Survival functions versus conditional aggregation-based survival functions on discrete space

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Abstract
In this paper we deal with conditional aggregation-based survival functions recently introduced by Boczek et al. (2020). The concept is worth to study because of its possible implementation in real-life situations and mathematical theory as well. The aim of this paper is the comparison of this new notion with the standard survival function. We state sufficient and necessary conditions under which the generalized and the standard survival function equal. The main result is the characterization of the family of conditional aggregation operators (on discrete space) for which these functions coincide.

Keywords: aggregation, survival function, nonadditive measure, visualization, size

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1. Introduction

We continue to study the novel survival functions introduced in [1] as a generalization of size-based level measure developed for the use in nonadditive analysis in [3, 12, 13]. The concept appeared initially in time-frequency analysis [8]. As the main result, in Theorem 4.7 we show that the generalized survival function is equal to the original notion (for any monotone measure and any input vector) just in very particular case. The concept of the novel survival function is useful in many real-life situations and pure theory as well. In fact, the standard survival function (also known in the literature as the standard level measure [13], strict level measure [5] or decumulative distribution function [10]) is the crucial ingredient of many definitions in mathematical analysis. Many well-known integrals are based on the survival function, e.g. the Choquet integral, the Sugeno integral, the Shilkret integral, the seminormed integral [5], universal integrals [14], etc. Also, the convergence of a sequence of functions in measure is based on the same concept. Hence a reasonable generalization of the survival function leads to the generalizations of all mentioned concepts. For more on applications of the generalized survival function, see [1, 8].
Due to the number of factors needed in the definition of the generalized survival function, it is quite difficult to understand this concept. In order to understand it more deeply, in the following we shall focus on the graphical visualization of inputs, see [4]. Moreover, the graphical representation will help us to formulate basic results of this paper. In the whole paper, we restrict ourselves to discrete settings. We consider finite basic set

\[ [n] := \{1, 2, \ldots, n\}, \ n \geq 1 \]

and a monotone measure \( \mu \) on \( 2^{[n]} \). If \( \mathbf{x} = (x_1, \ldots, x_n) \) is a nonnegative real-valued function on \( [n] \), i.e., a vector, then the survival function (or standard survival function) of the vector \( \mathbf{x} \) with respect to \( \mu \), see [1, 9], is defined by

\[
\mu(\{ \mathbf{x} > \alpha \}) := \mu(\{ i \in [n] : x_i \leq \alpha \}), \ \alpha \in [0, \infty).
\]

For the thorough exposition see Preliminaries. To avoid too abstract setting in the following visual representations, let us consider the input vector \( \mathbf{x} = (2, 3, 4) \) and the monotone measure \( \mu \) on \( 2^{[3]} \) defined in Table 1.

**The survival functions visual representation.** We begin with a nonstandard representation of standard survival function, as a stepping stone to its generalization. Before, let us introduce the following equivalent representation of survival function:

\[
\mu(\{ \mathbf{x} > \alpha \}) = \mu([n] \setminus \{ i \in [n] : x_i \leq \alpha \}) = \min \left\{ \mu(E^c) : (\forall i \in E) \ x_i \leq \alpha, \ E \in 2^{[n]} \right\} \\
= \min \left\{ \mu(E^c) : \max_{i \in E} x_i \leq \alpha, \ E \in 2^{[n]} \right\}, \quad (1)
\]

where \( E^c = [n] \setminus E \), see motivation problem 1 in [1]. Let us start the visualization with inputs from Table 1. Let us depict all maximal values of \( \mathbf{x} \) on \( E \), for each set \( E \in 2^{[3]} \) on the lower axis, see left image of Figure 1 in decreasing order and the corresponding values of monotone measure of complement, i.e. \( \mu(E^c) \), on the upper axis. In this picture of Figure 1 the number on lower axis is linked with the number on the upper one via a straight line once they correspond to the same set, i.e., a is linked with b if there is \( E \in 2^{[3]} \) such that

| \( E \) | \{1, 2, 3\} | \{2, 3\} | \{1, 3\} | \{1, 2\} | \{3\} | \{2\} | \{1\} | \emptyset |
|---|---|---|---|---|---|---|---|
| \( E^c \) | \emptyset | \{1\} | \{2\} | \{3\} | \{1, 2\} | \{1, 3\} | \{2, 3\} | \{1, 2, 3\} |
| \( \mu(E^c) \) | 0 | 0.25 | 0.25 | 0.4 | 0.75 | 0.75 | 0.75 | 1 |
| \( \max_{i \in E} x_i \) | 4 | 4 | 4 | 3 | 4 | 3 | 2 | 0 |
| \( \sum_{i \in E} x_i \) | 9 | 7 | 6 | 5 | 4 | 3 | 2 | 0 |

Table 1: Sample measure \( \mu \) and two conditional aggregation operators for vector \( \mathbf{x} = (2, 3, 4) \)
Finally, the value $\mu(\{x > \alpha\})$ at some $\alpha \in [0, \infty)$ can be read from the left image of Figure 1 considering the minimal value on the upper axis which is linked to a value smaller than $\alpha$ (i.e., right-hand side value) on the lower one. Thus considering an illustrative example in the left image of Figure 1, the value of survival function at 2.5 is 0.75. Indeed, there are just 2 values on the right hand side of 2.5, namely numbers 2 and 0. These are linked to 0.75 and 1, respectively. Hence, 0.75 is a smaller one. The graph of survival function is in the right image of Figure 1.

The generalized survival functions visual representation. In the modification of the survival function, the previously described computational procedure stays. However, we allow to use any conditional aggregation operator, not just maximum operator. The standard example of conditional aggregation is the sum of components of $x$, see the last line in Table 1 and the corresponding visualisation in Figure 2. Applying the described computational procedure we obtain the sum-based survival function of vector $x$, i.e., the generalized survival function of vector $x$ studied in [1, 3, 12, 13]. The formula linked to this procedure is the following:

$$\mu_{\text{sum}}(x, \alpha) = \min \{ \mu(E^c) : A^{\text{sum}}(x|E) \leq \alpha, E \in E' \}$$

with $A^{\text{sum}}(x|E) = \sum_{i \in E} x_i$ and $\{\emptyset\} \subseteq E' \subseteq 2^n$ (in the illustrative example $E' = 2^3$).

The corresponding graph is: Considering discrete space, the computation of the generalized survival function studied in [1, 8, 12, 13] may be always represented via the corresponding diagrams similar to those in Figures 1 and 2.

Except for a better understanding of survival functions, the visual representation may help us to answer the problem of their indistinguishability. With the introduction of novel survival function a natural question arises: When does the generalized survival function coincide with the survival function? The motivation for answering these questions is not only to know the relationship between mentioned concepts for given inputs, but it will help us to compare the corresponding integrals based on them, see [1, Definition 5.1, Definition 5.4]. In the literature, there are known some families of conditional aggregation operators.
Figure 2: Generalized survival function visualization for \( x = (2, 3, 4), \mu \text{ given in Table 1 and } A = A^{\text{sum}} \)

together with the collection \( \mathcal{E} \) when the generalized survival function equals to the survival function. In the following we list them:

- (cf. [13, Corollary 4.15]) \( \mathcal{A} = \mathcal{A}^{\text{size}} \) with size \( s \) being the weighted sum\(^2\) \( \mathcal{E} \) contains all singletons of \([n]\) and \( \mathcal{E}' = 2^{[n]} \);
- (cf. [1, Example 4.2] or [13, Section 5]) \( \mathcal{A} = \mathcal{A}^{\text{max}} \) with \( \mathcal{E} = 2^{[n]} \);
- (cf. [1, Proposition 4.6]) \( \mathcal{A} = \mathcal{A}^{\mu-\text{ess}} \) with \( \mathcal{E} = 2^{[n]} \).

Although the first two items appear to be different, in fact, under the above conditions, they are equal \( \mathcal{A}^{\text{size}} = \mathcal{A}^{\text{max}} \). Settings of above mentioned examples lead to the survival function regardless of the choice of monotone measure \( \mu \). However, the identity between generalized survival function and survival function may happen also for other families of conditional aggregation operators (a FCA for short), but with specific monotone measures, e.g. \( \mathcal{A}^{\text{sum}} \) with the weakest monotone measure\(^3\) shrinks to survival function for any input vector \( x \in [0, \infty)^{[n]} \) and \( \mathcal{E}' = 2^{[n]} \). In this paper we shall treat the following problems:

**Problem 1:** Let \( x \in [0, \infty)^{[n]} \), \( \mu \) be a monotone measure on \( 2^{[n]} \), and \( \mathcal{A} \) be FCA. What are sufficient and necessary conditions on \( x, \mu \) and \( \mathcal{A} \) to hold \( \mu_a(x, \alpha) = \mu_a(\{x > \alpha\}) \)?

**Problem 2:** Let \( x \in [0, \infty)^{[n]} \), \( \mathcal{A} \) be FCA. What are sufficient and necessary conditions on \( x \) and \( \mathcal{A} \) to hold \( \mu_a(x, \alpha) = \mu_a(\{x > \alpha\}) \) for any monotone measure \( \mu \)?

**Problem 3:** Let \( \mathcal{A} \) be FCA. What are sufficient and necessary conditions on \( \mathcal{A} \) to hold \( \mu_a(x, \alpha) = \mu_a(\{x > \alpha\}) \) for any monotone measure \( \mu \) and \( x \in [0, \infty)^{[n]} \)?

The paper is organized as follows. We continue with preliminary section containing needed definitions and notations. In Section 3 we solve Problem 1, see e.g. Corollary 3.7.

\(^2\)\( s_{\#p}(x)(E) = \left( \frac{1}{\#(E)} \cdot \sum_{x_i \in E} x_i^p \right)^{\frac{1}{p}} \) for \( E \neq \emptyset, s_{\#p}(x)(\emptyset) = 0 \) and \( p > 0 \).

\(^3\)\( \mu_* : 2^{[n]} \rightarrow [0, \infty) \) given by \( \mu_*(E) = \begin{cases} \mu([n]), & E = [n], \\ 0, & \text{otherwise}. \end{cases} \)
Corollary [3.11] Remark [3.12], Proposition [3.15] and Theorem [3.17]. In Section 4 we provide quite surprising result, see Theorem [4.7] that characterizes the family of conditional aggregation operators (in discrete setting) for which the generalized survival function coincides with the standard survival function. Thus we answer Problem 3. In Section 4 we also treat Problem 2, see Theorem [4.2] and Theorem [4.6]. Many our results are supported by appropriate examples.

2. Background and preliminaries

In order to be self-contained as far as possible, we recall in this section necessary definitions and all basic notations. In the whole paper, we restrict ourselves to discrete settings. As we have already mentioned, we shall consider a finite set

$$X = [n] := \{1, 2, \ldots, n\}, \ n \geq 1.$$  

We shall denote by $2^n$ the power set of $[n]$. A monotone or nonadditive measure on $2^n$ is a nondecreasing set function $\mu : 2^n \rightarrow [0, \infty)$, i.e., $\mu(E) \leq \mu(F)$ whenever $E \subseteq F$, with $\mu(\emptyset) = 0$. Moreover, we shall suppose $\mu([n]) > 0$. The set of monotone measures on $2^n$ we shall denote by $\mathcal{M}$. The monotone measure satisfying the equality $\mu([n]) = 1$ will be called the normalized monotone measure (also known as a capacity in [15]). In this paper we shall always work with monotone measures being defined on $2^n$, although, on several places the domain of $\mu$ can be smaller. Also, we shall need special properties of $\mu$ on a system $\mathcal{S} \subseteq 2^n$. The monotone measure $\mu \in \mathcal{M}$ with the property $\mu(E) \neq \mu(F)$ for any $E, F \in \mathcal{S} \subseteq 2^n$, $E \neq F$ will be called strictly monotone measure on $\mathcal{S}$. The counting measure will be denoted by $. Further, we put $\max \emptyset = 0$ and $\sum_{i \in \emptyset} x_i = 0$.

We shall work with nonnegative real-valued vectors, we shall use the notation $x = (x_1, \ldots, x_n)$, $x_i \in [0, \infty)$, $i = 1, 2, \ldots, n$. The set $[0, \infty)^n$ is the family of all nonnegative real-valued functions on $[n]$, i.e. vectors. For any $x = (x_1, \ldots, x_n) \in [0, \infty)^n$ we denote by $(\cdot)$ a permutation $(\cdot) : [n] \rightarrow [n]$ such that $x(1) \leq x(2) \leq \cdots \leq x(n)$ and $x(0) = 0$, $x(n+1) = \infty$ by convention. Let us remark that the permutation $(\cdot)$ need not be unique (this happens if there are some ties in the sample $(x_1, \ldots, x_n)$, see [7]). For a fixed input vector $x$ and a fixed permutation $(\cdot)$ we shall denote by $E_{(i)}$ the set of the form $E_{(i)} = \{(i), \ldots, (n)\}$ for any $i \in [n]$ with the convention $E_{(n+1)} = \emptyset$. By $1_E$ we shall denote the indicator function of a set $E \subseteq Y$, $Y \subseteq [0, \infty)$, i.e., $1_E(x) = 1$ if $x \in E$ and $1_E(x) = 0$ if $x \notin E$. Especially, $1_\emptyset(x) = 0$ for each $x \in Y$. We shall work with indicator function with respect to two different sets. We shall work with $Y = [n]$ when dealing with vectors (i.e. $1_E$ is a characteristic vector of $E \subseteq [n]$ in $\{0, 1\}^n$) and $Y = [0, \infty)$ when dealing with survival functions.

In the following we list several definitions (adopted to discrete settings). Firstly, the concept of the conditional aggregation operator is presented. Its crucial feature is that the validity of properties is required only on conditional set, not on the whole set. The inspiration for its introduction came from the conditional expectation, which is the fundamental notion of probability theory. Let us also remark that this operator generalizes the aggregation
operator introduced earlier by Calvo et al. in [6] Definition 1] and it is the crucial ingredient in the definition of the generalized survival function.

**Definition 2.1.** (cf. [1, Definition 3.1]) A map \( A(\cdot | B) : [0, \infty)^n \to [0, \infty) \) is said to be a *conditional aggregation operator* with respect to a set \( B \in 2^n \setminus \{\emptyset\} \) if it satisfies the following conditions:

i) \( A(x|B) \leq A(y|B) \) for any \( x, y \in [0, \infty)^n \) such that \( x_i \leq y_i \) for any \( i \in B \);

ii) \( A(1_B|B) = 0 \).

Let us compare the settings of the previous definition with the settings of the original definition, see [1, Definition 3.1]). We consider the greatest \( \sigma \)-algebra as the domain of \( \mu \) in comparison with the original arbitrary \( \sigma \)-algebra \( \Sigma \). Then all vectors are measurable and this assumption may be omitted from the definition. The measurability of each vector is desired property mainly from the application point of view. Because of the property \( A(x|B) = A(x1_B|B) \) for any \( x \in [0, \infty)^n \) with fixed \( B \in 2^n \setminus \{\emptyset\} \) the value \( A(x|B) \) can be interpreted as “an aggregated value of \( x \) on \( B \)”, see [1]. In the following we list several examples of conditional aggregation operators we shall use in this paper. For further examples and some properties of conditional aggregation operators we recommend [1, Section 3].

**Example 2.2.** Let \( x \in [0, \infty)^n \), \( B \in 2^n \setminus \{\emptyset\} \) and \( m \in M \).

i) \( A^{m-\text{ess}}(x|B) = \text{ess sup}_{m} (x1_B) \), where \( \text{ess sup}_{m}(x) = \min\{\alpha \geq 0 : \{x > \alpha\} \in \mathcal{M}_m\} \).

ii) \( A(x|B) = J(x1_B, m) \), (the multiplication of vectors is meant by components) where \( J \) is an integral defined in [2, Definition 2.2]. Namely,

\[ a) \quad A^{\text{Ch}m}(x|B) = \sum_{i=1}^{n} x_{(i)} \left( m(E_{(i)} \cap B) - m(E_{(i+1)} \cap B) \right); \]

\[ b) \quad A^{\text{Sh}m}(x|B) = \max_{i \in [n]} \left\{ x_{(i)} \cdot m(E_{(i)} \cap B) \right\}; \]

\[ c) \quad A^{\text{Su}m}(x|B) = \max_{i \in [n]} \left\{ \min\{x_{(i)}, m(E_{(i)} \cap B)\} \right\}. \]

iii) \( A(x|B) = \frac{\max_{i \in B}(x_{i} \cdot w_{i})}{\max_{i \in B} z_{i}} \), where \( w \in [0, 1]^n \) is a fixed weight vector, \( z \in (0, 1]^n \) is fixed vector such that \( \max_{i \in [n]} z_{i} = 1 \). We note, that for \( w = z = 1_{[n]} \) we get \( A^{\text{max}}(x|B) = \max_{i \in B} x_{i} \).

iv) \( A^{p-\text{mean}}(x|B) = \left( \frac{1}{\#(B)} \cdot \sum_{i \in B} (x_{i})^{p} \right)^{\frac{1}{p}} \) with \( p \in (0, \infty) \). For \( p = 1 \) we get the arithmetic mean.

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\(^4\) A set \( N \in 2^n \) is said to be a null set with respect to a monotone measure \( m \) if \( m(E \cup N) = m(E) \) for all \( E \in 2^n \). By \( \mathcal{N}_m \) we denote the family of null sets with respect to \( m \).
\textbf{v}) \( A_{\text{size}}(x|B) = \max_{D \in \mathcal{D}} s(x1_B)(D) \) with \( s \) being a size, see \[3, 12, 13\], is the outer essential supremum of \( x \) over \( B \) with respect to a size \( s \) and a collection \( \mathcal{D} \subseteq 2^{[n]} \). In particular, for the sum as a size, \( s \), \( s_{\text{sum}}(x)(G) = \sum_{i \in G} x_i \) for any \( G \in 2^{[n]} \) and for \( \mathcal{D} \) such that there is a set \( C \supseteq B \) for \( \mathcal{D} \) we get \( A_{\text{sum}}(x|B) = \sum_{i \in B} x_i \).

Observe that the empty set is not included in the Definition 2.1. The reason for that is the fact that the empty set does not provide any additional information for aggregation. However, in order to have the concept of the generalized survival function correctly introduced, it is necessary to add the assumption \( A(\cdot|\emptyset) = 0 \), see \[1, \text{Section 4}\]. From now on, we shall consider only these conditional aggregation operators. Let us remark, that all mappings from Example 2.2 with the convention “0/0 = 0” satisfy this property. In the following we shall provide the definition of the generalized survival function, see \[1, \text{Definition 4.1.}\]. Let us consider a collection \( \mathcal{E}, \{\emptyset\} \subseteq \mathcal{E} \subseteq 2^{[n]} \) and conditional aggregation operators on sets from \( \mathcal{E} \) with \( A(\cdot|\emptyset) = 0 \). The set of such aggregation operators we shall denote by

\[ \mathcal{A} = \{A(\cdot|E) : E \in \mathcal{E}\} \]

and we shall call it a family of conditional aggregation operators (FCA for short). For example, \( \mathcal{A}_{\text{sum}} = \{A_{\text{sum}}(\cdot|E) : E \in 2^{[n]}\} \), \( \mathcal{A}_{\text{max}} = \{A_{\text{max}}(\cdot|E) : E \in \{\emptyset, \{1\}, \{2\}, \ldots, \{n\}\}\} \), \( \mathcal{A}_{\text{max}} = \{A_{\text{max}}(\cdot|E) : E \in \{\emptyset\}\} \) or \( \mathcal{A} = \{A(\cdot|E) : E \in 2^{[n]}\}, n \geq 2 \), where

\[ A(x|E) = \begin{cases} A_{\text{max}}(x|E), & E \in \{\{1\}, \{2\}, \ldots, \{n\}\}, \\ 0, & E = \emptyset, \\ A_{\text{sum}}(x|E), & \text{otherwise} \end{cases} \]

for any \( x \in [0, \infty)^{[n]} \).

\textbf{Definition 2.3.} (cf. \[1, \text{Definition 4.1.}\]) Let \( x \in [0, \infty)^{[n]} \), \( \mu \in \mathcal{M} \). The \textit{generalized survival function} with respect to \( \mathcal{A} \) is defined as

\[ \mu_{\mathcal{A}}(x, \alpha) = \min \{\mu(E^c) : A(x|E) \leq \alpha, E \in \mathcal{E}\} \]

for any \( \alpha \in [0, \infty) \).

The presented definition is correct. Really, for any \( E \in \mathcal{E} \) it holds that \( E^c \in 2^{[n]} \) is a measurable set. Moreover, the set \( \{E \in \mathcal{E} : A(x|E) \leq \alpha\} \) is nonempty for all \( \alpha \in [0, \infty) \), because \( A(\cdot|\emptyset) = 0 \) by convention and \( \emptyset \in \mathcal{E} \). Immediately it is seen, that for \( \mathcal{E} = 2^{[n]} \) and \( \mathcal{A}_{\text{max}} \) we get the standard survival function, compare with \[1\]. When it will be necessary we shall emphasize the collection \( \mathcal{E} \) in the notation of generalized survival function, i.e. we shall use \( \mathcal{A}_{\mathcal{E}} \).

\footnote{Since \( A(\cdot|E) : [0, \infty)^{[n]} \to [0, \infty), \mathcal{A} \) is a family of operators parametrized by a set from \( \mathcal{E} \).}
On several places in this paper we shall work with the FCA that is *nondecreasing* w.r.t.
sets, i.e. the map \( E \mapsto A(|E|) \) will be nondecreasing. Many FCA satisfy this property, e.g. \( \mathcal{A}^{m-es} = \{ A^{m-es}(|E|) : E \in \mathcal{E} \} \), \( \mathcal{A}^{Chm} = \{ A^{Chm}(|E|) : E \in \mathcal{E} \} \), \( \mathcal{A}^{Shm} = \{ A^{Shm}(|E|) : E \in \mathcal{E} \} \), \( \mathcal{A}^{max} = \{ A^{max}(|E|) : E \in \mathcal{E} \} \), see Example 2.2 i), ii), iii).

3. Equality and inequalities of the generalized and standard survival function

In this section we shall treat Problem 1. We provide sufficient and necessary conditions
on \( x \), \( \mu \) and \( \mathcal{A} \) under which the generalized survival function and survival function coincide.
The important knowledge we use is the standard survival function formula. In what follows
we shall work with the expression of the survival function on a finite set in the form

\[
\mu(\{ x > \alpha \}) = \sum_{i=0}^{n-1} \mu \left( E_{i+1} \right) \cdot 1_{[x(i),x(i+1))}(\alpha)
\]

with the permutation \((\cdot)\) such that \( 0 = x(0) \leq x(1) \leq x(2) \leq \cdots \leq x(n) \) and \( E_i) = \{ (i) \} \) for \( i \in [n] \). However, one can easily see that some summands in the formula \((2)\) can be redundant. For example, for vectors with the property \( x(i) = x(i+1) \) for some \( i \in [n-1] \cup \{0\} \) we have \( \mu \left( E_{i+1} \right) \cdot 1_{[x(i),x(i+1))}(\alpha) = 0 \) for any \( \alpha \in [0,\infty) \), i.e., this summand does not change the values of survival function and can be omitted.

Let us consider an arbitrary (fixed) input vector \( x \) together with a permutation \((\cdot)\) such that \( 0 = x(0) \leq x(1) \leq x(2) \leq \cdots \leq x(n) \). Let us denote

\[
\Psi_x := \{ i \in [n-1] \cup \{0\} : x(i) < x(i+1) \} \cup \{n\}.
\]

For example, for the input vector \( x = (3,2,3,1) \) and the permutation \((\cdot)\) such that \( x(0) = 0 \), \( x(1) = 1 \), \( x(2) = 2 \), \( x(3) = 3 \), \( x(4) = 3 \), we get \( \Psi_x = \{0,1,2,4\} \). The following proposition includes the very basic properties of system \( \Psi_x \) needed for further results.

**Proposition 3.1.** Let \( x \in [0,\infty)^{[n]} \).

i) \( \Psi_x \) is independent on permutation \((\cdot)\) of \( x \), i.e., \( \Psi_x \) contains the same values for any
permutation \((\cdot)\) of \( x \) such that \( 0 = x(0) \leq x(1) \leq x(2) \leq \cdots \leq x(n) \).

ii) For any \( i \in [n] \) there exists \( k_i \in \Psi_x \setminus \{0\} \) such that \( x_i = x(k_i) \), i.e. \( \{x(k_i) : k_i \in \Psi_x \setminus \{0\} \} \)
contains all different values of \( x \).

iii) \( x_{\min \Psi_x}) = 0 \).

iv) \( \{x(k),x(k+1) : k \in \Psi_x \} \) is a decomposition of interval \([0,\infty)\) into nonempty pairwise
 disjoint sets.

**Proof.**
i) Let us consider two different permutations of $x$ (if they exist) ($\cdot)_1$ and ($\cdot)_2$ with the required property. Let us denote

\[
\Psi_x := \{i \in [n-1] \cup \{0\} : x_{(i)} < x_{(i+1)}\} \cup \{n\}, \\
\Phi_x := \{i \in [n-1] \cup \{0\} : x_{(i)} < x_{(i+1)}\} \cup \{n\}.
\]

We show that $\Psi_x = \Phi_x$. Indeed, $n \in \Psi_x$, $n \in \Phi_x$. If $i \in \Psi_x \setminus \{n\}$, then $x_{(i)} < x_{(i+1)}$. Because of nondecreasing rearrangement of $x$ with respect to ($\cdot)_1$, ($\cdot)_2$ we get $x_{(i)} < x_{(i+1)}$, therefore $i \in \Phi_x$ and $\Psi_x \subseteq \Phi_x$. By analogy it holds $\Phi_x \subseteq \Psi_x$.

ii) Since any $i \in [n]$ can be represented via permutation as $i = (j_i)$, $j_i \in [n]$, let us set

\[k_i = \max\{j_i \in [n] : x_i = x_{(j_i)}\}.
\]

As for any $k_i < n$ it holds that $x_{(k_i)} < x_{(k_i+1)}$, then $k_i \in \Psi_x \setminus \{0\}$. Moreover, $k_i = n \in \Psi_x$ because of the definition of $\Psi_x$, see (3).

iii) It follows immediately from the fact that $\min \Psi_x = \max\{i \in [n] \cup \{0\} : x_i = x(0) = 0\}$.

iv) It follows from part ii), iii) and from definition of system $\Psi_x$, since $x_{(k)} < x_{(k+1)}$ for any $k \in \Psi_x$ and $x_{(k_1)} \neq x_{(k_2)}$ for any $k_1, k_2 \in \Psi_x$. □

Since we have shown that the system $\Psi_x$ is independent of the chosen permutation, henceforward we shall not explicitly mention the permutation in assumptions of presented results. The following proposition states that the formula (2) can be rewritten by the system $\Psi_x$ in the simpler form, see part i). Moreover, in the second part of the proposition we show that for a fixed $x \in [0, \infty)^{[n]}$ it is $\mu_{\cdot,\text{max},\cdot}(x, \alpha) = \mu(\{x > \alpha\})$ with smaller collection $\mathcal{E}$ than the whole powerset $2^{[n]}$ (compare with the known result [1] Example 4.2) or see [1]). The collection $\mathcal{E}$ depends on $x$ (equivalently on $\Psi_x$).

**Proposition 3.2.** Let $x \in [0, \infty)^{[n]}$, $\mu \in M$.

i) Then

\[\mu(\{x > \alpha\}) = \sum_{k \in \Psi_x} \mu(E_{(k+1)}) \cdot 1_{[x_{(k)}, x_{(k+1)})}(\alpha)
\]

for any $\alpha \in [0, \infty)$ with the convention $x_{(n+1)} = \infty$.

ii) If $\mathcal{E} \supseteq \{E_{(k+1)} : k \in \Psi_x\}$, then

\[\mu_{\cdot,\text{max},\cdot}(x, \alpha) = \mu(\{x > \alpha\}).
\]

**Proof.**

i) For $k \in [n-1] \cup \{0\}$, $k \notin \Psi_x$, we have $x_{(k)} = x_{(k+1)}$. This leads to the fact that $\mu(E_{(k+1)}) \cdot 1_{[x_{(k)}, x_{(k+1)})}(\alpha) = 0$ for any $\alpha \in [0, \infty)$. Using the Proposition 3.1 iv) we have the required assertion.
Further, let us take the input vector \( x \) and generalized survival function (of input \( x \))

Indeed, in fact, it is enough to have a smaller domain of \( \mu \) being

\[ \text{dom}(\mu) = \{ E^c : E \in \mathcal{E} \} \]

Remark 3.3. Let us remark that in the whole paper we suppose \( \mu \) is defined on \( 2^n \). However, in fact, it is enough to have a smaller \( \text{dom}(\mu) \). For example, in part ii) of the previous proposition it is enough to consider the domain of \( \mu \) being \( \{ E^c : E \in \mathcal{E} \} \).

Let us note that in [4] the last summand is always equal to 0 because \( \mu(E_{(n+1)}) = \mu(\emptyset) = 0 \). However, it is useful to consider the form of survival function in [4] with sum over the whole set \( \Psi_\mathbf{x} \) not \( \Psi_\mathbf{x} \setminus \{n\} \) because of some technical details in presented proofs in this paper.

\textbf{Example 3.4.} Let us take \( \mathcal{A}^\text{max} = \{ A^\text{max}(\cdot|E) : E \in \mathcal{E} \} \) and normalized monotone measure \( \mu \) on \( 2^n \) given in the following table:

| \( E \) | \( \emptyset \) | \{1\} | \{2\} | \{3\} | \{1, 2\} | \{1, 3\} | \{2, 3\} | \{1, 2, 3\} |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|
| \( \mu(E) \) | 0   | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 1   |

Further, let us take the input vector \( x = (1, 2, 1) \) with the permutation \((1) = 1, (2) = 3, (3) = 2 \).

Then \( \Psi_\mathbf{x} = \{0, 2, 3\} \) and the collection guarantying the equality between survival function and generalized survival function (of input \( x \)) is according to Proposition 3.2 ii) e.g.

\[ \mathcal{E} = \{ E^c_{(k+1)} : k \in \Psi_\mathbf{x} \} = \{ E^c_{(1)}, E^c_{(2)}, E^c_{(3)} \} = \{ \emptyset, \{(1),(2)\}, \{(1),(2),(3)\} \} = \{ \emptyset, \{1, 3\}, \{1, 2, 3\} \} . \]

Indeed,

\[ \mu_{\mathcal{A}^\text{max}}(x, \alpha) = 1 \cdot 1_{[0, 1]}(\alpha) + 0.5 \cdot 1_{[1, 2]}(\alpha) = \mu(\{x > \alpha\}) . \]
From the previous result it follows that the standard survival function can be represented by the formula

$$\mu(\{x > \alpha\}) = \min \left\{ \mu(E^c) : A^\max(x|E) \leq \alpha, E \in \{E^c_{(k+1)} : k \in \Psi_x\} \right\}$$

with the system $\Psi_x$ given by the input vector $x$. This formula can be visualized by Figure 3. Let us remark that since $A^\max(x|E^c_{(k+1)}) = x(k)$ on the upper line we measure sets $(E^c_{(k+1)})^c$. The calculation of (generalized) survival function is processed as we have described in the Introduction. Let us remark that the essence of the following results is the pointwise comparison of the generalized survival function with the standard survival function having in mind the representation (5) together with its visualization, see Figure 3.

It is obvious that the equality of survival functions (standard and generalized) means that they have to achieve the same values, i.e., $\mu(E^c_{(k+1)})$, $k \in \Psi_x$, on the same corresponding intervals $[x(k), x(k+1))$, $k \in \Psi_x$. Having in mind the formula (4), the survival function representation given by (5) and the visualization, see Figure 3 we can formulate the following sufficient conditions. While (C1) ensures that the generalized survival function will be able to achieve the same values as the survival function, (C2) guarantees it. Let $\mathcal{A}$ be FCA.

(C1) For any $k \in \Psi_x$ there exists $G_k \in \mathcal{E}$ such that

$$A(x|G_k) = x(k) \quad \text{and} \quad \mu(G_k^c) = \mu(E^c_{(k+1)})$$

(C2) For any $k \in \Psi_x$ and for any $E \in \mathcal{E}$ it holds:

$$A(x|E) < x(k+1) \Rightarrow \mu(E^c) \geq \mu(E^c_{(k+1)})$$

The visualization of conditions (C1), (C2) via two parallel lines is drawn in Figure 4. Let us remark that for $k = n$ (C2) holds trivially. Also, for $k = \min \Psi_x$ (C1) holds trivially with $G_{\min \Psi_x} = \emptyset$.

**Remark 3.5.** In accordance with the above written, it can be easily seen that for $\mathcal{A}^\max$ with $\mathcal{E} \supseteq \{E^c_{(k+1)} : k \in \Psi_x\}$ it holds $G_k = E^c_{(k+1)}$ for any $k \in \Psi_x$ regardless of the choice of $\mu$ in (C1). Of course, for specific classes of monotone measures $\mu$ also other sets $G_k$ can satisfy (C1). Similarly, the validity of (C2) is clear. Indeed, if $A^\max(x|E) < x(k+1)$, then we have $E \subseteq E^c_{(k+1)}$, i.e., $E^c \supseteq E_{(k+1)}$. From the monotonicity of $\mu$ we have $\mu(E^c) \geq \mu(E_{(k+1)})$ for any $E \in \mathcal{E}$.
Figure 4: The visualization of conditions (C1) and (C2)

Conditions (C1) and (C2) guarantee inequalities between survival functions. Thus the equality of survival function is a consequence.

**Proposition 3.6.** Let $\mathbf{x} \in [0, \infty)^{|\mathcal{E}|}$, $\mu \in \mathcal{M}$, and let $\mathcal{A}$ be FCA.

i) If (C1) holds, then $\mu_{\mathcal{A}}(\mathbf{x}, \alpha) \leq \mu(\{\mathbf{x} > \alpha\})$ for any $\alpha \in [0, \infty)$.

ii) (C2) holds if and only if $\mu(\{\mathbf{x} > \alpha\}) \leq \mu_{\mathcal{A}}(\mathbf{x}, \alpha)$ for any $\alpha \in [0, \infty)$.

**Proof.** According to Proposition 3.1 (iv) let us divide interval $[0, \infty)$ into disjoint sets

$$[0, \infty) = \bigcup_{k \in \Psi_x} [x(k), x(k+1)).$$

Let us consider an arbitrary (fixed) $k \in \Psi_x$.

Let us prove part i). According to (C1) there exists the set $G_k \in \mathcal{E}$ such that $A(x|G_k) = x(k)$ and $\mu(G_c^k) = \mu(E(k+1))$. From the fact that $\mu(E(k+1)) = \mu(G_c^k) \in \{\mu(E^c) : A(x|E) \leq x(k)\}$ and since $\mu_{\mathcal{A}}(\mathbf{x}, \alpha)$ is nonincreasing (see [1, Proposition 4.3 (a)]) we have

$$\mu_{\mathcal{A}}(\mathbf{x}, \alpha) \leq \mu_{\mathcal{A}}(\mathbf{x}, x(k)) \leq \mu(E(k+1)) = \mu(\{\mathbf{x} > \alpha\})$$

for any $\alpha \in [x(k), x(k+1))$, where the last equality follows from [1].

Let us prove part ii). From (C2) it follows that for any $E \in \mathcal{E}$ if $A(x|E) < x(k+1)$, then $\mu(E^c) \geq \mu(E(k+1))$. Therefore,

$$\mu_{\mathcal{A}}(\mathbf{x}, \alpha) \geq \mu(E(k+1)) = \mu(\{\mathbf{x} > \alpha\})$$
for any $\alpha \in [x_{(k)}, x_{(k+1)})$, where the last equality follows from (4). It is enough to prove the implication $\Leftarrow$. Since

$$\mu_{\mathcal{A}}(x, \alpha) = \min \{ \mu(E^c) : A(x|E) \leq \alpha < x_{(k+1)}, E \in \mathcal{E} \} \geq \mu(E_{(k+1)}) = \mu(\{ x > \alpha \})$$

for any $\alpha \in [x_{(k)}, x_{(k+1)})$, then for any $E \in \mathcal{E}$ it has to hold: if $A(x|E) < x_{(k+1)}$, then $\mu(E^c) \geq \mu(E_{(k+1)})$. \hfill $\square$

**Corollary 3.7.** Let $x \in [0, \infty]^{|n|}$, $\mu \in \mathcal{M}$, and let $\mathcal{A}$ be FCA. If (C1) and (C2) are satisfied, then $\mu_{\mathcal{A}}(x, \alpha) = \mu(\{ x > \alpha \})$ for any $\alpha \in [0, \infty)$.

The application of the previous result is illustrated in the following example. The second example proves that (C1) and (C2) are only sufficient and not necessary.

**Example 3.8.** Let us consider $\mathcal{A}^\text{sum} = \{ A^\text{sum}(\cdot|E) : E \in 2^{[3]} \}$, and normalized monotone measure $\mu$ on $2^{[3]}$ with the following values:

| $E$      | $\emptyset$ | $\{1\}$ | $\{2\}$ | $\{3\}$ | $\{1,2\}$ | $\{1,3\}$ | $\{2,3\}$ | $\{1,2,3\}$ |
|----------|-------------|----------|----------|----------|-----------|-----------|-----------|------------|
| $\mu(E)$ | 0           | 0        | 0.5      | 0        | 0.5       | 0         | 0.7       | 1          |
| $A^\text{sum}(x|E)$ | 0        | 1        | 3        | 1        | 4         | 2         | 4         | 5          |

Further, let us take the input vector $x = (1, 3, 1)$ with the permutation $(1) = 1$, $(2) = 3$, $(3) = 2$. Then $x_{(0)} = 0$, $x_{(1)} = 1$, $x_{(2)} = 1$, $x_{(3)} = 3$, therefore $\Psi_x = \{0, 2, 3\}$ and

$$E_{(1)} = \{(1), (2), (3)\} = \{1, 2, 3\}, \quad E_{(3)} = \{(3)\} = \{2\}, \quad E_{(4)} = \emptyset.$$ 

We can see, that the assertion (C1) of Corollary 3.7 is satisfied with

$$G_0 = \emptyset, G_2 = \{3\}, G_3 = \{2\}.$$ 

Indeed, $A^\text{sum}(x|G_0) = 0 = x_{(0)}$ and $\mu(G_0^c) = \mu(E_{(1)})$. Further, $A^\text{sum}(x|G_2) = 1 = x_{(2)}$ and $\mu(G_2^c) = \mu(\{1,2\}) = \mu(E_{(3)})$. Finally, $A^\text{sum}(x|G_3) = 3 = x_{(3)}$ and $\mu(G_3^c) = \mu(\{1,3\}) = \mu(E_{(4)})$. The assertion (C2) is also satisfied, see the visualisation of generalized survival
function via parallel lines in Figure 5. Discussed survival functions coincide and take the form
\[ \mu(\{x > \alpha\}) = \mu_{\mathcal{A}^{\text{sum}}}(x, \alpha) = 1_{[0,1]}(\alpha) + 0.5 \cdot 1_{[1,3]}(\alpha) \]
for \( \alpha \in [0, \infty) \). The plot of (generalized) survival function is in Figure 5.

**Example 3.9.** Let us consider \( \mathcal{A}^{\text{sum}} = \{ \mathcal{A}^{\text{sum}}(\cdot | E) : E \in 2^{[3]} \} \), and normalized monotone measure \( \mu \) on \( 2^{[3]} \) with the following values:

| E     | \( \emptyset \) | \{1\} | \{2\} | \{3\} | \{1,2\} | \{1,3\} | \{2,3\} | \{1,2,3\} |
|-------|-----------------|-------|-------|-------|---------|---------|---------|---------|
| \( \mu(E) \) | 0               | 0     | 0.7   | 0     | 0.8     | 0.7     | 1       |
| \( \mathcal{A}^{\text{sum}}(x|E) \) | 0               | 2     | 3     | 4     | 5       | 6       | 7       | 9       |

Further, let us take the input vector \( x = (2,3,4) \) with the permutation being the identity. Then survival functions coincide
\[ \mu(\{x > \alpha\}) = \mu_{\mathcal{A}^{\text{sum}}}(x, \alpha) = 1_{[0,2]}(\alpha) + 0.7 \cdot 1_{[2,4]}(\alpha). \]
Here, \( G_0 = \emptyset \), \( G_1 = \{1\} \), \( G_2 = \{2\} \), \( G_3 = \{3\} \) are the only sets that satisfy the equality \( \mathcal{A}^{\text{sum}}(x|G_k) = x_{(k)} \) for \( k \in \Psi_x = \{0,1,2,3\} \). However,
\[ 0.8 = \mu(G_2) \neq \mu(E_{(3)}) = 0.7. \]
Thus, a sufficient condition in Corollary 3.7 is not a necessary condition.

Let us return to Proposition 3.6 While (C2) is the necessary and sufficient condition under which the generalized survival function is greater or equal to the survival function, (C1) is only sufficient for the reverse inequality. Since this condition seems too strict, let us define conditions (C3) and (C4) as follows:

(C3) For any \( k \in \Psi_x \) there exists \( F_k \in \mathcal{E} \) such that \( A(x|F_k) \leq x_{(k)} \) and \( \mu(F_k^c) \leq \mu(E_{(k+1)}). \)

(C4) For any \( k \in \Psi_x \) there exists \( F_k \in \mathcal{E} \) such that \( A(x|F_k) \leq x_{(k)} \) and \( \mu(F_k^c) = \mu(E_{(k+1)}). \)

The visualization of condition (C3) is drawn in Figure 6. In the following we show that exactly (C3) improves Proposition 3.6 ii). As a consequence we also get improvement of Corollary 3.7. Replacing (C1) with (C3), we obtain sufficient and necessary condition for equality between survival functions. However, it will turn out that under the (C2) assumption (C3) will be reduced to (C4).

**Proposition 3.10.** Let \( x \in [0, \infty)^n, \mu \in \mathcal{M}, \) and let \( \mathcal{A} \) be FCA. (C3) holds if and only if \( \mu_{\mathcal{A}}(x, \alpha) \leq \mu(\{x > \alpha\}) \) for any \( \alpha \in [0, \infty) \).
Proof. Let us prove the implication $\Rightarrow$. According to Proposition 3.1 (iv) let us divide interval $[0, \infty)$ into disjoint sets

$$[0, \infty) = \bigcup_{k \in \Psi_x} [x(k), x(k+1)).$$

Let us consider an arbitrary (fixed) $k \in \Psi_x$. Then by assumptions, there is $F_k \in \mathcal{E}$ such that $A(x|F_k) \leq x(k)$ and $\mu(F_k) \leq \mu(E(k+1))$. Thus $\mu(F_k) \in \{ \mu(E^c) : A(x|E) \leq \alpha, E \in \mathcal{E} \}$ for any $\alpha \in [x(k), x(k+1))$. Hence,

$$\mu_{\mathcal{E}}(x, \alpha) = \min \{ \mu(E^c) : A(x|E) \leq \alpha, E \in \mathcal{E} \} \leq \mu(F_k) \leq \mu(E(k+1)) = \mu(\{ x > \alpha \})$$

for any $\alpha \in [x(k), x(k+1))$.

Let us prove the reverse implication $\Leftarrow$. Let $\mu_{\mathcal{E}}(x, \alpha) \leq \mu(\{ x > \alpha \})$ for any $\alpha \in [0, \infty)$. Then from this fact and from (iv) it follows:

$$\mu_{\mathcal{E}}(x, x(k)) \leq \mu(\{ x > x(k) \}) = \mu(E(k+1))$$

for any $k \in \Psi_x$. As $\mu_{\mathcal{E}}(x, x(k)) = \min \{ \mu(E^c) : A(x|E) \leq x(k), E \in \mathcal{E} \}$, there exists $F_k \in \mathcal{E}$ such that $A(x|F_k) \leq x(k)$ and $\mu(F_k) \leq \mu(E(k+1))$. \hfill $\square$

Corollary 3.11. Let $x \in [0, \infty)^{|n|}$, $\mu \in \mathcal{M}$, and let $\mathcal{A}$ be FCA.

i) If (C2) holds, then (C3) is equivalent to (C4).

ii) (C2) and (C3) hold if and only if $\mu_{\mathcal{E}}(x, \alpha) = \mu(\{ x > \alpha \})$ for any $\alpha \in [0, \infty)$.

iii) (C2) and (C4) hold if and only if $\mu_{\mathcal{E}}(x, \alpha) = \mu(\{ x > \alpha \})$ for any $\alpha \in [0, \infty)$.

Proof. It is enough to prove part i), more precisely, the implication (C3) $\Rightarrow$ (C4). Let (C3) is satisfied, we show that $\mu(F_k^c) = \mu(E(k+1))$ holds for any $k \in \Psi_x$. Since for any $F_k \in \mathcal{E}$, $k \in \Psi_x$ we have $A(F_k|x) \leq x(k) < x(k+1)$, then from (C2) we have $\mu(F_k^c) \geq \mu(E(k+1))$. On the other hand, from (C3) we have $\mu(F_k) \leq \mu(E(k+1))$. \hfill $\square$
Remark 3.12. At the end of this main part let us remark that some above mentioned results are true also without constructing \( \Psi_\infty \) system. Let us denote:

\[(\widetilde{C}1)\] For any \( i \in [n] \cup \{0\} \) there exists \( G_i \in C \) such that \( A(x|G_i) = x(i) \) and \( \mu(F_i^e) = \mu(E(i+1)) \).

\[(\widetilde{C}2)\] For any \( i \in [n] \cup \{0\} \) and for any \( E \in C \) it holds: \( A(x|E) < x(i+1) \Rightarrow \mu(E^c) \geq \mu(E(i+1)) \).

\[(\widetilde{C}3)\] For any \( i \in [n] \cup \{0\} \) there exists \( F_i \in C \) such that \( A(x|F_i) \leq x(i) \) and \( \mu(F_i^c) \leq \mu(E(i+1)) \).

Then Proposition 3.6 and Corollary 3.11 (ii) remain to be true, although, requirements in (\( \widetilde{C}1 \)), (\( \widetilde{C}2 \)), (\( \widetilde{C}3 \)) will be for some \( i \in [n] \cup \{0\} \) redundant\(^6\). On the other hand, Corollary 3.11 (i), (iii) need not be satisfied in general.

Inequalities: Let \( x \in [0, \infty)^n \), \( \mu \in M \), and let \( C \) be FCA.

i) If (\( \widetilde{C}1 \)) holds, then \( \mu_C(x, \alpha) \leq \mu(\{x > \alpha\}) \) for any \( \alpha \in [0, \infty) \).

ii) (\( \widetilde{C}2 \)) holds if and only if \( \mu(\{x > \alpha\}) \leq \mu_C(x, \alpha) \) for any \( \alpha \in [0, \infty) \).

Sufficient and necessary condition: Let \( x \in [0, \infty)^n \), \( \mu \in M \), and let \( C \) be FCA. (\( \widetilde{C}2 \)) and (\( \widetilde{C}3 \)) hold if and only if \( \mu_C(x, \alpha) = \mu(\{x > \alpha\}) \) for any \( \alpha \in [0, \infty) \).

3.1. Equality of generalized survival function and standard survival function, further results

In this subsection we provide further results on indistinguishability of survival functions. Considering the formula of standard survival function (4) one can observe that the same value of monotone measure may be achieved on several intervals. These intervals can be joined together. Thus we obtain again a shorter formula of survival function, see Proposition 3.14 i), which allows us to formulate further results. Let us define system \( \Psi^*_\infty \subseteq \Psi_\infty \) as follows:

\[
\Psi_\infty^* := \{k \in \Psi_\infty \setminus \{\min \Psi_\infty\} : \mu(E_{j(k+1)}) > \mu(E_{j(k+1)}) + \min \Psi_\infty \}
\]

(compare with the definition of system \( \Psi_\infty \) which is analogous, however the main condition is concentrated on components of \( x \) instead of values of \( \mu \)). Let us give an example of the \( \Psi_\infty^* \) system calculation considering inputs from Example 3.9. For given input \( \Psi_\infty = \{0, 1, 2, 3\} \).

Then by definition of \( \Psi_\infty^* \) we have \( \min \Psi_\infty = 0 \in \Psi_\infty^* \). For \( k = 1, 3 \) the inequality \( \mu(E_{j(1+1)}) > \mu(E_{j(1+1)}) \), \( j < k \) holds, however, for \( k = 2 \) we have \( \mu(E_2) = \mu(E_3) \). Thus \( 2 \notin \Psi_\infty^* \). In summary, \( \Psi_\infty^* = \{0, 1, 3\} \).

For purpose of this subsection for any \( k \in \Psi_\infty^* \) let us denote

\[
l_k := \max\{j \in \Psi_\infty : \mu(E_{j(1+1)}) = \mu(E_{j(1+1)})\}.
\]

\(^6\)They will be redundant for \( i \in [n] \cup \{0\} \) such that \( x(i) = x(i+1) \), compare with the motivation of \( \Psi_\infty \) system introduction.
Proposition 3.13. Let $x \in [0, \infty)^n$, $\mu \in M$.

\( i \) $x_{(\min \Psi^*_x)} = 0.$

\( ii \) $\{[x(k), x_{(l_k+1)}] : k \in \Psi^*_x \}$ with $l_k$ given by (7) and with the convention $x_{(n+1)} = \infty$ is a decomposition of interval $[0, \infty)$ into nonempty pairwise disjoint sets.

\( iii \) $(\forall k \in \Psi^*_x \setminus \{\min \Psi^*_x\}) (\exists r \in \Psi^*_x, r < k)$ $x(k) = x_{(l_k+1)}$. Moreover, $\mu(E_{(l_k+1)}) < \mu(E_{(l_k+1)})$.

\( iv \) If $\mu$ is such that it is strictly monotone on $\{E_{(k+1)} : k \in \Psi^*_x \}$, then $\Psi^*_x = \Psi^*_x$.

Proof. Part i) follows from Proposition 3.1 iii). Since $\Psi^*_x \subseteq \Psi^*_x$ and $\min \Psi^*_x = \min \Psi^*_x$. The proof of ii) follows from Proposition 3.1 part iv) and from the fact that each partial interval $[x(k), x_{(l_k+1)}], k \in \Psi^*_x$ can be rewritten as follows

$$[x(k), x_{(l_k+1)}] = \bigcup_{j=k, j \in \Psi^*_x} [x(j), x_{(j+1)}].$$

The equality $x(k) = x_{(l_k+1)}$ in part iii) follows from ii) with $r = \max\{j \in \Psi^*_x : x(j) < x(k)\}$. Moreover, it holds $\mu(E_{(l_k+1)}) = \mu(E_{(r+1)}) > \mu(E_{(k+1)})$ where the first equality holds because of (7), the second inequality is true due to $r < k; r, k \in \Psi^*_x$. Part iv) follows from (6).  

Proposition 3.14. Let $x \in [0, \infty)^n$, $\mu \in M$.

\( i \) Then

$$\mu(\{x > \alpha\}) = \sum_{k \in \Psi^*_x} \mu(E_{(k+1)}) \cdot 1_{[x(k), x_{(l_k+1)}]}(\alpha)$$

for any $\alpha \in [0, \infty)$ with $l_k$ given by (7) and with the convention $x_{(n+1)} = \infty$.

\( ii \) If $\mathcal{E} \supseteq \{E_{c_{(k+1)}^c} : k \in \Psi^*_x \}$, then $\mu_{,\{\max(x, \alpha)\}} = \mu(\{x > \alpha\})$ for any $\alpha \in [0, \infty)$.

Proof. Part ii) can be proved analogously as Proposition 3.2 part ii). Part i) follows from the fact that each partial interval $[x(k), x_{(l_k+1)}], k \in \Psi^*_x$ can be rewritten as follows

$$[x(k), x_{(l_k+1)}] = \bigcup_{j=k, j \in \Psi^*_x} [x(j), x_{(j+1)}].$$

From the formula (4) and from definition of $l_k$ we get

$$\mu(\{x > \alpha\}) = \mu(E_{(j+1)}) = \mu(E_{(k+1)}).$$

for any $\alpha \in [x(j), x_{(j+1)}]$.

All results from the previous subsection will also be true under a slight modification of conditions (C1), (C2), (C3) and (C4) as follows:

(C1*) For any $k \in \Psi^*_x$ there exists $G_k \in \mathcal{E}$ such that $A(x|G_k) = x(k)$ and $\mu(G_k^c) = \mu(E_{(k+1)}).$
(C2*) For any \( k \in \Psi_x^* \) and for any \( E \in \mathcal{E} \) it holds: \( A(x|E) < x_{(l_k+1)} \Rightarrow \mu(E^c) \geq \mu(E_{(l_k+1)}) \).

(C3*) For any \( k \in \Psi_x^* \) there exists \( F_k \in \mathcal{E} \) such that \( A(x|F_k) \leq x_{(k)} \) and \( \mu(F_k^c) \leq \mu(E_{(k+1)}) \).

(C4*) For any \( k \in \Psi_x^* \) there exists \( F_k \in \mathcal{E} \) such that \( A(x|F_k) \leq x_{(k)} \) and \( \mu(F_k^c) = \mu(E_{(k+1)}) \).

In the following we summarize all modifications of results from the main part of this section. Since proofs of parts i) – vii) are based on the same ideas, we omit them. The comparison of these results with those obtained in the main part can be found in Remark 3.16.

**Proposition 3.15.** Let \( x \in [0, \infty)^n \), \( \mu \in M \), and let \( \mathcal{A} \) be FCA.

i) If (C1*) holds, then \( \mu_{\mathcal{A}}(x, \alpha) \leq \mu(\{x > \alpha\}) \) for any \( \alpha \in [0, \infty) \).

ii) (C2*) holds if and only if \( \mu(\{x > \alpha\}) \leq \mu_{\mathcal{A}}(x, \alpha) \) for any \( \alpha \in [0, \infty) \).

iii) If (C1*) and (C2*) are satisfied, then \( \mu_{\mathcal{A}}(x, \alpha) = \mu(\{x > \alpha\}) \) for any \( \alpha \in [0, \infty) \).

iv) (C3*) holds if and only if \( \mu_{\mathcal{A}}(x, \alpha) \leq \mu(\{x > \alpha\}) \) for any \( \alpha \in [0, \infty) \).

v) (C2*) and (C3*) hold if and only if \( \mu_{\mathcal{A}}(x, \alpha) = \mu(\{x > \alpha\}) \) for any \( \alpha \in [0, \infty) \).

vi) If (C2*) holds, then (C3*) is equivalent to (C4*).

vii) (C2*) and (C4*) hold if and only if \( \mu_{\mathcal{A}}(x, \alpha) = \mu(\{x > \alpha\}) \) for any \( \alpha \in [0, \infty) \).

viii) (C2) holds if and only if (C2*) holds.

**Proof.** The implication (C2) \( \Rightarrow \) (C2*) of part viii) is clear. We prove the reverse implication. Let us consider any set \( E \in \mathcal{E} \) such that \( A(x|E) < x_{(l_k+1)} \) for some \( k \in \Psi_x \). Let us define \( j_k = \min\{j \in \Psi_x : \mu(E_{(l_j+1)}) = \mu(E_{(k+1)})\} \).

It is easy to see that \( j_k \in \Psi_x^* \), \( l_{j_k} \geq k \geq j_k \). Moreover, \( \mu(E_{(l_{j_k}+1)}) = \mu(E_{(k+1)}) = \mu(E_{(j_k+1)}) \) and \( x_{(k+1)} \leq x_{(l_{j_k}+1)} \). Then from (C2*) we have \( \mu(E^c) \geq \mu(E_{(l_{j_k}+1)}) = \mu(E_{(k+1)}) \). \( \square \)

**Remark 3.16.** In comparison with results in the main part of this section, the advantage of previous statements lies in their efficiency for survival functions equality or inequality testing. In particular, Proposition 3.15 vii) requires to hold the same properties as Corollary 3.11 iii), however for a smaller number of sets, \( k \in \Psi_x^* \subseteq \Psi_x \). On the other hand, the equality (inequality) of survival functions implies more information than those included in the Proposition 3.15, the results are true for any \( k \in \Psi_x \) not only for \( k \in \Psi_x^* \). Moreover, system \( \Psi_x \) is also easier in definition.

We have seen in the main part of this section that (C1), (C2) are not necessary for equality between survival functions in general, see Corollary 3.7, Example 3.9. This result we have improved by replacing (C1) with (C4). Also, Corollary 3.7 can be improved as it follows.
Theorem 3.17. Let \( x \in [0, \infty)^n \), \( \mu \in \mathcal{M} \), and let \( \mathcal{A} \) be FCA. Then the following assertions are equivalent:

i) \((C1^*)\), \((C2^*)\) are satisfied.

ii) \( \mu_{\mathcal{A}}(x, \alpha) = \mu(\{x > \alpha\}) \) for any \( \alpha \in [0, \infty) \).

Proof. The implication i) \(\Rightarrow\) ii) follows from Proposition 3.15 iii). In order to prove the reverse implication, assume that survival functions are equal. Then \((C2^*)\) follows from Proposition 3.15 vii). It is enough to prove \((C1^*)\). From Proposition 3.15 vii) we have:

For any \( k \in \Psi_x^* \) there exists \( F_k \in \mathcal{E} \) such that \( A(x|F_k) \leq x(k) \) and \( \mu(F_k^c) = \mu(E(k+1)) \).

We show that \( A(x|F_k) = x(k) \). Indeed, for \( k = \min \Psi_x^* \) is the result immediate since \( 0 \leq A(x|F_{\min \Psi_x^*}) \leq x(\min \Psi_x^*) = 0 \), where the last inequality follows from Proposition 3.13 i).

Let \( k > \min \Psi_x^* \), \( k \in \Psi_x^* \). From Proposition 3.13 iii) there exists \( r \in \Psi_x^* \), \( r < k \) such that \( x(l+1) = x(k) \) and \( \mu(F_k^c) = \mu(E(l+1)) < \mu(E(k+1)) \). From contraposition to \((C2^*)\) we have \( A(x|F_k) \geq x(l+1) = x(k) \).

Corollary 3.18. Let \( x \in [0, \infty)^n \), \( \mu \in \mathcal{M} \) such that it is strictly monotone on \( \{E(k+1) : k \in \Psi_x\} \), and let \( \mathcal{A} \) be FCA. Then the following assertions are equivalent:

i) \((C1)\), \((C2)\) are satisfied.

ii) \( \mu_{\mathcal{A}}(x, \alpha) = \mu(\{x > \alpha\}) \) for any \( \alpha \in [0, \infty) \).

Proof. It follows from Proposition 3.13 iv) and Theorem 3.17 \(\square\)

A summary of relationships among some conditions as well as the summary of sufficient and necessary conditions under which survival functions coincide or under which they are pointwise comparable with respect to \( \leq, \geq \) can be found in the Appendix, see Table 2.

4. Equality characterization

Results of the previous section stated conditions depended on FCA \( \mathcal{A} \), input vector \( x \) and monotone measure \( \mu \) to hold equality between survival functions. Of course, when one changes the monotone measure and the other inputs stay the same, the equality can violate as the following example shows.

Example 4.1. Let us consider \( \mathcal{A}^{\text{sum}} = \{A^{\text{sum}}(\cdot|E) : E \in 2^{[3]}\} \), and normalized monotone measure \( \mu \) on \( 2^{[3]} \) with the following values:
Further, let us take the input vector $x = (1, 2, 1)$. Then we can see

$$
\mu_{\mathcal{A}, \mu}^\text{sum}(x, \alpha) = 1_{(0,1)}(\alpha) = \mu(\{x > \alpha\}), \quad \alpha \in [0, \infty),
$$

but

$$
\nu_{\mathcal{A}, \mu}^\text{sum}(x, \alpha) = 1_{(0,1)}(\alpha) + 0.5 \cdot 1_{[1,2]}(\alpha) = \nu(\{x > \alpha\}), \quad \alpha \in [0, \infty).
$$

In the following we shall find sufficient and necessary conditions on $\mathcal{A}$ and $x$ under which survival functions equal for any monotone measure. So, we answer Problem 2, see Theorem 4.2, Theorem 4.6. In the second step we characterize FCA for which survival functions equal for any monotone measure and any input vector. We answer Problem 3.

**Theorem 4.2.** Let $x \in [0, \infty)^{|n|}$, and $\mathcal{A}$ be FCA. Then the following assertions are equivalent:

i) $\mathcal{E} \supseteq \{E^c_{(k+1)} : k \in \Psi_x\}$ and $A(x|E) = A^\text{max}(x|E)$ for any $E = E^c_{(k+1)}$ with $k \in \Psi_x$, $A(x|E) \geq A^\text{max}(x|E)$ otherwise.

ii) For each $\mu \in \mathcal{M}$ such that it is strictly monotone on $\{E_{(k+1)} : k \in \Psi_x\}$ it holds $\mu_{\mathcal{A}, \mu}(x, \alpha) = \mu(\{x > \alpha\})$ for any $\alpha \in [0, \infty)$.

iii) For each $\mu \in \mathcal{M}$ it holds $\mu_{\mathcal{A}, \mu}(x, \alpha) = \mu(\{x > \alpha\})$ for any $\alpha \in [0, \infty)$.

**Proof.** The implication i) $\Rightarrow$ iii) we easily prove by Corollary 3.7. Indeed, for any $k \in \Psi_x$ (C1) is satisfied with $G_k = E^c_{(k+1)}$. If $A(x|E) < x_{(k+1)}$, $k \in \Psi_x$ and $E \in \mathcal{E}$, then from assumptions we have

$$
A^\text{max}(x|E) \leq A(x|E) < x_{(k+1)}.
$$

Then we get $E \subseteq E^c_{(k+1)}$, i.e., $E^c \supseteq E_{(k+1)}$ and for each monotone measure $\mu$ we have $\mu(E^c) \geq \mu(E_{(k+1)})$. Thus (C2) is also satisfied.

Let us prove the implication ii) $\Rightarrow$ i). Since the assumption holds for any $\mu : 2^{[n]} \rightarrow [0, \infty)$ such that it is strictly monotone measure on $\{E_{(k+1)} : k \in \Psi_x\}$, it holds for $\mu$ such that it is strictly monotone measure on $2^{[n]}$ (not only on $\{E_{(k+1)} : k \in \Psi_x\}$). From Corollary 3.18 (C1) holds. Moreover, since sets $E_{(k+1)}$ are the only sets with value equal to $\mu(E_{(k+1)})$, we get $G_k = E^c_{(k+1)}$ and $\mathcal{E} \supseteq \{E^c_{(k+1)} : k \in \Psi_x\}$. So, from (C1) we have

$$
A(x|E^c_{(k+1)}) = x_{(k)} = A^\text{max}(x|E^c_{(k+1)}).
$$

| $E$     | $\{1, 2, 3\}$ | $\{2, 3\}$ | $\{1, 3\}$ | $\{1, 2\}$ | $\{3\}$ | $\{2\}$ | $\{1\}$ | $\emptyset$ |
|---------|---------------|------------|------------|------------|---------|---------|---------|-----------|
| $E^c$   | $\emptyset$  | $\{1\}$   | $\{2\}$   | $\{3\}$   | $\{1, 2\}$ | $\{1, 3\}$ | $\{2, 3\}$ | $\{1, 2, 3\}$ |
| $\mu(E^c)$ | 0          | 0          | 0          | 0          | 0.5      | 0.5     | 0.5     | 1         |
| $\nu(E^c)$ | 0          | 0          | 0          | 0          | 0.5      | 0.5     | 0.5     | 1         |
| $A^\text{sum}(x|E)$ | 4          | 3          | 2          | 3          | 1        | 2       | 1       | 0         |
| $A^\text{max}(x|E)$  | 2          | 2          | 1          | 2          | 1        | 2       | 1       | 0         |
for any \( k \in \Psi_x \). Let us prove the second part of i). Again, if the equality between survival functions holds for any strictly monotone measure \( \mu \) on \( \{E_{i(k+1)}^r : k \in \Psi_x \} \), then it holds for \( \mu : 2^{[r]} \rightarrow [0, \infty) \) being strictly monotone on the above mentioned collection with values:

\[
\mu(E) = \mu(E_{i(k+1)}) \quad \text{for any set } E \quad \text{such that } A_{\max}(x|E^c) = x(k), \; k \in \Psi_x.7
\]

Let \( E \in \mathcal{F} \). Then according to Proposition 3.1 ii) there exists \( k \in \Psi_x \setminus \{0\} \) such that

\[
A_{\max}(x|E) = x(k).
\]

Since \( \mu \) is strictly monotone on \( \{E_{i(k+1)}^r : k \in \Psi_x \} \), then from Proposition 3.13 iv) we have \( \Psi_x = \Psi_x^* \). Further, from Proposition 3.13 i), if \( A_{\max}(x|E) = x_{(\min \Psi_x)} = 0 \) the result is trivial. Let \( k > \min \Psi_x^* \). Then from Proposition 3.13 iii) there exists \( r \in \Psi_x^*, k < r \) such that \( x_{(r+1)} = x(k) \) and \( \mu(E_{(k+1)}) < \mu(E_{(r+1)}) \). Therefore \( \mu(E^c) = \mu(E_{(k+1)}) < \mu(E_{(r+1)}) \) and from contraposition to \( (C^2) \) we have \( A(x|E) \geq x_{(r+1)} = x(k) = A_{\max}(x|E) \).

**Remark 4.3.** From the previous theorem one can see the other sufficient condition under which the standard and generalized survival functions coincide, i.e., the condition i). Let us remark that this sufficient condition is more strict than (C1) and (C2), i.e., if i) is satisfied then (C1), (C2) are true, however, the reverse implication need not be true in general, see Example 3.8.

According to previous result there are vectors for which the equality between survival functions (for any \( \mu \)) do not lead to \( A_{\max} \).

**Example 4.4.** Let us consider \( \mathcal{A} = \{A(\cdot|E) : E \in 2^{[r]} \} \) with conditional aggregation operator from Example 2.2 iii) with \( w = (0.5, 0.5, 1), \; z = (0.5, 0.25, 1) \). Let us take the input vector \( x = (2, 3, 4) \). The values of \( A(x|E), \; E \in 2^{[r]} \) are summarized in following table:

| \( E \) | \( \{1, 2, 3\} \) | \( \{2, 3\} \) | \( \{1, 3\} \) | \( \{1, 2\} \) | \( \{3\} \) | \( \{2\} \) | \( \{1\} \) | \( \emptyset \) |
|--------|----------------|----------------|----------------|----------------|--------------|--------------|--------------|--------------|
| \( E^c \) | \emptyset | \{1\} | \{2\} | \{3\} | \{1, 2\} | \{1, 3\} | \{2, 3\} | \{1, 2, 3\} |
| \( A(x|E) \) | 4 | 4 | 4 | 4 | 6 | 2 | 0 | \[3\] |

Then \( \Psi_x = \{0, 1, 2, 3\} \) and it holds

\[
\mu_{\mathcal{A}}(x, \alpha) = \mu(\{x > \alpha\}) = \mu(\{1, 2, 3\}) \cdot 1_{[0, 2]}(\alpha) + \mu(\{2, 3\}) \cdot 1_{[2, 3]}(\alpha) + \mu(\{3\}) \cdot 1_{[3, 4]}(\alpha)
\]

\[
= \mu(E_{(1)}) \cdot 1_{[0, 2]}(\alpha) + \mu(E_{(2)}) \cdot 1_{[2, 3]}(\alpha) + \mu(E_{(3)}) \cdot 1_{[3, 4]}(\alpha)
\]

for any \( \alpha \in [0, \infty) \) and monotone measure \( \mu \). So, we have shown that there is vector \( x \) and \( \mathcal{A} \neq \mathcal{A}_{\max} \) such that \( \mu_{\mathcal{A}}(x, \alpha) = \mu(\{x > \alpha\}) \) for any monotone measure \( \mu \). Indeed, \( 6 = A(x|\{2\}) > A_{max}(x|\{2\}) = 3 \) (\( A(x|E) = A_{max}(x|E) \) for any \( E \in 2^{[r]} \setminus \{\{2\}\} \)).

---

7Given set function is a monotone measure because \( A_{\max}(x|\emptyset^c) = x_{(n)} \), therefore \( \mu(E_{(n+1)}) = \mu(\emptyset) = 0 \). Further, let \( E_1 \subseteq E_2 \), i.e. \( E_1^c \supseteq E_2^c \). Then from Proposition 3.1 ii) there exist \( k_1, k_2 \in \Psi_x \) such that \( A_{\max}(x|E_1^c) = x_{(k_1)} \) and \( A_{\max}(x|E_2^c) = x_{(k_2)} \). It is easy to see that \( k_1 \geq k_2 \) and \( \mu(E_1) = \mu(E_{(k_1+1)}) \leq \mu(E_{(k_2+1)}) = \mu(E_2) \).
Lemma 4.5. Let \( \mathcal{A} = \{ A(\cdot|E) : E \in 2^{[n]} \} \) be FCA nondecreasing w.r.t. sets. If for \( x \in [0, \infty)^{[n]} \) it holds that \( A(x|E) = A^\max(x|E) \) for any \( E = E^c_{(k+1)} \) with \( k \in \Psi_x \) and \( A(x|E) \geq A^\max(x|E) \) otherwise, then \( A(x|E) = A^\max(x|E) \) for any \( E \in 2^{[n]} \).

Proof. Let us consider an arbitrary set \( E \in \mathcal{E} \) and let us denote \( A^\max(x|E) := x_s. \) Then according to Proposition 3.1 there exists \( k \in \Psi_x \) such that \( x_s = x((k_s)). \) Then \( E \subseteq E^c_{(k_s+1)}. \) From above mentioned and from Theorem 4.2 we have

\[
x_{(k_s)} = A^\max(x|E^c_{(k_s+1)}) = A(x|E^c_{(k_s+1)}) \geq A(x|E) \geq A^\max(x|E) = x_{(k_s)}.
\]

\( \square \)

Theorem 4.6. Let \( x \in [0, \infty)^{[n]} \), and \( \mathcal{A} \) be FCA nondecreasing w.r.t. sets. Then the following assertions are equivalent:

i) \( \mathcal{E} \supseteq \{ E^c_{(k+1)} : k \in \Psi_x \} \) and \( A(x|E) = A^\max(x|E) \) for any set \( E \in \mathcal{E} \).

ii) For each \( \mu \in \mathcal{M} \) it holds \( \mu(\mathcal{A}, x, \alpha) = \mu(\{ x > \alpha \}) \) for any \( \alpha \in [0, \infty) \).

Proof. The implication i) \( \Rightarrow \) ii) follows from Proposition 3.2 ii). The reverse implication follows from Theorem 4.2 and Lemma 4.3.

Let us return to Example 4.4. We have shown that for the input vector \( x = (2, 3, 4) \) with \( \mathcal{A} \) given in example \( \mu(\mathcal{A}, x, \alpha) = \mu(\{ x > \alpha \}) \) for any \( \mu \). However, for another vector, let us take \( y = (2, 5, 4) \) the equality can violate:

\[
\mu(\mathcal{A}, y, \alpha) = \mu(\{1, 2, 3\}) \cdot 1_{[0,2)}(\alpha) + \mu(\{2, 3\}) \cdot 1_{[2,4)}(\alpha),
\]

\[
\mu(\{ y > \alpha \}) = \mu(\{1, 2, 3\}) \cdot 1_{[0,2)}(\alpha) + \mu(\{2, 3\}) \cdot 1_{[2,4)}(\alpha) + \mu(\{2\}) \cdot 1_{[4,5)}(\alpha),
\]

i.e. \( \mu(\mathcal{A}, y, \alpha) = \mu(\{ y > \alpha \}) \) does not hold for any \( \mu \). In the following we shall ask for FCA \( \mathcal{A} \) for which the equality holds for any \( \mu \) and for any \( x \). As a last thus we solve Problem 3.

Theorem 4.7. Let \( \mathcal{A} \) be FCA. The following assertions are equivalent:

i) \( \mathcal{A} = \{ A^\max(\cdot|E) : E \in 2^{[n]} \} \).

ii) For each \( \mu \in \mathcal{M}, \) and for each \( x \in [0, \infty)^{[n]} \) it holds \( \mu(\mathcal{A}, x, \alpha) = \mu(\{ x > \alpha \}) \) for any \( \alpha \in [0, \infty) \).
Proof. The implication i) ⇒ ii) is immediate. We prove ii) ⇒ i). Since the equality holds for any \( x \), according to Theorem 4.2, we get

\[
\mathcal{E} = \bigcup_{x \in [0, \infty)^{[n]}} \mathcal{E}^\psi_{x}\text{-chain} = 2^{[n]}
\]

with \( \mathcal{E}^\psi_{x}\text{-chain} := \{E^c_{(k+1)} : k \in \Psi_x\} \).

Let \( x \in [0, \infty)^{[n]} \) be an arbitrary fixed vector. From Theorem 4.2, we have \( A(x|E) = \text{A}^{\max}(x|E) \) for any \( E \in \mathcal{E}^\psi_{x}\text{-chain} \) and \( A(x|E) \geq \text{A}^{\max}(x|E) \) for any \( E \in 2^{[n]} \setminus \mathcal{E}^\psi_{x}\text{-chain} \). However, we show that \( A(x|E) = \text{A}^{\max}(x|E) \) for any \( E \in 2^{[n]} \). Let us consider an arbitrary fixed \( E \in 2^{[n]} \setminus \mathcal{E}^\psi_{x}\text{-chain} \) and vector

\[
\hat{x} = x_1E + a1_{E^c}, \quad a > \max_{i \in E} x_i.
\]

The set \( [n] \in \mathcal{E}^\psi_{x}\text{-chain} \) by definition of \( \Psi_x \), therefore \( \hat{x} \neq x \). Moreover, there exists permutation \( (\cdot) \) such that \( 0 = \hat{x}(0) \leq \hat{x}(1) \leq \cdots \leq \hat{x}(\hat{k}) < \hat{x}(\hat{k}+1) = \cdots = \hat{x}(n) = a \) with \( \hat{k} = |E| \). Therefore \( \hat{k} \in \Psi_x \), and \( E = \{(1), \ldots, (\hat{k})\} \) = \( E^c_{(\hat{k}+1)} \in \mathcal{E}^\psi_{x}\text{-chain} \). Finally, from Theorem 4.2, and because of the property \( A(y|E) = A(y1_E|E) \) for any \( y \in [0, \infty)^{[n]} \), see [1], we have:

\[
A(x|E) = A(x1_E|E) = A(\hat{x}1_E|E) = A(\hat{x}|E) = \text{A}^{\max}(\hat{x}|E) = \text{A}^{\max}(\hat{x}1_E|E) = \text{A}^{\max}(x1_E|E) = \text{A}^{\max}(x|E).
\]

\( \square \)

5. Conclusion

In this paper we have solved three problems dealing with the question of equality between the survival function and the generalized survival function based on conditional aggregation operators introduced originally in [1] (the generalization of concepts of papers [8], [13]). We have restricted ourselves to discrete settings. The most interesting results are Corollary 3.11, Corollary 3.11, Proposition 3.15 and Theorem 3.17 (solutions of Problem 1), Theorem 4.2 and Theorem 4.6 (solution of Problem 2). Results were derived from the well-known formula of the standard survival function with a permutation \( (\cdot) \) playing a crucial role. As the main result, we have determined the family of conditional aggregation operators with respect to which the novel survival function is identical to the standard survival function regardless of the monotone measure and input vector, see Theorem 4.7.

We expect the future extension of our results into the area of integrals introduced with respect to novel survival functions, see [1] Definition 5.1. The relationship of studied survival functions (in the sense of equalities or inequalities) determines also the relationship of corresponding integrals (based on standard and generalized survival function). The interesting question for the future work is: Is \( \mathcal{A}^{sup} \) family of conditional operators also the only one that generates the standard survival function in case of arbitrary basic set \( X \) instead of \([n]\), i.e., is it true that
\[ \mu_{\mathcal{A}}(f, \alpha) = \mu(\{f > \alpha\}), \alpha \in [0, \infty) \text{ for any } \mu \text{ and for any } f \text{ if and only if } \mathcal{A} = \mathcal{A}^{\text{sup}}? \]

Up to now there are not known any other families except of \( \mathcal{A}^{\text{sup}} \) generating generalized survival function indistinguishable from survival function (for any \( \mu, f \)). We believe that new results will be beneficial in some applications, e.g. in the theory of decision making. The equality between survival functions of a given alternative \( x \) means that the overall score of it with respect to the Choquet integral and the \( \mathcal{A} \)-Choquet integral is the same. Also, immediately with decision making application the question of \((\mu, \mathcal{A})\)-indistinguishability arises, i.e. under which condition on \( \mu, \mathcal{A} \) it holds \( \mu_{\mathcal{A}}(x, \alpha) = \mu_{\mathcal{A}}(y, \alpha) \) for \( x, y \in [0, \infty)^n \).

Then alternatives \( x, y \) will be \( \mathcal{A} \)-Choquet integral indistinguishable, i.e. they achieve the same overall score.

**Appendix**

In this subsection we summarize all sufficient and necessary conditions for equality or inequality between survival functions, see Table 2.

| Condition Combination | Implication | Note |
|-----------------------|-------------|------|
| (C1) and (C2)         | \( \Rightarrow \) \( \mu_{\mathcal{A}}(x, \alpha) = \mu(\{x > \alpha\}) \) | Rem. 3.12 |
| (C2) and (C3)         | \( \iff \) \( \mu_{\mathcal{A}}(x, \alpha) = \mu(\{x > \alpha\}) \) | Rem. 3.12 |
| (C1) and (C2)         | \( \Rightarrow \) \( \mu_{\mathcal{A}}(x, \alpha) = \mu(\{x > \alpha\}) \) | Cor. 3.7 |
| (C2) and (C3)         | \( \iff \) \( \mu_{\mathcal{A}}(x, \alpha) = \mu(\{x > \alpha\}) \) | Cor. 3.11 |
| (C2) and (C4)         | \( \iff \) \( \mu_{\mathcal{A}}(x, \alpha) = \mu(\{x > \alpha\}) \) | Cor. 3.11 |
| (C1) and (C2)         | \( \Rightarrow \) \( \mu_{\mathcal{A}}(x, \alpha) = \mu(\{x > \alpha\}) \) | Cor. 3.18 |
| (C2) and (C3)         | \( \iff \) \( \mu_{\mathcal{A}}(x, \alpha) = \mu(\{x > \alpha\}) \) | Prop. 3.15 |
| (C2) and (C4)         | \( \iff \) \( \mu_{\mathcal{A}}(x, \alpha) = \mu(\{x > \alpha\}) \) | Prop. 3.15 |
| (C1) and (C2)         | \( \Rightarrow \) \( \mu_{\mathcal{A}}(x, \alpha) = \mu(\{x > \alpha\}) \) | Th. 3.17 |
| (C1)                  | \( \Rightarrow \) \( \mu_{\mathcal{A}}(x, \alpha) \leq \mu(\{x > \alpha\}) \) | Prop. 3.6 |
| (C3)                  | \( \iff \) \( \mu_{\mathcal{A}}(x, \alpha) \leq \mu(\{x > \alpha\}) \) | Prop. 3.15 |
| (C2)                  | \( \iff \) \( \mu_{\mathcal{A}}(x, \alpha) \leq \mu(\{x > \alpha\}) \) | Prop. 3.15 |
| (C2)                  | \( \iff \) \( \mu_{\mathcal{A}}(x, \alpha) \geq \mu(\{x > \alpha\}) \) | Prop. 3.6 |
| (C2)                  | \( \iff \) \( \mu_{\mathcal{A}}(x, \alpha) \geq \mu(\{x > \alpha\}) \) | Prop. 3.15 |

Table 2: Sufficient and necessary conditions for pointwise comparison of survival functions

From the Table 2 the following relationships between conditions (C1), (C2), (C3), (C4) and its * versions hold.
Corollary 5.1. Let $x \in [0, \infty)$, $\mu \in M$, and let $A$ be FCA. Then it holds:

$$((C1) \land (C2)) \Rightarrow ((C1^*) \land (C2^*)) \Leftrightarrow ((C2) \land (C4)) \Leftrightarrow ((\tilde{C}2) \land (\tilde{C}3))$$
$$\Leftrightarrow ((C2^*) \land (C3^*)) \Leftrightarrow ((C2^*) \land (C4^*)) \Leftrightarrow ((C1^*) \land (C2)).$$

Corollary 5.2. Let $x \in [0, \infty)$, $\mu \in M$, and let $A$ be FCA. If $(C2^*)$ holds, then

$$(C1^*) \Leftrightarrow (C3^*) \Leftrightarrow (C4^*).$$

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