6.30 the hypotheses of Proposition 6.16 are fulfilled. This proves the theorem. \( \square \)

**Remark 6.32.** [One dimensional case] Consider the function \( F : C_0[0,T] \to \mathbb{R} \) defined by \( F(x) = x_T \). \( F \) is a continuous function. By the contraction principle follows that the family of random variables \( (Z^n_T - x_0)_{n \in \mathbb{N}} \) (defined in (20)) satisfies a LDP on \( (\mathbb{R}, \mathcal{B}(\mathbb{R})) \), with the speed \( \varepsilon_n^{-2} \) and the good rate function \( I_{Z_T}(\cdot) \) given by

\[
I_{Z_T}(y) = \inf_{x \in H^0_0[0,T]; F(x) = y} \inf_{f \in H^2_0[0,T]} \left( \frac{1}{2} \|f\|^2_{H^2_0[0,T]} + \frac{1}{2} \int_0^T \left( \frac{\dot{f}(t) - \mu(\hat{f}(t))}{\rho \sigma(\hat{f}(t))} \right)^2 dt \right)
\]

\[
= \inf_{f \in H^2_0[0,T]} \inf_{x \in H^0_0[0,T]; F(x) = y} \left( \frac{1}{2} \|f\|^2_{H^2_0[0,T]} + \frac{1}{2} \int_0^T \left( \frac{\dot{f}(t) - \mu(\hat{f}(t))}{\rho \sigma(\hat{f}(t))} \right)^2 dt \right)
\]

\[
= \inf_{f \in H^2_0[0,T]} \inf_{x \in H^0_0[0,T]; F(x) = y} \left( \frac{1}{2} \|f\|^2_{H^2_0[0,T]} + J(x)|(f, \hat{f}) \right).
\]

So we have to calculate,

\[
\inf_{x \in H^0_0[0,T]; F(x) = y} \mathcal{J}(x)|(f, \hat{f}) = \inf_{x \in H^0_0[0,T]; F(x) = y} \frac{1}{2} \int_0^T \left( \frac{\dot{f}(t) - \mu(\hat{f}(t))}{\rho \sigma(\hat{f}(t))} \right)^2 dt. \tag{46}
\]
Recall that $\mathcal{J}(\cdot | (f, \hat{f}))$ is the rate function of the family $((Z_{t}^{n,(f,\hat{f})})_{t \in [0,T]})_{n \in \mathbb{N}}$ ($f \in H_{\beta}^{1}[0, T]$) defined in \cite{[33]}. This is a family of Gaussian diffusion processes, then we can apply the known results for Gaussian processes shown in Section 2. For every $n \in \mathbb{N}$, $((Z_{t}^{n,(f,\hat{f})})_{t \in [0,T]})_{n \in \mathbb{N}}$, is a Gaussian process with mean function,

$$m_{n,f}(t) = \int_{0}^{t} \left( \mu(\hat{f}(s)) - \rho(\hat{f}, \hat{f})(s) \right) ds, \quad t \in [0,T]$$

and covariance function,

$$k_{n,f}(t, s) = \text{Cov}(Z_{t}^{n,(f,\hat{f})}, Z_{s}^{n,(f,\hat{f})}) = \beta^{2} \beta_{n} \int_{0}^{\infty} \sigma(\hat{f}(u))^{2} du = \beta^{2} \beta_{n} k_{f}(t, s), \quad s, t \in [0,T].$$

It is not hard to prove that $((Z_{t}^{n,(f,\hat{f})})_{t \in [0,T]})_{n \in \mathbb{N}}$ satisfies (also) hypotheses of Theorem 2.10 and then the family $((Z_{t}^{n,(f,\hat{f})})_{t \in [0,T]})_{n \in \mathbb{N}}$ satisfies a LDP on $C_{0}[0, T]$ with the inverse speed $\beta_{n}^{2}$ and the good rate function $\mathcal{J}(\cdot | (f, \hat{f}))$ given by,

$$\mathcal{J}(x | (f, \hat{f})) = \begin{cases} \frac{1}{2} \| x - \int_{0}^{\infty} \left( \mu(\hat{f}(t)) + \rho(\hat{f}, \hat{f})(t) \right) dt \|_{\mathcal{H}^{f}}^{2} & \text{if } x \in \mathcal{H}^{f} \subset \mathcal{M}[0,T] \\ +\infty & \text{otherwise} \end{cases}$$

where $\mathcal{H}^{f}$ and $\| \cdot \|_{\mathcal{H}^{f}}$ denote, respectively, the reproducing kernel Hilbert space and the related norm associated to the covariance function $k_{f}$. The set of paths

$$y(u) = \int_{0}^{T} k_{f}(u, v) d\lambda(v), \quad u \in [0,T], \lambda \in \mathcal{M}[0,T],$$

is dense in $\mathcal{H}^{f}$. Therefore in the infimum \cite{[40]} we can consider the functions

$$x(u) - \int_{0}^{u} \left( \mu(\hat{f}(t)) + \rho(\hat{f}, \hat{f})(t) \right) dt = \int_{0}^{T} k_{f}(u, v) d\lambda(v), \quad u \in [0,T],$$

for some $\lambda \in \mathcal{M}[0,T]$, with the additional constraint that $x(T) = y$. Therefore we have to minimize the functional

$$\frac{1}{2} \int_{0}^{T} \int_{0}^{T} k_{f}(u, v) d\lambda(u) d\lambda(v),$$

(with respect to the measure $\lambda$) with the additional constraint

$$\int_{0}^{T} k_{f}(T, v) d\lambda(v) + \int_{0}^{T} \left( \mu(\hat{f}(t)) + \rho(\hat{f}, \hat{f})(t) \right) dt - y = 0.$$

By using the method of Lagrange multipliers the measure $\lambda$ must satisfy

$$\int_{0}^{T} \int_{0}^{T} k_{f}(u, v) d\lambda(u) d\eta(v) = \beta \int_{0}^{T} k_{f}(T, v) d\eta(v),$$

i.e.

$$\int_{0}^{T} \left( \int_{0}^{T} k_{f}(u, v) d\lambda(u) - \beta k_{f}(T, v) \right) d\eta(v) = 0$$

for every $\eta \in \mathcal{M}[0,T]$, for some $\beta \in \mathbb{R}$. Since \( \left( v \mapsto \int_{0}^{T} k_{f}(u, v) d\lambda(u) - \beta k_{f}(T, v) \right) \) is a continuous function, it must be

$$\int_{0}^{T} k_{f}(u, v) d\lambda(u) - \beta k_{f}(T, v) = 0, \quad (47)$$

27
for all \( v \in [0, T] \). Therefore the solution is
\[
\bar{\lambda} = \beta \delta_{\{T\}},
\]
\( \delta_{\{T\}} \) standing for the Dirac mass in \( T \). From equality (17) with \( v = T \), we find
\[
\beta = \frac{\int_0^T k^j(u, T)d\lambda(u)}{k^j(T, T)} = \frac{\int_0^T (\mu(\hat{f}(t)) + \rho \hat{\Psi}(f, \hat{f})(t)) dt}{k^j(T, T)}.
\]
Then
\[
\bar{\lambda} = \frac{y - \int_0^T (\mu(\hat{f}(t)) + \rho \hat{\Psi}(f, \hat{f})(t)) dt}{k^j(T, T)} \delta_{\{T\}},
\]
satisfies the Lagrange multipliers problem, and it is therefore a critique point for the functional we want to minimize. Since it is a strictly convex functional restricted on a linear subspace of \( \mathcal{M}[0, T] \), it is still strictly convex, and thus the critique point \( \bar{\lambda} \) is actually its unique point of minimum. Hence, we have
\[
I_{Z_T}(y) = \inf_{f \in H^1_0[0, T]} \left\{ \frac{1}{2} \|f\|^2_{H^1_0[0, T]} + \frac{1}{2} \left( x - \int_0^T (\mu(\hat{f}(t)) + \rho \hat{\Psi}(f, \hat{f})(t)) dt \right)^2 \right\}.
\]
The same result as in (17).

### 6.4 Asymptotic estimate for the crossing probability

Here we assume the following dynamics for the asset price process
\[
\begin{cases}
    dS_t = S_t \sigma(B_t) d(\rho W_t + \rho B_t), & 0 \leq t \leq T \\
    S_0 = 1
\end{cases}
\]
where the model has been normalized to have \( S_0 = 1 \) and \( \mu = 0 \). Then, the unique solution to the previous equation is given by
\[
S_t = \exp \left\{ -\frac{1}{2} \int_0^t \sigma(B_s)^2 ds + \tilde{\rho} \int_0^t \sigma(B_s) dW_s + \rho \int_0^t \sigma(B_s) dB_s \right\}
\]
for \( 0 \leq t \leq T \). We observe that the process \((S_t)_{t \in [0, T]} \) is a strictly positive local martingale, and hence a supermartingale. If we suppose that \( \sigma \) has sub-linear growth, then \((S_t)_{t \in [0, T]} \) is a martingale (for further details, see Lemma 9 in (17)). Then, in such a case, \( P \) is a risk-neutral measure.

Let us consider the case of an up-in bond, i.e. an option that pays one unit of numéraire if the underlying asset reached a given up-barrier \( U \) after \( T \). This is a path-dependent option whose pay-off is
\[
h^u_{\text{up}} = 1_{\{\sup_{t \in [0, T]} S_t \geq U \}} = 1_{\{\tau^U \leq T \}}, \quad \tau^U = \inf\{t \in [0, T] : S_t \geq U \}.
\]

Then, the up-in bond pricing functions in \( t = 0 \) is defined by
\[
E[1_{\{\sup_{t \in [0, T]} S_t \geq U \}}] = P(\sup_{t \in [0, T]} S_t \geq U) = P(\tau^U \leq T).
\]
As an application of the results of the previous section we obtain the asymptotic behavior of the small-noise up-in bond pricing function, that is

\[ P_n = \mathbb{E}\left[ \mathbf{1}_{\left\{ \sup_{t\in[0,T]} S^n_t \geq U \right\}} \right] = \mathbb{P}(\tau_n^U \leq T) \]

where \((S^n_t)_{t\in[0,T]}\) is the asset price process in the scaled model

\[
\begin{align*}
    dS^n_t &= \varepsilon_n S^n_t \sigma(\varepsilon_n \dot{B}_t) d(\bar{p}W_t + \rho B_t) \\
    S_0 &= 1
\end{align*}
\]

and

\[ \tau_n^U = \inf\{ t \in [0,T] : S^n_t \geq U \}. \]

This problem is nothing but the asymptotic estimate of level crossing for the family \((S^n_t)_{t\in[0,T]}\) for some quantity \(U > 0\). We have that

\[ \lim_{n\to+\infty} \frac{\varepsilon_n^2}{n} \log(P_n) = -I_U \tag{48} \]

for some quantity \(I_U > 0\). We have that

\[ \{ \tau_n^U \leq T \} = \left\{ \sup_{t\in[0,T]} S^n_t \geq U \right\} = \left\{ \sup_{t\in[0,T]} Z^n_t - \log U \geq 0 \right\} \]

where \((Z^n_t)_{t\in[0,T]}\) is the family of the log-price processes defined by

\[ Z^n_t = \left\{ -\frac{1}{2} \varepsilon_n^2 \int_0^t \sigma(\varepsilon_n \dot{B}_s)^2 ds + \varepsilon_n \rho \int_0^t \sigma(\varepsilon_n \dot{B}_s) dW_s + \varepsilon_n \rho \int_0^t \sigma(\varepsilon_n \dot{B}_s) dB_s \right\}. \]

We have already shown in Theorems 6.19 and 6.31 that \((Z^n_t)_{t\in[0,T]}\) satisfies a LDP with the speed \(\varepsilon_n^{-2}\) and the good rate function

\[ I_Z(x) = \left\{ \begin{array}{ll}
    \inf_{f \in H^0_1([0,T])} \mathcal{H}((f, \dot{f}), x) & x \in H^0_1([0,T]) \\
    +\infty & x \notin H^0_1([0,T])
\end{array} \right. \]

Then, we have

\[ -\inf_{x \in A} I_Z(x) \leq \liminf_{n \to +\infty} \frac{\varepsilon_n^2}{n} \log(P_n) \leq \limsup_{n \to +\infty} \frac{\varepsilon_n^2}{n} \log(P_n) \leq -\inf_{x \in \tilde{A}} I_Z(x) \]

where

\[ A = \tilde{A} = \left\{ x \in C_0([0,T]) : \sup_{t\in[0,T]} x(t) - \log U \geq 0 \right\}, \quad \tilde{A} = \left\{ x \in C_0([0,T]) : \sup_{t\in[0,T]} x(t) - \log U > 0 \right\}. \]

It is a simple calculation to show that

\[ \inf_{x \in A} I_Z(x) = \inf_{x \in \tilde{A}} I_Z(x), \]

therefore

\[ \lim_{n \to +\infty} \frac{\varepsilon_n^2}{n} \log(P_n) = -\inf_{x \in A} I_Z(x) = -I_U. \]

In what follows we will compute the quantity \(I_U\). We have to minimize the rate function (as in Remark 6.32). Define

\[ A_t = \{ x \in C_0[0,T] : x(t) - \log U = 0 \}, \]
then,
\[ A = \bigcup_{0 \leq t \leq T} A_t. \]
Therefore, with the same notations as in Remark 6.32,
\[
\inf_{x \in A} I_Z(x) = \inf_{t \in [0,T]} \inf_{x \in A_t} I_Z(x)
\]
\[
= \inf_{t \in [0,T]} \inf_{x \in A_t} \left\{ I(\hat{B}, \hat{\delta}) (f, \hat{f}) + J(x(f, \hat{f})) \right\}
\]
\[
= \inf_{f \in H^1_{\text{loc}}(0,T)} \inf_{t \in [0,T]} \inf_{x \in A_t} \left\{ \frac{1}{2} \| f \|^2_{H^1_{\text{loc}}(0,T)} + \frac{1}{2} \| x - \rho \Psi(f, \hat{f}) \|^2_{L^2} \right\}.
\]
The set set of paths
\[ y(u) = \int_0^T k^f(u,v) d\lambda(v) \quad u \in [0,T], \lambda \in \mathcal{M}[0,T] \]
is dense in \( \mathcal{H}^f \). Therefore in the infimum
\[
\inf_{x \in A_t} \left\{ \frac{1}{2} \| f \|^2_{H^1_{\text{loc}}(0,T)} + \frac{1}{2} \| x - \rho \Psi(f, \hat{f}) \|^2_{L^2} \right\}
\]
we can consider the functions
\[ x(u) = \rho \Psi(f, \hat{f})(u) + \int_0^T k^f(u,v) d\lambda(v), \quad u \in [0,T], \]
for some \( \lambda \in \mathcal{M}[0,T] \), with the additional constraint that \( x(t) = \log U \). The solution (calculations are the same as in Remark 6.32) is
\[ \hat{\lambda} = \frac{\log U - \rho \Psi(f, \hat{f})(t)}{k^f(t,t)} \delta(t), \]
and then
\[ I_U = \inf_{x \in A_t} I_Z(x) = \inf_{f \in H^1_{\text{loc}}(0,T)} \inf_{t \in [0,T]} \left\{ \frac{1}{2} \| f \|^2_{H^1_{\text{loc}}(0,T)} + \frac{1}{2} \left( \frac{\log U - \rho \Psi(f, \hat{f})(t)}{k^f(t,t)} \right)^2 \right\}
\]
\[
= \inf_{f \in H^1_{\text{loc}}(0,T)} \inf_{t \in [0,T]} \left( \frac{1}{2} \| f \|^2_{H^1_{\text{loc}}(0,T)} + \frac{1}{2} \left( \frac{\log U - \int_0^T \rho \Psi(f, \hat{f})(u) du}{\int_0^T \rho^2 \sigma^2(f(u)) du} \right)^2 \right)
\]

Remark 6.33. In this way we have established a large deviation estimation of the probability that the asset price process \( (S_t)_{t \in [0,T]} \) crosses the upper barrier \( U \). The same arguments can be applied if we consider a lower or a double barrier.

7 More general Volterra processes

In this section we extend the results obtained for the log-price process \( Z^n \) when \( \hat{B}^n = \varepsilon_n \hat{B} \) to a more general context in which \( (\hat{B}^n)_{n \in \mathbb{N}} \) is a family of Volterra processes which satisfies a LDP.

7.1 Uncorrelated case

Notice that we never used the fact that \( \hat{B}^n = \varepsilon_n \hat{B} \), therefore the same results can be easily deduced for more general families of Volterra processes. We consider a family of continuous Volterra type Gaussian processes (see Definition 2.8) of the form
\[ \hat{B}^n_t = \int_0^t K^n(t,s) dB_s, \quad 0 \leq t \leq T, \tag{49} \]
where $K^n$ is a suitable kernel. The covariance function of the process $\hat{B}^n$, for every $n \in \mathbb{N}$, is given by

$$
k^n(t, s) = \int_0^{t \wedge s} K^n(t, u)K^n(s, u) \, du \quad \text{for } t, s \in [0, T].
$$

Under suitable conditions on the covariance functions or on the kernels, the hypotheses of Theorem 2.6 are satisfied and a LDP holds.

**Assumption 7.1.** [Assumptions on the covariance]

(a) There exist an infinitesimal function $\varepsilon_n$ and an asymptotic covariance function $k$ (regular enough to be the covariance function of a continuous centered Gaussian process) such that

$$
k(t, s) = \lim_{n \to +\infty} \frac{k^n(t, s)}{\varepsilon_n^2}
$$

uniformly for $t, s \in [0, T]$.

(b) There exist constants $\beta, M > 0$, such that, for every $n \in \mathbb{N}$

$$
\sup_{s, t \in [0, T], s \neq t} \frac{|k^n(t, t) + k^n(s, s) - 2k^n(t, s)|}{\varepsilon_n^2|t - s|^{2\beta}} \leq M.
$$

**Assumption 7.2.** [Assumptions on the kernel]

(a) There exist an infinitesimal function $\varepsilon_n$ and a kernel $K$ (regular enough to be the kernel of a continuous Volterra process) such that

$$
\lim_{n \to +\infty} \frac{K^n(t, s)}{\varepsilon_n} = K(t, s)
$$

uniformly for $t, s \in [0, T]$.

(b) There exist constants $C, \beta > 0$ such that one has

$$
\frac{1}{\varepsilon_n^2} \int_0^T (K^n(t, u) - K^n(s, u))^2 \, du \leq C|t - s|^{2\beta}.
$$

Assumptions (a) guarantees that (2) holds (see, for example, [15]). Assumptions (b) guarantees exponential tightness of the family (see, for example, [23]). The following theorem holds.

**Theorem 7.3.** Let $n \in \mathbb{N}$, and $\hat{B}^n$ be a Volterra process as in (49). Suppose the family $(K^n)_{n \in \mathbb{N}}$ (respectively $(k^n)_{n \in \mathbb{N}}$) satisfies Assumption 7.2 (respectively Assumption 7.1). Then, the family of Volterra processes $((\hat{B}^n)_{t \in [0, T]})_{n \in \mathbb{N}}$ satisfies a large deviation principle on $C_0[0, T]$, with the inverse speed $\varepsilon_n^2$ and the good rate function

$$
I_{\hat{B}}(f) = \begin{cases} 
\frac{1}{2} ||f||^2_{\mathcal{H}_{\hat{B}}} & f \in \mathcal{H}_{\hat{B}} \\
+\infty & f \notin \mathcal{H}_{\hat{B}}
\end{cases}
$$

where $\mathcal{H}_{\hat{B}}$ and $||.||_{\mathcal{H}_{\hat{B}}}$ denote, respectively, the reproducing kernel Hilbert space and the related norm associated to the covariance function

$$
k(t, s) = \int_0^{t \wedge s} K(t, u)K(s, u) \, du \quad \text{for } t, s \in [0, T].
$$

In this case under hypotheses of Theorem 5.8 we have (from Theorem 3.3) that a large deviation principle with the speed $\varepsilon_n^{-2}$ and the good rate function

$$
I_X(x) = \begin{cases} 
\inf_{\varphi \in \mathcal{H}_{\hat{B}}} \left[ \frac{1}{2} ||\varphi||^2_{\mathcal{H}_{\hat{B}}} + \frac{1}{2} \int_0^T \left( \frac{\dot{\varphi}(t) - \mu(\varphi(t))}{\sigma(\varphi(t))} \right)^2 dt \right] & x \in \mathcal{H}_{\hat{B}} \\
+\infty & x \notin \mathcal{H}_{\hat{B}}
\end{cases}
$$

31
holds for the family \((X^n - x_0)_{n \in \mathbb{N}}\), where for every \(n \in \mathbb{N}\), \((X^n_t)_{t \in [0,T]}\) is defined from (10).

From Remark 2.13 if \(\hat{f}(t) = \int_0^t K(t,s) f(s) ds\), we have

\[
I_X(x) = \begin{cases}
\inf_{f \in H^1_0[0,T]} \left[ \frac{1}{2}\|f\|_{H^1_0[0,T]}^2 + \frac{1}{2} \int_0^T \left( \frac{\dot{x}(t) - \mu(f(t))}{\sigma(f(t))} \right)^2 dt \right] & x \in H^1_0[0,T] \\
+\infty & x \notin H^1_0[0,T]
\end{cases}
\]

7.2 Correlated case

Now we state a large deviation principle for the couple \((\varepsilon_n B, \hat{B}^n)_{n \in \mathbb{N}}\). First observe that \((\varepsilon_n B, \hat{B}^n)\) is a Gaussian process (for details see for example [13]) and therefore the following theorem is an application of Theorem 3.4.5 in [12].

**Theorem 7.4.** Let \(\varepsilon : \mathbb{N} \to \mathbb{R}_+\) be an infinitesimal function. Suppose Assumption (7.2) is fulfilled. Then \((\varepsilon_n B, \hat{B}^n)_{n \in \mathbb{N}}\) satisfies a large deviation principle on \(C_0[0,T]^2\) with the speed \(\varepsilon_n^{-2}\) and the good rate function

\[
I_{(B,\hat{B})}(f,g) = \begin{cases}
\frac{1}{2} \int_0^T \dot{f}(s)^2 ds & (f,g) \in \mathcal{H}_{(B,\hat{B})} \\
+\infty & (f,g) \notin C_0[0,T]^2 \setminus \mathcal{H}_{(B,\hat{B})}
\end{cases}
\]

where

\[
\mathcal{H}_{(B,\hat{B})} = \{(f,g) \in C_0[0,T]^2 : f \in H^1_0[0,T], g(t) = \int_0^t K(t,u) \dot{f}(u) du, 0 \leq t \leq T\}
\]

and \(K\) is defined from equation (17).

Notice that we used the fact that \(\hat{B}^n = \varepsilon_n \hat{B}\) only in the proof of the Proposition 6.18; in particular in proving (34) it has been used that

\[
\lim_{m \to +\infty} \lim_{n \to +\infty} \varepsilon_n^2 \log \mathbb{P} \left( \sup_{0 \leq t \leq T} \left| \hat{B}^n_t - \hat{B}_t \right| > \delta \right) = -\infty.
\]

This result is contained in Lemma 24 in [17]. The same results can be deduced for exponential tight families of Volterra processes.

**Lemma 7.5.** If \((\hat{B}^n)_{n \in \mathbb{N}}\) is an exponentially tight family at the inverse speed \(\varepsilon_n^2\), then for every \(\delta > 0\)

\[
\lim_{m \to +\infty} \lim_{n \to +\infty} \varepsilon_n^2 \log \mathbb{P} \left( \sup_{0 \leq t \leq T} \left| \hat{B}^n_t - \hat{B}_t \right| > \delta \right) = -\infty.
\]

**Proof.** From exponential tightness, for every \(R > 0\), there exists a compact set \(K_R\) (of equi-continuous functions) such that \(\sup_{n \to +\infty} \varepsilon_n^2 \log \mathbb{P} \left( \hat{B}^n \in K_R \right) \leq -R\). Therefore, for every \(\delta > 0\), there exists \(m_0 > 0\), such that for every \(m > m_0\) \lim_{n \to +\infty} \varepsilon_n^2 \log \mathbb{P} \left( \sup_{|s-t| \leq T/m} \left| \hat{B}^n_s - \hat{B}^n_t \right| > \delta \right) \leq -R\).

Since

\[
\lim_{m \to +\infty} \lim_{n \to +\infty} \varepsilon_n^2 \log \mathbb{P} \left( \sup_{0 \leq t \leq T} \left| \hat{B}^n_t - \hat{B}_t \right| > \delta \right) \leq \lim_{m \to +\infty} \lim_{n \to +\infty} \varepsilon_n^2 \log \mathbb{P} \left( \sup_{|s-t| \leq T/m} \left| \hat{B}^n_s - \hat{B}^n_t \right| > \delta \right),
\]

the claim follows. \(\square\)

Therefore Proposition 6.18 holds also in this more general case. In the hypotheses of Theorem 6.19 we have (from Theorem 5.8) that a large deviation principle with the speed \(\varepsilon_n^{-2}\) and the good rate function

\[
I_Z(x) = \begin{cases}
\inf_{(\psi,\varphi) \in \mathcal{H}(\alpha,\beta)} \left[ \frac{1}{2} \left\| (\psi,\varphi) \right\|_{\mathcal{H}(\alpha,\beta)}^2 + \frac{1}{2} \int_0^T \left( \frac{\dot{x}(t) - \mu(\varphi(t))}{\rho(\varphi(t))} - \hat{\rho}(\psi(t),\varphi(t)) \right)^2 dt \right] & x \in H^1_0[0,T] \\
+\infty & x \notin H^1_0[0,T]
\end{cases}
\]

32
holds for the family \((Z^n - x_0)_{n \in \mathbb{N}}\) \((Z^n_t)_{t \in [0,T]}\) is defined from \((\ref{eq:inf})\). If \(\hat{f}(t) = \int_0^t K(t,s) \tilde{f}(s)ds\), from Remark 2.13, we have
\[
I_Z(x) = \begin{cases}
\inf_{f \in H^1_0[0,T]} \left[ \frac{1}{2} \|f\|_{H^2_0[0,T]}^2 + \frac{1}{2} \int_0^T \left( \frac{\dot{x}(t) - \mu(\hat{f}(t)) - \rho \hat{\Psi}(f, \hat{f})(t)}{\bar{\rho} \sigma(\hat{f}(t))} \right)^2 dt \right] & x \in H^1_0[0,T] \\
\infty & x \notin H^1_0[0,T]
\end{cases}
\]

**Example 7.6.** Consider the sequence of processes \((U^n_{t, \epsilon})_{t \in [0,T], \epsilon \in \mathbb{N}} = (U^n_{t, \epsilon})_{t \in [0,T], \epsilon \in \mathbb{N}}\) where \((U^n_{t, \epsilon})_{t \in [0,T], \epsilon \in \mathbb{N}}\) is a fractional Ornstein-Uhlenbeck process. It is not a self similar process, therefore \((U^n_{t, \epsilon})_{t \in [0,T], \epsilon \in \mathbb{N}}\) is not equivalent to a scaled process \((\epsilon^n U^n_{t, \epsilon})_{t \in [0,T], \epsilon \in \mathbb{N}}\). Thanks to representation \((\ref{eq:representation})\), simple calculations show that, if \(k_H\) is the covariance function of a fractional Brownian motion, we have
\[
k_U(s,t) = \text{Cov}(U^n_{t, \epsilon}, U^n_{s, \epsilon}) = k_H(t-s) - a e^{-at} \int_0^t e^{au} k_H(u, s) du - a e^{-as} \int_0^s e^{av} k_H(t, v) dv + a^2 e^{-a(t+s)} \int_0^s \int_0^v e^{au} e^{av} k_H(u, v) dv du.
\]
Therefore the covariance function of the process \(U^n_{t, \epsilon}\) is \(k^n_{U^n_{t, \epsilon}}(t,s) = k_U(\epsilon n, s, \epsilon n, t)\). It is straightforward to prove that Assumption 7.1 is verified for the infinitesimal function \(\epsilon^n_H\) and limit covariance \(k_H\).

We will obtain a sample path large deviation principle for the family of processes \(((Z^n_t - x_0)_{t \in [0,T]})_{n \in \mathbb{N}}\) with the speed function \(\epsilon^{-2H}_n\) and the good rate function given by
\[
I_Z(x) = \begin{cases}
\inf_{f \in H^1_0[0,T]} \left[ \frac{1}{2} \|f\|_{H^2_0[0,T]}^2 + \frac{1}{2} \int_0^T \left( \frac{\dot{x}(t) - \mu(\hat{f}(t)) - \rho \hat{\Psi}(f, \hat{f})(t)}{\bar{\rho} \sigma(\hat{f}(t))} \right)^2 dt \right] & x \in H^1_0[0,T] \\
\infty & x \notin H^1_0[0,T]
\end{cases}
\]
where \(\hat{f}(t) = \int_0^t K_H(t,s) \tilde{f}(t)dt\) \((K_H\) is the kernel of the fractional Brownian motion defined in \((\ref{eq:kernel})\)).

**Acknowledgements.**

The authors wish to thank the Referee for her/his very useful comments which allowed us to improve the paper.

**References**

[1] Azencott R., (1980), *Grande Déviations et applications*, in École d’été de probabilités de St. Flour VIII, L.N.M. Vol 774, Springer, Berlin/Heidelberg/New York.

[2] Baxter J.R., Jain N.C., (1996), *An Approximation Condition for Large Deviations and some Applications*, Convergence in ergodic theory and probability (Columbus, OH, 1993), 5, 63–90.

[3] Berlinet A., Thomas-Agnan C., (2004), *Reproducing Kernel Hilbert Spaces in Probability and Statistics*, Kluwer Academic Publishers.

[4] Borkar S.V.,(1995), *Probability Theory*, Springer.

[5] Caramellino L., Pacchiarotti B., Salvadori S., (2015), *Large Deviation Approaches for the Numerical Computation of the Hitting Probability for Gaussian Processes*, Methodology and Computing in Applied Probability, 17, no. 2, 383-401.

[6] Chaganty N.R., (1997), *Large Deviations for Joint Distributions and Statistical Applications*, Sankhyā: The Indian Journal of Statistics, 59, no. 2, 147-166.
[7] Chen X., Li W.V., (2003), Quadratic functionals and small ball probabilities for the m-fold integrated Brownian motion, Annals of Probability, 31, 1052-1077.

[8] Cheridito P., Kawaguchi H., Maejima M., (2003), Fractional Ornstein-Uhlenbeck processes, Electronic Journal of Probability, 8, 1-14.

[9] Chiarini A., Fischer M., (2014), On large deviations for small noise Itô processes, Advances in Applied Probability, 46, no. 4, 1126-1147.

[10] Decreusefond L., Üstünel A.S., (1999), Stochastic Analysis of the Fractional Brownian Motion, Potential Analysis, 10, no. 2, 177-214.

[11] Dembo A., Zeitouni O., (1998), Large Deviations Techniques and Applications, Jones and Bartlett, Boston, MA.

[12] Deuschel J.D., Stroock D.W., (1989), Large Deviations, Academic Press, Boston, MA.

[13] Forde M. and Zhang H., (2017), Asymptotics for Rough Stochastic Volatility Models, SIAM Journal on Financial Mathematics, 8, 114-145.

[14] Gatheral, J., Jaisson, T., Rosenbaum, M., (2018), Volatility is rough, Quantitative Finance, 18, no. 6, 933-949.

[15] Giorgi, F., Pacchiarotti, B., (2017), Large deviations for conditional Volterra processes, Stochastic Analysis and Applications, 35, no. 2, 191-210.

[16] Gulisashvili A., (2012), Analytically Tractable Stochastic Stock Price Models, Springer-Verlag Berlin Heidelberg.

[17] Gulisashvili A., (2018), Large Deviation Principle for Volterra type Fractional Stochastic Volatility Models, SIAM Journal on Financial Mathematics, 9, no. 3, 1102-1136.

[18] Gulisashvili A., Gaussian stochastic volatility models: Scaling regimes, large deviations, and moment explosions, Stochastic Processes and Their Applications, Available online, 18 October 2019, https://doi.org/10.1016/j.spa.2019.10.005.

[19] Gulisashvili A., Viens F., Zhang X., (2018), Small-Time Asymptotics for Gaussian Self-Similar Stochastic Volatility Models, Applied Mathematics & Optimization, 1-41.

[20] Gulisashvili A., Viens F., Zhang X., (2018), Extreme-strike asymptotics for general Gaussian stochastic volatility models, Annals of Finance, 15, no. 1, 59-101.

[21] Hida T., Hitsuda M., (1993), Gaussian Processes, AMS Translations.

[22] Hult H., (2003), Approximating some Volterra type stochastic integrals with applications to parameter estimation, Stochastic Processes and their Applications, 105, no. 1, 1-32.

[23] Macci C., Pacchiarotti B., (2017), Exponential tightness for Gaussian processes with applications to some sequences of weighted means, Stochastics 89, no. 2, 469-484.

[24] Pacchiarotti B., (2019), Large deviations for generalized conditioned Gaussian processes and their bridges, Probability and Mathematical Statistics, 39, no. 1, 159-181.

[25] Pacchiarotti B., Pigliacelli A., (2018), Large deviations for conditionally Gaussian processes: estimates of level crossing probability, Modern Stochastics: Theory and Applications, 5, no. 4, 483-499.

[26] Revuz D., Yor M., (2004), Continuous Martingales and Brownian Motion, Springer, Berlin.
[27] Sottinen T., Viitasaari L. (2016), *Stochastic analysis of Gaussian processes via Fredholm representation*, International Journal of Stochastic Analysis.

[28] Zhang X., (2008), *Euler schemes and large deviations for stochastic Volterra equations with singular kernels*, Journal of Differential Equations, 244, 2226-2250.