THE SPACE OF D-NORMS REVISITED

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Abstract. The theory of D-norms is an offspring of multivariate extreme value theory: Replacing the spectral measure of a max-stable distribution $G$ by a $d$-dimensional random vector $Z$ allows the representation of $G$ via a norm on $\mathbb{R}^d$, called D-norm, whose generator is $Z$. We present recent results on D-norms. In the first part it is shown that the space of D-norms is a complete separable metric space, if equipped with the Wasserstein-metric in a suitable way. Secondly, multiplying a generator $Z$ with a doubly stochastic matrix yields another generator. An iteration of this multiplication provides a sequence of D-norms and we compute its limit. Finally, we consider a parametric family of D-norms, where we assume that $Z$ follows a symmetric Dirichlet distribution with parameter $\alpha > 0$. This family covers the whole range between complete dependence and independence.

1. Introduction

The theory of D-norms is an offspring of multivariate extreme value theory: Replacing the spectral measure of a max-stable distribution function (df) with standard negative margins by a $d$-dimensional random vector allows its representation via a norm on $\mathbb{R}^d$, called D-norm.

A norm $\| \cdot \|_D$ on $\mathbb{R}^d$ is a D-norm, if there exists a rv $Z = (Z_1, \ldots, Z_d)$ with $Z_i \geq 0$, $E(Z_i) = 1$, $1 \leq i \leq d$, such that

$$
\| x \|_D = E \left( \max_{1 \leq i \leq d} (|x_i| Z_i) \right) = E (\| xZ \|_\infty),
$$

$x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. In this case $Z$ is called generator of $\| \cdot \|_D$. By $\| x \|_\infty = \max_{1 \leq i \leq d} |x_i|$ we denote the usual sup-norm on $\mathbb{R}^d$; all operations on vectors such as $xZ = (x_1Z_1, \ldots, x_dZ_d)$ are meant componentwise.

Examples of D-norms are

(i) the sup-norm $\| x \|_\infty = \max_{1 \leq i \leq d} |x_i|$, which is generated by $Z = (1, \ldots, 1)$.
(ii) the $L_1$-norm $\|x\|_1 = \sum_{i=1}^d |x_i|$, generated by a random permutation of $(d, 0, \ldots, 0) \in \mathbb{R}^d$ with equal probability $1/d$.

(iii) the usual logistic-norm $\|x\|_\lambda = \left(\sum_{i=1}^d |x_i|^\lambda\right)^{1/\lambda}$, $1 < \lambda < \infty$. An explicit generator was only quite recently found: Let $X_1, \ldots, X_d$ be independent and identically Fréchet-distributed rv, i.e., $P(X_i \leq x) = \exp(-x^{-\lambda})$, $x > 0$, $\lambda > 1$. Then $Z = (Z_1, \ldots, Z_d)$ with

$$Z_i := \frac{X_i}{\Gamma(1 - p^{-1})}, \quad i = 1, \ldots, d,$$

generates $\|\cdot\|_\lambda$.

A df $G$ on $\mathbb{R}^d$ is a standard max-stable (sms) or standard extreme value df if

$$G(x) = G^n\left(\frac{x}{n}\right), \quad x \in \mathbb{R}^d, n \in \mathbb{N},$$

and for $1 \leq i \leq d$

$$G_i(x) := G(0, \ldots, 0, x_i, 0, \ldots, 0) = \exp(x_i), \quad x \leq 0 \in \mathbb{R}^d.$$

The following characterization of a standard max-stable df is a consequence of the results by Pickands (1975), de Haan and Resnick (1977) and Vatan (1985).

**Theorem 1.1** (Pickands, de Haan-Resnick, Vatan). A df $G$ on $\mathbb{R}^d$ is a standard max-stable df iff there exists a $D$-norm $\|\cdot\|_D$ on $\mathbb{R}^d$ such that

$$G(x) = \exp(-\|x\|_D), \quad x \leq 0 \in \mathbb{R}^d.$$

The generator $Z$ of a $D$-norm $\|\cdot\|_D$ is in general not uniquely determined, even its distribution is not, cf. Section 4. The sup-norm $\|\cdot\|_\infty$, for example, can be generated by every rv $Z = (Z, \ldots, Z)$ with constant entry $Z$ which is a positive rv with expectation 1.

The particular value

$$\|\mathbf{1}\|_D = \mathbb{E}\left(\max_{1 \leq i \leq d} Z_i\right)$$

of a $D$-norm on $\mathbb{R}^d$ with generator $Z$ is the generator constant or extremal coefficient, cf. Smith (1990), where $\mathbf{1} = (1, \ldots, 1)$. While a generator is in general not uniquely determined by the $D$-norm, the generator constant obviously is. It is a measure of dependence between the margins of the multivariate standard max-stable or extreme value df $G(x) := \exp(-\|x\|_D)$, $x \leq 0 \in \mathbb{R}^d$, see Falk et al. (2011, Section 4.4). We have by Takahashi’s (1988) theorem

$$\|\cdot\|_D = \|\cdot\|_1 \iff \|\mathbf{1}\|_D = d,$$

which is the case of independence of the margins of $G$, and

$$\|\cdot\|_D = \|\cdot\|_\infty \iff \|\mathbf{1}\|_D = 1,$$
which is the case of complete dependence of the margins. Note that we have

$$\| \cdot \|_\infty \leq \| \cdot \|_D \leq \| \cdot \|_1$$

for any $D$-norm, with the lower and the upper bound being $D$-norms themselves.

A rv $\eta$ that follows the smd df $G(x) = \exp(-\|x\|_D)$, $x \leq 0 \in \mathbb{R}^d$, can be generated in the following way. Consider a Poisson point process on $[0, \infty)$ with mean measure $r^{-2}dr$. Let $V_i, i \in \mathbb{N}$, be a realization of this point process. Consider independent copies $Z^{(1)}, Z^{(2)}, \ldots$ of a generator $Z$ of the $D$-norm $\| \cdot \|_D$, which are also independent of the Poisson process. Then we have

$$\eta = D - \frac{1}{\sup_{i \in \mathbb{N}} V_i Z^{(i)}}$$

which is a consequence of De Haan and Ferreira (2006, Lemma 9.4.7) and elementary computations.

Let $\| \cdot \|_{D_1}, \| \cdot \|_{D_2}$ be two $D$-norms on $\mathbb{R}^d$ with generators $Z^{(1)}, Z^{(2)}$. Suppose that these generators are independent. Then the product $Z := Z^{(1)} Z^{(2)}$, taken componentwise, defines the generator of a $D$-norm $\| \cdot \|_{D_1 \times D_2}$, say. This entails the definition of a multiplication type operation on the set of $D$-norms; note that this product $D$-norm does not depend on the special choice of generators. A $D$-norm $\| \cdot \|_D$ is called idempotent, if $\| \cdot \|_{D \times D} = \| \cdot \|_D$. The sup-norm $\| \cdot \|_\infty$ and the $L_1$-norm $\| \cdot \|_1$ are idempotent $D$-norms. Iterating the multiplication provides a track of $D$-norms, whose limit exists and is again a $D$-norm. If this iteration is repeatedly done on the same $D$-norm, then the limit of the track is idempotent, see Falk (2013), where also the set of idempotent $D$-norms is characterized.

In Section 2 of the present paper we define a metric on the space of $D$-norms such that it becomes a complete metric space. Convergence of $D$-norms is then equivalent with weak convergence of the corresponding generators. Multiplying a generator with a bistochastic or doubly stochastic matrix generates a new generator and, thus, another $D$-norm. Iterating the multiplication leads to a sequence of $D$-norms, whose limit is established in Section 3. A particularly interesting parametric model for generators is provided by the symmetric Dirichlet-distributions. In Section 4 we investigate this parametric family in detail.

2. Metrization of the Space of $D$-Norms

Denote by $Z_{\| \cdot \|_D}$ the set of all generators of a given $D$-norm $\| \cdot \|_D$ on $\mathbb{R}^d$. The proof of the de Haan-Resnick representation of a max-stable multivariate extreme value df as in Falk et al. (2011, Section 4.2) implies the following result.
Lemma 2.1. Each set $Z_{\|\cdot\|_D}$ contains a generator $Z$ with the additional property $\|Z\|_1 = d$. The distribution of this $Z$ is uniquely determined.

Let $\mathbb{P}$ be the set of all probability measures on $S_d := \{ x \geq 0 \in \mathbb{R}^d : \|x\|_1 = d \}$. We, thus, can identify the set $\mathcal{D}$ of $D$-norms on $\mathbb{R}^d$ with the subset $\mathbb{P}_D$ of those probability distributions $P \in \mathbb{P}$ which satisfy the additional condition $\int_{S_d} x_i P(dx) = 1$, $i = 1, \ldots, d$.

Denote by $d_W(P, Q)$ the Wasserstein metric between two probability distributions on $S_d$, i.e.,

$$d_W(P, Q) := \inf \{ E(\|X - Y\|_1) : X \text{ has distribution } P, Y \text{ has distribution } Q \}.$$

As $S_d$, equipped with an arbitrary norm $\|\cdot\|$, is a complete separable space, the metric space $(\mathbb{P}, d_W)$ is complete and separable as well; see, e.g., Bolley (2008).

Lemma 2.2. The subspace $(\mathbb{P}_D, d_W)$ of $(\mathbb{P}, d_W)$ is also separable and complete.

Proof. Let $P_n, n \in \mathbb{N}$, be a sequence in $\mathbb{P}_D$, which converges with respect to $d_W$ to $P \in \mathbb{P}$. We show that $P \in \mathbb{P}_D$. Let the rv $X$ have distribution $P$ and let $X^{(n)}$ have distribution $P_n, n \in \mathbb{N}$. Then we have

$$\sum_{i=1}^d \left| \int_{S_d} x_i P(dx) - 1 \right| = \sum_{i=1}^d \left| \int_{S_d} x_i P(dx) - \int_{S_d} x_i P_n(dx) \right|$$

$$= \sum_{i=1}^d E \left( X_i - X_i^{(n)} \right)$$

$$\leq E \left( \sum_{i=1}^d \left| X_i - X_i^{(n)} \right| \right)$$

$$= E \left( \left\| X - X^{(n)} \right\|_1 \right), \quad n \in \mathbb{N}.$$

As a consequence we obtain

$$\sum_{i=1}^d \left| \int_{S_d} x_i P(dx) - 1 \right| \leq d_W(P, P_n) \to_{n \to \infty} 0,$$

and, thus, $P \in \mathbb{P}_D$. The separability of $\mathbb{P}_D$ can be seen as follows. Let $\mathcal{P}$ be a countable and dense subset of $\mathbb{P}$. Identify each distribution $P$ in $\mathcal{P}$ with a random vector $Y$ on $S_d$ that follows this distribution $P$. Put $Z = Y/E(Y)$, where we can assume that each component of $Y$ has positive expectation. This yields a countable subset of $P_D$, which is dense. $\square$

We can now define the distance between two $D$-norms $\|\cdot\|_{D_1}, \|\cdot\|_{D_2}$ on $\mathbb{R}^d$ by

$$d_W(\|\cdot\|_{D_1}, \|\cdot\|_{D_2})$$
\[ \inf \left\{ E \left( \left\| Z^{(1)} - Z^{(2)} \right\|_1 \right) : Z^{(i)} \text{ generates } \| \cdot \|_{D_i}, \left\| Z^{(i)} \right\|_1 = d, i = 1, 2 \right\} \]

The space \( \mathbb{D} \) of \( D \)-norms on \( \mathbb{R}^d \), equipped with the distance \( d_W(\cdot, \cdot) \), is by Lemma 2.2 a complete and separable metric space.

For the rest of this section we restrict ourselves to generators \( Z \) of \( D \)-norms on \( \mathbb{R}^d \) that satisfy \( \| Z \|_1 = d \).

**Lemma 2.3.** Let \( \|\cdot\|_{D_n}, n \in \mathbb{N} \cup \{0\} \), be a sequence of \( D \)-norms on \( \mathbb{R}^d \) with corresponding generators \( Z^{(n)} \), \( n \in \mathbb{N} \cup \{0\} \). Then we have the equivalence

\[
  d_W \left( \|\cdot\|_{D_n}, \|\cdot\|_{D_0} \right) \to_{n \to \infty} 0 \iff Z^{(n)} \to_D Z^{(0)},
\]

where \( \to_D \) denotes ordinary convergence in distribution.

**Proof.** Convergence of probability measures \( P_n \) to \( P_0 \) with respect to the Wasserstein-metric is equivalent with weak convergence together with convergence of the moments

\[
  \int_{S_d} \|x\|_1 P_n(\,d x) \to_{n \to \infty} \int_{S_d} \|x\|_1 P_0(\,d x),
\]

see, e.g., Villani (2009). But as we have for each probability measure \( P \in \mathbb{P}_D \)

\[
  \int_{S_d} \|x\|_1 P(\,d x) = \int_{S_d} dP(\,d x) = d,
\]

convergence of the moments is automatically satisfied. \( \square \)

**Lemma 2.4.** We have for arbitrary \( D \)-norms \( \|\cdot\|_{D_1}, \|\cdot\|_{D_2} \) on \( \mathbb{R}^d \) the bound

\[
  \|x\|_{D_1} \leq \|x\|_{D_2} + \|x\|_\infty d_W \left( \|\cdot\|_{D_1}, \|\cdot\|_{D_2} \right)
\]

and, thus,

\[
  \sup_{x \in \mathbb{R}^d, \|x\|_\infty \leq r} \left| \|x\|_{D_1} - \|x\|_{D_2} \right| \leq r d_W \left( \|\cdot\|_{D_1}, \|\cdot\|_{D_2} \right), \quad r \geq 0.
\]

**Proof.** Let \( Z^{(i)} \) be a generator of \( \|\cdot\|_{D_i}, i = 1, 2 \). We have

\[
  \|x\|_{D_1} = E \left( \max_{1 \leq i \leq d} \left| x_i \right| Z_i^{(1)} \right) \nonumber
\]

\[
  = E \left( \max_{1 \leq i \leq d} \left( \left| x_i \right| \left( Z_i^{(2)} + Z_i^{(1)} - Z_i^{(2)} \right) \right) \right) \nonumber
\]

\[
  \leq E \left( \max_{1 \leq i \leq d} \left| x_i \right| Z_i^{(2)} \right) + \|x\|_\infty E \left( \max_{1 \leq i \leq d} \left| Z_i^{(1)} - Z_i^{(2)} \right| \right),
\]

which implies the assertion. \( \square \)
3. Doubly Stochastic Matrices

Denote by \( \mathbb{M} \) the set of all doubly stochastic (or bistochastic) \( d \times d \)-matrices. If \( Z \) is the generator of a \( D \)-norm \( \| \cdot \|_D \), then
\[
Z_M := MZ
\]
is the generator of a \( D \)-norm for each \( M \in \mathbb{M} \) as well.

Let, for instance, \( Z \) be a random permutation of the vector \((d, 0, \ldots, 0)^\top \in \mathbb{R}^d \) with equal probability \( 1/d \). The corresponding \( D \)-norm is \( \| \cdot \|_1 \), which is an upper bound for each \( D \)-norm. Let \( M_0 \) be the \( d \times d \)-matrix with constant entry \( 1/d \). Then we obtain
\[
Z_{M_0} = M_0Z = (1, \ldots, 1)^\top,
\]
which is the generator of the \( D \)-norm \( \| \cdot \|_\infty \). This \( D \)-norm is a lower bound for each \( D \)-norm. This example shows the influence that the multiplication of a generator with a doubly stochastic matrix can have. Note that actually \( M_0Z = (1, \ldots, 1)^\top \) for each generator \( Z \) satisfying \( \| Z \|_1 = d \).

By identifying a generator \( Z \) with its corresponding \( D \)-norm \( \| \cdot \|_{D(Z)} \), say, we define the function \( f : \mathbb{M} \times \mathbb{D} \to \mathbb{D} \) by
\[
f(M, \| \cdot \|_{D(Z)}) := \| \cdot \|_{D(MZ)} ;
\]
recall that the distribution of the generator \( Z \) of a \( D \)-norm is uniquely determined under the additional condition \( \| Z \|_1 = d \).

**Lemma 3.1.** If we equip \( \mathbb{M} \) with the metric \( \| M_1 - M_2 \|_1 = \sum_{i,j=1}^d |m_{ij}^{(1)} - m_{ij}^{(2)}| \), \( M_1, M_2 \in \mathbb{M} \), and the space \( \mathbb{D} \) of all \( D \)-norms on \( \mathbb{R}^d \) with the Wasserstein metric \( d_W \), then the function \( f \) is continuous, precisely,
\[
d_W \left( f \left( M_1, \| \cdot \|_{D(Z^{(1)})} \right) , f \left( M_1, \| \cdot \|_{D(Z^{(2)})} \right) \right)
\leq \| M_1 - M_2 \|_1 + d d_W \left( \| \cdot \|_{D(Z^{(1)})} , \| \cdot \|_{D(Z^{(2)})} \right).
\]

**Proof.** The triangular inequality implies
\[
d_W \left( f \left( M_1, \| \cdot \|_{D(Z^{(1)})} \right) , f \left( M_2, \| \cdot \|_{D(Z^{(2)})} \right) \right)
\leq d_W \left( f \left( M_1, \| \cdot \|_{D(Z^{(1)})} \right) , f \left( M_2, \| \cdot \|_{D(Z^{(1)})} \right) \right)
+ d_W \left( f \left( M_2, \| \cdot \|_{D(Z^{(1)})} \right) , f \left( M_2, \| \cdot \|_{D(Z^{(2)})} \right) \right)
\leq E \left( \| (M_1 - M_2)Z^{(1)} \|_1 \right) + E \left( \| M_2 (Z^{(1)} - Z^{(2)}) \|_1 \right)
\leq \| M_1 - M_2 \|_1 + dE \left( \| Z^{(1)} - Z^{(2)} \|_1 \right),
\]
which yields the assertion. \( \square \)
Let $\mathbf{Z}$ be a random permutation of the vector $(d,0,\ldots,0) \in \mathbb{R}^d$ with equal probability $1/d$ and set $M = (1/d) \in \mathbb{R}^{d \times d}$. Then we obtain from Lemma 3.1 the bound

$$dW(\|\cdot\|_{\infty}, \|\cdot\|_1) = dW(f(M, \|\cdot\|_1), f(Id, \|\cdot\|_1)) \leq \|M - Id\|_1 = 2(d - 1),$$

where $Id$ is the unit $d \times d$-matrix. Note that this bound is sharp by the fact that the distribution of a generator $\mathbf{Z}$ with $\|\mathbf{Z}\|_1 = d$ is uniquely determined and, thus, we compute $dW(\|\cdot\|_{\infty}, \|\cdot\|_1) = 2(d - 1)$.

The idea suggests itself to iterate the multiplication of a generator with a matrix and to consider

$$\mathbf{Z}^{(n)} := M^n \mathbf{Z}, \quad n \in \mathbb{N},$$

where $M^n$ denotes the ordinary $n$-times matrix product. The question, whether the sequence $\mathbf{Z}^{(n)}$, $n \in \mathbb{N}$, converges, can be answered by fundamental results from the theory of Markov chains. In particular we obtain the following result, which shows the sequence of $D$-norms $\|\cdot\|_{D(\mathbf{Z}^{(n)})}$, $n \in \mathbb{N}$ converges to $\|\cdot\|_1$ under mild conditions on the matrix $M$.

**Proposition 3.2.** Suppose that each entry $M^n(i,j)$ of the matrix $M^n$ is positive if $n$ is large. Then we obtain for an arbitrary generator $\mathbf{Z}$

$$\mathbf{Z}^{(n)} \to_{n \to \infty} (1, \ldots, 1)^T \in \mathbb{R}^d.$$  

The condition $M^n(i,j) > 0$ for each $i,j \in \{1, \ldots, d\}$ cannot be dropped in the preceding result; just set $M = Id$, the unit matrix, or let $M$ be that matrix with constant entry 1 on its secondary diagonal and zero elsewhere.

**Proof.** The matrix $M = (m(i,j))_{1 \leq i,j \leq d}$ can be viewed as a matrix of transition probabilities $p(j|i)$ from the state $i$ to the state $j$, where $i,j \in \{1, \ldots, d\}$, and, thus, the transition matrix $M$ defines a time-homogenous Markov chain on the state space $\{1, \ldots, d\}$. The condition that each entry of $M^n$ is positive for large $n$ is equivalent with the condition that $M$ is aperiodic and irreducible. It is well-known from the theory of Markov chains that in this case

$$M^n(i,j) \to_{n \to \infty} \mu(j), \quad i,j \in \{1, \ldots, d\},$$

where the (row) vector $\mu$ is the uniquely determined stationary distribution on $\{1, \ldots, d\}$, i.e., $\mu M = \mu$. As $M$ is bistochastic, we obtain $\mu(j) = 1/d$, $j = 1, \ldots, d$, which completes the proof. \qed
4. THE $D$-NORM GENERATED FROM A SYMMETRIC DIRICHLET DISTRIBUTION

Let in what follows $V_1, \ldots, V_d, d \geq 2$, be independent and identically gamma distributed rv with density $\gamma_\alpha(x) := x^{\alpha-1} \exp(-x)/\Gamma(\alpha), x > 0, \alpha > 0$. Then the rv $\tilde{Z} \in \mathbb{R}^d$ with components

$$\tilde{Z}_i := \frac{V_i}{V_1 + \cdots + V_d}, \quad i = 1, \ldots, d,$$

follows a symmetric Dirichlet distribution $\text{Dir}(\alpha)$ on the closed simplex $\tilde{S}_d = \{u \geq 0 \in \mathbb{R}^d : \sum_{i=1}^d u_i = 1\}$, see Ng et al. (2011, Theorem 2.1). By equation (2.6) in this reference we have $E(\tilde{Z}_i) = 1/d$ and, thus,

$$Z := d\tilde{Z}$$

is a generator of a $D$-norm $\|\cdot\|_{D(\alpha)}$ on $\mathbb{R}^d$, which we call the Dirichlet $D$-norm with parameter $\alpha$. We have in particular $\|Z\|_1 = d$.

Note that $\gamma_1(x) = \exp(-x), x > 0$, is the density of the standard exponential distribution, in which case $\tilde{Z}_i = D(U_i: d-1 - U_i: d-1)$, where $U_{1:d-1} \leq U_{2:d-1} \leq \cdots \leq U_{d-1:d-1}$ are the order statistics pertaining to $d-1$ independent and on $(0, 1)$ uniformly distributed rv, $U_{0:d-1} := 0, U_{d:d-1} := 1$, see Reiss (1989, Theorem 1.6.7). The distribution of the rv $\tilde{Z}$ with $\alpha = 1$ is, therefore, that of the vector of uniform spacings.

It is well-known that for a general $\alpha > 0$ the rv $(V_i/\sum_{j=1}^d V_j)^d$ and the sum $\sum_{j=1}^d V_j$ are independent, see, e.g., the proof of Theorem 2.1 in Ng et al. (2011). As $E(V_1 + \cdots + V_d) = d\alpha$, we obtain for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$

$$\|x\|_{D(\alpha)} = E\left(\max_{1 \leq i \leq d} (|x_i| Z_i)\right)$$

$$= d E\left(\frac{\max_{1 \leq i \leq d} (|x_i| V_i)}{V_1 + \cdots + V_d}\right)$$

$$= \frac{1}{\alpha} E(V_1 + \cdots + V_d) E\left(\frac{\max_{1 \leq i \leq d} (|x_i| V_i)}{V_1 + \cdots + V_d}\right)$$

$$= \frac{1}{\alpha} E\left(\max_{1 \leq i \leq d} (|x_i| V_i)\right).$$

(3)

Note that the independence of $V_i/(V_1 + \cdots + V_d)$ and $V_1 + \cdots + V_d, 1 \leq i \leq d$, is by Lukacs’ theorem a characteristic property of the gamma distribution; see, e.g., Ng et al. (2011, Section 2.6.1) for details.
The Dirichlet model for bivariate extreme value df was investigated by Coles and Tawn (1991, Section 4.3), Segers (2012, Example 3.6) studies the Dirichlet model in arbitrary dimension. Boldi and Davison (2007, Appendix A) show that each $D$-norm can be approximated by a $D$-norm generated by a mixture of Dirichlet distributions.

The symmetric Dirichlet distribution is also an appealing parametric model for a rv that follows a generalized Pareto distribution (GPD). Let $U$ be uniformly on $(0,1)$ distributed and independent of the generator $Z$ as defined in (2). Then

$$Y := -U \frac{1}{Z}$$

follows a GPD with

$$P(Y \leq x) = 1 - \frac{1}{\alpha} E \left( \max_{1 \leq i \leq d} (|x_i| V_i) \right)$$

for all $x \leq 0 \in \mathbb{R}^d$ with $\|x\|_\infty \leq 1/d$. Equally,

$$P(Y > x) = \frac{1}{\alpha} E \left( \min_{1 \leq i \leq d} (|x_i| V_i) \right).$$

In particular in the case $\alpha = 1$ we obtain from the min-stability of the exponential distribution on $[0, \infty)$

$$E \left( \min_{1 \leq i \leq d} V_i \right) = \frac{1}{d}$$

and, thus,

$$P(Y > -c) = \frac{c}{d}, \quad 0 \leq c \leq 1/d.$$

For an account of multivariate GPD we refer to Falk et al. (2011, Chapter 5).

We discuss in what follows the generator constant function

$$m(\alpha) := \|1\|_{D(\alpha)}, \quad \alpha > 0,$$

pertaining to the Dirichlet $D$-norms. We start with the bivariate case.

From the arguments in Coles and Tawn (1991, Section 4.3) we obtain the representation

$$\|(x,y)\|_{D(\alpha)} = |x| B \left( \alpha, \alpha + 1, \frac{|x|}{|x| + |y|} \right) + |y| B \left( \alpha, \alpha + 1, \frac{|y|}{|x| + |y|} \right),$$

where

$$B(a,b,x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^x u^{a-1}(1-u)^{b-1} du, \quad x \in [0,1],$$

denotes the normalized incomplete beta function. The next result follows from tedious but elementary computations.

**Proposition 4.1** (The bivariate case). We have for all $\alpha > 0$

$$m(\alpha) = 1 + \frac{\Gamma \left( \frac{\alpha}{2} \right)}{\sqrt{\pi} \Gamma(\alpha + 1)} = 1 + \frac{1}{\alpha B \left( \alpha, \frac{1}{2} \right)}.$$
The fact that the function $m(\alpha)$ is strictly decreasing and that it attains each value in the interval $(1, d)$ is shown for arbitrary dimension in what follows.

Denote by $F_\alpha$ the df of the gamma df with parameter $\alpha > 0$, i.e.,

$$F_\alpha(x) = \frac{\gamma(\alpha, x)}{\Gamma(\alpha)} = 1 - \frac{\Gamma(\alpha, x)}{\Gamma(\alpha)}, \quad x > 0,$$

where $\gamma(\alpha, x) = \int_0^x \exp(-t)t^{\alpha-1} dt$ and $\Gamma(\alpha, x) = \int_x^\infty \exp(-t)t^{\alpha-1} dt = \Gamma(\alpha) - \gamma(\alpha, x)$ are the lower and the upper incomplete gamma function.

**Lemma 4.2** (Arbitrary dimension). Let $m(\alpha) = \|1\|_{D(\alpha)}$ be the generator constant of the $d$-dimensional Dirichlet generator. Then we have $\lim_{\alpha \to 0} m(\alpha) = d$ and $\lim_{\alpha \to \infty} m(\alpha) = 1$.

**Proof.** We have by the formula

$$a^n - b^n = a - b \sum_{k=0}^{n-1} \left( \frac{b}{a} \right)^k, \quad a, b \in \mathbb{R}, \ a \neq 0,$$

$$m(\alpha) = \frac{1}{\alpha} \int_0^\infty 1 - F_\alpha(x)^d \ dx$$

$$= \frac{1}{\alpha} \int_0^\infty (1 - F_\alpha(x)) \sum_{k=0}^{d-1} F_\alpha(x)^k \ dx$$

$$= \int_0^\infty \frac{\Gamma(\alpha, x)}{\alpha \Gamma(\alpha)} \sum_{k=0}^{d-1} \left( \frac{\gamma(\alpha, x)}{\Gamma(\alpha)} \right)^k \ dx$$

$$= \sum_{k=0}^{d-1} \int_0^\infty \frac{\Gamma(\alpha, x)}{\Gamma(\alpha + 1)} \left( \frac{\gamma(\alpha, x)}{\Gamma(\alpha)} \right)^k \ dx.$$

Note that we have by Fubini’s theorem for all $\alpha \geq 0$

$$\int_0^\infty \Gamma(\alpha, x) \ dx = \int_0^\infty \int_x^\infty t^{\alpha-1} \exp(-t) \ dt \ dx$$

$$= \int_0^\infty t^\alpha \exp(-t) \ dt \ dx$$

$$= \int_0^\infty t^\alpha \ exp(-t) \ dt = \Gamma(\alpha + 1).$$

Therefore, we obtain

$$m(\alpha) = 1 + \sum_{k=1}^{d-1} \int_0^\infty \frac{\Gamma(\alpha, x)}{\Gamma(\alpha + 1)} \left( \frac{\gamma(\alpha, x)}{\Gamma(\alpha)} \right)^k \ dx.$$

We know from Lemma 4.1 that

$$\int_0^\infty \frac{\Gamma(\alpha, x)\gamma(\alpha, x)}{\Gamma(\alpha + 1)\Gamma(\alpha)} \ dx = \frac{\Gamma(\alpha + \frac{4}{2})}{\sqrt{\pi} \Gamma(\alpha + 1)} \to_{\alpha \to \infty} 0.$$

This implies $m(\alpha) \to_{\alpha \to 1} 1$ in arbitrary dimension by (5) since

$$\left( \frac{\gamma(\alpha, x)}{\Gamma(\alpha)} \right)^k \leq \frac{\gamma(\alpha, x)}{\Gamma(\alpha)}, \quad k = 1, \ldots, d - 1.$$
In order to prove the convergence $m(\alpha) \to_{\alpha \to 0} d$, check that for each $x > 0$ and $k = 1, \ldots, d - 1$

$$\lim_{\alpha \to 0} \frac{\Gamma(\alpha, x)}{\Gamma(\alpha + 1)} \left( \frac{\gamma(\alpha, x)}{\Gamma(\alpha)} \right)^k = \lim_{\alpha \to 0} \frac{\Gamma(\alpha, x)}{\Gamma(\alpha + 1)} \left( 1 - \frac{\Gamma(\alpha, x)}{\Gamma(\alpha)} \right)^k = \Gamma(0, x),$$

since the (incomplete) gamma function is continuous in $\alpha$ and $\Gamma(\alpha) \to_{\alpha \to \infty} \infty$.

Suppose for now that we are allowed to interchange limit and integration, then by (4), (5) and (6)

$$\lim_{\alpha \to 0} m(\alpha) = 1 + \sum_{k=1}^{d-1} \int_0^\infty \Gamma(0, x) \, dx = 1 + \sum_{k=1}^{d-1} \Gamma(1) = d.$$

Therefore, it remains to show that limit and integration are, actually, interchangeable. To this end, we write

$$\lim_{\alpha \to 0} \int_0^\infty \frac{\Gamma(\alpha, x)}{\Gamma(\alpha + 1)} \left( \frac{\gamma(\alpha, x)}{\Gamma(\alpha)} \right)^k \, dx = \lim_{\alpha \to 0} \int_0^1 \frac{\Gamma(\alpha, x)}{\Gamma(\alpha + 1)} \left( \frac{\gamma(\alpha, x)}{\Gamma(\alpha)} \right)^k \, dx + \lim_{\alpha \to 0} \int_1^\infty \frac{\Gamma(\alpha, x)}{\Gamma(\alpha + 1)} \left( \frac{\gamma(\alpha, x)}{\Gamma(\alpha)} \right)^k \, dx.$$

We have for every $\alpha \leq 1$ and $x \geq 1$

$$\frac{\Gamma(\alpha, x)}{\Gamma(\alpha + 1)} \left( \frac{\gamma(\alpha, x)}{\Gamma(\alpha)} \right)^k \leq 2\Gamma(\alpha, x)$$

$$= 2 \int_x^\infty t^{\alpha-1} \exp(-t) \, dt$$

$$\leq 2 \int_x^\infty \exp(-t) \, dt$$

$$= 2 \exp(-x),$$

and since $\exp(-x)$ is integrable on $[1, \infty)$, the second summand in (7) converges to $\int_1^\infty \Gamma(0, x) \, dx$ for $\alpha \to 0$ by the dominated convergence theorem. On the other hand, we have for all $\alpha \leq 1$ and all $x \in (0, 1]$

$$\frac{\Gamma(\alpha, x)}{\Gamma(\alpha + 1)} \left( \frac{\gamma(\alpha, x)}{\Gamma(\alpha)} \right)^k \leq 2\Gamma(\alpha, x)$$

$$= 2 \left( \int_x^1 t^{\alpha-1} \exp(-t) \, dt + \int_1^\infty t^{\alpha-1} \exp(-t) \, dt \right)$$

$$\leq 2 \left( \int_x^1 t^{-1} \exp(-t) \, dt + \int_1^\infty \exp(-t) \, dt \right)$$

$$= 2(\Gamma(0, x) - \Gamma(0, 1) + \exp(-1)).$$

This upper bound is integrable on $(0, 1]$, which yields the convergence of the first summand in (7) to $\int_0^1 \Gamma(0, x) \, dx$ for $\alpha \to 0$ by the dominated convergence theorem again. Hence, the proof of this lemma is complete. \qed
The following auxiliary result will be the crucial tool in the proof of the monotonicity of the Dirichlet-$D$-norm $\|\cdot\|_{D(\alpha)}$ with respect to the parameter $\alpha > 0$, see below. It might be of interest of its own.

**Lemma 4.3.** Let $V_{ij}$, $1 \leq i \leq d$, $1 \leq j \leq n$, $d \in \mathbb{N}$, $n \geq 2$, be an array of iid integrable rv. Then we have for arbitrary numbers $x_1, \ldots, x_d \in \mathbb{R}$

$$E \left( \max_{1 \leq i \leq d} \left( x_i \frac{\sum_{j=1}^{n} V_{ij}}{n} \right) \right) \leq E \left( \max_{1 \leq i \leq d} \left( x_i \frac{\sum_{j=1}^{n-1} V_{ij}}{n - 1} \right) \right).$$

**Proof.** The case $n = 2$ is obvious: We have

$$\max_{1 \leq i \leq d} (x_i(V_{1i} + V_{2i})) \leq \max_{1 \leq i \leq d} (x_iV_{1i}) + \max_{1 \leq i \leq d} (x_iV_{2i})$$

and, thus, by the identical distribution of $V_{1i}, V_{2i}$, $1 \leq i \leq d$,

$$E \left( \max_{1 \leq i \leq d} (x_i(V_{1i} + V_{2i})) \right) \leq 2E \left( \max_{1 \leq i \leq d} (x_iV_{1i}) \right).$$

The case $n = 3$ provides the crucial argument for a general $n$. Set

$$x_{i*} (V_{i*1} + V_{i*2} + V_{i*3}) = \max_{1 \leq i \leq d} (x_i (V_{i1} + V_{i2} + V_{i3})).$$

We have the obvious inequalities

$$x_{i*} (V_{i*1} + V_{i*2}) \leq \max_{1 \leq i \leq d} (x_i (V_{i1} + V_{i2})),
$$

$$x_{i*} (V_{i*1} + V_{i*3}) \leq \max_{1 \leq i \leq d} (x_i (V_{i1} + V_{i3})),
$$

$$x_{i*} (V_{i*2} + V_{i*3}) \leq \max_{1 \leq i \leq d} (x_i (V_{i2} + V_{i3})).$$

Summing up these inequalities we obtain

$$2x_{i*} (V_{i*1} + V_{i*2} + V_{i*3}) \leq \max_{1 \leq i \leq d} (x_i (V_{i1} + V_{i2}))) + \max_{1 \leq i \leq d} (x_i (V_{i1} + V_{i3})) + \max_{1 \leq i \leq d} (x_i (V_{i2} + V_{i3})).$$

Taking expectations on both sides yields

$$E \left( \max_{1 \leq i \leq d} (x_i (V_{i1} + V_{i2} + V_{i3})) \right) \leq \frac{3}{2} E \left( \max_{1 \leq i \leq d} (x_i (V_{i1} + V_{i2})) \right),$$

which proves the assertion for $n = 3$. Repeating the preceding arguments provides the assertion for a general $n$: Set

$$x_{i*} \sum_{j=1}^{n} V_{i* j} = \max_{1 \leq i \leq d} \left( x_i \sum_{j=1}^{n} V_{ij} \right).$$

We have for all subsets $T \subset \{1, \ldots, n\}$ with $n - 1$ elements, i.e., $|T| = n - 1$,

$$x_{i*} \sum_{j \in T} V_{i* j} \leq \max_{1 \leq i \leq d} \left( x_i \sum_{j \in T} V_{ij} \right).$$
Summing up these \( n \) inequalities we obtain
\[
(n - 1)x_i \sum_{j=1}^{n} V_{ij} \leq \max_{T \subseteq \{1, \ldots, n\}, |T| = n-1} \left( \sum_{j \in T} x_i \sum_{j=1}^{n} V_{ij} \right).
\]
Taking expectations on both sides now yields the assertion:
\[
E \left( \max_{1 \leq i \leq d} \left( x_i \sum_{j=1}^{n} V_{ij} \right) \right) \leq \frac{n}{n-1} E \left( \max_{1 \leq i \leq d} \left( x_i \sum_{j=1}^{n-1} V_{ij} \right) \right).
\]

**Corollary 4.4.** The Dirichlet \( D \)-norm \( \| \cdot \|_{D(\alpha)} \) is decreasing in \( \alpha > 0 \), i.e., we have for arbitrary \( x \in \mathbb{R}^d \)
\[
\| x \|_{D(\alpha_1)} \geq \| x \|_{D(\alpha_2)}, \quad 0 < \alpha_1 \leq \alpha_2.
\]

**Proof.** Choose \( x \in \mathbb{R}^d \) and put \( g(\alpha) := \| x \|_{D(\alpha)}, \alpha > 0 \). Note that the function \( g(\cdot) \) is continuous. We will prove the assertion by a contradiction. Suppose that there exists \( 0 < \alpha_1 < \alpha_2 \) with \( g(\alpha_1) < g(\alpha_2) \). By the continuity of \( g(\cdot) \) we can find \( \varepsilon > 0 \) and \( k, n \in \mathbb{N}, k < n \), such that \( g(\varepsilon k) < g(\varepsilon n) \).

Let \( V_{ij}, 1 \leq i \leq d, 1 \leq j \leq n \), be an array of independent and identically gamma distributed rv with parameter \( \varepsilon > 0 \). The convolution theorem for the gamma distribution now implies
\[
g(\varepsilon k) = E \left( \max_{1 \leq i \leq d} \left( x_i \sum_{j=1}^{k} V_{ij} \right) \right) < E \left( \max_{1 \leq i \leq d} \left( x_i \sum_{j=1}^{n} V_{ij} \right) \right),
\]
which contradicts Lemma 4.3. \( \square \)

**Proposition 4.5** (Arbitrary dimension). The function \( m(\alpha) = \| 1 \|_{D(\alpha)} \) is strictly monotone decreasing in \( \alpha \).

**Proof.** We know from Corollary 4.4 that the function \( m(\cdot) \) is decreasing. We will show that it is strictly increasing. Let in what follows \( V_1(\alpha), V_2(\beta), \ldots \) be independent random variables, each following a gamma distribution with parameter \( 0 < \alpha, \beta, \ldots \). The convolution theorem for the gamma distribution now implies
\[
V_1(\alpha) + V_1(\beta) = D V_1(\alpha + \beta)
\]
and, thus,
\[
m(\alpha + \beta) = E \left( \max_{1 \leq i \leq d} \frac{V_i(\alpha + \beta)}{\alpha + \beta} \right) = E \left( \max_{1 \leq i \leq d} \frac{V_i(\alpha) + V_i(\beta)}{\alpha + \beta} \right).
\]
\[ \leq E \left( \max_{1 \leq i \leq d} \frac{V_i(\alpha)}{\alpha + \beta} \right) + E \left( \max_{1 \leq i \leq d} \frac{V_i(\beta)}{\alpha + \beta} \right) \]
\[ = \frac{\alpha}{\alpha + \beta} m(\alpha) + \frac{\beta}{\alpha + \beta} m(\beta) \]

(8)

From Lemma 4.2 we know that \( m(\alpha) \to d \) as \( \alpha \to 0 \). Next we show that this together with the inequality (8) implies strict monotonicity of the function \( m(\alpha) \).

We know from Corollary 4.4 that the function \( m(\cdot) \) is decreasing. We will establish its strict monotonicity by a contradiction. Suppose that \( m(\alpha_1) = m(\alpha_2) \) for some \( 0 < \alpha_1 < \alpha_2 \). This implies

\[ m(\alpha_1 + \varepsilon) = m(\alpha_1), \quad \varepsilon \in [0, \alpha_2 - \alpha_1], \]

and, thus, we obtain from inequality (8) for \( 0 < \varepsilon \leq \min(\alpha_1, \alpha_2 - \alpha_1) \) and the monotonicity of \( m(\cdot) \)

\[ m(\alpha_1) = m(\alpha_1 + \varepsilon) \]
\[ \leq \frac{\alpha_1}{\alpha_1 + \varepsilon} m(\alpha_1) + \frac{\varepsilon}{\alpha_1 + \varepsilon} m(\varepsilon) \]
\[ \leq \frac{\alpha_1}{\alpha_1 + \varepsilon} m(\alpha_1) + \frac{\varepsilon}{\alpha_1 + \varepsilon} m(\alpha_1) \]
\[ = m(\alpha_1). \]

This implies \( m(\varepsilon) = m(\alpha_1) \) for \( 0 < \varepsilon \leq \alpha_1 \) and, thus,

\[ m(\alpha_1) = \lim_{\varepsilon \downarrow 0} m(\varepsilon) = d, \]
i.e.,

\[ E \left( \max_{1 \leq i \leq d} \frac{V_i(\alpha_1)}{\alpha_1} \right) = E \left( \sum_{i=1}^{d} \frac{V_i(\alpha_1)}{\alpha_1} \right). \]

This yields

\[ \sum_{i=1}^{d} V_i(\alpha_1) = \max_{1 \leq i \leq d} V_i(\alpha_1) \quad a.s. \]

which is a clear contradiction and completes the proof of Proposition 4.5. \( \square \)

**Lemma 4.6.** Let \( V_1, \ldots, V_d \) be iid standard exponential distributed rv. The Dirichlet D-norm \( \| \cdot \|_{D(1)} \) on \( \mathbb{R}^d \) with generator

\[ Z = d \left( \frac{V_i}{V_1 + \cdots + V_d} \right)_{i=1}^{d} \]

has generator constant

\[ \|1\|_{D(1)} = \sum_{k=1}^{d} \frac{1}{k}. \]
The generator constant of a general bivariate Dirichlet $D$-norm was computed in Lemma 4.1. To the best of our knowledge, the preceding result, with $\alpha = 1$, provides the only exact computation of $\|1\|_{D(\alpha)}$ for arbitrary dimension. Some representation is given by Nadarajah (2008).

Proof. The following argument is taken from Balakrishnan and Basu (1996, Section 33.3). Using the memoryless property of the exponential distribution one can generate order statistics $V_{1:d} \leq \cdots \leq V_{d:d}$ from the standard exponential distribution as follows:

- Generate $d$ independent standard exponential distributed rv $V_1, \ldots, V_d$.
- Then
  $$V_{i:d} := V_{i-1:d} + \frac{V_i}{d-i+1}, \quad i = 1, \ldots, d,$$
  with $V_{0:d} = 0$ are the required order statistics.

Hence we obtain

$$\|1\|_{D(1)} = E(V_{d:d}) = \sum_{i=1}^{d} \frac{E(V_i)}{i} = \sum_{i=1}^{d} \frac{1}{i}.$$ 

The fact that the function $m(\alpha)$ is continuous and strictly decreasing with $\lim_{\alpha \downarrow 0} m(\alpha) = d$, $\lim_{\alpha \uparrow \infty} m(\alpha) = 1$ shows that the family of symmetric Dirichlet distributions is a parametric family of generators of $D$-norms in arbitrary dimension, which attains each value between independence ($\|1\|_D = d$) and complete dependence ($\|1\|_D = 1$). The generator of the symmetric Dirichlet distribution is well-known and easy to simulate. This makes the family of symmetric Dirichlet distributions quite an attractive parametric model of $D$-norms.

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