An MGF-Based Unified Framework to Determine the Joint Statistics of Partial Sums of Ordered i.n.d. Random Variables

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Abstract—The joint statistics of partial sums of ordered random variables (RVs) are often needed for the accurate performance characterization of a wide variety of wireless communication systems. A unified analytical framework to determine the joint statistics of partial sums of ordered independent and identically distributed (i.i.d.) random variables was recently presented. However, the identical distribution assumption may not be valid in several real-world applications. With this motivation in mind, we consider in this paper the more general case in which the random variables are independent but not necessarily identically distributed (i.n.d.). More specifically, we extend the previous analysis and introduce a new more general unified analytical framework to determine the joint statistics of partial sums of ordered i.n.d. RVs. Our mathematical formalism is illustrated with an application on the exact performance analysis of the capture probability of generalized selection combining (GSC)-based RAKE receivers operating over frequency-selective fading channels with a non-uniform power delay profile.

Index Terms—Order statistics, joint statistics, non-identical distribution, moment generating function (MGF), probability density function (PDF), exponential distribution.

I. INTRODUCTION

The subject of order statistics deals with the properties and distributions of the ordered random variables (RVs) and their functions. It has found applications in many areas of statistical theory and practice [1], with examples in life-testing, quality control, radar, as well as signal and image processing [2]–[8]. Order statistics has made over the last decade an increasing number of appearances in the design and analysis of wireless communication systems, specifically for the performance analysis of advanced diversity techniques, adaptive transmission techniques, and multiuser scheduling techniques (see for example [9]–[22]). In these performance analysis exercises, the joint statistics of partial sums of ordered RVs are often necessary for the accurate characterization of system performance [12], [19], [23]. Note that even if the original unordered RVs are independently distributed, their ordered versions are dependent due to the inequality relations among them, which makes it challenging to such joint statistics. Recently, a successive conditioning approach was used to convert dependent ordered random variables into independent unordered ones [10], [11]. However, this approach requires some case-specific manipulations, which may not always be generalizable.

Motivated by these facts, we introduced in [24] a unified analytical framework to determine the joint statistics of partial sums of ordered independent and identically distributed (i.i.d.) RVs by extending the interesting results published in [4], [25], [26]. More specifically, our approach can be applied not only to the cases when all the \( N \) ordered RVs are involved but also to the cases when only the \( N_x \) \( (N_x < N) \) best RVs are considered. With the proposed approach, we can systematically derive the joint statistics of any partial sums of ordered statistics, in terms of the moment generating function (MGF) and the probability density function (PDF). These statistical results can be used for the performance analysis of various wireless communication systems over generalized fading channels [9]. However, the identical fading assumption on all diversity branches is not always valid in real-life applications. The average fading power may vary from one path to the other because the branches of a diversity system are sometimes unbalanced and the communication system is sometimes operating over frequency-selective channels with a non-uniform power delay profile or channel multipath intensity profile (i.e., the average SNR of the diversity paths are not necessary the same).

We therefore introduce in this paper an unified analytical framework to determine the joint statistics of partial sums of ordered independent non-identically distributed (i.n.d.) RVs by extending our previous work for i.i.d. fading scenarios [24]. More specifically, we use an MGF based systematic analytical approach to derive the joint statistics of any partial sums of ordered statistics for general i.n.d. fading, in terms of MGF and the PDF. We would like to emphasize that such generalization. The main challenge for generalizing the work in [24] to i.n.d. general fading cases is that joint PDF of ordered i.n.d. RVs is much more complicated than that of ordered
i.i.d. RVs. We need to carry out more detailed manipulation and introduce new mathematical representation to obtain the generic results (e.g., joint MGF and related joint PDF) for i.n.d. general cases in a compact form. In addition, we present the closed-form expressions for the exponential RV special case, which is most widely used in wireless literature. For other type of RVs, our approach will lead to much simpler results than the conventional approach involving multiple-fold integration. Furthermore, the exponential distribution is frequently used in the performance evaluation analysis of networks and telecommunication systems. It is also used to model the waiting times between occurrences of rare events, lifetimes of electrical or mechanical devices [2], [3], [27], [28]. Finally, as an application of our analytical framework, we generalize the performance results of GSC-based RAKE receivers in [23] by maintaining the assumption of independence among the diversity paths but relaxing the identically distributed assumption. We also discussed a couple of other sample applications of the generic results presented in this work. Due to space limitations, many proofs are omitted from this paper but are included in the companion technical report available on-line [29].

II. PROBLEM STATEMENT AND MAIN IDEA

Order statistics deals with the distributions and statistical properties of the new random variables obtained after ordering the realizations of some random variables. Let \( \{\gamma_i\} \) denote \( N \) i.n.d. nonnegative random variables with PDF \( p_i(\cdot) \) and CDF \( F_i(\cdot) \). Let \( u_i \) denote the random variable corresponding to the \( i \)-th largest observation of the \( N \) original random variables (also called \( i \)-th order statistics), such that \( u_1 \geq u_2 \geq \cdots \geq u_N \). The \( N \)-dimensional joint PDF of the ordered RVs \( \{u_i\}_{i=1}^N \) is given by [1]

\[
(1) \quad (u_1, u_2, \ldots, u_N) = \frac{1}{i_1! i_2! \cdots i_N!} \sum_{i_1, i_2, \ldots, i_N \leq N} p_{i_1}(u_{i_1}) p_{i_2}(u_{i_2}) \cdots p_{i_N}(u_{i_N}).
\]

Similarly, the \( N_s \)-dimensional joint PDF of \( \{u_i\}_{i=1}^{N_s} \) is given by [1]

\[
(2) \quad g_{N_s}(u_1, u_2, \ldots, u_{N_s}) = \frac{1}{i_1! i_2! \cdots i_{N_s}!} \sum_{i_1, i_2, \ldots, i_{N_s} \leq N_s} p_{i_1}(u_{i_1}) p_{i_2}(u_{i_2}) \cdots p_{i_{N_s}}(u_{i_{N_s}}) \times \prod_{j=N_s+1}^N P_j(u_N).
\]

The objective is to derive the joint PDF of partial sums involving either all \( N \) or the first \( N_s \) \((N_s < N)\) ordered RVs for the more general case in which the diversity paths are independent but not necessarily identically distributed. Similar to [24], we adopt a general two-step approach:

- **Step I:** Obtain the analytical expressions of the joint MGF of partial sums (not necessarily the partial sums of interest as will be seen later).
- **Step II:** Apply inverse Laplace transform to derive the joint PDF of partial sums (additional integration may be required to obtain the desired joint PDF).

In step I, by interchanging the order of integration, while ensuring each pair of limits is chosen to be as tight as possible, the multiple integral can be rewritten into compact equivalent representations. After obtaining the joint MGF in a compact form, we can derive joint PDF of selected partial sum through inverse Laplace transform. For most cases of our interest, the joint MGF involves basic functions, for which the inverse Laplace transform can be calculated analytically. In the worst case, we may rely on the Bromwich contour integral. In most of the case, the result involves a single one-dimensional contour integration, which can be easily and accurately evaluated numerically with the help of integral tables [30], [31] or using standard mathematical packages such as Mathematica and Matlab.

The above general steps can be directly applied when all \( N \) ordered RVs are considered and the RVs in the partial sums are continuous. When either of these conditions do not hold, we need to apply some extra steps in the analysis in order to obtain a valid joint MGF [24]. For example, when the RVs involved in one partial sum is not continuous, i.e., separated by the other RVs, we need to divide these RVs into smaller sums. For example in Fig. 1, we consider 3-dimensional joint PDF of \( \{\gamma_{k_1}, \gamma_{k_2}, \gamma_{k_3}\} \), \( \{\gamma_{k_4}, \gamma_{k_5}, \gamma_{k_6}\} \), and \( \{\gamma_{k_7}, \gamma_{k_8}, \gamma_{k_9}\} \) for \( K > 8 \). Note that the first group is not continuous. As a result, we will derive 5-dimensional joint MGF in step I, \( \{\gamma_{k_1}, \gamma_{k_2}\} \), \( \{\gamma_{k_3}, \gamma_{k_4}\} \), \( \{\gamma_{k_5}, \gamma_{k_6}\} \), \( \{\gamma_{k_7}\} \), \( \{\gamma_{k_8}\} \). After the joint PDF of the new substituted partial sums are derived with inverse Laplace transform in step II, we can transform it to a lower dimensional desired joint PDF with finite integration.

III. COMMON FUNCTIONS AND USEFUL RELATIONS

In the following sections, we present several examples to illustrate the proposed analytical framework. Our focus is on how
to obtain compact expressions of the joint MGFs for i.n.d. general fading conditions, which can be greatly simplified with the application of the following function and relations.

A. Common Functions

i) A mixture of a CDF and an MGF \( c_i(\gamma, \lambda) \):
\[
c_i(\gamma, \lambda) = \int_0^\gamma p_i(x) \exp(\lambda x) \, dx.
\] (3)

where \( p_i(x) \) denotes the PDF of the RV of interest. Note that \( c_i(\gamma, 0) = c_i(\gamma) \) is the CDF and \( c_i(\infty, \lambda) \) leads to the MGF. Here, the variable \( \gamma \) is real, while \( \lambda \) can be complex.

ii) A mixture of an exceedance distribution function (EDF) and an MGF, \( e_i(\gamma, \lambda) \):
\[
e_i(\gamma, \lambda) = \int_\gamma^\infty p_i(x) \exp(\lambda x) \, dx.
\] (4)

Note that \( e_i(\gamma, 0) = e_i(\gamma) \) is the EDF while \( e_i(0, \lambda) \) gives the MGF.

iii) An interval MGF \( \mu_i(\gamma_1, \gamma_2, \lambda) \):
\[
\mu_i(\gamma_1, \gamma_2, \lambda) = \int_{\gamma_1}^{\gamma_2} p_i(x) \exp(\lambda x) \, dx.
\] (5)

Note that \( \mu_i(0, \infty, \lambda) \) gives the MGF.

Note that the functions defined in (3), (4) and (5) are related as follows:
\[
\mu_i(\gamma_1, \gamma_2, \lambda) = c_i(\gamma_2, \lambda) - c_i(\gamma_1, \lambda) = e_i(\gamma_2, \lambda) - e_i(\gamma_1, \lambda).
\] (6)

B. Simplifying Relationship

i) Integral \( J_m \):
The integral \( J_m \), defined by
\[
J_m = \sum_{\{i_1, i_2, \ldots, i_m\} \in \mathcal{P}_m} \int_0^{u_m} p_{i_1}(u) \exp(\lambda u) \, du_i.
\] (8)

Here, the complicated summation notation used in (8) is simplified based on the following power set definition. We define index set \( I_N \) as \( I_N = \{1, 2, \ldots, N\} \). The subset of \( I_N \) with \( n \leq N \) elements is denoted by \( \mathcal{P}_n(I_N) \). The remaining index can be grouped in the set \( I_N - \mathcal{P}_n(I_N) \). Based on these definitions, a summation in (8) includes all possible subsets of the index set \( I_N \) \( \{i_1, i_2, \ldots, i_m\} \) with \( N - (m - 1) \) elements and these subsets with \( N - (m - 1) \) elements can be denoted by \( \mathcal{P}_n(I_N - \{i_1, i_2, \ldots, i_{m-1}\}) \).

ii) Integral \( J_{m'} \):
Following the similar derivation as given in ([29], Appendix II), the integral \( J_{m'} \), defined by
\[
J_{m'} = \sum_{\{i_1, i_2, \ldots, i_m\} \in \mathcal{P}_m(I_N - \{i_1, i_2, \ldots, i_{m-1}\})} \int_0^{u_m} p_{i_1}(u) \exp(\lambda u) \, du_i.
\] (10)

can be equivalently re-written in terms of the function \( e_i(\gamma, \lambda) \) as
\[
J_{m'} = \sum_{\{i_1, i_2, \ldots, i_m\} \in \mathcal{P}_m(I_N - \{i_1, i_2, \ldots, i_{m-1}\})} \prod_{i=1}^{m} e_{i_1}(u_{m+1}, \lambda).
\] (11)

iii) Integral \( J_{a,b} \):
The integral \( J_{a,b} \), defined as
\[
J_{a,b} = \sum_{\{i_1, i_2, \ldots, i_m\} \in \mathcal{P}_m(I_{N-b})} \int_0^{u_b} p_{i_1}(u) \exp(\lambda u) \, du_i.
\] (12)
can be re-written in terms of the function $\mu(\cdot, \cdot)$ in ([29], Appendix III) as

$$J'_{a b} = \sum_{\{i_1, \ldots, i_n-1\} \in P_{n-1} (J_0 - \{i_1, \ldots, i_n-1\})} \prod_{i=n+1}^{b-1} \mu_{i} (u_k, u_a, \lambda) \quad \text{for } b > a. \quad (13)$$

IV. SAMPLE CASES WHEN ALL $N$ ORDERED RVS ARE CONSIDERED

Theorem 4.1 (Joint PDF of $Z_1 = \sum_{n=1}^{m} u_n$ and $Z_2 = \sum_{n=m+1}^{N} u_n$): The 2-dimensional joint PDF of $Z = [Z_1, Z_2]$ can be derived as

$$p_Z(z_1, z_2) = \sum_{i=m+1}^{N} \int_{0}^{\infty} du_{m} p_{i_m}(u_m) \sum_{\{i_1, \ldots, i_{n-1}\} \in P_{n-1} (J_0 - \{i_m\})} \prod_{k=1}^{m-1} \epsilon_{i_k} (u_{m_k}, -S_1) \exp(-S_1 u_{m_k}) \times L_{S_1}^{-1} \left\{ \prod_{i=1}^{n-1} c_{i_k} (u_{m_k}, -S_2) \right\},$$

for $z_1 \geq \frac{m}{N - m} z_2. \quad (14)$

Proof: The second order MGF of $Z = [Z_1, Z_2]$ is given by the expectation

$$MGF_z(\lambda_1, \lambda_2) = \sum_{i_1, i_2, \ldots, i_N} \int_{0}^{\infty} du_{1} p_{i_1}(u_1) \exp(\lambda_1 u_1) \cdots \times \int_{0}^{\infty} du_{m} p_{i_m}(u_m) \exp(\lambda_1 u_m) \times \int_{0}^{\infty} du_{m+1} p_{i_{m+1}}(u_{m+1}) \exp(\lambda_2 u_{m+1}) \cdots \times \int_{0}^{\infty} du_{N} p_{i_N}(u_N) \exp(\lambda_2 u_N). \quad (15)$$

By applying (9) and ([24], (2)) and then (11) as shown in ([29], Appendix IV), we can obtain the second order MGF of $Z$ as (16), shown at the bottom of the page. Letting $\lambda_1 = -S_1$ and $\lambda_2 = -S_2$, we can obtain the desired 2-dimensional joint PDF of $Z_1 = \sum_{n=1}^{m} u_n$ and $Z_2 = \sum_{n=m+1}^{N} u_n$ by applying the inverse Laplace transform as

$$p_Z(z_1, z_2) = L_{S_1, S_2}^{-1} \{ MGF_Z(-S_1, -S_2) \} = \sum_{i=m+1}^{N} \int_{0}^{\infty} du_{m} p_{i_m}(u_m) \sum_{\{i_1, \ldots, i_{n-1}\} \in P_{n-1} (J_0 - \{i_m\})} \prod_{k=1}^{m-1} \epsilon_{i_k} (u_{m_k}, -S_1) \exp(-S_1 u_{m_k}) \times L_{S_1}^{-1} \left\{ \prod_{i=1}^{n-1} c_{i_k} (u_{m_k}, -S_2) \right\},$$

for $z_1 \geq \frac{m}{N - m} z_2. \quad (17)$

Theorem 4.2 (Joint PDF of $Z_1 = u_m$ and $Z_2 = \sum_{n=m+1}^{N} u_n$):

We can obtain the 2-dimensional joint PDF of $Z = [Z_1, Z_2]$ as (18), shown at the bottom of the next page.

Proof: Omitted (Please refer [29]).

V. SAMPLE CASES WHEN ONLY $N_s$ ORDERED RVS ARE CONSIDERED

Let us now consider the cases where only the best $N_s (\leq N)$ ordered RVs are involved.

$$MGF_z(\lambda_1, \lambda_2) = \sum_{i_1, i_2, \ldots, i_N} \int_{0}^{\infty} du_{1} p_{i_1}(u_1) \exp(\lambda_1 u_1) \cdots \times \sum_{\{i_1, \ldots, i_{n-1}\} \in P_{n-1} (J_0 - \{i_m\})} \prod_{k=1}^{m-1} \epsilon_{i_k} (u_{m_k}, \lambda_1) \times \sum_{\{i_{m+1}, \ldots, i_N\} \in P_{N-m} (J_N - \{i_{m+1}, \ldots, i_N\})} \prod_{k=m+1}^{N} c_{i_k} (u_{m_k}, \lambda_2). \quad (16)$$
Theorem 5.1 (PDF of $Z'$): The PDF of $Z'$ can be derived as

$$p_{Z'}(x) = p_{\sum_{n=1}^{N} u_n} \{ x \} = \int_{0}^{\frac{1}{\sqrt{2\pi}}} p_{Z}(x-z_2, z_2) dz_2 \quad \text{for } N > 2,$$

(19)

where

$$p_{Z}(z_1, z_2) = \sum_{i_{N_s} \rightarrow -}^{N} d u_{N_s} p_{i_{N_s}}(u_{N_s}) L_{S_{2}}^{-1} \{ \exp(-S_{2} u_{N_s}) \}$$

$$\times \prod_{i_{N_s}, \rightarrow +}^{i_{N_s} \rightarrow -} \prod_{u_{N_s} \rightarrow -}^{u_{N_s} \rightarrow +} P_{i_{N_s}}(u_{N_s})$$

$$\times \sum_{\{i_{1}, \ldots, i_{N_s} \} \in P_{N_s}, \{i_{N_s} \rightarrow 1, \ldots, i_{N_s} \}} e_{i_{1}}(u_{N_s} \rightarrow - S_{1}).$$

(20)

Proof: We only need to consider $u_{N_s}$ separately in this case. Let $Z_1 = \sum_{n=1}^{N} u_n$ and $Z_2 = u_{N_s}$. The target second order MGF of $Z = [Z_1, Z_2]$ is given by the expectation in

$$\text{MGF}_Z(\lambda_1, \lambda_2) = E[\exp(\lambda_1 z_1 + \lambda_2 z_2)]$$

$$= \sum_{i_{1}, \ldots, i_{N} \in \{0, 1\}} \int_{0}^{\infty} d u_{1} p_{i_{1}}(u_{1}) \exp(\lambda_1 u_{1})$$

(18)

$$\times \prod_{i_{2}, \ldots, \rightarrow -}^{i_{N_s} \rightarrow -} \prod_{u_{N_s} \rightarrow -}^{u_{N_s} \rightarrow +} P_{i_{N_s}}(u_{N_s})$$

$$\times \sum_{\{i_{1}, \ldots, i_{N_s} \} \in P_{N_s}, \{i_{N_s} \rightarrow 1, \ldots, i_{N_s} \}} e_{i_{1}}(u_{N_s} \rightarrow - S_{1}).$$

(23)

$$p_{Z}(z_1, z_2) = \sum_{i_{m} \rightarrow -}^{N} d u_{m} p_{i_{m}}(u_{m}) L_{S_{1}}^{-1} \{ \exp(-S_{1} u_{m}) \}$$

$$\times \prod_{\{i_{1}, \ldots, i_{m} \} \in P_{m-1}} \prod_{\{i_{m} \rightarrow 1, \ldots, i_{m} \}} e_{i_{1}}(u_{m} \rightarrow - S_{2}).$$

(17)

$$\text{MGF}_Z(\lambda_1, \lambda_2) = \sum_{i_{N_s} \rightarrow -}^{N} d u_{N_s} p_{i_{N_s}}(u_{N_s}) \exp(\lambda_2 u_{N_s})$$

$$\times \prod_{i_{N_s} \rightarrow +}^{i_{N_s} \rightarrow -} \prod_{u_{N_s} \rightarrow +}^{u_{N_s} \rightarrow -} P_{i_{N_s}}(u_{N_s})$$

$$\times \sum_{\{i_{1}, \ldots, i_{N_s} \} \in P_{N_s}, \{i_{N_s} \rightarrow 1, \ldots, i_{N_s} \}} \prod_{i_{1}, \ldots, i_{N_s} \rightarrow -}^{i_{1}, \ldots, \rightarrow -} e_{i_{1}}(u_{N_s} \rightarrow \lambda_1).$$

(22)
Finally, noting that $Z' = Z_1 + Z_2$, we can obtain the target PDF of $Z'$ with the following finite integration

$$p_{Z'}(x) = \int_0^x p_Z(x-z_2, z_2) \, dz_2. \tag{24}$$

**Theorem 5.2 (Joint PDF of $X = u_m$ and $Y = \sum_{n=1}^{N_s-1} u_n$ for $1 < m < N_s - 1$):** The joint PDF of $W = [X, Y]$ can be obtained as

$$p_W(x, y) = \int_0^x \int_{[m-1]x}^{y-(N_s-m)x} p_{u_m, u_m}(z_1, x - z_1 - z_4, z_4) \, dz_1 \, dz_4. \tag{25}$$

**Proof:** For the joint PDF of $u_m$ and $\sum_{n=1}^{N_s-1} u_n$, as one of the original groups is split by $u_m$, we should consider substituted groups for the split group instead of original groups as shown in Fig. 2. As a result, we will start by obtaining a four dimensional MGF.

The corresponding high dimensional joint PDF can then be used to find the desired 2-dimensional joint PDF of interest. Applying the results in [24, eq. (2)], (9), (11) and (13), we derive the joint MGF of $Z_1 = \sum_{m=1}^{N_s-1} u_m, Z_2 = u_{m-1} Z_3 = \sum_{m=1}^{N_s-1} u_m, Z_4 = u_{N_s-1}$ in ([29], Appendix V) as (26), shown at the bottom of the page. Starting from the MGF expressions given above, we apply inverse Laplace transforms to arrive at the joint PDFs in ([29], Appendix V) shown in (27) at the bottom of the page.

Note that (25) involves only finite integrations of joint PDFs. Therefore, while a generic closed-form expression is not possible, the desired joint PDF can be easily numerically evaluated with the help of integral tables [30], [31] or using standard mathematical packages, such as Mathematica or Matlab etc.

$$
MGF_{Z_1, Z_2, \ldots, Z_4}(\lambda_1, \lambda_2, \ldots, \lambda_4) = \sum_{i_1, \ldots, i_{N_s}} \int_{N_s}^{\infty} du_{N_s} n_{i_1 \ldots i_{N_s}} \exp(\lambda_4 u_{N_s}) \prod_{i_1, \ldots, i_{N_s}} P_{i_1}(u_{i_1}) \sum_{i_1, \ldots, i_{N_s}} \int_{[i_1 \ldots i_{N_s}-1]x}^{\infty} du_m n_{i_1 \ldots i_{N_s}} \exp(\lambda_2 u_m) \times \prod_{i_1, \ldots, i_{N_s}} \mu_{i_1}(u_{i_1}, u_m, \lambda_3) \times \prod_{i_1, \ldots, i_{N_s}} \nu_{i_1}(u_{i_1}, \lambda_4).
$$

$$
p_Z(z_1, z_2, z_3, z_4) = L_{S_1, S_2, S_3}^{-1} \{ \text{MGF}_Z(-S_1, -S_2, -S_3, -S_4) \} = \sum_{i_1, \ldots, i_{N_s}} \int_{[i_1 \ldots i_{N_s}-1]x}^{\infty} du_{N_s} n_{i_1 \ldots i_{N_s}} \exp(-S_4 u_{N_s}) \prod_{i_1, \ldots, i_{N_s}} P_{i_1}(u_{i_1}) \times \sum_{i_1, \ldots, i_{N_s}} \int_{[i_1 \ldots i_{N_s}-1]x}^{\infty} du_m n_{i_1 \ldots i_{N_s}} \exp(-S_2 u_m) \times \prod_{i_1, \ldots, i_{N_s}} \mu_{i_1}(u_{i_1}, u_m, -S_3) \times \prod_{i_1, \ldots, i_{N_s}} \nu_{i_1}(u_{i_1}, -S_1),
$$

for $z_4 < z_2, z_1 > (m-1)z_2$ and $(N_s-m-1)z_4 < z_3 < (N_s-m-1)z_2$. \tag{27}
Theorem 5.3 (Joint PDF of $X = \sum_{n=1}^{m} u_n$ and $Y = \sum_{n=m+1}^{N} u_n$): We can obtain the joint PDF of $W = [X, Y]$ as shown in (28) at the bottom of the page.

Proof: Omitted.

Note again that only the finite integration of high dimensional joint PDFs is involved.

VI. CLOSED-FORM EXPRESSIONS FOR EXPONENTIAL RV CASE

The above novel generic results are quite general and apply to any RVs. Now, we focus on obtaining the joint PDFs for i.n.d. exponential RV special cases in a ready-to-use form. The PDF and the CDF of the RVs are given by $p_{i}(x) = \frac{1}{\xi_{i}}\exp\left(-\frac{x}{\xi_{i}}\right)$ and $P_{i}(x) = 1 - \exp\left(-\frac{x}{\xi_{i}}\right)$ for $\gamma > 0$, respectively, where $\xi_{i}$ is the average of the $i$-th RV.

Noting that the generic joint statistics typically involves the products of functions as given in (9), (11) and (13), we first derive the close-form expression of these product for i.n.d. exponential RVs as shown in ([29], Appendix VI) [see (29)–(31), at the bottom of the page].

$$C_{l,n_{1},n_{2}} = \frac{1}{\prod_{l=n_{1}}^{n_{2}} (-\frac{1}{\xi_{l}})F'\left(\frac{1}{\xi_{l}}\right)},$$

$$F'(x) = \sum_{j_{1}=j_{0}+n_{1}}^{n_{2}-l+1} \cdots \sum_{j_{l-1}=j_{l-2}+1}^{n_{2}} \prod_{m=1}^{l} \frac{1}{\xi_{m}} + (n_{2} - n_{1} + 1)x^{n_{2}-n_{1}}.$$

(32)

(28)

$$p_{W}(x,y) = p_{\sum_{n=1}^{N} u_n, \sum_{n=m+1}^{N} u_n}(x,y) = \int_{0}^{\infty} \int_{0}^{\infty} \prod_{n=1}^{N} p_{u_{n}}(z_{n}) \prod_{n=m+1}^{N} p_{u_{n}}(z_{n}) \cdot (x - z_{2}, y - z_{4}, z_{4}) \, dz_{2} \, dz_{4}, \quad \text{for } x > \frac{m}{N_{s}} - y.$$

(33)

$$\prod_{l=n_{1}}^{n_{2}} C_{l}(x_{a}, \lambda) = \frac{1}{\prod_{l=n_{1}}^{n_{2}} (1 - \frac{1}{\xi_{l}})} \prod_{l=n_{1}}^{n_{2}} \left[ 1 - \exp \left( \left( \lambda - \frac{1}{\xi_{l}} \right) x_{a} \right) \right]$$

$$= \sum_{k=n_{2}-n_{1}}^{n_{2}} C_{k,n_{1},n_{2}} \times \left[ 1 + \left( \sum_{l=1}^{n_{2}-n_{1}+1} \exp l \cdot x_{a} \cdot \lambda \left( (-1)^{l} \sum_{j_{l}=j_{0}+n_{1}}^{n_{2}-l+1} \cdots \sum_{j_{l-1}=j_{l-2}+1}^{n_{2}} \exp \left( - \sum_{m=1}^{l} \frac{z_{a}}{\xi_{m}} \right) \right) \right) \right].$$

(29)

$$\prod_{l=n_{1}}^{n_{2}} C_{l}(x_{a}, \lambda) = \frac{1}{\prod_{l=n_{1}}^{n_{2}} (1 - \frac{1}{\xi_{l}})} \exp \left( \left( \sum_{l=n_{1}}^{n_{2}} \frac{x_{a}}{\xi_{l}} \right) \right)$$

$$= \sum_{k=n_{2}-n_{1}}^{n_{2}} C_{k,n_{1},n_{2}} \exp \left( - \sum_{l=n_{1}}^{n_{2}} \frac{x_{a}}{\xi_{l}} \right) \exp \left( (n_{2} - n_{1} + 1)x_{a} \lambda \right),$$

$$\prod_{l=n_{1}}^{n_{2}} \mu_{l}(z_{a}, z_{l}, \lambda) = \frac{1}{\prod_{l=n_{1}}^{n_{2}} (1 - \frac{1}{\xi_{l}})} \prod_{l=n_{1}}^{n_{2}} \left[ \exp \left( \left( \lambda - \frac{1}{\xi_{l}} \right) z_{a} \right) - \exp \left( \left( \lambda - \frac{1}{\xi_{l}} \right) z_{l} \right) \right]$$

$$= \sum_{k=n_{2}-n_{1}}^{n_{2}} C_{k,n_{1},n_{2}} \left[ \exp \left( (n_{2} - n_{1} + 1) \cdot x_{a} \cdot \lambda \right) \exp \left( - \sum_{l=n_{1}}^{n_{2}} \frac{x_{a}}{\xi_{l}} \right) \right]$$

$$\times \left[ 1 + \sum_{l=1}^{n_{2}-n_{1}+1} \exp l \cdot \left( \frac{z_{a}}{\xi_{l}} \right) \left( (-1)^{l} \sum_{j_{l}=j_{0}+n_{1}}^{n_{2}-l+1} \cdots \sum_{j_{l-1}=j_{l-2}+1}^{n_{2}} \exp \left( - \sum_{m=1}^{l} \frac{z_{a}}{\xi_{m}} \right) \right) \right].$$

(31)
After substituting (29), (30) and (31) into the derived expressions of the joint PDF of partial sums of ordered statistics presented in the previous sections, it is easy to derive the following closed-form expressions for the PDFs by applying the classical inverse Laplace transform pair and the property given in ([24], Appendix I). While some of these results have been derived using the successive conditioning approach previously, we list them here for the sake of convenience and completeness in (34)–(51).

1) Joint PDF of \( u_m \) and \( \sum_{n \neq m}^N u_n \) [see (34) at the bottom of the page], where [see (35)–(37) at the bottom of the page].

2) Joint PDF of \( \sum_{n=1}^m u_n \) and \( \sum_{n=m+1}^N u_n \): [see (38), at the bottom of the next page, where

\[
L_{S_2}^{-1} \left\{ \frac{1}{(-S_2 - \frac{1}{\gamma_2})} \right\} = - \exp \left( - \frac{z_2}{\gamma_2} \right), \tag{39}
\]

\[
p_Z(z_1, z_2) = L_{S_1, S_2}^{-1} \left\{ \mu_Z(-S_1, -S_2) \right\} = \sum_{i_1, \ldots, i_m=1}^{\infty} \int \frac{du_m}{\gamma_{1m}} \exp \left( - \frac{u_m}{\gamma_{1m}} \right) \left( L_{S_1}^{-1} \{ \exp(-S_1 u_m) \} \right)

\times \sum_{\{i_1, \ldots, i_m, \ldots, i_N\} \in \mathcal{P}_m} \sum_{k=1}^{m-1} \left\{ C_{k,1,m-1} \exp \left( - \sum_{l=1}^{m-1} \left( \frac{u_l}{\gamma_{1l}} \right) \right) \right\}

\times \sum_{\{i_{m+1}, \ldots, i_N\} \in \mathcal{P}_{N-m} \cap \mathcal{P}_m \cap \mathcal{P}_m \cap \mathcal{P}_m} \sum_{q=m+1}^{N} \left\{ C_{q,m+1,m} \exp \left( - \sum_{l=1}^{m} \left( \frac{u_l}{\gamma_{1l}} \right) \right) \right\}

\times \left[ \sum_{h=1}^{N-m} \left\{ (-1)^h \sum_{j_1=j_2+m+1}^{N-h} \cdots \sum_{j_N=j_{N-1}+1}^{N} \exp \left( - \sum_{l=1}^{h} \frac{u_{j_l}}{\gamma_{1l}} \right) \right\} \right]

\times \left( L_{S_2}^{-1} \left\{ \frac{\exp(-(m-1)u_m S_2)}{(-S_2 - \frac{1}{\gamma_2})} \right\} \right)

\times \left( L_{S_2}^{-1} \left\{ \frac{\exp(-(h+m-1)u_m S_2)}{(-S_2 - \frac{1}{\gamma_2})} \right\} \right)

\times \left( L_{S_2}^{-1} \left\{ \frac{\exp(-(h+m-1)u_m S_2)}{(-S_2 - \frac{1}{\gamma_2})} \right\} \right)

\times \left( L_{S_2}^{-1} \left\{ \frac{\exp(-(h+m-1)u_m S_2)}{(-S_2 - \frac{1}{\gamma_2})} \right\} \right), \tag{34}
\]

\[
L_{S_1}^{-1} \{ \exp(-S_1 u_m) \} = \delta(z_1 - u_m), \tag{35}
\]

\[
L_{S_2}^{-1} \left\{ \frac{\exp(-(m-1)u_m S_2)}{(-S_2 - \frac{1}{\gamma_2})} \right\} = \frac{\exp(-z_2 - (m-1)u_m)}{\left( \frac{1}{\gamma_2} + \frac{1}{\gamma_k} \right)} \exp \left( \frac{z_2}{\gamma_k} \right) \exp \left( \frac{(m-1)u_m}{\gamma_k} \right) \right\} U(z_2 - (m-1)u_m), \tag{36}
\]

\[
L_{S_2}^{-1} \left\{ \frac{\exp(-(h+m-1)u_m S_2)}{(-S_2 - \frac{1}{\gamma_2})} \right\} = \frac{\exp(-z_2 - (h+m-1)u_m)}{\left( \frac{1}{\gamma_2} + \frac{1}{\gamma_k} \right)} \exp \left( \frac{z_2}{\gamma_k} \right) \exp \left( \frac{(h+m-1)u_m}{\gamma_k} \right) \right\} U(z_2 - (h+m-1)u_m), \tag{37}
\]
\[ L_{Z_1} \left\{ \frac{\exp(-m/N)S_1}{-S_1 - \frac{1}{\tau_i}} \right\} = -\exp \left( \frac{z_1 - mu_m}{\tau_i} \right) U(z_1 - mu_m), \quad (40) \]
\[ L_{Z_1} \left\{ \frac{\exp(-N/N)S_1}{-S_1 - \frac{1}{\tau_i}} \right\} = -\exp \left( \frac{z_1 - (N - 1)u_{N_i}}{\tau_i} \right) U(z_1 - (N - 1)u_{N_i}). \quad (44) \]

3) PDF of \( \sum_{n=1}^{N} u_n \) [see (42), shown at the bottom of the page], where
\[ L_{Z_1} \left\{ \exp(-S_2 u_{N_i}) \right\} = \delta(z_2 - u_{N_i}), \quad (43) \]
\[ L_{Z_1} \left\{ \exp(-w_m S_2) \right\} = \delta(z_2 - u_m), \quad (47) \]

4) Joint PDF of \( u_m \) and \( \sum_{n=1}^{N} u_n \) for \( 1 < m < N - 1 \) [see (45), shown at the bottom of the next page], where
\[ L_{Z_1} \left\{ \exp(-N/N)S_4 \right\} = \delta(z_4 - u_{N_i}), \quad (46) \]

\[
p_Z(z_1, z_2) = L_{Z_1} \left\{ \mu Z(-S_1, -S_2) \right\} = \sum_{i=1}^{N} \int_{0}^{\infty} du_m \frac{1}{\tau_i} \exp \left( \frac{u_m}{\tau_i} \right) \sum_{\{i_1, \ldots, i_{m-1}\} \in P_{m-1}(I_N \setminus \{i_m\})} \sum_{k=1}^{m-1} C_{k,1,m-1} \times \exp \left( \frac{-u_m}{\tau_i} \right) L_{Z_1} \left\{ \frac{\exp(-m/N)S_1}{-S_1 - \frac{1}{\tau_i}} \right\} \sum_{\{i_{m+1}, \ldots, i_N\} \in P_{N-m}(I_N \setminus \{i_m\})} \sum_{k=1}^{m-1} C_{q,m+1,N} \times \exp \left( \frac{-u_m}{\tau_i} \right) L_{Z_1} \left\{ \frac{\exp(-N/N)S_4}{-S_1 - \frac{1}{\tau_i}} \right\} L_{Z_2} \left\{ \frac{\exp(-N/N)S_4}{-S_1 - \frac{1}{\tau_i}} \right\} \times \sum_{h=1}^{N-m} \left\{ (-1)^h \sum_{j_1 = j_0 + m + 1}^{N-h+1} \cdots \sum_{j_h = j_0 + h + 1}^{N} \exp \left( -\sum_{m=1}^{h} \frac{u_m}{\tau_i} \right) \right\} L_{Z_2} \left\{ \frac{\exp(-N/N)S_4}{-S_1 - \frac{1}{\tau_i}} \right\}, \quad (38) \]

\[
p_Z(z_1, z_2) = L_{Z_1} \left\{ \mu Z(-S_1, -S_2) \right\} = \sum_{i=1}^{N} \int_{0}^{\infty} du_N \frac{1}{\tau_i} \exp \left( \frac{-u_N}{\tau_i} \right) L_{Z_1} \left\{ \exp(-S_2 u_{N_i}) \right\} \times \sum_{i=1}^{N} \prod_{k=1}^{N_{i-1}} \prod_{j=1}^{N_{i} - 1} \frac{1}{\tau_i} \sum_{\{i_1, \ldots, i_{N_{i-1}}\} \in P_{N_{i-1}}(I_N \setminus \{i_1, \ldots, i_{N_{i-1}}\})} \prod_{q=1}^{N_{i} - 1} C_{q,1,N_{i} - 1} \times \exp \left( \frac{-u_N}{\tau_i} \right) L_{Z_1} \left\{ \frac{\exp(-N/N)S_4}{-S_1 - \frac{1}{\tau_i}} \right\} L_{Z_2} \left\{ \frac{\exp(-N/N)S_4}{-S_1 - \frac{1}{\tau_i}} \right\} \times \sum_{h=1}^{N_{i-1}} \left\{ (-1)^h \sum_{j_1 = j_0 + m + 1}^{N-h+1} \cdots \sum_{j_h = j_0 + h + 1}^{N} \exp \left( -\sum_{m=1}^{h} \frac{u_m}{\tau_i} \right) \right\} L_{Z_2} \left\{ \frac{\exp(-N/N)S_4}{-S_1 - \frac{1}{\tau_i}} \right\} \times \sum_{h=1}^{N_{i-1} - 1} \left\{ (-1)^h \sum_{j_1 = j_0 + m + 1}^{N-h+1} \cdots \sum_{j_h = j_0 + h + 1}^{N} \exp \left( -\sum_{m=1}^{h} \frac{u_m}{\tau_i} \right) \right\} L_{Z_2} \left\{ \frac{\exp(-N/N)S_4}{-S_1 - \frac{1}{\tau_i}} \right\} \times \sum_{h=1}^{N_{i-1} - 2} \left\{ (-1)^h \sum_{j_1 = j_0 + m + 1}^{N-h+1} \cdots \sum_{j_h = j_0 + h + 1}^{N} \exp \left( -\sum_{m=1}^{h} \frac{u_m}{\tau_i} \right) \right\} L_{Z_2} \left\{ \frac{\exp(-N/N)S_4}{-S_1 - \frac{1}{\tau_i}} \right\}, \quad (42) \]
Recently, we presented the exact performance analyses of the capture probability on GSC RAKE receivers in [23]. For analytical simplification, the fading was assumed both independent and identically distributed from path to path. However, the average SNR of each path (or branch) is different for most practical channel models, especially for wide-band SS signals since the average fading power may vary from one path to the other. For example, experimental measurements indicate that the radio channel is characterized by an exponentially decaying multipath intensity profile (MIP) for indoor office buildings [32] as well as urban [33] and suburban areas [34]. Based on this motivation in mind, with the help of our derived results in Section V, we can extend our previous result (a closed-form formula of the capture probability on GSC RAKE receivers) by maintaining the assumption of independence among the diversity paths but relaxing the identically distributed assumption.

The capture probability was defined as the probability that the power ratio of the combined paths to that of all available diversity paths exceeds a certain threshold. Let \( u_i \) be the order statistics obtained by arranging \( N \) \((N \geq 2)\) nonnegative i.n.d. RVs, \( \{\gamma_i\} \), in decreasing order of magnitude such that \( u_1 \geq u_2 \geq \cdots \geq u_N \). Based on the system model and definition in [23], the capture probability can be written as

\[
\text{Prob}_{GSC-cap} = \Pr \left[ \frac{\sum_{n=1}^{N} u_n}{\sum_{n=1}^{N} u_n} > T \right],
\]
where \(0 < T < 1\) and \(m < N\). If we define \(Z = [Z_1, Z_2]\), with \(Z_1 = \sum_{n=1}^{m} u_n\) and \(Z_2 = \sum_{n=m+1}^{N} u_n\), then (52) can be calculated in terms of the 2-dimensional joint PDF of \(Z_1\) and \(Z_2\), \(p_{Z(Z_1, Z_2)}\), as

\[
\text{Prob}_{\text{GSC-capture}} = \Pr \left[ \frac{Z_1}{Z_1 + Z_2} > T \right]
\]

\[
= \int_0^\infty \int_0^\infty \left( \frac{1}{Z_1 + Z_2} \right) p_{Z(Z_1, Z_2)}\, dZ_2 \, dZ_1. \tag{53}
\]

After substituting (38) into (53) and some tedious manipulations, the closed-form expression of (53) and related closed-form expression of each integral form in the expression presented in (55), at the bottom of the page, for i.n.d. Rayleigh fading conditions are shown at the end of this paper (refer to ([29], Appendix VII) for details).

### A. Finger Replacement Schemes for RAKE Receivers in the Soft Handover Region Over i.n.d. Fading Channels

Recently, new finger replacement techniques for RAKE reception in the soft handover (SHO) region [35] has been proposed and analyzed over independent and identical fading (i.i.d.) channel. The proposed schemes are basically based on the block comparison among groups of resolvable paths from different base stations and lead to the reduction of complexity while offering commensurate performance. If we let \(Y = L_n - L_i, W_1 = \sum_{i=1}^{L_n} u_i\) and \(W_n = \sum_{i=1}^{L_n} u_i^n\) (for \(n = 2, \ldots, N\)), where \(u_i (i = 1, 2, \ldots, I_i)\) and \(u_i^n (i = 1, 2, \ldots, I_i)\) are the order statistics obtained by arranging \(L_n\) nonnegative i.n.d. path SNRs corresponding to the \(n\)th base station \((2 \leq n \leq N)\) in descending order, then the RAKE combiner output SNR with GSC is given by \(Y + \max_n W_n\). Y and \(W_1\) are dependent but \(Y\) and \(W_n\) are independent. In

\[
\text{Prob}_{\text{GSC-capture}} = \sum_{i_m=1}^{N} \frac{1}{\Pi_{i_m-1 \in (I_m - (i_m - 1))}} \sum_{i_{m+1} \in (I_{m+1})} \sum_{h=1}^{m-1} C_{k,1,m-1}
\]

\[
\times \sum_{j_{m+1} \in (I_{m+1})} \sum_{j_m=1}^{N} \sum_{q=m+1}^{N} C_{q,m+1,N}
\]

\[
\times \left[ \left( \sum_{i=1}^{m} \left( \frac{1}{\eta_{i_k}} - \frac{m - h}{\eta_{k}} \right) \right) \int_0^\infty \int_0^\infty \left( \frac{1}{z_1 + z_2} \right) \exp \left( - \frac{z_1}{\eta_{i_k}} \right) \exp \left( - \frac{z_2}{\eta_{i_{k-1}}} \right) \, dZ_2 \, dZ_1 \right]
\]

\[
- \int_0^\infty \int_0^\infty \left( \frac{1}{z_1 + z_2} \right) \exp \left( - \frac{z_1}{\eta_{i_k}} \right) \exp \left( - \sum_{i=1}^{m} \left( \frac{1}{\eta_{i_k}} \right) \right) \frac{z_1}{m} \, dZ_2 \, dZ_1 \right]
\]

\[
+ \sum_{h=1}^{N-m} \left( -1 \right)^h \left[ \int_0^\infty \int_0^\infty \left( \frac{1}{z_1 + z_2} \right) \exp \left( - \frac{z_1}{\eta_{i_{k-1}}} \right) \exp \left( - \sum_{i=1}^{h} \left( \frac{1}{\eta_{i_k}} \right) \right) \frac{z_1}{h} \, dZ_2 \, dZ_1 \right]
\]

\[
- \int_0^\infty \int_0^\infty \left( \frac{1}{z_1 + z_2} \right) \exp \left( - \frac{z_1}{\eta_{i_{k-1}}} \right) \exp \left( - \sum_{i=1}^{h} \left( \frac{1}{\eta_{i_k}} \right) \right) \frac{z_1}{h} \, dZ_2 \, dZ_1 \right]
\]

\[
+ \sum_{h=1}^{N-m} \left( -1 \right)^h \left[ \int_0^\infty \int_0^\infty \left( \frac{1}{z_1 + z_2} \right) \exp \left( - \frac{z_1}{\eta_{i_{k-1}}} \right) \exp \left( - \sum_{i=1}^{h} \left( \frac{1}{\eta_{i_k}} \right) \right) \frac{z_1}{h} \, dZ_2 \, dZ_1 \right]
\]

\[
- \int_0^\infty \int_0^\infty \left( \frac{1}{z_1 + z_2} \right) \exp \left( - \frac{z_1}{\eta_{i_{k-1}}} \right) \exp \left( - \sum_{i=1}^{h} \left( \frac{1}{\eta_{i_k}} \right) \right) \frac{z_1}{h} \, dZ_2 \, dZ_1 \right], \tag{55}
\]
practice, the i.i.d. fading assumption on the diversity paths is not always realistic due to, for example, the different adjacent multipath routes with the same path loss. Although non-identical fading is important for practical implementation, [35] have only investigated the non-uniform power delay profile case only through computer simulation due to the high analysis complexity. The major difficulty in this problem is to derive the joint statistics of ordered exponential variates over non-identical fading assumptions, which can be obtained by applying Theorem 5.1 and 5.3 of Section V. Due to space limitation, the analytical details are omitted in this work.

B. Outage Probability of GSC RAKE Receivers Over i.n.d. Rayleigh Fading Channel Subject to Self-Interference

Recently, the outage probability of GSC RAKE receivers subject to self-interference over independent and identically distributed Rayleigh fading channels has been investigated in [23]. Let \( \gamma_i \) be the SNR of the \( i \)-th diversity path and \( u_i \) \( (i = 1, 2, \ldots, N) \) be the order statistics obtained by arranging \( N \) \( (N > 2) \) nonnegative i.n.d. RVs, \( \{\gamma_i\}_{i=1}^{N} \), in decreasing order of magnitude such that \( u_1 \geq u_2 \geq \cdots \geq u_N \). Then, the outage probability, denoted by \( P_{\text{Out}} \), is then defined as [23],

\[
P_{\text{Out}} = \Pr \left[ \frac{\sum_{n=1}^{N} u_n}{1 + \alpha \sum_{n=m+1}^{N} u_n} < T \right].
\] (54)

where \( T \) \( (0 \leq T) \) is the outage threshold and \( \alpha \) \( (0 \leq \alpha \leq 1) \) is the self-interference cancellation coefficient (in practice, each path may have the different value of \( \alpha \)). The closed-form expression for this outage probability over i.i.d. Rayleigh fading channels has been derived and compared to that of partial RAKE receivers. However, the average signal-to-noise ratio (SNR) of each path (or branch) is different for most practical channel models, especially for wide-band spread spectrum signals. As results, to evaluate the outage probability over i.n.d. fading channel subject to self-interference, the major difficulty is to derive the joint PDF of \( \sum_{n=1}^{m} u_n \) and \( \sum_{n=m+1}^{N} u_n \) for i.n.d. case. Fortunately, the target joint PDF can be obtained with the help of Theorem 4.1 in Section IV.

i) The first integral:

\[
\int_0^\infty \int_0^\infty \left( \frac{1}{\pi} \right)^{\frac{Z_1^2}{\pi_{\gamma_k}}} \left( \frac{1}{\pi} \right)^{\frac{Z_2^2}{\pi_{\gamma_k}}} \exp \left( -\frac{Z_1}{\pi_{\gamma_k}} \right) \exp \left( -\frac{Z_2}{\pi_{\gamma_k}} \right) dZ_2 dZ_1
\]

\[
= \frac{h}{\left( \sum_{m=1}^{h} \left( \frac{1}{\pi_{\gamma_k}} \right) + \sum_{l=1}^{m} \left( \frac{1}{\pi_{\gamma_k}} \right) - \frac{m}{\pi_{\gamma_k}} \right) \left( \sum_{m=1}^{h} \left( \frac{1}{\pi_{\gamma_k}} \right) + \sum_{l=1}^{m} \left( \frac{1}{\pi_{\gamma_k}} \right) - \frac{m}{\pi_{\gamma_k}} \right) \left( \sum_{m=1}^{h} \left( \frac{1}{\pi_{\gamma_k}} \right) + \sum_{l=1}^{m} \left( \frac{1}{\pi_{\gamma_k}} \right) - \frac{m}{\pi_{\gamma_k}} \right) \left( \sum_{m=1}^{h} \left( \frac{1}{\pi_{\gamma_k}} \right) + \sum_{l=1}^{m} \left( \frac{1}{\pi_{\gamma_k}} \right) - \frac{m}{\pi_{\gamma_k}} \right)}

\left[ 1 - U \left( \frac{1}{m} - \frac{1 - T}{\pi_{\gamma_k} \cdot h} \right) \right].
\] (59)

ii) The second integral:

\[
\int_0^\infty \int_0^\infty \left( \frac{1}{\pi} \right)^{\frac{Z_1^2}{\pi_{\gamma_k}}} \left( \frac{1}{\pi} \right)^{\frac{Z_2^2}{\pi_{\gamma_k}}} \exp \left( -\frac{Z_1}{\pi_{\gamma_k}} \right) \exp \left( -\frac{Z_2}{\pi_{\gamma_k}} \right) \left( \sum_{m=1}^{h} \left( \frac{1}{\pi_{\gamma_k}} \right) + \sum_{l=1}^{m} \left( \frac{1}{\pi_{\gamma_k}} \right) - \frac{m}{\pi_{\gamma_k}} \right) \left( \sum_{m=1}^{h} \left( \frac{1}{\pi_{\gamma_k}} \right) + \sum_{l=1}^{m} \left( \frac{1}{\pi_{\gamma_k}} \right) - \frac{m}{\pi_{\gamma_k}} \right) \left( \sum_{m=1}^{h} \left( \frac{1}{\pi_{\gamma_k}} \right) + \sum_{l=1}^{m} \left( \frac{1}{\pi_{\gamma_k}} \right) - \frac{m}{\pi_{\gamma_k}} \right) \left( \sum_{m=1}^{h} \left( \frac{1}{\pi_{\gamma_k}} \right) + \sum_{l=1}^{m} \left( \frac{1}{\pi_{\gamma_k}} \right) - \frac{m}{\pi_{\gamma_k}} \right)}

\left[ 1 - U \left( \frac{1}{m} - \frac{1 - T}{\pi_{\gamma_k} \cdot h} \right) \right] dZ_2 dZ_1
\]

\[
= \frac{h}{\left( \sum_{m=1}^{h} \left( \frac{1}{\pi_{\gamma_k}} \right) + \sum_{l=1}^{m} \left( \frac{1}{\pi_{\gamma_k}} \right) - \frac{m}{\pi_{\gamma_k}} \right) \left( \sum_{m=1}^{h} \left( \frac{1}{\pi_{\gamma_k}} \right) + \sum_{l=1}^{m} \left( \frac{1}{\pi_{\gamma_k}} \right) - \frac{m}{\pi_{\gamma_k}} \right) \left( \sum_{m=1}^{h} \left( \frac{1}{\pi_{\gamma_k}} \right) + \sum_{l=1}^{m} \left( \frac{1}{\pi_{\gamma_k}} \right) - \frac{m}{\pi_{\gamma_k}} \right) \left( \sum_{m=1}^{h} \left( \frac{1}{\pi_{\gamma_k}} \right) + \sum_{l=1}^{m} \left( \frac{1}{\pi_{\gamma_k}} \right) - \frac{m}{\pi_{\gamma_k}} \right)}

\left[ 1 - U \left( \frac{1}{m} - \frac{1 - T}{\pi_{\gamma_k} \cdot h} \right) \right].
\] (61)
ii) The third integral:
\[
\int_0^\infty \int_0^\infty \left( \frac{z_1}{z_1} \right) \exp \left( -\frac{z_1}{\gamma_{1k}} \right) \times \exp \left( -\frac{z_2}{\gamma_{2k}} \right) U \left( \frac{z_1}{m} - \frac{z_2}{h} \right) \, dz_2 \, dz_1
\]
\[
\times \left[ -\frac{\gamma_{1k}}{h} \left( \frac{1 - T}{T \cdot h} \right) - \frac{\gamma_{2k}}{h} \left( \frac{1 - T}{T \cdot h} \right) \right] . \tag{58}
\]

iv) The forth integral: [see (59), shown at the bottom of the previous page.]

v) The fifth integral:
\[
\int_0^\infty \int_0^\infty \left( \frac{z_1}{z_1} \right) \exp \left( -\frac{z_1}{\gamma_{1k}} \right) \exp \left( -\frac{z_2}{\gamma_{2k}} \right) \times \left[ 1 - U \left( \frac{z_1}{m} - \frac{z_2}{h} \right) \right] \, dz_2 \, dz_1
\]
\[
= \frac{\gamma_{1k}}{h} \left( \frac{1 - T}{T \cdot h} \right) - \frac{\gamma_{2k}}{h} \left( \frac{1 - T}{T \cdot h} \right) . \tag{60}
\]

vi) The sixth integral: [see (61), shown at the bottom of the previous page.]

VIII. CONCLUSION

In this paper, we presented an analytical framework to derive the joint statistics of partial sums of ordered i.n.d. random variables. We carried out a nontrivial generalization to the previous work on i.i.d. random variables. The resulting joint statistics can find applications in many research problems in signal processing and wireless communications. As an illustration, we analyze the capture probability of the GSC-based RAKE receivers. We also discussed a couple of other sample applications of the generic results.

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