NUMERICAL STUDY OF BLOW-UP IN SOLUTIONS TO GENERALIZED KORTEWEG-DE VRIES EQUATIONS

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Abstract. We present a numerical study of solutions to general Korteweg-de Vries equations with critical and supercritical nonlinearity. We study the stability of solitons, the small dispersion limit and the appearance of blow-up in the solution. It is shown that solitons are unstable against being radiated away and blow-up. In the cases with blow-up, we use a dynamic rescaling to identify the type of the singularity.

1. Introduction

The celebrated Korteweg-de Vries (KdV) equations provide an asymptotic description of one-dimensional waves in shallow water in the long wave-length limit. For shorter wavelengths the dispersion in the KdV equation is in general too strong compared to what is observed in applications. A possible approach to address this short-coming of the model equation KdV is to tilt the balance between nonlinearity and dispersion towards nonlinearity which leads to generalized KdV (gKdV) equations,

\[ u_t + u^n u_x + \epsilon^2 u_{xxx} = 0; \tag{1} \]

here the parameter \( n \) in the nonlinearity is taken to be a positive integer, KdV corresponds to \( n = 1 \); the parameter \( \epsilon \) is a small dispersion parameter which appears in the study of the small dispersion limit as in [8]. It can be understood to be introduced in the gKdV equation (1) with \( \epsilon = 1 \) by some initial data, which vary on large scales of order \( 1/\epsilon \). Studying the solution of gKdV for such initial data on large time scales of order \( 1/\epsilon \) can be achieved by a transformation \( x \rightarrow x/\epsilon, t \rightarrow t/\epsilon \) which takes gKdV with \( \epsilon = 1 \) to the form (1).

The introduction of a stronger nonlinearity in KdV equations increases its importance with respect to dispersion, but has the well-known effect that solutions to gKdV equations may have finite time blow-up for \( n \geq 4 \), i.e., a loss of regularity with respect to the initial data in the form of diverging norms of the solution. The global well-posedness of the Cauchy problem in \( H^1 \) was proven in [12, 13]. The case \( n = 4 \) is critical in the sense that solutions to gKdV for \( n < 4 \) for sufficiently smooth initial data \( u(x, 0) = u_0(x) \) are globally regular in time. The case \( n = 4 \) is also distinguished by the invariance of the mass, the \( L^2 \) norm of \( u \),

\[ M[u] = ||u||_2 \tag{2} \]

under rescalings of the form \( x \rightarrow x/\lambda, t \rightarrow t/\lambda^3 \) and \( u \rightarrow \lambda^{2/n} u \) with \( \lambda = \text{const} \) which leave equation (1) invariant. This case has been studied in detail in a series of papers, see [23, 24, 27, 25, 26, 32]. The discussion was recently completed in the three articles by Martel, Merle and Raphaël [28, 29, 30]. For more details we refer to those articles and references therein. A variational argument [36] implies that \( H^1 \) initial data with subcritical mass \( M[u_0] < M[Q] \) generate global in time solutions in \( H^1 \). In this paper we will always consider \( C^\infty \) initial data in \( L^2(\mathbb{R}) \) which will be treated as effectively periodic.

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The gKdV equations have localized travelling wave solutions of the form \( u = Q(x - x_0 - ct) \) with \( x_0, c = \text{const} \) and with

\[
Q(z) = \left( \frac{(n + 1)(n + 2)c}{2} \sech^2 \frac{\sqrt{n} z}{2c} \right)^{1/n}.
\]

It was shown in [4] that these solitons are linearly unstable for \( n \geq 4 \). For \( n = 4 \) it was proven in [28] that the solitons are both unstable against blow-up and against being radiated away towards infinity. More precisely it is shown that the solution for initial data close to the soliton, but with smaller mass, will not stay close to the soliton for small \( t \). In our numerical experiments, the initial data appear to be simply radiated away. The criterion to decide between these scenarios is given by the mass and the conserved energy for gKdV equations,

\[
E[u] = \int_{-\infty}^{\infty} \left( \frac{\epsilon^2}{2} u_x^2 - \frac{1}{(n + 1)(n + 2)} u^{n+2} \right) dx.
\]

For \( n = 4 \), the energy of the soliton vanishes. Perturbations of the soliton leading to positive energy initial data will be radiated away towards infinity, whereas initial data with negative energy and a mass larger than the soliton mass, \( M[u_0] > M[Q] \), lead to finite time blow-up. The same results apply to general localized initial data, see [28]. Moreover it was shown in [28] that the blow-up profile is asymptotically for \( t \) close to the blow-up time \( t^* \) given by a self-similar solution to gKdV in the form of the rescaled soliton (3). In the notation to be used in the following, this means that the gKdV solution close to blow-up is given by

\[
u(x, t) \sim \frac{1}{\sqrt{L(t)}} Q \left( \frac{x - x_m(t)}{L(t)} \right),
\]

with \( Q \) from (3), and where

\[
L(t) \sim l_0(t^* - t), \quad x_m(t) \sim \frac{1}{l_0^2(t^* - t)}, \quad l_0 = \text{const}.
\]

This implies that the \( L_2 \) norm of the gradient \( u_x \) diverges as \( 1/L(t) \). The reader is referred to [28] for details and proofs.

The picture is much less clear for the supercritical cases \( n > 5 \). There it is just known that blow-up is possible, but not for which initial data this is the case. It was also shown in [4] that the soliton is unstable in this case, but the mechanism of the instability or of a blow-up is not rigorously established. Therefore numerical experiments have been carried out in [1, 2, 3] to address this issue. The basic idea was to use cubic splines for the spatial dependence and an implicit fourth order Runge-Kutta method for the time dependence. This allowed to trace in [3] the issue. The basic idea was to use cubic splines for the spatial dependence and an implicit fourth order Runge-Kutta method for the time dependence. This allowed to trace in [3] the identification of the asymptotic behavior for the critical case \( n = 4 \) was not conclusive. In [7], the authors studied the gKdV equation for the case \( n = 5 \) with numerical and asymptotic methods. A similar numerical setup as in [3] was used. This allowed to conclude that a self-similar solution acts as an attractor for blow-up as in the critical case, but that this is not a rescaled soliton (3).

The goal of this paper is to present a careful numerical study of soliton stability in the critical and supercritical cases, of the small dispersion limit \( \epsilon \ll 1 \) as discussed in [8], and of the asymptotic behavior in the cases with blow-up. To this end we employ a dual approach: we always use a Fourier spectral method for the spatial dependence which is optimal for the smooth initial data we discuss here. Since these numerical approaches are independent, they provide a mutual check of their respective accuracy. In all cases, we first use as in [3] an implicit fourth order Runge-Kutta method for the time dependence which implies the use of two high order methods for space and time allowing for an efficient numerical solution of high accuracy. In case where blow-up is observed, we solve in addition a dynamically rescaled gKdV equation, where the scaling is motivated by (6). We follow the approach presented in [31, 22, 21, 33], as well as in the monograph [35], for numerical studies of the blow-up of solutions to the nonlinear Schrödinger equations (NLS). A similar approach was applied in [18, 37, 19], but there only the independent variables were rescaled. The rescaling allows a high resolution computation close to blow-up. Since the critical
case \( n = 4 \) is expected to show the behavior (5) near blow-up, this implies that the blow-up is expected for \( x_m \to \infty \). To address this problem, we solve the equation in a frame commoving with the maximum of the solution. In addition, the blow-up profile for \( n = 4 \) is reached algebraically in the rescaled time whereas an exponential dependence is expected for \( n > 4 \). Thus it is numerically extremely challenging to get close to the critical time \( t^* \). But as will be shown in the paper, we get close enough to allow for some fitting to the theoretical predictions. In [3, 7] a similar approach was used in the sense that the maximum was shifted every few time steps back to the center of the computational domain. But to the best of our knowledge, we present here the first implementation of a fully dynamical rescaling for gKdV equations.

The paper is organized as follows: In section 2 we present the used numerical methods and perform tests. The \( L_2 \) critical case \( n = 4 \) is discussed in section 3. We study perturbations of the soliton, the small dispersion limit and blow-up. The same examples are studied in section 4 for the supercritical case \( n = 5 \). We add some concluding remarks in section 5.

2. Numerical Methods

In this section we present the used numerical methods. Two approaches will be applied which complement each other: First we directly integrate the gKdV equation for given initial data. The time integration is carried out with a fourth order implicit Runge-Kutta method. With this code we study soliton stability and the small dispersion limit of gKdV. In the cases where there is an indication of blow-up, we use a dynamically rescaled version of the gKdV equation to study the blow-up in more detail and to identify the type of formed singularity. The spatial dependence will be treated with a Fourier spectral method in all cases. Both codes will be tested at the example of the explicitly known soliton solutions (3). Since the approaches are independent, they provide an additional test for their respective accuracy.

2.1. Direct numerical integration. The choice to use Fourier methods is based on the excellent approximation properties spectral methods have for smooth functions as the ones studied here. We will always consider initial data which are rapidly decreasing, and which can thus be analytically continued as periodic within the finite numerical precision if the computational domain is chosen large enough. Since we study here dispersive equations, it is also important that spectral methods minimize the introduction of numerical dissipation that might suppress dispersive effects. In addition it is well known that derivatives in physical space are equivalent to a multiplication of the Fourier coefficients \( \hat{u} \) with the corresponding wavenumber \( k \). This leads to a diagonal matrix corresponding to the third derivative in (1) after the spatial discretization. The method of lines approach implies that the partial differential equation (PDE) (1) will be approximated via a system of ordinary differential equations (ODEs) for the Fourier coefficients. The latter system is stiff\(^1\) because of the third derivative in \( x \), but has the advantage that the stiffness appears in the linear term of the equation,

\[
\hat{u}_t = \mathcal{F}(u) + \mathcal{L}\hat{u},
\]

where \( \mathcal{L} = c^2ik^3 \) and \( \mathcal{F}(u) = -iku^{n+1}/(n+1) \). For equations of the form (7) there are many efficient high-order time integrators, see e.g. [6, 11, 9, 14, 15], especially for diagonal \( \mathcal{L} \) as in the Fourier case. Thus the Fourier approach is not only very convenient for the spatial dependence, but allows also efficient time integration of high accuracy.

In [15], most of the studied integrators as in [11] are explicit and thus in general much more efficient than implicit integrators. It was shown, however, that an implicit Runge-Kutta of fourth order (IRK4), a two-stage Gauss scheme, could be competitive if the iteration is optimized. For the initial value problem \( y' = f(y, t) \), \( y(t_0) = y_0 \) and time steps \( t_m, m = 0, 1, \ldots \) with \( t_{m+1} - t_m = h \)

\(^1\)The term stiffness is used here to indicate that there are different timescales in the studied problem which make the use of explicit methods inefficient for stability reasons.
and \( y(t_m) = y_m \), this scheme has the form

\[
(8) \quad y_{m+1} = y_m + \frac{h}{2}(K_1 + K_2),
\]

\[
(9) \quad K_i = f \left( t_m + c_i h, y_m + h \sum_{j=1}^{s} a_{ij} K_j \right), \quad i = 1, 2,
\]

where \( c_1 = \frac{1}{2} - \sqrt{\frac{3}{6}} \), \( c_2 = \frac{1}{2} + \sqrt{\frac{3}{6}} \) and \( a_{11} = a_{22} = 1/4 \), \( a_{12} = \frac{1}{4} - \sqrt{\frac{3}{6}} \), \( a_{21} = \frac{1}{4} + \sqrt{\frac{3}{6}} \). The method is of classical order 4, but stage order 2. The task is to solve the implicit equations for \( K_1 \) and \( K_2 \) efficiently since it has to be done at every time step. Similar to [15] we apply a simplified Newton scheme in this context and solve

\[
(10) \quad \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} = \begin{pmatrix} 1 - ha_{11} \mathcal{L} & -ha_{12} \mathcal{L} \\ -ha_{21} \mathcal{L} & 1 - ha_{22} \mathcal{L} \end{pmatrix}^{-1} \left( \mathcal{F}(\hat{u}_m + h(a_{11} K_1 + a_{12} K_2)) + \mathcal{L} \hat{u}_m \right)
\]

iteratively. Since we use Fourier methods here, the operator \( \mathcal{L} \) is diagonal and the inverse matrix on the right-hand side of (10) can be given explicitly. In this form the iteration converges rapidly (at early times in 3 to 4 iterations) because the stiffness in this example appears in the linear part \( \mathcal{L} \) which is addressed explicitly. We apply this approach since it is robust up to an eventual blow-up of the solution whereas the explicit schemes used in [14] would require prohibitively small time steps to address stability issues to be discussed elsewhere.

Accuracy of the numerical solution is controlled as discussed in [14, 15] via the numerically computed energy (4) which will depend on time due to unavoidable numerical errors. We use the quantity

\[
(11) \quad \Delta = |E(t)/E(0) - 1|
\]

as an indicator of the numerical accuracy (if \( E(0) = 0 \) we just consider the difference \( |E(t) - E(0)| \)). It was shown in [14, 15] that the numerical accuracy of this quantity overestimates the norm of the difference between numerical and exact solution by two to three orders of magnitude. A precondition for the usability of this quantity is sufficient resolution in Fourier space, i.e., a large enough number of Fourier modes.

We generally choose the number \( N \) of Fourier modes high enough that they decrease to machine precision (10^{-16} here) for times much smaller than the time of blow-up. The number \( N \) thus depends on the size of the computational domain, \( x \in [-\pi, \pi]D \) where the real constant \( D \) is chosen large enough to ensure ‘periodicity’ of the initial data in the sense discussed above. An occurrence of blow-up in the code is indicated by an increase of the Fourier modes for the high wavenumbers which implies eventually a failure of the iterative solution of (10) to converge. We always choose the time step sufficiently small to ensure that the solution is obtained with machine precision for times much smaller than the blow-up time. A further (adaptive) reduction of the time step does not lead to a noticeable improvement of the accuracy of the solution and the convergence of the iteration, since the time step is always chosen small enough that a lack of resolution in Fourier space constrains the accuracy. Instead we use in these cases an adaptive rescaling of both independent variables \( x \) and \( t \) as well as the dependent variable \( u \) as discussed in the following subsection.

### 2.2. Dynamic rescaling.

The results in [28] for the critical case \( n = 4 \) and the numerical studies [3, 7] suggest that an \( L_\infty \) blow-up of solutions to gKdV is to be expected for initial data with negative energy and mass larger than the soliton mass. This can be numerically addressed by an adaptive approach, a rescaling of the coordinates and of the solution \( u \) by some time dependent factor \( L \), which is supposed to vanish at the blow-up time. In addition it is convenient to address the fact that the maximum which eventually develops into the blow-up is moving in contrast to NLS cases where such techniques have been applied previously. For the description of the method and the results in the NLS case, we refer to [35] and references therein. Thus we also shift the \( x \) coordinate by a factor \( x_m(t) \) in order to keep the maximum at a constant location in
the transformed coordinates. In the case \( n = 4 \), the quantity \( x_m \) is expected to tend to infinity at blow-up. With the coordinate change
\[
\xi = \frac{x - x_m}{L}, \quad \frac{d\tau}{dt} = \frac{1}{L^3}, \quad U = L^{2/n}u,
\]
we get for (1)
\[
U_{\tau} - a \left( \frac{\gamma}{n} U + \xi U_{\xi} \right) - v U_{\xi} + U^n U_{\xi} + \epsilon^2 U_{\xi\xi\xi\xi} = 0,
\]
with
\[
a = (\ln L)_{\tau}, \quad v = \frac{x_m, \tau}{L},
\]
where \( x_m, \tau \) denotes the derivative of \( x_m \) with respect to \( \tau \). The coordinate transformation (12) implies in the cases studied here that a blow-up occurs at \( \tau = \infty \). Thus the space and time scales are changed adaptively around blow-up, exactly in the way predicted by [28] for the critical case.

Equation (13) is also important for theoretical purposes and has been used in [23] and later works. In the limit \( \tau \to \infty \), i.e., at blow-up, the functions \( U, v \) and \( a \) are expected to become independent of \( \tau \) which is denoted by an index \( \infty \). For \( v \) this can be seen from the fact that \( x_m \propto 1/L \) for \( n = 4 \) and that we find numerically that \( x_m \sim \gamma L + x^* \) for \( n = 5 \) which implies that \( v_{\infty} \propto a_{\infty} \) in both cases. Thus (13) reduces in the limit \( \tau \to \infty \) to the ODE
\[
- a_{\infty} \left( \frac{\gamma}{n} U_{\infty, \xi} \right) - v_{\infty} U_{\infty, \xi} + U^n_{\infty, \xi} + \epsilon^2 U_{\infty, \xi\xi\xi\xi} = 0.
\]
There are in principle two different cases important in this context, an algebraic or an exponential decay of the scaling factor \( L \) with \( \tau \). In the algebraic case observed for \( n = 4 \) we have \( L(\tau) = C_1 \tau^{\gamma_1} \) with \( \gamma_1 < 0 \) and thus \( a_{\infty} = 0 \). In this case the ODE (15) reduces to the ODE for travelling wave solutions of the gKdV equation in a comoving frame which has the unique localized solution \( Q \).

For \( n > 4 \) exponential decay \( L(\tau) = C_2 e^{a_{\infty} \tau} \) with \( C_2 = \text{const} \) is observed which leads to \( a_{\infty} < 0 \). Relation (12) implies in this case
\[
L(t) = C(t^* - t)^{1/3}, \quad C = \text{const}.
\]
The ODE (15) was studied asymptotically for this case in [5] and numerically in [7] where the numerical gKdV solution was matched asymptotically to the rescaled solution of the ODE (15). Since we solve in this paper gKdV in the rescaled form (13) and since we observe the expected behavior of the scaling factor \( L(\tau) \), we solve in this sense asymptotically (15).

For the numerical implementation, the scaling factor \( L \) and the speed \( v \) have to be chosen in a convenient way. We choose \( \xi \) such that the maximum of \( |U| \) is located at \( \xi = \xi_0 \) which implies \( U_{\xi}(\xi_0, \tau) = 0 \). Therefore our approach is different from the one used in [3] and [7]. There, the authors also implement periodic boundary conditions, but to guarantee that the peak is located at the center of the domain, the solution is regularly shifted to the left, whereas this will be done automatically in our case by choosing \( v \) appropriately. The condition \( U_{\xi}(\xi_0, \tau) = 0 \) implies by differentiating (13) with respect to \( \xi \),
\[
v = -a\xi_0 + U(\xi_0, \tau)^n + \frac{\epsilon^2 U_{\xi\xi\xi\xi}(\xi_0, \tau)}{U_{\xi}(\xi_0, \tau)}.
\]
The scaling function \( L \) on the other hand can be chosen to keep certain norms constant, for instance the \( L_{\infty} \) norm,
\[
L^{2/n} = \frac{||U||_{\infty}}{||u||_{\infty}}.
\]
We get from (13) in this case
\[
a = \frac{ne^2 U_{\xi\xi\xi\xi}(\xi_0, \tau)}{2 U(\xi_0, \tau)}.
\]
Another possibility is to choose \( L \) such that the \( L^2 \) norm of \( U \), which reads
\[
L^{1/2 - 2/n} = \frac{||U||_2}{||u||_2},
\]
is constant. However, this is not convenient here since we also want to study the \( L^2 \) critical case \( n = 4 \) for which the \( L^2 \) norm is invariant under a rescaling of the form (12) (thus there is no condition on \( L \) in this case by imposing a constant \( L^2 \) norm of \( U \)). Alternatively one can impose that the \( L^2 \) norm of \( U_\xi \) is constant which is attractive since the blow-up theorems in [28] are formulated for this norm. In this case we have the relation
\[
L^{2/n+1/2} = \frac{||U_\xi||_2}{||u_x||_2}
\]
and we get
\[
a = \frac{2n}{(n+1)(n+4)||U_\xi||^2_2} \int_\mathbb{R} U^{n+1}U_{\xi\xi\xi} d\xi.
\]
For NLS computations with dynamic rescaling this approach proved to be numerically more stable than fixing the \( L_\infty \) norm to be constant, see [35] and references therein. We observe the same here. This appears to be due to the fact that condition (22) involves an integral over the whole computational domain, whereas condition (19) is local. In addition the appearance of the integral in (22) provides a global control of how well the numerical solution solves the rescaled equation (13). Therefore we will always apply the choice (22) in the following.

Thus all quantities in (13) can be expressed in terms of \( U \) alone. For a simpler reference later on, we combine (13), (22) and (17) into a single equation:
\[
\begin{cases}
U_\tau - a \left( \frac{2}{n} U^2 + \xi U_\xi \right) - v U_\xi + U^3 U_\xi + \epsilon^2 U_{\xi\xi\xi\xi} = 0 \\
a = \frac{2n}{(n+1)(n+4)||U_\xi||_2^2} \int_\mathbb{R} U^{n+1}U_{\xi\xi\xi} d\xi \\
v = - a_0 + U(\xi_0, \tau) + \epsilon^2 U_{\xi\xi\xi}(\xi_0, \tau) - U_{\xi\xi}(\xi_0, \tau).
\end{cases}
\]

The spatial dependence in equation (23) will be again treated with Fourier spectral methods for the reasons outlined above. In addition the appearance of a fourth derivative in (23) implies that spectral accuracy is needed in the computation of the derivatives. Since the coordinate \( \xi \) is dynamically rescaled with respect to \( x \), this implies that the computational domain has to be chosen large enough to avoid that the imposed periodicity of the solution affects the blow-up profile. As will be shown, this is possible in the studied examples except for the perturbed solitons where spurious oscillations in the quantity \( a(\tau) \) appear which can be clearly attributed to the periodic boundary conditions. But due to the comparatively slow decrease of the soliton solution towards infinity, this case is problematic with any approach. Nevertheless we still recover the predicted behavior [28] for \( n = 4 \). Note that the high spatial resolution also allows to avoid spurious oscillations as observed in [7] for the numerical study of blow-up in the supercritical case.

The numerically problematic term in (23) for the Fourier approach is \( \xi U_\xi^2 \). Since we have to choose large domains and since the solutions to (23) except for exact solitons have dispersive oscillations with slow decrease towards infinity, numerical errors at the boundaries for these dispersive tails lead to a pollution of the Fourier coefficients at the high frequencies. A damping of the oscillations at the boundaries is equivalent to imposing incorrect boundary conditions. These lead to reflections of the oscillations at the boundary which will eventually destroy the solution. A filtering in Fourier space to suppress an increase of the coefficients for high wavenumbers has a similar effect. Thus the only way to address this problem without affecting the numerical solution appears to be the use of sufficient resolution in Fourier space and high time resolution, thus effectively propagating the solution with machine precision for as long as possible.

\[2\text{It is exactly this term which requires a special fall-off condition for the initial data in [28].}\]
This is doable since equation (23) is in Fourier space again of the form (7). Since it is dynamically rescaled, there is no blow-up of the solution for finite $\tau$. Thus we can use an explicit integrator. It was shown in [14] that exponential time differencing (ETD) schemes perform best for KdV in the small dispersion limit, and in [15] that the performance is similar for different ETD schemes. The basic idea of these methods is to use a constant time step $h = t_{m+1} - t_m$ and to integrate (7) with an exponential factor,

$$\dot{u}(t_{m+1}) = e^{Ch}u(t_m) + \int_0^h d\theta e^{C(h-\theta))} F(\theta + t_m, \theta + t_m).$$

The different ETD schemes differ in the approximation of the integral. We use here the method by Cox and Matthews [6] of classical order four. An important aspect in implementing ETD schemes is the accurate and efficient computation of the so-called $\phi$-functions,

$$\phi_i = \frac{1}{(i-1)!} \int_0^1 d\theta e^{(1-\tau)\theta}\theta^{i-1}, \quad i = 1, 2, \ldots,$$

which appear in all ETD schemes. To avoid cancellation errors, we use contour integrals in the complex plane as in [11] in the enhanced version [34] as discussed in [14].

Note that the used ETD scheme is of classical order four (for a detailed discussion see [9, 10]). But the information on the quantity $a$ in (14) at the stages of the scheme is not of the same accuracy as at the full steps. Thus we only use $a$ at the time steps $t_n$ and obtain $\ln L$ via the trapezoidal rule which is of second order. Due to the high number of time steps we use in practice, $L$ is obtained with more than sufficient accuracy. In the same way we obtain the time $t$ from (12) and $x_m$ from (17). To control the accuracy of both the numerical solution $U$ and the computed factor $L$, we use that equation (13) has the conserved energy

$$(24) \quad E[U] = \frac{1}{L^{4/n+1}} \int_{-\infty}^{\infty} \left( \frac{\epsilon^2}{2} U^2 - \frac{1}{(n+1)(n+2)} U^{n+2} \right) d\xi.$$

Since $L(\tau)$ explicitly enters this quantity, $E[U]$ controls both $U$ and $L$ at the same time.

**Remark.** It would be possible to use a transformation of the form $x \to x - x_m(t)$ also for the direct integration of gKdV to take care of the propagation of the maximum of the solution and to choose a comoving frame of reference. In this article we will always study positive initial data, i.e., the case which includes solitons. These propagate to the right. But except for the exact soliton solution, there will be dispersive oscillations also propagating to the left of the initial hump localized at 0. Thus resolution for a not comoving frame is needed in practice, or for one where the soliton is fixed close to the right boundary. A comoving frame is mainly beneficial in the context of a dynamical rescaling and will not be used for the direct integration of gKdV. In the latter case, we will sometimes place the initial hump closer to the right boundary at $\xi = \xi_0$ to allow for maximal space for the dispersive oscillations propagating to the left.

### 2.3. Tests of the numerics.

Before using the above codes to study stability of the soliton solution (3) and the small dispersion limit of gKdV, we will first test with which accuracy the code can propagate the explicitly known soliton solution. It will be shown that this can be done essentially with machine precision which is important in view of the fact that the soliton is unstable against perturbations, which will be studied in the following sections. The fact that no instability is observed for the used methods on the studied timescales for the exact solution makes clear that a perturbation has to be considerably larger than the numerical error to lead to a visible effect for the considered times.

For the direct integration of gKdV for $\epsilon = 1$ with the IRK4 code, we use the initial data $Q(x+3)$ (3) with $c = 1$ (this choice for $c$ will be used through the whole article) and compute the solution until $t = 6$ for $x \in 10[-\pi, \pi]$. In this way we assure that the modulus of the Fourier coefficients for $N = 2^{10}$ Fourier modes decreases to $10^{-14}$ for $n = 4$ and $10^{-12}$ for $n = 5$, i.e., essentially machine precision, during the whole computation. If one wants to trace the solution for larger times with the same resolution, one has to use larger computational domains and a larger value of $N$. We use in both cases $N_t = 10^4$ time steps. Since the energy of the soliton vanishes for $n = 4$, we cannot
use the quantity $\Delta$ in (11) as an indicator of the numerical accuracy, but take the energy itself. The latter is of the order of $10^{-13}$ as is $\Delta$ for the computation with $n = 5$. The difference between numerical and exact solution is of the order of $10^{-13}$ for $n = 4$ and $10^{-12}$ for $n = 5$. Note that the energy cannot be computed to higher precision than the resolution in Fourier space. This is why it is almost of the same order as the $L_{\infty}$ norm of the difference between numerical and exact solution here. In general it overestimates the numerical accuracy by 2-3 orders of magnitude.

We also test the code with dynamical rescaling for the soliton though in this case the function $L(t)$ will be equal to one within numerical precision. Nonetheless this test checks whether the code can propagate the soliton for finite times with machine precision in a comoving frame (due to the condition that the maximum is localized at $\xi = \xi_0$ with $\xi_0 = 0$, the soliton is stationary in this setting). The code can be run up to $\tau = t = 100$ in this case, and the error is still of the order of $10^{-10}$. There is no indication of an instability due to the numerical error on this timescale. The computation is carried out with $N = 2^{14}$ Fourier modes and $N_t = 2.5 \cdot 10^5$ time steps for $x \in 100[-\pi, \pi]$. The computed energy is in this case of the order of $10^{-10}$ in accordance with the $L_{\infty}$ norm of the difference between numerical and exact solution. The $L_2$ norm of $U_\xi$ is constant to the order of $10^{-12}$.

Since the numerical approaches for the direct integration of gKdV (1) and the dynamical rescaled equation (23) are independent, they can be used to test each other's quality. To illustrate this at a concrete example we consider the case $n = 4$ for the initial data $u_0 = \text{sech}^2 x$ in the small dispersion limit $\epsilon = 0.1$, the example discussed in detail in section 3.2. In Fig. 13 the solution obtained with the dynamically rescaled code is shown for $\tau = 1700$ which corresponds to $t = 4.1786 \ldots$. We then use the code for the direct numerical integration of gKdV for the same initial data until exactly the same time (the parameters for the computation are given in section 3.2). In the latter case, the relative computed energy $\Delta$ is of the order of $10^{-7}$, and the modulus of the Fourier coefficients decreases to $10^{-5}$. Thus we estimate the solution to be accurate to the order of $10^{-4}$. In the former case the relative computed energy is of the order of $10^{-4}$, and the modulus of the Fourier coefficients decreases to the order of $10^{-8}$. Thus we expect the solution to be accurate to the order of $10^{-3}$. In fact if we plot both in dependence of $x$ in Fig. 1 by using the scaling (12), we see that both solutions cannot be really distinguished in the plot. Therefore we use cubic splines to interpolate from the grid in $\xi$ to the one in $x$ and present the difference of both solutions in the same figure. It can be seen that this difference is smaller than $10^{-3}$ as expected. This provides as already mentioned a strong test of the numerical accuracy of both codes since we even have to interpolate between to grids. Thus we are convinced that all shown solutions in this paper are correct to at least plotting accuracy.

![Figure 1. Solution to the gKdV equation (1) for $n = 4$, $\epsilon = 0.1$ and the initial data $u_0(x) = \text{sech}^2 x$ for $t = 4.1786 \ldots$ in blue, and the solution to the rescaled gKdV equation (23) for $\tau = 1700$ after inverting the scaling (12) in green on the right; on the left we show the difference between both solutions.](image-url)
3. The $L_2$ critical case $n = 4$

In this section we study numerically solutions to the gKdV equation (1) in the $L_2$ critical case $n = 4$. We first consider perturbations of the soliton and show that the results are in accordance with [28], i.e., perturbations with positive energy are radiated away, whereas perturbations with negative energy lead to finite time blow-up. In the small dispersion limit, too, initial data with positive energy are radiated away, whereas data with negative energy lead to blow-up. In cases where blow-up is observed, we use the dynamical rescaling which again allows within numerical precision to confirm the type of blow-up established in [28].

Quantities which can be used for comparing the numerical results with the theoretical description of the blow-up dynamics are the scaling factor $L$, the position $x_m$ of the maximum of the solution, i.e., the peak which leads to the blow-up and the $L_2$ norm $||u_x||_2$ of the first spatial derivative of the solution. In [28] the behavior of these quantities was determined to be given by

\begin{align}
L(t) &= C_1(u_0)(t^* - t)^{\gamma_1} \quad \text{as } t \nearrow t^* \\
-x_m(t) &= C_2(u_0)(t^* - t)^{\gamma_2} \quad \text{as } t \nearrow t^* \\
||u_x||_2(t) &= C_3(u_0)(t^* - t)^{\gamma_3} \quad \text{as } t \nearrow t^*
\end{align}

with $\gamma_1 = 1$ and $\gamma_2 = \gamma_3 = -1$. The constants $C_1$, $C_2$ and $C_3$ only depend on the initial data.

3.1. Perturbations of the soliton. We consider as initial data perturbations of the soliton (3) of the form $u(x, 0) = \sigma Q(x - x_0)$ with $\sigma$ a real constant slightly smaller or bigger than 1, and $x_0$ the initial position of the soliton. In the former case we find that the soliton is radiated away, in the latter that blow-up in finite time is observed. In this subsection, we put $\epsilon = 1$. In this case, the mass of the soliton is $M(Q) = 2.466\ldots$ and its energy vanishes, $E(Q) = 0$.

We first consider the initial data $u(x, 0) = \sigma Q(x + 3)$, with $\sigma = 0.99$, on a large domain, $x \in 100[-\pi, \pi]$ with $N = 2^{14}$ Fourier modes and $N_t = 10^4$ time steps. In this situation the mass is obviously smaller than the mass of the soliton and the energy is positive. It can be seen in Fig. 2 that dispersive oscillations propagating to the left form immediately. The amplitude of these oscillations decreases very slowly which is why we choose a large computational domain. Only part of it is shown in the figure.

It can be seen in Fig. 2 that the $L_\infty$ norm of the solution for the perturbed soliton decreases in this case monotonically. Thus it appears that the soliton will be just radiated away. The Fourier coefficients at the final time show that the solution is fully resolved in Fourier space. The energy is conserved in the computation to the order of $10^{-12}$.

The situation is completely different for initial data of the form $\sigma Q(x + 3)$ with $\sigma$ slightly larger than 1. For $\sigma = 1.01$ we use $N_t = 2 \cdot 10^5$ time steps for $t < 25$ whereas all other parameters are as above. Now the mass is larger than the soliton mass and the energy is negative. The code breaks at $t \approx 22.1814$ since the iteration no longer converges. But already at $t = 22.15$ the energy conservation used to check numerical accuracy drops below $10^{-3}$. Thus we suppress the solution for larger times due to a lack of reliability. It is shown for several times in Fig. 3. It can be seen that there will be again some dispersive oscillations propagating to the left. But more importantly the soliton becomes more and more peaked and thus travels at a higher and higher speed. At the same time it gets laterally compressed. Thus the solution behaves as predicted in [28] as a dynamically rescaled soliton.

The increase of the $L_\infty$ norm in Fig. 3 is even more obvious in Fig. 4. However the code with the used resolution is not able to get arbitrarily close to the blow-up time. As the Fourier coefficients at the final time in Fig. 4 show, this is not due to a lack of resolution in time but in space.

The slow increase of the $L_\infty$ norm of the solution implies that it will not be possible to get much closer to the blow-up even with considerably higher resolutions in both time and space. But we can numerically check whether the analytical prediction that the $L_\infty$ norm of the solution behaves close to blow-up as $||u||_\infty \sim (t^* - t)^\alpha$ with negative constant $\alpha$. By fitting $\ln ||u||_\infty$ to $\alpha \ln(t^* - t) + \kappa$ via the optimization algorithm [20] distributed with Matlab as fminsearch, we find $\alpha = -0.4923$, $\kappa = 2.2803$ and $t^* = 25.0302$. The value of $\alpha$ is compatible with the theoretically predicted $-1/2$. The difference between $\ln ||u||_\infty$ and $\alpha \ln(t^* - t) + \kappa$ is below 1% and largest at the
early times, but the values of the fitting parameters do not change much if the fitting is done only for larger times. This shows that the algebraic increase in time of the $L_\infty$ norm of the solution is followed already for $t \ll t^*$ in good approximation. The fitting indicates a blow-up roughly for $t = 25$.

The above results indicate as expected that the solution near blow-up is close to a rescaled soliton. Thus we solve the dynamically rescaled equation (23) for the same $\sigma = 1.01$ for $\tau \leq 500$ with $N_t = 2 \cdot 10^6$ time steps and $N = 2^{16}$ Fourier modes for $\xi \in 1000[-\pi,\pi]$. The solution can be seen in Fig. 5. Since the maximum is fixed at $\xi = 0$, the dispersive oscillations propagate more rapidly to the left than is the case in Fig. 3. The plot suggests that the solution will be the rescaled soliton for $\tau \to \infty$, and that the remainder of the initial data will be radiated to infinity. The Fourier coefficients of the solution at the final time are also shown in Fig. 5 indicating that there is sufficient resolution in Fourier space. The relative computed energy is conserved to better than $10^{-2}$.

Due to the slow increase of the factor $L$ with $\tau$, the dispersive oscillations reach the boundaries of the domain, and due to the imposed periodicity, affect the solution. This has visible effects as can be seen for the quantity $a(\tau)$ in Fig. 6. The slow increase of $a(\tau)$ to zero is superposed by oscillations due to the radiation emanating from the perturbed soliton. We only observe such spurious oscillations for perturbed soliton initial data since the latter and the generated dispersive oscillations decrease much slower than the sech$^2x$ initial data. This is also the reason why we use a large domain and higher resolution in Fourier space to treat this case. Integrating $a(\tau)$ we obtain $L(\tau)$ and $t(\tau)$. Since the spurious oscillations in $a(\tau)$ appear in both $L$ and $t$, the function $L(t)$ does not show this oscillatory behavior. For larger values of $t$, the scaling factor $L$ is as expected linear in $t$. We show in Fig. 7 a fit of $L$ to $t$ for $t > 5$ which yields the blow-up time $t^* \approx 25.0836$. 

Figure 2. Solution to the gKdV equation (1) with $\epsilon = 1$ for $n = 4$ and the perturbed soliton initial data $0.99 Q(x + 3)$ (3) for several values of $t$. 
Figure 3. Solution to the gKdV equation (1) with $\epsilon = 1$ for $n = 4$ and the perturbed soliton initial data $1.01Q(x + 3)$ (3) for several values of $t$.

Figure 4. $L_\infty$ norm of the solution to the gKdV equation (1) with $\epsilon = 1$ for $n = 4$ and the perturbed soliton initial data $1.01Q(x + 3)$ (3) in dependence of time on the left, and the modulus of the Fourier coefficients of the solution for $t = 21.5$ on the right.

The $L_\infty$ norm of the difference between $L$ and its fit is of the order $10^{-4}$. The fitting approach was the following: We wrote (25) in the form $L(t) = C_1(t^* - t) = mt + L_0$, assuming the linear relation between $L$ and $t$. Fitting the numerical data to this straight line led to the values $m = -0.03931$ and $L_0 = 0.9861$ and finally to the blow-up time $t^* = -L_0/m$. 
The solution $U(\xi, \tau)$ of the equation (23) for the initial data $U(\xi, 0) = 1.01 Q(\xi)$ (3) for $\tau = 500$ (physical time: $t = 21.0787$) on the left and the corresponding Fourier coefficients on the right.

The function $a(\tau)$ on the whole computed $\tau$-range (left) and a close-up of $a(\tau)$ in the interval $\tau \in [0, 1]$ (right) for the solution of Fig. 5.

In Fig. 7 we also present the physical time $t$ in dependence of the rescaled time $\tau$. The plot shows that we do not get arbitrarily close to the blow-up time in this example due to the fact that $a(\tau)$ tends slowly to 0. This phenomenon was also observed in the case of the critical NLS equation [31, 33, 35].

The position of the maximum $x_m(\tau)$ and thus ultimately the location of the blow-up is shown in Fig. 8. As theoretically predicted, it tends to infinity. In the same figure we also present $x_m$ in dependence of the physical time. A fit for small $t^* - t$ ($\ln(t^* - t) < 2.4$) of the form $\ln x_m \sim \gamma_2 \ln(t^* - t) + \ln(C_2)$ gives $\gamma_2 = -1.3282$ and $C_2 = 905.8$. It can be seen that the fitting for $x_m$ is not as good as for the scaling factor $L$ which is not surprising since the distance between the location of the blow-up and the dispersive tail is infinite. But it is compatible with the theoretical prediction $\gamma_2 = -1$ and the $L_\infty$ norm of the difference between $x_m$ and its fit is still of order $10^{-2}$. A similar procedure can be done for the $L_2$ norm of $u_x$. Our results from a fitting for $\|u_x\|_2$ lead to $\gamma_3 = -1.0004$ and $C_3 = 13.884$ and also match very well with theoretical prediction (27). The two control quantities, the $L_2$ norm of $U_\xi$ and the value of $U_\xi$ at $\xi = 0$ are preserved to the order $10^{-6}$ and $10^{-4}$ respectively.

3.2. Small dispersion limit. As in [8] we discuss the initial data $u_0 = \beta \text{sech}^2 x$ in the semiclassical limit $\epsilon \ll 1$. These data are motivated by the KdV soliton and have the advantage to be...
Figure 7. The scaling factor $L$ as a function of the physical time $t$ and its fit for $t > 2$ to a straight line (left). The right figure shows the physical time as a function of the rescaled time $\tau$. The black horizontal line in the right figure is the blow up time $t^*$ determined from the fitting of $L$. Both figures correspond to the situation shown in Fig. 5.

Figure 8. The position $x_m$ of the maximum of the solution of Fig. 5 as a function of the rescaled time $\tau$ on the left and as a function of $t$ and its fit for small values of $\ln(t^* - t)$ on the right.

more rapidly decreasing than the gKdV soliton (3). Thus it is possible to analytically continue the solution (within numerical precision) as a periodic function on a smaller computational domain. This allows to use effectively higher resolutions in Fourier space. The critical time for solutions to the generalized Hopf equation (gKdV (1) for $\epsilon = 0$) for these initial data can be given in closed form:

$$t_c = \frac{(1 + 2n)^{n+1/2}}{(2n)^{n+1/2} \beta^n}. \tag{28}$$

Note that we use a different scaling of the gKdV equation here with respect to [8] (a factor 6 is missing in front of the nonlinearity), which leads to a different critical time. The energy (4) is obviously always negative for fixed $\beta$ if $\epsilon$ is small enough. We concentrate here on the case $\epsilon = 0.1$. The mass of the soliton is $M[Q] = 0.779979$ in this case. We remind here that the energy of the soliton in the critical case vanishes independently of $\epsilon$.

For $\beta = 0.3$ the energy is positive, the mass again smaller than the soliton mass and the critical time is $t_c \approx 74.1577$. No blow-up is expected in this case. To directly integrate gKdV numerically for these initial data we use $N_t = 10^3$ time steps for $t \leq 2t_c$ and $N = 2^{12}$ Fourier modes for $x \in 40[-\pi, \pi]$. As can be seen in Fig. 9, the solution develops a tail of dispersive oscillations towards $-\infty$, the initial data appear to be simply radiated away. The $L_{\infty}$ norm of the solution
decreases monotonically. Note that the solution is resolved up to machine precision in Fourier space, and that the numerically computed energy is conserved to the order of $10^{-12}$.

The situation is completely different for $\beta = 1$ for which the energy is negative and the mass of the initial data is larger than the soliton mass $M[Q]$. The computation is carried out in this case with $N_t = 2 \cdot 10^5$ time steps for $t < 4.5$ and $N = 2^{14}$ Fourier modes for $x \in 10[-\pi, \pi]$. Due to the considerably smaller domain (reduced by a factor 10) than for the perturbed soliton, we have a higher resolution in Fourier space since we kept the same number of modes as there. The code is stopped at $t = 4.23$ when the energy conservation drops below $10^{-3}$ and the results are no longer reliable. The solution is shown for several times in Fig. 10. For small $t$ it is close to the solution of the generalized Hopf equation for the same initial data. The dispersive effects of the third derivative in the gKdV equation become important near the critical time of the generalized Hopf solution. A first oscillation forms at this time which then develops into a blow-up as for the perturbed soliton. This is to be compared to the subcritical initial data in Fig. 9. There the dispersive oscillations which also appear in Fig. 10 (where they are hardly visible) are dominant from the beginning, whereas they have only a negligible role in the blow-up case.

The increase of the $L_\infty$ norm in Fig. 10 can be clearly seen in Fig. 11. As for the perturbed soliton in the previous subsection, the code with the used resolution is not able to get close enough to the blow-up time. Again there is a lack of resolution in Fourier space as can be seen from the Fourier coefficients at the final time in Fig. 11.

It does not seem possible to get much closer to the blow-up even with considerably higher resolutions in both time and space. Thus we fit once more the $L_\infty$ norm of the solution close to blow-up to $||u||_\infty \sim e^{\kappa(t^* - t)^\alpha}$. Doing this for $t > 0.675$ (thus greater than the critical time

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig9}
\caption{Solution to the gKdV equation (1) with $\epsilon = 0.1$ for $n = 4$ and the initial data $u_0 = 0.3 \, \text{sech}^2 x$ for several values of $t$.}
\end{figure}
Figure 10. Solution to the gKdV equation (1) with $\epsilon = 0.1$ for $n = 4$ and the initial data $u_0 = \text{sech}^2 x$ for several values of $t$. The time of gradient catastrophe for the solution to the generalized Hopf equation is $t_c \approx 0.6007$.

Figure 11. $L_\infty$ norm of the solution to the gKdV equation (1) with $\epsilon = 0.1$ for $n = 4$ and the initial data $u_0 = \text{sech}^2 x$ in dependence of time on the left, and the modulus of the Fourier coefficients of the solution for $t = 4.23$ on the right.

$t_c \approx 0.6007$ of the generalized Hopf solution), we find $\alpha = -0.4279$, $\kappa = 0.7147$ and $t^* = 4.3554$. The fitting is less good than in the case of the soliton in the previous subsection (of the order of a few percent) as can be seen in Fig. 12. It indicates a blow-up roughly for $t = 4.35$. The blow-up rate is again compatible with $||u||_\infty \sim (t^* - t)^{-1/2}$ as predicted by [28].
To analyze the blow-up in more detail, we solve again the dynamically rescaled gKdV equation (23). With $N = 2^{14}$ Fourier modes for $\xi \in 120[-\pi, \pi]$ and $N_t = 6 \cdot 10^6$ time steps for $\tau \leq 1700$, we get the solution shown in Fig. 13. It can be seen that the initial data decompose into the rescaled soliton and a remainder which simply appears to be ultimately radiated away. The Fourier coefficients in Fig. 13 show the resolution up to machine precision of the solution. The computed relative energy is conserved to the order of $10^{-4}$.

The function $a(\tau)$ for the solution in Fig. 13 is shown in Fig. 14. Note that there are no spurious oscillations here in contrast to the perturbed soliton in Fig. 6. This is due to the more rapid decrease of the initial data with $x$ which implies the same behavior for the radiation which caused the oscillations in Fig. 6 due to the imposed periodicity in the computation. Note that the difference between $a(1700)$ and $a(1530)$ is of the order of $10^{-5}$. From $a(\tau)$ we get $L(\tau)$ as shown in Fig. 15 where also a least square fit is given for $L = mt + L_0$. We find $m = -0.1673$ and $L_0 = 0.7412$ and thus the blow-up time $t^* = 4.4316$. The difference of the fitted line and $L$ is of the order of $10^{-2}$ for $t > 2$. 
The physical time $t$ in dependence of $\tau$ can be seen in Fig. 15 on the right.

The position $x_m$ of the maximum of the solution in dependence of $t$ is shown in Fig. 16. For small $t^* - t$ it can be fitted to a straight line $\ln x_m = \gamma_2 \ln(t^* - t) + \ln(C_2)$. We find $\gamma_2 = -0.9117$ and $C_2 = 1.6683$. Thus in accordance with the theoretical prediction, the maximum shows the same scaling as the $L_2$ norm of $u_x$. The results of the fitting for $||u_x||_2$ are $\gamma_3 = -1.1921$ and $C_3 = 1.0921$ and therefore also match with the theoretical prediction $\gamma_3 = -1$. The conservation of the numerically computed energy and the $L_2$ norm of $U_\xi$ is of the same order as for the soliton perturbation.

**4. The supercritical case $n = 5$**

The $L_2$ critical blow-up studied in the previous section is characterized by an algebraic decrease of the $L_\infty$ norm of the solution and of the $L_2$ norm of the gradient with $t^* - t$. Numerical experiments [3, 7] indicate that the corresponding scalings in supercritical cases are exponential.

Since essentially the same behavior is expected for all $n > 4$, we concentrate here on the case $n = 5$ and study again perturbations of the soliton and the small dispersion limit.

Note that the energy of the soliton is not zero as for $n = 4$, but positive. It is known [4] that the soliton is unstable in the supercritical cases, but the precise mechanism (radiation, blow-up, . . . ) is unclear. It is also known that blow-up is to be expected, but there does not appear to be a precise criterion as for $n = 4$ that negative energy of the initial data and a mass larger than the
soliton mass implies blow-up, whereas initial data with positive energy do not lead to blow-up in finite time.

4.1. Perturbations of the soliton. As before we study slight perturbations of the soliton which can be numerically traced for the exact initial data with the used codes to machine precision. For \( n = 5 \) and \( \epsilon = 1 \), the energy of the soliton is \( E[Q] = 0.2763 \ldots \) and the mass is \( M[Q] = 2.230 \ldots \). Perturbing the soliton as for \( n = 4 \), i.e., considering initial data of the form \( u(x, 0) = \sigma Q(x + 3) \) on a large domain, \( x \in 100[-\pi, \pi] \) with \( N = 2^{14} \) Fourier modes and \( N_t = 10^4 \) time steps, we find for \( \sigma = 0.99 \) again that the soliton is radiated away. The energy of the perturbed initial data is larger than the one of the soliton, the mass smaller. The numerically computed energy is conserved to better than \( 10^{-12} \). It can be seen in Fig. 17 that dispersive oscillations propagating to the left form immediately and eventually appear to radiate the initial data away. The amplitude of these oscillations decreases very slowly which makes the use of a large computational domain necessary. In Fig. 17 again only part of the domain is shown. It can be seen that the \( L_\infty \) norm of the solution for the perturbed soliton decreases monotonically. The soliton is again unstable against radiation for perturbations leading to initial data with more positive energy than the soliton. The mass is obviously again smaller than the mass of the soliton.

For initial data with \( \sigma = 1.01 \), the soliton is as in the critical case unstable against blow-up though the energy in the considered example is positive, but smaller than the soliton energy and the mass is larger than the soliton mass. But this time the blow-up is approached much more rapidly as can be inferred from Fig. 18. We compute for \( x \in 100[-\pi, \pi] \) with \( N = 2^{14} \) Fourier modes and \( N_t = 2 \cdot 10^5 \) time steps. The code breaks at \( t \approx 1.885 \) since the iteration is not converging. The numerically computed energy is still conserved to better than \( 10^{-8} \) at \( t = 1.88 \).

It is clear from Fig. 19 that the resolution in Fourier space is no longer given near blow-up. Overall the way the blow-up is approached is very different to the case \( n = 4 \), the loss of resolution in both space and time happens on very small \( t^* - t \) scales. The \( L_\infty \) norm of the solution for the perturbed soliton is monotonically increasing which also indicates a blow-up, see Fig. 19.

Note that the type of blow-up is very different from the one in the \( L_2 \) critical case for the perturbed soliton in Fig. 3. There the \( L_2 \) norm is invariant under the rescaling (12), and the blow-up profile has the mass of the initial data. In the case \( n = 5 \) the \( L_2 \) norm of the part of the solution blowing up vanishes in the limit \( t \to t^* \) as follows from (20). This can be recognized already in Fig. 18 where the mass escapes to the left of the peak. To study the blow-up in more detail, we again use dynamic rescaling and solve \( gKdV \) in the form (23). With \( N = 2^{16} \) Fourier modes for \( \xi \in 800[-\pi, \pi] \) and \( N_t = 2 \cdot 10^6 \) time steps for \( \tau \leq 12 \), we obtain the solution shown in Fig. 20. Since the mass of the solution spreads out to the left of the peak, we shift the initial data to the right to allow longer simulation times (the code typically breaks if the modulus of the solution at the boundaries of the computational domain is of the order of \( 10^{-7} \)). We locate the maximum at \( \xi_0 = 1884.88 \). The Fourier coefficients which can be seen in the same figure indicate
that the solution is well resolved. The relative computed energy is conserved to the order of 4%. Our results here are in accordance with the ones by Bona et al. [3] as well as Dix and McKinney [7].

The corresponding function $a(\tau)$ can be seen in Fig. 21. As in the case $n = 4$ in Fig. 6, there are small oscillations in $a$ due to the periodic boundary conditions for $a$ which make it hard to read off a precise asymptotic value for $a(\tau)$. But since this value is definitely not zero, the function $L(\tau)$ given in Fig. 23 goes exponentially to zero as can be seen in the logarithmic plot.

If we consider $L$ as a function of the physical time $t$ the coordinate transform (12) implies a power law for $L(t)$, as we saw in formula (16). In Fig. 22 we can see that in a doubly logarithmic plot $L(t)$ approaches a linear regime. For values $\ln(t^*-t) < -8$ the systems runs into saturation and the corresponding values have to be neglected. We plot the linear part of $\ln L(t)$, $\ln(L) = \gamma \ln(t^*-t)+C$ and obtain: $\gamma = 0.3281$ and $C = -0.0256$ in accordance with the expectation $\gamma = 1/3$.

In contrast to the case $n = 4$, the blow-up time can directly be read off from the physical time in dependence of $\tau$ as can also be seen in Fig. 23. We get the final value $t^* = 1.8854$. The relative change compared to $\tau = 11$ is of the order $10^{-5}$.

A further difference to the $L_2$ critical case $n = 4$ is the location of the blow-up. Whereas it was infinite in the former case, it is clearly finite for $n = 5$ as can be seen in Fig. 24. Plotting $x_m$ in dependence of the scaling factor $L$ in Fig. 24, one can recognize an essentially linear dependence for small $L$ ($L < 0.4$), i.e., close to blow-up. We get $x_m = \gamma L + x^0_m$ with $\gamma = -5.0665$ and $x^0_m = 1891.08$. Due to the fact that $L$ goes to zero as we approach the blow-up, $x_m$ converges to $x^0_m$, which can therefore be interpreted as the blow-up position $x^*$. If we subtract the initial position of the maximum $x_m(0) = \xi_0 = 1884.88$ we get $\Delta x_m = 6.1983$. This is the distance between the initial position of the maximum and the blow-up position. The conservation of
Figure 18. Solution to the gKdV equation (1) with $\epsilon = 1$ for $n = 5$ and the perturbed soliton initial data $1.01 Q(x + 3)$ (3) for several values of $t$.

Figure 19. $L_\infty$ norm of the solution to the gKdV equation (1) with $\epsilon = 1$ for $n = 5$ and the perturbed soliton initial data $1.01 Q(x + 3)$ (3) in dependence of time on the left, and the modulus of the Fourier coefficients of the solution for $t = 1.88$ on the right.

the numerically computed quantities is here of order $10^{-5}$ for both the $L_2$ norm of $U_\xi$ and for the derivative $U_\xi$ at $\xi_0$. Since $a$ tends asymptotically to a more negative value than in the other considered examples (for $n = 4$ it tends to zero), comparatively small times $\tau$ are sufficient to come close to blow-up. This leads to better numerical results for the computed conserved quantities,
Figure 20. Solution $U(\xi, \tau)$ of the equation (23) with $n = 5$ for the initial data $U(\xi, 0) = 1.01 Q(\xi)$ (3) for $\tau = 12$ (physical time: $t = 1.8854$) (left) and the corresponding Fourier coefficients (right).

Figure 21. The function $a(\tau)$ on the whole $\tau$-range (left) and a detailed view of $a(\tau)$ in the interval $\tau \in [0, 1]$ (right) for the solution shown in Fig. 20.

Figure 22. The scaling factor $L$ in Fig. 20 as a function of the physical time $t$ and its corresponding fit.

but the effect is partially offset by the oscillations in $a$ due to the dispersive radiation and the periodic boundary conditions.
4.2. Small dispersion limit. As for the critical case $n = 4$ we discuss the initial data $u_0 = \beta \text{sech}^2 x$ in the semiclassical limit $\epsilon \ll 1$. For $\beta = 0.3$ the energy is smaller than the one of the soliton, but positive, the mass is smaller than the mass of the soliton and the critical time is $t_c \approx 219.8131$. For $\epsilon = 0.1$ the energy and the mass of the soliton are $E[Q] = 0.0276322$ and $M[Q] = 0.705252$ respectively. We use $N_t = 10^4$ time steps for $t \leq 2t_c$ and $N = 2^{12}$ Fourier modes for $x \in 100[-\pi, \pi]$ to directly integrate the gKdV equation for these initial data. As can be seen in Fig. 25, the solution develops again a tail of dispersive oscillations towards $-\infty$. The $L_\infty$ norm of the solution decreases monotonically.

For $\beta = 1$ the energy is again negative, and instead of dispersive radiation dominating as for $\beta = 0.3$, we observe blow-up. The mass of the initial data is again larger than the soliton mass. The computation is carried out in this case with $N_t = 10^5$ time steps for $t < 2.5$ and $N = 2^{14}$ Fourier modes for $x \in 5[-\pi, \pi]$. The code breaks at $t = 2.45$ since the iteration no longer converges. The solution is shown for several times in Fig. 26. For $t \ll t_c \approx 0.5341$ it is very close to the solution of the generalized Hopf equation for the same initial data. The dispersive effects of the third derivative in the gKdV equation become important near the critical time $t_c$. A first oscillation forms at this time which then develops into a blow-up as for the perturbed soliton. This is to be compared to the subcritical initial data in Fig. 25 where the initial data is just radiated away.

The monotonous increase of the $L_\infty$ norm in Fig. 26 is even more obvious from Fig. 27. In contrast to the critical case $n = 4$ we can get very close to the blow-up time. Even for $t \sim t^*$ there
Figure 25. Solution to the gKdV equation (1) with $\epsilon = 0.1$ for $n = 5$ and the initial data $u_0 = 0.3 \text{sech}^2 x$ for several values of $t$.

is still considerable resolution in Fourier space as can be seen from the Fourier coefficients at the final recorded time in Fig. 26.

To analyze the blow-up in more detail, we study the dynamically rescaled equation (23). We use $N = 2^{14}$ Fourier modes for $\xi \in 130[-\pi, \pi]$ and $N_t = 6 \cdot 10^6$ time steps for $\tau \leq 200$. The solution at the final time can be seen in Fig. 28 where also the Fourier coefficients are given. They indicate that the solution is well resolved. The relative computed energy is conserved to the order of 7%.

The corresponding function $a(\tau)$ tends here clearly to a negative constant as can be seen in Fig. 29. The final value for $a$ is considerably smaller than for the perturbed soliton, $a(200) = -0.0131$, but the relative change compared to $a(180)$ is of the order of $10^{-3}$. This indicates that the shown situation is close to blow-up. This can be also seen from the scaling $L(\tau)$ in Fig. 31 which follows for larger $\tau$ an exponential law.

As in the case of the soliton perturbation, we also perform a fitting of $L(t)$ to the expected behavior given in formula (16). In Fig. 30 we can see that in a doubly logarithmic plot $L(t)$ approaches the linear regime. For values $\ln(t^\ast - t) < -8$ the system runs also here into saturation, the corresponding values have to be neglected. We plot the linear part of $\ln L(t)$ and obtain: $\gamma = 0.3316$ and $C = -1.0585$ in accordance with the expectation $\gamma = 1/3$.

The dependence of the physical time on $\tau$ can be seen in Fig. 31 on the right. The final value can be again interpreted as the blow-up time $t^\ast = 2.4564$. The relative change compared to $t(180)$ is of the order of $10^{-5}$ which confirms that the shown situation is very close to the blow-up.

However this is not yet the case for the location $x_m$ of the maximum of the solution as can be seen in Fig. 32, where the asymptotic regime is not yet reached. This is in accordance with the above computations since $L$ showed asymptotic behavior always well before $x_m$. Plotting $x_m$
Figure 26. Solution to the gKdV equation (1) with $\epsilon = 0.1$ for $n = 5$ and the initial data $u_0 = \text{sech}^2 x$ for several values of $t$.

Figure 27. $L_\infty$ norm of the solution to the gKdV equation (1) with $\epsilon = 0.1$ for $n = 5$ and the initial data $u_0 = \text{sech}^2 x$ in dependence of time on the left, and the modulus of the Fourier coefficients of the solution for $t = 2.45$ on the right.

as a function of $L$ and fitting it for small $L$ ($L < 0.2$) to a straight line $x_m = \gamma L + x^0_m$, we find $\gamma = -3.3824$ and $x^0_m = 1.8656$. At the blow-up time $t^*$, the scaling factor $L$ vanishes, and therefore $x^0_m$ can be interpreted as the position of the blow-up $x^*$. The $L_2$ norm of $U_\xi$ and the value of $U_\xi(0, \tau) = 0$ are both preserved to the order of $10^{-4}$. 
Figure 28. The solution \( U(\xi, \tau) \) for the equation (23) for the initial data \( U(\xi, 0) = \text{sech}^2 \xi \) at \( \tau = 200 \) (physical time: \( t = 2.4564 \)) for \( n = 5 \) and \( \epsilon = 0.1 \) (left) and the corresponding Fourier coefficients (right).

Figure 29. The function \( a(\tau) \) on the whole computed \( \tau \)-range (left) and a close-up view in the interval \( \tau \in [0, 10] \) (right) for the solution of Fig. 28.

Figure 30. The scaling factor \( L \) in Fig. 28 as a function of the physical time \( t \) and its corresponding fit.

The results of Bona et al. [3] and Dix and McKinney [7] and the ones presented in this section can be summarized in the following
Conjecture. Let $Q$ be the soliton (3) and $u(x,t)$ be the solution of (1) for $n = 5$ for smooth initial data $u_0(x) \in L_2(\mathbb{R})$ with a single hump, $E[u_0] < E[Q]$ and $\|u_0\|_2 > \|Q\|_2$, then there is a a blow-up at the finite coordinates $(x^*, t^*)$ such that

1. the $L_\infty$ norm of the solution diverges
   \begin{equation}
   \|u\|_\infty(t) \to \infty \quad \text{as} \quad t \nearrow t^*
   \end{equation}

2. the scaling factor $L$ for constant $\|U_\xi\|_2$ follows the law
   \begin{equation}
   L(t) = C(t^* - t)^{1/3} \quad \text{as} \quad t \nearrow t^*
   \end{equation}
   where $C = C(u_0)$ is a constant depending on the initial data $u_0$

3. the position $x_m$ of the maximum has the asymptotic behavior
   \begin{equation}
   x_m(t) = \gamma L(t) + x^* \quad \text{as} \quad t \nearrow t^*
   \end{equation}
   where $\gamma = \gamma(u_0)$ is a constant depending on the initial data $u_0$. This implies with (30) and (31)

4. the solution is given asymptotically by a rescaled (according to (12)) solution to the ODE (15).
5. Outlook

In this paper we have shown that a combination of numerical approaches allowed to study blow-up in gKdV equations even in the presence of a dispersive shock. The basic idea was to identify blow-up scenarios with a spectral method in space and a fourth order method in time. In cases where blow-up was observed, we used a dynamically rescaled version of gKdV to allow for an adaptive high resolution treatment of the solution. This made it possible to identify the type of blow-up. The approaches using a direct integration of the gKdV equation and a dynamical rescaling complement each other and, since they are independent, test the respective accuracy. Thus they provide reliable and detailed information on the type of blow-up. In the $L_2$-critical case where the blow-up is approached algebraically in the rescaled time $\tau$, we managed to compute sufficiently long enough to match the computed quantities to the theoretical predictions. The method should make it possible to treat blow-up for more general initial data and for similar equations in the same way.

An interesting question in this context is the small dispersion limit. It was observed in this paper that the blow-up time $t^*$ is always larger than the critical time $t_c$ of the corresponding dispersionless equation. It would be interesting to study the relation of both times in dependence of $\epsilon$ in more detail.

Another important point is to extend the method to 2 + 1-dimensional cases as generalized Kadomtsev-Petviashvili equations studied numerically in [17]. There it was shown that localized initial data can have $L_\infty$ blow-up in a point, but the precise mechanism of the blow-up was not studied. Given the high resolution which was needed already in the 1 + 1-dimensional setting discussed here, it might be necessary to parallelize the code as has been done in [16].

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