On instability of Rayleigh–Taylor problem for incompressible liquid crystals under $L^1$-norm

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Abstract

We investigate the nonlinear Rayleigh–Taylor (RT) instability of a nonhomogeneous incompressible nematic liquid crystal in the presence of a uniform gravitational field. We first analyze the linearized equations around the steady state solution. Thus we construct solutions of the linearized problem that grow in time in the Sobolev space $H^4$, then we show that the RT equilibrium state is linearly unstable. With the help of the established unstable solutions of the linearized problem and error estimates between the linear and nonlinear solutions, we establish the nonlinear instability of the density, the horizontal and vertical velocities under $L^1$-norm.

Keywords: Rayleigh–Taylor instability; Liquid crystals; Steady state solution

1 Introduction

The instability arises when steady states of two fluid layers with different densities are accelerated in the direction toward the denser fluid [1]. This phenomenon was first studied by Rayleigh [2] and then Taylor [3], thus is called the Rayleigh–Taylor (RT) instability. In the last decades, this phenomenon has been extensively investigated from both physical and numerical aspects, see [4, 5] for examples. It has been also widely investigated how the RT instability evolves under the effects of other physical factors, such as elasticity [6–8], rotation [9], internal surface tension [10–12], magnetic fields [13–17], and so on. In particular, to the best of our knowledge, the linear Rayleigh–Taylor instability is well understood, see [5, 9], for instance; however, there are only few mathematical analysis results on nonlinear problems in the literature.

In this paper, we further mathematically prove the RT instability in incompressible liquid crystal materials in the presence of a uniform gravitational field in a bounded domain.

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\[
\Omega \subset \mathbb{R}^3:
\begin{align*}
\rho_t + u \cdot \nabla \rho = 0, \\
\rho u_t + \rho u \cdot \nabla u - \mu \Delta u + \nabla p &= -\Delta \cdot \nabla d - g \rho e_3, \\
d_t + u \cdot \nabla d &= \Delta d + |\nabla d|^2 d, \\
\text{div } u &= 0, \\
u(0, x) &= u^0(x), \quad d(0, x) = d^0(x), \\
(u, d)|_{\partial \Omega} &= (0, e_3).
\end{align*}
\tag{1.1}
\]

Here the unknown function \( \rho \) is the density of the nematic liquid crystals, \( u \) the velocity, and \( p \) the pressure, \( d \) represents the macroscopic average of the nematic liquid crystal orientation field. Also \( \mu > 0 \) is the coefficient of viscosity, \( g > 0 \) is the gravitational constant, \( e_3 = (0, 0, 1)^T \) is the vertical unit vector, and \( -\rho ge_3 \) is the gravitational force.

In this paper we study the Rayleigh–Taylor (RT) instability of system (1.1). To this purpose, we consider a density profile \( \bar{\rho} := \bar{\rho}(x_3) \in C^0(\bar{\Omega}) \), which satisfies
\[
\inf_{x \in \Omega} \| \bar{\rho} \| > 0, \tag{1.2}
\]
and an RT condition
\[
\bar{\rho}'(x^0_3) > 0 \quad \text{for some } x^0 \in \Omega, \tag{1.3}
\]
where \( x^0_3 \) denotes the third component of \( x^0 \). Then we further define a pressure \( \bar{p} \) (unique up to a constant) by the relation
\[
\nabla \bar{p} = -ge_3.
\]
The condition (1.3) means that there is a region in which the RT density profile has a larger density with increasing \( x_3 \) (height), thus leading to RT instability.

Obviously, \( R_C := (\bar{\rho}, 0, \bar{\rho}, e_3) \) is an RT equilibrium-state solution of the system (1.1). Now, we denote the perturbation by
\[
\begin{align*}
\varrho &= \rho - \bar{\rho}, \quad u = u - 0, \quad q = p - \bar{\rho}, \quad \sigma = d - e_3,
\end{align*}
\]
then, \( (\varrho, u, q, \sigma) \) satisfies the following perturbation equations:
\[
\begin{align*}
\varrho_t + u \cdot \nabla (\varrho + \bar{\rho}) &= 0, \\
(\varrho + \bar{\rho})u_t + (\varrho + \bar{\rho}) u \cdot \nabla u + \nabla q - \mu \Delta u &= -\Delta \sigma \cdot \nabla \sigma - g \rho e_3, \\
\sigma_t + u \cdot \nabla \sigma &= \Delta \sigma + |\nabla \sigma|^2 \sigma + |\nabla \sigma|^2 e_3, \\
\text{div } u &= 0,
\end{align*}
\tag{1.4}
\]
with the initial-boundary value conditions:
\[
(\varrho, u, q)|_{t=0} = (\varrho^0, u^0, q^0), \quad (u, \sigma)|_{\partial \Omega} = (0, 0), \tag{1.5}
\]
where $\Omega$ is a general bounded domain. In this article, the initial-boundary value problem (1.4)–(1.5) is called the LCRT problem.

If the perturbation is small, we omit the nonlinear terms, and thus get the linearized LCRT equations

$$
\begin{cases}
\rho_t + \bar{\rho} u_3 = 0, \\
\bar{\rho} u_t + \nabla q + g \bar{\rho} e_3 = \mu \Delta u, \\
\sigma_t = \Delta \sigma, \\
\text{div} \, u = 0.
\end{cases}
$$

The linearized equations (1.6) and the initial-boundary values (1.5) constitute the linearized LCRT problem. It is well-known that the linearized LCRT problem is convenient in mathematical analysis in order to have an insight into the physical and mathematical mechanisms of the instability.

1.1 Main results

Before stating our main result, we shall introduce some mathematical notations of Sobolev spaces:

$$\begin{align*}
\mathcal{L}^p &:= \int_{\Omega}, \quad L^p := L^p(\Omega) = W^{0,p}(\Omega), \quad \| \cdot \|_k := \| \cdot \|_{H^k}, \quad I_T := (0,T), \\
H^1_0 &:= \{ \eta \in H^1(\Omega) | \eta|_{\partial\Omega} = 0 \}, \quad H^1 := \{ \eta \in H^1(\Omega) | \text{div} \, \eta = 0 \}, \\
\| (u,q,\sigma) \|_{S,k} &:= \sqrt{\| (u,\sigma) \|_{H^{k+2}}^2 + \| q \|_{L^{k+1}}^2}, \quad H^1 :\{ \eta \in H^1 | \int_{\Omega} \eta \, dy = 0 \}, \\
\mathcal{E}(t) &:= \sum_{i=0}^2 \| \partial_t^i q \|_{L^2}^2 + \sum_{i=0}^1 \| \partial_t^i (u,q,\sigma) \|_{S,2-2i}^2 + \| (u_t,\sigma_{tt}) \|_{L^2}^2, \\
\mathcal{D}(t) &:= \sum_{i=0}^1 \| \partial_t^i (u,q,\sigma) \|_{S,3-3i}^2 + \| (u_t,\sigma_{tt}) \|_{L^2}^2.
\end{align*}$$

$a \lesssim b$ means that $a \leq cb$ for some positive constant $c$,

where $1 < p \leq \infty$, $k$ is a nonnegative integer, and the positive constant $c$ may depend on the domain occupied by the fluids and other known physical parameters such as $g$, $\sigma$, $\alpha$, and $\mu$, and vary from line to line.

Next we state the instability result in the LCRT problem

**Theorem 1** Let $\Omega$ be a $C^5$-bounded domain, and let the density profile $\bar{\rho} \in C^5(\Omega)$ satisfy (1.2)–(1.3). Then, the equilibrium $(\bar{\rho},0,\bar{q},e_3)$ to LCRT problem (1.4)–(1.5) is unstable in Hadamard sense, that is, there are positive constants $\Lambda, \delta_0, \epsilon_0, \delta_0$, and

$$\begin{align*}
(\bar{\rho}^0, \bar{u}^0, \bar{q}^0, u^0, q^0) \in H^4(\Omega) \times H^4_\epsilon(\Omega) \times H^3(\Omega) \times H^2(\Omega),
\end{align*}$$

such that, for any given $\delta \in (0, \delta_0)$, there is a unique classical solution $(\rho,u,\sigma) \in C^0(I_T, H^4(\Omega) \times H^4_\epsilon(\Omega) \times H^2(\Omega))$, with a unique associated (perturbation) pressure $q \in C^0(I_T, H^3)$,
to the LCRT problem with the initial data
\[
(\varrho^0, u^0, q^0) := \delta(\tilde{\varrho}^0, \tilde{u}^0, \tilde{q}^0) + \delta^2(0, u', q'),
\]
but the solution satisfies
\[
\left\| \varrho(T^*) \right\|_{L^1(\Omega)} + \left\| u_h(T^*) \right\|_{L^1(\Omega)} + \left\| u_3(T^*) \right\|_{L^1(\Omega)} \geq \varepsilon,
\]
for some escape time \( T^* := \frac{1}{\Lambda} \ln \frac{2\varepsilon}{m_0 \delta} \in I_T \). In addition, the initial data \( \varrho^0, u^0, q^0, \) and \( \sigma^0 \) satisfy the compatibility conditions:
\[
\text{div}((u^0 \cdot \nabla u^0 + (\nabla q^0 - \mu \Delta u^0) / (\varrho^0 + \bar{\rho})) = 0 \quad \text{in } \Omega,
\]
\[
\nabla q^0 - \mu \Delta u^0 + g e^0 e_3 + \Delta \sigma^0 \cdot \nabla \sigma^0 = 0 \quad \text{on } \partial \Omega,
\]
\[
\Delta \sigma^0 + |\nabla \sigma^0|^2 e_3 = 0 \quad \text{on } \partial \Omega.
\]

The proof of Theorem 1 is based on a bootstrap instability method, which has its origin in [18, 19]. We mention that many authors have established various versions of the bootstrap methods for mathematical proofs of various flow instabilities, see [20–22], for example. We complete the proof of Theorem 1 in four steps. Firstly, we introduce unstable solutions to the linearized LCRT problem, in view of the linearized LCRT problem, we can obtain a growing mode ansatz of solutions, i.e., for some \( \Lambda > 0 \),
\[
(\varrho, u, q, \sigma) := e^{\Lambda t}(-\bar{\rho}^\gamma \tilde{u}/\Lambda, \tilde{u}, 0),
\]
see Proposition 1. Secondly, by using the standard energy method, we establish a Gronwall-type energy inequality of the local-in-time solution of the LCRT problem, see Proposition 2. Thirdly, we use initial data of solutions of the linearized LCRT problem to construct initial data for the solutions of the LCRT problem, so that the modified initial data \( (\tilde{\varrho}^0, \tilde{u}^0, q^0) := \delta(\tilde{\varrho}^0, \tilde{u}^0, \tilde{q}^0) + \delta^2(0, u', q') \) belongs to \( H^4 \times H^2 \times H^3 \) and satisfies necessary compatibility condition, see Proposition 4. Finally, we introduce the error estimates between the solutions of the linearized and nonlinear LCRT problems, and then prove the nonlinear solution is unstable under \( L^1 \)-norm.

Now, we will introduce some well-known mathematical results, which will be used in the proof of Theorem 1.

**Lemma 1** (1) *Embedding inequalities* (see [23, Theorem 4.12]):
\[
\|f\|_{L^p} \lesssim \|f\|_1 \quad \text{for } 2 \leq p \leq 6,
\]
\[
\|f\|_{C^0(\Omega)} = \|f\|_{L^\infty} \lesssim \|f\|_2.
\]

(2) *Estimates of the product of functions in Sobolev spaces* (denoted as product estimates):
\[
\|fg\|_j \lesssim \begin{cases} \|f\|_1 \|g\|_1 & \text{for } j = 0; \\ \|f\|_1 \|g\|_2 & \text{for } 0 \leq j \leq 2; \\ \|f\|_2 \|g\|_j + \|f\|_j \|g\|_2 & \text{for } 3 \leq j \leq 5,
\end{cases}
\]
which can be easily verified by Hölder’s inequality and the embedding inequality (1.11)–(1.12).

(3) Interpolation inequality in $H^j$ (see [23, Theorem 5.2]):

$$
\|f\|_j \lesssim \|f\|_0^{1-j} \|f\|_i^j \leq C(\varepsilon)\|f\|_0 + \varepsilon\|f\|_i \quad \text{for any } 0 \leq j < i, \varepsilon > 0.
$$  \hspace{1cm} (1.14)

## 2 Linear instability

**Proposition 1** Under the assumptions of Theorem 1, the LCRT equilibrium state $R_C$ is linearly unstable, that is, there is an unstable solution in the form

$$
(\rho, u, q, \sigma) := e^{\Lambda t}(-\tilde{\rho} \tilde{u}_3/\Lambda, \tilde{u}, \tilde{q}, 0)
$$  \hspace{1cm} (2.1)

to (1.5)–(1.6), where $(\tilde{u}, \tilde{q}) \in H^4_\sigma \times H^3$ solves the following boundary problem:

$$
\begin{align*}
\Lambda^2 \tilde{\rho} \tilde{u} + \Lambda \nabla \tilde{q} &= \Lambda \mu \Delta \tilde{u} + g \tilde{\rho} \tilde{u}_3 \epsilon_3, \\
\text{div} \tilde{u} &= 0, \quad \tilde{u}|_{\partial \Omega} = 0,
\end{align*}
$$  \hspace{1cm} (2.2)

with the constant growth rate $\Lambda$ defined by

$$
\Lambda^2 = \sup_{w \in H^3_\sigma} \frac{g \int \tilde{\rho}' w^2 dx - \Lambda \mu \int |\nabla w|^2 dx}{\int \tilde{\rho} |w|^2 dx}.
$$  \hspace{1cm} (2.3)

Moreover, $\tilde{u}$ satisfies

$$
\tilde{\rho}' \tilde{u} \neq 0, \quad \tilde{u}_3 \neq 0, \quad \text{and} \quad \tilde{u}_3 \neq 0.
$$  \hspace{1cm} (2.4)

**Proof** Please refer to the proof of [24, Theorem 1.1]. \hfill \Box

## 3 Gronwall-type energy inequality of nonlinear solutions

We derive that any small solution of the LCRT problem enjoys a Gronwall-type energy inequality. We will derive such an inequality by the *a priori* estimate method for simplicity. Let $(\rho, u, \sigma)$ be a solution of LCRT problem such that

$$
\sup_{0 \leq t < T} \| (\rho(t), u(t), \sigma(t)) \|_4^2 \leq \delta \in (0, 1) \quad \text{for some } T > 0.
$$  \hspace{1cm} (3.1)

Moreover, the solution enjoys fine regularity, which makes valid the procedure of formal deduction. In addition, we rewrite (1.4) with the boundary-value condition in (1.5) as a nonhomogeneous form:

$$
\begin{align*}
\rho_t + \tilde{\rho}' u_3 &= N_1 := -u \cdot \nabla \rho, \\
\tilde{\rho} u_t + \nabla q - \mu \Delta u + g \tilde{\rho} \epsilon_3 &= N_2 := -\Delta \sigma \cdot \nabla \sigma - (\rho + \tilde{\rho}) u \cdot \nabla u - \rho u, \\
\sigma_t - \Delta \sigma &= N_3 := -u \cdot \nabla \sigma + |\nabla \sigma|^2 \sigma + |\nabla \sigma|^2 e_3, \\
\text{div} u &= 0, \\
u(0, x) &= \varphi^0(x), \\
\sigma(0, x) &= \sigma^0(x), \\
(u, \sigma)|_{\partial \Omega} &= (0, 0).
\end{align*}
$$  \hspace{1cm} (3.2)
Lemma 2  Under the assumption (3.1) with sufficiently small $\delta$, it holds that
\[
\frac{d}{dt} \left\| \sqrt{\rho} u \right\|_0^2 + c \| u \|_1^2 \lesssim \left\| (\varrho, u_3) \right\|_0^2 + \sqrt{\mathcal{E}} D. \tag{3.3}
\]

*Proof*  Multiplying $(3.2)_2$ by $u$ in $L^2$ and using integration by parts, we get
\[
\frac{1}{2} \frac{d}{dt} \left\| \sqrt{\rho} u \right\|_0^2 + \mu \| \nabla u \|_0^2 = -g \int \varrho u_3 \, dx + \int N_2 \cdot u \, dx. \tag{3.4}
\]

By (3.1) and product estimate, it holds that
\[
\int N_2 \cdot u \, dx \lesssim \sqrt{\mathcal{E}} D. \tag{3.5}
\]
Thus we immediately derive (3.3) from (3.4) and (3.5) by using the Young's and Friedrich's inequalities. \hfill \square

Lemma 3  Under the assumption (3.1) with sufficiently small $\delta$, it holds that
\[
\frac{d}{dt} \left\| \varrho \right\|_4^2 \lesssim \| \varrho \|_4 \| u \|_4. \tag{3.6}
\]

*Proof*  Let $\alpha := (\alpha_1, \alpha_2, \alpha_3)$ be a multiindex of order $|\alpha| := \alpha_1 + \alpha_2 + \alpha_3 \leq 4$, and $\partial^\alpha := \partial_{\alpha_1} \partial_{\alpha_2} \partial_{\alpha_3}$.

Using integration by parts, we can get
\[
\int u \cdot \nabla \partial^\alpha \varrho \, dx = 0,
\]
thus, applying $\partial^\alpha$ to $(3.2)_1$, and then multiplying the resulting identity by $\partial^\alpha \varrho$ in $L^2$, we get
\[
\frac{1}{2} \frac{d}{dt} \| \partial^\alpha \varrho \|_0^2 = -\int \partial^\alpha (\tilde{\rho}' u_3) \partial^\alpha \varrho \, dx + \int \left( u \cdot \nabla \partial^\alpha \varrho - \partial^\alpha (u \cdot \nabla \varrho) \right) \cdot \partial^\alpha \varrho \, dx := I_{1,\alpha}. \tag{3.7}
\]

By (3.1) and product estimate, it holds that
\[
I_{1,\alpha} \lesssim \| \varrho \|_4 \| u \|_4 \quad \text{for any } 0 \leq |\alpha| \leq 4. \tag{3.8}
\]
Thus putting (3.8) into (3.7), we immediately derive (3.6). \hfill \square

Lemma 4  Under the assumption (3.1) with sufficiently small $\delta$, it holds that
\[
\frac{d}{dt} \left\| \sqrt{\rho} + \tilde{\rho} u_t \right\|_0^2 + c \| u_t \|_1^2 \lesssim \| u_3 \|_0^2 + \sqrt{\mathcal{E}} D, \tag{3.9}
\]
\[
\frac{d}{dt} \left\| \sqrt{\rho} + \tilde{\rho} u_{tt} \right\|_0^2 + c \| u_{tt} \|_1^2 \lesssim \| \partial_t u_3 \|_0^2 + \sqrt{\mathcal{E}} D. \tag{3.10}
\]
Proof Let $1 \leq i \leq 2$. Applying $\partial_i t$ to (1.4)$_2$, (1.4)$_4$, and (3.2)$_6$, we get

\[
\begin{align*}
\partial_i t((\rho + \bar{\rho})u_t) + \partial_i t((\rho + \bar{\rho})u \cdot \nabla u) + \nabla \partial_i q \\
= \mu \Delta \partial_i u - \partial_i ((\Delta \sigma \cdot \nabla \sigma)) - g \partial_i \varrho e_3, \\
\text{div } \partial_i u = 0, \\
\partial_i |u|_{\partial \Omega} = 0.
\end{align*}
\]

(3.11)

Multiplying (3.11)$_1$ with $i = 2$ by $u_{tt}$ in $L^2$, and using the integration by parts and (3.2)$_1$, we can get that

\[
\frac{1}{2} \frac{d}{dt} (\|\sqrt{\rho + \bar{\rho}} u_{tt}\|_0^2) + \mu \|\nabla u_{tt}\|_0^2
\]

\[
= \int (g \partial_t (\bar{\rho} u_{tt} - N_1)) e_3 \cdot u_{tt} dx - \int \partial_i^2 ((\rho + \bar{\rho})u \cdot \nabla u) \cdot u_{tt} dx
\]

\[
+ \frac{1}{2} \int \varrho_i u_{tt}^2 dx + \int ((\rho + \bar{\rho})u_{tt} - \partial_i^2 ((\rho + \bar{\rho})u_t)) \cdot u_{tt} dx
\]

\[
+ \int \partial_i^2 (\Delta \sigma \cdot \nabla \sigma) \cdot u_{tt} dx
\]

\[
=: \sum_{i=1}^{5} I_i. \quad (3.12)
\]

By (3.1) and product estimate, it holds that

\[
\sum_{i=1}^{4} |I_i| \lesssim \|u_{tt}\|_0 (\|\partial_t u_3\|_0 + \sqrt{\mathcal{E}D}).
\]

Moreover, by using (3.1), integration by parts, and product estimate, we can obtain

\[
|I_5| \lesssim (\|u_{tt}\|_0 + \|u_{tt}\|_1) \sqrt{\mathcal{E}D}. \quad (3.13)
\]

Putting the above two estimates into (3.12), and then using Friedrichs's and Young's inequalities, we get (3.10). Similarly, we can easily derive (3.9) from (3.11) with $i = 1$. □

Lemma 5 Under the assumption (3.1) with sufficiently small $\delta$, it holds that

\[
\frac{d}{dt} \|\sigma\|_0^2 + c \|\sigma\|_1^2 \lesssim \sqrt{\mathcal{E}D}, \quad (3.14)
\]

\[
\frac{d}{dt} \|\sigma_t\|_0^2 + c \|\sigma_t\|_1^2 \lesssim \sqrt{\mathcal{E}D}, \quad (3.15)
\]

\[
\frac{d}{dt} \|\sigma_{tt}\|_0^2 + c \|\sigma_{tt}\|_1^2 \lesssim \sqrt{\mathcal{E}D}, \quad (3.16)
\]

Proof Let $1 \leq i \leq 2$, applying $\partial_i^i$ to (3.2)$_3$, we get

\[
\partial_i^{i+1} \sigma - \Delta \partial_i^i \sigma = \partial_i^i N_3, \quad (3.17)
\]
multiplying (3.17) by $\sigma$ in $L^2$ with $i = 0$, then using integration by parts, we get

$$\frac{1}{2} \frac{d}{dt} \|\sigma\|_0^2 + \|\nabla \sigma\|_0^2 = \int \mathcal{N}_3 \cdot \sigma \, dx.$$  \hspace{1cm} (3.18)

By (3.1), integration by parts and product estimate, it holds that

$$\int \mathcal{N}_3 \sigma \lesssim \sqrt{ED}.$$  

Putting the above estimate into (3.18), and then using Young’s inequality, we get (3.14). Similarly, we can easily derive (3.15) and (3.16) from (3.17) with $i = 1$ and $i = 2$, respectively. □

Lemma 6  Under the assumption (3.1) with sufficiently $\delta$, it holds that

$$\| (u, q) \|_{s, 2} \lesssim \|(\rho, u_t)\|_2 + \|(u_t, u_t)\|_3 + \|\nabla u_t\|_1 + \|\rho\|_2,$$  

$$\| \partial_i(u, q) \|_{s, 0} \lesssim \|(u, u_t)\|_0 + \|\rho\|_2 + \|\nabla u_t\|_1 + \|\rho\|_2 \|\sigma\|_3,$$  

$$\| (u, q) \|_{s, 3} \lesssim \|(\rho, u_t, \nabla u_t)\|_2 + \sqrt{ED},$$  

$$\| \partial_i(u, q) \|_{s, 1} \lesssim \|(u_t, u_t)\|_1 + \sqrt{ED}.$$  \hspace{1cm} (3.20)

Proof Applying $\partial_i^i$ to (3.2), we have

$$\rho \partial_i^i u + \nabla \partial_i^i q - \mu \Delta \partial_i^i u = \partial_i^i \mathcal{N}_2 - g \partial_i^i \rho e_3.$$  \hspace{1cm} (3.23)

By (3.11), (3.11)3, and (3.23), for $i = 0$ and $i = 1$, we get the following Stokes problem:

$$\begin{cases}
\nabla \partial_i^i q - \mu \Delta \partial_i^i u = \mathcal{M}_1,
\div \partial_i^i u = 0,
\partial_i^i u|_{\partial \Omega} = 0,
\end{cases}$$  \hspace{1cm} (3.24)

where we have defined

$$\mathcal{M}_1 = \begin{cases}
-g \rho e_3 - \bar{\rho} u_t + \mathcal{N}_2 \quad \text{for } i = 0,
g(\partial^i u_3 - \mathcal{N}_1) e_3 - \bar{\rho} u_{tt} + \partial_i \mathcal{N}_2 \quad \text{for } i = 1.
\end{cases}$$  \hspace{1cm} (3.25)

Applying the classical Stokes estimate to (3.24) yields

$$\| \partial_i^i (u, q) \|_{S^{2-2i}} \lesssim \begin{cases}
\|(\rho, u_t, \mathcal{N}_2)\|_2 \quad \text{for } i = 0,
\|(u_3, u_{tt}, \mathcal{N}_1, \partial_i \mathcal{N}_2)\|_0 \quad \text{for } i = 1,
\end{cases}$$  \hspace{1cm} (3.26)

and

$$\| \partial_i^i (u, q) \|_{S^{3-2i}} \lesssim \begin{cases}
\|(\rho, u_t, \mathcal{N}_2)\|_3 \quad \text{for } i = 0,
\|(u_3, u_{tt}, \mathcal{N}_1, \partial_i \mathcal{N}_2)\|_1 \quad \text{for } i = 1.
\end{cases}$$  \hspace{1cm} (3.27)
By (3.1), we can estimate that
\[
\| (N_1, \partial_t N_2) \|_0 \lesssim (\| \varrho \|_2 + \| \varrho t \|_0) \| u \|_1 + \| u_t \|_2 (\| \varrho \|_0 + \| u \|_1)
+ \| \varrho \|_2 \| u_t \|_0 + \| \sigma_t \|_2 \| \sigma \|_3,
\]
(3.28)
\[
\| N_2 \|_2 \lesssim \| u \|_2 \| u_t \|_3 + \| \varrho \|_2 \| u_t \|_2 + \| \sigma \|_4 \| \sigma \|_3,
\]
(3.29)
\[
\| N_2 \|_3 + \| (N_1, \partial_t N_2) \|_1 \lesssim \sqrt{ED}.
\]
(3.30)

In addition, using (3.1) and (3.2), we have
\[
\| \varrho_t \|_0 \lesssim \| u \|_0.
\]
(3.31)

By using Young's inequality and the four estimates above, we get (3.19)–(3.22) from (3.26) and (3.27).

**Lemma 7** Under the assumption (3.1) with sufficiently \( \delta \), it holds that

\[
\| \sigma \|_4 \lesssim \| N_3 \|_2 + \| \sigma_t \|_2, \quad (3.32)
\]
\[
\| \sigma \|_5 \lesssim \| N_3 \|_3 + \sqrt{ED}, \quad (3.33)
\]
\[
\| \sigma_t \|_2 \lesssim \| u_t \|_0 \| \sigma \|_3 + \| \sigma_{tt} \|_0, \quad (3.34)
\]
\[
\| \sigma_t \|_3 \lesssim \| \sigma_{tt} \|_1 + \sqrt{ED}. \quad (3.35)
\]

**Proof** Applying \( \partial_i^t \) to (3.2) and (3.2), we get the following elliptic equations:

\[
\begin{cases}
-\Delta \partial_i^t \sigma = \partial_i^t (N_3 - \sigma_t), \\
\partial_i^t \sigma \mid_{\partial\Omega} = 0.
\end{cases}
\]
(3.36)

Applying the classical regularity theory for an elliptic problem (3.36) with \( i = 0, 1, 2, 3 \), we have
\[
\| \partial_i^t \sigma \|_{i-2k+2} \lesssim \| \partial_i^t (N_3, \sigma_t) \|_{i-2k}.
\]
(3.37)

(1) Taking \( (i, k) = (2, 0) \) in (3.37), we immediately get
\[
\| \sigma \|_4 \lesssim \| N_3 \|_2 + \| \sigma_t \|_2.
\]
(3.38)

Now we estimate \( \| N_3 \|_2 \) and, by the definition of \( N_3 \), get
\[
\| N_3 \|_2 \lesssim \| u \cdot \nabla \sigma \|_2 + \| \nabla \sigma \|_2 + \| \nabla \sigma \|_2 \lesssim \| \sigma \|_3 \| u \|_2 + \| \sigma \|_0 \| \sigma \|_3 + \| \sigma \|_3^2,
\]
(3.39)
thus, by using (3.1), we get (3.32) from (3.38) and (3.39).

(2) Taking \( (i, k) = (3, 0) \) in (3.37), we immediately get
\[
\| \sigma \|_5 \lesssim \| N_3 \|_3 + \| \sigma_t \|_3.
\]
(3.40)
and similarly to (3.39), we get
\[ \| N_3 \|_3 \lesssim \| u \|_3 \| \sigma \|_4 + \| \sigma \|_2^2 \| \sigma \|_3 + \| \sigma \|_4^2, \]  
(3.41)
thus (3.33) follows by putting (3.41) into (3.40).

(3) Taking \((i, k) = (2, 1)\) in (3.37), we can obtain that
\[ \| \sigma_i \|_2 \lesssim \| \partial_t N_3 \|_0 + \| \sigma_t \|_0. \]  
(3.42)
By a simple calculation, we get
\[ \| \partial_t N_3 \|_0 \lesssim \| u_t \|_0 \| \sigma \|_3 + \| \sigma_t \|_2, \]  
(3.43)
thus by putting (3.43) into (3.42), we obtain (3.34).

(4) Taking \((i, k) = (3, 1)\) in (3.37), we obtain that
\[ \| \sigma_i \|_3 \lesssim \| \partial_t N_3 \|_1 + \| \sigma_t \|_1, \]  
(3.44)
similarly, we get that
\[ \| \partial_t N_3 \|_1 \lesssim \| u_t \|_1 \| \sigma \|_3 + \| u \|_2 \| \sigma_t \|_1 + \| \sigma_t \|_1 \| \sigma \|_3, \]  
(3.45)
thus, (3.35) follows from the above two estimates. \( \square \)

**Lemma 8** Under the assumption (3.1) with sufficiently \( \delta \), we have that
\[ \mathcal{E} \text{ is equivalent to } \| (\bar{\rho}, u, \sigma) \|_4^2. \]  
(3.46)

**Proof** By (3.19), (3.20), (3.32), and (3.34), to get (3.46), it suffices to derive, for sufficiently small \( \delta \),
\[ \| \rho_t \|_3 + \| \rho_{tt} \|_2 + \| u_t \|_0 + \| u_{tt} \|_0 + \| \sigma_{tt} \|_0 \lesssim \| (\bar{\rho}, u, \sigma) \|_4. \]  
(3.47)
Next, we verify (3.47).
Multiplying (3.23) with \( i = 1 \) by \( u_{tt} \) in \( L^2 \), we infer that
\[ \| \sqrt{\bar{\rho}} u_{tt} \|_0^2 = \int (g(\bar{\rho}' u_3 - N^2) v_3 + \mu \Delta u_t + \partial_t N_2) \cdot u_{tt} \, dy. \]
Using (1.2), (3.1), (3.28), (3.31), and Young's inequality, we can derive from the above identity that
\[ \| u_{tt} \|_0^2 \lesssim \| u \|_1 + \| u_t \|_2^2 + \| \sigma_t \|_2 \| \sigma \|_3. \]  
(3.48)
Next we shall estimate for \( \| u_i \|_2 \) and \( \| \sigma_i \|_2 \).
By using (3.29), we can obtain from (3.2) that
\[ \| u_i \|_2 \lesssim \| q \|_2 + \| q \|_3 + \| u \|_4 + \| \sigma \|_4 \]  
(3.49)
and
\[
\begin{align*}
\Delta q &= M_2 := \bar{\rho}' \partial_t u_3 + g \partial_3 \varrho + \text{div}((\varrho + \bar{\rho})u \cdot \nabla u + \Delta \sigma \cdot \nabla \sigma) + u_t \cdot \nabla \varrho, \\
\nabla q \cdot \mathbf{n} |_{\partial \Omega} &= M_3 := (\mu \Delta u - g \varrho e_3 - \Delta \sigma \cdot \nabla \sigma) \cdot \mathbf{n},
\end{align*}
\tag{3.50}
\]
where \( \mathbf{n} \) denotes the unit outer normal vector on \( \partial \Omega \). Applying the classical elliptic regularity theory to (3.50) yields that
\[
\|q\|_3 \lesssim \|M_2\|_1 + \|M_3\|_{H^{3/2}(\partial \Omega)} \lesssim \|(q, u, \sigma)\|_4 + \|u_t\|_1.
\]
Inserting the above estimate into (3.49), and then using interpolation inequality, we arrive at
\[
\|u_t\|_2 \lesssim \|(q, u, \sigma)\|_4 + \|u_t\|_0. \tag{3.51}
\]
Next we shall estimate \( u_t \).

We multiply (3.2) by \( u_t \) in \( L^2 \), and then use the integration by parts to obtain
\[
\left\| \sqrt{\bar{\rho}} u_t \right\|_0^2 = \int (\mu \Delta u + \mathcal{N}_2 - g \varrho e_3) \cdot u_t \, dx.
\]
Using (3.1) and Young’s inequality, we can derive from the above identity that
\[
\|u_t\|_0^2 \lesssim \|(q, u)\|_2^2 + \|\sigma\|_3^2. \tag{3.52}
\]
Thus, we derive from (3.48), (3.51), and (3.52) that, for sufficiently small \( \delta \),
\[
\|u_t\|_2 + \|u_{tt}\|_0 \lesssim \|(q, u, \sigma)\|_4^2 + \|\sigma_t\|_2^2. \tag{3.53}
\]
Next we estimate \( \|\sigma_t\|_2 \).

By (3.39), we can derive from (3.2) that
\[
\|\sigma_t\|_2 \lesssim \|(u, \sigma)\|_4. \tag{3.54}
\]
Now, we estimate \( \sigma_{tt} \). Multiplying (3.36) with \( i = 1 \) by \( \sigma_{tt} \) in \( L^2 \), we infer that
\[
\|\sigma_{tt}\|_0^2 = \int (\Delta \sigma + \partial_t \mathcal{N}_3) \cdot \sigma_{tt} \, dy. \tag{3.55}
\]
Using (3.1), (3.43), and Young’s inequality, we derive from the above identity
\[
\|\sigma_{tt}\|_0^2 \lesssim \|u_t\|_0^2 \|\sigma\|_3^2 + \|\sigma_t\|_2^2 + \|\sigma\|_2^2. \tag{3.56}
\]
Thus, by (3.52), (3.54), and (3.56), we can get
\[
\|\sigma_t\|_2 + \|\sigma_{tt}\|_0 \lesssim \|(q, u, \sigma)\|_4. \tag{3.57}
\]
Finally, by (3.2)1, we have
\[
\|\varrho_t\|_3 \lesssim \|u\|_3, \tag{3.58}
\]
\[
\|\varrho_{tt}\|_2 \lesssim \|u_t\|_2 + \|u\|_2 \|\varrho_t\|_3. \tag{3.59}
\]
Thus, we get (3.46) from (3.52) and (3.53), (3.57), and above two estimates. \qed

Proposition 2 There exist a constant \( \delta_1 \in (0,1) \) and \( C > 0 \) such that, for any \( \delta \leq \delta_1 \), if the solution \( (\varrho, u, \sigma) \) of LCRT problem satisfies (3.1), then for any \( t \in (0, T) \), the solution \( (\varrho, v, \sigma) \) satisfies the Gronwall-type energy inequality
\[
\mathcal{E} + \int_0^t \mathcal{D} \, d\tau \leq C_1 \left( \int_0^t \| (\varrho, v, \sigma) \|_0^2 \, d\tau + \| (\varrho^0, v^0, \sigma^0) \|_0^2 \right) + \Lambda \int_0^t \mathcal{E} \, d\tau. \tag{3.60}
\]

Proof We derive from Lemmas 2–5 that, for sufficiently large constant \( c_1 \),
\[
\frac{d}{dt} \mathcal{E}_1 + c \mathcal{D}_1 \leq c_1 \left( \| (\varrho, u_3) \|_0^2 + \| \sqrt{\mathcal{E}} \|_4 \right) / c,
\]
where we have defined
\[
\mathcal{E}_1 := \| \sqrt{\rho} u \|_2^2 + \| \sqrt{\varrho + \rho} (c_1 u_t, u_{tt}) \|_0^2 + \| (\sigma, \sigma_t, \sigma_{tt}) \|_1^2 + \| \sqrt{\mathcal{E}} \|_4^2,
\]
\[
\mathcal{D}_1 := \| (c_1 u_t, u_{tt}) \|_1^2 + \| u \|_1^2 + \| (\sigma, \sigma_t, \sigma_{tt}) \|_1^2.
\]
Integrating (3.61) over \( (0, t) \), and then using interpolation and Young’s inequalities, we get, for any given \( \varepsilon > 0 \),
\[
\mathcal{E}_1 + c \int_0^t \left( \mathcal{D}_1 + \| \varrho(\tau) \|_3^2 \right) \, d\tau \\
\leq \mathcal{E}_1(0) + \varepsilon \int_0^t \| \varrho \|_4^2 \, d\tau + \frac{1}{c} \int_0^t \left( \| (\varrho, u) \|_0^2 + \sqrt{\mathcal{E}} \mathcal{D} \right) \, d\tau,
\]
where the positive constant \( c \) depends on \( \varepsilon \).
Noting that, by (3.31), (3.58), (3.58), (3.59), and Lemmas 6–8, we easily derive that there exists a constant \( c \) such that, for sufficiently small \( \delta \),
\[
\mathcal{E}_1, \mathcal{E}, \text{ and } \| (\varrho, u, \sigma) \| \text{ are equivalent for any } t \geq 0,
\]
\[
\mathcal{D} \leq c \left( \mathcal{D}_1 + \| \varrho \|_3^2 \right).
\]
Consequently, we immediately derive (3.60) from (3.62)–(3.64). \qed

Proposition 3 (1) Let \( \bar{\rho} \in C^5(\bar{\Omega}) \). Then there are a sufficiently small \( \delta_2 \in (0,1) \) and \( K_1 > 0 \) such that if \( (\varrho^0, u^0, \sigma^0) \) satisfies
\[
\sqrt{\| (\varrho^0, u^0, \sigma^0) \|_4^2} < \delta_2, \tag{3.65}
\]
\[
\inf_{x \in \Omega} \left\{ (\bar{\rho} + \rho)(x) \right\} \geq K_1 > 0, \tag{3.66}
\]
and the compatibility conditions
\begin{align}
\text{div } u^0 &= 0 \quad \text{in } \Omega, \\
(\sigma, u)|_{t=0} &= 0 \quad \text{on } \partial \Omega,
\end{align}
(3.67)
(3.68)

then there exist a local existence time $T^{\text{max}} > 0$ (depending on $\delta_2$, the domain and the known parameters) and a unique local-in-time classical solution $(\varrho, u, \sigma, q) \in C^0(H^4 \times H^4_\sigma \times H^4 \times H^3)$ to the LCRT problem.

(2) In addition, if the solution $(\varrho, u, \sigma)$ further satisfies
\[
\sup_{t \in [0, T]} \sqrt{\| (\varrho, u, \sigma) \|^2_2} \leq \delta_1
\]
for some $T < T^{\text{max}}$, then $(\varrho, u, \sigma)$ enjoys the equivalent estimate (3.46) and the Gronwall-type energy inequality (3.60).

Remark 1 For any given initial data $(\varrho^0, u^0, \sigma^0) \in H^4 \times H^4_\sigma \times H^4$ satisfying (3.65)–(3.67) with sufficiently small $\delta_2$, there exists a unique local-in-time strong solution $(\varrho, u, \sigma, q) \in C^0([0, T), H^2 \times H^2_\sigma \times H^2 \times H^1)$. Moreover, the initial date of $q$ is a weak solution to
\[
\begin{cases}
\text{div}((\varrho^0 + \bar{\rho}) \sigma^0) = 0 & \text{in } \Omega, \\
\nabla q^0 \cdot \vec{n} = (\mu \Delta u^0 - g \sigma^0 \cdot \nabla \sigma^0) & \text{on } \partial \Omega.
\end{cases}
\]
(3.69)

If the condition (3.68) is further satisfied, i.e., $(\varrho^0, u^0, \sigma^0)$ satisfies
\[
\begin{cases}
\nabla q^0 = -\mu \Delta u^0 + g \sigma^0 \cdot \nabla \sigma^0 + \Delta \sigma^0 \cdot \nabla \sigma^0 & \text{on } \partial \Omega, \\
\Delta \sigma^0 + |\nabla \sigma^0|^2 e_3 = 0 & \text{on } \partial \Omega,
\end{cases}
\]
(3.70)

then we can improve the regularity of $(\varrho, u, \sigma)$ so that it is a classical solution for sufficiently small $\delta_2$.

Remark 2 For any classical solution $(\varrho, u, \sigma)$ constructed by Proposition 3, and for any given $t_0 \in (0, T^{\text{max}})$, we take $(\varrho, u, \sigma)|_{t=t_0}$ as a new initial datum. Then the new initial data can define a unique local-in-time classical solution $(\tilde{\varrho}, \tilde{u}, \tilde{\sigma}, \tilde{q})$ constructed by Proposition 3, moreover, the initial data of $\tilde{q}$ is equal to $q|_{t=t_0}$ by unique solvability of (3.69).

4 Construction of initial data for the nonlinear problem

For any given $\delta > 0$, let
\[
(\varrho^a, u^a, q^a) = \delta e^{\Lambda t} (\varrho^0, u^0, q^0),
\]
(4.1)
where $(\tilde{\varrho}, \tilde{u}, \tilde{q}) := (-\tilde{\rho} / \tilde{\rho}) \tilde{u}_3 / \Lambda, \tilde{u}, \tilde{q})$, and $(\tilde{u}, \tilde{q}) \in H^4_\sigma \times H^3$ comes from Proposition 1. Then $(\varrho^a, u^a, q^a)$ is a solution to the linearized RTLC equations, and enjoys the estimate, for any $i \geq 0$,
\[
\| \phi_i^a(u^a, q^a) \|_{S, 2} \leq c(i) \delta e^{\Lambda t}.
\]
(4.2)
Moreover, by (2.4),
\[ \|\tilde{\varrho}\|_{L^1} \|\tilde{u}_0\|_{L^1} \|\tilde{q}_0\|_{L^1} > 0. \]

Next we shall modify the initial data of the linear solutions.

**Proposition 4** Let \((\tilde{\varrho}^0, \tilde{u}^0, \tilde{q}^0)\) be the same as in (4.1), then is a constant \(\delta_3\), such that for any \(\delta \in (0, \delta_3)\), there exists \((u', q') \in H^4 \times H^3\) enjoying the following properties:

1. **The modified initial data**

   \[
   \left(\delta \tilde{\varrho}^0, \delta \tilde{u}^0, \delta \tilde{q}^0\right) := \delta \left(\tilde{\varrho}^0, \tilde{u}^0, \tilde{q}^0\right) + \delta^2 \left(0, u', q'\right) \tag{4.3}
   \]

   belongs to \(H^4 \times H^4 \times H^3\), and satisfies the compatibility conditions (3.69) and (3.70) with \((u_0, q_0)\) in place of \((u^0, q^0)\).

2. **The uniform estimate holds**

   \[ \sqrt{\|u'\|_{H^1}^2 + \|q'\|_{H^1}^2} \leq C_2, \tag{4.4} \]

   where the constant \(C_2 \geq 1\) depends on the domain, the density profile, and the known parameters, but is independent of \(\delta\).

**Proof** Recalling the construction of \((\tilde{u}^0, \tilde{q}^0)\), we can see that \((\tilde{u}^0, \tilde{q}^0)\) satisfies

\[
\begin{aligned}
\text{div } \tilde{u}^0 &= 0 \quad \text{in } \Omega, \\
\Lambda^2 \tilde{\varrho} \tilde{u}^0 + \Lambda \nabla \tilde{q}^0 - \Lambda \Delta \tilde{u}^0 &= g \tilde{\varrho}' \tilde{u}^0_3 \quad \text{in } \Omega, \\
\tilde{u}^0 &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\tag{4.5}
\]

If \((u', \sigma', q') \in H^4 \times H^4 \times H^3\) satisfies, for any given \(\delta\),

\[
\begin{aligned}
\text{div } u' &= 0 \quad \text{in } \Omega, \\
\nabla q' - \mu \Delta u' &= g \tilde{\varrho}' u'_3 e_3 - \Delta \sigma_0 \cdot \nabla \sigma_0 \\
&\quad + (\delta \tilde{\varrho}_0^0 + \tilde{\varrho}) (\nabla (\delta \tilde{u}_0^0 + \delta^2 u') \cdot \nabla (\delta \tilde{u}_0^0 + \delta^2 u')) / \delta^2 \quad \text{in } \Omega, \\
\text{div } \Upsilon &= -\delta \Lambda \text{div} (\delta \tilde{u}_0^0 / (\delta \tilde{\varrho}_0^0 + \tilde{\varrho})) \quad \text{in } \Omega, \\
(\Upsilon, u') &= 0 \quad \text{on } \Omega,
\end{aligned}
\tag{4.6}
\]

where \((\delta \tilde{\varrho}_0^0, \delta u'_0, \delta \sigma_0, \delta q_0)\) is given in the mode (4.3), then, by (4.5), it is easy to check that \((\delta \tilde{\varrho}_0^0, \delta u'_0, \delta \sigma_0, \delta q_0) \in H^4 \times H^4 \times H^3 \times H^3\), and it satisfies the compatibility conditions (3.69) and (3.70) with \((u_0, q_0)\) in place of \((u^0, q^0)\). Next we construct such \((u', \sigma', q')\) which satisfy (4.5) for sufficiently small \(\delta\).

Since \(H^2 \hookrightarrow L^\infty\), there exists a constant \(\delta_4 > 0\) such that

\[
\inf_{x \in \Omega} |w + \tilde{\varrho}| \geq \inf_{x \in \Omega} |\tilde{\varrho}| / 2 > 0 \quad \text{for any } w \text{ satisfying } \|w\|_2 \leq \delta_4. \tag{4.7}
\]

Moreover,
(1) There exists \((\Upsilon, q) \in H^2_0 \times H^1\) such that
\[
\begin{aligned}
\nabla q - \Delta \Upsilon &= 0, \quad \text{div } \Upsilon = -\delta \Lambda \text{div}(\rho_0^d \tilde{u}_0^d / (\rho_0^d + \tilde{\rho})) \quad \text{in } \Omega, \\
\Upsilon &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]
(4.8)
and
\[
\| (\Upsilon, q) \|_{S, 0} \leq \delta^2.
\] (4.9)

(2) There exists \((u', q') \in H^4_0 \times H^3\) such that
\[
\begin{aligned}
\text{div } u' &= 0 \quad \text{in } \Omega \\
\nabla q' - \mu \Delta u' &= g \rho' u_3 e_3 - \Delta \sigma_0^d \cdot \nabla \sigma_0^d \\
&\quad + (\rho_0^d + \tilde{\rho})(\Upsilon - (\delta \tilde{u}_0^d + \delta^2 u') \cdot \nabla (\delta \tilde{u}_0^d + \delta^2 u')) / \delta^2 \quad \text{in } \Omega, \\
u' &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]
(4.10)
and \(\| (u', q') \|_{S, 2} \leq c\), where \(\Upsilon\) is constructed in (4.8). We mention that the constant \(c\) above is independent of \(\delta\). Thus we can get Proposition 4 from the above arguments. 

\[\Box\]

5 Error estimates and existence of escape times

Let
\[
C_3 := \sqrt{\| (\tilde{\rho}_0^d, \tilde{u}_0^d) \|^2 + C_2} \geq 1,
\]
(5.1)
\[
\delta < \delta_0 := \min(\delta_1, \delta_2, 2C_3\delta_3, \delta_4)/2C_3 < 1,
\]
and \((u_0^d, q_0^d)\) be constructed by Proposition 4.

Noting that
\[
\sqrt{\| (\tilde{\rho}_0^d, u_0^d) \|^2} \leq C_3 \delta < 2C_3 \delta_0 < \delta_4,
\]
(5.2)
then
\[
\inf_{x \in \Omega} \{ \tilde{\rho}_0^d + \tilde{\rho} \} > 0.
\]

Thus, by the first assertion in Proposition 3, one sees that there is a (nonlinear) solution \((u, q)\) of the problem defined in some time interval \(I_{T_{\text{max}}}\) with the initial value \((\rho_0^d, u_0^d)\).

Moreover, we have \(\int q \, dx = \int q_0^d \, dx = 0\).

Let \(\varepsilon_0 \in (0, 1)\) be a constant, which will be defined in (5.9). We define
\[
T^\delta := (\Lambda)^{-1} \ln(\varepsilon_0/\delta) > 0, \quad \text{i.e., } \delta e^{\Lambda T^\delta} = \varepsilon_0,
\]
\[
T^* := \sup\{ t \in (0, T_{\text{max}}) \| (q, u)(\tau) \|_4 \leq 2C_3 \delta_0 \text{ for any } \tau \in [0, t] \},
\]
\[
T^{**} := \sup\{ t \in (0, T_{\text{max}}) \| (q, u)(\tau) \|_0 \leq 2C_3 \delta e^{\Lambda t} \text{ for any } \tau \in [0, t] \},
\]
(5.3)
where $T^{\text{max}}$ denotes the maximal time of existence of the solution $(\varrho, u) \in C([0, T^{\text{max}}), H^4 \times H^4)$. Obviously, $T^* T^{**} > 0$ and

\[
\| (\varrho, u)(T^*) \|_4 = 2C_3\delta_0, \quad \text{if } T^* < \infty, \tag{5.4}
\]

\[
\| (\varrho, u)(T^*) \|_0 = 2C_3\delta e^{\Lambda T^{**}}, \quad \text{if } T^{**} < T^{\text{max}}. \tag{5.5}
\]

We denote $T^{\text{min}} := \min\{T^*, T^{**}\}$. By the definition of $T^{**}$, we can deduce from the estimate (3.60) that, for all $t < T^{\text{min}},$

\[
E \leq C_4 \delta^2 e^{2\Lambda t} + \Lambda \int_0^t E \, dt. \tag{5.6}
\]

Applying Gronwall’s inequality to the above estimate, we arrive at, for some constant $C_4$,

\[
E \leq C_4 \delta^2 e^{2\Lambda t} \quad \text{for all } t < T^{\text{min}}. \tag{5.7}
\]

In addition, we have the following error estimate between the nonlinear solution $(\varrho, u)$ and the linear solution $(\varrho^a, u^a)$.

**Lemma 9** Let $\varepsilon_0 \leq \delta_4/C_4$. There exists a constant $C_5$ such that, for any $\delta \in (0, 1)$ and any $t \in I_{T^{\text{min}}}$,

\[
\| (\varrho^d, u^d) \|_{H^\infty} \leq C_5 \delta^3 e^{3\Lambda t}, \tag{5.8}
\]

where $(\varrho^d, u^d) := (\varrho, u) - (\varrho^a, u^a)$, $H = L^1$ or $L^2$, and $C_5$ is independent of $T^{\text{min}}$.

**Proof** Please refer to [25, Lemma 3.1]. We mention that the condition $\varepsilon_0 \leq \delta_4/C_4$ makes sure that $\inf_{x \in \Omega} \{\varrho + \bar{\rho}\} \geq \inf_{x \in \Omega} \{\bar{\rho}\}/2 > 0$. \hfill \square

Now we define that

\[
\varepsilon_0 := \min\left\{\frac{C_3\delta_0}{C_4}, \frac{C_3^2}{4C_3^2}, \frac{m_0^2}{4C_3^2} \right\} > 0, \tag{5.9}
\]

\[
m_0 := \min\{\| e^0 \|_{L^1}, \| \bar{u}^0 \|_{L^1}, \| (\bar{G}_1, \bar{G}_2^0) \|_{L^1} \}. \tag{5.10}
\]

It is easy to see that $m_0 > 0$ by (5.10). Now, we assert that

\[
T^d = T^{\text{min}} \neq T^* \text{ or } T^{**}, \tag{5.11}
\]

which can be proved by contradiction as follows:

(1) If $T^{\text{min}} = T^* < T^d$, then $T^* < \infty$. Moreover, $T^* \leq T^{\text{max}}$ by Proposition 3. Note that we can deduce from (5.3), (5.7), and (5.9) that

\[
\| (\varrho, u)(T^*) \|_4 \leq C_4 \delta e^{\Lambda T^d} = C_4 \varepsilon_0 < C_3 \delta_0,
\]

which contracts (5.4). Hence $T^{\text{min}} \neq T^*$. 

(2) If $T_{\min} = T^{**} < T^*$, then $T^{**} < T^* \leq T_{\max}$. Moreover, making use of (4.1), (5.1), (5.3), (5.8), and (5.9), we see that

$$\| (\varrho, u)(T^{**}) \|_0 \leq \| (\varrho^d, u^d)(T^{**}) \|_0 + \| (\varrho^d, u^d)(T^{**}) \|_0$$

$$\leq \delta C_3 e^{\lambda T^{**}} + C_5 \delta^{3/2} \lambda^{3/2} \lambda T^{**} \leq \delta e^{\lambda T^{**}} (C_3 + C_5 \sqrt{\varepsilon_0})$$

$$< 2\delta C_3 e^{\lambda T^{**}}, \quad (5.12)$$

which also contradicts (5.5). Therefore, $T_{\min} \neq T^{**}$. We immediately see that (5.11) holds. This completes the proof of claim (5.11).

Since $T^C < T^* \leq T_{\max}$, we can use (4.1), (5.8), and (5.9) to deduce that

$$\| \varrho(T^C) \|_{L^1} \geq \| \varrho^d(T^C) \|_{L^1} - \| \varrho^d(T^C) \|_{L^1}$$

$$\geq \delta e^{\lambda T^C} \| \tilde{\varrho} \|_{L^1} - C_5 \delta^{3/2} \lambda T^C \geq \varepsilon_0 \| \tilde{\varrho} \|_{L^1} - C_5 \delta^{3/2} \lambda T^C \geq m_0 \varepsilon_0 - C_5 \delta^{3/2} \lambda T^C \geq m_0 \varepsilon_0/2. \quad (5.13)$$

Similarly, we also have

$$\| u_3(T^C) \|_{L^1} \geq m_0 \varepsilon_0 - C_5 \delta^{3/2} \lambda T^C \geq m_0 \varepsilon_0/2,$$

and

$$\| (u_1, u_2)(T^C) \|_{L^1} \geq m_0 \varepsilon_0 - C_5 \delta^{3/2} \lambda T^C \geq m_0 \varepsilon_0/2,$$

where $u_i$ denote the $i$th component of $u(T^C)$ for $1 \leq i \leq 3$. This completes the proof of Theorem 1 by defining $\varepsilon := m_0 \varepsilon_0/2$. 

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Authors’ contributions
This work was carried out in collaboration between both authors. XL designed the study and guided the research. ML performed the analysis and wrote the first draft of the manuscript. XL and ML managed the analysis of the study. Both authors read and approved the final manuscript.

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