A numerical analysis for identification of flow transition in vortex generation in terms of local flow topology

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Abstract
A numerical analysis, based on a novel physical quantity of the topology, is presented to specify the key flow leading into a vortex. This analysis traces the flow transition into a vortical flow in terms of local flow geometry (topology) specified by the velocity gradient tensor, and specifies the important flow component for the vortex transition. The transition where a non-vortical flow becomes vortical can be identified by swirlity that represents the unidirectionality and intensity of the azimuthal flow in a plane. The swirl plane after the vortex transition can be predicted by an eigenplane of real eigenvalues of the velocity gradient tensor. Then the tensor components are represented associating with the predicted plane, and their relations to the flow topology are clarified. The analysis of their transitions enables to specify the important flow components that lead the flow into a vortical flow. This numerical analysis can be applied to various turbulent flows in order to clarify the mechanism or feature of the vortex transition, or suppress a specific vortex in engineering fields.

Keywords : Vortical flow, Topology, Swirlity, Swirl plane, Flow transition

1. Introduction

Vortical flow is an important feature in fluid dynamics and engineering. Vortices influence flow characteristics, and their generation (transition) is often an important subject to be estimated in flow design and fluid engineering. Various scales of vortices may construct a characteristic of the turbulence, and specific vortices may be imposed targets to be controlled in diverse fluid engineering fields such as power plants, wind turbines, and fluid machinery. In these fluid phenomena, the analysis of flow transition into a vortex is important for clarifying the mechanism of the transition or confining a vortex in flow control.

As for the vortex identification or definition of a vortex, no universal definition has been established. However, the popular definitions are based on the local flow geometry (topology) or pressure minimum feature where the velocity gradient tensor \( \nabla \mathbf{u} \) specifies the invariant local flow and the physical feature of a vortex (Chong et al., 1990, Hunt et al., 1988, Jeong and Hussain, 1995, Nakayama et al., 2014). Therefore, the analysis of flow transition into a vortex (vortex transition) is effective in terms of the local flow associated with the above vortex definitions. In the study of the topology, not only vorticity (Jiménez et al., 1993) but also the eigenvalues of \( \nabla \mathbf{u} \) have greatly contributed in various turbulent flows (Soria et al., 1994, Blackburn et al., 1996, Nakayama, 2017a). On the other hand, in the analysis of the vortex transition, a topological quantity is required to extract the azimuthal component in the local flow and specify its state or characteristic.

In this purpose of the analysis of the flow transition, the vorticity is difficult to specify the process of a vortical flow because it does not distinguish the topologies between shear and vortical flow with swirling. On the other hand, swirlity represents the unidirectionality and intensity of the azimuthal flow extracted from a local flow in a plane (Nakayama, 2014), and this property effectively specifies the topological transition (Nakayama et al., 2015, Nakayama, 2016). In
addition, it is shown that the swirl plane after the vortex transition can be predicted by the eigenplane associated with the two eigenvalues of $\nabla \mathbf{v}$ with the same sign (Nakayama et al., 2015). Then, if we define a coordinate system or its bases associated with the predicted swirl plane, it enables to associate the respective components of $\nabla \mathbf{v}$ with the topology, e.g., those associated with the shear or strain in the predicted plane or those corresponding to the normal direction of the plane. Furthermore, monitoring their transitions enables to specify the key flow that changes the non-vortical flow into a vortical flow, and how this key flow relates to the transition of the flow topology. This numerical analysis technique can be applied in the analysis of various turbulent flows to investigate mechanism of the vortex transition. In addition, this analysis may be effective for many fluid engineering fields to identify the key flow and predicted swirl plane, or to suppress specific vortices.

The present paper presents a numerical analysis of the transition of $\nabla \mathbf{v}$ components in the vortex transition to specify the key flow for the vortex. A specific coordinate system is defined associated with the eigenplane, say $\Sigma_\mathbf{v}$, that becomes the swirl plane, which enables to classify or relate each $\nabla \mathbf{v}$ component to the plane and flow topology. An application of the present analysis to an isotropic homogeneous turbulence in Direct Numerical Simulation (DNS) shows that this analysis captures the vortex transition and important $\nabla \mathbf{v}$ components in this transition. While the components that are not associated with the vortex transition exhibit less change and correlation coefficients with other ones, a (shear) component in $\Sigma_\mathbf{v}$ co-varies with the topology (swirlity), in order to have a swirling flow, i.e., azimuthal flow with the uniform direction. Furthermore, two strain components in $\Sigma_\mathbf{v}$ change to have the same value, which is consistent with the mathematical characteristic that the negative swirlity approaches to zero and becomes positive in the transition into a vortex (Nakayama, 2016).

2. Swirlity in vortex transition

We summarize the physical characteristics of the swirlity (Nakayama, 2014) applied for monitoring the flow transition into a vortex. In the velocity field $\mathbf{v}_i$ in a coordinate system $x_i$ ($i = 1, 2, 3$), the local flow around a point is expressed by $dx_i/dt = (\partial \mathbf{v}_i/\partial x_j)x_j$, where the summation convention is applied. Then the three dimensional local flow geometry is specified by the eigenvalues $\lambda_i$ and their eigenvectors $\hat{\mathbf{e}}_i$ ($i = 1, 2, 3$) of $\nabla \mathbf{v}$ that is Galilean invariant, i.e., independent of the inertial coordinate system (Chong et al., 1990, Nakayama, 2014). Here we focus on the azimuthal flow by decomposing the local flow in detail.

Because $\nabla \mathbf{v}$ generally has at least one real eigenvalue, say $\lambda_a$, and corresponding eigenvector $\hat{\mathbf{e}}_a$, we consider a coordinate system $\tilde{x}_i$ ($i = 1, 2, 3$) where the $\tilde{x}_3$ axis is parallel to $\hat{\mathbf{e}}_a$ and the $\tilde{x}_1$-$\tilde{x}_2$ plane composed of two orthonormal bases is an arbitrary plane linearly independent of (not parallel to) $\hat{\mathbf{e}}_a$. The matrix representation of $\nabla \mathbf{v}$ in this coordinate system, $\tilde{\mathbf{A}} = [\tilde{a}_{ij}] = [\partial \mathbf{v}_i/\partial \tilde{x}_j]$ ($i, j = 1, 2, 3$), can be expressed in the following form

$$\tilde{\mathbf{A}} = \begin{bmatrix} \tilde{a}_{11} & \tilde{a}_{12} & 0 \\ \tilde{a}_{21} & \tilde{a}_{22} & 0 \\ \tilde{a}_{31} & \tilde{a}_{32} & \epsilon_a \end{bmatrix}. \tag{1}$$

As for the local flow in the $\tilde{x}_1$-$\tilde{x}_2$ plane, where the velocity $\tilde{\mathbf{v}}'_i$ ($i = 1, 2$) is given by $\tilde{v}'_i = \tilde{a}_{ij}\tilde{x}_j$ ($i, j = 1, 2$), the decomposed azimuthal component $v_\theta$ is given by the inner product of $\tilde{v}'_i$ and the unit vector $\mathbf{e}_\theta = 1/|\tilde{x}'|(-\tilde{x}_2, \tilde{x}_1)$ in the azimuthal direction where $\tilde{x}' = (\tilde{x}_1, \tilde{x}_2)$. It follows

$$v_\theta = \frac{1}{|\tilde{x}'|} \tilde{x}' Q_\theta \mathbf{e}_\theta, \tag{2}$$

$$Q_\theta = \begin{bmatrix} \tilde{a}_{21} & -(\tilde{a}_{11} - \tilde{a}_{22})/2 \\ -(\tilde{a}_{11} - \tilde{a}_{22})/2 & -\tilde{a}_{12} \end{bmatrix}, \tag{3}$$

where the superscript $t$ before a vector denotes its transpose.

The eigenvalues of $Q_\theta$, $\lambda_{ti}$ ($i = 1, 2$), specify the direction and intensity of $v_\theta$. Then the swirlity $\phi$ represents the unidirectionality and intensity of $v_\theta$ in terms of their geometrical average. $\phi$ is defined as

$$\phi = \text{sgn}(\lambda_{t1} \lambda_{t2}) \sqrt{|\lambda_{t1} \lambda_{t2}|}, \tag{4}$$

where $\text{sgn}(\alpha)$ denotes the sign of $\alpha \in \mathbb{R}$ ($\alpha \neq 0$). $\phi$ can be expressed as

$$\phi = Q + \frac{3}{4} \epsilon_a^2, \tag{5}$$
where \( Q \) denotes the second invariant of \( \nabla \psi \) (Nakayama, 2014). Thus \( \phi \) is invariant in an arbitrary plane linearly independent of (not parallel to) \( \hat{\boldsymbol{e}}_a \).

If \( \phi < 0 \), it shows that two of \( \lambda_i \) have positive and negative values, i.e., clockwise and counterclockwise directions of \( \psi_0 \), thus the local flow does not have a swirling flow. If \( 0 < \phi \), then two of \( \lambda_i \) have the same sign, and \( \psi_0 \) has a uniform direction around a point. Thus the flow is vortical with a swirling flow. In this case, \( \nabla \psi \) has the complex conjugate eigenvalues \( \lambda_1, \lambda_2 = \varepsilon_R \pm i \psi \) (\( \psi > 0 \)), and it is mathematically shown that \( \phi = \psi \). Therefore, the transition of \( \phi \) specifies the feature of \( \psi_0 \), and the change of the sign of \( \phi \) clearly indicates the flow transition into a vortical flow.

Figure 1 shows the transition of the flow topology into a vortex followed by \( \phi \) as an example. In Fig. 1 (a) and (b), \( \phi \) is negative, and an azimuthal flow in the counterclockwise direction decreases in accordance with \( \phi \) approaching to 0. When \( \phi \) becomes positive, then the azimuthal flow has a uniform (clockwise) direction throughout, as shown in Fig. 1 (c).

As described in the previous section, the eigenvalues of \( \nabla \psi \) have been applied not only to the vortex definition but also topological condition in the identification method such as swirling condition in “sectional-swirl-and-pressure-minimum scheme” (Kida et al., 1998). On the other hand, the swirlity focuses on the feature of the azimuthal flow extracted from the local flow, and evaluates the unidirectionality and intensity of the azimuthal flow.

### 3. Predicted swirl plane and coordinate system for monitoring \( \nabla \psi \)

As described above, if \( \nabla \psi \) has a pair of complex conjugate and a real eigenvalues, i.e., \( \lambda_1, \lambda_2 = \varepsilon_R \pm i \psi \), and \( \lambda_3 = \varepsilon_a \), and their respective eigenvectors \( \xi_1, \xi_2 = \xi_{pl} \pm i \eta_{pl} \) and \( \xi_3 = \xi_a \), then the flow trajectory can be represented as \( \mathbf{x} = 2e^{\varepsilon_R t}[\xi_{pl} \cos(\psi t) - \eta_{pl} \sin(\psi t)] + e^{\varepsilon_a t} \xi_a \). It shows that the flow swirls with an intensity \( \psi \) (note \( \psi = \phi \)) in the plane defined by \( \xi_{pl} \) and \( \eta_{pl} \), hereafter referred to as the swirl plane \( \Sigma_S \), and proceeds (or approaches) along a vortical axis \( \xi_a, \xi_{pl} \) and \( \eta_{pl} \) can be orthogonal, i.e., \( \xi_{pl} \perp \eta_{pl} \), and the ratio \( c = ||\xi_{pl}||/||\eta_{pl}|| \) specified by the eigenequations of \( \nabla \psi \) represents the vortical flow symmetry (skewness) (Nakayama, 2014).

In an orthonormal coordinate system \( \hat{x}_i \) \( (i = 1, 2, 3) \) with the orthonormal bases \( \hat{e}_i \) \( (i = 1, 2, 3) \) where \( \hat{e}_1 \) and \( \hat{e}_2 \) are parallel to \( \xi_{pl} \) and \( \eta_{pl} \), respectively, so that the \( \hat{x}_1 - \hat{x}_2 \) plane is \( \Sigma_S \), \( \nabla \psi \) in this coordinate system, \( \hat{\mathbf{A}} = [\hat{a}_{ij}] \) \( (i, j = 1, 2, 3) \) can be expressed in the form (Nakayama, 2014)

\[
\hat{\mathbf{A}} = \begin{bmatrix}
\varepsilon_R & \varepsilon \psi & \hat{a}_{13} \\
-\psi/c & \varepsilon_R & \hat{a}_{23} \\
0 & 0 & \varepsilon_a
\end{bmatrix}.
\]

\( \hat{a}_{13} \) and \( \hat{a}_{23} \) can be expressed in terms of the vorticity vector \( \omega \) in this coordinate system, say \( \hat{\omega}_i \) \( (i = 1, 2, 3) \) (Nakayama,
2017a). Equation (6) can be rewritten as:

\[
\tilde{A} = \begin{bmatrix}
\varepsilon_R & e\psi & \tilde{\omega}_2 \\
-\psi/e & \varepsilon_R & -\tilde{\omega}_1 \\
0 & 0 & \varepsilon_c
\end{bmatrix}.
\] (7)

When \(\phi < 0\), the flow is non-vortical and all \(\lambda_i\) are real numbers (Nakayama, 2014). We set that real \(\lambda_1\) and \(\lambda_2\) (\(\lambda_1 < \lambda_2\)) become conjugate complex in the flow transition into a vortex. Because \(\nabla v\) has another real eigenvalue \(\lambda_3\), the eignequation of \(\nabla v, \Phi(\lambda)\), can be expressed as:

\[
\Phi(\lambda) = (\lambda_3 - \lambda)f(\lambda),
\]

\[
f(\lambda) = (\lambda^2 + \alpha \lambda + \beta),
\] (8) (9)

where \(\alpha\) and \(\beta\) are real constants. \(\lambda_1\) and \(\lambda_2\) are solutions of \(f(\lambda) = 0\), thus

\[
\lambda_1, \lambda_2 = -\alpha/2 \pm \sqrt{D}/2.
\] (10)

\[
D = \alpha^2 - 4\beta = (\lambda_1 - \lambda_2)^2.
\] (11)

When \(\lambda_1\) and \(\lambda_2\) become conjugate complex, \(D\) becomes zero in the process, that is, \(\lambda_1\) and \(\lambda_2\) become the same. Thus they have the same sign before the vortex transition (except the case that \(\lambda_1 = \lambda_2 = 0\)).

Here we derive a representation of \(\nabla v, \hat{A}\), with three real eigenvalues \(\lambda_i\) (\(i = 1, 2, 3\)). We express the eigenplane defined by \(\xi_1\) and \(\xi_2\) as \(\Sigma_N\) (note \(\lambda_1 < \lambda_2\)). Then we define an orthonormal coordinate system \(\hat{\xi}_i\), with unit bases \(\hat{\xi}_i\) (\(i = 1, 2, 3\)) where \(\hat{\xi}_1\) is parallel to \(\xi_1\), and the \(\hat{\xi}_1\)-\(\hat{\xi}_2\) plane is identical to \(\Sigma_N\). Because \(\hat{A}\hat{\xi}_1 = \lambda_1\hat{\xi}_1\) and \(\hat{\xi}_1 = (1, 0, 0)\), the components \(\hat{a}_{11}, \hat{a}_{21}, \text{and} \hat{a}_{31}\) are specified as:

\[
\hat{a}_{11} = \lambda_1, \quad \hat{a}_{21} = \hat{a}_{31} = 0.
\] (12)

Furthermore, because \(\xi_2 \in \Sigma_N\), it yields

\[
\hat{a}_{32} = 0, \quad \hat{a}_{22} = \lambda_2.
\] (13)

Then \(\hat{A}\) is expressed as follows

\[
\hat{A} = \begin{bmatrix}
\lambda_1 & \hat{a}_{12} & \hat{a}_{13} \\
0 & \lambda_2 & \hat{a}_{23} \\
0 & 0 & \hat{a}_{33}
\end{bmatrix}.
\] (14)

As for \(\lambda_3\) and \(\xi_3\), because \(\xi_3 \notin \Sigma_N\), \(\hat{a}_{33}\) is specified as:

\[
\hat{a}_{33} = \lambda_3.
\] (15)

On the other hand, in Eq. (14), \(\hat{a}_{12}, \hat{a}_{13}, \text{and} \hat{a}_{23}\) can be expressed in terms of the vorticity vector \(\hat{\omega}_i\) (\(i = 1, 2, 3\)) in this coordinate system. Therefore, Eqs. (14) and (15) give the following representation of \(\nabla v\):

\[
\tilde{A} = \begin{bmatrix}
\lambda_1 & -\hat{\omega}_3 & \hat{\omega}_2 \\
0 & \lambda_2 & -\hat{\omega}_1 \\
0 & 0 & \lambda_3
\end{bmatrix}.
\] (16)

\(\hat{A}\) and \(\tilde{A}\) in Eqs. (16) and (7) have similar representation except \(\hat{a}_{21} (\tilde{a}_{21})\) component. This difference is rational because this component determines the non-swirling or swirling flow in \(\Sigma_N\) or \(\Sigma_S\).

If \(\lambda_1\) and \(\lambda_2\) become the same and conjugate complex smoothly, then it can be supposed that the eigenplane defined by \(\xi_i\) and \(\xi_j\) become the plane defined by \(\xi_{pl}\) and \(\eta_{pl}\). It indicates that \(\Sigma_N\) becomes \(\Sigma_S\), that is, \(\Sigma_N\) is a predicted swirl plane (Nakayama et al., 2015).

We note that the components of \(\hat{\omega}_i\) (\(i = 1, 2, 3\)) can be expressed as invariant quantities. They are given by the following equations.

\[
\hat{\omega}_1 = (\hat{\omega}, \hat{\xi}_1) = (\hat{\omega}, \hat{\xi}_1)/|\hat{\xi}_1|.
\] (17)
\[ \hat{\omega}_2 = (\hat{\omega}, \hat{e}_2) \]
\[ = (\hat{\omega}, \hat{e}_3 \times \hat{e}_1) \]
\[ = \left( \hat{\omega}, \frac{(\xi_1 \times \xi_3) \times \xi_1}{|\xi_1| \times |\xi_3|} \right) \].

(18)

\[ \hat{\omega}_3 = (\hat{\omega}, \hat{e}_3) \]
\[ = \left( \hat{\omega}, \frac{(\xi_1 \times \xi_2)}{|\xi_1| \times |\xi_2|} \right) \].

(19)

This formulation shows that the all components of \( \hat{A} \) are invariant quantities. Before the vortex transition at a point, \( \vec{\nabla} \vec{v} \) can be expressed in Eq. (16), as an invariant coordinate system.

This coordinate system changes according to \( \xi_1 \) and \( \xi_2 \) that specify \( \Sigma_N \). If this coordinate system is defined at a preceding time step of the vortex transition, the change of velocity gradient or shear/strain components for a vortical flow can be identified relating to the predicted swirl plane and flow geometry. It indicates that monitoring (tracing) the components of \( \vec{\nabla} \vec{v} \) in this coordinate system specifies the key flow that leads the flow state into a vortical flow. Therefore, the present analysis specifies the \( \hat{\xi} \) coordinate system in a preceding time step of the vortex transition at a considered point and expresses \( \vec{\nabla} \vec{v} \) at arbitrary time steps in terms of this coordinate system. Then it enables to monitor the \( \vec{\nabla} \vec{v} \) components to find the key flow for the transition, relating to the topology.

4. Numerical analysis scheme

In order to analyse the flow transition into a vortex at a point, \( \vec{\nabla} \vec{v} \) in the point is analysed in a certain time period before its transition. The algorithm of this flow analysis to monitor \( \vec{\nabla} \vec{v} \) components during a series of a time interval is summarized in the following procedure.

(1) in the analytical region and instantaneous velocity field, vortices (nodes where \( \theta < \phi \)) are captured by the analysis of vortex identification. Then a point is set where \( \vec{\nabla} \vec{v} \) is to be monitored.

(2) time series of \( \vec{\nabla} \vec{v} \) and \( \phi \) at the point is analyzed with the several past velocity field data, and the preceding time step of the vortex transition is specified using \( \phi \).

(3) estimate \( \lambda_i \) and \( \xi_i \) with \( \vec{\nabla} \vec{v} \) at the preceding time step, and define \( \Sigma_N \) and the \( \hat{\xi}_i \) coordinate system as a fixed coordinate system in this analysis.

(4) evaluate \( \vec{\nabla} \vec{v} \) at past time steps with respect to the \( \hat{\xi}_i \) coordinate system by the linear transformation.

(5) construct a set of time series of \( \vec{\nabla} \vec{v} \) components in the \( \hat{\xi}_i \) coordinate system, during the time interval. Time series of a physical quantity can be constructed using the series of \( \vec{\nabla} \vec{v} \).

In (4) in the above procedure, \( \vec{\nabla} \vec{v} \) in the \( \hat{\xi}_i \) coordinate system is derived from the linear transformation of \( \vec{\nabla} \vec{v} \) in the space fixed coordinate system in numerical simulation (e.g., Direct Numerical Simulation) or experiment, say \( A \). \( \hat{A} \) is given by the following equation:

\[ \hat{A} = P^{-1} A P, \]

(20)

\[ P = [\hat{e}_1, \hat{e}_2, \hat{e}_3]. \]

(21)

The bases \( \hat{e}_i \) \( (i = 1, 2, 3) \) are specified by the following equations:

\[ \hat{e}_1 = \frac{\xi_1}{|\xi_1|} \]

(22)

\[ \hat{e}_2 = \hat{e}_3 \times \hat{e}_1 \]

(23)

\[ \hat{e}_3 = \frac{\xi_1 \times \xi_2}{|\xi_1 \times \xi_2|} \]

(24)

We note that, in this analysis, \( \vec{\nabla} \vec{v} \) is expressed in the form of Eq. (16) only at the preceding time step (last time step before the vortex transition), because \( \Sigma_N \) (or \( \xi_i \) specifying the coordinate system for the representation in Eq. (16)) differs at each time step.

5. Numerical analysis

The flow transition analysis is applied to an isotropic homogeneous decaying turbulence in DNS (Nakayama, 2017a), using the pseudo-spectral method in a region \( (2\pi)^3 \) composed of 256^3 nodes. For the wavenumber vector \( \vec{k} = (k_1, k_2, k_3) \),
Fig. 2  Flow transition into a vortical flow in $\Sigma_N (\phi < 0)$ or $\Sigma_S (0 < \phi)$ of a point at (a) $-10\Delta t$, (b) $-\Delta t$, and (c) $+20\Delta t$.

Fig. 3  Time series of $\phi$, $n_i (i = 1, 2, 3)$ of $\Sigma_N (\phi < 0)$ or $\Sigma_S (0 < \phi)$, and $|\omega|$ at a point. The red triangle shows the moment of the vortex transition.

$|k| < 121$ where $|k| = (k_i k_i)^{1/2}$, and the phase shifting method is used for dealiasing. The time step $\Delta t$ is 0.001 in the fourth-order Runge-Kutta method, and the initial Taylor-Reynolds number $Re_1 = 311$, Taylor microscale $\lambda_T = 0.59$, Kolmogorov length $\eta = 0.015$, and eddy turnover time $t_{edd} = 1.14$. The flow transition is analyzed at approximately 34000 nodes where vortices are generated in $Re_1 \cong 35$ (after the peak of the enstrophy). $\phi$, $\nabla \phi$ components, and vorticity $|\omega|$ are traced during an interval of the step $\Delta t$ while the Kolmogorov time $\tau = 0.07$. The specific time step when negative $\phi$ becomes positive at the next step is defined as the preceding time step before the vortex transition. It indicates that the vortex is generated during the time interval $\Delta t$ between the preceding and next time steps.

Figure 2 shows the flow transition into a vortical flow in a point at $-10\Delta t$, $-\Delta t$, and $+20\Delta t$, respectively. It is noted that $-\Delta t$ denotes the preceding time step of the vortex transition. In Fig. 2, the velocity field is shown in $\Sigma_N$ of the point, where the velocity component at the point is subtracted from the actual velocity field to facilitate observation of the flow feature. Through the transition of the flow topology in Fig. 2 (a) and (b), an elliptic vortex is formed. Then, after the vortex transition, the vortical flow symmetry increases with the development of swirling, i.e., $\phi$, as shown in Fig. 2 (c) (Nakayama, 2017a).

Figure 3 shows the time series of $\phi$, normal vector components $n_i (i = 1, 2, 3)$ of $\Sigma_N$ (before the vortex transition: $\phi < 0$) or $\Sigma_S$ (after the transition: $0 < \phi$), and $|\omega|$ at a point, in the time interval of $\pm 10\Delta t$ from the vortex transition. $\phi$ and $|\omega|$ are nondimensionalized by their root mean square values at the preceding time step of the vortex transition. It shows that $\phi$ has a particular feature that it steeply increases in the transition between non-vortical and vortical flow (Nakayama et al., 2015). On the other hand, the all $n_i$ components have almost no changes in the transition. It indicates that $\Sigma_N$ becomes $\Sigma_S$ after the generation smoothly, and that their normal vectors do not change drastically in the vortex transition.

The features of $\phi$ and $n_i$ in Fig. 3 are similar to the statistical characteristics of these quantities in the condition that $r = |A_n/A_b|$ or $|A_b/A_n|$ $(0 < r \leq 1)$ increases in this time interval (Nakayama et al., 2015). It is noted that $r$ is associated...
with \( c \) after the vortex transition; \( r = c^2 \) where \( c = c \) (0 < \( c \) ≤ 1) or 1/\( c \) (1 < \( c \)). After the transition, \( c' \) (flow symmetry) tends to increase with \( \phi \) (Nakayama, 2017a).

Figure 4 shows the time series of \( \nabla \vec{v} \) components through the vortex transition (time interval of ±10\( \Delta t \)) at a point. The components of \( \nabla \vec{v} \) are nondimensionalized by the root mean square value of \( |\vec{v}| \) at the preceding time step of the vortex transition. It shows that each component smoothly varies. Specifically \( \hat{a}_{21} \), \( \hat{a}_{31} \), and \( \hat{a}_{32} \) components become zero at the vortex transition, in accordance with the representation of \( \nabla \vec{v} \) in Eq. (16). On the other hand, although \( \hat{a}_{11} \) and \( \hat{a}_{22} \) approach throughout the time interval, they are not equal even after the vortex transition as shown in Eq. (7). This is because of the continuous transitions of \( \hat{a}_{31} \) or \( \hat{a}_{32} \), i.e., continuous transitions of \( \Sigma_N \) and \( \Sigma_S \). As \( \hat{a}_{31} \) or \( \hat{a}_{32} \) changes, consequently \( \Sigma_N \) and \( \Sigma_S \) change because their corresponding eigenvectors are associated with these components.

We analyse the joint probability density functions (JPDFs) of these components to examine their relationships in terms of statistical characteristics. Figure 5 shows the JPDFs of \( \hat{a}_{11} \) and \( \hat{a}_{22} \) at 10\( \Delta t \) and \( \Delta t \) before the vortex transition, i.e., at −10\( \Delta t \) and −\( \Delta t \). The distribution of the JPDF approaches to the line \( \hat{a}_{11} = \hat{a}_{22} \) before the transition. It indicates that \( \hat{a}_{11} \) and \( \hat{a}_{22} \) becomes the same toward the vortex transition, thus they have the same sign before the transition.

The JPDFs of the data (\( \hat{a}_{12}, \hat{a}_{21} \)) at −10\( \Delta t \), −\( \Delta t \), and +10\( \Delta t \) are shown in Fig. 6. It shows that \( \hat{a}_{12} \) and \( \hat{a}_{21} \) statistically tend to have the same sign before the vortex transition. Then \( \hat{a}_{21} \) approaches to zero toward the vortex transition and finally has the opposite sign to \( \hat{a}_{12} \) so that \( \hat{a}_{12}\hat{a}_{21} < 0 \). It is clear that a swirling motion is not generated as long as \( \hat{a}_{12} \) and \( \hat{a}_{21} \) have the same sign in the \( \hat{x}_1\hat{x}_2 \) (\( \hat{x}_1\cdot\hat{x}_2 \)) plane. In this point, the change of \( \hat{a}_{21} \) is an important key for the vortex transition.

As for the other components, they do not exhibit particular features. For example, the JPDFs of the data (\( \hat{a}_{11}, \hat{a}_{31} \)) are shown in Fig. 7. The feature of their correlation does not change through the flow transition, and the distribution of \( \hat{a}_{31} \) is symmetric to \( \hat{a}_{11} \). Thus \( \hat{a}_{31} \) has no correlation with \( \hat{a}_{11} \) that has the particular feature in the vortex transition. The correlation coefficients are less than 0.01 in both Fig. 7 (a) and (b). Similarly, the JPDF of the data (\( \partial\hat{a}_{23}/\partial t, \partial\hat{a}_{23}/\partial \tau \)) at −\( \Delta t \) is shown in Fig. 8. The time derivative is nondimensionalized by \( \tau \). The distribution is diffuse and symmetric to \( \partial\hat{a}_{23}/\partial t \). These features are nearly steady throughout the transition. Thus \( \partial\hat{a}_{23}/\partial t \) has no correlation with \( \partial\hat{a}_{23}/\partial \tau \). This correlation coefficient is 0.01.
6. Discussion

The result of the numerical analysis shows that $\phi$ effectively predicts or specifies the vortex transition while the $|\omega|$ is not appropriate for this purpose because it does not have any particular feature nor criterion in the transition. We note that, if we focus only on the detection of vortex transition, the criterion of the $\Delta$-definition (Chong et al., 1990) is possible to specify it. The criterion is expressed by the inequality: $\Delta = (Q/3)^3 + (R/2)^2 > 0$, where $R$ denotes the third invariant of $\nabla \phi$. $\Delta > 0$ is equivalent to the condition that $0 < \phi$ (Nakayama, 2014). However, $\Delta > 0$ is an algebraic condition that a cubic equation has conjugate complex solutions. While $\phi$ has a clear topological interpretation, $\Delta$ is difficult to be related to the topology. In this mathematical and physical point, $\phi$ seems to be an appropriate physical quantity to monitor the flow transition.

As for the predicted swirl plane, $\Sigma_N$ becomes $\Sigma_S$ in the transition smoothly, and the change of the normal vectors of these planes is small. Thus, $\Sigma_N$ is an appropriate predicted swirl plane, and the $\nabla \phi$ components analysed in the $\hat{x}_i$ coordinate system at the preceding time step before the vortex transition are related to the flow topology. Thus this specific coordinate system is appropriate to specify the key flow for the vortex transition.

Figure 5 shows that $\hat{a}_{11}$ and $\hat{a}_{22}$ have the same sign and become the same toward the vortex transition with intense correlation, where the correlation coefficients are over 0.9 in both Fig. 5 (a) and (b). Furthermore, it indicates an important feature that $\phi$ approaches to zero, because it is mathematically shown that $\phi = -|\lambda_1 - \lambda_2|/2$ (Nakayama, 2016). Thus this result is consistent of the physical and mathematical characteristics of these quantities.

In addition, $\hat{a}_{12}$ and $\hat{a}_{21}$ are strongly associated with the vortex transition. This is a reasonable characteristic because the sign of them must differ for the swirling flow in $\Sigma_N$ ($\Sigma_S$). In the vortex transition, $\hat{a}_{12}$ has a non-zero value, and the sign of $\hat{a}_{21}$ changes from the same to opposite, as shown in the sequence in Fig. 6. $\hat{a}_{12}$ and $\hat{a}_{21}$ have strong correlation in this transition, and their correlation coefficients before/after the vortex transition are 0.6 and $-0.6$ in Fig. 6 (a) and (c), respectively.

The present analysis with each component of $\nabla \phi$ elucidates detail characteristics of the flow transition, and enables to specify important $\nabla \phi$ components and clarify their relations and effects to the topology. The key flow ($\nabla \phi$) components may depend on the characteristics of a turbulent flow to be subjected. However, the flow feature of $\hat{a}_{21}$ described above
may be mandatory irrespective of turbulent flows to have swirling flow in \( \Sigma_N (\Sigma_S) \), because the feature to have the opposite sign to \( \hat{a}_{12} \) is necessary for a vortex.

This analysis is applicable to various turbulent flows and diverse engineering fields that have a concern with vortices. If the key flow for the vortex transition or swirl plane to be generated is specified, it contributes to the flow control or flow design. If the flow is decomposed into several scales, e.g., by band-pass filter or coarse graining, then the feature of the vortex transition in a subjected flow scale can be clarified (Nakayama, 2017b), and a state of the vortex transition is evaluated quantitatively using swirlity.

7. Conclusion

The numerical analysis of flow transition into a vortex has been presented that enables to relate the velocity gradient tensor (shear or strain) components to the local flow topology and identify the specific tensor components as key flow for the vortex transition. This analysis is applicable to various turbulent flows with several scales and fluid engineering fields for suppressing vortices or flow control.

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