Federated Functional Gradient Boosting

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Abstract

In this paper, we initiate a study of functional minimization in Federated Learning. First, in the semi-heterogeneous setting, when the marginal distributions of the feature vectors on client machines are identical, we develop the federated functional gradient boosting (FFGB) method that provably converges to the global minimum. Subsequently, we extend our results to the fully-heterogeneous setting (where marginal distributions of feature vectors may differ) by designing an efficient variant of FFGB called FFGB.C, with provable convergence to a neighborhood of the global minimum within a radius that depends on the total variation distances between the client feature distributions. For the special case of square loss, but still in the fully heterogeneous setting, we design the FFGB.L method that also enjoys provable convergence to a neighborhood of the global minimum but within a radius depending on the much tighter Wasserstein-1 distances. For both FFGB.C and FFGB.L, the radii of convergence shrink to zero as the feature distributions become more homogeneous. Finally, we conduct proof-of-concept experiments to demonstrate the benefits of our approach against natural baselines.

1 Introduction

Federated learning (FL) is a machine learning paradigm in which multiple clients cooperate to learn a model under the orchestration of a central server [McMahan et al., 2017]. In FL, clients tend to have very heterogeneous data, and this data is never sent to the central server due to privacy and communication constraints. Thus, it is necessary to offload as much computation as possible to the client devices. The challenge in FL is to train a single unified model via decentralized computation even though the clients have heterogeneous data.

Current research in FL mainly focuses on training parametric models where the variable to be optimized is typically a vector containing the parameters of a neural network. While convenient, the parametric model limits the feasible domain of the output function. Expert domain knowledge must be incorporated when designing the structure of the parametric model to ensure that a sufficiently good solution is feasible.

In this paper, we initiate a study of non-parametric functional minimization methods for FL as a means of overcoming the aforementioned limitations of parametric models. To the best of our knowledge, this is the first work to study the use of functional minimization methods in the context of FL. In functional minimization, the variable to be trained is the target function itself. The feasible domain of such a non-parametric learning paradigm can be as large as the family of the square-integrable functions, which allows tremendous flexibility in representing the desired models.

To solve the functional minimization problem, a powerful tool is the restricted functional gradient descent method (RFGD), also known as functional gradient boosting. Methods in this class select and aggregate simple functions (or the so-called weak learners) in an iterative manner and have been proved to perform well in terms of minimizing the training loss as well as generalization.
Contributions. This paper formulates the federated functional minimization problem and develops algorithms and convergence analyses for various settings of heterogeneity of clients. The main contributions are as follows:

1. Algorithms for the semi-heterogeneous setting. We first consider the semi-heterogeneous setting where the marginal distributions of the feature vectors on the clients are identical and heterogeneity arises due to differing conditional distributions of labels. In this setting, we propose the federated functional gradient boosting (FFGB) method. The algorithm relies on the clients running multiple RFGD steps locally, but augmented with residual variables that are crucial for proving global convergence of the algorithm.

2. Benefit of local steps in the semi-heterogeneous setting. Our analysis of FFGB suggests that the number of local steps of FFGB should be \( \Omega(\sqrt{T}) \) in order to have the best convergence rate where \( T \) is the number of global communication rounds. This is a rare result that shows the benefit of taking multiple local steps in FL.

3. Algorithms for the fully-heterogeneous setting. Next, we consider the fully heterogeneous setting where the entire joint distribution of features and labels varies across clients. In this setting, we propose two different algorithms, FFGB.C and FFGB.L, which are variants of FFGB. Our analysis shows that FFGB.C converges to a neighborhood of the global optimal solution, where the radius of the neighborhood depends on the heterogeneity of the data measured in terms of the total variation (TV) distance between the marginal distributions of the features of the clients. The algorithm FFGB.L operates under the square loss and our analysis shows a similar convergence result to that of FFGB.C, with the difference being that the degree of heterogeneity and radius of the neighborhood is measured in terms of the Wasserstein-1 distance, which can be significantly smaller than the TV distance. Importantly, for both FFGB.C and FFGB.L, the radius of the convergence neighborhood gracefully diminishes to zero when problem degenerates to the semi-heterogeneous case.

4. Experiments. While the main focus of this paper is to develop the theoretical foundations of federated functional minimization, we also performed some basic proof-of-concept experiments to provide evidence for the benefits of the methods developed in this paper on practical datasets. Our experiments show superior performance of our methods compared to natural baselines.

1.1 Related work

For the standard parametric optimization problem, FedAvg [McMahan et al., 2017] is the most popular algorithm in the federated setting. In every round of FedAvg, the server sends a global average parameter to the local machines. The local machines then perform multiple local steps (usually stochastic gradient descent) to update the received parameters. These improved local parameters are then aggregated by the server for the next round. It has been noted by several papers (see, e.g. [Karimireddy et al., 2020b] and the references therein) that FedAvg deteriorates in the presence of client heterogeneity. Federated optimization algorithms such as SCAFFOLD [Karimireddy et al., 2020b], FedProx [Li et al., 2020], MIME [Karimireddy et al., 2020a], etc. were designed to tackle this issue.

Boosting [Schapire and Freund, 2012] is a classical ensemble method for building additive models in a greedy, stagewise manner. In each stage, a new classifier is added to the current ensemble to decrease the training loss. In the regression setting, this is typically done by finding an approximation to the functional gradient of the loss in the base model class via a training procedure (also referred to as a “weak learning oracle”) for the base class (see, e.g., [Mason et al., 2000, Friedman, 2001]). Rigorous analysis with rates of convergence for procedures of this form were given by [Duffy and Helmbold, 2002, Rätsch et al., 2001, Zhang and Yu, 2005, Grubb and Bagnell, 2011]. The algorithms in this paper and their analysis build upon the prior work of [Grubb and Bagnell, 2011], with the main challenges coming from the restriction in the FL setting that no client data be shared with the server, as well as the heterogeneity in client data.

In the FL setting, stagewise boosting algorithms as described above have not been studied, to the best of our knowledge. An ensemble method in the FL setting called FedBOOST was developed by [Hamer et al., 2020].
to compute a convex combination of pre-trained classifiers to minimize the training loss. The algorithm is quite different from standard classical boosting methods since it doesn’t operate in a greedy stagewise manner, nor does it incorporate training of the base classifiers. Since this setting is quite different from the one considered in this paper, we do not provide further comparison with this method.

2 Preliminaries

In this section, we define the necessary notation as well as the functional minimization problem.

2.1 Notation.

For a positive integer $n$, we define $[n] := \{1, 2, \ldots, n\}$. For a vector $x \in \mathbb{R}^d$, we use $x_i$ to denote its $i^{th}$ entry. We use $\langle \cdot, \cdot \rangle$ to denote the standard Euclidean inner product and $\| \cdot \|$ for the corresponding standard Euclidean norm.

For some ground set $X \subseteq \mathbb{R}^d$, we use $\mathcal{M}_+^1(X)$ to denote the set of probability measures on $X$. For two measures $\alpha, \beta \in \mathcal{M}_+^1(X)$, we use $TV(\alpha, \beta)$ to denote their total variation distance and use $W_p(\alpha, \beta)$ to denote their Wasserstein-$p$ distance. The definition of these two distances are provided in Appendix A. Let $X \subseteq \mathbb{R}^d$ be an input space. For a fixed distribution $\alpha \in \mathcal{M}_+^1(X)$, we define the corresponding weighted $\mathcal{L}^p$ space $(1 \leq p < \infty)$ of functions from $X$ to $\mathbb{R}^c$ as follows:

$$\mathcal{L}^p(\alpha) \overset{\text{def}}{=} \left\{ f : X \to \mathbb{R}^c \mid \left( \int_X \| f(x) \|^p d\alpha(x) \right)^{\frac{1}{p}} < \infty \right\}.$$  

The space $\mathcal{L}^2(\alpha)$ is endowed with natural inner product and norm: For two functions $f, g \in \mathcal{L}^2(\alpha)$,

$$\langle f, g \rangle_\alpha = \int_X \langle f(x), g(x) \rangle d\alpha(x) \quad \text{and} \quad \| f \|_\alpha = \sqrt{\langle f, f \rangle_\alpha}.$$  

In the limiting case when $p \to \infty$, we define

$$\mathcal{L}^\infty(\alpha) \overset{\text{def}}{=} \left\{ f : X \to \mathbb{R}^c \mid \forall x \in \text{supp}(\alpha), \| f(x) \| < \infty \right\},$$  

where $\text{supp}(\alpha)$ denotes the support of $\alpha$. We define the $\alpha$-infinity norm of a function $f : X \to \mathbb{R}^c$ by

$$\| f \|_{\alpha, \infty} \overset{\text{def}}{=} \sup_{x \in \text{supp}(\alpha)} \| f(x) \|.$$  

We will also use

$$\mathcal{L}^\infty \overset{\text{def}}{=} \left\{ f : X \to \mathbb{R}^c \mid \forall x \in X, \| f(x) \| < \infty \right\},$$  

and define $\| f \|_{\infty} \overset{\text{def}}{=} \sup_{x \in X} \| f(x) \|$. We use $\mathcal{B}^\infty(R)$ to denote the $\mathcal{L}^\infty$ ball around 0 with radius $R$. For a Lipschitz continuous function $f : X \to \mathbb{R}^c$, we denote by $\| f \|_{\text{lip}}$ its Lipschitz constant, i.e.

$$\| f \|_{\text{lip}} \overset{\text{def}}{=} \inf \left\{ L : \forall x, x' \in X, \| f(x) - f(x') \| \leq L \| x - x' \| \right\}.$$  

For a function $f : X \to Y$ and $R > 0$, we define the clipping function $\Pi_R$ to be the projection on $\mathcal{B}^\infty(R)$, i.e.

$$\Pi_R(f)(x) \overset{\text{def}}{=} \max \{ \min \{ f(x), R \}, -R \}.$$  

For a given function $f : X \to \mathbb{R}$ and a probability measure $\alpha \in \mathcal{M}_+^1(X)$, their inner product is defined in the standard way: $\langle f, \alpha \rangle = \int_X f(x) d\alpha(x)$.

2.2 Functional Minimization in Weighted $\mathcal{L}^2$ Space

For some output space $Y$, let $\ell : \mathbb{R}^c \times Y \to \mathbb{R}$ be a loss function that is convex in the first argument. An important example of $\ell$ is the cross entropy loss, where $Y = [c]$, and

$$\ell(y', y) = -\log \left( \frac{\exp(y'_y)}{\sum_{i=1}^c \exp(y'_i)} \right). \quad (1)$$
Given a joint distribution $P$ on $\mathcal{X} \times \mathcal{Y}$, we use $\alpha$ to denote the marginal distribution of $P$ on $\mathcal{X}$, and for every $x \in \mathcal{X}$, we let $\beta^x \in \mathcal{M}_1^+(\mathcal{Y})$ be the distribution on $\mathcal{Y}$ under $P$ conditioned on $x$. We then define the risk functional $\mathcal{R} : \mathcal{L}^2(\alpha) \to \mathbb{R}$ as

$$\mathcal{R}[f] \overset{\text{def}}{=} \mathbb{E}_{(x,y) \sim P}[\ell(f(x), y)].$$

We consider the following Tikhonov regularized functional minimization problem:

$$\min_{f \in \mathcal{L}^2(\alpha)} \mathcal{F}[f] \overset{\text{def}}{=} \mathcal{R}[f] + \frac{\mu}{2}||f||^2.$$

Here, $\mu \geq 0$ is some regularization parameter. To solve such a problem, we define the subgradient $\nabla \mathcal{F}[f]$ of $\mathcal{F}$ at $f$ as the set of all $g \in \mathcal{L}^2(\alpha)$ such that

$$\mathcal{F}[g] \geq \mathcal{F}[f] + \langle g - f, \nabla \mathcal{F}[f] \rangle_{\alpha}.$$  

Since $\mathcal{L}^2(\alpha)$ is a Hilbert space, according to the Riesz representation theorem, the subgradient can also be represented by a function in $\mathcal{L}^2(\alpha)$. Concretely, the subgradient can be explicitly computed as follows:

$$h(x) = \mathbb{E}_{y \sim \beta^x}[\nabla_1 \ell(f(x), y)] + \mu f(x),$$

where $\nabla_1 \ell(y', y) = \partial \ell(y', y)/\partial y'$. In particular, when only empirical measures of $\alpha$ and $\beta^x$ are available, i.e. when $P$ is the empirical distribution on the set $\{(x_1, y_1), (x_2, y_2), \ldots, (x_M, y_M)\} \in \mathcal{X} \times \mathcal{Y}$, with $x_i \neq x_j$ for $1 \leq i < j \leq M$, the empirical version of the above subgradient computation (\ref{eq:subgradient}) is as follows: For all $j \in [M]$

$$h(x_j) = \nabla_1 \ell(f(x_j), y_j) + \mu f(x_j).$$

**Restricted Functional Gradient Descent (Boosting).** A standard approach to solve the functional minimization problem \ref{eq:functional_minimization} is the functional gradient descent method

$$f^{t+1} := f^t - \eta^t h^t, \ h^t \in \nabla \mathcal{R}[f^t].$$

In the empirical case \ref{eq:empirical_subgradient}, one can construct $h^t \in \nabla \mathcal{R}[f^t]$ by interpolating $\{x_j, \nabla_1 \ell(f^t(x_j), y_j)\}_{j=1}^M$. However, this choice of $h^t$ has at least two drawbacks: 1. Evaluating $h^t$ at a single point requires going through the whole dataset; 2. $h^t$ is constructed explicitly on the data points $(x_j, y_j)$ which provides no privacy protection. One alternative to the explicit functional gradient descent method is the restricted functional gradient descent, also known as Boosting [Mason et al. 2000] :

$$f^{t+1} := f^t - \eta^t h_{\text{weak}}^t, \ h_{\text{weak}}^t = \text{WO}_\alpha^2(\nabla \mathcal{R}[f^t]).$$

Here, $\text{WO}_\alpha^2$ is a weak learning oracle such that for any $\phi \in \mathcal{L}^2(\alpha)$, the output $h = \text{WO}_\alpha^2(\phi)$ is a function in $\mathcal{L}^2(\alpha)$ satisfying the following weak learning assumption:

$$||h - \phi||_{\alpha} \leq (1 - \gamma)||\phi||_{\alpha},$$

for some positive constant $0 < \gamma \leq 1$. Replacing the interpolating $h^t$ with $h_{\text{weak}}^t$ alleviates the aforementioned two drawbacks and the restricted functional gradient descent can be proved [Grubb and Bagnell 2011] to converge to the global optimal under standard regularity conditions.

### 3 Federated Functional Minimization

In this paper, we consider the federated functional minimization problem: We assume that there are $N$ client machines. Client machine $i$ draws samples from distribution $P_i$ over $\mathcal{X} \times \mathcal{Y}$. We denote the marginal distribution of $P_i$ on $\mathcal{X}$ by $\alpha_i$ and the conditional distribution on $\mathcal{Y}$ given $x$ by $\beta^x_i$. Due to the heterogeneous nature of the federated learning problems, the $P_i$’s differ across different clients.
Algorithm 1 Server procedure

1: procedure Server($f^0$, $T$, $C$)  
2:     for $t \leftarrow 0$ to $T - 1$ do  
3:         Sample set $S^t$ of clients.  
4:         $f^{t+1} = \frac{1}{|S^t|} \sum_{i \in S^t} \text{Client}(i, t, f^t)$  
5:     return $\text{Proj}_C(f^T)$.

We define $\alpha$ to be the “arithmetic” mean of the local input probability measures, i.e.\footnote{More precisely, for any Borel set subset to $X$, its measure under $\alpha$ is the average of the measures under $\alpha_i$’s.}

$$\alpha \overset{\text{def}}{=} \frac{1}{N} \sum_{i=1}^{N} \alpha_i.$$ \hfill (10)

It is easy to see that $L^2(\alpha) \subseteq L^2(\alpha_i)$ for all $i$. We define $\mathcal{R}_i : L^2(\alpha) \rightarrow \mathbb{R}$ to be $\mathcal{R}_i[f] \overset{\text{def}}{=} \mathbb{E}_{(x,y) \sim P_i}[\ell(f(x), y)]$ and denote $\mathcal{F}_i[f] = \mathcal{R}_i[f] + \frac{\ell}{2} \|f\|_{\alpha_i}^2$. The goal of federated learning is to minimize the average loss functional

$$\min_{f \in \mathcal{C}} \mathcal{F}[f] = \frac{1}{N} \sum_{i=1}^{N} \mathcal{F}_i[f],$$ \hfill (11)

where $\mathcal{C} \subseteq L^2(\alpha)$ is some convex set of functions. In the rest of the paper, we use $f^*$ to denote the optimizer of (11). We emphasize that in federated optimization, the local inner product structure $\langle \cdot, \cdot \rangle_{\alpha_i}$ as well as the subgradient $\mathcal{F}_i[f]$ cannot be shared during training phase.

We consider two notions of client heterogeneity:

**Semi-heterogeneous setting:** The local distributions on the input space is the same, but the conditional distributions on the output space varies among different machines, i.e. $\forall i \in [N], \alpha_i = \alpha$, but $\beta_i^x$ may not be equal to $\beta_i^x$ for $i \neq j$.

**Fully-heterogeneous setting:** The local distributions on the input space are different from the global average measure, i.e. $\alpha_i \neq \alpha$.

For the semi-heterogeneous setting, we propose a federated functional gradient boosting method, FFGB, that provably finds the global optimizer in sublinear time. For the fully-heterogeneous setting, we propose two variants FFGB.C and FFGB.L of FFGB that converge sublinearly to a neighborhood of the global optimizer. We emphasize that in both cases, in order to obtain the best convergence rate, the local machines need to take multiple local steps (specifically $O(\sqrt{T})$ where $T$ is the number of communication rounds) between every round of communication (see Theorems 3.1, 3.2, 3.3). Such results showing the benefits of taking multiple local steps are rare in the FL literature.

### 3.1 Semi-heterogeneous Federated Optimization

In this section, we consider the unconstrained case, i.e. $\mathcal{C} = L^2(\alpha)$. The FFGB shares a similar structure as FedAvg. After every round of global variable averaging, the server sends the global consensus function $f^t$ to the workers (see procedure Server in Algorithm 1), which is followed by $K$ local steps of the RFGD update\footnote{More precisely, for any Borel set subset to $X$, its measure under $\alpha$ is the average of the measures under $\alpha_i$’s.} tracked via the local variable $g_i^{k,t}$, for $k = 1, 2, \ldots, K$, with the initialization $g_i^{1,t} = f^t$. (All the line numbers refer to Algorithm 2 in the rest of this paragraph.) The crucial twist on the RFGD update is the use of an additional residual variable $\Delta_i$, which is initialized to the constant zero function (line 2). This residual variable accumulates the approximation error of the descent direction $h_i^{k,t}$ incurred by the weak oracle (line 6). Such a residual is then used to compensate the next functional subgradient (line 4) and is used in the query to the weak learning oracle. The local variable function $g_i^{k,t}$ is then refined by the approximated functional gradient $\left(h_i^{k,t} + \mu g_i^{k,t}\right)$ (line 5). Note that the residual $\Delta_i$ only tracks the error of estimating $\nabla \mathcal{R}_i$.
In the absence of the residual variable, the error accumulated by the RFGD updates via Assumption 3.1 holds, and suppose the weak learning oracle even at the global optimal solution due to client heterogeneity. This is in sharp contrast to the single non-zero step size for Assumption 3.1. Theorem 3.1 simply returns the output of FFGB.

The residual $\Delta_i$ used in FFGB (Algorithm 2) is crucial to prove the convergence to the global minimum (see Theorem 3.1): in the absence of the residual variable, the error accumulated by the RFGD updates via the calls to the weak learning oracle may not vanish, since in federated learning the local subgradient is non-zero even at the global optimal solution due to client heterogeneity. This is in sharp contrast to the single machine case where, at the global optimal solution, the subgradient vanishes and so does the approximation error incurred by the weak oracle, leading to convergence of RFGD. The design of the residual is inspired by the error-feedback technique of the SignSGD method [Karimireddy et al. 2019] to mitigate errors in gradient compression in distributed training. This technique has also been applied for functional minimization in the much simpler single machine setting by [Grubb and Bagnell 2011].

We now analyze the convergence of FFGB. We make the following standard regularity assumption, which is satisfied e.g. by the cross entropy loss [1].

**Assumption 3.1.** For all $i \in [N]$, the subgradients of $\mathcal{R}_i[f]$ are $G$-bounded under the $L^2(\alpha_i)$ norm, i.e. for any $f \in L^2(\alpha)$, we have $\|\nabla \mathcal{R}_i[f]\|_{\alpha_i} \leq G$.

Under this assumption, we have the convergence result for FFGB. For simplicity, in this section and subsequent ones for FFGB.C and FFGB.L, we present the convergence rate in the case when all $N$ clients are sampled in each round. The analysis easily extends to the case when only a few clients are sampled in each round, incurring a penalty for the variance in the sampling. Qualitatively the convergence bound stays the same. Details can be found in the appendix.

**Theorem 3.1 (Convergence result of FFGB).** Let $f^0$ be the global initializer function. Suppose that Assumption 3.1 holds, and suppose the weak learning oracle $\text{WO}_0$ satisfies (9) with constant $\gamma$. Using the step size $\eta_k = 2/(\mu(K+k+1))$, the output of FFGB satisfies

$$
\|f^T - f^*\|_2^2 = O \left( \frac{\|f^0 - f^*\|_2^2}{KT} + \frac{KG^2 \log(KT)}{T\gamma^2 \mu^2} + \frac{(1 - \gamma)G^2}{K\mu^2 \gamma^2} + \frac{(1 - \gamma)G^2 \log(KT)}{KT \mu^2 \gamma^2} \right).
$$

In the limit case when $\gamma = 1$, the weak oracle exactly approximates its input and hence the above result degenerates to the one of FedAvg [Li et al. 2019]:

$$
\|f^T - f^*\|_2^2 = O \left( \frac{\|f^0 - f^*\|_2^2}{KT} + \frac{KG^2 \log(KT)}{T\mu^2} \right).
$$

Note that when $\gamma < 1$, in order to have the best convergence rate (up to log factors), we need to set the number of local steps $K = \Omega(\sqrt{T})$, which leads to the following corollary.

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**Algorithm 2** Client procedure for Federated Functional Gradient Descent (FFGB)

1: **procedure** Client($i$, $t$, $f$)
2: $\Delta_i^{0,t} = 0$, $g_i^{1,t} = f^t$;
3: for $k \leftarrow 1$ to $K$ do
4: \hspace{0.5cm} $h_i^{k,t} := \text{WO}_0^{\alpha_i}(\Delta_i^{k-1,t} + \nabla \mathcal{R}_i[g_i^{k,t}])$
5: \hspace{0.5cm} $g_i^{k+1,t} := g_i^{k,t} - \eta^{k,t} \left( h_i^{k,t} + \mu g_i^{k,t} \right)$
6: \hspace{0.5cm} $\Delta_i^{k+1,t} := \Delta_i^{k-1,t} + \nabla \mathcal{R}_i[g_i^{k,t}] - h_i^{k,t}$
7: **return** $g_i^{K+1,t}$.
The subgradients of Algorithm 3

The fact that summable is because the radius \( r \) is important to our analysis. Of the clipping step is its non-expansiveness since it is exactly the projection operator onto the are uniformly bounded these operations ensure that during the entire optimization procedure, the (local and global) variable functions is consensus among all machines. Together with the clipping steps in lines 4 and 6 in the procedure is identical to that of Corollary 3.1. The following theorem states that we next analyze the convergence of We now present 3.2 Fully-heterogeneous Federated Optimization

The fact that FFGB is able to find the global optimal solution heavily relies on the consensus of the local inner product structure: \( \langle \cdot, \cdot \rangle_\alpha = \langle \cdot, \cdot \rangle_{\alpha_1} \), which is absent in the fully-heterogeneous setting. So we can only guarantee that the algorithm finds a solution within a neighborhood of the global optimal whose radius diminishes to zero with diminishing heterogeneity of the \( \alpha \)’s.

To achieve this goal, we consider minimizing the objective functional \( \mathcal{F}[f] \) of problem \( (11) \) over the ball \( B^\infty(B) \) for a given radius \( B > 0 \), i.e. \( \mathcal{C} = B^\infty(B) \).

This feasible domain allows us to exploit the variational formulation of the total variation (TV) distance in order to relate the local inner product structure to the global one:

\[
\|f - f^*\|_\alpha < 2\|f\|_\infty \|g\|_\infty TV(\alpha, \alpha_i). \tag{12}
\]

We now present FFGB.C, a variant of FFGB with additional clipping operations, in Algorithm 3 (the constants \( G^1 \) and \( G^2 \) are specified in Theorem 3.2). Only the Client procedure is shown; the Server procedure is identical to that of FFGB. FFGB.C relies on a different weak learning oracle \( \text{WO}^\infty_{\alpha_i} \) that satisfies an \( \mathcal{L}^\infty(\alpha_i) \) version of (9). Specifically, for any query \( h \in \mathcal{L}^\infty(\alpha_i) \), we assume that

\[
\|\text{WO}^\infty_{\alpha_i}(h) - h\|_{\alpha_i, \infty} \leq (1 - \gamma)\|h\|_{\alpha_i, \infty}, \tag{13}
\]

for some positive constant \( 0 < \gamma \leq 1 \).

The clipping step in line 2 of the Client procedure actually implements the global projection to \( \mathcal{C} \) as \( f^t \) is consensus among all machines. Together with the clipping steps in lines 4 and 6 in the Client procedure, these operations ensure that during the entire optimization procedure, the (local and global) variable functions are uniformly bounded\(^2\) in the whole domain \( \mathcal{X} \). Therefore, the bound (12) can be exploited. Another merit of the clipping step is its non-expansiveness since it is exactly the projection operator onto the \( \mathcal{L}^\infty \) ball. This is important to our analysis.

We next analyze the convergence of FFGB.C in the fully-heterogeneous setting. We need a slightly strengthened version of Assumption 3.1

**Assumption 3.2.** The subgradients of \( \mathcal{R}_i[f] \) are \( G \)-bounded under the \( \mathcal{L}^\infty(\alpha_i) \) norm, i.e.

\[
\forall f \in \mathcal{L}^\infty(\alpha_i), \|\nabla \mathcal{R}_i[f]\|_{\alpha_i, \infty} \leq G.
\]

The following theorem states that FFGB.C converges to a neighborhood of the global minimizer, with a radius \( r \) proportional to the average TV distance between \( \alpha \) and \( \alpha_i \), i.e. \( r = O(\frac{1}{N} \sum_{i=1}^{N} TV(\alpha, \alpha_i)) \).

\(^2\)Note that such a boundedness property is non-trivial even when the weak oracle only returns bounded functions. This is because the \( \mathcal{L}^\infty \) norm of the variable function potentially diverges as \( t \to \infty \) since standard choices of step sizes are not summable.
Algorithm 4 Client procedure for Federated Functional Gradient Descent for $\ell_2$ Regression among Lipschitz Continuous Functions (FFGB.L)

1: procedure CLIENT($i, t, f$)
2: $\Delta_i^0 = 0$, $g_i^{0,t} = f^t$;
3: for $k \leftarrow 1$ to $K$ do
4: $h_i^k := \text{WO}_{\alpha_i}(\Delta_i^{k-1} - u_i)$;
5: $g_i^{k+1,t} := g_i^{k,t} - \eta^{k,t}(g_i^{k,t} - h_i^k)$;
6: $\Delta_i^k := \Delta_i^{k-1} - u_i + h_i^k$;
7: return $g_i^{K,t}$.

Theorem 3.2 (Convergence result of FFGB.C). Let $f^0$ be the global initializer function. Let $\omega = \frac{1}{N} \sum_{i=1}^{N} \text{TV}(\alpha, \alpha_i)$. Set $G_1^\omega = \frac{1-\gamma^2}{\gamma^2} \cdot G$ and $G_2^\omega = \frac{2-\gamma^2}{\gamma^2} \cdot G$. Under Assumption 3.2 and supposing the weak learning oracle $\text{WO}^\omega_\alpha$ satisfies (13) with constant $\gamma$, using the step sizes $\eta^{k,t} = \frac{4}{\mu(K+3+T)}$, the output of FFGB.C satisfies

$$\|\Pi_B(f^T) - f^*\|_2^2 = O\left(\frac{\|f^0 - f^*\|_2^2}{KT} + \frac{K G^2 \log(KT)}{T \mu^2 \gamma^2} + \frac{(1 - \gamma^2) G^2}{K \mu^2 \gamma^2} + \frac{G B \omega}{\mu \gamma} + \frac{G^2 \log(TK) \omega}{\gamma^2 \mu^2 T}\right).$$

As previously discussed, the key to prove the above theorem is to ensure that the variable function remains bounded during the entire optimization procedure. Moreover, the choice of the constants $G_1^\omega$ and $G_2^\omega$ ensures that the clipping operations do not affect the value of $\Delta_i^{k,t}$ and $h_i^{k,t}$ on the support of $\alpha_i$. The existence of these two constants is due to the stronger weak learner oracle (13), while the standard $\mathcal{L}^2(\alpha)$ oracle (9) ensures the residual is reduced in average, it may still have large spiky values on $\text{supp}(\alpha_i)$.

In contrast to the semi-heterogeneous setting, even in the ideal case when $\gamma = 1$, FFGB.C does not converge to the global minimum due to the forth term in Theorem 3.2. This is because the fully-heterogeneous setting is fundamentally harder: local strong convexity due to Tikhonov regularization does not imply global strong convexity in the fully-heterogeneous setting which hinders the convergence to the global minimum of FFGB.C.

3.3 Federated Functional Least Squares Minimization

In this section, we show that when the loss $\ell$ is the square loss $\ell(y, y') = \frac{1}{2}||y - y'||^2$ for $y, y' \in \mathcal{Y} \subseteq \mathbb{R}^c$, we can improve the bound on radius of convergence $r$ by replacing the TV distance with the Wasserstein-1 distance, if the feasible domain is further restricted to the family of Lipschitz continuous functions. Note that even in the semi-heterogeneous case, the square loss requires special treatment as it does not satisfy Assumption 3.1.

For simplicity, we assume that the output domain $\mathcal{Y} \subseteq \mathbb{R}$, the extension to $\mathbb{R}^c$ is analogous. For some parameter $L > 0$ and a constant $\gamma$ (defined in (19)), we consider the federated functional least squares minimization problem

$$\min_{\|f\|_{l_{\text{lip}}} \leq L/\gamma} \mathcal{F}[f] = \frac{1}{N} \sum_{i=1}^{N} \mathcal{E}(x,y) \sim P_i \left[\frac{1}{2} f(x) - y^2\right].$$

(14)

The domain of optimization is the set $\mathcal{C} = \{f : \|f\|_{l_{\text{lip}}} \leq L/\gamma\}$. This feasible domain allows us to exploit the variational formulation of the Wasserstein-1 ($W_1$) distance in order to relate the local inner product structure to the global one: Denote $\xi = \|f\|_{l_{\text{lip}}} \|g\|_{\alpha,\infty} + \|g\|_{l_{\text{lip}}} \|f\|_{\alpha,\infty}$,

$$|\langle f, g \rangle_{\alpha_i} - \langle f, g \rangle_{\alpha_i}| \leq \xi W_1(\alpha, \alpha_i).$$

(15)

We assume in this section that $P_i$ is the empirical measure on a finite set of client data $\{(x_{i,j}, y_{i,j}) : j \in [M]\}$. We also assume that the data satisfies the following Lipschitzness property: for any $j, j' \in [M]$, we have

$$|y_{i,j} - y_{i,j'}| \leq L \|x_{i,j} - x_{i,j'}\|.$$
This assumption implicitly suggests that the labels are generated from the inputs via an \( L \)-Lipschitz function with no additive noise. The assumption implies that it is possible to construct an \( L \)-Lipschitz function \( u_i : \mathbb{R}^d \to \mathbb{R} \) that interpolates the data \( \{(x_{i,j}, y_{i,j}) : j \in [M]\} \); specifically, this Lipschitz extension is defined as
\[
u_i(x) \overset{\text{def}}{=} \min_{j \in [M]} (y_{i,j} + L\|x - x_{i,j}\|).
\] (17)

It is easy to check that \( u_i \) is an \( L \)-Lipschitz function such that \( u_i(x_{i,j}) = y_{i,j} \) for all \( j \in [M] \). If the output domain \( \mathcal{Y} \) is high dimensional, i.e. \( \mathcal{Y} \subseteq \mathbb{R}^c \), then the construction of \( u_i \) follows from Kirszbaum’s Lipschitz extension theorem [Schwartz 1969]. The function \( u_i \) is used in the CLIENT procedure in Algorithm 4 which describes a variant of FFGB named FFGB.L. It is designed to solve the federated least squares minimization problem over the Lipschitz continuous functions. The SERVER procedure for FFGB.L is identical to that of FFGB. Note that while \( \mathcal{C} \) is not the whole \( \mathcal{L}^2(\alpha) \) space, FFGB.L ensures the feasibility of \( f^T \) and hence the projection step in the SERVER procedure simply returns \( f^T \).

FFGB.L differs from FFGB as it exploits the following observation: for problem (14), the functional subgradient at \( g \) can be computed in any function \( h \) satisfying
\[
\forall j \in [M], h(x_j) = g(x_{i,j}) - y_{i,j}.
\] (18)

In particular, the function \( h = g - u_i \) is one such subgradient. Thus, given a function \( g \), to approximate \( g \) with \( u_i \), it suffices to approximate \( u_i \). In FFGB.L, we use the weak learning oracle to approximate \( u_i \) in (18) instead of approximating the whole subgradient like FFGB and FFGB.C. Note that FFGB.L uses a higher order weak learner oracle \( \text{WO}_h \), which will be described momentarily. The function \( h^k \) represents the approximation to \( u_i \) after \( k \) steps in the CLIENT procedure. We use \( (g^k_{i,t} - h^k_{i,t}) \) as ascent direction (line (3)) and the residual variable \( \Delta^k_i \) accumulates the error incurred from approximating \( u_i \) (lines (4) and (6)). We emphasize that the subgradient (18) cannot be directly used to update the local variable since \( u_i \) involves the data points.

Now we analyze the convergence of FFGB.L. We first clarify the weak oracle \( \text{WO}_h \) required by FFGB.L: on any query \( \phi \in \mathcal{L}^2(\alpha) \) with \( \|\phi\|_{\text{lip}} < \infty \), \( \text{WO}_h \) outputs a function \( h \) such that \( \|h\|_{\text{lip}} \leq \|\phi\|_{\text{lip}} \) and the following two conditions hold simultaneously
\[
\|h - \phi\|_{\alpha, \infty} \leq (1 - \gamma)\|\phi\|_{\alpha, \infty},
\] (19)
\[
\|h - \phi\|_{\text{lip}} \leq (1 - \gamma)\|\phi\|_{\text{lip}},
\] (20)
for some positive constants \( 0 < \gamma \leq 1 \). We can implement this oracle using the Sobolev training [Czarnecki et al. 2017] in the appendix.

Our analysis needs the following boundedness assumptions:

**Assumption 3.3.** For some parameter \( B > 0 \), all labels are \( B \)-bounded: i.e. \( \forall i \in [N], j \in [M], -B \leq y_{i,j} \leq B \).

For every pair of measures \( \alpha_i \) and \( \alpha_i' \), \( \forall x_{i,j} \in \text{supp}(\alpha_i), \exists x_{i',j'} \in \text{supp}(\alpha_i') \) such that \( \|x_{i,j} - x_{i',j'}\| \leq D \).

We now present the convergence result of FFGB.L.

**Theorem 3.3 (Convergence result of FFGB.L).** Let \( f^0 \) be the global initializer. Let \( G^2 = \frac{2L^2}{\mu} \sum_{i,s=1}^N W^2_1(\alpha_s, \alpha_i) + 2B^2 \) and \( \omega = \frac{1}{N} \sum_{i=1}^N W_1(\alpha, \alpha_i) \). Consider the federated functional least square minimization problem (14). Under Assumption 3.3 and supposing the weak learning oracle \( \text{WO}_h \) satisfies (19) and (20) with constant \( \gamma \), using the step sizes \( \eta^{s,i} = \frac{\mu}{\mu(K+K+1)} \), the output of FFGB.L satisfies
\[
\|f^T - f^\star\|_\alpha^2 = O \left( \frac{\|f^0 - f^\star\|^2}{KT} + \frac{K\gamma^2 (KT)}{T \mu^2 \gamma^2} + \frac{(1 - \gamma)^2 B^2}{\mu \gamma^2 K} + \frac{(DL^2 + BL)\omega}{\gamma^2 \mu} \right).
\]
The key component in the analysis of FFGB.L is to ensure that the variable function remains Lipschitz continuous along the entire optimization trajectory. However, unlike FFGB.C where the boundedness of the variable is a direct consequence of the clipping operation, maintaining the Lipschitz continuity of the variable in FFGB.L is more subtle and heavily relies on the structure of the update rule in line (5). We elaborate this in the appendix.

Remark 3.1. While we present FFGB.L in a structure similar to FedAvg, in fact, it can be implemented using only one round of communication: Lines 4 and 6 are independent of $t$. Therefore, we can compute $\{h^k_t\}_{t=1}^K$ once and send them to the server, where the update and the aggregation of the function ensembles are actually carried out.

4 Experiments

We conduct experiments on the two datasets CIFAR10 and MNIST and compare with FedAvg as baseline to showcase the advantage of our functional minimization formulation and the proposed algorithms. We will investigate the effect of the number of local steps, number of communication rounds, communication cost, as well as effect of the residual technique used in our algorithms.

4.1 Results on CIFAR10

Accuracy vs Communication Cost. To make a fair comparison between FFGB (which is a non-parametric method) and FedAvg (which is used for parametric models), we use the number of models exchanged (or equivalently, the number of parameters exchanged) to measure the communication cost. Recall that $N$ and $K$ are the number of local workers and local steps, respectively. The per-iteration cost of FFGB is $NK$: A client needs to upload its local increment which is an ensemble of $K$ models and it has to download $(N - 1)K$ models from the server. For FedAvg, this quantity is 2, i.e. the client uploads its own model and downloads the shared model from the server. Note that the models have the same number of parameters.

Setup. In our experiments, we consider the multiclass image classification problem using the cross entropy loss given in (1). The class of weak learners is a CNN similar to the one suggested by the PyTorch tutorial for CIFAR10. The heterogeneity across local datasets is controlled by dividing the dataset among $N = 56$ clients in the following manner: we randomly select a portion $s \times 100\%$ ($s \in [0, 1]$) of the data from the dataset and
allocate them equally to all clients; for the remaining \((1 - s) \times 100\%\) portion of the data, we sort the data points by their labels and assign them to the workers sequentially. This is a same scheme as employed in [Karimireddy et al., 2020b, Hsu et al., 2020] to enforce heterogeneity. In FedAvg, each worker takes 10 local steps and the step sizes are set to constants \(5 \times 10^{-4}\) (a larger step size leads to divergence).

**Comparing with FedAvg.** We present the results corresponding to CIFAR10 in Figure 1 where each figure corresponds to a different heterogeneity setting. We consider two values for \(s\), i.e. \(s = 0.1, 0.3\). For FFGB, the number of local steps \(K\) is 4, 3 for \(s = 0.1, 0.3\), respectively. A larger \(K\) leads to faster convergence rate according to our theory, however it also leads to higher communication complexity since the per-iteration communication cost of FFGB is \(NK\). The step size follows the scheme \(\eta^{k,t} = \frac{\eta_0}{K^{t+K-k-1}}\), where \(\eta_0\) is set to 10, 20 respectively for \(s = 0.1, 0.3\). In this experiment, we limit the total communication cost to be 2000 for FFGB and FedAvg (this corresponds to 1000 rounds for FedAvg and 2000/N/K rounds for FFGB). As we observe from Figure 1, even with the current implementation of FFGB which is quite communication-inefficient (i.e. it has a much high communication cost per round compared to FedAvg ), FFGB achieves a higher accuracy than FedAvg. We acknowledge that when \(N\) grows, a direct implementation of FFGB is non-ideal due to the high per-iteration communication complexity. We will discuss in Section 5 ways to improve the communication cost of FFGB using knowledge distillation methods.

**Residuals are necessary.** We compare FFGB with its no residual variant to show that the augmented residual is necessary to ensure a fast convergence. For both methods, we set the local steps \(K\) to 10 and set the initial step size \(\eta_0\) to 10 for \(s = 0.1\). We present the result in Figure 2(a) where we observe that FFGB consistently achieves a higher training accuracy over the no-residual counterpart.

**Local steps are necessary.** To show the necessity of the local steps, we vary the number of local steps \(K \in \{1, 5, 10, 20, 40\}\) in FFGB and compare the in Figure 2(b). We observe that a larger \(K\) leads to higher training accuracy.

4.2 Results on MNIST

In Figure 3 we report the testing accuracy of the two algorithms FFGB and FedAvg in terms of the number of communication rounds for the MNIST data set. As we observe from the figure, FFGB performs
significantly superior w.r.t. FedAvg under the same number of communication rounds. This shows how powerful functional minimization methods can be. Recall that the per-round communication complexity of FFGB is higher than FedAvg. We will provide the comparison w.r.t the communication cost in the appendix (with similar results as the previous section).

5 Towards a practical implementation

A direct implementation of the algorithms in this paper has one significant drawback in practice: it requires aggregation of a very large number of models. In each of the $T$ communication rounds, each of the $N$ clients computes $K$ functions that are communicated to the server, which aggregates all of them. Thus, the final ensemble has $TNK$ models, which is impractically large.

The focus of this paper is on developing the theory of federated functional minimization algorithms and the development of practical versions of our algorithms is a very important direction of future work. However, we now indicate how the technique of knowledge distillation [Hinton et al., 2015, Bucila et al., 2006] can be used to reduce the final ensemble to a more reasonable size. Knowledge distillation converts an ensemble of models into a single model by training a new model to mimic the ensemble’s predictions. This training procedure only needs access to unlabelled examples drawn from the appropriate marginal distribution. [Hinton et al., 2015] have shown this technique to be very effective (i.e. incurring minimal loss in performance) in practice when we have access to a large number of unlabelled examples.

The implication for our algorithms is that since clients have access to their local distributions $P_i$, they can distill the ensemble of $K$ models they compute in each round to a single model sent to the server. The server simply averages the $N$ models it receives from the clients, so the final ensemble size is reduced to $N$. This can be further reduced to a single model if we make the reasonable assumption that the server has access to the mean distribution $\alpha = \frac{1}{N} \sum_{i=1}^{N} \alpha_i$ on the feature vectors, because then the server can distill the aggregation of the $N$ models it receives from the clients down to a single model. Even if the server doesn’t have access to $\alpha$, it is still possible to reduce the ensemble size down to a single model by interleaving distillation rounds between boosting rounds. Specifically, after every boosting round, the server executes a distillation round via FedAvg. Distillation via FedAvg is possible the clients simply need to compute the gradients of the loss incurred by the single model under training on their own local data.

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A Total Variation Distance and Wasserstein-1 Distance between Probability Measures

Given two probability distributions $\alpha, \beta \in \mathcal{M}_1(X)$, the total variation distance between $\alpha$ and $\beta$ is

$$TV(\alpha, \beta) = \sup_{A \in \mathcal{F}} |\alpha(A) - \beta(A)|,$$  \hspace{1cm} (21)

where $\mathcal{F}$ is the Borel sigma algebra over $X$.

The $p$-Wasserstein metric between $\alpha$ and $\beta$ is defined as

$$W_p(\alpha, \beta) \overset{\text{def}}{=} \min_{\pi \in \Pi} \left( \int_X \|x - y\|^p d\pi(x, y) \right)^{1/p},$$  \hspace{1cm} (22)

where $\Pi(\alpha, \beta) \overset{\text{def}}{=} \{ \pi \in \mathcal{M}_1(X \times X) | \sharp_1 \pi = \alpha, \sharp_2 \pi = \beta \}$ is the set of joint distributions with given marginal distributions $\alpha$ and $\beta$. Here $\sharp_i$ denotes the marginalization.

B Experiment

The code can be found in the anonymous repo [https://anonymous.4open.science/r/279e682d-09e3-43cb-ba2d-f77eeac0d53e/](https://anonymous.4open.science/r/279e682d-09e3-43cb-ba2d-f77eeac0d53e/).

B.1 Structure of the Weak Learner

For MNIST, the weak learner is a multilayer perceptron with two hidden layers with size 32 and 32. The activation function is leaky relu with negative slope being 0.01. For CIFAR10, the weak learner is a CNN with the same structure as the pytorch tutorial [https://pytorch.org/tutorials/beginner/blitz/cifar10_tutorial.html](https://pytorch.org/tutorials/beginner/blitz/cifar10_tutorial.html). The only difference is that for the fully connected layers, the hidden sizes are changed from 120 and 84 to 32 and 32.

B.2 MNIST Result

In Figure 4, we present the results of MNIST with the y-axis denoting the testing accuracy and x-axis denoting the communication cost. We use the same step size setting for FFGB and the number of local steps $K$ is set to 2. For FedAvg the client uses 20% of the local data per local step, and takes 25 local steps. These is the suggested parameter setting in [Karimireddy et al., 2020b] for FedAvg. The local step size is set to 0.0003.

B.3 Implementing the Weak Learning Oracles

We now discuss the implementations of the weak learning oracles. In our experiments, we only use FFGB and hence only the weak learning oracle $WO^2_\alpha$ is actually implemented. We discuss the implementation of the oracles $WO^\infty_\alpha$ and $WO^{lip}_\alpha$ for completeness, but the suggested schemes may not be efficient in practice.

**Implementing $WO^2_\alpha$**. Let $\phi$ be the input to the oracle and let $h_\theta$ be the candidate weak learner to be trained. Here $h_\theta$ is a neural network with parameter $\theta$.

We can implement the oracle by solving

$$\min_{\theta} \sum_{x \in \text{supp}(\alpha)} \|\phi(x) - h_\theta(x)\|^2.$$ \hspace{1cm} (23)

In our experiment, we run Adam for 1000 steps to solve the above problem using initial step size 0.005.
Figure 4: Testing accuracy vs communication cost on MNIST.

**Implementing** $W_{O}^{\infty}$. Let $\phi$ be the input to the oracle and let $h_{\theta}$ be the candidate weak learner to be trained. Here $h_{\theta}$ is a neural network with parameter $\theta$.

We can implement the oracle by solving

$$
\min_{\theta} \max_{x \in \text{supp} (\alpha)} \| \phi(x) - h_{\theta}(x) \|^{2}.
$$

**Implementing** $W_{O}^{\lip}$. Let $\phi$ be the input to the oracle and let $h_{\theta}$ be the candidate weak learner to be trained. Here $h_{\theta}$ is a neural network with parameter $\theta$.

We can implement the oracle by solving

$$
\min_{\theta} \left( \max_{x \in \text{supp} (\alpha)} \| \phi(x) - h_{\theta}(x) \|^{2} \right) + \int_{X} \| \nabla_{x} \phi(x) - \nabla_{x} h_{\theta}(x) \|^{2} dx.
$$

Note that the gradient of the input $\phi$ is available in FFGB.L as we have the explicit expression of $\phi$ in line 4 of Algorithm 4 for all $k = 1, \ldots, K$. The above scheme is similar to the Sobolev training scheme (1) in [Czarnecki et al., 2017].

**C Proof of Theorem 3.1**

**Theorem C.1** (Theorem 3.1 restated.) Let $f^{0}$ be the global initializer function. Suppose that Assumption 3.1 holds, and suppose the weak learning oracle $W_{O}^{2}$ satisfies (9) with constant $\gamma$. Using the step size $\eta_{k,t} = \frac{2}{\mu(tK+k+1)}$, the output of FFGB satisfies

$$
\| f^{T} - f^{\star} \|_{\alpha}^{2} = O \left( \frac{\| f^{0} - f^{\star} \|_{\alpha}^{2}}{K T} + \frac{K G^{2} \log(KT)}{T \gamma^{2} \mu^{2}} + \frac{(1 - \gamma)G^{2} }{K \mu^{2} \gamma^{2}} + \frac{(1 - \gamma)G^{2} \log(KT)}{KT \mu^{2} \gamma^{2}} \right).
$$

**Proof.** Since we are considering the semi-heterogeneous case where $\alpha = \alpha_{i}$, we ignore the subscript $i$ and simply write $\| \cdot \|_{\alpha}$ and $\langle \cdot, \cdot \rangle_{\alpha}$ for the norm and the inner product in $L^{2}(\alpha)$.

For simplicity, we denote $\tilde{h}_{i}^{k,t} = h_{i}^{k,t} + \mu g_{i}^{k,t}$.

We define two hypothetical global average sequences $\tilde{g}^{k,t} = \frac{1}{N} \sum_{i=1}^{N} g_{i}^{k,t}$ and $\tilde{h}^{k,t} = \frac{1}{N} \sum_{i=1}^{N} \tilde{h}_{i}^{k,t}$. In particular, we have $\tilde{g}^{1,t} = f^{t}$. From the update rule in line 5 of Algorithm 2, we write

$$
\| \tilde{g}^{k+1,t} - f^{\star} \|_{\alpha}^{2} = \| \tilde{g}^{k,t} - f^{\star} \|_{\alpha}^{2} + (\eta^{k,t})^{2} \| \tilde{h}^{k,t} \|_{\alpha}^{2} - 2 \eta^{k,t} \langle \tilde{g}^{k,t} - f^{\star}, \tilde{h}^{k,t} \rangle_{\alpha}
$$

(26)
The last term of (26) can be split as
\[-2\langle g^{k,t} - f^*, \hat{h}^{k,t}_\alpha \rangle = -\frac{2}{N} \sum_{i=1}^{N} \langle g^{k,t} - f^*, \hat{h}^{k,t}_\alpha \rangle = -\frac{2}{N} \sum_{i=1}^{N} \langle g^{k,t} - g^{k,t}_i, \hat{h}^{k,t}_\alpha \rangle + \langle g^{k,t} - f^*, \hat{h}^{k,t}_\alpha \rangle + \langle f^* - g^{k,t}_i, \hat{h}^{k,t}_\alpha - \nabla F_i(g^{k,t}_i) \rangle \alpha.
\]

The second term of (27) can be bounded using the $\mu$-strong convexity of $F_i$
\[
\frac{2}{N} \sum_{i=1}^{N} \langle f^* - g^{k,t}_i, \nabla F_i(g^{k,t}_i) \rangle \alpha \leq -\frac{2}{N} \sum_{i=1}^{N} F_i(g^{k,t}_i) - F_i[f^*] - \frac{2}{N} \sum_{i=1}^{N} \frac{\mu}{2} \|f^* - g^{k,t}_i\|^2.
\]

Note that using the optimality of $f^*$ we have
\[
\sum_{i=1}^{N} F_i[g^{k,t}_i] - F_i[f^*] = \sum_{i=1}^{N} F_i(g^{k,t}_i) - F_i[f^*] + F_i[g^{k,t}_i] - F_i[f^*] = \sum_{i=1}^{N} F_i[g^{k,t}_i] - F_i[g^{k,t}_i]
\]
and that by recalling that $g^{k,t}_i = \frac{1}{N} \sum_{i=1}^{N} g^{k,t}_i$ and using the Cauchy–Schwarz inequality we have
\[
\frac{1}{N} \sum_{i=1}^{N} \|f^* - g^{k,t}_i\|^2 \geq \|f^* - g^{k,t}\|^2.
\]

Therefore, we can bound
\[
\frac{2}{N} \sum_{i=1}^{N} \langle f^* - g^{k,t}_i, \nabla F_i(g^{k,t}_i) \rangle \alpha \leq -\frac{2}{N} \sum_{i=1}^{N} \langle \nabla F_i[g^{k,t}_i], g^{k,t}_i - g^{k,t}_i \rangle \alpha - \frac{\mu}{2} \|f^* - g^{k,t}\|^2 - \frac{1}{N} \sum_{i=1}^{N} \frac{\mu}{2} \|f^* - g^{k,t}\|^2 \alpha \leq \frac{1}{N} \sum_{i=1}^{N} \eta^{k,t}_i \|\nabla F_i[g^{k,t}_i]\|^2 \alpha + \frac{1}{N} \eta^{k,t}_i \|\nabla F_i[g^{k,t}_i] - g^{k,t}_i\|^2 \alpha - \frac{\mu}{2} \|f^* - g^{k,t}\|^2 \alpha - \frac{1}{N} \sum_{i=1}^{N} \frac{\mu}{2} \|f^* - g^{k,t}\|^2 \alpha \leq \frac{1}{N} \sum_{i=1}^{N} \eta^{k,t}_i \|\nabla F_i[g^{k,t}_i]\|^2 \alpha + \frac{1}{N} \eta^{k,t}_i \|\nabla F_i[g^{k,t}_i] - g^{k,t}_i\|^2 \alpha - \frac{\mu}{2} \|f^* - g^{k,t}\|^2 \alpha - \frac{1}{N} \sum_{i=1}^{N} \frac{\mu}{2} \|f^* - g^{k,t}\|^2 \alpha,
\]
where we use Young’s inequality in the last inequality.

Besides, recall that $\|\nabla R_i[g]\|_\alpha \leq G$ from Assumption 3.1. Together with the property of the oracle, we have
\[
\|\Delta^{k,t}_i\|_\alpha \leq (1 - \gamma) \left(\|\Delta^{k-1,t}_i\|_\alpha + G\right) \quad \text{and} \quad \|\Delta^{0,t}_i\|_\alpha = 0 \Rightarrow \forall k, \|\Delta^{k,t}_i\|_\alpha \leq \frac{1 - \gamma}{\gamma} G.
\]

Consequently, we also have
\[
\|\tilde{h}^k\|_\alpha \leq \frac{1}{N} \sum_{i=1}^{N} \|h^{k,t}_\alpha\|_\alpha = \frac{1}{N} \sum_{i=1}^{N} \|\nabla R_i[g^{k,t}_i] + \Delta^{k-1}_i\|_\alpha \leq \frac{2 - \gamma}{\gamma} G.
\]

From line 3 of Algorithm 2 we have $g^{k+1,t}_i = (1 - \mu \eta^{k,t}) g^{k,t}_i + \eta^{k,t} h^{k,t}_\alpha$ and therefore
\[
\|g^{k+1,t}_i\|_\alpha \leq \frac{2G}{\gamma \mu} (1 - \mu \eta^{k,t}) \left(\|g^{k+1,t}_i\|_\alpha - \frac{2G}{\gamma \mu}\right),
\]
(29)
where we use \( \|h_{i,t}^{k,t}\|_\alpha \leq 2G/\gamma \). Therefore, if we have initially \( \|f^t\|_\alpha \leq \frac{3G}{\mu} \), we always have \( \|g^{k,t}\|_\alpha \leq \frac{2G}{\gamma} \) (hence so is \( f^{t+1} \) as it is the global average \( \hat{g}^{K+1,t} \)). Further, together with \( \|h_{i,t}^{k,t}\|_\alpha \leq 2G/\gamma \), we have \( \|\dot{h}_{i,t}^{k,t}\|_\alpha \leq 4G/\gamma \).

Additionally, \( \frac{1}{N} \sum_{i=1}^{N} \|g_{i,t}^{k,t} - \hat{g}_{i,t}^{k,t}\|_\alpha^2 \) can be bounded by (we use \( E[(X - E[X])^2] \leq E[X^2] \))

\[
\frac{1}{N} \sum_{i=1}^{N} \|g_{i,t}^{k,t} - \hat{g}_{i,t}^{k,t}\|_\alpha^2 = \frac{1}{N} \sum_{i=1}^{N} \|g_{i,t}^{k,t} - g_{i,t}^{1,t} + g_{i,t}^{1,t} - \hat{g}_{i,t}^{k,t}\|_\alpha^2 \\
\leq \frac{1}{N} \sum_{i=1}^{N} \|g_{i,t}^{k,t} - g_{i,t}^{1,t}\|_\alpha^2 \leq \sum_{k=1}^{K} 16(\eta_{\kappa,t})^2G^2/\gamma^2 \leq 64(\eta_{\kappa,t})^2K^2G^2/\gamma^2,
\]

where we use \( \eta_{\kappa,t} \leq 2\eta_{\kappa,t} \) for any \( t \geq 0 \) and \( 1 \leq \kappa \leq k \). Therefore we can bound the first term of (27) by

\[
\frac{2}{N} \sum_{i=1}^{N} (g_{i,t}^{k,t} - \hat{g}_{i,t}^{k,t})_\alpha \leq \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\eta_{\kappa,t}} \|g_{i,t}^{k,t} - \hat{g}_{i,t}^{k,t}\|_\alpha^2 + \frac{\eta_{\kappa,t}^{k,t}}{N} \sum_{i=1}^{N} \|\dot{h}_{i,t}^{k,t}\|_\alpha^2 \leq \frac{(64K^2 + 16)\eta_{\kappa,t}^2G^2/\gamma^2}{N}.
\]

Plug in the above results into (26) to yield

\[
\|g^{k+1,t} - f^*\|_\alpha^2 \leq (1 - \frac{\mu\eta_{\kappa,t}^{k,t}}{2})\|g^{k,t} - f^*\|_\alpha^2 + O\left((\eta_{\kappa,t}^2K^2G^2/\gamma^2)\right) \\
+ \frac{2\eta_{\kappa,t}^{k,t}}{N} \sum_{i=1}^{N} (f^* - g_{i,t}^{k,t} - \dot{h}_{i,t}^{k,t} - \nabla F_i[g_{i,t}^{k,t}])_\alpha - \frac{\eta_{\kappa,t}^{k,t}}{N} \sum_{i=1}^{N} \mu_2 \|f^* - g_{i,t}^{k,t}\|_\alpha^2.
\]

Recall that \( \eta_{\kappa,t} = \frac{1}{\mu^2(kT^2 + k + 1)} \) and multiply both sides by \( (kT + k + 1) \)

\[
(tK + k + 1)\|g^{k+1,t} - f^*\|_\alpha^2 \leq (tK + k)\|g^{k,t} - f^*\|_\alpha^2 + O\left(\frac{K^2G^2}{\gamma^2\mu^2(tK + k + 1)}\right) \\
+ \frac{4}{N} \sum_{i=1}^{N} (f^* - g_{i,t}^{k,t} - \dot{h}_{i,t}^{k,t} - \nabla F_i[g_{i,t}^{k,t}])_\alpha - \frac{2}{N} \sum_{i=1}^{N} \frac{\mu_2}{2} \|f^* - g_{i,t}^{k,t}\|_\alpha^2.
\]

Sum from \( k = 1 \) to \( K \)

\[
(tK + K + 1)\|g^{K+1,t} - f^*\|_\alpha^2 \leq (tK + 1)\|g^{1,t} - f^*\|_\alpha^2 + O\left(\frac{K^2G^2}{\gamma^2\mu^2} (\log(tK + K) - \log(tK + 1))\right) \\
+ \frac{4}{N} \sum_{i=1}^{N} \sum_{k=1}^{K} (f^* - g_{i,t}^{k,t} - \dot{h}_{i,t}^{k,t} - \nabla F_i[g_{i,t}^{k,t}])_\alpha - \frac{2}{N} \sum_{i=1}^{N} \sum_{k=1}^{K} \frac{\mu_2}{2} \|f^* - g_{i,t}^{k,t}\|_\alpha^2.
\]

(30)
We now focus on the first term of the second line above

\[
\sum_{k=1}^{K} \langle f^* - g_{i,t}^{k,t}, h_{i,t}^{k,t} - \nabla F_i[g_{i,t}^{k,t}] \rangle_\alpha = \sum_{k=1}^{K} \langle f^* - g_{i,t}^{k,t}, h_{i,t}^{k,t} - \nabla R_i[g_{i,t}^{k,t}] \rangle_\alpha
\]

\[
= \sum_{k=1}^{K} \langle f^* - g_{i,t}^{k,t}, \Delta_i^{k-1} \rangle_\alpha + \sum_{k=1}^{K} \langle f^* - g_{i,t}^{k,t}, \Delta_i^{k-1} \rangle_\alpha
\]

\[
= \sum_{k=1}^{K} \langle f^* - g_{i,t}^{k,t}, \Delta_i^{k} \rangle_\alpha + \sum_{k=1}^{K} \langle f^* - g_{i,t}^{k,t}, \Delta_i^{k-1} \rangle_\alpha
\]

\[
\leq \frac{\mu}{2} \| f^* - g_{i,t}^{K,t} \|^2 + \frac{1}{2\mu} \left( \frac{1}{\gamma^2} \right) G^2 + \frac{(1-\gamma)(2-\gamma)}{\gamma^2} G^2 \sum_{k=1}^{K-1} \eta_i^{k,t}
\]

\[
\leq \frac{\mu}{2} \| f^* - g_{i,t}^{K,t} \|^2 + \frac{1}{2\mu} \left( \frac{1}{\gamma^2} \right) G^2 + \frac{(1-\gamma)(2-\gamma)}{\gamma^2} G^2 (\log(tK + K) - \log(tK + 2)).
\]

Using this result, we obtain (we cancel \( \frac{\mu}{2} \| f^* - g_{i,t}^{K,t} \|^2 \) with the last term of (30))

\[
(K(t + 1) + 1) \| f^{t+1} - f^* \|^2_\alpha \leq (Kt + 1) \| f^t - f^* \|^2_\alpha + O\left( \frac{K^2 G^2 \log(Kt)}{\gamma^2 \mu^2} \right) + O\left( \frac{(1-\gamma)TG^2}{\mu^2 \gamma^2} \right) + O\left( \frac{(1-\gamma)G^2 \log(Kt)}{\mu \gamma^2} \right)
\]

Sum from \( t = 0 \) to \( T - 1 \) to yield

\[
(KT + 1) \| f^T - f^* \|^2_\alpha \leq \| f^0 - f^* \|^2_\alpha + O\left( \frac{K^2 G^2 \log(KT)}{\gamma^2 \mu^2} \right) + O\left( \frac{(1-\gamma)TG^2}{\mu^2 \gamma^2} \right) + O\left( \frac{(1-\gamma)G^2 \log(KT)}{\mu \gamma^2} \right),
\]

and hence

\[
\| f^T - f^* \|^2_\alpha \leq O\left( \frac{\| f^0 - f^* \|^2_\alpha}{KT} \right) + O\left( \frac{K^2 G^2 \log(KT)}{T \gamma^2 \mu^2} \right) + O\left( \frac{(1-\gamma)G^2}{KT \mu^2 \gamma^2} \right) + O\left( \frac{(1-\gamma)G^2 \log(KT)}{KT \mu \gamma^2} \right).
\]

\[\square\]

**D Proof of Theorem 3.2**

We restate FFGB.C in Algorithm 5 as there is a typo in Algorithm 3. Specifically, in line 5, we missed the term \( \mu g_{i,t}^{k,t} \). This is marked in red in Algorithm 5. Our result further requires that \( B \geq \frac{2\mu}{\gamma^2} \). Since the Tikhonov regularization ensures the feasibility of \( f^* \) (and \( g_{i,t}^{k,t} \)), we hence remove the clipping operation in line 2 of Algorithm 3 (see line 2 of Algorithm 5).

**Theorem D.1** (Theorem 3.2 restate). Let \( f^0 \) be the global initializer function. Let \( \omega = \frac{1}{N} \sum_{i=1}^{N} TV(\alpha, \alpha_i) \). Set \( G_1 \alpha = \frac{1-\gamma}{\gamma} \cdot G \) and \( G_2 \alpha = \frac{1-\gamma}{\gamma} \cdot G \). Under Assumption 3.2 and supposing the weak learning oracle \( W \)
Algorithm 5 Client procedure for Federated Functional Gradient Descent with Clipping (FFGB.C)

1: procedure Client(i, t, f)
2: \( \Delta_{i, t}^0 = 0 \), \( g_{i, t}^1 = f^t \)
3: for \( k \leftarrow 1 \) to \( K \) do
4: \( h_{i, k, t}^k := W_{\alpha}^N (\Delta_{i, k-1, t}^k + \nabla R_i [g_{i, t}^k]) \)
5: \( g_{i, k+1, t}^k := g_{i, t}^k - \eta_{k, t} \cdot (\Pi_{G^2} (h_{i, k, t}^k) + \mu g_{i, t}^k) \)
6: \( \Delta_{i, k, t}^k := \Pi_{G^2} \left( \Delta_{i, k-1, t}^k + \nabla R_i [g_{i, t}^k] - h_{i, k, t}^k \right) \)
7: return \( g_{i, K+1, t}^k \).

satisfies \([13]\) with constant \( \gamma \) and \( B \geq \frac{2G}{\mu \gamma} \), using the step sizes \( \eta_{k, t} = \frac{4}{\mu(K + k + 1)} \), the output of FFGB.C satisfies

\[
\| f^T - f^* \|^2 = O \left( \frac{\| f^0 - f^* \|^2}{K} + \frac{K^2 \log(KT)}{T \mu^2 \gamma^2} + \frac{(1 - \gamma)^2 G^2}{K \mu^2 \gamma^2} + \frac{GB \omega}{\mu \gamma} \right)
\]

Proof. For simplicity, in this proof, we define \( \hat{g}_{i, t}^k := \Pi_{G^2} (h_{i, t}^k) + \mu g_{i, t}^k \).

For a fixed communication round \( t \), we define two hypothetical sequences \( \tilde{g}_{i, t}^k = \frac{1}{N} \sum_{i=1}^N g_{i, t}^k \) and \( \tilde{h}_{i, t}^k = \frac{1}{N} \sum_{i=1}^N h_{i, k, t}^k \). Note that \( \tilde{g}_{i, t}^k = f^t \). From the update rule in line 5 of Algorithm 2, we write

\[
\| \tilde{g}_{i, t}^{k+1} - f^* \|^2 = \| \tilde{g}_{i, t}^k - f^* \|^2 + (\eta_{k, t})^2 \| \tilde{h}_{i, t}^k \|^2 - 2\eta_{k, t} (\tilde{g}_{i, t}^k - f^*, \tilde{h}_{i, t}^k)_{\alpha_i}.
\]

(32)

For the second term, we have \( \| \tilde{h}_{i, t}^k \|^2 \leq \| \tilde{h}_{i, t}^k \|_2^2 \leq 4G^2 / \gamma^2 = O(\frac{G^2}{\gamma^2}) \) due to the clip operation.

The last term of (32) can be split as

\[
-2 (\tilde{g}_{i, t}^k - f^*, \tilde{h}_{i, t}^k)_{\alpha_i} = -\frac{2}{N} \sum_{i=1}^N (\tilde{g}_{i, t}^k - f^*, \tilde{h}_{i, t}^k)_{\alpha_i} + (\tilde{g}_{i, t}^k - f^*, \tilde{h}_{i, t}^k)_{\alpha_i} - (\tilde{g}_{i, t}^k - f^*, \tilde{h}_{i, t}^k)_{\alpha_i}
\]

\[
= \frac{2}{N} \sum_{i=1}^N (g_{i, t}^k - \tilde{g}_{i, t}^k, \tilde{h}_{i, t}^k)_{\alpha_i} + (f^* - g_{i, t}^k, \nabla F_i [g_{i, t}^k])_{\alpha_i} + (f^* - g_{i, t}^k, \tilde{h}_{i, t}^k - \nabla F_i [g_{i, t}^k])_{\alpha_i}
\]

\[
+ (g_{i, t}^k - f^*, \tilde{h}_{i, t}^k)_{\alpha_i} - (g_{i, t}^k - f^*, \tilde{h}_{i, t}^k)_{\alpha_i}.
\]

(33)

We first introduce the following lemmas that characterize the boundedness of \( h_{i, t}^k, g_{i, t}^k \), and \( \tilde{h}_{i, t}^k \).

Lemma D.1. For a fixed \( i \in [N] \), we have the following property.

1. On the support of \( \alpha_i \), we have \( \Delta_{i, t}^k := \Delta_{i, k-1, t}^k + \nabla R_i [g_{i, t}^k] - h_{i, t}^k \), that is, on the support of \( \alpha_i \), the clip operation has no effect on the support of \( \alpha_i \) in line 6 of Algorithm 2.

2. On the support of \( \alpha_i \), we have \( \Pi_{G^2} (h_{i, t}^k) = h_{i, t}^k \), that is, on the support of \( \alpha_i \), the clip operation has no effect on the support of \( \alpha_i \) in line 5 of Algorithm 2.

Proof. Note that \( \Delta_{i, t}^0 = 0 \). Besides, for \( x \in \text{supp}(\alpha_i) \), in each iteration \( |\Delta_{i, t}^k(x)| \) is first increased at most by \( G \) after adding \( \nabla R_i [g_{i, t}^k] \) and is then reduced by at least \( 1 - \gamma \) after subtracting the weak learner \( h_{i, t}^k \). Consequently, \( |\Delta_{i, t}^k(x)| \leq \frac{1}{1-\gamma} G \) for \( x \in \text{supp}(\alpha_i) \):

\[
\| \Delta_{i, t}^k \|_{\alpha_i, \infty} \leq (1 - \gamma) \left( \| \Delta_{i, k-1, t}^k \|_{\alpha_i, \infty} + G \right) \text{ and } \| \Delta_{i, t}^0 \|_{\alpha_i, \infty} = 0 \Rightarrow \forall k, \| \Delta_{i, t}^k \|_{\alpha_i, \infty} \leq \frac{1 - \gamma}{\gamma} G = G_{\gamma} \cdot (34)
\]
Therefore, the Clip operation does not affect the values of $\Delta_i^{k,t}$ on $\text{supp}(\alpha_i)$ as they will never exceed $\frac{1 - \gamma}{\gamma}$. Further, $\|\Delta_i^{k-1,t} + \nabla R_i[g_i^{k,t}]\|_{\alpha_i,\infty} \leq G + \frac{1 - \gamma}{\gamma} G = \frac{G}{\gamma}$ and hence
\[
\|h_i^{k,t}\|_{\alpha_i,\infty} \leq \|\Delta_i^{k-1,t} + \nabla R_i[g_i^{k,t}] - h_i^{k,t}\|_{\alpha_i,\infty} + \|\Delta_i^{k-1,t} + \nabla R_i[g_i^{k,t}]\|_{\alpha_i,\infty} \leq \frac{2 - \gamma}{\gamma} G = G^2. \tag{35}
\]
Therefore, the Clip operation does not affect the values of $h_i^{k,t}$ on $\text{supp}(\alpha_i)$ neither as they will never exceed $G^2$.

**Lemma D.2.** Assume that the initial function satisfies $\|f^0\|_{\infty} \leq \frac{2G}{\gamma\mu}$. Then for all $1 \leq k \leq K$ and $t \geq 0$, $\|\bar{g}^{k,t}\|_{\infty} \leq \frac{2G}{\gamma\mu}$.

**Proof.** For $t = 0$, $\bar{g}^{1,0} = \Pi_B(f^0) = f^0$ (since $B \geq \frac{2G}{\gamma\mu}$) and hence $\|\bar{g}^{1,0}\|_{\infty} \leq \frac{2G}{\gamma\mu}$ due to the initialization. Now assume that for $t = \tau$ the statement holds. Therefore $\|f^{\tau+1}\|_{\infty} \leq \frac{2G}{\gamma\mu}$. So for $t = \tau + 1$, $\|\bar{g}^{k,t}\|_{\infty} \leq \frac{2G}{\gamma\mu}$. From the update rule in line (5), we have
\[
\bar{g}^{k+1,t} = (1 - \mu \eta^{k,t}) \bar{g}^{k,t} + \frac{1}{N} \sum_{i=1}^N \eta^{k,t} \Pi_{G^2_i} \langle h_i^{k,t} \rangle
\]
Recursively, we have
\[
\|\bar{g}^{k+1,t}\|_{\infty} \leq \frac{2G}{\gamma\mu} \leq (1 - \mu \eta^{k,t}) \left(\|\bar{g}^{k,t}\|_{\infty} - \frac{2G}{\gamma\mu}\right),
\]
which leads to the conclusion. \(\square\)

Combining the above two lemmas, we have the boundedness of $\hat{h}_i^{k,t}$ and $\nabla F_i[g_i^{k,t}]$.

**Lemma D.3.** $\|\hat{h}_i^{k,t}\|_{\infty} \leq \frac{4G}{\gamma}$ and $\|\nabla F_i[g_i^{k,t}]\|_{\infty} \leq \frac{4G}{\gamma}$.

Using the variational formulation of the TV norm, one has (clearly $\|\bar{g}^{k,t}\|_{\infty} \leq \frac{2G}{\gamma\mu} \leq B$
\[
\|\bar{g}^{k,t} - f^* - h_i^{k,t}\|_\alpha - \|\bar{g}^{k,t} - f^* - h_i^{k,t}\|_\alpha \leq O(BG/\gamma \cdot \text{TV}(\alpha, \alpha_i)). \tag{37}
\]
Denote $\omega \overset{def}{=} \frac{1}{N} \sum_{i=1}^N \text{TV}(\alpha, \alpha_i)$. We hence have
\[
\frac{2}{N} \sum_{i=1}^N \left(\|\bar{g}^{k,t} - f^* - h_i^{k,t}\|_\alpha - \|\bar{g}^{k,t} - f^* - h_i^{k,t}\|_\alpha\right) \leq \frac{2}{N} \sum_{i=1}^N \left(\|\bar{g}^{k,t} - f^*\|_\alpha - \|\bar{g}^{k,t} - f^*\|_\alpha\right) \leq \frac{2}{N} \sum_{i=1}^N \|\bar{g}^{k,t} - f^*\|_\alpha.
\]
\[
The first term of (33) can be bounded by (note that we simply use $\hat{h}_i^{k,t}$ since $\Pi_{G^2}(\hat{h}_i^{k,t}) = \hat{h}_i^{k,t}$ on $\text{supp}(\alpha_i)$ due to Lemma D.1)
\[
\frac{2}{N} \sum_{i=1}^N \langle \bar{g}^{k,t} - \bar{g}^{k,t}, \hat{h}_i^{k,t}\rangle_\alpha \leq \frac{1}{N} \sum_{i=1}^N \|\hat{h}_i^{k,t}\|_{\alpha_i}^2 + \frac{1}{\eta^{k,t}} \|\bar{g}^{k,t} - \bar{g}^{k,t}\|_\alpha^2 \leq \eta^{k,t} 4G^2/\gamma^2 + \frac{1}{\eta^{k,t}} \|\bar{g}^{k,t} - \bar{g}^{k,t}\|_\alpha^2 = O(\frac{4G^2}{\gamma^2}) + \frac{1}{\eta^{k,t}} \frac{1}{N} \sum_{i=1}^N \|\bar{g}^{k,t} - \bar{g}^{k,t}\|_\alpha^2.
\]
The second term of (33) can be bounded by using the $\mu$-strong convexity of $F_i$:
\[
\frac{2}{N} \sum_{i=1}^N \langle f^* - g_i^{k,t}, \nabla F_i[g_i^{k,t}]\rangle_\alpha \leq -\frac{2}{N} \sum_{i=1}^N \left(\|F_i[g_i^{k,t}] - F_i[f^*]\|_\alpha - \frac{1}{\mu} \|f^* - g_i^{k,t}\|_\alpha^2\right) - \frac{2}{N} \sum_{i=1}^N \frac{\mu}{2} \|f^* - g_i^{k,t}\|_\alpha^2.
\]
For the first term above, using the optimality of $f^*$, we have

$$-\frac{2}{N} \sum_{i=1}^{N} \left( F_i[g_i^{k,t}] - F_i[f^*] \right) = -\frac{2}{N} \sum_{i=1}^{N} \left( F_i[g_i^{k,t}] - F_i[g_i^{k,t}] + F_i[g_i^{k,t}] - F_i[f^*] \right) \leq \frac{2}{N} \sum_{i=1}^{N} \left( F_i[g_i^{k,t}] - F_i[g_i^{k,t}] \right).$$

For the second term, we have

$$\| f^* - g_i^{k,t} \|_{\alpha}^2 \leq 2 \| f^* - \bar{g}_i^{k,t} \|_{\alpha}^2 + 2 \| \bar{g}_i^{k,t} - g_i^{k,t} \|_{\alpha}^2.$$

Combine the above inequality to yield

$$\frac{2}{N} \sum_{i=1}^{N} \langle f^* - g_i^{k,t}, \nabla F_i[g_i^{k,t}] \rangle_{\alpha} \leq \frac{2}{N} \sum_{i=1}^{N} \left( F_i[g_i^{k,t}] - F_i[f^*] \right) \leq \frac{2}{N} \sum_{i=1}^{N} \left( \| f^* - g_i^{k,t} \|_{\alpha} \right)^2.$$

Note that $\| \bar{g}_i^{k,t} - f^* \|_{\alpha}^2 = \sum_{\alpha=1}^{k-1} \eta^{*,\alpha} h^{*,\alpha}_{\alpha}^2$ and $\eta^{*,\alpha} \leq 2 \eta^{*,\alpha}$ for $\alpha \leq k$. Therefore, $\frac{1}{N} \sum_{\alpha=1}^{N} \| g_i^{k,t} - h_i^{k,t} \|_{\alpha}^2$, can be bounded by (we use $\| \cdot \|_{\alpha} \leq \| \cdot \|_{\infty}$ in the following)

$$\frac{1}{N} \sum_{i=1}^{N} \| g_i^{k,t} - h_i^{k,t} \|_{\alpha}^2 = \frac{1}{N} \sum_{i=1}^{N} \| g_i^{k,t} - f^* + f^* - h_i^{k,t} \|_{\alpha}^2 \leq \frac{1}{N} \sum_{i=1}^{N} 2 \| g_i^{k,t} - f^* \|_{\alpha}^2 + 2 \| f^* - h_i^{k,t} \|_{\alpha}^2 \leq 36 \sum_{\alpha=1}^{k-1} \eta^{*,\alpha}^2 G^2 / \gamma^2 \leq 144 \eta^{*,k}^2 K^2 G^2 / \gamma^2 = O\left( \frac{(\eta^{*,k})^2 K^2 G^2}{\gamma^2} \right).$$

Plug in the above results into (32) to yield (note that $h_i^{k,t} - \nabla F_i[g_i^{k,t}] = h_i^{k,t} - \nabla R_i[g_i^{k,t}]$)

$$\| \bar{g}_i^{k+1,t} - f^* \|_{\alpha}^2 \leq \left( 1 - \frac{\eta^{*,k,t}}{4} \right) \| g_i^{k,t} - f^* \|_{\alpha}^2 + O\left( \frac{(\eta^{*,k})^2 K^2 G^2}{\gamma^2} \right) + \frac{2 \eta^{*,k,t}}{N} \sum_{i=1}^{N} \langle f^* - g_i^{k,t}, h_i^{k,t} - \nabla R_i[g_i^{k,t}] \rangle_{\alpha} + \eta^{*,k,t} \delta - \frac{\mu}{2N} \sum_{i=1}^{N} \| f^* - g_i^{k,t} \|_{\alpha}^2.$$

Recall that $\eta^{*,k} = \frac{4}{\mu(K+k+1)}$ and multiply both sides by $(Kt+k+1)$

$$(Kt+k+1)\| g_i^{k+1,t} - f^* \|_{\alpha}^2 \leq (Kt+k)\| g_i^{k,t} - f^* \|_{\alpha}^2 + O\left( \frac{K^2 G^2}{\mu^2 \gamma^2 (Kt+k+1)} \right) \frac{\mu}{2N} \sum_{i=1}^{N} \| f^* - g_i^{k,t} \|_{\alpha}^2.$$
Sum from $k = 1$ to $K$

\[
(Kt + K + 1)\|g^{k+1,t} - f^*\|^2 \leq (Kt + 1)\|g^{1,t} - f^*\|^2 + O\left(\frac{K^2G^2}{\mu^2\gamma^2} \log(Kt + K + 1) - \log(Kt + 1)\right) \\
+ 4\mu N \sum_{i=1}^N \sum_{k=1}^K (f^* - g^{k,t}_i, h^{k,t}_i - \nabla R_i[g^{k,t}_i])_{\alpha_i} + \frac{2\delta K}{\mu} - \frac{2}{\mu} \sum_{i=1}^N \sum_{k=1}^K \|f^* - g^{k,t}_i\|^2_{\alpha_i}.
\]

For the first on the second line above, the following equality holds for the same reason as \[3\] (note that the equality holds due to Lemma D.1 and the fact that the inner product only depends on values on supp($\alpha_i$))

\[
\sum_{k=1}^K (f^* - g^{k,t}_i, h^{k,t}_i - \nabla R_i[g^{k,t}_i])_{\alpha_i} = \mu f^* - g_i^{k,t}_i + O(G^2(1 - \gamma)^2) + O\left(\frac{G^2}{\mu^2\gamma^2} (\log(KT + K) - \log(Kt))\right).
\]

Using this result, we obtain

\[
(K(t + 1) + 1)\|f^{t+1} - f^*\|^2_{\alpha_i} \leq (Kt + 1)\|f^t - f^*\|^2_{\alpha_i} + O\left(\frac{K^2G^2}{\mu^2\gamma^2} \log(K(t + 1) + 1) - \log(Kt + 1)\right) \\
+ O(G^2(1 - \gamma)^2) + O\left(\frac{G^2}{\mu^2\gamma^2} (\log(Kt + 1) - \log(Kt))\right) + \frac{2\delta K}{\mu}
\]

Sum from $t = 0$ to $T - 1$ and use the non-expensiveness of the clip operation to yield

\[
(KT + 1)\|f^T - f^*\|^2_{\alpha_i} \leq (Kt + 1)\|f^0 - f^*\|^2_{\alpha_i} + O\left(\frac{K^2G^2\log(KT + 1)}{\mu^2\gamma^2}\right) + O\left(\frac{(1 - \gamma)G^2\log(TK)}{\mu^2\gamma^2}\right) \\
+ O(TG^2(1 - \gamma)^2) + O\left(\frac{GTBK\omega}{\mu^2\gamma}\right),
\]

and hence

\[
\|f^T - f^*\|^2_{\alpha_i} = O\left(\frac{\|f^0 - f^*\|^2_{\alpha_i}}{KT} + \frac{KG^2\log(KT)}{T\mu^2\gamma^2} + \frac{(1 - \gamma)G^2}{K\mu^2\gamma^2} + \frac{GB\omega}{\mu\gamma}\right).
\]

$\square$

## E Proof of Theorem 3.3

We first prove \[15\].

Recall \[22\] where the Wasserstein-1 distance between two discrete distribution $\mu$ and $\nu$ can be written as

\[
W_1(\mu, \nu) = \min_{\Pi \geq 0} \int_{\mathcal{X}^2} \|x - y\|d\Pi(x, y), \quad s.t. \sharp_1\Pi = \mu, \sharp_2\Pi = \nu.
\]

Note that the constraint of the above problem implies that supp($\Pi$) \(\subseteq\) supp($\mu$) \times supp($\nu$), otherwise $\Pi$ must be infeasible. The above minimization problem is equivalent to

\[
W_1(\mu, \nu) = \min_{\Pi \geq 0} \max_{\phi, \psi} \int_{\mathcal{X}^2} \|x - y\|d\Pi(x, y) + \int_{\mathcal{X}} \phi(x)d(\mu - \sharp_1\Pi)(x) + \int_{\mathcal{X}} \psi(y)d(\nu - \sharp_2\Pi)(y).
\]

Change the order of min-max to max-min (due to convexity) and rearrange terms:

\[
W_1(\mu, \nu) = \max_{\phi, \psi} \left\{ \min_{\Pi \geq 0} \int_{\mathcal{X}^2} \|x - y\| - \phi(x) - \psi(y)d\Pi(x, y) \right\} + \int_{\mathcal{X}} \phi(x)d\mu(x) + \int_{\mathcal{X}} \psi(y)d\nu(y).
\]
Therefore, we must have that for \((x, y) \in \text{supp}(\mathcal{P}) \subseteq \text{supp}(\mu) \times \text{supp}(\nu)\), \(\phi(x) + \psi(y) \leq \|x - y\|\) which leads to

\[
W_1(\mu, \nu) = \max_{\phi, \psi} \int_X \phi(x) d\mu(x) + \int_X \psi(y) d\nu(y), \quad \text{s.t.} \phi(x) + \psi(y) \leq \|x - y\|, \forall (x, y) \in \text{supp}(\mu) \times \text{supp}(\nu).
\]

Now, recall that every local distribution \(\alpha_i\) is described by a set of data feature points: \(\alpha_i = \frac{1}{M} \sum_{j=1}^{M} \delta_{x_i,j}\), where \(\delta_\cdot\) is the Dirac distribution; and the global distribution \(\alpha\) is described by the union of all these points: \(\alpha = \frac{1}{MN} \sum_{i,j=1}^{N,M} \delta_{x_i,j}\). Clearly we have \(\text{supp}(\alpha_i) \subseteq \text{supp}(\alpha)\). Using Proposition 6.1. of Peyré et al. 2019 with \(X = \text{supp}(\alpha)\), the above bi-variable problem is equivalent to the single-variable problem

\[
W_1(\mu, \nu) = \max_{\phi} \int_X \phi(x) d\mu(x) - \int_X \phi(y) d\nu(y), \quad \text{s.t.} \|\phi(x) - \phi(y)\| \leq \|x - y\|, \forall (x, y) \in \text{supp}(\mu) \times \text{supp}(\nu).
\]

Recall that in [15], \(\phi(x) = f(x)g(x)\) and \(\xi = \|g\|_{\text{lip}} \|f\|_{\infty} + \|f\|_{\text{lip}}\|g\|_{\infty}\). One can check that

\[
\|\phi(x)/\xi - \phi(y)/\xi\| = \|f(x) (g(x) - g(y)) + (g(y)/\xi)\| \leq (\|g\|_{\text{lip}} \|f\|_{\infty} + \|f\|_{\text{lip}}\|g\|_{\infty}) / \xi \|x - y\| = \|x - y\|.
\]

Therefore

\[
W_1(\mu, \nu) \geq \int_X f(x)g(x)/\xi d\alpha(x) - \int_X f(x)g(x)/\xi d\alpha_i(y),
\]

which is just [15].

**Theorem E.1 (Theorem 3.3 restated).** Let \(f^0\) be the global initializer function. Let \(\omega = \frac{1}{N} \sum_{i=1}^{N} W_1(\alpha, \alpha_i)\) and \(G^2 = \frac{2 \sqrt{2}}{\gamma} \sum_{i=1}^{N} W_2^2(\alpha_i, \alpha_i) + 2B^2\). Consider the federated functional least square minimization problem [14]. Under Assumption 3.3 and supposing the weak learning oracle \(\text{WO}_{\alpha_i}^\omega\) satisfies [19] and [20] with constant \(\gamma\), for a certain choice of step sizes \(\eta^{k\cdot t}\), the output of FFGB.L satisfies

\[
\|f^T - f^*\|^2 = O \left( \frac{\|f^0 - f^*\|^2}{KT} + \frac{K(LD + B)\omega + G^2 + B^2}{T\mu^2\gamma^2} \log(KT) + \frac{(1 - \gamma)^2 B^2}{\mu^2\gamma^2 K} + \frac{L(LD + B)\omega}{\gamma^2\mu} \right).
\]

(41)

**Proof.** For a fixed communication round \(t\), we define a hypothetical sequence \(\tilde{g}^{k\cdot t} = \frac{1}{N} \sum_{i=1}^{N} g_i^{k\cdot t}\). We also define \(\tilde{h}^{k\cdot t} = \frac{1}{N} \sum_{i=1}^{N} h_i^{k\cdot t}\). Note that \(\tilde{g}^{1\cdot t} = f^t\).

From the construction [17], we have \(\|u_t\|_{\text{lip}} \leq L\). Additionally, with property [20] of the oracle, the residual is inductively proved to be \(\frac{(1 - \gamma) L}{\gamma}\) L-Lipschitz continuous as follows. For the base case, note that \(\|\Delta^0\|_{\text{lip}} \equiv 0\). Now, assume that for some \(k \geq 1\), we have \(\|\Delta_{k-1}\|_{\text{lip}} \leq \frac{(1 - \gamma) L}{\gamma}\). Then

\[
\|\Delta_k\|_{\text{lip}} = \|h_i^k - (u_i - \Delta_{k-1})\|_{\text{lip}} \leq (1 - \gamma)\|u_i - \Delta_{k-1}\|_{\text{lip}} \leq (1 - \gamma)\|\Delta_{k-1}\|_{\text{lip}} + L \leq \frac{(1 - \gamma) L}{\gamma}.
\]

Therefore, the query to the weak learning oracle is also Lipschitz continuous: \(\|\Delta_{k-1} - u_k\|_{\text{lip}} \leq L/\gamma\), and so is the output, \(\|h_k^k\|_{\text{lip}} \leq L/\gamma\). Now, the update rule of \(g_i^{k\cdot t}\) (line [5]), and the boundedness of \(h_i^k\) imply the boundedness of \(g_i^{k\cdot t}\) for sufficiently small \(\eta^{k\cdot t}\):

\[
\|g_i^{k+1\cdot t}\|_{\text{lip}} = \|(1 - \eta^{k\cdot t})g_i^{k\cdot t} + \eta^{k\cdot t}h_i^k\|_{\text{lip}} \leq (1 - \eta^{k\cdot t})\|g_i^{k\cdot t}\|_{\text{lip}} + L/\gamma \cdot \eta^{k\cdot t}.
\]

**Lemma E.1.** The residual \(\Delta^k_i\) and the output \(h^k_i\) of the oracle \(\text{WO}_{\alpha_i}^\omega\) are bounded under the \(L^\infty(\alpha_i)\) norm:

\[
\|\Delta^k_i\|_{\alpha_i, \infty} \leq \frac{(1 - \gamma) B}{\gamma} \quad \text{and} \quad \|h^k_i\|_{\alpha_i, \infty} \leq B/\gamma.
\]
Proof. From property (19) of the weak learner oracle \( W_{\text{O}i}^{\text{lip}} \), we have
\[
||\Delta^k||_{\alpha_i,\infty} = ||\Delta^k_{i-1} - u_i + h^k||_{\alpha_i,\infty} \le (1 - \gamma)||\Delta^k_{i-1} - u_i||_{\alpha_i,\infty} \le (1 - \gamma)||\Delta^k_{i-1}||_{\alpha_i,\infty} + (1 - \gamma)B, \tag{42}
\]
where the second inequality uses the boundedness of \( y_{i, j} = f^i_t(x_{i, j}) \) in Assumption \( \text{D} \). We hence have
\[
||\Delta^k||_{\alpha_i,\infty} - \frac{(1 - \gamma)B}{\gamma} \le (1 - \gamma)\left(||\Delta^k_{i-1}||_{\alpha_i,\infty} - \frac{(1 - \gamma)B}{\gamma}\right) \Rightarrow ||\Delta^k||_{\alpha_i,\infty} \le \frac{(1 - \gamma)B}{\gamma}. \tag{43}
\]
The boundedness of \( ||h^k||_{\alpha_i,\infty} \) can be obtained from the above inequality: \( ||h^k||_{\alpha_i,\infty} \le ||u_i||_{\alpha_i,\infty} + ||\Delta^k||_{\alpha_i,\infty} \le B/\gamma \).

**Lemma E.2.** The local variable function \( g^{k,t}_i \) is bounded under the \( L^\infty(\alpha_i) \) norm: \( ||g^{k,t}_i||_{\alpha_i,\infty} \le B/\gamma \).

Proof. Using the update rule in line 6 of Algorithm 4, we have
\[
||g^{k+1,t}_i||_{\alpha_i,\infty} = ||(1 - \eta^{k,t})g^{k,t}_i + \eta^{k,t}h^k_i||_{\alpha_i,\infty} \le (1 - \eta^{k,t})||g^{k,t}_i||_{\alpha_i,\infty} + \eta^{k,t}||h^k_i||_{\alpha_i,\infty} \le (1 - \eta^{k,t})||g^{k,t}_i||_{\alpha_i,\infty} + \eta^{k,t}B/\gamma.
\]
Inductively, we have the boundedness of \( ||g^{k+1,t}_i||_{\alpha_i,\infty} \)
\[
||g^{k+1,t}_i||_{\alpha_i,\infty} - B/\gamma \le (1 - \eta^{k,t})\left(||g^{k,t}_i||_{\alpha_i,\infty} - B/\gamma\right) \Rightarrow ||g^{k,t}_i||_{\alpha_i,\infty} \le B/\gamma. \tag{44}
\]

**Lemma E.3.** The local variable function \( g^{k,t}_i \), the global average function \( \bar{g}^{k,t} \), and the output of the oracle \( W_{\text{O}i}^{\text{lip}} \) are \((L + B)/\gamma\)-bounded under the \( L^\infty(\alpha) \) norm.

Proof. From Lemmas E.2 and E.1, \( g^{k,t}_i \), \( \bar{g}^{k,t} \), and \( h^k_i \) are \( B/\gamma \) on the support of \( \alpha_i \). Using Assumption 3.3 together with the \( L/\gamma \)-Lipschitz continuity of \( g^{k,t}_i \), \( \bar{g}^{k,t} \), and \( h^k_i \), we have the results.

While the above lemma implies the boundedness of \( \bar{g}^{k,t} \) under the \( L^2(\alpha) \) norm, we can tighten the analysis with the following lemma. Important, the following result does not depend on the constant \( D \) in Assumption \( \text{D} \).

**Lemma E.4.** The hypothetical global sequences \( \bar{g}^{k,t} \) and \( \bar{h}^{k,t} \) are bounded under the local norm \( ||\cdot||_{\alpha} \): Denote \( G_s^2 = \frac{2L^2}{\gamma^2} \sum_{i=1}^N W_2^2(\alpha_s, \alpha_i) + 2B^2 \). We have that \( ||\bar{g}^{k,t}||_{\alpha_s} \le G_s^2/\gamma^2 \) and \( ||\bar{h}^{k,t}||_{\alpha_s} \le G_s^2/\gamma^2 \), where \( W_2(\alpha_s, \alpha_i) \) is the Wasserstein-2 distance between measures \( \alpha_s \) and \( \alpha_i \). Consequently, we have \( \bar{g}^{k,t} \) and \( \bar{h}^{k,t} \) are \( G \)-bounded under the \( L^2(\alpha) \) norm, where we further denote \( G^2 = \frac{1}{N} \sum_{s=1}^N G_s^2 \).

Proof. Let \( \Pi^{s,i} \in \mathbb{R}^{M \times M} \) be the Wasserstein-2 optimal transport plan (matrix) between \( \alpha_s \) and \( \alpha_i \). The entry \( \Pi^{s,j_1}_{j_2} \) denotes the portion of mass that should be transported from \( x_{s,j_1} \in \text{supp}(\alpha_s) \) to \( x_{j_2} \in \text{supp}(\alpha_i) \). Note that in \( \alpha_i \) and \( \alpha_s \), the entries \( x_{s,j_1} \) and \( x_{i,j_2} \) have uniform weight 1/M. As a transport plan, any row or column of \( \Pi^{s,i} \) sums up to 1/M. We now show that \( ||\bar{g}^{k,t}||_{\alpha_s} \) is bounded using the Lipschitz continuity of \( g^{k,t}_i \).

\[
||\bar{g}^{k,t}||_{\alpha_s}^2 = \frac{1}{M} \sum_{j=1}^M \frac{1}{N} \sum_{i=1}^N ||g^{k,t}_i(x_{s,j})||^2 \le \frac{1}{M} \sum_{j=1}^M \frac{1}{N} \sum_{i=1}^N ||g^{k,t}_i(x_{s,j})||^2 = \frac{1}{N} \sum_{i=1}^N \frac{1}{M} \sum_{j=1}^M ||g^{k,t}_i(x_{s,j})||^2. \tag{45}
\]
We analyze the summand as follows.

\[
\frac{1}{M} \sum_{j=1}^{M} \|g_{i}^{k,t}(x_{s,j})\|^2 = \sum_{j_1=1}^{M} \sum_{j_2=1}^{M} \Pi_{j_1,j_2}^{k,t} \|g_{i}^{k,t}(x_{s,j_1}) - g_{i}^{k,t}(x_{i,j_2}) + g_{i}^{k,t}(x_{i,j_2})\|^2 \\
\leq \sum_{j_1=1}^{M} \sum_{j_2=1}^{M} \Pi_{j_1,j_2}^{k,t} \left(2\|g_{i}^{k,t}(x_{s,j_1}) - g_{i}^{k,t}(x_{i,j_2})\|^2 + 2\|g_{i}^{k,t}(x_{i,j_2})\|^2 \right) \\
\leq \sum_{j_1=1}^{M} \sum_{j_2=1}^{M} \Pi_{j_1,j_2}^{k,t} \left(2L^2/\gamma^2 \cdot \|x_{s,j_1} - x_{i,j_2}\|^2 + 2\|g_{i}^{k,t}(x_{i,j_2})\|^2 \right) \\
= 2L^2/\gamma^2 \cdot W_2^2(\alpha_s, \alpha_i) + \frac{2}{M} \sum_{j_2=1}^{M} \|g_{i}^{k,t}(x_{i,j_2})\|^2 \\
= 2L^2/\gamma^2 \cdot W_2^2(\alpha_s, \alpha_i) + 2\|g_{i}^{k,t}\|_{\alpha_i}^2,
\]

where we used the definition of the Wasserstein-2 distance. Therefore, (45) can be bounded by

\[
\|g_{i}^{k,t}\|_{\alpha_i}^2 \leq \frac{1}{N} \sum_{i=1}^{N} 2L^2/\gamma^2 \cdot W_2^2(\alpha_s, \alpha_i) + 2\|g_{i}^{k,t}\|_{\alpha_i}^2 \leq \frac{2L^2}{N\gamma^2} \sum_{i=1}^{N} W_2^2(\alpha_s, \alpha_i) + 2B^2/\gamma^2. \tag{46}
\]

Following the similar proof above, we have the same bound for \(\|\tilde{h}_{i,k,t}\|_{\alpha_s}\) as \(h_{i,k,t}\) is also \(B/\gamma\)-bounded and \(L/\gamma\)-Lipschitz continuous:

\[
\|\tilde{h}_{i,k,t}\|_{\alpha_s}^2 \leq \frac{2L^2}{N\gamma^2} \sum_{i=1}^{N} W_2^2(\alpha_s, \alpha_i) + 2B^2/\gamma^2. \tag{47}
\]

We now present the convergence analysis of Algorithm 4. From the update rule in line 5 of Algorithm 4, we write

\[
\|\tilde{g}^{k+1,t} - f^*\|_{\alpha}^2 = \|\tilde{g}^{k,t} - f^*\|_{\alpha}^2 + (\eta^{k,t})^2 \|\tilde{g}^{k,t} - \tilde{h}_{k,t}\|_{\alpha}^2 - 2\eta^{k,t}(\tilde{g}^{k,t} - f^*, \tilde{g}^{k,t} - \tilde{h}_{k,t}). \tag{48}
\]

To bound the second term, note that

\[
\|\tilde{g}^{k,t} - \tilde{h}_{k,t}\|_{\alpha}^2 \leq \frac{1}{N} \sum_{i=1}^{N} \|g_{i}^{k,t} - h_{i}^{k,t}\|_{\alpha_i}^2. \tag{49}
\]

For each individual term on the R.H.S. of the above inequality, we have

\[
\|g_{i}^{k,t} - h_{i}^{k,t}\|_{\alpha_i}^2 = \left(\|g_{i}^{k,t} - h_{i}^{k,t}\|_{\alpha_i}^2 - \|g_{i}^{k,t} - h_{i}^{k,t}\|_{\alpha_i}^2\right) + \|g_{i}^{k,t} - h_{i}^{k,t}\|_{\alpha_i}^2 \leq O(L(LD + B)/\gamma^2 \cdot W_1(\alpha, \alpha_i)) + O(B^2/\gamma^2), \tag{50}
\]

where we use the variational formulation (15) of the Wasserstein-1 distance as well as the Lipschitz continuity and boundedness of \(g_{i}^{k,t}\) and \(h_{i}^{k,t}\) under the \(L^2(\alpha)\) norm. Therefore the second term is bounded by

\[
\|\tilde{g}^{k+1,t} - f^*\|_{\alpha}^2 \leq O(L(LD + B)/\gamma^2 \cdot \omega) + O(B^2/\gamma^2), \quad \omega = \frac{1}{N} \sum_{i=1}^{N} W_1(\alpha, \alpha_i). \tag{51}
\]
The third term of (48) can be split as
\[ -2\langle \bar{g}^{k,t} - f^*, \bar{g}^{k,t} - \bar{h}_i^k \rangle_{\alpha} \]
(52)
\[ = -2 \frac{N}{N} \sum_{i=1}^{N} (g_i^{k,t} - f^*, g_i^{k,t} - h_i^k)_{\alpha} + \left( \langle g_i^{k,t} - f^*, g_i^{k,t} - h_i^k \rangle - \langle g_i^{k,t} - f^*, g_i^{k,t} - h_i^k \rangle_{\alpha} \right) \]
\[ = 2 \frac{N}{N} \sum_{i=1}^{N} (g_i^{k,t} - g_i^{k,t}, g_i^{k,t} - h_i^k)_{\alpha} + (f^* - g_i^{k,t}, g_i^{k,t} - u_i)_{\alpha} + \langle f^* - g_i^{k,t}, u_i - h_i^k \rangle_{\alpha} \]
\[ + \left( \langle g_i^{k,t} - f^*, g_i^{k,t} - h_i^k \rangle_{\alpha} - \langle g_i^{k,t} - f^*, g_i^{k,t} - h_i^k \rangle_{\alpha} \right). \] (53)
The last term of R.H.S. of the above equality can be bounded using the Lipschitz continuity and the boundedness of \((g_i^{k,t} - f^*)\) and \((g_i^{k,t} - h_i^k)\) and the variational formulation of \(W_1\) (see (15)):
\[ \frac{1}{N} \sum_{i=1}^{N} \left( \langle g_i^{k,t} - f^*, g_i^{k,t} - h_i^k \rangle_{\alpha} - \langle g_i^{k,t} - f^*, g_i^{k,t} - h_i^k \rangle_{\alpha} \right) = O(L(D + B)/\gamma^2 \cdot \omega), \quad \omega = \frac{1}{N} \sum_{i=1}^{N} W_1(\alpha, \alpha_i). \] (54)
The first term of of the R.H.S. of (53) can be bounded by
\[ \frac{2}{N} \sum_{i=1}^{N} \langle f^* - g_i^{k,t}, \nabla R_i[g_i^{k,t}] \rangle_{\alpha_i} \]
\[ \leq O(\|f^* - g_i^{k,t}\|_{\alpha_i}^2) + O(\|g_i^{k,t} - f^*\|_{\alpha_i}^2) + O(\|g_i^{k,t} - h_i^k\|_{\alpha_i}^2) \]
\[ \leq O(\|f^* - g_i^{k,t}\|_{\alpha_i}^2) + O(\|f^* - g_i^{k,t}\|_{\alpha_i}^2) + O(\|g_i^{k,t} - f^*\|_{\alpha_i}^2) \]
\[ \leq O(\|f^* - g_i^{k,t}\|_{\alpha_i}^2) + O(\|f^* - g_i^{k,t}\|_{\alpha_i}^2) \]
\[ = O\left(\frac{(\eta^{k,t})^2 K^2 G^2}{\gamma^2} \right). \]

Note that \(\|f^* - g_i^{k,t}\|_{\alpha_i}^2 = \sum_{i=1}^{k} \|f^* - g_i^{k,t}\|_{\alpha_i}^2\) can be bounded by
\[ \frac{1}{N} \sum_{i=1}^{N} \|f^* - g_i^{k,t}\|_{\alpha_i}^2 = \frac{1}{N} \sum_{i=1}^{N} \|g_i^{k,t} - f^* + f^* - g_i^{k,t}\|_{\alpha_i}^2 \]
\[ \leq \frac{1}{N} \sum_{i=1}^{N} \|g_i^{k,t} - f^*\|_{\alpha_i}^2 + \|f^* - g_i^{k,t}\|_{\alpha_i}^2 \]
\[ = O(\sum_{i=1}^{k} (\eta^{k,t})^2 G^2 / \gamma^2) = O(\eta^{k,t})^2 K^2 G^2 / \gamma^2 \]
\[ = O\left(\frac{(\eta^{k,t})^2 K^2 G^2}{\gamma^2} \right). \]

Plug in the above results into (48) to yield
\[ \|g_i^{k+1,t} - f^*\|_{\alpha_i}^2 \leq (1 - \frac{\mu\eta^{k,t}}{2}) \|g_i^{k,t} - f^*\|_{\alpha_i}^2 + O\left(\frac{(\eta^{k,t})^2 K^2 G^2}{\gamma^2} \left( L(D + B)\omega + G^2 + B^2 \right) \right) + \frac{2\eta^{k,t}}{N} \sum_{i=1}^{N} \|f^* - g_i^{k,t} - u_i - h_i^k\|_{\alpha_i} \]
\[ + O(\eta^{k,t} L(D + B)/\gamma^2 \cdot \omega) - \frac{\mu\eta^{k,t}}{2N} \sum_{i=1}^{N} \|f^* - g_i^{k,t}\|_{\alpha_i}^2. \]
Set $\eta^{k,t} = \frac{4}{\mu(K(t+1) + 1)}$ and multiply both sides by $(Kt + k + 1)$

\[
(Kt + k + 1)\|g^{k+1,t} - f^*\|^2 \leq (Kt + k)\|g^{k,t} - f^*\|^2 + O\left(\frac{K^2 (L(D + B) + G^2 + B^2)}{\mu^2 \gamma^2 (Kt + k + 1)}\right)
\]

\[
+ \frac{4}{\mu N} \sum_{i=1}^{N} (f^* - g_i^{k,t}, u_i - h_i^k)_{\alpha_i} + O \left((L(D + B) \cdot \omega/(\gamma^2 \mu))\right) - \frac{1}{N} \sum_{i=1}^{N} \|f^* - h_i^{k,t}\|^2
\]

Sum from $k = 1$ to $K$

\[
(Kt + K + 1)\|g^{K+1,t} - f^*\|^2 \leq (Kt + 1)\|g^{1,t} - f^*\|^2 + O \left(\frac{K^2 (L(D + B) + G^2 + B^2)}{\mu^2 \gamma^2} (\log(Kt + K + 1) - \log(Kt + 1))\right)
\]

\[
+ \frac{4}{\mu N} \sum_{i=1}^{N} \sum_{k=1}^{K} (f^* - g_i^{k,t}, u_i - h_i^k)_{\alpha_i} + O \left(KL(D + B) \cdot \omega/(\gamma^2 \mu)\right) - \frac{1}{N} \sum_{k=1}^{K} \sum_{i=1}^{N} \|f^* - h_i^{k,t}\|^2_{\alpha_i}
\]

We now focus on the last term (note that the equality holds due to Lemma D.1 and the fact that the inner product only depends on values on supp(\(\alpha_i\)))

\[
\sum_{k=1}^{K} (f^* - g_i^{k,t}, u_i - h_i^k)_{\alpha_i} = \sum_{k=1}^{K} (f^* - g_i^{k,t}, \Delta_i^{k-1} - \Delta_i^0)_{\alpha_i}
\]

\[
= \sum_{k=1}^{K} (f^* - g_i^{k,t}, -\Delta_i^{k})_{\alpha_i} + \sum_{k=1}^{K} (f^* - g_i^{k,t}, \Delta_i^{k-1})_{\alpha_i} + (f^* - g_i^{1,t}, \Delta_i^{0})_{\alpha_i}
\]

\[
= \sum_{k=1}^{K} (f^* - g_i^{k,t}, -\Delta_i^{k})_{\alpha_i} + \sum_{k=1}^{K-1} (f^* - g_i^{k+1,t}, \Delta_i^{k})_{\alpha_i}
\]

\[
= \sum_{k=1}^{K} (f^* - g_i^{k,t}, -\Delta_i^{k})_{\alpha_i} + \sum_{k=1}^{K-1} (f^* - g_i^{k,t}, \Delta_i^{k})_{\alpha_i} + \sum_{k=1}^{K-1} (\eta_i^{k} (g_i^{k,t} - h_i^k), \Delta_i^{k})_{\alpha_i}
\]

\[
= \langle f^* - g_i^{k,t}, -\Delta_i^{K} \rangle_{\alpha_i} + \sum_{k=1}^{K-1} \langle \eta_i^{k} (g_i^{k,t} - h_i^k), \Delta_i^{k} \rangle_{\alpha_i}
\]

\[
\leq \mu \|f^* - g_i^{k,t}\|^2_{\alpha_i} + O((1 - \gamma)^2 B^2/(\gamma^2 \mu)) + O(G^2(1 - \gamma/\gamma^2) \sum_{k=1}^{K-1} \eta_i^{k,t})
\]

\[
= \mu \|f^* - g_i^{k,t}\|^2_{\alpha_i} + O((1 - \gamma)^2 B^2/(\gamma^2 \mu)) + O\left(G^2(1 - \gamma/\gamma^2) (\log(Kt + K) - \log(Kt))\right).
\]

Using this result, we obtain

\[
(Kt + 1 + 1)\|f^{t+1} - f^*\|^2
\]

\[
\leq (Kt + 1)\|f^t - f^*\|^2 + O \left(\frac{K^2 (L(D + B) + G^2 + B^2)}{\mu^2 \gamma^2} (\log(Kt + 1) - \log(Kt + 1))\right)
\]

\[
+ \frac{2}{\mu} O((1 - \gamma)^2 B^2/(\gamma^2 \mu)) + O \left(G^2(1 - \gamma/\gamma^2) (\log(Kt + 1) - \log(Kt))\right) + O \left(KL(D + B) \cdot \omega/(\gamma^2 \mu)\right)
\]

Sum from $t = 0$ to $T - 1$ and use the non-expensiveness of the projection operation (note that $C$ is a convex
Algorithm 2 for example. Similar arguments hold for Algorithms 3 and 4.

Recall that $E$ II of Lemma 5 in [Li et al., 2019]. However, unlike the setting of full device participation, we do not have

$$f \left( \frac{L(\omega \omega + G^2 + B^2)}{\mu^2 \mu^2} \log (KT + 1) \right) + O \left( \frac{(1 - \gamma)TB^2}{\mu^2 \gamma^2} \right) + \frac{(1 - \gamma)G^2 \log (TK)}{\mu \gamma^2} + O \left( TK L^2 \cdot \omega/(\gamma^2 \mu) \right),$$

and hence

$$\|f^T - f^*\|^2 = O(\frac{\|f^0 - f^*\|^2}{KT} + \frac{K \left( L(\omega \omega + G^2 + B^2) \log (KT) \right)}{T \mu^2 \gamma^2} + \frac{(1 - \gamma)B^2}{\mu^2 \gamma^2} + \frac{L(\omega \omega + G^2)}{\gamma^2 \mu}).$$

$$\text{(55)}$$

$\Box$

F Partial Device Participation

In this section, we consider the setting of partial device participation. In the following discussion, we take Algorithm 2 for example. Similar arguments hold for Algorithms 3 and 4.

In round $t$ of Algorithm 2 we randomly sample without replacement a subset $S_t \subseteq [N]$ of clients and only compute the average of their returns to update the global variable function. We assume that all $S_t$ has the same cardinality $m$. Conceptually, we can imagine all the clients are participating in the update, but we only utilize the results in the set $S_t$.

Similar to the proof for the setting of full device participation, we define the hypothetical global average function $\tilde{g}^{k,t} = \frac{1}{N} \sum_{i=1}^{N} g_i^{k,t}$. In particular, we have $\tilde{g}^{1,t} = f^t$. From the derivation therein (see Section C), we have

$$(tK + K + 1)\|\tilde{g}^{K+1,t} - f^*\|^2 \leq (tK + 1)\|f^1 - f^*\|^2 + O\left( \frac{K^2 \gamma^2 G^2}{\gamma^2 \mu^2} \log (tK + K + 1) - \log (tK + 1) \right)$$

$$+ \frac{4}{\mu} \left( \frac{1 - \gamma}{\gamma^2} G^2 + \frac{(1 - \gamma)(2 - \gamma)}{\gamma^2} G^2 \log (tK + K) - \log (tK) \right).$$

However, unlike the setting of full device participation, we do not have $f^{t+1} = \tilde{g}^{K+1,t}$. Instead, $f^{t+1} = \frac{1}{m} \sum_{i \in S_t} g_i^{K+1,t}$. We have the following simple but useful lemma. The proof of this lemma is similar to scheme II of Lemma 5 in [Li et al., 2019].

**Lemma F.1.** $E_{S_t} \|f^{t+1} - \tilde{g}^{K+1,t}\|^2 = O\left( \frac{N - m}{N - 1} \frac{\eta^{K,t} K^2 G^2}{\mu^2 m^2} \right)$.

Moreover, $f^{t+1}$ is an unbiased estimator of $\tilde{g}^{K+1,t}$. Therefore $E_{S_t}(f^{t+1} - \tilde{g}^{K+1,t}, \tilde{g}^{K+1,t} - f^*) = 0$ and

$$E_{S_t} \|f^{t+1} - f^*\|^2 = E_{S_t} \|f^{t+1} - \tilde{g}^{K+1,t}\|^2 + \|\tilde{g}^{K+1,t} - f^*\|^2. \tag{56}$$

Recall that $\eta^{k,t} = \frac{2}{\mu(tK+k+1)}$. Combining the above results, we have

$$(tK + 1)E_{S_t} \|f^{t+1} - f^*\|^2 \leq (tK + 1)\|f^t - f^*\|^2 + O\left( \frac{K^2 G^2}{\gamma^2 \mu^2} \log (tK + K + 1) - \log (tK + 1) \right)$$

$$+ \frac{4}{\mu^2} \left( \frac{1 - \gamma}{\gamma} G^2 + \frac{(1 - \gamma)(2 - \gamma)}{\gamma^2} G^2 \log (tK + K) - \log (tK) \right)$$

$$+ O\left( \frac{N - m}{N - 1} \frac{\eta^{K,t} K^2 G^2}{\mu^m \gamma^2} \right).$$

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Sum the above results from $t = 0$ to $T$, we have

\[
(T + 1)K + 1)E\|f_{T+1}^* - f^*\|_\alpha^2 \leq \|f_0 - f^*\|_\alpha^2 + O\left(\frac{K^2G^2\log(KT)}{\gamma^2\mu^2} + O\left(\frac{G^2}{\mu^2}(1 - \gamma)^2\right)
\]

\[
+ O\Bigg(\frac{(1 - \gamma)G^2\log(TK + K)}{T\gamma^2\mu^2} + O\left(\frac{N - mG^2\log T}{N - 1}\right)\Bigg)\]

\[
\Rightarrow E\|f_{T+1}^* - f^*\|_\alpha^2 = O\left(\frac{\|f_0 - f^*\|_\alpha^2}{KT} + \frac{KG^2\log(KT)}{T\gamma^2\mu^2} + \frac{(1 - \gamma)G^2}{K\mu^2\gamma^2} + \frac{(1 - \gamma)G^2\log(KT)}{KT\mu^2\gamma^2} + \frac{N - mG^2\log T}{N - 1}\right).
\]

**Theorem F.1.** Let $f_0$ be the global initializer function. Suppose that Assumption 3.1 holds, and suppose that the weak learning oracle \(WO^\gamma\) satisfies (9) with constant $\gamma$. We pick the step size $\eta^{k,t} = \frac{2}{m(TK + k + 1)}$ and in each round the server randomly selects a subset $S_t \subseteq [N]$ without replacement with $|S_t| = m$. The output of FFGB (Algorithm 2) satisfies

\[
\|f_T - f^*\|_\alpha^2 = O\left(\frac{\|f_0 - f^*\|_\alpha^2}{KT} + \frac{G^2\log(KT)}{T\gamma^2\mu^2} + \frac{(1 - \gamma)G^2}{K\mu^2\gamma^2} + \frac{G^2\log(KT)}{KT\mu^2\gamma^2} + \frac{N - mG^2\log T}{N - 1}\right).
\]

**Theorem F.2.** Let $f_0$ be the global initializer function. Let $\omega = \frac{1}{N}\sum_{i=1}^{N} TV(\alpha_i, \alpha_i)$. Set $G^1 = \frac{1 - \gamma}{\gamma} \cdot G$ and $G^2 = \frac{2 - \gamma}{\gamma} \cdot G$. We pick the step size $\eta^{k,t} = \frac{4}{m(TK + k + 1)}$ and in each round the server randomly selects a subset $S_t \subseteq [N]$ without replacement with $|S_t| = m$. Under Assumption 3.2, and supposing the weak learning oracle \(WO^\gamma\) satisfies (13) with constant $\gamma$, the output of FFGB.C satisfies

\[
\|f_T - f^*\|_\alpha^2 = O\left(\frac{\|f_0 - f^*\|_\alpha^2}{KT} + \frac{G^2\log(KT)}{T\gamma^2\mu^2} + \frac{(1 - \gamma)G^2}{K\mu^2\gamma^2} + \frac{GB\omega}{\mu^2\gamma^2} + \frac{G^2\log(KT)\omega}{\gamma^2\mu^2\gamma^2} + \frac{N - mG^2\log T}{N - 1}\right).
\]

**Theorem F.3.** Let $f_0$ be the global initializer function. Let $\omega = \frac{1}{N}\sum_{i=1}^{N} W_1(\alpha_i, \alpha_i)$ and $G^2 = \frac{2L^2}{N^2}\sum_{i=1}^{N} W_2(\alpha_i, \alpha_i) + 2B^2$. Consider the federated functional least square minimization problem (14). We pick the step size $\eta^{k,t} = \frac{4}{m(TK + k + 1)}$ and in each round the server randomly selects a subset $S_t \subseteq [N]$ without replacement with $|S_t| = m$. Under Assumption 3.3, and supposing the weak learning oracle \(WO^\gamma\) satisfies (19) and (20) with constant $\gamma$, the output of FFGB.L satisfies

\[
\|f_T - f^*\|_\alpha^2 = O\left(\frac{\|f_0 - f^*\|_\alpha^2}{KT} + \frac{K\left(L(D + B)\omega + G^2 + B^2\right)\log(KT)}{T\gamma^2\mu} + \frac{(1 - \gamma)B^2}{\mu^2\gamma^2K} + \frac{L(D + B)\omega}{\gamma^2\mu} + \frac{N - mG^2\log T}{N - 1}\right).
\]