On factorized groups with permutable subgroups of factors

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To the 75th anniversary of Mohamed Asaad

Abstract. The subgroups $A$ and $B$ of a group $G$ are called msp-permutable, if the following statements hold: $AB$ is a subgroup of $G$; the subgroups $P$ and $Q$ are mutually permutable, where $P$ is an arbitrary Sylow $p$-subgroup of $A$ and $Q$ is an arbitrary Sylow $q$-subgroup of $B$, $p \neq q$. In the present paper, we investigate groups that factorized by two msp-permutable subgroups. In particular, the supersolubility of the product of two supersoluble msp-permutable subgroups is proved.

Keywords. mutually permutable subgroups, Sylow subgroups, msp-permutable subgroups, supersoluble groups.

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Throughout this paper, all groups are finite and $G$ always denotes a finite group. We use the standard notations and terminology of \cite{3}. The notation $Y \leq X$ means that $Y$ is a subgroup of a group $X$.

The subgroups $A$ and $B$ of a group $G$ are called mutually (totally) permutable, if $UB = BU$ and $AV = VA$ (respectively, $UV = VU$) for all $U \leq A$ and $V \leq B$.

The idea of totally and mutually permutable subgroups was first initiated by M. Asaad and A. Shaalan in \cite{1}. This direction have since been subject of an in-depth study of many authors. An exhaustive report on this matter appears in \cite{3} chapters 4–5.
It is quite natural to consider a factorized group $G = AB$ in which certain subgroups of the factors $A$ and $B$ are mutually (totally) permutable. In this direction, V.S. Monakhov [7] obtained the solubility of a group $G = AB$ under the assumption that the subgroups $A$ and $B$ are soluble and the Carter subgroups (Sylow subgroups) of $A$ and of $B$ are permutable.

We introduce the following

**Definition.** The subgroups $A$ and $B$ of a group $G$ are called msp-permutable, if the following statements hold:

1. $AB$ is a subgroup of $G$;
2. the subgroups $P$ and $Q$ are mutually permutable, where $P$ is an arbitrary Sylow $p$-subgroup of $A$ and $Q$ is an arbitrary Sylow $q$-subgroup of $B$, $p \neq q$.

In the present paper, we investigate groups that factorized by two msp-permutable subgroups. In particular, the supersolubility of the product of two supersoluble msp-permutable subgroups is proved.

## 1 Preliminaries

In this section, we give some definitions and basic results which are essential in the sequel. A group whose chief factors have prime orders is called **supersoluble**. Recall that a $p$-closed group is a group with a normal Sylow $p$-subgroup and a $p$-nilpotent group is a group with a normal Hall $p'$-subgroup.

Denote by $G'$, $Z(G)$, $F(G)$ and $\Phi(G)$ the derived subgroup, centre, Fitting and Frattini subgroups of $G$, respectively; $\mathbb{P}$ the set of all primes. We use $E_{p^t}$ to denote an elementary abelian group of order $p^t$ and $Z_m$ to denote a cyclic group of order $m$. The semidirect product of a normal subgroup $A$ and a subgroup $B$ is written as follows: $A \rtimes B$.

The monographs [2], [5] contain the necessary information of the theory of formations. The formations of all nilpotent, $p$-groups and supersoluble groups are denoted by $\mathfrak{N}$, $\mathfrak{N}_p$ and $\mathfrak{U}$, respectively. A formation $\mathfrak{F}$ is said to be **saturated** if $G/\Phi(G) \in \mathfrak{F}$ implies $G \in \mathfrak{F}$. A **formation function** is a function $f$ defined on $\mathbb{P}$ such that $f(p)$ is a, possibly empty, formation. A formation $\mathfrak{F}$ is said to be **local** if there exists a formation function $f$ such that $\mathfrak{F} = \{G \mid G/F_p(G) \in f(p)\}$. Here $F_p(G)$ is the greatest normal $p$-nilpotent subgroup of $G$. We write $\mathfrak{F} = LF(f)$ and $f$ is a local definition of $\mathfrak{F}$. By [5, Theorem IV.3.7], among all possible local definitions of a local formation $\mathfrak{F}$ there exists a unique $f$ such that $f$ is integrated (i.e., $f(p) \subseteq \mathfrak{F}$ for all $p \in \mathbb{P}$) and full (i.e., $f(p) = \mathfrak{N}_p f(p)$ for all $p \in \mathbb{P}$). Such local definition $f$ is said to be **canonical local definition** of $\mathfrak{F}$. By [5, Theorem IV.4.6], a formation is saturated if and only if it is local.
A subgroup $H$ of a group $G$ is called $\mathbb{P}$-subnormal in $G$, see [12], if either $H = G$, or there is a chain subgroups

$$H = H_0 \leq H_1 \leq \ldots \leq H_n = G, \ |H_i : H_{i-1}| \in \mathbb{P}, \ \forall i.$$ 

A group $G$ is called w-supersoluble (widely supersoluble), if every Sylow subgroup of $G$ is $\mathbb{P}$-subnormal in $G$. Denote by $\mathfrak{wU}$ the class of all w-supersoluble groups, see [12]. In [12] Theorem 2.7, Proposition 2.8 proved that $\mathfrak{wU}$ is a subgroup-closed saturated formation and every group from $\mathfrak{wU}$ has an ordered Sylow tower of supersoluble type. By [8 Theorem B], [9 Theorem 2.6], [12] Theorem 2.13], $G \in \mathfrak{wU}$ if and only if $G$ has an ordered Sylow tower of supersoluble type and every metanilpotent (biprimary) subgroup of $G$ is supersoluble.

Denote by $\mathfrak{vU}$ the class of groups all of whose primary cyclic subgroups are $\mathbb{P}$-subnormal. In [8 Theorem B] proved that $\mathfrak{vU}$ is a subgroup-closed saturated formation and $G \in \mathfrak{vU}$ if and only if $G$ has an ordered Sylow tower of supersoluble type and every biprimary subgroup of $G$ with a cyclic Sylow subgroup is supersoluble. It is easy to verify that $\mathfrak{U} \subseteq \mathfrak{wU} \subseteq \mathfrak{vU} \subseteq \mathcal{D}$.

Here $\mathcal{D}$ is the formation of all groups which have an ordered Sylow tower of supersoluble type.

If $H$ is a subgroup of $G$, then $H_G = \bigcap_{x \in G} H^x$ is called the core of $H$ in $G$. If a group $G$ contains a maximal subgroup $M$ with trivial core, then $G$ is said to be primitive and $M$ is its primitivator. A simple check proves the following lemma.

**Lemma 1.1.** Let $\mathfrak{F}$ be a saturated formation and $G$ be a group. Assume that $G \notin \mathfrak{F}$, but $G/N \in \mathfrak{F}$ for all non-trivial normal subgroups $N$ of $G$. Then $G$ is a primitive group.

**Lemma 1.2.** ([5 Theorem 15.6]) Let $G$ be a soluble primitive group and $M$ is a primitivator of $G$. Then the following statements hold:

1. $\Phi(G) = 1$;
2. $F(G) = C_G(F(G)) = O_p(G)$ and $F(G)$ is an elementary abelian subgroup of order $p^n$ for some prime $p$ and some positive integer $n$;
3. $G$ contains a unique minimal normal subgroup $N$ and moreover, $N = F(G)$;
4. $G = F(G) \rtimes M$ and $O_p(M) = 1$.

**Lemma 1.3.** ([10 Lemma 2.16]) Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and $G$ be a group with a normal subgroup $E$ such that $G/E \in \mathfrak{F}$. If $E$ is cyclic, then $G \in \mathfrak{F}$.

**Lemma 1.4.** Let $\mathfrak{F}$ be a formation, $G$ group, $A$ and $B$ subgroups of $G$ such that $A$ and $B$ belong to $\mathfrak{F}$. If $[A, B] = 1$, then $AB \in \mathfrak{F}$.
Proof. Since

\[ [A, B] = \langle [a, b] \mid a \in A, b \in B \rangle = 1, \]

it follows that \( ab = ba \) for all \( a \in A, b \in B \). Let

\[ A \times B = \{ (a, b) \mid a \in A, b \in B \}, \]

be the external direct product of groups \( A \) and \( B \). Since \( A \in \mathcal{F}, B \in \mathcal{F} \) and \( \mathcal{F} \) is a formation, we have \( A \times B \in \mathcal{F} \). Let \( \varphi : A \times B \to AB \) be a function with \( \varphi((a, b)) = ab \). It is clear that \( \varphi \) is a surjection. Because

\[ \varphi((a_1, b_1)(a_2, b_2)) = \varphi((a_1a_2, b_1b_2)) = a_1a_2b_1b_2 = \]

\[ = a_1b_1a_2b_2 = \varphi((a_1, b_1))\varphi((a_2, b_2)), \]

it follows that \( \varphi \) is an epimorphism. The core \( \text{Ker} \varphi \) contains all elements \((a, b)\) such that \( ab = 1 \). In this case \( a = b^{-1} \in A \cap B \leq Z(G) \). By the Fundamental Homomorphism Theorem,

\[ A \times B / \text{Ker} \varphi \cong AB. \]

Since \( A \times B \in \mathcal{F} \) and \( \mathcal{F} \) is a formation, \( A \times B / \text{Ker} \varphi \in \mathcal{F} \). Hence \( AB \in \mathcal{F} \). \qed

Lemma 1.5. ([4]) Let a group \( G = HK \) be the product of subgroups \( H \) and \( K \). If \( L \) is normal in \( H \) and \( L \leq K \), then \( L \leq K_G \).

Lemma 1.6. Let \( G = P \rtimes M \) be a primitive soluble group, where \( M \) is a primitivator of \( G \) and \( P \) is a Sylow \( p \)-subgroup of \( G \). Let \( A \) and \( B \) be subgroups of \( M \) and \( M = AB \). If \( B \leq N_G(X) \) for every subgroup \( X \) of \( P \), then the following statements hold:

1. \( B \) is a cyclic group of order dividing \( p - 1 \);
2. \( [A, B] = 1 \).

Proof. We fix an element \( b \in B \). If \( x \in P \), then \( x^b \in \langle x \rangle \), since \( B \leq N_G(\langle x \rangle) \) by hypothesis. Hence \( x^b = x^{m_x} \), where \( m_x \) is a positive integer and \( 1 \leq m_x \leq p \). If \( y \in P \setminus \{ x \} \), then

\[ (xy)^b = (xy)^{m_{xy}} = x^{m_xy}\ y^{m_{xy}}, \ (xy)^b = x^b\ y^b = x^{m_x}\ y^{m_y}, \]

\[ x^{m_{xy}}\ y^{m_{xy}} = x^{m_{xy}}\ y^{m_{xy}}, \ x^{m_{xy}} = y^{m_{xy}} = 1, \ m_{xy} = m_x = m_y. \]

Therefore we can assume that \( x^b = x^{n_b} \) for all \( x \in P \), where \( 1 \leq n_b \leq p \) and \( n_b \) is a positive integer.
Assume that there exist $d \in B$ and $y \in P \setminus \{1\}$ such that $y^d = y$. Then $n_d = 1$ and $x^d = x$ for all $x \in P$, i.e. $d \in C_G(P) = P$ and $d = 1$. Consequently $B$ is a group automorphism of a group of order $p$. Hence $B$ is cyclic of order dividing $p - 1$.

Show that $[A, B] = 1$. We fix an element $[b^{-1}, a^{-1}] \in [A, B]$. Since $P$ is normal in $G$, it follows that $x^a \in P$ for any $a \in A$ and any $x \in P$. Hence

$$x^{[b^{-1}, a^{-1}]} = x^{bab^{-1}a^{-1}} = (x^b)^{a^{-1}} = (x^n)^{a^{-1}} = (x^a)^{b^{-1}a^{-1}} = (x^b)^{b^{-1}a^{-1}} = x.$$ 

Therefore $[b^{-1}, a^{-1}] \in C_G(P) = P$. Since $[A, B] \leq M$, we have $[b^{-1}, a^{-1}] \in M \cap P = 1$ and $[A, B] = 1$.

## 2 Properties of msp-permutable subgroups

We will say that a group $G$ satisfies the property:

- $E_\pi$ if $G$ has at least one Hall $\pi$-subgroup;
- $C_\pi$ if $G$ satisfies $E_\pi$ and any two Hall $\pi$-subgroups of $G$ are conjugate in $G$;
- $D_\pi$ if $G$ satisfies $C_\pi$ and every $\pi$-subgroup of $G$ is contained in some Hall $\pi$-subgroup of $G$.

Such a group is also called an $E_\pi$-group, $C_\pi$-group, and $D_\pi$-group, respectively.

**Lemma 2.1.** Let $A$ and $B$ be msp-permutable subgroups of $G$ and $G = AB$.

1. If $N$ is a normal subgroup of $G$, then $G/N = (AN/N)(BN/N)$ is the msp-permutable product of subgroups $AN/N$ and $BN/N$.

2. If $A \leq H \leq G$, then $H$ is the msp-permutable product of subgroups $A$ and $H \cap B$.

3. If $G \in D_\pi$, then there exist Hall $\pi$-subgroups $G_\pi$, $A_\pi$, and $B_\pi$ of $G$, of $A$ and, of $B$, respectively, such that $G_\pi = A_\pi B_\pi$ is the msp-permutable product of subgroups $A_\pi$ and $B_\pi$.

**Proof.** 1. Let $p \in \pi(AN/N)$, $X/N$ be a Sylow $p$-subgroup of $AN/N$ and $P$ be a Sylow $p$-subgroup of $A$. Then $PN/N = X/N$. Similarly, if $q \in \pi(BN/N)$ such that $q \neq p$, $Y/N$ is a Sylow $q$-subgroup of $BN/N$ and $Q$ is a Sylow $q$-subgroup of $B$. Then $QN/N = Y/N$. By hypothesis, $P$ and $Q$ are mutually permutable. Hence $X/N$ and $Y/N$ are mutually permutable.

2. By Dedekind’s identity, $H = A(H \cap B)$. Let $A_q$ be a Sylow $q$-subgroup of $A$, $R$ be a Sylow $r$-subgroup of $H \cap B$, where $q \neq r$, and $B_r$ be a Sylow
$r$-subgroup of $B$ containing $R$. Since $(H \cap B_r)$ is a Sylow $r$-subgroup of $H \cap B$
and $R \leq H \cap B_r$, it follows that $R = H \cap B_r$.

Because $A_q$ and $B_r$ are mutually permutable, we have $A_q U \leq G$ for every
subgroup $U$ of $R$.

Let $V$ be an arbitrary subgroup of $A_q$. Since $A_q$ and $B_r$ are mutually
permutable,

$$V B_r \leq G, \quad H \cap V B_r = V (H \cap B_r) = VR \leq G.$$ 

Hence $A_q$ and $R$ are mutually permutable.

3. By [3] Theorem 1.1.19, there are Hall $\pi$-subgroups $G_\pi$, $A_\pi$ and $B_\pi$ of $G$,
of $A$, and of $B$, respectively, such that $G_\pi = A_\pi B_\pi$. Since $A$ and $B$ are msp-
permutable, it follows that obviously, $A_\pi$ and $B_\pi$ are msp-permutable.

\begin{lemma}
Let $A$ and $B$ be msp-permutable subgroups of $G$ and $G = AB$.
Let $p, r \in \pi(G)$, $p$ be the greatest prime in $\pi(G)$ and $r$ be the smallest prime
in $\pi(G)$. Then the following statements hold:

1. if $A$ and $B$ are $p$-closed, then $G$ is $p$-closed;
2. if $A$ and $B$ are $r$-nilpotent, then $G$ is $r$-nilpotent;
3. if $A$ and $B$ have an ordered Sylow tower of supersoluble type, then $G$
has an ordered Sylow tower of supersoluble type.
\end{lemma}

\begin{proof}
1. By [3] Theorem 1.1.19, there are Sylow $p$-subgroups $P, P_1$ and $P_2$
of $G$, of $A$, and of $B$, respectively, such that $P = P_1 P_2$. By hypothesis, $P_1$
is normal in $A$ and $P_2$ is normal in $B$. Let $H_1$ and $H_2$ be Hall $p'$-subgroups
of $A$ and of $B$, respectively, and $Q$ be a Sylow $q$-subgroup of $H_1$, where $q \in \pi(H_1)$. Choose a chain of subgroups

$$1 = Q_0 < Q_1 < \ldots < Q_t-1 < Q_t = Q, \quad |Q_{i+1} : Q_i| = q.$$ 

Since $A$ and $B$ are msp-permutable, we have $P_2 Q_i$ is a subgroup of $G$ for
every $i$. Since $|P_2 Q_1 : P_2| = q$ and $p > q$, it follows that $P_2$ is normal in $P_2 Q_1$.
Then by induction, we have that $P_2$ is normal in $P_2 Q_i$. Because $q$ is an
arbitrary prime in $\pi(H_1)$, it follows that $P_2$ is normal in $P_2 H_1$ and $\langle H_1, H_2 \rangle \leq
N_G(P_2)$. Similarly, $\langle H_1, H_2 \rangle \leq N_G(P_1)$. Hence $P = P_1 P_2$ is normal in $G$.

2. Let $R, R_1$ and $R_2$ are Sylow $r$-subgroups of $G$, of $A$, and of $B$, respect-
ively, such that $R = R_1 R_2$. Let $K_1$ and $K_2$ be Hall $r'$-subgroups of $A$ and
of $B$. Let $q \in \pi(G) \setminus \{r\}$, $Q, Q_1$ and $Q_2$ be Sylow $q$-subgroups of $G$, of $A$,
and of $B$, respectively, such that $Q = Q_1 Q_2$. Choose a chain of subgroups

$$1 = V_0 < V_1 < \ldots < V_{i-1} < V_i = R_1, \quad |V_{i+1} : V_i| = r.$$ 

Since $A$ and $B$ are msp-permutable, $V_i Q_2$ is a subgroup of $G$ for every $i$.
Since $|V_1 Q_2 : Q_2| = r$ and $q > r$, it follows that $Q_2$ is normal in $V_1 Q_2$. Then
by induction, we have that $R_1 \leq N_G(Q_2)$. By hypothesis, $A$ is $r$-nilpotent, hence $R_1 \leq N_G(Q_1)$ and $R_1 \leq N_G(Q)$. Similarly, $R_2 \leq N_G(Q)$ and $G$ has a $r$-nilpotent Hall $\{r, q\}$-subgroup $RQ$. Since $q$ is an arbitrary prime in $\pi(G) \setminus \{r\}$, it follows that $G$ is soluble and $r$-nilpotent by \cite{11} Corollary.

3. By (1), we have that a Sylow $p$-subgroup $P$ is normal in $G$ for the greatest $p \in \pi(G)$. By Lemma\cite{2,1}(1), $G/P$ is the product of msp-permutable subgroups $AN/N$ and $BN/N$. By induction, $G/N$ has an ordered Sylow tower of supersoluble type, hence $G$ has an ordered Sylow tower of supersoluble type.

\[ \square \]

**Theorem 2.1.** Let $A$ and $B$ be msp-permutable subgroups of $G$ and $G = AB$. If $A$ and $B$ are soluble, then $G$ is soluble.

\[ \text{Proof.} \text{ We use induction on the order of } G \text{ and the method of proof from } \cite{7} \text{ Theorem 2]. Let } N \neq 1 \text{ be a soluble normal subgroup of } G. \text{ By Lemma\cite{2,1}(1), } G/P \text{ is the product of soluble msp-permutable subgroups } AN/N \text{ and } BN/N. \text{ By induction, } G/N \text{ is soluble, hence } G \text{ is soluble. In what follows, we assume that } G \text{ contains no non-trivial soluble normal subgroups.}

Since $A$ is soluble, $U = O_s(A) \neq 1$ for some $s \in \pi(A)$. If $B$ is an $s$-subgroup of $G$, then $G = AG_s$, $U \leq G_s$ and $U^G \leq (G_s)_G$ by Lemma\cite{1,5} a contradiction. Hence $B$ is not $s$-subgroup of $G$ and let $Q$ be an arbitrary Sylow $q$-subgroup of $B$, where $q \in \pi(B) \setminus \{s\}$. Since $A$ and $B$ are msp-permutable, $UQ = UQ^{ba} = U^a(Q^b)^a = (UQ^b)^a = (Q^bU)^a = QU$ for every $x = ba \in G$, where $b \in B$ and $a \in A$. By \cite{8} Theorem 7.2.5, $D = UQ \cap Q^U$ is subnormal in $G$. Since $UQ \leq UQ$ and $UQ$ is soluble, it follows that $D$ is a soluble subnormal subgroup of $G$ and $D = 1$. Hence $[U, Q] \leq [UQ, Q^U] \leq D = 1$.

This is true for any Sylow $q$-subgroup of $B$, therefore $[U, Q^B] = 1$.

Let $H = N_G(U)$. By Dedekind’s identity, $H = A(H \cap B)$. By Lemma\cite{2,1}(2), $H$ is the product of soluble msp-permutable subgroups $A$ and $H \cap B$. By induction, $H$ is soluble. Since $[U, Q^B] = 1$, we have $Q^B \leq N_G(U) = H$. Because $G = AB = HB$, $Q^B$ is normal in $B$ and $Q^B \leq H$, it follows that $Q^B \leq H_G = 1$ by Lemma\cite{1,5} a contradiction.

\[ \square \]

**Lemma 2.3.** Let $G = G_1G_2$ be the product of msp-permutable subgroups $G_1$ and $G_2$. If a Sylow $p$-subgroup $P$ of $G$ is normal in $G$ and abelian, then $P \cap G_i$ is normal in $G$ for every $i \in \{1, 2\}$. 

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Proof. Assume that \( i, j \in \{1, 2\} \) and \( i \neq j \). It is clear that \( P \cap G_i \) is a Sylow \( p \)-subgroup of \( G_i \) and \( P \cap G_i = (G_i)_p \) is normal in \( G_i \). Hence \( G_i \) has a Hall \( p' \)-subgroup \( (G_i)_{p'} \). Since \( G_i \) and \( G_j \) are msp-permutable, it follows that \( (G_i)_p(G_j)_{p'} \) is a subgroup of \( G \) and \( (G_j)_{p'} \leq N_G((G_i)_p) \), because every subgroup of \( G \) is \( p \)-closed. By hypothesis, \( P \) is abelian, therefore \( (G_i)_p \) is normal in \( P \) and

\[
G_j = (G_j)_p(G_j)_{p'} = (P \cap G_j)(G_j)_{p'} \leq N_G((G_i)_p).
\]

Hence \( (G_i)_p \) is normal in \( G = G_iG_j = G_1G_2 \) for every \( i \in \{1, 2\} \). \( \square \)

3 Proof of the main theorem

**Theorem 3.1.** Let \( \mathcal{F} \) be a subgroup-closed saturated formation such that \( U \subseteq \mathcal{F} \subseteq D \). Let \( G = G_1G_2 \) be the product of msp-permutable subgroups \( G_1 \) and \( G_2 \). If \( G_1, G_2 \in \mathcal{F} \), then \( G \in \mathcal{F} \).

**Proof.** By Lemma \( 2.2 \)(3), \( G \) has an ordered Sylow tower of supersoluble type. Let \( P \) be a Sylow \( p \)-subgroup of \( G \), where \( p \) is the greatest prime in \( \pi(G) \). Then \( P \) is normal in \( G \).

Assume that \( G \notin \mathcal{F} \). Let \( N \) be a non-trivial normal subgroup of \( G \). Hence

\[
G/N = (G_1N/N)(G_2N/N),
\]

By Lemma \( 2.3 \)(1), \( G_1N/N \) and \( G_2N/N \) are msp-permutable. Consequently, \( G/N \) satisfies the hypothesis of the theorem, and by induction, \( G/N \in \mathcal{F} \).

Since \( \mathcal{F} \) is saturated, \( G \) is primitive by Lemma \( 1.1 \). Hence \( \Phi(G) = 1 \), \( G = N \rtimes M \), where \( N = C_G(N) = F(G) = O_p(G) = P \) is a unique minimal normal subgroup of \( G \) by Lemma \( 1.2 \). Therefore \( M \) is a Hall \( p' \)-subgroup of \( G \) and \( M = (G_1)_{p'}(G_2)_{p'} \) for some Hall \( p' \)-subgroups \( (G_1)_{p'} \) and \( (G_2)_{p'} \) of \( G_1 \) and of \( G_2 \), respectively.

Suppose that \( p \) divides \( |G_1| \) and \( |G_2| \). By Lemma \( 2.4 \), \( P \leq G_1 \cap G_2 \). Let \( P_1 \leq P \) and \( |P_1| = p \). Since \( P \leq G_1 \) and \( P \) permutes with \( P_1 \) for every Sylow subgroup \( Q \) of \( G_2 \), we have \( P_1(G_2)_{p'} \leq G \) and \( (G_2)_{p'} \leq N_G(P_1) \). Similarly, since \( P \leq G_2 \) and \( P \) permutes with \( P_1 \) for every Sylow subgroup \( R \) of \( (G_1)_{p'} \), it follows that \( P_1(G_1)_{p'} \leq G \) and \( (G_1)_{p'} \leq N_G(P_1) \). Hence \( M = (G_1)_{p'}(G_2)_{p'} \leq N_G(P_1) \) and \( P_1 \) is normal in \( G \). By Lemma \( 1.3 \), \( G \in \mathcal{F} \), a contradiction.

Thus \( P \leq G_1 \) and \( G_2 \) is a \( p' \)-subgroup of \( G \). By Lemma \( 1.4 \)(1), \( G_2 \) is a cyclic group of order dividing \( p - 1 \). Hence \( G_2 \in g(p) \), where \( g \) is a
canonical local definition of a saturated formation $\mathcal{U}$. Since $\mathcal{U} \subseteq \mathcal{F}$, we have by Proposition IV.3.11, $g(p) \subseteq f(p)$, where $f$ is a canonical local definition of a saturated formation $\mathcal{F}$. Hence $G_2 \in f(p)$. Since $P \leq G_1$, it follows that $G_1 = P \times (G_1)_{p'}$. Because $G_1 \in \mathcal{F}$ and $F_p(G_1) = P$, we have $G_1/F_p(G_1) = G_1/P \cong (G_1)_{p'} \in f(p)$. By Lemma 1.4, $[(G_1)_{p'}, (G_2)_{p'}] = 1$. Since $(G_1)_{p'} \in f(p)$, $(G_2)_{p'} \in f(p)$ and $f(p)$ is a formation, it follows that by Lemma 1.3, $G/P \cong M = (G_1)_{p'}(G_2)_{p'} \in f(p)$. Because $P \in \mathcal{N}_p$, we have $G \in \mathcal{F}$, a contradiction. The theorem is proved.

**Corollary 3.1.** Let $G = G_1G_2$ be the product of msp-permutable subgroups $G_1$ and $G_2$.

1. If $G_1, G_2 \in \mathcal{U}$, then $G \in \mathcal{U}$.
2. If $G_1, G_2 \in w\mathcal{U}$, then $G \in w\mathcal{U}$.
3. If $G_1, G_2 \in v\mathcal{U}$, then $G \in v\mathcal{U}$.

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