Wigner quantum systems (Lie superalgebraic approach)

T. D. Palev
Institute for Nuclear Research and Nuclear Energy, 1784 Sofia, Bulgaria
tpalev@inrne.acad.bg

N. I. Stoilova
Mathematical Physics Group, Department of Physics, Technical University of Clausthal, Leibnizstrasse 10, D-38678 Clausthal-Zellerfeld, Germany
Institute for Nuclear Research and Nuclear Energy, 1784 Sofia, Bulgaria
ptns@pt.tu-clausthal.de, stoilova@inrne.bas.bg

Abstract

We present three groups of examples of Wigner Quantum Systems related to the Lie superalgebras $osp(1/6n)$, $sl(1/3n)$ and $sl(n/3)$ and discuss shortly their physical features. In the case of $sl(1/3n)$ we indicate that the underlying geometry is noncommutative.

1 Introduction

We present some examples of noncanonical nonrelativistic quantum systems, referred to as Wigner quantum systems. The corresponding to them statistics are the so called $A$- and $B$-statistics.

The justification for studying such more general quantum systems is based on two key issues. The first one stems from the ideas of Wigner in his 1950’s work *Do the equations of motion determine the quantum mechanical commutation relations?* [1]. In this paper he has shown that the canonical quantum statistics may, in principle, be generalized in a logically consistent way. Wigner demonstrated this on an example of a one-dimensional oscillator with a Hamiltonian $(m = \omega = \hbar = 1)$ $H = \frac{1}{2}(p^2 + q^2)$. Abandoning the canonical commutation relations (CCRs) $[p, q] = -i$, he was searching for all operators $q$ and $p$, such that the "classical" Hamiltonian equations $\dot{p} = -q$, $\dot{q} = p$ were identical with the Heisenberg equations $\dot{p} = -i[p, H]$, $\dot{q} = -i[q, H]$. Apart from the canonical solution he found infinitely many other solutions. In this relation Wigner noted that the Hamiltonian Eqs. and the Heisenberg Eqs. have a more direct physical significance than the CCRs. Therefore, concluded Wigner, it is logically justified to postulate them from the very beginning, instead of the CCRs. On the ground of these considerations, a Wigner quantum system (WQS) is defined as a quantum system for which the Hamiltonian Eqs. are identical to the Heisenberg Eqs.. Then, in general, the CCRs do not hold.

An important further step was the observation that all solutions found by Wigner turned to be different representations of one pair of para-Bose (pB) creation and annihilation operators (CAOs) [2]. Let us note that in 1950 the parastatistics was still unknown. It was introduced three years later by Green [3] as a possible generalization of Bose and Fermi statistics in quantum field theory.

The second key issue is based on the observation that any $n$ pairs of Bose operators $b_1^\pm, \ldots, b_n^\pm$ generate a representation, the Bose representation $\rho_b$, of the orthosymplectic Lie superalgebra (LS) $osp(1/2n) = B(0/n)$.

The Cartan-Weyl generators (in this representation) are the Bose operators and all of their anticommutators $\{b_1^\xi, b_2^\eta\}$, $\xi, \eta = \pm$. Denote by $B_1^\pm, \ldots, B_n^\pm$ those generators of
$B(0/n)$, which in the Bose representation coincide with the Bose operators, $\rho_b(B^\pm) = b^\pm$. Then the representation independent operators $B_1^\pm, \ldots, B_n^\pm$ generate $B(0/n)$ and satisfy the triple relations:

$$[B^\xi_k, B^\eta_j], B^\epsilon_k] = (\epsilon - \xi)\delta_{jk}B^\eta_j + (\epsilon - \eta)\delta_{jk}B^\xi_i. \quad (1)$$

The operators (1) are known in quantum field theory. These are the para-Bose operators, mentioned above, which generalize the statistics of the tensor fields. This observation is important. It casts a bridge between two very different algebraical structures: para-Bose statistics and the representation theory of Lie superalgebras. The more precise statement is that the representation theory of $n$ pairs of para-Bose CAOs is completely equivalent to the representation theory of the orthosymplectic Lie superalgebra $B(0/n)$.

A similar statement holds also for Fermi statistics and its generalization, the para-Fermi statistics: any $n$ pairs of para-Fermi (pF) operators $F_1^\pm, \ldots, F_n^\pm$ generate the orthogonal Lie algebra $B_n$. Note that both $B_n$ and $B(0/n)$ belong to the class $B$ of the basic Lie superalgebras. Therefore the pF and the pB statistics (and in particular the Fermi and the Bose statistics) can be called $B$-statistics.

On the ground of this observation statistics related to the other infinite classes of basic Lie superalgebras, namely $A$-, $C$- and $D$-statistics were introduced. So far only the $A$-statistics were studied in more detail. By definition the statistics of a quantum system (which has a classical analogue) is said to be $A$- (resp. $B$-, $C$-, $D$-) statistics if

- The system is a Wigner quantum system;
- The position and the momentum operators $r_\alpha$ and $p_\alpha$, $\alpha = 1, \ldots, n$, of the system generate (a representation of) a Lie superalgebra from the class $A$ (resp. $B$, $C$, $D$).

Throughout the paper we use the notation: $Z$ - the set of all integers; $Z_2 = \{0, 1\}$ - the ring of all integers modulo 2; $N$ - all positive integers. Furthermore:

$$[p; q] = \{p, p + 1, p + 2, \ldots, q - 1, q\}, \quad \text{for} \quad p \leq q \in Z; \quad (2)$$

$$\theta_i = \begin{cases} 0, & \text{if} \quad i = 0, 1, 2, \ldots, n, \\ 1, & \text{if} \quad i = n + 1, n + 2, \ldots, n + m; \end{cases} \quad \theta_{ij} = \theta_i + \theta_j; \quad (3)$$

$$[a, b] = ab - ba, \quad \{a, b\} = ab + ba, \quad [a, b] = ab - (-1)^{\deg(a)\deg(b)}ba, \quad (4)$$

where $\deg(a) \in Z_2$ is the degree of the homogeneous element $a$ from the superalgebra.

2 Wigner quantum systems. Classes of solutions

**Definition 1** A system with a Hamiltonian $H_{tot} = \sum_{k=1}^{N} \frac{p_{r_k}^2}{2m_k} + V(r_1, r_2, \ldots, r_N)$, is said to be a Wigner quantum system if the following conditions hold:

- **C1.** The state space $W$ is a Hilbert space. The observables are Hermitian (selfadjoint) operators in $W$.
- **C2.** The Hamiltonian equations coincide with the Heisenberg equations (as operator equations in $W$).
- **C3.** The description is covariant with respect to the Galilean group.

Here we consider a system of $n + 1$ particles with mass $m$ and a Hamiltonian:
\[ H_{\text{tot}} = \sum_{A=1}^{n+1} \left( \frac{P_A}{2m} \right)^2 + \frac{m\omega^2}{2(n+1)} \sum_{A<B=1}^{n+1} (R_A - R_B)^2. \] \hspace{1cm} (5)

We proceed to show how this system can be turned into a noncanonical WQS in three different ways, related to the Lie superalgebras \(osp(1/6n), sl(1/3n)\) and \(sl(n/3)\).

Introduce centre of mass (CM) and internal variables \(\{14\}\) \((\alpha = 1, \ldots, n)\):

\[ \begin{align*}
R &= \frac{\sum_{A=1}^{n+1} R_A}{n+1}, \quad P = \sum_{A=1}^{n+1} P_A, \quad r_\alpha &= \sum_{\beta=1}^{\alpha} R_\beta - \alpha R_{\alpha+1} \frac{1}{\sqrt{\alpha(\alpha+1)}}, \quad p_\alpha &= \sum_{\beta=1}^{\alpha} P_\beta - \alpha P_{\alpha+1} \frac{1}{\sqrt{\alpha(\alpha+1)}}.
\end{align*} \] \hspace{1cm} (6)

Independently of the fact that \(R, P, r_\alpha, p_\alpha\) are unknown operators, one has:

\[ H_{\text{tot}} = H_{\text{CM}} + H, \quad H_{\text{CM}} = \frac{P^2}{2m(n+1)}, \quad H = \sum_{\alpha=1}^{n} \left( \frac{p_\alpha^2}{2m} + \frac{m\omega^2}{2} r_\alpha^2 \right). \] \hspace{1cm} (7)

For the equations of motion one obtains:

- Heisenberg equations

\[ \begin{align*}
\dot{P} &= -\frac{i}{\hbar}[P, H_{\text{tot}}], \quad \dot{R} = -\frac{i}{\hbar}[R, H_{\text{tot}}],
\end{align*} \] \hspace{1cm} (8)

\[ \begin{align*}
\dot{p}_\alpha &= -\frac{i}{\hbar}[p_\alpha, H_{\text{tot}}], \quad \dot{r}_\alpha = -\frac{i}{\hbar}[r_\alpha, H_{\text{tot}}], \quad \alpha = 1, \ldots, n.
\end{align*} \] \hspace{1cm} (9)

- Hamiltonian equations

\[ \begin{align*}
\dot{P} &= 0, \quad \dot{R} = \frac{P}{m(n+1)},
\end{align*} \] \hspace{1cm} (10)

\[ \begin{align*}
\dot{p}_\alpha &= -m\omega^2 r_\alpha, \quad \dot{r}_\alpha = \frac{p_\alpha}{m}, \quad \alpha = 1, \ldots, n.
\end{align*} \] \hspace{1cm} (11)

In order to find some noncanonical solutions of the \(n+1\) particle quantum system we accept the following assumptions (which also hold for any canonical quantum system):

**Assumption 1.** The CM variables commute with the internal variables.

**Assumption 2.** The CM coordinates and momenta are canonical.

**Assumption 3.**
1. The angular momentum operators \(J = L + M\) are generators of the algebra \(so(3)\) of the space rotations (\(L\) and \(M\) are the angular momentum of the centre of mass and the internal angular momentum);
2. \(H_{\text{tot}} = H_{\text{CM}} + H\) is a generator of the translations in time \(t\);
3. The components of the total momentum \(P\) are generators of space translations;
4. \(K = m(n+1)R - Pt\) are generators of the acceleration.

**Assumption 4.** The components of \(M\) and \(H\) are generators of the algebra \(so(3) \oplus u(1)\). They are in the enveloping algebra of the internal position and momentum operators \(r_\alpha, p_\alpha\) and the following relations hold (\(\alpha = 1, \ldots, n, j, k, l = 1, 2, 3\)):

\[ \begin{align*}
[M_j, M_k] &= i\varepsilon_{jkl} M_l, \quad [M_j, r_\alpha] = i\varepsilon_{jkl} r_{\alpha l}, \quad [M_j, p_\alpha] = i\varepsilon_{jkl} p_{\alpha l}.
\end{align*} \] \hspace{1cm} (12)

Assumption 2 settles the compatibility between the CM variables. Therefore it remains to investigate a \(3n\)-dimensional Wigner quantum oscillator. Introduce in place of \(r_\alpha, p_\alpha\)
new unknown operators:

\[ a_{\alpha k}^\pm = \sqrt{\frac{c_n m \omega}{4\hbar}} r_{\alpha k} + \mu \sqrt{\frac{c_n}{4m\omega\hbar}} p_{\alpha k}, \quad k = 1, 2, 3, \alpha = 1, \ldots, n, \mu = \pm \text{ or } \mp, \quad c_n \in \mathbb{N}. \quad (13) \]

Then the internal Hamiltonian \( H \) and the compatibility condition of the internal Heisenberg equations \( \xi \) and the internal Hamiltonian equations \( \eta \) read:

\[ H = \frac{\omega \hbar}{c_n} \sum_{\alpha=1}^{n} \sum_{i=1}^{3} \{a_{\alpha i}^+, a_{\alpha i}^-\}, \quad (14) \]

\[ \sum_{\beta=1}^{n} \sum_{j=1}^{3} \{\{a_{\beta j}^+, a_{\beta j}^-\}, a_{\gamma k}^\pm\} = -\mu c_n a_{\alpha i}^\pm, \quad i = 1, 2, 3, \quad \alpha = 1, 2, \ldots, n. \quad (15) \]

We refer to the condition \( (15) \) as a main quantum condition (MQC). In order to be slightly more rigorous we introduce the following terminology.

**Definition 2** Definition The (free unital) associative algebra \( F(c_n) \) of the generators \( a_{\alpha i}^\pm, \alpha = 1, 2, \ldots, n, i = 1, 2, 3 \) and the relations \( (14) \) is said to be (free) oscillator algebra (FOA).

The problem is to find those representations of the FOA, for which the quantum conditions C1-C3 hold. In general this problem is an open one. Here we list three classes of solutions, which are closely related to three classes of Lie superalgebras.

### 2.1 Osp(1/6n) class of solutions

Let \( F(2) \) be the (free unital) associative superalgebra with odd generators \( a_{\alpha i}^\pm, \alpha = 1, 2, \ldots, n, i = 1, 2, 3 \) and relations \( (17) \)

\[ \{a_{\alpha i}^\xi, a_{\alpha j}^\eta\}, a_{\gamma k}^\epsilon\} = \delta_{\alpha \gamma} \delta_{\epsilon \xi} a_{\alpha j}^\eta + \delta_{\alpha \gamma} \delta_{\epsilon \eta} a_{\alpha j}^\xi. \quad (16) \]

The operators \( (14) \) are para-Bose operators. They satisfy Eqs. \( (13) \) with \( \mu = \mp \) and \( c_n = 2 \). Therefore we have:

**Conclusion 1.** \( F(2) \) is a factor algebra of \( F(c_n) \).

**Conclusion 2.** Any representation of \( F(2) \) is a representation of \( F(c_n) \).

**Remark 1.** The canonical Bose solution is from this class.

**Remark 2.** The solutions of Wigner belong also to this class.

**Proposition 1.** \( F(2) \) is the universal enveloping algebra \( U[osp(1/6n)] \) of the orthosymplectic Lie superalgebra \( osp(1/6n) \). The pB operators are odd elements generating a Lie superalgebra, which is isomorphic to \( B(0/3n) \equiv osp(1/6n) \)

**Conclusion 3.** Any representation of \( osp(1/6n) \) is a representation of the FOA; it is a candidate for a Wigner quantum system. The statistics of the latter is a B-statistics.

Unfortunately the representations of \( osp(1/6n) \) are still not known explicitly.
2.2 \(Sl(1/3n)\) class of solutions \cite{13}

Let \(F(3n - 1)\) be the (free unital) associative superalgebra with odd generators \(a_{\alpha i}^\pm, \alpha = 1, 2, \ldots, n, \ i = 1, 2, 3\) and relations

\[
\begin{align*}
\{a_{\alpha i}^+, a_{\beta k}^+\}, a_{\gamma k}^- &= -\delta_{j k}\delta_{\alpha\beta} a_{\alpha i}^+ + \delta_{i j}\delta_{\alpha\beta} a_{\gamma k}^+ - \delta_{i j}\delta_{\alpha\beta} a_{\gamma k}^+, \\
\{a_{\alpha i}^+, a_{\beta k}^-\}, a_{\gamma k}^+ &= \delta_{i k}\delta_{\alpha\beta} a_{\beta j}^+ - \delta_{i j}\delta_{\alpha\beta} a_{\gamma k}^+, \\
\{a_{\alpha i}^-, a_{\beta k}^+\}, a_{\gamma k}^- &= -\delta_{i k}\delta_{\alpha\beta} a_{\gamma j}^- + \delta_{i j}\delta_{\alpha\beta} a_{\beta k}^-, \\
\{a_{\alpha i}^-, a_{\beta k}^-\}, a_{\gamma k}^- &= -\delta_{i k}\delta_{\alpha\beta} a_{\gamma j}^- + \delta_{i j}\delta_{\alpha\beta} a_{\beta k}^+.
\end{align*}
\]

These operators also satisfy the MQC (\(\mu = \mp, \ c_n = 3n - 1\)) and therefore \(F(3n - 1)\) is a factor algebra of the FOA \(F(c_n)\).

Proposition 2. \(F(3n-1)\) is (isomorphic to) the universal enveloping algebra \(U[sl(1/3n)]\) of the LS \(sl(1/3n) \equiv A(0, 3n - 1)\) from the class \(A\) of the basic Lie superalgebras.

Any representation of \(sl(1/3n)\) gives a representation of the operators \(\{a_{\alpha i}^\pm\}\). Hence such operators do exist and the corresponding to them statistics is an \(A\)–statistics.

In \cite{13} a class of representations obtained by the usual Fock space technique were constructed. The conditions \(C1\)-\(C3\) lead to finite-dimensional irreducible \(sl(1/3n)\) modules \(W(n, p)\), where the number \(p = 1, 2, \ldots\), called an order of the statistics, characterizes the representation. Here is a list of some properties of these statistics.

- The internal energy \(E_k\) of the system takes \(\text{min}(3n, p) + 1\) different values: \(E_k = \frac{\hbar}{3n - 1}(3np - (3n - 1)k), \ k \equiv \sum_{n=1}^n \sum_{i=1}^3 \theta_{\alpha i} = 0, 1, 2, \ldots, \min(3n, p), \ \theta_{\alpha i} = 0, 1, 2, \ldots, n\).
- The internal angular momentum of the composite \((n + 1)\)-particle system takes all integer values between 0 and \(n\).
- The internal geometry is noncommutative: \([r_{\alpha i}, r_{\alpha j}] \neq 0, \ i \neq j, 1, 2, 3\). The positions of the particles cannot be localized. It turns out however that \([H, r_{\alpha i}^2] = 0, \ [r_{\alpha i}^2, r_{\alpha j}^2] = 0, \ \alpha, \beta = 1, \ldots, n\). Therefore the oscillating ”particles” move along spheres around the centre of mass with radiuses \(r_{\alpha i} = \sqrt{\hbar(3p - 3k + \sum_{i=1}^3 \theta_{\alpha i})}, \ k \equiv \sum_{n=1}^n \sum_{i=1}^3 \theta_{\alpha i} = 0, 1, 2, \ldots, \min(3n, p), \ \theta_{\alpha i} = 0, 1, 2, \ldots, n\). Setting in the last Eqs. \(k = 0\), one obtains the maximal radius. Hence the diameter of the oscillator is \(d = 2\sqrt{\frac{3hp}{(3n - 1)m\omega}}\). The different ”particles” can stay simultaneously on spheres with different radiuses. Their positions, however, cannot be localized because, as mentioned, the coordinate operators do not commute with each other.
- Turning to the initial \(n + 1\)-particle system one notes that the maximal distance between any two particles is \(d\). The system exhibits a nuclear kind structure: all particles are locked in a small volume \(V\) (with spatial dimension \(d\)) around the centre of mass. Since the coordinates do not commute, the particles are smeared with a certain probability within \(V\) and the geometry within this volume is noncommutative.

2.3 \(Sl(n/3)\) \((n \neq 3)\) class of solutions

Let \(F(n - 3)\) be the (free unital) associative superalgebra with odd generators \(a_{\alpha i}^\pm, \alpha = 1, 2, \ldots, n, \ i = 1, 2, 3\) and relations \((\xi, \eta, \epsilon = \pm)\)

\[
\begin{align*}
\{a_{\alpha i}^+, a_{\beta j}^+\}, a_{\gamma k}^- &= \delta_{jk}\delta_{\alpha\beta} a_{\gamma i}^+ - \delta_{i j}\delta_{\alpha\beta} a_{\gamma k}^+, \\
\{a_{\alpha i}^+, a_{\beta j}^-\}, a_{\gamma k}^+ &= -\delta_{jk}\delta_{\alpha\beta} a_{\gamma j}^- + \delta_{i j}\delta_{\alpha\beta} a_{\gamma k}^+, \\
\{a_{\alpha i}^-, a_{\beta j}^+\}, a_{\gamma k}^- &= -\delta_{i k}\delta_{\alpha\beta} a_{\gamma j}^- + \delta_{i j}\delta_{\alpha\beta} a_{\beta k}^+, \\
\{a_{\alpha i}^-, a_{\beta j}^-\}, a_{\gamma k}^- &= -\delta_{i k}\delta_{\alpha\beta} a_{\gamma j}^- + \delta_{i j}\delta_{\alpha\beta} a_{\beta k}^+.
\end{align*}
\]
It is straightforward to check that this operators satisfy the MQC with \( \mu = \mp, c_n = 3 - n \) for \( n = 1, 2 \) and \( \mu = \pm, c_n = n - 3 \) for \( n > 3 \). Therefore \( F([n-3]) \) is again a factor algebra of the FOA \( F(c_n) \).

**Proposition 3.** The operators \( a_{\alpha i}^\pm, \alpha = 1, \ldots, n, i = 1, 2, 3 \) constitute a basis in the odd subspace of the Lie superalgebra \( sl(n/3) \) and generate the whole algebra.

**Proof.** First we define the Lie superalgebra \( sl(n/m) \). The universal enveloping algebra \( U[gl(n/m)] \) of the general linear LS \( gl(n/m) \) is a \( \mathbb{Z}_2 \)-graded associative unital superalgebra generated by \((n+m)^2 \) \( \mathbb{Z}_2 \)-graded indeterminates \( \{e_{ij} | i, j \in [1; n+m]\} \), deg\((e_{ij}) = \theta_{ij} \), subject to the relations:

\[
\{e_{ij}, e_{il}\} = \delta_{jk}e_{il} - (-1)^\theta_{ij}\theta_{kl}e_{kj}, \quad i, j, k, l \in [1; n+m].
\]  

(19)

The LS \( gl(n/m) \) is a subalgebra of \( U[gl(n/m)] \), considered as a Lie superalgebra, with generators \( \{e_{ij} | i, j \in [1; n+m]\} \) and supercommutation relations (19). The LS \( sl(n/m) \) is a subalgebra of \( gl(n/m) \):

\[
sl(n/m) = \text{lin.env.}\{e_{ij}, (-1)^\theta e_{kk} - (-1)^\theta e_{jl} | i \neq j; \ i, j, k, l \in [1; n+m]\}.
\]  

(20)

The odd generators of \( sl(n/3) \), namely \( e_{\alpha,n+i}, e_{n+i,\alpha}, \alpha = 1, \ldots, n, i = 1, 2, 3 \) constitute a basis in the odd part \( sl_1(n/3) \). The even generators are all their anticommutators:

\[
\{e_{\alpha,n+i}, e_{n+j,\beta}\} = \delta_{ij}e_{\alpha\beta} + \delta_{\alpha\beta}e_{n+j,n+i}, \quad \text{satisfy Eqs. (18), constitute a basis in the odd subspace of the Lie superalgebra sl(n/3) and generate it. Clearly these properties hold for any other representation of sl(n/3), which completes the proof.}

**Conclusion 1.** The algebra \( F([n-3]) \) is (isomorphic) to the universal enveloping algebra \( U[sl(n/3)] \) of the LS \( sl(n/3) \equiv A(n-1/2) \) from the class \( A \) of the basic Lie superalgebras.

**Conclusion 2.** The operators \( a_{\alpha i}^\pm \) satisfy the condition C2 of the definition of a Wigner quantum system.

In terms of these operators \( H = \frac{\alpha}{n-1} \sum_{\alpha=1}^{n} \sum_{i=1}^{3} \{a_{\alpha i}^+, a_{\alpha i}^\mp\} \).

Set \( M_j = -\frac{\alpha}{n} \sum_{\alpha=1}^{n} \sum_{k,l=1}^{3} \varepsilon_{jkl}\{a_{\alpha k}^+, a_{\alpha l}^-\} \). Then \( [M_j, H] = 0, [M_j, A_k] = i\varepsilon_{jkl} A_l, A_k \in \{M_i, r_{\alpha i}, p_{\alpha i} | i = 1, 2, 3; \alpha = 1, \ldots, n\} \). In such a way we have

**Conclusion 3.** The condition C3 of the definition of a WQS is fulfilled.

In order to satisfy also C1 we have to define the position and the momentum operators \( r_{\alpha} \) and \( p_{\alpha} \), corresponding to the operators \( a_{\alpha i}^\pm \), as linear Hermitian operators in a Hilbert space \( W_{\text{int}} \). This means that the Hermitian conjugate to \( a_{\alpha k}^+ \) should be equal to \( a_{\alpha k}^- \), i.e.,

\[
(a_{\alpha k}^+)^\dagger = a_{\alpha k}^-.
\]  

(22)

The operators \( a_{\alpha i}^\pm \) have several representations. Here we consider a class of irreducible modules \( W(n,p)_{\text{int}} \) labeled by one positive integer \( p = 1, 2, \ldots \). Let

\[
|p; s \rangle \equiv |p; s_1, s_2, \ldots, s_{n+2} \rangle \in W(n,p)_{\text{int}}.
\]  

(23)
Postulate that the set of all vectors $|p; s\rangle$ with

$$s_i \in \mathbb{Z}_+ , \; i = 1, 2, \ldots, n-1; \; s_n, s_{n+1}, s_{n+2} \in \{0, 1\}, \; \sum_{l=1}^{n+2} s_l \leq p$$  \hspace{1cm} (24)$$

constitute an orthonormed basis in $W(n, p)_{\text{int}}$. Let $|p; s\rangle_{\pm i; \pm j}$ be a vector, obtained from $|p; s\rangle$ after a replacement of $s_i$ with $s_i \pm 1$ and $s_j$ with $s_j \pm 1$. Then the transformation of the basis under the action of the operators $a_{\alpha}^\pm$ read $(\alpha = 2, \ldots, n, \; i = 1, 2, 3)$:

$$a_{1i}^- |p; s\rangle = (-1)^{s_n+\ldots+s_{n+i-2}} s_{n-1+i} \sqrt{p - \sum_{l=1}^{n+2} s_l + 1} \; |p; s\rangle_{-(n-1+i)};$$  \hspace{1cm} (25)$$

$$a_{1i}^+ |p; s\rangle = (-1)^{s_n+\ldots+s_{n+i-2}} (1 - s_{n-1+i}) \sqrt{p - \sum_{l=1}^{n+2} s_l} \; |p; s\rangle_{+(n-1+i)};$$  \hspace{1cm} (26)$$

$$a_{\alpha i}^- |p; s\rangle = (-1)^{s_n+\ldots+s_{n+i-2}} s_{n-1+i} \sqrt{s_{\alpha-1}+1} \; |p; s\rangle_{-(\alpha-1);-(n-1+i)};$$  \hspace{1cm} (27)$$

$$a_{\alpha i}^+ |p; s\rangle = (-1)^{s_n+\ldots+s_{n+i-2}} (1 - s_{n-1+i}) \sqrt{s_{\alpha-1}} \; |p; s\rangle_{-(\alpha-1);+(n-1+i)}.$$  \hspace{1cm} (28)$$

It is straightforward to verify (22) and hence C1 holds.

The state space $W(n, p)_{\text{int}}$ is finite-dimensional. The internal Hamiltonian $H$ is diagonal in the basis (23)-(24)

$$H |p; s\rangle = \frac{\hbar \omega}{n-3} (3p + (n-3)(r_n + r_{n+1} + r_{n+2})) |p; s\rangle.$$  \hspace{1cm} (29)$$

In order to compute the eigenvalues of the internal angular momentum one has to decompose each $sl(n/3)$ module $W(p, n)_{\text{int}}$ according to the chain

$$sl(n/3) \supset sl(n) \oplus sl(3) \oplus u(1).$$  \hspace{1cm} (30)$$

This problem will be considered elsewhere.

In conclusion, we have considered three different examples of WQSs corresponding to one and the same Hamiltonian (5). The defining conditions C1, C2, C3 for the systems to be WQS were satisfied with position and momentum operators, which generate different representations of the Lie superalgebras $osp(1/6n)$ $sl(1/3n)$ and $sl(n/3)$. Some of the physical properties of these WQSs were shortly discussed.

**Acknowledgments**

N.I. Stoilova is thankful to Prof. H.D. Doebner for constructive discussions. The work was supported by the Humboldt Foundation and the Grant N П – 910 of the Bulgarian Foundation for Scientific Research.

**References**

[1] E.P. Wigner: Phys. Rev. **77**, 711 (1950).
[2] Y. Ohnuki and S. Kamefuchi: *Quantum Field Theory and Parastatistics*, University of Tokyo Press (1982); T.D. Palev, *J. Math. Phys.* 23, 1778 (1982).

[3] H.S. Green: *Phys. Rev.* 90, 270 (1953); See also Y. Ohnuki and S. Kamefuchi, *Quantum Field Theory and Parastatistics*, University of Tokyo Press, for general introduction in parastatistics (1982).

[4] A.Ch. Ganchev and T.D. Palev: *J. Math. Phys.* 21, 797 (1980).

[5] S. Kamefuchi and Y. Takahashi: *Nucl. Phys.* 36, 177 (1962); C. Ryan and E.C.G. Sudarshan: *Nucl. Phys.* 47, 207 (1963).

[6] V.G. Kac: *Lect. Notes Math.* 676, 597 (1978).

[7] T.D. Palev: *Lie algebraical aspects of the quantum statistics*, Habilitation thesis; Inst. Nuclear Research and Nucl. Energy, Sofia, (1977).

[8] T.D. Palev: *Lie algebraical aspects of quantum statistics. Unitary quantization (A-quantization)*, Preprint JINR E17-10550 (1977).

[9] T.D. Palev: *Czech. Journ. Phys.* B29, 91 (1979).

[10] T.D. Palev: *Rep. Math. Phys.* 18, 117, 129 (1980).

[11] T.D. Palev: *Czech. Journ. Phys.* B3, 680 (1982).

[12] T.D. Palev and J. Van der Jeugt: [hep-th/0009108](http://arxiv.org/abs/hep-th/0009108) and [hep-th/0010107](http://arxiv.org/abs/hep-th/0010107).

[13] T.D. Palev and N.I. Stoilova: *J. Math. Phys.* 38, 2506 (1997).

[14] M. Moshinsky: *The Harmonic Oscillator in Modern Physics: From Atoms to Quarks*, Gordon and Breach Sci. Publ., Berlin-Heidelberg-New York, (1969).