\textbf{ℓ}-modular representations of \( p \)-adic groups \( \text{SL}_n(F) \):
maximal simple \( k \)-types

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\textbf{Abstract}

Let \( p \) be an arbitrary prime number and \( k \) an algebraically closed field of characteristic \( l \neq p \). We establish the theory of maximal simple \( k \)-types of Levi subgroups \( M' \) of \( \text{SL}_n(F) \), where \( F \) is a non-archimedean locally compact field of residual characteristic \( p \). We prove the exhaustion property and the unicity property of weakly intertwining implying conjugacy for maximal simple \( k \)-types, extended maximal simple \( k \)-types and simple \( k \)-characters of \( M' \).

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1 Introduction

1.1 Backgrounds and motivations

Let $F$ be a non-archimedian locally compact field whose residue field is of characteristic $p$, and $A$ a connected algebraic reductive group defined over $F$. We say that $A$ is a $p$-adic group, if $A$ is the group $A(F)$ of the $F$-rational points of $A$, and we endow it with the locally pro-finite topology through $F$. Let $k$ be an algebraically closed field of characteristic $\ell$ with $\ell \neq p$ and $\ell \neq 0$, $W(k)$ the ring of Witt vectors of $k$ and $K$ an algebraic closure of the fractional field of $W(k)$ (the characteristic of $K$ is equal to 0). We call a $k$-representation of $A$ an $\ell$-modular representation. In this article, we consider the $\ell$-modular representations, and all representations are assumed to be smooth. We denote by $Rep_k(A)$ the category of smooth $k$-representations of $A$.

The theory of $k$-representations has great similarity with the theory of $K$-representations, but also has important differences. For examples: a $k$-representation of a compact group need not to be semisimple in general; there exist cuspidal representations of $A$ which are not supercuspidal, et cetera. For these reasons, only part of the methods applied in the study of $K$-representations are expected to be generalised to $k$-representations, and the theory of types is one of them. The theory of types consists of two parts: maximal simple $k$-types and $A$-covers. The first part comes from a conjecture which has been formulated for a long time, and the aim is to construct irreducible cuspidal representations (see Definition 1.1) from a compact open subgroup of $A$. The second part is to study the cuspidal support as well as the supercuspidal support of irreducible representations.

Now we focus on the first part. To summarise briefly, a maximal simple $k$-type is a pair $(K, \pi)$ consisting of an open compact subgroup $K$ of $A$ and an irreducible $k$-representation $\pi$ of $K$, which are constructed under technical conditions. For each maximal simple $k$-type, it is associated to a family of pairs $(\mathbb{K}, \Pi)$, where $\mathbb{K}$ is a subgroup of $A$, which is open compact modulo the centre containing $K$, and $\Pi$ is an extension of $\pi$ to $\mathbb{K}$, such that the compact induction $\rho := \text{ind}_K^A \Pi$ is irreducible and cuspidal (not necessarily supercuspidal). We say that $(K, \pi)$ constructs $\rho$, and $(\mathbb{K}, \Pi)$ is an extended maximal simple type of $\rho$. The set of maximal simple $k$-types is said to verify the exhaustion property, if each irreducible cuspidal $k$-representation can be constructed as above, and is said to verify the unicity property, if for each $\rho$ the maximal simple types that constructs $\rho$ is unique up to $A$-conjugation.

The theory of maximal simple types was firstly studied for representations of characteristic zero. It has been established in [BuKu] and [BuKuII] for $GL_n(F)$, and in [BuKul].
to establish category decompositions of \( \text{Rep}_k \) and its Levi subgroups, and in \([KS]\) for \( p \)-adic classical groups. In 2001, a construction of essentially tame cuspidal representations of characteristic zero through Bruhat-Tits buildings for general connected reductive groups is given in \([Yu]\). An equivalence of these two constructions while considering essentially tame cuspidal representations of \( \text{GL}_n(F) \) has been studied in \([Ma]\). For \( \ell \)-modular representations, Vignéras suggested in her book \([Vi]\) the method of reduction modulo \( \ell \), which gives a description of maximal simple \( \ell \)-modular types of \( \text{GL}_n(F) \). In \([MS]\) the maximal simple \( \ell \)-modular types of the inner forms of \( \text{GL}_n(F) \) were constructed. In \([Fin]\) a construction of essentially tame cuspidal \( \ell \)-modular representations of \( p \)-adic connected reductive groups is given. In the article, we establish the theory of maximal simple \( k \)-types of Levi subgroups \( M' \) of \( \text{SL}_n(F) \), which means we prove the exhaustion property as well as the unicity property. It is a generalisation of \([BuKuI]\), \([BuKuII]\), \([GoRo]\) and \([Ta]\) to the \( \ell \)-modular setting. With this relation in mind, in Section 2.4.1 we construct a family of maximal simple \( k \)-types \( (\tilde{J}_M', \tilde{\lambda}_M) \) of \( M' \) from a maximal simple \( k \)-types \( (J_M, \lambda_M) \) of \( M \). In particular, let \( \pi \) and \( \pi' \) be irreducible and cuspidal of \( M' \) and \( M \)

1.2 Main results

Now we state the main results in this article with more details. We first recall the definition of cuspidal \( k \)-representations and supercuspidal \( k \)-representations.

**Definition 1.1.** Let \( A \) be a \( p \)-adic group, and \( L \) a Levi subgroup of \( A \). Let \( \pi \) be an irreducible \( k \)-representation of \( A \), and \( i_L^A \) be the normalised parabolic induction from \( L \) to \( A \).

- We say that \( \pi \) is **cuspidal**, if for any proper Levi subgroup \( L \), any irreducible \( k \)-representation \( \rho \) of \( L \), \( \pi \) does not appear as a sub-representation or a quotient representation of \( i_L^A \rho \).
- We say that \( \pi \) is **supercuspidal**, if for any proper Levi subgroup \( L \) and any irreducible \( k \)-representation \( \rho \) of \( L \), \( \pi \) does not appear as a sub-quotient representation of \( i_L^A \rho \).

From now on, we always denote \( \text{GL}_n(F) \) by \( G \), and \( \text{SL}_n(F) \) by \( G' \). Let \( K \) be a closed subgroup of \( G \), we always denote the intersection \( K \cap G' = K' \). In particular, let \( M \) be a Levi subgroup of \( G \), then \( M' = M \cap G' \) is a Levi subgroup of \( G' \), which gives a bijection between Levi subgroups of \( G \) and \( G' \). A basic idea of this article is by applying the restriction functor \( \text{res}^M_{M'} \). It has been firstly studied in \([Ta]\) for \( K \)-representations of \( M' \). A key observation is that, for any irreducible representation \( \pi' \) of \( M' \), there exists a \( \pi \), irreducible of \( M \), such that \( \pi' \) appears as a direct component of the restriction \( \pi|_{M'} \), which holds true for \( \ell \)-modular setting. With this relation in mind, in Section 2.4.1 we construct a family of maximal simple \( k \)-types \( (\tilde{J}_M', \tilde{\lambda}_M) \) of \( M' \) from a maximal simple \( k \)-types \( (J_M, \lambda_M) \) of \( M \). In particular, let \( \pi \) and \( \pi' \) be irreducible and cuspidal of \( M' \) and \( M \).
respectively, such that \( \pi' \cong \pi|_{M'}. \) If \( \pi \) is constructed from \( (J_M, \lambda_M) \), then \( \pi' \) is constructed from \( (\tilde{J}_M', \tilde{\lambda}_M') \). In Section 2.4.2, we establish the theory of extended maximal simple \( k \)-types \( (N_{M'}(\tilde{\lambda}'_M), \tau_{M'}) \), which is to say that \( N_{M'}(\tilde{\lambda}'_M) \) is open compact modulo the centre and containing \( P_{M'} \), and that \( \tau_{M'} \) is an extension of \( \tilde{\lambda}'_M \).

We summarise Theorem 2.49, Theorem 4.3, and Definition 2.50 as below:

Theorem and Definition 1.2 (Exhaustion property). We have:

- The pairs \( (\tilde{J}_M', \tilde{\lambda}_M') \) are maximal simple \( k \)-types of \( M' \).
- The pairs \( (N_{M'}(\tilde{\lambda}_M'), \tau_{M'}) \) are extended maximal simple \( k \)-types of \( M' \), which means that \( \tau_{M'} \) is an extension of \( \tilde{\lambda}_M' \) and \( \text{ind}_{N_{M'}}^{M'}(\tilde{\lambda}'_M) \) is irreducible and cuspidal.
- The maximal simple \( k \)-types of \( M' \) verify the exhaustion property.

We always abbreviate \( (J_G, \lambda_G) \) and \( (\tilde{J}_G', \tilde{\lambda}_G') \) as \( (J, \lambda) \) and \( (\tilde{J}', \tilde{\lambda}') \) respectively.

Since a \( K \)-representation of a compact subgroup is always semisimple, being intertwined coincides with being weakly intertwined. However, they are not the same for \( \ell \)-modular setting (see Definition 2.7). In this article, we prove the following Theorem of weakly intertwining implying conjugacy, which says that two maximal simple \( k \)-types of \( M' \) are weakly intertwined if and only if they are conjugate and hence are intertwined. This theorem is essentially required in [C] to establish blocks of \( \text{Rep}_k(M') \), and it implies the unicity property of simple \( k \)-characters (resp. maximal simple \( k \)-types) contained in an irreducible cuspidal \( k \)-representation (Theorem 3.20).

Theorem and Definition 1.3 (Unicity property of weakly intertwining implying conjugacy, Theorem 3.23). Let \( (\tilde{J}'_i, \tilde{\lambda}'_i) \) be maximal simple \( k \)-types of \( M' \) for \( i = 1, 2 \).

- We say that \( \tilde{\lambda}'_1 \) is weakly intertwined with \( \tilde{\lambda}'_2 \), if there exists an \( m \in M' \) such that \( \tilde{\lambda}'_1 \) appears as a sub-quotient of \( \text{ind}_{m(\tilde{J}'_1 \cap J_1)}^{\tilde{J}'_1 \cap J_1} m(\tilde{\lambda}'_2) \) where \( m(\cdot) \) is the conjugation by \( m \).
- Suppose that \( (\tilde{J}'_1, \tilde{\lambda}'_1) \) is weakly intertwined with \( (\tilde{J}'_2, \tilde{\lambda}'_2) \), then they are conjugate in \( M' \).

1.3 The structure of this article

In this section, we introduce the structure of this article. We clarify the difference between characteristic zero setting and the \( \ell \)-modular setting, and introduce the specialties of \( M' \) from the technical point of view. The basic references are [BuKuII], [BuKuI], [GoRo] and [Kn].

This article has two parts: In Section 2, we establish maximal simple \( k \)-types of \( G' \) and \( M' \), and prove the exhaustion property; in Section 3 we prove the unicity properties.

For the first part (Section 2), the fact that the derived group of \( M' \) is non-trivial but is trivial of \( G' \) causes important differences in the theory of maximal simple types of \( M' \) and \( G' \). For an example, the extended maximal simple \( k \)-types coincide with maximal simple \( k \)-types for \( G' \), but it is not true for \( M' \). Hence we deal with \( M' \) separately in Section 2.4.

For both \( M' \) and \( G' \), the establishment of maximal simple \( k \)-types has significant differences comparing to that of general linear groups and classical groups. We start from the most crucial speciality. Let \( (J, \lambda) \) be a maximal simple \( k \)-type of \( G = \text{GL}_n(F) \), and \( J^1 \) the
useful properties on weakly intertwining. We end the preparation part by an important decomposition of $\operatorname{res}_G^\ell$ is a generalisation of Appendix in [BuKuII] to $G$, which is a preparation of the rest of this section. To be more precise, Section 2.2.1 consists of useful properties on weakly intertwining. We end the preparation part by an important decomposition of $\operatorname{res}_G^\ell \operatorname{ind}_J^G \lambda$ in Section 2.2.3 where $(J, \lambda)$ is a maximal simple $k$-type of $G$.

We come back to $G'$ from Section 2.2.4, where we introduce the group of projective normaliser $\tilde{J}$, which was firstly defined in [BuKuII] for characteristic zero setting. Section 2.2.5 is a key part, of which the purpose is to show that $\operatorname{ind}^{\tilde{J}}_J \lambda'$ is irreducible. When $\ell = 0$, the irreducibility is proved by showing that the intertwining set of $\tilde{X}$ in $G'$ is equal to $\tilde{J}$. However this condition is not sufficient for $\ell$-modular setting, and a second condition has to be added into the criteria of irreducibility (see Lemma 2.19), which was firstly introduced by Vigneras in [V3].

For $G'$, the maximal simple $k$-types are established in Section 2.2.2 and Section 2.2.3. In the first three sections 2.2.1, 2.2.2, and 2.2.3 we start by studying the maximal simple $k$-types of $G$, which is a preparation of the rest of this section. To be more precise, Section 2.2.1 is a generalisation of Appendix in [BuKuII] to $\ell$-modular setting; Section 2.2.3 consists of useful properties on weakly intertwining. We end the preparation part by an important decomposition of $\operatorname{res}_G^\ell \operatorname{ind}_J^G \lambda$ in Section 2.2.3 where $(J, \lambda)$ is a maximal simple $k$-type of $G$.

We come back to $G'$ from Section 2.2.4, where we introduce the group of projective normaliser $\tilde{J}$, which was firstly defined in [BuKuII] for characteristic zero setting. Section 2.2.5 is a key part, of which the purpose is to show that $\operatorname{ind}^{\tilde{J}}_J \lambda'$ is irreducible. When $\ell = 0$, the irreducibility is proved by showing that the intertwining set of $\tilde{X}$ in $G'$ is equal to $\tilde{J}$. However this condition is not sufficient for $\ell$-modular setting, and a second condition has to be added into the criteria of irreducibility (see Lemma 2.19), which was firstly introduced by Vigneras in [V3]. For $G$ and classical groups, this second condition is verified through Equation (1), by showing that the $\kappa|_{J_1}$-coinvariant of $\operatorname{ind}^{\tilde{J}}_J \lambda$ is semisimple. However, since Equation (1) does not exist in general for $G'$, it requires more technique to verify this condition: In Section 2.2.5 we pass the problem from $G'$ to $G$, and verify the second condition by using the decomposition of $\operatorname{res}_G^\ell \operatorname{ind}_J^G \lambda$ in Section 2.2.3. We complete the establishment of maximal simple $k$-types of $G'$ in Definition 2.4.0, Section 2.2.3.

In Section 2.2.4, we establish the maximal simple $k$-types of $M'$, where some results of $G'$ are applied. We emphasis that extended maximal simple $k$-types only appear when $M' \neq G'$ as explained above. We introduce this new object in Theorem 2.4.0 and Definition 2.5.0. By technical reason, we accomplish the establishment by showing an extended maximal simple $k$-type is an extension of a maximal simple $k$-type at the end of this article (see Theorem 1.3).

Now we arrive at the second part. In Section 3, the main purpose is proving the unicity property of weakly intertwining implying conjugacy (Theorem 3.24). We start by establishing the $\ell$-modular Clifford theory for $\pi|_{M'}$ (Section 3.1 and Section 3.2), of which the statement is different from the characteristic zero setting. Then we deduce a decomposition of $J_M^\ell$ (Theorem 3.19), which is the key of the proof of Theorem 3.24. In Section 3.4, except for the unicity property of weakly intertwining implying conjugacy, we also show the unicity property of the maximal simple $k$-types (resp. the simple $k$-characters) which are contained in a fixed irreducible cuspidal $k$-representation (Theorem 3.20). Finally in Section 3 we accomplish the establishment of extended maximal simple $k$-types, which ends this article.

It is worth noticing that we obtain an equation in Theorem 3.18 which is an intermediate step to prove Theorem 3.24. It is a generalisation of an equation of the length of $\pi|_{M'}$ for irreducible cuspidal $\pi$ in the characteristic zero setting. However, in the $\ell$-modular setting, the equation in Theorem 3.18 describes the $\ell$-prime part of the length of $\pi|_{M'}$. This
equation has a separate interest. In $\ell$-modular setting, it is hard to compute the length of a parabolically induced representation. In Example 3.11 we give a computation of this problem of $\text{SL}_2(F)$, which is an application of this equation and answers an open question at the end of [DaIII]. On the other hand, the length of $\pi|_{M'}$ is equal to the size of $L$-packet containing $\pi'$ when $\ell = 0$, and in [CLH] it is proved to be true in the $\ell$-modular setting when $n = 2$ by applying this equation, which shows potential use in the further study of $L$-packets of $M'$.

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2 Maximal simple $k$-types

2.1 Notation

We start by recalling some basic terminology and notations of theory of types given by Bushnell and Kutzko in [BuKu].

$F$ = a non-archimedean locally compact field, whose residue field is characteristic $p$;

$\mathfrak{o}_F$ = the ring of integers of $F$;

$\mathfrak{p}_F$ = the maximal ideal of $\mathfrak{o}_F$;

$\varpi_F$ = a uniformizer of $\mathfrak{p}_F$;

$k = \text{an algebraically closed field with characteristic } \ell \neq p$;

$W(k) = \text{the ring of Witt vectors of } k$, and $\mathcal{K} = \text{an algebraically closed field of the fractional field of } W(k)$.

ind = the compact induction functor; res = the restriction functor.

$\text{Rep}_k(A) = \text{the category of smooth } k\text{-representations of } A$, where $A$ is a $p$-adic group.

In this article, we denote by $G$ the group $\text{GL}_n(F)$, and $G'$ the group $\text{SL}_n(F)$, unless otherwise specified. Let $K$ be a subgroup of $G$, we always denote by $K'$ the intersection of $K$ and $G'$.

Let $V$ be an $F$-vector space with dimension $n$. Let $A$ be the $F$-algebra $\text{End}_F(V) \cong M_n(F)$. Let $\mathcal{L} = \{ L_i \}_{i \in \mathbb{Z}}$ be an $\mathfrak{o}_F$-lattice chain in $V$ such that for any $i \in \mathbb{Z}$ and any $x \in F$, $xL_i \in \mathcal{L}$. For any integer $m \in \mathbb{Z}$, we set

$$\mathfrak{M}^m = \text{End}_{\mathfrak{o}_F}^m(\mathcal{L}) = \{ x \in A; xL_i \subset L_{i=m}, i \in \mathbb{Z} \}.$$
A hereditary order $\mathfrak{A}$ is an $\sigma$-lattice as well as a subring of $A$ of the form $\text{End}_{\sigma_F}(\mathcal{L})$ for an $\sigma_F$-lattice chain of $V$ as above. We set

$$U(\mathfrak{A}) = \mathfrak{A}^\times,$$

which is a subgroup of $A^\times \cong \text{GL}_n(F)$, and write

$$U^m(\mathfrak{A}) = 1 + \mathfrak{P}^m, m \neq 1,$$

which are pro-$p$ subgroups of $A^\times$. Furthermore, we have

$$\bigcup_{m \in \mathbb{Z}} \mathfrak{P}^m = A,$$

and we set

$$\nu_{\mathfrak{A}}(x) = \max\{m \in \mathbb{Z}; x \in \mathfrak{P}^m\}.$$

for any $x \in A$.

Let $E/F$ be a field extension of $F$ such that there is an inclusion $E \hookrightarrow A$. We fix such an inclusion and view $V$ as an $E$-vector space, and write $B = \text{End}_E(V)$, then we have $B \subset A$. Assume that $E^\times$ normalises $\mathfrak{A}$, we denote by $\mathfrak{B} = B \cap \mathfrak{A}$.

A stratum in $A$ is a 4-tuple $[\mathfrak{A}, n, r, \beta]$ consisting of a hereditary order $\mathfrak{A}$, integers $n > r$, and an element $\beta \in A$ such that $\nu_{\mathfrak{A}}(\beta) \geq -n$ as defined in [BuKu §1.5.1]. A stratum $[\mathfrak{A}, n, r, \beta]$ is simple, if it is satisfied with the conditions in [BuKu Definition 1.5.5], in particular it is required that $E = F[\beta]$ is a field. Given a simple stratum $[\mathfrak{A}, n, 0, \beta]$, the subgroups $H^1(\beta) \subset J^1(\beta) \subset J(\beta)$ of $U(\mathfrak{A})$ are defined in [BuKu §3.1], with the property that the quotient $J(\beta)/J^1(\beta)$ is isomorphic to $\prod \text{GL}_f(k_E)$, where $f, e$ are integers and $k_E$ is the residue field of the field $E = F[\beta]$ inside $A$. We abbreviate them as $H^1, J^1$ and $J$ without causing ambiguity.

A $k$-character of $H^1$ is a **simple character** if it verifies the conditions in [BuKu Definition 3.2.3], and we denote by $C_k(\mathfrak{A}, 0, \beta)$ the set of simple $k$-characters of $H^1$. Let $\theta$ be a simple $k$-character of $H^1$, there is an unique irreducible $k$-representation $\eta(\theta)$ containing $\theta$ of $J^1$ ([BuKu §5.1.1]), we call it the **Heisenberg representation** of $\theta$ and we abbreviate it as $\eta$ without causing ambiguity. Furthermore, there are irreducible $k$-representations of $J$, which are extensions of $\eta(\theta)$ and their intertwining sets contain $B^\times$. We call such an irreducible $k$-representation a $\beta$-extension of $\eta(\theta)$, as defined in [BuKu Definition 5.2.1] (for complex representations) and [MS §2.4] (for modulo $\ell$ representations).

The definition of simple $k$-types and maximal simple $k$-types can be found in [BuKu Definition 5.5.10] in the case of complex representations and in [MS Definition 2.9] in the case of modulo $\ell$ representations. We give a brief summary. A simple $k$-type of $G$ is a pair $(J, \lambda)$, where $J$ is a compact subgroup of $G$ defined from a simple stratum $[\mathfrak{A}, n, 0, \beta]$, and $\lambda$ is of the form $\kappa \otimes \sigma$, where $\kappa$ is a $\beta$-extension of an irreducible $k$-representation $\eta(\theta)$ for a simple $k$-character $\theta$ in $C_k(\mathfrak{A}, 0, \beta)$, and $\sigma$ is the inflation of $\otimes^\beta \pi_0$ from $J/J^1$ to $J$, where $\pi_0$ is an irreducible cuspidal $k$-representation of $\text{GL}_f(k_E)$. A pair $(J, \lambda)$ is a maximal-simple $k$-type of $G$, if it is a simple $k$-type, and $e = 1$.

For a Levi subgroup $M$ of $G$, a maximal simple $k$-type of $M$ is isomorphic to a product of maximal simple $k$-types of general linear groups, and the same for simple $k$-characters.

### 2.2 Construction of cuspidal $k$-representations of $G'$

In this section, we want to prove that for any $k$-irreducible cuspidal representation $\pi'$ of $G'$, there exists an open compact subgroup $\tilde{J}'$ of $G'$ and an irreducible representation $\tilde{\chi}'$ of $\tilde{J}'$. 

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such that \( \pi' \) is isomorphic to \( \text{ind}^{G}_J \lambda' \) \( 2.29 \) and \( 2.30 \).

2.2.1 Types \((J, \lambda \otimes \chi \circ \det)\)

In this section, we assume that \( \ell \neq 0 \). Let \((J, \lambda)\) be a maximal simple \( k \)-type of \( G \) and \( \chi \) be a \( k \)-quasicharacter of \( F^\times \). We prove that the pair \((J, \lambda \otimes \chi \circ \det)\) is also a maximal simple \( k \)-type of \( G \), which will be used in the proof of Proposition \( 2.9 \). This has been proved in appendix of \([BuKuII]\) in the case of characteristic 0 by using the following two lemmas, of which we give a proof for the case of characteristic \( \ell \).

Let \( \mu_{p^\infty} \) be the group of roots of unities of powers of \( p \). We fix an injective morphism from \( \mu_{p^\infty} \) to \( K^\times \), which gives an injective morphism from \( \mu_{p^\infty} \) to \( k^\times \) by applying projection from \( W(k)^\times \) to \( k^\times \). Now we consider the Teichmuller character \( \iota_{p,k} \) from \( k^\times \) to \( K^\times \). Let \( K \) be a pro-\( p \) group and \( \theta_k \) be a smooth \( k \)-character of \( K \), then composing with \( \iota_{p,k} \) gives a mapping from the set of \( k \)-characters and \( K \)-characters of \( K \). Conversely, a \( K \)-character \( \theta_K \) of \( K \) takes value in \( W(k)^\times \), and by applying projection it gives a \( k \)-character of \( K \), which we call the reduction modulo \( \ell \) of \( \theta_K \). We have:

**Lemma 2.1.** Let \( K \) be a pro-\( p \) group, the composition with \( \iota_{p,k} \) gives a bijection between the set of smooth \( k \)-characters and the set of smooth \( K \)-characters of \( K \), of which the inverse is reduction modulo \( \ell \). Let \( \theta_k \) be a smooth \( k \)-character of \( K \), we call \( \theta_K = \iota_{p,k} \circ \theta_k \) the lifting of \( \theta_k \).

Let \([\mathfrak{A}, n, 0, \beta]\) be a simple stratum, as defined in \([MS] \) \( \S 2.2 \), the set of simple characters \( C_k(\mathfrak{A}, 0, \beta) \) consists of the reduction modulo \( \ell \) of simple \( K \)-characters of \( H^1(\beta) \).

From now on we fix a continuous additive character \( \psi_F \) from \( F \) to \( K^\times \), which is trivial on \( pF \) but non-trivial on \( pF \). Recall the equivalence (depending on the choice of \( \psi_F \))

\[
(U^{[\mathfrak{A}]\beta + 1}((\mathfrak{A})/U^{n+1}(\mathfrak{A})))^\wedge \cong \mathfrak{P}^{-n}/\mathfrak{P}^{-([\mathfrak{A}]\beta + 1)},
\]

where \((U^{[\mathfrak{A}]\beta + 1}((\mathfrak{A})/U^{n+1}(\mathfrak{A})))^\wedge\) denote the Pontrjagin dual. Let \( \beta \in \mathfrak{P}^{-n}/\mathfrak{P}^{-([\mathfrak{A}]\beta + 1)} \), we denote by \( \psi_{\beta,K} \) the character on \( U^{[\mathfrak{A}]\beta + 1}(\mathfrak{A})/U^{n+1}(\mathfrak{A}) \) given through the equivalence above (or consult \([BuKu]\) \( \S 1.6.6 \) for an explicit definition). Let \( \psi_{\beta,K} \) denote the reduction modulo \( \ell \) of \( \psi_{\beta,K} \).

**Lemma 2.2.** Let \([\mathfrak{A}, n, 0, \beta]\) be a simple stratum in \( A \) with \( \beta \notin F \), \( n \geq 1 \). Let \( c \in F^\times \), and \( n_0 = -v_{\mathfrak{A}}(c) \), \( n_1 = -v_{\mathfrak{A}}(\beta + c) \),

1. The stratum \([\mathfrak{A}, n_1, 0, \beta + c]\) is a simple stratum in \( A \), and we have \( \eta(\beta + c, \mathfrak{A}) = \eta(\beta, \mathfrak{A}) \) and \( \overline{\eta}(\beta + c, \mathfrak{A}) = \overline{\eta}(\beta, \mathfrak{A}) \).

2. Let \( \chi_K \) be a \( K \)-quasicharacter of \( F^\times \) such that \( \chi_K \circ \det \) is equivalent to \( \psi_{\beta,K} \) on \( U^{[\mathfrak{A}]\beta + 1}(\mathfrak{A}) \). Then we have an equivalence of simple \( K \)-characters:

\[
C_K(\mathfrak{A}, 0, \beta + c) = C_K(\mathfrak{A}, 0, \beta) \otimes \chi_K \circ \det.
\]

3. Let \( \chi_K \) be a \( k \)-quasicharacter of \( F^\times \) such that \( \chi_K \circ \det \) is equivalent to \( \psi_{\beta,K} \) on \( U^{[\mathfrak{A}]\beta + 1}(\mathfrak{A}) \). Then we have an equivalence of simple \( k \)-characters:

\[
C_K(\mathfrak{A}, 0, \beta + c) = C_K(\mathfrak{A}, 0, \beta) \otimes \chi_K \circ \det.
\]
Proof. The first two assertions are proved in Lemma of [BuKuII, appendix], so we only need to prove the last assertion. Let $\chi_{0, K}$ be the lifting of $\chi_k|_{\mathfrak{U}}$ to $K$, then $\chi_{0, K}$ can be extended to $F^\times$ and we denote also by $\chi_{0, K}$ an extension. By definition, $\chi_{0, K} \circ \det$ is equivalent to $\psi_{c, K}$. By Part 2, we have:

$$C_k(\mathfrak{A}, 0, \beta + c) = C_k(\mathfrak{A}, 0, \beta) \otimes \chi_{0, K} \circ \det.$$ 

Applying reduction modulo $\ell$ to both sides, by Lemma 2.1 we obtain

$$C_k(\mathfrak{A}, 0, \beta + c) = C_k(\mathfrak{A}, 0, \beta) \otimes \chi_k \circ \det.$$ 

\[\square\]

Corollary 2.3. Let $(J, \lambda)$ be a maximal simple $k$-type of $G$, and $\chi$ a $k$-quasicharacter of $F^\times$. Then the pair $(J, \lambda \otimes \chi \circ \det)$ is also a maximal simple $k$-type of $G$.

Proof. Let $[\mathfrak{A}, n, 0, \beta]$ be a simple stratum in the construction of $(J, \lambda)$. We have $\lambda \cong \kappa \otimes \sigma$ (see Section 2.1) where $\kappa$ is a $\beta$-extension of $\eta(\theta)$ of a simple $k$-character $\theta$ in $C_k(\mathfrak{A}, 0, \beta)$, hence $\lambda \otimes \chi \circ \det \cong (\kappa \otimes \chi \circ \det) \otimes \sigma$. From [BuKuII, appendix] and Part 1, 3 of Lemma 2.2 we know that there exists $c \in F^\times$ such that $\theta \otimes \chi \circ \det$ is a simple $k$-character of $C_k(\mathfrak{A}, 0, \beta + c)$. Hence $\eta(\theta) \otimes \chi \circ \det$ is the unique irreducible $k$-representation of $J^1$ containing $\theta \otimes \chi \circ \det$ after restricting to $H^1$. To end this proof, it is sufficient to prove that $\kappa \otimes \chi \circ \det$ is a $\beta$-extension of $\eta(\theta \otimes \chi \circ \det)$, which follows from the fact that the intertwining set of $\kappa \otimes \chi \circ \det$ is equal to that of $\kappa$. \[\square\]

Remark 2.4. Let $(J_M, \lambda_M)$ be a maximal simple $k$-type of $M$, where $M$ is a Levi subgroup of $G$. Then $\lambda_M \cong \lambda_1 \otimes \cdots \otimes \lambda_r$ for some $r \in \mathbb{N}_+$, where $(J_1, \lambda_1)$ are maximal simple $k$-types of $\text{GL}_{n_i}(F)$. Hence for any $k$-quasicharacter $\chi$ of $F^\times$, the tensor product $(J_M, \lambda_M \otimes \chi \circ \det)$ is also a maximal simple $k$-type of $M$.

2.2.2 Intertwining and weakly intertwining

Recall that $M$ is a Levi subgroup of $G$, and $M' = M \cap G'$ is a Levi subgroup of $G'$. Let $K$ be a subgroup of $M$, we always denote by $K'$ the intersection $K \cap M'$.

Proposition 2.5. Let $K$ be a compact subgroup of $M$, and $\rho$ an irreducible $k$-representation of $K$. Then the restriction $\text{res}^K_{K'} \rho$ is semisimple.

Proof. Let $O$ be the kernel of $\rho$, which is a normal subgroup of $K$. The subgroup $OK'$ is also a compact open normal subgroup of $K$, hence with finite index in $K$. We deduce that the restriction $\text{res}^K_{K'} \rho$ is semisimple by Clifford theory, and that the restriction $\text{res}^K_{K'} \rho$ is semisimple. \[\square\]

Proposition 2.6. Let $K$ be a compact open subgroup of $M$, $\rho$ an irreducible smooth representation of $K$, and $\rho'$ an irreducible component of the restriction $\text{res}^K_{K'} \rho$. Let $\overline{\rho}$ be an irreducible representation of $K$ such that $\text{res}^K_{K'} \rho$ contains $\rho'$ as well. Then there exists a $k$-quasicharacter $\chi$ of $F^\times$ such that $\rho \cong \chi \circ \det$.

Proof. Let $U$ be a pro-$p$ normal subgroup of $K$ contained in the kernel of $\rho$, which has finite index in $K$. The induction $\text{Ind}^M_U(1)$ is semisimple. Hence by Schur lemma it is a direct sum of characters of the form $(\chi \circ \det)|_U$, where $\chi$ is a $k$-quasicharacter of $F^\times$. The fact that $\text{res}^U_{K'} \rho$ contains the trivial character induces the same property for $\text{res}^U_{K'} \overline{\rho}$. By
Frobenius reciprocity, we know that \( \text{res}^K_U \mathbf{p} \) contains a character of the form \((\chi \circ \det)_U\), and the irreducibility implies that it is in fact a multiple of this character. We can hence assume that \( \mathbf{p} \) is trivial on \( U \).

By Clifford theory, the restriction of \( \rho \) (resp. \( \mathbf{p} \)) to \( K'U \) is semisimple. Hence the set \( \text{Hom}_{K'U}(\rho, \mathbf{p}) \) is non-trivial. Applying Frobenius reciprocity, we see that \( \rho \) is a subrepresentation of \( \text{ind}^K_{K'U} \text{res}^K_U \mathbf{p} \). The latter is equivalent to \( \mathbf{p} \otimes \text{ind}^K_{K'U}(1) \) by [VI] §1, 5.2 d), of which the Jordan-Hölder factors are \( \mathbf{p} \otimes \chi \circ \det \). We finish the proof.

\[ \square \]

**Definition 2.7.** Let \( A \) be a locally profinite group, and \( K_i \) be an open compact subgroup of \( A \) for \( i = 1, 2 \). Let \( \rho_i \) be an \( A \)-representation of \( K_i \), and \( x \) be an element in \( G \). Define \( i_{K_1,K_2}x(\rho_2) \) to be the induced \( A \)-representation

\[
\text{ind}_{K_1,\cap\chi(K_2)}^{K_1} \text{res}_{K_1,\cap\chi(K_2)}^x(\rho_2),
\]

where \( x(\rho_2) \) is the conjugate of \( \rho_2 \) by \( x \).

- We say that an element \( x \in A \) **weakly intertwines** \( \rho_1 \) with \( \rho_2 \), if \( \rho_1 \) is an irreducible subquotient of \( i_{K_1,K_2}x(\rho_2) \), and that \( \rho_1 \) is **weakly intertwined with** \( \rho_2 \) in \( A \), if \( \rho_1 \) is isomorphic to a subquotient of \( \text{res}^{A}_K \text{ind}^{A}_K \rho_2 \). We denote by \( I_{A}(\rho_1, \rho_2) \) the set of elements in \( A \), which weakly intertwines \( \rho_1 \) with \( \rho_2 \).

- We say that an element \( x \in A \) **intertwines** \( \rho_1 \) with \( \rho_2 \), if

\[
H_x(\rho_1, \rho_2) := \text{Hom}_{K_1}(\rho_1, i_{K_1,K_2}x(\rho_2)) \neq 0.
\]

The representation \( \rho_1 \) is **intertwined with** \( \rho_2 \) in \( A \), if

\[
H_A(\rho_1, \rho_2) := \text{Hom}_{A}(\text{ind}^{A}_{K_1} \rho_1, \text{ind}^{A}_{K_2} \rho_2) \neq 0.
\]

We denote by \( I_{A}(\rho_1, \rho_2) \) the set of elements in \( A \), which intertwine \( \rho_1 \) with \( \rho_2 \).

When \( K_1 = K_2 \), we use the abbreviation \( i_{K_1}x(\rho_1) \). When \( \rho_1 = \rho_2 \), we use the abbreviation \( I_{A}(\rho_1) \), \( H_{A}(\rho_1) \), and \( I_{A}(\rho_1) \) accordingly.

When \( \rho_1 \) is irreducible, we deduce directly from Mackey’s theory that \( \rho_1 \) is (weakly) intertwined with \( \rho_2 \) in \( A \) if and only if there exists an element \( x \in A \), such that \( x \) (weakly) intertwines \( \rho_1 \) with \( \rho_2 \).

**Proposition 2.8.** For \( i = 1, 2 \), let \( K_i \) be a compact open subgroup of \( M_i \), and \( \rho_i \) an irreducible representation of \( K_i \), and \( \rho_i' \) an irreducible component of \( \text{res}^{K_i}_{K_i'} \rho_i \). Let \( x \in M \) that weakly intertwines \( \rho_i' \) with \( \rho_2 \). Then there exists a \( k \)-quasicharacter \( \chi \) of \( F^\times \) such that \( x \) weakly intertwines \( \rho_i \) with \( \rho_2 \otimes \chi \circ \det \).

**Proof.** By Mackey’s theory, \( i_{K_1',K_2'}x(\rho_2') \) is a subrepresentation of \( i_{K_1,K_2}x(\rho_2) \). Since \( i_{K_1,K_2}x(\rho_2) \) has finite length, the uniqueness of Jordan-Hölder factors implies that there exists an irreducible subquotient \( \rho \) of \( i_{K_1,K_2}x(\rho_2) \), whose restriction to \( K_1' \) contains \( \rho_i' \) as a direct factors. By Proposition 2.6 \( \rho \) is isomorphic to \( \rho_i \otimes \chi^{-1} \circ \det \), which implies that \( \rho_1 \) is weakly intertwined with \( \rho_2 \otimes \chi \circ \det \) by \( x \).

Now we consider the maximal simple \( k \)-types of \( G = \text{GL}_n(F) \).
Let $(J, \lambda)$ be a maximal simple $k$-type of $G$, and $\chi$ a $k$-quasicharacter of $F^\times$. If $(J, \lambda \otimes \chi \circ \det)$ is weakly intertwined with $(J, \lambda)$, then they are intertwined, and conjugate under $U(\mathfrak{A})$, which is to say that there exists an element $x \in U(\mathfrak{A})$ such that $x(J) = J$ and $x(\lambda) \cong \lambda \otimes \chi \circ \det$.

Proof. Let $\theta_1$ and $\theta_2$ be the simple character contained in $\lambda$ and $\lambda \otimes \chi \circ \det$ respectively, and let $\eta_i$ be the Heisenberg representation of $\theta_i$ for $i = 1, 2$. There is a surjection from $\text{res}_{J, \lambda}^i \text{ind}_{K}^G \lambda$ onto $\theta_1$ (see Section 2.1 for $K$). By Frobenius reciprocity, there is an injection from $\lambda$ to $\text{ind}_{K}^i \text{res}_{J, \lambda}^1 \theta_1$, and hence an injection: $\text{res}_{J, \lambda}^i \text{ind}_{K}^G \lambda \hookrightarrow \text{res}_{J, \lambda}^i \text{ind}_{K}^G \theta_1$. By the assumption, $\text{res}_{J, \lambda}^i \lambda \otimes \chi \circ \det$ is a subquotient of $\text{res}_{J, \lambda}^i \text{ind}_{K}^G \theta_1$. From Corollary 2.3 we have $\text{H}^1(\beta + c) = \text{H}^1(\beta)$. Hence $\text{res}_{J, \lambda}^i (\lambda \otimes \chi \circ \det)$ is a multiple of $\theta_2$, from which we deduce that $\theta_2$ is a subquotient of $\text{res}_{J, \lambda}^i \text{ind}_{K}^G \theta_1$.

Notice that $\text{H}^1$ is a prop-$p$ group, and any smooth representation of $\text{H}^1$ is semisimple. It follows that $\theta_2$ is a sub-representation of $\text{res}_{J, \lambda}^i \text{ind}_{K}^G \theta_1$, which is equivalent to say that $\theta_2$ is intertwined with $\theta_1$ in $G$. The same property holds for the $\mathcal{K}$-lifting of $\theta_i$ for $i = 1, 2$, which implies that the nonsplit fundamental strata $[\mathfrak{A}, n, n - 1, \beta]$ and the nonsplit fundamental stratum $[\mathfrak{A}, m, m - 1, \beta + c]$ are intertwined. We deduce that $n = m$ from BuKu 2.3.4, 2.6.3. Then we apply Theorem 3.5.11 of BuKu: there exists $x \in U(\mathfrak{A})$ such that $x(\text{H}^1) = \text{H}^1$, $C(\mathfrak{A}, 0, \beta) = C(\mathfrak{A}, 0, x(\beta + c))$ and $x(\theta_2) = x(\theta_1) = 1$. In particular, $x(J)$ is a subset of $\mathcal{I}_{U(\mathfrak{A})}(\theta_1)$. Meanwhile, BuKu 2.3.3 and BuKu 3.1.15 implies that $\mathcal{I}_{U(\mathfrak{A})}(\theta_1) \cap U(\mathfrak{A}) = J$, then $x(J) = J$. Proposition 2.2 of MS shows the uniqueness of $\eta_1$, hence $x(\eta_2) \cong \eta_1$. From [V3, Corollary 8.4] we know that the $\eta_1$-isotypic part of $\text{res}_{J, \lambda}^i \text{ind}_{K}^G \lambda$ can be viewed as a representation of $J$, which is a direct factor of $\text{res}_{J, \lambda}^i \text{ind}_{K}^G \lambda$ and is a multiple of $\lambda$. Since $x(\lambda \otimes \chi \circ \det)$ is a subquotient of the $\eta_1$-isotypic part of $\text{res}_{J, \lambda}^i \text{ind}_{K}^G (\lambda)$, we conclude that $x(\lambda \otimes \chi \circ \det) \cong \lambda$ and $\text{Hom}_{U(J)}(\lambda \otimes \chi \circ \det, \text{res}_{J, \lambda}^i \text{ind}_{K}^G (\lambda)) \neq 0$.

Corollary 2.10. Let $g \in G$, if $g$ weakly intertwines $(J, \lambda \otimes \chi \circ \det)$ with $(J, \lambda)$, then $g$ intertwines $(J, \lambda \otimes \chi \circ \det)$ with $(J, \lambda)$.

Proof. Recall that $\eta_1$ and $\eta_2$ is contained in $\lambda$ and $\lambda \otimes \chi \circ \det$ respectively. By Proposition 2.8 there exists $x \in U(\mathfrak{A})$ such that $x(\eta_1) \cong \eta_2$, $x(\lambda) \cong \lambda \otimes \chi \circ \det$. By the assumption $\lambda \otimes \chi \circ \det$ is a subquotient of $i_{J, g(J)}^g (\lambda)$, hence a subquotient of $i_{J, g(J)}^g (\lambda)^{x(\eta_1)}$. The latter is a sub-representation of $(\text{res}_{J, \lambda}^i \text{ind}_{K}^G (\lambda))^{x(\eta_1)}$, hence a multiple of $x(\lambda)$. Then $\lambda \otimes \chi \circ \det^{x(\eta_1)}$ is a sub-representation of $(i_{J, g(J)}^g (\lambda))^{x(\eta_1)}$, and we finish the proof.

2.2.3 Decomposition of $\text{res}_{J, \lambda}^G \text{ind}_{K}^G \lambda$

In this section, the main purpose is to obtain the decomposition in Theorem 2.11 which plays a key role in the proof of Proposition 2.29. The latter consists half of the proof of Theorem 2.29.

Theorem 2.11. Let $(J, \lambda)$ be a maximal simple $k$-type of $G$. There exists an integer $m$ such that

$$\text{res}_{J, \lambda}^G \text{ind}_{K}^G \lambda \cong (\oplus_{i=1}^m x_i(\Lambda(\lambda))) \oplus W,$$

where $x_i \in U(\mathfrak{A})$, and put $x_1 = 1$. The representation $\Lambda(\lambda)$ is a multiple of $\lambda$. For each $x_i$, the representation $x_i(\Lambda(\lambda))$ is the $x_i$-conjugation of $\Lambda(\lambda)$. The elements $x_i$’s verify that $x_i(\eta) \neq x_j(\eta)$ for $i \neq j$ (see Section 2.1 for the definition of $\eta$). Furthermore, let $X$ be an irreducible sub-representation of $\text{res}_{J, \lambda}^G \lambda$, then $X$ is not equivalent to any irreducible subquotient of $\text{res}_{J, \lambda}^G W$. 

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Remark 2.12. From now on, we always denote $\oplus_{i=1}^{m} x_i(\Lambda(\lambda))$ by $\Lambda_\lambda$, and we write the decomposition in Theorem 2.11 above as:

$$\text{res}^{G}_{I}i^{J}_{I}\lambda \cong \Lambda_\lambda \oplus W.$$ 

Before prove Theorem 2.11 we first introduce the following very useful lemmas:

Lemma 2.13. Let $K_1, K_2$ be two compact open subgroups of $G$ such that $K_1 \subset K_2$. Then the compact induction $i^{K_2}_{K_1}$ respects infinite direct sum.

Proof. This can be checked directly from the definition of the compact induction functor. □

Lemma 2.14. Let $K$ be a compact open subgroup of $M$, and $K' = K \cap G'$. Let $\pi$ be a $K'$-representation of $K$. If $\tau'$ is an irreducible subquotient of the restriction $i^{K}_{K'}\pi$, then there exists an irreducible subquotient $\tau$ of $\pi$, such that $\tau'$ is an irreducible direct factor of $i^{K}_{K'}\tau$.

Proof. Let $H$ be a pro-$p$ open compact subgroup of $K$. The representation $i^{K}_{H}\pi$ is semisimple, which can be written as $\oplus_{i \in I}^{\pi_i}$, where $I$ is an index set. There is an injection from $\pi$ to $\text{ind}^{H}_{K}i^{K}_{H}\pi$. Lemma 2.13 implies that $i^{K}_{H}\pi \cong \oplus_{i \in I}^{\pi_i}i^{H}_{\pi_i}$.

Now we are ready to prove Theorem 2.11.

Proof. of Theorem 2.11. By [V3 Corollary 8.4], we can decompose $i^{G}_{J}i^{J}_{I}\lambda \cong \Lambda(\lambda) \oplus W_1$, where an irreducible subquotient of $W_1$ is not isomorphic to $\lambda$. Let $\lambda'$ be an irreducible direct component of $\text{res}^{G}_{J}i^{J}_{I}\lambda$. If $\lambda'$ is an irreducible subquotient of $i^{G}_{J}W_1$, by Lemma 2.14 and Proposition 2.6 there exists a $k$-quasicharacter $\chi$ of $P^\chi$ such that $\lambda \otimes \chi \circ \text{det}$ is an irreducible subquotient of $W_1$, which implies that $\lambda \otimes \chi \circ \text{det}$ and $\lambda$ are weakly intertwined in $G$. By Proposition 2.8, $\lambda$ is conjugate to $\lambda \otimes \chi \circ \text{det}$ by an element $x \in U(\mathfrak{A})$. The fact that $\lambda \otimes \chi \circ \text{det}$ is a subquotient of $W_1$ implies that $x(\eta) \not\cong \eta$. Thus we can decompose $W_1$ as $W_1^{x(\eta)} \oplus W_2$, and we have:

$$W_1^{x(\eta)} \cong x((\text{res}^{G}_{J}i^{J}_{I}\lambda)^{\eta}).$$

The latter is isomorphic to $x(\Lambda(\lambda))$, which is a direct sum of $x(\lambda)$. Now we obtain an isomorphism:

$$\text{res}^{G}_{J}i^{J}_{I}\lambda \cong \Lambda(\lambda) \oplus x(\Lambda(\lambda)) \oplus W_2,$$

where $W_2^{-x(\eta)} = 0$ and $W_2^{\eta} = 0$. It implies an irreducible subquotient of $W_2$ is not isomorphic to $\lambda$ neither to $x(\lambda)$. Now we can repeat the above steps to $W_2$. Notice that, any irreducible representation of $J$, whose restriction to $J'$ contains $\lambda'$ as a subrepresentation, is $U(\mathfrak{A})$-conjugate to $\lambda$. The quotient $U(\mathfrak{A})/J$ is finite, hence the set of irreducible representations $\{x(\lambda)\}_{x \in U(\mathfrak{A})}$ is finite, which means after repeating finite times, we obtain the required decomposition. □
2.2.4 Projective normalizer $\tilde{J}$ and its subgroups

In this section, we apply the same the definition of projective normalizer given in [BuKuII] for $K$-representations of $G'$, and we show that some properties in [BuKuII] still hold true in $\ell$-modular setting.

**Definition 2.15** (Bushnell,Kutzko). Let $\mathfrak{A}$ be the principal order attached to a maximal simple $k$-type $(J,\lambda)$. Then the projective normalizer $\tilde{J} = \tilde{J}(\lambda)$ of $(J,\lambda)$ is defined to be the group of all $x \in U(\mathfrak{A})$ such that:

- $xJx^{-1} = J$, and
- there exists a $k$-quasicharacter $\chi$ of $F^*$ such that $x(\lambda) \cong \lambda \otimes \chi \circ \det$.

**Proposition 2.16.** Let $(J,\lambda)$ be a maximal simple $k$-type of $G$, and $\chi$ a $k$-quasicharacter of $F^*$. The following are equivalent:

1. $\lambda \cong \lambda \otimes \chi \circ \det$,
2. $\chi \circ \det|_{J^1}$ is trivial and $\sigma \otimes \chi \circ \det|_{U(\mathfrak{A})} \cong \sigma$,
3. $\chi \circ \det|_{J^1}$ is trivial, and $\lambda, \lambda \otimes \chi \circ \det$ are intertwined in $G$.

**Proof.** The proof follows the same strategy as [BuKuII, Proposition 2.3]. We prove this proposition in the order of $2 \to 1 \to 3 \to 2$.

The implication $2 \to 1$ is immediate by $J/J' \cong U(\mathfrak{B}/U(\mathfrak{B}))$. For $1 \to 3$, we assume that $\lambda$ is equivalent to $\lambda \otimes \chi \circ \det$. Restricting to $H^1$, we have the equivalent of the simple characters $\theta \cong \theta \otimes \chi \circ \det$, which implies $\chi \circ \det|_{H^1}$ is trivial. For $3 \to 2$, we assume part 3 holds. Proposition 2.15 gives an element $x \in U(\mathfrak{A})$ such that $x(J) = J$ and $x(\lambda) = \lambda \otimes \chi \circ \det$. Hence the uniqueness of $\eta$ (see Proposition 2.2 of [MS]) implies that $x(\eta) \cong \eta$. In particular, $x \in \mathcal{S}_G(\theta) = J^1 B^* J^1$ by [V2, IV.1.1], hence $x \in J^1 B^* J^1 \cap U(\mathfrak{A}) = J$, then $\lambda \cong x(\lambda) \cong \lambda \otimes \chi \circ \det$. In other words, $\kappa \otimes \sigma \cong \kappa \otimes \sigma \otimes \chi \circ \det$. By Schur Lemma, we can apply the proof in [BuKuII, Proposition 5.3.2]:

Let $X$ be the representation space of $\kappa$ and $Y$ the representation space of $\sigma$, which can be identified with that of $\sigma \otimes \chi \circ \det$. Let $\phi$ be the isomorphism between $\kappa \otimes \sigma$ and $\kappa \otimes \sigma \otimes \chi \circ \det$. We may write $\phi = \sum_j S_j \otimes T_j$ where $S_j \in \text{End}_k(X)$ and $T_j \in \text{End}_k(Y)$, and where $\{T_j\}$ are linearly independent. Let $g \in J^1$, we have $\kappa \otimes \sigma(g) \circ \phi = \phi \circ (\kappa \otimes \sigma \otimes \chi \circ \det)(g)$. Since $J^1 \subset \ker(\sigma) = \ker(\sigma \otimes \chi \circ \det)$, this relation reads:

$$(\eta(g) \otimes 1) \circ \sum_j S_j \otimes T_j = \left(\sum_j S_j \otimes T_j\right) \circ (\eta(g) \otimes 1),$$

which is equivalent to say that:

$$\sum_j (\eta(g) \circ S_j - S_j \circ \eta(g)) \otimes T_j = 0.$$

The linear independence of $T_j$ implies that $S_j \in \text{End}_{k,J^1}(\eta) = k^*$, by the Schur lemma. Hence $\phi = 1 \otimes \sum_j S_j \cdot T_j$. Now note $T = \sum_j S_j \cdot T_j$ and take $g \in J$, the morphism relation reads:

$$(\kappa(g) \otimes \sigma(g)) \circ (1 \otimes T) = \kappa(g) \otimes (\sigma(g) \circ T) = \kappa(g) \otimes (T \circ \sigma \otimes \chi \circ \det(g))$$

$$= (1 \otimes T) \circ (\kappa(g) \otimes \sigma \otimes \chi \circ \det(g)),$$

which says $T \in \text{Hom}_{k,J}(\sigma, \sigma \otimes \chi \circ \det) \neq 0$. We finish the proof. 

\[\square\]
Corollary 2.17. Let \( x \in \tilde{J}(\lambda) \), and \( \chi \) a \( k \)-quasicharacter of \( F^* \) such that \( x(\lambda) \cong \lambda \otimes \chi \circ \det \). Then:

1. the map \( x \mapsto \chi \circ \det|_\Pi \) is an injective homomorphism from \( \tilde{J}/J \to (\det(J^1))^\wedge \). The latter is the \( k \)-dual group of \( \det(J^1) \);

2. \( \tilde{J}/J \) is a finite abelian \( p \)-group, where \( p \) is the residual characteristic of \( F \).

3. The exponent of \( \tilde{J}/J \) divides the rank \( n \) of \( G' \), where \( G' = \text{SL}_n(F) \).

Proof. For part 1, let \( x \in \tilde{J} \). Suppose that there exist two \( k \)-quasicharacters \( \chi_1, \chi_2 \) of \( F^* \), such that \( x(\lambda) \cong \lambda \otimes \chi_1 \circ \det \) and \( \lambda \otimes \chi_1 \circ \det \cong \lambda \otimes \chi_2 \circ \det \). This is equivalent to say that

\[
\lambda \cong \lambda \otimes (\chi_1 \otimes \chi_2^{-1}) \circ \det.
\]

Proposition 2.16 implies that \( \chi_1 \circ \det|_\Pi \cong \chi_2 \circ \det|_\Pi \), which implies that the group morphism is well defined. Now suppose that \( x \in \tilde{J} \) and \( \chi \) is trivial on \( \det(J^1) \), such that \( x(\lambda) \cong \lambda \otimes \chi \circ \det \). Then by Proposition 2.16 \( \lambda \cong \lambda \otimes \chi \circ \det \), and \( x \) intertwines \( \lambda \) to itself. Hence \( x \) belongs to \( JB^\infty J \cup U(\mathfrak{t}) = J \).

Since \( J^1 \) is a pro-\( p \)-group, part 2 is induced from part 1.

For part 3, let \( Z' \) be the centre of \( G' \). Since \( U(\mathfrak{a}_F)^n \subset \det(J^1 \cap Z') \), hence \( (\chi|_\Pi)^n \) is trivial, which implies that \( x^n \in J \) for \( x \in \tilde{J} \).

Remark 2.18. Part 2 and Part 3 of Corollary 2.17 imply that \( \tilde{J} = J \), when \( p \) does not divide \( n \).

2.2.5 Two conditions for irreducibility

In this section, let \((J, \lambda)\) be a maximal simple \( k \)-type of \( G \). We construct a compact subgroup \( M_\lambda \) of \( G' \) and a family of irreducible representations \( \lambda_{M_\lambda} \) of \( M_\lambda \), such that the induction \( \text{ind}^{G'}_{M_\lambda} \lambda_{M_\lambda} \) is irreducible and cuspidal (Theorem 2.29). In the next section, we will see that every irreducible cuspidal \( k \)-representation of \( G' \) can be constructed in this manner.

When \( \ell = 0 \), to show \( \text{ind}^{G'}_{M_\lambda} \lambda_{M_\lambda} \) is irreducible, it is sufficient to show that \( I_{G'}(\lambda_{M_\lambda}) = M_\lambda \). However, this intertwining condition is not sufficient in the \( \ell \)-modular setting, and a criteria of irreducibility has been established in [V3 Lemma 4.2]. For the reason of convenience, we present it here:

Lemma 2.19 (Criteria of irreducibility). Let \( K' \) be an open compact subgroup of \( G' \), and \( \pi' \) be an irreducible \( k \)-representation of \( K' \). The induction \( \text{ind}^{G'}_{K'} \pi' \) is irreducible, when

1. \( \text{End}_{K'}(\text{ind}^{G'}_{K'} \pi') = k \), and

2. for any irreducible \( k \)-representation \( \nu \) of \( G' \), if \( \pi' \) is contained in \( \text{res}^{G'}_{K'} \nu \) then there is a surjection from \( \text{res}^{G'}_{K'} \nu \) to \( \pi' \).

As shown in [V1 §1.8.3], the first condition is equivalent to \( I_{G'}(\pi') = K' \), where \( I_{G'}(\pi') \) is the intertwining set.

Corollary 2.20. Let \((J, \lambda)\) be a maximal simple \( k \)-type of \( G \). The induction \( \text{ind}^{J}_J \lambda \) is irreducible.
Proof. Lemma 2.19 can be applied by changing $G'$ to a locally pro-finite group. First, we compute $\text{End}_{\Lambda}(\text{ind}_{J}^{G'}\lambda)$, which is equal to $k$ since the intertwining group $I_{J}(\lambda) = J$. For the second condition, let $\nu$ be an irreducible $k$-representation of $\tilde{J}$, such that

$$\lambda \mapsto \text{res}_{J}^{\tilde{J}}\nu.$$ 

By [VI §1, 6.12] the restriction $\text{res}_{J}^{\tilde{J}}\nu$ is semisimple, hence $\lambda$ is a direct component.

Theorem 2.21. Let $\lambda'$ be a subrepresentation of $\text{res}_{J}^{G'}\lambda$. Then $\lambda'$ verifies the second condition of irreducibility: For any irreducible $k$-representation $\pi'$ of $G'$, if there is an injection: $\lambda' \hookrightarrow \text{res}_{J}^{G'}\pi'$, then there exists a surjection: $\text{res}_{J}^{G'}\pi' \twoheadrightarrow \lambda'$.

Proof. By Mackey’s theory, we have

$$\text{res}_{J}^{G'}\text{ind}_{J}^{G}\lambda \cong \bigoplus_{a \in J/G'}\text{ind}_{G'/\alpha(J)}^{G'}\text{res}_{\alpha(J)}^{a}(\lambda),$$

of which $\text{ind}_{J}^{G'}\text{res}_{J}^{\tilde{J}}\lambda$ is a direct factor. The assumption $\lambda' \hookrightarrow \text{res}_{J}^{G'}\pi'$ implies a surjection from $\text{ind}_{J}^{G'}\lambda'$ to $\pi'$ by Frobenius reciprocity. Since $\text{res}_{J}^{G'}\lambda$ is semisimple, we have a surjection

$$\text{ind}_{J}^{G'}\text{res}_{J}^{\tilde{J}}\lambda \twoheadrightarrow \pi'.$$

Hence we obtain

$$\text{res}_{J}^{G'}\text{ind}_{J}^{G}\lambda \twoheadrightarrow \pi'.$$

Now consider the surjection

$$\iota : \text{res}_{J}^{G'}\text{ind}_{J}^{G}\lambda \twoheadrightarrow \text{res}_{J}^{G'}\pi'.$$

By Theorem 2.11, we decompose $\text{res}_{J}^{G'}\text{ind}_{J}^{G}\lambda \cong \Lambda_{\lambda} \oplus W$. We have $\Lambda_{\lambda} \oplus W/\ker(\iota) \cong \text{res}_{J}^{G'}\pi'$. By the definition of $W$, the image of the composed morphism:

$$\lambda' \hookrightarrow \Lambda_{\lambda} \oplus W/\ker(\iota) \twoheadrightarrow \Lambda_{\lambda} \oplus W/(W + \ker(\iota)) \cong \Lambda_{\lambda}/(\Lambda_{\lambda} \cap (W + \ker(\iota)))$$

is non-trivial. Since $\text{res}_{J}^{G'}\Lambda_{\lambda}$ is semisimple, so is the quotient $\Lambda_{\lambda}/(\Lambda_{\lambda} \cap (W + \ker(\iota)))$, of which $\lambda'$ is an irreducible direct factor. This implies a surjection: $\text{res}_{J}^{G'}\pi' \twoheadrightarrow \lambda'$.

In the theorem above, we show that $\lambda'$ verifies the second condition of irreducibility in Lemma 2.15. Unfortunately, $(J', \lambda')$ does not satisfy the first condition, neither when $\ell = 0$ nor when $\ell \neq 0$. A natural idea is to construct an open compact subgroup bigger than $J'$, and to extend $\lambda'$ to this group. The aim of the rest of this section is to find such a group.

The first step is to introduce $M_{\lambda}$ (see Definition 2.24), of which a maximal simple $k$-types is defined. Then we prove that $M_{\lambda} = J = J \cap G'$ (see Proposition 2.39 and Definition 2.40).

Lemma 2.22. Let $L$ be a subgroup of $G'$ such that $J' \subset L \subset J'$, and $\lambda'$ an irreducible subrepresentation of $\lambda|_{J'}$. Then the induced representation $\text{ind}_{J'}^{L}\lambda'$ is semisimple.

Proof. By Mackey’s theory, the induced representation $\text{ind}_{J'}^{L}\lambda'$ is a subrepresentation of $\text{res}_{J'}^{L}\text{ind}_{J'}^{G'}\lambda'$, which is semisimple by Proposition 2.25 and Corollary 2.20. Then by Clifford theory, $\text{res}_{J}^{L}\text{ind}_{J}^{G'}\lambda$ is semisimple, of which $\text{ind}_{J'}^{L}\lambda'$ is a subrepresentation, hence is semisimple.

Proposition 2.23. Let $\lambda'$ be an irreducible subrepresentation of $\text{res}_{J'}^{G'}\lambda$, and $\lambda_{L}'$ an irreducible subrepresentation of $\text{ind}_{J'}^{L}\lambda'$. Then $\lambda_{L}'$ verifies the second condition of irreducibility. This is to say that for any irreducible representation $\pi'$ of $G'$, if there is an injection $\lambda_{L}' \hookrightarrow \text{res}_{L'}^{G'}\pi'$, then there exists a surjection $\text{res}_{L'}^{G'}\pi' \twoheadrightarrow \lambda_{L}'$, where $L' = L \cap G'$.
Proof. By Lemma 2.22, the injection from $\lambda'_L$ to $\text{res}^G_{J'}\pi'$ induces a non-trivial homomorphism $\text{ind}^J_{J'}\lambda' \to \text{res}^G_{J'}\pi'$. By Frobenius reciprocity, there is an injection from $\lambda'$ to $\text{res}^G_{J'}\pi'$. Thus there exists a non-trivial homomorphism $\text{res}^J_{J},\lambda \to \text{res}^G_{J'}\pi'$. By applying Frobenius reciprocity, we obtain a surjection:

$$\text{res}^G_{J'}\text{ind}^J_{J'}\text{res}^J_{J},\lambda \twoheadrightarrow \text{res}^G_{J'}\pi'.$$

By Mackey’s theory, the $k$-representation $\text{ind}^J_{J'}\text{res}^J_{J},\lambda$ is a direct factor of $\text{res}^G_{J'}\text{ind}^J_{J'}\lambda$, and hence $\text{res}^G_{J'}\text{ind}^J_{J'}\text{res}^J_{J},\lambda$ is a direct factor of $\text{res}^G_{J'}\text{ind}^J_{J'}\lambda$. Then the surjection above implies a non-trivial homomorphism:

$$\text{res}^G_{J'}\text{ind}^J_{J'}\lambda \twoheadrightarrow \text{res}^G_{J'}\pi'.$$

By Proposition 2.11 the left hand side is isomorphic to $\text{res}^J_{J},\Lambda_{\lambda} \oplus \text{res}^J_{J},W$, where the set of isomorphism class of irreducible subquotients of $\text{res}^J_{J},W$ is disjoint with that $\text{res}^J_{J},\lambda$. Now we consider the equivalence $\text{res}^J_{J},\pi' \cong (\text{res}^J_{J},\Lambda_{\lambda} \oplus \text{res}^J_{J},W)/U$, where $U$ is the kernel of the above surjection.

Define $\tau$ be the composed morphism as below:

$$\tau := \lambda'_L \hookrightarrow \text{res}^G_{J'}\pi' \hookrightarrow \text{ind}^J_{J'}\text{res}^G_{J'}\pi',$$

and the last factor is isomorphic to $\text{ind}^J_{J'}((\text{res}^J_{J},\Lambda_{\lambda} \oplus \text{res}^J_{J},W)/U)$.

Now we prove that $\text{ind}^J_{J'}((\text{res}^J_{J},W+U)/U)$ is contained in $\text{ker}(\tau)$. Suppose to the contrary, $\tau$ induces a non-trivial morphism from $\lambda'_L$ to $\text{ind}^J_{J'}((\text{res}^J_{J},W+U)/U)$. By Frobenius reciprocity, there is a non-trivial morphism from $\text{res}^J_{J},\lambda'_L$ to $(\text{res}^J_{J},W+U)/U$. Meanwhile, by the definition of $\tilde{J}$ we have

$$\text{res}^J_{J},\lambda'_L \hookrightarrow \text{res}^J_{J}',\text{ind}^J_{J'}\lambda \cong \oplus_{J,J'}\text{res}^J_{J},\lambda.$$

Thus there exists an irreducible component of $\text{res}^J_{J},\lambda$ appearing as a subquotient of $\text{res}^J_{J},W$, which contradicts to Theorem 2.11. So $\tau(\lambda'_L)$ is contained in $\text{ind}^J_{J'}((\text{res}^J_{J},\Lambda_{\lambda} + U)/U)$, which is a quotient $\text{ind}^J_{J'}\text{res}^J_{J},\Lambda_{\lambda}$, and the latter is semisimple by Lemma 2.22 and Lemma 2.14. Hence we conclude that $\text{ind}^J_{J'}((\text{res}^J_{J},\Lambda_{\lambda} + U)/U)$ is isomorphic, contains $\lambda'_L$ as a direct component. Since it is a quotient of $\text{res}^G_{J'}\pi'$, we finish the proof. 

\textbf{Definition 2.24.} Let $(J, \lambda)$ be a maximal simple $k$-type of $G$, and $\lambda'$ be an irreducible subrepresentation of $\text{res}^J_{J},\lambda$. Define $M_{\lambda}$ to be the subgroup of $\tilde{J}$ consisting of the elements $x \in \tilde{J}$, such that $x(\lambda') \equiv \lambda'$.

\textbf{Remark 2.25.} Since $M_{\lambda}$ normalizes $J$ and the intersection $M_{\lambda} \cdot J \cap G'$ is equal to $M_{\lambda}$, we have that $J$ normalizes $M_{\lambda}$. On the other hand, since the irreducible components of $\text{res}^J_{J},\lambda$ are $J$-conjugate, the group $M_{\lambda}$ is independent on the choice of $\lambda'$. We will prove in Proposition 2.29 that $M_{\lambda} = \tilde{J}$.

In Theorem 2.22, we prove that a pair $(M_{\lambda}, \lambda'_{M_{\lambda}})$ (we keep the notation in Proposition 2.22) verifies the criteria in Lemma 2.14 and hence a maximal simple $k$-type. The first criterion has been checked in Proposition 2.23. Now we compute its intertwining group in $G'$. We first show that $I_{G'}(\text{ind}^J_{J},(\mathfrak{A})' \lambda') \subset U(\mathfrak{A})'$ (Proposition 2.27), and then show that $I_{U(\mathfrak{A})'} \lambda'_{M_{\lambda}} = M_{\lambda}$ (Theorem 2.29).

\textbf{Lemma 2.26.} The induction $\text{ind}^J_{J}(\mathfrak{A})\lambda$ is irreducible.
Proof. Denote \( \text{ind}^J_{J′}(\lambda) \) by \( \rho \). Let \( \rho′ \) be an irreducible subrepresentation of \( \rho \), and \( E \) the field extension associated with \( (J, \lambda) \) as in Section 2.1. Let \( \Lambda \) be an extension of \( \lambda \) to \( E^×J \), then \( \text{ind}^J_{J′}(\Lambda) \) is irreducible. Let \( \omega \) be the central character of \( \Lambda \), and denote by \( \rho_\omega' \) the extension of \( \rho \) to \( E^×J \) by \( \omega \). Since \( E^×U(\mathfrak{A})/F^×U(\mathfrak{A}) \) is a finite cyclic group, \( \rho_\omega' \) can be extended to \( E^×U(\mathfrak{A}) \). By Frobenius reciprocity and Mackey’s theory, one of such extension \( \rho_\omega' \) is embedded in \( \text{ind}^J_{J′}(\lambda) \), which is irreducible. Hence this embedding is an equivalence. After restricting to \( U(\mathfrak{A}) \), it gives an equivalence from \( \rho′ \) to \( \rho \), and we finish the proof. \( \square \)

Proposition 2.27. Let \( \lambda′ \) be an irreducible subrepresentation of \( \text{res}^J_{J′}\lambda \), then the intertwining set \( I_G(\text{ind}^J_{J′}(\lambda′)) \) is contained in \( U(\mathfrak{A})′ \).

Proof. Let \( \rho \) be as above. The induction \( \text{ind}^J_{J′}(\lambda′) \) is embedding to \( \text{res}^J_{J′}(\rho) \), thus is semisimple of finite length, and write \( \text{ind}^J_{J′}(\lambda′) \cong \bigoplus_{i \in I} \rho_i′ \), where \( \rho_i′ \) are irreducible and \( I \) is a finite set. Let \( g \in G \), we have

\[
H_g(\text{ind}^J_{J′}(\lambda′)) = \bigoplus_{i \in I} \text{Hom}(\rho'_i, i_{U(\mathfrak{A})′}g(\rho'_i)).
\]

Hence we have:

\[
I_G(\text{ind}^J_{J′}(\lambda′)) = \bigoplus_{i \in I} H_G(\rho'_i, \rho'_i).
\]

Now we assume that \( g \in I_G(\rho'_i, \rho'_j) \), and we show that \( g \in U(\mathfrak{A})′ \).

By Proposition 2.8, there exists a \( \chi \)-quasicharacter \( \chi \) of \( E^× \) such that \( \rho \otimes \chi \circ \det \). Hence \( \text{res}^J_{J′}(\rho) \) is a subquotient of \( \text{res}^J_{J′}(\alpha \rho \otimes \chi \circ \det) \). Applying Mackey’s theory to \( \text{res}^J_{J′}(\alpha \rho \otimes \chi \circ \det) \), it is isomorphic to a finite direct sum, whose direct components are \( \text{ind}^J_{J′}(\lambda′) \) \( \text{res}^J_{J′}(\rho) \). More precisely,

\[
\text{res}^J_{J′}(\rho \otimes \chi \circ \det) = \bigoplus_{\beta} \text{ind}^J_{J′}(\beta \rho \otimes \chi \circ \det) \otimes \gamma(\lambda \otimes \chi \circ \det),
\]

where \( y = \beta \alpha \gamma \), and \( \beta \) (resp. \( \alpha \)) runs over a finite quotient of \( U(\mathfrak{A}) \) (resp. \( g(U(\mathfrak{A})) \)).

By the uniqueness of Jordan-Hölder factors, the representation \( \lambda \) is weakly intertwined with \( \chi \circ \det \) by a \( y_0 \in U(\mathfrak{A})g(U(\mathfrak{A})) \). Hence by Corollary 2.10, \( y_0 \) intertwines \( \lambda \) with \( \chi \circ \det \), and by Proposition 2.9 there exists \( x \in U(\mathfrak{A}) \) such that \( x(\lambda \otimes \chi \circ \det) = \lambda \). The element \( y_0x^{-1} \) intertwines \( \lambda \) to itself, and hence lies in \( E^×J \). Therefore \( g \in U(\mathfrak{A})E^×JU(\mathfrak{A}) \cap G′ \). The latter is equal to \( U(\mathfrak{A})′ \), since \( E^× \) normalises \( U(\mathfrak{A}) \) and for any \( e \in E^× \), \( \det(e) \in \mathfrak{o}_F^× \) if and only \( e \in \mathfrak{o}_E^× \), where \( \mathfrak{o}_F, \mathfrak{o}_E \) is the ring of integers of \( F, E \) respectively. We conclude that \( I_G(\text{ind}^J_{J′}(\lambda)) = U(\mathfrak{A})′ \). \( \square \)

Lemma 2.28. Let \( \lambda′ \) be an irreducible component of \( \text{res}^J_{J′}(\lambda) \), and \( x \in I_U(\mathfrak{A})(\lambda′) \) (see Definition 2.7), then \( x \in J′ \).

Proof. For \( x \in I_U(\mathfrak{A})(\lambda′) \), by Proposition 2.8, \( x \) weakly intertwines \( \lambda \) with \( \chi \circ \det \) for a quasicharacter \( \chi \). Then \( x \) intertwines \( \lambda \) with \( \chi \circ \det \), and hence by Proposition 2.9 there exists \( y \in U(\mathfrak{A}) \) such that \( y(J) = J \) and \( y(\lambda) = \lambda \circ \chi \circ \det \), which implies that \( y \in J′ \) by definition. The element \( xy^{-1} \) therefore intertwines \( \lambda \), then \( x \in E^×Jy \cap U(\mathfrak{A})′ \) by [23] §IV, 1.1. Since \( E^×Jy \cap U(\mathfrak{A}) = J′ \), we conclude that \( x \in Jy \cap U(\mathfrak{A})′ = J′ \). \( \square \)
Theorem 2.29. Recall that \( \lambda'_{M_A} \) is an irreducible subrepresentation of \( \text{ind}^{M'}_{M_A} \lambda' \). Then the induced representation \( \text{ind}^{G'}_{M_A} \lambda'_{M_A} \) is irreducible and cuspidal.

Proof. It is equivalent to show that \((M_A, \lambda'_{M_A})\) satisfies the two conditions in Lemma 2.19 where the second condition has been verified in Proposition 2.27. It is left to prove that \( I_G(\lambda'_{M_A}) = M_A \). We first show that \( I_{U(\mathfrak{a})'}(\lambda'_{M_A}) = M_A \), and then we show that \( I_G(\lambda'_{M_A}) \subset U(\mathfrak{a})' \).

By Lemma 2.28, we have \( I_{U(\mathfrak{a})'} \lambda' \subset \tilde{J}' \). Since \( J' \) normalizes \( J' \), then \( x \in I_{U(\mathfrak{a})'} \lambda' \) meaning \( x(\lambda') \cong \lambda' \). Hence \( I_{U(\mathfrak{a})'} \lambda' \subset M_A \). By [IV §1.8.10, Proposition 3], let \( g \in G' \) and \( X \) a finite set of \( G' \) such that \( M_A g M_A = \cup_{x \in X} J' x J' \), then there is an equivalence:

\[
\text{H}_{g^{-1}}(\text{ind}^{M}_{M_A} \lambda') \cong \oplus_{J \in X} \text{H}_{(gJ)^{-1}}(\lambda'),
\]

which implies

\[
I_{U(\mathfrak{a})'}(\text{ind}^{M}_{M_A} \lambda') = M_A,
\]

(3)

Hence \( I_{U(\mathfrak{a})'}(\lambda'_{M_A}) \subset M_A \).

We finish the proof by verifying the inclusion

\[
I_G(\lambda'_{M_A}) \subset U(\mathfrak{a})'.
\]

Notice that \( \text{ind}^{U(\mathfrak{a})'}_{U(\mathfrak{a})} \lambda'_{M_A} \) is a subrepresentation of \( \text{res}^{U(\mathfrak{a})}_{U(\mathfrak{a})} \rho \), where \( \rho = \text{ind}^{U(\mathfrak{a})}_{U(\mathfrak{a})} \lambda \) (as in the proof of Proposition 2.27). Since \( \text{res}^{U(\mathfrak{a})}_{U(\mathfrak{a})} \tau \) is semisimple, we have

\[
I_G(\text{ind}^{U(\mathfrak{a})'}_{U(\mathfrak{a})} \lambda'_{M_A}) \subset I_G(\text{res}^{U(\mathfrak{a})}_{U(\mathfrak{a})} \rho).
\]

Hence by Proposition 2.27, we have

\[
I_G(\text{ind}^{U(\mathfrak{a})'}_{U(\mathfrak{a})} \lambda'_{M_A}) \subset U(\mathfrak{a})'.
\]

(4)

As for equation (3), let \( h \in G' \) and \( Y \) a finite set of \( G' \) such that \( U(\mathfrak{a}) h U(\mathfrak{a}) = \cup_{y \in Y} M_A y M_A \), then there is an equivalence:

\[
\text{H}_{h^{-1}}(\text{ind}^{U(\mathfrak{a})}_{M_A} \lambda'_{M_A}) \cong \oplus_{s \in Y} \text{H}_{(hs)^{-1}}(\lambda'_{M_A}).
\]

We conclude that

\[
I_G(\lambda'_{M_A}) \subset I_G(\text{ind}^{U(\mathfrak{a})'}_{M_A} \lambda'_{M_A}).
\]

By equation (4), we obtain the desired result. \( \square \)

2.2.6 Cuspidal \( k \)-representations of \( G' \)

Recall that \( M \) is a Levi subgroup of \( G \), and \( M' = M \cap G' \). In this section, we consider the restriction functor \( \text{res}^{M'}_{M} \), which has been studied in [11] when \( \ell = 0 \). We show that an irreducible \( k \)-representation \( \pi' \) of \( M' \) is contained in \( \text{res}^{M'}_{M} \pi \) for an irreducible \( \pi \) of \( M \) (see Proposition 2.31), which coincides with a result in [11], and from which we deduce that any irreducible cuspidal \( \pi' \) can be constructed as in Theorem 2.29 (see Corollary 2.33).

Lemma 2.30. Let \( K \) be a locally pro-finite group, and \( K' \subset K \) a closed normal subgroup of \( K \) with finite index. Let \( (\pi, V) \) be an irreducible \( k \)-representation of \( K \), then the restriction \( \text{res}^{K}_{K'} \pi \) is semisimple of finite length.
Proof. The proof of [VII §I.6.12] can be applied here. There is a condition in [VII §I.6.12] that \([K : K']\) is invertible in \(k\), however it is not used in the proof.

The restriction \(\text{res}^K_{K'} \pi\) has finitely length, hence has an irreducible quotient. Let \(V_0\) be the sub-representation such that \(V/V_0\) is irreducible. Let \(\{k_1, ..., k_m\}, m \in \mathbb{N}\) be a family of representatives of the quotient \(K'/K\). Now we consider the kernel of the non-trivial projection from \(\text{res}^K_{K'} \pi\) to \(\oplus_{i=1}^m V/k_iV_0\), which is \(K\)-stable, hence is equal to 0. We deduce that \(\text{res}^K_{K'} \pi\) is a sub-representation of \(\oplus_{i=1}^m V/k_iV_0\) hence is semisimple.

\[\tag*{\Box} \]

**Proposition 2.31.** Let \(\pi\) be an irreducible \(k\)-representation of \(M\), then the restriction \(\text{res}^M_{M'} \pi\) is semisimple of finite length, and the irreducible direct components are \(M\)-conjugate. Conversely, let \(\pi'\) be an irreducible \(k\)-representation of \(M'\), then there exists an irreducible representation \(\pi\) of \(M\), such that \(\pi'\) is a direct factor of \(\text{res}^M_{M'} \pi\).

**Proof.** For the first part, the method of Silberger in [S] for the case when \(\ell = 0\) can be generalised to \(\ell\)-modular setting. At first we assume that \(\pi\) is cuspidal. Let \(Z\) be the center of \(M\), and the quotient \(M/ZM'\) is compact (it is finite when \(\text{char}(F) = 0\), but may be infinite when \(\text{char}(F) > 0\)). Since the stabiliser \(\text{Stab}_M(v)\) is open for any vector \(v\) in the representation space of \(\pi\), the image of \(\text{Stab}_M(v)\) has finite index in the quotient group \(M/ZM'\). Hence the restriction \(\text{res}_M^{M'} \pi\) is finitely generated. By [VI §II.2.7] the restriction is \(Z' = Z \cap M'\)-compact. Now we show that it is admissible.

Let \((v_1, ..., v_m), m \in \mathbb{N}\) be a family of generators of the representation space of \(\text{res}_M^{M'} \pi\). For a compact open subgroup \(K\) of \(M'\), which stabilises \(v_i, i = 1, ..., m\), we consider the maps

\[\alpha_i : g \mapsto e_K g v_i, i = 1, ..., m,\]

where \(e_K\) is the idempotent of the Heck algebra \(H(K)\) of \(K\). Apparently, the space \(V^K\) is generated by \(L = \{e_K g v_i, g \in M', i = 1, ..., m\}\). Suppose that \(V^K\) has infinite dimension. There is a subset \(L'\) of \(L\) which forms a basis of \(V^K\). In particular, there exists \(i_0 \in \{1, ..., m\}\) such that \(M_{i_0}' = \{g \in M ' : e_K g v_{i_0} \in L'\}\) is an infinite set. Furthermore, since \(K\) stabilises \(v_{i_0}\), \(M_{i_0}'\) is an infinite union of disjoint cosets in the form of \(gK\). On the other hand, since the centre \(Z'\) of \(M'\) acts as a character on \(\text{res}_M^{M'} \pi\), the cosets \(gZ'K, g \in M_{i_0}'\) are disjoint in the quotient \(M'\). Let \(v_{i_0}^*\) be a smooth \(k\)-linear form of \(V^K\), which maps each element in \(L'\) to \(1 \in k\). The support of coefficient \(\langle v_{i_0}^*, \cdot \rangle_{i_0}\) contains \(M_{i_0}'\), which is not compact in \(M'/Z'\). Hence \(\text{res}_M^{M'} \pi\) is admissible, and then has finite length.

Now we assume that \(\pi\) is irreducible of \(M\). We first prove that \(\text{res}_M^{M'} \pi\) has finite length, and then it is semisimple. For the first part, let \((L, \sigma)\) be a cuspidal pair in \(M\) such that \(\pi \hookrightarrow \iota_L^M \sigma\). Applying Theorem A.4 we have \(\text{res}_M^{M'} \iota_L^M \sigma \cong \iota_{L'}^M \text{res}_L^M \sigma\). By the first step and the fact that \(i_{L'}^M\) respects finite length, we deduce that \(\text{res}_M^{M'} \pi\) has finite length. For the semi-simplicity, let \(W\) be an irreducible sub-representation of \(\text{res}_M^{M'} \pi\), of which \(gW\) is also an irreducible sub-representation for \(g \in M\). Let \(W' = \sum_{g \in M} g(W)\), which is a semisimple (by the equivalence condition in §A.VII. of [Re]) sub-representation of \(\text{res}_M^{M'} \pi\). Obviously, \(M\) stabilises \(W'\), hence \(W' = \pi\).

For the second part, we apply the proof of [LM Proposition §2.2]. Let \(\pi'\) be irreducible of \(M'\), and \(S\) the subgroup of \(Z\) generated by the scalar matrix \(a_F\) (a uniformizer of \(a_F\)). It is clear that the intersection \(S \cap M' = \{1\}\). Hence we extend \(\pi'\) to \(SM'\) by acting trivially on \(S\), and denote it by \(\tilde{\pi}'\). The quotient \(M'/SM'\) is compact, hence the induction \(\text{ind}_{SM'}^{M'} \tilde{\pi}\) is admissible, and contains an irreducible subrepresentation \(\pi' \hookrightarrow \text{ind}_{SM'}^{M'} \tilde{\pi}'\). There is a surjective morphism \(\text{res}_M^{M'} \pi \rightarrow \tilde{\pi}'\), defined by \(f \mapsto f(1)\), which induces a surjective morphism \(\text{res}_M^{M'} \pi \rightarrow \pi'\). We finish the proof. \(\Box\)
Corollary 2.32. Let $\pi$ be irreducible of $M$. If the restriction $\text{res}^{M}_{M'}^M \pi$ contains an irreducible cuspidal subrepresentation of $M'$, then $\pi$ is cuspidal. In other words, an irreducible cuspidal $k$-representation of $M'$ is embedded in $\text{res}^{M}_{M'}^M \pi$ for an irreducible cuspidal $\pi$ of $M$.

Proof. Let $P' = L' \cdot U$ be a proper parabolic subgroup of $M'$, and $P = L \cdot U$ the proper parabolic of $M$ such that $P \cap M' = P'$ and $L \cap M' = L'$. By Proposition 2.31 the irreducible components of $\text{res}^{M}_{M'}^M \pi$ are $M$-conjugate, hence are $L$-conjugate. Let $\pi'_0$ be an irreducible component. The Jacquet module $\pi'_0(U) = 0$ if and only if the Jacquet module of each irreducible component of $\text{res}^{M}_{M'}^M \pi$ is equal to 0, if and only if the Jacquet module $\pi(U) = 0$. □

Corollary 2.33. Let $\pi'$ be an irreducible cuspidal $k$-representation of $G'$. There exists a maximal simple $k$-type $(J, \lambda)$ of $G$, and a direct factor $\lambda'_M$ of $\text{ind}_{J'}^{M}^G \lambda$ (see Definition 2.24 for $M\lambda$), such that $\pi' \cong \text{ind}_{J}^{G} \lambda'_M$.

Proof. By Corollary 2.32 let $\pi$ be irreducible cuspidal of $G'$ where $\pi'$ is embedded in. Let $(J_0, \lambda_0)$ be a maximal simple $k$-type of $G$ contained in $\pi$, and $(M\lambda_0, \lambda'_M)$ as in Theorem 2.29. Since $\pi \cong \text{ind}_{J}^{G} \lambda'_M$, where $\lambda_0$ is an extension of $\lambda$ to $E^n J_0$, and the intersection $E^n J_0 \cap G' = J_0'$. By Mackey’s theory $\text{ind}_{J}^{G} \lambda'_M$ is embedded in $\text{res}^{G}_{J} \pi$. Hence by Proposition 2.31 $\text{ind}_{J}^{G} \lambda'_M \cong g(\pi')$ for $g \in G$, then $\pi'$ contains $g^{-1}(\lambda'_M)$. Since $g^{-1}(M\lambda_0) = M_{g^{-1}(\lambda_0)}$ and $\lambda'_M \cong g^{-1}(\lambda'_M)$ is a direct factor of $\text{ind}_{J}^{G} \lambda'_M$, by Frobenius reciprocity and Theorem 2.29 we conclude that $\pi' \cong \text{ind}_{J}^{G} \lambda'_M$ as desired. □

2.3 Whittaker models and maximal simple $k$-types of $G'$

In this section, we show $M\lambda = \tilde{J}$ in Definition 2.30 (see Definition 2.24 for $M\lambda$), which completes the establishment of maximal simple $k$-types of $G'$. To be more precise, we show that for $x \in U(\mathfrak{g})$, if $x$ normalises $J$ and $x(\lambda) \cong \lambda \otimes \chi$ for a $k$-quasicharacter $\chi$, then $x(\lambda') \cong \lambda'$, for each irreducible direct component $\lambda'$ of $\lambda|\tilde{J}$. To do so, we study Whittaker models and define derivatives for a $k$-representation of $M'$, which requires Appendix A the $\ell$-modular version of geometric lemma, and can be viewed as a generalisation of [BeZ] to the representations of $G'_n$.

2.3.1 Uniqueness of Whittaker models

In this section, we denote $\text{GL}_n(F)$ by $G_n$ and $\text{SL}_n(F)$ by $G'_n$. Let $U = U_n(F)$ be the group consisting of unipotent upper triangular matrices in $G$. Let $\psi$ be a non-degenerate $k$-character of $U$ as defined in [V1] §III,1. Two non-degenerate $k$-characters of $U$ are conjugate by a diagonal matrix in $G$. Let $P_n = P_n(F)$ be the mirabolic subgroup of $\text{GL}_n(F)$, and $P'_n = P_n \cap \text{SL}_n(F)$. We denote by $V_{n-1}$ the unipotent radical of $P_n$, which is an abelian group isomorphic to the additive group $F^{n-1}$, and is also the unipotent radical of $P'_n$.

Definition 2.34. 1. $r_{id} : \text{Rep}_k(P_n) \to \text{Rep}_k(G_{n-1})$ the functor of $V_{n-1}$-coinvariants;

2. $r_{id'} : \text{Rep}_k(P'_n) \to \text{Rep}_k(G'_{n-1})$ the functor of $V_{n-1}$-coinvariants;

3. $r_{\psi} : \text{Rep}_k(P_n) \to \text{Rep}_k(G_{n-1})$ the functor of $(V_{n-1}, \psi)$-coinvariants;

4. $r'_{\psi} : \text{Rep}_k(P'_n) \to \text{Rep}_k(G'_{n-1})$ the functor of $(V_{n-1}, \psi)$-coinvariants.
Definition 2.35 (Bernstein and Zelevinsky). Let \( 1 \leq k \leq n, \) and \( \pi \in \text{Mod}_k P_n, \) \( \pi' \in \text{Mod}_k P'_n. \) Define \( \pi^{(k)} := r_{n,k}^{(k-1)} \pi \) to be the \( k \)-th derivative of \( \pi, \) and \( \pi^{(\psi,k)} := r_{n,k}^{(k-1)} \pi' \) to be the \( k \)-th derivative of \( \pi' \) relative to \( \psi. \)

Remark 2.36. There is an equivalence \( \text{res}^G_{\tilde{G}_{n-k}} \pi^{(k)} \cong (\text{res}^G_{\tilde{G}_{n}} \pi)^{(k)} \), where \( \pi \in \text{Mod}_n G_n. \)

Proposition 2.37. Let \( \pi \) be a cuspidal \( k \)-representation of \( G, \) then the restriction \( \text{res}^G_{\tilde{G}} \pi \) is multiplicity free.

Proof. By Proposition 2.31 we have \( \text{res}^G_{\tilde{G}} \pi \cong \bigoplus_{i=1}^m \pi'_i, \) where \( m \in \mathbb{N} \) and \( \pi'_i \) are irreducible and cuspidal of \( G'_n. \) By [VI, §III.1.7], we have that \( \dim(\pi^{(n)}) = 1. \) By Remark 2.36 we then have

\[
\dim(\text{res}^G_{\tilde{G}} \pi)^{(n)} = \bigoplus_{i=1}^m \dim(\pi'^{(n)}) = 1,
\]

which implies that there exists one unique \( \pi'_i, \) where \( 1 \leq i_0 \leq m, \) such that \( \pi'^{(n)} \) is non-trivial. Hence \( \text{res}^G_{\tilde{G}} \pi \) is multiplicity free by considering the \( G_n \)-conjugation of \( \psi. \)

Corollary 2.38. Let \( \pi' \) be an irreducible cuspidal \( \tilde{k} \)-representation of \( G'. \) Then there exists a non-degenerate character \( \psi \) of \( U, \) such that \( \dim(\pi'^{(\psi,n)}) = 1. \)

Proof. It is deduced from Corollary 2.32 and Proposition 2.37.

2.3.2 Maximal simple \( \tilde{k} \)-types of \( G' \)

We complete the establishment of maximal simple \( \tilde{k} \)-types of \( G'. \)

Proposition 2.39. The subgroup \( M_\lambda \) in Definition 2.27 is equal to \( \tilde{J}'. \)

Proof. Let \( \Lambda \) be an extension of \( \lambda \) to \( E^\psi J. \) Then \( \text{ind}^G_{E^\psi J} \lambda \) is irreducible and cuspidal of \( G, \) and we denote it by \( \pi. \) The restriction \( \text{res}^G_{\tilde{G}} \pi \) is semisimple and its direct components are cuspidal. By Theorem 2.29 a component \( \pi' \) is isomorphic to \( \text{ind}^G_{E^\psi J} \lambda'_M, \) for a \( \lambda'_M, \) as defined in Proposition 2.23. In the proof of Theorem 2.29 we show that \( I_{G'}(\lambda'_M) = M. \) Suppose \( \tilde{J}' \neq M. \) Let \( x \in \tilde{J}' \backslash M, \) we have that \( x(\lambda'_M) \not\equiv \lambda'_M. \) Meanwhile, since \( x(\pi') \cong \pi', \) then

\[
\text{ind}^G_{M} x(\lambda'_M) \cong \pi'.
\]

On the other hand, by the fact that \( \text{res}^M_{\tilde{J}'} x(\lambda'_M) \not\equiv x(\lambda'), \) and the definition of \( \tilde{J}, \) we have that \( x(\lambda') \not\equiv \text{res}^\psi_{\tilde{J}'} \lambda. \) Hence \( x(\lambda'_M) \not\equiv \text{ind}^M_{\tilde{J}'} \text{res}^\psi_{\tilde{J}'} \lambda. \)

By Mackey’s theory

\[
\text{ind}^G_{M} \text{res}^\psi_{\tilde{J}'} \lambda \not\equiv \text{res}^G_{\tilde{G}} \text{ind}^G_{E^\psi J} \lambda.
\]

We deduce that \( \text{ind}^G_{M} \lambda'_M \) and \( \text{ind}^G_{M} x(\lambda'_M) \) are two irreducible components of \( \text{res}^G_{\tilde{G}} \pi. \)

Since both of them are isomorphic to \( \pi, \) it is contradicted with Proposition 2.37.

Definition 2.40. Let \((J, \lambda)\) be a maximal simple \( \tilde{k} \)-type of \( G \) and \( \tilde{J}' = \tilde{J} \cap G' \) (see Definition 2.12). Let \( \tilde{\lambda}' \) be an irreducible direct component of \( \text{ind}^G_{J} \text{res}^\psi_{\tilde{J}'} \lambda. \) We define a pair in the form of \((\tilde{J}', \tilde{\lambda}')\) to be a maximal simple \( \tilde{k} \)-type of \( G'. \) By Corollary 2.33 and Proposition 2.39, for an irreducible cuspidal \( k \)-representation of \( G', \) there exists a maximal simple \( k \)-type \((\tilde{J}', \tilde{\lambda}')\) such that \( \pi' \cong \text{ind}^G_{\tilde{J}'} \tilde{\lambda}'. \)
2.4 Maximal simple $k$-types for Levi subgroups of $G'$

Since the structure of a Levi subgroup $M'$ of $G'$ is not a multiple of $p$-adic special linear groups, the establishment of maximal simple $k$-types of $G'$ can not be applied directly to $M'$. In this section, we establish the maximal simple $k$-types of $M'$. We show that the existence for extended maximal simple $k$-types of proper $M'$ (they coincide with maximal simple $k$-types when $M' = G'$), and that a cuspidal $k$-representation of $M'$ is obtained from an extended maximal simple $k$-type rather than a maximal simple $k$-type, which is different from the case of $G'$.

2.4.1 Intertwining and weakly intertwining

In this section, let $M \cong \operatorname{GL}_{n_1} \times \cdots \times \operatorname{GL}_{n_r}$ be a Levi subgroup of $G$. For a closed subgroup $H$ of $G$, we always denote $H \cap G'$ as $H'$. As in Section 2.2, we start by considering the maximal simple $k$-types of $M$. In particular, Proposition 2.25, Proposition 2.6, Definition 2.7, and Proposition 2.8 will be applied in this section.

Proposition 2.41. Let $(J_M, \lambda_M)$ be a maximal simple $k$-type of $M$, and $\chi$ a $k$-quasicharacter of $F^\times$. If $(J_M, \lambda_M \otimes \chi \circ \det)$ is weakly intertwined with $(J_M, \lambda_M)$, then they are intertwined in $M$, and there exists an element $x \in U(M) = U(A_1) \times \cdots \times U(A_r)$ such that $x(J_M) = J_M$ and $x(\lambda_M) \cong \lambda_M \otimes \chi \circ \det$, where $A_i$ is a hereditary order associated to $(J_i, \lambda_i) \ (i = 1, \ldots, r)$. Furthermore, for a $g \in G$, if $g$ weakly intertwines $(J_M, \lambda_M \otimes \chi \circ \det)$ with $(J_M, \lambda_M)$, then $g$ intertwines $(J_M, \lambda_M \otimes \chi \circ \det)$ with $(J_M, \lambda_M)$.

Proof. By definition, $J_M = J_1 \times \cdots \times J_r$ and $\lambda_M \cong \lambda_1 \times \cdots \times \lambda_r$, where $(J_i, \lambda_i)$ are $k$-maximal cuspidal simple type of $\operatorname{GL}_{n_i}$ for $i \in \{1, \ldots, r\}$. The group $U(M) = U(A_1) \times \cdots \times U(A_r)$. Hence the two results are directly deduced by 2.9 and 2.10.

Definition 2.42. Let $(J_M, \lambda_M)$ be a $k$-maximal cuspidal simple type of $M$. We define the group of projective normalizer $\tilde{J}_M$ a subgroup of $U(M)$, where $A_M = A_1 \times \cdots \times A_r$, such that $x \in \tilde{J}_M$, if and only if $x(J_M) = J_M$ and $x(\lambda_M) \cong \lambda_M \otimes \chi \circ \det$ for a $k$-quasicharacter $\chi$ of $F^\times$.

Since $\tilde{J}_M$ is a subgroup of $\tilde{J}_1 \times \cdots \times \tilde{J}_r$, the quotient $\tilde{J}_M / J_M$ is a finite abelian $p$-group. The proof of Corollary 2.20 can be applied to $(J_M, \lambda_M)$, hence the induction $\tilde{\lambda}_M := \operatorname{ind}^{\tilde{J}_M}_{J_M} \lambda_M$ is irreducible. By Proposition 2.8, the restriction $\operatorname{res}^{\tilde{J}_M}_{J_M} \tilde{\lambda}_M$ is semisimple, of which $\tilde{\lambda}_M$ is an irreducible component.

Lemma 2.43. Let $(J_M, \lambda_M)$ be a maximal simple $k$-type of $M$, and $\tilde{\lambda}'_{M,1}, \tilde{\lambda}'_{M,2}$ two irreducible components of $\operatorname{res}^{\tilde{J}_M}_{J_M} \tilde{\lambda}_M$. Then :

$$\Pi^M_{\lambda_M}(\tilde{\lambda}'_{M,1}, \tilde{\lambda}'_{M,2}) = \{m \in M' : m(\tilde{\lambda}'_{M,1}) \cong \tilde{\lambda}'_{M,2}\}.$$  

hence $\Pi^M_{\lambda_M}(\tilde{\lambda}'_{M,1}, \tilde{\lambda}'_{M,2}) = I_{\lambda_M}(\tilde{\lambda}'_{M,1}, \tilde{\lambda}'_{M,2})$. In particular, $I_{\lambda_M}(\tilde{\lambda}'_{M,2})$ is the normaliser group of $\tilde{\lambda}'_{M,2}$ in $M'$. Moreover, the group $I_{\lambda_M}(\tilde{\lambda}'_{M,2})$ depends on $(J_M, \lambda_M)$ but not the choice of $\tilde{\lambda}_M$.

Proof. Let $m \in M'$ weakly intertwines $\tilde{\lambda}'_{M,2}$ with $\tilde{\lambda}'_{M,1}$, then by Proposition 2.8 $m$ weakly intertwines $\tilde{\lambda}_M$ with $\tilde{\lambda}_M \otimes \chi \circ \det$ for a $\chi$. By Mackey’s theory

$$\tilde{\lambda}_M|_{\tilde{J}_M} \cong \oplus_{x \in \tilde{J}_M / J_M} x(\lambda_M) \cong \oplus_{x \in \tilde{J}_M / J_M} \lambda_M \otimes \chi_x \circ \det$$

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In this section, we construct maximal simple \( k \)-finite field extensions of \( k \).

### 2.4.2 Maximal simple \( k \)-types of \( M' \)

In this section, we construct maximal simple \( k \)-types of \( M' \), and extended maximal simple \( k \)-types (Definition 2.40). We show that every irreducible cuspidal \( k \)-representation of \( M' \) is constructed from an extended maximal simple \( k \)-type. We follow the same strategy as for the case of \( G' \), which is by checking the two conditions of irreducibility in Lemma 2.41.

**Lemma 2.45.** We have a decomposition:

\[
\text{res}^M_{J_M} \text{ind}^M_{J_M} \lambda_M \cong \Lambda_{\lambda_M} \oplus W_{\lambda_M},
\]

where \( \Lambda_{\lambda_M} \) is semisimple, of which each irreducible component is isomorphic to \( \lambda_M \otimes \chi \circ \det \) for a \( k \)-quasicharacter \( \chi \) of \( F^\times \), and non of irreducible subquotients of \( W_{\lambda_M} \) is contained in \( \Lambda_{\lambda_M} \).

**Proof.** This is directly deduced from the decomposition in Theorem 2.41.

**Proposition 2.46.** Let \( \tilde{\lambda}_M' \) be an irreducible subrepresentation of \( \text{res}^M_{J'_M} \lambda_M' \), where \( J'_M = J_M \cap M' \), then \( \tilde{\lambda}_M' \) satisfies the second condition of irreducibility in Lemma 2.41. In other words, for an irreducible representation \( \pi' \) of \( M' \), if there is an embedding \( \lambda_M' \hookrightarrow \text{res}_{J'_M}^{M'} \pi' \), then there exists a surjection \( \text{res}_{J'_M}^{M'} \pi' \twoheadrightarrow \lambda_M' \).

**Proof.** The proof of Proposition 2.23 can be applied here.

**Proposition 2.47.** Let \( \tau_{\lambda'} \) be an irreducible representation of \( N_{M'}(\lambda'_M) \) which contains \( \tilde{\lambda}_M' \).

Then \( \tau_{\lambda'} \) satisfies the second condition of irreducibility.

---

where \( \chi_x \) is a \( k \)-quasicharacter of \( F^\times \). Hence we have \( \tilde{\lambda}_M \otimes \chi_x \circ \det \cong \tilde{\lambda}_M \) for every \( x \in \tilde{J}_M/J_M \). It follows that for a \( g \in \tilde{J}_M \), the element \( gm \) weakly intertwines \( \lambda_M \) with \( \lambda_M \otimes \chi_x \circ \det \) for a \( x \in \tilde{J}_M/J_M \). By Proposition 2.41, the element \( gm \) intertwines \( \lambda_M \) with \( \lambda_M \otimes \chi_x \circ \det \), and \( y(\lambda_M) \cong \lambda_M \otimes \chi_x \circ \det \) for a \( y \in \tilde{J}_M \). By induction to \( \tilde{J}_M \), we obtain an isomorphism \( \tilde{\lambda}_M \cong \lambda_M \otimes \chi \circ \det \), hence \( m \in \text{Im}(\lambda_M) \).

Furthermore, the intertwining set \( \text{Im}(\lambda_M) = N_M(\lambda_M) \), and the latter is the normaliser group of \( \lambda_M \), which also normalises \( U(\mathfrak{M}) \), hence normalises \( \tilde{J}_M \). We deduce that \( \text{Im}(\lambda_M) = \tilde{J}_M N_M(\lambda_M) \). Then an element of \( \text{Im}(\lambda_M) = \tilde{J}_M, N_M(\lambda_M) \) normalises \( \tilde{\lambda}_M \), and \( \tilde{J}_M \), which implies the first two assertions.

To prove the last assertion that \( \text{Im}(\lambda'_M) \) is independent of the choice of \( \lambda'_M \), we observe at first that the irreducible components of \( \lambda'_M \) are \( \tilde{J}_M \)-conjugate. We have to show therefore that \( \tilde{J}_M \) normalises \( N_M(\lambda'_M) \).

The quotient group \( N_M(\lambda_M)/\tilde{J}_M \cong N_M(\lambda_M)/\tilde{J}_M \) is abelian, from which we deduce that \( N_M(\lambda'_M) \) is a subgroup of \( N_M(\lambda_M) \cap M' \). Let \( x \in \tilde{J}_M \) and \( y \in N_M(\lambda'_M) \), we have \( x^{-1}yx = y \cdot m \) for some \( m \in \tilde{J}_M \cap M' = \tilde{J}_M \). Therefore:

\[
x^{-1}yx(\lambda'_M) \cong y(\lambda'_M) \cong \lambda'_M,
\]

as required.

**Remark 2.44.** To be more detailed, we proved that the intertwining group \( \text{Im}(\lambda'_M) \) is the stabiliser group \( N_M(\lambda'_M) \), which is a subgroup of \( E_1 \times \cdots \times E_r \times J, \cap M' \), where \( E_i \)'s are finite field extensions of \( F \), hence a compact group modulo centre.
As in the proof of Lemma 2.14, this implies that there exists at least one multiple of $\tau_K$ is an subquotient of $\text{ind}_K^M$. Since $\bar{J}_M$ is the unique maximal open compact subgroup of $N_M$. Hence for $b, a \in M'$ we have $J_M^b \cap ba(N_{M'}) = J_M^b \cap ba(J_M^b)$. Then

$$\text{res}_{N_M}^{M'} \text{ind}_{N_M}^{M'} \tau_{M'} \cong \bigoplus_{b \in N_M \cap (a(N_{M'}) \cap J_M^b)} \text{res}_{N_M \cap (a(N_{M'}) \cap J_M^b)}^{a(N_{M'})} a(\tau_{M'}). \tag{6}$$

Notice that $\tilde{\lambda}_M$ is a finite multiple of $\tilde{\lambda}_M$. For $a \notin N_M$, by Lemma 2.43 we have $ba \notin I_M^w(\tilde{\lambda}_M)$. This implies that non of irreducible subquotients of $\text{ind}_{J_M^b \cap ba(N_{M'})}^{J_M^b \cap ba(N_{M'})} ba(\tau_{M'})$ is isomorphic to $\tilde{\lambda}_M$. Now combining with (6), we obtain that

$$\text{res}_{N_M}^{M'} \text{ind}_{N_M}^{M'} \tau_{M'} \cong \tau_{M'} \oplus W_{N_M},$$

where $\tilde{\lambda}_M$ is a finite multiple of $\tilde{\lambda}_M$.

Proof. Denote $N_M(\tilde{\lambda}_M)$ by $N_{M'}$, by Mackey’s theory we have

$$\text{res}_{N_{M'}}^{M'} \text{ind}_{N_{M'}}^{M'} \tau_{M'} \cong \bigoplus_{b \in N_{M'} \cap (a(N_{M'}) \cap J_{M'})} \text{res}_{N_{M'} \cap (a(N_{M'}) \cap J_{M'})}^{a(N_{M'})} a(\tau_{M'}). \tag{6}$$

Notice that $\tilde{\lambda}_M$ is the unique maximal open compact subgroup of $N_{M'}$. Hence for $b, a \in M'$ we have $J_{M'}^b \cap ba(N_{M'}) = J_{M'}^b \cap ba(J_{M'})$. Then

$$\text{res}_{J_{M'}}^{J_M} \text{ind}_{J_{M'}}^{J_M} \tau_{M'} \cong \bigoplus_{b \in J_{M'} \cap (a(N_{M'}) \cap J_{M'})} \text{res}_{J_{M'} \cap (a(N_{M'}) \cap J_{M'})}^{a(N_{M'})} a(\tau_{M'})$$

where $\oplus \tilde{\lambda}_M$ is a finite multiple of $\tilde{\lambda}_M$.

Lemma 2.48. Let $A$ be a locally pro-finite group, and $K_1, K_2$ two open subgroups of $A$, such that $K_1$ is the unique maximal open compact subgroup of $K_2$. Let $\pi$ be an irreducible $k$-representation of $K_2$, and $\tau$ an irreducible $k$-representation of $K_1$, such that $\pi|_{K_1}$ is a multiple of $\tau$. If $x \in I_A(\pi)$ (resp. $x \in I_A^k(\pi)$), then there exists an element $y \in K_2$ such that $yx \in I_A(\tau)$ (resp. $yx \in I_A^k(\tau)$).

Proof. Since $\pi$ is isomorphic to a subquotient of $\text{ind}_{K_2}^{K_2 \cap ax(K_2)} \text{res}_{K_2 \cap ax(K_2)}^{x(\pi)}$, the restriction $\text{res}_{K_2}^{K_2}$ is isomorphic to a subquotient of $\text{res}_{K_2}^{K_2 \cap ax(K_2)} \text{res}_{K_2 \cap ax(K_2)}^{x(\pi)}$. Applying Mackey’s theory, $R(\pi)$ is isomorphic to

$$\bigoplus_{a \in K_2 \cap ax(K_2)} \text{ind}_{K_1 \cap ax(K_2)}^{K_1} \text{res}_{K_1 \cap ax(K_2)}^{a(\pi)}.$$ 

Since $K_1 \cap ax(K_2)$ is open compact, and the maximal open compact subgroup of $ax(K_2)$ is unique, which implies that $K_1 \cap ax(K_2) = K_1 \cap ax(K_1)$. Meanwhile, since $\text{res}_{K_2}^{K_2} \pi$ is a multiple of $\tau$, and the functors $\text{ind}, \text{res}$ can change order with infinite direct sum, $R(\pi)$ is isomorphic to a multiple of

$$\bigoplus_{a \in K_2 \cap ax(K_2)} \text{ind}_{K_1 \cap ax(K_1)}^{K_1} \text{res}_{K_1 \cap ax(K_1)}^{a(\pi)}.$$ 

As in the proof of Lemma 2.13, this implies that there exists at least one $g \in K_2$ such that $\tau$ is a subquotient of $\text{ind}_{K_1 \cap ax(K_1)}^{K_1} \text{res}_{K_1 \cap ax(K_1)}^{g(\pi)}$. \qed
Theorem 2.49. The induction $\text{ind}_{N_M'}^{M'}(\tilde{\lambda}_M')$ is cuspidal and irreducible. Conversely, an irreducible cuspidal $k$-representation $\pi'$ of $M'$ is equivalent to $\text{ind}_{N_M'}^{M'}(\tilde{\lambda}_M')$, for a pair $(N_M'(\tilde{\lambda}_M'), \tau_{M'})$ as in Proposition 2.47 defined from a maximal simple $k$-type $(J_M, \lambda_M)$ of $M$.

Proof. For the first part, we first verify the two conditions in Lemma 2.19. The second condition has been checked in 2.17. By 2.48 and 2.43, the first condition is satisfied. Denote the irreducible induction $\text{ind}_{N_M'}^{M'}(\tilde{\lambda}_M')$ by $\pi'$, and let $\pi$ be irreducible of $M$ as in 2.20. We deduce from 2.14 that $\lambda_M \otimes \chi \circ \det$ is a subquotient of $\pi|_{J_M}$ for a $\chi$. Hence $\pi$ is cuspidal (by 2.3) and 2.30) which implies that $\pi'$ is cuspidal.

For the second part, let $\pi'$ be irreducible and cuspidal of $M'$, and $\pi$ be irreducible cuspidal of $M$ as in 2.31. Then there exists a maximal simple $k$-type $(J_M, \lambda_M)$, and an extension $\Lambda_M$ of $\lambda_M$ to $N_M(\lambda_M)$ such that $\pi \cong \text{ind}_{J_M}^{N_M} \Lambda_M$. Let $\tilde{\lambda}_M = \text{ind}_{J_M}^{N_M} \lambda_M$, and $N_M(\tilde{\lambda}_M)$ be the group of normalisers of $\tilde{\lambda}_M$ in $M$. By the transitivity of induction:

$$\pi \cong \text{ind}_{N_M(\tilde{\lambda}_M)}^{M} \circ \text{ind}_{N_M(\lambda_M)}^{N_M(\tilde{\lambda}_M)} \Lambda_M.$$

Denote $\text{ind}_{N_M(\tilde{\lambda}_M)}^{M} \Lambda_M$ by $\tilde{\Lambda}_M$, which is irreducible containing $\tilde{\lambda}_M$.

Till the end of this proof, we denote by $\tilde{\lambda}_M$ a direct component of $\tilde{\lambda}_M|_{J_M}$, and denote $N_M(\tilde{\lambda}_M)$ by $N$, $N \cap M'$ by $N'$, and $N_M(\tilde{\lambda}_M')$ by $N_M'$. Let $K$ be the kernel of $\tilde{\lambda}_M$, and $Z$ the centre of $M$, $Z' = Z \cap M'$ since $K$ is compact and $K$ open. Hence $\text{res}_K^{N} \tilde{\lambda}_M$ is a semisimple of finite length as in the proof of 2.31. We deduce that $\text{res}_{N_M'} \tilde{\lambda}_M$ is semisimple of finite length as well. By a conjugation of $M$, we can assume that $\pi'$ contains a direct factor of $\otimes_{J_M} \tilde{\lambda}_M$, which we denote by $\tilde{\lambda}_M$. By Mackey’s theory $\text{res}_{J_M}^{N_M} \text{ind}_{J_M}^{N_M} \tilde{\lambda}_M$ is a multiple of $\text{res}_{J_M}^{M'} \tilde{\lambda}_M$, and we can assume that $\tilde{\lambda}_M$ contains a $\tilde{\lambda}_M'$. On the other hand, $N_M'$ is a normal subgroup of finite index in $N'$. In fact, $N_M'$ contains $Z \otimes_{J_M} M$. We have showed in 2.43 that $N_M'$ is a subgroup of $E_1^{x_1} \cdots \times E_r^{x_r} J_M' \cap M'$ which is compact modulo the centre. Hence $\text{res}_{N_M'} \tilde{\lambda}_M'$ is semisimple of finite length, and there must be a direct component $\tau_{M'}$ that contains $\tilde{\lambda}_M'$. Hence we have

$$\pi' \cong \text{ind}_{N_M'}^{M'} \tau_{M'}.$$

\[ \square \]

Definition 2.50. Let $(J_M, \lambda_M)$ be a maximal simple $k$-type of $M$, and $\tilde{\lambda}_M'$ be an irreducible component of $\text{res}_{J_M}^{M'} \tilde{\lambda}_M$, where $\tilde{\lambda}_M$ and $\tilde{\lambda}_M$ are defined as in 2.44. We define the couples in forms of $(J_M', \tilde{\lambda}_M')$ are the maximal simple $k$-types of $M'$.

- Let $N_M'(\tilde{\lambda}_M')$ be the normaliser group of $\tilde{\lambda}_M'$ in $M'$, and $\tau_{M'}$ an irreducible $k$-representation of $N_M'(\tilde{\lambda}_M')$ containing $\tilde{\lambda}_M'$. We say a pair of the form $(N_M'(\tilde{\lambda}_M'), \tau_{M'})$ is an extended maximal simple $k$-type of $M'$.

Remark 2.51. We will prove in Theorem 4.4 that $\tau_{M'}$ is an extension of $\tilde{\lambda}_M'$, which is a parallel result for complex case.

3 Intertwining and conjugacy

In this section, we prove the unicity property of weakly intertwining implying conjugacy (Theorem 3.24), and the unicity property of maximal simple $k$-types (resp. simple $k$-
characters) contained in an irreducible cuspidal \( k \)-representation of \( M' \) (Theorem 3.29). The first one is a well-known result for representations of characteristic zero of \( M' \), and was proved in [BuKuII] for \( M' = G' \). The philosophy can be generalised to our case, with some technical difficulties that arise from the \( \ell \)-modular Clifford theory for \( \pi|_{G'} \) in Section 3.1. The main results are proved in Section 3.4.

### 3.1 A formula of the length of \( \pi|_{G'} \)

Recall that \( G = \mathrm{GL}_n(F) \) and \( G' = \mathrm{SL}_n(F) \). Let \( (\pi, V) \) be an irreducible \( k \)-representation of \( G \), we have seen in Proposition 2.31 that the restriction of \( \pi \) to \( G' \) is a finite direct sum of irreducible representations, that are conjugate under \( G \), and let \( (\pi', V') \) be one of them. We begin with a technical section to deal with the main difficulty while applying the method in [BuKuII] to our case, that is the inclusion in [BuKuII, Proposition 1.5] which is not true in the \( \ell \)-modular setting. It leads to the failure of [BuKuII, Corollary 1.6], which is the base stone in the proof of [BuKuII, Theorem 5.3]. Fortunately, we obtain a generalisation of this inclusion, which gives an interpretation of the \( \ell \)-prime part of the length of \( \pi|_{G'} \), as the \( \ell \)-modular version of [BuKuII, Corollary 1.6], and which is the key to prove the property of weakly intertwining implying conjugacy.

Write

\[
S(\pi) = \{ x \in G : x(\pi') \cong \pi' \}. \tag{7}
\]

The index of \( S(\pi) \) in \( G \) is finite. In fact, the group \( S(\pi) \) contains \( G' \) and the centre \( Z(G) \) of \( G \). On the other hand, denoting by \( V' \) the representation space of \( \pi' \), for any non-zero vector \( v' \in V' \), since \( G' \) is normal in \( G \) and \( v' \) generalises \( V' \) as \( G' \)-representation, hence a stabiliser of \( v' \) in \( G \) also normalises \( V' \), then it is contained in \( S(\pi) \), which implies that \( S(\pi) \) is open with finite index in \( G \).

Write

\[
\mathcal{G}(\pi) = \{ \chi : \pi \otimes \chi \circ \det \cong \pi \},
\]

where \( \chi \) ranges over the set of \( k \)-quasicharacters of \( F^\times \).

Write

\[
\mathcal{T}(\pi) = \bigcap_{\chi \in \mathcal{G}(\pi)} \ker(\chi \circ \det).
\]

The group \( \mathcal{T}(\pi) \) is closed in \( G \).

**Proposition 3.1** (Proposition 1.4 in [BuKuII]). The group \( \mathcal{G}(\pi) \) is finite. The subgroup \( \mathcal{T}(\pi) \) of \( G \) contains \( ZG' \), and is open of finite index, where \( Z \) is the centre of \( G \).

**Proof.** The original proof can be applied directly.

Let \( A \) be a finite abelian group. Let \( A_\ell \) be the subgroup of \( A \) consisting of elements whose order are powers of \( \ell \), and we say that \( A_\ell \) is the \( \ell \)-power part of \( A \). Meanwhile, let \( A_{\ell}' \) be the subgroup of \( A \) consisting of elements whose orders are prime to \( \ell \), and we say it is the \( \ell \)-prime part of \( A \).

**Lemma 3.2.** Let \( A \) be a finite abelian group, we consider the \( \ell \)-dual group \( A^\wedge \) of \( A \), which is the group of \( k \)-characters of \( A \). Then \( A^\wedge \) is isomorphic to \((A_{\ell}')^\wedge \).

**Proof.** Since \( A/A_\ell \cong A_{\ell}' \), and \( A_\ell \) is contained in the kernel of each \( k \)-character of \( A \), we deduce the result.
Composition with the determinant induces an isomorphism
\[ G(\pi) \cong (G/T(\pi))^{\wedge}. \]

The lemma above implies that the \( \ell \)-power part \((G/T(\pi))_{\ell}\) of the quotient group \(G/T(\pi)\) is trivial.

**Lemma 3.3.** Let \( G \) be a locally pro-finite group and \( H \) a normal subgroup of \( G \). Let \( (\pi, V) \) be an irreducible \( k \)-representation of \( H \). Assume that \( G/H \) is finite cyclic and \( G \) normalises \((\pi, V)\), then \((\pi, V)\) can be extended to \( G \).

**Proof.** This is a well-known result. For those who might be interested in, the construction in the first paragraph of the proof of Proposition 5.2.4 in [BuKu] can be applied. \( \square \)

Let \( (\pi, V) \) be an irreducible \( k \)-representation of \( G \), and \((\pi', V')\) an irreducible sub-representation of \((\pi|_{G'}), V)\). Let \( d \) be the multiplicity of \( \pi' \) in \( \pi|_{G'} \), which is independent of the choice of \( \pi' \). Let \( H(\pi', V') \) be the subgroup of \( G \), consisting of elements \( g \) such that \( \pi(g)V' = V' \). In other words, \( H(\pi', V') \) is the group of \( G \)-stabilizer of \((\pi', V')\). Hence \( H(\pi', V') \) is an open subgroup of \( S(\pi) \). Moreover, the quotient \( S(\pi)/H(\pi', V') \) is finite and abelian. Since the restriction of \( \pi \) to any normal subgroup with finite quotient in \( G \) is semisimple of finite length, the method of [BuKu] §1.16 can be applied, hence by choosing \( V' \), we can assume that \( H(\pi', V') \) is maximal, and write \( \pi_1 \) to be the \( H(\pi', V') \)-representation on \( V' \). Then we have,
\[ \pi \cong \text{ind}_{H(\pi', V')}^{G} \pi_1, \]
which implies that \( d = (S(\pi) : H(\pi', V')) \). The group \( H(\pi', V') \) is independent of the choice of \( \pi' \) and \( V' \), and we denote it as \( H(\pi) \).

**Remark 3.4.** Let \( \chi \) be a \( k \)-quasicharacter of \( F^{\times} \) which is trivial on \( \text{det}(H(\pi)) \), and \( \pi_1 \) as above, then \( \pi_1 \otimes \chi \circ \text{det} \cong \pi_1 \). It follows that \( \pi \otimes \chi \circ \text{det} \cong \pi \).

The purpose of Lemma 3.5 and Lemma 3.6 below is to prove Proposition 3.7, which is a generalisation of [BuKuII] Proposition 1.5 and the main result of this section. The complexity of Lemma 3.5 arises from the fact that the \( \ell \)-dual set of a non-trivial finite abelian group can be trivial, which is not possible for \( K \)-quasicharacters. Hence, the group \( T(\pi) \) given from \( k \)-characters will be larger than the setting of characteristic zero. However, the following two lemmas are sufficient for the later use.

**Lemma 3.5.** There is an equivalence between the \( \ell \)-dual groups \((G/T(\pi) \cap H(\pi))^{\wedge} \) and \((G/T(\pi))^{\wedge} \). Let \( H(\pi)_{\ell} \) (resp. \( H(\pi)_{c} \)) be a subgroup of \( H(\pi) \), consisting with the elements whose images by projection belong to \((H(\pi)/T(\pi) \cap H(\pi))_{\ell} \) (resp. \((H(\pi)/T(\pi) \cap H(\pi))_{c} \)). Then \( H(\pi)_{c} \) is equal to \( H(\pi) \).

**Proof.** First, we consider the quotient group
\[ \overline{T} = T(\pi)/T(\pi) \cap H(\pi) \cong T(\pi)H(\pi)/H(\pi). \]
The latter is a subgroup of the finite abelian group \( G/H(\pi) \). Hence a \( k \)-quasicharacter \( \theta \) of \( \overline{T} \) can be extended to \( G/H(\pi) \). Let \( \overline{\theta} \) be such an extension. The equivalence given in Remark 3.3 implies that \( \overline{\theta} \in \mathcal{G}(\pi) \), hence \( \theta \) is trivial by the definition of \( T(\pi) \). We deduce from Lemma 3.2 that \( \overline{T} \) only has \( \ell \)-power part, in other words, \( \overline{T}_{\ell} \) is equal to \( \overline{T} \). We conclude that, if a \( k \)-quasicharacter \( \chi \) of \( F^{\times} \) is trivial on \( \text{det}(\overline{T}(\pi) \cap H(\pi)) \), then it is trivial on \( \text{det}(T(\pi)) \), hence \( \chi \circ \text{det} \) belongs to \((G/T(\pi))^{\wedge}\).
Now consider the second part of this lemma. For any \( \chi \circ \det \in (G/T(\pi) \cap H(\pi))^\wedge \), we have an inclusion \( H(\pi)_\ell \subset \ker(\chi \circ \det) \), which implies that
\[
H(\pi)_\ell \subset T(\pi).
\]
Hence \( H(\pi)_\ell \) is equal to \( T(\pi) \cap H(\pi) \).

Write \( \pi_0 = \pi_1|_{T(\pi) \cap H(\pi)} \). The representation \( \pi_0 \) is an extension of \( \pi' \), hence is irreducible.

**Lemma 3.6.** Let \( \tau \) be an extension of \( \pi' \) to \( T(\pi) \cap H(\pi) \). Assume that \( \tau \) has the same central character as \( \pi_0 \). Then there exists a k-quasicharacter \( \phi \) of \( F^\times \) such that \( \tau \) is equivalent to \( \pi_0 \otimes \phi \circ \det \).

**Proof.** Let \( \pi_0' = \pi_0|_{ZG'} \), since the quotient group \( G/ZG' \) is compact, the assumption implies that \( \tau \) is a sub-representation of \( \text{ind}_{ZG'}^{T(\pi) \cap H(\pi)} \pi_0' \), which is equivalent to the tensor product \( \pi_0 \otimes \text{ind}_{ZG'}^{T(\pi) \cap H(\pi)} 1 \). The induction \( \text{ind}_{ZG'}^{T(\pi) \cap H(\pi)} 1 \) is isomorphic to a direct sum of \( k \)-representations of finite length. Moreover, an irreducible sub-quotient of \( \text{ind}_{ZG'}^{T(\pi) \cap H(\pi)} 1 \) is isomorphic to \( \phi \circ \det \), where \( \phi \) is a \( k \)-quasicharacter of \( F^\times \). We conclude that \( \text{ind}_{ZG'}^{T(\pi) \cap H(\pi)} \pi_0' \cong \bigoplus_{\chi \in I} \tau_\chi \), where \( \tau_\chi \) has finite length, and any irreducible sub-quotient of \( \tau_\chi \) is equivalent to \( \pi_0 \otimes \phi \circ \det \), which implies the result.

**Proposition 3.7.** Let \( \pi, V \) be an irreducible k-representation of \( G \) and \( \pi' \) an irreducible sub-representation of \( \pi|_{ZG'} \). Let \( d \) denote the multiplicity of \( \pi' \) in \( \pi \). Then the intersection \( T(\pi) \cap H(\pi) \) is contained in \( S(\pi) \) with index \( d^2 \).

**Proof.** It is clear that \( \pi \) contains \( \pi_0 \). Assume that \( \pi \) contains an extension \( \pi_2 \) of \( \pi' \) to \( T(\pi) \cap H(\pi) \), and is different from \( \pi_0 \), then \( \pi_2 \) has the same central character as \( \pi_0 \) and \( \pi_2 \) is isomorphic to \( \pi_0 \otimes \phi \circ \det \) by Lemma 3.6. Now adjusting \( \phi \) by a \( k \)-quasicharacter of \( F^\times \) which is trivial on \( \det(T(\pi) \cap H(\pi)) \), we deduce from Frobenius reciprocity that \( \pi \) contains \( \pi_1 \otimes \phi \circ \det \), which is an extension of \( \pi_2 \otimes \phi \circ \det \). Then by the definition of \( H(\pi) \), we have \( \pi \cong \pi_1 \otimes \phi \circ \det \), but \( \phi \circ \det \) is non-trivial on \( T(\pi) \cap H(\pi) \), which contradicts with the definition of \( T(\pi) \). Hence \( \pi_0 \) is the unique extension of \( \pi' \) occurring in \( \pi \).

Now we consider
\[
\text{ind}_{T(\pi) \cap H(\pi)}^{H(\pi)} \pi_0 \cong \pi_1 \otimes \text{ind}_{T(\pi) \cap H(\pi)}^{H(\pi)} 1.
\]
By Lemma 3.5, the orders of elements inside the quotient group \( H(\pi)/T(\pi) \cap H(\pi) \) are prime to \( \ell \). Hence the \( k \)-representation on the right-hand side is semisimple, and we have the equivalence
\[
\text{ind}_{T(\pi) \cap H(\pi)}^{H(\pi)} \pi_0 \cong \sum_{\chi \in (H(\pi)/T(\pi) \cap H(\pi))^\wedge} \pi_1 \otimes \chi \circ \det.
\]
Since by the first part of Lemma 3.5, each \( \chi \) can be viewed as the restriction of some element in \( G(\pi) \) and \( \text{ind}_{H(\pi)}^{G(\pi)} \pi_1 \) is equivalent to \( \pi_1 \), the induced representation \( \text{ind}_{T \cap H(\pi)}^{G(\pi)} \pi_0 \) is a multiple of \( \pi \) with multiplicity equal to the cardinality \( |(H(\pi)/T(\pi) \cap H(\pi))^\wedge| \), which is equal to \( (H(\pi): T(\pi) \cap H(\pi)) \) by Lemma 3.5. The multiplicity \( d \) of \( \pi' \) in \( \pi \) is equal to the dimension of \( \text{Hom}_{T(\pi) \cap H(\pi)}(\pi, \pi_0) \), and the Frobenius reciprocity implies that
\[
\text{Hom}_{T(\pi) \cap H(\pi)}(\pi, \pi_0) \cong \text{Hom}_{G(\pi)}(\pi, \text{ind}_{T(\pi) \cap H(\pi)} \pi_0).
\]
We conclude that \( d = (H(\pi): T(\pi) \cap H(\pi)) \).
The Corollary below is an equation of the $\ell$-prime part of the length of $\pi|_{G'}$.

**Corollary 3.8.** Let $\pi$ be an irreducible $k$-representation of $G$.

1. The length of $\pi|_{G'}$ is equal to $(G : S(\pi)) \cdot (S(\pi) : T(\pi) \cap H(\pi))^{\frac{1}{d}}$, where $d = \text{ind}_H^G(\pi)_{\ell}$.

2. The restriction $\pi|_{G'}$ is multiplicity-free if and only if $S(\pi) \subset T(\pi)$.

3. Assume the restriction $\pi|_{G'}$ is multiplicity-free, then we have an equation $\log(\pi|_{G'}) = |G(\pi)|$, where $\log(\pi|_{G'})$ denotes the length of $\pi|_{G'}$.

4. If $\pi'$ is an irreducible sub-representation of $\pi|_{G'}$, then there is a unique irreducible representation $\pi_0$ of $T(\pi) \cap H(\pi)$ which contains $\pi'$ on $G'$ and occurs in $\pi|_{T(\pi) \cap H(\pi)}$. Moreover, $\pi_0|_{G'} \cong \pi'$.

**Proof.** We have shown that the multiplicity $d = (S(\pi) : H(\pi))$. Parts 1, 2, 4 are deduced from Proposition 3.7. For part 3, since $\pi \cong \text{ind}_H^G(\pi)_{\ell}$, the length of $\pi|_{G'}$ is equal to $(G : H(\pi))$. When $d = 1$, $S(\pi) = H(\pi)$ and by Proposition 3.7 we have $S(\pi) \subset T(\pi)$. We conclude that when $d = 1$,

$$\log(\pi|_{G'}) = (G : H(\pi)) = (G : T(\pi)) \cdot (T(\pi) : H(\pi)).$$

Lemma 3.2 and Lemma 3.5 imply that $\log(\pi|_{G'}) = (G : T(\pi))$ and $\log(\pi|_{G'}) = (T(\pi) : S(\pi)).$

**In Section 3.2, when $\pi$ is cuspidal, we need the results above to give an equation on the $\ell$-prime part of the length of $\pi|_{G'}$ through type theory as in [BuKuII]. Now we consider a similar results for pro-finite groups. The results below is required in the study of the the maximal $k$-types contained in $\pi$.**

**Lemma 3.9.** Let $A$ be a pro-finite abelian group, and $K$ a closed subgroup of $A$. Let $\chi$ be a smooth $k$-character of $K$, then $\chi$ can be extended to $A$.

**Proof.** The kernel $\text{Ker}(\chi)$ is open in $K$. Hence there exists an open subgroup $K_0$ of $A$ such that $K_0 \cap K \subset \text{Ker}(\chi)$. We have $K_1 = \text{Ker}(\chi) \cdot K_0$ is an open subgroup of $A$ such that $K \cap K_1 = \text{Ker}(\chi)$. By the equivalence $K/\text{Ker}(\chi) \cong K \cdot K_1/K_1$, $\chi$ can be viewed as a smooth character of $K \cdot K_1/K_1$. Notice that $A/K_1$ is a finite abelian group, which is a direct sum of finite cyclic groups, and each element in $A$ normalises $\chi$. By repeating Lemma 3.3 we obtain the result.

**Proposition 3.10.** Let $G$ be a profinite group, and $N$ a closed normal subgroup of $G$ such that $G/N$ is abelian. Assume that there exists an open subgroup of $G$, whose pro-order is invertible in $k^*$. Let $\rho$ be an irreducible smooth representation of $G$, and $\rho'$ an irreducible component of $\rho|_{N}$. Define

$$\mathcal{G}(\rho) = \{ \phi \in (G/N)^\wedge : \rho \otimes \phi \cong \rho \},$$

$$\mathcal{T}(\rho) = \bigcap_{\phi \in (G/N)^\wedge} \text{Ker}(\phi).$$

The subgroup $\mathcal{T}(\rho)$ is open in $G$. Define $H(\rho)$ by the same manner of $H(\pi)$ as above. Then $H(\rho)$ is open of $G$ containing $N$, such that

$$(\mathcal{T}(\rho)/\mathcal{T}(\rho) \cap H(\rho))_{\ell} = \mathcal{T}(\rho)/\mathcal{T}(\rho) \cap H(\rho),$$
(H(\rho)/T(\rho) \cap H(\rho))_\nu = H(\rho)/T(\rho) \cap H(\rho).

Furthermore, there exists an irreducible k-representation \pi_1 of H(\pi), such that ind_{H(\pi)}^G \rho_1, and res_{H(\pi)}^G \rho_1 \cong \rho'. Suppose the restriction \rho|_N is multiplicity-free, then we have an inclusion H(\rho) \subset T(\rho) and \rho_1 is the unique extension of \rho' occurring in \rho|_N. We have \lg(\rho|_N)_\nu = |G(\rho)|, where \lg(\rho|_N) denotes the length of \rho|_N.

Proof. The proof is basically the same as the case above. We list some details which need to be modified while applying the proofs. We can define S(\rho) the same way as in (7). Similarly we can show that S(\rho) and H(\rho) are open with finite index. Moreover, T(\rho) is open with finite index. In fact, the k-representation \rho is finitely dimensional. There exists an open subgroup K of G stabilising \rho, which implies that \mathcal{G}(\rho) \subset (G/K)^\alpha. By applying Lemma 3.3, we can find an extension \rho_1 of \rho' to H(\rho).

Define \rho_0 as \rho_1|_{T(\rho) \cap H(\rho)}, and Propositon 3.7 is correct. By the assumption, there exists an open normal subgroup K_0 of G whose pro-order in invertible in k^*, then K_0N is an open subgroup, who is the inverse image of K_0/(K_0 \cap N), hence G/N contains an open subgroup whose pro-order is invertible in k^*. Then Lemma 3.3 can be applied to our case by replacing \phi \circ \det to \chi \in (T(\rho) \cap H(\rho)/N)^\alpha, furthermore, we can assume \chi \in (G/N)^\alpha according to Lemma 3.3. We deduce that \rho_0 is the unique extension of \rho' containing in \rho. The proof of Lemma 3.3 is valid, so is the proof of Proposition 3.7 of which this proposition can be deduced as Corollary 3.8.

We end this section with an example, which is an application of the length formula in Corollary 3.8. The length of parabolic induction is always a difficult problem while considering \ell-modular representations. Especially to estimate the upper bound, which is a difference comparing to the case of representations of characteristic zero. While in small ranks, many examples can be computed. In the example at the end of DaIII, the last case is left, and we give an answer of this problem.

**Example 3.11.** Let q be the cardinality of the residual field of F, and \alpha a square root of q^{-1}. Let G' = SL_2(F) and G = GL_2(F). Let M be the diagonal maximal torus in G, and M' = G \cap M, P the group of upper triangular matrices, with P = MU. Let \psi be a k-quasicharacter on M', such that for m' = (\alpha \psi, \alpha \psi^{-1}) (\alpha \psi is a uniformizer of F, m' \in M') \psi(m) = q^{-1} and q \equiv -1 (mod \ell). When \ell \neq 2, the induced representation i_{M'}^G \psi has length 4, and when \ell = 2, the length is 6.

Proof. As explained in the last example of DaIII, the induced representation i_{M'}^G \psi should have length 4 or 6. We compute the length in two ways: first we apply Corollary 3.8 with a condition that \ell \neq 2, second we compute through the theory of types.

**Method 1 (\ell \neq 2):** Assume that \ell \neq 2, and q^{-1} \equiv -1 (mod \ell). Let \delta_P be the mod character of P, and \mu be an k-quasicharacter of M, such that for an diagonal element m = (a, b) \in M, \mu(m) = \alpha^{val(a/b)}, which can be extended to P through P \rightarrow M. We have \mu^2 = \delta_P and \psi = \mu|_{M'}. Vignéras proved that i_{M'}^G \mu has length 3, with a quotient character and a sub-character. By geometric lemma, we have

i_{M'}^G \psi \cong \text{res}_{G}^G i_{M'}^G \mu.

Let \pi be the cuspidal sub-quotient of i_{M'}^G \mu, the length of \text{res}_{G}^G \pi should have length 2 or 4 as explained in DaIII. Now we apply Corollary 3.8 to \pi. The restriction \text{res}_{G}^G \pi is multiplicity-free, hence |G(\pi)| = \lg(\text{res}_{G}^G \pi)_\nu. Under the assumption \ell \neq 2, the equation above should be |G(\pi)| = \lg(\text{res}_{G}^G \pi).

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It’s left to compute $G(\pi)$. In fact, for a $k$-quasicharacter $\chi$ of $F^\times$, $\pi \otimes \chi \circ \det$ is a subquotient of $i_{G M}^G \mu \otimes \chi \circ \det$. If $\pi \cong \pi \otimes \chi \circ \det$, by the uniqueness of supercuspidal support, we obtain that $(M, \mu)$ and $(M, \mu \otimes \chi \circ \det)$ should belong to the same $G$-conjugacy class. There are only two element in the Weyl group $W_G$ relative to $M$. We have $W_G = \{1, w\}$, where $w(a, b) = (b, a)$, for any element $(a, b) \in M$. Hence $(M, \mu)$ and $(M, \mu \otimes \chi \circ \det)$ belong to the same $G$-conjugacy class, either $\mu \cong \mu \otimes \chi \circ \det$ or $w(\mu) \cong \mu \otimes \chi \circ \det$. For the first case, we deduce that $\chi = 1$. For the second case, we have $w(\mu) = \mu^{-1}$. Let $\theta$ be a $k$-quasicharacter of $M$, such that $w(\mu) \cong \mu \otimes \theta$, we have $\theta(a, b) = q^{-\text{val}(b) + \text{val}(\alpha)} = q^{\text{val}(ab)}$, since $q^{-1} \equiv q \equiv -1 \pmod{\ell}$. Hence $\theta$ factors through determinant. We deduce that $|G(\pi)| = 2$. Since the Jordan-Hölder components of $i_{G M}^G \mu$ are two characters as well as an infinitely dimensional representation $\pi$, we conclude that the length of $i_{G M}^G \psi$ is 4.

**Method 2** ($p \neq 2$): Now we consider the types: $\pi$ contains $(\text{GL}_2(\sigma_F), \text{St})$, where $\text{St}$ is the cuspidal Steinberg representation of $\text{GL}_2(k_F)$, $\sigma_F$ is the ring of integers of $F$, and $k_F$ is the residue field of $F$. By Frobenius reciprocity, we have

$$\text{res}_{G}^{G} \pi \cong \text{ind}_{\text{SL}_2(\sigma_F)}^{\text{SL}_2(k_F)} \text{St}_{\text{SL}_2(k_F)} \otimes \text{ind}_{\alpha(\text{SL}_2(\sigma_F))}^{\text{SL}_2(k_F)} \alpha(\text{St}) \text{St}_{\text{SL}_2(k_F)},$$

(8)

where $\alpha = (\omega_F, 1) \in M$. We determine the length of $\text{St}_{\text{SL}_2(k_F)}$ by looking at the character table in [BonII] which requires $p \neq 2$.

When $\ell = 2$ (hence $p \neq 2$), by [BonII] §9.4.4], we know that $\text{St}_{\text{SL}_2(k_F)}$ is cuspidal and semisimple with length two, containing two irreducible components $\text{St}_{\pm}$ and $\text{St}_{\mp}$ (the same notation as in [BonII]), which implies that $(\text{SL}_2(\sigma_F), \text{St}_{\pm})$ are maximal simple $k$-types of $G'$ (see Definition 2.40). Hence the length $l_{G}(\pi_{G'}) = 4$, and the length of $i_{G M}^G \psi$ is 6.

It is worth noticing that when $\ell \neq 2$ and $p \neq 2$, the length of $\pi_{G'}$ can be computed through the same equation [BonII] by applying [BonII] §9.4.4], which coincides with the computation in Method 1.

We determine the length of $i_{G M}^G \psi$ by combining the two methods above.

**3.2 The index of $G(\pi)$**

The third part of Corollary 5.13 gives an equation between $l_{G}(\pi_{G'})\ell'$ and the cardinality $|G(\pi)|$, while the latter will be expressed through type theory in this section (Theorem 5.15), which will be applied to prove Theorem 3.19.

Let $(J, \lambda)$ be a $k$-maximal simple type of $G$, and $t(\lambda)$ the length of $\lambda|_{J'}$. We divide the characters in $G(\pi)$ into two parts: those which are non-trivial on $\sigma_F^\ell$ (Proposition 3.12), Corollary 5.14, and those which are unramified (Proposition 3.15), Corollary 5.10, and we compute the size of these two parts individually.

**Proposition 3.12.** Let $(J, \lambda)$ be a maximal simple $k$-type of $G$. There are $[(J : J)t(\lambda)\ell']$ distinct $k$-quasicharacters of the group $\det(J)$, such that $\lambda$ and $\lambda \otimes \chi \circ \det$ are weakly intertwine in $G$ (Definition 2.7).

**Proof.** Let $\mathcal{I}(\lambda)$ be the set of $k$-quasicharacters $\chi$ of $\det(J)$ such that $\lambda$ and $\lambda \otimes \chi \circ \det$ are weakly intertwined in $G$. By Proposition 2.9 let $\chi \in \mathcal{I}(\lambda)$, then there exists an element $x \in U(\lambda)$ such that $x$ normalises $J$ and $x(\lambda) \equiv \lambda \otimes \chi \circ \det$. Let $G(\lambda)$ consists with the $k$-quasicharacter $\chi$ such that $\lambda$ is isomorphic to $\lambda \otimes \chi \circ \det$, which is a subgroup of $\mathcal{I}(\lambda)$. We compute firstly the cardinality of $G(\lambda)$. Since $\text{ind}_{J'}^{G} \text{res}_{J}^{G} \lambda$ is a subrepresentation of $\text{res}_{G}^{G} \pi$, and the latter is multiplicity-free, we have that $\text{res}_{J}^{G} \lambda$ is multiplicity-free. Hence by Proposition 3.10 the cardinality of $G(\lambda)$ is equal to $t(\lambda)\ell'$. Proposition 2.16 and Corollary 2.11 imply that $|\mathcal{I}(\lambda) : G(\lambda)| = (J : J)t(\lambda)\ell'$. We conclude that $|\mathcal{I}(\lambda)| = (J : J)t(\lambda)\ell'$.

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Proposition 3.13. Let \((J, \lambda)\) be a maximal simple \(K\)-type in \(G\), and \(\mathfrak{A}\) the attached principal order (see Section 2.1). Let \(\chi\) be a \(k\)-quasicharacter of \(F^\times\).

- Let \(\pi\) be an irreducible \(k\)-representation of \(G\) containing \(\lambda\). Suppose that \(\pi \otimes \chi \circ \det \cong \pi\).

Then there exists \(x \in J(\lambda)\) such that \(x(\lambda) \cong \lambda \otimes \chi \circ \det\).

- If \(\lambda\) and \(\lambda \otimes \chi \circ \det\) are weakly intertwined in \(G\), then there exists an unramified \(k\)-quasicharacter \(\phi\) of \(F^\times\), such that for any irreducible \(k\)-representation \(\pi\) of \(G\) which contains \(\lambda\), we have \(\pi \cong \pi \otimes (\chi \phi) \circ \det\).

Proof. If \(\pi\) contains \(\lambda\) and \(\lambda \otimes \chi \circ \det\), they are intertwined in \(G\). By Proposition 2.9, there exists \(x \in J\) such that \(x(\lambda) \cong \lambda \otimes \chi \circ \det\).

For the second part, by assumption there exists \(x \in J\) such that \(x(\lambda) \cong \lambda \otimes \chi \circ \det\). Suppose that \(\Lambda\) is an extension of \(\lambda\) to \(E^\times J\) such that \(\text{ind}_{E^\times J}^G \chi \cong \pi\). We have \(\text{ind}_{E^\times J}^G \chi(\Lambda) \cong \chi\) and \(\text{res}^{E^\times J}_{J} \chi(\Lambda) \cong \chi \otimes \chi \circ \det\). Hence \(x(\Lambda)\) and \(\Lambda \otimes \chi \circ \det\) are two extensions of \(\lambda \otimes \chi \circ \det\). It is sufficient to construct an unramified \(k\)-quasicharacter \(\phi\) of \(F^\times\) such that \(\Lambda \otimes (\chi \phi) \circ \det \cong x(\Lambda)\).

Let \(i\) be the canonical isomorphism of \(F^\times\) to the centre of \(G\), and \(\chi_0\) the central character of \(\pi\). We have \(i(F^\times) \cap J \cong i(\sigma_F^\times)\), and \(x(\Lambda)|_{i(\sigma_F^\times)}\) is a multiple of \(\chi_0\). Meanwhile the restriction \(\Lambda \otimes \chi \circ \det|_{i(\sigma_F^\times)}\) is a multiple of \(\chi_0 \otimes \chi \circ \det\), which implies that \(\chi_0 \otimes \chi \circ \det|_{i(\sigma_F^\times)} \cong \chi_0\).

By modifying an unramified \(k\)-character \(\theta_1\) of \(F^\times\), we obtain an extension \(\Lambda \otimes (\chi \theta_1) \circ \det\) of \(\Lambda \otimes \chi \circ \det\), such that \(x(\Lambda)|_{F^\times J} \cong \Lambda \otimes (\chi \theta_1) \circ \det|_{F^\times J}\).

Furthermore, we have

\[
\Lambda \otimes (\chi \theta_1) \circ \det \mapsto \text{ind}^{E^\times J}_{F^\times J} x(\Lambda) |_{F^\times J} \cong x(\Lambda) \otimes \text{ind}^{E^\times J}_{F^\times J} 1.
\]

We have an equivalence \(E^\times J/F^\times J \cong E^\times /F^\times \sigma_F^\times\), which is a cyclic group of order \(e(E|F)\). The inclusion above implies that \(\Lambda \otimes (\chi \theta_1) \circ \det \cong x(\Lambda) \otimes \theta_2\), where \(\theta_2\) is a \(k\)-quasicharacter of \(E^\times /F^\times \sigma_F^\times\). We have that \(\theta_2\) factors through \(\det\). In fact, the intersection \(E^\times J \cap G'\) is equal to \(J'\). Hence \(\theta_2\) can be extended to an unramified \(k\)-quasicharacter of \(G\) which factors through \(\det\).

We conclude that \(\Lambda \otimes (\chi \theta_1 \theta_2^{-1}) \circ \det \cong x(\Lambda)\), hence \(\pi \otimes (\chi \theta_1 \theta_2^{-1}) \circ \det \cong \pi\), where by construction \(\theta_1 \theta_2^{-1}\) satisfies the condition for \(\phi\) required in the statement.

Corollary 3.14. Let \((J, \lambda)\) be a maximal simple \(k\)-type of \(G\). Let \(\pi\) be an irreducible \(k\)-representation containing \(\lambda\), and \(\mathcal{G}_0(\pi)\) a subset of \(\mathcal{G}(\pi)\) consisting of unramified \(k\)-quasicharacters of \(F^\times\). Then

\[
(\mathcal{G}(\pi) : \mathcal{G}_0(\pi)) = (\sigma_F^\times : \det(J))_\mu(J(\lambda) : J)t(\lambda)_\mu.
\]

Proof. First we construct a map

\[
\tau_1 : (\sigma_F^\times / \det(J))^\wedge \rightarrow \mathcal{G}(\pi)/\mathcal{G}_0(\pi).
\]

Let \(\chi \in (\sigma_F^\times / \det(J))^\wedge\), which belongs to \((F^\times)^\wedge\) such that \(\chi(\sigma_F) = 1\). We have \(\lambda \cong \lambda \otimes \chi \circ \det\). Consider \(\tau_1(\chi)\) as the image of \(\chi \phi \circ \det\) in \(\mathcal{G}(\pi)/\mathcal{G}_0(\pi)\), where \(\phi\) is an unramified \(k\)-quasicharacter of \(F^\times\) satisfying the requirement in Proposition 3.13. The map \(\tau_1\) is an injective group morphism. Now we consider

\[
\mathcal{G}_1 := (\mathcal{G}(\pi)/\mathcal{G}_0(\pi))/\tau_1((\sigma_F^\times / \det(J))^\wedge).
\]

Recall that \(\mathcal{I}(\lambda)\) be the set of \(k\)-quasicharacters \(\chi\) of \(\det(J)\), such that \(\lambda \otimes \chi \circ \det\) and \(\lambda\) are weakly intertwined in \(G\). We consider a map \(\tau_2 : \mathcal{I}(\lambda) \rightarrow \mathcal{G}_1\), which maps \(\chi\) to the image
of $\chi \circ \det$, then $\tau_2$ is a group isomorphism. In fact, the restriction to $\det(J)^\times$ induces a left inverse of $\tau_2$, which must be injective. On the other hand, the second statement of Proposition 3.13 implies that $\tau_2$ is surjective. We conclude that

$$(G(\pi) : G_0(\pi)) = |(\sigma_F / \det(J))^\times| \cdot |I(\lambda)|.$$ 

Proposition 3.12 implies the equation we desire.

Now we consider the set $G_0(\pi)$. In [MS §III.5.7], there is a support preserving isomorphism of Hecke algebras:

$$H(G, \lambda) \cong H_k(1, q_{p_E}^f),$$

where $q_{p_E}^f$ is the cardinal of the residual field of the field $E$. The algebra $H(1, q_{p_E}^f)$ is commutative, generated as a $k$-algebra by $[\zeta]$, where $[\zeta]$ corresponding to the characteristic function supported on $J p_E^{-1} J$ in $H(G, \lambda)$ and $p_E$ is a uniformiser of the field extension $E$ associated to the maximal simple $k$-type $(J, \lambda)$. Let $M$ be a simple $H(1, q_{p_E}^f)$-module, the action of $H(1, q_{p_E}^f)$ is uniquely defined by $[\zeta]$, which acts on $M$ by a scalar. Let $\alpha \in k$, define $M_\alpha$ the simple $H(1, q_{p_E}^f)$-module, which is isomorphic to $M$ as $k$-vector space through $i$, such that $[\zeta]i(m) = \alpha [\zeta]m$ for any $m \in M$.

Theorem 4.2 and Proposition 4.8 in [MS] gives a bijection between the isomorphism classes of irreducible $k$-representations $\pi$ of $G$ such that $\Hom_J(\lambda, \res_J \pi) \neq 0$, and the isomorphic classes of simple $H(G, \lambda)$-modules. We occupy the notations as [BMKII]. Denote by $M(\pi)$ the corresponding simple $H(1, q_{p_E}^f)$-module.

**Proposition 3.15.** Let $\pi$ be an irreducible $k$-representation of $G$ containing the maximal simple $k$-type $(J, \lambda)$. Fix a support-preserving algebra isomorphism $H(G, \lambda) \cong H(1, q_{p_E}^f)$, and write $M(\pi)$ for the simple $H(1, q_{p_E}^f)$-module corresponding to $\pi$. Let $\chi$ be an unramified $k$-quasicharacter of $F^\times$, and $p_E$ a uniformiser of $F$. Then

$$M(\pi \otimes \chi \circ \det) \cong M(\pi)_\alpha,$$

where $\alpha = \chi(p_E)^{n/e(E/F)}$.

**Proof.** By definition, we have an equivalence of $k$-vector spaces

$$M(\pi) = \Hom_G(\ind_J^G \lambda, \pi) \cong \Hom_G(\lambda, \res_J^G \pi \cong \res_J^G \pi \otimes \chi \circ \det) = M(\pi \otimes \chi \circ \det).$$

Let $i$ be the above canonical equivalence. We determine the action of $[\zeta]$ on $M(\pi \otimes \chi \circ \det)$. Let $V$ be the representation space of $\lambda$, and $i_v$ the function of value $v \in V$ at the element $1 \in G$. Denote by $[\zeta]$ the morphism in $\Hom_G(\ind_J^G \lambda, \ind_J^G \lambda)$, determined by $[\zeta]i_v = p_E i_v$.

Let $h \in \Hom_G(\ind_J^G \lambda, \pi)$, and $H$ the element in $\Hom_J(\lambda, \res_J^G \pi)$. We have

$$(h \circ [\zeta])(i_v) = h(p_E i_v) = \pi(\pi) h(i_v),$$

where $\pi(p_E) h(i_v) = a h(i_v)$, with a scalar $a \in k^\times$. In other words,

$$(H \circ [\zeta]) v = a H(v).$$

Let $\chi$ be an unramified $k$-quasicharacter of $F^\times$. We have

$$(i(H) \circ [\zeta])(i_v) = \chi(\det(p_E)) a \chi(H) i_v,$$

which implies that $M(\pi \otimes \chi \circ \det) = M(\pi)_{\chi(\det(p_E))}$. Meanwhile, we have an equation $\det(p_E) = p_E^{n/e(E/F)}$, which ends the proof. \qed
Corollary 3.16. Let \( \chi \) be an unramified \( k \)-quasicharacter of \( F^\times \). Then \( \chi \in \mathcal{G}(\pi)_0 \), if and only if \( \chi(p_F)^{n_E/E[F]} = 1 \).

Lemma 3.17. Let \( \mathfrak{A} \) be a hereditary \( \sigma_F \)-order in \( \text{End}_F(V) \) and \( \mathcal{R}(\mathfrak{A}) \) be the set of normaliser of \( \mathfrak{A} \) in \( G \), where \( V \) is a \( n \)-dimensional \( F \)-vector space. Suppose that \( E^\times \subset \mathcal{R}(\mathfrak{A}) \), then \( \mathcal{R}(\mathfrak{A}) \cap \mathcal{R}(\mathfrak{B}) = \mathcal{R}(\mathfrak{A})U(\mathfrak{A}) \).

Proof. Let \( \mathcal{L} \) be the \( \sigma_F \)-lattice chain corresponding to \( \mathfrak{A} \), and \( L_0, ..., L_{e-1} \in \mathcal{L} \), where \( e = e(\mathfrak{A}) \) (see Section 2.1). The group \( \mathcal{R}(\mathfrak{A}) \) is generated by \( F^\times \), \( U(\mathfrak{A}) \) and a family of elements \( g_1, ..., g_{e-1} \in G \) such that \( g_iL_0 = L_i \) for \( 1 \leq i \leq e - 1 \). Since the field extension \( E^\times \subset \mathcal{R}(\mathfrak{A}) \), by [BuKu §1.2.1] each element in \( \mathcal{L} \) is an \( \sigma_E \)-lattice. Let \( B = \text{End}_E(V) \), then \( \mathfrak{B} = B \cap \mathfrak{A} \) is the hereditary \( \sigma_E \)-order corresponding to \( \mathcal{L} \) (viewed as an \( \sigma_E \)-lattice chain). Hence there exists a family of elements \( b_i \in \mathcal{R}(\mathfrak{B}) \) such that \( b_i L_0 = L_i \) for \( 0 \leq i \leq e - 1 \), which implies that \( \mathcal{R}(\mathfrak{A}) \subset \mathcal{R}(\mathfrak{B})U(\mathfrak{A}) \). The inverse inclusion is trivial. \( \Box \)

In our case, since \( (J, \lambda) \) is maximal simple, we have

\[ \mathcal{R}(\mathfrak{A}) = E^\times U(\mathfrak{A}), \]

and

\[ \text{det}(\mathcal{R}(\mathfrak{A})) = p_F^{n_E/E[F]} \sigma_F^\times. \]

Corollary 8.10 above implies that \( \chi \in \mathcal{G}_0(\pi) \), if and only if \( \ker(\chi) \subset \text{det}(\mathcal{R}(\mathfrak{A})) \), which implies \( \mathcal{G}_0(\pi) \) is in bijection with \( (F^\times/\text{det}(\mathcal{R}(\mathfrak{A})))^\times \). We conclude that

\[ |\mathcal{G}_0(\pi)| = (F^\times : \text{det}(\mathcal{R}(\mathfrak{A})))^e. \]

Theorem 3.18. Let \( \pi \) be an irreducible cuspidal \( k \)-representation of \( G \). We have the equation

\[ |\mathcal{G}(\pi)| = (F^\times : \text{det}(\mathcal{R}(\mathfrak{B}))^e(J : J)t(\lambda)^e. \]

Proof. By Corollary 8.14 and Equation (9), we have

\[ |\mathcal{G}(\pi)| = (F^\times : \text{det}(\mathcal{R}(\mathfrak{A})))^e(U(\sigma_F) : \text{det}(J))^e(J : J)t(\lambda)^e. \]

By Definition §3.1.8 and 3.1.14 of [BuKu], we have \( U(\mathfrak{B}) \subset J \). Hence \( \text{det}(U(\mathfrak{A})) \cap \text{det}(E^\times J) \) is equal to \( \text{det}(J) \). Lemma 8.17 implies

\[ (U(\sigma_F) : \text{det}(J)) = (\text{det}(\mathcal{R}(\mathfrak{A})) : \mathcal{R}(\mathfrak{B})). \]

We rewrite Equation (10)

\[ |\mathcal{G}(\pi)| = (F^\times : \text{det}(\mathcal{R}(\mathfrak{B}))^e(J : J)t(\lambda)^e. \]

3.3 A decomposition of \( \tilde{J} \)

The following theorem is a generalisation of [BuKuII Theorem 4.1] when \( (J, \lambda) \) is maximal simple, which is applied in the proof of Theorem 3.24.

Theorem 3.19. Let \( (J, \lambda) \) be a simple type in \( G \), with \( \lambda = \kappa \otimes \sigma \). Let \( t(\lambda) \) to be the length of \( \lambda|_F \). Then
1. the representation $\lambda|_{\mathcal{F}}$ is multiplicity-free;

2. $\tilde{J} = J\tilde{J}$;

3. an irreducible component $\lambda'$ of $\lambda|_{\mathcal{F}}$ extends in $(\tilde{J} : J)$ distinct ways to $\tilde{J}$.

Proof. Part 1 follows from the proof of Proposition 3.12. Let $\pi$ be an irreducible cuspidal $k$-representation of $G$ containing $(J, \lambda)$. To prove 2 and 3, we need to compute the length of $\pi|_{\mathcal{F}}$. By Corollary 3.14 Part 3 and Theorem 3.18, we have $\ell g(\pi|_{\mathcal{F}}) = |G(\pi)|$.

Let $G_1 = \mathfrak{R}G'$. Since the restriction $\pi|_{\mathcal{F}}$ is multiplicity-free, so is $\pi|_{G_1}$. We have an inequality $\ell g(\pi|_{G_1}) \leq (G : G_1) = (F^\times : \det(\mathfrak{R}A))$. Now we prove that $\ell g(\pi|_{G_1})$ divides $(G : G_1)$. By Corollary 3.14 we have $H(\pi) = S(\pi) \subset T(\pi)$. The fact that $\pi \cong \text{ind}_{H(\pi)}^{G_1} \pi_1$ (see Remark 3.14), implies that $\pi|_{T(\pi)G_1}$ is semisimple with length $(G : T(\pi)G_1)$. For any irreducible component $\tau$ of the restriction $\pi|_{T(\pi)G_1}$, its restriction to $G_1$ is multiplicity-free (since $\pi|_{G_1}$ is multiplicity-free). Then the length of $\tau|_{G_1}$ divides $(T(\pi)G_1 : G_1)$, in fact if we put $T_0 = \{g \in T(\tau)G_1, x(\tau') = \tau'\}$ where $\tau'$ a direct factor of $\tau|_{G_1}$, then the length $\tau|_{G_1} = (T(\pi)G_1 : T_0)$. On the other hand, the direct components of $\pi|_{T(\pi)G_1}$ are $G$-conjugate, hence their length are equivalent after restricting to $G_1$. We conclude that $\ell g(\pi|_{G_1}) = (G : T(\pi)G_1)(T(\pi)G_1 : T_0)$, which divides $(G : G_1)$. We need the lemma below to continue the proof.

Lemma 3.20. We occupy the notations as before, then we have $(T(\pi)G_1 : G_1)_{\mathcal{F}} = \{1\}$.

Proof. In other words, we need to proof the cardinality of the quotient group $T(\pi)G_1/G_1$ is an $\ell$-power. It is equivalent to prove its $k$-dual is trivial. In fact, let $\chi \circ \det \in (T(\tau)G_1/G_1)^\vee$. Since $\chi$ is trivial on $\det(E^\times J)$, we have $\pi \otimes \chi \circ \det \cong \pi$. Then the kernel of $\chi \circ \det$ contains $T(\pi)$, hence it is trivial on the quotient $T(\pi)G_1/G_1$.

Continue the proof of Theorem 3.19. The lemma above implies an equation below:

$$\ell g(\pi|_{G_1})_{\mathcal{F}} = (G : T(\pi)G_1)_{\mathcal{F}} = (G : G_1)_{\mathcal{F}}.$$  

Let $\Lambda$ be an extension of $\lambda$ to $E^\times J$ which is contained in $\pi$. The irreducible induction $\pi_1 := \text{ind}^{G_1}_{E^\times J} \Lambda$ is an irreducible component of $\pi|_{G_1}$.

Now we need to know the length of the restriction $\pi_1|_{\mathcal{F}}$. We write:

$$T = \text{ind}^{G(\mathfrak{R}A)}_{E^\times J} \Lambda,$$

$$\tau = \text{ind}^{U(\mathfrak{R}A)}_J \lambda.$$

The representation $T$ is irreducible, and the following lemma will show that $\tau$ is irreducible as well.

Lemma 3.21. The induction $\tau = \text{ind}^{U(\mathfrak{R}A)}_J \lambda$ is irreducible.

Proof. The intertwining group $\mathcal{I}_{U(\mathfrak{R}A)}(\lambda)$ is equal to $J$. It is sufficient to prove that $\lambda$ verifies the second condition of irreducibility (Lemma 2.19). Let $\nu$ be an irreducible $k$-representation of $U(\mathfrak{R}A)$, and $\lambda \lambda$ a sub-representation of $\nu|_{J}$. We need to show that $\lambda$ is also a quotient of $\nu|_{J}$. By Frobenius reciprocity and the exactness of the restriction functor, there is a surjection

$$\text{res}^{U(\mathfrak{R}A)}_J \text{ind}^{U(\mathfrak{R}A)}_J \lambda \to \nu|_{J}.$$  

(13)

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We have
\[ \text{res}_J^{U(\mathfrak{g})} \text{ind}_J^{U(\mathfrak{g})} \lambda \cong \bigoplus_{\alpha \in J, U(\mathfrak{g})/J} \text{ind}_{J^\alpha(J)}^{J^\alpha(J)} \text{res}_{J^\alpha(J)}^{\alpha(J)} \alpha(\lambda). \]

By Corollary 2.10, when \( \alpha \neq 1 \), there is not irreducible sub-quotient of \( \text{ind}_{J^\alpha(J)}^{J^\alpha(J)} \text{res}_{J^\alpha(J)}^{\alpha(J)} \alpha(\lambda) \) which is isomorphic to \( \lambda \). Hence the kernel of the morphism in Equation 14 is contained in the sub-representation \( \bigoplus_{\alpha \neq 1} \text{ind}_{J^\alpha(J)}^{J^\alpha(J)} \text{res}_{J^\alpha(J)}^{\alpha(J)} \alpha(\lambda) \), which implies the existence of a surjection from \( \nu|_J \) to \( \lambda \).

Continue the proof of Theorem 3.19. In our case, we have \( \mathfrak{g}(\mathfrak{a}) = E^\times U(\mathfrak{g}) \) and \( E^\times U(\mathfrak{g}) \cap G' = U(\mathfrak{g}) \cap G' \), and since we also have \( T|_{U(\mathfrak{g})} = \tau \), we conclude that
\[ \text{res}_{G'}^{G_1} \tau_1 \cong \text{ind}_{U(\mathfrak{g}) \cap G'}^{G_1} \text{res}_{U(\mathfrak{g}) \cap G'}^{\mathfrak{g}} \tau. \]

From now till the end of this proof, we abbreviate \( U(\mathfrak{g}) \) as \( U \) and \( U(\mathfrak{g}) \cap G' \) as \( U' \). We have
\[ \text{ind}_{U'}^{U} \text{res}_U^{U'} \tau \cong \bigoplus_{x \in U/U'} x(\text{ind}_{J' = J \cap G'}^{J'} \text{res}_{J'}^{J} \lambda). \]

The cardinality of the index set \( J'/U/U' \) is equal to \( (U(\mathfrak{a}) : \det(J)) \). Let \( \ell(\lambda) \) be the length of \( \lambda_{J'} \), and \( \tilde{\lambda}' \) an irreducible component of \( \text{ind}_{J'}^{J} \lambda' \), then \( \text{ind}_{J'}^{J} \tilde{\lambda}' \) is irreducible by Theorem 2.29. Hence \( \text{lg}(\text{ind}_{J'}^{J} \text{res}_U^{U'} \tau) \) is equal to \( \text{lg}(\text{res}_U^{U'}) \) and \( \text{lg}(\text{ind}_{J'}^{J} \text{res}_U^{U'} \lambda) \) is equal to \( \text{lg}(\text{ind}_{J'}^{J} \text{res}_U^{U'} \lambda) \).

Recall that \( \tau(\lambda) \) is the length of \( \text{res}_U^{U'} \lambda \), and the irreducible components of the latter are \( J \)-conjugate. Let \( \tilde{\lambda}' \) be one of them. We have \( \text{lg}(\text{ind}_{J'}^{J} \text{res}_U^{U'} \lambda) \) is equal to \( \text{lg}(\text{ind}_{J'}^{J} \text{res}_U^{U'} \lambda) \).

Meanwhile, we have
\[ (J' : J') = \text{lg}(\text{res}_U^{U'} \text{ind}_{J'}^{J} \lambda'). \]

Notice that \( \text{ind}_{J'}^{J} \lambda' \) is a sub-representation of \( \text{res}_U^{U'} \text{ind}_{J'}^{J} \lambda' \), whose irreducible components are \( J \)-conjugate, hence the length of \( \text{res}_U^{U'} \lambda' \) is indepand of the choice of irreducible component \( \lambda' \). We then obtain that
\[ (J' : J') = \text{lg}(\text{ind}_{J'}^{J} \lambda') \text{lg}(\text{res}_U^{U'} \lambda'), \]

which implies \( \text{lg}(\text{ind}_{J'}^{J} \lambda') (J' : J) \). The equation holds if and only if \( \tilde{\lambda}' \) is an extension of a \( \lambda' \). We conclude that
\[ \text{lg}(\tau|_{U'})(U(\mathfrak{a}) : \det(J))(J' : J') \ell(\lambda). \]

Combining this with Equation 12, we have
\[ \text{lg}(\tau|_{U'})^e(F^\times : \det(\mathfrak{g}(\mathfrak{a})))^e (U(\mathfrak{a}) : \det(J))^e (J' : J') \ell(\lambda). \]

On the other hand, by Corollary 3.3 and Equation 12, we have
\[ \text{lg}(\tau|_{U'})^e = (F^\times : \det(\mathfrak{g}(\mathfrak{a})))^e (U(\mathfrak{a}) : \det(J))^e (J : J') \ell(\lambda). \]

Hence we deduce that \((J' : J') = (J : J))\), and \(J = J' J\). Then the analysis above implies that \( \text{lg}(\text{ind}_{J'}^{J} \lambda') \) is equal to \((J' : J)\), hence \( \tilde{\lambda}' \) is an extension of \( \lambda' \). The distinction property follows from \( \text{res}_U^{U'} \lambda \) being multiplicity-free by Proposition 2.37.

A similar decomposition of \( \tilde{J}_M \) can be deduced when \( M \) is a Levi subgroup, which will be used in the proof of Theorem 4.21.

**Corollary 3.22.** Let \( M = \prod_{i=1}^n \mathbb{G}_m \) be a Levi subgroup of \( G \), and \((J_M, \lambda_M)\) a maximal simple \( k \)-type of \( M \). Then we have \( \tilde{J}_M = \tilde{J}_M J_M \), and \( \tilde{J}_M \subset J_M^* \prod_{i=1}^n \tilde{J}_i \).

**Proof.** This can be deduced from Theorem 3.19 Part 2 in the same way as [GoRo §1.4.1, §1.4.2].
3.4 Intertwining implies conjugacy

Recall that $M$ is a Levi subgroup of $G$ and $M' = M \cap G'$ a Levi subgroup of $G'$. In this section, we prove the unicity property of weakly intertwining implying conjugacy (Theorem 3.24). In the $\ell$-modular setting, there is a difference between intertwining and weakly intertwining. The unicity property of simple characters of $M'$ will be considered (Theorem 3.29), which is a new result for both $\ell$-modular and complex setting.

This section contains two parts. We study maximal simple $k$-types in Proposition 3.23 and Theorem 3.24 (for complex setting see [BuKuII] for $G'$, and see [GoRo] for $M'$). We study simple $k$-characters since Theorem 3.24.

Proposition 3.23. Let $(J_M, \lambda_M)$ be a maximal simple $k$-type of $M$. Let $\tilde{\lambda}_1'$ and $\tilde{\lambda}_2'$ be two irreducible components of $\lambda_M|\tilde{J}_M$, where $\lambda_M = \text{ind}_{\tilde{J}_M}^{J_M} \lambda_M$. Then $\tilde{\lambda}_1'$ and $\tilde{\lambda}_2'$ are weakly intertwined in $M'$, if and only if they are conjugate in $M'$. In particular, if $M = G$, then $\tilde{\lambda}_1'$ and $\tilde{\lambda}_2'$ are distinct, and they are never weakly intertwined in $G'$.

Proof. If $x \in M'$ weakly intertwines $\tilde{\lambda}_1'$ with $\tilde{\lambda}_2'$, then $x$ weakly intertwines $\tilde{\lambda}_M$ with $\tilde{\lambda}_M \otimes \theta \circ \text{det}$ for a $k$-quasicharacter $\chi$ of $F^*$. By Mackey’s theory, we have

$$\text{res}_{J_M}^{\tilde{J}_M} \text{ind}_{J_M}^{\tilde{J}_M} \chi(J_M) \otimes \chi \circ \text{det} \cong \bigoplus_{\alpha, \beta} \text{ind}_{J_M \cap \alpha(x)(J_M)}^{J_M} \chi(J_M) \otimes \chi \circ \text{det},$$

where $\alpha \in J_M \setminus \tilde{J}_M/\tilde{J}_M \cap x(\tilde{J}_M)$ and $\beta \in \alpha(x)(J_M) \setminus \alpha(x)(\tilde{J}_M)/\alpha(x)(\tilde{J}_M)$. This implies that $\beta \alpha x$ weakly intertwines $\lambda_M$ with $\lambda_M \otimes \chi \circ \text{det}$ and by Proposition 2.3 there exists $g \in \tilde{J}_M$ such that $g(\lambda_M) \cong \lambda_M \otimes \chi \circ \text{det}$ and by Proposition 2.3 $\beta \alpha x g$ intertwines $\lambda_M$ to itself, which means $\beta \alpha x g \in E_M^* M$ where $E_M^* = E_{1}^* \times \cdots \times E_{n}^*$ and $M = \text{GL}_{n_1} \times \cdots \times \text{GL}_{n_m}$. We deduce that $x \in E_M^* \tilde{J}_M \cap M'$, which normalises $\tilde{J}_M$, hence $\tilde{\lambda}_1' \cong x(\tilde{\lambda}_2')$. In particular, when $M = G$, we have $E_M^* \tilde{J}_M \cap M' = \tilde{J}_M$, which implies that $\tilde{\lambda}_1' \cong \tilde{\lambda}_2'$.

Theorem 3.24. Let $(J_i, \lambda_i)$ be an maximal simple $k$-type of $M$, $(\tilde{J}_i, \tilde{\lambda}_i)$ be an irreducible component of $\lambda_i|\tilde{J}_i$ for $i = 1, 2$ respectively. Suppose that $(\tilde{J}_1, \tilde{\lambda}_1')$ are weakly intertwined in $M'$, then they are conjugate in $M'$.

Proof. If $x \in M'$ weakly intertwines $(\tilde{J}_1, \tilde{\lambda}_1')$ with $(\tilde{J}_2, \tilde{\lambda}_2')$, we can assume that (up to twist a $k$-quasicharacter of $F^*$ on $\tilde{\lambda}_2'$) $x$ weakly intertwines $(\tilde{J}_1, \tilde{\lambda}_1')$ with $(\tilde{J}_2, \tilde{\lambda}_2')$. As in the proof of Proposition 3.23 this implies that $\beta \alpha x$ weakly intertwines $\lambda_1$ with $\lambda_2$, for $\alpha \in \tilde{J}_1$ and $\beta \in \alpha(x)(\tilde{J}_2)$. Hence there is an element $g \in M$ such that $g(J_2) = J_1$ and $g(\lambda_2) \cong \lambda_1$ by Proposition IV, 1.6 2) in [V2]. Furthermore, $g^{-1} \beta \alpha x$ weakly intertwines $\lambda_2$ to itself, hence $g^{-1} \beta \alpha x$ intertwines $\lambda_2$ to itself. Then $g^{-1} \beta \alpha x \in E_M^* J_2$ (see the proof of Proposition 3.23 for $E_M^*$), and $x \in \tilde{J}_1 g E_M^* J_2 \cap M' = g E_M^* J_2 \cap M'$. By Corollary 3.23 we have $\tilde{J}_2 = \tilde{J}_2 J_2$. After adjusting an element in $E_M^* J_2$ to the right of $g$, we can assume that $g J_2 \cap M'$ is non-trivial, hence $g \in M'$. Then we have $g(J_2) = \tilde{J}_1$ and $g(\lambda_2) \cong \lambda_1$. Therefore $g(\lambda_2')$ is an irreducible component of $\tilde{\lambda}_1'|\tilde{J}_1'$, and $\tilde{\lambda}_1'$ is weakly intertwined with $g(\tilde{\lambda}_2')$ in $M'$. By Proposition 3.23 this ensures that $\tilde{\lambda}_1'$ and $\tilde{\lambda}_1'$ are conjugate in $M'$.

Definition 3.25. A simple $k$-character in $M'$ is a $k$-character of $H^{1'} = H^1 \cap M'$ of the form $\theta_M|H^{1'}$, where $(H^1, \theta_M)$ is a simple $k$-character in $M$.
The simple $k$-characters in $M$ is in bijection with the simple $K$-characters in $M$, and the simple $k$-characters of $M'$ are given by the former. The above two relations are the base to study simple $k$-characters.

**Theorem 3.26.** Let $\pi$ be an irreducible cuspidal $k$-representation of $G = \mathrm{GL}_n(F)$, and $(H^{m+1}, \theta_i)$ for $i = 1, 2$ be two simple $k$-characters contained in $\pi$, where $[A_i, n_i, m_i, \beta_i]$ be the corresponding simple strata such that $\theta_i \in \mathcal{C}(A_i, m_i, \beta_i)$. We have that $A_1 \cong A_2$ as $\mathfrak{o}_F$-hereditary orders, $m_i = 0$ and $(H^{m+1}, \theta_i)$ are $G$-conjugate.

**Proof.** We give a proof by passing to the complex setting, and we require the parallel result for complex representations in [BuKu]. For this reason, we need the following lemma on a property of lifting and reduction modulo $\ell$.

**Lemma 3.27.** Let $\pi$ be an irreducible $k$-representation of $G$ containing a simple $k$-character $(H^{m+1}, \theta)$. Let $\pi_K$ be an irreducible $K$-representation of $G$ which is $\ell$-integral, and its reduction modulo $\ell$ contains $\pi$ as a sub-quotient. Let $(H^{m+1}, \theta_K)$ be the $K$-lifting of $(H^{m+1}, \theta)$. Then $\theta_K$ is a direct component of $\pi_K|_{H^{m+1}}$.

**Proof.** Since $H^{m+1}$ is a pro-$p$ subgroup for $m \in \mathbb{N}$, the reduction modulo $\ell$ gives a bijection between the set of simple $K$-characters and the set of simple $k$-characters in $G$. Let $O_{W(k)}$ be a $W(k)[G]$-lattice contained in $\pi_K$. Since $H^{m+1}$ is a pro-$p$ compact subgroup of $G$, there is a surjection from $O_{W(k)}|_{H^{m+1}}$ to $\theta$. On the other hand, let $\theta_K$ be the $K$-lifting of $\theta$, and $O_0$ be a $W(k)[H^{m+1}]$-lattice inside $\theta_K$. The lattice $O_0$ is projective in the category of smooth $W(k)[H^{m+1}]$-modules. Hence there is a non-trivial morphism from $O_0$ to $O_{W(k)|_{H^{m+1}}}$, which is injective since $O_0$ is rank $1$ and $O_{W(k)}$ is torsion-free. We conclude that

$$\theta_K \cong O_0 \otimes_{W(k)} K \hookrightarrow O_{W(k)}|_{H^{m+1}} \otimes_{W(k)} K \cong \pi_K|_{H^{m+1}}$$

as we desired.

**Continue the proof of Theorem 3.26.** The result can be deduced from the lemma above and [BuKu] §8. As in [V1] §III.5.10, there is an irreducible $\ell$-integral cuspidal $\pi_K$, of which the reduction modulo $\ell$ is isomorphic to $\pi$. Let $\theta_i, K$ be the $K$-lifting of $\theta_i$ for $i = 1, 2$. According to [BuKu] §3, a simple $K$-character $\theta_i, K$ is defined over a simple stratum $[A_i, n_i, m_i, \beta_i]$. By the proof of Theorem 8.5.1 of [BuKu], if $m_i \neq 0$, then $\pi_K$ contains a split type, which contradicts with the cuspidality of $\pi_K$ according to [BuKu] 8.2.5,8.3.3. Since for both $i = 1, 2$ we have $m_i = 0$. Let $J(\beta_i)$ be as in Section 2. Then $\pi_K|_{J(\beta_i)}$ contains an irreducible $K$-representation $\tau_i$ such that $\tau_i|_{J(\beta_i)}$ contains the Heisenberg representation of $\theta_i$. Then $\tau$ must be of the form $\kappa_i \otimes \xi_i$, where $\kappa_i$ is a $\beta$-extension of $\theta_i$ and $\xi_i$ is inflated from an irreducible $K$-representation $\tau_i$ of $J(\beta_i)/J^1(\beta_i)$ which is isomorphic to a direct product of finite groups of $GL(k)$, $k_i \in \mathbb{N}$. If $\tau_i$ is not cuspidal, by the proof of [BuKu] Theorem 8.1.5], $\pi_K$ contains a split type or a simple type $(J(\beta_3), \lambda_3)$ defined over a simple stratum $[A_3, n_3, 0, \beta_3]$, such that $J(\beta_3)/J^1(\beta_3)$ is a proper Levi of a finite group of $GL(k_3)$. We have explain above the it is impossible to contain a split type. Since $\pi_K$ contains a maximal $K$-type, by [BuKu] Theorem 6.2.4] deduces a contradiction with the non-maximality of $(J(\beta_3), \lambda_3)$, which implies that $\tau_i$ are both cuspidal. Hence $(J(\beta_i), \kappa_i \otimes \xi_i)$ are both simple types. By [BuKu] Lemma 6.2.5, Theorem 6.2.4], we have $A_i \cong A_2$ as $\mathfrak{o}_F$-hereditary orders, and there exists $g \in G$ such that $g(J(\beta_1)) = J(\beta_2)$ and $g(\kappa_1 \otimes \xi_1) \cong \kappa_2 \otimes \xi_2$. Then we obtain that $g(\theta_i, K) \cong \theta_2, K$, which is equivalent to $g(\theta_1) \cong \theta_2$ after reduction modulo $\ell$. 

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Remark 3.28. 1. Let $\pi$ be irreducible and cuspidal of $G$. Let $(J_1, \lambda_1)$ and $(J_2, \lambda_2)$ be two maximal simple $k$-types appear as sub-quotients of $\pi$. The result of Theorem 3.29 combining with the parahoric restriction of of [V2 Corollary 8.4,(1)] gives another proof of [V2 Proposition IV.1.6,(2)].

2. Let $\pi_M$ be an irreducible cuspidal $k$-representation of $M$, a Levi subgroup of $G$, and $(H_{1,M}, \theta_{1,M})$ for $i = 1, 2$ be two simple $k$-characters contained in $\pi_M$. Let $\mathfrak{A}_{1,M}$ be the direct product of $\varphi_F$-hereditary orders in the definition of $\theta_{1,M}$. We deduce from Theorem 3.29 that $\mathfrak{A}_{1,M} \cong \mathfrak{A}_{2,M}$, and $(H_{1,M}, \theta_{1,M})$ are $M$-conjugate.

3. Let $\pi_M$ be as above, and $(J_{i,M}, \lambda_{i,M})$ be simple $k$-types of $M$ appears as sub-quotients of $\pi_M$ for $i = 1, 2$. Part 2 implies that they are maximal and belong to the same $M$-conjugacy class.

Now we move on to the case of $M'$ and prove the desired result, which is the unicity of simple $k$-characters contained in a fixed cuspidal $k$-representation of $M'$.

Theorem 3.29. Let $\pi'$ be an irreducible cuspidal $k$-representation of $M'$ which is a Levi subgroup of $G' = \text{SL}_n(F)$.

1. Let $i = 1, 2$. Assume that $(\tilde{J}_i', \tilde{X}_i')$ is a maximal simple $k$-type which appears as a sub-quotient of $\pi'|_{\tilde{J}_i'}$, then there exists $g \in M'$, such that $g(\tilde{J}_1') = \tilde{J}_2'$ and $g(\tilde{X}_1') \cong \tilde{X}_2'$.

2. Assume $(H_i', \theta_i')$ for $i = 1, 2$ be two simple $k$-characters contained in $\pi'$, then there exists $h \in M'$, such that $h(H_1') = H_2'$ and $h(\theta_1') \cong \theta_2'$.

Proof. For part 1. Since $\pi'$ must contains a maximal simple $k$-type $(\tilde{J}_i', \tilde{X}_i')$ as a sub-representation. By Frobenius reciprocity and the exactness of the restriction functor, there is a surjection from $\text{res}^{M'}_{\tilde{J}_1'} \text{ind}^{M'}_{\tilde{J}_1'} \tilde{X}_1'$ to $\text{res}^{M'}_{\tilde{J}_1'} \pi'$, which implies that $\tilde{X}_i'$ is weakly intertwined with $\tilde{X}_0'$ in $M'$ for $i = 1, 2$. They are conjugate in $M'$ by Theorem 3.24.

For part 2, our strategy is to prove that there exists $k$-simple type $(\tilde{J}_i', \tilde{X}_i')$ for $i = 1, 2$ of $M'$, which contains $\theta_i'$ and appears in $\pi'$ as a sub-quotient, then we obtain the result by applying part 1. Let $\pi$ be an irreducible cuspidal $k$-representation of $M$, whose restriction on $M'$ contains $\pi'$ as a sub-representation. Since there is an irreducible sub-quotient $\tau_i'$ of $\pi'|_{\tilde{J}_i'}$ such that $\tau_i'|_{H_i'}$ contains $\theta_i'$. By Lemma 2.14 there is an irreducible sub-quotient $\tau_i$ of $\pi|_{\tilde{J}_i}$ such that $\tau_i|_{H_i} \cong \tau_i'$. Hence $\tau_i|_{H_i}$ must contains an $k$-character $\theta_i$, furthermore $(H_i, \theta_i)$ is a simple $k$-character in $M$ whose restriction $H_i'$ is isomorphic to $\theta_i'$. There is an irreducible sub-quotient $\lambda_i$ of $\tau_i|_{J_i}$ whose restriction on $H_i$ contains $\theta_i$. By [V3 §III, 4.18], a maximal simple $K$-type is always $\ell$-integral, and its reduction modulo $\ell$ is a maximal simple $k$-type in $M$, which implies that the proof of Theorem 3.29 can be applied here, and we conclude that $(J_i, \lambda_i)$ is a maximal simple $k$-type in $M$. Since $\pi$ must contains a maximal simple $k$-type as a sub-representation, the first part of Remark 3.28 implies that $(J_i, \lambda_i)$ appears as a sub-representation of $\pi|_{J_i}$ for $i = 1, 2$. Applying [V3 Corollary 8.4], we know that $\lambda_i$ is a sub-representation of $\tau_i|_{J_i}$. Hence we deduced from Corollary 2.20 that $\tau_i \cong \text{ind}^{J_i}_{J_i'} \lambda_i = \tilde{\lambda}_i$. We conclude that $\tau_i' \cong \tilde{\lambda}_i|_{\tilde{J}_i'}$ is a maximal simple $k$-type in $M$. We end the proof by applying part 1.

□

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### 4 Extended maximal simple $k$-types of $\tilde{M}'$

Recall that $M$ is a Levi subgroup of $G$. Write $M \cong \text{GL}_{n_1} \times \cdots \times \text{GL}_{n_m}$, for $m \in \mathbb{Z}$. Let $(J_M, \lambda_M)$ be a maximal simple $k$-type of $M$, and $\tilde{J}_M$ the group of projective normalisers of $(J, \lambda)$. Let $\tilde{\lambda}_M'$ be an irreducible component of $\tilde{\lambda}_M = \text{ind}_{\tilde{J}_M}^{J_M} \lambda_M$, and $N_M(\tilde{\lambda}_M')$ be the normaliser of $\tilde{\lambda}_M'$ in $M'$. We have, as in Remark 2.44, $N_M(\tilde{\lambda}_M')$ is a subgroup of $E_M^\times \tilde{J}_M \cap M'$.

**Lemma 4.1.** Let $(J_M, \lambda_M)$ be as above. Then

1. $E_M^\times \tilde{J}_M \cap M'$ is compact modulo centre. Furthermore, the quotient $E_M^\times \tilde{J}_M \cap M'/N_M(\tilde{\lambda}_M')$ is finite and abelian.

2. Denote $\text{ind}_{E_M^\times \tilde{J}_M}^{E_M^\times J_M} \lambda_M$ by $\tilde{\lambda}_M$. Then $\tilde{\lambda}_M$ is an extension of $\tilde{\lambda}_M'$.

3. The group $N' = N_M(\tilde{\lambda}_M')$ is normal in $E_M^\times \tilde{J}_M$. In particular, $N'$ is independent of the choice of irreducible component $\tilde{\lambda}_M'$. Furthermore, for any irreducible $k$-representation of $N'$, if it contains $\tilde{\lambda}_M'$ as a sub-representation, then it is a multiple of $\tilde{\lambda}_M'$ after restricted to $J_M$.

4. An irreducible $k$-representation of $E_M^\times \tilde{J}_M$ that contains $\tilde{\lambda}_M$ is an extension of $\tilde{\lambda}_M$. In particular, an extension of $\tilde{\lambda}_M$ can be written as $\text{ind}_{E_M^\times \tilde{J}_M}^{E_M^\times J_M} \lambda_M$.

**Proof.** For Part 1, the property that $E_M^\times \tilde{J}_M \cap M'$ is compact modulo centre follows from the facts that $E_M^\times \tilde{J}_M'/Z_M$ is compact and that $E_M^\times \tilde{J}_M \cap M'/Z_M \cap M'$ is closed inside $E_M^\times \tilde{J}_M/Z_M$, where $Z_M$ is the centre of $M$. Recall that the centre $Z_M$ of $M'$ is equal to $Z_M \cap M'$. Since

$$E_M^\times \tilde{J}_M \cap M'/N_M(\tilde{\lambda}_M') \cong (E_M^\times \tilde{J}_M \cap M'/Z_M)/(N_M(\tilde{\lambda}_M')/Z_M),$$

combining with the fact that $N_M(\tilde{\lambda}_M')$ is open, we know that the quotient $E_M^\times \tilde{J}_M \cap M'/N_M(\tilde{\lambda}_M')$ is finite, which is abelian, since $E_M^\times \tilde{J}_M \cap M'/\tilde{J}_M' \subset E_M^\times \tilde{J}_M/\tilde{J}_M$, and the latter is abelian.

For Part 2, we consider $\text{res}_{E_M^\times \tilde{J}_M}^{E_M^\times J_M} \text{ind}_{E_M^\times \tilde{J}_M}^{E_M^\times J_M} \lambda_M$, and the result can be deduced by Mackey’s theory.

For Part 3, since $\prod_{i=1}^m \tilde{J}_i' \subset N'$, to show that $N'$ is normal in $E_M^\times \tilde{J}_M$, it is sufficient to show that the quotient $\prod_{i=1}^m E_M^\times \tilde{J}_i/\tilde{J}_i'$ is abelian, which is deduced from the fact that $E_M^\times \tilde{J}_i \cap \text{SL}_{n_i}(F) = \tilde{J}_i'$. Since the irreducible components of $\lambda_M|_{\tilde{J}_i'}$ are $\tilde{J}_M$-conjugate, the second part can be deduced from the first part. The last part comes from the definition of $N'$.

For Part 4, we start from the second part. Let $\tilde{\lambda}_0$ be an extension of $\tilde{\lambda}_M$ to $E_M^\times \tilde{J}_M$. We know that $\text{res}_{E_M^\times \tilde{J}_M}^{E_M^\times J_M} \tilde{\lambda}_0$ is semisimple of finite length, and write it as $\oplus_{s \in S} \lambda_s$, where $S$ is a finite index set. Since $\text{res}_{\tilde{J}_M}^{J_M} \tilde{\lambda}_0$ is semisimple of finite length, containing $\lambda_M$ as a sub-representation, and

$$\text{Hom}(\lambda_M, \bigoplus_{s \in S} \Lambda_s) \cong \bigoplus_{s \in S} \text{Hom}(\lambda_M, \Lambda_s),$$

we deduce that there is an $s_0 \in S$ such that $\text{res}_{\tilde{J}_M}^{J_M} \Lambda_{s_0}$ contains $\lambda_M$ as a sub-representation, hence $\Lambda_{s_0}$ is an extension of $\lambda_M$. Meanwhile $\Lambda_s$ are $E_M^\times \tilde{J}_M$-conjugate, hence the induction
ind_{E^\Pi_M J_M \Lambda} \Lambda_s is independent of the choice of \Lambda_s. Since \bar{\Lambda}_0 \mapsto \oplus_{s \in \text{ind}_{E^\Pi_M J_M \Lambda}} E^\Pi_M J_M \Lambda_s, we conclude that \bar{\Lambda}_0 \cong \text{ind}_{E^\Pi_M J_M \Lambda} \Lambda_s, where \Lambda_s is an extension of \Lambda_M. We deduce the result from Part 2.

For the first part, let \pi be an irreducible k-representation of \text{res}_{E^\Pi_M J_M} \tilde{\Lambda}_M and \bar{\Lambda}_M appears as a sub-representation. It is left to prove \pi is an extension of \bar{\Lambda}_M. Since \text{res}_{E^\Pi_M J_M} \tilde{\Lambda}_M is normal in \text{res}_{E^\Pi_M J_M} \pi, the restriction \text{res}_{E^\Pi_M J_M} \tilde{\Lambda}_M is semisimple and has finite length, and we write it as \oplus_{i \in I} \pi_i. For each \pi_i, there is a \kappa-quasicharacter \chi of \pi^\kappa such that \pi_i contains \lambda_M \otimes \chi \circ \text{det} as a sub-representation. In fact, by Frobenius reciprocity, there is a surjection

$$\text{res}_{E^\Pi_M J_M} \text{ind}_{J_M} \lambda_M \to \text{res}_{E^\Pi_M J_M} \pi_i.$$  

By Mackey's theory, \text{res}_{E^\Pi_M J_M} \text{ind}_{J_M} \lambda_M is semisimple of finite components \lambda_M \otimes \chi \circ \text{det} for a family of \kappa-quasicharacters \chi of \pi^\kappa. We deduce that \pi_i is an extension of \lambda_M \otimes \chi \circ \text{det}. Meanwhile, there is an injection

$$\pi \mapsto \oplus_{i \in I} \text{ind}_{E^\Pi_M J_M} \pi_i.$$  

By the unicity of Jordan-Hölder components, there is an \pi_i \in I such that \pi \cong \text{ind}_{E^\Pi_M J_M} \pi_i, and the latter is an extension of \bar{\Lambda}_M \otimes \chi \circ \text{det} by Part 2. Notice that \lambda_M \otimes \chi \circ \text{det} and \bar{\Lambda}_M are \tilde{J}_M-conjugate, hence \bar{\Lambda}_M \otimes \chi \circ \text{det} \cong \bar{\Lambda}_M.

**Proposition 4.2.** The restriction \text{res}_{E^\Pi_{N_M}(\bar{\Lambda}_M)} \text{ind}_{E^\Pi_M J_M} \lambda_M is semisimple and multiplicity-free. Furthermore, each irreducible component of \text{res}_{E^\Pi_{N_M}(\bar{\Lambda}_M)} \text{ind}_{E^\Pi_M J_M} \lambda_M is an extension of an irreducible component of \bar{\Lambda}_M|_{\tilde{J}_M}'.

**Proof.** In this proof, we denote \text{N}_{M'}(\bar{\Lambda}_M') by \text{N}'. Let \lambda_M be an extension of (J_M, \lambda_M) to \text{E}^\kappa_M J_M. By Frobenius reciprocity and Corollary 3.22 we have

$$\text{res}_{E^\Pi_M J_M} \text{ind}_{E^\Pi_M J_M} \lambda_M \cong \text{ind}_{E^\Pi_M J_M} \text{res}_{J_M} \lambda_M,$$

while the latter is semisimple and multiplicity-free. In fact

$$\text{ind}_{E^\Pi_M J_M} \lambda_M \cong \text{res}_{E^\Pi_M J_M} \lambda_M,$$

by Corollary 3.22 and the induced representation \text{ind}_{J_M} \lambda_M is irreducible by Corollary 2.20. Hence \text{res}_{E^\Pi_M J_M} \lambda_M is semisimple by Proposition 2.5. For the multiplicity-free part, first we consider the case when \text{M} = \text{G}. In this case \text{ind}_{E^\Pi_M J_M} \lambda_M is a sub-representation of \text{res}_{E^\Pi_M J_M} \lambda_M which is semisimple and multiplicity-free, hence the same for \text{res}_{E^\Pi_M J_M} \lambda_M, as well as a sub-representation of \text{res}_{E^\Pi_M J_M} \lambda_M. For Levi subgroup M, we write \text{J}_M = \prod_{i=1}^m J_i, \lambda_M = \prod_{i=1}^m \lambda_i where \text{(J}_i, \lambda_i) a maximal simple k-type of GL_{\lambda_i}(F). We deduce from the case of \text{G}, that \text{res}_{E^\Pi_M J_M} \lambda_M is semisimple and multiplicity-free. Since \tilde{J}_M \subset \prod_{i=1}^m J_i and \prod_{i=1}^m \tilde{J}_i \subset \bar{J}_M, we obtain the same property for \text{res}_{E^\Pi_M J_M} \lambda_M.

For the second part, we denote the irreducible induction \text{ind}_{E^\Pi_M J_M} \lambda_M by \bar{\Lambda}_M in this proof. The restriction \text{res}_{E^\Pi_{N'}} \lambda_M is semisimple of finite length, which is deduced from
the fact that $\text{res}_{E_M^\infty}^{\tilde{J}_M} \tilde{\lambda}_M$ is semisimple. We deduce that $\text{res}_{N'_M}^{E_M^\infty} \tilde{\Lambda}_M$ is semisimple of finite length by Lemma 4.1 Part 1 and Clifford theory. By Lemma 4.1 Part 2, we have $\text{res}_{J_M}^{E_M^\infty} \tilde{\Lambda}_M \cong \text{res}_{J_M}^{\tilde{J}_M} \tilde{\Lambda}_M$, which is semisimple and multiplicity-free as discussed above. On the other hand, the restriction to $\tilde{J}_M$ of an irreducible component of $\text{res}_{N'_M}^{E_M^\infty} \tilde{\lambda}_M$ contain an irreducible component of $\tilde{\lambda}_M|_{\tilde{J}_M}$. By Lemma 4.1 Part 3 and the fact that $\text{res}_{J_M}^{E_M^\infty} \text{ind}_{E_M^\infty}^{E_M^\infty} \tilde{\lambda}_M$ is multiplicity-free, its restriction must be irreducible, which ends the proof.

Lemma 4.3. Let $\tau'$ be an irreducible $k$-representation of $(E_M^\infty \tilde{J}_M)' = E_M^\infty \tilde{J}_M \cap M'$, then there is an irreducible $k$-representation $\tau$ of $E_M^\infty \tilde{J}_M$ such that $\tau'$ is isomorphic to a quotient of $\tau|_{E_M^\infty \tilde{J}_M \cap M'}$.

Proof. The proof of Proposition 2.2 of [13] can be applied here. The construction of $\tau$ will appear in the proof of Theorem 4.4 for the convenience reason we state the part that we need here. Let $S_M$ be a subgroup of the centre $Z_M$, consisting of elements whose coefficients on diagonal are powers of $p_F$, where $p_F$ is an uniformiser of $F$. Then $S_M \cong \mathbb{Z}^m$, where $M$ is a product of $m$-blocks. Denote $S_0 = S_M \cap (E_M^\infty \tilde{J}_M)'$ and $S_1$ a maximal subgroup of $S_0$ such that $S_1 \cap S_0 = \{1\}$, hence the rank of $S_1$ equals to $m$, and the quotient $E_M^\infty \tilde{J}_M/S_1(E_M^\infty \tilde{J}_M)'$ is compact. We extend $\tau'$ to $S_1(E_M^\infty \tilde{J}_M)'$ by acting $S_1$ trivially. The induction $\text{ind}_{S_1(E_M^\infty \tilde{J}_M)'}^{E_M^\infty \tilde{J}_M} \tau'$ is admissible, which has an irreducible sub-representation and denote it by $(V, \tau)$. Consider the map $f \rightarrow f(1)$ from $\tau$ to $\tau'$, which is a non-trivial morphism of $(E_M^\infty \tilde{J}_M)'$-representation. Hence $\tau'$ is a quotient of $\tau$.

Theorem 4.4. Let $\tau_{M'}$ be an irreducible $k$-representation of $N_{M'}(\tilde{\lambda}_M)$ containing an irreducible component $\tilde{\lambda}_M$ of $\tilde{\lambda}_M = \text{ind}_{J_M}^{\tilde{J}_M} \tilde{\lambda}_M$. Then $\tau_{M'}$ is an extension of $\tilde{\lambda}_M$.

Proof. Let $N'$ denote $N_{M'}(\tilde{\lambda}_M)$. The induced $k$-representation $\tau' = \text{ind}_{N'}^{E_M^\infty} \tau_{M'}$ is irreducible by Theorem 2.49. Let $\tau$ be as in Lemma 4.3 which is a sub-representation of $\text{ind}_{S_1(E_M^\infty \tilde{J}_M)'}^{E_M^\infty \tilde{J}_M} \tau'$, where $\tau'$ is extended to $S_1(E_M^\infty \tilde{J}_M)'$. By Mackey’s theory, we have

$$\text{res}_{J_M}^{E_M^\infty} \text{ind}_{S_1(E_M^\infty \tilde{J}_M)'}^{E_M^\infty} \tau' \cong \bigoplus_{\alpha \in S_1(E_M^\infty \tilde{J}_M)'} \text{ind}_{J_M}^{\tilde{J}_M} \text{res}_{J_M}^{E_M^\infty} \text{ind}_{S_1(E_M^\infty \tilde{J}_M)'}^{E_M^\infty} \alpha(\tau').$$

By Lemma 4.1 Part 3 and the fact that $E_M^\infty \tilde{J}_M$ normalises $\tilde{\lambda}_M$, we obtain that for any $\alpha$ as above, $\text{res}_{J_M}^{E_M^\infty} \alpha(\tau')$ is a multiple of an irreducible component $\tilde{\lambda}_M|_{J_M}$. Meanwhile, the induced representation $\text{ind}_{J_M}^{\tilde{J}_M} \tilde{\lambda}_M$ is a sub-representation of $\tilde{\lambda}_M \otimes \text{ind}_{J_M}^{\tilde{J}_M} \tilde{\lambda}_M \cong \text{ind}_{J_M}^{\tilde{J}_M} \tilde{\lambda}_M$. We deduce that, an irreducible sub-quotient of $\text{res}_{J_M}^{E_M^\infty} \text{ind}_{S_1(E_M^\infty \tilde{J}_M)'}^{E_M^\infty} \tau'$ is isomorphic to an irreducible component of $\tilde{\lambda}_M|_{J_M}$.

Now we come back to $\tau$. As in the proof of Lemma 2.13 the restriction $\text{res}_{J_M}^{E_M^\infty} \tau$ is embedding to a direct sum of finite length representations. Hence $\text{res}_{J_M}^{E_M^\infty} \tau$ has an irreducible sub-representation $\tau_0$. As we has explained in the previous paragraph, $\tau_0|_{J_M}$ contains an irreducible component of $\tilde{\lambda}_M|_{J_M}$, hence after twisting an $k$-quasicharacter of $F^\times$ to $\tilde{\lambda}_M$, we can assume that $\tau_0 \cong \tilde{\lambda}_M$ by Proposition 2.6. Then $\tau$ is an extension of $\tilde{\lambda}_M$ to $E_M^\infty \tilde{J}_M$ and is isomorphic to $\text{ind}_{E_M^\infty}^{E_M^\infty} \tilde{\lambda}_M$, for an extension $\Lambda_M$ of $\lambda_M$ by Lemma 4.1 (4).
other words, $\tau_{M'}$ is an irreducible component of $\text{res}^{E_{M'}}_{N'}\text{ind}^{E_{M'}}_{M}$. We apply Proposition 4.2 and obtain the result. 

Remark 4.5. In other words, Theorem 4.4 states that an extended maximal simple $k$-types defined in Definition 2.50 is an extension of a maximal simple $k$-type.

Theorem 4.6. Let $(N_i, \tau_i)$ be two extended maximal simple $k$-types of $M'$ for $i = 1, 2$. Assume that there exists an element $g \in M'$ such that $g$ weakly intertwines $\tau_1$ with $\tau_2$ then they are in the same $M'$-conjugacy class.

Proof. Let $(\tilde{J}', \tilde{\lambda}')$ be the maximal simple $k$-types contained in $(N_i, \tau_i)$ for $i = 1, 2$. The assumption implies that $\alpha g$ weakly intertwines $\tilde{\lambda}_1'$ to $\tilde{\lambda}_2'$, according to Theorem 3.24, they are $M'$-conjugate. Now up to conjugate $(N_2, \tau_2)$ by an element in $M'$, we can deduce the problem to the case where $\tilde{J}_1' = \tilde{J}_2'$ and $\tilde{\lambda}_1' = \tilde{\lambda}_2'$. Then there exists an element $\beta \in N_1$ such that $\beta g$ weakly intertwines $\tilde{\lambda}_1'$ to itself. By Lemma 2.43, $\beta g \in N_1$ hence $g \in N_1$, which implies that $\tau_2 \cong \tau_1$. \hfill $\square$

A Geometric lemma of Bernstein and Zelevinsky

We need [BeZe] Theorem 5.2 in the $\ell$-modular setting. In fact, the proof in [BeZe] is given by the theory of sheaves, which can be translated to a representation theoretical proof and be applied to our case. To be self-contained, we rewrite the proof following the same idea as [BeZe].

Let $G$ be a locally compact totally disconnected group, $P$, $M$, $U$, $Q$, $N$, $V$ are closed subgroups of $G$, and $\theta, \psi$ be $k$-characters of $U$ and $V$ respectively. Suppose that they verify conditions (1) – (4) in [BeZe] §5.1, and denote $P\backslash G$ by $X$. The numbering that we choose in condition (3) is $Z_1, \ldots, Z_k$ which are $Q$-orbits on $X$, and for an orbit $Z \subset X$, we choose $\varpi \in G$ and $\omega$ as in condition (3) of [BeZe].

We introduce condition ($\ast$):

(*) The characters $\omega(\theta)$ and $\psi$ coincide when restricted to $\omega(U) \cap V$.

We define $\Phi_2$ to be 0 if ($\ast$) does not hold, and define $\Phi_2$ as in [BeZe] §5.1 if ($\ast$) holds.

Definition A.1. Let $M, U$ be closed subgroups of $G$, and $M \cap U = \{e\}$, and the subgroup $P = MU$ is closed in $G$. Let $\theta$ be a $k$-character of $U$ normalised by $M$.

- Define functor $i_{U, \theta}: \text{Rep}_k(M) \to \text{Rep}_k(G)$, which maps $\rho \in \text{Rep}_k(M)$, to $\text{ind}^G_U \rho_{U, \theta}$, where $\rho_{U, \theta} \in \text{Rep}_k(P)$, such that
  \[ \rho_{U, \theta}(mu) = \theta(u)\text{mod}^M_U(m)\rho(m), \]
  and mod is the module, and $u \in U, m \in M$.

- Define functor $r_{U, \theta}: \text{Rep}_k(G) \to \text{Rep}_k(M)$, which maps $(\pi, W) \in \text{Rep}_k(G)$ to $\text{res}^G_U \pi | (\text{res}^G_U \pi)(U, \theta)$ of $(\text{res}^G_U \pi)(U, \theta)$ is generated by $\pi(u)w - \theta(u)w$, for $w \in W$.

Remark A.2. By replacing $\text{ind}$ to $\text{Ind}$, we have $i_{U, \theta}$. Notice that $r_{U, \theta}$ is the left adjoint to $I_{U, \theta}$.

Proposition A.3. The functors $i_{U, \theta}$ and $r_{U, \theta}$ commute with inductive limits.
Proof. The functor \( r_{U, \theta} \) commutes with inductive limits since it has a right adjoint as in A.2.

For \( i_{U, \theta} \), let \((\pi_\alpha, \alpha \in \mathcal{C})\) be an inductive system, where \( \mathcal{C} \) is a directed pre-ordered set. We prove that \( i_{U, \theta}(\lim_{\alpha} \pi_\alpha) \cong \lim_{\alpha} (i_{U, \theta} \pi_\alpha) \). The inductive limit \( \lim_{\alpha} \pi_\alpha \) is defined as \( \oplus_{\alpha \in \mathcal{C}} \pi_\alpha / \sim \), where \( \sim \) is an equivalent relation as below. When \( \alpha < \beta, x \in W_\alpha, y \in W_\beta, \ x \sim y \) if \( \phi_{\alpha, \beta}(x) = y \), where \( W_\alpha \) is the space of \( k \)-representation \( \pi_\alpha \), and \( \phi_{\alpha, \beta} \) is the morphism from \( \pi_\alpha \) to \( \pi_\beta \) in the inductive system.

First, we prove that \( i_{U, \theta} \) commutes with direct sum. By definition, \( \oplus_{\alpha \in \mathcal{C}} i_{U, \theta} \pi_\alpha \) is a sub-representation of \( i_{U, \theta} \oplus_{\alpha \in \mathcal{C}} \pi_\alpha \), and the natural embedding is a morphism of \( k \)-representations of \( G \). We will prove that the natural embedding is actually surjective. For any \( f \in \pi := i_{U, \theta} \oplus_{\alpha \in \mathcal{C}} \pi_\alpha \), there exists an open compact subgroup \( K \) of \( G \) such that \( f \) is constant on each right \( K \) coset of \( M \setminus G \). Furthermore, the function \( f \) is non-trivial on finitely many right \( K \) cosets. Hence there exists a finite index subset \( J \subset C \), such that \( f(g) \in \oplus_{j \in J} W_j \), which means \( f \in i_{U, \theta} \oplus_{j \in J} \pi_j \). Since \( i_{U, \theta} \) commutes with finite direct sum, we finish this case.

The functors \( i_{U, \theta} \) is exact, we have:

\[
i_{U, \theta}(\lim_{\alpha} \pi_\alpha) \cong i_{U, \theta}(\oplus_{\alpha \in \mathcal{C}} \pi_\alpha) / i_{U, \theta}(x - y) z \sim y.
\]

Notice that \( \lim_{\alpha} i_{U, \theta} \pi_\alpha \cong \oplus_{\alpha \in \mathcal{C}} i_{U, \theta} \pi_\alpha / \sim \), where \( \sim \) is the equivalent relation as below. When \( \alpha < \beta, f_\alpha \in V_\alpha, f_\beta \in V_\beta, \) where \( V_\alpha \) is the representation space of \( i_{U, \theta} \pi_\alpha \), then \( f_\alpha \sim f_\beta \) if \( i_{U, \theta}(\phi_{\alpha, \beta})(f_\alpha) = f_\beta \), which is equivalent to say that \( \phi_{\alpha, \beta}(f_\alpha(g)) = f_\beta(g) \) for any \( g \in G \). It is left to prove that the natural isomorphism from \( \oplus_{\alpha \in \mathcal{C}}(i_{U, \theta} \pi_\alpha) \) to \( i_{U, \theta}(\oplus_{\alpha \in \mathcal{C}} \pi_\alpha) \), induces an isomorphism from \( (f_\alpha - f_\beta)_{f_\alpha \sim f_\beta} \) to \( i_{U, \theta}(x - y) z \sim y \). This can be checked directly through definition.

Theorem A.4 (Bernstein, Zelevinsky). Under the conditions above, the functor \( F = r_{V, \psi} \circ i_{U, \theta} : \text{Rep}_k(M) \to \text{Rep}_k(N) \) is glued from the functor \( Z \) runs through all Q-orbits on \( X \). More precisely, if orbits are numerated so that all sets \( Y_i = \bigcup_{i = 1}^k \bigcup_{j = 1}^n \) are open in \( X \), then there exists a filtration \( 0 = F_0 \subset F_1 \subset \ldots \subset F_k = F \) such that \( F_i/F_{i-1} \cong \Phi_{Z_i} \).

The quotient space \( X = \mathbb{P} \setminus G \) is locally compact totally disconnected. Let \( Y \) be a \( Q \)-invariant open subset of \( X \). We define the subfunctor \( F_Y \subset F \). Let \( \rho \) be a \( k \)-representation of \( M \), and \( W \) be its representation space. We denote \( i(W) \) the representation \( k \)-space of \( i_{U, \theta}(\rho) \). Let \( i_Y(W) \subset i(W) \) the subspace consisting of functions which are equal to 0 outside the set \( P \cdot Y \), and \( \tau_Y \) be the \( k \)-representations of \( Q \) on \( i(W) \) and \( i_Y(W) \). Put \( F_Y(\rho) = r_{V, \psi}(\tau_Y) \), which is a \( k \)-representation of \( N \). The functor \( F_Y \) is a subfunctor of \( F \) due to the exactness of \( r_{V, \psi} \).

Proposition A.5. Let \( Y, Y' \) be two \( Q \)-invariant open subset in \( X \), we have:

\[
F_{Y \cap Y'} = F_Y \cap F_{Y'}, \quad F_{Y \cup Y'} = F_Y + F_{Y'}, \quad F_\emptyset = 0, \quad F_X = F.
\]

Proof. Since \( r_{V, \psi} \) is exact, it is sufficient to prove similar formulae for \( \tau_Y \). The only non-trivial one is the equality \( \tau_{Y \cup Y'} = F_Y + F_{Y'} \). As in §1.3 [BeZe], let \( K \) be a compact open subgroup of \( Y \cup Y' \), there exists \( \varphi \) and \( \varphi' \), which are idempotent \( k \)-function on \( Y \) and \( Y' \), such that \( (\varphi + \varphi')|_K = 1 \). We deduce the result from this fact.

Let \( Z \) be any \( Q \)-invariant locally closed set in \( X \), we define the functor

\[
\Phi_Z : \text{Rep}_k(M) \to \text{Rep}_k(N)
\]
to be the functor $F_{Y/Z}/F_Y$, where $Y$ can be any $Q$-invariant open set in $X$ such that $Y \cup Z$ is open and $Y \cap Z = \emptyset$. Let $Z_1, \ldots, Z_k$ be the numbering of $Q$-orbits on $X$ as in (A.3) and let $F_i = F_{Y_i}$ ($i = 1, \ldots, k$), which is a filtration of the functor $F$ be the definition. To prove Theorem (A.4) it is sufficient to prove that $F_{Z_i} \cong \Phi_{Z_i}$.

By replace $P$ to $\omega(P)$, we can assume that $\omega = 1$. Now we consider the diagram in figure BZ. This is the same diagram as in §5.7 [BeZe], in which a group $a$ point $H$ means $\text{Rep}_k(H)$, an arrow $H \mapsto$ means the functor $i_{H, \theta}$, an arrow $\subseteq$ means the functor $i_{H, \psi}$, and an arrow $\varepsilon$ means the functor $\varepsilon$ (consult §5.1 [BeZe] for the definition of $\varepsilon$). Notice that $G \twoheadrightarrow Q$ does not mean any functor, but the functor $P \rightarrow G \twoheadrightarrow Q$ is well-defined as explained above (A.3).

The composition functors along the highest path is $F_{Z_i}$, and if the condition $(\star)$ holds, the composition functors along the lowest path is $\Phi_{Z_i}$. We prove Theorem (A.4) by showing that this diagram is commutative if condition $(\star)$ holds, and $F_{Z_i}$ equals 0 otherwise. Notice that parts I, II, III, IV are four cases of (A.3) and we prove the statements through verifying them under the four cases respectively.

Let $\rho$ be any $k$-representation of $M$, and $W$ is its representation space. We use $\pi$ to denote $F_{Z_i}(\rho)$, and $\tau$ to denote $\Phi_{Z_i}(\rho)$.

Case I: $P = G, V = \{e\}$. The $k$-representations $\pi$ and $\tau$ act on the same space $W$, and the quotient group $M \backslash (P \cap Q)$ is isomorphic to $(M \cap Q) \backslash (P \cap Q)$. We verify directly by definition that $\pi \cong \tau$.

Case II: $P = G = Q$. The representation space of $\pi$ is still $W$. We have the equation:

$$r_{V, \psi}(W) \cong r_{V \cap M, \psi}(r_{V \cap U, \theta}(W)).$$

If $\theta|_{U \cap V} \neq \psi|_{U \cap V}$, then $\pi = 0$ since $U \cap V = U \cap Q \cap V \cap P$ and $r_{V \cap U, \theta}(W) = 0$. This means that after proving the diagrams of cases I, III, IV are commutative, the functor $F_{Z_i}$ equals 0 if condition $(\star)$ does not hold.

Now we assume that $(\star)$ holds. The $k$-representations $\pi$ and $\tau$ act on the same space $W/W(V \cap M, \psi)$, because the fact $r_{V \cap U, \theta}(r_{V \cap U, \theta}(W)) = W$ and the equation above. Notice that we have equations for $k$-character mod:

$$\text{mod}_U = \text{mod}_{U \cap M} \cdot \text{mod}_{U \cap V}, \quad \text{mod}_V = \text{mod}_{V \cap N} \cdot \text{mod}_{V \cap U},$$

from which we deduce that $\pi \cong \tau$ when condition $(\star)$ holds.
Figure 2: Case IV

Case III: $U = V = \{e\}$. Let $i(W)$ be the representation space of $i_o^\nu \rho$, then $\pi$ acts on a quotient space $W_1$ of $i(W)$. Let:

$$E = \{ f \in i(W) | f(PQ) = 0 \},$$

$$E' = \{ f \in i(W) | f(PQ) = 0 \},$$

then $W_1 \cong E/E'$. The $k$-representation $\tau$ acts on $i(W)'$, which is the representation space of $i_{P \cap Q}^\nu \rho$. By definition,

$$i(W)' = \{ h : Q \to W | h(pg) = \rho(p)h(q), p \in P \cap Q, q \in Q \}.$$

We define a morphism $\gamma$ from $W_1$ to $i(W)'$, by sending $f$ to $f|_Q$, which respects $Q$-actions and is actually a bijection. For injectivity, let $f_1, f_2 \in W_1$ and $f_1|_Q = f_2|_Q$, then $f_1 - f_2$ is trivial on $PQ$, hence $f_1 - f_2$ is trivial on $\overline{PQ}$ by the definition of $E$. This means $f_1 - f_2 = 0$ in $W_1$. Now we prove $\gamma$ is surjective. Let $h \in i(W)'$, there exists an open compact subgroup $K'$ of $(P \cap Q)\setminus Q$ such that $h$ is constant on the right $K'$ cosets of $(P \cap Q)\setminus Q$, and denote $S$ the compact support of $h$. Let $K$ be an open compact subgroup of $P \setminus G$ such that $(P \cap Q)\setminus (Q \cap K) \subset K'$, and $S \cdot K \cap \overline{(PQ)/PQ} = \emptyset$. We define $f$ such that $f$ is constant on the right $K$ cosets of $P \setminus G$, and $f|_{(P \cap Q)\setminus Q} = h$. The function $f$ is smooth with compact support on the complement of $\overline{PQ}/PQ$, hence belongs to $E$, and $\gamma(f) = h$ as desired.

Case IV: $U = \{e\}, Q = G$. We divide this case into two cases IV1 and IV2 as in the diagram of figure CaseIV.

Case IV1: $U = \{e\}, Q = G, V \subset M = P$. The $k$-representation $\pi$ acts on $i(W)^+ = i(W)/i(W)(V, \psi)$, where

$$i(W)(V, \psi) = \langle vf - \psi(v)f, \forall f \in i(W), v \in V \rangle.$$

The $k$-representations $\tau$ acts on $i(W)^+$, which is the smooth functions with compact support on $(M \cap N)\setminus N$ defined as below:

$$\{ h : N \to W/W(V, \psi)|f(\tau) = \rho(m)f(n), \forall m \in M \cap N, n \in N \}.$$

There is a surjective projection from $i(W)$ to $i(W)^+$, which projects $f(n)$ in $W^+ = W/W(V, \psi)$, for any $f \in i(W)$. In fact, let $h \in i(W)^+$, there exists an open compact subgroup $K$ of $P \setminus G \cong (M \cap N)\setminus N$, such that $f = \sum_{i=1}^m h_i$, $m \in N$, where $h_k \in i(W^+)$ is nontrivial on one right $K$ coset $a_kK$ of $P \setminus G$. We have $h_i \equiv \overline{w_i}$ on $a_iK$, where $w_i \in W$ and $\overline{w_i} \in W^+$. Define $f = \sum_{i=1}^m f_i$, where $f_i \equiv w_i$ on $a_iK$, and equals 0 otherwise. The function $f \in i(W)$, and the projection image is $h$.

It is clear that this projection induces a morphism from $i(W)^+$ to $i(W^+)$, and we prove this morphism is injective. Let $f, f' \in i(W)^+$, and $f = f'$ in $i(W^+)$. As in the proof
above, there exists an open compact subgroup \( K_0 \) of \( P \backslash G \), and \( f_j \in i(W)^+ \) such that \( f_j \) is non-trivial on one right \( K_0 \) coset of \( P \backslash G \) and \( f - f' = \sum_{j=1}^{s} f_j \). Furthermore, the supports of \( f_j \)'s are two-two disjoint. Hence the image of \( f_j \) on its support is contained in \( W^+ \), since \( f_j \) is constant on its support, it equals 0 in \( i(W)^+ \), whence \( f - f' \) equals 0 in \( i(W)^+ \). We conclude that this morphism is bijection, and the diagram case IV\(_1\) is commutative.

Case IV\(_2\): \( U = \{ e \} \), \( G = Q, N \subset M \). In this case:

\[
X = NV \backslash NV \cong V'V,
\]

where \( V' = V \cap M \). We choose one Haar measures \( \mu \) of \( X \) (the existence see §I, 2.8, [VI]). Let \( W^+ \) denote the quotient \( W/W(V', \psi) \) and \( p \) the canonical projection \( p : W \to W^+ \). Let \( i(W) \) be the space of \( k \)-representation \( \tau = i_{(e)}(\rho) \).

Define \( \mathcal{A} \) a morphism of \( k \)-vector spaces from \( i(W) \) to \( W^+ \) by:

\[
\mathcal{A}f = \int_{V \backslash V'} \psi^{-1}(v)p(f(v))d\mu(v).
\]

This is well defined since the function \( \psi^{-1}f \) is locally constant with compact support of \( V'/\mathbb{Q} \), and the integral is in fact a finite sum. Since \( \mu \) is stable by right translation, we have for any \( v \in V' \):

\[
\mathcal{A}(\tau(v, f)) = \psi(v)\mathcal{A}(f).
\]

Hence \( \mathcal{A} \) induces a morphism of \( k \)-vector spaces:

\[
A : i(W)/i(W)(V, \psi) \to W^+.
\]

Now we justify that \( A \in \text{Hom}_{k[\mathbb{Q}]}(\pi, \tau) \), where \( k \)-representations \( \pi = r_{V, \psi}(\tau) \) equals \( F(\rho) \), and \( \tau = \varepsilon_2 \cdot r_{V, \psi}(\rho) \) equals \( \Phi(\rho) \). For any \( n \in \mathbb{N} \):

\[
A(\pi(n)f) = \text{mod}_{V'}^N(n) \int_{V'/\mathbb{Q}} \psi^{-1}(v)p(f(vn))d\mu(v) \tag{15}
\]

\[
= \text{mod}_{V'}^N(n)\text{mod}_{V, V'}(n)^{-1} \cdot \int_{V'/\mathbb{Q}} \psi^{-1}(v)p(f(n^{-1}vn))d\mu(v) \tag{16}
\]

By replacing \( v' = n^{-1}vn \), the equation above equals to:

\[
\sigma(n) \int_{V'/\mathbb{Q}} \psi^{-1}(v')p(f(v'))d\mu(v') = \sigma(n)A(f).
\]

Therefore \( A \) belongs to \( \text{Hom}_{k[\mathbb{Q}]}(\pi, \tau) \), and hence a morphism from functor \( F \) to \( \Phi \). Now we prove that \( A \) is an isomorphism.

Let \( \rho' \) be the trivial representations of \( \{ e \} \) on \( W \), then \( i(W)' \) the space of \( k \)-representation \( \text{ind}_{V, \psi}^{V'}(\rho) \) is isomorphic to \( i(W) \) the space of \( k \)-representation \( i_{(e)}(\rho) \). Meanwhile, the diagram \( \square_{IV2} \) is commutative, where \( A \) indicates the morphism of \( k \)-vector spaces associated to the functor \( A \). Hence it is sufficient to suppose that \( N = \{ e \}, M = V' \). Replacing \( \rho \) by \( \psi^{-1}\rho \), we can suppose that \( \psi = 1 \).

First of all, we consider \( \rho = i_{(e)}(1) = \text{ind}_{V, \psi}^{V}1 \) the regular \( k \)-representation on \( S(V') \), which is the space of locally constant functions with compact support on \( V' \). Then \( \tau = i_{(e)}(\rho) \) is the regular \( k \)-representation of \( V \) on \( S(V) \) by the transitivity of induction functor. Any \( k \)-linear form on \( r_{V, 1}(S(V')) \) gives a Haar measure on \( V' \), and conversely any Haar measure gives a \( k \)-linear form on \( S(V') \), whose kernel is \( S(V')(V', 1) \), hence the two spaces is isomorphic,
and the uniqueness of Haar measures implies that the dimension of $r_{V',1}(S(V'))$ equals one. We obtain the same result to $r_{V,1}(S(V'))$. Since in this case the morphism $A$ is non-trivial, then it is an isomorphism. The functors $i_{(e),1}, r_{V,\psi}, r_{V',\psi}$ commute with direct sum (as in A.3), and the morphism $A$ between $k$-vector spaces also commutes with direct sum, hence $A : \pi \to \tau$ is an isomorphism when $\rho$ is free, which means $\rho$ is a direct sum of regular $k$-representations of $V'$. Notice that any $\rho$ can be viewed as a module over Heck algebra, then $\rho$ is a quotient of some free $k$-representation. Hence $\rho$ has a free resolution. The exactness of $F$ and $\Phi$ implies that $A : F(\rho) \to \Phi(\rho)$ is an isomorphism for any $\rho$.

References

[AKMSS] U. K. Anandavardhanan, R. Kurinczuk, N. Matringe, V. Sécherre et S. Stevens, Galois self-dual cuspidal types and Asai local factors. J. Eur. Math. Soc. 23 (2021), No. 9, 3129-3191.

[AMS] A.-M. Aubert; A. Moussaoui; M. Solleveld, Generalizations of the Springer correspondence and cuspidal Langlands parameters. Manuscripta Math. 157 (2018), no. 1-2, 121-192.

[BeZe] I.N. Bernstein; A.V. Zelevinsky, Induced representations of reductive $p$-adic groups. I. Ann. Sci. École Norm. Sup. (4) 10 (1977).

[Bon] C. Bonnafé, Sur les caractères des groupes réductifs finis à centre non connexe: applications aux groupes spéciaux linéaires et unitaires. Astérisque No. 306, 2006.

[BonII] C. Bonnafé, Representations of $\text{SL}_2(\mathbb{F}_q)$. Algebra and Applications, 13. Springer-Verlag London, (2011).

[BrMi] M. Broué; J. Michel, Blocs et séries de Lusztig dans un groupe réductif fini. J. Reine Angew. Math. 395 (1989), 56-67.

[BuHe] C.J. Bushnell; G. Henniart, Modular local Langlands correspondence for $\text{GL}_n$. Int. Math. Res. Not. 15 (2014), 4124-4145.

[BuKu] C.J. Bushnell, P.C. Kutzko, The admissible dual of $\text{GL}(N)$ via compact open subgroups. Annals of Mathematics Studies, 129. Princeton University Press (1993).

[BuKuI] C.J. Bushnell; P.C. Kutzko, The admissible dual of $\text{SL}(N)$. I. Ann. Sci. École Norm. Sup. (4) 26 (1993), no. 2, 261-280.

[BuKuII] C.J. Bushnell; P.C. Kutzko, The admissible dual of $\text{SL}(N)$. II. Proc. London Math. Soc. (3) 68 (1994), no. 2, 317-379.
[BuKuIII] C.J. Bushnell; P.C. Kutzko, Smooth representations of reductive $p$-adic groups: structure theory via types. Proc. London Math. Soc. (3) 68 (1994), no. 2, 317-379.

[BSS] P. Broussous; V. Sécherre; S. Stevens, Smooth representations of $GL_n(D)$ V: Endo-
classes. Doc. Math. 17 (2012), 23-77.

[C] P. Cui, Category decomposition of $\text{Rep}_k(\text{SL}_n(F))$. Journal of Algebra, Vol. 602 (2022), 130-153.

[CLH] P. Cui; T. Lanard; H. Lu, Modulo $\ell$ distinction problems for $GL_2$ and $SL_2$. arXiv:2203.14788.

[Da] J.-F. Dat, Simple subquotients of big parabolically induced representations of $p$-adic
groups, J. Algebra 510 (2018), 499-507, with an Appendix: Non-uniqueness of super-
cuspidal support for finite reductive groups by O. Dudas.

[DaII] J.-F. Dat, A functoriality principle for blocks of linear $p$-adic groups. Contemporary
Mathematics, 691 (2017), 103-131.

[DaIII] J.-F. Dat, $\nu$-tempered representations of $p$-adic groups. I. $\ell$-adic case. , Duke Math.
J. 126 (2005), no. 3, 397-469.

[DeLu] P. Deligne; G. Lusztig, Representations of reductive groups over finite fields. Ann.
of Math. (2) 103 (1976), no. 1, 103-161.

[DiFl] R. Dipper; P. Fleischmann, Modular Harish-Chandra theory. II Arch. Math. 62
(1994).

[DiMi] F. Digne; J. Michel, Representations of finite groups of Lie type. London Mathematical
Society Student Texts, 21. Cambridge University Press, Cambridge (1991).

[Fin] J.Fintzen, Tame cuspidal representations in non-defining characteristics.
arXiv:1905.06374.

[Geck] M. Geck, Modular Harish-Chandra series, Hecke algebras and (generalized) $q$-Schur
algebras. Modular representation theory of finite groups (Charlottesville, VA, 1998),
1966, de Gruyter, Berlin, 2001.

[GoRo] D. Goldberg; A. Roche, Types in $SL_n$. Proc. London Math. Soc. (3) 85 (2002), no.
1, 119-138.

[GrHi] J. Gruber; G. Hiss, Decomposition numbers of finite classical groups for linear
primes. J. Reine Angew. Math. 485 (1997), 55-91.

[HT] M. Harris and R. Taylor. The geometry and cohomology of some simple Shimura va-
rieties. Number 151 in Ann. of Math. studies. Princeton Univ. Press (2001).

[Helm] D. Helm, The Bernstein center of the category of smooth $W(k)[GL_n(F)]$-modules. Forum Math. Sigma 4 (2016).

[Hen] G. Henniart. Une preuve simple des conjectures de Langlands pour $GL(n)$ sur un corps
$p$-adique. Invent. Math., 139 (2000): 439-455.

[Hiss] G. Hiss, Supercuspidal representations of finite reductive groups. J. Algebra, 184
(1996), no. 3, 839-851.
[V3] M.-F. Vignéras, *Irreducible modular representations of a reductive p-adic group and simple modules for Hecke algebras*. European Congress of Mathematics, Vol. I (Barcelona, 2000), 117-133, Progr. Math., 201, Birkhäuser, (2001).

[V4] M.-F. Vignéras, *Correspondance de Langlands semi-simple pour GL(n, F) modulo ℓ ≠ p*. Invent. Math. 144 (2001), no. 1, 177-223.

[ Yu] J.-K. Yu, *Construction of tame supercuspidal representations*. J. Amer. Math. Soc. 14 (2001), no. 3, 579-622.

[Ze] A.V. Zelevinsky, *Induced representations of reductive p-adic groups. II. On irreducible representations of GL(n)*. Ann. Sci. École Norm. Sup. (4) 13 (1980), no. 2, 165-210.