On the application of Sylvester’s law of inertia to QUBO formulations for systems of linear equations

Sun Woo Park\textsuperscript{a} and Kyungtaek Jun\textsuperscript{b,*}

\textsuperscript{a}Department of Mathematics, University of Wisconsin-Madison, 480 Lincoln Dr., Madison, WI 53706, USA / National Institute for Mathematical Sciences, 463-1 Jeonmin-dong, Yuseong-gu, Daejon, 34047, Republic of Korea.

\textsuperscript{b}Research Center, Innovation on Quantum and Computed Tomography, Seoul, Republic of Korea

ABSTRACT

Previous research on quantum annealing methods focused on effectively modeling systems of linear equations by utilizing quadratic unconstrained binary optimization (QUBO) formulations. These studies take part in enhancing quantum computing algorithms, which extract properties of quantum computers suitable for improving classical computational models. In this paper, we further develop the QUBO formulations of systems of linear equations by applying Sylvester’s law of inertia, which explores matrix congruence of any real symmetric matrix to a diagonal matrix. We expect that the proposed algorithm can effectively implement higher dimensional systems of linear equations on a quantum computer. In particular, the proposed algorithm hints the linear correspondence between the number of unknown variables of a linear system and the number of qubits supported on a quantum annealing device. Further experimental verification of the proposed QUBO models as well as their comparisons to classical algorithms are also made.

Keywords: Quantum Annealing, QUBO, Sylvester’s law of inertia, Systems of linear equations

1. INTRODUCTION

Recent research on quantum computers focus on devising an effective and efficient algorithm which surpasses or complements previously studied algorithms. Among the newly proposed quantum algorithms, quantum annealing method focuses on devising an optimization problem which minimizes an energy level function induced from pre-existent classical models.\textsuperscript{1,2} The problem of solving a system of linear equations, which is the focus of this paper, can be also reformulated as an energy minimization problem. In particular, quadratic unconstrained binary optimization (QUBO) formulations, proposed by Borle and Lomonaco,\textsuperscript{3} can be utilized to constructing a quadratic energy function for solving a system of linear equations.\textsuperscript{2,3} Quantum annealing processors effectively parallelize the procedure of solving the minimization problem by utilizing the base 2 representation of real numbers.\textsuperscript{2,3} This has potential to greatly reduce computational complexity in solving higher dimensional systems of linear equations compared to previously studied methods. However, physical restraints of contemporary quantum computers may hinder such potential of QUBO formulations. These include partial connectivity of qubits and the limited number of implementable qubits on a quantum computer.\textsuperscript{3,4} It is therefore a crucial question to ask what quantum algorithmic simplifications are required to surpass classical algorithms in solving a system of linear equations with large number of variables.

Recent research on such QUBO formulations focuses on utilizing matrix congruence to obtain a new simplified model for solving linear systems of equations.\textsuperscript{5} This congruence relation is known as “Sylvester’s law of inertia”, which guarantees the transformations of real symmetric matrices to diagonal matrices, regardless of whether the given symmetric matrices are invertible or not.\textsuperscript{6} We further examine the proposed methodology to show that Sylvester’s law can significantly extend the implementable number of variables of a system of linear equations on
a quantum computer. This feature is obtained from the fact that the proposed QUBO formulation substantially decreases the occurrences of required entanglements among utilized qubits, which reduces the required qubit connectivity of quantum computers.

We expect that the proposed feasible optimization technique opens up numerous possibilities to effectively and efficiently implement a wide range of systems of linear equations on a quantum annealing device. One particular merit of the proposed method, which we will primarily focus on this paper, is that Sylvester’s law of inertia hints the linear correspondence between the number of unknown variables of a linear system and the number of qubits supported on a quantum annealing device. For instance, we conjecture that the upcoming D-wave 7000Q quantum annealing device can process systems of linear equations with \( \frac{7}{2} \) times more unknown variables than those implementable on the D-wave 2000Q quantum computer. Careful constructions and examinations of some examples are provided for verification of the potential effectiveness, efficiency, and extensiveness of the proposed algorithm.

### 2. METHOD

#### 2.1 Sylvester’s Law of Inertia

We recall how Sylvester’s Law of Inertia simplifies the QUBO formulation of systems of linear equations.\(^5\)

Let \( A = (a_{i,j})_{i,j=1}^{n,n} \in \mathbb{R}^{n \times n} \) be an invertible real matrix, and \( b = (b_i)_{i=1}^n \in \mathbb{R}^n \) a real column vector. We denote the vector of \( n \) unknown variables as \( x = (x_i)_{i=1}^n \in \mathbb{R}^n \). Suppose we have the system of linear equations given by

\[
Ax = b. \tag{1}
\]

The solution which minimizes the \( l^2 \)-norm of the equation

\[
\|Ax - b\|^2 = x^T A^T Ax - 2b^T Ax + bb^T \tag{2}
\]

solves (1).

Using the \( l^2 \)-norm, we define the energy level function:

\[
f(x) = x^T A^T Ax - 2b^T Ax. \tag{3}
\]

Then the \( l^2 \)-norm minimizing solution of (2) is the solution \( x^* \in \mathbb{R}^n \) which satisfies

\[
f(x^*) = -bb^T. \tag{4}
\]

Recall that \( A^T A \) is a positive semidefinite symmetric matrix over \( \mathbb{R} \). Sylvester’s law of inertia, which is stated below, implies that \( A^T A \) can be transformed into a diagonal matrix \( D \in \mathbb{R}^{n \times n} \) using matrix congruence relations.

**Theorem 2.1 (Sylvester’s Law of Inertia).** Let \( S \in \mathbb{R}^{n \times n} \) be any real symmetric matrix. Then there exists a diagonal matrix \( D = (d_{i,j})_{i,j=1}^n \in \mathbb{R}^{n \times n} \) and a non-singular real matrix \( R = (r_{i,j})_{i,j=1}^n \in \mathbb{R}^{n \times n} \) such that

\[
D = R^T SR. \tag{5}
\]

Furthermore, the number of non-zero diagonal entries of \( D \) is equal to the rank of the matrix \( S \).

Using (5), the equation (2) can be written as:

\[
\|Ax - b\|^2 = (R^{-1}x)^T R^T A^T AR(R^{-1}x) - 2b^T AR(R^{-1}x) + bb^T \tag{6}
\]

Let \( y \in \mathbb{R}^n \) be a new vector of \( n \) unknown variables defined as

\[
y = R^{-1}x. \tag{7}
\]
In this subsection, we demonstrate that the proposed QUBO model and Borle and Lomonaco.

The above formulation allows $y_i$ to take both positive and negative values. In fact, the set of qubits satisfy the condition that for any digits $-m \leq l_1, l_2 \leq m$,

$$q_{i,l_1}^+, q_{i,l_2}^- = 0.$$  

Under this choice of representation, we can approximate (10) as follows. The first term of (10) reduces to:

$$\sum_{i=1}^{n} d_{i,i} y_i^2 \approx \sum_{i=1}^{n} d_{i,i} \left( \sum_{l=-m}^{m} 2^l q_{i,l}^+ - \sum_{l=-m}^{m} 2^l q_{i,l}^- \right)^2$$

$$= \sum_{i=1}^{n} d_{i,i} \left( \left( \sum_{l=-m}^{m} 2^l q_{i,l}^+ \right)^2 + \left( \sum_{l=-m}^{m} 2^l q_{i,l}^- \right)^2 \right)$$

$$= \sum_{i=1}^{n} d_{i,i} \left( \sum_{l=-m}^{m} 2^{2l} \left( q_{i,l}^+ \right)^2 + \left( q_{i,l}^- \right)^2 \right) + \sum_{l_1 < l_2} 2^{l_1+l_2+1} \left( q_{i,l_1}^+ q_{i,l_2}^- + q_{i,l_1}^- q_{i,l_2}^+ \right)$$

$$= \sum_{i=1}^{n} d_{i,i} \left( \sum_{l=-m}^{m} 2^{2l} \left( q_{i,l}^+ + q_{i,l}^- \right) + \sum_{l_1 < l_2} 2^{l_1+l_2+1} \left( q_{i,l_1}^+ q_{i,l_2}^- + q_{i,l_1}^- q_{i,l_2}^+ \right) \right)$$

The second term of (10) reduces to:

$$2 \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{m} b_k a_{k,j} r_{j,i} y_i \approx \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{m} \sum_{l=-m}^{m} 2^{l+1} b_k a_{k,j} r_{j,i} \left( q_{i,l}^+ - q_{i,l}^- \right).$$

2.2 QUBO models

Quantum annealing method can be used to approximate the $l^2$-norm minimizing solution of (8) by utilizing a combination of qubits $q_{i,j} \in \{0, 1\}$.

In this subsection, we demonstrate that the proposed QUBO model greatly reduces the complexity of the energy function associated to system of linear equations, compared to that obtained from previously studied quantum annealing approaches.

The number of qubits required for implementing the QUBO model depends on the choice of the representations of variables. In this manuscript, we consider two types of representations, as proposed by O’Malley and Vesselinov, and Borle and Lomonaco.

We start with the base-2 representation of the column vector $y^2$:

$$y_i \approx \sum_{l=-m}^{m} 2^l q_{i,l}^+. \sum_{l=-m}^{m} 2^l q_{i,l}^-$$

The energy function from (3) can be reformulated as:

$$f(y) = y^T D y - 2b^T A R y = \sum_{i=1}^{n} d_{i,i} y_i^2 - 2 \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{m} b_k a_{k,j} r_{j,i} y_i.$$
The associated QUBO model is the summation of (13) and (14), i.e.

$$f(y) = \sum_{i=1}^{n} d_{i,i} \left( \sum_{l=-m}^{m} 2^{2l} \left( q_{i,l}^{+} + q_{i,l}^{-} \right) + \sum_{l_{1} < l_{2}} 2^{l_{1}+l_{2}+1} \left( q_{i,l_{1},l_{2}}^{+} + q_{i,l_{1},l_{2}}^{-} \right) \right)$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{m} 2^{l+1} b_{k} a_{k,j} r_{j,i} \left( q_{i,l}^{+} - q_{i,l}^{-} \right)$$

(Borle and Lomonaco’s approximation of $y$ on the other hand, takes the following form:

$$y_{i} \approx -2^{m+1} q_{i}^{-} + \sum_{l=-m}^{m} 2^{l} q_{i,l}^{+}$$

This approach leads to a new QUBO model which further reduces the number of utilized qubits. The first term of (9) is approximated by:

$$\sum_{i=1}^{n} d_{i,i} y_{i}^{2} \simeq \sum_{i=1}^{n} d_{i,i} \left( -2^{m+1} q_{i}^{-} + \sum_{l=-m}^{m} 2^{l} q_{i,l}^{+} \right)^{2}$$

$$= \sum_{i=1}^{n} d_{i,i} \left( 2^{2m+2} \left( q_{i}^{-} \right)^{2} + \sum_{l=-m}^{m} \left( 2^{2l} \left( q_{i,l}^{+} \right)^{2} - 2^{m+l+2} q_{i,l}^{-} q_{i,l}^{+} \right) + \sum_{l_{1} < l_{2}} 2^{l_{1}+l_{2}+1} q_{i,l_{1},l_{2}}^{+} q_{i,l_{1},l_{2}}^{-} \right)$$

(17)

The second term of (9) simplifies to:

$$2 \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} b_{k} a_{k,j} r_{j,i} y_{i} \approx \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} b_{k} a_{k,j} r_{j,i} \left( -2^{m+2} q_{i}^{-} + \sum_{l=-m}^{m} 2^{l+1} q_{i,l}^{+} \right)$$

(18)

Under this approximation technique, the QUBO model can be summarized as the summation of (17) and (18).

$$f(y) = \sum_{i=1}^{n} d_{i,i} \left( 2^{2m+2} q_{i}^{-} + \sum_{l=-m}^{m} 2^{2l} q_{i,l}^{+} - \sum_{l=-m}^{m} 2^{m+l+2} q_{i,l}^{-} q_{i,l}^{+} + \sum_{l_{1} < l_{2}} 2^{l_{1}+l_{2}+1} q_{i,l_{1},l_{2}}^{+} q_{i,l_{1},l_{2}}^{-} \right)$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{m} b_{k} a_{k,j} r_{j,i} \left( -2^{m+2} q_{i}^{-} + \sum_{l=-m}^{m} 2^{l+1} q_{i,l}^{+} \right)$$

(19)

After the solution $y^{*}$ for the $l^{2}$-norm minimization problem is obtained, we use the matrix $R$ to obtain the original solution to (1):

$$x^{*} = R y^{*}$$

(20)

2.3 Example: System of invertible linear equations in 2 variables

Throughout the upcoming sections, we represent the variables of the linear system of equations using base-2 digits as given by O’Malley and Vesselinov (11). We also assume that $A$, from the equation $Ax = b$, is an invertible matrix.

We illustrate with an example to demonstrate the significance of applying Sylvester’s law of inertia. This section closely follows the previous work by authors of this paper and their collaborators. Suppose we have a linear system $Ax = b$ given by:

$$\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x_{1} \\ x_{2} \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}.$$  

(21)
The symmetric matrix $A^T A$ can be diagonalized to matrix $D$ using the invertible matrix $R$.

$$D = \begin{pmatrix} \frac{8}{5} & 0 \\ 0 & \frac{98}{125} \end{pmatrix}, \quad R = \begin{pmatrix} \frac{8}{5} & -\frac{1}{5} \\ 0 & \frac{1}{5} \end{pmatrix}$$  \hspace{1cm} (22)$$

We use base-2 digits to approximate $y = (y_1, y_2)^T = R^{-1}x$:

$$y_i = \sum_{l=1}^{3} 2^{l-1} q_{i,l} - \sum_{l=1}^{3} 2^{l-1} q_{i,l+3}$$  \hspace{1cm} (23)$$

In accordance to (12), the set of qubits $\{q_{i,j}\}$ satisfy:

$$\begin{cases} 
q_{i,l_1, l_2} = 0 \\
q_{i,l_2} = q_{i,l_1} 
\end{cases}$$  \hspace{1cm} (24)$$

where $i \in \{1,2\}$ denotes the components of the variable $y$, $j \in \{1,2,\ldots,6\}$ the digits used in the base 2 representation of $y_i$, $1 \leq l_1 \leq 3$ the positive digits, and $4 \leq l_2 \leq 6$ the negative digits.

For a 2-dimensional system of linear equations, the energy level function from (10) is given as

$$f(y) = \sum_{i=1}^{2} d_{i,j} y_i^2 - 2 \sum_{i=1}^{2} \sum_{j=1}^{2} \sum_{k=1}^{2} b_{k} a_{k,j} r_{j,i} y_i,$$  \hspace{1cm} (25)$$

where the column vector $y \in \mathbb{R}^n$ which satisfies

$$f(y) = -b^T b = -26$$  \hspace{1cm} (26)$$

solves the desired system of linear equations. Substituting (23) to (25) and further reducing the terms using conditions on products of qubits from (12) and (24), we obtain:

$$f(y) = 8 q_{11} + \frac{32}{5} q_{11} q_{12} + \frac{64}{5} q_{11} q_{13} + \frac{96}{5} q_{12} q_{13} + \frac{128}{5} q_{12} q_{14} + \frac{256}{5} q_{13} - \frac{24}{5} q_{14} + \frac{32}{5} q_{14} q_{15} + \frac{64}{5} q_{14} q_{16} - \frac{32}{5} q_{15}$$

$$+ \frac{128}{5} q_{15} q_{16} - \frac{882}{125} q_{21} + \frac{392}{125} q_{21} q_{22} - \frac{1568}{125} q_{22} + \frac{784}{125} q_{22} q_{23} + \frac{1568}{125} q_{22} q_{23} + \frac{2352}{125} q_{23} + \frac{1078}{125} q_{24}$$

$$+ \frac{392}{125} q_{24} q_{25} + \frac{2352}{125} q_{25} + \frac{784}{125} q_{24} q_{26} + \frac{1568}{125} q_{25} q_{26} + \frac{5488}{125} q_{26}.$$  \hspace{1cm} (27)$$

The above energy function gives rise to the $12 \times 12$ matrix $\hat{Q}$, defined as in (28).

$$\hat{Q} = \begin{pmatrix}
8 & 6.4 & 12.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 19.2 & 25.6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 51.2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -4.8 & 6.4 & 12.8 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -6.4 & 25.6 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -7.056 & 3.136 & 6.272 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -12.544 & 12.544 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -18.816 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8.624 & 3.136 & 6.272 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 18.816 & 12.544 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 43.904 
\end{pmatrix}.$$  \hspace{1cm} (28)$$

The qubits utilized in the QUBO model are given by the column vector $q_y = [q_{11}, q_{12}, \cdots, q_{26}]^T$, which approximates the unknown variables $y = (y_1, y_2)^T$ using base-2 digits. The cost function $f(y)$ can be reformulated in terms of $q_y$ as:

$$f(y) = q_y^T \hat{Q} q_y = y^T D y - 2b^T A R y.$$  \hspace{1cm} (29)$$
We note that the equality holds under a set of conditions specified in (24). Entanglements among pairs of qubits required for approximating \( y_i \)'s are represented as non-zero entries in the matrix \( \hat{Q} \). Finding the vector of utilized qubits \( q_{y_i} \) which satisfies \( q_{y_i}^T \hat{Q} q_{y_i} = -26 \) solves the given linear system.

We clearly observe that Sylvester’s law of inertia and a number of conditions on qubit entanglements (24) forces a substantial portion of upper triangular matrix entries of \( \hat{Q} \) to be equal to zero. This is unfortunately not the case for base-2 digit approximations of the variable \( x \) from (2). As before, we use \( q_x = [q_{11}, q_{12}, \ldots, q_{26}]^T \) to represent the set of qubits utilized for approximating the variable \( x = (x_1, x_2)^T \) using base-2 digits. The energy function formulating the system of linear equations gives rise to the characterizing matrix \( \hat{Q}' \):

\[
\hat{Q}' = \begin{pmatrix}
26 & 40 & 80 & -20 & -40 & -80 & 2 & 4 & 8 & -2 & -4 & -8 \\
0 & 72 & 160 & -40 & -80 & -160 & 4 & 8 & 16 & -4 & -8 & -16 \\
0 & 0 & 224 & -80 & -160 & -320 & 8 & 16 & 32 & -8 & -16 & -32 \\
0 & 0 & 0 & -6 & 40 & 80 & -2 & -4 & -8 & 2 & 4 & 8 \\
0 & 0 & 0 & 0 & 8 & 160 & -4 & -8 & -16 & 4 & 8 & 16 \\
0 & 0 & 0 & 0 & 0 & 96 & -8 & -16 & -32 & 8 & 16 & 32 \\
0 & 0 & 0 & 0 & 0 & 0 & -13 & 20 & 40 & -10 & -20 & -40 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 & 80 & -20 & -40 & -80 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & -40 & -80 & -160 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 23 & 20 & 40 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 56 & 80 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 152
\end{pmatrix},
\]  

where \( x^T \hat{Q}' x = x^T A^T A x - 2b^T A x \) up equivalence relation \( q_{y_i}^2 = q_{i,j} \). Then, we have

\[
q_{y_i}^T \hat{Q}' q_{x} = x^T A^T A x - 2b^T A x
\]  

The qubit entanglements required for representing \( x_i \)'s correspond to upper triangular entries of the matrix \( \hat{Q}' \). The solution to the quadratic equation \( q_{y_i}^T \hat{Q}' q_{x} = -26 \) is the solution to the linear system (21).

3. IMPLEMENTATION

This section verifies how Sylvester’s law of inertia significantly boosts both effectiveness and extensiveness of QUBO formulations of systems of linear equations on a quantum annealing device. For all experiments we use the D-Wave 2000Q quantum annealer to process the system of linear equations. We approximate the unknown variables of a given system of linear equations using O’Malley and Vesselov’s technique to effectively compare the proposed QUBO formulation to previously studied models.

3.1 Effectiveness

As shown in previous research, we process the system of linear equations in 2 variables from Section 2.3 on the D-Wave 2000Q quantum annealer. We perform 3 trials of both QUBO models (29, 31) on the D-Wave system using 10,000 anneals to verify the strength of utilizing Sylvester’s theorem.

3.1.1 Vanilla model

The vanilla QUBO model (31) computes all possible combinations of qubits \( q_x \) for \((x_1, x_2) = (-1, 2)\), which are solutions obtained from the 2-dimensional linear system without using Sylvester’s law of inertia. There are 7 qubit combinations for \( x_1 = -1 \), and 6 qubit combinations for \( x_2 = 2 \) under the base-2 representation from (23).

\[
(q_{11}, q_{12}, q_{13}, q_{14}, q_{15}, q_{16}) \in \left\{ (0, 0, 0, 1, 0, 0), (0, 1, 0, 1, 1, 0), \right. \\
(0, 0, 1, 1, 0, 1), (0, 1, 1, 1, 1), \\
(0, 0, 0, 0, 1), (1, 0, 1, 0, 1), \\
(1, 1, 0, 0, 0, 1) \right\}
\]

\[
(q_{21}, q_{22}, q_{23}, q_{24}, q_{25}, q_{26}) \in \left\{ (0, 0, 1, 0, 1, 0), (0, 1, 0, 0, 0, 0), \right. \\
(0, 1, 1, 0, 0, 1), (1, 0, 1, 1, 0), \\
(1, 1, 0, 1, 0, 0), (1, 1, 1, 0, 1) \right\}
\]
The implementation results of the vanilla model is provided in Table 1. We abbreviate the set of all possible qubit combinations for $x_2 = 2$. Each row lists the number of anneals which achieves the given combination of qubits and minimizes the energy function. The vanilla model minimizes the energy function 887, 1181, and 1065 times out of 10,000 anneals.

### Table 1. Number of occurrences which minimizes the energy function obtained from (30).

| $q_{11}$ | $q_{12}$ | $q_{13}$ | $q_{14}$ | $q_{15}$ | $q_{16}$ | $q_{21}$ | $q_{22}$ | $q_{23}$ | $q_{24}$ | $q_{25}$ | $q_{26}$ | Energy | # Occurrences |
|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|---------|--------------|
| 0        | 0        | 0        | 1        | 0        | 0        | 0        | 1        | 0        | 0        | 0        | 0        | -26.0   | 203       |
| 0        | 1        | 0        | 1        | 1        | 0        | 0        | 0        | 0        | 0        | 0        | 0        | -26.0   | 77       |
| 0        | 0        | 1        | 0        | 0        | 1        | 0        | 0        | 0        | 0        | 0        | 0        | -26.0   | 131      |
| 0        | 1        | 1        | 1        | 1        | 1        | 0        | 0        | 0        | 0        | 0        | 0        | -26.0   | 71       |
| 1        | 0        | 0        | 0        | 0        | 1        | 0        | 0        | 0        | 0        | 0        | 0        | -26.0   | 75       |
| 1        | 0        | 1        | 0        | 1        | 0        | 0        | 0        | 0        | 0        | 0        | 0        | -26.0   | 83       |
| 1        | 1        | 0        | 0        | 0        | 0        | 1        | 0        | 0        | 0        | 0        | 0        | -26.0   | 116      |
| 1        | 1        | 0        | 0        | 0        | 0        | 0        | 1        | 0        | 0        | 0        | 0        | -26.0   | 51       |
| 1        | 0        | 0        | 0        | 0        | 1        | 0        | 0        | 0        | 0        | 0        | 0        | -26.0   | 71       |
| 1        | 0        | 1        | 0        | 1        | 0        | 0        | 0        | 0        | 0        | 0        | 0        | -26.0   | 116      |
| 1        | 1        | 0        | 0        | 0        | 0        | 1        | 0        | 0        | 0        | 0        | 0        | -26.0   | 51       |
|          |          |          |          |          |          |          |          |          |          |          |          |         | Total      |
|          |          |          |          |          |          |          |          |          |          |          |          |         | 887       |
|          |          |          |          |          |          |          |          |          |          |          |          |         | 1181      |
|          |          |          |          |          |          |          |          |          |          |          |          |         | 1065      |

The column “Zero Terms” specifies whether the zero terms of the characterizing matrix are explicitly implemented on the D-wave system. The list of all possible qubit combinations for $x_2 = 2$ is omitted from the table.

### Table 2. Number of occurrences which minimizes the energy function obtained from (28).

| $q_{11}$ | $q_{12}$ | $q_{13}$ | $q_{14}$ | $q_{15}$ | $q_{16}$ | $q_{21}$ | $q_{22}$ | $q_{23}$ | $q_{24}$ | $q_{25}$ | $q_{26}$ | Energy | Zero Terms | # Occurrences |
|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|---------|------------|--------------|
| 0        | 0        | 0        | 0        | 1        | 0        | 1        | 0        | 0        | 0        | 0        | 0        | -26.0   | Yes        | 1526         |
| 0        | 0        | 0        | 0        | 0        | 1        | 0        | 1        | 0        | 0        | 0        | 0        | -26.0   | No         | 2103         |
| 0        | 0        | 0        | 0        | 0        | 1        | 0        | 1        | 0        | 0        | 0        | 0        | -26.0   |            |              |

Table 2. Number of occurrences which minimizes the energy function obtained from (28). The column “Zero Terms” specifies whether the zero terms of the characterizing matrix are explicitly implemented on the D-wave system.

The implementation results of the vanilla model is provided in Table 1. We abbreviate the set of all possible qubit combinations for $x_2 = 2$. Each row lists the number of anneals which achieves the given combination of qubits and minimizes the energy function. The vanilla model minimizes the energy function 887, 1181, and 1065 times out of 10,000 anneals.

#### 3.1.2 Proposed model

Sylvester’s law of inertia transforms the solution of the linear system of equations (21) to $(y_1, y_2) = (-2, 5)$. The objective of the new simplified QUBO model (29) is to obtain the unique combination of qubits $y_i = q_{i1} + 2q_{i2} + 4q_{i3} - q_{i4} - 2q_{i5} - 4q_{i6}$:

\[
(q_{11}, q_{12}, q_{13}, q_{14}, q_{15}, q_{16}) = (0, 0, 0, 1, 0, 0)
\]
\[
(q_{21}, q_{22}, q_{23}, q_{24}, q_{25}, q_{26}) = (1, 0, 1, 0, 0, 0)
\]

Specifying the zero terms of the matrix characterizing the system of linear equations may affect the performance of the quantum algorithm on the D-wave computer. To verify this, the proposed model can be executed in two ways. One way is to explicitly code the zero terms appearing on the upper triangular region of the characterizing matrix $\hat{Q}$ to the D-wave system; the other way is to omit them. We display the number of occurrences of the desired qubits in Table 2. The former minimizes the energy function 1526, 2495, and 2063 times out of 10,000 anneals. On the other hand, the latter obtains the lowest energy level for 2103, 4441, and 1727 anneals out of 10,000 anneals.

#### 3.2 Extensiveness

We shall now empirically assess the comparative extensiveness of the proposed QUBO model (10) to that of the vanilla QUBO model (3) in processing a system of linear equations. Consider the following 4 systems of linear equations in 2, 3, 4, and 5 variables:

\[
\begin{pmatrix}
3 & 1 \\
-1 & 2
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
=
\begin{pmatrix}
-1 \\
5
\end{pmatrix}
\]

\[
\begin{pmatrix}
5 & 0 & 1 \\
-1 & 2 & 1 \\
3 & 2 & 3
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
=
\begin{pmatrix}
-65 \\
-15 \\
-80
\end{pmatrix}
\]
Table 3. Number of qubits implementable on a D-wave 2000Q quantum annealer using the vanilla QUBO model (3). Experimental results suggest that the vanilla QUBO model can utilize at most 64 qubits.

| Linear system | # variables | # base-2 digits | # utilized qubits | Compile |
|---------------|-------------|-----------------|------------------|---------|
| (34)          | 2           | 20              | 40               | ○       |
| (35)          | 3           | 20              | 60               | ○       |
| (36)          | 4           | 20              | 80               | ×       |
| (34)          | 2           | 6               | 12               | ○       |
| (34)          | 2           | 32              | 64               | ○       |

Table 4. Number of qubits implementable on a D-wave 2000Q quantum annealer using the proposed QUBO model (10). Experimental results suggest that the proposed QUBO model can utilize at most 132 qubits.

| Linear system | # variables | # base-2 digits | # utilized qubits | Compile |
|---------------|-------------|-----------------|------------------|---------|
| (34)          | 2           | 20              | 40               | ○       |
| (35)          | 3           | 6              | 12               | ○       |
| (36)          | 4           | 20             | 60               | ○       |
| (37)          | 5           | 20             | 80               | ○       |
| (34)          | 2           | 6              | 12               | ○       |
| (34)          | 2           | 60             | 120              | ○       |
| (34)          | 2           | 66             | 132              | ○       |
| (34)          | 2           | 68             | 136              | ×       |
| (34)          | 2           | 70             | 140              | ×       |
| (34)          | 2           | 80             | 160              | ×       |
| (34)          | 2           | 100            | 200              | ×       |

We determine the extensiveness of both QUBO models (3, 10) by verifying whether the D-wave quantum annealer using 1,000 anneals compiles the given system of linear equations (34, 35, 36, 37) with predetermined number of base-2 digits representing the unknown variables. For example, a system of linear equation in 2 variables where each variable is approximated with 20 base-2 digits requires the QUBO model to utilize $20 \times 2 = 40$ qubits on the D-wave quantum annealer. Exemplary compilations of these systems of linear equations can be found in the following github repository: https://github.com/ktfriends/Quantum_Computing/tree/main/Sylvester_Theorem

3.2.1 Vanilla model

As shown in Table 3, the D-wave quantum annealer compiles the vanilla QUBO model (3) which utilizes at most 64 qubits. We omitted the experimental results obtained for QUBO models utilizing more than 80 qubits, for all these attempts failed to compile on the D-wave quantum annealer.

3.2.2 Proposed model

Experimental results suggest that the D-wave quantum annealer compiles the proposed QUBO model (10) utilizing at most 132 qubits, as demonstrated in Table 4.
4. DISCUSSION

To examine whether the proposed model is effective for processing high dimensional systems of linear equations, the experimental results should verify the following key criteria:

- **Effectiveness**: Guarantees enhanced and cost-efficient performance with the simplified QUBO model
- **Extensiveness**: Extends implementable QUBO model with reduced qubit connectivity

In this section, we carefully examine and assess how the implementation results from the previous section achieve all three criteria for extending the number of implementable variables of systems of linear equations. We expect that the proposed model effectively enhances efficiency, accuracy, and extensiveness of previously studied QUBO models for solving a system of linear equations on a quantum annealing device.

4.1 Effectiveness and the simplified characterization matrix

As demonstrated in Section 3.1, both implementations of the proposed QUBO model (29) clearly surpasses the vanilla QUBO model (31). A list of average number of occurrences and the average probability of obtaining the lowest energy levels obtained from all three experiments is provided in Table 5.

| # Trial | Vanilla QUBO model (Table 1) | New QUBO model (Table 2, row 1) | New QUBO model (Table 2, row 2) |
|---------|-----------------------------|---------------------------------|---------------------------------|
| Run 1   | 887                         | 1526                            | 2103                            |
| Run 2   | 1181                        | 2495                            | 4441                            |
| Run 3   | 1065                        | 2063                            | 1727                            |

Average # Occurrences: 1044, 2028, 2758
Average Probability: 10.44%, 20.28%, 27.58%

Table 5. A summary of the average number of occurrences that minimize the energy level function from Section 3.1 for all three models.

The new QUBO model minimizes the energy function which formulates (21) with average probability ranging between 20.28% and 27.58%. These values are at least twice to that obtained from the vanilla QUBO model, which is 10.44%. Among the two implementations of the new QUBO model, omitting the zero entries from the code significantly improves the accuracy of the algorithm.

The block diagonalization of the matrix \( Q \) characterizing the QUBO model is a key contributor to the outperformance of the new QUBO model. Suppose we want to obtain a solution of a system of \( n \) linear equations with \( n \) variables. We further assume that we use \( 2m \) digits to represent the variables using base-2 digits (11). The matrix \( Q \) characterizing the QUBO model of the systems of linear equations is a \( 2mn \times 2mn \)-matrix.

The number of non-zero entries of \( Q_{\text{vanilla}} \) characterizing the vanilla QUBO model, as in (31), satisfies the following inequality:

\[
\#\text{non-zero entries of } Q_{\text{vanilla}} \leq \frac{2mn(2mn+1)}{2} = mn(2mn+1). \tag{38}
\]

Sylvester’s law of inertia allows \( Q_{\text{vanilla}} \) to be decomposed into \( 2n \) block matrices of dimension \( m \times m \), all of which are upper triangular. The number of non-zero entries of the newly obtained matrix \( Q_{\text{new}} \), as in (29), satisfies

\[
\#\text{non-zero entries of } Q_{\text{new}} \leq 2n \times \frac{m(m+1)}{2} = mn(m+1). \tag{39}
\]

Therefore, assuming that both matrices \( Q_{\text{vanilla}} \) and \( Q_{\text{new}} \) have maximal numbers of non-zero entries, Sylvester’s law of inertia significantly reduces the number of non-zero entries by at least a factor of \( 1/2n^2 \), and at most a factor of \( 1/2n-1 \):

\[
\frac{1}{2n} \leq \frac{\#\text{non-zero entries of } Q_{\text{new}}}{\#\text{non-zero entries of } Q_{\text{vanilla}}} = \frac{m+1}{2mn+1} \leq \frac{1}{2n-1}. \tag{40}
\]

For instance, when \( n = 2 \) and \( m = 3 \), there are at most 78 non-zero entries in \( Q_{\text{vanilla}} \), \( \hat{Q} \) from (30). However, there are at most 24 non-zero entries in \( Q_{\text{new}} = \hat{Q} \) from (30), the number of which is less than a third of 78.
4.2 Extensiveness and reduced qubit connectivity

The strength of the proposed QUBO model can be also found from its extensiveness in compiling a wide variety of systems of linear equations. Further implementations of linear systems on D-wave quantum annealer, as demonstrated in Section 3.2, clearly show that the proposed model allows compilations of higher dimensional systems of linear equations that previous models were not able to achieve. In fact, experimental verification suggests that the proposed model doubles the utilisable number of qubits compared to other QUBO formulations.

Such expansions in a range of implementable system of equations originates from substantial reduction in the connectivity of qubits required to perform the QUBO model on quantum computers. The key philosophy behind the proposed methodology is to transform a dense matrix characterizing the systems of linear equations to a sparse matrix using matrix congruence relations. This allows room for implementing a wider range of systems of linear equations on a quantum computer without substantially increasing the number of utilized qubits.

We demonstrate the aforementioned advantage with the characterizing matrices $\hat{Q}$ and $\hat{Q}'$ from (30) and (28). For now, let’s assume that all qubit quadratic terms appearing in the energy function (29) let the corresponding pair of qubits to be simultaneously entangled. The matrix $\hat{Q}'$ from the vanilla QUBO model (30), requires simultaneous couplings of all pairs of 12 qubits, as shown in the graph on the left of Figure 1 (We note that the graphs are drawn in a similar fashion to those appearing on a previous work by Zbinden et al.\textsuperscript{11}).

On the other hand, our proposed model is characterized by the matrix $\hat{Q}$ from (28), which can be decomposed into 4 blocks of $3 \times 3$ upper triangular matrices. This implies that any two qubits which process different blocks of matrices do not have to be entangled. Therefore, we require that 3 out of 12 qubits have to be interconnected on the quantum computer. For example, the following diagram of qubit connections is sufficient to simultaneously process the matrix $\hat{Q}$. We see that only a set of three qubits (\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}, \{10, 11, 12\}) are fully connected. The diagram of qubit connections displayed in the graph on the right of Figure 1 is far simpler than the one required for implementing the vanilla QUBO model. The stark difference between the connectivity of qubits required for two QUBO formulations becomes clearly noticeable for system of linear equations utilizing 40 qubits, see Figure 2 for instance.

In general, suppose that we want to solve a system of $n$ linear equations with $n$ variables. We further assume that we use $m$ positive digits and $m$ negative digits to approximate unknown variables using base 2 digits. The
Figure 2. A graph of connectivity of 40 qubits required to simultaneously process the characterizing matrix of the vanilla QUBO model (left) and that of the proposed QUBO model (right). The QUBO model formulates a system of linear equations in 2 variables where each variable is approximated by 20 base-2 digits. Note that 10 out of the 20 qubits are used to represent positive digits, whereas the other 10 are used to represent negative digits.

| QUBO model | Qubit Connectivity |
|------------|--------------------|
| Vanilla    | Fully connected network of $2mn$ qubits |
| Proposed   | $2n$ sets of fully connected network of $m$ qubits |

Table 6. A table summarizing the required qubit connectivity for simultaneously processing the characterizing matrix of the QUBO model for a system of linear equations with $n$ variables.

The underlying topology of connectivity of qubits on the D-wave 2000Q quantum annealer is a key factor that determines the upper bound on the number of qubits the given QUBO formulation can execute. The D-wave 2000Q quantum annealer uses the Chimera topology $C_{16,16,4}$ to model the connectivity of 2048 qubits.\textsuperscript{12,14} Given a positive number $m$, the space $C_{m,m,4}$, comprised of $4m^2$ qubits, is an interconnected $m \times m$ grid of bipartite graphs $K_{4,4}$. There exists a natural embedding of the complete graph with $4m$ nodes $K_{4m}$ inside $C_{m,m,4}$ as a graph minor, corresponding to a triangle on the $m \times m$ grid consisting of $4m$ chains of length $m + 1$.\textsuperscript{12,13} As for the topology $C_{16,16,4}$, the space has a naturally embedded complete graph $K_{64}$ inside $C_{16,16,4}$ as a graph minor.

$$K_{64} \hookrightarrow C_{16,16,4} \quad (41)$$

Observant readers may have already noticed that the number of nodes of the complete graph that can be embedded in $C_{16,16,4}$ is precisely the empirical maximum number of qubits the vanilla QUBO model can process. As Figure 1 and 2 illustrate, the vanilla QUBO model utilizing $N$ qubits can be represented as a complete graph with $N$ nodes $K_N$. There doesn’t exist a natural embedding of complete graphs with number of nodes exceeding 64 into the Chimera topology $C_{16,16,4}$. Thus, the D-wave quantum annealer can process the previously studied QUBO models which utilizes up to 64 qubits.
Meanwhile, the proposed model significantly bounds the connectivity of qubits regardless of the number of variables used in the system of linear equations. As previously assumed, suppose the proposed model formulates a linear system in \( n \) variables, with each variable represented by \( m \) base-2 positive digits and \( m \) base-2 negative digits. As shown in Figure 1 and 2, the model can be represented as a union of 4 complete graphs with \( m \) nodes \( K_m \). Experimental results suggest that the D-wave 2000Q quantum annealer succeeds in embedding 4 copies of complete graphs with 33 nodes \( K_{33} \) into the space \( C_{16,16,4} \) (Note that \( K_{64} \), which includes \( K_{33} \), can be). This gives an explanation of why the proposed model can process \( 33 \times 4 = 132 \) qubits instead of the 64 qubits the previously studied QUBO models can process.

\[
\bigcup_{i=1}^{4} K_{33} \hookrightarrow C_{16,16,4} \quad (42)
\]

Even better, the proposed model establishes a linear relation between the implementable dimension of the system of linear equations on the quantum annealer and the number of qubits on the quantum annealing device. This achievement is made possible because the connectivity of qubits is solely determined from the number of base-2 digits used to represent each variable, regardless of the number of unknown variables used in the system of linear equations. For example, given a positive number \( m > 0 \), suppose the quantum annealing device supports the connectivity of \( 2048m^2 \) qubits using the Chimera topology \( C_{16m,16m,4} \). Previously researched QUBO models can process the QUBO model which utilizes \( 64m \) qubits.\(^{12,13}\) In particular, the vanilla QUBO model establishes a square-root relation between the number of qubits on the quantum annealer and the number of qubits executed by the QUBO model.

\[
\# \text{ implementable qubits (vanilla QUBO)} \leq \sqrt{2 \times (\# \text{ qubits on } C_{16m,16m,4})} \quad (43)
\]

On the other hand, the new QUBO model can process \( 132m^2 \) qubits using the same topology. This is because the topology \( C_{16m,16m,4} \) contains \( m^2 \) copies of \( C_{16,16,4} \), each space of which can process 132 qubits using the proposed model. Thus, we obtain:

\[
\# \text{ implementable qubits (proposed QUBO)} \leq \frac{33}{512} \times (\# \text{ qubits on } C_{16m,16m,4}). \quad (44)
\]

Prospective quantum annealing devices are expected to support more number of qubits by enlarging the dimensions of planar grids of the Chimera topology. In other words, one shall expect linear growth in the number of variables of systems of linear equations implementable on quantum annealing devices, as shown in (44). For example, the D-wave 7000Q quantum annealer aims to enact qubit connections on Zephyr topology \( Z_m \), obtained as a quotient space of the Chimera topology \( C_{2m+1,2m+1,8} \).\(^{14}\) We conjecture that the number of implementable unknown variables supported by the D-wave 7000Q quantum annealer will be \( \frac{7}{2} \) times more than that supported by the D-wave 2000Q quantum annealer.

4.3 Complexity

Given the two matrices \( D \) and \( R \) associated to \( A^T A \) from Sylvester’s law of inertia, our proposed method ensures that the computational complexity of solving the QUBO model is in the order of \( mn^2 \). Evaluating (14) dominates the computational burden of utilizing our proposed method. Because \( b \in \mathbb{R}^n \) is a column vector of dimension \( n \),

\[
\text{# calculations for computing } \sum_{i,j,k=1}^{n} b_k a_{k,j} r_{j,i} \left( q_{i,l}^+ - q_{i,l}^- \right) \leq 4n^2 \quad (45)
\]

In light of the observation that the range of \( l \) is between \(-m\) and \( m\),

\[
\text{# calculations for computing (14)} \leq 4(2m + 1)n^2. \quad (46)
\]

Our model hence provides polynomial speedup in solving the QUBO model in comparison to previously studied quantum annealing methods, whose computational complexities are in the order of \( mn^3 \).\(^{3,8}\)
4.4 Limitations and relation to classical algorithms

A central question which is left unexplored in this manuscript is determining the computational complexity of obtaining the two matrices $D$ and $R$ from Sylvester’s law of inertia. Suppose that $A$ is a non-singular matrix. Then the two matrices can be obtained from the $QR$ decomposition of $A$. Indeed, we may factorize $A = QR'$ where $Q \in \mathbb{R}^{n \times n}$ is a real orthogonal matrix, and $R' \in \mathbb{R}^{n \times n}$ is an upper triangular matrix. This yields the following equation:

$$(R'^{-1})^T A^T A R'^{-1} = Q^T Q = I. \quad (47)$$

The diagonal matrix $D$ can be then chosen to be the identity matrix, and $R$ the matrix $R'^{-1}$.

One limitation of the proposed method is the potential difficulty in computing the matrices $D$ and $R$ for any given real matrix $A$. Assuming that both $D$ and $R$ are obtained, however, the QUBO model can effectively parallelize the computational process of obtaining solutions to systems of linear equations.

Note that Sylvester’s law of inertia does not necessitate the use of $QR$ factorization of $A$. One only needs to guarantee that the column vectors of the real matrix $AR' \in \mathbb{R}^{n \times n}$, whose norms are the diagonal entries of $D$, are mutually orthogonal. In other words, comparative advantages that quantum algorithms possess to classical computational methods can prove to be effective in enhancing the cost efficiency of solving systems of linear equations on a quantum computer. For example, parallel computations of solutions of linear systems can be executed, which is not yet achievable on classical implementations such as QR decomposition using householder transformations. Meanwhile, QR decomposition methods can be implemented in a cost-efficient manner by utilizing intrinsic properties of quantum algorithms, including computational bases and synthesis of quantum circuits. These methods are expected to achieve polynomial speedup in computing an orthogonal basis of a vector space in comparison to previously researched classical algorithms. Thus, with prospective effective quantum implementations of Sylvester’s law of inertia, we shall expect to achieve an accurate, efficient, and feasible quantum algorithm for solving systems of linear equations with large number of variables.

Declaration of Interests

The authors have no competing interests which may have influenced the work shown in this manuscript.

Contributions

S.W.P. conceived and designed the theoretical experiments, and the theoretical solutions. K.J. obtained the experimental results in a quantum annealer D-wave 2000. All authors wrote the paper.

REFERENCES

[1] S.B. Rønnow, T. and Isakov, S., "Evidence for quantum annealing with more than one hundred qubits," Nat. Phys 10, 218-224 (2014).
[2] O’Malley, D. and Vesselinov, V. V., "Toq. jl: A high-level programming language for d-wave machines based on julia," 2016 IEEE High Performance Extreme Computing Conference (HPEC) 1-7. (2016)
[3] Borle, A. and Lomonaco, S. J., "Analyzing the quantum annealing approach for solving linear least squares problems," International Workshop on Algorithms and Computation, 289-301 (2019).
[4] Jun, K., Conley, R., Huang, Y., Lim, H., and Yu, K., "Solving linear systems by quadratic unconstrained binary optimization on D-Wave quantum annealing device,” Quantum Information Science, Sensing, and Computation XIII 11726 117260C (2021).
[5] Park, S. W., Lee, H. J., Kim, B. C., Woo, Y., and Jun, K. "On the application of Sylvester’s law of inertia to QUBO formulations for systems of linear equations,” Accepted to the Proceedings of the 12th International Conference on ICT Convergence, 1363–1367 (2021).
[6] Ayres, F., "Theory and problems of matrices," Schaum’s Outline Series 115-124 (1962).
[7] Sylvester, J. J., "XIX. A demonstration of the theorem that every homogeneous quadratic polynomial is reducible by real orthogonal substitutions to the form of a sum of positive and negative squares,” The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science 4 (23) 138–142 (1852).
[8] Jun, K. "QUBO formulations for system of linear equations” arXiv preprint arXiv:2106.10819 (2021).
[9] Ma, G., Li, H., and Zhao, J. "Quantum QR decomposition in the computational basis," *Quantum Information Processing* **19**(8), 1–16 (2020).

[10] de Brugièreme, T. G., Baboulin, M., Valiron, B., and Allouche, C. "Quantum circuits synthesis using Householder transformations," *Computer Physics Communications* **248**, 107001 (2020).

[11] Zbinden, S., B’artschi, A., Djidjev, H., and Eidenbenz, S. "Embedding algorithms for quantum annealers with Chimera and Pegasus connection topologies," *International Conference on High Performance Computing* 187–206 (2020).

[12] Boothby, K., King, A.D., and Roy, A. "Fast clique minor generation in Chimera qubit connectivity graphs," *Quantum Information Processing* **15**, 495–508 (2016).

[13] Choi, V. "Minor-embedding in adiabatic quantum computation: II. Minor-universal graph design," *Quantum Information Processing* **10** (3), 343–353 (2011).

[14] Boothby, K., King, A.D., and Raymond, J. "Zephyr Topology of D-Wave Quantum Processors" Technical report available at https://www.dwavesys.com/media/fawfas04/14-1056a-a_zephyr_topology_of_d-wave_quantum_processors.pdf 1–18 (2021).