Equivalent definitions of Arthur packets for real quasisplit unitary groups

N. Arancibia Robert, P. Mezo

April 25, 2022

Abstract

Mok has defined Arthur packets for quasisplit unitary groups. His definition follows Arthur’s work on classical groups and relies on harmonic analysis. For real groups there is an alternative definition of Arthur packets, due to Adams-Barbasch-Vogan. It relies on sheaf-theoretic techniques instead of harmonic analysis. We prove that these two definitions of Arthur packets are equivalent in the case of real quasisplit unitary groups.

1 Introduction

In an effort to characterize the automorphic spectrum of a connected reductive group, Arthur introduced a set of parameters together with a collection of conjectures concerning them ([A1], [A2]). The parameters are commonly called Arthur-parameters or simply A-parameters. These A-parameters are global objects. Conjecturally, each global A-parameter gives rise to a local A-parameter for every valuation of the underlying field.

In the present work, we study local A-parameters only for a real valuation. Moreover, our main theorem concerns only real quasisplit unitary groups. A real quasisplit unitary group is a real form of a general linear group, which we denote here as $G(\mathbb{R})$. An A-parameter for $G$ is a group homomorphism $\psi_G : W_{\mathbb{R}} \times SL_2 \rightarrow \mathcal{V} G^\Gamma$ in which, $\Gamma = Gal(\mathbb{C}/\mathbb{R})$, $\mathcal{V} G^\Gamma = \mathcal{V} G \times \Gamma$ is the Galois form of the L-group of $G$, and $W_{\mathbb{R}}$ is the real Weil group. In addition, $\psi_G |_{W_{\mathbb{R}}}$ is a tempered L-parameter and $\psi_G |_{SL_2}$ is algebraic (see [B] for the definitions of these terms). Arthur conjectured the existence of a stable virtual character $\eta_{\psi_G}$ of $G(\mathbb{R})$ with several properties. The set of irreducible characters appearing in $\eta_{\psi_G}$ with non-zero multiplicity are, by definition, the A-packet $\Pi_{\psi_G}$, and Arthur conjectured them to be characters of irreducible unitary representations.

In this limited scope, Arthur’s conjectures have been proven to a great extent. However, there have been two seemingly disparate methods employed, and it has been unknown whether the two methods lead to the same conclusions. The first method relies on global harmonic analysis ([R], [A3], [KMSW], [M3]). We denote the stable virtual character defined by Mok in [M3] by $\eta_{\psi_G}^{Mok}$. The second method is local and relies on sheaf theory ([ABV]). We denote the stable virtual character defined by Adams, Barbasch and Vogan in [ABV] by $\eta_{\psi_G}^{ABV}$. Our main theorem is

$$\eta_{\psi_G}^{Mok} = \eta_{\psi_G}^{ABV}$$

(Theorem 9.3). An immediate consequence of this theorem is that the sets of irreducible characters appearing in each of the virtual characters are the same, that is

$$\Pi_{\psi_G}^{Mok} = \Pi_{\psi_G}^{ABV}.$$ 

The irreducible characters in $\Pi_{\psi_G}^{Mok}$ are all unitary ([M3, Theorem 3.2.1 (b)]), whereas this was not known for $\Pi_{\psi_G}^{ABV}$.

Further conjectures of Arthur which are consolidated by the main theorem are presented in Section 10.
The principal difficulty in proving (2) lies in the disparate manner in which the two virtual characters are defined. Let us examine the definitions beginning with the virtual character $\eta_{\psi, G}^{Mok}$. The main idea here is to express the unitary group $G$ as a twisted endoscopic group of a pair $(R_\mathbb{C}/R \text{GL}_N, \vartheta)$. In this pair $R_\mathbb{C}/R \text{GL}_N$ is the algebraic group obtained from the general linear group $\text{GL}_N$ by restriction of scalars from $\mathbb{C}$ to $\mathbb{R}$ ([S6, Proposition 11.4.22], [B, I.5]). It may be regarded as $\text{GL}_N \times \text{GL}_N$ together with a real structure whose real points determine the real form $\text{GL}_N(\mathbb{C})$. The other member of the pair is an automorphism $\vartheta$ of $R_\mathbb{C}/R \text{GL}_N$. It is defined by

$$\vartheta(g_1, g_2) = (\tilde{J}(g_2^{-1})^T \tilde{J}^{-1}, \tilde{J}(g_1^{-1})^T \tilde{J}^{-1}), \quad g_1, g_2 \in \text{GL}_N,$$

where $\tilde{J}$ is the anti-diagonal matrix

$$\tilde{J} = \begin{bmatrix}
0 & -1 \\
-1 & 0 \\
0 & \cdots & 0
\end{bmatrix}.$$  

Clearly, $\vartheta$ is an automorphism of order two, and as a real form, the semidirect product $\text{GL}_N(\mathbb{C}) \rtimes (\vartheta)$ is a disconnected algebraic group. The group $G$ is attached to the pair $(R_\mathbb{C}/R \text{GL}_N, \vartheta)$ in that $^\vartheta G$ is isomorphic to the identity component of the fixed-point set $(^\vartheta R_\mathbb{C}/R \text{GL}_N)^\vartheta$. This furnishes an inclusion

$$\epsilon : ^\vartheta G^\Gamma \hookrightarrow ^\vartheta R_\mathbb{C}/R \text{GL}_N^\Gamma$$

which allows us to define the A-parameter

$$\psi = \epsilon \circ \psi_G$$

for $R_\mathbb{C}/R \text{GL}_N$ using (1). Since the real form $\text{GL}_N(\mathbb{C})$ is particularly well-understood, there is an obvious choice of stable virtual character $\eta_{\psi, G}^{Mok}$. It is in fact a single irreducible character

$$\eta_{\psi, G}^{Mok} = \pi_\psi$$

of $\text{GL}_N(\mathbb{C})$. Furthermore, as a representation, $\pi_\psi$ is stable under composition with $\vartheta$. This allows one to extend $\pi_\psi$ to a representation $\pi_\psi^\sim$ of the disconnected group $R_\mathbb{C}/R \text{GL}_N(\mathbb{C})$. At this stage some care must be taken, as the extension is only unique up to a sign. If we ignore this wrinkle for the time being, then we obtain the twisted character $\pi_\psi^\sim$ as the restriction of the character of $\pi_\psi^\sim$ to the non-identity component $\text{GL}_N(\mathbb{C}) \rtimes \vartheta$. If we identify $\pi_\psi^\sim$ with its twisted character then the stable virtual character $\eta_{\psi, G}^{Mok}$ is uniquely determined by the twisted endoscopic transfer identity

$$\pi_\psi^\sim = \text{Trans}_{G_\mathbb{R}(G)}^{\text{GL}_N(\mathbb{C}) \times \vartheta}((\eta_{\psi, G}^{Mok}))$$

([M3, Proposition 8.2.1]). The twisted endoscopic transfer map $\text{Trans}_{G_\mathbb{R}(G)}^{\text{GL}_N(\mathbb{C}) \times \vartheta}$ is defined on the space of stable virtual characters of $G(\mathbb{R})$. It is defined for real reductive groups in [S4], [M1] and [M2].

Turning now to the definition of $\eta_{\psi, G}^{ABV}$, we come upon completely different methods. A remarkable innovation of Adams, Barbasch and Vogan is their introduction of a complex variety $X(^\vartheta G^\Gamma)$, together with a $^\vartheta G$-action, such that the $^\vartheta G$-orbits are in bijection with the equivalence classes of L-parameters ([ABV, Section 6]). The orbits stratify the variety. Thus, one may consider $^\vartheta G$-equivariant local systems on the orbits, and $^\vartheta G$-equivariant constructible sheaves or perverse sheaves on $X(^\vartheta G)$. We define a complete geometric parameter to be a pair

$$\xi = (S, \mathcal{V})$$

consisting of an orbit $S \subset X(^\vartheta G^\Gamma)$, together with a $^\vartheta G$-equivariant local system $\mathcal{V}$ on $S$ ([ABV, Definition 7.6]). The set of complete geometric parameters is denoted by $\Xi(^\vartheta G^\Gamma)$. This definition ignores the more general local systems in [ABV], which are equivariant for an algebraic cover of $^\vartheta G$. By [ABV, Theorem 10.11] there is a canonical bijection

$$\Xi(^\vartheta G^\Gamma) \longleftrightarrow \Pi(G/\mathbb{R})$$

(7)
which may be regarded as a more precise version of the local Langlands correspondence. The set on the right is the set of (equivalence classes of) irreducible representations of pure forms of $G$. The pure forms include the quasisplit form $G(\mathbb{R})$. We write bijection (7) as

$$
\xi \mapsto \pi(\xi).
$$

Let $K\Pi(G/\mathbb{R})$ be the Grothendieck group of the finite-length representations appearing in $\Pi(G/\mathbb{R})$. This Grothendieck group has $\{\pi(\xi)\}$ as a $\mathbb{Z}$-basis. It contains the virtual characters of $G(\mathbb{R})$ as a subgroup.

There is a similar picture for sheaves for the dual group $^\vee G$. Suppose $\xi = (S, \psi) \in \Xi(\mathcal{Y}G\Gamma)$. The local system $\mathcal{Y}$ is a $^\vee G$-equivariant sheaf on $S \subset X(\mathcal{Y}G\Gamma)$. Applying the functor of intermediate extension to the closure of $S$, and then taking the direct image to $X(\mathcal{Y}G)$ produces an irreducible $^\vee G$-equivariant perverse sheaf $\mathcal{P}(\xi)$, and a bijection

$$
\xi \mapsto \mathcal{P}(\xi)
$$

([ABV, Section 7]). Let $KX(\mathcal{Y}G\Gamma)$ be the Grothendieck group of the category of $^\vee G$-equivariant perverse sheaves on $X(\mathcal{Y}G\Gamma)$. It has $\{\mathcal{P}(\xi)\}$ as a $\mathbb{Z}$-basis.

There is a perfect pairing

$$
\langle \cdot, \cdot \rangle_G : K\Pi(G/\mathbb{R}) \times KX(\mathcal{Y}G\Gamma) \to \mathbb{Z}
$$

which satisfies

$$
\langle \pi(\xi), \mathcal{P}(\xi') \rangle_G = e(\xi) (-1)^{d(\xi)} \delta_{\xi, \xi'}, \xi, \xi' \in \Xi(\mathcal{Y}G\Gamma). (10)
$$

([ABV, Theorem 1.24]). Here, $d(\xi)$ is the dimension of the orbit $S$ in $\xi = (S, \psi)$, $e(\xi)$ is the Kottwitz sign ([ABV, Definition 15.8]), and $\delta_{\xi, \xi'}$ is the Kronecker delta.

Using pairing (9) we shall identify virtual characters as $\mathbb{Z}$-valued linear functionals on $KX(\mathcal{Y}G\Gamma)$. The theory of microlocal geometry provides a family of linear functionals

$$
\chi_{\mathcal{S}}^{\text{mic}} : KX(\mathcal{Y}G\Gamma) \to \mathbb{Z}
$$

parameterized by the $^\vee G$-orbits $S \subset X(\mathcal{Y}G\Gamma)$. These microlocal multiplicity maps appear in the theory of characteristic cycles ([ABV, Chapter 19], [BGK+]), and are associated with $^\vee G$-equivariant local systems on a conormal bundle over $X(\mathcal{Y}G\Gamma)$ ([ABV, Section 24], [GM]). The virtual characters associated by the pairing to these linear functionals are stable ([ABV, Theorems 1.29 and 1.31]).

The stable virtual character $\eta^{\text{ABV}}_{\psi_G}$ is defined from (11) as follows. There is an L-parameter $\phi_{\psi_G}$ associated to $\psi_G$ ([ABV, Definition 22.4]). Let $S_{\psi_G} \subset X(\mathcal{Y}G\Gamma)$ be the unique $^\vee G$-orbit associated to $\phi_{\psi_G}$, and $\eta^{\text{mic}}_{\psi_G}$ be the unique virtual character satisfying

$$
\langle \eta^{\text{mic}}_{\psi_G}, \mu \rangle_G = \chi_{\mathcal{S}_{\psi_G}}^{\text{mic}}(\mu), \mu \in KX(\mathcal{Y}G\Gamma).
$$

As a distribution, the stable virtual character $\eta^{\text{mic}}_{\psi_G}$ is supported on real forms of $G$ which include the quasisplit form $G(\mathbb{R})$. For our purposes, it suffices to consider the restriction

$$
\eta^{\text{ABV}}_{\psi_G} = \eta^{\text{mic}}_{\psi_G} |_{G(\mathbb{R})}
$$

(12)

to the quasisplit form.

The definitions of $\eta^{\text{Mok}}_{\psi_G}$ and $\eta^{\text{ABV}}_{\psi_G}$ are encapsulated in (5) and (12) respectively. One might hope to find a bridge between the two definitions by working in a theory of twisted endoscopy for $\eta^{\text{ABV}}_{\psi_G}$. Fortunately, the theory of standard endoscopy already appears in [ABV] and is extended to the twisted setting in [CM]. There are two main tasks in this extension.

The first task is the definition of a meaningful pairing between the $\mathbb{Z}$-modules of twisted characters $K\Pi(\mathcal{GL}_N(\mathbb{C}), \vartheta)$ and twisted sheaves $KX(\mathcal{Y}R_{G/\mathbb{R}}\mathcal{GL}_N^\Gamma, \vartheta)$ ([LV, Section 2.3])

$$
\langle \cdot, \cdot \rangle : K\Pi(\mathcal{GL}_N(\mathbb{C}), \vartheta) \times KX(\mathcal{Y}\mathcal{GL}_N^\Gamma, \vartheta) \to \mathbb{Z}.
$$

(13)

This is a serious task to which we will return later in the introduction.
One of the favourable features of the Atlas parameters in this special context is the existence of $\mathbb{Z}$ and provides the construction for the related concept of the $\mathbb{Z}$-module of twisted characters of $GL_N(\mathbb{C}) \rtimes \langle \vartheta \rangle$. This is not that serious, for there is an inverse image functor on sheaves

$$L \rightarrow \text{Trans}_{G(\mathbb{R})}^{\mathbb{C}} \vartheta$$

which allows us to define $\text{Lift}_0$ by the identity

$$\langle \text{Lift}_0(\eta), \mu \rangle = \langle \eta, \varepsilon^* (\mu) \rangle_G, \quad \mu \in KX(\mathbb{C}^G)$$

In this identity both pairings (13) and (9) are used.

The equality between $\eta^{\text{Mok}}_{\vartheta}$ and $\eta^{\text{ABV}}_{\vartheta}$ may then be established by returning to (5), proving

$$\text{Lift}_0 = \text{Trans}_{G(\mathbb{R})}^{GL_N(\mathbb{C}) \rtimes \vartheta}$$

and

$$\text{Lift}_0(\eta^{\text{ABV}}_{\vartheta}) = \pi^{\vartheta}_{\vartheta}$$

Equation (17) ends up being simple consequence of the definition of $\text{Lift}_0$ and [AMR1, (1.0.3)]. Equation (18) is to a large extent proven in [ABV, Theorem 26.25]. From these two equations it follows that

$$\text{Lift}_0(\eta^{\text{Mok}}_{\vartheta}) = \text{Lift}_0(\eta^{\text{ABV}}_{\vartheta})$$

and then an injectivity result yields the main theorem (2).

The proof we have sketched follows [AAM] entirely. The classical groups in [AAM] are twisted endoscopic groups of $GL_N$. In the present work the classical groups are replaced by unitary groups, and $GL_N$ is replaced by $R_{\mathbb{C}/\mathbb{R}}GL_N$. The good properties of $GL_N$ which were harnessed in [AAM] (e.g. connected centralizers) are also properties of $R_{\mathbb{C}/\mathbb{R}}GL_N$. In fact, both the structure and the representation theory of $GL_N(\mathbb{C}) = R_{\mathbb{C}/\mathbb{R}}GL_N(\mathbb{R})$ are simpler than for $GL_N(\mathbb{R})$. We have made efforts to highlight the simplifications. Where we have not been able to improve on preliminary material, we have copied passages from [AAM].

We now give the main points of each section. Section 2.1 is a review of the preliminary material of [ABV]. Two important objects appearing here are the extended group $G^\Gamma$ which mirrors the L-group $G^\Gamma$ and the complex variety $X' G^\Gamma$. The term $\mathcal{O}$ is a semisimple $G$-orbit in the complex Lie algebra $\mathfrak{g}$, and is to be thought of as an infinitesimal character. This infinitesimal character accompanies all of the arguments in the sequel and for the most part is assumed to be regular. The notions of pure inner form and pure strong involution are also introduced and compared. The section culminates with a description of the revised local Langlands correspondence (7).

In Section 2.2 the specifics of some of this preliminary material are given for quasisplit unitary groups and $R_{\mathbb{C}/\mathbb{R}}GL_N$. In particular it is proven that $R_{\mathbb{C}/\mathbb{R}}GL_N$ has only one pure strong involution which corresponds to the sole inner form, namely $GL_N(\mathbb{C})$.

Section 2.3 presents an alternative set of parameters to the complete geometric parameters (6). They are the parameters introduced in [AdC] and [AV], and so we call them Atlas parameters. The Atlas parameters are more convenient for the computations appearing in later sections. The Atlas parameters also have an easily discernible involution which pertains to Vogan duality, a tool used later as well.

A third advantage to the Atlas parameters is that they may be extended to provide a parameterization for the irreducible representations of $GL_N(\mathbb{C}) \rtimes \langle \vartheta \rangle$. This is the subject of Section 2.4. One of the favourable features of the Atlas parameters in this special context is the existence of a preferred extension $\pi(\xi)^{\vartheta}$ to $GL_N(\mathbb{C}) \rtimes \langle \vartheta \rangle$ of any $\vartheta$-stable irreducible representation $\pi(\xi)$ of $GL_N(\mathbb{C})$. We refer to this preferred extension as the Atlas extension of $\pi(\xi)$.

Section 2.5 lays out the notation for the Grothendieck group of admissible representations and provides the construction for the related concept of the $\mathbb{Z}$-module of twisted characters of $GL_N(\mathbb{C}) \rtimes \langle \vartheta \rangle$. The second task is to define a twisted endoscopic lifting map

$$\text{Lift}_0 : KII(G(\mathbb{R}))^{\vartheta} \rightarrow K(GL_N(\mathbb{C}), \vartheta)$$

from stable virtual characters to twisted characters, i.e. the counterpart of $\text{Trans}_{G(\mathbb{R})}^{GL_N(\mathbb{C}) \rtimes \vartheta}$ in (5).
Section 3 is devoted to the pairings, (9) and (13), and the definitions of the virtual characters, $\eta_{\nu_G}$ and $\nu_{\nu}$, which are defined through them. The values of the ordinary pairing (9) were given in (10) for irreducible representations and perverse sheaves. It is equally important to understand the values of this pairing on standard representations and irreducible constructible sheaves. Before saying why, we recall that any irreducible representation $\pi(\xi)$ in (8) is the unique Langlands quotient of a standard representation, which we denote by $M(\xi)$. If the parameter $\xi$ is $\vartheta$-stable then there is an Atlas extension $M(\xi)^{+}$ which contains $\pi(\xi)^{+}$ as a quotient. On the other hand, if one replaces the intermediate extension with extension by zero in the construction of $P(\xi)$ above, then one arrives at an irreducible $^G\!G$-equivariant constructible sheaf $\mu(\xi)$. The Grothendieck group of the $^G\!G$-equivariant constructible sheaves on $X(\mathcal{O}$, $G)$ is isomorphic to the Grothendieck group $K_{X}(\mathcal{O}$, $G)$ for the perverse sheaves ([ABV, Lemma 7.8]). It therefore makes sense to evaluate the pairing on these two objects. As a matter of fact pairing (9) is defined by

$\langle M(\xi), \mu(\xi') \rangle = e(\xi) \delta_{\xi, \xi'}, \ \xi, \xi' \in \Xi(\mathcal{O}, ^G\!G)$

and the content of [ABV, Theorem 1.24] is (10). It is important to know the values of the pairing on these objects, since there is a well-known basis for the stable virtual characters (cf. (14)) given in terms of standard representations ([S3]). In addition, the inverse image functor (15) is computed relative to constructible sheaves.

Much of the technical work in this paper is spent on defining the twisted pairing (13) and proving that its values on standard representations and constructible sheaves are related to its values on irreducible representations and perverse sheaves as in the ordinary case. The first undertaking is to define preferred $(^G\!R_{C/\mathbb{R}}GL_{N} \times \{\vartheta\})$-equivariant sheaves $\mu(\xi)^{+}$ and $P(\xi)^{+}$ which restrict to $\rho(\xi)$ and $P(\xi)$ respectively as $^G\!R_{C/\mathbb{R}}GL_{N}$-equivariant sheaves. These are the twisted sheaves referred to above. There is a $\mathbb{Z}$-module $K(X(\mathcal{O}$, $^G\!R_{C/\mathbb{R}}GL_{N}), \vartheta)$ of twisted sheaves akin to the module of twisted characters $KH((\mathcal{O}$, $GL_{N})\mathbb{C}, \vartheta)$. We then define the twisted pairing (13) by setting

$\langle M(\xi)^{+}, \mu(\xi')^{+} \rangle = (-1)^{i(\xi)-i(\xi')} \delta_{\xi, \xi'}.$

(19)

The definition of the signs on the right appear in (57) and (58). As shall be seen soon, these signs are crucial in making comparisons with other extensions $M(\xi)^{\sim}$ and $\pi(\xi)^{\sim}$. The principal result pertaining to the twisted pairing is

$\langle \pi^{\sim}, P(\xi')^{+} \rangle = (-1)^{d(\xi)}(-1)^{i(\xi)} \delta_{\xi, \xi'}.$

(20)

(Section 3.5).

The sole objective of Section 4 is to prove (20). Our proof is an adaptation of the proof of the equivalence for ordinary pairings ([ABV, Sections 16-17]) using the tools of [AV]. Hecke operators are among these tools. There is a difference between [ABV] and [AV] in the objects upon which the Hecke operators act. In [AV] Hecke operators are defined on both characters and sheaves. By contrast, the Hecke operators of [AV, Section 7] are defined only on (twisted) characters. The links between characters and sheaves in the Hecke actions are the Riemann-Hilbert and Beilinson-Bernstein correspondences ([ABV, Theorems 7.9 and 8.3]). In Section 4.1 we describe these correspondences as a bijection

$P(\xi) \leftrightarrow \pi(\xi), \ \xi \in \Xi(\mathcal{O}$, $G),$

where $\pi(\xi)$ is the Vogan dual of $\pi(\xi)$ (as the equivalence class of a Harish-Chandra module) (6.1 [AV]). For $R_{C/\mathbb{R}}GL_{N}$ the correspondence is extended to

$P(\xi)^{+} \leftrightarrow \pi(\xi)^{+}$

for $\vartheta$-fixed complete geometric parameters $\xi$. Once sheaves are aligned with characters in this manner, the rest of the proof of (20) follows [ABV] and [AAM, Section 4].

In Section 5 we describe the theory of endoscopy, both standard and twisted, for $R_{C/\mathbb{R}}GL_{N}$. The standard theory of endoscopy is included to motivate the twisted theory and is also used in Section 6. The twisted theory of endoscopy in Subsection 5.2 is a specialization of [CM, Section 5.4]. We compute the values of the twisted endoscopic lifting map $Lift_{0}$ on a basis of the stable
virtual characters. This is instrumental in proving (17) and in proving that \( \text{Lift}_0 \) is injective. The value \( \text{Lift}_0(\eta_{G,N}^{ABV}) \) is described as an element \( \eta_{G,N}^{ABV} \in K\Pi^{(\psi, \O, \text{GL}_N(\C), \vartheta)} \), which reduces to \( \eta_{G,N}^{ABV} \) upon restriction to \( \text{GL}_N(\C) \) (Theorem 5.6).

In Section 6 we prove that for any A-parameter \( \psi \) of \( \text{RC}/\R \text{GL}_N \) (not necessarily of the form (4)) \( \eta_{G,N}^{ABV} = \pi_\psi \). The proof begins under the assumption that \( \psi \) is an A-parameter studied by Adams and Johnson ([AJ]). Adams and Johnson defined A-packets for these parameters, and it is easily shown that their packets are singletons for \( \text{RC}/\R \text{GL}_N \). The equality of the Adams-Johnson packets with the ABV-packets is proven in [AR]. The proof that ABV-packets are singletons follows from a decomposition of \( \psi \) in terms of Adams-Johnson A-parameters of smaller general linear groups, and an application of standard endoscopic lifting from the direct product of these smaller general linear groups (Proposition 6.3).

In section 7 we describe another extension \( \pi(\xi)^- \) of the \( \vartheta \)-stable irreducible representations \( \pi(\xi) \) of \( \text{GL}_N(\C) \) – the so-called Whittaker extension. Regarded as a twisted character \( \pi(\xi)^- \) differs from the Atlas extension \( \pi(\xi)^+ \) by at most a sign. For special complete geometric parameters \( \xi \) it is shown that the Whittaker extension agrees with the Atlas extension. The key to determining the sign for other complete geometric parameters is to compute the sign when \( \pi(\xi) \) is generic, \textit{i.e.} has a Whittaker model. Indeed, the Whittaker extensions are built from extensions of generic representations and irreducible generic representations occur as subrepresentations of any standard representation. If one knows the (signed) multiplicity with which an irreducible twisted generic representation \( \pi(\xi_0)^+ \) appears in the decomposition of a twisted standard representation \( M(\xi)^+ \), then one can use this knowledge together with the agreement at the special complete geometric parameters to prove that

\[
\pi(\xi)^+ = (-1)^{l(\xi) - l_0(\xi)} \pi(\xi)^-.
\]

Observe that the sign on the right appears on the right of (19) and (20). Replacing \( \pi(\xi)^+ \) with \((-1)^{l(\xi) - l_0(\xi)} \pi(\xi)^+ \) yields a cosmetic simplification to the pairing. More importantly, it is the Whittaker extension which appears in (5). With the substitution of the Whittaker extensions into the computed values of \( \text{Lift}_0 \), the identities (17) and (18) are established.

These last observations are spelled out in Section 8. This short section assembles the essential ingredients already outlined in the introduction in proving the main theorem (2). However, it works on the assumption that the infinitesimal character is regular in \( \text{RC}/\R \text{GL}_N \). This assumption is removed in Section 9 by applying \textit{Jantzen-Zuckerman translation}. There is nothing novel in this approach and the ideas are all present in [ABV, Section 16].

In the final section we examine \textit{standard} endoscopy for \( G \). There are formulae given in both [M3] and [ABV] which describe standard endoscopic lifting in terms of representations appearing in the definition of \( \eta_{G,N}^{Mok} = \eta_{G,N}^{ABV} \). We prove that these representations are equal.

Let us briefly mention a different approach to obtaining the equality between the A-packets (2). Mœglin and Renard give an explicit description of the representations in \( \Pi_{\psi_G}^{Mok} \). More precisely, by [MgR, Equation (5.1)], \( \psi \) in (4) decomposes as \( \psi = \psi_0 \otimes \psi_1 \), where \( \psi_0 : \R \times \text{SL}_2 \rightarrow \text{RC}/\R \text{GL}_{N_i}^G, \quad i = 0, 1, \quad N_0 + N_1 = N. \)

The A-parameter \( \psi_0 \) corresponds to an irreducible unitary representation \( \pi_0 \) of \( \text{GL}_{N_i}(\C) \). In addition, the A-parameter \( \psi_1 \) factors to an A-parameter \( \psi_{G_1} \) of a smaller rank unitary group \( G_1 \)

\[
\psi_1 : \R \times \text{SL}_2 \xrightarrow{\psi_{G_1}} \text{GL}_{N_1}^G \quad \leftrightarrow \quad \text{RC}/\R \text{GL}_{N_1}^G.
\]

According to ([MgR, Proposition 5.2]), every irreducible representation in \( \Pi_{\psi_G}^{Mok} \) is parabolically induced from a representation in

\[
\left\{ \pi_0 \otimes \pi_1 : \pi_1 \in \Pi_{\psi_{G_1}}^{Mok} \right\}.
\]

Furthermore, each \( \pi_1 \in \Pi_{\psi_{G_1}}^{Mok} \) is cohomologically induced from a character of an inner form of \( G_1 \) determined by \( \psi_{G_1} \) ([MgR, Theorem 4.1]). One should also be able to prove that each representation in \( \Pi_{\psi_G}^{ABV} \) is parabolically induced from a representation in \( \left\{ \pi_0 \otimes \pi_1 : \pi_1 \in \Pi_{\psi_{G_1}}^{ABV} \right\} \) by imitating Proposition 5.1. This would reduce the proof of the equality of A-packets (2) to showing that
\( \Pi_{ABV} = \Pi_{Mok} \). The proof of this last equality should follow the proof of [AR, Theorem 4.16], which asserts the equality of ABV-packets and packets defined by Adams-Johnson ([AJ]). Although to do this, one must first extend the framework in [AR] to include unitary groups and \( A \)-parameters with singular infinitesimal character.

In closing, we note that the A-packets in [M3] are only defined for quasi-split unitary groups. In [MR] and [KMSW] the definition of Mok’s packets have been extended to include pure inner forms. The microlocal packets of [ABV] apply to any inner form. Hence, a subject for future research is to compare the A-packets of [MR], for pure inner forms of unitary groups, with the microlocal packets of [ABV].

2 The local Langlands correspondence

Unless otherwise stated, the group \( G \) in this section may be taken to be an arbitrary connected complex reductive algebraic group. Our goal is to review the local Langlands correspondence as developed in [ABV]. Our review differs in two ways from [ABV]. First, we replace the notion of strong real form with the equivalent notion of strong involution ([AdC]). Second, we limit the theory to the set of pure strong involutions (or equivalently pure strong real forms). This limitation simplifies the review, while retaining the necessary information for quasi-split forms of \( G \).

2.1 Extended groups and complete geometric parameters

The L-group of \( G \) is an essential feature of the local Langlands correspondence. The dual group \( \check{G} \) is an index two subgroup of (the Galois form of) the L-group of \( G \). One of the innovations in the local Langlands correspondence of [ABV] is the introduction of an extended group for \( G \), which mirrors the L-group in that \( G \), not \( \check{G} \), appears as an index two subgroup.

We begin the definition of an extended group for \( G \) by fixing a pinning \((B, H, \{X_\alpha\})\) in which \( B \subset G \) is a Borel subgroup, \( H \subset B \) is a maximal torus and \( \{X_\alpha\} \) is a set of simple root vectors relative to the positive root system \( R^+(G, H) = R(B, H) \) of \( R(G, H) \). We fix an inner class of real forms for \( G \), or equivalently, an algebraic involution \( \delta_0 \) of \( G \) fixing the pinning (Proposition 2.12 [ABV]). The (weak) extended group defined by the inner class is

\[
G^\Gamma = G \rtimes \langle \delta_0 \rangle
\]

([AdC, Definition 5.1], cf. [ABV, Definition 2.13]).

A strong involution is an element \( \delta \in G^\Gamma - G \) such that \( \delta^2 \in Z(G) \) is central and has finite order ([AdC, Definition 5.5]). Two strong involutions are equivalent if they are \( G \)-conjugate. There is a surjective map

\[
\delta \mapsto G(\mathbb{R}, \delta)
\]

from (equivalence classes of) strong involutions to (isomorphism classes of) real inner forms of \( G \) ([AdC, Lemma 5.7]). There is a well-known bijection between the real inner forms of \( G \) and \( H^1(\mathbb{R}, G/Z(G)) \) ([S6, 12.3.7]). Using this bijection one may also think of (21) as a surjection onto \( H^1(\mathbb{R}, G/Z(G)) \). It is natural to juxtapose this surjection with the the quotient map

\[
H^1(\mathbb{R}, G) \to H^1(\mathbb{R}, G/Z(G)).
\]

The cohomology set \( H^1(\mathbb{R}, G) \) is known as the set of pure inner forms ([V5, Section 2]). The pure inner forms may be realized as strong involutions in the following fashion. Let \( \sigma \in \Gamma \) be the nontrivial element of the Galois group. Set

\[
\vee \rho = \frac{1}{2} \sum_{\alpha \in R^+(G, H)} \vee \alpha.
\]

To any 1-cocycle \( z \in Z^1(\mathbb{R}, G) \), one may attach the strong involution

\[
z(\sigma) \exp(\pi i \vee \rho) \delta_0 \in G^\Gamma
\]
(exp(πiνρ)δ₀ is the large involution in [AV, (11f)-(11h)]). This assignment sends classes in H₁(R, G) to equivalence classes of strong involutions. The (equivalence classes of) pure strong involutions are defined as the image of this map.

The quasisplit real form corresponds to the trivial cocycle of H₁(R, G). It lies in the image of (22) as the image of the trivial cocycle in H₁(R, G). Equivalently, the quasisplit real form is the image under (21) of the pure strong involution

\[ δ_q = \exp(πiνρ)δ₀. \]  

(23)

Given a strong involution δ we set

\[ K = K_δ \subset G \]  

(24)

to be the fixed-point subgroup of Int(δ). The real form G(R, δ) contains

\[ K(R) = G(R, δ) \cap K \]

as a maximal compact subgroup and is determined by K (AV, (5f)-(5g)). By a representation of G(R, δ) we usually mean an admissible (g, K)-module, although we will need veritable admissible group representations ([V3, Definition 1.1.5]) in Section 7. A representation of a strong involution is a pair (π, δ) in which δ is a strong involution and π is an admissible (g, K)-module. We let Π(G(R, δ)) be the set of equivalence classes of irreducible representations (π', δ') of strong involutions in which δ' is equivalent to δ. Let

\[ Π(G/R) = \coprod_δ Π(G(R, δ)) \]  

(25)

be the disjoint union over the (equivalence classes of) pure strong involutions δ.

Returning to the more familiar territory of L-groups, we fix a pinning

\[ (\gamma B, \gamma H, \{X\gamma α\}) \]

of \( \gamma G \). The previous two pinnings and the involution δ₀ fix an involution \( \gamma δ₀ \) of \( \gamma G \) as prescribed in [AV, (12)]. The group

\[ \gamma G^{π} = \gamma G \rtimes (\gamma δ₀) \]

is the L-group of our inner class.

Suppose λ is a semisimple element of the complex Lie algebra \( \gamma g \). After conjugating by \( \gamma G \) we may assume λ ∈ \( \gamma h \). Using the canonical isomorphism \( \gamma h \simeq h^* \) we identify λ with an element of \( h^* \), and hence via the Harish-Chandra homomorphism, with an infinitesimal character for G. This construction depends only on the \( \gamma G \)-orbit of λ. We refer to a semisimple element λ ∈ \( \gamma g \), or a \( \gamma G \)-orbit \( \gamma O \subset \gamma g \) of semisimple elements, as an infinitesimal character for G. Let

\[ Π(\gamma O, G/R) \subset Π(G/R) \]

be the representations (of pure strong involutions) with infinitesimal character \( \gamma O \).

Let \( P(\gamma G^{π}) \) be the set of quasadimissible homomorphisms \( ϕ : W_R → \gamma G^{π} \) ([ABV, Definition 5.2]). There is an infinitesimal character associated to ϕ in \( P(\gamma G^{π}) \) ([ABV, Proposition 5.6]). Let

\[ P(\gamma O, \gamma G^{π}) \]  

(26)

be the set of quasadimissible homomorphisms with infinitesimal character \( \gamma O \). The group \( \gamma G \) acts on \( P(\gamma O, \gamma G^{π}) \) by conjugation. It is the set of \( \gamma G \)-orbits

\[ P(\gamma O, \gamma G^{π})/\gamma G \]  

(27)

that the Langlands correspondence, in its original form, assigns L-packets of representations.

Another great innovation of [ABV] is the introduction of the complex variety \( X(\gamma O, \gamma G^{π}) \) of geometric parameters, which lies between (26) and (27) ([ABV, Definition 6.9]). It may be regarded

\[ ^1 \text{Warning! Identical notation is used in [ABV] in which δ runs over all, not necessarily pure, involutions.} \]
as a set of equivalence classes in \( P(\gamma, G^\Gamma) \) upon which \( \gamma G \) still acts by conjugation with finitely many orbits ([AAM, Section 2.2], [ABV, Proposition 6.16]). Furthermore, the quotient map

\[
P(\gamma, G^\Gamma) \rightarrow X(\gamma, G^\Gamma)
\]

passes to a bijection at the level of \( \gamma G \)-orbits ([ABV, Proposition 6.17]).

Let \( S \subset X(\gamma, G^\Gamma) \) be a \( \gamma G \)-orbit, and for \( p \in S \) let \( \gamma G_p = \text{Stab}_{\gamma G}(p) \). A pure complete geometric parameter for \( X(\gamma, G^\Gamma) \) is a pair \( (S, \tau_S) \) where \( \tau_S \) is (an equivalence class of) an irreducible representation of the component group \( \gamma G_p/\gamma G_p^0 \) ([AV, Definitions 7.1 and 7.6]). We denote the set of pure complete geometric parameters for \( X(\gamma, G^\Gamma) \) by \( \Xi(\gamma, G^\Gamma) \).

A special case of the local Langlands correspondence as stated in [AV, Theorem 10.11] is a bijection

\[
\Pi(\gamma, G/R) \leftrightarrow \Xi(\gamma, G^\Gamma)
\]

between representations of pure strong involutions and pure complete geometric parameters. It is important to bear in mind that the left-hand side of (29) contains the subset \( \Pi(\gamma, G(R, \delta_0)) \) of representations of the quasisplit form of \( G \).

### 2.2 Extended groups for unitary groups and the complex general linear group

We specialize the discussion of the previous section to two groups, each with a fixed inner class. The first group is \( G_N \). When \( N \) is even we fix the inner class whose quasisplit form is the indefinite unitary group \( U(N/2, N/2) \). When \( N \) is odd we fix the inner class to contain the quasisplit form \( U((N - 1)/2, (N + 1)/2) \). The second group is

\[
R_{C/R}G_N = G_N \times G_N
\]

together with the inner class whose quasisplit form is \( G_N(C) \).

Let us begin with the unitary groups. Fix the usual pinning for \( G_N \) in which \( B \) is the upper-triangular Borel subgroup, \( H \) is the diagonal subgroup, and \( X_\alpha \) is the matrix with 1 in the entry corresponding to the root \( \alpha \) and zeroes elsewhere. Regardless of whether \( N \) is even or odd, the specified inner classes contain the compact real form \( U(N) = U(N, 0) \). This implies that \( \delta_0 \) acts as the trivial automorphism on \( G_N \) ([AV, (5)]). Consequently, the extended group for the inner class of unitary groups is

\[
\gamma G_N = G_N \times (\delta_0) \cong G_N \times \mathbb{Z}/2\mathbb{Z}.
\]

It follows from [AV, (12c)] that the involution \( \gamma \delta_0 \) acts on \( \gamma G_N \) as

\[
\gamma \delta_0(g) = \tilde{J}(g^{-1})\tilde{J}^{-1}, \quad g \in \gamma G_N.
\]

The corresponding L-group is

\[
\gamma G_N^\Gamma = \gamma G_N \times (\gamma \delta_0).
\]

The strong involutions and pure strong involutions of real unitary groups are presented in [A].

For the group \( R_{C/R}G_N \) and the inner class containing \( G_N(C) \), we shall follow [M3, Section 2.1] and begin with the L-group first. The L-group is defined as

\[
\gamma R_{C/R}G_N^\Gamma = \gamma (G_N \times G_N)^\Gamma = (\gamma G_N \times \gamma G_N) \times (\gamma \delta_0),
\]

where the involution \( \gamma \delta_0 \) is now defined by

\[
\gamma \delta_0(g_1, g_2) = (g_2, g_1), \quad g_1, g_2 \in \gamma G_N.
\]

Let us now fix the pinning \( (B, H, \{X_\alpha\}) \) for \( G_N \times G_N \) (and its dual) by taking \( B \) to be the direct product of the upper-triangular subgroups, \( H \) to be the direct product of the diagonal subgroups and \( \{X_\alpha\} \) to be the union of simple root vectors for each of the two factors in the direct product. Then, by the prescription [AV, (5)], the involution \( \delta_0 \) on \( R_{C/R}G_N = G_N \times G_N \) is defined by

\[
\delta_0(g_1, g_2) = (\tilde{J}(g_2^{-1})\tilde{J}^{-1}, \tilde{J}(g_1^{-1})\tilde{J}^{-1}), \quad g_1, g_2 \in G_N.
\]
and the extended group of $R_{C/R} GL_N$ is

$$R_{C/R} GL_N^\Gamma = (GL_N \times GL_N)^\Gamma = (GL_N \times GL_N) \rtimes \langle \delta_0 \rangle. \quad (33)$$

It is coincidental that $\delta_0$ defines the same automorphism as $\vartheta$ (3). The real form $\sigma$ associated to $\delta_0$ is given by composing $\delta_0$ with the compact real form $([AV, (5)])$. This turns out to be

$$\sigma(g_1, g_2) = (\tilde{J}g_2\tilde{J}^{-1}, \tilde{J}g_1\tilde{J}^{-1}), \quad g_1, g_2 \in GL_N. \quad (34)$$

The group $R_{C/R} GL_N(\mathbb{R}, \delta_0)$ from (21) is by definition the fixed-point subgroup of $\sigma$,

$$R_{C/R} GL_N(\mathbb{R}, \delta_0) = \{(g, \tilde{J}g\tilde{J}^{-1}) : g \in GL_N\} \cong GL_N(\mathbb{C}).$$

In what follows we will reserve the notation $GL_N(\mathbb{C})$ for this particular real form, and reserve the notation $GL_N$ for the absolute theory of reductive groups.

**Lemma 2.1.** The Galois cohomology sets

$$H^1(\mathbb{R}, R_{C/R} GL_N) \text{ and } H^1(\mathbb{R}, R_{C/R} GL_N/Z(R_{C/R} GL_N))$$

are both trivial. In particular, $GL_N(\mathbb{C})$ is the only real form in its inner class, there is only one equivalence class of pure strong involutions, and this equivalence class corresponds to $GL_N(\mathbb{C})$ via (22).

**Proof.** We begin with a cocycle $z \in Z^1(\mathbb{R}, R_{C/R} GL_N)$. Here, the implicit action of the non-trivial element $\sigma \in \Gamma$ on $R_{C/R} GL_N$ is given by (34). Suppose $z(\sigma) = (g_1, g_2)$. Then by definition,

$$(1, 1) = z(\sigma^2) = z(\sigma) \sigma(z(\sigma)) = (g_1\tilde{J}g_2\tilde{J}^{-1}, g_2\tilde{J}g_1\tilde{J}^{-1}),$$

which implies

$$z(\sigma) = (g_1, \tilde{J}g_1^{-1}\tilde{J}^{-1}).$$

The cocycle $z$ is trivial in $H^1(\mathbb{R}, R_{C/R} GL_N)$ since

$$z(\sigma) = (g_1, 1) \sigma((g_1, 1)^{-1}).$$

This proves the triviality of $H^1(\mathbb{R}, R_{C/R} GL_N)$. The proof of the triviality of the second cohomology set follows in the same manner. \qed

Lemma 2.1 reduces the set of representations in (25) to the unique real form $GL_N(\mathbb{C})$. Thus, we write

$$\Pi(\mathcal{O}, (R_{C/R} GL_N)/R) = \Pi(\mathcal{O}, GL_N(\mathbb{C})). \quad (35)$$

This is an opportune moment to bring up two peculiarities of $R_{C/R} GL_N$ that will be of importance later. The first is that we could equally well have reversed the roles of the $L$-group and extended group by defining the $L$-group as (33) and the extended group as (31). Indeed, with this reversal the (unique) real form corresponding to the extended group is the fixed-point set of

$$(g_1, g_2) \mapsto ((\bar{g_1}^\dagger)^{-1}, (\bar{g_2}^\dagger)^{-1})$$

([AV, (5)]), which is easily seen to be

$$\{(g, (\bar{g}^\dagger)^{-1}) : g \in GL_N\} \cong GL_N(\mathbb{C}).$$

The recovery of the same inner class, namely $GL_N(\mathbb{C})$, under this reversal shall be useful when we explore Vogan duality in Section 4.1.

The second peculiarity has to do with the connectedness of centralizers in the dual group $^\vee R_{C/R} GL_N = ^\vee GL_N \times ^\vee GL_N$. It is well-known that the centralizer of the image of any $L$-parameter for the general linear group $GL_N(\mathbb{R})$ is connected. This peculiarity is shared by $R_{C/R} GL_N(\mathbb{R}) =
GL\(_N(\mathbb{C})\) and may be seen as follows. To lighten the notation take \(N = 2\). It is a simple exercise to show that after conjugation any \(L\)-parameter \(\phi \in P(\text{^g R}_{C/R}\text{GL}_2)\) may be taken to have the form

\[
\phi(z) = \left(\begin{array}{cc}
z \lambda_1 z \lambda_1' & 0 \\
0 & z \lambda_2 z \lambda_2'
\end{array}\right), \quad z \in \mathbb{C}^\times
\]

\[
\phi(j) = \left(\begin{array}{cc}
(-1)^{(\lambda_1 - \lambda_1')/2} & 0 \\
0 & (-1)^{(\lambda_1 - \lambda_1')/2}
\end{array}\right),
\]

for some \(\lambda_1, \lambda_1', \lambda_2, \lambda_2' \in \mathbb{C}\). It is not difficult to see that the centralizer of \(\phi(j)\) in \(^g\text{GL}_2 \times ^g\text{GL}_2\) reduces the further computation of the centralizer of \(\phi(\mathbb{C}^\times)\) to the well-known case of \(^g\text{GL}_2\), where it is known to be connected. The connectedness of the centralizer of \(\phi\) is the same as the triviality of the component group, that is

\[
(^g\text{R}_{C/R}\text{GL}_N)_\phi / (^g\text{R}_{C/R}\text{GL}_N)_\phi^0 = \{1\}.
\]

### 2.3 Atlas parameters for \(\text{R}_{C/R}\text{GL}_N\)

We introduce another set of parameters for representations of \(\text{R}_{C/R}\text{GL}_N\) which may be used in place of the complete geometric parameters of Section 2.1. These **Atlas parameters** are convenient for Hecke algebra computations and are well suited to a description of Vogan duality (see Section 4.1). The main references for this section are [AdC] and [AV, Section 3].

We start by working in the context of the extended group (33). We take \(H \subset \text{R}_{C/R}\text{GL}_N\) to be the direct product of the diagonal subgroups as in the pinning of the previous section. Following [AV, Section 3] we set

\[
\mathcal{X}_\rho = \left\{ \delta \in \text{Norm}(\text{GL}_N, \text{GL}_N)_{\delta_0}(H) : \delta^2 = \exp(2\pi i \mathcal{V}\rho) \right\} / H
\]

where the quotient is by the conjugation action of \(H\). This is a set of \(H\)-conjugacy classes of strong involutions with infinitesimal cocharacter \(\mathcal{V}\rho\) ([AV, (16e)], cf. (23)). By Lemma 2.1, these strong involutions are all pure and correspond to the real form \(\text{GL}_N(\mathbb{C})\).

We fix a \(\vartheta\)-fixed, regular, integrally dominant element \(\lambda \in \mathcal{V}\mathfrak{h}\). This means

\[
\vartheta(\lambda) = \lambda
\]

\[
(\lambda, \mathcal{V}\alpha) \neq 0, \quad \alpha \in \text{R}(\text{R}_{C/R}\text{GL}_N, H)
\]

\[
(\lambda, \mathcal{V}\alpha) \notin \{-1, -2, -3, \ldots\}, \quad \alpha \in \text{R}^+(\text{R}_{C/R}\text{GL}_N, H).
\]

The \(^g\text{R}_{C/R}\text{GL}_N\)-orbit \(^g\mathcal{O}\) of \(\lambda\) will be the infinitesimal character of our representations of \(\text{GL}_N(\mathbb{C})\). The assumption of integral dominance is harmless ([AV, Lemma 4.1]). We shall remove the regularity assumption at the beginning of Section 9.

The action of \(\delta_0\) in the extended group (33) induces an action on the Weyl group \(W(\text{R}_{C/R}\text{GL}_N, H)\). Consider the set

\[
\mathcal{X}_\rho^w = \left\{ x \in \mathcal{X}_\rho : x h x^{-1} = w \delta_0 \cdot h, \quad \text{for all } h \in H \right\}.
\]

Turning to the \(L\)-group (31), we have an analogous set in which the infinitesimal cocharacter \(\mathcal{V}\rho\) is replaced by the infinitesimal character \(\lambda\), and \(\delta_0\) is replaced by \(\mathcal{V}\delta_0\), namely

\[
\mathcal{X}_\lambda = \left\{ \delta \in \text{Norm}_{\text{GL}_N} \mathcal{V}\delta_0(\mathcal{V}H) : \delta^2 = \exp(2\pi i \mathcal{V}\lambda) \right\} / \mathcal{V}H.
\]

The analogue of the set (38) is

\[
\{ w \in (\text{^g R}_{C/R}\text{GL}_N, \mathcal{V}H) : w \mathcal{V}\delta_0(w) = 1 \}
\]

and there is an obvious analogue of (39), which we denote by \(\mathcal{X}_\lambda^w\).
If we identify \( ^\vee \text{GL}_N \) with \( \text{GL}_N \) then the actions of \( \delta_0 \) and \( ^\vee \delta_0 \) on \( H \) are related by

\[
^\vee \delta_0(h) = w_0 \delta_0(h^{-1}), \quad h \in H,
\]

where \( w_0 \) is the long Weyl group element ([AV, (12c)]). From this it is easily verified that for all \( h \in H \) and \( w \in W(\text{R}_{\mathbb{C}/\mathbb{R}} \text{GL}_N, H) \)

\[
w \delta_0 \cdot h = w \delta_0 \cdot (w h w_0 \delta_0) \cdot h = w w_0 \delta_0 \cdot h.
\]

It follows that

\[
w \mapsto w w_0
\]

defines a bijection from (38) onto (40). This map allows us to pair any set \( ^\vee X_{w}^\rho \) with the set \( ^\vee X_{ww_0}^\lambda \).

The next result follows from [AdC], [ABV] and [AV, Theorem 3.11]. Our proof follows that of [AAM, Lemma 2.2].

**Lemma 2.2.** There is a canonical bijection

\[
\bigwedge \{ w \in W : w \delta_0 = 1 \} ^\vee X_{w}^\rho \times ^\vee X_{w w_0}^\lambda \leftrightarrow \Xi \left( ^\vee \text{O}, ^\vee \text{R}_{\mathbb{C}/\mathbb{R}} \text{GL}_N \right).
\]

**Proof.** First of all \( |X_{w}^\rho| = 1 \) for all \( w \). This follows from [AdC, Proposition 12.19(5)] which equates the cardinality with that of the component group of a dual Cartan subgroup. As indicated in the first peculiarity near the end of Section 2.2, the dual inner class consists of (products of) complex general linear groups. The dual Cartan subgroup is therefore isomorphic to \( N \) copies of \( \mathbb{C}^n \) and is evidently connected, so its component group is trivial.

The lemma is now reduced to defining a canonical bijection

\[
\bigwedge \{ w \in W : w \delta_0 = 1 \} ^\vee X_{w w_0}^\lambda \leftrightarrow \Xi \left( ^\vee \text{O}, ^\vee \text{R}_{\mathbb{C}/\mathbb{R}} \text{GL}_N \right).
\]

As explained in [AAM, Lemma 2.2], the triviality of the component groups (36) further reduces the task to defining a canonical bijection with \( ^\vee \text{R}_{\mathbb{C}/\mathbb{R}} \text{GL}_N \)-orbits in \( X \left( ^\vee \text{O}, ^\vee \text{R}_{\mathbb{C}/\mathbb{R}} \text{GL}_N \right) \), or equivalently, with \( ^\vee \text{R}_{\mathbb{C}/\mathbb{R}} \text{GL}_N \)-orbits of quasiadmissible homomorphisms. The definition of the latter bijection is almost identical to the one in [AAM, Lemma 2.2] and is left to the reader. \( \square \)

Together with (29) this gives

**Theorem 2.3.** Let \( ^\vee \text{O} \) be the \( ^\vee \text{R}_{\mathbb{C}/\mathbb{R}} \text{GL}_N \)-orbit of \( \lambda \). There are canonical bijections

\[
\bigwedge \{ w \in W : w \delta_0 = 1 \} ^\vee X_{w}^\rho \times ^\vee X_{w w_0}^\lambda \leftrightarrow \Xi \left( ^\vee \text{O}, ^\vee \text{R}_{\mathbb{C}/\mathbb{R}} \text{GL}_N \right) \leftrightarrow \Xi (\text{O}, \text{GL}_N(\mathbb{C})).
\]

As in [AV, Theorem 3.11] the bijection of Theorem 2.3 is written as

\[
^\vee X_{w}^\rho \times ^\vee X_{w w_0}^\lambda \ni (x, y) \mapsto J(x, y, \lambda) \quad (41)
\]

We call the pair \( (x, y) \) on the left the Atlas parameter of the irreducible representation \( J(x, y, \lambda) \). By Lemma 2.2, the Atlas parameter \( (x, y) \) is equivalent to a unique complete geometric parameter \( \xi \in \Xi (\text{O}, ^\vee \text{R}_{\mathbb{C}/\mathbb{R}} \text{GL}_N) \), and accordingly we define

\[
\pi(\xi) = J(x, y, \lambda).
\]

The representation \( \pi(\xi) \) is the Langlands quotient of a standard representation which we denote by \( M(\xi) \) or \( M(x, y) \).
2.4 Twisted Atlas parameters for $R_{C/R}GL_N$

We next describe the generalization of Theorem 2.3 to the $\vartheta$-twisted setting, which involves representations of the group $GL_N(\mathbb{C}) \ltimes \langle \vartheta \rangle$. We specialize the results of [AV, Sections 3-5] to this case. Some of the more complicated issues that arise in [AV] do not occur for $R_{C/R}GL_N$.

We continue with the hypotheses of (37). Recall that both $\rho$ and $\lambda$ are fixed by $\vartheta$. By Clifford theory, an irreducible representation of $GL_N(\mathbb{C}) \ltimes \langle \vartheta \rangle$ restricted to $GL_N(\mathbb{C})$ is either an irreducible $\vartheta$-fixed representation, or the direct sum of two irreducible representations which are exchanged by the action of $\vartheta$. Since we shall be restricting our attention to twisted characters, we only need representations of the first type.

By [CM, Theorem 4.1] and Lemma 2.2, the map (41) is $\vartheta$-equivariant. Therefore $J(x, y, \lambda)$ is $\vartheta$-stable if and only if $(x, y) \in X^{\vartheta}_{\rho} \times \bigwedge^w x^{\vartheta}_{\vartheta}$ is fixed by $\vartheta$. Let

$$\Pi(\bigwedge^\vartheta O, GL_N(\mathbb{C}))^\vartheta \subset \Pi(\bigwedge^\vartheta O, GL_N(\mathbb{C}))$$

be the subset of $\vartheta$-stable irreducible representations and set

$$W(\delta_0, \vartheta) = \{ w \in W(R_{C/R}GL_N, H) : w\delta_0(w) = 1, w = \vartheta(w) \}$$

(cf. (38)). The $\vartheta$-equivariance of (41) allows us to restrict Theorem 2.3 to these sets and we obtain

**Corollary 2.4.** Suppose $\lambda$ satisfies the hypotheses of (37) and let $\bigwedge^\vartheta O$ be its $\bigwedge^\vartheta R_{C/R}GL_N$-orbit. Then there is a canonical bijection

$$\prod_{\{ w \in W(\delta_0, \vartheta) \}} X^{\vartheta}_{\rho} \times \bigwedge^w X^{\vartheta}_{\vartheta} \longleftrightarrow \Pi(\bigwedge^\vartheta O, GL_N(\mathbb{C}))^\vartheta$$

written $(x, y) \mapsto J(x, y, \lambda)$.

We now introduce the *extended parameters* of [AV, Sections 3-5], and summarize some facts. Fix $w \in W(\delta_0, \vartheta)$. An extended parameter for $w$ is a set

$$E = (\lambda, \tau, \ell, t), \quad \lambda, \tau \in X^*(H), \quad \ell, t \in X_*(H)$$

satisfying certain conditions depending on $w$ (see [AV, Definition 5.4]). There is a surjective map

$$E \mapsto (x(E), y(E))$$

(42)

carrying extended parameters for $w$ to $X^{\vartheta}_{\rho} \times \bigwedge^w \bigwedge^\vartheta_{\vartheta}$. This map only depends on $\lambda$ and $\ell$. In addition,

$$J(x(E), y(E), \lambda) \in \Pi(\bigwedge^\vartheta O, GL_N(\mathbb{C}))^\vartheta,$$

and every $\vartheta$-fixed irreducible representation arises this way. The remaining parameters $\tau$ and $t$ in $E$ define an irreducible representation $J(E, \lambda)$ of $GL_N(\mathbb{R}) \ltimes \langle \vartheta \rangle$ satisfying

$$J(E, \lambda)|_{GL_N(\mathbb{C})} = J(x(E), y(E), \lambda).$$

The representation $J(x(E), y(E), \lambda)$ is determined by a quasicharacter of a Cartan subgroup of $GL_N(\mathbb{C})$. The representation $J(E, \lambda)$ is determined by the semidirect product of this Cartan subgroup with an element $h\vartheta \in R_{C/R}GL_N \ltimes \vartheta$ ([AV, (24e)]) and a choice of extension of the quasicharacter to the semidirect product. The value of the extended quasicharacter on the element $h\vartheta$ depends on a choice of sign [AV, Definition 5.2], and the square root of this sign is given by

$$z(E) = \sqrt{}(\tau, (1+w)\ell) \cdot (-1)^{2(\lambda, t)}.$$

The preceding discussion is a specialization of a general framework to $R_{C/R}GL_N \ltimes \langle \vartheta \rangle$. One of the special properties of $R_{C/R}GL_N$ is that the preimage of any $(x, y) \in X^{\vartheta}_{\rho} \times \bigwedge^w X^{\vartheta}_{\vartheta}$ under (42) has a preferred extended parameter of the form

$$(\lambda, \tau, 0, 0).$$

---

2Warning! The symbols $\lambda$ and $\tau$ here are not to be confused with symbols $\lambda$ and $\tau$ appearing elsewhere. Note the slight difference in font. We have chosen to use $\lambda$ and $\tau$ for ease of comparison with [AV].
This comes down to the fact that $X^\vartheta_\rho$ is a singleton (see the proof of Lemma 2.2). In taking $t = 0$ we see $z(\lambda, \tau, 0, 0) = 1$, and this in turn amounts to taking the aforementioned semidirect product of the Cartan subgroup with $h\vartheta = \vartheta$, and setting the value of the extended quasicharacter at $\vartheta$ equal to 1. In this way, the preferred extended parameter defines a canonical extension

$$J(x, y, \lambda)^+ = J((\lambda, \tau, 0, 0), \lambda) \quad (44)$$

of $J(x, y, \lambda)$ to $GL_N(\mathbb{C}) \rtimes \langle \vartheta \rangle$. We call this extension the *Atlas extension* of $J(x, y, \lambda)$.

Going back to Theorem 2.3 and Corollary 2.4, we may formulate the result as follows.

**Corollary 2.5.** There is a natural bijection of $\vartheta$-fixed sets

$$\prod_{\{w \in W(\delta_\vartheta, \vartheta)\}} \mathcal{X}_\vartheta^w \times ^\vartheta \mathcal{A}_\lambda^\text{unw} \longleftrightarrow \Xi(\mathcal{W}_C, ^\vartheta \mathcal{R}_C GL_N^R) \longleftrightarrow \Pi(\mathcal{W}_O, GL_N(\mathbb{C})^\vartheta).$$

Furthermore, if $\xi \in \Xi(\mathcal{W}_C, ^\vartheta \mathcal{R}_C GL_N^R)^\vartheta$ is identified with $(x, y)$ under the first bijection, then there is a canonical representation

$$\pi(\xi)^+ = J(x, y, \lambda)^+$$

extending $\pi(\xi)$ to $GL_N(\mathbb{C}) \rtimes \langle \vartheta \rangle$.

The irreducible representation $\pi(\xi)^+$ is defined as the unique (Langlands) quotient of a representation $M(\xi)^+$ such that $M(\xi)^+_{GL_N(\mathbb{C})} = M(\xi)$. We call $\pi(\xi)^+$ and $M(\xi)^+$ the *Atlas extensions* of $\pi(\xi)$ and $M(\xi)$ respectively.

### 2.5 Grothendieck groups of characters and twisted characters

The setting for studying characters of reductive groups is the Grothendieck group of admissible representations. There is a corresponding notion in the twisted setting. In this section we establish the notation for these objects.

For the moment let $G$ be an arbitrary complex connected reductive group. Fix a semisimple connected orbit $^\vartheta g \subset ^\vartheta g$. Recall from Section 2.1 that $\Pi(\mathcal{W}_O, G/\mathbb{R})$ is the set of equivalence classes of irreducible representations $(\pi, \delta)$ of pure strong involutions with infinitesimal character $^\vartheta \mathcal{O}$. We define $K\Pi(\mathcal{W}_O, G/\mathbb{R})$ to be the Grothendieck group of admissible representations of pure strong involutions with infinitesimal character $^\vartheta \mathcal{O}$ (see [ABV, (15.5)-(15.6)]). We identify this with the $\mathbb{Z}$-span of distribution characters of the irreducible representations in $\Pi(\mathcal{W}_O, G/\mathbb{R})$, and refer to elements of this $\mathbb{Z}$-module as virtual characters.

For the unitary groups we often refer to the submodule of stable characters for the quasisplit form. So we define

$$K\Pi(\mathcal{W}_O, G(\mathbb{R}, \delta_\vartheta))^\text{st} \subset K\Pi(\mathcal{W}_O, G(\mathbb{R}, \delta_\vartheta))$$

to be the subspace spanned by the (strongly) stable virtual characters. If we identify virtual characters with functions on $G(\mathbb{R}, \delta_\vartheta)$ these are the virtual characters $\eta$ which satisfy $\eta(g) = \eta(g')$ whenever strongly regular semisimple elements $g, g' \in G(\mathbb{R}, \delta_\vartheta)$ are $G$-conjugate. See [S3, Section 5] or [ABV, Definition 18.2].

We now turn the twisted setting of the inner class $GL_N(\mathbb{C})$ of $R_{C/\mathbb{R}}GL_N$, equipped with the involution $\vartheta$. An immediate consequence of (35) is the expression of the Grothendieck group

$$K\Pi(R_{C/\mathbb{R}}GL_N/\mathbb{R}) = K\Pi(GL_N(\mathbb{C}))$$

in terms of the unique real form. We define

$$K\Pi(\mathcal{W}_O, GL_N(\mathbb{C})^\vartheta) \subset K\Pi(\mathcal{W}_O, GL_N(\mathbb{C}))$$

to be the submodule spanned by $\Pi(\mathcal{W}_O, GL_N(\mathbb{C})^\vartheta)$. This is not the Grothendieck group of $\vartheta$-stable representations of $GL_N(\mathbb{C})$, but we retain the “$K$” to help align the object with its ambient Grothendieck group. On the other hand we let

$$K\Pi(\mathcal{W}_O, GL_N(\mathbb{C}) \rtimes \langle \vartheta \rangle) \quad (45)$$
be the veritable Grothendieck group of admissible representations of $GL_N(\mathbb{C}) \rtimes \langle \vartheta \rangle$ with infinitesimal character $\vartheta$.

We are ready to construct the $\mathbb{Z}$-module of twisted characters of $GL_N(\mathbb{C})$. An irreducible character in $KII(\nu O, GL_N(\mathbb{C}) \rtimes \langle \vartheta \rangle)$ is the usual distribution on $GL_N(\mathbb{C}) \rtimes \langle \vartheta \rangle$ given by $\text{Tr} \pi$, where $\pi$ is an irreducible representation of $GL_N(\mathbb{C}) \rtimes \langle \vartheta \rangle$. The restriction of such a distribution character to the non-identity component $GL_N(\mathbb{C}) \rtimes \vartheta$ (when non-zero) is what we we mean by an irreducible twisted character of $GL_N(\mathbb{C})$ (cf. [AAM, (42)]). We define

$$KII(\nu O, GL_N(\mathbb{C}), \vartheta)$$

to be the $\mathbb{Z}$-module generated by the irreducible twisted characters of $GL_N(\mathbb{C})$ of infinitesimal character $\nu O$.

As noted in Section 2.4, an irreducible representation of $GL_N(\mathbb{C}) \rtimes \langle \vartheta \rangle$ restricts either to an irreducible $\vartheta$-fixed representation of $GL_N(\mathbb{C})$, or to a direct sum $\pi \oplus (\pi \circ \vartheta)$ of inequivalent irreducible representations. In the second case the twisted character is 0, so we only need to consider the first case. The first case describes the irreducible representations in $KII(\nu O, GL_N(\mathbb{C}))^\vartheta$. If $\pi \in \Pi(\nu O, GL_N(\mathbb{C}))^\vartheta$ then it has two extensions $\pi^\pm$ to $GL_N(\mathbb{C}) \rtimes \langle \vartheta \rangle$, satisfying

$$\pi^-(\vartheta) = - \pi^+(\vartheta). \quad (46)$$

Consequently the twisted characters of $\pi^\pm$ agree up to sign. If we set $U_2 = \{ \pm 1 \}$ then it follows that the homomorphism

$$KII(\nu O, GL_N(\mathbb{C}))^\vartheta \otimes_{\mathbb{Z}} \mathbb{Z}[U_2] \to KII(\nu O, GL_N(\mathbb{C}), \vartheta),$$

which restricts the distribution character of $\pi(\xi)^+ \vartheta$ to the non-identity component, is surjective. By (46), the homomorphism passes to an isomorphism

$$KII(\nu O, GL_N(\mathbb{C}), \vartheta) \cong KII(\nu O, GL_N(\mathbb{C}))^\vartheta \otimes_{\mathbb{Z}} \mathbb{Z}[U_2]/((\pi \otimes 1) + (\pi \otimes -1)), \quad (47)(a)$$

where the quotient runs over $\pi \in \Pi(\nu O, GL_N(\mathbb{C}))^\vartheta$. The map carrying $\pi(\xi) \in \Pi(\nu O, GL_N(\mathbb{C}))^\vartheta$ to the twisted character of the Atlas extension $\pi(\xi)^+ \vartheta$ extends to an isomorphism

$$KII(\nu O, GL_N(\mathbb{C}))^\vartheta \cong KII(\nu O, GL_N(\mathbb{C}), \vartheta).$$

We again remind the reader that the $\mathbb{Z}$-modules appearing in (47) are not Grothendieck groups in any natural fashion, notwithstanding the appearance of the "$K$".

3 Sheaves, Pairings and Characteristic Cycles

Suppose $\psi_G$ is an Arthur parameter for $G$ as in (1). In this section we give more details on the definition of the ABV-packet $\Pi_{\psi_G}^{ABV}$ and its stable virtual character $\eta_{\psi_G}^{ABV}$ (2). The results apply in the more general context of complex connected reductive groups $G$ ([ABV, Sections 19, 22]). However, for this section $G$ will be $GL_N$ or $R_{C/\mathbb{R}}GL_N$, with the setup of Section 2.2.

The definitions depend on a pairing between characters and sheaves. We also define a pairing between twisted characters and twisted sheaves for $R_{C/\mathbb{R}}GL_N$ ([CM, Sections 5-6], [AAM, Section 3]). The key properties of this twisted pairing are listed in this section and shall be proved in Section 4.

3.1 The pairing and the ABV-packets in the non-twisted case

Let $\phi_{\psi_G}$ be the Langlands parameter associated to $\psi_G$ ([ABV, Definition 22.4]). $\nu O$ be the infinitesimal character of $\phi_{\psi_G}$, and $S_{\psi_G} \subset X(\nu O, \nu G^F)$ be the corresponding orbit (28). Recall that $\Xi(\nu O, \nu G^F)$ is the set of pure complete geometric parameters.$

Let $C(X(\nu O, \nu G^F))$ be the category of $\nu G$-equivariant constructible sheaves of complex vector spaces on $X(\nu O, \nu G^F)$. This is an abelian category and its simple objects are parameterized by the set of complete geometric parameters $\xi = (S, \tau_S) \in \Xi(\nu O, \nu G^F)$ as follows. Choose $p \in S$, let
\[ \mathcal{V} \circ \mathcal{V}_p = \text{Stab}_{\mathcal{V}}(p), \] and choose a character \( \tau_\xi \) of the component group of \( \mathcal{V}_p \) so that \( (p, \tau_\xi) \) is a representative of \( \xi \). Then \( \tau_\xi \) pulled back to \( \mathcal{V}_p \) defines an algebraic vector bundle

\[ \mathcal{V}_p \times \mathcal{V}_p \rightarrow S. \] (48)

The sheaf of sections of this vector bundle is, by definition, an \( \mathcal{V} \)-equivariant local system on \( S \) ([ABV, Section 7, Lemma 7.3]). Extend this local system to the closure \( \tilde{S} \) by zero and then take the direct image into \( X(\mathcal{O}, \mathcal{V}^G) \) to obtain an irreducible (i.e. simple) \( \mathcal{V} \)-equivariant constructible sheaf denoted by \( \mu(\xi) \) ([ABV, (7.10)(c)]).

Now let \( \mathcal{P}(X(\mathcal{O}, \mathcal{V}^G)) \) be the abelian category of \( \mathcal{V} \)-equivariant perverse sheaves of complex vector spaces on \( X(\mathcal{O}, \mathcal{V}^G) \) ([BL, Section 5]). The simple objects of \( \mathcal{P}(X(\mathcal{O}, \mathcal{V}^G)) \) are defined from \( \xi = (S, \tau_\xi) \in \Xi(\mathcal{V}^G, \mathcal{O}) \) and the algebraic vector bundle (48) by taking the intermediate extension [BBD, Section 2] to the closure \( \tilde{S} \) instead of the extension by zero. This is denoted \( P(\xi) \) ([ABV, (7.10)(d)]). It is an irreducible \( \mathcal{V} \)-equivariant perverse sheaf on \( X(\mathcal{O}, \mathcal{V}^G) \).

The Grothendieck groups of the two categories \( \mathcal{C}(X(\mathcal{O}, \mathcal{V}^G)) \) and \( \mathcal{P}(X(\mathcal{O}, \mathcal{V}^G)) \) are canonically isomorphic ([BBD], [ABV, Lemma 7.8]). We identify the two Grothendieck groups via this isomorphism and denote them by \( KX(\mathcal{O}, \mathcal{V}^G) \). This Grothendieck group has two natural bases

\[ \{ \mu(\xi) : \xi \in \Xi(\mathcal{O}, \mathcal{V}^G) \} \quad \text{and} \quad \{ P(\xi) : \xi \in \Xi(\mathcal{O}, \mathcal{V}^G) \}. \]

Suppose \( \xi = (S, \tau) \in \Xi(\mathcal{O}, \mathcal{V}^G) \). We define two invariants associated to \( \xi \). First, let \( d(\xi) \) be the dimension of \( S \). Second, associated to \( \xi \) is the representation \( \pi(\xi) \) of a pure strong involution of \( G \) (29). Let \( \epsilon(\xi) = \pm 1 \) be the Kottwitz invariant of the underlying real form of this strong involution ([ABV, Definition 15.8]).

As stated in the introduction, we define a perfect pairing

\[ \langle \cdot, \cdot \rangle : K\Pi(\mathcal{O}, G/\mathbb{R}) \times KX(\mathcal{O}, \mathcal{V}^G) \rightarrow \mathbb{Z} \] (49)

by

\[ \langle M(\xi), \mu(\xi') \rangle = \epsilon(\xi) \delta_{\xi, \xi'}. \]

The pairing takes a deceptively simple form relative to the bases given by \( \pi(\xi) \) and \( P(\xi') \) ([ABV, Theorem 1.24, Sections 15-17]). We state it as a theorem.

**Theorem 3.1.** The pairing (49) satisfies

\[ \langle \pi(\xi), P(\xi') \rangle = (-1)^{d(\xi)} \epsilon(\xi) \delta_{\xi, \xi'}, \quad \xi, \xi' \in \Xi(\mathcal{O}, \mathcal{V}^G). \]

This pairing allows us to regard elements of \( K\Pi(\mathcal{O}, G/\mathbb{R}) \) as \( \mathbb{Z} \)-linear functionals of \( KX(\mathcal{O}, \mathcal{V}^G) \). The microlocal multiplicity maps \( \chi^\text{mic} \) discussed in (11) are \( \mathbb{Z} \)-linear functionals on \( KX(\mathcal{O}, \mathcal{V}^G) \).

Before making the connection with the pairing (49), we review some facts needed to define \( \chi^\text{mic} \). To begin, we consider the category of \( \mathcal{V} \)-equivariant coherent \( \mathcal{D} \)-modules on \( X(\mathcal{O}, \mathcal{V}^G) \). We denote this category by \( \mathcal{D}(X(\mathcal{O}, \mathcal{V}^G)) \). Here, \( \mathcal{D} \) is the sheaf of algebraic differential operators on \( X(\mathcal{O}, \mathcal{V}^G) \) ([BGK+, VIII.14.4], [ABV, Section 7]). Convenient references for equivariant \( \mathcal{D} \)-modules are [HTT] and [ST].

The equivariant Riemann-Hilbert correspondence ([BGK+, Theorem VIII.14.4]) induces an isomorphism

\[ DR : K\mathcal{D}(X(\mathcal{O}, \mathcal{V}^G)) \rightarrow KX(\mathcal{O}, \mathcal{V}^G). \] (50)

For simplicity we write \( X = X(\mathcal{O}, \mathcal{V}^G) \), and \( DX = \mathcal{D}(X(\mathcal{O}, \mathcal{V}^G)) \).

The sheaf \( \mathcal{D} \) is filtered by the order of the differential operators, and the associated graded ring is canonically isomorphic to \( \mathcal{O}_{\mathcal{F}^\mathcal{D}(X)} \), the coordinate ring of the cotangent bundle of \( X \) ([HTT, Section 1.1]). Suppose \( M \in DX \). Then \( M \) has a filtration such that the resulting graded sheaf \( grM \) is a coherent \( \mathcal{O}_{\mathcal{F}^\mathcal{D}(X)} \)-module ([HTT, Section 2.1]).

The support of \( grM \) is a closed subvariety of \( T^*(X) \) ([ABV, Definition 19.7]). Each minimal \( \mathcal{V} \)-invariant component of this closed subvariety is the closure of a conormal bundle \( T^*_S(X) \), where \( S \subset X \) is a \( \mathcal{V} \)-orbit ([ABV, Proposition 19.12(c)]). Therefore to each conormal bundle \( T^*_S(X) \) we may attach a non-negative integer, denoted by \( \chi^\text{mic}(M) \), which (when nonzero) is the length of the module \( grM \) localized at \( T^*_S(X) \) ([HTT, Section 16.2]).
The characteristic cycle of $\mathcal{M}$ is defined as

$$\text{Ch}(\mathcal{M}) = \sum_{S \in X/\gamma G} \chi^\text{mic}_S(\mathcal{M}) \overline{T_S}(X).$$

For a given $\gamma G$-orbit $S$ we may regard $\chi^\text{mic}_S$ as a function on $\mathcal{D}$-modules which is additive for short exact sequences ([ABV, Proposition 19.12(e)]). It therefore defines a homomorphism $K\mathcal{D}(X(\gamma \mathcal{O}, \gamma G^T)) \to \mathbb{Z}$, called the microlocal multiplicity along $S$. Using the isomorphism (50), we interpret this as a homomorphism

$$\chi^\text{mic}_S : KX(\gamma \mathcal{O}, \gamma G^T) \to \mathbb{Z}.$$

We now return to the pairing (49) and its relationship to $\chi^\text{mic}_S$. This relationship defines $\eta^\text{ABV}_{\gamma G}$. We first define $\eta^\text{mic}_{\gamma G} \in K\Pi(\gamma \mathcal{O}, \mathbb{G}/\mathbb{R})$ to be the element of $K\Pi(\gamma \mathcal{O}, \mathbb{G}/\mathbb{R})$ corresponding via the pairing to the $\mathbb{Z}$-linear functional $\chi^\text{mic}_S$ on $KX(\gamma \mathcal{O}, \gamma G^T)$. Explicitly working through the identifications in the definition we see

$$\eta^\text{mic}_{\gamma G} = \sum_{\xi \in \Xi(\gamma \mathcal{O}, \gamma G^T)} (-1)^{d(S_{\xi})-d(S_{\gamma G})} \chi^\text{mic}_{\gamma S_{\gamma G}}(P(\xi)) \pi(\xi). \quad (51)$$

An important result of Kashiwara and Adams-Barbasch-Vogan is

**Proposition 3.2** ([ABV, Theorem 1.31, Corollary 19.16]). The virtual character $\eta^\text{mic}_{\gamma G}$ is stable.

The microlocal packet $\Pi^\text{mic}_{\gamma G}$ of $\psi_G$ is defined to be the irreducible representations in the support of $\eta^\text{mic}_{\gamma G}$. In other words

$$\Pi^\text{mic}_{\gamma G} = \{ \pi(\xi) : \xi \in \Xi(\gamma \mathcal{O}, \gamma G^T), \chi^\text{mic}_{\gamma S_{\gamma G}}(P(\xi)) \neq 0 \}.$$

This is a set of irreducible representations of pure strong involutions of $G$. We are primarily interested in the packet for the quasisplit strong involutions. We therefore define

$$\eta^\text{ABV}_{\gamma G} = \eta^\text{mic}_{\gamma G}(\delta_q) \quad (52)$$

to be the restriction of $\eta^\text{mic}_{\gamma G}$ to the submodule of $K\Pi(\gamma \mathcal{O}, \mathbb{G}/\mathbb{R})$ generated by the representations in $\Pi(\gamma \mathcal{O}, \mathbb{G}(\mathbb{R}, \delta_q))$. The ABV-packet $\Pi^\text{ABV}_{\gamma G}$ is defined as the support of $\eta^\text{ABV}_{\gamma G}$, that is

$$\Pi^\text{ABV}_{\gamma G} = \{ \pi(\xi) : \xi \in \Xi(\gamma \mathcal{O}, \gamma G^T), \chi^\text{mic}_{\gamma S_{\gamma G}}(P(\xi)) \neq 0, \pi(\xi) \in \Pi(\mathbb{G}(\mathbb{R}, \delta_q)) \}. \quad (53)$$

We conclude this section with a restatement of Theorem 3.1 which will be valuable later on. Define the representation-theoretic transition matrix $m_r$ by

$$M(\xi) = \sum_{\xi' \in \Xi(\gamma \mathcal{O}, \gamma G^T)} m_r(\xi', \xi) \pi(\xi').$$

Define the geometric “transition matrix” $c_g$ by

$$P(\xi) = \sum_{\xi' \in \Xi(\gamma \mathcal{O}, \gamma G^T)} (-1)^{d(\xi)} c_g(\xi', \xi) \mu(\xi').$$

(see [ABV, (7.11)(c)]). Then [ABV, Corollary 15.13] says

**Proposition 3.3.** Theorem 3.1 is equivalent to the identity

$$m_r(\xi', \xi) = (-1)^{d(\xi)-d(\xi')} c_g(\xi, \xi').$$

This equation relates the decomposition of characters with the decomposition of sheaves.
3.2 The pairing in the twisted case

In the previous section the pairing (49) plays a fundamental role in the definition of ABV-packets. We now develop a twisted version of this pairing for $R_{C/R}\text{GL}_N$.

We replace $K\Pi(\mathcal{O},\text{GL}_N(\mathbb{C}))$ with the $\mathbb{Z}$-module $K\Pi(\mathcal{O},\text{GL}_N(\mathbb{C}),\vartheta)$ of twisted characters (47). Associated to $\xi \in \Xi(\mathcal{O},\text{GL}_{C/R}\text{GL}_N^\vartheta)\vartheta$ are an irreducible representation $\pi(\xi) \in \Pi(\mathcal{O},\text{GL}_N(\mathbb{C}))\vartheta$ as well as the Atlas extension $\pi(\xi)^+ \rightarrow \text{GL}_N(\mathbb{C}) \rtimes \langle \vartheta \rangle$ (Corollary 2.5). The twisted character of $\pi(\xi)$ is an element of $K\Pi(\mathcal{O},\text{GL}_N(\mathbb{C}),\vartheta)$, the $\mathbb{Z}$-module of twisted characters, and this gives a basis of $K\Pi(\mathcal{O},\text{GL}_N(\mathbb{C}),\vartheta)$ parameterized by $\Xi(\mathcal{O},\text{GL}_{C/R}\text{GL}_N^\vartheta)$, see (45) and the end of Section 2.4.

The twisted characters are to be paired with twisted sheaves which are elements in a $\mathbb{Z}$-module generalizing $KX(\mathcal{O},\mathcal{G}^\vartheta)$. The twisted objects for this pairing are given in [ABV, (25.7)] (see also [CM, Section 5.4]). We provide a short summary.

The automorphism $\vartheta$ acts on $X(\mathcal{O},\mathcal{R}_{C/R}\text{GL}_N^\vartheta)$ in a manner which is compatible with its $\mathcal{R}_{C/R}\text{GL}_N^\vartheta$-action ([ABV, (25.1)]), and so also acts on $\mathcal{R}_{C/R}\text{GL}_N^\vartheta$-equivariant sheaves. Let

\[ \mathcal{P}(X(\mathcal{O},\mathcal{R}_{C/R}\text{GL}_N^\vartheta); \vartheta) \]

be the category of $\mathcal{R}_{C/R}\text{GL}_N^\vartheta$-equivariant perverse sheaves with a compatible $\vartheta$-action. An object in this category is a pair $(P, \vartheta_P)$ in which $P$ is an equivariant perverse sheaf and $\vartheta_P$ is an automorphism of $P$ which is compatible with $\vartheta$ ([CM, Section 5.4]). Similarly, we define

\[ \mathcal{C}(X(\mathcal{O},\mathcal{R}_{C/R}\text{GL}_N^\vartheta); \vartheta) \]

to be the category of $\mathcal{R}_{C/R}\text{GL}_N^\vartheta$-equivariant constructible sheaves with a compatible $\vartheta$-action. An object in this category is a pair $(\mu, \vartheta_\mu)$ in which $\mu$ is an equivariant constructible sheaf and $\vartheta_\mu$ is an automorphism of $\mu$ which is compatible with $\vartheta$.

The Grothendieck groups of these two categories are isomorphic ([CM, (35)]). We identify them and denote their Grothendieck groups by $K(X(\mathcal{O},\mathcal{R}_{C/R}\text{GL}_N^\vartheta); \vartheta)$. This is the sheaf-theoretic analogue of $K\Pi(\text{GL}_N(\mathbb{C}) \times \langle \vartheta \rangle)$.

As with the representations (see (44)), we seek a canonical choice of extension of $P(\xi)$, i.e., a canonical automorphism $\vartheta_{P(\xi)}$ of $P(\xi)$. This is achieved by the following lemma, whose proof follows exactly as for [AAM, Lemma 3.4] by virtue of (36).

Lemma 3.4. Let $\mathcal{G} = \mathcal{R}_{C/R}\text{GL}_N$, $\xi = (S, \tau_S) \in \Xi(\mathcal{O},\mathcal{R}_{C/R}\text{GL}_N^\vartheta)$, $p \in S$, and (48) be the equivariant vector bundle representing $\mu(\xi)$.

(a) Suppose $p' \in S$ and $p' = a \cdot p$ for some $a \in \mathcal{R}_{C/R}\text{GL}_N$. Then the maps

\begin{align*}
(g, v) &\mapsto (ga^{-1}, v) \\
g \cdot p &\mapsto (ga^{-1}) \cdot p'
\end{align*}

define an isomorphism of equivariant vector bundles

\[ \mathcal{G} \times_{\mathcal{G}_p} V \rightarrow \mathcal{G} \times_{\mathcal{G}_{p'}} V. \]

which is independent of the choice of $a$.

(b) There exist canonical choices of pairs

\[ \mu(\xi)^+ = (\mu(\xi), \vartheta_{\mu(\xi)}^+) \in \mathcal{C}(X(\mathcal{O},\mathcal{R}_{C/R}\text{GL}_N^\vartheta); \vartheta), \]

\[ P(\xi)^+ = (P(\xi), \vartheta_{P(\xi)}^+) \in \mathcal{P}(X(\mathcal{O},\mathcal{R}_{C/R}\text{GL}_N^\vartheta); \vartheta) \]

such that if $p \in S$ is fixed by $\vartheta$ then $\vartheta_{\mu(\xi)}^+$ (and $\vartheta_{P(\xi)}^+$) acts trivially on the stalk of $\mu(\xi)$ (and $P(\xi)$) in $KX(\mathcal{O},\mathcal{R}_{C/R}\text{GL}_N^\vartheta)$ at $p$. 

18
We now imitate the definition of $K^{\vee}O, GL_N(C), \vartheta)$ (47) for the sheaves appearing in Lemma 3.4. Attached to $\xi \in \Xi(\vee O, \vee R_{C/R}GL_N^\vartheta)$ are perverse sheaves $P(\xi)^\pm$, where $P(\xi)^+$ is defined in Lemma 3.4, and $P(\xi)^-$ is the unique other choice of extension. Furthermore, the microlocal traces of $P(\xi)^\pm$ differ by sign ([ABV, (25.1)(j)]). Similar comments apply to $\mu(\xi)^\pm$. The microlocal traces are used in Proposition 5.5 and are the analogues of twisted characters.

We are interested only in irreducible sheaves with non-vanishing microlocal trace. We consequently follow the definition of (47) in defining the quotient

$$KX(\vee O, \vee R_{C/R}GL_N^\vartheta, \vartheta) = K(X(\vee O, \vee R_{C/R}GL_N^\vartheta))^\vartheta \otimes \mathbb{Z}[U_2]/((P(\xi) \otimes 1) + (P(\xi) \otimes -1))$$

(54)

where the quotient runs over $\xi \in \Xi(\vee O, \vee R_{C/R}GL_N^\vartheta)^\vartheta$.

This is the $\mathbb{Z}$-module which we shall pair with $K^{\vee}O, GL_N(C), \vartheta)$ in Section 4. We call the elements of this module twisted sheaves, and remind the reader that these modules are not naturally Grothendieck groups, even though we have kept the “$K$” in the notation.

For reasons that will only become clear in Section 7, the definition of our twisted pairing involves some additional signs. The signs depend on the integral lengths of parameters, which may be described as follows.

From now on we assume $\lambda \in \vee O$ satisfies the regularity condition (37). Let $\xi \in \Xi(\vee O, \vee R_{C/R}GL_N^\vartheta)^\vartheta$. Lemma 2.2 tells us that associated to $\xi$ is an element $x \in X_{\vartheta}$. Set

$$\theta_x = \text{Int}(x)_{\vartheta}.$$ 

(55)

Let

$$R(\lambda) = \{ \alpha \in R(R_{C/R}GL_N, H) : \langle \lambda, \vee \alpha \rangle \in \mathbb{Z} \}$$

(56)(a)

be the $\lambda$-integral roots, with positive $\lambda$-integral roots

$$R^+(\lambda) = \{ \alpha \in R(\lambda) : \langle \lambda, \vee \alpha \rangle > 0 \}.$$ 

(56)(b)

Define the integral length, following [ABV, (16.16)], as

$$l_I(\xi) = -\frac{1}{2} \left( \left| \{ \alpha \in R^+(\lambda) : \theta_x(\alpha) \in R^+(\lambda) \} \right| + \dim(H^{\vartheta_+}) \right) + \frac{N}{2}. $$

(57)

The integral length takes values in the non-positive integers.

Furthermore define

$$R_0^+(\lambda) = \{ \alpha \in R(R_{C/R}GL_N^\vartheta)^\vartheta, (H^{\vartheta_+})^\vartheta : \langle \lambda, \vee \alpha \rangle \in \mathbb{Z}_{>0} \}.$$ 

We define the $\vartheta$-integral length by

$$l_0^I(\xi) = -\frac{1}{2} \left( \left| \{ \alpha \in R_0^+(\lambda) : \theta_x(\alpha) \in R_0^+(\lambda) \} \right| + \dim((H^{\vartheta_+})^\vartheta) \right) + \frac{[N/2]}{2},$$

(58)

where $[N/2]$ is the least integer greater than or equal to $N/2$. This is the integral length for the fixed-point group $(R_{C/R}GL_N)^\vartheta \cong GL_N$, or more precisely, for a quasisplit unitary group of Section 2.2.

Now we define a perfect pairing (under the assumption (37))

$$\langle \cdot, \cdot \rangle : K^{\vee}O, GL_N(C), \vartheta \times KX(\vee O, \vee R_{C/R}GL_N^\vartheta, \vartheta) \to \mathbb{Z}$$

(59)

by setting

$$\langle M(\xi)^+, \mu(\xi')^+ \rangle = (-1)^{l_0^I(\xi) - l_0^I(\xi')} \delta_{\xi, \xi'}$$

(60)

for $\xi, \xi' \in \Xi(\vee O, \vee R_{C/R}GL_N^\vartheta)^\vartheta$. The analogue of Theorem 3.1 is
Theorem 3.5. Suppose $\lambda \in {}^\vee \mathcal{O}$ satisfies (37). Define the pairing (59) by (60). Then

$$\langle \pi(\xi)^+, P(\xi')^+ \rangle = (-1)^d(\xi) (-1)^{\nu(\xi')-\nu(\xi)} \delta_{\xi,\xi'}$$

where $\xi, \xi' \in {}^\vee \mathcal{O}, {}^\vee \mathcal{R}_{\mathcal{C}/R} \mathcal{GL}_N^\Gamma$.

The proof of this theorem is the primary purpose of Section 4. Its proof is modelled on the proof of Theorem 3.1 in [ABV, Sections 15-17].

We conclude this section by giving a twisted analogue of Proposition 3.3. This analogue will only be needed in Sections 7 and 9, so the reader may wish to skip this discussion and return to it later.

For $\xi, \xi' \in {}^\vee \mathcal{O}, {}^\vee \mathcal{R}_{\mathcal{C}/R} \mathcal{GL}_N^\Gamma$, define $m_\nu(\xi,\xi')$ to be the multiplicity of the representation $\pi(\xi')^+$ in $M(\xi)^+$ as elements of the Grothendieck group $K_{\mathcal{X}}(\mathcal{O}, \mathcal{GL}_N(\mathcal{C}) \times \langle \vartheta \rangle)$ (Section 2.5). In other words

$$M(\xi)^+ = \sum_{\xi' \in {}^\vee \mathcal{O}, \mathcal{R}_{\mathcal{C}/R} \mathcal{GL}_N^\Gamma} m_\nu(\xi',\xi) \pi(\xi')^+ + \cdots$$

where the omitted summands are irreducible representations of $\mathcal{GL}_N(\mathcal{C}) \times \langle \vartheta \rangle$ which restrict to the sum of two irreducible representations of $\mathcal{GL}_N(\mathcal{C})$. Define the twisted multiplicity of $\pi(\xi')^+$ in $M(\xi)^+$ by

$$m_\nu^\varrho(\xi',\xi) = m_\nu(\xi',\xi_+) - m_\nu(\xi',\xi_-), \quad \xi, \xi' \in {}^\vee \mathcal{O}, {}^\vee \mathcal{R}_{\mathcal{C}/R} \mathcal{GL}_N^\Gamma$$

(cf. [AvLTV, (19.3d)]). By construction, the image of $M(\xi)^+$ in $K_{\mathcal{X}}(\mathcal{O}, \mathcal{GL}_N(\mathcal{C}), \vartheta)$ (47) decomposes as

$$M(\xi)^+ = \sum_{\xi' \in {}^\vee \mathcal{O}, \mathcal{GL}_N^\Gamma} m_\nu^\varrho(\xi',\xi) \pi(\xi')^+.$$

The matrix given by (61) is invertible ([AAM, Lemma 3.6]).

In a parallel fashion, we define $c_\varrho(\xi',\xi)$ for $\xi, \xi' \in {}^\vee \mathcal{O}, \mathcal{R}_{\mathcal{C}/R} \mathcal{GL}_N^\Gamma$ by

$$P(\xi)^+ = \sum_{\xi' \in {}^\vee \mathcal{O}, \mathcal{R}_{\mathcal{C}/R} \mathcal{GL}_N^\Gamma} (-1)^d(\xi) c_\varrho(\xi',\xi_+) \mu(\xi')^+ + (-1)^d(\xi') c_\varrho(\xi',\xi_+) \mu(\xi')^- + \cdots$$

in the Grothendieck group $K_{\mathcal{X}}(\mathcal{O}, \mathcal{R}_{\mathcal{C}/R} \mathcal{GL}_N, \vartheta)$ of Section 3.2. Setting

$$c_\varrho(\xi',\xi) = c_\varrho(\xi',\xi_+) - c_\varrho(\xi',\xi_-).$$

we see that the image of $P(\xi)^+$ in $K_{\mathcal{X}}(\mathcal{O}, \mathcal{R}_{\mathcal{C}/R} \mathcal{GL}_N, \vartheta)$ is

$$\sum_{\xi' \in {}^\vee \mathcal{O}, \mathcal{R}_{\mathcal{C}/R} \mathcal{GL}_N^\Gamma} (-1)^d(\xi') c_\varrho(\xi',\xi) \mu(\xi')^+.$$ Just as Theorem 3.1 is equivalent to Proposition 3.3. We have the following equivalence.

Proposition 3.6 ([AAM, Proposition 3.7]). Theorem 3.5 is equivalent to the identity

$$m_\nu^\varrho(\xi',\xi) = (-1)^{\nu(\xi')-\nu(\xi)} c_\varrho(\xi',\xi)$$

for all $\xi, \xi' \in {}^\vee \mathcal{O}, {}^\vee \mathcal{R}_{\mathcal{C}/R} \mathcal{GL}_N^\Gamma$.

4 The proof of Theorem 3.5

We continue working with an infinitesimal character $\lambda \in {}^\vee \mathcal{O}$ which satisfies the hypothesis of (37). The proof of Theorem 3.5 is nearly identical to the proof of [AAM, Theorem 3.5]. In fact, the proof of Theorem 3.5 is somewhat simpler. In order to convince the reader of these claims we review the proof of [AAM, Theorem 3.5] in our context, calling attention to the points which are simplified.

The general idea of the proof is already given in the non-twisted context of [ABV, Sections 15-17]. In the proof one first extends the $\mathbb{Z}$-modules appearing in Theorem 3.5 to Hecke modules.
acted upon by a Hecke algebra. The extended pairing is then shown to furnish an isomorphism between one of the Hecke modules and the dual of the other. Special bases are chosen for the Hecke modules. The values of the special bases are explicitly computed under the pairing. Theorem 3.5 then follows by restricting these values to the setting of the original Z-modules.

There are numerous Hecke module computations underlying this proof, and many of them have been completed in [LV] and [AV]. The computations of [AV] are given in representation-theoretic language and are therefore suitable when working with $K\Pi(\mathcal{O}, GL_N(C), \vartheta)$. In order to adapt the computations of [AV] to $KX(\mathcal{O}, GL_N(C), \vartheta)$ we replace this module with an equivalent module of representations. In so doing, we touch on the notion of Vogan duality ([AV, Section 6.1]). We attend to this preliminary work in the next section.

### 4.1 Vogan duality and $KX(\mathcal{O}, R_{C/R}GL_N, \vartheta)$

We wish to replace the sheaf-theoretic module $KX(\mathcal{O}, R_{C/R}GL_N, \vartheta)$ with an equivalent module of representations. In the non-twisted setting this is achieved by [ABV, Theorem 8.5]. This theorem relies on two correspondences. The first correspondence is the Riemann-Hilbert correspondence, which furnishes an equivalence between the category of equivariant perverse sheaves on $X(\mathcal{O}, R_{C/R}GL_N)$ and a category of equivariant D-modules on the same space ([ABV, Theorem 7.9], [BBD, Theorem VIII.14.4]). The second correspondence is the Beilinson-Bernstein localization theorem, which furnishes an equivalence between the category of equivariant D-modules and a category of Harish-Chandra modules ([ABV, Theorem 8.3], [V1, Proposition 1.2]).

Combining the two correspondences produces a bijection between a set of irreducible perverse sheaves and a set of irreducible representations (Harish-Chandra modules). The set of irreducible perverse sheaves is

$$\{P(\xi) : \xi \in \Xi(\mathcal{O}, R_{C/R}GL_N)\}$$

(Section 3.1). We denote the set of corresponding representations by $\Pi(\mathcal{O}, GL_N(C))$, so that the bijection may be written as

$$\{P(\xi) : \xi \in \Xi(\mathcal{O}, R_{C/R}GL_N)\} \leftrightarrow \Pi(\mathcal{O}, GL_N(C)). \quad (64)$$

The notation for the set of representations on the right hints at some manner of duality with $\Pi(\mathcal{O}, GL_N(C))$. The particulars of this duality are given in [AAM, Section 4.2] for $GL_N(R)$. The arguments there apply just as well to $GL_N(C)$ and are summarized as follows. By Lemma 2.2 a complete geometric parameter $\xi \in \Xi(\mathcal{O}, R_{C/R}GL_N)$ corresponds to a unique Atlas parameters $(x, y) \in \mathcal{X}_\rho \times \mathcal{X}_\lambda$. By reversing the order of the entries in the Atlas parameter, one obtains an Atlas parameter $(y, x) \in \mathcal{X}_\lambda \times \mathcal{X}_\rho$ for the group

$$\mathcal{R}_{C/R}GL_N(\lambda) = \text{centralizer in } \mathcal{R}_{C/R}GL_N \text{ of } \exp(2\pi i \lambda) \quad (65)$$

(cf. [AAM, Lemma 4.2]). The irreducible representation $J(y, x, \varrho)$ (cf. (41)) is a $((\mathcal{R}_{C/R}GL_N)_{(\lambda)} \times \mathcal{K}_y)$-module, where $\mathcal{K}_y$ is a two-fold cover of $\mathcal{K}_y$ (cf. (24)). It has infinitesimal character $\varrho$. This representation may be made plainer by computing that $\mathcal{R}_{C/R}GL_N(\lambda)$ is a product of groups $\prod_i \mathcal{R}_{C/R}GL_{n_i}$. As noted in Section 2.2, this product has only one real form, namely $\prod_i GL_{n_i}(C)$. Consequently, $J(y, x, \varrho)$ is a representation of a double-cover of $\prod_i GL_{n_i}(C)$.

Bijection (64) is given by

$$P(\xi) \mapsto J(y, x, \varrho)$$

([AAM, Proposition 4.3]). We streamline the notation by the identification $\xi = (x, y)$ using Lemma 2.2, and defining the dual parameter $\varrho$ by

$$\varrho \xi = (y, x) \iff \xi = (x, y). \quad (66)$$

We denote the dual representation by

$$\pi(\varrho \xi) = J(y, x, \varrho).$$
In this way, bijection (64) takes the form
\[ P(\xi) \leftrightarrow \pi(\xi'), \quad \xi \in \Xi(\mathcal{O}, \mathcal{R}_{C/\mathbb{R}} \mathbf{GL}_N^\Gamma). \]

Every representation \(\pi(\xi')\) is the unique irreducible (Langlands) quotient of a standard module, which we denote by \(M(\xi')\) ([AV, (20)]). Bijection (64) extends to an isomorphism of Grothendieck groups
\[ \mathcal{K}_X(\mathcal{O}, \mathcal{R}_{C/\mathbb{R}} \mathbf{GL}_N^\Gamma) \cong K^{\mathcal{V}} \Pi(\mathcal{O}, \mathbf{GL}_N(\mathbb{C})). \]

which satisfies
\[ (-1)^{d(\xi)} \mu(\xi) \to M(\xi), \quad \xi \in \Xi(\mathcal{O}, \mathcal{R}_{C/\mathbb{R}} \mathbf{GL}_N^\Gamma) \]
([AAM, Proposition 4.3]).

The isomorphism (67) may be generalized to the twisted setting as well. Recall from (44) that the Atlas extension \(\pi(\xi)^+\) is defined from a preferred extended parameter. The same extended parameter also determines a unique extension \(\pi(\xi)^+\) of \(\pi(\xi')\) ([AV, (39h)]). It is a \((\mathcal{V}_{\mathcal{N}} \times \mathfrak{gl}_{\mathcal{N}}(\lambda), \mathcal{K}_{\mathcal{X}} \rtimes \langle \theta \rangle)\)-module, which is the unique irreducible quotient of an extension \(M(\xi)^+\) of \(M(\xi')\). Define a bijection
\[ P(\xi)^+ \leftrightarrow \pi(\xi')^+, \quad \xi \in \Xi(\mathcal{O}, \mathcal{R}_{C/\mathbb{R}} \mathbf{GL}_N^\Gamma)^\theta. \]

The bijection induces an isomorphism of \(\mathcal{Z}\)-modules
\[ \mathcal{K}_X(\mathcal{O}, \mathcal{R}_{C/\mathbb{R}} \mathbf{GL}_N^\Gamma, \mathcal{V}) \cong K^{\mathcal{V}} \Pi(\mathcal{O}, \mathbf{GL}_N(\mathbb{C}), \mathcal{V}) \]
which satisfies
\[ (-1)^{d(\xi)} \mu(\xi) \to M(\xi)^+, \quad \xi \in \Xi(\mathcal{O}, \mathcal{R}_{C/\mathbb{R}} \mathbf{GL}_N^\Gamma)^\theta \]
([AAM, Proposition 4.5]).

Making this replacement in Theorem 3.5, and taking into account the fixed relationship between \(d(\xi)\) and \(\ell(\xi)\) ([AMR2, Proposition B.1]), we obtain

**Lemma 4.1** ([AAM, Lemma 4.6]). **Theorem 3.5** is equivalent to the following assertion. The pairing
\[ \langle \cdot, \cdot \rangle : K^{\mathcal{V}} \Pi(\mathcal{O}, \mathbf{GL}_N(\mathbb{C}), \mathcal{V}) \times K^{\mathcal{V}} \Pi(\mathcal{O}, \mathbf{GL}_N(\mathbb{C}), \mathcal{V}) \to \mathcal{Z} \]
defined by
\[ \langle M(\xi)^+, M(\xi')^+ \rangle = (-1)^{\ell(\xi)} \delta_{\xi, \xi'} \]
satisfies
\[ \langle \pi(\xi)^+, \pi(\xi')^+ \rangle = (-1)^{\ell(\xi)} \delta_{\xi, \xi'}, \]
where \(\xi, \xi' \in \Xi(\mathcal{O}, \mathcal{R}_{C/\mathbb{R}} \mathbf{GL}_N^\Gamma)^\theta.\)

### 4.2 Twisted Hecke modules

The proof of Theorem 3.5 relies on a Hecke algebra and Hecke modules, which we introduce in the context of (68). In the twisted setting, Lusztig and Vogan define a Hecke algebra which we denote by \(\mathcal{H}(\lambda)\) ([LV, Section 3.1]). This Hecke algebra acts on the Hecke modules
\[ K^{\mathcal{V}} \Pi(\mathcal{O}, \mathbf{GL}_N(\mathbb{C}), \mathcal{V}) \cong K^{\mathcal{V}} \Pi(\mathcal{O}, \mathbf{GL}_N(\mathbb{C}), \mathcal{V}) \otimes_{\mathbb{Z}} \mathbb{Z}[q^{1/2}, q^{-1/2}] \]
and
\[ K^{\mathcal{V}} \Pi(\mathcal{O}, \mathbf{GL}_N(\mathbb{C}), \mathcal{V}) \cong K^{\mathcal{V}} \Pi(\mathcal{O}, \mathbf{GL}_N(\mathbb{C}), \mathcal{V}) \otimes_{\mathbb{Z}} \mathbb{Z}[q^{1/2}, q^{-1/2}] \]
as in [LV, Section 7]. We extend the pairing (68) to these Hecke modules
\[ \langle \cdot, \cdot \rangle : K^{\mathcal{V}} \Pi(\mathcal{O}, \mathbf{GL}_N(\mathbb{C}), \mathcal{V}) \times K^{\mathcal{V}} \Pi(\mathcal{O}, \mathbf{GL}_N(\mathbb{C}), \mathcal{V}) \to \mathbb{Z}[q^{1/2}, q^{-1/2}], \]
by setting
\[ \langle M(\xi)^+, M(\xi')^+ \rangle = (-1)^{\ell(\xi)} q^{(\ell(\xi)+\ell(\xi'))/2} \delta_{\xi, \xi'} \]
for all $\xi, \xi' \in \Xi(\wp\wp,^\wp\wp R_{C/\mathbb{R}}^\wp GL_N^\wp)$. In view of the Kronecker delta, the term $q^{1/2(t'(\xi)+t'(\xi'))}$ in the pairing could be replaced by $q^{1/2(t'(\xi)+t'(\xi'))}$ or $q^{1/2(t'(\xi)+t'(\xi'))}$. In fact, both of the latter terms are independent of $\xi$ or $\xi'$ ([AAM, Lemma 4.7]).

To say more about the Hecke algebra $\mathcal{H}(\lambda)$, we must examine the set of integral roots $R(\lambda)$ (56). Let $\kappa$ be a $\vartheta$-orbit on the set of simple roots of $R^+(\lambda)$. The orbit $\kappa$ is equal to one of the following:

\begin{align*}
\text{one root } \{ \alpha = \vartheta(\alpha) \} & \quad \text{(type 1)} \\
\text{two roots } \{ \alpha, \beta = \vartheta(\alpha) \}, & \quad \langle \alpha, \wp \beta \rangle = 0 \quad \text{(type 2)} \quad (70) \\
\text{two roots } \{ \alpha, \beta = \vartheta(\alpha) \}, & \quad \langle \alpha, \wp \beta \rangle = -1 \quad \text{(type 3)}.
\end{align*}

It is clear that our automorphism $\vartheta$ renders all orbits $\kappa$ to be of type 2. This is a notable simplification in our setting.

Write $W(\lambda)$ for the Weyl group of the integral roots $R(\lambda)$, and let

$$W(\lambda) = \{ w \in W(\lambda) : \vartheta(w) = w \}.$$ 

The group $W(\lambda)$ is a Coxeter group ([LV, Section 4.3]) with generators

$$w_\kappa = s_\alpha s_{\vartheta(\alpha)}, \quad \kappa = \{ \alpha, \vartheta(\alpha) \}.$$ 

(71)

The Hecke algebra $\mathcal{H}(\lambda)$ ([AV, Section 10], [LV, Section 4.7]) is a free $\mathbb{Z}[q^{1/2}, q^{-1/2}]$-algebra with basis

$$\{ T_w : w \in W(\lambda) \}.$$ 

It is a consequence of [LV, Equation 4.7 (a)] that $\mathcal{H}(\lambda)$ is generated by the operators $T_\kappa := T_{w_\kappa}$, where $\kappa$ is a $\vartheta$-orbit as in (70).

The action of $T_\kappa$ is defined in terms of the types listed in (70), which for us are only of type 2. The action also depends on the relationship of $\kappa$ relative to a fixed parameter $\xi \in \Xi(\wp\wp \wp \wp,^\wp\wp R_{C/\mathbb{R}}^\wp GL_N^\wp)$. To say a bit more about this dependence, recall that the parameter $\xi$ is equivalent to an Atlas parameter $(x, y)$ as in Lemma 2.2. The adjoint action of $x$ acts as an involution on $R(\lambda)$ (see (39)). This action separates the $\vartheta$-orbits of roots $\kappa$ into various types, e.g. real, imaginary, etc. Lusztig and Vogan combine this information with the types of (70) and also with the types defined by Vogan in [V3, Section 8.3]. The list of combined types may be found in [LV, Section 7] or [AV, Table 1].

Very few of the types that appear in these lists are relevant for $GL_N(\mathbb{C})$. We have already observed that only type 2 orbits appear in the sense of (70). Furthermore, $GL_N(\mathbb{C})$ has only complex roots relative to $x$ (32). The only relevant types for $GL_N(\mathbb{C})$ in [AV, Table 1] are labelled as

$$2C_+, 2C_-, 2C_1, 2C_r.$$ 

(72)

That only these four types are relevant to our setting is another notable simplification. Any $\vartheta$-orbit $\kappa$ also has a type relative to the dual parameter $^\wp(\xi)$ (66). The dual parameter is equivalent to the Atlas parameter $(y, x)$ and the adjoint action of $y$ is essentially the negative of the adjoint action of $x$ ([AV, Definition 3.10]). In consequence, it is easy to compute the types and see that we again recover exactly those listed in (72).

In [LV, Section 4] and [AV, Section 7] the Hecke algebra action on $\mathcal{K}(\wp\wp \wp \wp, GL_N(\mathbb{C}), \vartheta)$ is given by defining the action of the operators $T_\kappa$ on the generating set $\{ M(\xi) : \xi \in \Xi(\wp\wp \wp \wp,^\wp\wp R_{C/\mathbb{R}}^\wp GL_N^\wp) \}$. The actions are presented in terms of extended Atlas parameters in [AV, Proposition 10.4]. A case-by-case summary of the actions is given in [AV, Table 5].

There are obvious parallel constructions for (65) which define a Hecke algebra $^\wp \mathcal{H}(\lambda)$ and a Hecke module structure for $^\wp \mathcal{K}(\wp, GL_N(\mathbb{C}), \vartheta)$. The Hecke algebra $^\wp \mathcal{H}(\lambda)$ for (65) is generated by Hecke operators $T_{\kappa,\kappa}$, where $^\wp \kappa$ runs over the simple coroots corresponding to $\kappa$. The bijection between the two sets of operators

$$\{ T_\kappa : \kappa \in R^+(\lambda) \text{ simple} \} \longleftrightarrow \{ T_{\kappa,\kappa} : \kappa \in R^+(\lambda) \text{ simple} \}$$

extends to an isomorphism $\mathcal{H}(\lambda) \cong ^\wp \mathcal{H}(\lambda)$. For this reason, we also regard $^\wp \mathcal{K}(\wp, GL_N(\mathbb{C}), \vartheta)$ as an $\mathcal{H}(\lambda)$-module.
4.3 A Hecke module isomorphism

Let
\[ K^\vee \Pi(\gamma O, GL_N(\mathbb{C}), \vartheta)^* = \text{Hom}_{\mathbb{Z}[q^{1/2}, q^{-1/2}]} \left( K^\vee \Pi(\gamma O, GL_N(\mathbb{C}), \vartheta), \mathbb{Z}[q^{1/2}, q^{-1/2}] \right). \]

The extended pairing (69) induces a \( \mathbb{Z}[q^{1/2}, q^{-1/2}] \)-module isomorphism
\[ K\Pi(\gamma O, GL_N(\mathbb{C}), \vartheta) \rightarrow K^\vee \Pi(\gamma O, GL_N(\mathbb{C}), \vartheta)^* \]
\[ M(\xi)^\perp \mapsto (M(\xi)^\perp)^* \tag{73} \]

We endow \( K^\vee \Pi(\gamma O, GL_N(\mathbb{C}), \vartheta)^* \) with the Hecke module structure given in [AV, Section 11] (cf. [AAM, Section 4.5]). Specifically, for any \( \mu \in K^\vee \Pi(\gamma O, GL_N(\mathbb{C}), \vartheta)^* \) and \( \vartheta \)-orbit \( \kappa \) as in (70)
\[ T_w^* \cdot \mu = -(T_w)^\dagger \cdot \mu + (q^{|w\kappa|} - 1)\mu, \tag{74} \]
where \( l(w) \) is the usual length of \( w \) with respect to the simple reflections, and \( (T_w)^\dagger \) is the transpose of \( T_w \). Since both the domain and codomain of (73) are \( \mathcal{H}(\lambda) \)-modules it is natural to ask whether (73) extends to a \( \mathcal{H}(\lambda) \)-module isomorphism.

**Proposition 4.2.** The map (73) is an isomorphism of \( \mathcal{H}(\lambda) \)-modules.

The proof of this proposition is a simplified version of the proof of [AAM, Proposition 4.8]. We provide a sketch. In view of (74), Proposition 4.2 is equivalent to
\[ \langle T_w \cdot M(\xi_1)^+, M(\gamma \xi_2)^+ \rangle = \langle M(\xi_1)^+, -T_w \cdot M(\gamma \xi_2)^+ + (q^{|w\kappa|} - 1)M(\gamma \xi_2)^+ \rangle \tag{75} \]
for all \( \xi_1, \xi_2 \in \Xi(\gamma O, \gamma R_{C/\mathbb{R}} GL_N^\gamma) \) and \( w_0 \) as in (71). Looking back to the definition of (69), the left-hand-side of (75) may be expressed as
\[ \langle T_w \cdot M(\xi_1)^+, M(\gamma \xi_2)^+ \rangle = (-1)^{l_w(\xi_2)}q^{l_w(\xi_2) + l(\gamma \xi_2)/2} \cdot \text{(the coefficient of } M(\gamma \xi_2)^+ \text{ in } T_w \cdot M(\xi_1)^+) \rangle. \]

Similarly, the right-hand-side of (75) may be expressed as the product of \( (-1)^{l_w(\xi_1)}q^{l_w(\xi_1) + l(\gamma \xi_1)/2} \) with

the coefficient of \( M(\gamma \xi_1)^+ \) in \( -T_w \cdot M(\gamma \xi_2)^+ + (q^{|w\kappa|} - 1)M(\gamma \xi_2)^+ \).

As in [AAM, Lemma 4.7],
\[ l(\gamma \xi_1) + l(\gamma \xi_2) = l(\xi_1) + l(\xi_2), \]
so Equation (75) is equivalent to
\[ (-1)^{l_w(\xi_2) - l_w(\xi_1)} \cdot \text{(the coefficient of } M(\xi_2)^+ \text{ in } T_w \cdot M(\xi_1)^+) \rangle = \text{the coefficient of } M(\gamma \xi_1)^+ \text{ in } -T_w \cdot M(\gamma \xi_2)^+ + (q^{|w\kappa|} - 1)M(\gamma \xi_2)^+. \tag{76} \]

The values of \( T_w \cdot M(\xi_1)^+ \) and \( T_w \cdot M(\gamma \xi_2)^+ \) are known ([AV, Table 5], [AAM, Proposition 4.9]). The proof of (76) may therefore be achieved by a case-by-case analysis of the types of \( \kappa \) relative to \( \xi_1, \xi_2 \), \( \gamma \xi_1 \) and \( \gamma \xi_2 \) (72).

We provide some more detail in the case that \( \kappa = \{ \alpha, \beta \} \) is of type 2C1 relative to \( \xi_1 \), leaving the remaining cases to the reader. We identify \( \xi_1 \) with its corresponding Atlas parameter \( (x_1, y_1) \) (Lemma 2.2) and do likewise for all other parameters to come. Type 2C1 implies \( \beta = x_1 \cdot \alpha \).

Let us start by computing the left-hand side of (76). According to [AV, Proposition 9.1 and Proposition 10.4], we have
\[ T_w \cdot M(\xi_1)^+ = qM(\xi_1)^+ + (q + 1)M(\xi)^+, \]
where \( \xi = (x, y) \in \Xi(\gamma O, \gamma R_{C/\mathbb{R}} GL_N^\gamma) \) is a parameter satisfying
\[ x = s_\alpha x_1 s_\alpha^{-1} = s_\alpha s_\beta x_1 = x_1 s_\alpha s_\beta \]
\[ (77) \]

24
Atlas extensions. Consequently, the coefficient of \( M(\xi_2)^+ \) in \( T_{\psi*}M(\xi_1)^+ \) is
\[
\begin{cases} 
q, & \text{if } \xi_2 = \xi_1 \\
(q+1), & \text{if } \xi_2 = \xi \\
0, & \text{otherwise.}
\end{cases}
\]

To compute the \( \vartheta \)-integral lengths (58), observe that
\[
x \cdot \{\alpha, \beta\} = s_\alpha s_\beta x_1 \cdot \{\alpha, \beta\} = \{-\alpha, -\beta\}.
\]
and for any positive root \( \gamma \in R^+(\lambda) \) with \( \gamma \neq \alpha, \beta \), we have
\[
x \cdot \{\gamma, \vartheta \gamma\} = x_1 s_\alpha s_\beta \{\gamma, \vartheta \gamma\} = x_1 \cdot \{\gamma', \vartheta \gamma'\},
\]
where \( \gamma' \) is a positive root in \( R^+(\lambda) \). Hence, after identifying the roots in \( R_0^+(\lambda) \) with \( \vartheta \)-orbits in \( R^+(\lambda) \), we deduce
\[
|\{\gamma \in R_0^+(\lambda) : x \cdot \gamma \in R_0^+(\lambda)\}| = |\{\gamma \in R_0^+(\lambda) : x_1 \cdot \gamma \in R_0^+(\lambda)\}| - 1,
\]
and
\[
\dim ((H^\theta)^+) = \dim ((H^\theta)^{x_1}) - 1.
\]
It follows that
\[
l_0^\theta(\xi) = l_0^\theta(\xi_1) - 1 \quad \text{and} \quad (-1)^{l_0^\theta(\xi) - l_0^\theta(\xi_1)} = -1.
\]
The left-hand side of (76) is therefore equal to
\[
\begin{cases} 
q, & \text{if } \xi_2 = \xi_1 \\
(q+1), & \text{if } \xi_2 = \xi \\
0, & \text{otherwise.}
\end{cases}
\]
Let us consider the right-hand side of (76), in which \( \gamma \) is of type \( 2C_r \) relative to \( \gamma \xi_1 \). According to [AV, Table 5], \( M(\gamma \xi_1)^+ \) occurs in \( T_{\psi*}M(\gamma \xi_2)^+ \) only if one of the following holds
1. \( M(\gamma \xi_1)^+ = M(\gamma \xi_2)^+ \),
2. \( M(\gamma \xi_1)^+ = s\nu_\alpha \times M(\gamma \xi_2)^+ \),
3. \( M(\gamma \xi_1)^+ = w_{\psi \kappa} \times M(\gamma \xi_2)^+ \).

The third equation holds if and only if \( M(\gamma \xi_2)^+ = w_{\psi \kappa} \times M(\gamma \xi_1)^+ \), and for \( \psi \kappa \) of type \( 2C_r \) relative to \( \gamma \xi_1 \) one obtains
\[
w_{\psi \kappa} \times M(\gamma \xi_1)^+ = s\nu_\alpha \times (s\nu_\beta \times M(\gamma \xi_1)^+) = M(\gamma \xi_1)^+.
\]
Therefore the third equation is equivalent to the first one.

The second equation is equivalent to \( \gamma \xi_2 \) being equal to \( \gamma \xi \) (77), [AAM, Proposition 4.9]). Moreover, \( \psi \kappa \) is of type \( 2C_1 \) relative to \( \gamma \xi \). The results [AV, Proposition 9.1 and Proposition 10.4] indicate that
\[
T_{w_{\psi \kappa}} M(\gamma \xi_1)^+ = (q^2 - q - 1)M(\gamma \xi_1)^+ + (q^2 - q)M(\gamma \xi)^+, \\
T_{w_{\psi \kappa}} M(\gamma \xi)^+ = qM(\gamma \xi)^+ + (q + 1)M(\gamma \xi_1)^+.
\]
Therefore, the right-hand side of (76) equals
\[
\begin{cases} 
q, & \text{if } \gamma \xi_2 = \gamma \xi_1 \\
(q+1), & \text{if } \gamma \xi_2 = \gamma \xi \\
0, & \text{otherwise.}
\end{cases}
\]
In all cases we have equality with (78).
4.4 Special bases and the proof of Theorem 3.5

In addition to the $\mathcal{H}(\lambda)$-action on $\mathcal{K}(\mathcal{O}, \text{GL}_N(\mathbb{C}), \vartheta)$, there is a $\mathbb{Z}$-linear involution $D$ on $\mathcal{K}(\mathcal{O}, \text{GL}_N(\mathbb{C}), \vartheta)$ satisfying

$$D(q^{1/2} M(\xi)^+) = q^{-1/2} D((\xi)^+)$$

$$D(T_\kappa + 1)M(\xi)^+ = q^{-(w_\kappa)}(T_\kappa + 1)D(M(\xi)^+)$$ \hspace{1cm} (79)

for all $\xi \in \Xi(\mathcal{O}, \mathcal{O}_C \odot \text{GL}_N^\vartheta)$. This is the Verdier duality map \cite[(LV, 4.8 (e)-(f))]{LV}. The Verdier duality map is uniquely determined by the additional conditions given in [LV, 8.1 (a)].

The special basis we seek for $\mathcal{K}(\mathcal{O}, \text{GL}_N(\mathbb{C}), \vartheta)$ is a basis of eigenvectors for $D$. It is defined in terms of the twisted KL-polynomials $P^\vartheta(\xi', \xi) \in \mathbb{Z}[q^{1/2}, q^{-1/2}]$ defined in [LV, Section 0.1]. Its definition and characteristics are summarized in following theorem.

Theorem 4.3 \cite[(LV, Theorem 5.2)]{LV}. For every $\xi \in \Xi(\mathcal{O}, \mathcal{O}_C \odot \text{GL}_N^\vartheta)$, define

$$C^\vartheta(\xi) = \sum_{\xi' \in \Xi(\mathcal{O}, \mathcal{O}_C \odot \text{GL}_N^\vartheta)} (-1)^{(l(\xi') - l(\xi))} P^\vartheta(\xi', \xi) M(\xi')^+,$$ \hspace{1cm} (80)

an element in $\mathcal{K}(\mathcal{O}, \text{GL}_N(\mathbb{C}), \vartheta)$. Then

1. $D(C^\vartheta(\xi)) = q^{-l(\xi)} C^\vartheta(\xi)$
2. $P^\vartheta(\xi, \xi) = 1$
3. $P^\vartheta(\xi', \xi) = 0$ if $\xi' \not\sim \xi$
4. $\deg P^\vartheta(\xi', \xi) \leq (l(\xi) - l(\xi') - 1)/2$ if $\xi' \preceq \xi$.

Conversely suppose \{C(\xi', \xi)\} and \{P(\xi', \xi)\} satisfy (80) and (1)-(4) above. Then $P(\xi', \xi) = P^\vartheta(\xi', \xi)$ and $C(\xi', \xi) = C^\vartheta(\xi', \xi)$ for all $\xi, \xi' \in \Xi(\mathcal{O}, \mathcal{O}_C \odot \text{GL}_N^\vartheta)$.

The third property of this theorem uses a partial order on $\Xi(\mathcal{O}, \mathcal{O}_C \odot \text{GL}_N^\vartheta)$, the Bruhat order, which is defined in [LV, Section 5.1] \cite{LV}.

There is also a Verdier duality map $\mathcal{V}$ for the module $\mathcal{K}(\mathcal{O}, \text{GL}_N(\mathbb{C}), \vartheta)$ which satisfies obvious analogues of (79) and the additional conditions of [LV, 8.1 (a)]. Furthermore, there is an obvious analogue Theorem 4.3 for the dual basis elements

$$C^\vartheta(\xi) = \sum_{\xi' \in \Xi(\mathcal{O}, \mathcal{O}_C \odot \text{GL}_N^\vartheta)} (-1)^{(l(\xi') - l(\xi))} \mathcal{V} P^\vartheta(\xi', \xi) M(\xi')^+.$$

According to [AAM, Proposition 4.14], by setting $q = 1$ in the polynomials of the basis elements, one obtains

$$C^\vartheta(\xi)(1) = \pi(\xi)^+ \quad \text{and} \quad C^\vartheta(\xi')(1) = \pi(\xi')^+ \hspace{1cm} (81)$$

for all $\xi \in \Xi(\mathcal{O}, \mathcal{O}_C \odot \text{GL}_N^\vartheta)$. It is immediate from Equation (81) and Lemma 4.1 that Theorem 3.5 is proved by the following theorem.

Theorem 4.4. Pairing (69) satisfies

$$\langle C^\vartheta(\xi), C^\vartheta(\xi') \rangle = (-1)^{l(\xi)} q^{(l(\xi) + l(\xi'))/2} \delta_{\xi, \xi'}.$$ 

for all $\xi, \xi' \in \Xi(\mathcal{O}, \mathcal{O}_C \odot \text{GL}_N^\vartheta)$.

Proof. The isomorphism (73) allows us to define unique elements $C^\vartheta(\xi) \in \mathcal{K}(\mathcal{O}, \text{GL}_N(\mathbb{C}), \vartheta)$, $\xi \in \Xi(\mathcal{O}, \mathcal{O}_C \odot \text{GL}_N^\vartheta)$, satisfying

$$\langle C^\vartheta(\xi), C^\vartheta(\xi') \rangle = (-1)^{l(\xi')} q^{(l(\xi) + l(\xi'))/2} \delta_{\xi, \xi'}.$$ 

The proof amounts to showing that

$$C^\vartheta(\xi) = C^\vartheta(\xi'), \quad \xi \in \Xi(\mathcal{O}, \mathcal{O}_C \odot \text{GL}_N^\vartheta)$$
and we may do so by showing that $\mathcal{C}^\vartheta(\xi)$ satisfies each of the four properties in Theorem 4.3. We outline the proof of the first property of Theorem 4.3, as it is the most involved. We leave the reader to verify the remaining three properties by consulting [AAM, Section 4.7].

The first step is to prove that the two Verdier dualities are related by

$$
\langle D(M(\xi)), M(\vartheta^\vartheta(\xi^\vartheta)) \rangle = \langle M(\xi), \vartheta D(M(\vartheta^\vartheta(\xi^\vartheta))) \rangle, \quad \xi, \xi^\vartheta \in \Xi(\vartheta, \mathcal{O}, \vartheta R_{C/\mathcal{R}_1}^N)\vartheta
$$

where

$$\vartheta : \mathbb{Z}[q^{1/2}, q^{-1/2}] \to \mathbb{Z}[q^{1/2}, q^{-1/2}]$$

is the unique automorphism sending $q^{1/2}$ to $q^{-1/2}$. To do this, one may define $\vartheta D'$ by

$$\langle M(\xi), \vartheta D'(M(\vartheta^\vartheta(\xi^\vartheta))) \rangle = \langle D(M(\xi)), M(\vartheta^\vartheta(\xi^\vartheta)) \rangle, \quad \xi, \xi^\vartheta \in \Xi(\vartheta, \mathcal{O}, \vartheta R_{C/\mathcal{R}_1}^N)\vartheta$$

and prove that $\vartheta D' = \vartheta D$ using the properties of (79) and [LV, 8.1 (a)] which characterize the Verdier duality (cf. [V4, Lemma 13.4]). Proposition 4.2 plays a key role in proving the property

$$\vartheta D'((T_\kappa + 1)M(\vartheta^\vartheta(\xi^\vartheta)^+) = q^{-l(\kappa w_\kappa)}(T_\kappa + 1)^{\vartheta}D'(M(\vartheta^\vartheta(\xi^\vartheta)^+))$$

of (79), so it seems fitting to supply the arguments for it. We are to prove that

$$\langle M(\xi), \vartheta D'((T_\kappa + 1)M(\vartheta^\vartheta(\xi^\vartheta)^+)) \rangle = \langle M(\xi), q^{-l(\kappa w_\kappa)}(T_\kappa + 1)^{\vartheta}D'(M(\vartheta^\vartheta(\xi^\vartheta)^+)) \rangle$$

for all $\xi, \xi^\vartheta \in \Xi(\vartheta, \mathcal{O}, \vartheta R_{C/\mathcal{R}_1}^N)\vartheta$. According to the definition of $\vartheta D'$, and Equation (75) of Proposition 4.2, this is equivalent to

$$(-T_\kappa + q^{l(\kappa w_\kappa)})D(M(\xi)) = D((-T_\kappa + q^{l(\kappa w_\kappa)})q^{-l(\kappa w_\kappa)}M(\xi)).$$

Beginning with the right-hand side and following ([V4, Lemma 13.4]) with (79), we compute

$$
\begin{align*}
- D \left( (T_\kappa + 1)q^{-l(\kappa w_\kappa)}M(\xi) \right) + D \left( (q^{l(\kappa w_\kappa)} + 1)q^{-l(\kappa w_\kappa)}M(\xi) \right) \\
= -q^{-l(\kappa w_\kappa)}(T_\kappa + 1)D(q^{-l(\kappa w_\kappa)}M(\xi) + 1 + q^{l(\kappa w_\kappa)}D(M(\xi)) \\
= -(T_\kappa + 1)D(M(\xi)) + (1 + q^{l(\kappa w_\kappa)})D(M(\xi)) \\
= (-T_\kappa + q^{l(\kappa w_\kappa)})D(M(\xi)).
\end{align*}
$$

This proves (83). The remaining properties ensuring that $\vartheta D' = \vartheta D$ are also easily read from the proof of [V4, Lemma 13.4].

We now take for granted Equation (82), and may prove that $\mathcal{C}^\vartheta(\xi)$ satisfies the first property of Theorem 4.3 as follows.

$$
\langle D\mathcal{C}^\vartheta(\xi), C^\vartheta(\vartheta^\vartheta(\xi^\vartheta)) \rangle = \langle \mathcal{C}^\vartheta(\xi), \vartheta D\mathcal{C}^\vartheta(\vartheta^\vartheta(\xi^\vartheta)) \rangle = q^{l(\vartheta^\vartheta(\xi^\vartheta))} \langle \mathcal{C}^\vartheta(\xi), C^\vartheta(\vartheta^\vartheta(\xi^\vartheta)) \rangle = (-1)^{l(\vartheta^\vartheta(\xi^\vartheta))} q^{l(\vartheta^\vartheta(\xi^\vartheta))} q^{-l(\vartheta^\vartheta(\xi^\vartheta))]^2 \delta_{\xi, \xi^\vartheta} \\
\langle q^{-l(\vartheta^\vartheta(\xi^\vartheta))} \mathcal{C}^\vartheta(\xi), C^\vartheta(\vartheta^\vartheta(\xi^\vartheta)) \rangle.
$$

Since the elements $C^\vartheta(\vartheta^\vartheta(\xi^\vartheta))$ form a basis we conclude that

$$D\mathcal{C}^\vartheta(\xi) = q^{-l(\vartheta^\vartheta(\xi^\vartheta))} \mathcal{C}^\vartheta(\xi)$$

and the first property of Theorem 4.3 is proved. As mentioned, the remaining properties of Theorem 4.3 follow easily from [AAM, Section 4.7].

5 Endoscopic lifting for complex general linear groups following Adams-Barbasch-Vogan

We proceed with a review of standard endoscopy and twisted endoscopy from the perspective of [ABV], but restricted only to the particular case of the group $R_{C/\mathcal{R}_1}^\gamma N$. A similar review was made for $GL_N$ in [AAM, Section 5]. The present review contains nothing new and so we shall refer to [AAM, Section 5] liberally. The background material for this section is found in [ABV, Section 26] and [CM, Section 5].
5.1 Standard endoscopy

In this section we do not place any restrictions on the infinitesimal characters. Let \( \mathcal{R}_{C/B}^{\Gamma} = (\mathcal{V}GL_N \times GL_N) \times (\mathcal{V}^\delta_0) \) be as in Section 2.2. An endoscopic datum for \( \mathcal{R}_{C/B}^{\Gamma} \) is a pair

\[
(s, \mathcal{V}G^\Gamma)
\]

which satisfies

1. \( s \in \mathcal{R}_{C/B}^{\Gamma} \) is semisimple
2. \( \mathcal{V}G^\Gamma \subset \mathcal{R}_{C/B}^{\Gamma} \) is open in the centralizer of \( s \) in \( \mathcal{R}_{C/B}^{\Gamma} \)
3. \( \mathcal{V}G^\Gamma \) is an E-group for a group \( G \) ([ABV, Definition 4.6]).

This is a specialization of [ABV, Definition 26.15] to \( \mathcal{R}_{C/B}^{\Gamma} \). The groups \( \mathcal{V}G \) and \( G \) here are isomorphic Levi subgroups. They are products of smaller general linear groups. Consequently, \( \mathcal{V}G \) and \( \mathcal{R}_{C/B}^{\Gamma} \) share the maximal torus \( \mathcal{V}H \), which is two copies of the diagonal subgroup. Similarly, \( G \) and \( \mathcal{R}_{C/B}^{\Gamma} \) share the maximal torus \( H \). We shall abusively denote by \( \delta_0 \) the strong involution on both \( G \) and \( \mathcal{R}_{C/B}^{\Gamma} \) which correspond to the quasisplit real forms. The group \( G \) in this definition is called the endoscopic group. We do not require the concept of an E-group in this section. From now on we assume that \( \mathcal{V}G^\Gamma = \mathcal{V}G \times (\mathcal{V}^\delta_0) \). In other words, \( \mathcal{V}G^\Gamma \) is an L-group for \( G \).

There is a notion of equivalence for endoscopic data, and using this equivalence we may assume without loss of generality that \( s \in \mathcal{V}H \). We fix \( \lambda \in \mathcal{V}H \). Let \( \mathcal{O}_G \) be the \( \mathcal{V}G \)-orbit of \( \lambda \) and \( \mathcal{O} \) be the \( \mathcal{R}_{C/B}^{\Gamma} \)-orbit of \( \lambda \). The second property of the endoscopic datum above allows us to define the inclusion

\[
e : \mathcal{V}G^\Gamma \hookrightarrow \mathcal{R}_{C/B}^{\Gamma}.
\]

This inclusion induces another map ([ABV, Corollary 6.21]), which we abusively also denote as

\[
e : X (\mathcal{V}O_G, \mathcal{V}G^\Gamma) \to X (\mathcal{V}O, \mathcal{R}_{C/B}^{\Gamma} N^{\Gamma}).
\]

It is easily verified that the \( \mathcal{V}G \)-action on \( X(\mathcal{V}O_G, \mathcal{V}G^\Gamma) \) is compatibly carried under \( \epsilon \) to the \( \mathcal{R}_{C/B}^{\Gamma} \)-action on \( X(\mathcal{V}O, \mathcal{R}_{C/B}^{\Gamma} N^{\Gamma}) \) ([ABV, (7.17)]). As a result, the map \( \epsilon \) induces a map from the orbits of the space \( X(\mathcal{V}O_G, \mathcal{V}G^\Gamma) \) to the orbits of \( X(\mathcal{V}O, \mathcal{R}_{C/B}^{\Gamma} N^{\Gamma}) \).

The inverse image functor of \( \epsilon \) on equivariant constructible sheaves induces a homomorphism

\[
e^* : K(\mathcal{V}O_G, \mathcal{R}_{C/B}^{\Gamma} N^{\Gamma}) \to K(\mathcal{V}O, \mathcal{V}G^\Gamma)
\]

([ABV, Proposition 7.18]). When \( \ne^* \) is combined with the pairings of Theorem 3.1, we obtain a map

\[
\epsilon_* : K_C(\mathcal{V}O_G, G/R) \to K_C(\mathcal{V}O, GL_N(C))
\]

defined on \( \eta_G \in K_C(\mathcal{V}O_G, G/R) \) by

\[
\langle \epsilon_* \eta_G, \mu(\xi) \rangle = \langle \eta_G, \epsilon_* \mu(\xi) \rangle_G, \quad \xi \in \Xi(\mathcal{V}O, \mathcal{R}_{C/B}^{\Gamma} N^{\Gamma}).
\]

Here, \( K_C = C \otimes_R K \) and we have placed a subscript \( G \) beside the pairing on the right to distinguish it from the pairing for \( R_{C/B}^{\Gamma} N^{\Gamma} \) on the left.

The endoscopic lifting map \( \text{Lift}_0 \) is defined to be the restriction of \( \epsilon_* \) to the submodule

\[
K_C(\mathcal{V}O_G, G(R, \delta_0)) \subset K_C(\mathcal{V}O_G, G/R)
\]

of stable virtual characters of the quasisplit form \( G(R, \delta_0) \). Since \( G(R, \delta_0) \) is a product of complex general linear groups, stability is not an issue and we have

\[
K_C(\mathcal{V}O_G, G(R, \delta_0))^{st} = K_C(\mathcal{V}O_G, G/R).
\]

This equality will not hold for twisted endoscopic groups in Section 5.2, and so it is better two write the endoscopic lifting map as

\[
\text{Lift}_0 : K_C(\mathcal{V}O_G, G(R, \delta_0))^{st} \to K_C(\mathcal{V}O, GL_N(C)).
\]
An argument of Shelstad ([S3], [ABV, Lemma 18.11]) provides a basis for $K_C \Pi(\gamma \mathcal{O}_G, G(\mathbb{R}, \delta_q))^{st}$. The basis elements are of the form

$$\eta_{S_1}^{loc}(\delta_q) = \sum_{\tau_{S_1}} M(S_1, \tau_{S_1}),$$

where $(S_1, \tau_{S_1}) \in \Xi(\gamma \mathcal{O}_G, G^\Gamma)$ runs over those complete geometric parameters which correspond to the strong involution $\delta_q$ under the local Langlands correspondence (29). As indicated at the end of Section 2.2, the relevant component groups for complex general linear groups are trivial. Therefore the representations $\tau_{S_1}$ are all trivial for $G$ and (88) reduces to

$$\eta_{S_1}^{loc}(\delta_q) = M(S_1, 1),$$

a single standard representation. The following proposition describes the image of $\eta_{S_1}^{loc}(\delta_q)$ under endoscopic lifting. Its proof follows from [AAM, Proposition 5.1] by replacing $\text{GL}_N$ with $\text{R}_{C/\mathbb{R}} \text{GL}_N$.

**Proposition 5.1** ([AAM, Proposition 5.1]). (a) Suppose $S_1 \subset X(\gamma \mathcal{O}_G, G^\Gamma)$ is a $\gamma G$-orbit which is carried to the $\gamma R_{C/\mathbb{R}} \text{GL}_N$-orbit $S$ under $\epsilon$. Then

$$\text{Lift}_0(\eta_{S_1}^{loc}(\delta_q)) = \eta_S^{loc},$$

or equivalently,

$$\text{Lift}_0(M(S_1, 1)) = M(S, 1).$$

(b) The endoscopic lifting map $\text{Lift}_0$ is equal to the parabolic induction functor $\text{ind}^{\text{GL}_N(\mathbb{C})}_{G(\mathbb{R}, \delta_q)}$ on $K_C \Pi(\gamma \mathcal{O}_G, G(\mathbb{R}, \delta_q))^{st}$.

It is much more difficult to compute the value of $\text{Lift}_0$ on the stable virtual character $\eta^{\text{mic}}_{\psi_G}$ given in (51). Let $\psi = \epsilon \circ \psi_G$. According to [ABV, Theorem 26.25]

$$\text{Lift}_0(\eta^{\text{mic}}_{\psi_G}) = \sum_{\xi \in \Xi(\gamma \text{GL}_N^\Gamma, \gamma \mathcal{O})} (-1)^{\delta(S_\psi) - d(S_\psi)} \chi_{S_\psi}(P(\xi)) \pi(\xi) = \eta^{\text{mic}}_{\psi}. \tag{89}$$

Recall from (53) that the ABV-packets $\Pi^{\text{ABV}}_{\psi_G}$ and $\Pi^{\text{ABV}}_{\psi}$ are defined from $\eta^{\text{mic}}_{\psi_G}$ and $\eta^{\text{mic}}_{\psi}$ respectively. We shall see in Section 6 that these ABV-packets are singletons.

### 5.2 Twisted endoscopy

Our aim in this section is to lay out the twisted versions of the concepts presented in the previous section. We define twisted endoscopic data relevant to unitary groups, the twisted endoscopic version of $\text{Lift}_0$ (87), compute twisted variants of $\text{Lift}_0(\eta^{\text{loc}}_{S_1})$ for $S \in X(\gamma \mathcal{O}_G, \gamma G^\Gamma)$, and compute twisted variants of $\text{Lift}_0(\eta^{\text{mic}}_{\psi_G})$. We shall work under the assumption of (37) on the infinitesimal characters. This assumption is made only to accommodate the definition of Atlas extensions.

An endoscopic datum for $(\gamma \text{R}_{C/\mathbb{R}} \text{GL}_N^\Gamma, \theta)$ is a pair

$$(s, \gamma G^\Gamma)$$

which satisfies

1. $s \in \gamma \text{R}_{C/\mathbb{R}} \text{GL}_N$ is $\theta$-semisimple (see [KS, (2.1.3)])
2. $\gamma G^\Gamma \subset \gamma \text{R}_{C/\mathbb{R}} \text{GL}_N^\Gamma$ is open in the fixed-point set of $\text{Int}(s) \circ \theta$ in $\gamma \text{R}_{C/\mathbb{R}} \text{GL}_N^\Gamma \rtimes \langle \theta \rangle$
3. $\gamma G^\Gamma$ is an $E$-group for a group $G$ ([ABV, Definition 4.6]).

This is a special case of [CM, Definition 5.1] to $\gamma \text{R}_{C/\mathbb{R}} \text{GL}_N^\Gamma$. There is a notion of equivalence for these endoscopic data ([CM, Definition 5.6], [KS, (2.1.5)-(2.1.6)]).

Let us take $s = 1$ in the endoscopic pair above. Then the fixed-point subgroup

$$(\gamma \text{R}_{C/\mathbb{R}} \text{GL}_N^\Gamma)^\theta = \{g, \tilde{J}(g^\Gamma)^{-1} \tilde{J}^{-1} : g \in \gamma \text{GL}_N\} \cong \gamma \text{GL}_N$$
is a legitimate dual group for an endoscopic datum. Furthermore, by setting
\[ \gamma G^\delta = (\gamma R_{C/R}GL_N)^\delta \times (\gamma \delta_0) \]
with \( \gamma \delta_0 \) as in (32) we see that \( \gamma G^\delta \) is isomorphic to the L-group of a quasisplit unitary group as in (30). We have just shown that
\[ (1, \gamma G^\delta) = (1, \gamma \text{GL}^\Gamma_N) \]
is an endoscopic datum for \( (\gamma R_{C/R}GL_N^\Gamma, \vartheta) \) and that the corresponding endoscopic group is the rank \( N \) quasisplit unitary group. This is the only endoscopic datum of interest to us here.

Unlike the previous section, we must distinguish between maximal tori in \( \gamma R_{C/R}GL_N \) and \( \gamma G^\delta \). We let \( \gamma H \) be the diagonal maximal torus in \( \gamma R_{C/R}GL_N \), and \( \gamma H_G \) be a maximal torus in \( \gamma G \equiv \gamma \text{GL}_N \). The two tori are related by
\[ \gamma H_G = (\gamma H)^\delta. \]
We fix \( \lambda \in \gamma h^\theta \). Let \( \gamma \mathcal{O}_G \) be the \( \gamma G \)-orbit of \( \lambda \) and \( \gamma \mathcal{O} \) be the \( \gamma R_{C/R}GL_N \)-orbit of \( \lambda \).

The \( \epsilon \) maps of (84)-(85) have obvious analogues and are equally valid in the twisted setting. The analogue of (84) is quite transparent as it takes the form
\[ \epsilon(g) = (g, \tilde{J}^{-1}(g\tilde{J})^{-1} \tilde{J}^{-1}), \quad g \in \gamma G \]
and carries the element \( \gamma \delta_0 \) in (30) to the element \( \gamma \delta_0 \) in (31).

The crucial point in the twisted setting is to include the action of \( \vartheta \) into the objects pertinent to endoscopy. In particular we must extend the sheaf theory of [ABV] for \( \gamma R_{C/R}GL_N \) to the disconnected group \( \gamma R_{C/R}GL_N \times (\vartheta) \). This mimics the extension of the representation theory of \( GL_N(C) \) to the disconnected group \( GL_N(C) \times (\vartheta) \) in Section 2.4. Rather than viewing the sheaves in \( C(\gamma \mathcal{O}, \gamma R_{C/R}GL_N; \vartheta) \) as \( \gamma R_{C/R}GL_N \)-equivariant with compatible \( \vartheta \)-action (Section 3.2), we view them simply as \( \gamma R_{C/R}GL_N \times (\vartheta) \)-equivariant sheaves and apply the theory of [ABV] which is valid in this generality ([CM, Section 5.4]).

Let \( \xi = (S, 1) \in \Xi(\gamma \mathcal{O}, \gamma R_{C/R}GL_N^\Gamma)^\delta \) and \( p \in S \). Here, 1 is the trivial representation of the trivial group \( (\gamma R_{C/R}GL_N)_p / ((\gamma R_{C/R}GL_N)_p)^\delta \times \langle \vartheta \rangle \) with representation space \( V \equiv C \) as in (48). We define \( 1^+ \) on
\[ (\gamma R_{C/R}GL_N)_p / ((\gamma R_{C/R}GL_N)_p)^\delta \times \langle \vartheta \rangle \]
by
\[ 1^+(\vartheta) = \vartheta \mu(\xi)^+ = 1 \]  \hspace{1cm} (90)
In this way, \( 1^+ \) defines the local system underlying the irreducible \( (\gamma R_{C/R}GL_N \times (\vartheta)) \)-equivariant constructible sheaf \( \mu(\xi)^+ \) (Lemma 3.4, [ABV, p. 83]).

In a similar, but completely vacuous, fashion we may include the trivial action of \( \vartheta \) on \( \mu(\xi_1) \in C(\gamma \mathcal{O}_G, \gamma G) \) with \( \xi_1 = (S_1, \tau_1) \) and \( p_1 \in S_1 \). In other words, we may regard \( \mu(\xi_1) \) as a \( (\gamma G \times (\vartheta)) \)-equivariant sheaf whose underlying local system is defined by a quasicharacter \( \tau_1^+ \) on
\[ \gamma G_{p_1} / ((\gamma G_{p_1})^\delta \times \langle \vartheta \rangle \]
by \( \tau_1^+(\vartheta) = 1 \).

The inverse image functor
\[ \epsilon^* : KX(\gamma \mathcal{O}, \gamma R_{C/R}GL_N^\Gamma, \vartheta) \to KX(\gamma \mathcal{O}_G, \gamma G^\delta) \]
in the present twisted setting is defined on \( (\gamma R_{C/R}GL_N \times (\vartheta)) \)-equivariant sheaves (Section 3.2). As in standard endoscopy, we combine \( \epsilon^* \) with a pairing, namely the pairing of Theorem 3.5, to define
\[ \epsilon_* : K_C \Pi(\gamma \mathcal{O}_G, G/R) \to K_C \Pi(\gamma \mathcal{O}, GL_N(C), \vartheta). \]
To be precise, the image of any \( \eta \in K_C \Pi(\gamma \mathcal{O}_G, G/R) \) under \( \epsilon_* \) is determined by
\[ \langle \epsilon_* \eta, \mu(\xi)^+ \rangle = \langle \eta, \epsilon^* \mu(\xi)^+ \rangle_G, \quad \xi \in \Xi(\gamma \mathcal{O}, \gamma R_{C/R}GL_N^\Gamma)^\delta. \]  \hspace{1cm} (92)
The twisted endoscopic lifting map

\[ \text{Lift}_0 : K_C \Pi(\mathcal{O}_G, G(\mathbb{R}, \delta_\eta))^{st} \to K_C \Pi(\mathcal{O}, GL_N(\mathbb{C}), \vartheta) \]  

is the restriction of \( \epsilon_\eta \) to the stable submodule \( K_C \Pi(\mathcal{O}_G, G(\mathbb{R}, \delta_\eta))^{st} \).

Now, we wish to evaluate \( \text{Lift}_0 \) on the basis elements (88) of \( K_C \Pi(\mathcal{O}_G, G(\mathbb{R}, \delta_\eta))^{st} \). To maintain ease of comparison with [ABV] we evaluate \( \text{Lift}_0 \) on the virtual representations \( \eta_{S_1}^{loc}(\vartheta)(\delta_\eta) \) ([ABV, p. 279]). These virtual characters are defined by

\[ \eta_{S_1}^{loc}(\vartheta)(\delta_\eta) = \sum \tau_1 \text{Tr}(\tau_1^\circ(\vartheta)) M(S_1, \tau_1) = \sum \tau_1 M(S_1, \tau_1), \]

where \( \tau_1 \) runs over all quasicharacters of \( \mathcal{O}(G_{P_1}/(\mathcal{O}(G_{P_1})^0 \) as in (91) which correspond to the strong involution \( \delta_\eta \) ([ABV, Definition 18.9]) under (29). It is immediate from the definition of \( \tau_1^\circ \) following (91) that

\[ \eta_{S_1}^{loc}(\vartheta)(\delta_\eta) = \eta_{S_1}^{loc}(\delta_\eta) \]

and so this virtual character is stable ([ABV, Lemma 18.10]). The proof of the following proposition is the same as the proof of Proposition 5.3 [AAM] once \( \text{GL}_N \) is replaced by \( \text{R}_{\mathbb{C}/\mathbb{R}} \text{GL}_N \).

**Proposition 5.2** (Proposition 5.3 [AAM]). Suppose \( S_1 \subset X(\mathcal{O}_G, \mathcal{O}(G)) \) is a \( \mathcal{O}(G) \)-orbit which is carried to a \( \mathcal{O}(R_{\mathbb{C}/\mathbb{R}} \text{GL}_N) \)-orbit \( S \) under \( \epsilon \). Then

\[ \text{Lift}_0 \left( \eta_{S_1}^{loc}(\vartheta)(\delta_\eta) \right) = (-1)^{f(S_1,1)-f(S,1)} M(S,1)^+ \]

**Proposition 5.3.** The twisted endoscopic lifting map \( \text{Lift}_0 \) is injective.

Proof. Suppose \( S_1, S_2 \subset X(\mathcal{O}_G, \mathcal{O}(G)) \) are \( \mathcal{O}(G) \)-orbits which are carried to the same \( \mathcal{O}(R_{\mathbb{C}/\mathbb{R}} \text{GL}_N) \)-orbit under \( \epsilon \). Then, after identifying these orbits with \( \text{L}(\vartheta) \) parameters ([ABV, Proposition 6.17]), [GGP, Theorem 8.1] implies \( S_1 = S_2 \) (cf. [M3, Lemma 2.2.1]). It now follows from Proposition 5.2 that \( \text{Lift}_0 \) sends the basis

\[ \left\{ \eta_{S_1}^{loc}(\delta_\eta) : S_1 \text{ a } \mathcal{O}(G) \text{-orbit of } X(\mathcal{O}_G, \mathcal{O}(G)) \right\} \]

of \( K_C \Pi(\mathcal{O}_G, G(\mathbb{R}, \delta_\eta))^{st} \) bijectively onto the linearly independent subset

\[ \left\{ (-1)^{f(S_1,1)-f(S,1)} M(S_1,1)^+ : S_1 \text{ a } \mathcal{O}(G) \text{-orbit of } X(\mathcal{O}_G, \mathcal{O}(G)) \right\} \]

of \( K_C \Pi(\mathcal{O}, GL_N(\mathbb{C}), \vartheta) \).

The next and final goal of this section is to provide the twisted analogue of the endoscopic lifting of the virtual characters attached to \( A \)-parameters as in (89). As a guiding principle, it helps to remember that in moving from \( \eta_{S_1}^{loc} \) to \( \eta_{S_1}^{loc}(\vartheta)(\delta_\eta) \) we extended the component groups by \( \langle \vartheta \rangle \) to obtain (91), and then extended the quasicharacters \( \tau_1 \) defined on the original component groups. We shall follow the same process with \( \eta_{\psi_G}^{mic} \), doing our best to avoid the theory of microlocal geometry.

The stable virtual character (51) for the endoscopic group \( G \) is

\[ \eta_{\psi_G}^{mic} = \sum_{\xi \in \Xi(\mathcal{O}_G, \mathcal{O}(G))} (-1)^{d(S_\xi)-d(S_\psi)} \chi_{S_\psi G}^{mic}(P(\xi)) \pi(\xi) \in K_P(\mathcal{O}_G, G(\mathbb{R}))^{st}. \]

Here, \( S_\psi G \subset X(\mathcal{O}_G, \mathcal{O}(G)) \) is the \( \mathcal{O}(G) \)-orbit determined by the \( \text{L} \)-parameter \( \psi_G \), and \( \xi = (S_\xi, \tau_{S_\xi}) \).

We may rewrite \( \eta_{\psi_G}^{mic} \) using the following deep theorem in microlocal analysis. It is a summary of [ABV, Theorem 24.8, Corollary 24.9, Definition 24.15].

**Theorem 5.4.** For each \( \xi \in \Xi(\mathcal{O}_G, \mathcal{O}(G)) \) there is a representation \( \tau_{S_\psi G}^{mic}(P(\xi)) \) of \( \mathcal{O}(G(\psi_G)) \), the component group of the centralizer in \( \mathcal{O}(G) \) of the image of \( \psi_G \), which satisfies the following properties.
(a) $\tau_{S_{\psi_G}}^{\text{mic}}(P(\xi))$ represents a (possibly zero) $\forall^G$-equivariant local system $Q^{\text{mic}}(P(\xi))$ of complex vector spaces.

(b) The degree of $\tau_{S_{\psi_G}}^{\text{mic}}(P(\xi))$ is equal to $\chi_{S_{\psi_G}}^{\text{mic}}(P(\xi))$.

(c) If $\xi = (S_{\psi}, \tau_{S_{\psi}})$ then $\tau_{S_{\psi}}^{\text{mic}}(P(\xi)) = \tau_{S_{\psi}}^{\text{mic}} \circ i_{S_{\psi}}^\tau$, where $i_{S_{\psi}}^\tau : \forall^G_{\psi G}/(\forall^G_{\psi G})^0 \rightarrow \forall^G_{\psi G}/(\forall^G_{\psi G})^0$

is a surjective homomorphism for $p \in S_{\psi_G}$.

By Theorem 5.4 (b), we may rewrite $\eta_{\psi_G}^{\text{mic}}$ as

$$\eta_{\psi_G}^{\text{mic}} = \sum_{\xi \in \Xi(\forall^G_{\psi G}, \forall^G_{\psi G})} (-1)^{d(S_{\psi}) - d(S_{\psi_G})} \Tr(\tau_{S_{\psi_G}}^{\text{mic}}(P(\xi))(1)) \pi(\xi). \quad (94)$$

Next, we extend $\forall^G_{\psi G}/(\forall^G_{\psi G})^0$ trivially to

$$\forall^G_{\psi G}/(\forall^G_{\psi G})^0 \times \{\vartheta\}, \quad (95)$$

and extend $\tau_{S_{\psi_G}}^{\text{mic}}(P(\xi))$ trivially to (95) by defining $\tau_{S_{\psi_G}}^{\text{mic}}(P(\xi))(\vartheta)$ to be the identity map. We define

$$\eta_{\psi_G}^{\text{mic}}(\vartheta) = \sum_{\xi \in \Xi(\forall^G_{\psi G}, \forall^G_{\psi G})} (-1)^{d(S_{\psi}) - d(S_{\psi_G})} \Tr(\tau_{S_{\psi_G}}^{\text{mic}}(P(\xi))(\vartheta)) \pi(\xi)$$

$$= \sum_{\xi \in \Xi(\forall^G_{\psi G}, \forall^G_{\psi G})} (-1)^{d(S_{\psi}) - d(S_{\psi_G})} \dim(\tau_{S_{\psi_G}}^{\text{mic}}(P(\xi))) \pi(\xi).$$

Clearly

$$\eta_{\psi_G}^{\text{mic}}(\vartheta) = \eta_{\psi_G}^{\text{mic}}. \quad (96)$$

Finally, define

$$\eta_{\psi_G}^{\text{mic}}(\vartheta)(\delta_{\xi}) = \sum_{(S_{\xi}, \tau_{S_{\xi}})} (-1)^{d(S_{\xi}) - d(S_{\psi_G})} \Tr(\tau_{S_{\psi_G}}^{\text{mic}}(P(\xi))(\vartheta)) \pi(\xi)$$

$$= \sum_{(S_{\xi}, \tau_{S_{\xi}})} (-1)^{d(S_{\xi}) - d(S_{\psi_G})} \dim(\tau_{S_{\psi_G}}^{\text{mic}}(P(\xi))) \pi(\xi)$$

in which the sum runs over only those $\xi = (S_{\xi}, \tau_{S_{\xi}}) \in \Xi(\forall^G_{\psi G}, \forall^G_{\psi G})$ in which $\tau_{S_{\xi}}$ corresponds to the strong involution $\delta_{\xi}$ under (29). Therefore, by (52)

$$\eta_{\psi_G}^{\text{mic}}(\vartheta)(\delta_{\xi}) = \eta_{S_{\psi_G}}^{\text{mic}}(\delta_{\xi}) = \eta_{S_{\psi_G}}^{\text{ABV}}.$$

The virtual character $\eta_{\psi_G}^{\text{mic}}(\vartheta)(\delta_{\xi})$ is a summand of the stable virtual character $\eta_{\psi_G}^{\text{mic}}$ and is therefore also stable ([ABV, Theorem 18.7]). Consequently, $\eta_{\psi_G}^{\text{mic}}(\vartheta)(\delta_{\xi})$ lies in the domain of Lift$_{\vartheta}$. In addition, the ABV-packet $\Pi_{\psi_G}^{\text{ABV}}$ consists of the irreducible characters in the support of $\eta_{\psi_G}^{\text{mic}}(\vartheta)(\delta_{\xi})$

(53).

What we have done for $\eta_{\psi_G}^{\text{mic}}$ we begin to do for $\eta_{\psi}^{\text{mic}^+}$, which we define as

$$\eta_{\psi}^{\text{mic}^+} = \sum_{\xi \in \Xi(\forall^G_{\psi G}, \forall^R_C \gg \text{GL}_N)^{\vartheta}} (-1)^{d(S_{\psi}) - d(S_{\psi_G})} \Tr(\chi_{S_{\psi_G}}^{\text{mic}}(P(\xi))) (-1)^{\ell(\xi) - \ell_{\vartheta}(\xi)} \pi(\xi)^\vartheta \quad (97)$$

for

$$\psi = \varepsilon \circ \psi_{\vartheta}.$$

The main difference now is that $\vartheta$ does not act trivially on $\forall^R_C \gg \text{GL}_N$ and so the extensions require more attention. The properties of Theorem 5.4 hold for $\psi$ and $\text{R}_C \gg \text{GL}_N$ as they do for $\psi_{\vartheta}$ and $G$.  

32
The first step is writing
\[ \eta^{\text{mic}^+}_{\psi} = \sum_{\xi \in \Xi(\mathcal{O}, \mathcal{V}_{\mathcal{C}/\mathcal{R}GL_N})^\theta} (-1)^{d(S_\psi) - d(S_v)} \text{Tr}(\tau^{\text{mic}}_S(P(\xi))(1)) (-1)^{d(S_v)} \pi(\xi)^+. \]
This holds from Theorem 5.4 (b) as (94) did for the endoscopic group \( G \). What is simpler here is that the component group \( (\mathcal{V}_{\mathcal{C}/\mathcal{R}GL_N})^\theta \) is trivial ([A1, Section 2.3]). It follows that \( \tau^{\text{mic}}_{S_v}(P(\xi)) \) is either trivial or zero.

Let us digress briefly to examine Theorem 5.4 (c) for \( \xi = (S_\psi, \tau_{S_v}) \). Since the component group \( (\mathcal{V}_{\mathcal{C}/\mathcal{R}GL_N})^\theta \) is trivial, the representation \( \tau_{S_v} \) is trivial. It follows that
\[ \tau^{\text{mic}}_{S_v}(P(S_\psi, \tau_{S_v})) = \tau_{S_v} \circ i_{S_v} = 1 \circ i_{S_v} = 1 \neq 0. \]
In particular, \( \pi(S_\psi, 1) \) is in the support of \( \eta^{\text{mic}}_\psi \) and belongs to \( \Pi^S_\psi \). In the next section we will prove that this is the only representation in \( \Pi^S_\psi \).

Returning to the matter of extensions, there is an obvious extension
\[ (\mathcal{V}_{\mathcal{C}/\mathcal{R}GL_N})_\psi / ((\mathcal{V}_{\mathcal{C}/\mathcal{R}GL_N})^\theta \times \langle \theta \rangle) \]
of the trivial component group, as \( \theta \) fixes the image of \( \psi \). We wish to extend the representation \( \tau^{\text{mic}}_{S_v}(P(\xi)) \) to this group for \( \xi \in \Xi(\mathcal{O}, \mathcal{V}_{\mathcal{C}/\mathcal{R}GL_N})^\theta \). The action of \( \theta \) on \( P(\xi) \in \mathcal{P}(\mathcal{O}, \mathcal{V}_{\mathcal{C}/\mathcal{R}GL_N})^\theta \) determines an action on the stalks of the local system \( Q^{\text{mic}}(P(\xi)) \) as in (5.4) ([ABV, (25.1)]).

We may extend \( i_{S_v} \) in Theorem 5.4 (c) to include the products with \( \langle \theta \rangle \). Definitions (90) and (98) are compatible in that
\[ \tau^{\text{mic}}_{S_v}(P(S_\psi, 1)) = 1^+ \circ i_{S_v}. \]

Finally, we define
\[ \eta^{\text{mic}^+}_{\psi}(\theta) = \sum_{\xi \in \Xi(\mathcal{O}, \mathcal{V}_{\mathcal{C}/\mathcal{R}GL_N})^\theta} (-1)^{d(S_\psi) - d(S_v)} \text{Tr}(\tau^{\text{mic}}_S(P(\xi)^+)(\theta)) (-1)^{d(S_v)} \pi(\xi)^+. \]
It is clear from definition (97) that \( \eta^{\text{mic}^+}_{\psi}(\theta) = \eta^{\text{mic}^+}_{\psi} \).

The obvious definition of the representation \( \tau^{\text{mic}}_{S_v}(P(\xi)^+) \) is to take
\[ \tau^{\text{mic}}_{S_v}(P(\xi)^-)(\theta) = -1. \]

With this definition in place the following proposition is a consequence of [ABV, Corollary 24.9].

**Proposition 5.5.** The functor \( \tau^{\text{mic}}_{S_v}(-) \), from \( (\mathcal{V}_{\mathcal{C}/\mathcal{R}GL_N} \times \langle \theta \rangle) \)-equivariant perverse sheaves to representations of \( (\mathcal{V}_G^{\psi} / (\mathcal{V}_G^{\psi})^0 \times \langle \theta \rangle) \), induces a map from the Grothendieck group \( K(X(\mathcal{O}, \mathcal{V}_{\mathcal{C}/\mathcal{R}GL_N}); \theta) \) to the space of virtual representations. Furthermore the microlocal trace map
\[ \text{Tr} \left( \tau^{\text{mic}}_{S_v}(-)(\theta) \right) \]
induces a homomorphism from \( K(X(\mathcal{O}, \mathcal{V}_{\mathcal{C}/\mathcal{R}GL_N}); \theta) \) (as in (54)) to \( \mathbb{C} \).

A similar statement is true for \( \tau^{\text{mic}}_{S_v} \) and the \( (\mathcal{V}_G \times \langle \theta \rangle) \)-equivariant sheaves defined earlier. The proof of the next theorem is identical to the proof of [AAM, Theorem 5.6].
Theorem 5.6. [AAM, Theorem 5.6]

(a) As a function on $K(X(\mathcal{O},^\circ R_{C/R}GL_N^F), \vartheta)$ we have
\[
\left\langle \eta^{mic+}_\psi(\vartheta), \cdot \right\rangle = (-1)^{d(S_\psi)} \text{Tr} \left( \tau^{mic}_S(\cdot)(\vartheta) \right).
\]

(b) The stable virtual character $\eta^{mic+}_\psi(\vartheta)$ is equal to
\[
(-1)^{d(S_\psi)} \sum_{\xi \in \Xi(\mathcal{O},^\circ R_{C/R}GL_N^F)^{\psi}} \text{Tr} \left( \tau^{mic}_S(\mu(\xi))(\vartheta) \right) (-1)^{\ell(\xi)} \ell_0(\xi) M(\xi)^{\psi}.
\]

(c) Lifts $\left( \eta^{mic+}_\psi(\vartheta)(\delta_0) \right) = \eta^{mic+}_\psi(\vartheta)$.

6. ABV-packets for complex general linear groups

In this section we prove that any ABV-packet for $R_{C/R}GL_N = GL_N(\mathbb{C})$ consists of a single (equivalence class of an) irreducible representation. This implies that such an ABV-packet is equal to its corresponding L-packet ([ABV, Theorem 22.7 (a)]). From the classification of the unitary dual of $GL_N(\mathbb{C})$ we shall deduce that the single representation in the packet is unitary.

In this section we let $\psi : W_R \times SL_2 \to ^\circ R_{C/R}GL_N$ be an arbitrary A-parameter for $R_{C/R}GL_N$. The description of the ABV-packet $\Pi^{ABV}_\psi$ will be achieved in three steps. We follow the same proof as in [AAM, Section 6]. First, we treat the case of an irreducible A-parameter. Second, we compute the ABV-packet for a Levi subgroup of $R_{C/R}GL_N$, whose dual group contains the image of $\psi$ minimally. The final result is obtained from the second step by considering the Levi subgroup as an endoscopic group of $R_{C/R}GL_N$ and applying the endoscopic lifting (89).

According to [M3, Section 2.3], any A-parameter $\psi$ for $R_{C/R}GL_N$ may be decomposed as a formal direct sum of A-parameters
\[
\psi = \bigoplus_{i=1}^r \psi_i,
\]
with $\ell_i \in \mathbb{N}, \psi_i$ being an A-parameter of $R_{C/R}GL_{N_i}$, and $N = \sum_{i=1}^r \ell_i N_i$. We may identify real L-parameters of $R_{C/R}GL_N$ with their corresponding complex L-parameters of $GL_{N_i}$ ([B, Section I.5]). This correspondence extends in an evident way to an analogous identification between A-parameters. The complex A-parameter of $GL_{N_i}$ corresponding to each $\psi_i$ is given by
\[
\mu_i \otimes \nu_{N_i},
\]
where $\nu_{N_i}$ is the unique irreducible representation of $SL_2$ of dimension $N_i$, and $\mu_i$ is an irreducible representation $\mathbb{C}^\times$.

The parameter $\psi$ in (100) is said to be irreducible if $r = 1$ and $\ell_1 = 1$. For any irreducible A-parameter $\psi$ of $R_{C/R}GL_N$ the corresponding representation $\nu_N$ of $SL_2$ is irreducible and of dimension $N$. As a consequence the image of any unipotent subgroup of $SL_2$ under $\nu_N$ is principally unipotent (i.e. regular and unipotent) in $^\circ GL_N$. Equivalently, the image of any unipotent subgroup of $SL_2$ under the real A-parameter $\psi$ is principally unipotent in $^\circ GL_N \times ^\circ GL_N$. [AR, Theorem 4.11 (d)] therefore implies the following result (cf. [ABV, Theorem 27.18]).

Proposition 6.1. Suppose $\psi$ is an irreducible A-parameter of $R_{C/R}GL_N$. Then $\Pi^{ABV}_\psi$ consists of a single unitary character.

Let us proceed to the case of a general A-parameter $\psi$ as in Equation (100). Define
\[
^\circ G = \prod_{i=1}^r (^\circ G_i)^{\ell_i} \cong \prod_{i=1}^r (^\circ GL_{N_i} \times ^\circ GL_{N_i})^{\ell_i}
\]
to be the obvious Levi subgroup of $\overset{\vee}{\text{GL}}_N \times \overset{\vee}{\text{GL}}_N$ containing the image of $\psi$ minimally. Set $\overset{\vee}{G}^\psi = \overset{\vee}{G} \times (\overset{\vee}{\delta})$, a subgroup of $\overset{\vee}{\text{R}}_{C/R} \text{GL}_N$. It is immediate that $\psi$ factors through an $A$-parameter

$$\psi : W_R \times SL_2 \overset{\psi_G}{\rightarrow} \overset{\vee}{G}^\psi \hookrightarrow \overset{\vee}{\text{R}}_{C/R} \text{GL}_N,$$

where $\psi_G = \times_{\nu} \ell_i \psi_{G_i}$ and each $\psi_{G_i}$ is an irreducible $A$-parameter of $\overset{\vee}{G}^\psi_i = \overset{\vee}{\text{R}}_{C/R} \text{GL}_N$. The description of the ABV-packet corresponding to $\psi_G$ is a straightforward consequence of Proposition 6.1. We must only remind ourselves that the direct product of (101) translates into a tensor product of ABV-packets as it passes through the process defining the packets in Section 3.1. The proof follows exactly as for Corollary 6.2 [AAM].

Corollary 6.2. The ABV-packet $\Pi_{\psi \in \pi}^{\text{ABV}}$ consists of a single irreducible unitary representation $\pi(S_{\psi_G}, 1)$.

Finally, take $\overset{\vee}{G}$ as in (101), and take $s \in Z(\overset{\vee}{G}) \subseteq \overset{\vee}{\text{GL}}_N \times \overset{\vee}{\text{GL}}_N$ to be as regular as possible so that its centralizer in $\overset{\vee}{\text{GL}}_N \times \overset{\vee}{\text{GL}}_N$ is equal to $\overset{\vee}{G}$. Then $(s, \overset{\vee}{G})$ is an endoscopic datum for $\overset{\vee}{\text{R}}_{C/R} \text{GL}_N$ (Section 5.1). According to (89), Corollary 6.2, and Proposition 5.1 we have

$$\eta^{\text{mic}}_{\psi} = \text{Lift}_0 \left( \eta^{\text{mic}}_{\psi_G} \right) = \text{Lift}_0 (\pi(S_{\psi_G}, 1)) = \text{ind}_{G(R, \delta)}^{G_{C(R, \delta)}} \pi(S_{\psi_G}, 1).$$

Consequently, $\Pi_{\psi \in \pi}^{\text{ABV}}$ consists of a single representation, and this representation is parabolically induced from the single unitary representation of $\Pi_{\psi \in \pi}^{\text{ABV}}$. Since parabolic induction for general linear groups takes irreducible unitary representations to irreducible unitary representations ([T, Proposition 2.1]), we have just proved

Proposition 6.3. Let $\psi$ be an $A$-parameter for $\overset{\vee}{\text{R}}_{C/R} \text{GL}_N$ as in (100). Then the ABV-packet $\Pi_{\psi \in \pi}^{\text{ABV}}$ consists of a single irreducible unitary representation $\pi(S_{\psi_G}, 1)$.

As a corollary, we have the next result. Its proof is the same as that of [AAM, Corollary 6.4].

Corollary 6.4. The stable virtual character $\eta^{\text{mic}}_{\psi}(\vartheta)$ defined in (99) is equal to $(-1)^{t'(\xi) - t'(\xi)} \pi(\xi)^+$, where $\xi = (S_{\psi}, 1)$. In particular,

$$\text{Lift}_0 \left( \eta^{\text{mic}}_{\psi_G}(\vartheta) \right) = (-1)^{t'(\xi) - t'(\xi)} \pi(\xi)^+.$$

7 Whittaker extensions and their relationship to Atlas extensions

Recall that (44) defines a preferred extension to $\text{GL}_N(\mathbb{C}) \times (\vartheta)$ of any irreducible representation of $\text{GL}_N(\mathbb{C})$. These are the extensions we have called “Atlas extensions”. Since $\text{GL}_N(\mathbb{C})$ is quasisplit as a real group, there is another choice of preferred extension, which depends on a Whittaker datum. This extension is introduced by Arthur ([A3, Section 2.2]) and followed by Mok ([M3, Section 3.2]). We call this alternative extension the Whittaker extension.

Here is a summary of the definition of Whittaker extensions. We fix a unitary character $\chi$ on the upper-triangular unipotent subgroup

$$U(\mathbb{R}) \subset \overset{\vee}{\text{R}}_{C/R} \text{GL}_N(\mathbb{R}) = \text{GL}_N(\mathbb{C}),$$

which satisfies $\chi \circ \vartheta = \chi$. In this manner $(U, \chi)$ is a $\vartheta$-fixed Whittaker datum. We work under the hypothesis of (37) on an infinitesimal character $\lambda \in \overset{\vee}{\mathfrak{h}}$, set $\overset{\vee}{\mathfrak{o}}$ to be its $\overset{\vee}{\text{R}}_{C/R} \text{GL}_N$-orbit. Let $\xi \in \Xi(\overset{\vee}{\mathfrak{o}}, \overset{\vee}{\text{R}}_{C/R} \text{GL}_N)^{\vartheta}$ so that $\pi(\xi)$ is an infinitesimal equivalence class of an irreducible representation of $\text{GL}_N(\mathbb{C})$. In defining Whittaker extensions we must work with an actual admissible group representation in this equivalence class which we also denote by $(\pi(\xi), V)$. If $\pi(\xi)$ is tempered then up to a scalar there is a unique Whittaker functional $\omega : V \rightarrow \mathbb{C}$ satisfying

$$\omega(\pi(\xi)(u)v) = \chi(u) \omega(v), \quad u \in U(\mathbb{R}),$$

(102)
for all smooth vectors $v \in V$. It follows that there is a unique operator $I^\sim$ which intertwines $\pi(\xi) \circ \vartheta$ with $\pi(\xi)$ and also satisfies $\omega \circ I^\sim = \omega$. We extend $\pi(\xi)$ to a representation $\pi(\xi)^\sim$ of $GL_N(\mathbb{C}) \rtimes \langle \vartheta \rangle$ by setting $\pi(\xi)^\sim(\vartheta) = I^\sim$. We call this extension $\pi(\xi)^\sim$ the **Whittaker extension** of $\pi(\xi)$.

If $\pi(\xi)$ is not tempered then we express it as the Langlands quotient of a representation $M(\xi)$ induced from an essentially tempered representation of a Levi subgroup. The $\vartheta$-stability of $\pi(\xi)$ and the uniqueness statement in the Langlands classification together imply the $\vartheta$-stability of the essentially tempered representation. The earlier argument for tempered representations has an obvious analogue for the essentially tempered representation of the Levi subgroup. We may argue as above to extend the essentially tempered representation to the semi-direct product of the Levi subgroup with $\langle \vartheta \rangle$. One then induces this extended representation to $GL_N(\mathbb{C}) \rtimes \langle \vartheta \rangle$. The unique irreducible quotient of this representation is the canonical extension of $\pi(\xi)$, namely the **Whittaker extension** $\pi(\xi)^\sim$ of $\pi(\xi)$. If one omits the Langlands quotient in this argument then we obtain, by definition, the Whittaker extension $M(\xi)^\sim$ of the standard representation $M(\xi)$. A Whittaker functional for the tempered representation of the Levi subgroup induces a Whittaker functional for $M(\xi)$ ([S2, Proposition 3.2]). It is a simple exercise to prove that $M(\xi)^\sim(\vartheta)$ is the unique intertwining operator that fixes an induced Whittaker functional as in (102).

How does the Whittaker extension $\pi(\xi)^\sim$ differ from the Atlas extension $\pi(\xi)^+ \in (44)$? The operators $\pi(\xi)^\sim(\vartheta)$ and $\pi(\xi)^+(\vartheta)$ are involutive, and both intertwine $\pi(\xi) \circ \vartheta$ with $\pi(\xi)$. Therefore they are equal up to a sign, i.e.

$$\pi(\xi)^\sim(\vartheta) = \pm \pi(\xi)^+(\vartheta). \quad (103)$$

A direct comparison of the two extensions is problematic in that the Whittaker extension is essentially analytic in nature, and the Atlas extension is essentially algebraic.

Happily, there is a special type of parameter $\xi \in \Xi(\wp, \wp^* R_{\mathbb{C}/\mathbb{R}} GL^I_N)$ for which the construction of the two extensions is directly comparable. To describe this type, we must examine the construction of $\pi(\xi)$ in (44) in more detail. In this construction we identify $\xi$ with its equivalent Atlas parameter $(x, y)$ (Lemma 2.2), or its equivalent preferred extended parameter (43). Recall that $\theta_x$ is an automorphism of $H$ (55). To the parameter $(x, y)$ one associates the irreducible $(\mathfrak{h}, \mathfrak{h}^{\psi} \times \langle \vartheta \rangle)$-module $\pi^0_\theta$ of differential $\lambda$ such that $\pi^0_\theta(\vartheta) = 1$ ([AV, (20b), Lemma 5.1, Definition 5.6]). Then to the module $\pi^0_\theta$ one applies the functor [AV, (20b)] to obtain $\pi(\xi)^+ = J(x, y, \lambda)^+$ (cf. [KV, (11.54b), (11.116b)]). The functor depends on $\theta_x$. In the particular circumstance that $\theta_x$ sends all positive roots (determined by the Borel subgroup $B$ in the pinning of $R_{\mathbb{C}/\mathbb{R}} GL_N$) to negative roots, the Borel subalgebra $b$ is real relative to $\theta_x$ and $B$ is a real parabolic subgroup ([AvLT, Proposition 13.12 (2)]). Moreover, in this circumstance, the functor is equivalent to the (normalized) parabolic induction functor $\text{ind}_{B(\mathbb{R}) \rtimes \langle \vartheta \rangle}^{GL_N(\mathbb{C}) \rtimes \langle \vartheta \rangle} [KV, Proposition 11.47]$, and so

$$M(\xi)^+ = \text{ind}_{B(\mathbb{R}) \rtimes \langle \vartheta \rangle}^{GL_N(\mathbb{C}) \rtimes \langle \vartheta \rangle} \pi^0_\theta. \quad (104)$$

The following lemma shows that the assumption of the $\vartheta$-stability of the parameter $\xi$ is not necessary when $\theta_x$ sends all positive roots to negative roots, and asserts the equality of the Atlas and Whittaker extensions.

**Lemma 7.1.** Suppose $\xi = (x, y) \in \Xi(\wp, \wp^* R_{\mathbb{C}/\mathbb{R}} GL^I_N)$ and $\theta_x$ sends all positive roots to negative roots. Then

(a) $M(\xi)$ and $\pi(\xi)$ are $\vartheta$-stable so that $\xi = (x, y) \in \Xi(\wp, \wp^* R_{\mathbb{C}/\mathbb{R}} GL^I_N)^\theta$.

(b) The Whittaker and Atlas extensions of $M(\xi)$ and $\pi(\xi)$ are equal, i.e. $M(\xi)^\sim = M(\xi)^+$ and $\pi(\xi)^\sim = \pi(\xi)^+$.

**Proof.** The assertions for $\pi(\xi)$ follow by definition from the assertions for $M(\xi)$. For $M(\xi)$ we imitate the process described before the lemma and thereby obtain an irreducible $(\mathfrak{h}, H^{\psi} \circ \vartheta)$-module $\pi_0$ of differential $\lambda$. In addition,

$$M(\xi) = \text{ind}_{B(\mathbb{R}) \rtimes \vartheta}^{GL_N(\mathbb{C})} \pi_0.$$  

As a representation of the connected Lie group $H(\mathbb{R}) \cong (\mathbb{C}^\times)^N$, $\pi_0$ is completely determined by its differential $\lambda$. Obviously, $\pi_0 \circ \vartheta$ has differential $\vartheta(\lambda) = \lambda$ (37). Consequently $\pi_0 \circ \vartheta = \pi_0$ and one
This proves (a).

For (b) we may return to (104) and note that, vacuously, \( \pi_0^+ (\vartheta) = 1 \) determines the Whittaker extension of \( \pi_0 \), i.e. \( \pi_0^+ = \pi_0 \). Therefore, by definition, \( M(\xi)^\sim = (104) \). (The reader may wonder in what sense (104) is “standard”. An explanation may be found in [KV, Theorem 11.129 (a)]).

Lemma 7.1 provides a simple solution for the comparison of Atlas and Whittaker extensions of some special representations. As we shall soon see, an arbitrary irreducible representation appears as a subquotient of one of these special standard representations. Remarkably, the twisted multiplicity (61) with which this subquotient appears may be used to determine the sign in (103).

We begin this line of reasoning by recalling how irreducible generic representations appear in the characters of standard representations. Recall that a representation is generic if it admits a non-zero Whittaker functional as in (102).

**Lemma 7.2.** Suppose \( \xi \in \Xi(\mathcal{O}, \mathcal{O}_{\mathbb{C}/\mathbb{R}} GL_N^\Gamma)^\vartheta \). Then

(a) (up to infinitesimal equivalence) there is a unique irreducible generic representation \( \pi(\xi_0) = M(\xi_0) \) which occurs in \( M(\xi) \) as a subquotient. Moreover \( \pi(\xi_0) \) is \( \vartheta \)-stable and occurs as a subquotient with multiplicity one;

(b) (any representative in the class of) \( \pi(\xi_0) \) embeds as a subrepresentation of (any representative in the class of) \( M(\xi) \);

(c) (any representative in the class of) \( \pi(\xi_0)^\sim \) embeds as a subrepresentation of (any representative in the class of) \( M(\xi)^\sim \).

**Proof.** A result due to Vogan and Kostant states that every standard representation \( M(\xi) \) contains a unique generic irreducible subquotient occurring with multiplicity one ([K, Theorems E and L], [V2, Corollary 6.7]). In the rest of the proof we write \( \pi(\xi_0) \) for the actual generic representation (not the equivalence class) for some \( \xi_0 \in \Xi(\mathcal{O}, \mathcal{O}_{\mathbb{C}/\mathbb{R}} GL_N^\Gamma) \). It is straightforward to verify that \( \pi(\xi_0) \circ \vartheta \) satisfies (102), just as \( \pi(\xi_0) \) does. Therefore \( \pi(\xi_0) \circ \vartheta \) is the unique irreducible generic subquotient of \( M(\xi) \circ \vartheta = M(\xi) \). By uniqueness, \( \pi(\xi_0) \circ \vartheta \cong \pi(\xi_0) \) and so \( \xi_0 \in \Xi(\mathcal{O}, \mathcal{O}_{\mathbb{C}/\mathbb{R}} GL_N^\Gamma)^\vartheta \). The equality \( \pi(\xi_0) = M(\xi_0) \) follows from [V2, Theorem 6.2 (f)].

For part (b), assume that \( M(\xi) \) is an actual representation (not an equivalence class) with infinitesimal character \( \lambda \in \mathfrak{h} \) satisfying the assumption of (37). As a standard representation, we may write it as

\[
M(\xi) = \text{ind}_{P(\mathbb{R})}^{GL_N(\mathbb{C})} (\pi_M \otimes e^{\nu}) .
\]

Here, \( P \) is a standard real parabolic subgroup with Levi subgroup \( M \), \( M \) has Langlands decomposition \( M(\mathbb{R}) = M^1(\mathbb{R}) A(\mathbb{R}) \), \( \pi_M \) is an irreducible tempered representation of \( M^1(\mathbb{R}) \), and \( \nu \in \mathfrak{a}^\ast \) has dominant real part. The Levi subgroup \( M(\mathbb{R}) \) is a product of smaller complex general linear groups ([B, Section 5.2]). Since \( M(\mathbb{R}) \) is a direct product of complex general linear groups, [T, Proposition 2.1] allows us to write the tempered representation as a parabolically induced representation from a discrete series representation on the unique standard cuspidal Levi subgroup \( H(\mathbb{R}) \). By induction in stages we may write

\[
M(\xi) = \text{ind}_{B(\mathbb{R})}^{GL_N(\mathbb{C})} (\pi_0 \otimes e^{\nu})
\]

where we now regard \( \nu \) as an element in the Lie algebra of the split component of \( H(\mathbb{R}) \). The differential of \( \pi_0 \otimes e^{\nu} \) is \( \lambda \). Since \( \lambda \) is integrally dominant and \( \text{Re} \nu \) is dominant, \( M(\xi) \) satisfies the properties of [V2, Theorem 6.2 (e) (i)-(ii)]. From [V2, Theorem 6.2 (e)] we know that any irreducible subrepresentation of \( M(\xi) \) is generic. By part (a), we conclude that \( M(\xi) \) has a unique irreducible subrepresentation and this subrepresentation is equivalent to \( \pi(\xi_0) \).

For part (c) we consider the standard representation \( M(\xi) \) which has a Whittaker functional \( \omega \) induced from \( \pi_M \) above ([S1, Proposition 3.2]). The functional \( \omega \) restricts to a non-zero Whittaker functional as in (102).
functional on $\pi(\xi_0)$. It is simple to verify that $M(\xi)(\vartheta)$ is the intertwining operator which satisfies $\omega \circ M(\xi)(\vartheta) = \omega$. Restricting this equation to the subrepresentation $\pi(\xi_0)$ yields in turn that

$$
\pi(\xi_0)^\sim(\vartheta) = M(\xi)^\sim(\vartheta) \mid_{\pi(\xi_0)} \text{ and } \pi(\xi_0)^\sim \hookrightarrow M(\xi)^\sim.
$$

(105)

Lemma 7.2 tells us that the multiplicity of $\pi(\xi_0)^\sim$ in $M(\xi)^\sim$ is one. On the other hand the twisted multiplicity $m_\varphi^\psi(\xi_0, \xi)$ of (61) tells us about the “signed multiplicity” of $\pi(\xi_0)^\sim$ in $M(\xi)^\sim$. We investigate $m_\varphi^\psi(\xi_0, \xi)$ further before comparing the two kinds of multiplicities.

**Proposition 7.3.** Suppose $\xi \in \Xi(\mathcal{O}, \mathcal{V}R_{C/R}GL_N^\Gamma)^\varphi$ and $\pi(\xi_0)$ is the generic subrepresentation of $M(\xi)$ (Lemma 7.2). Then

$$m_\varphi^\psi(\xi_0, \xi) = (-1)^{t(\xi) - t_0(\xi) + t(\xi_0) - t_0(\xi_0)}.
$$

(106)

**Proof.** It follows from the definition of the KLV-polynomials ([LV, Section 0.1]), (63) and [AMR2, Proposition B.1] that

$$
\mathcal{V}P^\psi(\mathcal{V}\xi', \mathcal{V}\xi)(1) = (-1)^{d(\xi') - d(\xi')} \mathcal{C}^\psi(\xi', \xi) = (-1)^{t(\xi) - t(\xi_0) - t_0(\xi_0)} \mathcal{C}^\psi(\xi', \xi).
$$

(Note that the definition of $P(\xi)$ in [LV] differs from ours by a shift in degree $d(\xi)$ cf. [ABV, (7.10)(d)].) According to Proposition 3.6 the above equation may be written as

$$\mathcal{V}P^\psi(\mathcal{V}\xi', \mathcal{V}\xi)(1) = (-1)^{t(\xi) - t_0(\xi_0)} \mathcal{C}^\psi(\xi', \xi)
$$

Therefore the proposition is equivalent to proving

$$\mathcal{V}P^\psi(\mathcal{V}\xi', \mathcal{V}\xi_0)(1) = 1.
$$

This equation follows from [AAM, Proposition 4.17] once we establish that $\mathcal{V}\xi_0$ satisfies the stated hypotheses. Indeed, the proof of [AAM, Proposition 4.17] depends on the group $R_{C/R}GL_N$ only in terms of the types of roots (72). The proof relies crucially on the absence of some nuisance types, which fortunately do not appear in (72) either.

The hypotheses of [AAM, Proposition 4.17] refer to the block of $\xi$ ([V4, Definition 1.14]). The hypotheses of [AAM, Proposition 4.17] are satisfied if we can prove that $\mathcal{V}\xi_0$ is the unique maximal parameter in the block of $\pi(\mathcal{V}\xi)$ with respect to the (dual) Bruhat order. This is equivalent to establishing that $\xi_0$ is the unique minimal parameter in the block of $\pi(\xi)$ ([V4, Theorem 1.15]). We use [ABV, Proposition 1.11] to convert the Bruhat order for the representations of $GL_N^\Gamma$ into a closure relation between $R_{C/R}GL_N$-orbits of $X(\mathcal{V}\mathcal{O}, \mathcal{V}R_{C/R}GL_N^\Gamma)$. Moreover, this proposition implies that the minimality of $\xi_0 = (S_0, 1)$ is equivalent to the $R_{C/R}GL_N$-orbit $S_0 \subset X(\mathcal{V}\mathcal{O}, \mathcal{V}R_{C/R}GL_N^\Gamma)$ being maximal. The uniqueness and maximality of this orbit follows from [ABV, p. 19].

Equation (105) gives us information about the multiplicity of the Whittaker extension of a generic representation, and Equation (106) gives us information about the multiplicity of the Atlas extension of a generic representation. In the next theorem we combine this information with Lemma 7.1 to determine the sign between the two extensions of a generic representation. This may then be leveraged to determine the sign between the two extensions of an arbitrary irreducible representation.

**Theorem 7.4.** Suppose $\xi \in \Xi(\mathcal{V}\mathcal{O}, \mathcal{V}R_{C/R}GL_N^\Gamma)^\varphi$. Then

$$M(\xi)^\sim(\vartheta) = (-1)^{t(\xi) - t_0(\xi)} M(\xi)^\sim(\vartheta)
$$

and

$$\pi(\xi)^\sim(\vartheta) = (-1)^{t(\xi) - t_0(\xi)} \pi(\xi)^\sim(\vartheta).
$$

38
Proof. We first prove that there exists \( \xi_p = (x_p, y_p) \in \Xi(\text{AdC}) \) satisfying the hypothesis of Lemma 7.1 such that \( \pi(\xi_p) \) lies in the block of \( \pi(\xi) \). Let \( (x, y) \) be the Atlas parameter corresponding to \( \xi \) (Lemma 2.2) and \( \theta_x \) as in (55). A short exercise using (38) and (39) shows that
\[
\theta_x = (w, w_0 w^{-1} w_0^{-1}) \delta_0
\]
for some Weyl group element \( w \) for \( \text{R}_{C/R} \text{GL}_N \), the long Weyl group element \( w_0 \) for \( \text{R}_{C/R} \text{GL}_N \), and \( \delta_0 \) as in (32). According to [V4, Theorem 8.8] there is a transitive action on the block of \( \pi(\xi) \) given by cross actions (there are no Cayley transforms for \( \text{R}_{C/R} \text{GL}_N \)). Cross actions are recorded in the Atlas parameter \( (x, y) \) by conjugating both entries with an appropriate Weyl group element ([AdC, (9.11)(f)]). By taking the cross action of \( (x, y) \) with respect to \( (w_0 w^{-1}, 1) \in W(\text{R}_{C/R} \text{GL}_N, H) \) we arrive at an Atlas parameter \( (x_p, y_p) \) such that
\[
\theta_{x_p} = (w_0, w_0) \delta_0.
\]
Evidently, \( \theta_{x_p} \) sends all positive roots to negative roots. Setting \( \xi_p = (x_p, y_p) \), we see that \( \pi(\xi_p) \) lies in the block of \( \pi(\xi) \) and satisfies the hypothesis of Lemma 7.1. Consequently, \( M(\xi_p) = M(\xi_p)^+ \). It is straightforward to show \( \ell^1(\xi_p) = \ell^1(\xi_p) + 0 \). By (62) and (106)
\[
M(\xi_p)^+ = m^{\theta}_\rho(\xi_0, \xi) \pi(\xi_0)^+ + \sum_{\xi' \notin \xi_0} m^{\theta}_\rho(\xi', \xi_0) \pi(\xi')^+
\]
\[
= (-1)^{t(\xi_0)} + t(\xi_0) \pi(\xi_0)^+ + \sum_{\xi' \notin \xi_0} m^{\theta}_\rho(\xi', \xi_0) \pi(\xi')^+.
\]
According to (105), with \( \xi = \xi_p \), this equation implies
\[
(-1)^{t(\xi_0) - l(\xi_0)} \pi(\xi_0)^+ (\vartheta) = \pi(\xi_0)^- (\vartheta).
\] (107)
Thus, the theorem holds for \( \xi = \xi_0 \).

It remains to prove that the theorem holds when \( \xi \neq \xi_0 \). We compute, using (106) and (107), that
\[
M(\xi)^+ = \sum_{\xi' \notin \xi_0} m^{\theta}_\rho(\xi', \xi) \pi(\xi')^+ + m^{\theta}_\rho(\xi_0, \xi) \pi(\xi_0)^+
\]
\[
= \sum_{\xi' \notin \xi_0} m^{\theta}_\rho(\xi', \xi) \pi(\xi')^+ + (-1)^{t(\xi) - l(\xi) + t(\xi_0) - l(\xi_0)} \pi(\xi_0)^+
\]
\[
= \sum_{\xi' \notin \xi_0} m^{\theta}_\rho(\xi', \xi) \pi(\xi')^+ + (-1)^{t(\xi) - l(\xi)} \pi(\xi_0)^-.
\]
This equation and Lemma 7.2 imply
\[
(-1)^{t(\xi) - l(\xi)} \pi(\xi_0)^- (\vartheta) = M(\xi)^+ (\vartheta) |_{\pi(\xi_0)}.
\]
Combining this equation with (105), we see in turn that
\[
(-1)^{t(\xi) - l(\xi)} M(\xi)^+ (\vartheta) |_{\pi(\xi_0)} = \pi(\xi_0)^- (\vartheta) = M(\xi)^- (\vartheta) |_{\pi(\xi_0)}
\]
and
\[
(-1)^{t(\xi) - l(\xi)} M(\xi)^+ (\vartheta) = M(\xi)^- (\vartheta). \]
By taking Langlands quotients we obtain
\[
(-1)^{t(\xi) - l(\xi)} \pi(\xi)^+ (\vartheta) = \pi(\xi)^- (\vartheta).
\]

Theorem 7.4 allows us to replace the Atlas extensions with signed Whittaker extensions in the twisted pairing (59) and the endoscopic lifting (93). The results are recorded in the following two corollaries. The proofs are obtained from the analogous corollaries indicated in [AAM].

**Corollary 7.5.** [AAM, Corollary 7.9] The pairing (59), defined by (60), satisfies
\[
\langle M(\xi)^-, \mu(\xi')^+ \rangle = \delta_{\xi, \xi'}
\]

39
and

\[ \langle \pi(\xi'), P(\xi') \rangle = (-1)^{d(\xi)} \delta_{\xi, \xi'} \]

for \( \xi, \xi' \in \Xi(O, ^\vee R_{C/\mathbb{R}} GL_N^\Gamma) \). Equivalently (cf. Proposition 3.6),

\[ m_{\pi}^{-}(\xi', \xi) = (-1)^{d(\xi') - d(\xi)} c^\delta_\psi(\xi, \xi'), \quad (108) \]

where \( m_{\pi}^{-}(\xi', \xi) \) is defined by the decomposition

\[ M(\xi) = \sum_{\xi' \in \Xi(O, ^\vee R_{C/\mathbb{R}} GL_N^\Gamma)^o} m_{\pi}^{-}(\xi', \xi) \pi(\xi')^{-} \]

in \( K_{\Pi}(O, GL_N(\mathbb{C}), \psi) \).

**Corollary 7.6.** [AAM, Corollary 7.10] Suppose that \( S_G \subset X(O_G, ^\vee G^\Gamma) \) is a \(^\vee G\)-orbit and let \( \epsilon(S_G) \subset X(O, ^\vee R_{C/\mathbb{R}} GL_N^\Gamma) \) be the \(^\vee R_{C/\mathbb{R}} GL_N\)-orbit of the image of \( S_G \) under \( \epsilon \) \((85)\). Then

(a) \[ \text{Lift}_0(\eta_{S_G}^{loc}(\sigma)(\delta_q)) = M(\epsilon(S_G), 1)^{-}, \]

(b) \[ \text{Lift}_0 = \text{Trans}_{G(\mathbb{R})}(GL_N(\mathbb{C}) \times \theta) \]

on \( K_{\Pi}(O_G, G(\mathbb{R}, \delta_q))^{st} \).

8 The equality of \( \Pi_{\psi_G}^{\text{Mok}} \) and \( \Pi_{\psi_G}^{\text{ABV}} \) for regular infinitesimal character

In this section we prove the equality of the stable virtual characters \( \eta_{\psi_G}^{\text{Mok}} \) and \( \eta_{\psi_G}^{\text{ABV}} \) under a regularity condition on the infinitesimal character. We shall work using the framework of Section 5.2. In particular, \( \psi_G \) and \( \psi = \epsilon \circ \psi_G \) are \( A \)-parameters with respective infinitesimal characters \( ^\vee \mathcal{O}_G \) and \( ^\vee \mathcal{O} \). The assumption on the infinitesimal characters is that they are regular with respect to \( R_{C/\mathbb{R}} GL_N \). This assumption shall be removed in the next section.

The definition of \( \eta_{\psi_G}^{\text{Mok}} \) was outlined in the introduction. Let us follow [M3] and provide a few more details. All we need is contained in the following lemma, which is a version of Lemma 8.1 [AAM] for unitary groups.

**Lemma 8.1.** Let \( S_\phi \subset X(\mathcal{V} \mathcal{O}, ^\vee R_{C/\mathbb{R}} GL_N^\Gamma) \) be the \(^\vee R_{C/\mathbb{R}} GL_N\)-orbit corresponding to \( \phi_\psi \) ([ABV, Proposition 6.17]).

(a) There exist integers \( n_S \) such that

\[ \pi(S_\phi, 1)^{-} = \sum_{(S, 1) \in \Xi(\mathcal{V} \mathcal{O}, ^\vee R_{C/\mathbb{R}} GL_N^\Gamma)^o} n_S M(S, 1)^{-} \quad (109) \]

in \( K_{\Pi}(^\vee \mathcal{O}, GL_N(\mathbb{C}), \theta) \).

(b) For every \( S \) such that \( n_S \neq 0 \) in \( (109) \) there exists a unique \(^\vee G\)-orbit \( S_G \subset X(\mathcal{V} \mathcal{O}_G, ^\vee G^\Gamma) \) which is carried to \( S \) under \( \epsilon \).

(c) Writing

\[ S = \epsilon(S_G) \]

for the orbits in part (b), we have

\[ \pi(S_\phi, 1)^{-} = \text{Trans}_{G(\mathbb{R})}(GL_N(\mathbb{C}) \times \theta) \left( \sum_{S_G} n_{\epsilon(S_G)} \eta_{S_G}^{loc}(\delta_q) \right) \]

\[ = \text{Lift}_0 \left( \sum_{S_G} n_{\epsilon(S_G)} \eta_{S_G}^{loc}(\delta_q) \right) \quad (110) \]
Proof. The proof follows almost exactly as for [AAM, Lemma 8.1]. The only difference is that the existence of the orbit $S_G$ in part (b) is established in the proof of [M3, Proposition 8.2.1].

Let us recall that Mok defines $\eta_{\psi_G}^{Mok}$ through the identity

$$\pi(S_\psi, 1)^{\sim} = \text{Trans}_{G(R)}^{GL_N(\mathbb{C}) \rtimes \theta} (\eta_{\psi_G}^{Mok}) .$$

Since $\text{Trans}_{G(R)}^{GL_N(\mathbb{C}) \rtimes \theta} = \text{Lift}_0$ (7.6) and Lift$_0$ is injective (Proposition 5.3), it follows by Equation (110) that

$$\eta_{\psi_G}^{Mok} = \sum_{S_G} \eta_{\psi_G}^{\text{loc}}(S_G(\delta_\eta)) \in K_G(\mathfrak{O}_G, G(R, \delta_\eta))^{st}$$

(111)

(cf. [M3, Proposition 8.2.1]). By definition, the A-packet $\Pi_{\psi_G}^{Mok}$ consists of those irreducible representations in $\Pi(\mathfrak{O}_G, G(R, \delta_\eta))$ which occur with non-zero multiplicity when (111) is expressed as a linear combination of irreducible representations.

**Theorem 8.2.** Let $\psi_G$ be an A-parameter for $G$ with regular infinitesimal character. Then

$$\Pi_{\psi_G}^{Mok} = \Pi_{\psi_G}^{ABV}$$

Proof. The proof is the same as that of [AAM, Theorem 8.2 (a)]. We repeat it here in the case of unitary groups for the convenience of the reader. Let $\xi = (S_\psi, 1)$ as in Corollary 6.4.

$$\text{Lift}_0(\eta_{\psi_G}^{\text{mic}}(\delta_\eta)) = \text{Lift}_0(\eta_{\psi_G}^{\text{mic}}(\delta_\eta)) = \sum_{S_G} \eta_{\psi_G}^{\text{mic}}(S_G(\delta_\eta)) \in K_G(\mathfrak{O}_G, G(R, \delta_\eta))^{st}$$

(111)

The equality of the stable virtual characters follows from the injectivity of Lift$_0$ (Proposition 5.3). The equality of packets follows immediately.

9 The equality of $\Pi_{\psi_G}^{Mok}$ and $\Pi_{\psi_G}^{ABV}$ for singular infinitesimal character

To conclude our comparison of stable virtual characters, we retain the setup of the previous section, but without the hypothesis of regularity on the infinitesimal character. In other words, the orbits $\mathfrak{O}_G$ and $\mathfrak{O}$ are now allowed to be orbits of singular infinitesimal characters and the reader should think of them as such. In order to prove $\eta_{\psi_G}^{Mok} = \eta_{\psi_G}^{ABV}$ for singular $\lambda \in \mathfrak{O}_G$, we must extend the pairing of Theorem 3.5 and extend the twisted endoscopic lifting (93) to include representations with singular infinitesimal character. This was done in Section 9 of [AAM] through the use of the Jantzen-Zuckerman translation principle, to which we refer from now on simply as translation. The same arguments can be used to extend the results of the previous sections to singular representations of $GL_N(\mathbb{C})$. For the convenience of the reader we are including it here again, adding the needed modifications to the context of the current paper.

In essence, translation allows one to transfer results for regular infinitesimal character to results for singular infinitesimal character. Applying this principle to the results of the previous section will allow us to compare $\Pi_{\psi_G}$ with $\Pi_{\psi_G}^{\text{ABV}}$ with no restriction on the infinitesimal character.

The translation principle begins with the existence of a regular orbit $\mathfrak{O}' \subset \mathfrak{gl}_N \times \mathfrak{gl}_N$ and a translation datum $T$ from $\mathfrak{O}$ to $\mathfrak{O}'$ ([ABV, Definition 8.6, Lemma 8.7]). As $\mathfrak{O}$ is the $\mathfrak{R}_{C/R}GL_N$-orbit of $\lambda \in \mathfrak{h}$, we may take $\mathfrak{O}'$ to be the $\mathfrak{R}_{C/R}GL_N$-orbit of $\lambda' = \lambda + \lambda_1 \in \mathfrak{h}$

(112)
where \( \lambda_1 \in X_+(H) \) is regular and dominant with respect to the positive system of \( R^+(R_{C/R}GL_N, H) \). We may and shall take \( \lambda_1 \) to be the sum of the positive roots. In this way, each of \( \lambda, \lambda_1 \) and \( \lambda' \) are fixed by \( \vartheta \). The translation datum \( T \) induces a \( \gamma R_{C/R}GL_N \)-equivariant morphism

\[
 f_T : X(\gamma O', \gamma R_{C/R}GL_N) \to X(\gamma O, \gamma R_{C/R}GL_N) \tag{113}
\]

of geometric parameters ([ABV, Proposition 8.8]). The morphism has connected fibres of fixed dimension, a fact we shall use when comparing orbit dimensions. The \( \gamma R_{C/R}GL_N \)-equivariance of (113) is tantamount to a coset map commuting with left-multiplication by \( \gamma R_{C/R}GL_N \) ([ABV, (6.10)(b)]). Since both \( \lambda \) and \( \lambda' \) are fixed by \( \vartheta \), it is just as easy to see that the action of \( \vartheta \) commutes with the same coset map. We leave this exercise to the reader, taking for granted the resulting \( \gamma R_{C/R}GL_N \)-equivariance of (113).

According to [ABV, Proposition 7.15], the morphism \( f_T \) induces an inclusion

\[
f_T^\ast : \Xi(\gamma O, \gamma R_{C/R}GL_N) \hookrightarrow \Xi(\gamma O', \gamma R_{C/R}GL_N) \tag{114}
\]

of complete geometric parameters. The \( \vartheta \)-equivariance of (113) implies that this inclusion restricts to an inclusion (denoted by the same symbol)

\[
f_T^\ast : \Xi(\gamma O, \gamma R_{C/R}GL_N)^\vartheta \hookrightarrow \Xi(\gamma O', \gamma R_{C/R}GL_N)^\vartheta .
\]

The (Jantzen-Zuckerman) translation functor ([AvLT, (17.8)])

\[
 T^\lambda_N = T^\lambda_{\lambda_1},
\]

is an exact functor on a category of Harish-Chandra modules, which we shall often regard as a homomorphism

\[
 T^\lambda_{\lambda_1} : KII(\gamma O', GL_N(C) \rtimes \langle \vartheta \rangle) \to KII(\gamma O, GL_N(C) \rtimes \langle \vartheta \rangle) \tag{115}
\]

of Grothendieck groups. It is surjective ([AvLT, Corollary 17.9.8]). This translation functor is an extended version of the usual translation functor ([AvLT, (16.8f)]), which we also denote by

\[
 T^\lambda_{\lambda_1} : KII(\gamma O', GL_N(C)) \to KII(\gamma O, GL_N(C)).
\]

Let us take a moment to make (114) more precise. The sum of the positive roots \( \lambda_1 \) is the infinitesimal character of a finite-dimensional representation of \( GL_N(C) \). Therefore, \( \lambda_1 \) is the differential of a \( \vartheta \)-fixed quasicharacter \( \Lambda_1 \) of the real diagonal torus \( H(\mathbb{R}) \), which matches the weight of this finite-dimensional representation. The quasicharacter \( \Lambda_1 \) may be extended to a quasicharacter \( \Lambda_1^\ast \) of the semi-direct product \( H(\mathbb{R}) \rtimes \langle \vartheta \rangle \) by setting

\[
 \Lambda_1^\ast(\vartheta) = 1.
\]

We define translation in the extended setting of (114) using this representation of the extended group. Since the extension is evident here we continue to write \( T^\lambda_{\lambda_1} \) instead of \( T^\lambda_{\lambda_1} \).

In the ordinary setting of (115) we have

\[
 \pi(\xi) = T^\lambda_{\lambda_1} (\pi(f_T^\ast(\xi))), \quad M(\xi) = T^\lambda_{\lambda_1} (M(f_T(\xi))), \quad \xi \in \Xi(\gamma O, \gamma R_{C/R}GL_N) \tag{116}
\]

([AvLT, Corollary 16.9.4, 16.9.7 and 16.9.8], or [ABV, Theorem 16.4 and Proposition 16.6]). We define the Atlas extensions of \( \pi(\xi) \) and \( M(\xi) \), with \( \xi \in \Xi(\gamma O, \gamma R_{C/R}GL_N)^\vartheta \), by

\[
 \pi(\xi)^+ = T^\lambda_{\lambda_1} (\pi(f_T^\ast(\xi))^+), \quad M(\xi)^+ = T^\lambda_{\lambda_1} (M(f_T^\ast(\xi))^+).
\]

With the definition of Atlas extensions in place, the discussion of Section 2.5 is valid, and we see that \( T^\lambda_{\lambda_1} \) factors to a homomorphism of \( KII(\gamma O, GL_N(C), \vartheta) \) (see (47)). We use the same notation \( T^\lambda_{\lambda_1} \) to denote the functor of Harish-Chandra modules, and either of the earlier homomorphisms. The reader will be reminded of the context when it is important.

The definition of a Whittaker extension does not depend on the regularity of the infinitesimal character. The following proposition shows that translation sends Whittaker extensions to Whittaker extensions.
**Proposition 9.1.** Suppose $\xi \in \Xi(\mathcal{O}, \mathcal{O}_{G}^{T})$. Then (as Harish-Chandra modules)

$$T_{\lambda + \lambda'_1} (M(f^+_T(\xi))) = M(\xi),$$

and

$$T_{\lambda + \lambda'_1} (\pi(f^+_T(\xi))) = \pi(\xi).$$

**Proof.** The proof runs along the same lines as that of [AAM, Proposition 9.1]. The proof is even simpler in the current context, since every $\vartheta$-invariant irreducible representation of $G(N)$ appears as a subquotient of a principal series representation as given in Lemma 7.1.

Our translation datum $T$ for $R_{G}^{\mathcal{O}, \mathcal{O}_{G}^{T}}$ is defined by (112), in which both $\lambda$ and $\lambda'$ are fixed by $\vartheta$. For this reason (112) also determines a translation datum $T_{G}$ for $\gamma G$-orbits $\mathcal{O}_{G}$ to $\mathcal{O}_{G}^{T}$ for the twisted endoscopic group $G$ ([ABV, Definition 8.6 (e)]). Just as for $R_{G}^{\mathcal{O}, \mathcal{O}_{G}^{T}}$, we have maps

$$f_{T_G} : X(\mathcal{O'}, \gamma G^{T}) \rightarrow X(\mathcal{O}, \gamma G^{T})$$

and the translation functor $T_{\lambda + \lambda'_1}$ which satisfies

$$\pi(\xi) = T_{\lambda + \lambda'_1} (\pi(f^+_T(\xi))) , \quad \xi \in \Xi(\mathcal{O}, \mathcal{O}_{G}^{T})$$

([ABV, Proposition 16.6], [AvLT, Section 16]).

The translation data $T$ and $T_{G}$ allow us to transport properties of our pairings at regular infinitesimal character to the same properties for pairings at singular infinitesimal character. More precisely, as explained at the end of [ABV, page 178], the translation datum $T_{G}$ applied to Proposition 3.3 allows us to extend Theorem 3.1 to any infinitesimal character. In a similar fashion, the translation datum $T$ applied this time to Equation (108), allows us to transport Corollary 7.5 from regular to singular infinitesimal character. These two procedures have the following result.

**Proposition 9.2** ([AAM, Proposition 9.2]). Define the pairing

$$\langle \cdot, \cdot \rangle : K(\mathcal{O}, \mathcal{O}_{G}^{T}, \vartheta) \times K X(\mathcal{O}, \mathcal{O}_{G}^{T}, \vartheta) \rightarrow \mathbb{Z}$$

by

$$\langle M(\xi), \mu(\xi') \rangle = \delta_{\xi, \xi'}. \quad (116)$$

Then

$$\langle \pi(\xi), P(\xi') \rangle = (-1)^{d(\xi)} \delta_{\xi, \xi'},$$

where $\xi, \xi' \in \Xi(\mathcal{O}, \mathcal{O}_{G}^{T})$.

Proposition 9.2 is the final version of the twisted pairing, and we use it to extend the definition of endoscopic lifting $\text{Lift}_0$ to include singular infinitesimal characters ((92), (93)). In fact, all of the remaining results used in Section 8 easily carry over to the more general setting. In particular, using the pairing (116) in the proof of Proposition 5.2, we see that for any $\gamma G$-orbit $S_G \subset X(\mathcal{O}, \gamma G^{T})$ we still have

$$\text{Lift}_0 (\sigma_{\text{loc}}^{\vartheta}(\delta_\vartheta))(\delta_\vartheta) = M(\varepsilon(S_G), 1)^{-\vartheta}.$$ 

Finally, since the injectivity of $\text{Lift}_0$ still holds for singular infinitesimal character, the same argument used in the proof of Theorem 8.2 allows us to conclude

**Theorem 9.3.** Let $\psi_G$ be an $A$-parameter for $G$. Then

$$\eta^{\text{Mok}}_{\psi_G} = \eta^{\text{mic}}_{\psi_G}(\delta_\vartheta) = \eta^{\text{ABV}}_{\psi_G}, \quad \Pi^{\text{Mok}}_{\psi_G} = \Pi^{\text{ABV}}_{\psi_G}.$$
10 Some consequences for standard endoscopy

In this section we explore standard endoscopic lifting from an endoscopic group $G'$ of our quasisplit unitary group $G$. Both [ABV] and [M3] provide formulae for this lifting, and it is not obvious why the two types of formulae should be equal. Our goal is to show how Theorem 9.3 implies the equality of the formulae, arguing as in [AAM, Section 10].

A brief review of the definition of $G'$ may be found in [ABV, Section 26]. The endoscopic group $G'$ is defined to be a quasisplit form of a complex reductive group whose dual $\check{G}'$ is the identity component of the centralizer in $\hat{G}$ of a semisimple element $s \in \hat{G}$ (cf. Section 5). Furthermore, the element $s$ is taken to centralize the image of $\psi_G$, and there is a natural embedding $\epsilon': \hat{G}' \hookrightarrow \hat{G}$ ($\epsilon' = \zeta_\chi$ with trivial $\chi$ in [M3, (2.1.13)])

Let us first look at endoscopic lifting from the perspective of [ABV]. Let $\text{Lift}_G$ denote the endoscopic lifting map from stable virtual characters of the quasisplit group $G'$ to the virtual characters of $G(\mathbb{R}) = G(\mathbb{R}, \delta)$ as given in [ABV, Definition 26.18]. According to [ABV, Theorem 22.7 and Theorem 26.25] there exists a stable virtual character $\psi^{ABV}_{G'}$ of $G'(\mathbb{R})$ such that

$$\text{Lift}_G^G(\eta_{\psi_{G'}}^{ABV}) = \sum_{\pi \in \Pi_{\psi_{G}}} (-1)^{d(\pi)-d(S_{\psi_{G}})} \text{Tr} \left( \tau_{\psi_{G}}(\pi)(\bar{s}) \right) \pi.$$  

(117)

Here, $d(S_{\psi_{G}})$ is the dimension of the $\hat{G}$-orbit $S_{\psi_{G}}$, and $\bar{s}$ is the coset of $s$ in $A_{\psi_{G}}$.

The analogue of formula (117) from Mok’s perspective is [M3, (8.2.4)]. It describes the endoscopic lifting map $\text{Trans}_{G(\mathbb{R})}(\eta_{\psi_{G}}^{Mok})$ defined by Shelstad ([S5]) on a stable virtual character $\eta_{\psi_{G}}^{Mok}$ of $G(\mathbb{R})$. With this notation Mok’s formula is

$$\text{Trans}_{G(\mathbb{R})}(\eta_{\psi_{G}}^{Mok}) = \sum_{\sigma \in \Sigma_{\psi_{G}}} \langle s_{\psi_{G}} \bar{s}, \sigma \rangle \sigma.$$  

(118)

Here, $\Sigma_{\psi_{G}}$ is a finite set of non-negative integral linear combinations

$$\sigma = \sum_{\pi \in \Pi_{\text{min}}(G(\mathbb{R}))} m(\sigma, \pi) \pi$$

of irreducible unitary characters of $G(\mathbb{R})$. Furthermore, there is an injective map from $\Sigma_{\psi_{G}}$ into the set of those quasicharacters of $A_{\psi_{G}}$ which are trivial on the centre of $\hat{G}$. The injection is denoted by

$$\sigma \mapsto \langle \cdot, \sigma \rangle.$$  

The element $s_{\psi_{G}}$ is the image of

$$\psi_{G} \left(1, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right)$$

in $A_{\psi_{G}}$.

If one takes $s = 1$ in (118) then one recovers $G' = G$ and the equation reduces to

$$\eta_{\psi_{G}}^{Mok} = \sum_{\sigma \in \Sigma_{\psi_{G}}} \langle s_{\psi_{G}}, \sigma \rangle \sigma$$

$$= \sum_{\sigma \in \Sigma_{\psi_{G}}} \langle s_{\psi_{G}}, \sigma \rangle \sum_{\pi \in \Pi_{\text{min}}(G(\mathbb{R}))} m(\sigma, \pi) \pi$$

$$= \sum_{\pi \in \Pi_{\psi_{G}}} \left( \sum_{\sigma \in \Sigma_{\psi_{G}}} m(\sigma, \pi) \langle s_{\psi_{G}}, \sigma \rangle \right) \pi.$$
For each $\pi \in \Pi_{\psi_G}$ define
\[
\tau_{\psi_G}(\pi) = \sum_{\sigma \in \Sigma_{\psi_G}} m(\sigma, \pi) \langle \cdot, \sigma \rangle,
\]
which is apparently a finite sum of quasicharacters of $A_{\psi_G}$. Mœglin and Renard have proven that $\tau_{\psi_G}(\pi)$ is actually irreducible ([MgR, Théorème 1.3]). The previous equation becomes
\[
\eta_{\psi_G}^{Mok} = \sum_{\pi \in \Pi_{\psi_G}} \tau_{\psi_G}(\pi)(s_{\psi_G}) \pi,
\]
and more generally (118) becomes
\[
\text{Trans}_{G'_{\psi_G}(R)}^{G(R)}(\eta_{\psi_G}^{Mok}) = \sum_{\pi \in \Pi_{\psi_G}} \tau_{\psi_G}(\pi)(s_{\psi_G}\bar{s}) \pi. \tag{119}
\]

Our goal is to show that the two equations, (119) and (117), are identical. In order to compare the two equations, we must choose the element $s \in \mathcal{Y}G$ defining the endoscopic group $G'$ carefully. We may suppose $\bar{s} \in A_{\psi_G}$ is not trivial. Recall that $s$ lies in the centralizer of the image of $\psi_G$. An explicit description of this centralizer is given in [M3, (2.4.13)] (following [GGP, Section 4]). In consideration of this description, it is not difficult to choose for each, $\bar{s} \in A_{\psi_G}$, a diagonal representative $\bar{s}$ in the centralizer with eigenvalues $\pm 1$. The endoscopic group $G'(\bar{s})$ determined by $\bar{s}$ is a direct product $G'_1(\bar{s}) \times G'_2(\bar{s})$ in which each of the two factors is a quasisplit unitary group whose rank is less than $G$ (cf. [R, Proposition 6.4.1]). The $A$-parameter $\psi_{G'(\bar{s})}$ decomposes accordingly as a product $\psi_{G'_1(\bar{s})} \times \psi_{G'_2(\bar{s})}$ of $A$-parameters. Similarly, Mok’s stable virtual character $\eta_{\psi_{G'(\bar{s})}}^{Mok}$ is defined as the tensor product $\eta_{\psi_{G'_1(\bar{s})}}^{Mok} \otimes \eta_{\psi_{G'_2(\bar{s})}}^{Mok}$. For these choices of endoscopic data, (119) reads as
\[
\text{Trans}_{G'_{\psi_G}(R)}^{G(R)}(\eta_{\psi_{G'(\bar{s})}}^{Mok} \otimes \eta_{\psi_{G'(\bar{s})}}^{Mok}) = \sum_{\pi \in \Pi_{\psi_G}} \tau_{\psi_G}(\pi)(s_{\psi_G}\bar{s}) \pi. \tag{120}
\]

We now turn to rewriting the left-hand side of (120) so as to match it with the left-hand side of (117). First, it is noted on [ABV, p. 289] that $\text{Trans}_{G'_{\psi_G}(R)}^{G(R)} = \text{Lift}_{G'_1}$. Second, using the arguments in the proof of Corollary 6.2, we see that
\[
\eta_{\psi_{G'(\bar{s})}}^{ABV} = \eta_{\psi_{G'_1(\bar{s})}}^{ABV} \otimes \eta_{\psi_{G'_2(\bar{s})}}^{ABV}.
\]

Third, since $G'_1(\bar{s})$ and $G'_2(\bar{s})$ are both quasisplit unitary groups, Theorem 9.3 tells us that
\[
\eta_{\psi_{G'_j(\bar{s})}}^{Mok} = \eta_{\psi_{G'_j(\bar{s})}}^{ABV}, \quad j = 1, 2.
\]

Taking these three observations together we conclude
\[
\text{Trans}_{G'_{\psi_G}(R)}^{G(R)}(\eta_{\psi_{G'(\bar{s})}}^{Mok} \otimes \eta_{\psi_{G'(\bar{s})}}^{Mok}) = \text{Lift}_{G'_1} (\eta_{\psi_{G'_1(\bar{s})}}^{Mok} \otimes \eta_{\psi_{G'_2(\bar{s})}}^{Mok}) \\
= \text{Lift}_{G'_1} (\eta_{\psi_{G'_1(\bar{s})}}^{ABV} \otimes \eta_{\psi_{G'_2(\bar{s})}}^{ABV}) \\
= \text{Lift}_{G'_1} (\eta_{\psi_{G'_1(\bar{s})}}^{ABV}) 
\]

It is now immediate from (120) and (117) that
\[
\sum_{\pi \in \Pi_{\psi_G}} \tau_{\psi_G}(\pi)(s_{\psi_G}\bar{s}) \pi = \sum_{\pi \in \Pi_{\psi_G}} (-1)^{d(\pi)-d(S_{\psi_G})} \text{Tr} (\tau_{\psi_G}^{ABV}(\pi)(\bar{s})) \pi
\]
for any $s \in A_{\psi}$. By the linear independence of characters on $G(R)$
\[
\tau_{\psi_G}(\pi)(s_{\psi_G}\bar{s}) = (-1)^{d(\pi)-d(S_{\psi_G})} \text{Tr} (\tau_{\psi_G}^{ABV}(\pi)(\bar{s}))
\]
for any $s \in A_{\psi}$. This may be regarded as an equality between virtual characters on $A_{\psi}$. By appealing to the linear independence of these characters we conclude that
\[
\tau_{\psi_G}(\pi)(s_{\psi_G}) = (-1)^{d(\pi)-d(S_{\psi_G})}
\]
and
\[
\tau_{\psi_G}(\pi) = \tau_{\psi_G}^{ABV}(\pi).
\]
In particular, $\tau_{\psi_G}^{ABV}(\pi)$ is irreducible. This completes our comparison of (117) and (119).
List of Symbols

$(\pi, \delta)$ representation of a strong involution, page 8
$(\lambda, \tau, \ell, t)$ extended parameter, page 13
$(S, \tau_S)$ pure complete geometric parameter, page 8
$(U, \chi)$ Whittaker datum, page 35
$(x, y)$ Atlas parameter, page 12
$\chi$ unitary character of $U(\mathbb{R})$, page 35
$\chi^\text{mic}$ microlocal multiplicity map, page 16
$\delta$ strong involution, page 7
$\delta_0$ strong involution defining extended group, page 7
$\delta_q$ strong involution corresponding to quasisplit real form, page 8
$\epsilon$ endoscopic maps, page 28
$\epsilon^*$ inverse image functor, page 28
$\epsilon_*$, page 28
$\eta^\text{ABV}_{\psi}$ stable virtual character defining the ABV-packet, page 17
$\eta_{\psi}^\text{loc}$ virtual character of a pseudopacket, page 28
$\eta^\text{mic}_{\psi}$ stable virtual character, page 16
$\eta^\text{Mok}_{\psi}$ stable virtual character defining Arthur’s A-packet, page 40
$\eta^\text{mic}_{\psi}(\delta_q)$, page 17
$\eta^\text{Mok}_{\psi}$ Mok’s stable virtual character, page 1
$\eta^\text{mic+}_{\psi}$ virtual twisted character, page 32
$\eta^\text{mic+}_{\psi}(\vartheta)$, page 33
$\eta_{\psi}^\text{loc}(\vartheta)(\delta_q)$, page 30
$\Gamma$ Galois group of $\mathbb{C}/\mathbb{R}$, page 1
$\kappa$ $\vartheta$-orbit of simple roots, page 22
$\mathcal{C}(X(\vee \mathcal{O}, \check{\vee} \mathcal{R}_{\mathbb{C}/\mathbb{R}} \mathcal{G}^\Gamma_N); \vartheta)$ category of constructible sheaves with a compatible $\vartheta$-action, page 18
$\mathcal{C}(X(\vee \mathcal{O}, \check{\vee} \mathcal{G}^\Gamma))$ category of constructible sheaves, page 15
$\mathcal{D}(X(\vee \mathcal{O}, \check{\vee} \mathcal{G}^\Gamma))$ category of $\mathcal{D}$-modules, page 16
$\mathcal{H}(\lambda)$ twisted Hecke algebra, page 22
$\mathcal{KII}(\vee \mathcal{O}, \mathcal{G}^\Gamma_N(\mathbb{C}), \vartheta)$ twisted Hecke module, page 22
$\mathcal{K}(\vee \mathcal{O}, \check{\vee} \mathcal{G}^\Gamma_N(\mathbb{C}))$ dual twisted Hecke module, page 22
$\mathcal{P}(X(\vee \mathcal{O}, \check{\vee} \mathcal{R}_{\mathbb{C}/\mathbb{R}} \mathcal{G}^\Gamma_N); \vartheta)$ category of perverse sheaves with a compatible $\vartheta$-action, page 17
$\mathcal{P}(X(\vee \mathcal{O}, \check{\vee} \mathcal{G}^\Gamma))$ category of perverse sheaves, page 15

46
\( \mathcal{T} \) translation datum, page 41
\( \mathcal{X}_{\psi_R} \) set of conjugacy classes of strong involutions with infinitesimal cocharacter \( \psi_R \), page 11
\( \mathcal{X}_{\psi_R}^w \), page 11
\( \text{Ch}(\mathcal{M}) \) characteristic cycle of a \( \mathcal{D} \)-module \( \mathcal{M} \), page 16
\( \text{gr}\mathcal{M} \) graded sheaf of a \( \mathcal{D} \)-module \( \mathcal{M} \), page 16
\( \text{Lift}_0 \) endoscopic lifting map, page 28
\( \text{Lift}_0 \) twisted endoscopic lifting map, page 30
\( \text{R}_{\mathbb{C}/\mathbb{R}} \text{GL}_N \) restriction of scalars of \( \text{GL}_N \), page 9
\( \mu_r \boxtimes \mu_n \), page 34
\( \mu(\xi) \) irreducible constructible sheaf, page 15
\( \mu(\xi)^+ \) twisted constructible sheaf, page 18
\( \pi(\xi) \) (equivalence class of the) irreducible Harish-Chandra module (representation) associated to \( \xi \), page 12
\( \pi(\xi)^+ \) (equivalence class of) the Atlas extension of \( \pi(\xi) \), page 13
\( \pi(\xi)^\sim \) (equivalence class of) the Whittaker extension of \( \pi(\xi) \), page 35
\( \Pi(G/\mathbb{R}) \) set of equivalence classes of irreducible representations, page 8
\( \Pi(\psi \mathcal{O},\text{GL}_N(\mathbb{C})) \) set of infinitesimal equivalence classes of irreducible representations of \( \text{GL}_N(\mathbb{C}) \) with infinitesimal character \( \psi \mathcal{O} \), page 10
\( \Pi(\psi \mathcal{O}, G/\mathbb{R})^0 \) \( \psi \)-fixed equivalence classes of representations, page 12
\( \Pi(\psi \mathcal{O}, G/\mathbb{R})^\psi \) set of equivalence classes of representations with infinitesimal character \( \psi \mathcal{O} \), page 8
\( \pi^- \), page 14
\( \Pi^{\text{ABV}}_{\psi_G} \) ABV-packet, page 17
\( \Pi^{\text{mic}}_{\psi_G} \) microlocal packet, page 17
\( \Pi^{\text{Mok}}_{\psi_G} \) Mok’s A-packet, page 40
\( \psi \) A-parameter for \( \text{R}_{\mathbb{C}/\mathbb{R}} \text{GL}_N \), page 2
\( \psi_G \) A-parameter for \( G \), page 1
\( \tau_{S_{\psi_G}} (P(\xi)) \) representation of component group, page 31
\( \tau_S \) irreducible representation of component group, page 8
\( \tau_{S_{\psi_G}} (P(\xi)^+) \), page 33
\( \tilde{J} \) anti-diagonal matrix, page 2
\( C^d(\xi) \), page 26
\( \vartheta \) automorphism of \( \text{R}_{\mathbb{C}/\mathbb{R}} \text{GL}_N \), page 2
\( \vartheta \) automorphism of \( \text{R}_{\mathbb{C}/\mathbb{R}} \text{GL}_N \), page 17
\( \Xi(\psi \mathcal{O}, \psi G^\vartheta) \) set of pure complete geometric parameters, page 9
\( C^d(\xi) \), page 25
$C^0(\check{\xi})$, page 26

c_{g}(\xi',\xi)$ signed sheaf-theoretic multiplicity, page 17

c_{g}^0(\xi',\xi)$, page 20

$D$ Verdier duality map, page 25

d(\xi)$ dimension of the orbit attached to a complete geometric parameter, page 15

d(S)$ dimension of S, page 16

$DR$ Riemann-Hilbert correspondence, page 16

e(\xi)$ Kottwitz invariant attached to a complete geometric parameter, page 15

$f_{\tau}^*$, page 41

$\Gamma$ extended group, page 7

$i_{\nu\alpha}$ homomorphism of component groups, page 31

$J(x,y,\lambda)^+$ Atlas extension, page 13

$J(x,y,\lambda)$ Langlands quotient, page 12

$K(X(\check{\mathcal{O}},\check{\mathcal{R}}_{C/R}GL_{N}^\Gamma,\vartheta))$ $\mathbb{Z}$-module of twisted sheaves, page 18

$K\Pi(GL_{N}(\mathbb{C}))$, page 14

$K\Pi^d(GL_{N}(\mathbb{C}))$, page 21

$K'$ fixed-point set of a strong involution, page 8

$K(X(\check{\mathcal{O}},\check{\mathcal{R}}_{C/R}GL_{N}^\Gamma,\vartheta))$ Grothendieck group, page 18

$K\Pi(\check{\mathcal{O}},GL_{N}(\mathbb{C}),\vartheta)$ virtual twisted characters, page 14

$K\Pi(\check{\mathcal{O}},GL_{N}(\mathbb{C}))^d$, page 14

$K\Pi(\check{\mathcal{O}},G(\mathbb{R},\delta_q))^s$ stable virtual characters, page 14

$K\Pi(\check{\mathcal{O}},G/\mathbb{R})$ Grothendieck group of representations of pure strong involutions, page 14

$K_{\mathbb{C}}\Pi^d(\check{\mathcal{O}}_{G},G/\mathbb{R})$, page 28

$K(\check{\mathcal{O}},\check{\mathcal{R}})$ Grothendieck group, page 15

$K\Pi^d(\check{\mathcal{O}},GL_{N}(\mathbb{C}))$ dual Grothendieck group, page 21

$l(w)$ length in Weyl group, page 23

$l^i(\xi)$ integral length, page 19

$l^i_{\vartheta}(\xi)$ $\vartheta$-integral length, page 19

$M(\xi)$, $M(x,y)$ (equivalence class of the) standard module (representation) associated to $\xi$, page 12

$M(\xi)^+$ (equivalence class of) the Atlas extension of $M(\xi)$, page 13

$M(\xi)^-$ (equivalence class of) the Whittaker extension of $M(\xi)$, page 36

$m_{\vartheta}^d(\xi',\xi)$, page 19

$m_{\vartheta}^d(\xi',\xi)$ representation-theoretic multiplicity, page 17
References

[A] Jeffrey Adams, *Discrete series and characters of the component group*, On the stabilization of the trace formula, 2011, pp. 369–387.

[AAM] J. Adams, N. Arancibia, and P. Mezo, *Equivalent definitions of Arthur packets for real classical groups*. arXiv:2108.05788.

[A1] James Arthur, *On some problems suggested by the trace formula*, Lie group representations, II (College Park, Md., 1982/1983), 1984, pp. 1–49.

[A2] James Arthur, *Unipotent automorphic representations: conjectures*, Astérisque 171-172 (1989), 13–71. Orbites unipotentes et représentations, II. MR1021499

[A3] James Arthur, *The endoscopic classification of representations*, American Mathematical Society Colloquium Publications, vol. 61, American Mathematical Society, Providence, RI, 2013. Orthogonal and symplectic groups. MR3185650
On certain $L$-functions, Amer. J. Math. 103 (1981), no. 2, 297–355.

D. Shelstad, Characters and inner forms of a quasi-split group over $\mathbf{R}$, Compositio Math. 39 (1979), no. 1, 11–45.

On geometric transfer in real twisted endoscopy, Ann. of Math. (2) 176 (2012), no. 3, 1919–1985.

Diana Shelstad, Orbital integrals, endoscopic groups and $L$-indistinguishability for real groups, Conference on automorphic theory (Dijon, 1981), 1983, pp. 135–219. MR723184 (85i:22019)

T. A. Springer, Linear algebraic groups, Second, Progress in Mathematics, vol. 9, Birkhäuser Boston, Inc., Boston, MA, 1998. MR1642713

Laura Smithies and Joseph L. Taylor, An analytic Riemann-Hilbert correspondence for semi-simple Lie groups, Represent. Theory 4 (2000), 398–445.

Marko Tadić, $\text{GL}(n,\mathbb{C})^\sim$ and $\text{GL}(n,\mathbb{R})^\sim$, Automorphic forms and $L$-functions II. Local aspects, 2009, pp. 285–313.

David A. Vogan, Irreducible characters of semisimple Lie groups. III. Proof of Kazhdan-Lusztig conjecture in the integral case, Invent. Math. 71 (1983), no. 2, 381–417.

David A. Vogan Jr., Gelfand-Kirillov dimension for Harish-Chandra modules, Invent. Math. 48 (1978), no. 1, 75–98.

Representations of real reductive Lie groups, Progress in Mathematics, vol. 15, Birkhäuser, Boston, Mass., 1981.

Irreducible characters of semisimple Lie groups. IV. Character-multiplicity duality, Duke Math. J. 49 (1982), no. 4, 943–1073. MR683010

The local Langlands conjecture, Representation theory of groups and algebras, 1993, pp. 305–379. MR1216197 (94e:22031)