Robust Semiparametric Efficient Estimators in Elliptical Distributions

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Abstract—Covariance matrices play a major role in statistics, signal processing and machine learning applications. This paper focuses on the semiparametric covariance/scatter matrix estimation problem in elliptical distributions. The class of elliptical distributions can be seen as a semiparametric model where the finite-dimensional vector of interest is given by the mean vector and by the (vectorized) covariance/scatter matrix, while the density generator represents an infinite-dimensional nuisance function. The main aim of this work is then to provide possible estimators of the finite-dimensional parameter vector able to reconcile the two dichotomic concepts of robustness and (semiparametric) efficiency. An $R$-estimator satisfying these requirements has been recently proposed by Hallin, Oja and Paindaveine for real-valued elliptical data by exploiting the Le Cam's theory of Local Asymptotic Normality (LAN) and the rank-based statistics. In this paper, we firstly provide a survey about the building blocks needed to derive such a robust and semiparametric efficient $R$-estimator, then its extension to complex-valued data is proposed. Finally, through numerical simulations, its estimation performance is investigated in finite-sample regime by comparing its Mean Squared Error with the Semiparametric Cramér-Rao Bound in different scenarios.

Index Terms—Semiparametric models, robust estimation, elliptically symmetric distributions, scatter matrix estimation, Le Cam’s one-step estimator, ranks statistics.

I. INTRODUCTION

Semiparametric inference is the branch of theoretical and applied statistics dealing with point estimation or hypothesis testing in semiparametric model. In short, a semiparametric model is a family of probability density functions (pdfs) parameterized by a finite-dimensional parameter vector of interest, say $\phi \in \Omega \subseteq \mathbb{R}^d$, and by an infinite-dimensional parameter, say $g \in G$, where $G$ is a suitable set of functions (see e.g. the monograph [1] or [2,3] and references therein). In the vast majority of applications where semiparametric models are used, the infinite-dimensional parameter $g$ plays the role of a nuisance function.

Despite of their generality and practical relevance, the use of semiparametric models in Signal Processing (SP) applications is still limited to very few cases. To name some examples, we refer to [4] for a semiparametric approach to blind source separation, to [5] for robust non-linear regression and to [6] for empirical likelihood methods applied to covariance estimation. More recently, in [7,8], the class of the Real and Complex Elliptically Symmetric (RES and CES) distributions has been revised from a semiparametric standpoint (see also [10]–[14] in the statistical literature). The family of ES distributions is in fact a typical example of semiparametric model where the finite-dimensional parameter vector of interest is given by the mean vector $\mu$ and by the (vectorized version of) the covariance (scatter) matrix $\Sigma$, while the density generator $g$ can be considered as a nuisance function. In particular, in [7] the RES class has been framed in the context of semiparametric group models, then a Semiparametric Cramér-Rao Bound (SCRB) for the joint estimation of $\mu$ and $\Sigma$ in the presence of the nuisance density generator $g$ has been derived. The second work [8] extended the previously obtained SCRB to semiparametric estimation of complex parameters in CES distributed data. A semiparametric version of the celebrated Slepian-Bangs formula has been also proposed.

However, the following fundamental question has not been addressed in [7,8] which were focused on lower bounds: is it possible to derive a robust and semiparametric efficient estimator of the covariance (scatter) matrix $\Sigma$ of a set of ES distributed observations? This paper aims at filling this gap.

Let us now take a closer look to the two main feature that this estimator should have. Firstly, it should be semiparametric efficient, at least asymptotically. In other words, we require that the error covariance matrix of such estimator should be equal to the SCRB given in [7,8] as the number of observations goes to infinity. The second desirable feature is the distributional robustness. As said before, a semiparametric model allows for the presence of a nuisance function that, in the case of ES distributed observations, is the unknown density generator $g$ characterizing the shape of their actual distribution. So, a distributionally robust estimator is basically a “distribution-free” estimator of $\Sigma$ whose statistical properties do not rely on $g$, and consequently on the actual ES distribution of the data. It is worth to underline that, even if robust estimators of covariance matrices are already available in the statistics and SP literature ([9,15]–[19], [20, Ch. 4] and references therein), they fail to be semiparametric efficient as shown in [7,8].

A good candidate for the estimator that we are looking for is the one proposed by Hallin, Oja and Paindaveine in their seminal paper [12]. Building upon their previous work [11], in [12] the Authors propose an estimator of the constrained, real-valued scatter matrix $\Sigma$ in RES distributed data that meets the two requirements of semiparametric efficiency and distributional robustness. To achieve the requirement of semiparametric efficiency, the Local Asymptotic Normality (LAN) property of the family of ES distributions has been exploited in [12]. The LAN property has been introduced for the first time by Le Cam in his fundamental work [21] (see also [22, Ch. 6]) and it has since established itself as a milestone in modern...
statistics. Leaving aside the deep theoretical implications that the LAN property has for a given family of distributions, there is at least one outcome of great interest for any practitioner working in SP and related fields. As Le Cam showed, if a statistical model is Locally Asymptotic Normal, then it is possible to derive asymptotically efficient estimators that, unlike the Maximum Likelihood (ML) one, do not search for the maxima of the log-likelihood function. This fact is of great importance in practical applications, where the ML estimator can present computational difficulties in the resulting optimization problem or even existence/uniqueness issues [23, Ch. 6].

The second requirement of distributional robustness has been addressed in [12] using a rank-based approach [24], [25, Ch. 13]. Originally developed in the context of order statistics, rank-based methods has been used in robust statistics to derive distribution-free estimators and tests that are usually referred to as R-estimators and R-tests [26, Ch. 3].

Since the required theory is far from being straightforward, we firstly provide a review of the methodology used in [12] to derive a semiparametric efficient R-estimator of the constrained, real-valued scatter matrix Σ in Real ES distributed data. In particular, in Sections II and III, we introduce the Le Cam’s one-step estimator for Σ and then, in Section IV, the rank-based procedure needed to “robustify” it. The main theorems underlying the proposed method will be presented in a way that can help to understand them easily without having a deep knowledge of the Le Cam’s theory. The only price that we have to pay in doing so is in the required regularity conditions, that are more stringent that the one assumed by Le Cam in [21]. However, this does not represent a big limitation, since in most of the practical applications such stringent, à la Cramér [27, Ch. 32 and 33], regularity conditions are satisfied. Due to their highly technical nature, the details of the proofs are not reported here to not confuse the presentation of the main statistical concepts. However, for the interested readers, we will always indicate where to find them in the relevant statistical literature\(^1\). Section V focuses on the extension of the previously derived outcomes to the complex-valued parameter case with Complex ES distributed data. In Section VI the effectiveness of the proposed semiparametric efficient R-estimator will be investigated through numerical simulations in a “finite-sample” regime. The theoretical analysis, in fact, can only provide us with asymptotic guarantees on the good behavior of an estimator but, since in practice the number of available observation is always finite, a “finite-sample” performance characterization is necessary as well. To this end, the error covariance matrix of the proposed R-estimator (evaluated using independent Monte Carlo runs) will be compared with the SCRB in [7,8] in different scenarios.

In the rest of the paper, we assume that the reader is already familiar with the basic concepts of the semiparametric theory, and in particular with the definition of efficient score function and efficient semiparametric Fisher Information Matrix (SFIM) that can be found in the relevant statistical literature (e.g. in the monographs [1,28]) or in [7,8,29] where they are introduced in a more familiar way for the SP community.

\(^1\)Even if the proofs can be found in various papers and books, where possible, we will refer to the monograph [1] that provides a consistent treatise of the Le Cam’s theory for both parametric and semiparametric models.

\textbf{Algebraic notation:} Throughout this paper, italics indicates scalar quantities (a), lower case and upper case boldface indicate column vectors (a) and matrices (A), respectively. Each entry of a matrix A is indicated as \( a_{ij} \triangleq [A]_{ij} \). \( \mathbf{I}_N \) defines the \( N \times N \) identity matrix. The superscripts \( * \), \( T \) and \( H \) indicate the complex conjugation, the transpose and the Hermitian operators respectively, then \( A^H \triangleq (A^*)^T \). Moreover, \( A^{-T} \triangleq (A^{-1})^T = (A^T)^{-1} \), \( A^{-*} \triangleq (A^{-1})^* = (A^*)^{-1} \) and \( A^{-H} \triangleq (A^{-1})^H = (A^H)^{-1} \). The Euclidean norm of a vector a is indicated as \( ||a|| \). The determinant and the Frobenius norm of a matrix A are indicated as \( |A| \) and \( ||A||_F \), respectively. The symbol vec indicates the standard vectorization operator that maps column-wise the entry of an \( N \times N \) matrix A in an \( N^2 \)-dimensional column vector vec(A). The operator vec(A) defines the \( N^2 - 1 \) dimensional vector obtained from vec(A) by deleting its first element, i.e. vec(A) \( \triangleq [a_{11}, \text{vec}(A)^T]^T \). A matrix A whose first top-left entry is constrained to be equal to 1, i.e. \( a_{11} \triangleq 1 \), is indicated as \( A_1 \).

For any \( N \times N \) symmetric matrix A:

- vecs(A) indicates the \( N(N+1)/2 \)-dimensional vector of the entries of the lower (or upper) sub-matrix of A.
- According to the notation previously introduced, \( \text{vec}(A) \triangleq [a_{11}, \text{vec}(A)^T]^T \).
- If \( a_{11} = 0 \), then \( \mathbf{M}_N = \text{the } (N+1)/2 \times N^2 \) matrix such that (s.t.) \( \mathbf{M}_N \text{vec}(A) = \text{vec}(A) \). Note that \( \mathbf{M}_N \) can be obtained from the duplication matrix \( \mathbf{D}_N \) [30,31] by removing its first column.

\textbf{Statistical notation:} Let \( x_l \) be a sequence of random variables in the same probability space. We write:

- \( x_l = o_P(1) \) if \( \lim_{l \to \infty} \Pr \{ |x_l| \geq \epsilon \} = 0, \forall \epsilon > 0 \) (convergence in probability to 0).
- \( x_l = O_P(1) \) if for any \( \epsilon > 0 \), there exists a finite \( M > 0 \) and a finite \( L > 0 \), s.t. \( \Pr \{ |x_l| > M \} < \epsilon, \forall l > L \) (stochastic boundedness).

The cumulative distribution function (cdf) and the related probability density function (pdf) of a random variable \( x \) or a random vector \( x \) are indicated as \( P_X \) and \( p_X \), respectively. For random variables and vectors, \( =_d \) stands for “has the same distribution as”. The symbol \( \sim \) indicates the convergence in distribution. According to the notation introduced in [7,8,29], we indicate the true pdf as \( p_0(x) \triangleq p_X(x|\phi_0,g_0) \), where \( \phi_0 \) and \( g_0 \) indicate the true parameter vector to be estimated and the true nuisance function, respectively. We define as \( E_{\phi,g} \{ f(x) \} = \int f(x)p_X(x|\phi,g)dx \) the expectation operator of a measurable function \( f \) of a random vector \( x \). Moreover, we simply indicate as \( E_0 \{ \cdot \} \) the expectation with respect to (w.r.t.) the true pdf \( p_0(x) \). The superscript * indicates a \( \sqrt{L} \)-consistent, preliminary, estimator \( \phi^* \) of \( \phi_0 \), s.t. \( \sqrt{L}(\phi^* - \phi_0) = O_P(1) \).

\section{II. THE SEMIPARAMETRIC SHAPE MATRIX ESTIMATION}

Let \( \{ x_l \}_{l=1}^L \) be a set of \( N \)-dimensional, real-valued, independent and identically distributed (i.i.d.) observation vectors.
Each observation is assumed to be sampled from a real elliptical pdf \[9,32,33\] of the form:
\[ p_X(x_t|\mu, \Sigma, g) = 2^{-N/2} |\Sigma|^{-1/2} g \left( (x_t - \mu)^T \Sigma^{-1} (x_t - \mu) \right) , \]
where \( \mu \in \mathbb{R}^N \) is a location parameter, \( \Sigma \in \mathcal{M}_N^R \) is a \( N \times N \) scatter matrix in the set \( \mathcal{M}_N^R \) of the symmetric, positive definite, real matrices. The function \( g \in \mathcal{G} \) is the density generator, an infinite-dimensional parameter that characterizes the specific distribution in the RES family. In order to guarantee the integrability of the pdf in (1), the set of all the possible density generators is defined as \( \mathcal{G} = \{ g : \mathbb{R}_0^+ \rightarrow \mathbb{R}_+ \mid \int_0^\infty t^{N/2-1} g(t) dt < \infty, \int p_X = 1 \} [32, Sec. 4] \). Each random vector whose pdf is given by (1), say \( x \sim R E S_N(\mu, \Sigma, g) \), admits the following stochastic representation [9,32]:
\[ x = \mu + R \Sigma^{1/2} u, \quad \text{where} \quad u \sim \mathcal{U}(\mathbb{R}^N) \quad \text{is a real random vector uniformly} \]
distributed on the unit \( N \)-sphere, \( R = \sqrt{Q} \) is called modular variate while \( Q \), usually referred to as 2nd-order modular variate, is such that (s.t.)
\[ Q = d (x_t - \mu)^T \Sigma^{-1} (x_t - \mu) = Q_1, \forall l. \]
Moreover, \( Q \) has pdf given by:
\[ p_Q(q) = (\pi/2)^{N/2} \Gamma(N/2)^{-1} q^{N/2-1} g(q), \]
where \( \Gamma(\cdot) \) stands for the Gamma function.

The expression of the elliptical pdf in (1) and the stochastic representation in (2) are not uniquely defined due to the well-known scale ambiguity between the scatter matrix \( \Sigma \) and the density generator \( g \). Specifically, from (1), it is immediate to verify that \( R E S_N(\mu, \Sigma, g(t)) = R E S_N(\mu, c\Sigma, g(t/c)), \forall c > 0 \).
In an equivalent way, from (2), we have that \( x = d \mu + R \Sigma^{1/2} u = d \mu + (c^{-1}R)(c\Sigma^{1/2})u, \forall c > 0 \). This readily implies that \( \Sigma \) is identifiable only up to a scale factor and consequently only a scaled version of \( \Sigma \) can be estimated. To avoid this identifiability problem, following [9,13,14], let us define the symmetric and positive definite shape matrix \( V \) as:
\[ V \triangleq \Sigma/s(\Sigma), \]
where \( s : \mathcal{M}_N^R \rightarrow \mathbb{R}_0^+ \) is a scalar functional on \( \mathcal{M}_N^R \) satisfying the following assumptions [13,14]:
\begin{align*}
\text{A1} & \quad \text{Homogeneity: } s(c \cdot \Sigma) = c \cdot s(\Sigma), \forall c > 0, \\
\text{A2} & \quad \text{Differentiability over } \mathcal{M}_N^R, \text{with } \frac{\partial s(\Sigma)}{\partial \Sigma}_{1,1} \neq 0, \text{ and } \frac{\partial s(\Sigma)}{\partial \Sigma}_{1,1} = 1.
\end{align*}
Typical examples of this class of scale functional are \( s(\Sigma) = |\Sigma|_{1,1} \), \( s(\Sigma) = tr(\Sigma)/N \) and \( s(\Sigma) = |\Sigma|^{1/N} \). Each scale functional \( s \) correspond to a differentiable constraint on the shape matrix \( V \). As an example, the constraints induced by the three above-mentioned scale functionals are \( v_{11} = 1 \), \( tr(V) = N \) and \( |V|^{1/N} = 1 \). It is easy to verify that, under A1, A2 and A3, the first top-left entry of \( V \), i.e. \( v_{11} \), can always be expressed as function of the other entries. This consideration, along with the fact that \( V \) is symmetric by definition, suggests us that, to avoid the identifiability problem, in the semiparametric estimation problem, we just need to consider the vector \( \text{vec}(V) \) as unknown. Moreover, as discussed in [13] and verified here, in Sec. VI the optimality properties of the proposed semiparametric estimator of the shape matrix does not depend on the particular scale functional. Consequently, in order to avoid tedious matrix calculation that may confuse the derivation of the algorithm, we choose the simple scale functional \( s(\Sigma) = |\Sigma|_{1,1} \), i.e. the one that constrains the shape matrix \( V \) to have its first top-left entry equal to 1. In the rest of the paper, a generic shape matrix satisfying this constraint is indicated as \( V_1 \) according with the notation previously introduced.

Having said that, we can formally state the semiparametric estimation problem that we are going to analyze in the following sections. Let \( \Omega \subseteq \mathbb{R}^q \) be a parameter space of dimension \( q = N(N+3)/2-1 = N+N(N+1)/2-1 \) where the “−1” term is due to the 1-dimensional scale constraint).

Each element of \( \Omega \) is a vector \( \phi \) of the form:
\[ \phi \triangleq \left( \mu^T, \text{vec}(V_1) \right)^T, \]
where \( \mu \in \mathbb{R}^N \) and \( V_1 \in \mathcal{M}_N^R \). Let us define the RES semiparametric model as the following set of (uniquely defined) pdfs:
\[ P_{\phi,g} = \left\{ p_X(p_X(x|\phi, g) = 2^{-N/2} |V_1|^{-1/2} \times \right. \]
\[ \left. g \left( (x_t - \mu)^T V_1^{-1} (x_t - \mu) \right) ; \phi \in \Omega, g \in \mathcal{G} \right\}. \]
The semiparametric estimation problem that we want to address is then to find a robust and semiparametric efficient estimator of a true parameter vector \( \phi_0 \in \Omega \) in the presence of a nuisance function \( g_0 \in \mathcal{G} \).

### III. SEMIPARAMETRIC EFFICIENT ESTIMATOR OF SHAPE MATRICES

In their seminal paper [12], Hallin, Oja and Paindaveine introduced a semiparametric efficient estimator of \( \phi \in \Omega \) in (6) based on the Le Cam’s theory of Local Asymptotic Normality (LAN) of models. Following the results obtained in [11], [12], [13], [34] and without any claim of completeness, the aim of this section is to provide a tutorial discussion on how the Le Cam’s theory can be applied to address the semiparametric estimation problem introduced in Sec. II.

We start this Section by defining in a proper mathematical way the LAN property. Then, in Theorem 1, a methodology to construct asymptotically efficient estimators for the parameter vector \( \phi \) in (6) in the RES parametric only model is provided. Finally, Theorem 2 extends this result to the semiparametric estimation problem discussed in Sec. II. We note, in passing, that even if Theorems 1 and 2 are presented here having in mind the specific estimation problem at hand (i.e. the estimation of the mean vector and of the shape matrix in RES distributed data), they remain valid in any other estimation problem involving parametric and semiparametric models that satisfy the LAN property.

#### A. LAN property and ES distributions

The LAN is a property defined for parametric models. Consequently, let us start by assuming to know the actual
density generator \( g_0 \) of the observations. Then, the resulting parametric RES model is given by:

\[
P_{\phi} = \left\{ p_X|x_\phi(x) = 2^{-N/2}|V_1|^{-1/2} \times \right. \\
g_0 \left((x_\phi - \mu)^T V_1^{-1} (x_\phi - \mu); \phi \in \Omega \right) \}
\]

Proposition 1. Let \( \{x_i\}_{i=1}^L \) be a set of \( N \)-dimensional, real-valued, i.i.d. observations sampled from a RES pdf \( p_0(x) \in \mathcal{P}_\phi \) in (8). Let \( \Delta_\phi(x_1, \ldots, x_L) \) be a random vector, usually referred to as central sequence, defined as:

\[
\Delta_\phi(x_1, \ldots, x_L) \equiv \Delta_\phi \triangleq L^{-1/2} \sum_{i=1}^L s_\phi(x_i),
\]

where \( s_\phi(x_i) \) is the usual score vector given by:

\[
s_\phi(x_i) = \nabla_\phi \ln p_X(x_i|\phi) = \left( s_\mu(x_i) \right),
\]

\( \phi \in \Omega \). Let \( I(\phi) \) be the Fisher Information Matrix (FIM):

\[
I(\phi) \triangleq \mathcal{E}_{\phi,0} \left( s_\phi(x) s_\phi^T(x) \right).
\]

Then, any \( p_X(x|\phi) \in \mathcal{P}_\phi \) satisfies the following LAN property:

\[
\ln \prod_{i=1}^L p_X(x_i|\phi + L^{-1/2}h) \prod_{i=1}^L p_X(x_i|\phi)
\]

\[
= h^T \Delta_\phi - \frac{1}{2} h^T I(\phi) h + o_P(1), \quad \forall \phi, h \in \Omega.
\]

Moreover, \( \Delta_\phi \) satisfies the following two properties:

C1 Asymptotic differentiability: for all \( \phi, h \in \Omega \)

\[
\Delta_{\phi + L^{-1/2}h} - \Delta_\phi = -I(\phi) h + o_P(1),
\]

C2 Asymptotic normality:

\[
\Delta_\phi \sim \mathcal{N}(0, I(\phi)), \quad \forall \phi \in \Omega.
\]

Remark: Proposition 1 is a “simplified” version of Proposition 2.1 provided in [11]. The proof of this result is rather technical and it can be found in [11, Appendix 1]. However, it is worth mentioning here that our version of this Proposition requires more stringent conditions with respect to the ones assumed in [11]. In particular, we require the existence of the gradient w.r.t. \( \phi \) of the log-likelihood function in (10) for every possible \( \phi \in \Omega \), while in [11] a much weaker definition of differentiability, usually referred to as Hellyger differentiability [1, Ch. 2, Def. 1], is adopted. Under the restrictive regularity conditions assumed here, the expansion in (12) can be though as the second-order Taylor approximation of the log-likelihood function [27].

Proposition 1 assures us that the RES parametric model \( \mathcal{P}_\phi \) in (8) satisfies the LAN property. This is of great practical importance because, as proved by Le Cam in [21], [22, Ch. 6], if a parametric model is Local Asymptotic Normal, then asymptotically efficient estimators of the parameter of interest \( \phi \) can be built using a “one-step linear correction” to any preliminary \( \sqrt{L} \)-consistent estimator \( \hat{\phi}^* \) of \( \phi \).

B. Efficient one-step parametric estimators

In parametric setting, the standard procedure to derive efficient estimators is given by the Maximum Likelihood theory. Specifically, given a set of i.i.d. data \( \{x_i\}_{i=1}^L \), an asymptotically efficient estimate of the true parameter vector \( \phi_0 \in \Omega \subset \mathbb{R}^q \), if it exists, can be obtained as:

\[
\hat{\phi}_{ML} = \arg\max_{\phi \in \Omega} \sum_{i=1}^L \ln p_X(x_i|\phi).
\]

As every practitioner knows, solving the optimization problem in (15) may result to be a prohibitive task and, in some cases, \( \hat{\phi}_{ML} \) may not even exist or may not be unique [23, Ch. 6]. So, it would be useful to figure out a different methodology to derive efficient estimates.

Under Cramér-type regularity conditions [27, Sec. 33.3], if \( \hat{\phi}_{ML} \) exists, then it satisfies

\[
\Delta_\phi(x_1, \ldots, x_M)|_{\phi=\hat{\phi}_{ML}} \equiv \Delta_{\hat{\phi}_{ML}} = 0,
\]

(16)

where \( s_\phi(x) \) is the score vector defined in (10). Eq. (16) can be thought as a set of \( q \) nonlinear equations and consequently, we can define a new estimator \( \tilde{\phi} \) given by the one-step Newton-Raphson approximate solution of (16) as:

\[
\tilde{\phi} = \hat{\phi} - \left[ \nabla_\phi^T \Delta_\phi \right]^{-1} \Delta_\phi,
\]

(17)

where \( \hat{\phi} \) is a “good” starting point and \( \nabla_\phi^T \Delta_\phi \) indicates the Jacobian matrix of \( \Delta_\phi \) evaluated at \( \hat{\phi} \). Note that the approximation in (17) is valid even if \( \hat{\phi}_{ML} \) does not exist. In [21] and [22, Ch. 6], Le Cam formalized and generalized this intuitive procedure by providing an asymptotic characterization of the class of efficient, “one-step”, estimators. The following Theorem specializes the Le Cam’s result in the estimation problem that we are dealing with.

Theorem 1. Let \( \{x_i\}_{i=1}^L \) be a set of i.i.d. observations sampled from a RES pdf \( p_0(x) \in \mathcal{P}_\phi \) in (8). Let \( \hat{\phi}^* \) any preliminary \( \sqrt{L} \)-consistent estimator of the true parameter vector \( \phi_0 \triangleq \left( \mu_0^T, \text{vec}(V_1) \right)^T \). Then, the “one-step” estimator

\[
\tilde{\phi} = \hat{\phi}^* + \sqrt{L} \text{vec}(I(\hat{\phi}^*))^{-1} \Delta_\phi^*,
\]

(18)

has the following properties:

P1 \( \sqrt{L} \)-consistency

\[
\sqrt{L} \left( \hat{\phi} - \phi_0 \right) \sim o_P(1),
\]

(19)

P2 Asymptotic normality and efficiency

\[
\sqrt{L} \left( \hat{\phi} - \phi_0 \right) \sim \mathcal{N}(0, I(\phi_0)^{-1}),
\]

(20)

where \( I(\phi_0)^{-1} \equiv \text{CCRB}((\mu_0, V_1)) \) and the constrained CCRB (CCRB) [35,36] is evaluated for the constraint \( \{V_1 \}_{11} = 1 \).

Proof: Let us start by showing that the expression defining the “one-step” estimator in (18) can be derived directly from the Newton-Raphson approximation in (17), using the asymptotic differentiability property C1 in (13) of the central sequence. Specifically, in analogy with the definition of Jacobian matrix, we have

\[
\nabla_\phi^T \Delta_\phi = -L^{1/2} I(\hat{\phi}^*) + o_P(1), \quad \forall \phi \in \Omega.
\]

(21)
Finally, substituting (21) in (17), and noticing that \( \hat{\phi}^* \) is a good starting point since it is, by definition, in the \( \sqrt{L} \)-neighborhood of \( \phi_0 \), yields the expression (18).

The proof of the \( \sqrt{L} \)-consistency property P1 of \( \hat{\phi} \) can be found in [1, Sec. 2.5, Th. 2]. To prove the property P2, we start from the intermediate result provided in [1, Sec. 2.3, Th. 1], that is \( I(\phi)^{-1} \Delta_{\phi} L \sim N(0, I(\phi)^{-1}) \). Consequently, using the fact that \( \hat{\phi}^* \) is \( \sqrt{L} \)-consistent, the asymptotic normality and efficiency of \( \phi \) in (18) follows form a direct application of the Slutsky’s theorem [25, Lemma 2.8]. Note that the same warning raised up for Proposition 1 holds here for Theorem 1. In fact, in [1, Sec. 2.3, Th. 1 and Sec. 2.5, Th. 2] only the Hellinger differentiability is required, while here we need to assume the existence of the gradient (w.r.t. \( \phi \in \Omega \)) of the log-likelihood function.

As preliminary \( \sqrt{L} \)-consistent estimator we may use:
\[
\hat{\phi}^* \triangleq \left( \hat{\mu}_{SM}, \text{vecs}(\hat{V}_{1,T_y})^T \right)^T, \tag{22}
\]
where \( \hat{\mu}_{SM} \triangleq L^{-1} \sum_{i=1}^{L} x_t \) is the sample mean estimator, while \( \hat{V}_{1,T_y} \) is the Tyler estimator [15] constrained to have \( \hat{V}_{1,T_y}[11] = 1 \).

The result in Theorem 1 would be enough to derive original, asymptotically efficient, estimators of the mean vector \( \mu_0 \) and of the shape matrix \( V_{1,0} \) in the classical parametric context.

In this paper however, we want to go one step further towards the semiparametric framework.

C. Clairvoyant, one-step, semiparametric estimators

In practical applications, we generally do not have any \textit{a priori} information on the actual density generator \( g_0 \) characterizing the specific RES distribution of the observations. Under this scenario, the parametric modeling of the estimation problem can be dropped in favor of the semiparametric one. As already discussed in the dedicated statistical literature (see e.g. [1,28,34]) and in our recent works [7,8,29], the semiparametric counterpart of the score vector \( s_{\phi} \) is the efficient score vector \( \bar{s}_{\phi} \) defined as (see [29] and [7, Th. IV.1]):
\[
s_{\phi}(x) \equiv \bar{s}_{\phi} \triangleq \Pi(s_{\phi} \mid T_{g_0}), \tag{23}
\]
where \( \Pi(s_{\phi} \mid T_{g_0}) \) is the orthogonal projection of the score vector \( s_{\phi} \) in (10) on the semiparametric nuisance tangent space \( T_{g_0} \) [37]. The semiparametric counterpart of the FIM \( I(\phi) \) is the efficient semiparametric FIM (SFIM) [29],[7, Th. IV.1]:
\[
\bar{I}(\phi)_{g_0} \triangleq E_{\phi,g_0}(s_{\phi}(x) \bar{s}_{\phi}(x))^T. \tag{24}
\]

Let us now introduce the \textit{efficient} central sequence \( \Delta_{\phi} \) and its asymptotic properties.

**Proposition 2.** Let \( \{x_t\}_{t=1}^{L} \) be a set of i.i.d. observations sampled from a RES pdf \( p_0(x) \in \mathcal{P}_{\phi,g} \) in (7). Let \( s_{\phi}(x) \) and \( \bar{I}(\phi)_{g_0} \) be the efficient score vector and the efficient SFIM defined in (23) and (24), respectively. Then, the efficient central sequence:
\[
\Delta_{\phi}(x_1, \ldots, x_L) \equiv \Delta_{\phi} \triangleq L^{-1/2} \sum_{t=1}^{L} s_{\phi}(x_t), \tag{25}
\]

satisfies the following two properties:

1. **Asymptotic differentiability:** for all \( \phi, h \in \Omega \)
\[
\Delta_{\phi + L^{-1/2}h} - \Delta_{\phi} = -\bar{I}(\phi)_{g_0}h + o_P(1), \tag{26}
\]

2. **Asymptotic normality**
\[
\Delta_{\phi} \overset{L \to \infty}{\sim} N(0, \bar{I}(\phi)_{g_0}^{-1}), \qquad \forall \phi \in \Omega. \tag{27}
\]

**Remark:** The proof can be found in [11, Sec. 3].

The results in Proposition 2 suggest us that it may be possible to derive asymptotically efficient, semiparametric estimators using a procedure similar to the one provided in Theorem 1, simply by substituting the parametric score vector and FIM with their semiparametric counterparts. This intuition is formalized by the next theorem.

**Theorem 2.** Let \( \{x_t\}_{t=1}^{L} \) be a set of i.i.d. observations sampled from a RES distribution whose pdf \( p_0(x) \in \mathcal{P}_{\phi,g} \) in (7). Let \( \hat{\phi}^* \) be any preliminary \( \sqrt{L} \)-consistent estimator of the true parameter vector \( \phi_0 \). Then, the semiparametric “one-step” estimator
\[
\hat{\phi}_s = \hat{\phi}^* + L^{-1/2}\bar{I}(\hat{\phi}^*)_{g_0}^{-1}\Delta_{\phi}^*, \tag{28}
\]
has the following properties:

1. **\( \sqrt{L} \)-consistency**
\[
\sqrt{L} \left( \hat{\phi}_s - \phi_0 \right) = O_P(1), \tag{29}
\]

2. **Asymptotic normality and efficiency**
\[
\sqrt{L} \left( \hat{\phi}_s - \phi_0 \right) \overset{L \to \infty}{\sim} N(0, \bar{I}(\phi_0)_{g_0}^{-1}), \tag{30}
\]

where \( \bar{I}(\phi_0)_{g_0}^{-1} \equiv \text{CSRFB}(\phi_0|g_0) = \text{CSRFB}(\mu_0, V_{1,0}|g_0) \) and the constrained semiparametric CRB (CSRFB) [7] is evaluated for the constraint \( V_{1,0}[11] = 1 \).

**Proof:** The expression of the semiparametric one-step estimator in (28) can be obtained using the same arguments discussed in Theorem 1. The proof of the \( \sqrt{L} \)-consistency property P1 of \( \hat{\phi}_s \) can be found in [1, Sec. 7.8, Th. 1]. To prove the asymptotic normality, we start from the intermediate result, given in [1, Sec. 3.3, Th. 2], that \( \bar{I}(\phi_0)_{g_0}^{-1}\Delta_{\phi} \overset{L \to \infty}{\sim} N(0, \bar{I}(\phi_0)_{g_0}^{-1}) \). Then, from the expression (28) and from the fact that \( \hat{\phi}^* \) is \( \sqrt{L} \)-consistent, the asymptotic normality and efficiency property PS2 of \( \hat{\phi}_s \) follows from a direct application of the Slutsky’s theorem (see also [1, Sec. 7.8, Cor. 1]). Again, here we need to assume the existence of the gradient (w.r.t. \( \phi \in \Omega \)) of the log-likelihood function, while in the proof [1, Sec. 7.8, Th. 1] only the Hellinger differentiability is required.

Before we move on, an important clarification is in order. As we can see from its closed form expression in (28), the “clairvoyant” estimator \( \hat{\phi}_s \) relies on the true density generator \( g_0 \), so it is not useful for inference problems in the semiparametric model (7) where the density generator is an unknown nuisance function. However, it has the fundamental role to link the parametric, one-step, Le Cam’s estimator in (18) with the distributionally robust estimator discussed ahead in Section IV. Before addressing the crucial issue of robustness, we provide a “tangible” expression of the clairvoyant estimator of \( V_1 \).
D. Semiparametric clairvoyant estimator of shape matrices

As for the parametric case, to construct \( \hat{\phi}_s \) in (28) we only need three elements: the efficient score vector \( \hat{s}_\mu = (\hat{s}_\mu^T, \hat{s}_{\text{vecs}}(V_1))^T \), the efficient SFIM \( \hat{I}(\phi|g_0) \) and a preliminary \( \sqrt{L} \)-consistent estimators \( \hat{\phi}^* \) of \( \phi_0 \). Building upon the results in our previous work [7], \( \hat{s}_\mu \) and \( \hat{s}_{\text{vecs}}(V_1) \) can be expressed as [7, eq. (53)]:

\[
\hat{s}_\mu \equiv s_\mu = -2\sqrt{Q_0}(Q_l)V_1^{-1/2}u_l, \tag{31}
\]

\[
\hat{s}_{\text{vecs}}(V_1) = -Q_l\psi_0(Q_l)KV_{1}, \text{vec}(u_lu_l^T), \tag{32}
\]

where \( Q_l \) is defined in (3) and

\[
KV_{1} \triangleq M_N \left ( V_1^{1/2} \otimes V_1^{-1/2} \right ) \Pi_{\text{vecs}(I_N)}, \tag{33}
\]

\[
u_l \triangleq V_1^{-1/2}(x_l - \mu), \tag{34}
\]

\[
\psi_0(t) \triangleq \frac{1}{g_0(t)} \frac{dg_0(t)}{dt}, \tag{35}
\]

\[
\Pi_{\text{vecs}(I_N)} = I_{N^2} - N^{-1} \text{vec}(I_N) \text{vec}(I_N)^T, \tag{36}
\]

where \( M_N \) is defined in the notation section. Before moving forward, some comments are in order. As already proved in [7], the efficient score vector \( s_\mu \) in (31) of the mean vector is equal to the parametric score vector \( s_\mu \), or in other words, \( s_\mu \) is orthogonal to the nuisance tangent space \( T_{g_0} \). This implies that, knowing or not knowing the true density generator \( g_0 \) does not have any impact on the asymptotic performance of an estimator of \( \mu \). The expression of the efficient score vector for the shape matrix in eq. (32) of this paper comes directly from eq. (53) of [7]. Even if clearly related, the main difference between these two expressions is in the fact that, while in eq. (53) of [7] the gradient is taken w.r.t. \( \text{vecs}(\Sigma_0) \) where \( \Sigma_0 \) is the unconstrained scatter matrix, in this paper the gradient is taken w.r.t. \( \text{vecs}(V_1) \) where \( V_1 \) is the constrained shape matrix s.t. \( [V_1]_{11} = 1 \). This is the reason why we have the matrix \( M_N \) instead of the duplication matrix \( D_N \) as in eq. (53) of [7]. Moreover, eq. (32) follows from eq. (53) of [7] through basic matrix algebra and the fact that \( \text{tr}(u_lu_l^T) = ||u_l||^2 = 1, \forall l \) and allows us to write a more compact expression for \( s_{\text{vecs}}(V_1) \).

The efficient SFIM \( \hat{I}(\phi|g_0) \) in (24) can be immediately obtained from the results in (31) and (32) and from the expression given in [7, eq. (54)] as:

\[
\hat{I}(\phi|g_0) = E_{\phi|g_0} \{ \hat{s}_\phi(x) \hat{s}_\phi(x)^T \} = \begin{pmatrix} \hat{I}(\mu|g_0) & 0 \\ 0 & \hat{I}(\text{vecs}(V_1)|g_0) \end{pmatrix}, \tag{37}
\]

As already noticed in [7], the block-diagonal structure of \( \hat{I}(\phi|g_0) \) in (37) implies that a lack of a priori knowledge about the mean vector \( \mu \) does not have any impact on the asymptotic performance of an estimator of the shape matrix \( V_1 \). In other words, the estimate of \( \mu \) and the one of \( V_1 \) are asymptotically decorrelated. This and the above-mentioned fact that \( \hat{s}_\mu \perp T_{g_0} \) allow us to considered the estimation of \( \mu \) and the one of \( V_1 \) as two separate problems. For this reason, from now on, we will focus our attention only on the estimation of \( V_1 \).

From (32) and building upon the expression already derived in eq. (56) of [7], we have that:

\[
\hat{I}(\text{vecs}(V_1)|g_0) = \frac{2E\{Q^2\psi_0(Q)^2\}}{N(N + 2)} K_{V_1} K_{V_1}^T. \tag{38}
\]

By substituting the expression of \( \hat{s}_{\text{vecs}}(V_1) \) given in (32) in the definition of the efficient central sequence in (25), we get:

\[
\Delta V_{1} = -L^{-1/2}K_{V_1} \sum_{l=1}^{L} Q_{l} \psi_{0}(Q_{l}) \text{vec}(u_{l}u_{l}^{T}). \tag{39}
\]

Finally, we just need to put (39) and the expression of \( \hat{I}(\text{vecs}(V_1)|g_0) \) given in (38) in the definition of one-step estimator in (28). This yields the following estimator:

\[
\text{vecs}(\hat{V}_{1,s}) = \text{vecs}(\hat{V}_{1}) - \frac{N(N + 2)}{2LE\{Q^2\psi_0(Q)^2\}} \left \{ K_{\hat{V}_1} K_{\hat{V}_1}^T \right \}^{-1} K_{\hat{V}_1} \times \sum_{l=1}^{L} \hat{Q}_{l}^{*} \psi_{0}(Q_{l}) \text{vec}(\hat{u}_{l}^{*}(\hat{u}_{l}^{*})^{T}), \tag{40}
\]

where:

\[
\hat{Q}_{l}^{*} \triangleq (x_{l} - \hat{\mu}^{*})^{T} [\hat{V}_{1}^{*}]^{-1} (x_{l} - \hat{\mu}^{*}), \tag{41}
\]

\[
\hat{u}_{l}^{*} \triangleq (\hat{Q}_{l}^{*})^{-1/2} (x_{l} - \hat{\mu}^{*}), \tag{42}
\]

while, as the notation suggests, the matrix \( K_{\hat{V}_1} \) is obtained from \( K_{V_1} \) in (33) by substituting \( V_1 \) with its preliminary estimator \( \hat{V}_{1} \).

The last thing to do is to choose preliminary estimators for the mean vector and for the shape matrix. To this end, we can use the joint Tyler’s shape and mean vector estimator [38, eq. (6)], i.e. \( \hat{\mu}^{*} = \hat{\mu}_{T_{y}} \) and \( \hat{V}_{1}^{*} = \hat{V}_{1,T_{y}} \) with the constraint \( [\hat{V}_{1,T_{y}}]_{11} = 1 \). This is a good choice since such \( \hat{\phi}^{*} \) is \( \sqrt{L} \)-consistent under any possible density generator \( g \in \mathcal{G} \). However, other choices are also possible, e.g. the one proposed in [39].

As previously said, the clairvoyant estimators provided in (40) cannot be directly applied for semiparametric inference since it still depends on the true density generator \( g_0 \) from two different standpoints:

i) Statistical dependence. The estimator \( \hat{V}_{1,s} \) in (40) relies on the random variables \( \{Q_{l}^{*}\}_{l=1}^{L} \) whose pdf depends on \( g_0 \) through the one of the data \( \{x_{l}\}_{l=1}^{L} \) (see eq. (41)).

ii) Functional dependence. The expectation \( E\{Q^2\psi_0(Q)^2\} \) depends on \( g_0 \) through the function \( \psi_0 \) in (35) and the pdf of \( Q \) in (4).

A closer look at Theorem 2 readily reveals that the dependence on \( g_0 \) of the semiparametric efficient estimator in (40) can be traced back to the efficient central sequence and to its asymptotic differentiability property CS2 in (26). This implies that, to obtain distributionally robust estimators, we need a “distribution-free” version of the efficient central sequence.

IV. A SEMIPARAMETRIC EFFICIENT R-ESTIMATOR FOR SHAPE MATRICES

This section relies on the deep theoretical findings proved by Hallin, Oja and Paindaveine in their seminal work [12] and aims at providing an intuitive interpretation of the latter. No claim to completeness or rigoroussness is made here. To fully understand the theory underlying the results in [12] in fact, a
strong knowledge of the Le Cam theory and of its invariance-based extension to semiparametric framework, provided in [34], is required. Since the goal of this paper is to supply any SP practitioner with a “ready-to-use” estimator, we drop the mathematical rigorosity in favor of a more intuitive exposition that leads directly to an easy implementation of the theoretical findings.

We start by introducing a robust version of the efficient central sequence, then we shown that it satisfies an asymptotic differentiability property similar to the one in (26). Finally, building upon these results, an implementable expression of the robust and semiparametric efficient estimator of shape matrices is provided.

It is important to stress that, even if the theory may appear a bit convoluted, the resulting estimator can be easily implemented in few lines of code and its computational complexity is comparable to the one of Tyler’s estimator. For the interested reader, our Matlab implementation of the R-estimator can be found at [40].

A. Preliminaries on rank-based statistics

As a prerequisite for the derivation of the robust version of the efficient central sequence in (39), we have to introduce some basic concepts and properties of the ranks.

The ranks of a set of relevant random variables are a useful tool in non-parametric statistics and numerous works can be found on this topic (see e.g. [24], [41], [42], [25, Ch. 13] and references therein). Far be it from us to propose a comprehensive overview of the use of ranks in robust statistics, in the following we limit ourselves to introduce their definition and some of their main properties.

Let \( \{x_l\}_{l=1}^L \) be a set of \( L \) continuous i.i.d. random variables with pdf \( p_X \). We define the vector of the order statistics as \( v_X \) \( \triangleq [x_{L(1)}, x_{L(2)}, \ldots, x_{L(L)}]^{T} \) whose entries \( x_{L(1)} < x_{L(2)} < \cdots < x_{L(L)} \) are the values of \( \{x_l\}_{l=1}^L \) ordered in an ascending way. Then, the rank \( r_l \in \mathbb{N}/\{0\} \) of \( x_l \) is the position index of \( x_l \) in \( v_X \). Finally, we define \( r_X \) \( \triangleq [r_1, \ldots, r_L]^{T} \in \mathbb{N}^L \) as the vector collecting the ranks.

**Lemma 1.** Let \( \{x_l\}_{l=1}^L \) be a set of i.i.d. random variables s.t. \( x_l \sim p_X, \forall l \). Let \( K \) be the family of scoring functions \( K : (0, 1) \to \mathbb{R} \) that are continuous, square integrable and that can be expressed as the difference of two monotone increasing functions. Then, we have:

1) The vectors \( v_X \) and \( r_X \) are independent.
2) Regardless the actual pdf \( p_X \), the rank vector \( r_X \) is uniformly distributed on the set of all \( L! \) permutations on \( \{1, 2, \ldots, L\} \) and \( ! \) stands for the factorial notation.
3) For each \( l = 1, \ldots, L \)

\[
K\left(\frac{r_l}{L+1}\right) = K(u_l) + o_p(1),
\]

where \( K \in \mathcal{K} \) and \( u_l \sim \mathcal{U}(0, 1) \) is a random variable uniformly distributed in \( (0, 1) \).

Remark: The proof can be found in [25, Ch. 13] or in [42, Ch. 3].

To understand why Lemma 1 is useful to derive a distributionally robust and semiparametric efficient estimator of the shape matrix we should take a step back.

B. A distribution-free approximation of \( \overline{\Delta}_{V_1} \)

From the stochastic representation in (2), there is a one-to-one correspondence between a RES distributed observation vector \( x_l \sim \text{RES}_{\mathcal{K}}(\mu, \Sigma, g_0) \) and the couple \((Q_l, u_l)\), where \( Q_l \sim \mathcal{Q} \) is defined in (3) and whose pdf \( p_\mathcal{Q} \) is given in (4), while \( u_l \sim \mathcal{U}(\mathbb{R}^{S^N}) \). According to the notation introduced in Lemma 1, let us define \( r_\mathcal{Q} \) as the vector of the ranks of the set \( \{Q_l\}_{l=1}^L \). We can now exploit the results of Lemma 1 for our goal. In fact, Point 2) in the Lemma 1 tells us that the distribution of \( r_\mathcal{Q} \) is invariant w.r.t. the pdf \( p_\mathcal{Q} \) in (4) that depends on the actual, and generally unknown, density generator \( g_0 \in \mathcal{G} \). This feature is very attractive for robust inference since it allows us to derive rank-based (or \( R \)-) estimators and tests that are distribution-free. Point 3) of Lemma 1 provides us with the missing piece to obtain a distributionally robust approximation of the efficient central sequence \( \overline{\Delta}_{V_1} \). Specifically, let

\[
P_{\mathcal{Q},0}(q) \triangleq (\pi/2)^{N/2}\Gamma(N/2)^{-1}\int_{0}^{q} t^{N/2-1}g_0(t)dt
\]

be the true, and generally unknown, cdf of 2nd-order modular variates whose pdf is given in (4). Let us now recall the basic fact that (see e.g. [43, Th. 2.1.10])

\[
P_{\mathcal{Q},0}^{-1}(u_l) = Q_l, \quad u_l \sim \mathcal{U}(0, 1), \quad Q_l \sim P_{\mathcal{Q},0} \forall l
\]

where \( P_{\mathcal{Q},0}^{-1} \) indicates the inverse function of the cdf. Finally, we have to introduce the “true” score function

\[
K_0(u) \triangleq P_{\mathcal{Q},0}^{-1}(u)\psi_0(P_{\mathcal{Q},0}^{-1}(u)), \quad u \in (0, 1),
\]

that can be shown to belong to the set \( \mathcal{K} \) [44]. Note that \( K_0 \) depends on the true density generator \( g_0 \) through \( \psi_0 \) in (35) and \( P_{\mathcal{Q},0} \) in (44). From Point 3) of Lemma 1 and by using the relation (45) we have

\[
K_0\left(\frac{r_l}{L+1}\right) = Q_l\psi_0(Q_l) + o_p(1).
\]

Consequently, substituting (47) in (32) yields to the following approximation of the efficient central sequence in (39):

\[
\overline{\Delta}_{V_1} = -L^{-1/2}K_{V_1}\sum_{l=1}^{L}K_0\left(\frac{r_l}{L+1}\right)\text{vec}(u_lu_l^T)+o_p(1).
\]

The expression in (48) depends “statistically” only on the ranks \( r_l \) and on the random vectors \( u_l \) whose both distributions are invariant w.r.t. the actual RES distribution of the data. However, we still have a functional dependence from \( g_0 \) due to the score function \( K_0 \). To get rid of this dependence, we can adopt a common procedure in robust statistics: since \( K_0 \) is unknown, let us choose another function \( K \in \mathcal{K} \) that “approximates” \( K_0 \) and providing, at the same time, a certain
amount of robustness. A classical example is the set of the power score functions defined as:
\[
\mathcal{K}_a = \{ K_a : (0, 1) \to \mathbb{R} | K_a(u) = -N(a + 1)u^a, a \geq 0 \}.
\]
(49)
The celebrated sign, Wilcoxon, and Spearman score functions belong to \( \mathcal{K}_a \) in (49) and are obtained for \( a = 0, a = 1 \) and \( a = 2 \) [45].

We may also use a “misspecified approach” [46]: since we do not know which is the true density generator \( g \), we may adopt a possibly misspecified, \( g \in \mathcal{G} \) instead of the unknown \( g_0 \). Consequently, a distributionally robust and semiparametric efficient estimator of \( \alpha \) has the same role that the efficient SFIM has in (26).

As a useful example of score function \( g \), we may cite the van der Waerden score function \( K_{G} \) [47] that can be obtained by choosing the Gaussian density generator \( g_0(t) = \exp(-t/2) \). With this choice, it is immediate to verify that, from (46)
\[
K_G(u) = -\Psi^{-1}(u)/2,
\]
(51)
where \( \Psi^{-1}(u) \) indicates the inverse function of the cdf of a \( \chi \)-squared random variable with \( N \) degrees of freedom. Clearly, a misspecification of the density generator will bring to a loss in semiparametric efficiency. Remarkably, as we will see in Sec. VI of this work, such performance loss are small.

C. The asymptotic differentiability of \( \Delta_{V_1} \)

In the previous subsection, we have introduced a distribution-free approximation \( \Delta_{V_1} \) of the efficient central sequence \( \Delta_{V_1} \). The missing piece for a derivation of a distributionally robust and semiparametric efficient estimator of the shape matrix is to establish an asymptotic differentiability property for \( \Delta_{V_1} \) similar to the one given in (26) for \( \Delta_{V_1} \). The result that we need is given in [12, Prop. 2.1 and Sec. 4]:

**Proposition 3.** Let \( H^0 \) be any symmetric matrix whose first top-left entry is equal to 0, i.e., \( H^0_{1,1} = 0 \). Then, we have:
\[
\Delta_{V_1 + L^{-1/2}H^0} - \Delta_{V_1} = -\Psi(V_1|g_0,g)\text{vecs}(H^0) + o_p(1),
\]
(52)
where the matrix
\[
\Psi(V_1|g_0,g) \triangleq \alpha_{0,g}K_{V_1}K_{\hat{V}_1}^T
\]
has the same role that the efficient SFIM has in (26) and \( \alpha_{0,g} \) is a scalar that depends on the true density generator \( g_0 \) and on the assumed one \( g \in \mathcal{G} \). Consequently, a consistent estimator of \( \alpha_{0,g} \) can be obtained as:
\[
\hat{\alpha} = \frac{|\Delta_{V_1 + L^{-1/2}H^0} - \Delta_{V_1}|}{|K_{\hat{V}_1}K_{\hat{V}_1}^T\text{vecs}(H^0)|},
\]
(54)
where \( \hat{V}_1 \) is a preliminary \( \sqrt{L} \)-consistent estimator of the shape matrix \( V_1 \). Consequently, an estimator of \( \Psi(V_1|g_0,g) \) in (53) can be obtained as:
\[
\hat{\Psi} \triangleq \hat{\alpha}K_{\hat{V}_1}K_{\hat{V}_1}^T.
\]
(55)

**Remark:** As it is clear from (54), the consistent estimator \( \hat{\alpha} \) depends on the “small perturbation” matrix \( H^0 \) that can be considered as an hyper-parameter to be defined by the user. Some consideration on the choice of \( H^0 \) will be provided in Sec. VI where a numerical analysis of the performance of the proposed shape matrix estimator is presented.

Note that the estimator \( \hat{\alpha} \) in (54) is only an example of a possible estimator for \( \alpha_{0,g} \), but other procedures may be adopted as well. In [12, Sec. 4.2] for example, an ML-based approach is implemented to derive a consistent and efficient estimator for \( \alpha_{0,g} \). However, such ML-based estimator requires the solution of an optimization problem that may become computationally heavy as the matrix dimension increases.

D. A distributionally robust and semiparametric efficient estimators for shape matrices

We finally have at our disposal all the ingredients required to implement the desired \( R \)-estimator for shape proposed by Hallin, Oja and Paidaveine in [12, Prop. 3.1]. We recall that in Subsec. IV-B, we introduced a distribution-free approximation \( \Delta_{V_1} \) of the efficient central sequence \( \Delta_{V_1} \) in (39). Moreover, in Subsec. IV-C, the asymptotic differentiability property \( \Delta_{V_1} \) has been discussed. Then, by relying on the above-mentioned findings, the same “one-step” procedure used in Theorems 1 and 2 to obtain parametric and semiparametric efficient estimators can be applied here. In particular, a “one-step” \( R \)-estimator for \( V_1 \) can be obtained as:
\[
\text{vecs}(\hat{\mathbf{V}}_{1,r}) = \text{vecs}(\hat{\mathbf{V}}_1) + L^{-1/2}{\hat{\Psi}}^{-1}\Delta_{\hat{V}_1}.
\]
(56)
Finally, by substituting in (56) the expressions of \( \hat{\Psi} \) and of \( \Delta_{\hat{V}_1} \) given in eqs. (55) and (50) respectively, we get:
\[
\text{vecs}(\hat{\mathbf{V}}_{1,r}) = \text{vecs}(\hat{\mathbf{V}}_1) - \frac{1}{L\hat{\alpha}} \left[ K_{\hat{V}_1}K_{\hat{V}_1}^T \right]^{-1} K_{\hat{V}_1} \times \sum_{l=1}^{L} K_g(\frac{r_l^T}{L+1}) \text{vecs}(\hat{u}_l^T)^T,
\]
(57)
where \( \{r_l^T\}_{l=1}^L \) are the ranks of the random variables \( \{\hat{Q}_l^T\}_{l=1}^L \) defined in (41), while \( \hat{u}_l^T \) is defined in (42). Again, as preliminary estimator of the (constrained) shape matrix we may use the Tyler’s estimator \( \hat{\mathbf{V}}_1 = \hat{\mathbf{V}}_{1,T} \).

V. EXTENSION TO COMPLEX ES DISTRIBUTIONS

Building upon the previously obtained results, this section aims at providing an extension of the \( R \)-estimator in (57) to the (complex-valued) shape matrix estimation problem in CES-distributed data. As already shown in [9], [20, Ch. 4] and [8, Def. II.1], there exists a one-to-one mapping between the set of the CES distributions and a subset of the RES ones. This implies that the theory already developed for the real-valued case can be applied straight to complex-valued data. However, the use of a real representation of complex quantities usually leads to a loss in the clarity and even in the interpretability of the results. Best practice is then to work directly in the complex field by means of the Wirtinger calculus [48]–[53]. Basically, the Wirtinger calculus generalizes the concept of complex derivative to non-holomorphic, real-valued functions of complex variables. In our recent paper
[8], the Wirtinger calculus has been exploited to derive the SCRB for the joint estimation of the (complex) mean vector and scatter matrix of a set of CES distributed data. In particular, the complex-valued counterparts of the efficient score vector and of the SFIM for shape matrices in CES data have been evaluated in [8]. As for the real-valued case, among all the possible scale functionals, we have that, due to the strong similarity between the properties of the CES and RES distributed random vectors, in the following we will mostly reuse the same notation introduced in Section II for the corresponding entities.

**A. Preliminaries on CES distributed data**

Let \( \{z_i\}_{i=1}^N \in \mathbb{C}^N \) be a set of complex i.i.d. observation vectors. Let \( G_C \) be the following set of functions \( G_C = \{ h : \mathbb{R}_+^N \to \mathbb{R}^+ \mid \int_0^\infty t^{N-1} h(t) dt < \infty, \int p_Z = 1 \} \) [9]. Moreover, we indicate with \( M_N^C \) the set of the Hermitian, positive definite, \( N \times N \) complex matrices.

Any CES-distributed random vector \( z_i = x_{R,i} + jx_{I,i} \sim CES(\mu, \Sigma, h) \) satisfies the properties ([9],[8, Sec. II]):

- \( z_i \in \mathbb{C}^N \) is CES distributed iff \( [x_{R,i}^T, x_{I,i}^T]^T \in \mathbb{R}^{2N} \) has a \( 2N \)-variate RES distribution,
- Its pdf \( p_Z \) is fully specified by the mean vector \( \mu \in \mathbb{C}^N \), by the scatter matrix \( \Sigma \in M_N^C \) and by the density generator \( h \in G_C \) and it can be expressed as:
  \[
  p_Z(z_i | \mu, \Sigma, h) = |\Sigma|^{-1/2} h((z_i - \mu)^H \Sigma^{-1} (z_i - \mu)).
  \]
  (58)
- **Stochastic representation:** \( z_i = d \mu + R \Sigma^{1/2} u_i \), where \( R \) is the modular variate and \( u_i \sim U(\mathbb{C}^N) \) is a complex random vector uniformly distributed on the unit N-sphere.
- The 2nd-order modular variate \( Q \triangleq R^2 \) is s.t.
  \[
  Q = d (z_i - \mu)^H \Sigma^{-1} (z_i - \mu) \triangleq Q_i, \forall i,
  \]
  and it admits a pdf \( p_Q \) of the form:
  \[
  p_Q(q) = \pi^N \Gamma(N)^{-1} q^{N-1} h(q).
  \]
  (60)

Exactly as for the real-valued case, the complex scatter matrix \( \Sigma \) is not identifiable and only a scaled version of it can be estimated. Then, the shape matrix \( V \triangleq \Sigma/s(\Sigma) \) has to be introduced, where \( s(\cdot) \) is a scalar functional on \( M_N^C \) satisfying conditions A1, A3 and A3 given in Sec. II. As for the real case, among all the possible scale functionals, we choose \( s(\Sigma) = |\Sigma|_{1,1} \) for simplicity.

At first, we need to define the unknown complex-valued parameter vector \( \phi \) to be estimated. As shown in [8] and in analogy with the real-valued case, the estimation of the mean vector and of the shape matrix are asymptotically decorrelated. Consequently, we focus only on the complex shape matrix estimation from the “centered” data set \( \{z_i - \mu\}_{i=1}^N \), where \( \mu^* \) is any \( \sqrt{N} \)-consistent estimator of \( \mu \in \mathbb{C}^N \). According to the basics of the Wirtinger calculus, \( \phi \) has to be constructed stacking in a single vector the unknown parameters and their complex conjugate [48,52]. Then, according to the detailed discussion provided in [8, Sec. III.A] and [52, Sec. 6.5.5], we have that

\[
\phi = \text{vec}(V_1).
\]
  (61)

As shown in Theorem 2, the basic building blocks for a semiparametric efficient estimators are the semiparametric efficient score vector \( \bar{s}_\phi \equiv \bar{s}_{\text{vec}(V_1)} \) and the efficient SFIM \( I(\text{vec}(V_1)|h_0) \). Both \( \bar{s}_{\text{vec}(V_1)} \) and \( I(\text{vec}(V_1)|h_0) \) have been already introduced in full details in our previous work [8] and their expressions are recalled here for clarity. Let us start by defining the following matrices:

\[
P = [\mathbf{e}_2 | \mathbf{e}_3 | \cdots | \mathbf{e}_{N^2}], \quad (62)
\]

where \( e_i \) is the \( i \)-th vector of the canonical basis of \( \mathbb{R}^{N^2} \),

\[
L_{V_1} \triangleq P \left( \mathbf{V}_1^{-T/2} \otimes \mathbf{V}_1^{-1/2} \right) \Pi_{-\text{vec}(I_N)}^+,
\]

(63)

and \( \Pi_{-\text{vec}(I_N)}^+ \) has already been defined in (36). Then, from the calculation in [8, Sec. III.B],4 using some matrix algebra, we obtain the following expression for the complex efficient semiparametric score vector

\[
\bar{s}_{\text{vec}(V_1)} = -Q_1 \psi_0(Q_1) L_{V_1} \text{vec}(u_i u_i^H),
\]

(64)

where \( \psi_0(t) \triangleq \frac{1}{h_0(t)} \frac{dh_0(t)}{dt} \) and

\[
u_i \triangleq V_1^{-1/2}(z_i - \mu),
\]

(65)

and \( Q_1 \) has been defined in (59). Note that the function \( \psi_0 \) here is defined by means of the true density generator \( h_0 \) related to the CES pdf in (58). Moreover, from [8, eq. (29)]:

\[
I(\text{vec}(V_1)|h_0) = \frac{E \{ Q^2 \psi_0(Q)^2 \}}{N(N + 1)} L_{V_1} L_{V_1}^H.
\]

(66)

It is worth to underline that the matrix (62) has been introduce in order to take into account the fact that the first top-left entry of \( V_1 \), i.e. \( [V_1]_{1,1} = 1 \), and it does not has to be estimated.

**B. A complex-valued R-estimator**

The derivation of the complex-valued R-estimator mimics the one proposed in Subsections IV-B, IV-C and IV-D for the real case. In particular, an approximation of the complex-valued efficient central sequence can be obtained as:

\[
\hat{\Delta}_{V_1} \triangleq -L^{-1/2} L_{V_1} \sum_{l=1}^L K_h \left( \frac{r_l}{L+1} \right) \text{vec}(u_i u_i^H),
\]

(67)

where \( \{r_l\}_{l=1}^L \) are the ranks of the \( L \), positive and real-valued random variables \( \{Q_l\}_{l=1}^L \) defined in (59) and related to the i.i.d. CES-distributed observations \( \{z_i\}_{i=1}^N \). Moreover, the score function \( K_h(\cdot) \) is the “complex” counterpart of the one defined in (46). Specifically, \( K_h(\cdot) \) can be obtained from the expression (46) by evaluating \( F_Q^{-1}(\cdot) \) and \( \psi_0 \) by means of an assumed, and possibly misspecified, \( h \in G_C \) instead of its real counterpart \( g \in G \). For example, the “complex” version of the van der Waerden score function in (51) can be obtained from (46) by noticing that the complex circular Gaussian distribution has a density generator given by \( h_{CG}(t) = \exp(-t) \) while \( Q \sim \text{Gamma}(N, 1) \) [9]. Consequently, the “complex” van der Waerden score function is:

\[
K_{CG}(u) \triangleq -\Phi^{-1}_G(u),
\]

(68)

4Not that in [8, eq. (25)] there is a typo. In fact, a minus “−” is missing in front of the right-hand side.
where $\Phi^{-1}$ indicates the inverse function of the cdf of a Gamma-distributed random variable with parameters $(N, 1)$.

The complex-valued version of the matrix $\tilde{\mathbf{Y}}$ in (55) can be obtained as:

$$\tilde{\mathbf{Y}}_C \triangleq \hat{\alpha}_C \mathbf{L}_{\tilde{\mathbf{Y}}_1} \mathbf{L}^H_{\tilde{\mathbf{Y}}_1},$$

where:

$$\hat{\alpha}_C = \frac{||\hat{\Delta}^{C}_{\tilde{\mathbf{Y}}_1} + L^{-1/2} \mathbf{H}^{E1} - \hat{\Delta}^{C}_{\tilde{\mathbf{Y}}_1}||}{||\mathbf{L}_{\tilde{\mathbf{Y}}_1} \mathbf{L}^H_{\tilde{\mathbf{Y}}_1} \vec{\mathbf{c}}(\mathbf{H}^{C2})||},$$

$\hat{\Delta}^{C}_{\tilde{\mathbf{Y}}_1}$ is any $\sqrt{L}$-consistent estimator of the (complex-valued) shape matrix and $\mathbf{H}^{C2}$ is a “small perturbation”, Hermitian, matrix s. t. $[\mathbf{H}^{C2}]_{1,1} = 0$. Finally, putting together the previous results, the complex extension of the, “one-step”, $R$-estimator in (56) can be obtained as:

$$\vec{\mathbf{V}}_{1,R} = \vec{\mathbf{V}}_{1} + L^{-1/2} \hat{\Gamma}_C^{-1} \hat{\Delta}^{C}_{\tilde{\mathbf{Y}}_1},$$

In the following, the pseudocode to implement the proposed $R$-estimator is reported, while its related Matlab code can be found at [40]. A good preliminary estimator of the constrained, complex-valued shape matrix, may be Tyler’s estimator $\hat{\mathbf{V}}_{1} = \hat{\mathbf{V}}_{1,T,y}$. We analyze two different cases:

1. The true RES distribution is a $t$-distribution,
2. The true CES distribution is a (complex) Generalized Gaussian (GG) distribution.

\[6.1\] Numerical Analisys

In this section, we investigate the semiparametric efficiency of the $R$-estimator given in (57) for the RES data case and in (71) for the CES one in a finite-sample regime. In particular, the Mean Squared Error (MSE) of $\hat{\mathbf{V}}_{1,R}$ will be compared with the one of the constrained Sample Covariance Matrix (CSCM) and of the constrained Tyler’s (C-Tyler) estimator. For the sake of consistency with the SP literature on scatter matrix estimation, in the figures, we re-normalized $\hat{\mathbf{V}}_{1,R}$ in order to have $\text{tr}(\hat{\mathbf{V}}_{1,R}) = N$. According to the discussion on Sec. II, we can define the re-scaled estimator as:

$$\hat{\mathbf{V}}_{R} \triangleq N\hat{\mathbf{V}}_{1,R}/\text{tr}(\hat{\mathbf{V}}_{1,R}).$$

Plotting the MSE of this re-scaled estimator will allow us to underline the fact that the semiparametric efficiency property of the derived $R$-estimator does not depend on the particular scale functional adopted. As a reference, in the figures we also report the Constrained Semiparametric CRB (CSCRB) derived in [7] for the RES data case and in [8] for the CES one along with the classical Constrained CRB (CCRB). As performance index for the shape matrix estimators, we use

$$\varepsilon_{\gamma} \triangleq ||E\{\vec{\mathbf{V}}_{\gamma} - \mathbf{V}_0\}||_F,$$

for the real and complex case respectively, where $\gamma = \{\text{CSCM, C-Tyler}; R\}$ indicates the particular estimator under test. Similarly, as performance bounds, we adopt the indices:

$$\varepsilon_{\text{CSCRB}} \triangleq ||[\text{CSCRB}(\Sigma_0)]||_F,$$

$$\varepsilon_{\text{CCRB}} \triangleq ||[\text{CCRB}(\Sigma_0, g_0)]||_F.$$
the data are highly non-Gaussian while, as $\lambda \to \infty$, the distribution collapses into the Gaussian one. The simulation parameters for this study case are:

- $\Sigma_0|_{i,j} = \rho^{|i-j|}, i,j = 1, \ldots, N; \rho = 0.8$ and $N = 8$.
- The “small perturbation” matrix $H^0$ is chosen to be a symmetric random matrix s.t. $H^0 = (G + G^T)/2$ where $[G|_{i,j}] \sim N(0,v^2)$, $[G|_{1,1} = 0$ and $v = 0.01$. Note that $v$ should be small enough to guarantee that $\bar{V}_i^* + L^{-1/2}H^0 \in M^R_N$.
- The van der Waerden score $K_{CG}$ in (51) is used to build the R-estimator.

As shown in Fig. 1 the R-estimator performs better than the SCM and the Tyler’s one for every value of the shape parameter $\lambda \in (2, \infty)$. Moreover, $\lambda \to \infty$ the MSE of the R-estimator tends to be equal to the one of the SCM, as expected. We can also see that the R-estimator is (almost) efficient even in the “finite sample” regime ($L = 5N$). It is worth to underline that the semiparametric efficiency property is preserved even after the re-normalization in (74). This suggests us that the choice of the particular scale functional in (5) does not have any impact on the optimality properties of the R-estimator. Clearly, the classical parametric CCRB remains unreachable. In fact, to derive efficient estimators w.r.t. the CCRB, the exact knowledge of the true density generator $g_0$ should be available.

B. Case 2: the complex Generalized Gaussian distribution

The density generator of the complex Generalized Gaussian (GG) distribution is:

$$h_0(t) = \frac{s}{\pi N \Gamma(N/s)} \exp(-t^2/b^2), \quad t \in \mathbb{R}^+ \tag{80}$$

and, according to the value of the shape parameter $s > 0$, it can model a distribution with both heavier tails ($0 < s < 1$) and lighter tails ($s > 1$) compared to the Gaussian distribution ($s = 1$). The simulation parameters for this study case are:

- $\Sigma_0$ is a Toeplitz Hermitian matrix whose first column is given by $[1, \rho, \ldots, \rho^{N-1}]^T$; $\rho = 0.8e^{i2\pi/5}$ and $N = 8$.
- The “small perturbation” matrix $H^0_C$ is chosen to be a symmetric random matrix s.t. $H^0_C = (G_G + G^H_G)/2$ where $[G_G|_{i,j} \sim CN(0,v^2), [G_G|_{1,1} = 0$ and $v = 0.01$. As for the real-case, note that $v$ has to be small enough to guarantee that $\bar{V}_i^* + L^{-1/2}H^0_\mathbb{C} \in M^C_N$.
- The “complex” van der Waerden score $K_{CG}$ in (68) is used to build the R-estimator.

The simulation results related to the complex GG-distributed data proposed in Fig. 2 confirm all the observation already done for the real-valued $t$-distributed one. The only difference here is that the R-estimator experiences some efficiency loss in the sub-Gaussian case, i.e. for $s > 1$. However, it still has better performance w.r.t. the Tyler’s estimator for every values of the shape parameter $s$.

A last comment on the “small perturbation” matrix $H^0$ is in order now. The theory, in fact, does not provide us with any hint for the optimal selection of this hyper-parameter. Fortunately, simulation results seems to suggest that the R-estimator is quite robust w.r.t. the choice of $H^0$ for various density generators and various levels of non-Gaussianity. However, the choice of $H^0$ could be sensible to the data dimension $N$ and to the number of observations $L$. We left to future works an in-depth and analysis of this importation aspect.

VII. CONCLUSIONS

In this paper, a robust and semiparametric efficient R-estimator of the shape matrix in Real and Complex ES distributions has been discussed and analyzed. This estimator has been firstly proposed by Hallin, Oja and Paindaveine in their seminal paper [12] where the Le Cam’s theory of Local Asymptotic Normality and the properties of rank-based statistics have been exploited as basic building blocks for its derivation. In the first part of this paper, a survey of the main statistical concepts underlying such R-estimator has been proposed for the case of RES-distributed data. Then, its extension to CES distributions has been derived by means of the Wirtinger calculus. Finally, the finite-sample performance of the R-estimator has been investigated in different scenarios and its MSE compared with the SCRB. However, a number of fundamental issues still remain to be fully addressed. In our opinion, the most important one is related to the estimation of $\alpha_{0,y}$ in (53). The estimator in (54) in fact is consistent but it does not satisfy any optimal property. Moreover, it depends on an hyper-parameter, i.e. the “small perturbation” matrix $H^0_C$ (or $H^0_\mathbb{C}$ in the complex-valued case), that has to be defined by the user in an heuristic way and, currently, without any theoretical guidelines. A possible improvement w.r.t. the estimator in (54) is discussed in [12, Sec. 4.2] and it will be the subject of future works. Other important open questions are related to the asymptotic distribution of the proposed R-estimator and to its robustness to outliers, i.e. to the presence of non-elliptically distributed observations.

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Fig. 1: MSE indices for the shape matrix estimators and lower bounds as function of $\lambda$ for $t$-distributed data ($L = 5N$).
Fig. 2: MSE indices for the shape matrix estimators and lower bounds as function of $s$ for $GG$-distributed data ($L = 5N$).