Algebraic Aspects of Tremblay-Turbiner-Winternitz Hamiltonian Systems

J A Calzada¹, E. Celeghini², M. A. del Olmo³ and M A Velasco³

¹Departamento de Matemática Aplicada, Escuela Superior de Ingenieros Industriales, Universidad de Valladolid, 47011 Valladolid, Spain
²Departimento di Fisica, Università di Firenze and INFN–Sezione di Firenze, I50019 Sesto Fiorentino, Firenze, Italy
³Departamento de Física Teórica, Atómica y Optica, Facultad de Ciencias, Universidad de Valladolid, 47011 Valladolid, Spain
E-mail: juacal@eis.uva.es, celeghini@fi.infn.it, olmo@fta.uva.es

Abstract. Using the factorization method we find a hierarchy of Tremblay-Turbiner-Winternitz Hamiltonians labeled by discrete indices. The shift operators (those connecting eigenfunctions of different Hamiltonians of the hierarchy) as well the ladder operators (they connect eigenstates of a determined Hamiltonian) obtained in this way close different algebraic structures that are presented here.

1. Introduction
In 2009 Tremblay, Turbiner and Winternitz introduced a family of Hamiltonian systems [1] depending on four real parameters \((k, \omega, \alpha, \beta)\)

\[
H_{k,\omega,\alpha,\beta} = -\frac{\partial^2_r}{r} - \frac{1}{r} \frac{\partial_r}{r} + \frac{1}{r^2} \frac{\partial^2_\varphi}{\varphi^2} + \omega^2 r^2 + \frac{k^2}{r^2} \left( \frac{\alpha(\alpha - 1)}{\cos^2 k\varphi} + \frac{\beta(\beta - 1)}{\sin^2 k\varphi} \right),
\]

where \(0 \leq r < \infty\) and \(0 < \varphi < \pi/2k\). When \(k\) is integer or rational number the corresponding Hamiltonians are superintegrable in the classical [2] as well in the quantum case [3,4]. In this way the restricted list of superintegrable systems is enlarged [5]. Moreover, for particular values of \(k\) some well known Hamiltonian systems are recovered. So, when \(k = 1\) we get the Smorodinsky-Winternitz system [5–7]; for \(k = 2\) the rational \(BC_2\) model [8] and for \(k = 3\) the 3-particle Calogero model with extra three-body interaction [9].

An interesting property, that we will profit in this work, is that the Tremblay-Turbiner-Winterniz (TTW) -Hamiltonian (1) is separable in the coordinates \((r, \varphi)\). Effectively, the corresponding Schrödinger equation for stationary states associated to the TTW–Hamiltonian reads

\[ H_{k,\omega,\alpha,\beta} \Psi(r, \varphi) = E \Psi(r, \varphi). \]
It admits a variable separation in two equations if we consider $\Psi(r, \varphi) = \psi(r) \phi(\varphi)$:

$$H^{r}_{k,\omega,kM} \psi(r) \equiv \left( -\partial^{2}_{r} - \frac{1}{r} \partial_{r} + \omega^{2} r^{2} + \frac{k^{2} M^{2}}{r^{2}} \right) \psi(r) = E \psi(r), \quad (2)$$

$$H^{\varphi}_{k,\alpha,\beta} \phi(\varphi) \equiv \left( -\partial^{2}_{\varphi} + k^{2} \frac{\alpha(\alpha - 1)}{\cos^{2} k\varphi} + k^{2} \frac{\beta(\beta - 1)}{\sin^{2} k\varphi} \right) \phi(\varphi) = k^{2} M^{2} \phi(\varphi), \quad (3)$$

where $(kM)^{2} = \varepsilon^{2}$ is the separation constant. In the following we will consider the parameter $\varepsilon$ instead of $kM$. It is worth noticing that the Hamiltonian (3) was obtained (starting from the Smorodinsky-Winternitz system and using some trigonometric identities) and its solvability revealed in [10] some years before the TTW Hamiltonian was introduced.

We can consider after the separation two Hamiltonians: $H^{r}_{k,\omega,\varepsilon}$ (2) and $H^{\varphi}_{k,\alpha,\beta}$ (3). The first one can be seen as a radial oscillator Hamiltonian [11] and the second one as a generalized Pöschl–Teller Hamiltonian [12, 13].

Although this TTW-system is well known, because it has deserved a lot of attention since its introduction, and can be dealt with standard procedures we present a different point of view based on the properties of intertwining operators, i.e. a form of Darboux transformations [14]. This approach, founded on the factorization of Hamiltonians [15], gives a simple explanation of the main features of these physical systems from a group-theoretical point of view. The relationship between this procedure and group theory was established by variable separation and special functions [16]. Intertwining operators and integrable Hamiltonians have been studied previously, for instance, in [11,17,18]. The intertwining operators, here involved, are first order differential operators connecting different Hamiltonians in the same class (hierarchy). We will consider two classes of intertwining operators: shift operators (connecting pairs of eigenstates belonging to two consecutive Hamiltonians of the hierarchy) and ladder operators (connecting eigenstates of the same Hamiltonian with different eigenvalues) [19]. Our purpose is to study the algebraic properties of this kind of operators connecting TTW-Hamiltonians (1) by factorizing independently the Hamiltonians $H^{r}_{k,\omega,\varepsilon}$ (2) and $H^{\varphi}_{k,\alpha,\beta}$ (3).

The shift operators here involved are also called shape-invariant operators since they change the parameters in the same family of potentials. In this way, they span ‘potential algebras’ [20]. On the other hand, the ladder operators determine ‘spectrum generating algebras’ [21]. Since we will put together both kinds of actions we will refer these mixed algebras as ‘dynamical algebras’.

The energy spectra of these Hamiltonians can differ in the number of levels (they are equivalent Hamiltonians). The existence of quantum systems with identical (up to one level) energy spectra is accounted by a hidden symmetry: supersymmetry [22]. The supersymmetric origin of the equivalence between quantum Hamiltonians emerges from the fact that equivalent Hamiltonians are the components of a supersymmetric Hamiltonian (i.e. the so-called Supersymmetric Quantum Mechanics, see [23] and references therein).

2. Factorization of the radial Hamiltonian $H^{r}_{k,\omega,\varepsilon}$

In [11] the radial oscillator Hamiltonian, $H(r) = -\partial^{2}_{r} + l(l + 1)/r^{2} + r^{2}$, was studied generalizing results by [19] and recovering results by [21].

The radial Hamiltonian that we have here (2)

$$H^{r}_{k,\omega,\varepsilon} = -\partial^{2}_{r} - \frac{1}{r} \partial_{r} + \omega^{2} r^{2} + \frac{\varepsilon^{2}}{r^{2}} \quad (4)$$
can be factorized as a product of first order operators \( \mathcal{H}_{r,k,\omega,\epsilon} = \mathcal{A}_0^+ \mathcal{A}_0^- + \lambda_0 \), where
\[
\begin{align*}
\mathcal{A}_0^+ &= -\partial_r + \omega r - \frac{\epsilon + 1}{r}, \\
\mathcal{A}_0^- &= +\partial_r + \omega r - \frac{\epsilon}{r}, \\
\lambda_0 &= 2\omega(\epsilon + 1).
\end{align*}
\] (5)

The fundamental relation
\[
\mathcal{H}_{r,k,\omega,\epsilon} \equiv \mathcal{H}_0^r = \mathcal{A}_0^+ \mathcal{A}_0^- + \lambda_0 = \mathcal{A}_{-1}^+ \mathcal{A}_{-1}^- + \lambda_{-1}
\] (6)
allows us to get a hierarchy of Hamiltonians \( \{ \mathcal{H}_{r,k,\omega,\epsilon}^m = \mathcal{H}_m^r, m \in \mathbb{Z} \} \) with
\[
\begin{align*}
\mathcal{A}_m^+ &= -\partial_r + \omega r - \frac{(\epsilon + m + 1)}{r}, \\
\mathcal{A}_m^- &= +\partial_r + \omega r - \frac{(\epsilon + m)}{r}, \\
\lambda_m &= 2\omega(\epsilon + 2m + 1),
\end{align*}
\] (7)
such that
\[
\mathcal{H}_m^r = \mathcal{A}_m^+ \mathcal{A}_m^- + \lambda_m = \mathcal{A}_{m-1}^+ \mathcal{A}_{m-1}^- + \lambda_{m-1}
\] (8)
The Hamiltonians \( \mathcal{H}_m^r \) and the operators \( \mathcal{A}_m^\pm \) satisfy the following recurrence relations:
\[
\begin{align*}
\mathcal{A}_{m}^- \mathcal{H}_m^r &= \mathcal{H}_{m+1}^r \mathcal{A}_{m}^+, & \mathcal{A}_{m}^+ \mathcal{H}_{m+1}^r &= \mathcal{H}_m^r \mathcal{A}_{m+1}^+.
\end{align*}
\]
So, \( \mathcal{A}_m^\pm \) act as shape-invariant intertwining operators. It is worthy noticing that under the action of \( \mathcal{A}_m^- \) the eigenfunctions of \( \mathcal{H}_m^r \) become eigenfunctions of \( \mathcal{H}_{m+1}^r \) and, conversely, the eigenfunctions of \( \mathcal{H}_{m+1}^r \) are transformed on eigenfunctions of \( \mathcal{H}_m^r \) by \( \mathcal{A}_m^+ \). However, in both cases the eigenfunctions have the same eigenvalue.

From \( \{ \mathcal{H}_m^r, \mathcal{A}_m^\pm \}_{m \in \mathbb{Z}} \) we define index-free operators \( \{ \mathcal{H}_m^r, \mathcal{A}_m^\pm \} \) acting on the eigenfunctions \( \psi_{\epsilon,m} \) of \( \mathcal{H}_m^r \) as
\[
\begin{align*}
\mathcal{H}_m^r \psi_{\epsilon,m}^n &= \mathcal{H}_m^r \psi_{\epsilon,m}^n, \\
\mathcal{A}_-^\epsilon \psi_{\epsilon,m}^n &= \frac{1}{\sqrt{\omega}} \mathcal{A}_-^\epsilon \psi_{\epsilon,m}^n, \\
\mathcal{A}_+^\epsilon \psi_{\epsilon,m}^n &= \frac{1}{\sqrt{\omega}} \mathcal{A}_+^\epsilon \psi_{\epsilon,m}^n.
\end{align*}
\] (9)
The operators \( \mathcal{A}_m^\pm \) close the Heisenberg–Weyl algebra \( \mathfrak{h}_A \) together with the identity operator \( \mathbb{I} \):
\[
[A^-, A^+] = \mathbb{I}, \quad [\mathcal{A}_m^\pm, \mathbb{I}] = 0.
\]
Note that if we also consider the “number operator” \( \mathcal{N}_A = \mathcal{A}_+^\epsilon \mathcal{A}_-^\epsilon \) then the operators \( \mathcal{A}_m^\pm \), \( \mathcal{N} \) and \( \mathbb{I} \) generate a boson algebra since from (8) and (9) we have
\[
[\mathcal{N}_A, A^+] = A^+, \quad [\mathcal{N}_A, A^-] = -A^-.
\]
In this case one can proof that \( (\mathcal{A}_m^\pm)^\dagger = \mathcal{A}_m^\mp \). However, in other factorizations this is not true, but these systems are realizations of the concept of “pseudo-boson” operator recently introduced in [24].
Fundamental states
An eigenstate $\psi_{\varepsilon,m}^0$ of $H_{m}^r$ is a fundamental (highest or lowest weight) vector if it verifies the equation

$$ A_{m}^- \psi_{m}^0(r) = 0. $$

Hence, the explicit form of $\psi_{\varepsilon,m}^0$ is

$$ \psi_{\varepsilon,m}^0(r) = N_{\varepsilon,m}^0 e^{-\omega r^2/2} r^{\varepsilon+m}, $$

where $N_{\varepsilon,m}^0$ is a normalization constant. Its associated eigenvalue is

$$ E_{\varepsilon,m}^0 \equiv \lambda_m = 2\omega(\varepsilon + 2m + 1). $$

These functions are continuous and square-integrable (with measure $d\mu(r) = 2r dr$) when $m \geq -\varepsilon$. The value of the normalization constant is

$$ N_{\varepsilon,m}^0 = \sqrt{\omega^{\varepsilon+m+1}/\Gamma(\varepsilon + m + 1)}. $$

Excited states
The excited states of $\{H_{m}^r\}_{m \geq -\varepsilon}$ are obtained by the recursive application of the operator $A^+$ on the ground states $\psi_{\varepsilon,m}^0$ in the following way:

$$ \psi_{\varepsilon,m}^n = N_{\varepsilon,m}^n A_{m+1}^+ \cdots A_{m+n-2}^+ A_{m+n-1}^+ \psi_{\varepsilon,m}^0. $$

The explicit expression of $\psi_{\varepsilon,m}^n$ is

$$ \psi_{\varepsilon,m}^n(r) = N_{\varepsilon,m}^n r^{\varepsilon+m} e^{-\omega r^2/2} P_{n}^{\varepsilon+m}(\omega r^2), $$

where

$$ N_{\varepsilon,m}^n = (-1)^n \sqrt{\omega^{\varepsilon+m+1} n! / \Gamma(\varepsilon + m + 1 + n)}, $$

and $P_{n}^{\varepsilon}(\omega r^2)$ is a modified Laguerre polynomial. The eigenvalue associated to the state $\psi_{\varepsilon,m}^n$ is

$$ E_{\varepsilon,m}^n = E_{\varepsilon,m+n}^0 \equiv \lambda_{m+n} = 2\omega(\varepsilon + 2m + 2n + 1). $$

Note that for $m = 0$ we recover the eigenfunctions and eigenvalues of the original Hamiltonian $H_{k,\omega,\varepsilon}^r$.

2.1. More factorizations
A new hierarchy $\{B_{m}^\pm, H_{m}^r\}_{m \in \mathbb{Z}}$ can be obtained in such a way that

$$ H_{m}^r = B_{m}^+ B_{m}^- + \lambda_m = B_{m-1}^- B_{m-1}^+ + \lambda_{m-1} $$

$$ = -\partial_r^2 - \frac{1}{r} \partial_r + \omega^2 r^2 + \frac{(\varepsilon - m)^2}{r^2} + 2m \omega, $$

$$ B_{m}^+ = -\partial_r + \omega r + \frac{\varepsilon-(m+1)}{r}, $$

$$ B_{m}^- = \partial_r + \omega r + \frac{\varepsilon-m}{r}, $$

$$ \lambda_m = 2\omega(-\varepsilon + 2m + 1). $$
Also for $m = 0$ we recover the original Hamiltonian $H^r_{k,\omega,\varepsilon}$. The following relation between the Hamiltonians of the two factorizations (8) and (10)

$$\hat{H}^r_{-m} + 4m\omega = H^r_m$$

allows us to obtain a new factorization for the original Hamiltonian $H^r_{k,\omega,\varepsilon}$

$$H^r_m = B^+_m \ B^-_m + \hat{\lambda}^-_m + 4m\omega = B^+_{-(m+1)} \ B^-_{-(m+1)} + \hat{\lambda}^-_{(m+1)} + 4m\omega.$$ (12)

Similarly to the previous case we have the intertwining relations

$$H^r_{m-1} B^-_m = B^-_m (H^r_m - 4\omega), \quad H^r_{m+1} B^+_m = B^+_m (H^r_m + 4\omega).$$

From these intertwining relations we see that $B^+_{-(m+1)} \psi^n_{\varepsilon,m}$ and $B^-_m \psi^n_{\varepsilon,m}$ are eigenfunctions of $H^r_{m+1}$ and $H^r_{m-1}$ with eigenvalues $E^m_{m+1} = E^n_m + 4\omega$ and $E^m_{m-1} = E^n_m - 4\omega$, respectively.

Index-free operators $\{B^\pm\}$ acting on the wavefunctions $\psi_{\varepsilon,m}$ are defined by

$$B^+ \psi^n_{\varepsilon,m} := \frac{1}{2\sqrt{\omega}} B^+_{-(m+1)} \psi^n_{\varepsilon,m}, \quad B^- \psi^n_{\varepsilon,m} := \frac{1}{2\sqrt{\omega}} B^-_m \psi^n_{\varepsilon,m}.$$ (13)

The operators $B^\pm$ also close a Heisenberg-Weyl Lie algebra ($\mathfrak{h}_B$)

$$[B^-, B^+] = \mathbb{I}, \quad [B^\pm, \mathbb{I}] = 0,$$

and they also determine a boson algebra like in the previous case of the operators $A^\pm$. Moreover

$$[A^\pm, B^\pm] = 0, \quad [A, B] = 0, \quad [A, B^\pm] = 0, \quad [B, A^\pm] = 0.$$

Hence, we have a direct sum of Lie algebras

$$\mathfrak{h}_A \oplus \mathfrak{h}_B.$$

The hierarchy of Hamiltonians $H^r_m$ as well as their energy levels $E^n_{\varepsilon,m}$ can be represented in a plane by the points $(m, n)$ of the lattice $\mathbb{Z} \times \mathbb{Z}$. Associating to the point $(m, n)$ the state $\psi_{\varepsilon,m}$ the action of the operators $A^\pm$ and $B^\pm$ is displayed in Figure 1.

### 2.2. Composing intertwining operators

We can construct “à la Schwinger” two new sets of (second order differential) operators by composing the intertwining operators $A^\pm$ and $B^\pm$ as follows:

**Ladder operators**

$$L^- = A^- B^-, \quad L^+ = A^+ B^+, \quad L^3 = \frac{1}{2}(A^- A^+ + B^+ B^-).$$ (13)

These operators verify the commutations relations

$$[L^3, L^\pm] = \pm L^\pm, \quad [L^+, L^-] = 2L^3,$$

that correspond to $su(1, 1)$. 

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The action of the operators \( \{L^\pm, L^3\} \) on the eigenfunctions \( \psi_{\varepsilon,m}^n \) can be computed making use of their definition (13):

\[
L^- \psi_{\varepsilon,m}^n \equiv L^- \psi_{\varepsilon,m}^n = \frac{1}{4\omega} A_{m-1}^- B_{-m}^- \psi_{\varepsilon,m}^n,
\]

\[
L^+ \psi_{\varepsilon,m}^n = \frac{1}{4\omega} A_{m}^+ B_{-(m+1)}^+ \psi_{\varepsilon,m}^n,
\]

\[
L^3 \psi_{\varepsilon,m}^n = \frac{1}{8\omega} (A_{-(m-1)}^- A_{m-1}^+) + B_{m-1}^+ B_{-m}^- \psi_{\varepsilon,m}^n.
\]

This action is displayed in Figure 1.

The explicit differential form of these operators can be easily computed from (7) and (11)

\[
L^- m = \frac{1}{4\omega} \left( \partial_r^2 + (2\omega r + \frac{1}{r}) \partial_r - \frac{(\varepsilon + m)^2}{r^2} + \omega^2 r^2 + 2\omega \right),
\]

\[
L^+ m = \frac{1}{4\omega} \left( \partial_r^2 + (-2\omega r + \frac{1}{r}) \partial_r - \frac{(\varepsilon + m)^2}{r^2} + \omega^2 r^2 - 2\omega \right),
\]

\[
L^3 m = \frac{1}{4\omega} \left( -\partial_r^2 - \frac{1}{r} \partial_r + \frac{(\varepsilon + m)^2}{r^2} + \omega^2 r^2 \right).
\]

Taking into account the expression of the Hamiltonian (8) the operators \( L^\pm \) can be rewritten as first order differential operators, and the action of \( L^3 \) on the state \( \psi_{\varepsilon,m}^n \) can be easily computed

\[
L^3 \psi_{\varepsilon,m}^n = \frac{1}{2} (\varepsilon + m + 2n + 1) \psi_{\varepsilon,m}^n.
\]

**Shift operators**

\[
J^- = A^- B^+, \quad J^+ = A^+ B^- , \quad J^3 = \frac{1}{2} (A^- A^+ - B^- B^+).
\]

They close a \( su(2) \) algebra since their Lie commutators are

\[
[J^3, L^\pm] = \pm J^\pm, \quad [J^+, J^-] = 2J^3.
\]

The action of the operators \( J^\pm \) is displayed in Figure 1.

**2.3. Dynamical algebras**

All the operators found before can be grouped in different sets closing in all the cases Lie algebras. Thus, we have three dynamical algebras:

\[
\langle L^\pm, L^3, J^\pm, J^3, (A^\pm)^2, (B^\pm)^2 \rangle \simeq sp(4, \mathbb{R}),
\]

\[
\langle L^\pm, L^3, J^\pm, J^3, (A^\pm)^2, (B^\pm)^2, A^\pm, B^\pm, \text{I} \rangle \simeq isp(4, \mathbb{R}),
\]

\[
\langle L^\pm, L^3, J^3, A^\pm, B^\pm, \text{I} \rangle \simeq \text{Schrödinger}(2 + 1).
\]

Note that Schrödinger(2 + 1) as well \( isp(4, \mathbb{R}) \) are subalgebras of \( isp(4, \mathbb{R}) \).
3. Factorization of the angular Hamiltonian $H_{k,\alpha,\beta}^\varphi$

The angular Hamiltonian $H_{k,\alpha,\beta}^\varphi$ is a two-parametric (or generalized) Pöschl-Teller Hamiltonian

$$H_{k,\alpha,\beta}^\varphi = -\partial_\varphi^2 + k^2 \frac{\alpha(\alpha - 1)}{\cos^2 k\varphi} + k^2 \frac{\beta(\beta - 1)}{\sin^2 k\varphi}, \quad k, \alpha, \beta \in \mathbb{R}, \quad 0 < \varphi < \pi/2k. \quad (15)$$

Recently a complete study of its dynamical algebras has been made in [13].

From the factorization

$$H_{k,\alpha,\beta}^{\varphi} = C_0^+ C_0^- + \mu_0,$$

where

$$C_0^+ = \partial_\varphi - k\alpha \tan(k\varphi) + k\beta \cot(k\varphi),$$

$$C_0^- = -\partial_\varphi - k\alpha \tan(k\varphi) + k\beta \cot(k\varphi),$$

$$\mu_0 \equiv \varepsilon_0^2 = k^2(\alpha + \beta)^2,$$

we obtain the hierarchy of Hamiltonians $\{H_{k,\alpha,\beta}^{\varphi, m}, C_m\}_{m \in \mathbb{Z}}$, where

$$H_{k,\alpha,\beta}^{\varphi, m} \equiv H_{\alpha+m, \beta+m}^{\varphi} = C_m^+ C_m^- + \mu_m = C_{m-1}^- C_{m+1}^+ + \mu_{m-1}$$

$$= -\partial_\varphi^2 + k^2 \frac{(\alpha + m)(\alpha + m - 1)}{\cos^2 (k\varphi)} + k^2 \frac{(\beta + m)(\beta + m - 1)}{\sin^2 (k\varphi)},$$

$$C_m^\pm = \pm \partial_\varphi - k(\alpha + m) \tan(k\varphi) + k(\beta + m) \cot(k\varphi),$$

$$\mu_m \equiv \varepsilon_m^2 = k^2(\alpha + \beta + 2m)^2.$$
The intertwining action of $C^\pm_m$ on the Hamiltonian hierarchy $\{H_{\alpha+\beta,m+1}^\varphi\}_{m \in \mathbb{Z}}$ is given by

$$C^-_m H_{\alpha+\beta,m+1}^\varphi = H_{\alpha+\beta,m+1+1}^\varphi C^-_m, \quad C^+_m H_{\alpha+\beta,m+1}^\varphi = H_{\alpha+\beta,m+1}^\varphi C^+_m.$$ 

Hence, the action on the eigenfunctions $\phi_{\alpha+\beta,m+1}^\varphi(n)$ of the Hamiltonian $H_{\alpha+\beta,m+1}^\varphi$ is

$$\phi_{\alpha+\beta,m+1}^\varphi(n) \rightarrow C^-_m \phi_{\alpha+\beta,m+1}^\varphi(n), \quad \phi_{\alpha+\beta,m+1}^\varphi(n) \rightarrow C^+_m \phi_{\alpha+\beta,m+1}^\varphi(n),$$

in such a way that the eigenfunctions $\phi_{\alpha+\beta,m+1}^\varphi(n)$ and $\phi_{\alpha+\beta,m+1}^\varphi(n-1)$ have the same eigenvalue

$$\mu_{m+n} = (\varepsilon_m)^2 = k^2(\alpha + \beta + 2m + 2n)^2.$$  

(16)

Index-free operators, $C^\pm$, acting on the wavefunctions $\phi_{\alpha,\beta}^m(\varphi)$ are defined by

$$C^- \phi_{\alpha+\beta,m}^\varphi = \frac{1}{2} C^- \phi_{\alpha+\beta,m}^\varphi, \quad C^+ \phi_{\alpha+\beta,m}^\varphi = \frac{1}{2} C^+ \phi_{\alpha+\beta,m}^\varphi.$$ 

Their Lie commutators are

$$[C^+, C^-] = 2C, \quad [C, C^\pm] = \pm C^\pm,$$ 

(17)

where

$$C \phi_{\alpha+\beta,m}^\varphi = -\frac{1}{2}(\alpha + \beta + 2m - 1) \phi_{\alpha+\beta,m}^\varphi.$$

So, the operators $C^\pm$ and $C$ close a $su(2)$ algebra.

The ground state $\phi_{\alpha+\beta,m+1}^\varphi(\alpha,\beta) = 0.$

The excited states $\phi_{\alpha+\beta,m+1}^\varphi$ of $H_{\alpha+\beta,m+1}^\varphi$ are obtained from the reiterative action of the operators $C^+$

$$\phi_{\alpha+\beta,m}^\varphi(n) = N_{\alpha,\beta,m}^n C^+_m C^+_{m+1} \cdots C^+_{m+n-2} C^+_{m+n-1} \phi_{\alpha+\beta,m+n}^\varphi(\alpha,\beta),$$

with $N_{\alpha,\beta,m}^n$ a normalization constant. The explicit expressions of the eigenfunctions and their associated eigenvalues are

$$\phi_{\alpha,\beta,m}^\varphi(\alpha,\beta) = N_{\alpha,\beta,m}^n \cos^{\alpha+m}(k\varphi) \sin^{\beta+m}(k\varphi),$$

$$\phi_{\alpha+\beta,m}^\varphi(n) = N_{\alpha,\beta,m}^n \cos^{\alpha+m}(k\varphi) \sin^{\beta+m}(k\varphi) P_{\varphi}^{\beta+m-1/2,\alpha+1/2}(\cos(2k\varphi)),$$

$$\mu_{m+n} = (\varepsilon_m)^2 = k^2(\alpha + \beta + 2m + 2n)^2,$$

where

$$N_{\alpha,\beta,m}^n = \sqrt{\frac{2k^n n! (\alpha + \beta + 2m + 2n)}{\Gamma(1/2 + \alpha + m)\Gamma(1/2 + \beta + m)}}.$$


3.1. Discrete symmetries of \( H_{\alpha + m, \beta + m}^\varepsilon \)

Under the following discrete transformations of the parameters \((\alpha, \beta)\) the Hamiltonians \(H_{\alpha + m, \beta + m}^\varepsilon\) remain invariant

\[
\begin{align*}
I_\alpha : (\alpha, \beta) \rightarrow (-\alpha + 1, \beta), \quad I_\beta : (\alpha, \beta) \rightarrow (\alpha - \beta + 1).
\end{align*}
\]

These discrete symmetries provide us a new set of shift operators defined by conjugation

\[
D^\pm := I_\alpha C^\pm I_\alpha,
\]

whose explicit expressions are

\[
D^\pm_m = \pm \partial_\varphi + k(\alpha - 1 - n) \tan(k \varphi) + k(\beta + n) \cot(k \varphi).
\]

Hence, in this way we obtain a new Hamiltonian hierarchy \(\{H_{\alpha - m, \beta + m}^\varepsilon\}_{m \in \mathbb{Z}}\) with eigenvalues \(\varepsilon^{\pm}_m = k^2(-\alpha + 1 + \beta + 2m + 2n)^2\). The operators \(D^\pm_m\) act on the hierarchy of Hamiltonians and on their eigenfunctions by

\[
\begin{align*}
D^-_m : & \quad H_{\alpha - m, \beta + m}^\varepsilon \rightarrow H_{\alpha - m - 1, \beta + m + 1}^\varepsilon; & \quad D^+_m : & \quad H_{\alpha - m - 1, \beta + m + 1}^\varepsilon \rightarrow H_{\alpha - m, \beta + m}^\varepsilon, \\
\phi_{\alpha - m, \beta + m}^n \rightarrow & \quad \phi_{\alpha - m - 1, \beta + m + 1}^{n-1}; & \quad \phi_{\alpha - m - 1, \beta + m + 1}^{n-1} \rightarrow & \quad \phi_{\alpha - m, \beta + m}^n.
\end{align*}
\]

The corresponding index-free operators associated to these operators \(D^\pm_m\) are defined by

\[
\begin{align*}
D^- \phi_{\alpha - m, \beta + m}^n := & \quad \frac{1}{2} D^-_m \phi_{\alpha - m, \beta + m}^n, \\
D^+ \phi_{\alpha - m, \beta + m}^n := & \quad \frac{1}{2} D^+_m \phi_{\alpha - m, \beta + m}^n, \\
D \phi_{\alpha - m, \beta + m}^n := & \quad -\frac{1}{2} (-\alpha + \beta + 2m) \phi_{\alpha - m, \beta + m}^n(\varphi),
\end{align*}
\]

and they span a \(su(2)\) algebra with commutators like (17).

3.2. Ladder operators

The next shift operators that we present in this section change the eigenvalue \(\varepsilon\) together with one of the other two parameters \(\alpha\) or \(\beta\). In this sense they are ladder operators.

Let us consider the Schrödinger equation (3) \(H_{\alpha, \beta}^\varepsilon \phi_{\alpha, \beta} = \varepsilon \phi_{\alpha, \beta}\) that we rewrite as

\[
H_{\alpha, \beta, \varepsilon}^\varepsilon \phi_{\alpha, \beta, \varepsilon} = \varepsilon \phi_{\alpha, \beta, \varepsilon}
\]

by introducing \(\varepsilon\) among the indexes that label the Hamiltonian and their eigenfunctions. Multiplying by \(\cos^2 k \varphi\) the previous eq. (3) we get

\[
(- \cos^2 k \varphi \partial_\varphi^2 - \varepsilon^2 \cos^2 k \varphi + k^2 \beta(\beta - 1) \cot^2 k \varphi + k^2 \alpha(\alpha - 1)) \phi_{\alpha, \beta, \varepsilon} = 0. \tag{18}
\]

Factorizing this new eq. (18) we can obtain a new set of shift operators \(X^\pm\) transforming the indexes \((\alpha, \beta, \varepsilon)\) as follows

\[
(\alpha, \beta, \varepsilon) X^{\pm} \rightarrow (\alpha, \beta \mp 1, \varepsilon \mp 1)
\]

and a new Hamiltonian hierarchy. Moreover, \(X^\pm\) together with the operator \(X\) close a \(su(1, 1)\) algebra, with

\[
X \phi_{\alpha, \beta, \varepsilon}^n := X_n \phi_{\alpha, \beta, \varepsilon}^n = -\frac{1}{2} k(\alpha + 2 \beta + 2n) \phi_{\alpha, \beta, \varepsilon}^n = -\frac{1}{2} k(\beta + \varepsilon) \phi_{\alpha, \beta, \varepsilon}^n.
\]
Note that in the last equality we have used expression (16) with \( m = 0 \).

In a similar way, but now multiplying the original equation (3) by \( \sin^2 k \varphi \), we get a new set of operators

\[ \{ Y^\pm, Y \} \]

acting non-trivially on the indexes \((\alpha, \varepsilon)\).

Taking into account the reflections keeping invariant the Hamiltonian (18)

\[ I_\alpha : (\alpha, \beta, \varepsilon) \rightarrow (-\alpha + 1, \beta, \varepsilon), \quad I_\beta : (\alpha, \beta, \varepsilon) \rightarrow (\alpha, -\beta + 1, \varepsilon), \quad I_\varepsilon : (\alpha, \beta, \varepsilon) \rightarrow (\alpha, \beta, -\varepsilon) \]

we obtain by conjugation two new set of operators \( \{ W^\pm, W \} \) and \( \{ Z^\pm, Z \} \)

\[
W^\pm = I_\beta X^\pm I_\beta, \quad W = I_\beta X I_\beta, \\
Z^\pm = I_\alpha Y^\pm I_\alpha, \quad Z = I_\alpha Y I_\alpha,
\]

closing each of them, obviously, a \( su(1,1) \) algebra.

3.3. The dynamical algebra

All these shift and leader operators determine together a Lie algebra

\[
\langle C^\pm, C, D^\pm, D, X^\pm, X, Y^\pm, Y, W^\pm, W, Z^\pm, Z \rangle \simeq so(4,2) \simeq su(2,2).
\]

However, the diagonal operators \( \{ C, D, X, Y, W, Z \} \) are not independent, but they can be written as linear combinations of the diagonal operators \( D_\alpha, D_\beta \) and \( D_\varepsilon \) defined by

\[
D_\alpha \phi_{\alpha, \beta, \varepsilon} := \alpha \phi_{\alpha, \beta, \varepsilon}, \quad D_\beta \phi_{\alpha, \beta, \varepsilon} := \beta \phi_{\alpha, \beta, \varepsilon}, \quad D_\varepsilon \phi_{\alpha, \beta, \varepsilon} := \varepsilon \phi_{\alpha, \beta, \varepsilon}.
\]

Thus,

\[
\langle C^\pm, D^\pm, X^\pm, Y^\pm, W^\pm, Z^\pm, D_\alpha, D_\beta, D_\varepsilon \rangle \simeq so(4,2) \simeq su(2,2). \quad (19)
\]

4. The dynamical algebra for the TTW-Hamiltonian

Fixed the values of the parameters \( \alpha, \beta, \omega \) and \( k \) from the variable separation \( \Psi(r, \varphi) = \psi(r) \phi(\varphi) \) we can write

\[
\Psi_{m,m'}^{n,n'}(r, \varphi) = \psi(r)_{\varepsilon,m}^{n} \otimes \phi_{m'}^{n'}(\varphi)
\]

that belongs to the state space of

\[
H_{k, \omega, \varepsilon + m}^r \otimes H_{k, \alpha + m', \beta + m'}^\omega,
\]

where \( \varepsilon = k(\alpha + \beta + 2m' + 2n') \) and eigenvalue

\[
E_{k, \omega, m, \alpha + m', \beta + m'}^{n,n'} = 2\omega(\varepsilon + 2m + 2n + 1) = 2\omega(k(\alpha + \beta + 2m' + 2n') + 2m + 2n + 1)
\]

with degeneration

\[
k(\alpha + \beta + 2m' + 2n') + 2m + 2n = \text{integer}
\]

Note that for \( m = m' = 0 \) we recover the original TTW-Hamiltonian (1).

The dynamical algebra for the total system (1) has to have as subalgebras the dynamical algebras of the two Hamiltonians (2) and (3). Since the operators acting on the radial part depend of the previous action of the operators acting on the angular part (note that \( \varepsilon/k \equiv M = \alpha + \beta + 2m' + 2n' \)) the total algebra is not a direct sum of both algebras (14) and
It is well known that a “dynamical algebra” of the Smorodinsky-Winternitz system in \( N \) dimensions [25, 26] is

\[ \mathfrak{h}(N) \oplus_{s} sp(2N, \mathbb{R}), \]

where \( \mathfrak{h}(N) \) is the Heisenberg-Weyl algebra in \( N \) dimensions. Hence, for the \( k = 1 \) TTW-system this “dynamical algebra” would be

\[ \mathfrak{h}(2) \oplus_{s} sp(4, \mathbb{R}). \]  

(20)

So, we conjecture that in our case the dynamical algebra could be

\[ \mathfrak{i}sp(4, \mathbb{R}) \oplus_{s} su(2, 2). \]  

(21)

This dynamical algebra (21) is bigger than the other one (20) since we have considered operators acting not only on a Hamiltonian but in all a hierarchy of them. Also we have take into account discrete symmetries of the Hamiltonian.

5. Final remarks

We have constructed dynamical algebras (independent of the values of the parameters \( k, \omega, \alpha, \beta \)) for both Hamiltonians in which the TTW-Hamiltonian separates: \( \mathfrak{i}sp(4, \mathbb{R}) \) for the radial Hamiltonian and \( su(2, 2) \) for the angular Hamiltonian.

An open problem is to find the complete dynamical algebra of the TTW-Hamiltonian that we conjecture to be \( \mathfrak{i}sp(4, \mathbb{R}) \oplus_{s} su(2, 2) \). Work is in progress in this direction.

Another open problem is to classify the eigenfunctions of the hierarchies of the TTW-Hamiltonians in terms of irreducible representations (IRREP) of its complete dynamical algebra. Work in this sense have been made in the last years for a family of superintegrable quantum systems living in spaces of constant curvature [27, 28]. More recently, for the hierarchy of angular Hamiltonians (3) it has been proved that their eigenfunctions \( \psi_{\alpha, \beta, \varepsilon} \) span the support space of a IRREP of its dynamical algebra \( su(2, 2) \) [13].

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