Extreme eigenvalues of large-dimensional spiked Fisher matrices with application

Qinwen Wang and Jianfeng Yao

Qinwen Wang
Department of Mathematics
Zhejiang University
e-mail: wqw8813@gmail.com

Jianfeng Yao
Department of Statistics and Actuarial Science
The University of Hong Kong
Pokfulam, Hong Kong
e-mail: jeffyao@hku.hk

Abstract: Consider two \( p \)-variate populations, not necessarily Gaussian, with covariance matrices \( \Sigma_1 \) and \( \Sigma_2 \), respectively, and let \( S_1 \) and \( S_2 \) be the sample covariances matrices from samples of the populations with degrees of freedom \( T \) and \( n \), respectively. When the difference \( \Delta \) between \( \Sigma_1 \) and \( \Sigma_2 \) is of small rank compared to \( p, T \) and \( n \), the Fisher matrix \( F = S_2^{-1} S_1 \) is called a spiked Fisher matrix. When \( p, T \) and \( n \) grow to infinity proportionally, we establish a phase transition for the extreme eigenvalues of \( F \): when the eigenvalues of \( \Delta \) (spikes) are above (or under) a critical value, the associated extreme eigenvalues of the Fisher matrix will converge to some point outside the support of the global limit (LSD) of other eigenvalues; otherwise, they will converge to the edge points of the LSD. Furthermore, we derive central limit theorems for these extreme eigenvalues of the spiked Fisher matrix. The limiting distributions are found to be Gaussian if and only if the corresponding population spike eigenvalues in \( \Delta \) are simple. Numerical examples are provided to demonstrate the finite sample performance of the results. In addition to classical applications of a Fisher matrix in high-dimensional data analysis, we propose a new method for the detection of signals allowing an arbitrary covariance structure of the noise. Simulation experiments are conducted to illustrate the performance of this detector.

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1. Introduction

Consider two \( p \)-variate populations with covariance matrices \( \Sigma_1 \) and \( \Sigma_2 \), respectively, and let \( S_1 \) and \( S_2 \) be the sample covariances matrices from samples of the populations with degrees of freedom \( T \) and \( n \), respectively. Specifically, if both populations are Gaussian, \( TS'_1 \) and \( nS_2 \) are distributed as Wishart \( W_P(T, \Sigma_1) \) and \( W_P(n, \Sigma_2) \), respectively. For testing the equality hypothesis \( H_0 : \Sigma_1 = \Sigma_2 \), the likelihood ratio statistic relies on the \( p \) characteristic
roots of the determinental equation

$$|S_1 - lS_2| = 0, \quad l \in \mathbb{R}. \quad (1.1)$$

Here and throughout the paper, the determinant of a matrix $A$ is denoted by either $|A|$ or $\text{det}(A)$. As a famous story in multivariate analysis of last century, the joint distribution of these characteristic roots for Gaussian populations was simultaneously and independently published in 1939 by R. A. Fisher, S. N. Roy, P. L. Hsu and M. A. Girshick. When $S_2$ is invertible, these roots are simply the eigenvalues of the matrix $F = S_2^{-1}S_1$, widely known as a Fisher matrix in the literature, which generalises the one-dimensional Fisher ratio.

Another breakthrough is the work of Wachter (1980) where he finds a deterministic limit, the celebrated Wachter distribution, for the empirical measure of these roots when the dimension $p$ grows to infinity proportionally to the degrees of freedom $T$ and $n$ (under the Gaussian assumption). Wachter’s result has been later extended to non-Gaussian populations in what is now called the random matrix theory and two early examples of such extensions are Silverstein (1985) and Bai et al. (1987). It is also important to notice that the determinental equation (1.1) arises not only in the classical hypothesis testing problem mentioned above, it indeed covers also similar equations arising in important fields of multivariate analysis such as discriminant analysis, canonical correlation analysis and MANOVA, see Wachter (1980).

Needless to say that such limiting results allowing large values of dimension $p$ comparable to the degrees of freedom (i.e. sample sizes) are going to have much impact on today’s high-dimensional data analysis. A particularly important question is to investigate the properties of the characteristic roots under an alternative of form

$$H_1 : \Sigma_1 = \Sigma_2 + \Delta, \quad (1.2)$$

where $\Delta$ is a nonnegative definite matrix of rank $M$. When $p$, $T$ and $n$ are all large, the discrimination between the null hypothesis and the alternative is not difficult if the rank difference $M$ is all large. The real challenge here lies in detecting a small rank-$M$ alternative. In this perspective and assuming $M$ is a fixed integer while $p$, $T$ and $n$ grow to infinity proportionally, the empirical measure of the $p$ characteristic roots of (1.1) will be affected by a difference of order $M/p$ which vanishes, so that its limit remains the same as in the null hypothesis, i.e. the Wachter distribution. In other words, such global limit from all the characteristic roots will be of little help for distinguishing the two hypotheses.

It happens that the useful information to detect a small rank alternative is encoded in a few largest characteristic roots of (1.1). In a recent preprint Dharmawansa et al. (2014), by assuming both population are Gaussian and $M = 1$, these authors show that, when the norm of the rank-1 difference $\Delta$ (spike) exceeds a phase transition threshold, the asymptotic behaviour of the log-ratio of the joint density of these characteristic roots under a local deviation from the spike depends only on the largest characteristic root $l_{p,1}$ and the statistical experiment of observing all the characteristic roots is locally asymptotically normal (LAN). As a by-product of their analysis, the authors also establish joint asymptotic normality of a few of the largest roots when the corresponding spikes in $\Delta$ (with $M > 1$)
exceed the phase transition threshold. As it can be guessed, the analysis given in this reference highly rely on the Gaussian assumption so that the joint density function of the characteristic roots has indeed an explicit form under both the null and the alternative, and the main results are obtained via an accurate analytic approximation of the log-ratio of these density functions when the dimension $p$, $T$ and $n$ grow to infinity proportionally.

Intrigued by these findings, in this paper, we explore the same questions for general populations without Gaussian assumption. It is thus apparent that the joint density of the characteristic roots no more exist and new techniques are needed to solve the questions. Our approach relies on the tools borrowed from the theory of random matrices. This theory is closely connected to modern high-dimensional statistics, and has provided in recent years many efficient estimation and testing procedures for high-dimensional data analysis. Excellent introduction and surveys on this approach can be found in Bai (2005), Johnstone (2007), Johnstone and Titterington (2009) and Paul and Aue (2014). A methodology particularly successful both in theory and applications within this approach relies on the spiked population model coined in Johnstone (2001). This model deals with one population only with a unit population covariance matrix $I_p$ and the hypotheses are simply $H_0 : \Sigma_1 = I_p$ versus $H_1 : \Sigma_1 = I_p + \Delta$ where $\Delta$ is a rank-$M$ difference as in (1.2). Again for small rank $M$, the discrimination between both hypotheses will rely on the extreme eigenvalues of the sample covariance matrix $S_1$. Important results have been obtained in the last decade on the behaviour of these extreme eigenvalues. For example, the fluctuation of largest eigenvalues of a sample covariance matrix from a complex spiked Gaussian population is studied in Baik et al. (2005). These authors uncover a phase transition phenomenon: the weak limit and the scaling of these extreme eigenvalues are different depending on whether the eigenvalues of $\Delta$ (spikes) are above, equal or below a critical value, situations refereed as super-critical, critical and sub-critical, respectively. In Baik and Silverstein (2006), the authors consider the spiked population model with general populations (not necessarily Gaussian). For the almost sure limits of the extreme sample eigenvalues of $S_1$, they find that if a population spike (in $\Delta$) is large or small enough, the corresponding sample spike eigenvalues will converge to a limit outside the support of the limiting spectrum (outliers). In Paul (2007), a CLT is established for these outliers, i.e. the super-critical case, under the Gaussian assumption and assuming that population spikes are simple (multiplicity 1). The CLT for super-critical outliers with general populations and arbitrary multiplicity numbers is developed in Bai and Yao (2008). This theory has been later extended for generalised spiked population model in Bai and Yao (2012).

In summary, from the perspective of spiked population model, the Fisher matrix $F = S_2^{-1}S_1$ under the alternative (1.2) can be viewed as a spiked Fisher matrix and it is important to establish a theory for this two-population Fisher matrix in the vein of the results discussed above on the one-population spiked covariance matrix $S_1$. As said before, in Dharmawansa et al. (2014), the authors have already identified the transition phenomenon for the extreme eigenvalues under the Gaussian assumption, and these eigenvalues are proved to be asymptotic normal assuming that the spike eigenvalues in $\Delta$ are simple. The main contributions of the paper are the following. We prove that this phase transition phenomenon for extreme eigenvalues of a spiked Fisher matrix is universal, valid
for general populations under some suitable moment conditions. Next, we provide a general CLT for the extreme sample eigenvalues of $F$ in the super-critical regime: the limiting distributions are not necessarily Gaussian; they are Gaussian if and only if the population spikes in $\Delta$ are simple.

In addition to the motivations given so far on the importance of a spiked Fisher matrix, we are able to implement an application of the general theory developed in this paper in the context of a signal detection problem with a large number of detectors, see Section 7. Indeed, this problem has its own interests and even with quite limited experiments, we show that our implementation can lead to very reliable solutions.

Finally, within the theory of random matrices, the techniques we use in this paper for spiked models are closely connected to other random matrix ensembles through the concept of small-rank perturbations. The goal is again to examine the effect caused on the extreme sample eigenvalues by such perturbations. Theories on perturbed Wigner matrices can be found in Pêché (2006), Féral and Pêché (2007), Capitaine et al. (2009), Pizzo et al. (2013) and Renfrew and Soshnikov (2013). In a more general setting of finite-rank perturbation including both the additive and the multiplicative one, point-wisely convergence of extreme eigenvalues is established in Benaych-Georges and Nadakuditi (2011) while their fluctuations are studied in Benaych-Georges et al. (2011). In addition, Benaych-Georges and Nadakuditi (2011) contain also results on spiked eigenvectors.

The rest of the paper is organised as follows. First, the exact setting of the spiked Fisher matrix $F = S_2^{-1}S_1$ is introduced in Section 2. Then in Section 3, we establish the phase transition phenomenon for the extreme eigenvalues of $F$ where the transition boundary is explicitly obtained. Next, CLTs for those extreme eigenvalues fluctuating around some outliers (i.e. the super-critical case) are established first in Section 4 for one group of sample eigenvalues corresponding to a same population spike, and then in Section 6 for all the groups jointly. Section 5 contains numerical illustrations that demonstrate the finite sample performance of our results. In Section 7, we develop in details a signal detection technique with prewhitening. Proofs of the main theorems are included in these sections while some technical lemmas are postponed into the Appendix A.

2. Spiked Fisher matrix and preliminary results

In what follows, we will assume that $\Sigma_2 = I_p$. This assumption does not loss any generality since the eigenvalues of the Fisher matrix $F = S_2^{-1}S_1$ are invariant under the transformation $S_1 \mapsto \Sigma_2^{-1/2}S_1\Sigma_2^{-1/2}$, $S_2 \mapsto \Sigma_2^{-1/2}S_2\Sigma_2^{-1/2}$. Also we will write $\Sigma_p$ for $\Sigma_1$ to signify the dependence on the dimension $p$. Therefore, the sample covariance matrices $S_1$ and $S_2$ that make up the Fisher matrix $F = S_2^{-1}S_1$ are assumed to have the following structure. Let

$$Z = (z_1, \ldots, z_n) = (z_{ij})_{1 \leq i \leq p, 1 \leq j \leq n} \quad (2.1)$$

and

$$W = (w_1, \ldots, w_T) = (w_{kl})_{1 \leq k \leq p, 1 \leq l \leq T} \quad (2.2)$$
be two independent arrays, with respective size $p \times n$ and $p \times T$, of independent real-valued random variables with mean 0 and variance 1. The sample covariance matrix $S_2$ is

$$S_2 = \frac{1}{n} \sum_{j=1}^{n} z_j z_j^* = \frac{1}{n} Z Z^*. \quad (2.3)$$

Next, $\Sigma_p$ is a rank $M$ perturbation of $I_p$; therefore, we can assume that it has the spiked structure of form

$$\Sigma_p = \begin{pmatrix} \Omega_M & 0 \\ 0 & I_{p-M} \end{pmatrix}, \quad (2.4)$$

where $\Omega_M$ is an $M \times M$ covariance matrix, $M$ being a fixed constant, containing $k$ spike eigenvalues ($a_i$), ($a_1, \ldots, a_1, \ldots, a_k, \ldots, a_k$), of respective multiplicity numbers ($n_i$) ($n_1 + \cdots + n_k = M$). That is, $\Omega_M = U \text{diag}(a_1, \ldots, a_1, \ldots, a_k, \ldots, a_k) U^*$, where $U$ is an $M \times M$ orthogonal matrix. Consider a sample $x_1, \ldots, x_T$ of size $T$ that can be expressed as $x_l := \Sigma_p^{1/2} w_l$ and let $X = (x_1, \ldots, X_T) = \Sigma_p^{1/2} W$. The sample covariance matrix $S_1$ is

$$S_1 = \frac{1}{T} \sum_{l=1}^{T} x_l x_l^* = \frac{1}{T} X X^* = \Sigma_p^{1/2} \left( \frac{1}{T} W W^* \right) \Sigma_p^{1/2}. \quad (2.5)$$

Throughout the paper, we consider an asymptotic regime of Marčenko-Pastur type, i.e.

$$p \wedge n \wedge T \to \infty, \quad y_p := p/n \to y \in (0, 1), \quad \text{and} \quad c_p := p/T \to c > 0. \quad (2.6)$$

Recall that the empirical spectral distribution (ESD) of a $p \times p$ matrix $A$ with eigenvalues $\{\lambda_j\}$ is the distribution $p^{-1} \sum_{j=1}^{p} \delta_{\lambda_j}$, where $\delta_a$ denotes the Dirac mass at $a$. Since the total rank $M$ generated by the $k$ spikes is fixed, the ESD of $F$ will have the same limit (LSD) as there were no spikes. This limiting spectral distribution, the celebrated Wachter distribution, has been known for a long time.

**Proposition 2.1.** For the Fisher matrix $F = S_2^{-1} S_1$ with the sample covariance matrices $S_i$’s given in (2.3)-(2.5), assume that the dimension $p$ and the two sample sizes $n, T$ grow to infinity proportionally as in (2.6). Then almost surely, the ESD of $F$ weakly converges to a deterministic distribution $F_{c,y}$ with a bounded support $[b_1, b]$ and a density function given by

$$f_{c,y}(x) = \begin{cases} \frac{(1-y)\sqrt{(b-x)(x-b_1)}}{2\pi x (c+xy)} & , \quad \text{when } b_1 \leq x \leq b \\ 0 & , \quad \text{otherwise} \end{cases}, \quad (2.7)$$

where

$$b_1 = \left( \frac{1 - \sqrt{c + y - cy}}{1 - y} \right)^2 \quad \text{and} \quad b = \left( \frac{1 + \sqrt{c + y - cy}}{1 - y} \right)^2. \quad (2.8)$$
Furthermore, if $c > 1$, then $F_{c,y}$ has a point mass $1 - 1/c$ at the origin. Also, the Stieltjes transform $s(z)$ of $F_{c,y}$ equals:

$$s(z) = \frac{1}{zc} - \frac{1}{z} - \frac{c(z(1-y) + 1 - c) + 2zy - c\sqrt{(1 - c + z(1-y))^2 - 4z}}{2zc(c + zy)}, \quad z \notin [b_1, b].$$

(2.9)

**Remark 2.1.** Assuming both populations are Gaussian, (Wachter, 1980, Theorem 3.1) derives the limiting distribution for roots of the determinental equation,

$$|TS_1 - x^2(TS_1 + nS_2)| = 0, \quad x \in \mathbb{R}.$$  

The continuous component of the distribution has a compact support $[A^2, B^2]$ with density function proportional to $\{(x - A^2)(B^2 - x)^{1/2}/\{y(1 - x^2)\}$. It can be readily checked that by the change of variable $z = cx^2/\{y(1 - x^2)\}$, the density of the continuous component of the LSD of $F$ is exactly (2.7). The validity of this limit for general populations (non necessarily Gaussian) is due to Silverstein (1985) and Bai et al. (1987).

For a complex number $z \notin [b_1, b]$, we define the following integrals with respect to $F_{c,y}(x)$:

$$s(z) := \int \frac{1}{x - z} dF_{c,y}(x), \quad m_1(z) := \int \frac{1}{(z - x)^2} dF_{c,y}(x),$$
$$m_2(z) := \int \frac{x}{z - x} dF_{c,y}(x), \quad m_3(z) := \int \frac{x}{(z - x)^2} dF_{c,y}(x),$$
$$m_4(z) := \int \frac{x^2}{(z - x)^2} dF_{c,y}(x).$$

(2.10)

3. Phase transition of the extreme eigenvalues of $F = S_2^{-1}S_1$

In this section, we establish a phase transition phenomenon for the extreme eigenvalues of $F = S_2^{-1}S_1$, that is, when a population spike $a_i$ with multiplicity $n_i$ is larger (or smaller) than a critical value, a packet of $n_i$ corresponding sample eigenvalues of $F$ will jump outside the support $[b_1, b]$ of its LSD $F_{c,y}$ and converge all to a fixed limit. Otherwise, these associated sample eigenvalues will converge to one of the edges $b_1$ and $b$. For notation convenience, let $\gamma = 1/(1 - y) \in (1, \infty)$. Define the function

$$\phi(x) = \frac{\gamma x(x - 1 + c)}{x - \gamma}, \quad x \neq \gamma,$$

(3.1)

which is a rational function with a single pole $\gamma$. An example is depicted in Figure 1 with parameters $(c, y) = (\frac{1}{5}, \frac{1}{2})$. The function has an asymptote of equation $g(x) = \gamma(x + c - 1 + \gamma)$ when $|x| \to \infty$.

By assumption, the $k$ population spike eigenvalues $\{a_i\}$ are all positive and non unit. We order them with their multiplicities in descending order together with the $p - M$ unit
Figure 1: Example of the $\phi$ function with $(c, y) = (\frac{1}{5}, \frac{1}{2})$ and pole $\gamma = 2$. The asymptote has equation $y = 2x + \frac{12}{5}$. The boundary points are $A(0.450, 0.203)$ and $B(3.549, 12.597)$ meaning that critical values for spikes are 0.450 and 3.549 while the support of the LSD is $[0.203, 12.597]$.

eigenvalues as

$$a_1 = \cdots = a_1 > a_2 = \cdots = a_2 > \cdots > a_{k_0} = \cdots = a_{k_0} > 1 = \cdots = 1 > a_{k_0+1} = \cdots = a_{k_0+1} > \cdots > a_k = \cdots = a_k. \quad (3.2)$$

That is, $k_0$ of these spike eigenvalues are larger than 1 while the other $k - k_0$ are smaller. Let

$$J_i = \begin{cases} [n_1 + \cdots + n_{i-1} + 1, n_1 + \cdots + n_i], & 1 \leq i \leq k_0, \\ [p - (n_i + \cdots + n_k) + 1, p - (n_{i+1} + \cdots + n_k)], & k_0 < i \leq k. \end{cases}$$

Notice that the cardinality of each $J_i$ is $n_i$. Next, the sample eigenvalues $\{l_{p,j}\}$ of the Fisher matrix $S_2^{-1}S_1$ are also sorted in the descending order as $l_{p,1} \geq l_{p,2} \geq \cdots \geq l_{p,p}$. Therefore, for each spike eigenvalue $a_i$, there are $n_i$ associated sample eigenvalues $\{l_{p,j}, \ j \in J_i\}$.

**Theorem 3.1.** For the Fisher matrix $F = S_2^{-1}S_1$ with the sample covariance matrices $S_i$’s given in (2.3)-(2.5), assume that the dimension $p$ and the two sample sizes $n, T$ grow to infinity proportionally as in (2.6). Then for any spike eigenvalue $a_i$ ($i = 1, \cdots, k$), it holds
that for all $j \in J_i$, $l_{p,j}$ almost surely converges to a limit

$$
\lambda_i = \begin{cases} 
\phi(a_i), & |a_i - \gamma| > \gamma \sqrt{c+y-cy}, \\
b, & 1 < a_i \leq \gamma \{1 + \sqrt{c+y-cy}\}, \\
b_1, & \gamma \{1 - \sqrt{c+y-cy}\} \leq a_i < 1.
\end{cases}
$$

(3.3)

Basically, the theorem establishes a phase transition phenomenon for the largest and smallest sample eigenvalues of a Fisher matrix. Consider again the example shown in Figure 1. The transition boundary is indicated with the boundary points $A$ and $B$ with respective coordinates

$$
A(\gamma \{1 - \sqrt{c+y-cy}\}, b_1) \quad \text{and} \quad B(\gamma \{1 + \sqrt{c+y-cy}\}, b).
$$

When the spike is large enough or small enough, the corresponding sample eigenvalues converge to $\phi(a_i)$ located outside the support $[b_1, b]$ of the LSD of $F$. Otherwise, they converge to one of its edges $b_1$ and $b$.

It is worth observing that when $y \to 0$, the $\phi(x)$ function tends to the function well-known in the literature for similar transition phenomenon of a spiked sample covariance matrix, i.e.

$$
\lim_{y \to 0} \phi(x) = x + \frac{cx}{x-1}, \quad x \neq 1,
$$

(3.4)

see e.g. the $\psi$-function on Figure 4 of Bai and Yao (2012). These functions share a same shape; however the pole here equals $\gamma = 1/(1-y)$ which is larger than the pole 1 for the case of a spiked sample covariance matrix.

As said in Introduction, this transition phenomenon has already been established in a preprint Dharmawansa et al. (2014) (their Proposition 5) under Gaussian assumption and using a completely different approach. Theorem 3.1 proves that such a phase transition phenomenon is indeed universal.

Proof. (of Theorem 3.1) The proof is divided into the following three steps:

- Step 1: we derive the almost sure limit of an outlier eigenvalue of $S_2^{-1}S_1$;
- Step 2: we show that in order for the extreme eigenvalue of $S_2^{-1}S_1$ to be an outlier, the population spike $a_i$ should be larger (or smaller) than a critical value;
- Step 3: if not so, the extreme eigenvalue of $S_2^{-1}S_1$ will converge to one of the edge points $b$ and $b_1$.

**Step 1:** Let $l_{p,j}$ ($j \in J_i$) be the outlier eigenvalue of $S_2^{-1}S_1$ corresponding to the population spike $a_i$. Then $l_{p,j}$ must satisfy the following equation:

$$
|l_{p,j}I_p - S_2^{-1}S_1| = 0,
$$

and it is equivalent to

$$
|l_{p,j}S_2 - S_1| = 0.
$$

(3.5)
Now we make some short-hands. Denote \( Z = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \), where \( Z_1 \) is the \( n \) observations of its first \( M \) coordinates and \( Z_2 \) the remaining. We partition \( X \) accordingly as \( X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \), where \( X_1 \) is the \( T \) observations of its first \( M \) coordinates and \( X_2 \) the remaining. Using such a representation, we have

\[
S_1 = \frac{1}{T} XX^* = \frac{1}{T} \begin{pmatrix} X_1X_1^* & X_1X_2^* \\ X_2X_1^* & X_2X_2^* \end{pmatrix}, \quad S_2 = \frac{1}{n} ZZ^* = \frac{1}{n} \begin{pmatrix} Z_1Z_1^* & Z_1Z_2^* \\ Z_2Z_1^* & Z_2Z_2^* \end{pmatrix}. \tag{3.6}
\]

Then, (3.5) could be written in the block form:

\[
\begin{pmatrix}
\frac{l_{p,j}}{n} Z_1Z_1^* - \frac{1}{T}X_1X_1^* & \frac{l_{p,j}}{n} Z_1Z_2^* - \frac{1}{T}X_1X_2^* \\
\frac{l_{p,j}}{n} Z_2Z_1^* - \frac{1}{T}X_2X_1^* & \frac{l_{p,j}}{n} Z_2Z_2^* - \frac{1}{T}X_2X_2^*
\end{pmatrix} = 0. \tag{3.7}
\]

Since \( l_{p,j} \) is an outlier, it holds \( |l_{p,j} - \frac{1}{n}Z_2Z_2^* - \frac{1}{T}X_2X_2^*| \neq 0 \), and for block matrix, we have \( \det \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \det D \cdot \det(A - BD^{-1}C) \) when \( D \) is invertible. Therefore, (3.7) reduces to

\[
\begin{align*}
\left| \frac{l_{p,j}}{n} Z_1Z_1^* - \frac{1}{T}X_1X_1^* \\
- \left( \frac{l_{p,j}}{n} Z_1Z_2^* - \frac{1}{T}X_1X_2^* \right) \left( \frac{l_{p,j}}{n} Z_2Z_2^* - \frac{1}{T}X_2X_2^* \right)^{-1} \left( \frac{l_{p,j}}{n} Z_2Z_1^* - \frac{1}{T}X_2X_1^* \right) \right| &= 0.
\end{align*}
\]

More specifically, we have

\[
\begin{align*}
\det & \left( \frac{l_{p,j}}{n} Z_1 \left[ I_n - Z_2 \left( \frac{l_{p,j}I_p}{n} - \left( \frac{1}{n}Z_2Z_2^* \right)^{-1} \frac{1}{T}X_2X_2^* \right)^{-1} \left( \frac{1}{n}Z_2Z_2^* \right)^{-1} \frac{l_{p,j}Z_2}{n} \right] Z_1^* \right) \\
&- \frac{1}{T}X_1 \left[ I_T + X_2^* \left( \frac{l_{p,j}I_p}{n} - \left( \frac{1}{n}Z_2Z_2^* \right)^{-1} \frac{1}{T}X_2X_2^* \right)^{-1} \left( \frac{1}{n}Z_2Z_2^* \right)^{-1} \frac{1}{T}X_2 \right] X_1^* \\
&+ \frac{l_{p,j}}{n} Z_2 \left( \frac{l_{p,j}I_p}{n} - \left( \frac{1}{n}Z_2Z_2^* \right)^{-1} \frac{1}{T}X_2X_2^* \right)^{-1} \left( \frac{1}{n}Z_2Z_2^* \right)^{-1} \frac{l_{p,j}Z_1}{n} \\
&+ \frac{1}{T}X_1 \left( \frac{l_{p,j}I_p}{n} - \left( \frac{1}{n}Z_2Z_2^* \right)^{-1} \frac{1}{T}X_2X_2^* \right)^{-1} \left( \frac{1}{n}Z_2Z_2^* \right)^{-1} \frac{l_{p,j}Z_2}{n}
\end{align*}
\]

\[
= 0. \tag{3.8}
\]

In all the following, we denote by \( S \) the Fisher matrix \( \left( \frac{1}{n}Z_2Z_2^* \right)^{-1} \frac{1}{T}X_2X_2^* \), which has a LSD \( F_{c,y}(x) \). And in order to find the limit of \( l_{p,j} \), we simply find the limit on the left hand
side of (3.8), then it will generate an equation. Solving this equation will give the value of its limit.

First, consider the terms (III) and (IV). Since \((Z_1, X_1)\) is independent of \((Z_2, X_2)\), using Lemma A.2, we see these two terms will converge to some constant multiplied by the covariance matrix between \(X_1\) and \(Z_1\). On the other hand, \(X_1\) is also independent of \(Z_1\), we have

\[
\text{Cov}(X_1, Z_1) = \mathbb{E}X_1 Z_1 - \mathbb{E}X_1 \mathbb{E}Z_1 = \mathbb{E}X_1 \mathbb{E}Z_1 - \mathbb{E}X_1 \mathbb{E}Z_1 = 0_{M \times M}.
\]

Therefore, these two terms will both tend to a zero matrix \(0_{M \times M}\) almost surely.

So the remaining task is to find the limit of (I) and (II). We recall the expression of \(X_1\) and \(Z_1\) that

\[
\text{Cov}(X_1) = U \text{diag}(a_1, \ldots, a_1, \ldots, a_k, \ldots, a_k) U^*, \quad \text{Cov}(Z_1) = I_M.
\]

According to Lemma A.2, we have

\[
(I) = \frac{l_{p,j}}{n} Z_1 \left[ I_n - Z_2^* (l_{p,j} I_p - S)^{-1} \left( \frac{1}{n} Z_2 Z_2^* \right)^{-1} \frac{l_{p,j}}{n} Z_2 \right] Z_1^* \\
\rightarrow \frac{\lambda_i}{n} \left\{ \mathbb{E} \text{tr} \left[ I_n - Z_2^* (\lambda_i I_p - S)^{-1} \left( \frac{1}{n} Z_2 Z_2^* \right)^{-1} \frac{\lambda_i}{n} Z_2 \right] \right\} \cdot I_M \\
= \lambda_i (1 + y \lambda_i s(\lambda_i)) \cdot I_M,
\]

here, we denote \(\lambda_i\) as the limit of the outlier \(\{l_{p,j}, j \in J_i\}\). For the same reason,

\[
(II) = -\frac{1}{T} X_1 \left[ I_T + X_2^* (l_{p,j} I_p - S)^{-1} \left( \frac{1}{n} Z_2 Z_2^* \right)^{-1} \frac{1}{T} X_2 \right] X_1^* \\
\rightarrow -\frac{1}{T} \left\{ \mathbb{E} \text{tr} \left[ I_T + X_2^* (\lambda_i I_p - S)^{-1} \left( \frac{1}{n} Z_2 Z_2^* \right)^{-1} \frac{1}{T} X_2 \right] \right\} \cdot U \begin{pmatrix} a_1 \\ \vdots \\ a_k \end{pmatrix} U^* \\
= U \left( 1 - 1 + c + c \lambda_i s(\lambda_i) \right) \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_k \end{pmatrix} U^*.
\]

Therefore, combining (3.8), (3.9) and (3.10), we have the determinant of the following \(M \times M\) matrix

\[
U \begin{pmatrix} \lambda_i (1 + y \lambda_i s(\lambda_i)) + (-1 + c + c \lambda_i s(\lambda_i)) a_1 & 0 \\ \vdots & \ddots & \vdots \\ 0 & \lambda_i (1 + y \lambda_i s(\lambda_i)) + (-1 + c + c \lambda_i s(\lambda_i)) a_k \end{pmatrix} U^*
\]
equal to zero, which is also to say that \( \lambda_i \) satisfies the equation:

\[
\lambda_i (1 + y \lambda_i s(\lambda_i)) + (-1 + c + c \lambda_i s(\lambda_i)) a_i = 0 .
\] (3.11)

Finally, together with the expression of the Stieltjes transform of a Fisher matrix in (2.9), we have

\[
\lambda_i = \frac{a_i (a_i + c - 1)}{a_i - a_i y - 1} = \phi(a_i),
\]

where the function \( \phi(x) \) is defined in (3.4).

**Step 2:** Define \( s(z) \) as the Stieltjes transform of the LSD of \( \frac{1}{T} X_2^* (\frac{1}{n} Z_2 Z_2^*)^{-1} X_2 \), who shares the same non-zero eigenvalues as \( S_2^{-1} S_1 \). Then we have the relationship:

\[
s(z) + \frac{1}{z} (1 - c) = cs(z) .
\] (3.12)

Recall the expression of \( s(z) \) in (2.9), we have

\[
s(z) = -\frac{c(z(1 - y) + 1 - c) + 2zy - c\sqrt{(1 - c + z(1 - y))^2 - 4z}}{2z(c + zy)} .
\] (3.13)

On the other hand, due to (3.11) and (3.12), we have the value for \( s(\lambda_i) \):

\[
s(\lambda_i) = \frac{yc - y - c}{y \lambda_i + a_ic} .
\] (3.14)

Since \( \lambda_i \) is outside the support of the LSD, we have

\[
s^{-1} \left( \frac{yc - y - c}{y \lambda_i + a_ic} \right) = \lambda_i > b \quad \text{or} \quad s^{-1} \left( \frac{yc - y - c}{y \lambda_i + a_ic} \right) = \lambda_i < b_1
\]

which is also to say that

\[
s(b) < \frac{yc - y - c}{y \lambda_i + a_ic} ,
\] (3.15)

or

\[
s(b_1) > \frac{yc - y - c}{y \lambda_i + a_ic} .
\] (3.16)

Then (3.15) says that \( s(b) \) must be smaller than the minimum value on its right hand side, whose minimum value is attained when \( \lambda_i = b \) (the right hand side of (3.15) is a decreasing function of \( \lambda_i \)). Similarly, (3.16) says that \( s(b_1) \) must be larger than the maximum value on its right hand side, which is attained when \( \lambda_i = b_1 \). Therefore, the condition for \( \lambda_i \) be an outlier is:

\[
s(b) < \frac{yc - y - c}{yb + a_ic}, \quad \text{or} \quad s(b_1) > \frac{yc - y - c}{yb_1 + a_ic} .
\] (3.17)
Finally, using (3.13) together with the value of $b$ and $b_1$, we have:

$$a_i > \frac{1 + \sqrt{c + y - cy}}{1 - y}, \quad \text{or} \quad a_i < \frac{1 - \sqrt{c + y - cy}}{1 - y},$$

which is equivalent to say that (recall the expression of $\gamma$ that $\gamma = 1/(1 - y)$):

$$|a_i - \gamma| > \gamma \sqrt{c + y - cy}.$$

**Step 3:** In this step, we show that if the condition in Step 2 is not fulfilled, then the extreme eigenvalues of $S_2^{-1}S_1$ will tend to one of the edge points $b_1$ and $b$. For simplicity, we only show the convergence to the right edge $b$; the proof for the convergence to the left edge $b_1$ is similar. Thus suppose all the $a_i > 1$ for $i = 1, \ldots, k$. For now, we make some short-hands. Let

$$S_1 = \frac{1}{T}XX^* = \frac{1}{T} \begin{pmatrix} X_1X_1^* & X_1X_2^* \\ X_2X_1^* & X_2X_2^* \end{pmatrix} := \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

and

$$S_2 = \frac{1}{n}ZZ^* = \frac{1}{n} \begin{pmatrix} Z_1Z_1^* & Z_1Z_2^* \\ Z_2Z_1^* & Z_2Z_2^* \end{pmatrix} := \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where $B_{11}$ and $A_{11}$ are the corresponding blocks with size $M \times M$. Using the inverse formula for block matrix, the $(p - M) \times (p - M)$ major sub-matrix of $S_2^{-1}S_1$ is

$$-(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}A_{21}A_{11}^{-1}B_{12} + (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}B_{22} := C. \quad (3.18)$$

The part

$$-(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}A_{21}A_{11}^{-1}B_{12} = -(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}A_{21}A_{11}^{-1} \cdot \frac{1}{T}X_1X_2^*$$

is of rank $M$; besides, we have

$$\text{tr} \left\{ (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}A_{21}A_{11}^{-1} \frac{1}{T}X_1X_2^* \right\} \to 0,$$

since $X_1$ is independent of $X_2$. Therefore, the $M$ nonzero eigenvalues of the matrix $-(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}A_{21}A_{11}^{-1}B_{12}$ will all tend to zero (so is its largest one). Then consider the second part of (3.18) as follows.

$$A_{22} - A_{21}A_{11}^{-1}A_{12} = \frac{1}{n}Z_2 \left[ I_n - Z_1^* \left( \frac{1}{n}Z_1Z_1^* \right)^{-1} \frac{1}{n}Z_1 \right] Z_2^* := \frac{1}{n}Z_2PZ_2.$$
Since $P = I_n - Z_1^*(\frac{1}{n}Z_1Z_1^*)^{-1}\frac{1}{n}Z_1$ is a projection matrix of rank $p - M$, it has the spectral decomposition:

$$P = V \begin{pmatrix} 0 & & & \
& & & \
& & & \
& & & \
0 & & & 1_{n-M} \end{pmatrix} V^*,$$

where $V$ is a $n \times n$ orthogonal matrix. Since $M$ is fixed, the ESD of $P$ tends to $\delta_1$, which leads to the fact that the LSD of the matrix $\frac{1}{n}Z_2PZ_2^*$ is the standard Marčenko-Pastur law. Then the matrix $(\frac{1}{n}Z_2PZ_2^*)^{-1}B_{22}$ is a standard Fisher matrix, and its largest eigenvalues (finitely many) will tend to the right edge $b$ of the Wachter distribution. It follows then the two largest eigenvalues of $C$, say $\alpha_1(C)$ and $\alpha_2(C)$, also tend to $b$.

Next since $C$ is the $(p - M) \times (p - M)$ major sub-matrix of $S_1^{-1}S_1$, we have by Cauchy interlacing theorem

$$\alpha_2(C) \leq l_{p,M+1} \leq \alpha_1(C) \leq l_{p,1}.$$ 

Thus $l_{p,M+1} \to b$ either. On the other hand, we have

$$l_{p,1} = \|S_2^{-1}S_1\|_{op} \leq \|S_2^{-1}\|_{op} \cdot \|S_1\|_{op},$$

so that for some positive constant $\theta$, lim sup $l_{p,1} \leq \theta$. Consequently, almost surely,

$$b \leq \lim \inf l_{p,M} \leq \cdots \leq \lim \sup l_{p,1} \leq \theta < \infty;$$

in particular the whole family $\{l_{p,j}, 1 \leq j \leq M\}$ is bounded. Now let $1 \leq j \leq M$ be fixed and assume that a subsequence $(l_{p,k,j})_k$ converges to a limit $\beta \in [b, \theta]$. Either $\beta = \phi(a_i) > b$ or $\beta = b$. However, according to Step 2, $\beta > b$ implies that $a_i > \gamma \{1 + \sqrt{c + y - cy}\}$, and otherwise, we have $a_i \leq \gamma \{1 + \sqrt{c + y - cy}\}$. Therefore, accordingly to one of these two conditions, all subsequences converge to a same limit $\phi(a_i)$ or $b$, which is thus also the unique limit of the whole sequence $(l_{p,j})_p$. The proof of Theorem 3.1 is complete.

4. Central limit theorem for the outlier eigenvalues of $S_2^{-1}S_1$

The aim of this section is to give a CLT for the $n_i$-packed outlier eigenvalues:

$$\sqrt{p} \left\{ l_{p,j} - \phi(a_i), j \in J_i \right\}.$$ 

Denote $U = (U_1 \ U_2 \ \cdots \ \ U_k)$, where each $U_i$ is a $M \times n_i$ matrix that corresponds to the $n_i$-packed spike eigenvalue $a_i$.

**Theorem 4.1.** Assume the same assumptions as in Theorem 3.1 and in addition, the variables $(z_{ij})$ (in (2.1)) and $(w_{kl})$ (in (2.2)) have the same first four moments and denote $v_4$ as their common fourth moment:

$$v_4 = \mathbb{E}|z_{ij}|^4 = \mathbb{E}|w_{kl}|^4, \quad 1 \leq i, k \leq p, \ 1 \leq j \leq n, \ 1 \leq l \leq T.$$
Then for any population spike \(a_i\) satisfying \(|a_i - \gamma| > \gamma \sqrt{c + y - cy}\), the normalised \(n_i\)-packed outlier eigenvalues of \(S_2^{-1}S_1: \sqrt{P} \{l_{p,j} - \phi(a_i), j \in J_i\}\) converge weakly to the distribution of the eigenvalues of the random matrix \(-U_i^*R(\lambda_i)U_i/\Delta(\lambda_i)\). Here,

\[
\Delta(\lambda_i) = \frac{(1 - a_i - c)(1 + a_i(y - 1))}{(a_i - 1)(-1 + 2a_i + c + a_i^2(y - 1))},
\]

\(R(\lambda_i) = (R_{mn})\) is a \(M \times M\) symmetric random matrix, made with independent Gaussian entries of mean zero and variance

\[
\text{Var}(R_{mn}) = \begin{cases} 
2\theta_i + (v_4 - 3)\omega_i, & m = n, \\
\theta_i, & m \neq n,
\end{cases}
\]

where

\[
\omega_i = \frac{a_i^2(a_i + c - 1)^2(c + y)}{(a_i - 1)^2},
\]

\[
\theta_i = \frac{a_i^2(a_i + c - 1)^2(cy - c - y)}{-1 + 2a_i + c + a_i^2(y - 1)}.
\]

Numerical illustrations of this theorem are detailed in the next section.

**Remark 4.1.** Notice that the result above involves the \(i\)-th block \(U_i\) of the eigen-matrix \(U\). When the spike \(a_i\) is simple, \(U_i\) is unique up to its sign, then \(U_i^*R(\lambda_i)U_i\) is uniquely determined. But when \(a_i\) has multiplicities greater than 1, \(U_i\) is not unique; actually, any rotation of \(U_i\) can be an eigenvector matrix corresponding to \(a_i\). Therefore, Lemma A.1 in the Appendix states that, such a rotation will not affect the eigenvalues of the matrix \(U_i^*R(\lambda_i)U_i\).

**Proof.** (proof of Theorem 4.1)

**Step 1: Convergence to the eigenvalues of the random matrix \(-U_i^*R(\lambda_i)U_i/\Delta(\lambda_i)\).**

We start from (3.8). First we make some short hands. Define

\[
\begin{align*}
A(\lambda) &= I_n - Z_2^*\left[\lambda I_p - \left(\frac{1}{n}Z_2Z_2^*\right)^{-1} \frac{1}{T}X_2X_2^*\right]^{-1} \left(\frac{1}{n}Z_2Z_2^*\right)^{-1}\frac{1}{n}Z_2, \\
B(\lambda) &= I_T + X_2^*\left[\lambda I_p - \left(\frac{1}{n}Z_2Z_2^*\right)^{-1} \frac{1}{T}X_2X_2^*\right]^{-1} \left(\frac{1}{n}Z_2Z_2^*\right)^{-1}\frac{1}{T}X_2, \\
C(\lambda) &= Z_2^*\left[\lambda I_p - \left(\frac{1}{n}Z_2Z_2^*\right)^{-1} \frac{1}{T}X_2X_2^*\right]^{-1} \left(\frac{1}{n}Z_2Z_2^*\right)^{-1}\frac{1}{T}X_2, \\
D(\lambda) &= X_2^*\left[\lambda I_p - \left(\frac{1}{n}Z_2Z_2^*\right)^{-1} \frac{1}{T}X_2X_2^*\right]^{-1} \left(\frac{1}{n}Z_2Z_2^*\right)^{-1}\frac{1}{n}Z_2,
\end{align*}
\]

then (3.8) could be written as

\[
\det \left(\frac{l_{p,j}}{n}Z_1A(l_{p,j})Z_1^* - \frac{1}{T}X_1B(l_{p,j})X_1^* + \frac{l_{p,j}}{n}Z_1C(l_{p,j})X_1^* + \frac{l_{p,j}}{T}X_1D(l_{p,j})Z_1^*\right) = 0.
\]
The remaining is to find second order approximation of the four terms on the left hand side of (4.6).

Using Lemma A.5 in the appendix, we have

\[ (i) = \frac{\lambda_i}{n} Z_1 A(l_i) Z_1^* + \frac{l_{p,j}}{n} Z_1 A(l_{p,j}) Z_1^* - \mathbb{E} \frac{\lambda_i}{n} Z_1 A(l_i) Z_1^* \]
\[ = (\lambda_i + y \lambda_i^2 s(\lambda_i)) \cdot I_M + \frac{l_{p,j} - \lambda_i}{n} Z_1 A(l_{p,j}) Z_1^* + \frac{\lambda_i}{n} Z_1 A(l_i) Z_1^* - \mathbb{E} \frac{\lambda_i}{n} Z_1 A(l_i) Z_1^* \]
\[ = (\lambda_i + y \lambda_i^2 s(\lambda_i)) \cdot I_M + \frac{l_{p,j} - \lambda_i}{n} Z_1 A(l_{p,j}) Z_1^* + \frac{\lambda_i}{n} Z_1 (A(l_{p,j}) - A(l_i)) Z_1^* \]
\[ + \frac{\lambda_i}{\sqrt{n}} \left[ \frac{1}{n} Z_1 A(l_i) Z_1^* - \mathbb{E} \frac{1}{\sqrt{n}} Z_1 A(l_i) Z_1^* \right] \]
\[ \rightarrow (\lambda_i + y \lambda_i^2 s(\lambda_i)) \cdot I_M + (l_{p,j} - \lambda_i) \cdot (1 + 2y \lambda_i s(\lambda_i) + \lambda_i^2 y m_1(\lambda_i)) \cdot I_M \]
\[ + \frac{\lambda_i}{\sqrt{n}} \left[ \frac{1}{n} Z_1 A(l_i) Z_1^* - \mathbb{E} \frac{1}{\sqrt{n}} Z_1 A(l_i) Z_1^* \right], \tag{4.7} \]

\[ (ii) = \frac{1}{T} X_1 B(l_i) X_1^* + \frac{1}{T} X_1 B(l_{p,j}) X_1^* - \mathbb{E} \frac{1}{T} X_1 B(l_i) X_1^* \]
\[ = U (1 - c - c \lambda_i s(\lambda_i)) \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_k \end{pmatrix} U^* + \frac{1}{T} X_1 (B(l_{p,j}) - B(l_i)) X_1^* \]
\[ + \frac{1}{\sqrt{T}} \left[ \frac{1}{\sqrt{T}} X_1 B(l_i) X_1^* - \mathbb{E} \frac{1}{\sqrt{T}} X_1 B(l_i) X_1^* \right] \]
\[ \rightarrow U (1 - c - c \lambda_i s(\lambda_i)) \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_k \end{pmatrix} U^* - U(l_{p,j} - \lambda_i) \cdot cm_3(\lambda_i) \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_k \end{pmatrix} U^* \]
\[ + \frac{1}{\sqrt{T}} \left[ \frac{1}{\sqrt{T}} X_1 B(l_i) X_1^* - \mathbb{E} \frac{1}{\sqrt{T}} X_1 B(l_i) X_1^* \right], \tag{4.8} \]

\[ (iii) = \frac{l_{p,j}}{n} Z_1 C(l_{p,j}) X_1^* - \mathbb{E} \frac{\lambda_i}{n} Z_1 C(l_i) X_1^* \]
\[ = \frac{l_{p,j}}{n} Z_1 C(l_{p,j}) X_1^* - \frac{\lambda_i}{n} Z_1 C(l_i) X_1^* + \frac{\lambda_i}{n} Z_1 C(l_i) X_1^* - \mathbb{E} \frac{\lambda_i}{n} Z_1 C(l_i) X_1^* \]
\[ = \frac{l_{p,j}}{n} Z_1 (C(l_{p,j}) - C(l_i)) X_1^* + \frac{l_{p,j} - \lambda_i}{n} Z_1 C(l_i) X_1^* + \frac{\lambda_i}{n} \cdot \left[ Z_1 C(l_i) X_1^* - \mathbb{E} Z_1 C(l_i) X_1^* \right] \]
\[ \rightarrow \frac{\lambda_i}{n} \cdot \left[ Z_1 C(l_i) X_1^* - \mathbb{E} Z_1 C(l_i) X_1^* \right], \tag{4.9} \]
(iv) $= \frac{l_{p,j}}{T} X_1 D(l_{p,j}) Z_1^* - \frac{\lambda_i}{T} X_1 D(\lambda_i) Z_1^*

= \frac{l_{p,j}}{T} X_1 D(l_{p,j}) Z_1^* - \frac{\lambda_i}{T} X_1 D(\lambda_i) Z_1^* + \frac{\lambda_i}{T} X_1 D(\lambda_i) Z_1^* - \frac{\lambda_i}{T} X_1 D(\lambda_i) Z_1^*

= \frac{l_{p,j}}{T} (X_1 D(l_{p,j}) - D(\lambda_i)) Z_1^* + \frac{l_{p,j} - \lambda_i}{T} X_1 D(\lambda_i) Z_1^* + \frac{\lambda_i}{T} \left[ X_1 D(\lambda_i) Z_1^* - \mathbb{E} X_1 D(\lambda_i) Z_1^* \right]

\rightarrow \frac{\lambda_i}{T} \left[ X_1 D(\lambda_i) Z_1^* - \mathbb{E} X_1 D(\lambda_i) Z_1^* \right]. \quad (4.10)

Denote

$$R_n(\lambda_i) = \lambda_i \sqrt{\frac{p}{n}} \left[ \frac{1}{\sqrt{n}} Z_1 A(\lambda_i) Z_1^* \right] - \frac{\sqrt{p}}{T} \left[ \frac{1}{\sqrt{T}} X_1 B(\lambda_i) X_1^* \right] + \lambda_i \sqrt{\frac{p}{n}} \left[ \frac{1}{\sqrt{n}} Z_1 C(\lambda_i) X_1^* \right]$$

$$+ \lambda_i \sqrt{\frac{p}{T}} \left[ \frac{1}{\sqrt{T}} X_1 D(\lambda_i) Z_1^* \right] - \mathbb{E}[\cdot], \quad (4.11)$$

where $\mathbb{E}[\cdot]$ denotes the total expectation of all the preceding terms in the equation, and

$$\Delta(\lambda_i) = 1 + 2y\lambda_i s(\lambda_i) + \lambda_i^2 y m_1(\lambda_i) + acm_3(\lambda_i).$$

Combining (4.6), (4.7), (4.8), (4.9), (4.10) and considering the diagonal block that corresponds to the row and column index in $J_i \times J_i$ leads to:

$$\left| \sqrt{p} (l_{p,j} - \lambda_i) \cdot \Delta(\lambda_i) \cdot I_{n_i} + [U^* R_n(\lambda_i) U]_{i} \right| \rightarrow 0. \quad (4.12)$$

Furthermore, it will be established in Step 2 below that

$$[U^* R_n(\lambda_i) U]_{i} \rightarrow [U^* R(\lambda_i) U]_{i} \quad \text{in distribution}, \quad (4.13)$$

for some random matrix $R(\lambda_i)$. Using the device of Skorokhod strong representation (Skorokhod, 1956; Hu and Bai, 2014), we may assume that this convergence hold almost surely by considering an enlarged probability space. Under this device, (4.12) is equivalent to say that $\sqrt{p}(l_{p,j} - \lambda_i)$ tends to an eigenvalue of the matrix $-[U^* R(\lambda_i) U]/\Delta(\lambda_i) (= -U^*_i R(\lambda_i) U_i/\Delta(\lambda_i))$. Finally, as the index $j$ is arbitrary over the set $J_i$, all the $n_i$ random variables

$$\{ \sqrt{p} (l_{p,j} - \lambda_i), j \in J_i \}$$

converge almost surely to the set of eigenvalues of the random matrix $-U^*_i R(\lambda_i) U_i/\Delta(\lambda_i)$. Besides, due to Lemma A.3, we have

$$\Delta(\lambda_i) = 1 + 2y\lambda_i s(\lambda_i) + \lambda_i^2 y m_1(\lambda_i) + acm_3(\lambda_i)$$

$$= \frac{(1 - a_i - c)(1 + a_i(y - 1))^2}{(a_i - 1)(-1 + 2a_i + c + a_i^2(y - 1))}.$$
Step 2: Proof of the convergence (4.13) and structure of the random matrix $R(\lambda_i)$. In the second step, we aim to find the matrix limit of the block random matrix $[U^* R_n(\lambda_i)U]_i$. First, we show $[U^* R_n(\lambda_i)U]_i$ equals to another random matrix $[U^* \tilde{R}_n(\lambda_i)U]_i$, here $\tilde{R}_n(\lambda_i)$ is the type of random sesquilinear form. Then using the results in Bai and Yao (2008) (Proposition 3.1 and Remark 1), we are able to find the matrix limit of $\tilde{R}_n(\lambda_i)$.

By assumption (b) that $x_i = \Sigma_p^{1/2} s_i$, we have its first $M$ components

$$X_1 = \Omega_p^{1/2} S_1 = U \begin{pmatrix} \sqrt{a_1} \\ \vdots \\ \sqrt{a_k} \end{pmatrix} U^* S_1 .$$

Recall the definition of $R_n(\lambda_i)$ in (4.11), we have

$$U^* R_n(\lambda_i)U$$

$$= U^* \frac{\sqrt{p\lambda_i}}{n} Z_1 A(\lambda_i) Z_1^* U - \frac{\sqrt{p}}{T} \begin{pmatrix} \sqrt{a_1} \\ \vdots \\ \sqrt{a_k} \end{pmatrix} U^* S_1 B(\lambda_i) S_1^* U \begin{pmatrix} \sqrt{a_1} \\ \vdots \\ \sqrt{a_k} \end{pmatrix}$$

$$+ U^* \frac{\sqrt{p\lambda_i}}{n} Z_1 C(\lambda_i) S_1^* U \begin{pmatrix} \sqrt{a_1} \\ \vdots \\ \sqrt{a_k} \end{pmatrix} + \frac{\sqrt{p}}{T} \begin{pmatrix} \sqrt{a_1} \\ \vdots \\ \sqrt{a_k} \end{pmatrix} U^* S_1 D(\lambda_i) Z_1^* U$$

$$- \mathbb{E}[:].$$

Therefore, if we consider its $i$-th block that corresponds to the row and column index in the set $J_i \times J_i$:

$$[U^* R_n(\lambda_i)U]_i$$

$$= \lambda_i \sqrt{\frac{p}{n}} \left[ \lambda_i \sqrt{n} U^* Z_1 A(\lambda_i) Z_1^* U \right]_i - a_i \sqrt{\frac{p}{T}} \left[ \frac{1}{\sqrt{T}} U^* S_1 B(\lambda_i) S_1^* U \right]_i$$

$$+ \lambda_i \sqrt{\frac{p}{n}} \left[ \frac{1}{\sqrt{n}} U^* Z_1 C(\lambda_i) S_1^* U \right]_i + \lambda_i \sqrt{\frac{p}{T}} \left[ - \frac{1}{\sqrt{T}} U^* S_1 D(\lambda_i) Z_1^* U \right]_i - \mathbb{E}[:]$$

$$= \left[ \frac{\lambda_i \sqrt{n}}{n} U^* Z_1 A(\lambda_i) Z_1^* U - a_i \frac{\sqrt{T}}{T} U^* S_1 B(\lambda_i) S_1^* U \right.$$  

$$+ \lambda_i \sqrt{\frac{p}{n}} U^* Z_1 C(\lambda_i) S_1^* U + \lambda_i \sqrt{\frac{p}{T}} U^* S_1 D(\lambda_i) Z_1^* U \right]_i$$

$$- \mathbb{E}[:].$$

$$= \left[ U^* \begin{pmatrix} Z_1 & S_1 \end{pmatrix} \begin{pmatrix} \frac{\lambda_i \sqrt{n} A(\lambda_i)}{\lambda_i \sqrt{n} D(\lambda_i)} & \frac{\lambda_i \sqrt{n} C(\lambda_i)}{\lambda_i \sqrt{n} D(\lambda_i)} \\ \frac{\lambda_i \sqrt{n} B(\lambda_i)}{\lambda_i \sqrt{n} D(\lambda_i)} & \frac{\lambda_i \sqrt{n} D(\lambda_i)}{\lambda_i \sqrt{n} D(\lambda_i)} \end{pmatrix} \begin{pmatrix} Z_1^* \\ S_1^* \end{pmatrix} U - \mathbb{E}[:]. \right]_i$$

$$= [U^* \tilde{R}_n(\lambda_i)U]_i$$

$$= U^*_i \tilde{R}_n(\lambda_i)U_i ,$$

(4.15)
where
\[ \tilde{R}_n(\lambda_i) := (Z_1 \ S_1) \begin{pmatrix} \frac{\lambda_i \sqrt{P}(\lambda_i)}{n} & \frac{\lambda_i \sqrt{P}(\lambda_i)}{n} \\ \frac{\lambda_i \sqrt{Q}(\lambda_i)}{T} & -a_i \sqrt{P}(\lambda_i) \end{pmatrix} \left( \begin{array}{c} Z_1^* \\ S_1^* \end{array} \right) - \mathbb{E}[\cdot]. \]

Finally, using Lemma A.6 in the appendix leads to the result. The proof of Theorem 4.1 is complete.

Next we consider a special case where \( \Omega_p \) is diagonal, whose eigenvalues being all simple. In other words, we have \( M = k \) and \( n_i = 1 \) for all \( 1 \leq i \leq M \). Hence \( U = I_M \). Following Theorem 4.1, we can derive the asymptotic normality for the normalised outlier eigenvalues of \( S_2^{-1}S_1 \) when \( |a_i - \gamma| > \gamma \sqrt{c + y - cy} \).

**Proposition 4.1.** Under the same assumptions as in Theorem 3.1, with additional conditions that \( \Omega_p \) is diagonal and all its eigenvalues \( a_i \) (\( 1 \leq i \leq M \)) are simple, we have when \( |a_i - \gamma| > \gamma \sqrt{c + y - cy} \), the outlier eigenvalue \( l_i \) of \( S_2^{-1}S_1 \) is asymptotically Gaussian:

\[ \sqrt{p} \left( l_i - \frac{a_i(a_i - 1 + c)}{a_i - 1 - a_i y} \right) \Rightarrow N(0, \sigma_i^2), \]

where
\[ \sigma_i^2 = \frac{2a_i^2(cy - c - y)(a_i - 1)^2(-1 + 2a_i + c + a_i^2(y - 1))}{(1 + a_i(y - 1))^4} + (v_4 - 3) \cdot \frac{a_i^2(c + y)(-1 + 2a_i + c + a_i^2(y - 1))^2}{(1 + a_i(y - 1))^4}. \]

**Remark 4.2.** Notice that when the data are standard Gaussian, we have \( v_4 = 3 \), then the above theorem reduces to

\[ \sqrt{p} \left( l_i - \frac{a_i(a_i - 1 + c)}{a_i - 1 - a_i y} \right) \Rightarrow N \left( 0, \frac{2a_i^2(a_i - 1)^2(cy - c - y)(-1 + 2a_i + c + a_i^2(y - 1))}{(1 + a_i(y - 1))^4} \right), \]

which is exactly the result in Dharmawansa et al. (2014), see setting 1 in their Proposition 11.

**Proof.** (of Proposition 4.1) Under the above assumptions, the random matrix \(-[U^* R(\lambda_i) U]_i\) reduces to \(-[R(\lambda_i)]_i\). And since all the \( n_i = 1 \), we have \(-[R(\lambda_i)]_i\) equals the \((i, i)\)-th element of \(-R(\lambda_i)\), which is a Gaussian random variable with mean zero and variance

\[ 2\theta_i + (v_4 - 3) \omega_i = \frac{2a_i^2(a_i + c - 1)^2(cy - c - y)}{-1 + 2a_i + c + a_i^2(y - 1)} + (v_4 - 3) \cdot \frac{a_i^2(a_i + c - 1)^2(c + y)}{(a_i - 1)^2}. \]

Therefore, combining with (4.1) we have

\[ \sqrt{p} \left( l_i - \frac{a_i(a_i - 1 + c)}{a_i - 1 - a_i y} \right) \Rightarrow N(0, \sigma_i^2), \]
where

$$
\sigma_i^2 = \frac{2a_i^2(cy - c - y)(a_i - 1)^2(-1 + 2a_i + c + a_i^2(y - 1))}{(1 + a_i(y - 1))^4}
$$

$$
+ (v_4 - 3) \cdot \frac{a_i^2(c + y)(-1 + 2a_i + c + a_i^2(y - 1))^2}{(1 + a_i(y - 1))^4}.
$$

The proof of Proposition 4.1 is complete.

5. Numerical illustrations

In this section, numerical results are provided to illustrate the results of our Theorem 4.1 and Proposition 4.1. We fix $p = 200$, $T = 1000$, $n = 400$ with 1000 replications, thus $y = 1/2$ and $c = 1/5$. The critical interval is then $[\gamma - \gamma \sqrt{c + y - cy}, \gamma + \gamma \sqrt{c + y - cy}] = [0.45, 3.55]$ and the limiting support $[b_1, b] = [0.2, 12.6]$. Consider $k = 3$ spike eigenvalues $(a_1, a_2, a_3) = (20, 0.2, 0.1)$ with respective multiplicity $(n_1, n_2, n_3) = (1, 2, 1)$. Let $l_1 \geq \cdots \geq l_p$ be the ordered eigenvalues of the Fisher matrix $S_1^{-2}S_2$. We are particularly interested in the distributions of $l_1$, $(l_{p-2}, l_{p-1})$ and $l_p$, which corresponds to the spike eigenvalues $a_1$, $a_2$ and $a_3$, respectively.

5.1. Case of $U = I_4$

In this subsection, we consider a simple case that $U = I_4$. Therefore, following Theorem 4.1, we have

- for $j = 1, p$, $\sqrt{p}(l_j - \phi(a_i)) \rightarrow N(0, \sigma_i^2)$. Here, for $j = 1$, $i = 1$, $\phi(a_1) = 42.67$ and $\sigma_1^2 = 4246.8 + 1103.5(v_4 - 3)$; and for $j = p$, $i = 3$, $\phi(a_3) = 0.07$ and $\sigma_3^2 = 7.2 \times 10^{-3} + 3.15 \times 10^{-3}(v_4 - 3)$.

- for $j = p - 2, p - 1$ and $i = 2$, the two dimensional random vector $\sqrt{p}(l_j - \phi(a_2))$ converges to the eigenvalues of the random matrix $-\frac{R_{mn}}{\Delta(\lambda_2)}$. Here, $\phi(a_2) = 0.13$, $\Delta(\lambda_2) = 1.45$ and $R_{mn}$ is the $2 \times 2$ symmetric random matrix, made with independent Gaussian entries of mean zero and variance

$$
\text{Var}(R_{mn}) = \begin{cases} 
2\theta_2 + (v_4 - 3)\omega_2 & \text{if } m = n \ , \\
\theta_2 & \text{if } m \neq n ,
\end{cases}
$$

Simulations are conducted to compare the distributions of the empirical extreme eigenvalues with their limits.

5.1.1. Gaussian case

First, we assume all the $z_{ij}$ and $w_{ij}$ are i.i.d. standard Gaussian, thus $v_4 - 3 = 0$. And according to (5.1), $R_{mn}/\sqrt{0.04}$ is the standard $2 \times 2$ Gaussian Wigner matrix (GOE). Therefore, we have
Figure 2: Upper panels show the empirical densities of $l_1$ and $l_p$ (solid lines, after centralisation and scaling) compared to their Gaussian limits (dashed lines). Lower panels show contour plots of empirical joint density function of $(l_{p-2}, l_{p-1})$ (left plot, after centralisation and scaling) and contour plots of their limits (right plot). Both the empirical and limit joint density functions are displayed using the two-dimensional kernel density estimates. Samples are from i.i.d. standard Gaussian distribution with $U = I_4$ with 1000 independent replications.
\begin{itemize}
  \item \( \sqrt{p}\{l_1 - 42.67\} \to N(0, 4246.8) \),
  \item \( \sqrt{p}\{l_p - 0.07\} \to N(0, 7.2 \times 10^{-3}) \),
  \item The two-dimensional random vector \( \sqrt{p}\{l_{p-2} - 0.13, l_{p-1} - 0.13\} \) converges to the eigenvalues of the random matrix \(-0.138 \cdot W\), here \( W \) is a \( 2 \times 2 \) GOE.
\end{itemize}

Figure 2, upper panels, show the empirical kernel density estimates (in solid lines) of \( \sqrt{p}\{l_1 - 42.67\} \) and \( \sqrt{p}\{l_p - 0.07\} \) from 1000 independent replications, compared to their Gaussian limits \( N(0, 4246.8) \) and \( N(0, 7.2 \times 10^{-3}) \), respectively (dashed lines). When considering the empirical distribution of the two-dimensional random vector \( \sqrt{p}\{l_{p-2} - 0.13, l_{p-1} - 0.13\} \), we run the two-dimensional kernel density estimation from 1000 independent replications and display their contour lines, see the lower-left panel of the figure, while the lower-right panel plot shows the contour lines of the kernel density estimation of the eigenvalues of the \( 2 \times 2 \) random matrix \(-0.138 \cdot GOE \) (their limits).

5.1.2. Binary case

Second, we assume all the \( z_{ij} \) and \( w_{ij} \) are i.i.d. binary variables taking values \( \{1, -1\} \) with probability \( 1/2 \), and in this case we have \( v_4 = 1 \). Similarly, we have
\begin{itemize}
  \item \( \sqrt{p}\{l_1 - 42.67\} \to N(0, 2039.8) \),
  \item \( \sqrt{p}\{l_p - 0.07\} \to N(0, 9 \times 10^{-4}) \),
  \item The two-dimensional random vector \( \sqrt{p}\{l_{p-2} - 0.13, l_{p-1} - 0.13\} \) converges to the eigenvalues of the random matrix \(-R_{mn}/1.45\). Here, \( R_{mn} \) is the \( 2 \times 2 \) symmetric random matrix, made with independent Gaussian entries of mean zero and variance
  \[ \text{Var}(R_{mn}) = \begin{cases} 0.008, & m = n, \\ 0.02, & m \neq n. \end{cases} \]
\end{itemize}

Figure 3, upper panels, show the empirical kernel density estimates of \( \sqrt{p}\{l_1 - 42.67\} \) and \( \sqrt{p}\{l_p - 0.07\} \) from 1000 independent replications (in solid lines), compared to their Gaussian limits (in dashed lines). Also, the lower panel on the figure show the contour lines of the empirical joint density of the \( \sqrt{p}\{l_{p-2} - 0.13, l_{p-1} - 0.13\} \) (the left plot), with the right plot displaying the contour lines of their limit.

5.2. Case of general \( U \)

In this subsection, we consider the following non unit orthogonal matrix
\[ U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}, \]
Figure 3: Upper panels show the empirical densities of $l_1$ and $l_p$ (solid lines, after centralisation and scaling) compared to their Gaussian limits (dashed lines). Lower panels show contour plots of empirical joint density function of $(l_{p-2}, l_{p-1})$ (left plot, after centralisation and scaling) and contour plots of their limits (right plot). Both the empirical and limit joint density functions are displayed using the two-dimensional kernel density estimates. Samples are from i.i.d. binary distribution with $U = I_4$ and 1000 independent replications.
i.e., we have

\[
U_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & \sqrt{2} \\ 0 & \sqrt{2} \end{pmatrix}, \quad U_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \sqrt{2} \end{pmatrix}.
\]

Since Gaussian distribution is invariant under orthogonal transformation, we only consider the case that all the \(z_{ij}\) and \(w_{ij}\) to be i.i.d. binary variables taking values \(\{1, -1\}\) with probability \(1/2\), with all the other settings fixed as in Section 5.1. Then according to Theorem 4.1, we have

- \(\sqrt{p}\{l_1 - 42.67\} \rightarrow N(0, 2039.8)\),
- \(\sqrt{p}\{l_p - 0.07\} \rightarrow N(0, 0.004)\),
- The two-dimensional random vector \(\sqrt{p}\{l_{p-2} - 0.13, l_{p-1} - 0.13\}\) converges to the eigenvalues of the random matrix \(-U_2^* R(\lambda_2) U_2/1.45\). Here, \(R(\lambda_2)\) is the \(4 \times 4\) symmetric random matrix, made with independent Gaussian entries of mean zero and variance

\[
\text{Var}(R_{mn}) = \begin{cases} 0.008, & m = n, \\ 0.02, & m \neq n. \end{cases}
\]

Figure 4, upper panels, show the empirical kernel density estimates of \(\sqrt{p}\{l_{1} - 42.67\}\) and \(\sqrt{p}\{l_{p} - 0.07\}\) from 1000 independent replications (in solid lines), compared to their Gaussian limits (in dashed lines). Also, the lower panel of the figure shows the contour lines of the empirical joint density of \(\sqrt{p}\{l_{p-2} - 0.13, l_{p-1} - 0.13\}\) (the lower-left plot), with the lower-right plot showing the contour lines of their limit.

6. Joint distribution of the outlier eigenvalues

In the previous section, we have obtained the following result for the outlier eigenvalues: the \(n_1\)-dimensional real random vector \(\sqrt{p}\{l_{p,j} - \lambda_i, j \in J_1\}\) converges to the distribution of the eigenvalues of random matrix \(-U_1^* R(\lambda_i) U_1/\Delta(\lambda_i)\). It is in fact possible to derive their joint distribution, i.e. the limit of the \(M\)-dimensional real random vector

\[
\begin{pmatrix} \sqrt{p}\{l_{p,j_1} - \lambda_1, j_1 \in J_1\} \\ \vdots \\ \sqrt{p}\{l_{p,j_k} - \lambda_k, j_k \in J_k\} \end{pmatrix}.
\]

(6.1)

Such joint convergence results are useful for inference procedures where consecutive sample eigenvalues are used such as their differences or ratios, see e.g. Onatski (2009) and Passemier and Yao (2014).

**Theorem 6.1.** Assume the same condition as in Theorem 4.1 and that all the population spikes \(a_i\) satisfy the condition \(|a_i - \gamma| > \gamma \sqrt{c + y - cy}\). Then the \(M\)-dimensional vector in
Figure 4: Upper panels show the empirical densities of $l_1$ and $l_p$ (solid lines, after centralisation and scaling) compared to their Gaussian limits (dashed lines). Lower panels show contour plots of empirical joint density function of $(l_{p-2}, l_{p-1})$ (left plot, after centralisation and scaling) and contour plots of their limits (right plot). Both the empirical and limit joint density functions are displayed using the two-dimensional kernel density estimates. Samples are from i.i.d. binary distribution with $U$ given by (5.2) and 1000 independent replications.
(6.1) converges in distribution to the eigenvalues of the $M \times M$ random matrix

$$
\begin{pmatrix}
-\frac{U_i^* R(\lambda_i) U_1}{\Delta(\lambda_1)} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & -\frac{U_i^* R(\lambda_i) U_k}{\Delta(\lambda_k)}
\end{pmatrix},
$$

(6.2)

where the matrices $R(\lambda_i)$, made with zero-mean independent Gaussian random variables, are defined in Theorem 4.1, with the following covariance function between different blocks ($l \neq s$): for $1 \leq i \leq j \leq M$,

$$
\text{Cov}(R(\lambda_l)(i,j), R(\lambda_s)(i,j)) = \begin{cases} 
\theta(l,s), & i \neq j \\
\omega(l,s)(v_4 - 3) + 2\theta(l,s), & i = j,
\end{cases}
$$

where

$$
\theta(l,s) = \lim_{n \to \infty} \frac{1}{n + T} \text{tr} A_n(\lambda_l)A_n(\lambda_s),
$$

$$
\omega(l,s) = \lim_{n \to \infty} \frac{1}{n + T} \sum_{i=1}^{n+T} A_n(\lambda_l)(i,i)A_n(\lambda_s)(i,i),
$$

and $A_n(\lambda)$ is defined in (A.16).

The proof of this theorem is very close to that of Theorem 2.3 in Wang et al. (2014), thus omitted.

In principle, the limiting parameters $\theta(l,s)$ and $\omega(l,s)$ can be completely specified for a given spiked structure. However, this will lead to quite complex formulas. Here, we prefer to explain a simple case where $\Omega_p$ is diagonal whose eigenvalues $|a_i - \gamma| > \gamma \sqrt{c + y} - cy$ ($i = 1, \cdots, M$) are all simple, we have $U = I_M$, $M = k$ and $n_i = 1$ ($i = 1, \cdots, M$). Therefore, $U_i^* R(\lambda_i) U_i$ in (6.2) reduces to the $(i,i)$-th element of $R(\lambda_i)$, which is a Gaussian random variable. Besides, from Theorem 6.1, we see that the random variables $\{R(\lambda_i)(i,i)\}_{i=1,\cdots,M}$ are jointly independent since the index sets $(i,i)$ are disjoint. Finally, we have the following joint distribution of the $M$ outlier eigenvalues of $S_2^{-1}S_1$.

**Proposition 6.1.** Under the same assumptions as in Theorem 4.1, when $\Omega_p$ is diagonal with all its eigenvalues $|a_i - \gamma| > \gamma \sqrt{c + y} - cy$ being simple, the $M$ outlier eigenvalues $l_{p,i}$ ($j = 1, \cdots, M$) of $S_2^{-1}S_1$ are asymptotically independent Gaussian:

$$
\begin{pmatrix}
\sqrt{p}(l_{p,1} - \lambda_1) \\
\vdots \\
\sqrt{p}(l_{p,M} - \lambda_M)
\end{pmatrix} \Rightarrow \mathcal{N}
\begin{pmatrix}
0_M, & \begin{pmatrix}
\sigma_1^2 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \sigma_M^2
\end{pmatrix}
\end{pmatrix},
$$

where

$$
\sigma_i^2 = \frac{2a_i^2(cy - c - y)(a_i - 1)^2(-1 + 2a_i + c + a_i^2(y - 1))}{(1 + a_i(y - 1))^4} + (v_4 - 3) \cdot \frac{a_i^2(c + y)(-1 + 2a_i + c + a_i^2(y - 1))^2}{(1 + a_i(y - 1))^4}.
$$
7. Application to large-dimensional signal detection

In this section, we develop an application of the previous results to an inference problem where spiked Fisher matrices arise naturally. In a signal detection equipment, records are of form

$$x_i = As_i + e_i, \quad i = 1, \ldots, T$$

(7.1)

where $x_i$ is $p$-dimensional, $s_i$ is a $k \times 1$ low-dimensional signal ($k \ll p$) with unit covariance matrix, $A$ a $p \times k$ mixing matrix, and $(e_i)$ is an i.i.d. noise with covariance matrix $\Sigma_2$. Therefore, the covariance matrix of $x_i$ can be considered as a $k$-dimensional perturbation of $\Sigma_2$, denoted as $\Sigma_p$ in the following. Notice that none of the quantities in the r.h.s. of (7.1) is observed. One of the fundamental problem here is to estimate $k$, the number of signals present in the system. This problem is challenging when the dimension $p$ is large, say has a comparable magnitude with the sample size $T$. When the noise has the simplest covariance structure, i.e. $\Sigma_2 = \sigma^2_e I_p$, this problem has been much investigated recently and several solutions are proposed, see e.g. Kritchman and Nadler (2008), Nadler (2010), Passemier and Yao (2012, 2014). However the problem with an arbitrary noise covariance matrix $\Sigma_2$, say diagonal to simplify, remains unsolved in the large-dimensional context (to the best of our knowledge). Nevertheless, there exists an astute engineering device where the system can be tuned in a signal-free environment, for example in laboratory: that is we can directly record a sequence of pure-noise observations $z_j$, $j = 1, \ldots, n$, which have the same distribution as the $(e_i)$ above. These signal-free records can then be used to whiten the observations $(x_i)$ as follows. Let $S_1 = T^{-1} \sum_{i=1}^{T} x_i x_i^*$, $S_2 = n^{-1} \sum_{i=1}^{n} z_i z_i^*$ and $l_i, i = 1, \ldots, p$ be the eigenvalues of $S_2^{-1} S_1$. Notice that the eigenvalues $\{l_i\}$ are invariant under the transformation $S_1 \mapsto \Sigma_2^{-1/2} S_1 \Sigma_2^{-1/2}$, $S_2 \mapsto \Sigma_2^{-1/2} S_2 \Sigma_2^{-1/2}$; they are in fact independent of $\Sigma_2$. Therefore, these eigenvalues can be thought as if $\Sigma_2 = I_p$, that is $S_2^{-1} S_1$ becomes a spiked Fisher matrix as introduced in Section 2. This is actually the reason why the two sample procedure developed here can deal with an arbitrary covariance matrix of the noise while the existing one-sample procedures cannot. Based on Theorem 3.1, we propose our estimator of the number of signals as the number of eigenvalues of $S_2^{-1} S_1$ larger than the right edge point of the support of its LSD:

$$\hat{k} = \max\{i : l_i \geq b + d_n\},$$

(7.2)

where $(d_n)$ is a sequence of vanishing constants.

**Theorem 7.1.** Assume all the spike eigenvalues $a_i$ ($i = 1, \ldots, k$) satisfy $a_i > \gamma + \gamma \sqrt{c + y - cy}$. Let $d_n$ be a sequence of positive numbers such that $d_n \to 0$, $\sqrt{p} \cdot d_n \to 0$ and $p^{2/3} \cdot d_n \to +\infty$ as $p \to +\infty$, then the estimator $\hat{k}$ is constant, i.e. $\hat{k} \to k$ in probability as $p \to +\infty$.

**Remark 7.1.** Notice here that there’s no need for those spikes $a_i$ to be simple, the only requirement is that they should be properly strong enough $(a_i > \gamma + \gamma \sqrt{c + y - cy})$ for detection.
Proof. (of Theorem 7.1). Since 
\[ \{ \hat{k} = k \} = \{ k = \max \{ i : l_i \geq b + d_n \} \} = \{ \forall j \in \{ 1, \cdots , k \}, l_j \geq b + d_n \} \cap \{ l_{k+1} < b + d_n \} , \]
we have
\[
P(\hat{k} = k) = P \left( \bigcap_{1 \leq j \leq k} \{ l_j \geq b + d_n \} \cap \{ l_{k+1} < b + d_n \} \right)
= 1 - P \left( \bigcup_{1 \leq j \leq k} \{ l_j < b + d_n \} \cup \{ l_{k+1} \geq b + d_n \} \right)
\geq 1 - \sum_{j=1}^{k} P(l_j < b + d_n) - P(l_{k+1} \geq b + d_n) . \quad (7.3)
\]
For \( j = 1, \cdots , k \),
\[
P(l_j < b + d_n) = P \left( \sqrt{p}(l_j - \phi(a_j)) < \sqrt{p}(b + d_n - \phi(a_j)) \right)
\to P \left( \sqrt{p}(l_j - \phi(a_j)) < \sqrt{p}(b - \phi(a_j)) \right) , \quad (7.4)
\]
which is due to the assumption that \( \sqrt{p} \cdot d_n \to 0 \). Then the part \( \sqrt{p}(b - \phi(a_j)) \) in (7.4) will tend to \( -\infty \) since we have always \( \phi(a_j) > b \) when \( a_i > \gamma + \gamma \sqrt{c + y - cy} \). On the other hand, by Theorem 4.1, \( \sqrt{p}(l_j - \phi(a_j)) \) in (7.4) has a limiting distribution; it is then bounded in probability. Therefore, we have
\[
P(l_j < b + d_n) \to 0 \quad \text{for } j = 1, \cdots , k . \quad (7.5)
\]
Also
\[
P(l_{k+1} \geq b + d_n) = P \left( p^{2/3}(l_{k+1} - b) \geq p^{2/3} \cdot d_n \right) ,
\]
and the part \( p^{2/3}(l_{k+1} - b) \) is asymptotically Tracy-Widom distributed (see Bao et al. (2015) where the Tracy-Widom distribution for the largest eigenvalue of general sample covariance matrix is derived). As \( p^{2/3} \cdot d_n \) tend to infinity as assumed, we have
\[
P(l_{k+1} \geq b + d_n) = 0 . \quad (7.6)
\]
Combine (7.3), (7.5) and (7.6), we have \( P(\hat{k} = k) \to 1 \) as \( p \to +\infty \). The proof of Theorem 7.1 is complete.

We conduct a short simulation to illustrate the performance of our estimator. We fix \( y = 1/2 \) and \( c = 1/5 \) as in Section 5, and the value of \( p \) varies from 50 to 250, therefore, the critical value for \( a_i \) in the model (2.4) (after whitening) is \( a_i > \gamma\{1 + \sqrt{c + y - cy}\} = 3.55 \). For each given pair of \( (p, n, T) \), we repeat 1000 times. The tuning parameter \( d_n \) is chosen to be \((\log \log p)/p^{2/3}\).
Next, suppose $k = 3$ and $A$ is a $p \times 3$ matrix of form $A = (\sqrt{c_1}v_1, \sqrt{c_2}v_2)$, where $c_1 = 10$, $c_2 = 5$,

$$v_1 = (1 \ 0 \ \cdots \ 0)^* \quad \text{and} \quad v_2 = \begin{pmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 & \cdots & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} & 0 & \cdots & 0 \end{pmatrix}^*.$$

Besides, assume $\text{Cov}(s_i) = I_k$. In this setting, we have two spike eigenvalues $c_1 = 10$, $c_2 = 5$ (before whitening) with multiplicity $n_1 = 1$, $n_2 = 2$, respectively. Finally, we choose $\text{Cov}(e_i)$ to be either diagonal or non-diagonal as below.

| Table 1 | Frequency of our estimator in Model 1. |
|---------|--------------------------------------|
| $p$     | 50 100 150 200 250                    |
| $n$     | 100 200 300 400 500                   |
| $T$     | 250 500 750 1000 1250                 |
| $\hat{k} = 1$ | 0.038 0.003 0 0 0.001     |
| $\hat{k} = 2$ | 0.578 0.317 0.166 0.103 0.047       |
| $\hat{k} = 3$ | **0.381** 0.675 **0.818** 0.883 **0.937** |
| $\hat{k} = 4$ | 0.003 0.005 0.016 0.014 0.015       |

- For Model 1: set $\text{Cov}(e_i) = \text{diag}(1, \cdots, 1, 2, \cdots, 2)$ . In this case, we have the three non-zero eigenvalues of $(c_1 v_1 v_1^* + c_2 v_2 v_2^*) \cdot [\text{Cov}(e_i)]^{-1}$ equal 10, 5, 5, respectively, which are all larger than the critical value $3.55 - 1$, therefore, the number of detectable signals is three;

- For Model 2: set $\text{Cov}(e_i)$ be compound symmetric with all the diagonal elements equal 1 and all the off-diagonal elements equal 0.1. In this case, we have for each given $p$, the three non-zero eigenvalues of $(c_1 v_1 v_1^* + c_2 v_2 v_2^*) \cdot [\text{Cov}(e_i)]^{-1}$ are all larger than $5.36 (> 3.55 - 1)$. The number of detectable signals is again three.

| Table 2 | Frequency of our estimator in Model 2. |
|---------|--------------------------------------|
| $p$     | 50 100 150 200 250                    |
| $n$     | 100 200 300 400 500                   |
| $T$     | 250 500 750 1000 1250                 |
| $\hat{k} = 1$ | 0.016 0 0 0 0     |
| $\hat{k} = 2$ | 0.475 0.186 0.053 0.028 0.008       |
| $\hat{k} = 3$ | **0.505** 0.806 **0.926** 0.950 **0.971** |
| $\hat{k} = 4$ | 0.004 0.008 0.021 0.022 0.021       |

Tables 1 and 2 report the empirical frequency of our estimator $\hat{k} = 1, 2, 3, 4$ in Model 1 and Model 2, respectively, where the true number of signals is $k = 3$. Also, Figure 5 shows more clearly the trends of the frequency of correct estimation in both cases. We can see the frequency both increase as $p$ gets larger, which confirms the consistency of our estimator.
Figure 5: Frequency of true estimation $\hat{k} = k = 3$.

References

Bai, Z. (2005). High dimensional data analysis. *Cosmos*, 1(1), 17–27.
Bai, Z. and Ding, X. (2012). Estimation of spiked eigenvalues in spiked models. *Random Matrices Theory Appl.*, 1(2), 1150011, 21.
Bai, Z. and Yao, J. (2008). Central limit theorems for eigenvalues in a spiked population model. *Ann. Inst. Henri Poincaré Probab. Stat.*, 44(3), 447–474.
Bai, Z. and Yao, J. (2012). On sample eigenvalues in a generalized spiked population model. *J. Multivariate Anal.*, 106, 167–177.
Bai, Z., Yin, Y.Q. and Krishnaiah P.R., (1987). On the limiting empirical distribution function of the eigenvalues of a multivariate $F$-matrix. *Probab. Theo. Appli.* 32, 490–500.
Bai, J., Ben Arous, G., and Péché, S. 2005. Phase transition of the largest eigenvalue for nonnull complex sample covariance matrices. *Ann. Probab.*, 33(5), 1643–1697.
Baik, J. and Silverstein, J.W. (2006). Eigenvalues of Large Sample Covariance Matrices of Spiked Population Models. *J. Multivariate Anal.*, 97, 1382–1408.
Bao, Z. G., Pan, G.M. and Zhou, W. 2014. Universality for the largest eigenvalue of sample covariance matrices with general population. *Ann. Statistics*, 43(1), 382–421.
Benaych-Georges, F., Guionnet, A. and Maida, M. (2011). Fluctuations of the extreme eigenvalues of finite rank deformations of random matrices. *Electron. J. Probab.* 16(60), 1621–1662.
Benaych-Georges, F. and Nadakuditi, R.R. (2011). The eigenvalues and eigenvectors of
finite, low rank perturbations of large random matrices. *Adv. Math.*, **227**(2), 494–521.

Capitaine, M., Donati-Martin, C. and Féra, D. (2009). The largest eigenvalues of finite rank deformation of large Wigner matrices: convergence and nonuniversality of the fluctuations. *Ann. Probab.*, **37**(1), 1–47.

Dharmawansa, P., Johnstone, I.M. and Onatski, A. (2014). Local asymptotic normality of the spectrum of high-dimensional spiked F-ratios. *Preprint*, available at arXiv:1411.3875.

Féra, D. and Péché, S. (2007). The largest eigenvalue of rank one deformation of large Wigner matrices. *Comm. Math. Phys.*, **272**(1), 185–228.

Hu, J. and Bai, Z.D. (2014). Strong representation of weak convergence. *Science China (Mathematics)* **57**(11), 2399-2406.

Johnstone, I. (2001). On the distribution of the largest eigenvalue in principal components analysis. *Ann. Statistics*, **29**(2), 295–327.

Johnstone, I. (2007). High dimensional statistical inference and random matrices. Pages 307–333 of: *International Congress of Mathematicians. Vol. I*. Eur. Math. Soc., Zürich.

Johnstone, I. and Titterington, D. (2009). Statistical challenges of high-dimensional data. *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, **367**(1906), 4237–4253.

Kritchman, S. and Nadler, B. (2008). Determining the number of components in a factor model from limited noisy data. *Chem. Int. Lab. Syst.*, **94**, 19–32.

Nadler, B. (2010). Nonparametric detection of signals by information theoretic criteria: performance analysis and an improved estimator. *IEEE Trans. Signal Process.*, **58**(5), 2746–2756.

Onatski, A. (2009). Testing hypotheses about the numbers of factors in large factor models. *Econometrica* **77**, 1447–1479.

Passemier, D. and Yao, J. (2012). On determining the number of spikes in a high-dimensional spiked population model. *Random Matrix: Theory and Applications*, doi, 10.1142/S201032631150002X.

Passemier, D. and Yao, J. 2014. On the detection of the number of spikes, possibly equal, in the high-dimensional case. *J. Multivariate Analysis*, **127**, 173–183.

Paul, D. (2007). Asymptotics of sample eigenstructure for a large dimensional spiked covariance mode. *Statistica Sinica*, **17**, 1617–1642.

Paul, D. and Aue, A. (2014). Random matrix theory in statistics: A review. *J. Statist. Planning and Inference* **150**, 1-29.

Péché, S. (2006). The largest eigenvalue of small rank perturbations of Hermitian random matrices. *Probab. Theory Related Fields*, **134**(1), 127–173.

Pizzo, A., Renfrew, D. and Soshnikov, A. (2013). On finite rank deformations of Wigner matrices. *Ann. Inst. Henri Poincaré Probab. Stat.*, **49**(1), 64–94.

Renfrew, D., and Soshnikov, A. (2013). On finite rank deformations of Wigner matrices II: Delocalized perturbations. *Random Matrices Theory Appl.*, **2**(1), 1250015, 36.

Silverstein, J. W. (1985). The limiting eigenvalue distribution of a multivariate F matrix. *SIAM J. Math. Anal.* **16**(3), 641–646.

Skorokhod A. V. (1956). Limit theorems for stochastic processes. *Theory Probab. Appli.*, **1**, 261–290.

Wachter K. W. (1980). The limiting empirical measure of multiple discriminant ratios.
Lemma A.1. Let $R$ be a $M \times M$ real-valued matrix, $U = (U_1 \cdots U_k)$ and $V = (V_1 \cdots V_k)$ are two orthogonal bases of some subspace $E \subseteq \mathbb{R}^M$ of dimension $M$, where both $U_i$ and $V_i$ are of size $M \times n_i$, satisfying $n_1 + \cdots + n_k = M$. Then the two $n_i \times n_i$ matrices $U_i^*RU_i$ and $V_i^*RV_i$ have the same eigenvalues.

Proof. (of Lemma A.1) It is sufficient to prove that there exists an $n_i \times n_i$ orthogonal matrix $A$, such that $V_i = U_i \cdot A$. If it is true, then $V_i^*RV_i = A^*(U_i^*RU_i)A$, and since $A$ is orthogonal, we have the eigenvalues of $V_i^*RV_i$ and $U_i^*RU_i$ are the same. Now let $U_i = (u_1 \cdots u_{n_i})$ and $V_i = (v_1 \cdots v_{n_i})$. Define $A = (a_{is})_{1 \leq i,s \leq n_i}$, such that

\[
\begin{align*}
  v_1 &= a_{11}u_1 + \cdots + a_{n_1,i}u_{n_1} \\
  \vdots \\
  v_{n_i} &= a_{1n_i}u_1 + \cdots + a_{n_i,i}u_{n_i}
\end{align*}
\]

Put in matrix form:

\[
\begin{pmatrix} v_1 & \cdots & v_{n_i} \end{pmatrix} = \begin{pmatrix} u_1 & \cdots & u_{n_i} \end{pmatrix} \begin{pmatrix} a_{11} & \cdots & a_{1n_i} \\
  \vdots & \ddots & \vdots \\
  0 & \cdots & a_{n_i,n_i} \end{pmatrix},
\]

i.e. $V_i = U_i \cdot A$. Since $< v_i, v_j > = < a_{i}, a_{j} >$ by orthogonality of $\{u_j\}$, where $a_k = (a_{ik})_{1 \leq k \leq n_i}$, therefore, the matrix $A$ is orthogonal. \hfill $\square$

Lemma A.2. Suppose $X = (x_1, \cdots, x_n)$ is a $p \times n$ matrix, whose columns $\{x_i\}$ are independent random vectors. $Y = (y_1, \cdots, y_n)$ is also similarly defined. Let $\Sigma_p$ be the covariance matrix of $x_i$ and $y_i$, $A$ is a deterministic matrix, then we have

\[XAY^* \rightarrow tr A \cdot \Sigma_p.\]

Moreover, if $A$ is random but independent of $X$ and $Y$, then we have

\[XAY^* \rightarrow \mathbb{E} tr A \cdot \Sigma_p.\] (A.1)
Proof. We consider the \((i,j)\)-th entry of \(XAY^*\):

\[XAY^*(i,j) = \sum_{k,l=1}^{n} X(i,k)A(k,l)Y^*(l,j) = \sum_{k,l=1}^{n} X_i k Y_j A_{k l} . \tag{A.2}\]

Since \(X_i k Y_j \rightarrow \Sigma_p(i,j)\) when \(k = l\). Therefore, the right hand side of (A.2) tends to \(\Sigma_p(i,j) \cdot \sum_{k=1}^{n} A_{k k}\), which is equivalent to say that

\[XAY^* \rightarrow \text{tr} A \cdot \Sigma_p .\]

Then (A.1) is simply due to the conditional expectation. The proof of Lemma A.2 is complete. \(\square\)

In all the following, \(\lambda\) refers to the outlier limit that

\[\lambda = \frac{a(a - 1 + c)}{a - 1 - ay} .\]

Lemma A.3. We have

\[s(\lambda) = \frac{a(y - 1) + 1}{(a - 1)(a + c - 1)} ,\]
\[m_1(\lambda) = \frac{(a(y - 1) + 1)^2(-1 + 2a + a^2(y - 1) + y(c - 1))}{(a - 1)^2(a + c - 1)^2(-1 + 2a + c + a^2(y - 1))} ,\]
\[m_2(\lambda) = \frac{1}{a - 1} ,\]
\[m_3(\lambda) = \frac{-(a(y - 1) + 1)^2}{(a - 1)^2(-1 + 2a + c + a^2(y - 1))} ,\]
\[m_4(\lambda) = \frac{-1 + 2a + c + a^2(-1 + c(y - 1))}{(a - 1)^2(-1 + 2a + c + a^2(y - 1))} .\]

Proof. (sketch of the proof of Lemma A.3) In this short proof, we skip all the detailed calculations. Recall the definition of \(s(z)\) in (3.13), its value at \(\lambda\) is

\[s(\lambda) = \frac{a(y - 1) + 1}{(a - 1)(a + c - 1)} . \tag{A.3}\]

Also, (3.13) says that \(s(z)\) is the solution of the following equation:

\[z(c + zy)s^2(z) + (c(z(1 - y) + 1 - c) + 2zy)s(z) + c + y - cy = 0 . \tag{A.4}\]

Taking derivatives on both sides of (A.4) and combing with (A.3) will give the value of \(s'(\lambda)\). On the other hand, since it holds

\[s(z) + \frac{1}{z}(1 - c) = cs(z) , \tag{A.5}\]

see (3.12), taking derivatives on both sides again will give the value of \(s'(\lambda)\). Finally, the above five values is just a combination of \(s(\lambda)\) and \(s'(\lambda)\).

The proof of Lemma A.3 is complete. \(\square\)
Lemma A.4. Under assumptions (a)-(d),
\[
\frac{1}{p} \text{tr} \left\{ \left( \lambda \cdot \frac{1}{n} Z_2 Z_2^* - \frac{1}{T} X_2 X_2^* \right)^{-1} \right\} \xrightarrow{a.s.} \frac{1}{a + c - 1}.
\]

Proof. (of Lemma A.4) We first fix \( \frac{1}{n} Z_2 Z_2^* \), then we can use the result in Zheng et al. (2013) (Lemma 4.3), which says that
\[
\frac{1}{p} \text{tr} \left( \frac{1}{z} \cdot \frac{1}{n} Z_2 Z_2^* - \frac{1}{T} X_2 X_2^* \right)^{-1} \rightarrow \tilde{m}(z), \quad \text{a.s.}
\]

where \( \tilde{m}(z) \) is the unique solution to the equation
\[
\tilde{m}(z) = \int \frac{1}{x - \frac{1}{1 - c \tilde{m}(z)}} dF_y(x)
\]  
(A.6)
satisfying
\[
\Im(z) \cdot \Im(\tilde{m}(z)) \geq 0,
\]
where \( F_y(x) \) is the LSD of \( \frac{1}{n} Z_2 Z_2^* \) (deterministic), which is the standard M-P law with parameter \( y \). Besides, if we denote its Stieltjes transform as \( s(z) := \int \frac{1}{x - z} dF_y(x) \), then (A.6) could be written as
\[
\tilde{m}(z) = \int \frac{z}{x - \frac{1}{1 - c \tilde{m}(z)}} dF_y(x) = z \cdot s \left( \frac{z}{1 - c \tilde{m}(z)} \right).
\]  
(A.7)

Since we know that the Stieltjes transform of the LSD of a standard sample covariance matrix satisfies:
\[
s(z) = \frac{1}{1 - y - y z s(z) - z},
\]  
(A.8)
then we bring (A.7) into (A.8) leads to
\[
\frac{\tilde{m}(z)}{z} = \frac{1}{1 - y - y \cdot \frac{z}{1 - c \tilde{m}(z)}}, \quad \tilde{m}(z) - \frac{z}{1 - c \tilde{m}(z)} = \frac{z}{1 - c \tilde{m}(z)},
\]
whose nonnegative solution is unique, which is
\[
\tilde{m}(z) = \frac{-1 + y + z - z c + \sqrt{(1 - y - z + z c)^2 + 4 z (y c - y - c)}}{2 (y c - y - c)}.
\]  
(A.9)

Therefore, we have for fixed \( \frac{1}{n} Z_2 Z_2^* \),
\[
\frac{1}{p} \text{tr} \left( \lambda \cdot \frac{1}{n} Z_2 Z_2^* - \frac{1}{T} X_2 X_2^* \right)^{-1} \rightarrow \tilde{m} \left( \frac{1}{\lambda} \right) = \frac{1}{a + c - 1}
\]
almost surely. Finally, due to the fact that for each \( \omega \), the ESD of \( \frac{1}{n}Z_2Z_2^*(\omega) \) will tend to the same limit (standard M-P distribution), which is independent of the choice of \( \omega \). Therefore, we have for all \( \frac{1}{n}Z_2Z_2^* \) (not necessarily deterministic but independent of \( \frac{1}{T}X_2X_2^* \)),

\[
\frac{1}{p} \text{tr} \left( \lambda \cdot \frac{1}{n}Z_2Z_2^* - \frac{1}{T}X_2X_2^* \right)^{-1} \to \frac{1}{a + c - 1}
\]

almost surely.

The proof of Lemma A.4 is complete. \( \square \)

\textbf{Lemma A.5.} \( A(\lambda), B(\lambda), C(\lambda) \) and \( D(\lambda) \) are defined in (4.5), then

\[
(l - \lambda) \cdot \frac{1}{n}Z_1A(l)Z_1^* \to (l - \lambda) \cdot (1 + y\lambda s(\lambda)) \cdot I_M , \quad (A.10)
\]

\[
\frac{\lambda}{n}Z_1[A(l) - A(\lambda)]Z_1^* \to (l - \lambda) \cdot (\lambda y s(\lambda) + \lambda^2 y m_{1}(\lambda)) \cdot I_M , \quad (A.11)
\]

\[
\frac{1}{T}X_1(B(l) - B(\lambda))X_1^* \to -(l - \lambda) \cdot cm_{3}(\lambda) \cdot U \begin{pmatrix} a_1 \\ \vdots \\ a_k \end{pmatrix} U^* , \quad (A.12)
\]

\[
\frac{1}{n}Z_1(C(l) - C(\lambda))X_1^* + \frac{l - \lambda}{n}Z_1C(\lambda)X_1^* \to (l - \lambda) \cdot 0_{M \times M} , \quad (A.13)
\]

\[
\frac{l}{T}X_1(D(l) - D(\lambda))Z_1^* + \frac{l - \lambda}{T}X_1D(\lambda)Z_1^* \to (l - \lambda) \cdot 0_{M \times M} . \quad (A.14)
\]

\textbf{Proof.} (of Lemma A.5)

\textbf{Proof of (A.10):} Since \( Z_1 \) is independent of \( A \) and \( \text{Cov}(Z_1) = I_M \), we combine this fact with Lemma A.2:

\[
(l - \lambda) \cdot \frac{1}{n}Z_1A(l)Z_1^* \to (l - \lambda) \cdot \frac{1}{n} \text{E tr } A(l) \cdot I_M . \quad (A.15)
\]

Considering the expression of \( A(l) \), we have

\[
\frac{1}{n} \text{E tr } A(\lambda) = \frac{1}{n} \text{E tr } \left[ I_n - Z_2^*(\lambda I_p - S)^{-1}\left( \frac{1}{n}Z_2Z_2^* \right)^{-1}\lambda Z_2 \right]
\]

\[
= 1 - \frac{\lambda}{n} \text{E tr } (\lambda I_p - S)^{-1}
\]

\[
= 1 - y\lambda \int \frac{1}{\lambda - x} dF_{c,y}(x)
\]

\[
= 1 + y\lambda s(\lambda) .
\]

Therefore, combine with (A.15), we have

\[
(l - \lambda) \cdot \frac{1}{n}Z_1A(l)Z_1^* \to (l - \lambda)(1 + y\lambda s(\lambda)) \cdot I_M .
\]
Proof of (A.11): Bringing the expression of $A(l)$ into consideration, we first have

$$A(l) - A(\lambda) = Z_2^*(\lambda I_p - S)^{-1}\left(\frac{1}{n}Z_2Z_2^*\right)^{-1}\lambda Z_2 - Z_2^*(UI_p - S)^{-1}\left(\frac{1}{n}Z_2Z_2^*\right)^{-1}\frac{1}{n}Z_2$$

Then using Lemma A.2 for the same reason, we have

$$\frac{\lambda}{n}Z_1[A(l) - A(\lambda)]Z_1^* \rightarrow \frac{\lambda}{n}\{\mathbb{E} \text{tr}(A(l) - A(\lambda))\} \cdot I_M,$$

and

$$\frac{1}{n} \mathbb{E} \text{tr}(A(l) - A(\lambda)) = (l - \lambda) \cdot \left[ -\frac{1}{n} \mathbb{E} \text{tr}\left\{Z_2^*(\lambda I_p - S)^{-1}\left(\frac{1}{n}Z_2Z_2^*\right)^{-1}\frac{1}{n}Z_2\right\} \right. + \frac{1}{n} \mathbb{E} \text{tr}\left\{Z_2^*(\lambda I_p - S)^{-1}(UI_p - S)^{-1}\left(\frac{1}{n}Z_2Z_2^*\right)^{-1}\frac{1}{n}Z_2\right\} \right]$$

$$= (l - \lambda) \cdot \left[ -\frac{1}{n} \mathbb{E} \text{tr}(\lambda I_p - S)^{-1} + \frac{\lambda}{n} \mathbb{E} \text{tr}(\lambda I_p - S)^{-2} + o(1) \right]$$

$$= (l - \lambda) \cdot \left[ y \int \frac{1}{x - \lambda} dF_{c, y}(x) + \lambda y \int \frac{1}{(\lambda - x)^2} dF_{c, y}(x) + o(1) \right]$$

$$= (l - \lambda) \cdot \left[ y\lambda s(\lambda) + \lambda^2ym_1(\lambda) + o(1) \right].$$

Therefore, we have

$$\frac{\lambda}{n}Z_1[A(l) - A(\lambda)]Z_1^* \rightarrow (l - \lambda) \cdot (y\lambda s(\lambda) + \lambda^2ym_1(\lambda)) \cdot I_M.$$

Proof of (A.12):

First recall the fact that $\text{Cov}(X_1) = U \begin{pmatrix} a_1 \\ \vdots \\ a_k \end{pmatrix} U^*$ and $X_1$ is independent of $B$. Using Lemma A.2, we have

$$\frac{1}{T}X_1(B(l) - B(\lambda))X_1^* \rightarrow \frac{1}{T} \mathbb{E} \text{tr}(B(l) - B(\lambda)) \cdot U \begin{pmatrix} a_1 \\ \vdots \\ a_k \end{pmatrix} U^*.$$
The part

\[ \frac{1}{T} \mathbb{E} \text{tr}(B(l) - B(\lambda)) \]
\[ = \frac{1}{T} \mathbb{E} \text{tr} \left\{ X_2^* \left[ (lI_p - S)^{-1} - (\lambda I_p - S)^{-1} \right] \left( \frac{1}{n} Z_2 Z_2^* \right)^{-1} \frac{1}{T} X_2 \right\} \]
\[ = (l - \lambda) \cdot \left(-\frac{1}{T} \mathbb{E} \text{tr} \left\{ (\lambda I_p - S)^{-2} S \right\} + o(1) \right) \]
\[ = (l - \lambda) \cdot \left(-c \int \frac{x}{(\lambda - x)^2} dF_{c,y}(x) + o(1) \right) \]
\[ = (l - \lambda) \cdot (-cm_3(\lambda) + o(1)) . \]

Therefore, we have

\[ \frac{1}{T} X_1^* (B(l) - B(\lambda)) X_1 \rightarrow -c(l - \lambda)m_3(\lambda) \cdot U \begin{pmatrix} a_1 \\ \vdots \\ a_k \end{pmatrix} U^*. \]

**Proof of (A.13) and (A.14):** (A.13) and (A.14) are derived simply due to the fact that Cov\(\{x_1, z_1\} = 0_{M\times M} .\) 
The proof of Lemma A.5 is complete. **□**

**Lemma A.6.** Define

\[ \tilde{R}_n(\lambda_i) := \begin{pmatrix} Z_1 & S_1 \end{pmatrix} \begin{pmatrix} \frac{\lambda_i \sqrt{n} A(\lambda_i)}{\theta_i} & \frac{\lambda_i \sqrt{n} \sqrt{p} C(\lambda_i)}{\theta_i} \\ \frac{\lambda_i \sqrt{n} \sqrt{p} D(\lambda_i)}{\theta_i} & -\frac{\lambda_i \sqrt{n} \sqrt{p} B(\lambda_i)}{\theta_i} \end{pmatrix} \begin{pmatrix} Z_1^* \\ S_1^* \end{pmatrix} - \mathbb{E}[] . \]

then \( \tilde{R}_n(\lambda_i) \) weakly converges to a \( M \times M \) symmetric random matrix \( R(\lambda_i) = (R_{mn}) \), which is made with independent Gaussian entries of mean zero and variance

\[ \text{Var}(R_{mn}) = \begin{cases} 2\theta_i + (v_4 - 3)\omega_i, & m = n, \\ \theta_i, & m \neq n, \end{cases} \]

where

\[ \omega_i = \frac{a_i^2 (a_i + c - 1)^2 (c + y)}{(a_i - 1)^2}, \]
\[ \theta_i = \frac{a_i^2 (a_i + c - 1)^2 (cy - c - y)}{-1 + 2a_i + c + a_i^2(y - 1)}. \]

**Proof.** Since \( Z_1 \) and \( S_1 \) are independent, having the same first four moments, both are made with i.i.d. components, we can now view \( \begin{pmatrix} Z_1 & S_1 \end{pmatrix} \) as a \( M \times (n + T) \) table \( \xi \), made
with i.i.d elements of mean 0 and variance 1. Besides, we can rewrite the expression of $A(\lambda)$, $B(\lambda)$, $C(\lambda)$ and $D(\lambda)$ as follows:

$$
A(\lambda) = I_n - Z_2^* \left( \frac{1}{n} Z_2 Z^*_2 - \frac{1}{T} X_2 X^*_2 \right)^{-1} \frac{1}{n} Z_2 ,
$$

$$
B(\lambda) = I_T + X_2^* \left( \frac{1}{n} Z_2 Z^*_2 - \frac{1}{T} X_2 X^*_2 \right)^{-1} \frac{1}{T} X_2 ,
$$

$$
C(\lambda) = Z_2^* \left( \frac{1}{n} Z_2 Z^*_2 - \frac{1}{T} X_2 X^*_2 \right)^{-1} \frac{1}{n} Z_2 .
$$

$$
D(\lambda) = X_2^* \left( \frac{1}{n} Z_2 Z^*_2 - \frac{1}{T} X_2 X^*_2 \right)^{-1} \frac{1}{n} Z_2 .
$$

It holds

$$
A(\lambda)^* = A(\lambda) , \quad B(\lambda)^* = B(\lambda) , \quad T \cdot C(\lambda)^* = n \cdot D(\lambda) ,
$$

therefore, the matrix

$$
\begin{pmatrix}
\frac{\lambda_i \sqrt{\beta} A(\lambda_i)}{n} & \frac{\lambda_i \sqrt{\alpha p C(\lambda_i)}}{n} \\
\frac{\lambda_i \sqrt{\alpha p D(\lambda_i)}}{n} & -\frac{\lambda_i \sqrt{\beta B(\lambda_i)}}{n}
\end{pmatrix}
$$

is symmetric. Define

$$
A_n(\lambda_i) = \sqrt{n + T} \cdot \begin{pmatrix}
\frac{\lambda_i \sqrt{\beta} A(\lambda_i)}{n} & \frac{\lambda_i \sqrt{\alpha p C(\lambda_i)}}{n} \\
\frac{\lambda_i \sqrt{\alpha p D(\lambda_i)}}{n} & -\frac{\lambda_i \sqrt{\beta B(\lambda_i)}}{n}
\end{pmatrix} .
$$

(A.16)

Now we can apply the results in Bai and Yao (2008) (Proposition 3.1 and Remark 1), which says that $\tilde{R}_n(\lambda_i)$ weakly converges to a $M \times M$ symmetric random matrix $R(\lambda_i) = (R_{mn})$, which is made with i.i.d. Gaussian entries of mean zero and variance

$$
\mathbb{V} \text{ar}(R_{mn}) = \begin{cases}
2\theta_i + (\nu_4 - 3)\omega_i , & m = n , \\
\theta_i , & m \neq n ,
\end{cases}
$$

The following is devoted to the calculation of the values of $\theta_i$ and $\omega_i$.

**Calculating of $\theta_i$:** From the definition of $\theta$ (see Bai and Yao (2008) for details), we have

$$
\theta_i = \lim \frac{1}{n + T} \operatorname{tr} A_n^2(\lambda_i)
$$

$$
= \lim \operatorname{tr} \left( \frac{\lambda_i \sqrt{\beta} A(\lambda_i)}{n} \frac{\lambda_i \sqrt{\alpha p C(\lambda_i)}}{n} \right) \left( \frac{\lambda_i \sqrt{\beta} A(\lambda_i)}{n} \frac{\lambda_i \sqrt{\alpha p C(\lambda_i)}}{n} \right)
$$

$$
= \lim \operatorname{tr} \left( \frac{p\lambda_i^2}{n^2} A^2(\lambda_i) + \frac{\lambda_i^2 a p}{n T} C(\lambda_i) D(\lambda_i) \right)
$$

$$
= \lim \left[ \frac{p\lambda_i^2}{n^2} \operatorname{tr} A^2(\lambda_i) + \frac{2\lambda_i^2 a p}{n T} \operatorname{tr} C(\lambda_i) D(\lambda_i) + \frac{a^2 p}{T^2} \operatorname{tr} B^2(\lambda_i) \right].
$$

(A.17)
Calculating of $A$

In the following, we will show that

\[ \text{tr} A^2(\lambda_i) = \text{tr} \left[ I_n + Z_2^* (\lambda_i I_p - S)^{-1} \left( \frac{1}{n} Z_2 Z_2^* \right)^{-1} \lambda_i \frac{1}{n} Z_2 Z_2^* (\lambda_i I_p - S)^{-1} \left( \frac{1}{n} Z_2 Z_2^* \right)^{-1} \lambda_i \frac{1}{n} Z_2 \right] - 2 \text{tr} (\lambda_i I_p - S)^{-1} \left( \frac{1}{n} Z_2 Z_2^* \right)^{-1} \frac{1}{n} \lambda_i \frac{1}{n} Z_2 \]

\[ = n + \lambda_i^2 \text{tr} (\lambda_i I_p - S)^{-2} - 2 \lambda_i \text{tr} (\lambda_i I_p - S)^{-1} \]

\[ = n + p \lambda_i^2 m_1(\lambda_i) + 2 p \lambda_i s(\lambda_i), \tag{A.18} \]

\[ \text{tr} C(\lambda_i) D(\lambda_i) = \text{tr} \left\{ Z_2^* (\lambda_i I_p - S)^{-1} \left( \frac{1}{n} Z_2 Z_2^* \right)^{-1} \frac{1}{T} X_2 X_2^* (\lambda_i I_p - S)^{-1} \left( \frac{1}{n} Z_2 Z_2^* \right)^{-1} \frac{1}{n} Z_2 \right\} \]

\[ = \text{tr} (\lambda_i I_p - S)^{-1} S (\lambda_i I_p - S)^{-1} = pm_3(\lambda_i) \tag{A.19} \]

\[ \text{tr} B^2(\lambda_i) = \text{tr} \left[ I_T + Z_2^* (\lambda_i I_p - S)^{-1} \left( \frac{1}{n} Z_2 Z_2^* \right)^{-1} \frac{1}{T} X_2 X_2^* (\lambda_i I_p - S)^{-1} \left( \frac{1}{n} Z_2 Z_2^* \right)^{-1} \frac{1}{T} X_2 \right] + 2 X_2^* (\lambda_i I_p - S)^{-1} \left( \frac{1}{n} Z_2 Z_2^* \right)^{-1} \frac{1}{T} X_2 \]

\[ = T + \text{tr} (\lambda_i I_p - S)^{-1} F (\lambda_i I_p - S)^{-1} S + 2 \text{tr} (\lambda_i I_p - S)^{-1} S \]

\[ = T + pm_4(\lambda_i) + 2 pm_2(\lambda_i), \tag{A.20} \]

Combining (A.17), (A.18), (A.19) and (A.20), we have

\[ \theta_i = \lambda_i^2 y (1 + y \lambda_i^2 m_1(\lambda_i) + 2 y \lambda_i s(\lambda_i)) + 2 \lambda_i^2 a_i c y m_3(\lambda_i) + a_i^2 c (1 + cm_4(\lambda_i) + 2 cm_2(\lambda_i)) \]

\[ = \frac{a_i^2 (a_i + c - 1)^2 (cy - c - y)}{-1 - 2a_i + c + a_i^2 (y - 1)}. \]

Calculating of $\omega_i$:

\[ \omega_i = \lim_{n \to \infty} \frac{1}{n+T} \sum_{i=1}^{n+T} (A_n(\lambda_i)(i,i))^2 = \lim \left[ \sum_{i=1}^{n} \frac{\lambda_i^2 p}{n^2} A^2(i,i) + \sum_{i=1}^{T} \frac{a_i^2 p}{T^2} B^2(i,i) \right]. \tag{A.21} \]

In the following, we will show that $A(i,i)$ and $B(i,i)$ both tend to some limits that is independent of $i$.

\[ A(i,i) = 1 - \left[ Z_2^* \left( \lambda_i I_p - \left( \frac{1}{n} Z_2 Z_2^* \right)^{-1} \frac{1}{T} X_2 X_2^* \right)^{-1} \left( \frac{1}{n} Z_2 Z_2^* \right)^{-1} \lambda_i \frac{1}{n} Z_2 \right](i,i) \]

\[ = 1 - \frac{\lambda_i}{n} \left[ Z_2^* \left( \lambda_i \frac{1}{n} Z_2 Z_2^* - \frac{1}{T} X_2 X_2^* \right)^{-1} Z_2 \right](i,i) \tag{A.22} \]
If we denote $\eta_i$ as the $i$-th column of $Z_2$, we have
\[
\frac{1}{n} Z_2 Z_2^* = \frac{1}{n} (\eta_1 \cdots \eta_n) \begin{pmatrix} \eta_1^* \\ \vdots \\ \eta_n^* \end{pmatrix} = \frac{1}{n} \eta_i \eta_i^* + \frac{1}{n} Z_{2i} Z_{2i}^*,
\]
where $Z_{2i}$ is independent of $\eta_i$. Since
\[
\left( \frac{1}{n} Z_2 Z_2^* - \frac{1}{T} X_2 X_2^* \right)^{-1} - \left( \frac{1}{n} Z_2 Z_2^* - \frac{1}{T} X_2 X_2^* \right)^{-1}
= - \left( \frac{1}{n} Z_2 Z_2^* - \frac{1}{T} X_2 X_2^* \right)^{-1} \frac{\lambda_i}{n} \eta_i \eta_i^* \left( \frac{1}{n} Z_{2i} Z_{2i}^* - \frac{1}{T} X_2 X_2^* \right)^{-1},
\]
we have
\[
\left( \frac{1}{n} Z_2 Z_2^* - \frac{1}{T} X_2 X_2^* \right)^{-1} = \frac{\left( \frac{1}{n} Z_2 Z_2^* - \frac{1}{T} X_2 X_2^* \right)^{-1}}{1 + \frac{\lambda_i}{n} \eta_i \left( \frac{1}{n} Z_{2i} Z_{2i}^* - \frac{1}{T} X_2 X_2^* \right)^{-1} \eta_i}. \tag{A.23}
\]
Bringing (A.23) into (A.22),
\[
A(i, i) = 1 - \frac{\lambda_i}{n} \eta_i \left[ \frac{1}{n} Z_2 Z_2^* - \frac{1}{T} X_2 X_2^* \right]^{-1} \eta_i \\
= 1 - \frac{\lambda_i}{n} \eta_i \left( \frac{1}{n} Z_{2i} Z_{2i}^* - \frac{1}{T} X_2 X_2^* \right)^{-1} \eta_i \\
= \frac{1}{1 + \frac{\lambda_i}{n} \eta_i \left( \frac{1}{n} Z_{2i} Z_{2i}^* - \frac{1}{T} X_2 X_2^* \right)^{-1} \eta_i},
\]
whose denominator of (A.24) equals
\[
1 + \frac{\lambda_i}{n} \text{tr} \left( \frac{1}{n} Z_{2i} Z_{2i}^* - \frac{1}{T} X_2 X_2^* \right)^{-1} \eta_i \eta_i^*. \tag{A.24}
\]
Since $\eta_i$ is independent of $\left( \frac{1}{n} Z_{2i} Z_{2i}^* - \frac{1}{T} X_2 X_2^* \right)^{-1}$, (A.24) converges to the value $1 + \lambda_i y \cdot \frac{1}{a_i + c - 1}$ according to Lemma A.4. Therefore, we have
\[
A(i, i) \to \frac{1}{1 + \lambda_i y \cdot \frac{1}{a_i + c - 1}}, \tag{A.25}
\]
which is independent of the choice of $i$.

For the same reason, we have
\[
B(i, i) = 1 + \left[ X_2^* \left[ \lambda_i I_p - \left( \frac{1}{n} Z_2 Z_2^* \right)^{-1} \frac{1}{T} X_2 X_2^* \right]^{-1} \left( \frac{1}{n} Z_2 Z_2^* \right)^{-1} \frac{1}{T} X_2 \right] (i, i) \\
= 1 + \left[ X_2^* \left[ \lambda_i \frac{1}{n} Z_2 Z_2^* - \frac{1}{T} X_2 X_2^* \right]^{-1} \frac{1}{T} X_2 \right] (i, i). \tag{A.26}
\]
If we denote $\delta_i$ as the $i$-th column of $X_2$, then we have

$$\frac{1}{T}X_2X_2^* = \frac{1}{T} (\delta_1 \cdots \delta_T) \left( \begin{array}{c} \delta_1^* \\ \vdots \\ \delta_T^* \end{array} \right) = \frac{1}{T} \delta_i \delta_i^* + \frac{1}{T} X_{2i} X_{2i}^*, $$

and

\begin{align*}
(\lambda_i \cdot \frac{1}{n} Z_2 Z_2^* - \frac{1}{T} X_2 X_2^*)^{-1} - (\lambda_i \cdot \frac{1}{n} Z_2 Z_2^* - \frac{1}{T} X_2 X_2^*)^{-1} \\
= (\lambda_i \cdot \frac{1}{n} Z_2 Z_2^* - \frac{1}{T} X_2 X_2^*)^{-1} \frac{1}{T} \delta_i \delta_i^* (\lambda_i \cdot \frac{1}{n} Z_2 Z_2^* - \frac{1}{T} X_2 X_2^*)^{-1} .
\end{align*}

So we have

$$\left( \lambda_i \cdot \frac{1}{n} Z_2 Z_2^* - \frac{1}{T} X_2 X_2^* \right)^{-1} = \frac{1}{1 - \frac{1}{T} \delta_i \delta_i^* (\lambda_i \cdot \frac{1}{n} Z_2 Z_2^* - \frac{1}{T} X_2 X_2^*)^{-1} \delta_i} . \tag{A.27}$$

Combine (A.26) and (A.27), we have

\begin{align*}
B(i, i) = 1 + \delta_i^* \left[ \lambda_i \cdot \frac{1}{n} Z_2 Z_2^* - \frac{1}{T} X_2 X_2^* \right]^{-1} \frac{1}{T} \delta_i & \\
= 1 + \frac{\frac{1}{T} \delta_i \delta_i^* (\lambda_i \cdot \frac{1}{n} Z_2 Z_2^* - \frac{1}{T} X_2 X_2^*)^{-1} \delta_i}{1 - \frac{1}{T} \delta_i \delta_i^* (\lambda_i \cdot \frac{1}{n} Z_2 Z_2^* - \frac{1}{T} X_2 X_2^*)^{-1} \delta_i} & \\
= \frac{1}{1 - \frac{1}{T} \delta_i \delta_i^* (\lambda_i \cdot \frac{1}{n} Z_2 Z_2^* - \frac{1}{T} X_2 X_2^*)^{-1} \delta_i} . \tag{A.28}
\end{align*}

Using the independence between $\delta_i$ and $(\lambda_i \cdot \frac{1}{n} Z_2 Z_2^* - \frac{1}{T} X_2 X_2^*)^{-1}$ and Lemma A.4 again, we have

$$\frac{1}{T} \delta_i \delta_i^* (\lambda_i \cdot \frac{1}{n} Z_2 Z_2^* - \frac{1}{T} X_2 X_2^*)^{-1} \delta_i \rightarrow c \cdot \frac{1}{a_i + c - 1} .$$

Therefore, we have

$$B(i, i) \rightarrow \frac{1}{1 - \frac{c}{a_i + c - 1}} ,$$

which is also independent of the choice of $i$.

Finally, taking the definition of $\omega_i$ in (A.21) into consideration, we have

$$\omega_i = \frac{\lambda_i^2 y}{(1 + y \lambda_i \cdot \frac{1}{a_i + c - 1})^2} + \frac{a_i^2 c}{(1 - \frac{c}{a_i + c - 1})^2} = a_i^2 (a_i + c - 1)^2 (c + y) . \tag{A.29}$$

The proof of Lemma A.6 is complete.