GENERALIZED DOUBLE AFFINE HECKE ALGEBRAS OF HIGHER RANK

PAVEL ETINGOF, WEE LIANG GAN, AND ALEXEI OBLOMKOV

1. Introduction

Double affine Hecke algebras (DAHA) appeared in the work of Cherednik [Ch], as a tool to prove Macdonald’s conjectures; since that time they have been in the center of attention of many representation theorists. In particular, in [Sa], Sahi extended them to root systems of type $C_C^n$, and used this extension to establish Macdonald’s conjectures for Koornwinder polynomials.

DAHA have a rich algebraic structure, which relates them to algebraic geometry and the theory of integrable systems. For instance, it is shown in [Ob1] that Cherednik’s DAHA of type $A_{n-1}$ is a quantization of the relativistic Calogero-Moser space (the space of states for the Ruijsenaars-Schneider integrable system). Also, it is shown in [Ob2] that Sahi’s DAHA of rank 1 is a quantization of a generic cubic surface with three lines forming a triangle removed.

Motivated by this, P.E., A.O., and Eric Rains introduced generalized DAHA (GDAHA) of rank 1, attached to any star-shaped affine Dynkin diagram, i.e. $\tilde{D}_4, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$, [EOR]. It was shown in [EOR] that the PBW theorem holds for these algebras, and that they provide quantizations of del Pezzo surfaces (with a singular genus one curve removed). In the case of $\tilde{D}_4$, the GDAHA is the same as the Sahi algebra (of rank 1), so one recovers the results of [Ob2].

Later, it was pointed out in [ER] that the definition of GDAHA of rank 1 makes sense and the PBW property remains true for any star-shaped graph $D$ which is not a Dynkin diagram of finite type. Such GDAHA are flat deformations of group algebras of polygonal Fuchsian groups acting on the Euclidean (in the affine case) or the Lobachevsky plane.

The main goal of this paper is to introduce and begin to study GDAHA of higher rank $n > 1$, attached to any star-shaped graph that is not a finite Dynkin diagram. These algebras are deformations of the semidirect products of the symmetric group $S_n$ with the $n$-th tensor power of the rank 1 GDAHA. Like the original DAHA, GDAHA are quotients of group algebras of appropriate braid groups, and reduce to the Sahi algebras (of rank $n$) in the case of the graph $\tilde{D}_4$. 

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This paper is organized as follows. In Section 2, we study degenerate, or rational GDAHA of higher rank. These algebras are not really new, as they are “spherical” subalgebras of the algebras introduced by W.L.G. and V. Ginzburg in [GG] (Def. 1.2.3), associated to the idempotent of the branching vertex; in the affine case they are also “spherical” subalgebras of the wreath product symplectic reflection algebras introduced in [EG]. However, we give a new presentation of rational GDAHA by generators and relations, which is a higher rank generalization of Theorem 1 in [Me] (see also [MOV]). Using this presentation, we give a parametrization of irreducible representations of rational GDAHA for affine $D$ when the quantum parameter $\hbar$ vanishes. This parametrization is by the space of solutions of a certain additive Deligne-Simpson problem, which turns out to be a smooth algebraic variety of dimension $2n$. Since for affine $D$ and generic parameters GDAHA are Morita equivalent to symplectic reflection algebras, this parametrization is not really new, and essentially coincides with the parametrization of representations by generalized Calogero-Moser spaces, i.e. by quiver-theoretical data given in [EG], Theorem 11.16; however, our new presentation is somewhat simpler.

In Section 3, we define GDAHA and prove a formal PBW theorem for them. The proof is based on the fact that formal GDAHA are a special case of Hecke algebras of orbifolds introduced in [E], for which the formal PBW theorem holds in a very general situation. On the other hand, the algebraic PBW theorem (i.e. the freeness of the algebra as a module over the ring of coefficients) unfortunately remains a conjecture.

In Section 4, we introduce the Knizhnik-Zamolodchikov connection with coefficients in the degenerate GDAHA, and use it to construct the monodromy functor from the category of finite dimensional representations for the degenerate GDAHA to that for the nondegenerate one; this gives a large supply of finite dimensional representations of GDAHA in the affine case, by applying the monodromy functor to the representations from [EM, M, Ga]. This connection also allows us to construct a Riemann-Hilbert homomorphism between completions of the nondegenerate and the degenerate GDAHA, which gives another (more elementary) proof of the formal PBW theorem. We note however that although the formal PBW theorem for GDAHA is a purely algebraic statement, both proofs we give are based on the Riemann-Hilbert correspondence and therefore use complex analysis; we don’t know a purely algebraic proof.

In Section 5, we study GDAHA in the case of affine $D$. Unfortunately, outside of type $\tilde{D}_4$, we are unable to establish any of the important properties of GDAHA (proved for usual DAHA of type A in [Ob1]), and they are stated as conjectures. Basically, we expect that GDAHA are quantizations of spaces of Calogero-Moser type, which are (topologically trivial) deformations of Hilbert schemes of affine del Pezzo surfaces described above.

The main result of Section 5 is the construction of the parametrization of irreducible representations of GDAHA for $q = 1$ by points of generalized
relativistic Calogero-Moser spaces, which are defined as spaces of solutions of a certain multiplicative Deligne-Simpson problem. This parametrization is the non-degenerate analog of the parametrization in the degenerate case constructed in Section 2, and we conjecture it to be a bijection. With respect to these two parametrizations, the monodromy functor induces the usual Riemann-Hilbert map between solutions of the additive and multiplicative Deligne-Simpson problems (see e.g. [CB2]).

We note that if \( m \) is the number of “legs” of \( D \), then the monodromy maps discussed above depend on a choice of \( m \) points \( \alpha_1, \ldots, \alpha_m \) on \( \mathbb{CP}^1 \), modulo fractional linear transformations. Thus, in the case \( D_4^{\tilde{\cdot}} \), there is an essential parameter – the cross-ratio \( \kappa \) (while in the other affine cases there is no such parameter). If one keeps the value of the monodromy fixed and varies \( \kappa \), one gets a flow on the space of solutions of the additive Deligne-Simpson problem (which is \( 2n \)-dimensional). In the case \( n = 1 \) this is the Painlevé VI flow (see e.g. [EOR], Section 7), so for \( n > 1 \) this flow should be regarded as a higher rank analogue of Painlevé VI. Note that this flow for \( n > 1 \) has an additional parameter (so it has 5 rather than 4 parameters); it would be interesting to study this flow in more detail, for instance write it down explicitly.

We expect that for any non-Dynkin graph there exist multiplicative analogs of the Gan-Ginzburg algebras ([GG], Definition 1.2.3), which are higher rank generalizations of multiplicative preprojective algebras introduced in [CBS]. Moreover, we expect that GDAHA are “spherical” subalgebras of the multiplicative Gan-Ginzburg algebras corresponding to the idempotent of the branching vertex. In the rank 1 case, this is shown by Crawley-Boevey and Shaw in the appendix to [EOR].

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Let $\gamma = (\gamma_{kj})$, $k = 1, \ldots, m$, $j = 1, \ldots, d_k$, be a collection of variables, and $\nu$ an additional variable. It is easy to show that there exist unique $\mu_i = \mu_i(\gamma)$, $i \in I$, and $\xi_k = \xi_k(\gamma)$, $k = 1, \ldots, m$ such that $\xi_1 + \ldots + \xi_m = 0$ and

\begin{equation}
\gamma_{kj} = \sum_{p=1}^{j-1} \mu_{ip} + \frac{\mu_{i0}}{m} + \xi_k
\end{equation}

for all $j, k$.

### 2.2. The definition of the rational GDAHA.

**Definition 2.2.1.** The rational (or degenerate) generalized DAHA of rank $n$ attached to $D$ is the algebra $B_n$ generated over $\mathbb{C}[\gamma, \nu]$ by elements $Y_{i,k}$ (where $i = 1, \ldots, n$; $k = 1, \ldots, m$) and the symmetric group $S_n$, with the following defining relations: for any $i, j, h \in [1, n]$ with $i \neq j$, and $k, l \in [1, m]$,

- $s_{ij} Y_{i,k} = Y_{j,k} s_{ij}$,
- $s_{ij} Y_{h,k} = Y_{h,k} s_{ij}$ if $h \neq i, j$,
- $\prod_{j=1}^{d_k} (Y_{i,k} - \gamma_{kj}) = 0$,
- $Y_{i,1} + Y_{i,2} + \cdots + Y_{i,m} = \nu \sum_{j \neq i} s_{ij}$,
- $[Y_{i,k}, Y_{j,k}] = \nu (Y_{i,k} - Y_{j,k}) s_{ij}$,
- $[Y_{i,k}, Y_{j,l}] = 0$, $k \neq l$,

where $s_{ij} \in S_n$ denotes the transposition $i \leftrightarrow j$.

**Remark 2.2.2.** It is obvious that the algebra $B_n(\gamma, \nu)$ does not change (up to an isomorphism) under the transformations of parameters $\gamma_{kj} \rightarrow \gamma_{kj} + \sigma_k$, where $\sigma_k \in \mathbb{C}$, $\sigma_1 + \ldots + \sigma_m = 0$ (the required isomorphism is given by $Y_{i,k} \rightarrow Y_{i,k} + \sigma_k$). So the essential parameters of $B_n(\gamma, \nu)$ are $\mu_i$ and $\nu$, and there are $m$ “redundant” parameters $\xi_k$. However, it is convenient to keep the redundant parameters to simplify the presentation. A similar remark applies to GDAHA defined in Section 3 below.

### 2.3. GDAHA and the Gan-Ginzburg algebras.

We now recall some definitions from [GG]. Let $k$ be a commutative ring. Let $Q$ be a quiver, and denote by $I$ the set of vertices of $Q$. The double $\overline{Q}$ of $Q$ is the quiver obtained from $Q$ by adding a reverse edge $j \rightarrow a_i$ for each edge $i \rightarrow a_j$ in $Q$. If $i \rightarrow j$ is an edge in $\overline{Q}$, we call $t(a) := i$ its tail, and $h(a) := j$ its head.

Let $R := \bigoplus_{i \in I} k$, and $E$ be the free $k$-module with basis formed by the set of edges $\{a \in Q\}$. Thus, $E$ is naturally a $R$-bimodule and $E = \bigoplus_{i,j \in I} E_{i,j}$, where $E_{i,j}$ is spanned by the edges $a \in Q$ with $h(a) = i$ and $t(a) = j$. The path algebra of $\overline{Q}$ is $\mathbb{C}\overline{Q} := T_R E = \bigoplus_{n \geq 0} T^n_R E$, where $T^n_R E = E \otimes_R \cdots \otimes_R E$
is the \( n \)-fold tensor product. The trivial path for the vertex \( i \) is denoted by \( e_i \), an idempotent in \( R \).

Let \( n \) be a positive integer. Let \( R := R^{\otimes n} \). For any \( \ell \in [1, n] \), define the \( R \)-bimodules
\[
E_\ell := R^{\otimes (\ell - 1)} \otimes E \otimes R^{\otimes (n - \ell)} \quad \text{and} \quad E := \bigoplus_{1 \leq \ell \leq n} E_\ell.
\]
The natural inclusion \( E_\ell \hookrightarrow R^{\otimes (\ell - 1)} \otimes T_R E \otimes R^{\otimes (n - \ell)} \) induces a canonical identification \( T_R E_\ell = R^{\otimes (\ell - 1)} \otimes T_R E \otimes R^{\otimes (n - \ell)} \). Given two elements \( \varepsilon \in E_\ell \) and \( \varepsilon' \in E_m \) of the form
\begin{align}
\varepsilon &= e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes a \otimes \cdots \otimes h(b) \otimes \cdots \otimes e_{i_n}, \\
\varepsilon' &= e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes t(a) \otimes \cdots \otimes b \otimes \cdots \otimes e_{i_n},
\end{align}
where \( \ell \neq m \), \( a, b \in \overline{Q} \) and \( i_1, \ldots, i_n \in I \), we define
\[
[\varepsilon, \varepsilon'] := (e_{i_1} \otimes \cdots \otimes a \otimes \cdots \otimes h(b) \otimes \cdots \otimes e_{i_n})(e_{i_1} \otimes \cdots \otimes t(a) \otimes \cdots \otimes b \otimes \cdots \otimes e_{i_n}) - (e_{i_1} \otimes \cdots \otimes h(a) \otimes \cdots \otimes b \otimes \cdots \otimes e_{i_n})(e_{i_1} \otimes \cdots \otimes a \otimes \cdots \otimes t(b) \otimes \cdots \otimes e_{i_n}).
\]
Note that \( [\varepsilon, \varepsilon'] \) is an element in \( T_R^2 E \).

**Definition 2.3.3** ([GG], Defn. 1.2.3). For any \( \mu = (\mu_i)_{i \in I} \), where \( \mu_i \in k \), and \( \nu \in k \), define the algebra \( A_n = A_{n, \mu, \nu} \) to be the quotient of \( T_R^2 E \times k[S_n] \) by the following relations.

(i) For any \( i_1, \ldots, i_n \in I \) and \( \ell \in [1, n] \):
\[
e_{i_1} \otimes \cdots \otimes \left( \sum_{\{a \in Q \mid h(a) = i_j\}} a \cdot a^* - \sum_{\{a \in Q \mid t(a) = i_i\}} a^* \cdot a - \mu_i e_{i_i} \right) \otimes \cdots \otimes e_{i_n} = \nu \sum_{\{j \neq \ell \mid j = i_i\}} (e_{i_1} \otimes \cdots \otimes e_{i_j} \otimes \cdots \otimes e_{i_n}) s_{j\ell}.
\]

(ii) For any \( \varepsilon, \varepsilon' \) of the form (2.3.1)–(2.3.2):
\[
[\varepsilon, \varepsilon'] = \begin{cases} 
\nu(e_{i_1} \otimes \cdots \otimes h(a) \otimes \cdots \otimes t(a) \otimes \cdots \otimes e_{i_n}) s_{\ell m} & \text{if } b \in Q, \ a = b^*, \\
-\nu(e_{i_1} \otimes \cdots \otimes h(a) \otimes \cdots \otimes t(a) \otimes \cdots \otimes e_{i_n}) s_{\ell m} & \text{if } a \in Q, \ b = a^*, \\
0 & \text{else}.
\end{cases}
\]

Now, given a star-like graph \( D \), we let \( Q = Q(D) \) be the quiver obtained from \( D \) by assigning an orientation to the edges of \( D \) so that they look away from the node. Let \( e_i \) be the idempotent in \( C^I \) that corresponds to the vertex \( i \). Denote by \( A_i \) the algebra over \( k := C[\mu, \nu] \) associated to \( Q \) defined in Definition 2.3.3.

Let \( \xi = (\xi_1, \ldots, \xi_m) \) be a set of variables such that \( \xi_1 + \ldots + \xi_m = 0 \).

**Proposition 2.3.4.** There is a natural isomorphism \( \varphi : B_n \simeq e_{i_0}^{\otimes n} A_n e_{i_0}^{\otimes n} \otimes C[\xi] \).
Proof. In the case when \( n = 1 \) and \( D \) is affine, the proof is given in [Me], see also [MOV] and [EOR] Prop. 7.2. In general, the proof is analogous.

Namely, note that the algebra \( B_n \) has a natural filtration defined by the condition \( \deg(Y_{i,k}) = 2 \); similarly \( A_n \) has a filtration defined by giving the edges of \( Q \) degree 1. Let \( h_k \) be the edge of \( Q \) that starts at \( i_0 \) and goes along the \( k \)-th leg of \( Q \). Let \( h_k^* \) be the edge of \( Q \) opposite to \( h_k \). Then we can define a filtration preserving homomorphism \( \varphi : B_n \to e_{i_0}^{\otimes n}A_ne_{i_0}^{\otimes n} \otimes \mathbb{C}[\xi] \) by the formula

\[
\varphi(Y_{i,k}) = e_{i_0}^{\otimes i-1} \otimes (h_k^*h_k + (\xi_k + \frac{\mu_{i_0}}{m})e_{i_0}) \otimes e_{i_0}^{\otimes n-i}
\]

for all \( i, k \). It is clear that \( \text{gr}(B_n) \) is a quotient of \( B_n(0,0)[\gamma, \nu] \), and it is shown in [GG] that \( \text{gr}A_n = A_n(0,0)[\mu, \nu] \). Also, by Theorem 1 in [Me], the specialization of \( \varphi \) at \( (0,0) \) is an isomorphism. Thus \( \text{gr} \varphi \) is an isomorphism and hence \( \varphi \) is an isomorphism, and the proposition is proved. \( \Box \)

Proposition 2.3.4 and the results of [GG] imply the following corollary.

Corollary 2.3.5. The natural homomorphism \( B_n(0,0)[\gamma, \nu] \to \text{gr}B_n \) is an isomorphism, and thus \( B_n \) is a free \( \mathbb{C}[\gamma, \nu] \)-module.

Assume now that \( D \) is affine. Let \( H(\Gamma_n) \) be the symplectic reflection algebra associated (as in [EG]) to the wreath product group \( \Gamma_n = S_n \rtimes \Gamma^n \), where \( \Gamma \) is the finite subgroup of \( SL(2, \mathbb{C}) \) corresponding to \( D \). Let \( V_i \) be the representation of \( \Gamma \) corresponding to the vertex \( i \in I \) under the McKay correspondence, and let \( e_i \in \mathbb{C}[\Gamma] \) be a primitive idempotent of this representation. Let \( p = \sum e_i \). Then \( p^{\otimes n} \in \mathbb{C}[\Gamma_n] \).

It is shown in [GG] that \( p^{\otimes n}H(\Gamma_n)p^{\otimes n} = A_n \). This fact and Proposition 2.3.4 imply the following.

Corollary 2.3.6. There is a natural isomorphism \( \psi : B_n \to e_{i_0}^{\otimes n}H(\Gamma_n)e_{i_0}^{\otimes n} \otimes \mathbb{C}[\xi] \).

2.4. The degenerate cyclotomic Hecke algebra. In this subsection we define the degenerate version of the Ariki-Koike cyclotomic Hecke algebra.

Let \( \ell \) be a positive integer, and \( \lambda = (\lambda_1, \ldots, \lambda_\ell) \) be a collection of variables.

Definition 2.4.1. The degenerate cyclotomic Hecke algebra \( B_{n,\ell} \) is the algebra over \( \mathbb{C}[\lambda, \nu] \), generated by \( S_n \) and additional generators \( Y_1, \ldots, Y_n \), with defining relations

\[
\begin{align*}
    s_{ij}Y_i &= Y_js_{ij}, \\
    s_{ij}Y_h &= Y_hs_{ij} \quad \text{if} \ h \neq i, j, \\
    (Y_i - \lambda_1) \cdots (Y_i - \lambda_\ell) &= 0, \\
    [Y_i, Y_j] &= \nu(Y_i - Y_j)s_{ij}.
\end{align*}
\]

Note that \( B_{n,\ell}(\lambda, 0) = \mathbb{C}[S_n] \rtimes R_\lambda^{\otimes n} \), where \( R_\lambda = \mathbb{C}[Y]/(\prod (Y_i - \lambda_j)) \).

For each \( k = 1, \ldots, m \), we have a homomorphism \( \eta_k : B_{n,\ell_k} \to B_n \), such that \( \eta_k(\lambda) = \gamma_k \), where \( (\gamma_k)_j := \gamma_{kj} \), and \( \eta_k(\nu) = \nu \). This homomorphism
is given by the formulas $\eta_k(Y_i) = Y_{i,k}$, $\eta_k(s_{ij}) = s_{ij}$. It is easy to check that the specialization of $\eta_k$ at $(0,0)$ is injective. By Corollary 2.3.5, this implies that the natural map $B_{n,\ell}(0,0)\{\lambda,\nu\} \to \text{gr} B_{n,\ell}$ is an isomorphism, and hence $B_{n,\ell}$ is a free module over $\mathbb{C}[\lambda,\nu]$ of rank $n!\ell^n$. Thus $B_{n,\ell}(\lambda,\nu)$ is an algebra of dimension $n!\ell^n$ for all $\lambda,\nu$, which is semisimple for generic values of parameters.

The algebra $B_{n,\ell}(\gamma,\nu)$ has a 1-dimensional representation $\chi$ given by the formula $\chi(s_{ij}) = 1$, $\chi(Y_i) = \gamma \ell$. We call this representation the trivial representation. For generic parameters $\lambda,\nu$, the representation $\chi$ defines an idempotent in $B_{n,\ell}(\lambda,\nu)$. We will denote this idempotent by $e$.

For any representation $V \in \text{Rep} B_{n,\ell}(\gamma,\nu)$, we denote by $V^{B_{n,\ell}(\gamma,\nu)}$ the space of homomorphisms of representations $\chi \to V$. Obviously, this space can be naturally regarded as a subspace of $V$. In the generic case, this subspace is equal to $eV$.

The algebra $B_{n,\ell}(\gamma,\nu)$ contains an obvious subalgebra $B_{n-1,\ell}(\gamma,\nu)$, generated by $Y_i$ and $s_{ij}$ with $i,j < n$. For any representation $V$ of $B_{n,\ell}(\gamma,\nu)$, denote by $V'$ the space $V^{B_{n-1,\ell}(\gamma,\nu)}$.

Consider the element $x := Y_n - \nu \sum_{j=1}^{n-1} s_{nj}$. It is easy to check that $x$ commutes with $B_{n-1,\ell}(\gamma,\nu)$, hence $x$ preserves the space $V'$.

Let $T$ be the $n$-by-$n$ matrix such that $T_{ij} = 1 - \delta_{ij}$. Note that $T + 1$ has rank 1.

**Lemma 2.4.2.** Let $V$ be the regular representation of $B_{n,\ell}(\gamma,\nu)$. Then for generic parameters, the space $V'$ has dimension $n\ell$, and $x|_{V'}$ is conjugate to

$$(\lambda \ell \text{Id}_n - \nu T) \oplus \text{diag}(\lambda_1, \ldots, \lambda_{\ell-1}) \otimes \text{Id}_n.$$

**Proof.** Let $\pi$ be the idempotent in $B_{n-1,\ell}(\gamma,\nu) \subset B_{n,\ell}(\gamma,\nu)$ corresponding to the character $\chi$. Let $P_j(x) := \prod_{p \neq j} \frac{x - \lambda_p}{\lambda_j - \lambda_p}$. It is easy to check (by considering the case $\nu = 0$) that the elements $v_{ij} := \pi P_j(Y_n)s_{ni}$ (where we agree that $s_{nn} = 1$) form a basis of $V'$. Let us compute the action of $x$ in this basis. If $i \neq n$, we have

$$xv_{ij} = \pi x P_j(Y_n)s_{ni} = \lambda_j v_{ij} - \nu \sum_{p \neq n} \pi s_{np}P_j(Y_n)s_{ni} =$$

$$\lambda_j v_{ij} - \nu \sum_{p \neq n} \pi P_j(Y_p)s_{np}s_{ni} = \lambda_j v_{ij} - \nu \delta_{j\ell} \sum_{p \neq n} \pi s_{np}s_{ni} =$$

$$\lambda_j v_{ij} - \nu \delta_{j\ell} \pi \left( \sum_{p \neq n,i} s_{ip}s_{np} + 1 \right) =$$

$$\lambda_j v_{ij} - \nu \delta_{j\ell} \pi \sum_{p \neq i} \sum_{q=1}^{\ell} P_q(Y_n)s_{np} = \lambda_j v_{ij} - \nu \delta_{j\ell} \sum_{p \neq i} \sum_{q=1}^{\ell} v_{pq}.$$


If \( i = n \), we have the same result:
\[
 x v_{nj} = \pi x P_j(Y_n) = \lambda_j v_{nj} - \nu \sum_{p \neq n} \pi s_{np} P_j(Y_n) = \\
\lambda_j v_{nj} - \nu \sum_{p \neq n} \pi P_j(Y_p) s_{np} = \lambda_j v_{nj} - \nu \delta_{j\ell} \sum_{p \neq n} \pi s_{np} = \\
\lambda_j v_{nj} - \nu \delta_{j\ell} \pi \sum_{p \neq n} \sum_{q=1}^\ell P_q(Y_n) s_{np} = \lambda_j v_{nj} - \nu \delta_{j\ell} \sum_{p \neq n} \sum_{q=1}^\ell v_{pq}.
\]

This implies the required statement. \( \square \)

2.5. The affine case. Consider now the affine case, i.e. \( D = \bar{D}_4, \bar{E}_6, \bar{E}_7, \bar{E}_8 \). Then \( m = 3, 4 \) and the numbers \( d_k \) are the following (up to ordering): \( (2, 2, 2), (3, 3, 3), (2, 4, 4), \) and \( (2, 3, 6) \), respectively. We let \( \delta_i \) be the coordinates of the basic imaginary root \( \delta \) of \( D \) in the basis of simple roots. Also, let \( 0 \in I \) be the vertex corresponding to the trivial representation of \( \Gamma \) under the McKay correspondence.

Let us assume that \( \ell := d_m \) is the largest of the \( d_k \). In this case \( \ell \) is divisible by \( d_k \) for all \( k \). Set

\[
h = h(\gamma) := \ell \sum_{k,j} \frac{\gamma_{kj}}{d_k}.
\]

The main properties of \( B_n \) in the affine case are summarized in the following theorem.

**Theorem 2.5.1.** (i) The Gelfand-Kirillov dimension of \( B_n(\gamma, \nu) \) is \( 2n \).

(ii) The algebra \( B_n(\gamma, \nu) \) is PI if and only if \( h = 0 \). If \( h = 0 \), this algebra is PI of degree \( n! \ell^n \).

(iii) If \( h = 0 \), then \( B_n(\gamma, \nu) \) is finitely generated over its center \( Z(B_n(\gamma, \nu)) \). Moreover, for generic \( \gamma, \nu \) (with \( h = 0 \)) the map \( Z(B_n(\gamma, \nu)) \to eB_n(\gamma, \nu)e \) given by \( z \mapsto ze \) is an isomorphism. In particular, \( eB_n(\gamma, \nu)e \) is a commutative algebra.

(iv) If \( h = 0 \) and otherwise \( (\gamma, \nu) \) are generic then \( B_n(\gamma, \nu) \) is an Azumaya algebra, and \( \mathcal{R}_{n, \gamma, \nu} := \text{Spec}(Z(B_n(\gamma, \nu))) \) is a smooth affine algebraic variety of dimension \( 2n \). In this case, every irreducible representation of \( B_n(\gamma, \mu) \) restricts (via the map \( \eta_m \)) to the regular representation of \( \mathcal{B}_{n, \ell}(\gamma_m, \nu) \).

**Proof.** (i) By Corollary 2.3.6 the associated graded algebra of \( B_n \) under its natural filtration is

\[
e_i^{\otimes n}(\mathbb{C}[\Gamma_n] \times \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]) e_i^{\otimes n}.
\]

This implies that the Gelfand-Kirillov dimension of \( B_n \) is \( 2n \).

(ii) If \( h = 0 \) and otherwise \( \gamma, \nu \) are generic, then by Theorem 16.1 of [EG] and Corollary 2.3.6 \( B_n(\gamma, \nu) \) is an Azumaya algebra, whose fibers are matrix algebras of size \( n! \ell^n \). Thus, \( B_n(\gamma, \nu) \) is PI of this degree. Hence it is PI of degree \( \leq n! \ell^n \) for any \( \gamma, \nu \) with \( h = 0 \). But the associated graded of \( B_n \) is clearly PI of degree exactly \( n! \ell^n \), so the statement follows.
On the other hand, let $\hbar \neq 0$. The algebra $B_n(\gamma, \nu)$ induces a Poisson bracket on the center $Z_0 = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ of the algebra $B_n(0, 0)$. It follows from [EG], Section 2, that this Poisson bracket is the one induced by a symplectic form on $\mathbb{C}^{2n}$. The Poisson center of $Z_0$ under this bracket consists only of scalars. This implies that the center $Z(B_n(\gamma, \nu))$ is trivial, and hence $B_n(\gamma, \nu)$ is not PI. (iii) By Corollary 2.3.6, the center $Z(B_n)$ of $B_n$ coincides with the center of the symplectic reflection algebra $H(\Gamma_n)$. It is proved in [EG] that $H(\Gamma_n)$ is finite over its center, so the first statement of (iii) follows. The rest follows from the proof of (ii) (the Azumaya property of $B_n$).

(iv) The first two statements follow from the Azumaya property of $B_n$ and Section 11 of [EG]. To prove the last statement, note that since $B_n,\ell(\gamma_m, \nu)$ is a semisimple algebra for generic parameters, it is sufficient to prove the statement for $\nu = 0$ and generic representations. In this case, the result follows easily from the rank 1 case, see [CBH].

2.6. Representations of $B_n(\gamma, \nu)$ for $h = 0$. Assume that $h = 0$ and otherwise $(\gamma, \nu)$ are generic. Theorem 11.16 of [EG] furnishes an isomorphism of algebraic varieties $\Phi_{EG} : \mathcal{R}_{n, \gamma, \nu} \to \mathcal{M}_{n, \gamma, \nu}$ of $\mathcal{R}_{n, \gamma, \nu}$ onto a certain explicitly described variety $\mathcal{M}_{n, \gamma, \nu}$ (the Calogero-Moser space attached to $D$), which is a deformation of the Hilbert scheme of the desingularization of the Kleinian singularity $\mathbb{C}^2/\Gamma$. By the definition, $\mathcal{M}_{n, \gamma, \nu}$ is the variety of isomorphism classes of representations of the doubled quiver $\mathcal{Q}$ with dimension vector $n\delta$ such that $\sum_{a \in Q} [a, a^*] = \sum \mu_i e_i - \nu T e_0$ (here $\mu_i$ are related to $\gamma_{k,j}$ by formula (2.1.1)).

It is convenient for us to give a slightly different description of the variety $\mathcal{M}_{n, \gamma, \nu}$.

**Definition 2.6.1.** Define $\mathcal{M}_{n, \gamma, \nu}$ to be the variety of conjugacy classes of $m$-tuples $(x_1, \ldots, x_m) \in \mathfrak{gl}_n(\mathbb{C})^m$ satisfying the following equations:

\begin{align*}
(2.6.2) & \quad x_1 + x_2 + \ldots + x_m = 0, \\
(2.6.3) & \quad x_k \sim \text{diag}(\gamma_{k1}, \ldots, \gamma_{kd_k}) \otimes \text{Id}_{n\ell/d_k}, \quad k = 1, \ldots, m - 1, \\
(2.6.4) & \quad x_m \sim (\gamma_{m\ell}\text{Id}_n - \nu T) \oplus \text{diag}(\gamma_{m1}, \ldots, \gamma_{m,\ell-1}) \otimes \text{Id}_n.
\end{align*}

That is, $\mathcal{M}_{n, \gamma, \nu}$ is the categorical quotient of the variety $\tilde{\mathcal{M}}_{n, \gamma, \nu}$ of $m$-tuples as above by the action of the group $\text{PGL}_n(\mathbb{C})$.

Here, if $x, y \in \mathfrak{gl}_N(\mathbb{C})$, we use the notation $x \sim y$ to say that $x$ and $y$ are in the same conjugacy class.

**Remark 2.6.5.** Thus, $\mathcal{M}_{n, \gamma, \nu}$ is defined as the variety of solutions of an appropriate additive Deligne-Simpson problem.

**Proposition 2.6.6.** For generic parameters $\mathcal{M}_{n, \gamma, \nu}$ is a smooth variety, of dimension $2n$.

**Proof.** The proof is standard and analogous to the proof of Proposition 5.2.8 below. \[\square\]
Proposition 2.6.7. There exists a regular map $\beta : M_{n,\gamma,\nu} \to M_{n,\gamma,\nu}$ which sends a representation of $\mathcal{Q}$ from $M_{n,\gamma,\nu}$ to the collection of operators $h_k h_k + \xi_k + \frac{\mu_{i,a}}{m}$, where $h_k, h_k^*$ are defined in the proof of Proposition 2.3.4. This map is an isomorphism.

Proof. The proof is based on the following lemma from linear algebra, due to Crawley-Boevey.

Let $\Lambda_i \in \mathbb{C}$ for $i = 1, \ldots, N$, such that $\Lambda_i + \ldots + \Lambda_j \neq 0$ for any $1 \leq i \leq j \leq N$. Let $V_i$, $i = 0, \ldots, N + 1$, $N \geq 0$, be finite dimensional complex vector spaces of dimensions $D_i$, $D_{i-1} < D_i$. Let $\mathcal{O}$ be a conjugacy class in $\mathfrak{gl}(V_0)$, such that $-\Lambda_1 - \ldots - \Lambda_p$ is not an eigenvalue of an element of $\mathcal{O}$ for any $0 \leq p \leq N$.

Let $M_N$ be the set of collections $(a, b)$ of linear maps $a_i : V_i \to V_{i+1}, b_i : V_{i+1} \to V_i$, $i = 0, \ldots, N$, such that $b_i a_i - a_{i-1} b_{i-1} = \Lambda_i \text{Id}_{V_i}$ for $i = 1, \ldots, N$, and $b_0 a_0 \in \mathcal{O}$.

Let $x(a, b) := a_N b_N \in \mathfrak{gl}(V_{N+1})$. The group $G_N := \prod_{i=0}^N GL(V_i)$ acts naturally on $M_N$ preserving the function $x(a, b)$.

Lemma 2.6.8. (CBL) (i) Let $(a, b) \in M_N$, and $x(a, b) = C$. Then $C$ is conjugate to

$$C_0 \oplus \oplus_{i=1}^{N+1} (\Lambda_N + \ldots + \Lambda_i) \text{Id}_{D_i-D_{i-1}},$$

where $C_0 - (\Lambda_N + \ldots + \Lambda_1) \in \mathcal{O}$ (here the subscripts $D_i-D_{i-1}$ denote matrix sizes)\footnote{We agree that if $i = N + 1$ then $\Lambda_N + \ldots + \Lambda_i = 0$}.

(ii) For any $C \in \mathfrak{gl}(V_{N+1})$ as in (i), there exists an element $(a, b) \in M_N$ such that $x(a, b) = C$. Moreover, any two such elements are conjugate under $G_N$.

Proof. The proof is by induction in $N$. The base of induction ($N = 0$) is easy. Now assume that $N$ is arbitrary, and the statement is known for $N-1$.

To prove (i), note that by the induction assumption $a_{N-1} b_{N-1}$ is conjugate to

$$C_0' \oplus \oplus_{i=1}^{N} (\Lambda_{N-1} + \ldots + \Lambda_i) \text{Id}_{D_i-D_{i-1}},$$

where $C_0' - (\Lambda_{N-1} + \ldots + \Lambda_1) \in \mathcal{O}$. But we have $a_{N-1} b_{N-1} = b_N a_N - \Lambda_N$, so $b_N a_N$ is conjugate to

$$C_0 \oplus \oplus_{i=1}^{N} (\Lambda_N + \ldots + \Lambda_i) \text{Id}_{D_i-D_{i-1}},$$

where $C_0 - (\Lambda_N + \ldots + \Lambda_1) \in \mathcal{O}$. By our assumption, this implies that $b_N a_N$ is invertible, hence $a_N b_N$ is conjugate to the direct sum of $b_N a_N$ and the zero matrix of size $D_{N+1} - D_N$, and (i) follows.

To prove (ii), pick $C' \in \mathfrak{gl}(V_N)$ conjugate to

$$C_0' \oplus \oplus_{i=1}^{N} (\Lambda_{N-1} + \ldots + \Lambda_i) \text{Id}_{D_i-D_{i-1}},$$

where $C_0' - (\Lambda_{N-1} + \ldots + \Lambda_1) \in \mathcal{O}$. By the induction assumption, there exist unique up to conjugation by $G_{N-1}$ operators $(a_0, \ldots, a_{N-1}, b_0, \ldots, b_{N-1}) \in$
implies the required statement. 

2.6.6, \( M \)

V

D

part (i) of Lemma 2.6.8, by applying the Lemma separately to each leg of

C

action of the centralizer of \( B \)

commute with

Thus, these elements define linear operators on

We have Proposition 2.6.9.

Proof. A simple deformation argument from the case \( \nu = 0 \) shows that the spectral decompositions of \( x_1, ..., x_{m-1} \) are as required. The fact that the spectral decomposition of \( x_m \) is as required follows from Lemma 2.4.2.

Proposition 2.6.10. \( \Phi = \beta \circ \Phi_{\text{EG}}. \)

Remark 2.6.11. Note that as a by-product Proposition 2.6.10 gives another proof of Proposition 2.6.9.

Proof. Let \( W \) be an irreducible representation of \( H(\Gamma_n) \). In this case by Corollary 2.3.6 the corresponding representation \( V \) of \( B_n \) is \( e_{i_0}^{\otimes n} W \). On the other hand, the subspace \( W S_{n-1}^{*} \Gamma^{n-1} \) considered in \( \text{EG} \) equals

\[ e(n-1)(e_0^{\otimes (n-1)} \otimes 1) W, \]

where \( e(n-1) \) is the symmetrizer of \( S_{n-1} \).

We need to construct an isomorphism \( \zeta : (e_{i_0})^n W S_{n-1}^{*} \Gamma^{n-1} \rightarrow V', \) where

\( (e_{i_0})^n \)

is the element \( 1^{\otimes (n-1)} \otimes e_{i_0} \). This isomorphism is defined as follows.

Let \( a_0, ..., a_{\ell-1} \) be the opposites of the edges of \( Q \) belonging to the \( m \)-th leg \( (a_{\ell-1} = h^*_{m-1}) \).

Define \( \zeta \) by the formula

\[ \zeta(w) = ((a_{\ell-1}...a_1 a_0)^{\otimes (n-1)} \otimes 1) w, \]

where \( (a_{\ell-1}...a_1 a_0)^{\otimes (n-1)} \otimes 1 \) denotes an element of \( A_n \). We note that this element is well defined, because by Definition 2.3.3 the elements \( (b_i^*)_p \) \( (b_i^*) \)

\( \text{EG} \) the subgroup \( S_{n-1} \subset S_n \) is taken to be the stabilizer of 1, while here it is taken to be the stabilizer of \( n \).
in the $p$-th tensor component) and $(b_2^*)^q$ commute for any edges $b_1, b_2 \in Q$ and $p \neq q$.

We claim that $\zeta(w)$ belongs to $V'$. Indeed, $\zeta(w)$ is clearly invariant under $S_{n-1}$. Also, for any $i < n$ we have $Y_{i,m} \zeta(w) = \gamma_{m}\zeta(w)$. To see this, recall that $Y_{i,m} = (a_{i-1}a_i^*)_i + \xi_m + \mu_0/m$; thus the statement follows from the relation (i) of Definition 2.3.3 and formula (2.1.1).

It is easy to show that $\zeta$ is injective. Since $\zeta$ is a morphism between spaces of the same dimension, it is an isomorphism.

It is now easy to check that if the spaces $(e_i)_{n} W S_{n-1} \rtimes \Gamma_{n-1}$ and $V'$ are identified using $\zeta$, then the quiver-theoretical data of [EG], Section 11 and the matrices $x_1, ..., x_m$ introduced above are related by the map $\beta$. The proposition is proved. □

Corollary 2.6.12. $\Phi$ is an isomorphism.

Proof. The corollary follows from Proposition 2.6.10 and Proposition 2.6.7. □

3. Generalized double affine Hecke algebras

3.1. Generalized DAHA of rank 1. First, let us recall the constructions in [EOR, ER]. Consider the group $G$ with generators $U_k, k = 1, ..., m$, and defining relations

$$ U_k^{d_k} = 1, \quad k = 1, ..., m, \quad \text{and} \quad \prod_{k=1}^{m} U_k = 1. $$

This group is a discrete group of motions of the Euclidean plane, or Lobachevsky plane, generated by rotations by the angles $2\pi/d_k$ around the vertices of the $m$-gon with angles $\pi/d_k, k = 1, ..., m$. Thus $G$ is a Euclidean crystallographic group $\mathbb{Z}_\ell \rtimes \mathbb{Z}^2$ where $\ell = 2, 3, 4, 6$ for $D$ being affine ($\tilde{D}_4, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$ respectively), and a hyperbolic motion group otherwise.

The general double affine Hecke algebra of rank 1 associated to $D$ will be denoted by $H_1$. It is an algebra over $\mathbb{C}[u^{\pm 1}]$, where $u$ denotes the collection of variables

$$ u_{11}, ..., u_{1d_1}, ..., u_{m1}, ..., u_{md_m}. $$

The algebra $H_1$ is generated over $\mathbb{C}[u^{\pm 1}]$ by elements $U_k, k = 1, ..., m$, with defining relations

$$ \prod_{j=1}^{d_k}(U_k - u_{kj}) = 0, \quad k = 1, ..., m, \quad \text{and} \quad \prod_{k=1}^{m} U_k = 1. $$

The group algebra $\mathbb{C}[G]$ is isomorphic to the quotient of $H_1$ by the two-sided ideal generated by $u_{kj} - e^{2\pi ij/d_k}$, for all $k, j$. In other words, $\mathbb{C}[G]$ is the specialization of $H_1$ at the values $u_{kj} = e^{2\pi ij/d_k}$. Thus, $H_1$ is a deformation of $\mathbb{C}[G]$. 
3.2. Generalized DAHA of higher rank. We will now generalize the definition of $H_1$ to the higher rank case. Fix a positive integer $n > 1$. Let $t$ be an additional invertible variable.

**Definition 3.2.1.** The generalized double affine Hecke algebra $H_n$ of rank $n$ associated to $D$ is the algebra generated over $\mathbb{C}[u^\pm, t^\pm]$ by invertible elements

$$U_1, \ldots, U_m, T_1, \ldots, T_{n-1},$$

with defining relations

$$(U_1 \cdots U_m)(T_1 T_2 \cdots T_{n-2} T_n^{-2} T_{n-1}) = 1,$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad i = 1, \ldots, n-2,$$

$$[T_i, T_j] = 0, \quad |i - j| > 1,$$

$$[U_j, T_i] = 0, \quad i = 2, \ldots, n-1, \quad j = 1, \ldots, m,$$

$$[U_j, T_i U_j T_1] = 0, \quad j = 1, \ldots, m,$$

$$[U_k, T_1^{-1} U_j T_1] = 0, \quad 1 \leq k < j \leq m,$$

$$\prod_{j=1}^{dk} (U_k - u_{kj}) = 0, \quad k = 1, \ldots, m,$$

$$T_i - T_i^{-1} = t - t^{-1}, \quad i = 1, \ldots, n-1.$$

**Remark 3.2.2.** The specialization of $H_n$ at the value $t = 1$ is the semidirect product $\mathbb{C}[S_n] \rtimes H^\otimes_1$.

**Remark 3.2.3.** If we eliminate the last two groups of relations, we get a presentation for the n-th braid group $Br_{n,m}$ of $\mathbb{C}P^1$ without $m$ points. Thus the algebra $H_n$ can be viewed as a quotient of the group algebra $\mathbb{C}[u^\pm, t^\pm][Br_{n,m}]$ by the last two groups of relations.

3.3. The case $D = \tilde{D}_4$. In the case when $D$ is of type $\tilde{D}_4$, the algebra $H_n$ is essentially the same as the algebra $H_n$ introduced by Sahi [Sa, §3], which we now recall.

**Definition 3.3.1.** $H_n$ is the algebra generated over $\mathbb{C}[t_0^\pm, t_n^\pm, u_0^\pm, u_n^\pm, t^\pm, q^\pm]$ by elements $T_i^\pm, \quad i = 0, \ldots, n$, and elements $X_i^\pm, \quad i = 1, \ldots, n$, subject to
the relations
\[
T_0T_1T_0T_1 = T_1T_0T_1T_0, \\
X_iX_j = X_jX_i, \ 1 \leq i < j \leq n, \\
T_{n-1}T_nT_{n-1} = T_nT_{n-1}T_nT_{n-1}, \\
T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \ i = 1, \ldots, n-2, \\
[T_i, T_j] = 0, \ |i - j| > 1, \\
T_i - T_i^{-1} = t - t^{-1}, \ i = 1, \ldots, n-1, \\
T_0^{-1} = t_0 - t_0^{-1}, \ T_n - T_n^{-1} = t_n - t_n^{-1}, \\
T_iX_j = X_jT_i, \text{ if } |i - j| > 1, \text{ or if } i = n \text{ and } j = n-1, \\
T_iX_i = X_{i+1}T_i^{-1}, \ i = 1, \ldots, n-1, \\
T_o^\vee - (T_n^\vee)^{-1} = u_n - u_n^{-1}, \text{ where } T_o^\vee = X_n^{-1}T_n^{-1}, \\
T_0^\vee - (T_0^\vee)^{-1} = u_0 - u_0^{-1}, \text{ where } T_0^\vee = q^{-1}T_0^{-1}X_1.
\]

Let $H'_n$ be the specialization of $H_n$ defined by
\[
u_{11} = qt_0, \ nu_{12} = -qt_0^{-1}, \nu_{21} = u_0, \nu_{22} = -u_0^{-1}, \\
u_{31} = u_n, \nu_{32} = -u_n^{-1}, \nu_{41} = t_n, \nu_{42} = -t_n^{-1}.
\]

(This specialization is generic, in the sense that any set of parameter values can be obtained from this one by the rescaling transformations $u_{kj} \to u_{kj}w_k$.)

**Proposition 3.3.2.** There is an isomorphism $\phi : H'_n \to \mathcal{H}_n$, given by
\[
\phi(U_1) = qT_0, \ \phi(U_2) = T_0^\vee, \ \phi(U_3) = ST_n^\vee S^{-1}, \ \phi(U_4) = ST_nS^{-1}, \\
\phi(T_i) = T_i, \ i = 1, \ldots, n-1,
\]
where $S = T_1T_2 \ldots T_{n-1}$.

The proof of the proposition is by a direct computation.

**Remark 3.3.3.** The $T_i$’s from $\mathcal{S}_a$ are denoted by $V_i$’s in $\mathcal{S}_t$. Also, the following relations should be added in $\mathcal{S}_t$ Theorem 3.4:
\[
[V_0^\vee, V_n] = [V_0, V_n^\vee] = 0.
\]
This is very minor misprint and it does not affect any other results of the paper $\mathcal{S}_t$ and the subsequent papers.

### 3.4. The flatness theorem.

**Conjecture 3.4.1.** The algebra $H_n$ is a free module over $\mathbb{C}[u^{\pm 1}, t^{\pm 1}]$.

In the case $n = 1$, this conjecture is proved in [FOR]. Also, by Proposition 3.3.2 and the results of $\mathcal{S}_a$, the conjecture is true for the affine diagram $\tilde{D}_4$ for all $n$.

We can prove only a weaker version of this conjecture, which is the following theorem. Let $\tilde{H}_n$ be the completion of $H_n$ with respect to the ideal generated by $t - 1$, and let $\nu = \frac{1}{\pi t} \log t$. 
Theorem 3.4.2. The algebra $\hat{H}_n$ is a flat 1-parameter deformation of the algebra $S_n \times H_1^{\otimes n}$ with deformation parameter $\nu$ (i.e., $\hat{H}_n = S_n \times H_1^{\otimes n}[[\nu]]$ with deformed multiplication).

To prove this theorem, note first that by a general deformation argument, it is sufficient to show the following:

Proposition 3.4.3. The completion $(\hat{H}_n)_{u_0}$ of $\hat{H}_n$ at some point $u_0$ is a flat deformation of the completion $(S_n \times H_1^{\otimes n})_{u_0}$.

We will give two proofs of this fact, using two different choices of the point $u_0$.

First proof of Proposition 3.4.3. Let $Y$ be the Euclidean or hyperbolic plane which carries the action of the group $G$. Let $G_n = S_n \times G^n$. Then $G_n$ acts properly discontinuously on $Y^n$, so following [E], we can define the Hecke algebra $H_\tau(Y^n, G_n)$ attached to the orbifold $Y^n/G_n$. Using the explicit description of the braid group $Br_{n,m}$ of $Y^n/G_n$ given above, we find that the algebra $(\hat{H}_n)_{u_0}$ for $(u_0)_{kj} = e^{2\pi ij/d_k}$ is a specialization of the algebra $H_\tau(Y^n, G_n)$, $\tau = (\gamma, \nu)$. On the other hand, since $\pi_2(Y^n) = 0$, one of the main results of [E] says that the algebra $H_\tau(Y^n, G_n)$ is flat over $\mathbb{C}[[\tau]]$. This implies the theorem.

A second proof of Theorem 3.4.2, which does not use the results of [E], will be given in the next section. It is based on a variant of Knizhnik-Zamolodchikov equations.

4. Knizhnik-Zamolodchikov equations

4.1. KZ equations. Let $\alpha_1, \ldots, \alpha_m$ be distinct points in $\mathbb{C}$. Consider the connection $\nabla$ on the trivial bundle over $(\mathbb{C}P^1)^n$ with fiber $B_n$, defined by the system of Knizhnik-Zamolodchikov (KZ) differential equations

$$\frac{\partial F}{\partial z_i} = A_i F, \quad i = 1, \ldots, n,$$

where

$$A_i := \sum_k Y_{i,k} z_i - \alpha_k - \sum_{p \neq i} \nu s_{ip} z_i - z_p.$$

Lemma 4.1.2. The connection $\nabla$ is flat.

Proof. To show that the curvature is zero, we have to check that

$$\partial_i A_j - \partial_j A_i + [A_i, A_j] = 0.$$

This follows from the following computations:

$$\partial_j A_i = -\frac{\nu s_{ij}}{(z_i - z_j)^2} = \partial_i A_j,$$

---

3Here is the place where we use the fact that $D$ is not of finite Dynkin type.
\[ [A_i, A_j] = \sum_{k,l} \frac{[Y_{i,k}, Y_{j,l}]}{(z_i - \alpha_k)(z_j - \alpha_l)} - \sum_{k,q \neq j} \frac{[Y_{i,k}, \nu_{s_{jq}}]}{(z_i - \alpha_k)(z_j - z_q)} \]

\[ - \sum_{k,p \neq i} \frac{[\nu_{sip}, Y_{j,k}]}{(z_i - z_p)(z_j - \alpha_k)} + \sum_{p \neq i, q \neq j} \frac{[\nu_{sip}, \nu_{s_{jq}}]}{(z_i - z_p)(z_j - z_q)} \]

\[ = \sum_k \frac{\nu(Y_{i,k} - Y_{j,k})s_{ij}}{(z_i - \alpha_k)(z_j - \alpha_k)} - \sum_k \frac{\nu(Y_{i,k} - Y_{j,k})s_{ij}}{(z_i - \alpha_k)(z_j - z_i)} \]

\[ - \sum_k \frac{(z_i - z_j)(z_i - \alpha_k)}{(z_i - z_j)(z_j - z_i)} + \sum_{q \neq i, j} \frac{[\nu_{s_{ij}}, \nu_{s_{jq}}]}{(z_i - z_j)(z_j - z_q)} + \sum_{p \neq i, j} \frac{[\nu_{sip}, \nu_{s_{jq}}]}{(z_i - z_p)(z_j - z_p)} \]

\[ = 0. \]

\[ \square \]

4.2. The monodromy representation of the KZ equations. Taking quotient by the \( S_n \)-action, we get a flat connection, which we will also denote by \( \nabla \), on the configuration space \( Conf_n(\mathbb{C}^1 \setminus \{\alpha_1, \ldots, \alpha_m\}) \). Note that we may replace the trivial bundle over \( (\mathbb{C}^1)^n \) with fiber \( B_n \) by the trivial bundle whose fiber is a \( B_n \)-module \( M \), and this also gives a flat connection \( \nabla_M \) on \( Conf_n \).

When the \( B_n \)-module \( M \) is finite dimensional, it acquires an action of the monodromy operators. Namely, given a base point \( z_0 \in Conf_n \), we can define the End\( M \)-valued solution \( F_0 \) of the KZ equations such that \( F_0(z_0) = 1 \). Then, given \( \sigma \in \pi_1(Conf_n, z_0) \), we let \( F_\sigma \) be the analytic continuation of \( F_0 \) along \( \sigma \), and \( L_\sigma \in \text{End} M \) by \( F_\sigma = F_0 L_\sigma \). Then the monodromy representation \( \rho : \pi_1(Conf_n, z_0) \to \text{Aut}(M) \) is defined\(^4\) by the formula \( \rho(\sigma) = L_\sigma \).

For convenience let us choose \( z_0 = (z_{01}, \ldots, z_{0n}) \) to be such that \( z_{0j}, \alpha_p \) are real and \( \alpha_1 < \ldots < \alpha_m < z_{01} < \ldots < z_{0n} \). In this case we can identify \( \pi_1(Conf_n, z_0) \) with \( Br_{n,m} \) as follows: \( T_i \) is the path in which the points \( z_i, z_{i+1} \) move counterclockwise to exchange positions, and other points don’t move; \( U_k \) is the path in which \( z_1 \) moves counterclockwise around \( \alpha_k \) (passing \( \alpha_{k+1}, \ldots, \alpha_m \) from below). Thus \( \rho \) may be viewed as a representation of the group \( Br_{n,m} \) on \( M \).

Moreover, we claim that that the operators \( T_i \) and \( U_k \) in this representation satisfy the relations

\[ T_i - T_i^{-1} = t - t^{-1}, \]

and

\[ (U_k - u_{k1}) \cdots (U_k - u_{kd_k}) = 0, \]

\(^4\)Our convention for the multiplication of loops in \( \pi_1 \) is as follows: to obtain \( \sigma \sigma' \), first trace \( \sigma' \), then \( \sigma \).
where

\[(4.2.1) \quad u_{kj} := \exp(2\pi i \gamma_{kj}), \quad t = e^{-\pi i \nu}.\]

Indeed, to prove the equation for $U_k$ it suffices to consider the KZ equation for the derivative with respect to $z_1$ with other variables fixed, and look at the eigenvalues of the residue of the connection at the point $z_1 = \alpha_k$. On the other hand, to prove the equation for $T_i$, transform the KZ equations by the change of variables $z_{i,i+1}^+ = \frac{z_i + z_{i+1}^+}{2}$, $z_{i,i+1}^- = \frac{z_{i+1}^+ - z_i^-}{2}$, (leaving the variables other than $z_i, z_{i+1}$ unchanged); then the loop $T_i$ can be realized as a semicircle in which $z_{i,i+1}^-$ goes from some (small) positive value $\zeta$ to $-\zeta$ counterclockwise, and other variables (including $z_{i,i+1}^-$) are unchanged.

Looking at the eigenvalues of the residue of the connection at $z_{i,i+1}^- = 0$, we deduce the equation for $T_i$. Therefore, the monodromy representation of $Br_{n,m}$ on $M$ is in fact a representation of the algebra $H_n$ with parameters $u, t$ as above. Let us denote this representation of $H_n$ by $F(M)$. Thus we have obtained the following result.

**Proposition 4.2.2.** The monodromy of the KZ equations defines a functor $F : \text{Rep}_{f}B_n \rightarrow \text{Rep}_{f}H_n$ between the categories of finite dimensional representations of $B_n$ and $H_n$, under which the parameters $\gamma, \nu$ and $u, t$ are related as above.

**Remark 4.2.3.** This functor, of course, depends on the choice of $\alpha_k$, but only up to fractional-linear transformations.

We note that this proposition allows us to construct a large supply of finite dimensional representations of $H_n$ in the case when $D$ is affine. Indeed, a large supply of finite dimensional representations for $A_n$ and $H(\Gamma_n)$ (and hence for $B_n$) is constructed in [Ga, EM, M], and we can apply the functor $F$ to these representations to obtain representations of $H_n$.

**Second proof of Proposition 3.4.3**

Let $\tilde{B}_n$ be the formal completion of $B_n$ at the point $\gamma = 0, \nu = 0$, and let $\tilde{H}_n$ be the formal completion of $H_n$ at the “unipotent point” $t = 1, \ u_{kj} = 1$.

The monodromy of the KZ equation defines a morphism $f : \tilde{H}_n \rightarrow \tilde{B}_n$, where parameters are related as above (the Riemann-Hilbert homomorphism). It is clear that this homomorphism is in fact an isomorphism (since the relations of $B_n$ are infinitesimal versions of the relations of $H_n$). Thus, Proposition 3.4.3 follows from Corollary 2.3.5.

### 4.3. Cyclotomic Hecke algebras.

Let $H_{n,\ell}$ be the Ariki-Koike cyclotomic Hecke algebra (see e.g. [Ma]). It is an algebra over $\mathbb{C}[v^\pm 1, t^\pm 1]$ (where
Formula \( \chi \) presentation. For generic parameters \( v, t \) \( idempotent \) in \( B \) relations for \( U \) \( \chi \) space of homomorphisms of representations (i.e., it is a flat deformation of the group algebra \( C \) over \( C \) can be naturally regarded as a subspace of \( V \) subspace is equal to \( e \)).

For any representation \( V \), \( T_1, T_2, \ldots, T_{n-1} \) in \( H \) \( \ell \) for all \( v, t \) (i.e., it is a flat deformation of the group algebra \( C[S_n \ltimes (Z/\ell Z)^n]) \).

The algebra \( H_{n, \ell}(v, t) \) has a 1-dimensional representation \( \chi \) given by the formula \( \chi(T_i) = t, \chi(U) = v \). We call this representation the trivial representation. For generic parameters \( v, t \), the representation \( \chi \) defines an idempotent in \( B_{n, \ell}(\lambda, \nu) \). We will denote this idempotent by \( e \).

For any representation \( V \in \text{Rep} H_{n, \ell}(v, t) \), we denote by \( V^{H_{n, \ell}(v, t)} \) the space of homomorphisms of representations \( \chi \to V \). Obviously, this space can be naturally regarded as a subspace of \( V \). In the generic case, this subspace is equal to \( eV \).

Consider the subalgebra \( H_{n-1, \ell} \) of \( H_{n, \ell} \) generated by \( U, T_1, T_2, \ldots, T_{n-2} \). For any representation \( V \) of \( H_{n, \ell}(v, t) \), denote by \( V' \) the space \( V^{H_{n-1, \ell}(v, t)} \).

Let \( X := T_{n-1} \ldots T_1 UT_1 \ldots T_{n-1} \in H_{n, \ell}(v, t) \). In the braid group, the element \( X \) corresponds to the point \( z_n \) making a counterclockwise loop around 0, \( z_1, \ldots, z_{n-1} \); thus \( X \) commutes with \( H_{n-1, \ell}(v, t) \). Therefore, \( X \) acts on the space \( V' \) for any representation \( V \) of \( H_{n, \ell}(v, t) \).

**Lemma 4.3.1.** Let \( V \) be the regular representation of \( H_{n, \ell}(v, t) \). Then for generic parameters the space \( V' \) has dimension \( n \ell \), and the operator \( X |_{V'} \) is conjugate to

\[ v_1^{\ell 2T} \oplus \text{diag}(v_1, \ldots, v_{\ell-1}) \otimes \text{Id}_n, \]

where \( T \) is as in Subsection 2.2.

**Proof.** Let \( V_0 \) be the regular representation of the degenerate cyclotomic Hecke algebra \( B_{n, \ell}(\lambda, \nu) \). Consider the following KZ differential equations for a function \( F(z_1, \ldots, z_n) \) of complex variables \( z_1, \ldots, z_n \) with values in \( V_0 \) (introduced by Cherednik [Ch1]):

\[ \frac{\partial F}{\partial z_i} = \left( \frac{Y_i}{z_i} - \sum_{p \neq i} \frac{\nu_{ip}}{z_i - z_p} \right) F. \]
(these are essentially equations 2.1.1 with $\alpha_m = 0, \alpha_i = \infty$ for $i < m$). Let $V$ be the monodromy representation of this differential equation, with base point $z_0 = (z_{01}, \ldots, z_{0n})$, $0 < z_{01} < z_{02} < \ldots < z_{0n}$. This is a representation of the braid group $Br_{n,2}$ which obviously factors through the Hecke algebra $H_{n,\ell}(v, t)$, where $t = e^{-\pi i v}$ and $v_j = e^{2\pi i \lambda_j}$. It is clear that generically $V$ is the regular representation.

The space $V$ can be thought of as the space of local solutions of the KZ equations around the base point. The space $V^{H_{n-1,\ell}(v, t)}$ then can be viewed as the subspace of solutions $f$ of the form
\[
f = \prod_{j<i<n} (z_i - z_j)^{-\nu}\prod_{i<n} z_i^{\lambda_i} f_0,
\]
where $f_0$ analytically continues to a meromorphic function in the region defined by the inequalities $|z_n - z_{0n}| < \varepsilon$, $|z_i| < z_{0n} - \varepsilon$ for $i < n$ (for some small $\varepsilon$). The operator $X$ acts on the space of such solutions by taking their monodromy around the loop $\sigma$ in which $z_n$ goes counterclockwise around $0, z_1, \ldots, z_{n-1}$. Tending $|z_i|$ to 0, we find that the $n$-th KZ equation tends to the equation $\frac{\partial F}{\partial z_n} = \frac{Y_n - \nu(s_n + \ldots + s_{n,n-1})}{z_n} F$. Therefore, the monodromy around $\sigma$ on $V'$ is conjugate to $e^{2\pi i Y_n(v(s_n + \ldots + s_{n,n-1}))}|_{V'}$. Thus the required statement follows from Lemma 2.4.2.

Here is another, purely algebraic proof of Lemma 4.3.1.

**Proof.** Let $\pi \in H_{n-1,\ell}(v, t)$ be the idempotent of the trivial representation (it exists since generically the algebra $H_{n-1,\ell}(v, t)$ is semisimple); we have $V' = \pi V$.

Let $P_k(x) = \prod_{j=1 \neq k}^{\ell} \frac{x - v_j}{(x - v_k)}$. Let $U_n = T^{-1}_{n-1} \ldots T^{-1}_1 U T_1 \ldots T_{n-1}$. We have $[\pi, U_n] = 0$, and hence $V' = \oplus V_k'$, where $V_k' := P_k(U_n)V'$. Note that $V_k' = y_k V'$, where $y_k = \pi T^{-1}_{n-1} \ldots T^{-1}_1 P_k(U)$, and that $\dim V_k' = n$ for all $k$.

**Lemma 4.3.3.** $X|_{V_k'} = y_k \text{Id}$ for any $k \neq \ell$.

**Proof.** It suffices to show that
\[
X y_k = v_k y_k, \quad k = 1, \ldots, \ell - 1.
\]
Since $X \pi = \pi X$, we find
\[
X y_k = v_k \pi T_{n-1} \ldots T_2 T_1 P_k(U) = v_k \pi T_{n-1} \ldots T_2 T^{-1}_1 P_k(U) + v_k (t - t^{-1}) \pi T_{n-1} \ldots T_2 P_k(U).
\]
Since $P_k(U)$ commutes with $T_{n-1}, \ldots, T_2$ and $\pi P_k(U) = 0$, the last summand is zero. Thus we have
\[
X y_k = v_k \pi T_{n-1} \ldots T_2 T^{-1}_1 P_k(U) = v_k \pi T_{n-1} \ldots T_2 T^{-1}_1 P_k(U) + v_k (t - t^{-1}) \pi T_{n-1} \ldots T_3 T^{-1}_1 P_k(U),
\]
and again the last term is zero. Continuing in this way, we will find that
\[
X y_k = v_k \pi T_{n-1}^{-1} \ldots T^{-1}_2 T^{-1}_1 P_k(U) = v_k y_k.
\]

□
Now let $H_n(t)$ be the usual Hecke algebra of type $A_{n-1}$, and consider the homomorphism $\theta : H_n,t(v,t) \to H_n(t)$ given by $T_i \to T_i$, $U \to v_t$. We have $\theta(X) = v_tT_{n-1}T_2T_1^2T_2...T_n$. Also, $\theta(Y') = 0$ for $k \neq \ell$, while $\theta|_Y$ is injective, and its image $J$ is the space of all elements $y$ in $H_n(t)$ such that $T_iy = ty$ for $i \neq n-1$. By the result of Subsection 4.3 in [Ob1], the operator $T_{n-1}...T_2T_1^2T_2...T_n|_J$ is conjugate to $t^{2T}$. This statement, together with Lemma 4.3.3 implies Lemma 4.3.1.

5. The affine case

5.1. The algebra $H_n$ for affine $D$. From now on let us consider the affine case, i.e. $D = \tilde{D}_4, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$. We keep the notation of subsection 2.5.

It is natural to expect that in this case the algebra $H_n(u,t)$ has properties similar to those of the algebra $B_n(\gamma, \nu)$ stated in Theorem 2.5.1. Unfortunately, we are unable to establish any of these properties, and we are going to state them as conjectures.

Set

$$q = q(u) := \prod_{k,j} u_{kj}^{-t/d_k}.$$  

We have a homomorphism $\eta_m : H_{n,t}(u_m, t) \to H_n(u,t)$, where $u_m := (u_{m_{kj}})$, given by the formulas $\eta_m(T_i) = T_i, \eta_m(U) = U_m$.  

**Conjecture 5.1.1.** (i) The Gelfand-Kirillov dimension of $H_n(u,t)$ is $2n$.

(ii) The algebra $H_n(u,t)$ is PI if and only if $q$ is a root of unity. More precisely, it is PI of degree $n!(N!)^n$ if $q$ is a root of unity of order $N$.

(iii) If $q$ is a root of unity, then $H_n(u,t)$ is finitely generated over its center $Z(H_n(u,t))$.

(iv) If $q$ is a root of unity and otherwise $(u,t)$ are generic then $H_n(u,t)$ is an Azumaya algebra, and $S(u,t) := \text{Spec}(Z(H_n(u,t)))$ is a smooth affine algebraic variety of dimension $2n$.

(v) If $q = 1$ then the map $Z(H_n(u,t)) \to eH_n(u,t)e$ given by $z \mapsto ze$ is an isomorphism. In particular, $eH_n(u,t)e$ is a commutative algebra.

(vi) If $q = 1$ and otherwise $(u,t)$ are generic then every irreducible representation of $H_n(u,t)$ restricts (via the map $\eta_m$) to the regular representation of $H_{n,t}(u_m, t)$.

**Remark 5.1.2.** For $n = 1$, this conjecture follows from the paper [EOR]. Also, for $D = \tilde{D}_4$, because of Proposition 3.3.2 this conjecture can be attacked using the methods of [Sa] and [Ob1] (for example, parts (i)-(iii) and the second statement of (v) follow rather easily from [Sa]); this will be done in a subsequent paper. Finally, using the Riemann-Hilbert homomorphism discussed in Section 4, it can be shown that parts (i)-(iii) and (v) of the conjecture hold for the completed algebra $\hat{H}_n$. On the other hand, for the algebra $H_n$ of types $\tilde{E}_l$, $l = 6, 7, 8$, it is unclear to us how to attack any of the above questions (basically, because we don’t know how to construct
a basis or at least a well behaved filtration of $H_n$, similar to those used in \cite{EOR} for $H_1$).

5.2. **Representations of $H_n(u,t)$ for $q = 1$.** Assume that $q = 1$ and otherwise $(u,t)$ are generic. In this case, the algebra $H_n(u,t)$ has a $2n$-parameter family of representations of dimension $n!\ell^n$, which are constructed as follows. Let $\gamma, \nu$ satisfy equations \((4.2.1)\), and $h = 0$ (this is possible since $q = e^{-h}$). In this case, by Theorem \textbf{2.6.1} $B_n(\gamma, \nu)$ is an Azumaya algebra, so all irreducible representations of $B_n(\gamma, \nu)$ have dimension $n!\ell^n$, and are parametrized by a smooth connected $2n$-dimensional algebraic variety $\mathcal{R}_{n,\gamma,\nu}$. Thus for any $M \in \mathcal{R}_{n,\gamma,\nu}$, we can define a representation $F(M)$ of $H_n(u,t)$, of dimension $n!\ell^n$ (see Subsection 4.2).

**Proposition 5.2.1.** For generic $M \in \mathcal{R}_{n,\gamma,\nu}$, $F(M)$ is irreducible, and $\eta^*_n F(M)$ is the regular representation of $H_n(u_m, t)$.

**Proof.** In the case $\nu = 0$ the statement reduces to the rank 1 case and hence follows from the results of \cite{EOR}. Therefore, the statement holds for generic parameters and generic $M$. \hfill $\square$

Let $\mathbb{R}_{n,u,t}$ be the set of equivalence classes of irreducible representations of $H_n(u,t)$ which restrict (via the map $\eta_m$) to the regular representation of $H_n(u_m,t)$. This is an affine algebraic variety. By Proposition 5.2.1 for generic $M$ as above, $F(M) \in \mathbb{R}_{n,u,t}$.

**Remark 5.2.2.** As we mentioned in the previous subsection, we conjecture that all irreducible representations of $H_n(u,t)$ (for $q = 1$ and otherwise generic $u,t$) restrict to the regular representation of $H_n(u_m,t)$ and thus belong to $\mathbb{R}_{n,u,t}$.

We now want to parametrize irreducible representations of $H_n(u,t)$, by constructing a map $\Phi : \mathbb{R}_{n,u,t} \to \mathbb{M}_{n,u,t}$ of $\mathbb{R}_{n,u,t}$ into some explicitly described algebraic variety $\mathbb{M}_{n,u,t}$, similarly to the map $\Phi$ for $B_n(\gamma, \nu)$ discussed in Subsection 2.6.

The variety $\mathbb{M}_{n,u,t}$ is defined as follows.

**Definition 5.2.3.** $\mathbb{M}_{n,u,t}$ is the variety of conjugacy classes of $m$-tuples $(X_1, \ldots, X_m) \in GL_{n\ell}(\mathbb{C})^m$ satisfying the following equations:

\begin{align*}
  (5.2.4) & \quad X_1X_2 \ldots X_m = 1, \\
  (5.2.5) & \quad X_k \sim \text{diag}(u_{k1}, \ldots, u_{kd_k}) \otimes \text{Id}_{n\ell/d_k}, \quad k = 1, \ldots, m - 1, \\
  (5.2.6) & \quad X_m \sim u_{n\ell\ell^2T} \oplus \text{diag}(u_{m1}, \ldots, u_{m,\ell-1}) \otimes \text{Id}_n.
\end{align*}

That is, $\mathbb{M}_{n,u,t}$ is the categorical quotient of the variety $\tilde{\mathbb{M}}_{n,u,t}$ of $m$-tuples as above by the action of the group $PGL_{n\ell}(\mathbb{C})$.

**Remark 5.2.7.** Thus, $\mathbb{M}_{n,u,t}$ is defined as the variety of solutions of an appropriate multiplicative Deligne-Simpson problem.
Proposition 5.2.8. For generic parameters $\mathcal{M}_{n,u,t}$ is a smooth variety, of dimension $2n$.

Proof. The proof is standard (see also [CBS]). First of all, for generic parameters, any matrices $X_1, \ldots, X_m$ satisfying equations 5.2.4, 5.2.5, 5.2.6 form an irreducible family. Indeed it is easy to see by computing determinants of both sides of (5.2.4) using equations (5.2.5, 5.2.6) that the only nonzero invariant subspace for $X_1, \ldots, X_m$ is the whole space. This implies that the group $PGL_{n\ell}(\mathbb{C})$ acts freely on the variety $\tilde{\mathcal{M}}_{n,u,t}$.

It remains to show that the variety $\tilde{\mathcal{M}}_{n,u,t}$ is smooth, of dimension $2n + n^2\ell^2 - 1$. To do so, let $C_1, \ldots, C_m$ denote the conjugacy classes of $X_1, \ldots, X_m$. We have a map $\mu : C_1 \times \ldots \times C_m \to SL_{n\ell}(\mathbb{C})$ given by $(X_1, \ldots, X_m) \mapsto X_1 \ldots X_m$, and $\tilde{\mathcal{M}}_{n,u,t} = \mu^{-1}(1)$. We have

$$\dim C_k = n^2\ell^2(1 - 1/d_k), \quad k = 1, \ldots, m-1; \quad \dim C_m = 2n - 2 + n^2\ell^2(1 - 1/d_m).$$

Since for affine $D$, $\sum_k (1 - 1/d_k) = 2$, we have

$$\dim(C_1 \times \ldots \times C_m) = 2n - 2 + n^2\ell^2.$$

Thus, to prove the proposition, it suffices to show that 1 is a regular value for the map $\mu$, i.e. that for any $X = (X_1, \ldots, X_m) \in \tilde{\mathcal{M}}_{n,u,t}$, the differential $d\mu_X$ is surjective.

A tangent vector to $X$ in $C_1 \times \ldots \times C_m$ is of the form $([P_1, X_1], \ldots, [P_m, X_m])$, where $P_k$ are some matrices. We have

$$d\mu_X([P_1, X_1], \ldots, [P_m, X_m]) = \sum_{k=1}^m X_1 \ldots X_{k-1}[P_k, X_k]X_{k+1} \ldots X_m =$$

$$\sum_{k=0}^m X_1 \ldots X_k(P_{k+1} - P_k)X_{k+1} \ldots X_m$$

(where we agree that $P_0 = P_{m+1} = 0$). Let $Q_k := P_{k+1} - P_k, \quad k = 1, \ldots, m$ (they can be arbitrary matrices). Then we get

$$d\mu_X([P_1, X_1], \ldots, [P_m, X_m]) = \sum_{k=1}^m [X_1 \ldots X_k, Q_k]X_{k+1} \ldots X_m =$$

$$\sum_{k=1}^m \text{Ad}(X_1 \ldots X_k)(Q_k) - Q_k$$

(as $X_1 \ldots X_m = 1$). Now, since $X_1, \ldots, X_m$ is an irreducible family, we have $\cap_{k=1}^m \text{Ker}(\text{Ad}(X_1 \ldots X_k) - 1) = \mathbb{C}$, and hence dually $\sum_{k=1}^m \text{Im}(\text{Ad}(X_1 \ldots X_k) - 1) = \mathfrak{sl}_{n\ell}(\mathbb{C})$. Thus, $d\mu_X$ is surjective and we are done. □

Remark 5.2.9. We have not shown that the variety $\mathcal{M}_{n,u,t}$ is nonempty. This will follow from the existence of the map $\Phi$ defined below, and also follows from the results of [CBS].
Let us explain the construction of the map $\Phi : \mathbb{R}_{n,u,t} \to \mathbb{M}_{n,u,t}$. Let $V \in \mathbb{R}_{n,u,t}$. Using the map $\eta_m$ we can regard $V$ as a representation of $H_{n,t}(u_m,t)$, which is isomorphic to the regular representation. It is easy to see that the elements $\bar{U}_i := T_{n-1} \ldots T_1 T_i T_{i+1} \ldots T_n$ commute with $H_{n-1,t}(u_m,t)$. The same is true about $\bar{U}_m := T_{n-1} \ldots T_1 T_m T_1 \ldots T_{n-1}$. Thus the operators $\bar{U}_1, \ldots, \bar{U}_{m-1}, \bar{U}_m$ preserve the space of $V' := V^{H_{n-1,t}(u_m,t)}$. We have $\dim V' = n \ell$. We define the map $\Phi$ by the formula:

$$
\Phi(V) = (\bar{U}_1|_{V'}, \ldots, \bar{U}_{m-1}|_{V'}, \bar{U}_m|_{V'}).$

**Proposition 5.2.10.** We have $\Phi : \mathbb{R}_{n,u,t} \to \mathbb{M}_{n,u,t}$.

**Proof.** An easy deformation argument from the group algebra case shows that equation (5.2.8) is satisfied for $X_k = \bar{U}_k|_{V'}$, $k < m$. Equation (5.2.6) is also clearly satisfied. Finally, the fact that equation (5.2.6) holds follows from Lemma 4.3.11. 

**Conjecture 5.2.11.** $\Phi$ is an isomorphism of algebraic varieties.

Recall now that there is a Riemann-Hilbert map between the spaces of solutions of the additive and the multiplicative Deligne-Simpson problems: $RH : \mathcal{M}_{n,\gamma,\nu} \to \mathbb{M}_{n,u,t}$, defined as follows.

Given $x := (x_1, \ldots, x_m) \in \mathcal{M}_{n,\gamma,\nu}$, consider the Fuchsian differential equation

$$(5.2.12) \quad \frac{dF}{dz} = \sum_{k=1}^{m} \frac{x_k F}{z - \alpha_k}. $$

Assume that $z_0 \in \mathbb{R}$ is a base point, and $\alpha_1 < \ldots < \alpha_m < z_0$. Then $RH(x) = X := (X_1, \ldots, X_m)$, where $X_k$ is the monodromy matrix of this differential equation around the loop, in which $z$ goes counterclockwise around $\alpha_k$ passing $\alpha_{k+1}, \ldots, \alpha_m$ from below. This map, of course, depends on the choice of $\alpha_k$, but only up to fractional-linear transformations.

**Proposition 5.2.13.** One has $\Phi \circ F = RH \circ \Phi$.

**Proof.** The proof is similar to the first proof of Lemma 4.3.11. If $V \in \mathcal{R}_{n,\gamma,\nu}$, then $\Phi \circ F(V)$ is the collection of operators $\bar{U}_k$ on the subspace $F(V)'$ of $F(V)$. This subspace can be viewed as the space of solutions of the KZ equations which become single-valued near $z_i = \alpha_m$ and $z_i = z_j$ ($i, j < n$) after division by $\prod_{j<i<n}(z_i - z_j)^{-\nu} \prod_{i<n}(z_i - \alpha_m)^{-\ell}$, and $\bar{U}_k$ are the monodromy operators for such solutions around the loops $\sigma_k$, in which $z_n$ goes counterclockwise around $\alpha_k$ passing $\alpha_{k+1}, \ldots, \alpha_m, z_1, \ldots, z_{n-1}$ from below for $k < m$, and goes around $\alpha_m, z_1, \ldots, z_{m-1}$ for $k = m$. To compute the spectral type of $\bar{U}_k$, we may send $z_i$ with $i < n$ to zero. In this case, the $n$-th KZ equation tends to equation (5.2.12). This implies the required statement.
In conclusion we would like to discuss the dependence of the map $RH$ on the parameters $\alpha_k$. In the $E_i$-cases, there is no such dependence, since there are only three parameters $\alpha_k$, and all collections of them are projectively equivalent. On the other hand, in the $D_4$ case, we have an essential parameter, which is the cross-ratio $\kappa$ of $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. Thus we have a 1-parameter family of holomorphic maps $RH_\kappa : M_{n,\gamma,\nu} \to M_{n,u,t}$. If we fix $X \in M_{n,u,t}$, we can (locally) implicitly solve for $x \in M_{n,\gamma,\nu}$ such that $RH_\kappa(x) = X$. This gives a function $x = x(\kappa, X)$, which defines a flow on the $2n$-dimensional complex manifold $M_{n,\gamma,\nu}$. In the case $n = 1$, this is the Painlevé VI flow; so in general this flow should be regarded as a higher rank version of Painlevé VI. Note that the higher rank Painlevé VI flow has an additional parameter $\nu$, so it has 5 parameters, rather than 4 for the usual Painlevé VI; if $\nu = 0$, the higher rank Painlevé VI flow decouples into a (symmetric) product of $n$ copies of the usual Painlevé VI flows. It would be interesting to write this differential equation explicitly using an appropriate coordinate system on $M_{n,\gamma,\nu}$.

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Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, U.S.A.

E-mail address: etingof@math.mit.edu

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, U.S.A.

E-mail address: wlgan@math.mit.edu

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, U.S.A.

E-mail address: oblomkov@math.mit.edu