The splitting algorithms by Ryu and by Malitsky-Tam applied to normal cones of linear subspaces converge strongly to the projection onto the intersection

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Abstract

Finding a zero of a sum of maximally monotone operators is a fundamental problem in modern optimization and nonsmooth analysis. Assuming that resolvents of the operators are available, this problem can be tackled with the Douglas-Rachford algorithm. However, when dealing with three or more operators, one must work in a product space with as many factors as there are operators. In groundbreaking recent work by Ryu and by Malitsky and Tam, it was shown that the number of factors can be reduced by one. These splitting methods guarantee weak convergence to some solution of the underlying sum problem; strong convergence holds in the presence of uniform monotonicity.

In this paper, we provide a case study when the operators involved are normal cone operators of subspaces and the solution set is thus the intersection of the subspaces. Even though these operators lack strict convexity, we show that striking conclusions are available in this case: strong (instead of weak) convergence and the solution obtained is (not arbitrary but) the projection onto the intersection. Numerical experiments to illustrate our results are also provided.

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1 Introduction

Throughout the paper, we assume that

\[ X \text{ is a real Hilbert space} \]  

with inner product \( \langle \cdot, \cdot \rangle \) and induced norm \( \| \cdot \| \). Let \( A_1, \ldots, A_n \) be maximally monotone operators on \( X \). (See, e.g., [7] for background on maximally monotone operators.) One central problem in modern optimization and nonsmooth analysis asks to

\[ \text{find } x \in X \text{ such that } 0 \in (A_1 + \cdots + A_n)x. \]  

In general, solving (2) may be quite hard. Luckily, in many interesting cases, we have access to the firmly nonexpansive resolvents \( J_{A_i} := (\text{Id} + A_i)^{-1} \) which opens the door to employ splitting algorithms to solve (2). The most famous instance is the Douglas-Rachford algorithm [15] whose importance for this problem was brought to light in the seminal paper by Lions and Mercier [17]. However, the Douglas-Rachford algorithm requires that \( n = 2 \); if \( n \geq 3 \), one may employ the Douglas-Rachford algorithm to a reformulation in the product space \( X^n \) [13, Section 2.2]. In recent breakthrough work by Ryu [20], it was shown that for \( n = 3 \) one may formulate an algorithm that works in \( X^2 \) rather than \( X^3 \). We will refer to this method as Ryu’s algorithm. Very recently, Malitsky and Tam proposed in [18] an algorithm for a general \( n \geq 3 \) that is different from Ryu’s and that operators in \( X^{n-1} \). (No algorithms exist in product spaces featuring fewer factors than \( n - 1 \) factors in a certain technical sense.) We will review these algorithms in Section 3 below. Both Ryu’s and the Malitsky-Tam algorithm are known to produce some solution to (2) via a sequence that converges weakly. Strong convergence holds in the presence of uniform monotonicity.

The aim of this paper is provide a case study for the situation when the maximally monotone operators \( A_i \) are normal cone operators of closed linear subspaces \( U_i \) of \( X \). These operators are not even strictly monotone. Our main results show that the splitting algorithms by Ryu and by Malitsky-Tam actually produce a sequence that converges strongly and we are able to identify the limit to be the projection onto the intersection \( U_1 \cap \cdots \cap U_n \). The proofs of these results rely on the explicit identification of the fixed point set of the underlying Ryu and Malitsky-Tam operators. Moreover, a standard translation technique gives the same result for affine subspaces of \( X \) provided their intersection is nonempty.

The paper is organized as follows. In Section 2, we collect various auxiliary results for later use. The known convergence results on Ryu splitting and on Malitsky-Tam splitting
are reviewed in Section 3. Our main results are presented in Section 4. Matrix representations of the various operators involved are provided in Section 5. These are useful for our numerical experiments in Section 6. Finally, we offer some concluding remarks in Section 7.

The notation employed in this paper is standard and follows largely [7]. When \( z = x + y \) and \( x \perp y \), then we also write \( z = x \oplus y \) to stress this fact. Analogously for the Minkowski sum \( Z = X + Y \), we write \( Z = X \oplus Y \) as well as \( P_Z = P_X \oplus P_Y \) if \( X \perp Y \).

## 2 Auxiliary results

In this section, we collect useful properties of projection operators and results on iterating linear/affine nonexpansive operators. We start with projection operators.

### 2.1 Projections

**Fact 2.1.** Suppose \( U \) and \( V \) are nonempty closed convex subsets of \( X \) such that \( U \perp V \). Then \( U \oplus V \) is a nonempty closed subset of \( X \) and

\[
P_{U \oplus V} = P_U \oplus P_V
\]

**Proof.** See [7, Proposition 29.6].

Here is a well known illustration of Fact 2.1 which we will use repeatedly in the paper (sometimes without explicit mentioning).

**Example 2.2.** Suppose \( U \) is a closed linear subspace of \( X \). Then

\[
P_{U^\perp} = \text{Id} - P_U.
\]

**Proof.** The orthogonal complement \( V := U^\perp \) satisfies \( U \perp V \) and also \( U + V = X \); thus \( P_{U + V} = \text{Id} \) and the result follows.

**Fact 2.3 (Anderson-Duffin).** Suppose that \( X \) is finite-dimensional and that \( U, V \) are two linear subspaces of \( X \). Then

\[
P_{U \cap V} = 2P_U(P_u + P_V)^+ P_V,
\]

where \( "^+" \) denotes the Moore-Penrose inverse of a matrix.


**Proof.** See, e.g., [7, Corollary 25.38] or the original [1].

**Corollary 2.4.** Suppose that $X$ is finite-dimensional and that $U, V, W$ are three linear subspaces of $X$. Then

$$P_{U \cap V \cap W} = 4P_U(P_U + P_V)^\dagger P_V(2P_U(P_U + P_V)^\dagger P_V + P_W)^\dagger P_W.$$  \hfill (6)

**Proof.** Use Fact 2.3 to find $P_{U \cap V}$, and then use Fact 2.3 again on $(U \cap V, W)$. \hfill ■

**Corollary 2.5.** Suppose that $X$ is finite-dimensional and that $U, V$ are two linear subspaces of $X$. Then

$$P_{U + V} = \text{Id} - 2P_{U \perp + P_V \perp}^\dagger P_{V \perp}$$  \hfill (7a)

$$= \text{Id} - 2(\text{Id} - P_U)(2\text{Id} - P_U - P_V)^\dagger (\text{Id} - P_V).$$  \hfill (7b)

**Proof.** Indeed, $U + V = (U \perp \cap V \perp) \perp$ and so $P_{U + V} = \text{Id} - P_{U \perp \cap V \perp}$. Now apply Fact 2.3 to $(U \perp, V \perp)$ followed by Example 2.2. \hfill ■

**Fact 2.6.** Let $Y$ be a real Hilbert space, and let $A: X \to Y$ be a continuous linear operator with closed range. Then

$$P_{\text{ran } A} = AA^\dagger.$$  \hfill (8)

**Proof.** See, e.g., [7, Proposition 3.30(ii)]. \hfill ■

### 2.2 Linear (and affine) nonexpansive iterations

We now turn results on iterating linear or affine nonexpansive operators.

**Fact 2.7.** Let $L: X \to X$ be linear and nonexpansive, and let $x \in X$. Then

$$L^k x \to P_{\text{Fix } L}(x) \iff L^k x - L^{k+1} x \to 0.$$  \hfill (9)

**Proof.** See [3, Proposition 4], [4, Theorem 1.1], [8, Theorem 2.2], or [7, Proposition 5.28]. (The versions in [3] and [4] are much more general.) \hfill ■

**Fact 2.8.** Let $T: X \to X$ be averaged nonexpansive with $\text{Fix } T \neq \emptyset$. Then $(\forall x \in X) \ T^k x - T^{k+1} x \to 0.$

**Proof.** See Bruck and Reich’s paper [12] or [7, Corollary 5.16(ii)]. \hfill ■
Corollary 2.9. Let $L : \mathcal{H} \to \mathcal{H}$ be linear and averaged nonexpansive. Then

$$(\forall x \in \mathcal{H}) \quad L^k x \to P_{\text{Fix}L}(x).$$

\textit{Proof.} Because $0 \in \text{Fix} L$, we have $\text{Fix} L \neq \emptyset$. Now combine Fact 2.7 with Fact 2.8. ■

Fact 2.10. Let $L$ be a linear nonexpansive operator and let $b \in X$. Set $T : X \to X : x \mapsto Lx + b$ and suppose that $\text{Fix} T \neq \emptyset$. Then $b \in \text{ran}(\text{Id} - L)$, and for every $x \in X$ and $a \in (\text{Id} - L)^{-1}b$, the following hold:

(i) $b = a - La \in \text{ran}(\text{Id} - L)$.
(ii) $\text{Fix} T = a + \text{Fix} L$.
(iii) $P_{\text{Fix} T}(x) = P_{\text{Fix} L}(x) + P_{(\text{Fix} L)^\perp}(a)$.
(iv) $T^k x = L^k(x - a) + a$.
(v) $L^k x \to P_{\text{Fix} L} x \iff T^k x \to P_{\text{Fix} T} x$.

\textit{Proof.} See [9, Lemma 3.2 and Theorem 3.3]. ■

Remark 2.11. Consider Fact 2.10 and its notation. If $a \in (\text{Id} - L)^{-1}b$ then $P_{(\text{Fix} L)^\perp}$ is likewise because $b = (\text{Id} - L)a = (\text{Id} - L)(P_{\text{Fix} L}(a) + P_{(\text{Fix} L)^\perp}(a)) = (\text{Fix} L)^\perp(a)$; moreover, using [16, Lemma 3.2.1], we see that

$$(\text{Id} - L)^\dagger b = (\text{Id} - L)^\dagger (\text{Id} - L)a = P_{(\ker(\text{Id} - L))^\perp}(a) = P_{(\text{Fix} L)^\perp}(a),$$

where again $"^\dagger"$ denotes the Moore-Penrose inverse of a continuous linear operator (with possibly nonclosed range). So given $b \in X$, we may concretely set

$$a = (\text{Id} - L)^\dagger b \in (\text{Id} - L)^{-1}b;$$

with this choice, (iii) turns into the even more pleasing identity

$$P_{\text{Fix} T}(x) = P_{\text{Fix} L}(x) + a.$$  

3 Known results on Ryu and on Malitsky-Tam splitting

In this section, we present the precise form of Ryu’s and the Malitsky-Tam algorithms and review known convergence results.
3.1 Ryu splitting

We start with Ryu’s algorithm. In this subsection, \( A, B, C \) are maximally monotone operators on \( X \), with resolvents \( J_A, J_B, J_C \), respectively.

The problem of interest is to

\[
\text{find } x \in X \text{ such that } 0 \in (A + B + C)x,
\]

and we assume that (15) has a solution. The algorithm pioneered by Ryu [20] provides a method for finding a solution to (15). It proceeds as follows. Set

\[
M: X \times X \to X \times X \times X: \left( \begin{array}{c} x \\ y \end{array} \right) \mapsto \left( \begin{array}{c} J_A(x) \\ J_B(J_A(x) + y) \\ J_C(J_A(x) - x + J_B(J_A(x) + y) - y) \end{array} \right).
\]

Next, denote by \( Q_1: X \times X \times X \to X: (x_1, x_2, x_3) \mapsto x_1 \) and similarly for \( Q_2 \) and \( Q_3 \). We also set \( \Delta := \{(x, x, x) \in X^3 \mid x \in X \} \). We are now ready to introduce the Ryu operator

\[
T := T_{Ryu}: X^2 \to X^2: z \mapsto z + ((Q_3 - Q_1)Mz, (Q_3 - Q_2)Mz).
\]

Given a starting point \((x_0, y_0) \in X \times X\), the basic form of Ryu splitting generates a governing sequence via

\[
(\forall k \in \mathbb{N}) \quad (x_{k+1}, y_{k+1}) := (1 - \lambda)(x_k, y_k) + \lambda T(x_k, y_k).
\]

The following result records the basic convergence properties by Ryu [20], and recently improved by Aragón-Artacho, Campoy, and Tam [2].

**Fact 3.1 (Ryu and also Aragon-Artacho-Campoy-Tam).** The operator \( T_{Ryu} \) is nonexpansive with

\[
\text{Fix } T_{Ryu} = \{(x, y) \in X \times X \mid J_A(x) = J_B(J_A(x) + y) = J_C(R_A(x) - y)\}
\]

and

\[
\text{zer}(A + B + C) = J_A(Q_1 \text{ Fix } T_{Ryu}).
\]

Suppose that \( 0 < \lambda < 1 \) and consider the sequence generated by (18). Then there exists \((x, y) \in X \times X\) such that

\[
(x_k, y_k) \rightharpoonup (x, y) \in \text{Fix } T_{Ryu},
\]

\[1\]We will express vectors in product spaces both as column and as row vectors depending on which version is more readable.
\[ M(x_k, y_k) \rightarrow M(\bar{x}, \bar{y}) \in \Delta, \]  

and

\[ ((Q_3 - Q_1)M(x_k, y_k), (Q_3 - Q_2)M(x_k, y_k)) \rightarrow (0, 0). \]  

In particular,

\[ J_A(x_k) \rightarrow J_A \bar{x} \in \text{zer}(A + B + C). \]  

**Proof.** See [20] and [2]. □

### 3.2 Malitsky-Tam splitting

We now turn to the Malitsky-Tam algorithm. In this subsection, let \( n \in \{3, 4, \ldots \} \) and let \( A_1, A_2, \ldots, A_n \) be maximally monotone operators on \( X \). The problem of interest is to

\[ \text{find } x \in X \text{ such that } 0 \in (A_1 + A_2 + \cdots + A_n)x, \]  

and we assume that (25) has a solution. The algorithm proposed by Malitsky and Tam [18] provides a method for finding a solution to (25). Now set\(^2\)

\[ M: X^{n-1} \rightarrow X^n: \left(\begin{array}{c} z_1 \\ \vdots \\ z_{n-1} \end{array}\right) \mapsto \left(\begin{array}{c} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{array}\right), \quad \text{where} \]

\[ (\forall i \in \{1, \ldots, n\}) \quad x_i = \begin{cases} J_{A_1}(z_1), & \text{if } i = 1; \\ J_{A_i}(x_{i-1} + z_i - z_{i-1}), & \text{if } 2 \leq i \leq n - 1; \\ J_{A_n}(x_1 + x_{n-1} - z_{n-1}), & \text{if } i = n. \end{cases} \]  

As before, we denote by \( Q_1: X^n \rightarrow X: (x_1, \ldots, x_{n-1}, x_n) \mapsto x_1 \) and similarly for \( Q_2, \ldots, Q_n \). We also set \( \Delta := \{ (x, \ldots, x) \in X^n \mid x \in X \} \), the diagonal in \( X^n \). We are now ready to introduce the Malitsky-Tam (MT) operator

\[ T := T_{MT}: X^{n-1} \rightarrow X^{n-1}: z \mapsto z + \begin{pmatrix} (Q_2 - Q_1)Mz \\ (Q_3 - Q_2)Mz \\ \vdots \\ (Q_n - Q_{n-1})Mz \end{pmatrix}. \]  

\(^2\)Again, we will express vectors in product spaces both as column and as row vectors depending on which version is more readable.
Given a starting point $z_0 \in X^{n-1}$, the basic form of MT splitting generates a governing sequence via

$$(\forall k \in \mathbb{N}) \quad z_{k+1} := (1 - \lambda)z_k + \lambda Tz_k. \quad (28)$$

The following result records the basic convergence.

**Fact 3.2 (Malitsky-Tam).** The operator $T_{MT}$ is nonexpansive with

$$\text{Fix } T_{MT} = \{ z \in X^{n-1} \mid Mz \in \Delta \}, \quad (29)$$

$$\text{zer}(A_1 + \cdots + A_n) = J_{A_1}(Q_{1} \text{Fix } T_{MT}). \quad (30)$$

Suppose that $0 < \lambda < 1$ and consider the sequence generated by (28). Then there exists $\bar{z} \in X^{n-1}$ such that

$$z_k \rightharpoonup \bar{z} \in \text{Fix } T_{MT}, \quad (31)$$

$$Mz_k \rightharpoonup M\bar{z} \in \Delta, \quad (32)$$

and

$$(\forall (i, j) \in \{1, \cdots, n\}^2) \quad (Q_i - Q_j)Mz_k \to 0. \quad (33)$$

In particular,

$$J_{A_1}Q_1Mz_k \rightharpoonup J_{A_1}Q_1M\bar{z} \in \text{zer}(A_1 + \cdots + A_n). \quad (34)$$

*Proof. See [18].*

## 4 Main Results

We are now ready to tackle our main results. We shall find useful descriptions of the fixed point sets of the Ryu and the Malitsky-Tam operators. These description will allow us to deduce strong convergence of the iterates to the projection onto the intersection.

### 4.1 Ryu splitting

In this subsection, we assume that

$$U, V, W \text{ are closed linear subspaces of } X. \quad (35)$$

We set

$$A := N_U, \ B := N_V, \ C := N_W. \quad (36)$$
Then
\[ Z := \text{zer}(A + B + C) = U \cap V \cap W. \] (37)

Using linearity of the projection operators, the operator \( M \) defined in (16) turns into
\[
M: X \times X \to X \times X \times X: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} P_U x \\ P_V P_U x + P_V y \\ P_W P_U x + P_W P_V P_U x - P_W x + P_W y - P_W y \end{pmatrix}, \] (38)

while the Ryu operator is still (see (17))
\[
T := T_{\text{Ryu}}: X^2 \to X^2: z \mapsto z + ((Q_3 - Q_1)Mz, (Q_3 - Q_2)Mz). \] (39)

We now determine the fixed point set of the Ryu operator.

**Lemma 4.1.** Let \((x, y) \in X \times X\). Then
\[
\text{Fix } T = \left( Z \times \{0\} \right) \oplus \left( (U^\perp \times V^\perp) \cap (\Delta^\perp + (\{0\} \times W^\perp)) \right), \] (40)

where \(\Delta = \{(x, x) \in X \times X \mid x \in X\}\). Consequently, setting
\[
E = (U^\perp \times V^\perp) \cap (\Delta^\perp + (\{0\} \times W^\perp)), \] (41)

we have
\[
P_{\text{Fix } T}(x, y) = (P_Z x, 0) \oplus P_E(x, y) \in (P_Z x \oplus U^\perp) \times V^\perp. \] (42)

**Proof.** Note that \((x, y) = (P_W y + (x - P_W y), P_W y + P_W y) = (P_W y, P_W y) + (x - P_W y, P_W y) \in \Delta + (X \times W)\). Hence
\[
X \times X = \Delta + (X \times W) \text{ is closed}; \] (43)

consequently, by, e.g., [7, Corollary 15.35],
\[
\Delta^\perp + (\{0\} \times W^\perp) \text{ is closed}. \] (44)

Next, using (19), we have the equivalences
\[
(x, y) \in \text{Fix } T_{\text{Ryu}} \iff P_U x = P_V (P_U x + y) = P_W (R_U x - y) \] (45a)
\[
\iff P_U x \in Z \land y \in V^\perp \land P_U x = P_W (P_U x - P_U^\perp x - y) \] (45b)
\[
\iff x \in Z + U^\perp \land y \in V^\perp \land P_U^\perp x + y \in W^\perp. \] (45c)
Now define the linear operator
\[ S: X \times X \to X: (x, y) \mapsto x + y. \tag{46} \]

Hence
\[
\text{Fix } T_{\text{Ryu}} = \left\{ (x, y) \in (Z + U^\perp) \times V^\perp \mid P_U x + y \in W^\perp \right\} \tag{47a}
\]
\[
= \left\{ (z + u^\perp, v^\perp) \mid z \in Z, u^\perp \in U^\perp, v^\perp \in V^\perp, u^\perp + v^\perp \in W^\perp \right\} \tag{47b}
\]
\[
= (Z \times \{0\}) \oplus ((U^\perp \times V^\perp) \cap S^{-1}(W^\perp)). \tag{47c}
\]

On the other hand, \( S^{-1}(W^\perp) = (\{0\} \times W^\perp) + \ker S = (\{0\} \times W^\perp) + \Delta^\perp \) is closed by (44). Altogether,
\[
\text{Fix } T_{\text{Ryu}} = (Z \times \{0\}) \oplus ((U^\perp \times V^\perp) \cap ((\{0\} \times W^\perp) + \Delta^\perp)), \tag{48}
\]
i.e., (40) holds. Finally, (42) follows from Fact 2.1. \[\blacksquare\]

We are now ready for the main convergence result on Ryu’s algorithm.

**Theorem 4.2 (main result on Ryu splitting).** Given \( 0 < \lambda < 1 \) and \((x_0, y_0) \in X \times X\), generated the sequence \((x_k, y_k)_{k \in \mathbb{N}}\) via \(^3\)
\[
(\forall k \in \mathbb{N}) \quad (x_{k+1}, y_{k+1}) = (1 - \lambda)(x_k, y_k) + \lambda T(x_k, y_k). \tag{49}
\]
Then
\[
M(x_k, y_k) \to (P_Z(x_0), P_Z(x_0), P_Z(x_0)); \tag{50}
\]
in particular,
\[
P_U x_k \to P_Z(x_0). \tag{51}
\]

**Proof.** Set \( T_\lambda := (1 - \lambda)\text{Id} + \lambda T \) and observe that \((x_k, y_k)_{k \in \mathbb{N}} = (T_\lambda^k(x_0, y_0))_{k \in \mathbb{N}}\). Hence, by Corollary 2.9 and (42)
\[
(x_k, y_k) \to P_{\text{Fix } T_\lambda}(x_0, y_0) = P_{\text{Fix } T}(x_0, y_0) \tag{52a}
\]
\[
= (P_Z x_0, 0) + P_E(x_0, y_0) \in (P_Z x_0) \oplus U^\perp \times V^\perp, \tag{52b}
\]
where \( E \) is as in Lemma 4.1. Hence
\[
Q_1 M(x_k, y_k) = P_U x_k \to P_U(P_Z x_0) = P_Z x_0. \tag{53}
\]
Now (23) yields
\[
\lim_{k \to \infty} Q_1 M(x_k, y_k) = \lim_{k \to \infty} Q_2 M(x_k, y_k) = \lim_{k \to \infty} Q_3 M(x_k, y_k) = P_Z x_0, \tag{54}
\]
i.e., (50) and we’re done. \[\blacksquare\]

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\(^3\)Recall (38) and (39) for the definitions of \( M \) and \( T \).
4.2 Malitsky-Tam splitting

Let \( n \in \{3, 4, \ldots \} \). In this subsection, we assume that \( U_1, \ldots, U_n \) are closed linear subspaces of \( X \). We set

\[
(\forall i \in \{1, 2, \ldots, n\}) \quad A_i := N_{U_i} \text{ and } P_i := P_{U_i}.
\] (55)

Then

\[
Z := \text{zer}(A_1 + \cdots + A_n) = U_1 \cap \cdots \cap U_n.
\] (56)

The operator \( M \) defined in (26) turns into

\[
M: X^{n-1} \rightarrow X^n: \begin{pmatrix} z_1 \\ \vdots \\ z_{n-1} \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix}, \quad \text{where}
\]

\[
(\forall i \in \{1, \ldots, n\}) \quad x_i = \begin{cases} P_1(z_1), & \text{if } i = 1; \\ P_i(x_{i-1} + z_i - z_{i-1}), & \text{if } 2 \leq i \leq n - 1; \\ P_n(x_1 + x_{n-1} - z_{n-1}), & \text{if } i = n.
\end{cases}
\] (57b)

and the MT operator remains (see (27))

\[
T := T_{MT}: X^{n-1} \rightarrow X^{n-1}: z \mapsto z + \begin{pmatrix} (Q_2 - Q_1)Mz \\ (Q_3 - Q_2)Mz \\ \vdots \\ (Q_n - Q_{n-1})Mz \end{pmatrix}.
\] (58)

We now determine the fixed point set of the Malitsky-Tam operator.

**Lemma 4.3.** The fixed point set of the MT operator \( T = T_{MT} \) is

\[
\text{Fix } T = \{ (z, \ldots, z) \in X^{n-1} \mid z \in Z \} \oplus E,
\] (59)

where

\[
E := \text{ran } \Psi \cap (X^{n-2} \times U_n) \quad \text{(60a)}
\]

\[
\subseteq U_1 \perp \times \cdots \times (U_1 \perp + \cdots + U_{n-2} \perp) \times ((U_1 \perp + \cdots + U_{n-1} \perp) \cap U_n) \quad \text{(60b)}
\]

and

\[
\Psi: U_1 \perp \times \cdots \times U_{n-1} \rightarrow X^{n-1} \quad \text{(61a)}
\]
\[(y_1, \ldots, y_{n-1}) \mapsto (y_1, y_1 + y_2, \ldots, y_1 + y_2 + \cdots + y_{n-1}) \quad (61b)\]

is the continuous linear partial sum operator which has closed range.

Let \(z = (z_1, \ldots, z_{n-1}) \in X^{n-1}\), and set \(\bar{z} = (z_1 + z_2 + \cdots + z_{n-1}) / (n-1)\). Then

\[P_{\text{Fix} T} z = (P_Z \bar{z}, \ldots, P_Z \bar{z}) \oplus P_{\bar{Z}} z \in X^{n-1}\]  
(62)

and hence

\[P_1(Q_1 P_{\text{Fix} T}) z = P_Z \bar{z}.\]  
(63)

Proof. Assume temporarily that \(z \in \text{Fix} T\) and set \(x = Mz = (x_1, \ldots, x_n)\). Then \(\bar{x} := x_1 = \cdots = x_n\) and so \(\bar{x} \in Z\). Now \(P_1z_1 = x_1 = \bar{x} \in Z\) and thus

\[z_1 \in \bar{x} + U_1^\perp \subseteq Z + U_1^\perp.\]  
(64)

Next, \(\bar{x} = x_2 = P_2(x_1 + z_2 - z_1) = P_2 x_1 + P_2(z_2 - z_1) = P_2 \bar{x} + P_2(z_2 - z_1) = \bar{x}\), which implies \(P_2(z_2 - z_1) = 0\) and so \(z_2 - z_1 \in U_2^\perp\). It follows that

\[z_2 \in z_1 + U_2^\perp.\]  
(65)

Similarly, by considering \(x_3, \ldots, x_{n-1}\), we obtain

\[z_3 \in z_2 + U_3^\perp, \ldots, z_{n-1} \in z_{n-2} + U_{n-1}^\perp.\]  
(66)

Finally, \(\bar{x} = x_n = P_n(x_1 + x_{n-1} - z_{n-1}) = P_n(\bar{x} + \bar{x} - z_{n-1}) = 2\bar{x} - P_n z_{n-1}\), which implies \(P_n z_{n-1} = \bar{x}\), i.e., \(z_{n-1} \in \bar{x} + U_n^\perp\). Combining with (66), we see that \(z_{n-1}\) satisfies

\[z_{n-1} \in (z_{n-2} + U_{n-1}^\perp) \cap (P_1 z_1 + U_n^\perp).\]  
(67)

To sum up, our \(z \in \text{Fix} T\) must satisfy

\[z_1 \in Z + U_1^\perp\]  
(68a)
\[z_2 \in z_1 + U_2^\perp\]  
(68b)
\[\vdots\]  
(68c)
\[z_{n-2} \in z_{n-3} + U_{n-2}^\perp\]  
(68d)
\[z_{n-1} \in (z_{n-2} + U_{n-1}^\perp) \cap (P_1 z_1 + U_n^\perp).\]  
(68e)

We now show the converse. To this end, assume now that our \(z\) satisfies (68). Note that \(Z^\perp = U_1^\perp + \cdots + U_n^\perp\). Because \(z_1 \in Z + U_1^\perp\), there exists \(z \in Z\) and \(u_1^\perp \in U_1^\perp\) such that \(z_1 = z \oplus u_1^\perp\). Hence \(x_1 = P_1z_1 = P_1z = z\). Next, \(z_2 \in z_1 + U_2^\perp\), say \(z_2 = z_1 + u_2^\perp = z_1 + \bar{x} + u_2^\perp = \bar{x} + u_2^\perp\). Similarly, \(z_3, \ldots, z_{n-1}\) satisfy (68d), and finally, \(z_n = \bar{x} + u_n^\perp\). Therefore, \(z \in \text{Fix} T\).
Consequently, if \( x \in X \), then \( x = P_2(x_1 + z_2 - z_1) = P_2(z + u_2^1) = P_2z = z \). Similarly, there exists also \( u_3^1 \in U_3^1 \) such that \( x_3 = \cdots = x_{n-1} = z \) and \( z_i = z + (u_3^1 + \cdots + u_i^1) \) for \( 2 \leq i \leq n-1 \). Finally, we also have \( z_n = z + u_n^1 \) for some \( u_n^1 \in U_n^1 \). Thus \( x_n = P_n(x_1 + x_{n-1} - z_{n-1}) = P_n(2z - (z + u_n^1)) = P_nz = z \). Altogether, \( z \in \text{Fix} T \). We have thus verified the description of \( \text{Fix} T \) announced in (59), using the convenient notation of the operator \( \Psi \) which is easily seen to have closed range.

Next, we observe that

\[
D := \{ (z, \ldots, z) \in X^{n-1} \mid z \in Z \} = Z^{n-1} \cap \Delta,
\]

where \( \Delta \) is the diagonal in \( X^{n-1} \) which has projection \( P_\Delta(z_1, \ldots, z_n) = (z, \ldots, z) \) (see, e.g., [7, Proposition 26.4]). By convexity of \( Z \), we clearly have \( P_\Delta(Z^{n-1}) \subseteq Z^{n-1} \). Because \( Z^{n-1} \) is a closed linear subspace of \( X^{n-1} \), [14, Lemma 9.2] and (69) yield \( P_D = P_{Z^{n-1}}P_\Delta \) and therefore

\[
P_Dz = P_{Z^{n-1}}P_\Delta z = (P_Z\bar{z}, \ldots, P_Z\bar{z}).
\]

Combining (59), Fact 2.1, (69), and (70) yields (62).

Finally, observe that \( Q_1(P_Ez) \in U_1^+ \) by (60). Thus \( Q_1(P_{\text{Fix} T}z) \in P_Z\bar{z} + U_1^+ \) and therefore (63) follows.

We are now ready for the main convergence result on the Malitsky-Tam algorithm.

**Theorem 4.4 (main result on Malitsky-Tam splitting).** Given \( 0 < \lambda < 1 \) and \( z_0 = (z_{0,1}, \ldots, z_{0,n-1}) \in X^{n-1} \), generate the sequence \( (z_k)_{k \in \mathbb{N}} \) via

\[
(\forall k \in \mathbb{N}) \quad z_{k+1} = (1 - \lambda)z_k + \lambda Tz_k.
\]

Set

\[
p := \frac{1}{n-1}(z_{0,1} + \cdots + z_{0,n-1}).
\]

Then there exists \( \bar{z} \in X^{n-1} \) such that

\[
\bar{z}_k \to \bar{z} \in \text{Fix} T,
\]

and

\[
Mz_k \to M\bar{z} = (P_Zp, \ldots, P_Zp) \in X^n.
\]

In particular,

\[
P_1(Q_1z_k) = Q_1Mz_k \to P_Z(p) = \frac{1}{n-1}P_Z(z_{0,1} + \cdots + z_{0,n-1}).
\]

Consequently, if \( x_0 \in X \) and \( z_0 = (x_0, \ldots, x_0) \in X^{n-1} \), then

\[
P_1Q_1z_k \to P_Zx_0.
\]

\[4\] Recall (57) and (58) for the definitions of \( M \) and \( T \).
Proof. Set $T_\lambda := (1 - \lambda) \text{Id} + \lambda T$ and observe that $(z_k)_{k \in \mathbb{N}} = (T_k^\lambda z)_{k \in \mathbb{N}}$. Hence, by Corollary 2.9 and Lemma 4.3,

$$z_k \to P_{\text{Fix } T_\lambda} z_0 = P_{\text{Fix } T} z_0 = (P_Z p, \ldots, P_Z p) \oplus P_E(z_0), \tag{77a}$$

where $E$ is as in Lemma 4.3. Hence, using also (63),

$$Q_1 M z_k = P_1 Q_1 z_k \to P_1 Q_1((P_Z p, \ldots, P_Z p) \oplus P_E(z_0)) \tag{78a}$$

$$= P_1 (P_Z p + Q_1 (P_E(z_0))) \tag{78b}$$

$$\in P_1 (P_Z p + U_1^\perp) \tag{78c}$$

$$= \{P_1 P_Z p\} \tag{78d}$$

$$= \{P_Z p\}, \tag{78e}$$

i.e., $Q_1 M z_k \to P_Z p$. Now (33) yields $Q_i M z_k \to P_Z p$ for every $i \in \{1, \ldots, n\}$. This yields (74) and (75).

Finally, the “Consequently” part is clear because when $z_0$ has this special form, then $p = x_0$. ■

4.3 Extension to the consistent affine case

In this subsection, we comment on the behaviour of the splitting algorithms by Ryu and by Malitsky-Tam in the consistent affine case. To this end, we shall assume that $V_1, \ldots, V_n$ are closed affine subspaces of $X$ with nonempty intersection:

$$V := V_1 \cap V_2 \cap \cdots \cap V_n \neq \emptyset. \tag{79}$$

We repose the problem of finding a point in $Z$ as

$$\text{find } x \in X \text{ such that } 0 \in (A_1 + A_2 + \cdots + A_n)x, \tag{80}$$

where each $A_i = N_{V_i}$. When we consider Ryu splitting, we also impose $n = 3$. Set $U_i := V_i - V_i$, which is the parallel space of $V_i$. Now let $v \in V$. Then $V_i = v + U_i$ and hence $J_{N_{V_i}} = P_{V_i} = P_{v+U_i}$ satisfies $P_{v+U_i} = v + P_{U_i}(x - v) = P_{U_i} x + P_{U_i}^\perp(v)$. Put differently, the resolvents from the affine problem are translations of the the resolvents from the corresponding linear problem which considers $U_i$ instead of $V_i$.

The construction of the operator $T \in \{T_{\text{Ryu}}, T_{\text{MT}}\}$ now shows that it is a translation of the corresponding operator from the linear problem. And finally $T_\lambda = (1 - \lambda) \text{Id} + \lambda T$ is
a translation of the corresponding operator from the linear problem which we denote by $L_\lambda \::\: L_\lambda = (1-\lambda) \text{Id} + \lambda L$, where $L$ is either the Ryu operator of the Malitsky-Tam operator of the parallel linear problem, and there exists $b \in X^{n-1}$ such that

$$T_\lambda(x) = L_\lambda(x) + b.$$  \hfill (81)

By Fact 2.10 (applied in $X^{n-1}$), there exists a vector $a \in X^{n-1}$ such that

$$(\forall k \in \mathbb{N}) \quad T_\lambda^k x = a + L_\lambda^k (x - a).$$ \hfill (82)

In other words, the behaviour in the affine case is essentially the same as in the linear parallel case, appropriately shifted by the vector $a$. Moreover, because $L_\lambda^k \to P_{\text{Fix} L}$ in the parallel linear setting, we deduce from Fact 2.10 that

$$T_\lambda^k \to P_{\text{Fix} T}$$ \hfill (83)

By (82), the rate of convergence in the affine case are identical to the rate of convergence in the parallel linear case. Thus, if $(x_k, y_k)_{k \in \mathbb{N}}$ is the governing sequence generated by Ryu splitting, then

$$P_{V_1} x_k \to P_V(x_0).$$ \hfill (84)

And if $z_k = (z_{k,1}, \ldots, z_{k,n-1})_{k \in \mathbb{N}}$ is the sequence generated by Malitsky-Tam splitting, then

$$P_{V_1} Q_1 z_k \to \frac{1}{n-1} P_V(z_{0,1} + \cdots + z_{0,n-1}).$$ \hfill (85)

To sum up this subsection, we note that in the consistent affine case, Ryu’s and the Malitsky-Tam algorithm exhibit the same pleasant convergence behaviour as their linear parallel counterparts!

It is, however, presently quite unclear how these two algorithms behave when $V = \emptyset$.

### 5 Matrix representation

In this section, we assume that $X$ is finite-dimensional, say

$$X = \mathbb{R}^d.$$ \hfill (86)

The two splitting algorithms considered in this paper are of the form

$$T_\lambda^k \to P_{\text{Fix} T}, \quad \text{where } 0 < \lambda < 1 \text{ and } T_\lambda = (1-\lambda) \text{Id} + \lambda T.$$ \hfill (87)

Note that $T$ is a linear operator; hence, so is $T_\lambda$ and by [9, Corollary 2.8], the convergence in (87) is linear because $X$ is finite-dimensional. What can be said about this rate? By [6,
Theorem 2.12(ii) and Theorem 2.18], a (sharp) lower bound for the rate of linear convergence is the spectral radius of $T_\lambda - P_{\operatorname{Fix} T}$, i.e.,

$$\rho(T_\lambda - P_{\operatorname{Fix} T}) = \max \{|\text{spectral values of } T_\lambda - P_{\operatorname{Fix} T}|\},$$  

while an upper bound is the operator norm

$$\|T_\lambda - P_{\operatorname{Fix} T}\|.$$  

The lower bound is optimal and close to the true rate of convergence, see [6, Theorem 2.12(i)]. Both spectral radius and operator norms of matrices are available in programming languages such as Julia [11] which features strong numerical linear algebra capabilities. In order to compute these bounds for the linear rates, we must provide matrix representations for $T$ (which immediately gives rise to one for $T_\lambda$) and for $P_{\operatorname{Fix} T}$. In the previous sections, we casually switched back and forth being column and row vector representations for readability. In this section, we need to get the structure of the objects right. To visually stress this, we will use square brackets for vectors and matrices.

For the remainder of this section, we fix three linear subspaces $U, V, W$ of $\mathbb{R}^d$, with intersection

$$Z = U \cap V \cap W.$$  

We assume that the matrices $P_U, P_V, P_W$ in $\mathbb{R}^{d\times d}$ are available to us (and hence so are $P_{U^\perp}, P_{V^\perp}, P_{W^\perp}$ and $P_Z$, via Example 2.2 and Corollary 2.4, respectively).

### 5.1 Ryu splitting

In this subsection, we consider Ryu splitting. First, the block matrix representation of the operator $M$ occurring in Ryu splitting (see (38)) is

$$\begin{bmatrix} P_U & 0 \\ P_V P_U & P_V \\ P_W P_U + P_W P_V P_U - P_W & P_W P_V - P_W \end{bmatrix} \in \mathbb{R}^{3d\times 2d}. \quad (91)$$

Hence, using (39), we obtain the following matrix representation of the Ryu splitting operator $T = T_{\operatorname{Ryu}}$:

$$T = \begin{bmatrix} \text{Id} & 0 \\ 0 & \text{Id} \end{bmatrix} + \begin{bmatrix} -\text{Id} & 0 & \text{Id} \\ 0 & -\text{Id} & \text{Id} \end{bmatrix} \begin{bmatrix} P_U & 0 \\ P_V P_U & P_V \\ P_W P_U + P_W P_V P_U - P_W & P_W P_V - P_W \end{bmatrix}.$$  

$$\quad (92a)$$

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Next, we set, as in Lemma 4.1,

\[
\Delta = \{ [x, x]^{\top} \in \mathbb{R}^{2d} \mid x \in X \},
\]

\[
E = (U^\perp \times V^\perp) \cap (\Delta^\perp + (\{0\} \times W^\perp))
\]

so that, by (42),

\[
P_{\text{Fix}} T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} P_Z x \\ 0 \end{bmatrix} + P_E \begin{bmatrix} x \\ y \end{bmatrix}.
\]

With the help of Corollary 2.4, we see that the first term, \([P_Z x, 0]^\top\), is obtained by applying the matrix

\[
\begin{bmatrix} P_Z & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 4P_U(P_U + P_V)^\dagger P_V (2P_U(P_U + P_V)^\dagger P_V + P_W)^\dagger P_W & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{2d \times 2d}
\]

to \([x, y]^\top\). Let’s turn to \(E\), which is an intersection of two linear subspaces. The projector of the left linear subspace making up this intersection, \(U^\perp \times V^\perp\), has the matrix representation

\[
P_{U^\perp \times V^\perp} = \begin{bmatrix} \text{Id} - P_U & 0 \\ 0 & \text{Id} - P_V \end{bmatrix}.
\]

We now turn to the right linear subspace, \(\Delta^\perp + (\{0\} \times W^\perp)\), which is a sum of two subspaces whose complements are \(\Delta^{\perp \perp} = \Delta\) and \(((\{0\} \times W^\perp)^\perp = X \times W\), respectively. The projectors of the last two subspaces are

\[
P_\Delta = \frac{1}{2} \begin{bmatrix} \text{Id} & \text{Id} \\ \text{Id} & \text{Id} \end{bmatrix}
\]

and

\[
P_{X \times W} = \begin{bmatrix} \text{Id} & 0 \\ 0 & P_W \end{bmatrix},
\]

respectively. Thus, Corollary 2.5 yields

\[
P_{\Delta^\perp + (\{0\} \times W^\perp)}
\]

\[
= \begin{bmatrix} \text{Id} & 0 \\ 0 & \text{Id} \end{bmatrix} - 2 \cdot \frac{1}{2} \begin{bmatrix} \text{Id} & \text{Id} \\ \text{Id} & \text{Id} \end{bmatrix} \left( \frac{1}{2} \begin{bmatrix} \text{Id} & \text{Id} \\ \text{Id} & \text{Id} \end{bmatrix} + \begin{bmatrix} \text{Id} & 0 \\ 0 & P_W \end{bmatrix} \right)^\dagger \begin{bmatrix} \text{Id} & 0 \\ 0 & P_W \end{bmatrix}
\]

\[
= \begin{bmatrix} \text{Id} & 0 \\ 0 & \text{Id} \end{bmatrix} - 2 \begin{bmatrix} \text{Id} & \text{Id} \\ \text{Id} & \text{Id} \end{bmatrix} \begin{bmatrix} 3\text{Id} & \text{Id} \\ \text{Id} & \text{Id} + 2P_W \end{bmatrix}^\dagger \begin{bmatrix} \text{Id} & 0 \\ 0 & P_W \end{bmatrix}.
\]

To compute \(P_E\), where \(E\) is as in (93b), we combine (96), (98) under the umbrella of Fact 2.3 — the result does not seem to simplify so we don’t typeset it. Having \(P_E\), we simply add it to (95) to obtain \(P_{\text{Fix}} T\) because of (94).
5.2 Malitsky-Tam splitting

In this subsection, we turn to Malitsky-Tam splitting for the current setup — this corresponds to Section 4.2 with \( n = 3 \) and where we identify \((U_1, U_2, U_3)\) with \((U, V, W)\).

The block matrix representation of \( M \) from (57) is
\[
\begin{bmatrix}
P_U & 0 \\
-P_V(\Id - P_U) & P_V \\
P_W(P_U + P_V P_U - P_V) & -P_W(\Id - P_V)
\end{bmatrix} \in \mathbb{R}^{3d \times 2d}.
\]

Thus, using (58), we obtain the following matrix representation of the Malitsky-Tam splitting operator \( T = T_{MT} \):
\[
T = \begin{bmatrix}
\Id & 0 \\
0 & \Id & 0
\end{bmatrix} + \begin{bmatrix}
-\Id & \Id & 0 \\
0 & -\Id & \Id
\end{bmatrix} \begin{bmatrix}
P_U & 0 \\
-P_V(\Id - P_U) & P_V \\
P_W(P_U + P_V P_U - P_V) & -P_W(\Id - P_V)
\end{bmatrix}
\]
\[
= \begin{bmatrix}
\Id - P_U - P_V(\Id - P_U) \\
P_V(\Id - P_U) + P_W(P_U + P_V P_U - P_V) & \Id - P_V - P_W(\Id - P_V)
\end{bmatrix} \in \mathbb{R}^{2d \times 2d}.
\]

Next, in view of (62), we have
\[
P_{\text{Fix}} T = \frac{1}{2} \begin{bmatrix} P_Z & P_Z \\ P_Z & P_Z \end{bmatrix} + P_E,
\]
where (see (60) and (61))
\[
E = \text{ran} \Psi \cap (X \times W^\perp)
\]
and
\[
\Psi : U^\perp \times V^\perp \to X^2 : \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \mapsto \begin{bmatrix} y_1 \\ y_1 + y_2 \end{bmatrix}.
\]

We first note that
\[
\text{ran} \Psi = \text{ran} \begin{bmatrix} \Id & 0 \\ \Id & \Id \end{bmatrix} \begin{bmatrix} P_U & 0 \\ 0 & P_V \end{bmatrix} = \text{ran} \begin{bmatrix} P_U & 0 \\ P_U & P_V \end{bmatrix}.
\]

We thus obtain from Fact 2.6 that
\[
P_{\text{ran}} \Psi = \begin{bmatrix} P_U & 0 \\ P_U & P_V \end{bmatrix} \begin{bmatrix} P_U & 0 \\ P_U & P_V \end{bmatrix}^\dagger.
\]
On the other hand,

\[ P_{X \times W^\perp} = \begin{bmatrix} \text{Id} & 0 \\ 0 & P_{W}^\perp \end{bmatrix} \]  

(106)

In view of (102) and Fact 2.3, we obtain

\[ P_E = 2P_{\text{ran} \Psi} \left( P_{\text{ran} \Psi} + P_{X \times W^\perp} \right)^\dagger P_{X \times W^\perp}. \]  

(107)

We could now use our formulas (105) and (106) for \( P_{\text{ran} \Psi} \) and \( P_{X \times W^\perp} \) to obtain a more explicit formula for \( P_E \) — but we refrain from doing so as the expressions become unwieldy. Finally, plugging the formula for \( P_Z \) from Corollary 2.4 into (101) as well as plugging (107) into (101) yields a formula for \( P_{\text{Fix} T} \).

6 Numerical experiments

We now outline a few experiments conducted to observe the performance of the algorithms outlined in Section 5. Each instance of an experiment involves 3 subspaces \( U_i \) of dimension \( d_i \) for \( i \in \{1, 2, 3\} \) in \( X = \mathbb{R}^d \). By [19, equation (4.419) on page 205],

\[ \dim(U_1 + U_2) = d_1 + d_2 - \dim(U_1 \cap U_2). \]  

(108)

Hence

\[ \dim(U_1 \cap U_2) = d_1 + d_2 - \dim(U_1 + U_2) \geq d_1 + d_2 - d. \]  

(109)

Thus \( \dim(U_1 \cap U_2) \geq 1 \) whenever

\[ d_1 + d_2 \geq d + 1. \]  

(110)

Similarly,

\[ \dim(Z) \geq \dim(U_1 \cap U_2) + d_3 - d \geq d_1 + d_2 - d + d_3 - d = d_1 + d_2 + d_3 - 2d. \]  

(111)

Along with (110), a sensible choice for \( d_i \) satisfies

\[ d_i \geq 1 + \lceil 2d/3 \rceil \]  

(112)

because then \( d_1 + d_2 \geq 2 + 2 \lceil 2d/3 \rceil \geq 2 + 4d/3 > 2 + d \). Hence \( d_1 + d_2 \geq 3 + d \) and \( d_1 + d_2 + d_3 > 3 + 3 \lceil 2d/3 \rceil \geq 3 + 2d \). The smallest \( d \) that gives proper subspaces is \( d = 6 \), for which \( d_1 = d_2 = d_3 = 5 \) satisfy the above conditions.

We now describe our set of 3 numerical experiments designed to observe different aspects of the algorithms.
6.1 Experiment 1: Bounds on the rates of linear convergence

As shown in Section 5, we have lower and upper bounds on the rate of linear convergence of the operator $T_\lambda$. We conduct this experiment to observe how these bounds change as we increase $\lambda$. To this end, we generate 1000 instances of sets of linear subspaces $U_1$, $U_2$ and $U_3$. This can be done by randomly generating sets of 3 matrices $B_1, B_2, B_3$ in $\mathbb{R}^{5 \times 6}$. These can be used to define the range spaces of these subspaces, which in turn will give us the projection onto $U_i$ using [7, Proposition 3.30(ii)],

$$P_{U_i} = B_i B_i^\dagger.$$  \hspace{1cm} (113)

For each instance, algorithm and $\lambda \in \{0.01 \cdot k \mid k \in \{1, 2, \ldots, 99\}\}$, we obtain the operators $T_\lambda$ and $P_{\text{Fix}_T}$ as outlined in Section 5 and compute the spectral radius and operator norm of $T_\lambda - P_{\text{Fix}_T}$. Figure 1 reports the average of the spectral radii and operator norms for each $\lambda$. While Ryu sees a decline in the lower bound for the rate of convergence, MT sees a minimizer around 0.9.
Figure 2: Experiment 2: number of iterations for the governing sequence

Figure 3: Experiment 2: number of iterations for the shadow sequence
6.2 Experiment 2: Number of iterations to achieve prescribed accuracy

Because we know the limit points of the governing as well as shadow sequences, we investigate how changing λ affects the number of iterations required to approximate the limit to a given accuracy. For these experiments, we fix 100 instances of sets of subspaces \( \{U_1, U_2, U_3\} \). We also fix 100 different starting points in \( \mathbb{R}^6 \). For each instances of the subspaces, starting point \( z_0 \) and \( \lambda \in \{0.01 \cdot k \mid k \in \{1, 2, \ldots, 99\}\} \), we obtain the number of iterations (up to a maximum of \( 10^4 \) iterations) required to achieve \( \varepsilon = 10^{-6} \) accuracy.

For the governing sequence, the limit \( P_{\text{Fix}Tz_0} \) is used to determine the stopping condition. Figure 2 reports the median number of iterations required for each \( \lambda \) to achieve the given accuracy. For the shadow sequence, we compute the median number of iterations required to achieve \( \varepsilon = 10^{-6} \) accuracy for the shadow sequence \( Mz_k \) with respect to its limit \( (P_{\text{Z}z_0}, P_{\text{Z}z_0}, P_{\text{Z}z_0}) \). Here \( M \) for Ryu and MT can be obtained from (91) and (99) respectively. See Figure 3 for results.

For both the algorithms and experiments, increasing values of \( \lambda \) result in a decreasing number of median iterations required. As is evident from the maximum number of iterations required for a fixed lambda, the shadow sequence converges before the governing sequence for larger values of \( \lambda \). One can also see that Ryu requires fewer median iterations for both the governing and the shadow sequence to achieve the same accuracy as MT for a fixed lambda.

6.3 Experiment 3: Convergence plots of shadow sequences

In this experiment, we measure the distance of the shadow sequence from the limit point for each iteration to observe the approach of the iterates of the algorithm to the solution. We pick the \( \lambda \) with respect to which the iterates converge the fastest, which is \( \lambda = 0.99 \) for both the algorithms because of Figure 3. Similar to the setup of the previous experiment, we fix 100 starting points and 100 sets of subspaces \( \{U_1, U_2, U_3\} \). We now run the algorithms for 150 iterations for each starting point and each set of subspaces. We measure \( \|Mz_n - (P_{\text{Z}z_0}, P_{\text{Z}z_0}, P_{\text{Z}z_0})\| \) for each iteration. Figure 4 reports the median of \( \|Mz_i - (P_{\text{Z}z_0}, P_{\text{Z}z_0}, P_{\text{Z}z_0})\| \) for each iteration \( i \in \{1, \ldots, 150\} \).

As can be seen in Figure 4, Ryu converges faster to the solution compared to MT. Both show faint “rippling” akin to the one known to occur for the Douglas-Rachford algorithm.
In this paper, we investigated the recent splitting methods by Ryu and by Malitsky-Tam in the context of normal cone operators for subspaces. We discovered that both algorithms find not just some solution but in fact the projection of the starting point onto the intersection of the subspaces. Moreover, convergence of the iterates is strong even in infinite-dimensional settings. Our numerical experiments illustrated that Ryu’s method seems to converge faster although Malitsky-Tam splitting is not limited in its applicability to just 3 subspaces.

Two natural avenues for future research are the following. Firstly, when \( X \) is finite-dimensional, we know that the convergence rate of the iterates is linear. While we illustrated this linear convergence numerically in this paper, it is open whether there are natural bounds for the linear rates in terms of some version of angle between the subspaces involved. For the prototypical Douglas-Rachford splitting framework, this was carried out in [5] in terms of the Friedrichs angle. Secondly, what can be said in the inconsistent affine case? Again, the Douglas-Rachford algorithm may serve as a guide to what the expected results and complications might be; see, e.g., [10].
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