Solution by numerical methods of the heat equation in engineering applications. A case of study: Cooling without the use of electricity

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Abstract. The physical model proposing the explanation of the phenomenon of cooling without electricity consists of a pot-in-pot refrigerator. This device consists of an external porous ceramic pot in which a smaller waterproof pot containing the material to be cooled. The space between the pots is filled with wet sand. The cooling effect in the device is produced by the evaporation of water, the heat transfer from the material to the sand and the influence of the air. An experimental model of the cooling effect investigates the influence of the main physical parameters and an analytical model of the heat transfer was proposed as a result of the experiments. In this article the analytical model of the heat transfer of a pot-pot refrigerator is solved by explicit and implicit numerical methods. The numerical solutions were compared with the analytical solution and it was observed that in the case of the explicit and implicit method, the results fit very well with the expected evolution system. Finally, a new way to continue the research is proposed.

1. Introduction
The growth of armed conflict in the region of Catatumbo have a negative effect in your economy [1]. This situation reflected in the high poverty rates indicates the need to propose solutions to this problem. The development of research around new technologies would be interesting to generate welfare in the inhabitants of Catatumbo, Colombia. In this direction the study of the electricity-free devises is important in places of poor economic development. The study of new cooling methods and their impact on different places is now a source of increasing interest [2-5].

The study of phenomena related to heat transfer in firing process has had a growing increase in the municipality of Ocaña, Colombia; this interest is directly related to the seminal work that some researchers are carrying out at Universidad Francisco de Paula Santander, Seccional Ocaña. A rigorous study of this situation can be found in the works [6,7]. An electricity-free cooling device consists of two pots as described in reference in [8]. The external pot is made of a porous material that houses another pot made of a water-permeable material. The inner pot contains the material to be cooled. The space between the pots is filled with wet sand. The cooling effect in the device is produced by the evaporation of water, the transmission of heat from the material to the sand and the influence of the air.
2. Methodology
In order to simplify the physical model of the pot-in-pot refrigerator, in [8] design an experiment on a set of long cylindrical containers which contain water or ethanol, simulating the material to be cooled. The cylinders are covered with paper towels that are moistened with a liquid.

In order to construct a theoretical model relevant to the study of the cooling process, in [8] use cylinders of greater height in relation to their diameter; in addition, the theoretical model under consideration is in two dimensions. The upper part of the cylinder is covered to ensure that the temperature gradient is radial and to reduce the convection effect inside the fluid. From now on, the pot-in-pot cooling model will be studied on a cylinder.

3. Mathematical model for the cooling without electricity
To study the transference phenomenon of the cooling process modeling in reference [8] is relevant to change the rectangular coordinate system to the heat equation to a system in polar coordinates, we follow the methodology of the reference [9]. The goal of this section is to generate the differential equation of heat in polar coordinates and to find an analytical solution associated to the problem related to [8]. In the first place, we will work in a general context, in which we will limit to the problem under study.

Be Ω a region in the space where the heat is flowing and T(x, y, z, t) the temperature function at the point (x, y, z) in the region Ω at the time t. Suppose that the region is homogeneous and is characterized by a specific heat c and a density ρ. Now let B be a ball contained in the region Ω. The amount of heat in small volume is ρcTdV; therefore, the amount of the calorific value contained in B is given by the triple integral see Equation (1).

\[ \int_B \rho c T dV \quad (1) \]

Then, the heat transfer function \( \phi(x, y, z, t) \) is introduced. The rate of change at which the flow crosses an oblique surface element dA in the direction of the unit vector \( \mathbf{n} \) see Equation (2):

\[ \phi \cdot \mathbf{n} \, dA \quad (2) \]

Consequently, the reason for the change in caloric energy crossing the B border, denoted by \( \partial(B) \), see Equation (3).

\[ \int \phi \cdot \mathbf{n} \, dA \quad (3) \]

Applying the law of conservation of energy, it is had (Equation (4)):

\[ \frac{d}{dt} \int_B \rho c T dV = - \int \phi \cdot \mathbf{n} \, dV \quad (4) \]

Applying the divergence theorem to the right side of Equation (4) the following is obtained, Equation (5):

\[ \frac{d}{dt} \int_B \rho c T dV = - \int_{\partial B} \text{div}(\phi) \, dV \quad (5) \]

By entering the differential in Equation (5) and applying energy balance, the following Equation (6) is generated:
ρcT_t = − div(ϕ) \quad (6)

It is noteworthy that Fourier's law states that \( ϕ = −K \nabla(T) \), where \( K \) is the thermal conductivity constant. Substituting in Equation (6) and applying the definition of \( \text{div} \) (Equation (7)),

\[ T_t = k(T_{xx} + T_{yy} + T_{zz}) \quad (7) \]

Where \( k = \frac{K}{\rho c} \) is the diffusion constant. Due to the selection of the model of the experiment and the simplifications of the cooling model proposed by [1], it is convenient to work in cylindrical coordinates. For this reason when replacing \( x = r \cos \theta, y = r \sin \theta, z = z \), in Equation (7), the following differential Equation (8) is generated:

\[ T_t = T_{rr} + \frac{1}{r}T_r + \frac{1}{r^2}T_{\theta\theta} + T_{zz} \quad (8) \]

3.1. Analytical solution of the heat diffusion equation for a solid cylinder

The analytical solutions of Equation (8) have been a line of research of broad study, a current work in that direction is found in [10]. Consider heat diffusion in a solid cylinder, diameter \( a \), height \( h \), density \( h \), specific heat \( c \), thermal conductivity \( \lambda \), initial temperature \( T_U \), placed in an environment at temperature \( T_V \). The Equation (8) of heat in transient state with radial diffusion, with the appropriate boundary conditions is Equation (9) to Equation (12).

\[ T_t = k(T_{rr} + \frac{1}{r}T_r) \quad (9) \]
\[ T(0, R) = T_0 \quad (10) \]
\[ \frac{∂T}{∂t}(r = 0, t) = T_1^k \quad (11) \]
\[ T(r, 0) = T_0, \quad 0 < r < R \quad (12) \]

The technique for solving the equation is to assume that \( T(r, t) = y(r)g(t) \), the equation decomposes into, Equation (13):

\[ \frac{g'(t)}{kg(t)} = \frac{y''(r) + \frac{1}{r}y'(r)}{y(r)} = −\lambda. \quad (13) \]

The solution for \( g(t) \) is solved in the usual way and \( g(t) = e^{-\lambda kt} \) is obtained. The solution for the equation in \( y \) is Equation (14).

\[ y''(r) + \frac{1}{r}y'(r) = −\lambda y(r) \quad (14) \]

Applying the boundary conditions, the following solutions are obtained Equation (15).

\[ T(r, t) = \sum_{n=1}^{\infty} c_n e^{-\lambda n^2 t} J_0 \left( \frac{zn}{R} \right) \quad (15) \]
The coefficients are obtained by applying the initial conditions. An approximation of Equation (9) to Equation (12) is generated by having the first term which is Equation (16).

\[
T(r, t) = c_t e^{-\lambda t} I_0 \left( \frac{2 \pi r}{R} \right)
\]  

(16)

Where the function \( f(t) = c_t e^{-\lambda t} \) represents the centerline temperature. This means that a suitable approach to studying the model of the temperature change inside the cylinder is to study the temperature in the center of the cylinder. Equation (9) can be written as Equation (17).

\[
f(t) = (T_1 - T_0) e^{-\frac{t}{\tau}}
\]

(17)

Where \( \tau = \frac{\Delta s^2}{2c} \). The Equation (17) is the analytical model associated to the model of the cooling process for the pot in pot refrigerator. In the next section we are working with a difference perspective. The complexity of Equation (17) and the simplicity and elegance of the numerical method in the context of heat transfer is the reason to study heat equation with finite difference. A complete study of this subject is found in reference [11].

4. Finite difference for the heat diffusion equation for a solid cylinder

The idea of finite difference method for the diffusion equation is related to replace the partial derivatives in the equation by their difference quotient approximations [12,13]. We are dealing with two differences scheme of solution of the Equation (9) to Equation (12).

4.1. An explicit scheme

The explicit scheme to solve the Equation (9) consists in replacing the derivatives of the function \( T(r, t) \) by the approximations that are generated by the use of Taylor's theorem, therefore we have Equation (18) and Equation (19):

\[
T_t(r_j, t_n) = \frac{T(r_{j,n+1}) - T(r_{j,n})}{\Delta t}
\]

(18)

\[
T_{rr}(r_j, t_n) = \frac{T(r_{j-1}, t_n) - 2T(r_{j,n}) + T(r_{j+1}, t_n)}{(\Delta r)^2}
\]

(19)

Replacing Equation (18) and Equation (19) in Equation (7) gives an equation in divided differences, Equation (20).

\[
\frac{T^{n+1}_j - T^n_j}{\Delta t} = k \frac{T^n_{j-1} - 2T^n_j + T^n_{j+1}}{(\Delta r)^2}
\]

(20)

The symbols \( T^{n+1}_j, T^n_1, T^n_{-1}, T^n_{+1} \) represent the temperature function \( T(r_{j,n+1}), T(r_{j,n}), T(r_{j,n-1}) \). The approximation generated by the Equation (20) to calculate the temperature function are Equation (21) to Equation (23):

\[
T^{k+1}_1 = T^k_1 + \frac{k \Delta t}{2 \pi \Delta r} \left[ T^k_0 - T^k_1 \right] + \frac{k \Delta t}{(\Delta r)^2} \left[ T^k_0 - 2T^k_1 + T^k_2 \right]
\]

(21)
\[ T_j^{k+1} = T_j^k + \frac{k \Delta t}{2\pi r} [T_{j+1}^k - T_{j-1}^k] + \frac{k \Delta t}{(\Delta r)^2} [T_{j+1}^k - 2T_j^k + T_{j-1}^k] \] (22)

\[ T_n^{k+1} = T_n^k + \frac{k \Delta t}{2\pi r} [T_{n+1}^k - T_{n-1}^k] + \frac{k \Delta t}{(\Delta r)^2} [T_{n+1}^k - 2T_n^k + T_{n-1}^k] \] (23)

We let \( s = \frac{k \Delta t}{(\Delta r)^2} \) and we arrive to the matrix equation for the vector unknown temperatures \( T^k \) for \( n=3 \) (Equation (24)).

\[
\begin{pmatrix}
1 - \frac{s}{2j} - s & \frac{s}{2j} + s & 0 \\
-s - \frac{s}{2j} & 1 - 2s & s + \frac{s}{2j} \\
0 & s - \frac{s}{2j} & 1 - 2s
\end{pmatrix}
\begin{pmatrix}
T_{1}^{k+1} \\
T_{2}^{k+1} \\
T_{3}^{k+1}
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
20(2s + s)
\end{pmatrix}
\] (24)

4.2. An implicit scheme

We want to derive an implicit scheme for the Equation (9), from the explicit scheme, we apply the following approximation, Equation (25):

\[
\frac{T_j^{n+1} - T_j^n}{\Delta t} = k \frac{T_{j+1}^{n+1} - 2T_j^{n+1} + T_{j-1}^{n+1}}{(\Delta r)^2}
\] (25)

Using the Equation (24) our discretized PDE becomes Equation (26) [9].

\[
T_j^{k+1} \left[ -s - \frac{s}{2j} \right] + T_j^{k+1} \left[ -s + \frac{s}{2j} \right] + T_j^{k+1} \left[ 1 + 2s \right] = T_j^k
\] (26)

Where the matrix equation for the vector unknown temperatures \( T^k \) for \( n = 3 \), Equation (27).

\[
\begin{pmatrix}
1 + \frac{s}{2j} + s & -\frac{s}{2j} - s & 0 \\
-s + \frac{s}{2j} & 1 + 2s & -s - \frac{s}{2j} \\
0 & -s - \frac{s}{2j} & 1 + 2s
\end{pmatrix}
\begin{pmatrix}
T_{1}^{k} \\
T_{2}^{k} \\
T_{3}^{k}
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
20\left( -\frac{s}{2j} - s \right)
\end{pmatrix}
\] (27)

5. Results

In order to validate the explicit and implicit schemes, consider a solid 30mm diameter PVC cylinder initially heated to a temperature of 52 ºC and placed in an air stream at a temperature of 20 ºC. For the above case of study, the analytical solution with the numerical solutions of Equation (9) generated by explicit and implicit numerical will be compared.

For the explicit and implicit method, we have written a computer code using the matrix Equation (25) for the explicit method and the matrix Equation (27) for the implicit method. After running the computer codes, both methods provide the same accuracy of the results with the analytical solution, which are in very good agreement. The graphs below show the curves of the temperature at the center of the cylinder within the explicit and the implicit method.
Figure 1. Temperature at the centre of the cylinder within the explicit method.

Figure 2. Temperature within the explicit method.

Figure 3. Temperature at the centre of the cylinder within the implicit method.

Figure 4. Temperature within the implicit method.

Figure 1 and Figure 3 explain the temperature evolution in the center of the cylinder described by the analytic Equation (17) and the numerical approximation of the explicit and implicit method. We observe that the results are consistent with the physical model described [11,12]. Figure 2 and Figure 3 explain the modeling of the heat transfer for different times, again the results are accurate with the physical model. The numerical methods, explicit and implicit, works well in this situation because the simplicity of boundaries condition of Equation (9) to Equation (12). When the model of pot in pot refrigerator is more realistic, thermodynamics phenomena will be included in consideration [14]. For example, the evaporation rate of the evaporation liquid and energy loss to environment are factors important in the modeling in the cooling process of pot in pot cooler. For this reason, is necessary to change the boundaries condition of Equation (9) to Equation (12).

Guerrero in references [6,7] proposes a model in relation with firing process in ladrillera Ocaña, Colombia. We believe that the application of the differential equation of firing process to free-cooler devises would be a good approach for next step of the study of pot in pot refrigerator.

6. Conclusion
The results of the application of the methods explicit and explicit applicate to the case of the model of pot in pot refrigerator proved a good accuracy as verified by the Figure 1 to Figure 4. The numerical approach used in this work was more natural that the analytical solution formulate in the article [8]. In order to continue with the next stage of the research, two possible lines of work are proposed. Modify
the differential Equation (9) to Equation (12) in its boundary and initial conditions to adapt more precisely to the experiment of the article of [8]. Note that if the geometry is modified for a truncated cone, the boundary conditions depend on the axial direction. The numerical methods exposed in this work would be easily extended for this more general situation.

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