EXTREME VALUES FOR ITERATED INTEGRALS OF THE LOGARITHM OF THE RIEMANN ZETA-FUNCTION

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Abstract. In this paper, we give an approximate formula for the measure of extreme values for the logarithm of the Riemann zeta-function and its iterated integrals. The result recovers the unconditional best result for the Ω-result of $S_1(t)$ for the part of minus of Tsang.

1. Introduction and the statement of results

In this paper, we discuss the value distribution of $	ilde{\eta}_m(s)$ in the critical strip. Here, the function $	ilde{\eta}_m(s)$ is defined by the recurrence equation

$$
\tilde{\eta}_m(\sigma + it) = \int_{\sigma}^{\infty} \tilde{\eta}_{m-1}(\alpha + it) d\alpha,
$$

where $\tilde{\eta}_0(\sigma + it) = \log \zeta(\sigma + it)$, and $\zeta(s)$ is the Riemann zeta-function. Here, we decide the branch of the logarithm of the Riemann zeta-function as follows. Let $s = \sigma + it \in \mathbb{C}$ with $\sigma, t \in \mathbb{R}$. When $t$ is equal to neither zero nor the ordinate of nontrivial zeros of $\zeta(s)$, we choose the branch by the continuation with the initial condition $\lim_{\sigma \to +\infty} \log \zeta(\sigma + it) = 0$. If $t = 0$, then $\log \zeta(\sigma) = \lim_{\varepsilon \to 0} \log \zeta(\sigma + i\varepsilon)$. If $t$ is the ordinate of a nontrivial zero $\rho = \beta + i\gamma$ of the Riemann zeta-function, then $\log \zeta(\sigma + i\gamma) = \lim_{\varepsilon \to 0} \log \zeta(\sigma + i(\gamma - \text{sgn}(\gamma)\varepsilon))$.

The function $\tilde{\eta}_m$ is related to the well known function $S_m(t)$, which has been studied by many mathematicians including [12], [14]. Here, $S_m(t)$ is defined by the recurrence equation, for $m \in \mathbb{Z}_{\geq 1}$,

$$
S_m(t) = \int_{0}^{t} S_{m-1}(u) du + b_m.
$$

Here, $S_0(t) = \frac{1}{\pi} \text{Im} \log \zeta(\frac{1}{2} + it)$, and $b_m = \frac{1}{\pi (m-1)!} \text{Im} \int_{\frac{1}{2}}^{\infty} (\alpha - \frac{1}{2})^{m-1} \log \zeta(\alpha) d\alpha$.

To explain the relation between $\tilde{\eta}_m(\sigma + it)$ and $S_m(t)$, we also define the function $\eta_m(s)$ by the recurrence equation

$$
\eta_m(\sigma + it) = \int_{0}^{t} \eta_{m-1}(\sigma + iu) du + c_m(\sigma),
$$

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where \( \eta_0(\sigma + it) = \log \zeta(\sigma + it) \), and \( c_m(\sigma) = \frac{1}{(m-1)!} \int_\sigma^\infty (\alpha - \sigma)^{m-1} \log \zeta(\alpha) d\alpha \).

Then the function \( S_m(t) \) is clearly equal to \( \frac{1}{\pi} \text{Im} \eta_m(1/2 + it) \). In the following, \( \rho = \beta + i\gamma \) means a nontrivial zero of the Riemann zeta-function. Under this notation, the relation between \( \tilde{\eta}_m(s) \) and \( \eta_m(s) \) is understood by the equation, for \( m \geq 1 \),

\[
\eta_m(\sigma + it) = i^m \tilde{\eta}_m(\sigma + it) + 2\pi \sum_{k=0}^{m-1} \frac{i^{m-1-k}}{(m-k)!k!} \sum_{\beta > \sigma \atop 0 < \gamma < t} (\beta - \sigma)^{m-k} (t - \gamma)^k.
\]

This can be obtained by Lemma 1 of [10] and the fact \( \tilde{\eta}_m(\sigma + it) = \frac{1}{(m-1)!} \int_\sigma^\infty (\alpha - \sigma)^{m-1} \log (\alpha + it) d\alpha \) that can be easily obtained by integration by parts. Hence, it holds that \( S_1(t) = \pi^{-1} \text{Re} \tilde{\eta}_1(1/2 + it) \), and additionally if the Riemann Hypothesis is true, then \( S_m(t) = \frac{1}{\pi} \text{Im} i^m \tilde{\eta}_m(1/2 + it) \) for any \( m \in \mathbb{Z}_{\geq 0} \). The difference between \( \eta_m(1/2 + it) \) and \( \tilde{\eta}_m(1/2 + it) \) was firstly studied by Fujii [6], and he gave a statement for the magnitude of \( \eta_m(1/2 + it) \) which is equivalent to the Riemann Hypothesis. From this perspective, the function \( \tilde{\eta}_m(s) \) is an interesting object.

The study of the value distribution of \( \tilde{\eta}_m(s) \) is important because it is directly related to the Lindelöf Hypothesis. It is known that (cf. Theorems 13.6 (B) and 13.8 in [18]) the Lindelöf Hypothesis is equivalent to the estimate \( \text{Re} \tilde{\eta}_1(1/2 + it) = \pi S_1(t) = o(\log t) \) as \( t \to +\infty \). Additionally, we can generalize this fact to the following proposition.

**Proposition 1.** Let \( m \in \mathbb{Z}_{\geq 1} \). The Lindelöf Hypothesis is equivalent to the estimate \( \text{Re} \tilde{\eta}_m(1/2 + it) = o(\log t) \) as \( t \to +\infty \).

We omit the proof of this proposition in this paper because it can be proved by almost the same method as Theorems 13.6 (B) and 13.8 in [18]. In view of Proposition 1, it is desirable to understand the exact behavior of \( \tilde{\eta}_m(\sigma + it) \). Incidentally, we can show the estimate \( \tilde{\eta}_m(1/2 + it) \ll_m \log (|t| + 2) \) for \( m \geq 1 \) by the standard way.

Recently, \( \Omega \)-estimates on \( S_m(t) \) have been developed by some studies such as [2], [4], [5] under the Riemann Hypothesis. Those results were shown by the resonance method due to Bondarenko and Seip [2], [3]. On the other hand, as mentioned in [2], it is desired that those could be shown unconditionally by proving a stronger result for the measure of extreme values like Soundararajan’s result [16, Theorem 1]. In this paper, the author shows a result toward this problem.

Now, we define the set \( \mathcal{I}_{m,\theta}(T, V; \sigma) \) by

\[
\mathcal{I}_{m,\theta}(T, V; \sigma) := \{ t \in [T, 2T] \mid \text{Re}(e^{-i\theta} \tilde{\eta}_m(\sigma + it)) > V \}.
\]

The symbol \( \text{meas}(\cdot) \) stands for the Lebesgue measure on \( \mathbb{R} \). Then we show the following theorem.
Theorem 1. Let $m \in \mathbb{Z}_{\geq 1}$, $\theta \in \mathbb{R}$ be fixed. There exists a positive constant $a_1 = a_1(m)$ such that, for any large numbers $T$, $V$ with $V \leq a_1 \left( \frac{\log T}{(\log \log T)^{2m+1}} \right)^{2m+1}$, we have

$$\frac{1}{T} \operatorname{meas}(\mathcal{S}_m, \theta(T, V; 1/2)) = \exp \left( -2m4^mV^2(\log V)^{2m}(1 + R) \right),$$

(1.2)

where the error term $R$ satisfies

$$R \ll mV^2(\log V)^{2m}(1 + R).$$

This theorem contains the unconditional best result $S_1(t) = \Omega_{-} \left( \frac{(\log t)^{1/3}}{(\log \log t)^{4/3}} \right)$ due to Tsang [20]. Actually, we can immediately obtain the following corollary.

Corollary 1. Let $m \in \mathbb{Z}_{\geq 1}$, $\theta \in \mathbb{R}$ be fixed. Then we have

$$\operatorname{Re} e^{-i\theta} \tilde{\eta}_m(1/2 + it) = \Omega_{\pm} \left( \frac{(\log t)^{m+1}}{(\log \log t)^{2m+1}} \right) + \text{contribution from zeros}.$$ (1.4)

If the Riemann Hypothesis is true, we can improve this corollary. Actually, assuming the Riemann Hypothesis, Bondarenko-Seip [2], Chirre [4], and Chirre-Matahab [5] showed that, for certain $\theta$,

$$\operatorname{Re} e^{-i\theta} \tilde{\eta}_m(1/2 + it) = \Omega_{+} \left( \frac{\log t \log \log \log t}{(\log \log t)^{m+1/2}} \right).$$

Moreover, by using Tsang’s method [20], we can prove that for any fixed $\theta \in \mathbb{R}$,

$$\operatorname{Re} e^{-i\theta} \tilde{\eta}_m(1/2 + it) = \Omega_{+} \left( \frac{\log t}{(\log \log t)^{m+1/2}} \right).$$

(1.3)

under the Riemann Hypothesis. The author cannot find a suitable reference for the latter $\Omega$-result, but it is not difficult to check it. Furthermore, Tsang [19] showed this $\Omega$-estimate unconditionally in the case $\theta = 0$, $m = 1$. As we mentioned above, it seems desirable to establish a stronger result for the measure of extreme values of $\tilde{\eta}_m(1/2 + it)$ corresponding to the above $\Omega$-results. Therefore, we hope to prove asymptotic formula (1.2) for larger $V$, but the author was not able to prove it. In the following, we observe this matter. To prove Theorem 1, we in this paper use the fact that $\tilde{\eta}_m(s)$ looks roughly like

$$\tilde{\eta}_m(s) \approx \sum_{p \leq X} \frac{1}{p^s(\log p)^m} + \text{contribution from zeros}.$$ (1.4)

This is an analogue of the fact obtained by the hybrid formula of Gonek, Hughes, and Keating [8]. Hence, we need to understand the value distribution of the Dirichlet polynomial $\sum_{p \leq X} \frac{1}{p^s(\log p)^m}$ and to estimate the contribution from zeros. For the value distribution of the Dirichlet polynomial, we can obtain the following proposition.
Proposition 2. Let $m \in \mathbb{Z}_{\geq 1}$, $\theta \in \mathbb{R}$ be fixed. There exist positive constants $a_2 = a_2(m)$, $a_3 = a_3(m)$ such that for large numbers $T, V, X$ with $V \leq a_2 \frac{\sqrt{\log T}}{(\log \log T)^{m+1/2}}$, and $V^4 \leq X \leq T^{a_3/V^2(\log V)^{2m}}$, we have

$$\frac{1}{T} \text{meas} \left\{ t \in [T, 2T] \mid \Re e^{-it} \sum_{p \leq X} \frac{1}{p^{1/2+it}(\log p)^{m}} > V \right\}$$

$$= \exp \left(-\frac{2m^4V^2(\log V)^{2m}}{1 - \left(\frac{\log V^2}{\log X}\right)^{m}} \left(1 + O_{m} \left(\sqrt{\frac{\log \log V}{\log V}}\right)\right)\right).$$

This proposition contains an $\Omega$-result for Dirichlet polynomials corresponding to $\Omega$-result (1.3). Actually, it follows from Proposition 2 that, for $X = (\log T)^4$,

$$\max_{t \in [T, 2T]} \Re e^{-it} \sum_{p \leq X} \frac{1}{p^{1/2+it}(\log p)^{m}} \geq c \frac{\sqrt{\log T}}{(\log \log T)^{m+1/2}}$$

for some constant $c > 0$. Hence, for Dirichlet polynomials, we can prove unconditionally a result for the measure that contains the $\Omega$-result corresponding to (1.3). On the other hand, we use the result of the previous work of the author [10, Theorem 5] to estimate the contribution from zeros based on the zero density estimate of Selberg [15, Theorem 1]. However, it is difficult to obtain the satisfactory estimate of the contribution from zeros since there are many zeros near the critical line, and so the author has been not yet able to prove Theorem 1 for larger $V$. Additionally, he was not able to prove it even under the Riemann Hypothesis.

So far, we described the results in the case $\sigma = 1/2$. On the other hand, the method of the proof of the above assertions can be also applied to the case $\frac{1}{2} < \sigma < 1$. Moreover, we can estimate the contribution from zeros in this case satisfactorily. Thanks to that, we can obtain a theorem which is an analogue of the works due to Lamzouri [11]. We define $A_m(\sigma)$ by

$$A_m(\sigma) = \left(\frac{\sigma^{2\sigma}}{(1-\sigma)^{2\sigma-1}}G(\sigma)\right)^{1/\sigma}.$$

Here, $G(\sigma) = \int_0^\infty \log I_0(u)u^{-1-\frac{\sigma}{2}}du$, and $I_0$ is the modified 0-th Bessel function defined by $I_0(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(z \cos \theta) d\theta = \sum_{n=0}^{\infty} (z/2)^{2n}/(n!)^2$.

Theorem 2. Let $m \in \mathbb{Z}_{\geq 0}$, $\frac{1}{2} < \sigma < 1$, and $\theta \in \mathbb{R}$ be fixed. There exists a positive constant $a_4 = a_4(\sigma, m)$ such that, for any large numbers $T, V$ with $V \leq a_4 \frac{(\log T)^{1-\frac{\sigma}{2}}}{(\log \log T)^{m+1/2}}$, we have

$$\frac{1}{T} \text{meas}(\mathcal{A}_{m, \theta}(T; V; \sigma)) = \exp \left(-A_m(\sigma)V^{1/\sigma} \frac{\log V}{\log T}^m (1+R)\right),$$
where the error term $R$ satisfies the estimate

$$R \ll_{\sigma, m} \sqrt{\frac{1 + m \log \log V}{\log V}}.$$  \hspace{1cm} (1.5)

When $m = 0$, the asymptotic formula of this type was firstly proved by Hattori and Matsumoto$^1$ [9]. They showed that, for $\frac{1}{2} < \sigma < 1$,

$$\lim_{T \to +\infty} \frac{1}{T} \text{meas} \left( \bigcup_{j=0}^{3} \mathcal{S}_{0, \frac{\pi}{2j}}(T, V; \sigma) \right) = \exp \left( -A_0(\sigma) V^{\frac{1}{1-\sigma}} (\log V)^{\frac{\sigma}{1-\sigma}} (1 + o(1)) \right)$$

as $V \to +\infty$. Note that the parameter $V$ in their asymptotic formula is not effective with respect to $T$. Theorem 2 can recover this asymptotic formula effectively. Actually, we see that

$$\frac{1}{T} \text{meas} (\mathcal{S}_{0,0}(T, V; \sigma)) \leq \frac{1}{T} \text{meas} \left( \bigcup_{j=0}^{3} \mathcal{S}_{0, \frac{\pi}{2j}}(T, V; \sigma) \right) \leq \frac{1}{T} \sum_{j=0}^{3} \text{meas} (\mathcal{S}_{0, \frac{\pi}{2j}}(T, V; \sigma)),$$

and both sides are equal to $\exp \left( -A_0(\sigma) V^{\frac{1}{1-\sigma}} (\log V)^{\frac{\sigma}{1-\sigma}} (1 + R) \right)$ from Theorem 2. Here, the error term $R$ satisfies (1.5). Hence, we can improve (1.6) to the effective form. On the other hand, it seems this improvement has been essentially obtained by Lamzouri’s work [11]. After the study of Hattori-Matsumoto, Lamzouri [11] showed an effective asymptotic formula in the case $\theta = 0$ only. Though he did not mention, we can also prove his theorem for any $\theta \in \mathbb{R}$ by just using his method. Therefore, we may say that the above improvement has been already given by Lamzouri.

Now, we state the proposition corresponding to Proposition 2, which plays an important role in Theorem 2.

**Proposition 3.** Let $m \in \mathbb{Z}_{\geq 0}$, $\frac{1}{2} < \sigma < 1$, and $\theta \in \mathbb{R}$ be fixed. There exist positive constants $a_5 = a_5(\sigma, m)$, $a_6 = a_6(\sigma, m)$ such that for large numbers $T, X, V$ with $V \leq a_5 \left( \frac{(\log T)^{1-\sigma}}{(\log \log T)^{m+1}} \right)$ and $V^{\frac{4\sigma}{1-\sigma}} \leq X \leq T^{a_6/V^{\frac{1}{1-\sigma}} (\log V)^{\frac{1+\sigma}{1-\sigma}}}$, we have

$$\frac{1}{T} \text{meas} \left\{ t \in [T, 2T] \bigg| \text{Re} \left( e^{-i\theta} \sum_{p \leq X} \frac{1}{p^{\sigma+it} (\log p)^m} \right) > V \right\}$$

$$= \exp \left( -A_m(\sigma) V^{\frac{1}{1-\sigma}} (\log V)^{\frac{\sigma}{1-\sigma}} \left( 1 + O_{\sigma, m} \left( \sqrt{\frac{1 + m \log \log V}{\log V}} \right) \right) \right).$$

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$^1$There is a difference of the range of $t$ between ours and theirs, but it seems not essential. Precisely, our range of $t$ is $t \in [T, 2T]$, and theirs is $t \in [-T, T]$. 
As we mentioned above, we can obtain a good estimate of the contribution from zeros, and so Theorem 2 is proved in the same range as Proposition 3.

Here, we describe the method of the proofs of Theorem 1 and Theorem 2 roughly. These theorems are analogues of Lamzouri’s result, but we cannot adopt directly his method. He used the Euler product of the Riemann zeta-function and the generalized divisor function to estimate a Dirichlet polynomial. However, \( \tilde{\eta}_m(s) \) does not have the representation of Euler product when \( m \geq 1 \), and so we cannot apply directly his method. To avoid this obstacle the author uses (1.4), and estimates the Dirichlet polynomial by using Radziwill’s method [13].

2. Preliminaries

In this section, we prepare some lemmas.

**Lemma 1.** Let \( \theta \in \mathbb{R} \) be fixed. For any \( n \in \mathbb{Z}_{\geq 2} \), we write \( n = q_1^{\alpha_1} \cdots q_r^{\alpha_r} \), where \( q_j \) are distinct prime numbers. Then we have

\[
\frac{1}{T} \int_T^{2T} \prod_{j=1}^r (\cos(t \log q_j + \theta))^{\alpha_j} \, dt = f(n) + O \left( \frac{n}{T} \right)
\]

for any \( T > 0 \). Here, \( f \) is the multiplicative function defined by \( f(p^\alpha) = 2^{-\alpha} \left( \frac{\alpha}{\alpha/2} \right) \) for a prime power \( p^\alpha \), and we regard that \( \left( \frac{\alpha}{\alpha/2} \right) = 0 \) if \( \alpha \) is odd.

**Proof.** We find that

\[
(\cos(t \log q_j + \theta))^{\alpha_j} = \frac{1}{2^{\alpha_j}} \left( e^{i(t \log q_j + \theta)} + e^{-i(t \log q_j + \theta)} \right)^{\alpha_j} = \frac{1}{2^{\alpha_j}} \sum_{\varepsilon_1, \ldots, \varepsilon_{\alpha_j} \in \{-1, 1\}} e^{i(\varepsilon_1 + \cdots + \varepsilon_{\alpha_j})(t \log q_j + \theta)}
\]

\[
= \frac{1}{2^{\alpha_j}} \left( \frac{\alpha_j}{\alpha_j/2} \right) + \frac{1}{2^{\alpha_j}} \sum_{\varepsilon_1, \ldots, \varepsilon_{\alpha_j} \in \{-1, 1\}} \varepsilon_{\alpha_j} \neq 0 e^{i(\varepsilon_1 + \cdots + \varepsilon_{\alpha_j})(t \log q_j + \theta)}.
\]

Therefore, we obtain

\[
\prod_{j=1}^r (\cos(t \log q_j + \theta))^{\alpha_j} = f(n) + E,
\]

where \( E \) is the sum whose the number of terms is less than \( 2^{\Omega(n)} \), and the form of each term is \( \delta e^{it(\beta_1 \log q_1 + \cdots + \beta_s \log q_s)} \). Here, \( \delta \) is a complex number independent of \( t \) satisfying \( |\delta| \leq 2^{-\Omega(n)} \), and \( \beta_j \)'s are integers with \( 0 \leq |\beta_j| \leq \alpha_j \) and \( \beta_s \neq 0 \) for some \( 1 \leq s \leq r \). Since \( |\beta_1 \log q_1 + \cdots + \beta_r \log q_r| \gg n^{-1} \), the integral of each term of \( E \) is estimated by \( \ll n 2^{-\Omega(n)} \). As the number of such terms \( \ll 2^{\Omega(n)} \), we have \( \int_T^{2T} E dt \ll n \). Thus, by this estimate and equation (2.1), we obtain this lemma. \( \square \)
Lemma 2. Let $m \in \mathbb{Z}_{\geq 0}$, $\frac{1}{2} \leq \sigma < 1$ be fixed. Let $X \geq 3$, and $T$ be large. Then, for any positive integer $k$, we have

$$\frac{1}{T} \int_T^{2T} \left( \text{Re} \left( e^{-i\theta} \sum_{p \leq X} \frac{1}{p^{\sigma + it} (\log p)^m} \right) \right)^k dt$$

$$= \frac{k!}{2\pi i} \oint_{|w|=R} \prod_{p \leq X} I_0 \left( \frac{w}{p^\sigma (\log p)^m} \right) dw + O \left( \frac{X^{2k}}{T} \right).$$

Here, $R$ is any positive number, and $I_0$ is the modified 0-th order Bessel function.

Proof. Define the multiplicative function $g_X(n)$ as, for every prime number $p$ and $\alpha \in \mathbb{Z}_{\geq 1}$, $g_X(p^\alpha) = 1/\alpha!(\log p)^{\alpha m}$ if $p \leq X$, and $g_X(p^\alpha) = 0$ otherwise. By Lemma 1, we find that

$$\frac{1}{T} \int_T^{2T} \left( \text{Re} \sum_{p \leq X} \frac{1}{p^{\sigma + it} (\log p)^m} \right)^k dt$$

$$= \frac{1}{T} \sum_{p_1, \ldots, p_k \leq X} \int_T^{2T} \cos(t \log p_1 + \theta) \cdots \cos(t \log p_k + \theta) dt$$

$$= \sum_{p_1, \ldots, p_k \leq X} \frac{f(p_1 \cdots p_k)}{(p_1 \cdots p_k)^\sigma (\log p_1 \cdots \log p_k)^m} + O \left( \frac{X^{2k}}{T} \right).$$

From this equation and the definition of $g_X$, we have

$$\frac{1}{T} \int_T^{2T} \left( \text{Re} \sum_{p \leq X} \frac{1}{p^{\sigma + it} (\log p)^m} \right)^k dt = k! \sum_{\Omega(n)=k} \frac{f(n)}{n^\sigma} g_X(n) + O \left( \frac{X^{2k}}{T} \right).$$

By Cauchy’s integral formula, the above is equal to

$$\frac{k!}{2\pi i} \oint_{|w|=R} \sum_{n=1}^\infty \frac{f(n)}{n^\sigma} g_X(n) w^{\Omega(n)} dw \frac{w^{k+1}}{w^{k+1}} + O \left( \frac{X^{2k}}{T} \right).$$

Since the functions $f$, $g_X$, and $w^{\Omega(n)}$ are multiplicative, this main term is

$$= \frac{k!}{2\pi i} \oint_{|w|=R} \prod_{p \leq X} \left( \sum_{l=0}^\infty \left( \frac{w/2p^\sigma (\log p)^m}{(l)!^2} \right)^{2l} \right) dw$$

$$= \frac{k!}{2\pi i} \oint_{|w|=R} \prod_{p \leq X} I_0 \left( \frac{w}{p^\sigma (\log p)^m} \right) dw,$$

which completes the proof of this lemma. □
Lemma 3. Let $m$ be a fixed positive integer. For $x \geq 3$, $X \geq x^3$, we have

$$\prod_{p \leq X} I_0 \left( \frac{x}{\sqrt{p} (\log p)^m} \right) = \exp \left( \frac{x^2}{8m(2 \log x)^{2m}} \left( 1 - \left( \frac{\log x^2}{\log X} \right)^{2m} + O \left( \frac{\log \log x}{\log x} \right) \right) \right).$$

Proof. By the Taylor expansion of $I_0$ and the prime number theorem, we find that

$$\prod_{\frac{x^2}{(\log x)^{2m}} < p \leq X} I_0 \left( \frac{x}{\sqrt{p} (\log p)^m} \right) = \exp \left( \sum_{\frac{x^2}{(\log x)^{2m}} < p \leq X} \left( \frac{x^2}{4p(\log p)^{2m}} + O_m \left( \frac{x^4}{p^2(\log p)^{4m}} \right) \right) \right).$$

$$= \exp \left( \frac{x^2}{8m(2 \log x)^{2m}} \left( 1 - \left( \frac{\log x^2}{\log X} \right)^{2m} + O_m \left( \frac{\log \log x}{\log x} \right) \right) \right). \quad (2.2)$$

On the other hand, by using the inequality $I_0(x) \leq \exp(x)$ and the prime number theorem, it holds that

$$\prod_{p \leq \frac{x^2}{(\log x)^{2m}}} I_0 \left( \frac{x}{\sqrt{p} (\log p)^m} \right) \leq \exp \left( x \sum_{p \leq \frac{x^2}{(\log x)^{2m}}} \frac{1}{\sqrt{p} (\log p)^m} \right) \leq \exp \left( O_m \left( \frac{x^2}{(\log x)^{2m+1}} \right) \right).$$

From this estimate and equation (2.2), we obtain this lemma. \qed

Lemma 4. Let $\frac{1}{2} < \sigma < 1$, $m \in \mathbb{Z}_{\geq 0}$ be fixed. Then, for large $x$, $X \geq x^3$, we have

$$\prod_{p \leq X} I_0 \left( \frac{x}{p^\sigma (\log p)^m} \right) = \exp \left( \frac{\sigma m}{\sigma - \sigma} G(\sigma) \frac{x^{\frac{1}{\sigma}}}{(\log x)^{\frac{2m}{\sigma} + 1}} \left( 1 + O \left( \frac{1 + m \log \log x}{\log x} \right) \right) \right).$$

Proof. We take the numbers $y_0, y_1$ as satisfying the equations $y_0^m (\log y_0)^m = x^{1/2}$, $y_1^m (\log y_1)^m = x^{3/2}$, respectively. Then, it holds that $y_0 \asymp_m x^{\frac{1}{2\sigma}} (\log x)^{-\frac{m}{\sigma}}$, $y_1 \asymp_m x^{\frac{3}{2\sigma}} (\log x)^{-\frac{m}{2}}$, and the estimate $X \gg y_1$ also holds. By the Taylor expansion of $I_0$ and the prime number theorem, we find that

$$\sum_{p \leq X} \log I_0 \left( \frac{x}{p^\sigma (\log p)^m} \right) = \sum_{p \leq y_1} \log I_0 \left( \frac{x}{p^\sigma (\log p)^m} \right) + O_m, \frac{1 - 2\sigma}{x^{\frac{2m}{\sigma} + 1}}.$$
By the inequality $I_0(x) \leq \exp(x)$, it holds that
\[
\sum_{p \leq y_0} \log I_0 \left( \frac{x}{p^\sigma (\log p)^m} \right) \leq \sum_{p \leq y_0} \frac{x}{p^\sigma (\log p)^m} \ll_{m, \sigma} \frac{x^{\frac{1-\sigma}{\sigma} (\log x)^{\frac{m}{\sigma}+1}}.
\]

From these estimates, one has
\[
\sum_{y_0 < p \leq y_1} \log I_0 \left( \frac{x}{p^\sigma (\log p)^m} \right) + O_{m, \sigma} \left( \frac{1}{(\log x)^{\frac{m}{\sigma}+1}} \right) \left( x^{\frac{1+\sigma}{2\sigma}} + x^{\frac{3-2\sigma}{2\sigma}} \right).
\]

By using partial summation and estimates of $I_0$, we obtain
\[
\sum_{y_0 < p \leq y_1} \log I_0 \left( \frac{x}{p^\sigma (\log p)^m} \right) = - \int_{y_0}^{y_1} \pi(\xi) \left( \frac{dx}{d\xi} \log I_0 \left( \frac{x}{\xi^\sigma (\log \xi)^m} \right) \right) d\xi + O_m \left( \frac{x^{\frac{1+\sigma}{2\sigma}} + x^{\frac{3-2\sigma}{2\sigma}}}{(\log x)^{\frac{m}{\sigma}+1}} \right).
\]

Applying the basic formula \( \pi(\xi) = \int_2^\xi \frac{du}{\log u} + O(\xi e^{-c\sqrt{\log \xi}}) \), we find that the first term on the right hand side is equal to
\[
\int_{y_0}^{y_1} \log I_0 \left( \frac{x}{\xi^\sigma (\log \xi)^m} \right) \log \xi d\xi + O \left( \int_{y_0}^{y_1} e^{-c\sqrt{\log \xi}} \log I_0 \left( \frac{x}{\xi^\sigma (\log \xi)^m} \right) d\xi \right).
\]

Note that we used the monotonicity of $I_0$ in the above deforming. By the estimate $I_0(x) \leq \exp(x)$ and the Taylor expansion of $I_0(z)$, we find that
\[
\int_{y_0}^{y_1} e^{-c\sqrt{\log \xi}} \log I_0 \left( \frac{x}{\xi^\sigma (\log \xi)^m} \right) d\xi \ll m \int_{y_0}^{y_1} \frac{dx}{\xi^\sigma (\log \xi)^{2m+3}} + x^2 \int_{x^{1/\sigma}}^{\infty} \frac{d\xi}{\xi^\sigma (\log \xi)^{2m+3}} \ll \frac{x^{\frac{1}{\sigma} + \frac{3}{\sigma} + \frac{1}{\sigma} (\log \log x)}}{(\log x)^{\frac{2m}{\sigma} + 2}}.
\]

Finally, we consider the first term of (2.5). By making the change of variables $u = \frac{x}{\xi^\sigma (\log \xi)^m}$, hard but not difficult calculations can lead that the first term of (2.5) is equal to
\[
\sigma^{m/\sigma} x^{1/\sigma} \int_{x^{-1/\sigma}}^{x^{1/\sigma}} (1 + O_m (\frac{m \log \log x}{\log x})) \log I_0(u)u^{1+\frac{m}{\sigma} (\log (x/u))^{\frac{m}{\sigma}+1}} du
\]
\[
= \sigma^{m/\sigma} x^{1/\sigma} \int_{x^{-1/\sigma}}^{x^{1/\sigma}} \log I_0(u)u^{1+\frac{m}{\sigma} (\log (x/u))^{\frac{m}{\sigma}+1}} du + O_{m, \sigma} \left( \frac{m x^{1/\sigma} \log \log x}{(\log x)^{\frac{m}{\sigma}+2}} \right).
\]
Lemma 6. Let $\frac{1}{(\log (x/u))^{m/\sigma+1}} = 1+O_m(\log u/\log x)$ for $x^{-1/2} \leq u \leq x^{1/2}$, we find that

$$\int_{x^{-1/2}}^{x^{1/2}} \frac{\log I_0(u)}{u^{1+\frac{1}{\sigma}}(\log (x/u))^{\frac{m}{\sigma}+1}} \, du = \int_{x^{-1/2}}^{x^{1/2}} \frac{\log I_0(u)}{u^{1+\frac{1}{\sigma}}} \, du + O_m \left( \frac{1}{(\log x)^{\frac{m}{\sigma}+2}} \int_{x^{-1/2}}^{x^{1/2}} \frac{\log I_0(u)|\log u|}{u^{1+\frac{1}{\sigma}}} \, du \right).$$

Moreover, by $I_0(x) \leq \exp(x)$ and the Taylor expansion of $I_0$, it holds that

$$\int_{x^{-1/2}}^{x^{1/2}} \frac{\log I_0(u)}{u^{1+\frac{1}{\sigma}}} \, du = \int_0^\infty \frac{\log I_0(u)}{u^{1+\frac{1}{\sigma}}} \, du + O_{\sigma} \left( x^{\frac{1-2\sigma}{2\sigma}} + x^{\frac{\sigma-1}{2\sigma}} \right),$$

and that

$$\int_{x^{-1/2}}^{x^{1/2}} \frac{\log I_0(u)|\log u|}{u^{1+\frac{1}{\sigma}}} \, du \ll_{\sigma} 1$$

for $\frac{1}{2} < \sigma < 1$. From the above calculations, equation (2.4) is

$$= \frac{\sigma^{\frac{m}{\sigma}} G(\sigma)x^{\frac{1}{\sigma}}}{(\log x)^{\frac{m}{\sigma}+1}} \left( 1 + O \left( \frac{1+m \log \log x}{\log x} \right) \right).$$

Hence, by estimates (2.3), (2.4), (2.5), we obtain this lemma. \(\Box\)

Lemma 5. Let $T$ be large, and let $3 \leq X \leq T$. Let $k$ be a positive integer such that $X^k \leq T/\log T$. For any complex numbers $a(p)$ we have

$$\int_T^{2T} \left| \sum_{p \leq X} a(p)p^{1/2+it} \right|^{2k} \, dt \ll TK! \left( \sum_{p \leq X} \frac{|a(p)|^2}{p} \right)^k.$$

Proof. This is Lemma 3 in [17]. \(\Box\)

Lemma 6. Let $m \in \mathbb{Z}_{\geq 0}$, $\frac{1}{2} \leq \sigma < 1$ be fixed with $(m, \sigma) \neq (0, 1/2)$. Let $T$, $W$ be large numbers. Put $\kappa(\sigma) = 0$ if $\sigma = 1/2$, $\kappa(\sigma) = \sigma$ otherwise. Define the set

$$\mathcal{A} = \mathcal{A}(T, X, W; \sigma, m)$$

by

$$\mathcal{A} = \left\{ t \in [T, 2T] \left| \left| \sum_{p \leq X} \frac{1}{p^{\sigma+it}(\log p)^{m}} \right| \leq W \right. \right\}. \quad (2.6)$$

Then, there exists a small positive constant $b_1 = b_1(\sigma, m) \leq 1$ such that for any $3 \leq X \leq T^{1/(1-\sigma)(\log W)^{\frac{m}{1-\sigma}}}$,

$$\frac{1}{T} \meas([T, 2T] \setminus \mathcal{A}) \ll \exp \left( -b_1 W^{1/(1-\sigma)}(\log W)^{\frac{m}{1-\sigma}} \right).$$

Proof. Using the prime number theorem, we can obtain

$$\sum_{p \leq k(\log k)^{2-\kappa(\sigma)}} \frac{1}{p^{\sigma+it}(\log p)^{m}} \ll_m \frac{k^{1-\sigma}}{(\log k)^{m+\kappa(\sigma)}}.$$
By Lemma 5, we have

\[
\frac{1}{T} \int_{T}^{2T} \left| \sum_{k(\log k)^{2-\kappa(\sigma)} < p \leq X} \frac{1}{p^{\sigma + it} \log^m p} \right|^2 dt \ll k! \left( \sum_{p > k(\log k)^{2-\kappa(\sigma)}} \frac{1}{p^{2\sigma} \log^2 p} \right)^k \leq \left( C_1 \frac{k^{1-\sigma}}{\log^m k} \right)^{2k}
\]

for \( X^k \leq T^{1/2} \), where \( C_1 = C_1(\sigma, m) \) is a positive constant. Therefore, when \( X^k \leq T^{1/2} \) it holds that

\[
\frac{1}{T} \int_{T}^{2T} \left| \sum_{p \leq X} \frac{1}{p^{\sigma + it} \log^m p} \right|^2 dt \leq \left( C_2 \frac{k^{1-\sigma}}{\log^m k} \right)^{2k}
\]

for some constant \( C_2 = C_2(\sigma, m) > 0 \). Hence, we have

\[
\frac{1}{T} \text{meas}([T, 2T] \setminus A) \leq \left( C_2 \frac{k^{1-\sigma}}{W(\log^m k)} \right)^{2k}.
\]

Choosing \( k = [cW^{\frac{1}{1-\sigma}}(\log W)^{\frac{m+\kappa(\sigma)}{1-\sigma}}] \) with \( c = c(\sigma, m) \) a suitably small constant, we obtain this lemma. \( \square \)

**Lemma 7.** Assume the same situation as in Lemma 6. There exists a small positive constant \( b_2 = b_2(\sigma, m) \) such that for \( 3 \leq x \leq b_2 W^{\frac{1}{1-\sigma}}(\log W)^{\frac{m+\kappa(\sigma)}{1-\sigma}} \), \( x^3 \leq X \leq T^{1/2} W^{\frac{1}{1-\sigma}}(\log W)^{\frac{m+\kappa(\sigma)}{1-\sigma}} \), we have

\[
\frac{1}{T} \int_A \exp \left( x \text{ Re} \left( e^{-i\theta} \sum_{p \leq X} \frac{1}{p^{\sigma + it} \log^m p} \right) \right) dt = \prod_{p \leq X} I_0 \left( \frac{x}{p^{\sigma} \log^m p} \right) + O(\exp(-xW)).
\]

**Proof.** By the definition of \( A \) and the Stirling formula, we have

\[
\int_A \exp \left( x \text{ Re} \left( e^{-i\theta} \sum_{p \leq X} \frac{1}{p^{\sigma + it} \log^m p} \right) \right) dt = \sum_{k \leq Y} \frac{x^k}{k!} \int_A \left( \text{ Re} \sum_{p \leq X} e^{-i\theta} \frac{1}{p^{\sigma + it} \log^m p} \right)^k dt + O \left( T \sum_{k > Y} \frac{1}{\sqrt{k}} \left( \frac{eW}{k} \right)^k \right),
\]

where \( Y = e^{2}xW \). Here, an easy calculation for geometric sequence shows that the above \( O \)-term is \( \ll T \exp(-e^2xW) \). By using the Cauchy-Schwarz inequality,
we find that
\[
\int_A \left( \text{Re} \sum_{p \leq X} \frac{e^{-it}}{p^\sigma + it (\log p)^m} \right)^k dt = \int_T \left( \text{Re} \sum_{p \leq X} \frac{e^{-it}}{p^\sigma + it (\log p)^m} \right)^k dt +
\]
\[
+ O \left( \left( \text{meas}([T, 2T] \setminus \mathcal{A}) \right)^{1/2} \left( \int_T \left| \sum_{p \leq X} \frac{1}{p^\sigma + it (\log p)^m} \right|^{2k} dt \right)^{1/2} \right).
\]

When \( b_2 \leq e^{-2} \), from estimate (2.7) and Lemma 6, this \( O \)-term is
\[
\ll T \exp \left( -\frac{b_1}{2} W \frac{1}{1-\sigma} (\log W)^{\frac{m+\kappa(\sigma)}{1-\sigma}} \right) C_2 \left( \frac{k^{1-\sigma}}{\log k} \right)^m
\]
for \( k \leq Y \), where \( C_2 = C_2(\sigma, m) \) is a positive constant. Also, it holds that
\[
\sum_{0 \leq k \leq Y} \frac{x^k}{k!} \left( \frac{C_2}{\log k} \right)^m \leq \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{Y^{1-\sigma}}{\log Y} \right)^m
\]
\[
\leq \exp \left( 2b_2^{2-\sigma} C_2 W \frac{1}{1-\sigma} (\log W)^{\frac{m+\kappa(\sigma)}{1-\sigma}} \right)
\]
for any sufficiently large \( W \). Therefore, choosing \( b_2 \) suitably small, we find that the right hand side is \( \leq \exp \left( \frac{b_1}{6} W \frac{1}{1-\sigma} (\log W)^{\frac{m+\kappa(\sigma)}{1-\sigma}} \right) \). Hence, we obtain
\[
\sum_{k \leq Y} \frac{x^k}{k!} \int_A \left( \text{Re} \sum_{p \leq X} \frac{e^{-it}}{p^\sigma + it (\log p)^m} \right)^k dt
\]
\[
= \sum_{k \leq Y} \frac{x^k}{k!} \int_T \left( \text{Re} \sum_{p \leq X} \frac{e^{-it}}{p^\sigma + it (\log p)^m} \right)^k dt +
\]
\[
+ O \left( T \exp \left( -\frac{b_1}{3} W \frac{1}{1-\sigma} (\log W)^{\frac{m+\kappa(\sigma)}{1-\sigma}} \right) \right).
\]

From these estimates, the left hand side of (2.8) is equal to
\[
\sum_{k \leq Y} \frac{x^k}{k!} \int_T \left( \text{Re} \sum_{p \leq X} \frac{e^{-it}}{p^\sigma + it (\log p)^m} \right)^k dt + O \left( T \exp \left( -e^2 x W \right) \right) \quad (2.9)
\]
for any sufficiently large \( W \) when \( b_2 \) is suitably small. By Lemma 2, this main term is equal to
\[
\frac{T}{2\pi i} \oint_{|w| = \epsilon x} \sum_{k \leq Y} \frac{x^k}{w^{k+1}} \prod_{p \leq X} I_0 \left( \frac{w}{p^\sigma (\log p)^m} \right) dw.
\]
(2.10)

By Lemmas 3 and 4, there exists a constant \( C_4 = C_4(\sigma, m) > 0 \) such that
\[
\left| \prod_{p \leq X} I_0 \left( \frac{w}{p^\sigma (\log p)^m} \right) \right| \leq I_0 \left( \frac{R}{p^\sigma (\log p)^m} \right) \leq \exp \left( C_4 \frac{x \frac{1}{\sigma}}{(\log x)^{\frac{m+\kappa(\sigma)}{1-\sigma}}} \right).
\]
Choosing $b_2$ as a suitably small constant, the right hand side is $\ll \exp(xW)$. Moreover, since we see that
\[
\left| \sum_{k>Y} \frac{x^k}{w^{k+1}} \right| \ll \exp\left( -e^2 xW \right),
\]
it holds that
\[
\left| \sum_{k>Y} \frac{x^k}{w^{k+1}} \prod_{p \leq X} I_0 \left( \frac{w}{\sqrt{p(\log p)^m}} \right) \right| \leq \exp (-xW)
\]
for $|w| = ex$. Hence, (2.10) is equal to
\[
\frac{T}{2\pi i} \oint_{|w|=ex} \sum_{k>Y} \frac{1}{w-x} \prod_{p \leq X} I_0 \left( \frac{w}{p^\sigma (\log p)^m} \right) \, dw + O \left( T \exp (-xW) \right).
\]
Thus, by this formula and equation (2.9) and using Cauchy's integral formula, we obtain
\[
\frac{1}{T} \int_A \exp \left( x \operatorname{Re} \left( e^{-it} \sum_{p \leq X} \frac{1}{p^{\sigma+i\theta}(\log p)^m} \right) \right) \, dt
\]
\[
= \prod_{p \leq X} I_0 \left( \frac{x}{p^\sigma (\log p)^m} \right) + O \left( \exp (-xW) \right),
\]
which completes the proof of this lemma. \qed

**Lemma 8.** Let $m \in \mathbb{Z}_{\geq 1}$, $\frac{1}{2} \leq \sigma < 1$ be fixed. Let $T$ be large, $X \geq 3$, and $\Delta > 0$. Define the set $\mathcal{B} = \mathcal{B}(T, X, \Delta; \sigma)$ by
\[
\mathcal{B} = \left\{ t \in [T, 2T] \left| \left| \tilde{\eta}_m(\sigma+it) - \sum_{2 \leq n \leq X} \frac{\Lambda(n)}{n^{\sigma+it}(\log n)^{m+1}} \right| \leq \Delta X^{1/2-\sigma} \right. \right\}.
\]
Then, for $0 < \Delta \leq \left( \frac{\log T}{(\log X)^{2m+1}} \right)^{\frac{m}{2m+1}}$, we have
\[
\frac{1}{T} \operatorname{meas}([T, 2T] \setminus \mathcal{B}) \leq \exp \left( -b_3 \Delta^2 (\log X)^{2m} \right),
\]
and for $\left( \frac{\log T}{(\log X)^{2m+1}} \right)^{\frac{m}{2m+1}} \leq \Delta \leq \frac{\log T}{(\log X)^{m+1}}$, we have
\[
\frac{1}{T} \operatorname{meas}([T, 2T] \setminus \mathcal{B}) \leq \exp \left( -b_4 (\Delta(\log T)^m)^{1/(m+1)} \right).
\]
Here, $b_3$, $b_4$ are absolute positive constants.
Proof. By equation (1.1) and Theorem 5 of [10], we have
\[
\frac{1}{T} \int_T^{2T} \left| \tilde{\eta}_m(\sigma + it) - \sum_{2 \leq n \leq X} \frac{\Lambda(n)}{n^{\sigma + it} (\log n)^{m+1}} \right|^{2k} dt \\
\ll C^k k! \frac{X^{k(1-2\sigma)}}{(\log X)^{2km}} + C^k k^{2k(m+1)} \frac{T^{1+2\sigma}}{(\log T)^{2km}}
\]
for \(3 \leq X \leq T^{1/35}\), where \(C\) is an absolute positive constant. Therefore, we obtain
\[
\frac{1}{T} \text{meas}([T, 2T] \setminus B) \ll \frac{C k^{1/2}}{(\Delta (\log X)^m)}^{m/k} + \frac{C k^{m+1}}{\Delta (\log T)^m}^{m/k}
\]
When \(\Delta \leq \left( \frac{\log T}{(\log X)^{2(m+1)}} \right)^{m/k} \), putting \(k = \left[ c \Delta^2 (\log X)^{2m} \right] + 1\) with \(c\) a suitably small constant, we have
\[
\frac{1}{T} \text{meas}([T, 2T] \setminus B) \leq \exp (-b_3 \Delta^2 (\log X)^{2m})
\]
for some absolute constant \(b_3 > 0\). When the inequality \(\left( \frac{\log T}{(\log X)^{2(m+1)}} \right)^{m/k} \leq \Delta \leq \frac{\log T}{(\log X)^{m+1}} \) holds, by choosing \(k = \left[ c (\Delta (\log T)^m)^{1/(m+1)} \right] + 1\) with \(c\) a suitably small constant, we have
\[
\frac{1}{T} \text{meas}([T, 2T] \setminus B) \leq \exp (-b_4 (\Delta (\log T)^m)^{1/(m+1)})
\]
for some absolute constant \(b_4 > 0\). Thus, we obtain this lemma. \(\square\)

3. Proofs of Proposition 2 and Theorem 1

In this section, we prove Proposition 2 and Theorem 1.

Proof of Proposition 2. Let \(m \in \mathbb{Z}_{\geq 1}, \theta \in \mathbb{R}\) be fixed. Let \(T, V\) be large numbers with \(V \leq a_2 \frac{\sqrt{\log T}}{\log \log T} m^{1/2}\), and let \(X\) be a real parameter with \(V^4 \leq X \leq T^{a_3/V^2 (\log V)^{2m}}\). Here, \(a_2 = a_2(m), a_3 = a_3(m)\) are positive constants to be chosen later. Moreover, let \(W > 0, 3 \leq x \leq b_2 W (\log W)^{2m}\) be numbers to be chosen later, where \(b_2 = b_2(1/2, m)\) is the same constant as in Lemma 7. Put
\[
\mathscr{A}^* (T, V) := \left\{ t \in \mathcal{A} \left| \text{Re} \left( e^{-iy} \sum_{p \leq X} \frac{1}{p^{1/2 + it} (\log p)^m} \right) > V \right\} \right. \}
\]
Here, the set \(\mathcal{A} = \mathcal{A}(T, X, W; 1/2, m)\) is defined by (2.6). Then, for \(x > 0\), we have
\[
\int_{\mathcal{A}} \exp \left( x \text{Re} \left( e^{-iy} \sum_{p \leq X} \frac{1}{p^{1/2 + it} (\log p)^m} \right) \right) dt = x \int_{-\infty}^{\infty} e^{xv} \text{meas}(\mathscr{A}^* (T, v)) dv.
\]
By this equation and Lemma 7, it holds that
\[
\frac{1}{T} \int_{-\infty}^{\infty} e^{xv} \text{meas}(\mathcal{S}^*(T, v))\,dv = \frac{1}{x} \prod_{p \leq X} I_0 \left( \frac{x}{p^\sigma \log p} \right) + O \left( \frac{1}{x} \exp (-xW) \right)
\]
when \( x^3 \leq X \leq T^{1/W^2(\log W)^2m} \). Therefore, by Lemma 3, we obtain
\[
\frac{1}{T} \int_{-\infty}^{\infty} e^{xv} \text{meas}(\mathcal{S}^*(T, v))\,dv = \exp \left( \frac{x^2}{8m(2 \log x)^2m} \left( 1 - \left( \frac{\log x^2}{\log X} \right)^{2m} + O_m \left( \frac{\log \log x}{\log x} \right) \right) \right)
\]
for \( x^3 \leq X \leq T^{1/W^2(\log W)^2m} \). Now, we decide the parameters \( x, W \) as satisfying the equations
\[
V = \frac{2x}{8m(2 \log x)^2m} \left( 1 - \left( \frac{\log x^2}{\log X} \right)^{2m} \right),
\]
and \( W = 8m4^m K_1 V \), respectively. The constant \( K_1 = K_1(m) \) is defined as \( K_1 = \max\{b_1^{-1}, b_2^{-1}\} \), and \( b_1 \) is the same constant as in Lemma 6. Then, this \( x \) satisfies
\[
x = \frac{4m4^m}{1 - (\log V^2/\log X)^{2m}} V (\log V)^{2m} \left( 1 + O_m (\log \log V / \log V) \right),
\]
and hence we can take out \( x \) from the range \( 3 \leq x \leq b_2 W (\log W)^{2m} \) for any large \( V \). Also, when \( a_2, a_3 \) are suitably small, the inequalities \( x^3 \leq T^{1/W^2(\log W)^2m} \) and \( x^3 \leq X \leq T^{1/W^2(\log W)^2m} \) hold for any large \( V \). Moreover, by using Lemma 6, the inequality \( \text{meas}([T, 2T] \setminus A) \leq T \exp (-8m4^m V^2 (\log V)^{2m}) \) holds. Therefore, we obtain
\[
\frac{1}{T} \text{meas} \left\{ t \in [T, 2T] \left| \Re \left( \sum_{p \leq X} \frac{1}{p^{1/2+it} \log p} \right) > V \right. \right\} = \frac{1}{T} \text{meas}(\mathcal{S}^*(T, V)) + O \left( \frac{1}{T} \text{meas}([T, 2T] \setminus A) \right) = \frac{1}{T} \text{meas}(\mathcal{S}^*(T, V)) + O \left( \exp (-8m4^m V^2 (\log V)^{2m}) \right).
\]
Put \( \epsilon = K_2 \sqrt{\log \log x / \log x} \) with \( K_2 = K_2(m) \) a sufficiently large constant. Then, by using equation (3.1), we find that

\[
\int_{-\infty}^{V(1-\epsilon)} e^{xv} \text{meas}(\mathcal{F}^*(T, v)) dv \leq e^{xV(1-\epsilon)} \int_{-\infty}^{\infty} e^{x(1-\epsilon)u} \text{meas}(\mathcal{F}^*(T, v)) dv
\]

\[
= T \exp \left( \frac{x^2}{8m(2 \log x)^2} \left( 1 - \left( \frac{\log x^2 \log X}{\log X} \right)^{2m} - \frac{\epsilon^2}{3} + O_m \left( \frac{\log \log x}{\log x} \right) \right) \right)
\]

\[
\leq \frac{1}{3} \int_{-\infty}^{\infty} e^{xv} \text{meas}(\mathcal{F}^*(T, v)) dv.
\]

Similarly, we find that

\[
\int_{V(1+\epsilon)}^{\infty} e^{xv} \text{meas}(\mathcal{F}^*(T, v)) dv \leq e^{-xV(1+\epsilon)} \int_{-\infty}^{\infty} e^{x(1+\epsilon)u} \text{meas}(\mathcal{F}^*(T, v)) dv
\]

\[
= T \exp \left( \frac{x^2}{8m(2 \log x)^2} \left( 1 - \left( \frac{\log x^2 \log X}{\log X} \right)^{2m} + \frac{\epsilon^2}{3} + O_m \left( \frac{\log \log x}{\log x} \right) \right) \right)
\]

\[
\leq \frac{1}{3} \int_{-\infty}^{\infty} e^{xv} \text{meas}(\mathcal{F}^*(T, v)) dv.
\]

Hence, we have

\[
\frac{1}{T} \int_{V(1-\epsilon)}^{V(1+\epsilon)} e^{xv} \text{meas}(\mathcal{F}^*(T, v)) dv
\]

\[
= \exp \left( \frac{x^2}{8m(2 \log x)^2} \left( 1 - \left( \frac{\log x^2 \log X}{\log X} \right)^{2m} + O_m \left( \frac{\log \log x}{\log x} \right) \right) \right).
\]

Moreover, since \( \text{meas}(\mathcal{F}^*(T, v)) \) is a nonincreasing function and \( \int_{V(1-\epsilon)}^{V(1+\epsilon)} e^{xv} dv = \exp(xV(1 + O(\epsilon))) \), it holds that

\[
\frac{1}{T} \text{meas}(\mathcal{F}^*(T, V(1 + \epsilon)))
\]

\[
\leq \exp \left( -\frac{x^2}{8m(2 \log x)^2} \left( 1 - \left( \frac{\log x^2 \log X}{\log X} \right)^{2m} + O_m \left( \frac{\log \log x}{\log x} \right) \right) \right)
\]

\[
\leq \frac{1}{T} \text{meas}(\mathcal{F}^*(T, V(1 - \epsilon))).
\]

In particular, since \( x \) satisfies

\[
x = 4mV(2 \log V)^{2m} \left\{ (1 + (\log x^2 / \log X)^{2m})^{-1} + O_m(\log \log V / \log V) \right\},
\]

the second term of the above inequalities is equal to

\[
\exp \left( -\frac{2m4^m}{1 - \left( \frac{\log V^2}{\log X} \right)^m} V^2(\log V)^{2m} \left( 1 + O_m \left( \frac{\log \log V}{\log V} \right) \right) \right).
\]
Additionally, if we change the above $V$ to $V(1 + O(\varepsilon))$, the above form does not change. Hence, we obtain

\[
\frac{1}{T} \text{meas}(\mathcal{S}^*(T, V)) = \exp \left( -\frac{2m^4 V^2 (\log V)^{2m}}{1 - \left( \frac{\log V}{\log \log V} \right)^2} \left( 1 + O_m \left( \sqrt{\frac{\log \log V}{\log V}} \right) \right) \right).
\]

By this equation and (3.2), we complete the proof of Proposition 2. \(\square\)

**Proof of Theorem 1.** Let $T, V$ be sufficiently large parameters satisfying $V \leq a_1 \left( \frac{\log T}{(\log \log T)^{2m+2}} \right)^{\frac{m}{2m+1}}$, where $a_1 = a_1(m)$ is a suitably small constant to be chosen later. Let $a_3, b_4$ be the same constants as in Proposition 2 and Lemma 8. Put $X = T^{b_6 V^2 (\log V)^{2m}}$ with $b_6 = \min\{a_3, b_4(4m^4)^{-1}\}$. Note that this $X$ satisfies the inequality $X \geq \exp \left( \left( \frac{\log T}{2^m + 1} \right)^{\frac{m}{2m+1}} \right) \geq V^4$ when $T$ is large. Then, applying Lemma 8 as $\Delta = \frac{\log T}{(\log X)^{m+1}} = V^{2m+2 (\log V)^{2m(m+1)}} b_5^{m+1 (\log T)^{m+1}}$, we find that there exists a set $B \subset [T, 2T]$ such that $\text{meas}([T, 2T] \setminus B) \leq T \exp \left( -4m^4 V^2 (\log V)^{2m} \right)$, and for all $t \in B$

\[
\left| \tilde{\eta}_m(1/2 + it) - \sum_{p \leq X} \frac{1}{p^{1/2 + it} \log p^m} \right| \leq \left( \frac{V^{2m+1} (\log V)^{2m(m+1)}}{b_6^{m+1} (\log T)^m} + \frac{c}{V} \right) V =: \delta_m V,
\]

say. Here the constant $c$ indicates the value $\sum_{p^k, k \geq 2} \frac{1}{p^{k/(\log p)^m}}$. Now, we decide the number $a_1$ such that $\delta_m \leq 1/2$. Then, it holds that

\[
\text{meas} \left\{ t \in B \mid \text{Re} e^{-it} \sum_{p \leq X} \frac{1}{p^{1/2 + it} \log p^m} > V(1 + \delta_m) \right\} 
\leq \text{meas} \left\{ t \in B \mid \text{Re} e^{-it} \tilde{\eta}_m(1/2 + it) > V \right\} 
\leq \text{meas} \left\{ t \in B \mid \text{Re} e^{-it} \frac{1}{p^{1/2 + it} \log p^m} > V(1 - \delta_m) \right\}.
\]

Hence, by these inequalities and Proposition 2, we have

\[
\frac{1}{T} \text{meas} \left\{ t \in B \mid \text{Re} e^{-it} \tilde{\eta}_m(1/2 + it) > V \right\} = 
\exp \left( -2m^4 V^2 (\log V)^{2m} \left( 1 + O_m \left( \frac{V^{2m+1} (\log V)^{2m(m+1)}}{(\log T)^m} + \sqrt{\frac{\log \log V}{\log V}} \right) \right) \right).
\]

Thus, by this equation and $\text{meas}([T, 2T] \setminus B) \leq T \exp \left( -4m^4 V^2 (\log V)^{2m} \right)$, we complete the proof of Theorem 1. \(\square\)
4. Proofs of Proposition 3 and Theorem 2

Some parts in the proof of Proposition 3 are written briefly because many points are similar to the proof of Proposition 1.

**Proof of Proposition 3.** Let \( m \in \mathbb{Z}_{\geq 0}, \frac{1}{2} < \sigma < 1 \) be fixed. Let \( T, V \) be large numbers with \( V \leq a_5 \frac{(\log T)^{1-\sigma}}{(\log \log T)^{m+1}}, \) and let \( X \) be a real parameter with \( V^{\frac{1}{1-\sigma}} \leq X \leq T^{a_5/V^{\frac{1}{1-\sigma}}}(\log V)^{\frac{m+\sigma}{1-\sigma}}. \) Here \( a_5 = a_5(\sigma, m), a_6 = a_6(\sigma, m) \) are positive constants to be chosen later. Moreover, let \( W > 0, 3 \leq x \leq b_2 W^{\frac{\sigma}{1-\sigma}}(\log W)^{\frac{m+\sigma}{1-\sigma}} \) be numbers to be chosen later. Here, \( b_2 = b_2(\sigma, m) \) is the same constant as in Lemma 7. Put

\[
\mathcal{J}_\sigma^*(T, V) := \left\{ t \in A \mid \text{Re} \left( e^{-\iota t} \sum_{p \leq X} \frac{1}{p^{\sigma + \iota \log p} m} \right) > V \right\},
\]

where \( A = A(T, X, V; \sigma, m) \) is the set defined by (2.6). Using Lemmas 4, 7, and the equation

\[
\int_A \exp \left( x \text{Re} \left( e^{-\iota t} \sum_{p \leq X} \frac{1}{p^{\sigma + \iota \log p} m} \right) \right) = x \int_{-\infty}^{\infty} e^{xv} \text{meas}(\mathcal{J}_\sigma^*(T, v))dv,
\]

we obtain

\[
\frac{1}{T} \int_{-\infty}^{\infty} e^{xv} \text{meas}(\mathcal{J}_\sigma^*(T, v))dv = \exp \left( \frac{\sigma \pi G(\sigma)x^{\frac{1}{\sigma}}}{(\log x)^{\frac{m+\sigma}{m+1}}} \left( 1 + O \left( \frac{1 + m \log \log x}{\log x} \right) \right) \right)
\]

for \( 3 \leq X \leq T^{1/W^{\frac{1}{1-\sigma}}}(\log W)^{\frac{m+\sigma}{1-\sigma}}. \) Here, we decide the parameters \( x, W \) as the numbers satisfying the equations

\[
V = \frac{\sigma \pi G(\sigma)x^{\frac{1}{\sigma}}}{(\log x)^{\frac{m+\sigma}{m+1}}},
\]

and \( W = \left( 2^\frac{4A_m(\sigma)}{1-\sigma} K_3 \right)^{\frac{1-\sigma}{1-\sigma}} V, \) respectively. The constant \( K_3 = K_3(\sigma, m) \) is defined as \( K_3 = \max\{b_1, b_2\}, \) where \( b_1 \) is the same constant as in Lemma 6. Then, this \( x \) satisfies \( x = \frac{A_m(\sigma)}{1-\sigma} V^{\frac{\sigma}{1-\sigma}} (\log V)^{\frac{m+\sigma}{1-\sigma}} (1 + O(\log \log V/\log V)), \) and so we can pick up this \( x \) from the range \( 3 \leq x \leq b_2 W^{\frac{\sigma}{1-\sigma}}(\log W)^{\frac{m+\sigma}{1-\sigma}} \) for any large \( V. \) Also, choosing \( a_5, a_6 \) as suitably small constants, we find that the inequalities \( x^3 \leq T^{1/W^{\frac{1}{1-\sigma}}}(\log W)^{\frac{m+\sigma}{1-\sigma}} \) and \( x^3 \leq X \leq T^{1/W^{\frac{1}{1-\sigma}}}(\log W)^{\frac{m+\sigma}{1-\sigma}} \) hold for any large \( V. \) Moreover, by Lemma 6, the inequality \( \text{meas}([T, 2T] \setminus A) \leq T \exp \left( -2A_m(\sigma) V^{\frac{1}{\sigma}} (\log V)^{\frac{m+\sigma}{1-\sigma}} \right) \) holds.
Putting $\varepsilon = K_4 \sqrt{1 + m \log \log x}$ with $K_4 = K_4(\sigma, m)$ a suitably large constant and using equation (4.1), we have
\[\int_{-\infty}^{V(1-\varepsilon)} e^{xv} \text{meas}(\mathcal{S}^*(T, v; X)) dv \leq \frac{1}{3} \int_{-\infty}^{\infty} e^{xv} \text{meas}(\mathcal{S}^*(T, v; X)) dv,\]
and
\[\int_{V(1+\varepsilon)}^{\infty} e^{xv} \text{meas}(\mathcal{S}^*(T, v; X)) dv \leq \frac{1}{3} \int_{-\infty}^{\infty} e^{xv} \text{meas}(\mathcal{S}^*(T, v; X)) dv.\]
Therefore, we obtain
\[\frac{1}{T} \int_{V(1-\varepsilon)}^{(1+\varepsilon)V} e^{xv} \text{meas}(\mathcal{S}^*(T, v)) dv = \exp \left( \frac{\sigma}{\log x} \left( 1 + O \left( \frac{1 + m \log \log x}{\log x} \right) \right) \right).\]
Moreover, since $\text{meas}(\mathcal{S}^*(T, v))$ is a nonincreasing function and $\int_{V(1-\varepsilon)}^{V(1+\varepsilon)} e^{xv} dv = \exp(xV(1 + O(\varepsilon)))$, it holds that
\[\frac{1}{T} \text{meas}(\mathcal{S}^*(T, V(1 + \varepsilon); X)) \leq \exp \left( -\frac{1 - \sigma}{\sigma} \frac{\sigma^m G(\sigma) x^{1/\sigma}}{(\log x)^{m+1}} (1 + O(\varepsilon)) \right) \leq \frac{1}{T} \text{meas}(\mathcal{S}^*(T, V(1 - \varepsilon); X)).\]
In particular, as $x$ is the solution of equation (4.1), the above second term is equal to
\[\exp \left( -A_m(\sigma) V^{1-\sigma} (\log V)^{\frac{m+\sigma}{1-\sigma}} (1 + R) \right),\]
where
\[R \ll \sqrt{\frac{1 + m \log \log x}{\log x}} \ll \sqrt{\frac{1 + m \log \log V}{\log V}}.\]
Additionally, if we change the above $V$ to $V(1 + O(\varepsilon))$, the above form does not change. Thus, we obtain
\[\frac{1}{T} \text{meas}(\mathcal{S}^*(T, V; X)) = \exp \left( -A_m(\sigma) V^{1/\sigma} (\log V)^{\frac{m+\sigma}{1-\sigma}} \left( 1 + O \left( \sqrt{\frac{1 + m \log \log V}{\log V}} \right) \right) \right).\]
By this equation and $\text{meas}([T, 2T] \setminus A) \leq T \exp \left( -2A_m(\sigma) V^{1-\sigma} (\log V)^{\frac{m+\sigma}{1-\sigma}} \right)$, we obtain Proposition 3. $\square$
Proof of Theorem 2. We show only the case \( m \geq 1 \) because the case \( m = 0 \) can be shown similarly by use of Lemma 2.2 in [7] instead of Lemma 8.

Let \( m \in \mathbb{Z}_{\geq 1} \), \( 1/2 < \sigma < 1 \). Let \( a_5, a_6, \) and \( b_4 \) be the same constants as in Proposition 3 and Lemma 8. Let \( T, V \) be sufficiently large positive numbers satisfying the inequality \( V \leq a_4 \frac{(\log T)^{1-\sigma}}{(\log \log T)^{m+1}} \), where \( a_4 = a_4(\sigma, m) \) is a suitably small constant less than \( a_5 \) to be chosen later. Put \( X = T^{b_4/V} \frac{1}{4} (\log V)^{m+\sigma} \) with \( b_6 = \min\{a_6, b_4(2A_m(\sigma))^{-1}\} \). Then we decide the number \( a_4 \) satisfying \( X^{\sigma-1/2} \geq (\log T)^{6/5} \). Applying Lemma 8 as \( \Delta = \frac{\log T}{(\log X)^{m+1}} = \frac{(\log V)^{\frac{m+\sigma}{1-\sigma}}}{b_6^{m+1}(\log T)^{m}} \), we find that there exists a set \( B \subset [T, 2T] \) such that \( \text{meas}([T, 2T] \setminus B) \leq T \exp \left( -2A_m(\sigma)V^\frac{1}{\tau-\sigma}(\log V)^{\frac{m+\sigma}{\tau-\sigma}} \right) \), and for all \( t \in B \)

\[
\left| \tilde{\eta}_m(\sigma + it) - \sum_{p \leq X} \frac{1}{p^{\sigma+it}(\log p)^m} \right| \leq \left( \frac{V^{\frac{1}{\tau-\sigma}}(\log V)^{\frac{m+\sigma}{\tau-\sigma}}} {X^{\sigma-1/2}b_6^{m+1}(\log T)^m} \right)^m + c.
\]

Here, \( c = \sum_{p^k, k \geq 2} \frac{A(p^k)}{p^{\sigma}(\log p)^{m+\tau}} \). Therefore, the right hand side is \( \leq K_4 \) with \( K_4 = K_4(m, \sigma) \) a positive constant. Then, it holds that

\[
\text{meas} \left\{ t \in B \mid \text{Re} e^{-it} \sum_{p \leq X} \frac{1}{p^{\sigma+it}(\log p)^m} > V(1 + K_4V^{-1}) \right\} \leq \text{meas} \left\{ t \in B \mid \text{Re} e^{-it} \tilde{\eta}_m(\sigma + it) > V \right\} \leq \text{meas} \left\{ t \in B \mid \text{Re} e^{-it} \sum_{p \leq X} \frac{1}{p^{\sigma+it}(\log p)^m} > V(1 - K_4V^{-1}) \right\}.
\]

Hence, by these inequalities and Proposition 2, we have

\[
\frac{1}{T} \text{meas} \left\{ t \in B \mid \text{Re} e^{-it} \tilde{\eta}_m(\sigma + it) > V \right\} = \exp \left( -A_m(\sigma)V^\frac{1}{\tau-\sigma}(\log V)^{\frac{m+\sigma}{\tau-\sigma}} \left( 1 + O \left( \frac{1 + m \log \log V}{\log V} \right) \right) \right).
\]

By this equation and \( \text{meas}([T, 2T] \setminus B) \leq T \exp \left( -2A_m(\sigma)V^\frac{1}{\tau-\sigma}(\log V)^{\frac{m+\sigma}{\tau-\sigma}} \right) \), we complete the proof of Theorem 2. \( \square \)

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