AN ISOPERIMETRIC CONSTANT ASSOCIATED TO HORIZONS IN $S^3$ BLOWN-UP AT TWO POINTS.

MATTIAS DAHL AND EMMANUEL HUMBERT

Abstract. Let $g$ be a metric on $S^3$ with positive Yamabe constant. When blowing up $g$ at two points, a scalar flat manifold with two asymptotically flat ends is produced and this manifold will have compact minimal surfaces. We introduce the $\Theta$-invariant for $g$ which is an isoperimetric constant for the cylindrical domain inside the outermost minimal surface of the blown-up metric. Further we find relations between $\Theta$ and the Yamabe constant and the existence of horizons in the blown-up metric on $\mathbb{R}^3$.

Contents

1. Introduction 1
2. Preliminaries 2
2.1. Asymptotically flat 3-manifolds 2
2.2. Inverse mean curvature flow 3
2.3. The Yamabe operator and the Green’s function 4
3. The $\Theta$-invariant, definition and basic properties 6
4. An upper bound for $\Theta$ 7
5. Metrics with large $\Theta$-invariant 11
Appendix A. Evaluation of integrals 15
Appendix B. Mean curvature computations 16
References 17

1. Introduction

Let $(N, h)$ be a 3-manifold with an asymptotically flat end. An outermost minimal surface is a compact minimal surface which encloses all other compact minimal surfaces. As long as $N$ is not diffeomorphic to $\mathbb{R}^3$, a result due to Meeks, Simon, and Yau [9] guarantees the existence of a compact minimal surface. Using the asymptotic flatness one then finds an outermost minimal surface.

Let $(M, g)$ be a compact Riemannian 3-manifold with positive Yamabe constant and fix $p \in M$. We denote by $G_p$ the Green’s function at $p$ for the Yamabe operator. The manifold $(M \setminus \{p\}, G_p^4 g)$ is asymptotically flat and scalar flat. If $M$ is not diffeomorphic to $S^3$ then $M^3 \setminus \{p\}$ is not diffeomorphic to $\mathbb{R}^3$ and hence the result mentioned above gives the existence of an outermost minimal surface.

Date: November 2, 2009.
2000 Mathematics Subject Classification. 53A30, 53C20 (Primary) 58J50 .
Key words and phrases. Asymptotically flat manifolds, inverse mean curvature flow, Yamabe invariant.
in \((M \setminus \{p\}, G_p^4 g)\). If \(M = S^3\) the existence of an outermost minimal surface in 
\((S^3 \setminus \{p\}, G_p^4 g)\) depends on \(g\). For instance, if \(g\) is the standard round metric of \(S^3\) 
then the corresponding asymptotically flat metric is \(\mathbb{R}^3\) equipped with its standard 
Euclidean metric and hence does not possess any compact minimal surface. On 
the other hand, if \(g\) is close enough to a scalar flat metric, then the corresponding 
asymptotically flat metric will have an (outermost) minimal surface, see \([3]\), \([13]\), 
and Section 5. To characterize the metrics \(g\) on \(S^3\) for which 
\((M \setminus \{p\}, G_p^4 g)\) have 
a minimal surface is an open problem.

One the contrary, if \(g\) est a metric on \(S^3\) blown-up at two points, then it always 
contains an horizon. In other words, if \(g\) is a metric on \(S^3\), and if \(p, q \in S^3\) are 
distinct points of \(S^3\) then 
\((S^3 \setminus \{p, q\}, (G_p + G_q)^4 g)\) is asymptotically flat and scalar 
flat but possesses an outermost minimal surface since \(S^3 \setminus \{p, q\}\) is not diffeomorphic 
to \(\mathbb{R}^3\). The existence of this outermost minimal surface allows us to apply powerful 
tools such as the weak inverse mean curvature flow developed by Huisken and 
Ilmanen \([7]\).

In this paper, we define the invariant \(\Theta\) by

\[
\Theta_p^g(q) := \frac{|\Omega|}{|\Sigma|^{3/2}},
\]

where \(\Sigma\) is the only outermost minimal surface in \((S^3 \setminus \{p, q\}, (G_p + G_q)^4 g)\) bounding 
a cylindrical domain \(\Omega\) diffeomorphic to \(S^2 \times (a, b)\) for some \(a, b \in \mathbb{R}, a \leq b\). Here 
the volume of \(\Omega\) and the area of \(\Sigma\) are computed in the metric \((G_p + G_q)^4 g\). We 
show that \(\Theta_p^g\) has several interesting properties, in particular it is related to the 
Yamabe constant of \(g\).

Beside these interesting properties, the motivation for studying such an isoperi-
metric quotient comes from the following observation: the metric \((G_p + G_q)^4 g\) 
tends to \(16G_p^4 g\) in all \(C^k\) on all compact sets \(K \subset S^3 \setminus \{p\}\) when \(q\) tends to \(p\). It 
then seems natural to study such metrics blown up in two points to get information 
on the metrics blown up in one point. We expect to get results of this kind by 
studying the behavior of \(\Theta_p^g(q)\) as \(q\) tends to \(p\).

## 2. Preliminaries

In this section we recall some well-known facts about asymptotically flat 3-
manifolds, the inverse mean curvature flow, and the Yamabe operator. We begin 
by establishing some notational conventions.

The standard euclidean metric on \(\mathbb{R}^3\) is denoted by \(\xi\) and the round metric on 
\(S^3\) of constant sectional curvature 1 is denoted by \(\sigma\). For a Riemannian manifold 
\((M, g)\) with a point \(p \in M\) we denote by \(B_p^g(\delta)\) the open ball of all points of distance 
less than \(\delta\) to \(p\). The gradient of a function \(u\) is denoted by \(\nabla u\) or \(\nabla g\), since it 
usually only appears in norm \(|\nabla u|_g\) there is no risk of confusion when omitting the 
Riemannian metric from the notation. For an open subset \(\Omega\) in the Riemannian 
3-manifold \((M, g)\) we denote the volume by \(|\Omega|_g\) and for a surface \(\Sigma\) in \(M\) we denote 
the area by \(|\Sigma|_g\).

### 2.1. Asymptotically flat 3-manifolds.

**Definition 2.1.** Let \((M, g)\) be a Riemannian 3-manifold.

- An *asymptotically flat end* of \((M, g)\) is an open set \(E\) of \(M\) diffeomorphic 
to the complement of a compact set in \(\mathbb{R}^3\). In the coordinates given by
this diffeomorphism the metric $g$ is required to satisfy
\[
|g_{ij} - \xi_{ij}| \leq \frac{C}{|x|}, \quad |\partial_k g_{ij}| \leq \frac{C}{|x|^2}, \quad \text{Ric}^g \geq -\frac{C}{|x|^2} g,
\]
for large $|x|$.

\[\star\] The Riemannian manifold $(M, g)$ is said to be asymptotically flat if $(M, g)$ with a compact set removed is a union of asymptotically flat ends.

The simplest example of an asymptotically flat manifold is $(\mathbb{R}^3, \xi)$ which has one end. Another example which plays a central role in many problems is the spatial Schwarzschild manifold defined by
\[
(S, g_S) := \left(\mathbb{R}^3 \setminus \{0\}, \left(1 + \frac{m^2}{2|x|}\right)^4 \xi\right).
\]
(1)

This is an asymptotically flat manifold with two ends. Note that it possesses an involutive isometry fixing the sphere of radius $m/2$ (with respect to $\xi$) centered at the origin. In the Schwarzschild metric this sphere has area $16\pi m^2$.

Let $E$ be an asymptotically flat end of $(M, g)$. Then, Arnowitt, Deser, and Misner [1] introduced the ADM mass given by
\[
m^g(E) := \lim_{r \to \infty} \frac{1}{16\pi} \int_{S_r} \left(\partial_j g_{ii} - \partial_i g_{ij}\right) \nu^j \, da^\xi,
\]
where $S_r$ is the sphere centered at the origin and of radius $r$ in $\mathbb{R}^3$ and where $da^\xi$ is the area element induced by $\xi$ on $S_r$. This quantity does not depend on the coordinates and is finite when
\[
\int_E |\text{Scal}^g| \, dv^g < \infty.
\]
(2)

See for instance Bartnik [2] for further discussion. A fundamental result concerning the mass is the Positive Mass Theorem.

**Theorem 2.2.** Let $(M, g)$ be an asymptotically flat 3-manifold whose scalar curvature is non-negative and satisfies (2). Then
\[
m^g(E) \geq 0
\]
for each end $E$ with equality if and only if $(M, g)$ is isometric to $(\mathbb{R}^3, \xi)$.

This theorem was first proved by Schoen and Yau [11], notable among the other proofs available is the one of Witten [12] which uses spin geometry.

A compact minimal surface in an asymptotically manifold is called a horizon. A minimal surface is called outermost [4] if it is not contained entirely inside another minimal surface.

2.2. Inverse mean curvature flow. In their proof of the Penrose inequality [7], Huisken and Ilmanen introduced the "weak inverse mean curvature flow". The standard inverse mean curvature flow may develop singularities and is therefore difficult to use. On the contrary, the weak inverse mean curvature flow gives a flow for "almost all $t$" and provides a powerful technique in many situations. As an example, Bray and Neves [5] used this tool to show that the Yamabe constant of $\mathbb{R}P^3$ is attained by the constant curvature metric. In Section 4 we will use the method of Bray and Neves to prove Theorem 4.1. We recall some basic facts about
the weak inverse mean curvature flow. First, if $\Sigma$ is a $C^1$ surface of a Riemannian 3-manifold $(N, h)$, we say that $H \in L^1_{loc}(\Sigma)$ is the weak mean curvature of $\Sigma$ if
\[
\int_\Sigma \text{div}^h(X) \, da^h = \int_\Sigma H h(X, \nu) \, da^h
\]
for all compactly supported vector fields $X$, where $\nu$ is the outer normal vector field on $\Sigma$. This definition coincides with the usual one as soon as $\Sigma$ is smooth.

**Definition 2.3.** Let $\Sigma$ be a compact $C^1$ hypersurface $\Sigma$ with weak mean curvature $H$ in $L^2(\Sigma)$. The Hawking mass of $\Sigma$ is defined by
\[
m_H(\Sigma) := \sqrt{\frac{|\Sigma|^h}{(16\pi)^3}} \left(16\pi - \int_\Sigma H^2 \, da^h\right).
\]
Here $|\Sigma|^h$ is the area of $\Sigma$ computed using the metric $h$.

We collect the main properties of the inverse mean curvature flow as in [5, Theorem 5.2].

**Theorem 2.4.** [7] Let $(N, h)$ be an asymptotically flat 3-manifold with non-negative scalar curvature. We assume that $N$ is diffeomorphic to $\mathbb{R}^3 \setminus B$ where $B$ is the unit ball in $\mathbb{R}^3$ and that $\partial N = S^2$ is an outermost horizon. Then, there exists a precompact locally Lipschitz function $\Phi$ satisfying
\begin{itemize}
  \item for all $t \geq 0$, $\Sigma_t := \partial\{\Phi < t\}$ defines an increasing family of $C^{1,\alpha}$ surfaces such that $\Sigma_0 = \Sigma$;
  \item for almost all $t \geq 0$, the weak mean curvature of $\Sigma_t$ is $|\nabla \Phi|^h$;
  \item for almost all $t \geq 0$, $|\nabla \Phi|^h \neq 0$ on $\Sigma_t$ for almost all $x \in \Sigma_t$ (with respect to the surface measure) and
  \[
  |\Sigma_t|^h = |\Sigma_0|^h e^t
  \]
  for all $f \geq 0$;
  \item The Hawking mass $m_H(\Sigma_t)$ is a non-decreasing function of $t \geq 0$ provided the Euler characteristic $\chi(\Sigma_t) \leq 2$ for all $t \geq 0$.
\end{itemize}

2.3. **The Yamabe operator and the Green’s function.** Let $g$ be a Riemannian metric on the 3-sphere $S^3$. We set
\[
L^g := 8\Delta^g + \text{Scal}^g.
\]
This self-adjoint elliptic operator is called the Yamabe operator and is conformally invariant in the following sense. If $h = u^4 g$ where $u$ is a smooth positive function is a metric conformal to $g$ then the Yamabe operators of $g$ and $h$ are related by
\[
L^h f = u^{-5} L^g (uf)
\]
and the scalar curvature of $h$ is given by
\[
\text{Scal}^h = u^{-5} \text{Scal}^g.
\]

The Yamabe constant of the metric $g$ is defined by
\[
\mu(g) := \inf_{u \in C^\infty(S^3) : u \neq 0} \frac{\int_{S^3} u L^g u \, dv^g}{\left(\int_{S^3} u^6 \, dv^g\right)^{1/3}} = \inf_{u \in C^\infty(S^3) : u \neq 0} \frac{\int_{S^3} (8|\nabla u|^2 + \text{Scal}^g u^2) \, dv^g}{\left(\int_{S^3} u^6 \, dv^g\right)^{1/3}}.
\]

The number $\mu(g)$ is conformally invariant and it is known that $\mu(g) > 0$ (resp. $\mu(g) = 0$, resp. $\mu(g) < 0$) if and only if there exists a metric in the conformal class of $g$ with positive (resp. identically zero, resp. negative) scalar curvature.
Assume from now on that the metric $g$ has a positive Yamabe constant. Then $L^g$ is invertible and if $p \in S^3$ is a fixed point, this allows to construct the unique Green’s function $G_p$ for $L^g$ at $p$, see [5] Lemma 6.1. We recall that $G_p$ is smooth on $S^3 \setminus \{p\}$, satisfies
\[ L^g G_p = \delta_p \] (4)
in the sense of distributions, and has the expansion
\[ G_p = \frac{1}{d^g(p, \cdot)} + \alpha_p \] (5)
at $p$, where $\alpha_p$ is a smooth function defined in a neighborhood of $p$. Set $S_p := S^3 \setminus \{p\}$ and $g_p := G^4 g$. Then the Riemannian manifold $(S_p, g_p)$ is asymptotically flat with one end $E_p = S_p$, by [3] and [4] it is scalar flat, and one can deduce from [4] that its mass is given by [5] Lemma 9.7
\[ m^{g_p}(E_p) = \alpha_p(p). \]

As an example, if $g$ is the round metric on $S^3$, one easily checks that $(S_p, g_p)$ is isometric to $\mathbb{R}^3$ equipped with its standard Euclidean metric.

A question which will interest us here is the existence of horizons for the metric $g_p$, that is of compact minimal surfaces in the Riemannian manifold $(S_p, g_p)$. First note that the same construction on any 3-manifold not diffeomorphic to $S^3$ always gives rise to a horizon. Note also that, as an application of the techniques from [5], Miao finds that a necessary condition for the existence of a horizon is that $\mu(g) \leq \mu(\sigma)^{2/3}$ where $\sigma$ is the round metric on $S^3$, see [10].

Now, let us define $g_{p,q} := (G_p + G_q)^2 g$ and $S_{p,q} := S^3 \setminus \{p, q\}$ for $p, q \in S^3$. Then $(S_{p,q}, g_{p,q})$ is also asymptotically flat and scalar flat but has two ends $E_p$ and $E_q$, whereas $(S_p, g_p)$ has only one end. As an example, if $g = \sigma$ is the round metric on $S^3$ and if $q = -p$, then $(S_{p,q}, g_{p,q})$ is isometric to the Schwarzschild manifold $(\mathbb{R}^3 \setminus \{0\}, (1 + |x|^{-1})^4 \xi)$. Since
\[ G_p + G_q = \frac{1}{d^g(p, \cdot)} + \alpha_p + G_q \]
near $p$ and since
\[ G_p + G_q = \frac{1}{d^g(q, \cdot)} + \alpha_q + G_p \]
near $q$, one checks that the masses $m^{g_{p,q}}(E_p)$ and $m^{g_{p,q}}(E_q)$ of the manifold $(S_{p,q}, g_{p,q})$ at the ends $E_p$ and $E_q$ are given by [5] Lemma 9.7
\[ m^{g_{p,q}}(E_p) = \alpha_p(p) + G_q(p) \quad \text{and} \quad m^{g_{p,q}}(E_q) = \alpha_q(q) + G_p(q). \] (6)

Another important difference compared to the case of one blow-up point is that $(S_{p,q}, g_{p,q})$ has always an horizon. We deal with the outermost horizon. More precisely, there exists a compact minimal surface $\Sigma_{p,q}$ (not necessarily connected) in $(S_{p,q}, g_{p,q})$ bounding a bounded domain $\Omega_{p,q}$ (maybe empty) such that any other compact horizon lies inside $\Omega_{p,q}$. One sees that the connected components of $\Sigma_{p,q}$ are diffeomorphic to $S^2$ (see Lemma 4.1 in [7]). At least one and at most two of them divide $S_{p,q}$ in two non-compact parts. Let $\Sigma_{p,q}$ be this (or these) dividing sphere.

If the surface $\Sigma_{p,q}$ has two connected components then it bounds a domain $\Omega_{p,q}$ diffeomorphic to $S^2 \times (a, b)$ (with $a < b$). If $\Sigma_{p,q}$ has only one connected component, then $\Omega_{p,q}$ is empty, and can be viewed as a limit case: $\Omega_{p,q}$ is diffeomorphic to
Figure 1. $\tilde{\Sigma}_{p,q} = \Sigma_{p,q} \cup \Sigma_1 \cup \Sigma_2$, $\tilde{\Omega}_{p,q} = \Omega_{p,q} \cup \Omega_1 \cup \Omega_2$

$S^2 \times (a, b)$ with $a = b$ so that by extension, we say that $\partial \Omega_{p,q} = \Sigma_{p,q}$ even if $\Omega_{p,q}$ is empty. Finally, one can see that

$$\tilde{\Sigma}_{p,q} = \Sigma_{p,q} \cup \bigcup_{i=1}^{p} \Sigma_i$$

where $p$ can be zero and where for all $i$, $\Sigma_i$ is a 2-sphere bounding a 3-ball $\Omega_i$. We get that

$$\tilde{\Omega}_{p,q} = \Omega_{p,q} \cup \bigcup_{i=1}^{p} \Omega_i,$$

(7)

see Figure 1 for an illustration.

3. The Θ-invariant, definition and basic properties

Let $g$ be a metric on $S^3$ with positive Yamabe constant. We fix a point $p \in S^3$ and define $(S_{p,q}, g_{p,q})$ as in Subsection 2.3. Then, we define the Θ-invariant by

$$\Theta^g_{p}(q) := \frac{|\Omega_{p,q}|_{g_{p,q}}}{|\Sigma_{p,q}|_{g_{p,q}}^{3/2}}$$

for $q \in S^3 \setminus \{p\}$. The goal of this paper is to explore properties of $\Theta^g_{p}$. We start with some basic properties.

**Proposition 3.1.**

(i) The function $\Theta$ is conformally invariant. In other words, if $g$ and $g'$ are conformal then $\Theta^g_{p}(q) = \Theta^{g'}_{p}(q)$ for all $p, q \in S^3$, $p \neq q$.

(ii) The function $\Theta$ is symmetric in $p$ and $q$, that is $\Theta^g_{p}(q) = \Theta^{g}_{q}(p)$ for all $g$, $p, q \in S^3$, $p \neq q$.

(iii) If $\sigma$ stands for the round metric on $S^3$, then $\Theta^{\sigma}_{p} = 0$.

**Proof.** (i) If $g' = u^4 g$ is a metric conformal to $g$ then the Green’s functions $G'_x$ and $G_x$ for $L^g$ and $L^\sigma$ are related by $G'_p = \frac{G_x p}{u(p) u(p)}$ for all $p \in S^3$. As a consequence the metrics $g_{p,q} = (G_p + G_q)^4 g$ and $g'_{p,q} = (G'_p + G'_q)^4 g'$ are proportional, more precisely we have

$$g'_{p,q} = \left(\frac{u(p) + u(q)}{u(p) u(q)}\right)^4 g_{p,q}.$$
Let $\Sigma_{p,q}'$ be the outermost horizon bounding the domain $\Omega_{p,q}'$ in the metric $g_{p,q}'$. Then
$$|\Sigma_{p,q}'|_{g_{p,q}'} = \frac{(u(p) + u(q))^4}{u(p)^4u(q)^4} |\Sigma_{p,q}|_{g_{p,q}}$$
and
$$|\Omega_{p,q}'|_{g_{p,q}'} = \frac{(u(p) + u(q))^6}{u(p)^6u(q)^6} |\Omega_{p,q}|_{g_{p,q}}.$$
Here the notation $| \cdot |_h$ indicates that the area (or volume) is computed using the metric $h$. We get
$$\frac{|\Omega_{p,q}'|_{g_{p,q}'}^\frac{3}{2}}{|\Sigma_{p,q}'|_{g_{p,q}'}^\frac{3}{2}} = \frac{|\Omega_{p,q}|_{g_{p,q}}^\frac{3}{2}}{|\Sigma_{p,q}|_{g_{p,q}}^\frac{3}{2}},$$
and hence $\Theta_{p,q}'(q) = \Theta_{p,q}'(q)$ which proves Property (i).

(ii) Obvious from the definition of $\Theta_{p,q}'$.

(iii) Let $p,q \in S^3$ be fixed. Clearly, there exists a conformal diffeomorphism $\alpha$ of $(S^3, \sigma)$ with $\alpha(p) = p$ and $\alpha(q) = -p$. By Property (i) we can then assume that $q = -p$. From Subsection 2.3 we see that $(S_{p,q}, g_{p,q})$ is isometric to $\mathbb{R}^3 \setminus \{0\}$ equipped with the Schwarzschild metric. In particular, $\Omega_{p,q}$ is empty from which Property (iii) follows.

4. An upper bound for $\Theta$

In this section we prove the following result.

**Theorem 4.1.** For all $p,q \in S^3$, $p \neq q$, and all metrics $g$ such that $\mu(g) > 0$ we have
$$\mu(g) \left( 1 + \frac{4}{\sqrt{\pi}} \Theta_{p,q}'(q) \right)^{1/3} \leq \mu(\sigma)$$
where $\sigma$ is the standard round metric on $S^3$.

**Corollary 4.2.** The function $\Theta_{p,q}'$ is bounded on $S^3 \setminus \{p\}$.

Note that Theorem 4.1 provides an alternative proof of Property (iii) in Proposition 3.1.

In our mind, the main interest of this result, as well as Theorem 5.1 in next Section, is to exhibit how the $\Theta$-invariant is closely related with the Yamabe constant. Its proof relies on convexity inequalities combined with Bray and Neves techniques [5] using the weak inverse mean curvature flow. The trick here is to apply these techniques on $(S_{p,q} \setminus \Omega_{p,q}, g_{p,q})$ which consists in two connected components, each of them being an asymptotically flat manifold.

We begin with a technical lemma.

**Lemma 4.3.** Let $(N, h)$ be an asymptotically flat manifold whose boundary is the outermost compact minimal surface. Let $(S, g_S)$ be one half of the spatial Schwarzschild manifold with $m = 2$, see [7], whose boundary is the minimal sphere $\{|x| = 1\}$. Let also $\Phi$ and $\Phi^S$ be the functions given by Theorem 2.4 and associated to the weak inverse mean curvature flow on $(N, h)$ and $(S, g_S)$.

Finally, denote by $\Sigma_t := \{\Phi = t\}$ and $\Sigma_t^S := \{\Phi^S = t\}$ the level sets of $\Phi$ and $\Phi^S$. Then,
$$\int_{\Sigma_t} |\nabla \Phi|_h \, da_h \leq \sqrt{\frac{\Sigma_0||h}{\Sigma_0|g_S|} \int_{\Sigma_t^S} |\nabla \Phi^S|_{g_S} \, da_{g_S}}$$

(8)
and

\[
\int_{\Sigma_t} \frac{1}{|\nabla \Phi|^h} \, da^h \geq \left( \frac{\left| \Sigma_0 \right|_h}{|\Sigma_0|_{g_S}} \right)^{3/2} \int_{\Sigma_t} \frac{1}{|\nabla \Phi|^h} \, da^{g_S}
\]

for almost all \( t \). Here if \( \Sigma \) is a compact surface, \( |\Sigma|_h \) denotes its area in the metric \( h \).

The proof of this lemma is entirely contained in the work of Bray and Neves [5], but not stated in this way. So, we recall the proof here. The integrals above are not obviously convergent since \( \nabla \Phi \) can have zeros, the existence of the integrals is carefully justified in [5] and we do not recall all details of these arguments.

**Proof.** Let

\[
m_H(\Sigma_t) = \sqrt{\frac{|\Sigma_t|_h}{16\pi}} \left( 16\pi - \int_{\Sigma_t} |\nabla \Phi|^h \, da^h \right)
\]

be the Hawking mass of \( \Sigma_t \). By Theorem 2.4 we have

\[
m_H(\Sigma_t) \geq m_H(\Sigma_0) = \sqrt{\frac{|\Sigma_0|_h}{16\pi}}.
\]

This gives

\[
\int_{\Sigma_t} |\nabla \Phi|^h \, da^h \leq 16\pi \left( 1 - e^{-t/2} \right)
\]

since \( |\Sigma_t|_h = |\Sigma_0|_h e^t \). By the Cauchy-Schwarz inequality,

\[
\int_{\Sigma_t} |\nabla \Phi|^h \, da^h \leq \sqrt{|\Sigma_t|_h} \left( \int_{\Sigma_t} |\nabla \Phi|^h \, da^h \right)^{1/2}
\]

\[
\leq \sqrt{|\Sigma_0|_h} e^{t/2} \sqrt{16\pi(1 - e^{-t/2})}
\]

and finally

\[
\int_{\Sigma_t} |\nabla \Phi|^h \, da^h \leq \sqrt{16\pi|\Sigma_0|_h(e^t - e^{t/2})}.
\]

Observe that the Hawking mass is constant for the inverse mean curvature flow on \((S, g_S)\) and also that \( |\nabla \Phi|_{g_S} \) is constant on the corresponding \( \Sigma_t^h \). So the above reasoning is still valid on \((S, g_S)\) but all inequalities become equalities. In other words we have

\[
\int_{\Sigma_t^h} |\nabla \Phi|_{g_S} \, da^{g_S} = \sqrt{16\pi|\Sigma_0|_{g_S}(e^t - e^{t/2})}.
\]

Together with Inequality (10) we get Inequality (9).

The Hölder inequality tells us that

\[
\int_{\Sigma_t} \frac{1}{|\nabla \Phi|^h} \, da^h \geq \frac{|\Sigma_t|^2}{\int_{\Sigma_t} |\nabla \Phi|^h \, da^h},
\]

and

\[
\int_{\Sigma_t^h} \frac{1}{|\nabla \Phi|_{g_S}} \, da^{g_S} = \frac{|\Sigma_t^h|^2_{g_S}}{\int_{\Sigma_t^h} |\nabla \Phi|_{g_S} \, da^{g_S}}
\]

since \( |\nabla \Phi|_{g_S} \) is constant on \( \Sigma_t^h \). Using this together with the observation

\[
\frac{|\Sigma_t|_h}{|\Sigma_t^h|_h} = \frac{|\Sigma_0|_{g_S}}{|\Sigma_0^h|_{g_S}}
\]

we conclude that (9) holds. \( \square \)
Lipschitz. This implies derivatives are also in $L^C$. Let $\eta$ be a function such that $\int_{\Sigma_1} (8|\nabla \eta w|^2 + \text{Scal}^g(\eta w)^2)\,dv^g = \frac{\int_{\Sigma_3} (8|\nabla w|^2 + \text{Scal}^g w^2)\,dv^g}{(\int_{\Sigma_3} w^6\,dv^g)^{1/3}}$.

By the definition of $\mu(g)$, we then get that
$$\mu(g) = \inf \frac{\int_{\Sigma_3} (8|\nabla w|^2 + \text{Scal}^g w^2)\,dv^g}{(\int_{\Sigma_3} w^6\,dv^g)^{1/3}}$$
where the infimum is taken over all smooth non-zero functions $w$ which are identically zero in a neighborhood of $p$ and $q$. Since $\mu$ is conformally invariant and since $C^\infty$ is dense in $C^8$, we have
$$\mu(g) = \inf \frac{8\int_{S_{p,q}} |\nabla u|^2_{g_{p,q}}\,dv^{g_{p,q}}}{\left(\int_{S_{p,q}} u^6\,dv^{g_{p,q}}\right)^{1/3}}$$
where the infimum is taken over all non-zero functions locally Lipschitz functions $u \in H^2(S_{p,q}, g_{p,q})$. Here $H^2(S_{p,q}, g_{p,q})$ denotes the set of functions in $L^2$ whose derivatives are also in $L^2$. It follows from [5] that $u \in H^2(S_{p,q}, g_{p,q})$ is locally Lipschitz. This implies
$$\mu(g) \leq \frac{8\int_{S_{p,q}} |\nabla u|^2_{g_{p,q}}\,dv^{g_{p,q}}}{\left(\int_{S_{p,q}} u^6\,dv^{g_{p,q}}\right)^{1/3}}. \tag{11}$$

We have,
$$\int_{S_{p,q}} |\nabla u|^2_{g_{p,q}}\,dv^{g_{p,q}} = \int_{M_1} f'(\Phi_1)^2|\nabla \Phi_1|^2_{g_{p,q}}\,dv^{g_{p,q}} + \int_{M_2} f'(\Phi_2)^2|\nabla \Phi_2|^2_{g_{p,q}}\,dv^{g_{p,q}}.$$
Let \((S, g_s)\) be one half of the spatial Schwarzschild manifold and let \(\Phi^S\) be the function associated to the weak inverse mean curvature flow on \((S, g_s)\) and whose level sets will be denoted by \(\Sigma^S_t\). We set \(a_0 := |\Sigma^S_0|_{g_s}\) and \(a_i := |\Sigma^S_i|_{g_{p,q}}\) for \(i = 1, 2\).

One can compute that

\[
|\nabla \Phi^S|_{g_s} \equiv \sqrt{\frac{16\pi}{a_0}} \sqrt{e^t - e^{t/2}} \sqrt{e^t}
\]

on \(\Sigma^S_t\). By the coarea formula, Inequality (8), and the fact that \(|\Sigma^S_t|_{g_s} = a_0 e^t\) we have

\[
\int_{M_1} f'(\Phi^1)^2 |\nabla \Phi^1|_{g_{p,q}}^2 \, dv_{g_{p,q}} = \int_0^\infty f'(t)^2 \left( \int_{\Sigma^S_t} \frac{1}{|\nabla \Phi^1|_{g_{p,q}}} \, da_{g_{p,q}} \right) \, dt
\]

\[
\leq \sqrt{\frac{a_1}{a_0}} \int_0^\infty f'(t)^2 \left( \int_{\Sigma^S_t} |\nabla \Phi^S|_{g_s} \, da_{g_s} \right) \, dt
\]

\[
= \sqrt{16\pi} \sqrt{a_1} I
\]

where \(I\) is defined as

\[
I := \int_0^\infty f'(t)^2 \sqrt{e^t - e^{t/2}} \, dt.
\]

Doing the same on \(M_2\) and inserting the value of \(I\) which is computed in Lemma A.1 in Appendix A we obtain

\[
\int_{S_{p,q}} |\nabla u|_{g_{p,q}}^2 \, dv_{g_{p,q}} = \frac{3\pi^{3/2}}{8} \left( \sqrt{a_1} + \sqrt{a_2} \right).
\]

(13)

By the coarea formula, Inequality (9), and Equation (12) we get

\[
\int_{M_1} f(\Phi^1)^6 \, dv_{g_{p,q}} = \int_0^\infty f(t)^6 \left( \int_{\Sigma^1_t} \frac{1}{|\nabla \Phi^1|_{g_{p,q}}} \, da_{g_{p,q}} \right) \, dt
\]

\[
\geq \left( \frac{a_1}{a_0} \right)^{3/2} \int_0^\infty f(t)^6 \left( \int_{\Sigma^S_t} \frac{1}{|\nabla \Phi^S|_{g_s}} \, da_{g_s} \right) \, dt
\]

\[
= a_1^{3/2} \frac{1}{\sqrt{16\pi}} J
\]

where \(J\) is defined as

\[
J := \int_0^\infty \frac{f(t)^6 e^{2t}}{(e^t - e^{t/2})^2} \, dt.
\]

Hence, doing the same on \(M_2\) and using the value of \(J\) from Lemma A.1 we obtain

\[
\int_{S_{p,q}} u^6 \, dv_{g_{p,q}} \geq \frac{\sqrt{\pi}}{8} (a_1^{3/2} + a_2^{3/2}) + |\Omega_{p,q}|_{g_{p,q}}.
\]

(14)

Plugging (13) and (14) in (11), we conclude

\[
\mu(g) \leq 8 \frac{3\pi^{1/2}}{8} \left( \sqrt{a_1} + \sqrt{a_2} \right)
\]

\[
\left( \frac{\sqrt{\pi}}{8} (a_1^{3/2} + a_2^{3/2}) + |\Omega_{p,q}|_{g_{p,q}} \right)^{1/3}.
\]
It follows that
\[ \mu(g) \leq \frac{3\pi^{3/2}A_1}{\left(\frac{\sqrt{\pi}}{8}A_2 + \frac{\Omega_{p,q}|g_{p,q}}{(a_1 + a_2)^{3/2}}\right)^{1/3}} \]  
(15)
where
\[ A_1 := \frac{\sqrt{a_1} + \sqrt{a_2}}{\sqrt{a_1} + a_2} \quad \text{and} \quad A_2 := \frac{a_1^{3/2} + a_2^{3/2}}{(a_1 + a_2)^{3/2}}. \]
Elementary arguments show that
\[ A_1 \leq \sqrt{2} \quad \text{and} \quad A_2 \geq \frac{1}{\sqrt{2}}. \]  
(16)
Note that if \( \Omega_{p,q} \neq \emptyset \), then \( a_1 + a_2 \leq |\Sigma_{p,q}|g_{p,q} \). If \( \Omega_{p,q} = \emptyset \), then the boundaries of \( M_1 \) and \( M_2 \) are exactly \( \Sigma_{p,q} \) which is a connected component of \( \Sigma_{p,q} \) and hence \( a_1 + a_2 = 2|\Sigma_{p,q}|g_{p,q} \). In both cases, \( a_1 + a_2 \leq 2|\Sigma_{p,q}|g_{p,q} \) and
\[ \frac{\Omega_{p,q}|g_{p,q}}{(a_1 + a_2)^{3/2}} \geq \frac{\Theta_p^p(q)}{2^{3/2}} \]  
(17)
Plugging (16) and (17) in (15) we get
\[ \mu(g) \leq \frac{3\pi^{3/2}\sqrt{2}}{\left(\frac{\sqrt{\pi}}{8} + \frac{\Theta_p^p(q)}{2^{3/2}}\right)^{1/3}} = \frac{6(2\pi^2)^{2/3}}{\left(1 + \frac{1}{\sqrt{\pi}}\Theta_p^p(q)\right)^{1/3}}, \]
and Theorem 4.1 follows since \( \mu(\sigma) = 6(2\pi^2)^{2/3} \).

5. Metrics with large \( \Theta \)-invariant

Note that the upper bound of \( \Theta_p^p(q) \) provided by Theorem 4.1 will tend to infinity if the metric \( g \) tends to a metric with vanishing Yamabe constant. In the following theorem we prove that \( \Theta_p^p(q) \) itself tends to infinity in that situation.

**Theorem 5.1.** Let \( g^\infty \) be a Riemannian metric on \( S^3 \) and assume that \( (g^k) \) is a sequence of Riemannian metrics with positive Yamabe constant tending to \( g^\infty \) in all \( C^l, l \in \mathbb{N} \) as \( k \to \infty \). Let \( p, q \in S^3, p \neq q \). Then
\[ \mu(g^\infty) = 0 \iff \lim_{k \to \infty} \Theta_p^p(q) = \infty. \]

This theorem implies in particular that if the metric \( g \) is close enough to a metric of zero Yamabe constant then \( \Theta_p^p \neq 0 \). The proof is inspired by an argument of Beig and O’Murchadha [3]. The main result of [3] is that \( (S_p, g^k_p) := (S^3 \setminus \{p\}, (G_p^k)^4g^k) \) contains a trapped compact minimal surface if \( \mu(g^\infty) = 0 \) and \( k \) is large enough.

**Proof.** First, Theorem 4.1 tells us that
\[ \lim_{k \to \infty} \Theta_p^p(q) = \infty \Rightarrow \mu(g^\infty) = 0. \]
To show the opposite implication we assume that \( \mu(g^\infty) = 0 \). Since \( \mu \) and \( \Theta_p^p \) are conformally invariant, we can further assume that \( \text{Scal}^p \equiv 0 \) and therefore there exists a sequence \( (\epsilon_k) \) tending to 0 such that
\[ \|\text{Scal}^p\|_{L^\infty} \leq \epsilon_k. \]  
(18)
Let \( \eta \in C^\infty([0, \infty)) \) be a cut-off function satisfying \( 0 \leq \eta \leq 1, \eta \equiv 1 \) on \( [0, \delta) \), \( \eta \equiv 0 \) on \( [2\delta, \infty) \), \( \delta \) being a fixed small number. Denote by \( G_p^k \) the Green’s function for
$L^g$ at a point $p \in S^3$. Then, by (5), there exists a function $\alpha^k_p \in C^\infty(S^3)$ such that

$$G^k_p = \frac{\eta(r_k)}{r_k} + \alpha^k_p$$

where we use the notation $r_k = d^g_k(p, \cdot)$. We start by proving the following result.

**Lemma 5.2.** There is a subsequence of $(g^k)$ for which the corresponding functions $\alpha^k_p$ can be decomposed as

$$\alpha^k_p = a^k_p + \beta^k_p$$

where $(a^k_p)$ is a sequence of real numbers tending to $\infty$ and where $\beta^k_p \in C^\infty$ is a smooth function such that

$$\int_{S^3} \beta^k_p dv^g = 0$$

and such that

$$\|\beta^k_p\|_{C^1} = o(a^k_p).$$  \hspace{1cm} (19)

Here the notation in the last claim means that $\|\beta^k_p\|_{C^1}/a^k_p$ tends to zero as $k \to \infty$. In the following proof $C > 0$ stands for a constant which is independent of $k$ but may change from line to line.

**Proof of Lemma 5.2.** First we prove that

$$\limsup_{k \to \infty} \int_{S^3} \alpha^k_p dv^g = \infty. \hspace{1cm} (20)$$

From the definition of $G^k_p$ together with (5) we have

$$1 = \int_{S^3} G^k_p(Lg^k 1) dv^g = \int_{S^3} G^k_p dv^g \leq \|Lg^k 1\|_{L^\infty} \int_{S^3} G^k_p dv^g \leq \|\text{Scal}^g\|_{L^\infty} \left( \int_{B^g_{2\delta}(2\delta)} r^{-1}_k dv^g + \int_{S^3} a^k_p dv^g \right).$$

By Lebesgue’s Theorem

$$\lim_{k \to \infty} \int_{B^g_{2\delta}(2\delta)} r^{-1}_k dv^g = \int_{B^g_{\delta}(2\delta)} r^{-1}_\delta dv^g < \infty$$

where $r_\delta = d^g_\delta(\cdot, \cdot)$. To get a contradiction we assume that (20) does not hold. Then $\int_{S^3} a^k_p dv^g$ is bounded as $k$ goes to $\infty$. We conclude that

$$1 \leq C \|\text{Scal}^g\|_{L^\infty}.$$

By Equation (18), the right-hand side of this inequality tends to zero which is not possible. This proves (20).

Next we set

$$a^k_p := \int_{S^3} a^k_p dv^g / \int_{S^3} dv^g$$

and $\beta^k_p := \alpha^k - a^k_p$ so that

$$\int_{S^3} \beta^k_p dv^g = 0.$$
AN ISOPERIMETRIC CONSTANT ASSOCIATED TO HORIZONS IN S^3 BLOWN-UP AT TWO POINTS

Since by assumption \( g^k \to g^\infty \) in all \( C^l, \ l \in \mathbb{N} \), the Sobolev inequality
\[
\| \beta^k_p \|_{C^1} \leq C \| \beta^k_p \|_{H^2}
\]
holds with a constant \( C \) independent of \( k \). Here, for any \( s > 1 \), \( H^s \) is the space of \( L^s \) functions whose derivatives of first and second order belong to \( L^s \). Let now \( l > 1 \). By standard regularity result, see for example [8, Theorem 2.4],
\[
\| \beta^k_p \|_{H^l} \leq C \left( \| \Delta g^k \beta^k_p \|_{L^l} + \| \beta^k_p \|_{L^l} \right).
\]
Using (18) we write
\[
\| 8 \Delta g^k \beta^k_p \|_{L^l} = \| \Delta g^k \beta^k_p - \text{Scal} g^k \beta^k_p \|_{L^l}
\leq \left( \| \Delta g^k \beta^k_p \|_{L^l} + \epsilon k \| \beta^k_p \|_{L^l} \right).
\]
From the definition of \( \beta^k_p \) together with (18) we get
\[
\| L^k \beta^k_p \|_{L^l} = \left\| L^k G^k_p - L^k \frac{\eta(r_k)}{r_k} - L^k a^k_p \right\|_{L^l}
\leq \| \delta_p - \eta(r_k) \Delta g^k r_k^{-1} \|_{L^l} + \left\| \Delta g^k \frac{\eta(r_k)}{r_k} \right\|_{L^l}
+ 2 \left\| g^k |\nabla g^k \eta(r_k), \nabla g^k r_k^{-1} | \right\|_{L^l} + C \epsilon k \| \beta^k_p \|_{L^l} + C \epsilon k a^k_p.
\]
Since the derivatives of \( \eta(r_k) \) are supported in \( M \setminus B^k_p(\delta) \), the second and third terms of the right hand side in the expression above are bounded by some constant \( C > 0 \) independent of \( k \). One can also compute [8, Section 6]
\[
\| \delta_p - \eta(r_k) \Delta g^k r_k^{-1} \|_{L^\infty} \leq C.
\]
Finally, we obtain
\[
\| \Delta g^k \beta^k_p \|_{L^l} \leq C(1 + \epsilon k \| \beta^k_p \|_{H^l} + \epsilon_k a^k_p).
\]
In particular, we easily deduce from (22) that
\[
\| \beta^k_p \|_{H^l} \leq C \left( (1 + \epsilon_k a^k_p) + \| \beta^k_p \|_{L^l} \right).
\]
Let \( \lambda_k \) denote the first eigenvalue of \( \Delta g^k \). Since \( \int_M \beta^k_p dv^{g^k} = 0 \) the Cauchy-Schwarz inequality tells us that
\[
\int_M (\beta^k_p)^2 dv^{g^k} \leq \frac{1}{\lambda_k} \int_M |\nabla \beta^k_p|^2 dv^{g^k}
\leq \frac{1}{\lambda_k} \int_M \beta^k_p \Delta g^k \beta^k_p dv^{g^k}
\leq \frac{1}{\lambda_k} \left( \int_M (\beta^k_p)^2 dv^{g^k} \right)^{1/2} \left( \int_M (\Delta g^k \beta^k_p)^2 dv^{g^k} \right)^{1/2}.
\]
The sequence \( (\lambda_k) \) has a non-zero limit since the metrics \( (g^k) \) converges, and hence the sequence \( (\lambda_k) \) is bounded. Together with (23) applied with \( l = 2 \), one gets
\[
\| \beta^k_p \|_{L^2} \leq C(1 + \epsilon_k a_k).
\]
Returning to (24), we obtain
\[
\| \beta^k_p \|_{H^2} \leq C(1 + \epsilon_k a_k).
\]
In particular, by the Sobolev embedding theorem, we get that
\[ \|\beta^k_p\|_{L^4} \leq C(1 + \epsilon_k a_k). \]
Setting \( l = 4 \) and inserting this inequality in (24), we get
\[ \|\beta^k_p\|_{H^2_4} \leq C(1 + \epsilon_k a_k). \]
Together with (21) this ends the proof of Lemma 5.2. \( \square \)

Let us return to the proof of Theorem 5.1. We fix points \( p, q \in S^3, p \neq q, \) and to get a contradiction we assume that \( \Theta_{g^k_p}^k(q) \) has a bounded subsequence. Define \( G^k := G^k_p + G^k_q \) so that \( g^k_{p,q} = G^k_p g^k \). For \( r > 0 \) small let \( S^k_p(r) \) be the sphere defined by \( r = r_k \), where again \( r_k = d^k_g(p, \cdot) \). Using the transformation formula for mean curvature under a conformal change of the metric (see for example [6, Equation 1.4]) one can compute that the mean curvature of \( S^k_p(r) \) in the metric \( g^k_{p,q} \) is
\[ H_k = \frac{1}{G^k} \left( 2g^k(\nabla^g r_k, \nabla^g G_k) + \left( \frac{1}{r} + O(r) \right) G_k \right) \]
where the constant involved in the bound of the ordo term is independent of \( k \). Apply Lemma 5.2 to \( G^k_p \) and \( G^k_q \) and take further subsequences to get the corresponding \( a_k^p \) and \( a_k^q \) which tend to infinity. Set \( a_k := a_k^p + a_k^q \), by (19) we have
\[ G_k = r_k^{-1} + a_k + o(a_k) \]
near \( p \). It follows that the mean curvature of \( S^k_p(r) \) satisfies
\[ H_k = \frac{1}{G_k r_k^2} \left( -1 + (a_k + o(a_k))r + r^2 o(a_k) \right), \]
see Appendix B for further details. In particular, the sphere \( S^k_p(2/a_k) \) has positive mean curvature whereas the sphere \( S^k_p(1/2a_k) \) has negative mean curvature when \( k \) is large. By standard existence results, there exists a minimal 2-sphere \( \Sigma^k_p \) lying between these two spheres. Doing the same near \( q \), we get the existence of a minimal 2-sphere \( \Sigma^k_q \). Clearly, \( \Sigma_{p,q} = \Sigma_{p,q}^k = \Sigma^k_p \cup \Sigma^k_q \). We also get
\[ S^3 \setminus \left( D^k_p(2/a_k) \cup B^k_q(2/a_k) \right) \subset \Omega_{p,q}. \]
(25)
Since \( \Sigma^k_p \) is minimal and since \( G_k \leq C a_k \) on \( S^k_p(1/2) \) the area of \( \Sigma^k_p \) satisfies
\[
|\Sigma^k_p|_{g^k_{p,q}} \leq \left| S^k_p(1/2a_k) \right|_{g^k_{p,q}}
\]
\[
= \int_{S^k_p(1/2a_k)} d\sigma^k_{p,q}
\]
\[
= \int_{S^k_p(1/2a_k)} G^k da^k
\]
\[
\leq C a_k^4 \int_{S^k_p(1/2a_k)} d\sigma^k_{p,q}
\]
\[
\leq C a_k^2.
\]
Doing the same for \( \Sigma^k_q \) we get that
\[ |\Sigma_{p,q}|_{g^k_{p,q}} \leq C a_k^2. \] (26)
Using (25), we have
\[
|\Omega_{p,q}|_{g^{k}_{p,q}} \geq \int_{S^3 \setminus (B^k_g(2/a_k) \cup B^k_q(2/a_k))} \sigma^k_{p,q} \, dv^{g_{k}}
\]
\[
\geq \int_{S^3 \setminus (B^k_g(2/a_k) \cup B^k_q(2/a_k))} \sigma^k_{q} \, dv^{g_{k}}.
\]
Estimate (19) implies that
\[
G_{k} \geq C_{a_k} \text{ on } S^3 \setminus (B^k_g(2/a_k) \cup B^k_q(2/a_k)).
\]
This leads to
\[
|\Omega_{p,q}|_{g^{k}_{p,q}} \geq C_{a_k} \int_{S^3 \setminus (B^k_g(2/a_k) \cup B^k_q(2/a_k))} \sigma^k_{p,q} \, dv^{g_{k}} \geq C_{a_k}^6.
\]
Together with (26), we get
\[
\Theta^g_{k}(q) = \frac{|\Omega_{p,q}|_{g^{k}_{p,q}}}{|\Sigma_{p,q}|_{g^{k}_{p,q}}} \geq C_{a_k}^3
\]
and hence \(\Theta^g_{k}(q)\) for the subsequence cannot be bounded. This proves Theorem 5.1. □

Appendix A. Evaluation of integrals

Here we indicate how to evaluate two definite integrals needed in the proof of Theorem 4.1. Compare the discussion in [5], pages 421-422.

Lemma A.1. Let
\[
f(t) := \frac{1}{\sqrt{2e^t - e^{t/2}}},
\]
for \(t \in (0, \infty)\) and set
\[
I := \int_{0}^{\infty} f'(t)^2 \sqrt{e^t - e^{t/2}} \, dt \quad \text{and} \quad J := \int_{0}^{\infty} \frac{f(t) e^{2t}}{(e^t - e^{t/2})} \, dt.
\]
Then
\[
I = \frac{3\pi}{32} \quad \text{and} \quad J = \frac{\pi}{2}.
\]
Proof. Observe that
\[
I = \frac{1}{16} \int_{0}^{\infty} \frac{(4e^{t/2} - 1)^2}{(2e^{t/2} - 1)^3} \sqrt{e^{t/2} - 1} \, dt.
\]
Through the change of variables \(s = \sqrt{\frac{e^{t/2} - 1}{e^{t/2}}},\) that is \(e^{t/2} = \frac{1}{1 - s^2}\) and \(dt = \frac{4ds}{1 - s^2},\) we get
\[
I = \frac{1}{4} \int_{0}^{1} \frac{(3 + s^2)^2 s^2}{(s^2 + 1)^3} \, ds.
\]
Writing
\[
\frac{(3 + s^2)^2 s^2}{(s^2 + 1)^3} = 1 + \frac{3}{s^2 + 1} - \frac{4}{(s^2 + 1)^3},
\]
one gets
\[
I = \frac{1}{4} + \frac{3}{4} \left[ \arctan t \right]_{0}^{\infty} - \left[ \frac{3}{8} \arctan t + \frac{3}{8} \frac{t}{1 + t^2} + \frac{1}{4} \frac{t^2}{1 + t^2} \right]_{1}^{\infty} = \frac{3\pi}{32}.
\]
In the same way, observe that

\[ J = \int_0^\infty \frac{1}{(2e^{t/2} - 1)^2} \sqrt{e^{t/2} - 1} \, dt. \]

Using the change of variables \( s = \sqrt{e^{t/2} - 1} \), that is \( e^{t/2} = \frac{s^2}{s^2 - 1} \) and \( dt = -\frac{4ds}{s(s^2 - 1)} \), we get

\[ J = \int_1^\infty \frac{4(s^2 - 1)^2}{(s^2 + 1)^3} \, ds. \]

From

\[ \frac{4(s^2 - 1)^2}{(s^2 + 1)^3} = \frac{4}{s^2 - 1} - \frac{16}{(s^2 - 1)^2} + \frac{16}{(s^2 - 1)^3}, \]

we have

\[ J = 4 \left[ \arctan \left( \frac{1}{t} \right) \right]_1^\infty - 16 \left[ \frac{1}{2} \arctan t + \frac{1}{2} \frac{t}{1 + t^2} \right]_1^\infty + 16 \left[ \frac{3}{8} \arctan t + \frac{3}{8} \frac{t}{1 + t^2} + \frac{1}{4} \frac{t^2}{1 + t^2} \right]_1^\infty = \frac{\pi}{2}. \]

This proves Lemma A.1. \( \Box \)

Appendix B. Mean curvature computations

In [6, Equation 1.4] we find the conformal transformation formula for mean curvature. If \( \tilde{g} = u^4 g \) then the mean curvatures for \( \tilde{g} \) and \( g \) are related by

\[ \tilde{h} = \frac{2}{u^3} \left( \frac{\partial}{\partial \eta} + \frac{1}{2} \hat{h} \right) u. \]

Here \( \frac{\partial}{\partial \eta} \) is the normal outward derivative with respect to the metric \( g \). If \( r = d^g(p, \cdot) \) then

\[ \frac{\partial}{\partial \eta} u = g(\nabla^g r, \nabla^g u) \]

where \( \nabla^g \) denotes the gradient and we get

\[ \tilde{h} = \frac{1}{u^3} \left( 2g(\nabla^g r, \nabla^g u) + hu \right). \]

If we apply this with our notation \( g_{p,q}^k = G_k^p G^q \) etc, then we conclude

\[ H_k = \frac{1}{G_k^p} \left( 2g^k(\nabla^g r_k, \nabla^g G_k) + H^g G_k \right). \]

In our situation we have that the sphere \( S^k_p(r) \) is close to a round sphere of radius \( r \) in flat \( \mathbb{R}^3 \) for small \( r \), therefore

\[ H^g = \frac{1}{r} + O(r), \]

where the constant involved in the ordo term is independent of \( k \) since \( g_k \) tends to \( g_\infty \). We get

\[ H_k = \frac{1}{G_k^p} \left( 2g^k(\nabla^g r_k, \nabla^g G_k) + \left( \frac{1}{r} + O(r) \right) G_k \right). \]
We now insert the expansion of $G_k$ which we get from (19),

$$G_k = \frac{1}{r_k} + a_k + o(a_k).$$

Then

$$\nabla^{g^k} G_k = -\frac{1}{r^2_k} \nabla^{g^k} r_k + o(a_k)$$

and

$$H_k = \frac{1}{G_k^3} \left( 2g^k(\nabla^{g^k} r_k, \nabla^{g^k} G_k) + \left( \frac{1}{r} + O(r) \right) G_k \right)$$

$$= \frac{1}{G_k^3} \left( 2g^k (\nabla^{g^k} r_k, \frac{1}{r^2} \nabla^{g^k} r_k + o(a_k) + \left( \frac{1}{r} + O(r) \right) \left( \frac{1}{r} + a_k + o(a_k) \right) \right) \right)$$

$$= \frac{1}{G_k^3} \left( -\frac{1}{r^2} + \frac{1}{r} (a_k + o(a_k)) + o(a_k) \right)$$

$$= \frac{1}{G_k^3 r^2} \left( -1 + (a_k + o(a_k))r + r^2 o(a_k) \right),$$

since $g^k(\nabla^{g^k} r_k, \nabla^{g^k} r_k) = 1.$

References

1. R. Arnowitt, S. Deser, and C. W. Misner, Coordinate invariance and energy expressions in general relativity., Phys. Rev. (2) 122 (1961), 997–1006.
2. R. Bartnik, The mass of an asymptotically flat manifold, Comm. Pure Appl. Math. 39 (1986), no. 5, 661–693.
3. R. Beig and N. O' Murchadha, Trapped surfaces due to concentration of gravitational radiation, Phys. Rev. Lett. 66 (1991), no. 19, 2421–2424.
4. H. L. Bray, Proof of the Riemannian Penrose inequality using the positive mass theorem, J. Differential Geom. 59 (2001), no. 2, 177–267.
5. H. L. Bray and A. Neves, Classification of prime 3-manifolds with Yamabe invariant greater than $\mathbb{RP}^3$, Ann. of Math. (2) 159 (2004), no. 1, 407–424.
6. J. F. Escobar, Conformal deformation of a Riemannian metric to a scalar flat metric with constant mean curvature on the boundary, Ann. of Math. (2) 136 (1992), no. 1, 1–50.
7. G. Huisken and T. Ilmanen, The inverse mean curvature flow and the Riemannian Penrose inequality, J. Differential Geom. 59 (2001), no. 3, 353–437.
8. J. M. Lee and T. H. Parker, The Yamabe problem, Bull. Amer. Math. Soc. (N.S.) 17 (1987), no. 1, 37–91.
9. W. Meeks, L. Simon, and S. T. Yau, Embedded minimal surfaces, exotic spheres, and manifolds with positive Ricci curvature, Ann. of Math. (2) 116 (1982), no. 3, 621–659.
10. P. Miao, A note on existence and non-existence of horizons in some asymptotically flat 3-manifolds, Math. Res. Lett. 14 (2007), no. 3, 395–402.
11. R. Schoen and S. T. Yau, On the proof of the positive mass conjecture in general relativity, Comm. Math. Phys. 65 (1979), no. 1, 45–76.
12. E. Witten, A new proof of the positive energy theorem, Comm. Math. Phys. 80 (1981), no. 3, 381–402.
13. Y. Yan, The existence of horizons in an asymptotically flat 3-manifold, Math. Res. Lett. 12 (2005), no. 2-3, 219–230.

E-mail address: doughmath.kth.se

E-mail address: ehumbert@iecn.u-nancy.fr