Equation governing the evolution of axial-perturbations in space-time of a non-rotating uncharged primordial black hole and conditions of instability from it

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Abstract

We derive the equation governing the axial-perturbations in the space-time of a non-rotating uncharged primordial black hole (PBH), produced in early Universe, whose metric is taken as generalized McVittie metric. The generalized McVittie metric is a cosmological black hole metric, proposed by V. Faraoni and A. Jacques in 2007 [1]. This describes the space-time of a Schwarzschild black hole embedded in FLRW-Universe, while allowing its mass-change. Our derivation is quite similar to the procedure of derivation of S. Chandrasekhar, for deriving the Regge-Wheeler equation for Schwarzschild metric [2]. The equation we derive, is the equivalent counterpart of the Regge-Wheeler equation, in case of generalized McVittie metric. We show that after applying some approximations which are very well valid in the early radiation-dominated Universe, the overall equation governing the axial perturbations can be separated into radial-cum-temporal and angular parts, among which the radial-cum-temporal part is the intended one, as the angular part is identical to the case of Schwarzschild metric as expected. Using this equation, we have analyzed the conditions for which the corresponding modes are unstable or grow exponentially with time.

I. INTRODUCTION:

Perturbations in black hole space-times has been an interesting topic of research for the last few decades. At present, after the first direct detection of gravitational waves and subsequent series of detections of gravitational waves from various similar sources [3-8]; it is even more important for observational estimation of different parameters of gravitational wave sources, related to black holes, specially newly born black holes in ‘ring-down phase’ after merging of a binary of black holes. Black hole perturbation theory is one of the most important aspects for accurately determining characteristics of this type of gravitational wave sources, through gravitational wave astronomy. The perturbations in a black hole’s space-time can be created by various means e.g. (i) perturbations can be generated in the space-time of a black hole due to inspiralling motion of a comparatively very smaller particle (i.e. test-particle) around it; (ii) the merging of two black holes in a binary creates perturbations in the newly born resultant black hole; (iii) infall or interaction of gravitational waves from other sources can also create perturbations in a black hole’s space-time etc. In each of these different cases, black hole perturbation theory is pivotal to study the evolution of perturbations in the space-time of the black hole. In 1957, Tullio Regge and John A. Wheeler derived the equation describing the behaviour or evolution of axial-perturbations in Schwarzschild metric [9] and using this, they studied the stability of the Schwarzschild geometry, when it is subjected to small non-spherical perturbations. This equation is known as ‘Regge-Wheeler’ equation after their name. This can be called effectively the birth of ‘black hole perturbation theory’. Later S. Chandrasekhar derived the same equation in a different procedure and in a more general way[2]. Later in 1970, Zerilli extended the analysis to polar-perturbations in the Schwarzschild space-time [10, 11]. He showed that the equations governing the perturbations can be expressed as a pair of Schrödinger-like equations and he applied the formalism to study the gravitational radiation emitted by infalling test-particles into black holes. The Regge-Wheeler and the Zerilli equations or perturbation techniques developed by them, and their similar counterparts for other different types of black holes, paved the way to investigate the stability of perturbations for certain modes of vibration of the black holes. Instability or stability of perturbations means whether the perturbations grow with time, thereby becoming too large to be handled by the linear perturbation theory or those decay gradually with time respectively. C. V. Visheshwarna analyzed the stability of a Schwarzschild black hole, through a numerical experiment by slightly perturbing it with an infalling wave packet, thereby observing the scattered wave [12]. He found that the scattered wave is a sum of damped sinusoids, whose frequencies and damping times are the ‘quasinormal modes’ i.e. the modes of free vibration of the black hole. The damping implies that the concerned black hole is stable i.e. they return into a stationary state after being perturbed. The outcome can be different for different types of black hole metrics and even can be different for same black hole metric with different environments, i.e. presence of some other matter near it.

In the present work, we derive the equation governing axial-perturbations in the space-time of a cosmological black hole metric, called ‘generalized McVittie metric’, proposed by V. Faraoni and A. Jacques in 2007 [1]. This is a generalization over the original ‘McVittie metric’, given by G. C. McVittie [13]. This metric describes the space-time geometry of a Schwarzschild black hole embedded in FLRW-Universe, while allows change of mass of the black hole. The reason for the choice of this metric in our work, for describing the space-time around non-rotating uncharged primordial black
holes (PBHs) has been described in the section II. PBHs are thought to be produced in early Universe by direct gravitational collapse of regions with sufficient over-density after the horizon re-entry. Our main motivation is to study the evolution of the perturbations in the space-time of a PBH in early radiation-dominated Universe, which has two distinct differences in comparison with the space-time of any astrophysical black hole in present late-time Universe. These differences are: (i) most of the PBHs in early Universe were subjected to rapid rate of change of mass due to either spherical accretion of the surrounding high-density radiation \([14-17]\) or due to Hawking evaporation and (ii) the effect of expansion of the Universe on the space-time around a PBH was significant due to the robust value of Hubble-parameter at that early era, in comparison to the late Universe. Due to these two differences, it is expected that the evolution of metric-perturbations around a PBH in early radiation-dominated Universe would be different than that of around an astrophysical black hole in the Universe of present era and that can not be explained by the usual equations viz. ‘Regge-Wheeler equations’ and ‘Zerilli equations’ for Schwarzschild metric or ‘Tuckolsky master equations’ for Kerr metric. The equation we derive, is clearly the equivalent counterpart of the ‘Regge-Wheeler equation’ in case of generalized McVittie metric. From this equation, we derive the conditions of instability of axial-perturbations in generalized McVittie metric.

The paper is organized as follows: in the section II, we describe the reason for our choice of the generalized McVittie metric for describing the space-time around a non-rotating, uncharged PBH, which is subject to change of mass in the early radiation-dominated Universe. In this section we also describe the convenient forms of the line-element of this metric in different coordinates. In the section III, we describe the derivation of the equation governing the axial-perturbations in generalized McVittie metric to a certain level, and in the section IV we simplify that equation by applying some approximations. We also separate the radial-cum-temporal and angular parts of the equation with the application of separation of variables technique, where the radial-cum-temporal part is the intended equation, as the angular part is identical to that of the Schwarzschild metric. In the section V, we derive the conditions of instability of the axial-perturbations in generalized McVittie metric, using the equation derived in the previous section. In the last section VI, we give final remarks of our work. Besides these, there are four Appendices to complement and clarify various aspects of this work. It is to be noted that we have mainly followed the natural system of units in the analytical calculations, where ‘G’ and ‘c’ are set to 1 or omitted. But, during discussing some numerical-orders, we use G and c in their necessary places in the expressions.

II. THE REASON FOR CHOICE OF THE GENERALIZED MCVITTIE METRIC AND ITS FORMS IN DIFFERENT COORDINATE SYSTEMS:

To derive the equation governing the axial-perturbations in space-time around a PBH in early Universe, first of all the correct metric describing the space-time is to be chosen. As in the early Universe, when the PBHs were produced and the era in which we are interested, the rate of expansion of the Universe i.e. the Hubble-parameter was very high, in comparison with the present era. Therefore, this effect of robust cosmological expansion on the local space-time around the PBHs can not be neglected. There were series of efforts to describe this effect and hence to get a resultant metric, which can describe the space-time of a black hole embedded in an expanding Universe, when the effect of expansion of the Universe on the space-time around the black hole is significant. This type of metrics are usually referred as ‘cosmological black hole metrics’.

One of the first attempts was by McVittie \([13]\). The solution, given by him, is known as ‘McVittie metric’. But, this metric can describe the intended space-time, provided the mass of the black hole is not changing with time. However, in the case of PBHs in early Universe, this condition of constancy of mass would be impractical because, there was highly dense radiation almost everywhere in the radiation-dominated era and hence this high-density radiation was subject to spherical accretion by the PBHs, leading to the growth of masses of the PBHs. Also, as PBHs spanned an enormous mass-range, from the end of inflation \((10^{-32} \text{ s})\) up to the big bang nucleosynthesis \((\sim 1 \text{ s})\) \([18]\) and a large fraction of the PBHs, created in early Universe, were of smaller mass than the Solar-mass ; Hawking-evaporation was a prominent phenomenon for those. Some PBHs were in the mass-range such that for them the rate of loss of mass due to Hawking radiation should be very dominant, such that the rate of gain of mass due to spherical accretion of the surrounding radiation would be insignificant in comparison to the mass-loss rate due to Hawking evaporation. Again, there were some PBHs in the mass rage such that for them the rate of loss of mass due to Hawking evaporation was negligible with respect to the rate of mass-gain by spherical accretion of high-density radiation. So, there is no doubt that most of the PBHs were in the mass range such that they were undergone mass change with time, whether it was gain of mass or loss of mass. So, the condition of constancy of mass is not justified at all for those PBHs. That is why there were more works following the work of McVittie, trying to give a more generalized solution where there is not any restriction on the change of mass. One of the metrics, which has been subject of much interest, is the ‘Schwarschild-De Sitter metric’. When the background metric is chosen to be De Sitter, the McVittie metric reduces to Schwarzschild-De Sitter metric. The ‘Schwarschild-Anti De Sitter metric’ is also of similar interest. But, these are vacuum solutions of Einstein’s equations and hence these do not allow the mass-change of PBHs due to accretion of surrounding radiation, in the early radiation-dominated era of the Universe.

Another two attempts, to describe such black hole metrics, are the Sultana-Dyer solution \([19]\) and McClure-Dyer solution \([20]\). But, each of these have their own shortcomings to describe the metric of a black hole embedded in an expanding FRBW- Universe. Their deficiencies have been briefly described in references \([1, 21]\).

In 2007, V. Faraoni and A. Jacques \([1]\) gave a new solution, better to say they proposed a more generalized extension of the McVittie metric. They showed that it can describe mass-change of a Schwarzschild black hole embedded in an expanding FRBW- Universe, due to spherically accreting surrounding cosmic-fluid (viz. radiation in this case). They called it the ‘Generalized McVittie metric’. More explanation and emphasis on this metric was given in a subsequent work \([22]\).
For the present purpose, we choose this new metric, as it does not require any imposed condition on the change of mass of the black holes, while also it does not seem to have any major theoretical drawbacks. This metric is written in isotropic coordinates as:

\[
ds^2 = -\frac{B^2(t, r)}{A(t, r)} \, dt^2 + a^2(t)A^4(t, r)(dr^2 + r^2d\Omega^2),
\]

where, \(r\) and \(t\) are respectively the isotropic radial and time coordinates. The quantities \(A\) and \(B\) are given by:

\[
A(t, r) = 1 + \frac{m(t)}{2r} \quad \text{and} \quad B(t, r) = 1 - \frac{m(t)}{2r},
\]

where \(m(t)\) is the mass of the PBH and it is time-varying. The \(d\Omega^2 = r^2(d\theta^2 + \sin^2\theta d\phi^2)\) is the usual angular part.

For convenience, the authors of reference [1] introduced the quantity:

\[
C = \left(\frac{\dot{a}}{a} + \frac{\dot{m}}{rA}\right) = \frac{M_H}{M_H} - \frac{\dot{m}B}{mA},
\]

where \(M_H\) represents the ‘Hawking-Hayward quasi-local mass’ of the black hole and \(a(t)\) is the scale-factor of the background FLRW- Universe. The ‘dot’ denotes differentiation with respect to the isotropic time coordinate \(t\).

This metric can be transformed to a Schwarzschild-like coordinate system. C. Gao et al have described this process of transforming the metric into a Schwarzschild-like form, in their work [22]. Yet we are mentioning it briefly as the definitions and inter-relations of corresponding coordinates are required in our work. First by defining the areal radius

\[
\tilde{r} = r \left(1 + \frac{M_H(t)}{2a(t)}\right)^2
\]

and using \(m(t) = M_H(t)/a(t)\), and then introducing the co-moving radial coordinate \(R = a\tilde{r}\), in terms of which \(d\tilde{r} = \frac{dR}{a} - H\tilde{r}dt\), equation 1 is turned into the Painleve-Gullstrand form:

\[
ds^2 = -\left\{1 - \frac{2M_H}{R} - \frac{H\tilde{r} + \dot{m}a}{\tilde{r}}\right\}dt^2 + \frac{1}{1 - \frac{2M_H}{R}}dR^2 + R^2d\Omega^2 - \frac{2}{1 - \frac{2M_H}{R}}\left(H\tilde{r} + \dot{m}a\right)\tilde{r}dtdR.
\]

Further setting

\[
A(t, r) = 1 - \frac{2M_H}{R},
\]

\[
C(t, r) = H\tilde{r} + \dot{m}a\sqrt{\frac{\tilde{r}}{r}},
\]

and defining the time coordinate \(\tilde{t}\) as

\[
\tilde{t} = \frac{1}{F}\left(\frac{dt}{\sqrt{A(A - C^2/d^2)}},
\]

where \(F(t, r)\) is an integrating-factor that makes \(d\tilde{t}\) an exact-differential, one gets

\[
ds^2 = -\left(\frac{A^2 - C^2}{A}\right)\left\{F^2d\tilde{t}^2 + \frac{C^2dR^2}{(A^2 - C^2)^2} - \frac{2FB}{(A^2 - C^2)^2}d\tilde{t}dR\right\} + \frac{dR^2}{A} + R^2d\Omega^2 - \frac{2C}{A}dR\left(Fd\tilde{t} - \frac{C}{(A^2 - C^2)}dR\right).
\]

The cross terms containing \(d\tilde{t}dR\) cancel out and the squared line-element in the new ‘Nolan-gauge’ becomes

\[
ds^2 = -A\left(1 - \frac{C^2}{A^2}\right)F^2d\tilde{t}^2 + A^{-1}\left(1 - \frac{C^2}{A^2}\right)R^2d\Omega^2 + R^2d\Omega^2.
\]

We are using this form 9 in our work. There may be a question why we choose to work with this form 9 in Nolan gauge, which is a Schwarzschild-like form, instead of using the form 1 in isotropic coordinate system, despite the fact that working in isotropic-coordinate system seems to be comparatively simpler. The main reason is that we want to utilize the symmetry of the form 9 viz. if we see the time coordinate as \(t\) such that \(d\tilde{t} = Fd\tilde{t}\), then \(g_{00} = -g_{11}\), where the indices ‘0’ and ‘1’ stand for temporal and radial coordinate respectively. Furthermore, the isotropic radial coordinate \(r\) does not always faithfully represent radial distances. This fact is very disturbing while interpreting the results with practical cases.

III. DERIVATION OF EQUATIONS DESCRIBING THE AXIAL PERTURBATIONS IN GENERALIZED MCVITTIE METRIC:

The square of line-element of a generalized metric can be written as:

\[
ds^2 = -e^{2\nu}d\tau^2 + e^{2\nu}(d\phi - \omega d\tau - q_2dx_2 - q_3dx_3)^2 + e^{2\mu_2}dx_2^2 + e^{2\mu_3}dx_3^2,
\]

where the quantities \(\nu, \psi, \mu_2, \mu_3, \omega, q_2, q_3\) are functions of the coordinates \(\tau, x_2, x_3\). Now we compare the form of the metric given in the equation 10 with the generalized McVittie metric proposed by V. Faraoni et al [1] in Nolan Gauge, as given in equation 9, with the corresponding coordinates being \(\tau \equiv \tilde{\tau}, x_2 \equiv R, x_3 \equiv \theta\) and \(\phi \equiv \psi\) (We follow the index designation : 0, 1, 2, 3 stand for respectively \(\tilde{\tau}, \phi, R, \theta\)). Comparing the metric in equation 10 with the generalized McVittie metric in Nolan Gauge, as given in equation 9, we see that the coefficients \(\nu, \psi, \mu_2, \mu_3, \omega, q_2, q_3\), in case of generalized McVittie metric are given by the set of equations:

\[
e^{-2\nu} + e^{2\psi}e^{2\alpha} = \left(1 - \frac{2M_H}{R}\right)\left(1 - \frac{C^2}{A^2}\right)F^2,
\]

\[
e^{2\nu}q_2^2 + e^{2\psi}q_3^2 = \left(1 - \frac{2M_H}{R}\right)\left(1 - \frac{C^2}{A^2}\right)^{-1},
\]

\[
e^{2\nu}q_2 + e^{2\psi}q_3 = R^2,
\]

\[
e^{2\nu} = R^2\sin^2\theta.
\]

\(^1\) While the absence of cross-terms (i.e. terms with \(dx_2dx_3, d\tilde{r}dx_1, dx_1d\phi\ etc.)\) in the metric 9 indicate that the zeroth order or unperturbed values of the coefficients causing the cross terms are zero even comparing with the metric given in equation 10 : \(\omega = 0, q_2 = 0, q_3 = 0\).

\(^1\) One issue is to be noted here that the metric signatures of the metric in form 10 is opposite to that used in reference [2]. For this reason in each of the expressions we are using here, there will be a negative multiplicity with \(e^{2\alpha}\) (\(\alpha = \nu, \psi, \mu_2, \mu_3\)). While except this sign change, there would not be any other change in the expressions of the Ricci tensors.
Then solving the above set of equations 11 to 14 for the coefficients $\nu$, $\psi$, $\mu_2$ and $\mu_3$, we obtain:

$$
\nu = \frac{1}{2} \ln \left\{ \left(1 - \frac{2M_R}{R} \right) \left(1 - \frac{C^2}{A^2} \right) \right\},
$$

$$
\mu_2 = \frac{1}{2} \ln \left\{ \left(1 - \frac{2M_R}{R} \right)^{-1} \left(1 - \frac{C^2}{A^2} \right) \right\},
$$

$$
\mu_3 = \frac{1}{2} \ln (R^2),
$$

$$
\psi = \frac{1}{2} \ln (R^2 \sin^2 \theta).
$$

Now, we have to get the expressions of the Ricci tensor components $R_{12}$ and $R_{13}$, up to first order perturbations of the quantities $\omega$, $q_2$ and $q_3$. Here we denote the first order or linear perturbations in $\omega$, $q_2$ and $q_3$ as $\delta \omega$, $\delta q_2$, and $\delta q_3$ respectively. But, as we have already stated that in this case the background or zeroth order values of these quantities are zero: $\omega = q_2 = q_3 = 0$, hence the overall quantities can be given by $\delta \omega$, $\delta q_2$, and $\delta q_3$. The axial perturbations (as defined in reference [2]) are characterized by the non-zero values of $\delta \omega$, $\delta q_2$, and $\delta q_3$. Any general perturbation of this metric 10 would generate the perturbations $\delta \omega$, $\delta q_2$, and $\delta q_3$, with zero unperturbed values for the case without any cross-terms. While the non-zero unperturbed values of the quantities $\nu$, $\psi$, $\mu_2$, $\mu_3$ would experience first order perturbations $\delta \nu$, $\delta \psi$, $\delta \mu_2$, and $\delta \mu_3$ respectively. But, as in the case of Schwarzschild metric, in this case of generalized McVittie metric too, these two sets of perturbations have completely different effects. As argued in reference [2], the set of perturbations $\delta \omega$, $\delta q_2$, and $\delta q_3$ induce a dragging of the inertial frame thereby imparting a rotation on the black hole. But, the other set has no such rotational effects. For this reason they are respectively called as Axial and Polar perturbations in reference [2], on the basis of effect of sign-reversal of $\phi$ on the metric. On the basis of this different behaviour, we can physically interpret that they must decouple.

For this reason instead of getting the equations, which govern all the perturbations in a metric, in detail i.e. where both the sets of perturbations will be present, we can extract the part containing the set $\delta \omega$, $\delta q_2$, and $\delta q_3$, and then the part containing the other. The part containing the set of perturbations $\delta \omega$, $\delta q_2$, and $\delta q_3$, will contain the background values of quantities $\nu$, $\psi$, $\mu_2$, $\mu_3$.

Following reference [2], we use the definitions:

$$
Q_{AB} = \delta q_{A,B} - \delta q_{B,A},
$$

$$
Q_{A0} = \delta q_{A,0} - \delta q_{0,A},
$$

where the indices $A$, $B = 2, 3$ in this case.

Using the expressions given in the reference [2], the components of Ricci tensor $R_{12} + \delta \omega_{q_2} R_{12}$ and $R_{13} + \delta \omega_{q_3} R_{13}$ (these give the first order perturbations only, because the background values of these Ricci tensor components $R_{12}$ and $R_{13}$ are zero) in our case are given by:

$$
R_{12} + \delta \omega_{q_2} R_{12} = \frac{1}{2} e^{-2\nu} \left( e^{-2\nu} e^{-2\mu_3} \right)^{1/2} \left[ \left( e^{3\psi+\nu-\omega=2\mu_3} \right) Q_{22} \right],
$$

and

$$
R_{13} + \delta \omega_{q_3} R_{13} = \frac{1}{2} e^{-2\nu} \left( e^{-2\nu} e^{-2\mu_3} \right)^{1/2} \left[ \left( e^{3\psi+\nu-\omega=2\mu_3} \right) Q_{33} \right] + \left( e^{3\psi-\nu-\omega=2\mu_3} \right) Q_{03}.0.
$$

Before proceeding we need the values of the perturbation to the metric components $g_{12}$ and $g_{13}$ i.e. $\delta \omega_{q_2} g_{12}$ and $\delta \omega_{q_3} g_{13}$. It is to be noted that the metric is given in covariant form in the line-element 10. So, the perturbation to the metric components w.r.t. $\omega$, $q_2$ and $q_3$ are given by:

$$
\delta \omega_{q_2} g_{12} = - e^{2\psi} \delta q_2 \text{ and } \delta \omega_{q_3} g_{13} = - e^{2\psi} \delta q_3.
$$

Now, we start getting the equations describing the axial perturbations for the generalized McVittie metric. The origin of the equations has been discussed in APPENDIX 2 IX.

The equation $\omega_{q_2} R_{12} = \frac{1}{2} (\delta \omega_{q_2} g_{12}) R = 0$ is:

$$
-\frac{1}{2} e^{-2\psi} \left( e^{-2\nu} e^{-2\mu_3} \right)^{1/2} \left[ \left( e^{3\psi+\nu-\omega=2\mu_3} \right) Q_{23} \right],+ \left( e^{3\psi-\nu-\omega=2\mu_3} \right) Q_{02}.0 = \frac{1}{2} \left( e^{-2\psi} \delta q_2 \right) R.
$$

and the equation $\omega_{q_3} R_{13} = -\frac{1}{2} (\delta \omega_{q_3} g_{13}) R = 0$ is:

$$
-\frac{1}{2} e^{-2\psi} \left( e^{-2\nu} e^{-2\mu_3} \right)^{1/2} \left[ \left( e^{3\psi+\nu-\omega=2\mu_3} \right) Q_{33} \right] - \left( e^{3\psi-\nu-\omega=2\mu_3} \right) Q_{03},0 = \frac{1}{2} \left( e^{-2\psi} \delta q_3 \right) R.
$$

The $R$ in the above equations is the Ricci-scalar. Although we shall see later, that $R$ vanishes in the scenario of our interest, yet we keep the $R$ in the equations to a certain stage, so that the appearance of the $R$ in the desired final equation can be checked. In any more general case, where the $R$ does not vanish, this may be utilised.

After shifting the factor $e^{-2\psi}$ from LHS to RHS of the equations 23 and 24, we write these as respectively:

$$
-\frac{1}{2} e^{-2\nu} \left( e^{-2\psi} e^{-2\mu_3} \right)^{1/2} \left[ \left( e^{3\psi+\nu-\omega=2\mu_3} \right) Q_{23} \right],+ \left( e^{3\psi-\nu-\omega=2\mu_3} \right) Q_{02}.0 = \frac{1}{2} e^{4\psi} \left( -\delta q_2 \right) R,
$$

and

$$
\frac{1}{2} e^{-2\nu} \left( e^{-2\psi} e^{-2\mu_3} \right)^{1/2} \left[ \left( e^{3\psi+\nu-\omega=2\mu_3} \right) Q_{33} \right] - \left( e^{3\psi-\nu-\omega=2\mu_3} \right) Q_{03},0 = \frac{1}{2} e^{4\psi} \left( -\delta q_3 \right) R.
$$

Before proceeding we define some quantities for brevity and compactness of the upcoming equations, as was defined in reference [2] for Schwarzschild metric, as below:

$$
Q(\xi, R, \theta) = Q_{23} \Delta \sin^2 \theta,
$$

where the quantity $\Delta$ is given by:

$$
\Delta = R^2 - 2M_R R.
$$

Hence,

$$
\left( 1 - \frac{2M_R}{R} \right) = \left( \frac{R^2 - 2M_R R}{R^2} \right) = \Delta / R^2.
$$

Following expressions of the unperturbed coefficients present in the metric of form 10 for the metric 9 are useful:

$$
e^{3\psi} = R^3 \sin^2 \theta,
$$

$$
e^\nu = \left( 1 - \frac{2M_R}{R} \right)^{1/2} \left( 1 - \frac{C^2}{A^2} \right)^{1/2} F,
$$

$$
e^{\mu_2} = \left( 1 - \frac{2M_R}{R} \right)^{1/2} \left( 1 - \frac{C^2}{A^2} \right)^{1/2} F.
$$
Now, substituting the expressions of $e^{2\alpha}$ ($\alpha = \nu, \psi, \mu_2, \mu_3$), in the equations 25 and 26, we obtain the quantities present in these equations for the metric given in equation 9. According to the convention and notation of S. Chandrasekhar in reference [2], the quantity $Q_{02}$ is given by:

$$Q_{02} = -Q_{20} = (\delta \omega, -\delta q_{2,0}).$$ (30)

Hence, the equation 25 in our case (rearranging it in a form of our convenience) can be written as:

$$\frac{F^2}{R^4 \sin^3 \theta} \left(1 - \frac{C^2}{A^2}\right) \frac{\partial Q}{\partial \theta} = F^2 R \sin \theta \left(1 - \frac{2M}{R}\right) \frac{1}{2} \left(1 - \frac{C^2}{A^2}\right) \frac{\partial \delta q_{2}}{\partial R}$$

$$- \left(\frac{\partial^2 \delta \omega}{\partial R \partial R} - \frac{\partial^2 \delta q_{2}}{\partial R} \right) - \left(\frac{\partial \delta \omega}{\partial R} \frac{\partial \delta q_{2}}{\partial R} \right) \frac{F}{R^2} \frac{\partial R}{\partial F} \left(\frac{R}{F}\right).$$

(31)

On the other hand, equation 26 in our case, for the metric given in equation 9, can be written as below:

$$\frac{\Delta}{R^4 \sin^3 \theta} \left\{ \frac{\partial Q}{\partial R} \left(1 - \frac{C^2}{A^2}\right) F^{-1} \frac{\partial}{\partial R} \left(F \left(1 - \frac{C^2}{A^2}\right)\right) \right\}$$

$$- \left(1 - \frac{C^2}{A^2}\right)^{-2} F^{-1} \frac{\partial}{\partial R} \left(\frac{\partial \delta \omega}{\partial R} \frac{\partial \delta q_{2}}{\partial R}\right)$$

$$- \frac{\Delta}{R^4 \sin^3 \theta} \left(1 - \frac{C^2}{A^2}\right) F^{-1} \left(\frac{\partial \delta \omega}{\partial R} \frac{\partial \delta q_{2}}{\partial R}\right)$$

$$= -\delta q_{3} \Delta \sin \theta \left(1 - \frac{C^2}{A^2}\right)^{-1} R.$$ (32)

At this point, we shall face difficulty if we proceed in the usual way as followed by S. Chandrasekhar for Schwarzschild metric [2] i.e. if we assume that the perturbations have time-dependence proportional to $e^{i\sigma t}$, where $\sigma$ is a constant, then the resulting terms will be complicated as in the coordinate system, which we are using, the radial coordinate $R$ and the temporal coordinate $T$ are inter-related. For the same reason, in the term $\frac{\partial^2 \delta q_{2}}{\partial R \partial t}$, the partial derivatives w.r.t. $R$ and $T$ can not be commuted. So, to avoid this complication in the calculation, we shall assume that the perturbations have time-dependence proportional to $e^{i\sigma t}$ and they can be expressed as: $\delta \alpha \sim \delta \alpha (r, \theta) e^{i\sigma t}$, where $\alpha = \omega, q_2, q_3$.

Before inserting this time-dependence into the equations 31 and 32, we calculate the double-partial derivatives of the perturbations in terms of $r$ and $t$:

$$\frac{\partial^2 \delta \omega}{\partial R \partial R} = \left\{ F \left(1 - \frac{C^2}{A^2}\right) \frac{\partial}{\partial t} \left(a^{-1} \left(1 - \frac{M^2}{A^2}\right)\right) \right\}$$

$$\frac{\partial \delta \omega}{\partial t} + \left\{ F \left(1 - \frac{C^2}{A^2}\right) \left(a^{-1} \left(1 - \frac{M^2}{A^2}\right)\right) \right\} i \sigma \frac{\partial \delta \omega}{\partial t},$$

(33)

$$\frac{\partial^2 \delta q_{2}}{\partial R \partial R} = \left\{ F \left(1 - \frac{C^2}{A^2}\right) \frac{\partial}{\partial t} F \left(1 - \frac{C^2}{A^2}\right) \right\} i \sigma \delta q_{2}$$

$$+ F^2 \left(1 - \frac{C^2}{A^2}\right)^2 (i \sigma)^2 \delta q_{2}.$$ (34)

Inserting the above expression of the partial derivatives of $\delta \omega$ and $\delta q_{2}$ w.r.t. $r$ and $t$ in the equation 31, we obtain:

$$\frac{F^2}{R^4 \sin^3 \theta} \left(1 - \frac{C^2}{A^2}\right) \frac{\partial Q}{\partial \theta} =$$

$$F^2 R \sin \theta \left(1 - \frac{2M}{R}\right) \frac{1}{2} \left(1 - \frac{C^2}{A^2}\right) \frac{\partial \delta q_{2}}{\partial R}$$

$$- \left\{ a^{-1} \left(1 - \frac{M^2}{A^2}\right) \frac{\partial \delta \omega}{\partial r} - F \left(1 - \frac{C^2}{A^2}\right) i \sigma \delta q_{2} \right\} F \left(1 - \frac{C^2}{A^2}\right) \frac{\partial R}{\partial F}$$

$$- \left[ F \left(1 - \frac{C^2}{A^2}\right) \frac{\partial}{\partial t} \left(a^{-1} \left(1 - \frac{M^2}{A^2}\right)\right) \right] \frac{\partial \delta \omega}{\partial r}$$

$$+ \left\{ F \left(1 - \frac{C^2}{A^2}\right) \left(a^{-1} \left(1 - \frac{M^2}{A^2}\right)\right) \right\} i \sigma \frac{\partial \delta \omega}{\partial t} -$$

$$\left\{ F \left(1 - \frac{C^2}{A^2}\right) \frac{\partial}{\partial t} \left(F \left(1 - \frac{C^2}{A^2}\right)\right) \right\} i \sigma - F^2 \left(1 - \frac{C^2}{A^2}\right)^2 (i \sigma)^2 \delta q_{2}. \right\}$$

(35)

Similarly, after converting the partial derivatives of the perturbations w.r.t. $T$ into that w.r.t. $t$ and then inserting those in the equation 32, we get:

$$\frac{\Delta}{R^4 \sin^3 \theta} F \left(1 - \frac{C^2}{A^2}\right) \frac{\partial}{\partial R} \left\{ F \left(1 - \frac{C^2}{A^2}\right) Q \right\}$$

$$= -\delta q_{3} \left(1 - \frac{C^2}{A^2}\right) \Delta \sin \theta F^2 R + \left[ i \sigma F \left(1 - \frac{C^2}{A^2}\right) \right]$$

$$+ \frac{\Delta}{R^4 \sin^3 \theta} F \left(1 - \frac{C^2}{A^2}\right) \frac{\partial}{\partial t} \left\{ \frac{R^4 \sin^3 \theta}{\Delta} F^{-1} \left(1 - \frac{C^2}{A^2}\right) \right\} \frac{\partial \delta \omega}{\partial \theta}$$

$$+ \left[ -i \sigma F \left(1 - \frac{C^2}{A^2}\right) \frac{\partial}{\partial t} \left(F \left(1 - \frac{C^2}{A^2}\right)\right) - F^2 \left(1 - \frac{C^2}{A^2}\right)^2 \frac{\partial}{\partial \theta} \left(R^4 \sin^3 \theta \right)^{-1} \right] F^2 \left(1 - \frac{C^2}{A^2}\right)^2 (i \sigma)^2 \delta q_{3}. \right\}$$

(36)

Now, we have to eliminate $\delta \omega$ from these two equations 35 and 36, so that we may obtain an equation describing the perturbations $\delta q_{2}$ and $\delta q_{3}$ only. For this, we start by partially differentiating these two equations w.r.t. $\theta$ and $r$ respectively, so that we may have the quantities $\frac{\partial^2 \delta \omega}{\partial \theta \partial \theta} = \frac{\partial^2 \delta \omega}{\partial r \partial r}$ in both the differentiated equations and we can replace that. For brevity we denote $\frac{\partial}{\partial \theta} \left(\frac{\partial}{\partial \theta}\right) = \partial_{\theta \theta}$. Hence, partially differentiating both sides of the equation 35 w.r.t. $\theta$, we obtain the equation:

$$\frac{F^2}{R^4} \left(1 - \frac{C^2}{A^2}\right) \frac{\partial}{\partial \theta} \left(\frac{1}{\sin^3 \theta} \frac{\partial Q}{\partial \theta}\right) =$$

$$F^2 R \left(1 - \frac{2M}{R}\right) \frac{1}{2} \left(1 - \frac{C^2}{A^2}\right) \frac{\partial \delta q_{2}}{\partial R}$$

$$\left[ F \left(1 - \frac{C^2}{A^2}\right) \frac{\partial}{\partial t} \left(a^{-1} \left(1 - \frac{M^2}{A^2}\right)\right) \right] + F \left(1 - \frac{C^2}{A^2}\right)^2 \frac{\partial^2 \delta \omega}{\partial r^2}$$

$$+ i \sigma \left\{ F \left(1 - \frac{C^2}{A^2}\right) \frac{\partial}{\partial t} \left(F \left(1 - \frac{C^2}{A^2}\right)\right) + F \left(1 - \frac{C^2}{A^2}\right)^2 \frac{\partial \delta q_{2}}{\partial t} \right\} \right\}.$$ (37)

Next, partially differentiating both sides of the equation 36 w.r.t. $r$, we obtain:
\[
\frac{1}{\sin^3 \theta} \frac{\partial}{\partial r} \left[ \frac{\Delta}{R^4} F \left( 1 - \frac{C^2}{A^2} \right) \frac{\partial}{\partial r} \left\{ F \left( 1 - \frac{C^2}{A^2} \right) Q \right\} \right] = -\sin \theta \frac{\partial}{\partial \theta} \left\{ \delta q_3 \left( 1 - \frac{C^2}{A^2} \right) \Delta F^2 R \right\} \\
+ \left\{ \frac{\partial}{\partial \theta} \left[ \frac{\Delta}{R^4} F \left( 1 - \frac{C^2}{A^2} \right) \frac{\partial}{\partial \theta} \left\{ R^4 \sin^3 \theta F^{-1} \left( 1 - \frac{C^2}{A^2} \right)^{-1} \right\} \right] \right\} \delta \omega \spired \delta \omega \\
+ \left\{ \sigma F \left( 1 - \frac{C^2}{A^2} \right) + \frac{\Delta}{R^4 \sin^3 \theta} F \left( 1 - \frac{C^2}{A^2} \right) \frac{\partial}{\partial \theta} \left\{ \frac{R^4 \sin^3 \theta}{\Delta} F^{-1} \left( 1 - \frac{C^2}{A^2} \right)^{-1} \right\} \right\} \delta q_1 \frac{\partial}{\partial r} \\
+ F \left( 1 - \frac{C^2}{A^2} \right) \left[ -\sigma \frac{\partial}{\partial t} \left( F \left( 1 - \frac{C^2}{A^2} \right) \right) - \sigma \frac{\Delta}{R^4 \sin^3 \theta} F \left( 1 - \frac{C^2}{A^2} \right) \frac{\partial}{\partial \theta} \left\{ \frac{R^4 \sin^3 \theta}{\Delta} F^{-1} \left( 1 - \frac{C^2}{A^2} \right)^{-1} \right\} + F \left( 1 - \frac{C^2}{A^2} \right) \left( \sigma \right) \right] \delta q_3 \frac{\partial}{\partial r} \\
+ \left\{ \delta q_1 \frac{\partial}{\partial r} \left( F \left( 1 - \frac{C^2}{A^2} \right) \right) - \sigma \frac{\Delta}{R^4 \sin^3 \theta} F \left( 1 - \frac{C^2}{A^2} \right) \frac{\partial}{\partial \theta} \left\{ \frac{R^4 \sin^3 \theta}{\Delta} F^{-1} \left( 1 - \frac{C^2}{A^2} \right)^{-1} \right\} + F \left( 1 - \frac{C^2}{A^2} \right) \right\} \delta q_3 \frac{\partial}{\partial r} \right\} \frac{\partial}{\partial \theta} \left\{ (42) \right\}
\]

For brevity, from now we designate the quantity \( \frac{\Delta}{R^4} F \left( 1 - \frac{C^2}{A^2} \right) F \left( 1 - \frac{C^2}{A^2} \right) \) as \( F \). Now we have to substitute the quantity \( \frac{\partial q_3}{\partial \theta} \) from the equation 37 in the equation 38. But, the equation, which is obtained after substituting the expression of \( \frac{\partial q_3}{\partial \theta} \) from the equation 37, in the equation 38, would have a term containing \( \frac{\partial q_3}{\partial \theta} \). Therefore, in that equation, we have to again substitute the \( \frac{\partial q_3}{\partial \theta} \) from the equation 36, to make in completely in-terms of the perturbations \( \delta q_1 \) and \( \delta q_2 \). Doing these substitutions, we obtain :

\[
\frac{1}{\sin^3 \theta} \frac{\partial}{\partial r} \left[ \frac{\Delta}{R^4} F \left( 1 - \frac{C^2}{A^2} \right) \frac{\partial}{\partial r} \left\{ F \left( 1 - \frac{C^2}{A^2} \right) Q \right\} \right] = \\
- \left\{ \frac{\partial}{\partial \theta} \left[ \frac{\Delta}{R^4} F \left( 1 - \frac{C^2}{A^2} \right) \frac{\partial}{\partial \theta} \left\{ F \left( 1 - \frac{C^2}{A^2} \right) Q \right\} \right] \right\} \right\} \right\} \left\{ \frac{\Delta}{R^4 \sin^3 \theta} F \left( 1 - \frac{C^2}{A^2} \right) \frac{\partial}{\partial \theta} \left\{ F \left( 1 - \frac{C^2}{A^2} \right) Q \right\} + \delta q_3 \left( 1 - \frac{C^2}{A^2} \right) \Delta \sin \theta F^2 R \right\} \\
- \frac{\partial}{\partial \theta} \left\{ \frac{\partial}{\partial \theta} \left( F \left( 1 - \frac{C^2}{A^2} \right) \right) + F + i \sigma F \left( 1 - \frac{C^2}{A^2} \right) \right\} \right\} \right\} \right\} \right\} \right\} \delta q_3 .
\]

At this point, some simplification is required before further proceeding with the equation 39. It is to be noted that some terms on the RHS of the equation 39 contain the Ricci scalar \( R \). We check the value of the Ricci scalar in this regard. Now, we show that at any arbitrary distance from a PBH, described by an arbitrary spherically symmetric and diagonal metric, the Ricci scalar vanishes when the cosmic fluid is radiation i.e. has the equation-of-state parameter 1/3. This can be shown using the Einstein’s equation in the following way. The Einstein’s equation gives :

\[
R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} = (8\pi) T_{\mu \nu} .
\]

Contracting both sides of the above equation 40 with \( g^{\mu \nu} \), we get

\[
\nabla^2 R = -4 R = (8\pi) T_{\mu \nu} g^{\mu \nu} .
\]

Now, we substitute the stress-energy tensor component for an imperfect fluid :

\[
T_{\mu \nu} = (p + \rho) u_{\mu} u_{\nu} + p g_{\mu \nu} + (\gamma_{\mu} u_{\nu} + \gamma_{\nu} u_{\mu}) + \Pi_{\mu \nu} ,
\]

in the RHS of the above equation 41. Here, \( \gamma_{\mu} \) is the heat-flux vector and \( \Pi_{\mu \nu} \) is the viscous-shear tensor for the concerned fluid; while \( u_{\mu} \) denotes the four-velocity of the fluid. Then, on the RHS of the equation 41, we get, after contracting \( T_{\mu \nu} \) with \( g^{\mu \nu} \):

\[
R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} = (8\pi) T_{\mu \nu} ,
\]
\[ T_{\mu\nu}g^{\mu\nu} = (\rho + p)u_{\mu}u_{\nu}g^{\mu\nu} + pg_{\mu\nu}g^{\mu\nu} + (\gamma_m u_\nu + \gamma_\nu u_m)g^{\mu\nu} + \Pi_{\mu\nu}g^{\mu\nu} \]
\[ = -(\rho + p) + 4p + (\gamma_m u_\nu + \gamma_\nu u_m) + \Pi_{\mu\nu}g^{\mu\nu} \]  
\[ = -\rho + 3p + 2\gamma_m u_\nu + \Pi_{\mu\nu}g^{\mu\nu}. \]  
\[ (43) \]

In our case the concerned cosmic fluid is radiation, then \( p = \rho/3 \). Again, as the heat-flux vector is transverse to the world-lines, \( \gamma_m u^\nu = 0 \). Using these finally we obtain,
\[ T_{\mu\nu}g^{\mu\nu} = \Pi_{\mu\nu}g^{\mu\nu}. \]  
\[ (44) \]

Therefore, if we assume that the radiation is viscosity free or \( \Pi_{\mu\nu} = 0 \), then \( T_{\mu\nu}g^{\mu\nu} = 0 \). Then, the equation 41 gives \( R = 0 \). Now, using the fact that, in the case of our interest the Ricci scalar vanishes everywhere around the black hole, the equation 39 simplifies to:
\[ \frac{1}{\sin^3 \theta} \frac{\partial}{\partial r} \left[ \frac{\Delta}{R^2} F \left( 1 - \frac{C^2}{A^2} \right) \frac{\partial}{\partial R} \left\{ F \left( 1 - \frac{C^2}{A^2} \right) Q \right\} \right] = - \left\{ \frac{i\sigma F \left( 1 - \frac{C^2}{A^2} \right) + F}{i\sigma F \left( 1 - \frac{C^2}{A^2} \right) + \mathcal{F} + \mathcal{F}} \right\} \left\{ \frac{\partial}{\partial \theta} \left( \frac{\partial q_3}{\partial \theta} \right) - F \left( \frac{\partial q_3}{\partial \theta} \right) \right\} + \left\{ \frac{\partial}{\partial \theta} \left( \frac{\partial q_3}{\partial \theta} \right) - F \left( \frac{\partial q_3}{\partial \theta} \right) \right\} \delta q_3. \]  
\[ (45) \]

At this stage, we note that the above equation 45 can yet not be expressed completely in terms of the perturbation variable \( Q \). But, we see that after applying certain approximations, the above equation 45 can be expressed w.r.t. \( Q \) only and subsequently the separation of variables technique can be applied to it. We describe this in the next section.

IV. THE SIMPLIFIED EQUATION AFTER APPLYING THE APPROXIMATIONS, SEPARATING THE EQUATIONS FOR RADIAL AND ANGULAR PARTS OF \( Q \):

After applying the approximations described in the APPENDIX-4 i.e. section XI, including the approximation
\[ \frac{i\sigma F \left( 1 - \frac{C^2}{A^2} \right) + \mathcal{F}}{i\sigma F \left( 1 - \frac{C^2}{A^2} \right) + \mathcal{F} + \mathcal{F}} \approx 1, \]
writing the partial derivatives w.r.t. \( r \) in terms of that w.r.t. \( R \), then cancelling the quantity \( a \left( 1 - \frac{M_0}{4\pi r^3 a^2} \right) \) from both sides of the equation and expressing the quantities \( Q_{23} \) in terms of \( Q \), the equation 45 can be written as:
\[ \frac{\partial}{\partial R} \left\{ \frac{\Delta}{R^2} F \left( 1 - \frac{C^2}{A^2} \right) \frac{\partial}{\partial R} \left\{ F \left( 1 - \frac{C^2}{A^2} \right) Q \right\} \right\} = - \left\{ \frac{F^2}{R^2} \left( 1 - \frac{C^2}{A^2} \right) \left( -3C\cot \theta \frac{\partial Q}{\partial \theta} + \frac{\partial^2 Q}{\partial \theta^2} \right) \right\} + \left\{ \frac{i\sigma F \left( 1 - \frac{C^2}{A^2} \right) + F}{i\sigma F \left( 1 - \frac{C^2}{A^2} \right) + \mathcal{F} + \mathcal{F}} \right\} \left\{ \frac{\partial}{\partial \theta} \left( \frac{\partial q_3}{\partial \theta} \right) - F \left( \frac{\partial q_3}{\partial \theta} \right) \right\} + \left\{ \frac{\partial}{\partial \theta} \left( \frac{\partial q_3}{\partial \theta} \right) - F \left( \frac{\partial q_3}{\partial \theta} \right) \right\} \delta q_3. \]  
\[ (46) \]

Thereafter we express the quantity \( Q \) as the multiplication of two parts \( Q_R(R,T) \equiv Q_R(r,t) \) and \( Q_\theta(\theta) \) as \( Q = Q_RQ_\theta \), where the part \( Q_R \), called the radial part, is a function of \( r \) and \( t \) or of \( R \) and \( T \); while the part \( Q_\theta \), called the angular part, is a function of \( \theta \). We now substitute \( Q = Q_RQ_\theta \) in the equation 46, to implement the procedure known as separation of variables.
technique and thus we obtain:

\[
\frac{1}{Q_R} R^4 (1 - \frac{C^2}{A^2})^{-1} \frac{\partial}{\partial R} \left[ \frac{\Delta}{R^4} F \left( 1 - \frac{C^2}{A^2} \right) \frac{\partial}{\partial R} \left\{ F \left( 1 - \frac{C^2}{A^2} \right) Q_R \right\} \right] + \frac{R^4}{F} \left\{ \left\{ F \left( 1 - \frac{C^2}{A^2} \right) \sigma - i \sigma \frac{\partial}{\partial \theta} \left( F \left( 1 - \frac{C^2}{A^2} \right) \right) \right\} - i \sigma \mathcal{F} \right\} \left( \frac{1}{\Delta} \right) \]  

(47)

\[
- \frac{1}{Q_R} \frac{\partial}{\partial R} \left\{ i \sigma F \left( 1 - \frac{C^2}{A^2} \right) + \mathcal{F} \right\} A \frac{\partial}{\partial \theta} \left\{ F \left( 1 - \frac{C^2}{A^2} \right) Q_R \right\} = - \frac{1}{Q_R} \left( - 3 C \cot \theta \frac{\partial Q_R}{\partial \theta} + \frac{\partial^2 Q_R}{\partial \theta^2} \right) = K.
\]  

(48)

Now, in the above equation 47, the LHS is a function of R and \( \bar{t} \) (or, \( r \) and \( t \)), while the RHS is a function of \( \theta \). Therefore, we can say that in the above equation 47, the LHS = RHS = constant, which is independent of \( r \), \( t \) and \( \theta \). Let this constant be \( K \). Then, we can write two equations resulting from the equation 47 as:

\[
- \frac{1}{Q_R} \left( - 3 C \cot \theta \frac{\partial Q_R}{\partial \theta} + \frac{\partial^2 Q_R}{\partial \theta^2} \right) = K
\]  

(49)

So, we see that the equation satisfied by the angular part \( Q_{\theta} \) i.e. the equation 48, is same as that is obtained in case of the Schwarzschild metric, as expected. Hence, the angular part in this case too, like the Schwarzschild metric, can be taken as: \( Q_{\theta} \propto C_{l+1/2}^{-3/2} \) [2]; where \( C_{l+1/2} \) is the ‘Gegenbauer function’. This function \( C_{l+1/2} \) satisfies the equation:

\[
\left[ \frac{d}{d \theta} \sin^{2\nu} \theta \frac{d}{d \theta} + n(2 \nu) \sin^{2\nu} \theta \right] C_{l+1/2}^{2\nu} = 0.
\]  

(50)

It may be noted that the ‘Gegenbauer function’ \( C_{l+1/2}^{-3/2} \), associated with this case is related to the ‘Legendre Polynomial function’ \( \Phi_l(\theta) \) by the formulae:

\[
C_{l+1/2}^{-3/2} (\theta) = \sin^{3/2} \theta \frac{d}{d \theta} \left( \frac{1}{\sin \theta} \frac{d}{d \theta} \Phi_l(\theta) \right).
\]  

(51)

The angular part \( Q_{\theta} \) being proportional to the \( C_{l+1/2}^{-3/2} \), sets the constant \( K \) and hence, the equation satisfied by the radial part \( Q_{R} \) becomes:

\[
\frac{R^4}{F^2} \left( 1 - \frac{C^2}{A^2} \right)^{-1} \frac{\partial}{\partial R} \left[ \frac{\Delta}{R^4} \right] \left\{ F \left( 1 - \frac{C^2}{A^2} \right) \frac{\partial}{\partial R} \left\{ F \left( 1 - \frac{C^2}{A^2} \right) Q_R \right\} \right\} - \frac{\frac{\partial}{\partial R} \left\{ i \sigma F \left( 1 - \frac{C^2}{A^2} \right) + \mathcal{F} \right\} A \frac{\partial}{\partial \theta} \left\{ F \left( 1 - \frac{C^2}{A^2} \right) Q_R \right\} \]  

\[+ R^4 \left\{ F \left( 1 - \frac{C^2}{A^2} \right) \sigma - i \sigma \frac{\partial}{\partial \theta} \left( F \left( 1 - \frac{C^2}{A^2} \right) \right) \right\} - i \sigma \mathcal{F} \right\} \left( \frac{Q_R}{\Delta} \right) = \mathcal{M}^2 Q_R,
\]  

(52)

where the quantity \( \mathcal{M} = (l + 2)(l - 1) \), where \( l \) is a positive integer \( \geq 2 \), describes the angular dependence. Now, we shall transform the radial coordinate from \( R \) to \( R_* \), where the new coordinate \( R_* \) is defined as:

\[
\frac{\partial}{\partial R_*} = \frac{\Delta}{R^2} \frac{\partial}{\partial R}.
\]  

(53)

Again, we define a new variable \( \tilde{Q}_R = \left( 1 - \frac{C^2}{A^2} \right) F Q_R \).

Using the new radial coordinate \( R_* \) and the new variable \( \tilde{Q}_R \), putting \( \mathcal{F} \) in place of \( F \) (as \( F \approx \mathcal{F} \), the approximation which we
have already applied), the above equation 52 can be written as:

\[
\frac{\partial^2 \tilde{Q}_R}{\partial R_*^2} + \left[ \frac{\partial}{\partial R_*} \left\{ \frac{1}{R^2} F \left( 1 - \frac{C^2}{A^2} \right) \right\} - i \sigma \frac{\partial}{\partial R_*} \left\{ \sigma F \left( 1 - \frac{C^2}{A^2} \right) + F \right\} \right] \frac{\partial \tilde{Q}_R}{\partial R_*} \\
+ \left[ \sigma^2 - i \sigma \frac{\partial}{\partial R_*} \left( \frac{F \left( 1 - \frac{C^2}{A^2} \right)}{F \left( 1 - \frac{C^2}{A^2} \right)} \right) - i \sigma \frac{F \left( 1 - \frac{C^2}{A^2} \right) + F}{F \left( 1 - \frac{C^2}{A^2} \right)} \right] \tilde{Q}_R = \mathcal{M}^2 \Delta R^4 \left( 1 - \frac{C^2}{A^2} \right)^{-1} \tilde{Q}_R.
\]

(54)

The above equation 54 (or, its different forms) can be called the equivalent-counterpart of the ‘Regge-Wheeler equation’ for generalized McVittie metric: the equation governing the ‘Axial perturbations’ in the space-time of a non-rotating uncharged primordial black hole embedded in FLRW-Universe, described by the generalized McVittie metric. Sometimes it may be necessary to make the equation 54 free of the single-derivative term. So, we briefly describe the transformation of the equation 54 into the form containing only second-order derivative term. The process of elimination of first-order derivative term, from a general second-order differential-equation, is quite well-known. Let us first write the equation 54 with the following notations for brevity:

\[
\frac{\partial^2 \tilde{Q}_R}{\partial R_*^2} + \xi(R, \bar{T}) \frac{\partial \tilde{Q}_R}{\partial R_*} + \zeta(R, \bar{T}) \tilde{Q}_R = 0,
\]

(55)

where the quantities \( \xi(R, \bar{T}) \) and \( \zeta(R, \bar{T}) \) stand for respectively:

\[
\xi(R, \bar{T}) = \frac{\partial}{\partial R_*} \left\{ \frac{1}{R^2} F \left( 1 - \frac{C^2}{A^2} \right) \right\} - \frac{\partial}{\partial R_*} \left\{ \sigma F \left( 1 - \frac{C^2}{A^2} \right) + F \right\},
\]

(56)

and

\[
\zeta(R, \bar{T}) = \left[ \sigma^2 - i \sigma \frac{\partial}{\partial R_*} \left( \frac{F \left( 1 - \frac{C^2}{A^2} \right)}{F \left( 1 - \frac{C^2}{A^2} \right)} \right) - i \sigma \frac{F \left( 1 - \frac{C^2}{A^2} \right) + F}{F \left( 1 - \frac{C^2}{A^2} \right)} \right]

- \mathcal{M}^2 \Delta R^4 \left( 1 - \frac{C^2}{A^2} \right)^{-1}.
\]

(57)

Now, we use the substitution:

\[
\Psi_R = \exp \left\{ \int^{R_*} R \xi(R') dR' \right\} \tilde{Q}_R(R_*)
\]

(58)

in the equation 55. With this substitution, after some calculations the equation 55 is transformed into the form:

\[
\frac{\partial^2 \Psi_R}{\partial R_*^2} + \left\{ \xi - \frac{1}{2} \frac{\partial \xi}{\partial R_*} - \frac{1}{4} \xi^2 \right\} \Psi_R = 0.
\]

(59)

V. ANALYZING THE STABILITY FOR THE AXIAL-PERTURBATIONS: CONDITIONS FOR EXISTENCE OF UNSTABLE MODES OF PERTURBATIONS

After multiplying the equation 54 with the complex-conjugate of \( \tilde{Q}_R \), denoted by \( \tilde{Q}_R^* \) and integrating the resultant equation from \( R_* = R_*^c \), the black hole horizon to theoretically infinite distance, we carry out prolong calculations, which are basically similar to that followed in the reference [23], but quite complicated and lengthy due to comparative complexity of the generalized
McVittie metric. Thus we obtain a complex-equation, whose real-part is given by:

\[ \int_{R_+}^{\infty} \frac{F}{R^2} \left( 1 - \frac{C^2}{\Delta} \right) \frac{\partial \tilde{Q}_R}{\partial R_*} |\tilde{Q}_R|^2 dR_* + \int_{R_+}^{\infty} i^2 F \frac{\Delta}{R^2} |\tilde{Q}_R|^2 dR_* = \int_{R_+}^{\infty} \text{RE} \left( \tilde{Q}_R \frac{\partial \tilde{Q}_R}{\partial R_*} \right) F \left( 1 - \frac{C^2}{\Delta} \right) \left\{ \frac{(\sigma^2)}{R^2} \frac{\partial}{\partial R_*} F \left( 1 - \frac{C^2}{\Delta} \right) \right\} dR_* + i(\sigma + \bar{\sigma}) \int_{R_+}^{\infty} (\tilde{Q}_R \frac{\partial \tilde{Q}_R}{\partial R_*}) \frac{F}{2R^2} \left\{ F \left( 1 - \frac{C^2}{\Delta} \right) \frac{\partial}{\partial R_*} F \left( 1 - \frac{C^2}{\Delta} \right) \right\} dR_* + i\tilde{\sigma} \int_{R_+}^{\infty} \left( \frac{F}{R^2} \right) \frac{\partial}{\partial R_*} F \left( 1 - \frac{C^2}{\Delta} \right) |\tilde{Q}_R|^2 dR_* - i\bar{\sigma} \int_{R_+}^{\infty} \left( \frac{F}{R^2} \right) \frac{\partial}{\partial R_*} F \left( 1 - \frac{C^2}{\Delta} \right) |\tilde{Q}_R|^2 dR_* + (\sigma^2 - \sigma^2) \int_{R_+}^{\infty} \left( \frac{F}{R^2} \right) |\tilde{Q}_R|^2 dR_* + \sigma \int_{R_+}^{\infty} \left( \frac{F}{R^2} \right) \frac{\partial}{\partial R_*} F \left( 1 - \frac{C^2}{\Delta} \right) |\tilde{Q}_R|^2 dR_* + i\tilde{\sigma} \int_{R_+}^{\infty} \left( \frac{F}{R^2} \right) \frac{\partial}{\partial R_*} F \left( 1 - \frac{C^2}{\Delta} \right) |\tilde{Q}_R|^2 dR_* + i\bar{\sigma} \int_{R_+}^{\infty} \left( \frac{F}{R^2} \right) \frac{\partial}{\partial R_*} F \left( 1 - \frac{C^2}{\Delta} \right) |\tilde{Q}_R|^2 dR_* + (\sigma^2 - \sigma^2) \int_{R_+}^{\infty} \left( \frac{F}{R^2} \right) |\tilde{Q}_R|^2 dR_* = 0. \]

On the other-hand, the imaginary part of the equation is given by:

\[ \int_{R_+}^{\infty} \text{IM} \left( \tilde{Q}_R \frac{\partial \tilde{Q}_R}{\partial R_*} \right) \frac{F}{2R^2} \left\{ \frac{(\sigma^2)}{R^2} \frac{\partial}{\partial R_*} F \left( 1 - \frac{C^2}{\Delta} \right) \right\} dR_* + (\sigma - \bar{\sigma}) \int_{R_+}^{\infty} \left( \frac{F}{R^2} \right) \frac{\partial}{\partial R_*} F \left( 1 - \frac{C^2}{\Delta} \right) |\tilde{Q}_R|^2 dR_* + \sigma \int_{R_+}^{\infty} \left( \frac{F}{R^2} \right) \frac{\partial}{\partial R_*} F \left( 1 - \frac{C^2}{\Delta} \right) |\tilde{Q}_R|^2 dR_* + 2\sigma R \int_{R_+}^{\infty} \left( \frac{F}{R^2} \right) |\tilde{Q}_R|^2 dR_* + \sigma R \int_{R_+}^{\infty} \left( \frac{F}{R^2} \right) \frac{\partial}{\partial R_*} F \left( 1 - \frac{C^2}{\Delta} \right) |\tilde{Q}_R|^2 dR_* = 0. \]

In the above equations 60 and 61, and in any following equation or expression, the \( \text{RE} \) and \( \text{IM} \) denote respectively the real and imaginary part of the concerned complex-quantity with parentheses ‘( )’. \( \sigma_R \) and \( \sigma_I \) are respectively the real and imaginary parts of the frequency \( \sigma \). \( \bar{\sigma} \) is the complex-conjugate of the frequency \( \sigma \). The \( | \cdot | \) indicates modulus of the complex-quantity within it. In the above equations 60 and 61, the integrals have been performed from \( R_+ \) to \( \infty \), where \( R_+ \) is the black hole horizon in the new radial coordinate, that we have defined in the previous section. The quantity \( D \) is given by:

\[ D = - \left[ (\sigma_R F \left( 1 - \frac{C^2}{\Delta} \right)^2 + \{ -\sigma_I F \left( 1 - \frac{C^2}{\Delta} \right) + \mathcal{F} \} \right]^2. \]

It is to be noted that on the LHS of the equation 60, the integrals are real-quantities. Moreover, as it can be checked numerically that the quantity \( F \) is positive outside the horizon \( R_+ \), the integrands \( F \left( 1 - \frac{C^2}{\Delta} \right) |\tilde{Q}_R|^2 \) and \( i^2 F \frac{\Delta}{R^2} |\tilde{Q}_R|^2 \) are also positive outside the horizon \( R_+ \), and hence are the integrals. Now, on the RHS of the equation 60, the first integral and the last two integrals are clearly real quantities. Therefore, the rest integrals must also constitute a real quantity. Similarly, on the LHS of the equation 61, the first integral and the last two integrals are real quantities and hence the rest three integrals must constitute a real quantity.

So, we can apply the property of a real quantity, viz. a real quantity and its conjugate are same, to these parts separately. Applying this property to the three parts on the RHS of equation 60 and to the LHS of equation 61 respectively, we obtain the equations:

\[ \int_{R_+}^{\infty} \text{RE} \left( \tilde{Q}_R \frac{\partial \tilde{Q}_R}{\partial R_*} \right) \frac{F}{2R^2} \left\{ \frac{(\sigma^2)}{R^2} \frac{\partial}{\partial R_*} F \left( 1 - \frac{C^2}{\Delta} \right) \right\} dR_* = \frac{1}{2} (\bar{\mathcal{I}}_1 + \mathcal{I}_2), \]

and

\[ \int_{R_+}^{\infty} \text{RE} \left( \tilde{Q}_R \frac{\partial \tilde{Q}_R}{\partial R_*} \right) \frac{F}{2R^2} \left\{ \frac{(\sigma^2)}{R^2} \frac{\partial}{\partial R_*} F \left( 1 - \frac{C^2}{\Delta} \right) \right\} dR_* = -\frac{1}{2} (\mathcal{I}_1 + \bar{\mathcal{I}}_2), \]

where the \( \mathcal{I}_1 \) and \( \mathcal{I}_2 \) represents the following integrals:
Now, adding and subtracting the equations 63 and 64, we can have the expressions of the integrals:
\[ \int_{R_+}^{\infty} \mathcal{R}_{R_+} \mathcal{R}_{R_*} \left( F \left( 1 - \frac{C^2}{\Delta^2} \right) \right)^2 dR_* \]
\[ + \int_{R_+}^{\infty} \mathcal{R}_{R_+} \mathcal{R}_{R_*} \left( F \left( 1 - \frac{C^2}{\Delta^2} \right) \right)^2 dR_* = \int_{R_+}^{\infty} \mathcal{R}_{R_+} \mathcal{R}_{R_*} \left( F \left( 1 - \frac{C^2}{\Delta^2} \right) \right)^2 dR_* \]
\[ + i^2 (\sigma_1) \int_{R_+}^{\infty} \frac{\partial}{\partial R_*} \left( F \left( 1 - \frac{C^2}{\Delta^2} \right) \right)^2 dR_* - i^2 \sigma_1 \int_{R_+}^{\infty} \frac{\partial}{\partial R_*} \left( F \left( 1 - \frac{C^2}{\Delta^2} \right) \right)^2 dR_* \]
\[ + (\sigma_R^2 - \sigma_1^2) \int_{R_+}^{\infty} \frac{\partial}{\partial R_*} \left( F \left( 1 - \frac{C^2}{\Delta^2} \right) \right)^2 dR_* + \sigma \int_{R_+}^{\infty} \frac{\partial}{\partial R_*} \left( \frac{C}{\Delta} |\mathcal{R}_R|^2 \right) \left( \frac{C}{\Delta} |\mathcal{R}_R|^2 \right) dR_* \]
\[ (65) \]
Similarly, the equation 61 can be written as:
\[ \int_{R_+}^{\infty} \mathcal{R}_{R_+} \mathcal{R}_{R_*} \left( F \left( 1 - \frac{C^2}{\Delta^2} \right) \right)^2 dR_* \]
\[ + \sigma_R \int_{R_+}^{\infty} \frac{\partial}{\partial R_*} \left( F \left( 1 - \frac{C^2}{\Delta^2} \right) \right)^2 dR_* - \sigma \int_{R_+}^{\infty} \frac{\partial}{\partial R_*} \left( F \left( 1 - \frac{C^2}{\Delta^2} \right) \right)^2 dR_* \]
\[ + 2\sigma_R |\mathcal{R}_R|^2 dR_* + \sigma \int_{R_+}^{\infty} \frac{\partial}{\partial R_*} \left( \frac{C}{\Delta} |\mathcal{R}_R|^2 \right) \left( \frac{C}{\Delta} |\mathcal{R}_R|^2 \right) dR_* = 0. \]
As the LHS of the equation 65 is positive, hence the RHS must also be positive, which gives:
\[ \sigma_1 \left\{ \int_{R_+}^{\infty} \frac{\partial}{\partial R_*} \left( F \left( 1 - \frac{C^2}{\Delta^2} \right) \right)^2 dR_* + \int_{R_+}^{\infty} \frac{\partial}{\partial R_*} \left( F \left( 1 - \frac{C^2}{\Delta^2} \right) \right)^2 dR_* \right\} \]
\[ \geq + \frac{\partial}{\partial R_*} \left( \frac{C}{\Delta} |\mathcal{R}_R|^2 \right) \left( \frac{C}{\Delta} |\mathcal{R}_R|^2 \right) \][66]
Now, on the LHS of the above inequality 67, we apply integration-by-parts on the term \[ \int_{R_+}^{\infty} \frac{\partial}{\partial R_*} \left( \frac{C}{\Delta} |\mathcal{R}_R|^2 \right) \left( \frac{C}{\Delta} |\mathcal{R}_R|^2 \right) dR_* \]
thereby getting a surface-term \[ -F \left( 1 - \frac{C^2}{\Delta^2} \right) \frac{\partial}{\partial R_*} \left( \frac{C}{\Delta} |\mathcal{R}_R|^2 \right) \]
vanishes at the limits. As a result, we get the term
\[ 61 \] Using this property too, we can have the equations 63 and 64.

3 As on the RHS of the equation 60, the second, third and fourth integrals together constitute a real quantity, so the imaginary part of it must be vanish. Similar argument is valid for the equation 61.
\[
\int_{R_+}^\infty \frac{\tilde{Q}_R}{2} |\tilde{Q}_R|^2 \frac{\partial}{\partial R_*} \left( F \left( 1 - \frac{C^2}{A^2} \right) \right) \, dR_* \].
\]
Hence, we write the inequality as:

\[
\sigma_1 \left[ \int_{R_+}^\infty \left[ \frac{\tilde{Q}_R}{2} \left( F \left( 1 - \frac{C^2}{A^2} \right) \right) \right] \, dR_* \right] + \left\{ \frac{C}{\Delta} \frac{\partial}{\partial R_*} \left( \left( F \left( 1 - \frac{C^2}{A^2} \right) \right) + \frac{\mathcal{F}}{R^2} \right) \right\} |\tilde{Q}_R|^2 \, dR_* \geq 0
\]

\[
- \int_{R_+}^\infty \Re \left( \tilde{Q}_R \frac{\partial}{\partial R_*} \right) F \left( 1 - \frac{C^2}{A^2} \right) \left\{ |\sigma|^2 \frac{\partial}{\partial R_*} \left( F \left( 1 - \frac{C^2}{A^2} \right) \right)^2 \right\} \, dR_* - (\sigma_R^2 - \sigma_I^2) \int_{R_+}^\infty \frac{F \left( 1 - \frac{C^2}{A^2} \right) \frac{\partial}{\partial R_*} \left( \frac{\mathcal{F}}{F \left( 1 - \frac{C^2}{A^2} \right)} \right)}{2R^2} |\tilde{Q}_R|^2 \, dR_* .
\]

(68)

On the RHS of the above inequality 68, we see that the real and imaginary parts of the complex-quantity \( \tilde{Q}_R \frac{\partial}{\partial R_*} \), can be expressed as respectively:

\[
\Re \left( \tilde{Q}_R \frac{\partial}{\partial R_*} \right) = |\tilde{Q}_R| \frac{\partial}{\partial R_*} |\cos \varphi | \tag{69}
\]

and

\[
\Im \left( \tilde{Q}_R \frac{\partial}{\partial R_*} \right) = |\tilde{Q}_R| \frac{\partial}{\partial R_*} |\sin \varphi | \tag{70}
\]

where the quantity \( \varphi = \tan^{-1} \frac{\Im \left( \tilde{Q}_R \frac{\partial}{\partial R_*} \right)}{\Re \left( \tilde{Q}_R \frac{\partial}{\partial R_*} \right)} \), which is the argument of the complex-quantity \( \tilde{Q}_R \frac{\partial}{\partial R_*} \).\footnote{In this context the value of \( \tan \varphi \) is quite important. In principle it may have value ranging from \(-\infty\) to \(+\infty\), but practical cases, its value is expected to be finite and it would depend on the certain mode under consideration. Yet, the value of \( \tan \varphi \) would be much wide-ranging for various modes.}

Writing the real and imaginary parts of \( \tilde{Q}_R \frac{\partial}{\partial R_*} \) in this style, we can express the RHS of the inequality 68 as:

\[
\int_{R_+}^\infty \left| \tilde{Q}_R \right| \frac{\partial}{\partial R_*} F \left( 1 - \frac{C^2}{A^2} \right) \left\{ (\cos \varphi) \left\{ \left( |\sigma|^2 \right)^2 \frac{\partial}{\partial R_*} \left( F \left( 1 - \frac{C^2}{A^2} \right) \right)^2 \right\} + 2 \sigma_R (\sin \varphi) F^2 \left( 1 - \frac{C^2}{A^2} \right)^2 \frac{\partial}{\partial R_*} \left\{ \frac{\mathcal{F}}{F \left( 1 - \frac{C^2}{A^2} \right)} \right\} \right\} \, dR_* .
\]

For brevity, let us denote:

\[
\left[ (\cos \varphi) \left\{ \left( |\sigma|^2 \right)^2 \frac{\partial}{\partial R_*} \left( F \left( 1 - \frac{C^2}{A^2} \right) \right)^2 \right\} + 2 \sigma_R (\sin \varphi) F^2 \left( 1 - \frac{C^2}{A^2} \right)^2 \frac{\partial}{\partial R_*} \left\{ \frac{\mathcal{F}}{F \left( 1 - \frac{C^2}{A^2} \right)} \right\} \right] = \varepsilon . \tag{71}
\]

Therefore, the inequality 68, gives:

\[
\sigma_1 \geq \int_{R_+}^\infty \left| \tilde{Q}_R \right| \frac{\partial}{\partial R_*} F \left( 1 - \frac{C^2}{A^2} \right) \, dR_* - (\sigma_R^2 - \sigma_I^2) \int_{R_+}^\infty \frac{F \left( 1 - \frac{C^2}{A^2} \right) \frac{\partial}{\partial R_*} \left( \frac{\mathcal{F}}{F \left( 1 - \frac{C^2}{A^2} \right)} \right)}{2R^2} |\tilde{Q}_R|^2 \, dR_* , \tag{72}
\]

\[
\sigma_1 \geq \int_{R_+}^\infty \left| \tilde{Q}_R \right| \frac{\partial}{\partial R_*} \left( F \left( 1 - \frac{C^2}{A^2} \right) \right) \, dR_* + \left\{ \frac{C}{\Delta} \frac{\partial}{\partial R_*} \left( F \left( 1 - \frac{C^2}{A^2} \right) \right) + \frac{\mathcal{F}}{R^2} \right\} |\tilde{Q}_R|^2 \, dR_* \, dR_* \tag{73}
\]

\[
\sigma_1 \geq \int_{R_+}^\infty \left| \tilde{Q}_R \right| \frac{\partial}{\partial R_*} \left( F \left( 1 - \frac{C^2}{A^2} \right) \right) \, dR_* + \left\{ \frac{C}{\Delta} \frac{\partial}{\partial R_*} \left( F \left( 1 - \frac{C^2}{A^2} \right) \right) + \frac{\mathcal{F}}{R^2} \right\} |\tilde{Q}_R|^2 \, dR_* \]
According the convention of the time-dependence of the Fourier-modes of perturbations used here, as they are proportional to $e^{i\sigma t}$, the axial perturbations will be unstable if $\sigma_I < 0$ viz. when the imaginary part of the frequency $\sigma_I$ is negative. 5 As, $\sigma_I < 0 \Rightarrow -\sigma_I > 0$, so the corresponding condition in this case, as can be concluded from the condition 73, is :

$$-\int_{R_{+}}^{\infty} |\tilde{Q}_R R| \frac{\partial \tilde{Q}_R}{\partial R_*} \left\{ F(1 - \frac{C^2}{A^2}) \frac{\partial}{\partial R_*} \left\{ F(1 - \frac{C^2}{A^2}) \right\} \right\} + \left\{ C \frac{\partial}{\partial R_*} \left\{ F(1 - \frac{C^2}{A^2}) \right\} + \frac{\partial \tilde{F}}{\partial R_*} \right\} \right| R^2 \, dR_* > 0.$$  

The above inequality-condition can be simplified in two different cases: (i) When the denominator in the term involved in the inequality 74 is positive,

$$\int_{R_{+}}^{\infty} \left[ \frac{\partial}{\partial R_*} \left\{ F(1 - \frac{C^2}{A^2}) \frac{\partial}{\partial R_*} \left\{ F(1 - \frac{C^2}{A^2}) \right\} \right\} + \left\{ C \frac{\partial}{\partial R_*} \left\{ F(1 - \frac{C^2}{A^2}) \right\} + \frac{\partial \tilde{F}}{\partial R_*} \right\} \right| R^2 \, dR_* > 0,$$

On the other-hand, (ii) when the denominator is negative,

$$\int_{R_{+}}^{\infty} \left[ \frac{\partial}{\partial R_*} \left\{ F(1 - \frac{C^2}{A^2}) \frac{\partial}{\partial R_*} \left\{ F(1 - \frac{C^2}{A^2}) \right\} \right\} + \left\{ C \frac{\partial}{\partial R_*} \left\{ F(1 - \frac{C^2}{A^2}) \right\} + \frac{\partial \tilde{F}}{\partial R_*} \right\} \right| R^2 \, dR_* < 0,$$

So ultimately, the quantities $\epsilon$, defined in the equation 71, and $(\sigma_R^2 - \sigma_I^2)$ decide whether the conditions 75 or 76 would be satisfied or not. This is because the fact that the rest of the quantities in the integrands of the integrals involved in the inequalities 75 and 76 are either strictly positive or negative viz. $|\tilde{Q}_R \frac{\partial \tilde{Q}_R}{\partial R_*}|, |\tilde{Q}_R|^2 > 0$, $\frac{\partial}{\partial R_*} \left\{ F(1 - \frac{C^2}{A^2}) \right\} > 0, D > 0 < \frac{\partial}{\partial R_*} \left\{ F(1 - \frac{C^2}{A^2}) \right\}$.

Now, for the satisfaction of the inequality 75, there may be different types of situations. We are describing these successively.

(I) The first among these will be that both sides of this inequality 75 are positive. On the RHS, as the integral $\int_{R_{+}}^{\infty} \frac{\partial}{\partial R_*} \left\{ F(1 - \frac{C^2}{A^2}) \right\} \, dR_* > 0$. So, for RHS > 0, the required condition : $\sigma_R^2 > \sigma_I^2$. While the condition for LHS > 0 have to be $\epsilon < 0$ ; as in the integrand of LHS $|\tilde{Q}_R \frac{\partial \tilde{Q}_R}{\partial R_*} | F(1 - \frac{C^2}{A^2}) < 0$. Both sides being positive is a necessary condition, but not a sufficient one. Whether the inequality 75 is satisfied or not, will depend on the magnitudes of both sides.

(II) If the RHS is positive i.e. $\sigma_R^2 > \sigma_I^2$, then one sufficient condition for satisfaction of the inequality is $\epsilon > 0$, as this makes the LHS < 0.

(III) The third possibility is that when RHS is negative i.e. $\sigma_R^2 < \sigma_I^2$, and the LHS < 0 too, or in other way to say, $\epsilon > 0$, then the inequality 75 will be satisfied, but this will depend on the magnitudes of the both sides. This is also a necessary but not sufficient condition like the case-(I).

The situations for the satisfaction of the inequality 76 are similar as above. Now, we have to check the implications of $\epsilon \geq 0$.

Using its expression in the equation 71, we can get :

$$\epsilon \geq 0 \Rightarrow |\sigma|^2 \leq \left[ \frac{2(\sigma_R \tan \varphi) F^2(1 - \frac{C^2}{A^2}) \frac{\partial}{\partial R_*} \left\{ \frac{\partial \tilde{F}}{\partial R_*} \right\} - \frac{\partial \tilde{F}^2}{\partial R_*}}{\frac{\partial}{\partial R_*} \left\{ F(1 - \frac{C^2}{A^2}) \right\}^2} \right].$$  

5 Substituting $\sigma = \sigma_R + i\sigma_I$ in the $e^{i\sigma t}$, one gets $e^{i\sigma t} = e^{i(\sigma_R + \sigma_I)t} = e^{i\sigma_R t} e^{-\sigma_I t}$. 
Or, we may write the condition (77) as:

\[ \varepsilon \geq 0 \Rightarrow \sigma_R^2 \leq -\sigma_R^2 + \frac{2(\sigma_R \tan \varphi) F^2 \left(1 - \frac{C^2}{A^2}\right)^2 \partial \mathcal{F} \left(\frac{\mathcal{F}}{F\left(1 - \frac{C^2}{A^2}\right)}\right) - \partial \mathcal{F}^2}{\partial \mathcal{R}_R \left(F\left(1 - \frac{C^2}{A^2}\right)^2\right)} . \] (78)

We have already argued and explained in criterion (I), that the set of conditions \( \varepsilon < 0 \) and \( \sigma_R^2 > \sigma_I^2 \) is a necessary condition for the satisfaction of \( \sigma_I < 0 \). So, \( \sigma_R^2 > \sigma_I^2 \) and \( \varepsilon < 0 \) simultaneously give:

\[ \sigma_R^2 > \sigma_I^2 > -\sigma_R^2 + \frac{2(\sigma_R \tan \varphi) F^2 \left(1 - \frac{C^2}{A^2}\right)^2 \partial \mathcal{F} \left(\frac{\mathcal{F}}{F\left(1 - \frac{C^2}{A^2}\right)}\right) - \partial \mathcal{F}^2}{\partial \mathcal{R}_R \left(F\left(1 - \frac{C^2}{A^2}\right)^2\right)} , \] (79)

which can be simplified to:

\[ \sigma_R^2 > \frac{2 \partial \mathcal{F} \left(\frac{\mathcal{F}}{F\left(1 - \frac{C^2}{A^2}\right)}\right) - \partial \mathcal{F}^2}{\partial \mathcal{R}_R \left(F\left(1 - \frac{C^2}{A^2}\right)^2\right)} , \] (80)

or,

\[ 2 \frac{\partial \mathcal{F}}{\partial \mathcal{R}_R} \left(F\left(1 - \frac{C^2}{A^2}\right)^2\right) \sigma_R - 2 \tan(\varphi) F^2 \left(1 - \frac{C^2}{A^2}\right)^2 \frac{\partial \mathcal{F}}{\partial \mathcal{R}_R} \left(\frac{\mathcal{F}}{F\left(1 - \frac{C^2}{A^2}\right)}\right) \sigma_R > 0 . \] (81)

On the other hand, the criterion (II) gives that the set of conditions \( \varepsilon > 0 \) and \( \sigma_R^2 > \sigma_I^2 \) is a sufficient condition for the satisfaction of \( \sigma_I < 0 \). Using \( \sigma_R^2 > \sigma_I^2 \) and \( \varepsilon > 0 \) simultaneously, we get:

\[ \sigma_I^2 < \sigma_R^2 < -\sigma_I^2 + \frac{2(\sigma_R \tan \varphi) F^2 \left(1 - \frac{C^2}{A^2}\right)^2 \partial \mathcal{F} \left(\frac{\mathcal{F}}{F\left(1 - \frac{C^2}{A^2}\right)}\right) - \partial \mathcal{F}^2}{\partial \mathcal{R}_R \left(F\left(1 - \frac{C^2}{A^2}\right)^2\right)} , \] (82)

or, simply as,

\[ 2 \frac{\partial \mathcal{F}}{\partial \mathcal{R}_R} \left(F\left(1 - \frac{C^2}{A^2}\right)^2\right) \sigma_I - 2 \tan(\varphi) F^2 \left(1 - \frac{C^2}{A^2}\right)^2 \frac{\partial \mathcal{F}}{\partial \mathcal{R}_R} \left(\frac{\mathcal{F}}{F\left(1 - \frac{C^2}{A^2}\right)}\right) \sigma_I < 0 . \] (83)

Again, another constraint can be imposed for this criterion-(II), as \( |\sigma|^2 > 0 \), for \( \varepsilon > 0 \) i.e. \( |\sigma|^2 < \text{RHS} \) in the condition (77), we can say that \( \text{RHS} > 0 \) or,

\[ \frac{\partial \mathcal{F}}{\partial \mathcal{R}_R} \left(F\left(1 - \frac{C^2}{A^2}\right)^2\right) > 0 . \] (84)

While for \( |\sigma|^2 > \text{RHS} \) in the condition (77), no such certain constraint can be imposed on the RHS.

It is to be noted that all the above criteria I to III discussed above are for the case-(i), where the denominator of the term in the inequality (74) is positive. As, \( |Q_R|^2 \geq 0 \), this can be said as:

\[ \frac{\partial}{\partial \mathcal{R}_R} \left\{ \frac{F\left(1 - \frac{C^2}{A^2}\right)}{2R^2D} \frac{\partial}{\partial \mathcal{R}_R} \left(F\left(1 - \frac{C^2}{A^2}\right)\mathcal{F}\right) + \left(\frac{C}{A} \frac{\partial}{\partial \mathcal{R}_R} \left(\frac{F\left(1 - \frac{C^2}{A^2}\right)}{2R^2D}\right) + \frac{\mathcal{F}}{R^2}\right) \right\} > 0 . \]

We are not discussing the criterion (III), as it is quite similar to the criterion (I) and is a necessary but not sufficient condition.

Therefore, we see that the conditions required for \( \sigma_I < 0 \) are not impossible or unphysical, and can be achieved in certain cases. Satisfaction of these conditions depends mainly on the relative values of the real and imaginary parts of the complex frequency.
viz. \( \sigma_R \) and \( \sigma_I \), belonging to any certain mode, and on the values of the quantities \( \left( F \left( 1 - \frac{c^2}{a^2} \right) \right)^2 \), \( \frac{\partial \sigma_I}{\partial R} \), \( \frac{\sigma_R}{F \left( 1 - \frac{c^2}{a^2} \right)} \) and their derivatives w.r.t. \( R \). So, the unstable frequencies would depend on the initial mass of a PBH i.e. time of its production and the time instant in early Universe, at which the stability is being examined.

VI. CONCLUSION AND DISCUSSION:

In this work, we have derived the equation governing the axial perturbations in the generalized McVittie metric, which can well describe the space-time around a non-rotating uncharged PBH, created in the early radiation-dominated Universe, where the effect of expansion of the Universe on the local space-time of the PBH was significant due to very high value of Hubble-parameter at that time and the PBHs were of continuously changing masses due to spherical accretion of the surrounding high-density radiation.

We have analyzed the conditions of existence of unstable modes of axial perturbations using this equation. We see that unlikely to the cases of many usual black hole metrics, the unstable modes corresponding to \( \sigma_I < 0 \) in this case, for axial perturbations in generalized McVittie metric, can not be ruled out. The existence of unstable modes in this metric mainly depends on the relative values of the real and imaginary parts of frequency of any certain mode. We know that for some of the usual black hole metrics viz. Schwarzschild, Schwarzschild-Anti-De Sitter etc., the unstable mode does not exist i.e. perturbations, exponentially growing with time, are not practically possible. As the generalized McVittie metric is physically the space-time around a Schwarzschild black hole of varying mass, embedded in an expanding FLRW-metric, it can be interpreted that it is due to the expansion of the background FLRW-Universe and the time-variation of mass of the black hole, which are making the existence of the unstable modes possible.

For any unstable mode, as the corresponding axial-perturbation grows exponentially with time, at a certain stage the linear perturbation theory breaks down if the growing perturbation becomes sufficiently large so that it can not be treated by linear perturbation theory. In other way to say, these unstable modes indicate the possibility of non-linear instabilities have been proposed in fast dynamical effects in these perturbations. In some earlier works, the possibility of non-linear instabilities have been proposed in fast spinning black holes, with a similarity of turbulence in hydrodynamics [24]. But till date, there is hardly any highly-energetic astrophysical or cosmological phenomena studied in numerical general relativity (e.g. the merging of two black holes in a binary), which can generate observationally important non-linear effects. So in case of generalized McVittie metric, it will be interesting to investigate whether the unstable modes can give rise to significant non-linear effects. If there exists significant non-linear instability, then that might even leave imprint on the stochastic gravitational wave background produced due to the vibration of perturbed PBHs in the early Universe.

In some cases, non-zero stress-energy surrounding any black hole, which is then called ‘dirty black hole’, can affect the linear stability of that black hole [25]. This stress-energy can be due a shell of matter or a planet. So, this indicates a concordance with the case of generalized McVittie metric, where the mass of the black hole is time-varying due to spherical accretion of the surrounding radiation in the early radiation-dominated era. Here, the surrounding radiation should provide the non-zero stress-energy, which can affect the black hole’s linear stability. Although in practical case, there are thought to be many ways to perturb those PBHs, as we have argued in the introduction (section I).

The equation derived by us, is the preliminary step for calculation of quasi-normal modes of PBHs of changing masses, described by the generalized McVittie metric, in the early radiation-dominated Universe. Though the similar counterpart for polar-perturbations is required too. We plan to numerically find out the quasi-normal modes of PBHs described by generalized McVittie metric in a future work.

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VIII. APPENDIX 1 : CLARIFICATION ABOUT SOME DIMENSIONAL ISSUES

We have to be clear about the presence of the two fundamental constants G (Universal Gravitational constant) and c (Speed of light in vacuum) in all the expressions, which are generally omitted according to the natural units’ convention of taking G.c as 1 (unity). The convention of natural units is okay for purely analytical i.e. non-numerical calculations, but this is not suitable as we need to get exact numerical orders of several quantities.

For the second part of the quantity \( C \) i.e. \( \dot{M}_H \), we first evaluate \( \dot{M}_H \) in terms of the Hawking-Hayward Quasilocal mass \( M_H \), which is related with the former as \( M_H (t) = m(t) a(t) \):

\[
\dot{M}_H = \frac{\dot{a}}{a^2} M_H ,
\]

\[
\Rightarrow \dot{M}_H = M_H - H M_H . \tag{85}
\]

To estimate an approximate order of the first term on the RHS in the equation 85 i.e. \( \dot{M}_H \), we use its expression derived in the reference [1]. This gives the time-rate of change of the Hawking-Hayward Quasi-local mass in terms of cosmic fluid density at a finite radial distance from the black hole (in isotropic coordinates):

\[
\dot{M}_H = - \frac{G}{2} a \mathcal{B}^2 \sqrt{1 + a^2 \mathcal{A}^2 r^{-2} (P(r) + \rho(r))} \mathcal{A}' r' , \tag{86}
\]

where the concerned quantities \( \mathcal{A} \) and \( \mathcal{B} \) have already been defined earlier. \( \mathcal{A}' \) is the contra-varient radial component of the four-velocity of cosmic fluid getting spherically accreted by the black hole, \( \mathcal{A}' = 4 \pi \mathcal{A} a^2 r^2 \) is the area of the spherical surface of isotropic radius \( r \), \( P(r) \) and \( \rho(r) \) are respectively the density and pressure of the cosmic fluid at that isotropic radial distance \( r \).
But, it is easy to verify that this expression of $M_H$, when converted to the Schwarzschild-like coordinate in Nolan gauge, it gives:

$$M_H = 4\pi c R_0^2 (1 + w)\rho,$$

which is simply the time rate of accretion of cosmic-fluid into the black hole, through the apparent black hole horizon $R_0$. Here, $w$ is the equation-of-state parameter of the cosmic-fluid and as it is radiation in our case, $w = 1/3$.

From the metric given in equation 9, it is clear that the quantity $C = HR + m\alpha \sqrt{\frac{r}{T}}$, written in the convention of natural units, must be dimensionless. In this context it is also to be noted that the corresponding ratio $M_H/R$ in the metric 9, is actually $GM_H/c^2R$ i.e. dimensionless, as the quantity $G/c^2$ remain omitted when we use the convention of natural units, as has been stated already. Hence, both the quantities $HR$ and $m\alpha \sqrt{\frac{r}{T}}$ must be actually dimensionless. We here use the notations viz. $[L],[M]$ and $[T]$ for the dimensions of length, mass and time respectively.

In the second term of $C$, in $m\alpha$, including the $G/c^2$ that occurs with the mass $m$ to make it a length-scale, we get $Gm\sqrt{c^2}$, which has the dimension of length/time : $[LT^{-1}]$ (Dimension of $G : [G] = [L^2T^{-2}M^{-1}]$). So, although it seems that $[C] \equiv [HR] = [m\alpha \sqrt{\frac{r}{T}}] = [LT^{-1}]$, it is not the actual dimension, as $C$ must be dimensionless. So, for being dimensionless there must be a $c^{-1}$ multiplied with these and which is evident as we shall show later that $C = \frac{HR}{c} + \frac{Gm\alpha \sqrt{\frac{r}{T}}}{c}$. Therefore, without erasing $G$ and $c$ the actual expression of $C$ is:

$$C(t, R) = \frac{HR}{c} + \frac{Gm\alpha \sqrt{\frac{r}{T}}}{c}. (88)$$

It can be easily verified that $C$ is dimensionless by substituting the dimensions of corresponding quantities.

The confusion for the expression of $M_H$ given in equation 86 is deeper if we do not write the omitted $G$, $c$ in proper places because the author in reference [1] has kept the ‘$G$’ in Einstein’s equations there, while has omitted the ‘$c$’s there and also omitted those ‘$G$’, ‘$c$’s present with the masses in the ratios $M_H/R \equiv GM_H/c^2R$ in metric coefficients. It can be easily checked that for $M_H$ to be dimensionless there must be $Ge^{-3}$ with it. This can be checked by substituting the dimensions of the corresponding quantities. Thus, keeping the ‘$G$’ and ‘$c$’s in proper places the actual expressions are:

$$\frac{GM_H}{c^3} = \frac{aB^2}{2c^3} \sqrt{1 + a^2A^4 \left(\frac{u^2}{c}\right) (P(r) + \rho(r))} du, (89)$$

Or, canceling $Ge^{-3}$ from both sides,

$$M_H = \frac{1}{2} aB^2 \sqrt{1 + a^2A^4 \left(\frac{u^2}{c}\right) (P(r) + \rho(r))} du. (90)$$

### IX. APPENDIX 2 : BASIC EQUATIONS DESCRIBING THE AXIAL PERTRUBATIONS

Here, the origin of the equations governing the perturbations on a metric given by equation 10, has been described. In this case, the corresponding unperturbed components of the Ricci tensors are:

$$R_{12} = \frac{1}{2} e^{-2\psi} (e^{-2\nu} e^{-2\rho})^{1/2} \left( [((\epsilon^{\psi+\nu-(\mu_2+\mu_3)} Q_{32})_{33} - \left( (e^{3\psi-\nu+\mu_2+\mu_3}) Q_{02} \right)_{30}, (91)$$

and

$$R_{13} = \frac{1}{2} e^{-2\psi} (e^{-2\nu} e^{-2\rho})^{1/2} \left( [((\epsilon^{3\psi+\nu-(\mu_2+\mu_3)} Q_{23})_{30} - \left( (e^{3\psi-\nu+\mu_2+\mu_3}) Q_{03} \right)_{0}, (92)$$

where $Q_{AB} = q_{A,B} - q_{B,A}$ and $Q_{00} = q_{A,0} - \omega_A$ are defined similarly as of $Q_{AB}$ and $Q_{00}$, but they contain the unperturbed values of the quantities $q_2, q_3$ and $\omega$ instead of their linear perturbations.

As for the metric given in equation 10, in our case, $q_2 = q_3 = \omega = 0$, therefore $Q_{AB} = Q_{00} = 0$. Hence, the unperturbed Ricci tensor components $R_{12} = R_{00}$ and $R_{13} = R_{00}$ are zero (0).

The origin of the equations governing the perturbations is from Einstein’s equations for those components. The Einstein’s equation for these components, taken to first order axial perturbations, is given by:

$$R_{ij} + \delta_{0,2,q_3} R_{ij} - \frac{1}{2} (g_{ij} + R \delta_{0,2,q_3} g_{ij} + g_{ij} \delta_{0,2,q_3} R) = (8\pi) (T_{ij} + \delta_{0,2,q_3} T_{ij}), (93)$$

where $i,j$ are indices denoting spatial coordinates. (For avoiding confusion with the radial coordinate $R$, we denote the Ricci scalar by $R$.)

Subtracting the unperturbed Einstein’s equation from the above equation 93 with the linear perturbation, we obtain:

$$\delta_{0,2,q_3} R_{ij} - \frac{1}{2} (g_{ij} + R \delta_{0,2,q_3} g_{ij} + g_{ij} \delta_{0,2,q_3} R) = \frac{1}{2} (8\pi) \delta_{0,2,q_3} T_{ij} . (94)$$

As there is no cross components in the metric, hence for this case i.e. $i=1$ and $j=2,3$ ; $g_{12} = g_{13} = 0$, which reduces the above equation to:

$$\delta_{0,2,q_3} R_{ij} - \frac{1}{2} (8\pi) \delta_{0,2,q_3} T_{ij} . (95)$$

We have already shown in the section III that if the cosmic-fluid is radiation having equation-of-state parameter $w = 1/3$, then the associated Ricci scalar vanishes. Hence, the equation 95 further reduces to:

$$\delta_{0,2,q_3} R_{ij} = (8\pi) \delta_{0,2,q_3} T_{ij} . (96)$$

While it can be shown that in a flat FLRW-Universe, for the part of the energy-momentum tensor belonging to a perfect fluid, the concerned components of the linear perturbations of energy-momentum tensor of the cosmic-fluid are zero : $\delta T_{ij,p} = 0$, where $i \neq j$ and the suffix ‘p’ in the energy-momentum tensor component represents it is due to the ‘perfect’ part of the fluid. But, in this case of generalized McVitie metric, a perfect fluid can not describe the surrounding cosmic-fluid. As it has been already shown and explained in the reference [1] that a single perfect cosmic-fluid can not describe a physical solution of a spherically symmetric black hole embedded in an expanding FLRW-Universe. We need at least one imperfection parameter in it to describe the solution physically. As was chosen by the authors in reference [1], we also choose this imperfection parameter to be the heat-flux vector $\gamma_\mu$. Only one component of the heat-flux vector suffices in this case and we can take it to be the radial component, in accordance with the radial mass-flow into the accreting black hole. So, due to the heat-flux vector there would be
an additional imperfect part in the energy-momentum tensor of the cosmic-fluid i.e. radiation in this case, which is $T_{ij} = \gamma u_j u_i + \gamma_j u_i$ (where ‘ip’ represents ‘imperfect’). So, the perturbation to this is given by (we write $\delta$ in place of $\delta_{\nu_2 \nu_3}$ for brevity):

$$\delta T_{ij} = u_j \delta \gamma_i + \gamma_i \delta u_j + u_i \delta \gamma_j + \gamma_j \delta u_i. \tag{97}$$

The components, with which we have to deal with are: $\delta T_{\phi R} = \delta T_{\phi ip}$. For the component $\delta T_{\phi ip}$, the associated components of the four-velocity $u_{\phi}, u_t$ are zero; and also the associated components of the heat-flux vectors $\gamma_{\phi}, \gamma_t$ are zero. Hence, the $\delta T_{\phi ip}$ vanishes. On the other hand, for the component $\delta T_{\phi R} = \gamma R \delta u_\phi + u_t \delta \gamma_\phi$. Hence,

$$\delta T_{\phi R} = \gamma R \delta u_\phi + u_t \delta \gamma_\phi. \tag{98}$$

As a non-zero velocity perturbation in the $\phi$-direction would imply presence of angular momentum in that direction, in the cosmic-fluid being accreted by the black hole, then the accretion would no longer remain spherical and that would result in the formation of accretion-disk around the black hole. So, to avoid this complexity we assume that $\delta u_\phi$ can be neglected. The same argument also holds for the $\gamma_\phi$. So, with this assumption $\delta T_{\phi R} = 0$. Therefore, the equations reduce to:

$$\delta_{\nu_2 \nu_3} R_{ij} = 0. \tag{99}$$

We begin our calculation from these equations.

**X. APPENDIX 3: DETERMINING SOME ESSENTIAL RELATIONS REGARDING THE GENERALIZED MCVITTIE METRIC**

In the present work we need various relations between different quantities present in the line-element of the metric i.e. the metric-coefficients and transformation rules to shift from differentiation w.r.t. one coordinate system to the other. In this section, we are giving these relations and formula which have been used in our work.

The integrating factor $F$ in the equation 7 satisfies the differential equation [22]:

$$\frac{\partial F}{\partial R} \left( \frac{1}{F} \right) = \frac{\partial \beta}{\partial t} \left( \frac{1}{F} \right), \tag{100}$$

where $\beta$ is the quantity $\frac{\dot{C}}{\sqrt{C^2 - \dot{a}^2}}$. To simplify this equation, first of all we express the partial derivative w.r.t. the radial coordinate $R(t, r)$ in Nolan-gauge, in terms of the partial derivative w.r.t. isotropic time coordinate $t$. As already stated the coordinate $R$ is given by:

$$R = a(t) r \left( 1 + \frac{M_H(t)}{2a(t)r} \right)^2. \tag{101}$$

So, the partial derivative of $R$ w.r.t. $t$ is given by:

$$\frac{\partial R}{\partial t} = a(t) \left( 1 + \frac{M_H(t)}{2a(t)r} \right)^2 + 2ar \left( 1 + \frac{M_H(t)}{2a(t)r} \right) \left( \frac{1}{2} \frac{M_H}{a^2} \dot{a} \right). \tag{102}$$

(It is quite clear that as the scale-factor $a(t)$ and Hawking-Hayward quasi-local mass $M_H(t)$ are the functions of time $t$ only, $\partial a/\partial t = \dot{a}$ and $\partial M_H/\partial t = \dot{M}_H$.)

On simplifying the above expression of $\partial R/\partial t$, we obtain:

$$\frac{\partial R}{\partial t} = \left\{ HR + M_H \left( 1 + \frac{M_H}{2a^2} \right) \frac{M_H}{M_H - H} \right\}. \tag{103}$$

As, $M_H(t) = m(t)a(t)$, it is easy to check that this can be written as:

$$\frac{\partial R}{\partial t} = \left\{ HR + ma \sqrt{\frac{\dot{a}}{a}} \right\} = C(t, r). \tag{104}$$

Therefore, we can say,

$$\frac{\partial R}{\partial t} = \frac{C(t, r)}{\partial t} \tag{105}$$

Hence, the equation 100 can be written as:

$$\frac{\partial F}{\partial t} \frac{1}{F} = \frac{\partial \beta}{\partial t} \frac{1}{F}, \tag{106}$$

$$\Rightarrow -F \frac{\partial \beta}{\partial F} = \frac{C}{1 - C \beta}. \tag{107}$$

Where,

$$\frac{C}{1 - C \beta} = \frac{C}{A^2 - 2C^2} = \frac{C}{A^2 - 2C^2}. \tag{108}$$

The above equation 105 has to be solved for getting the solution $F$.

Again, we have to determine the relation between $\frac{\partial}{\partial R}$ and $\frac{\partial}{\partial t}$. We have already shown that

$$\frac{\partial R}{\partial t} = C(t, R), \tag{109}$$

and the time-coordinate $\dot{t}$, we are working with, is given by:

$$\dot{t} = \frac{1}{F} \left( \dot{t} + \frac{C}{A^2 - C^2} \frac{dR}{dt} \right). \tag{110}$$

From the above equation 106 we obtain:

$$\frac{\partial t}{\partial R} = \frac{1}{F} \frac{\partial R}{\partial t} \frac{1}{F} + \frac{C}{F(A^2 - C^2)} \tag{111}$$

$$\Rightarrow \frac{\partial t}{\partial R} = \frac{1}{FC(1 - \frac{C^2}{A^2})}. \tag{112}$$

Hence, multiplying both sides with the differential operator $\frac{\partial}{\partial R}$, we obtain:

$$\frac{\partial}{\partial R} \frac{\partial}{\partial R} \frac{\partial}{\partial R} = \frac{1}{FC(1 - \frac{C^2}{A^2})} \frac{\partial}{\partial t}. \tag{113}$$

Inserting the relation between $\frac{\partial}{\partial R}$ and $\frac{\partial}{\partial t}$, we see:

$$\frac{1}{C} \frac{\partial}{\partial t} = \frac{1}{FC(1 - \frac{C^2}{A^2})} \frac{\partial}{\partial R}. \tag{114}$$

or,

$$\frac{\partial}{\partial t} = \frac{1}{F(1 - \frac{C^2}{A^2})} \frac{\partial}{\partial R}. \tag{115}$$
XI. APPENDIX 4 : ANALYZING THE QUANTITIES $\frac{\partial (R, C, A)}{\partial (R, C, A)}$, $\frac{\partial (R, F)}{\partial (R, F)}$ AND THE APPROXIMATIONS APPLICABLE TO THESE

A. Expressing the ratio $\frac{\partial (R, T)}{\partial (R, T)}$, conveniently in terms of $C, A, \Delta, F$ and their derivatives :

In this sub-section, we are going to express the quantities $F$ and $\mathcal{F}$ in terms of $C, A, \Delta, F$ and their derivatives ; and then in the next sub-section we shall investigate some approximations, which will be applicable to our calculations. The quantity $\frac{\partial (R, T)}{\partial (R, T)}$ can be expressed as :

$$\frac{\mathcal{F}(R, T)}{\mathcal{F}(R, T)} = \frac{\frac{\Delta}{R^2} \left( 1 - \frac{C^2}{A^2} \right) F \frac{\partial}{\partial t} \left( 1 - \frac{C^2}{A^2} \right) F^{-1}}{\frac{F}{R^2} \frac{\partial}{\partial t} \left( \frac{R}{F^2} \right)}, \quad (110)$$

$$\Rightarrow \frac{\mathcal{F}(R, T)}{\mathcal{F}(R, T)} = 1 + \frac{\Delta \left( 1 - \frac{C^2}{A^2} \right) \frac{\partial}{\partial t} \left( \frac{R}{F^2} \right)}{\frac{F}{R^2} \frac{\partial}{\partial t} \left( \frac{R}{F^2} \right)}. \quad (111)$$

The denominator in the additional term with 1 on the RHS of the above equation 111 can be written as :

$$\frac{1}{R^2} \frac{\partial}{\partial t} \left( \frac{R}{F^2} \right) = \left\{ F \frac{\partial}{\partial t} \left( \frac{1}{F} \right) + \frac{1}{R^2} \frac{\partial}{\partial t} R^2 \right\}^{-1}. \quad (112)$$

Using the differential relation satisfied by $F$ given in 100 we can obtain the following relation :

$$F \left\{ \frac{\partial}{\partial t} \left( \frac{1}{F} \right) \right\} = \left( \frac{1}{C} - \beta \right)^{-1} \frac{\partial \beta}{\partial t}, \quad (113)$$

where the quantity $\beta$ is given by $\beta = \frac{C}{A^2 C^2} - \Delta$; and using the relations between $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial t}$, derived in the appendix 3 i.e. section X, we can easily get :

$$\frac{1}{R^2} \frac{\partial}{\partial t} R^2 = \frac{4}{R} FC \left( 1 - \frac{C^2}{A^2} \right). \quad (114)$$

We write the detailed expressions of $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial t}$ respectively as :

$$\frac{\partial \beta}{\partial t} = \frac{1}{(A^2 - C^2)^2} \left\{ (A^2 + C^2) \frac{\partial C}{\partial t} - 2CA \frac{\partial A}{\partial t} \right\}. \quad (115)$$

and

$$\frac{\partial A}{\partial t} = \frac{\partial}{\partial t} \left( 1 - \frac{2M_H}{R} \right) = -2 \left\{ \frac{M_H \partial R}{R^2} + \frac{1}{R} \frac{\partial M_H}{\partial t} \right\}. \quad (116)$$

Here, $\frac{\partial M_H}{\partial t}$ can be expressed as :

$$\frac{\partial M_H}{\partial t} = F \left( 1 - \frac{C^2}{A^2} \right) \frac{\partial M_H}{\partial t} = M_H F \left( 1 - \frac{C^2}{A^2} \right) \left( H + \frac{\dot{m}}{m} \right).$$

Substituting the expression of the $\frac{\partial M_H}{\partial t}$ in the equation 116, we obtain :

$$\frac{\partial A}{\partial t} = -2 M_H F \left( 1 - \frac{C^2}{A^2} \right) \left\{ -C + HR + \frac{\dot{m}}{m} \right\}. \quad (117)$$

As, $C = HR + \frac{\dot{m}}{m}$, substituting it in the RHS of the above equation 117 we obtain :

$$\frac{\partial A}{\partial t} = -2 M_H F \left( 1 - \frac{C^2}{A^2} \right) \left\{ \sqrt{\frac{r}{r}} + \frac{\dot{r}}{m} \right\}. \quad (118)$$

Substituting the expressions :

$$\sqrt{\frac{r}{r}} = \left( 1 + \frac{M_H}{2 \alpha} \right) = \left( 1 + \frac{m}{2r} \right)$$

and

$$\frac{\dot{r}}{m} = \frac{r}{m} \left( 1 + \frac{m}{2r} \right)^2 = \frac{(r + m)^2}{2r},$$

on the RHS of the equation 118, we get :

$$\frac{\partial A}{\partial t} = -2 M_H F \left( 1 - \frac{C^2}{A^2} \right) \left( \frac{r}{m} - \frac{m}{4r} \right). \quad (119)$$

Calculating the detailed expression of the quantity $\frac{\partial C}{\partial t}$ we obtain :

$$\frac{\partial C}{\partial t} = F \left( 1 - \frac{C^2}{A^2} \right) \left\{ -H^2 R + 2 \dot{m} \alpha \left( 1 + \frac{m}{2r} \right) + \frac{\dot{m}^2 \alpha}{2r} + \hat{m} \alpha \left( 1 + \frac{m}{2r} \right) \right\}. \quad (120)$$

Hence, the denominator in the additive term with 1 on the RHS of the equation 111 can be written as :

$$\frac{F}{R^2} \frac{\partial}{\partial t} \left( \frac{R^2}{F^2} \right) = \left( \frac{1}{C} - \beta \right)^{-1} \frac{\partial \beta}{\partial t} + \frac{4}{R} FC \left( 1 - \frac{C^2}{A^2} \right)$$

$$= C \left( 1 - \frac{C^2}{A^2} \right) \left\{ \left( 1 - 2 \frac{C^2}{A^2} \right)^{-1} \frac{\partial \beta}{\partial t} + \frac{4}{R} FC \right\}. \quad (121)$$

While the numerator of that term is given by :

$$\Delta \left( 1 - \frac{C^2}{A^2} \right) \frac{\partial}{\partial t} \left( \Delta \left( 1 - \frac{C^2}{A^2} \right) \right)^{-1}.$$

Hence, the additive term with 1 on the RHS of the equation 111 can be written as :

$$\Delta \left( 1 - \frac{C^2}{A^2} \right) \frac{\partial}{\partial t} \left( \Delta \left( 1 - \frac{C^2}{A^2} \right) \right)^{-1}$$

$$= \left( 1 - \frac{C^2}{A^2} \right)^{-2} \frac{\partial}{\partial t} \left( \Delta \left( 1 - \frac{C^2}{A^2} \right) \right)^{-1}$$

$$\left\{ \left( 1 - 2 \frac{C^2}{A^2} \right)^{-1} \frac{\partial \beta}{\partial t} + \frac{4}{R} FC \right\}. \quad (121)$$

Writing the detailed expressions of numerator and denominator in the additive term with 1 on the RHS of the equation 111, in terms of $C, A, \Delta, F$ and their derivatives w.r.t. $\bar{t}$, we obtain :
\[
\frac{\Delta\left(1 - \frac{C^2}{A^2}\right) \frac{\partial}{\partial t} \left(\Delta\left(1 - \frac{C^2}{A^2}\right)\right)^{-1}}{F^2 R^3 \frac{\partial}{\partial t} \left(\frac{R^2}{F}\right)} = \frac{\left(\frac{-2}{A^2} \frac{\partial C}{\partial t} + \frac{2C}{A^2} \frac{\partial A}{\partial t}\right) + \left(1 - \frac{C^2}{A^2}\right) \frac{1}{A^2} \frac{\partial \Delta}{\partial t}}{\left(1 - \frac{2C^2}{A^2}\right)^{-1} \left(1 + \frac{C^2}{A^2}\right) \frac{1}{A^2} \frac{\partial A}{\partial t} - \frac{2C}{A^2} \frac{\partial A}{\partial t}} + \left(1 - \frac{C^2}{A^2}\right) \frac{1}{R^2} \frac{\partial F}{\partial t}.
\]

(122)

B. Checking the order of different quantities present in the ratio \( \frac{C}{HR/c} \) and applying approximation

If we examine the ratio given in equation 122 in the previous sub-section, then in the numerator and denominator of the ratio on the RHS of the equation 122, the quantities \( \left(\frac{-2}{A^2} \frac{\partial C}{\partial t} + \frac{2C}{A^2} \frac{\partial A}{\partial t}\right) \) and \( \left(1 - \frac{2C^2}{A^2}\right)^{-1} \left(1 + \frac{C^2}{A^2}\right) \frac{1}{A^2} \frac{\partial A}{\partial t} - \frac{2C}{A^2} \frac{\partial A}{\partial t}\) should have same order of magnitude within finite distance from the black hole \(^6\). Therefore, the order of the magnitude of the ratio given in equation 122 would depend mainly on the quantities \( \frac{1}{A^2} \frac{\partial A}{\partial t} \) and \( F/R \), if their order of magnitude is higher than the former quantities.

Therefore, now we have to check the relative significance of different quantities in the ratio given in equation 122 for having an idea on its overall order of magnitude. First of all, we check the relative significance of the quantities : \( \left(1 - \frac{C^2}{A^2}\right) \frac{1}{A^2} \frac{\partial A}{\partial t} \) and \( \left(1 - \frac{C^2}{A^2}\right) \frac{2}{A^2} \frac{\partial F}{\partial t} \). Their ratio can be expressed as :

\[
\left(1 - \frac{C^2}{A^2}\right) \frac{1}{A^2} \frac{\partial A}{\partial t} = \frac{2C(R - M_H) - 2M_H R}{R^2 - 2M_H R} \left(1 - \frac{C^2}{A^2}\right) \frac{1}{A^2} \frac{\partial A}{\partial t} = \frac{1}{2} \left\{ C \left(1 - \frac{M_H}{R}\right) - \frac{M_H}{R} \right\} \frac{1}{2} \left\{ 1 - \frac{M_H}{R}\right\} \frac{M_H}{R} \frac{1}{R}.
\]

Hence, if magnitude of \( C < < 1 \) and magnitude of \( M_H < < 1 \), then the magnitude of the above ratio is also \( < < 1 \).

To establish the fact that the order of magnitude of \( C \) and \( M_H \) are \(< < 1 \), we plot these w.r.t. time from \( 10^{-25} \) s to 100 s after Big-bang. We separately plot the two parts of \( C \) viz. \( HR/c \) and \( \frac{\sqrt{\frac{G}{c^2} \rho a}}{\sqrt{r}} \); as these vary with time in different ways. It is to be noted that the radial-distance \( R \) is arbitrary in this case, but that does not imply that it can be taken to theoretically-infinite or asymptotic distance. This is because at theoretically-infinite distance the generalized McVittie metric would reduce to FLRW-metric. The evolution of perturbations, around the PBH, is mainly to be studied within finite distance from the PBH. To set a characteristic length-scale for plotting \( C \) w.r.t. time, we use the comoving Schwarzschild length-scale \( R_s \) for the PBHs of maximum mass available at a certain instant of time in early Universe, which is just the horizon-mass \( m_H \) at that time. The horizon-mass \( m_h \) at a time \( t \) seconds after Big-bang is given by \( m_h = 10^{25} t \text{ kg} \) \([26, 27]\). In this way, we depict the maximum possible value of the comoving Schwarzschild length-scale \( R_s \) at that time. The plot of \( HR_s/c \) w.r.t. time has been shown in the figure 1.

On the other hand, for \( \sqrt{\frac{G}{c^2} \rho a} \sqrt{r} \), the quantity \( \sqrt{r} \) for the Schwarzschild length-scale would be 2 ; as for the Schwarzschild radius, isotropic radial-coordinate \( r = \frac{\rho a}{\sqrt{G}} \). Again, we use \( \rho a \) at \( m = m_h \), thereby taking the maximum possible value of PBH-mass at a certain instant of time. The plot of \( \sqrt{\frac{G}{c^2} \rho a} \sqrt{r} \) w.r.t. time has been shown in the figure 2. In this case, it is worth mentioning that here we are considering the mass-range of PBHs such that their mass-change due to Hawking-evaporation would be negligible and the only significant way of mass-change is the spherical accretion of the surrounding radiation. Another issue is to be noted in these cases, that we have shown these plots for time till 100 s after Big-bang, mainly because this is the order of time (at which Big-bang nucleosynthesis occurred), at which new PBH-production, by direct gravitational-collapse of sufficiently deep density-perturbations, is predicted to be stopped.

Next, we show the plot of \( \sqrt{\frac{G}{c^2} \rho a} \sqrt{r} \).
would be

with 1 can be neglected making the ratio

is very smaller than 1. This

ratio would have the order of

is of order 100.000. Therefore, from these plots, it is clear that the approximation based on $C << 1$ and $M_H << 1$ are very well valid in the specified range of time.

We now check the relative significance of the quantities 

$$
(1 - \frac{c^2}{\Delta t}) \frac{1}{2} \frac{\partial a}{\partial t} \quad \text{and} \quad (\frac{\Delta M}{\Delta t} + \frac{\Delta a}{\Delta t}).
$$

the second term of the expression, on the RHS of the equation 123, w.r.t. time in the figure 3. \(^7\)

So, we see that all the three quantities $\frac{H R a}{c^2} \frac{2 c^2}{\Delta t} a \tilde{m}_{124}$ and $\frac{\Delta M}{\Delta t} [M_H]_{m=m_B}$ have magnitudes within order of $10^{-23}$ to $10^{-9}$, for the range of time from $10^{-25}$ s to $100$ s after Big-bang. \(^8\)

Now, it is to be noted that in the ratio given in the above equation 124, the quantities containing $\tilde{m}$, $\tilde{m}$ or $M_H$, have the constant $G c^{-3}$ with each of them, as we have described in the APPENDIX-1 viz. section VIII. Furthermore, these quantities have $a$ or $\tilde{a}$. Hence, all of these quantities would be of very much smaller order in the scenario of our interest, where the radial distance from the PBH is finite. The dominant term in the numerator of the ratio given in equation 124 would be $H^2 R$. Therefore, the overall ratio would have the order of $\sim -2 \frac{H^2 R}{\Delta t} = -2 \frac{H R}{\Delta t}$ or $\frac{2 c^2}{\Delta t} \tilde{a} \tilde{r}$. As we are interested in the case where the radial distance from the PBH is finite, $A \sim 1$ and in the early radiation dominated era, \(^9\) $\tilde{a} << 1$, the resultant order of the ratio in equation 124 is very smaller than 1. This implies that, in the scenario of our interest the quantity

$$(\frac{-2 \frac{\partial G}{\partial t} + 2 \frac{c^2}{\Delta t} \frac{\partial A}{\partial t}}{1 - \frac{c^2}{\Delta t} \frac{\partial A}{\partial t}})$$

may be neglected with respect to the quantity

$$(1 - \frac{c^2}{\Delta t}) \frac{1}{2} \frac{\partial a}{\partial t}.$$ 

Again, we have previously shown that the quantity

$$(1 - \frac{c^2}{\Delta t}) \frac{1}{2} \frac{\partial a}{\partial t}$$

is negligible in comparison with the quantity

$$(1 - \frac{c^2}{\Delta t})^2 \frac{\partial a}{\partial t}.$$ 

It is quite clear that the quantity

$$(1 - \frac{c^2}{\Delta t})^{-1} \left\{ \left(1 + \frac{c^2}{\Delta t}ight) \frac{1}{2} \frac{\partial a}{\partial t} \right\} \sim \left(\frac{\Delta M}{\Delta t} + \frac{\Delta a}{\Delta t}\right).$$

Hence, the ratio in the equation 122 is of order $<< 1$. Thus, this analysis implies that the additive term in the equation 111 with 1 can be neglected making the ratio $F/F \approx 1$ or, $F \approx F$. Again, in the quantity $\frac{\mathcal{F} + i \sigma}{\mathcal{F} + i \sigma}$, if the frequency of the mode is not too small (i.e. if we neglect the ultra-low frequency modes), then too, the quantity $\frac{\mathcal{F} + i \sigma}{\mathcal{F} + i \sigma}$ would be $\approx 1$.

\(^7\) One part in $\frac{\Delta M}{\Delta t} [M_H]_{m=m_B}$ viz. the $\frac{\Delta M}{\Delta t} a \tilde{m}_{124}$, has already been plotted in figure 2, with the factor 2. So, if the other part $\frac{\Delta M}{\Delta t} \tilde{a} \tilde{m}_{124}$ is less than or equal to the former, then the order of the whole $\frac{\Delta M}{\Delta t} [M_H]_{m=m_B}$ would be same as that of $\frac{\Delta M}{\Delta t} a \tilde{m}_{124}$; and in fact this is happening in this case.

\(^8\) Another issue is also to be noted that the overall order of the ratio given in equation 123 may be lesser (even before $10^{-23}$ s) as in the expression derived in equation 123, the term containing $M_H$ is substracted from the term containing $C$.

\(^9\) the only case where $\tilde{a}$ may be near order 1 is the time just after the end of inflation.
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