ψ-Pascal and \( q_\psi \)-Pascal matrices - an accessible factory of one source identities and resulting applications

Andrzej K. Kwaśniewski

Higher School of Mathematics and Applied Informatics
Kamienna 17, PL-15-021 Białystok, Poland

Summary
Recently the author proposed two extensions of Pascal and \( q \)-Pascal matrices defined here also - in the spirit of the Ward “Calculus of sequences” [1] promoted in the framework of the \( \psi \)- Finite Operator Calculus [2,3]. Specifications to \( q \)-calculus case and Fibonomial calculus case are made explicit as an example of abundance of new possibilities being opened. In broader context the \( \psi \)-Pascal \( P_\psi [x] \) and \( q_\psi \)-Pascal \( P_{q_\psi} [x] \) matrices appear to be as natural as standard Pascal matrix \( P[x] \) already is known to be [4]. Among others these are a one source factory of streams of identities and indicated resulting applications.

1 I. On the usage of references

The papers of main reference are: [1-3]. We shall take here notation from [2,3] (see below) and the results from [1] as well as from [2,3] - for granted. For other respective references see: [2,3]. The acquaintance with “The matrices of Pascal and other greats” [4] is desirable. Further relevant references of the present author are: [5] on extended finite operator calculus of Rota and quantum groups and other [6-7]. The reference to \( q \)-Pascal matrix is [8] Further Pascal matrix references for further readings are [9-14]. One may track down there among others relations: the Pascal Matrix versus Classical Polynomials. The book [15] is recommended and the recent reference [16] is useful for further applications. Very recent \( \psi \)-Pascal matrix reference is [17] and also recent further Pascal matrices references ( far more not complete list of them ) are to be found in [18-21]. The book of Kassel Christian [22] - makes an intriguing link to the advanced world of related mathematics.

Before to proceed we anyhow explain -for the reader convenience - some of the very basic of the intuitively useful \( \psi \)-notation promoted by the author [2,3,5,6]. Here \( \psi \) denotes an extension of \( \langle \frac{1}{n} \rangle_{n \geq 0} \) sequence to quite arbitrary one (the so called - admissible) and the specific choices are for example: Fibonomialy-extended \( \langle \frac{1}{F_n} \rangle_{n \geq 0} \) (here \( \langle F_n \rangle \) denotes the Fibonacci sequence ) or Gauss \( q \)-extended \( \langle \frac{1}{F_n(q)} \rangle_{n \geq 0} \) admissible sequences of extended umbral operator calculus or just ”the usual” \( \langle \frac{1}{n} \rangle_{n \geq 0} \) common choice. We get used to write these \( q \) - Gauss and other extensions in mnemonic convenient upside down notation [2,3,5,6]

\[
\begin{align*}
(1) \quad & \psi_n \equiv n_\psi, x_\psi \equiv \psi(x) \equiv \psi_x, n_\psi! = n_\psi(n - 1)_\psi!, n > 0, \\
(2) \quad & x_\psi^k = x_\psi(x - 1)_\psi(x - 2)_\psi \ldots (x - k + 1)_\psi \\
(3) \quad & x_\psi(x - 1)_\psi \ldots (x - k + 1)_\psi = \psi(x)\psi(x - 1)\ldots \psi(x - k - 1).
\end{align*}
\]

The corresponding \( \psi \)-binomial symbol and \( \partial_\psi \) difference linear operator on \( F[[x]] \) (F - any field of zero characteristics) are below defined accordingly where following Roman [3,3,5,6] we shall call \( \psi = \{ \psi_n(q) \}_{n \geq 0}; \psi_n(q) \neq 0; n \geq 0 \) and \( \psi_0(q) = 1 \) an admissible sequence.
Definition 1  The $\psi$-binomial symbol is defined as follows:

$$(\binom{n}{k})_\psi = \frac{n!_\psi}{k!_\psi (n-k)!_\psi} = \binom{n}{n-k}_\psi$$

Definition 2  Let $\psi$ be admissible. Let $\partial_\psi$ be the linear operator lowering degree of polynomials by one defined according to $\partial_\psi x^n = n_\psi x^{n-1}; n \geq 0$. Then $\partial_\psi$ is called the $\psi$-derivative.

You may consult [2,3,5,6] and references therein for further development and use of this notation “$q$-commuting variables” - included.

2 II. Towards $\psi$-Pascal matrix factory of identities

Let us define analogously to [4,9,10] define the $\psi$-Pascal matrix as

$$P_\psi[x] = \exp_\psi \{xK_\psi\}$$

where ($Z_n$ denotes the additive cyclic group)

$$K_\psi = (j+1)_\psi \delta_{i,j+1}$$

therefore

$$P_\psi[x] = \binom{x}{i-j}_\psi$$

due to: $\partial_\psi P_\psi[x] = K_\psi P_\psi[x]$ where $\psi P_\psi[x]|_{x=0} = K_\psi$.

Explicitly (see [8] for $q$-case) $K_\psi$ matrix is of the form

$$K_\psi = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1_\psi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2_\psi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 3_\psi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 4_\psi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 5_\psi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 6_\psi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 7_\psi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 8_\psi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9_\psi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10_\psi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 11_\psi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (n-1)_\psi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \end{bmatrix}$$

Fig.1. The $K_\psi$ matrix

Naturally $K_\psi^n = 0; K_\psi^k \neq 0$ for $0 \leq k \leq (n-1)$. Hence we have

$$P_\psi[x] = \exp_\psi \{xK_\psi\} = \sum_{k \in Z_n} \frac{x^k K_\psi^k}{k!_\psi}$$
the result $P_\psi[x]$ of $\psi$-exponentiation above being shown on the Fig.2.

$$
P_\psi[x] = 
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x^1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
x^2 & 2_\psi x & 1 & 0 & 0 & 0 & 0 & 0 \\
x^3 & 3_\psi x^2 & 3_\psi x & 1 & 0 & 0 & 0 & 0 \\
x^4 & 4_\psi x^3 & 6_\psi x^2 & 4_\psi x & 1 & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
x^{n-1} & 0 & 0 & 0 & 0 & 0 & (n-1)_\psi x & 1
\end{bmatrix}
$$

**Fig. 2. The $P_\psi[x]$ matrix**

Immediately we see that the $\psi$-Pascal matrix $P_\psi[x] = \exp_\psi \{x K_\psi\}$ is also the source of many important identities. Here below there are the examples correspondent to those from [4] which are accordingly inferred from the $\psi$-additivity property (non-group property in general):

$$P_\psi[x]P_\psi[y] = P_\psi[x + \psi y].$$

**Warning:** for not normal sequences : see: [1,2,3,5,6,8] - the one parameter family $\{P_\psi[x]\}_{x \in F}$ is *not a group*! since for not normal sequences $\{1 - \psi 1\}^{2k} \neq 0$ thought $[x + \psi (-x)]^{2k+1} = 0$.

In general we are dealing with abelian semigroup with identity which becomes the group only for normal sequences. And so coming back to identities we have for example:

$$\sum_{j \leq k \leq i} \binom{i}{k}_\psi \binom{k}{j}_\psi = (1 + \psi 1)^{i-j} \binom{i}{j}_\psi, i \geq j \iff P_\psi[1]P_\psi[1] = P_\psi[1 + \psi 1].$$

$$\sum_{j \leq k \leq i} (-1)^k \binom{i}{k}_\psi \binom{k}{j}_\psi = (1 - \psi 1)^{i-j} \binom{i}{j}_\psi, i \geq j \iff P_\psi[1]P_\psi[-1] = P_\psi[1 - \psi 1].$$

The above identities after the choice $\psi = \binom{1}{n}_\psi$ coincide with the corresponding ones from [4]. There are much more examples of this nature.

We shall now try also to find out a kind of $\psi$-extended version of the $q$-identity (6)

$$\sum_{0 \leq k \leq i} \binom{i}{k}_q^2 = \binom{2i}{i}_q \iff P_\psi[1]P_\psi^T[1] = F_q[1].$$

where we have defined the $q$-Fermat matrix as follows

$$F_q[1] = \binom{i + j}{i}_q, i,j \in \mathbb{Z}_n.$$  

For $q=1$ case- name Fermat - see [15] for this Fermat called Pascal symmetric Matrix for $q=1$ see: [4,9]. For $q$-binomial - see below in **Important**.

In order to find out a kind of $\psi$-extended version of the Pascal-Fermat $q$-identity identity (6) we shall proceed as in [16]. There the Cauchy $\hat{q}_\psi$- identity and $\hat{q}_\psi$-Fermat
matrix were introduced due to the use of the $\hat{q}_\psi$-muting variables from Extended Finite Operator Calculus [3,5]. The linear $\hat{q}_\psi$-mutator operator was defined in [3,5,16] as follows for $F$ - field of characteristic zero and $F[x]$ - the linear space of polynomials.

$$\hat{q}_\psi : F[x] \rightarrow F[x]; \quad \hat{q}_\psi x^n = \frac{(n + 1)q^n - 1}{nq^n} x^n; \quad n \geq 0.$$

**Important.** With the Gaussian choice of admissible sequence [3,5] $\psi = \{\psi_n(q)\}_{n \geq 0}, \psi_n(q) = [n_q!]^{-1}, n_q = \frac{1 - q^n}{1 - q}, n_q! = n_q(n - 1)_q!, 1_q! = 1, \hat{q}_\psi x^n = q^n x^n$

and the $\hat{q}_\psi$-Pascal and $\hat{q}_\psi$-Fermat matrices from [16] (see next section) coincide with $q$-Pascal and $q$-Fermat matrices correspondingly **which is not the case** for the general case - for example Fibonomial $F$-Pascal matrix is different from $\hat{q}_\psi$-Pascal matrix - see next section.

In [16] in analogy to the standard case [9,10,4] the matrices with operator valued matrix elements

$$x^{i-j} \binom{i}{j}_{\hat{q}_\psi} = \binom{i + j}{j}_{\hat{q}_\psi}, \quad i, j \in \mathbb{Z}_n$$

were named the $\hat{q}_\psi$-Pascal $P[x]$ and $\hat{q}_\psi$-Fermat $F[1]$ matrices - correspondingly i.e.

$$P_{\hat{q}_\psi}[x] = \left( x^{i-j} \binom{i}{j}_{\hat{q}_\psi} \right)_{i, j \in \mathbb{Z}_n}$$

The $\hat{q}_\psi$-P[1] Pascal and $\hat{q}_\psi$-F[1] Fermat matrices from [16] are related via the following identity for operator valued matrix elements

$$\sum_{k \geq 0} \hat{q}_\psi^{(r-k)(j-k)} \binom{i}{k}_{\hat{q}_\psi} \binom{j}{k}_{\hat{q}_\psi} = \binom{i + j}{j}_{\hat{q}_\psi}.$$

The relation (8) is the one being looked for to extend the Pascal-Fermat $q$-identity (6). Here - following [16]- we use the new $\hat{q}_\psi$-Gaussian symbol with operator valued matrix elements.

**Definition 3** We define $\hat{q}_\psi$-binomial symbol i.e. $\hat{q}_\psi$-Gaussian coefficients as follows:

$$\binom{n}{k}_{\hat{q}_\psi} = \frac{n_{\hat{q}_\psi}!}{k_{\hat{q}_\psi}!(n-k)_{\hat{q}_\psi}!} = \binom{n}{n-k}_{\hat{q}_\psi} \quad \text{where} \quad n_{\hat{q}_\psi}! = n_{\hat{q}_\psi}(n-1)_{\hat{q}_\psi}!, 1_{\hat{q}_\psi}! = 1$$

and $n_{\hat{q}_\psi} = 1 - q^n_{\hat{q}_\psi}$ for $n > 0$.

3 III. Specifications : $q$-umbral and umbral Fibonomial cases

**III-q** $q$-umbral calculus case [1,2,3,5-8]

Let us make the $q$-Gaussian choice [2,3,5,6,8] of the admissible sequence $\psi = \{\psi_n(q)\}_{n \geq 0}$. Then the $\psi$-Pascal matrix becomes the $q$-Pascal matrix from [8] and we arrive mnemonic at the corresponding to $q = 1$ case numerous $q$-identities and
other "$q$-applications". Specifically in the $q$-case we have (see: Proposition 4.2.3 in [22])

\[
\sum_{k \geq 0} q^{(r-k)(j-k)} \binom{r}{k}_q \binom{s}{j-k}_q = \binom{r+s}{j}_q
\]

hence from this Cauchy $q$-identity we obtain the following easy to find out formula for the symmetric Pascal (or Fermat) matrix elements:

\[
\sum_{k \geq 0} q^{(r-k)(j-k)} \binom{i}{k}_q \binom{j}{k}_q = \binom{i+j}{j}_q.
\]

Naturally we are dealing now with not normal sequences i.e. not with a one parameter $q$-Pascal group [8] since for $(1-q)2^k \neq 0$ though $[x+q(-x)]^{2k+1} = 0$ ; see: [1] and then [2,3,5,6,8]. If $q$-Pascal matrix $P_q[1] = exp_q\{xK_q\}|_{x=1}$ is considered also for $q \in GF(q)$ field then $q = p_m$ where $p$ is prime and $\binom{n}{k}_q$ becomes the number of $k$-dimensional subspaces in $n-th$ dimensional space over Galois field $GF(q)$. Also $q$ real and $-1 < q < +1$ cases are exploited in vast literature on the so-called $q$-umbral calculus (for Cigler, Roman and Others see: [3,23] and references therein- links to thousands in [23]). It is not difficult to notice that the $\hat{q}_\psi$-Pascal and $\hat{q}_\psi$-Fermat matrices under the $q$-Gaussian choice of the admissible sequence $\psi$ - coincide with $q$-Pascal and $q$-Fermat matrices correspondingly which is meaningful magnificent exception and which is not the case in general.

III-F FFOC-umbral calculus case [6-7]

In straightforward analogy to the $q$-case above consider now the Fibonomial coefficients (see: FFOC = Fibonomial Finite Operator Calculus Example 2.1 in [6]) where $F_n$ denote the Fibonacci numbers and $\psi_n(q) = [F_n]^{-1}$. 

\[
\binom{n}{k}_F = \frac{F_n!}{F_k!F_{n-k}!} \equiv \frac{n_kF}{k_F!}, \quad n_F \equiv F_n \neq 0,
\]

where we make an analogy driven [6,5,3,2] identifications ($n > 0$):

\[
n_F! \equiv n_F(n-1)_F(n-2)_F(n-3)_F \ldots 2_F 1_F; \quad 0_F! = 1; \quad n_F^k = n_F(n-1)_F \ldots (n-k+1)_F.
\]

**Information** In [7] a partial ordered set was defined in such a way that the Fibonomial coefficients count the number of specific finite "birth-self-similar" sub-posets of this infinite non-tree poset naturally related to the Fibonacci tree of rabbits growth process.

The $\psi$-Pascal matrix becomes then the $F$-Pascal matrix and we arrive at the corresponding $F$-identities (mnemonic replacement of $\psi$ by $F$) and other "$F$-applications" - hoped to be explored soon.

Naturally we are now dealing with **not normal** sequences : see: [1,2,3,5,6,8] - i.e. we have no $F$-Pascal group since for $(1-F1)^{2k} \neq 0$ though $[x+F(-x)]^{2k+1} = 0$. For example: $(x+Fy)^2 = x^2 + F_2xy + y^2, (x+Fy)^3 = x^3 + F_4x^3y + F_4F_3x^2y^2 + F_4xy^3 + y^4$.

Here in the Fibonomial choice case the semi-group generating matrix matrix $K_F$ is of the form
The $K_{\hat{q}}$-Pascal $P_{K_{\hat{q}}}[x]$ and $K_{\hat{q}_{\psi}}$-Fermat matrix do not coincide with $F$-Pascal and $F$-Fermat matrices correspondingly as indicated earlier though in our friendly mnemonic notation they look so much alike. Namely, the corresponding $K_{\hat{q}_{\psi}}$ matrix with the Fibonomial choice $\psi_n(q) = [F_n]^{-1}$ is now of the form
The perspective of numerous applications are opened. Apart from being the natural one source factory of identities we indicate in explicit also the origins of the $\hat{q}_F$-Pascal and $\hat{q}_F$-Fermat matrices factory of mnemonic attainable identities (compare via [16] with [9-14,18-21,4]). From operator identities involving the $\hat{q}_F$-Pascal $P_{\hat{q}_F}[x]$ and $K_{\hat{q}_F}$-Fermat matrix we obtain identities in terms of objects on which the $\hat{q}_F$ (or $\hat{q}_F$ from [3,5,6,16]) act and these are polynomials from $F[x]$ or in more general setting [6,5,3] from formal series algebra $F[[x]]$ where $F$ denotes any field of zero characteristics. In order to get such countless realizations of operator identities in terms of formal series it is enough to act by both sides of a given operator identity on the same element from $F[[x]]$.

4 IV. Remark on perspectives

The perspective of numerous applications are opened. Apart from being the natural one source factory of identities $\psi$-Pascal $P_{\psi}[x]$ and $\hat{q}_F$-Pascal $P_{\hat{q}_F}[x]$ and $K_{\hat{q}_F}$-Fermat...
matrices as well appear to be the similar way natural objects and tools as the Pascal matrix $P[x]$ is in the already mentioned and other applications - (see [4,18]- for example). Just to indicate few more of them: the considerations and results of [4] concerned with Bernoulli polynomials might be extended to the case of $\psi$-basic Bernoulli-Ward polynomials introduced in [1] and investigated recently in [17] in the framework of the $\psi$- Finite Operator Calculus [2,3,5-7] due to the use of the $\psi$- integration proposed in [2,6] . The same applies equally well to the case of $\psi$-basic Hermite-Ward polynomials and other examples of $\psi$-basic generalized Appell polynomials [3,2,5-6] which - being of course $\psi$- Sheffer are characterized equivalently by the familiar $\psi$-Sheffer identity [3,2]

\[ A_n(x + \psi y) = \sum_{k \geq 0} \binom{n}{k} \psi A_k(y)x^{n-k}. \]  \hspace{1cm} (11)

For further possibilities - see references [8-14,18-21] and many other ones not known for the moment to the present author.

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