Distributed Nonlinear MPC of Multi-Agent Systems with Data Compression and Random Delays - Extended Version

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Abstract

This is an extended version of a technical note accepted for publication in IEEE Transactions on Automatic Control. The note proposes an Input to State practically Stable (ISpS) formulation of distributed nonlinear model predictive controller (NMPC) for formation control of constrained autonomous vehicles in presence of communication bandwidth limitation and transmission delays. Planned trajectories are compressed using neural networks resulting in considerable reduction of data packet size, while being robust to propagation delays and uncertainty in neighbors’ trajectories. Collision avoidance is achieved by means of spatially filtered potential field. Analytical results proving ISpS and generalized small gain conditions are presented for both strongly- and weakly-connected networks, and illustrated by simulations.

I. INTRODUCTION

Cooperation between autonomous vehicles has shown promising advantages in terms of robustness, adaptivity, reconfigurability, and scalability. A prevalent technique for formation control is MPC for its inherent ability to handle constraints and uncertainty. Dunbar et al [1] considered distributed NMPC for synchronization of agents by broadcasting state error trajectories to the immediate neighbors. A generalized framework for distributed NMPC for cooperative control is proposed in [2], where asymptotic stability is ensured by terminal constraint set. A framework for quasi-parallel NMPC without restriction of terminal set, extended to the multi-agent case

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recently is shown to be asymptotically stable [3]. Distributed NMPC was considered for a
group of strongly connected agents receiving delayed input from their neighbors in [4]-[5].
The delayed information is projected in the prediction horizon using either a *time-based forward
forgetting-factor* or by *linear recurrence*, respectively. Collision avoidance (CA) within MPC
framework is well studied for linear systems [6], but similar work in nonlinear MPC setting
is still rare. CA among multiple vehicles is achieved by adding a repelling potential field to
local NMPC cost function and transmitting the entire planned trajectory [7]. Priority strategy for
CA in NMPC framework, using neighbors’ randomly delayed information has been proposed
in [5]. Hierarchical multi-level control is considered in [8] by combining potential field with
linear MPC, such that only the first step of the trajectory is optimized and linear recursion
is used to predict the trajectory over the remaining horizon. Stability proofs are unavailable
in most of these CA works. In this note, we address fleet control with collision avoidance of
constrained autonomous vehicles subject to limited network throughput and propagation delays
by employing distributed NMPC control. Each agent performs local optimization based on an
estimate of planned trajectories received from neighboring agents. Since network throughput
is assumed limited, exchanged trajectories are compressed using neural networks (NN) as a
universal approximator. This property is crucial in our stability analysis, since the impact of
estimation error on system dynamics is considered as a bounded non-vanishing (persistent)
disturbance. Correction for propagation delays is achieved by time-stamping each communication
packet [9]. Collision avoidance is achieved by formulating a new spatially-filtered repelling
potential field which is activated in a “gain-scheduling” type of approach to avoid transforming
the problem into mixed-integer nonlinear programming. We prove this distributed control strategy
to be ISpS for heterogeneous agents connected in strongly- or weakly-connected network, robust
to uncertainty in neighbors’ planned trajectories. This algorithm is an improvement over [4]
and contributes to the literature with the following original results: (a) Only an approximation
of planned trajectories is transmitted; (b) NN-based data compression algorithm is used in
compressing the planned trajectories; (c) collision avoidance by using a spatially filter potential
function with rigorous stability proofs; (d) new ISpS and generalized small gain conditions are
derived to ensure stability of proposed algorithm; (e) stability results are extended even to weakly
connected networks.
II. Preliminaries

Let $L_2$ Euclidean norm be denoted by $|·|$ and let $|·|_\infty$ be the $L_\infty$ norm. The identity function is denoted by $I : \mathbb{R} \to \mathbb{R}$, functional composition of two functions $\gamma_1$ and $\gamma_2$ by $\gamma_1 \circ \gamma_2$ and function inverse of function $\alpha$ by $\alpha^{-1}$. For a set $A \subseteq \mathbb{R}^n$, the point to set distance from $\zeta \in \mathbb{R}$ to $A$ is denoted by $d(\zeta, A) \triangleq \inf \{ |\eta - \zeta|, \eta \in A \}$. The difference between two sets $A, B \subseteq \mathbb{R}^n$ is denoted by $A \setminus B \triangleq \{ x : x \in A, x \notin B \}$. An indicator function of vector $x$ defined as $1_{x>0} = \{1$ if $x > 0, 0$ otherwise}, where $\succ$ is element-wise inequality. We also use class $\mathcal{K}, \mathcal{K}_\infty$ and $\mathcal{KL}$ comparison functions \cite{10}. Consider the discrete-time nonlinear system $x_{t+1} = f(x_t, w_t)$ with $f(0, 0) = 0$, where $x_t \in \mathbb{R}^n$ and $w_t \in \mathbb{R}^r$ are state and external input respectively. If $x_t \in \Xi, \forall t > t_0$ whenever $x_{t_0} \in \Xi$ and bounded input $w_t \in W$, then $\Xi$ is called a Robust Positively Invariant (RPI) set. Moreover, if $\Xi$ is compact, RPI and contains the origin as an interior point, the system $x_{t+1} = f(x_t, w_t)$ is said to be regionally Input-to-State practically Stable (ISpS) in $\Xi$ for $x_0 \in \Xi$ and $w \in W$, if there exists $\mathcal{KL}$-function $\beta$, $\mathcal{K}$-function $\gamma$ and constant $c > 0$ such that

$$|x_t| \leq \beta (|x_0|, t) + \gamma (|w|_\infty) + c \quad (1)$$

If $c \equiv 0$, then the system is said to be regionally Input-to-State Stable (ISS) in $\Xi$ \cite{10}. Function $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is an ISpS Lyapunov function in $\Xi$, if for suitable functions $\alpha_{1,2,3}, \sigma_3 \in \mathcal{K}_\infty$, $\sigma_{1,2} \in \mathcal{K}$ and constants $\bar{c}, \bar{\bar{c}} > 0$, there exists a compact and RPI set $\Xi$ and another set $\Omega \subset \Xi$ with origin as an interior-point ($\Omega$ is also RPI), such that the following conditions hold,

$$V(x_t, w_t) \geq \alpha_1 (|x_t|), \quad \forall x_t \in \Xi \quad (2)$$

$$V (f (x_t, w_t), w_{t+1}) - V (x_t, w_t) \leq -\alpha_2 (|x_t|) + \sigma_1 (|w_t|) + \sigma_2 (|w_{t+1}|) + \bar{c}, \quad \forall x_t \in \Xi \quad (3)$$

$$V (x_t, w_t) \leq \alpha_3 (|x_t|) + \sigma_3 (|w_t|) + \bar{\bar{c}}, \quad \forall x_t \in \Omega \quad (4)$$

The relation between ISpS Lyapunov functions and ISpS is shown in Theorem \cite{11}. ISS implies ISpS, but converse is not true, since an ISS system with $0$–input, i.e. $w_k = 0, \forall k \geq 0$ implies asymptotic stability to the origin, while for an ISpS system, $0$–input implies asymptotic stability to a compact set (ball of radius $c$) containing the origin. In this paper, the stability analysis will demonstrate that according to the proposed control approach, closed-loop dynamics is ISpS, not ISS, due to uncertainty resulting from data compression. Thus, in this study, $c$ in equation
is not zero but function of bounded error in NN estimation. Information exchange among networked vehicles is conveniently modeled by general mixed graphs (directed and undirected edges). An information graph is a set of nodes $A^i$ and edges connecting node pairs $E(A^i, A^j)$. Define connectivity matrix as $\Gamma = [\bar{\gamma}_{ij}]$, where $\bar{\gamma}_{ij} > 0$ if $(A^i, A^j) \in E$ and 0 otherwise (by convention $\bar{\gamma}_{ii} = 0$). Neighborhood of a node $A^i$ is $G^i := \{A^j : \bar{\gamma}_{ij} > 0\} \cup \{A^j : \bar{\gamma}_{ji} > 0\}$. A network is said to be strongly connected if there is an undirected path from any node to any other node in the network. In this case, connectivity gain matrix $\Gamma$ is irreducible. A network is said to be weakly connected if there are at least two nodes for which a directed path connecting them does not exist. For weakly connected networks, connectivity gain matrix $\Gamma$ can be reduced to upper block triangular form \textcircled{1}. Next, we will formulate the distributed multi-agent problem.

A. Distributed Multi-Agent NMPC with Collision Avoidance

Consider a set of $N$ agents $A^i$ each having nonlinear discrete-time dynamics:

$$x^i_{t+1} = f^i(x^i_t, u^i_t), \quad \forall t \geq 0, \quad i = 1, \ldots, N \quad (5)$$

Local states $x^i_t$ and control inputs $u^i_t$ belong to constrained sets $x^i_t \in X^i \subset \mathbb{R}^{n^i}, \quad u^i_t \in U^i \subset \mathbb{R}^{m^i}$. Agents are decoupled from each other in open loop. On the other hand, closed-loop control takes into account the neighbors’ states and therefore couples the dynamics. Let $\bar{w}^i_t$ be the approximation of trajectories $w^i_t = \{x^j_t\}, \forall j \in G^i$ of neighbors of $A^i$, such that $w^i_t \in W^i \subset \mathbb{R}^{p^i}$. For each agent $A^i$, the general finite-horizon cost function is defined as:

$$J^i_t = \sum_{t = t}^{t + N^i_p - 1} [h^i(x^i_t, u^i_t) + q^i(x^i_t, \bar{w}^i_t)] + h^i_f(x^i_{t+N^i_c}) \quad (6)$$

where $N^i_p$ and $N^i_c$ are prediction and control horizons respectively. Distributed cost (6) consists of local transition cost $h^i_t$, local terminal cost $h^i_f$ and interaction cost $q^i_t$, see \textcircled{2} for details. We define an agent $A^i$ to be on collision course with at least one other agent if

$$\sum_{j \in G^i} 1(R^i_{\min} - d^i_{jk}) > 0, \forall t \leq k \leq (t + N^i_p) > 0,$$

where $R^i_{\min}$ is the safety zone of an agent and $d^i_{jk}$ is the Euclidean distance between agent $A^i$ and $A^j$. Repelling potential can be formulated as:

$$\Phi^i_t = \sum_{j \in G^i} \lambda R^i_{\min} 1(R^i_{\min} - d^i_{jk}) > 0, \forall t \leq k \leq (t + N^i_p) \frac{1}{\sum_{k=t}^{t+N^i_p} \lambda(d^i_{jk})d^i_{jk}} \quad (7)$$
where \( 0 < \lambda_{\text{min}} \leq \lambda(d_{ij}) \leq \lambda_{\text{max}} \) are positive weights of a filter and are strictly decreasing in their argument, such that \( \bar{\lambda} \triangleq \sum_{k=t}^{t+N_p^i} \lambda(d_{k}^{ij}) \). If at any instant \( t \leq k \leq (t + N_p^i) \) in the prediction horizon, an agent \( A^i \) has a feasible trajectory which falls within \( R_{\text{min}}^j \) of agent \( A^j \), the repelling potential (7) becomes non-zero. To cater for collision course, cost (6) is modified as

\[
\dot{J}_i^t = J_i^t (1 + \Phi_i^t) \tag{8}
\]

Strength of potential field (7) is inversely proportional to the weighted average distance between the two agents \( \bar{d}_{ij}^t = \sum_{k=t}^{t+N_p^j} \lambda(d_k^{ij})d_k^{ij} / \bar{\lambda} \). The weights \( \lambda \), strictly decreasing with \( d_k^{ij} \), ensure that the smallest separation between two agents gets the highest weight. On the other hand, taking a simple average (i.e. \( \lambda \equiv 1 \)) or a time-based forgetting factor (\( \lambda \) is strictly decreasing with \( k \), the time index), results in poor performance in collision avoidance, as trajectories which enter very late in zone \( R_{\text{min}}^j \) (i.e. \( R_{\text{min}}^j - d_k^{ij} > 0, k \rightarrow t + N_p^i \)) have a small repelling potential (7), and hence not prevented from very early on. Such strategy results in agents getting very close before they start repelling each other to avoid collision. However, with cost (8), trajectories are immediately penalized upon falling within zone \( R_{\text{min}}^j \) and are obviously avoided in the NMPC optimization. The indicator function in (7) acts as a “gain-scheduled” binary (0-1) variable depending on whether a feasible trajectory falls within \( R_{\text{min}}^j \). We define successful collision avoidance to occur if weighted average distance between the agents on collision course increases i.e.

\[
\sum_{k=t}^{t+N_p^j} \lambda(d_k^{ij})d_k^{ij} < \sum_{k=t+1}^{t+N_p^j+1} \lambda(d_k^{ij})d_k^{ij} \tag{9}
\]

Control sequence \( u_{t,t+N_p^i}^i \) consists of \( u_{t,t+N_c^i-1}^i \) and \( u_{t+N_c^i,t+N_p^i-1}^i \). The latter part is generated by local auxiliary control law \( u_t^i = k_j^i(x_t^i) \) for \( t \geq N_c^i \), while the former is the distributed optimal control \( u_{t,t+N_c^i}^i \) which is the solution of the Problem 1. Suboptimal \( u_{t,t+N_c^i-1}^i \) satisfying all constraints is called feasible control.

**Problem 1.** At every instant \( t \geq 0 \) for each agent, given horizons \( N_p^i \) and \( N_c^i \), and auxiliary control \( k_j^i \), find the optimal control sequence \( u_{t,t+N_c^i-1}^{i,*} \) which minimizes distributed finite horizon cost (6) (or (8) for collision avoidance), satisfies state and input constraints and system dynamics (5), such that the terminal state is constrained to a terminal set, i.e. \( x_{t,N_c^i}^i \in X_j^i \). In the receding horizon strategy, only the first element of \( u_{t,t+N_c^i-1}^{i,*} \) is implemented at each instant, such that the
closed loop dynamics becomes

\[ x_{i,t+1}^i = f_i(x_{i,t}^i, u_i^*(x_{i,t}^i, w_{i,t}^i)) = \tilde{f}_i(x_{i,t}^i, w_{i,t}^i) \]  \hspace{1cm} (10)

B. Data Compression

For cooperation, agents transmit their planned state trajectories, \( x_{i,t,t+N_i^p}^i \in \mathbb{R}^{n_i \times N_i^p} \), but reception occurs after some delay \( \Delta_{ji} \). To reduce packet size, trajectory containing \( n_i \times N_i^p \) floating points is compressed by approximating it with neural network \( N^i \) of \( q^i \) weights and biases, with compression factor of \( 1 - (q^i + \text{overhead size})/(n_i \times N_i^j) \). Overhead size accounts for agent identity \( i \), time-stamp \( (T_i^s) \) and sampling time \( T_i^s \) etc. The leader also communicates formation geometry and way-points to followers. It is assumed that there exists a mechanism for synchronizing clocks, which allows delay \( \Delta_{ji} \) to be estimated. NN at \( A^i \) is trained using state trajectory as output and \( N_i^j \) discrete instants as input. Using sampling rate \( T_j^i \) and prediction horizon \( N_p^j \) at \( A^j \), re-sampled trajectory \( \tilde{w}_i^j \in W^j \subset \mathbb{R}^{n_j \times N_p^j} \) is generated using received neural network \( N^i \). If horizon is sufficiently long, states can be extrapolated with bounded error. If packet is delayed by more than a threshold \( \bar{\Delta} \), the packet is deemed to be lost. Any smooth function \( w(t) \) can be approximated arbitrarily closely on a compact set using a NN with appropriate weights and activation functions \cite{12}. Let \( w(\tau) \) be a set of smooth functions, then we can show \( \tilde{w}(\tau) = w(\tau) + \xi \), where \( \tilde{w}(\tau) \) is approximation of \( w(\tau) \) by NN, and \( \tau \triangleq \text{col}(t, t \ldots t) \) is the stack of \( t \) vector \( n_i \) times and \( \xi \) is NN approximation error which is inversely proportional to hidden-layer size \( H_L \). Error \( \xi^i_t \) in prediction also depends on the delay \( \Delta_{ij}^t \) in information received from \( A^j \) due to extrapolation of trajectory tail \( \tilde{w}_{i,t+N_i^p+\Delta_{ij}}^i \). If the error (or delay) is greater than an upper bound, i.e. \( \xi^i_t > \bar{\xi} \), a feasible control for avoiding collision may not exist. This means that agents will get too close due to error \( \xi^i_t \), such that there is not enough time to maneuver for avoiding collision. Consequently, we assume an upper bound on the permissible delay \( \Delta_{ij}^t \leq \bar{\Delta} \), which is the worst case scenario of two agents on a direct collision course at maximum permissible speed and with minimum separation between them, i.e. \( \bar{\Delta} \triangleq R_{\text{min}}/v_{\text{max}} \). With this conservative (can be relaxed) bound on \( \Delta_{ij}^t \), there is always enough time to execute collision avoidance maneuvers.
Algorithm 1 DNMPC Algorithm with Collision Avoidance

1: Given $A^i, A^i \leftarrow x_0^i, d^{bi}, d^{bi}, g^i$ \quad $\triangleright i = 1 \triangleq$ Leader, $t = 0$

2: Solve Problem $[2]$ offline for $Q^j_f$ and $K^j_f$

3: procedure COLLISION FREE DISTRIBUTED NMPC

4: Design Spatially filter potential $[13]$

5: Solve Problem $[1]$ at $A^i$ for $u^i_{t,t+N_p-1}$

6: Train NN Train Neural network for $x^i_{t,t+N_p}$

7: Implement first element/block of $u^i_{t,t+N_p-1}$

8: Transmit/Receive data packets

9: Estimate time delay $\Delta_{ij}$

10: Reconstruct $\bar{w}^{t+N_p}_{t}$ with received NN

\hspace{1cm} Increment time by one sample

\hspace{1cm} $\triangleright t^i = t^i + T_s$

11: end procedure

$\triangleright$ End CF-DNMPC Alg.

III. Stability Analysis

We first state an important new result in regional input-to-state practical stability. This general result will form the cornerstone of later development.

**Theorem 1.** If system $x_{t+1} = f(x_t, w_t)$ admits an ISpS-Lyapunov function in $\Xi$, then it is regional ISpS and satisfies condition (7), with $\beta(r, s) \triangleq \alpha_1^{-1}(3\beta(3\alpha_3(r), s)), \gamma(s) \triangleq \alpha_1^{-1}(3(\gamma(3\sum_{i=1}^{3} s_i(s)) + \beta(3\sigma_3(s), 0)))$ and $c \triangleq \alpha_1^{-1}(3(\beta(3\bar{c} + d), 0) + \alpha_1^{-1}\gamma(\mu(3\bar{c})) + \alpha_1^{-1}\gamma(3\bar{c}))$, where $\mu, \gamma \in K_{\infty}$ while $\beta \in K_{\mathcal{L}}$ (d is defined in proof).

**Proof:** for all $w_{t+1} \in W$. Since $\Omega$ is RPI, therefore for $x_1 \in \Xi \setminus \Omega$ and $x_2 \in \Omega$, there exists $d > 0$ such that $V(x_1, w_1) \leq V(x_2, w_2) + d$ for $w_{1,2} \in W$. Letting $\bar{\alpha}_3(s) \triangleq \alpha_3(s) + \sigma_3(s) + s, \bar{\alpha}_2(s) \triangleq \min(\alpha_2(s/3), \sigma_3(s/3), \mu(s/3)), \alpha_4(s) \triangleq \alpha_2(s) \circ \bar{\alpha}_3^{-1}(s), \bar{\omega} \triangleq \max(|w_{t}|_{\infty}, |w_{t+1}|_{\infty}), \omega(\bar{\omega}, \bar{c}, \bar{c}) \triangleq \sum_{i=1}^{3} \sigma_i(\bar{\omega}) + \mu(\bar{c}) + \bar{c}$ and selecting $\rho \in K_{\infty}$ such that $(I - \rho) \in K_{\infty}$, we can define a compact set $D \subset \Omega \subset \Xi$ containing the origin: $D \triangleq \{ x | d(x, d\Omega) > d_1, V(x_1, w_1) \leq \gamma(\omega) \}$, where $\gamma \triangleq \alpha_4 \circ \rho^{-1}$. With these definitions and using steps similar to equations (14)-(17) in proof of Theorem 4.1 of [4], we can show that $D$ is RPI. Moreover, $D$ can also be shown to be asymptotically attractive for state starting in $\Xi \setminus D$ using arguments similar to equations (18)-(23).
of [4]. Hence, a state $x_t$ starting in $\Xi$ will enter $\Omega \setminus D$ in finite time, and from there it will enter $D$ in finite time as well, where it shall remain as $D$ is RPI. Using a standard comparison lemma [13], $\exists \beta(r, s) \in \mathcal{KL}$ such that $V(x_t, w_t) \leq \max(\beta(V(x_0, w_0), t), \dot{\gamma}(\omega(|w_t|_\infty, \bar{e}, \bar{e})), \forall x_t \in \Xi, w_t \in W$. Using a property for $\mathcal{K}$ functions: $\alpha(r_1 + r_2 + r_3) \leq \alpha(3 \max(r_1, r_2, r_3)) \leq \alpha(3r_1) + \alpha(3r_2) + \alpha(3r_3)$, we can show that system $x_{t+1} = f(x_t, w_t)$ is regional ISpS in $\Xi, \forall x_t \in \Xi, w_t \in W$. 

We will now particularize this result for Algorithm 1. Stability is analyzed in two stages. First, individual agents are shown to be ISpS and robust to communication delays and trajectory approximation error in a subset of $X^i$, followed by generalized small gain condition for team stability.

A. Stability of Individual Agents without Collision Avoidance

Asymptotic stability (ISS) for MPC schemes can be shown in case of additive and vanishing disturbance, but only ultimate boundedness (or ISpS) can be guaranteed in case of non-vanishing (not decaying with state) uncertainties [14]. In the proposed approach, the uncertainty in trajectory approximation $\xi$ is non-vanishing and one can only guarantee ISpS. We consider first the stability of individual agent $A^i$ with respect to the information received from other agents, by exploiting Theorem 1. At this stage the interconnections are ignored, and information from neighbors is considered as external input. We assume at this stage that agents generate conflict free trajectories.

**Theorem 2.** Let terminal set $X^i_f \subset X^i$ be RPI and let $k^i_f(x^i_t), f^i(x^i_t, k^i_f(x^i_t)), w^i_{t+1}, h^i(x^i_t, u^i_t), q^i(x^i_t, w^i_t)$ $h^i_f(x^i_t, u^i_t)$ be locally Lipschitz with respect to $x^i_t, u^i_t$ and $w^i_t$ in $X^i \times U^i \times W^i$, with the following Lipschitz constants $L^i_{k_f}, L^i_f, L^i_{gw}, L^i_{hx}, L^i_{hu}, L^i_{qx}, L^i_{qw}$ and $L^i_{hf}$. Moreover, there exist nonlinear bounds $\alpha_{1,f}, \alpha_{2,f}, \xi^i \in \mathcal{K}_\infty$ such that $\xi^i(|x^i_t)| \leq h^i(x^i_t, u^i_t)$ and $\alpha_{1,f}(|x^i_t|) \leq h^i_f(x^i_t) \leq \alpha_{2,f}(|x^i_t|), \forall x^i_t \in X^i$. Now, if the neural network trajectory approximation error is bounded $|\tilde{w}_t| \leq |w_t| + \xi_t$, and the following holds for $x^i_t \in X^i_f$ and $w^i_t \in W^i$

$$h^i_f(f^i(x^i_t, k^i_f(x^i_t))) - h^i_f(x^i_t) \leq -h^i(x^i_t, k^i_f(x^i_t)) - q^i(x^i_t, \tilde{w}_t) + \psi^i(|\tilde{w}_t|)$$

(11)

for some $\psi^i \in \mathcal{K}$, then agent $A^i$ under NMPC optimal $u^{i,*}$ and terminal $k^i_f(x^i_t)$ control laws admits ISpS Lyapunov function $V(x^i_t, w^i_t, u^i_t) = J^i_f(x^i_t, u^i_t, w^i_t, N^i_{t+1})$ and is therefore ISpS with robust output feasible set $X^i_{MPC} \subseteq X^i$, which is the set of initial states for which the Problem 7 is feasible.
Proof: We need to prove that \( V(x^i_t, u^i_t, w^i_t) = J^*_i(x^i_{t,*}, u^i_{t,t+N^i_p}, w^i_t) \) is an ISpS Lyapunov function. The lower bound on \( V(x^i_t, w^i_t) \) is obviously given by \( \mathcal{L}^i(|x^i_t|) \leq V(x^i_t, w^i_t), \forall x^i_t \in X^i, w^i_t \in W^i \). Local control \( \tilde{u}^i_{t,t+N^i_p-1} = [k^i_1(x^i_t), \ldots, k^i_{N^i_p-1}(x^i_{t+N^i_p-1})]^T \) is feasible but suboptimal \( \forall x^i_t \in X^i_t \), i.e., \( V(x^i_t, \tilde{w}^i_t) \leq J^*_i(x^i_t, \tilde{w}^i_t, \tilde{u}^i_{t,t+N^i_p}) \). Using assumptions in Theorem 2, we get \( V(x^i_t, w^i_t) \leq \alpha_2(|x^i_t|) + \sigma_2(|w^i_t|) + \tilde{c}, \) where \( \alpha_2(s) = \alpha^i_2 \left( \mathcal{L}^i_N s + \tilde{b}^i \right), \sigma_2(s) = \tilde{b}^i + \tilde{c} = N^i_p \left( L^i_{gw} \right) (\hat{\xi}_i) \). The constants are \( \tilde{b}^i = (L^i_i + L^i_{hu} L^i_{k_j} + L^i_{b_0})(\mathcal{L}^i_{N^i_p} - 1) - 1 \) and \( \tilde{b}^i = L^i_{gw} (L^i_{gw} N^i_p - 1) - 1 \). Clearly, \( \tilde{u}^i_{t+1,t+N^i_p} = [u^i_{t+1,t+N^i_p}, k^i_t(x^i_{t+N^i_p})]^T \) is also feasible control for \( x^i_t \in X^i_{MPC} \) which gives \( V(x^i_{t+1}, w^i_{t+1}) \leq \sum_{t=1}^{t+1} \{ h(x^i_t, \tilde{w}^i_t) + q(x_t, \tilde{w}^i_t) \} + h^i(f(x^i_{t+N^i_p}, k^i_t(x^i_{t+N^i_p}))) \), where, \( \tilde{w}^i_t \) is NN approximation of \( w^i_{t+1} \) and \( \tilde{u}^i_t \) is approximation of \( u^i_t \), hence \( \tilde{w}^i_t \neq \tilde{u}^i_t \). Canceling common terms, we get \( V(x^i_{t+1}, w^i_{t+1}) = V(x^i_t, w^i_t) \leq -\alpha_2(|x^i_t|) + \sigma_2(|w^i_t|) + \sigma_2(|w^i_{t+1}|) + \tilde{c}^i \), where \( \sigma_1(s) = \sigma_2(s) + \psi(s) \), \( \sigma_2(s) = \tilde{b}^i \tilde{c} = \sigma_2(\hat{\xi}_i) \) and \( \tilde{b}^i = L^i_{gw} (L^i_{gw} N^i_p - 1) - 1 \). Hence, from Theorem 1, the system \( \mathcal{S} \) under NMPC is ISpS.

A method for terminal control law design (by solving (11) is given: Let \( h^i_t = x^i_t T^i Q^i x^i_t + u^i_t R^i u^i_t, \) \( q^i_t \leq x^i_t S^i x^i_t + \psi(|w^i_t|) \) and \( h^i_f = x^i f Q^i x^i f \), where \( Q^i, R^i, Q^f \) and \( S^i \) are positive definite matrices for \( i = 1, \ldots, M \) agents. Let \( k^i_f(x^i_t) = K^i x^i_t \) exist, such that \( A^i_c = A^i_o + B^i_o K^i \) is stable, where \( A^i_o \) and \( B^i_o \) are the Jacobians of system \( \mathcal{S} \). The terminal set is defined as \( X^i_f \triangleq (x^i)^T Q^i x^i \leq a \) for some \( a \in \mathbb{R}_\geq 0 \) which satisfies constraints \( x^i \in X^i \) and \( u^i = K^i x^i \in U^i \). Let \( Q^i_f \) be the solution of the convex problem.

\textbf{Problem 2.}

\[ \min_{Q^i_f, K^i_f} \left[ - \log \left( \det \left( a Q^i_f \right) \right) \right] \]  
\text{subject to the Lyapunov inequality} \[ A^i_c^T Q^i_f A^i_c - Q^i_f + Q^i + K^i R^i K^i + (N - 1) S^i \leq 0, \text{ and } Q^i_f > 0 \]

\textbf{B. Stability of Individual Agents with Collision Avoidance}

Results of the previous section will now be extended to prove stability of the agents under the collision avoidance scheme described in Section II-A. Let \( V(x^i_t, w^i_t) = J^*_i(x^i_{t,*}, u^i_t) \) be the local ISpS Lyapunov function for \( A^i \) without collision avoidance. Let \( x^i_{t,t+N^i_p} \) be the optimal solution of the cost (6) and \( \hat{x}^i_{t,t+N^i_p} \) be the optimal solution of the modified cost (8). We will prove that \( \hat{V}(\hat{x}^i_t, w^i_t) = J^*_i(\hat{x}^i_{t,*}, w^i_t) \) is an ISpS Lyapunov function. It is obvious that \( d_{ij}(k) \neq 0 \) for at least one instant \( t \leq k \leq t + N^i_p \), since otherwise would mean that the current position
as well planned optimal trajectories of two agents coincide exactly, which is impossible. We assume that $|\dot{x}^i| \leq |x^i| \leq |\dot{x}^i|$ for some constants $\kappa^i, \kappa^i \geq 0$, since both $x^i$ and $\dot{x}^i$ are finite. This leads to bounds on potential function, i.e. $\Phi^i \leq \Phi^i \leq \Phi^i$ for some constants $\Phi^i, \Phi^i \geq 0$.

**Theorem 3.** For an agent on collision course, the optimal trajectory $\dot{x}^{i*}_{t, t+ N_p}$ for modified cost (8) not only guarantees collision avoidance with other agents in the sense of (9), but also maintains input-to-state practical stability, if its repulsive spatial filter weights $\lambda(k)$ are chosen at each instant such that

$$\frac{\lambda^i_{\text{max}, t}}{\lambda^i_{\text{min}, t}} < \frac{\ell^i(|x_t|)}{\left(N^i_p R^i_{\text{min}} + N^i_p (N^i_p - 1) v^i_{\text{max}} \right) - 1} \triangleq \tilde{a}_t$$

**(Proof):** The proof consists of two parts. We first show that negative gradient of modified cost (8) lies in the direction of expanding weighted average distance $\bar{d}^i_{\text{ij}}$ between agents on collision course. Hence, the optimal trajectory $\dot{x}^{i*}_{t, t+ N_p}$ reaches the terminal set by avoiding collision in the sense of (9). Next, we will show that the optimal trajectory in that direction is also ISpS stable. From (8), we can see that $\frac{\partial J^i_t}{\partial d^i_{\text{ij}}^j} = \frac{\partial J^i_t}{\partial d^i_{\text{ij}}^j} (1 + \Phi^i) + J^i_t \frac{\partial \Phi^i}{\partial d^i_{\text{ij}}^j}$. Since $\partial \Phi^i / \partial d^i_{\text{ij}}^j = -\Phi^i / d^i_{\text{ij}}^j < 0$ and $J^i_t, \Phi^i > 0$, in order to have $\partial J^i_t / \partial d^i_{\text{ij}}^j < 0$, we have $\frac{\partial J^i_t}{\partial d^i_{\text{ij}}^j} < \frac{\Phi^i}{1 + \Phi^i d^i_{\text{ij}}^j} < \frac{J^i_t}{d^i_{\text{ij}}^j}$. Since $J^i_t, d^i_{\text{ij}}^j > 0$, this condition can be satisfied if $\max \left| \frac{\partial J^i_t}{\partial d^i_{\text{ij}}^j} \right| < \frac{\min(J^i_t)}{\max(d^i_{\text{ij}}^j)}$ for RHS, note that by chain rule of differentiation and using triangle inequality, $\left| \frac{\partial J^i_t}{\partial d^i_{\text{ij}}^j} \right| < \sum_{k=t}^{t + N_p} \left| \frac{\partial J^i_t}{\partial d^i_{\text{ij}}^j} \right| < \frac{1}{\lambda_{\text{min}, t}} \sum_{k=t}^{t + N_p} \left| \frac{\partial J^i_t}{\partial d^i_{\text{ij}}^j} \right|$. With slight abuse of notation we can write $d^i_{\text{ij}}^j = |x^i_k - w^j_k|$. For given neighbor trajectory $w^j_k = x^j_k, \forall j \in G^i$, we have $\partial d^i_{\text{ij}}^j / \partial x^i_k = (x^i_k - w^j_k) / d^i_{\text{ij}}^j$ such that $|\partial d^i_{\text{ij}}^j / \partial x^i_k| = 1$. Similarly, $\partial d^i_{\text{ij}}^j / \partial d^i_{\text{ij}}^j = \lambda^i_k$, which results in

$$\max \left| \frac{\partial J^i_t}{\partial d^i_{\text{ij}}^j} \right| < \frac{(N^i_p - 1)(L^i_h + L^i_h) + L^i_h}{\lambda_{\text{max}, t}}$$

Now, maximum $\tilde{d}^i_{\text{ij}}^j$ can occur when the minimum distance between agents on collision course is $R^i_{\text{min}}$, and then move away from each other at $v^i_{\text{max}}$, i.e. $\max(\tilde{d}^i_{\text{ij}}^j) = \sum_{k=t}^{t + N_p} \lambda^i_k (R^i_{\text{min}} + 2(k - t) v^i_{\text{max}}) < \lambda^i_{\text{max}, t} (N^i_p R^i_{\text{min}} + N^i_p (N^i_p - 1) v^i_{\text{max}})$. Also, as noted in Theorem 2, $\min(J^i_t) \leq v^i_t \leq \ell^i(|x^i_t|)$. This can be combined with (14) to result in the condition specified in (13). Hence, the minimum of modified cost lies in the direction of collision avoidance in the sense of (9). Since any feasible trajectory for cost (6) is also feasible for modified cost (8) and the reachable set is compact, an
optimum almost always exists, unless there is not enough time to maneuver (to cater for which we have placed a conservative bound on $\Delta^i_j \leq \Delta$).

For the next part of this proof, note that $\dot{J}(x^i_t, w^i_t) \leq \tilde{J}(x^i_t, w^i_t)$ and $J(x^i_t, w^i_t) \leq J(x^i_t, w^i_t)$, since $x^{i,s*}_{t,t+N_p^i}$ is feasible but suboptimal control for minimization of (6) and $x^{i,s*}_{t,t+N_p^i}$ is suboptimal for (8). For conciseness, we will ignore the difference between $V$ and $J$ in this section and also drop the $\star$ symbol. From Theorem 2 we have $\alpha^i_1(|x^i_t|) \leq V(x^i_t, w^i_t)$, which gives $\alpha^i_1(|x^i_t|) \leq V(x^i_t, w^i_t) \leq V(x^i_t, w^i_t)$. Combining this with (8) and defining $\dot{\alpha}^i_1(s) \triangleq (1 + \Phi^i)\alpha^i_1(k^i s) \in K_{\infty}$, we get $\dot{\alpha}^i_1(|x^i_t|) \leq \tilde{V}(x^i_t, w^i_t)$. Let $\dot{V}(x^i_t, w^i_t) = V(x^i_t, w^i_t) \leq \alpha^i_1(|x^i_t|)$ for some constant $\alpha^i_1 > 0$. Defining $\dot{\alpha}^i_2(s) \triangleq (1 + \Phi^i)\alpha^i_1(k^i s) \in K_{\infty}$, $\dot{\sigma}^i_3(s) \triangleq (1 + \Phi^i)\sigma^i_3(s) \in K_{\infty}$ and $\dot{\sigma}^i_4 \triangleq (1 + \Phi^i)(\alpha^i_1 + \alpha^i_2)$, we get $\dot{V}(x^i_t, w^i_t) \leq 3(1)|x^i_t| + \dot{\sigma}^i_3(|w^i_t|) + \dot{\sigma}^i_4$. Using (8), and defining $\dot{\alpha}^i_2(s) \triangleq (1 + \Phi^i)\alpha^i_2(k^i s) \in K_{\infty}$, $\dot{\sigma}^i_1(s) \triangleq (1 + \Phi^i)\sigma^i_1(s) \in K_{\infty}$, $\dot{\sigma}^i_1 \triangleq (1 + \Phi^i)(\alpha^i_2 + \alpha^i_3)$, we get $\gamma^i_{t+1} \dot{V}(x^i_{t+1}, w^i_{t+1}) - \dot{V}(x^i_t, w^i_t) \leq -\alpha^i_2(|x^i_t|) + \dot{\sigma}^i_1(|w^i_t|) + \dot{\sigma}^i_2(|w^i_{t+1}|) + \dot{\sigma}^i_4$, where $\gamma^i_{t+1} \triangleq \frac{1 + \Phi^i}{1 + \Phi^i}$. From (7), $\gamma^i_{t+1} \geq 1$ if (9) holds and we can write $\dot{V}(x^i_{t+1}, w^i_{t+1}) - \dot{V}(x^i_t, w^i_t) \leq -\alpha^i_2(|x^i_t|) + \dot{\sigma}^i_1(|w^i_t|) + \dot{\sigma}^i_2(|w^i_{t+1}|) + \dot{\sigma}^i_4$. Hence, agent $A^i$ is ISpS according to Theorem 1 and moves towards its goal in an optimal manner while avoiding collision with other agents.

**Corollary 1.** If spatial filter for collision avoidance is shaped as a geometric progression $\lambda^i_{kl,t} = \lambda^i_{\max,t} r^i_t$ such that $d^i_{kl,t} > d^i_{kl,t+1}$ for $l = 0, \ldots N_p^i - 1$, then the filter can be designed by specifying $\tilde{b} > 1$, $\lambda^i_{\max,t}$ and calculating $r^i_t = (\tilde{b} \alpha^i_t)^{1/(N_p^i-1)}$ from (13).

**C. Stability of Team of Agents under NMPC**

We will establish a generalized small gain condition to prove stability of the interconnected system, for both strongly- and weakly-connected network topologies. The result is general, not limited by the number of subsystems and the way in which subsystem gains are distributed is arbitrary.

**Theorem 4.** For a team of agents $A^i$ (10), each with local ISpS Lyapunov function $V(x^i_t, w^i_t)$, there exists $\bar{\alpha} \in K_{\infty}$ such that $V(x^i_{t+1}, w^i_{t+1}) - V(x^i_t, w^i_t) \leq \bar{\alpha}^i(|x^i_t|)$. Let the ISpS Lyapunov gain from $A^i$ to $A^j \in G^i$ be denoted by the function $\bar{\gamma}^i_{ij}(s) : \mathcal{R}_{\geq 0} \rightarrow \mathcal{R}_{\geq 0}$ and given by

$$\bar{\gamma}^i_{ij}(s) \triangleq \alpha^i_1 \circ (\bar{\alpha}^i)^{-1} \circ \sigma^i_1 \circ (\alpha^i_1)^{-1}(s),$$

then the team of agents is ISpS stable if the network is at least weakly connected, as long as
the following small gain condition is satisfied

\[ V(x^i_t, w^i_t) > \max_{j \in G^i, j \neq i} \{ \tilde{\gamma}_{ij}(V(x^j_t, w^j_t)) \} \]  \hspace{1cm} (16)

**Proof:** Consider \( \bar{\rho}^i \in K_\infty \). Let \( V(x^i_{t+1}, w^i_{t+1}) - V(x^i_t, w^i_t) \leq -\alpha^i_2(|x^i_t|) + \sigma^i_1(|w^i_t|) + \sigma^i_2(|w^i_{t+1}|) + \bar{c}^i \leq \bar{\rho}^i \circ \alpha^i_2(|x^i_t|) \) for \( x^i_t \in X^i \setminus \mathcal{B}^\alpha(e^i) \) and for \( \bar{\rho}^i \in K_\infty \) constructed such that \( \sigma^i_1(|w^i_t|) + \sigma^i_2(|w^i_{t+1}|) + \bar{c}^i \leq (I + \bar{\rho}^i)^\alpha^i_2(|x^i_t|) \). Then, in view of (2) and letting \( \tilde{\alpha}^i = (I + \bar{\rho}^i) \circ \alpha^i_2 \), we get \( V(x^i_t, w^i_t) \geq \alpha^i_1 \circ (\tilde{\alpha}^i)^{-1} \circ \sigma^i_1(|w^i_t|) \). Now, since \( w^i_t = \text{col}(x^j_{t,t+N^p_i}) \), then \( |w^i_t| \geq \max_j |x^j_t| \geq |x^i_t|, \forall j \in G^i \). Hence, \( V(x^i_t, w^i_t) \geq \max_j (\alpha^i_1 \circ (\tilde{\alpha}^i)^{-1} \circ \sigma^i_1(|x^j_t|)) \). But, \( V(x^i_t, w^i_t) \geq \alpha^i_1(|x^i_t|) \Rightarrow (\tilde{\alpha}^i)^{-1}(V(x^i_t, w^i_t)) \geq |x^i_t|, \text{ and hence } V(x^i_t, w^i_t) \geq \max_j (\alpha^i_1 \circ (\tilde{\alpha}^i)^{-1} \circ \sigma^i_1 \circ (\alpha^i_1)^{-1}(V(x^i_t, w^i_t)) \).

If gain \( \tilde{\gamma}_{ij} \) is defined as in (15), then (16) is obtained. From recent results in [11], it can be shown that this is equivalent to having an ISpS Lyapunov function for the network.

**Remark 1**

One way to design \( \tilde{\alpha}^i \) is by choosing \( \bar{\rho}^i(s) = \bar{k}^i s, \forall \bar{k}^i > 0 \), since it was shown that \( V^i_{t+1} - V^i_t < 0 \). This choice results in stable network, provided that individual agents are locally ISpS. We take the case of agents not on collision course first.

**Agents not on collision course:** Continuing from proof of Theorem 2, and letting \( \lambda_{\Pi_{\max}} \) and \( \lambda_{\Pi_{\min}} \) be the maximum and minimum eigenvalues of a p.d. matrix \( \Pi \), respectively, then,

\[ \sigma^i_1(r) = \sigma^i_2(r) + \psi^i(r) = \frac{L^i_{gw}(L^i_{gw} N_p^{i-1} - 1)}{L^i_{gw} - 1} r + (M - 1) \lambda_{S_{\max}} r^2, \forall L^i_{gw} \neq 1, \]

For \( L^i_{gw} = 1 \), the results need trivial modifications, by replacing \( L^i_{gw}(L^i_{gw} N_p^{i-1} - 1) (L^i_{gw} - 1)^{-1} \) with \( \sum_{l=0}^{l=N_p^i-2} L^i_{gw} l = N_p^i - 1 \). Similarly,

\[ \alpha^i_{1,2}^{-1}(r) = \sqrt{\frac{r}{\lambda_{\min} Q^j}}, \]

\[ \tilde{\alpha}^i_{1,2}^{-1}(r) = \alpha^i_{1,2}^{-1} \circ (I + \bar{\rho}^i)^{-1}(r) \]

and

\[ L_{gw} = \lambda_{S_{\max}} |\bar{u}_m^i| \]

We mentioned that one choice of \( \bar{\rho}^i \) could be \( \bar{\rho}^i(r) = \bar{k}^i r \) for all \( \bar{k}^i > 0 \). Therefore,

\[ \tilde{\alpha}^i_{1,2}^{-1}(r) = \alpha^i_{1,2}^{-1} \left( \frac{1}{\bar{k}^i - 1} r \right) \]
It is also worth noting that we showed in the proof of Theorem 2 that $\alpha_2(r) = \alpha_2(r)$. Since 
$\tilde{\gamma}_{ij}(r) = \alpha_1 \circ \alpha_1^{-1} \circ \sigma_1 \circ \alpha_1^{-1} (r)$, we can obtain 
$$
\tilde{\gamma}_{ij}(r) = \frac{1}{\bar{k}^i + 1} \left( N_p - 1 \right) \lambda_{S_{\max}} \bar{w}_{\max} \sqrt{\frac{r}{\lambda_{\min} Q_j}} + \frac{1}{\bar{k}^i + 1} \left( M - 1 \right) \lambda_{S_{\max}} \left( \frac{r}{\lambda_{\min} Q_j} \right)
$$
Hence, (16) can be written as 
$$
\tilde{\gamma}_{ij}(V(x^i, \tilde{x}^j)) = \left[ \frac{1}{\bar{k}^i + 1} \left( N_p - 1 \right) \bar{w}_{\max} \sqrt{\lambda_{S_{\max}}} \right] \sqrt{V(x^i, \tilde{x}^j)} + \left[ \frac{1}{\bar{k}^i + 1} \left( M - 1 \right) \left( \frac{\lambda_{S_{\max}}}{\lambda_{\min} Q_j} \right) \right] V(x^i, \tilde{x}^j)
$$
Therefore, by choosing a suitable value of $\lambda_{S_{\max}} / \lambda_{\min} Q_j$ and $\bar{k}^i > 0$, the small gain condition (16) can be satisfied.

Agents on collision course: For agents on collision course, similar results can be reproduced as all functions have corresponding counterparts in collision avoidance case, see proof of Theorem 3. Therefore we can write (15) as 
$$
\tilde{\gamma}'_{ij}(r) \triangleq \alpha_1 \circ (\alpha_1^{-1} \circ \sigma_1 \circ \alpha_1^{-1}) (r)
$$
Thus, we get
$$
\tilde{\gamma}'_{ij}(r) = \frac{(1 + \phi^j)}{(\bar{k}^i + 1) \bar{k}^j} \left( N_p - 1 \right) \bar{w}_{\max} \frac{\lambda_{S_{\max}}}{\sqrt{(1 + \phi^j) \lambda_{\min} Q_j}} (r)^{\frac{1}{2}} + \frac{(M - 1) \lambda_{S_{\max}}}{\bar{k}^j \left( 1 + \phi^j \right) \lambda_{\min} Q_j \bar{k}^i} r
$$
Hence, even with collision avoidance, it possible to find $\tilde{k}^i > 0$ which satisfies the small gain condition (see IV). As far as the small gain condition for weakly connected networks is concerned, we show in Remark 2 that the small gain condition is equivalent to that for strongly connected networks. It should be noted that there is no need to find the exact numerical values for construction of controller. As long as there exists some $\tilde{k}^i > 0$, we can be assured of ISpS stability of the team. See Section IV.

1) Strongly Connected Network: We will now particularize the result of Theorem 4 to the case of strongly connected network.

Lemma 1. A team of $N$ agents connected with a strongly connected network is ISpS stable if each agent $A^i$ has an ISpS Lyapunov function $V(x_t^i, w_t^i)$, edge gain $\bar{\gamma}_{ij}$ is defined as in (15) and the following small gain condition is achieved:

$$
V(x_t^i, w_t^i) > \max_j (\bar{\gamma}_{ij}(V(x_t^j, w_t^j))), \quad \forall j \neq i, j = 1, \ldots, N - 1
$$
(17)
This can be simply stated as: the diagonal is either upper block triangular form by appropriate re-indexing of agents, such that each upper block on the diagonal is either 0 or irreducible. Hence, we can now rewrite the gain matrix as:

\[
\Gamma = \begin{bmatrix}
0 & \tilde{\gamma}_{12} & \cdots & \tilde{\gamma}_{1,M} \\
0 & \ddots & \ddots & \tilde{\gamma}_{2,M} \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & \tilde{\gamma}_{N-1,M} \\
0 & \cdots & 0 & 0
\end{bmatrix}
\]

According to the recent generalized small gain theorems of [11], if a strongly connected network obeys the following small gain condition (SGC): \( G > \Gamma_{\tilde{\mu}} \), then it is stable in the ISS sense (see Theorem 5.3 of [11]). Now, \( \tilde{\mu} = \max \Gamma_{\tilde{\mu}} \) is a monotone aggregation function ([15]). Let \( r = (V(x_i^1, w_i^1), \ldots, V(x_i^N, w_i^N)) \), then the SGC is satisfied if:

\[
\begin{bmatrix}
V(x_i^1, w_i^1) \\
\vdots \\
V(x_i^N, w_i^N)
\end{bmatrix} \succ
\begin{bmatrix}
\max(\bar{\gamma}_{12}(V(x_2^1, w_2^1)), \ldots, \bar{\gamma}_{1,N}(V(x_N^1, w_N^1))) \\
\vdots \\
\max(\bar{\gamma}_{N,1}(V(x_1^1, w_1^1)), \ldots, \bar{\gamma}_{N,N-1}(V(x_{N-1}^1, w_{N-1}^1)))
\end{bmatrix}
\]

This can be simply stated as:

\[
V(x_i^1, w_i^1) > \max_j(\bar{\gamma}_{ij}(V(x_j^1, w_j^1))), \quad \forall j \neq i, j = 1, \ldots, N - 1
\]

2) Weakly Connected Network: We will now focus on the case of a network of agents, in which not all agents are connected to every other agent.

**Lemma 2.** A team of cooperating agents connected with a weakly connected network is ISpS stable if each agent \( A_i \) has an ISpS Lyapunov function \( V(x_i^1, w_i^1) \), edge gain \( \bar{\gamma}_{ij} \) is defined as in (15) and the following small gain condition is achieved:

\[
V(x_i^1, w_i^1) > \max_j(\bar{\gamma}_{ij}(V(x_j^1, w_j^1))), \quad \forall j \neq i, j \in G_i
\]

**Proof:** The connectivity gain matrix for a weakly connected network can be brought in upper block triangular form by appropriate re-indexing of agents, such that each upper block on the diagonal is either 0 or irreducible. Hence, we can now rewrite the gain matrix as:

\[
\Gamma = \begin{bmatrix}
0 & \tilde{\gamma}_{12} & \cdots & \tilde{\gamma}_{1,M} \\
0 & \ddots & \ddots & \tilde{\gamma}_{2,M} \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & \tilde{\gamma}_{N-1,M} \\
0 & \cdots & 0 & 0
\end{bmatrix}
\]
where \( \bar{M} \triangleq \max_i M^i \) is the size of neighborhood of the most connected agent. According to Proposition 6.2 of \cite{[11]}, the interconnected system is stable if each upper diagonal block satisfies the SGC: \( \mathcal{I} > \Gamma_{\mu} \). Now, the upper diagonal blocks are:

\[
\bar{\Gamma}_1 = 0, \quad \bar{\Gamma}_2 = \begin{bmatrix} 0 & \bar{\gamma}_{12} \\ 0 & 0 \end{bmatrix}, \quad \bar{\Gamma}_3 = \begin{bmatrix} 0 & \bar{\gamma}_{12} & \bar{\gamma}_{13} \\ 0 & 0 & \bar{\gamma}_{23} \\ 0 & 0 & 0 \end{bmatrix}
\]

\[
\bar{\Gamma}_d = \begin{bmatrix} 0 & \bar{\gamma}_{12} & \ldots & \bar{\gamma}_{1,d} \\ 0 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \ldots & 0 & 0 \end{bmatrix}, \quad \bar{\Gamma}_N = \Gamma
\]

Then stability is assured if each of the above blocks obey the SGC iteratively, i.e.

\[
\begin{align*}
r_1 > \Gamma_{\mu_1}(r_1) & \Rightarrow V(x^1_t, w^1_t) > 0 \\
r_2 > \Gamma_{\mu_2}(r_2) & \Rightarrow V(x^1_t, w^1_t) > \bar{\gamma}_{12}(V(x^2_t, w^2_t)), V(x^2_t, w^2_t) > 0 \\
r_3 > \Gamma_{\mu_3}(r_3) & \Rightarrow V(x^1_t, (x^1_t, w^1_t)) > \max(\bar{\gamma}_{12}(V(x^2_t, w^2_t)), \bar{\gamma}_{13}(V(x^3_t, w^3_t))) \\
& \quad \quad \quad V(x^2_t, w^2_t) > \bar{\gamma}_{23}(V(x^3_t, w^3_t)), V(x^3_t, w^3_t)) > 0 \\
& \quad \quad \vdots
\end{align*}
\]

(19)

This iterative procedure reduces to (16). Hence, the team is stable irrespective of the network topology as long as it is at least weakly connected, provided it obeys certain small gain conditions.

\[\blacksquare\]

**IV. SIMULATION RESULTS**

Consider a fleet of 5 autonomous vehicles moving in the horizontal plane, with the following continuous-time models (discretized at \( T=0.1s \)) having similar dynamics (for simplicity):

\[
m^i \ddot{x}^i = -\mu_1^i \dot{x}^i + (u^i_R + u^i_L) \cos \theta^i, \quad m^i \ddot{y}^i = -\mu_1^i \dot{y}^i + (u^i_R + u^i_L) \sin \theta^i, \quad J^i \ddot{\theta}^i = -\mu_2^i \dot{\theta}^i + (u^i_R + u^i_L) r_v
\]

where \( m^i, J^i, \mu_{1,2} \) and \( r_v \) are parameters specified in \cite{[4]}. Constraints on inputs and states are \( 0 \leq |u^i_{R,L}| \leq 6 |\dot{\theta}| \leq 1 \text{ rad/s} \). Uniformly distributed communication delay is bounded by \( T \leq \Delta_{ij} \leq \)
with extra delay due to multiple hops, i.e. verification of small gain condition (16) from Theorem 4 for random delays.

It is evident that the proposed algorithm performs well despite large delays due to collision avoidance. The effect of delay is manifest in lag in synchronization, while temporary divergence is avoided throughout the trajectory. Synchronization of states is achieved quickly, as shown in Fig. 2. Executing sharp turns, such as right angle turns when transitioning between WPs puts agents on the inside of the turn (A2, A4) at risk of collision. Also, A4,5 receive WP information, with extra delay due to multiple hops, i.e. Δ4 = Δ5 = 2Δ. However, collision is successfully avoided throughout the trajectory. Synchronization of states is achieved quickly, as shown in Fig. 2. The condition for only Agent 1 (connected to Agents 2 and 3 in the weakly connected network) is shown.

In the given example, cost function and corresponding gain is shown in Fig. 3 to illustrate verification of small gain condition (16) from Theorem 4 for k = 5 × 10^3. The condition for only Agent 1 (connected to Agents 2 and 3 in the weakly connected network) is shown. However, small gain conditions hold for all the other agents (results not shown in interest of...
V. CONCLUSION

We presented distributed NMPC framework for formation control of constrained agents robust to uncertainty due to data compression and propagation delays. Collision avoidance is ensured by means of spatially filtered potential field. Rigorous proofs are provided ensuring practical stability regardless of network topology. Simulations illustrate good performance of the proposed scheme in both strongly- and weakly-connected networks. Future research directions include the need...
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