Crossed Andreev reflection at ferromagnetic domain walls

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We investigate several factors controlling the physics of hybrid structures involving ferromagnetic domain walls (DWs) and superconducting (S) metals. We discuss the role of non-collinear magnetizations in S/DW junctions in a spin $\otimes$ Nambu $\otimes$ Keldysh formalism. We discuss transport in S/DW/N and S/DW/S junctions in the presence of inelastic scattering in the domain wall. In this case transport properties are similar for the S/DW/S and S/DW/N junctions and are controlled by sequential tunneling of spatially separated Cooper pairs across the domain wall. In the absence of inelastic scattering we find that a Josephson current circulates only if the size of the ferromagnetic region is smaller than the elastic mean free path meaning that the Josephson effect associated to crossed Andreev reflection cannot be observed under usual experimental conditions. Nevertheless a finite dc current can circulate across the S/DW/S junction due to crossed Andreev reflection associated to sequential tunneling.

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I. INTRODUCTION

A simple way of obtaining correlated pairs of electrons in solid state devices is to extract Cooper pairs from a BCS superconductor. Devices based on this principle have focused an important interest recently. For instance entangled pairs of electrons can be manipulated in double dot experiments. Other devices involving a larger number of quantum dots have been proposed recently as a quantum teleportation experiment. Devices involving several ferromagnetic electrodes connected to a superconductor have been investigated recently. Noise correlations can also provide useful information about quantum entanglement.

Many phenomena are involved in the proximity effect at ferromagnet / superconductor (F/S) interfaces. For instance it is well established that the pair amplitude induced in a ferromagnetic metal oscillates in space. An interesting consequence is the possibility of fabricating S/F/S $\pi$-junctions in which the Josephson relation is $I = I_c \sin (\phi + \pi)$. In F/S/F trilayers the superconducting transition temperature is larger in the antiferromagnetic alignment of the ferromagnetic electrodes because a finite exchange field is induced in the superconductor in the ferromagnetic alignment. On the other hand there exist “non local” superconducting correlations coupling the two ferromagnetic electrodes that favor $\Delta_F > \Delta_{AF}$ (the zero-temperature superconducting order parameter is larger in the ferromagnetic alignment). It is also well known that the superconducting transition temperature of F/S multilayers oscillates as the thickness of the ferromagnetic layers is increased. Several recent works have investigated new phenomena taking place in diffusive F/S heterostructures. Other recent works were devoted to understand the interplay between Andreev reflection and spin polarization at a single F/S interface.

In a recent article M. Giroud et al. have proposed on the basis of experiments that the proximity effect at F/S interfaces could be strongly modified by the presence of Cooper pair-like states propagating along domain walls (DWs). These Cooper pair-like states correspond to pair states in which the spin-up and spin-down electrons propagate in a neighboring spin-up and spin-down magnetic domain. This proximity effect is not strictly speaking equivalent to the proximity effect a N/S interfaces. The reason is that the pair correlations induced in the N side of a N/S interface have entangled orbital and spin degrees of freedom. By contrast for half-metal ferromagnets the wave function associated to the propagation of superconducting correlations along domain walls is given by the product state $|e, \alpha, \uparrow\rangle \otimes |e, \beta, \downarrow\rangle$, where $\alpha$ and $\beta$ represent two points in neighboring magnetic domains. Another difference between a N/S interface and a multiterminal hybrid structure is that the incoming electron and the Andreev reflected hole propagate in different electrodes in multiterminal structures. As a consequence the Andreev reflected hole cannot follow the same trajectory as the incoming electron. This has important consequences regarding disorder averaging.

The purpose of our article is to investigate theoretically the mechanisms by which the Cooper pair-like state $|e, \alpha, \uparrow\rangle \otimes |e, \beta, \downarrow\rangle$ can propagate along a ferromagnetic domain wall and to investigate several new situations that may be the object of experiments in the future. In section we discuss the perturbative transport formula of a S/DW junction in which the domain wall consists of many independent channels in parallel having a rotating magnetization.
To discuss this model we use the spin \( \otimes \) Nambu \( \otimes \) Keldysh formalism described in section II. For the sake of obtaining analytical results we restrict the discussion to the transport formula obtained within lowest order perturbation theory.

If propagation in the ferromagnet is phase coherent then the pair state \( | e, \alpha, \uparrow \rangle \otimes | e, \beta, \downarrow \rangle \) injected at one end of the domain wall can propagate to the other end. On the other hand if the phase coherence length \( l_{\phi} \) is small compared to the size of the ferromagnetic region then inelastic scattering processes are strong and there are just a spin-up and a spin-down electron propagating independently in the spin-up and spin-down magnetic domains. There is no Josephson current but there exists crossed Andreev reflection taking place locally at each F/S interface, so that the conductance is larger in the presence of the domain wall.

In section IV we discuss the S/DW/N and S/DW/S junctions in a regime where transport properties are dominated by inelastic scattering in the domain wall. The domain wall is represented by two channels in parallel, with an opposite magnetization. This schematic model of domain wall is expected to capture the essential physics, and can be a useful experimental conditions. Final remarks are given in section VI.

To end up the introductory section we note that the theory of inhomogeneous ferromagnets with non collinear magnetizations in contact with a superconductor was already elaborated in Refs. 35-38 in connection with the long-range proximity effect associated to the triplet component of the superconducting condensate. In our article the emphasis is put on other aspects of this problem (the transport of spatially separated Cooper pairs). Both effects may play a relevant role in experiments. Finally a recent preprint appeared in which the conductance of a S/DW junction was calculated independtly from our work.

II. PRELIMINARIES

A. Spin \( \otimes \) Nambu \( \otimes \) Keldysh formalism

The direction of the magnetization is rotating in a ferromagnetic domain wall. To describe superconducting correlations in the presence of non collinear magnetizations we use a spin \( \otimes \) Nambu \( \otimes \) Keldysh formalism. The advanced Green’s function is a \( 4 \times 4 \) matrix:

\[
\hat{G}_{a,j}^A(t,t') = -i\theta(t-t') \begin{bmatrix}
\langle c^+_j(t'), c_{i,j}(t) \rangle & \langle c^+_j(t'), c_{i,j}(t) \rangle & \langle c^+_j(t'), c_{i,j}(t) \rangle & \langle c^+_j(t'), c_{i,j}(t) \rangle \\
\langle c^+_j(t'), c_{i,j}(t) \rangle & \langle c^+_j(t'), c_{i,j}(t) \rangle & \langle c^+_j(t'), c_{i,j}(t) \rangle & \langle c^+_j(t'), c_{i,j}(t) \rangle \\
\langle c^+_j(t'), c_{i,j}(t) \rangle & \langle c^+_j(t'), c_{i,j}(t) \rangle & \langle c^+_j(t'), c_{i,j}(t) \rangle & \langle c^+_j(t'), c_{i,j}(t) \rangle \\
\langle c^+_j(t'), c_{i,j}(t) \rangle & \langle c^+_j(t'), c_{i,j}(t) \rangle & \langle c^+_j(t'), c_{i,j}(t) \rangle & \langle c^+_j(t'), c_{i,j}(t) \rangle \\
\end{bmatrix}.
\]
The Dyson equation relates the Green’s functions of the connected system to the Green’s functions of the disconnected system. In a compact notation the Dyson equation takes the form \( \hat{G} = \hat{g} + \hat{g} \otimes \hat{\Sigma} \otimes \hat{G} \), where the symbol \( \otimes \) includes a summation over the sites of the network and a convolution over time variables. Since we consider stationary transport the convolution over time variables becomes a simple product after a Fourier transform is carried out. The Dyson equation for the Keldysh Green’s function \( \hat{G}^{+, -} \) is given by
\[
\hat{G}^{+, -} = \left[ \hat{I} + \hat{G}^0 \otimes \hat{\Sigma} \right] \otimes \hat{g}^{+, -} \otimes \left[ \hat{I} + \hat{\Sigma} \otimes \hat{G}^0 \right],
\]
where the self-energy \( \hat{\Sigma} \) contains all the couplings present in the tunnel Hamiltonian. The tunnel Hamiltonian corresponding to Fig. 1 takes the form
\[
\mathcal{W} = \sum_{\sigma} \left[ t_{a,\sigma} c_{a,\sigma}^+ + t_{a,\sigma} c_{a,\sigma} \right] + \left[ t_{\beta,\sigma} c_{\beta,\sigma}^+ + t_{\beta,\sigma} c_{\beta,\sigma} \right].
\]
The current through the link \( a - \alpha \) is given by
\[
I_{a,\alpha} = \frac{e}{2\hbar} \int \text{Tr} \left\{ \hat{\sigma}_z \left[ \hat{\mathcal{I}}_{a,\alpha} \hat{G}^{+, -}_{a,\alpha} - \hat{\mathcal{I}}_{a,\alpha} \hat{G}^{+, -}_{a,\alpha} \right] \right\} d\omega,
\]
where the matrix \( \hat{\sigma}_z \) is given by
\[
\hat{\sigma}_z = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix}
\]
and the Nambu representation of the hopping matrix elements is given by \( \hat{\mathcal{I}}_{a,\alpha} = t_{a,\alpha} \hat{\sigma}_z \), \( \hat{\mathcal{I}}_{a,\alpha} = t_{a,\alpha} \hat{\sigma}_z \), \( \hat{\mathcal{I}}_{\beta,\beta} = t_{\beta,\beta} \hat{\sigma}_z \), \( \hat{\mathcal{I}}_{\beta,\beta} = t_{\beta,\beta} \hat{\sigma}_z \).

### B. Green’s function of a ferromagnetic metal

Now we give the expressions of the Green’s functions of a ferromagnetic metal. We first suppose that the spin quantization axis is parallel to the direction of the magnetization. The Green’s function takes the form
\[
\hat{g}(R, \omega) = \begin{bmatrix}
g_{1,1}(R, \omega) & 0 & 0 & 0 \\
0 & g_{2,2}(R, \omega) & 0 & 0 \\
0 & 0 & g_{3,3}(R, \omega) & 0 \\
0 & 0 & 0 & g_{4,4}(R, \omega)
\end{bmatrix}.
\]
The four diagonal elements are given by
\[
g_{1,1}(R, \omega) = -\frac{m_{1} a_0^2}{\hbar^2} \frac{a_0}{2\pi R} \exp \left\{ -i \left( k_F^+ + \frac{\omega}{v_F} \right) R \right\} \exp \left\{ - \left( \frac{R}{l_0} \right) \right\}
\]
(7)
\[
g_{2,2}(R, \omega) = \frac{m_{1} a_0^2}{\hbar^2} \frac{a_0}{2\pi R} \exp \left\{ i \left( k_F^- - \frac{\omega}{v_F} \right) R \right\} \exp \left\{ - \left( \frac{R}{l_0} \right) \right\}
\]
(8)
\[
g_{3,3}(R, \omega) = -\frac{m_{1} a_0^2}{\hbar^2} \frac{a_0}{2\pi R} \exp \left\{ -i \left( k_F^+ + \frac{\omega}{v_F} \right) R \right\} \exp \left\{ - \left( \frac{R}{l_0} \right) \right\}
\]
(9)
\[
g_{4,4}(R, \omega) = \frac{m_{1} a_0^2}{\hbar^2} \frac{a_0}{2\pi R} \exp \left\{ i \left( k_F^- - \frac{\omega}{v_F} \right) R \right\} \exp \left\{ - \left( \frac{R}{l_0} \right) \right\},
\]
(10)
where we have introduced a Fermi wave vector mismatch as well as a mismatch between the spin-up and spin-down Fermi velocities. The parameter \( a_0 \) is equal to the distance between neighboring sites on the cubic lattice. For generality we introduced a different mass for the spin-up and spin-down electrons, meaning that the spin-up density of states is different from the spin-down density of states. The local propagators are defined by
\[
g_{1,1}^{\text{loc}} = g_{4,4}^{\text{loc}} = i \frac{a_0 k_F^+ m_{1} a_0^2}{\hbar^2} = i \pi \rho_F \left( \frac{1 + P}{2} \right)
\]
(11)
\[
g_{2,2}^{\text{loc}} = g_{3,3}^{\text{loc}} = i \frac{a_0 k_F^- m_{1} a_0^2}{\hbar^2} = i \pi \rho_F \left( \frac{1 - P}{2} \right).
\]
(12)
We also introduced phenomenologically in (7) – (10) an exponential decay of the correlations due to the presence of a finite coherence length \( l_\phi \) in the ferromagnet. \( l_\phi \) is usually smaller than the dimension of the ferromagnetic metal. In this case ferromagnetism can be treated semi-classically like in the theoretical description of the giant magnetoresistance.\(^{41,42,43}\) However Aharonov-Bohm oscillations in a ferromagnetic nanoring have been reported recently.\(^{44}\) The inner diameter of the Fe-Ni nanoring in Ref.\(^{42}\) is 420 Å and the outer diameter is 500 Å.

We will use in section III the expression of the local Green’s functions of a ferromagnetic metal with the quantization axis not parallel to the magnetization. We suppose that the direction of the exchange field is rotated by an angle \( \theta \) around the \( z \) axis. We do not incorporate a rotation of angle \( \varphi \) around the \( x \) axis since this rotation just introduces simple phase factors. The local Green’s function of the rotated ferromagnet takes the form

\[
\hat{g}_{\text{loc}} = i\pi \hat{\rho} \begin{bmatrix}
1 + P \cos \theta & 0 & -iP \sin \theta & 0 \\
0 & 1 - P \cos \theta & 0 & -iP \sin \theta \\
iP \sin \theta & 0 & 1 - P \cos \theta & 0 \\
iP \sin \theta & 0 & 1 + P \cos \theta & 0
\end{bmatrix},
\]

where \( \hat{\rho} = (\rho_\uparrow + \rho_\downarrow)/2 \) is the average density of states at the Fermi level and \( P = (\rho_\uparrow - \rho_\downarrow)/ (\rho_\uparrow + \rho_\downarrow) \) is the spin polarization at the Fermi level.

We will also use in section V B the expression of the full propagator \( \hat{g}(R, \omega) \) of a rotated ferromagnet. The Green’s function takes the form

\[
\hat{g}(R, \omega) = \begin{bmatrix}
\tilde{g}_{1,1} & 0 & \tilde{g}_{1,3} & 0 \\
0 & \tilde{g}_{2,2} & 0 & \tilde{g}_{2,4} \\
\tilde{g}_{3,1} & 0 & \tilde{g}_{3,3} & 0 \\
0 & \tilde{g}_{4,2} & 0 & \tilde{g}_{4,4}
\end{bmatrix},
\]

where the diagonal elements are given by

\[
\tilde{g}_{1,1} = \frac{1}{2}(g_{1,1} + g_{3,3}) + \frac{1}{2} \cos \theta (g_{1,1} - g_{3,3})
\]

\[
\tilde{g}_{2,2} = \frac{1}{2}(g_{2,2} + g_{4,4}) + \frac{1}{2} \cos \theta (g_{2,2} - g_{4,4})
\]

\[
\tilde{g}_{3,3} = \frac{1}{2}(g_{3,3} + g_{1,1}) + \frac{1}{2} \cos \theta (g_{3,3} - g_{1,1})
\]

\[
\tilde{g}_{4,4} = \frac{1}{2}(g_{4,4} + g_{2,2}) + \frac{1}{2} \cos \theta (g_{4,4} - g_{2,2}),
\]

where \( g_{1,1}, g_{2,2}, g_{3,3} \) and \( g_{4,4} \) are given by (7) – (10). The extra-diagonal elements are given by

\[
\tilde{g}_{1,3} = -\tilde{g}_{3,1} = \frac{i}{2} \sin \theta (g_{3,3} - g_{1,1})
\]

\[
\tilde{g}_{2,4} = -\tilde{g}_{4,2} = \frac{i}{2} \sin \theta (g_{2,2} - g_{4,4}).
\]

**C. 4 × 4 Green’s functions of a superconductor**

The Green’s function of a superconductor takes the form

\[
\hat{g}^{A,R}(R, \omega) = \begin{bmatrix}
g(R, \omega) & f(R, \omega) & 0 & 0 \\
f(R, \omega) & g'(R, \omega) & 0 & 0 \\
0 & 0 & g(R, \omega) & -f(R, \omega) \\
0 & 0 & -f(R, \omega) & g'(R, \omega)
\end{bmatrix}.
\]

The matrix elements of the Green’s function are given by

\[
g(R, \omega) = \frac{ma^2}{\hbar^2} \frac{a_0}{2\pi R} \exp \left( -\frac{R}{\xi(\omega)} \right) \left\{ \sin \left( k_F R \right) \frac{-\omega}{\sqrt{\Delta^2 - \omega^2}} - \cos \left( k_F R \right) \right\}
\]

\[
g'(R, \omega) = \frac{ma^2}{\hbar^2} \frac{a_0}{2\pi R} \exp \left( -\frac{R}{\xi(\omega)} \right) \left\{ \sin \left( k_F R \right) \frac{-\omega}{\sqrt{\Delta^2 - \omega^2}} + \cos \left( k_F R \right) \right\}
\]

\[
f(R, \omega) = \frac{ma^2}{\hbar^2} \frac{a_0}{2\pi R} \exp \left( -\frac{R}{\xi(\omega)} \right) \sin \left( k_F R \right) \frac{\Delta}{\sqrt{\Delta^2 - \omega^2}}
\]

where we supposed that \( \omega < \Delta \). The coherence length is given by \( \xi(\omega) = \hbar v_F / \sqrt{\Delta^2 - \omega^2} \).
D. $4 \times 4$ Green’s functions of a superconductor in a uniform magnetic field

A uniform magnetic field $h_S$ can penetrate in a superconductor if the superconductor is in a thin film geometry and the magnetic field is parallel to the direction of the superconducting film. The effect of the magnetic field is a Zeeman splitting of the spin-up and spin-down quasiparticle bands. Let us suppose that the quantization axis is parallel to the orientation of the magnetic field. The $4 \times 4$ Green’s function takes the form

$$g^{A,R}(R, \omega) = \begin{bmatrix} g_+(R, \omega) & f_+(R, \omega) & 0 & 0 \\ f_+(R, \omega) & g_+(R, \omega) & 0 & 0 \\ 0 & 0 & g_-(R, \omega) & -f_-(R, \omega) \\ 0 & 0 & -f_-(R, \omega) & g_-(R, \omega) \end{bmatrix},$$

(25)

with $g_+(R, \omega) = g(R, \omega + h_S)$, $g_-(R, \omega) = g(R, \omega + h_S)$, $f_+(R, \omega) = f(R, \omega + h_S)$, $g_-(R, \omega) = g(R, \omega - h_S)$, $g'_-(R, \omega) = g'(R, \omega - h_S)$, $f_-(R, \omega) = f(R, \omega - h_S)$.

III. CROSSED ANDREEV REFLECTION AND ELASTIC COTUNNELING WITH NON COLLINEAR MAGNETIZATIONS

A. Transport formula

In this section we evaluate the transport formula corresponding to the device on Fig. 1 in which the magnetization of electrode “a” (“b”) makes an angle $\theta_a$ ($\theta_b$) with the z-axis. Using the formalism described in section II we obtain the current per conduction channel through electrode “a” to lowest order in $t_{a,\alpha}$ and $t_{b,\beta}$:

$$I_{a,\alpha} = \frac{e}{h} \int d\omega 8\pi^2 \alpha^2 \rho_{a0}(1 - P_a^2) f_{1a}(\omega) [n_F(\omega - eV_a) - n_F(\omega + eV_a)]$$

(26)

$$+ \frac{e}{h} \int d\omega 4\pi \alpha^2 \beta^2 \rho_{a0} \rho_{b0} [1 + P_a P_b \cos (\theta_a - \theta_b)] \langle \langle g_{a,\alpha} g_{b,\beta} \rangle \rangle$$

$$\times [n_F(\omega - eV_a) - n_F(\omega + eV_a) - n_F(\omega - eV_b) + n_F(\omega + eV_b)]$$

$$+ \frac{e}{h} \int d\omega 4\pi \alpha^2 \beta^2 \rho_{a0} \rho_{b0} [1 - P_a P_b \cos (\theta_a - \theta_b)] \langle \langle f_{a,\alpha} f_{b,\beta} \rangle \rangle$$

$$\times [n_F(\omega - eV_a) - n_F(\omega + eV_a) + n_F(\omega - eV_b) - n_F(\omega + eV_b)].$$

We have supposed that electrodes “a” and “b” are made of a large number of independent conduction channels in parallel so that we make an averaging over the microscopic phases in the propagators. Now if we consider that the same voltage is applied on both electrodes the conductance is given by local Andreev reflection and crossed Andreev reflection:

$$G = 32\pi^2 \alpha^2 \beta^2 \rho_{a0} \rho_{b0} (1 - P_a^2) f_{1a} + 32\pi^2 \alpha^2 \beta^2 \rho_{a0} \rho_{b0} [1 - P_a P_b \cos (\theta_a - \theta_b)] \langle \langle f_{a,\alpha} f_{b,\beta} \rangle \rangle.$$

(27)

B. Conductance associated to a domain wall

Let us now consider the situation on Fig. 2 representing a S/DW contact between a superconductor and a magnetic domain wall. We suppose that the ferromagnetic metal is made of a collection of independent channels. The magnetization is rotating inside the domain wall meaning that the angle $\theta$ is a function of $z$: $\theta = \theta(z)$. We want to evaluate the difference $G^{(DW)} - G^{(0)}$ between the conductances $G^{(DW)}$ in the presence of the domain wall and $G^{(0)}$ in the absence of the domain wall. To obtain the conductance we sum the contributions of the different channels (see Fig. 2) and we obtain

$$G^{(DW)} - G^{(0)} = \frac{4\pi^2}{h} \alpha^2 \beta^2 P^2 \left( \frac{m a_0}{h^2} \right)^2 F(\xi_0, D),$$

(28)

with

$$F(\xi_0, D) = \int \frac{d(\Delta y)}{a_0} \int \frac{dz_a}{a_0} \int \frac{dz_b}{a_0} \frac{a_0^2}{(\Delta y)^2 + (z_a - z_b)^2} \sin^2 \left( \frac{\theta(z_a) - \theta(z_b)}{2} \right) \exp \left( -2 \sqrt{(\Delta y)^2 + (z_a - z_b)^2} \frac{\xi_0}{\xi_0} \right).$$

(29)
FIG. 2: The device involving a S/DW junction between a superconductor and a ferromagnetic domain wall. In the ferromagnet the local magnetization makes an angle $\theta(z)$ with the $z$-axis. The $x$- and $z$-axis are in the plane of the figure. The $y$-axis is perpendicular to the plane of the figure.

FIG. 3: Variation of $F(\xi_0, D)$ as a function of $\xi_0$ for different values of $D$. We used the domain wall profile given by (30) in a three-dimensional geometry. The distance between neighboring channels is $a_0 = 1$. We introduced a cut-off $-L_y/2 \leq \Delta y \leq L_y$, $-L_z/2 \leq z_a, z_b \leq L_z/2$ in the expression of $F(\xi_0, D)$ (see Eq. (29)). The solid lines correspond to $L_y = L_z = 500$ and the dashed line (almost superimposed with the solid line) correspond to $L_y = L_z = 600$.

where $\xi_0 = \hbar v_F/\Delta$ is the BCS coherence length at zero energy and $D$ is the width of the domain wall. $L_y$ is equal to the dimension of the junction in the $y$ direction and we used the notation $\Delta y = y_a - y_b$. To obtain (29) we have supposed that the width of the domain wall is much larger than the Fermi wave-length so that we can average over the microscopic phase variables in the propagator $f_{\alpha,\beta}$ (see Eq. (24)).

Crossed Andreev reflection cannot take place between the channels separated by a distance much smaller than the width $D$ of the domain wall because such channels have an almost parallel magnetization. Crossed Andreev reflection cannot take place either between channels separated by a distance much larger than the superconducting coherence length because of the exponential decay of the propagator $f_{\alpha,\beta}$. As a consequence the value of $G^{(DW)} - G^{(0)}$ is the largest if the width of the domain wall is small compared to the BCS coherence length. This is illustrated on Fig. 3 where we have represented the variation of the conductance as a function of $\xi_0$ for different values of $D$ and for the domain wall profile given by

$$\theta(z) = \arctan\left(\frac{z}{D}\right). \tag{30}$$

C. Exchange field in the superconductor due to the proximity effect

Now we come back to a system in which two ferromagnetic electrodes are connected to a superconductor. An exchange field can be generated in the superconductor because of the proximity effect. This was first observed in Ref. 16 in the case of insulating ferromagnets. An exchange field in a superconductor is a pair breaking perturbation. As
a consequence in the F/S/F trilayer with insulating ferromagnets the order parameter is larger in the antiferromagnetic alignment of the ferromagnetic electrodes. This was well verified in experiments with insulating ferromagnets. The same effect is present with metallic ferromagnets but in this case there exists also pair correlations induced in the ferromagnetic electrodes that can modify the value of the self consistent order parameter.

We suppose that the magnetizations in electrodes “a” and “b” make an angle $\theta_a$ and $\theta_b$ and that an exchange field $h_S$ is induced in the superconductor. Without loss of generality we suppose that the direction of the exchange field in the superconductor is parallel to the quantization axis. In terms of the $g_+, f_+, g_- f_-$ introduced in section the transport formula is found to be

$$I_{a,\alpha} = \frac{e}{\hbar} \int d\omega 4\pi^2 t^2_{a,\alpha} \rho a \left[ (f^2_a + f^2) (1 - P^2_a \cos^2 \theta_a) - 2f_a P_a \sin^2 \theta_a \right] [n_F(\omega - eV_a) - n_F(\omega + eV_a)]$$

$$+ \frac{e}{\hbar} \int d\omega 2\pi^2 t^2_{a,\alpha} \rho a \tilde{\rho} b \left[ \left\langle g\alpha,\beta g\alpha,\beta \right\rangle + \left\langle g\alpha,\beta g\alpha,\beta \right\rangle \right] [1 + P_a P_b \cos \theta_a \cos \theta_b]$$

$$+ 2 \left\langle g\alpha,\beta g\beta,\alpha \right\rangle P_a P_b \sin \theta_a \sin \theta_b \right\} [n_F(\omega - eV_a) - n_F(\omega + eV_a) - n_F(\omega - eV_b) + n_F(\omega + eV_b)]$$

$$+ \frac{e}{\hbar} \int d\omega 2\pi^2 t^2_{b,\beta} \rho b \tilde{\rho} a \left[ \left\langle f\alpha,\beta f\alpha,\beta \right\rangle + \left\langle f\alpha,\beta f\alpha,\beta \right\rangle \right] [1 - P_a P_b \cos \theta_a \cos \theta_b]$$

$$- 2 \left\langle f\alpha,\beta f\beta,\alpha \right\rangle P_a P_b \sin \theta_a \sin \theta_b \right\} [n_F(\omega - eV_a) - n_F(\omega + eV_a) + n_F(\omega - eV_b) - n_F(\omega + eV_b)] .$$

The term corresponds to local Andreev reflection. The term corresponds to elastic cotunneling and the term corresponds to crossed Andreev reflection. The term corresponds to a process in which a spin-up electron travels from electrode $a$ to electrode $b$ and comes back to electrode $a$ as a spin-up electron. The term corresponds to a process in which a spin-up electron travels from electrode $a$ to electrode $b$, undergoes a spin precession in electrode $b$ and comes back as a spin-down electron traveling from electrode $b$ to electrode $a$.

Replacing the propagators involved in Eqs. and by their expressions given in section leads to the transport formula to lowest order in $h_S$ and $\omega$:

$$I_{a,\alpha} = 8\pi^2 t^2_{a,\alpha} \left( \frac{m_{\alpha}}{2\pi R_0} \right)^2 \left( \frac{a_0}{2\pi R_{a,\beta}} \right)^2 \left( \frac{\omega^2 + h_S^2}{\Delta^2} \right) \left[ 1 - P_a \right] [n_F(\omega - eV_a) - n_F(\omega + eV_a)]$$

$$+ 2 \pi^2 t^2_{a,\alpha} \rho a \tilde{\rho} b \left( \frac{m_{\alpha}}{2\pi R_{a,\beta}} \right)^2 \left( \frac{a_0}{2\pi R_{a,\beta}} \right)^2 \exp \left( \frac{-2R_{a,\beta}}{\xi(\omega)} \right) \left\{ 1 + \left[ 1 + \frac{\omega^2}{\Delta^2} \right] P_a P_b \cos (\theta_a - \theta_b) \right\}$$

$$+ \frac{h_S^2}{\Delta^2} P_a P_b \cos (\theta_a + \theta_b) \right\} [n_F(\omega - eV_a) - n_F(\omega + eV_a) - n_F(\omega - eV_b) + n_F(\omega + eV_b)]$$

$$+ 2 \pi^2 t^2_{b,\beta} \rho b \tilde{\rho} a \left( \frac{m_{\beta}}{2\pi R_{a,\beta}} \right)^2 \left( \frac{a_0}{2\pi R_{a,\beta}} \right)^2 \exp \left( \frac{-2R_{a,\beta}}{\xi(\omega)} \right) \left\{ 1 + \frac{\omega^2 + h_S^2}{\Delta^2} \right\} \left[ 1 - P_a P_b \cos (\theta_a - \theta_b) \right\}$$

$$\times [n_F(\omega - eV_a) - n_F(\omega + eV_a) + n_F(\omega - eV_b) - n_F(\omega + eV_b)] ,$$

where $k_F R_0 = \pi/2$ is the ultraviolet cut-off used to define the local propagator involved in local Andreev reflection. We see that the crossed Andreev reflection term given by is not identical to the elastic cotunneling term given by. This shows that the symmetry between elastic cotunneling and crossed Andreev reflection is broken by the exchange field in the superconductor. This can be illustrated by considering that electrode “b” is a normal metal: $P_b = 0$. The crossed conductance at zero voltage is finite if the exchange field $h_S$ in the superconductor takes a finite value:

$$G_{a,b} = \frac{\partial I_{a}}{\partial V_b} = 4\pi^2 t^2_{a,\alpha} \rho a \tilde{\rho} b \left( \frac{m_{\alpha}}{2\pi R_{a,\beta}} \right)^2 \left( \frac{a_0}{2\pi R_{a,\beta}} \right)^2 \exp \left( \frac{-2R_{a,\beta}}{\xi(\omega)} \right) \left( \frac{h_S}{\Delta} \right)^2 .$$

By comparison we have $G_{a,b} = 0$ if $h_S = 0$ because of a cancellation between the crossed Andreev reflection and elastic cotunneling conductances. We thus see that a crossed Andreev reflection experiment with a ferromagnetic and a normal metal electrode can give information about the existence of an induced exchange field in the superconductor. We see also from Eqs. and that there is no precession of the electron spin around the direction of the exchange field in the superconductor. The absence of spin precession in the superconducting case can be contrasted with the metallic case (see Appendix A).
IV. SEQUENTIAL TUNNELING OF COOPER PAIRS THROUGH A MAGNETIC DOMAIN WALL

Now we consider the junction on Fig. 4 in which a ferromagnetic wire is inserted in between two superconductors. In the absence of a domain wall in the ferromagnetic wire (see Fig. 4(a)) the junction is just a S/F/S junction. In the presence of a domain wall (see Fig. 4(b)) Cooper pair-like states arising from crossed Andreev reflection can be transmitted through the junction. As a consequence the conductance is larger in the presence of a magnetic domain wall. We consider two limiting cases:

(i) Transport is dominated by inelastic scattering in the ferromagnetic domains. Because of inelastic scattering the distribution functions in the ferromagnetic domains relax to the Fermi distribution. This case is discussed in sections IV B, IV C and IV D.

(ii) Transport through the domain wall is phase-coherent and there is a Josephson current circulating between the two ferromagnetic electrodes. This case is discussed in section V.

A. The different time scales

Similarly to Ref. we notice that three time scales are involved in out-of-equilibrium transport through a ferromagnetic domain wall:

(i) The transport dwell time $\tau_d$ being the time taken by an electron to travel through one of the magnetic domains.

(ii) The energy relaxation time $\tau_E$. Because of inelastic scattering the distribution function in the out-of-equilibrium conductor relaxes to the Fermi distribution. This relaxation takes place on a time scale $\tau_E$. 
(iii) The spin-flip time $\tau_{sf}$ being the time above which spin-flip scattering is relevant.

We suppose in this section that $\tau_E \ll \tau_d \ll \tau_{sf}$. The distribution function in the intermediate magnetic domains is thus well approximated by a Fermi distribution. The chemical potential of spin-up electrons is different from the chemical potential of spin-down electrons.

### B. Perturbative transport formula

In this section we discuss the perturbative transport formula of the S/DW/N junction on Fig. 5. The full transport formula to order $t^4$ is evaluated in Appendix B. The expression of $I(t)$ contains two kinds of terms: the terms $B1$ – $B3$ describe processes taking place locally at the interfaces between the superconductor and the ferromagnetic electrodes without propagation in the ferromagnetic electrodes. The terms $B4$ – $B7$ involve a propagation in the ferromagnetic electrodes. The two kinds of terms would contribute if electrodes $(a, a')$ and $(b, b')$ were single channel electrodes. We consider here that electrodes $(a, a')$ and $(b, b')$ are multichannel electrodes and we average the current over the microscopic phases. Once this averaging is done only the “local” terms survive in the transport formula given by

$$I_{t,\alpha} = -4\pi^2 t_{\alpha}^2 \rho_{\alpha}^2 f_{loc} [1 - P_{\alpha}^2] \left\{ \left[ n_F(\omega + \mu_{\alpha,\uparrow}) - n_F(\omega + \mu_{\alpha,\downarrow}) \right] \right\}$$ (38)

The term $B3$ corresponds to elastic cotunneling through the superconductor and the term $B4$ corresponds to crossed Andreev reflection.

A similar calculation can be carried out at interface $(a', \alpha')$. Once the average over the microscopic phase variables is carried out we find

$$I_{t',\alpha'} = -4\pi^2 t_{\alpha'}^2 \rho_{\alpha'}^2 [1 + P_{\alpha'}] \left\{ \left[ n_F(\omega - \mu_{\alpha',\uparrow}) - n_F(\omega - \mu_{\alpha',\downarrow}) \right] \right\}$$ (41)

where $B1$ and $B2$ describe electron tunneling from the electrode $(a, a')$ into the normal metal and $B3$ describes elastic cotunneling from electrode $(b, b')$ to electrode $(a, a')$.

### C. Sequential tunneling through the S/DW/N junction

In this section we discuss out-of-equilibrium transport in a S/DW/N junction on the basis of the two-channel model shown on Fig. 5. We suppose that a voltage $V = 0$ is applied on the superconductor and a voltage $V'$ is applied on the normal metal. The spin-up and spin-down chemical potentials in the two magnetic domains $(a, a')$ and $(b, b')$ are determined in such a way that current is conserved. In general there are four unknown chemical potentials ($\mu_{\alpha,\uparrow}$, $\mu_{\alpha,\downarrow}$, $\mu_{\beta,\uparrow}$ and $\mu_{\beta,\downarrow}$) that can be determined from four equations for current conservation. There exist two cases in which the $4 \times 4$ system of equations can be reduced to a $2 \times 2$ system of equations:

(i) Half-metal ferromagnets where there is only one spin population in each of the ferromagnetic electrodes $(a, a')$ and $(b, b')$. This case is treated in the main body of the article.

(ii) The symmetric case where the two electrodes $(a, a')$ and $(b, b')$ have identical density of states and where $t_{\alpha} = t_{\beta}$ and $t_{\alpha'} = t_{\beta'}$. This case is treated in Appendix C.

Let us consider half-metal ferromagnets: $P_a = 1$, $P_b = -1$. The transport formula is found to be

$$\frac{I_{tot}}{V'} = \frac{16\pi^2}{D} t_{\alpha}^2 t_{\alpha'}^2 t_{\beta}^2 t_{\beta'}^2 \rho_{\alpha,\uparrow} \rho_{\alpha,\downarrow} (\rho')^2 (f_{a,\alpha}^2) \left[ 1 - 2\pi^2 t_{\alpha}^2 \rho_{\alpha,\uparrow} \right] \left[ 1 - 2\pi^2 t_{\beta}^2 \rho_{\alpha,\downarrow} \right] \left[ 1 + P_a \right] \left[ 1 + P_b \right] \left[ n_F(\omega + \mu_{\alpha,\uparrow}) - n_F(\omega + \mu_{\alpha,\downarrow}) \right] \left[ n_F(\omega + \mu_{\beta,\uparrow}) - n_F(\omega + \mu_{\beta,\downarrow}) \right]$$

with

$$D = t_{\alpha}^2 t_{\alpha'}^2 t_{\beta}^2 t_{\beta'}^2 \rho_{\alpha,\uparrow} \rho_{\alpha,\downarrow} (f_{a,\alpha}^2) \left[ 1 - 2\pi^2 t_{\alpha}^2 \rho_{\alpha,\uparrow} \right] \left[ 1 + P_a \right] \left[ 1 - P_b \right] (f_{a,\alpha}^2) \left[ 1 + P_a \right] \left[ 1 + P_b \right] \left[ n_F(\omega + \mu_{\alpha,\uparrow}) - n_F(\omega + \mu_{\alpha,\downarrow}) \right] \left[ n_F(\omega + \mu_{\beta,\uparrow}) - n_F(\omega + \mu_{\beta,\downarrow}) \right]$$

(43)
We note \( \rho_N \) a typical value of the density of states either in the superconductor or in the ferromagnetic and normal metal electrodes. We first suppose that \( t_{\alpha,\beta}^2 \rho_N \) and \( t_{\alpha,\beta}^2 \rho_N \) are small compared to \( t_{\alpha'} \) and \( t_{\beta'} \). The transport formula takes the same form as in the case where the ferromagnetic electrodes \((a,a')\) and \((b,b')\) are in equilibrium:

\[
\frac{I_{\text{tot}}}{V'} = 16\pi^2 t_{\alpha,\beta}^2 \rho_a \gamma \rho_{b,\perp} \langle \langle f_{\alpha,\beta}^2 \rangle \rangle. \tag{46}
\]

In the other limiting case where \( t_{\alpha,\beta}^2 \rho_N \) and \( t_{\alpha,\beta}^2 \rho_N \) are large compared to \( t_{\alpha'} \) and \( t_{\beta'} \) we find

\[
\frac{I_{\text{tot}}}{V'} = 16\pi^2 \rho' t_{\alpha,\beta}^2 \rho_a \gamma \rho_{b,\perp} + t_{\alpha,\beta}^2 \rho_{b,\perp}. \tag{47}
\]

We note \( g_a = 16\pi^2 t_{\alpha,\beta}^2 \rho_a \gamma \rho' \) and \( g_b = 16\pi^2 t_{\alpha,\beta}^2 \rho_{b,\perp} \rho' \) the conductances associated to the interfaces \((a',a')\) and \((b',b')\). The total conductance is given by \( 1/G_{\text{tot}} = 1/g_a + 1/g_b \). The two interfaces are thus in series which is because transport is mediated by crossed Andreev reflection: a spin-up electron from the normal metal is transferred at site \( a' \), travels to site \( a \), is reflected as a spin-down hole at site \( b \). The spin-down hole travels to site \( b' \) and is transferred in the normal metal at site \( b' \). As a consequence of this transport process the two interfaces \((a',a')\) and \((b',b')\) are coupled in series.

**D. Sequential tunneling through the S/DW/S junction**

We consider the same model as in the preceding section but now the electrode on the right is superconducting (see Fig. 4). We show that the properties of the S/DW/S junction are similar to the properties of the S/DW/N junction. We suppose that a voltage \( V \) is applied on the left electrode and a voltage \( V' \) is applied on the right electrode. We consider a regime in which inelastic scattering in the ferromagnetic electrodes is strong enough so that the transport dwell time is much larger than the energy relaxation time (see section IV A). Moreover we suppose that inelastic scattering is strong enough so that there is no Josephson effect.

We consider that the ferromagnetic electrodes \((a,a')\) and \((b,b')\) are half-metal ferromagnets: \( P_a = 1, P_b = -1 \). The total current is given by

\[
\frac{I_{\text{tot}}}{V' - V} = 16\pi^2 \rho_a \gamma \rho_{b,\perp} t_{\alpha,\beta}^2 t_{\alpha',\beta'}^2 \langle \langle f_{\alpha,\beta}^2 \rangle \rangle \langle \langle f_{\alpha',\beta'}^2 \rangle \rangle. \tag{48}
\]

If we note \( g = 16\pi^2 t_{\alpha,\beta}^2 \rho_a \gamma \rho_{b,\perp} \langle \langle f_{\alpha,\beta}^2 \rangle \rangle \) and \( g' = 16\pi^2 t_{\alpha,\beta}^2 t_{\alpha',\beta'}^2 \langle \langle f_{\alpha',\beta'}^2 \rangle \rangle \) the conductances associated to crossed Andreev reflection at the contacts with each of the two superconductors we see that the total conductance is such that

\[
\frac{1}{G_{\text{tot}}} = \frac{1}{g} + \frac{1}{g'}. \tag{49}
\]

which is the expected result since electron pairs travel in series through the two superconductors.

**V. JOSEPHSON EFFECT IN THE S/DW/S JOSEPHSON JUNCTION**

**A. A two-channel model**

Now we consider the S/DW/S junction on Fig. 5 in which the right electrode is superconducting. We suppose that \( \tau_E \approx \tau_d \ll \tau_{sf} \) or \( \tau_E \ll \tau_d \ll \tau_{sf} \) so that the two superconductors can be coupled coherently through the two ferromagnetic channels. We look for the condition under which a Josephson current can circulate across the junction. We suppose in this section that the two electrodes \((a,a')\) and \((b,b')\) are half-metal ferromagnets with antiparallel spin orientations. The case of a partial spin polarization and non collinear spin orientations will be discussed in section \( IV B \).

The Nambu representation of the hopping matrix elements is given by

\[
\hat{t}_{\alpha,a} = \begin{bmatrix} t_{\alpha} e^{-i(\varphi - \chi)/4} & 0 \\ 0 & -t_{\alpha} e^{i(\varphi - \chi)/4} \end{bmatrix} \tag{50}
\]

\[
\hat{t}_{\beta,b} = \begin{bmatrix} t_{\beta} e^{-i(\varphi + \chi)/4} & 0 \\ 0 & -t_{\beta} e^{i(\varphi + \chi)/4} \end{bmatrix} \tag{51}
\]
a uniform \( \theta \) connected to the superconductors. The local magnetization makes an angle under usual experimental conditions since the size of the ferromagnetic region is usually much larger than the elastic mean free path. The opposite limit of small disorder is considered in Appendix D.

\[
\begin{align*}
\hat{t}_{\alpha',\alpha} & = \begin{bmatrix} t_{\alpha'} & 0 \\ 0 & -t_{\alpha'} \end{bmatrix} e^{-i(\varphi-\chi)/4} \\
\hat{t}_{\alpha',\beta} & = \begin{bmatrix} t_{\beta} & 0 \\ 0 & -t_{\beta} \end{bmatrix} e^{i(\varphi+\chi)/4}
\end{align*}
\]

(52)

(53)

where \( \varphi \) is the difference between the superconducting phases in the right and left electrode and \( \chi \) is the magnetic flux through the loop. We have the relations \( \hat{t}_{\alpha,\alpha} = (\hat{t}_{\alpha,\alpha})^* \), \( \hat{t}_{\alpha,\beta} = (\hat{t}_{\beta,\alpha})^* \), \( \hat{t}_{\alpha',\alpha'} = (\hat{t}_{\alpha',\alpha'})^* \) and \( \hat{t}_{\beta',\beta'} = (\hat{t}_{\beta',\beta'})^* \). The equilibrium current flowing from site \( \alpha \) to site \( a \) is given by

\[
I_{\alpha,a} = \frac{e}{h} \int d\omega f(\omega) \text{Tr} \left\{ \hat{\sigma} \left[ \hat{t}_{\alpha,a} \left( \hat{g}_{a,a}^{A} - \hat{g}_{a,a}^{R} \right) - \hat{t}_{\alpha,a} \left( \hat{g}_{a,a}^{A} - \hat{g}_{a,a}^{R} \right) \right] \right\}. 
\]

(54)

The Green’s functions are \( 2 \times 2 \) matrices since we do not discuss non-collinear magnetizations for the moment.

We deduce from (52) that to order \( t^4 \) the spin-up current through electrode \( (a,a') \) is given by

\[
I_{\alpha,a}^{(4)} = -2t e \int_{0}^{t} d\omega f_{\alpha,\beta}(\omega) f_{\alpha',\beta}(\omega) \left\{ \frac{g_{a,a'}^{A} g_{b,b'}^{R} - g_{a,a'}^{R} g_{b,b'}^{R}}{\text{Det}[I - K^{A}]} - \frac{g_{a,a'}^{A} g_{b,b'}^{R}}{\text{Det}[I - K^{R}]} \right\},
\]

(55)

where \( K^{A,R} \) is the \( 4 \times 4 \) matrix involved in the Dyson equation \( [I - K^{A,R} \hat{G}_{A,R}] = \hat{g}_{A,R} \):

\[
\begin{bmatrix}
1 - K_{1,1}^{A} & -K_{1,2}^{A} & -K_{2,1}^{A} & -K_{2,2}^{A} \\
-K_{1,1}^{R} & 1 - K_{1,2}^{R} & -K_{2,1}^{R} & -K_{2,2}^{R} \\
-K_{1,1}^{A} & -K_{1,2}^{A} & 1 - K_{2,1}^{A} & -K_{2,2}^{A} \\
-K_{1,1}^{R} & -K_{1,2}^{R} & -K_{2,1}^{R} & 1 - K_{2,2}^{R}
\end{bmatrix}
= \begin{bmatrix}
G_{1,1}^{A,a} & 0 & 0 & G_{1,2}^{A,a} \\
0 & G_{1,1}^{A,a} & G_{1,2}^{A,a} & 0 \\
G_{1,1}^{A,a} & 0 & G_{1,2}^{A,a} & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\]

(56)

where we used the notation \( K_{1,1}^{A,a} = \bar{t}_{1,1}^{A,a} g_{1,1}^{A,a} \), \( K_{1,2}^{A,a} = \bar{t}_{1,2}^{A,a} g_{1,2}^{A,a} \), \( K_{2,1}^{A,a} = \bar{t}_{2,1}^{A,a} g_{2,1}^{A,a} \), \( K_{2,2}^{A,a} = \bar{t}_{2,2}^{A,a} g_{2,2}^{A,a} \), \( G_{1,1}^{A,a} = \bar{t}_{1,1}^{A,a} g_{1,1}^{A,a} \), etc. The role of disorder can be included in a straightforward fashion. Since the spin-up and spin-down electrons of the Cooper pair propagate in different electrodes we should replace \( g_{A,a}^{1,1} \) and \( g_{B,b'}^{A,2,2} \) by their average over disorder which decay exponentially with distance over a length scale equal to the elastic mean free path. We conclude that a Josephson current cannot be observed under usual experimental conditions since the size of the ferromagnetic region is usually much larger than the elastic mean free path. The opposite limit of small disorder is considered in Appendix D.

B. Multichannel effects

1. Transport formula

We consider in this section the multichannel S/DW/S junction on Fig. 6 in which the ferromagnetic metal is multiconnected to the superconductors. The local magnetization makes an angle \( \theta(z) \) with the \( z \)-axis. The situation with a uniform \( \theta \) corresponds to the multichannel \( \pi \)-junction.
The supercurrent is given by
\[ I_S = \frac{1}{2} e \int_0^{+\infty} d\omega \sum_n \text{Tr} \left\{ \hat{\sigma}^z \left[ \hat{G}^A_{\alpha_n,\alpha_n} - \hat{G}^R_{\alpha_n,\alpha_n} \right] - \hat{G}^A_{\alpha_n,\alpha_n} \right\} , \]  
where the Green's functions are 4 × 4 trajectories parallel to the interface that we can take into account in our calculation. The supercurrent is given by
\[ I_S = -2i \frac{e}{\hbar} \int_0^{+\infty} d\omega \sum_{k.l.m,n} f_{\alpha_m,\alpha_n} f_{\beta_k,\beta_l} \]  
\[ \times \left\{ \left[ \begin{array}{cc} g^{2,1,1}_{\alpha_n,\alpha_m,\beta_k,\beta_l} - g^{2,1,1}_{\alpha_n,\alpha_m,\beta_k,\beta_l} \frac{\text{Det}[I - K^A]}{\text{Det}[I - K^R]} \right] + \left[ \begin{array}{cc} g^{4,3,1,1}_{\alpha_n,\alpha_m,\beta_k,\beta_l} - g^{4,3,1,1}_{\alpha_n,\alpha_m,\beta_k,\beta_l} \frac{\text{Det}[I - K^A]}{\text{Det}[I - K^R]} \right] \right\} , \]  
where \( g^{i,j}_{\alpha_k,\beta_m} \) is \( i \)-th Nambu component of the propagator connecting the two ends of the ferromagnetic metal at sites \( a_k \) and \( m \). The first two terms in the r.h.s. of Eq. (68) correspond to a propagation without spin-flip in the ferromagnetic region whereas the last term correspond to a propagation with spin flip. We deduce from Eq. (68) the same conclusions as in the two-channel model. Namely an average Josephson current can circulate only if the size of the ferromagnetic region is smaller than the elastic mean free path in the ferromagnetic metal, a condition that is not usually realized in experiments.

2. Limit of small disorder

To obtain the supercurrent in the ballistic limit we replace the propagators by their expressions and sum over all channels. The propagators of a ferromagnetic metal with a rotating magnetization are not known in general. This is why we discuss here only the situation where the width of the domain wall is vanishingly small and the ferromagnets are half-metal ferromagnets. In this case there is no spin precession in the ferromagnetic region but there exist trajectories parallel to the interface to take into account in our calculation. The supercurrent is given by
\[ I_S = 8\pi \frac{e}{\hbar} L_y t^2 \sin \varphi \left( \frac{m a_0^2}{\hbar^2} \right)^4 G(k_F) , \]  
with
\[ G(k_F) = \frac{1}{L_y} \int_{-\infty}^{+\infty} \frac{dz_m}{a_0} \int_{-\infty}^{+\infty} \frac{dy_n}{a_0} \int_{-\infty}^{+\infty} \frac{dz_l}{a_0} \int_{-\infty}^{+\infty} \frac{dy_m}{a_0} \int_{-\infty}^{+\infty} \frac{dz_k}{a_0} \int_{-\infty}^{+\infty} \frac{dy_k}{a_0} \]  
\[ \frac{2\pi R_{\alpha_m,\alpha_n}}{2\pi R_{\beta_k,\beta_l} \sin [k_F R_{\alpha_m,\alpha_n}] \sin [k_F R_{\beta_k,\beta_l}] \cos [k_F (R_{\alpha_m,\alpha_n} - R_{\beta_k,\beta_l})]} , \]  
and with
\[ R_{\alpha_m,\alpha_n} = \sqrt{(z_m - z_n)^2 + (y_m - y_n)^2} \]  
\[ R_{\beta_k,\beta_l} = \sqrt{(z_k - z_l)^2 + (y_k - y_l)^2} \]  
\[ R_{\alpha_m,\beta_k} = \sqrt{R^2 + (z_m - z_k)^2 + (y_m - y_k)^2} \]  
\[ R_{\alpha_n,\beta_l} = \sqrt{R^2 + (z_n - z_l)^2 + (y_n - y_l)^2} , \]  
where \( R \) is the longitudinal dimension of the junction (see Fig. 6). We have shown on Fig. 7 the variation of \( G(k_F) \) as a function of \( k_F \). We see that strong finite size effects are present but still we can make a comparison between (i) a calculation in which all trajectories are taken into account and (ii) a calculation in which only the trajectories perpendicular to the interface are taken into account. We see that for small values of \( k_F \) (typically \( k_F \) smaller than \( 1/a_0 \), where \( a_0 \) is the lattice parameter) the summation (i) is larger than (ii) whereas the opposite is true for larger values of \( k_F \). This shows that trajectories parallel to the interface play a relevant role in the determination of the supercurrent.

VI. CONCLUSIONS

To conclude we have presented a detailed investigation of several mechanisms involved in transport across several junctions involving ferromagnetic domain walls (S/DW, S/DW/N and S/DW/S junctions) The role of non collinear
magnetization was studied for the S/DW junctions. Using a spin $\otimes$ Nambu $\otimes$ Keldysh formalism we have derived the form of lowest order transport formula valid for an arbitrary profile of magnetization. We find that the conductance is a scaling function of $\xi_0/D$, where $\xi_0$ is the zero-energy BCS correlation length and $D$ is the width of the domain wall. Because of the proximity effect an exchange field can be induced in the superconductor. Neglecting the spatial variation of the exchange field, we have derived the transport formula and shown that there was no spin precession around the axis of the exchange field. We discussed the transport formula of the S/DW/N junction. We have shown that to lowest order only the processes taking place locally at each interface played a role. These processes are: elastic cotunneling through the superconductor, crossed Andreev reflection, electron tunneling from the ferromagnet to the normal metal and elastic cotunneling through the normal metal. We described the transport of Cooper pairs across the S/DW/N and S/DW/S junctions in a regime where transport is dominated by inelastic scattering but spin is conserved. With these assumptions the local distribution function within the domain wall is a Fermi distribution with a different spin-up and spin-down chemical potential. This model provides a detailed description of the sequential tunneling of Cooper pairs across the S/DW/N and S/DW/S junctions. We described the Josephson effect in a S/DW/S junction. Diffusion is usually strong in a ferromagnet and disorder is thus expected to play a relevant role. In particular the Josephson current decays exponentially with the longitudinal dimension of the junction. The characteristic length is equal to the mean free path in the ferromagnetic metal. This means that a Josephson current cannot be observed in usual conditions. Nevertheless there can exist a finite current due to crossed Andreev reflection associated to elastic cotunneling in the ferromagnetic region.

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APPENDIX A: SPIN PRECESSION IN THE METALLIC CASE

In this appendix we consider the junction on Fig. 8 in which two ferromagnetic electrodes with non collinear magnetizations are connected to a normal metal\cite{52,53}. Our goal is to provide a comparison with the superconducting case presented in section III C. We suppose that a magnetic field $h$ is applied on the normal metal and that the only effect of the magnetic field is to generate Zeeman splitting. The crossed conductance $G_{a,b} = \partial I_a/\partial V_b$ associated to elastic cotunneling takes the form

$$G_{a,b} = 8\pi^2t_{\alpha\beta}^2\tilde{\rho}_a\tilde{\rho}_b \left(\frac{ma^2}{\hbar^2}\right)^2 \left(\frac{a_0}{2\pi R_{\alpha\beta}}\right)^2 \left\{1 + P_aP_b \cos \theta_a \cos \theta_b (A1) + P_aP_b \sin \theta_a \sin \theta_b \cos \left\{[k_{F,\uparrow} - k_{F,\downarrow}] R_{\alpha\beta} + eV_b \left[\frac{1}{v_{F,\uparrow}} - \frac{1}{v_{F,\downarrow}}\right] R_{\alpha\beta}\right\}\right\}. $$

Spin precession can have two origins: (i) the term $[k_{F,\uparrow} - k_{F,\downarrow}] R_{\alpha\beta}$ describes oscillations of the conductance due to a mismatch in the Fermi wave-vectors; (ii) the term $\left[\frac{1}{v_{F,\uparrow}} - \frac{1}{v_{F,\downarrow}}\right] R_{\alpha\beta}$ describes oscillations in the conductance due
to a mismatch in the Fermi velocities.

APPENDIX B: PERTURBATIVE TRANSPORT FORMULA OF THE TWO CHANNEL S/DW/N JUNCTION

In this appendix we provide a derivation of the transport formula of the S/DW/N model represented on Fig. 5.

1. Transport at interface \((a, a)\)

The current through each link of the network on Fig. 5 is given by the transport formula (4). The spin-up current through the link \(a - a\) is found to be

\[
I_{a,a}^{(\uparrow)} = -4\pi^2 t_{a,a}^2 \rho_{a,a}^R \left[ 1 - P_{a}^2 \right] \left[ n_F(\omega - \mu_{a,\uparrow}) - n_F(\omega + \mu_{a,\downarrow}) \right]
\]

(\text{B1})

\[
-4\pi^2 t_{a,b}^2 \tilde{\rho}_{a,b} \tilde{g}_{a,b}^2 \left[ 1 + P_{a} \right] \left[ n_F(\omega - \mu_{a,\uparrow}) - n_F(\omega - \mu_{b,\uparrow}) \right]
\]

(\text{B2})

\[
-4\pi^2 t_{a,b}^2 \tilde{\rho}_{a,b} \tilde{g}_{a,b}^2 \left[ 1 - P_{a} \right] \left[ n_F(\omega - \mu_{a,\uparrow}) - n_F(\omega + \mu_{b,\downarrow}) \right]
\]

(\text{B3})

\[
-4\pi^2 t_{a,a}^2 \tilde{\rho}_{a,a} \tilde{g}_{a,a}^2 \left[ 1 - P_{a} \right] \left[ n_F(\omega - \mu_{a,\uparrow}) - n_F(\omega + \mu_{a,\downarrow}) \right]
\]

(\text{B4})

where \(\mu_{a,\uparrow}\) and \(\mu_{a,\downarrow}\) are the spin-up and spin-down chemical potentials in electrode \((a, a)\), \(\mu_{b,\uparrow}\) and \(\mu_{b,\downarrow}\) are the spin-up and spin-down chemical potentials in electrode \((b, b)\), and \(\mu'\) is the chemical potential in the normal metal. After phase averaging we obtain three contributions to the transport formula: local Andreev reflection given by (B8), elastic cotunneling through the superconductor given by (B9) and crossed Andreev reflection given by (B10).

2. Transport at interface \((a', a')\)

The same calculation can be carried out at interface \((a', a')\). The transport formula is found to be

\[
I_{a',a'}^{(\uparrow)} = 4\pi^2 t_{a',a'}^2 \tilde{\rho}_{a',a'}^A \left[ 1 + P_{a} \right] \left[ n_F(\omega - \mu_{a,\uparrow}) \right]
\]

(\text{B8})

\[
-8\pi^4 t_{a',a'}^2 \tilde{\rho}_{a',a'}^A \left[ 1 + P_{a} \right] \left[ n_F(\omega - \mu') \right]
\]

\[
-4\pi^2 t_{a',a'}^2 \tilde{\rho}_{a',a'}^A \left[ 1 + P_{a} \right] \left[ n_F(\omega - \mu_{a,\uparrow}) - n_F(\omega - \mu') \right]
\]

\[
+8\pi^4 t_{a',a'}^2 \tilde{\rho}_{a',a'}^A \left[ 1 + P_{a} \right] \left[ n_F(\omega - \mu_{a,\uparrow}) - n_F(\omega - \mu') \right]
\]

(\text{B8})

-8\pi^2 t_{a',a'}^2 \tilde{\rho}_{a',a'}^A \left[ 1 + P_{a} \right] \left[ n_F(\omega - \mu_{a,\uparrow}) \right]

(\text{B9})
identical in the two channels: We suppose that the two channels have an identical density of states: \( \tilde{\rho} \) confirm the results obtained in the main body of the article for the asymmetric junction with half-metal ferromagnets.

After averaging over the phase variables we obtain the transport formula given by Eqs. (11) – (13) that contains only two processes: tunneling from site \( a' \) to site \( a' \) and elastic cotunneling from site \( b' \) to site \( a' \).

**APPENDIX C: TRANSPORT FORMULA OF THE SYMMETRIC TWO-CHANNEL S/DW/N AND S/DW/S JUNCTIONS**

In this appendix we consider S/DW/N and S/DW/S junctions with two symmetric channels. With this model we confirm the results obtained in the main body of the article for the asymmetric junction with half-metal ferromagnets.

We suppose that the two channels have an identical density of states: \( \tilde{\rho} = \tilde{\rho}_b \), that the tunnel matrix elements are identical in the two channels: \( t = t_a = t_b, t' = t_a' = t_b' \), and that the two channels have an opposite spin polarization: \( P_a = P \) and \( P_b = -P \). Then there exists a simple symmetry relation between the chemical potentials in the two ferromagnetic electrodes: \( \mu_{a,\uparrow} = \mu_{b,\downarrow} \) and \( \mu_{a,\downarrow} = \mu_{b,\uparrow} \).

1. The S/DW/N junction

In the limiting case \( t \ll t' \) we have \( \mu_{a,\uparrow} \simeq V' \). The transport formula is identical to the case where electrodes \( (a, a') \) and \( (b, b') \) are in equilibrium:

\[
\frac{I_{\text{tot}}}{V'} = 32\pi^2 t^4 \tilde{\rho}^2 (1 - P^2) f_{\text{loc}}^2 + 32\pi^2 t^4 \tilde{\rho}^2 (1 + P^2) \langle \langle f_{a,\beta}^2 \rangle \rangle. \tag{C1}
\]

In the case of half-metal ferromagnets \( (P = 1) \) only the term corresponding to crossed Andreev reflection is non-zero and Eq. (C1) is equivalent to Eq. (46).

In the limiting case \( \rho_N t' \ll \rho_N t \ll 1 \) and \( t' \ll \rho_N t^2 \) the current is the sum of a contribution due to local Andreev reflection and a contribution due to crossed Andreev reflection: \( I_{\text{tot}} = I_{\text{AR}} + I_{\text{CAR}} \), with

\[
\frac{I_{\text{AR}}}{V'} = \frac{32\pi^2 (t')^2 \tilde{\rho}^2 (1 - P^2) f_{\text{loc}}^2 \left( \langle \langle f_{a,\beta}^2 \rangle \rangle + \langle \langle g_{a,\beta}^2 \rangle \rangle \right)}{(1 - P^2) \left[ f_{\text{loc}}^2 \langle \langle g_{a,\beta}^2 \rangle \rangle + (1 + P^2) \langle \langle f_{a,\beta}^2 \rangle \rangle \right]} \tag{C2}
\]

\[
\frac{I_{\text{CAR}}}{V'} = \frac{16\pi^2 (t')^2 \tilde{\rho}^2 \left( \langle \langle f_{a,\beta}^2 \rangle \rangle + \langle \langle g_{a,\beta}^2 \rangle \rangle \right) + 2P^2 \left( f_{\text{loc}}^2 - \langle \langle f_{a,\beta}^2 \rangle \rangle \right)}{(1 - P^2) \left[ f_{\text{loc}}^2 + (1 + P^2) \langle \langle f_{a,\beta}^2 \rangle \rangle \right]} \tag{C3}
\]

In the case of half-metal ferromagnets (2) and (3) are equivalent to (17).

2. The S/DW/S junction

In the case of the S/DW/S junction the total current is the sum of the local Andreev reflection and crossed Andreev reflection terms:

\[
\frac{I_{\text{AR}}}{V'} = \frac{128\pi^2 t^4 (t')^4 \tilde{\rho}^2 f_{\text{loc}}^2 (1 - P^2) \left\{ \left[ t^4 \langle \langle g_{a,\beta}^2 \rangle \rangle + (t')^4 \langle \langle g_{a',\beta'}^2 \rangle \rangle \right] \left[ (1 - P^2) f_{\text{loc}}^2 + (1 + P^2) \langle \langle f_{a',\beta'}^2 \rangle \rangle \right] \right\}}{\left\{ \left[ t^4 \langle \langle f_{a,\beta}^2 \rangle \rangle + (t')^4 \langle \langle f_{a',\beta'}^2 \rangle \rangle \right] \left[ (1 + P^2) f_{\text{loc}}^2 + (1 - P^2) \langle \langle f_{a',\beta'}^2 \rangle \rangle \right] \right\}} \tag{C4}
\]
\[
\frac{I_{\text{CAR}}}{V' - V} = \frac{128\pi^2}{\mathcal{D}'} t^4 (t')^4 \bar{\rho}^4 \langle \langle f_{a,\beta}^2 \rangle \rangle (1 + P^2) \left\{ t^4 \langle \langle g_{a,\beta}^2 \rangle \rangle + (t')^4 \langle \langle g_{a',\beta'}^2 \rangle \rangle \right\} \left[ (1 - P^2) f_{\text{loc}}^2 + (1 + P^2) \langle \langle f_{a,\beta}^2 \rangle \rangle \right] \] (C5)
+ \left[ t^4 \langle \langle f_{a,\beta}^2 \rangle \rangle + (t')^4 \langle \langle f_{a',\beta'}^2 \rangle \rangle \right] \left[ (1 + P^2) f_{\text{loc}}^2 + (1 - P^2) \langle \langle f_{a',\beta'}^2 \rangle \rangle \right],
\]

with
\[
\mathcal{D}' = 4\bar{\rho}^2 \left\{ (1 - P^2) \left[ f_{\text{loc}}^2 \langle \langle g_{a,\beta}^2 \rangle \rangle + \langle \langle f_{a,\beta}^2 \rangle \rangle \right] + (1 + P^2) \left[ f_{\text{loc}}^2 \langle \langle g_{a',\beta'}^2 \rangle \rangle + \langle \langle f_{a',\beta'}^2 \rangle \rangle \right] \right\}
+ 4(t')^8 \bar{\rho}^2 \left\{ (1 - P^2) \left[ f_{\text{loc}}^2 \langle \langle g_{a',\beta'}^2 \rangle \rangle + \langle \langle f_{a',\beta'}^2 \rangle \rangle \right] + (1 + P^2) \left[ f_{\text{loc}}^2 \langle \langle g_{a',\beta'}^2 \rangle \rangle + \langle \langle f_{a',\beta'}^2 \rangle \rangle \right] \right\}
+ 4t^4(\bar{t}')^4 \bar{\rho}^2 \left\{ (1 - P^2) \left[ f_{\text{loc}}^2 \langle \langle g_{a',\beta'}^2 \rangle \rangle + \langle \langle f_{a',\beta'}^2 \rangle \rangle \right] + (1 + P^2) \left[ f_{\text{loc}}^2 \langle \langle g_{a',\beta'}^2 \rangle \rangle + \langle \langle f_{a',\beta'}^2 \rangle \rangle \right] \right\}
+ \left[ (1 + P^2) \left[ f_{\text{loc}}^2 \langle \langle f_{a',\beta'}^2 \rangle \rangle + f_{\text{loc}}^2 \langle \langle f_{a',\beta'}^2 \rangle \rangle + \langle \langle f_{a',\beta'}^2 \rangle \rangle \right] \right].
\]

If the contacts with the two superconductors are identical we have \( t = t' \), \( f_{\alpha,\beta} = f_{\alpha',\beta'} \) and \( g_{\alpha,\beta} = g_{\alpha',\beta'} \) from what we deduce
\[
\frac{I_{\text{AR}}}{V' - V} = 4\pi^2 t^4 \bar{\rho}^2 f_{\text{loc}}^2 (1 - P^2)
\]
\[
\frac{I_{\text{CAR}}}{V' - V} = 4\pi^2 t^4 \bar{\rho}^2 \langle \langle f_{\alpha,\beta}^2 \rangle \rangle (1 + P^2),
\]
where we used the notation \( \bar{\rho} = \rho / 2 \) for the spin-up or spin-down density of state in the ferromagnetic electrodes. Eqs. (C7) and (C8) in the limit \( P = 1 \) are in agreement with Eq. (15) in the limit of a symmetric contact. In the symmetric case the conductance is thus equal to the conductance associated to a single superconductor divided by two, in agreement with Eq. (19).

**APPENDIX D: JOSEPHSON EFFECT IN A TWO-CHANNEL BALLISTIC S/DW/S JUNCTION**

In this appendix we describe the Josephson effect with a ballistic propagation in the ferromagnetic electrodes. In the limit of a long junction \( R \gg a_0 \) the matrix \( J - K A^{1,R} \) given by (36) is block-diagonal because the Andreev bound states do not couple the two superconductors. There exist two bound states associated to the interfaces \( (\alpha, a) \) and \( (\beta, b) \) and two bound states associated to the interfaces \( (\alpha', a') \) and \( (\beta', b') \). The secular equation for the bound states living at the interfaces \( (\alpha, a) \) and \( (\beta, b) \) takes the form
\[
1 + \sqrt{\pi^2 \rho_F \rho_N} (t'^2 + t^2) \frac{\omega_0}{\sqrt{\Delta^2 - \omega_0^2}} + (\pi^2 \rho_F \rho_N)^2 t'^2 t_\beta t_\alpha \frac{z^2 \Delta^2 - \omega_0^2}{\Delta^2 - \omega_0^2} = 0,
\]
where \( \rho_N \) and \( \rho_F \) are the density of states in the superconductor and in the half-ferromagnetic electrodes, and where we used the notation \( z = \sin [k_F R] / (k_F R) \). In the case \( t = t_\alpha = t_\beta \) and in the tunnel limit \( \pi t^2 \rho_N \rho_F \ll 1 \) the solution of (D1) takes the form
\[
\omega_0^2 = \Delta^2 \left[ 1 + (\pi^2 \rho_N \rho_F)^2 (1 + z)^2 \right].
\]

The supercurrent is easily deduced from (35):
\[
I_S = \frac{16 \pi^9}{(a_0 k_F)^2} \frac{e}{\hbar} \Delta t^4 \rho_F \rho_N^2 (1 + P^2)^2 \frac{a_0}{2 \pi R_{a,\beta}} \frac{a_0}{2 \pi R_{a',\beta'}} \frac{a_0}{2 \pi R_{a,a'}} \frac{a_0}{2 \pi R_{b,b'}} \sin [k_F R_{a,\beta}] \sin [k_F R_{a',\beta'}] \]
\[
\exp \left\{ - \left( \frac{R_{a,a'} + R_{b,b'}}{l_\phi} \right) \right\} f(z, z') \cos \alpha \sin \phi,
\]
where \( \alpha \) is defined by
\[
\alpha = k_F^2 (R_{a,a'} - R_{b,b'}) + \frac{\Delta}{v_F} (R_{a,a'} + R_{b,b'}),
\]
and where \( f(z, z') \) is a geometrical prefactor of order unity.

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