Hairy black holes in theories with massive gravitons

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Abstract This is a brief survey of the known black hole solutions in the theories of ghost-free bigravity and massive gravity. Various black holes exist in these theories, in particular those supporting a massive graviton hair. However, it seems that solutions which could be astrophysically relevant are the same as in General Relativity, or very close to them. Therefore, the no-hair conjecture essentially applies, and so it would be hard to detect the graviton mass by observing black holes.

1 Black holes and the no-hair conjecture

More than 40 year ago J.A. Wheeler summarized the progress in the area of black hole physics at the time by his famous phrase: black holes have no hair [1]. More precisely, this means that

• All stationary black holes are completely characterized by their mass, angular momentum, and electric charge measurable from infinity.
• Black holes cannot support hair = external fields distributed close to the horizon but not seen from infinity.

Therefore, according to the ho-hair conjecture, the only allowed characteristics of stationary black holes are those associated with the Gauss law. The logic behind this is the following. Black holes are formed in the gravitational
collapse, which is so violent a process that it breaks all the usual conservation laws not related to the exact symmetries. For example, the chemical content, the baryon number, etc. are not conserved during the collapse – the black hole ‘swallows’ all the memory of them. Everything that can be absorbed by the black hole gets absorbed. Only few exact local symmetries, such as the local Lorentz or local U(1), can survive the gravitational collapse. Associated to them conserved quantities – the mass, angular momentum, and electric charge – cannot be absorbed by the black hole and remain attached to it as parameters. They give rise to the Gaussian fluxes that can be measured at infinity.

The no-hair conjecture essentially implies that the only asymptotically flat black holes in Nature should be those described by the Kerr-Newman solutions. And indeed, a number of the uniqueness theorems \[2, 3, 4\] confirm that all stationary and asymptotically flat electrovacuum black holes with a non-degenerate horizon should belong to the Kerr-Newman family.

The electrovacuum uniqueness theorems do not directly apply to systems with matter fields other than the electromagnetic field. The field equations for such systems read schematically

\[ G_{\mu\nu} = 8\pi G T_{\mu\nu}(\Psi), \quad \Box\Psi = V(\Psi), \quad (1) \]

where \(\Psi\) denotes the matter field, or several interacting matter fields. One can wonder if these equations admit asymptotically flat black hole solutions with the curvature bounded everywhere outside the black hole horizon. According to the no-hair conjecture, the answer should be negative, but to prove this requires considering each matter type separately. In view of this, a number of the no-hair theorems have been proven to confirm the absence of static black hole solutions of Eqs.(1) in the cases where \(\Psi\) denotes scalar, spinor, etc. fields \[5, 6, 7, 8, 9, 10\]. The common feature in all these cases is that if \(\Psi\) does not vanish, then the field equations require that it should diverge at the black hole horizon, where the curvature diverges too. Therefore, to get regular black holes one is bound to set \(\Psi = 0\), but then the solution is a vacuum black hole belonging to the Kerr-Newman family\(^1\). All this confirms the non-existence of hairy black holes.

The first explicit evidence against the no-hair conjecture was found 20 years after its formulation, in the context of the Einstein-Yang-Mills theory with gauge group SU(2). This theory contains all the electrovacuum solutions, hence all Kerr-Newman black holes \[13\], because the electromagnetic U(1) gauge group is contained in SU(2). However, it also admits static black holes supporting a non-trivial Yang-Mills field which asymptotically decays as \(1/r^3\), so that the corresponding Gaussian flux is zero \[14, 15\]. Close to the horizon the geometry deviates from the Schwarzschild one, but the deviations rapidly

\(^1\) It has recently been shown that these arguments can be circumvented for fine-tuned black hole mass and angular momentum \[11\]. This allows one to construct spinning hairy black holes which do not admit a static limit \[12\].
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Hairy black holes decay with distance and cannot be seen from infinity. Therefore, such black holes support a hair.

Subsequent developments have revealed that the Einstein-Yang-Mills black holes can be generalized to include scalar fields, as for example a Higgs field, which leads to a variety of new solutions describing hairy black holes [16]. In particular, it turns out that regular gravitating solitons, as for example gravitating magnetic monopoles or gravitating Skyrmions, can be generalized to contain inside a small black hole. This gives rise to black holes with a 'solitonic hair'. However, when the black hole size exceeds a certain critical value, the black hole 'swallows the soliton' and 'looses its hair', becoming a Kerr-Newman black hole [16].

Yet more hairy black holes can be obtained in models inspired by string theory and including a dilaton [17], the curvature corrections and so on [16]. Adding a cosmological term, positive or negative, gives asymptotically (anti)-de Sitter hairy black holes [18]. Summarizing, one can say that hairy black holes arise generically in physical models. However, large hairy black holes are typically unstable and loose the hair when perturbed, whereas the stable ones are typically very small [16]. As a result, despite a large number of solutions describing hairy black holes in various systems, it seems that the no-hair conjecture essentially holds for the astrophysical black holes, all of which should be of the Kerr-Newman type.

In what follows we shall be considering black holes in theories with massive gravitons – the ghost-free bigravity and massive gravity. Some of these black holes are of the known Kerr-Newman(-de Sitter) type, but there are also black holes supporting a massive graviton hair. However, the hairy black holes turn out to be either asymptotically anti-de Sitter (AdS), or cosmologically large, which contradicts the observations. Therefore, the astrophysical black holes should be described by the Kerr-Newman(-de Sitter) metrics, possibly with small corrections in the near-horizon region, so that the no-hair conjecture essentially holds.

2 Theories with massive gravitons

The idea that gravitons could have a tiny mass was proposed long ago [19], but it attracted a particular interest after the recent discovery of the special massive gravity theory by de Rham, Gabadadze, and Tolley (dRGT) [20] (see [21],[22] for a review). Before this discovery it had been known that the massive gravity theory generically had six propagating degrees of freedom (DoF). Five of them could be associated with the polarizations of the massive graviton, while the sixth one, usually called Boulware-Deser (BD) ghost, is unphysical, because it has a negative kinetic energy and renders the whole theory unstable [23]. The specialty of the dRGT theory is that it contains two Hamiltonian constraints which eliminate one of the six DoF
Therefore, there remain just the right number of DoF to describe massive gravitons and so the theory is referred to as ghost-free. This does not mean that all solutions are stable in this theory, since there could be other instabilities, which should be checked in each particular case. However, since the most dangerous BD ghost instability is absent, the theory of [20] and its bigravity generalization [29] can be considered as healthy physical models for interpreting the observational data.

These theories can be used to explain the current cosmic acceleration [30, 31]. This acceleration could be accounted for by introducing a cosmological term in Einstein equations, however, this would pose the problem of explaining the origin and value of this term. An alternative possibility is to consider modifications of General Relativity (GR), and theories with massive gravitons are natural candidates for this, since the graviton mass can effectively manifest itself as a small cosmological term [32].

Theories with massive gravitons are described by two metrics, $g_{\mu\nu}$ and $f_{\mu\nu}$. In massive gravity theories the f-metric is non-dynamical and is usually chosen to be flat, although other choices are also possible, while the dynamical g-metric describes massive gravitons. In bigravity theories [29] both metrics are dynamical and describe together two gravitons, one massive and one massless. The theory contains two gravitational couplings, $\kappa_g$ and $\kappa_f$, and in the $\kappa_f \rightarrow 0$ limit the f-metric decouples and can be chosen to be flat. Therefore, the bigravity theory is more general, while the massive gravity theory can be viewed as its special case.

All known bigravity black holes were obtained in Ref. [33] (see also [34]), with the exception of special solutions discovered in Ref. [35]. These black holes can be divided into three types. First, there are solutions for which the two metrics are proportional, $f_{\mu\nu} = C^2 g_{\mu\nu}$ with a constant $C$, where $g_{\mu\nu}$ fulfills the Einstein equations with a cosmological term $\Lambda(C) \propto m^2$. If $C = 1$ then $\Lambda = 0$ and one obtains all solutions of the vacuum GR, in particular the vacuum black holes. For other values of $C$ one has $\Lambda(C) \neq 0$, which gives rise to black holes with a cosmological term. None of these solutions fulfill equations of the massive gravity theory with a flat $f$.

Secondly, imposing spherical symmetry, there are black holes described by two metrics which are not simultaneously diagonal. They formally decouple one from the other and each of them fulfills its own set of Einstein equations with its own cosmological term. The g-metric is Schwarzschild-de Sitter, whereas the f-metric can be chosen to be AdS, with $\Lambda_f \sim \kappa_f^2$, and it becomes flat when $\kappa_f \rightarrow 0$, in which limit the dRGT massive gravity is naturally recovered. Therefore, these solutions exist both in the bigravity and dRGT massive gravity theories. In the latter case they exhaust all known black hole solutions.

Solutions of the third type are obtained when the two metrics are both diagonal but not proportional. One obtains in this case more complex solutions describing static black holes with a massive graviton hair, which can be
either asymptotically AdS [33], or asymptotically flat [35], although in the latter case their size should be comparable with the Hubble radius.

A more detailed description of the currently known bigravity and massive gravity black holes is given below.

### 3 Ghost-free bigravity

The theory of the ghost-free bigravity [29] is defined on a four-dimensional spacetime manifold equipped with two metrics, $g_{\mu\nu}$ and $f_{\mu\nu}$, which describe two interacting gravitons, one massive and one massless. The kinetic term for each metric is chosen to be of the standard Einstein-Hilbert form, while the interaction between them is described by a local potential $\mathcal{U}[g, f]$ which does not contain derivatives and is expressed by a scalar function of the tensor $\gamma^\mu{}_{\nu} = \sqrt{g^{\mu\alpha}f_{\alpha\nu}}$.

$$\gamma^\mu{}_{\nu} = \sqrt{g^{\mu\alpha}f_{\alpha\nu}}. \quad (2)$$

Here $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$ and the square root is understood in the matrix sense, i.e.

$$(\gamma^2)^\mu{}_{\nu} \equiv \gamma^\mu{}_{\alpha}\gamma^\alpha{}_{\nu} = g^{\mu\alpha}f_{\alpha\nu}. \quad (3)$$

The action is (with the metric signature $-+++$)

$$S[g, f] = \frac{1}{2\kappa_g^2} \int d^4x \sqrt{-g} R(g) + \frac{1}{2\kappa_f^2} \int d^4x \sqrt{-f} R(f) - \frac{m^2}{\kappa^2} \int d^4x \sqrt{-g}\mathcal{U}[g, f], \quad (4)$$

where $R$ and $\mathcal{R}$ are the Ricci scalars for $g_{\mu\nu}$ and $f_{\mu\nu}$, respectively, $\kappa_g^2 = 8\pi G$ and $\kappa_f^2 = 8\pi G$ are the corresponding gravitational couplings, while $\kappa^2 = \kappa_g^2 + \kappa_f^2$ and $m$ is the graviton mass. The interaction between the two metrics is given by

$$\mathcal{U} = \sum_{k=0}^{4} b_k \mathcal{U}_k(\gamma), \quad (5)$$

where $b_k$ are parameters, while $\mathcal{U}_k(\gamma)$ are defined by the relations

$$\mathcal{U}_0(\gamma) = 1, \quad \mathcal{U}_1(\gamma) = \sum_A \lambda_A = [\gamma],$$

$$\mathcal{U}_2(\gamma) = \sum_{A<B} \lambda_A \lambda_B = \frac{1}{2!} ([\gamma]^2 - [\gamma^2]),$$

$$\mathcal{U}_3(\gamma) = \sum_{A<B<C} \lambda_A \lambda_B \lambda_C = \frac{1}{3!} ([\gamma]^3 - 3[\gamma][\gamma^2] + 2[\gamma^3]).$$
\[ U_4(\gamma) = \lambda_0\lambda_1\lambda_2\lambda_3 = \frac{1}{4\pi} ([\gamma]^4 - 6[\gamma]^2[\gamma^2] + 8[\gamma][\gamma^3] + 3[\gamma^2]^2 - 6[\gamma^4]). \] (6)

Here \( \lambda_A \) (\( A = 0, 1, 2, 3 \)) are the eigenvalues of \( \gamma^\mu_\nu \), and, using the hat to denote matrices, one has defined \( [\gamma] = \text{tr} (\hat{\gamma}) \equiv \gamma^\mu_\mu, [\gamma^k] = \text{tr} (\hat{\gamma}^k) \equiv (\gamma^k)^\mu_\mu \). The (real) parameters \( b_k \) could be arbitrary, however, if one requires flat space to be a solution of the theory, and \( m \) to be the Fierz-Pauli mass of the graviton [19], then the five \( b_k \)'s are expressed in terms of two free parameters \( c_3, c_4 \) as follows:

\[
\begin{align*}
  b_0 &= 4c_3 + c_4 - 6, \\
  b_1 &= 3 - 3c_3 - c_4, \\
  b_2 &= 2c_3 + c_4 - 1, \\
  b_3 &= -(c_3 + c_4), \\
  b_4 &= c_4. 
\end{align*}
\] (7)

The theory (4) propagates \( 7=5+2 \) Dof corresponding to the polarizations of two gravitons, one massive and one massless. Before this theory was discovered [29], more general bigravity models, sometimes called f-g theories, had been considered [36]. In these models the potential \( U \) is a scalar function of \( H^\mu_\nu = \delta^\mu_\nu - g^{\alpha\nu}f_{\alpha\nu} \) of the form

\[ U = \frac{1}{8} (H^\mu_\nu H^\nu_\mu - (H^\mu_\mu)^2) + \ldots, \] (8)

where the dots denote all possible higher order scalars made of \( H^\mu_\nu \). A particular choice of these terms leads to (5). The generic f-g theories propagate \( 7+1 \) Dof, the additional one being the BD ghost [23].

Introducing the mixing angle \( \eta \) such that \( \kappa_g = \kappa \cos \eta, \kappa_f = \kappa \sin \eta \) and varying the action (4) gives the field equations

\[
\begin{align*}
  G^\mu_\nu &= m^2 \cos^2 \eta T^\mu_\nu, \\
  G^\mu_\nu &= m^2 \sin^2 \eta T^\mu_\nu, 
\end{align*}
\] (9, 10)

where \( G^\mu_\nu \) and \( G^\mu_\nu \) are the Einstein tensors for \( g^\mu_\nu \) and \( f^\mu_\nu \). The graviton energy-momentum tensors obtained by varying the interaction \( U \) are

\[
\begin{align*}
  T^\mu_\nu &= \tau^\mu_\nu - U \delta^\mu_\nu, \\
  T^\mu_\nu &= -\sqrt{-g} \sqrt{f} \tau^\mu_\nu, 
\end{align*}
\] (11)

where

\[
\begin{align*}
  \tau^\mu_\nu &= \{ b_1 U_0 + b_2 U_1 + b_3 U_2 + b_4 U_3 \} \gamma^\mu_\nu \\
  &- \{ b_2 U_0 + b_3 U_1 + b_4 U_2 \} (\gamma^2)^\mu_\nu \\
  &+ \{ b_3 U_0 + b_4 U_1 \} (\gamma^3)^\mu_\nu \\
  &- b_4 U_0 (\gamma^4)^\mu_\nu, 
\end{align*}
\] (12)

with \( U_k \equiv U_k(\gamma) \). The Bianchi identities for (9) and (10) imply that
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\[ \nabla^\text{(g)}_\mu T^\mu_\nu = 0, \quad \nabla^\text{(f)}_\mu T^\mu_\nu = 0, \quad (13) \]

where \( \nabla^\text{(g)} \) and \( \nabla^\text{(f)} \) are the covariant derivatives with respect to \( g_{\mu\nu} \) and \( f_{\mu\nu} \). In fact, the latter of these conditions is not independent and follows from the former one in view of the diffeomorphism invariance of the interaction term.

If \( \eta \to 0 \) and \( \sin^2 \eta T^\mu_\nu \to 0 \), then equations \((10)\) for the \( f \)-metric decouple and their solution enters the \( g \)-equations \((9)\) as a fixed reference metric. The \( g \)-equations describe in this case a massive gravity theory. If \( f \) becomes flat for \( \eta \to 0 \), then one recovers the dRGT theory \([20]\). Therefore, the massive gravity theory is contained in the bigravity.

4 Proportional backgrounds

The simplest solutions of the bigravity equations are obtained by assuming the two metrics to be proportional \([33],[37]\),

\[ f_{\mu\nu} = C^2 g_{\mu\nu}. \quad (14) \]

The energy-momentum tensors \((11)\) then become

\[ T^\mu_\nu = -A_g(C) \delta^\mu_\nu, \quad T^\mu_\nu = -A_f(C) \delta^\mu_\nu, \quad (15) \]

with

\[ A_g(C) = m^2 \cos^2 \eta \left( b_0 + 3b_1 C + 3b_2 C^2 + b_3 C^3 \right), \]

\[ A_f(C) = m^2 \frac{\sin^2 \eta}{C^3} \left( b_1 + 3b_2 C + 3b_3 C^2 + b_4 C^3 \right). \quad (16) \]

Since the energy-momentum tensors should be conserved, it follows that \( C \) is a constant. As a result, one obtains two sets of Einstein equations,

\[ G^\mu_\nu + A_g(C) \delta^\mu_\nu = 0, \quad G^\mu_\nu + A_f(C) \delta^\mu_\nu = 0. \quad (17) \]

Since one has \( G^\mu_\nu = G^\mu_\nu/C^2 \), it follows that \( A_f = A_g/C^2 \), which gives an algebraic equation for \( C \). If the parameters \( b_k \) are chosen according to Eq.\((7)\), then this equation reads

\[ 0 = (C - 1) \left[ (c_3 + c_4)C^3 + (3 - 5c_3 + (\chi - 2)c_4)C^2 \
+ ((4 - 3\chi)c_3 + (1 - 2\chi)c_4 - 6)C + (3c_3 + c_4 - 1)\chi \right], \quad (18) \]

with \( \chi = \tan^2 \eta \), while the cosmological constant is

\[ \frac{A_g}{m^2 \cos^2 \eta} = (1 - C)\left( (c_3 + c_4)C^2 + (3 - 5c_3 - 2c_4)C + 4c_3 + c_4 - 6 \right). \quad (19) \]
Depending on values of $c_3, c_4, \eta$, Eq. (18) can have up to four real roots, so that there can be solutions with four different values of the cosmological constant, which can be positive, negative, or zero.

One solution of (18) is $C = 1$, in which case the two metrics coincide, $g_{\mu\nu} = f_{\mu\nu}$, while $A_g = 0$, so that the vacuum GR is recovered. Therefore, the black hole solutions obtained in this case are either Kerr, or Kerr-de Sitter, or Kerr-AdS. None of these solutions admit the massive gravity limit with a flat f-metric.

### 5 Solutions with non-bidiagonal metrics

Let us assume both metrics to be invariant under spatial $SO(3)$ rotations. Since the theory is invariant under diffeomorphisms, one can choose the spacetime coordinates such that the g-metric is diagonal. However, the f-metric will in general contain an off-diagonal term, so that the two metrics can be parameterized as

$$
\begin{align*}
\text{ds}^2_g &= -N^2 dt^2 + \frac{dr^2}{\Delta} + R^2 d\Omega^2, \\
\text{ds}^2_f &= -(aN dt + \frac{c}{\Delta} dr)^2 + \left( cN dt - \frac{b}{\Delta} dr \right)^2 + u^2 R^2 d\Omega^2,
\end{align*}
$$

with $d\Omega^2 = d\vartheta^2 + \sin^2 \vartheta d\phi^2$. The amplitudes $N, \Delta, R$ depend on $r$, while $a, b, c, u$ can in general depend on $t, r$. It is straightforward to check that the matrix square root is

$$
\gamma_{\mu\nu} = \sqrt{g^{\alpha\beta} f_{\alpha\beta}} = \begin{pmatrix}
a & c/(\Delta N) & 0 & 0 \\
-c\Delta N & b & 0 & 0 \\
0 & 0 & u & 0 \\
0 & 0 & 0 & u
\end{pmatrix},
$$

whose eigenvalues are

$$
\lambda_{0,1} = \frac{1}{2} \left( a + b \pm \sqrt{(a - b)^2 - 4c^2} \right), \quad \lambda_2 = \lambda_3 = u.
$$

Inserting this to (6) gives

$$
\begin{align*}
U_1 &= a + b + 2u, \quad U_2 = u(u + 2a + 2b) + ab + c^2, \\
U_3 &= u(au + bu + 2ab + 2c^2), \quad U_4 = u^2(ab + c^2).
\end{align*}
$$

Although the eigenvalues (22) can be complex-valued, the $U_k$’s are always real. It is straightforward to compute the energy-momentum tensors $T^\mu_\nu$ and $T^\mu_\rho$ defined by Eqs. (11), (12). In particular, one finds
Since the g-metric is static, there is no radial energy flux, and so \( T^0_r \) should be zero. Therefore, either \( c \) should vanish, or the expression in brackets in (24) vanishes. The former option will be considered in the next Section, while presently let us assume that \( c \neq 0 \) and

\[
b_1 + 2b_2u + b_3u^2 = 0. \tag{25}\]

This yields

\[
u = \frac{1}{b_3} \left( -b_2 \pm \sqrt{b_2^2 - b_1b_3} \right). \tag{26}\]

Notice that \( u \) was a priori a function of \( t, r \), but now it is restricted to be a constant. Using this, one finds that

\[
T^0_0 = T^r_r = -\lambda_g \tag{27}\]

\[
T^0_0 = T^\varphi_\varphi = -\lambda_f \tag{28}\]

The conditions \( \nabla_\rho T^\rho_\lambda = 0 \) reduce in this case to the requirement that \( T^0_0 - T^\varphi_\varphi \) should vanish. On the other hand, one finds

\[
T^0_0 - T^\varphi_\varphi = (b_2 + b_3u)[(u - a)(u - b) + c^2], \tag{29}\]

and since this has to vanish, either the first or the second factor on the right should be zero. Let us assume that one of these conditions is fulfilled. Then one has \( T^0_0 = T^\varphi_\varphi \) and \( T^0_0 = T^\varphi_\varphi \), hence both energy-momentum tensors are proportional to the unit tensor, \( T^\mu_\nu = -\lambda_g \delta^\mu_\nu \) and \( T^\mu_\nu = -\lambda_f \delta^\mu_\nu \). The field equations (9) then reduce to

\[
G^\rho_\lambda + \Lambda_g \delta^\rho_\lambda = 0, \tag{29}\]

\[
G^\rho_\lambda + \Lambda_f \delta^\rho_\lambda = 0, \tag{30}\]

where

\[
\Lambda_g = m^2 \cos^2 \eta \lambda_g, \quad \Lambda_f = m^2 \sin^2 \eta \lambda_f. \tag{31}\]

As a result, the two metrics decouple one from the other, and the graviton mass gives rise to the two cosmological terms. If the parameters \( b_k \) are chosen according to (7), then \( \lambda_g + u^2 \lambda_f = -(u - 1)^2 \), therefore, if \( \Lambda_g > 0 \) then \( \Lambda_f < 0 \).

Since we want the g-metric to describe a black hole geometry, the solution of (29) is the Schwarzschild-de Sitter metric. On the other hand, as the cosmological term for the f-metric is negative, the solution of (30) can be chosen to be AdS. Therefore,

\[
ds_g^2 = -\Sigma(r) dt^2 + \frac{dr^2}{\Sigma(r)} + r^2 d\Omega^2, \quad \Sigma(r) = 1 - \frac{2M}{r} - \frac{A_g}{3} r^2,
\]
\[ ds_f^2 = -D(U)\, dt^2 + \frac{dU^2}{D(U)} + U^2\, d\Omega^2, \quad D(U) = 1 - \frac{\Lambda_f}{3} U^2, \] \tag{32}

with \( U = ur \). It is worth noting that, since \( \Lambda_f \sim \sin^2 \eta \to 0 \) when \( \eta \to 0 \), the \( f \)-metric becomes flat in this limit. Therefore, the solutions apply both in the bigravity theory and in the dRGT massive gravity.

### 5.1 Imposing the consistency condition

The solution (32) is not yet complete, since the two metrics are expressed in two different coordinate systems, \( t, r \) and \( T, U \), whose relation to each other is not known. One has \( U = ur \) but the function \( T(t, r) \) is still undetermined. We therefore remember that up to now we have not considered the consistency condition, which requires that the expression in (28) should vanish. This condition will be fulfilled in either of the following two cases:

I: \( (b_2 + b_3 u) = 0; \) \tag{33}

II: \( (u - a)(u - b) + c^2 = 0. \) \tag{34}

In case I, since \( u \) is already expressed in terms of \( b_1, b_2, b_3 \) by Eq. (26), the condition (33) imposes a constraint on values of these parameters. Therefore, this condition is possible only for the special subclass of the theory characterized by the restricted values of \( b_k \). Within this subclass the consistency condition is fulfilled without specifying \( T(t, r) \). Therefore, the function \( T(t, r) \) in (32) remains arbitrary, which can probably be traced to a some kind of hidden gauge invariance.

In case II no restrictions on the coefficients \( b_k \) arise, so that this case is generic. The coefficients \( a, b, c \) can be obtained by comparing the line element \( ds_f^2 \) in (20) with that in (32), which gives

\[ a^2 - c^2 = \frac{D\dot{T}^2}{\Sigma}, \quad b^2 - c^2 = \Sigma \left( \frac{u^2}{D} - D\dot{T}'^2 \right), \quad c(a + b) = D\dot{T}T'. \] \tag{35}

Resolving these relations with respect to \( a, b, c \) and inserting the result to (34) yields the equation,

\[ \frac{D}{\Sigma} \dot{T}^2 + \frac{\Sigma D}{\Sigma - D} T'^2 = 1, \] \tag{36}

with \( T = u T \). A simple solution can be obtained by separating the variables,

\[ T = t + \int \frac{dr}{\Sigma} - \int \frac{dr}{D} \equiv t + r^{*}_\Sigma - r^{*}_D. \] \tag{37}
One can think that this solution is singular, since the tortoise coordinate \( r^*_\Sigma \) diverges at the black hole and cosmological horizons, where \( \Sigma \) vanishes. However, introducing the light-like coordinate

\[
V = t + r^*_\Sigma = T + r^*_\Sigma,
\]

both metrics can be written in the Eddington-Finkelstein form

\[
d s^2_g = -\Sigma d V^2 + 2dV dr + r^2 d\Omega^2,
\]
\[
\frac{1}{u^2} d s^2_f = -D d V^2 + 2dV dr + r^2 d\Omega^2,
\]

from where it is obvious that the solution is regular. This solution is valid for all values of the parameters \( b_k \). All the above solutions have been obtained in the ghost-free bigravity context in \([33]\) (see also \([34]\)), but in fact solutions of this type were considered already long ago in the generic f-g bigravity theories \([38, 39, 40]\). The generalization for a nonzero electric charge was considered in \([41]\).

Since the f-metric becomes flat for \( \eta \to 0 \), the solutions describe in this limit black holes in the dRGT massive gravity. In this context they were studied in Refs.\([42, 43]\) for the special case I, and in Refs.\([44, 45]\) for the generic case II. These solutions and their generalization for a nonzero electric charge \([42, 43, 46]\) exhaust all static, spherically symmetric black holes in the dRGT theory.

### 6 Hairy black holes, lumps, and stars

Black holes considered in the previous two sections are described by the known GR metrics. New black holes are obtained in the case where the two metrics are simultaneously diagonal \([33]\),

\[
d s^2_g = N^2 dt^2 - \frac{dr^2}{\Delta} - r^2 d\Omega^2, \quad d s^2_f = A^2 dt^2 - \frac{U'^2}{Y^2} dr^2 - U^2 d\Omega^2.
\]

Here \( N, \Delta, Y, U, A \) are 5 functions of \( r \) which fulfill the equations

\[
G_0^0 = m^2 \cos^2 \eta T_0^0, \quad G_r^r = m^2 \cos^2 \eta T_r^r, \\
G_0^r = m^2 \sin^2 \eta T_0^r, \quad G_r^0 = m^2 \sin^2 \eta T_r^0, \\
T_r^r' + \frac{N'}{N}(T_r^r - T_0^0) + \frac{2}{r}(T_0^0 - T_r^r) = 0.
\]

The simplest solutions are obtained if \( f_{\mu\nu} = C^2 g_{\mu\nu} \), where \( g_{\mu\nu} \) fulfills (17) while \( C, A_g(C) \) are defined by (16), (18). Since \( A_g \) can be positive, negative, or
zero, there are the Schwarzschild, Schwarzschild-de Sitter, and Schwarzschild-AdS black holes. Let us call them background black holes.

More general solutions are obtained by numerically integrating Eqs. (41). It turns out [33] that the equations for the three amplitudes $\Delta, Y, U$ comprise a closed system. Its local solution near the horizon,

$$\Delta^2 = \sum_{n \geq 1} a_n (r - r_h)^n, \quad Y^2 = \sum_{n \geq 1} b_n (r - r_h)^n, \quad U = ur_h + \sum_{n \geq 1} c_n (r - r_h)^n,$$

contains only one free parameter $u = U(r_h)/r_h$, which is the ratio of the horizon radius measured by $f_{\mu\nu}$ to that measured by $g_{\mu\nu}$. The horizon is common for both metrics, in addition, its surface gravities and temperatures determined with respect to both metrics are the same [47].

Choosing a value of $u$ and integrating numerically the equations from $r = r_h$ towards large $r$, the result is as follows [33]. If $u = C$ where $C$ is a root of the algebraic equation (18), then the solution is one of the background black holes. If $u = C + \delta u$ then one can expect the solution to be the background black hole slightly deformed by a massive graviton ‘hair’ localized in the horizon vicinity. This is indeed confirmed for the Schwarzschild-AdS type solutions ($A_g < 0$), which can support a short massive hair and show deviations from the pure Schwarzschild-AdS in the horizon vicinity, but far away from the horizon the deviations tend to zero (see Fig. 1). Therefore, there are asymptotically AdS hairy black holes in the theory.

However, the procedure goes differently for $A_g \geq 0$. When one deforms the Schwarzschild background by setting $u = r_h + \delta u$, then the solutions first stay very close to Schwarzschild. However, at large $r$ they deviate away and show a completely different asymptotic behavior at infinity, characterized by a quasi-AdS g-metric and a compact f-metric [33]. Therefore, the only

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**Fig. 1** Hairy deformations of the Schwarzschild-AdS background, where $A_0, N_0, \Delta_0, Y_0$ correspond to the undeformed solution.
asymptotically flat black hole one finds is the pure Schwarzschild, while its hairy deformations loose the asymptotic flatness. Similarly, trying to deform the Schwarzschild-de Sitter background produces a curvature singularity at a finite proper distance away from the black hole horizon, hence the only asymptotically de Sitter black hole is the pure Schwarzschild-de Sitter.

The conclusion is that there are hairy black holes in the theory, but they are not asymptotically flat. The following argument helps to understand this. Let us require the solution to be asymptotically flat. Then one should have at large \( r \)

\[
\Delta = 1 - \frac{A \sin^2 \eta}{r} + B \cos^2 \eta \frac{mr + 1}{r} e^{-mr} + \ldots,
\]

\[
U = r + B \frac{m^2 r^2 + mr + 1}{m^2 r^2} e^{-mr} + \ldots,
\]

\[
Y = 1 - \frac{A \sin^2 \eta}{r} - B \sin^2 \eta \frac{1 + mr}{r} e^{-mr} + \ldots,
\]

where \( A, B \) are integration constants. Suppose that one wants to find black hole solutions with this asymptotic behavior using the multiple shooting method. In this method one tries to match the asymptotics (42) and (43) by integrating the equations starting from the horizon towards large \( r \), and at the same time starting from infinity towards small \( r \). The two solutions should match at some intermediate point, which gives three matching conditions for \( \Delta, Y, U \). These conditions should be fulfilled by adjusting the free parameters \( A, B, u \) in Eqs. (42),(43). Solutions of this problem may exist at most for discrete sets of values of \( A, B, u \), hence one cannot vary continuously the horizon parameter \( u \). Therefore, there could be no continuous, asymptotically flat hairy deformations of the Schwarzschild solution. However, this does not exclude isolated solutions, and in fact they exist, but to find them requires a good initial guess for \( A, B, u \).

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**Fig. 2** Hairy deformations of the Schwarzschild background.
It is interesting to see what happens to the hairy black holes when one changes the horizon radius \( r_h \). It turns out that in the \( r_h \to 0 \) limit, where the black hole disappears, its ‘hair’ survives and becomes a static ‘lump’ made of massive field modes. Such lumps are described by globally regular solutions for which the event horizon is replaced by a regular center at \( r = 0 \), while at infinity the asymptotic behavior is the same as for the black holes [33]. None of the lumps are asymptotically flat. Neither lumps nor hairy black holes admit the dRGT limit, they exist only in the bigravity theory.

It is worth mentioning in this context that there are asymptotically flat solutions with a matter [33]. Such solutions describe regular stars, and for them one can take limits where one of the two metrics becomes flat. Suppose that the f-sector is empty, while the g-sector contains \( T^{\mu \nu} = \text{diag}[\rho(r), P(r), P(r), P(r)] \) with \( \rho = \rho_0 \theta(r_* - r) \), corresponding to a ‘star’ with a constant density \( \rho_0 \) and a radius \( r_* \). Adding this source to the field equations (41) and assuming a regular center at \( r = 0 \), one finds solutions for which both metrics approach Minkowski metric at infinity according to (43). Introducing the mass functions \( M_g, M_f \) via \( g^{rr} = \Delta^2 = 1 - 2M_g(r)/r \) and \( f^{rr} = Y^2/U'^2 = 1 - 2M_f(r)/r \), one finds that \( M_g, M_f \) rapidly increase inside the star, while outside they approach the same asymptotic value \( M_g(\infty) = M_f(\infty) \sim \sin^2 \eta \) (see Fig. 3). For \( \eta = \pi/2 \) the g-metric is coupled only to the matter and is described by the GR Schwarzschild solution, \( M_g(r) = \rho_0 r_3^3/6 \) for \( r < r_* \) and \( M_g(r) = \rho_0 r_*^3/6 \equiv M_{\text{ADM}} \) for \( r > r_* \). For \( \eta < \pi/2 \) the star mass \( M_{\text{ADM}} \) is partially screened by the negative graviton energy. For \( \eta = 0 \) (dRGT theory) the f-metric becomes flat, so that \( M_f = 0 \), while \( M_g \) asymptotically approaches zero and the star mass is totally screened, because the massless graviton decouples and there could be no \( 1/r \) terms in the metric.

![Fig. 3 Profiles of the asymptotically flat star solution sourced by a regular matter distribution.](image-url)
If the graviton mass is very small, then the $m^2 T_{\mu\nu}$ contribution to the equations is small as compared to $T^{[m]}_{\mu\nu}$, and for this reason $M_g$ rests approximately constant for $r_* < r < r_V \sim (M_{ADM}/m^2)^{1/3}$. This illustrates the Vainshtein mechanism of recovery of General Relativity in a finite region [48]. This mechanism has also been confirmed by the numerical analysis within the generic massive gravity theory with the BD ghost [49, 50], and also in the dRGT theory [51]. The approximate analytical solutions in the weak field limit were considered in Refs.[44, 45, 52] within the dRGT theory and in Ref.[53] within the bigravity theory.

7 Black hole stability and new hairy black holes

As discussed in Section 4, if the two metrics coincide, $g_{\mu\nu} = f_{\mu\nu}$, then the bigravity theory reduces to the vacuum GR, hence one can choose the Schwarzschild metric as a solution. This solution is known to be linearly stable in the GR context, but one can wonder if it is stable also within the bigravity theory. Let us consider small perturbations around this solution,

$$
g_{\mu\nu} = g_{\mu\nu}^{(0)} + \delta g_{\mu\nu}, \quad f_{\mu\nu} = f_{\mu\nu}^{(0)} + \delta f_{\mu\nu},$$

where $g_{\mu\nu}^{(0)}$ is the Schwarzschild metric. If one sets $\delta g_{\mu\nu} = \delta f_{\mu\nu}$, then the GR result will be recovered. However, the perturbations of the two metrics need not be the same in general. Linearizing the bigravity field equations with respect to the perturbations, it turns out that the linear combinations

$$
h_{\mu\nu} = \cos \eta \delta g_{\mu\nu} + \sin \eta \delta f_{\mu\nu}, \quad h_{\mu\nu}^{(0)} = \cos \eta \delta f_{\mu\nu} - \sin \eta \delta g_{\mu\nu}$$

(45)

decouple from each other and can be identified with the massive and massless gravitons, respectively. Equations for the massless graviton are the same as in GR, while for the massive graviton one obtains [54]

$$\Box h_{\mu\nu}^{(0)} + 2 R_{\mu\alpha\nu\beta} h^{\alpha\beta} = m^2 h_{\mu\nu},$$

(46)

$$\nabla_{\mu} h_{\nu}^{\mu} = 0, \quad h_{\mu}^{\mu} = 0.$$

An interesting observation [54] is that these equations have exactly the same structure as those describing perturbations of the black strings – Schwarzschild black holes uplifted to five spacetime dimensions. At the same time, it is known that the black strings are prone to the Gregory-Laflamme instability [55]. Specifically, setting $h_{\mu\nu} = e^{i\omega t} H_{\mu\nu}(r, \vartheta, \varphi)$, it turns out that Eqs. (46) admit a bound state solution with $\omega^2 < 0$ in the spherically-symmetric sector, provided that [56]
It follows that small black holes are unstable, since the frequency $\omega$ is imaginary and so the perturbations grow in time [54]. The condition of smallness is not crucial, since all usual black holes are small compared to the Hubble radius and so fulfill the bound (47), so that all of them should be unstable. On the other hand, since the frequency $|\omega| \propto m$, this instability is very mild, as it needs a Hubble time $\sim 1/m$ to develop. Therefore, even if real astrophysical black hole were described by the bigravity theory, their instability would be largely irrelevant and they would actually be stable for all practical purposes over a cosmologically long period of time.

A similar instability was found also for the Kerr black holes [56] and for the Schwarzschild-de Sitter black holes with proportional metrics described in Section 4 [57]. Interestingly, it was found in the latter case that the instability disappears in the partially massless limit, where the graviton mass is related to the cosmological constant as $m^2 = 2\Lambda/3$ [57].

As discussed in Section 5 above, the Schwarzschild-de Sitter solution in the bigravity theory can exist also in a different version, for which the two metrics are not simultaneously diagonal. The linear stability of this solution was studied with respect to all possible perturbations, but only in the restricted case (33) [58], and also in the generic case (34), but only with respect to spherically symmetric perturbations [59]. In both cases the solution was found to be stable.

Getting back to the unstable Schwarzschild black holes, it turns out that their instability can be used to find new black holes which support hair and which are asymptotically flat. As was explained above, asymptotically flat solutions subject to the boundary conditions (42),(43) may exist, but to find them requires to fine-tune the parameters $A, B, u$ in (42),(43), for which an additional information is needed. Now, the existence of the black hole instability provides such an information [35].

Indeed, Eqs.(46) admit solutions with $\omega^2 < 0$ only for $mr_h < 0.86$, while for $mr_h > 0.86$ all solutions have $\omega^2 > 0$. This means that for $mr_h \approx 0.86$ there is a zero mode: a static solution of (46) with $\omega = 0$. This zero mode can be viewed as approximating a new black hole solution which exists for $mr_h < 0.86$ and which merges with the Schwarzschild solution for $mr_h \approx 0.86$. Close to the merging point the deviations of the new solution from the Schwarzschild are small and can be described by the linear theory. Therefore, one can use the linear zero mode to read-off the values of the parameters $A, B, u$ in (42),(43), after which one can iteratively decrease $r_h$ to obtain the ‘fully-fledged’ non-perturbative hairy black holes. This was done in Ref.[35].

The conclusion is that there are asymptotically flat black holes with a massive hair in the bigravity theory. However, it seems that their parameter $mr_h$ cannot be too small (unless for $c_3 = -c_4 = 2$) [35], which means that
these black holes are cosmologically large, their size being comparable with the Hubble radius. Such solutions are unlikely to be relevant.

All described above black holes have been obtained in the theory either without a matter source or in the theory with an electromagnetic field. At the same time, the perturbative analysis of Ref. [60] predicts that hairy black holes should generically exist in the massive gravity theory coupled to a matter with a non-vanishing trace of the energy-momentum tensor. It would be very interesting to test this prediction by fully non-perturbative calculations.

8 Concluding remarks

Summarizing the above discussion, all possible static, spherically symmetric black holes in the dRGT massive gravity theory are described by the Schwarzschild-de Sitter metrics. They belong to the type studied in Section 5 and they are probably stable. One may wonder why one does not find asymptotically flat black holes. However, our universe is actually in the de Sitter phase, and the main motivation for considering theories with massive gravitons is to describe this fact. Hence, it is not astonishing that the solutions are not asymptotically flat.

One finds more solutions in the bigravity theory, as for example the hairy black holes. However, these seem to be not very relevant, since they are either asymptotically AdS, which contradicts the observations, or they are too large. There are also asymptotically flat or asymptotically de Sitter black hole solutions, but they are unstable. However, they can describe astrophysical black holes, since the instability takes cosmologically long times to develop. One can also wonder what these black holes decay to, and one possibility is that their instability actually implies that there is a slow accretion of massive graviton modes to the horizon [61]. If this is true, then the black holes should be almost exactly Kerr (Kerr-de Sitter), apart from small corrections in the near-horizon region where the accretion takes place.

Some aspects of the graviton mass can be captured within a simplified description in the context of the Galileon theory [62]. This is essentially the General Relativity coupled to a self-interacting scalar field that mimics the scalar polarization mode of the massive graviton. It turns out that black holes in these theory are described by the GR metrics [63], and a no-hair theorem can be proven in this case [64].

To recapitulate, even if the gravitons are indeed massive, this would be hard to detect by observing black holes.

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References

1. R. Ruffini and J.A. Wheeler. Introducing the black hole. *Phys. Today*, 24:30–41.
2. W. Israel. Event horizons in static vacuum space-times. *Phys. Rev.*, 164:1776–1779, 1967.
3. B. Carter. Black hole equilibrium states. In C. DeWitt, B.S. DeWit, editor, *Black Holes*. Gordon and Breach, 1973.
4. P.O. Mazur. Proof of uniqueness of the Kerr-Newman black hole solution. *J. Phys.*, A15:3173–3180, 1982.
5. J.D. Bekenstein. Transcendence of the law of baryon-number conservation in black hole physics. *Phys. Rev. Lett.*, 28:452–455, 1972.
6. J.D. Bekenstein. Nonexistence of baryon number for static black holes. *Phys. Rev.*, D5:1239–1246, 1972.
7. J.D. Bekenstein. Nonexistence of baryon number for black holes. II. *Phys. Rev.*, D5:2403–2412, 1972.
8. J.D. Bekenstein. Novel ‘no scalar hair’ theorem for black holes. *Phys. Rev.*, D51:6608–6611, 1995.
9. A.E. Mayo and J.D. Bekenstein. No hair for spherical black holes: Charged and non-minimally coupled scalar field with selfinteraction. *Phys. Rev.*, D54:5059–5069, 1996.
10. J.D. Bekenstein. Black hole hair: 25 - years after. 1996.
11. S. Hod. Stationary Scalar Clouds Around Rotating Black Holes. *Phys. Rev.*, D86:104026, 2012.
12. C.A.R. Herdeiro and E. Radu. Kerr black holes with scalar hair. *Phys. Rev. Lett.*, 112:221101, 2014.
13. P.B. Yasskin. Solutions for gravity coupled to massless gauge fields. *Phys. Rev.*, D12:2212–2217, 1975.
14. M.S. Volkov and D.V. Galtsov. Non-Abelian Einstein Yang-Mills black holes. *JETP Lett.*, 50:346–350, 1989.
15. M.S. Volkov and D.V. Galtsov. Black holes in Einstein Yang-Mills theory. *Sov. J. Nucl. Phys.*, 51:747–753, 1990.
16. M.S. Volkov and D.V. Galtsov. Gravitating non-Abelian solitons and black holes with Yang-Mills fields. *Phys. Rept.*, 319:1–83, 1999.
17. B. Kleihaus and J. Kunz. Static axially symmetric Einstein Yang-Mills dilaton solutions. 2. Black hole solutions. *Phys. Rev.*, D57:6138–6157, 1998.
18. S.S. Gubser and S.S. Pufu. The Gravity dual of a p-wave superconductor. *JHEP*, 0811:033, 2008.
19. M. Fierz and W. Pauli. On relativistic wave equations for particles of arbitrary spin in an electromagnetic field. *Proc. Roy. Soc. Lond.*, A173:211–232, 1939.
20. C. de Rham, G. Gabadadze, and A.J. Tolley. Resummation of massive gravity. *Phys. Rev. Lett.*, 106:231101, 2011.
21. K. Hinterbichler. Theoretical aspects of massive gravity. *Rev. Mod. Phys.*, 84:671–710, 2012.
22. C. de Rham. Massive Gravity. 2014.
23. D.G. Boulware and S. Deser. Can gravitation have a finite range? *Phys. Rev.*, D6:3368–3382, 1972.
24. S.F. Hassan and R.A. Rosen. Resolving the ghost problem in non-linear massive gravity. *Phys. Rev. Lett.*, 108:041101, 2012.
25. S.F. Hassan and R.A. Rosen. Confirmation of the secondary constraint and absence of ghost in massive gravity and bimetric gravity. *JHEP*, 1204:123, 2012.
26. J. Kluson. Non-linear massive gravity with additional primary constraint and absence of ghosts. *Phys. Rev.*, D86:044024, 2012.
27. D. Comelli, M. Crisostomi, F. Nesti, and L. Pilo. Degrees of freedom in massive gravity. *Phys. Rev.*, D86:101502, 2012.
28. D. Comelli, F. Nesti, and L. Pilo. Massive gravity: a general analysis. 2013.
29. S.F. Hassan and R.A. Rosen. Bimetric gravity from ghost-free massive gravity. *JHEP*, 102:126, 2012.
30. A.G. Riess et al. Observational evidence from supernovae for an accelerating universe and a cosmological constant. *Astron. Journ.*, 116(3):1009, 1998.
31. S. Perlmutter et al. Measurements of and from 42 high-redshift supernovae. *Astrophys. Journ.*, 517(2):565, 1999.
32. T. Damour, I.I. Kogan, and A. Papazoglou. Nonlinear bigravity and cosmic acceleration. *Phys. Rev. D*, 66:104025, Nov 2002.
33. M.S. Volkov. Hairy black holes in the ghost-free bigravity theory. *Phys.Rev.*, D85:124043, 2012.
34. D. Comelli, M. Crisostomi, F. Nesti, and L. Pilo. Spherically symmetric solutions in ghost-free massive gravity. *Phys.Rev.*, D85:024044, 2012.
35. R. Brito, V. Cardoso, and P. Pani. Black holes with massive graviton hair. *Phys.Rev.*, D88:064006, 2013.
36. C.J. Isham, A. Salam, and J.A. Strathdee. F-dominance of gravity. *Phys.Rev.*, D3:867–873, 1971.
37. M.S. Volkov. Self-accelerating cosmologies and hairy black holes in ghost-free bigravity and massive gravity. *Class.Quant.Grav.*, 30:184009, 2013.
38. A. Salam and J.A. Strathdee. A class of solutions for the strong gravity equations. *Phys.Rev.*, D16:2668, 1977.
39. C.J. Isham and D. Storey. Exact spherically symmetric classical solutions for the f-g theory of gravity. *Phys.Rev.*, D18:1047, 1978.
40. Z. Berezhiani, D. Comelli, F. Nesti, and L. Pilo. Exact spherically symmetric solutions in massive gravity. *JHEP*, 0807:130, 2008.
41. E. Babichev and A. Fabbri. A class of charged black hole solutions in massive (bi)gravity. 2014.
42. Th.M. Nieuwenhuizen. Exact Schwarzschild-de Sitter black holes in a family of massive gravity models. *Phys.Rev.*, D84:024038, 2011.
43. L. Berezhiani, G. Chkareuli, C. de Rham, G. Gabadadze, and A.J. Tolley. On black holes in massive gravity. *Phys.Rev.*, D85:044024, 2012.
44. K. Koyama, G. Niz, and G. Tasinato. Analytic solutions in non-linear massive gravity. *Phys.Rev.Lett.*, 107:131101, 2011.
45. K. Koyama, G. Niz, and G. Tasinato. Strong interactions and exact solutions in non-linear massive gravity. *Phys.Rev.*, D84:064033, 2011.
46. Yi-Fu Cai, D.A. Easson, C. Gao, and E.N. Saridakis. Charged black holes in nonlinear massive gravity. *Phys.Rev.*, D87(6):064001, 2013.
47. C. Deffayet and T. Jacobson. On horizon structure of bimetric spacetimes. *Class.Quant.Grav.*, 29:065009, 2012.
48. A.I. Vainshtein. To the problem of nonvanishing gravitation mass. *Phys.Lett.*, B39:393–394, 1972.
49. E. Babichev, C. Deffayet, and R. Ziour. Recovering General Relativity from massive gravity. *Phys.Rev.Lett.*, 103:201102, 2009.
50. E. Babichev, C. Deffayet, and R. Ziour. The recovery of General Relativity in massive gravity via the Vainshtein mechanism. *Phys.Rev.*, D82:104008, 2010.
51. A. Gruzinov and M. Mirbabayi. Stars and black holes in massive gravity. *Phys.Rev.*, D84:124019, 2011.
52. F. Sbisa, G. Niz, K. Koyama, and G. Tasinato. Characterising Vainshtein solutions in massive gravity. *Phys.Rev.*, D86:024033, 2012.
53. E. Babichev and M. Crisostomi. Restoring General Relativity in massive bi-gravity theory. *Phys.Rev.*, D88:084002, 2013.
54. E. Babichev and A. Fabbri. Instability of black holes in massive gravity. *Class.Quant.Grav.*, 30:152001, 2013.
55. R. Gregory and R. Laflamme. Black strings and p-branes are unstable. *Phys.Rev.Lett.*, 70:2837–2840, 1993.
56. R. Brito, V. Cardoso, and P. Pani. Massive spin-2 fields on black hole spacetimes: Instability of the Schwarzschild and Kerr solutions and bounds on the graviton mass. \textit{Phys. Rev.}, D88(2):023514, 2013.

57. R. Brito, V. Cardoso, and P. Pani. Partially massless gravitons do not destroy general relativity black holes. \textit{Phys. Rev.}, D87(12):124024, 2013.

58. H. Kodama and I. Arraut. Stability of the Schwarzschild-de Sitter black hole in the dRGT massive gravity theory. 2013.

59. E. Babichev and A. Fabbri. Stability analysis of black holes in massive gravity: a unified treatment. \textit{Phys. Rev.}, D89:081502, 2014.

60. S. Deser and A. Waldron. Non-Einstein source effects in massive gravity. \textit{Phys. Rev.}, D89:027503, 2014.

61. M. Mirbabayi and A. Gruzinov. Black hole discharge in massive electrodynamics and black hole disappearance in massive gravity. \textit{Phys. Rev.}, D88:064008, 2013.

62. A. Nicolis, R. Rattazzi, and E. Trincherini. The Galileon as a local modification of gravity. \textit{Phys. Rev.}, D79:064036, 2009.

63. E. Babichev and C. Charmousis. Dressing a black hole with a time-dependent Galileon. 2013.

64. L. Hui and A. Nicolis. No-hair theorem for the Galileon. \textit{Phys. Rev. Lett.}, 110(24):241104, 2013.