Energy dissipation of moved magnetic vortices

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Abstract – A two-dimensional easy-plane ferromagnetic substrate, interacting with a dipolar tip which is magnetised perpendicularly with respect to the easy plane is studied numerically by solving the Landau-Lifshitz Gilbert equation. The dipolar tip stabilises a vortex structure which is dragged through the system and dissipates energy. An analytical expression for the friction force in the \( v \to 0 \) limit based on the Thiele equation is presented. The limitations of this result which predicts a diverging friction force in the thermodynamic limit, are demonstrated by a study of the size dependence of the friction force. While for small system sizes the dissipation depends logarithmically on the system size, it saturates at a specific velocity-dependent value. This size can be regarded as an effective vortex size and it is shown how this effective vortex size agrees with the infinite extension of a vortex in the thermodynamic limit. A magnetic friction number is defined which represents a general criterion for the validity of the Thiele equation and quantifies the degree of nonlinearity in the response of a driven spin configuration.

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Introduction. – Vortices in magnetic layers have been known for a long time [1–7]. Although a vortex represents a strong excitation with a high energy, it has a long life time for topological reasons: Each vortex can be characterised by the vorticity, which is a conserved quantity for the entire system. To annihilate a vortex in a closed system, an antivortex (which has a negative vorticity and also cannot be created spontaneously) is required. In an open system, (anti)vortices can be created at the system boundary. A second quantity related to vortices is the polarisation, the out-of-plane magnetisation of the vortex core, which may adopt two states. Therefore a vortex state represents a bit, which can be easily probed, e.g., with GMR sensors, as those used in reading heads of magnetic hard disks. It can be manipulated at very short timescales (down to picoseconds) by magnetic field pulses [8,9], alternating magnetic fields [10] or spin-polarised currents [11], making magnetic vortices promising candidates for non-volatile storage concepts. In a previous work, it has been shown that vortex states may also be generated or annihilated by a magnetic tip scanning a magnetic substrate, when the interaction strength between tip and substrate as well as the scanning velocity is appropriately adjusted [12]. The manipulation of vortices by the tip of a magnetic force microscope has been observed experimentally in type-II superconductors [13]. Recently, also the switching of single skyrmions (magnetic vortices with the tail magnetisation pointing antiparallel to the core magnetisation, [14]) at thin PdFe films has been realised [15]. The switching is induced by a spin-polarised current, injected by a scanning-tunnelling microscopy tip which is positioned above the substrate.

In this work, the focus lies on the energy dissipation occurring when a vortex is dragged through the ferromagnet, which leads to a friction force decelerating the magnetic tip. This magnetic friction force is a direct consequence of the non-equilibrium nature. A prototype for studies of magnetic friction is the Ising model, subdivided into (at least) two subsystems, one of which is shifted with respect to the other one with a rate representing a velocity [16–19]. Magnetic friction in the Potts model [20] as well as in sheared geometries [21] has also been observed. Experimental evidence of magnetic friction forces has been provided by magnetic exchange force microscopy experiments recently [22,23], where a magnetic tip dragged a single magnetic atom across a magnetic surface. Such a system has been modelled in our earlier works [24–29]. The microscopic mechanism leading to a friction force, which depends linearly on the scanning velocity of the tip [25,28] was identified, as well as the influence of

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temperature on the friction force \[26,27\]. An explicit comparison and unification of results in the Ising model and results in the Heisenberg model was provided in \[29\]. A field-theoretical treatment is presented in \[30\]. In this letter the magnetic friction force for a system containing a magnetic vortex is explicitly calculated.

**System.** – The system consists of \(L^2\) classical Heisenberg spins \(S_i = \mu_i/\mu_s\) on a square grid, where \(\mu_s\) is a material specific saturation magnetisation. The Hamiltonian contains two terms, corresponding to a substrate and a tip part,

\[
\mathcal{H} = \mathcal{H}_\text{sub} + \mathcal{H}_\text{tip},
\]

To describe the substrate I use an isotropic exchange be-

\[
\mathcal{H}_\text{sub} = -J \sum_{\langle i,j \rangle} S_i \cdot S_j - d_z \sum_i S_i^2, \tag{2}
\]

where \(d_z<0\) leads to an easy plane (spins tend to align in the xy-plane) and stabilises magnetic vortices. The value \(d_z = -0.1J\) is used in this letter. The explicit value of \(d_z\) only influences the vortex core radius, but not the phenomenology described below.

The moved tip interacts with the substrate via a dipolar interaction,

\[
\mathcal{H}_\text{tip} = -w \sum_i \frac{3}{R_i^3} (S_i \cdot e_i)(S_{\text{tip}} \cdot e_i) - S_i \cdot S_{\text{tip}}, \tag{3}
\]

where \(R_i = |\mathbf{R}_i|\) denotes the norm of the position of spin \(i\) relative to the tip \(\mathbf{r}_i = \mathbf{r}_\text{tip}\), and \(e_i\) its unit vector \(\mathbf{e}_i = \mathbf{R}_i/R_i\). \(\mathbf{r}_i\) and \(\mathbf{r}_\text{tip}\) are the position vectors of the substrate spins and the tip, respectively. \(w\) is a free parameter that quantifies the dipole-dipole-coupling between the substrate spins and the tip, thus controlling the strength of the tip. I use \(S_{\text{tip}} = (0,0,-1)\). The tip is moved with constant velocity \((v,0,0)\) two lattice constants above the substrate, along the middle line between two spin rows. In a previous work, ref. \[12\], a regime of \(w\) - and \(v\)-values has been identified, leading to stable vortices dragged through the system. Results presented here are restricted to this regime. The height \(z\) at which the tip is dragged above the substrate as well as the value \(w\) and the fact that a dipolar interaction is used (and not, e.g., a pseudo-pole approximation as discussed in \[31\]) are of little relevance for the structure of the vortices, as well as for the results presented below.

Open boundary conditions are used in \(y\)-direction. Co-

moving open boundaries are implemented in \(x\)-direction: When the tip is moved by exactly one lattice constant (one cycle), the foremost spin row is duplicated, and the last one is deleted, see also \[25\]. In this way the simulation can go on indefinitely. The substrate spins follow the Landau- Lifshitz-Gilbert (LLG) equation \[32,33\],

\[
\frac{\partial}{\partial t} \mathbf{S}_i = -\frac{\gamma}{(1+\alpha^2)\mu_s} \left[ \mathbf{S}_i \times \mathbf{h}_i + \alpha \mathbf{S}_i \times (\mathbf{S}_i \times \mathbf{h}_i) \right], \tag{4}
\]

with saturation magnetisation \(\mu_s\), gyromagnetic ratio \(\gamma\), the phenomenological damping constant \(\alpha\) (high damping values \(\alpha = 0.5\) and \(\alpha = 0.3\) are used in this letter to reach a steady state in a “short” simulation time) and the local field \(\mathbf{h}_i = -\partial \mathcal{H}/\partial \mathbf{S}_i\). It produces Larmor precession with frequency \(|\mathbf{h}_i|\gamma/\mu_s\), and a damping in the direction of the local field. One may define a characteristic frequency \(\omega = \gamma J/\mu_s\). To solve the LLG equation the Heun integration scheme \[34\] is used. To calculate the energy dissipation, the expression \[25\]

\[
P_{\text{diss}} = -\sum_i \mathbf{h}_i \cdot \partial_t \mathbf{S}_i = \frac{\gamma}{\mu_s} \frac{\alpha}{1+\alpha^2} \sum_i (\mathbf{S}_i \times \mathbf{h}_i)^2 \tag{5}
\]

is used, which leads to the correct magnetic friction force at zero temperature,

\[
F = \frac{P_{\text{diss}}}{v}, \tag{6}
\]

after averaging over at least one cycle \(a/v\) in the steady state. Note that reaching of the steady state, especially for large system sizes, requires long simulation runs (the data points presented below are determined after \(10^8\) integration steps). In the following natural units are used (time \(\omega^{-1} = \mu_s/(\gamma J)\), energy \(J\) and length \(a\)).

**Friction in the quasi-static limit.** – The energy dissipation, occurring in a moved magnetic structure has been addressed by Thiele \[35\], under the assumption that the magnetic structure remains invariant when motion with velocity \(v\) sets in. Using the LLG equation, Thiele derived in a continuum approximation

\[
\mathbf{F}_{\text{ext}} + \mathbf{G} \times \mathbf{v} + \mathbf{D} \cdot \mathbf{v} = 0, \tag{7}
\]

with \(\mathbf{F}_{\text{ext}}\) representing an external force initiating and keeping the motion, \(\mathbf{G}\) being the gyrovector, and \(\mathbf{D}\) a diadic tensor representing the dissipation. For a magnetisation configuration given in spherical coordinates
(S = (cos \varphi \sin \theta, \sin \varphi \sin \theta, \cos \theta)) these quantities read\(^1\)

\[
G = -\frac{1}{2} \int d^2r \sin \theta (\nabla \vartheta \times \nabla \varphi) \quad \text{and} \quad (8a)
\]

\[
\mathcal{D} = -\frac{\alpha}{2} \int d^2r \left( \nabla \theta \otimes \nabla \theta + \sin^2 \theta \nabla \varphi \otimes \nabla \varphi \right), \quad (8b)
\]

with \(\otimes\) the dyadic product. Thus, once the \(v = 0\) configuration is known, the energy dissipation occurring in the driven out-of-equilibrium system can be calculated using the \(v = 0\) result. In the following, I will derive the friction force emerging from a magnetic vortex. For a vortex configuration, \(\nabla \vartheta\) points radially from the vortex core to the tail, while \(\nabla \varphi\) circulates tangentially around the vortex core. The resulting orientation of \(G\) is sketched in fig. 1. The symmetry results in vanishing off-diagonals and \(z\)-components of \(\mathcal{D}\). Assuming cylindrical coordinates in space \((x, y, z) = (\rho \cos \phi, \rho \sin \phi, z)\), the dissipation tensor simplifies to

\[
\mathcal{D}_0 = \mathcal{D}_{xx} = \mathcal{D}_{yy} = -\alpha \pi \int \rho \left( \left( \frac{\partial \theta(\rho)}{\partial \rho} \right)^2 + \frac{\sin^2 \theta(\rho)}{\rho^2} \right) d\rho. \quad (9)
\]

Thus, in a cylindrically symmetric configuration a dissipative force acts against the direction of motion \(v\). This is puzzling at first sight, as the external force which is applied by the tip on the substrate contains also a vertical component \(F_z\). This component counterbalances the gyroscopic term \(G \times v\) which is not dissipative.

In the following, the equilibrium configuration for a continuum approximation of the present system is derived to illustrate the assumptions and stress the universality of the results. The impatient reader may directly proceed to the result, eq. (16). Equations (2), (3) read for cylindrical space coordinates and spherical spin components (I skip the dependence of \(\theta\) and \(\varphi\) on \(r\) for brevity here) in the continuum approximation (cf. ref. [36])

\[
\mathcal{H}^c = \int d^2r \left\{ \frac{1}{2} \left( \nabla \vartheta \cdot \nabla \vartheta + \sin^2 \theta \nabla \varphi \cdot \nabla \varphi \right) - dz \cos^2 \theta \right\} + w \int d^2r \frac{(2z^2 - \rho^2) \cos \theta + 3z \rho \cos(\phi - \varphi) \sin \theta}{(z^2 + \rho^2)^{3/2}}. \quad (10)
\]

As I am not interested in the dynamics in this section, I consider the undamped case, \(\alpha = 0\). Then, the equations of motion for \(\theta\) and \(\varphi\) read (cf. ref. [36])

\[
\dot{\theta} = \frac{1}{\sin \theta} \frac{\partial \mathcal{H}^c}{\partial \varphi} \quad \text{and} \quad \dot{\varphi} = -\frac{1}{\sin \theta} \frac{\partial \mathcal{H}^c}{\partial \theta}. \quad \text{(11)}
\]

Setting \(\dot{\theta} = 0\) and \(\dot{\varphi} = 0\) and solving for \(\theta\) and \(\varphi\) yields the ground-state configuration. The resulting differential equations read

\[
0 = -2 \cos \theta \nabla \vartheta \cdot \nabla \varphi - \sin \theta \Delta \varphi + \frac{3wz \rho \sin(\phi - \varphi)}{(\rho^2 + z^2)^{3/2}}, \quad (12)
\]

\[
0 = \frac{\Delta \varphi}{\sin \theta} - (2dz + \nabla \varphi \cdot \nabla \varphi) \cos \theta + w \frac{\rho^2 - 2z^2 + 3z \rho \cos(\phi - \varphi) \cot \theta}{(\rho^2 + z^2)^{3/2}}. \quad (13)
\]

Because a general solution is not possible, the vortex core length \(\rho_0 = 1/\sqrt{-2dz}\) simplifies eq. (13) to

\[
0 = \frac{1}{\sin \theta} \left( \frac{\partial^2}{\partial \rho^2} + \frac{\partial^2}{\rho \partial \rho} \right) \cos \theta - \left( \frac{1}{\rho^2} + \frac{1}{\rho_0^2} \right) \cos \theta + w \frac{\rho^2 - 2z^2 + 3z \rho \cot \theta}{(\rho^2 + z^2)^{3/2}}. \quad (14)
\]

This equation is analytically still not solvable —even for \(w = 0\), which is assumed from now on. However, if the two limiting cases that the magnetisation points out of the plane in the core region (\(\theta \approx 0\) for \(\rho \ll \rho_0\)) and in plane in the tail region (\(\theta \approx \pi/2\) for \(\rho \gg \rho_0\)) are assumed, eq. (14) reads

\[
0 = \rho^2 \frac{\partial^2 \theta}{\partial \rho^2} + \rho \left( \frac{\partial \theta}{\partial \rho} \right) + \left( \frac{\rho^2}{\rho_0^2} - 1 \right) \theta, \quad \text{for} \ \rho \ll \rho_0,
\]

\[
0 = \rho^2 \frac{\partial^2 \theta}{\partial \rho^2} + \rho \left( \frac{\partial \theta}{\partial \rho} \right) + \left( \frac{\rho^2}{\rho_0^2} + 1 \right) (\theta - \frac{\pi}{2}), \quad \text{for} \ \rho \gg \rho_0. \quad (15)
\]

These are Bessel differential equations and their solutions up to the integration constants are

\[
\theta(\rho) = \begin{cases} c_1 J_1(\rho/\rho_0), & \rho \ll \rho_0, \\ c_2 K_1(\rho/\rho_0) + \pi/2, & \rho \gg \rho_0. \end{cases} \quad \text{(16)}
\]
The constants can be determined from a comparison with numerical results of eq. (14) (using \(w = 0\), \(c_1 \approx 2.233\) and \(c_2 \approx -2.081\)). The result is plotted in fig. 2.

The \(v = 0\) result can be now plugged into eq. (9). Because only in the core region the out-of-plane component of the magnetisation is significant, it is reasonable to rewrite eq. (9),

\[
\mathcal{D}_0 = -\alpha \pi \int \left( \rho \left( \frac{\partial \theta}{\partial \rho} \right)^2 - \frac{\cos^2 \theta}{\rho} + \frac{1}{\rho} \right) \, d\rho. \tag{17}
\]

The first two terms in the integrand of eq. (17) only contribute in the core region and quickly converge. The third term leads to a logarithmic dependence of the dissipation on the system size and a divergence at the vortex core \((\rho \to 0)\). The latter issue is resolved by introducing a cutoff, which is also physical as the original lattice model has the lattice constant as a lower bound for the integration. One may summarise the continuum result by

\[
F = -\mathcal{D}_0 v = \alpha \pi v \log \frac{L}{L_0}, \tag{18}
\]

where \(L\) is the system size, and \(L_0\) a constant which contains the cutoff at the core, the vortex-core contribution to the integral (which weakly depends on \(d_z\)) as well as a geometrical correction as the continuum is calculated for a disk, whereas our lattice model is a square. For \(d_z = -0.1\) one gets \(L_0 \approx 0.7\) from a final numerical integration of eq. (17) using eq. (16).

In fig. 3 it is observable that this result, which has been derived for any vortex dragged by some not specified external force is also valid for the dipolar tip, as long as \(w\) is large enough to stabilise a vortex. In practice a weak dependence of the vortex size \(\rho_0\) on \(w\) can be observed.

But such minor corrections are always dominated by the anisotropy, and even the shape of the tip field does not influence \(\rho_0\) significantly. Furthermore the core contribution in eq. (17) is always dominated by the tail part which diverges with the system size. One may summarise this section that in the quasi-static limit, \(v \to 0\), the Thiele equation provides a good estimate for the energy dissipation.

**System size dependence of friction.** – On the one hand the logarithmic dependence of the friction force on the system size leads to a diverging force in the thermodynamic limit. On the other hand, the quasi-static motion of the ground state is unphysical in the thermodynamic limit, as the excitations imposed by the tip have to travel to the system boundaries first. This fact has been already mentioned in ref. [3], where the condition \(\alpha = 0\) used in that work is made responsible for the logarithmic divergence in eq. (18). Accordingly, size effects are important when the system size is increased, and I calculated the friction force for several velocities and system sizes in computer simulations.

In fig. 4 one finds evidence that eq. (7) is only valid up to a certain system size, which is called an effective vortex size \(R\). Above \(R\) the friction force saturates, and from the value in the large-\(L\) limit one may fit the dependence of \(R\) on the dynamic non-equilibrium parameters \(v\) and \(\alpha\), resulting in

\[
R \approx \frac{5 \alpha^2 \omega}{\alpha v}. \tag{19}
\]

This can be understood in terms of a macrospin model, introduced in ref. [25]. If a single Heisenberg spin is driven by an external field, it tries to follow the field with a lag proportional to \(\alpha v\), where \(v\) is the rate of the drive. Approaching equilibrium (corresponding to a tip at rest, \(v \to 0\)), one gets a lag \(\alpha v \to 0\), which corresponds to
Fig. 5: (Color online) Rescaled friction vs. logarithmic rescaled system size, for $\alpha = 0.5$ (filled symbols) and $\alpha = 0.3$ (empty symbols).

A vortex with infinite effective size $R$. The validity of eq. (19) leads to a good data collapse, cf. Fig. 5, where both, velocity and damping constant, have been varied.

What does a finite effective vortex size mean? To provide a better understanding, in Fig. 6 the non-equilibrium steady state for a system with $L < R(\alpha, v = 0.01)$, now called system (a), as well as for a system with $L > R(\alpha, v = 0.2)$ (system (b)) is plotted. Both systems contain exactly one vortex and no antivortex, with the core directly under the tip, and thus contain the same vorticity. But while configuration (a) is nearly symmetric with respect to the $y$-axis, system (b) shows a strong deformation. Spins at $|y| > R/2$ (the vortex core is at $y = 0$) are only little influenced by the vortex. As a consequence, this part of the system contributes only a negligibly small $\partial_t S_x$-value to the overall dissipation (cf. eq. (5)). In other words: Depending on the parameters $\alpha$ and $v$, the vortex dragged through the system by the tip has a finite cross-section with a diameter $R$.

Generalising these observations, I propose a dimensionless “magnetic friction number”, analogous to a magnetic Reynolds number,

$$\mathcal{M} = vL \frac{\alpha}{\omega a^2},$$

that quantifies the degree of nonlinearity in the response of a driven spin configuration. If $\mathcal{M}$ is smaller than unity the Thiele equation applies yielding the correct dissipation. This statement should remain valid for structures not containing a vortex, but, e.g., a domain wall. However, if $\mathcal{M}$ is greater than unity, the moved structure is subject to dynamical changes depending on dynamical parameters $\alpha$ and $v$. Then the detailed microscopic out-of-equilibrium behaviour and the nonlinearity gains importance. At the same time, in this limit size effects vanish. For the case of a vortex configuration this leads to the observed reduced vortex size $R < L$.

The fraction in eq. (20) is a material specific constant, and corresponds to about $\mathcal{M}/(vL) \approx 10^4 \text{s/m}^2$ for the magnetic transition metals cobalt, iron and nickel. While nanometre-sized systems should be in the low-$\mathcal{M}$ regime, where the equilibrium configuration is relevant and the effects of finite system size become apparent, fast moving structures in micron-sized systems make a crossover into the high-$\mathcal{M}$ regime where non-equilibrium gains importance.

**Conclusion.** – Energy dissipation for a magnetic vortex, dragged by a dipole tip through a substrate has been calculated analytically and recovered in simulations. These results are valid for a vortex driven by any external force, thus also the drive by a spin-polarised current or an external magnetic field are possible. Limitations of the assumptions which are essential for the analytical result have been discussed. In a study of size effects a finite vortex size has been observed which defines the limit of validity of the analytical result. As the vortex size is a macroscopic quantity which directly depends on the microscopic damping constant, the results offer an alternative to determine the damping constant experimentally or verify the underlying model, including the verification of the observed dissipation mechanism.

Based on the observations in the presented system, a magnetic friction number as a criterion for the validity of the Thiele equation has been proposed. Its relevance for different system setups, magnetisation structures as well as, e.g., in the weak damping limit should be clarified in future studies.

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