A note on complex-hyperbolic Kleinian groups

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Abstract

Let Γ be a discrete group of isometries acting on the complex hyperbolic $n$-space $\mathbb{H}^n_C$. In this note, we prove that if Γ is convex-cocompact, torsion-free, and the critical exponent $\delta(\Gamma)$ is strictly lesser than 2, then the complex manifold $\mathbb{H}^n_C/\Gamma$ is Stein. We also discuss several related conjectures.

The theory of complex hyperbolic manifolds and complex-hyperbolic Kleinian groups (i.e. discrete holomorphic isometry groups of complex hyperbolic spaces $\mathbb{H}^n_C$) is a rich mixture of Riemannian and complex geometry, topology, dynamics, symplectic geometry and complex analysis. The purpose of this note is to discuss interactions of the theory of complex-hyperbolic Kleinian groups and the function theory of complex-hyperbolic manifolds. Let Γ be a discrete group of isometries acting on the complex-hyperbolic $n$-space, $\mathbb{H}^n_C$, the unit ball $B^n \subset \mathbb{C}^n$ equipped with the Bergmann metric. A fundamental numerical invariant associated with Γ is the critical exponent $\delta(\Gamma)$ of Γ, defined by

$$\delta(\Gamma) = \inf \left\{ s : \sum_{\gamma \in \Gamma} e^{-s d(x, \gamma x)} < \infty \right\},$$

where $x \in \mathbb{H}^n_C$ is any¹ point. The critical exponent measures the rate of exponential growth the Γ-orbit $\Gamma x \subset \mathbb{H}^n_C$; it also equals the Haussdorff dimension of the conical limit set of Γ, see [6] and [7].

Our main result is:

**Theorem 1.** Suppose that $\Gamma \subset \text{Aut}(B^n)$ is a convex-cocompact, torsion-free discrete subgroup satisfying $\delta(\Gamma) < 2$. Then $M_\Gamma = B^n/\Gamma$ is Stein.

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¹$\delta(\Gamma)$ does not depend on the choice of $x \in \mathbb{H}^n_C$. 
The condition on the critical exponent in the above theorem is sharp since, for a complex Fuchsian subgroup \( \Gamma < \text{Aut}(\mathbb{B}^n) \), \( \delta(\Gamma) = 2 \), but the quotient \( M_\Gamma = \mathbb{B}^n / \Gamma \) is non-Stein because the convex core of \( M_\Gamma \) is a complex curve, see Example 4. On the other hand, if \( \Gamma \) is a torsion-free real Fuchsian subgroup or a small deformation of such (see Example 3), then \( \Gamma \) satisfies the condition of the above theorem.

The main ingredients in the proof of Theorem 1 are Proposition 11 and Theorem 15. The condition “convex-cocompact” is only used in Proposition 11, whereas Theorem 15 holds for any torsion-free discrete subgroup \( \Gamma < \text{Aut}(\mathbb{B}^n) \) satisfying \( \delta(\Gamma) < 2 \).

**Conjecture 2.** Theorem 1 holds if we omit the “convex-cocompact” assumption on \( \Gamma \).

In section 4 we discuss other conjectural generalizations of Theorem 1 and supporting results.

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## 1 Preliminaries

In this section, we recall some definitions and basic facts about the \( n \)-dimensional complex hyperbolic space, we refer to [8] for details.

Consider the \( n \)-dimensional complex vector space \( \mathbb{C}^{n+1} \) equipped with the pseudo-hermitian bilinear form

\[
\langle z, w \rangle = -z_0 \bar{w}_0 + \sum_{k=1}^{n} z_k \bar{w}_k
\]

and define the quadratic form \( q(z) \) of signature \((n,1)\) by \( q(z) := \langle z, z \rangle \). Then \( q \) defines the *negative light cone* \( V_- := \{z : q(z) < 0\} \subset \mathbb{C}^{n+1} \). The projection of \( V_- \) in the projectivization of \( \mathbb{C}^{n+1}, \mathbb{P}^n \), is an open ball which we denote by \( \mathbb{B}^n \).

The tangent space \( T_{[z]}\mathbb{P}^n \) is naturally identified with \( z \perp \), the orthogonal complement of \( \mathbb{C}z \) in \( V \), taken with respect to \( \langle \cdot, \cdot \rangle \). If \( z \in V_- \), then the restriction of \( q \) to \( z \perp \) is positive-definite, hence, \( \langle \cdot, \cdot \rangle \) project to a hermitian metric \( h \) (also denoted \( \langle \cdot, \cdot \rangle_h \)) on \( \mathbb{B}^n \). The *complex hyperbolic \( n \)-space* \( \mathbb{H}^n \) is \( \mathbb{B}^n \) equipped with the hermitian metric \( h \). The boundary \( \partial \mathbb{B}^n \) of \( \mathbb{B}^n \) in \( \mathbb{P}^n \) gives a natural compactification of \( \mathbb{B}^n \).

In this note, we usually denote the complex hyperbolic \( n \)-space by \( \mathbb{B}^n \). The real part of the hermitian metric \( h \) defines a Riemannian metric \( g \) on \( \mathbb{B}^n \). The sectional curvature of \( g \) varies between \(-4\) and \(-1\). We denote the distance function on \( \mathbb{B}^n \) by \( d \). The distance function satisfies

\[
\cosh^2(d(0, z)) = (1 - |z|^2)^{-1}.
\]

A real linear subspace \( W \subset \mathbb{C}^{n+1} \) is said to be *totally real* with respect to the form (1) if for any two vectors \( z, w \in W \), \( \langle z, w \rangle \in \mathbb{R} \). Such a subspace is automatically totally real in the usual sense: \( JW \cap W = \{0\} \), where \( J \) is the almost complex structure on \( V \). *(Real)*
geodesics in $\mathbb{B}^n$ are projections of totally real indefinite (with respect to $q$) 2-planes in $\mathbb{C}^{n+1}$ (intersected with $V_-$). For instance, geodesics through the origin $0 \in \mathbb{B}^n$ are Euclidean line segments in $\mathbb{B}^n$. More generally, totally-geodesic real subspaces in $\mathbb{B}^n$ are projections of totally real indefinite subspaces in $\mathbb{C}^{n+1}$ (intersected with $V_-$). They are isometric to the real hyperbolic space $\mathbb{H}^n_\mathbb{R}$ of constant sectional curvature $-1$.

Complex geodesics in $\mathbb{B}^n$ are projections of indefinite complex 2-planes. Complex geodesics are isometric to the unit disk with the hermitian metric

$$\frac{dz d\bar{z}}{(1 - |z|^2)^2},$$

which has constant sectional curvature $-4$. More generally, $k$-dimensional complex hyperbolic subspaces $\mathbb{H}^k_\mathbb{C}$ in $\mathbb{B}^n$ are projections of indefinite complex $(k + 1)$-dimensional subspaces (intersected with $V_-$).

All complete totally-geodesic submanifolds in $\mathbb{B}^n$ are either real or complex hyperbolic subspaces.

The group $U(n, 1) \cong U(q)$ of (complex) automorphisms of the form $q$ projects to the group $\text{Aut}(\mathbb{B}^n) \cong \text{PU}(n, 1)$ of complex (biholomorphic, isometric) automorphisms of $\mathbb{B}^n$. The group $\text{Aut}(\mathbb{B}^n)$ is linear, its matrix representation is given, for instance, by the adjoint representation, which is faithful since $\text{Aut}(\mathbb{B}^n)$ has trivial center.

A discrete subgroup $\Gamma$ of $\text{Aut}(\mathbb{B}^n)$ is called a complex-hyperbolic Kleinian group. The accumulation set of any orbit $\Gamma \sigma$ in $\partial \mathbb{B}^n$ is called the limit set of $\Gamma$ and denoted by $\Lambda(\Gamma)$. The complement of $\Lambda(\Gamma)$ in $\partial \mathbb{B}^n$ is called the domain of discontinuity of $\Gamma$ and denoted by $\Omega(\Gamma)$. The group $\Gamma$ acts properly discontinuously on $\mathbb{B}^n \cup \Omega(\Gamma)$.

For a (torsion-free) complex-hyperbolic Kleinian group $\Gamma$, the quotient $\mathbb{B}^n/\Gamma$ is a Riemannian orbifold (manifold) equipped with push-forward of the Riemannian metric of $\mathbb{B}^n$. We reserve the notation $M_\Gamma$ to denote this quotient. The convex core of $M_\Gamma$ is the the projection of the closed convex hull of $\Lambda(\Gamma)$ in $\mathbb{B}^n$. The subgroup $\Gamma$ is called convex-cocompact if the convex core of $M_\Gamma$ is a nonempty compact subset. Equivalently (see [3]), $M_\Gamma = (\mathbb{B}^n \cup \Omega(\Gamma))/\Gamma$ is compact.

Below are two interesting examples of convex-cocompact complex-hyperbolic Kleinian groups which will also serve as illustrations our results.

**Example 3** (Real Fuchsian subgroups). Let $\mathbb{H}^2_\mathbb{R} \subset \mathbb{B}^n$ be a totally real hyperbolic plane. This inclusion is induced by an embedding $\rho : \text{Isom}(\mathbb{H}^2_\mathbb{R}) = \text{PSL}(2, \mathbb{R}) \to \text{Aut}(\mathbb{B}^n)$ whose image preserves $\mathbb{H}^2_\mathbb{R}$. Let $\Gamma' < \text{Isom}(\mathbb{H}^2_\mathbb{R})$ be a uniform lattice. Then $\Gamma = \rho(\Gamma')$ preserves $\mathbb{H}^2_\mathbb{R}$ and acts on it cocompactly. Such subgroups $\Gamma < \text{Aut}(\mathbb{B}^n)$ will be called real Fuchsian subgroups. The compact surface-orbifold $\Sigma = \mathbb{H}^2_\mathbb{R}/\Gamma$ is the convex core of $M_\Gamma$. The critical exponent $\delta(\Gamma)$ is 1.

Let $\Gamma_t$, $t \geq 0$, be a continuous family of deformations of $\Gamma_0 = \Gamma$ in $\text{Aut}(\mathbb{B}^n)$ such that $\Gamma_t$, for $t > 0$, are convex-cocompact but not real Fuchsian. Such deformation exist as long as $\Gamma_t$ is, say, torsion-free, see e.g. [13]. The groups $\Gamma_t$, $t > 0$, are called real quasi-Fuchsian subgroups. The critical exponents of such subgroups are strictly greater than 1.
Example 4 (Complex Fuchsian subgroups). In the previous example, we replace the totally-real hyperbolic plane $\mathbb{H}^2_\mathbb{R}$ by a complex line $\mathbb{H}^1_\mathbb{C}$ and let $\Gamma$ be a discrete subgroup of $\text{Aut}(\mathbb{B}^n)$ obtained by a similar procedure. Such subgroups $\Gamma$ will be called complex Fuchsian subgroups. In this case, the convex core of $M_\Gamma$, $\Sigma = \mathbb{H}^1_\mathbb{C}/\Gamma$, is also a complex curve in $M_\Gamma$. The critical exponent $\delta(\Gamma)$ is 2.

2 Generalities on complex manifolds

By a complex manifold with boundary $M$, we mean a smooth manifold with (possibly empty) boundary $\partial M$ such that $\text{int}(M)$ is equipped with a complex structure and that there exists a smooth embedding $f : M \to X$ to an equidimensional complex manifold $X$, biholomorphic on $\text{int}(M)$. A holomorphic function on $M$ is a smooth function which admits a holomorphic extension to a neighborhood of $M$ in $X$.

Let $X$ be a complex manifold and $Y \subset X$ is a codimension 0 smooth submanifold with boundary in $X$. The submanifold $Y$ is said to be strictly Levi-convex if every boundary point of $Y$ admits a neighborhood $U$ in $X$ such that the submanifold with boundary $Y \cap U$ can be written as

$$\{ \phi \leq 0 \},$$

for some smooth submersion $\phi : U \to \mathbb{R}$ satisfying $\text{Hess}(\phi) > 0$, where $\text{Hess}(\phi)$ is the holomorphic Hessian:

$$\left( \frac{\partial^2 \phi}{\partial z_i \partial z_j} \right).$$

Definition 5. A strongly pseudoconvex manifold $M$ is a complex manifold with boundary which admits a strictly Levi-convex holomorphic embedding in an equidimensional complex manifold.

Definition 6. An open complex manifold $Z$ is called holomorphically convex if for every discrete closed subset $A \subset Z$ there exists a holomorphic function $Z \to \mathbb{C}$ which is proper on $A$.

Alternatively, one can define holomorphically convex manifolds as follows: For a compact $K$ in a complex manifold $M$, the holomorphic convex hull $\hat{K}_M$ of $K$ in $M$ is

$$\hat{K}_M = \{ z \in M : |f(z)| \leq \sup_{w \in K} |f(w)|, \forall f \in \mathcal{O}_M \}.$$

In the above, $\mathcal{O}_M$ denotes the ring of holomorphic functions on $M$. Then $M$ is holomorphically convex iff for every compact $K \subset M$, the hull $\hat{K}_M$ is also compact.

Theorem 7 (Grauert [9]). The interior of every compact strongly pseudoconvex manifold $M$ is holomorphically convex.

$^2$and this is the standard definition
Definition 8. A complex manifold $M$ is called Stein if it admits a proper holomorphic embedding in $\mathbb{C}^n$ for some $n$.

Equivalently, $M$ is Stein iff it is holomorphically convex and holomorphically separable. That is, for every distinct points $x, y \in M$, there exists a holomorphic function $f : M \to \mathbb{C}$ such that $f(x) \neq f(y)$. We will use:

Theorem 9 (Rossi [11], Corollary on page 20). If a compact complex manifold $M$ is strongly pseudoconvex and contains no compact complex subvarieties of positive dimension, then $\text{int}(M)$ is Stein.

We now discuss strong quasiconvexity and Stein property in the context of complex-hyperbolic manifolds. A classical example of a complex submanifold with Levi-convex boundary is a closed round ball $\overline{B^n}$ in $\mathbb{C}^n$. Suppose that $\Gamma < \text{Aut}(B^n)$ is a discrete torsion-free subgroup of the group of holomorphic automorphisms of $B^n$ with (nonempty) domain of discontinuity $\Omega = \Omega(\Gamma) \subset \partial B^n$. The quotient

$$\overline{M}_\Gamma = (B^n \cup \Omega)/\Gamma$$

is a smooth manifold with boundary.

Lemma 10. $\overline{M}_\Gamma$ is strongly pseudoconvex.

Proof. We let $T_\Lambda$ denote the union of all projective hyperplanes in $P^n_\mathbb{C}$ tangent to $\partial B^n$ at points of $\Lambda$, the limit set of $\Gamma$. Let $\hat{\Omega}$ denote the connected component of $P^n_\mathbb{C} \setminus T_\Lambda$ containing $B^n$. It is clear that $B^n \cup \Omega \subset \hat{\Omega}$ is strictly Levi-convex. By the construction, $\Gamma$ preserves $\hat{\Omega}$. It is proven in [5, Thm. 7.5.3] that the action of $\Gamma$ on $\hat{\Omega}$ is properly discontinuous. Hence, $X := \hat{\Omega}/\Gamma$ is a complex manifold containing $\overline{M}_\Gamma$ as a strictly Levi-convex submanifold with boundary.

Specializing to the case when $\overline{M}_\Gamma$ is compact, i.e. $\Gamma$ is convex-cocompact, we obtain:

Proposition 11. Suppose that $\Gamma$ is torsion-free, convex-cocompact and $n > 1$. Then:

1. $\partial \overline{M}_\Gamma$ is connected.
2. If $\text{int}(\overline{M}_\Gamma) = M_\Gamma$ contains no compact complex subvarieties of positive dimension, then $M_\Gamma$ is Stein.

For example, as it was observed in [4], the quotient-manifold $B^2/\Gamma$ of a real-Fuchsian subgroup $\Gamma < \text{Aut}(B^2)$ is Stein while the quotient-manifold of a complex-Fuchsian subgroup $\Gamma < \text{Aut}(B^2)$ is non-Stein.
3 Proof of Theorem 1

In this section, we construct certain plurisubharmonic functions on $M_{\Gamma}$, for each finitely generated, discrete subgroup $\Gamma < \text{Aut}(B^n)$ satisfying $\delta(\Gamma) < 2$. We use these functions to show that $M_{\Gamma}$ has no compact subvarieties of positive dimension. At the end of this section, we prove the main result of this paper.

Let $X$ be a complex manifold. Recall that a continuous function $f : X \to \mathbb{R}$ is called plurisubharmonic\(^3\) if for any homomorphic map $\phi : V(\subset \mathbb{C}) \to X$, the composition $f \circ \phi$ is subharmonic. Plurisubharmonic functions $f$ satisfy the maximum principle; in particular, if $f$ restricts to a nonconstant function on a connected complex subvariety $Y \subset X$, then $Y$ is noncompact.

Now we turn to our construction of plurisubharmonic functions. Let $\Gamma < \text{Aut}(B^n)$ be a discrete subgroup. Consider the Poincaré series

$$\sum_{\gamma \in \Gamma} (1 - |\gamma(z)|^2), \quad z \in B^n. \quad (3)$$

**Lemma 12.** Suppose that $\delta(\Gamma) < 2$. Then (3) uniformly converges on compact sets.

**Proof.** Since $\delta(\Gamma) < 2$, the Poincaré series

$$\sum_{\gamma \in \Gamma} e^{-2d(0,\gamma(z))}$$

uniformly converges on compact subsets in $B^n$. By (2), we get

$$e^{-2d(0,\gamma(z))} \leq (1 - |\gamma(z)|^2) \leq 4e^{-2d(0,\gamma(z))}. \quad (4)$$

Then, the result follows from the upper inequality. \qed

**Remark 13.** Note that when $\delta(\Gamma) > 2$, or when $\Gamma$ is of divergent type (e.g., convex-cocompact) and $\delta(\Gamma) = 2$, then (3) does not converge. This follows from the lower inequality of (4).

Assume that $\delta(\Gamma) < 2$. Define $F : B^n \to \mathbb{R}$,

$$F(z) = \sum_{\gamma \in \Gamma} (|\gamma(z)|^2 - 1).$$

Since $F$ is $\Gamma$-invariant, i.e., $F(\gamma z) = F(z)$, for all $\gamma \in \Gamma$ and all $z \in B^n$, $F$ descends to a function $f : M_{\Gamma} \to \mathbb{R}$.

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\(^3\)There is a more general notion of plurisubharmonic functions; for our purpose, we only consider this restrictive definition.
Lemma 14. The function $f : M_{\Gamma} \to \mathbb{R}$ is plurisubharmonic.

Proof. Enumerate $\Gamma$ as $\Gamma = \{\gamma_1, \gamma_2, \ldots\}$. Consider the sequence of partial sums of the series $F$,

$$S_k(z) = \sum_{j \leq k} (|\gamma_j(z)|^2 - 1).$$

Since each summand in the above is plurisubharmonic$^4$, $S_k$ is plurisubharmonic for each $k \geq 1$. Moreover, the sequence of functions $S_k$ is monotonically decreasing. Thus, the limit $F = \lim_{k \to \infty} S_k$ is also plurisubharmonic, and hence so is $f$. \hfill \square

Note, however, that at this point we do not yet know that the function $f$ is nonconstant.

Now we prove the main result of this section.

Theorem 15. Let $\Gamma$ be a torsion-free discrete subgroup of $\text{Aut}(B^n)$. If $\delta(\Gamma) < 2$, then $M_{\Gamma}$ contains no compact complex subvarieties of positive dimension.

Proof. Suppose that $Y$ is a compact connected subvariety of positive dimension in $M_{\Gamma}$. Since $\pi_1(Y)$ is finitely generated, so is its image $\Gamma'$ in $\pi_1(M_{\Gamma})$. Since $\delta(\Gamma') \leq \delta(\Gamma)$, by passing to the subgroup $\Gamma'$ we can (and will) assume that the group $\Gamma$ is finitely generated.

We construct a sequence of functions $F_k : B^n \to \mathbb{R}$ as follows. For $k \in \mathbb{N}$, let $\Sigma_k \subset \Gamma - \{1\}$ denote the subset consisting of $\gamma \in \Gamma$ satisfying $d(0, \gamma(0)) \leq k$. Since $\Gamma$ is a finitely generated linear group, it is residually finite and, hence, there exists a finite index subgroup $\Gamma_k \subset \Gamma$ disjoint from $\Sigma_k$. For each $k \in \mathbb{N}$, define $F_k : B^n \to \mathbb{R}$ as the sum

$$F_k(z) = \sum_{\gamma \in \Gamma_k} (|\gamma(z)|^2 - 1).$$

Since

$$\bigcap_{k \in \mathbb{N}} \Gamma_k = \{1\},$$

the sequence of functions $F_k$ converges to $(|z|^2 - 1)$ uniformly on compact subsets of $B^n$. As before, each $F_k$ is plurisubharmonic (cf. Lemmata 12, 14).

Let $\tilde{Y}$ be a connected component of the preimage of $Y$ under the projection map $B^n \to M_{\Gamma}$. Since $\tilde{Y}$ is a closed, noncompact subset of $B^n$, the function $(|z|^2 - 1)$ is nonconstant on $\tilde{Y}$. As the sequence ($F_k$) converges to $(|z|^2 - 1)$ uniformly on compacts, there exists $k \in \mathbb{N}$ such that $F_k$ is nonconstant on $\tilde{Y}$. Let $f_k : M_k = M_{\Gamma_k} \to \mathbb{R}$ denote the function obtained by projecting $F_k$ to $M_k$, and $Y_k$ be the image of $\tilde{Y}$ under the projection map $B^n \to M_k$. Since $M_k$ is a finite covering of $M_{\Gamma}$, the subvariety $Y_k \subset M_k$ is compact. Moreover, $f_k$ is a nonconstant plurisubharmonic function on $Y_k$ since $F_k$ is such a function on $\tilde{Y}$. This contradicts the maximum principle. \hfill \square

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$^4$This follows from the fact that the function $|z|^2$ is plurisubharmonic.
Remark 16. Regarding Remark 13: The failure of convergence of the series (3) as pointed out in Remark 13 is not so surprising. In fact, if $\Gamma$ is a complex Fuchsian group, then $\delta(\Gamma) = 2$ and the convex core of $M_\Gamma$ is a compact Riemann surface, see Example 4. Thus, our construction of $F$ must fail in this case.

We conclude this section with a proof of the main result of this paper.

Proof of Theorem 1. By Theorem 15, $M_\Gamma$ does not have compact complex subvarieties of positive dimensions. Then, by the second part of Proposition 11, $M_\Gamma$ is Stein. \qed

4 Further remarks

In relation to Theorem 1, it is also interesting to understand the case when $\delta(\Gamma) = 2$, that is: For which convex-cocompact, torsion-free subgroups $\Gamma$ of $\text{Aut}(B^n)$ satisfying $\delta(\Gamma) = 2$, is the manifold $M_\Gamma$ Stein? It has been pointed out before that a complex Fuchsian subgroup $\Gamma < \text{Aut}(B^n)$ satisfies $\delta(\Gamma) = 2$, but the manifold $M_\Gamma$ is not Stein. In fact, the convex core of $M_\Gamma$ is a complex curve, see Remark 16. We conjecture that complex Fuchsian subgroups are the only such non-Stein examples.

Conjecture 17. Let $\Gamma < \text{Aut}(B^n)$ be a convex-cocompact, torsion-free subgroup such that $\delta(\Gamma) = 2$. Then, $M_\Gamma$ is non-Stein if and only if $\Gamma$ is a complex Fuchsian subgroup.

We illustrate this conjecture in the following very special case: Let $\phi : \pi_1(\Sigma) \to \text{Aut}(B^n)$ be a faithful convex-cocompact representation where $\Sigma$ is a compact Riemann surface of genus $g \geq 2$. Then $\phi$ induces a (unique) equivariant harmonic map

$$F : \tilde{\Sigma} \to B^n.$$ which descends to a harmonic map $f : \Sigma \to M_\Gamma$.

Proposition 18. Suppose that $F$ is a holomorphic immersion. Then $\Gamma = \phi(\pi_1(\Sigma))$ satisfies $\delta(\Gamma) \geq 2$. Moreover, if $\delta(\Gamma) = 2$, then $\Gamma$ preserves a complex line. In particular, $\Gamma$ is a complex Fuchsian subgroup of $\text{Aut}(B^n)$.

Proof. Noting that $M_\Gamma$ contains a compact complex curve, namely $f(\Sigma)$, the first part follows directly from Theorem 1.

For the second part, we let $Y$ denote the surface $\tilde{\Sigma}$ equipped with the Riemannian metric obtained via pull-back of the Riemannian metric $g$ on $B^n$. The entropy\footnote{The \textit{volume entropy} of a simply connected Riemannian manifold $(X, g)$ is defined as $\lim_{r \to \infty} \frac{\log \text{Vol}(B(r, x))}{r}$, where $x \in X$ is a chosen base-point and $B(r, x)$ denotes the ball of radius $r$ centered at $x$. This limit exists and is independent of $x$, see [10].} $h(Y)$ of $Y$ is bounded above by $\delta(\Gamma)$, i.e.

$$h(Y) \leq 2. \tag{5}$$
This can be seen as follows: The distance function $d_Y$ on $Y$ satisfies
\[ d_Y(y_1, y_2) \geq d(F(y_1), F(y_2)). \]

Therefore, the exponential growth-rate $\delta_Y$ of $\pi_1(\Sigma)$-orbits in $Y$ satisfies $\delta_Y \leq \delta(\Gamma)$. On the other hand, the quantity $\delta_Y = h(Y)$ since $\pi_1(\Sigma)$ acts cocompactly on $Y$.

Assume that $\Sigma$ is endowed with a conformal Riemannian metric of constant $-4$ sectional curvature. Since $\Sigma$ is a symmetric space, we have
\[ h^2(Y) \text{Area}(Y/\Gamma) \geq h^2(\Sigma) \text{Area}(\Sigma), \]
see [1, p. 624]. The inequality (5) together with the above implies that $\text{Area}(Y/\Gamma) \geq \text{Area}(\Sigma)$.

On the other hand, since $f : Y/\Gamma \to M_\Gamma$ is holomorphic, $4 \cdot \text{Area}(Y/\Gamma)$ equals to the Toledo invariant $c(\phi)$ (see [12]) of the representation $\phi$. Since $c(\phi) \leq 4\pi(g - 1)$, the inequality $\text{Area}(Y/\Gamma) \geq \text{Area}(\Sigma) = \pi(g - 1)$ shows that $\text{Area}(Y/\Gamma) = \pi(g - 1)$ or, equivalently, $c(\phi) = 4\pi(g - 1)$. By the main result of [12], $\Gamma$ preserves a complex-hyperbolic line in $B^n$. \hfill \Box

**Remark 19.** The assumption that $F$ is an immersion can be eliminated: Instead of working with a Riemannian metric, one can work with a Riemannian metric with finitely many singularities.

Motivated by Theorem 15, we also make the following conjecture.

**Conjecture 20.** If $\Gamma < \text{Aut}(B^n)$ is discrete, torsion-free, and $\delta(\Gamma) < 2k$, then $M_\Gamma$ does not contain compact complex subvarieties of dimension $\geq k$.

We conclude this section with a verification of this conjecture under a stronger hypothesis.

**Proposition 21.** If $\Gamma < \text{Aut}(B^n)$ is discrete, torsion-free, and $\delta(\Gamma) < 2k - 1$, then $M_\Gamma$ does not contain compact complex subvarieties of dimension $\geq k$.

**Proof.** Note that if $\Gamma$ is elementary (i.e., virtually abelian), then $\delta(\Gamma) = 0$. In this case, the result follows from Theorem 15. For the rest, we assume that $\Gamma$ is nonelementary.

By [2, Sec. 4], there is a natural map $f : M_\Gamma \to M_\Gamma$ homotopic to the identity map $\text{id}_{M_\Gamma} : M_\Gamma \to M_\Gamma$ and satisfying
\[ |\text{Jac}_p(f)| \leq \left( \frac{\delta(\Gamma) + 1}{p} \right)^p, \quad 2 \leq p \leq 2n, \]
where $\text{Jac}_p(f)$ denotes the $p$-Jacobian of $f$. When $\delta(\Gamma) < 2k - 1$, we have $|\text{Jac}_p(f)| < 1$, for $p \in [2k, 2n]$. This means that $f$ strictly contracts the volume form on each $p$-dimensional tangent space at every point $x \in M_\Gamma$, for $p \in [2k, 2n]$. 

9
Let $Y \subset M_{\Gamma}$ be a compact complex subvariety of dimension $\geq k$ (real dimension $\geq 2k$). Then, $Y$ is also a volume minimizer in its homology class. Since $f$ strictly contracts volume on $Y$, $f(Y)$ has volume strictly less than that of $Y$. However, $f$ being homotopic to $\text{id}_{M_{\Gamma}}$, $f(Y)$ belongs to the homology class of $Y$. This is a contradiction to the fact that $Y$ minimizes volume its homology class. 

Remark 22. Note that Proposition 21 gives an alternative proof of Theorem 15 (hence Theorem 1) under a stronger hypothesis, namely $\delta(\Gamma) \in (0, 1)$. However, this method fails to verify Theorem 15 in the case when $\delta(\Gamma) \in [1, 2)$.

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