AN INHOMOGENEOUS EVOLUTION EQUATION INVOLVING THE NORMALIZED INFINITY LAPLACIAN WITH A TRANSPORT TERM

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Abstract. In this paper, we prove the uniqueness and stability of viscosity solutions of the following initial-boundary problem related to the random game named tug-of-war with a transport term

\[
\begin{align*}
    u_t - \Delta_N^\infty u - \langle \xi, Du \rangle &= f(x,t), & \text{in } Q_T, \\
    u &= g, & \text{on } \partial \Omega \cap Q_T,
\end{align*}
\]

where \( \Delta_N^\infty u = \frac{1}{|Du|^2} \sum_{i,j=1}^n u_{x_i} u_{x_j} u_{x_i} u_{x_j} \) denotes the normalized infinity Laplacian, \( \xi : Q_T \to \mathbb{R}^n \) is a continuous vector field, \( f \) and \( g \) are continuous. When \( \xi \) is a fixed field and the inhomogeneous term \( f \) is constant, the existence is obtained by the approximate procedure. When \( f \) and \( \xi \) are Lipschitz continuous, we also establish the Lipschitz continuity of the viscosity solutions. Furthermore, we establish the comparison principle of the generalized equation with the first order term with initial-boundary condition

\[
\begin{align*}
    u_t(x,t) - \Delta_N^\infty u(x,t) - H(x,t, Du(x,t)) &= f(x,t), & \text{in } Q_T,
\end{align*}
\]

where \( H(x,t,p) : Q_T \times \mathbb{R}^n \to \mathbb{R} \) is continuous, \( H(x,t,0) = 0 \) and grows at most linearly at infinity with respect to the variable \( p \). And the existence result is also obtained when \( H(x,t,p) = H(p) \) and \( f \) is constant for the generalized equation.

1. Introduction. In this paper, we are interested in the quasilinear degenerate parabolic normalized infinity Laplacian equation with a transport term,

\[
u_t - \Delta_N^\infty u - \langle \xi, Du \rangle = f(x,t), \quad \text{in } Q_T, \tag{1} \]

where \( Q_T = \Omega \times (0,T), \Omega \subset \mathbb{R}^n \) is a bounded domain, \( \xi : Q_T \to \mathbb{R}^n \) is a vector-valued function, \( f \) is a function in \( Q_T \) and the normalized infinity Laplacian

\[
\Delta_N^\infty u = \frac{1}{|Du|^2} \langle D^2u Du, Du \rangle.
\]

The infinity Laplacian was first studied in relation with the absolutely minimizing Lipschitz extension problem, see [6] for a survey on this subject. The infinity harmonic functions are equivalent to the property of the comparison with cone functions, see the survey [9] and the references therein. Due to the high degeneracy and

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singularity of the operator, the regularity of solutions is still an open problem as far as we know, see [16, 17] etc. The normalized infinity Laplacian is closely related to the game theory named tug-of-war [35, 36]. It has many applications in image processing [1, 8] and optimal mass transportation problems [14, 18].

The study of the singular inhomogeneous normalized infinity Laplace equation with a transport term (1) in this work was inspired by recent works, [23, 34] and [2, 3, 4, 10, 20, 24, 38, 39], on the normalized counterpart with a transport term by the game theory, and on the parabolic equations associated with infinity Laplacian by the partial differential equation theory.

In [23] the elliptic normalized infinity Laplacian with a transport term

\[
\begin{aligned}
-\Delta^\infty_N u - \langle \xi, Du \rangle &= 0, \quad \text{in } \Omega, \\
u &= g, \quad \text{on } \partial \Omega 
\end{aligned}
\] (2)

was first studied and the existence of viscosity solutions as the continuous value of the modified tug-of-war game is obtained by probabilistic approach when \( \xi \) is a Lipschitz vector field. While for a continuous gradient vector field \( \xi \), the existence and uniqueness of viscosity solutions were obtained by the \( p \)-Laplace approximation. Let us briefly recall from [23] the two-player, random-turn, tug-of-war game with a transport term. Set \( G \) is the final payoff function defined in a narrow strip around the boundary \( \partial \Omega \). The tug-of-war game with a transport term is played with two stages. First we toss an unfair coin, which has head probability \( 0 < C(\varepsilon) < 1 \), and tail probability \( 1 - C(\varepsilon) \). If we have obtained a head, we then toss a new (fair) coin and the winner moves the token to any new position \( x_1 \in B_\varepsilon(x_0) \). But if in the first (unfair) coin toss we obtain a tail, the token is moved to \( x_0 + \xi(x_0) \varepsilon \), where \( \xi(x) : \Omega \rightarrow \mathbb{R}^n \) is the vector field that appears (that is assumed to be Lipschitz).

Note that there is no strategies of the players involved if we get a tail in the first coin toss. The game continues until the first time the token arrives to \( x_\tau \in \mathbb{R}^n \setminus \Omega \) and then Player I earns \( G(x_\tau) \), and thus Player II earns \( -G(x_\tau) \), where \( G \) is the extension of \( g \) from \( \partial \Omega \) to a small strip \( \Gamma_\varepsilon = \{ x \in \mathbb{R}^n \setminus \Omega : \text{dist}(x, \partial \Omega) < \varepsilon \| \xi \|_\infty \} \) and gives the final payoff of the game.

Since the normalized infinity Laplacian was first introduced from the point of game theory named tug-of-war in [35], some kinds of modified tug-of-war have received a lot of attention. A biased tug-of-war was introduced in [37] and some results were established via a comparison property with exponential cones, see also [5, 27]. A tug-of-war with noise was studied in [36] by the game theory, [12, 22, 31, 32] by the methods of the partial differential equations, and [12, 13] in image processing.

In this paper we are devoted to prove the wellposedness and Lipschitz regularity of the viscosity solutions to the following problem

\[
\begin{aligned}
u_t - \Delta^\infty_N u - \langle \xi, Du \rangle &= f(x, t), \quad \text{in } Q_T, \\
u &= g, \quad \text{on } \partial_p Q_T.
\end{aligned}
\] (3)

And in fact we establish the uniqueness for a slightly more general equation with a first order term. Therefore, we consider

\[
u_t(x, t) - \Delta^\infty_N u(x, t) - H(x, t, Du(x, t)) = f(x, t),
\] (4)

where \( H : Q_T \times \mathbb{R}^n \rightarrow \mathbb{R} \) is a continuous function, \( H(x, t, 0) = 0 \) for any point \( (x, t) \in Q_T \) and grows at most linearly at infinity with respect to the variable \( p \), that is, there exists \( M > 0 \) such that \( |H(x, t, p)| \leq M|p| \). Now let us see two special cases for \( H(x, t, p) \).
Suppose Theorem 1.2.

(i) \( H(x, t, p) = \langle \xi, p \rangle \), (4) reduces to the infinity Laplacian with a transport term (1).

(ii) \( H(x, t, p) = \beta|p|, \Delta^N_{\infty}u + \beta|Du| \) is the \( \beta \)-biased infinity Laplacian related to the \( \beta \)-biased tug-of-war [37], where \( \beta \) is a fixed constant.

Notice that (1) and its generalized version (4) not only are degenerate parabolic, singular and not in divergence form but also have wide applications to image processing and mass transportation etc. They constitute a class of equations with particular properties. And our current work also helps to build a further connection between the partial differential equation theory and the random differential game theory about the infinity Laplacian.

Now our first result is

**Theorem 1.1.** Let \( H \) be as above, \( f \) be continuous in \( \overline{Q}_T \), and let \( g \in C(R^{n+1}) \). Suppose also that \( f \geq 0 \) or \( f \leq 0 \) in \( Q_T \). Then there exists at most one viscosity solution \( u \in C(\overline{Q}_T) \) to the following initial-boundary problem

\[
\begin{cases}
    u_t - \Delta^N_{\infty}u - H(x, t, Du) = f(x, t), & \text{in } Q_T, \\
    u = g, & \text{on } \partial_p Q_T.
\end{cases}
\]

Based on the comparison principle and the 1-homogeneity of the parabolic operator \( Pu = u_t - \Delta^N_{\infty}u - \langle \xi, Du \rangle \), we can obtain the following stability result.

**Theorem 1.2.** Suppose \( \xi \in C(Q_T), f \in C(Q_T) \) such that either \( f \geq 0 \) or \( f \leq 0 \) in \( Q_T \). Suppose \( c_j \to 0 \), \( g_j, g \in C(\partial_p Q_T) \) such that \( \|g_j - g\|_{L^\infty(\partial_p Q_T)} \to 0 \), and \( u_j, u \in C(Q_T) \) is the viscosity solution of the following problems

\[
\begin{cases}
    (u_j)_t - \Delta^N_{\infty}u_j - \langle \xi, Du_j \rangle = (1 + c_j)f(x, t), & \text{in } Q_T, \\
    u_j = g_j, & \text{on } \partial_p Q_T,
\end{cases}
\]

and

\[
\begin{cases}
    u_t - \Delta^N_{\infty}u - \langle \xi, Du \rangle = f(x, t), & \text{in } Q_T, \\
    u = g, & \text{on } \partial_p Q_T,
\end{cases}
\]

respectively. Then

\[ u_j \to u, \quad \text{uniformly in } Q_T, \quad \text{as } j \to \infty. \]

When \( H(x, t, p) = H(p) \) and the inhomogeneous term is constant, the equation is translation invariant. Therefore we can also establish the following existence theorem by the approximate procedure.

**Theorem 1.3.** Let \( H \) be as above and \( H(x, t, p) = H(p) \), \( f \) be constant, and let \( g \in C(R^{n+1}) \). Then there exists a viscosity solution \( u \in C(\overline{Q}_T) \) to the initial-boundary problem (5).

For \( f \equiv 0 \) and \( H \equiv 0 \), the uniqueness and existence results were obtained by Juutinen and Kawohl [20]. The evolution equations involved in the infinity Laplacian have been studied in a number of works. See \([2, 3, 4, 10, 24, 38, 39]\) for details. In these references they all consider the homogeneous parabolic equation involving infinity Laplacian. Here we are interested in the inhomogeneous equation (4) and we prove the comparison principle when the inhomogeneous term does not change its sign, i.e. \( f \geq 0 \) or \( f \leq 0 \). For the uniqueness, the method we use is the classical perturbation argument for viscosity solutions. Due to the difficulty of the first order and inhomogeneous terms we must construct suitable double variables function. And thanks to the parabolic term the uniqueness is valid for \( f < 0 \) or
$f > 0$ [5, 21, 28, 29, 30, 26, 35] for the elliptic case, and counterexamples show that the wellposedness fails if $f$ changes its sign [29, 35]. From this point of view, our uniqueness result is optimal. As for the existence of viscosity solutions, due to the high degeneracy and singularity of (1) we adopt the regularized approximation method. The key point is to establish the uniform estimates to the solutions of the regularized equations. The barrier function arguments are also employed to establish the equi-continuity of approximate solutions and we follow the argument in [3, 12, 20, 38, 39]. Note that in order to deal with the first order and inhomogeneous terms we must construct suitable barrier functions and calculate carefully.

**Remark 1.** When $H(x, t, p) = H(p)$ and $f$ is constant in (5), the equation is translation invariant. Then due to the translation invariance and the approximate procedure, we can establish the existence result. And for the general first order term and $f(x, t)$, we will study later in a different paper. Especially, during the approximation procedure we can also obtain the Lipschitz regularity in time and the Hölder continuity in space variable.

By Theorem 1.3, we can immediately obtain the existence for the problem with a transport term:

**Theorem 1.4.** Let $\xi$ be a fixed field, $f$ be constant, and let $g \in C(R^{n+1})$. Then there exists a viscosity solution $u \in C(\overline{Q_T})$ to the initial-boundary problem (3).

Besides the wellposedness results we also obtain the Lipschitz estimate to the viscosity solutions of the infinity Laplacian equation with a transport term (1) by approximation method. It should be pointed out that the Lipschitz regularity in time can be obtained by barrier method, but the Lipschitz property in space variable can not be obtained by the same method, and we adopt the Bernstein’s method to deal with it.

**Theorem 1.5.** Suppose that $\xi$ and $f$ are Lipschitz continuous in $\overline{Q_T}$ and $f \geq 0$($f \leq 0$). If $u \in C(\overline{Q_T})$ is a viscosity solution of the equation (1) in $Q_T$, then there exists a positive constant $C$ depending only on the dimension $n$ such that

$$
|Du(x, t)| \leq C(n)\|u\|_{\infty} \left(1 + \|\xi\|_{\infty} + \|D\xi\|_{\infty} + \frac{1 + \|\xi\|_{\infty}}{(\text{dist}(x, t, \partial_\eta Q_T))^2} + C(n) (\|f\|_{\infty} + \|Df\|_{\infty}) \right)
$$

for almost every $(x, t) \in Q_T$.

When $\xi \equiv 0$, this slightly extends the result in [20], because of regarding the extra source terms $\langle \xi, Du \rangle$ and $f$. The proof is based on the Bernstein’s method similar to the argument in [12] or [20]. But in order to deal with the additional first order and inhomogeneous terms we must estimate carefully. We clarify that in fact one can also obtain the Lipschitz estimate to the viscosity solutions of the generalized equation (4) without much additional cost.

The rest of the paper will be organized as follows. In Section 2, we give the equivalent definition of viscosity solutions to the equation (4). We give the comparison principle of the viscosity solutions of the initial-boundary problem of (5) and some stability results of the (1) in Section 3. In Section 4, we prove the existence of the viscosity solutions to the problem (5) by approximation method when $H(x, t, p) = H(p)$ and the inhomogeneous term is constant. In Section 5, we furthermore establish the Lipschitz regularity in space variable of the viscosity solutions to the equation (1) by the Bernstein’s method.
2. **Viscosity solutions.** In this section, we give the definition of the viscosity solutions to (4). In this paper, $Q_T = \Omega \times (0, T)$ denotes the space-time cylinder with the parabolic boundary

$$\partial_p Q_T = \{ \partial \Omega \times [0, T] \} \cup \{ \Omega \times \{0\} \}.$$  

We denote by $(D^2 \varphi(x, t))$ the second order Hessian matrix with space variable for a smooth function $\varphi$. $\lambda_{\max} (D^2 \varphi(x, t))$ and $\lambda_{\min} (D^2 \varphi(x, t))$ denote the largest and the smallest of the eigenvalues to the symmetric matrix $(D^2 \varphi(x, t))$ respectively.

Due to the singularity of the infinity Laplacian, we should give a reasonable explanation when the gradient vanishes. Here we use the definition of viscosity solutions based on semi-continuous extension, and we refer to the reader to [15, 19] etc.

**Definition 2.1.** Suppose that $H$ is as above, $f \in C(Q_T)$, and $u : Q_T \to \mathbb{R}$ is upper semi-continuous. If for every $(x_0, t_0)^{\top} \in Q_T$ and any $C^2(Q_T)$ test function $\varphi$ such that $u - \varphi$ has a strict local maximum at point $(x_0, t_0)$, that is $u(x_0, t_0) = \varphi(x_0, t_0)$ and $u(x, t) < \varphi(x, t)$ in a neighborhood of $(x_0, t_0)$, there holds

$$\begin{cases}
\varphi_t(x_0, t_0) - \Delta_N^\infty \varphi(x_0, t_0) - H(x_0, t_0, D\varphi(x_0, t_0)) \leq f(x_0, t_0), & \text{if } D\varphi(x_0, t_0) \neq 0, \\
\varphi_t(x_0, t_0) - \lambda_{\max} (D^2 \varphi(x_0, t_0)) \leq f(x_0, t_0), & \text{if } D\varphi(x_0, t_0) = 0,
\end{cases}$$  

(7)

then we say that $u$ is a viscosity sub-solution of (4).

Similarly, suppose $u : Q_T \to \mathbb{R}$ is lower semi-continuous. If for every $(x_0, t_0) \in Q_T$ and any $C^2(Q_T)$ test function $\varphi$ such that $u - \varphi$ has a strict local minimum at point $(x_0, t_0)$, that is $u(x_0, t_0) = \varphi(x_0, t_0)$ and $u(x, t) > \varphi(x, t)$ in a neighborhood of $(x_0, t_0)$, there holds

$$\begin{cases}
\varphi_t(x_0, t_0) - \Delta_N^\infty \varphi(x_0, t_0) - H(x_0, t_0, D\varphi(x_0, t_0)) \geq f(x_0, t_0), & \text{if } D\varphi(x_0, t_0) \neq 0, \\
\varphi_t(x_0, t_0) - \lambda_{\min} (D^2 \varphi(x_0, t_0)) \geq f(x_0, t_0), & \text{if } D\varphi(x_0, t_0) = 0,
\end{cases}$$  

(8)

then we say that $u$ is a viscosity super-solution of (4).

If $u \in C(Q_T)$ is both a viscosity sub-solution and a viscosity super-solution, then we say that $u$ is a viscosity solution of (4).

**Remark 2.** If $u$ is twice differentiable with respect to $x$ at point $(x, t)$, then the normalized infinity Laplacian of $u$ at $(x, t)$ is defined to be the closed interval

$$\Delta_N^\infty u(x, t) = \left[ \lambda_{\min} (D^2 u(x, t)) , \lambda_{\max} (D^2 u(x, t)) \right].$$

When $Du(x, t) \neq 0$, $\Delta_N^\infty u(x, t)$ contains only one real number and it is clear

$$\Delta_N^\infty u(x, t) = \frac{1}{|Du(x, t)|} \langle D^2 u(x, t) Du(x, t), Du(x, t) \rangle.$$  

In fact, if the gradient of a test function vanishes, one may assume that $D^2 \varphi = 0$, and thus $\lambda_{\max} (D^2 \varphi) = \lambda_{\min} (D^2 \varphi) = 0$. This means that if $D\varphi = 0$, then $\Delta_N^\infty \varphi = 0$. The proof of this fact is based on the well-known perturbation argument in [15, 20, 21] etc. But in order to deal with the difficulty of the first order term and the inhomogeneous term we should perturb twice. In fact the following lemma exhibits an equivalent definition to Definition 2.1. The idea of such a statement, together with the strategy of the proof, come from [20], however, some of the details are quite different due to the first order and inhomogeneous terms.
Lemma 2.2. Suppose that $H$ is as above, $f \in C(Q_T)$, $f \geq 0 (f \leq 0)$ and $u : Q_T \to \mathbb{R}$ is an upper semi-continuous function with the property that for every $(x_0, t_0) \in Q_T$ and any $C^2(Q_T)$ test function $\varphi$ such that $u - \varphi$ has a strict local maximum at point $(x_0, t_0)$, the following holds:

\[
\begin{cases}
\varphi_t(x_0, t_0) - \Delta^N \varphi(x_0, t_0) - H(x_0, t_0, D\varphi(x_0, t_0)) \leq f(x_0, t_0), & \text{if } D\varphi(x_0, t_0) \neq 0, \\
\varphi_t(x_0, t_0) \leq f(x_0, t_0), & \text{if } D\varphi(x_0, t_0) = 0 \text{ and } D^2\varphi(x_0, t_0) = 0.
\end{cases}
\]

Then $u$ is a viscosity sub-solution of (4).

Proof. We consider the case for $f \geq 0$ and leave the details for $f \leq 0$ to the reader. We argue by contradiction. Suppose that $u$ is not a viscosity sub-solution but satisfies the assumption of the lemma. Then there exist $(x_0, t_0) \in Q_T$ and $\varphi \in C^2(Q_T)$ such that $u - \varphi$ has a strict local maximum at point $(x_0, t_0)$, $D\varphi(x_0, t_0) = 0$, $D^2\varphi(x_0, t_0) \neq 0$, and

\[
\varphi_t(x_0, t_0) - \lambda_{\max} \left( D^2\varphi(x_0, t_0) \right) > f(x_0, t_0).
\]

By $D\varphi(x_0, t_0) = 0$ and $H(x_0, t_0, 0) = 0$, we also have

\[
\varphi_t(x_0, t_0) - \lambda_{\max} \left( D^2\varphi(x_0, t_0) \right) - H(x_0, t_0, D\varphi(x_0, t_0)) > f(x_0, t_0). \tag{9}
\]

Let $u_\delta(x, t) = (1 - \delta)u(x, t)$ with $0 < \delta < 1$. We double the variables

\[
w_j(x, t, y, s) = u_\delta(x, t) - \varphi(y, s) - \frac{j}{4}|x - y|^4 - \frac{j}{2}(t - s)^2 \tag{10}
\]

and let $(x_j, t_j, y_j, s_j)$ be a maximum point of $w_j$ in $Q_T \times \bar{Q}_T$. Since $(x_0, t_0)$ is a strict local maximum for $u - \varphi$, there exists a strict local maximum $(x_0^\delta, t_0^\delta)$ for $u_\delta - \varphi$ and small enough $\delta > 0$ such that $(x_0^\delta, t_0^\delta) \to (x_0, t_0)$ as $\delta \to 0$. By first choosing a small enough $\delta > 0$ and then large enough $j$, we have $(x_j, t_j), (y_j, s_j) \in Q_T$, and

\[
(x_j, t_j), (y_j, s_j) \to (x_0^\delta, t_0^\delta), \quad \text{as } j \to \infty. \tag{11}
\]

Denote

\[
\phi(y, s) = \frac{j}{4}|x_j - y|^4 - \frac{j}{2}(t_j - s)^2,
\]

then $\varphi - \phi$ has a local minimum at $(y_j, s_j)$. By (9) and continuity of $f$ and

\[(x, t) \mapsto \varphi_t(x, t) - \lambda_{\max} \left( D^2\varphi(x, t) \right) - H(x, t, D\varphi(x, t)), \]

we obtain

\[
\varphi_t(y_j, s_j) - \lambda_{\max} \left( D^2\varphi(y_j, s_j) \right) - H(y_j, s_j, D\varphi(y_j, s_j)) > f(y_j, s_j) + \delta
\]

for $j$ large enough and $\delta > 0$ sufficiently small.

Since

\[
D^2\varphi(y_j, s_j) \geq D^2\phi(y_j, s_j), D\varphi(y_j, s_j) = D\phi(y_j, s_j), \quad \text{and } \varphi_t(y_j, s_j) = \phi_t(y_j, s_j),
\]

we also have

\[
\phi_t(y_j, s_j) - \lambda_{\max} \left( D^2\phi(y_j, s_j) \right) - H(y_j, s_j, D\phi(y_j, s_j)) > f(y_j, s_j) + \delta. \tag{12}
\]

Similarly, set

\[
\psi(x, t) = \frac{j}{4}|x - y_j|^4 + \frac{j}{2}|t - s_j|^2,
\]

then $u_\delta - \psi$ has a local maximum at $(x_j, t_j)$. We consider the two cases: either $x_j \neq y_j$ or $x_j = y_j$ for all $j$ large enough.

Case 1: If $x_j = y_j$, (12) implies

\[
j(t_j - s_j) > f(y_j, s_j) + \delta. \tag{13}
\]
On the other hand, since \( u_\delta - \psi \) has a local maximum at \((x_j, t_j)\), we have
\[
 j(t_j - s_j) \leq (1 - \delta) f(x_j, t_j),
\] (14)
where we have used the assumption of the lemma. Due to (11), the continuity of \( f \) and \( f \geq 0 \) in \( Q_T \), we obtain that (14) contradicts to (13).

**Case 2:** If \( x_j \neq y_j \), we use jets and the parabolic maximum principle for semi-continuous functions [11, 33]. Then by the parabolic theorem of sums for \( w_j \) there exist \( n \times n \) symmetric matrices \( X_j, Y_j \) such that \( Y_j - X_j \) is positive semi-definite and
\[
 (j(t_j - s_j), \eta_j, X_j) \in \tilde{P}^{2,+} u_\delta(x_j, t_j), \quad (j(t_j - s_j), \eta_j, Y_j) \in \tilde{P}^{2,-} \phi(y_j, s_j),
\]
where \( \eta_j = j|x_j - y_j|^2 (x_j - y_j) \) and \( \eta_j \to 0 \) as \( j \to \infty \). Using (12) and the fact that
\[
 \lambda_{\text{max}}(D^2 \phi(y_j, s_j)) \geq \left\langle D^2 \phi(y_j, s_j) \frac{x_j - y_j}{|x_j - y_j|}, \frac{x_j - y_j}{|x_j - y_j|} \right\rangle,
\]
we have
\[
 j(t_j - s_j) - \left\langle Y_j \frac{x_j - y_j}{|x_j - y_j|}, \frac{x_j - y_j}{|x_j - y_j|} \right\rangle - H(y_j, s_j, \eta_j) > f(y_j, s_j) + \delta. \tag{15}
\]
Since \( u_\delta - \psi \) has a local maximum at \((x_j, t_j)\) and the assumption of the lemma, we have
\[
 j(t_j - s_j) - \left\langle X_j \frac{x_j - y_j}{|x_j - y_j|}, \frac{x_j - y_j}{|x_j - y_j|} \right\rangle - (1 - \delta) H \left( x_j, t_j, \frac{1}{1 - \delta} \eta_j \right) \leq (1 - \delta) f(x_j, t_j). \tag{16}
\]
With the help of (15) and (16), we have for large enough \( j \) and \( \delta \) sufficiently small that
\[
 f(y_j, s_j) + \delta - (1 - \delta) f(x_j, t_j)
\]
\[
 < j(t_j - s_j) - \left\langle Y_j \frac{x_j - y_j}{|x_j - y_j|}, \frac{x_j - y_j}{|x_j - y_j|} \right\rangle - H(y_j, s_j, \eta_j)
\]
\[
 - j(t_j - s_j) + \left\langle X_j \frac{x_j - y_j}{|x_j - y_j|}, \frac{x_j - y_j}{|x_j - y_j|} \right\rangle + (1 - \delta) H \left( x_j, t_j, \frac{1}{1 - \delta} \eta_j \right)
\]
\[
 = \left\langle (X_j - Y_j) \frac{x_j - y_j}{|x_j - y_j|}, \frac{x_j - y_j}{|x_j - y_j|} \right\rangle - H(y_j, s_j, \eta_j) + (1 - \delta) H \left( x_j, t_j, \frac{1}{1 - \delta} \eta_j \right). \tag{17}
\]
Letting \( j \to \infty \), we have
\[
 f(x_0^\delta, t_0^\delta) + \delta - (1 - \delta) f(x_0^\delta, t_0^\delta) \leq 0,
\]
where we have used the continuity of the functions \( f, H \) and \( H(\cdot, 0) = 0 \). This is impossible because \( f \geq 0 \) in \( Q_T \).

We get a contradiction. Now we have completed the proof. \( \square \)

**Remark 3.** When \( f \equiv 0 \), the above Lemma 2.2 is still valid if we modify some details in the proof. In fact in this case we can take \( \delta = 0 \) in the function of double variables (10).

**Remark 4.** Lemma 2.2 is also valid for \( f \leq 0 \) if we replace \( 1 - \delta \) by \( 1 + \delta \) throughout the argument.

**Remark 5.** A similar result is valid for viscosity super-solutions.
3. Comparison principle. In this section we prove the comparison principle of viscosity solutions of the initial-boundary problem of (5). The method is also the perturbation argument similar to the method in Section 2. Notice that we should consider the δ–perturbation in order to deal with the inhomogeneous and the first order terms. For more comparison results about the degenerate partial differential equations one can see [3, 12, 19, 20].

**Theorem 3.1.** Suppose that H is as above, f ∈ C( QT), and f ≥ 0. Suppose also that u is a viscosity sub-solution and v a viscosity super-solution to (4) and u ≤ v on ∂p QT. Then u ≤ v in the whole cylinder QT.

Furthermore, if u and v are viscosity solutions of (4), then

\[
\max_{Q_T} |u - v| \leq \max_{\partial_p Q_T} |u - v|.
\]

**Proof.** Notice that we can also assume that v is a strict viscosity super-solution. Otherwise we consider \( \tau = v + \frac{1}{\lambda} \) with \( \lambda > 0 \) sufficiently small. Then \( \tau \) is a strict viscosity super-solution and \( \tau > u \) on \( \partial_p QT - \varepsilon(\lambda) \).

We argue by contradiction. Suppose that there exists \((x_0, t_0) \in Q_T\) such that

\[
u(x_0, t_0) - v(x_0, t_0) = \sup_{Q_T} (u - v) > 0. \tag{18}\]

Let \( u_{\delta}(x, t) = (1 - \delta)u(x, t) \), where \( 0 < \delta < 1 \). We double the variables as above

\[
w_j(x, t, y, s) = u_{\delta}(x, t) - v(y, s) - \frac{j}{4} |x - y|^4 - \frac{j}{2}(t - s)^2
\]

and denote by \((x_j, t_j, y_j, s_j)\) the maximum point of \( w_j \) in \( \bar{Q}_T \times \bar{Q}_T \). Since \((x_0, t_0)\) is a strict local maximum for \( u - v \), there exists a strict local maximum \((x_0^{\delta}, t_0^{\delta})\) for \( u_\delta - v \) and small enough \( \delta > 0 \) such that \((x_0^{\delta}, t_0^{\delta}) \to (x_0, t_0) \) as \( \delta \to 0 \). By first choosing a small enough \( \delta > 0 \) and then large enough \( j \), we have \((x_j, t_j, y_j, s_j) \in Q_T\), and

\[
(x_j, t_j, y_j, s_j) \to (x_0^{\delta}, t_0^{\delta}, x_0^{\delta}, t_0^{\delta}), \quad \text{as } j \to \infty. \tag{19}\]

Denote

\[
\varphi(y, s) = -\frac{j}{4} |x_j - y|^4 - \frac{j}{2}(t_j - s)^2
\]

and

\[
\phi(x, t) = \frac{j}{4} |x - y_j|^4 + \frac{j}{2}|t - s_j|^2.
\]

Then \( v - \varphi \) has a local minimum at \((y_j, s_j)\) and \( u_\delta - \phi \) has a local maximum at \((x_j, t_j)\).

We consider the two cases: either \( x_j \neq y_j \) for all \( j \) large enough or \( x_j = y_j \) infinitely often.

**Case 1:** If \( x_j = y_j \), then \( D\varphi(y_j, s_j) = 0 \) and \( D^2\varphi(y_j, s_j) = 0 \). By the definition of strict viscosity super-solutions, we obtain

\[
j(t_j - s_j) > f(y_j, s_j) + \delta, \tag{20}\]

where we have used the equivalent definition of strict viscosity super-solutions. Since \( u_\delta - \phi \) has a local maximum at \((x_j, t_j)\) and \( D\phi(y_j, s_j) = 0, D^2\phi(x_j, t_j) = 0 \), we have

\[
j(t_j - s_j) \leq (1 - \delta)f(x_j, t_j), \tag{21}\]

where we have used Lemma 2.2. Due to (19), the continuity of \( f \) and \( f \geq 0 \) in \( Q_T \), we obtain that (21) contradicts to (20).
Case 2: If $x_j \neq y_j$, by the parabolic theorem of sums for $w_j$ again there exist $n \times n$ symmetric matrices $X_j, Y_j$ such that $Y_j - X_j$ is positive semi-definite and

\[ j(t) = \langle X_j \frac{x_j - y_j}{|x_j - y_j|}, \frac{x_j - y_j}{|x_j - y_j|} \rangle - H(y, \eta, \sigma) \geq f(y, s, \eta) \]

where $\eta_j = j|x_j - y_j|^2(x_j - y_j)$ and $\eta_j \to 0$ as $j \to \infty$. By the definition of strict viscosity super-solutions, we get

\[ j(t) - \phi \leq f(x, t, \delta) \]

by the definition of viscosity sub-solutions. With the help of (22) and (23), we have

\[ \delta + f(y, s) = \langle X_j \frac{x_j - y_j}{|x_j - y_j|}, \frac{x_j - y_j}{|x_j - y_j|} \rangle - H(y, s, \eta) \]

\[ \leq j(t) - \phi \leq f(x, t, \delta) \]

\[ \leq (1 - \delta) f(x, t) - H(y, s, \eta) + (1 - \delta) H \left( x, t, \frac{1}{1 - \delta} \right) \]

Letting $j \to \infty$, we have

\[ \delta + f(x, t) \leq (1 - \delta) f(x, t) \]

where we have used the continuity of the functions $f, H$ and $H(\cdot, 0) = 0$. This is impossible because $f \geq 0$ in $Q_T$. This finishes the proof.

**Remark 6.** Theorem 3.1 is also valid for $f \leq 0$ in $Q_T$, if we replace $1 - \delta$ by $1 + \delta$ throughout the argument.

The uniqueness Theorem 1.1 follows immediately as a direct consequence of the comparison principle Theorem 3.1.

Based on the comparison principle and the 1-homogeneity of the parabolic operator $Pu = u_t - \Delta_N u - \langle \xi, Du \rangle$, we can get the following stability result.

**Theorem 3.2.** Suppose $\xi \in C(Q_T)$, $f \in C(Q_T)$ and either $f \geq 0$ or $f \leq 0$ in $Q_T$. For $j = 1, 2$, suppose $c_j > 0$, $g_j \in C(\partial Q_T)$ and $u_j \in C(Q_T)$ is the viscosity solution of the initial-boundary problem

\[ \begin{cases} 
    (u_j)_t - \Delta_N u_j - \langle \xi, Du_j \rangle = c_j f(x, t), & \text{in } Q_T, \\
    u_j = g_j, & \text{on } \partial Q_T.
\end{cases} \]

Then

\[ \frac{u_1 - u_2}{c_1 - c_2} \leq \frac{u_1 - u_2}{c_2} \leq \frac{u_1 - u_2}{c_1} \]

Especially, if $g_1 = g_2 = g \in C(\partial Q_T)$, then

\[ \frac{1}{c_1} - \frac{1}{c_2} \leq \frac{1}{c_1} - \frac{1}{c_2} \leq \frac{1}{c_2} \]
Proof. Set
\[ v_j = \frac{1}{c_j} u_j. \]
Then \( v_j \) is a viscosity solution of the following initial-boundary problem
\[
\begin{cases}
(v_j)_t - \Delta_N v_j - \langle \xi, Dv_j \rangle = f(x, t), & \text{in } Q_T, \\
v_j = \frac{1}{c_j} g_j & \text{on } \partial_p Q_T,
\end{cases}
\]
for \( j = 1, 2 \). By the comparison principle and Remark 6, we obtain
\[
\|v_1 - v_2\|_{L^\infty(Q_T)} \leq \left\| \frac{g_1}{c_1} - \frac{g_2}{c_2} \right\|_{L^\infty(\partial_p Q_T)},
\]
which means the desired inequality. \( \square \)

With the above theorem in hand we can show the stability property of the solutions to (3), i.e. Theorem 1.2.

Proof of Theorem 1.2. By Theorem 3.2, we obtain
\[
\left\| \frac{u_j}{1 + c_j} - u \right\|_{L^\infty(Q_T)} \leq \left\| \frac{g_j}{1 + c_j} - g \right\|_{L^\infty(\partial_p Q_T)}.
\]
By the triangle inequality we have
\[
\frac{1}{1 + c_j} \|u_j - u\|_{L^\infty(Q_T)} - \frac{|c_j|}{1 + c_j} \|u\|_{L^\infty(Q_T)} \\
\leq \frac{1}{1 + c_j} \|g_j - g\|_{L^\infty(\partial_p Q_T)} + \frac{|c_j|}{1 + c_j} \|g\|_{L^\infty(\partial_p Q_T)}.
\]
Hence
\[
\|u_j - u\|_{L^\infty(Q_T)} \leq |c_j| \|u\|_{L^\infty(Q_T)} + \|g_j - g\|_{L^\infty(\partial_p Q_T)} + |c_j| \|g\|_{L^\infty(\partial_p Q_T)}.
\]
So
\[
\lim_{j \to \infty} \|u_j - u\|_{L^\infty(Q_T)} = 0.
\]
\( \square \)

4. Existence theorem. The main purpose of this section is to prove the existence of viscosity solutions to (4) with the initial and boundary data \( g \). The method we adopt is the classical approximation procedure and we follow the argument in [3, 20, 38, 39]. For clarity we will present them schematically. Due to the degeneracy and singularity of the equation, we start with the approximation
\[
u_t - L^{e, \delta} u - H(x, t, Du) = f, \tag{25}\]
where
\[
L^{e, \delta} u = e \Delta u + \frac{1}{|Du|^2 + \delta^2} \langle D^2 u Du, Du \rangle = \sum_{i,j=1}^n a_{ij}^{e, \delta} (Du) u_{ij} \tag{26}\]
with \( 0 < e \leq 1, 0 < \delta \leq \frac{1}{\sqrt{3}} \) and
\[
a_{ij}^{e, \delta} (p) = e \delta_{ij} + \frac{p_i p_j}{|p|^2 + \delta^2}. \tag{27}\]
This means that the coefficient matrix \( A^{e, \delta} (p) = \left( a_{ij}^{e, \delta} (p) \right) = e I + \frac{p_i p_j}{|p|^2 + \delta^2} \) is uniformly elliptic. Since the approximate equation (25) is uniformly parabolic, the
existence of a smooth solution $u_{\varepsilon, \delta}$ is guaranteed by classical results in [25] under the condition that initial-boundary data $g$ is smooth. In the following we will prove the uniform estimates for $u_{\varepsilon, \delta}$ independent of $0 < \varepsilon \leq 1, 0 < \delta \leq \frac{1}{\lambda}$. Then by the compactness method and the stability property of viscosity solutions we obtain that the limit function of $u_{\varepsilon, \delta}$ as $\varepsilon, \delta \to 0$ is a viscosity solution of (4). These estimates we require will be obtained by the standard barrier methods.

**Theorem 4.1** (Boundary regularity at $t = 0$). Let $H(x, t, p)$ as above, $f$ be continuous in $\overline{Q}_T$, and let $g \in C^{2,1}(\overline{Q}_T)$. Suppose that $u_{\varepsilon, \delta}$ is a smooth solution satisfying

\[
\begin{cases}
    u_t - \Delta^{\varepsilon, \delta} u - H(x, t, Du) = f, & \text{in } Q_T, \\
    u = g, & \text{on } \partial_p Q_T.
\end{cases}
\]

Then there exists a constant $C \geq 0$ depending on $\|g\|_\infty$, $\|Dg\|_\infty$, $\|\nabla g\|_\infty$, $\|\xi\|_\infty$ and $\|f\|_\infty$ but independent of $0 < \varepsilon < 1$ and $1 < \delta < 1$ such that

$$|u_{\varepsilon, \delta}(x, t) - g(x, 0)| \leq Ct \text{ in } Q_T.$$ 

Moreover, if $g$ is only continuous in $x$ and bounded in $t$, then the modulus of continuity of $u_{\varepsilon, \delta}$ on $\Omega \times [0, T]$ (for small $T$) can be estimated in terms of $\|g\|_\infty$, $\|f\|_\infty$ and the modulus of continuity of $g(x, 0)$ in $x$.

**Proof.** Step 1. Suppose first that $g \in C^{2,1}(\overline{Q}_T)$ and we consider the upper barrier function

$$w^+(x, t) = g(x, 0) + \lambda t,$$

where $\lambda > 0$ is to be determined. We have

$$w^+_t - \Delta^{\varepsilon, \delta} w^+ - H(x, t, Dw^+) = \lambda - \varepsilon \Delta g(x, 0) + \frac{1}{\|Dg(x, 0)\|^2 + \varepsilon^2} \langle D^2 g(x, 0) Dg(x, 0), Dg(x, 0) \rangle + H(x, t, Dg(x, 0))$$

\[\geq \lambda - (1 + \varepsilon n) \|D^2 g\|_\infty - M \|Dg\|_\infty\]

\[\geq f\|_\infty + M \|Dg\|_\infty \geq f(x, t),\]

if $\lambda \geq (1 + n) \|D^2 g\|_\infty + \|f\|_\infty + M \|\xi\|_\infty$. Therefore $w^+$ is a super-solution.

Clearly, $w^+(x, 0) = g(x, 0)$ for all $x \in \Omega$. Moreover, for $x \in \partial \Omega$ and $t > 0$,

$$w^+(x, t) = g(x, 0) + \lambda t \geq g(x, t) + (\lambda - \|g_t\|_\infty) t \geq g(x, t),$$

if $\lambda \geq \|g_t\|_\infty$. That is, $w^+ \geq u_{\varepsilon, \delta}$ on $\partial_p Q_T$.

Thus, by the classical comparison principle, we obtain

$$u_{\varepsilon, \delta}(x, t) \leq w^+(x, t) = g(x, 0) + \lambda t$$

for every $(x, t) \in Q_T$. Similarly, by considering also the lower barrier function

$$w^-(x, t) = g(x, 0) - \lambda t,$$

we obtain the symmetric inequality, and hence the Lipschitz estimate

$$|u_{\varepsilon, \delta}(x, t) - g(x, 0)| \leq Ct$$

for $0 < t < T$ and

$$C = \max \left\{ \|g_t\|_\infty, (1 + n) \|D^2 g\|_\infty + \|f\|_\infty + M \|Dg\|_\infty \right\}.$$ 

Step 2. Suppose now that $g$ is only continuous in $x$ and let $\mu(\cdot)$ be its modulus of continuity. Let us fix a point $x_0 \in \Omega$ by choosing $0 < \rho < \text{dist}(x_0, \partial \Omega)$, then
we have \(|g(x, 0) - g(x_0, 0)| < \mu\) whenever \(|x - x_0| < \rho\). Let us consider the smooth functions
\[
g^\pm(x, t) = g(x_0, 0) \pm \mu(\rho) \pm \frac{2\|g\|_\infty}{\rho^2}|x - x_0|^2.
\]

It is easy to check that \(g^- \leq g \leq g^+\) on the parabolic boundary \(\partial_p Q_T\). Thus if \(u^\pm\) are the unique classical solutions to (25) with boundary and initial data \(g^\pm\), respectively, we have \(u^- \leq u_{\varepsilon, \delta} \leq u^+\) in \(Q_T\) by the classical comparison principle again. Since \(g^\pm\) are smooth, we can use estimate (28) to conclude that
\[
|u^\pm(x_0, t) - g^\pm(x_0, 0)| \\
\leq Ct = C(g^\pm)t \\
= t \max \left\{ \|g^\pm\|_\infty, \|f\|_\infty + (1 + n) \|D^2 g^\pm\|_\infty + \|M \|Dg^\pm\|_\infty \right\} \\
\leq t \left( \|f\|_\infty + (1 + n) \frac{4\|g\|_\infty}{\rho^2} + M \frac{4\|g\|_\infty(\rho + \text{diam}\Omega)}{\rho^2} \right) + \mu
\]
Then by calculation we get
\[
|u_{\varepsilon, \delta}(x_0, t) - g(x_0, 0)| \\
\leq |u_{\varepsilon, \delta}(x_0, t) - u^-(x_0, t)| + |u^-(x_0, t) - g^-(x_0, 0)| + |g^-(x_0, 0) - g(x_0, 0)| \\
\leq u^-(x_0, t) - u_{\varepsilon, \delta}(x_0, t) \\
+ t \left( \|f\|_\infty + (1 + n) \frac{4\|g\|_\infty}{\rho^2} + M \frac{4\|g\|_\infty(\rho + \text{diam}\Omega)}{\rho^2} \right) + \mu
\]
and
\[
|u_{\varepsilon, \delta}(x_0, t) - g(x_0, 0)| \\
\leq |u_{\varepsilon, \delta}(x_0, t) - u^-(x_0, t)| + |u^-(x_0, t) - g^-(x_0, 0)| + |g^-(x_0, 0) - g(x_0, 0)| \\
\leq u_{\varepsilon, \delta}(x_0, t) - u^-(x_0, t) \\
+ t \left( \|f\|_\infty + (1 + n) \frac{4\|g\|_\infty}{\rho^2} + M \frac{4\|g\|_\infty(\rho + \text{diam}\Omega)}{\rho^2} \right) + \mu.
\]
With these two inequalities (29) and (30) we have
\[
|u_{\varepsilon, \delta}(x_0, t) - g(x_0, 0)| \\
\leq \frac{1}{2} (u^+(x_0, t) - u^-(x_0, t)) \\
+ t \left( \|f\|_\infty + (1 + n) \frac{4\|g\|_\infty}{\rho^2} + M \frac{4\|g\|_\infty(\rho + \text{diam}\Omega)}{\rho^2} \right) + \mu
\]
\[
\leq \frac{1}{2} |u^+(x_0, t) - g^+(x_0, 0)| + \frac{1}{2} |g^+(x_0, 0) - g^-(x_0, 0)| + \frac{1}{2} |g^-(x_0, 0) - u^-(x, t)| \\
+ t \left( \|f\|_\infty + (1 + n) \frac{4\|g\|_\infty}{\rho^2} + M \frac{4\|g\|_\infty(\rho + \text{diam}\Omega)}{\rho^2} \right) + \mu
\]
\[
\leq 2t \left( \|f\|_\infty + (1 + n) \frac{4\|g\|_\infty}{\rho^2} + M \frac{4\|g\|_\infty(\rho + \text{diam}\Omega)}{\rho^2} \right) + 2\mu.
\]
This inequality concludes the proof.

Noting that the equation (25) is translation invariant when \(H(x, t, p) = H(p)\) and the inhomogeneous term \(f\) is constant. Hence the full Lipschitz estimate in
time now follows easily with the aid of the comparison principle and the fact that the equation (25) is translation invariant.

**Theorem 4.2 (full Lipschitz regularity in time).** If $H(x,t,p) = H(p)$ and $f$ is constant, $g \in C^{2,1}(\overline{Q}_T)$ and $u = u_{\varepsilon,\delta}$ is as in Theorem 4.1, then there exists a constant $C \geq 0$ depending on $\|g_t\|_\infty, \|\xi\|_\infty$ and $\|D^2g\|_\infty$ but independent of $0 < \varepsilon \leq 1$ such that

$$|u(x,t) - u(x,s)| \leq C|t - s|$$

for all $x \in \Omega$ and $t, s \in (0, T)$. Moreover, if $g$ is only continuous, then the modulus of continuity of $u$ on $Q_T$ can be estimated in terms of $\|g\|_\infty$ and the modulus of continuity of $g$.

**Proof.** Let $\tilde{u}(x,t) = u(x,t + \sigma), \sigma > 0$. Then both $u$ and $\tilde{u}$ are smooth solutions to (25) in $Q_\sigma = \Omega \times (0, T - \sigma)$, and hence if $g \in C^{2,1}(\overline{Q}_T)$, we have

$$\sup_{Q_\sigma}|u - \tilde{u}| = \sup_{\partial \Omega} |u - \tilde{u}|$$

$$\leq \max \left\{ \|u(\cdot, \sigma) - u(\cdot, 0)\|_\infty, \sup_{x \in \partial \Omega} \|u(x, \cdot + \sigma) - u(x, \cdot)\|_{(0, T - \sigma)} \right\}$$

$$\leq \max \{ C\sigma, \|g_t\|_\infty \sigma \} = C\sigma$$

by the classical comparison principle and Theorem 4.1. This implies the Lipschitz estimate asserted above, and the proof for the case when $g$ is only continuous is analogous. \hfill \Box

**Remark 7.** It should be pointed out that we need not invoke the translation invariance property of the equation in the proof of the boundary regularity at $t = 0$. But in order to obtain the interior Lipschitz regularity with respect to $t$ we add the condition that $H(x,t,p) = H(p)$ and $f$ is constant to guarantee the translation invariance property of the equation.

**Theorem 4.3 (Hölder regularity at the lateral boundary).** Let $f$ is continuous in $\overline{Q}_T$, and let $g \in C^{2,1}(\overline{Q}_T)$. Suppose that $u_{\varepsilon,\delta}$ is a smooth solution satisfying

\[
\begin{cases}
  u_t - C^{\varepsilon,\delta}u - H(x,t,Du) = f, & \text{in } Q_T, \\
  u = g, & \text{on } \partial_p Q_T.
\end{cases}
\]

Then for each $0 < \alpha < 1$, there exists a constant $C_\alpha \geq 1$ depending on $\alpha$, $\|g\|_\infty$, $\|g_t\|_\infty$, $\|f\|_\infty$ and $\|D^2g\|_\infty$ but independent of $\varepsilon$ and $\delta$ sufficiently small such that

$$|u_{\varepsilon,\delta}(x,t_0) - g(x_0,t_0)| \leq C_\alpha |x - x_0|^{\alpha},$$

for all $(x_0,t_0) \in \partial \Omega \times (0, T)$ and $x \in \Omega \cap B_r(x_0)$, where $r$ depending only on $\alpha$ and $M$. Moreover, if $g$ is only continuous, then the modulus of continuity of $u_{\varepsilon,\delta}$ in $x$ can be estimated in terms of $\|g\|_\infty, \|f\|_\infty, M$ and the modulus of continuity of $g$ in $x$.

**Proof.** Step 1. For every $(x_0,t_0) \in \partial \Omega \times (0, T)$ and $0 < \alpha < 1$, let

$$w^+(x,t) = g(x_0,t_0) + C_\alpha |x - x_0|^{\alpha} + \lambda(t_0 - t),$$
where \( C_* \geq 1, \lambda > 0 \) are to be determined. Then a straightforward computation gives

\[
\begin{align*}
   w^+_t - \mathcal{L}^{\varepsilon, \delta} w^+ - H(x, t, Dw^+) \\
   = -\lambda - C_* \alpha |x - x_0|^{\alpha - 2} \left[ \varepsilon(n + \alpha - 2) + \frac{\alpha - 1}{1 + \frac{\delta}{C_* \alpha |x - x_0|^{\alpha - 1}}} \right] \\
   - H \left( x, t, C_* \alpha |x - x_0|^{\alpha - 2} (x - x_0) \right) \\
   \geq -\lambda - C_* \alpha |x - x_0|^{\alpha - 2} \left[ \varepsilon(n + \alpha - 2) + \frac{\alpha - 1}{1 + \frac{\delta}{C_* \alpha |x - x_0|^{\alpha - 1}}} + M|x - x_0| \right].
\end{align*}
\]

If \(|x - x_0| \leq 1\) and \( C_* \geq 1\), we have

\[
-\varepsilon(n + \alpha - 2) - \frac{\alpha - 1}{1 + \frac{\delta}{C_* \alpha |x - x_0|^{\alpha - 1}}} - M|x - x_0| \geq \frac{1 - \alpha}{4} - M|x - x_0|
\]

for \( \delta < \alpha \) and for \( 1 < \varepsilon < \frac{1 - \alpha}{4(\alpha + \alpha - 2)} \) if \( n > 1 \) and for any \( \varepsilon > 0 \) if \( n = 1 \). Now if we choose \( r = \min \left\{ \frac{1 - \alpha}{4 \alpha M}, 1 \right\} \), then for \(|x - x_0| \leq r\), we have \( M|x - x_0| \leq \frac{1 - \alpha}{8} \).

Therefore

\[
\begin{align*}
   w^+_t - \mathcal{L}^{\varepsilon, \delta} w^+ - H(x, t, Dw^+) &\geq -\lambda - C_* \alpha |x - x_0|^{\alpha - 2} \cdot \frac{1 - \alpha}{8} \\
   &\geq -\lambda + \frac{C_* \alpha (1 - \alpha)}{8} \cdot |x - x_0|^{\alpha - 2} \\
   &\geq \|f\|_{\infty} \\
   &\geq f(x, t),
\end{align*}
\]

if \( C_* \geq \max \left\{ \frac{8}{\alpha(1 - \alpha)} \left( \lambda + \|f\|_{\infty} \right), 1 \right\} \).

We have shown that \( w^+ \) is a super-solution of (25).

Step 2. Let \( Q_* = (\Omega \cap B_r(x_0)) \times (t_0 - t_*, t_0) \), where \( t_* = \min \{1, t_0\} \). We want to prove first \( w^+ \geq u_{\varepsilon, \delta} \) on \( \partial Q^* \).

Case 1. If \( x \in \partial \Omega \cap B_r(x_0) \), then

\[
\begin{align*}
   w^+(x, t) &= g(x_0, t_0) + C_* |x - x_0|^\alpha + \lambda (t_0 - t) \\
   &\geq g(x_0, t_0) + C_* |x - x_0| + \lambda (t_0 - t) \\
   &\geq -\|g_t\|_{\infty} (t_0 - t) - \|Dg\|_{\infty} |x - x_0| + g(x, t) + C_* |x - x_0| + \lambda (t_0 - t) \\
   &\geq g(x, t) \\
   &= u_{\varepsilon, \delta}(x, t),
\end{align*}
\]

provided \( C_* \geq \|Dg\|_{\infty} \) and \( \lambda \geq \|g_t\|_{\infty} \).

Case 2. If \( x \in \Omega \cap \partial B_r(x_0) \), it is easy to see that \( v(x, t) = \|f\|_\infty t + \|g\|_{\infty} \) is a super-solution of (25) in \( Q_T \) and \( v \geq g \) on \( \partial \Omega Q_T \). Hence, we have

\[
\begin{align*}
   w^+(x, t) &= g(x_0, t_0) + C_* r^\alpha + \lambda (t_0 - t) \\
   &\geq g(x_0, t_0) + C_* r^\alpha \\
   &\geq \|g\|_{\infty} + \|f\|_{\infty} T \\
   &\geq \|g\|_{\infty} + \|f\|_{\infty} \cdot t \\
   &= v(x, t)
\end{align*}
\]
Therefore, we have
\[ w \geq u_{\varepsilon,\delta}(x,t), \]
provided \( C_* \geq r^{-\alpha}(2\|g\|_\infty + \|f\|_\infty T) \), and in the last inequality we have used the comparison principle.

Step 3. To prove \( w^+ \geq u_{\varepsilon,\delta} \) on \( \partial_0Q^* \).

Case 1. If \( t_* = t_0 \), then \( Q_* = (\Omega \cap B_r(x_0)) \times (t_0, t_0) \), and notice that since \( u_{\varepsilon,\delta} = g \) on the bottom of this cylinder,
\[
\begin{align*}
  w^+(x,0) &= g(x_0, t_0) + C_*|x - x_0|^\alpha + \lambda_0 \\
  &\geq g(x_0, t_0) + C_*|x - x_0| + \lambda_0 \\
  &\geq g(x_0, t_0) + \|Dg\|_\infty |x - x_0| + \|g\|_T t_0 \\
  &\geq g(x,0),
\end{align*}
\]
if \( C_* \geq \|Dg\|_\infty \) and \( \lambda \geq \|g\|_\infty \).

Case 2. If \( t_* = t_0 - 1 \), then \( Q_* = (\Omega \cap B_r(x_0)) \times (t_0 - 1, t_0) \). Using the comparison principle again, we have
\[
\begin{align*}
  u^+(x, t_0 - 1) &= g(x_0, t_0) + C_*|x - x_0|^\alpha + \lambda \\
  &\geq g(x_0, t_0) + \lambda \\
  &\geq g(x_0, t_0) + \|g\|_\infty + \|f\|_\infty(T - 1) \\
  &\geq \|g\|_\infty + \|f\|_\infty(t_0 - 1) \\
  &= v(x, t_0 - 1) \\
  &\geq u_{\varepsilon,\delta}(x, t_0 - 1),
\end{align*}
\]
if \( \lambda \geq 2\|g\|_\infty + \|f\|_\infty(T - 1) \).

Step 4. In conclusion, we have shown that \( w^+ \geq u_{\varepsilon,\delta} \) on \( \partial_0Q_* \), if we choose
\[
\begin{align*}
  \lambda &\geq \max\{\|g\|_\infty, 2\|g\|_\infty + \|f\|_\infty(T - 1)\}, \\
  C_* &\geq \max\left\{\|Dg\|_\infty, r^{-\alpha}(2\|g\|_\infty + \|f\|_\infty T), \frac{8}{\alpha(1-\alpha)r^{\alpha-2}}(\lambda + \|f\|_\infty)\right\}, \\
  r &\leq \min\left\{\frac{1-\alpha}{8M}, 1\right\}.
\end{align*}
\]
Therefore, we have \( w^+ \geq u_{\varepsilon,\delta} \) in \( Q_* \) by the comparison principle. In particular,
\[ u(x, t_0) \leq w^+(x, t_0) = g(x_0, t_0) + C_*|x - x_0|^\alpha \]
for \( x \in \Omega \cap B_r(x_0) \). Using the lower barrier
\[ w^-(x,t) = g(x_0, t_0) - C_*|x - x_0|^\alpha - \lambda(t_0 - t), \]
we get the symmetric inequality. When \( g \) is only continuous, the argument is analogous to the previous case. This finishes the proof. \( \square \)

Using again the translation invariance of the equation and the comparison principle, we can extend the Hölder estimate to the interior of the domain, cf. [20, 38, 39] etc.

**Theorem 4.4** (full Hölder regularity in space). Let \( f \) be constant, and let \( g \in C^{2,1}(\overline{Q_T}) \). Suppose that \( u_{\varepsilon,\delta} \) is a smooth solution satisfying
\[
\begin{cases}
  u_t - \mathcal{L}^c \delta u - H(Du) = f, & \text{in } Q_T, \\
  u = g, & \text{on } \partial_0Q_T.
\end{cases}
\]
Then for each $0 < \alpha < 1$, there exists a constant $C \geq 1$, depending on $\alpha$, $\|g\|_\infty$, $\|g_t\|_\infty$ and $\|Dg\|_\infty$ but independent of $\varepsilon$ and $\delta$ sufficiently small such that
\[
|u_{\varepsilon,\delta}(x,t) - u_{\varepsilon,\delta}(y,t)| \leq C|x-y|^\alpha,
\]
for all $(x,y) \in \Omega$. Moreover, if $g$ is only continuous, then the modulus of continuity of $u_{\varepsilon,\delta}$ in $x$ can be estimated in terms of $\|g\|_\infty$ and the modulus of continuity of $g$ in $x$.

Notice that we can not obtain the Lipschitz estimate of solutions to the regularized equation (25) by the barrier method. But it is interesting that we can get the Lipschitz estimate when we remove the viscous term from the approximate equation (25), cf. [12, 20, 38, 39].

**Theorem 4.5** (Lipschitz regularity at the lateral boundary when $\varepsilon = 0$). Let $f$ be continuous in $\overline{Q}_T$, and $g \in C^{2,1}(\overline{Q}_T)$. Suppose that $u = u_\delta$ is a smooth solution satisfying
\[
\begin{aligned}
& u_t - L^0u - H(x, t, Du) = f, \quad \text{in} \; Q_T, \\
& u = g, \quad \text{on} \; \partial_p Q_T. 
\end{aligned}
\]
Then for each $0 < \alpha < 1$, there exist constants $C_* \geq 1$ and $r > 0$ depending on $\|g\|_\infty$, $\|g_t\|_\infty$, $\|f\|_\infty$ and $\|Dg\|_\infty$ but independent of $\delta$ sufficiently small such that
\[
|u(x, t_0) - g(x_0, t_0)| \leq C_*|x-x_0|,
\]
for all $(x_0, t_0) \in \partial \Omega \times (0, T)$ and $x \in \Omega \cap B_r(x_0)$. Moreover, if $g$ is only continuous, then the modulus of continuity of $u$ on $\partial \Omega \times (0, T)$ can be estimated in terms of $\|g\|_\infty$, $\|f\|_\infty$ and the modulus of continuity of $g$.

**Proof.** Step 1. Assume first that $g \in C^{2,1}(\overline{Q}_T)$. For every $(x_0, t_0) \in \partial \Omega \times (0, T)$ and $0 < \alpha < 1$, set
\[
w^+(x,t) = g(x_0, t_0) + C_*|x-x_0| + \lambda(t_0 - t) - K|x-x_0|^2,
\]
where $C_* \geq 1, \lambda, K > 0$ are to be determined. Then by a straightforward computation we obtain
\[
w^+_t - L^0w^+ - H(x, t, Dw^+)
= - \lambda - \left(\frac{C_*}{|x-x_0|} - 2K\right)^2|x-x_0|^2 - C_*\left(\frac{C_*}{|x-x_0|} - 2K\right)^2|x-x_0| \\
- H\left(x, t, C_*\frac{x-x_0}{|x-x_0|} - 2K(x-x_0)\right)
= - \lambda + \frac{2K}{1 + \left(\frac{C_* - 2K|x-x_0|}{|x-x_0|}\right)^2} - H\left(x, t, C_*\frac{x-x_0}{|x-x_0|} - 2K(x-x_0)\right)
\geq - \lambda + \frac{2K}{1 + \left(\frac{1}{C_* - 2K|x-x_0|}\right)^2} - M|C_* - 2K|x-x_0||^2
\geq - \lambda + K - M
\geq f(x,t),
\]
(34)
if we choose $|x - x_0| \leq r = \min\{1, \frac{1}{2K}\}$, $0 < \delta \leq 1$, $C_* \geq 2$, $K \geq \lambda + \| f \|_\infty + M$.

That is $w^+$ is a super-solution.

Step 2. Let $Q_* = (\Omega \cap B_r(x_0)) \times (t_0 - t_*, t_0)$, where $t_* = \min \{1, t_0\}$. We want to prove $w^+ \geq u$ on $\partial_h Q^*$.

Case 1. If $x \in \partial \Omega \cap B_r(x_0)$, noting that $2K|x - x_0| \leq 1$, then

$$w^+(x, t) = g(x_0, t_0) + C_*|x - x_0| + \lambda (t_0 - t) - K|x - x_0|^2$$

$$\geq g(x_0, t_0) + (C_* - \frac{1}{2})|x - x_0| + \lambda (t_0 - t)$$

$$\geq -\|g_t\|_\infty (t_0 - t) - \|Dg\|_\infty |x - x_0| + g(x, t) + (C_* - \frac{1}{2})|x - x_0| + \lambda (t_0 - t)$$

$$\geq g(x, t) = v(x, t),$$

provided $C_* \geq \frac{1}{2} + \|Dg\|_\infty$ and $\lambda \geq \|g_t\|_\infty$.

Case 2. If $x \in \Omega \cap \partial B_r(x_0)$, it is easy to see that $v(x, t) = \|f\|_\infty t + \|g_t\|_\infty$ is a super-solution of

$$u_t - \mathcal{L}^{0,8} u - H(x, t, Du) = f(x, t)$$

in $Q_T$ and $v \geq g$ on $\partial_h Q_T$. Hence, we have

$$w^+(x, t) = g(x_0, t_0) + C_*|x - x_0| + \lambda (t_0 - t) - K|x - x_0|^2$$

$$\geq g(x_0, t_0) + \lambda (t_0 - t) + (C_* - \frac{1}{2})|x - x_0|$$

$$\geq \|g\|_\infty + \|f\|_\infty T$$

$$\geq \|g\|_\infty + \|f\|_\infty t$$

$$= v(x, t)$$

$$\geq v(x, t),$$

provided $C_* \geq \frac{1}{2} + 2\|g\|_\infty + \|f\|_\infty T$.

Step 3. To prove $w^+ \geq u$ on $\partial_h Q^*$.

Case 1. If $t_* = t_0$, then $Q_* = (\Omega \cap B_r(x_0)) \times (0, t_0)$, and notice that since $u = g$ on the bottom of this cylinder,

$$w^+(x, 0) = g(x_0, t_0) + C_*|x - x_0| + \lambda t_0 - K|x - x_0|^2$$

$$\geq g(x_0, t_0) + (C_* - K)|x - x_0| + \lambda t_0$$

$$\geq g(x_0, t_0) + \|Dg\|_\infty |x - x_0| + \|g_t\| t_0$$

$$\geq g(x, 0),$$

if $C_* \geq K + \|Dg\|_\infty$ and $\lambda \geq \|g_t\|_\infty$.

Case 2. If $t_* = t_0 - 1$, then $Q_* = (\Omega \cap B_r(x_0)) \times (t_0 - 1, t_0)$. Using the comparison principle again, we have

$$w^+(x, t_0 - 1) = g(x_0, t_0) + C_*|x - x_0| + \lambda - K|x - x_0|^2$$

$$\geq g(x_0, t_0) + \lambda + (C_* - K)|x - x_0|$$

$$\geq g(x_0, t_0) + \lambda$$

$$\geq \|g\|_\infty + \|f\|_\infty (T - 1)$$

$$\geq \|g\|_\infty + \|f\|_\infty (t_0 - 1)$$

$$= v(x, t_0 - 1)$$
\[ \geq u(x, t_0 - 1), \]

if \( \lambda \geq 2 \| g \|_{\infty} + \| f \|_{\infty}(T - 1) \).

Step 4. In conclusion, we have shown that \( w^+ \geq u \) on \( \partial T Q_* \) and \( w^+ \) is a supersolution of (35), if we choose

\[
\lambda \geq \max \left\{ \| g \|_{\infty}, 2 \| g \|_{\infty} + \| f \|_{\infty}(T - 1) \right\},
\]

\[
C_* \geq \max \left\{ 2, \frac{1}{2} + \| Dg \|_{\infty}, \frac{1}{2} + 2 \| g \|_{\infty} + \| f \|_{\infty}T \right\},
\]

\[
K \geq \lambda + \| f \|_{\infty} + M.
\]

Therefore, the comparison principle implies \( w^+ \geq u \) in \( Q_* \). In particular,

\[ u(x, t_0) \leq w^+(x, t_0) = g(x_0, t_0) + C_*|x - x_0| - K|x - x_0|^2 \leq g(x_0, t_0) + C_*|x - x_0| \]

for \( x \in \Omega \cap B_r(x_0) \). Using the lower barrier

\[ w^-(x, t) = g(x_0, t_0) - C_*|x - x_0|^n - \lambda(t_0 - t) + K|x - x_0|^2, \]

we get the symmetric inequality.

Assume now that \( g \) is only continuous. Let us fix a point \( (x_0, t_0) \in \Omega \times (0, T) \) and for a given \( \mu > 0 \), choose \( 0 < \tau < t_0 \) such that \( |g(x, t) - g(x_0, t_0)| < \mu \) whenever \( |x - x_0| + |t - t_0| < \tau \). Define the smooth functions

\[ g^+(x, t) = g(x_0, t_0) \pm \mu \pm \frac{4\| g \|_{\infty}}{\tau^2}|x - x_0|^2 \pm \frac{4\| g \|_{\infty}}{\tau}|t - t_0|. \]

It is easy to check that

\[ g^-(x, t) \leq g(x_0, t_0) - \mu < g(x, t) < g(x_0, t_0) + \mu \leq g^+(x, t) \]

if \( |x - x_0| + |t - t_0| < \tau \) and

\[ g^-(x, t) \leq -\| g \|_{\infty} \leq g(x, t) \leq \| g \|_{\infty} \leq g^+(x, t) \]

otherwise, we have \( g^- \leq g \leq g^+ \) on the parabolic boundary \( \partial_T Q_T \).

Thus if \( u^\pm \) are the unique classical solutions to (35) with boundary and initial data \( g^\pm \), respectively, we have \( u^- \leq u \leq u^+ \) in \( Q_T \) by the classical comparison principle again. Since \( g^\pm \) are smooth, we can use estimate (33) to conclude that

\[ |u^+(x, t_0) - g^+(x_0, t_0)| \leq C^*_+ |x - x_0|. \]

Because

\[ u^-(x, t_0) - \mu - g^-(x_0, t_0) = u^- - g(x_0, t_0) \]

\[ \leq u(x, t_0) - g(x_0, t_0) \]

\[ \leq u^+(x, t_0) - g(x_0, t_0) \]

\[ = u^+(x, t_0) + \mu - g^+(x_0, t_0), \]

we get

\[ |u(x, t_0) - g(x_0, t_0)| \leq |u(x, t_0) - u^+(x, t_0)| \]

\[ + |u^+(x, t_0) - g^+(x_0, t_0)| + |g^+(x_0, t_0) - g(x_0, t_0)| \]

\[ \leq \frac{1}{2} |u^+(x, t_0) - g^+(x_0, t_0)| + C^*_+ |x - x_0| + \mu \]

\[ \leq \frac{1}{2} |u^+(x, t_0) - g^+(x_0, t_0)| + \frac{1}{2} |g^+(x_0, t_0) - g^-(x_0, t_0)| \]

\[ + \frac{1}{2} |u^-(x, t_0) - g^-(x_0, t_0)| + C^*_+ |x - x_0| + \mu \]
This finishes the proof. \[ \square \]

**Theorem 4.6.** Let \( H(x,t,p) = H(p) \) and \( f \) be constant and let \( g \in C^{2,1}(\Omega_T) \). Suppose that \( u_\delta \) is a smooth solution satisfying (32). Then there exists a constant \( C_* \geq 1 \) depending on \( \|g\|_\infty, \|g_t\|_\infty \) and \( \|Dg\|_\infty \) but independent of \( \delta \) sufficiently small such that

\[
|u_\delta(x,t) - u_\delta(y,t)| \leq C_*|x - y|
\]

for all \((x,y) \in \Omega \) and \( t \in (0,T) \). Moreover, if \( g \) is only continuous, then the modulus of continuity of \( u_\delta \) in \( x \) on \( \Omega \times (0,T) \) can be estimated in terms of \( \|g\|_\infty \) and the modulus of continuity of \( g \) in \( x \) and \( t \).

The existence theorem follows now easily by piecing out the results in Theorems 4.2, 4.4 and 4.6, and using the standard compactness arguments and the stability properties of viscosity solutions.

**Proof of Theorem 1.3:** If \( g \in C^{2,1}(\Omega_T) \) and \( u_{\varepsilon,\delta} \) is the unique smooth solution to

\[
\begin{cases}
u_t - \mathcal{L}^{\varepsilon,\delta} u - H(Du) = f, & \text{in } Q_T, \\
u = g, & \text{on } \partial_p Q_T,
\end{cases}
\]

Theorems 4.2 and 4.4 and the comparison principle imply that the family of functions \( \{u_{\varepsilon,\delta}\} \) is equi-continuous and uniformly bounded. Therefore, up to a subsequence, \( u_{\varepsilon,\delta} \to u_\delta \) as \( \varepsilon \to 0 \) and \( u_\delta \) is the unique viscosity solution to (32) by the stability properties of viscosity solutions. Next by Theorems 4.2 and 4.6 we conclude that \( u_\delta \to u \) uniformly as \( \delta \to 0 \) and \( u \) is a viscosity solution to (4) with initial and boundary data \( g \) by the stability properties of viscosity solutions again.

The existence for a general continuous data \( g \) follows by approximating the data by smooth functions and using Theorems 4.2 and 4.6 and the stability properties of viscosity solutions again.

**Remark 8.** Theorems 1.1 and 1.3 are also valid for unbounded domain \( \Omega \). In fact, we can let \( \Omega_r = \Omega \cap B_r(0), \ Q_r = \Omega_r \times (0,T) \) and \( g_r : \partial_p Q_r \to R \) with

\[
g_r(x,t) = \begin{cases} 0, & \text{if } |x| = r, \\
\chi_r(x)g(x,t), & \text{if } (x,t) \in \partial_p Q_T \cap Q_r,
\end{cases}
\]

where \( \chi_r(x) = \chi(\frac{x}{r}) \) and \( \chi \in C^\infty_c(R^n) \) satisfies \( \chi(x) = 1 \) if \( |x| \leq \frac{1}{2} \), \( \chi(x) = 0 \) if \( |x| \geq 1 \). Then in \( Q_r \) there exists a unique viscosity solution with initial-Dirichlet data \( g_r \). From the assumptions on \( g_r \), the estimates for \( u_r \) and the stability property of the viscosity solutions we can obtain the result by letting \( r \to \infty \).

5. **Lipschitz regularity.** In this section, we first derive the interior gradient estimate for smooth solutions of the approximating equation

\[
u_t - \mathcal{L}^{\varepsilon,\delta} u - \langle \xi, Du \rangle = f,
\]

where \( \mathcal{L}^{\varepsilon,\delta} \) is as in (26). Then we use the compactness method to establish the interior Lipschitz estimate for viscosity solutions of equation (1). The idea comes from [12, 20], but the details are quite different due to the transport and inhomogeneous terms.
Theorem 5.1. Let \( \xi \in C^1(\bar{Q}_T, \mathbb{R}^n) \), \( f \in C^1(\bar{Q}_T) \). If \( u_{\varepsilon, \delta} \in C^1(\bar{Q}_T) \cap C^2(Q_T) \) is a smooth solution of the approximating equation (37) in \( Q_T \), then there exists a positive constant \( C \) depending only on \( n \), which is independent of \( \varepsilon \in (0, 1] \) and \( \delta \in (0, \frac{1}{\sqrt{n}}) \), such that the estimate

\[
|Du_{\varepsilon, \delta}(x, t)| \leq C(n)\|u_{\varepsilon, \delta}\|_\infty \left(1 + \|\xi\|_\infty + \|D\xi\|_\infty + \frac{1 + \|\xi\|_\infty}{(\text{dist}(x, t), \partial_p Q_T)^2}\right)
\]

+\( C(n) (\|f\|_\infty + \|Df\|_\infty) \quad (38) \)

holds for all \((x, t) \in Q_T\).

Proof. For simplicity we denote \( u_{\varepsilon, \delta} \) by \( u \). We construct an auxiliary function of the form

\[
w = \zeta v + \kappa u^2, \quad (39)\]

where \( v = \sqrt{|Du|^2 + \delta^2} \), \( \kappa > 0 \) to be determined, and \( \zeta \) is a smooth positive cut-off function that vanishes on \( \partial_p Q_T \). Let \((x_0, t_0)\) be a point where \( w \) attains its maximum in \( Q_T \). We first assume \((x_0, t_0) \notin \partial_p Q_T \). At the maximum point \((x_0, t_0)\), we have

\[
Dw = 0, \quad \text{and} \quad \sum_{i,j=1}^{n} a_{ij}^{\varepsilon, \delta}(Du)w_{ij} \leq 0, \quad (40)\]

where \( a_{ij}^{\varepsilon, \delta} \) is the operator defined in (27). The equality in (40) shows that

\[
Dv = -\frac{1}{\zeta}[vD\zeta + 2\kappa uDu] \quad (41)\]

at \((x_0, t_0)\). Since \((x_0, t_0) \notin \partial_p Q_T \), the denominator \( \zeta(x_0, t_0) \neq 0 \) so that the equality (41) makes sense at \((x_0, t_0)\). We next calculate the inequality in (40). Since the matrix \( \left( a_{ij}^{\varepsilon, \delta}(Du(x, t)) \right) \) is positive definite for all points \((x, t)\), by a direct calculation, we have

\[
0 \leq w_t - \sum_{i,j=1}^{n} a_{ij}^{\varepsilon, \delta}(Du)w_{ij}
\]

\[
= \zeta \left( v_t - \sum_{i,j=1}^{n} a_{ij}^{\varepsilon, \delta}(Du)v_{ij} \right) + v \left( \zeta_t - \sum_{i,j=1}^{n} a_{ij}^{\varepsilon, \delta}(Du)\zeta_{ij} \right)
\]

\[
+ 2\kappa u \left( u_t - \sum_{i,j=1}^{n} a_{ij}^{\varepsilon, \delta}(Du)u_{ij} \right) - 2 \sum_{i,j=1}^{n} a_{ij}^{\varepsilon, \delta}(Du)\zeta_{ij}v_i - 2\kappa \sum_{i,j=1}^{n} a_{ij}^{\varepsilon, \delta}(Du)u_iu_j
\]

\[
= \zeta \left( v_t - \sum_{i,j=1}^{n} a_{ij}^{\varepsilon, \delta}(Du)v_{ij} \right) + v \left( \zeta_t - \sum_{i,j=1}^{n} a_{ij}^{\varepsilon, \delta}(Du)\zeta_{ij} \right)
\]

\[
+ 2\kappa u (\langle \xi, Du \rangle + f) - 2 \sum_{i,j=1}^{n} a_{ij}^{\varepsilon, \delta}(Du)\zeta_{ij}v_i - 2\kappa |Du|^2 \left( \varepsilon + \frac{|Du|^2}{v^2} \right) \quad (42)\]
Therefore by a straightforward calculation we have
\[
v \left( \frac{\partial}{\partial x} - \sum_{i,j=1}^n a_{ij}^\varepsilon Du \partial \xi_{ij} \right) \leq v \left( |\xi_t| + (n\varepsilon + 1)|D^2\xi| \right)
\leq \frac{k}{4} \varepsilon^2 + \frac{1}{k} \left( |\xi_t| + (n+1)|D^2\xi| \right)^2.
\] (43)

By (41), we have
\[
-2 \sum_{i,j=1}^n a_{ij}^\varepsilon (Du) \xi_{ij} v_i
= -2 \varepsilon D\xi \cdot Dv - \frac{2}{v^2} (Du \cdot D\xi)(Du \cdot Dv)
\leq \frac{2\varepsilon}{\zeta} \left( |D\xi|^2 + 2\kappa u Du \cdot D\xi \right) + \frac{2}{\zeta v} (Du \cdot D\xi)^2 + \frac{4\kappa v\varepsilon Du^2}{v^2} Du \cdot D\xi
\leq \frac{2\varepsilon}{\zeta} (1 + \varepsilon)|D\xi|^2 + \frac{4\kappa v}{\zeta} (1 + \varepsilon) v|D\xi|
\leq \frac{4\varepsilon (1 + \varepsilon)}{\zeta} (|D\xi|^2 + \kappa^2 u^2),
\] (44)
in the last inequality we have used Young’s inequality and $0 < \varepsilon < 1$.

By direct calculation it is easy to show that $v_t = \frac{1}{v} \sum_{k=1}^n u_k u_{kk}$, $v_k = \frac{1}{v^3} \sum_{i=1}^n u_{i,k}$, and the second derivatives of $v$ are
\[
v_{ij} = \frac{1}{v^3} \sum_{l=1}^n u_{i,l} u_{j,l} + \frac{1}{v^2} \sum_{l=1}^n u_{i,l} u_{i,j} - \frac{1}{v^3} \sum_{l=1}^n (u_{i,l} u_{i,k}) \sum_{k=1}^n (u_{i,k} u_{k,j}).
\]
Therefore by a straightforward calculation we have
\[
v_t - \sum_{i,j=1}^n a_{ij}^\varepsilon (Du) u_{ij} = \frac{1}{v^3} \sum_{j=1}^n \sum_{i=1}^n (u_{i,j})^2 - \frac{1}{v^3} \sum_{i,k=1}^n u_{i,k} u_{i,k} + \frac{\varepsilon}{v^3} \sum_{j=1}^n u_{i,j}^2 + \frac{\varepsilon}{v^3} \sum_{k=1}^n \sum_{i=1}^n u_{i,k}^2 - \frac{\varepsilon}{v} \sum_{j=1}^n u_{i,j}^2 + \frac{\varepsilon}{v^3} \sum_{k=1}^n \sum_{i=1}^n u_{i,k}^2 - \frac{\varepsilon}{v^3} \sum_{i=1}^n u_{i,k}^2 - \frac{\varepsilon}{v^3} \sum_{i=1}^n u_{i,k}^2.
\] (45)

We differentiate the equation (37) with respect to $x_k$ to obtain
\[
(u_{k,t}) - D_k(a_{ij}^\varepsilon (Du)) u_{ij} - a_{ij}^\varepsilon (Du)(u_{k})_{ij} = \xi_{i,k} u_i + \xi_{i,k} u_i + f_k,
\] (46)
where $\xi_{i,k} = \frac{\partial \xi_i}{\partial x_k}$. Then we multiply $u_{k}/v$ in (46) and sum with respect to $k$ from 1 to $n$ to get
\[
\sum_{k=1}^n \frac{u_{k}}{v} (u_{k,t}) - \sum_{i,j,k=1}^n \frac{u_{k}}{v} D_k(a_{ij}^\varepsilon (Du)) u_{ij} - \sum_{i,j,k=1}^n \frac{u_{k}}{v} a_{ij}^\varepsilon (Du) (u_{k})_{ij}
= \sum_{i,k=1}^n \xi_i \frac{u_{i,k} u_k}{v} + \sum_{i,k=1}^n \xi_i \frac{u_{i,k} u_k}{v} + \sum_{k=1}^n \frac{u_{k}}{v} f_k.
\]
Then we obtain
\[ v_t - \sum_{i,j,k=1}^{n} \frac{u_k}{v} D_k(a_{ij}^{\varepsilon,\delta}(Du))u_{ij} - \sum_{i,j,k=1}^{n} a_{ij}^{\varepsilon,\delta}(Du) \left( v_{ij} - \frac{u_{ki}u_{kj}}{v} + \frac{(u_ku_{ki})(u_iu_{ij})}{v^3} \right) \]
\[ = \sum_{i=1}^{n} \xi_i v_i + \sum_{i,k=1}^{n} \xi_{i,k} u_k u_{ki} + \sum_{k=1}^{n} \frac{u_k}{v} f_k. \]

Noting that
\[ D_k(a_{ij}^{\varepsilon,\delta}(Du)) = D_k \left( \varepsilon \delta_{ij} + \frac{u_iu_{ij}}{v^2} \right) = \frac{2}{v^2} u_i u_{kj} - \frac{2}{v^3} u_i u_j v_k, \]
then we obtain
\[ v_t - \sum_{i,j=1}^{n} a_{ij}^{\varepsilon,\delta}(Du)v_{ij} \]
\[ = \frac{\varepsilon}{v^3} \sum_{i=1}^{n} \left( \sum_{k=1}^{n} u_k u_{ki} \right)^2 - \frac{\varepsilon}{v^3} \sum_{k,i=1}^{n} (u_{ki})^2 + \frac{1}{v^3} \sum_{k=1}^{n} \left( \sum_{i=1}^{n} u_i u_{ki} \right)^2 \]
\[ - \frac{1}{v^3} \sum_{i=1}^{n} \left( \sum_{k=1}^{n} u_i u_{ki} \right)^2 + \sum_{k=1}^{n} \xi_k v_k + \sum_{i,k=1}^{n} \xi_{i,k} u_i u_{ki} + \frac{1}{v} \sum_{k=1}^{n} u_k f_k \] (47)
\[ \leq \frac{1 + \varepsilon}{v^3} \sum_{i=1}^{n} \left( \sum_{k=1}^{n} u_i u_{ki} \right)^2 + \xi \cdot Dv + \frac{1}{v} \langle D\xi Du, Du \rangle + \frac{Du \cdot Df}{v} \]
\[ \leq (1 + \varepsilon) \frac{|Dv|^2}{v} + ||\xi||_{\infty}|Dv| + ||D\xi||_{\infty}|v| + |Df|. \]

By (41), we have
\[ (1 + \varepsilon) \frac{|Dv|^2}{v} + ||\xi||_{\infty}|Dv| \]
\[ = \frac{(1 + \varepsilon)}{v \zeta^2} \left( |vD\zeta| + 2\kappa|Du|^2 \right) + \frac{|\xi||Dv|}{\zeta} + 2\kappa|u||Du| \]
\[ \leq \frac{2(1 + \varepsilon)}{v \zeta^2} \left( |v|^2 |D\zeta|^2 + 4\kappa^2 u^2 |Du|^2 \right) + \frac{|\xi||Dv|}{\zeta} (v|D\zeta| + 2\kappa|u||Du|) \] (48)
\[ \leq \frac{2(1 + \varepsilon)}{\zeta^2} \left( |D\zeta|^2 + 4\kappa^2 u^2 \right) + \frac{|\xi||Dv|}{\zeta} (|D\zeta| + 2\kappa|u|). \]

Substituting (48) into (47), then we obtain
\[ \zeta \left( v_t - \sum_{i,j=1}^{n} a_{ij}^{\varepsilon,\delta}(Du)v_{ij} \right) \leq \frac{2(1 + \varepsilon)}{\zeta} \left( |D\zeta|^2 + 4\kappa^2 u^2 \right) \]
\[ + ||\xi||_{\infty}|v| (|D\zeta| + 2\kappa|u|) + \zeta ||D\xi||_{\infty}|v| + \zeta |Df|. \] (49)

Substituting (43), (44) and (49) into (42), we get
\[ 2\kappa|Du|^2 \left( \varepsilon + \frac{|Du|^2}{v^2} \right) \]
\[ \leq \frac{4v(1 + \varepsilon)}{\zeta} \left( |D\zeta|^2 + \kappa^2 u^2 \right) \]
\[ + \frac{\kappa}{4} v^2 + \frac{1}{\kappa} \left( |\zeta_t| + (n + 1)|D^2 \zeta| \right) \]
\[ + \frac{2(1 + \varepsilon)}{\zeta} \left( |D\zeta|^2 + 4\kappa^2 u^2 \right) + ||\xi||_{\infty}|v| (|D\zeta| + 2\kappa|u|) + \zeta ||D\xi||_{\infty}|v| + \zeta |Df| \]
\[
\leq \frac{12\kappa(1 + \varepsilon)}{\zeta} (|D\zeta|^2 + \kappa^2 u^2) + \frac{\kappa}{4} u^2 + \frac{1}{\kappa} (|\zeta| + (n + 1)|D^2\zeta|)^2 \\
+ 2\kappa|u| (\|\xi\|_{\infty} + \|f\|_{\infty}) + 2|\xi||\|v\|_{\infty} (|D\zeta| + \kappa|u|) + \zeta|D\xi||_{\infty} v + \zeta|Df|
\]
\[
\leq \frac{3}{4} \kappa v^2 + \frac{1152}{\kappa^2} (|D\zeta|^2 + \kappa^2 u^2)^2 + \frac{1}{\kappa} (|\zeta| + (n + 1)|D^2\zeta|)^2 + 8\kappa u^2 \|\xi\|_{\infty} \\
+ 2\kappa|u||f||_{\infty} + \frac{8|\xi|\|v\|_{\infty}}{\kappa} (|D\zeta| + \kappa|u|)^2 + \frac{1}{\kappa} \|D\xi||_{\infty}^2 + \zeta|Df| \tag{50}
\]

at \((x_0, t_0)\).

Now we consider the first case if \(|Du(x_0, t_0)| \geq 1\). Then for \(\delta \in (0, \frac{1}{\sqrt{3}}]\), we have
\[
2\kappa|Du|^2 \left( \frac{\varepsilon + |Du|^2}{v^2} \right) = 2\kappa v^2 \frac{|Du|^2}{v^2} \left( \frac{\varepsilon + |Du|^2}{v^2} \right) \geq 2\kappa v^2 \left( \frac{|Du|^2}{v^2} \right)^2 \geq \frac{9}{8} \kappa v^2.
\]

Combining with (50) and (51), we get
\[
\frac{9}{8} \kappa v^2 \leq \frac{3}{4} \kappa v^2 + \frac{1152}{\kappa^2} (|D\zeta|^2 + \kappa^2 u^2)^2 + \frac{1}{\kappa} (|\zeta| + (n + 1)|D^2\zeta|)^2 + 8\kappa u^2 \|\xi\|_{\infty} \\
+ 2\kappa|u||f||_{\infty} + \frac{8|\xi|\|v\|_{\infty}}{\kappa} (|D\zeta| + \kappa|u|)^2 + \frac{1}{\kappa} \|D\xi||_{\infty}^2 + \zeta|Df|.
\]

That is
\[
(v\zeta)^2 \leq C(n) \zeta^2 \left( u^2 \|\xi\|_{\infty} + \|u||f||_{\infty} \right) + \frac{\varepsilon^3}{\kappa} |Df| \\
+ \frac{C(n)}{\kappa^2} \left( (|D\zeta|^2 + \kappa^2 u^2)^2 + (|\zeta| + |D^2\zeta|)^2 + \|\xi||_{\infty}^2 (|D\zeta| + \kappa|u|)^2 + \zeta^2 \|D\xi||_{\infty}^2 \right) \tag{52}
\]
at the point \((x_0, t_0)\). We fix a point \((x, t) \in Q_T\) and choose the cut-off function \(\zeta\) such that
\[
\zeta(x, t) = 1,
\]
\[
\max\{|D\zeta||_{\infty}, |\xi||_{\infty}\} \leq \frac{1}{\text{dist}((x, t), \partial_p Q_T)}
\]
and
\[
\|D^2\zeta\|_{\infty} \leq \frac{1}{(\text{dist}((x, t), \partial_p Q_T))^2}.
\]

Because \(w\) attains its maximum at \((x_0, t_0) \in Q_T\), we have from (52)
\[
\begin{align*}
|Du(x, t)| & \leq w(x, t) \\
& \leq w(x_0, t_0) + (\zeta v)(x_0, t_0) + \kappa u^2(x_0, t_0) \\
& \leq C(n) \|u\|_{\infty} \left( 1 + \|\xi||_{\infty} + \frac{1}{\text{dist}((x, t), \partial_p Q_T)^2} + \frac{1}{\text{dist}((x, t), \partial_p Q_T)} + \|D\zeta||_{\infty} \right) \\
& + C(n) \|f||_{\infty} + |Df||_{\infty},
\end{align*}
\]

where we have chosen \(\kappa = \frac{1}{\|u\|_{\infty}}\) and the constant \(C(n)\) depends only on the dimension \(n\). Now we consider two cases. First if \(\text{dist}((x, t), \partial_p Q_T) \leq 1\), then by (53) we
have
\[ |Du(x,t)| \leq C(n)\|u\|_\infty \left(1 + \|\xi\|_\infty + \|D\xi\|_\infty + \frac{2(1 + \|\xi\|_\infty)}{(\text{dist}((x,t), \partial_p Q_T))^\gamma} \right) \]
\[ + C(n) (\|f\|_\infty + \|Df\|_\infty) \]
\[ \leq C(n)\|u\|_\infty \left(1 + \|\xi\|_\infty + \|D\xi\|_\infty + \frac{1 + \|\xi\|_\infty}{(\text{dist}((x,t), \partial_p Q_T))^\gamma} \right) \]
\[ + C(n) (\|f\|_\infty + \|Df\|_\infty), \]

where for simplicity \(C(n)\) denotes a generic constant. While if \(\text{dist}((x,t), \partial_p Q_T) \geq 1\), then we have
\[ |Du(x,t)| \leq C(n)\|u\|_\infty \left(1 + \|\xi\|_\infty + \|D\xi\|_\infty + \frac{1 + \|\xi\|_\infty}{\gamma} \right) \]
\[ + C(n) (\|f\|_\infty + \|Df\|_\infty) \]

On the other hand, if \(|Du(x_0,t_0)| < 1\), since \(\|\xi\|_\infty = 1\) and \(\delta \in (0, \frac{1}{\sqrt{3}}]\), then we have
\[ |Du(x,t)| \leq w(x,t) \leq w(x_0,t_0) = (\zeta v)(x_0,t_0) + \kappa u^2(x_0,t_0) \leq \frac{2}{\sqrt{3}} + \|u\|_\infty. \]

Now, we consider the case \((x_0,t_0) \in \partial_p Q_T\). That is the maximum point \((x_0,t_0)\) of \(w\) over \(Q_T\) is attained on the parabolic boundary. Since \(\zeta = 0\) on \(\partial_p Q_T\), we have
\[ |Du(x,t)| \leq w(x,t) \leq w(x_0,t_0) = \kappa u^2(x_0,t_0) \leq \|u\|_\infty. \]

Now we have finished the proof of the interior gradient estimate (38) for a smooth solution. \(\square\)

**Remark 9.** It is obvious that the estimate (38) is valid for almost every \((x,t) \in Q_T\) when \(f\) and \(\xi\) are Lipschitz continuous in a bounded domain.

Noting that the interior gradient estimate (38) in Theorem 5.1 does not depend on \(\varepsilon\) and \(\delta\), it provides us a uniform estimate. Based on the compactness argument and the stability properties of the viscosity solutions, we obtain the Theorem 1.5.

**Proof of Theorem 1.5.** For every \((\overline{x}, \overline{t}) \in Q_T\), we choose a ball of radius \(r > 0\) such that \(B_r(\overline{x}, \overline{t}) \subset Q_T\). Let \(u_{\varepsilon, \delta}\) satisfy
\[
\begin{cases}
(u_{\varepsilon, \delta})_t - \mathcal{L}^{\varepsilon, \delta} u_{\varepsilon, \delta} - \langle \xi, Du_{\varepsilon, \delta} \rangle = f, \text{ in } B_r(\overline{x}, \overline{t}), \\
u_{\varepsilon, \delta}(x,t) = u(x,t), \text{ on } \partial_p B_r(\overline{x}, \overline{t}),
\end{cases}
\]
where \(\varepsilon \in (0, 1]\) and \(\delta \in (0, \frac{1}{\sqrt{3}}]\). Then by Theorem 5.1 and the comparison principle, we have
\[ |Du_{\varepsilon, \delta}(x,t)| \leq C(n)\|u\|_\infty \left(1 + \|\xi\|_\infty + \|D\xi\|_\infty + \frac{1 + \|\xi\|_\infty}{\gamma} \right) \]
\[ + C(n) (\|f\|_\infty + \|Df\|_\infty) \]
for any \((x,t) \in B_r(\overline{x}, \overline{t})\) with the constant \(C(n)\) independent of \(\varepsilon\) and \(\delta\). By Ascoli-Arzelà compactness theorem, we get that the functions \(u_{\varepsilon, \delta}\) converge locally uniformly as \(\varepsilon \to 0\) and \(\delta \to 0\) to a locally Lipschitz continuous function \(\hat{u}\). Then by
Rademacher’s Theorem, we have the gradient $D\tilde{u}(x,t)$ exists almost everywhere. Then it is clear that

$$|D\tilde{u}(x,t)| \leq C(n)\|u\|_\infty \left(1 + \|\xi\|_\infty + \|D\xi\|_\infty + \frac{1 + \|\xi\|_\infty}{\min_{(x,t) \in B_r(\bar{x},\bar{t})} (\text{dist}(\bar{x},\bar{t}), \partial_p Q_T)^2} \right)$$

$$+ C(n) (\|f\|_\infty + \|DF\|_\infty)$$

$$= C(n)\|u\|_\infty \left(1 + \|\xi\|_\infty + \|D\xi\|_\infty + \frac{1 + \|\xi\|_\infty}{\min_{(x,t) \in B_r(\bar{x},\bar{t})} (\text{dist}(\bar{x},\bar{t}), \partial_p Q_T)^2} \right)$$

for almost every $(x,t) \in B_r(\bar{x},\bar{t})$. Due to the stability properties of the viscosity solutions [11], we obtain that $\tilde{u}$ satisfies

$$\begin{cases}
\tilde{u}_t - \Delta^\infty \tilde{u} - \langle \xi, D\tilde{u} \rangle = f, & \text{in } B_r(\bar{x},\bar{t}), \\
\tilde{u}(x,t) = u(x,t), & \text{on } \partial_p B_r(\bar{x},\bar{t})
\end{cases}$$

in the viscosity sense. By the comparison principle Theorem 3.1, we have $\tilde{u} = u$ in $B_r(\bar{x},\bar{t})$. Hence, we have

$$|Du(x,t)| \leq C(n)\|u\|_\infty \left(1 + \|\xi\|_\infty + \|D\xi\|_\infty + \frac{1 + \|\xi\|_\infty}{\min_{(x,t) \in B_r(\bar{x},\bar{t})} (\text{dist}(\bar{x},\bar{t}), \partial_p Q_T)^2} \right)$$

$$+ C(n) (\|f\|_\infty + \|DF\|_\infty)$$

for a.e. $(x,t) \in B_r(\bar{x},\bar{t})$. Then by Lebesgue-Besicovitch Differentiation Theorem, we obtain

$$|Du(\bar{x},\bar{t})| \leq \lim_{r \to 0} \frac{1}{|B_r(\bar{x},\bar{t})|} \int_{B_r(\bar{x},\bar{t})} |Du(x,t)| dx dt$$

$$\leq C(n)\|u\|_\infty \left(1 + \|\xi\|_\infty + \|D\xi\|_\infty + \frac{1 + \|\xi\|_\infty}{\min_{(x,t) \in B_r(\bar{x},\bar{t})} (\text{dist}(\bar{x},\bar{t}), \partial_p Q_T)^2} \right)$$

$$+ C(n) (\|f\|_\infty + \|DF\|_\infty)$$

$$\leq C(n)\|u\|_\infty \left(1 + \|\xi\|_\infty + \|D\xi\|_\infty + \frac{1 + \|\xi\|_\infty}{(\text{dist}(\bar{x},\bar{t}), \partial_p Q_T)^2} \right)$$

$$+ C(n) (\|f\|_\infty + \|DF\|_\infty)$$

for almost every point $(\bar{x},\bar{t}) \in Q_T$. The Lipschitz estimate asserted above follows.

\[\square\]

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