HEIGTHS OF CM CYCLES AND DERIVATIVES OF $L$-SERIES

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ABSTRACT. We extend the work of S. Zhang and Yuan-Zhang-Zhang to obtain a Gross-Zagier formula for modular forms of even weight in terms of an arithmetic intersection pairing of CM-cycles on Kuga-Sato varieties over Shimura curves. Combined with a result of the first author and de Vera-Piquero adapting Kolyvagin’s method of Euler systems to this setting, we bound the associated Selmer and Tate-Shafarevich groups, assuming the non-vanishing of the derivative of the $L$-function at the central point.

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1. INTRODUCTION

Given an imaginary quadratic field, the classical Gross-Zagier formula [9] relates the first derivative of the $L$-function of a modular form of weight two to the Néron-Tate height pairing of a suitable Heegner point on the modular curve, whose existence is guaranteed by a certain Heegner hypothesis. This result was generalized by S. Zhang [23] to the setting of modular forms of positive even weight, and by Yuan-Zhang-Zhang [21] to the setting of imaginary quadratic extensions of totally real number fields, with a relaxed Heegner hypothesis. These results contribute to the conjecture of Birch and Swinnerton-Dyer and its generalization due to Beilinson, Bloch and Kato. Combined with Kolyvagin’s method of Euler systems [11] subsequently adapted by Nekovář in both settings [13, 14], they imply a bound on the rank of the associated Selmer and Tate-Shafarevich groups, assuming the non-vanishing of the derivative at the central point of the $L$-function.

In this paper, we simultaneously extend the results of [21] and [23] to obtain a Gross-Zagier formula for Hilbert modular forms of even parallel weight, under a relaxed Heegner hypothesis. More precisely, we relate the central value of the
first derivative of the associated Rankin-Selberg $L$-function to the intersection of suitable CM cycles on Kuga-Sato varieties over Shimura curves. Combining this with a recent result of the first author and de Vera-Piquero adapting Kolyvagin and Nekovar’s Euler system argument to this setting [7], we provide a bound on the rank of the associated Selmer and Tate-Shafarevich groups.

Let $F$ be a totally real number field. Let $\pi$ be a cuspidal automorphic representation of an indefinite quaternion algebra $B$ over a totally real number field $F$ associated, by the Jacquet-Langlands correspondence, to a cuspidal automorphic representation $\sigma$ of parallel weight $2k$, and $\tilde{\pi}$ its contragredient representation. Let $K$ be an imaginary quadratic extension of $F$ with character $\eta$, such that finite primes dividing the discriminant of $B$ are inert in $K$ over $F$. Given a Hecke character $\chi$ of $K$, we denote by $Z_K$, the $\chi$-eigencomponent of a suitable CM-cycle $Z$ on the Kuga-Sato variety $W$ over the Shimura curve over $F$, and by $T$ a suitable map from $\pi \otimes \tilde{\pi}$ to the group of homomorphisms of the $k-1$-th Chow group of $W$ as in (2.24).

Taking first $F=\mathbb{Q}$, our generalization for the Gross-Zagier formula can be stated as follows:

**Theorem 1.0.1.** For $f_1 \otimes f_2$ in $\pi \otimes \tilde{\pi}$, the following equality holds

\begin{equation}
\langle T(f_1 \otimes f_2)Z_\chi, Z_{\chi^{-1}} \rangle_{\text{GS}} = \frac{c_F(2)L'\left(\frac{1}{2}, \pi, \chi\right)}{4L(1,\eta)^2L(1, \pi, \text{ad})} \alpha(f_1 \otimes f_2),
\end{equation}

where

\begin{equation}
T: \bigoplus_{\pi \in A(B^\times)} \pi \otimes \tilde{\pi} \mapsto \text{Hom}(\text{CH}^{k-1}(W), \text{CH}^{k-1}(W))_\mathbb{Q},
\end{equation}

is a suitable correspondence defined in Lemma (2.7.2), $\langle \cdot, \cdot \rangle_{\text{GS}}$ is the Gillet-Soulé pairing on arithmetic Chow groups, $\alpha(\cdot \otimes \cdot)$ is the Petersson inner product, and $L(s, \pi, \text{ad})$ is the usual adjoint $L$-function.

The following two corollaries correspond to the prediction of the Beilinson, Bloch and Kato conjectures. Still assume $F=\mathbb{Q}$, and consider $f$ in $\pi$. Let $p$ be an odd prime not dividing $N(2k!)$, and let $\wp$ be a prime ideal dividing $p$ in the number field $E$ generated over $\mathbb{Q}$ by the Fourier coefficients of $f$. We denote by $O_{\wp}$ the ring of integers of $E$, localized at $\wp$. Consider the $p$-adic Abel–Jacobi map

$$
\Phi: \text{CH}^k(W_{k-1}/K)_0 \otimes O_{\wp} \to \text{Sel}_{\wp}^\infty(f, K),
$$

where $\text{CH}^k(W_{k-1}/K)_0$ is the $k$-th Chow group of a suitable Kuga-Sato variety $W_{k-1}$ over $K$, and $\text{Sel}_{\wp}^\infty(f, K)$ is the Selmer group associated to $f$ and $K$ defined by (7.1.1). Its cokernel

$$
\text{III}_{\wp\infty} := \text{coker}(\Phi) = \text{Sel}_{\wp}^\infty(f, K)/\text{Im}(\Phi).
$$

is the $\wp$-primary part of the Shafarevich–Tate group.

On the one hand, assuming the injectivity of the $p$-adic Abel Jacobi map, if the order of vanishing of the $L$-functions associated to $\pi$ and $\chi$ at the central point is 1, then $\text{Im}(\Phi)$ has rank 1 over $\mathbb{Q}$ as in the next corollary.

**Corollary 1.0.2.** Assume that the $p$-adic Abel Jacobi map $\Phi$ (7.3) is injective. Then

\begin{equation}
L'(\frac{1}{2}, \pi, \chi) \neq 0 \implies \text{III}_{\wp\infty} \text{ is finite, and } \text{Im}(\Phi) \otimes \mathbb{Q} \text{ has rank } 1.
\end{equation}
On the other hand, assuming the Gillet-Soulé conjecture, if \( \text{Im}(\Phi) \) has rank 1 over \( \mathbb{Q} \), then the \( L \)-function associated to \( \pi \) and \( \chi \) has order of vanishing 1 at the central point.

**Corollary 1.0.3.** Suppose the Gillet-Soulé conjecture (7.2.2) holds. Then

\[
(1.4) \quad \text{Im}(\Phi) \otimes \mathbb{Q} \text{ has rank 1 } \implies L\left(\frac{1}{2}, \pi, \chi\right) \neq 0.
\]

Our work also sets the stage for the general case of \( F \) a totally real field, depending on the existence of suitable CM cycles. Granting the existence of such cycles, our methods give the analogous Gross-Zagier formula in the form of Theorem 1.0.1 with little modifications. Also, assuming, as one would expect, that the CM cycles, under the \( p \)-adic Abel-Jacobi map form an Euler system as in \([7]\), the analogues of Corollaries 1.0.2 and 1.0.3 will also then follow for the associated Selmer groups over \( F \).

1.1. **Outline.** In Section (2), we introduce the basic objects of our study and give a complete statement of Theorem (1.0.1). Our proof of the theorem follows closely the method of [21], which in turn follows the proof of the Waldspurger formula [19], inspired by the philosophy of Kudla.

In Section (3), we construct the analytic and geometric kernels respectively corresponding to the right and left sides of the main identity (1.1). We reformulate it as a kernel identity,

\[
(1.5) \quad 2(\tilde{Z}(\cdot, \chi, \Phi), \varphi)_{\text{Pet}} = (I'(0, \cdot, \chi, \Phi), \varphi)_{\text{Pet}}
\]

which can be viewed as an arithmetic theta lifting. In particular, the geometric kernel relates to a generating series of CM cycles. In Section (4), we show that this generating series defines an automorphic form on \( GL_2(\mathbb{A}) \) as this is an essential ingredient in the comparison of kernels. We also prove an arithmetic theta lifting,

\[
(1.6) \quad \tilde{Z}(\Phi \otimes \varphi) = \frac{L(1, \pi, \text{ad})}{2\zeta_F(2)} T(\theta(\Phi \otimes \varphi)),
\]

relating the generating series to the correspondence. The key observation is that the automorphy of the generating series can be essentially deduced from the weight 2 case, while the arithmetic theta lifting argument relies on the local Siegel-Weil formulae, which are independent of weight at nonarchimedean places.

In Section (5), we decompose the global kernels into local components. In Section (6), we prove the kernel identity. We use the Gillet-Soulé intersection theory and Green’s currents as developed in [23]. We restrict ourselves to the class of degenerate Schwartz functions introduced in [21], which simplifies the local computations. This completes the proof of the main identity, which is combined in Section (7) with [7, Theorem 1.1] to prove Corollary (1.0.2), and with the Gillet-Soulé conjecture (7.2.2) to prove Corollary (1.0.3).

1.2. **Notation and measures.** We fix the following notation:

- \( F \) is a totally real number field with adele ring \( \mathbb{A}_F = \mathbb{A} \), and a fixed embedding \( \tau : F \hookrightarrow \mathbb{C} \). Also, \( F_\infty = \prod_{v|\infty} F_v \), and \( \mathbb{A}_F^\infty = \prod_{v \not\in S} F_v \).
- If \( \pi \) is an irreducible admissible complex representation of \( B^\times \), then by the Jacquet-Langlands correspondence we associate a cuspidal automorphic representation \( \sigma \) of \( GL_2(\mathbb{A}) \), discrete of parallel weight \( 2k \) at all infinite places.
• $K$ is an imaginary quadratic extension of $F$ with adele ring $\mathbb{A}_K$ and quadratic character $\eta : K^\times \backslash \mathbb{A}_K^\times \rightarrow \mathbb{C}^\times$, such that finite primes dividing the discriminant of $\mathcal{B}$ are inert in $K$ over $F$. Also consider a character $\chi : \text{Gal}(K^{ab}/K) \rightarrow \mathbb{C}^\times$.
• $\text{CH}^k(X)$ is the Chow group of codimension $k$ cycles of the variety $X$.
• $V$ is the orthogonal space $\mathcal{B}$ with reduced norm $q$.
• For any algebraic group $T$, we denote the adelic quotient by $[T] = T_F \backslash T_{\mathbb{A}} / Z_{\mathbb{A}}$, and fix $G = GL(2)$.

1.2.1. Local measures. In general, our choice of measures will be fixed so as to be consistent with that of [21]. Given a local field $F_v$, we will choose the additive Haar measure $dx$ to be self-dual, that is, $\hat{f}(x) = f(-x)$ with respect to the Fourier transform

$$\hat{f}(y) = \int_{F_v} f(x) \psi(xy) dx,$$

where the additive character $\psi(x)$ is defined by

$$\psi(x) = \begin{cases} 
\exp(2\pi i x) & F_v \simeq \mathbb{R} \\
\exp(4\pi i \text{Re}(x)) & F_v \simeq \mathbb{C} \\
\exp(-2\pi i \text{Tr}_{F_v/Q_v}(x)) & F_v \text{ finite extension of } Q_v.
\end{cases}$$

Thus, $dx$ is the Lebesgue measure if $F_v \simeq \mathbb{R}$, twice the Lebesgue measure if $F_v \simeq \mathbb{C}$, and such that $\text{vol}(O_v) = |\mathfrak{o}|^{1/2}$ where $\mathfrak{o}$ is the different of $F_v$ over $Q_v$, if $F_v$ is a finite extension of $Q_v$. Fix the multiplicative measure $d^\times x$ on $F_v^\times$ to be $d^\times x = \zeta_{F_v}(1)|x|^{-1} dx$, where $\zeta_{F_v}(s) = \pi^{-s/2}\Gamma(s/2)$ for $F_v \simeq \mathbb{R}$, $2(2\pi)^{-s}\Gamma(s)$ for $F_v \simeq \mathbb{C}$, and $(1 - q_v^{-s})^{-1}$ for $F_v$ nonarchimedean, with residue field of cardinality $q_v$.

Let $(V, q)$ be a quadratic space of a local field $F_v$. The self-dual measure on $V$ with respect to $\psi$ is the unique Haar measure $dx$ such that the Fourier transform

$$\hat{f}(y) = \int_{F_v} f(x) \psi(\langle x, y \rangle) dx$$

satisfies $\hat{f}(x) = f(-x)$, where $\langle x, y \rangle = q(x + y) - q(x) - q(y)$ is the associated bilinear pairing.

Let $E$ be a quadratic étale algebra extension, or a quaternion algebra over $F_v$ with reduced norm $q$. Endow $K$ with the self-dual Haar measure with respect to $(E, q)$, and $E^\times$ with the multiplicative Haar measure

$$d^\times x = \zeta_E(1)|q(x)|^{-j} dx$$

where $j = 1$ if $E$ is a quadratic étale algebra extension, and $j = 2$ if it is a quaternion algebra. Let $E^3$ be set of elements in $E^\times$ with reduced norm 1. If $E = K$ a quadratic extension, set

$$\text{vol}(K^1) = \begin{cases} 
2 & K = \mathbb{C}, F_v = \mathbb{R} \\
|D|^{1/2} & K/F_v \text{ nonsplit, unramified,} \\
2|D|^{1/2}|\mathfrak{o}|^{1/2} & K/F_v \text{ ramified}
\end{cases}$$

where $D$ is the discriminant of $K$ in $F_v$, and $\text{vol}(O_K) = |D|^{1/2}|\mathfrak{o}|$.

If $E$ is a quaternion algebra, we have for $F_v$ nonarchimedean,

$$\text{vol}(G(O_v)) = \zeta_{F_v}(2)^{-1}\text{vol}(O_v)^4, \quad \text{vol}(SL(2, O_v)) = \zeta_{F_v}(2)^{-1}\text{vol}(O_v)^3,$$
whereas for \( F_v = \mathbb{R} \) and \( E \) equal to the Hamiltonian algebra, we have \( \text{vol}(E^1) = 4\pi^2 \).

### 1.2.2. Global measures.

Let \( G \) be a reductive group over a global field \( F \), with center \( Z \). Assume that the volume of the adelic quotient \( [G] = G(F)\backslash G(\mathbb{A}_F)/Z(\mathbb{A}_F) \) is finite under some Haar measure \( dg \) of \( G(\mathbb{A}_F)/Z(\mathbb{A}_F) \). Denote by \( G^1 \) the norm one subgroup of \( G \). Hence, taking the Tamagawa measure we have \( \text{vol}(K^1\backslash A^1_K) = 2L(1, \eta) \). On the other hand, the Tamagawa measure on the quaternion algebra \( B \) over \( F \) induces

\[
(1.13) \quad \text{vol}(B^1\backslash B^1) = 1, \quad \text{and} \quad \text{vol}(B^\times\backslash B^\times / Z(A)) = 2,
\]

where \( B^1 \) denotes the elements of \( B^\times \) with reduced norm one.

Define the modulus function

\[
(1.14) \quad \delta_v : \ P(F_v) \rightarrow \mathbb{R}^\times : \begin{pmatrix} a & b \\ d & 1 \end{pmatrix} \mapsto \left| \frac{a}{d} \right|_{v}^{1/2},
\]

where \( P(F_v) \) is the parabolic group of \( F_v \). This extends to a map \( \delta_v \) from \( G_2(F_v) \) to \( \mathbb{R}^\times \). Let \( \delta = \prod_v \delta_v \). Define the regularized integral

\[
(1.15) \quad \int_{[G]} f(g)dg = \int_{[G]} \frac{1}{\text{vol}(Z(A))} \int_{Z(A)} f(zg)dz \, dg,
\]

for any automorphic function \( f \) on \( G(A) \) invariant under \( Z(F_\infty) \).

Also, denote by \( (f_1, f_2)_{\text{Pet}} \) the Petersson pairing for automorphic forms \( f_1, f_2 \) on \( G(A) \) with respect to the Tamagawa measure on \( G \)

\[
(1.16) \quad (f_1, f_2)_{\text{Pet}} = \int_{Z(A)G(F)\backslash G(A)} f_1(g)f_2(g)dg,
\]

which is defined when at least one of the two automorphic forms is cuspidal.

### 2. Kuga-Sato varieties over Shimura curves

In this section we introduce the main objects of our study and some of their properties, and we give a more precise statement of Theorem 1.0.1.

#### 2.1. Quaternion algebras.

Recall that \( F \) is a totally real number field with adele ring \( \mathbb{A} = \mathbb{A}_F \). Let \( S \) be a finite set of places of \( F \) of odd cardinality containing the archimedean places \( S_\infty \). Let \( B \) be a (incoherent) quaternion algebra over \( \mathbb{A} \), such that \( B_v = B \otimes \mathbb{A}_F \) is isomorphic to \( M_2(F_v) \) if \( v \not\in S \) and to the unique division quaternion algebra over \( F_v \) for \( v \in S \). Note that \( B \) is totally definite at infinity, that is, \( B_v \) is the Hamiltonian algebra for every \( v \in S_\infty \).

For a place \( v \) of \( F \), we denote by \( B(v) \) the nearby quaternion algebra over \( F \) obtained from \( B \) by changing the Hasse invariant at \( v \), and we fix an identification \( B(v) \otimes_F \mathbb{A}_F \simeq B_f \). The ramification set of \( B(v) \) has distance one from \( S \), that is, it is either \( S \cup \{v\} \) or \( S - \{v\} \) depending on whether \( B \) is ramified at \( v \).

#### 2.2. The Kuga-Sato variety.

For every open compact subgroup \( U \) of \( B_f^\times = (B \otimes \mathbb{A}_F \mathbb{A}_F)^\times \), we have a compactified quaternionic Shimura curve \( X_U \) over \( F \). Over \( C \), given an embedding \( \tau : F \hookrightarrow C \), and the quaternion algebra \( B(\tau) \) over \( F \) ramified exactly at \( S - \{\tau\} \), we have a uniformization

\[
(2.1) \quad X_U = (B(\tau)^\times \backslash \mathbb{H}^\pm \times B_f^\times)/U
\]
where $\mathbf{H}^\pm = \mathbf{C} - \mathbf{R}$. The set of cusps is empty (and hence $X_U$ is compact) since we exclude the setting where $F = \mathbf{Q}$ and $S = \{\infty\}$ already considered in [23], in which $X_U$ is the classical modular curve.

For $F = \mathbf{Q}$, consider the projective limit
\begin{equation}
X = \lim_{\leftarrow} X_U,
\end{equation}
over compatible systems of open compact subgroups $U$. The Shimura curve $X_U$ admits a model over $\mathbf{Q}$ which is the coarse moduli scheme classifying abelian varieties $A$ with quaternionic multiplication (QM) by $U$, that is, an optimal embedding $\iota_U : U \rightarrow \text{End}(A)$. Given an integer $M \geq 3$, consider the fine moduli scheme classifying pairs $(A, \iota)$ endowed with level $M$ structure, that is, an embedding $v : U/MU \rightarrow A[M]$, and let $\mathcal{A}_U$ be a universal abelian variety with QM by $U$. Let $\mathcal{A} = \lim_{U} \mathcal{A}_U$.

The Kuga-Sato variety $W_U$ over the Shimura curve $X_U$ is defined as the $(k-1)$-th fibre product of $\mathcal{A}_U$ over $X_U$,
\begin{equation}
W_U = \mathcal{A}_U \times_{X_U} \cdots \times_{X_U} \mathcal{A}_U \rightarrow X_U.
\end{equation}
It is a nonsingular projective variety of dimension $2k - 1$. In the same way, we define the Kuga-Sato variety $W$ over the Shimura curve $X$ to be the $(k-1)$-fold fibre product of $\mathcal{A}$ over $X$, or, equivalently,
\begin{equation}
W = \lim_{U} W_U.
\end{equation}

For $F$ totally real, the Shimura curve $X_U$ is not a moduli space of abelian varieties, but is related to one that is, i.e., the Shimura curve associated to the group $GU$ of unitary similitudes, as in [5], and over these the analogue of the Kuga-Sato variety was introduced by T. Saito in [17, Section 6.3].

2.3. CM cycles over Shimura curves. Complex multiplication cycles on Kuga-Sato varieties arose in the work of S. Zhang [23] over the classical modular curve and in the work of Iovita and Spieß, Besser, Chida, and de Vera Piquero and the first author [10, 2, 6, 7] over the Shimura curve. In this subsection, we describe the construction of CM cycles in the case $F = \mathbf{Q}$.

Recall that $K_c$ denotes the ring class field of $K$ of conductor $c$. An abelian surface $A$ with QM has by $U$ has complex multiplication (CM) by $K_c$ if the ring of endomorphisms of $A$ commuting with the optimal embedding $\tau : U \rightarrow \text{End}(A)$ denoted by $\text{End}(A, \tau)$ is isomorphic to $K_c$. The complex multiplication points of conductor $c$ of the Shimura curve $X$ are in bijection with isomorphism classes of abelian surfaces $A$ with QM by $U$ and CM by $K_c$. Such an abelian surface is isomorphic to a product of elliptic curves with CM by $K_c$
\begin{equation}
A \cong E \times E_a,
\end{equation}
where $E(\mathbf{C}) = C/K_c, E_a(\mathbf{C}) = C/a$, and $a$ is an ideal of $K_c$ defined by $U = K_c + ea$ for some $e$ in $B$.

Given a map $\gamma : E \rightarrow E_a$, we denote by
\begin{equation}
Z_\gamma = \text{graph}(\gamma) - (E \times 0) - \text{deg} \gamma (0 \times E_a) \subseteq E \times E_a.
\end{equation}
Writing \( a = Za + Z\sqrt{-D} \) for \( a \) in \( Z \), the Néron-Severi group \( NS(A) \) of \( A \) is given by
\[
(2.7) \quad NS(A) = \langle E \times 0, \ 0 \times E_a, \ Z_a, \ Z\sqrt{-D} \rangle.
\]
We define \( Z_A = Z\sqrt{-D} \subseteq A \). Then \( Z = Z_A^{k-1} \) is a fiber of \( W \) with dimension \( k - 1 \).
Note that its construction depends on a given CM point on \( X \), which arises from an embedding of the imaginary quadratic field \( K \to B^\times \). This embedding exists because primes dividing the discriminant of \( B \) are inert in \( K \).

### 2.4. Chow groups and height pairings

Given a regular arithmetic scheme \( Y \) of dimension \( d \) over \( \mathcal{O}_F \), and an integer \( p \geq 0 \), we denote by \( A^{p,p}(Y) \) the real vector space of real differential forms of type \( (p,p) \) on \( C \), such that \( \tau^*\alpha = (-1)^p\alpha \). Here, \( \tau \) stands for complex conjugation. Similarly, \( D^{p,p}(Y) \) refers to currents satisfying these properties. A cycle \( Z = \sum a_\alpha Z_\alpha \), where \( Z_\alpha \) are closed irreducible subvarieties of codimension \( p \) on \( Y \) defines a current of integration \( \delta_Z \). For a form \( \eta \) of complementary degree, \( \delta_Z(\eta) = \sum a_\alpha \int_{Z_\alpha}(\eta) \), where \( Z_\alpha^{ns} = Z_\alpha - Z_\alpha^{sing} \) is the smooth part of \( Z_\alpha \). A green current for \( Z \) is any current \( g \) in \( D^{p-1,p-1}(Y_{\mathbb{R}}) \) such that the curvature \( h_Z = \delta_Z - \frac{\delta\delta}{\pi i}g \) is a smooth form in \( A^{p,p}(Y) \).

Consider the Chow group \( \widehat{CH}^k(X)_0 \) of codimension \( k \) null-homologous algebraic cycles on \( X \) modulo rational equivalence. Let \( \widehat{CH}^k(X)_0 \) be the group generated by pairs \( (Z,g) \) where \( Z \) is in \( CH^k(X)_0 \), and \( g \) is a green current for \( Z \). There is an associative and commutative intersection product due to Gillet and Soulé
\[
(2.8) \quad \widehat{CH}^k(X)_0 \times \widehat{CH}^{k'}(X)_0 \to \widehat{CH}^{k+k'}(X)_0
\]
mapping cycles \( \hat{Z}_1 = (Z_1, g_1) \) in \( \widehat{CH}^k(X)_0 \) and \( \hat{Z}_2 = (Z_2, g_2) \) in \( \widehat{CH}^{k'}(X)_0 \) with codim(\( Z_1 \cap Z_2 \)) = \( k+k' \) to \( (\int Z_1 \cdot Z_2, g_2\delta Z_1, h_2g_1) \). This defines a height pairing by
\[
(2.9) \quad \langle Z_1, Z_2 \rangle_{GS} = (-1)^k \hat{Z}_1 \cdot \hat{Z}_2,
\]
where the dot indicates taking intersection in \( CH^k(X) \times CH^{k'}(X) \). In Section (7) we discuss the nondegeneracy of the Gillet-Soulé pairing.

When the arithmetic cycles \( \hat{Z}_1 = (Z_1, g_1) \) and \( \hat{Z}_2 = (Z_2, g_2) \) have codimension \( p \) and \( d-p \) in \( Y \), with \( Z_1 \) and \( Z_2 \) irreducible intersecting properly, then
\[
(2.10) \quad \hat{Z}_1 \cdot \hat{Z}_2 = \log |\Gamma(Z_1 \cdot Z_2, 0)| + \int_{Z_1}(C) g_1 + \int_{Y(C)} g_2 h_{Z_1}.
\]

### 2.5. Hodge structures

Denote by \( \mathcal{A}(B^\times) \) the set of equivalence classes of irreducible admissible representations \( \pi \) of \( B^\times \) that correspond by Jacquet-Langlands to cuspidal automorphic representations of \( GL_2(A_F) \), discrete of parallel weight \( 2k \) at infinity. By \cite[Theorem 5.9]{[2]}, we have a Hodge decomposition of the étale cohomology of \( W_U \) depending on the fixed embedding \( \tau \) of \( F \),
\[
(2.11) \quad H^{k-1}(W_U(C), Q) \otimes Q C \simeq \bigoplus_{1 \leq i \leq k-1} H^i(W_U(C), R^{k-1-i}p_{U*}Q) \otimes Q C,
\]
where \( p_{U*} : W_U \to X_U \), and a natural isomorphism
\[
(2.12) \quad H^{k-1,0}(W_U, C) \simeq \bigoplus_{\pi \in \mathcal{A}(B^\times)} \pi^U,
\]
where \( \pi^U \) are the fixed vectors of \( \pi \) by \( U \). Note that \( \pi^U \) is nonzero for only finitely many \( \pi \), and appears with multiplicity at most one. By complex conjugation we also have a decomposition of \( H^{0,k-1}(W_U, \mathbb{C}) \) into \( \bigoplus_{\pi \in A(B^\times)} \pi^U \), whereas the con-travariant representation gives a decomposition into the dual spaces \( \bigoplus_{\pi \in A(B^\times)} \pi^U \), with the usual pairing

\[
(2.13) \quad \pi \times \bar{\pi} \rightarrow \mathbb{C}.
\]

Taking direct limits, we obtain similar decompositions of the cohomology of \( W \) in terms of \( \pi, \bar{\pi} \) and \( \pi \).

2.6. Hodge class. The action of \( U \) on \( \mathcal{A}_U \) induces an action of \( B^\times \) on \( R^q \pi_* Q_p \), for \( q \geq 1 \). Consider the \( p \)-adic sheaf

\[
\mathcal{L}_U := \bigcap_{b \in B} \ker(b - n(b) : R^2 \pi_* Q_p \rightarrow R^2 \pi_* Q_p),
\]

which is a 3-dimensional local system on the Shimura curve. The non-degenerate pairing

\[
(\cdot, \cdot) : \mathcal{L}_U \otimes \mathcal{L}_U \rightarrow R^2 \pi_* Q_p \otimes R^2 \pi_* Q_p \rightarrow R^4 \pi_* Q_p \rightarrow Q_p(-2)
\]

induces a Laplacian operator

\[
\Delta_{k-1} : \text{Sym}^{k-1} \mathcal{L}_U \rightarrow (\text{Sym}^{k-3} \mathcal{L}_U)(-2),
\]

and \( \mathcal{L}_{U,2k-2} \subset \text{Sym}^{k-1} \mathcal{L}_U \) is defined as the kernel of \( \Delta_{k-1} \).

We consider the projector constructed in [2, Theorem 5.8(iii)],

\[
(2.14) \quad \mathcal{P}_U \in \text{Corr}^0(W_U, W_U) = \text{CH}^{2k-1}(W_U \times W_U)
\]

encoding the construction of the line bundle \( \mathcal{L}_{U,2k-2} \). Its global sections are holomorphic modular forms of weight \( 2k \) and level \( U \) [10, Section 10.1]. We denote by \( \deg(\mathcal{P}_U) \) the degree of the correspondence.

In what follows, we index the connected components of \( W_U \) by \( \alpha \) in \( \pi_0(W_U, F) \), and denote by \( \mathcal{P}_{U,\alpha} \) the restriction of \( \mathcal{P}_U \) to a connected component \( \alpha \) of \( W_U \).

2.7. Hecke correspondences. Let \( R_x \) be the action of \( x \) in \( B^\times \) on \( X \) by right translation, so that \( X_U \) is the quotient of \( X \) by the action of \( \{ R_x : x \in U \} \). This gives an isomorphism \( X_{x^{-1}Ux} \rightarrow X_U \) for each \( U \), inducing a map \( W \rightarrow W \) which we still denote by \( R_x \). The Hecke algebra \( \mathcal{H} = C^\infty(B^\times_U) \) is defined as the smooth and compactly supported functions from \( B^\times_U \) to \( \mathbb{C} \). Given \( \phi \in \mathcal{H} \), define a correspondence on \( W \) given by

\[
(2.15) \quad R(\phi) = \int_{B^\times_U} \phi(x) R_x \, dx.
\]

At level \( U \), define the Hecke algebra \( \mathcal{H}_U = \{ \phi \in \mathcal{H} | \phi(UxU) = \phi(x), \ x \in B^\times_U \} \).

Let \( Z(x) \subset W \times W \) be the graph of \( R_x \), and \( Z(x)_U \) its image in \( W_U \times W_U \). Define the linear combination

\[
(2.16) \quad R(\phi)_U = \sum_{x \in U \backslash B^\times/U} \phi(x) Z(x)_U
\]

in \( \text{CH}^{2k-1}(W_U \times W_U)_C \). We have a map

\[
(2.17) \quad \text{CH}^{2k-1}(W_U \times W_U) \rightarrow \text{Hom}(\text{CH}^{k-1}(W_U), \text{CH}^{k-1}(W_U)),
\]
given by pushforward of correspondences,

\[ (2.18) \quad c \mapsto (c_1 \mapsto \text{pr}_2^*(\text{pr}_1^{-1}(c_1) \cdot c)). \]

The direct systems are compatible as \( U \) varies by the same argument as in [21, Lemma 3.1.2], observing that \([12, \text{Prop} 1.13]\) holds in our setting. Consider the diagonal subspace

\[ (2.19) \quad \iota_U : \bigoplus_{\pi \in \mathcal{A}(\Bbbk^\times)} \pi^U \otimes \pi^U \hookrightarrow \text{Hom}(H^{k-1,0}(W_U, \mathcal{C}), H^{k-1,0}(W_U, \mathcal{C})). \]

The following lemma allows us to identify its image with an induced morphism on Chow groups.

**Lemma 2.7.1.** For \( \pi \) in \( \mathcal{A}(\Bbbk^\times) \), there is a function \( \phi \) in \( \mathcal{H}_U \) such that

\[ (2.20) \quad \iota_U(\pi^U \otimes \pi^U) = R(\phi)_U. \]

The proof of the lemma is as in [21, Prop 3.14(1)]. By the lemma, it follows that the image of \( \iota_U \) is contained in the image of the inclusion

\[ (2.21) \quad \text{Hom}_F(CH^{k-1}(W_U), CH^{k-1}(W_U))_C \rightarrow \text{Hom}(H^{k-1,0}(W_U, \mathcal{C}), H^{k-1,0}(W_U, \mathcal{C})), \]

since the image of (2.19) is represented by the induced morphism \( R(\phi)_U \) on (2.17). This defines a map

\[ (2.22) \quad T_U : \bigoplus_{\pi \in \mathcal{A}(\Bbbk^\times)} \pi^U \otimes \pi^U \rightarrow \text{Hom}_F(CH^{k-1}(W_U), CH^{k-1}(W_U))_C. \]

**Lemma 2.7.2.** We have a direct system

\[ (2.23) \quad T = \{ \text{deg} (\mathcal{P}_U) T_U \}_U, \]

defining a map

\[ (2.24) \quad T : \bigoplus_{\pi \in \mathcal{A}(\Bbbk^\times)} \pi \otimes \pi \rightarrow \text{Hom}^0(CH^{k-1}(W'), CH^{k-1}(W'))_C. \]

**Proof.** The key property of the normalizing volume factor \( \text{vol}(X_U) \) in [21, Prop 3.14(2)] is that it is computed by the degree of the Hodge bundle \( \text{deg}(L_U) \). We normalize instead by \( \text{deg}(\mathcal{P}_U) \) and use the pullback and pushforward maps \( p_U^*, \pi_U \) on the spaces \( H^{k-1,0}(W_U, \mathcal{C}) \) and \( H^{k-1,0}(W_U, \mathcal{C}) \) corresponding to the inclusion of open compact sets \( U' \subset U \), the lemma follows.

---

### 2.8. The main identity

We formulate the main identity in terms of projectors. Recall that \( K \) is an imaginary quadratic extension of \( F \) with a fixed embedding \( \mathbb{A}_K \hookrightarrow \Bbbk^\times \) over \( \mathbb{A}_F \) defining a torus \( T \) over \( F \) and an associated quadratic character \( \eta : K^\times \backslash \mathbb{A}_K^\times \rightarrow \mathbb{C}^\times \). Also, \( \chi \) is a character \( \chi : \text{Gal}(K^{ab}/K) \rightarrow \mathbb{C}^\times \rightarrow \mathbb{C}^\times \) with \( \omega_\pi \cdot \chi|_{\mathbb{A}_F} = 1 \), where \( \omega_\pi \) is the central character of \( \pi \). By the reciprocity law, it can be viewed as a character of \( K^\times \backslash \mathbb{A}_K^\times \rightarrow \mathbb{C}^\times \).

Denote by \( \langle , \cdot \rangle : \pi_v \otimes \pi_v \rightarrow \mathbb{C} \) the pairing of contragredient representations. Consider the period integral \( \alpha = \otimes_v \alpha_v : \pi \otimes \tilde{\pi} \rightarrow \mathbb{C} \) where \( \alpha_v \) is defined by the linear forms

\[ (2.25) \quad \alpha_v(f_1, f_2) = \frac{L(1, \eta_v)L(1, \pi_v, \text{ad})}{\zeta_F(2)L(k - \frac{1}{2}, \pi_v, \chi_v)} \int_{K^\times \backslash F^\times} (\pi_v(t) f_1, f_2) \chi_v(t) dt \]

for \( f_1 \) in \( \pi_v \) and \( f_2 \) in \( \tilde{\pi}_v \).
Consider the CM-cycle $Z$ arising from the CM point of conductor 1 on the Shimura curve corresponding to the embedding $K \hookrightarrow \mathcal{B}^\times$. Viewing $R_t$ as a correspondence in $\text{CH}^{2k-1}(W \times W)$, the $\chi$-eigencomponent $Z_\chi$ of $Z$, given by

$$Z_\chi = \text{vol}^{-1}(\mathcal{T}) \int_\mathcal{T} R_t(Z)(t) dt,$$

belongs to $\text{CH}^{2k-1}(W)_h(\bar{F})_\mathbb{C}$.

We state the main theorem.

**Theorem 2.8.1.** For $f_1 \otimes f_2$ in $\pi \otimes \tilde{\pi}$, we have

$$L'(\frac{1}{2}, \pi, \chi) = \frac{(2k-1)!}{2^{2k-2}} \langle \mathcal{T}(f_1 \otimes f_2)Z_\chi, Z_{\chi^{-1}} \rangle_{\text{GS}},$$

where $\langle \cdot, \cdot \rangle_{\text{GS}}$ is the Gillet-Soulé pairing on arithmetic Chow groups.

The Gross-Zagier formula obtained in [23, Corollary 0.3.1] under the Heegner hypothesis is given by

$$L'(k, f, \chi) = \frac{(2k-1)!}{2^{2k-2}} \langle s_{X,f}^{\prime \chi}, s_{X,f}^{\prime \chi} \rangle_{\text{GS}},$$

where $f$ is a weight $2k$ modular form and $s_{X,f}^{\prime \chi}$ is a suitable Heegner cycle on the Kuga-Sato variety fibered over the classical modular curve. It can be compared to our formula by a careful use of the explicit computations in [4] for the fudge factors in the Gross-Zagier and Waldspurger formulae. More precisely, the classical Petersson inner product on weight $2k$ modular forms $(f,f)_{\text{Pet}}$ is proportional to $L(1, \pi, \text{ad})/\zeta_F(2)$ (see [4, Proposition 1.11]), the constant $(2k-2)!$ arises from the Gamma factors of the archimedean $L$-values, and the arithmetic constants are given by the usual class number formula for $L(1, \eta)$.

### 3. Analytic and geometric kernels

#### 3.1. Preliminaries

In this section, we describe the Schwartz and Fock spaces, Waldspurger’s Weil representation, and Shimizu’s theta lift which we use to define the analytic and geometric kernels.

Let $(V, q)$ be an even-dimensional quadratic space over $\mathbb{A}$, where $q$ is the reduced norm. Similarly, denote by $(V_v, q_v)$ the local quadratic spaces. Given an embedding $\mathbb{A}_K \hookrightarrow \mathcal{B}$, we have an orthogonal decomposition

$$\mathcal{B} = \mathbb{A}_K + j\mathbb{A}_K,$$

where $j^2 \in \mathbb{A}_K$. The space $V$ is coherent if it has a model over $F$, and incoherent otherwise. Note that $V_1 = \mathbb{A}_K$ is coherent, while $V_2 = j\mathbb{A}_K$ is incoherent if and only if $S$ is odd.

#### 3.1.1. The Weil representation

Let $F_v$ be a nonarchimedean local field, and consider $(V_v, q_v)$ over $F_v$. Given $u \in F_v^\times$, we can associate to $F_v$ a quadratic space with new norm $(V_v, uq_v)$. Let $GO(V)$ be the general orthogonal group of $V$ with similitude character $\nu : GO(V) \to G_m$, and denote by $S(V_v \times F_v^\times)$ the Schwartz space, that is, compactly supported, locally constant, complex-valued functions on $V_v \times F_v^\times$.

We denote

$$n(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$
where $\chi_{(V,\omega_v)}$ is the quadratic character attached to $V_v$, $\gamma_{(V,\omega_v)}$ is a fourth root of unity depending on $V_v$, and $\hat{\Phi}$ is the Fourier transform of $\Phi(x, u)$ in the $x$ variable, with respect to the self-dual measure determined by $\psi(1.8)$. Set $\mathscr{S}(V_v \times F_v^\times) = S(V_v \times F_v^\times)$ in this case.

Consider $F_v$ real archimedean, and assume $V_v$ is positive definite. Define $S(V_v \times \mathbb{R}^\times)$ to be the Fock space, which is the space of functions spanned by

$$P(x)e^{-2\pi|x||q_v(x)|}H(u),$$

where $H$ is a compactly supported smooth function on $\mathbb{R}^\times$ and $P$ is a complex polynomial function on $V_v$. This space is not stable under the action of $G(\mathbb{R})$, but under the restriction of the induced $(\mathfrak{gl}_2(\mathbb{R}), O_2(\mathbb{R}))$ action on the usual Schwartz space $[21, \S2.1.2]$. Also define the reduced Fock space $\mathscr{S}(V_v \times \mathbb{R}^\times)$, spanned by functions

$$P_1(uq_v(x)) + \text{sgn}(u)P_2(uq_v(x))e^{-2\pi|x||q_v(x)|},$$

where $P_1, P_2$ are polynomial functions, which admits a surjective quotient map $S(V_v \times \mathbb{R}^\times) \rightarrow \mathscr{S}(V_v \times \mathbb{R}^\times)$.

Suppose that $V_{\infty} = \prod_{v \mid \infty} V_v$ is positive-definite. We define the restricted tensor product $S(V \times A^\times) = \otimes_v S(V_v \times F_v^\times)$ with respect to the spherical functions

$$(3.9) \quad \Phi_v(x, u) = 1_{V_v}(x)1_{\mathbb{R}^\times}(u)$$

for nonarchimedean $v$, where $V_v \subset V_v$ is an $O_{F_v}$-lattice, $n_v$ is the level of $\psi_v$, and $\omega_v$ is a uniformizer of $O_{F_v}$. We call $\Phi_v(x, u)$ the standard Schwartz function at $v$. The reduced space $\mathscr{S}(V_v \times \mathbb{R}^\times)$ is similarly defined. The Weil representation gives an action of $(\mathfrak{gl}_2(F_v), O_2(F_v)) \times G(A_f) \times GO(V_f)$ on both spaces.

We introduce in particular at real archimedean places the Schwartz function $\Phi_k(x) = P_k(x)e^{-2\pi|x||q_v(x)|}$ of weight $-2k$,

$$(3.10) \quad P_k(x) = \sum_{j=0}^{k} \binom{\frac{k}{2}}{j} \frac{(-4\pi|x||j!}{(2j)!} \left(x_1 - x_2\right)^{2j}$$

so that $r(k_\theta)\Phi_k(x) = e^{-2\pi ik_\theta} \Phi_k(x)$ for all $k_\theta \in SO_2(\mathbb{R})$ [15, Proposition 2.5.5].

Remark 3.1.1. If $V$ is the orthogonal space $(B, q)$, we can identify $S(V \times A^\times)$ with the space of Schwartz functions bi-invariant under an open compact subgroup $U$ of $B^\times$, and we identify $\mathscr{S}(V \times A^\times)$ with the maximal $B^\times \times B^\times$ quotient. Given an element $\Phi$ in $S(V \times A^\times)$, its image in $\mathscr{S}(V \times A^\times)$ can be obtained by the integration

$$(3.11) \quad \int_{F_v^\times \backslash (B^\times \times B^\times_v)} r(1, h)\Phi(x, u)dh_\infty,$$
where \( F_\infty \) is diagonally embedded into \( B_\infty^* \times B_{\tau}^* \).

3.1.2. Shimizu’s theta lift. Let \( \sigma \) be a cuspidal automorphic representation of \( G(A) \) which is a discrete series at all places in \( S \), and let \( \pi \) be the representation of \( B^* \) associated to \( \sigma \) by the Jacquet-Langlands correspondence. Consider the embedding of \( (B^* \times B^*)/A^* \) in \( GO(V) \) where \( h = (h_1, h_2) \in (B^* \times B^*)/A^* \) acts on \( x \) in \( V \) by \( (h_1, h_2)x = h_1xh_2^{-1} \). For \( g \) in \( G(A) \), \( h \) in \( (B^* \times B^*)/A^* \), and \( \phi \) in \( S(V \times A^*) \), define the theta series

\[
\theta(g, h, \phi) = \sum_{u \in F^*} \sum_{x \in V} r(g, h) \phi(x, u).
\]

The incoherent (global) Shimizu lift of \( \varphi \) in \( \sigma \), given by

\[
\theta(\varphi \otimes \Phi)(h) = \frac{\zeta_F(2)}{2L(1, \pi, \text{ad})} \int_{G(F) \backslash G(A)} \varphi(g) \theta(g, h, \Phi) dg,
\]

is an element in \( \text{Hom}_{G(A) \times B \times B}(\sigma \otimes S(V \times A^*), \pi \otimes \tilde{\pi}) \).

3.2. Analytic kernel. Our expression for the analytic kernel follows from the derivation of [21, §2], based on the Waldspurger formula [19], by observing that the proofs of the necessary statements hold for general cuspidal automorphic forms in \( GL_2 \).

For \( g \) in \( G(A) \), and \( \Phi \) in \( S(V \times A^*) \), define the mixed Eisenstein-theta series

\[
I(s, g, \Phi) = \sum_{\gamma \in P(F) \backslash G(F)} \delta(\gamma g)^s \sum_{(x_1, x_2) \in V^2 \times F^2} r(\gamma g, 1) \Phi(x_1, x_2).
\]

The analytic kernel is its \( \chi \)-component

\[
I(s, g, \chi, \Phi) = \int_{T(F) \backslash T(A)} I(s, g, r(t, 1) \Phi) \chi(t) dt.
\]

Given \( \varphi \in \sigma \), consider the Petersson inner product

\[
P(s, \chi, \Phi, \varphi) := \langle I(s, g, \chi, \Phi), \varphi(g) \rangle_{\text{pet}} = \int_{[G]} \varphi(g) I(s, g, \chi, \Phi) dg.
\]

By a result of Waldspurger [21, Prop 2.5], we have \( P(s, \chi, \Phi, \varphi) = \prod_v P_v(s, \chi_v, \Phi_v, \varphi_v) \), where \( P_v(s, \chi_v, \Phi_v, \varphi_v) \) is given by the local integral representation

\[
P_v(s, \chi_v, \Phi_v, \varphi_v) = \int_{Z_v \backslash T_v} \chi(t) dt \int_{\mathbb{A}^\times_v \backslash G_v} \delta(g)^s W_{-1, v}(g) r(g, 1) \Phi_v(t^{-1}, q_v(t)) dg,
\]

and \( W_{-1, v} \) is the Whittaker function of \( \varphi_v \) with respect to \( \tilde{\psi}_v \). Moreover, for \( v \) unramified, we have

\[
P_v(s, \chi_v, \Phi_v, \varphi_v) L(1 + s, \eta_v) = L\left(\frac{1 + s}{2}, \pi_v, \chi_v\right).
\]

Hence, we normalize the local integrals by the functional \( \alpha_v \) in \( \text{Hom}(\pi_v \otimes \tilde{\pi}_v, \mathbb{C}) \), given by the integration of matrix coefficients:

\[
\alpha_v(f_1 \otimes f_2) = \frac{L(1, \eta_v) L(1, \pi_v, \text{ad})}{\zeta_F(2) L(\frac{1}{2}, \pi_v, \chi_v)} \int_{T(F_v) \backslash T(F)} (\pi_v(t)f_1, f_2) \chi_v(t) dt,
\]

where as usual the torus \( T \) is defined by the quadratic extension \( K \).

This leads to the integral representation of the \( L \)-function:
Proposition 3.2.1 ([21, Prop 2.6]). If $|S|$ is odd, then $L(t, \pi, \chi) = 0$ and the value of the derivative of $P(s, \chi, \Phi, \varphi)$ at $s = 0$ is

$$L'(1, \pi, \chi) \prod_v \alpha_v(\theta(\Phi_v \otimes \varphi_v)).$$

This analytic kernel will be compared with the geometric kernel, which we introduce next.

3.3. The geometric kernel. In order to define the geometric kernel, we first define the generating series of algebraic cycles, which will be the focus of our study.

3.3.1. The generating series. Our goal is to define a generating series in

$$\text{Hom}_{\mathcal{B} \times \mathcal{B} \times G(\text{A})}(S(V, \text{A}^\times), C^\infty(G(F) \setminus G(\text{A}), \text{CH}^{2k-1}(W \times W))).$$

First, observe that $\text{CH}^{2k-1}(W \times W)$ is invariant under $\mathcal{B}_\infty \times \mathcal{B}_\infty$. Thus we may work with functions in the quotient $\mathcal{Z}(V \times \text{A}^\times)$.

For $g$ in $GL_2(\text{A}), u$ in $F^\times$, and $\Phi$ in $\mathcal{Z}(V \times \text{A}^\times)$, define the Siegel Eisenstein series by

$$E(s, g, u, \Phi) = \sum_{\gamma \in F^g(\text{SL}_2(F))} \delta(\gamma g) \delta(\gamma g, 1) \Phi(0, u).$$

The constant term of its Fourier expansion is given in terms of Whittaker functions

$$E_0(s, g, u, \Phi) = \delta(g) r(g, 1) \Phi(0, u) + W_0(s, g, u, \Phi),$$

where

$$W_a(s, g, u, \Phi) = \int_{N(\text{A})} \delta(wng) \delta(wng, 1) \Phi(0, u) \psi(-an) dn$$

for $a \in F^\times$. Here we have written $n = n(b)$. The so-called intertwining part $W_0(s, g, u, \Phi)$ has analytic continuation to $s = 0$, is nonzero only if $F = \mathbb{Q}$ and $S = \{\infty\}$, and satisfies

$$W_0(0, g, u, \Phi) = W_0(0, 1, u, r(g, 1) \Phi).$$

Let $\mu_U = F^\times \cap U$ which can be viewed as a finite index subgroup of $O_f^\times$, and let $w_U = \{|-1, 1] \cap U|$ which is equal to 1 if $U$ is small enough, otherwise it is 2.

We define the generating series

$$Z(g, \Phi)_U = Z_0(g, \Phi)_U + Z_* (g, \Phi)_U,$$

where

$$Z_*(g, \Phi)_U = w_U \sum_{a \in F^\times} \sum_{x \in U \setminus \mathcal{B}_f^x / U} r(g) \Phi(x, aq(x^{-1})) Z(x)_U,$$

and

$$Z_0(g, \Phi)_U = -\sum_{\alpha \in \mu_U^2 \setminus F^\times} \sum_{\alpha^{-1} u} E_0(0, 1, \alpha^{-1} u, r(g) \Phi) \mathcal{P}_{U, \alpha}.$$

We refer to $Z_0(g, \Phi)_U$ as the constant part of $Z(g, \Phi)_U$.

We normalize the generating series as follows

$$Z(g, \Phi)_U = \frac{2^{[F:\mathbb{Q}]} h_F |d_F|^{-1/2}}{[O_f^\times : \mu_U^2]} Z(g, \Phi)_U,$$
where $h_F$ is the class number of $F$, and $d_F$ the discriminant of $F$. Note that the quotient equals 2 if $F = \mathbb{Q}$. Also, the denominator is finite since $\mu_F^2$ is a subgroup of finite index, and ensures that the right-hand side is compatible under pull-back from different levels.

**Lemma 3.3.1.** The system $\tilde{Z}(g, \Phi) = \{ \tilde{Z}(g, \Phi)_U \}_U$ is compatible with pull-back and defines an element of $\text{CH}^{2k-1}(W \times W)_C$.

**Proof.** The proof follows [21, Lemma 3.18] with the pull-back maps obtained instead from the morphism of Kuga-Sato varieties

\[(3.30) \quad W_U \times W_U^\vee \to W_U \times W_U^\vee \]

induced from right multiplication of $(h_1, h_2)$ in $B_f \times B_f^\times$ on the underlying Shimura curves, under the condition that $h_i h_i^{-1} U \subset U$ and $U$ acts trivially on $\Phi$. \(\square\)

**3.3.2. The geometric kernel.** Define the height series

\[(3.31) \quad \tilde{Z}(g, (h_1, h_2), \Phi) = \tilde{Z}(g, \Phi) R_{h_1}(Z), R_{h_2}(Z))_{\text{GS}} \]

for $h_1, h_2$ in $B^\times$. As in [21, Lemma 3.19], the height series $\tilde{Z}(g, (h_1, h_2), \Phi)$ is independent of $Z$ and invariant under the left action of $T_F \times T_F$ on $(h_1, h_2)$, and it is a cuspidal form on $GL_2(A)$. The geometric kernel is the $\chi$-component of $\tilde{Z}(g, (h_1, h_2), \Phi)$ given by

\[(3.32) \quad \tilde{Z}(g, \chi, \Phi) = \int_{[}\tilde{Z}(g, (t, 1), \Phi) \chi(t) dt. \]

**Lemma 3.3.2.** The geometric kernel $\tilde{Z}(g, \chi, \Phi)$ is a cuspidal automorphic representation of $GL_2(A)$, and satisfies

\[(3.33) \quad \tilde{Z}(g, \chi, r(t_1, t_2)\Phi) = \chi(t_1^{-1} t_2) \tilde{Z}(g, \chi, \Phi) \]

for $t_1, t_2 \in A_1^\times$.

**Proof.** See [21, Lemma 3.20]. \(\square\)

**3.4. The kernel identity.** We describe the arithmetic theta lifting and use it to show that the main theorem is equivalent to a kernel identity that can be viewed as an arithmetic Siegel-Weil formula according to the philosophy of Kudla. We give two versions, the first in terms of a function $\Phi$ in $S(V \times A^\times)$, and the second in terms of its image $\phi$ in $\tilde{\mathcal{H}}(V \times A^\times)$.

**3.4.1. Kernel identity I.** Shimizu’s theta lift gives a map $\theta : S(V \times A^\times) \times \sigma \to \pi \otimes \tilde{\pi}$, which we compose with a Hecke correspondence $T : \pi \otimes \tilde{\pi} \to \text{Hom}_F(\text{CH}^{k-1}, \text{CH}^{k-1})$. For $\varphi$ in $\sigma$ and $\Phi$ in $S(V \times A^\times)$, the arithmetic theta lifting asserts that

\[(3.34) \quad \tilde{Z}(\varphi \otimes \Phi) := \int_{[}\varphi(g) \tilde{Z}(g, \Phi) dg \]

is proportional to the Shimizu lift

\[(3.35) \quad \tilde{Z}(\varphi \otimes \Phi) = \frac{L(1, \pi, \text{ad})}{2 \zeta_F(2)} T(\theta(\varphi \otimes \Phi)), \]

see Theorem (4.2.1) for a precise formulation.

Then the first from of the kernel identity to be established is the following:
Theorem 3.4.1. Consider \( \Phi \) in \( S(V \times A^x) \). Then the identity
\[
(3.36) \quad (I'(0, \cdot, \chi, \Phi), \varphi)_{\text{Pet}} = 2(\tilde{Z}(\cdot, \chi, \Phi), \varphi)_{\text{Pet}}
\]
for all \( \varphi \) in \( \sigma \), together with the arithmetic theta lifting (3.35), implies Theorem (2.8.1).

Proof. The Shimizu lift \( \theta \) is surjective since \( \pi \otimes \tilde{\pi} \) is an irreducible representation of \( B^x \times B^\times \). Therefore, by linearity, we may assume that
\[
(3.37) \quad f_1 \otimes f_2 = \theta(\varphi \otimes \Phi)
\]
for some \( \varphi \) in \( \sigma \) and \( \Phi \) in \( S(B \times A^x) \). Applying Proposition (3.2.1) to the kernel identity, we have
\[
(3.38) \quad \frac{L(\frac{1}{2}, \pi, \chi)}{2L(1, \eta)} \alpha(\theta(\varphi \otimes \Phi)) = 2(\tilde{Z}(\cdot, \chi, \Phi), \varphi)_{\text{Pet}}.
\]

It then suffices to prove that
\[
(3.39) \quad (\tilde{Z}(\cdot, \chi, \Phi), \varphi)_{\text{Pet}} = \frac{L(1, \pi, \text{ad})L(1, \eta)}{\zeta_F(2)} (T(f_1 \otimes f_2)Z_{\chi}, Z_{\chi^{-1}})_{\text{GS}}.
\]

Using the relation
\[
(3.40) \quad (\tilde{Z}(g, \Phi)R_t(Z), R_t(Z))_{\text{GS}} = (\tilde{Z}(g, \Phi)R_{t'}(Z), R_{t'}(Z))_{\text{GS}}
\]
for \( t' \) in \( T(A) \), which follows by viewing the action of \( T(F) \setminus T(A) \simeq K^x \setminus A^x_K \) on the cycle \( Z \) as a Galois action by class field theory, and after a change of variables, we obtain
\[
(3.41) \quad \tilde{Z}(g, \chi, \Phi) = 2L(1, \eta)(\tilde{Z}(g, \Phi)Z_{\chi}, Z_{\chi^{-1}})_{\text{GS}},
\]
where \( 2L(1, \eta) \) arises from the class number formula for \( T(F) \).

Hence, we have
\[
(3.42) \quad (\tilde{Z}(\cdot, \chi, \Phi), \varphi)_{\text{Pet}} = 2L(1, \eta)(\tilde{Z}(\varphi \otimes \Phi)Z_{\chi}, Z_{\chi^{-1}})_{\text{GS}}.
\]
The main identity follows from the arithmetic theta lift (3.35). \( \square \)

3.4.2. Kernel identity II. We restate the kernel identity for \( \phi \) in \( \hat{\mathcal{S}}(V \times A^x) \) after rewriting the analytic kernel in terms of \( \phi \) (the geometric kernel does not change).

We define the mixed theta-Eisenstein series for \( \phi \) with respect to \( U \),
\[
(3.43) \quad I(s, g, \phi)_U = w_U \sum_{\gamma \in P(F) \setminus SL_2(F)} \delta(\gamma g)^s \sum_{x_1 \in E \setminus \mu_U \setminus F^x} r(\gamma g, 1)\phi(x_1, u),
\]
where \( \mu_U = F^x \cap U \), as before. It defines an automorphic form on \( GL_2(A) \). If \( w_U = 1 \), which is the case if \( U \) is small enough, we may rewrite it as
\[
(3.44) \quad \sum_{\gamma \in P(F) \setminus GL_2(F)} \delta(\gamma) r(\gamma g, 1)\phi(x_1, u).
\]

Define the twisted average
\[
(3.45) \quad I(s, g, \chi, \phi)_U = \int_{[T]} I(s, g, r(t, 1)\phi)_U \chi(t)dt.
\]

Note that both \( \chi(t) \) and \( I(s, g, r(t, 1), \phi)_U \) are invariant under \( T(F) \) and \( T(F_\infty) \) as functions of \( t \) in \( T(A) \), so that the regularized integral is well-defined. Also note that for different levels \( U' \subset U \), we have the relation
\[
(3.46) \quad I(s, g, \phi)_{U'} = [\mu_{U'} : \mu_U]^2 I(s, g, \phi)_U,
\]
and similarly for $I(s, g, \chi, \phi)_{U'}$.

We state the second version of the kernel identity, which is the one we prove:

**Theorem 3.4.2.** Let $\phi$ be a Schwartz function in $\mathcal{H}(\mathbf{V} \times \mathbf{A}^\times)$ bi-invariant under a fixed $U \subset \mathbf{B}_F^+$, and let $\Phi$ be its preimage in $S(\mathbf{V} \times \mathbf{A}^\times)$. Then

$$
(3.47) 
(I'(0, \cdot, \chi, \phi)_{U'}, \varphi)_{\text{Pet}} = 2(\tilde{Z}(\cdot, \chi, \phi), \varphi)_{\text{Pet}}
$$

is equivalent to (3.36).

**Proof.** The equivalence follows from the comparison of normalizations, namely from the identity

$$
(3.48) 
I(s, g, \chi, \Phi) = \frac{2|\mathcal{Q}|^{-1} h_F}{|\mathcal{O}_F^\times : \mu_U|^2 \sqrt{d_F}} I(s, g, \chi, \Phi)_{U'},
$$

clear from the definitions. \hfill \Box

4. Modularity and arithmetic theta lift

In this section, we establish two key properties of the generating series of CM cycles. First, we show that the series defines an automorphic form on $G(\mathbf{A})$. This plays a key role in the comparison of local analytic and geometric terms at bad finite primes. For this argument, we essentially use the modularity proved on the Shimura curve by [21]. Second, we prove the arithmetic theta lift identity, which will relate the geometric kernel to the correspondence $I(f_1 \otimes f_2)$ that appears in the main identity. In this case, our argument still follows that of [21], but is complicated by the appearance of higher cohomological degrees.

4.1. Modularity of the generating series. We say that the generating series $Z(r(g)\Phi)_{U'}$ is modular if for any linear functional $\ell : \text{CH}^{k-1}(W_U \times W_U)_{\mathbf{C}} \to \mathbf{C}$, $\ell(Z(r(g)\Phi)_{U'})$ converges absolutely and defines an automorphic form on $G(\mathbf{A})$ with coefficients in $\text{CH}^{2k-1}(W_U \times W_U)_{\mathbf{C}}$.

A version of modularity for a generating series of CM-cycles over Kuga-Sato varieties over Shimura curves was proven in the setting $F = \mathbf{Q}$ by [22] using Borcherd’s method of the singular theta lift, which is at present not available for totally real fields $F$. Our method is closer to Kudla’s method of arithmetic theta lifting.

**Theorem 4.1.1** (Modularity of generating series). Given $g$ in $G(\mathbf{A})$ and $\phi$ in $\mathcal{H}(\mathbf{V} \times \mathbf{A}^\times)$, the generating series $Z(g, \phi)_{U'}$ converges absolutely, and is an automorphic form on $GL_2(\mathbf{A}_F)$ with values in $\text{CH}^{2k-1}(W_U \times W_U)_{\mathbf{C}}$.

**Proof.** We use the modularity of the generating series constructed in [21]. Given $g$ in $G(\mathbf{A})$ and $\phi$ in $\mathcal{H}(\mathbf{V} \times \mathbf{A}^\times)$, we recall the generating series of special divisors on $X_U \times X_U$ taking values in $\text{Pic}(X_U \times X_U)_{\mathbf{C}}$

$$
Z'(g, \phi)_{U'} = -\sum_{\alpha} \sum_{u \in \mu_U^\times} E_0(\alpha^{-1} u, r(g)\phi)_{\mathcal{L}_{K, \alpha}}
$$

$$
+ w_U \sum_{a \in \mathbf{F}_U^\times} \sum_{x \in K \setminus \mathbf{B}_U^x} r(g, 1)\phi(x, aq(x)^{-1})Z'(x)_{U'},
$$

(4.1)

where $K = U \times U$, and $\mathcal{L}_K$ is the Hodge bundle on $X_U \times X_U$.

We use the fact that the non-constant terms in $Z'(g, \phi)_{U'}$ are constructed from $Z'(x)_{U'}$, where $Z'(x)_{U'}$ is defined by the correspondence in $\text{Pic}(X_U \times X_U)$ induced
by the action of $R_x$ on $X_U$. In our setting, the correspondence $Z(x)_U$ is induced by the same action $R_x$ on $W_U$. Consider the map
\begin{equation}
Z'(g, \phi)_U \mapsto Z(g, \phi)_U,
\end{equation}
for fixed $g$ and $\phi$, obtained by
\begin{equation}
Z'(x)_U \mapsto Z(x)_U, \quad \mathcal{P}_{K, \alpha} \mapsto \mathcal{P}_{K, \alpha}.
\end{equation}
Extending it linearly on multiples of $Z'(g, \phi)_U$ in $\text{Pic}(X_U \times X_U)$, and by 0 otherwise, we see that the map is well-defined. Then given a linear functional $\ell$ on $\text{CH}^{2k-1}(W_U \times W_U)$, denote by $\ell'$ the composition of $\ell$ with the map above. It defines a linear functional on $\text{Pic}(X_U \times X_U)$. Thus
\begin{equation}
\ell(Z(g, \phi)) = \ell'(Z'(g, \phi))
\end{equation}
is an automorphic form on $G(A)$ by the modularity of the left-hand side [21, Theorem 3.4.1].

4.2. **Arithmetic theta lifting.** Let $G$ be an open compact subgroup of $GO(A_f)$ such that the restriction $\Phi_f$ of $\Phi$ to $S(V \times A_f^\times)$ is invariant under the action of $G$ by Weil representation. Let $\mu_G = F^\times \cap G$, and define a theta series as follows
\begin{equation}
\theta(g, \Phi) = \sum_{u \in P_{k, \mu_G}^\times \setminus F^\times} \sum_{x \in V} r(g) \Phi(x, u).
\end{equation}
For convenience, we recall the geometric kernel,
\begin{equation}
\tilde{Z}(\Phi \otimes \varphi) = (\tilde{Z}(g, \Phi), \varphi)_{\text{Pet}} = \int_{[G]} \tilde{Z}(r(g) \Phi) \varphi(g) dg.
\end{equation}
We prove the arithmetic theta lifting for the geometric kernel.

**Theorem 4.2.1.** For $\Phi$ in $S(V \times A^\times)$ and $\varphi$ in $\sigma$, we have
\begin{equation}
\tilde{Z}(\Phi \otimes \varphi) = \frac{L(1, \pi, \text{ad})}{2\zeta_F(2)} T(\theta(\Phi \otimes \varphi))
\end{equation}
in $\text{Hom}_F(\text{CH}^{k-1}, \text{CH}^{k-1})_C$.

**Proof.** First, observe that the identity is true up to a constant. Indeed, both $\tilde{Z}$ and $T \circ \theta$ are invariant under the diagonal action of $GL_2(A)$ on $\Phi \otimes \varphi$, and the action of $B^\times \times B^\times$ on $\Phi$. Therefore, they must be multiples of the Shimizu lift, and hence belong to the one-dimensional space
\begin{equation}
\text{Hom}_{GL_2(A) \times B^\times \times B^\times} (\mathcal{S}(V \times A^\times) \otimes \sigma, \pi \otimes \tilde{\pi}).
\end{equation}
Viewing this as an identity of operators on $H^{2k-1,0}(W_U) = H^{2k-1,0}(W_U, \zeta(C), C)$, it suffices to compare the traces of each operator.

The computation of the trace of the projector follows that of [21, Section 4.4]. By definition, we have
\begin{equation}
T(f_1 \otimes f_2) = \text{deg}(\mathcal{P}_U) T(f_1 \otimes f_2)_U
\end{equation}
where the action of $T(f_1 \otimes f_2)_U$ on $H^{k,0}(W_U)$ is given by $f \mapsto (f, f_2)f_1$. It follows that
\begin{equation}
\text{tr}(T(\theta(\Phi \otimes \varphi)))|H^{k,0}(W_U)) = \text{deg}(\mathcal{P}_U) \mathcal{F}\theta(\Phi \otimes \varphi),
\end{equation}
where \( \mathcal{F} = (\cdot, \cdot) : \pi \otimes \tilde{\pi} \to \mathbb{C} \). The Shimizu lift is normalized \cite[Proposition 5]{19} so that

\[
\mathcal{F} \theta(\Phi \otimes \varphi) = \prod_v \frac{\zeta_v(2)}{L_v(1, \pi, \text{ad})} \int_{N_v \setminus G_v} W_{\varphi_v, -1}(g) r(g, 1) \Phi_v(1, 1) dg.
\]

We compute the trace of the correspondence \( \tilde{Z}(\Phi \otimes \varphi)_U \) by an application of the Lefschetz trace formula

\[
\deg \Delta^* \tilde{Z}(\Phi \otimes \varphi)_U = \sum_{i=0}^{2k-1} (-1)^i \text{tr}(\tilde{Z}(g, \Phi)_U|H^i(W_U)),
\]

where \( \Delta : W_U \to W_U \times W_U \) is the diagonal embedding. The left-hand side is equal to the fixed points of the correspondence \( \tilde{Z}(\Phi \otimes \varphi)_U \).

By the inclusions (2.17) and (2.21), the correspondence \( \tilde{Z}(\Phi \otimes \varphi)_U \) only acts in middle degree, so that \( \text{tr}(\tilde{Z}(g, \Phi)_U|H^i(W_U)) = 0 \) unless \( i = k - 1 \). Indeed, it acts on the top filtration

\[
H^{k-1,0}(W_U) = \Gamma(W_U, \Omega^{k-1}_{W_U}) \otimes_F \mathbb{C},
\]

where \( \Gamma(W_U, \Omega^{k-1}_{W_U}) \) are the global sections of the \((k-1)\)-th tensor power of the canonical bundle \( \Omega_{W_U} \) over \( W_U \). Since \( F \) is totally real, the action is fixed by complex conjugation, and is equal to the trace on \( H^{0,k-1}(W_U) \). The Lefschetz formula reduces to

\[
\text{tr}(\tilde{Z}(\Phi \otimes \varphi)_U|H^{k-1,0}(W_U)) = -\frac{1}{2} \deg \Delta^* \tilde{Z}(\Phi \otimes \varphi)_U.
\]

Note that the vanishing of the trace on the top and bottom degrees can also be computed explicitly as in \cite[Proposition 4.3]{21}.

We show next that

\[
\deg \Delta^* \tilde{Z}(\Phi \otimes \varphi)_U = -\deg(\mathcal{P}_U) \frac{L(1, \pi, \text{ad})}{\zeta_F(2)} \mathcal{F} \theta(\Phi \otimes \varphi).
\]

Combined with the identity (4.10), this completes the proof of the theorem. \( \square \)

4.2.1. Degree of the pullback. The rest of this section is devoted to proving the formula (4.15). The key strategy is to express the degree of the pullback in terms of an incoherent Eisenstein series. It suffices to consider the case \( |S| > 1 \) in which our computations are simplified by compactness, in contrast with \cite{21}.

The next proposition implies Formula (4.15) by carefully unfolding the integrals as in the argument following \cite[Proposition 4.5]{21}.

**Proposition 4.2.2.** Let \( \phi \in \tilde{S}(A \times A^\times) \) be invariant under \( U \times U \). For \( g \) in \( G(A) \),

\[
\deg \Delta^* Z(\Phi \otimes \varphi)_U = -\deg(\mathcal{P}_U) J(0, g, \phi)_U,
\]

where

\[
J(s, g, \phi)_U = \sum_{\gamma \in P_F \setminus G_F} \delta(\gamma g)^s \sum_{u \in \mu_F \setminus F^\times} \sum_{x \in F} r(\gamma g, 1) \phi(x, u)
\]

is the mixed Eisenstein-theta series.

In order to prove the proposition, we require an explicit parametrization of the CM cycles on \( W_U \), which by construction depend on CM cycles on the base Shimura curve. Using this, we obtain a formula for the pullback, then compute its degree.
Let $B_0$ be the set of elements of $B$ of trace 0, and denote by $Ad$ the conjugation action of $B^\times$ on $B_0$ so that $Ad(h)x = hxh^{-1}$ for $h$ in $B^\times$ and $x$ in $B_0$. There is an orthogonal decomposition of quadratic spaces $B = A \oplus B_0$. Fixing an archimedean place $\tau$ of $F$, we denote by $B = B(\tau)$ the nearby quaternion algebra, and define similarly the local decomposition $B = F \oplus B_0$.

For $y$ in $B_0$ with $q(y) \neq 0$, we denote by $B_y$ the centralizer of $y$ in $B$. Then $B_0 = F[y]$ is a quadratic extension of $F$ which is CM if and only if $y$ is in the subset $B_{0,+}$ of elements of $B_0$ with totally positive norm. Let $\tau_y$ be the unique point of $H^+$ fixed by the action of $B_y^+ \subset B(\mathbf{R})$, and $\bar{\tau}_y$ the fixed point by the action of $B_y^\times$ in $H^-$. For $h$ in $B_f^\times$, consider the pushforward $c(y,h)_U$

$$
(4.18) \quad \tau_y^+ \times B^\times \backslash B_{f,y}^+ / U_h \longrightarrow X_U = B^\times \backslash H^+ \times B_f^\times / U : (\tau,b) \mapsto (\tau, bh),
$$

where $U_h = B_{f,y}^\times \cap hUh^{-1}$.

Consider the image $C(y,h)_U$ of $c(y,h)_U$ by the map associating to each CM point $c(y,h)_U$ a CM cycle in $CH^1(B_f)$ by the construction in Section (2.3).

By [21, Lemma 4.6], given $y$ in $B_{f,0}^+ = \{x \in B_{f,0} \mid q(x) \in F_y^+\}$, we may define

$$
C(y)_U = C(y_0, h)_U, \quad y = h^{-1}y_0h, \quad y_0 \text{ in } B_{0,+}, \quad h \text{ in } B_f^+.
$$

We have the following pull-back formula.

**Proposition 4.2.3.** Let $\phi = \phi^0 \otimes \phi_0$ under the orthogonal decomposition $B = A \oplus B_0$. Then

$$
(4.19) \quad \Delta^* Z(g, \phi)_U = \sum_{u \in \mu_f^0 \backslash F^\times} \theta(g, u, \phi^0)C(g, u, \phi_0)_U,
$$

where $C(g, u, \phi_0)_U$ is given by

$$
(4.20) \quad - r(g)\phi_0(0, u)\mathcal{P}_U + \sum_{y \in Ad(U) \backslash B_{f,0}^+} [F[y]^\times \cap U : \mu_U]^{-1} r(g)\phi_0(y, u)C(y)_U.
$$

Here $F[y] = F + Fy$ is the CM extension of $F$ generated by $y$, and $r(g)\phi_0(y, u) = r(g)\phi(1, uq(y)x))$.

**Proof.** Let $x$ be in $B_f^\times$. We first claim that if $x$ is in $F^\times U$, then $\Delta^* Z(x)_U = -\mathcal{P}_U$, and if $x$ is not in $F^\times U$, then

$$
(4.21) \quad \Delta^* Z(x)_U = \sum_{y \in Ad(U) \backslash B_{f,0}^+ / F^\times} [F[y]^\times \cap U : U]C(y)_U.
$$

Indeed, if $x \in F^\times U$, then $Z(x)_U = Z(1)_U = \Delta$, and the identity follows from the definition of $\mathcal{P}_U$, which encodes the construction of the bundle $\mathcal{L}_{U,2k-2}$. If $x \notin F^\times U$, then $\Delta \cdot Z(x)_U$ is a proper intersection, and the proof follows that of [21, Lemma 4.7(2)].

We divide the summation into the constant term, the sum over $x$ in $F^\times U$ and the sum over $x$ not in $F^\times U$, and we write the pullback as $\Delta^* Z(g, \phi)_U = P + Q + R.$
where

\begin{align*}
P &= - \sum_{\alpha} \sum_{u \in \mu_2 \setminus F^\times} \phi(0, \alpha^{-1} u) \Delta^* \mathcal{P}_{K, \alpha} \\
Q &= w_U \sum_{a \in F^\times} \sum_{x \in F^\times \setminus U} \phi(x)_a \Delta^* Z(x)_U \\
R &= w_U \sum_{a \in F^\times} \sum_{x \in U \setminus (B^*_0 F^\times U)/U} \phi(x)_a \Delta^* Z(x)_U.
\end{align*}

On the one hand, the pullback $\Delta^* \mathcal{P}_{K, \alpha}$ is nontrivial if and only if $\mathcal{P}_{K, \alpha}$ lies in the same component as the diagonal $\Delta$, that is, if $\alpha$ is the connected component of the identity. Then, since $\Delta^* \mathcal{P}_{K, 1} = \mathcal{P}_U$, it follows that

\begin{equation}
P = - \sum_{u \in \mu_2 \setminus F^\times} \phi(0, u) \mathcal{P}_U.
\end{equation}

On the other hand, by the preceding claim, we have

\begin{equation}
Q = - \sum_{u \in \mu_2 \setminus F^\times} \sum_{x \in F^\times} \phi(x, u) \mathcal{P}_U,
\end{equation}

so that the decomposition $\phi = \phi^0 \otimes \phi_0$ implies

\begin{equation}
P + Q = - \sum_{u \in \mu_2 \setminus F^\times} \theta(g, u, \phi^0) \phi_0(0, u) \mathcal{P}_U.
\end{equation}

As for the $R$ term, we choose for simplicity $U$ small enough so that the all the ramification indices $[F[y]^\times \cap U : \mu_U]$ equal 1. Then following [21, Proposition 4.8], by Equality (4.21),

\begin{equation}
R = \sum_{u \in \mu_2 \setminus F^\times} \sum_{y \in \text{Ad}(U) \setminus B^*_0 / F^\times \setminus x \in F^\times \setminus y} \phi(z, u) C(y)_U
\end{equation}

leading to formula (4.19). \qed

We use the formula for the pullback to compute the degree. We compute the degree for $g$ in $\text{SL}_2(A)$, which extends to $G(A)$ by the diagonal action of the center $Z(A)$.

**Proposition 4.2.4.** Assume $|S| > 1$. We have

\begin{equation}
\deg C(g, u, \phi_0)_U = - \deg(\mathcal{P}_U) E(0, g, u, \phi_0),
\end{equation}

where $E(0, g, u, \phi_0)$ is the Eisenstein series given by

\begin{equation}
E(s, g, u, \phi_0) = \sum_{\gamma \in P^1(F)/\text{SL}_2(F)} \delta(\gamma g)^s r(\gamma g) \phi_0(0, u),
\end{equation}

and $g$ is in $\hat{\text{SL}}_2(A)$.

**Proof.** By the pullback formula (4.21) on $W_U \times W_U$ and the modularity of $Z(g, \phi)_U$ of Theorem (4.1.1), it follows that $C(g, u, \phi_0)_U$ also defines an automorphic form on $\hat{\text{SL}}_2(A)$ with coefficients in $\text{CH}^{k-1}(W_U)_C$. Both $E(0, g, u, \phi_0)$ and $\deg C(g, u, \phi_0)_U$ define elements in the one-dimensional space

\begin{equation}
\text{Hom}_{SO(B_0) \times \hat{\text{SL}}_{2k}(A)}(S(B_0), C^\infty(SL_2(F) \setminus \hat{\text{SL}}_{2k}(A))).
\end{equation}
They are hence proportional and related by the quotient of their constant terms. The formula
\begin{equation}
E(0, g, u, \phi_0) = r(g)\phi_0(0, u) - \sum_{a \in F^\times} L(1, \eta_{-ua}) \int_{B \times \mathbb{B}}^* r(g, (h, h))\phi_0(ya, u)dh
\end{equation}
for the incoherent Eisenstein series of weight $\frac{3}{2}$ given in [21, Proposition 2.10(3)(a)] implies
\begin{equation}
\deg C_0(g, u, \phi_0)_U = - \deg(\mathcal{P}_U) E_0(0, g, u, \phi_0).
\end{equation}

Remark 4.2.5. Though not used in this paper, we briefly note that the above argument can also be extended to the classical case of the Kuga-Sato variety over the modular curve, that is, where $k > 1$, $F = \mathbb{Q}$, and $|S| = 1$. In this setting, the Kuga-Sato variety is again compact after desingularization and compactification, but the incoherent Eisenstein series has now an additional non-holomorphic part, described in [21, Proposition 2.10(3)(b)].

The contribution of the non-holomorphic part to the degree makes up one part of the archimedean contribution of the extra series $B(g, \phi)$ described in [21, Theorem 4.15], but we observe that this contribution is in fact a Schwartz function and in particular nonsingular, so that we may indeed use the general fact that the theta lift of any cuspidal automorphic representation of $G(A)$ to $GSO(V_{hyp}(A))$ is zero. Thus this contribution is zero.

5. Local decomposition of kernels

In this section, we describe the class of degenerate Schwartz functions. We show that if the kernel identity holds for this class of functions, then it holds in general. Then, we decompose the analytic and geometric kernels into local components, and make case by case computations. In the next section, these kernels will be compared term to term at each place.

5.1. Degenerate Schwartz functions. We consider degenerate Schwartz functions introduced in [21, §5.2].

Write the set of all places $S_F$ of $F$ as
\begin{equation}
S_F = S_\infty \cup S_{\text{nonsplit}} \cup S_{\text{split}},
\end{equation}
where $S_\infty$ is the set of archimedean places of $F$, $S_{\text{nonsplit}}$ is the set of non-archimedean places of $F$ non-split in $E$, and $S_{\text{split}}$ is the set of non-archimedean places of $F$ split in $E$. Denote by $S_1$ a finite subset of $S_{\text{nonsplit}}$ containing all places ramified over $\mathbb{Q}$, in $\mathcal{B}$, $K$, $\sigma$ or $\chi$, and by $S_2$ the two places in $S_{\text{split}}$ at which $\chi$ and $\sigma$ are ramified.

The standard Schwartz function $\phi$ in $\mathcal{H}(V \times \mathbb{R}^\times)$ is the Gaussian
\begin{equation}
\phi(x, u) = P_k(x)e^{-2\pi au(x)} \mathbf{1}_{\mathbf{R}^+}(u),
\end{equation}
where $1_{\mathbf{R}^+}$ is the characteristic function of $(0, \infty)$. The space $\mathcal{H}(V \times \mathbb{R}^\times)$ gives the weight $2k$ discrete series of $G(\mathbb{R})$. The standard Schwartz function corresponds to the element of lowest weight. The action of $g_{\ell_2}(\mathbb{R})$ on it generates the entire space, so that it is sufficient to consider it.

We make the following assumptions on the Schwartz functions, which simplify the computations at many places.
• The Schwartz function \( \phi = \otimes \phi_v \in \mathcal{S}(B \times \mathbb{A}^\times) \) is a pure tensor and \( \phi_v \) is standard for any \( v \) in \( S_\infty \).

• For \( v \in S_1 \), \( \phi_v \) lies in \( \mathcal{S}(B_v \times F_v^\times) \) defined by

\[
\{ \phi_v \in \mathcal{S}(B_v \times F_v^\times) | \phi_v(x, u) = 0 \text{ if } v(uq(x)) \geq -v(d_v) \text{ or } v(uq(x_2)) \geq -v(d_v) \},
\]

where \( dv \) is the local different of \( F \) at \( v \), and \( x_2 \) denotes the orthogonal projection of \( x \) in \( V_2 = E_v \).

• For any \( v \) in \( S_2 \), \( \phi_v \) lies in \( \mathcal{S}(B_v \times F_v^\times) \) defined by

\[
\{ \phi_v \in \mathcal{S}(B_v \times F_v^\times) | \forall g \in G(F_v) \text{ and } u \in F_v^\times \},
\]

• Let \( U = \prod_v U_v \) be an open compact subgroup of \( B_1^\times \) such that:
  - \( \phi \) is invariant under the action of \( U \times U \),
  - \( \chi \) is invariant under the action of \( U_T = U \cap T(A_1) \),
  - \( U_v \) is of the form \( 1 + wv \mathcal{O}_{B_v}^\times \) for some \( r \geq 0 \) for every finite place \( v \),
  - \( U_v \) is maximal for all \( v \) in \( S_{\text{non-split}} - S_1 \), and \( v \) in \( S_2 \),
  - \( U \) does not contain -1,
  - \( U \) is small enough that that each connected component of the complex points of \( X_U \) is an unramified quotient of \( H \) by the complex uniformization.

**Theorem 5.1.1.** The kernel identity is true for all \((\phi, U)\) if and only if its true for all \((\phi, U)\) satisfying the assumptions.

**Proof.** It suffices to show that there exists a Schwartz function \( \Phi \) in \( \mathcal{S}(B \times \mathbb{A}^\times) \) such that its image \( \phi \) in \( \mathcal{S}(B \times \mathbb{A}^\times) \) satisfies the assumptions and the pairing \( \alpha(\hat{\phi}_f \otimes \varphi_f) \) is nonzero for some \( \varphi \) in \( \sigma \). These properties follow from Proposition 5.11 and Proposition 5.15 of [21] for \( v \) in \( S_1 \) and \( v \) in \( S_2 \) respectively. \( \square \)

We record the following properties of the degenerate Schwartz function which we make use of.

**Corollary 5.1.2.** Consider \( \phi_v \) in \( \mathcal{S}(B_v \times F_v^\times) \).

(1) Let \( v \) be in \( S_{\text{non-split}} \), assume moreover there is a constant \( c > 0 \) such that

\[
\phi_v(x, u) = 0 \text{ for all } (x, u) \text{ with } v(uq(x_2)) \geq 2.
\]

Then

\[
\phi_v = \sum_{i=1}^{r} \phi^i_{1,v} \otimes \phi^i_{2,v}
\]

where for each \( i \), \( \phi^i_{1,v} \) in \( \mathcal{S}(E_v \times F_v^\times) \) and \( \phi^i_{2,v} \) in \( \mathcal{S}(E_v \times F_v^\times) \) satisfy \( \phi^i_{2,v}(x_2, u) = 0 \) for all \((x_2, u) \text{ with } v(uq(x_2)) \geq c \).

(2) Let \( v \) be a finite place of \( F \). Then \( r(g)\phi_v(0, u) = 0 \) for all \( g \) in \( G(F_v) \), and \( u \) in \( F_v^\times \) if and only if the average

\[
\int_{B_v(a)} r(g, h)\phi_v(x, u)dx = 0
\]

for all \( g \) in \( G(F_v) \), \( h \) in \( B_v^\times \times B_v^\times \), and \( a, u \) in \( F_v^\times \).

**Proof.** See Lemma 5.9 and Lemma 5.10 of [21]. \( \square \)
5.2. Derivative of the analytic kernel. In this section, we decompose the derivative of the analytic kernel $I(s, g, \phi)$ into a sum of local components. As discussed, the analytic kernel is essentially the one considered in [21, §6], which follows Waldspurger’s integral representation for the $L$-function. We summarize the main results we use.

5.2.1. Decomposition of the test function. Using the decomposition $V = V_1 \oplus V_2$, we may write $\phi$ in $\mathcal{F}(V \times \mathbb{A}^\times)$ as a finite linear combination of

$$
\phi_1 \otimes \phi_2 \in \mathcal{F}(V_1 \times \mathbb{A}^\times) \times \mathcal{F}(V_2 \times \mathbb{A}^\times),
$$

where $(\phi_1 \otimes \phi_2)(x_1 + x_2, u) := \phi_1(x_1, u)\phi_2(x_2, u)$. This decomposition preserves the Weil representation, that is,

$$
r(g, (t_1, t_2))(\phi_1 \otimes \phi_2)(x, u) = r_1(g, (t_1, t_2))\phi_1(x_1, u)r_2(g, (t_1, t_2))\phi_2(x_2, u)
$$

for $g$ in $GL_2(A)$, and $(t_1, t_2)$ in $A_E^\times \times A_F^\times$. Here, $r_1, r_2$ denote the Weil representations associated to $V_1, V_2$ respectively, and $(t_1, t_2)$ acts on $x_i \in V_i$ by $(t_1, t_2)x_i = t_1x_it_2^{-1}$.

As a consequence, we can express the analytic kernel (3.43) as a product of a theta series and an Eisenstein series

$$
I(s, g, \phi)_U = \sum_{u \in \mu_{F}^\times \setminus F^\times} \theta(g, u, \phi_1)E(s, g, u, \phi_2),
$$

for $g$ in $GL_2(A)$, where

$$
\theta(g, u, \phi_1) = \sum_{x_1 \in E} r(g, 1)\phi_1(x_1, u)
$$

and $E(s, g, u, \phi_2)$ is defined as in (3.22). Note that this definition of $\theta(g, u, \phi_1)$ differs from that given in (3.12), as the summation over $u$ is dropped here.

5.2.2. Decomposition of the derivative. Given $u$ in $F^\times$ and a place $v$ of $F$, denote by $F_u(v)$ the set of $a$ in $F^\times$ represented by $(V_2^\times, uq)$. Note that $F_u(v)$ is non-empty only if $E$ is non-split at $v$, and $W_{g, v}(0, g, u, \phi) = 0$ for $a$ in $F_u(v)$.

Fix a place $v$ of $F$ non-split in $E$. For $a$ in $F_u(v)$, we can evaluate the derivative $E'(s, g, u, \phi_2)$ at $s = 0$ using the Fourier expansion of $E(s, g, u, \phi_2)$ in terms of Whittaker functions to obtain

$$
E'(0, g, u, \phi_2) = E'_0(0, g, u, \phi_2) - \sum_{v \text{ non-split}} E'_v(0, g, u, \phi_2),
$$

where $E'_0(0, g, u, \phi_2)$ is the derivative at $s = 0$ of the usual constant term of the Siegel Eisenstein series $E_0(s, g, u, \phi_2)$, and

$$
E'_v(0, g, u, \phi_2) := \sum_{a \in F_u(v)} W_{a, v}(0, g, u, \phi_2)W_{a}^v(0, g, u, \phi_2)
$$

for suitable local Whittaker functions $W_{a, v}(0, g, u, \phi_2)$ and $W_{a}^v(0, g, u, \phi_2)$ translated by the Weil index $\gamma_{a, v}$.

The decomposition of the analytic kernel follows from the preceding definitions. We separate the places $v$ in $F$ that are non-split and split in $E$.

$$
I'(0, g, \phi)_U = - \sum_{v \text{ non-split}} I'_v(0, g, \phi)_U + \sum_{u \in \mu_{F}^\times \setminus F^\times} \theta(g, u, \phi_1)E'_0(0, g, u, \phi_2)
$$
Proof. See (5.21) (Pr(\text{form Pr}(f for all } \phi \in \mathcal{S}(A)) in (5.17) \text{ where } (5.18) \text{ where } (5.19) \text{ where } (5.20) \text{ from the space of automorphic forms of central character } \omega \text{ to the space of holomorphic cusp forms of parallel weight } 2k \text{ mapping } f \in \mathcal{A}(G(A), \omega) \text{ to the unique form } \text{Pr}(f) \in \mathcal{A}_0^{(2k)}(G(A), \omega) \text{ such that } (5.21) \text{ for all } \phi \in \mathcal{A}_0^{(2k)}(G(A), \omega). \text{ Refer to [9, Proposition 5.1] for a classical formulation of the holomorphic projection onto weight } 2k \text{ forms.}

\text{Note that in [21], the analogous function is denoted by } \mathcal{K}_\phi^{(v)} \text{. Our choice of notation is intended to simplify the following formulae.}
For any automorphic form \( f \) for \( GL(2, \mathbb{A}) \), we define a Whittaker function \( f_{\psi,s}(g) \) as follows:

\[
W_{\psi,s}(g_{\infty}) \int_{Z_{\mathbb{A}}N_{\mathbb{A}}G_{\mathbb{A}}} \delta(h)^s f_{\psi}(gh) W^{(2k)}(h) dh,
\]

where \( W^{(2k)} \) is the standard holomorphic Whittaker function of weight \( 2k \) at infinity, and \( f_{\psi} \) is the Whittaker function of \( f \) with respect to the additive character \( \psi \). That is,

\[
W^{(2k)}(g) = \begin{cases} 
\omega(z)|y|^{k}e^{2ik\theta}e^{2\pi i(x+iy)} & y > 0 \\
0 & y \leq 0
\end{cases}
\]

as usual

\[
g = z \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}
\]

by the Iwasawa decomposition, with \( y > 0 \) and \( z \in \mathbb{R}^\times \).

The following growth condition is a key ingredient in our projection formula.

**Lemma 5.2.2.** Consider \( f \) in \( \mathcal{A}(G(\mathbb{A}), \omega) \) such that

\[
f \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g \right) = O(|a|^{1-\epsilon})
\]
as \( |a| \to \infty \) for \( a \) in \( \mathbb{A}^\times \) and some \( \epsilon > 0 \). The holomorphic projection \( \text{Pr}(f) \) has Whittaker function given by

\[
\text{Pr}(f)_{\psi}(g) = \lim_{s \to 0} f_{\psi,s}(g).
\]

**Proof.** The proof follows that of [21, Proposition 6.12], except that we use the Whittaker function of weight \( 2k \) inducing instead

\[
\int_{Z(\mathbb{R})N(\mathbb{R})\backslash G(\mathbb{R})} |W^{(2k)}(g)|^2 dg = \int_0^\infty y^{2k-1} e^{-4\pi y} dy = 4\pi^{-2k+1}(2k-1)!
\]
at each archimedean place. \( \square \)

We introduce the notation

\[
\text{Pr}'(f)(g) = \sum_{a \in F^\times} \lim_{s \to 0} f_{\psi,s} \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g \right).
\]

Under the growth condition (5.25), we have an equality \( \text{Pr}(f) = \text{Pr}'(f) \).

We decompose the derivative of the analytic kernel into local components.

**Proposition 5.2.3.** Under Assumptions (5.3) and (5.4), the holomorphic projection of the analytic kernel is given by

\[
\text{Pr}'I'(0, g, \chi, \phi) = \int_{[T]} \text{Pr}'I'(0, g, r(t, 1)\phi) \chi(t) dt,
\]

where \( \text{Pr}'I'(0, g, \phi) \) can be expressed as

\[
\text{Pr}'I'(0, g, \phi) = -\sum_{\nu \text{ nonsplit}} 2\text{vol}([T])^{-1} \int_{[T]} K_{\phi}^{(v)}(g, (t,t)) dt,
\]
and $\mathcal{K}_\phi^{(v)}(g, (t, t))$ is respectively described by (5.18) and (5.17) for archimedean and non-archimedean $v$.

**Proof.** For non-archimedean $v$, under Assumption (5.3), we have

$$I'(0, g, \phi) = - \sum_{v \text{ non-split}} I'_v(0, g, \phi)$$

for $g \in P(F_{S_0})GL_2(A^{S_0})$ by [21, Proposition 6.7], and Identity (5.31) follows.

For archimedean $v$, we first observe that under Assumption (5.4), our kernel function satisfies the condition

$$I'(0, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, g, \chi, \phi) = O(e^{-\epsilon_v|a|})$$

for some $\epsilon_v > 0$ as in [21, Proposition 6.14]. [21, Proposition 6.12] then leads to Identity (5.30). More precisely, we express the integrand as in (5.31) by [21, Prop 6.15] for $\mathcal{K}_\phi^{(v)}(g, (t, t))$ given by Lemma (5.2.1). We replace $k_{\phi_0}(g, y, u)$ by

$$\tilde{k}_{\phi_0, s}(g, y, u) = \frac{4\pi^{2k-1}}{\Gamma(2k-1)} W^{(2k)}(g_v) \int_{F_{v,+}} a^{s+k-1} e^{-2\pi a} k_{\phi_0}\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, y, u\right) \frac{da}{a},$$

and take the quasi-limit $\tilde{\ }$. Assume $u q(y) = 1$ and $u > 0$ for simplicity. Consider the function

$$q_k(a) = \int_1^\infty e^{-at} (t - 1)^k t^{-k-1} dt$$

for $a > 0$ defined in [9, Proposition 3.3(e)]. Then [20, Lemma 2.3(3)] implies that for $a$ in $F_u(v)$, or equivalently, for $u a < 0$, we have

$$W_{a,v}'(0, 1, u) = -\pi e^{2\pi a} q_k(-4\pi a)$$

if $u > 0$, and $W_{a,v}'(0, 1, u) = 0$ otherwise. Note that $q_0(a) = -\text{Ei}(-a)$ is the exponential integral. By properties of the Weil representation, we have

$$k_{\phi_0}\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, y, u\right) = -\frac{1}{2} q_k(4\pi u q(y_2) a) |a|^k e^{2\pi u q(y_2) a}.$$

It suffices to consider $u q(y) = 1$. Taking $u > 0$, we evaluate the integral as follows:

$$\int_{F_{v,+}} a^{s+k-1} e^{-2\pi a} k_{\phi_0}\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, y, u\right) \frac{da}{a}$$

$$= -\frac{1}{2} \int_0^\infty a^{s+k-1} e^{-2\pi a} q_k(4\pi u q(y_2) a) a^k e^{2\pi a} \frac{da}{a}$$

$$= \frac{1}{2} \int_0^\infty a^{s+2k-1} e^{-4\pi a} \int_1^\infty e^{-4\pi u q(y_2) at} (t - 1)^k t^{-k-1} dt \frac{da}{a}$$

$$= \frac{1}{2} \int_1^\infty (t - 1)^k t^{-k-1} \int_0^\infty a^{s+2k-1} e^{-4\pi (1-u q(y_2)) t} \frac{da}{a} dt.$$
which equals \(W^{(2k)}(g_v)k_{\nu,v}(y)\) by definition.

5.3. **Decomposition of the geometric kernel.** Consider arithmetic cycles \(\hat{Z}_1 = (Z_1, g_1)\) and \(\hat{Z}_2 = (Z_2, g_2)\) with codimension \(p\) and \(d - p\) in a regular arithmetic scheme \(Y\) of dimension \(d\), with \(Z_1\) and \(Z_2\) irreducible intersecting properly. If \(Z_1, Z_2\) are disjoint at the generic fiber then the intersection \((Z_1, Z_2)\) with support defines an element in \(\text{CH}^d_{\{Z_1\cap Z_2\}}(Y)\), as in \([23, \S 1.2]\). So we can write \((Z_1, Z_2) = \sum_v x_v \mathcal{P}_v\), for \(x_v \in \text{CH}^1_{\{Y\cap \hat{Z}_v\}}(Y)\). Define

\[
(Z_1, Z_2)_v = \text{deg}(x_v) \quad \text{if } v \text{ is finite},
\]

\[
(Z_1, Z_2)_v = \int_{\mathcal{Z}_v(C)} g_1 + \int_{\mathcal{Y}_v(C)} g_2 h_1 \quad \text{if } v \text{ is infinite},
\]

where \(Y_v(C) = Y \otimes_{\mathcal{O}_F, \sigma} \mathcal{C}\), \(\sigma : F \to \mathcal{C}\) inducing \(v\) and \(Z_{2v}(C)\) is the pullback on \(Z_2\) of \(Y_v\).

If \(Z_1, Z_2\) are not disjoint at the generic fiber, assume there is a morphism

\[
\pi : Y \to X,
\]

where \(X\) is a regular arithmetic surface such that both \(Z_1\) and \(Z_2\) are contained in a fiber \(Y_D\) of \(Y\) over an integral divisor \(D\) of \(X\), and the morphism \(\pi_D : Y_D \to D\) is smooth. Let \(t\) be a local coordinate for \(D_F\) in \(X_F\). Let \(\eta\) be a differential form such that \(d\eta = dt = 0\) on each fiber \(Y(C)q\) when \(q\) is near \(|D|(C)\) and with restriction \(\delta_{Z_2}(C)\) on \(Y(C)_{|D(C)}\) modulo \(\text{Im}(\delta)\) and \(\text{Im}(\bar{\delta})\). Let \(\text{div}(dDt)\) denote \((\frac{1}{e} \text{div}(dt^e) - D)|_D\) and let \(\text{ord}_v dDt\) be the rational number such that the pushforward of the 0-cycle \(\text{div}(dD)\) to \(\text{Spec}(\mathcal{O}_F)\) is \(\sum_v \text{ord}_v dDt(v)\). Then

\[
\begin{cases}
(Z_1, Z_2)_v = -(Z_1, F, Z_2, F)_{Y_D,r} \text{ord}_v dDt & \text{if } v \text{ is finite},

(Z_1, Z_2)_v = G(D_v(C)) + \int_{\mathcal{Y}_v(C)} g_2 h Z_1 & \text{if } v \text{ is infinite},
\end{cases}
\]

where

\[
G(p) = \lim_{q \to p} \left( \int_{Y(C)q} g_1 \eta + \log |t(q)|(Z_1, F, Z_2, F)_{Y_D,F} \right).
\]

5.3.1. **Decomposition.** We use the local decomposition of the arithmetic intersection pairing to decompose the geometric kernel as follows. Recall that by Lemma (3.3.2), the generating series \(Z(g, (t_1, t_2), \phi)_U\) is a cuspidal automorphic form on \(G(\mathcal{A})\), so only its nonconstant part \(Z_*\) is nonzero. We have

\[
Z(g, (t_1, t_2), \phi)_U = (Z_*(g, \phi)_U R_{t_1} Z_{U_1} R_{t_2} Z_{U_2})_{GS},
\]

for \(t_1, t_2\) in \(B^X\). By Assumption (5.4), the pairing \(\langle Z_*(g, \phi)_U R_{t_1} Z_{U_1} R_{t_2} Z_{U_2}\rangle_{GS}\) has no self-intersections for any \(g\) in \(1_{S_1}(G(\mathcal{A}^{B^1}))\). It decomposes into a local sum

\[
\sum_v i_v(Z_*(g, \phi)_U t_{1,2}) = \sum_v \text{vol}(\{T\}) \int_{[T]} i_v(Z_*(g, \phi)_U t_{1,2}) dt
\]

By class field theory, we have a surjection \(T(F)\backslash T(\mathcal{A}) \to \text{Gal}(K^{ab}/K)\). Hence, we obtain the following expression for the local decomposition of the geometric kernel.

**Proposition 5.3.1.**

(5.50) \(Z(g, (t_1, t_2), \phi)_U = -\sum_v \text{vol}(\{T\}) \int_{[T]} i_v(Z_*(g, \phi)_U t_{1,2}) dt\)}
where
\[(5.51) \quad i_v(Z_*(g, \phi)t_1, t_2) = \sum_{a \in F^\times} \sum_{x \in B_f^T/U} r(g) \phi(x, a g(x)^{-1}) \sum_v i_v(t_1 x, t_2), \]
and \(i_v(t_1 x, t_2) = (t_1 x, t_2)_v.\)

6. Proof of kernel identity

In this section, we use the local decomposition of the analytic and geometric kernels to prove the kernel identity under the assumptions on the Schwartz functions in (5.1). The structure of our arguments follows that of [21], except that we must use the arithmetic intersection pairings on the geometric side, rather than the Néron-Tate height pairings.

6.1. The main local theorem. The kernel identity will follow from the local computations listed below. We first show that they imply the main theorem, then the rest of the section is devoted to these local assertions.

**Theorem 6.1.1.** Suppose the assumptions of Section (5.1) are satisfied. For all \(t_1, t_2 \in T(A_f),\) and \(g \in \text{I}_S, G(A^S),\) suppose that the following conditions are satisfied.

1. If \(v \in S_{\text{split}},\) then \(i_v(Z_*(g, \phi)t_1, t_2) = 0.\)
2. If \(v \in S_{\infty},\) then \(\mathcal{K}_\phi(v)(g, (t_1, t_2)) = i_v(Z_*(g, \phi)t_2, t_2).\)
3. If \(v \in S_{\text{nonsplit}} - S_1,\) then \(\mathcal{K}_\phi(v)(g, (t_1, t_2)) = i_v(Z_*(g, \phi)t_2, t_2) \log N_v.\)
4. If \(v \in S_1,\) then there exist Schwartz functions \(k_{\phi_v}, m_{\phi_v} \in \tilde{\mathcal{F}}(B(v)_v \times F_v^\times)\) depending on \(\phi_v\) and \(U_v,\) such that

\[(6.1) \quad \mathcal{K}_\phi(v)(g, (t_1, t_2)) = \theta(g, (t_1, t_2), k_{\phi_v} \otimes \phi^v),\]
\[(6.2) \quad i_v(Z_*(g, \phi)t_1, t_2) = \theta(g, (t_1, t_2), m_{\phi_v} \otimes \phi^v).\]

Then, the kernel identity (3.47) holds.

**Proof.** By holomorphic projection, it suffices to prove
\[(6.3) \quad (\text{Pr} I'(0, g, \chi, \phi) - 2Z(g, \chi, \phi), \varphi(g))_{\text{ret}} = 0\]

Decomposing the analytic kernel using Proposition (5.2.3), and the geometric kernel using Proposition (5.3.1), we have
\[(6.4) \quad \text{Pr} I'(0, g, \chi, \phi) - 2Z(g, \chi, \phi) = 2\text{vol}([T])^{-1} \int_{[T]} \int_{[T]} D(g, (t_2 t_1, t_2), \phi)\chi(t_1)dt_1 dt_2\]

where \(D(g, (t_1, t_2), \phi)\) is given by
\[(6.5) \quad \sum_{v < \infty} \mathcal{K}_\phi(v)(g, (t_1, t_2)) + \sum_{v < \infty} \mathcal{K}_\phi(v)(g, (t_1, t_2)) - \sum_v i_v(Z_*(g, \phi)t_1, t_2) \log N_v.\]

By the assumptions on the Schwartz function \(\phi,\) and using the identities (1) to (4), only the terms corresponding to \(v \in S_1\) remain, hence
\[(6.6) \quad D(g, (t_1, t_2), \phi) = \sum_{v \in S_1} \theta(g, (t_1, t_2), d_{\phi_v} \otimes \phi^v)\]

where \(d_{\phi_v} = k_{\phi_v} - m_{\phi_v} \log N_v\) is a Schwartz function in \(\tilde{\mathcal{F}}(B(v)_v \times F_v^\times).\) The equality is true for \(g\) in \(\text{I}_S, G(A^S).\) By modularity, it extends to \(g\) in \(G(F)G(A^S),\) and by density, it extends to \(G(A).\)
We define $\Phi(v) = \Phi_\infty \otimes d_{\phi_v} \otimes \phi^s_v$ in $S(B(v) \times A^\times)$, where $\Phi_\infty$ is such that its image in $\tilde{S}(B_\infty \times F_\infty^\times)$ is the standard Schwartz function. Working with this Schwartz function instead, we define

$$I(0, g, \chi, \Phi(v)) = \int_{T(F) \backslash T(A)} \int_{[T]} \theta(g, (t_2t_1, t_2), \Phi(v))\chi(t_1)dt_1dt_2,$$

where

$$\theta(g, (t_1, t_2), \Phi(v)) = \sum_{u \in F^\times} \sum_{y \in B(v)} r(g, (t_1, t_2))\Phi(v)(y, u).$$

Since $I(0, g, \chi, \Phi(v))$ and $I(0, g, \chi, d_{\phi_v} \otimes \phi^s_v)$ are equal up to a constant, it suffices to show that

$$I(0, g, \chi, \Phi(v)) \varphi(g)_{\text{Pet}} = 0$$

for all $v$ in $S_1$. The series $\theta(g, (t_1, t_2), \Phi(v))$ is defined on the nearby quaternion algebra. Therefore, by the same argument as in [21, §7.4.3], we interpret the left-hand side as an integral of a Shimizu lifting. Since the ramification set of $B(v)$ is different from $S$, the right-hand sum is perpendicular to $\sigma$ by an application of the theorem of Tunnell and Saito [18, 16] stating that

$$\text{Hom}_{K^\times}(\pi(v) \otimes \chi, C) = 0.$$

Hence, Equality (6.9) follows. This proves the kernel identity. \hfill \Box

Next, we prove the local assertions of Theorem (6.1.1). We separate our analysis according to the reduction type of the Shimura variety at various places, and treat archimedean places disjointly. The computations at archimedean and ordinary primes are essentially those of [21] and [23]. At supersingular and superspecial primes, the computations involve computations of the intersection pairing rather than the Néron-Tate height. In all cases, we use the notion of geometric pairing with multiplicity function introduced in [24], which depends only on the relative position of the CM cycles.

6.2. Archimedean Case. As in [23, §3.4], consider the Legendre function of the second kind defined for $t > 1$ by

$$Q_{s+k}(t) = \int_0^\infty (t + \sqrt{t^2 - 1} \cosh u)^{-k-s}du.$$

For $z_1, z_2$ in $H$ and $\text{Re}(s) > 0$, let

$$m_s(z_1, z_2) = -2Q_{s+k}(1 + \frac{|z_1 - z_2|^2}{2\text{Im}(z_1)\text{Im}(z_2)}),$$

where the argument of $Q_{s+k}$ is the hyperbolic cosine of the distance between $z_1$ and $z_2$. Let $B(v)$ be the adelic nearby quaternion algebra. For $(z_1, \beta_1), (z_2, \beta_2)$ in $H \times B(v)^\times$, the local height of the induced CM cycles is defined by

$$i_\circ((z_1, \beta_1), (z_2, \beta_2)) = \lim_{s \to 0} \sum_{\gamma \in \mu_U \setminus B(v)^\times} m_s(z_1, \gamma z_2)1_U(\beta_1^{-1}\gamma\beta_2).$$

In particular, the archimedean pairing used in [24, Proposition 3.4.1] is the same as in the weight 2 case, except that $s$ is replaced with $s + k$. In the case of CM
points, we have for any \( \gamma \in B^\infty_{v,+} - E^\infty_v \),
\[
1 + \frac{|z_0 - \gamma z_0|^2}{2 \text{Im}(z_0) \text{Im}(\gamma z_0)} = 1 - 2\lambda(\gamma)
\]
where \( \lambda(\gamma) = q(\gamma_2)/q(\gamma) \). Thus we denote \( m_s(\gamma) = Q_{s+k}(1 - 2\lambda(\gamma)) \).

**Lemma 6.2.1.** Let \( v \) be an infinite place of \( F \). Then
\[
K_{\phi}^{(v)}(g, (t_1, t_2)) = i_v(Z_s(g) t_1, t_2).
\]

**Proof.** For two distinct CM points given by \((z_1, \beta_1), (\gamma z_1, \beta_2)\), we can write
\[
\sum_{\gamma \in \mu_v \setminus B^\infty_v} m_s(z_1, \gamma z_2) 1_U(\beta_1^{-1} \gamma \beta_2) = \sum_{\gamma \in \mu_v \setminus B^\infty_v} m_s(\gamma) 1_U(\beta_1^{-1} \gamma \beta_2).
\]
We may replace the summation index by \( B^\infty_v - E^\infty_v \) given in Proposition (6.2), and
\[
\lim_{s \to 0} \sum_{\gamma \in \mu_v \setminus B^\infty_v} m_s(\gamma) 1_U(\beta_1^{-1} \gamma \beta_2).
\]
By [21, Proposition 8.1], we obtain
\[
i_v(Z_s(g) t_1, t_2) = \sum_{a \in F^\times} \lim_{s \to 0} \sum_{\gamma \in \mu_v \setminus B^\infty_v} r(g, (t_1, t_2)) \phi(\gamma + aq(\gamma)^{-1}) \gamma m_s(\gamma).
\]
To compare this to
\[
K_{\phi}^{(v)}(g, (t_1, t_2)) = \sum_{a \in F^\times} \lim_{s \to 0} \sum_{\gamma \in \mu_v \setminus B^\infty_v} r(g, (t_1, t_2)) \phi(\gamma + aq(\gamma)^{-1}) k_{v,s}(\gamma)
\]
given in Proposition (5.2.3), it is enough to compare \( m_s(\gamma) \) with \( k_{v,s}(\gamma) \). Recall that
\[
k_{v,s}(\gamma) = \frac{\Gamma(s + 2k - 1)}{2(4\pi)^s \Gamma(2k - 1)} \int_1^\infty (t - 1)^{k-1} t^{-\lambda(y)t + \gamma} dt.
\]
Referring to the computation in [9, p.293], we may rewrite the integrand by partial fraction decomposition as a linear combination of \( t^{-j} \) and \((1 + \lambda(y)t)^{-s-j}\) with \( 1 \leq j \leq k \). Then setting \( z = 1 + \lambda(y) \) and making the substitution \( t = 1 + \sqrt{2 - 2^{-1} e^u} \), we obtain
\[
\int_1^\infty (t - 1)^{k-1} t^{-k}(1 + z - 2^{-1} x)^{-k} dt = \int_{-\infty}^\infty (z + \sqrt{2 - 2^{-1} x})^{-k} du = 2Q_k(z)
\]
The equality follows. This proves Assumption (2) of Theorem (6.1.1).

In the non-archimedean setting, given a fixed place \( v \) of \( F \), the multiplicity function \( m \) on CM points \( x, y \) in \( K^\infty \setminus B^\infty / U \) will be defined by
\[
m(x, y) = m_0(x, y) m_1(x, y),
\]
where \( m_0(x, y) \) is the multiplicity function at \( v \) defined on CM points in [23, Section 5], and \( m_1(x, y) \) denotes the local intersection at \( v \) of the CM cycles \( S_k(x) \) and \( S_k(y) \) attached to \( x \) and \( y \).
6.3. **Supersingular Case.** Let \( v \) be a finite prime of \( F \) nonsplit in \( E \) but split in \( B \). The local pairing depends on the local integral model of the Shimura curve. We first study the multiplicity function \( m_1 \).

6.3.1. **Local intersections.** The Shimura curve \( X \) can be viewed as the fine moduli space classifying triples \((A, \iota, v)\) where \( A \) is an abelian surface endowed with maps
\[
\iota : U \to A, \quad \text{and} \quad v : (U/MU)_S \to A[M],
\]
and \( A[M] \) denotes the \( M \)-division points of \( A \). Let \( W \) be a complete discrete valuation ring with characteristic zero field of fraction. Let \( x, y \) be in \( X \otimes W \) and let \( W_x, W_y \) be the normalizations of their structure rings. Denote by \( w \) a uniformizer of \( W_x \). Define \( \text{Isom}_n(x, y) \) as the set of couples \((f, g)\) of an embedding \( f : \text{Spec} \ W_x/w^n \to \text{Spec} \ W_y \) over \( W \), and a homomorphism \( g \) of group schemes over \( W_x/w^n \) making the following diagrams commutative.

\[
\begin{array}{ccc}
U & \xrightarrow{\iota_x} & A_x \\
\downarrow{\iota_y} & & \downarrow{g} \\
A_y & \xrightarrow{v_y} & A_y[M]
\end{array}
\quad \begin{array}{ccc}
U/MU & \xrightarrow{v_x} & A_x[M] \\
\downarrow{v_y} & & \downarrow{g} \\
A_y & \xrightarrow{v_y} & A_y[M]
\end{array}
\]

As in [23, Proposition 4.1.1(a)] and [23, Equation 3.3.1], one computes
\[
\langle S_k(x), S_k(y) \rangle_v = \frac{(-1)^k}{2} \sum_{\psi \in \text{Isom}_n(x, y)} S_k(y)_0 \cdot \psi_0^* S_k(x)_0
\]
(6.24)
\[
= \langle x, y \rangle_v \cdot S_k(y)_0 \cdot \psi_0^* S_k(x)_0,
\]
(6.25)
where \( \psi_0^* S_k(x)_0 \) is the pullback of \( S_k(x)_0 \), the operation \( \cdot \) indicates taking intersection product in \( A_y \otimes g(W_x/w) \), and \( \langle x, y \rangle_v \) is the local length at \( v \) of the intersection of \( x \) with \( y \). Next, we write \( S_k(y) \cdot S_k(x) \) for \( S_k(y)_0 \cdot \psi_0^* S_k(x)_0 \) to simplify notation. This is possible because \( |\text{Isom}_0(x, y)| = 1 \). We remind the reader that by our assumptions we need not consider self-intersections.

6.3.2. **Multiplicity function.** Given an isogeny between \( x \) and \( y \), we have an induced endomorphism \( \phi^* \) on the Shimura curve \( X \) given by a matrix
\[
M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\]

**Lemma 6.3.1.** Let \( P_{k-1}(t) \) denote a constant multiple of \( \frac{dt^{k-1}}{dt^{k-1}}(t^2 - 1)^{k-1} \) such that \( P_{k-1}(1) = 1 \), then
\[
S_k(x)_0 \cdot \phi^* S_k(y)_0 = (-\det M)^{k-1} P_{k-1}(\frac{bc + ad}{\det M}).
\]
(6.26)

**Proof.** It suffices to follow the proof in [23, Proposition 3.3.3] using the moduli interpretation of the Shimura curve \( X \) in terms of abelian surfaces as in Section (6.3.1), and the fact that \( \dim H^1_\text{et}(A, \mathbb{Q}_l) = 2 \) for a smooth abelian surface \( A \). \( \square \)

Given CM points \( x \) and \( y \) in \( K^\times \backslash B_f^\times /U \) with same reduction in \( F^\times_\nu \backslash B_f^\times /U \), the map
\[
K^\times \backslash B_f^\times /U \to B \backslash F^\times_\nu \times B_f^\times /U : g \to (\det g_v, g^\nu)
\]
(6.27)
associates to them the supersingular points \((\det g_v, g^v)\) and \((\det h_v, h^v)\) in \(B \setminus F^\times_v \times B_{f_v}^v/U\). Since \(x\) and \(y\) lie in the same connected component, there is \(\gamma\) in \(B_{f}^v\) such that
\[
\gamma = \gamma_v \gamma^v, \quad h^v = \gamma^v g^v, \quad \det(h_v) = \det(\gamma_v) \det(g_v),
\]
where \(\gamma_v\) lies in \(B_v^\times\), and \(\gamma^v\) lies in \(B_{f}^v\). Let \(B_0\) be the trace zero elements of \(B^\times\).

The isomorphism
\[
B_0 \bmod B_{f}^v/U^v \cong B \bmod B_f/U,
\]
where \(U' = O_{B_v}^\times \cdot U^v\), allows to view \(\gamma^v\) in \(B \bmod B_f/U\) and further localize it at \(v\) to obtain the element \(g_1\) in \(B_v^\times\). The intersection \(S_k(x) \cdot S_k(y)\) of the CM cycles associated to \(x\) and \(y\) at \(v\) only depends on \(g_1\) and \(g_2 = g_v h_v^{-1}\). We will hence denote it by \(m_1(g_1, g_2)\).

Given a pair \((a, b)\) in
\[
\mathcal{H}_{U_v} := B_v^\times \times_{K_v^\times} B_{f_v}^v / U_v
\]
satisfying \(\det(a) \det(b) = 1\), we denote by \(m_0(a, b)\) the multiplicity function defined in [23, Section 5.5]. The multiplicity function \(m\) is an intersection pairing on \(\mathcal{H}_{U_v}\) defined by
\[
m(a, b) = m_0(a, b) m_1(a, b).
\]
Using Definition (6.31), we have by [21, Lemma 8.2] the expression
\[
i_\phi(\beta_1, \beta_2) = \sum_{\gamma \in \mu_v \bmod B^\times} m(\gamma/\beta_2, \beta_1^{-1}) 1_{U_v}((\beta_1^{-1}\gamma/\beta_2)^{-1}).
\]
As a consequence, by [21, Proposition 8.4], we obtain
\[
i_\phi(Z_v(g) t_1, t_2) = \sum_{u \in \mu_v \bmod F_{x}^\times} \sum_{y \in B - E} r(g, (t_1, t_2)) \phi^v(y, u) m_{r(g, (t_1, t_2))\phi_v}(y, u),
\]
for \(g \in P(F_{S_1}) G(\mathbb{A}^S_1)\), where
\[
m_{\phi_v}(y, u) = \int_{B_{f_v}^v / U_v} m(y, x^{-1}) \phi_v(x, u q(y)/q(x)) dx,
\]
for \(y \in B_v - E_v\), and \(u \in F_v^\times\).

As in [21, Section 8.2], in the unramified case, one can rewrite \(m_{\phi_v}(y, u)\) as
\[
m_{\phi_v}(y, u) = \sum_{x \in M_2(O_{F_v}) / U_v} m(y^{-1}, x),
\]
where \(M_2(O_{F_v}) / U_v\) denotes the set of integral matrices whose determinants have valuation \(n = \text{ord}_v(q(y))\). Consider the decomposition
\[
GL_2(F_v) = \coprod_{c = 0}^{\infty} K_v^\times h_c GL_2(O_{F_v}),
\]
where \(h_c = \begin{pmatrix} 1 & 0 \\ 0 & w^c \end{pmatrix}\). The set \(K_v^\times h_c GL_2(O_{F_v}) \cap M_2(O_{F_v}) / U_v\) is non-empty only if \(n - c\) is non-negative and even. It is given then by \(w^{\frac{n-c}{2}} O_{K_v^\times} h_c U_v\). Therefore,
letting
\[ x_c = \begin{pmatrix} w^{-n/c} & 0 \\ 0 & w^{n/c} \end{pmatrix}, \]
we have
\[ m_{\phi_v}(y, u) = \sum_{c=0}^{\infty} m(y^{-1}, x_c) \text{vol}(K_v^x h_c GL_2(O_{F_v}) \cap M_2(O_{F_v}))_. \]

Consider \( \psi = \begin{pmatrix} w^{-1} & 0 \\ 0 & w \end{pmatrix} \). Then \( \psi^{n/c} = 1 \) is an isogeny between \( x_c \) and \( x_n \) in \( G(F_v) \). Hence,
\[ m_1(y, x_c) = m_1(y, x_n) \]
by Lemma (6.3.1). Consider the isogeny \( \psi_n = \begin{pmatrix} w^{-n/c} & 0 \\ 0 & w \end{pmatrix} \) between \( w^{-n/c} x_n \) and \( 1 \) in \( B_v^x \), then
\[ m_1(y, x_n) = m_1(w^{n/c} y, w^{-n/c} x_n) = m_1(w^{n/c} y, 1) = 1. \]
The first equality follows from the property \( m(y^{-1}, a^{-1} x) = m((ay)^{-1}, x) \) for \( a \) in \( F_v^x \), and the second and third equalities follow from Lemma (6.3.1) respectively applied to the matrices \( \psi_n \) and \( 1 \).

As explained in [21, Section 8.2.3], for \( c \neq 0 \),
\[ m_0(y^{-1}, x_n) \text{vol}(K_v^x h_c GL_2(O_{F_v}) \cap M_2(O_{F_v})) = 1, \]
implying that
\[ m(y^{-1}, x_n) \text{vol}(K_v^x h_c GL_2(O_{F_v}) \cap M_2(O_{F_v})) = 1 \]
by the identities (6.39) and (6.40). Hence, we have
\[ m_{\phi_v}(y, u) = \frac{1}{2} (\text{ord}_v(q(y_2)) + 1), \]
as in [21, Proposition 8.7].

6.3.3. Local comparisons. We remind the reader that the assumptions on the degenerate Schwartz functions are in force. We divide the cases into the unramified case where \( \phi_v \) is the characteristic function of \( M_2(O_v) \times O_{F_v}^x \) and \( U_v \) is the maximal compact subgroup of \( G(O_v) \), and the ramified case where \( U_v \) is general. These cases correspond to nonsplit \( v \) not in \( S_1 \) and \( v \) in \( S_1 \) respectively.

**Lemma 6.3.2.** Let \( v \) be nonsplit in \( E \) but split in \( B \). In the unramified case, we have
\[ k_{r(t_1, t_2)\phi_v}(g, y, u) = m_{r(g(t_1, t_2))\phi_v}(y, u) \log N_v, \]
and thus
\[ K_{\phi}^{(v)}(g, (t_1, t_2)) = i_\psi(Z_s(g)t_1, t_2) \log N_v. \]
In the ramified case, given \( \phi_v \) in \( \mathcal{N}^1(B_v \times F_v^x) \) invariant under the right action of \( U_v \), the function \( m_{\phi_v}(y, u) \) extends to a Schwartz function on \( B_v \times F_v^x \), and thus
\[ i_\psi(Z_s(g, \phi)t_1, t_2) = \theta(g, (t_1, t_2), m_{\phi_v} \otimes \phi^\vee). \]
for all \( g \) in \( 1_{S_1} G(A^{S_1}). \)
Proof. We first consider the unramified setting. The case \((g, t_1, t_2) = (1, 1, 1)\) follows from (6.43) and the formula

\[
k_{\phi_v}(1, y, u) = 1_{O_{B_v} \times O_{F_v}}(y, u) \frac{1}{2} (1 + v(y_2)) \log N_v
\]

by [21, Proposition 6.8], since the Whittaker functions at non-archimedean places are the same as in the weight 2 case. The result extends to \(g \in G(O_{F_v})\) since for standard Schwartz functions \(\phi_v\), the kernel functions are the same. More generally, apply the action of \(P(F_v)\) and \(E \times F_v \times E \times F_v\), observing that \(k_{\phi_v}\) and \(m_{\phi_v}\) transform like the Weil representation respectively by [21, Lemma 6.6] and by directly verifying that

\[
m_{r(g, (t_1, t_2))\phi_v}(y, u) = r(g, (t_1, t_2)) m_{\phi_v}(y, u)
\]

for \(g \in P(F_v)\) and \(t_1, t_2 \in E \times F_v\). Note that this uses the property

\[
m(y^{-1}, ax) = m(ay^{-1}, x)
\]

for \(a \in F_v^\times\).

As for the ramified case, we defer the proof to Lemma (6.4.1).

This completes the proof of (3) in Theorem (6.1.1).

6.4. Superspecial Case. The approach in the superspecial case is very similar to the one in the supersingular case. Let \(v\) be nonsplit in \(E\) and \(B\). As in the supersingular case, we will view the multiplicity function \(m\) as a function on \(\mathcal{H}_{U_v} = B_v^\times \times E_v^\times \times B_v^\times /U_v\).

Lemma 6.4.1. Given \(\phi_v\) in \(\hat{S}(B_v \times F_v^\times)\) invariant under the right action of \(U_v\), the function \(m_{\phi_v}(y, u)\) extends to a Schwartz function on \(B_v \times F_v^\times\), thus

\[
i_v(Z_v(g, \phi)t_1, t_2) = \theta(g, (t_1, t_2), m_{\phi_v} \otimes \phi_v),
\]

for \(g \in 1_{S_v}G(A^{S_1})\).

Proof. The lemma follows from the key observation that the multiplicity function \(m(\beta_1, \beta_2)\) vanishes if \(m_0(\beta_1, \beta_2)\) vanishes. That is, we have an inclusion of supports

\[
supp(m(\beta_1, \beta_2)) \subset supp(m_0(\beta_1, \beta_2)).
\]

Then, by the arguments of [21, Proposition 8.9] and [21, Proposition 8.13], \(m_{\phi_v}(y, u)\) extends to a Schwartz function on \(B_v \times F_v^\times\). This allows us to approximate

\[
i_v(Z_v(g, \phi)t_1, t_2)
\]

by the theta series

\[
\theta(g, (t_1, t_2), m_{\phi_v} \otimes \phi_v) = \sum_{u \in F_v^\times} \sum_{y \in v} r(g, (t_1, t_2)) m_{\phi_v}(y, u) r(g, (t_1, t_2)) \phi_y(u, y),
\]

giving rise to Identity (6.50).

This completes the proof of (4) in Theorem (6.1.1).
6.5. **Ordinary Case.** Let $v$ be a finite prime of $F$ split in $K$. The reduction map from CM points to ordinary CM points is given by

$$K^x \setminus B_f^\vee /U \to K^x \setminus [N(F_v) \setminus B_f^\vee] \times B_f^{x\vee} /U,$$

where $N(F_v)$ is the unipotent radical of $B^x(F_v)$. Hence, ordinary CM points $x$ and $y$ in the same connected component can be represented by $g, h$ in $B_f^\vee$ such that $h^\vee = g^\vee$ and $h_v = ng_v$ for $n \in N(F_v)$, see [23, Section 5] for more details. In this case, we do not need an explicit form of the multiplicity function $m$. We fix an identification $B_v \simeq M_2(F_v)$ under which $K_v = \text{diag}(F_v, F_v)$. Consider the prime $v_1$ lying in $K$ above $v$ such that $v_1$ corresponds to the ideal $\text{diag}(F_v, 0)$. The local height of two CM points $\beta_1, \beta_2 \in K^x \setminus B_f^\vee /U$ is given by

$$i_{v_1}(\beta_1, \beta_2) = \sum_{\gamma \in \mu \setminus \overline{K}^x} m_{v_1}(\beta_1, \gamma \beta_2) \cdot _{1U} (\beta_1^{-1} \gamma \beta_2).$$

The summation is independent of the nearby quaternion algebra, so using Assumption (5.3), by the same argument as in [21, Proposition 8.15], we have

$$i_{v}(Z_s(g) t_1, t_2) = 0.$$ 

This proves (1) of Theorem (6.1.1), and concludes the proof.

### 7. Bounding the Selmer group

In this section, we relate the order of vanishing of the $L$-function of $\pi$ at the central point to the size of the associated Selmer and Shafarevich-Tate groups. Under the conjectures on the injectivity of the $p$-adic étale Abel-Jacobi map (7.3) and the Gillet-Soulé conjecture (7.2.2), we obtain an equivalence between order of vanishing 1 for the $L$-function and rank 1 for the Selmer group.

#### 7.1. Selmer and Tate-Shafarevich groups

We define the Selmer and Tate-Shafarevich groups, quote the Beilinson-Bloch-Kato conjectures, and state a result of the first author and de Vera Piquero on the size of the Selmer group, assuming the non-vanishing of the CM cycle $Z$.

Fix $F = \mathbb{Q}$, and consider $f$ in $\pi$. Let $p$ be an odd prime not dividing $N \cdot (2k)!$, and let $\wp$ be a prime ideal dividing $p$ in the number field $E$ generated over $\mathbb{Q}$ by the Fourier coefficients of $f$. We view the Galois representation $V_\wp(f)$ attached to $f$, that is a 2-dimensional $E_\wp$-vector space, as a factor of the middle étale cohomology of the Kuga-Sato variety over the Shimura curve $[2, 10]$. Let $Y := V_\wp \otimes \mathbb{Q}_p / \mathbb{Z}_p$ and $Y_s := Y_{\wp}^s = V_\wp / p^s V_\wp$ for $s \geq 1$.

**Definition 7.1.1.** The $(p^s$-th) Selmer group $\text{Sel}^{(s)}_\wp(f, K) \subseteq H^1(K, Y_s)$ is defined as

$$\{ x \in H^1(K, Y_s) : x_v \in H^1_{\text{ur}}(K_v, Y_s) \text{ for all } v \nmid Np \text{ and } x_v \in H^1_{\text{ur}}(K_v, Y_s) \text{ for } v \mid p \},$$

where

$$H^1_{\text{ur}}(K_v, Y_s) = H^1(\bar{K_v} / K_v^{ur}, Y_s), \text{ and } H^1_{\text{ur}}(K_v, Y_s) = \text{finite part of } H^1(K_v, Y_s).$$

The $p$-adic étale Abel-Jacobi map induces a Hecke and Galois-equivariant map

$$\text{CH}^k(W_{k-1} / K_0) \otimes E_\wp \to H^1(K, V_\wp(f))$$
and factors through the Selmer group

\[(7.3) \quad \text{CH}^k(W_{k-1}/K)_0 \otimes O_\wp/p^sO_\wp \longrightarrow \text{Sel}_\wp^{(s)}(f, K).\]

The Beilinson conjectures predict that the rank of the Chow group of a smooth projective variety \(X\) of dimension \(2i + 1\) over a number field \(K\) corresponds to the order of vanishing of the \(L\)-function attached to its étale cohomology

\[(7.4) \quad \text{ord}_{s=i+1} L(H^{2i+1}_{et}(X), s) = \dim_{\QQ} \text{CH}^{i+1}(X)_0,\]

(see [1, Conjecture 5.9]). This conjecture can be refined by applying a suitable projector \(e\) to both sides of the equality. Letting \(M\) denote \(H^{2i+1}_{et}(X)\), Bloch and Kato predict that the Abel-Jacobi map

\[(7.5) \quad e \text{CH}^{i+1}(X)_0 \otimes \QQ_p \longrightarrow \text{Sel}(K, e M)\]

is an isomorphism [3, Conjecture 5.3]. Hence, one expects that

\[(7.6) \quad \text{ord}_{s=i+1} L(e M, s) = \text{rank} \text{Sel}(K, e M).\]

We denote

\[(7.7) \quad \text{Sel}_\wp^{(\infty)}(f, K) := \lim_{\text{←}} \text{Sel}_\wp^{(s)}(f, K).\]

Taking the inductive limit of the Abel–Jacobi maps (7.3) one obtains a map

\[(7.8) \quad \Phi : \text{CH}^k(W_{k-1}/K)_0 \otimes O_\wp \longrightarrow \text{Sel}_\wp^{(\infty)}(f, K) \subseteq H^1(K, V_\wp).\]

Its cokernel is by definition the \(\wp\)-primary part of the Shafarevich-Tate group,

\[(7.9) \quad \text{III}_\wp^{\infty} := \text{coker}(\Phi) = \text{Sel}_\wp^{(\infty)}(f, K)/\text{Im}(\Phi).\]

**Theorem 7.1.2.** [7, Theorem 1.1] Suppose \(Z\) is non-torsion. Then \(\text{Im}(\Phi)\) has rank 1 and \(\text{III}_\wp^{\infty}\) is finite. More precisely, we have

\[(\text{Im}(\Phi))^\vee = 0 \quad \text{and} \quad (\text{Im}(\Phi))^{-\vee} = E_\wp \cdot y_0.\]

In [7], the authors construct an Euler system of Kolyvagin cohomology classes \(c_n\) in \(\text{Sel}_\wp^{(s)}(f, K_n)\) where \(K_n\) is a ring class field of conductor \(n\), that is a collection of cohomology classes satisfying suitable global and local compatibility properties. Kolyvagin’s method is subsequently applied to bound the size of \(\text{Sel}_\wp^{(\infty)}(f, K)\) and \(\text{III}_\wp^{\infty}\).

**7.2. Bridging analytic and algebraic invariants.** We combine Theorem (7.1.2) with our main theorem (2.8.1) to link analytic and algebraic invariants associated to \(\pi\), under suitable assumptions.

**Theorem 7.2.1.** Assume that the \(p\)-adic étale Abel-Jacobi map \(\Phi\) is injective. Then

\[(7.10) \quad L'(k, \pi) \neq 0 \implies \text{Im}(\Phi) \otimes \QQ \text{ has rank 1, and } |\text{III}_\wp^{\infty}| < \infty.\]

**Proof.** If \(L'(k, \pi) \neq 0\), then \(Z\) is non-torsion in \(\text{CH}^k(W_{k-1}/K)_0\) by Theorem (2.8.1) hence

\[(7.11) \quad Z_0 = \text{cor}_{K_1/K} Z_\wp\]

is non-torsion in \(H^1(K, V_\wp)\). The injectivity of the \(p\)-adic Abel Jacobi map predicted by Bloch and Kato [3] implies that \(\Phi(Z_0)\) is non-torsion. By Theorem (7.1.2), we conclude that \(\text{Im}(\Phi) \otimes \QQ\) has rank 1, and \(\text{III}_\wp^{\infty}\) is finite. \(\Box\)
Next, we deduce analytic rank 1 from algebraic rank 1, under the Gillet-Soulé conjecture [8, Conjecture 2] which implies that the height pairing $\langle \cdot, \cdot \rangle_{GS}$ is non-degenerate in our setting.

We state the Gillet-Soulé conjecture. Consider a line bundle $L$ on a regular arithmetic scheme $Y$ of dimension $d$ over $\text{Spec}(O_F)$, equipped with a smooth hermitian metric invariant under the complex conjugation $F_\infty$. One can associate to $L$ a first Chern class $c_1(L)$ in $\widehat{\text{CH}}(Y)^R$, namely the class $(\text{div}(s) - \log ||s||)$, for a nonzero rational section $s$ of $L$ on $Y$. Denote by $L$ the map given by

\begin{equation}
L : \widehat{\text{CH}}^p(Y)^R \to \widehat{\text{CH}}^{p+1}(Y)^R : Y \mapsto (Y, c_1(L)).
\end{equation}

The line bundle $L$ is positive if the following conditions are satisfied:
- it is ample on $Y$,
- the curvature $c_1(L)$ is a positive $1-1$ form on $Y(C)$, and
- for any subvariety $Y'$ of $Y$ of dimension $n$, flat over $\text{Spec}(Z)$, we have

\begin{equation}
(c_1(L)^n, Y') > 0.
\end{equation}

**Conjecture 7.2.2** (Gillet-Soulé). Let $L$ be a positive hermitian line bundle on $Y$ and $x$ an arithmetic Chow cycle of codimension $p$. If $2p \leq d$, $x \neq 0$ and $L^{d-2p+2}(x) = 0$, then

\begin{equation}
(-1)^p \deg(xL^{d-2p+1}x) > 0.
\end{equation}

In our setting, $d = 2k - 1$ and $p = k$, so that $d - 2p + 2 = 1$. The conjecture hence implies that if $x \neq 0$ and $(x, c_1(L)) = 0$, then $(-1)^k \deg(x \cdot x) > 0$, namely, the height pairing is non-degenerate.

**Proposition 7.2.3.** For CM cycles $Z_A$, the intersection $c_1(L) \cdot Z_A$ is zero in $\widehat{\text{CH}}(W)$.

**Proof.** Consider a line bundle $L$ that is the sum of pull-backs $L_i$, $1 \leq i \leq 2k-2$, of line bundles on $A$ with a divisor supported on the unit section. The proof follows the proof of [23, Proposition 3.1.1] where the next lemma substitutes Lemma 3.1.3. □

**Lemma 7.2.4.** We have

\begin{align}
\langle Z_A, \{0\} \times E \rangle &= 0, \\
\langle Z_A, E \times \{0\} \rangle &= 0.
\end{align}

**Theorem 7.2.5.** Suppose the Gillet-Soulé conjecture (7.2.2) holds. Then

\begin{equation}
\text{Im}(\Phi) \otimes Q \text{ has rank } 1 \implies L'(k, \pi) \neq 0.
\end{equation}

**Proof.** Let $g$ be a generator of $\text{Im}(\Phi) \otimes Q$. Then $T(f_1 \otimes f_2)Z$ and $Z$ can be written in terms of $g$. Since the height pairing $\langle \cdot, \cdot \rangle_{GS}$ is non-degenerate by Conjecture (7.2.2) and Proposition (7.2.3), we have that $\langle T(f_1 \otimes f_2)Z, Z \rangle_{GS}$ is non-zero. Hence, $L'(k, \pi) \neq 0$ by Theorem (2.8.1).

As a consequence, we obtain the following corollary.

**Corollary 7.2.6.** Assume that the $p$-adic Abel-Jacobi map $\Phi$ is injective and the Gillet-Soulé conjecture (7.2.2) holds. Then

\begin{equation}
\text{ord}_{s=k} L(s, \pi) = 1 \iff \text{rank } (\text{Im}(\Phi) \otimes Q) = 1.
\end{equation}
Remark 7.2.7. Chida [6] used the method of Euler systems to prove that, under certain assumptions, the non-vanishing of the central value of the Rankin-Selberg $L$-function associated with a modular form $f$ and a ring class character implies the finiteness of the associated Selmer group.

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