Lyapunov Functions for Time-Scale Dynamics on Riemannian Geometries of the Simplex

Marc Harper · Dashiell Fryer

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Abstract We combine incentive, adaptive, and time-scale dynamics to study multipopulation dynamics on the simplex equipped with a large class of Riemannian metrics, simultaneously generalizing and extending many dynamics commonly studied in dynamic game theory and evolutionary dynamics. Each population has its own geometry, method of adaptation (incentive), and time-scale (discrete, continuous, and others). Using information-theoretic measures of distance we give a widely-applicable Lyapunov result for the dynamics.

Keywords Evolutionary dynamics · Evolutionary stability · Lyapunov functions · Riemannian geometry · Time-scale calculus

1 Introduction

Evolutionary dynamics now includes the study of many discrete and continuous dynamical systems such as the replicator [32], best reply [25], projection [26] [30] [22], and logit dynamics [14], to name a few (see also [31] and [17]). Modified population growth dynamics incorporating a scale-invariant power law parameter, commonly used in generalized statistical physics [33], have recently been applied to human populations [16] and are closely related to the dynamical systems described in [15]. We explore the unification of all these dynamics with the game-theoretically motivated incentive dynamic introduced in [13], in discrete and continuous forms using time-scale calculus [5], and with Riemannian geometries on the simplex. The geometry is specified by a geometrically motivated functional parameter called an escort [15] that allows some dynamics to be realized in multiple ways and by a more arbitrary Riemannian metric in general. We show that the incentive dynamic and the replicator dynamics...
dynamic are equivalent on the interior of the simplex, through a mapping that yields insight into the stability of the aforementioned dynamics, and explains clearly how to separate the selective action from the underlying geometry. Ultimately we define the time-scale escort incentive dynamic and time-scale metric dynamics (which in the continuous case correspond to a special case of the adaptive dynamics of [18]), building up through a series of examples, and prove a general stability theorem for a large class of discrete and continuous dynamics. For general introductions to evolutionary dynamics see [9,19,20].

In this paper we show that the Kullback–Leibler information divergence (KL-divergence, also called relative entropy) and natural generalizations serve as Lyapunov functions in a variety of contexts. The use of the KL-divergence and similar information-theoretic quantities in evolutionary dynamics goes back at least to [6] and is developed further in [34]. A geometrically-motivated generalization from information geometry [1,2] was introduced in [15]. To the reader familiar with information geometry this should come as little surprise since the Shahshahani metric of evolutionary game theory can be identified with the Fisher information metric, which is in some sense a local variant of the KL-divergence. The projection dynamic [27] can similarly be described in terms of the Euclidean geometry [22,24] and its stability in terms of the Euclidean distance which can be realized as a generalized information geometry [15] through the introduction of a functional parameter called an escort. We show in this work that these geometries can simplify the stability theory of some evolutionary dynamics (particularly for non-linear landscapes), define novel dynamics, and are compatible with the formulation of the incentive dynamic and the time-scale calculus. Finally, we introduce an information divergence for a large class of Riemannian metrics to construct an explicit Lyapunov function, extending the adaptive dynamics of [18] to discrete time-scales. Together these ingredients yield a very general stability result.

2 Incentive Dynamics

Let us first introduce the incentive dynamics of Fryer [13]. Motivated by game-theoretic considerations, the incentive dynamics is a differential equation the proportions of population types, much like the replicator equation, and takes the form

$$\dot{x}_i = \varphi_i(x) - x_i \sum_j \varphi_j(x), \quad (1)$$

where $\varphi(x)$ is an incentive, a typically non-negative function that governs the interaction between the population state and the fitness landscape. Table 1 lists incentive functions for some common dynamics. Fryer shows in [13] that any game dynamic can be written as an incentive dynamic and gives a stability theorem as follows. Define an incentive stable state (ISS) to be a state $\hat{x}$ such that in an open neighborhood of $\hat{x}$ the following inequality holds

$$\sum_i \hat{x}_i \varphi_i(x) > \sum_i x_i \varphi_i(x). \quad (2)$$

It is then shown that given an internal ISS, the KL-divergence is a local Lyapunov function for the incentive dynamic (Eq. 1). The incentive $\varphi_i(x) = x_i f_i(x)$ captures the known result for the replicator dynamics [6,19], with the definition of ISS being exactly that of an evolutionarily stable state (ESS): $\hat{x} \cdot f(x) > x \cdot f(x)$. Moreover, a short proof shows that for the best reply incentive, an ISS is again an ESS [12]. This Lyapunov theorem will generalize to many dynamics.
Table 1  Incentives for some common dynamics, where $\bar{f}(x) = \sum_i x_i f(x) = x \cdot f(x)$ is the mean fitness, $BR_i(x)$ is the best reply to state $x$, $S(f(x), x)$ is the set of all strategies in the support of $x$ as well as any collection of pure strategies in the complement of the support that maximize the average.

| Dynamics     | Incentive                                                                 |
|--------------|---------------------------------------------------------------------------|
| Replicator   | $\varphi_i(x) = x_i \left(f_i(x) - \bar{f}(x)\right)$                   |
| Best reply   | $\varphi_i(x) = BR_i(x) - x_i$                                           |
| Logit        | $\varphi_i(x) = \frac{\exp\left(\eta^{-1} f_i(x)\right)}{\sum_j \exp\left(\eta^{-1} f_j(x)\right)} \sum_{j \in S(f(x), x)} \frac{f_j(x)}{|S(f(x), x)|}$ |
| Projection   | $\varphi_i(x) = \begin{cases} f_i(x) - \frac{1}{|S(f(x), x)|} \sum_{j \in S(f(x), x)} f_j(x) & \text{if } i \in S(f(x), x) \\ 0 & \text{otherwise} \end{cases}$ |

Note that on the interior of the simplex the projection incentive is just $\varphi_i(x) = f_i(x) - 1/n \sum_j f_j(x)$. For more examples see Table 1 in [13]. Note that multiple incentives can yield the same continuous dynamic, e.g. $\varphi_i(x) = x_i f_i(x)$ also yields the replicator dynamic. For the logit incentive we assume that $\eta \neq 0$.

Although incentives generalize payoff structures in game theory, they produce dynamics similar to those motivated by geometric and agent-based considerations. The continuous incentive dynamics is essentially the same as the ray-projection dynamics and an ISS is a TESS (truly-ESS) as defined in [22]. The interested reader should consult [23] and particularly [21] for a discussion of the relationships between many various stability concepts. There is also a similar family of dynamics derived from target and revision protocols [29], though it is not clear that the two families are precisely the same.

2.1 The Incentive Dynamic is the Replicator Dynamic

Observe that we can transform any incentive dynamic into a replicator dynamic on the interior of the simplex (the behavior of incentives near the boundary of the simplex is another matter altogether and we will not consider it here). We simply solve for $f_i$ in the equation $\varphi_i(x) = x_i f_i(x)$ so that to every incentive $\varphi_i$ we define an effective fitness landscape $f_i(x) = \frac{\varphi_i(x)}{x_i}$, which is well-defined at least on interior trajectories and possibly on the boundary simplices, such as is the case for forward-invariant dynamics. (Note that some incentives take particular care on the boundary, such as the projection incentive above, from [30].) The summation term $\sum_j \varphi_j(x)$ in Eq. (1) is the mean of the effective fitness landscape; for the replicator incentive, this term is the mean fitness $\bar{f}(x) = \sum_i x_i f(x) = x \cdot f(x)$.

The ISS condition for an incentive is the same as the ESS condition for the effective fitness landscape, and this shows that the ISS stability theorem is equivalent to the analogous theorem for ESS and the replicator dynamic. This does not of course imply ESS for any fitness landscape used in the definition of an incentive (such as a best reply incentive using a landscape $f$). Typical fitness landscapes in evolutionary dynamics are linear and given by $f(x) = Ax$ where $A$ is a game matrix. In this formulation, one will encounter a much larger class of landscapes. For the best reply dynamic, the effective landscape is $f_i(x) = BR_i(x)/x_i - 1$, and while it is clear that such a function may not be well-defined on the boundary simplex, since the dynamic is forward-invariant we will not concern ourselves further.
2.2 Example: q-Replicator Incentive

From the preceding section it is tempting to suspect that the concepts of ISS and ESS are equivalent, but this is not the case. Consider the *q*-replicator incentive $\phi_i(x) = x^q_i f_i(x)$ for a fitness landscape $f$. Further, assume that the fitness function is of the form $f(x) = Ax$ where $A$ is the rock-scissors-paper matrix:

$$ f(x) = \begin{pmatrix} 0 & -b & a \\ a & 0 & -b \\ -b & a & 0 \end{pmatrix} x. $$

Several curves for various values of $q$ are plotted in Fig. 1. For the replicator incentive ($q = 1$) the trajectory converges to an interior ESS; for other values of $q$, the trajectories may either converge or diverge. This shows that an ESS for the fitness landscape need not be an ISS for the incentive. Note also that whether a curve converges or not depends on the initial point. While for $q = 1$ the Lyapunov function is global, this is not the case for other values of $q$ (see Fig. 2). Figure 1 shows that an ESS need not be an ISS and Fig. 3 shows that an ISS need not be an ESS.

3 Time-Scale Incentive Dynamics

3.1 Time-Scale Definitions

To study dynamics at different time-scales we use the time-scale derivative from time-scale calculus (also known as the delta derivative) \[4,5\]. For time-scales $T = h\mathbb{Z}$, the time-scale derivative of a function $g$ is given by the standard difference quotient:

$$ g^\Delta(z) = \frac{g(z+h) - g(z)}{h}. $$

For the time-scale $T = \mathbb{R}$, the time-scale derivative is the standard derivative. While there are other common time-scales, this paper restricts attention to $\mathbb{R}$ and $h\mathbb{Z}$ for $0 < h \leq 1$.

![Fig. 1](image-url) Phase portraits for q-Replicator incentive for the powers $q \in \{0.5, 1, 1.5, 2, 2.5, 4\}$ with colors blue, green, red, cyan, magenta, and yellow, respectively. The game matrix is the RSP matrix with $a = -1$, $b = -2$. This shows that an ESS need not be an ISS since some of these trajectories converge and some do not, depending on the incentive (value of $q$) and initial point. **a** Initial point: $(1/10, 1/10, 8/10)$, **b** Initial point: $(1/83, 2/83, 80/83)$.
3.2 Time-Scale Replicator Equation

Let us consider an illustrative example. Suppose briefly that \( f_i(x) > 0 \) for all \( i \) and define the time-scale replicator equation as:

\[
x_i^\Delta = \frac{x_i \left( f_i(x) - \bar{f}(x) \right)}{\bar{f}(x)}.
\]
Note that $\sum_i x_i^\Delta = 0$. For $h = 1$, this is the discrete replicator dynamic, where

$$x_i^\Delta = \frac{x_i(t+\Delta t) - x_i(t)}{\Delta t} := x_i' - x_i,$$

which can be easily seen to be equivalent to the common form

$$x_i' = \frac{x_i f_i(x)}{\bar{f}(x)}.$$

For the case $h \to 0$, we have the following:

$$\dot{x}_i = \frac{x_i \left( f_i(x) - \bar{f}(x) \right)}{\bar{f}(x)}.$$

This equation is trajectory equivalent to the replicator equation (up to a change in velocity and transformation of $f(x)$ so that $\bar{f}(x) > 0$) [9,19]. The definitions of the time-scale calculus can unify the description of continuous and discrete dynamics in evolutionary dynamics, as well as set the stage for dynamics on other time-scales, through the delta derivative.

### 3.3 Time-Scale Best Reply and Fictitious Play Dynamic

In [13], a best reply dynamic is shown to result from the incentive $\varphi_i(x) = BR_i(x) - x_i$, and the ISS condition is shown to reduce to the ESS condition. Now consider the case of $T = h\mathbb{Z}$ for $h \in (0, 1)$. This time-scale yields a discrete best reply that is algebraically equivalent to

$$x_i' = (1 - h)x_i + hBR_i(x).$$

Here we see that the time-scale appears as a weighting between the best reply and the current state of the dynamic. In other words, $h$ is the proportion of the population that adopts the best reply. If $h = 1$, then entire population switches to the best reply, and the dynamic cycles through the corner points of the simplex for the landscape defined by an RSP matrix as above.

The center of the simplex is not an equilibrium point for the time-scale $T = h\mathbb{Z}$. To see this, suppose the fitness landscape $f$ has an interior ESS at the barycenter (say for an RSP matrix). On the interior of the simplex, the dynamic is stationary if and only if $x_k+1 = x_k$ for some $k$, which implies that either $h = 0$ or $x_i \in \{0, 1\}$ for all $i$, which is impossible for an interior ESS. However, if the time-step $h$ is not fixed, this may not be the case (see Fig. 4). On a variable time-scale the dynamic is (in vector form)

$$x_{k+1} = (1 - h_k)x_k + h_kBR(x_k),$$

which is similar to the fictitious play dynamic in the case that the time-scale weights decrease over time [7]. Eventually such a time-scale has $h_k \to 0$, so the above discussion does not apply, and this dynamic can converge.

### 3.4 Time-Scale Lyapunov Function

Lyapunov stability is a frequently-used technique in evolutionary dynamics. These techniques have been ported to the time-scale calculus [3,10,11]. A time-scale Lyapunov function is a positive definite function defined on the trajectories of a dynamic with negative-semi-definite (negative-definite) time-scale derivative which then implies stability (asymptotic stability). The interested reader should see [3] for a presentation that mirrors the traditional approach with the appropriate changes necessary to extend the stability theory to arbitrary time-scales. In particular, we will use Theorems 5.1 and 5.4 in [3], which have two technical conditions.
Fig. 4 Sample trajectories for the discrete best reply dynamic for $h = 1/3$ (blue, outer) and a variable time-scale with $h_k = 1/(k + 1)$ (green, inner) for an RSP matrix. Only the latter converges to the interior ESS, hence time-scales with non-constant steps can affect convergence (Color figure online)

that will be familiar to the reader aware of the details of the continuous-time Lyapunov stability theorems. Here we give the necessary time-scale definitions and discuss the technical conditions; for more detail, see [3]. Define the open balls $K_h(\hat{x}) = \{x \in \mathbb{R}^n : ||x - \hat{x}|| < h\}$. A function $c : [0, h] \rightarrow [0, \infty)$ belongs to class $K$ if it is continuous, increasing, and if $c(0) = 0$. A function $v : K_h \times \mathbb{T} \rightarrow \mathbb{R}$ is called decrescent if there exists a function $c$ of class $K$ and $t_0 \in \mathbb{T}$ such that for all $t \geq t_0$ and $x \in K_h$, $v(x, t) \leq c(||x - \hat{x}||)$. Similarly, if there exists $c$ of class $K$ such that $v(x, t) \geq c(||x - \hat{x}||)$, $v$ is called positive definite. (The functions $c$ need not be the same in the two definitions.) We will take the definition of time-scale Lyapunov function to include both conditions on the appropriate time-scale; all such Lyapunov functions in this paper satisfy both conditions.

We now define a time-scale generalization of the incentive dynamics and generalize the Lyapunov results of [13] and [15] to cover more general time-scale dynamics. The KL-divergence between two points on the interior of the simplex is defined as

$$D_{KL}(y, x) = \sum_i y_i \log y_i - y_i \log x_i.\)$$

**Theorem 1** For the time-scale incentive dynamic with $0 \leq h \leq 1$ and $\mathbb{T} = h\mathbb{Z}$, a state $\hat{x}$ is an ISS if and only if $D(\hat{x}) := D_{KL}(\hat{x}, x)$ is a time-scale Lyapunov function.

This theorem is a special case of Theorem 2 and so the proof is deferred. A direct proof for the replicator equation is a straightforward exercise.

4 Escorts

The stability theorem for the incentive dynamic reduces the problem of finding a Lyapunov function to the problem of proving the ISS condition valid for a candidate equilibrium. We now introduce geometric methods for this problem. Consider the projection dynamic, with incentive $\phi_i(x) = f_i(x) - \frac{1}{n} \sum_j f_j(x)$. From [24] and [30], we know that $||\hat{x} - x||^2$ is a Lyapunov function if $\hat{x}$ is ESS, so one would hope that ISS reduces to ESS in this case as well. The ISS condition $(\hat{x}/x) \cdot f(x) > 1 \cdot f(x)$, however, is not obviously the same as the ESS condition. If we assume a two-type linear fitness landscape

$$f(x) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} x,$$

it is a straightforward derivation to show that the ISS condition leads to

$$[(a - c)x_1 + (b - d)x_2](x_2\hat{x}_1 - x_1\hat{x}_2) > 0,$$
which along with the constraint that \( x_1 + x_2 = 1 \) is the well-known condition for the existence of an internal ESS (and more generally, a Nash equilibrium). Similarly, we can give a family of examples that includes the projection dynamic. Consider the escort dynamics defined in [15], where \( \sigma \) is a positive non-decreasing function:

\[
\dot{x}_i = \sigma_i(x) \left( f_i(x) - \frac{\sum_j \sigma_j(x) f_j(x)}{\sum_j \sigma_j(x)} \right).
\]

For these dynamics, an incentive is given by the right-hand-side of Eq. 3 and the ISS condition is

\[
\sum_i \frac{\dot{x}_i}{x_i} \sigma_i(x) f_i(x) > \sum_i \frac{x_i \sigma_i(x) f_i(x)}{x_i},
\]

which reduces to ESS for the replicator dynamic (\( \sigma(x) = x \)), or if the fitness landscape factors appropriately. Nevertheless, it was shown in [15] that an ESS, if it exists, is asymptotically stable for these dynamics, so it is desirable to have a generalization of Theorem 1 that captures this family as well, as it also includes the projection dynamic.

To this end we introduce a functional parameter \( \sigma \) called an escort and related quantities. We will also need these definitions in a later section when we incorporate Riemannian metrics. An escort is a function \( \sigma \) that is nondecreasing and strictly positive on \((0, 1]\). A vector-valued function \( \sigma(x) = (\sigma_1(x), \ldots, \sigma_n(x)) \) is called an escort if each component is an escort. We denote the normalized escort vector by \( \hat{\sigma} \), i.e. normalized such that \( \hat{\sigma}(x) \) is in the simplex.

Generalized information divergences are defined by generalizing the natural logarithm using an escort function. See [15] and [28] for more details (note that the references use the symbol \( \phi \) rather than \( \sigma \)).

**Definition 1** (Escort Logarithm) Define the escort logarithm

\[
\log_\sigma(x) = \int_1^x \frac{1}{\sigma(v)} \, dv.
\]

The escort logarithm shares several properties with the natural logarithm: it is negative and increasing on \((0, 1]\) and concave on \((0, 1]\) if \( \sigma \) is strictly increasing.

**Definition 2** (Escort Divergence) Define the escort divergence

\[
D_\sigma(x, y) = \sum_{i=1}^n \int_{y_i}^{x_i} \log_{\sigma_i}(u) - \log_{\sigma_i}(y_i) \, du.
\]

Since the logarithms are increasing on \((0, 1]\), this divergence satisfies the usual properties of an information divergence on the simplex: \( D(x, y) > 0 \) if \( x \neq y \) and \( D(x, x) = 0 \).

5 The Time-Scale Escort Incentive Dynamic

Now we are able to define the time-scale escort incentive dynamic and give the main theorems.

**Definition 3** Define the time-scale escort incentive dynamic to be

\[
x_i^\Delta = \varphi_i(x) - \hat{\sigma}_i(x) \sum_j \varphi_j(x).
\]
We also need a definition of ISS that incorporates the escort parameter. This definition can be understood in terms of inner products of a Riemannian metric $g_{ij}(x) = \delta_{ij}/\sigma_i(x)$.

**Definition 4** Define $\hat{x}$ to be an escort ISS (or EISS) for an incentive $\varphi$ and an escort $\sigma$ if for all $x$ in some neighborhood of $\hat{x}$ the following inequality holds:

$$\sum_i \frac{\hat{x}_i \varphi_i(x)}{\sigma_i(x)} > \sum_i \frac{x_i \varphi_i(x_i)}{\sigma_i(x)}.$$

For one example of a choice of escorts leading to interesting dynamics, consider the escort $\sigma(x) = \beta x$. This introduces an intensity of selection parameter that alters the velocities of the dynamic (but not the stable point, if any). In fact, each type can have its own intensity of selection, and its own geometry. A popular choice of escort in information geometry is $\sigma(x) = x^q$, which gives $q$-analogs of logarithms, exponentials, and divergences [28]. See [15] for more examples. The corresponding $q$-divergence is given by:

$$D_q(x, y) = \begin{cases} 
\frac{1}{2}||x - y||^2 & \text{if } q = 0 \\
D_{KL}(x, y) & \text{if } q = 1 \\
\sum_i \left[ \log \frac{x_i}{y_i} + \frac{y_i - x_i}{x_i} \right] & \text{if } q = 2 \\
\frac{1}{1-q} \sum_i \left[ \frac{x_i^{2-q} - y_i^{2-q}}{2-q} - y_i^{1-q} (x_i - y_i) \right] & \text{if } q \geq 0, q \neq 1, 2
\end{cases}$$

As in the case of the incentive dynamic, the continuous-time escort incentive dynamic is a special case of the escort replicator dynamic. Consider the escort exponential $\exp_\sigma$, which is the functional inverse of the escort logarithm. It has the following crucial property:

$$\frac{d}{dx} \exp_\sigma x = \sigma(\exp_\sigma x).$$

Now let $x_i = \exp_\sigma_i (v_i - G)$ and consider the derivative $\dot{x}_i = \sigma(\exp_\sigma_i (v_i - G))(\dot{v}_i - \dot{G}) = \sigma_i(x)(\dot{v}_i - \dot{G})$. Then if we can solve the following two auxiliary equations we can also solve the escort incentive dynamic:

$$\dot{v}_i = \frac{\varphi_i(x)}{\sigma_i(x)} \quad \text{and} \quad \dot{G} = \frac{\sum_j \varphi_j(x)}{\sum_j \sigma_j(x)}.$$

**Fig. 5** Sample trajectories of $q$-escort dynamics for various $q$, with replicator incentive given by a linear fitness landscape with RSP game matrix ($a = 1, b = 2$). Left $q \in \{0.2, 0.8, 1.0, 2.0\}$ with colors blue, green, red, and cyan respectively. Right $q = 0.5$. The standard replicator equation diverges for $q = 1$ but this is not always the case for the $q$-escort, showing that the geometry can affect stability (Color figure online).
Fig. 6 Trajectories of Eq. 4 for various $q$, having factors of $x_i^q$ in both the incentive and the geometry (escort), which cancel in the divergence function. Unlike in Fig. 5, all the interior sample trajectories converge. Left $q = 0.5, 1.0, 1.5, 2.0$ for blue, green, red, cyan, magenta, respectively. Right $q = 4$. For these dynamics, the geometric structure factors out of the incentive, so the EISS is an ESS, and the associated $q$-divergences are Lyapunov functions (Color figure online).

This also shows how we can translate the escort incentive into the escort replicator equation. Given the escort incentive

$$\dot{x}_i = \varphi_i(x) - \hat{\sigma}_i(x) \sum_j \varphi_j(x),$$

define a fitness landscape

$$f_i(x) = \frac{\varphi_i(x)}{\sigma_i(x)}.$$ 

Then the escort incentive equation is an escort replicator equation, and the ESS condition $\hat{x} \cdot f(x) > x \cdot f(x)$ is the EISS condition

$$\sum_i \hat{x}_i \varphi_i(x) \sigma_i(x) > \sum_i x_i \varphi_i(x) \sigma_i(x).$$

That an $\hat{x}$ is ESS iff the escort divergence is a Lyapunov function was shown in [15] for the continuous escort replicator dynamic. What remains to be shown now is the extension of this result to time-scales $T = h\mathbb{Z}$ for $0 < h \leq 1$, and the relationship between the concepts of EISS and ESS. Before doing so, consider the following examples. In Fig. 5, replicator incentives for RSP matrices are plotted with various $q$-incentives. Notice that the equilibrium is not an ESS for the fitness landscape $f$ yet the dynamic converges to the center of the simplex for some choices of the escort. In Fig. 6 we have phase plots for dynamics with the same landscape but with both $q$-replicator incentives and $q$-escorts. In this case the conditions for EISS and ESS are the same, so all the dynamics converge. The KL-divergences are not monotonically decreasing in all cases, but the corresponding escort-divergences are as shown in Fig. 7. The dynamic in this case is given by:

$$\dot{x}_i = x_i^q f_i(x) - \frac{x_i^q}{\sum_j x_j^q} \sum_j x_j^q f_j(x).$$ (4)

Now we state and prove the main theorem of this section.

**Theorem 2** Let $\varphi$ be an incentive function and let $\sigma$ be an escort. Then $\hat{x}$ is an EISS iff $D(x) = D_\sigma(\hat{x}, x)$ is a local time-scale Lyapunov function for the time-scale escort incentive dynamics.
The proof follows easily from the established facts and the following lemma.

**Lemma 1**

\[ D^\Delta(x) \leq -\sum_i \frac{\dot{x}_i - x_i}{\sigma_i(x)} x_i^\Delta. \]

*Equality holds in the limit that \( h \to 0 \), i.e. for \( T = \mathbb{R} \).*

**Proof** Since escort functions are nondecreasing and positive on \((0, 1]\), we have the following two facts:

\[ \int_a^b \frac{du}{\sigma(u)} \leq \frac{b - a}{\sigma(a)}, \quad \text{and} \]

\[ \int_a^b \log_\sigma(u) du \leq (b - a) \log_\sigma(b). \]

Now we have

\[ h D^\Delta(x) = \sum_i \int_{x_i}^{x_i'} \log_\sigma(u) du - \sum_i \left[ (x_i' \log_\sigma(x_i') - \dot{x}_i \log_\sigma(x_i')) - (x_i \log_\sigma(x_i) - \dot{x}_i \log_\sigma(x_i)) \right] \]

\[ \leq \sum_i (x_i' - x_i) \log_\sigma(x_i') - \sum_i x_i' \log_\sigma(x_i') - \dot{x}_i \log_\sigma(x_i') - x_i \log_\sigma(x_i) + \dot{x}_i \log_\sigma(x_i) \]

\[ = \sum_i (\dot{x}_i - x_i) \left( \log_\sigma(x_i') - \log_\sigma(x_i) \right) \]

\[ = -\sum_i (\dot{x}_i - x_i) \int_{x_i}^{x_i'} \frac{du}{\sigma_i(u)} \leq -\sum_i (\dot{x}_i - x_i) \frac{x_i' - x_i}{\sigma_i(x)}. \]

Bringing \( h \) to the right hand side completes the lemma for \( h > 0 \). Equality for \( h = 0 \) (i.e. \( T = \mathbb{R} \)) can be directly verified with differentiation (and is given in [15]).

To complete the proof of Theorem 2 we need only apply the lemma to the respective dynamics and use the definition of EISS.
Proof of Theorem 2 Using the lemma and substituting the right-hand side of Eq. 3 gives:

\[ D^\Delta(x) \leq - \sum_i \frac{\hat{x}_i - x_i}{\sigma_i(x)} x_i^\Delta \]

\[ = - \sum_i \frac{\hat{x}_i - x_i}{\sigma_i(x)} \left( \varphi_i(x) - \hat{\sigma}_i(x) \sum_j \varphi_j(x) \right) \]

\[ = (x_i - \hat{x}_i) \left( f_i(x) - \bar{f}(x) \right) = x \cdot f(x) - \hat{x} \cdot f(x). \]

where \( f_i(x) = \varphi_i(x)/x_i \) is the effective landscape of the incentive. If \( \hat{x} \) is an EISS, then the right-hand side is negative. \( \square \)

Throughout this paper, it may appear that all the examples we have given are for continuous dynamics. In fact, every phase plot in this paper is for a discrete time-scale with \( h \approx 1/100 \to 1/1000 \) unless otherwise indicated, and in all previously known cases, the results have coincided with the expectation for the continuous dynamics. Hence all the examples in this paper illustrate the main theorem for these particular time-scales.

Let us return to the example of the projection dynamic. Previously we saw that the orthogonal projection dynamic was obtainable from the incentive \( \varphi_i(x) = f_i(x) - (1/n) \sum_j f_j(x) \) and the escort \( \sigma(x) = x \) on interior trajectories, but the ISS condition did not appear to be the same as ESS (though we were able to argue for two-player linear landscapes that they are equivalent). With the generalized theorem, we can also obtain the dynamic from the incentive \( \varphi_i(x) = f_i(x) \) and \( \sigma(x) = 1 \), and now the EISS condition reduces immediately to ESS. Moreover, the Lyapunov function given by the escort information divergence is, remarkably, one-half the Euclidean distance: \( D(x) = (1/2)||\hat{x} - x||^2 \), capturing the known result up to a constant factor of one-half [24,27].

From the these results it should be clear that if the incentive factors as \( \varphi_i(x) = \sigma_i(x)g_i(x) \) then the condition for EISS will be the condition for ESS for the landscape \( g \). So in particular if the function \( g \) is payoff monotonic, these dynamics have similar interior dynamics (though with different Lyapunov functions). If the incentive factors as \( \varphi_i(x) = x_i\sigma_i(x_i)g_i(x) \), then the ESS criterion is the ISS criterion for the incentive \( g \). So while the theorem covers all combinations of valid incentives and escorts, it is clear that careful choices may lead to simplifications in the stability criterion. The projection dynamic is somewhat special in that it can be described equivalently by multiple choices of escort and incentive. It is also clear that while there exist known Lyapunov functions for some dynamics that are not the KL-divergence (or a generalization) such as the best reply dynamic, the incentive dynamics approach yields a commonly-derived and motivated local Lyapunov function.

6 Time-Scale Lyapunov Functions for Adaptive Dynamics

As suggested in the last section, we now consider the adaptive dynamics for a Riemannian metric \( G \) on the simplex, defined in [18]. Let \( g = G^{-1}1 \) and \( C = G^{-1} - (g^T)^{-1}gg^T \). Then the time-scale adaptive dynamics for the metric \( G \) and a fitness landscape \( f \) is \( x_i^\Delta = \sum_j C_{ij} f_j \). The adaptive dynamics includes the escort dynamics as a special case. A geodesic approximation is shown to be a local Lyapunov function in [18] for the continuous dynamic; to obtain an explicit time-scale Lyapunov function we use a global divergence.
Define
\[ D_G(x) := D_G(\hat{x}, x) = \sum_{i,j} \int_{\hat{x}_i}^{x_i} (\log G)_{ij}(v) - (\log G)_{ij}(\hat{x}_i) \, dv, \]
where
\[ (\log G)_{ij}(x) = \int_{1}^{x} G_{ij}(v) \, dv. \]

**Theorem 3** Let \( G \) be a Riemannian metric and assume that the reciprocal of each component \( G_{ij} \) is an escort function. Then

1. \( D_G \) is an information divergence; that is \( D_G(\hat{x}) = 0 \) and \( D_G(x) > 0 \) for \( x \neq \hat{x} \).
2. \( D \) is a local time-scale Lyapunov function for the adaptive dynamics iff \( \hat{x} \) is an ESS.

The proof is almost identical to the proof of Theorem 2 (just sum over the additional index) and so is omitted. Note that \( D_G(x) \leq (x - \hat{x})^{T} G(x) (x - \hat{x}) \). One can now play the same game as before, identifying the incentive in terms of the fitness landscape, and forming best reply, logit, projection, or any other incentive dynamics with respect to particular Riemannian geometries. Indeed, we have that \( \varphi_i(x) = \sum_j G^{-1}_{ij} f_j(x) \) and the adaptive dynamics can be written as
\[
\Delta x_i = \varphi_i(x) - \left( G^{-1} \right)_{ii} \sum_j \varphi_j(x).
\]

(5)

Call this the **metric-incentive dynamic**. As before, we can identify the idea of an ESS with that of a GISS, i.e. a state \( \hat{x} \) is a GISS if for all \( x \) in a neighborhood of \( \hat{x} \), the inequality \( \varphi(x) \cdot G(x)(\hat{x} - x) > 0 \). With these definitions, we can restate Theorem 3 as follows.

**Theorem 4** Let \( G \) be a Riemannian metric and assume that the reciprocal of each component \( G_{ij} \) is an escort function. Then \( D_G \) is a local time-scale Lyapunov function for the metric-incentive dynamics iff \( \hat{x} \) is a GISS.

7 Multiple Populations

Following Eq. 5 we can formulate a multiple population dynamic in which each population operates on its own incentive, time-scale, and geometry. Let \( \hat{G}(x) = \frac{(G^{-1})}{\sum_j (G^{-1})_j} \) be the vector of coefficients in Eq. 5. Then the multiple population time-scale metric incentive dynamic (with populations indexed by \( \alpha \)) is :
\[
\Delta x^\alpha_i = \varphi_i(x^\alpha) - \hat{G}_{i,\alpha}(x^\alpha) \sum_j \varphi_j(x^\alpha).
\]

(6)

We give two examples in Figs. 8 and 9. The only difference between the two examples is the incentive for the second population: in the first case, the incentive is logit, and in the second, \( q \)-replicator with \( q = 2 \). Nevertheless, the resulting dynamics are very different.

In the spirit of [13] and [8], if there is a multiple-population GISS where the time-scales do not differ for each population then we can find a time-scale Lyapunov function for the system by summing \( D_G \) for each population. The proof is again analogous to Theorem 2 (sum over the additional index), and can easily be verified with differentiation for the continuous case.
Fig. 8  Two population dynamic with time step $h = 1/10$ for both populations, fitness landscape given by $a = -1, b = -2$. Both populations converge to the center (500 iterations shown). Left Replicator incentive, Shahshahani geometry, initial point $(1/5, 1/5, 3/5)$. Right Logit incentive, $\eta = 0.4$, Euclidean geometry, initial point $(3/5, 1/5, 1/5)$.

Fig. 9  Two population dynamic with time-scale $h = 1/10$ for both populations, fitness landscape given by $a = -1, b = -2$. Neither population converges after 10,000 iterations. Left Replicator incentive, Shahshahani geometry, initial point $(1/5, 1/5, 3/5)$. Right $q$-Replicator incentive with $q = 2$, Euclidean geometry, initial point $(3/5, 1/5, 1/5)$.

Theorem 5  Suppose each population in system 6 is of the same time-scale ($\mathbb{T} = h\mathbb{Z}, 0 \leq h \leq 1$, or $\mathbb{T} = \mathbb{R}$). Let $L = \sum DG_\alpha(\hat{x}_\alpha, x_\alpha)$. $L$ is a local time-scale Lyapunov function for the system 6 iff

$$\sum_\alpha \varphi_\alpha(x_\alpha) \cdot G_\alpha(x_\alpha)(\hat{x}_\alpha - x_\alpha) > 0$$

for some neighborhood of $\hat{x}_\alpha$.

8 Discussion

We have shown that a vast array of dynamics can be analyzed in terms of time-scales, incentives, Riemannian metrics, and divergence functions, the last of which allows the stability analysis of many discrete dynamics associated to the same geometric and game-theoretic constructions. In the process, we introduced a new information divergence defined in terms of a Riemannian metric. We have shown that choice of incentive and Riemannian metric can affect the presence of equilibria, their stability, and that decomposition of dynamics into incentives, escorts, and Riemannian metrics can lead directly to Lyapunov functions for particular dynamics. In some cases, such as for multiple populations with different geometries and incentives, complex Lyapunov functions can easily be found with our methodology.

In our approach to evolutionary dynamics we focused attention on a few special cases that yield a particularly nice set of examples. It is possible to define these dynamics on arbitrary
time-scales and achieve some analogous results. It may be possible to formulate analogous results for multipopulation dynamics where each population evolves according to a different time-scale, which would require an expansion of the current state of stability theorems on time-scales.

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