Abstract. The article deals with the problem of finding vertex-minimal graphs with a given automorphism group. We exhibit two undirected 16-vertex graphs having automorphism groups $A_4$ and $A_5$. It improves the Babai’s bound for $A_4$ and the graphical regular representation bound for $A_5$. The graphs are constructed using projectivisation of the vertex-face graph of icosahedron.

Key words. graph, icosahedron, hemi-icosahedron, automorphism group, alternating group.

AMS subject classifications. 05C25, 05E18, 05C35.

1. Introduction.

1.1. Outline. This article addresses a problem in graph representation theory of finite groups - finding undirected graphs with a given automorphism group and minimal number of vertices.

Denote by $\mu(G)$ the minimal number of vertices of undirected graphs having automorphism group isomorphic to $G$, $\mu(G) = \min_{\Gamma: \text{Aut}(\Gamma) \cong G} |V(\Gamma)|$. It is known [1] that $\mu(G) \leq 2|G|$, for any finite group $G$ which is not cyclic of order 3, 4 or 5. See Babai [2] and Cameron [4], for expositions of this area. There are groups which admit a graphical regular representation, for such groups $\mu(G) \leq |G|$. For some recent work see [6], [7], [9].

For alternating groups $A_n$, $\mu(A_n)$ is known for $n \geq 13$, see Liebeck [10]. If $n \equiv 0 \text{ or } 1(\text{mod } 4)$, then $\mu(A_n) = 2^n - n - 2$. Additionally, for $n \geq 5$ $A_n$ admits a graphical regular representation, see [13]. Thus for $A_5$ the best published estimate until now seemed to be $\mu(A_5) \leq 60$.

In this paper we exhibit graphs $\Gamma_i = (V, E_i)$, $i \in \{4, 5\}$, such that $|V| = 16$ and $\text{Aut}(\Gamma_i) \cong A_i$. The graph $\Gamma_5$ (also denoted $\Pi_5$) is listed in [5] together with order of its automorphism group. These $\mu$ values are less than the Babai’s bound for groups $A_4$ and $A_5$. For $A_5$ our graph has fewer vertices than the graphical regular representation. The new graphs are based on projectivisation of vertex-face incidence relation of icosahedron.

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1.2. Notations. We use standard notations for undirected graphs, see Diestel [8]. A bipartite graph $\Gamma$ with vertex partition sets $V_1$ and $V_2$ is denoted as $\Gamma = (V_1, V_2, E)$.

Given a polyhedron $P$, we denote its vertex, edge and face sets as $V = V(P)$, $E = E(P)$ and $F = F(P)$, respectively. We can think of $P$ as the triple $(V, E, F)$.

If $S$ is a subset of $\mathbb{R}^3$ not containing the origin, then its image under a projectivisation map to $P(\mathbb{R}^3)$ is denoted by $\pi(S)$ or $[S]$, $[S] = \bigcup_{x \in S} [x]$.

2. Main results.

2.1. Vertex-face graphs of polyhedra.

**Definition 2.1.** Let $P = (V, E, F)$ be a polyhedron. An undirected bipartite graph $\Gamma_P = (V, F, I)$ is the vertex-face graph of $P$ if $v \sim f$ iff $v \in V$, $f \in F$ and $v \in f$. In other words, $\Gamma_P$ corresponds to the vertex-face incidence relation in $V \times F$.

**Definition 2.2.** Let $S = (V, E, F)$ be a centrally symmetric polyhedron. Let $S$ be positioned in $\mathbb{R}^3$ so that its center is at $(0, 0, 0)$. We call the undirected bipartite graph $\Pi_S = ([V], [F], I_P)$ projective vertex-face graph if for any $v_p \in [V]$, $f_p \in [F]$ we have $v_p \sim f_p$ iff $v \in f$ for some $v \in \pi^{-1}(v_p)$ and $f \in \pi^{-1}(f_p)$.

2.2. Projective vertex-face graph of icosahedron and $A_5$.

Let $I = (V, E, F)$ be a regular icosahedron. Denote by $\text{Rot}(I) \leq SO(3)$ the group of rotational symmetries of $I$ - rotations of $\mathbb{R}^3$ preserving $V$ and $E$. It is known that $\text{Rot}(I) \simeq A_5$. $\Pi_I$ is shown in Fig.1. We note that $\Pi_I$ can be interpreted as the vertex-face graph of hemi-icosahedron, see [12].

![Fig.1. - $\Pi_I$.](image)

**Proposition 2.3.** Let $I$ be regular icosahedron. Then $\text{Aut}(\Pi_I) \simeq A_5$. 
Proof. The stated fact can be checked using an appropriate software, such as Magma, see [3]. Nevertheless we give a proof based on the geometric construction. We prove that \( \text{Rot}(I) \cong \text{Aut}(\Pi_I) \) in two steps.

First we prove that there is a subgroup in \( \text{Aut}(\Pi_I) \) isomorphic to \( \text{Rot}(I) \). We show that there is an injective group morphism \( f_1 : \text{Rot}(I) \to \text{Aut}(\Gamma_I) \to \text{Aut}(\Pi_I) \). \( f_1 : \text{Rot}(I) \to \text{Aut}(\Gamma_I) \) maps every \( \rho \in \text{Rot}(I) \) to \( f_1(\rho) \in \text{Aut}(\Gamma_I) \) which is the permutation of \( V \cup F \) induced by \( \rho \): \( f_1(\rho)(x) = \rho(x) \) for any \( x \in V \cup F \). Rotations of \( I \) preserve the vertex-face incidence relation and \( f_1 \) is a group morphism. \( f_2 : \text{Aut}(\Gamma_I) \to \text{Aut}(\Pi_I) \) maps every \( \varphi \in \text{Aut}(\Gamma_I) \) to \( \varphi_P \in \text{Aut}(\Pi_I) \) defined by the rule \( \varphi_P([x]) = [\varphi(x)] \) for any \( x \in V(\Gamma_I) \). Projectivization and composition commute therefore \( f_2 \) is a group morphism. \( f \) is injective since there is no nontrivial rotation of \( I \) sending each vertex to another vertex in the same projective class.

In the second step we show that \( |\text{Aut}(\Pi_I)| \leq 60 \) by a counting argument. Every vertex \( v \in [V] \) is contained in a subgraph \( \sigma(v) \) shown in Fig.2.

![Fig.2. - \( \sigma(v) \).](image)

All \( \Pi_I \)-vertices in \([V]\) have degree 5, all \( \Pi_I \)-vertices in \([F]\) have degree 3. It follows that \([V]\) and \([F]\) both are unions of \( \text{Aut}(\Pi_I) \)-orbits. \( v \) can be mapped by a \( \Pi_I \)-automorphism in at most 6 possible ways. After fixing the image of \( v \) it follows again by \( \text{Aut}(\Pi_I) \)-invariance of \([V]\) that the subgraph \( \sigma(v) \) can be mapped in at most 10 ways. Any permutation of \([V]\) by an automorphism determines a unique permutation of \([F]\). Thus \( |\text{Aut}(\Pi_I)| \leq 60 \). We have shown that \( \text{Aut}(\Pi_I) = f(\text{Rot}(I)) \cong A_5 \). \( \square \)

Remark 2.4. A graph isomorphic to \( \Pi_I \) is listed without discussion of automorphism group in [5] as one of connected edge-transitive bipartite graphs, ET16.5.

2.3. A modification of the projective vertex-face graph of icosahedron and \( A_4 \).

Since \( A_5 \) has subgroups isomorphic to \( A_4 \), we can try to modify \( \Pi_I \) so that the automorphism group of the modified graph is isomorphic to \( A_4 \). We find generators for a subgroup \( H \leq \text{Rot}(I) \), such that \( H \cong A_4 \), and add 3 extra edges to \( \Pi_I \) which are permuted only by elements of \( H \).
Denote by $I_1$ the polyhedral (1-skeleton) graph of $I$, $Aut(I_1) \cong Sym(I) \cong A_5 \times Z_2$. An isomorphism $Sym(I) \to Aut(I_1)$ takes a symmetry $f$ to $f|_{I_1}$.

**Proposition 2.5.** Choose a 6-subset of vertices $W = \{O, A, B, C, D, E\} \subseteq V(I)$ such that $I_1[W]$ is isomorphic to the 5-wheel, see Fig.3.

![Fig.3 - $I_1[W]$](image)

Define an undirected graph $\Xi_I = ([V] \cup [F], I_p \cup J)$ by adding 3 edges to $\Pi_I$: $J = \{[A] \sim [C], [B] \sim [O], [D] \sim [E]\}$, see Fig.4. Then $Aut(\Xi_I) \cong A_4$.

![Fig.4 - extra edges.](image)

**Proof.** Consider the subgroup $H = \langle r_1, r_2 \rangle \leq Rot(I)$ generated by two rotations: $r_1$ - rotation by $\frac{2\pi}{3}$ radians around the line passing through the center of the face $OCD$ and the center of $I$, $r_2$ - rotation by $\pi$ radians around the line passing through the center of the edge $OB$ and the center of $I$.

It can be checked that $H \cong A_4$. Note that the vertices $O, A, B, C, D, E$ in Fig.3 represent the 6 projective classes of $V$.

We have to show that $Aut(\Xi_I) \cong H$. First we show that $H \leq Aut(\Xi_I)$. $\Xi_I$ differs from $\Pi_I$ by 3 extra edges. It suffices to note by direct inspection that $r_1$ permutes these extra edges and $r_2$ fixes each of them. To show that $Aut(\Xi_I) \leq H$ we observe that any additional rotation $r'$ does not permute these three new edges and thus $r' \notin Aut(\Xi_I)$. $\square$

**Remark 2.6.** If $D$ is dodecahedron then $\Pi_D \cong \Pi_I \cong A_5$.

**Acknowledgements.** We used Magma, see Bosma et al. [3], and Nauty, available at [http://cs.anu.edu.au/~bdm/data/](http://cs.anu.edu.au/~bdm/data/), see McKay and Piperno [11]. The author thanks Valentina Beinarovica for her assistance.
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