Coherent bremsstrahlung in a bent crystal vs. experiments on radiation at volume reflection

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Abstract. Coherent bremsstrahlung in a bent silicon crystal in the orientation (111) is calculated. The radiation spectrum is averaged over the angular distribution of particles in the incident beam. Comparison of the dipole radiation theory predictions with the available experimental data is made.

1. Introduction
Recent experiments [1, 2] on radiation from relativistic electrons and positrons in thin bent crystals were searching for spectrum features inherent to radiation at volume reflection. It was suggested that the spectrum from the side of high frequencies may be treated as a coherent bremsstrahlung in a bent crystal (CBBC), which is simpler than radiation from vicinity of the volume reflection point, and a formula for the CBBC spectrum from a single electron was derived in [3]. However, the found spectrum profile appears to be sensitive to the angle of the particle entrance to the crystal, on the scale of the crystal bending angles, which necessitates taking into account angular divergence of the beam in experiments [1, 2]. In addition, [3] explicitly dealt with the simplest case of (110) orientation, involving only a single-well inter-planar continuous potential, whereas in experiments [1, 2] crystal orientation with the active plane (111) was employed. In the present note it is desired to adapt the theory of [3] to the experimental conditions, and to draw a comparison with the data.

2. CBBC from a single particle in a crystal with orientation (111)
2.1. Generic formula for CBBC [3]
The spectral density of dipole radiation (emitted in the forward cone ∼ 1/γ) on a bent crystal with inter-planar continuous potential of arbitrary profile results from Eq. 40 of [3] by an obvious substitution $\frac{F_{\max}^2}{\gamma^{3+2\epsilon}} \rightarrow n \left( \frac{2V_n}{d} \right)^2$:

$$\frac{dE_{\text{CBBC}}}{d\omega} = \frac{\pi e^2 R E'^2}{m^2 d} \sum_{n=1}^{\infty} nV_n^2 \left\{ \Theta (nq_+ - q_{\min}) D \left( \frac{q_{\min}}{nq_+}, \frac{\omega}{E} \right) + \Theta (nq_+ + q_{\min}) \Theta (nq_+ - q_{\min}) D \left( \frac{q_{\min}}{nq_+}, \frac{\omega}{E} \right) + \Theta (-nq_- - q_{\min}) \left[ D \left( \frac{q_{\min}}{nq_-}, \frac{\omega}{E} \right) - D \left( \frac{q_{\min}}{n|q_-|}, \frac{\omega}{E} \right) \right] \right\}.$$ (1)
Here \( d \) is the distance between active crystalline planes, \( L \) and \( R \) are the crystal thickness and the active plane bending radius, \( m \) and \( e \) – the electron mass and charge, \( E \) and \( \omega \) – the initial electron’s and the photon’s energies

\[
q_{\text{min}}(\omega) = \frac{\omega m^2}{2EE'}, \quad E' = E - \omega, \quad q_{\pm} = \frac{2\pi}{d} \left( \frac{L}{2R} \pm |\theta_0| \right),
\]

\( \Theta(v) \) – the Heavyside unit-step function, and

\[
D(v, \omega/E) = (1 - v) \left( \frac{2 - v + 2v^2}{3} + \frac{\omega^2}{2EE'} \right), \quad (0 \leq v \leq 1)
\]

Finally, \( V_n \) are the Fourier series coefficients for the inter-planar continuous potential in a straight crystal, i.e.

\[
V(x) = \sum_{n=1}^{\infty} V_n \cos \frac{2\pi nx}{d} \quad \Leftrightarrow \quad V_n = \frac{2}{d} \int_{-d/2}^{d/2} dx V(x) \cos \frac{2\pi nx}{d},
\]

(presuming the origin of \( x \)-coordinate to be chosen in a point of symmetry of the potential).

### 2.2. Silicon inter-planar continuous potential at orientation (111)

Silicon has a lattice of diamond type, with the lattice constant 5.431 Å, thus the period in direction \( \langle 111 \rangle \) equals \( d = \frac{5.431}{\sqrt{3}} \approx 3.1355 \) Å. However, for this orientation there are two atomic planes falling within a period. Packing densities of those planes are equal, hence the potential derivative discontinuities, as well potentials themselves, are equal at each plane, and the potential function is symmetrical with respect to center of any of the wells. The well width ratio is \( \frac{d_L}{d_S} = 3 \), and the sum of the widths \( d_L + d_S = d \). As for the well depths and shape, we will infer them from [4], presuming the well shapes to be parabolic (neglecting here the thermal smear-out of the potential) – see Fig. 1.

\[
V_L \approx 26.5eV, \quad V_S \approx 7.5eV.
\]

**Figure 1.** Continuous potential of silicon crystal in planar orientation \((111)\).

### 2.3. Potential decomposition to Fourier series

Having specified the periodic potential, we can evaluate its Fourier series coefficients. Calculation of the integral in (4) with our potential function yields

\[
V_n = (-1)^n(d_L + d_S) \left( \frac{2}{\pi n} \right)^2 \left\{ \left( \frac{V_L}{d_L} + \frac{V_S}{d_S} \right) \cos \frac{\pi nd_S}{d_L + d_S} + \frac{d_L + d_S}{\pi nd_S} \left( \frac{V_L d_S}{d_L} - \frac{V_S}{d_S} \right) \sin \frac{\pi nd_S}{d_L + d_S} \right\}
\]

\[
= (-1)^n \left( \frac{4}{\pi n} \right)^2 \left\{ \left( \frac{V_L}{3} + V_S \right) \cos \frac{\pi n}{4} + \frac{4}{\pi n} \left( \frac{V_L}{9} - V_S \right) \sin \frac{\pi n}{4} \right\}.
\]
With numerical values (5), one estimates

\[
V_1 \approx -12.1 \text{ eV},
\]

\[
2 \left( \frac{V_2}{V_1} \right)^2 \approx 0.02, \quad 3 \left( \frac{V_3}{V_1} \right)^3 \approx 0.11, \quad 4 \left( \frac{V_4}{V_1} \right)^2 \approx 0.075, \quad 5 \left( \frac{V_5}{V_1} \right)^2 \approx 0.02, \quad \sum_{n=2}^{\infty} n \left( \frac{V_n}{V_1} \right)^2 \approx 0.25.
\]

Therefore, at typical radiation frequencies the first Fourier harmonic of the potential dominates, but within its range the 3rd and the 4th harmonics in sum contribute \( \sim 20\% \). This is about as much as contributes the 2nd harmonics in case of orientation (110).

In investigations of particle dynamics and radiation in the crystal the continuous potential properties enter through two basic parameters: the critical radius \( R_c \) and the non-dipole radiation angle \( \theta_V \). For perturbative treatment of the particle passage, the potential shape-independent definitions of \( R_c, \theta_V \) are

\[
R_c = \frac{E}{|V_1| \pi^2 / d} \approx \frac{E}{4 \text{GeV/cm}}
\]

(concordant with \( 4V_L/d_L \approx 4.5 \text{GeV/cm}, 4V_S/d_S \approx 3.8 \text{GeV/cm} \)), and

\[
\tilde{\theta}_V = \frac{2 \sqrt{2}}{m} |V_1| \approx 0.67 \cdot 10^{-4} \text{rad}
\]

(cf. \( \theta_V = \frac{V_L}{m} \approx 0.5 \cdot 10^{-4} \text{rad} \)).

3. Beam and bent crystal parameters in IHEP and CERN SPS experiments

The parameters of experiments [1, 2] are listed in Table 1. As one realizes immediately, the beam spread in all cases is significant compared to the crystal half-bending angle entering (2), so it is necessary to average intensity (1) with distribution

\[
f(\theta_0) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\theta_0^2/2\sigma^2}.
\]

Table 1. Bent crystal and the beam parameters

| Experiment | Energy, \( E \) | Beam divergence | Critical radius, \( R_c \) | Critical angle, \( \theta_c \) = \( \frac{\sqrt{d}}{2R_c} \) | Crystal thickness, \( L \) | Bending radius, \( R \) | Bending angle, \( L/R \) |
|------------|-----------------|-----------------|------------------|-----------------|-----------------|-----------------|-----------------|
| IHEP \( e^+ \) | 10 GeV | \( \Gamma = 0.1 \text{ mrad} \) | 2.5 cm | 80 \( \mu \text{rad} \) | 0.65 mm | 1.3 m | 0.5 \( \mu \text{rad} \) |
| CERN \( e^+ \) | 180 GeV | \( \sigma = 25 \text{ \( \mu \text{rad} \)} \) | 0.5 m | 20 \( \mu \text{rad} \) | 0.84 mm | 12 m | 70 \( \mu \text{rad} \) |
| CERN \( e^- \) | 180 GeV | \( \sigma = 25 \text{ \( \mu \text{rad} \)} \) | 0.5 m | 20 \( \mu \text{rad} \) | 0.90 mm | 8 m | 110 \( \mu \text{rad} \) |

To judge about the reliability of CBBC theory, it is expedient to form dimensionless ratios quantifying various effects ignored by Eq. (1) (see [3]). Those are gathered in Table 2.

Parameter \( R/R_c \) is made large by setup, to sharpen manifestation of the volume reflection effect, and thus, the deflection is highly non-perturbative. Nonetheless, a lot of radiation is generated away from the volume reflection area, given the significant crystal thickness, and may be viewed as emitted by a perturbatively deflected particle (smallness of numbers in the third
Table 2. Robustness of the dipole CBBC theory

| Experiment | Steering strength, $R/R_c$ | Coherence intervals per crystal, $L/R_c$ | Trajectory wigging on the lattice scale, $R/R_c$ | Non-dipole degree of effect on radiation, $R/R_c$ | Multiple scattering strength, intervals wiggling on the degree of effect on radiation, $R/R_c$ |
|------------|---------------------------|------------------------------------------|----------------------------------|-----------------------------|---------------------------------|
| IHEP $e^+$ | 50                        | 23                                       | 0.01                             | 0.13                        | 3.5                             |
| CERN $e^+$ | 27                        | 10                                       | 0.03                             | 0.95                        | 0.86                            |
| CERN $e^-$ | 18                        | 13                                       | 0.01                             | 0.6                         | 1.25                            |

Non-dipole and multiple scattering spoiling effects are rather moderate, except that the multiple scattering is too large in the IHEP experiment.

In any case, it is interesting to confront the dipole CBBC theory with the data from both experiments. In advance, let us assess main parameters of the radiation spectrum [3], which are collected in Table 3. One notices that the turnover frequency, related with the volume reflection, is substantially lower than the spectrum extent, thus the domain of CBBC must be formidable. Also, the BH background is seen to be subdominant. Let us now average intensity (1) over the charged particle incidence angles and look over the correspondence with the data (Figs. 2-4).

Table 3. Radiation spectrum dimensions

| Experiment | End-of-spectrum at normal incidence, $\omega_{\text{end}}$ | Smear of the end-point, $\omega_{\text{v.r.}} = 4\gamma^2 \sqrt{2 \pi c d}$ | Turnover at $\omega \to 0$, $\omega_{\text{v.r.}} = 4\gamma^2 \sqrt{2 \pi c d}$ | CBBC Coherent radiation, $dE_{\text{CBBC}}(\omega_{\text{v.r.}}) = \frac{L}{4} \frac{d}{\gamma^2 \theta_{\text{v.r.}}} (\omega_{\text{v.r.}})$ |
|------------|--------------------------------------------------------|-------------------------------------------------|--------------------------------------------------|---------------------------------|
| IHEP $e^+$ | 0.77GeV                                                | 0.3GeV                                          | 0.15GeV                                           | 0.07                            | 0.009                           | 0.01                          |
| CERN $e^+$ | 35GeV                                                  | 25GeV                                           | 12GeV                                            | 0.37                            | 0.03 [2]                        | 0.19                          |
| CERN $e^-$ | 56GeV                                                  | 25GeV                                           | 12GeV                                            | 0.35                            | 0.03 [2]                        | 0.075                         |

The agreement with CERN data is encouraging in the high-frequency part, and a turnover occurs approximately in the place predicted by the theory. Still, to some degree, there can be non-dipole effects. In the case of IHEP, we encounter a big normalization disagreement, yet there is no signature of a turnover in the expected place. These data also disagree with the calculation of [5]. One may attribute the discrepancy to a destruction of the radiation coherence by multiple scattering, but quantitative theory of such influence is lacking.

4. Conclusions

The dipole CBBC theory appears to be a satisfactory first approximation for radiation from relativistic charged particle passing through bent crystals, and owing to its simplicity it may be used as a guide at the initial stage of experiment planning. Admittedly, it does not take into account many effects, which need more dedicated theoretical investigations.
To verify the expectation that the high-energy part of CERN experiments is actually CBBC, it would be desirable to perform additional measurements, in the same energy range, but with a better collimated beam, and with a variation of the target orientation. That might check the existence of a sharp spectrum end and its dependence on the beam and crystal parameters.

References
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