Abstract
We introduce the theory of enrichment over an internal monoidal category as a common generalization of both the standard theories of enriched and internal categories. Then, we contextualize the new notion by comparing it to another known generalization of enrichment: that of enrichment for indexed categories. It turns out that the two notions are closely related.

Keywords Category theory · Enriched categories · Internal categories · Indexed categories

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Contents
1 Introduction .............................................. 947
2 Internal Categories ........................................... 949
3 Internal Monoidal Categories ..................................... 951
4 Indexed Categories ........................................... 953
5 Externalization of Internal Categories ................................. 954
6 Internal Enriched Categories ...................................... 958
7 Indexed Enriched Categories ...................................... 963
References ........................................................................ 967

1 Introduction
Categories, even large ones, are said to be complete if they have limits for merely all small diagrams, and various size issues originate from this. Small categories avoid this pitfall, but, unfortunately, the only small complete categories are the complete preorders (a well-known, though unpublished, result by Freyd).
Internalizing the notion of small category yields that of internal category, which is thus well-behaved with respect to size. Moreover, the internal logic of realizability toposes allows for small complete categories that are not preorders, as Freyd’s degeneracy result relies on classical logic. For example, the category of modest sets in the effective topos (or, more precisely, in its subcategory of assemblies) is complete despite not being a preorder [7–9].

Unfortunately, the theory of internal categories is not as expressive as we might like. For example, it’s generally not possible to formulate the notion of presheaf in the internal context, because internal categories do not come with an enriching category the way that locally small categories are enriched over Set. To overcome these limitations, this paper presents and develops a theory of internal enrichment which combines the pleasantness of internal categories with respect to size issues with the expressivity of enriched categories.

The paper starts by recalling some background material. Section 2 briefly introduces the fundamental notions of internal category theory, by use of the internal language of the ambient category in an informal style.

Section 3 presents the notion of monoidal structure for internal categories and discusses the theory of internal monoidal categories, which will be used as the enriching categories for the proposed notion of internal enriched category.

Section 4 quickly recalls the notion of indexed category, together with some basic element of the theory of fibrations and their relation to indexed categories via the Grothendieck construction, with a focus on the case of indexed monoidal categories and monoidal fibrations.

Section 5 presents the construction of the externalization of an internal category, which yields an indexed category over the ambient category. We focus in particular on the externalization of internal monoidal categories, which yields indexed monoidal categories.

Section 6 introduces the theory of internal enrichment over an internal monoidal category, obtained by internalizing the standard theory of enrichment over a monoidal category. The section presents the definitions of internal enriched category, i.e., of an internal category enriched over an internal monoidal category, of functor between internal enriched category and of natural transformation between such functors. It discusses the 2-category of internal enriched categories, functors and natural transformations, and issues of change of base.

Finally, Sect. 7 places internal enrichment in the landscape of notions of generalized enrichment by showing that internal enriched categories are closely related to enriched indexed categories [15, 16] via externalization. This ties together the notion of internal enriched category with all of the background material presented in the early sections of the paper.

From an application perspective, we believe internal enrichment can be a valuable tool in the study of categorical models for polymorphism, in theoretical computer science. Indeed, Eugenio Moggi originally suggested that the effective topos contains a small complete subcategory as a way to understand how realizability toposes give rise to models for impredicative polymorphism, and concrete versions of such models first appeared in [5]. Moreover, [14] noted that set theory is inadequate to treat polymorphism, and [13] overcame the issue by internalizing the model in a suitable topos. Finally, [6] has presented a model for higher-order polymorphic lambda-calculus based on enrichment over the category of partial equivalence relations, and noticed that inconveniently this category is incomplete, although it is internally complete in the effective topos [8]. Thus, internal and enriched categories are essential tools in the treatment of polymorphism. Internal enriched categories, being their common generalization and carrying the benefits of both, would remove the necessity of picking one and allow for a natural extension of the known models.
This paper is based on the author’s doctoral research [3]. Further developments on the theory of internal enrichment, particularly in the direction of a theory of completeness, are already contained in the dissertation.

2 Internal Categories

In this section we quickly set some notation with regard to internal categories, without actually discussing their theory. Such notation is mostly standard, as it is fundamentally similar to that used in well-known textbooks [1, 11]. A brief account of the theory of internal categories adopting the notation of this section can be found in [4].

In the context of this paper, let \( \mathcal{E} \) be a category with finite limits, which we regard as our ambient category. To be precise, in particular we require \( \mathcal{E} \) to have a cartesian monoidal structure, that is, a monoidal structure given by a functorial choice of binary products \( \times: \mathcal{E} \times \mathcal{E} \to \mathcal{E} \) and a chosen terminal object \( \mathbb{1} \). As a category with finite limits, \( \mathcal{E} \) is a model for cartesian logic, or finite limit logic. We will frequently use the internal language of \( \mathcal{E} \) to ease the notation, although in an informal style. At the cost of readability, the given formulas could be elaborated either into the language of Freyd’s essentially algebraic theories [2], or into Johnstone’s cartesian logic [10], or into the partial Horn logic of Palmgren and Vickers [12], other than into commutative diagrams.

We start by giving the definition of internal category in \( \mathcal{E} \).

**Definition 1** (internal category) An internal category \( \mathbf{A} \) in \( \mathcal{E} \) is a diagram

\[
\begin{array}{ccc}
A_0 & \xleftarrow{\text{id}_A} & A_1 \\
\downarrow{\text{s}_A} & & \downarrow{\text{t}_A} \\
A_1 & \xleftarrow{\text{a}_A} & A_1 \times_{A_0} A_1
\end{array}
\]

in \( \mathcal{E} \) (where \( A_1 \times_{A_0} A_1 \) is the pullback of \( s_A \) and \( t_A \)) satisfying the usual axioms for categories, which can be expressed in the internal language of \( \mathcal{E} \) by the formulae

\[
a: A_0 \vdash s_A(\text{id}_A(a)) = a \land t_A(\text{id}_A(a)) = a
\]

\[
(\text{g}, f): A_1 \times_{A_0} A_1 \vdash s_A(g \circ_A f) = s_A(f) \land t_A(g \circ_A f) = t_A(g)
\]

\[
f: A_1 \vdash f \circ_A \text{id}_A(s_A(f)) = f \land \text{id}_A(t_A(f)) \circ A f = f
\]

\[
(\text{h}, g, f): A_1 \times_{A_0} A_1 \times_{A_0} A_1 \vdash h \circ_A (g \circ_A f) = (h \circ_A g) \circ_A f
\]

where \( A_1 \times_{A_0} A_1 \) is the pullback of \( s_A \) and \( t_A \), and \( A_1 \times_{A_0} A_1 \) is the iterated pullback. Notice that, by the universal property of the pullback, \((g, f): A_1 \times_{A_0} A_1\) if and only if \( f: A_1 \to A_1\), \( g: A_1\), and \( t_A(f) = s_A(g)\), in the sense that local sections \( X \to A_1 \times_{A_0} A_1 \) correspond bijectively to pairs of local sections \( f, g: X \to A_1 \) such that \( t_A f = s_A g \).

We now define functors of internal categories. Then internal categories and their functors form a category \( \text{Cat}_\mathcal{E} \).

**Definition 2** (internal functor) Let \( \mathbf{A} \) and \( \mathbf{B} \) be internal categories in \( \mathcal{E} \). A functor of internal categories \( F: \mathbf{A} \to \mathbf{B} \) is given by a pair of arrows \( F_0: A_0 \to B_0 \) and \( F_1: A_1 \to B_1 \) such that

\[
f: A_1 \vdash s_B(F_1(f)) = F_0(s_A(f)) \land t_B(F_1(f)) = F_0(t_A(f))
\]

\[
a: A_0 \vdash F_1(\text{id}_A(a)) = \text{id}_B(F_0(a))
\]
\[(g, f) : A_1 \times_{A_0} A_1 \vdash F_1(g \circ_A f) = F_1(g) \circ_{B_1} F_1(f)\]

The composition \(G \circ_{\text{Cat}_\mathcal{E}} F : \text{A} \rightarrow \text{C}\) (shortened in \(G\) for brevity) of two consecutive internal functors \(F : \text{A} \rightarrow \text{B}\) and \(G : \text{B} \rightarrow \text{C}\), and the identity functor \(\text{id}_{\text{Cat}_\mathcal{E}} (\text{A}) : \text{A} \rightarrow \text{A}\) for an internal category \(\text{A}\), are defined in the usual way.

The following result is the internal version of the standard set-theoretic one, and it can be proved by a completely routine application of the internal language of \(\mathcal{E}\).

**Proposition 1** The category \(\text{Cat}_\mathcal{E}\) has finite limits induced point-wise by the corresponding limits in \(\mathcal{E}\). In particular, there is a terminal internal category \(\text{1}_{\text{Cat}_\mathcal{E}}\) and a binary product \(\times_{\text{Cat}_\mathcal{E}}\) of internal categories making \(\text{Cat}_\mathcal{E}\) a cartesian monoidal category.

Let \(\mathcal{E}'\) be another category with finite limits, and \(F : \mathcal{E} \rightarrow \mathcal{E}'\) a functor preserving finite limits. Then, there is a change-of-base functor \(F_* : \text{Cat}_\mathcal{E} \rightarrow \text{Cat}_{\mathcal{E}'}\) applying \(F\) to the underlying graph of internal categories. In the following remark, we notice some useful properties of \(\text{Cat}_\mathcal{E}\) in relation to slicing and change of base.

**Remark 1** Let \(i : J \rightarrow I\) be an arrow in \(\mathcal{E}\). Then, there is an adjunction \(i_! \dashv i^* : \mathcal{E}/J \rightarrow \mathcal{E}/I\) where the functor \(i_! : \mathcal{E}/J \rightarrow \mathcal{E}/I\) is given by post-composition with \(i\), and the functor \(i^* : \mathcal{E}/I \rightarrow \mathcal{E}/J\) is given by pullback along \(i\). This adjunction extends to internal categories, yielding \((i_!)_* \dashv (i^*)_* : \text{Cat}_{\mathcal{E}/J} \rightarrow \text{Cat}_{\mathcal{E}/I}\). In particular, the unique arrow \(! : I \rightarrow \text{1}\) yields an adjunction \((I_!)_* \dashv (I^*)_* : \text{Cat}_\mathcal{E} \rightarrow \text{Cat}_{\mathcal{E}/I}\).

**Proposition 2** Let \(F : \mathcal{E} \rightarrow \mathcal{E}'\) be a functor between categories with finite limits which preserves such finite limits. Then, the induced change-of-base functor \(F_* : \text{Cat}_\mathcal{E} \rightarrow \text{Cat}_{\mathcal{E}'}\) preserves finite limits.

There is also an obvious objects functor \(U : \text{Cat}_\mathcal{E} \rightarrow \mathcal{E}\) sending an internal category to its underlying object-of-objects. This functor preserves the cartesian monoidal structure. We then mention a few remarkable examples of internal categories:

- \(\text{dis}\ A\), the discrete category over an object \(A\).
- \(\text{ind}\ A\), the indiscrete category over an object \(A\).
- \(A^{\text{op}}\), The opposite category of an internal category \(A\).

Then, we have the following adjunctions:

- \(\text{dis} \dashv U \dashv \text{ind}\).
- \((-)^{\text{op}} \dashv (-)^{\text{op}}\).

An alternative, more abstract way to look at the discrete category with respect to Remark 1 is to notice that \(\text{dis}\ A\) is (isomorphic to) \((A_!)_*(A^*)_*\text{1}_{\text{Cat}_\mathcal{E}}\).

We now define natural transformations of internal categories. Internal categories, together with their functors and the natural transformations between them, form a 2-category \(\text{Cat}_\mathcal{E}\) (denoted in the same way as its underlying 1-category with abuse of notation; context usually suffices to distinguish whether we are referring to the underlying 1-category or the 2-category).

**Definition 3** (internal natural transformation) Let \(F, G : \text{A} \rightarrow \text{B}\) be functors of internal categories in \(\mathcal{E}\). A natural transformation of internal functors \(\alpha : F \rightarrow G : \text{A} \rightarrow \text{B}\) is given by an arrow \(\alpha : A_0 \rightarrow B_1\) such that

\[a : A_0 \vdash s_B(\alpha(a)) = F_0(a) \land t_B(\alpha(a)) = G_0(a)\]
Vertical and horizontal compositions of natural transformations and the identity natural transformation are defined in the usual way.

To clarify a subtlety of the notation, notice that id\(_A\) denotes the identity of the category \(A\), while id\(_{\text{Cat}}\)\(_{(A)}\) denotes the identity functor on \(A\) and id\(_{\text{Cat}}\)\(_{(A,B)}\)\((F)\) denotes the identity natural transformation on \(F : A \rightarrow B\). When it is clear from the context, we might omit the subscript of id and write, for example, id\((A)\) in place of id\(_{\text{Cat}}\)\(_{(A)}\) and id\((F)\) in place of id\(_{\text{Cat}}\)\(_{(A,B)}\)\((F)\).

Sometimes, and especially when using the internal language, the notation denoting the object or morphism components of a functor can be cumbersome. Thus, in the following sections we shall adopt a common convention in category theory and omit to make such components explicit when it is clear from the context which one we are referring to. For example, given an internal functor \(F : A \rightarrow B\), in context \(a : A_0, f : A_1\) we shall write \(F(a)\) for \(F_0(a)\) and \(F(f)\) for \(F_1(f)\).

### 3 Internal Monoidal Categories

We could not conceivably present a notion of enrichment without a suitable notion of monoidal category to enrich over. We introduce here the definitions, in the internal language of \(\mathcal{E}\), of the notions of monoidal category, functor and natural transformation.

**Definition 4** (internal monoidal category) An **internal monoidal category** is an internal category \(\mathcal{V}\) in \(\mathcal{E}\) equipped with functors

- Monoidal product: \(\otimes : \mathcal{V} \times \text{Cat}_\mathcal{E} \mathcal{V} \rightarrow \mathcal{V}\), and
- Monoidal unit: \(\mathbb{I}_\mathcal{V} : \mathbb{I}_{\text{Cat}_\mathcal{E}} \rightarrow \mathcal{V}\),

and natural isomorphisms

- Associator: \(\alpha_\mathcal{V} : (\pi_1 \otimes_{\mathcal{V}} \pi_2) \otimes_{\mathcal{V}} \pi_3 \rightarrow \pi_1 \otimes_{\mathcal{V}} (\pi_2 \otimes_{\mathcal{V}} \pi_3) : \mathcal{V} \times \text{Cat}_\mathcal{E} \mathcal{V} \times \text{Cat}_\mathcal{E} \mathcal{V} \rightarrow \mathcal{V}\),
- Left unitor: \(\lambda_\mathcal{V} : \mathbb{I}_\mathcal{V} \otimes_{\mathcal{V}} \text{id}(\mathcal{V}) \rightarrow \text{id}(\mathcal{V}) : \mathcal{V} \rightarrow \mathcal{V}\), and
- Right unitor: \(\rho_\mathcal{V} : \text{id}(\mathcal{V}) \otimes_{\mathcal{V}} \mathbb{I}_\mathcal{V} \rightarrow \text{id}(\mathcal{V}) : \mathcal{V} \rightarrow \mathcal{V}\),

such that, in context \(a, b, c, d : V_0\), the axioms

\[
\begin{align*}
(a \otimes_{\mathcal{V}} \mathbb{I}_\mathcal{V}) \otimes_{\mathcal{V}} b & \xrightarrow{\alpha_\mathcal{V}(a, \mathbb{I}_\mathcal{V}, b)} a \otimes_{\mathcal{V}} (\mathbb{I}_\mathcal{V} \otimes_{\mathcal{V}} b) & (a \otimes_{\mathcal{V}} b) \otimes_{\mathcal{V}} (c \otimes_{\mathcal{V}} d) & \xrightarrow{\alpha_\mathcal{V}(a \otimes_{\mathcal{V}} b, c, d)} (a \otimes_{\mathcal{V}} b) \otimes_{\mathcal{V}} (c \otimes_{\mathcal{V}} d) \\
\rho_\mathcal{V}(a) \otimes_{\mathcal{V}} \text{id}(\mathcal{V})(b) & \xrightarrow{\text{id}(\mathcal{V})(a) \otimes_{\mathcal{V}} \lambda_\mathcal{V}(b)} a \otimes_{\mathcal{V}} b & \text{id}(\mathcal{V})(a) \otimes_{\mathcal{V}} \lambda_\mathcal{V}(b) & \xrightarrow{\alpha_\mathcal{V}(a \otimes_{\mathcal{V}} b, c, d)} (a \otimes_{\mathcal{V}} b) \otimes_{\mathcal{V}} (c \otimes_{\mathcal{V}} d) \\
(a \otimes_{\mathcal{V}} (b \otimes_{\mathcal{V}} c)) \otimes_{\mathcal{V}} d & \xrightarrow{\alpha_\mathcal{V}(a, b \otimes_{\mathcal{V}} c, d)} (a \otimes_{\mathcal{V}} (b \otimes_{\mathcal{V}} c)) \otimes_{\mathcal{V}} d & \text{id}(\mathcal{V})(a) \otimes_{\mathcal{V}} \alpha_\mathcal{V}(b, c, d) & \xrightarrow{\alpha_\mathcal{V}(a \otimes_{\mathcal{V}} b, c, d)} (a \otimes_{\mathcal{V}} b) \otimes_{\mathcal{V}} (c \otimes_{\mathcal{V}} d)
\end{align*}
\]

hold. Here, the diagrams are just convenient representations of formulae in the internal language of the ambient category \(\mathcal{E}\). For example, Equation 1 represents the following formula.

\[
a : V_0, b : V_0 \vdash (\rho_\mathcal{V}(a) \otimes_{\mathcal{V}} \text{id}(\mathcal{V})(b)) \circ_{\mathcal{V}} \alpha_\mathcal{V}(a, \mathbb{I}_\mathcal{V}, b) = \text{id}(\mathcal{V})(a) \otimes_{\mathcal{V}} \lambda_\mathcal{V}(b)
\]
Such translation is straightforward, but we favour the diagrammatic representation as it is more intuitive and familiar to the category theorist.

The previous definition is a direct internalization of the standard definition of monoidal category, and that alone should suffice to persuade us of its correctness. If we were still skeptical, though, it could also be argued that, since small monoidal categories are pseudomonoids in the 2-category of categories, then internal monoidal categories in $E$ must be pseudomonoids in the 2-category $\text{Cat}_E$, which is what our definition amounts to.

We then proceed with the definition of monoidal functor.

**Definition 5** (internal monoidal functor) An internal monoidal functor $(F, \epsilon, \mu) : V \to W$ is given by an internal functor $F : V \to W$ and coherence natural isomorphisms

$$\epsilon : 1_W \to F 1_V : 1_{\text{Cat}_E} \to W$$

and

$$\mu : F \otimes W F \to F(- \otimes V -) : V \times_{\text{Cat}_E} V \to W$$

such that, in context $a, b, c : V_0$, the axioms

$$\alpha_{W(F(a), F(b), F(c))} \mu(a, b \otimes W c) \mu(a \otimes b, c) \alpha(a \otimes b, c) \mu(a \otimes b, c) \mu(a, b \otimes W c)$$

hold.

Then, we define natural transformations of monoidal functors.

**Definition 6** (internal monoidal natural transformation) An internal monoidal natural transformation $\alpha : (F, \epsilon, \mu) \to (G, \epsilon, \mu) : V \to W$ is a natural transformation $\alpha : F \to G : V \to W$ such that, in context $a, b : V_0$, the axioms

$$\alpha \mu(a, b) \mu(a, b) \mu(a, b) \alpha(a, b) \mu(a, b)$$

hold.
It is routine to check in the internal language that the data above gives 2-categories, so we can give the following definitions.

**Definition 7** (category of internal monoidal categories) Internal monoidal categories and monoidal functors in $\mathcal{E}$ form a category $\text{Mon}_{\mathcal{E}}$. Moreover, monoidal natural transformations in $\mathcal{E}$ give $\text{Mon}_{\mathcal{E}}$ the structure of a 2-category (denoted in the same way as its underlying 1-category with abuse of notation; context usually suffices to distinguish whether we are referring to the underlying 1-category or the 2-category).

As for internal categories (see Proposition 2), we can transport the monoidal structure along a functor changing the base category.

**Proposition 3** Let $F : \mathcal{E} \to \mathcal{E}'$ be a functor between categories with finite limits which preserves such finite limits. Then, there is an induced change-of-base functor $F_* : \text{Mon}_{\mathcal{E}} \to \text{Mon}_{\mathcal{E}'}$, which also preserves finite limits.

Finally, notice that there is an underlying-internal-category 2-functor

$$U_{\text{Cat},\mathcal{E}} : \text{Mon}_{\mathcal{E}} \to \text{Cat}_{\mathcal{E}}$$

sending monoidal categories, functors and natural transformations to, respectively, their underlying internal categories, functors and natural transformations.

## 4 Indexed Categories

Indexed categories, while not playing a direct role in the definition of internal enrichment, will be essential to their understanding in relation to other notions of enrichment.

Indexed categories have been treated extensively in the literature, and the main ideas are long established. However, we shall refer to the recent exposition given in [15, 16], since these sources are also needed in regard to the notion of enriched indexed category.

To begin with, we state the definition of indexed category.

**Definition 8** (indexed category) An $\mathcal{E}$-indexed category is a pseudofunctor $\mathcal{E}^{\text{op}} \to \text{Cat}$, where $\text{Cat}$ is the 2-category of categories, functors, and natural transformations.

Consider the following notable example, which will turn out to be useful later on.

**Example 1** The self-indexing of $\mathcal{E}$ is the $\mathcal{E}$-indexed category whose fiber over an object $X$ is the slice category $\mathcal{E}_{/X}$ and where the reindexing along $f : X \to Y$ is given by pullback along $f$.

Next, we state the definition of fibration.
Definition 9 (fibration) An arrow \( l: A' \to A \) in \( \mathcal{F} \) is cartesian with respect to a functor \( P: \mathcal{F} \to \mathcal{E} \) if, for any other arrow \( h: A'' \to A \) in \( \mathcal{F} \) and \( g: P(A'') \to P(A') \) in \( \mathcal{E} \) such that \( P(l)g = P(h) \), there exists a unique \( k: A'' \to A' \) in \( \mathcal{F} \) such that \( lk = h \) and \( P(k) = g \).

A functor \( P: \mathcal{F} \to \mathcal{E} \) is a fibration if for any \( A \) in \( \mathcal{F} \) and \( f: X \to P(A) \) there is a cartesian arrow \( l: A' \to A \) such that \( P(l) = f \), called the cartesian lifting of \( f \). Moreover, a fibration is cloven if it comes with a choice of cartesian liftings.

There is a strict relation between the theory of indexed categories and that of fibrations, as established by the following, classic result.

Theorem 1 An \( \mathcal{E} \)-indexed category \( \mathcal{C}: \mathcal{E}^{\text{op}} \to \text{Cat} \) is, via the Grothendieck construction, equivalent to a cloven fibration \( \int \mathcal{C} \to \mathcal{E} \). More precisely, there is an equivalence of bicategories \( [\mathcal{E}^{\text{op}}, \text{Cat}] \simeq \text{Fib}(\mathcal{E}) \) (and, in fact, an equivalence of strict 2-categories [10]), where \( \text{Fib}(\mathcal{E}) \) is the 2-category of cloven fibrations over \( \mathcal{E} \), strong morphisms of fibrations, and transformations of fibrations.

Now we want to extend the previous ideas to the context of monoidal categories. We begin by giving the notion of indexed monoidal categories.

Definition 10 (indexed monoidal category) An \( \mathcal{E} \)-indexed monoidal category is a pseudofunctor \( W: \mathcal{E}^{\text{op}} \to \text{Mon} \), where \( \text{Mon} \) is the 2-category of monoidal categories, strong monoidal functors, and monoidal transformations.

A suitable notion of monoidal fibration is required to establish a relation with indexed monoidal categories, so we recall that in the following definition.

Definition 11 (monoidal fibration, Definition 12.1 [15]) Let \( \mathcal{V} \) be a monoidal category. A monoidal fibration is a cloven fibration \( \mathcal{V} \to \mathcal{E} \) such that the underlying functor is strict monoidal (with \( \mathcal{E} \) regarded as cartesian monoidal) and the tensor product in \( \mathcal{V} \) preserves cartesian arrows.

For a general monoidal base category the notions of indexed monoidal category and of monoidal fibration do not correspond under the Grothendieck construction. Indeed, if \( W: \mathcal{E}^{\text{op}} \to \text{Mon} \) is an \( \mathcal{E}' \)-indexed monoidal category, then, in the cloven fibration \( \int W \to \mathcal{E}' \), it is evident that \( \int W \) has tensor products only for objects in the same fiber, and the result is still an object in that fiber. On the other hand, if \( \mathcal{V} \to \mathcal{E}' \) is a monoidal fibration, and \( A \) and \( B \) are objects of \( \mathcal{V} \) lying over the objects \( X \) and \( Y \) of \( \mathcal{E}' \) respectively, then the tensor product \( A \otimes_{\mathcal{V}} B \) lies over \( X \otimes_{\mathcal{E}'} Y \). However, in case the monoidal structure on \( \mathcal{E}' \) is given by the product, i.e., \( \mathcal{E}' \) is cartesian monoidal, such as our ambient category \( \mathcal{E} \) is, then there is a correspondence (for details, see Theorem 12.7 from [15]). We recall this result in the following theorem.

Theorem 2 An \( \mathcal{E} \)-indexed monoidal category \( W \) is, via the Grothendieck construction, equivalent to a monoidal fibration \( \int W \to \mathcal{E} \). More precisely, there is an equivalence of bicategories \( [\mathcal{E}^{\text{op}}, \text{Mon}] \simeq \text{MonFib}(\mathcal{E}) \), where \( \text{MonFib}(\mathcal{E}) \) is the 2-category of monoidal fibrations over \( \mathcal{E} \), strong monoidal morphisms of fibrations, and monoidal transformations of fibrations.

5 Externalization of Internal Categories

The last piece of background material that we present concerns the relationship between internal and indexed categories, and makes an essential use of the theory of indexed categories.
from Sect. 4. For the central notion of externalization of an internal category, we shall follow the exposition of [8, 9].

Let $A$ be a category in $\mathcal{E}$ and $X$ an object of $\mathcal{E}$. We regard an arrow $X \to A_0$ as representing an indexed family of objects of $A$ over the indexing object $X$. Given two such indexed families $x_0, x_1 : X \to A_0$, consider the pullback

$$(x_0, x_1)^* A_1 \longrightarrow A_1$$

$$(x_0, x_1) \downarrow \downarrow \downarrow^{(s_{A_1}, t_A)}$$

$$(x_0, x_1) \longrightarrow A_0 \times A_0.$$

Then the sections of $p$ represent indexed families of arrows of $A$ with domain $x_0$ and codomain $x_1$. Given another family $x_2 : X \to A_0$, the composition in $A$ restricts to an indexed composition

$$o_{A|x_0, x_1, x_2} : (x_1, x_2)^* A_1 \times (x_0, x_1)^* A_1 \to (x_0, x_2)^* A_1$$

inducing a composition of indexed families of arrows: given two families of arrows $s_0 : X \to (x_0, x_1)^* A_1$ and $s_1 : X \to (x_1, x_2)^* A_1$, their composition is defined as

$$s_1 \circ o_{A|x_0, x_1, x_2} s_0 = X \xrightarrow{(s_1, s_0)} (x_1, x_2)^* A_1 \times (x_0, x_1)^* A_1 \xrightarrow{o_{A|x_0, x_1, x_2}} (x_0, x_2)^* A_1.$$

Moreover, a family of objects $x : X \to A_0$ induces a family of identity arrows $id_{A|x^x} : X \to (x, x)^* A_1$. These data form the category $[A]^X$ of indexed families of objects and morphisms of $A$ over $X$.

Given a reindexing $u : X' \to X$, precomposition reindexes a family of objects $x : X \to A_0$ over $X$ the family $xu$ over $X'$; a family of arrows $s : X \to (x_0, x_1)^* A_1$ is reindexed to $u^* s : X' \to (ux_0, ux_1)^* A_1$ by pulling back the section $s$ along $(x_0, x_1)$. That gives a functor $u^* : [A]^X \to [A]^{X'}$.

The above discussion leads to the following result.

**Proposition 4** For $A$ an internal category in $\mathcal{E}$, there is an indexed category $[A]$ given by $[A](X) := [A]^X$ and $[A](u) := u^*$.

**Remark 2** Notice that the indexed category arising from an internal one is given by a strict functor $\mathcal{E}^{op} \to \text{Cat}$ (rather than merely a pseudofunctor). Then, evidently, internal categories yield rather special indexed categories, and not all indexed categories can be obtained from an internal one.

The construction extends to the monoidal context, as shown in the following proposition.

**Proposition 5** Let $V$ be an internal monoidal category in $\mathcal{E}$. Then, $[V]$ is an indexed monoidal category on $\mathcal{E}$.

**Proof** Let $X$ be an object in $\mathcal{E}$. Then, $[V]^X$ has a monoidal structure induced by that of $V$. The monoidal product on objects is defined as

$$(X \xrightarrow{x} V_0) \otimes_{[V]^X} (X' \xrightarrow{x'} V_0) := X \xrightarrow{(x, x')} V_0 \times V_0 \xrightarrow{\otimes_{V}} V_0.$$

To define the monoidal product of arrows, let

$$(X \xrightarrow{x_0} V_0) f : X \to (x_0, x_1)^* V_1 \xrightarrow{(X \xrightarrow{x_1} V_0)}$$
and
\[(X \xrightarrow{x'_0} V_0) \xrightarrow{f': X \to (x'_0, x'_1)^*V_1} (X \xrightarrow{x'_1} V_0)\]
be arrows of \([V]^X\), and notice that \(\otimes_V\) restricts to
\[(x_0, x_1)^*V_1 \times (x'_0, x'_1)^*V_1 \to (x_0 \otimes_{[V]^X} x'_0, x_1 \otimes_{[V]^X} x'_1)^*V_1.\]
Then the monoidal product of arrows \(f \otimes_{[V]^X} f'\) is given by the arrow
\[X \xrightarrow{(f, f')} (x_0, x_1)^*V_1 \times (x'_0, x'_1)^*V_1 \otimes_V (x_0 \otimes_{V} x'_0, x_1 \otimes_{V} x'_1)^*V_1.\]
The monoidal unit \(\mathbb{1}_{[V]^X}\) is defined by the constant family indexed by \(X\) on the monoidal unit of \(V\). The structural isomorphisms, associator and unitors are defined point-wise. Moreover, reindexing preserves the monoidal product. \(\Box\)

A more conceptual explanation of Proposition 5, pointed out by the anonymous reviewer, is that \([-\] is a finite product preserving 2-functor, and thus it preserves pseudomonoids.

**Remark 3** The strictness of the monoidal products of the fibers of the indexed monoidal category \([V]\) obtained from an internal monoidal category \(V\) is the same as that of the original monoidal product of \(V\), so it will generally not be strict monoidal. Still, the reindexing functors for \([V]\) strictly preserve the monoidal structure, regardless of how strict the monoidal product of \(V\) is. That means that the (actually strict) functor \([V]: \mathcal{E} \to \text{Mon}\) factorizes through the 2-category of (non-necessarily-strict) monoidal categories, strict monoidal functors and monoidal natural transformations. Such a category is quite uncommon, since normally there is little use for strict monoidal functors, especially between non-strict monoidal categories. Nonetheless, this shows that the indexed monoidal categories arising from internal monoidal categories are rather special ones.

**Remark 4** For every internal category \(A\), the fiber \([A]^X\) over an object \(X\) is enriched over \(\mathcal{E}/X\):

- Homset: \(\text{Hom}_{[A]^X}(x_0, x_1) := (x_0, x_1)^*A_1 \xrightarrow{p} X.\)
- Composition: \(\circ_{[A]^X}(x_0, x_1, x_2) := \circ_{A|_{x_0, x_1, x_2}}.\)
- Identity: \(\text{id}_{[A]^X}(x)\).

Reindexing is compatible with this structure, in that the reindexing of the homset is the same as the homset of the reindexing. More explicitly, given a reindexing \(u: X' \to X\), by pullback-pasting we have that
\[u^*(x_0, x_1)^*A_1 \cong (x_0u, x_1u)^*A_1.\]
In fact, the reindexing functor \(u^*: [A]^X \to [A]^{X'}\) is a fully-faithful functor of enriched categories. Then \([A]\) is an indexed enriched category over the self-indexing of \(\mathcal{E}\) (see Example 1).

As stated in Theorem 1, indexed categories are equivalent to cloven fibrations. So, we can give an abstract definition of the externalization of an internal category as follows.

**Definition 12** (externalization) The *externalization* of an internal category \(A\) is the total category for the fibration associated to the indexed category \([A]\). With abuse of notation, we denote the externalization of \(A\) with \([A]\), and context will usually suffice to distinguish between the use of the notation as a fibration or as an indexed category.
For practical purposes, it is useful to make the previous definition more explicit. The externalization of \( \mathcal{A} \) is the category given by the data

- Objects: families of objects of \( \mathcal{A} \) indexed over objects of \( \mathcal{E} \), that is, arrows \( X \to A_0 \) with \( X \) in \( \mathcal{E} \).
- Morphisms: an arrow \( (x : X \to A_0) \to (y : Y \to A_0) \) is given by a reindexing \( u : X \to Y \) and a family of arrows \( x \to y u \), that is, a section of the projection \( p : (x, y u)^* A_1 \to X \).
- Composition: the composition is given by

\[
(X \to A_0) \to (Y \to A_0) \mapsto (x \to y z, y z u g_{y z}) A_1 \to (Z \to A_0).
\]

\[:(X \to A_0) \mapsto (X \to Y \to A_0) \to (Y \to Z \to A_0) \to (Z \to A_0).
\]

- Identity: the family of identity arrows.

Let \( F : \mathcal{A} \to \mathcal{B} \) be a functor of internal categories. Then, there is a functor of fibered categories \([F] : [\mathcal{A}] \to [\mathcal{B}]\) defined on objects as

\[
[F](X \to A_0) := X \to A_0 \to F_0 \to B_0
\]

and on morphisms as

\[
[F]\left((X \to A_0) \to (Y \to A_0)\right) := (X \to Y \to A_0) \to (Y \to F_0 \to B_0)
\]

which restricts to a functor on the fibers \([F]^X : [\mathcal{A}]^X \to [\mathcal{B}]^X\).

Let \( \alpha : F \to G : \mathcal{A} \to \mathcal{B} \) be a natural transformation. Then, there is a natural transformation of fibered categories \([\alpha] : [F] \to [G] : [\mathcal{A}] \to [\mathcal{B}]\), defined as

\[
[\alpha](X \to A_0) := (X \to A_0) \to (F_0 \to B_0) := (\text{id}_X, X \to A_0 \to (F_0 \to B_0)) \to (X \to A_0 \to (F_0 \to B_0))
\]

which restricts to a natural transformation on the fibers \([\alpha]^X : [F]^X \to [G]^X\).

**Remark 5** If \( V \) is a monoidal category in \( \mathcal{E} \), then \([V]\) is an indexed monoidal category by Proposition 5. By Theorem 2, it follows that the externalization \([V]\) has an induced monoidal structure. Explicitly, the monoidal product on objects is given by

\[
(X \to V_0) \otimes [V] (Y \to V_0) := X \times Y \xrightarrow{X \times Y} V_0 \otimes V_0 \xrightarrow{V_0} V_0.
\]

The monoidal product on arrows

\[
(X \to V_0) \otimes [V] (Y \to V_0) := (X \to Y \to V_0)
\]

and

\[
(Z \to V_0) \otimes [V] (W \to V_0) := (Z \to W \to V_0)
\]

is the arrow \( (X \to V_0) \otimes [V] (Z \to V_0) \to (Y \to V_0) \otimes [V] (W \to V_0) \) indexed by \( u \times v : X \times Z \to Y \times W \) and given by

\[
f \otimes_V g : X \times Z \to (x \otimes_V z, y u \otimes_V w v)^* V_1.
\]
The monoidal unit is $\mathbb{I}_V : 1_\mathcal{E} \to V_0$. Finally, the structural isomorphisms are induced by those of $V$.

### 6 Internal Enriched Categories

We are finally ready to introduce the main topic of this paper: the theory of internal enrichment. We shall derive the necessary notions by the process of internalization of the theory of standard enrichment. Substantially, that amounts to translating the definitions from enriched category theory into the internal language of the ambient category. In other words, we take advantage of the fact that the axioms of the theory of enrichment are expressible in the internal language.

From now on, let $V$ be an internal monoidal category in $\mathcal{E}$. We define the notion of enrichment in $V$ internal to the ambient category $\mathcal{E}$, making use of the following notation to ease the use of the internal language in relation to internal categories, by bringing it closer to the standard notation of category theory. Given terms $v : V_0$, $w : V_0$ and $f : V_1$, we write $f : v \to w$ instead of (the conjunction of) the formulae $s_V(f) = v$ and $t_V(f) = w$.

**Definition 13** (internal enriched category) An internal $V$-enriched category $X$ in $\mathcal{E}$, or $V$-category, is given by the following data.

- Underlying object: an object $X$ of $\mathcal{E}$.
- Internal hom: a morphism $\text{Hom}_X : X \times X \to V_0$.
- Composition: a morphism $\circ_X : X \times X \times X \to V_1$ such that, in context $x_0$, $x_1$, $x_2 : X$,
  
  $\circ_X(x_0, x_1, x_2) : \text{Hom}_X(x_1, x_2) \otimes V \text{Hom}_X(x_0, x_1) \to \text{Hom}_X(x_0, x_2)$.

- Identity: a morphism $\text{id}_X : X \to V_1$ such that, in context $x : X$,
  
  $\text{id}_X(x) : \mathbb{I}_V \to \text{Hom}_X(x, x)$.

Moreover, it has to satisfy the following axioms (in context $x_0$, $x_1$, $x_2$, $x_3 : X$).

\[
\begin{align*}
\text{Hom}_X(x_2, x_3) \otimes_V \text{Hom}_X(x_1, x_2) &\to \text{Hom}_X(x_0, x_1) \\
\text{Hom}_X(x_2, x_3) \otimes_V \text{Hom}_X(x_1, x_2) \otimes_V \text{Hom}_X(x_0, x_1) &\to \text{Hom}_X(x_0, x_3) \\
\text{Hom}_X(x_1, x_1) \otimes V \text{Hom}_X(x_0, x_1) &\to \text{Hom}_X(x_0, x_1) \\
\text{Hom}_X(x_0, x_1) \otimes V \text{Hom}_X(x_1, x_1) &\to \text{Hom}_X(x_0, x_1) \\
\text{Hom}_X(x_0, x_1) \otimes V \text{Hom}_X(x_0, x_1) &\to \text{Hom}_X(x_0, x_1)
\end{align*}
\]
Notice how the conventions on the internal language of $\mathcal{E}$ allow one to express those axioms in a form very close to that used to define standard enriched categories.

**Example 2** Let $\mathcal{V}$ be a small monoidal category. Then, $\mathcal{V}$ is an internal category in $\text{Set}$, and internal $\mathcal{V}$-enriched categories in $\text{Set}$ are standard (small) $\mathcal{V}$-enriched categories.

Continuing in the style of the previous definition, we give a notion of internal enriched functor, by translating the standard definition into the internal language.

**Definition 14** (internal enriched functor) Let $X$ and $Y$ be $V$-enriched categories. A $V$-enriched functor, or $V$-functor, $F: X \to Y$ is given by the following data.

- Objects component: an arrow $F_0: X \to Y$.
- Morphisms component: an arrow $F_1: X \times X \to V_1$ such that, in context $x_0, x_1: X$,

$$F_1(x_0, x_1): \text{Hom}_X(x_0, x_1) \to \text{Hom}_Y(F_0(x_0), F_0(x_1)).$$

Moreover, it has to satisfy the following axioms (in context $x_0, x_1, x_2: X$).

$$\text{Hom}_X(x_1, x_2) \otimes_Y \text{Hom}_X(x_0, x_1) \xrightarrow{\alpha_X(x_0,x_1,x_2)} \text{Hom}_X(x_0, x_2) \quad \text{Hom}_Y(F_0(x_1), F_0(x_2)) \otimes_Y \text{Hom}_Y(F_0(x_0), F_0(x_1)) \xrightarrow{\alpha_Y(F_0(x_0), F_0(x_1), F_0(x_2))} \text{Hom}_Y(F_0(x_0), F_0(x_2))$$

$$F_1 \text{id}_X = \text{id}_Y F_0 \quad (6)$$

We would expect the definition above to provide a category $V \text{Cat}_E$ of internal $V$-enriched categories and functors. We present the relevant data for that.

The composition of $V$-functors $F: X \to Y$ and $G: Y \to Z$ is defined, in context $x_0, x_1: X$, as follows.

$$(GF)_0 := G_0 F_0: X \to Z$$

$$\text{Hom}_X(x_0, x_1) \xrightarrow{F_1(x_0, x_1)} \text{Hom}_Z(G_0 F_0(x_0), G_0 F_0(x_1))$$

$$(GF)_1(x_0, x_1) := \text{Hom}_Y(F_0(x_0), F_0(x_1)) \xrightarrow{G_1(F_0(x_0), F_0(x_1))} \text{Hom}_Z(G_0 F_0(x_0), G_0 F_0(x_1))$$

The identity $V$-functor $\text{id}_{V \text{Cat}_E}(X): X \to X$ on $X$ is defined as follows.

$$(\text{id}_{V \text{Cat}_E}(X))_0 := \text{id}(X) \xrightarrow{\text{id}(X)} X$$

$$(\text{id}_{V \text{Cat}_E}(X))_1 := X \times X \xrightarrow{\text{Hom}_X} V_0 \xrightarrow{\text{id}_Y} V_1$$

It is just an exercise in the internal language to prove that the data so defined yield a category, as stated in the following proposition.

**Proposition 6** Composition and identity of internal enriched functors strictly satisfy associativity and unitarity. Thus, $V$-enriched categories and functors form a category $V \text{Cat}_E$. 

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**Example 3** There is an underlying-object functor $U : \text{V Cat}_\mathcal{E} \to \mathcal{E}$ sending $\text{V}$-enriched categories to their underlying object, and $\text{V}$-enriched functors to their object-component.

**Example 4** Let $X$ be an object of $\mathcal{E}$. The indiscrete $\text{V}$-enriched category $\text{ind}(X)$ on $X$ is given by

$$\text{Hom}_{\text{ind}(X)} := X \times X \xrightarrow{1} \mathcal{E} \xrightarrow{\text{V}} V_0.$$  

The rest of the structure follows from that. Analogously, a morphism $f : X \to Y$ induces an indiscrete $\text{V}$-enriched functor $\text{ind}(f) : \text{ind}(X) \to \text{ind}(Y)$. Then, there is a functor $\text{ind} : \mathcal{E} \to \text{V Cat}_\mathcal{E}$.

**Remark 6** To define the discrete $\text{V}$-enriched category over an object of $\mathcal{E}$, we would need to assume some extra hypothesis. Firstly, we would need to be able to tell whether two elements of the underlying object of the $\text{V}$-enriched category are equal. Secondly, we would need an initial object in $\text{V}$ to be the homset of non-equal elements of the underlying object. Both hypothesis do not hold in general. For example, the first one does not hold in the effective topos.

Finally, again by translating the standard definition into the internal language, we give the definition of internal enriched natural transformation.

**Definition 15** (internal enriched natural transformation) Let $X$ and $Y$ be $\text{V}$-enriched categories, and $F$ and $G$ be $\text{V}$-enriched functors $X \to Y$. A $\text{V}$-enriched natural transformation, or $\text{V}$-natural transformation, $\alpha : F \times G : X \to Y$ is given by an arrow $\alpha : X \to V_1$ such that, in context $x : X$,

$$\alpha(x) : I_V \to \text{Hom}_\text{V} \left( F_0(x), G_0(x) \right)$$

and satisfying, in context $x_0, x_1 : X$, the following axiom.

$$\begin{array}{c}
\xymatrix{ 
\text{Hom}_X(x_0, x_1) \ar[r]^{\varphi(x_0, x_1)} & \text{Hom}_Y \left( F_0(x_0), G_0(x_1) \right) \ar[r]^{\varphi(x_0, x_1)} & \text{Hom}_Y \left( F_0(x_1), G_0(x_0) \right) \\
\text{Hom}_X(x_0, x_1) \ar[r]_\varphi & \text{Hom}_Y \left( F_0(x_0), G_0(x_1) \right) \ar[r]_\varphi & \text{Hom}_Y \left( F_0(x_1), G_0(x_0) \right) }
\end{array}\tag{8}$$

We would expect the definition above to provide a 2-category of internal $\text{V}$-enriched categories, functors and natural transformations. We present the relevant data for that.

Consider $\text{V}$-categories, $\text{V}$-functors and $\text{V}$-natural transformations as shown in the diagram:

$$\begin{array}{ccc}
W & \xrightarrow{L} & X \\
\downarrow \varphi \uparrow & & \downarrow \varphi \\
\downarrow \psi & & \downarrow \psi \\
G & \xrightarrow{R} & Z
\end{array}$$

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Vertical composition of $\mathbf{V}$-natural transformations $\beta \circ \alpha : F \to H : X \to Y$ is defined, in context $x : X$ by

$$
\begin{align*}
\beta \circ \alpha(x) := & \quad \Hom_X\left(G_0(x), H_0(x)\right) \otimes \Hom_X\left(F_0(x), G_0(x)\right) \\
& \quad \downarrow_{\beta(x) \otimes \alpha(x)} \\
& \quad \Hom_X\left(F_0(x), H_0(x)\right)
\end{align*}
$$

The left whiskering $\alpha L : FL \to GL : W \to Y$ is defined, in context $w : W$, as $(\alpha L)(w) := \alpha(L_0(w))$. The right whiskering $R\beta : R \circ G \to R \circ H : X \to Z$ is defined, in context $x : X$, as follows.

$$
\begin{align*}
(R\beta)(x) := & \quad \Hom_Y\left(G_0(x), H_0(x)\right) \\
& \quad \downarrow_{R_1\left(G_0(x), H_0(x)\right)} \\
& \quad \Hom_Z\left(R_0G_0(x), R_0H_0(x)\right)
\end{align*}
$$

The identity $\mathbf{V}$-natural transformation $\text{id}_F : F \to F : X \to Y$ is defined as $\text{id}_F := \text{id}_Y F_0$. It is just an exercise in the internal language to prove that vertical composition and identity of internal enriched natural transformation strictly satisfy associativity and unitarity. Thus, a pair of $\mathbf{V}$-enriched categories yield a category of functors and natural transformations, as stated in the following proposition.

**Proposition 7** Given $\mathbf{V}$-enriched categories $X$ and $Y$, the $\mathbf{V}$-enriched functors $X \to Y$ and natural transformations between them form a category $\mathbf{V}\text{Cat}_{\mathcal{E}}(X, Y)$.

Moreover, horizontal and vertical composition of $\mathbf{V}$-enriched natural transformations strictly satisfy the interchange laws, thus yielding an enrichment in $\text{Cat}$. Equivalently, internal enriched categories, functors and natural transformations form a 2-category, as stated in the following result, whose proof is again an exercise in the internal language.

**Proposition 8** $\mathbf{V}$-enriched categories, functors, and natural transformations form a strict 2-category $\mathbf{V}\text{Cat}_{\mathcal{E}}$.

By abuse of notation, we call $\mathbf{V}\text{Cat}_{\mathcal{E}}$ both the category of $\mathbf{V}$-enriched categories and their functors, and the 2-category of $\mathbf{V}$-enriched categories, their functors and their natural transformations. As a consequence, given two $\mathbf{V}$-enriched categories $X$ and $Y$, we will denote by $\mathbf{V}\text{Cat}_{\mathcal{E}}(X, Y)$ both the hom-set of $\mathbf{V}$-enriched functors $X \to Y$ and the hom-category of $\mathbf{V}$-enriched functors $X \to Y$ and their natural transformations. Context will usually suffice to determine in which sense the notation is being used.

**Remark 7** Let $X$ be a $\mathbf{V}$-enriched category. There is an underlying $\mathcal{E}$-category $U(X)$, such that $U(X)_0 := X$ and $U(X)_1$ is the subobject of $X \times X \times V_1$ defined by the formula

$$(x_0, x_1, f) : X \times X \times V_1 \vdash f : I_{\mathbf{V}} \to \Hom_X(x_0, x_1)$$
(which can be explicitly defined via an equalizer) with the first and second projections as source and target. The composition is defined, in context \((x_1, x_2, g), (x_0, x_1, f) : U(X)_1 \times X U(X)_1\), as follows.

\[
(x_1, x_2, g) \circ_{U(X)} (x_0, x_1, f) := (x_0, x_2, \circ_X (x_0, x_1, x_2) \circ_{V} (g \otimes_{V} f))
\]

Let \(F : X \rightarrow Y\) be a \(V\)-enriched functor. There is an underlying functor \(U(F) : U(X) \rightarrow U(Y)\) in \(\mathcal{E}\), with \(U(F)_0\) defined as \(F_0\) and \(U(F)_1(x_0, x_1, f)\), in context \((x_0, x_1, f) : U(X)_1\), as the tuple

\[
\left( F_0(x_0), F_0(x_1), \exists_{V} \xrightarrow{f} \text{Hom}_X(x_0, x_1) \xrightarrow{F_1(x_0,x_1)} \text{Hom}_X(F_0(x_0), F_0(x_1)) \right)
\]

in \(U(Y)_1\).

Let \(\alpha : F \rightarrow G : X \rightarrow Y\) be a \(V\)-enriched natural transformation. There is an underlying natural transformation \(U(\alpha) : U(F) \rightarrow U(G) : U(X) \rightarrow U(Y)\) in \(\mathcal{E}\), defined, in context \(x : U(X)_0\) as \(U(\alpha)(x) := (F_0(x), G_0(x), \alpha(x))\).

Those data yield the underlying-category-in-\(\mathcal{E}\) 2-functor \(U : \mathcal{V} \text{Cat}_{\mathcal{E}} \rightarrow \text{Cat}_{\mathcal{E}}\).

We now consider the issue of the change of base. In this context, though, there are two sensible such notions, one coming from internal category theory and one from enriched category theory. Indeed, we can change both the ambient category and the enriching category. To begin, let’s state the internal version of the standard result changing the enriching category.

**Proposition 9** Let \(V'\) be another monoidal category in \(\mathcal{E}\), and \(F : \mathcal{V} \rightarrow \mathcal{V}'\) a monoidal functor. Then there is an induced 2-functor \(F_* : \mathcal{V} \text{Cat}_\mathcal{E} \rightarrow \mathcal{V}' \text{Cat}_\mathcal{E}\).

**Proof** Let \(X\) be a \(V\)-category. Define a \(V'\)-category \(F_* (X)\) on \(X\) given by the following data.

- Internal hom: \(\text{Hom}_{F_* (X)} := X \times X \xrightarrow{\text{Hom}_X} V_0 \xrightarrow{F_0} V_0'\).
- Composition: \(\circ_{F_* (X)} := X \times X \times X \xrightarrow{\circ_X} V_1 \xrightarrow{F_1} V_1'\).
- Identity: \(\text{id}_{F_* (X)} := X \xrightarrow{\text{Hom}_X} V_1 \xrightarrow{F_1} V_1'\).

Let \(G : X \rightarrow Y\) be a \(V\)-functor. Define a \(V'\)-functor \(F_* (G) : F_* (X) \rightarrow F_* (Y)\), with the same object component as \(G\) and arrow component given by \((F_* (G))_1 := F_1 G_1\).

Let \(\alpha : G \rightarrow G' : X \rightarrow Y\) be a \(V\)-natural transformation. Define a \(V'\)-natural transformation \(F_* (\alpha) : F_* (G) \rightarrow F_* (G') : F_* (X) \rightarrow F_* (Y)\) as \(F_* (\alpha) := F_1 \alpha\).

The axioms for the above definitions hold because of the functoriality of \(F\). \(\square\)

Finally, let’s check that changing the ambient category induces a 2-functorial operation on internal enriched categories, just as it does on internal categories (see Proposition 2).

**Proposition 10** Let \(\mathcal{E}'\) be another finitely complete category and \(F : \mathcal{E} \rightarrow \mathcal{E}'\) a functor preserving finite limits. By Proposition 3, there is an induced monoidal category \(F_* (\mathcal{V})\) in \(\mathcal{E}'\). Then \(F\) induces a 2-functor \(F_* : \mathcal{V} \text{Cat}_\mathcal{E} \rightarrow F_* (\mathcal{V}) \text{Cat}_{\mathcal{E}'}\).

**Proof** Let \(X\) be a \(V\)-category. Define a \(F_* (\mathcal{V})\)-category \(F_* (X)\) on \(F(X)\) by applying the functor \(F\) to the structural arrows \(\text{Hom}_X, \circ_X\) and \(\text{id}_X\) of \(X\). That gives a \(F_* (\mathcal{V})\)-enriched category because \(F\) preserves finite-limit logic, in terms of which internal enriched categories are defined. Analogously, define \(F_*\) on \(V\)-enriched functors and natural transformations. \(\square\)
7 Indexed Enriched Categories

In Sect. 4 we recalled the notion of indexed monoidal category. In [16], Shulman introduces a notion of enrichment over such a category which is a fibrational generalization of the standard enrichment. This comes in two versions: a general indexed version and a version which Shulman calls “small”. The latter is in a sense a hybrid notion, having an internal as well as an indexed aspect. We shall then compare both of them to internal enrichment, and find that they are closely related.

First, we give an outline of the notions of small \( \mathcal{W} \)-category, of functors between such categories, and of natural transformations between such functors. For brevity we will omit some diagrammatic axioms, referring to [16] for those. In this section, let \( \mathcal{W} \) be an \( \mathcal{E} \)-indexed monoidal category. Moreover, if \( f : B \to A \) is a morphism in \( \mathcal{E} \) and \( H \) is an object in \( \mathcal{W}(A) \), we shall write \( H(f) \) as a convenient notation for the object \( \mathcal{W}(f)(H) \) of \( \mathcal{W}(B) \).

Definition 16 (small \( \mathcal{W} \)-category) A small \( \mathcal{W} \)-category \( \mathbf{A} \) consists of the following data:

- An object \( A \) of \( \mathcal{E} \).
- An object \( \text{Hom}_A \) of \( \mathcal{W}(A \times A) \).
- A morphism \( \text{id}_A : I_{\mathcal{W}(A)} \to \text{Hom}_A(\Delta) \) where \( \Delta : A \to A \times A \) is the diagonal.
- A morphism of \( \mathcal{W}(A \times A \times A) \)

\[
\circ_A : \text{Hom}_A(\pi_2, \pi_3) \otimes \mathcal{W}(A \times A \times A) \text{Hom}_A(\pi_1, \pi_2) \to \text{Hom}_A(\pi_1, \pi_2)
\]

where \( \pi_1, \pi_2, \pi_3 : A \times A \times A \to A \) are projections.

Moreover, it has to satisfy the associativity and unitarity axioms from [16].

Definition 17 (functor of small \( \mathcal{W} \)-categories) A functor of small \( \mathcal{W} \)-categories \( F : \mathbf{A} \to \mathbf{B} \) consists of the following data:

- A morphism \( F_0 : A \to B \) of \( \mathcal{E} \).
- A morphism \( F_1 : \text{Hom}_A \to \text{Hom}_B(F_0, F_0) \) of \( \mathcal{W}(A \times A) \).

Moreover, it has to satisfy the functoriality axioms from [16].

Definition 18 (natural transformation of small \( \mathcal{W} \)-categories) A natural transformation of small \( \mathcal{W} \)-categories \( \alpha : F \to G : \mathbf{A} \to \mathbf{B} \) consists of a morphism

\[
\alpha : I_{\mathcal{W}(A)} \to \text{Hom}_B((F_0, G_0)\Delta)
\]
satisfying the naturality axiom from [16].

We shall denote with \( \mathcal{W} \text{SCat}_{\mathcal{E}} \) the (2-)category of small \( \mathcal{W} \)-categories and their functors (and the natural transformation between those).

Recall from Sect. 5 that the externalization of an internal monoidal category \( \mathbf{V} \) is a monoidal indexed category \([\mathbf{V}]\) over \( \mathcal{E} \). Thus, we can investigate the relationship between \( \mathbf{V} \)-enriched categories and small \([\mathbf{V}]\)-categories, and it turns out that they are the same thing in a very strict sense: their definitions coincide!

Proposition 11 To give a \( \mathbf{V} \)-enriched category (functor, natural transformation) is to give a small \([\mathbf{V}]\)-category (functor, natural transformation). Thus, the 2-categories \( \mathbf{V \text{Cat}_{\mathcal{E}}} \) and \([\mathbf{V}] \text{SCat}_{\mathcal{E}} \) are isomorphic.

Proof A small \([\mathbf{V}]\)-category \( \mathbf{X} \) is given by the following data:
An object $X$ of $\mathcal{E}$.

An object $\text{Hom}_X: X \times X \to V_0$ of $[V]^{X \times X}$.

A morphism of $[V]^{X \times X}$

$$(X \to x_\mathcal{E} \to V_0) \xrightarrow{(X \to X \times X \xrightarrow{id} \text{Hom}_X(\pi_2, \pi_3) \otimes \text{Hom}_X(\pi_1, \pi_2) \to V_0)} (X \times X \to V_0).$$

A morphism of $[V]^{X \times X}$

$$(X \times X \times X \xrightarrow{\text{Hom}_X(\pi_2, \pi_3) \otimes \text{Hom}_X(\pi_1, \pi_2) \to V_0)} (X \times X \to V_0)$$

over $X \times X \times X \xrightarrow{(\pi_1, \pi_3)} X \times X$, given by

$$X \times X \times X \xrightarrow{\cap_X} (\text{Hom}_X(\pi_2, \pi_3) \otimes \text{Hom}_X(\pi_1, \pi_2), \text{Hom}_X(\pi_1, \pi_3))^* V_1.$$

Moreover, such data have to satisfy associativity and unitarity axioms. But these are precisely the same data that yield an internal $V$-enriched category.

Analogously, to give a functor or a natural transformation of small $[V]$-categories is to give a functor or a natural transformation of internal $V$-enriched categories. \qed

Now we present the notion of indexed category enriched in an indexed monoidal category [16]. For that, we shall extend a notation that we have consistently used in the internal context to standard enriched categories: if $F: \mathcal{V} \to \mathcal{V}'$ is a lax monoidal functor and $\mathcal{A}$ is a $\mathcal{V}'$-enriched category, then $F_* (\mathcal{A})$ is the induced $\mathcal{V}'$-enriched category.

**Definition 19** (indexed $\mathcal{V}$-category) An indexed $\mathcal{V}$-category $\mathcal{B}$ consists of the following data:

- For each $X$ object of $\mathcal{E}$, a $\mathcal{V}(X)$-category $\mathcal{B}^X$.
- For each $f: X \to Y$ in $\mathcal{E}$, a fully faithful $\mathcal{V}(X)$-functor $f^*: (f^*)_*(\mathcal{B}^Y) \to \mathcal{B}^X$.
- For each $f: X \to Y$ and $g: Y \to Z$ in $\mathcal{E}$, a $\mathcal{V}(X)$-natural isomorphism $(gf)^* \cong f^* (f^*)_*(g^*)$ (where we implicitly identify $(f^*)_*(g^*)^* \equiv B^Z$ in the domains of these functors).
- For each $X$ object of $\mathcal{E}$, a $\mathcal{V}(X)$-natural isomorphism $(\text{id}_X)^* \cong \text{id}_{\mathcal{B}^X}$.

Moreover, for every $f: X \to Y$, $g: Y \to Z$ and $h: Z \to K$ in $\mathcal{E}$, it has to satisfy the axioms for associativity and unitarity, analogous to those for ordinary indexed categories, by making the following diagrams of isomorphisms commute:

$$
\begin{array}{c}
(hg)^* \Rightarrow f^* (f^*)_* ((hg)^*) \\
\downarrow \\
(gf)^* \circ ((gf)^*)_*(h^*) \\
\downarrow \\
f^* \circ (f^*)_*(g^*) \circ ((gf)^*)_*(h^*) \\
\downarrow \\
f^* \circ (f^*)_*(g^*) \circ (f^*)_* ((id_Y)^*) \\
\downarrow \\
(f \text{id}_X)^* \circ ((id_X)^*)_*(f^*) \\
\downarrow \\
f^* (id_Y f)^* \\
\downarrow \\
f^*
\end{array}
$$
Definition 20 (functor of indexed \( \mathcal{W} \)-categories) An indexed \( \mathcal{W} \)-functor \( \mathcal{T} : \mathcal{B} \to \mathcal{B}' \) consists, for every object \( X \) of \( \mathcal{E} \), of a \( \mathcal{W}(X) \)-enriched functor \( \mathcal{T}^X : \mathcal{B}^X \to \mathcal{B}'^X \) together with, for every \( f : X \to Y \), an isomorphism \( \mathcal{T}^X \circ f^* \cong f^* \circ (f^*)_*(\mathcal{T}^Y) \). Such data have to satisfy the functoriality axioms by making the following diagrams of isomorphisms commute, for every \( f : X \to Y \) and \( g : Y \to Z \) in \( \mathcal{E} \).

\[
\begin{array}{ccc}
\mathcal{T}^I \circ (gf)^* & \xrightarrow{\sim} & \mathcal{T}^X \circ f^* \circ (f^*)_*(g^*) \\
\downarrow & & \downarrow \\
(gf)^* \circ ((gf)^*)_*(\mathcal{T}^Z) & \xrightarrow{\sim} & f^* \circ (f^*)_*(\mathcal{T}^Y \circ g^*) \\
\downarrow & & \downarrow \\
f^* \circ (f^*)_*(g^*) \circ ((g^*)_*(\mathcal{T}^Z)) & \xrightarrow{\sim} & f^* \circ (f^*)_*(\mathcal{T}^Y) \\
\end{array}
\]

\[
\mathcal{T}^X \circ (\text{id}_X)^* \xrightarrow{\sim} (\text{id}_X)^* \circ ((\text{id}_X)^*)_*(\mathcal{T}^X)
\]

Definition 21 (natural transformation of indexed \( \mathcal{W} \)-categories) An indexed \( \mathcal{W} \)-natural transformation \( \alpha : \mathcal{T} \to \mathcal{T}' : \mathcal{B} \to \mathcal{B}' \) consists, for every object \( X \) of \( \mathcal{E} \), of a \( \mathcal{W}(X) \)-natural transformation \( \alpha^X : \mathcal{T}^X \to \mathcal{T}'^X : \mathcal{B}^X \to \mathcal{B}'^X \), satisfying naturality axioms by making the following diagram commute, for every \( f : X \to Y \).

\[
\begin{array}{ccc}
\mathcal{T}^X \circ f^* & \xrightarrow{\sim} & f^* \circ (f^*)_*(\mathcal{T}^Y) \\
\downarrow & & \downarrow \\
\mathcal{T}^X \circ f^* & \xrightarrow{\sim} & f^* \circ (f^*)_*(\mathcal{T}^Y)
\end{array}
\]

With the data thus defined (plus the obvious notions of compositions and identities) we can define a 2-category of indexed enriched categories.

Definition 22 (category of indexed \( \mathcal{W} \)-categories) We denote with \( \mathcal{W} \text{ICat}_\mathcal{E} \) the 2-category of indexed \( \mathcal{W} \)-categories, their functors and the natural transformations between them.

By abuse of notation, we shall denote with \( \mathcal{W} \text{ICat}_\mathcal{E} \) also the mere 1-category of indexed \( \mathcal{W} \)-categories and their functors. Usually, the context is sufficient to distinguish when the notation is being used referring to the 1-category or the 2-category.

In general, \( \mathcal{W} \text{SCat}_\mathcal{E} \) embeds into \( \mathcal{W} \text{ICat}_\mathcal{E} \) as a full sub-2-category (even though this fact is omitted from [16] and, according to the author’s knowledge, not addressed in the literature at large). Of course, since \( \mathcal{V} \text{SCat}_\mathcal{E} \) and \( \mathcal{V} \text{Cat}_\mathcal{E} \) are isomorphic by Proposition 11, \( \mathcal{V} \text{Cat}_\mathcal{E} \) is a full sub-2-category of \( \mathcal{V} \text{ICat}_\mathcal{E} \) as well. We will now describe concretely the 2-embedding \([\cdot] : \mathcal{V} \text{Cat}_\mathcal{E} \hookrightarrow \mathcal{V} \text{ICat}_\mathcal{E} \).

First, let \( \mathcal{X} \) be a \( \mathcal{V} \)-enriched category and let us define an indexed \( \mathcal{V} \)-category \( [\mathcal{X}] \). Given an indexing object \( I \) of \( \mathcal{E} \), define the \( \mathcal{V}^I \)-enriched category \( [\mathcal{X}]^I \) as follows:

- Objects: \( I \)-indexed families \( x : I \to X \) of elements of \( X \).
- Internal hom: \( \text{Hom}_{[\mathcal{X}]^I}(x_0, x_1 : I \to X) : = I \times X \times X \xrightarrow{\text{Hom}_X} V_0 \).
- Composition: $\circ_{[X]^I}(x_0, x_1, x_2) := I \xrightarrow{(x_0, x_1, x_2)} X \times X \times X \xrightarrow{\alpha_X} V_1$.
- Identity: $\text{id}_{[X]^I}(x) := I \xrightarrow{x} X \xrightarrow{\text{id}_X} V_1$.

Let $f : I \to J$ be a re-indexing. Define the $[V]^I$-functor $f^* : \langle f^* \rangle : ([X]^I) \to [X]^I$ as follows.

$$f_0^*(J \xrightarrow{\alpha} X) := I \xrightarrow{f} J \xrightarrow{\alpha} X$$

$$f_1^*(J \xrightarrow{\alpha} X, J \xrightarrow{\lambda} Y) := \text{Hom}_{\langle f^* \rangle_\alpha ([X]^I)}(x_0, x_1) \xrightarrow{\text{id}} \text{Hom}_{[X]^I}(x_0 f, x_1 f)$$

Since $f_1^*(x_0, x_1)$ is the identity of $\text{Hom}_X(x_0 f, x_1 f)$ as an object of $[V]^I$, then $f^*$ is full and faithful, as required by the definition. The rest of the structure is given by canonical isomorphisms verifying the axioms.

Secondly, let $F : [X] \to [Y]$ be a $V$-enriched functor and let us define an indexed $[V]$-functor $[F]^I : [X]^I \to [Y]^I$ as follows:

- Objects component: $[F]^I(I \xrightarrow{x} X) := I \xrightarrow{x} X \xrightarrow{F_0} Y$.
- Morphisms component: $[F]^I(I \xrightarrow{x} X, I \xrightarrow{\lambda} Y) := I \xrightarrow{(x_0, x_1)} X \times X \xrightarrow{F_1} V_1$.

Notice that, for any reindexing $f : I \to J$, we have an equality $[F]^I \circ f^* = f^* \circ ([F]^I)$, meaning that the axioms for indexed $[V]$-functors are automatically satisfied.

Finally, let $\alpha : F \to G : [X] \to [Y]$ be a $V$-enriched natural transformation and let us define an indexed $[V]$-natural transformation $[\alpha]^I : [F]^I \to [G]^I : [X]^I \to [Y]^I$ as $[\alpha]^I : [F]^I \to [G]^I$ as $[\alpha]^I(x : I \to X) := \alpha x$. The naturality condition for indexed $[V]$-natural transformations is trivially satisfied because the defining isomorphisms of indexed $[V]$-functors $[F]$ and $[G]$ are identities.

**Remark 8** Indexed $[V]$-categories don’t canonically induce internal $V$-enriched categories. In particular, the categories $[X]^I$ are small, as their object of objects is the homset $\mathcal{E}(I, X)$, but that is not generally the case for indexed $[V]$-categories.

To conclude, we look at the interplay between externalizations and the underlying categories (see Remark 7 for the definition of underlying internal category of an internal enriched category).

**Proposition 12** Let $X$ be a $V$-enriched category. Then, there is a natural isomorphism of indexed categories $U([X]^I) \cong [U(X)]^I$ between the underlying indexed category of the indexed $[V]$-category $[X]$ and the externalization of the underlying $\mathcal{E}$-category $U(X)$.

**Proof** Let $I$ be an indexing object. We need to prove that there is an isomorphism $U([X]^I) \cong [U(X)]^I$ between the underlying standard category of the $[V]^I$-enriched category $[X]^I$ and the fiber over $I$ of the externalization of the underlying $\mathcal{E}$-category $U(X)$. Moreover, for any reindexing $f : J \to I$ in $\mathcal{E}$, the square

$$U([X]^I) \xrightarrow{\cong} [U(X)]^I$$

$$\downarrow\quad U(f^*([X])) \quad \downarrow f^*([U(X)])$$

$$U([X]^J) \xrightarrow{\cong} [U(X)]^J$$

has to commute.
For both categories, the objects are $I$-indexed families of objects of $X$, and the arrows $(I \xrightarrow{x_0} X) \rightarrow (I \xrightarrow{x_1} X)$ are the sections of the projection

$$(\mathcal{V}_X, \text{Hom}_X(x_0, x_1))^*V_1 \rightarrow I,$$

so that the categories are clearly isomorphic to each other, and the square commutes trivially.

\[\square\]

The previous result can be extended to the following proposition.

**Proposition 13** The following diagram of 2-functors commutes.

\[
\begin{array}{ccc}
\mathbf{V} \text{Cat}_{\mathcal{E}} & \xrightarrow{U} & \text{Cat}_{\mathcal{E}} \\
[-1] & \xleftarrow{[-1]} & \xrightarrow{[-1]} \\
[V] \text{ICat}_{\mathcal{E}} & \xrightarrow{U} & \text{ICat}_{\mathcal{E}}
\end{array}
\]

The above discussion suggests that the indexed $[V]$-category $[X]$ should yield a notion of externalization of an internal enriched category $X$. That would be defined as the large $[V]$-category obtained by $[X]$ via the functor $\Theta$ [16, Section 6], analogously to how one gets the total category of an indexed category via the Grothendieck construction (Theorem 1).

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**References**

1. Borceux, F.: Handbook of Categorical Algebra, vol. 2. Cambridge University Press, Cambridge (1994)
2. Freyd, P.: Aspects of topoi. Bull. Aust. Math. Soc. 7(1), 1–76 (1972)
3. Ghiorzi, Enrico: Internal enriched categories. PhD thesis, University of Cambridge (2019)
4. Ghiorzi, E.: Complete internal categories (2020)
5. Girard, J.-Y.: Interprétation fonctionnelle et élimination des coupures de l’arithmétique d’ordre supérieur. PhD thesis, Université Paris VII (1972)
6. Hasegawa, R.: Relational limits in general polymorphism. Publicat. Res. Inst. Math. Sci. 30(4), 535–576 (1994)
7. Hyland, J.M.E.: The effective topos. In Toelstra, A.S., van Dalen, D., editors. The L. E. J. Brouwer Centenary Symposium. Vol. 110, pp. 165–216, Amsterdam. North-Holland (1982)
8. Hyland, J.M.E.: A small complete category. Ann. Pure Appl. Logic 40(2), 135–165 (1988)
9. Hyland, J.M.E., Robinson, E.P., Rosolini, G.: The discrete objects in the effective topos. Proc. Lond. Math. Soc. 3(1), 1–36 (1990)
10. Johnstone, P.T.: Sketches of an elephant: A topos theory compendium: Volumes 1 and 2. Number 43 in Oxford Logic Guides. Oxford Science Publications (2002)
11. Lane, S.M.: Categories for the working mathematician. Number 7 in Graduate Texts in Mathematics, 2nd edn. Springer (1989)
12. Palmgren, E., Vickers, S.J.: Partial horn logic and cartesian categories. Ann. Pure Appl. Logic 145(3), 314–353 (2007)
13. Pitts, A.M.: Polymorphism is set theoretic, constructively. In: Category Theory and Computer Science, pp. 12–39. Springer, New York (1987)
14. Reynolds, J.C.: Polymorphism is not set-theoretic. In: International Symposium on Semantics of Data Types, pp. 145–156. Springer, New York (1984)
15. Shulman, M.: Framed bicategories and monoidal fibrations. Theory Appl. Categ. 20(18), 650–738 (2008)
16. Shulman, M.: Enriched indexed categories. Theory Appl. Categ. 28(21), 616–695 (2013)

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