A method to solve nonlinear Schrödinger equation using Riccati equation

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November 14, 2014

Abstract

A method to find exact solutions to nonlinear Schrödinger equation, defined on a line and on a plane, is found by connecting it with second order linear ordinary differential equation. The connection is essentially made using Riccati equation. Generalisation of several known solutions is found using this method, in case of nonlinear Schrödinger equation defined on a line. This method also yields non-singular and singular vortex solutions, when applied to nonlinear Schrödinger equation on a plane.

1 Introduction

Nonlinear Schrödinger equation (NLSE) is one of the well studied nonlinear partial differential equations (PDE) in the literature. It manifests in a variety of physical problems, ranging from optical fibres [1], superfluidity [2, 3] and ocean waves [4] amongst others. When defined on a line, it is found to be integrable [5], as was shown by Zakharov and Shabat [6], using inverse scattering transform. The inverse scattering transform has yielded several interesting solutions to this system including the soliton solutions [5]. This system has also been studied intensely using Hirota’s direct method [7] and a number of solutions have been obtained [8]. Connections of NLSE with second order nonlinear ordinary differential equations, which are satisfied by elliptic functions, are also well known and well studied [9]. Further, its relation with Painleve II and IV equations is also known [10, 11].

In this paper, we show that NLSE, defined on a line and on a plane, in certain cases can be mapped to Riccati equation, a first order nonlinear ordinary differential equation (ODE). Since Riccati system can be mapped onto a second order linear ODE, using Cole-Hopf map [12, 13], this effectively gives one a connection between second order linear ODE and NLSE. Generalised versions of solitonic, periodic and rational solutions are found for NLSE defined on a line, using this straightforward map. Similar technique is extended to study NLSE defined on a plane, which yields non-singular and singular vortex solutions.
2 NLSE on a line

Nonlinear Schrödinger equation, defined on a line, reads [9]:

\[
i \frac{\partial \psi}{\partial t} + \frac{\partial^2 \psi}{\partial x^2} - g|\psi|^2\psi + \mu \psi = 0,
\]

where both \( \mu \) and \( g \) are assumed to be positive definite constants. Two non-dynamical trivial solutions of above equation are, \( \psi = 0 \) and \( \psi = \pm \sqrt{\frac{\mu}{g}} \). In order to find a non-trivial dynamical solution to above equation, we choose an ansatz:

\[
\psi(x, t) = a \cos \theta + ib \sin \theta + d \cos \left( \frac{x - ut}{\xi} \right) \cos \left( \frac{\tau}{\xi} \right),
\]

where \( a, b, d, \xi \) and \( \theta \) are real constants, with \( f(\tau) \) being a real function of travelling variable: \( \tau = (x-ut)\frac{\cos \theta}{\xi} \). Above ansatz solves equation (1) provided following two equations are obeyed by \( \psi \).

\[
\begin{align*}
\tau &= \frac{ud \cos^2 \theta}{\xi} f_{\tau} = gb \sin \theta \left( a^2 \cos^2 \theta + b^2 \sin^2 \theta + d^2 \cos^2 \theta f^2 + 2adf \cos^2 \theta \right) - \mu b \sin \theta \quad (3) \\
\tau &= \frac{d \cos^3 \theta}{\xi^2} f_{\tau\tau} = \left( g(a^2 \cos^2 \theta + b^2 \sin^2 \theta + d^2 \cos^2 \theta f^2 + 2ADF \cos^2 \theta f) - \mu \right)(a \cos \theta + d \cos \theta f). \quad (4)
\end{align*}
\]

Here, \( f_{\tau} \) stands for derivative of \( f(\tau) \) with respect to \( \tau \). Compatibility of above two equations requires that \( u = \sqrt{2gb^2 \sin \theta} \). With this condition, one observes that for all \( f(\tau) \) which solve equation (3), also solve equation (1). Hence, by solving for equation (3), a Riccati equation [12], one obtains a corresponding solution to equation (1). Interestingly, this choice of ansatz has mapped the problem of solving a nonlinear PDE problem given by (1), to a problem of solving a Riccati equation. Riccati equation (3) can be generally written as,

\[
f_{\tau} = Af^2 + Bf + C, \quad (5)
\]

where \( A = -\text{sgn}(b)\sqrt{\frac{2}{\xi}}d\xi, B = -\text{sgn}(b)af\sqrt{2y} \) and \( C = -\frac{b|\xi|\sqrt{2y}}{d \cos^2 \theta} (ga^2 \cos^2 \theta + gb^2 \sin^2 \theta - \mu) \). It is well known that, Riccati equation can be mapped on to a second order linear ODE via Cole-Hopf map [12, 13]:

\[
f(\tau) = -\frac{1}{A} u'(\tau), \quad (6)
\]

where \( u(\tau) \) solves:

\[
u'' - Bu' + ACu = 0. \quad (7)
\]

Above is a second order linear ODE, whose general solution is can be written in terms of two linearly independent solutions, weighted by two arbitrary constants.

**Case I:** \( B^2 > 4AC \)

In this case, equation (7) is solved by two linearly independent real solutions \( u(\tau) = e^{\lambda_\pm \tau} \), where

\[
\lambda_\pm = \frac{B}{2} \pm \frac{1}{2} \sqrt{B^2 - 4AC}. \quad (8)
\]

Hence, the general solution to equation (5) is given by:

\[
f(\tau) = -\frac{1}{A} \left( \frac{c_1\lambda_+e^{\lambda_+\tau} + c_2\lambda_-e^{\lambda_-\tau}}{c_1e^{\lambda_+\tau} + c_2e^{\lambda_-\tau}} \right). \quad (9)
\]

Appearance of two arbitrary constants in above solution may appear unpleasant; however it is to be noted that, these two will be fixed by appropriate boundary conditions, under which equation (1) is solved. In case
when, \( a = b = \sqrt{\frac{2}{3}} \), and \( c_1 = c_2 = 1 \), one finds that above solution reduces to the well known grey soliton solution:

\[
\psi = a \cos \theta \tanh(a \lambda \tau) + ia \sin \theta.
\] (10)

This shows that the solution (10) is a generalised version of known soliton solutions of NLSE \( [6, 14] \).

**Case II:** \( B^2 < 4AC \)

In this case, solution space of equation (7) can be constructed out of linearly independent solutions\( u(\tau) = e^{\lambda \pm \lambda} \), which are in general complex, since

\[
\lambda = \frac{B}{2} \pm \frac{i}{2} \sqrt{4AC - B^2}.
\] (11)

The general solution to equation (5) correspondingly is given by:

\[
f(\tau) = -\frac{1}{A} \left( \frac{c_1 e^{i\lambda \sqrt{4AC - B^2} \tau} + c_2 e^{-i\lambda \sqrt{4AC - B^2} \tau}}{c_1 e^{i\lambda \sqrt{4AC - B^2} \tau} + c_2 e^{-i\lambda \sqrt{4AC - B^2} \tau}} \right).
\] (12)

Note that, reality of \( f \) in this case can be ensured by appropriate choice of values of constants \( c_1, c_2 \). As is evident, this is a generalised periodic solution of NLSE \( [14] \).

**Case III:** \( B^2 = 4AC \)

In this case, equation (7) admits two linearly independent solutions: \( e^{B \tau} \) and \( \tau e^{B \tau} \). The corresponding solution to (5) is given by:

\[
f(\tau) = -\frac{1}{A} \left( \frac{c_1 + c_2 + c_2 \tau}{c_1 + c_2 \tau} \right).
\] (13)

This is a generalised version of known rational solution of NLSE \( [8] \).

Above one saw, that the ansatz based mapping, defined by (2), from NLSE problem to a second order linear ODE problem, gives rise to a variety of generalised solutions to NLSE. Further, it also shows that solitonic, periodic and rational solutions to NLSE exist in mutually exclusive parameter space.

### 3 NLSE on a plane

This approach can also be extended to study NLSE defined on a plane. Time independent NLSE in two spatial dimensions reads:

\[
-\nabla^2 \psi + g(x, y)|\psi|^2 \psi - \mu(x, y)\psi + V(x, y)\psi = 0,
\] (14)

where field \( \psi \), coupling constant \( g \), chemical potential \( \mu \) and trapping potential \( V \) are all assumed to be functions of coordinates but not of time. In what follows, it will be assumed that one is dealing with spherically symmetric problems, in which case \( g, V \) and \( \mu \) are all function of \( r = \sqrt{x^2 + y^2} \) only. Being interested in solutions which are isotropic, the following ansatz for \( \psi \) is chosen:

\[
\psi(r, \theta) = R(r) e^{in\theta},
\] (15)

where \( n \) is an integer, and \( R(r) \) is a real function. In light of this ansatz, equation (14) now reads:

\[
- \left( \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2 R}{\partial r^2} - \frac{n^2}{r^2} R \right) + g(r)R^3 + (V(r) - \mu(r))R = 0.
\] (16)

Above equation can be rewritten in terms of \( \rho \), where \( R = \rho^{\frac{1}{3}} \), as

\[
\frac{\partial^2 \rho}{\partial r^2} = \frac{g(r)}{r} \rho^3 + \left( \frac{4n^2 - 1}{4r^2} - \mu(r) + V(r) \right) \rho.
\] (17)
Often in many physical problems, one imposes boundary conditions: $\rho(r \to \infty) \to \text{const.}$ and $\rho'(r \to \infty) \to 0$, which shall be assumed to be true here as well. Equation (17) is seen to be equivalent to following Riccati equation:

$$\frac{\partial \rho}{\partial r} = \alpha(r)\rho^2 + \beta(r)\rho + \gamma(r),$$

(18)

provided following conditions are satisfied:

$$\beta = -\frac{\alpha'}{3\alpha},$$

(19)

$$\beta = -\frac{\gamma'}{\gamma},$$

(20)

$$\alpha^2 = \frac{g(r)}{2r},$$

(21)

$$2\alpha \gamma + \beta' + \beta^2 = \frac{4n^2 - 1}{4r^2} - \mu(r) + V(r).$$

(22)

These imply that:

$$g(r) = 2r\alpha^2(r),$$

(23)

$$\alpha(r) = \alpha_0 e^{-\int r \beta(t) \, dt},$$

(24)

$$\gamma(r) = \gamma_0 e^{-\int r \beta(t) \, dt},$$

(25)

$$\beta'(r) + \beta^2(r) + 2\alpha_0 \gamma_0 e^{-\int r \beta(t) \, dt} = V(r) + \frac{4n^2 - 1}{4r^2} - \mu(r).$$

(26)

Here, $\alpha_0, \gamma_0$ are constants of integration and need to be fixed, so as to be compatible with boundary conditions.

Below two cases are considered, where the above conditions are explicitly solved, to yield nontrivial solutions to equation (14).

**Case - I**: $\beta = 0, \gamma_0 \neq 0$

When $\beta = 0$, one finds that the above conditions imply,

$$2\alpha_0 \gamma_0 = V(r) - \mu(r) + \frac{4n^2 - 1}{4r^2},$$

(27)

which is easily satisfied, if the external potential is $V(r) = \frac{1}{4r^2}$, and the chemical potential is given by $\mu(r) = -2\alpha_0 \gamma_0$, where constants $\alpha_0$ and $\beta_0$ are nonzero. This also means that coupling constant $g(r) = 2\alpha_0^2 r$.

The Riccati equation (18):

$$\frac{\partial \rho}{\partial r} = \alpha_0 \rho^2 + \gamma_0,$$

(28)

can now be solved by mapping it to second order linear ODE:

$$y'' + \alpha_0 \gamma_0 y = 0,$$

(29)

using Cole-Hopf transformation, $\rho = -\frac{1}{\alpha_0} \frac{y'}{y}$. When $\alpha_0 \gamma_0 < 0$, the above linear ODE admits a solution:

$$y(r) = \cosh(\sqrt{\alpha_0 \gamma_0} |r|).$$

(30)

Correspondingly $\rho$ is given by:

$$\rho(r) = -\frac{\sqrt{|\alpha_0 \gamma_0|}}{\alpha_0} \tanh(\sqrt{\alpha_0 \gamma_0} |r|),$$

(31)
using which $\psi$ comes out to be:

$$\psi(r, \theta) = \frac{-1}{\sqrt{r}} \frac{\sqrt{|\alpha_0 \gamma_0|}}{\alpha_0} \tanh(\sqrt{|\alpha_0 \gamma_0|} r) e^{in\theta}. \quad (32)$$

Note this solution actually describes a vortex of charge $n$, since gradient of phase possesses non zero circulation: phase function $\chi(\theta) = n\theta$, is such that $\oint \nabla \chi \cdot dl = 2\pi n$ along any closed curve enclosing the origin \[15, 3\]. It should be also be noted that density at the core of vortex is zero since $|\psi(r \to 0)|^2 \sim r$, and hence is a non-singular vortex.

Case - II: $\beta = 0$, $\gamma_0 = 0$

In this case, one finds that $V(r) = \frac{1}{4} \frac{4n^2}{r^2}$, $\mu = 0$ and $g(r) = 2\alpha_0^2 r$. Riccati equation \[18\] now reads:

$$\frac{\partial \rho}{\partial r} = \alpha_0 \rho^2. \quad (33)$$

It is easy to see that, this equation possesses a power law solution:

$$\rho(r) = -\frac{1}{\alpha_0 r^n}. \quad (34)$$

complying with the boundary conditions. Correspondingly, the expression for $\psi$ reads:

$$\psi(r, \theta) = \frac{-1}{\alpha_0 r^{\frac{3}{2}}} e^{in\theta}. \quad (35)$$

Akin to the previous case, this solution also describes a vortex of charge $n$, since here also gradient of phase possesses non-zero circulation. However unlike above vortex, in this case, density at the core diverges since $|\psi(r \to 0)|^2 \sim \frac{1}{r}$. Also note that, unlike earlier case, where a length scale $\frac{1}{\sqrt{|\alpha_0 \gamma_0|}}$ was associated with the vortex, often called vortex radius \[3\], here no such scale is present.

This results would have relevance for the discussion regarding various kinds of possible variation of nonlinearity, like cases of constant nonlinearity and bounded nonlinearity, which has been well explored for vortex solutions \[16, 17\].

4 Conclusion

In this paper, it is shown that the problem of solving NLSE, in certain cases, can be mapped onto a problem of solving a second order linear ODE. The mapping procedure critically exploits the property of Riccati equation. It is observed that, for NLSE defined on a line, generalisation of several known solutions is straightforwardly obtained. In case of NLSE defined on a plane, two different types of vortex solutions are found. It must be noted that, a completely different method, employing the Riccati equation for finding solutions of NLSE type problem also exists in literature \[18, 19\]. In the present treatment, we have not dealt with NLSE with a possible source term \[20, 21\]. One wonders, if the present approach can be generalised appropriately to yield solutions to such inhomogeneous problems as well.

References

[1] Govind P Agrawal. *Nonlinear fiber optics*. Academic press, 2007.

[2] Alexander L Fetter and John Dirk Walecka. *Quantum theory of many-particle systems*. Dover Publications, 2003.

[3] Kerson Huang. *Quantum field theory: From operators to path integrals*. John Wiley & Sons, 2008.
[4] D H Peregrine. Water waves, nonlinear Schrödinger equations and their solutions. *The Journal of the Australian Mathematical Society. Series B. Applied Mathematics*, 25(01):16–43, 1983.

[5] Ashok Das. *Integrable models*, volume 30. World Scientific, 1989.

[6] V E Zakharov and A B Shabat. Exact theory of two-dimensional self-focussing and one-dimensional self-modulating waves in nonlinear media. *Sov. Phys. JETP*, 34, 1972.

[7] Ryogo Hirota. *The direct method in soliton theory*, volume 155. Cambridge University Press, 2004.

[8] Akira Nakamura and Ryogo Hirota. A new example of explode-decay solitary waves in one-dimension. *Journal of the Physical Society of Japan*, 54(2):491–499, 1985.

[9] Muthusamy Lakshmanan and Shanmuganathan Rajasekar. *Nonlinear dynamics: integrability, chaos and patterns*. Springer, 2003.

[10] Peter A Clarkson. Painlevé equations - nonlinear special functions. In *Orthogonal polynomials and special functions*, pages 331–411. Springer, 2006.

[11] L A Toikka, J Hietarinta, and K-A Suominen. Exact soliton-like solutions of the radial Gross-Pitaevskii equation. *Journal of Physics A: Mathematical and Theoretical*, 45(48):485203, 2012.

[12] Henry Thomas and Herbert Piaggio. *An elementary treatise on differential equations and their applications*. Bell, 1952.

[13] Fred Cooper, Avinash Khare, and Uday Sukhatme. *Supersymmetry in quantum mechanics*. World Scientific, 2001.

[14] Ludwig D Faddeev and Leon A Takhtajan. *Hamiltonian methods in the theory of solitons*. Springer, 2007.

[15] Hiroomi Umezawa. *Advanced field theory*. AIP, 1993.

[16] L Wu, L Li, J F Zhang, D Mihalache, B A Malomed, and W M Liu. Exact solutions of the Gross-Pitaevskii equation for stable vortex modes in two-dimensional Bose-Einstein condensates. *Physical Review A*, 81(6):061805, 2010.

[17] L A Toikka, J Hietarinta, and K-A Suominen. Exact soliton like solutions of the radial Gross-Pitaevskii equation. *Journal of Physics A: Mathematical and Theoretical*, 45(48), 2012.

[18] Rajneesh Atre, Prasanta K Panigrahi, and G S Agarwal. Class of solitary wave solutions of the one-dimensional Gross-Pitaevskii equation. *Phys. Rev. E*, 73(5):056611, 2006.

[19] Amit Goyal, Rama Gupta, Shally Loomba, and C N Kumar. Riccati parameterized self-similar waves in tapered graded-index waveguides. *Phys. Lett. A*, 376(45):3454–3457, 2012.

[20] T Solomon Raju, C Nagaraja Kumar, and Prasanta K Panigrahi. On exact solitary wave solutions of the nonlinear Schrödinger equation with a source. *J. Phys. A: Math. Gen.*, 38(16):L271, 2005.

[21] Vivek M Vyas, T Solomon Raju, C Nagaraja Kumar, and Prasanta K Panigrahi. Soliton solutions of driven nonlinear Schrödinger equation. *J. Phys. A: Math. Gen.*, 39(29):9151, 2006.