PROPER TRAJECTORIES OF TYPE $\mathbb{C}^*$ OF A POLYNOMIAL VECTOR FIELD ON $\mathbb{C}^2$

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Abstract. We prove that if a polynomial vector field on $\mathbb{C}^2$ has a proper and non-algebraic trajectory analytically isomorphic to $\mathbb{C}^*$ all its trajectories are proper, and except at most one which is contained in an algebraic curve of type $\mathbb{C}$ all of them are of type $\mathbb{C}^*$. As corollary we obtain an analytic version of Lin-Zaidenberg Theorem for polynomial foliations.

1. Introduction

We shall consider from now on polynomial vector fields on $\mathbb{C}^2$ with isolated zeroes. Such vector fields $X$ define a foliation by curves $\mathcal{F}_X$ in $\mathbb{C}^2$ with a finite number of singularities (zeros of $X$) that extends to $\mathbb{CP}^2 = \mathbb{C}^2 \cup L_\infty$ (see [6]). Each trajectory $C_z$ of $X$ through a $z \in \mathbb{C}^2$ with $X(z) \neq 0$ is contained in a leaf $L$ of this extended foliation, and its limit set $\lim(C_z)$ is defined as $\cap_{m \geq 1} L \setminus K_m$, where $K_m \subset K_{m+1} \subset L$ is a sequence of compact subsets with $\cup_{m \geq 1} K_m = L$. We say that a trajectory $C_z$ is proper if its topological closure $\overline{C_z}$ defines an analytic curve in $\mathbb{C}^2$ of pure dimension one, i.e. if the inclusion of $\overline{C_z}$ in $\mathbb{C}^2$ is a proper map. For a proper trajectory $C_z$ its $\lim(C_z)$ is either a finite set of points, and $C_z$ is said to be algebraic, or it contains $L_\infty$, and $C_z$ is said to be non-algebraic. In what follows, transcendental will mean proper and non-algebraic.

The important work of Marco Brunella on the trajectories of a polynomial vector field with a transcendental planar isolated end [2] has a remarkable corollary: If $X$ is a polynomial vector field on $\mathbb{C}^2$ with a transcendental trajectory $C_z$ of type $\mathbb{C}$ ("of type" means analytically isomorphic to) the foliation $\mathcal{F}_X$ in $\mathbb{C}^2$ is equal to the foliation defined by a constant vector field after an holomorphic automorphism [2 Corollaire]. In particular any proper immersion $\gamma$ of $\mathbb{C}$ in $\mathbb{C}^2$ whose image is contained in a leaf of a polynomial foliation is equal to $\gamma(t) = (t, 0)$ modulo a holomorphic automorphism. That result can be considered as an Abhyankar-Moh and Suzuki Theorem ([1] and [12]) for polynomial foliations [2, p. 1230]. In this note we will study the case of a polynomial vector field with a transcendental trajectory of type $\mathbb{C}^*$. We will start with [2 Théorème] and apply some previous results of [3] and [5] to determine these vector fields. The main result is the following:

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Theorem. If a polynomial vector field \( X \) on \( \mathbb{C}^2 \) has a transcendental trajectory of type \( \mathbb{C}^* \), all its trajectories are proper, and except at most one which is contained in an algebraic curve of type \( \mathbb{C} \) all of them are of type \( \mathbb{C}^* \).

2. Corollaries

Corollary 1. Any polynomial vector field \( X \) on \( \mathbb{C}^2 \) with a transcendental trajectory of type \( \mathbb{C}^* \) has a meromorphic first integral of type \( \mathbb{C}^* \) which modulo a holomorphic automorphism is of the form

\[
(x^\ell y + p(x))^n
\]

where \( m \in \mathbb{Z}^*, n \in \mathbb{N}^* \) with \((m, n) = 1 \) if \( \ell > 0 \) or \( p(x) \equiv 0 \) if \( \ell = 0 \).

Proof. According to Masakazu Suzuki [14, Théorème II] for a vector field on \( \mathbb{C}^2 \) with proper parabolic trajectories there is always a meromorphic first integral. In particular for \( X \) this integral must be of type \( \mathbb{C}^* \) and it can be explicitly written applying Saito-Suzuki Theorem [13, p. 527], [10].

Remark 1. It follows from Corollary 1 that if \( X \) is a polynomial vector field on \( \mathbb{C}^2 \) with a transcendental trajectory of type \( \mathbb{C}^* \) after a holomorphic change of coordinates \( \phi \), the corresponding vector field \( \phi^* X \) (maybe not polynomial) has a rational first integral of the form (1). Removing the poles and zeros of codimension one of the differential of (1) one obtains that \( \phi^* X \) must be of the form

\[
\phi^* X = f \cdot Y = f \cdot \left\{ nx^{\ell+1} \frac{\partial}{\partial x} - ((m + nl)x^\ell y + mp(x) + nxp(x)) \frac{\partial}{\partial y} \right\},
\]

where \( f \) is a holomorphic function that never vanishes; and \( m, n, \ell \) and \( p(x) \) are as in (1). In particular, any foliation \( \mathcal{F}_X \) generated by a polynomial vector field \( X \) on \( \mathbb{C}^2 \) with a transcendental trajectory of type \( \mathbb{C}^* \) corresponds to the algebraic foliation generated by the polynomial vector field \( Y \) of (2) after a holomorphic automorphism.

Analytic version of Lin-Zaidenberg Theorem for polynomial vector fields

Lin-Zaidenberg Theorem [15] asserts that any irreducible algebraic curve of type \( \mathbb{C} \) in \( \mathbb{C}^2 \) is of the form \( y^r - ax^s = 0 \), with \((r, s) = 1 \) and \( a \in \mathbb{C}^* \), after a polynomial change of coordinates. From our Theorem we obtain the analytic version of this theorem for polynomial foliations:

Corollary 2. Let \( C \) be an irreducible transcendental curve in \( \mathbb{C}^2 \) of type \( \mathbb{C} \). If there is a point \( p \in C \) such that \( C \setminus \{ p \} \) defines a trajectory of a polynomial vector field then \( C = \{ y^r - ax^s = 0 \}, r, s \in \mathbb{N}^+, (r, s) = 1, a \in \mathbb{C}^* \), up to a holomorphic automorphism.

Proof. As \( C \setminus \{ p \} \) is a trajectory of type \( \mathbb{C}^* \) of a polynomial vector field it must be contained in a level set of (1) by Corollary 1. If the level is over \( a \neq 0 \), as it is of type \( \mathbb{C} \), \( \ell = 0 \) and \( m < 0 \). It is enough define \( r = n \) and \( s = -m \). If the level set is
over zero, necessarily it is a line: \( \{ x = 0 \} \) or also \( \{ y = 0 \} \) if \( \ell = 0 \), which has the required form with \( r = s = 1 \) after a rotation.

\[ \square \]

**Remark 2.** The classification of H. Saito in [11] contains polynomials of this form:

\[
P = 4((xy + 1)^2 + y)(x(xy + 1) + 1)^2 + 1
\]

Such a \( P \) has two singular fibers: \( P^{-1}(0) \) and \( P^{-1}(1) \). One of them, \( P^{-1}(0) \), is a disjoint union of two curves of type \( C^* \), and another, \( P^{-1}(0) \), is an irreducible curve of type \( C^* \). The generic fiber of \( P \) is of type \( \mathbb{C} \setminus \{0, 1\} \). In particular, our Theorem implies that if there is a polynomial vector field with a holomorphic first integral of the form \( P \circ \varphi \) with \( \varphi \) a holomorphic automorphism then either \( \varphi \) is a polynomial automorphism or \( (P \circ \varphi)^{-1}(0) \) and \( (P \circ \varphi)^{-1}(1) \) are contained in algebraic curves.

### 3. Proof of Theorem

Let \( C_z \) be the transcendental trajectory of \( X \) of type \( C^* \). It defines a leaf \( L \) of \( \mathcal{F}_X \) of type \( C^* \) with a transcendental planar isolated end \( \Sigma \) (see [2] Lemma 4.1]). We can apply [2, Théoreme] and conclude that there exists a polynomial \( P \) with generic fiber of type \( C \) or \( C^* \) (that we will call of type \( C \) or \( C^* \), respectively) such that \( \mathcal{F}_X \) is \( P \)-complete. Let us recall from [2] that \( \mathcal{F}_X \) is is \( P \)-complete if there exists a finite set \( Q \subseteq \mathbb{C} \) such that for all \( t \notin Q \) \( (i) \) \( P^{-1}(t) \) is transverse to \( \mathcal{F}_X \), and \( (ii) \) there is a neighborhood \( U_t \) of \( t \) in \( \mathbb{C} \) such that \( P : P^{-1}(U_t) \to U_t \) is a holomorphic fibration and the restriction of \( \mathcal{F}_X \) to \( P^{-1}(U_t) \) defines a local trivialization of this fibration.

As noted in [2, p.1229] (see also [3] Remark 2.2)) the set \( Q \) associated to \( P \) consists of the critical values of \( P \) together with the regular values of \( P \) in which some of the components of the corresponding fiber are not transversal to \( \mathcal{F}_X \), and then they are invariant by \( \mathcal{F}_X \). Thus every leaf of \( \mathcal{F}_X \) is either disjoint from \( P^{-1}(Q) \) or else is contained in it.

#### 3.1. \( P \) of type \( \mathbb{C} \).

If \( \mathcal{F}_X \) is \( P \)-complete with \( P \) of type \( \mathbb{C} \) it can be determined explicitly. According to Abhyankar-Moh and Suzuki Theorem (11 and 12), up to a polynomial automorphism, we assume that \( P = x \). It is pointed out in [2, pp.1230] (see also [3] Lemma 2.6]) that a foliation \( \mathcal{F}_X \) on \( \mathbb{C}^2 \) which is \( x \)-complete is generated by a vector field of the form:

\[
a(x) \frac{\partial}{\partial x} + [b(x)y + c(x)] \frac{\partial}{\partial y}, \ a, b, c \in \mathbb{C}[x].
\]

As \( C_z \) is covered by \( \mathbb{C} \) the projection of the universal covering map by \( P \) defines a map from \( \mathbb{C} \) to \( a(x) \neq 0 \), and according Picard Theorem we may assume \( a(x) = \lambda x^N \) with \( \lambda \in \mathbb{C}^* \). Remark that \( C_z \not\subseteq \{x = 0\} \) since \( C_z \) is not algebraic. In fact as \( C_z \) is of type \( C^* \) it holds \( N > 0 \).

**Lemma 1.** If \( L \) is the leaf of \( \mathcal{F}_X \) defined by \( C_z \), the leaves of \( \mathcal{F}_X \) different from the one contained in \( \{x = 0\} \) are defined by the sets \( f_\alpha(L) \), where \( f_\alpha \) are the translations in \( \mathbb{C}^2 \) of the form: \( (x, y) \rightarrow (x + \alpha, y) \), \( \alpha \in \mathbb{C} \).
Proof. Let us divide \( \text{(3)} \) by \( \lambda x^N \). The system obtained can be integrated explicitly as a linear equation: For a fixed \( z = (x, y) \in \mathbb{C}^2 \), from the first equation \( x(t) = t + x \).

By substitution of it in the second equation if \( y = uv \) we get

\[
(uv)' = uv' + u'v = \bar{b}(x(t))uv + \bar{c}(x(t)),
\]

with \( \bar{b}(x) = b(x)/\lambda x^N \) and \( \bar{c}(x) = c(x)/\lambda x^N \). If \( v' = \bar{b}(x(t))v \) then \( v(t) = e^{\int \bar{b}(x(s))ds} \) and \( u'v = \bar{c}(x(t)) \). Hence

\[
u(t) = \mu + \int \bar{c}(x(u)) e^{-\int \bar{b}(x(s))ds} du,
\]

\( \mu \in \mathbb{C} \).

The trajectories of \( X \) different from one contained in \( \{ x = 0 \} \) are the subsets in \( \mathbb{C}^2 \) defined by the images \( \gamma_{(x,y)}(\mathbb{C} \setminus \{ -x \}) \) of the (mulivaluated) parametrizations

\[
\gamma_{(x,y)}(t) = \left( t + x, \left\{ y + \int_0^t \bar{c}(u + x) e^{-\int \bar{b}(s+x)ds} du \right\} e^{\int \bar{b}(s+x)ds} \right).
\]

Let \( L' \) be a leaf of \( \mathcal{F}_X \) such that \( L' \neq L \) and \( L' \not\subset \{ x = 0 \} \). There is at least one (in fact there are lots of them) \( z_1 = (x_1, y_1) \in \mathbb{C}_z \) such that \( \{ y = y_1 \} \cap L' \neq \emptyset \). If \( z_2 = (x_2, y_1) \in \{ y = y_1 \} \cap L' \) then \( L' = C_{z_2} = \gamma_{(x_2, y_1)}(\mathbb{C} \setminus \{ -x_2 \}) \). As \( L = \gamma_{(x_1, y_1)}(\mathbb{C} \setminus \{ -x_1 \}) \) since \( C_{z_1} = C_z \) we see that \( L' = f_\alpha(L) \) with \( \alpha = x_1 - x_2 \). \( \square \)

As \( L \) is proper by hypothesis and the maps \( f_\alpha \) are linear automorphisms the leaves of \( \mathcal{F}_X \) different from the one defined by \( \{ x = 0 \} \) are proper and biholomorphic to \( L \), i.e. of type \( \mathbb{C}^* \).

### 3.2. \( P \) of type \( \mathbb{C}^* \)

The situation is completely different to the previous one, since in this case there are many distinct polynomials of type \( \mathbb{C}^* \) after a polynomial automorphism. According to Saito and Suzuki (\cite{10} and \cite{13}), up to a polynomial automorphism, we may assume that \( P = x^m(x^\ell y + p(x))^n \), where \( m, n \in \mathbb{N}^* \) with \( (m, n) = 1, \ell \in \mathbb{N}, p \in \mathbb{C}[x] \) of degree \( \ell \) with \( p(0) \neq 0 \) if \( \ell > 0 \) or \( p(x) \equiv 0 \) if \( \ell = 0 \).

New coordinates. By the relations \( x = a^n \) and \( x^\ell y + p(x) = v u^{-m} \), it is enough to take the rational map \( H \) from \( u \neq 0 \) to \( x \neq 0 \) defined by

\[
(u, v) \mapsto (x, y) = (a^n, u^{-(m+n)}[v - u^m p(u^n)])
\]

in order to get \( P \circ H(u, v) = v^m \).

It follows from the proof of \cite{3} Proposition 3.2 that \( H^* \mathcal{F} \) is a Riccati foliation \( v \)-complete having \( u = 0 \) as invariant line. Still more, according to \cite{5} Lemma 2 at least one of the irreducible components of \( P \) over 0 must be a \( \mathcal{F}_X \)-invariant line. Therefore we may assume that \( \{ x = 0 \} \) is invariant by \( \mathcal{F}_X \). As \( H \) is a finite regular covering map from \( u \neq 0 \) to \( x \neq 0 \), it implies that each component of \( H^{-1}(C_z) \) is of type \( \mathbb{C}^* \) and then covered by \( \mathbb{C} \). Thus according to Picard’s Theorem

\[
H^* X = u^k \cdot Z
\]

\[
= u^k \cdot \left\{ a(v) v \frac{\partial}{\partial u} + cu^r \frac{\partial}{\partial v} \right\},
\]

where \( k \in \mathbb{Z}, a \in \mathbb{C}[v], c \in \mathbb{C}, \) and \( N \in \mathbb{N}^+ \).
The global one form of times. Let us take the one-form $\eta$ obtained when we remove the codimension one zeros and poles of $dP(x, y)$. The contraction of $\eta$ by $X$, $\eta(X)$, is a polynomial, which vanishes only on components of fibres of $P$ since $X$ has only isolated singularities. Then, up to multiplication by constants:

$$\eta(X) = x^\alpha \cdot (x^\beta y + p(x))$$

where $\alpha \in \mathbb{N}^+$ (since $x = 0$) is invariant and $\beta \in \mathbb{N}$. If we define $\tau = [1/\eta(X)] \cdot \eta$, this one-form on $\eta(X) \neq 0$ coincides locally along each trajectory of $X$ with the differential of times given by its complex flow. It is called the global one-form of times for $X$. Moreover $\tau$ can be easily calculated attending to (6) as

$$\tau = \frac{x(x^\beta y + p(x))}{\eta(X)} \cdot \frac{dP}{P}.$$ 

In $(u, v)$ coordinates we then get

$$\varrho = H^* \tau = \frac{u^{m(\beta-1)-n(\alpha-1)}}{v^{\beta-1}} \cdot \frac{dv^n}{v^n}.$$ 

It holds that $\varrho(H^* X) \equiv 1$. Since $\varrho - 1/(u^k \cdot cv^N) \, dv$ contracted by $H^* X$ is identically zero and we can assume that there is no rational first integral, up to multiplication by constants

$$\varrho = 1/(u^k \cdot cv^N) \, dv.$$ 

Therefore, (5) and (9) must be equal and thus $k$ of (5) can be explicitly calculated: $k = n(\alpha - 1) - m(N - 1)$. Finally, let us observe that for any path $\epsilon$ contained in a trajectory of $X$ from $p$ to $q$ that can be lifted by $H$ as $\tilde{\epsilon}$, $\int_\epsilon \varrho$ represents the complex time required by the flow of $X$ to travel from $p$ to $q$.

Existence of a meromorphic first integral. Our aim is to prove that there is an explicit meromorphic first integral for $X$. We will obtain that as a consequence of the following lemmas:

Lemma 2. It holds that $n|k$, $n|[(N - 1)$ if $N > 1$, and $a \in \mathbb{C}[z^n]$. 

Proof. We assume that $\beta = N$ and $\alpha \in \mathbb{N}^+$ in (5). Let us observe that $X$ can be explicitly calculated as

$$X = u^k \cdot H_s(a(v)u \frac{\partial}{\partial u} + cv^N \frac{\partial}{\partial v}) = u^k \cdot DH(u, v) \cdot \left( \frac{a(v)u}{cv^N} \right)$$

where

$$DH(u, v) = \begin{pmatrix} nu^{n-1} & 0 \\ n\ell u^n p(u^n) - u^{n+m}p'(u^n) - (m + n\ell)v & \frac{1}{u^{m+n\ell}} \end{pmatrix}$$

and $u = x^1/n$ and $v = x^{m/n} (x^\beta y + p(x))$.

Remark that $a(0) \neq 0$. Otherwise $X$ had not isolated singularities since $N > 0$. The first component $n_a x^{(k+n)/n} a(x^{m/n} (x^\beta y + p(x)))$ of (10) must be a polynomial.
Proof. The one-form of (11), that we denote by \( X \), has a fraction expansion
\[
a(z) = \frac{a(z)}{cz^N} dz = \left( s(z) + \frac{A_1}{z} + \frac{A_2}{z^2} + \cdots + \frac{A_N}{z^N} \right) dz,
\]
where \( s(z) \in C[z] \), and \( A_i \in C^* \), for \( 1 \leq i \leq N \). Let us fix
\[
\Gamma(z) = e^{s(z)} \cdot e^{\lambda_1 \log z + \frac{\lambda_2}{z} + \cdots + \frac{\lambda_N}{z^N}}
\]
where \( s(z) = \int^z s(t) dt \), and \( \lambda_1 = A_1 \) and \( \lambda_i = A_i/(-i + 1) \) for \( 2 \leq i \leq N \). If we substitute (12) in (11), after explicit integration of \( \omega \), one has that \( \sigma(w, t) \) is of the form
\[
\int w \cdot \Gamma(t)/\Gamma(w_0), t)
\]
Then \( n|k \). On the other hand \( n|(N - 1) \) when \( N > 1 \) since \( k = n(\alpha - 1) - m(N - 1) \) and \( (m, n) = 1 \). It implies that \( a \in C[z^n] \). \( \Box \)

\textbf{Lemma 3.} Let \( v_0 \neq 0 \). The trajectories of \( H^*X \) except the horizontal ones and the line \( \{ u = 0 \} \) are parameterized by maps \( \sigma(w_0, t) \), where \( w_0 \) is a fixed point and \( \sigma \) is a multivalued holomorphic map defined on \( C^* \times C^* \) of the form
\[
\sigma(w, t) = (u(w, t), v(w, t)) = (we^{\int_{v_0}^{w} \frac{a(z)}{cz^N} dz}, t).
\]

Proof. Let us take the local solution through \((u(w_0, v_0), v(w_0, v_0))\), with \( w_0 \in C^* \), of \( 1/c(w) \cdot Z \) extending by analytic continuation along paths in \( C^* \). This map is defined as \( \sigma(w_0, t) \) with \( \sigma \) equals (11) (see [4, Section 2]). \( \Box \)

\textbf{Lemma 4.} \( X \) has a multivalued meromorphic first integral.

Proof. The one-form of (11), that we denote by \( \omega \), has a fraction expansion
\[
a(z) = \frac{a(z)}{cz^N} dz = \left( s(z) + \frac{A_1}{z} + \frac{A_2}{z^2} + \cdots + \frac{A_N}{z^N} \right) dz,
\]
where \( s(z) \in C[z] \), and \( A_i \in C^* \), for \( 1 \leq i \leq N \). Let us fix
\[
\Gamma(z) = e^{s(z)} \cdot e^{\lambda_1 \log z + \frac{\lambda_2}{z} + \cdots + \frac{\lambda_N}{z^N}}
\]
where \( s(z) = \int^z s(t) dt \), and \( \lambda_1 = A_1 \) and \( \lambda_i = A_i/(-i + 1) \) for \( 2 \leq i \leq N \). If we substitute (12) in (11), after explicit integration of \( \omega \), one has that \( \sigma(w, t) \) is of the form
\[
\int w \cdot \Gamma(t)/\Gamma(w_0), t)
\]
is a first integral of \( H^*X \). Finally, we can express (14) in terms of \( x \) and \( y \) by (4),
\[
G(x, y) = \frac{x^{1/n}}{\Gamma(x^{m/n}) \cdot (x^ny + p(x))},
\]
and thus obtain a (multivalued meromorphic) first integral of \( X \). \( \Box \)

\textbf{Lemma 5.} \( N = 1 \), \( \lambda_1 = p/q \in Q \) and \( s \in C[z^n] \)

Proof. When \( N > 1 \) the function \( \Gamma(v) \) has an essential singularity at \( v = 0 \) (for definition of essential singularity of a multivalued map see [7, p. 7]). On the other hand, (12) and (13) imply that \( \Gamma(v) \) is solution of the differential equation
\[
\frac{w'}{w} = \frac{v^N s(v) + v^{N-1} A_1 + \cdots + A_N}{v'}
\]
This differential equation is of the form
\[
v^{N} w' = \frac{R(v, w)}{S(v, w)}
\]
with \( R(v, w) = w(v^N s(v) + v^{N-1} A_1 + \cdots + A_N) \) and \( S(v, w) \equiv 1 \) verifying: a) \( R(v, w) \) is a polynomial in \( w \) whose coefficients are holomorphic around \( v = 0 \), b) \( R(0, w) \) and \( S(0, w) \) are not identically zero, and c) \( R(v, w) \) and \( S(v, w) \) have no common roots when \( v = 0 \). From [7, Théorème 1, p. 99] then \( \Gamma(v) \) verifies the \textit{Picard's Property}: \( \Gamma(v) \) takes in any punctured disk centered at \( v = 0 \) all the values in \( C \) except the zero, which corresponds with the unique \textit{principle characteristic
value of (15) [7, p. 34] given by the solutions of \( R(0, w) = 0 \). Therefore each level of (14), and then each component of \( H^{-1}(C_z) \), accumulates \( v = 0 \). It implies that \( C_z \) accumulates \( x^\ell y + p(x) = 0 \) by the equations of \( H(4) \) what is impossible due to properness of \( C_z \). Hence \( N = 1 \).

Let us show that \( \lambda_1 \in \mathbb{Q} \). From (12) as \( \omega \) has a pole of order one at \( v = 0 \) we can assume that it is \( \lambda_1/z \, dz \) after a biholomorphism in a neighborhood of \( v = 0 \) fixing it [9]. This way we may suppose that \( F(u, v) = u/v^{\lambda_1} \).

• If \( \lambda_1 \in \mathbb{R} \setminus \mathbb{Q} \) each component of \( H^{-1}(C_z) \) is contained in a real subvariety of dimension three [8, p. 120]. Hence \( C_z \) is not proper projecting by \( H \).

• If \( \lambda_1 \in \mathbb{C} \setminus \mathbb{R} \) each component of \( H^{-1}(C_z) \) must accumulate \( \{u = 0\} \) and \( \{v = 0\} \) [8, p. 120]. In particular \( C_z \) accumulates \( x^\ell y + p(x) = 0 \) by the equations of \( H(4) \) what again gives us a contradiction with properness of \( C_z \).

Finally, \( z s(z) = a(z) - a(0) \) implies \( s \in \mathbb{C}[z^n] \) since \( a \in \mathbb{C}[z^n] \) by Lemma 2. □

As a consequence of the above lemmas taking \( \lambda_1 = p/q \) we obtain that

\[
G^{nq} = \frac{\bar{s}^{nq}}{e^{nq \bar{s}(x^\ell y + p(x))^n} [x^m(x^\ell y + p(x))^n]^p}
\]

with \( x^m(x^\ell y + p(x))^n \) as in (11) is a meromorphic first integral of type \( C^* \) for \( X \) up to a polynomial automorphism. Therefore all the trajectories of \( X \) are proper, and except at most the one contained in \( x = 0 \) all of them are of type \( C^* \).

**Remark 3.** According to §3.2 any polynomial vector field \( X \) with a transcendental trajectory of type \( C^* \) defining a foliation \( P \)–complete with \( P \) of type \( C^* \) must be proportional to a complete vector field. It is enough to take in (10) \( k = 0 \) to obtain complete vector fields in the cases (i.2) and (i.3) of [3] Theorem 1.1.

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