BERRY-ESSÉEN BOUND FOR THE PARAMETER ESTIMATION OF FRACTIONAL ORNSTEIN-UHLENBECK PROCESSES WITH THE HURST PARAMETER $H \in (0, \frac{1}{2})$

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Abstract. For an Ornstein-Uhlenbeck process driven by a fractional Brownian motion with Hurst parameter $H \in (0, \frac{1}{2})$, one shows the Berry-Esséen bound of the least squares estimator of the drift parameter. Thus, a problem left in the previous paper (Chen, Kuang and Li, Stochastics and Dynamics, 2019+) is solved, where the Berry-Esséen bound of the least squares estimator is proved for $H \in \left[\frac{1}{2}, \frac{3}{4}\right]$. An approach based on Malliavin calculus given in (Kim and Park, Journal of Multivariate Analysis 155, P284-304) is used.

Keywords: Berry-Esséen bound; Fourth Moment theorems; fractional Ornstein-Uhlenbeck process; Malliavin calculus.

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1. Introduction

The statistical aspects of the following 1-dimensional Ornstein-Uhlenbeck process has been intensively studied by some authors recently.

\[ dX_t = -\theta X_t dt + dB^H_t, \quad X_0 = 0, \quad 0 \leq t \leq T, \]

(1.1)

where $B^H_t$ be a 1-dimensional fractional Brownian motion with Hurst parameter $H \in (0, 1)$. Suppose that $H$ is fixed and known, then there are several types of estimators to the drift coefficient.

Based on the continuous observation, the following maximum likelihood estimator is proposed:

\[ \hat{\theta}_{\text{MLE}} = -\left\{ \int_0^T Q^2(s)dw^H_s \right\}^{-1} \int_0^T Q(s)dz(s), \]

where

\[ Q(t) = \frac{d}{dw^H_t} \int_0^t k_H(t,s)X_s ds, \quad Z(t) = \int_0^t k_H(t,s)dx_s, \]

\[ k_H(t,s) = \kappa_H^{-1} s^{\frac{1}{2} - H} (t - s)^{\frac{1}{2} - H}, \quad w^H_t = \lambda_H^{-1} t^{2 - 2H} \]

with constants $\kappa_H, \lambda_H$ depending on $H$. Please refer to [17] and [21] where the almost sure convergence of both the MLE and a version of the MLE using discrete observations for all $H \in (0, 1)$ is shown. Later on, the central limit theorem of $\hat{\theta}_{\text{MLE}}$ is shown in [2] and [5].

The least squares estimator of the drift coefficient is given by a ratio of two Gaussian functionals [11]:

\[ \hat{\theta}_T = -\frac{\int_0^T X_t dX_t}{\int_0^T X_t^2 dt} = \theta - \frac{\int_0^T X_t dB^H_t}{\int_0^T X_t^2 dt}. \]

(1.2)
where the integral with respect to $B^H$ is interpreted in the Skorohod sense (or say a divergence-type integral). In case of $H \in (0, \frac{1}{2}]$, the strong consistency and asymptotic normality of the estimator $\hat{\theta}_T$ are shown in [11, 12]. It is worth noting that several crucial computations in this paper come from that given in [12].

It is well known that $\hat{\theta}_T$ cannot be computed from the path of $X$ since the translation between divergence and Young integrals relies on the parameter $\theta$ that being estimated. This makes many authors study other more practical and difficult parameter estimate based on discrete observations [1, 8, 19]. For example, in [7], a discrete time least squares estimator

$$\hat{\theta}_n := -\frac{\sum_{i=1}^n X_{i+1} (X_i - X_{i-1})}{\Delta_n \sum_{i=1}^n X_i^2},$$

where $t_i = i\Delta_n$, is proposed and an upper Berry-Esseen-type bound in the Kolmogorov distance for $\hat{\theta}_n$ is shown when $\Delta_n \to 0$ and $n \to \infty$. Moreover, the so-called “polynomial variation” estimator is proposed and an upper Berry-Esseen-type bound in the Wasserstein distance is shown in [14]. It is also found out that to discretize the continuous-time estimator will lost the estimator’s interpretation as a least square optimizer [14].

But it is still meaningful to study the property of $\hat{\theta}_T$ because it is a first step to understand the problem of parameter estimate for the 1-dimensional fractional Ornstein-Uhlenbeck process (1.1) such as its Berry-Esseen behavior. Recently, by using an approach based on Malliavin calculus given by Kim and Park [16], it is shown in [6] that as $T \to \infty$, when $H \in \left[\frac{1}{2}, \frac{3}{4}\right)$, the Berry-Esseen bound of $\sqrt{T} (\hat{\theta}_T - \theta)$ in the Kolmogorov distance is $\frac{1}{\sqrt{T^{1-2H}}}$. When $H = \frac{1}{4}$, the Berry-Esseen bound of $\sqrt{T} (\hat{\theta}_T - \theta)$ in the Kolmogorov distance is $\frac{1}{\log T}$. In fact, when $H = \frac{1}{2}$, the Berry-Esseen bound of $\sqrt{T} (\hat{\theta}_T - \theta)$ in the Kolmogorov distance is well known, please refer to [3, 4] and the references therein.

Since it involves much more complicated method to calculate the inner product of the Hilbert space associated to the fractional Brownian motion in the case of $H \in \left(0, \frac{1}{2}\right)$ than $H \in \left[\frac{1}{2}, \frac{3}{4}\right]$ (see (2.1)-(2.2) below), the Berry-Esseen bound of $\sqrt{T} (\hat{\theta}_T - \theta)$ is still unknown for $H \in \left(0, \frac{1}{2}\right)$. In this paper, we will give an affirmative answer to this question. The main result of the present paper is as follows.

**Theorem 1.1.** Let $Z$ be a standard Gaussian random variable. When $H \in \left(0, \frac{1}{2}\right)$, there exists a constant $C_{\theta, H}$ such that when $T$ is large enough,

$$\sup_{z \in \mathbb{R}} \left| P\left(\sqrt{\frac{T}{\theta \sigma^2_H}} (\hat{\theta}_T - \theta) \leq z\right) - P(Z \leq z) \right| \leq \frac{C_{\theta, H}}{T^{(1-2H)/2}},$$

where $\sigma^2_H$ is given in [12] as follows:

$$\sigma^2_H = (4H - 1) \frac{2\Gamma(2 - 4H) \Gamma(4H)}{\Gamma(2H) \Gamma(1 - 2H)}.$$  

Proof of Theorem 1.1 will be given in Section 3. The main idea to show Theorem 1.1 will be given in Section 2.

Theorem 1.1 implies that when $H \in \left(0, \frac{1}{4}\right]$, the Berry-Esseen bound is $\frac{C_{\theta}}{\sqrt{T}}$. When $H \in \left(\frac{1}{4}, \frac{1}{2}\right)$, the Berry-Esseen bound is $\frac{C_{\theta}}{\sqrt{T^{1-2H}}}$. But it is known that when $H = \frac{1}{2}$, the optimal Berry-Esseen
bound is $\frac{\varepsilon}{\sqrt{H}}$ [15, 16]. Thus, it is reasonable to conjecture that in case of $H \in (\frac{1}{2}, \frac{3}{4})$, a better bound should be $\frac{\varepsilon}{\sqrt{H}}$ (see Remark 3.8). This improving topic will be investigated in other works. In the remaining part of this paper, $\varepsilon$ will be a generic positive constant whose values may differ from line to line.

2. Preliminary

The fractional Brownian motion (fBm) $B^H = \{ B^H_t, t \in \mathbb{R} \}$ with Hurst parameter $H \in (0, 1)$ is a zero mean Gaussian process, defined on a complete probability space $(\Omega, \mathcal{F}, P)$, with the following covariance function

$$\mathbb{E}(B^H_t B^H_s) = R_H(t, s) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t - s|^{2H}) .$$

Let $\mathcal{E}$ denote the space of all real valued step functions on $\mathbb{R}$. The Hilbert space $\mathcal{H}$ is defined as the closure of $\mathcal{E}$ endowed with the inner product

$$\langle 1_{[a,b]}, 1_{[c,d]} \rangle_{\mathcal{H}} = \mathbb{E}((B^H_b - B^H_a)(B^H_d - B^H_c)).$$

If $f, g \in \mathcal{H}$ and $g$ is a continuously differentiable function with compact support, one can use step functions in $\mathcal{E}$ to approximate $f$ and $g$ and by a limiting argument one deduces that [10]

$$\langle f, g \rangle_{\mathcal{H}} = -\int_{\mathbb{R}^2} f(t)g'(s) \frac{\partial R_H(t, s)}{\partial t} dt ds . \quad (2.1)$$

In addition, if $g'$ is interpreted as the distributional derivative, then the identity (2.1) is still valid for the functions such as $g = h \cdot 1_{[a,b]}$ with $h$ a continuously differentiable function. This is a key identity what one really uses in the present paper.

It is well known that when $H \in (\frac{1}{2}, 1)$, for any $f, g \in L^2([0, T])$, if one extends $f$ and $g$ to be zero on $\mathbb{R} \cap [0, T]^c$, then $f, g \in \mathcal{H}$ and (2.1) is equal to a simple identity

$$\langle f, g \rangle_{\mathcal{H}} = H(2H - 1) \int_{[0, T]^2} f(u)g(v) |u - v|^{2H - 2} du dv . \quad (2.2)$$

As one points out before, it is the difference between (2.1) and (2.2) that leads to the case of $H \in (0, \frac{1}{2})$ much more complicated than the case of $H \in [\frac{1}{2}, \frac{3}{4}]$.

A Gaussian isonormal process associated with $\mathcal{H}$ is given by Wiener integrals with respect to a fBm $B^H$ for any deterministic kernel $f \in \mathcal{H}$:

$$B^H(f) = \int_0^\infty f(s) dB^H_s .$$

Let $H_n$ be the $n$-th Hermite polynomial. The closed linear subspace $H_n$ of $L^2(\Omega)$ generated by $\{ H_n(B^H(f)) : f \in \mathcal{H}, \| f \|_{\mathcal{H}} = 1 \}$ is called the $n$-th Wiener-Ito chaos. The linear isometric mapping $I_n : \mathcal{H}^\otimes n \to H_n$ given by $I_n(h_1 \otimes \cdots \otimes h_n) = H_n(B^H(f))$ is called the $n$-th multiple Wiener-Ito integral. For any $f \otimes 1_{[0,T]^c}$, define $I_n(f) = I_n(\tilde{f})$ where $\tilde{f}$ is the symmetrization of $f$.

Given $f \in \mathcal{H}^\otimes p$ and $g \in \mathcal{H}^\otimes q$ and $r = 1, \cdots, p \wedge q$, $r$-th contraction between $f$ and $g$ is the element of $\mathcal{H}^\otimes (p + q - 2r)$ defined by

$$f \otimes_r g(t_1, \ldots, t_{p+q-2r}) = \langle f(t_1, \ldots, t_r), g(t_{r+1}, \ldots, t_{p+q-2r}) \rangle_{\mathcal{H}^\otimes r}.$$
One will make use of the following estimate of the Kolmogrov distance between a nonlinear Gaussian functional and the standard normal (see Corollary 1 of [16]).

**Theorem 2.1** (Kim, Y. T., & Park, H. S). Suppose that \( \varphi_T(t, s) \) and \( \psi_T(t, s) \) are two functions on \( \mathbb{H} \otimes^2 \). Let \( b_T \) be a positive function of \( T \) such that \( I_2(\psi_T) + b_T > 0 \) a.s. If \( \Psi_i(T) \to 0, i = 1, 2, 3 \) as \( T \to \infty \), then there exists a constant \( c \) such that for \( T \) large enough,

\[
\sup_{z \in \mathbb{R}} \left| P \left( \frac{I_2(\varphi_T)}{I_2(\psi_T)} + b_T \leq z \right) - P(Z \leq z) \right| \leq c \times \max_{i=1,2,3} \Psi_i(T),
\]

(2.3)

where

\[
\Psi_1(T) = \frac{1}{b_T^2} \sqrt{\left[ b_T^2 + 2 \| \varphi_T \|_{\mathbb{H} \otimes^2}^2 \right]^2 + 8 \| \varphi_T \otimes_1 \varphi_T \|_{\mathbb{H} \otimes^2}^2},
\]

\[
\Psi_2(T) = \frac{2}{b_T^2} \sqrt{2 \| \varphi_T \otimes_1 \psi_T \|_{\mathbb{H} \otimes^2}^2 + \langle \varphi_T, \psi_T \rangle_{\mathbb{H} \otimes^2}^2},
\]

\[
\Psi_3(T) = \frac{2}{b_T^2} \sqrt{\| \varphi_T \|_{\mathbb{H} \otimes^2}^4 + 2 \| \varphi_T \otimes_1 \psi_T \|_{\mathbb{H} \otimes^2}^2}.
\]

It follows from Eq. (1.2) and the product formula of multiple integrals that

\[
\sqrt{\frac{T}{\theta \sigma_H^2}} (\hat{\theta}_T - \theta) = \frac{I_2(f_T)}{I_2(g_T)} + b_T,
\]

(2.4)

where

\[
f_T(t, s) = \frac{1}{2 \sqrt{\theta \sigma_H^2 T}} e^{-\theta |t-s|} 1_{\{0 \leq s, t \leq T\}},
\]

(2.5)

\[
g_T(t, s) = \sqrt{\frac{\sigma_H^2}{\theta T}} f_T - \frac{1}{2 T} b_T,
\]

(2.6)

\[
h_T(t, s) = e^{-\theta (T-t) - \theta (T-s)} 1_{\{0 \leq t, s \leq T\}},
\]

(2.7)

\[
b_T = \frac{1}{T} \int_0^T \left\| e^{-\theta (t-\cdot)} 1_{[0, t]}(\cdot) \right\|^2_{\mathbb{H}} dt.
\]

(2.8)

The reader can also refer to Eq. (17)-(19) of [15] for details. By Theorem 2.1 and the identity (2.4), to obtain the Berry-Esséen bound of \( \hat{\theta}_T \), one need only to estimate the right hand side of (2.3) which are just several integrals. This is the main idea of the present paper and the previous paper [6].

## 3. Proof of Theorem 1.1

One divides the estimate of the right hand side of (2.3) into several lemmas. The following estimate is cited from the inequality (3.17) of [12].

**Lemma 3.1.** When \( H \in (0, \frac{1}{2}) \), there exists a constant \( C_{\theta, H} \) such that

\[
\| f_T \otimes_1 f_T \|_{\mathbb{H} \otimes^2} \leq \frac{C_{\theta, H}}{\sqrt{T}}.
\]

(3.1)
Although the estimate (3.1) is crucial to the present paper, one will not use this method to compute the inner product any more in this paper.

Denote by $\delta_a(\cdot)$ the Dirac delta function centered at a point $a$. The Heaviside step function $H(x)$ is defined as

$$H(x) = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x < 0. \end{cases}$$

The distributional derivative of the Heaviside step function is the Dirac delta function:

$$\frac{dH(x)}{dx} = \delta_0(x).$$

Hence, one has that for any $-\infty < a < b < \infty$,

$$\frac{d}{dx}1_{[a,b]}(x) = \frac{d}{dx}[H(x-a) - H(x-b)] = \delta_a(x) - \delta_b(x). \quad (3.2)$$

**Lemma 3.2.** When $H \in (0, 1)$, the speed of $b_T \to H \Gamma(2H)\theta^{-2H}$ as $T \to \infty$ is at least $\frac{1}{T^{1-\epsilon}}$ for any $\epsilon \in (0, 1)$.

**Proof.** The identity (2.1), the symmetry and the identity (3.2) imply that one can write $b_T$ as

$$b_T = -\frac{\theta H}{T} \int_0^T e^{-2\theta t} dt \int \frac{\partial}{\partial u} [e^{\theta(u+v)} 1_{[0,1]}(u, v)] \frac{\partial R_H(u, v)}{\partial u} du dv$$

$$= \frac{\theta H}{T} \int_0^T e^{-2\theta t} dt \int e^{\theta(u+v)} (\text{sgn}(u-v) |u-v|^{2H-1} - |u|^{2H-1}) du dv$$

$$+ \frac{H}{T} \int_0^T e^{-2\theta t} dt \int e^{\theta(u+v)} 1_{[0,1]}(u)(\delta_0(v) - \delta_1(v))(\text{sgn}(u-v) |u-v|^{2H-1} - |u|^{2H-1}) du dv$$

$$= \frac{H}{T} \int_0^T e^{-2\theta t} dt \left[ (\theta - \theta) \int_0^T e^{\theta(u+v)} u^{2H-1} du dv + \int_0^t e^{\theta(u+t)} u^{2H-1} du \right]$$

$$= \frac{H}{T} \int_0^T e^{-2\theta t} dt \left[ \int_0^t e^{\theta(u+t)} u^{2H-1} du + \int_0^t e^{\theta(u+t)} (t-u)^{2H-1} du \right]$$

$$:= B_1 + B_2, \quad (3.3)$$

where $B_1, B_2$ and their convergence speeds are given respectively as follows.

First, L’Hospital’s rule implies that for any $\epsilon \in (0, 1)$,

$$\lim_{T \to \infty} T^{1-\epsilon} B_1 = \lim_{T \to \infty} \frac{H}{T} \int_0^T e^{-2\theta t} dt \int_0^t e^{\theta u} u^{2H-1} du$$

$$= \lim_{T \to \infty} \frac{H}{T^{2\epsilon - 1}} \int_0^T e^{\theta u} u^{2H-1} du$$

$$= 0 \quad (3.4)$$
Second, by letting $z = t - u$, it is clear that
\[
\lim_{T \to \infty} B_2 = \lim_{T \to \infty} \frac{H}{T} \int_0^T e^{-\theta t}dt \int_0^t e^{\theta u}(t-u)^{2H-1}du
\]
\[
= \lim_{T \to \infty} \frac{H}{T} \int_0^T dt \int_0^t e^{-\theta z}z^{2H-1}dz
\]
\[
= \lim_{T \to \infty} H \int_0^T e^{-\theta z}z^{2H-1}dz
\]
\[
= H \Gamma(2H)\theta^{-2H},
\]
and one has that
\[
\lim_{T \to \infty} T^{1-\varepsilon} \left( B_2 - H \Gamma(2H)\theta^{-2H} \right)
\]
\[
= \lim_{T \to \infty} \frac{H}{T^{\varepsilon}} \left( \int_0^T dt \int_0^t e^{-\theta z}z^{2H-1}dz - \Gamma(2H)\theta^{-2H}T \right)
\]
\[
= \lim_{T \to \infty} \frac{H}{T^{\varepsilon-1}\varepsilon} \left( \int_0^T e^{-\theta z}z^{2H-1}dz - \Gamma(2H)\theta^{-2H} \right)
\]
\[
= - \lim_{T \to \infty} \frac{H}{T^{\varepsilon-1}\varepsilon} \int_0^T e^{-\theta z}z^{2H-1}dz = 0
\]
(3.5)
Combining the limits (3.4) and (3.5) with the equality (3.3), one has that the convergence speed of $b_T \to H \Gamma(2H)\theta^{-2H}$ is at least $\frac{1}{T^{\varepsilon-\varepsilon}}$. □

**Remark 3.3.** In the case of $H \in \left[\frac{1}{2}, \frac{3}{4}\right]$, the same conclusion is shown in [6]. The proof in the present paper is suited to all $H \in (0, 1)$.

**Lemma 3.4.** Let $h_T$ be given as in (2.7) and $H \in (0, \frac{3}{4})$. Then as $T \to \infty$,
\[
\frac{1}{\sqrt{T}} h_T \to 0, \quad \text{in} \quad \mathcal{F}_{\mathcal{S}^2}.
\]

**Proof.** Without loss of generality, we can assume that $\theta = 1$. Denote $\vec{t} = (t_1, t_2)$, $\vec{s} = (s_1, s_2)$. The identity (2.1) implies that
\[
\frac{1}{T} \|h_T\|_{\mathcal{B}_{\mathcal{S}^2}}^2 = \frac{1}{Te^{4T}} \int_{\mathbb{R}^4} \frac{\partial^2}{\partial t_1 \partial s_2} \left[ e^{t_1+s_1+t_2+s_2} 1_{[0,T]}(t_1, s_1, t_2, s_2) \right] \frac{\partial R_H(t_1, t_2)}{\partial t_2} \frac{\partial R_H(s_1, s_2)}{\partial s_1} d\vec{t} d\vec{s}
\]
\[
= \frac{1}{Te^{4T}} \int_{\mathbb{R}^4} 1_{[0,T]}(t_1, t_2) e^{t_1+s_1+t_2+s_2} \frac{\partial R_H(t_1, t_2)}{\partial t_2} \frac{\partial R_H(s_1, s_2)}{\partial s_1} d\vec{t} d\vec{s}
\]
\[
\times \left[ 1_{[0,T]}(t_1, s_2) + 1_{[0,T]}(t_1) (\delta_0(s_2) - \delta_T(s_2))
\right.
\]
\[
+ 1_{[0,T]}(s_2) (\delta_0(t_1) - \delta_T(t_1)) + (\delta_0(s_2) - \delta_T(s_2))(\delta_0(t_1) - \delta_T(t_1)) \right] d\vec{t} d\vec{s}
\]
\[
:= I_1 + I_2 + I_3 + I_4.
\]
(3.7)
By the symmetry and the L’Hospital’s rule, one has that
\[
\lim_{T \to \infty} I_1 = \lim_{T \to \infty} \frac{1}{Te^{4T}} \int_{[0,T]^4} e^{t_1+s_1+t_2+s_2} \frac{\partial R_H(t_1, t_2)}{\partial t_2} \frac{\partial R_H(s_1, s_2)}{\partial s_1} d\vec{t} d\vec{s}
\]
Lemma 3.7. When

\[ I_2 = I_3 = \frac{H}{T e^{2T}} \int_{[0,T]^3} e^{t_1 + t_2 + t} (t^{2H-1} + (T-t)^{2H-1}) \frac{\partial R_H(s_1, s_2)}{\partial s_1} dt ds_1 ds_2 \]

\[ \rightarrow 0. \]

Finally, one has that as \( T \to \infty \),

\[ I_4 = \frac{H^2}{T^2 e^{2T}} \int_{[0,T]^2} e^{t+s} (t^{2H-1} + (T-t)^{2H-1}) (s^{2H-1} + (T-s)^{2H-1}) ds dt \]

\[ = \frac{H^2}{T e^{2T}} \left[ \int_0^T e^t (t^{2H-1} + (T-t)^{2H-1}) dt \right]^2 \]

\[ \rightarrow 0. \]

Combining the above three limits with the equality (3.7), one has that \( \frac{1}{T} \| h_T \|_{\mathbb{B}^{2\theta}} \to 0 \) as \( T \to \infty \).

\[ \square \]

Remark 3.5. In the case of \( H \in \left[ \frac{1}{2}, 1 \right) \), the same conclusion is shown in [6]. The proof in the present paper is suited to the case of \( H \in \left( 0, \frac{1}{2} \right) \).

Based on Lemma 3.4, one can obtain the following corollary whose proof is almost the same as Lemma 3.4 of [6].

Corollary 3.6. Let \( g_T \) and \( \sigma_H^2 \) be given as in (2.6) and (1.4) respectively. Denote by \( \delta_H = H^2 \Gamma^2(2H) \sigma_H^2 \). When \( H \in (0, \frac{1}{2}) \), we have that as \( T \to \infty \),

\[ T \| g_T \|_{\mathbb{B}^{2\theta}}^2 \to \frac{\delta_H}{2^{\theta+4H}}, \quad T \langle f_T, g_T \rangle_{\mathbb{B}^{2\theta}}^2 \to \frac{\delta_H^2}{4^{\theta+8H} \sigma_H^2}, \]

\[ T \| f_T \|_{\mathbb{B}^{2\theta}}^2 \to 0, \quad T \| g_T \|_{\mathbb{B}^{2\theta}}^2 \to 0. \]

Lemma 3.7. When \( H \in (0, \frac{1}{2}) \), the convergence speed of \( 2 \| f_T \|_{\mathbb{B}^{2\theta}}^2 \to \left[ H \Gamma(2H) \theta^{-2H} \right]^2 \) is at least \( T^{2H-1} \) as \( T \to \infty \).

Proof. Without loss of generality, one can assume that \( \theta = 1 \). One divide the proof into several steps.

Step 1. Similarly to obtain (3.7), one has that

\[ 2 \| f_T \|_{\mathbb{B}^{2\theta}}^2 = \frac{1}{2T \sigma_H^2} \int_{\mathbb{R}^2} \frac{\partial^2}{\partial t_1 \partial s_2} \left[ e^{-|t_1 - s_1| - |t_2 - s_2|} \mathbb{1}_{[0,T)}(t_1, s_1, t_2, s_2) \right] \frac{\partial R_H(t_1, t_2)}{\partial t_2} \frac{\partial R_H(s_1, s_2)}{\partial s_1} dt ds \]
\[
= \frac{1}{2T \sigma_H^2} \int_{\mathbb{R}^2} e^{-|t_1-s_1|-|t_2-s_2|} 1_{[0,T]^2}(s_1, t_2) \frac{\partial R_H(t_1, t_2)}{\partial t_2} \frac{\partial R_H(s_1, s_2)}{\partial s_1} \\
\times \left[ 1_{[0,T]^2}(t_1, s_2) \text{sgn}(t_1 - s_1) \text{sgn}(s_2 - t_2) - 1_{[0,T]^2}(t_1) \text{sgn}(t_1 - s_1)(\delta_0(s_2) - \delta_T(s_2)) \\
- 1_{[0,T]^2}(s_2) \text{sgn}(s_2 - t_2)(\delta_0(t_1) - \delta_T(t_1)) + (\delta_0(s_2) - \delta_T(s_2))(\delta_0(t_1) - \delta_T(t_1)) \right] d\vec{s} d\vec{\tau}
\]
\begin{align*}
&:= I_1(T) + I_2(T) + I_3(T) + I_4(T).
& (3.8)
\end{align*}
Step 2. It is known \([12]\) that as \(T \to \infty\),
\[
I_1(T) = \frac{1}{2T \sigma_H^2} \int_{[0,T]^2} e^{-|t_1-s_1|-|t_2-s_2|} \frac{\partial R_H(t_1, t_2)}{\partial t_2} \frac{\partial R_H(s_1, s_2)}{\partial s_1} \text{sgn}(t_1 - s_1) \text{sgn}(s_2 - t_2) d\vec{s} d\vec{\tau} \\
\to \left[ H \Gamma(2H) \right]^2.
\]

Next one will estimate the convergence speed of the above limit. The symmetry and the L'Hospital's rule \([20]\) imply that
\[
\lim_{T \to \infty} T^{1-2H} \frac{\sigma_H^2}{H^2} \left| I_1(T) - \left( H \Gamma(2H) \right)^2 \right| \\
= \lim_{T \to \infty} \frac{\sigma_H^2}{T^{2H}} \frac{1}{2} \int_{[0,T]^2} e^{-|t_1-s_1|-|t_2-s_2|} (t_2^{2H-1} - \text{sgn}(t_2 - t_1)|t_2 - t_1|^{2H-1}) \\
\times (s_1^{2H-1} - \text{sgn}(s_1 - s_2)|s_1 - s_2|^{2H-1}) \text{sgn}(t_1 - s_1) \text{sgn}(s_2 - t_2) d\vec{s} d\vec{\tau} - \Gamma^2(2H) \sigma_H^2 T
\]
\begin{align*}
&\leq \frac{\sigma_H^2}{2H} \lim_{T \to \infty} T^{1-2H} \left| I_{11}(T) + I_{12}(T) - \Gamma^2(2H) \sigma_H^2 \right| 
& (3.9)
\end{align*}
where
\[
I_{11}(T) = \int_{[0,T]^3} e^{t_1 - T - |s - t_2|} \text{sgn}(t_2 - s)(T^{2H-1} - (T - s)^{2H-1})(t_2^{2H-1} - \text{sgn}(t_2 - t_1)|t_2 - t_1|^{2H-1}) ds d\vec{t},
\]
\[
I_{12}(T) = \int_{[0,T]^3} e^{s_1 - T - |s - t_2|} \text{sgn}(s_2 - t)(T^{2H-1} + (T - t)^{2H-1})(s_1^{2H-1} - \text{sgn}(s_1 - s_2)|s_1 - s_2|^{2H-1}) dt d\vec{s},
\]
\begin{align*}
& (3.10)
\end{align*}
please refer to (6.30)-(6.31) of \([12]\).

Setp 2.1. After dividing the domain of integration \(I_{11}\) into two domains according to \(s > t_2\) or not and doing a change of variables as in (6.32) of \([12]\), one has an expansion of \(I_{11}\) as follows.

\[
I_{11}(T) = \int_{[0,T]^3, x \leq t} e^{-u - x(T^{2H-1} - t^{2H-1})}((T - t + x)^{2H-1} - \text{sgn}(x + u - t)|x + u - t|^{2H-1}) dudx dt \\
- \int_{[0,T]^3, x \leq t} e^{-u - x(T^{2H-1} - (t - x)^{2H-1})}((T - t)^{2H-1} - \text{sgn}(u - t)|u - t|^{2H-1}) dudx dt
\]
In fact, it is clear that there exists a constant $c$ such that as $T \to \infty$,

$$\int_{[0,T]^2} e^{-u-x} \sum_{i=1}^{3} \varphi_i \, dx \, du \to 0$$

where

$$\varphi_1(x) = \int_x^T ((t-x)^{2H-1} - t^{2H-1})((T-t)^{2H-1} - (T-t+x)^{2H-1}) \, dt,$$

$$\varphi_2(x) = \int_x^T ((t-x)^{2H-1} - t^{2H-1})(T-t+x)^{2H-1} \, dt,$$

$$\varphi_3(x, u) = T^{2H-1} \int_x^T (\text{sgn}(u-t)|u-t|^{2H-1} - \text{sgn}(x+u-t)|x+u-t|^{2H-1}) \, dt,$$

$$\varphi_4(x, u) = \int_x^T (t^{2H-1}\text{sgn}(x+u-t)|x+u-t|^{2H-1} - (t-x)^{2H-1}\text{sgn}(u-t)|u-t|^{2H-1}) \, dt.$$  

Step 2.2. For any fixed $\varepsilon \in (0, \frac{1}{4})$, denote $I_1 = [0, T\varepsilon]^2$ and $I_2 = [0, T]^2 \setminus I_1$. Then it is obvious that as $T \to \infty$,

$$\int_{I_2} e^{-u-x} \sum_{i=1}^{3} |\varphi_i| \, dx \, du \to 0$$

with an exponential rate, and

$$\int_{I_1} e^{-u-x}(|\varphi_1| + |\varphi_3|) \, dx \, du \to 0$$

with a speed at least $T^{2H-1}$. The reader can also refer to Lemma 13-15 of [12] for details.

Moreover, one claims that there exists a constant $c > 0$ such that

$$0 < \int_{I_1} e^{-u-x} \varphi_2(x) \, dx \, du = (1 - e^{-T\varepsilon}) \int_0^{T\varepsilon} e^{-x} \varphi_2(x) \, dx$$

$$\leq \int_0^{T\varepsilon} e^{-x} \varphi_2(x) \, dx$$

$$\leq cT^{2H-1}. \quad (3.12)$$

In fact, it is clear that there exists a constant $c > 0$ such that

$$0 < (1-z)^{2H-1} - 1 < c e^z, \quad \forall z \in (0, \frac{1}{2}]. \quad (3.13)$$

One can divides the domain of integral of $\varphi_2(x)$ into three parts as follows.

$$\int_0^{T\varepsilon} e^{-x} \varphi_2(x) \, dx$$

$$= \int_0^{T\varepsilon} e^{-x} \left[ \int_x^{2x} + \int_{2x}^{2T} + \int_{2T}^{T} \right] ((t-x)^{2H-1} - t^{2H-1}) (T-t+x)^{2H-1} \, dt$$

$$:= J_1 + J_2 + J_3. \quad (3.14)$$

The inequality (3.13) and the monotonicity of the function $t^{2H-1}$ imply that

$$J_1 \leq ((1-e)T)^{2H-1} \int_0^{T\varepsilon} e^{-x} \, dx \frac{(2-2^{2H})}{2H} x^{2H} < cT^{2H-1},$$
\[ J_2 \leq ((1 - 2\epsilon)T^{2H-1} - T^e) e^{-x} dx \int_{2^T}^{2T} c_\mu x t^{2H-1} dt < cT^{2H-1}, \]

\[ J_3 \leq \int_0^T e^{-x} x^{2H-1} dx \int_{2T}^T c_\mu x t^{2H-1} dt < cT^{2H-1}. \]

Substituting the above three inequalities into the identity (3.14), one has the inequality (3.12).

Hence, one has that
\[
\lim_{T \to \infty} T^{1-2H} \int_{[0,T]^2} e^{-u-x} \sum_{i=1}^3 |\varphi_i| du \]
\[ = \lim_{T \to \infty} T^{1-2H} \int_{I_2} e^{-u-x} \sum_{i=1}^3 |\varphi_i| dx du + \lim_{T \to \infty} T^{1-2H} \int_{I_1} e^{-u-x} \sum_{i=1}^3 |\varphi_i| dx du \]
\[ = \lim_{T \to \infty} T^{1-2H} \int_{I_1} e^{-u-x} \sum_{i=1}^3 |\varphi_i| dx du < \infty. \tag{3.15} \]

Step 2.3. It is known that as \( T \to \infty, \)
\[
\int_{[0,T]^2} e^{-u-x} \varphi_d dx du \to \frac{1}{2} \Gamma^2(2H) \sigma_H^2, \tag{3.16} \]
please refer to Lemma 14 of [12]. The integral can be decomposed as follows:
\[
\int_{[0,T]^2} e^{-u-x} \varphi_d dx du := L_1(T) - L_2(T) + L_3(T), \]
where
\[
L_1(T) = \int_{[0,T]^2} e^{-u-x} dx du \int_{x}^{x+u} t^{2H-1} (x + u - t)^{2H-1} dt, \]
\[
L_2(T) = \int_{[0,T]^2, x<u} e^{-u-x} dx du \int_{x}^{u} (t-x)^{2H-1} (u-t)^{2H-1} dt, \]
\[
L_3(T) = \int_{[0,T]^2} e^{-u-x} dx du \left[ \int_{x+u}^{T} (t-x)^{2H-1} (t-u)^{2H-1} dt - \int_{x+u}^{T} t^{2H-1} (t-x-u)^{2H-1} dt \right]. \]

please refer to the proof of Lemma 14 of [12]. Hence, one has that
\[
\lim_{T \to \infty} T^{3-4H} \left| \int_{[0,T]^2} e^{-u-x} \varphi_d dx du - \frac{1}{2} \Gamma^2(2H) \sigma_H^2 \right| = \sum_{i=1}^3 \lim_{T \to \infty} T^{3-4H} |L_i(T) - L_i(\infty)|. \tag{3.17} \]

One has that
\[
0 < L_1(\infty) - L_1(T) \]
\[ = \int_{\mathbb{R}^2 \setminus [0,T]^2} e^{-u-x} dx du \int_{x}^{x+u} t^{2H-1} (x+u-t)^{2H-1} dt, \]
\[ = \int_{\mathbb{R}^2 \setminus [0,T]^2} e^{-u-x} dx du \int_{0}^{u} (x+s)^{2H-1} (u-s)^{2H-1} ds, \]
\[ = \left[ \int_{0}^{T} dx \int_{0}^{T} ds \int_{-\infty}^{T} du + \int_{0}^{T} dx \int_{0}^{T} ds \int_{-\infty}^{T} du + \int_{0}^{T} dx \int_{0}^{T} ds \int_{-\infty}^{T} du \right] \]
\times e^{-u-x} (x+s)^{2H-1} (u-s)^{2H-1}.
which imply that
\[ L < e^{-\frac{1}{2}H^2(t-x)}(t-x)^{-2H-1}(u-T)^{-2H-1}du < ce^{-T}. \]

It is clear that
\[ L_{11} < \int_0^T dx \int_0^T ds \int_T^\infty e^{-u-x}(x+s)^{2H-1}(u-T)^{-2H-1}du < cTe^{-T}, \]
\[ L_{12} < \int_0^T dx \int_T^\infty ds \int_0^\infty e^{-u-s-x}e^{2H-1}du < cT^2e^{-T}, \]
\[ L_{13} < \int_T^\infty dx \int_T^\infty ds \int_0^\infty e^{-u-s-x}e^{2H-1}du < ce^{-T}, \]

which imply that \( L_1(T) \rightarrow L_1(\infty) \) with an exponential rate as \( T \rightarrow \infty \). It is obvious that \( L_2(T) \rightarrow L_2(\infty) \) also with an exponential rate as \( T \rightarrow \infty \). In fact,
\[ 0 < L_2(\infty) - L_2(T) \]
\[ = \int_{K^1_2 \setminus [0, T]^2} e^{-u-x}du \int_x^u (t-x)^{2H-1}(u-t)^{2H-1}dt \]
\[ = B(2H, 2H) \int_T^\infty dx \int_x^\infty e^{-u-x}(u-x)^{4H-1}du \]
\[ = B(2H, 2H)\Gamma(2H) \int_T^\infty e^{-2x}dx < ce^{-2T}. \]

Since \( (t-x)(t-u) \geq t(t-x-u) \) for \( x, u > 0 \), the symmetry and the monotonicity of the function \( t^{2H-1} \) imply that
\[ \frac{1}{2}|L_3(\infty) - L_3(T)| \]
\[ = \left| \int_{0<u<x<t, t<x+u, t>T} e^{-u-x}(t-x)^{2H-1}(t-u)^{2H-1}du \right| \]
\[ = \left( -\int_{0<u<x<t, t>x+u, t>T} e^{-u-x}(t-x)^{2H-1}(t-u)^{2H-1}du \right) \]
\[ + \int_{0<u<x, t>x+u, t>T} e^{-u-x}e^{2H-1}(t-x-u)^{2H-1}du \]
\[ := |K_1(T) - K_2(T)| < K_1(T) + K_2(T). \]

It is clear that
\[ K_1(T) = \int_{0<u<x<t+x+u, t>T} e^{-u-x}(t-x)^{2H-1}(t-u)^{2H-1}du \]
\[ = \int_T^\infty dt \int_0^t \int_x^{t-x} e^{-u-x}(t-x)^{2H-1}(t-u)^{2H-1}du \]
\[ < \int_T^\infty e^{-t}dt \int_0^t (t-x)^{2H-1}dx \int_x^{t-x} (t-u)^{2H-1}du \]
\[ < cT^{4H}e^{-T}, \]

and
\[ K_2(T) = \int_T^\infty dt \int_0^t du \int_u^{t-u} e^{-u-x}[e^{2H-1}(t-x-u)^{2H-1} - (t-x)^{2H-1}(t-u)^{2H-1}] \]
\[\lim_{T \to \infty} \frac{2(3-4H)}{T^{4H-3}} K_2(T)\]

\[= \lim_{T \to \infty} \frac{2}{e^{2T/4H-2}} \left[ e^{T/2H-1} \int_0^T e^{-u} e^{z/2H-1} dz - \int_0^T e^{y} e^{2H-1} dy \right] \]

\[= \lim_{T \to \infty} \frac{1}{e^{T/2H-1}} \left[ (T + 2H - 1) \int_0^T e^z e^{z/2H-1} dz - \frac{T}{2} \int_0^T e^{z/2H-1} dz \right] \]

\[= (1 - 2H).\]

Hence it follows from (3.17) that one has that

\[\lim_{T \to \infty} T^{3-4H} \left| \int_{[0,T]^2} e^{-u-x} \varphi_4 dx du - \frac{1}{2} \Gamma^2(2H)\sigma_H^2 \right| < \frac{1-2H}{3-4H} < \infty. \tag{3.18}\]

Combining (3.18) and (3.15) with (3.11), one has that

\[\lim_{T \to \infty} T^{1-2H} \left| I_{11}(T) - \frac{1}{2} \Gamma^2(2H)\sigma_H^2 \right| < \frac{1-2H}{3-4H} < \infty. \tag{3.19}\]

Step 2.4. It follows from (6.36) of [12] that

\[I_{12}(T) = \int_{[0,T]^2} e^{-u-x}(\varphi_4 + \varphi_5) dx du, \tag{3.20}\]

where

\[\varphi_5 = \int_x^T (\text{sgn}(x + u - t) |x + u - t|^{2H-1} (T - t)^{2H-1} - \text{sgn}(u - t) |u - t|^{2H-1} (T - t + x)^{2H-1}) dt.\]

Similar to Step 2.2, it follows from Lemma 13-15 of [12] that as \(T \to \infty\),

\[\int_{I_2} e^{-u-x} \varphi_5 dx du \to 0\]

with an exponential rate, and

\[\int_{I_3} e^{-u-x} \varphi_5 dx du \to 0\]
with a speed at least $T^{2H-1}$. Hence one has that
\[
\lim_{T \to \infty} T^{1-2H} \int_{[0,T]^2} e^{-u-x} |\varphi_5| \, dx \, du < \infty. \tag{3.21}
\]
Combining (3.18) and (3.21) with (3.20), one has that
\[
\lim_{T \to \infty} T^{1-2H} \left| I_{12}(T) - \frac{1}{2} \Gamma^2(2H) \sigma_H^2 \right| \leq \lim_{T \to \infty} T^{1-2H} \int_{[0,T]^2} e^{-u-x} |\varphi_5| \, dx \, du < \infty. \tag{3.22}
\]
Combining (3.19) and (3.22) with (3.9), one has that
\[
\lim_{T \to \infty} T^{1-2H} \left| I_{11}(T) - \frac{1}{2} \Gamma^2(2H) \sigma_H^2 \right| + \lim_{T \to \infty} T^{1-2H} \left| I_{12}(T) - \frac{1}{2} \Gamma^2(2H) \sigma_H^2 \right| < \infty.
\]
That is to say, the speed of $I_1 \to \left[ H \Gamma(2H) \right]^2$ is at least $T^{2H-1}$.

Step 3. One has that
\[
\frac{2\sigma_H^2}{H^2} I_2(T) = \frac{1}{T} \int_{[0,T]^3} e^{t_2-|t_1|} \text{sgn}(s-t_1)(s^{2H-1} + (T-s)^{2H-1})(t_2^{2H-1} - \text{sgn}(t_2-t_1)|t_2-t_1|^{2H-1}) \, ds \, dt. \tag{3.23}
\]
Comparing (3.23) with the identity (3.10), one has that
\[
\frac{2\sigma_H^2}{H^2} I_2(T) = \frac{1}{T} I_{12}(T).
\]
Hence, it follows from the equalities (3.16) and (3.20) that there exists a constant $c$ such that for $T$ large enough,
\[
|I_2(T)| \leq \frac{c}{T}.
\]
By the symmetry, one has that $I_2 = I_3$.

Step 4. It is clear that there exists a constant $c > 0$ such that as $T$ large enough,
\[
I_4(T) = \frac{H^2}{2\sigma_H^2} \left[ \int_0^T e^t (t^{2H-1} + (T-t)^{2H-1}) \, dt \right]^2 \leq \frac{c}{T}.
\]
Finally, substituting the convergence speeds obtained at Step 2-4 to (3.8), one has the desired conclusion. \qed

**Remark 3.8.** (i) The convergence speed given by Lemma 3.7 is essential to the main result of the present paper. It involves much more complicated computation than Lemma 3.5 of \cite{6} where the convergence speed of the same limit $2 \| f_T \|_{\mathcal{H}_2}^2 \to \left[ H \Gamma(2H) \theta^{-2H} \right]^2$ in the case of $H \in \left[ \frac{1}{4}, \frac{1}{2} \right]$ is obtained.

(ii) In the above proof, one has shown that there exists a constant $c > 0$ such that for $T$ large enough,
\[
\int_{I_1} e^{-u-x}(|\varphi_1| + |\varphi_2| + |\varphi_3| + |\varphi_5|) \, dx \, du \leq \frac{c}{T^{1-2H}},
\]
\[ 2 \| f_T \|^2_{\mathcal{H}^{2H}} - \left[ H \Gamma(2H) \theta^{-2H} \right]^2 \leq \frac{c}{T^{1-2H}}. \]

One conjectures that the following estimates
\[ \left| \int_{I_1} e^{-u^2/2} \left( \varphi_1 + \varphi_2 + \varphi_3 \right) du \right| \leq \frac{c}{T^{1-4H}}, \]
\[ 2 \| f_T \|^2_{\mathcal{H}^{2H}} - \left[ H \Gamma(2H) \theta^{-2H} \right]^2 \leq \frac{c}{T^{1-4H}}, \]
also hold for \( T \) large enough. If the conjecture is valid, then the Berry-Esséen bound (1.3) can be improved to \( \frac{c}{T^{1-(3-2H)/2}} \). This question is difficult and will be investigated in the future works.

After showing the above three lemmas, the proof of Theorem 1.1 is almost the same as that of the case of \( H \in \left[ \frac{1}{2}, \frac{3}{4} \right] \) [6]. But for the reader’s convenience, one still writes it here.

**Proof of Theorem 1.1.** It follows from Theorem 2.1, Lemma 3.2 and Eq. (2.4)-(2.8) that there exists a constant \( C_{\theta,H} \) such that for \( T \) large enough,
\[ \sup_{z \in \mathbb{R}} \left| P \left( \sqrt{T \sigma_H^2 (\hat{\theta}_T - \theta)} \leq z \right) - P(Z \leq z) \right| \leq C_{\theta,H} \times \max \left\{ \left| b_2^2 - 2 \| f_T \|^2 \right|, \| f_T \otimes_1 f_T \|, \| f_T \otimes_1 g_T \|, \langle f_T, g_T \rangle, \| g_T \|^2, \| g_T \otimes_1 g_T \| \right\}. \]

Denote \( a = H \Gamma(2H) \theta^{-2H} \). Lemma 3.2 and Lemma 3.7 imply that there exists a constant \( c \) such that for \( T \) large enough,
\[ \left| b_2^2 - 2 \| f_T \|^2 \right| \leq \left| b_2^2 - a^2 \right| + \left| 2 \| f_T \|^2 - a^2 \right| \leq c \times \frac{1}{T^{1-2H}}. \]

Lemma 3.6 implies that there exists a constant \( c \) such that for \( T \) large enough,
\[ \| f_T \otimes_1 g_T \|, \langle f_T, g_T \rangle, \| g_T \otimes_1 g_T \| \leq c \times \frac{1}{\sqrt{T}}, \quad \| g_T \|^2 \leq c \times \frac{1}{T}. \]

Combining (3.1) with the above inequalities, one obtains that the Berry-Esséen bound (1.3) holds. \qed

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