Weight vs Magnetization Enumerator for Gallager Codes

Jort van Mourik\textsuperscript{1}, David Saad\textsuperscript{1} and Yoshiyuki Kabashima\textsuperscript{2}

\textsuperscript{1} Neural Computing Research Group, Aston University, Birmingham B4 7ET, UK
\textsuperscript{2} Department of Computational Intelligence and Systems Science, Tokyo Institute of Technology, Yokohama 2268502, Japan

Abstract. We propose a method to determine the critical noise level for decoding Gallager type low density parity check error correcting codes. The method is based on the magnetization enumerator ($M$), rather than on the weight enumerator ($W$) presented recently in the information theory literature. The interpretation of our method is appealingly simple, and the relation between the different decoding schemes such as typical pairs decoding, MAP, and finite temperature decoding (MPM) becomes clear. Our results are more optimistic than those derived via the methods of information theory and are in excellent agreement with recent results from another statistical physics approach.

1 Introduction

Triggered by active investigations on error correcting codes in both of information theory (IT) and statistical physics (SP) communities, there is a growing interest in the relationship between IT and SP. As the two communities investigate similar problems, one may expect that standard techniques known in one framework would bring about new developments in the other, and vice versa. Here we present a direct SP method to determine the critical noise level for Gallager type low density parity check codes which allows us to focus on the differences between the various decoding criteria and their approach for defining the critical noise level for which decoding, using Low Density Parity Check (LDPC) codes, is theoretically feasible.

2 Gallager code

In a general scenario, the $N$ dimensional Boolean message $s^v \in \{0,1\}^N$ is encoded to the $M(>N)$ dimensional Boolean vector $t^v$, and transmitted via a noisy channel, which is taken here to be a Binary Symmetric Channel (BSC) characterized by an independent flip probability $p$ per bit; other transmission channels may also be examined within a similar framework. At the other end of the channel, the corrupted codeword is decoded utilizing the structured codeword redundancy.
The error correcting code that we focus on here is Gallager’s linear code [6]. Gallager’s code is a low density parity check code defined by the a binary \((M-N) \times M\) matrix \(A = [C_1 | C_2]\), concatenating two very sparse matrices known to both sender and receiver, with the \((M-N) \times (M-N)\) matrix \(C_2\) being invertible. The matrix \(A\) has \(K\) non-zero elements per row and \(C\) per column, and the code rate is given by \(R = 1 - C/K = 1 - N/M\). Encoding refers to multiplying the original message \(s^o\) with the \((M \times N)\) matrix \(G^T\) (where \(G = [I_N | C_2^{-1}]\)), yielding the transmitted vector \(t^o\). Note that all operations are carried out in \((\text{mod } 2)\) arithmetic. Upon sending \(t^o\) through the binary symmetric channel (BSC) with noise level \(p\), the vector \(r = t^o + n^o\) is received, where \(n^o\) is the true noise.

Decoding is carried out by multiplying \(r\) by \(A\) to produce the syndrome vector \(z = Ar\) (= \(An^o\), since \(AG^T = 0\)). In order to reconstruct the original message \(s^o\), one has to obtain an estimate \(\hat{n}\) for the true noise \(n^o\). First we select all \(\hat{n}\) that satisfy the parity checks \(A\hat{n} = An^o\):

\[
T_{\text{pc}}(A, n^o) \equiv \{ n \mid An = z \}, \text{ and } T_{\text{pc}}^r(A, n^o) \equiv \{ n \in T_{\text{pc}}(A, n^o) \mid n \neq n^o \}, \quad (1)
\]

the (restricted) parity check set. Any general decoding scheme then consists of selecting a vector \(n^s\) from \(T_{\text{pc}}(A, n^o)\) on the basis of some noise statistics criterion. Upon successful decoding \(n^o\) will be selected, while a decoding error is declared when a vector \(n^s \in T_{\text{pc}}^r(A, n^o)\) is selected. An measure for the error probability is usually defined in the information theory literature [3] as

\[
P_e(p) = \langle \Delta \left( \exists n \in T_{\text{pc}}^r(A, n^o) : w(n) \leq w(n^o) \mid n^o \right) \rangle_{A, n^o}, \quad (2)
\]

where \(\Delta(\cdot)\) is an indicator function returning 1 if there exists a vector \(n \in T_{\text{pc}}^r(A, n^o)\) with lower weight than that of the given noise vector \(n^o\). The weight of a vector is the average sum of its components \(w(n) \equiv \frac{1}{M} \sum_{j=1}^{M} n_j\). To obtain the error probability, one averages the indicator function over all \(n^o\) vectors drawn from some distribution and the code ensemble \(A\) as denoted by \(\langle \cdot \rangle_{A, n^o}\).

Carrying out averages over the indicator function is difficult, and the error probability [3] is therefore upper-bounded by averaging over the number of vectors \(n\) obeying the weight condition \(w(n) \geq w(n^o)\). Alternatively, one can find the average number of vectors with a given weight value \(w\) from which one can construct a complete weight distribution of noise vectors \(n\) in \(T_{\text{pc}}^r(A, n^o)\). From this distribution one can, in principle, calculate a bound for \(P_e\) and derive critical noise values above which successful decoding cannot be carried out.

A natural and direct measure for the average number of states is the entropy of a system under the restrictions described above, that can be calculated via the methods of statistical physics.

It was previously shown (see e.g. [4] for technical details) that this problem can be cast into a statistical mechanics formulation, by replacing the field \(\{0, 1\}, (+\text{mod}(2))\) by \(\{1, -1\}, \times\), and by adapting the parity checks correspondingly. The statistics of a noise vector \(n\) is now described by its magnetization \(m(n) \equiv \frac{1}{M} \sum_{j=1}^{M} n_j\), \((m(n) \in [1, -1])\), which is inversely linked to the vector weight in the \([0, 1]\) representation. With this in mind, we introduce the conditioned magnetization enumerator, for a given code and noise, measuring the
noise vector magnetization distribution in $I_{pc}(A, n^o)$

$$M_{A,n^o}(m) = \frac{1}{M} \ln \left[ \sum_{n \in I_{pc}(A, n^o)} \delta(m(n) - m) \right].$$ (3)

To obtain the magnetization enumerator $M(m)$

$$M(m) = \langle M_{A,n^o}(m) \rangle_{A,n^o},$$ (4)

which is the entropy of the noise vectors in $I_{pc}(A, n^o)$ with a given $m$, one carries out uniform explicit averages over all codes $A$ with given parameters $K, C$, and weighted average over all possible noise vectors generated by the BSC, i.e.,

$$P(n^o) = \prod_{j=1}^{M} \left( (1-p) \delta(n^o_j-1) + p \delta(n^o_j+1) \right).$$ (5)

It is important to note that, in calculating the entropy, the average quantity of interest is the magnetization enumerator rather than the actual number of states. For physicists, this is the natural way to carry out the averages due to three main reasons: a) The entropy obtained in this way is believed to be self-averaging, i.e., its average value (over the disorder) coincides with its typical value. b) This quantity is extensive and grows linearly with the system size. c) This averaging distinguishes between annealed variables that are averaged or summed for a given set of quenched variables, that are averaged over later on. In this particular case, summation over all $n$ vectors is carried for a fixed choice of code $A$ and noise vector $n^o$; averages over these variables are carried out at the next level.

One should point out that in somewhat similar calculations, we showed that this method of carrying out the averages provides more accurate results in comparison to averaging over both sets of variables simultaneously.

A positive magnetization enumerator, $M(m) > 0$ indicates that there is an exponential number of solutions (in $M$) with magnetization $m$, for typically chosen $A$ and $n^o$, while $M(m) \to 0$ indicates that this number vanishes as $M \to \infty$ (note that negative entropy is unphysical in discrete systems).

Another important indicator for successful decoding is the overlap $\omega$ between the selected estimate $n^*$, and the true noise $n^o$: $\omega(n, n^o) = \frac{1}{M} \sum_{j=1}^{M} n_j n^o_j$, $(\omega(n, n^o) \in [-1, 1])$, with $\omega = 1$ for successful (perfect) decoding. However, this quantity cannot be used for decoding as $n^o$ is unknown to the receiver. The (code and noise dependent) overlap enumerator is now defined as:

$$W_{A,n^o}(\omega) = \frac{1}{M} \ln \left[ \sum_{n \in I_{pc}(A, n^o)} \delta(\omega(n, n^o) - \omega) \right],$$ (6)

and the average quantity being

$$W(\omega) = \langle W_{A,n^o}(\omega) \rangle_{A,n^o}.$$ (7)
This measure is directly linked to the weight enumerator \( W(x) \), although according to our notation, averages are carried out distinguishing between annealed and quenched variables unlike the common definition in the IT literature. However, as we will show below, the two types of averages provide identical results in this particular case.

3 The statistical physics approach

Quantities of the type \( Q(c) = \langle Q_y(c) \rangle_y \), with \( Q_y(c) = \frac{1}{M} \ln [Z_y(c)] \) and \( Z_y(c) = \text{Tr}_x \delta(c(x, y) - M c) \), are very common in the SP of disordered systems; the macroscopic order parameter \( c(x, y) \) is fixed to a specific value and may depend both on the disorder \( y \) and on the microscopic variables \( x \). Although we will not prove this here, such a quantity is generally believed to be self-averaging in the large system limit, i.e., obeying a probability distribution \( P(Q_y(c)) = \delta(Q_y(c) - Q(c)) \). The direct calculation of \( Q(c) \) is known as a quenched average over the disorder, but is typically hard to carry out and requires using the replica method \[9\]. The replica method makes use of the identity \( \langle \ln Z \rangle = \langle \lim_{n \to 0} \frac{Z^n - 1}{n} \rangle \), by calculating averages over a product of partition function replicas. Employing assumptions about replica symmetries and analytically continuing the variable \( n \) to zero, one obtains solutions which enable one to determine the state of the system.

To simplify the calculation, one often employs the so-called annealed approximation, which consists of performing an average over \( Q_y(c) \) first, followed by the logarithm operation. This avoids the replica method and provides (through the convexity of the logarithm function) an upper bound to the quenched quantity:

\[
Q_a(c) = \frac{1}{M} \ln [\langle Z_y(c) \rangle_y] \geq Q_q(c) = \frac{1}{M} \langle \ln [Z_y(c)] \rangle_y = \lim_{n \to 0} \frac{\langle Z^n_y(c) \rangle_y - 1}{nM}.
\]

The full technical details of the calculation will be presented elsewhere, and those of a very similar calculation can be found in e.g. \[3\]. It turns out that it is useful to perform the gauge transformation \( n_j \rightarrow n_j n^o_j \), such that the averages over the code \( A \) and noise \( n^o \) can be separated, \( \mathcal{W}_{A,n^o} \) becomes independent of \( n^o \), leading to an equality between the quenched and annealed results, \( \mathcal{W}(m) = \mathcal{M}_a(m)|_{p=0} = \mathcal{M}_q(m)|_{p=0} \). For any finite noise value \( p \) one should multiply \( \exp[\mathcal{W}(\omega)] \) by the probability that a state obeys all parity checks \( \exp[-K(\omega, p)] \) given an overlap \( \omega \) and a noise level \( p \) \[3\]. In calculating \( \mathcal{W}(\omega) \) and \( \mathcal{M}_{a/q}(m) \), the \( \delta \)-functions fixing \( m \) and \( \omega \), are enforced by introducing Lagrange multipliers \( \hat{m} \) and \( \hat{\omega} \).

Carrying out the averages explicitly one then employs the saddle point method to extremize the averaged quantity with respect to the parameters introduced while carrying out the calculation. These lead, in both quenched and annealed calculations, to a set of saddle point equations that are solved either analytically or numerically to obtain the final expression for the averaged quantity (entropy).
The final expressions for the annealed entropy, under both overlap ($\omega$) and magnetization ($m$) constraints, are of the form:

$$Q_a = -\frac{C}{K} \left[ \ln(2) + (K-1) \ln(1+q_1^K) \right] + \ln \left( \text{Tr}_{n=\pm 1} \frac{e^{(n\hat{\omega} + \hat{m}n)}}{(1+nq_1^{K-1})^C} \right)_{n^0} - \hat{\omega} \omega - \hat{m} m ,$$

(9)

where $q_1$ has to be obtained from the saddle point equation $\frac{\partial Q_a}{\partial q_1} = 0$. Similarly, the final expression in the quenched calculation, employing the simplest replica symmetry assumption [9], is of the form:

$$Q_q = -C \int dx d\hat{x} \pi(x) \hat{\pi}(\hat{x}) \ln(1+x\hat{x}) + \frac{C}{K} \int \left\{ \prod_{k=1}^{K} dx_k \pi(x_k) \right\} \ln \left[ \frac{1}{2} \left( 1 + \prod_{k=1}^{K} x_k \right) \right] + \int \left\{ \prod_{c=1}^{C} dx_c \hat{\pi}(\hat{x}_c) \right\} \ln \left( \text{Tr}_{n=\pm 1} \frac{\exp(n(\hat{\omega} + \hat{m}n))}{\prod_{c=1}^{C} (1+n\hat{x}_c)} \right)_{n^0} - \hat{\omega} \omega - \hat{m} m ,$$

(10)

The probability distributions $\pi(x)$ and $\hat{\pi}(\hat{x})$ emerge from the calculation; the former represents a probability distribution with respect to the noise vector local magnetization [10], while the latter relates to a field of conjugate variables which emerge from the introduction of $\delta$-functions while carrying out the averages (for details see [5]). Their explicit forms are obtained from the functional saddle point equations $\frac{\delta Q_a}{\delta \pi(x)} = 0$, $\frac{\delta Q_q}{\delta \hat{\pi}(\hat{x})} = 0$, and all integrals are from $-1$ to $1$. Enforcing a $\delta$-function corresponds to taking $\hat{\omega}$, $\hat{m}$ such that $\frac{\partial Q_a}{\partial \hat{\omega}} = 0$, while not enforcing it corresponds to putting $\hat{\omega}$, $\hat{m}$ to $0$. Since $\omega$, $m$ follow from $\frac{\partial Q_a}{\partial \omega} = 0$, $\frac{\partial Q_q}{\partial m} = 0$, all the relevant quantities can be recovered with appropriate choices of $\hat{\omega}$, $\hat{m}$.

4 Qualitative Picture

We now discuss the qualitative behaviour of $M(m)$, and the interpretation of the various decoding schemes. To obtain separate results for $M(m)$ and $W(m)$ we calculate the results of Eqs.(9) and (10), corresponding to the annealed and quenched cases respectively, setting $\hat{\omega} = 0$ for obtaining $M(m)$ and $\hat{m} = 0$ for obtaining $W(m)$ (that becomes $W(m)$ after gauging). In Fig. 1, we have qualitatively plotted the resulting function $M(m)$ for relevant values of $p$. $M(m)$ (solid line) only takes positive values in the interval $[m-(p), m+(p)]$; for even $K$, $M(m)$ is an even function of $m$ and $m-(p) = -m+(p)$. The maximum value of $M(m)$ is always $(1-R) \ln(2)$. The true noise $n^0$ has (with probability 1) the typical magnetization of the BSC: $m(n^0) = m_0(p) = 1-2p$ (dashed-dotted line).

The various decoding schemes can be summarized as follows:

- **Maximum likelihood (MAP) decoding** - minimizes the block error probability [11] and consists of selecting the $n$ from $I_{PC}(A, n^0)$ with the
highest magnetization. Since the probability of error below $m_\pm(p)$ vanishes, $P(\mathbf{n} \in \mathcal{I}_{pc}: m(\mathbf{n}) > m_\pm(p)) = 0$, and since $P(m(\mathbf{n}) = m_\pm(p)) = 1$, the critical noise level $p_c$ is determined by the condition $m_\pm(p_c) = m_\pm(p_c)$. The selection process is explained in Fig.1(a)-(c).

- **Typical pairs decoding** - is based on randomly selecting a $\mathbf{n}$ from $\mathcal{I}_{pc}$ with $m(\mathbf{n}) = m_\pm(p)[1]$; an error is declared when $\mathbf{n}^0$ is not the only element of $\mathcal{I}_{pc}$. For the same reason as above, the critical noise level $p_c$ is determined by the condition $m_\pm(p_c) = m_\pm(p_c)$.

- **Finite temperature (MPM) decoding** - An energy $-F_m(\mathbf{n})$ (with $F = 1/2 \ln(1/p)$) according to Nishimori’s condition is attributed to each $\mathbf{n} \in \mathcal{I}_{pc}$, and a solution is chosen from those with the magnetization that minimizes the free energy [2]. This procedure is known to minimize the bit error probability [1]. Using the thermodynamic relation $F = \mathcal{U} - \frac{1}{\beta} \mathcal{S}$, $\beta$ being the inverse temperature (Nishimori’s condition corresponds to setting $\beta = 1$), the free energy of the sub-optimal solutions is given by $F(\mathbf{m}) = -F_m - \frac{1}{\beta} \mathcal{M}(\mathbf{m})$ (for $\mathcal{M}(\mathbf{m}) \geq 0$), while that of the correct solution is given by $-F_m(\mathbf{n}_0)$ (its entropy being 0). The selection process is explained graphically in Fig.1(a)-(c).

The free energy differences between sub-optimal solutions relative to that of the correct solution in the current plots, are given by the orthogonal distance between $\mathcal{M}(\mathbf{m})$ and the line with slope $-\beta F$ through the point $(m_\pm(p), 0)$. Solutions with a magnetization $m$ for which $\mathcal{M}(\mathbf{m})$ lies above this line, have a lower free energy, while those for which $\mathcal{M}(\mathbf{m})$ lies below, have a higher free energy. Since negative entropy values are unphysical in discrete systems, only sub-optimal solutions with $\mathcal{M}(\mathbf{m}) \geq 0$ are considered. The lowest $p$ value for which there are sub-optimal solutions with a free energy equal to $-F_m(\mathbf{n})$ is the critical noise level $p_c$ for MPM decoding. In fact, using the convexity of $\mathcal{M}(\mathbf{m})$ and Nishimori’s condition, one can show that the slope $\partial \mathcal{M}(\mathbf{m})/\partial m > -\beta F$ for any value $m < m_\pm(p)$ and any $p$, and equals $-\beta F$ only at $m = m_\pm(p)$; therefore, the critical noise level for MPM decoding $p = p_c$ is identical to that of MAP, in agreement with results obtained in the information theory community [12].

The statistical physics interpretation of finite temperature decoding corresponds to making the specific choice for the Lagrange multiplier $\hat{m} = \beta F$ and considering the free energy instead of the entropy. In earlier work on MPM decoding in the SP framework [1], negative entropy values were treated by adopting different replica symmetry assumptions, which effectively result in changing the inverse temperature, i.e., the Lagrange multiplier $\hat{m}$. This effectively sets $m = m_\pm(p)$, i.e. to the highest value with non-negative entropy. The sub-optimal states with the lowest free energy are then those with $m = m_\pm(p)$.

The central point in all decoding schemes, is to select the correct solution only on the basis of its magnetization. As long as there are no sub-optimal solutions

---

1 This condition corresponds to the selection of an accurate prior within the Bayesian framework.
with the same magnetization, this is in principle possible. As shown here, all
three decoding schemes discussed above, manage to do so. To find whether at a
given $p$ there exists a gap between the magnetization of the correct solution and
that of the nearest sub-optimal solution, just requires plotting $M(m)(>0)$ and
$m_0(p)$, thus allowing a graphical determination of $p_c$. Since MPM decoding is
done at Nishimori’s temperature, the simplest replica symmetry assumption is
sufficient to describe the thermodynamically dominant state [9]. At $p_c$ the states
with $m_+(p_c) = m_0(p_C)$ are thermodynamically dominant, and the $p_c$ values that
we obtain under this assumption are exact.

5 Critical noise level - results

Some general comments can be made about the critical MAP (or typical set)
values obtained via the annealed and quenched calculations. Since $M_q(m) \leq
M_a(m)$ (for given values of $K$, $C$ and $p$), we can derive the general inequality
$p_{c,q} \geq p_{c,a}$. For all $K$, $C$ values that we have numerically analyzed, for both
annealed and quenched cases, $m_+(p)$ is a non increasing function of $p$, and $p_c$
is unique. The estimates of the critical noise levels $p_{c,a/q}$, based on $M_{a/q}$, are
obtained by numerically calculating $m_{c,a/q}(p)$, and by determining their inter-
section with $m_0(p)$. This is explained graphically in Fig. 2(a). As the results for
MPM decoding have already been presented elsewhere [13], we will now concen-
trate on the critical results $p_c$ obtained for typical set and MAP decoding; these
are presented in Fig. 2(b), showing the values of $p_{c,a/q}$ for various choices of $K$
and $C$ are compared with those reported in the literature.

From the table it is clear that the annealed approximation gives a much more
pessimistic estimate for $p_c$. This is due to the fact that it overestimates $M$ in
the following way. $M_a(m)$ describes the combined entropy of $n$ and $n^o$ as if $n^o$
were thermal variables as well. Therefore, exponentially rare events for $n^o$ (i.e.
$m(n^o) \neq m_0(p)$) still may carry positive entropy due to the addition of a positive
entropy term from $n$. In a separate study [14] these effects have been taken care
of by the introduction of an extra exponent; this is not necessary in the current
formalism as the quenched calculation automatically suppresses such contribu-
tions. The similarity between the results reported here and those obtained in [8]
is not surprising as the equations obtained in quenched calculations are similar
to those obtained by averaging the upper-bound to the reliability exponent using
a methods presented originally by Gallager [6]. Numerical differences between
the two sets of results are probably due to the higher numerical precision here.

6 Conclusions

To summarize, we have shown that the magnetization enumerator $M(m)$ plays a
central role in determining the achievable critical noise level for various decoding
schemes. The formalism based on the magnetization enumerator $M$ offers a
intuitively simple alternative to the weight enumerator formalism as used in
typical pairs decoding [3,14], but requires invoking the replica method given
the very low critical values obtained by the annealed approximation calculation. Although we have concentrated here on the critical noise level for the BSC, both other channels and other quantities can also be treated in our formalism. The predictions for the critical noise level are more optimistic than those reported in the IT literature, and are up to numerical precision in agreement with those reported in [4]. Finally, we have shown that the critical noise levels for typical pairs, MAP and MPM decoding must coincide, and we have provided an intuitive explanation to the difference between MAP and MPM decoding.

Support by Grants-in-aid, MEXT (13680400) and JSPS (YK), The Royal Society and EPSRC-GR/N00562 (DS/JvM) is acknowledged.

References

1. MacKay, D.J.C.: Good Error-correcting Codes Based on Very Sparse Matrices: IEEE Transactions on Information Theory 45 (1999) 399-431.
2. Richardson, T., Shokrollahi, A., Urbanke, R.: Design of Provably Good Low-density Parity-check Codes: IEEE Transactions on Information Theory (1999) in press.
3. Aji, S., Jin, H., Khandekar, A., MacKay, D.J.C., McEliece, R.J.: BSC Thresholds for Code Ensembles Based on “Typical Pairs” Decoding: In: Marcus, B., Rosenthal, J. (eds): Codes, Systems and Graphical Models. Springer Verlag, New York (2001) 195-210.
4. Kabashima, Y., Murayama, T., Saad, D.: Typical Performance of Gallager-type Error-Correcting Codes: Phys. Rev. Lett. 84 (2000) 1355-1358. Murayama, T., Kabashima, Y., Saad, D., Vicente, R.: The Statistical Physics of Regular Low-Density Parity-Check Error-Correcting Codes: Phys. Rev. E 62 (2000) 1577-1591.
5. Nishimori, H., Wong, K.Y.M.: Statistical Mechanics of Image Restoration and Error Correcting Codes: Phys. Rev. E 60 (1999) 132-144.
6. Gallager, R.G.: Low-density Parity-check Codes: IRE Trans. Info. Theory IT-8 (1962) 21-28. Gallager, R.G.: Low-density Parity-check Codes, MIT Press, Cambridge, MA. (1963).
7. Gallager, R.G.: Information Theory and Reliable Communication: Weily & Sons, New York (1968).
8. Kabashima, Y., Suzuka, N., Nakamura, K., Saad, D.: Tighter Decoding Reliability Bound for Gallager’s Error-Correcting Code: Phys. Rev. E (2001) in press.
9. Nishimori, H.: Statistical Physics of Spin Glasses and Information Processing. Oxford University Press, Oxford UK (2001).
10. Opper, M., Saad, D.: Advanced Mean Field Methods - Theory and Practice. MIT Press, Cambridge MA (2001).
11. Iba, Y.: The Nishimori Line and Bayesian Statistics : Jour. Phys. A 32 (1999) 3875-3888.
12. MacKay, D.J.C.: On Thresholds of Codes: “http://wol.ra.phy.cam.ac.uk/~mackay/CodesTheory.html” (2000) unpublished.
13. Vicente, R., Saad, D., Kabashima, Y.: Error-correcting Code on a Cactus - a Solvable Model: Europhys. Lett. 51, (2000) 698-704.
14. Kabashima, Y., Nakamura, K., van Mourik, J.: Statistical Mechanics of Typical Set Decoding: cond-mat/0106323 (2001) unpublished.
15. Shannon, C.E.: A Mathematical Theory of Communication: Bell Sys. Tech. J., 27 (1948) 379-423, 623-656.
Fig. 1. The qualitative picture of $\mathcal{M}(m) \geq 0$ (solid lines) for different values of $p$. For MAP, MPM and typical set decoding, only the relative values of $m_+(p)$ and $m_0(p)$ determine the critical noise level. Dashed lines correspond to the energy contribution of $-\beta F$ at Nishimori’s condition ($\beta = 1$). The states with the lowest free energy are indicated with •. 

a) Sub-critical noise levels $p < p_c$, where $m_+(p) < m_0(p)$, there are no solutions with higher magnetization than $m_0(p)$, and the correct solution has the lowest free energy. 

b) Critical level $p = p_c$, where $m_+(p) = m_0(p)$. The minimum of the free energy of the sub-optimal solutions is equal to that of the correct solution at Nishimori’s condition. 

c) Over-critical noise levels $p > p_c$, where many solutions have a higher magnetization than the true typical one. The minimum of the free energy of the sub-optimal solutions is lower than that of the correct solution.
Fig. 2. a) Determining the critical noise levels $p_{c,a/q}$ based on the function $M_{a/q}$, a qualitative picture. b) Comparison of different critical noise level ($p_c$) estimates. Typical set decoding estimates have been obtained via the methods of IT [3], based on having a unique solution to $W(m) = K(m, p_c)$, as well as using the methods of SP [14]. The numerical precision is up to the last digit for the current method. Shannon’s limit denotes the highest theoretically achievable critical noise level $p_c$ for any code [15].