On the Integrability of Ablowitz-Ladik models with local and nonlocal reductions

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Abstract. We briefly consider the Ablowitz-Ladik model (ALM) with special nonlocal reduction introduced in [4]. The nonlocal ALM also allows Lax representation. Next the expansions over the ‘squared solutions’ developed for the local case, are naturally extended to the nonlocal one. This allows to derive all fundamental properties of the nonlocal ALM including its Hamiltonian properties and action-angle variables.

1. Introduction

Besides the numerous integrable nonlinear evolution equations (NLEE) (see [1, 7, 17]) there also exist, a number of physically important difference nonlinear evolution equations (DEE), solvable by the inverse scattering transform (IST) [2, 3] known today as Ablowitz-Ladik model (ALM). The ALM allows Lax pair of the form [2, 3]:

\[ \psi(n+1, z) = L_n(z) \psi(n,z), \quad L_n(z) = E(z) + Q(n), \quad \frac{i}{\partial t} \frac{\partial \psi(n)}{\partial t} + M(n,t) \psi(n,t) = 0, \]  

\[ M(n,t) = -f(z) \sigma_3 + \sigma_3 Q(n) Q(n-1) + \sigma_3 Q(n-1) E(z) - \sigma_3 E(z) Q(n). \]  

Here \( E(z) = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \), \( Q(n) = \begin{pmatrix} 0 & q(n) \\ r(n) & 0 \end{pmatrix} \) and \( f(z) = \frac{1}{2} (z - z^{-1})^2 \) is the dispersion law of ALM. The compatibility condition is

\[ i \frac{\partial L(n,z)}{\partial t} + L(n,z) M(n,z) - M(n+1,z) L(n,z) = 0. \]  

It holds true identically with respect to \( z \) and for \( r(n) = \epsilon q(n)^* \), \( \epsilon = \pm 1 \) provides the ALM:

\[ i \frac{\partial q(n)}{\partial t} - q(n+1,t) + 2 q(n,t) - q(n-1,t) + \epsilon |q(n,t)|^2 (q(n-1,t) + q(n+1,t)) = 0. \]  

\[ i \frac{\partial q(n)}{\partial t} - q(n+1,t) + 2 q(n,t) - q(n-1,t) + |q(n,t)|^2 (q(n-1,t) + q(n+1,t)) = 0. \]
In [14, 15] the results on ALM were substantially extended. These include: i) the Hamiltonian formulation and the hierarchy of Hamiltonian structures of ALM; ii) the proof that AKNS idea of treating the ISM as generalized Fourier transform holds true also for ALM; iii) the proof of the complete integrability of ALM and derivation of its action-angle variables; iv) derivation of the classical and quantum $R$-matrices for ALM.

The present paper is a further development of our preprint [14] and the paper [15] proving that the results i)–iii) for ALM can be extended to the non-local version of ALM (nALM) recently introduced in [4]. In Section 2 we have collected the main results of the properties i)–iii) of ALM. Special attention here should be paid to the symplectic basis of squared solutions recently introduced in [4].

This basis allows one to treat the expansion coefficients of $\{P_{n,z}, Q(n,z)\}$ of the Lax operator (1.1) generalizing the one for the continuous case [13]. This basis allows one to treat the expansion coefficients of $\sigma_3\delta\hat{Q}(n,z)$ over $\{P(n,z), Q(n,z)\}$ as the variations of the action-angle variables ALM. In the next Section 3 we briefly outline how the results i)–iii) can be extended to the nonlocal ALM. An important consequence of this is the explicit transformation of the symplectic form (3.12) into canonical form (3.13) thus allowing one to obtain the action-angle variables of nALM in terms of the scattering data. The last Section 4 contains discussion and conclusions.

2. Preliminaries

2.1. Lax representation of ALM

The class of integrable DEE is by now rather well studied [15, 5, 6, 23, 24]. The group of reductions of these DEE, unlike the reductions of NLEE [22] are not so well known. This is specially true for their nonlocal reductions [4], which recently started to attract attention. In this Section we remind the main results for ALM following [2, 3, 15].

Let us start with some known facts about the direct and inverse scattering problem for the system (1.1). In order to make the exposition simpler, we consider the case when the potential $Q(n) \in S(\mathbb{Z}, \mathbb{C}^2)$, the space of complex-valued off-diagonal matrix sequences such that $\lim_{n \to \pm \infty} n^k Q(n) = 0$ for all $k = 0, 1, 2, \ldots$. Imposing the condition $0 < \prod_{k=\pm \infty} |h(k)| < \infty$, $h(k) = 1 - q(k)r(k)$ one is able to prove the existence and the analyticity properties of the Jost solutions of (1.1), introduced by

$$\lim_{n \to \infty} \psi(n, z) E^{-n}(z) = I, \quad \lim_{n \to \infty} \phi(n, z) E^{-n}(z) = I,$$

$$\psi(n, z) = ||\psi^+, \psi^-||, \quad \phi(n, z) = ||\phi^+, \phi^-||,$$

where $\psi^+, \psi^-$, $(\phi^-, \phi^-)$ are analytic for $|z| > 1$ (or $|z| < 1$). The scattering matrix is introduced by

$$\phi(n, z) = \psi(n, z) T(z), \quad T(z) = \begin{pmatrix} a^+ & -b^- \\ b^+ & a^- \end{pmatrix}, \quad \det T(z) = v = \prod_{k=\pm \infty} h(k).$$

We will need also the symmetry properties of $T(z, t)$. It is easy to check, that for generic $Q(n)$ we have $L(n, z) = -\sigma_3 L(n, -z) \sigma_3$, which implies that the Jost solutions satisfy $\psi(n, z) = (-1)^n \sigma_3 \psi(n, -z) \sigma_3$. Thus the matrix elements of $T(z, t)$ satisfy:

$$a^+(z) = a^+(-z), \quad b^+(z) = -b^+(-z).$$

We will request that $r(n) = \epsilon Q^*(n)$, i.e. $Q(n) = \epsilon Q^1(n)$; it is this last constraint that renders the compatibility condition (1.3) to the form (1.4). This constraint can be written down as

$$\sigma_\epsilon L(n, z^*) \sigma_\epsilon^{-1} = L(n, \epsilon/z), \quad \sigma_\epsilon = \begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix}, \quad \epsilon = \pm 1.$$
The consequences of the second symmetry on the scattering matrix and its matrix elements are:
\[ \sigma_e T^*(z^*)\sigma_e^{-1} = T(\epsilon/z), \quad a^+(z) = a^{-\epsilon}(\epsilon/z^*), \quad b^+(z) = eb^{-\epsilon}(\epsilon/z^*). \] (2.5)

An immediate consequence of the Lax representation (1.3) is the fact that the \( t \)-dependence of the scattering matrix \( T(z, t) \) is given by:
\[ i \frac{\partial T(z, t)}{\partial t} + f(z)[\sigma_3, T(z, t)] = 0. \] (2.6)

Thus we have shown that the nonlinear DEE (1.4) becomes linear evolution equation for the scattering matrix. In components eq. (2.6) becomes:
\[ i \frac{\partial a^+(z)}{\partial t} = 0, \quad i \frac{\partial b^+(z)}{\partial t} = \mp 2f(z)b^+(z, t). \] (2.7)

An immediate consequence of eq. (2.7) is that ALM possesses an infinite set of integrals of motion. Indeed, both \( a^+(z) \) and \( a^{-\epsilon}(z) \) can be viewed as the generating functionals of such integrals. In fact \( a^+(z) \) and \( a^{-\epsilon}(z) \) are analytic functions of \( z \) for \( |z| > 1 \) and \( |z| < 1 \) respectively. Skipping the details (see, e.g. [14, 15]), we can express them via the dispersion relations:
\[ \ln a^+(z) = \frac{1}{4\pi i} \frac{d\zeta^2}{\zeta^2 - z^2} \ln(1 + \epsilon|\rho^+(\zeta)|^2) + \sum_{j=1}^N \ln \frac{z^2 - z_{j,+}^2}{z^2 - z_{j,-}^2}, \quad |z| > 1, \]
\[ -\ln a^{-\epsilon}(z) = \frac{1}{4\pi i} \frac{d\zeta^2}{\zeta^2 - z^2} \ln(1 + \epsilon|\rho^+(\zeta)|^2) + \sum_{j=1}^N \ln \frac{(z^2 - z_{j,+}^2)z_{j,-}^2}{(z^2 - z_{j,-}^2)z_{j,+}^2}, \quad |z| < 1, \] (2.8)

where
\[ \rho^\pm(\zeta) = \frac{b^\pm(\zeta)}{a^\pm(\zeta)}, \quad |\zeta| = 1, \quad z_{j,-} = 1/z_{j,+}. \] (2.9)

The points \( \pm z_{j,+}, \pm z_{j,-}, j = 1, \ldots, N \) are the discrete eigenvalues of \( L \).

2.2. The ISM is a generalized Fourier transform
The idea of AKNS [1] that the inverse scattering method is a generalized Fourier transform was extended to a large class of differential Lax operators [8]. It was extended also to the DEE generated by the discrete Lax operator (1.1) [14, 15]. To this end we introduce the fundamental analytic solutions by \( \chi^\pm \) of (1.1); they are analytic for \( |z| > 1 \) and \( |z| < 1 \):
\[ \chi^+(n, z) = ||\phi^+, \psi^+||, \quad \chi^-(n, z) = ||\psi^-, \phi^-||, \]
\[ \chi^+(n, z) = \psi(n, z)T^-(z) = \phi(n, z)S^+(z), \quad \chi^-(n, z) = \psi(n, z)T^+(z) = \phi(n, z)S^-(z), \] (2.10)

where
\[ S^+(z) = \begin{pmatrix} 1 & -b^+/v \\ 0 & a^+/v \end{pmatrix}, \quad S^-(z) = \begin{pmatrix} a^- & 0 \\ b^- & 1 \end{pmatrix}, \quad T^+(z) = \begin{pmatrix} 1 & -b^- \\ 0 & a^- \end{pmatrix}, \quad T^-(z) = \begin{pmatrix} a^- & 0 \\ -b^- & v \end{pmatrix}. \]

Here \( v = \prod_{k=-\infty}^{\infty} h(k) \) and obviously \( S^-\hat{T}^+ = T^+\hat{T}^- = T(z) \); by \( \hat{X} \) we have denoted the matrix inverse to \( X \), i.e., \( \hat{X} \equiv X^{-1} \).

The analytic properties of \( a^\pm(z) \) allow one to reconstruct them from the set of reflection coefficients \( s^\pm(\zeta) = b^\pm/a^\pm(\zeta) \) or \( \tau^\pm(\zeta) = b^\mp/a^\pm(\zeta) \). Thus the minimal sets of scattering data which determine both the potential \( Q(n) \) and the scattering matrix \( T(z) \) consist of two parts:
the reflection coefficients on the continuous spectrum (the unit circle) \( z \in S^1 \) and additional data, characterizing the discrete spectrum of \( L(n, z) \). More precisely:
\[
\begin{align*}
\mathcal{T}_1 &= \{ \rho^\pm(\zeta), \ z \in S^1, \ \pm z_j, \ c_j^\pm, \ j = 1, \ldots, N \}, \\
\mathcal{T}_2 &= \{ \tau^\pm(\zeta), \ z \in S^1, \ \pm z_j, \ m_j^\pm, \ j = 1, \ldots, N \},
\end{align*}
\]
(2.11)

where \( c_j^\pm \) and \( m_j^\pm \) are the norming constants of the discrete eigenfunctions of \( L(n, z) \); for more details see \([14, 15]\).

Obviously, the scattering matrix \( T(z) \) is recovered uniquely from either of the sets \( \mathcal{T}_i \) using the dispersion relations (2.8). The recovery of the potential \( Q(n) \) is based on the study of the mapping \( F_i: \mathcal{M} \rightarrow \mathcal{T}_i \), where \( \mathcal{M} \) is the manifold of allowed potentials \( Q(n) \). Such study is based on the Wronskian relations and is deeply related to the interpretation of the ISM to a generalized Fourier transform. These facts are detailed in \([14, 15]\). Here we will formulate the main results in gauge covariant formulation (see, \([8, 18]\)). The mappings \( F_i \) are based on the expansions of the ‘squared solutions’ of \( L(n, z) \) which can be introduced by:
\[
\begin{align*}
\Psi^\pm(n, z) &= v a^\pm(z) P_0(\chi^\pm \sigma \mp \chi^\mp(n, z)), \\
\Phi^\pm(n, z) &= v a^\pm(z) P_0(\chi^\pm \sigma \mp \chi^\mp(n, z)),
\end{align*}
\]
\[
\begin{align*}
\sigma_+ &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, & \sigma_- &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, & \sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\end{align*}
\]
(2.12)

where \( P_0(X) \) are the projectors onto the off-diagonal part of the \( 2 \times 2 \) matrix \( X \):
\[
P_0(X) = \frac{1}{4} [\sigma_3, [\sigma_3, X]].
\]
(2.13)

Skipping the details we formulate the main results of \([14, 15]\) in explicitly gauge covariant form.

The first important result there is the fact that the set of ‘squared solutions’ is a complete set of functions on \( \mathcal{M} \). Therefore any elements of \( \mathcal{M} \), including the potential \( Q(n) \) and its variation \( \sigma_3 \delta Q(n) \) can be expanded over the squared solutions. The result is:
\[
\begin{align*}
Q(n) &= \frac{i}{2\pi} \int_{S^1} \frac{dz}{z} (\rho^+(z)\Psi^+(n, z) + \rho^-(z)\Psi^-(n, z)), \\
\sigma_3 \delta Q(n) &= -\frac{i}{2\pi} \int_{S^1} \frac{dz}{z} (\delta \rho^+(z)\Psi^+(n, z) - \delta \rho^-(z)\Psi^-(n, z)).
\end{align*}
\]
(2.14)

where we have neglected the terms corresponding to the discrete spectrum and
\[
\rho^\pm(z) = [[Q(n), \Phi^\pm(n, z) h^{-1}(n)]] , \quad \delta \rho^\pm(z) = [[\sigma_3 \delta Q(n), \Phi^\pm(n, z) h^{-1}(n)]].
\]
(2.15)

By \([X(n), Y(n)] = -[Y(n), X(n)] \) above we have denoted is the skew-scalar product:
\[
[[X(n), Y(n)] = \sum_{n=\infty}^{\infty} \text{tr} (X(n), [\sigma_3, Y(n)]).
\]
(2.16)

Note that the sets of squared solutions are bi-orthogonal sets of functions with respect to the skew-scalar product \([15]\). We can also introduce the so-called symplectic basis \([13, 15]\):
\[
\begin{align*}
\mathcal{P}(n, z) &= \frac{1}{2\pi} (\rho^+ \Psi^+ + \rho^- \Psi^-)(n, z) = -\frac{1}{2\pi v} (\tau^+ \Phi^+ + \tau^- \Phi^-)(n, z), \\
\mathcal{Q}(n, z) &= \frac{i}{2b+b^-} (v \rho^+ \Psi^+ + \tau^+ \Phi^+)(n, z) = \frac{i}{2b+b^-} (\tau^- \Phi^- + v \rho^- \Psi^-)(n, z),
\end{align*}
\]
(2.17)
which is orthogonal with respect to the skew-scalar product, i.e. for \( z, z' \in S^1 \) we have
\[
\left[ [ \mathcal{P}(n, z), \mathcal{P}(n, z') ] h^{-1}(n) \right] = 0, \quad \left[ [ \mathcal{Q}(n, z), \mathcal{Q}(n, z') ] h^{-1}(n) \right] = 0, \quad (2.18)
\]

The expansions over the symplectic basis take the form:
\[
\mathcal{P}(n, z) = \int_{S^1} dz \mathcal{P}(n, z), \quad \mathcal{Q}(n, z) = \int_{S^1} dz \mathcal{Q}(n, z), \quad (2.19)
\]
where
\[
\sigma(z) = -\frac{1}{2\pi} \ln(1 + \rho^+ \rho^- (z)) = \frac{1}{2\pi} \ln(1 + \rho^+ (z) a^{+,*} (z^*)),
\]
\[
\kappa(z) = -\frac{i}{2} \ln \left( \frac{b^+(z, t)}{b^-(z, t)} \right) = \text{arg}(b^+(z, t)), \quad v = \sum_{k=-\infty}^{\infty} h(k), \quad (2.20)
\]

and \( z \in S^1 \). As we shall see below, \( \pi(z) \) and \( \kappa(z) \) provide the action-angle variables for ALM.

### 2.3. Hamiltonian Properties and Complete integrability of ALM

The simplest nontrivial Hamiltonian formulation of ALM is based on non-canonical Poisson brackets \([14,15]\) and the Hamiltonian:
\[
\{q(n), q^*(m)\} = (1 - \epsilon |q(n)|^2) \delta_{nm}, \quad H_{ALM} = \sum_{n=-\infty}^{\infty} \left( \epsilon |q(n)q^*(n+1) - q(n+1)q^*(n)| \right) \ln(1 - \epsilon |q(n)|^2). \quad (2.21)
\]

The generating functional of integrals of motion is:
\[
\ln a^+(z) = -\frac{1}{2\pi} \ln(1 + \epsilon |\rho^+(z)|^2) = \sum_{p=0}^{\infty} C_p z^{-2p}. \quad (2.22)
\]

The first two integrals of motion are:
\[
C_0 = \ln v = \sum_{n=-\infty}^{\infty} \ln(1 - \epsilon |q(n)|^2), \quad C_1 = \sum_{n=-\infty}^{\infty} (q(n)q^*(n+1) - q(n+1)q^*(n)). \quad (2.23)
\]

The Poisson brackets (2.21) are dual to the symplectic form \([15]\):
\[
\Omega^{(0)} = \sum_{n=-\infty}^{\infty} \frac{\delta q(n) \wedge \delta q^*(n)}{1 - \epsilon |q(n)|^2} = \left[ [\delta q(n) \wedge \delta q^*(n) ] \right]. \quad (2.24)
\]

Using the expansion of \( \sigma_3 \delta Q(n) \) (2.19) over the ‘symplectic basis one finds \([14,15]\):
\[
\Omega^{(0)} = 2i \int_{S^1} \frac{dz}{z} \delta \pi(z) \wedge \delta \kappa(z). \quad (2.25)
\]

Thus we conclude that the action-angle variables of the ALM can be expressed in terms of the scattering data as follows:
\[
\pi(z) = -\frac{1}{2\pi} \ln(1 + \epsilon |\rho^+(z)|^2), \quad \kappa(z) = -\frac{i}{2} \ln \left( \frac{b^+(z)}{b^-(z)} \right) = \text{arg}(b^+(z)), \quad z \in S^1. \quad (2.26)
\]

In addition we conclude also that
\[
\{C_n, C_m\} = 0. \quad (2.27)
\]
3. Nonlocal Difference Nonlinear Equations

Here, on the example of the ALM we will describe the effect of the nonlocal reductions on the Lax representations of the DEE mentioned above. One can check that if $r(n) = eq^*(-n)$, i.e.

$$Q(n) = eQ^1(-n), \quad h(n) = \det L(n) = 1 - eq(n)q^*(-n) = h^*(-n).$$

(3.1)

then the Lax operator satisfies

$$\sigma_i(L(n,z^*))^*\sigma_i^{-1} = L(-n,1/z) = h(-n)L^{-1}(-n,-z).$$

(3.2)

Therefore for the scattering matrix elements we get (see also [4]):

$$a^+(z) = e^{2i\theta}a^+(z^*), \quad a^-(z) = e^{2i\theta}a^-(z^*), \quad b^+(z) = \pm e^{2i\theta}b^-(z^*),$$

(3.3)

where $\theta = \arg(v)$, $\rho^\pm(z) = \frac{b^\pm(z)}{a^\pm(z)}$ and $\tau^\pm(z) = \frac{b^\mp(z)}{a^\pm(z)}$. The nonlocal reduction (3.1) does not influence the analyticity properties of the Jost solutions and the minimal sets of scattering data (2.2). One can again introduce the FAS as in (2.10) and analyze the Wronskian relations.

**Remark 1** Of course, one should keep in mind that all these quantities: the scattering matrix $T(z)$, FAS $\chi^\pm(n,z)$, reflection coefficients $\rho^\pm(z)$, $\tau^\pm(z)$ etc. are substantially different from the their ‘local’ analogs in the previous Section, because of the nonlocal reduction (3.1).

Thus most of the constructions and results for the ‘local’ case can be extended to the nonlocal one, keeping in mind Remark 1 above. The first important one is the expansions over the ‘squared solutions’ of $L(n,z)$ which again are introduced by:

$$\Psi^\pm(n,z) = va^\pm(z)P_0(\chi^\pm\sigma^\pm\chi^\pm(n,z)), \quad \Phi^\pm(n,z) = va^\pm(z)P_0(\chi^\pm\sigma^\pm\chi^\pm(n,z)),$$

(3.4)

and over the symplectic basis, which is formally provided by (2.17). The projector $P_0(X)$ is the same as in (2.13). The ‘squared solutions’ (3.4), along with the symplectic basis, preserve their main property, i.e. they are complete set of functions on $\mathcal{M}_{sl}$, which of course has different complex structure as compare to $\mathcal{M}$. We can derive the expansions

$$Q(n) = \frac{i}{2\pi} \int_{S^1} \frac{dz}{z} \left(\rho^+(z)\Psi^+(n,z) + \rho^-(z)\Psi^-(n,z)\right),$$

(3.5)

$$\sigma_3\delta Q(n) = -\frac{i}{2\pi} \int_{S^1} \frac{dz}{z} \left(\delta\rho^+(z)\Psi^+(n,z) - \delta\rho^-(z)\Psi^-(n,z)\right).$$

where

$$\rho^\pm(z) = \left[\left(Q(n),\Phi^\pm(n,z)h^{-1}(n)\right)\right], \quad \delta\rho^\pm(z) = \left[\sigma_3\delta Q(n),\Phi^\pm(n,z)h^{-1}(n)\right],$$

(3.6)

where $\left[[X(n),Y(n)]\right]$ is the skew-scalar product in (2.16). Similar expansions over the dual set of squared solutions $\Phi^\pm(n,z)$ with coefficients $\tau^\pm(z)$ hold true.

Like in the local case, one can interpret the mapping of $Q(n)$ onto the set of reflection coefficients $\rho^\pm(z)$ or the dual set $\tau^\pm(z)$ as generalized Fourier transform. Of course the expansions (3.5) are derived assuming that the Lax operator $L(n,z)$ has no discrete eigenvalues.

One can derive also the expansions over the symplectic basis with the results:

$$Q(n) = -i \int_{S^1} \frac{dz}{z} P(n,z), \quad \sigma_3\delta Q(n) = \int_{S^1} \frac{dz}{z} \left(\delta\pi(z)Q(n,z) - \delta\kappa(z)P(n,z)\right),$$

(3.7)

where

$$\pi(z) = -\frac{1}{2\pi} \ln(1 + \rho^+\rho^-(z)) = -\frac{1}{2\pi} \ln(1 + \tau^+\tau^-(z)) = \frac{1}{2\pi} \ln\left(\frac{v}{a^+z\bar{a}^-}\right),$$

$$\kappa(z) = -\frac{i}{2} \ln\left(\frac{b^+}{b^-}(z)\right) = \arg(b^+(z)), \quad z \in S^1.$$
3.1. Hamiltonian Properties and Complete integrability of the nonlocal ALM

The simplest nontrivial Hamiltonian formulation of the nonlocal ALM (nALM) is based on non-canonical and nonlocal Poisson brackets and the Hamiltonian:

$$\{q(n), q^*(m)\} = (1 - \epsilon q(n)q^*(-n))\delta_{nm},$$

$$H_{n\text{ALM}} = \sum_{n=-\infty}^{\infty} (\epsilon(q(n)q^*(-n + 1) - q(-n + 1)q^*(n) + \ln(1 - \epsilon q(n)q^*(-n))).$$

(3.9)

The generating functional of integrals of motion is, as in the local case:

$$\ln a^+(z) = -\frac{1}{2\pi} \ln(1 + \epsilon |\rho^+(z)|^2) = \sum_{p=0}^{\infty} \tilde{C}_p z^{-2p}.$$  

(3.10)

The first two integrals of motion are:

$$\tilde{C}_0 = \sum_{n=-\infty}^{\infty} \ln(1 - \epsilon q(n)q^*(-n)), \quad \tilde{C}_1 = \sum_{n=-\infty}^{\infty} (q(n)q^*(-n + 1) - q(n + 1)q^*(-n)).$$

(3.11)

The Poisson brackets (3.9) are dual to the symplectic form:

$$\tilde{\Omega}^{(0)} = \sum_{n=-\infty}^{\infty} \frac{\delta q(n) \wedge \delta q^*(-n)}{1 - \epsilon q(n)q^*(-n)} = \left[ \sigma_3 \delta Q(n) \wedge \sigma_3 \delta Q(n) \right].$$

(3.12)

Using the expansion of $\sigma_3 \delta Q(n)$ over the symplectic basis (3.7) one finds:

$$\Omega^{(0)} = 2i \int_{S^1} \frac{dz}{z} \delta \pi(z) \wedge \delta \kappa(z).$$

(3.13)

Thus we conclude that the action-angle variables of the ALM can be expressed in terms of the scattering data as follows:

$$\pi(z) = \frac{1}{2\pi} \ln \left( \frac{v}{a^+(z)a^-(z)} \right), \quad \kappa(z) = -\frac{i}{2} \ln \left( \frac{b^+(z)}{b^-(z)} \right) = \arg(b^+(z)), \quad z \in S^1.$$

(3.14)

Formally the expressions for the action-angle variables are similar to the ones for the local case. Of course one should keep in mind the Remark 1. Again we conclude also that

$$\{\tilde{C}_n, \tilde{C}_m\} = 0.$$  

(3.15)

**Discussion and Conclusions**

We formulated ideas that will be developed and elaborated in next publications. They can be extended in several directions.

First, like the ALM model, the nALM possesses a hierarchy of Hamiltonian structures which is generated by the recursion operators $\Lambda_{\pm}$ which have the squared solutions as eigenfunctions, see [14, 15]. Combined with the expansions of $Q(n)$ and $\sigma_3 \delta Q(n)$ over the squared solutions one is able to derive all fundamental properties of nALM in terms of the recursion operators $\Lambda_{\pm}$. It seems that nALM, like ALM, possesses also classical and quantum $R$ matrices.

Another challenge is related to the multicomponent models. Indeed, ALM can be viewed as integrable discrete approximation to the nonlinear Schrödinger (NLS) equation [16]. Recently it was demonstrated, see that several classes of multicomponent NLS and $N$-wave equations such as [11, 10], also allow integrable nonlocal reductions [12, 9, 20, 21, 19]. Constructing integrable discrete multicomponent analogs of ALM is a difficult task, though some results in this direction are already known [23, 24]. Constructing nonlocal multicomponent analogs of ALM however is a bigger challenge.
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