POLYUNSATURATED POSETS AND GRAPHS
AND THE GREENE-KLEITMAN THEOREM

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ABSTRACT. A partition of a finite poset into chains places a natural upper bound on the size of a union of $k$ antichains. A chain partition is $k$-saturated if this bound is achieved. Greene and Kleitman [9] proved that, for each $k$, every finite poset has a simultaneously $k$- and $k + 1$-saturated chain partition. West [15] showed that the Greene-Kleitman Theorem is best-possible in a strong sense by exhibiting, for each $c \geq 4$, a poset with longest chain of cardinality $c$ and no $k$- and $l$-saturated chain partition for any distinct, nonconsecutive $k, l < c$. We call such posets polyunsaturated. We give necessary and sufficient conditions for the existence of polyunsaturated posets with prescribed height, width, and cardinality. We prove these results in the more general context of graphs satisfying an analogue of the Greene-Kleitman Theorem. Lastly, we discuss analogous results for antichain partitions.

1. Introduction

Let $P$ be a finite partially ordered set. A theorem of Dilworth [4] states that the cardinality of the largest antichain in $P$ is equal to the least number of chains into which $P$ can be partitioned.

Greene and Kleitman [9] generalized this to unions of antichains. A $k$-family in $P$ is a union of $k$ antichains; we denote the size of the largest $k$-family by $d_k(P)$. Given a partition $\mathcal{C}$ of a set $S$, we define the $k$-norm of $\mathcal{C}$, denoted $m_k(\mathcal{C})$, as follows:

$$m_k(\mathcal{C}) := \sum_{C \in \mathcal{C}} \min\{k, |C|\}.$$  

For each chain partition $\mathcal{C}$ of $P$, $m_k(\mathcal{C})$ is an upper bound for $d_k(P)$; a chain partition $\mathcal{C}$ is $k$-saturated if this bound is achieved. Dilworth’s Theorem says that every finite poset $P$ has a 1-saturated chain partition. Greene and Kleitman [9, Thm. 3.11] proved that for each $k$, $P$ has a simultaneously $k$- and $k + 1$-saturated chain partition. Other proofs of this result can be found in [2, 6, 7, 10, 11, 12].

It is natural to ask whether a stronger result is possible: can we always find a $k$-, $k + 1$-, and $k + 2$-saturated chain partition? Greene and Kleitman showed that the
answer to this question is “no” by exhibiting a poset (the first poset in Figure 1) with no 1- and 3-saturated partition.

The height $h(P)$ of a poset $P$ is the maximum size of a chain in $P$. West [13] generalized the above example by exhibiting a poset of each height $c \geq 4$ in which no chain partition is $k$- and $l$-saturated for any distinct, nonconsecutive $k, l < c$ (Figure 1). We say such a poset is polyunsaturated. Since every chain partition of $P$ is $k$-saturated for each $k \geq h(P)$, West’s examples show that the Greene-Kleitman Theorem is best-possible in a strong sense.

West’s examples have exponential width as a function of their heights. In Section 2 we construct much narrower polyunsaturated posets (Figure 2). We show later that these posets have the smallest possible width and cardinality for polyunsaturated posets with the same height.

We denote the comparability graph of a poset $P$ by $G(P)$. The chains and antichains of $P$ are precisely the cliques and independent sets of $G(P)$. In Section 3,
we use these ideas to extend the concept of polyunsaturation to graphs that satisfy an analogue of the Greene-Kleitman Theorem, even though they may not be comparability graphs. Thus, while our results are primarily of interest in the context of posets, we prove them in a more general graph-theoretic form.

Our main result (Theorem 3.3) characterizes, in graph-theoretic form, the possible values of \( d_k(P) \) for a polyunsaturated poset \( P \). In Section 4, we obtain corollaries of Theorem 3.3 that give necessary and sufficient conditions for the existence of polyunsaturated posets and graphs with certain parameters. In Section 5, we discuss results analogous to those above for partitions of a poset into antichains.

2. The Construction

We now construct the narrowest polyunsaturated posets, relative to height. When \( x < y \) is a cover relation we write \( x \prec y \). We denote a finite sequence of positive integers by an underlined letter, and we use subscripted letters to name the elements of the sequence. Given a sequence \( \underline{b} = (b_1, b_2, \ldots, b_t) \), the difference sequence of \( \underline{b} \) is the sequence \( \Delta \underline{b} \), where \( \Delta b_i = b_i - b_{i-1} \), with the convention that \( \Delta b_1 = b_1 \).

**Definition 2.1 (posets \( P_j \)).** We define a sequence of posets inductively. Let \( P_1 \) be a chain \( Q_1 \) of three elements, \( u < s_1 < r_1 \), and let \( T_1 = \{u\} \) (see Figure 3). For \( j > 1 \), suppose \( P_{j-1} \) is defined and contains the element \( s_{j-1} \). We define \( P_j \) to be the disjoint union of \( P_{j-1} \) with a chain \( Q_j \) of \( j + 1 \) elements, plus one new cover relation. Let \( r_j \) be the maximal element of \( Q_j \), and let \( s_j \) be the next-greatest element. The set of all remaining elements in \( Q_j \) will be called \( T_j \). We add the cover relation \( s_{j-1} \prec s_j \) and extend by transitivity.

![Figure 3. The first four \( P_j \)'s.](image-url)
Next we list some properties of the $P_j$'s. A ranked poset $P$ has the strong Sperner property if, for each positive integer $k$, the union of the $k$ largest ranks of $P$ is a maximum-sized $k$-family. A poset has order dimension at most 2 if and only if the complement of its comparability graph is also a comparability graph [5, Thm. 3.61] (see also [14, p. 62]).

**Lemma 2.2.** For the poset $P_j$ of Definition 2.1 the following all hold.

1. $P_j$ is a ranked poset with width $j$, height $j+2$, and cardinality $\binom{j+2}{2}$.
2. $P_j$ has the strong Sperner property.
3. $\Delta d_1(P_j) = j$, $\Delta d_{j+2}(P_j) = 1$, and, for $k = 2, \ldots, j+1$, $\Delta d_k(P_j) = j+2-k$, and
4. $P_j$ has dimension at most 2.

**Proof.** (1) This is immediate from the construction, with $u < s_1 < \cdots < s_j < r_j$ a maximum chain and $\{r_i\}$ a maximum antichain.

(2) Let $k$ be a positive integer. Let $S$ be the union of the $k$ largest ranks of $P_j$. It suffices to exhibit a chain partition $C_k$ of $P_j$ such that $|S| = m_k(C_k)$. Let $C_k$ consist of the chain $C = \{u, s_1, \ldots, s_k, r_k\}$ and the chains $Q_i \setminus C$ for $1 \leq i \leq k$ (see Figure 4 for examples). We claim that $C_k$ is the required partition. Each chain $C \in C_k$ either is contained in $S$ or contains one element of each rank of $S$. In the former case, $C$ contributes $|C|$ to $m_k(C_k)$; in the latter case, $C$ contributes $k$. In both cases, $C$ contributes $\min\{k, |C|\}$, and so $m_k(C_k) = |S|$. (In fact, $C_k$ is both $k$- and $k+1$-saturated.)

(3) This follows from Statement (2) and the sequence of rank sizes of $P_j$.

(4) We define two linear extensions of $P_j$:

- $T_1, s_1, r_1, T_2, s_2, r_2, \ldots, T_j, s_j, r_j$,
- $T_1, \ldots, T_2, T_1, s_1, s_2, \ldots, s_j, r_j, \ldots, r_2, r_1$,

where the elements of $T_i$ are in ascending order. The intersection of these two total orders is the partial order on $P_j$, i.e., each related pair appears in the proper order in both extensions, and each incomparable pair appears in opposite orders in the two extensions. Thus, $P_j$ has dimension at most 2. □

**Lemma 2.3.** For each positive integer $j$, $P_j$ is a polyunsaturated poset.

**Proof.** We claim that, if $2 \leq k \leq j$ and $C$ is a $k$-saturated chain partition of $P_j$, then $u$ lies on the same chain in $C$ as either $r_{k-1}$ or $r_k$. Also, in every 1-saturated chain partition, $u$ lies on the same chain as $r_1$, and in every $j+1$-saturated chain partition, $u$ lies on the same chain as $r_j$. Since in $C$ the element $u$ can be on the same chain with only one $r_i$, this prevents $C$ from being $k$-saturated for nonconsecutive values of $k$.

Let $2 \leq k \leq j$, and let $C$ be a $k$-saturated chain partition of $P_j$. The ranks containing $s_1, \ldots, s_j$, in order, are the $j$ largest ranks. Thus there exists a union of $k$ largest ranks of $P_j$ that omits both $u$ and $r_k$. By Statement (2) of Lemma 2.2, a union of $k$ largest ranks of $P_j$ is a maximum $k$-family. Thus, the chain in $C$ that contains $u$ must contain at least one element from each of those ranks, and the chain containing $r_k$ must contain at least one element from each of those ranks. Since $r_{k-1}$
and $s_k$ are the only elements of the $k$th largest rank that are comparable to $u$, one of the two must be on its chain. Similarly, $s_k$ must be on the same chain as $r_k$ in $C$. Since either $r_{k-1}$ or $s_k$ must be on the chain containing $u$, we see that either $r_{k-1}$ or $r_k$ must be on this chain.

The $k = 1$ and $k = j + 1$ cases of the claim are proven similarly. Thus, we have established the claim. □

We will see (Remark 4.4) that the posets $P_j$ have the minimum cardinality among all polyunsaturated posets with the same height. The first two, $P_1$ and $P_2$, are unique; thus, our $P_2$ is isomorphic to the first of West’s posets and to the Greene-Kleitman example. However, the rest of the $P_j$’s are not unique. Figure 5 shows $P_4$ and two other polyunsaturated posets having the same parameters.

**Figure 4.** $P_4$ shown with chain partitions $C_1$, $C_2$, $C_3$, and $C_4$, as defined in the proof of Lemma 2.2.
3. POLYUNSATURATED GRAPHS AND THE MAIN RESULT

We denote the vertex set of a (finite, simple, undirected) graph $G$ by $V(G)$. The independence number of $G$, denoted $\alpha(G)$, is the maximum size of an independent set in $G$. The clique number of $G$, denoted $\omega(G)$, is the maximum size of a clique of $G$. We refer to [1, 16] for graph-theoretic terminology not defined here.

The antichains in a poset $P$ are precisely the independent sets in its comparability graph $G(P)$; the chains in $P$ are the cliques in $G(P)$. This allows us to extend poset properties to graphs.

Given a graph $G$ and a positive integer $k$, we denote the maximum size of a union of $k$ independent sets in $G$ by $\alpha_k(G)$. A partition $\mathcal{C}$ of $V(G)$ into cliques is $k$-saturated if $\alpha_k(G) = m_k(\mathcal{C})$. We say that $G$ is a strong Greene-Kleitman graph (SGK graph) if every induced subgraph of $G$ has a $k$- and $k+1$-saturated clique partition, for each positive integer $k$. An SGK graph $G$ with clique number $c$ is polyunsaturated if $G$ has no $k$- and $l$-saturated clique partition for any distinct, nonconsecutive $k, l < c$ (note that every clique partition is $k$-saturated for each $k \geq c$).

A chain partition $\mathcal{C}$ of a poset $P$ is $k$-saturated if and only if $\mathcal{C}$ is a $k$-saturated clique partition of $G(P)$. Thus, $G(P)$ is an SGK graph, by the Greene-Kleitman Theorem. A poset $P$ is polyunsaturated if and only if $G(P)$ is polyunsaturated.

In this and the following sections, we will be proving necessary and sufficient conditions for the existence of polyunsaturated posets. We will state our results in the more general graph-theoretic form. However, for every graph result in the remainder of this paper, there is an analogous poset result, which can be obtained by replacing graph-theoretic terminology with the corresponding poset terminology; e.g., replace clique number with height, $\alpha$ with $d$, etc.

The following lemma is due to Greene [8, remark after Thm. 3.1] (see also [13, Thm. 4.14]).

**Lemma 3.1** (Greene 1976). For every SGK graph $G$, $\Delta_{\alpha}(G)$ is a nonincreasing sequence of positive integers. □

This follows from comparing the expressions $\alpha_k(G) = m_k(\mathcal{C})$, $\alpha_{k+1}(G) = m_{k+1}(\mathcal{C})$, and $\alpha_{k+2}(G) \leq m_{k+2}(\mathcal{C})$ for a $k$- and $k+1$-saturated partition $\mathcal{C}$ and noting that $\Delta m(\mathcal{C})$ is nonincreasing.

Conversely, for each nonincreasing finite sequence $b$ of positive integers, there is an SGK graph $G$ with $\Delta_{\alpha}(G) = b$; for example, we can let $G$ be a disjoint union
of cliques of the proper sizes. Thus we have characterized those sequences that are the $\Delta \alpha$ sequence of an SGK graph. Our main result (Thm. 3.3) characterizes sequences of an SGK graph. Our main result (Thm. 3.3) characterizes those nonincreasing sequences that are the $\Delta \alpha$ sequence of a polyunsaturated SGK graph.

The following lemma will be used in this characterization. A poset version of this lemma was proven by Greene and Kleitman [1, Lemma 3.7].

**Lemma 3.2.** Let $G$ be an SGK graph, and let $k$ be a positive integer. If $\Delta \alpha_k(G) = \Delta \alpha_k+1(G)$, and $C$ is a $k$-saturated clique partition, then $C$ is $k+1$-saturated.

**Proof.** For all $i$, $\alpha_i(G) \leq m_i(C)$. Thus, since $C$ is $k$-saturated, we have

1. $\alpha_{k-1}(G) \leq m_{k-1}(C)$,
2. $\alpha_k(G) = m_k(C)$, and
3. $\alpha_{k+1}(G) \leq m_{k+1}(C)$.

By the definition of $m(C)$, $\Delta m_i(C)$ is the number of cliques in $C$ with at least $i$ vertices. Thus, $\Delta m$ is nonincreasing, and we have

$$\Delta \alpha_{k+1}(G) \leq \Delta m_{k+1}(C) \leq \Delta m_k(C) \leq \Delta \alpha_k(G) = \Delta \alpha_{k+1}(G).$$

Thus, $\Delta \alpha_{k+1}(G) = \Delta m_{k+1}(C)$. Adding corresponding sides to [2], we see that $\alpha_{k+1}(G) = m_{k+1}(C)$, and so $C$ is $k+1$-saturated. □

Our main result is the following.

**Theorem 3.3.** Let $c$ be a positive integer and let $\mathbf{b} = (b_1, b_2, \ldots, b_c)$ be a nonincreasing sequence of positive integers. There exists a polyunsaturated SGK graph $G$ with clique number $c$ and $\Delta \alpha(G) = \mathbf{b}$ if and only if $b_2 > b_3 > \cdots > b_{c-1}$. Moreover, we may require both $G$ and $\overline{G}$ to be comparability graphs.

**Proof.** ($\Rightarrow$) Let $G$ be a polyunsaturated SGK graph with clique number $c$ and $\Delta \alpha(G) = \mathbf{b}$. We show that $b_2 > b_3 > \cdots > b_{c-1}$.

By Lemma 3.1 the sequence is nonincreasing. If $\Delta \alpha_k(G) = \Delta \alpha_{k+1}(G)$ with $2 \leq k \leq c-2$, then Lemma 3.2 implies that every $k-1$- and $k$-saturated clique partition is also $k+1$-saturated. The hypothesis that $G$ is polyunsaturated forbids this, so $b_k > b_{k+1}$ throughout.

($\Leftarrow$) Let $\mathbf{b} = (b_1, \ldots, b_c)$ be a nonincreasing sequence of positive integers with $b_2 > b_3 > \cdots > b_{c-1}$. It suffices to show that there exists a polyunsaturated poset $R$ of dimension at most 2 and height $c$ such that $\Delta d(R) = \mathbf{b}$; $G(R)$ will be the required graph. If $c < 3$, then every poset with height $c$ is polyunsaturated, and we may let $R$ be a disjoint union of chains of the appropriate sizes. Thus, we may assume $c \geq 3$.

Since $b_{c-1} \geq b_c \geq 1$, we have $b_{c-2} \geq 2$; for $1 < i < c$, we have $b_i \geq c - i$. Also, $b_1 \geq b_2 \geq c - 2$. Thus,

$$\sum_{i=1}^{c} b_i \geq c - 2 + \left[ \sum_{i=2}^{c-1} c - i \right] + 1 = \binom{c}{2}.$$
We proceed by induction on $\sum b_i$, with the base case being $\sum b_i = \binom{c}{2}$. If $\sum b_i = \binom{c}{2}$, then the elements of $b$ must all equal the lower bounds found above. Let $R = P_{c-2}$. By Statement [3] of Lemma 2.2, we have $\Delta d(R) = b$. By Lemma 2.3, $R$ is polyunsaturated. By Statement [4] of Lemma 2.2, $R$ has dimension at most 2, and so $R$ is the required poset.

Now suppose $\sum b_i > \binom{c}{2}$. Let $t$ be maximal such that $b_t$ exceeds the lower bound found above. We define a new sequence $b' = (b'_1, \ldots, b'_c)$ as follows:

$$b'_i = \begin{cases} b_i - 1, & i \leq t; \\ b_i, & i > t. \end{cases}$$

This new sequence satisfies $b'_1 \geq b'_2 > b'_3 > \cdots > b'_{c-1} \geq b'_c \geq 1$. By the induction hypothesis, there is a polyunsaturated poset $R'$ of dimension at most 2 and height $c$ such that $\Delta d(R') = b'$. Let $R$ be the disjoint union of $R'$ and a chain of $t$ elements. Being a disjoint union of posets of dimension at most 2, $R$ has dimension at most 2. Since $t \leq c$, $R$ has height $c$. If $k$ and $l$ are distinct, nonconsecutive, positive integers less than $c$, then every $k$- and $l$-saturated chain partition of $R$ will give such a partition of $R'$. Since $R'$ is polyunsaturated, no such partition exists, and so $R$ is polyunsaturated. Lastly, $\Delta d(R) = b$, and so $R$ is the required poset. □

Lemma 3.1 and Theorem 3.3 characterize those sequences $b$ with $b = \Delta \alpha(G)$ for some polyunsaturated SGK graph $G$. However, it is not true that every SGK graph $G$ with $b = \Delta \alpha(G)$ must be polyunsaturated. Indeed, as noted after Lemma 3.1, every nonincreasing sequence $(b_1, \ldots, b_c)$ equals $\Delta \alpha(G)$ for some graph $G$ that is a disjoint union of cliques. When $c \geq 4$, no such graph is polyunsaturated.

Theorem 3.3 gives necessary and sufficient conditions for the existence of a polyunsaturated SGK graph $G$ with certain parameters and then notes that we may require $G$ to be a comparability graph. Thus, an analogous result holds for posets. Translating Theorem 3.3 into the language of posets, we obtain the following.

**Corollary 3.4.** Let $c$ be a positive integer and let $b = (b_1, b_2, \ldots, b_c)$ be a nonincreasing sequence of positive integers. There exists a polyunsaturated poset $P$ with height $c$ and $\Delta d(P) = b$ if and only if $b_2 > b_3 > \cdots > b_{c-1}$. Moreover, we may require $P$ to have dimension at most 2. □

4. Independence and Clique Numbers

Using Theorem 3.3, we can find necessary and sufficient conditions for the existence of a polyunsaturated SGK graph with prescribed clique number, independence number, and number of vertices. As noted earlier, for each result in this section, an analogous poset result holds.

**Corollary 4.1.** Let $n$, $c$, $a$ be positive integers, with $c \geq 3$. There exists an $n$-vertex polyunsaturated SGK graph $G$ with clique number $c$ and independence number $a$ if and only if all of the following conditions hold:

1. $a \geq c - 2$,
2. $n \geq a + 1 + \binom{c-1}{2}$, and
Moreover, we may require both $G$ and $\overline{G}$ to be comparability graphs.

**Proof.** By Lemma 3.1 and Theorem 3.3, for every sequence $(b_1, b_2, \ldots, b_c)$ of positive integers, there is a polyunsaturated SGK graph $G$ with $\Delta\alpha(G) = b$ if and only if

$$b_1 \geq b_2 > b_3 > \cdots > b_{c-1} \geq b_c,$$

and this remains true if we require both $G$ and $\overline{G}$ to be comparability graphs. For every $n$-vertex SGK graph $G$ with clique number $c$ and independence number $a$, we have $n = \sum \Delta\alpha(G)$ and $a = \Delta\alpha_1$. Thus, to prove Corollary 4.1, it suffices to show that, for positive integers $n$, $c$, and $a$, there exists a sequence $b = (b_1, b_2, \ldots, b_c)$ of positive integers such that

$$b_1 \geq b_2 > b_3 > \cdots > b_{c-1} \geq b_c, \quad n = \sum_{i=1}^{c} b_i, \quad \text{and} \quad a = b_1$$

if and only if conditions (1)–(3) hold.

($\Rightarrow$) Let $n$, $c$, $a$, and $b$ be as above. As in the proof of Theorem 3.3, since $b_{i-1} \geq b_i \geq 1$, we have $b_i \geq c - i$, for $1 < i < c$. In particular, $a = b_1 \geq b_2 \geq c - 2$. Thus (1) holds. Noting that $b_1 \geq a$ (since the two are equal), we can sum all these lower bounds to obtain (2):

$$n = \sum_{i=1}^{c} b_i \geq a + \left[ \sum_{i=2}^{c-1} c - i \right] + 1 = a + 1 + \binom{c - 1}{2}.$$  

We prove (3) similarly, by summing upper bounds. Since $b_2 \leq b_1 \leq a$, we have $b_3 \leq a - 1$, and we have $b_i \leq a - i + 2$, for $1 < i < c$. In particular, $b_c \leq b_{c-1} \leq a - c + 3$. Summing these upper bounds, we obtain

$$n = \sum_{i=1}^{c} b_i \leq a + \left[ \sum_{i=2}^{c-1} a - i + 2 \right] + a - c + 3 = ca + 1 - \binom{c - 1}{2}.$$  

($\Leftarrow$) Let $n$, $c$, and $a$ be positive integers satisfying conditions (1)–(3). As shown above, if all elements of $b$ equal their lower bounds—which requires (1)—then $\sum b_i = a + 1 + \binom{c - 1}{2}$. Similarly, if all elements of $b$ equal their upper bounds—which also requires (1)—then $\sum b_i = ca + 1 - \binom{c - 1}{2}$. By (2) and (3), $n$ is between these two values, inclusive. Thus, we may construct a sequence $b$ that satisfies the required conditions. We begin by letting each $b_i$ equal its lower bound. We increase $b_2$ until either $b_2$ reaches its upper bound, or $\sum b_i = n$. Then we increase $b_3$ in a similar manner, and so on. □

**Corollary 4.2.** Let $c$ and $a$ be positive integers. There exists a polyunsaturated SGK graph $G$ with clique number $c$ and independence number $a$ if and only if $a \geq c - 2$. Moreover, we may require both $G$ and its complement $\overline{G}$ to be comparability graphs.

**Proof.** The cases $c = 1$ and $c = 2$ are easy. We assume $c \geq 3$. The necessity follows immediately from Corollary 4.1. To show the sufficiency, note that if $a \geq c - 2$, then
\[ a + 1 + \left( \frac{c-1}{2} \right) \leq ca + 1 - \left( \frac{c-1}{2} \right). \] Thus, we can choose an \( n \) satisfying the conditions of Corollary 4.1. \( \square \)

The following result also follows easily from Corollary 4.1.

**Corollary 4.3.** Let \( n \) and \( c \) be positive integers, with \( c \geq 3 \). There exists an \( n \)-vertex polyunsaturated SGK graph \( G \) with clique number \( c \) if and only if \( n \geq \binom{c}{2} \). Moreover, we may require both \( G \) and \( \overline{G} \) to be comparability graphs. \( \square \)

**Remark 4.4.** By the above corollaries, the posets \( P_j \) of Definition 2.1 have the minimum width and cardinality over all polyunsaturated posets with the same height. \( \square \)

5. **Partitions into Independent Sets**

Reversing the roles of cliques and independent sets in our definitions, we may partition \( V(G) \) into independent sets (such a partition is called a proper coloring) and consider the natural upper bound placed on \( \omega_k(G) \), the maximum number of vertices in a union of \( k \) cliques. We call a proper coloring \( C \) \( k \)-saturated if this bound is achieved, that is, if \( \omega_k(G) = m_k(C) \). We ask if similar results to those above hold for \( k \)-saturated colorings.

A proper coloring \( C \) of \( G \) is \( k \)-saturated if and only if \( C \) is a \( k \)-saturated clique partition of \( G \). Let us call \( G \) co-polyunsaturated if \( \overline{G} \) is polyunsaturated. A result of Greene [8, Thm. 3.1] states that the complement of an SGK graph is also an SGK graph. Thus, for each of our results in Sections 3 and 4, there is a dual result. Some of these are given below.

As in previous sections, for each result below there is an analogous poset result dealing with partitions of a poset into antichains.

**Corollary 5.1.** Let \( a \) be a positive integer and let \( b = (b_1, b_2, \ldots, b_a) \) be a nonincreasing sequence of positive integers. There exists a co-polyunsaturated SGK graph \( G \) with independence number \( a \) and \( \Delta \omega(G) = b \) if and only if \( b_2 > b_3 > \cdots > b_{a-1} \). Moreover, we may require both \( G \) and \( \overline{G} \) to be comparability graphs. \( \square \)

**Corollary 5.2.** Let \( n, a, c \) be positive integers, with \( a \geq 3 \). There exists an \( n \)-vertex polyunsaturated SGK graph \( G \) with independence number \( a \) and clique number \( c \) if and only if all of the following conditions hold:

1. \( c \geq a - 2 \),
2. \( n \geq c + 1 + \binom{a-1}{2} \), and
3. \( n \leq ac + 1 - \binom{a-1}{2} \).

Moreover, we may require both \( G \) and \( \overline{G} \) to be comparability graphs.

**Corollary 5.3.** Let \( a \) and \( c \) be positive integers. There exists a co-polyunsaturated SGK graph \( G \) with independence number \( a \) and clique number \( c \) if and only if \( c \geq a - 2 \). Moreover, we may require both \( G \) and \( \overline{G} \) to be comparability graphs. \( \square \)
POLYUNSATURATED POSETS AND GRAPHS

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