GENERALIZED MACMAHON FORMULA FOR CYLINDRIC PARTITIONS AND PERIODIC MACDONALD PROCESSES

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ABSTRACT. We present a generalized MacMahon formula for cylindric partitions, which is a \((q, t)\)-deformation of the one given by Borodin. Our proof relies on a periodic Macdonald process and follows the idea of Vuletić for the generalized MacMahon formula for plane partitions. The technical tool is the trace of a vertex operator on the Fock space, relying on which we also derive a trace formula for Macdonald refined topological vertices as a generalization of a result for topological vertices obtained by Bryan, Kool and Young. It is known that a Macdonald process has many applications to stochastic models. As for this aspect, we construct a periodic stochastic process whose marginal law at a fixed time is a periodic Macdonald measure and discuss observables of a periodic Macdonald measure.

1. Introduction

A plane partition is a collection of non-negative integers \(\pi = (\pi_{i,j})_{i,j \in \mathbb{N}}\) such that \(\pi_{i,j} \geq \pi_{i,j+1}\) and \(\pi_{i,j} \geq \pi_{i+1,j}\) for all \(i, j \in \mathbb{N}\) and \(|\pi| = \sum_{i,j \in \mathbb{N}} \pi_{i,j} < \infty\). When a plane partition is represented as

\[
\begin{array}{cccc}
\pi_{1,1} & \pi_{1,2} & \pi_{1,3} & \cdots \\
\pi_{2,1} & \pi_{2,2} & \pi_{2,3} & \cdots \\
\pi_{3,1} & \pi_{3,2} & \pi_{3,3} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{array}
\]

it is naturally understood as a collection of boxes in \((\mathbb{R}_{\geq 0})^3\). Namely, there are \(\pi_{i,j}\) boxes of unit linear length piled vertically at the position \((i, j)\) \(\in \mathbb{R}^2\). Let us denote the collection of plane partitions by \(\mathcal{P}\). Then the generating function of plane partitions is given by (see e.g. [Sta99])

\[
\sum_{\pi \in \mathcal{P}} s^{|\pi|} = \prod_{n \geq 1} \left( \frac{1}{1 - s^n} \right)^n
\]

known as the MacMahon formula.

A two parameter \((q, t)\)-deformation of the MacMahon formula was proposed by [Vul09]. Let us fix \((i, j) \in \mathbb{N}^2\) and define partitions \(\lambda, \mu\) and \(\nu\) by

\[
\lambda = (\pi_{i,j}, \pi_{i+1,j+1}, \ldots), \quad \mu = (\pi_{i,j+1}, \pi_{i+1,j+2}, \ldots), \quad \nu = (\pi_{i+1,j}, \pi_{i+2,j+1}, \ldots).
\]

Associated with these data, we set

\[
F_\pi(i, j; q, t) := \prod_{m=0}^{\infty} \frac{f(q^{\lambda_1 - \lambda_{m+1} t^m}) f(q^{\lambda_1 - \lambda_{m+2} t^m})}{f(q^{\lambda_1 - \mu_{m+1} t^m}) f(q^{\lambda_1 - \nu_{m+1} t^m})}
\]

where

\[
f(a) = \prod_{m=0}^{\infty} (1 - a t^m)^{-1}.
\]
where \( f(u) = \frac{(tuq)_{\infty}}{(quq)_{\infty}} \). The product reduces to a finite product since the factor is unity for sufficiently large \( m \). Then the weight for a plane partition \( \pi \) is defined by
\[
F_{\pi}(q, t) = \prod_{(i, j) \in \mathbb{N}^2} F_{\pi}(i, j; q, t).
\]
Since, if \( \pi_{i,j} = 0 \), \( F_{\pi}(i, j; q, t) = 1 \), the product over \( \mathbb{N}^2 \) can be replaced by a product over the support of \( \pi \). The generalized MacMahon formula by [Vul09] reads
\[
\sum_{\pi \in \mathcal{P}} F_{\pi}(q, t)s|\pi| = \prod_{n=1}^{\infty} \left( \frac{(ts^n; q)_{\infty}}{(s^n; q)_{\infty}} \right)^n.
\]
Note that when we set \( q = t \), it recovers the MacMahon formula.

**Remark 1.1.** Let us set, for \( n, m \in \mathbb{Z}_{\geq 0} \),
\[
\tilde{f}(n, m) = \begin{cases} \prod_{i=0}^{n-1} \frac{1-q^{i+1}}{1-q^{i+1}m}, & n \geq 1, \\ 1, & n = 0. \end{cases}
\]
Then we have
\[
\tilde{f}(n, m) = \frac{(tm+1; q)_{\infty}}{(tmq; q)_{\infty}} \frac{1}{f(qn tm)}.
\]
Therefore the weight for each box \( \mathcal{L} \) is also written as
\[
F_{\pi}(i, j; q, t) = \prod_{m=0}^{\infty} \frac{\tilde{f}(\lambda_1 - \mu_{m+1}, m) \tilde{f}(\lambda_1 - \nu_{m+1}, m)}{\tilde{f}(\lambda_1 - \lambda_{m+1}, m) \tilde{f}(\lambda_1 - \lambda_{m+2}, m)},
\]
which is the same as the one used in [Vul09] (The symbol \( \tilde{f} \) here is denoted as \( f \) in [Vul09]).

In this note, we propose this kind of generalized MacMahon formula for cylindric partitions. To define a cylindric partition, it is convenient to regard a plane partition as a sequence of partitions. There are several ways to it. Namely, we can define a sequence \( (\lambda^i : i = 1, 2, \ldots) \) of partitions by \( \lambda^i = (\pi_{i,1}, \pi_{i,2}, \ldots) \) or \( \lambda^i = (\pi_{1,i}, \pi_{2,i}, \ldots) \), or even by \( \lambda^i = \{ (k, l) \in \mathbb{N}^2 : \pi_{k,l} \geq i \} \) from a plane partition \( \pi \). The relevant way for us is, however, to identify a plane partition \( \pi \) with a bilateral sequence \( (\lambda^i : i \in \mathbb{Z}) \) of partitions each of which is a diagonal slice of the plane partition defined by
\[
\lambda^0 = (\pi_{1,1}, \pi_{2,2}, \ldots), \quad \lambda^1 = (\pi_{1,0}, \pi_{2,1}, \pi_{3,2}, \ldots), \quad \lambda^{-n} = (\pi_{n+1,1}, \pi_{n+2,2}, \ldots), \quad n > 0.
\]
It is immediate that a bilateral sequence \( (\lambda^i : i \in \mathbb{Z}) \) of partitions determines a plane partition if and only if the relations
\[
\ldots < \lambda^{-i} < \lambda^{-i+1} < \cdots \lambda^{-1} < \lambda^0 > \lambda^1 > \cdots > \lambda^i > \lambda^{i+1} > \ldots
\]
hold. Here, we write \( \lambda \succ \mu \) for two partitions \( \lambda, \mu \) if \( \mu \subset \lambda \) and the skew-partition \( \lambda/\mu \) is a horizontal strip, i.e., it has at most single box in each column.

We further extend the notion of a plane partition introducing a boundary profile. For a partition \( \nu \), a plane partition with boundary profile \( \nu \) is a collection \( \pi = (\pi_{i,j})_{i,j \in \mathbb{N}, j \geq \nu_{i+1}} \) such that \( \pi_{i,j} \geq \pi_{i,j+1}, \pi_{i,j} \geq \pi_{i+1,j} \) for all \( i \in \mathbb{N}, j \geq \nu_{i+1} \) and \( |\pi| = \sum_{i,j} \pi_{i,j} < \infty \). To associate a bilateral sequence of partitions to a plane partition with boundary profile \( \nu \), we extend \( \pi \in \mathcal{P}(\nu) \) to \( \pi = (\pi_{i,j})_{i,j \in \mathbb{N}} \) by setting \( \pi_{i,j} = \infty \) if \( (i, j) \in \nu \), where
we identified a partition \( \nu \) with its support \( \nu = \{ (i, j) : i = 1, \ldots, \ell(\nu), j = 1, \ldots, \nu_i \} \). Then we set

\[
\tilde{\lambda}_0^0 = (\pi_{1,1}, \pi_{2,2}, \ldots), \quad \tilde{\lambda}_n^0 = (\pi_{1,n+1}, \pi_{2,n+2}, \ldots), \quad \tilde{\lambda}_n^{-n} = (\pi_{n+1,1}, \pi_{n+2,2}, \ldots), \quad n > 0.
\]

For each \( n \in \mathbb{Z} \), we write \( p_n = \min \{ j \in \mathbb{N} : \tilde{\lambda}_n^j < \infty \} \) and define \( \lambda^n = (\tilde{\lambda}_{p_n}^n, \tilde{\lambda}_{p_n}^{n+1}, \ldots) \). Then we have obtained a bilateral sequence \( (\lambda^n : n \in \mathbb{Z}) \) of partitions from a plane partition with boundary profile \( \nu \). To recover a plane partition with boundary profile \( \nu \) from a bilateral sequence of partitions, we define the Maya diagram of \( \nu \) by

\[
\mathbb{M}(\nu) = \{ \lambda_i - i : i \in \mathbb{N} \} \subset \mathbb{Z}.
\]

Then a bilateral sequence \( (\lambda^n : n \in \mathbb{Z}) \) of partitions determines a plane partition with boundary profile \( \nu \) if and only if the relations

\[
\begin{cases}
\lambda_n < \lambda^{n+1} & n \in \mathbb{M}(\nu), \\
\lambda_n > \lambda^{n+1} & n \notin \mathbb{M}(\nu)
\end{cases}
\]

for all \( n \in \mathbb{Z} \) hold. When \( \nu = \emptyset \), this condition reduces to the one for a plane partition since \( \mathbb{M}(\emptyset) = \mathbb{Z}_{\leq -1} \).

We can define a cylindric partition as a generalization of a plane partition with a boundary profile viewed in the above manner. Let us fix \( N \in \mathbb{N} \) and write \( [1, N] = \{ 1, \ldots, N \} \). A cylindric partition of periodicity \( N \) and boundary profile \( \mathbb{M} \subset [1, N] \) is a sequence \( \lambda = (\lambda^1, \ldots, \lambda^N) \) of partitions such that

\[
\begin{cases}
\lambda^k < \lambda^{k+1}, & k \in \mathbb{M}, \\
\lambda^k > \lambda^{k+1}, & k \notin \mathbb{M}
\end{cases}
\]

for all \( k = 1, \ldots, N \), where we identify \( \lambda^{N+1} = \lambda^1 \). We write \( \mathcal{CP}(N, \mathbb{M}) \) be the collection of cylindric partition of periodicity \( N \) and boundary profile \( \mathbb{M} \). For a cylindric partition \( \lambda \in \mathcal{CP}(N, \mathbb{M}) \), we call the number \( |\lambda| = \sum_{k=1}^N |\lambda^k| \) its weight.

We define a weight for a cylindric partition \( \lambda \in \mathcal{CP}(N, \mathbb{M}) \). For fixed \( k \in [1, N] \) and \( j \in \mathbb{N} \), we define partitions

\[
\lambda = (\lambda^k_j, \lambda^k_{j+1}, \ldots), \quad \mu = (\lambda^{k+1}_j + \chi[k \in \mathbb{M}!], \lambda^{k+1}_{j+1}, \ldots), \quad \nu = (\lambda^{k-1}_j + \chi[k-1 \notin \mathbb{M}!], \lambda^{k-1}_{j+1}, \ldots),
\]

where \( \chi[\cdot] \) is the indicator defined by \( \chi[P] = 1 \) if \( P \) is true and \( \chi[P] = 0 \) if \( P \) is false and set

\[
F_\lambda(k, j; q, t) := \prod_{m=0}^\infty \frac{f(q^{\lambda^1_j - \lambda^1_{m+1}}t^m)f(q^{\lambda^1_j - \lambda^1_{m+2}}t^m)}{f(q^{\lambda^1_j - \mu^1_{m+1}}t^m)f(q^{\lambda^1_j - \nu^1_{m+1}}t^m)}
\]

and

\[
F_\lambda := \prod_{k=1}^N \prod_{j \in \mathbb{N}} F_\lambda(k, j).
\]

In this note the following \( n \)-fold \( q \)-Pochhammer symbol is extensively used:

\[
(a; p_1, \ldots, p_n)_\infty = \prod_{i_1, \ldots, i_n = 0}^{\infty} (1 - ap_1^{i_1} \cdots p_n^{i_n}).
\]
Our generalized MacMahon formula for cylindrical partitions reads as follows.

**Theorem 1.2.** Let $N \in \mathbb{N}$ and let $\mathcal{M} \subset [1, N]$ be a boundary profile. Then

$$
\sum_{\lambda \in \mathcal{C}(N, \mathcal{M})} F_{\lambda}(q, t)s^{\lambda} = \frac{1}{(s^{N}; s^{N})_{\infty}} \prod_{k \in \mathcal{M}, l \notin \mathcal{M}} \frac{(ts^{l-k}; q, s^{N})_{\infty}}{(s^{l-k}; q, s^{N})_{\infty}},
$$

where we set $\lambda = (\lambda_{1}, \ldots, \lambda_{N})$, $\mu = (\mu_{1}, \ldots, \mu_{N})$, and $\rho = (\rho_{1}, \ldots, \rho_{N})$ of partitions the following weight:

$$
(1.2) \quad W_{q, t; u}^{\rho^{+}, \rho^{-}}(\lambda, \mu) = u^{\mu_{N}} \prod_{i=1}^{N} Q^{\lambda_{i}/\mu_{i}}(\rho_{i}^{-}; q, t) P_{\lambda_{i+1}/\mu_{i}}(\rho_{i}^{+}; q, t),
$$

where we understand $\lambda^{N+1} = \lambda^{1}$ and $\rho^{+}_{N} = \rho^{+}_{N}$. It is obvious that the weight vanishes unless the partitions satisfy the following inclusion properties:

$$
\mu^{N} = \mu^{0} \subset \lambda^{1} \supset \mu^{1} \subset \lambda^{2} \supset \cdots \supset \mu^{N-1} \subset \lambda^{N} \supset \mu^{N}.
$$

In Sect. 4 we will prove Theorem 1.2 as a corollary of the following fundamental identity.

**Proposition 1.3.** We have

$$
\Pi_{q, t; u}(\rho^{+}, \rho^{-}) := \sum_{\lambda, \mu \in \mathcal{C}^{N}} W_{q, t; u}^{\rho^{+}, \rho^{-}}(\lambda, \mu) = \frac{1}{(u; u)_{\infty}} \prod_{i=0}^{N-1} \prod_{j=1}^{N} \tilde{\Pi}_{q, t; u}(\rho_{i}^{+}, \rho_{j}^{-}).
$$

where

$$
\tilde{\Pi}_{q, t; u}(X; Y) = \prod_{i, j \geq 1} \frac{(tx_{i}y_{j}; q, u)_{\infty}}{(x_{i}y_{j}; q, u)_{\infty}}.
$$

Though this could be proved by a direct computation, we show it, in Sect. 3 by identifying the ring of symmetric functions with a Fock representation of a Heisenberg algebra and applying free field computation. The relevant formula is the trace of a vertex operator (in an extended sense) shown in Subsect. 2.2, which is also applied to the trace of a Macdonald refined topological vertex.

A topological vertex is a combinatorial object that counts the number of plane partitions with prescribed asymptotic partitions. In the paper by Foda and Wu [FW17], the authors proposed a generalization of a topological vertex called a Macdonald refined topological vertex that unifies the refinement of a topological vertex [AK05] [IKV09] [AK09] [AFS12] and the Macdonald deformation [Vu09]. For a triple of partitions $\lambda$, $\mu$, and $\nu$, the corresponding Macdonald refined topological vertex is

$$
\mathcal{V}_{\lambda \mu \nu}(x, y; q, t) = \prod_{s \in \nu} \frac{(tx_{s}^{(s+1)y_{s}}; q)_{\infty}}{(x_{s}^{(s+1)y_{s}}; q)_{\infty}} \sum_{\eta \in \mathcal{Y}} P_{\lambda/\eta}(y^{\rho-1}x^{-\nu}; q, t) Q_{\mu/\eta}(x^{\rho-1}y^{-\nu}; q, t),
$$
where the specializations are defined by \( y^{\rho-1}x^{-\nu'} : x_i \mapsto y^{i-1}x^{-\nu'} \) and \( x^{\nu}y^{-\nu} : x_i \mapsto x^iy^{-\nu}, i \geq 1 \). When we set \( q = t \), it reduces to the refined topological vertex and when we further set \( x = y \), it reduces to the topological vertex [AKMV05]. In [BKY18], the authors presented several trace identities for topological vertices. A generalization of one of their results to a Macdonald refined topological vertex that will be proved in Sect. 5 reads as follows:

**Theorem 1.4.** For any partition \( \nu \in \mathcal{Y} \), we have

\[
\sum_{\lambda \in \mathcal{Y}} u^{[\lambda]} \mathcal{Y}_{\lambda\mu}(x, y; q, t) = \frac{1}{(u; u)_{\infty}} \prod_{s \in \nu} \frac{(tx^{1}_{\nu}(s)+1y^{a_{s}(s)}; q)_{\infty}}{(x^{1}_{\nu}(s)+1y^{a_{s}(s)}; q)_{\infty}} \prod_{s \in \nu} \frac{(tx^{1}_{\nu}(s)y^{-a_{s}(s)-1}; q, u)_{\infty}}{(x^{1}_{\nu}(s)y^{-a_{s}(s)-1}; q, u)_{\infty}}
\]

\[
\times \prod_{s \in \partial^{1}_{R}\nu} \frac{(tx^{1}_{\nu}(s)+1y^{j(s)-1}; q, u, x)_{\infty}}{(x^{1}_{\nu}(s)+1y^{j(s)-1}; q, u, x)_{\infty}} \prod_{s \in \partial^{1}_{R}\nu} \frac{(tx^{1}_{\nu}(s)y^{a_{s}(s)}; q, u, y)_{\infty}}{(x^{1}_{\nu}(s)y^{a_{s}(s)}; q, u, y)_{\infty}}
\]

Here \( re(\nu) = (v_{1}^{1}(\nu) \right) is the rectangular envelop of \( \nu \) and \( \partial^{1}_{R}\nu = \{ (i, \nu_{i}) : i = 1, \ldots, \ell(\nu) \} \) and \( \partial^{1}_{R}\nu = \{ (\nu_{i}^{j}, j) : j = 1, \ldots, \nu_{i} \} \) are the right and bottom boundaries of \( \nu \).

Motivated by several applications of Macdonald processes to stochastic models [BCT14], we define a probability measure on \( \mathcal{Y}^{N} \) from the weight in (1.2) by

\[
\mathbb{P}^{\rho^{+}, \rho^{-}}_{\lambda^{1}, \cdots, \lambda^{N}} = \frac{1}{\Pi_{q,t;u}(\rho^{+}; \rho^{-})} \sum_{\mu \in \mathcal{Y}^{N}} W^{\rho^{+}, \rho^{-}}_{\lambda^{1}, \cdots, \lambda^{N}}(\lambda, \mu)
\]

and call it an \( N \)-step periodic Macdonald process. In this note, we, in particular, focus on the case when \( N = 1 \) and call it an periodic Macdonald measure on \( \mathcal{Y} \). In Sect. 6 we will define a continuous time periodic stochastic process on \( \mathcal{Y} \) called a stationary periodic Macdonald–Plancherel process and show that the marginal law at a fixed time is described by a periodic Macdonald measure specified by Plancherel specializations. This stochastic process reduces to the one considered in [BB19] at \( q = t \). We also study observables for a periodic Macdonald measure. In [Shi06, FHH+09], the authors showed that several operators that are diagonalized by the Macdonald symmetric functions admit expressions using vertex operators on a Fock space. In the previous work [Kos19], we applied their construction to a Macdonald process, recognized that earlier results regarding Macdonald processes are recovered in the free field approach and furthermore found that the determinantal structure of Macdonald processes is manifest at the operator level. It is obvious that the free field approach is also applicable for a periodic Macdonald measure. We considered four series of observables for a periodic Macdonald measure and computed their expectation value. Consequently, we found that an each expectation value is given by a multiple integral of product of a determinant and a function that is independent of specializations in contrast to that, for a Macdonald measure, an expectation value of an observable is expressed as a multiple integral of a single determinant.

This note is organized as follows: The following Sect. 2 is devoted to preliminaries, where we will review some notions of symmetric functions and Fock representations of
a Heisenberg algebra. We will also present the trace formula of a vertex operator in an extended sense and recall the free field realization of operators diagonalized by the Macdonald symmetric functions. In Sect. 3 we introduce a periodic Macdonald process and apply the trace formula of a vertex operator in Sect. 2 to the computation of a partition function to prove Proposition 1.3. In Sect. 4, we prove Theorem 1.2 as an application of Proposition 1.3. In Sect. 5, we deal with a Macdonald refined topological vertex and prove Theorem 1.4. In Sect. 6, after recalling a Plancherel specialization of the Macdonald symmetric functions, we define a periodic continuous process and show that its marginal law at a fixed time is a periodic Macdonald measure. In Sect. 7, we present several observables of a periodic Macdonald measure and compute their expectation value. We will find that, in contrast to a Macdonald measure, for which an expectation value is written as integral of a determinant, one for a periodic Macdonald measure is modified by a universal measure independent of specializations. We have two appendices. In Appendix A, we present another attempt to generalize a result of [BKY18] concerning the trace of a Macdonald refined topological vertex than one in Sect. 5. In Appendix B, we consider the shift-mixed version of a periodic Macdonald measure. For a periodic Schur measure, the determinantal structure appears for its shift-mixed version. For a shift-mixed periodic Macdonald measure, though, an expectation value does not admit a purely determinantal expression, which we also see that reduces to a simple determinant at the Schur limit $q \to t$. We also show that the Macdonald operators at the Schur limit are written in terms of free fermions, which gives the origin of well-known determinantal structure of Macdonald processes.

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2. Preliminaries

2.1. Symmetric functions. Here we recall the basic notion of symmetric functions. A detail can be found in [Mac95]. Let us begin with recalling that a partition is a non-decreasing sequence of non-negative integers $\lambda = (\lambda_1, \lambda_2, \ldots)$ such that its weight is finite: $|\lambda| = \sum_{i \geq 1} \lambda_i < \infty$. We denote the collection of partitions by $\mathcal{Y}$.

Let $\mathbb{F} = \mathbb{Q}(q, t)$ be the field of rational functions of $q$ and $t$. The ring of symmetric polynomials of $n$ variables is denoted as $\Lambda^{(n)} = \mathbb{F}[x_1, \ldots, x_n]^{S_n}$ and the ring of symmetric functions is defined by $\Lambda := \varprojlim_n \Lambda^{(n)}$ in the category of graded rings. In case we specify the variable $X = (x_1, x_2, \ldots)$ of symmetric functions, we write $\Lambda_X$ for the corresponding ring of symmetric functions. For a symmetric function $F \in \Lambda$, its $n$-variable reduction, i.e. the image of the canonical projection $\Lambda \to \Lambda^{(n)}$ is denoted by $F^{(n)}$.

We recall some series of symmetric functions. For $r \in \mathbb{N}$, the $r$-th power sum symmetric function is $p_r(X) = \sum_{i \geq 1} x_i^r$ and the $r$-th elementary symmetric function is $e_r(X) = \sum_{i_1 < \cdots < i_r} x_{i_1} \cdots x_{i_r}$. For a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ we set $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_{\ell(\lambda)}}$. 


and $e_{\lambda} = e_{\lambda_1} \cdots e_{\lambda_\ell(\lambda)}$. Then the collections $\{p_{\lambda} : \lambda \in \mathbb{Y}\}$ and $\{e_{\lambda} : \lambda \in \mathbb{Y}\}$ are both bases of $\Lambda$. Take a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ of length less than or equal to $n$. Then the corresponding monomial symmetric polynomial of $n$-variable is defined by

$$m_{\lambda}^{(n)}(x_1, \ldots, x_n) = \sum_{\sigma \in S_n, \text{distinct}} x_1^{\lambda_{\sigma(1)}} \cdots x_n^{\lambda_{\sigma(n)}},$$

where the sum runs over distinct permutations $\sigma$ of $(\lambda_1, \ldots, \lambda_n)$. Then the sequence $\cdots \rightarrow m_{\lambda}^{(\ell(\lambda)+2)} \rightarrow m_{\lambda}^{(\ell(\lambda)+1)} \rightarrow m_{\lambda}^{(\ell(\lambda))}$ defines a unique symmetric function $m_{\lambda}(X) \in \Lambda$ called the monomial symmetric function corresponding to $\lambda$. The collection $\{m_{\lambda} : \lambda \in \mathbb{Y}\}$ is known to form a basis of $\Lambda$.

To the aim of introducing the Macdonald symmetric functions, let us define a bilinear form $\langle \cdot, \cdot \rangle_{q,t} : \Lambda \times \Lambda \rightarrow \mathbb{F}$ by

$$\langle p_{\lambda}, p_{\mu} \rangle_{q,t} = z_{\lambda}(q,t) \delta_{\lambda,\mu},$$

where

$$z_{\lambda}(q,t) := \prod_{i \geq 1} m_{\lambda}(i) i! m_{\lambda}(\ell(\lambda)) \prod_{i=1}^{\ell(\lambda)} \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}}.$$ 

Here we denote the multiplicity of a number $i$ in a partition $\lambda$ by $m_{\lambda}(i)$ so that a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ is equivalently expressed as $\lambda = (1^{m_{\lambda}(1)} 2^{m_{\lambda}(2)} \ldots)$. Recall that the dominance order on $\mathbb{Y}$ is defined so that $\lambda \geq \mu$ if

$$\lambda_1 + \cdots + \lambda_k \geq \mu_1 + \cdots + \mu_k$$

holds for all $k = 1, 2, \ldots$. The Macdonald symmetric functions $\{P_{\lambda}(q,t)\}$ form a unique basis of $\Lambda$ specified by the following properties:

$$P_{\lambda}(q,t) = m_{\lambda} + \sum_{\mu : \mu < \lambda} c_{\lambda\mu} m_{\mu}, \quad c_{\lambda\mu} \in \mathbb{F},$$

$$\langle P_{\lambda}(q,t), P_{\mu}(q,t) \rangle_{q,t} = 0, \quad \lambda \neq \mu.$$ 

We set $Q_{\lambda}(q,t) = (\langle P_{\lambda}(q,t), P_{\mu}(q,t) \rangle_{q,t})_{\mu \neq \lambda}$ and define $\{Q_{\lambda}(q,t)\}$ so that $\langle P_{\lambda}(q,t), Q_{\mu}(q,t) \rangle_{q,t} = \delta_{\lambda,\mu}$.

The Macdonald symmetric functions are also characterized as simultaneous eigenfunctions of a family of commuting operators. For a fixed $n < \infty$, let us introduce the Macdonald difference operators $D_r^{(n)}$, $r = 1, \ldots, N$ on $\Lambda^{(n)}$ by

$$D_r^{(n)} = t^{\binom{n}{2}} \sum_{I \subset [1,n], |I| = r} \prod_{i \in I} \frac{tx_i - x_j}{x_i - x_j} \prod_{i \in I} T_{q,x_i},$$

where $T_{q,x_i}$ is the $q$-shift operator defined by $(T_{q,x_i}f)(x_1, x_2, \ldots) = f(qx_1, x_2, \ldots)$. Then a Macdonald symmetric polynomial $P_{\lambda}^{(n)}(q,t)$ is an eigenpolynomial such that

$$D_r^{(n)} P_{\lambda}^{(n)}(q,t) = e_{\lambda}^{(n)}(q^{\lambda_1} t^{n-1}, \ldots, q^{\lambda_n}) P_{\lambda}^{(n)}(q,t), \quad r = 1, \ldots, n.$$ 

Obviously, the Macdonald difference operators do not extend to operators on $\Lambda$ since their eignenvalues explicitly depend on the number of variables $n$. Instead, we define renormalized operators

$$F_r^{(n)} := \sum_{k=0}^{r} \frac{t^{-nr} - (r-k+1)}{(t^{-1}; t^{-1})_{r-k}} D_k^{(n)}, \quad r = 1, \ldots, n,$$
with $D_{0}^{(n)} = 1$. Then the projective limit $E_r = \lim_{n \to \infty} E_r^{(n)}$ exists for all $r \in \mathbb{N}$ and is diagonalized by the Macdonald symmetric functions so that

$$E_r P_\lambda(q,t) = e_r(q^\lambda t^{-\rho}) P_\lambda(q,t), \; \lambda \in \mathbb{Y}.$$ 

Here $q^\lambda t^{-\rho}$ is a specialization $\Lambda \to \mathbb{F}$ defined by $x_i \mapsto q^\lambda t^{-i}$, $i \geq 1$.

Let us describe some Pieri rules for the Macdonald symmetric functions. For a partition $\lambda$ and a box $s = (i,j) \in \lambda$, we write $a_\lambda(s) = \lambda_i - j$ and $l_\lambda(s) = \lambda'_j - i$ for the arm length and the leg length, respectively. Using these notions, we set

$$b_\lambda(s;q,t) = \frac{1 - q^\lambda(s)q^{\lambda(s)+1}}{1 - q^\lambda(s)+1q^{\lambda(s)}}.$$ 

For a skew partition $\lambda/\mu$, we denote the union of rows (resp. columns) containing boxes in $\lambda/\mu$ by $R_{\lambda/\mu}$ (resp. $C_{\lambda/\mu}$). Assume that $\mu \prec \lambda$ and set

$$\psi_{\lambda/\mu}(q,t) := \prod_{s \in R_{\lambda/\mu}-C_{\lambda/\mu}} \frac{b_\mu(s;q,t)}{b_\lambda(s;q,t)}, \quad \varphi_{\lambda/\mu}(q,t) := \prod_{s \in C_{\lambda/\mu}} \frac{b_\lambda(s;q,t)}{b_\mu(s;q,t)}.$$ 

For $r \in \mathbb{N}$, we write $g_r(q,t) = Q_r(q,t)$. Then we have

$$P_\mu(q,t)g_r(q,t) = \sum_{\lambda, \mu \prec \lambda | \lambda| - |\mu| = r} \varphi_{\lambda/\mu}(q,t) P_\lambda(q,t),$$

$$Q_\mu(q,t)g_r(q,t) = \sum_{\lambda, \mu \prec \lambda | \lambda| - |\mu| = r} \psi_{\lambda/\mu}(q,t) Q_\lambda(q,t).$$

Though we have been regarding $q$ and $t$ as indeterminates, they have to be thought of real parameters for application to probability theory. Let $\Lambda_\mathbb{R} = \varprojlim \Lambda_\mathbb{F}^{(n)}$, $\Lambda_\mathbb{R}^{(n)} = \mathbb{R}[x_1, \ldots, x_n]^{S_n}$ be the ring of symmetric functions over $\mathbb{R}$. For $q,t \in \mathbb{R}$ such that $|q|, |t| < 1$, we write, by abuse of notation, $P_\lambda(q,t) \in \Lambda_\mathbb{R}$ for the image of $P_\lambda(q,t) \in \Lambda$ under the specification map $\bigoplus_{\lambda \in \mathbb{Y}} Q_\lambda(q,t) P_\lambda(q,t) \to \Lambda_\mathbb{R}$. A specialization $\theta : \Lambda_\mathbb{R} \to \mathbb{R}$ is said to be $(q,t)$-Macdonald positive if $P_\lambda(\theta(q,t)) \geq 0$ for all $\lambda \in \mathbb{Y}$. Note that, for a specialization $\theta$, its evaluation at $F \in \Lambda_\mathbb{R}$ is often written as $F(\theta)$ instead of $\theta(F)$. The following theorem is due to [Mat19].

**Theorem 2.1.** For fixed $q,t \in \mathbb{R}$ such that $|q|, |t| < 1$, a specialization $\theta : \Lambda_\mathbb{R} \to \mathbb{R}$ is $(q,t)$-Macdonald positive if and only if there exists $(\alpha, \beta) = (\alpha_1 \geq \alpha_2 \geq \cdots \geq 0, \beta_1 \geq \beta_2 \geq \cdots \geq 0)$ and $\gamma \geq 0$ such that $\sum_{i \geq 1} (\alpha_i + \beta_i) < \infty$ and

$$p_1(\theta) = \sum_{i=1}^{\infty} \alpha_i + \frac{1-q}{1-t} \left( \sum_{i=1}^{\infty} \beta_i + \gamma \right),$$

$$p_k(\theta) = \sum_{i=1}^{\infty} \alpha_i^k + (-1)^{k-1} \frac{1-q^k}{1-t^k} \sum_{i=1}^{\infty} \beta_i^k, \; k \geq 2.$$ 

Let $X$ and $Y$ be two sets of infinite variables. Then they are combined to be a single set of infinite variables $(X,Y)$ and a Macdonald symmetric function $P_\lambda(X,Y)$ of $(X,Y)$ corresponding to a partition $\lambda$ makes sense. The Macdonald symmetric functions corresponding to skew-partitions $\lambda$ makes sense. The Macdonald symmetric functions defined on $P_\lambda(X,Y)$ with respect to the Macdonald symmetric functions of $Y$. Namely, $P_{\lambda/\mu}(X)$ is defined
by \( P_{\lambda/\mu}(X, Y) = \sum_{\mu \in \gamma} P_{\lambda/\mu}(X)P_{\mu}(Y) \). Similarly, we define \( Q_{\lambda/\mu}(X) \) by \( Q_{\lambda}(X, Y) = \sum_{\mu \in \gamma} Q_{\lambda/\mu}(X)Q_{\mu}(Y) \).

2.2. Fock representations of a Heisenberg algebra. The Heisenberg algebra we work with is an associative algebra \( U \) over \( \mathbb{F} \) generated by \( a_n, n \in \mathbb{Z} \setminus \{0\} \) subject to relations

\[ [a_m, a_n] = m \frac{1 - q^{|m|}}{1 - t^{|m|}} \delta_{m+n,0}, \quad m, n \in \mathbb{Z} \setminus \{0\}. \]

Note that the Heisenberg algebra admits Poincaré–Birkhoff–Witt decomposition \( U = U_+ \otimes_{\mathbb{F}} U_- \), where \( U_\pm \) is a subalgebra generated by \( a_{\pm n}, n > 0 \).

Let \( \mathbb{F}[0] \) be a one dimensional representation of \( U_+ \) defined by \( U_+ [0] = 0 \). Then the Fock representation is defined by means of induction so that \( \mathcal{F} = U \otimes_{U_-} \mathbb{F}[0] \simeq U_- \otimes_{\mathbb{F}} \mathbb{F}[0] \). Obviously, it admits a basis labeled by partitions. For a partition \( \lambda = (\lambda_1, \ldots, \lambda_\ell(\lambda)) \in \gamma \), we set \( |\lambda\rangle := a_{-\lambda_1} \cdots a_{-\lambda_\ell(\lambda)} |0\rangle \). Then the collection \( \{ |\lambda\rangle : \lambda \in \gamma \} \) forms a basis of \( \mathcal{F} \). The dual Fock representation is a right representation of \( U \) defined analogously. Let \( \mathbb{F}[0] \) be a one dimensional representation of \( U_- \) such that \( 0 U_- = 0 \) and define a right representation by \( \mathcal{F}^\dagger = \mathbb{F}[0] \otimes_{U_-} U \simeq \mathbb{F}[0] \otimes_{\mathbb{F}} U_+ \). For a partition \( \lambda = (\lambda_1, \ldots, \lambda_\ell(\lambda)) \in \gamma \), we set \( \langle \lambda | = \langle 0 | a_{\lambda_1} \cdots a_{\lambda_\ell(\lambda)} \). Then the collection \( \{ \langle \lambda | : \lambda \in \gamma \} \) forms a basis of \( \mathcal{F}^\dagger \).

We define a bilinear paring \( \langle \cdot, \cdot \rangle : \mathcal{F}^\dagger \times \mathcal{F} \to \mathbb{F} \) by two properties, \( 0 \langle 0 | 0 \rangle = 1 \) and \( \langle u | a_n \cdot |v\rangle = \langle u | a_n |v\rangle \) for all \( u \in \mathcal{F}^\dagger, |v\rangle \in \mathcal{F} \) and \( n \in \mathbb{Z} \setminus \{0\} \). Then the assignments \( i : \mathcal{F} \to \Lambda \) and \( i^\dagger : \mathcal{F}^\dagger \to \Lambda \) defined by \( i(|\lambda\rangle) = i^\dagger(\langle \lambda |) = p_\lambda, \lambda \in \gamma \) are compatible with the bilinear parings so that \( \langle \lambda | |\mu\rangle = \langle p_\lambda, p_\mu \rangle_{q,t}, \lambda, \mu \in \gamma \). Therefore, the Fock spaces are identified with the space of symmetric functions equipped with a bilinear paring.

A relevant class of operators in this note is that of vertex operators (in an extended sense). For commuting symbols \( \gamma_n, n \in \mathbb{Z} \setminus \{0\} \), we write

\[ V(\gamma) = \exp \left( \sum_{n>0} \frac{\gamma_n}{n} a_{-n} \right) \exp \left( \sum_{n>0} \frac{\gamma_n}{n} a_n \right), \]

where \( a_n \in \text{End}(\mathcal{F}) \) is the action of the corresponding Heisenberg generator on \( \mathcal{F} \). In this note, we will adopt following two manners to understand a vertex operator.

(1) There is a \( \mathbb{Z} \)-graded commutative algebra \( R = \bigoplus_{n \in \mathbb{Z}} R_n \) over \( \mathbb{F} \) such that \( \gamma_n \in R_n, n \in \mathbb{Z} \setminus \{0\} \). We define a completed tensor product \( \text{End}(\mathcal{F}) \hat{\otimes} R := \prod_{n \in \mathbb{Z}} \text{End}(\mathcal{F}) \hat{\otimes} R_n \). Then the vertex operator is understood as \( V(\gamma) \in \text{End}(\mathcal{F}) \hat{\otimes} R \). A particular example of \( R \) is the ring of Laurent polynomials \( \mathbb{F}[z, z^{-1}] \).

(2) There are \( \mathbb{N} \)-graded commutative algebras \( R^i = \bigoplus_{n \in \mathbb{N}} R^i_n, i = 1, 2 \) over \( \mathbb{F} \) such that \( \gamma_n \in R^1_n, \gamma_{-n} \in R^2_n, n > 0 \). The tensor product of them also admits an \( \mathbb{N} \)-gradation \( R^1 \otimes R^2 = \bigoplus_{n \in \mathbb{N}} (R^1 \otimes R^2)_n \). Then we can understand \( V(\gamma) \in \text{End}(\mathcal{F}) \hat{\otimes} (R^1 \otimes R^2) \). An example is the case that \( R^1 = R^2 = \Lambda \).

The isomorphisms \( i \) and \( i^\dagger \) are realized as computation of matrix elements of vertex operators. Let us introduce

\[ \Gamma(X)_\pm = \exp \left( \sum_{n>0} \frac{1 - t^n}{1 - q^n} p_n(X) a_{\pm n} \right). \]
Notice that these are vertex operators specified by $\gamma_{\pm n} = \frac{1-t^n}{1-q^n}p_n(X) \in \Lambda_X$, $\gamma_{\mp n} = 0$, $n > 0$. We will rely on extensive use of the following fact.

**Proposition 2.2.** For any $|v\rangle \in \mathcal{F}$ or $\langle v| \in \mathcal{F}^\dagger$, its image under $\iota$ or $\iota^\dagger$ is realized as

$$\iota(|v\rangle) = \langle 0|\Gamma(X)_+ |v\rangle \in \Lambda_X, \quad \iota^\dagger(\langle v|) = \langle v|\Gamma(X)_- |0\rangle \in \Lambda_X.$$ 

For a symmetric function $F \in \Lambda$, we write $|F\rangle := \iota^{-1}(F) \in \mathcal{F}$ and $\langle F| := (\iota^\dagger)^{-1}(F) \in \mathcal{F}^\dagger$. It immediately follows that

$$P_{\lambda/\mu}(X) = \langle Q_{\mu}|\Gamma(X)_+ |P_\lambda \rangle = \langle P_\lambda|\Gamma(X)_- |Q_\mu \rangle, \quad Q_{\lambda/\mu}(X) = \langle P_{\mu}|\Gamma(X)_+ |Q_\lambda \rangle = \langle Q_\lambda|\Gamma(X)_- |P_\mu \rangle,$$

for any skew-partition $\lambda/\mu$.

In this note, trace of a vertex operator plays a significant role. Let $D \in \text{End}(\mathcal{F})$ be the degree operator defined by $D|\lambda\rangle = |\lambda| |\lambda\rangle$, $\lambda \in \mathbb{Y}$. The following formula was essentially discussed in e.g. [DM00, CW07].

**Proposition 2.3.** Let $\gamma_n$, $n \in \mathbb{Z}\setminus \{0\}$ be commuting symbols and let $u$ be yet another formal variable. Then we have

$$\text{Tr}_\mathcal{F}(u^D V(\gamma)) = \frac{1}{(u; u)_\infty} \exp \left( \sum_{n>0} \frac{1-q^n}{1-u^n} \gamma_n \right).$$

**Remark 2.4.** The both sides are understood as follows.

1. If $\gamma_n \in R_n$, $n \in \mathbb{Z}\setminus \{0\}$ with a $\mathbb{Z}$-graded commutative algebra $R$, the trace lies in $R[[u]]$, where $R = \prod_{n \in \mathbb{Z}} R_n$.

2. If $\gamma_n \in R_n^1$ and $\gamma_n \in R_n^2$, $n > 0$ with $\mathbb{N}$-graded commutative algebras $R^1$, $R^2$, the trace lies in $R^1 \otimes R^2[[u]]$.

**Proof.** For a partition $\lambda \in \mathbb{Y}$, we denote by $\pi_\lambda : \mathcal{F} \to \mathbb{F}|\lambda\rangle$ the projection with respect to a basis $\{|\lambda\rangle : \lambda \in \mathbb{Y}\}$. Then we can see that

$$\pi_\lambda V(\gamma) |\lambda\rangle = \prod_{n>0} \left( \sum_{m_n(\lambda)} \frac{1}{k!} \gamma_n \right)^k (a_n)^k |\lambda\rangle.$$

Here, note that

$$(a_n)^k (a_n)^k (a_n)^k |0\rangle = k! \left( \sum_{m_n(\lambda)} \right) \left( \frac{1-q^n}{1-t^n} \right)^k |\lambda\rangle$$

to find

$$\pi_\lambda V(\gamma) |\lambda\rangle = \prod_{n>0} \left( \sum_{m_n(\lambda)} \frac{1}{k!} \gamma_n \right)^k |\lambda\rangle.$$

Since $|\lambda\rangle = \sum_{n>0} m_n(\lambda)n$, the trace becomes

$$\text{Tr}_\mathcal{F}(u^D V(\gamma)) = \prod_{n>0} \left( \sum_{k=0}^\infty \frac{1}{k!} \left( \frac{1-q^n}{1-t^n} \right)^k \sum_{m=k}^\infty \left( \frac{m}{k} \right) u^m \right).$$
Notice
\[
\sum_{m=k}^{\infty} \binom{m}{k} x^m = \frac{x^k}{(1-x)^{k+1}}.
\]
which implies
\[
\text{Tr}_y \left( u^D V(\gamma) \right) = \frac{1}{(u; u)_{\infty}} \prod_{n>0} \left( \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{1 - q^n}{1 - t^n} \frac{u^n}{n} \gamma_n \gamma_n \right)^k \right)
\]
\[
= \frac{1}{(u; u)_{\infty}} \exp \left( \sum_{n>0} \frac{1}{n} \left( \frac{1 - q^n}{1 - t^n} \frac{u^n}{n} \gamma_n \gamma_n \right) \right)
\]
completing the proof.

Associated with \( \gamma^i = (\gamma^i_n : n \in \mathbb{Z} \setminus \{ 0 \}) \), \( i = 1, \ldots, N \), the normally ordered product of corresponding vertex operators is defined by
\[
:V(\gamma^1) \cdots V(\gamma^N): = \exp \left( \sum_{n>0} \sum_{i=1}^N \frac{\gamma^i_n}{n} a_{-n} \right) \exp \left( - \sum_{n>0} \sum_{i=1}^N \frac{\gamma^i_n}{n} a_n z^{-n} \right).
\]
Namely, we put positive modes of the Heisenberg algebra on the right and negative modes on the left. Also, note that, in a normally ordered product, vertex operators are commutative.

2.3. Free field realization of operators. Since we have identified the Fock space \( \mathcal{F} \) with the space of symmetric functions, we can also identify an operator \( T \in \text{End}(\Lambda) \) with one \( T := e^{-1} \circ T \circ e \in \text{End}(\mathcal{F}) \), the free field realization of \( T \). We introduced a family of renormalized Macdonald operators \( E_r, r \in \mathbb{N} \). To describe their free field realization, we introduce a vertex operator
\[
\eta(z) = V(\gamma) = \exp \left( \sum_{n>0} \frac{1 - t^{-n}}{n} a_{-n} z^n \right) \exp \left( - \sum_{n>0} \frac{1 - t^n}{n} a_n z^{-n} \right)
\]
associated with \( \gamma_n = -\frac{1-t^n}{n} z^{-n} \in \mathbb{F}[z, z^{-1}], n \in \mathbb{Z} \setminus \{ 0 \} \).

Proposition 2.5 (Sho06, FHH+09, Kos19). For each \( r \in \mathbb{N} \), the free field realization of the \( r \)-th renormalized Macdonald operator is given by
\[
\hat{E}_r = \frac{t^{-r}}{r!} \int \left( \prod_{i=1}^{r} \frac{dz_i}{2\pi\sqrt{-1}} \right) \det \left( \frac{1}{z_i - t^{-1} z_j} \right)_{1 \leq i, j \leq r} : \eta(z_1) \cdots \eta(z_r) :.
\]
Here the integral \( \int \left( \prod_{i=1}^{r} \frac{dz_i}{2\pi\sqrt{-1}} \right) \) is understood as a linear functional taking the residue of the integrand and a rational function of the form \( \frac{1}{x-y} := \sum_{k=0}^{\infty} x^{-k-1} y^k \) is always expanded in \( \mathbb{F}[y, y^{-1}]((x^{-1})) \) (Therefore, \( \frac{1}{x-y} = -\frac{1}{y-x} \)).

Owing to the property \( P_3(q, t) = P_3(q^{-1}, t^{-1}) \) of a Macdonald symmetric function, a renormalized Macdonald operator with inverted parameters \( E'_r := E_r(q^{-1}, t^{-1}) \) is also diagonalized by the Macdonald symmetric functions. Let us introduce another vertex operator
\[
\xi(z) = V(\gamma) = \exp \left( - \sum_{n>0} \frac{1 - t^{-n}}{n} (t/q)^{n/2} a_{-n} z^n \right) \exp \left( \sum_{n>0} \frac{1 - t^n}{n} (t/q)^{n/2} a_n z^{-n} \right)
\]
associated with $\gamma_n = \frac{1 - r^n}{n}(t/q)^{n/2}z^{-n} \in \mathbb{Q}(q^{1/2}, t^{1/2})[z, z^{-1}], \ n \in \mathbb{Z}\ \setminus \{0\}$.

**Proposition 2.6** ([Shi06][FHH+09][Kos19]). For each $r \in \mathbb{N}$, we have

$$\hat{E}_r^\prime = \frac{r^r}{r!} \int \left( \prod_{i=1}^{r} \frac{dz_i}{2\pi\sqrt{-1}} \right) \det \left( \frac{1}{z_i - tz_j} \right)_{1 \leq i, j \leq r} :\xi(z_1) \cdots \xi(z_r):.$$  

Then, by definition, these operators are diagonalized by the Macdonald basis so that

$$\hat{E}_r^\prime |P_\lambda(q, t)\rangle = e_r(q^{-\lambda}t^\rho) |P_\lambda(q, t)\rangle, \ r \in \mathbb{N}, \ \lambda \in \mathbb{Y}.$$  

Let us see yet other series of operators diagonalized by the Macdonald basis following [Shi06][FHH+09][Kos19]. For $r \in \mathbb{N}$, we introduce the following operator

$$\hat{G}_r := \frac{(-1)^r}{r!} \int \left( \prod_{i=1}^{r} \frac{dz_i}{2\pi\sqrt{-1}} \right) \det \left( \frac{1}{z_i - q^{-1}z_j} \right)_{1 \leq i, j \leq r} :\eta(z_1) \cdots \eta(z_r):.$$  

Then we have

$$\hat{G}_r |P_\lambda(q, t)\rangle = g_r(q^\lambda t^{-\rho}; q, t) |P_\lambda(q, t)\rangle, \ r \in \mathbb{N}, \ \lambda \in \mathbb{Y}.$$  

The other series corresponds to $\hat{G}_r$, $r \in \mathbb{N}$ with inverted parameters.

$$\hat{G}_r^\prime = \frac{(-1)^r}{r!} \int \left( \prod_{i=1}^{r} \frac{dz_i}{2\pi\sqrt{-1}} \right) \det \left( \frac{1}{z_i - qz_j} \right)_{1 \leq i, j \leq r} :\xi(z_1) \cdots \xi(z_r):, \ r \in \mathbb{N}.$$  

Then they are diagonalized so that

$$\hat{G}_r^\prime |P_\lambda(q, t)\rangle = g_r(q^{-\lambda}t^\rho; q^{-1}, t^{-1}) |P_\lambda(q, t)\rangle, \ r \in \mathbb{N}, \ \lambda \in \mathbb{Y}.$$  

Note, here, that the eigenvalue is the same as $g_r(q^{-\lambda}t^\rho; q^{-1}, t^{-1}) = g_r(q^{-\lambda+1}t^{\rho-1}; q, t)$.

### 3. Periodic Macdonald Process

The aim of this section is to prove Proposition 1.3. Let us define a universal version of the weight (1.2). Let $N \in \mathbb{N}$. We write $X = (X^0, \ldots, X^{N-1})$ and $Y = (Y^1, \ldots, Y^N)$ for $N$-tuples of infinite variables. For $\lambda = (\lambda^1, \ldots, \lambda^N), \mu = (\mu^1, \ldots, \mu^N) \in \mathbb{Y}^N$, we set

$$W_{q,t;u}^{X,Y}(\lambda, \mu) = \mathcal{W}^N \prod_{i=1}^{N} Q_{X^i/\mu^i}(Y^i; q, t) P_{\lambda^{i+1}/\mu^i}(X^i; q, t) \in \left( \bigotimes_{i=0}^{N-1} \Lambda_{X^i} \otimes \Lambda_{Y^{i+1}} \right)^{\mu^N}.$$  

When we fix parameters $q, t \in \mathbb{R}$ such that $|q|, |t| < 1$, adopt specialization at these parameters and take $N$-tuples of $(q, t)$-Macdonald positive specializations $\rho^+ = (\rho^+_0, \ldots, \rho^+_N)$, $\rho^- = (\rho^-_1, \ldots, \rho^-_N)$, the weight (1.2) is the image of $W_{q,t;u}^{X,Y}(\lambda, \mu)$ under $\bigotimes_{i=0}^{N-1} \rho^+_i \otimes \rho^-_{i+1}$, where each specialization acts as $\rho^+_i : \Lambda_{X^i} \to \mathbb{R}, \rho^-_i : \Lambda_{Y^{i+1}} \to \mathbb{R}$. Note that, at the limit $u \to 0$, the weight $W_{q,t;u}^{X,Y}(\lambda, \mu)$ vanishes unless $\mu^N = \emptyset$ recovering the $N$-step Macdonald process since

$$\mathcal{W}^u \to \begin{cases} 1, & \mu = \emptyset, \\ 0, & \text{otherwise}, \end{cases}$$  

as $u \to 0$. 
Proof of Proposition 2.3. Our goal is to show

\[ \Pi_{q,t;u}(X;Y) := \sum_{\lambda, \mu \in \mathcal{Y}} W_{q,t;u}^{X,Y}(\lambda, \mu) = \frac{1}{(u; u)_{\infty}} \prod_{i=0}^{N-1} \prod_{j=1}^{N} \tilde{\Pi}_{q,t;u}(X^i; Y^j). \]

Let us first write the weight by means of matrix elements of vertex operators:

\[ W_{q,t;u}^{X,Y}(\lambda, \mu) = u^{\mu N} \prod_{i=1}^{N} \langle Q_{\lambda}(q,t)|\Gamma(Y^i)_{-}|-P_{\mu}(q,t)\rangle \langle Q_{\mu}(q,t)|\Gamma(X^i)_{+}|P_{\lambda+1}(q,t)\rangle. \]

Recalling the property \( \text{Id}_F = \sum_{\lambda \in \mathcal{Y}} \langle P_{\lambda}(q,t) \rangle \langle Q_{\lambda}(q,t) \rangle \) and the definition of the degree operator \( D \), we can see that

\[ \Pi_{q,t;u}(X;Y) = \text{Tr}_F (u^D \Gamma(X^0)_{+} \Gamma(Y^1)_{-} \Gamma(X^1)_{+} \cdots \Gamma(Y^{N-1})_{-} \Gamma(X^{N-1})_{+} \Gamma(Y^{N})_{-}). \]

To apply the trace formula in Proposition 2.3, we rearrange the operators in the trace in the normally ordered manner. By a standard computation for vertex operators, we have

\[ \Gamma(X)_{+} \Gamma(Y)_{-} = \exp \left( \sum_{n>0} \frac{1-t^n}{1-q^n} p_n(X)p_n(Y) \right) \Gamma(Y)_{-} \Gamma(X)_{+} = \tilde{\Pi}_{q,t;0}(X;Y) \Gamma(Y)_{-} \Gamma(X)_{+}. \]

Repeating this kind of reordering, we can make the operators normally ordered so that

\[ \Gamma(X^0)_{+} \Gamma(Y^1)_{-} \Gamma(X^1)_{+} \cdots \Gamma(Y^{N-1})_{-} \Gamma(X^{N-1})_{+} \Gamma(Y^{N})_{-} = \left( \prod_{i<j} \tilde{\Pi}_{q,t;0}(X^i; Y^j) \right) V(\gamma), \]

where we set

\[ \gamma_n = \frac{1-t^n}{1-q^n} \sum_{i=0}^{N-1} p_n(X^i), \quad \gamma_{-n} = \frac{1-t^n}{1-q^n} \sum_{i=1}^{N} p_n(Y^i), \quad n > 0. \]

Now we can apply Proposition 2.3 to obtain

\[ \text{Tr}_F (u^D V(\gamma)) = \frac{1}{(u; u)_{\infty}} \prod_{i=0}^{N-1} \prod_{j=1}^{N} \exp \left( \sum_{n>0} \frac{1-t^n}{1-q^n} \frac{u^n}{1-u^n} p_n(X^i)p_n(Y^j) \right). \]

Notice that

\[ \tilde{\Pi}_{q,t;0}(X;Y) \exp \left( \sum_{n>0} \frac{1-t^n}{1-q^n} \frac{u^n}{1-u^n} p_n(X)p_n(Y) \right) = \exp \left( \sum_{n>0} \frac{1-t^n}{1-q^n} \frac{p_n(X)p_n(Y)}{n} \right) = \tilde{\Pi}_{q,t;u}(X;Y) \]

Therefore, we obtain the desired result. □

Let us investigate some limiting cases.
• When \( N = 1 \), we have
\[
\sum_{\lambda, \mu \in \mathcal{Y}} u^{\lambda} P_{\mu/\lambda}(X) Q_{\mu/\lambda}(Y) = \Pi_{q,t;u}(X; Y) := \frac{1}{(u; u)_\infty} \prod_{i,j=1}^{\infty} \frac{(tx_i y_j; q, u)_\infty}{(x_i y_j; q, u)_\infty},
\]
which was proved in [RW18].

• At the limit \( u \to 0 \), it reduces to the partition function for a Macdonald process [BC14, BCGS16]:
\[
\Pi_{q,t;0}(X; Y) = \prod_{i<j} \tilde{\Pi}_{q,t;0}(X_i; Y_j).
\]

• At the Schur-limit \( q \to t \), the partition function for a periodic Schur process [Bor07] is recovered:
\[
\tilde{\Pi}_{t,t;u}(X; Y) = \prod_{i,j} 1_{x_i y_j; \infty}(u),
\]

4. Generalized MacMahon formula for cylindric partitions

This section is devoted to a proof of Theorem 1.2. Given a boundary profile \( \mathbb{M} \subset [1, N] \), we apply the following specializations to the weight \( W_{q,t;u}^{p^+, p^-}(\lambda, \mu) \) in Eq. (1.2). For \( k = 1, \ldots, N - 1 \), if \( k \in \mathbb{M} \), then
\[
\rho_k^+ : x_1 = s^{-k}, \quad x_2 = x_3 = \cdots = 0,
\]
\[
\rho_k^- : x_1 = x_2 = \cdots = 0,
\]
and if \( k \not\in \mathbb{M} \), then
\[
\rho_k^+ : x_1 = x_2 = \cdots = 0,
\]
\[
\rho_k^- : x_1 = s^k, \quad x_2 = x_3 = \cdots = 0.
\]
The specializations \( \rho_0^+ \) and \( \rho_N^- \) are defined so that if \( N \in \mathbb{M} \), then
\[
\rho_0^+ : x_1 = 1, \quad x_2 = x_3 = \cdots = 0,
\]
\[
\rho_N^- : x_1 = x_2 = \cdots = 0,
\]
and if \( N \not\in \mathbb{M} \), then
\[
\rho_0^+ : x_1 = x_2 = \cdots = 0,
\]
\[
\rho_N^+ : x_1 = s^N, \quad x_2 = x_3 = \cdots = 0.
\]

Lemma 4.1. Under the above specializations, the weight \( W_{q,t;u}^{p^+, p^-}(\lambda, \mu) \) vanishes unless \( \mu^k = \lambda^k \) when \( k \in \mathbb{M} \) and \( \mu^k = \lambda^{k+1} \) when \( k \not\in \mathbb{M} \) for all \( k = 1, \ldots, N \) and \( \lambda \) is a cylindric partition of periodicity \( N \) and boundary profile \( \mathbb{M} \). Moreover, for such \( \mu \), we have
\[
W_{s_N^+; p^-}(\lambda, \mu) = \Phi_{\lambda}(q, t) s^{|\lambda|},
\]
where
\[
\Phi_{\lambda}(q, t) := \prod_{i=1}^{N} J_{\lambda^i, \lambda^{i+1}}(q, t), \quad J_{\lambda^i, \lambda^{i+1}}(q, t) := \begin{cases} 
\psi_{\lambda^i, \lambda^{i+1}}(q, t), & i \in \mathbb{M}, \\
\varphi_{\lambda^i, \lambda^{i+1}}(q, t), & i \not\in \mathbb{M}.
\end{cases}
\]
Proof. Since for each \( k = 1, \ldots, N \), either \( \rho_k^+ \) or \( \rho_k^- \) is the zero specialization, the weight vanishes unless \( \mu^k = \lambda^k \) when \( k \in \mathbb{M} \) and \( \mu^k = \lambda^{k+1} \) when \( k \not\in \mathbb{M} \). Recall that Chapter VI, (7.14) and (7.14′)]

\[
P_{\lambda/\mu}(a, 0, \ldots) = \chi[\lambda | \mu] \psi_{\lambda/\mu}(q, t)a^{[\lambda] | [\mu]},
\]

\[
Q_{\lambda/\mu}(a, 0, \ldots) = \chi[\lambda | \mu] \varphi_{\lambda/\mu}(q, t)a^{[\lambda] | [\mu]},
\]

which verifies that the weight \( W_{a^{\rho^+}^{\emptyset}^{-\rho}}(\lambda, \mu) \) is nonzero only if \( \lambda \in \mathbb{C}\mathbb{P}(N, \mathbb{M}) \). For \( k = 1, \ldots, N - 1 \), we have

\[
Q_{\lambda^k/\mu^k}(\rho_k^+)P_{\lambda^k+1/\mu^k}(\rho_k^+) = \begin{cases} 
\chi[\mu^k = \lambda^k] \psi_{\lambda^k+1/\lambda^k}(q, t)s^{-k((\lambda^k+1)-[\lambda^k])}, & k \in \mathbb{M}, \\
\chi[\mu^k = \lambda^{k+1}] \varphi_{\lambda^k+1/\lambda^{k+1}}(q, t)s^{k((\lambda^k+1)-[\lambda^{k+1}])}, & k \not\in \mathbb{M}.
\end{cases}
\]

and

\[
s^{N|\mu'|}Q_{\lambda'|\mu'|}(\rho_N^+) = \begin{cases} 
\chi[\mu^N = \lambda^N] \psi_{\lambda^N/\lambda^N}(q, t)s^{N|\lambda^N|}, & N \in \mathbb{M}, \\
\chi[\mu^N = \lambda^N] \varphi_{\lambda^N/\lambda^N}(q, t)s^{N|\lambda^N|}, & N \not\in \mathbb{M}.
\end{cases}
\]

Therefore, we can see that

\[
W_{s^{\rho^+}^{\emptyset}^{-\rho}}(\lambda, \mu) = \Phi_{\lambda}(q, t)s^{\sum_{k=1}^{N-1} k((\lambda^k+1)-[\lambda^k])+N|\lambda^N|} = \Phi_{\lambda}(q, t)s|\lambda|,
\]

where \( \mu^k = \lambda^k \) if \( k \in \mathbb{M} \) and \( \mu^k = \lambda^{k+1} \) if \( k \not\in \mathbb{M} \), \( k = 1, \ldots, N \). \( \square \)

It is obvious from the definition of the weight \( W_{a^{\rho^+}^{\emptyset}^{-\rho}}(\lambda, \mu) \) that

\[
\sum_{\lambda \in \mathbb{C}\mathbb{P}(N, \mathbb{M})} \Phi_{\lambda}(q, t)s^{\lambda|} = \sum_{\lambda, \mu \in \mathbb{C}\mathbb{P}(N, \mathbb{M})} W_{s^{\rho^+}^{\emptyset}^{-\rho}}(\lambda, \mu) = \Pi_{q, t, s^{\emptyset}}(\rho^+; \rho^-).
\]

Now the right hand side can be further computed by means of Proposition [13] to be

\[
\Pi_{q, t, s^{\emptyset}}(\rho^+; \rho^-) = \frac{1}{(s^{N}; s^{N})\infty} \prod_{k \leq l, \ell \in \mathbb{M}} (ts^{l-k}; q, s^{N})_{\infty} \prod_{k > l, \ell \in \mathbb{M}} (ts^{N-k-l}; q, s^{N})_{\infty}
\]

\[
= \frac{1}{(s^{N}; s^{N})\infty} \prod_{k \in \mathbb{M}, \ell \in \mathbb{M}} (ts^{l-k}; q, s^{N})_{\infty}
\]

It remains to show the following property.

Lemma 4.2. For every \( \lambda \in \mathbb{C}\mathbb{P}(N, \mathbb{M}) \), we have

\[
F_{\lambda}(q, t) = \Phi_{\lambda}(q, t).
\]

Proof. We show this by induction in terms of numbers of boxes. For the empty cylindric partition \( \emptyset = (\emptyset, \ldots, \emptyset) \), it is obvious that \( F_{\emptyset}(q, t) = 1 = \Phi_{\emptyset}(q, t) \). Given \( \lambda \in \mathbb{C}\mathbb{P}(N, \mathbb{M}) \), let us fix \( k \in \{1, \ldots, N\} \) and write \( \lambda = \lambda^k, \lambda^L = \lambda^{k-1}, \lambda^R = \lambda^{k+1} \), where we understand \( \lambda^0 = \lambda^N \) and \( \lambda^{N+1} = \lambda^1 \). We let \( \lambda' \) be the partition made from \( \lambda \) by removing the last row, i.e., \( \lambda' = (\lambda_1, \ldots, \lambda_{|\lambda|-1}, 0) \) and assume that \( \lambda' = (\lambda^1, \ldots, \lambda^L, \lambda', \lambda^R, \ldots, \lambda^N) \in \mathbb{C}\mathbb{P}(N, \mathbb{M}) \). Then it suffices to show that

\[
F_{\lambda'}(q, t) = \frac{F_{\lambda'}(q, t)}{F_{\lambda}(q, t)} = \Phi_{\lambda}(q, t).
\]
On one hand, we have

\[
\frac{F_{\lambda}(q,t)}{F_{\lambda}(q,t)} = \left( \prod_{\ell \neq k} \prod_{j=1}^{\ell(\lambda)} \frac{F_{\lambda}(l, j; q, t)}{F_{\lambda}(l, j; q, t)} \right) \left( \prod_{j=1}^{\ell(\lambda)-1} \frac{F_{\lambda}(k, j; q, t)}{F_{\lambda}(k, j; q, t)} \right) \frac{1}{F_{\lambda}(k, \ell(\lambda); q, t)}
\]

\[
= \prod_{i=1}^{\ell(\lambda)} f(q^{\lambda_i - \lambda(\ell(\lambda))} q^{\ell(\lambda)_i - i}) \prod_{i=1}^{\ell(\lambda)} f(q^{\lambda_i - \lambda(\ell(\lambda))} q^{\ell(\lambda)_i - i}) \frac{f(q^{\lambda_i} q^{\ell(\lambda)_i - i})}{f(1)}
\]

\[
= \prod_{i=1}^{\ell(\lambda)} f(q^{\lambda_i} q^{\ell(\lambda)_i - i}) \prod_{i=1}^{\ell(\lambda)} f(q^{\lambda_i} q^{\ell(\lambda)_i - i}) \frac{f(q^{\lambda_i})}{f(1)}
\]

To derive the other side, we note the following expressions of \(\psi_{\lambda/\mu}(q, t)\) and \(\varphi_{\lambda/\mu}(q, t)\) [Mac95 Chapter VI, Section 6]:

\[
\psi_{\lambda/\mu}(q, t) = \prod_{1 \leq i \leq j \leq \ell(\mu)} \frac{f(q^{\mu_i - \mu_j} t^{j-i}) f(q^{\lambda_i - \lambda_j} t^{j-i})}{f(q^{\mu_i - \mu_j} t^{j-i}) f(q^{\lambda_i - \lambda_j} t^{j-i})}
\]

\[
\varphi_{\lambda/\mu}(q, t) = \prod_{1 \leq i \leq j \leq \ell(\lambda)} \frac{f(q^{\lambda_i - \lambda_j} t^{j-i}) f(q^{\mu_i - \mu_j} t^{j-i})}{f(q^{\lambda_i - \lambda_j} t^{j-i}) f(q^{\mu_i - \mu_j} t^{j-i})}
\]

We can think of the following four cases:

1. When \(\lambda^L < \lambda < \lambda^R\). In this case, \(\ell(\lambda^L) = \ell(\lambda') = \ell(\lambda) - 1\) and \(\ell(\lambda^R) = \ell(\lambda)\) and

\[
\frac{\Phi_{\lambda}(q, t)}{\Phi_{\lambda}(q, t)} = \frac{\psi_{\lambda/\lambda^L}(q, t)}{\psi_{\lambda^L}(q, t)} \frac{\psi_{\lambda}/\lambda^R(q, t)}{\psi_{\lambda^R}(q, t)}
\]

The first factor in the product is computed as

\[
\frac{\psi_{\lambda/\lambda^L}(q, t)}{\psi_{\lambda^L}(q, t)} = \prod_{i=1}^{\ell(\lambda^L)} \frac{f(q^{\lambda_i - \lambda(\ell(\lambda))} q^{\ell(\lambda^L)_i - i})}{f(1)} \prod_{i=1}^{\ell(\lambda^L) - 1} \frac{f(q^{\lambda_i} q^{\ell(\lambda^L)_i - i})}{f(q^{\lambda_i} q^{\ell(\lambda^L)_i - i})}
\]

while the second factor is

\[
\frac{\psi_{\lambda^R}/\lambda(q, t)}{\psi_{\lambda^R}/\lambda(q, t)} = \prod_{i=1}^{\ell(\lambda^R)} \frac{f(q^{\lambda_i - \lambda(\ell(\lambda))} q^{\ell(\lambda^R)_i - i})}{f(1)} \prod_{i=1}^{\ell(\lambda^R) - 1} \frac{f(q^{\lambda_i} q^{\ell(\lambda^R)_i - i})}{f(q^{\lambda_i} q^{\ell(\lambda^R)_i - i})}
\]

2. When \(\lambda^L < \lambda > \lambda^R\). In this case, \(\ell(\lambda^L) = \ell(\lambda^R) = \ell(\lambda) - 1\) and

\[
\frac{\Phi_{\lambda}(q, t)}{\Phi_{\lambda}(q, t)} = \frac{\psi_{\lambda^L}/\lambda(q, t)}{\psi_{\lambda^L}/\lambda(q, t)} \frac{\varphi_{\lambda^R}/\lambda(q, t)}{\varphi_{\lambda^R}/\lambda(q, t)}
\]
the second factor of which is
\[
\frac{\varphi_{\lambda'/\lambda R}(q,t)}{\varphi_{\lambda/\lambda R}(q,t)} = \prod_{i=1}^{\ell(\lambda^R)} \frac{f(q^{\lambda^R_{-i}}) f(\ell(\lambda^R)-i)}{f(q^{\lambda^R_i}) f(\ell(\lambda^R)-i)} \prod_{i=1}^{\ell(\lambda)} \frac{f(q^{\lambda_{-i}}) f(\ell(\lambda)-i)}{f(q^{\lambda_i}) f(\ell(\lambda)-i)}.
\]

(3) When \( \lambda^t \succ \lambda \sim \lambda^R \). In this case, \( \ell(\lambda^t) = \ell(\lambda^R) = \ell(\lambda) \) and
\[
\frac{\Phi_{\lambda}(q,t)}{\Phi_{\lambda}(q,t)} = \frac{\varphi_{\lambda'/\lambda R}(q,t)}{\varphi_{\lambda'/\lambda}(q,t)} \psi_{\lambda'/\lambda R}(q,t),
\]
the first factor of which is
\[
\frac{\varphi_{\lambda'/\lambda R}(q,t)}{\varphi_{\lambda'/\lambda}(q,t)} = \prod_{i=1}^{\ell(\lambda^R)} \frac{f(q^{\lambda^R_{-i}}) f(\ell(\lambda^R)-i)}{f(q^{\lambda^R_i}) f(\ell(\lambda^R)-i)} \prod_{i=1}^{\ell(\lambda)-1} \frac{f(q^{\lambda_{-i}}) f(\ell(\lambda)-i)}{f(q^{\lambda_i}) f(\ell(\lambda)-i)}.
\]

(4) When \( \lambda^t \succ \lambda \succ \lambda^R \). In this case, \( \ell(\lambda^t) = \ell(\lambda), \ell(\lambda^R) = \ell(\lambda) - 1 \) and
\[
\frac{\Phi_{\lambda}(q,t)}{\Phi_{\lambda}(q,t)} = \frac{\varphi_{\lambda'/\lambda R}(q,t)}{\varphi_{\lambda'/\lambda}(q,t)} \frac{\varphi_{\lambda'/\lambda R}(q,t)}{\varphi_{\lambda'/\lambda}(q,t)}.
\]

In any case, we can see (4.1) and, therefore, the desired relation \( F_{\lambda}(q,t) = \Phi_{\lambda}(q,t), \lambda \in \mathcal{CP}(N, M) \) is valid.

Let us investigate some limiting cases. Obviously, at the Schur-limit \( q \to t \), the weight \( F_{\lambda}(q,t) \) reduces to the unity and our formula recovers the MacMahon formula for cylindric partitions presented in [Bor07].

**Corollary 4.3.** Let \( N \in \mathbb{N} \) and let \( M \) be a boundary profile. Then
\[
\sum_{\lambda \in \mathcal{CP}(N, M)} s^{\lambda} = \frac{1}{(s^N; s^N)_\infty} \prod_{k \in \mathbb{Z}, l \in \mathbb{M}} \frac{1}{(s^{l-k}; s^N)_\infty}.
\]

The generalized MacMahon formula for plane partitions [Vnl09] is recovered at the infinite period limit \( N \to \infty \). To see this, we translate a boundary profile \( M \in \mathbb{Z} \) and understand \( M \subseteq \{-N, -N+1, \ldots, N-2, N-1\} \). Therefore, it is a boundary profile of cylindric partitions of periodicity \( 2N \). In particular, if we take \( M = \{-N, -N+1, \ldots, -1\} \), we have
\[
\sum_{\lambda \in \mathcal{CP}(2N, M)} F_{\lambda}(q,t) s^{\lambda} = \frac{1}{(s^{2N}; s^{2N})} \prod_{k=-N}^{-1} \prod_{l=0}^{N-1} \frac{(ts^{l-k}; q, s^{2N})_\infty}{(s^{l-k}; q, s^{2N})_\infty}.
\]

At the limit \( N \to \infty \), the collection \( \mathcal{CP}(2N, M) \) of cylindric partitions approaches the collection of plane partitions \( \mathcal{P} \) and, assuming \( s \in (0, 1) \), the right hand side converges to
\[
\prod_{k=-\infty}^{-1} \prod_{l=0}^{\infty} \frac{(ts^{l-k}; q, s^{2N})_\infty}{(s^{l-k}; q, s^{2N})_\infty} = \prod_{n=1}^{\infty} \left( \frac{(ts^n; q, s^{2N})_\infty}{(s^n; q, s^{2N})_\infty} \right)^n.
\]

To describe the Hall–Littlewood limit \( q \to 0 \), we introduce some notions. Let us fix a cylindric partition \( \lambda \in \mathcal{CP}(N, M) \) of periodicity \( N \) and boundary profile \( M \). For a position \( (k, j) \in [1, N] \times \mathbb{N} \), the level \( h(k, j) \) is defined by
\[
h(k, j) = \min \{ h \in \mathbb{N} \mid \lambda^k_j + h < \lambda^k_j \}.
\]
We also say that a position \((k, j)\) is adjacent to \((k + 1, j - \chi[k \not\in \mathcal{M}])\) and \((k + 1, j + \chi[k \in \mathcal{M}])\). Then the support of \(\lambda\), the collection of positions with nonzero entries, is decomposed into a disjoint union of connected components, each of which consists of positions of the same level. Obviously, such a connected component contains at most \(N\) boxes and, if it indeed consists of \(N\) boxes, it winds around the cylinder. In contrast to such a global connected component, we call a connected component of a fixed level \(B\) is local if \(|B| < N\). Now, to the cylindric partition \(\lambda\), we associate a weight
\[
A_\lambda(t) = \prod_{B \subseteq \text{Supp}(\lambda) : \text{local}} (1 - t^{|B|}),
\]
where the product runs over local connected components of fixed levels and \(h(B)\) is the level of boxes constituting \(B\). In fact, the following proposition is the Hall–Littlewood limit of Theorem 1.2.

**Proposition 4.4.** Let \(N \in \mathbb{N}\) and let \(\mathcal{M}\) be a boundary profile. Then
\[
\sum_{\lambda \in \mathcal{CP}(N, \mathcal{M})} A_\lambda(t)^{s(\lambda)} = \frac{1}{(s^N; s^N)_\infty} \prod_{k \in \mathcal{M}, s \not\in \mathcal{M}} (t^{s[k-k]; s^N})_\infty.
\]

**Proof.** Our goal is to show that \(F_\lambda(0, t) = A_\lambda(t)\) for all \(\lambda \in \mathcal{CP}(N, \mathcal{M})\). Fix \(\lambda\) and assume that \((k, j) \in [1, N] \times \mathbb{N}\) is of level \(h\) and belongs to a connected component \(B\) of the same level. The weight \(F_\lambda(k, j; 0, t)\) is decomposed into three parts:
\[
F_\lambda(k, j; q, t) = \prod_{m=0}^{h-2} \frac{f(q^{\lambda_1 - \lambda_{m+1} + \lambda m}) f(q^{\lambda_1 - \lambda_{m+2} + \lambda m})}{f(q^{\lambda_1 - \mu_{m+1} + \lambda m}) f(q^{\lambda_1 - \nu_{m+1} + \lambda m})} \times \frac{f(q^{\lambda_1 - \lambda_{h} + \lambda h - 1}) f(q^{\lambda_1 - \lambda_{h+1} + \lambda h - 1})}{f(q^{\lambda_1 - \mu_{h} + \lambda h - 1}) f(q^{\lambda_1 - \nu_{h} + \lambda h - 1})} \times \prod_{m=0}^{h-2} \frac{f(q^{\lambda_1 - \lambda_{m+1} + \lambda m}) f(q^{\lambda_1 - \lambda_{m+2} + \lambda m})}{f(q^{\lambda_1 - \mu_{m+1} + \lambda m}) f(q^{\lambda_1 - \nu_{m+1} + \lambda m})}.
\]
For the definition of partitions \(\lambda, \mu, \nu\), see Sect. 1. The first part is identical to unity because, by definition of the level, we have
\[
\lambda_1 \geq \mu_1 \geq \cdots \geq \mu_{h-1} \geq \lambda_h = \lambda_1, \quad \lambda_1 \geq \nu_1 \geq \cdots \geq \nu_{h-1} \geq \lambda_h = \lambda_1.
\]
Since \(f(u) \to 1 - tu\) at the Hall–Littlewood limit \(q \to 0\), the third parts converge to unity at that limit. Only the second part has nontrivial limit so that
\[
F_\lambda(k, j; 0, t) = \frac{1 - t^h}{(1 - th)^{\chi[k = \mu_h]/(1 - th)w(k = \nu_h)}} = \frac{1 - t^{h}}{(1 - t^{h})^{1-c(k, j)}},
\]
where we defined
\[
c(k, j) = \chi[(k - 1, j + \chi[k - 1 \not\in \mathcal{M}] \in B] + \chi[(k + 1, j + \chi[k \in \mathcal{M}] \in B].
\]
Notice that
\[
\sum_{(k, j) \in B} c(k, j) = \begin{cases} |B| - 1 & B : \text{local}, \\ |B| = N & B : \text{global}. \end{cases}
\]
Therefore, we have
\[ \prod_{(k,j) \in B} F_\lambda(k, j; 0, t) = \begin{cases} 1 - t^h & B : \text{local}, \\ 1 & B : \text{global}, \end{cases} \]
which proves \( F_\lambda(0, t) = A_\lambda(t) \).

We say that a cylindric partition \( \lambda \in CP(N, \mathbb{M}) \) is a strict cylindric partition if there is no local connected component of level larger than 1. We denote \( SCP(N, \mathbb{M}) \) is the collection of strict cylindric partitions in \( CP(N, \mathbb{M}) \). Since the weight \( A_\lambda(t) \) is defined multiplicatively, it vanishes at \( t = -1 \) if \( \lambda \) contains a local connected component of level larger than 1. Therefore, we have
\[ A_\lambda(-1) = \chi[\lambda \in SCP(N, \mathbb{M})] 2^{k(\lambda)}, \]
where \( k(\lambda) \) is the number of local connected components (of level 1) in the support of \( \lambda \). In [FW07, Vul07], the authors found the shifted MacMahon formula for strict plane partitions. The following corollary of Proposition 4.4 is an analogue of their formula for cylindric partitions.

**Corollary 4.5.** Let \( N \in \mathbb{N} \) and let \( \mathbb{M} \) be a boundary profile. Then
\[ \sum_{\lambda \in SCP(N, \mathbb{M})} 2^{k(\lambda)} s_{\lambda[N]} = \frac{1}{(s^N; s^N)_\infty} \prod_{k \in \mathbb{M}, l \in \mathbb{N}} (-s^{[l-k]}; s^N)_\infty. \]

5. **Trace of Macdonald refined topological vertices**

In this section, we prove Theorem 1.4. Let us write a Macdonald refined topological vertex in terms of a matrix element of vertex operators so that
\[ V_{\lambda\mu\nu}(x, y; q, t) = \prod_{s \in \nu} \frac{(t x^l(s)+1 y^{a\nu}(s); q)_\infty}{(x^l(s)+1 y^{a\nu}(s); q)_\infty} (P_\lambda(q, t)|\Gamma(x^{-\nu'} y^{\rho'-1})_+ \Gamma(x^{\rho'} y^{-\nu})_+ |Q_\mu(q, t)). \]

Therefore, the left hand side of Theorem 1.4 is simply
\[ \sum_{\lambda \in \mathbb{Y}} u^{[\lambda]} V_{\lambda\mu\nu}(x, y; q, t) = \prod_{s \in \nu} \frac{(t x^l(s)+1 y^{a\nu}(s); q)_\infty}{(x^l(s)+1 y^{a\nu}(s); q)_\infty} \text{Tr}_\nu \left( u^D \Gamma(x^{\rho'} y^{-\nu})_+ \Gamma(x^{\rho} y^{\nu'})_+ \right) \]
\[ = \prod_{s \in \nu} \frac{(t x^l(s)+1 y^{a\nu}(s); q)_\infty}{(x^l(s)+1 y^{a\nu}(s); q)_\infty} \Pi_{q, t; u}(x^{\rho'} y^{-\nu}; x^{\rho} y^{\nu'}). \]

For those specializations, we have
\[ p_n(x^{\rho} y^{\nu}) = \sum_{i=1}^{\ell(\nu)} (x^i y^{-\nu_i})^n + \frac{x^n(\ell(\nu)+1)}{1 - x^n}, \]
\[ p_n(x^{-\nu'} y^{\rho'-1}) = \sum_{i=1}^{\nu_1} (x^{-\nu'_i} y^{j_1})^n + \frac{y^{\nu_1}}{1 - y^n}. \]
functions are given by

\[ \Pi_{\nu}(x^n y^\nu; x^{-\nu} y^{\nu-1}) = \prod_{i=1}^{\ell(\nu)} \frac{(tx^{-\nu_i} y^{\nu_i - 1}; q, u)_\infty}{(x^{-\nu_i} y^{\nu_i - 1}; q, u)_\infty} \]

\[ \times \prod_{i=1}^{\ell(\nu)} \frac{(tx^y y^{\nu_i - \nu_i}; q, u, y)_\infty}{(x^y y^{\nu_i - \nu_i}; q, u, y)_\infty} \prod_{j=1}^{\nu_i}(tx^{y_j + \ell(\nu+1} y^{j-1}; q, u, x)_\infty \]

\[ \times \frac{(tx^{\ell(\nu)+1} y^{\nu_i}; q, u, x, y)_\infty}{(x^{\ell(\nu)+1} y^{\nu_i}; q, u, x, y)_\infty}. \]

For a box \( s = (i, j) \), we write \( i(s) := i \) and \( j(s) = j \). Recall that the arm length and leg length are defined as \( a_{\nu}(s) = \nu_i - j \) and \( b_{\nu}(s) = \nu_j - i \). The product in the first line is understood as one over the rectangular envelop \( \text{re}(\nu) = (\nu_1^{\ell(\nu)}) \) of the partition \( \nu \). The products in the second line are one over the right boundary \( \partial_R \nu = \{(i, \nu_i) : i = 1, \ldots, \ell(\nu)\} \) and one over the bottom boundary \( \partial_B \partial = \{(\nu_j', j) : j = 1, \ldots, \nu_1\} \). Therefore, we finally find that

\[
\sum_{\lambda \in \mathbb{V}} y_{\lambda}^{(x, y; q, t)} = \frac{1}{(u; u)_\infty} \prod_{s \in \nu} \frac{(tx^{a_{\nu}(s)}y^{a_{\nu}(s)}; q)_\infty}{(x^{a_{\nu}(s)}y^{a_{\nu}(s)}; q)_\infty} \prod_{s \in \nu} \frac{(tx^{\nu_i(s)}y^{\nu_i(s); q, u, x)}{q, u, x)_\infty} \prod_{s \in \partial_B \nu} \frac{(tx^{(\nu(s)+1)}y^{\nu(s)}; q, u, y)_\infty}{(x^{(\nu(s)+1)}y^{\nu(s)}; q, u, y)_\infty} \]

which is just the desired result.

When \( \nu = \emptyset \), the formula gets simpler.

**Corollary 5.1.** We have

\[
\sum_{\lambda \in \mathbb{V}} y_{\lambda}^{(x, y; q, t)} = \frac{(tx; q, u, x, y)_\infty}{(x; q, u, x, y)_\infty}. \]

In Appendix A we will present yet another attempt to generalize a result by [BKY18].

### 6. Stationary periodic Macdonald Plancherel process

For a positive number \( \xi > 0 \), we write \( \rho_{\xi} \) for the Plancherel specialization of parameter \( \xi \). Namely, it is a specialization \( \rho_{\xi} : \Lambda \rightarrow \mathbb{R} \) defined by

\[ \rho_{\xi}(p_n) = \xi \delta_{n,1}, \ n \in \mathbb{N}. \]

**Proposition 6.1.** For \( \xi > 0 \), the Plancherel specialization of the Macdonald symmetric functions are given by

\[ P_{\lambda/\mu}(\rho_{\xi}; q, t) = \frac{\xi^{\lambda-\mu} \dim_{q,t}(\mu, \lambda)}{(\lambda - \mu)!} \]

\[ Q_{\lambda/\mu}(\rho_{\xi}; q, t) = \frac{\xi^{\lambda-\mu} \dim'_{q,t}(\mu, \lambda)}{(\lambda - \mu)!}. \]
Here we write \( \dim_{q,t}(\mu, \lambda) \) and \( \dim'_{q,t}(\mu, \lambda) \) for deformed dimensions defined by
\[
\dim_{q,t}(\mu, \lambda) = \sum_{\mu^{\succ} \nu^{\succ} \cdots \nu_{|\lambda|-|\mu|-1}^{\succ} \lambda} \prod_{i=0}^{|\lambda|-|\mu|-1} \psi_{\nu_{i+1}/\nu_{i}}(q,t),
\]
\[
\dim'_{q,t}(\mu, \lambda) = \sum_{\mu^{\succ} \nu^{\succ} \cdots \nu_{|\lambda|-|\mu|-1}^{\succ} \lambda} \varphi_{\nu_{i+1}/\nu_{i}}(q,t),
\]
where the sum runs over paths from \( \mu \) to \( \lambda \) in the Young graph and we wrote \( \nu_{0} = \mu \), \( \nu_{|\lambda|-|\mu|} = \lambda \).

**Proof.** The Pieri rules say
\[
P_{\lambda}(q,t)g_{1}(q,t) = \sum_{\mu^{\succ} \lambda^{\succ} \mu} \varphi_{\mu/\lambda}(q,t)P_{\mu}(q,t), \quad Q_{\lambda}(q,t)g_{1}(q,t) = \sum_{\mu^{\succ} \lambda^{\succ} \mu} \psi_{\mu/\lambda}(q,t)Q_{\mu}(q,t).
\]
Noting \( g_{1}(q,t) = \frac{1-t}{1-q}P_{1} \), we have
\[
\frac{1-t}{1-q} a_{-1} [P_{\lambda}(q,t)] = \sum_{\mu^{\succ} \lambda^{\succ} \mu} \varphi_{\mu/\lambda}(q,t) [P_{\mu}(q,t)],
\]
\[
\frac{1-t}{1-q} a_{-1} [Q_{\lambda}(q,t)] = \sum_{\mu^{\succ} \lambda^{\succ} \mu} \psi_{\mu/\lambda}(q,t) [Q_{\mu}(q,t)].
\]
On the other hand, a Plancherel specialization of the Macdonald symmetric functions is computed as
\[
P_{\lambda/\mu}(\rho; q, t) = \langle P_{\lambda}(q,t)\Gamma(\rho)_{-} | P_{\mu}(q,t) \rangle, \quad Q_{\lambda/\mu}(\rho; q, t) = \langle Q_{\lambda}(q,t)\Gamma(\rho)_{-} | Q_{\mu}(q,t) \rangle,
\]
where
\[
\Gamma(\rho)_{-} = \exp \left( \frac{\xi}{1-q} a_{-1} \right).
\]
Then the desired results can be verified combinatorially. \( \square \)

For parameters \( \gamma > 0 \) and \( u \in (0, 1) \), we define an operator on \( \mathcal{F} \)
\[
\mathcal{F}(\gamma)(u) := e^{\frac{u}{1-q} \gamma^{2}(u-1)} \Gamma(\rho_{\gamma}(1-u)) u^{D} \Gamma(\rho_{\gamma}(1-u)).
\]

**Proposition 6.2.** For a fixed \( \gamma > 0 \) and parameters \( u, v \in (0, 1) \), we have
\[
\mathcal{F}(\gamma)(u)\mathcal{F}(\gamma)(v) = \mathcal{F}(\gamma)(uv).
\]
Moreover, \( \text{Tr}_{\mathcal{F}}(\mathcal{F}(\gamma)(u)) = 1/(u; u)_{\infty} \).

**Proof.** This is verified by a direct computation:
\[
\mathcal{F}(\gamma)(u)\mathcal{F}(\gamma)(v) = e^{\frac{u}{1-q} \gamma^{2}(u+v-2)} \Gamma(\rho_{\gamma}(1-u)) u^{D} \Gamma(\rho_{\gamma}(1-u)) + \Gamma(\rho_{\gamma}(1-v)) - v^{D} \Gamma(\rho_{\gamma}(1-v)) +
\]
\[
= e^{\frac{u}{1-q} \gamma^{2}(u-1)} \Gamma(\rho_{\gamma}(1-u)) u^{D} \Gamma(\rho_{\gamma}(1-u)) - \Gamma(\rho_{\gamma}(1-v)) + v^{D} \Gamma(\rho_{\gamma}(1-v)) +
\]
\[
= e^{\frac{v}{1-q} \gamma^{2}(v-1)} \Gamma(\rho_{\gamma}(1-u)) - \Gamma(\rho_{\gamma}(1-v)) - (uv)^{D} \Gamma(\rho_{\gamma}(1-u)) + \Gamma(\rho_{\gamma}(1-v)) +
\]
\[
= e^{\frac{1}{1-q} \gamma^{2}(uv-1)} \Gamma(\rho_{\gamma}(1-u)) - (uv)^{D} \Gamma(\rho_{\gamma}(1-u)).
\]
Therefore, the desired property holds. \( \square \)
For $\lambda, \mu \in \mathcal{Y}$, we write a matrix element of $\mathcal{T}^{(\gamma)}(u)$ as

$$T^{(\gamma)}_{\lambda \mu}(u) = \langle P_\lambda | \mathcal{T}^{(\gamma)}(u) | Q_\mu \rangle.$$ 

**Definition 6.3.** Let $\gamma > 0$ and $\beta > 0$. A stationary $\beta$-periodic Macdonald Plancherel process of intensity $\gamma$ is a stochastic process $(\lambda(t) : t \in \mathbb{R})$ in $\mathcal{Y}$ such that $\lambda(t + \beta) = \lambda(t)$, a.s., $t \in \mathbb{R}$ whose finite dimensional reduction measure for each $0 = b_0 < b_1 < \cdots < b_n < b_{n+1} = \beta$ is given by

$$\text{Prob}(\lambda(0), \lambda(b_1), \ldots, \lambda(b_n)) = (e^{-\beta}; e^{-\beta})_\infty \prod_{i=0}^{n} T^{(\gamma)}_{\lambda(b_i+1), \lambda(b_i)}(e^{-(b_{i+1}-b_i)}).$$

**Proposition 6.4.** For a stationary $\beta$-periodic Macdonald Plancherel process of intensity $\gamma$, the marginal law for $\lambda(0)$ is a periodic Macdonald measure $P^{\rho^+, \rho^-}_{q,t,e^{-\beta}}$.

**Proof.** First, we recall that a periodic Macdonald measure is written in the framework of the free field theory as

$$P^{\rho^+, \rho^-}_{q,t,u}(\lambda) = \frac{\text{Tr}_F (u^D \Gamma(\rho^+) | P_\lambda(q,t) \rangle \langle Q_\lambda(q,t) | \Gamma(\rho^-)_-)}{\text{Tr}_F (u^D \Gamma(\rho^+)_+ \Gamma(\rho^-)_-)}.$$

The marginal law of $\lambda(0)$ is just given by

$$(e^{-\beta}; e^{-\beta})_\infty < P_\lambda(0) | \mathcal{T}^{(\gamma)}(e^{-\beta}) | Q_\lambda(0) > = (e^{-\beta}; e^{-\beta})_\infty \text{Tr}_F < e^{-\beta D \Gamma(\rho_{(1-e^{-\beta})})_+} | P_\lambda(0) \rangle \langle Q_\lambda(0) | \Gamma(\rho_{(1-e^{-\beta})})_- >,$$

which is exactly the desired periodic Macdonald measure. \qed

**7. Observables of periodic Macdonald measures**

In this section, the following function of $z = (z_1, \ldots, z_n)$ is used repeatedly.

$$\Delta_{p_1, p_2; u}(z) = \prod_{i,j=1}^{n} \frac{(u z_i / z_j; u)_\infty (p_1 p_2 u z_i / z_j; u)_\infty}{(p_1 u z_i / z_j; u)_\infty (p_2 u z_i / z_j; u)_\infty}.$$

For a random variable $f : \mathcal{Y} \to \mathbb{F}$, we define an operator $\mathcal{O}(f) \in \text{End}(\mathcal{F})$ by

$$\mathcal{O}(f) := \sum_{\lambda \in \mathcal{Y}} f(\lambda) | P_\lambda \rangle \langle Q_\lambda |.$$

Then the expectation value of $f$ under a periodic Macdonald measure $P^{\rho^+, \rho^-}_{q,t,u}$ is

$$\mathbb{E}^{\rho^+, \rho^-}_{q,t,u}[f] = \sum_{\lambda \in \mathcal{Y}} f(\lambda) P^{\rho^+, \rho^-}_{q,t,u}(\lambda) = \frac{\text{Tr}_F (u^D \Gamma(\rho^+)_+ \mathcal{O}(f) \Gamma(\rho^-)_-)}{\text{Tr}_F (u^D \Gamma(\rho^+)_+ \Gamma(\rho^-)_-)}.$$

**7.1. First series of observables.** For $r \in \mathbb{N}$, we consider a random variable $\mathcal{E}_r : \mathcal{Y} \to \mathbb{F}$ defined by $\mathcal{E}_r(\lambda) := e_r(q^\lambda t^{-\rho^+})$, $\lambda \in \mathcal{Y}$. Comparing the values of this random variable and the eigenvalues of the $r$-th Macdonald operator, we can conclude that $\mathcal{O}(\mathcal{E}_r) = t^r \hat{\mathcal{E}}_r$. 

Theorem 7.1. For $r \in \mathbb{N}$, the expectation value of $E_r$ under a periodic Macdonald measure $F_{q,t,u}^{\rho^+, \rho^-}$ is computed as

$$E_{q,t,u}^{\rho^+, \rho^-}[E_r] = \frac{1}{r!} \int \left( \prod_{i=1}^{r} \frac{dz_i}{2\pi \sqrt{-1}} \right) \det \left( K_{q,t;u; \xi}(z_i, z_j) \right)_{1 \leq i,j \leq r} \Delta_{q,t;u}(z),$$

where

$$K_{q,t;u; \xi}(z, w) = \frac{1}{z - t^{-1} w} \prod_{n>0} \exp \left( \frac{1 - t^{-n}}{1 - u^n} p_n(\rho^+) \frac{z^n}{n} - \frac{1 - t^n}{1 - u^n} p_n(\rho^-) z^{-n} \right).$$

Proof. From Proposition 2.3 it follows that

$$O(\varepsilon_r) = \frac{1}{r!} \int \left( \prod_{i=1}^{r} \frac{dz_i}{2\pi \sqrt{-1}} \right) \det \left( \frac{1}{z_i - t^{-1} z_j} \right)_{1 \leq i,j \leq r} :\eta(z_1) \cdots \eta(z_r):.$$ 

To make the presentation simpler, we write

$$:\eta(z_1) \cdots \eta(z_r): = \eta(z) - \eta(z)_+, \ \eta(z)_+ = \exp \left( \pm \sum_{n>0} \frac{1 - t^{\pm n}}{n} a_{\pm n} \sum_{i=1}^{r} z_i^{\pm n} \right).$$

By a standard computation, we have

$$\Gamma(\rho^+) \eta(z)_- \Gamma(\rho^-)_- = \prod_{r=0}^{\infty} \exp \left( \sum_{n>0} \frac{1 - t^{-n}}{n} p_n(\rho^+) z^n \right) \exp \left( - \sum_{n>0} \frac{1 - t^n}{n} p_n(\rho^-) z^{-n} \right) \eta(z)_- \Gamma(\rho^-)_- \Gamma(\rho^+) \eta(z)_+.$$

The normally ordered operators in the right hand side form a vertex operator:

$$V(\gamma) = \eta(z)_- \Gamma(\rho^-)_- \Gamma(\rho^+) \eta(z)_+,$$

where

$$\gamma_n = - \frac{1 - t^n}{n} \sum_{i=1}^{r} z_i^{-n} + \frac{1 - t^n}{n} p_n(\rho^+) \frac{z^n}{n},$$

$$\gamma^-_n = - \frac{1 - t^{-n}}{n} \sum_{i=1}^{r} z_i^{n} + \frac{1 - t^n}{n} p_n(\rho^-) \frac{z^{-n}}{n}$$

for $n > 0$. Applying the trace formula for a vertex operator in Proposition 2.3 we see

$$\text{Tr}_\gamma \left( u^D \Gamma(\rho^+) \eta(z)_- \eta(z)_+ \Gamma(\rho^-)_- \right) = \prod_{i,j=1}^{r} \frac{u z_i / z_j; u}{(u z_i / z_j; u)_{\infty}} \frac{(q t^{-1} u z_i / z_j; u)_{\infty}}{(q z_i / z_j; u)_{\infty}} \frac{(t^{-1} u z_i / z_j; u)_{\infty}}{(t^{-1} u z_i / z_j; u)_{\infty}} \exp \left( \sum_{n>0} \frac{1 - t^{-n}}{n} p_n(\rho^+) \frac{z^n}{n} \right) \exp \left( - \sum_{n>0} \frac{1 - t^n}{n} p_n(\rho^-) \frac{z^{-n}}{n} \right).$$

Obviously, the partition function $\Pi_{q,t;u}(\rho^+, \rho^-)$ is divided by the denominator in Eq. (7.2) and the product in the second line is just $\Delta_{q,t;u}(z)$. The quantity in the third line is understood as a multiplication by a diagonal matrix to the matrix in the determinant. Therefore, the desired result follows. \qed
Note that, in the formula (7.2), the measure part \( \Delta_{q,t^{-1};u}(z) \) is universal in the sense that it is independent of specializations \( \rho^+, \rho^- \) and reduces to unity at \( u \to 0 \) so that the expectation value is written as integral of a single determinant. Therefore, at \( u = 0 \), the expectation value of the generating function of \( \mathcal{E}_r, r \in \mathbb{N} \) admits an expression as a Fredholm determinant. If \( u > 0 \), however, the integrand is modified by the measure \( \Delta_{q,t^{-1};u}(z) \).

Let us see a particular case where the specializations \( \rho^+ \) and \( \rho^- \) are both a Plancherel specialization \( \rho_{\gamma(1-u)} \) for \( \gamma > 0 \). Then the kernel of the determinant becomes

\[
K_{q,t;u;\mathcal{E}}^{\rho_{\gamma(1-u)} \rho_{\gamma(1-u)}}(z, w) = \frac{e^{\gamma(1-t^{-1})z - \gamma(1-t)z^{-1}}}{z - t^{-1}w}.
\]

**Example 7.2.** Let us compute the specific case \( r = 1 \). In this case,

\[
\mathbb{E}_{q,t;u}^{\rho_{\gamma(1-u)} \rho_{\gamma(1-u)}}[\mathcal{E}_1] = \frac{1}{1 - t^{-1}} \int \frac{dz}{2\pi \sqrt{-1}z} e^{\gamma(1-t^{-1})z - \gamma(1-t)z^{-1}}
\]

\[
= \frac{1}{1 - t^{-1}} (u; u)_\infty (qt^{-1}u; u)_\infty \sum_{n=0}^{\infty} \frac{(\gamma^2 (t^{-1}-1) (1-t))^n}{(n!)^2}.
\]

Recall that the 0-th order modified Bessel function of the first kind is defined by

\[
I_0(z) = \sum_{n=0}^{\infty} \frac{(z^2/4)^n}{(n!)^2}.
\]

Then the desired expectation values is expressed as

\[
\mathbb{E}_{q,t;u}^{\rho_{\gamma(1-u)} \rho_{\gamma(1-u)}}[\mathcal{E}_1] = \frac{I_0(2\gamma \sqrt{(1-t)(t^{-1}-1)})}{1 - t^{-1}} \frac{(u; u)_\infty (qt^{-1}u; u)_\infty}{(qu; u)_\infty (t^{-1}u; u)_\infty}.
\]

Let us further take the Hall–Littlewood limit \( q \to 0 \). Under this limit, we have

\[
\mathcal{E}_1(\lambda) = \sum_{i \geq 1} q^{\lambda_i} t^{-i+1} \to \sum_{i = \ell(\lambda) + 1}^{\infty} t^{-i+1} = \frac{t^{-\lambda^\prime}}{1 - t^{-1}}.
\]

Therefore, we can see that

\[
\mathbb{E}_{0,t;u}^{\rho_{\gamma(1-u)} \rho_{\gamma(1-u)}}\left[ t^{-\lambda^\prime} \right] = I_0(2\gamma \sqrt{(1-t)(t^{-1}-1)}) \frac{(u; u)_\infty}{(t^{-1}u; u)_\infty}.
\]

**7.2. Second series of observables.** The next series of observables is obtained from the first one by inverting parameters \( q \) and \( t \). For \( r \in \mathbb{N} \), we consider a random variable \( \mathcal{E}'_r : \mathbb{Y} \to \mathbb{F} \) defined by \( \mathcal{E}'_r(\lambda) = e_r(q^{-\lambda} t^{\rho_{\gamma(1-u)}}, \lambda \in \mathbb{Y} \). Then the corresponding operator is \( \mathcal{O}(\mathcal{E}'_r) = t^{-r} \mathcal{E}'_r \).

**Theorem 7.3.** For \( r \in \mathbb{N} \), the expectation value of \( \mathcal{E}'_r \) under a periodic Macdonald measure \( \mathbb{E}_{q,t;u}^{\rho^+, \rho^-} \) is computed as

\[
\mathbb{E}_{q,t;u}^{\rho^+, \rho^-}[\mathcal{E}'_r] = \frac{1}{r!} \int \left( \prod_{i=1}^{r} \frac{dz_i}{2\pi \sqrt{-1}} \right) \det \left( K_{q,t;u;\mathcal{E}'}^{\rho^+, \rho^-}(z_i, z_j) \right) \Delta_{q^{-1}, t; u}(z),
\]
where

\[
K_{q,t;w;E'}^{\rho^+,\rho^-}(z, w) = \frac{1}{z - tw} \prod_{n>0} \exp \left( - \frac{1 - t^{-n} (t/q)^{n/2} p_n(\rho^+) z^n}{1 - w^n} \right)
\]

+ \frac{1 - t^n (t/q)^{n/2} p_n(\rho^-) z^{-n}}{1 - w^n}.

Proof. We write \(\xi(z_1) \cdots \xi(z_r) = \xi(z) \cdot \xi(z)^+_r\), where

\[
\xi(z) = \exp \left( \sum_{n>0} \frac{1 - t^{-n}}{n} (t/q)^{n/2} a_n \sum_{i=1}^r z_i^n \right).
\]

Then, by a standard computation, we have

\[
\Gamma(\rho^+)_+ \xi(z) \cdot \Gamma(\rho^-)_- = \tilde{P}_{q,t;0}(\rho^+; \rho^-) \exp \left( - \sum_{n>0} \frac{1 - t^{-n}}{n} (t/q)^{n/2} p_n(\rho^+) \sum_{i=1}^r z_i^n \right)
\]

\[
\times \exp \left( \sum_{n>0} \frac{1 - t^n}{n} (t/q)^{n/2} p_n(\rho^-) \sum_{i=1}^r z_i^{-n} \right) \times \xi(z) \cdot \Gamma(\rho^-)_- \Gamma(\rho^+_r) \cdot \xi(z)_+. \]

We can see that the operator in the right hand side is

\[
V(\gamma) = \xi(z) \cdot \Gamma(\rho^-)_- \Gamma(\rho^+_r) \cdot \xi(z)_+ _r,
\]

where

\[
\gamma_n = \frac{1 - t^n}{n} (t/q)^{n/2} \sum_{i=1}^r z_i^{-n} + \frac{1 - t^n p_n(\rho^+)}{1 - q^n},
\]

\[
\gamma_n = - \frac{1 - t^{-n}}{n} (t/q)^{n/2} \sum_{i=1}^r z_i^n + \frac{1 - t^n p_n(\rho^-)}{1 - q^n}
\]

for \(n > 0\). The trace formula in Proposition 23 gives the desired result. \(\square\)

If we take the Plancherel specialization \(\rho^\pm = \rho_{\gamma(1-u)}\) with \(\gamma > 0\), the kernel function becomes

\[
K_{q,t;w;E'}^\rho(z, w) = \frac{e^{\gamma (t/q)^{1/2} (t^{-1} - 1) z + (1-t) z^{-1}}}{z - tw}.
\]

Example 7.4. In case of \(r = 1\), we have

\[
I_0^\rho(1-u)^{\rho(1-u)} [E] = \frac{I_0(2\gamma(1-t) \sqrt{q^{-1}} (u; u)_\infty (tq^{-1}u; u)_\infty)}{(tu; u)_\infty (q^{-1}u; u)_\infty}.
\]

In the \(q\)-Whittaker limit, \(t \to 0\), we have

\[
E'(\lambda) = \sum_{i \geq 1} q^{-\lambda_i} t^{i-1} \to q^{-\lambda_1}.
\]

Therefore,

\[
I_0^\rho(1-u)^{\rho(1-u)} [q^{-\lambda}] = I_0(2\gamma \sqrt{q^{-1}}) \frac{(u; u)_\infty}{(q^{-1}u; u)_\infty}.
\]
7.3. Third series of observables. For $r \in \mathbb{N}$, we consider $S_r : \mathbb{Y} \to \mathbb{F}$ defined by $S_r(\lambda) = g_r(q^{\lambda}t^{-\rho}; q, t)$. Following the argument in Subsect. 2.3, we see that the corresponding operator is
\[ \mathcal{O}(S_r) = \hat{G}_r = \frac{(-1)^r}{r!} \int \left( \prod_{i=1}^{r} \frac{dz_i}{2\pi \sqrt{-1}} \right) \det \left( \frac{1}{z_i - q^\rho j} \right)_{1 \leq i, j \leq r} \eta(z_1) \cdots \eta(z_r), \]
which is almost the same as $\mathcal{O}(E_r)$ except for the determinant part and the overall sign. Therefore, the following theorem immediately follows.

**Theorem 7.5.** For $r \in \mathbb{N}$, the expectation value of $S_r$ under a periodic Macdonald measure $\mathbb{P}^{\rho, \rho^{-}}_{q, t; u}$ is
\[ \mathbb{E}^{\rho, \rho^{-}}_{q, t; u} [S_r] = \frac{(-1)^r}{r!} \int \left( \prod_{i=1}^{r} \frac{dz_i}{2\pi \sqrt{-1}} \right) \det \left( K^{\rho, \rho^{-}}_{q, t; u}(z_i, z_j) \right)_{1 \leq i, j \leq r} \Delta_{q, t^{-1}; u}(z), \]
where
\[ K^{\rho, \rho^{-}}_{q, t; u}(z, w) = \frac{1}{z - qw} \prod_{n>0} \exp \left( \frac{1-t^{-n} p_n(\rho^+)}{1-u^n} z^n - \frac{1-t^n p_n(\rho^-)}{1-u^n} z^{-n} \right) \]

7.4. Fourth series of observables. The final series of observable we consider is $S'_r : \mathbb{Y} \to \mathbb{F}$, $r \in \mathbb{N}$ defined by $S'_r(\lambda) = g_r(q^{-\lambda}t^\rho; q^{-1}, t^{-1}) = g_r(q^{-\lambda+1}t^{-\rho-1}; q, t)$, $\lambda \in \mathbb{Y}$. The corresponding operator is
\[ \mathcal{O}(S'_r) = \hat{G}'_r = \frac{(-1)^r}{r!} \int \left( \prod_{i=1}^{r} \frac{dz_i}{2\pi \sqrt{-1}} \right) \det \left( \frac{1}{z_i - q^{-1} j} \right)_{1 \leq i, j \leq r} \xi(z_1) \cdots \xi(z_r), \]
which is almost the same as $\mathcal{O}(E'_r)$. Therefore, we have the following.

**Theorem 7.6.** For $r \in \mathbb{N}$, the expectation value of $S'_r$ under a periodic Macdonald measure $\mathbb{P}^{\rho, \rho^{-}}_{q, t; u}$ is
\[ \mathbb{E}^{\rho, \rho^{-}}_{q, t; u} [S'_r] = \frac{(-1)^r}{r!} \int \left( \prod_{i=1}^{r} \frac{dz_i}{2\pi \sqrt{-1}} \right) \det \left( K^{\rho, \rho^{-}}_{q, t; u}(z_i, z_j) \right)_{1 \leq i, j \leq r} \Delta_{q^{-1}, t; u}(z), \]
where
\[ K^{\rho, \rho^{-}}_{q, t; u}(z, w) = \frac{1}{z - q^{-1} w} \prod_{n>0} \exp \left( -\frac{1-t^{-n} (t/q)^{n/2} p_n(\rho^+)}{1-u^n} z^n + \frac{1-t^n (t/q)^{n/2} p_n(\rho^-)}{1-u^n} z^{-n} \right). \]

**Appendix A. More on the Macdonald refined topological vertex**

To write another formula regarding the trace of a Macdonald refined topological vertex, we denote the set of nonnegative signatures as
\[ \text{Sign}^+ = \bigsqcup_{N \geq 0} \text{Sign}^+_N, \quad \text{Sign}^+_N = \{ \lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{Z}^N : \lambda_1 \geq \cdots \geq \lambda_N \geq 0 \}. \]
We set $\ell(\lambda) = N$ if $\lambda \in \text{Sign}^+_N$. Note that, in contrast to a partition, $\lambda_{\ell(\lambda)}$ can be zero. As in the case for a partition, we also write $m_r(\lambda) = \# \{ i : \lambda_i = r \}$ and $|\lambda| = \sum_i \lambda_i$. 
Theorem A.1. Let \( \hat{q} \) and \( \hat{t} \) be parameters possibly different from other ones \( q,t,x,y \).
We have the following identity:

\[
\sum_{\lambda \in \mathcal{Y}} u^{|\lambda|} \mathcal{V}_{\lambda \mathbb{0}}(x, y; q, t) \frac{\mathcal{V}_{\square \mathbb{0} \lambda}(q, t; \hat{q}, \hat{t})}{\mathcal{V}_{\mathbb{0} \lambda}(q, t; \hat{q}, \hat{t})} = \frac{1}{1 - t} \left( \frac{u; u \infty (tq^{-1}u; u \infty)}{(x; q, u, x, y \infty)} (x; q, u, x, y \infty) \right) \times \sum_{N=0}^{\infty} t^{-N} \sum_{\lambda, \mu \in \text{Sign}_N} \prod_{r \geq 0} \frac{(t; u)_{m_{\lambda}(r)} (t; u)_{m_{\mu}(r)} (x; q, t; \hat{q}, \hat{t})}{(u; u)_{m_{\lambda}(r)} (u; u)_{m_{\mu}(r)}} |x; q, t; \hat{q}, \hat{t}|. 
\]

Proof. First, notice that

\[
\mathcal{V}_{\square \mathbb{0} \lambda}(q, t; \hat{q}, \hat{t}) = \mathcal{P}_{\square}(q^{-1} \lambda^{\rho-1}; \hat{q}, \hat{t}) = e_1(q^{-1} \lambda^{\rho-1}) = E_1(\lambda').
\]

Therefore, it follows that

\[
\sum_{\lambda \in \mathcal{Y}} u^{|\lambda|} \mathcal{V}_{\lambda \mathbb{0}}(x, y; q, t) \frac{\mathcal{V}_{\square \mathbb{0} \lambda}(q, t; \hat{q}, \hat{t})}{\mathcal{V}_{\mathbb{0} \lambda}(q, t; \hat{q}, \hat{t})} = \text{Tr}_{\mathbb{Y}} \left( u^D \Gamma(x; q, t; \hat{q}, \hat{t}) \mathcal{V}_{\square \mathbb{0} \lambda}(q, t; \hat{q}, \hat{t}) \right),
\]

which has been already computed in Subsection 7.2. Thus we have

\[
\sum_{\lambda \in \mathcal{Y}} u^{|\lambda|} \mathcal{V}_{\lambda \mathbb{0}}(x, y; q, t) \frac{\mathcal{V}_{\square \mathbb{0} \lambda}(q, t; \hat{q}, \hat{t})}{\mathcal{V}_{\mathbb{0} \lambda}(q, t; \hat{q}, \hat{t})} = \frac{1}{1 - t} \int \frac{dz}{2\pi \sqrt{-1} z} \prod_{q, t, u} (x^p, y^{p-1}) \left( \frac{u; u \infty (tq^{-1}u; u \infty)}{(x; q, u, x, y \infty)} (x; q, u, x, y \infty) \right) \times \exp \left( - \sum_{n \geq 1} \frac{1 - t^{-n}}{1 - u^n} \frac{(t/q)^{n/2} p_n(x^p)}{n} z^n \right) \times \exp \left( \sum_{n \geq 1} \frac{1 - t^{-n}}{1 - u^n} \frac{(t/q)^{n/2} p_n(y^{p-1})}{n} z^n \right)
\]

\[
= \frac{1}{1 - t} \left( \frac{u; u \infty (tq^{-1}u; u \infty)}{(x; q, u, x, y \infty)} (x; q, u, x, y \infty) \right) \times \frac{dz}{2\pi \sqrt{-1} z} \left( \frac{(t/q)^{1/2} x; u, x \infty}{(t/q)^{1/2} x; u, x \infty} \frac{(t/q)^{1/2} t^{-1} z; u, y \infty}{(t/q)^{1/2} t^{-1} z; u, y \infty} \right)
\]

Now we notice the following lemma:

Lemma A.2. We have

\[
\frac{(az; x, y \infty)}{(z; x, y \infty)} = \sum_{\lambda \in \text{Sign}_+} \prod_{r \geq 0} (a; x)_{m_{\lambda}(r)} y^{|\lambda|} z^\ell(\lambda).
\]

Proof. It follows from a direct computation. Using the \( q \)-binomial theorem, we have

\[
\frac{(az; x, y \infty)}{(z; x, y \infty)} = \prod_{r=0}^{\infty} \frac{(y^r a z; x \infty)}{(y^r z; x \infty)} = \prod_{r=0}^{\infty} \sum_{n=0}^{\infty} (a; x)_n y^n z^n.
\]

Commuting the sum and the product, we see a sum over nonnegative signatures. \( \square \)
This lemma allows us to evaluate the constant term to obtain the desired result. □

**APPENDIX B. SHIFT-MIXED MEASURES AND THE SCHUR-LIMIT**

In this appendix, we consider the shift-mixed version of a periodic Macdonald measure, which is a probability measure on \( \mathbb{Y} \times \mathbb{Z} \) extended from a periodic Macdonald measure. In the Schur case [Bor07, BB19], it is known that several determinantal formulas are only available after extending a periodic Schur measure to the shift-mixed one. For a periodic Macdonald measure, on the other hand, we will see that the expectation values of natural generalization of the observables introduced in the previous section do not admit fully determinantal expression even when we consider the shift-mixed version.

**B.1. Definition.** Let \( q, t, u \in (0, 1) \). To define a shift-mixed periodic Macdonald measure, we also take an additional parameter \( \zeta \in (0, 1) \).

**Definition B.1.** Let \( \rho^+, \rho^- : \Lambda \to \mathbb{R} \) be Macdonald positive specialization. Then the corresponding shift-mixed periodic Macdonald measure is a probability measure \( \mathbb{P}_{q,t,u,\zeta}^{\rho^+,-\rho^-} \) on \( \mathbb{Y} \times \mathbb{Z} \) defined by

\[
\mathbb{P}_{q,t,u,\zeta}^{\rho^+,\rho^-}(\lambda, n) \propto u^{n^2/2} \zeta^n \sum_{\mu \in \mathbb{Y}} u^{[\lambda]} P_{\lambda/\mu}(\rho^+) Q_{\lambda/\mu}(\rho^-).
\]

The partition function is immediately given by

\[
\Pi_{q,t,u,\zeta}(\rho^+; \rho^-) := \sum_{n \in \mathbb{Z}} u^{n^2/2} \zeta^n \sum_{\lambda, \mu \in \mathbb{Y}} u^{[\lambda]} P_{\lambda/\mu}(\rho^+) Q_{\lambda/\mu}(\rho^-) = \vartheta_3(\zeta; u) \Pi_{q,t,u}(\rho^+; \rho^-),
\]

where \( \vartheta_3(\zeta; u) = \sum_{n \in \mathbb{Z}} u^{n^2/2} \zeta^n \) is a Jacobi theta function.

It is immediate that a shift-mixed periodic Macdonald measure admits an interpretation in terms of Fock spaces. We set \( \Xi = \mathcal{F} \otimes \mathbb{C}[e^{\pm \alpha}] \) and \( \Xi^\dagger = \mathcal{F} \otimes \mathbb{C}[e^{\pm \alpha}] \). We often write \( |v \otimes e^{\alpha}| : = |v\rangle \otimes e^{\alpha} \in \Xi, |v\rangle \in \mathcal{F}, n \in \mathbb{Z} \) and \( \langle v \otimes e^{\alpha}| = \langle v| \otimes e^{\alpha} \in \Xi^\dagger, \langle v| \in \mathcal{F}^\dagger, n \in \mathbb{Z} \). A paring \( \langle \mathcal{F}^\dagger \times \mathcal{F} \to \mathbb{C} \) is naturally defined by \( \langle v \otimes e^{\alpha}|w \otimes e^{\alpha}| = \langle v|w\rangle \delta_{m,n}, (v) \in \mathcal{F}^\dagger, |w\rangle \in \mathcal{F}, m, n \in \mathbb{Z} \). We define the charge operator \( a_0 \in \text{End}(\Xi) \) by \( a_0 |v \otimes e^{\alpha}| = n |v \otimes e^{\alpha}| \), \( |v\rangle \in \mathcal{F}, n \in \mathbb{Z} \) and the energy operator \( H = D + a_0^2/2 \). Then the weight of a shift-mixed periodic Macdonald measure is expressed as

\[
\mathbb{P}_{q,t,u,\zeta}^{\rho^+,\rho^-}(\lambda, n) = \frac{\text{Tr}_{\Xi} (u^H \zeta^{a_0} \Gamma(\rho^+) + |P_{\lambda} \otimes e^{\alpha}| \langle Q_{\lambda} \otimes e^{\alpha}| \Gamma(\rho^-) - )}{\text{Tr}_{\Xi} (u^H \zeta^{a_0} \Gamma(\rho^+) + \Gamma(\rho^-) - )}.
\]

**B.2. Observables.** We only consider an analogue of the first series of observables. For \( r \in \mathbb{N} \), we write \( \tilde{E}_r : \mathbb{Y} \times \mathbb{Z} \to \mathbb{R} \) for a random variable defined by \( \tilde{E}_r(\lambda, n) = t^{-rn} \tilde{E}_r(\lambda) \). Obviously, it is just the spectrum of an operator

\[
\tilde{E}_r = \frac{1}{r} \int \left( \prod_{i=1}^{r} \frac{dz_i}{2\pi \sqrt{-1}} \right) \det \left( \frac{1}{z_i - t^{-1} z_j} \right)_{1 \leq i, j \leq r} t^{-ra_0} : \eta(z_1) \cdots \eta(z_r) : \in \text{End}(\Xi),
\]
which we call the $r$-th extended Macdonald operator. Therefore, the expectation value is computed as

$$
\mathbb{E}_{q,t;u,z}^{\rho_+\rho_-} [\tilde{E}_r] = \sum_{\lambda \in \mathbb{Y}, n \in \mathbb{Z}} \tilde{E}_r(\lambda, n) \mathbb{E}_{q,t;u,z}^{\rho_+\rho_-}(\lambda, n) = \frac{\text{Tr}_\Xi \left( u^H \zeta \Gamma(\rho^+) + \tilde{E}_r \Gamma(\rho^-) \right)}{\text{Tr}_\Xi \left( u^H \zeta \Gamma(\rho^+) + \Gamma(\rho^-) \right)}.
$$

**Theorem B.2.** For each $r \in \mathbb{N}$, we have

$$
\mathbb{E}_{q,t;u,z}^{\rho_+\rho_-} [\tilde{E}_r] = \frac{1}{r!} \int \left( \prod_{i=1}^{r} \frac{dz_i}{2\pi\sqrt{-1}} \right) \det \left( \tilde{K}_{q,t;u,z}^{\rho_+\rho_-} (z_i, z_j) \right) \left( \frac{\theta_3(z; \zeta u)}{\theta_3(z; \zeta)} \right)^{\Delta q,t;u(z)}.
$$

**Proof.** It follows straightforwardly that

$$
\text{Tr}_\Xi \left( u^H \zeta \Gamma(\rho^+) + \tilde{E}_r \Gamma(\rho^-) \right) = \theta_3(\zeta^{-r}; u) \text{Tr}_\mathcal{F} \left( u^D \Gamma(\rho^+) + \tilde{E}_r \Gamma(\rho^-) \right),
$$

while the trace over the Fock space $\mathcal{F}$ has been computed in the proof of Theorem 7.1. □

**B.3. The Schur-limit.** We fix $t$ and consider the Schur-limit $q \to t$. Let us introduce charged free fermion:

$$
\psi(z) = e^{-\alpha z - \alpha_0} \exp \left( \sum_{n > 0} \frac{a_n}{n} z^n \right) \exp \left( - \sum_{n > 0} \frac{a_n}{n} z^{-n} \right),
$$

$$
\psi^*(z) = e^{\alpha z + \alpha_0} \exp \left( - \sum_{n > 0} \frac{a_n}{n} z^n \right) \exp \left( \sum_{n > 0} \frac{a_n}{n} z^{-n} \right).
$$

According to the boson-fermion correspondence (see e.g. [KRR13, Lecture 5]), the space $\Xi$ is isomorphic to a fermion Fock space with respect to defined fermion fields at the Schur-limit $q \to t$. For $r \in \mathbb{N}$, set

$$
\mathcal{E}_r(t) = \frac{1}{r!} \int \left( \prod_{i=1}^{r} \frac{dz_i}{2\pi\sqrt{-1}} \right) \mathcal{E}_r(z, t), \quad \mathcal{E}_r(z, t) = \psi(z_1) \cdots \psi(z_r) \psi^*(t^{-1} z_r) \cdots \psi^*(t^{-1} z_1).
$$

**Proposition B.3.** For each $r \in \mathbb{N}$, the extended Macdonald operator $\tilde{E}_r$ reduces, at the Schur-limit $q \to t$, to $\mathcal{E}_r(t)$.

**Proof.** Owing to the fermionic Wick formula or the Cauchy determinant formula, we see that

$$
\mathcal{E}_r(z, t) = \det \left( \frac{1}{z_i - t^{-1} z_j} \right)_{1 \leq i, j \leq r} t^{-\alpha_0} \cdot \eta(z_1) \cdots \eta(z_r),
$$

which implies the desired result. □

Note that the operator $\mathcal{E}_1(t)$ is essentially the same as the one introduced in [OP06]. Due to Proposition B.3, we can say that the Macdonald operators are deformation of the operators $\mathcal{E}_r(t)$, $r \in \mathbb{N}$, each of which is realized by means of charged free fermion. More precisely, we apply deformation of the Heisenberg algebra only after rearranging the operator $\mathcal{E}_r(t)$ in normally ordered manner with respect to the bosonic modes to obtain the corresponding Macdonald operator. Following this recognition about the Macdonald operators, we can understand the determinant found in the free field realization of the Macdonald operators, which is also the origin of the determinantal structure of
Macdonald processes, as remnant from when they were written in terms of free fermions before the deformation.

We can see that the boson-fermion correspondence implies the following identity.

**Proposition B.4.** Let $x_1, \ldots, x_r, y_1, \ldots, y_r$ be indeterminates and let $u, \zeta \in (0, 1)$ be parameters. We have

$$\frac{\prod_{i<j}(x_i - x_j) \prod_{i>j}(y_i - y_j)}{\prod_{i,j=1}(x_i - x_j)} \frac{\theta_3 \left( \frac{y_1 \cdots y_i}{x_1 \cdots x_r}; u \right)}{\theta_3(\zeta; u)} = \frac{\prod_{i,j=1}^{r}(u x_i/y_j; u)_{\infty}(u y_i/x_j; u)_{\infty}}{\prod_{i,j=1}^{r}(u y_i/x_j; u)_{\infty}(u y_i/x_j; u)_{\infty}} \quad 1 \leq i, j \leq r.$$  

**Proof.** We compute the correlation function

$$\langle \psi(x_1) \cdots \psi(x_r) \psi^{*}(y_r) \cdots \psi^{*}(y_1) \rangle_{u, \zeta} := \frac{\text{Tr}_{\Xi} \left( u^{H(\zeta)} \psi(x_1) \cdots \psi(x_r) \psi^{*}(y_r) \cdots \psi^{*}(y_1) \right)}{\text{Tr}_{\Xi} (u^{H(\zeta)} a^{0})}$$

in two different ways, namely, in the bosonic and fermionic methods. On the bosonic side, we have

$$\psi(x_1) \cdots \psi(x_r) \psi^{*}(y_r) \cdots \psi^{*}(y_1) = \prod_{i<j}(x_i - x_j) \prod_{i>j}(y_i - y_j) \prod_{i,j=1}^{r}(u x_i/y_j; u)_{\infty}(u y_i/x_j; u)_{\infty}$$

where

$$\gamma_n = \frac{n}{n!} \sum_{i=1}^{r} (x_i^n - y_i^n), \quad n \in \mathbb{Z}\{0\}.$$  

The trace over $\Xi$ is evaluated by decomposing it into the sum over charges and the trace over $\mathcal{F}$ that can be computed by applying Proposition [2.3]. Consequently, we have

$$\langle \psi(x_1) \cdots \psi(x_r) \psi^{*}(y_r) \cdots \psi^{*}(y_1) \rangle_{u, \zeta} = \prod_{i<j}(x_i - x_j) \prod_{i>j}(y_i - y_j) \frac{\theta_3 \left( \frac{y_1 \cdots y_i}{x_1 \cdots x_r}; u \right)}{\theta_3(\zeta; u)} \prod_{i,j=1}^{r}(u x_i/y_j; u)_{\infty}(u y_i/x_j; u)_{\infty}.$$  

On the other hand, the fermionic Wick formula (see [BB19, Appendix B]) allows us to have

$$\langle \psi(x_1) \cdots \psi(x_r) \psi^{*}(y_r) \cdots \psi^{*}(y_1) \rangle_{u, \zeta} = \det \left( \langle \psi(x_i) \psi^{*}(y_j) \rangle_{u, \zeta} \right)_{1 \leq i, j \leq r},$$

where the two point function is obviously

$$\langle \psi(x) \psi^{*}(y) \rangle_{u, \zeta} = \frac{1}{x-y} \frac{\theta_3(y/x; u)}{\theta_3(\zeta; u)} \frac{u_{\infty}^{2}}{(u y/x; u)_{\infty}}$$

as a special case of (B.1) at $r = 1$. Comparing (B.1) and (B.2), we can see the desired identity.  

As an application of Proposition B.4, we can check that the formula in Theorem B.2 reduces to integral of a single determinant at the Schur-limit $q \to t$, which is of course expected since the observable $E_r$ reduces to $\delta_r$ written by means of fermions.
Corollary B.5. For each $r \in \mathbb{N}$, we have
\[
E_{\rho^+, \rho^-}^{\tau, \bar{\rho}, \bar{\xi}, \bar{\nu}, \bar{\zeta}}[\tilde{E}_r] = \frac{1}{r!} \int \left( \prod_{i=1}^{r} \frac{dz_i}{2\pi \sqrt{-1}} \right) \det \left( K_{\rho^+, \rho^-}^{\tau, \bar{\rho}, \bar{\xi}, \bar{\nu}, \bar{\zeta}}(z_i, z_j) \right) \bigg|_{1 \leq i, j \leq r},
\]
where
\[
K_{\rho^+, \rho^-}^{\tau, \bar{\rho}, \bar{\xi}, \bar{\nu}, \bar{\zeta}}(z, w) = K_{\tau, \bar{\rho}, \bar{\xi}, \bar{\nu}, \bar{\zeta}}^{\rho^+, \rho^-}(z, w) \frac{\theta_3(\zeta t^{-1} w/z; u)}{\theta_3(\zeta^{-1} u; \bar{u})} \frac{(u; u)^2_{\infty}}{(tuz/w; u)_{\infty} (t^{-1}uw/z; u)_{\infty}^{2}}.
\]

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