3-uniform hypergraphs without a cycle of length five

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Abstract

In this paper we show that the maximum number of hyperedges in a 3-uniform hypergraph on \( n \) vertices without a (Berge) cycle of length five is less than \( (0.254 + o(1))n^{3/2} \), improving an estimate of Bollobás and Győri.

We obtain this result by showing that not many 3-paths can start from certain subgraphs of the shadow.

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1 Introduction

A hypergraph \( H = (V, E) \) is a family \( E \) of distinct subsets of a finite set \( V \). The members of \( E \) are called hyperedges and the elements of \( V \) are called vertices. A hypergraph is called 3-uniform if each member of \( E \) has size 3. A hypergraph \( H = (V, E) \) is called linear if every two hyperedges have at most one vertex in common.

A Berge cycle of length \( k \geq 2 \), denoted Berge-\( C_k \), is an alternating sequence of distinct vertices and distinct edges of the form \( v_1, h_1, v_2, h_2, \ldots, v_k, h_k \) where \( v_i, v_{i+1} \in h_i \) for each \( i \in \{1, 2, \ldots, k-1\} \) and \( v_k, v_1 \in h_k \). (Note that if a hypergraph does not contain a Berge-C_2, then it is linear.) This definition of a hypergraph cycle is the classical definition due to Berge. More generally, if \( F = (V(F), E(F)) \) is a graph and \( Q = (V(Q), E(Q)) \) is a hypergraph, then we say \( Q \) is Berge-\( F \) if there is a bijection \( \phi : E(F) \to E(Q) \) such that
Given a family of graphs $\mathcal{F}$, we say that a hypergraph $\mathcal{H}$ is Berge-$\mathcal{F}$-free if for every $F \in \mathcal{F}$, the hypergraph $\mathcal{H}$ does not contain a Berge-$F$ as a subhypergraph. The maximum possible number of hyperedges in a Berge-$\mathcal{F}$-free 3-uniform hypergraph on $n$ vertices is the Turán number of Berge-$\mathcal{F}$, and is denoted by $\text{ex}_3(n, \mathcal{F})$. When $\mathcal{F} = \{F\}$ then we simply write $\text{ex}_3(n, F)$ instead of $\text{ex}_3(n, \{F\})$.

Determining $\text{ex}_3(n, \{C_2, C_3\})$ is basically equivalent to the famous (6, 3)-problem. This was settled by Ruzsa and Szemerédi in their classical paper [23], showing that $n^{2 - \frac{1}{\sqrt{\log n}}} < \text{ex}_3(n, \{C_2, C_3\}) = o(n^2)$ for some constant $c > 0$. An important Turán-type extremal result for Berge cycles is due to Lazebnik and Verstraëte [21], who studied the maximum number of hyperedges in an $r$-uniform hypergraph containing no Berge cycle of length less than five (i.e., girth five). They showed the following.

**Theorem 1** (Lazebnik, Verstraëte [21]).

$$\text{ex}_3(n, \{C_2, C_3, C_4\}) = \frac{1}{6} n^{3/2} + o(n^{3/2}).$$

The systematic study of the Turán number of Berge cycles started with the study of Berge triangles by Győri [15], and continued with the study of Berge five cycles by Bollobás and Győri [1] who showed the following.

**Theorem 2** (Bollobás, Győri [1]).

$$(1 + o(1)) \frac{n^{3/2}}{3\sqrt{3}} \leq \text{ex}_3(n, C_5) \leq \sqrt{2}n^{3/2} + 4.5n.$$

The following example of Bollobás and Győri proves the lower bound in Theorem 2.

**Bollobás-Győri Example.** Take a $C_4$-free bipartite graph $G_0$ with $n/3$ vertices in each part and $(1 + o(1))(n/3)^{3/2}$ edges. In one part, replace each vertex $u$ of $G_0$ by a pair of two new vertices $u_1$ and $u_2$, and add the triple $u_1u_2v$ for each edge $uv$ of $G_0$. It is easy to check that the resulting hypergraph $H$ does not contain a Berge cycle of length 5. Moreover, the number of hyperedges in $H$ is the same as the number of edges in $G_0$.

In this paper, we improve Theorem 2 as follows.

**Theorem 3.**

$$\text{ex}_3(n, C_5) < (1 + o(1)) 0.254n^{3/2}.$$

Roughly speaking, our main idea in proving the above theorem is to analyze the structure of a Berge-$C_5$-free hypergraph, and use this structure to efficiently bound the number of paths of length 3 that start from certain dense subgraphs (e.g., triangle, $K_4$) of the 2-shadow. This bound is then combined with the lower bound on the number of paths of length 3 provided by the Blakley-Roy inequality [2]. We prove Theorem 3 in Section 2.
Ergemlidze, Győri and Methuku [3] considered the analogous question for linear hypergraphs and proved that \( \text{ex}_3(n, \{C_2, C_3\}) = n^{3/2}/3\sqrt{3} + o(n^{3/2}) \). Surprisingly, even though their lower bound is the same as the lower bound in Theorem 2, the linear hypergraph that they constructed in [3] is very different from the hypergraph used in the Bollobás-Győri example discussed above – the latter is far from being linear. In [3], the authors also strengthened Theorem 1 by showing that \( \text{ex}_3(n, \{C_2, C_3, C_4\}) \sim \text{ex}_3(n, \{C_2, C_4\}) \).

Recently, \( \text{ex}_3(n, C_4) \) was studied in [5]. See [6] for results on the maximum number of hyperedges in an \( r \)-uniform hypergraph of girth six.

Győri and Lemons [16, 17] generalized Theorem 2 to Berge cycles of any given length and proved bounds on \( \text{ex}_r(n, C_{2k+1}) \) and \( \text{ex}_r(n, C_{2k}) \). These bounds were improved by Füredi and Özkahya [9], Jiang and Ma [19], Gerbner, Methuku and Vizer [11]. Recently Füredi, Kostochka and Luo [7] started the study of the maximum size of an \( n \)-vertex \( r \)-uniform hypergraph without any Berge cycle of length at least \( k \). This study has been continued in [8, 18, 20, 4].

General results for Berge-\( F \)-free hypergraphs have been obtained in [12, 13, 10] and the Turán numbers of Berge-\( K_{2,t} \) and Berge cliques, among others, were studied in [24, 22, 11, 14, 10].

**Notation**

We introduce some important notations and definitions used throughout the paper.

- Length of a path is the number of edges in the path. We usually denote a path \( v_0, v_1, \ldots, v_k \), simply as \( v_0v_1 \ldots v_k \).
- For convenience, an edge \( \{a, b\} \) of a graph or a pair of vertices \( a, b \) is referred to as \( ab \). A hyperedge \( \{a, b, c\} \) is written simply as \( abc \).
- For a hypergraph \( H \) (or a graph \( G \)), for convenience, we sometimes use \( H \) (or \( G \)) to denote the edge set of the hypergraph \( H \) (or \( G \) respectively). Thus the number of edges in \( H \) is \( |H| \).
- Given a graph \( G \) and a subset of its vertices \( S \), let the subgraph of \( G \) induced by \( S \) be denoted by \( G[S] \).
- For a hypergraph \( H \), let \( \partial H = \{ab \mid ab \subset e \in E(H)\} \) denote its 2-shadow graph.
- For a hypergraph \( H \), the *neighborhood* of \( v \) in \( H \) is defined as \( N(v) = \{x \in V(H) \setminus \{v\} \mid v, x \in h \text{ for some } h \in E(H)\} \).
- For a hypergraph \( H \) and a pair of vertices \( u, v \in V(H) \), let \( \text{codeg}(v, u) \) denote the number of hyperedges of \( H \) containing the pair \( \{u, v\} \).

**2 Proof of Theorem 3**

Let \( H \) be a hypergraph on \( n \) vertices without a Berge 5-cycle and let \( G = \partial H \) be the 2-shadow of \( H \). First we introduce some definitions.
Definition 4. A pair $xy \in \partial H$ is called thin if $\text{codeg}(xy) = 1$, otherwise it is called fat.
We say a hyperedge $abc \in H$ is thin if at least two of the pairs $ab, bc, ac$ are thin.

Definition 5. We say a set of hyperedges (or a hypergraph) is tightly-connected if it can be obtained by starting with a hyperedge and adding hyperedges one by one, such that every added hyperedge intersects with one of the previous hyperedges in 2 vertices.

Definition 6. A block in $H$ is a maximal set of tightly-connected hyperedges.

Definition 7. For a block $B$, a maximal subhypergraph of $B$ without containing thin hyperedges is called the core of the block.

Let $K_3^4$ denote the complete 3-uniform hypergraph on 4 vertices. A crown of size $k$ is a set of $k \geq 1$ hyperedges of the form $abc_1, abc_2, \ldots, abc_k$. Below we define 2 specific hypergraphs:

- Let $F_1$ be a hypergraph consisting of exactly 3 hyperedges on 4 vertices (i.e., $K_3^4$ minus an edge).
- For distinct vertices $a, b, c, d$ and $o$, let $F_2$ be the hypergraph consisting of hyperedges $oab, obc, ocd$ and $oda$.

Lemma 8. Let $B$ be a block of $H$, and let $\mathcal{B}$ be a core of $B$. Then $\mathcal{B}$ is either $\emptyset, K_3^4, F_1, F_2$ or a crown of size $k$ for some $k \geq 1$.

Proof. If $\mathcal{B} = \emptyset$, we are done, so let us assume $\mathcal{B} \neq \emptyset$. Since $\mathcal{B}$ is tightly-connected and it can be obtained by adding thin hyperedges to $\mathcal{B}$, it is easy to see that $\mathcal{B}$ is also tightly-connected. Thus if $\mathcal{B}$ has at most two hyperedges, then it is a crown of size 1 or 2 and we are done. Therefore, in the rest of the proof we will assume that $\mathcal{B}$ contains at least 3 hyperedges.

If $\mathcal{B}$ contains at most 4 vertices then it is easy to see that $\mathcal{B}$ is either $K_3^4$ or $F_1$. So assume that $\mathcal{B}$ has at least 5 vertices (and at least 3 hyperedges). Since $\mathcal{B}$ is not a crown, there exists a tight path of length 3, say $abc, bcd, cde$. Since $abc$ is in the core, one of the pairs $ab$ or $ac$ is fat, so there exists a hyperedge $h \neq abc$ containing either $ab$ or $ac$. Similarly there exists a hyperedge $f \neq cde$ and $f$ contains $ed$ or $ec$. If $h = f$ then $\mathcal{B} \supseteq F_2$. However, it is easy to see that $F_2$ cannot be extended to a larger tightly-connected set of hyperedges without creating a Berge 5-cycle, so in this case $\mathcal{B} = F_2$. If $h \neq f$ then the hyperedges $h, abc, bcd, cde, f$ create a Berge 5-cycle in $H$, a contradiction. This completes the proof of the lemma.

Observation 9. Let $B$ be a block of $H$ and let $\mathcal{B}$ be the core of $B$. If $\mathcal{B} = \emptyset$ then the block $B$ is a crown, and if $\mathcal{B} \neq \emptyset$ then every fat pair of $B$ is contained in $\partial \mathcal{B}$.

**Edge Decomposition of $G = \partial H$.** We define a decomposition $\mathcal{D}$ of the edges of $G$ into paths of length 2, triangles and $K_4$’s such as follows:

Let $B$ be a block of $H$ and $\mathcal{B}$ be its core.
If $B = \emptyset$, then $B$ is a crown-block $\{abc_1, abc_2, \ldots, abc_k\}$ (for some $k \geq 1$); we partition $\partial B$ into the triangle $abc_1$ and paths $ac_ib$ where $2 \leq i \leq k$.

If $B \neq \emptyset$, then our plan is to first partition $\partial B \setminus \partial B$. If $abc \in B \setminus \partial B$, then $abc$ is a thin hyperedge, so it contains at least 2 thin pairs, say $ab$ and $bc$. We claim that the pair $ac$ is in $\partial B$. Indeed, $ac$ has to be a fat pair, otherwise the block $B$ consists of only one hyperedge $abc$, so $B = \emptyset$ contradicting the assumption. So by Observation 9, $ac$ has to be a pair in $\partial B$. For every $abc \in B \setminus \partial B$ such that $ab$ and $bc$ are thin pairs, add the 2-path $abc$ to the edge decomposition $D$. This partitions all the edges in $\partial B \setminus \partial B$ into paths of length 2. So all we have left is to partition the edges of $\partial B$.

- If $B$ is a crown $\{abc_1, abc_2, \ldots, abc_k\}$ for some $k \geq 1$, then we partition $\partial B$ into the triangle $abc_1$ and paths $ac_ib$ where $2 \leq i \leq k$.
- If $B = F_1 = \{abc, bcd, acd\}$ then we partition $\partial B$ into 2-paths $abc, bcd$ and $cad$.
- If $B = F_2 = \{oab, obc, ocd, oda\}$ then we partition $\partial B$ into 2-paths $obo, bco, cdo$ and $dao$.
- Finally, if $B = K_3^4 = \{abc, abd, acd, bcd\}$ then we partition $\partial B$ as $K_4$, i.e., we add $\partial B = K_4$ as an element of $D$.

Clearly, by Lemma 8 we have no other cases left. Thus all of the edges of the graph $G$ are partitioned into paths of length 2, triangles and $K_4$’s.

**Observation 10.**

(a) If $D$ is a triangle that belongs to $D$, then there is a hyperedge $h \in H$ such that $D = \partial h$.

(b) If $abc$ is a 2-path that belongs to $D$, then $abc \in H$. Moreover $ac$ is a fat pair.

(c) If $D$ is a $K_4$ that belongs to $D$, then there exists $F = K_3^3 \subseteq H$ such that $D = \partial F$.

Let $\alpha_1 |G|$ and $\alpha_2 |G|$ be the number of edges of $G$ that are contained in triangles and 2-paths of the edge-decomposition $D$ of $G$, respectively. So $(1 - \alpha_1 - \alpha_2) |G|$ edges of $G$ belong to the $K_4$’s in $D$.

**Claim 11.** We have,

$$|H| = \left(\frac{\alpha_1}{3} + \frac{\alpha_2}{2} + \frac{2(1 - \alpha_1 - \alpha_2)}{3}\right) |G|.$$ 

**Proof.** Let $B$ be a block with the core $B$. Recall that for each hyperedge $h \in B \setminus B$, we have added exactly one 2-path or a triangle to $D$.

Moreover, because of the way we partitioned $\partial B$, it is easy to check that in all of the cases except when $B = K_3^3$, the number of hyperedges of $B$ is the same as the number of elements of $D$ that $\partial B$ is partitioned into; these elements being 2-paths and triangles. On the other hand, if $B = K_3^3$, then the number of hyperedges of $B$ is 4 but we added only one element to $D$ (namely $K_4$).
This shows that the number of hyperedges of $H$ is equal to the number of elements of $D$ that are 2-paths or triangles plus the number of hyperedges which are in copies of $K_4^3$ in $H$, i.e., 4 times the number of $K_4$'s in $D$. Since $\alpha_1 |G|$ edges of $G$ are in 2-paths, the number of elements of $D$ that are 2-paths is $\alpha_1 |G|/2$. Similarly, the number of elements of $D$ that are triangles is $\alpha_2 |G|/3$, and the number of $K_4$'s in $D$ is $(1 - \alpha_1 - \alpha_2) |G|/6$. Combining this with the discussion above finishes the proof of the claim.

The link of a vertex $v$ is the graph consisting of the edges $\{uw \mid uw \in H\}$ and is denoted by $L_v$.

Claim 12. $|L_v| \leq 2 |N(v)|$.

Proof. First let us notice that there is no path of length 5 in $L_v$. Indeed, otherwise, there exist vertices $v_0, v_1, \ldots, v_5$ such that $vv_{i-1}v_i \in H$ for each $1 \leq i \leq 5$ which means there is a Berge 5-cycle in $H$ formed by the hyperedges containing the pairs $vv_1, v_1v_2, v_2v_3, v_3v_4, v_4v$, a contradiction. So by the Erdős-Gallai theorem $|L_v| \leq \frac{\alpha_1}{2} |N(v)|$, proving the claim.

Lemma 13. Let $v \in V(H)$ be an arbitrary vertex, then the number of edges in $G[N(v)]$ is less than $8 |N(v)|$.

Proof. Let $G_v$ be a subgraph of $G$ on a vertex set $N(v)$, such that $xy \in G_v$ if and only if there exists a vertex $z \neq v$ such that $xyz \in H$. Then each edge of $G[N(v)]$ belongs to either $L_v$ or $G_v$, so $|G[N(v)]| \leq |L_v| + |G_v|$. Combining this with Claim 12, we get $|G[N(v)]| \leq |G_v| + 2 |N(v)|$. So it suffices to prove that $|G_v| < 6 |N(v)|$.

First we will prove that there is no path of length 12 in $G_v$. Let us assume by contradiction that $P = v_0, v_1, \ldots, v_{12}$ is a path in $G_v$. Since for each pair of vertices $v_i, v_{i+1}$, there is a hyperedge $v_i v_{i+1} x$ in $H$ where $x \neq v$, we can conclude that there is a subsequence $u_0, u_1, \ldots, u_6$ of $v_0, v_1, \ldots, v_{12}$ and a sequence of distinct hyperedges $h_1, h_2, \ldots, h_6$, such that $u_{i-1}u_i \subset h_i$ and $v \not\subset h_i$ for each $1 \leq i \leq 6$. Since $u_0, u_3, u_6 \in N(v)$ there exist hyperedges $f_1, f_2, f_3 \in H$ such that $uu_0 \subset f_1$, $uu_3 \subset f_2$ and $uu_6 \subset f_3$. Clearly, either $f_1 \neq f_2$ or $f_2 \neq f_3$. In the first case the hyperedges $f_1, h_1, h_2, h_3, f_2$, and in the second case the hyperedges $f_2, h_4, h_5, h_6, f_3$ form a Berge 5-cycle in $H$, a contradiction.

Therefore, there is no path of length 12 in $G_v$, so by the Erdős-Gallai theorem, the number of edges in $G_v$ is at most $\frac{12 - 1}{2} |N(v)| < 6 |N(v)|$, as required.

2.1 Relating the hypergraph degree to the degree in the shadow

For a vertex $v \in V(H) = V(G)$, let $d(v)$ denote the degree of $v$ in $H$ and let $d_G(v)$ denote the degree of $v$ in $G$ (i.e., $d_G(v)$ is the degree in the shadow).

Clearly $d_G(v) \leq 2d(v)$. Moreover, $d(v) = |L_v|$ and $d_G(v) = |N(v)|$. So by Claim 12, we have

$$\frac{d_G(v)}{2} \leq d(v) \leq 2d_G(v). \quad (1)$$

Let $\bar{d}$ and $\bar{d}_G$ be the average degrees of $H$ and $G$ respectively.

Suppose there is a vertex $v$ of $H$, such that $d(v) < \bar{d}/3$. Then we may delete $v$ and all the edges incident to $v$ from $H$ to obtain a graph $H'$ whose average degree is more than
3(\(n \bar{d}/3 - \bar{d}/3\))/(n - 1) = \bar{d}. Then it is easy to see that if the theorem holds for \(H'\), then it holds for \(H\) as well. Repeating this procedure, we may assume that for every vertex \(v\) of \(H\), \(d(v) \geq \bar{d}/3\). Therefore, by (1), we may assume that the degree of every vertex of \(G\) is at least \(\bar{d}/6\).

### 2.2 Counting paths of length 3

**Definition 14.** A 2-path in \(\partial H\) is called bad if both of its edges are contained in a triangle of \(\partial H\), otherwise it is called good.

**Lemma 15.** For any vertex \(v \in V(G)\) and a set \(M \subseteq N(v)\), let \(P\) be the set of the good 2-paths \(vxy\) such that \(x \in M\). Let \(M' = \{y \mid vxy \in P\}\) then \(|P| < 2|M'| + 48d_G(v)\).

**Proof.** Let \(B_P = \{xy \mid x \in M, y \in M', xy \in G\}\) be a bipartite graph, clearly \(|B_P| = |P|\). Let \(E = \{xyz \in H \mid x, y \in N(v), \text{codeg}(x, y) \leq 2\}\). By Lemma 13, \(|E| \leq 2 \cdot 8|N(v)|\) so the number of edges of 2-shadow of \(E\) is \(|\partial E| \leq 48|N(v)|\). Let \(B = \{xy \in B_P \mid \exists z \in V(H), xyz \in H \setminus E\}\). Then clearly,

\[|B| \geq |B_P| - |\partial E| \geq |P| - 48|N(v)| = |P| - 48d_G(v).\]  

Let \(d_B(x)\) denote the degree of a vertex \(x\) in the graph \(B\).

**Claim 16.** For every \(y \in M'\) such that \(d_B(y) = k \geq 3\), there exists a set of \(k - 2\) vertices \(S_y \subseteq M'\) such that \(\forall w \in S_y\) we have \(d_B(w) = 1\). Moreover, \(S_y \cap S_z = \emptyset\) for any \(y \neq z \in M'\) (with \(d_B(y), d_B(z) \geq 3\)).

**Proof.** Let \(yx_1, yx_2, \ldots, yx_k \in B\) be the edges of \(B\) incident to \(y\). For each \(1 \leq j \leq k\) let \(f_j \in H\) be a hyperedge such that \(vx_j \subseteq f_j\). For each \(yx_i \in B\) clearly there is a hyperedge \(yx_iw_i \in H \setminus E\).

We claim that for each \(1 \leq i \leq k\), \(w_i \in M'\). It is easy to see that \(w_i \in N(v)\) or \(w_i \in M'\) (because \(vx_iw_i\) is a 2-path in \(G\)). Assume for a contradiction that \(w_i \in N(v)\), then since \(yx_iw_i \notin E\) we have, \(\text{codeg}(x_i, w_i) \geq 3\). Let \(f \in H\) be a hyperedge such that \(vw_i \subseteq f\). Now take \(j \neq i\) such that \(x_j \neq w_i\). If \(f_j \neq f\) then since \(\text{codeg}(x_i, w_i) \geq 3\) there exists a hyperedge \(h \supseteq x_iw_i\) such that \(h \neq f\) and \(h \neq x_iw_i\), then the hyperedges \(f, h, x_iw_i, yx_jw_j, f_j\) form a Berge 5-cycle. So \(f_j = f\), therefore \(f_j \neq f_i\). Similarly in this case, there exists a hyperedge \(h \supseteq x_iw_i\) such that \(h \neq f_i\) and \(h \neq x_iw_i\), therefore the hyperedges \(f_i, h, x_iw_i, yx_jw_j, f_j\) form a Berge 5-cycle, a contradiction. So we proved that \(w_i \in M'\) for each \(1 \leq i \leq k\).

**Claim 17.** For all but at most 2 of the \(w_i\)'s (where \(1 \leq i \leq k\)), we have \(d_B(w_i) = 1\).

**Proof.** If \(d_B(w_i) = 1\) for all \(1 \leq i \leq k\) then we are done, so we may assume that there is \(1 \leq i \leq k\) such that \(d_B(w_i) \neq 1\).

For each \(1 \leq i \leq k\), \(w_i \in M'\) and \(x_iw_i \in \partial(H \setminus E)\) (because \(x_iw_iy \in H \setminus E\)), so it is clear that \(d_B(w_i) \geq 1\). So \(d_B(w_i) > 1\). Then there is a vertex \(x \in M \setminus \{x_i\}\) such that \(w_ix \in B\). Let \(f, h \in H\) be hyperedges with \(w_ix \in h\) and \(xv \in f\). If there are \(j, l \in \{1, 2, \ldots, k\}\) such that \(x, x_j\) and \(x_l\) are all different from each other, then
clearly, either \( f \neq f_j \) or \( f \neq f_i \), so without loss of generality we may assume \( f \neq f_j \). Then the hyperedges \( f, h, w_i x_j y, y w_j x_j, f_j \) create a Berge cycle of length 5, a contradiction. So there are no \( j, l \in \{1, 2, \ldots, k\} \setminus \{i\} \) such that \( x, x_j \) and \( x_l \) are all different from each other. Clearly this is only possible when \( k < 4 \) and there is a \( j \in \{1, 2, 3\} \setminus \{i\} \) such that \( x = x_j \). Let \( l \in \{1, 2, 3\} \setminus \{i, j\} \). If \( f_j \neq f_l \) then the hyperedges \( f_j, h, w_i x_j y, y w_j x_j, f_l \) form a Berge 5-cycle. Therefore \( f_j = f_l \). So we proved that \( d_B(w_i) \neq 1 \) implies that \( k = 3 \) and for \( \{j, l\} = \{1, 2, 3\} \setminus \{i\} \), we have \( f_j = f_l \).

Since the minimum degree in \( S \) is at least \( 160 \) and \( \sum_{i \in S} \deg_i \) is the number of \( 2 \)-paths
\[ \sum_{i \in S} \deg_i = \sum_{i \in S} \deg_i x_i y_i . \]

For each \( x \in M' \) with \( d_B(x) = k \geq 3 \), let \( S_x \) be defined as in Claim 16. Then the average of the degrees of the vertices in \( S_x \cup \{x\} \) in \( B \) is \( (k + |S_x|)/ (k - 1) = (2k - 2)(k - 1) = 2 \). Since the sets \( S_x \cup x \) (with \( x \in M' \), \( d_B(x) \geq 3 \)) are disjoint, we can conclude that the average degree of the set \( M' \) is at most 2. Therefore \( 2 |M'| \geq |B| \). So by (2) we have \( 2 |M'| \geq |B| > |P| - 48d_G(V) \), which completes the proof of the lemma.

Claim 18. We may assume that the maximum degree in the graph \( G \) is less than \( 160 \sqrt{n} \) when \( n \) is large enough.

Proof. Let \( v \) be an arbitrary vertex with \( d_G(v) = C \bar{d} \) for some constant \( C > 0 \). Let \( P \) be the set of the good \( 2 \)-paths starting from the vertex \( v \). Then applying Lemma 15 with \( M = N(v) \) and \( M' = \{y \mid vxy \in P \} \), we have \( |P| < 2 |M'| + 48d_G(v) < 2n + 48 \cdot C \bar{d} \). Since the minimum degree in \( G \) is at least \( \bar{d}/6 \), the number of (ordered) \( 2 \)-paths starting from \( v \) is at least \( d(v) \cdot (\bar{d}/6 - 1) = C \bar{d} \cdot (\bar{d}/6 - 1) \). Notice that the number of (ordered) bad \( 2 \)-paths starting at \( v \) is the number of \( 2 \)-paths \( vxy \) such that \( x, y \in N(v) \). So by Lemma 13, this is at most \( 2 \cdot 8 |N(v)| = 16 C \bar{d} \), so the number of good \( 2 \)-paths is at least \( C \bar{d} \cdot (\bar{d}/6 - 17) \). So \( |P| \geq C \bar{d} \cdot (\bar{d}/6 - 17) \). Thus we have
\[ C \bar{d} \cdot (\bar{d}/6 - 17) \leq |P| < 2n + 48C \bar{d} . \]
So $C \bar{d}(\bar{d}/6 - 65) < 2n$. Therefore, $6C(\bar{d}/6 - 65)^2 < 2n$, i.e., $\bar{d} < 6\sqrt{n/3C} + 390$, so $|H| = n\bar{d}/3 < 2n\sqrt{n/3C} + 130n$. If $C \geq 36$ we get that $|H| \leq \frac{n^{3/2}}{3\sqrt{3}} + 130n = \frac{n^{3/2}}{3\sqrt{3}} + O(n)$, proving Theorem 3. So we may assume $C < 36$.

Theorem 2 implies that

$$|H| = n\bar{d}/3 \leq \sqrt{2}n^{3/2} + 4.5n,$$

so $\bar{d} \leq 3\sqrt{2}\sqrt{n} + 13.5$. So combining this with the fact that $C < 36$, we have $d_G(v) = C\bar{d} < 108\sqrt{2}\sqrt{n} + 486 < 160\sqrt{n}$ for large enough $n$.

Combining Lemma 15 and Claim 18, we obtain the following.

**Lemma 19.** For any vertex $v \in V(G)$ and a set $M \subseteq N(v)$, let $\mathcal{P}$ be the set of good 2-paths $vxy$ such that $x \in M$. Let $M' = \{y \mid vxy \in \mathcal{P}\}$ then $|\mathcal{P}| < 2|M'| + 7680\sqrt{n}$ when $n$ is large enough.

**Definition 20.** A 3-path $x_0, x_1, x_2, x_3$ is called *good* if both 2-paths $x_0, x_1, x_2$ and $x_1, x_2, x_3$ are good 2-paths.

**Claim 21.** The number of (ordered) good 3-paths in $G$ is at least $n\bar{d}^3_G - C_0n^{3/2}\bar{d}_G$ for some constant $C_0 > 0$ (for large enough $n$).

**Proof.** First we will prove that the number of (ordered) 3-walks that are not good 3-paths is at most $5440n^{3/2}\bar{d}_G$.

For any vertex $x \in V(H)$ if a path $yzx$ is a bad 2-path then $zy$ is an edge of $G$, so the number of (ordered) bad 2-paths whose middle vertex is $x$, is at most 2 times the number of edges in $G[N(x)]$, which is less than $2 \cdot 8|N(x)| = 16d_G(x)$ by Lemma 13. The number of 2-walks which are not 2-paths and whose middle vertex is $x$ is exactly $d_G(x)$. So the total number of (ordered) 2-walks that are not good 2-paths is at most $\sum_{x \in V(H)} 17d_G(x) = 17n\bar{d}_G$.

Notice that, by definition, any (ordered) 3-walk that is not a good 3-path must contain a 2-walk that is not a good 2-path. Moreover, if $xyz$ is a 2-walk that is not a good 2-path, then the number of 3-walks in $G$ containing it is at most $d_G(x) + d_G(z) < 320\sqrt{n}$ (for large enough $n$) by Claim 18. Therefore, the total number of (ordered) 3-walks that are not good 3-paths is at most $17n\bar{d}_G \cdot 320\sqrt{n} = 5440n^{3/2}\bar{d}_G$.

By the Blakley-Roy inequality, the total number of (ordered) 3-walks in $G$ is at least $n\bar{d}^3_G$. By the above discussion, all but at most $5440n^{3/2}\bar{d}_G$ of them are good 3-paths, so letting $C_0 = 5440$ proves the proof of the claim.

**Claim 22.** Let $\{a, b, c\}$ be the vertex set of a triangle that belongs to $\mathcal{D}$. (By Observation 10 (a) $abc \in H$.) Then the number of good 3-paths whose first edge is $ab$, $bc$ or $ca$ is at most $8n + C_1\sqrt{n}$ for some constant $C_1$ and for large enough $n$.

**Proof.** For each $\{x, y\} \subset \{a, b, c\}$, let $S_{xy} = N(x) \cap N(y) \setminus \{a, b, c\}$. For each $x \in \{a, b, c\}$, let $S_x = N(x) \setminus (N(y) \cup N(z) \cup \{a, b, c\})$ where $\{y, z\} = \{a, b, c\} \setminus \{x\}$.
For each \( x \in \{a, b, c\} \), let \( \mathcal{P}_x \) be the set of good 2-paths \( xuv \) where \( u \in S_x \). Let \( S'_x = \{v \mid xw \in \mathcal{P}_x\} \). For each \( \{x, y\} \subset \{a, b, c\} \), let \( \mathcal{P}_{xy} \) be the set of good 2-paths \( xuv \) and \( yuv \) where \( u \in S_{xy} \). Let \( S''_{xy} = \{v \mid xw \in \mathcal{P}_{xy}\} \).

Let \( \{x, y\} \subset \{a, b, c\} \) and \( z = \{a, b, c\} \setminus \{x, y\} \). Notice that each 2-path \( yuv \in \mathcal{P}_{xy} \) (\( xw \in \mathcal{P}_{xy} \)), is contained in at most one good 3-path \( zyuv \) (respectively \( zxuv \)) whose first edge is in the triangle \( abc \). Indeed, since \( u \in S_{xy}, xyuv \) (respectively \( yxuv \)) is not a good 3-path. Therefore, the number of good 3-paths whose first edge is in the triangle \( abc \), and whose third vertex is in \( S_{xy} \) is at most \( |\mathcal{P}_{xy}| \). The number of paths in \( \mathcal{P}_{xy} \) that start with the vertex \( x \) is less than \( 2 |S'_{xy}| + 7860\sqrt{n} \), by Lemma 19. Similarly, the number of paths in \( \mathcal{P}_{xy} \) that start with the vertex \( y \) is less than \( 2 |S'_{xy}| + 7860\sqrt{n} \). Since every path in \( \mathcal{P}_{xy} \) starts with either \( x \) or \( y \), we have \( |\mathcal{P}_{xy}| < 4 |S'_{xy}| + 15360\sqrt{n} \). Therefore, for any \( \{x, y\} \subset \{a, b, c\} \), the number of good 3-paths whose first edge is in the triangle \( abc \), and whose third vertex is in \( S_{xy} \) is less than \( 4 |S'_{xy}| + 15360\sqrt{n} \).

In total, the number of good 3-paths whose first edge is in the triangle \( abc \) and whose third vertex is in \( S_{ab} \cup S_{bc} \cup S_{ac} \) is at most

\[
4(|S'_{ab}| + |S'_{bc}| + |S'_{ac}|) + 46080\sqrt{n}. \tag{4}
\]

Let \( x \in \{a, b, c\} \) and \( \{y, z\} = \{a, b, c\} \setminus \{x\} \). For any 2-path \( xuv \in \mathcal{P}_x \) there are 2 good 3-paths with the first edge in the triangle \( abc \), namely \( yxuv \) and \( zxuv \). So the total number of 3-paths whose first edge is in the triangle \( abc \) and whose third vertex is in \( S_a \cup S_b \cup S_c \) is \( 2(|\mathcal{P}_a| + |\mathcal{P}_b| + |\mathcal{P}_c|) \), which is at most

\[
4(|S'_a| + |S'_b| + |S'_c|) + 46080\sqrt{n}, \tag{5}
\]

by Lemma 19.

Now we will prove that every vertex is in at most 2 of the sets \( S'_a, S'_b, S'_c, S'_{ab}, S'_{bc}, S'_{ac} \). Let us assume by contradiction that a vertex \( v \in V(G) \setminus \{a, b, c\} \) is in at least 3 of them. We claim that there do not exist 3 vertices \( u_a \in N(a) \setminus \{b, c\}, u_b \in N(b) \setminus \{a, c\} \) and \( u_c \in N(c) \setminus \{a, b\} \) such that \( xuv \) is a good 3-path for each \( x \in \{a, b, c\} \). Indeed, otherwise, consider hyperedges \( h_a, h_a' \) containing the pairs \( au_a \) and \( au_v \) respectively (since \( au_v \) is a good 2-path, note that \( h_a \neq h_a' \)), and hyperedges \( h_b, h_b', h_c, h_c' \) containing the pairs \( bu_a, bu_v, cu_c, u_v \) respectively. Then either \( h_a' \neq h_b' \) or \( h_a' \neq h_c' \), say \( h_a' \neq h_b' \) without loss of generality. Then the hyperedges \( h_a, h_a', h_b, h_c, abc \) create a Berge 5-cycle in \( H \), a contradiction, proving that it is impossible to have 3 vertices \( u_a \in N(a) \setminus \{b, c\}, u_b \in N(b) \setminus \{a, c\} \) and \( u_c \in N(c) \setminus \{a, b\} \) with the above mentioned property. Without loss of generality let us assume that there is no vertex \( u_a \in N(a) \setminus \{b, c\} \) such that \( au_a \) is a good 2-path – in other words, \( v \notin S'_a \cup S'_{ab} \cup S'_{ac} \). However, since we assumed that \( v \) is contained in at least 3 of the sets \( S'_a, S'_b, S'_c, S'_{ab}, S'_{bc}, S'_{ac} \), we can conclude that \( v \) is contained in all 3 of the sets \( S'_b, S'_c, S'_{bc} \), i.e., there are vertices \( u_b \in S_b, u_c \in S_c, u \in S_{bc} \) such that \( vu_b, vu_c, vub, vuc \) are good 2-paths. Using a similar argument as before, if \( vu_b \in h, vu_c \in h_b \) and \( vu_b, vuc \in h_c \), without loss of generality we can assume that \( h \neq h_b \), so the hyperedges \( abc, h, h_b \) together with hyperedges containing \( uc \) and \( ub \) form a Berge 5-cycle in \( H \), a contradiction.
So we proved that

\[2 |S'_a \cup S'_b \cup S'_c \cup S'_{ab} \cup S'_{bc} \cup S'_{ac}| \geq |S'_a| + |S'_b| + |S'_c| + |S'_{ab}| + |S'_{bc}| + |S'_{ac}|\]

This together with (4) and (5), we get that the number of good 3-paths whose first edge is in the triangle \(abc\) is at most

\[8 |S'_a \cup S'_b \cup S'_c \cup S'_{ab} \cup S'_{bc} \cup S'_{ac}| + 92160\sqrt{n} < 8n + C_1\sqrt{n}\]

for \(C_1 = 92160\) and large enough \(n\), finishing the proof of the claim. \(\square\)

**Claim 23.** Let \(P = abc\) be a 2-path and \(P \in \mathcal{D}\). (By Observation 10 (b) \(abc \in H\).) Then the number of good 3-paths whose first edge is \(ab\) or \(bc\) is at most \(4n + C_2\sqrt{n}\) for some constant \(C_2 > 0\) and large enough \(n\).

**Proof.** First we bound the number of 3-paths whose first edge is \(ab\). Let \(S_{ab} = N(a) \cap N(b)\). Let \(S_a = N(a) \setminus (N(b) \cup \{b\})\) and \(S_b = N(b) \setminus (N(a) \cup \{a\})\). For each \(x \in \{a, b\}\), let \(P_x\) be the set of good 2-paths \(xuv\) where \(u \in S_x\), and let \(S'_x = \{v \mid xuv \in P_x\}\). The set of good 3-paths whose first edge is \(ab\) is \(P_a \cup P_b\), because the third vertex of a good 3-path starting with an edge \(ab\) can not belong to \(N(a) \cap N(b)\) by the definition of a good 3-path.

We claim that \(|S'_a \cap S'_b| \leq 160\sqrt{n}\). Let us assume by contradiction that \(v_0, v_1, \ldots, v_k \in S'_a \cap S'_b\) for \(k > 160\sqrt{n}\). For each vertex \(v_i\) where \(0 \leq i \leq k\), there are vertices \(a_i \in S_a\) and \(b_i \in S_b\) such that \(aa_iv_i, bb_iv_i\) are good 2-paths. For each \(0 \leq i \leq k\), the hyperedge \(a_i v_i b_i\) is in \(H\), otherwise we can find distinct hyperedges containing the pairs \(aa_i, a_i v_i, v_i b_i, b_i b\) and these hyperedges together with \(abc\), would form a Berge 5-cycle in \(H\), a contradiction.

We claim that there are \(j, l \in \{0, 1, \ldots, k\}\) such that \(a_j \neq a_l\), otherwise there is a vertex \(x\) such that \(x = a_i\) for each \(0 \leq i \leq k\). Then \(xv_i \in G\) for each \(0 \leq i \leq k\), so we get that \(d_G(x) > k > 160\sqrt{n}\) which contradicts Claim 18.

So there are \(j, l \in \{0, 1, \ldots, k\}\) such that \(a_j \neq a_l\) and \(a_j v_j b_j, a_l v_l b_l \in H\). By observation 10 (b), there is a hyperedge \(h \neq abc\) such that \(ac \subset h\). Clearly either \(a_j \notin h\) or \(a_l \notin h\). Without loss of generality let \(a_j \notin h\), so there is a hyperedge \(h_a\) with \(aa_j \subset h_a \neq h\). Let \(h_b \supset b, b, h, h\), then the hyperedges \(abc, h, h, a_i v_i b_j, h_b\) form a Berge 5-cycle, a contradiction, proving that \(|S'_a \cap S'_b| \leq 160\sqrt{n}\).

Notice that \(|S'_a| + |S'_b| = |S'_a \cup S'_b| + |S'_a \cap S'_b| \leq n + 160\sqrt{n}\). So by Lemma 19, we have

\[|P_a| + |P_b| \leq 2(|S'_a| + |S'_b|) + 2 \cdot 7680\sqrt{n} \leq 2(n + 160\sqrt{n}) + 2 \cdot 7680\sqrt{n} = 2n + 15680\sqrt{n}\]

for large enough \(n\). So the number of good 3-paths whose first edge is \(ab\) is at most \(2n + 15680\sqrt{n}\). By the same argument, the number of good 3-paths whose first edge is \(bc\) is at most \(2n + 15680\sqrt{n}\). Their sum is at most \(4n + 2C_2\sqrt{n}\) for \(C_2 = 31360\) and large enough \(n\), as desired. \(\square\)

**Claim 24.** Let \(\{a, b, c, d\}\) be the vertex set of a \(K_4\) that belongs to \(\mathcal{D}\). Let \(F = K_4^4\) be a hypergraph on the vertex set \(\{a, b, c, d\}\). (By Observation 10 (c) \(F \subseteq H\).) Then the number of good 3-paths whose first edge belongs to \(\partial F\) is at most \(6n + C_3\sqrt{n}\) for some constant \(C_3 > 0\) and large enough \(n\).
Proof. First, let us observe that there is no Berge path of length 2, 3 or 4 between distinct vertices \( x, y \in \{a, b, c, d\} \) in the hypergraph \( H \setminus F \), because otherwise this Berge path together with some edges of \( F \) will form a Berge 5-cycle in \( H \). This implies, that there is no path of length 3 or 4 between \( x \) and \( y \) in \( G \setminus \partial F \), because otherwise we would find a Berge path of length 2, 3 or 4 between \( x \) and \( y \) in \( H \setminus F \).

Let \( S = \{ u \in V(H) \setminus \{a, b, c, d\} \mid \exists \{x, y\} \subseteq \{a, b, c, d\}, u \in N(x) \cap N(y)\} \). For each \( x \in \{a, b, c, d\} \), let \( S_x = N(x) \setminus (S \cup \{a, b, c, d\}) \). Let \( \mathcal{P}_S \) be the set of good 2-paths \( xuv \) where \( x \in \{a, b, c, d\} \) and \( u \in S \). Let \( S' = \{ v \mid xuv \in \mathcal{P}_S \} \). For each \( x \in \{a, b, c, d\} \), let \( \mathcal{P}_x \) be the set of good 2-paths \( xuv \) where \( u \in S_x \), and let \( S_x' = \{ v \mid xuv \in \mathcal{P}_x \} \).

Let \( v \in S' \). By definition, there exists a pair of vertices \( \{x, y\} \subseteq \{a, b, c, d\} \) and a vertex \( u \), such that \( xuv \) and \( yuv \) are good 2-paths.

Suppose that \( zu'v \) is a 2-path different from \( xuv \) and \( yuv \) where \( z \in \{a, b, c, d\} \). If \( u' = u \) then \( z \notin \{x, y\} \) so there is a Berge 2-path between \( x \) and \( y \) or between \( x \) and \( z \) in \( H \setminus F \), which is impossible. So \( u \neq u' \). Either \( z \neq x \) or \( z \neq y \), without loss of generality let us assume that \( z \neq x \). Then \( zu'vux \) is a path of length 4 in \( G \setminus \partial F \), a contradiction.

So for any \( v \in S' \) there are only 2 paths of \( \mathcal{P}_a \cup \mathcal{P}_b \cup \mathcal{P}_c \cup \mathcal{P}_d \cup \mathcal{P}_S \) that contain \( v \) as an end vertex – both of which are in \( \mathcal{P}_S \) – which means that \( v \notin S'_a \cup S'_b \cup S'_c \cup S'_d \), so \( S' \cap (S'_a \cup S'_b \cup S'_c \cup S'_d) = \emptyset \). Moreover, \( |\mathcal{P}_S| \leq 2|S'| \). \( (6) \)

We claim that \( S'_a \) and \( S'_b \) are disjoint. Indeed, otherwise, if \( v \in S'_a \cap S'_b \) there exists \( x \in S_a \) and \( y \in S_b \) such that \( vxa \) and \( vyb \) are paths in \( G \), so there is a 4-path \( axvyb \) between vertices of \( F \) in \( G \setminus \partial F \), a contradiction. Similarly we can prove that \( S'_a \), \( S'_b \), \( S'_c \) and \( S'_d \) are pairwise disjoint. This shows that the sets \( S', S'_a, S'_b, S'_c \) and \( S'_d \) are pairwise disjoint. So we have

\[
|S' \cup S'_a \cup S'_b \cup S'_c \cup S'_d| = |S'| + |S'_a| + |S'_b| + |S'_c| + |S'_d| . \tag{7}
\]

By Lemma 19, we have \( |\mathcal{P}_a| + |\mathcal{P}_b| + |\mathcal{P}_c| + |\mathcal{P}_d| \leq 2(|S'_a| + |S'_b| + |S'_c| + |S'_d|) + 4 \cdot 7680 \sqrt{n} \). Combining this inequality with (6), we get

\[
|\mathcal{P}_S| + |\mathcal{P}_a| + |\mathcal{P}_b| + |\mathcal{P}_c| + |\mathcal{P}_d| \leq 2|S'| + 2(|S'_a| + |S'_b| + |S'_c| + |S'_d|) + 4 \cdot 7680 \sqrt{n} . \tag{8}
\]

Combining (7) with (8) we get

\[
|\mathcal{P}_S| + |\mathcal{P}_a| + |\mathcal{P}_b| + |\mathcal{P}_c| + |\mathcal{P}_d| \leq 2|S' \cup S'_a \cup S'_b \cup S'_c \cup S'_d| + 30720 \sqrt{n} < 2n + 30720 \sqrt{n} , \tag{9}
\]

for large enough \( n \).

Each 2-path in \( \mathcal{P}_S \cup \mathcal{P}_a \cup \mathcal{P}_b \cup \mathcal{P}_c \cup \mathcal{P}_d \) can be extended to at most three good 3-paths whose first edge is in \( \partial F \). (For example, \( awv \in \mathcal{P}_a \) can be extended to \( bauvw \) and \( dauv \).) On the other hand, every good 3-path whose first edge is in \( \partial F \) must contain a 2-path of \( \mathcal{P}_a \cup \mathcal{P}_b \cup \mathcal{P}_c \cup \mathcal{P}_d \cup \mathcal{P}_S \) as a subpath. So the number of good 3-paths whose first edge is in \( \partial F \) is at most \( 3(|\mathcal{P}_S| + |\mathcal{P}_a| + |\mathcal{P}_b| + |\mathcal{P}_c| + |\mathcal{P}_d|) \) which is at most \( 6n + C_3 \sqrt{n} \) by (9), for \( C_3 = 92160 \) and large enough \( n \), proving the desired claim. \( \square \)
2.3 Combining bounds on the number of 3-paths

Recall that \(\alpha_1 |G|, \alpha_2 |G|, (1 - \alpha_1 - \alpha_2) |G|\) are the number of edges of \(G\) that are contained in triangles, 2-paths and \(K_4\)'s of the edge-decomposition \(D\) of \(G\), respectively. Then the number of triangles, 2-paths and \(K_4\)'s in \(D\) is \(\alpha_1 |G|/3, \alpha_2 |G|/2\) and \((1 - \alpha_1 - \alpha_2) |G|/6\) respectively. Therefore, using Claim 22, Claim 23 and Claim 24, the total number of (ordered) good 3-paths in \(G\) is at most

\[
\frac{\alpha_1}{3} |G| (8n + C_1 \sqrt{n}) + \frac{\alpha_2}{2} |G| (4n + C_2 \sqrt{n}) + \frac{(1 - \alpha_1 - \alpha_2)}{6} |G| (6n + C_3 \sqrt{n}) \leq |G| n \left( \frac{8\alpha_1}{3} + 2\alpha_2 + (1 - \alpha_1 - \alpha_2) \right) + (C_1 + C_2 + C_3) \sqrt{n} |G| =
\]

\[
= \frac{n^2 \overline{d}_G}{2} \left( \frac{5\alpha_1 + 3\alpha_2 + 3}{3} \right) + (C_1 + C_2 + C_3) \frac{n^{3/2} \overline{d}_G}{2}.
\]

Combining this with the fact that the number of good 3-paths is at least \(n \overline{d}_G^3 - C_0 n^{3/2} \overline{d}_G\) (see Claim 21), we get

\[
nd_G^3 - C_0 n^{3/2} \overline{d}_G \leq \frac{n^2 \overline{d}_G}{2} \left( \frac{5\alpha_1 + 3\alpha_2 + 3}{3} \right) + (C_1 + C_2 + C_3) \frac{n^{3/2} \overline{d}_G}{2}.
\]

Rearranging and dividing by \(n \overline{d}_G\) on both sides, we get

\[
\overline{d}_G^2 \leq \left( \frac{5\alpha_1 + 3\alpha_2 + 3}{6} \right) n + \frac{1}{2} \sqrt{n}((C_1 + C_2 + C_3) + 2C_0).
\]

Since \((5\alpha_1 + 3\alpha_2 + 3)/6 \geq 1/2\), we may replace \(1/2\) with \((5\alpha_1 + 3\alpha_2 + 3)/6\) in the above inequality to obtain

\[
\overline{d}_G^2 \leq \left( \frac{5\alpha_1 + 3\alpha_2 + 3}{6} \right) n \left( 1 + \frac{(C_1 + C_2 + C_3) + 2C_0}{\sqrt{n}} \right).
\]

So letting \(C_4 = (C_1 + C_2 + C_3) + 2C_0\) we have,

\[
\overline{d}_G \leq 1 + \frac{C_4}{\sqrt{n}} \sqrt{\frac{5\alpha_1 + 3\alpha_2 + 3}{6}} \sqrt{n} < \left( 1 + \frac{C_4}{2\sqrt{n}} \right) \sqrt{\frac{5\alpha_1 + 3\alpha_2 + 3}{6}} \sqrt{n}, \quad \text{(10)}
\]

for large enough \(n\). By Claim 11, we have

\[
|H| \leq \frac{\alpha_1}{3} |G| + \frac{\alpha_2}{2} |G| + \frac{2(1 - \alpha_1 - \alpha_2)}{3} |G| = \frac{4 - 2\alpha_1 - \alpha_2}{6} n \overline{d}_G.
\]

Combining this with (10) we get

\[
|H| \leq \left( 1 + \frac{C_4}{2\sqrt{n}} \right) \frac{(4 - 2\alpha_1 - \alpha_2)}{12} \sqrt{\frac{5\alpha_1 + 3\alpha_2 + 3}{6}} n^{3/2},
\]
for sufficiently large \( n \). So we have

\[
\text{ex}_3(n, C_5) \leq (1 + o(1)) \frac{(4 - 2\alpha_1 - \alpha_2)}{12} \sqrt{\frac{5\alpha_1 + 3\alpha_2 + 3}{6} n^{3/2}}.
\]

The right hand side is maximized when \( \alpha_1 = 0 \) and \( \alpha_2 = 2/3 \), so we have

\[
\text{ex}_3(n, C_5) \leq (1 + o(1)) \frac{4 - 2/3}{12} \sqrt{\frac{5}{6} n^{1.5}} < (1 + o(1))0.2536n^{3/2}.
\]

This finishes the proof.

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