A note on non-linear electrodynamics, regular black holes and the entropy function.

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Abstract

We examine four dimensional magnetically charged extremal black holes in certain non-linear $U(1)$ gauge theories coupled to two derivative gravity. For a given coupling, one can tune the magnetic charge (or vice versa) so that the curvature singularity at the centre of the space-time is cancelled. Since these solutions have a horizon but no singularity, they have been called regular black holes. Contrary to recent claims in the literature, we find that the entropy function formalism reproduces the near horizon geometry and gives the correct entropy for these objects.

1 Introduction

The Penrose cosmic censorship hypothesis states that, if singularities predicted by General Relativity occur in nature, they must be dressed by event horizons [1]. Behind the veil of an event horizon, there is no causal contact from the interior to the exterior of a black hole, so the pathologies occurring at the singular region can have no influence on an external observer. However, the converse of the hypothesis is apparently not true — a horizon does not necessarily hide a singularity. Solutions with a horizon but no singularity have been called regular black holes.

The holographic principle, [2, 3], states that the number of degrees of freedom describing the black hole is bounded by the area of the horizon. A stronger statement is that degrees of freedom living on the horizon can describe the physics of the interior completely. While the holographic principle is essentially a proposed feature of quantum gravity, one might wonder whether having a classically regular or singular solution has any quantitative or qualitative effect on the entropy and the physics at the horizon.

Earlier work on regular black hole models can be found in [4–8] These regular solutions are referred to as Bardeen black holes [9]. In addition, regular black hole solutions to Einstein equations with various physical sources were reported in [10] and [11]. Among known regular black hole solutions, are the solutions to the coupled equations of nonlinear electrodynamics and general relativity found by Ayón-Beato and García [12] and by Bronnikov [13]. The latter describes magnetically charged black hole, and provides an interesting example of the system that could be both regular and extremal. In this note we are specially interested in the near horizon geometry of an extremal magnetically charged black hole non-linearly coupled to a $U(1)$ gauge field. For a given magnetic charge, one can tune the non-linear coupling so that the solution is regular.

In [14], Matyjasek found the near horizon, $AdS_2 \times S^2$ geometry of a particular magnetically charged extremal black hole. The entropy function formalism of Sen [15, 16], is particularity useful for discussing

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the entropy of extremal black holes especially when non-linear or high derivative terms make a full analysis
difficult. Since, the formalism is equivalent to solving Einsteins equations for the near horizon region, a priori,
and assuming the near horizon geometry decouples, one would expect to be able to reproduce the results of
[14] using Sen’s approach. This issue has been studied recently, [17, 18], and authors reported that, even at
the level of two derivative gravity, the entropy function approach does not lead to the correct Bekenstein-
Hawking. To account for this discrepancy, they claim that the entropy function approach is sensitive to
whether the nature of the central region of the black hole is regular (linear) or singular (nonlinear).

Contrary to the claims of, [17, 18], in this note we find that a straight forward application of the entropy
function formalism reproduces the results of [14]. The equation of motion derived from extremizing the
entropy function are exactly the same as equation of motion at horizon found by extremizing the action,
since the entropy function (up to Legendre transformation) is the Lagrangian at the horizon. The fact that
the entropy is the value of entropy function at its extremum is derived from the Wald entropy formula,
using the near horizon symmetries. Both of these results, just coming from careful consideration of the near
horizon symmetries and have nothing to do with the regularity of the solution inside the horizon. Further
more, we find that by varying the non-linear coupling, the regular solution can be smoothly connected to
the extremal Reisner-Nordstrom solution of Einstein-Maxwell theory.

The paper is organised as follows. In section 2 we review a particular regular black hole solution of
interest. Then, in section 3, we review the entropy function formalism and apply it to Einstein gravity
coupled to non-linear electrodynamics. In section 4 we consider the special case of the formalism applied
to a regular black hole solution. Finally we end with the conclusion in section 5, having relegated some
technical details about the large charge, small coupling expansion of the entropy to appendix A.

2 Regular Black holes

In this section we review a magnetically charged regular black hole solution of Einstein gravity coupled to
non-linear electrodynamics and its extremal limit [12–14, 19], mainly following [14] with slightly different
notation.

We consider an action given by,

\[ S = \frac{1}{16\pi} \int d^4x \sqrt{-g} (R - \mathcal{L}_F (F^2)) , \]

(2.1)

where \( F^2 = F_{\mu\nu} F^{\mu\nu} \) and the non-linear \( U(1) \) gauge field Lagrangian, \( \mathcal{L}_F \), is,\(^1\)

\[ \mathcal{L}_F = F^2 \cosh^{-2} \left( \left( \lambda^2 F^2 / 2 \right)^{1/4} \right) . \]

(2.2)

The equations of motion corresponding to the metric and gauge field and the Bianchi identity are,

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{\partial \mathcal{L}_F}{\partial (F^2)} 2 F_{\mu\lambda} F^\lambda_{\nu} - \frac{1}{2} \mathcal{L}_F g_{\mu\nu} , \]

(2.3)

\[ \partial_{\mu} \left( \sqrt{-g} \frac{\partial \mathcal{L}_F}{\partial (F^2)} F^{\mu\nu} \right) = 0 , \]

(2.4)

\[ \partial_{[\mu} F_{\alpha\beta]} = 0 . \]

(2.5)

For a magnetically charged black hole, the equation of motion for the gauge field and the Bianchi identity
can be solved by,

\[ F_{\theta\phi} = P \sin \theta , \]

(2.6)

where \( P \) is the magnetic charge of the black hole. A static, spherically symmetric ansatz for the metric:

\[ ds^2 = -a^2(r) dt^2 + \frac{dr^2}{a^2(r)} + r^2 d\Omega_2^2 , \]

(2.7)

can solve Einstein equations with,

\[ a^2(r) = 1 - \frac{2m(r)}{r} , \]

(2.8)

\(^1\)The coupling \( a = \sqrt{\lambda} \) is commonly used in the literature. With out loss of generality, we can take \( \lambda > 0. \)
where,

\[ m(r) = m_\infty - \frac{|P|}{2|\lambda/P|^{1/2}} \tanh \frac{|\lambda/P|^{1/2}}{r/|P|}. \]  

(2.9)

The parameter, \( m_\infty \), is an integration constant which can be fixed by employing the boundary condition \( m(\infty) = M \), where \( M \) is the black hole mass. Moreover demanding of the regularity of the line element as \( r \to 0 \), yields,

\[ M = \frac{|P|}{2|\lambda/P|^{1/2}}, \]  

(2.10)

and consequently, \( m(r) \) reads,

\[ m(r) = M \left( 1 - \tanh \frac{P^2}{2Mr} \right) = \frac{|P|}{2|\lambda/P|^{1/2}} \left( 1 - \tanh \frac{|\lambda/P|^{1/2}}{r/|P|} \right). \]  

(2.11)

The location of the inner and outer horizons, \( r_\pm \), which are given by equation \( a(r) = 0 \), can be expressed in terms of the real branches of the Lambert function, \( W_i(x) \), as follows,

\[ r_+ = -\frac{p^2}{W_0(-pe^{p^2/4}p^2/4) - p^2/4}, \quad r_- = -\frac{p^2}{W_{-1}(-pe^{p^2/4}p^2/4) - p^2/4}, \]  

(2.12)

where, \( p = P/M \), is the magnetic charge-to-mass ratio. The Lambert function\(^2\), is defined by the formula,

\[ e^{W(x)}W(x) = x. \]  

(2.13)

This function has two real branches, called \( W_0 \) and \( W_{-1} \), with the branch point at \( x = -1/e \). Since the value of the principal branch of the Lambert function, \( W_0 \), at \( 1/e \), plays an important role in our discussion, we define \( w_0 = W_0(1/e) \).

When \( p = p_{\text{ext}} = 2w_0^{1/2} \), \( r_+ = r_- \), and the two horizons merge into a degenerate horizon giving an extremal solution. Since we will be considering the near horizon geometry, we will eliminate the mass from our formulae as it is defined asymptotically. Using (2.10) we can express the condition for extremality and regularity, \( p_{\text{ext}} = 2w_0^{1/2} \), as,

\[ \frac{\lambda}{P} = w_0. \]  

(2.14)

In other words for an extremal black hole to be regular we must tune the charge to coupling ratio to a particular value. A generic extremal, but not necessarily regular, solution to (2.1) will still have a degenerate horizon but presumably with a different charge to coupling ratio.

One can write the near horizon limit, found by [14] as:

\[ ds^2 = v_1 \left( -\rho^2 dt^2 + \frac{d\rho^2}{\rho^2} \right) + v_2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right). \]  

(2.15)

with,

\[ v_2 = \frac{4w_0}{(1 + w_0)^2} P^2 \approx 0.68P^2; \]  

(2.16)

\[ v_1 = \frac{8w_0}{(1 + w_0)^2} P^2 \approx 1.07P^2; \]  

(2.17)

\[ \frac{v_2}{v_1} = \frac{1}{2}(1 + w_0) \approx 0.64; \]  

(2.18)

and the Bekenstein-Hawking entropy is,

\[ S_{BH} = \frac{1}{4} A = \pi v_2 = \frac{4\pi w_0}{(1 + w_0)^2} P^2. \]  

(2.19)

\(^2\)See [20] for a nice review of the properties of the Lambert function.
3 Entropy function Analysis

In this section we briefly review the entropy function formalism of Sen [15, 16] and subsequently apply it to magnetically charged extremal solutions of (2.1).

Assuming a gauge and diffeomorphism invariant Lagrangian and a near horizon $AdS_2 \times S^2$ geometry, the entropy function is defined as the Legendre transform, with respect to the electric charges, of the reduced Lagrangian evaluated at the horizon:

$$\mathcal{E}(\vec{u}, \vec{v}, \vec{Q}, \vec{P}) = 2\pi \left( e^i Q_i - f(\vec{u}, \vec{v}, \vec{e}, \vec{P}) \right) = 2\pi \left( e^i Q_i - \int_H d\theta d\phi \sqrt{-G} \mathcal{L} \right),$$  \tag{3.1}

where $e^i$ are the electric fields, $Q_i = \partial f / \partial e^i$, are the electric charges conjugate to the electric field, $\vec{u}$ are the values of the scalar moduli at the horizon, and $v_1$, $v_2$ are the sizes of the $AdS_2$ and $S^2$, respectively. The near horizon equations of motion for a black hole carrying electric charges $\vec{Q}$ and magnetic charges $\vec{P}$, are equivalent to the extremisation of $\mathcal{E}$ with respect to $\vec{u}, \vec{v}$ and $\vec{e}$:

$$\frac{\partial \mathcal{E}}{\partial \vec{u}} = 0, \quad \frac{\partial \mathcal{E}}{\partial v_1} = 0, \quad \frac{\partial \mathcal{E}}{\partial \vec{e}} = 0. \tag{3.2}$$

Furthermore, the Wald entropy associated with the black hole is given by $\mathcal{E}$ at the extremum (3.2). If $\mathcal{E}$ has no flat directions, then the extremization of $\mathcal{E}$ determines $\vec{u}$, $v_1$, and $\vec{e}$, in terms of $Q$ and $P$. The extremal value of the Wald entropy, $S = \mathcal{E}(\vec{Q}, \vec{P})|_{\text{extr}}$, is independent of the asymptotic values of the scalar fields. This neatly demonstrates the attractor mechanism, [21–23], with out requiring supersymmetry [24]. The formalism can even be extended to rotating black holes which have less near horizon symmetry [25]. However, since it only involves the near horizon geometry, a weakness of the formalism is that one implicitly assumes that the full solution exists, which is not always the case [26].

We now specialise our discussion to the case of interest. Since the regular black hole solution has an extremal limit, one can use the entropy function formalism to find the near horizon geometry and the entropy. We take the near horizon $AdS_2 \times S^2$ metric to be given by (2.15). From the definition (3.1), using the Lagrangian (2.1), the entropy function evaluates to,

$$\mathcal{E} = \pi \left( v_2 - v_1 + \frac{1}{2} v_1 v_2 \mathcal{L}_F(2P^2/v_2^2) \right). \tag{3.3}$$

By extremizing this entropy function with respect to $v_1$ and $v_2$, we find following equations,

$$0 = -1 + \frac{1}{2} v_2 \mathcal{L}_F(2P^2/v_2^2), \tag{3.4}$$

$$0 = 1 + \frac{1}{2} v_1 \frac{\partial}{\partial v_2} \left[ v_2 \mathcal{L}_F(2P^2/v_2^2) \right]. \tag{3.5}$$

Substituting (3.4) into (3.3) gives,

$$\mathcal{E} = \pi v_2 = \frac{1}{4} A, \tag{3.6}$$

which is just the Bekenstein-Hawking entropy. This result, which is independent of the form of $\mathcal{L}_F$, is to be expected, since, in the absence of higher derivative terms, the Bekenstein-Hawking and Wald entropies coincide.

Now, the equations of motion allow us to determine $v_1$ and $v_2$ in terms of $P$ and the coupling $\lambda$. The first equation, (3.4), determines $v_2$, and consequently the entropy, in terms of $P$ (and $\lambda$). Having found $v_2$, (3.5) allows us to determine $v_1$ in terms of $v_2$. Consequently, we see that extremising the entropy function completely determines the entropy and near horizon geometry in terms of $P$ (and $\lambda$).

We now consider explicitly finding $v_1$ and $v_2$ for a particular Lagrangian. Using the Lagrangian (2.2), (3.4) and (3.5) become,

$$0 = -1 + (P^2/v_2) \cosh^{-2}(\sqrt{\lambda P/v_2}), \tag{3.7}$$

$$0 = 1 - v_1 (P/v_2)^2 \cosh^{-2}(\sqrt{\lambda P/v_2}) + v_1 \sqrt{\lambda} (P/v_2)^{5/2} \cosh^{-3}(\sqrt{\lambda P/v_2}) \sinh(\sqrt{\lambda P/v_2}), \tag{3.8}$$
which agrees with the near horizon equations of motion found directly in [14].

To solve (3.7), it is convenient to rewrite it as,

\[ \cosh \xi = \gamma \xi, \]  

(3.9)

where,

\[ \xi = \sqrt{\frac{\lambda P}{v^2}}, \quad \gamma = \left(\frac{\lambda}{P}\right)^{-1/2}. \]  

(3.10)

One can then graphically solve (3.7) by finding the intersection of \( \cosh \xi \) and \( \gamma \xi \) for various values of \( \gamma \). We illustrate this procedure in figure 1. It is not hard to see that as we increase \( \gamma \), there are either zero, one or two solutions to (3.9). One can also see from figure 1, that, as \( \lambda/P \to 0 \) (i.e. \( \gamma \to \infty \)), the two possible values for \( \xi \) are,

\[ \xi_{\lambda/P \to 0} \to \left\{ \begin{array}{c} \infty \\ 0 \end{array} \right. \]  

(3.11)

Notice that, since \( \cosh x \geq 1 \), (3.7) also implies,

\[ v^2 \leq P^2. \]  

(3.12)

![Figure 1: This figure illustrates the graphical solution of (3.9) which is given by the intersection of \( \cosh \xi \) and \( \gamma \xi \). As we increase the gradient, \( \gamma = \sqrt{P/\lambda} \), one obtains either no solutions, a tangential point (denoted by a red square above) or two solutions. The first point of a double intersection, labelled with a brown dot, corresponds to a point on what we call, for reasons that will be clear later, the large branch and the second intersection, labelled with a green triangle, is on the small branch.]

Now, we can define a (multi-valued) function, \( F(x) \), by,

\[ \frac{F(x)}{\cosh F(x)} = \sqrt{x}, \]  

(3.13)

so that we can formally write down a solution to (3.9) as,

\[ \xi = F(\gamma^{-2}) = F(\lambda/P). \]  

(3.14)

Then letting,

\[ G(x) = \frac{x}{F^2(x)} \]

and using (3.10), we can write,

\[ v^2 = \left( \frac{\lambda}{\xi^2} \right) P^2 = G \left( \frac{\lambda}{P} \right) P^2. \]  

(3.16)
which is of the generic form expected by dimensional analysis. Since (3.9) may have two solutions, \( F \) and \( G \) both have two branches. Substituting (3.11) into (3.15) we find that, in the limit that the non-linear coupling goes to zero (or the charge becomes very large),

\[
G(0) = 1/\cosh^2(F(0)) = \begin{cases} 0 & \text{(small branch, } G_S) \\ 1 & \text{(large branch, } G_L) \end{cases}.
\]

We call the two branches of \( G \), the small and large branch. While it seems challenging to find an analytical expression for \( G \), it is very easy to evaluate it numerically. We have plotted \( v_2/\sqrt{P^2} \), or in other words \( G \), as a function of \( \lambda/P \) in figure 2. We note that \( G \) decreases monotonically on the large branch, so that, for a fixed charge, the \( \lambda = 0 \) solution is the most entropic.

![Figure 2](attachment:image.png)

**Figure 2:** This figure shows \( v_2/\sqrt{P^2} \) as a function of \( \lambda/P \) found by numerically solving (3.9). Specifically we plot, \( v_2/\sqrt{P^2} = \xi^{-2}\gamma^{-2} = G(\lambda/P) \). The large branch, \( G_L \), is plotted in brown and the small branch, \( G_S \), is plotted in green. We should exclude the shaded region, defined by \( \gamma^2\xi^2 - \gamma^2 < 1 \), in which, using (3.19), \( v_1 \) is negative. Since it is entirely contained within the shaded region, the small branch is unphysical. The regular black hole, denoted by a blue dot, is found on the big branch at \( \lambda/P = w_0 \). At the place where the branches meet, denoted by a red square, \( v_1 \to \infty \) (or \( -\infty \) if we approach from below).

Having determined \( v_2 \) (at least in principle), we can find \( v_1 \) by substituting (3.7) into (3.8) and using (3.16), we get,

\[
v_1 = v_2(1 - \gamma^{-1}\sqrt{\xi^2\gamma^2 - 1})^{-1},
\]

(3.18)

with (3.12) ensuring reality.

For \( v_1 \) to be positive and finite, on sees that from (3.18), we require \( \xi^2\gamma^2 - 1 > \gamma^2 \). Now, at the branch point, the function \( f(\xi) = \cosh \xi - \gamma \xi \) has a single zero, so we require that \( f'(\xi) = 0 \) when \( f(\xi) = 0 \). In other words, in addition to (3.9) the branch point is determined by,

\[
\sinh \xi = \gamma.
\]

(3.20)

Combining (3.9)and (3.20) gives,

\[
\gamma^2\xi^2 - 1 = \gamma^2,
\]

(3.21)

which, using (3.18), implies that as we approach the branch point, \( v_1 \to \infty \) or in other words the \( AdS_2 \) approaches flat space. From figure 2, we see that the small branch lies entirely in the region \( \gamma^2\xi^2 - \gamma^2 < 1 \), and consequently, \( v_1 \) is always negative on it, making it unphysical. Discarding the small branch, we have used (3.19) to plot \( v_2/v_1 \) as a function of \( \lambda/P \) in figure 3. We see that \( v_1/v_2 \) increases monotonically, eventually diverging at the branch point.
Finally, one can actually obtain a large charge, small coupling expansion for $G_L$. Assuming $G_L$, has a nice Taylor expansion about zero, using (3.17) as a starting point, and by taking successive derivatives of (3.13) and (3.15) one can recursively expand $G_L(x)$ about zero. As discussed in appendix A, we find that,

$$G_L(x) = 1 - x - \frac{1}{2} x^2 + O(x^3),$$

so that we can write a large charge, small coupling expansion for the entropy,

$$\mathcal{E} = \pi P^2 \left( 1 - \frac{\lambda}{P} - \frac{1}{3} \frac{\lambda^2}{P^2} + O\left(\frac{\lambda^3}{P^3}\right) \right).$$

For completeness, we mention that on the small branch, as discussed in appendix A, for $x$ small, we get,

$$G_S(x) \approx x \left[W_{-1}(-\frac{1}{2}\sqrt{x})\right]^{-2}.$$  \hspace{1cm} (3.24)

where $W_{-1}$ is the non-principal real branch of the Lambert function.

As a check we note that on the large branch, taking $\lambda \rightarrow 0$, we recover the usual near horizon extremal Einstein-Maxwell Reisner-Nordstrom solution with,

$$v_1 = v_2 = P^2.$$  \hspace{1cm} (3.25)

4 Entropy function and the regular black hole

In this section we confirm that the entropy function analysis of the regular black hole reproduces the near horizon geometry of the known solution found in [14]. This merely entails considering the results of the previous section with the appropriate value of $\lambda/P$.

As discussed in section 2, the regular black hole corresponds to the point $\lambda/P = w_0$, so that (3.9) becomes,

$$\cosh \xi = w_0^{-1/2} \xi.$$ \hspace{1cm} (4.1)

One can analytically check, using the property $w_0 e^{w_0} = e^{-1}$, that (4.1) has a solution,

$$\xi = \frac{w_0 + 1}{2},$$ \hspace{1cm} (4.2)

or in other words $F(w_0) = \frac{1}{2}(w_0 + 1)$. We have plotted the position of the regular solution as a blue dot in figures 2 and 3, from which we observe that it is on the large branch.

Finally, one can check that substituting the solution, (4.2), into (3.16) and (3.18) reproduces (2.16) and we are done.
Conclusion

In this paper we examine entropy function formalism for regular magnetically charged black hole solution in Einstein-Hilbert gravity coupled with a certain non-linear $U(1)$ gauge field. The mass and charge of the full solution can be tuned so that it has no curvature singularity at the centre. In the extremal limit this corresponds to a particular charge to non-linear coupling ratio with an $AdS_2 \times S^2$ near horizon geometry. Unsurprisingly we find that the entropy function analysis match with the exact solution found by solving the full Einstein equations. This is in contrast with the claim in the recent papers [17, 18].

Indeed in the entropy function formalism, the equation of motion, which follow from extremizing the entropy function, are exactly the same as equation of motion at horizon found by extremizing the action, simply because the entropy function (up to Legendre transformation) is the Lagrangian at the horizon. The fact that the entropy is the value of entropy function at its extremum is derived from Wald entropy formula, using the near horizon symmetries. Both of these results apparently have nothing to do with the regularity of the solution inside the horizon.

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A Large charge/small coupling expansion of the entropy

In this appendix we discuss the expansion of $G(x)$ about zero.

Taking the derivative of (3.13) with respect to $x$ and solving for $F'$ we find,

$$F'(x) = \frac{\cosh(F(x))}{2\sqrt{F(x)}(\sqrt{F(x)} - 1)} ,$$

(A.1)

while taking of derivative of (3.15) gives,

$$G'(x) = -2\text{sech}^2(F(x)) \tanh(F(x)) F'(x) .$$

(A.2)

Now using (3.13) and (A.1) we can rewrite (A.2) as,

$$G'(x) = \frac{\tanh(F)}{F(F \tanh(F) - 1)} ,$$

(A.3)

and using (3.11) and taking the limit $x \to 0$, we get,

$$G'(0) = \begin{cases} 0 & \text{(small branch)} \\ -1 & \text{(large branch)} \end{cases} .$$

(A.4)

Taking another set of derivative, after some algebra we obtain,

$$G''(x) = \frac{F \cosh(2F) - \cosh(F) \sinh(F)}{2F^3(F \tanh(F) - 1)^3} ,$$

(A.5)

and once again taking the $x \to 0$ limit, we get,

$$G''(0) = \begin{cases} \infty & \text{(small branch)} \\ -\frac{2}{3} & \text{(large branch)} \end{cases} .$$

(A.6)

We found that the second order Taylor expansion of $G_l(x)$, (3.22), agrees well with our numerical plot for $x$ small. For example, at $x = w_0$, we find that they differ by about 2%.

On the other hand, we see that the small branch does not have a nice Taylor expansion about the origin. However, on the small branch, we see from figure 1, that when $\xi$ is large, $\gamma$ is also large and consequently, $\lambda/P = \gamma^{-2}$, is small. For $\xi \gg 1$, we can approximate (3.9) by,

$$\frac{1}{2}e^\xi \approx \gamma\xi + O(e^{-\xi}) ,$$

(A.7)
which can approximately be solved by,
\[ \xi \approx -W\left(-\frac{1}{2}\gamma^{-1}\right). \] (A.8)

where \( W \) is the Lambert function defined in (2.13). For, \(-e^{-1} < x < 0\), the two real branches of \( W \) satisfy \( W_0(x) \geq -1 \) and \( W_{-1}(x) \leq -1 \), [20], consequently since we are assuming that \( \xi \gg 1 \), we should take the branch \( W_{-1} \). So, using (3.14,3.15), we obtain,
\[ G_S(\lambda/P) \approx (\lambda/P)\left[W_{-1}\left(-\frac{1}{2}\sqrt{\lambda/P}\right)\right]^{-2}, \] (A.9)

which we found agrees well with our numerical results shown in figure 2, for \( \lambda/P \) small.

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