Robust Event-Triggered Control Subject to External Disturbance*

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Abstract: This paper studies the event-triggered control problem for systems subject to both dynamic uncertainties and external disturbances. To overcome infinitely fast sampling caused by the disturbance, a new event-triggering mechanism is proposed, which uses not only the measurable system state but also an estimate of the unmeasurable state and the external disturbance. The inter-sampling intervals can be proved to be lower bounded by a positive constant, which is independent on the magnitudes of the unmeasurable state and the external disturbance. With the proposed design, the closed-loop event-triggered system can be input-to-state stabilized with the external disturbance as the input. Refined tools of input-to-state stability (ISS) and the small-gain theorem are employed in solving the problem.

Keywords: Event-Triggered Control, Input-to-State Stabilization, External Disturbances, Dynamic Uncertainty.

1. INTRODUCTION

Event-triggered control has attracted a lot of attention from the controls community. Compared with the periodic time-triggered sampling, the event-triggered sampling time instants are determined by an event generator that depends on the system state, to take into account the system behavior during the sampling intervals. Early papers, e.g., Årzen (1999); Aström and Bernhardsson (1999), have shown the advantages of event-triggered control over periodic time-triggered control.

In the past decade, great effort has been spent in developing the event-triggered control theory; see, e.g., Anta and Tabuada (2010); Gawthrop and Wang (2009); Heemels et al. (2012); Heemels and Donkers (2015); Lunze and Lehmann (2010); Tabuada (2007); Marchand et al. (2013); Brunner et al. (2015) and the references therein. In event-triggered control, threshold signals are usually employed to generate the events of data sampling. In the earlier results, e.g., Heemels et al. (2008); Henningsson et al. (2008), constant threshold signals are used, while in the recent results, e.g., Dimarogonas and Johansson (2009); Wang and Lemmon (2009, 2008), the threshold signals are designed to depend on real-time state and even the historical data. Also, refined classical control design methods have been employed for event-triggered control. For example, the concept of input-to-state stability (ISS) is used to characterize the robustness with respect to sampling errors, and the threshold signal is designed according to the margin of robustness, for (asymptotic) stability of the closed-loop event-triggered system Tabuada (2007); Girard (2015). For practical implementation of event-triggered control, infinitely fast sampling should be avoided, i.e., the sampling intervals are required to be lower bounded by some positive constant; see e.g., the discussions in Lemmon (2010). One special case of infinitely fast sampling is the Zeno behavior, i.e., there is an infinite number of sampling time instants which converge to a finite value; see e.g., Goebel et al. (2009) for the Zeno behavior of hybrid systems. In most of the existing results, the event-triggered control problem is transformed into the problem of choosing appropriate threshold signals to avoid infinitely fast sampling. In the presence of unmeasurable external disturbances, an intuitive design to avoid infinitely fast sampling is to introduce a constant threshold signal, by paying the price of losing asymptotic convergence. The event-triggered control problem of continuous-time linear systems in the presence of external disturbances is discussed in Lunze and Lehmann (2010) in which the disturbance behaviour of a feedback loop is the main concern of the paper. However, in Lunze and Lehmann (2010), the smallest inter-sampling is related to the upper bound of disturbances. A self-triggered sampling strategy for a class of nonlinear systems subject to external disturbances is proposed in

* This work was supported in part by NSF grant ECCS-1501044, in part by NSFC grants 61374042, 61522305, 61653007 and 61533007, in part by the Fundamental Research Funds for the Central Universities under Grants N130108001 and N140805001 in China, and in part by State Key Laboratory of Intelligent Control and Decision of Complex Systems at BIT.

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Liu and Jiang (2015) in which the upper bound of external disturbances is assumed to be known a priori.

The main purpose of this paper is to refine the existing results of robust event-triggered control for systems with both dynamic uncertainties and external disturbances. This paper does not assume a known upper bound of the external disturbance. To avoid infinitely fast sampling, a new event-triggering mechanism is proposed, which uses not only the measurable system state but also an estimate of the unmeasurable state and the external disturbance. Interestingly, it is proved that the event-triggered sampling intervals are lower bounded by a positive constant, which is independent of both the dynamic uncertainty and the external disturbance. ISS of the closed-loop event-triggered system is also proved by considering the external disturbance as the input. In the design, refined tools of ISS theory (Jiang et al. (1994); Liu et al. (2014) are used. Moreover, as a by-product of the main result, this paper proposes a small-gain result for a new class of interconnected systems induced by event-triggered control.

The rest of the paper is organized as follows. Section 2 gives the problem formulation of event-triggered control for continuous-time systems subject to both dynamic uncertainties and external disturbances. In Section 3, we propose a new event-triggering mechanism, and prove that the resulted closed-loop event-triggered system is ISS with the external disturbance as the input. An example with numerical simulation is employed to demonstrate the implementation and the effectiveness of the proposed design. Section 5 contains some concluding remarks.

To make the paper self-contained, some notations are given here. We use $\mathbb{Z}_+$ to denote the set of all nonnegative integers. For any vector $x \in \mathbb{R}^n$, $|x|$ represents its Euclidean norm. For any function $\psi : \mathbb{Z}_+ \to \mathbb{R}^n$, mean $\frac{1}{b-a} \int_a^b \psi(t) \, dt$, where $b > a$.

2. PROBLEM FORMULATION

We consider an event-triggered control system in the following form:

\[\begin{align*}
\dot{z}(t) &= h(z(t), x(t), \delta(t)) \\
\dot{x}(t) &= f(x(t), z(t), u(t), \delta(t)) \\
u(t) &= v(x(t))
\end{align*}\]

where $x \in \mathbb{R}^n$ is the measurable system state, $z \in \mathbb{R}^m$ is the unmeasurable portion of the state, $u \in \mathbb{R}^m$ is the control input, $h : \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ represents the external disturbance, $f : \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^n$ represents system dynamics, $v : \mathbb{R}^m \to \mathbb{R}^n$ represents the feedback control law based on event-triggered data-sampling. In this paper, the $z$-subsystem is considered as dynamic uncertainty, while the $x$-subsystem is called the nominal system. The time sequence $\{t_i\} \subseteq \mathbb{Z}_+$ is determined online by an appropriately designed event-triggering mechanism with $S = \{0, 1, 2, \ldots\} \subseteq \mathbb{Z}_+$ being the set of the indices of the sampling times and $t_0 = 0$.

Define $w(t) = x(t_i) - x(t)$, $t \in [t_i, t_{i+1})$, $i \in S$ as the sampling error, and rewrite

\[u(t) = v(x(t) + w(t)).\] (5)

Then, the controlled $x$-subsystem can be represented by

\[\begin{align*}
\dot{x}(t) &= f(x(t), z(t), v(x(t) + w(t)), \delta(t)) \\
&=: g(x(t), z(t), w(t), \delta(t)).
\end{align*}\] (6)

According to the definition of sampling error $w$ in (4), the closed-loop event-triggered system can be represented as an interconnected system, as shown in Figure 1.

![Fig. 1. The block diagram of the closed-loop event-triggered system composed of (1) and (6), where $S$ represents the event-triggered sampler.](image)

In this paper, we study the case in which the system dynamics $g$ is globally Lipschitz.

**Assumption 1.** There exist constants $L_x, L_z, L_w, L_\delta > 0$ such that

\[|g(x, z, w, \delta)| \leq L_x|x| + L_z|z| + L_w|w| + L_\delta|\delta|\]

holds for all $x, z, w, \delta$.

Also, it is assumed that the dynamic uncertainty, i.e., the $z$-subsystem, is ISS with $x$ and $\delta$ as the inputs, and there exists a $\nu$ such that the nominal system, i.e., the $x$-subsystem, is ISS with $z, w$ and $\delta$ as the inputs.

**Assumption 2.** System (1) is ISS with $x$ and $\delta$ as the inputs, and admits an ISS-Lyapunov function $V_z : \mathbb{R}^m \to \mathbb{R}_+$ satisfying the following conditions:

(1) There exist constants $\alpha_\nu, \overline{\alpha}_z > 0$ such that

\[\alpha_\nu |z|^2 \leq V_z(z) \leq \overline{\alpha}_z |z|^2, \quad \forall z;\]

(2) There exist constants $\alpha_z, k_{z}^2, k_\delta^2 > 0$ such that

\[\nabla V_z(z)h(z, x, \delta) \leq -\alpha_z V_z(z) + \max\{k_{z}^2 |x|^2, k_\delta^2 |\delta|^2\}, \quad \forall z, x, \delta.\]

**Assumption 3.** System (6) is ISS with $z, w$ and $\delta$ as inputs, and admits an ISS-Lyapunov function $V_z : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the following conditions:

(1) There exist constants $\overline{\alpha}_\nu, \overline{\alpha}_x > 0$ such that

\[\alpha_\nu |x|^2 \leq V_z(x) \leq \overline{\alpha}_x |x|^2, \quad \forall x;\]

(2) There exist constants $\alpha_x, k_z^2, k_w^2, k_\delta^2 > 0$ such that

\[\nabla V_z(x)g(x, z, w, \delta) \leq -\alpha_x V_z(x) + \max\{k_z^2 |z|^2, k_w^2 |w|^2, k_\delta^2 |\delta|^2\}, \quad \forall x, z, w, \delta.\]

Our objective is to develop a new event-triggering mechanism, which fully takes into consideration the external disturbance and the dynamic uncertainty such that the following two objectives are achieved:

**Objective 1:** There exists a constant $\Delta > 0$, independent of the unmeasurable state $z$ and the external disturbance $\delta$, such that

\[t_{i+1} - t_i \geq \Delta\] (12)
holds for all $i, i + 1 \in \mathbb{S}$.

**Objective 2:** The closed-loop event-triggered system composed of (1)-(3) is ISS with respect to the external disturbance $\delta$.

### 3. Robust Event-Triggered Control

The main contribution of this paper lies in a new event-triggering mechanism. Significantly different from most of the published results, an estimation of the unmeasurable state and the external disturbance is introduced, so that the threshold signal depends not only on the system state but also on the magnitudes of the unmeasurable state and the external disturbance. In this way, the technical challenge caused by the dynamic uncertainty and the external disturbance is solvable at the same time.

Specifically, the new event-triggering mechanism is proposed in the form of

$$t_{i+1} = \min \{ t \geq t_i + t_\Delta : |w(t)| \leq \max \{ k_w^x|x(t)|, k_w^\delta|\zeta(t)| \},$$

where $t_\Delta > 0$, $k_w^x > 0$ and $k_w^\delta > 0$ are constants to be designed later, $\zeta(t)$ represents an estimate of the influence of the unmeasurable state and the external disturbance.

#### 3.1 An Example as Motivation

Before presenting the main result, we employ a simple example to show how an estimation of the external disturbance can be made to guarantee the positive lower bound of the inter-sampling intervals.

Consider a first-order linear system in the form of

$$\dot{x}(t) = x(t) + u(t) + \delta(t)$$

where $x \in \mathbb{R}$ is the state, $u \in \mathbb{R}$ is the control input, and $\delta \in \mathbb{R}$ represents external disturbances.

The event-triggered control law is designed as

$$u(t) = -2x(t_i)$$

for $t_i \leq t < t_{i+1}, i \in \mathbb{S}$.

By using (4), (14) and (15), we have

$$\dot{x}(t) = -x(t) - 2w(t) + \delta(t).$$

Also, according to the definition of sampling error $w$, we have

$$w(t) = \dot{x}(t) = -w(t) - x(t_i) + \delta(t)$$

for $t_i \leq t < t_{i+1}, i \in \mathbb{S}$. Note that $w(t_i) = x(t_i) - x(t_i) = 0$.

For $t_i \leq t < t_{i+1}$ with $i \in \mathbb{S}$, the solution of system (17) with initial condition $w(t_i) = 0$ is

$$w(t) = e^{-(t-t_i)}w(t_i) + \int_{t_i}^{t} e^{-(t-t)}\delta(v) - x(t_i)) \, dv$$

and thus,

$$|x(t) - x(t_i)| \leq (1 - e^{-(t-t_i)})|x(t_i)| + \int_{t_i}^{t} |\delta(v)| \, dv \quad (19)$$

holds for $t_i \leq t \leq t_{i+1}, i \in \mathbb{S}$. Here, (19) holds for $t = t_{i+1}$ because of the continuity of both sides of the inequality.

Clearly, the term $\int_{t_i}^{t} |\delta(v)| \, dv$ represents the influence of the external disturbance $\delta$. For $t \in (t_i, t_{i+1}], a lower bound of $\int_{t_i}^{t} |\delta(v)| \, dv$ can be calculated as

$$\eta(t) = \max \{|x(t) - x(t_i)| - (1 - e^{-(t-t_i)})|x(t_i)|, 0\} \quad (20)$$

It can be observed that $\eta(t)$ is left-continuous at the discontinuous points.

Then, direct calculation yields:

$$|x(t) - x(t_i)| \leq (1 - e^{-(t-t_i)})|x(t_i)| + \eta(t)$$

for $t_i < t \leq t_{i+1}, i \in \mathbb{S}$.

To represent the “average” influence of $\delta(t)$, define

$$\zeta(t) = \begin{cases} 0, & \text{for } t = t_0 = 0; \\ \frac{\eta(t)}{t - t_i}, & \text{for } t \in (t_i, t_{i+1}]. \end{cases}$$

With an estimation in form of (22), it can be guaranteed that

$$|x(t) - x(t_i)| \leq (1 - e^{-(t-t_i)})|x(t_i)| + \max \{|x(t_i)|, \zeta(t)| \}$$

for $t_i < t \leq t_{i+1}, i \in \mathbb{S}$.

On the other hand, the event trigger (13) guarantees that

$$|x(t) - x(t_i)| \leq \min \{ k_w^x/(1 + k_w^\delta), k_w^\delta \} \max \{|x(t_i)|, \zeta(t)| \}$$

holds for $t_i < t \leq t_{i+1}, i \in \mathbb{S}$.

Define

$$t_\Delta = \min \{ t > t_i : 2\max \{ t - t_i, 1 - e^{-(t-t_i)} \} \}

= \min \{ k_w^x/(1 + k_w^\delta), k_w^\delta \} - t_i$$

for $t \geq t_i, i \in \mathbb{S}$. That is, a positive lower bound of the inter-sampling intervals is guaranteed. Also, it can be directly proved that

$$|w(t)| \leq \max \{ k_w^x|x(t)|, k_w^\delta|\zeta(t)| \}$$

for all $t \in [t_i, t_i + t_\Delta], i \in \mathbb{S}$.

Moreover, in the following discussion, it is proved that the convergence of $\delta$ leads to the convergence of $\zeta$, to satisfy the necessary condition for input-to-state stabilization.

Figure 2 shows the structure of the estimation-based event-triggering mechanism.

![Fig. 2. The block diagram of the closed-loop event-triggered system (14)-(15), where E represents the estimation (22) of the external disturbance.](image)
3.2 ISS of Event-Triggered Controlled x-Subsystem

In this subsection, we focus on the event-triggered control of the x-subsystem. In accordance with the design in Subsection 3.1, an estimation of the external disturbance $\delta$ and the unmeasurable state $z$ is defined as

$$\zeta(t) = \frac{1}{(t-t_i) L^\delta_g E(t, t_i)} \max \{ |x(t) - x(t_i)| - L_g (E(t, t_i) - 1) |x(t_i)|, 0 \} \quad (28)$$

for all $t \in [t_i, t_{i+1}), i \in \mathbb{S}$, where $L_g^\delta = 2 \max \{ L_g^\delta, L_g^\delta \}$, $E(t, t_i) = e^{(L_g^\delta + L_w^\delta)(t-t_i)}$, $L_g = L_g^\delta / (L_g^\delta + L_w^\delta)$ and $\tau$ is a positive constant to be determined later.

Theorem 4 presents the main result on ISS of the event-triggered controlled x-subsystem.

**Theorem 4.** Consider the closed-loop event-triggered system (6) with event-triggering mechanism (13) and (28). Under Assumptions 1 and 3, by choosing constant $k_w^\delta > 0$ satisfying

$$k_w^\delta (k_w^\delta)^2 / (1 - \sigma_\delta) \alpha_\delta \alpha_x < 1, \quad (29)$$

and constant $t_\delta$ as

$$t_\delta = \{ t : \varphi(t) < \min \} \quad (30)$$

with $0 < \varphi(t) < 1$, $\varphi(t_\delta) = 2 \max \{ L_g e^{L_g^\delta t_\delta} - L_g^\delta, e^{L_w^\delta t_\delta} \} L_g^\delta$, $L_x^w = L_g^\delta + L_w^\delta$, $\min \{ k_w^\delta (1 + k_w^\delta), k_w^\delta \}$ and $k_w^\delta$ is to be designed later, Objective 1 is achieved, and the event-triggered controlled x-subsystem (6) is ISS with $\delta$ and $z$ as the inputs.

**Proof.** In this proof, we first find a positive $t_\delta$, and then prove the ISS of the event-triggered controlled x-subsystem (6).

(a) The Existence of $t_\delta > 0$ The definition of $w$ in (4) implies

$$\dot{w}(t) = x(t) \quad (31)$$

for $t \geq t_i, i \in \mathbb{S}$. Then, by using Assumption 1, we have

$$\dot{w}(t) \leq L_g^\delta |x(t)| + L_g^\delta |z(t)| + L_g^\delta |w(t)| + L_g^\delta |\delta(t)|$$

$$\leq L_g^\delta |x(t_i) - x(t)| + L_g^\delta |z(t)| + L_g^\delta |w(t)| + L_g^\delta |\delta(t)|$$

$$\leq L_g^\delta |w(t)| + L_g^\delta |x(t_i)| + L_g^\delta \max \{|z(t)|, |\delta(t)|\} \quad (32)$$

for $t \geq t_i, i \in \mathbb{S}$, where $L_g^\delta = L_g^\delta + L_w^\delta$ and $L_g^\delta = 2 \max \{ L_g^\delta, L_g^\delta \}$.

Without loss of generality, we consider $|w(t)|$ continuously differentiable for $t_i \leq t < t_{i+1}, i \in \mathbb{S}$, then

$$\frac{d}{dt} |w(t)| \leq |\dot{w}(t)| \quad (33)$$

for $t_i \leq t < t_{i+1}, i \in \mathbb{S}$. Note that $|w(t_i)| = |x(t_i)| = 0$. For $t_i \leq t < t_{i+1}$ with $i \in \mathbb{S}$, the solution of $\frac{d}{dt} |w(t)|$ with initial condition $|w(t_i)| = 0$ is

$$|w(t)| \leq L_g^\delta |x(t_i)| + \int_{t_i}^{t} e^{L_g^\delta (t-t)} L_g^\delta |x(t)| dt$$

$$+ L_g^\delta \max \{|z(\tau)|, |\delta(\tau)|\} d\tau$$

$$\leq (L_g e^{L_g^\delta (t-t_i)} - L_g^\delta) |x(t_i)|$$

$$+ e^{L_g^\delta (t-t_i)} L_g^\delta \int_{t_i}^{t} \max \{|z(\tau)|, |\delta(\tau)|\} d\tau \quad (34)$$

for $t_i \leq t \leq t_i, i \in \mathbb{S}$.

By substituting (35) into (34), we have

$$|x(t) - x(t_i)| \leq (L_g e^{L_g^\delta (t-t_i)} - L_g^\delta) |x(t_i)| + e^{L_g^\delta (t-t_i)} L_g^\delta |\hat{\eta}(t)| \quad (35)$$

for $t_i \leq t \leq t_{i+1}, i \in \mathbb{S}$. Here, (36) holds for $t = t_{i+1}$ because of the continuity of both sides of the inequality.

Clearly, the term $\hat{\eta}$ represents the influence of the unmeasurable state $z$ and the external disturbance $\delta$. For $t \in (t_i, t_{i+1})$, a lower bound of $\eta$ can be calculated as

$$\hat{\eta}(t) = \frac{1}{e^{L_g^\delta (t-t_i)} L_g^\delta} \max \{|x(t) - x(t_i)| - (L_g e^{L_g^\delta (t-t_i)} - L_g^\delta) |x(t_i)|, 0 \}).$$

Then, it can be proved that

$$\hat{\eta}(t) \leq \eta(t) \quad (37)$$

and

$$|x(t) - x(t_i)| \leq (L_g e^{L_g^\delta (t-t_i)} - L_g^\delta) |x(t_i)| + e^{L_g^\delta (t-t_i)} L_g^\delta \hat{\eta}(t) \quad (38)$$

for $t_i \leq t \leq t_{i+1}, i \in \mathbb{S}$.

Specifically, property (37) is proved as follows. First consider the case of $|x(t) - x(t_i)| - (L_g e^{L_g^\delta (t-t_i)} - L_g^\delta) |x(t_i)| \geq 0$. In this case, according to the definition of $\hat{\eta}$, we have

$$\hat{\eta}(t) = \frac{1}{e^{L_g^\delta (t-t_i)} L_g^\delta} ((x(t) - x(t_i)) - (L_g e^{L_g^\delta (t-t_i)} - L_g^\delta) |x(t_i)|)$$. Thus, $\hat{\eta}(t) \leq \eta(t)$. For the case of $|x(t) - x(t_i)| - (L_g e^{L_g^\delta (t-t_i)} - L_g^\delta) |x(t_i)| \leq 0$, we have $\hat{\eta}(t) \leq 0 \leq \eta(t)$.

The validity of property (38) can be proved by directly using the definition of $\hat{\eta}(t)$.

In accordance with the design in Subsection 3.1, to represent the “average” influence of $z$ and $\delta$, define

$$\zeta(t) = \begin{cases} 0, & \text{for } t = t_0 = 0; \\ \frac{\hat{\eta}(t)}{t-t_i}, & \text{for } t \in (t_i, t_{i+1}). \end{cases} \quad (39)$$

Property (38) and (39) means that the system dynamics satisfy

$$|x(t) - x(t_i)| \leq (L_g e^{L_g^\delta (t-t_i)} - L_g^\delta) |x(t_i)| + e^{L_g^\delta (t-t_i)} L_g^\delta \zeta(t) (t-t_i)$$

$$\leq 2 \max \{ L_g e^{L_g^\delta (t-t_i)} - L_g^\delta, e^{L_g^\delta (t-t_i)} L_g^\delta \}$$

$$\times \max \{|x(t_i)|, \zeta(t)|\} \quad (40)$$

for $t_i \leq t \leq t_{i+1}, i \in \mathbb{S}$.

On the other hand, the event trigger (13) guarantees that property (24) and (25) holds for $t_i < t < t_{i+1}, i \in \mathbb{S}$.

Suppose that there exists a $t_\Delta > 0$ such that for all $t_i < t < t_i + t_\Delta$,
\[
\min \left\{ \frac{k_x^2}{1 + k_w^2}, k_w^2 \right\} \max \{|x(t_i), \zeta(t_i)| \geq \max \{|x(t_i), \zeta(t_i)| \}
\times 2 \max \{L_y e^{\sigma w}(t-t_i) - L_y e^{\sigma w}(t-t_i) I_d(t-t_i) \}. \quad (41)
\]

Then, from the discussions above, \([t_i, t_i + t_\Delta] \subset [t_i, t_i+1].\)

Thus, such \(t_\Delta\) satisfies
\[
t_\Delta \geq t_\Delta \quad \text{(42)}
\]

with \(t_\Delta^2\) satisfying
\[
2 \max \{L_y e^{\sigma w} t_\Delta - L_y e^{\sigma w} t_\Delta I_d(t_\Delta) \} = \min \left\{ \frac{k_x^2}{1 + k_w^2}, k_w^2 \right\}.
\]

Since the left-hand side of the equality is nonnegative and strictly increasing with respect to \(t_\Delta\) and the right-hand side of the equality is positive, \(t_\Delta^2 > 0\). Clearly, \(t_\Delta\) can be considered as a lower bound of the inter-sampling intervals, i.e., \(t_{i+1} - t_i \geq t_\Delta\) for all \(i, i+1 \in \mathbb{S}\), and it is independent on the external disturbance \(\delta\) and the unmeasurable state \(z\).

(b) ISS

The event-triggering mechanism (13) can guarantee that
\[
|x(t) - x(t_i)| \leq \max \{k_x^2 |x(t_i)|, k_w^2 \zeta(t) \}
\]
for \(t \in (t_i, t_i+1), i \in \mathbb{S}\).

With the satisfaction of (29) and (43), (11) implies
\[
\nabla V_x(x(t))g(x(t), z(t), w(t), \delta(t)) \leq -\sigma_1 \alpha_x V_x(x(t)) + \max \{|k_x^2 z(t)|, k_w^2 (k_x^2) \zeta(t)^2, k_w^2 \delta(t)^2\}
\]
for \(t \in (t_i, t_i+1), i \in \mathbb{S}\).

From (35), (37) and (39), we have
\[
\zeta(t) \leq \frac{\int_{t_i}^{t} \max \{|z(\tau)|, \delta(\tau)|\} \, d\tau}{t - t_i}
\]
\[
\leq \max \{1.5 \text{mean } |z(\tau)|, 3 \text{mean } |\delta(\tau)|\}
\]
for all \(t \in (t_i, t_i+1], i \in \mathbb{S}\).

Substituting (45) into the right-hand side of (44) yields
\[
\nabla V_x(x(t))g(x(t), z(t), w(t), \delta(t)) \leq -\sigma_1 \alpha_x V_x(x(t)) + \max \{|k_x^2 z(t)|, 2.3 k_w^2 (k_x^2) \zeta(t)^2, 9 k_w^2 \delta(t)^2\}
\]
\[
\leq -\sigma_1 \alpha_x V_x(x(t)) + \max \{|k_x^2 z(t)|, 2.3 k_w^2 (k_x^2) \zeta(t)^2, k_w^2 \delta(t)^2\}
\]
for all \(t \in (t_i, t_i+1], i \in \mathbb{S}, k_x^2 = 2.3 k_w^2 (k_x^2) \zeta^2, k_w^2 = 9 k_w^2 (k_x^2) \zeta^2\).

Property (46) means (delayed) ISS of the closed-loop event-triggered system (6) with respect to \(\delta\) and \(z\). This ends the proof of Theorem 4.

By the proof of Theorem 4, the event-triggered mechanism (13) and (29) satisfying (29) and (30) can achieve input-to-state stabilization of the closed-loop system (6). However, it cannot be directly clarified that the stability of the whole system composed of (1) and (6).

3.3 ISS of the Closed-Loop Event-Triggered System

The \(x\)-subsystem and the \(z\)-subsystem are feedback interconnected with each other. In this subsection, a condition for ISS of the closed-loop event-triggered system is given by using the small-gain idea.

Theorem 5. Consider the \((x, z)\)-system in (1) and (6). Under Assumptions 1, 2 and 3, if the event-triggering mechanism composed of (13) and (28) satisfies (29) and (30), and
\[
\chi_{xz} \cdot \chi_{xz} < 1, \quad \sigma_1 \chi_{xz} < 1
\]
then, ISS property with respect to external disturbances for the whole system composed of (1)-(3) is achieved.

Proof. By combining (8) and (46), we have
\[
\nabla V_x(x(t))g(x(t), z(t), w(t), \delta(t)) \leq -\sigma_1 \alpha_x V_x(x(t)) + \max \{|k_x^2 z(t)|, 2.3 k_w^2 (k_x^2) \zeta(t)^2, 9 k_w^2 \delta(t)^2\}
\]
\[
\Rightarrow \nabla V_x(x(t))g(x(t), z(t), w(t), \delta(t)) \leq -\sigma_1 \alpha_x V_x(x(t))
\]
for \(t \in (t_i, t_i+1], i \in \mathbb{S}\). Then, the gain margin formulation of the ISS-Lyapunov functions can be obtained by modifying property (50) as follows
\[
V_x(x(t)) \geq \max \{\chi_{xz} V_x(x(t)), \chi_{xz} \text{mean } V_x(x(t))\}
\]
\[
\chi_{xz} \max \{|\delta(t)|^2\}
\]
\[
\Rightarrow \nabla V_x(x(t))g(x(t), z(t), w(t), \delta(t)) \leq -\sigma_1 \alpha_x V_x(x(t))
\]
for all \(t \geq 0\), where \(\sigma_1 \alpha_x, \chi_{xz}, \chi_{xz} \delta^2\) are chosen as \(\alpha_x = \sigma_1 \sigma_2 \alpha_x\), \(\chi_{xz} = k_x^2 (1 - \sigma_1 \sigma_2 \alpha_x)\) and \(\chi_{xz} = k_x^2 (1 - \sigma_2 \sigma_1 \alpha_x)\).

An ISS-Lyapunov function candidate for the interconnected system composed of (1) and (6) is defined as
\[
V(x, z) = \max \{\sigma V_x(x, z)\}
\]
where \(\sigma\) is defined in (49).

According to Appendix A, by using conditions (47)-(49), we have
\[
\nabla V(x(t), z(t)) \geq \max \{\sigma \chi_{xz} \text{mean } V(x(t), z(t))\}
\]
\[
\chi_{xz} \text{mean } |\delta(t)|^2
\]
\[
\Rightarrow \nabla V(x(t), z(t))F(x(t), z(t), w(t), \delta(t)) \leq -\sigma_1 V(x(t), z(t))
\]
for \(t \in (t_i, t_i+1], i \in \mathbb{S}, \chi_{xz} = \sigma_1 \chi_{xz}^2, F(x, z, w, \delta) = \left[ h^T(x, z, \delta), g^T(x, z, w, \delta) \right]^T \) and \(\alpha = \min \{\frac{1}{2}\alpha_x^2, \frac{1}{2}\alpha_x^2\}\).

We now define
\[
V^*(x(t), z(t)) = \max \{V(x(t), z(t)), \sigma \chi_{xz} \text{mean } V(x(t), z(t))\}
\]
\[
\chi_{xz} \text{mean } V(x(t), z(t))
\]
for \(t \in (t_i, t_i+1], i \in \mathbb{S}, V(x, z)\) satisfies condition (53). To calculate the derivative of \(V^*(x, z)\) along the trajectories of interconnected system composed of
We have the case, we have for where $t \leq \tau \leq t$. Consequently, we have for all $t \leq \tau \leq t$.

**Case 1:** $V(x(t), z(t)) > \sigma_{\chi_2} \text{mean} V(x(\tau), z(\tau))$ and $V(x(t), z(t)) < \sigma_{\chi_2} \text{mean} V(x(t), z(t))$ for $t \in (t_i, t_{i+1}], i \in S$, respectively.

**Case 2:** $V(x(t), z(t)) < \sigma_{\chi_2} \text{mean} V(x(\tau), z(\tau))$. In this case, we consider the effectiveness of the obtained results in this paper. We consider the system

$$
z(t) = -z(t) + 0.1 \frac{z(t)}{1 + z^2(t)} x(t) + 0.4 \delta(t)$$

$$\dot{x}(t) = x(t) + 0.4 \sin(x(t)) z(t) + u(t) + \delta(t)$$

where $x \in \mathbb{R}$ is the measurable system state, $z \in \mathbb{R}$ is the unmeasured portion of the state, $u \in \mathbb{R}$ is the control input, $\delta \in \mathbb{R}$ represents the external disturbance.

The control law is chosen as

$$u(t) = -2x(t_i)$$

for $t_i \leq t < t_{i+1}, i \in S$.

With the sampling error $w$ defined in (4), by substituting (64) into (63), we obtain the closed-loop x-system

$$\dot{x}(t) = -x(t) - 2w(t) + 0.4 \sin(x(t)) z(t) + \delta(t)$$

Then, it can be directly proved that

$$|g(x, z, w, \delta)| \leq |x| + 0.4|z| + 2|w| + |\delta|$$

holds for all $x, z, w, \delta$.

To verify the satisfaction of Assumption 2 and 3, we define $V_{x}(x) = \frac{1}{2} x^2$ and $V_{z}(z) = \frac{1}{2} z^2$. It holds that

$$\nabla V_{x}(x) g(x, z, w, \delta) \leq -0.5V_{x}(x) + \max\{0.24|z|^2, 14.4|w|^2, 18|\delta|^2\}$$

for all $x, z, w, \delta$, and

$$\nabla V_{z}(z) h(z, x, \delta) \leq -V_{z}(z) + \max\{0.03|z|^2, 0.48|\delta|^2\}$$

for all $z, x, \delta$.

By using conditions (29), (30) and (48), we choose constants $k^w, k^\Delta, k^\delta$ as follows

$$k^w = 0.117, \quad k^\Delta = 0.0207, \quad k^\delta = 0.089.$$}

Then, we have

$$t_{i+1} = \min\{t_i \geq t_{i+1}, 0.0207 : |w(t)| = \max\{0.117|x(t)|, 0.089\zeta(t)\}$$

where $\zeta(t)$ satisfies

$$\zeta(t) = \frac{1}{(t - t_i)2e^{3(t-t_i)}} \times \max\{0.117|x(t)|, 0.089\zeta(t_i)\}$$

for all $t \in [t_i + 0.0207, t_{i+1}], i \in S$.

The simulation result with initial states $x(0) = -1, z(0) = 1$ and external disturbances $\delta(t) = (\cos(7t) + \sin(5t) + \sin(9t) + \sin(11t))/6$ is shown in Figs 3, 4 and 5. Compared with the existing event-triggering mechanism in the form of (72), the event-triggering mechanism (13) designed in this paper can increase sampling intervals, see Fig.5

$$t_{i+1} = \min\{t_i \geq t_{i+1} : |w(t)| = 0.117|x(t)|\}.$$
5. CONCLUSIONS

This paper has proposed a new event-triggering mechanism for systems subject to both dynamic uncertainties and external disturbances. Significantly different from the existing designs, an estimation of the unmeasurable state and the external disturbance is introduced, to avoid infinitely fast sampling. Interestingly, it is proved that the inter-sampling intervals are lower bounded by a positive constant, which is independent on the magnitudes of the unmeasurable state and the external disturbance. With the proposed design, the closed-loop event-triggered system has also been guaranteed to be ISS with the external disturbance as the input. Following this line of research, more general systems (e.g., systems formulated by input-to-output stability) and event-triggered control design problems would be of interest to study in the future.

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Appendix A. A SMALL-GAIN RESULT FOR A CLASS OF INTERCONNECTED SYSTEMS INDUCED BY EVENT-TRIGGERED CONTROL

For an interconnected system as following
\[ \dot{x}_1(t) = f_1(x_1(t), x_2(t), \dot{x}_2(t)) \]
\[ \dot{x}_2(t) = f_2(x_1, x_2) \]  
(A.1)

where \( x_i \in \mathbb{R}^{n_i} \) is the state for \( i = 1, 2 \), \( \dot{x}_2(t) = \int_0^t x_2(\tau)\,d\tau/t, t > 0 \). We assume that each \( x_i \)- subsystem for \( i = 1, 2 \) admits a continuously differentiable ISS-Lyapunov function \( V_i : \mathbb{R}^{n_i} \to \mathbb{R} \), satisfying the following:

(1) There exist constants \( \alpha_i, \bar{\alpha}_i > 0 \) such that
\[ \alpha_i |x_i|^2 \leq V_i(x_i) \leq \bar{\alpha}_i |x_i|^2, \quad \forall x_i. \]
(A.2)

(2) There exist constants \( \chi_{i(3-i)}, \alpha_i, \chi_1^m > 0 \) and \( \chi_2^m = 0 \), such that
\[ V_i(x_i) \geq \max\{\chi_{i(3-i)}V_3-1(x_{3-i}), \chi_1^m \bar{V}_2(x_2)\} \]
\[ \Rightarrow \nabla V_i(x_i)f_i \leq -\alpha_i V_i(x_i), \quad \forall x_i, \]
(A.3)

where \( \bar{V}_2(x_2) = \int_0^t \bar{V}_2(x_2(\tau))\,d\tau/t \).

For \( \chi_{12}, \chi_{21} \) satisfying \( \chi_{12} \chi_{21} < 1 \), we find a constant \( \sigma \) such that it satisfies
\[ 1/\sigma > \chi_{12}, \quad \sigma > \chi_{21}. \]
(A.4)

If \( V \) satisfies
\[ V(x) = \max\{\sigma V_1(x_1), V_2(x_2)\}. \]

Then, there exist constants \( \chi, \alpha > 0 \) such that \( V \) satisfies
\[ V(x) \geq \chi \bar{V}(x) \]
\[ \Rightarrow \nabla V(x)f(x) \leq -\alpha V(x), \]
(A.6)

where \( x = [x_1^T, x_2^T]^T \), \( \bar{V}(x) = \int_0^t \bar{V}(x(\tau))\,d\tau/t \) and \( f = [f_1^T, f_2^T]^T \).

To this purpose, we define the following sets
\[ A = \{ (x_1, x_2) : V_2(x_2) < \sigma V_1(x_1) \}; \]
\[ B = \{ (x_1, x_2) : V_2(x_2) > \sigma V_1(x_1) \}; \]
\[ C = \{ (x_1, x_2) : V_2(x_2) = \sigma V_1(x_1) \}. \]

Now fix any point \( p = (p_1, p_2) \neq (0, 0) \). There are three cases.

**Case 1:** \( p \in A \). In this case, \( V(x) = \sigma V_1(x_1) \) in a neighborhood of \( p \), and consequently
\[ \nabla V(p)f(p) = \sigma \nabla V_1(p_1)f_1 \]
(A.7)

For \( p \in A \), it holds that \( V_2(p_2) < \sigma V_1(p_1) \), and thus \( V_1(p_1) > \chi_{12} V_2 \). This then implies
\[ \nabla V_1(p_1)f_1 \leq -\alpha_1 V_1(p_1) \]
whenver \( V_1(p_1) \geq \chi_1^m \bar{V}_2(p_2) \). It follows from this that, for \( p \in A \),
\[ \nabla V(p)f(p) \leq -\alpha_1 V(p) \]
(A.9)

whenever \( V(p) \geq \chi_1^m \bar{V}(p) \).

**Case 2:** \( p \in B \). In this case, using exactly the same arguments as in Case 1, one shows that
\[ \nabla V(p)f(p) \leq -\alpha_2 V(p). \]
(A.10)

**Case 3:** \( p \in C \). First note that it holds for the locally Lipschitz function \( V \) that
\[ \nabla V(p)f(p) = \frac{d}{dt}|_{t=0} V(\varphi(t)) \]
for almost all \( p \), with \( \varphi(t) = [\varphi_1^T(t), \varphi_2^T(t)]^T \) being the solution of the initial-value problem
\[ \dot{\varphi}(t) = f(\varphi(t)), \quad \varphi(0) = p. \]
(A.12)

In this case, assume \( p = (p_1, p_2) \neq (0, 0) \) and \( V_1(p_1) \geq \chi_1^m \bar{V}_2(p_2) \)
(A.13)

Then, by using similar arguments as for Cases 1 and 2, one has
\[ \sigma \nabla V_1(p_1)f_1 \leq -\alpha_1 V(p) \]
\[ \nabla V_2(p_2)f_2 \leq -\alpha_2 V(p) \]
(A.15)

Note that in this case \( p_1 \neq 0 \) and \( p_2 \neq 0 \). Then, because of the continuous differentiability of \( V_1 \) and \( V_2 \), and the continuity of \( f \), there exist neighborhoods \( X_1 \) of \( p_1 \) and \( X_2 \) of \( p_2 \) such that
\[ \sigma \nabla V_1(p_1)f_1 \leq -\frac{1}{2}\alpha_1 V(p) \]
\[ \nabla V_2(p_2)f_2 \leq -\frac{1}{2}\alpha_2 V(p) \]
(A.16)
\[ \nabla V_2(p_2)f_2 \leq -\frac{1}{2}\alpha_2 V(p) \]
(A.17)

for all \( x \in X_1 \times X_2 \). Note also that there exists \( \delta > 0 \) such that \( \varphi(t) \in X_1 \times X_2 \) for all \( 0 \leq t \leq \delta \).

Now pick \( \Delta t \in (0, \delta) \). If \( \varphi(\Delta t) \in A \cup O \), then
\[ V(\varphi(\Delta t)) - V(p) = \sigma V_1(\varphi_1(\Delta t)) - \sigma V_1(p_1) \]
\[ \leq -\frac{1}{2}\alpha_1 V(p)\Delta t. \]
(A.18)

Similarly, if \( \varphi(\Delta t) \in B \cup O \), then
\[ V(\varphi(\Delta t)) - V(p) = V_2(\varphi_2(\Delta t)) - V_2(p_2) \]
\[ \leq -\frac{1}{2}\alpha_2 V(p)\Delta t. \]
(A.19)

Hence, if \( V \) is differentiable at \( p \), then
\[ \nabla V(p)f(p) \leq -\alpha V(p) \]
where \( \alpha = \min\{\alpha_1/2, \alpha_2/2\} \) for \( s \in \mathbb{R}_+ \). Note that conditions (A.13) can be guaranteed by
\[ V(p) \geq \sigma \chi_1^m \bar{V}(p) \]
(A.20)

By combining the three cases, it can be concluded that
\[ V(p) \geq \chi \bar{V}(p) \Rightarrow \nabla V(p)f(p) \leq -\alpha V(p), \]
(A.22)

where \( \chi = \sigma \chi_1^m \). Since \( V \) is continuously differentiable almost everywhere, (A.22) holds for almost all \( p \).