Multi-mode correlations and the entropy of turbulence in shell models

Gregory Falkovich\textsuperscript{1,2}, Yotam Kadish\textsuperscript{1} and Natalia Vladimirova\textsuperscript{2,3}

\textsuperscript{1}Weizmann Institute of Science, Rehovot 76100 Israel
\textsuperscript{2}Landau Institute for Theoretical Physics, 142432 Chernogolovka, Russia
\textsuperscript{3}Brown University, Providence, RI 02912, USA

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We suggest a new focus for turbulence studies — multi-mode correlations — which reveal the hitherto hidden nature of turbulent state. We apply this approach to shell models describing basic properties of turbulence. The family of such models allows one to study turbulence close to thermal equilibrium, which happens when the interaction time weakly depends on the mode number. As the number of modes increases, the one-mode statistics approaches Gaussian (like in weak turbulence), the occupation numbers grow, while the three-mode cumulant describing the energy flux stays constant. Yet we find that higher multi-mode cumulants grow with the order. We derive analytically and confirm numerically the scaling law of such growth. The sum of all squared dimensionless cumulants is equal to the relative entropy between the full multi-mode distribution and the Gaussian approximation of independent modes; we argue that the relative entropy could grow as the logarithm of the number of modes, similar to the entanglement entropy in critical phenomena. Therefore, the multi-mode correlations give the new way to characterize turbulence states and possibly divide them into universality classes.

INTRODUCTION

We define turbulence as a state where many degrees of freedom are deviated from thermal equilibrium. This usually happens when wavenumbers or frequencies of the modes excited are vastly different from those of the modes that dissipate. A statistically steady state is then a long cascade of excitations.

Fluctuations in a wide interval of scales and times is the property shared by turbulence and critical phenomena. That analogy was discussed either in terms of power-law behavior of correlation functions \cite{1,2} or in terms of the probability distribution of macroscopic quantities \cite{3}. However, the analogy was not advanced to reach deeper understanding of turbulence, in particular, reproduce the spectacular success of statistical physics in defining the universality classes of critical phenomena.

We suggest to focus on the entropic characteristics of turbulence, following the approach developed for critical phenomena by Wilczek, Kitaev, Cardy and others \cite{4–6}. While the entropy itself is usually linear in the number of degrees of freedom, the quantum entanglement entropy or classical mutual information between different parts of the system or between a system and its environment could be logarithmic at criticality. The factor in front of the logarithm is the central charge of the conformal field theory, which determines the universality class of the transition \cite{4}. That approach to critical phenomena brought most progress in two dimensions, here we shall apply it to one-dimensional model of turbulence.

We believe that an entropic approach is also natural for turbulence, which must have much lower entropy than thermal equilibrium (at the same energy). How does the entropy deficit depend on the number of modes deviated from equilibrium or, in other terms, on the Reynolds number? Since the entropy deficit is information, where it is encoded?

Two natural possibilities exist here: correlations between points in space or between modes in Fourier space. The multi-point spatial correlations were first discovered in the simplest case of non-equilibrium imposed by a spatial gradient (of temperature, velocity, etc). So-called Dorfman-Cohen anomalies manifest themselves as infrared divergencies in the density expansion of kinetic coefficients (thermal conductivity, viscosity, diffusivity) \cite{7–9}. However, turbulence deviates from equilibrium by forcing some modes and dissipating other, imposing different conditions in Fourier space rather than in real space. Therefore, the cascade nature of turbulence must manifest itself in inter-mode correlations.

Since the very existence of a cascade hinges on interaction between modes, the traditional approach is usually focused on the lowest two nonzero moments: the occupation numbers and the flux \cite{10–15}, which is typically a three- or four-mode correlation function. Yet it was argued recently from an entropic analysis that even for a weak wave turbulence two moments are not sufficient to describe the statistics of a multi-mode system, which could be far from Gaussian \cite{16}.

We suggest that the nature of a turbulent state is reflected in multi-mode correlations which can be quantified by the mutual information between upscale and downside parts of the cascade or between all modes. We hope that the mutual information can play here the role that the entanglement entropy plays in critical phenomena. For the formidable task of analyzing the multi-mode correlations, it is natural to start from the class of systems where they are weak and can be treated perturbatively. We find one such class where occupation numbers are close to thermal equipartition, the single-mode statistics is close to Gaussian, and the (dimensionless) flux is small. We demonstrate that nonzero multi-mode dimen-
sionless cumulants can be all of the same order in such systems. When this is the case, the mutual information must grow as a logarithm of the total number of modes in the cascade, thus establishing a direct quantitative link with classical and quantum critical phenomena.

Developed turbulence with the same scaling as in equilibrium occurs in many systems. One class is both direct and inverse cascades in the universal Nonlinear Schrödinger model in two dimensions [12], which describes cold atoms and all spectrally narrow wave spectra. Another example is a joint turbulence of interacting high and low-frequency waves abundant in geophysics and plasma physics [11, 17].

DEFINITION OF THE MODELS

We consider a very wide class of systems with strong interaction, described by the equations with quadratic nonlinearity,

\[ i \dot{a}_j = \sum_{sq} (2V_{js}^2 a_s^* a_q + V_{sq}^2 a_s a_q), \]

conserving a quadratic integral of motion, \( E = \sum \omega_j |a_j|^2 \), as long as \( V_{sq} \neq 0 \) if \( \omega_s + \omega_q = \omega_j \). Such are the Euler equations for incompressible fluid flows and for solid body rotations, as well as the family of two-dimensional hydrodynamic models from geophysics, astrophysics and plasma physics, where a scalar field \( \alpha \) (vorticity, temperature, potential) is transported by the velocity whose stream function \( \psi \) is exactly Gaussian statistics in thermal equilibrium \([20, 21]\) due to conservation of \( E \). The right hand side is the discrete divergence of the flux \( \Pi_j \approx \sum_{s,q} V_{js}^2 \omega_j (C_\beta(j, s, q)) \), where \( C_\beta(j, s, q) = \Im \langle a_j^2 a_s a_q \rangle \) and the brackets \( \langle \cdot \rangle \) denote time averaging. We also denote by \( C_m \) with \( m = 3, 4, \ldots, N \) generic multiplications of \( m \) modes, for which the averages are the correlation functions with no reducible parts (zero for Gaussian statistics). For scale-invariant systems, stationarity requires the flux independence on \( j \), which sets the scaling \( a_j^3 \propto \Pi \omega_j^{1-\alpha} \). Strong turbulence of resonantly interacting waves is different from weak turbulence determined by quasi-resonances, where \( a_j^4 \propto \Pi \) \([11]\).

The rest of the article is devoted to the case of \( \alpha = 1/2 \), as it allows us to explore the difference between statistics of turbulence and of thermal equilibrium. One might expect a minimal difference, since the scaling of turbulence and equipartition coincide, \( a_j \propto \omega_j^{(1+\alpha)/3} = \omega_j^{-1/2} \). Yet, turbulence carries the flux from excess to scarcity, so the decrease of \( n_q \) towards the damping must be logarithmic \([11]\). For a direct cascade, we denote \( \Lambda_q \equiv \log(\omega_j/\omega_k) \) and assume \( \omega_j n_k = \Lambda_k^2 \), with \( \xi \) to be determined. Considering \( j-s \approx j-q \ll j \) we expand \( n_s, n_q \) around \( n_j \) (see the Supplement for the details), which gives: \( \langle C_4(j) \rangle \approx 2V_{sq}^2 \omega_j^2 n_q^2 / \Lambda_j \propto \Lambda_q^{2-1} \) for \( \Lambda_j \gg 1 \). Stationarity of \( C_4 \) gives respectively \( \langle C_5 \rangle \approx V n \langle C_4 \rangle \), \( \langle C_6 \rangle \approx V n \langle C_4 \rangle \approx V n^3 / \Lambda, \) and generally \( \langle C_m \rangle \) \( \approx V n^{m/2} / \Lambda \propto \omega_j^{-m/2} \Lambda^{m/2-1} \). The flux law requires that \( C_3 \) has no logarithm, which gives \( \xi = 2/3 \). We thus see that the dimensionless cumulants of all orders are suppressed by the same factor: \( \langle C_m \rangle n_j^{-m/2} \approx \log^{-1}(\omega_j/\omega_j) \). Cumulants are of order unity at \( j \approx d \) \([21]\). As \( d \approx N \) increases, the cumulants at \( j \ll d \) are getting uniformly smaller but their number increases, since the cumulants of orders up to \( m \approx d - j \) involve the mode \( j \).
COMPUTATION OF CUMULANTS

We now present a direct computation for two models, defined respectively by the Hamiltonians [20, 21]:

\[ \mathcal{H}_2 = \sum_j 2^{i\phi_j} (b_j^+ a_{j+1}^- + a_j^+ b_{j+1}^-), \]

\[ \mathcal{H}_3 = \sum_j F_j^0 \left( a_j^+ a_{j+1}^+ a_{j+2} + a_j a_{j+1} a_{j+2}^+ \right). \]

The first (doubling) model describes a chain of modes resonantly interacting with their second harmonics. The second (Fibonacci) model describes a chain of resonantly interacting triplets: here \( F_j \) are the Fibonacci numbers, defined by \( F_j + F_{j+1} = F_{j+2} \). The Hamiltonians are invariant with respect to the gauge transformation with \( \omega_j = 2^j \) and \( \omega_j = F_j \) respectively. Since \( F_j = \lfloor \phi^j - (-\phi)^{-j} \rfloor / \sqrt{5} \), then at \( j \gg 1 \), the wave frequency also depends exponentially on the mode number, \( F_j \propto \phi^j \), where \( \phi = (1 + \sqrt{5})/2 \) is the golden mean. Both (4,5) are so-called shell models defined in logarithmically discretized Fourier space [20–23].

To study turbulence, we pass to \( b_j = a_j \omega^{(1+\alpha)/3} \), which makes the triple moment \( j \)-independent in the transparency window. We also renormalize time to bring the dynamical equations to the universal form (see numerical facts in the Supplement):

\[ i\partial_t b_j = 2^{i\phi} \left( 2^{i\phi} b_j^+ b_{j+1}^+ + b_j^- b_{j+1}^- \right), \]

\[ i\partial_t b_j = \phi^{i\phi} \left[ b_{j-1} b_{j+2}^+ + b_{j-2}^+ b_{j+1}^- + \phi^{-1} b_j^+ b_{j+2}^- \right]. \]

Remarkably, both equations have an exact stationary solution \( b_j = \eta_j \), \( \forall j \). It corresponds to the turbulent scaling \( a_j \propto \omega^{-\alpha(1+\alpha)/3} \). The solution is linearly unstable and cannot be matched with pumping and damping, yet we find that turbulence fluctuates around it. The prefactors \( \omega^{i(2\alpha-1)/3} = 2^{i(2\alpha-1)/3}, \phi^{i(2\alpha-1)/3} \) determine the dependence of the interaction time on the mode number \( j \). For \( \alpha = 1/2 \), the equations (6,7) are translation invariant along \( j \), the interaction time is the same for all modes, and scaling laws of turbulence and equilibrium coincide. Both direct and inverse cascades exist in this case [20, 21].

Analytic derivations of cumulants for (4) are presented here and for (5) in the Supplement. Consider the products \( C_{0,\infty}(n) = b_{j-k} b_{j-2} b_{j-3} \ldots b_j^* \), having non-zero mean values when gauge invariant: \( 2^{-k} + 2^{-l} - 2^{-m} + \ldots - 1 = 0 \). Every such product of order \( m \) can be obtained from that of \( m - 1 \) replacing \( b_j \) by either \( b_j^* \) or \( b_{j-1} b_{j+1} \). Denote \( A_m(n) = \langle \{ b_j^m \} \rangle \). Stationarity of \( A_2 \) similar to (2) gives \( \langle C_4 \rangle = \langle C_{1100} \rangle = 1 \). Next,

\[ d(b_j^+ b_{j+1}^+) / dt = 2 \langle \{ b_j b_{j+1} \}^2 \rangle - \langle \{ b_j^4 \} + 2 \langle C_{2210} (j + 1) \rangle - \langle C_{2210} (j + 2) \rangle = 2n_j (n_{j+1} - n_j) + \langle \{ C_4 \} \rangle = 0. \]

where we denoted the combination of the fourth-order cumulants: \( \langle \{ C_4 \} \rangle = \langle C_{2210} \rangle + \langle \{ 2 | b_j b_{j+1} |^2 - | b_j |^4 \} \rangle \). Double brackets denote subtraction of the reducible parts. Assuming self-consistently that the statistics is close to Gaussian and \( n_j \propto \log^2 \langle 2^{-d} \rangle \| j - d \| \), we obtain \( \langle \{ C_4 \} \rangle \approx 2n_j (n_{j+1} - n_{j+1}) \approx 2 \xi n_j^2 / \| j - d \|. \) Numerics described below confirm that. Next, stationarity of \( \langle \{ b_j^4 \} \rangle \) gives \( \langle \{ b_j^2 C_3(j) \} \rangle = \langle \{ b_j^2 C_3(j + 1) \} \rangle \).

Since the reducible contributions are respectively 3 and 2 times \( n_j \), then the difference of the cumulants is \( \langle \{ b_j^2 C_3(j) - C_3(j + 1) \} \rangle = A_2(j)(C_4) = n_j \), that is the fifth-order cumulants are comparable to \( n_j \). Numerics give \( \langle | b_j^2 C_3(j) \rangle \rangle = \langle | b_j^2 C_3(j + 1) \rangle \rangle \approx 2.25 n_j \). Stationarity of \( \langle \{ b_j b_{j-k} \} \rangle \) by induction over \( k \) imposes the identities: \( \langle | b_j^2 C_3(j + k) \rangle \rangle = 0 \), the cumulants are negligible for \( k > 1 \) when there is no overlap (see Supplement). Next orders give \( \langle C_6 \rangle \approx A_2 \langle \{ C_4 \} \rangle \approx A_2 A_4 | j - d \| \sim A_6 | j - d \| \) and \( \langle C_{2m} \rangle \approx A_2 | j - d \| \) supporting the above estimates in the general case. Since \( A_6 \propto n_j^{-m}/2 \), then \( \langle C_{2m} \rangle \rangle = n_j^{-m}/2 | j - d | \propto | j - d |^{-m}/2 \). Applying it to \( \langle C_3 \rangle = 1 \) we obtain \( \xi = 2/3 \), that is \( n_j \propto | j - d |^{-2/3}, A_m(j) \approx n_j^{-m}/2 \propto | j - d |^{-m}/2 \) and \( \langle C_{2m} \rangle \rangle \approx A_m(j) | j - d | \propto | j - d |^{-m}/2 \). We see that cumulants with \( m > 3 \) actually grow with increasing the length of the cascade. The dimensionless cumulants \( D_m = \langle C_{2m} \rangle / \langle C_{2m} \rangle \rangle \) all decay by the same law:

\[ D_{kIS...}(j) = \frac{\langle \{ b_{j-k} b_{j-l} \} \langle b_{j-k} b_{j-l} \} \rangle}{\langle \{ b_{j-k} b_{j-l} \} \langle b_{j-k} b_{j-l} \} \rangle^{1/2}} \approx \frac{1}{| j - d |}. \]

NUMERICAL RESULTS

To check the assumptions in our derivations, we solved numerically (6,7) for \( \alpha = 1/2 \) and both direct and inverse cascades. Exponentially large \( \omega \) do not appear for the variables \( b_i \), which allows us to enlarge \( N \) without shortening the time step. Compared with [20–22], this novel approach gives much larger transparency window, up to a record \( N = 200 \). Figure 1 shows that the single-mode probability distribution is close to Gaussian for all modes in the transparency window, see also [20, 21].

It also shows the dependence of the single-mode moments \( A_m(j) \) on \( j \) counted from dissipation. The occupation numbers are of order unity at \( j \approx d \); away from dissipation, the moments are indeed Gaussian, that is the renormalized moments \( A_m(j) A_2^{-m/2}(j)/\Gamma(1 + m/2) \) are all equal to unity, independent of \( j \). That means that the single-mode distribution is scale invariant, the scaling is indeed \( A_m(j) \propto | j - d |^{-m}/2 \). The same is true for the doubling model.

Both large transparency window and massive statistics are crucial for the first-ever computation of high-order cumulants in turbulence presented in Figure 2. Due to symmetry \( b_j \rightarrow -b_j^* \), the cumulants with even \( m \) are real and those with the odd \( m \) are imaginary (they are
non-zero due the breakdown of time reversibility in turbulence). The cumulants of $b' j = ib j$ are all real and proportional to the flux, that is have opposite signs for direct and inverse cascades (see also Supplement). Figure 2 shows dimensionless cumulants multiplied by $|j - d|$. The data confirm the scaling (9) for all pure cumulants, like $⟨⟨b_j^3⟩⟩$, and those comparable to the reducible part, like $⟨|b_j|^2 C_3(j)⟩$. The few cumulants due to amplitude-only correlations, like $⟨⟨|b_j b_{j-1}|^2⟩⟩$, are small differences of large values; the data are irregular, don’t contribute the mutual information, and left out of the consideration.

Numerics give the squared flux $⟨C_j^2⟩ = ⟨⟨b_j^2 b'_j⟩⟩^2 ≈ 2.44n^3/|j - d|$ for (4). Two things are noteworthy. First, $⟨C_j^2⟩$ grows strongly, its variance grows unbounded with $N$, while its mean stays constant. The same is true for (5), and reproduces the experimental and numerical data on direct and inverse cascades in fluid turbulence [24]. Second, for both models, we find no systematic decay of cumulants with $m$. We cannot presently prove this for $m → ∞$, nor we are able to classify all gauge-invariant cumulants of an arbitrary order. We managed it up to the seventh order: (4) gives 26 distinct types of cumulants and (5) gives 70 for $m = 7$ (see the tables in the Supplement). For example, cumulants of the type $⟨⟨C_3 C_4⟩⟩$ are among the largest and are comparable to $⟨⟨C_3⟩⟩^2 = C_4$ — another sign of no decay with $m$ (see also the Table).

To summarize the findings: the dimensionless cumulants are uniformly small far from dissipation: $D_m(j) \approx |j - d|^{-1}$. In the thermodynamic limit $d = N → ∞$ (direct cascade), all dimensionless cumulants tend to zero and we have an asymptotic equipartition with the temperature $T = N^{2/3}$ for all finite $j$. Yet the number of cumulants grows with $N$, so it is not clear if the full multimode statistics approaches Gaussian in the limit.

ENTROPIC CONSIDERATION

Let us argue that the growth of cumulants makes the full multi-mode probability distribution $P\{b_j\}$ very different from the Gaussian distribution of independent modes with the same occupation numbers, $P_G\{b_j\} = Π_j n_j^{-1} exp (−|b_j|^2/n_j)$. Since the dimensionless cumulants are small, the probability distribution can be approximated as follows:

$$P\{b_j\} = Z^{-1} exp \left[ -\sum_{j=1}^N α_j |b_j|^2 + B \right],$$  \hspace{1cm} (10)

$$Z \approx (1 + ⟨B^2⟩_0)/2 Π_j α_j^{-1}. \hspace{1cm} (11)$$

The average $⟨\ldots⟩_0$ is with $P_G$, and $B$ is the sum of all products giving nonzero cumulants, symbolically

$$B = \sum_m C_m(C_m^*) + C_m^*(C_m) / ⟨|C_m|^2⟩. \hspace{1cm} (12)$$

Define $S_N = ⟨ln 1/P⟩$ as the total entropy. Relative entropy, $D(P/P_G) = ⟨ln(P/P_G)⟩$ measures the difference between $P$ and $P_G$. Since the latter is a product, $P_G = Π_j exp (−α_j |b_j|^2)$, then $D = \sum_{j=1}^N S_j - S_N$ is also the difference of the entropies, that is the multi-mode mutual information $I_N$, quantifying correlations in the system.

In the leading order in $1/|j - d|$, $D$ is the sum of all squared dimensionless cumulants. If we had only triple cumulant, then $D$ would be given by a converging series and thus independent of the number of modes: $D = \sum_{j=3}^N D(j) / 4 \approx Π_{j=3}^N (N - j)^{-2} ≈ 1$. The contributions of multi-mode cumulants makes $D$ a double sum. Our computations suggest that the sum of squared dimensionless cumulants do not decrease with the order: Table I shows quite irregular dependence on $m$, but no overall decay: contributions of $m = 4$ is comparable to $m = 6$, while $m = 5$ to that of $m = 7$. This is all the more remarkable since only nearest neighbors interact. Assuming that indeed the sum of squared cumulants do not decay with the order and asymptotically saturates to some number (which we cannot yet compute)
The next step will be to study multi-mode correlations for \( \alpha \neq 1/2 \), when non-Gaussianity of a single mode grows exponentially along the cascade: \( A_{2m}/A_2^m \propto \omega_{j-p}^{2m} \) [20–22]. Preliminary (short-interval) data show a linear growth without \( |j-p| \) for \( I(b_{j-1}, b_j) \) and \( I(b_{j-1}, b_j, b_{j+1}) \) [20, 21], which bound \( I(b_1, \ldots, b_N) \) from below due to monotonicity. It is tempting to treat \( \alpha = 1/2 \) logarithmic case as that of critical phenomena in dimension 4 [26] and develop a Wilson-type \( \epsilon \)-expansion in \( \epsilon = \alpha - 1/2 \) [27]. That could be non-trivial, since the perturbation expansions, regular in thermal equilibrium, tend to be singular in non-equilibrium states, especially in turbulence [7, 8, 16].

Note the dramatic difference between our turbulence and nonuniform dilute gases described in [7–9]. There, even though naive expansion encounters divergencies, renormalized expansion gives higher cumulants proportional to higher powers of the small parameter (density), which leaves the multi-particle mutual information small. Our (9) gives all cumulants proportional to the same (first) power of the small parameter, which may lead to logarithmically large multi-mode mutual information.

Our models deal only with resonant modes and respective cumulants. In a general wave turbulence, both resonant and non-resonant interactions are present, so the analysis of multi-mode correlations will be more complicated. It was argued in [16] that cumulants might be substantial for resonant modes in the weak turbulence, even when the statistics of mode amplitudes is close to Gaussian. It is likely that in a continuous limit, determined by quasi-resonances rather than resonances, the higher cumulants are proportional to higher powers of the small parameter, and the mutual information is small. As far as real-world fluid turbulence is concerned, we believe that our work shows importance of measuring multi-mode correlations and the entropy of multi-mode distributions. While only treatment of a moderate number of modes is feasible, it may provide an important insight.

### CONCLUSION

We have shown that turbulent cascade necessary involves multi-mode correlations even in systems with local interaction and close to equilibrium. The entropy-lowering information about turbulence is encoded in the overlapping sets of multi-mode correlations. As far as we were able to compute, the degree of correlation does not decay with the number of modes. If true, multi-mode cumulants would make the multi-mode statistics very different from Gaussian; the logarithmic growth of the relative entropy with the number of modes establishes a promising analogy with critical phenomena. It remains to be seen how universal are multi-mode correlations across classes of turbulent systems and whether they can be related to coherent structures.

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Supplement: Multi-mode correlations and the entropy of turbulence

Gregory Falkovich\textsuperscript{1,2}, Yotam Kadish\textsuperscript{1} and Natalia Vladimirova\textsuperscript{2,3}

\textsuperscript{1}Weizmann Institute of Science, Rehovot 76100 Israel
\textsuperscript{2}Landau Institute for Theoretical Physics, 142432 Chernogolovka, Russia
\textsuperscript{3}Brown University, Providence, RI 02912, USA

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Here we list some technical details pertaining to analytic and numerical computations of multi-mode correlations in doubling and Fibonacci models defined in the main text.

DERIVATION OF HIGHER ORDER CUMULANTS IN A GENERAL TRIPLET CASCADE

Consider the three-wave Hamiltonian:

\[ H_w = \sum_{j,s,q} V_{sq}^j (a_s^* a_q^* a_j + a_s a_q a_j^*) \]  

which admits the following dynamical equations:

\[ i \dot{a}_j = \sum_{s,q} (2 V_{sq}^j a_s^* + V_{sq}^j a_q) \]

Expression for the flux

The energy flux through site \( j \) is the rate of change of the total energy in sites \( \ell \leq j \) (opposite direction for the inverse cascade). We compute this quantity explicitly from (2):

\[ \Pi_j = -\sum_{\ell \leq j} \frac{d \omega_{\ell} \langle |a_{\ell}|^2 \rangle}{dt} = \]

\[ -\sum_{\ell \leq j} \omega_{\ell} (\dot{a}_{\ell}^* a_{\ell} + a_{\ell}^* \dot{a}_{\ell}) = \sum_{s,q} i (2 \omega_{\ell} V_{sq}^s a_s^* a_q^* + \omega_{\ell} V_{sq}^q a_q^* a_s^*) a_{\ell} + c.c \]

\[ = -2 \sum_{\ell \leq j} \sum_{s,q} \omega_{\ell} [2 V_{sq}^s C_3(\ell, q; s) - V_{sq}^q C_3(s, q; \ell)] = \]

\[ = -4 \sum_{\ell \leq j} \sum_{s} \sum_{q \leq s} \omega_{\ell} [V_{sq}^s C_3(\ell, q; s) - V_{sq}^q C_3(s, q; \ell)] \]

In the final row, the right expression is exactly all the terms of the form \( V_{sq}^s C_3(s, q; \ell) \) such that \( \ell \leq j \) and \( \omega_s + \omega_q = \omega_{\ell} \). The left expression also contains these, but also the terms \( V_{sq}^s C_3(\ell, q; s) \) such that \( s > j \), \( \ell \leq j \) and \( \omega_{\ell} + \omega_q = \omega_s \). So we are left exactly with those terms, i.e.:

\[ \Pi_j = -4 \sum_{\ell \leq j} \sum_{s > j} \sum_{q} \omega_{\ell} V_{sq}^s C_3(\ell, q; s) \]

We see that \( \Pi_j - \Pi_{j-1} \propto \omega_j^{-1-\alpha} \). If each site interacts with only a small number of relatively near-by sites, then this sum has only few terms, and the flux is close to \( \Pi_j \simeq \omega_j V_j C_3(j) \).
Estimation of $C_4$

Under assumption of interaction locality, it is enough to consider interacting triplets only sites $j,s,q$ such that $j-q \simeq j-s \ll j$ while writing the equation for the fourth order cumulants $\langle C_4 \rangle$:

\[
\text{Im} \frac{d \langle a_q a_s a_q^* \rangle}{dt} = \text{Im} \langle a_q a_s a_q^* + a_q a_s a_j^* + a_q a_j a_q^* \rangle \\
= \text{Re} \left\{ - \left( \sum_{kl} V_{kl}^q a_k a_l + 2 V_{kl}^q a_k^* a_q \right) a_s a_j^* - \text{Re} \left( \sum_{kl} V_{kl}^q a_k a_l + 2 V_{kl}^q a_k^* a_q \right) a_j^* a_q a_s \left( \sum_{kl} V_{kl}^q a_k^* a_l^* + 2 V_{kl}^q a_k a_l \right) \right\} \\
= \langle C_4 \rangle + 2 V_{sq}^q [n_q n_s - n_j n_q - n_j n_s] \\
\]

From stationarity this equation gives a steady state value for the fourth order cumulants in terms of differences between Gaussian moments which are non-zero in thermal equilibrium, but their differences are zero (this is a general feature of the dynamics, for cumulants of even orders). Stationarity gives:

\[
\langle C_4 \rangle = -2 V_{sq}^j [n_q n_s - n_j n_q - n_j n_s] = -2 V_{sq}^j n_q n_s n_j [n_j^{-1} - n_s^{-1} - n_q^{-1}] = \\
= -2 V_{sq}^j n_q n_s n_j \left[ \omega_j \Lambda_j^{-\xi} - \omega_s \Lambda_s^{-\xi} - \omega_q \Lambda_q^{-\xi} \right] = -2 V_{sq}^j n_q n_s n_j \left[ \omega_j \Lambda_j^{-\xi} - \omega_s \Lambda_s^{-\xi} - (\omega_i - \omega_s) \Lambda_q^{-\xi} \right] \approx \omega_j \Lambda_j^{-\xi} - \omega_s \Lambda_s^{-\xi} - (\omega_i - \omega_s) \Lambda_q^{-\xi} \\
= -2 V_{sq}^j n_q n_s n_j \omega_j \Lambda_j^{-\xi - 1} \left[ 1 - \frac{\omega_q^2 + \omega_j^2}{\omega_j^2} \right] = 4 V_{sq}^j n_q n_s n_j \frac{\omega_j E_j}{\Lambda_j} \approx 4 V_{sq}^j n_j^2 \Lambda_j^{-1} \\
\]

Here we substituted $\omega_k n_k = \log \frac{\omega_k}{\omega_n} \equiv \Lambda_k^\xi$ and $\Lambda_s^{-\xi} \approx \Lambda_j^{-\xi} + \frac{\xi}{\omega_j} \Lambda_j^{-\xi - 1}(\omega_s - \omega_j)$.

**NUMERICAL SETUPS AND DIAGNOSTICS**

**Fibonacci model**

The flux, $\Pi_j = -\sum_{j} F_{j+k-1} \frac{d |a_j|^2}{dt} = 2 F_{j+1} V_{j-1} J_{j+1} + 2 F_j V_j J_{j+2}$, is expressed via the triple cumulant, $J_j = \text{Im} \langle a_j^* a_j^* a_j \rangle$. In a steady state, the flux is constant in the transparency window, where $P_j = \gamma_j = 0$, which determines $J_j = (P F_p/2)^{\alpha/2} \phi^{-j(1+\alpha)+1+2\alpha}$. The sign of the flux coincides with the sign of $2\alpha - 1$; when $\alpha \neq 1/2$ the cascade goes from slow to fast modes. For the normalized time $t \rightarrow t (\phi^{1-\alpha} 5^{-\alpha} P F_p/2)^{1/3}$ and amplitudes, $b_j = a_j \left[ \phi^{j(1+\alpha)+1-2\alpha} 2/P F_p \right]^{1/3} 5^{-\alpha/6}$, the dynamic equation takes the form:

\[
b_j^2 = -i \left[ \phi b_{j-2} b_{j-1} + b_{j-1} b_{j+1} + \phi^{-1} b_{j+1} b_{j+2} \right] \phi^{j(2\alpha-1)/3} - \gamma_j b_j + \xi_j, \\
\]

The system above is evolved numerically using the **ode** solver [A.C. Hindmarsh, ACM Signum Newsletter, 15(4), 10, (1980), K. Radhakrishnan and A. C. Hindmarsh, the Livermore solver for ordinary differential equations (1993)] The forcing is applied to a single mode, $p$. It is implemented to provide solutions independent of time step, $\Delta t$, in the limit of small $\Delta t$,

\[
\xi_p = \frac{\sqrt{P_p}}{\sqrt{\Delta t}} (r_{re} + i r_{im}), \quad \text{with} \quad P_p = 2 \sqrt{5} \phi^{(2\alpha-1)/3-(1+\alpha)}, \\
\]

where $r_{re}$ and $r_{im}$ are random numbers taken from distribution $P = \pi^{-1/2} \exp(-r^2)$ with $\langle r_{re}^2 \rangle = \langle r_{im}^2 \rangle = \sqrt{2}/2$, so that $\langle |r|^2 \rangle = 1$. Damping $\gamma = \gamma_d = \gamma_{d+1}$ is applied only to the two modes at the end of interval where the flux is going, that is to modes 1 and 2 for inverse cascades, and to modes 199 and 200 for the direct cascade. All simulations are done in the system of 200 modes with $\alpha = 1/2$. Pumping is applied to mode $p = 10$ and to mode $p = 190$ for the direct and inverse cascades respectively. Note that we do not control the amplitude of pumping since it is absorbed into rescaled time. Thus, $\gamma = 0.67$ is the only empirical parameter in simulations; it is selected to minimize the built-up in occupation numbers at the end of the interval. Simulations are done at time step $\Delta t = 0.01$, while complex amplitudes of all modes are recorded every 10th step. We have collected $5 \times 10^8$ snapshots over multiple realizations.
Doubling model

The frequency doubling Hamiltonian for $\alpha = \frac{1}{2}$ is

$$\mathcal{H} = \sum_{j=1}^{N} 2^j \left( a_j^* a_{j+1} + a_j^2 a_{j+1}^* \right),$$

hence the Hamiltonian equations of motion are $\dot{a}_j = -i2^{1+\frac{j}{2}} a_j^* a_{j+1} - i2^{\frac{j-1}{2}} a_j^2 a_{j-1}^*$. To initiate a turbulent state in the system, we introduce white-in-time Gaussian pumping at site $p$ and a linear dissipation at sites $d$. For direct cascade, $p = 10$, and for inverse cascade, we set $p = N - 10$, where $N$ is the number of modes. The dissipated sites are always $1, 2, N - 1$ and $N$. The dynamical equations now read

$$\dot{a}_j = -i2^{1+\frac{j}{2}} a_j^* a_{j+1} - i2^{\frac{j-1}{2}} a_j^2 a_{j-1}^* - \sum_d \gamma_j a_j \delta_{jd} + \sqrt{P} \xi_j(t) \delta_{jp},$$

where $\langle \xi_p(t) \xi_p(t') \rangle = \delta(t-t')$. The energy flux through the modes equals the rate of energy pumping into the system:

$$\omega_p P = \Pi = \frac{d}{dt} \sum_{m=p}^{j} \omega_j |a_j|^2 = 4\omega_j V_j J_{j+1},$$

where $\left\{ \begin{array}{l} \omega_j = 2^j \\ V_j = 2^{\frac{j}{2}} \end{array} \right.$, and $J_j = \text{Im} \{a_{j-1}^2 a_j^* \}$. So we estimate $a_j \sim (\omega_p P)^{\frac{j}{4}} 2^{-\frac{j+1}{4}}$.

We transform to dimensionless fields $b_j = (\omega_p P)^{-\frac{j}{4}} 2^{\frac{j+1}{4}} a_j$ and get the following equations for the $b_j$:

$$(\omega_p P)^{-\frac{j}{4}} b_j = -i (b_j^* b_{j+1} + b_j^2 b_{j-1}^* ) + (\omega_p P)^{-\frac{j}{4}} \sqrt{2} \xi_j(t) \delta_{jp}.$$  \hspace{1cm} (8)

After rescaling time by $\tau = (\omega_p P)^{\frac{j}{4}} t$, the forcing term changes to $\langle \xi_p(0) \xi_p^*(t) \rangle = (\omega_p P)^{\frac{j}{4}} \delta(\tau)$. We define a dimensionless noise term $\bar{\xi}_p = (\omega_p P)^{-\frac{j}{4}} \xi_p$, such that $\langle \bar{\xi}_p(0) \bar{\xi}_p^*(\tau) \rangle = \sqrt{2} \delta_{t_0} \delta_{jp} \delta(\tau)$. Now the equations take the form

$$b_j' = -i (b_j^* b_{j+1} + b_{j-1}^2 ) - \sum_d \gamma_j \delta_{jd} + \bar{\xi}_j \delta_{jp}. \hspace{1cm} (8)$$

The only free parameter in equation 8 is the value of dissipation. Similar to the Fibonacci chain, this value is empirically chosen to minimize the temperature build-up near the dissipation scale: for direct cascade $\gamma = 1$ and in inverse cascade $\gamma = 0.275$. The noise distribution is $\bar{\xi} \sim \frac{1+\xi}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$. The time increment is $dt = 0.01$, and the number of time steps is $4 \times 10^6$ for each of the $10^4$ iterations over which we average. The system is integrated by a $4^\text{th}$ order stochastic RK integrator, which has been written for this purpose.

Computation of cumulants

For every polynomial, defined above as $C_{ki\tilde{s}...}(i) = b_{i-k} b_{i-1} \bar{b}_{i-s}^* \ldots \bar{b}_{i}^*$, the average $\langle C_{ki\tilde{s}...}(j) \rangle$ is computed in post-processing. Angular brackets denote time-averaging. As mentioned in the main paper, due to the symmetry $b_j \mapsto -ib_j$ of the Hamiltonian parts in equations (7-8), for odd (even) orders only imaginary (real) parts of $\langle C_{ki\tilde{s}...}(j) \rangle$ are nonzero. By double brackets we denote the cumulants, that is the mean values of the polynomials with all reducible contributions (if they exist) subtracted. For example, if two polynomials of orders $m_1$ and $m_2$ have nonzero mean values, from the mean value of the product polynomial $C_m = C_{m_1} C_{m_2}$ we need to subtract the product of the mean values multiplied by an appropriate combinatorial factor. As explained in the main paper, the ratio $\langle |C| \rangle / \langle \bar{C} \rangle$ scales as $1/|j-d|$ with the distance to damping. Thus, we introduce compensated normalized cumulants,

$$E_m(j) = |j-d| \left( \frac{C_m(j)}{C_m(j)} - c_{\text{comb}}(C_{m_1}(j)) C_{m_2}(j) \right) - \langle \ldots \rangle.$$
Here \(\langle \ldots \rangle\) denotes all possible decompositions into irreducible moments and \(c_{\text{comb}}\) is the combinatorial factor of each decomposition. For instance, in Fibonacci model, the moment \(3211100\) breaks down into \(311\) and \(21\), since each of two \(0\) and each of three \(1\) can go into \(21\). For unique decompositions we use notations \(E_{m_1m_2}\), such as \(E_{43}\); for moments with multiple decompositions, \(m = m_1 + m_2 = m_3 + m_4\), we use \(E_{m_1m_2,m_3m_4}\), such as \(E_{43,52}\), etc; notation \(E_7\) stands for irreducible cumulants. The quantity in denominator is a normalization factor, based on the Gaussian fit \(A_m(j) \approx (1.13 \Gamma(\frac{m}{2} + 1) |j - d|)^{m/3}\), discussed in the main paper,

\[
\tilde{C}_{\text{kis}...}(j) = 1.13^{m/3} \sqrt{\mu} (|j - k - d||j - l - d||j - s - d| \ldots |j - d|)^{1/3},
\]

where \(\mu\) is the product of factorials of numbers of appearances of each multiplier in the product, and \(\langle n_p \rangle = 1.13\) is the value of occupation number at the pumping scale. This normalization can also be expressed through time-averaged occupation numbers, \(n_j = \langle |b_j|^2 \rangle\), as \(\tilde{C}_{\text{kis}...}(j) \approx \langle |b_{j-k}b_{j-k}b_{j-s} \ldots b_{j}^2| \rangle^{1/2} \approx (\mu n_{j-k} n_{j-l} n_{j-s} \ldots n_j)^{1/2}\). We have confirmed that three methods of normalization give similar numerical results.

Finding all the cumulants for an arbitrary \(m\) is a nontrivial mathematical problem. In Fibonacci model, for \(m = 2\) there is only one nonzero moment \(0\), the occupation numbers. The only triple moment, \(21\), is responsible for \(E\) independent of \(j\). The quantity in denominator is a normalization factor, based on the Gaussian fit \(A_m(j) \approx (1.13 \Gamma(\frac{m}{2} + 1) |j - d|)^{m/3}\), discussed in the main paper,

\[
\tilde{C}_{\text{kis}...}(j) = 1.13^{m/3} \sqrt{\mu} (|j - k - d||j - l - d||j - s - d| \ldots |j - d|)^{1/3},
\]

where \(\mu\) is the product of factorials of numbers of appearances of each multiplier in the product, and \(\langle n_p \rangle = 1.13\) is the value of occupation number at the pumping scale. This normalization can also be expressed through time-averaged occupation numbers, \(n_j = \langle |b_j|^2 \rangle\), as \(\tilde{C}_{\text{kis}...}(j) \approx \langle |b_{j-k}b_{j-k}b_{j-s} \ldots b_{j}^2| \rangle^{1/2} \approx (\mu n_{j-k} n_{j-l} n_{j-s} \ldots n_j)^{1/2}\). We have confirmed that three methods of normalization give similar numerical results.

Analytic consideration in the main text predict that \(E_m(j)\) are constant in the inertial range of \(j\) for the doubling model. Let us give respective arguments for the Fibonacci model. Nonzero correlation functions appear as the mean value of the polynomials \(C_{\text{kis}...}(j) = b_{j-k}b_{j-k}b_{j-s} \ldots b_{j}^2\), which are gauge invariant: \(F_{j-k} + F_{j-l} - F_{j-s} + \ldots = F_j\). Overbar in subscript indicates complex conjugation for a mode in the product.

Stationarity of the flux, \(C_{21\bar{0}}\), gives:

\[
\frac{d}{dz} (C_{21\bar{0}}(j)) = \phi^{-1} \langle |b_{j-1}b_j|^2 \rangle + \langle |b_{j-2}b_j|^2 \rangle - \phi \langle |b_{j-2}b_{j-1}|^2 \rangle
\]

\[
+ \langle C_{431\bar{0}}(j) \rangle \phi - \langle C_{431\bar{0}}(j + 2) \rangle \phi^{-1} + \langle C_{322\bar{0}}(j) \rangle \phi
\]

\[
+ \langle C_{322\bar{0}}(j + 1) \rangle (\phi^{-1} - 1) + C_{311\bar{0}}.
\]

Using \(\phi^2 = 1 + \phi\), we approximate the first line:

\[
n_{j-2}(n_j - n_{j-1}) + \frac{n_{j-1}}{\phi}(n_j - n_{j-2}) \approx \left(1 + \frac{2}{\phi}\right) \frac{2\xi n_j^3}{|j - d|},
\]

which is an estimate for the combination of cumulants of the fourth order, \(\langle \langle C_4 \rangle \rangle\). Computing time derivatives of higher \(\langle C_m \rangle\), we come self-consistently to the same estimate for the dimensionless cumulants \(D_m \propto |j - d|^{-1}\) and \(E_m\) independent of \(j\).

This is supported by numerics, even though we see mode-to-mode fluctuations, especially for higher order moments, due to limited statistics. With this said, averaging over \(i\) in the inertial range (we use \(20 < i < 180\)) provides reasonable metrics for \(E_m\). These values are shown in the tables below and used in the final “sum of squares” \(\sum |E_m|^2\).
DETAILS ON THE EVALUATION OF CUMULANTS

Estimations for the statistics - Numerical results

In this section we define \( n_j = |a_j|^2 \) and \( J_j = b_j^2 - b_{j-1}^2 \), i.e., without averages.

Logarithmic dependence of temperature on distance from dissipation

As discussed in the main text, the spectrum shows a logarithmic dependence on the distance from the damping scale, with scaling \( \xi = \frac{2}{3} \), i.e. \( \langle n_j \rangle \propto |j - d|^{-\frac{2}{3}} \). This result is verified numerically and shown in the left panel of Fig. 1 for both cascades in Doubling chain. Corresponding data for Fibonacci chain are included in main document.

![Graph showing temperature vs distance](image)

**FIG. 1.** Left: Temperature dependence on distance from dissipation in both cascades in doubling model. Right: Correlation function \( D5(k) = \langle n_j J_{j+k} \rangle \), normalized by \( \langle n_j \rangle \overline{\langle J \rangle} \), where bar stands for averaging over the inertial range, for doubling model.

Irreducibility of energy density and flux correlations in doubling cascade

Consider \( D5(k) = \langle n_j J_{j+k} \rangle \). If it was reducible for all \( k \), one expects \( D5(1) = 3D5(0)/2 = 3 \langle n_j \rangle \langle J_{j+1} \rangle \). Meanwhile, from the equation of motion for the fourth-order correlation function it is straightforward to derive \( D5(0) = D5(1) = 3D5(2) \). The right panel in Fig. 1 shows the true values of the correlation functions: \( D5(0) = D5(1) = 3D5(2) \approx 2.2 \langle n_j \rangle \langle J_{j+k} \rangle \).

From stationarity of the \( \langle n_j^{m+1} \rangle \) moments, we get the identity:

\[
\langle n_j^m J_{j+1} \rangle = \langle n_j^m J_j \rangle \tag{10}
\]

This identity is clearly non-trivial as already seen above — for \( m = 1 \) it defies the basic expectation; for the polynomial PDF seen in the text in equations (10)-(12) we would estimate \( \langle n_j^m J_{j+1} \rangle \sim \left( \frac{m+2}{m+1} \right) \langle n_j \rangle^{m+1} \overline{\langle J \rangle} = \frac{m+2}{2} \) which is greater than 1 for all \( m > 0 \). In general, it puts a strong constraint on the PDF, from which the consequences are yet to be fully extracted.

In the left panel of Fig. 2 we validate identity (10) for \( m = 0, 1, 2, 3 \) in doubling system. In this figure, we also show that the irreducible parts of the \( \langle n_j^m J_j \rangle \) cumulants tend to zero when there is no overlap. This decay is not monotonic with \( k \) but has a small minimum for \( k = 2 \), unlike the the decay of \( \langle J_{j-k} J_j \rangle \) seen below. However, similar to the decay of \( \langle J_{j-k} J_j \rangle \), the tails decay exponentially with \( k \), and with a similar decay rate. It could be that on top of \( |j - d|^{-1} \) dependence, there is an even weaker dependence on the interval width, see Fig. 2, right panel.
Flux fluctuations in doubling model

The energy flux in turbulence is a stationary yet highly fluctuating quantity. From the dynamical equations on the fifth order cumulants, we get an estimation for the flux’s variance, \( \sigma_\Pi^2 = \langle J_j^2 \rangle \propto n^3_j \approx |j - d| \). As shown in Fig. 3, this estimation holds for both direct and inverse cascades, with the same proportionality constant \( \approx 2.4 \). In the normalized variables, \( \Pi = \frac{\Pi}{N} (\Pi = \frac{N - p}{N}) \) in direct(inverse) cascade, therefore \( \sigma_\Pi \gg \Pi \). This result and other flux-flux correlations, which are seen in Fig. 3, show that the flux fluctuations are much larger than flux mean.

FIG. 3. Left: The estimation \( \langle J_j^2 \rangle \approx \frac{2.4n^3_j}{|j - d|} \). Right: Values for the flux-flux correlators in direct cascade of doubling system.
Decay of flux-flux fluctuations

In Figure 4 we display the rate of decay of site-shifted flux-flux correlations for doubling and Fibonacci models. As we expected, the energy fluxes become exponentially less correlated with increasing shift in indices. In both models, we observe similar but not identical coefficients of exponential decay.

In particular, the main contribution into the sum of sixth-order cumulants is given by squared flux terms. Also for the doubling model, $\langle JJ \rangle$ terms give 0.1269, while all the other sixth-order cumulants give 0.0733. Single-time correlations between flux values at different points beyond overlap decrease exponentially with the shift $k$ as $e^{-k/2}$, except $JJ(1)$. For Fibonacci model, $JJ(1) = \langle C_{210}(j)C_{210}(j+1) \rangle = (1+\phi)\langle C_{210}^2(j) \rangle/2$, which we derived analytically (from stationarity of $\langle n_j(J_j-J_{j+1}) \rangle$) and confirmed numerically.

Relative entropy — sum of squares in each order

The main difference between entropies of turbulence and thermal equilibrium is simply due to the distortion of equipartition, which lowers the entropy at a given total energy $\sum_j n_j$: the entropy of the Gaussian $P_G(n_j)$ having distribution $P_G(n_j)$ with $n_j = 5(j/N)^{2/3}$ is lower by $N[2/3 - \ln(5/3)]$ than the entropy of thermal equilibrium with $n_j = 1$. More informative is the difference between the turbulent distribution and the Gaussian distribution with the same occupation numbers. That difference is expressed in irreducible correlation functions called cumulants, which are zero for a Gaussian distribution.

As explained in the main text, the sum of squares $\sum E_{rs}^2$ measures the relative entropy between the Gaussian PDF ($\ln P_G \propto -E$) and the polynomial PDF (equations (10)–(12) in the main text). In Tables I and II we count the multiplicity of such cumulants for each order and show their contribution to the sum of squares. That is the main result of our work.

In Table III we show the irreducible-normalized-squared form of all the cumulants in the doubling chain, i.e. $\langle C_{C(r)} \rangle/e_{C(r)}^2$, where $C_{C(r)}$ is the sum of irreducible contributions to $C$ and $e_{C\text{max}}$ is the mean-squared value of the cumulant for the Gaussian PDF. In the last column of the table, the mean value of the cumulant is to be substituted instead of the letter “$C$”. The cumulants are specified uniquely by a symbol of the form $(kl...\bar{s})$ which corresponds to the number $b_{j-k}b_{j-l}...b_{j-s}$. $J$ is always defined by the complex number $J_j \equiv b_{j-k}^2$, and a complex number of the form $b_{j-k}b_{j-l}...b_{j-s}b_{j}^*\text{ }$ such that $2^{-k} + 2^{-l} + ...2^{-s} = 1$, is denoted by $C_{kl...\bar{s}}$ (these are the irreducible cumulants).
|   | count | inverse  | direct     |
|---|-------|----------|------------|
| $m = 3$ | $E_3$ | 1        | 0.05635 0.06343 |
| $m = 4$ | $E_4$ | 1        | 0.02540 0.02743 |
|       | $E_{22}$ | 3       | 0.07599 0.04292 |
|       | total  | 4        | 0.10140 0.07035 |
| $m = 5$ | $E_5$ | 2        | 0.00020 0.00026 |
|       | $E_{32}$ | 4       | 0.00518 0.00568 |
|       | total  | 6        | 0.00539 0.00594 |
| $m = 6$ | $E_6$ | 3        | 0.00449 0.00466 |
|       | $E_{222}$ | 2      | 0.14040 0.06946 |
|       | $E_{42}$ | 4       | 0.00060 0.00063 |
|       | $E_{33}$ | 10      | 0.16785 0.18357 |
|       | $E_{42.33}$ | 1      | 0.00034 0.00033 |
|       | $E_{222.33}$ | 1      | 0.02502 0.03109 |
|       | total  | 21       | 0.33869 0.28974 |
| $m = 7$ | $E_7$ | 5        | 0.00112 0.00115 |
|       | $E_{322}$ | 9      | 0.01032 0.01118 |
|       | $E_{52}$ | 7       | 0.00023 0.00027 |
|       | $E_{43}$ | 5       | 0.00861 0.00946 |
|       | $E_{43.52}$ | 1      | 0.0001 0.00001 |
|       | $E_{43.322}$ | 2     | 0.00302 0.00315 |
|       | total  | 29       | 0.02332 0.02521 |
| $m = 8$ | $E_8$ | 9        | 0.00106 0.00089 |

TABLE I. Partial sums of squares of cumulants computed in numerical simulations for the Doubling chain. For reducible moments count is the number of moments computed for partial sums.

|   | count | inverse  | direct     |
|---|-------|----------|------------|
| $m = 3$ | $E_3$ | 1        | 0.189922 0.179861 |
| $m = 4$ | $E_4$ | 3        | 0.227484 0.226358 |
| $m = 5$ | $E_5$ | 8        | 0.048491 0.049622 |
|       | $E_{32}$ | 9      | 0.022531 0.021377 |
|       | total  | 17       | 0.071022 0.070999 |
| $m = 6$ | $E_6$ | 24       | 0.021736 0.021942 |
|       | $E_{42}$ | 15     | 0.018720 0.017911 |
|       | $E_{33}$ | 12     | 0.069856 0.067788 |
|       | $E_{42.33}$ | 3     | 0.010556 0.010644 |
|       | total  | 54       | 0.120878 0.118285 |
| $m = 7$ | $E_7$ | 70       | 0.008809 0.009171 |
|       | $E_{43}$ | 56     | 0.027466 0.027042 |
|       | $E_{52}$ | 40     | 0.008649 0.008244 |
|       | $E_{43.52}$ | 14    | 0.004217 0.004037 |
|       | $E_{43.322}$ | 2     | 0.000010 0.000019 |
|       | $E_{43.322}$ | 6     | 0.015772 0.014467 |
|       | total  | 189      | 0.066995 0.065005 |

TABLE II. Partial sums of squares of cumulants computed in numerical simulations for the Fibonacci chain. For reducible moments count show the number of moments computed for partial sums.
| order | symbol | definition | normalization |
|-------|--------|------------|--------------|
| 3     | (110)  | Im \( b_{j-1}^* b_j^* \) | \( C^2/2n_j^2, n_j \) |
| 4     | (2210) | Re \( b_{j-1}^* b_j^* b_j^* b_{j-1} \) | \( C^2/2n_j^2, n_j \) |
|       | (0000) | \( \langle n_j^2 \rangle \) | \( (C - 2n_j^2)^2/24n_j^2 \) |
|       | (1100) | \( \langle n_j \rangle \langle n_j \rangle \) | \( (C - n_j^2, n_j^2)^2/4n_j^2 \) |
|       | (2200) | \( \langle n_j \rangle \langle n_j \rangle \) | \( (C - n_j^2, n_j^2)^2/4n_j^2 \) |
| 5     | (33210) | Im \( b_{j-2}^* b_j^* b_j^* b_{j-1} \) | \( C^2/2n_j^2, n_j \) |
|       | (22200) | Im \( b_{j-1}^* b_j^* \) | \( C^2/24n_j^2 \) |
|       | (11000) | Im \( \langle n_j \rangle \langle J_j \rangle \) | \( (C - 2n_j J_j)^2/12n_j^2 \) |
|       | (11110) | Im \( \langle n_j \rangle \langle J_{j-1} \rangle \) | \( C - 3n_j J_{j-1}^2/12n_j^2 \) |
| 6     | (443210) | Re \( b_{j-1}^* b_j^* b_{j-2} b_j^* b_{j-1} \) | \( C^2/2n_j^3, n_j \) |
|       | (333310) | Re \( b_{j-1}^* b_j^* b_{j-1}^* \) | \( C^2/24n_j^3 \) |
|       | (332220) | Re \( b_{j-2}^* b_j^* b_j^* \) | \( C^2/12n_j^3 \) |
|       | (111100) | Re \( \langle J_j \rangle \) | \( (C - 6n_j^2 J_j)^2/48n_j^2 \) |
|       | (221110) | Re \( \langle J_{j-1} \rangle \) | \( (C - 221J_{j-1} - 2 J_{j-1}^2)^2/12n_j \) |
|       | (332110) | Re \( \langle J_j \rangle \langle J_{j-2} \rangle \) | \( (C - J_j^2)^2/12n_j \) |
|       | (332110) | Re \( \langle J_j \rangle \langle J_{j-2} \rangle \) | \( (C - J_{j-2}^2)^2/4n_j \) |
|       | (221000) | Re \( C221 \langle n_j \rangle \) | \( C - 221n_j J_j^2/12n_j \) |
|       | (222110) | Re \( C221 \langle J_j \rangle \) | \( C - 3C221J_j^2/24n_j \) |
| 7     | (5543210) | Im \( b_{j-1}^* b_j^* b_{j-3} b_j^* b_{j-1} \) | \( C^2/2n_j^3, n_j \) |
|       | (4444210) | Im \( b_{j-2}^* b_j^* b_{j-3} b_j^* \) | \( C^2/24n_j^3 \) |
|       | (4433310) | Im \( b_{j-3}^* b_j^* b_{j-1}^* b_{j-1} \) | \( C^2/12n_j^3 \) |
|       | (4432200) | Im \( b_{j-2}^* b_j^* b_j^* b_{j-2} \) | \( C^2/24n_j^3 \) |
|       | (3333220) | Im \( b_{j-2}^* b_j^* \) | \( C^2/48n_j^3 \) |
|       | (1100000) | Im \( \langle n_j J_j \rangle \) | \( (C - 6n_j^2 J_j)^2/240n_j^3 \) |
|       | (1111000) | Im \( \langle n_j \rangle \langle n_{j-1} \rangle \) | \( (C - 6n_j^2 J_{j-1})^2/144n_j^3 \) |
|       | (1111110) | Im \( \langle n_j^2 \rangle \) | \( (C - 12n_j J_j) \) |
|       | (2211110) | Im \( \langle n_j \rangle \langle n_{j-2} \rangle \) | \( (C - 3n_j J_{j-2})^2/48n_j^3 \) |
|       | (2222110) | Im \( \langle n_{j-2} \rangle \) | \( (C - 2n_j J_j + 2C221J_j)^2/48n_j^3 \) |
|       | (1100100) | Im \( \langle n_j \rangle \langle n_{j-1} \rangle \) | \( (C - 2n_j n_{j-1} J_j + 2C221J_j)^2/24n_j^3 \) |
|       | (3321100) | Im \( \langle n_j \rangle \langle n_{j-1} \rangle \) | \( (C - n_j J_{j-1} J_j)^2/8n_j^3 \) |
|       | (3321200) | Im \( \langle n_j \rangle \langle n_{j-2} \rangle \) | \( (C - 2n_j J_{j-2})^2/24n_j^3 \) |
|       | (4432200) | Im \( \langle n_{j-1} \rangle \langle n_{j-3} \rangle \) | \( (C - 2n_j J_{j-3})^2/8n_j^3 \) |
|       | (4433300) | Im \( \langle n_j \rangle \langle n_{j-3} \rangle \) | \( (C - 2n_j J_{j-3})^2/24n_j^3 \) |
|       | (2222000) | Im \( \langle n_{j-1} \rangle \) | \( (C - 2n_j^2 J_{j-1})^2/144n_j^3 \) |
|       | (2222110) | Im \( \langle n_{j-1} \rangle \) | \( (C - 2n_j J_{j-1}^2 + 6C221J_j^2)^2/48n_j^3 \) |
|       | (2222220) | Im \( \langle n_{j-2} \rangle \) | \( (C - 5n_j^2 J_{j-2})^2/720n_j^5 \) |
|       | (3322200) | Im \( \langle n_j \rangle \langle n_{j-3} \rangle \) | \( (C - n_j J_{j-3} J_j)^2/48n_j^3 \) |
|       | (3321100) | Im \( \langle n_j \rangle \langle n_{j-2} \rangle \) | \( (C - 2n_j J_{j-2} J_j)^2/12n_j^3 \) |
|       | (3322100) | Im \( \langle n_{j-1} \rangle \langle n_{j-3} \rangle \) | \( (C - 2n_j^2 J_{j-3} J_j)^2/12n_j^3 \) |
|       | (3333210) | Im \( \langle n_{j-2} \rangle \langle n_{j-3} \rangle \) | \( (C - 3n_j J_{j-3} J_j)^2/24n_j^3 \) |
|       | (2211100) | Im \( \langle n_j \rangle \) | \( (C - 6C221J_j J_j)^2/24n_j^3 \) |
|       | (3322100) | Im \( \langle n_{j-1} \rangle \langle n_{j-3} \rangle \) | \( (C - 3n_j^2 J_{j-3} J_j)^2/24n_j^3 \) |
|       | (4432210) | Im \( \langle n_j \rangle \langle n_{j-3} \rangle \) | \( (C - 3n_j J_{j-3} J_j)^2/24n_j^3 \) |
|       | (4432210) | Im \( \langle n_{j-1} \rangle \langle n_{j-3} \rangle \) | \( (C - 3n_j^2 J_{j-3} J_j)^2/24n_j^3 \) |
| 9     | (1111110000) | Im \( J_j \) | \( (C - 90J_j^2)^2/4320n_j^3 \) |
| 12    | (111111110000) | Re \( J_j \) | \( (C - 2520J_j^2)^2/48n_j^3 \) |
APPENDIX: TABLES OF CORRELATORS FOR FIBONACCI CHAIN

In the tables we use "text" notations where the digits after dash stands for conjugated modes, for example 611-440 is equivalent to 644110.

| IRREDUCIBLE MOMENTS | DIRECT | INVERSE |
|---------------------|--------|---------|
| -- IRREDUCIBLE E3 -- |        |         |
| 21-0                | -0.4241 +/- 0.0040 | +0.4358 +/- 0.0045 |
| SUM OF SQUARES E3 (1): | 0.179861 | 0.189922 |
| -- IRREDUCIBLE E4 -- |        |         |
| 322-0               | -0.1634 +/- 0.0048 | +0.1707 +/- 0.0049 |
| 431-0               | -0.0879 +/- 0.0040 | +0.0869 +/- 0.0038 |
| 11-30               | +0.4381 +/- 0.0047 | -0.4368 +/- 0.0050 |
| SUM OF SQUARES E4 (3): | 0.226358 | 0.227484 |
| -- IRREDUCIBLE E5 -- |        |         |
| 4332-0              | -0.0712 +/- 0.0033 | +0.0722 +/- 0.0027 |
| 5422-0              | -0.0538 +/- 0.0036 | +0.0589 +/- 0.0045 |
| 5441-0              | -0.0171 +/- 0.0036 | +0.0152 +/- 0.0033 |
| 6531-0              | -0.0214 +/- 0.0036 | +0.0224 +/- 0.0036 |
| 411-20              | -0.1346 +/- 0.0040 | +0.1279 +/- 0.0037 |
| 331-50              | -0.0146 +/- 0.0035 | +0.0144 +/- 0.0036 |
| 222-40              | -0.0351 +/- 0.0032 | +0.0354 +/- 0.0034 |
| 11-540              | +0.1461 +/- 0.0031 | -0.1458 +/- 0.0041 |
| SUM OF SQUARES E5 (8): | 0.0496218 | 0.0484914 |
| -- IRREDUCIBLE E6 -- |        |         |
| 44333-0             | -0.0260 +/- 0.0041 | +0.0264 +/- 0.0037 |
| 54432-0             | -0.0404 +/- 0.0035 | +0.0418 +/- 0.0039 |
| 65332-0             | -0.0107 +/- 0.0036 | +0.0089 +/- 0.0036 |
| 65522-0             | -0.0225 +/- 0.0037 | +0.0277 +/- 0.0045 |
| 76422-0             | -0.0140 +/- 0.0039 | +0.0148 +/- 0.0034 |
| 76631-0             | -0.0101 +/- 0.0032 | +0.0120 +/- 0.0036 |
| 63333-0             | +0.0094 +/- 0.0037 | -0.0098 +/- 0.0041 |
| 65541-0             | -0.0085 +/- 0.0041 | +0.0082 +/- 0.0037 |
| 76441-0             | +0.0009 +/- 0.0036 | -0.0017 +/- 0.0037 |
| 87531-0             | -0.0039 +/- 0.0041 | +0.0036 +/- 0.0038 |
| 3332-50             | +0.0298 +/- 0.0037 | -0.0265 +/- 0.0037 |
| 4422-60             | +0.0223 +/- 0.0038 | -0.0211 +/- 0.0036 |
| 4441-60             | +0.0054 +/- 0.0041 | -0.0075 +/- 0.0043 |
| 5531-70             | +0.0042 +/- 0.0037 | -0.0049 +/- 0.0035 |
| 6511-20             | -0.0462 +/- 0.0039 | +0.0438 +/- 0.0037 |
| 6331-40             | +0.0245 +/- 0.0038 | -0.0281 +/- 0.0040 |
| 5222-30             | +0.0309 +/- 0.0045 | -0.0398 +/- 0.0062 |
| 4111-00             | +0.0629 +/- 0.0037 | -0.0617 +/- 0.0034 |
| 611-440             | -0.0503 +/- 0.0041 | +0.0483 +/- 0.0036 |
| 311-520             | +0.0468 +/- 0.0048 | -0.0370 +/- 0.0047 |
| 331-760             | -0.0188 +/- 0.0038 | +0.0181 +/- 0.0038 |
| 222-650             | +0.0385 +/- 0.0038 | -0.0400 +/- 0.0038 |
| 11-7640             | +0.0334 +/- 0.0036 | -0.0329 +/- 0.0032 |
| 11-6550             | +0.0423 +/- 0.0037 | -0.0433 +/- 0.0038 |
| SUM OF SQUARES E6 (24): | 0.0219424 | 0.0217464 |
| Column     | Value 1          | Value 2          | Value 3          |
|------------|------------------|------------------|------------------|
| 544433-0   | -0.0215 +/- 0.0036 | +0.0222 +/- 0.0040 |
| 554442-0   | -0.0114 +/- 0.0033 | +0.0116 +/- 0.0032 |
| 554433-0   | -0.0005 +/- 0.0034 | -0.0002 +/- 0.0035 |
| 555432-0   | -0.0163 +/- 0.0036 | +0.0178 +/- 0.0034 |
| 654333-0   | -0.0023 +/- 0.0037 | +0.0020 +/- 0.0039 |
| 655432-0   | -0.0114 +/- 0.0033 | +0.0116 +/- 0.0032 |
| 665551-0   | -0.0005 +/- 0.0034 | -0.0002 +/- 0.0035 |
| 665432-0   | -0.0163 +/- 0.0036 | +0.0178 +/- 0.0034 |
| 674442-0   | -0.0023 +/- 0.0037 | +0.0020 +/- 0.0039 |
| 766332-0   | -0.0114 +/- 0.0033 | +0.0116 +/- 0.0032 |
| 766522-0   | -0.0005 +/- 0.0034 | -0.0002 +/- 0.0035 |
811-6660 $-0.0067 +/- 0.0038$ $+0.0069 +/- 0.0039$
11-76650 $+0.0216 +/- 0.0029$ $-0.0218 +/- 0.0037$
11-87550 $+0.0067 +/- 0.0039$ $-0.0075 +/- 0.0033$
11-87740 $+0.0084 +/- 0.0034$ $-0.0081 +/- 0.0036$
11-90660 $+0.0055 +/- 0.0036$ $-0.0066 +/- 0.0037$

**SUM OF SQUARES E7 (70):**

| Moment | Direct | Inverse |
|--------|--------|---------|
| $E_{32}$ | **REDUCIBLE** | |
| $E_{42}$ | **REDUCIBLE** | |
| $E_{32}$ | **REDUCIBLE** | |
| $E_{42}$ | **REDUCIBLE** | |
| $E_{33}$ | **REDUCIBLE** | |
| $E_{33}$ | **REDUCIBLE** | |
| Expression                | Result                  |
|---------------------------|-------------------------|
| (11-30)+(21-0) x6        | (-0.0220) - (-0.0063) = -0.0157 +/- 0.0034 |
| (11-30)+(32-1) below     | (+0.0110) - (-0.0065) = +0.0175 +/- 0.0032 |
| (11-30)+(43-2) below     | (-0.0360) - (-0.0023) = -0.0337 +/- 0.0035 |
| (11-30)+(54-3) x2        | (+0.0707) - (-0.0034) = +0.0704 +/- 0.0035 |
| (11-30)+(65-4) x1        | (-0.0328) - (-0.0024) = -0.0304 +/- 0.0035 |
| (11-30)+(76-5) x1        | (-0.0160) - (-0.0023) = -0.0137 +/- 0.0034 |
| (11-30)+(87-6) x1        | (-0.0053) - (-0.0023) = -0.0030 +/- 0.0039 |
| (11-30)+(98-7) x1        | (-0.0057) - (-0.0023) = -0.0035 +/- 0.0036 |
| (11-30)+(0-21) below     |                                                                 |
| (11-30)+(1-32) x6        | (+0.0096) - (-0.0062) = +0.0034 +/- 0.0036 |
| (11-30)+(2-43) below     |                                                                 |
| (11-30)+(3-54) below     |                                                                 |
| (11-30)+(4-65) x1        | (-0.0388) - (-0.0024) = -0.0364 +/- 0.0035 |
| (11-30)+(5-76) x1        | (-0.0195) - (-0.0023) = -0.0172 +/- 0.0034 |
| (11-30)+(6-87) x1        | (-0.0056) - (-0.0023) = -0.0033 +/- 0.0034 |
| (11-30)+(7-98) x1        | (-0.0046) - (-0.0023) = -0.0023 +/- 0.0033 |
| (11-30)+(0-21) below: (11-30)+(0-21) = 110-3210 |                                                                 |
| (11-30)+(21-0) below     |                                                                 |
| (11-30)+(32-1) x6        | (-0.0360) - (-0.0023) = -0.0337 +/- 0.0039 |
| (11-30)+(43-2) below     |                                                                 |
| (11-30)+(54-3) below     |                                                                 |
| (11-30)+(65-4) x1        | (+0.0095) - (-0.0009) = +0.0086 +/- 0.0034 |
| (11-30)+(76-5) x1        | (+0.0083) - (-0.0009) = +0.0074 +/- 0.0033 |
| (11-30)+(87-6) x1        | (+0.0004) - (-0.0008) = +0.0000 +/- 0.0035 |
| (11-30)+(98-7) x1        | (+0.0017) - (-0.0008) = +0.0009 +/- 0.0034 |
| (11-30)+(0-21) below: (32-2)+(0-21) = 3220-210 |                                                                 |
| (11-30)+(1-32) x6        | (-0.0141) - (-0.0009) = -0.0132 +/- 0.0037 |
| (11-30)+(2-43) below     |                                                                 |
| (11-30)+(3-54) below     |                                                                 |
| (11-30)+(4-65) x1        | (+0.0156) - (-0.0013) = +0.0143 +/- 0.0041 |
| (11-30)+(5-76) x1        | (+0.0037) - (-0.0009) = +0.0028 +/- 0.0038 |
| (11-30)+(6-87) x1        | (+0.0004) - (-0.0008) = +0.0000 +/- 0.0035 |
| (11-30)+(7-98) x1        | (+0.0018) - (-0.0008) = +0.0009 +/- 0.0035 |
| (11-30)+(0-21) below: (32-2)+(0-21) = 3220-210 |                                                                 |
| (11-30)+(3-54) below     |                                                                 |
| (11-30)+(4-65) x1        | (-0.0122) - (-0.0009) = -0.0113 +/- 0.0038 |
| (11-30)+(5-76) x1        | (+0.0037) - (-0.0009) = +0.0028 +/- 0.0038 |
| (11-30)+(6-87) x1        | (+0.0004) - (-0.0008) = +0.0000 +/- 0.0035 |
| (11-30)+(7-98) x1        | (+0.0018) - (-0.0008) = +0.0009 +/- 0.0035 |
| (11-30)+(0-21) below: (32-2)+(0-21) = 3220-210 |                                                                 |
| (11-30)+(3-54) below     |                                                                 |
| (11-30)+(4-65) x1        | (-0.0141) - (-0.0009) = -0.0132 +/- 0.0037 |
| (11-30)+(5-76) x1        | (+0.0037) - (-0.0009) = +0.0028 +/- 0.0038 |
| (11-30)+(6-87) x1        | (+0.0004) - (-0.0008) = +0.0000 +/- 0.0035 |
| (11-30)+(7-98) x1        | (+0.0018) - (-0.0008) = +0.0009 +/- 0.0035 |
| (11-30)+(0-21) below: (32-2)+(0-21) = 3220-210 |                                                                 |
| (11-30)+(3-54) below     |                                                                 |
| (11-30)+(4-65) x1        | (+0.0156) - (-0.0013) = +0.0143 +/- 0.0041 |
| (11-30)+(5-76) x1        | (+0.0037) - (-0.0009) = +0.0028 +/- 0.0038 |
| (11-30)+(6-87) x1        | (+0.0004) - (-0.0008) = +0.0000 +/- 0.0035 |
| (11-30)+(7-98) x1        | (+0.0018) - (-0.0008) = +0.0009 +/- 0.0035 |
| (11-30)+(0-21) below: (32-2)+(0-21) = 3220-210 |                                                                 |
| (11-30)+(3-54) below     |                                                                 |
| (11-30)+(4-65) x1        | (-0.0122) - (-0.0009) = -0.0113 +/- 0.0038 |
| (11-30)+(5-76) x1        | (+0.0037) - (-0.0009) = +0.0028 +/- 0.0038 |
| (11-30)+(6-87) x1        | (+0.0004) - (-0.0008) = +0.0000 +/- 0.0035 |
| (11-30)+(7-98) x1        | (+0.0018) - (-0.0008) = +0.0009 +/- 0.0035 |
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4111-210 (411-20)+(1-1) x3 (-0.1011) - (-0.1174) = +0.0164 +/- 0.0019 (+0.0936) - (+0.1109) = -0.0173 +/- 0.0019
4211-220 (411-20)+(2-2) below
4311-320 (411-20)+(3-3) below
4411-420 (411-20)+(4-4) x2 (-0.0911) - (-0.1107) = +0.0196 +/- 0.0020 (+0.0867) - (+0.1046) = -0.0179 +/- 0.0024
5411-520 (411-20)+(5-5) x1 (-0.0729) - (-0.0959) = +0.0230 +/- 0.0026 (+0.0684) - (+0.0906) = -0.0222 +/- 0.0028
6411-620 (411-20)+(6-6) x1 (-0.0919) - (-0.0959) = +0.0040 +/- 0.0028 (+0.0862) - (+0.0906) = -0.0044 +/- 0.0023
3310-500 (331-50)+(0-0) x2 (-0.0170) - (-0.0120) = -0.0050 +/- 0.0022 (+0.0174) - (+0.0118) = +0.0057 +/- 0.0020
3311-510 (331-50)+(1-1) x2 (-0.0246) - (-0.0120) = -0.0126 +/- 0.0024 (+0.0219) - (+0.0118) = +0.0101 +/- 0.0021
3321-520 (331-50)+(2-2) below
3331-530 (331-50)+(3-3) x3 (-0.0187) - (-0.0127) = -0.0059 +/- 0.0017 (+0.0184) - (+0.0125) = +0.0059 +/- 0.0018
4331-540 (331-50)+(4-4) below
5331-550 (331-50)+(5-5) x2 (-0.0079) - (-0.0120) = +0.0041 +/- 0.0020 (+0.0076) - (+0.0118) = -0.0042 +/- 0.0020
6331-650 (331-50)+(6-6) x1 (+0.0005) - (-0.0104) = +0.0109 +/- 0.0027 (+0.0076) - (+0.0118) = -0.0042 +/- 0.0020
2220-400 (222-40)+(0-0) x2 (-0.0465) - (-0.0289) = -0.0176 +/- 0.0025 (+0.0443) - (+0.0289) = +0.0154 +/- 0.0018
2221-410 (222-40)+(1-1) below
2222-420 (222-40)+(2-2) x4 (-0.0024) - (-0.0027) - (+0.0024) - (+0.0027)
3222-430 (222-40)+(3-3) below
4222-440 (222-40)+(4-4) x2 (-0.0024) - (-0.0027) - (+0.0024) - (+0.0027)
5222-450 (222-40)+(5-5) x1 (-0.0013) - (-0.0025) - (+0.0014) - (+0.0025) - (+0.0014) - (+0.0025)
6222-460 (222-40)+(6-6) x1 (-0.0166) - (-0.0250) - (+0.0185) - (+0.0251) - (+0.0185) - (+0.0251)
11-5400 (11-540)+(0-0) x2 (+0.0996) - (+0.1202) = -0.0206 +/- 0.0020 (-0.0999) - (-0.1192) = +0.0193 +/- 0.0023
11-5410 (11-540)+(1-1) x3 (+0.1219) - (+0.1274) = -0.0055 +/- 0.0020 (-0.1201) - (-0.1264) = +0.0063 +/- 0.0021
211-5420 (11-540)+(2-2) below
311-5430 (11-540)+(3-3) below
411-5440 (11-540)+(4-4) x2 (+0.0994) - (+0.1201) = -0.0252 +/- 0.0027 (-0.0927) - (-0.1192) = +0.0265 +/- 0.0019
511-5540 (11-540)+(5-5) x2 (+0.1042) - (+0.1201) = -0.0159 +/- 0.0022 (-0.1024) - (-0.1193) = +0.0169 +/- 0.0020
611-6540 (11-540)+(6-6) x1 (+0.0842) - (+0.1040) = -0.0199 +/- 0.0025 (-0.0832) - (-0.1033) = +0.0201 +/- 0.0027

SUM OF SQUARES E52 (40): 0.00824359 0.00864855

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MOMENTS WITH MULTIPLE DECOMPOSITION

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DIRECT INVERSE

-- REDUCIBLE E42.33 --

3221-10 (-0.2099) - (-0.1140) = -0.0959 (+0.2207) - (+0.1235) = +0.0972
(21-0)+(32-1) x2 - (+0.0024) - (+0.0027)
(32-0)+(1-1) x1 - (-0.1164) - (-0.1208)

4321-20 (-0.0758) - (-0.0609) = -0.0149 (+0.0775) - (+0.0635) = +0.0140
(21-0)+(43-2) x1 - (+0.0017) - (+0.0020)
(43-0)+(2-2) x1 - (-0.0626) - (-0.0615)

211-320 (+0.3494) - (+0.3144) = +0.0350 (-0.3367) - (-0.3065) = -0.0302
(21-0)+(1-32) x2 - (+0.0024) - (+0.0027)
(11-30)+(2-2) x1 - (+0.3120) - (+0.3092)

SUM OF SQUARES E42.33 (3): 0.0106438 0.0105559

-- REDUCIBLE E43.52 --

211-5420 (+0.1006) - (+0.1045) = -0.0039 (-0.0993) - (-0.1027) = 0.0034
(1-542)+(21-0) x2 - (+0.0005) - (+0.0005)
(11-540)+(2-2) x1 - (+0.1040) - (+0.1032)

2221-410 (+0.0042) - (-0.0281) = +0.0323 (-0.0080) - (+0.0218) = -0.0298
(22-41)+(21-0) x3 - (-0.0031) - (-0.0033)
(222-40)+(1-1) x1 - (-0.0250) - (+0.0251)

311-5430 (+0.0896) - (+0.1023) = -0.0127 (-0.0897) - (-0.1053) = +0.0156
(11-30)+(3-54) x1 - (-0.0017) - (-0.0020)
(11-540)+(3-3) x1 - (+0.1040) - (-0.1032)
322-430  (+0.0087) - (-0.0239) = +0.0326  (-0.0087) - (+0.0264) = -0.0351
(322-0)+(-2-43) x3  (-0.0011)  - (+0.0250)  - (+0.0251)
(222-40)+(3-3)  x1  (-0.0250)  - (+0.0251)

332-50  (-0.0056) - (-0.0121) = +0.0065  (+0.0025) - (+0.0082) = -0.0057
(33-52)+(-21-0)  x1  (-0.0017)  - (-0.0104)  - (+0.0102)
(331-50)+(2-2)  x1  (-0.0104)  - (+0.0102)

421-20  (-0.1106) - (-0.1128) = +0.0022  (+0.1010) - (+0.1024) = -0.0014
(41-22)+(-21-0)  x2  (-0.0021)  - (-0.1107)  - (+0.1046)
(411-20)+(-2-2)  x2  (-0.1107)  - (+0.1046)

4331-540  (+0.0178) - (-0.0099) = +0.0277  (-0.0170) - (+0.0010) = -0.0278
(431-0)+(-3-54)  x2  (+0.0005)  - (-0.0104)  - (+0.0102)
(331-50)+(-4-4)  x1  (-0.0104)  - (+0.0102)

4332-20  (-0.0491) - (-0.0579) = +0.0088  (+0.0481) - (+0.0599) = -0.0118
(322-0)+(-43-2)  x2  (+0.0007)  - (+0.0586)  - (+0.0590)
(4332-0)+(2-2)  x2  (+0.0586)  - (+0.0590)

542-10  (-0.0651) - (-0.0378) = -0.0273  (+0.0693) - (+0.0422) = 0.0271
(542-1)+(-21-0)  x2  (+0.0005)  - (+0.0383)  - (+0.0417)
(5422-0)+(-1-1)  x1  (+0.0383)  - (+0.0417)

5442-20  (-0.0095) - (+0.0116) = +0.0021  (+0.0095) - (+0.0115) = -0.0020
(544-2)+(-21-0)  x1  (+0.0006)  - (+0.0122)  - (+0.0107)
(5441-0)+(2-2)  x1  (+0.0122)  - (+0.0107)

5443-30  (-0.0083) - (-0.0117) = +0.0034  (+0.0065) - (+0.0113) = -0.0048
(431-0)+(-54-3)  x2  (+0.0005)  - (+0.0122)  - (+0.0107)
(5441-0)+(3-3)  x1  (+0.0122)  - (+0.0107)

6532-20  (-0.0226) - (-0.0148) = -0.0078  (+0.0244) - (+0.0163) = 0.0081
(653-2)+(-21-0)  x1  (+0.0004)  - (+0.0152)  - (+0.0159)
(6531-0)+(2-2)  x1  (+0.0152)  - (+0.0159)

6543-40  (-0.0165) - (-0.0149) = -0.0016  (+0.0175) - (+0.0163) = +0.0012
(431-0)+(-65-4)  x1  (+0.0003)  - (+0.0152)  - (+0.0159)
(6531-0)+(4-4)  x1  (+0.0152)  - (+0.0159)

SUM OF SQUARES E43.52 (14):  0.00403747  0.00421724
-- REDUCIBLE E43.43 --

211-4330  (+0.0022) - (-0.0022) = +0.0044  (-0.0056) - (-0.0024) = -0.0032
(11-30)+(-2-43)  x2  (-0.0035)  - (-0.0039)  - (+0.0015)
(1-433)+(-21-0)  x2  (-0.0035)  - (+0.0015)

SUM OF SQUARES E43.43 (1):  0.00001936  0.00001024
-- REDUCIBLE E43.43.52 --

4311-320  (-0.0633) - (-0.0971) = +0.0338  (+0.0554) - (+0.0891) = -0.0337
(431-0)+(-1-32)  x2  (+0.0005)  - (+0.0005)  - (+0.0005)
(11-30)+(-43-2)  x1  (-0.0017)  - (-0.0020)  - (+0.0006)
(411-20)+(-3-3)  x1  (-0.0959)  - (+0.0906)

43321-10  (-0.0792) - (-0.0495) = -0.0297  (+0.0831) - (+0.0525) = +0.0306
(433-1)+(-21-0)  x1  (+0.0007)  - (+0.0008)  - (+0.0005)
(431-0)+(-32-1)  x2  (+0.0005)  - (+0.0005)  - (+0.0005)
(4332-0)+(-1-1)  x1  (-0.0507)  - (+0.0611)

SUM OF SQUARES E43.43.52 (2):  0.00202453  0.00207205
-- REDUCIBLE E43.322 --
| 3211-310 | (-0.2078) - (-0.2493) = +0.0415 | (+0.2073) - (+0.2512) = -0.0439 |
| 3220-210 | (-0.2811) - (-0.2460) = +0.0365 | (+0.2920) - (+0.2563) = +0.0357 |
| 4310-210 | (-0.1790) - (-0.2092) = +0.0302 | (+0.1989) - (+0.2253) = -0.0264 |
| 4321-430 | (-0.1591) - (-0.2148) = +0.0557 | (+0.1587) - (+0.2188) = -0.0601 |

SUM OF SQUARES E43.322 (6): 0.0144673 0.015772

Moments of order 9 and 12 built from flux J
(direct cascade only)

| m=9, "I" stands for conj(J), last character is shift |
| JJ0: +0.0034 +/- 0.0039 0 "bb-*" pairs |
| IJ0: -0.1953 +/- 0.0055 3 |
| II0: -0.1953 +/- 0.0055 3 |
| JJ1: -0.0511 +/- 0.0040 2 |
| IJ1: +0.0181 +/- 0.0039 2 |
| II1: -0.0491 +/- 0.0048 2 |
| JJ2: -0.0166 +/- 0.0033 2 |
| IJ2: +0.0011 +/- 0.0039 1 |
| II2: +0.0016 +/- 0.0034 0 |
| JJ3: +0.0067 +/- 0.0034 0 |
| IJ3: +0.0062 +/- 0.0037 0 |
| II3: -0.0090 +/- 0.0040 0 |
| JJ4: -0.0095 +/- 0.0039 0 |

m=12, no shift:
| JJ3: +0.0002 +/- 0.0037 |

In the last portion of the table above we consider 9th (and one 12th) order moments made out of shifted triplets. Here, J stand for $J = h_{i-2}h_{i-1}h_i$, while I stands for $J^*$, and the last characte rindicates the shift between triplets. So, IJJ2 means $J_{i-4}J_{i-3}J_i$. The table shows simple computed averages with no subtraction. We can evaluate subtracted components for moments with decomposition $m = 9 = 2 + 2 + 5$, such as JJ1 or $J_{i-2}J_{i-1}^*J_i$: $432 + 321 + 210 = 22 + 33 + 42110$. The set does not account for every shift with overlap. In particular, the combinations with non-equal shifts, such as $J_{i-3}J_{i-1}J_i$, are not considered.