A NOTE ON THE MULTIPLE-RECURSIVE MATRIX METHOD FOR GENERATING PSEUDORANDOM VECTORS

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Abstract. The multiple-recursive matrix method for generating pseudorandom vectors was introduced by Niederreiter (Linear Algebra Appl. 192 (1993), 301-328). We propose an algorithm for finding an efficient primitive multiple-recursive matrix method. Moreover, for improving the linear complexity, we introduce a tweak on the contents of the primitive multiple-recursive matrix method.

1. Introduction

In most of the modern stream ciphers, we generally use linear feedback shift registers (LFSRs) as basic building blocks that produce only one new bit per step. Such ciphers are often referred to as bit-oriented. Bit-oriented ciphers not only have large period and good statistical properties, but also have low cost of implementation in hardware and thus are quite useful in applications like wireless communications. However, in many situations such as high speed link encryption, an efficient software encryption is required and bit-oriented ciphers do not provide adequate efficiency.

The question arises: How to design feedback shift registers (FSRs) that output a word instead of a bit per clock? A very natural and obvious way is to consider FSRs over extension fields, but then field multiplication being an expensive operation, it would not really make our life easy in terms of software efficiency. The other way is to exploit word operations – logic operations and arithmetic operations – of modern computer processors in designing FSRs so as to enhance their efficiency in software implementation. In fact, Preneel in [32] poses the question of whether one can design fast and secure FSRs with the help of the word operations of modern processors and the techniques of parallelism.

Interestingly, a solution to Preneel’s problem was already available in the literature even before it was formally stated and was given by Niederreiter in a series of papers [27, 28, 29, 30] by introducing the multiple-recursive matrix method (MRMM) for generating pseudorandom vectors. This method involves matrix multiplication which is again an expensive operation as far as software efficiency is concerned. Zeng et al. [38] resolve this problem by imposing restriction on the choice of matrices used in the multiple-recursive matrix method. In fact, Zeng et al. [38] introduce the notion of word-oriented σ-LFSR and it turns out that the seemingly new notion of σ-LFSR is essentially equivalent to the multiple-recursive
matrix method for generating pseudorandom vectors. They also gave a conjectural formula for the number of primitive $\sigma$-LFSRs. This conjecture has been proved in the affirmative and the reader is referred to [7, 11, 12, 13, 21, 22] for more details. Throughout this paper, we shall use the acronym MRMM instead of $\sigma$-LFSR.

It may be noted that Tsaban and Vishne [36] also addressed the problem of Preneel by introducing the notion of transformation shift registers (TSR). It turns out that TSR is a special case of the multiple-recursive matrix method. One may refer to [8, 17, 19, 33] for some recent progress concerning TSRs.

By choosing matrices at random from a special set of matrices that are compatible with word operations of modern processors, a search algorithm for finding some efficient primitive MRMM was proposed in [38, Algorithm 1]. We would like to reiterate here that by efficient, we mean that the matrices used in MRMM can simply be replaced by word operations while computing the feedback. For all practical purposes where software efficiency is of paramount importance, we need an algorithm for explicitly constructing efficient primitive MRMM. It may be noted that a method for constructing primitive MRMM can be gleaned from the proof of [11, Theorem 6.1], but it neither constructs all the primitive MRMM nor does it produce an efficient primitive MRMM. Lachaud [23] and Krishnaswamy et al. [22] also proposed nice methods for constructing all of the primitive MRMM. In this paper, however, we are not really focusing on constructing all of the primitive MRMM. In fact, we are interested in constructing only some efficient ones so that the problem of software efficiency in various applications is resolved. Very recently, M. A. Golvanitsa has drawn our attention to [14, Section 2.2] which also discusses a similar construction for non-linearized skew MP-polynomials. However, it appears that our techniques are completely different from those discussed in [14].

As we know, linear complexity plays a crucial role in determining the security of the keystream generated by FSRs. In order to enhance linear complexity of sequences generated by the multiple-recursive matrix method, one might consider employing some nonlinear functions on its contents. In fact, in [17], a nonlinear scheme based on Langford arrangement was employed on sequences generated by primitive TSRs. We replicate a similar, yet slightly different tweak, for the sequences generated by primitive MRMM along the similar lines. Since this “little tweak” has not yet been reported in the literature, we thought of including it in the form of the tweaked primitive multiple-recursive matrix method for the sake of completeness.

The paper is organized as follows. In Section 2, we recall some definitions and results concerning the multiple-recursive matrix method that are needed in this work. We develop some mathematical theory for constructing efficient MRMM in Section 3. We propose an algorithm for finding efficient primitive MRMM in Section 4. In Section 5, we discuss implementation issues of MRMM obtained through our algorithm. Finally, in Section 6, we discuss the tweaked primitive MRMM based on Langford arrangement.

2. The Multiple-Recursive Matrix Method

We denote by $\mathbb{F}_q$ the finite field with $q$ elements, where $q$ is a prime power and by $\mathbb{F}_q[X]$ the ring of polynomials in one variable $X$ with coefficients in $\mathbb{F}_q$. Also we denote by $M_d(\mathbb{F}_q)$ the set of all $d \times d$ matrices with entries in $\mathbb{F}_q$. We now recall
some definitions and results from [11, 28] concerning the multiple-recursive matrix method.

In what follows, we fix positive integers \( m \) and \( n \), and a vector space basis \( \{ \alpha_0, \ldots, \alpha_{m-1} \} \) of \( \mathbb{F}_q^m \) over \( \mathbb{F}_q \). Given any \( s \in \mathbb{F}_q^m \), there are unique \( s_0, \ldots, s_{m-1} \in \mathbb{F}_q \) such that \( s = s_0 \alpha_0 + \cdots + s_{m-1} \alpha_{m-1} \), and we shall denote the corresponding co-ordinate vector \((s_0, \ldots, s_{m-1})\) of \( s \) by \( s \). Evidently, the association \( s \to s \) gives a vector space isomorphism of \( \mathbb{F}_q^m \) onto \( \mathbb{F}_q^m \). Elements of \( \mathbb{F}_q^m \) may be thought of as column vectors and so \( CS \) is a well-defined element of \( \mathbb{F}_q^m \) for any \( s \in \mathbb{F}_q^m \) and \( C \in M_m(\mathbb{F}_q) \).

**Definition 2.1.** Let \( C_0, C_1, \ldots, C_{n-1} \in M_m(\mathbb{F}_q) \). Given any \( n \)-tuple \((s_0, \ldots, s_{n-1})\) of elements of \( \mathbb{F}_q^m \), let \( (s_i)_{i=0}^\infty \) denote the infinite sequence of elements of \( \mathbb{F}_q^m \) determined by the following linear recurrence relation:

\[
(1) \quad s_{i+n} = C_0 s_i + C_1 s_{i+1} + \cdots + C_{n-1} s_{i+n-1}, \quad i = 0, 1, \ldots.
\]

The system (1) is called the multiple-recursive matrix method (MRMM) of order \( n \) over \( \mathbb{F}_q^m \), while the sequence \((s_i)_{i=0}^\infty\) is referred to as the sequence generated by the MRMM (1). The \( n \)-tuple \((s_0, s_1, \ldots, s_{n-1})\) is called initial state of the MRMM (1) and the polynomial \( I_m X^n - C_{n-1} X^{n-1} - \cdots - C_1 X - C_0 \) with matrix coefficients is called the matrix polynomial of the MRMM (1). The sequence \((s_i)_{i=0}^\infty\) is ultimately periodic if there are integers \( r, n_0 \) with \( r \geq 1 \) and \( n_0 \geq 0 \) such that \( s_{i+j} = s_i \) for all \( j \geq n_0 \). The least positive integer \( r \) with this property is called the period of \((s_i)_{i=0}^\infty\) and the corresponding least nonnegative integer \( n_0 \) is called the preperiod of \((s_i)_{i=0}^\infty\). The sequence \((s_i)_{i=0}^\infty\) is said to be periodic if its preperiod is 0.

The following proposition [11, Proposition 4.2] gives some basic facts about MRMM.

**Proposition 2.2.** For the sequence \((s_i)_{i=0}^\infty\) generated by the MRMM (1) of order \( n \) over \( \mathbb{F}_q^m \), we have

(i) \((s_i)_{i=0}^\infty\) is ultimately periodic, and its period is no more than \( q^{mn} - 1 \);

(ii) if \( C_0 \) is nonsingular, then \((s_i)_{i=0}^\infty\) is periodic; conversely, if \((s_i)_{i=0}^\infty\) is periodic whenever the initial state is of the form \((b, 0, \ldots, 0)\), where \( b \in \mathbb{F}_q^m \) with \( b \neq 0 \), then \( C_0 \) is nonsingular.

An MRMM of order \( n \) over \( \mathbb{F}_q^m \) is primitive if for any choice of nonzero initial state, the sequence generated by that MRMM is periodic of period \( q^{mn} - 1 \).

In view of Proposition 2.2 if \( I_m X^n - C_{n-1} X^{n-1} - \cdots - C_1 X - C_0 \in M_m(\mathbb{F}_q) [X] \) is the matrix polynomial of a primitive MRMM, then the matrix \( C_0 \) is necessarily nonsingular.

Corresponding to a matrix polynomial \( I_m X^n - C_{n-1} X^{n-1} - \cdots - C_1 X - C_0 \in M_m(\mathbb{F}_q) [X] \), we can associate a \((m, n)\)-block companion matrix \( T \in M_{mn}(\mathbb{F}_q) \) of the following form

\[
(2) \quad T = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & 0 & C_0 \\
I_m & 0 & 0 & \cdots & 0 & 0 & C_1 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\end{pmatrix},
\]
where $I_m$ denotes the $m \times m$ identity matrix over $\mathbb{F}_q$, while $0$ indicates the zero matrix in $M_m(\mathbb{F}_q)$. The set of all such $(m,n)$-block companion matrices $T$ over $\mathbb{F}_q$ shall be denoted by $\text{MRMM}(m,n,q)$. Using a Laplace expansion or a suitable sequence of elementary column operations, we conclude that if $T \in \text{MRMM}(m,n,q)$ is given by (2), then $\det T = \pm \det(C_0)$. Consequently,

(3) \hspace{1cm} T \in \text{GL}_{mn}(\mathbb{F}_q) \iff C_0 \in \text{GL}_m(\mathbb{F}_q),

where $\text{GL}_m(\mathbb{F}_q)$ is the general linear group of all $m \times m$ nonsingular matrices over $\mathbb{F}_q$.

It may be noted that the block companion matrix (2) is the state transition matrix for the MRMM (1). Indeed, the $k$-th state $S_k := (s_k, s_{k+1}, \ldots, s_{k+n-1}) \in \mathbb{F}^n_{q^m}$ of the MRMM (1) is obtained from the initial state $S_0 := (s_0, s_1, \ldots, s_{n-1}) \in \mathbb{F}^n_{q^m}$ by $S_k = S_0 T_k$, for any $k \geq 0$. We can identify MRMM (1) with block companion matrix (2).

The following lemma [11 Lemma 5.1] reduces the calculation of an $mn \times mn$ determinant to an $m \times m$ determinant.

**Lemma 2.3.** Let $T \in \text{MRMM}(m,n,q)$ be given as in (2) and also let $M(X) \in M_m(\mathbb{F}_q[X])$ be defined by $M(X) := I_mX^n - C_{n-1}X^{n-1} - \cdots - C_1X - C_0$. Then the characteristic polynomial of $T$ is equal to $\det (M(X))$.

The following characterization of primitive MRMM can be easily extracted from the results given in [11] (see also [27 Theorem 4]).

**Proposition 2.4.** Let $C_0 \in \text{GL}_m(\mathbb{F}_q)$. Then the following are equivalent:

(i) an MRMM (1) of order $n$ over $\mathbb{F}_{q^m}$ is primitive;
(ii) $o(T) = q^{mn} - 1$, where $o(T)$ denotes the multiplicative order of $T$ in $\text{GL}_{mn}(\mathbb{F}_q)$;
(iii) $\det (M(X))$ is a primitive polynomial over $\mathbb{F}_q$ of degree $mn$, where $M(X)$ is same as defined in Lemma 2.3.

We recall a lemma [28 Lemma 1] that enables us to determine the linear complexity of sequences generated by primitive MRMM.

**Lemma 2.5.** Let

$$s_i = \left(s_i^{(1)}, \ldots, s_i^{(m)}\right) \in \mathbb{F}_{q^m}^m \cong \mathbb{F}_{q^m} \quad i = 0, 1, \ldots,$$

be an arbitrary recursive vector sequence and let $h(X) \in \mathbb{F}_q[X]$ be the characteristic polynomial of the matrix $T$ in (2). Then for each $1 \leq j \leq m$ the sequence $s_0^{(j)}, s_1^{(j)}, \ldots$ of the $j$-th coordinates is a linear recurring sequence in $\mathbb{F}_q$ with characteristic polynomial $h(X)$.

The following corollary trivially follows from Lemma 2.3 and gives the component-wise linear complexity of the sequences generated by primitive MRMM.

**Corollary 2.6.** Let

$$s_i = \left(s_i^{(1)}, \ldots, s_i^{(m)}\right) \in \mathbb{F}_{q^m}^m \cong \mathbb{F}_{q^m} \quad i = 0, 1, \ldots,$$

be a sequence generated by a primitive MRMM of order $n$ over $\mathbb{F}_{q^m}$. Then for each $1 \leq j \leq m$, the linear complexity of the $j$-th coordinate sequence $s_0^{(j)}, s_1^{(j)}, \ldots$ over $\mathbb{F}_q$ is $mn$. 

An alternative statement of the Corollary 2.6 can be found in [38, Theorem 3]. Moreover, in view of Corollary 2.6, it is clear that if a sequence over $\mathbb{F}_q$ generated by a primitive MRMM of order $n$ is viewed as a sequence over $\mathbb{F}_q$, then its linear complexity is $m^2n$. In effect, a primitive MRMM of order $n$ over $\mathbb{F}_q$ is same as $m$ parallel primitive LFSRs of order $mn$.

3. Construction of the Multiple-Recursive Matrix Method

As alluded to in the introduction, Zeng et al. proposed a search algorithm [38, Algorithm 1] for generating efficient primitive MRMM. Their algorithm begins by randomly choosing some matrices that are compatible with word operations and then testing the primitivity of a polynomial obtained by computing the determinant of a matrix using Lemma 2.3.

We begin this section by defining the notion of generalized Horner’s form corresponding to a given polynomial. We then use it to construct an efficient primitive MRMM. It may be noted that the idea of using Horner’s form in the context of LFSR may not be common, but it has been used to make jumping efficient as can be seen in [15].

**Definition 3.1.** Let $f(X) = \sum_{i=0}^{d} a_i X^i$ be a polynomial of degree $d$ over $\mathbb{F}_q$. For any given positive integer $n \leq d$, we can find integers $m$ and $r$ such that $d = mn + r$, where $0 \leq r < n$. We express $f(X)$ in the following form

$$f(X) = f_0 + X^n (f_1 + X^n (f_2 + \cdots + X^n (f_{m-1} + X^n f_m) \cdots)),$$

where,

$$f_i(X) = \sum_{k=\text{in}}^{(i+1)n-1} a_k X^{k-in} \text{ for } i = 0, 1, \ldots, (m-1) \text{ and } f_m(X) = \sum_{k=mn}^{d} a_k X^{k-mn}.$$

The representation of $f(X)$ in (4) is referred to as $n$-Horner’s form of $f(X)$.

**Example 3.2.** Consider the polynomial $f(X) = 1 + X^2 + X^3 + X^7 + X^{10} + X^{11} + X^{12} \in \mathbb{F}_2[X]$ of degree 12. Here $d = 12$. For $n = 3$, we have $m = 4$ and $r = 0$. Then 3-Horner’s form of $f(X)$ is given by

$$f(X) = f_0 + X^3 (f_1 + X^3 (f_2 + X^3 (f_3 + X^3 f_4))),$$

where $f_0(X) = (1 + X^2)$, $f_1(X) = X^2$, $f_2(X) = X$, $f_3(X) = (X + X^2)$ and $f_4(X) = 1$.

Corresponding to the $n$-Horner’s form of a given polynomial of degree $mn$ over $\mathbb{F}_q$, we can associate an $m \times m$ matrix as defined below. This matrix would play a crucial role in the construction of efficient multiple-recursive matrix method of order $n$ over $\mathbb{F}_q$ for generating pseudorandom vectors.

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1. The 1-Horner’s form is indeed the usual Horner’s form of a polynomial used for computing the polynomial value with less number of multiplications.
**Definition 3.3.** Let \( m \) and \( n \) be positive integers and let \( f(X) = \sum_{i=0}^{mn} a_i X^i \) be a polynomial of degree \( mn \) over \( \mathbb{F}_q \). The \( m \times m \) matrix

\[
\begin{pmatrix}
X^n & 0 & 0 & \cdots & 0 & f_0 \\
-1 & X^n & 0 & \cdots & 0 & f_1 \\
0 & -1 & X^n & \cdots & 0 & f_2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & X^n & f_{m-2} \\
0 & 0 & 0 & \cdots & -1 & f_{m-1} + f_m X^n \\
\end{pmatrix}
\]

(5)

is referred to as the \( n \)-Horner’s matrix corresponding to the polynomial \( f(X) \) and denoted as \( H_m(n, f) \).

For each \( j = 0, 1, \ldots, n-1 \), let \( C_j \) denotes the \( m \times m \) matrix whose entries are the coefficients of \( X^j \) in the matrix \( H_m(n, f) \). It is easy to see that the matrix \( H_m(n, f) \) can be written as

\[
H_m(n, f) = I_m X^n + C_{n-1} X^{n-1} + \cdots + C_1 X + C_0,
\]

(6)

provided \( f \) is monic, that is, \( f_m = 1 \). It is clear from (6) that we can associate the multiple-recursive matrix method of order \( n \) over \( \mathbb{F}_q \) corresponding to these \( m \times m \) matrices \( C_0, C_1, \ldots, C_{n-1} \).

It is interesting to note that the matrix \( C_j (1 \leq j \leq n-1) \) has the following form

\[
C_j = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & a_j \\
0 & 0 & 0 & \cdots & 0 & a_{n+j} \\
0 & 0 & 0 & \cdots & 0 & a_{2n+j} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & a_{(m-2)n+j} \\
0 & 0 & 0 & \cdots & 0 & a_{(m-1)n+j} \\
\end{pmatrix}
\]

whose first \( (m-1) \) columns are zero. Moreover, the matrix \( C_0 \) has the following form

\[
C_0 = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & a_0 \\
-1 & 0 & 0 & \cdots & 0 & a_n \\
0 & -1 & 0 & \cdots & 0 & a_{2n} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & a_{(m-2)n} \\
0 & 0 & 0 & \cdots & -1 & a_{(m-1)n} \\
\end{pmatrix}
\]

It is due to this special structure of these matrices that we are able to construct an efficient multiple-recursive matrix method. In Section 5 we shall see in greater detail why such a construction is fast and efficient.

The following lemma gives the determinant of the matrix \( H_m(n, f) \) and will be used in the sequel.

**Lemma 3.4.** Let \( H_m(n, f) \) be \( n \)-Horner’s matrix corresponding to the polynomial \( f(X) \) of degree \( mn \) over \( \mathbb{F}_q \) as defined in (5). Then \( \det (H_m(n, f)) \) is equal to \( f(X) \).

**Proof.** Add \( X^n \) times the \( n \)th row to the \((n-1)\)th row of the matrix \( H_m(n, f) \). This will remove the \( X^n \) in the \((n-1)\)th row and it will not alter the determinant. Next, add \( X^n \) times the new \((n-1)\)th row to the \((n-2)\)th row. Continue successively...
until all of the $X^n$'s on the main diagonal have been removed. The result is the matrix

$$
\begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & f_0 + X^n (f_1 + X^n (f_2 + \cdots + X^n (f_{m-1} + X^n f_m) \cdots)) \\
-1 & 0 & 0 & \cdots & 0 & f_1 + X^n (f_2 + \cdots + X^n (f_{m-1} + X^n f_m) \cdots) \\
0 & -1 & 0 & \cdots & 0 & f_2 + \cdots + X^n (f_{m-1} + X^n f_m) \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & f_{m-2} + X^n (f_{m-1} + X^n f_m) \\
0 & 0 & 0 & \cdots & -1 & f_{m-1} + X^n f_m
\end{pmatrix}
$$

which has the same determinant as $H_m(n,f)$. We can clean up the last column by adding to it appropriate multiples of the other columns so as to obtain

$$
\det (H_m(n,f)) = \det \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & f(X) \\
-1 & 0 & 0 & \cdots & 0 & 0 \\
0 & -1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & -1 & 0
\end{pmatrix}.
$$

Finally, we can slide the last column to the first by successive column interchanges. We need $(m-1)$ interchanges, and so the determinant changes by $(-1)^{(m-1)}$. Further, if we pull out the negative sign in each of the rows in all except the first row, then the determinant gets multiplied by $(-1)^{(m-1)}$. It follows that the determinant of $H_m(n,f)$ is $(-1)^{2(m-1)}$ times the determinant of the diagonal matrix $\operatorname{diag}(f(X), 1, \ldots, 1)$ and this proves the lemma.

We can construct an efficient MRMM regardless of whether it’s reducible, irreducible or primitive. However, in view of cryptographic applications, we shall focus only in the construction of efficient primitive MRMM.

From Proposition 2.4, it is clear that an MRMM is primitive if the characteristic polynomial $\det(M(X))$ of its transition matrix $T$ is primitive of degree $mn$ over $\mathbb{F}_q$. We shall denote by $\operatorname{MRMMP}(m,n,q)$, the set of all those block companion matrices in $\operatorname{MRMM}(m,n,q)$ whose characteristic polynomial is primitive and by $\mathcal{P}(d,q)$, the set of primitive polynomials in $\mathbb{F}_q[X]$ of degree $d$. Then the characteristic map

$$
\Psi : \operatorname{M}_{mn}(\mathbb{F}_q) \rightarrow \mathbb{F}_q[X] \quad \text{defined by} \quad \Psi(T) := \det(XI_{mn} - T),
$$

if restricted to the set $\operatorname{MRMMP}(m,n,q)$ yields the following map

$$
\Psi_p : \operatorname{MRMMP}(m,n,q) \rightarrow \mathcal{P}(mn,q).
$$

By using the structure of Horner’s matrix, we prove the surjectivity of the map $\Psi_p$ in the following theorem. The proof of this theorem would enable us a way to construct efficient primitive MRMM.

**Theorem 3.5.** The map $\Psi_p : \operatorname{MRMMP}(m,n,q) \rightarrow \mathcal{P}(mn,q)$ is surjective.

**Proof.** Let $f(X) = \sum_{i=0}^{mn} a_i X^i \in \mathcal{P}(mn,q)$. Clearly, $f$ is a monic polynomial i.e. $a_{mn} = 1$. Therefore, as in (3), the $n$-Horner’s matrix $H_m(n,f)$ of $f(X)$ can be
expressed in the following form
\begin{equation}
H_m(n, f) = I_m X^n + C_{n-1} X^{n-1} + \cdots + C_1 X + C_0,
\end{equation}
where \( C_i \) denotes the \( m \times m \) matrix whose entries are coefficients of \( X^i \) in the Horner’s matrix \( H_m(n, f) \). Let \( \tilde{T} \in \text{MRMM}(m, n, q) \) denotes the block companion matrix corresponding to the matrix polynomial (7). Then by Lemma 2.3 and Lemma 3.4 it follows that
\( \Psi_P(\tilde{T}) = \det(XI_{mn} - \tilde{T}) = \det(H_m(n, f)) = f(X) \),
as desired. \( \Box \)

**Remark 3.6.** By taking the proof of Theorem 3.5 a step further, a short and elementary proof of [11, Theorem 6.1] follows immediately. In fact, it is easy to see that \( \det(C_0) = \pm a_0 \). Since \( f(X) \) is primitive, we have \( a_0 \neq 0 \) and hence \( C_0 \in \text{GL}_m(\mathbb{F}_q) \). Thus in view of \( \tilde{T} \in \text{GL}_{mn}(\mathbb{F}_q) \). Moreover, since characteristic polynomial \( f(X) \) of \( \tilde{T} \) is primitive, it follows from Proposition 2.4 that \( o(\tilde{T}) = q^{mn} - 1 \).

**Remark 3.7.** It may also be interesting to note that a short and elementary proof of [12, Proposition 2.2] can be derived by considering \( f(X) \) to be irreducible and following the similar lines as in the proof of Theorem 3.5.

4. The Algorithm

In this section, we present an algorithm to find an efficient primitive MRMM of order \( n \) over \( \mathbb{F}_{q^m} \). In view of the proof of Theorem 3.5 we shall begin by first finding a primitive polynomial \( f(X) \) of degree \( mn \) over \( \mathbb{F}_q \) so as to obtain a primitive MRMM of order \( n \) over \( \mathbb{F}_{q^m} \).

It may be remarked that for checking primitivity of a polynomial of degree \( mn \) over \( \mathbb{F}_q \), one needs to know the distinct prime factors of \( q^{mn} - 1 \) beforehand. The computational complexity of finding distinct prime factors of \( q^{mn} - 1 \) is very large. In fact, the factors of \( q^{mn} - 1 \) can not be computed in polynomial time in general. However, for smaller values of \( q \) (note that in most of applications \( q \) is 2), many thanks to the Cunningham project [5, 37], the factorization of \( q^{mn} - 1 \) is known for reasonable values of \( mn \) that are needed in most of practical applications. Our algorithm is based on the assumption that the distinct prime factors of \( q^{mn} - 1 \) are already known. All the sequential steps are described in Algorithm 1.

In Step 2 of the algorithm, one may use Ben-Or’s algorithm [1, 9] for irreducibility test, which is quite efficient in practice. It is pointed out in [9] that by using fast multiplication [6, 34, 35], the worst case complexity of Ben-Or’s algorithm is \( O(m^2 n^2 \log mn \log \log mn \log q) \). As noted in [10, Section 1], in polynomial basis representation of \( \mathbb{F}_{q^{mn}} \) over \( \mathbb{F}_q \), the exponentiation can be done with \( O(m^2 n^2 \log mn \log \log mn \log q) \) operations in \( \mathbb{F}_q \), with fast multiplication and repeated squaring. Thus the cost of Step 3 is \( O(km^2 n^2 \log mn \log \log mn \log q) \). Let \( \alpha \) denote the probability that a given random monic polynomial of degree \( mn \) be primitive. Since the number of primitive polynomials of degree \( mn \) over \( \mathbb{F}_q \) is \( \phi(q^{mn} - 1)/mn \), where \( \phi \) is Euler’s totient function. The value of \( \alpha \) is given by \( \phi(q^{mn} - 1)/(mnq^{mn}) \). It is clear that the expected number of times the Algorithm 1 is iterated to find a primitive MRMM is \( 1/\alpha \). So the expected number of times Step 2 to be executed is \( 1/\alpha \). It is well-known that the probability of a random monic polynomial of degree \( mn \) in \( \mathbb{F}_q[X] \) being irreducible over \( \mathbb{F}_q \) is close...
Algorithm 1: Finding an efficient primitive MRMM

Input: Positive integers $m$ and $n$, the prime power $q$, the distinct prime factors $p_1, p_2, \ldots, p_k$ of $q^{mn} - 1$.

Output: An efficient primitive MRMM of order $n$ over $\mathbb{F}_{q^m}$.

1. Choose at random a monic polynomial $f \in \mathbb{F}_q[X]$ of degree $mn$. This is done by randomly selecting $mn$ elements $a_0, a_1, \ldots, a_{mn-1}$ in $\mathbb{F}_q$ with $a_0 \neq 0$ and taking $f(X) = X^{mn} + a_{mn-1}X^{mn-1} + \cdots + a_1X + a_0$.
2. Verify if $f$ is irreducible. If $f$ is not irreducible then go to Step 1 otherwise go to Step 2.
3. Verify if $f$ is primitive. If $f$ is not primitive then go to Step 1 otherwise go to Step 3. The primitivity test is done as follows: Compute $h(X) = X^{(q^{mn} - 1)/p_i} \mod f(X)$ for each $i$. If $h(X) \neq 1$ for all $k$ distinct prime factors $p_i$ then $f$ is primitive.
4. Express $f(X)$ in its $n$-Horner’s form.
5. Construct $n$-Horner’s matrix $H_m(n, f)$ of $f(X)$.
6. Express $H_m(n, f)$ in the form of matrix polynomial as described in (6), i.e.,

$$H_m(n, f) = I_mX^n + C_{n-1}X^{n-1} + \cdots + C_1X + C_0,$$

where $C_i$ denotes the $m \times m$ matrix whose entries are coefficients of $X^i$ in the Horner’s matrix $H_m(n, f)$.
7. Return $C_0, C_1, \ldots, C_{n-1}$.

to $1/mn$. So the expected number of times Step 3 to be executed is $1/(mnq)$. Thus the expected run time of Algorithm 1 is $\alpha^{-1}O(m^2n^2 \log mn \log \log mn \log mnq) + (mnq)^{-1}O(km^2n^2 \log mn \log \log mn \log q)$, which, after simplification, can be seen to be equal to $O(\alpha^{-1}mn \log mn \log \log mn(q \log mnq + k \log q))$. In view of the fact that for a given number $N$, the number of distinct prime factors of $N$ is asymptotically $\log \log N$ [16, p. 51], we can simply omit the second term “$k \log q$” inside the big Oh notation. As a consequence, the expected run time of Algorithm 1 is $O(\alpha^{-1}m^2n^2 \log mn \log \log mn \log mnq)$. Further, by using the well-known lower bound on Euler’s totient function due to Landau [24, Theorem 3.4.2] (see also [26, Fact 2.102]), it follows that the expected run time of Algorithm 1 is given by $O(m^3n^3 \log \log qmn \log mn \log \log mn \log mnq)$.

Remark 4.1. It is clear that our algorithm finds an efficient primitive MRMM of order $n$ over $\mathbb{F}_{q^m}$ only for small values of $mn$ for which the factorization of $q^{mn} - 1$ is known. For large values of $mn$, it would be computationally infeasible to generate all primitive polynomials of degree $mn$ over $\mathbb{F}_q$ due to the rapid growth of the Euler’s totient function. For instance, the number of primitive polynomials of degree 100 over $\mathbb{F}_2$ is already $5.70767634 \times 10^{87}$. Even at the conservative estimate of three bytes per polynomial, this would exceed the total amount of data stored digitally in the world, which was estimated to be 264 exabytes in 2007 [18]. On the other hand, primitive polynomials of large degree over $\mathbb{F}_2$ are known; see, for example, some recent papers due to Brent and Zimmerman [2, 3, 4]. Thus if one intends to use above algorithm to generate primitive MRMM corresponding to all primitive polynomials of degree $mn$ (for small values of $mn$), then it is not an efficient way to do so. In fact, there are faster algorithms; see, for example, an algorithm due
to Porto, Guida and Montolivo [31], which generate all primitive polynomials of degree $D$ given a single primitive polynomial of degree $D$.

**Example 4.2.** Let us consider the same polynomial $f(X)$ as given in Example 3.2. One can verify that $f(X)$ is a primitive over $\mathbb{F}_2$. The 3-Horner’s matrix of $f(X)$ is given by

$$H_4(3,f) = \begin{pmatrix} X^3 & 0 & 0 & (1 + X^2) \\ 1 & X^3 & 0 & X^2 \\ 0 & 1 & X^3 & X \\ 0 & 0 & 1 & (X^3 + X^2 + X) \end{pmatrix}.$$  

We can express $H_4(3,f)$ in the form of a following matrix polynomial

$$H_4(3,f) = I_3X^3 + C_2X^2 + C_1x + C_0,$$

where $C_0 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, C_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$ and $C_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. It is clear that $C_0 \in \mathrm{GL}_4(\mathbb{F}_2)$. If $\widetilde{T}$ denotes the block companion matrix corresponding to the matrix polynomial, that is, $\widetilde{T} = \begin{pmatrix} 0 & 0 & C_0 \\ I_3 & 0 & C_1 \\ 0 & I_3 & C_2 \end{pmatrix}$, then $\widetilde{T} \in \mathrm{GL}_{12}(\mathbb{F}_2)$ and $o(\widetilde{T}) = 2^{12} - 1$. Moreover, $\Psi \rho(\widetilde{T}) = f(X)$. Now corresponding to these $C_0, C_1,$ and $C_2$, we can associate a primitive MRMM of order 3 over $\mathbb{F}_{2^4}$.

### 5. Efficient Implementation

In this section, we shall restrict ourselves to only binary fields and their extensions. However, all the results can be emulated over an arbitrary finite field. As pointed out in Section 3, the efficiency of MRMM constructed through Algorithm 1 is due to the special structure of the matrices $C_0, C_1, \ldots, C_{n-1}$. It was also noted in Section 3 that the first $(m-1)$ columns of matrix $C_j$ ($1 \leq j \leq n-1$) are zero. Moreover, the matrix $C_0$ has a special structure. In fact, it is easy to see that $C_0 = R + \widehat{C}_0$, where $R$ is right shift operator given by the matrix

$$R = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 \end{pmatrix}_{m \times m},$$

and $\widehat{C}_0$ has all its columns zero except the $m^{th}$ column, which is essentially the last column of $C_0$. The structure of $\widehat{C}_0$ is exactly same as $C_j$, $j \geq 1$.

The following lemma makes implementation of MRMM fast and efficient.

**Lemma 5.1.** For any matrix $A \in M_m(\mathbb{F}_2)$ having all the columns zero except the $m^{th}$ column and for any vector $s = [s_0, s_1, \ldots, s_{m-1}]^t \in \mathbb{F}_{2^m}$, we have

$$A\mathbf{s} = s_{m-1}\mathbf{v}_m$$

where $\mathbf{v}_m$ represents the $m^{th}$ column of the matrix $A$. 

The operation as the component-wise addition of the vectors.

Let \( s_i \) define a sequence over \( \mathbb{F}_q \). It is clear that (8) can be computed by using only one right shift operation and at most \( n \) bitwise XOR operations instead of matrix multiplications and thus, provides an efficient software realization.

6. Tweaked Multiple-Recursive Matrix Method

As we know that in [17], a tweak based on Langford arrangement was introduced for the sequences generated by TSRs. In this section, however, we shall consider a slightly different tweak, but based on Langford arrangement itself for the sequences generated by the multiple-recursive matrix method along the similar lines.

We recall the definition of Langford arrangement [25] of a sequence of numbers, which is an important object of study in combinatorics.

**Definition 6.1.** Arrange the numbers 11223344 in a sequence such that between equal numbers \( h \) there are exactly \( h \) other numbers. This type of arrangement of numbers is known as a Langford arrangement.

**Example 6.2.** For \( g = 4 \) and \( g = 8 \), the Langford arrangements are 41312432 and 6751814657342832, respectively.

We define the notion of tweaked primitive multiple-recursive matrix method based on Langford arrangement as follows.

**Definition 6.3.** Let \( s_i = \left( s_i^{(1)}, \ldots, s_i^{(m)} \right) \in \mathbb{F}_q^m \simeq \mathbb{F}_q^n \), \( i = 0, 1, \ldots \), be the sequence over \( \mathbb{F}_q^m \) generated by a primitive MRMM of order \( 2g \), where \( g \) is a positive integer. Suppose there exists a Langford arrangement for the number \( g \), and let \( \ell_k \) and \( r_k \), respectively, denote the left and right positions of the number \( k \) in the Langford arrangement of \( g \) from the left. Then \( r_k = \ell_k + k + 1 \).

We define a sequence \( t^\infty = t_0, t_1, \ldots \) over \( \mathbb{F}_q^m \) obtained from \( (s_i)_{i=0}^\infty \) by the following recurrence relation:

\[
(9) \quad t_i = \sum_{j=0}^{i} u_j \quad \text{for} \quad i = 0, 1, \ldots
\]

\[
(10) \quad u_j = \sum_{k=1}^{g} s_{2g+j-\ell_k} \ast s_{2g+j-r_k}.
\]

The operation \( \ast \) denotes the component-wise multiplication of the vectors defined as \( s_{2g+i-\ell_k} \ast s_{2g+i-r_k} = \left( s_{2g+i-\ell_k}^{(1)}, \ldots, s_{2g+i-r_k}^{(m)} \right) \) and \( \sum \) denotes the component-wise addition of the vectors.

The system (9) is called the **tweaked primitive MRMM based on a Langford arrangement** of order \( 2g \) over \( \mathbb{F}_q^m \), while the sequence \( (t_i)_{i=0}^\infty \) is referred to as the **sequence generated by the tweaked primitive MRMM based on a Langford arrangement**.

**Proof.** Proof is obvious. \( \square \)

By invoking Lemma 5.1, the recurrence relation (10) can be written as follows:

\[
(8) \quad s_{i+n} = R s_i + s_{i+1} = s_0 v_0^i + s_1 v_1^i + \cdots + s_{n-1} v_{n-1}^i,
\]

where \( s_i \) is the least significant bit (LSB) of \( i \), \( v_m^i \) is the \( m \)th column of the matrix \( C_i \) (0 ≤ \( i \) ≤ \( n - 1 \)).

It is clear that (8) can be computed by using only one right shift operation and at most \( n \) bitwise XOR operations instead of matrix multiplications and thus, provides an efficient software realization.
Example 6.4. We consider the Langford arrangement for the number \( g = 4 \) given by 41321432. In this case, the values of \( \ell_i \)'s and \( r_i \)'s are: \( \ell_1 = 2, r_1 = 4, \ell_2 = 5, r_2 = 8, \ell_3 = 3, r_3 = 7, \ell_4 = 1, r_4 = 6. \)

Let \( (s_i)_{i=0}^\infty \) be a sequence generated by a primitive MRMM of length 8 over \( \mathbb{F}_{2^m} \). Then the sequence \( (t_i)_{i=0}^\infty \) generated by a tweaked primitive MRMM based on the above Langford arrangement of length 8 over \( \mathbb{F}_{2^m} \) is given by

\[
t_0 = u_0, t_1 = u_0 + u_1, \text{ and so on,}
\]

where \( u_0 = s_6 * s_4 + s_3 * s_0 + s_5 * s_1 + s_7 * s_2, u_1 = s_7 * s_5 + s_4 * s_1 + s_6 * s_2 + s_8 * s_3, \) and so on.

The following theorem gives the component-wise linear complexity of the auxiliary sequence \( (u_i)_{i=0}^\infty \) as defined in (10).

Theorem 6.5. Let

\[
u_i = \left( u_i^{(1)}, \ldots, u_i^{(m)} \right) \in \mathbb{F}_q^m \cong \mathbb{F}_{q^m}, i = 0, 1, \ldots,
\]

be a sequence as defined in (10). Then for each \( 1 \leq j \leq m \), the linear complexity of the \( j \)th coordinate sequence \( u_0^{(j)}, u_1^{(j)}, \ldots \) is given by \( mn/(mn+1) \).

Proof. For each \( 1 \leq j \leq m \), it follows from (10) that

\[
u_i^{(j)} = \sum_{k=1}^g s_i^{(j)}(2g+i-\ell_k) s_i^{(j)}(2g+i-r_k), \quad i = 0, 1, \ldots.
\]

The Corollary 6.6 ensures that the linear complexity of the component sequences \( s_i^{(j)}, i = 0, 1, \ldots, \) is \( mn \). Now \( s_i^{(j)}, i = 0, 1, \ldots, \) can be thought of as a sequence generated by a primitive LFSR of order \( mn \) and thus it follows from [20] Section III that the linear complexity of the sequence \( (u_i^{(j)})_{i=0}^\infty \) is \( mn/(mn+1) \).

In view of Theorem 6.5, the component-wise linear complexity of the sequence \( (t_i)_{i=0}^\infty \) generated by tweaked primitive MRMM based on Langford arrangement of order \( 2g \) over \( \mathbb{F}_{q^m} \) is of the order of \( mn(mn+1)/2 \), which is \( (mn+1)/2 \) times more than that of the sequences generated by the usual primitive MRMM.

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