Multi-qubit compensation sequences

Y Tomita, J T Merrill and K R Brown

Schools of Chemistry and Biochemistry; Computational Science and Engineering; and Physics, Georgia Institute of Technology, Atlanta, GA 30332, USA
E-mail: ken.brown@chemistry.gatech.edu

New Journal of Physics 12 (2010) 015002 (13pp)
Received 17 August 2009
Published 19 January 2010
Online at http://www.njp.org/
doi:10.1088/1367-2630/12/1/015002

Abstract. The Hamiltonian control of \( n \) qubits requires precision control of both the strength and timing of interactions. Compensation pulses relax the precision requirements by reducing unknown but systematic errors. Using composite pulse techniques designed for single qubits, we show that systematic errors for \( n \)-qubit systems can be corrected to arbitrary accuracy given either two non-commuting control Hamiltonians with identical systematic errors or one error-free control Hamiltonian. We also examine composite pulses in the context of quantum computers controlled by two-qubit interactions. For quantum computers based on the XY interaction, single-qubit composite pulse sequences naturally correct systematic errors. For quantum computers based on the Heisenberg or exchange interaction, the composite pulse sequences reduce the logical single-qubit gate errors but increase the errors for logical two-qubit gates.

\(^{1}\) Author to whom any correspondence should be addressed.
1. Introduction

The control of quantum bits for quantum computation requires a high degree of accuracy. Aside from coherence and random noise, systematic errors limit our ability to control these quantum systems. These errors include slow fluctuations in control parameters relative to the experimental time and slight imperfections in fabrication. Overcoming these systematic errors will be crucial to achieve the potentially high-accuracy gates required for fault-tolerant quantum computation with local gates [1]–[3]. The problem of unknown systematic errors has been studied extensively in nuclear magnetic resonance (NMR) [4]. In NMR, a large collection of spins is addressed by a radio frequency (RF) field with an unknown spatial variation. To overcome this variation, broadband composite pulses were introduced [5, 6].

In principle, compensating pulses can be used to correct unknown systematic errors in single-qubit gates to arbitrary order [7]. In a real experimental situation, other errors begin to accumulate and higher-order pulses may be of limited use [8]. The second-order broadband pulse devised by Wimperis (BB1) [9] is the standard of compensation and has been extended to two-qubit couplings by Jones [10]. In this paper, compensation pulses for multi-qubit systems and Hamiltonians are examined using BB1 as an example pulse. BB1 and the higher-order pulse sequences of [7, 11] are fully compensating; the pulses do not require a specific input state of the system and can be used to replace single pulses that are part of a larger sequence.

The paper is organized as follows: section 2 describes how the control theory (and related geometry) of multiple qubits is suited for the type of compensation pulses used on single qubits. Section 3 introduces our notation and a generalized BB1 sequence. Section 4 re-examines the two-qubit pulse sequence of Jones [10] in the case of multiple systematic errors. Different methods of creating BB1-style sequences are compared. Section 5 generalizes to \( n \) qubits and proves inductively that only two systematic errors need to be correlated to achieve arbitrary correction in all systematic errors. Section 6 examines the cases with sufficient control for universal quantum computation but not full control of the \( n \)-qubit space. Finally, we conclude in section 7.

New Journal of Physics 12 (2010) 015002 (http://www.njp.org/)
2. Control theory and geometry of $n$ qubits

The model we consider here is $n$ qubits and $M$-dimensionless Hamiltonians denoted as $H_m$. We define a pulse as applying $H_m$ with constant strengths $\Omega_m$ for a time $t$, where $\Omega_m$ is bound between $-\Omega_{\text{max}}$ and $\Omega_{\text{max}}$. The resulting unitary evolution is $U(t) = \exp(-i\sum_m \Omega_m H_m t)$. The applied pulse may not create the desired evolution due to systematic errors in the control strength $\Omega_m = \Omega_m (1 + \delta_m)$ and the timing $t' = t (1 + \delta_t)$. In this model, timing errors are correlated, whereas the individual strengths could have independent errors. The source of the errors will not be considered and we will examine unitaries of the form $U([\theta], \{\epsilon\}) = \exp(-i\sum_m \theta_m (1 + \epsilon_m) H_m)$.

The quantum system is universally controllable without unknown errors if the set of $H_m$ generates the entire control algebra $\text{su}(2^n)$ by addition and the Lie bracket [12]. The very same technique can be used to determine if a composite pulse sequence exists [13]. Additionally, the Lie bracket can be used to constructively build pulses, e.g. the Solovay–Kitaev composite pulse sequences in [7].

For $n$ qubits, the corresponding Lie algebra is $\text{su}(2^n)$. We choose as a convenient representation of the generators of the algebra, $\eta_j = \frac{1}{2} \bigotimes_{k=1}^n \sigma_{(j \mod 4^k)/4^{k-1}}$, where $\sigma_0 = I$ is the identity on the qubit and $\sigma_1 = X$, $\sigma_2 = Y$, and $\sigma_3 = Z$ are the single-qubit Pauli operators.

There are $4^n - 1$ operators since the generator of the global phase $\eta_0 = \frac{1}{2} \bigotimes_{k=1}^n I$ is outside of the algebra $\text{su}(2^n)$. For any two generators $\eta_i$ and $\eta_j$, we find that either they commute $[\eta_i, \eta_j] = 0$ or $[\eta_i, \eta_j] = i\epsilon_{ijk} \eta_k$. If they do not commute, the two operators generate a representation of $\text{su}(2)$.

The Lie algebra then imposes that given Pauli-operator generators with the same systematic control error, arbitrarily accurate composite pulses can be created, if and only if they do not commute. Furthermore, if they do not commute, the resulting pulse sequence will have the same form as a single-qubit pulse sequence [13]. A geometrical interpretation is that controlling two elements that do not commute is homomorphic to rotations on a sphere, whereas the space for commuting elements is a 2-torus [14, 15].

3. Notation and BB1 revisited

The goal is to create accurate multi-qubit unitaries in the presence of systematic errors in $\theta_j$. For each case, we will start by defining the set of generators we control, $\{H_l\}$, and denote the unitary transformations as

$$
U_l(\theta_l) = \exp(-i\theta_l H_l),
U_{l,m}(\theta_l, \theta_m) = \exp(-i(\theta_l H_l + \theta_m H_m)),
U_{l,m,n}(\theta_l, \theta_m, \theta_n) = \exp(-i(\theta_l H_l + \theta_m H_m + \theta_n H_n)),
U_{l,m,n,p}(\theta_l, \theta_m, \theta_n, \theta_p) = \ldots .
$$

We will be particularly interested in sets of three generators that have commutation relations equivalent to $\text{su}(2)$. In this case, rotations around the sphere can be used to guide the mathematics. The compensation pulses we present require that we can perform both positive and negative rotations. Physically this corresponds to inverting applied fields and changing the sign of multi-qubit interactions.
A useful metric for evaluating the effects of control errors is the infidelity, \(1 - F(U, V)\), where \(F\) is the fidelity,

\[
F(U, V) = \min\sqrt{\langle \psi | U^* V | \psi \rangle \langle \psi | V^* U | \psi \rangle},
\]

where \(U\) is the ideal unitary and \(V\) is the actual operation affected by the systematic error \(\epsilon\). We choose this measurement over the distance, \(D(U, V) = \|U - V\|\), to avoid complications due to a global phase, e.g. \(U = X\) and \(V = -X\). For \(U = U_1(\theta_1)\) and \(V = U_1(\theta_1(1 + \epsilon_1))\), the distance scales as \(O(\epsilon)\) and the infidelity scales as \(O(\epsilon^2)\) [16, 17].

Imagine we would like to perform \(U_1(\theta)\) but our systematic control errors limit us to control of the form \(U_{1,2}(\theta_1(1 + \epsilon_1), \theta_2(1 + \epsilon_2))\). Compensation sequences minimize the effect of these errors by applying successive error-prone pulses that cancel the leading error terms. In this notation, the BB1 sequence [9] is

\[
V_W(\theta, H_1, H_2) = U_1(\theta(1 + \epsilon_1))T_W(\phi, H_1, H_2),
\]

where \(T_W(\phi, H_1, H_2)\) is the correction sequence with \(\phi = \cos(-\theta / 4\pi)\) and is given as

\[
T_W(\phi, H_1, H_2) = U_{1,2}(\pi \cos(\phi)(1 + \epsilon_1), \pi \sin(\phi)(1 + \epsilon_2))
\]

\[
\times U_{1,2}(2\pi \cos(3\phi)(1 + \epsilon_1), 2\pi \sin(3\phi)(1 + \epsilon_2))
\]

\[
\times U_{1,2}(\pi \cos(\phi)(1 + \epsilon_1), \pi \sin(\phi)(1 + \epsilon_2)).
\]

We refer to this sequence as BB1-W and when \(\epsilon_1 = \epsilon_2 = \epsilon\) the sequence yields an infidelity that scales as \(\epsilon^6\), \(1 - F(V_W(\theta, H_1, H_2), U_1(\theta)) = O(\epsilon^6)\), or a distance that scales as \(\epsilon^3\). Details in the appendix. An infidelity that scales as \(\epsilon^n\) corresponds to a distance that scales as \(\epsilon^n\) [7]. The fine control of the relative amplitude or phase \(\phi\) allows for the correction; the compensation of higher-order terms relies on increasingly finer control.

The BB1 pulse sequence was derived in the context of single spins in NMR where \(H_1 = \frac{1}{2}X\) and \(H_2 = \frac{1}{2}Y\) [9]. In many controlled quantum systems, the control occurs in a rotating frame and the difference between applying the generator \(H_1\) or \(\cos(\phi)H_1 + \sin(\phi)H_2\) is phase shifting the applied oscillating field relative to the rotating frame [18]. As a result, for single-qubit gates it is often reasonable to assume that \(\epsilon_1 = \epsilon_2\).

4. Two qubits and multiple errors

Jones applied BB1 to two-qubit gates [10]. His construction assumes that the single-qubit gates are without error. In the context of NMR, the natural two-qubit Hamiltonian is \(H_1 = \frac{1}{2}Z_1Z_2\). The error in the control of \(H_1\) is unrelated to the error in \(H_2 = \frac{1}{2}X_1\), in this case no error. The direct application of BB1-W by simultaneous \(H_1\) and \(H_2\) pulses would fail to correct the errors. However, the error-free rotations about \(X_1\) allows us to construct unitaries that are generated by \(H_3 = \frac{1}{2}Y_1Z_2\). Since the algebra is equivalent to rotations, we can use a \(y\) \((H_2)\) rotation to rotate the \(x\)-axis \((H_1)\) to an axis in the \(x-z\) plane \((\epsilon_1H_1 + \epsilon_2H_2)\), yielding

\[
U_2(\phi)U_1(\theta)U_2(-\phi) = U_{1,3}(\theta \cos(\phi), \theta \sin(-\phi)).
\]

This identity was used by Jones [10] to create an alternative pulse sequence we will refer to as BB1-J. BB1-J transforms the requirement of relative amplitude-control (BB1-W) into the accurate control of a rotation. We note that in NMR, the sign of the ZZ Hamiltonian \(H_1\) is
determined by the molecule [5]. This shows that not all of the control Hamiltonians require invertible couplings in order to compensate. The correction sequence is then

\[
V_1(\theta, H_1, H_2) = U_1(\theta(1+\epsilon_1))U_2(\phi(1+\epsilon_2))U_1(\pi(1+\epsilon_1))U_2^\dagger(\phi(1+\epsilon_2))U_1(2\pi(1+\epsilon_1))U_2(3\phi(1+\epsilon_2))
\]

\[
\times U_1(\pi(1+\epsilon_1))U_2^\dagger(3\phi(1+\epsilon_2))U_2(\phi(1+\epsilon_2))U_1(\pi(1+\epsilon_1))U_2^\dagger(\phi(1+\epsilon_2)).
\]

This sequence yields an infidelity that scales as \(\epsilon_1^2\) when \(\epsilon_2 = 0\) [10]. The scaling for the case when \(\epsilon_2 \neq 0\) is examined in the appendix.

The utility of any fully compensating pulse sequence is that it can be used to replace single pulses in a sequence. If \(X_1\) and \(Y_1\) have the same systematic error, we can correct the \(X_1\) rotation by BB1-W before correcting the \(Z_1Z_2\) transformation by BB1-J. The sequence of BB1-WJ is

\[
V_{WJ}(\theta, H_1, H_2, H_4) = U_1(\theta(1+\epsilon_1))V_W(\phi, H_2, H_4)U_1(\pi(1+\epsilon_1))V_W^\dagger(\phi, H_2, H_4)
\]

\[
\times V_W(3\phi, H_2, H_4)U_1(2\pi(1+\epsilon_1))V_W^\dagger(3\phi, H_2, H_4)
\]

\[
\times V_W(\phi, H_2, H_4)U_1(\pi(1+\epsilon_1))V_W^\dagger(\phi, H_2, H_4),
\]

where

\[
V_W^\dagger(\phi, H_1, H_2) = T_{W,H_1,H_2}^\dagger(\cos(-\phi/4\pi)U_1^\dagger(\phi(1+\epsilon_1)).
\]

This sequence replaces error-prone \(U_2(\phi(1+\epsilon_2))\) pulse with the corrected rotation \(V_W\) generated by the BB1-W sequence. Here, \([H_1, H_2, H_3 = -i[H_1, H_2]]\) is a representation of su(2) and \([H_2, H_4 = \frac{1}{2}Y, H_5 = -i[H_2, H_4]]\) is also a representation of su(2). The assumption is that the errors of \(H_2\) and \(H_4\) are equivalent, \(\epsilon_2 = \epsilon_4\). The infidelity then scales as \(|\alpha\epsilon_1^4 + \beta\epsilon_1\epsilon_2^3|^2\) where \(\alpha\) and \(\beta\) are constants that depend on \(\theta, H_1, H_2\) and \(H_4\).

For fixed \(\epsilon_2\), the infidelity at small \(\epsilon_1\) scales as \(\epsilon_1^2\) in \(\epsilon_1\). This is the same order as the uncorrected pulse in \(\epsilon_1\), although with a substantially smaller infidelity. In the case of \(H_1 = \frac{1}{2}Z_1Z_2, H_2 = \frac{1}{2}X_1\) and \(H_4 = \frac{1}{2}Y_1\), where \(\epsilon_2 = \epsilon_X = 0.01\), the infidelity in this regime is a factor of \(10^8\) smaller than the uncorrected pulse (see figure 1). For \(V_{WJ}\), the infidelity scales as \(\epsilon_1^2\) when \(\epsilon_1 < \frac{\beta^2}{\alpha^4}\epsilon_2^{3/2}\). However, we can replace the BB1-W sequences \(V_W\) in \(V_{WJ}\) with higher-order pulse sequences, for example the \(Bn\) sequences where \(B2 = BB1\) [7]. In this case, the infidelity will scale as \(|\alpha\epsilon_1^4 + \gamma_\epsilon_1\epsilon_2^3|^2\), where \(\gamma_\epsilon\) is a constant that depends on \(\theta\) and \(Bn\). As a result, the value of \(\epsilon_1\) where the scaling changes from \(\epsilon_1^2\) to \(\epsilon_1^4\) becomes smaller and smaller. In figure 1, we compare the scaling properties of the BB1-WJ and the higher-order BB1-WJ, where we have replaced the \(V_W\) BB1 sequence with the \(B4\) sequence [7, 8]. As expected, the error \(\epsilon_1 = \epsilon_Z\), where the scaling changes from \(\epsilon_6^{1/2}\) to \(\epsilon_2^{1/2}\) changes from \(\approx 10^{-2}\) for BB1-WJ to \(\approx 10^{-4}\) for BB1-WJ. In principle, given a target infidelity and systematic errors \(\epsilon < 1\) [7], we can construct a pulse sequence with an infidelity guaranteed below the target infidelity. We note that in practice other errors including random control errors and decoherence typically limit the fidelity.

These sequences are each optimized for different correlations in the errors. BB1-W performs well when errors in the control of \(Z_1Z_2\) and \(X_1\) are correlated, whereas BB1-J is optimized for the case when one control has no error. BB1-WJ combines both strategies by first correcting the correlated errors and then correcting the independent error.

In figure 2, we compare the ideal unitary \(U = U_{ZZ}(\pi/4) = \exp(-i\frac{\pi}{8}Z_1Z_2)\) to the approximate unitaries \(V\) assuming equivalent errors in \(X_1, Y_1\) and uncorrelated errors in \(Z_1Z_2\). BB1-J \((V = V_1(\pi/4, Z_1Z_2/2, X_1/2))\) outperforms BB1-W \((V = V_W(\pi/4, Z_1Z_2/2, X_1/2))\)
Figure 1. Comparison of BB1-WJ and the higher-order BB1-\tilde{W}J pulse sequences applied to a $U_{ZZ}(\pi/4)$ operation. For a fixed $X_1$ and $Y_1$ error $\epsilon_X$, the infidelity after a BB1-WJ correction scales as $|\alpha\epsilon^3_{ZZ} + \beta\epsilon_{ZZ}\epsilon^3_X|^2$ (see the text). For the same $\epsilon_X$, the BB1-\tilde{W}J sequence scales as $|\alpha\epsilon^3_{ZZ} + \gamma\epsilon_{ZZ}\epsilon^5_X|^2$, extending the regime where the infidelity scales as $\epsilon^6_{ZZ}$.

Figure 2. Comparison of (a) BB1-W, (b) BB1-J and (c) BB1-WJ pulse sequences applied on $U_{ZZ}(\pi/4)$ operation on a pair of qubits. BB1-WJ assumes that $X_1$ and $Y_1$ have equivalent systematic errors. when either error is low. BB1-W is preferable when the systematic errors are identical. BB1-WJ ($V = V_{WJ}(\pi/4, Z_1Z_2/2, X_1/2, Y_1/2)$) results in low errors over the range of two errors. Initial compensation of the $X_1$ pulses results in better compensation of $Z_1Z_2$.

5. Extension to many qubits

Given a control operator with a systematic error and a perfect rotation that transforms that operator to an orthogonal independent operator, we can perform compensation, e.g. BB1-J. Given two control operators with correlated errors that are generators of su(2), we can perform compensation, e.g. BB1-W. As a result, in principle one can perform arbitrarily accurate composite pulses on a controllable quantum system where all the controls have independent errors except two.
Figure 3. Compensation of $U_{X_n}(\pi/4)$ by application of BB1-WJ$^{2(n-1)}$. Compensation of $U_{X_n}$ by BB1-W pulses using $Y_n$ works only when the errors are correlated. Anti-correlated errors between $X_n$ and $Y_n$ increase the infidelity. BB1-WJ$^{2(n-1)}$ uses the correlated errors of $X_1$ and $Y_1$ and a chain of $Z_jZ_{j+1}$ interactions to compensate for the errors in the $X_n$ rotation. The results for $X_2$, $X_4$ and $X_6$ are shown.

As an example, imagine $n$ qubits in a row with single-qubit operators and tunable Ising couplings. The Hamiltonians are $X_j$, $Y_j$ on each qubit and $Z_jZ_{j+1}$ between neighbours. If for the qubit $n$, $X_n$ and $Y_n$ have uncorrelated error, there does not exist a compensation pulse [13]. However, if the $X$ and $Y$ systematic errors are correlated on the first qubit but otherwise independent, the following sequence can be used to generate an arbitrarily accurate $X$ rotation on the $n$th qubit.

For the initial qubit with correlated $X_1$ and $Y_1$ errors, BB1-W is used. To correct $Z_1Z_2$, BB1-J is used with BB1-W corrected $X_1$ pulses. This is the sequence BB1-WJ. $X_2$ on the second qubit is then corrected via BB1-J using BB1-WJ-corrected $Z_1Z_2$ pulses. We denote this sequence as BB1-WJJ or BB1-WJ$^2$. Errors on the $n$th qubit can be compensated by repeated use of BB1-J along the chain, first correcting $X_j$, then $Z_jZ_{j+1}$ and then $X_{j+1}$ until $X_n$ is reached. The total sequence correcting the $n$th $X$ rotation is denoted as BB1-WJ$^{2(n-1)}$.

Figure 3 compares correcting a $\pi/4X$ rotation as a function of chain length assuming equal magnitude errors for all operators but with a random sign except for $X_1$ and $Y_1$. The correlated and anti-correlated lines serve as references. If $X_n$ and $Y_n$ have correlated errors, then local BB1-W greatly reduces the infidelity. In the worst case scenario, the errors are anti-correlated and the compensation pulses add additional error to the initial overrotation. $X_n$ rotations can still be corrected using BB1-WJ$^{2(n-1)}$, if only $X_1$ and $Y_1$ are correlated. The error increases with position (comparing BB1-WJ$^2$ with BB1-WJ$^{10}$) on the chain for large errors but approaches an equivalent fidelity for small errors. Asymptotically, the correction of $X_n$ rotations by sequential correction (BB1-WJ$^{2(n-1)}$) is equivalent to the BB1-W correction composed of correlated $X_n$ and $Y_n$ rotations. Replacing BB1 with the pulse sequences from [7] allows for the creation of arbitrarily accurate pulse sequences.

Although this is not practical on a large scale, it can lead to a constant reduction in the number of gates that need to be calibrated at the beginning of an experiment for a large quantum
system. Per region of computation, only a few highly reliable quantum gates can be used to reduce systematic errors in their neighbours.

6. Limited universality

An interesting theoretical proposal with potential applications for quantum dots [19, 20], superconducting qubits [21] and trapped ions [22] is the use of only two-qubit interactions for quantum computation [23]. These two-qubit interactions are chosen to generate a sufficiently large algebra to create universal computation on a subspace of the total Hilbert space. We examine composite pulses for $XY$ and Heisenberg interaction-based quantum computers. The pulse sequences require that the sign of the two-qubit interactions can be inverted. Coupled quantum dots have been shown to exhibit reversible exchange couplings, which can be controlled by an external magnetic field [25, 26], and may be promising candidates for this encoding.

6.1. $XY$

A Hamiltonian made of $XY$ interactions, $A_{i,j} = \frac{1}{2}(X_i X_j + Y_i Y_j)$, has been shown to be universal over three-qubits encoded into one-qubit [23, 24]. The $XY$ interaction preserves the projection of angular momentum along the $z$-axis, but does not preserve total angular momentum. The qubit is encoded in a subspace of the qutrit defined by $m_z = 1/2$ or $m_z = -1/2$.

Irrespective of the encoding, compensation is possible if the systematic errors are shared because

\[
A_{(1,2)} = \frac{1}{2} (X_1 X_2 + Y_1 Y_2) \quad (9)
\]

\[
A_{(2,3)} = \frac{1}{2} (X_2 X_3 + Y_2 Y_3) \quad (10)
\]

\[
A' = [A_{(1,2)}, A_{(2,3)}] / (i) = \frac{1}{2} (X_1 Z_2 Y_3 - Y_1 Z_2 X_3) \quad (11)
\]

is a representation of $su(2)$. For three qubits, we can block diagonalize the operators into four irreducible representations. For $m_z = \pm 3/2$, the irreducible representation is one-dimensional. For $m_z = \pm 1/2$, the irreducible representation is three-dimensional.

The operators $\bar{X}$, $\bar{Y}$ and $\bar{Z}$ act as the Pauli matrices on the encoded space and can be performed with pulses utilizing the $XY$ interaction alone. For example, $\bar{Z}$ rotations on the encoded qubit are implemented by the five-pulse $P_3$ sequence, which uses $XY$ interactions between each of the physical qubit pairs [24, 27].

\[
P_3(\theta, \epsilon) = U_{[1,2]} \left( -\frac{\pi}{4}(1+\epsilon) \right) U_{[2,3]} \left( -\frac{\pi}{2}(1+\epsilon) \right) U_{[1,3]} \left( -\frac{\theta}{2}(1+\epsilon) \right)
\]

\[
\times U_{[2,3]} \left( \frac{\pi}{2}(1+\epsilon) \right) U_{[1,2]} \left( \frac{\pi}{4}(1+\epsilon) \right).
\]

When $\epsilon = 0$, $P_3(\theta, \epsilon)$ is equivalent to a rotation about the $z$-axis in the code space, $U_3(\theta)$, up to a global phase. The $P_3$ sequence explicitly requires invertible couplings between physical qubits, which may limit the types of systems that an $XY$ computer can be built from. Assuming that the errors are proportional for each $H_{[i,j]}$, we can correct the timing error using BB1-W for each pulse; each $U_{[k,j]}(\theta(1+\epsilon))$ is replaced by $V_W(\theta, A_{[k,j]}, A_{(j,i\neq k)})$. The results of using the correction are shown in figure 4.
The remarkable part of the $XY$ interaction is that the su(2) algebra of neighbouring $XY$ operators is independent of our choice of encoded qubit. The exact same methods can be used to compensate the two-qubit gate sequences. Furthermore, we can apply our results for the $ZZ$ chain from section 5 to show that only two neighbouring $XY$ interactions need to have identical systematic errors.

6.2. Heisenberg

The Heisenberg Hamiltonian $G_{\{i,j\}} = (X_i X_j + Y_i Y_j + Z_i Z_j)$ has also been shown to be universal [28]–[30]. Furthermore, for certain arrangements of spins and exchanges it serves to protect errors by both energetics and symmetries [31]. It is more convenient to write this as $G_{\{i,j\}} = (-I + 2E(i, j))$, where $E(i, j)$ exchanges the states of qubits $i$ and $j$. As a result $[G_{\{i,j\}}, G_{\{j,k\}}] = 4(P(i, j, k) - P(i, k, j))$, where $P(i, j, k)$ is the cyclic permutation of $i, j$ and $k$. This does not result in a representation of su(2).

The exchange Hamiltonian $E(i, j)$ preserves both total angular momentum and the projection along $z$. For a single logical qubit made of three spins [30], any two exchange terms represent the algebra $u(1) \oplus su(2)$. The qubit is encoded into the $su(2)$ block corresponding to the total angular momentum, $S = 1/2$, and projection, $m_z = 1/2$. The exchange Hamiltonian generates $su(2)$ on the code space with $E(1, 2)$ and $\frac{1}{\sqrt{3}}(E(1, 2) + 2E(2, 3))$ corresponding to $\bar{Z}$ and $\bar{X}$, respectively. Up to a global phase, $G_{\{i,j\}}$ and $2E(i, j)$ generate the same unitary evolution. Assuming that $G_{\{1,2\}}$ and $G_{\{2,3\}}$ have equivalent systematic errors and the interaction strengths can change sign, compensation is then possible on the code space using BB1-W.

It is not clear whether the two-qubit gates can be corrected since the state has support on the $u(1)$ and $su(2)$ blocks. In figure 5, we calculate the three-qubit fidelity after compensation. If we limit ourselves to states in the code space, the compensation works as expected. Allowing states outside of the code space, the compensating pulses are worse than the uncompensated
pulse for low errors. This result is expected since the operators do not form an su(2) algebra that the BB1 sequence depends on.

7. Conclusions

We have shown that arbitrarily accurate compensation is possible with a fully controllable system if either two non-commuting Hamiltonians that generate su(2) have equivalent systematic errors or if a single Hamiltonian is error-free. In the case of two non-commuting Hamiltonians with equivalent systematic errors, pulses of both positive and negative amplitudes are required. The underlying pulse sequences are equivalent to sequences for qubits. Furthermore, we re-emphasize the importance of the algebra and show that the same compensation pulses work for universal XY quantum computation but not for universal Heisenberg quantum computation with a three-qubit encoding.

Compensation pulses are well-suited for single qubits controlled by interaction with electromagnetic waves in the rotating frame. In this case, the difference between X and Y Hamiltonians is simply a change in the phase of the electromagnetic wave. Uncertainty in the amplitude of the applied field naturally leads to an unknown but equivalent error in the two Hamiltonians.

For systems based on two-qubit interactions, the requirement of positive and negative couplings can present a challenge. Furthermore, for solid-state systems, the gate couplings will most likely have independent systematic errors. Our work presents a possible solution. For the XY case, a single, invertible error-free XY interaction can be used to build accurate gates. Using the identity that \( \exp(-iA_{1,2}A_{2,3}) \exp(iA_{1,2}) = -A_{2,3} \), we can create effective negative couplings for neighbouring XY interactions. We can then generate arbitrarily accurate unitary gates locally using BB1-J-type sequences. These can interact with their neighbours, etc. This is impractical but it does suggest that a system with a few low-error invertible couplings could efficiently compensate neighbouring high-error couplings.
The su(2) algebra underlying these compensating pulse provides additional incentive to continue development of single-qubit compensation pulses. Shaped pulse sequences or continuous time control can lead to further improvements [32]. The question remains how to develop composite pulses that do not rely on an su(2) or so(3) subalgebra. The Lie algebraic technique of Li and Khaneja [13] rules out composite pulses with the Heisenberg coupling. However, we know that over an encoded space at least space single-qubit compensation pulses are possible. The development of compensation pulses that do not use the geometry of the sphere and the development of techniques for identifying compensation compatible subspaces are both interesting challenges.

Acknowledgments

This work was supported by Georgia Tech and by IARPA through the Army Research Office grant no. W911NF-08-1-0515. YT acknowledges the support of an Emerson Fellowship. JTM acknowledges the support of a Georgia Tech Presidential Fellowship and an Emerson-Williams Fellowship.

Appendix. Analytical evaluation of errors in BB1 and BB1-J

The original BB1 sequence [9] is BB1-W where both errors are equivalent

\[ V(\theta, H_1, H_2) = U_1(\theta (1 + \epsilon))T(\phi, H_1, H_2), \quad (A.1) \]

where \( T(\phi, H_1, H_2) \) is the correction sequence with \( \phi = \cos(-\theta/4\pi) \)

\[ T(\phi, H_1, H_2) = U_{1,2}(\pi \cos(\phi)(1 + \epsilon), \pi \sin(\phi)(1 + \epsilon)) \]
\[ \times U_{1,2}(2\pi \cos(3\phi)(1 + \epsilon), 2\pi \sin(3\phi)(1 + \epsilon)) \]
\[ \times U_{1,2}(\pi \cos(\phi)(1 + \epsilon), \pi \sin(\phi)(1 + \epsilon)). \quad (A.2) \]

The \( \pi \) rotations toggle \( 3\phi \) to -\( \phi \) (see [9]). After removing identities, we can rewrite \( T(\phi, H_1, H_2) \) as

\[ T(\phi, H_1, H_2) = U_{1,2}(\pi \epsilon \cos(\phi), \pi \epsilon \sin(\phi)) \]
\[ \times U_{1,2}(2\pi \epsilon \cos(\phi), -2\pi \epsilon \sin(\phi)) \]
\[ \times U_{1,2}(\pi \epsilon \cos(\phi), \pi \epsilon \sin(\phi)). \quad (A.3) \]

We use the Magnus expansion [33] to combine the three unitary operators,

\[ T(\phi, H_1, H_2) = \exp\left(ie\theta H_1 + i\epsilon^3 M_3(\phi, H_1, H_2) + O(\epsilon^5)\right) \]
\[ = U_1(-\epsilon\theta) \left[1 + i\epsilon^3 M_3(\phi, H_1, H_2) + O(\epsilon^4)\right], \quad (A.4) \]

where

\[ M_3(\phi, H_1, H_2) = \frac{2\pi^3}{3} \cos(\phi) \sin^2(\phi)H_1 + 2\pi^3 \cos^2(\phi) \sin(\phi)H_2. \quad (A.5) \]

The second-order term vanishes due to the symmetry of the pulse sequence [7, 9]. As a result,

\[ V(\theta, H_1, H_2) = U_1(\theta (1 + \epsilon))U_1(-\epsilon\theta) \left[1 + i\epsilon^3 M_3(\phi, H_1, H_2) + O(\epsilon^4)\right] \]
\[ = U_1(\theta) \left[1 + i\epsilon^3 M_3(\phi, H_1, H_2) + O(\epsilon^4)\right]. \quad (A.6) \]
This shows that $V(\theta, H_1, H_2)$ is an approximation of $U_1(\theta)$ that scales as $\epsilon^3$ in distance [7] and $\epsilon^6$ in infidelity [10]. Both the distance and infidelity depend on the specific $H_1$ and $H_2$. We perform a similar analysis for BB1-J. Starting from equation (6), we find

$$V_j(\theta, H_1, H_2) = U_1(\theta(1 + \epsilon_1))T_j(\Phi, H_1, H_2),$$

(A.7)

where $T_j(\Phi, H_1, H_2)$ is the correction sequence with $\Phi = a\cos(\theta/4\pi)(1 + \epsilon_2) = \phi(1 + \epsilon_2)$

$$T_j(\phi, H_1, H_2) = U_{1,3}(\pi \cos(\Phi)(1 + \epsilon_1), -\pi \sin(\Phi)(1 + \epsilon_1))$$

$$\times U_{1,3}(2\pi \cos(3\Phi)(1 + \epsilon_1), -2\pi \sin(3\Phi)(1 + \epsilon_1))$$

$$\times U_{1,3}(\pi \cos(\Phi)(1 + \epsilon_1), -\pi \sin(\Phi)(1 + \epsilon_1)).$$

(A.8)

with $H_3 = i[H_1, H_2]$ for the unitary $U_{1,3}$. Following the steps above, we find that

$$V_j(\theta, H_1, H_2) = U_1(\theta(1 + \epsilon_1))U_1(4\epsilon_1 \pi \cos(\Phi)) \left[ 1 + i\epsilon_2^3 M_3(\Phi, H_1, -H_3) + O(\epsilon_1^3) \right].$$

(A.9)

Assuming $\epsilon_2$ is small,

$$\cos(\Phi) = \cos(\phi) \cos(\phi \epsilon_2) - \sin(\phi) \sin(\phi \epsilon_2)$$

$$\approx \cos(\phi) - \sin(\phi) \phi \epsilon_2$$

(A.10)

and

$$V_j(\theta, H_1, H_2) \approx U_1(\theta) \left[ 1 - i\epsilon_1 \epsilon_2 4\pi \phi \sin(\phi) + i\epsilon_2^3 M_3(\Phi, H_1, -H_3) \right].$$

(A.11)

The key result is that for fixed $\epsilon_2$ the infidelity scales as $\epsilon_2^6$ when $\epsilon_1$ is large and $\epsilon_2^3 \epsilon_1^2$ when $\epsilon_1$ is small. If we can improve the $U_2$ pulses by compensation, we can reduce the error to higher order in $\epsilon_2$. This is exactly how the BB1-WJ sequence (equation (7)) is constructed, resulting in a fidelity that scales as $\epsilon_1^6$ when $\epsilon_1$ is large and $\epsilon_1^2 \epsilon_2^6$ when $\epsilon_1$ is small.

References

[1] Gottesman D 2000 J. Mod. Opt. 47 333
[2] Svore K M, DiVincenzo D P and Terhal B M 2007 Quant. Inf. Comput. 7 297
[3] Clark C R, Metodi T S, Gasster S D and Brown K R 2009 Phys. Rev. A 79 062314
[4] Freeman R 1999 Spin Choreography (Oxford: Oxford University Press)
[5] Levitt M H 1986 Prog. NMR Spectrosc. 18 61
[6] Tycko R 1983 Phys. Rev. Lett. 51 775
[7] Brown K R, Harrow A W and Chuang I L 2004 Phys. Rev. A 70 052318
[8] Xiao L and Jones J A 2006 Phys. Rev. A 73 032334
[9] Wimperis S 1994 J. Magn. Reson. A 109 221
[10] Jones J A 2003 Phys. Rev. A 67 012317
[11] Alway W G and Jones J A 2007 J. Magn. Reson. A 189 114
[12] Huang G M, Tan T J and Clark J W 1983 J. Math. Phys. 24 2608
[13] Li J-S and Khaneja N 2006 Phys. Rev. A 73 030302
[14] Zhang J, Vala J, Sastry S and Whaley K B 2003 Phys. Rev. A 67 042313
[15] Khaneja N and Glaser S J 2001 Chem. Phys. 267 11
[16] Dawson C M and Nielsen M A 2006 Quant. Inf. Comput. 6 81
[17] Gilchrist A, Langford N K and Nielsen M A 2005 Phys. Rev. A 71 062310
[18] Rakreungdet W, Lee J H, Lee K F, Mischuck B E, Montano E and Jessen P S 2009 Phys. Rev. A 79 022316
[19] Weinstein Y S and Hellberg C S 2005 Phys. Rev. A 72 022319

New Journal of Physics 12 (2010) 015002 (http://www.njp.org/)
[20] Weinstein Y S, Hellberg C S and Levy J 2005 Phys. Rev. A 72 020304
[21] Storcz M J, Vala J, Brown K R, Kempe J, Wilhelm F K and Whaley K B 2005 Phys. Rev. B 72 064511
[22] Brown K R, Vala J and Whaley K B 2003 Phys. Rev. A 67 012309
[23] Kempe J, Bacon D, DiVincenzo D P and Whaley K B 2001 Quant. Inf. Comput. 1 33
[24] Kempe J and Whaley K B 2002 Phys. Rev. A 65 052330
[25] Burkard G, Loss D and DiVincenzo D P 1999 Phys. Rev. B 59 2070
[26] Zumbühl D M, Marcus C M, Hanson M P and Gossard A C 2004 Phys. Rev. Lett. 93 256801
[27] Lidar D A and Wu L A 2001 Phys. Rev. Lett. 88 017905
[28] Bacon D, Kempe J, Lidar D A and Whaley K B 2000 Phys. Rev. Lett. 85 1758
[29] Kempe J, Bacon D, Lidar D A and Whaley K B 2001 Phys. Rev. A 63 042307
[30] DiVincenzo D P, Bacon D, Kempe J, Burkard G and Whaley K B 2000 Nature 408 339
[31] Bacon D, Brown K R and Whaley K B 2001 Phys. Rev. Lett. 87 247902
[32] Khaneja N, Reiss T, Kehlet C, Schulte-Herbruggen T and Glaser S J 2005 J. Magn. Reson. 172 296
[33] Magnus W 1954 Commun. Pure Appl. Math. 7 649