Invariant weakly positive semidefinite kernels with values in topologically ordered ∗-spaces

by

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Abstract. We consider weakly positive semidefinite kernels valued in ordered ∗-spaces with or without certain topological properties, and investigate their linearisations (Kolmogorov decompositions) as well as their reproducing kernel spaces. The spaces of realisations are of VE (Vector Euclidean) or VH (Vector Hilbert) type, more precisely, vector spaces that possess gramians (vector valued inner products). The main results refer to the case when the kernels are invariant under certain actions of ∗-semigroups and show under which conditions ∗-representations on VE-spaces, or VH-spaces in the topological case, can be obtained. Finally, we show that these results unify most of dilation type results for invariant positive semidefinite kernels with operator values as well as recent results on positive semidefinite maps on ∗-semigroups with values operators from a locally bounded topological vector space to its conjugate Z-dual space, for Z an ordered ∗-space.

1. Introduction. Dilation theory, initiated in the seminal articles of M. A. Naĭmark [22], [23], consists today of an extraordinarily large diversity of results that may look, at first glance, as having next to nothing in common; see e.g. N. Aronszajn [1], W. B. Arveson [2], S. D. Barreto et al. [5], D. Gašpar and P. Gašpar [10], [9], A. Gheondea and B. E. Uğurcan [12], J. Görniak and A. Weron [13], [14], J. Heo [15], G. G. Kasparov [16], R. M. Loynes [18], G. J. Murphy [21], M. Skeide [28], W. F. Stinespring [29], F. H. Szafraniec [30], [31], B. Sz.-Nagy [32], to cite but a few.

In a series of recent articles [11], [3], and [4], a unification of this theory under some general results on operator valued positive semidefinite kernels that are invariant under actions of ∗-semigroups has been initiated. Histori-
cally, making use of some classical results on scalar kernels of J. Mercer [20] and A. N. Kolmogorov [17], positive semidefinite kernels invariant under group actions were used more than forty years ago in mathematical models of quantum physics by D. E. Evans and J. T. Lewis [8] and in probability theory by K. R. Parthasarathy and K. Schmidt [24] and turned out to be successful even beyond positive semidefiniteness, as in [7].

On the other hand, positive semidefiniteness of a scalar valued kernel $k : X \times X \to \mathbb{C}$, defined as

$$\sum_{i,j=1}^{n} \alpha_i \alpha_j k(x_i, x_j) \geq 0, \quad n \in \mathbb{N}, \alpha_1, \ldots, \alpha_n \in \mathbb{C}, \ x_1, \ldots, x_n \in X,$$

has many different ways of generalisation when it comes to operator valued kernels, and consequently the diversity of dilation results increases considerably. From the point of view of unification of dilation theory there is a challenge: are there a concept of positive semidefiniteness and a concept of vector space where these kernels take values, that can yield dilation theorems that are sufficiently general to contain all of (or most of) the dilation theorems for operator valued kernels or maps? Clearly, such a concept of “weak” positive semidefiniteness must refer simply to the bare situation as in (1.1), while the concept of vector space should be an ordered $\ast$-space, and consequently the spaces of dilation that we expect should be, in the nontopological case, of VE (Vector Euclidean) type or, in the topological case, of VH (Vector Hilbert) type, in the sense of R. M. Loynes [18], [19]. So far, “weak” positive semidefiniteness have been rarely considered, e.g. W. L. Paschke [25] has a remark on maps on $C^\ast$-algebras and, for the special purposes of reproducing kernel spaces, it was first considered in [9] and then used in [26] as well.

The aim of this article is to develop a systematic study of invariant weakly positive semidefinite kernels with values in ordered $\ast$-spaces and to show that most of the previous dilation results as in [11], [3], [4], and hence most of the known dilation theory, can be recovered in this setting. The main results are Theorems 4.3, 4.6, and 4.7 from which we then derive special cases concerning different kinds of “stronger” positive semidefiniteness. Of course, since these dilation theorems are so general, in each particular case we expect some additional technical difficulties, but here the main idea is of unification and that is the price to pay.

In the following we briefly present the contents of this article. In Section 2 we briefly recall the general terminology of ordered $\ast$-spaces and their topological versions, of VE-spaces and VH-spaces and their operator theory. In Lemma 2.2 we prove a surrogate of the Schwarz inequality which turns out to be useful. This inequality with constant 2 has been claimed in [6], and
stated without proof or reference in [26], but since the proofs we have seen until now turned out to be flawed, we give a detailed proof with constant 4.

Section 3 contains a detailed study of linearisations and reproducing kernel spaces associated to weakly positive semidefinite kernels on VE-spaces or VH-spaces. Since the geometry of these spaces is so badly behaved, a careful treatment is necessary from the point of view of minimality and of the equivalence of linearisation with reproducing kernel space. The main results are contained in Section 4. In Theorem 4.3 we obtain the nontopological fabric of dilation theorems for invariant weakly positive semidefinite kernels with values in ordered ∗-spaces and then we obtain two topological versions, Theorem 4.6 for bounded operators and Theorem 4.7 for continuously adjointable operators. As expected, both these variants refer to a variant of the B. Sz.-Nagy boundedness condition, but it is interesting to observe that a second boundedness condition which refers to an anomaly of operator theory for continuously adjointable operators on VH-spaces related to the continuity of the adjoint (see condition (c) in Theorem 5.3), does not show up.

Finally, in Section 5 we show that the main theorems contain the dilation results obtained in [11], [3], and [4], and hence most of dilation theory, by explicitly showing how to set the stage in each case. A special observation is that for the reproducing kernel space versions, which is one of the main tool we use, there are some technical difficulties related to missing a version of Riesz’s Representation Theorem in VH-spaces or VE-spaces and which is solved by carefully using identifications. In addition, we show how the recent results of F. Pater and T. Binzar [26] on positive semidefinite maps on ∗-semigroups with values operators from a vector space to its conjugate Z-dual space, for Z an ordered ∗-space, that generalise previous results of J. Górniak and A. Weron [14], can be recovered by our main results, with actually stronger statements.

2. Notation and preliminary results. In this section we review some of the definitions and theorems on ordered ∗-spaces, topologically ordered ∗-spaces, admissible spaces, VE-spaces, topologically VE-spaces and VH-spaces, and their operator theory (see R. M. Loynes [18], [19]; for a modern treatment of the subject and some proofs, we also refer to [3] and [4]).

2.1. Topologically ordered ∗-spaces. A complex vector space Z is called an ordered ∗-space if:

(a1) Z has an involution *, that is, a map Z ⊆ z → z* ∈ Z that is conjugate linear ((sx + ty)* = sx* + ty* for all s, t ∈ C and all x, y ∈ Z) and involutive ((z*)* = z for all z ∈ Z).

(a2) In Z there is a convex cone Z_+ (sx + ty ∈ Z_+ for all numbers s, t ≥ 0
and all \(x, y \in Z_+\) that is strict \((Z_+ \cap -Z_+ = \{0\})\), and consists of selfadjoint elements only \((z^* = z\) for all \(z \in Z_+)\). This cone is used to define a partial order in \(Z\): \(z_1 \geq z_2\) if \(z_1 - z_2 \in Z_+\).

The complex vector space \(Z\) is called a topologically ordered \(*\)-space if it is an ordered \(*\)-space and:

(a3) \(Z\) is a Hausdorff separated locally convex space.
(a4) The cone \(Z_+\) is closed with respect to this topology.
(a5) The topology of \(Z\) is compatible with the partial ordering in the sense that there exists a base of the topology, linearly generated by a family \(\{N_j\}_{j \in J}\) of neighbourhoods of the origin that are all absolutely convex and solid, that is, whenever \(x \in N_j\) and \(0 \leq y \leq x\) then \(y \in N_j\).

It can be proven that axiom (a5) is equivalent to the following \([4]\):

(a5') There exists a collection \(\{p_j\}_{j \in J}\) of seminorms defining the topology of \(Z\) that are increasing, that is, \(0 \leq x \leq y\) implies \(p_j(x) \leq p_j(y)\).

We denote by \(S(Z)\) the collection of all increasing continuous seminorms on \(Z\). The space \(Z\) is called admissible if, in addition to (a1)–(a5),

(a6) The topology on \(Z\) is complete.

2.2. Vector Euclidean spaces and their linear operators. Given a complex linear space \(E\) and an ordered \(*\)-space \(Z\), a \(Z\)-valued inner product or \(Z\)-gramian is, by definition, a mapping \(E \times E \ni (x, y) \mapsto [x, y] \in Z\) with the following properties:

(ve1) \([x, x] \geq 0\) for all \(x \in E\), and \([x, x] = 0\) if and only if \(x = 0\).
(ve2) \([x, y] = [y, x]^*\) for all \(x, y \in E\).
(ve3) \([x, ay_1 + by_2] = a[x, y_1] + b[x, y_2]\) for all \(a, b \in \mathbb{C}\) and all \(x_1, x_2 \in E\).

A complex linear space \(E\) on which a \(Z\)-valued inner product \([\cdot, \cdot]\) is specified, for a certain ordered \(*\)-space \(Z\), is called a VE-space (Vector Euclidean space) over \(Z\).

In any VE-space \(E\) over \(Z\) the familiar polarisation formula

\[
(2.1) \quad 4[x, y] = \sum_{k=0}^{3} i^k [(x + i^k y, x + i^k y)], \quad x, y \in E,
\]

holds, which shows that the \(Z\)-valued inner product is completely determined by the \(Z\)-valued quadratic map \(E \ni x \mapsto [x, x] \in Z\).

The concept of VE-space isomorphism is also naturally defined: this is just a linear bijection \(U: E \rightarrow F\), for two VE-spaces over the same ordered \(*\)-space \(Z\), which is isometric, that is, \([Ux, Uy]_F = [x, y]_E\) for all \(x, y \in E\).

The following lemma is useful for the constructions in this paper.
Lemma 2.1 (Loynes [18]). Let $Z$ be an ordered $\ast$-space, $\mathcal{E}$ a complex vector space and $[\cdot,\cdot]: \mathcal{E} \times \mathcal{E} \to Z$ a positive semidefinite sesquilinear map, that is, $[\cdot,\cdot]$ is linear in the second variable, conjugate linear in the first variable, and $[x,x] \geq 0$ for all $x \in \mathcal{E}$. If $f \in \mathcal{E}$ is such that $[f,f] = 0$, then $[f,f'] = [f',f] = 0$ for all $f' \in \mathcal{E}$.

Given two VE-spaces $E$ and $F$, over the same ordered $\ast$-space $Z$, one can consider the vector space $L(E,F)$ of all linear operators $T: E \to F$. The operator $T$ is called bounded if there exists $C \geq 0$ such that

$$[Te,Te]_F \leq C^2[e,e]_E, \quad e \in \mathcal{E}. \tag{2.2}$$

Note that the inequality (2.2) is in the sense of the order of $Z$ uniquely determined by the cone $Z_+$ (see axiom (a2)). The infimum of the scalars $C$ is denoted by $\|T\|$ and it is called the operator norm of $T$:

$$\|T\| = \inf\{C > 0 \mid [Te,Te]_F \leq C^2[e,e]_E \text{ for all } e \in \mathcal{E}\}. \tag{2.3}$$

Let $\mathcal{B}(\mathcal{E}, \mathcal{F})$ denote the collection of all bounded linear operators $T: \mathcal{E} \to \mathcal{F}$. Then $\mathcal{B}(\mathcal{E}, \mathcal{F})$ is a linear space and $\| \cdot \|$ is a norm on it [19, Theorem 1]. In addition, if $T$ and $S$ are bounded linear operators acting between appropriate VE-spaces over the same ordered $\ast$-space $Z$, then $\|TS\| \leq \|T\| \|S\|$, in particular $TS$ is bounded. If $\mathcal{E} = \mathcal{F}$ then $\mathcal{B}(\mathcal{E}) := \mathcal{B}(\mathcal{E}, \mathcal{E})$ is a normed algebra, that is, the operator norm is submultiplicative.

A linear operator $T \in L(\mathcal{E}, F)$ is called adjointable if there exists $T^* \in L(F, \mathcal{E})$ such that

$$[Te,f]_F = [e,T^*f]_\mathcal{E}, \quad e \in \mathcal{E}, \ f \in \mathcal{F}. \tag{2.4}$$

The operator $T^*$, if it exists, is uniquely determined by $T$ and called its adjoint. Since there is no analog of the Riesz Representation Theorem for VE-spaces, in general not all operators are adjointable. We denote by $L^*(\mathcal{E}, F)$ the vector space of all adjointable operators from $L(\mathcal{E}, F)$. Note that $L^*(\mathcal{E}) := L^*(\mathcal{E}, \mathcal{E})$ is a $\ast$-algebra with respect to taking the adjoint.

An operator $A \in L(\mathcal{E})$ is called selfadjoint if

$$[Ae,f] = [e, Af], \quad e, f \in \mathcal{E}. \tag{2.5}$$

Clearly, any selfadjoint operator $A$ is adjointable and $A = A^*$. By the polarisation formula (2.1), $A$ is selfadjoint if and only if

$$[Ae,e] = [e, Ae], \quad e \in \mathcal{E}. \tag{2.6}$$

An operator $A \in L(\mathcal{E})$ is positive if

$$[Ae,e] \geq 0, \quad e \in \mathcal{E}. \tag{2.7}$$

Since the cone $Z_+$ consists of selfadjoint elements only, any positive operator is selfadjoint and hence adjointable.

Let $\mathcal{B}^*(\mathcal{E})$ denote the collection of all adjointable bounded linear operators $T: \mathcal{E} \to \mathcal{E}$. Then $\mathcal{B}^*(\mathcal{E})$ is a pre-$C^*$-algebra, that is, it is a normed
$\ast$-algebra with the property
\begin{equation}
\|A^*A\| = \|A\|^2, \quad A \in B^*(\mathcal{E})
\end{equation}
(see [19, Theorem 4]). In particular, the involution $\ast$ is isometric on $B^*(\mathcal{E})$, that is, $\|A^*\| = \|A\|$ for all $A \in B^*(\mathcal{E})$.

If $A \in B^*(\mathcal{E})$ can be factored as $A = T^*T$ for some $T \in B^*(\mathcal{E})$, then $A$ is positive. If, in addition, $B^*(\mathcal{E})$ is complete, and hence a $C^*$-algebra, and $A \in B^*(\mathcal{E})$ is positive, then $A = T^*T$ for some $T \in B^*(\mathcal{E})$ [19, Lemma 2].

2.3. Vector Hilbert spaces and their linear operators. If $Z$ is a topologically ordered $\ast$-space, any VE-space $\mathcal{E}$ can be made in a natural way into a Hausdorff separated locally convex space by considering the weakest locally convex topology on $\mathcal{E}$ that makes the mapping $E \ni h \mapsto [h,h] \in Z$ continuous, more precisely, if $\{N_j\}_{j \in \mathcal{J}}$ is the collection of convex and solid neighbourhoods of the origin in $Z$ as in axiom (a5), then the collection of sets
\begin{equation}
U_j = \{x \in \mathcal{E} \mid [x,x] \in N_j\}, \quad j \in \mathcal{J},
\end{equation}
is a base of neighbourhoods of the origin that linearly generates the weakest locally convex topology on $\mathcal{E}$ that makes all mappings $\mathcal{E} \ni h \mapsto [h,h] \in Z$ continuous [18, Theorem 1]. In terms of seminorms, this topology can be defined in the following way: Let $\{p_j\}_{j \in \mathcal{J}}$ be a family of increasing seminorms defining the topology of $Z$ as in axiom (a5') and let
\begin{equation}
\tilde{p}_j(h) = p_j([h,h])^{1/2}, \quad h \in \mathcal{E}, \quad j \in \mathcal{J}.
\end{equation}
Then each $\tilde{p}_j$ is a seminorm on $\mathcal{E}$ and the topology of $\mathcal{E}$ is fully determined by the family $\{\tilde{p}_j\}_{j \in \mathcal{J}}$ [4, Lemma 1.3]. With this topology, $\mathcal{E}$ is a topological VE-space over $Z$.

We first prove a surrogate of the Schwarz inequality that we will use several times.

**Lemma 2.2.** Let $\mathcal{E}$ be a topological VE-space over the topologically ordered $\ast$-space $Z$ and let $p \in S(Z)$. Then
\begin{equation}
p([e,f]) \leq 4p([e,e])^{1/2}p([f,f])^{1/2} = 4\tilde{p}(e)\tilde{p}(f), \quad e, f \in \mathcal{E}.
\end{equation}

**Proof.** For any $h, k \in \mathcal{E}$ we have
\[ [h \pm k, h \pm k] = [h, h] + [k, k] \pm [h, k] \pm [k, h] \geq 0, \]
in particular
\[ [h, k] + [k, h] \leq [h, h] + [k, k], \]
and
\begin{equation}
0 \leq [h + k, h + k] \leq [h - k, h - k] + [h + k, h + k] = 2([h, h] + [k, k]).
\end{equation}
Since \( p \in S(Z) \) is increasing, from (2.12) it follows that

(2.13) \[ p([h + k, h + k]) \leq 2(p([h, h]) + p([k, k])). \]

Let now \( e, f \in \mathcal{E} \). By the polarisation formula (2.1) and (2.13),

\[
p([e, f]) = p\left(\frac{1}{4} \sum_{k=0}^{3} i^k[e + i^k f, e + i^k f]\right) \leq \frac{1}{4} \sum_{k=0}^{3} p([e + i^k f, e + i^k f])
\]

\[
\leq \frac{2}{4} \sum_{k=0}^{3} (p([e, e]) + p([i^k f, i^k f])) = 2(p([e, e]) + p([f, f])).
\]

Picking \( \lambda > 0 \) and replacing \( e \) with \( \sqrt{\lambda} e \) and \( f \) with \( f/\sqrt{\lambda} \) in the previous inequality, we get

\[
p([e, f]) \leq 2(\lambda p([e, e]) + \lambda^{-1} p([f, f]));
\]

since the left hand side does not depend on \( \lambda \), it follows that

\[
p([e, f]) \leq \inf_{\lambda > 0} 2(\lambda p([e, e]) + \lambda^{-1} p([f, f])) = 4p([e, e])^{1/2}p([f, f])^{1/2}. \]

Let \( \mathcal{E} \) and \( \mathcal{F} \) be topological VE-spaces over the same topologically ordered \(*\)-space \( Z \). Clearly, any bounded linear operator \( T : \mathcal{E} \to \mathcal{F} \), with norm defined as in (2.3), is continuous.

If both \( Z \) and \( \mathcal{E} \) are complete in their locally convex topologies, then \( \mathcal{E} \) is called a VH-space (Vector Hilbert space). Any topological VE-space \( \mathcal{E} \) on a topologically ordered \(*\)-space can be embedded as a dense subspace of a VH-space \( \mathcal{H} \), uniquely determined up to isomorphism [18, Theorem 2]. Note that, given VH-spaces \( \mathcal{H} \) and \( \mathcal{K} \) over the same admissible space \( Z \), any isomorphism \( U : \mathcal{H} \to \mathcal{K} \) in the sense of VE-spaces is automatically bounded and adjointable, hence \( U \in B(\mathcal{H}, \mathcal{K}) \), and it is natural to call such an operator unitary.

If \( \mathcal{E} \) and \( \mathcal{F} \) are topological VE-spaces over the same admissible space \( Z \) and \( \mathcal{F} \) is complete, that is, a VH-space, then \( B(\mathcal{E}, \mathcal{F}) \) is a Banach space with the operator norm. In particular, if \( \mathcal{E} \) is a VH-space, then \( B^*(\mathcal{E}) \) is a \( C^* \)-algebra. Note that, in this case, the usual notion of \( C^* \)-algebra positive elements in \( B^*(\mathcal{E}) \) coincides with that of positive operators in the sense of (2.7) [18].

For topological VE-spaces \( \mathcal{E} \) and \( \mathcal{F} \) over the same topologically ordered \(*\)-space \( Z \), we denote by \( \mathcal{L}_c(\mathcal{E}, \mathcal{F}) \) the space of all continuous linear operators \( T : \mathcal{E} \to \mathcal{F} \), and \( \mathcal{L}_c(\mathcal{E}, \mathcal{E}) := \mathcal{L}_c(\mathcal{E}) \). The \(*\)-algebra of all continuous and continuously adjointable linear operators \( T : \mathcal{E} \to \mathcal{F} \) is denoted by \( \mathcal{L}_c^*(\mathcal{E}, \mathcal{F}) \), and \( \mathcal{L}_c^*(\mathcal{E}) := \mathcal{L}_c^*(\mathcal{E}, \mathcal{E}) \) [4].

A subspace \( \mathcal{M} \) of a VH-space \( \mathcal{H} \) is orthocomplemented or accessible if every element \( x \in \mathcal{H} \) can be written as \( x = y + z \) where \( y \in \mathcal{M} \) and \( z \) is
such that \( [z,m] = 0 \) for all \( m \in \mathcal{M} \), that is, \( z \) is in the orthogonal companion \( \mathcal{M}^\perp \) of \( \mathcal{M} \). If such a decomposition exists, it is unique. Also, any orthocomplemented subspace is closed. A closed subspace \( \mathcal{M} \) of \( \mathcal{H} \) is orthocomplemented if and only if it is the range of a selfadjoint projection, that is, an adjointable linear operator \( P: \mathcal{H} \to \mathcal{H} \) such that \( P^2 = P = P^* \). Note that any selfadjoint projection is a contraction, in particular it is bounded.

3. Weakly positive semidefinite kernels

3.1. Hermitian kernels. Let \( X \) be a nonempty set and \( Z \) an ordered \(*\)-space. A map \( k: X \times X \to Z \) is called a \( Z \)-valued kernel on \( X \). If no confusion can arise, we simply say that \( k \) is a kernel. The adjoint kernel \( k^*: X \times X \to Z \) is defined by \( k^*(x,y) = k(y,x)^* \) for \( x,y \in X \). The kernel \( k \) is Hermitian if \( k^* = k \).

Consider \( \mathbb{C}^X \), the complex vector space of all functions \( f: X \to \mathbb{C} \), as well as its subspace \( \mathbb{C}_0^X \) consisting of functions with finite support. Given a \( Z \)-valued kernel \( k \) on \( X \), a pairing \( \langle \cdot, \cdot \rangle_k: \mathbb{C}_0^X \times \mathbb{C}_0^X \to Z \) can be defined by

\[
\langle f, g \rangle_k = \sum_{x,y \in X} \overline{f(x)}g(y)k(x,y), \quad f, g \in \mathbb{C}_0^X.
\]

This pairing is linear in the second variable and conjugate linear in the first variable. If, in addition, \( k = k^* \), then \( \langle \cdot, \cdot \rangle_k \) is Hermitian:

\[
\langle f, g \rangle_k = \langle g, f \rangle_{k^*}, \quad f, g \in \mathbb{C}_0^X.
\]

Conversely, if \( \langle \cdot, \cdot \rangle_k \) is Hermitian then \( k = k^* \).

A convolution operator \( K: \mathbb{C}_0^X \to \mathbb{Z}^X \), where \( \mathbb{Z}^X \) is the complex vector space of all functions \( g: X \to \mathbb{Z} \), can be associated to the \( Z \)-kernel \( k \) by

\[
(Kg)(x) = \sum_{y \in X} g(y)k(x,y), \quad f \in \mathbb{C}_0^X.
\]

Clearly, \( K \) is a linear operator. There is a natural relation between the pairing \( \langle \cdot, \cdot \rangle_k \) and the convolution operator \( K \):

\[
\langle f, g \rangle_k = \sum_{x \in X} \overline{f(x)}(Kg)(x), \quad f, g \in \mathbb{C}_0^X.
\]

Hence, it is easy to see that the kernel \( k \) is Hermitian if and only if the pairing \( \langle \cdot, \cdot \rangle_k \) is Hermitian.

Given a natural number \( n \), a \( Z \)-valued kernel \( k \) is called weakly \( n \)-positive if for all \( x_1, \ldots, x_n \in X \) and all \( t_1, \ldots, t_n \in \mathbb{C} \) we have

\[
\sum_{j,k=1}^n \overline{t_k}t_j k(x_k, x_j) \geq 0.
\]

The kernel \( k \) is weakly positive semidefinite if it is weakly \( n \)-positive for all \( n \in \mathbb{N} \).
Lemma 3.1. Let $k$ be a weakly 2-positive $Z$-kernel on $X$. Then:

1. $k$ is Hermitian.
2. If, for some $x \in X$, $k(x, x) = 0$, then $k(x, y) = 0$ for all $y \in X$.
3. There exists a unique decomposition $X = X_0 \cup X_1$ with $X_0 \cap X_1 = \emptyset$ such that $k(x, y) = 0$ for all $x, y \in X_0$ and $k(x, x) \neq 0$ for all $x \in X_1$.

Proof. (1) Clearly, weak 2-positivity implies weak 1-positivity: $k(x, x) \geq 0$ for all $x \in X$. Let $x, y \in X$. Since $k$ is weakly 2-positive, for any $s, t \in \mathbb{C}$ we have

$$|s|^2k(x, x) + |t|^2k(y, y) + \bar{s}t k(x, y) + s\bar{t}k(y, x) \geq 0.$$  \hfill (3.6)

Since the sum of the first two terms in (3.6) is in $Z_+$, and $Z_+$ consists of selfadjoint elements only, it follows that the sum of the last two terms in (3.6) is selfadjoint, that is,

$$\bar{s}t k(x, y) + s\bar{t}k(y, x) = \bar{t}s k(x, y)^* + \bar{s}t k(y, x)^*.$$  

Letting $s = t = 1$ and then $s = 1$ and $t = i$, it follows that $k(y, x) = k(x, y)^*$. (2) Assume that $k(x, x) = 0$ and let $y \in X$. From (3.6) it follows that for all $s, t \in \mathbb{C}$ we have

$$\bar{s}t k(x, y) + s\bar{t}k(y, x) \geq -|t|^2k(y, y).$$  \hfill (3.7)

We claim that for all $s, t \in \mathbb{C}$ we have

$$\bar{s}t k(x, y) + s\bar{t}k(y, x) = 0.$$  \hfill (3.8)

To prove this, note that (3.8) is trivially true for $t = 0$. If $t \in \mathbb{C} \setminus \{0\}$, we distinguish two cases: if $k(y, y) = 0$, then from (3.7) it follows that $\bar{s}t k(x, y) + s\bar{t}k(y, x) \geq 0$; changing $t$ to $-t$, we see that the opposite inequality holds; hence we have (3.8). If $k(y, y) \neq 0$, we observe that the right hand side in (3.7) does not depend on $s$; hence, a routine reasoning in which $s$ is replaced by $ns$, $n \in \mathbb{Z}$, shows that (3.8) must hold as well.

Finally, in (3.8) we first let $s = 1 = t$ and then $s = 1$ and $t = i$ and solve for $k(x, y)$, which turns out to be 0.

3. Denote $X_0 = \{x \in X \mid k(x, x) = 0\}$ and let $X_1 = X \setminus X_0$. Then use (2) in order to obtain $k(x, y) = 0$ for all $x, y \in X_0$. \hfill $\blacksquare$

3.2. Weak linearisations. Given an ordered $\ast$-space $Z$ and a $Z$-valued kernel $k$ on a nonempty set $X$, a weak VE-space linearisation, or weak Kolmogorov decomposition of $k$, is, by definition, a pair $(E; V)$ subject to the following conditions:

1. $E$ is a VE-space over $Z$.
2. $V : X \to E$ satisfies $k(x, y) = [V(x), V(y)]_E$ for all $x, y \in X$. 

If, in addition,
(vel3) \( \text{Lin } V(X) = \mathcal{E} \),
then the weak VE-space linearisation \((\mathcal{E}; V)\) is called minimal.

Two weak VE-space linearisations \((V; \mathcal{E})\) and \((V'; \mathcal{E}')\) of the same kernel \(k\) are called unitarily equivalent if there exists a unitary operator \(U : \mathcal{E} \to \mathcal{E}'\) such that \(UV(x) = V'(x)\) for all \(x \in X\).

Remarks 3.2. (a) Note that any two minimal weak VE-space linearisations \((E; V)\) and \((E'; V')\) of the same \(Z\)-kernel \(k\) are unitarily equivalent. The proof is standard: if \((E'; V')\) is another minimal weak VE-space linearisation of \(k\), then for any \(x_1, \ldots, x_m, y_1, \ldots, y_n \in X\) and \(t_1, \ldots, t_m, s_1, \ldots, s_n \in \mathbb{C}\), we have

\[
\left[ \sum_{j=1}^{m} t_j V(x_j), \sum_{k=1}^{n} s_k V(y_k) \right]_{\mathcal{E}}
= \sum_{j=1}^{m} \sum_{k=1}^{n} s_k \overline{t_j} [V(x_j), V(y_k)]_{\mathcal{E}} = \sum_{j=1}^{m} \sum_{k=1}^{n} s_k \overline{t_j} k(x_j, y_k)
= \sum_{j=1}^{m} \sum_{k=1}^{n} s_k \overline{t_j} [V'(x_j), V'(y_k)]_{\mathcal{E}'} = \left[ \sum_{j=1}^{m} t_j V'(x_j), \sum_{k=1}^{n} s_k V'(y_k) \right]_{\mathcal{E}'}
\]

hence \(U : \text{Lin } V(X) \to \text{Lin } V'(X)\) defined by

(3.9) \( \sum_{j=1}^{m} t_j V(x_j) \mapsto \sum_{j=1}^{m} t_j V'(x_j)\), \(x_1, \ldots, x_m \in X, t_1, \ldots, t_m \in \mathbb{C}, m \in \mathbb{N}\),

is a correctly defined linear operator, isometric, everywhere defined, and onto. Thus, \(U\) is a VE-space isomorphism \(U : \mathcal{E} \to \mathcal{E}'\) and \(UV(x) = V'(x)\) for all \(x \in X\), by construction.

(b) From any weak VE-space linearisation \((\mathcal{E}; V)\) of \(k\) one can make a minimal one, \((\mathcal{E}_0; V_0)\), in a canonical way: let \(\mathcal{E}_0 = \text{Lin } V(X)\) and define \(V_0 : X \to \mathcal{E}_0\) by \(V_0(x) = V(x)\) for \(x \in X\).

Let us assume now that \(Z\) is an admissible space and \(k\) is a \(Z\)-kernel on a set \(X\). A weak VH-space linearisation of \(k\) is a linearisation \((\mathcal{H}; V)\) of \(k\) such that \(\mathcal{H}\) is a VH-space. The weak VH-space linearisation \((\mathcal{H}; V)\) is called topologically minimal if

(vhl3) \(\text{Lin } V(X)\) is dense in \(\mathcal{H}\).

Two weak VH-space linearisations \((\mathcal{H}; V)\) and \((\mathcal{H}'; V')\) of the same \(Z\)-kernel \(k\) are called unitarily equivalent if there exists a unitary operator \(U \in \mathcal{B}^*(\mathcal{H}, \mathcal{H}')\) such that \(UV(x) = V'(x)\) for all \(x \in X\).
**Remarks 3.3.** (a) Any two topologically minimal weak VH-space linearisations of the same Z-kernel are unitarily equivalent. Indeed, letting \((\mathcal{H}; V)\) and \((\mathcal{H}'; V')\) be minimal weak VH-space linearisations of the Z-kernel \(k\), we proceed as in Remark 3.2(a) and define \(U : \text{Lin} V(X) \to \text{Lin} V'(X)\) as in (3.9). Since \(U\) is isometric, it is bounded in the sense of (2.2), hence continuous, and then \(U\) can be uniquely extended to an isometric operator \(U : \mathcal{H} \to \mathcal{H}'\). Since \(\text{Lin} V'(X)\) is dense in \(\mathcal{H}'\) and \(U\) has closed range, it follows that \(U\) is surjective, hence \(U \in B^*(\mathcal{H}, \mathcal{H}')\) is unitary and, by (3.9), we have \(UV(x) = V'(x)\) for all \(x \in X\).

(b) From any weak VH-space linearisation \((\mathcal{H}; V)\) of \(k\) one can make, in a canonical way, a topologically minimal weak VH-space linearisation \((\mathcal{H}_0; V_0)\) by letting \(\mathcal{H}_0 = \text{Lin} V(X)\) and \(V_0(x) = V(x)\) for all \(x \in X\).

**Theorem 3.4.** (a) Given an ordered \(*\)-space \(Z\) and a \(Z\)-valued kernel \(k\) on a nonempty set \(X\), the following assertions are equivalent:

1. \(k\) is positive semidefinite.
2. \(k\) admits a weak VE-space linearisation \((\mathcal{E}; V)\).

Moreover, if a weak VE-space linearisation \((\mathcal{E}; V)\) exists, it can always be chosen such that \(\mathcal{E} \subseteq Z^X\), that is, consisting of functions \(f : X \to Z\) only, and minimal.

(b) If, in addition, \(Z\) is an admissible space and \(k : X \times X \to Z\) is a kernel, then (1) and (2) are each equivalent to

3. \(k\) admits a weak VH-space linearisation \((\mathcal{H}; V)\).

Moreover, if a weak VH-space linearisation \((\mathcal{H}; V)\) exists, it can always be chosen such that \(\mathcal{H} \subseteq Z^X\) and topologically minimal.

**Proof.** (1) \(\Rightarrow\) (2). If \(k\) is positive semidefinite, then by Lemma 3.1(1) it is Hermitian, that is, \(k(x, y)^* = k(y, x)\) for all \(x, y \in X\). With notation as in Subsection 3.1, we consider the convolution operator \(K : \mathbb{C}_0^X \to Z^X\) and let \(Z_K^X\) be its range:

\[
Z_K^X = \{ f \in Z^X \mid f = Kg \text{ for some } g \in \mathbb{C}_0^X \} = \left\{ f \in \mathcal{F} \mid f(x) = \sum_{y \in X} g(y)k(x, y) \text{ for some } g \in \mathbb{C}_0^X \text{ and all } y \in X \right\}.
\]

A pairing \([\cdot, \cdot]_\mathcal{E} : Z_K^X \times Z_K^X \to Z\) can be defined by

\[
[e, f]_\mathcal{E} = [g, h]_k = \sum_{x, y \in X} \overline{g(x)}h(y)k(x, y),
\]
where $f = Kh$ and $e = Kg$ for some $g, h \in \mathbb{C}_0^X$. We observe that

$$[e, f]_{\mathcal{E}} = \sum_{x \in X} g(x)f(x) = \sum_{x, y \in X} g(x)k(x, y)h(y) = \sum_{x, y \in X} h(y)g(x)k(y, x)^* = \sum_{x \in X} h(y)e(y)^*,$$

which shows that the definition in (3.11) is correct, that is, independent of the choice of $g$ and $h$.

We claim that $[\cdot, \cdot]_{\mathcal{E}}$ is a $\mathbb{Z}$-valued inner product, that is, it satisfies (ve1)–(ve3). The only fact that needs proof is that $[f, f]_{\mathcal{E}} = 0$ implies $f = 0$. To see this, we use Lemma 2.1 and first get $[f, f']_{\mathcal{E}} = 0$ for all $f' \in Z_K^X$. For each $x \in X$, let $\delta_x \in \mathbb{C}_0^X$ denote the $\delta$-function with support $\{x\}$,

$$\delta_x(y) = \begin{cases} 1 & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases}$$

and let $f' = K\delta_x$. Then

$$0 = [f, f']_{\mathcal{E}} = \sum_{y \in X} \delta_x f(y) = f(x);$$

since $x \in X$ is arbitrary, it follows that $f = 0$.

Thus, $(Z_K^X; [\cdot, \cdot]_{\mathcal{E}})$ is a VE-space. For $x \in X$ define $V(x) \in Z_K^X \subseteq \mathcal{E}$ by

$$V(x) = K\delta_x.$$  

Actually, there is an even more explicit way of expressing $V(x)$:

$$V(x)(y) = (K\delta_x)(y) = \sum_{z \in X} \delta_x(z)k(y, z) = k(y, x), \quad x \in X.$$  

On the other hand, for any $x, y \in X$, by (3.13) and (3.14) we have

$$[V(x), V(y)]_{\mathcal{E}} = (V(y))(x) = k(x, y),$$

hence $(\mathcal{E}; V)$ is a linearisation of $k$. We prove that it is minimal. To see this, note that for any $g \in \mathbb{C}_0^X$, with notation as in (3.12),

$$g = \sum_{x \in \text{supp}(g)} g(x)\delta_x,$$

hence, by (3.13), the linear span of $V(X)$ equals $Z_K^X$.

$(2) \Rightarrow (1)$. This is proven exactly as in the classical case.

$(3) \Rightarrow (2)$. Clear.

$(1) \Rightarrow (3)$. Assuming that $Z$ is an admissible space, let $k$ be positive semidefinite, and let $(\mathcal{E}; V)$ be the weak VE-space linearisation of $k$. Then $\mathcal{E}$ is naturally equipped with a Hausdorff locally convex topology (see Subsection 2.3), and then completed to a VH-space $\mathcal{H}$. Thus, $(\mathcal{H}; V)$ is a weak
VH-space linearisation of $k$ and it is easy to see that it is topologically minimal. The fact that this completion can be made within $Z^X$ will follow from Proposition 3.8.

3.3. Reproducing kernel spaces. Let $Z$ be an ordered $\ast$-space and let $X$ be a nonempty set. As in Subsection 3.1, we consider the complex vector space $Z^X$ of all functions $f : X \to Z$. A VE-space $\mathcal{R}$ over $Z$ is called a weak $Z$-reproducing kernel VE-space on $X$ if there exists a Hermitian kernel $k : X \times X \to Z$ such that:

(rk1) $\mathcal{R}$ is a linear subspace of $Z^X$.

(rk2) For all $x \in X$, the $Z$-valued map $k_x = k(\cdot, x) : X \to Z$ belongs to $\mathcal{R}$.

(rk3) For all $f \in \mathcal{R}$ we have $f(x) = [k_x, f]_\mathcal{R}$ for all $x \in X$.

Axiom (rk3) is called the reproducing property and implies

\begin{equation}
  k(x, y) = k_y(x) = [k_x, k_y]_\mathcal{R}, \quad x, y \in X.
\end{equation}

A weak $Z$-reproducing kernel VE-space $k$ on $X$ is called minimal if

(rk4) $\text{Lin}\{k_x \mid x \in X\} = \mathcal{R}$.

If $Z$ is an admissible space, a weak $Z$-reproducing kernel VE-space $\mathcal{R}$ that is a VH-space is called a weak $Z$-reproducing kernel VH-space. Such an $\mathcal{R}$ is called topologically minimal if

(rk4)' $\text{Lin}\{k_x \mid x \in X\}$ is dense in $\mathcal{R}$.

Remark 3.5. Let $\mathcal{R}$ be a weak $Z$-reproducing kernel VH-space with respect to some admissible space $Z$. In general, the closed subspace $\overline{\text{Lin}\{k_x \mid x \in X\}} \subseteq \mathcal{R}$ may or may not be orthocomplemented in $\mathcal{R}$ (see Subsection 2.3). This anomaly causes some differences from the classical theory of reproducing kernel spaces, as is the case in closely related situations in [3] and [4] as well.

Proposition 3.6. A weak $Z$-reproducing kernel VH-space $\mathcal{R}$ with respect to some admissible space $Z$ is topologically minimal if and only if $\overline{\text{Lin}\{k_x \mid x \in X\}}$ is orthocomplemented in $\mathcal{R}$.

Proof. If $\mathcal{M} := \overline{\text{Lin}\{k_x \mid x \in X\}}$ is orthocomplemented then, as a consequence of (rk3), $\mathcal{R}$ is topologically minimal in the sense of (rk4)'. Indeed, let $f \in \mathcal{R}$. Since $\mathcal{M}$ is orthocomplemented, there exist $f_1 \in \mathcal{M}$ and $f_2 \in \mathcal{M}^\perp$ with $f = f_1 + f_2$. By (rk3) we obtain $0 = [k_x, f] = f_2(x)$ for all $x \in \mathcal{R}$, and so $f_2 = 0$. It follows that $f \in \mathcal{M}$ and $\mathcal{M} = \mathcal{R}$, i.e. $\overline{\text{Lin}\{k_x \mid x \in X\}}$ is dense in $\mathcal{R}$. The converse is trivial.

We first consider the relation between weak $Z$-reproducing kernel VE/VH-spaces and their reproducing kernels.
Proposition 3.7. (a) Let \( \mathcal{R} \) be a weak \( Z \)-reproducing kernel VE-space on \( X \), with respect to some ordered *-space \( Z \) and with kernel \( k \). Then:

(i) \( k \) is positive semidefinite and uniquely determined by \( \mathcal{R} \).
(ii) \( \mathcal{R}_0 = \text{Lin}\{k_x \mid x \in X\} \subseteq \mathcal{R} \) is a minimal weak \( Z \)-reproducing kernel VE-space on \( X \) and is a unique such space determined by \( k \).
(iii) The gramian \([\cdot,\cdot]_\mathcal{R}\) is uniquely determined by \( k \) on \( \mathcal{R}_0 \).

(b) Assume that \( Z \) is admissible and \( \mathcal{R} \) is a weak \( Z \)-reproducing kernel VH-space. Then:

(i) \( \mathcal{R}_0 \) is a topologically minimal \( Z \)-reproducing kernel VH-space in \( \mathcal{R} \).
(ii) The gramian \([\cdot,\cdot]_\mathcal{R}\) is uniquely determined by \( k \) on \( \mathcal{R}_0 \subseteq \mathcal{R} \).
(iii) If \( \mathcal{R} \) is topologically minimal then it is unique with this property.

Proof. (a) Let \( t_1, \ldots, t_n \in \mathbb{C} \) and \( x_1, \ldots, x_n \in X \). By (3.15),
\[
\sum_{j,k=1}^n t_j t_k k(x_j, x_k) = \sum_{j,k=1}^n t_j t_k [k_{x_j}, k_{x_k}]_\mathcal{R} = \left[ \sum_{j=1}^n t_j k_{x_j}, \sum_{k=1}^n t_k k_{x_k} \right]_\mathcal{R} \geq 0,
\]
hence \( k \) is positive semidefinite. On the other hand, (rk3) implies that for all \( x \in X \) the functions \( k_x \) are uniquely determined by \( (\mathcal{R}; [\cdot,\cdot]_\mathcal{R}) \), hence \( k(y, x) = k_x(y) \), \( x, y \in X \), are uniquely determined. Thus (i) is proven; (ii) is clear by inspecting the definitions; and (iii) is now clear by (rk3) (see (3.15)).

(b) The subspace \( \mathcal{R}_0 \) of \( \mathcal{R} \) is a topologically minimal \( Z \)-reproducing kernel VH-space, by definition. Using (a)(ii) and the continuity of \([\cdot,\cdot]_\mathcal{R}\), it follows that it is uniquely determined by \( k \) on \( \mathcal{R}_0 \).

Assume that \( \mathcal{R} \) is topologically minimal and let \( \mathcal{R}' \) be another topologically minimal weak \( Z \)-reproducing kernel VH-space on \( X \) with the same kernel \( k \). By (rk2) and (rk4), \( \mathcal{R}_0 = \text{Lin}\{k_x \mid x \in X\} \) is a linear space that lies and is dense in both \( \mathcal{R} \) and \( \mathcal{R}' \). By (rk3), the \( Z \)-valued inner products \([\cdot,\cdot]_\mathcal{R}\) and \([\cdot,\cdot]_{\mathcal{R}'}\) coincide on \( \mathcal{R}_0 \) and then, due to the special way in which the topologies on VH-spaces are defined (see (2.9) and (2.10)), \( \mathcal{R} \) and \( \mathcal{R}' \) induce the same topology on \( \mathcal{R}_0 \); hence, as \( \mathcal{R}_0 \) is dense in both \( \mathcal{R} \) and \( \mathcal{R}' \), we actually have \( \mathcal{R} = \mathcal{R}' \) as VH-spaces.

Consequently, given a weak \( Z \)-reproducing kernel VE-space \( \mathcal{R} \) on \( X \), without ambiguity we can talk about the \( Z \)-reproducing kernel \( k \) corresponding to \( \mathcal{R} \).

As a consequence of Proposition 3.7 weak positive semidefiniteness is an intrinsic property of the reproducing kernel of any weak reproducing kernel VE-space. In the following we clarify in an explicit fashion the relation between weak VE/VH-linearisations and weak reproducing kernel VE/VH-spaces associated to positive semidefinite kernels.
Proposition 3.8. Let \( k \) be a weakly positive semidefinite kernel on \( X \) and with values in the ordered \( * \)-space \( Z \).

(a) Any weak reproducing kernel VE-space \( \mathcal{R} \) associated to \( k \) gives rise to a weak VE-space linearisation \( (\mathcal{E}; V) \) of \( k \), where \( \mathcal{E} = \mathcal{R} \) and
\[
V(x) = k_x, \quad x \in X.
\]

If \( \mathcal{R} \) is minimal, then \( (\mathcal{E}; V) \) is minimal.

(b) Any minimal weak VE-space linearisation \( (\mathcal{E}; V) \) of \( k \) gives rise to the minimal weak reproducing kernel VE-space \( \mathcal{R} \), where
\[
\mathcal{R} = \{ [V(\cdot), h]_\mathcal{E} | h \in \mathcal{E} \},
\]
that is, \( \mathcal{R} \) consists of all functions \( X \ni x \mapsto [V(x), e]_\mathcal{E} \in Z \), for all \( e \in \mathcal{E} \); in particular, \( \mathcal{R} \) is a linear subspace of \( Z^X \).

Proof. (a) Assume that \( (\mathcal{R}; [\cdot, \cdot]_\mathcal{R}) \) is a weak \( Z \)-reproducing kernel VE-space on \( X \), with reproducing kernel \( k \). We let \( \mathcal{E} = \mathcal{R} \) and define \( V \) as in (3.16).

Note that \( V(x) \in \mathcal{E} \) for all \( x \in X \). Also, by (3.15) we have
\[
[V(x), V(y)]_\mathcal{E} = k(x, y), \quad x, y \in X.
\]

Thus, \( (\mathcal{E}; V) \) is a weak VE-space linearisation of \( k \).

(b) Let \( (\mathcal{E}; V) \) be a minimal weak VE-space linearisation of \( k \). Let \( \mathcal{R} \) be defined by (3.17), that is, \( \mathcal{R} \) consists of all functions \( X \ni x \mapsto [V(x), h]_\mathcal{E} \in Z \), in particular \( \mathcal{R} \) is a linear subspace of \( Z^X \).

The correspondence
\[
\mathcal{E} \ni h \mapsto Uh = [V(\cdot), h]_\mathcal{E} \in \mathcal{R}
\]
is clearly surjective. In order to verify that it is injective as well, let \( h, g \in \mathcal{E} \) be such that \( [V(\cdot), h]_\mathcal{E} = [V(\cdot), g]_\mathcal{E} \). Then, for all \( x \in X \),
\[
[V(x), h]_\mathcal{E} = [V(x), g]_\mathcal{E},
\]
or equivalently
\[
[V(x), h - g]_\mathcal{E} = 0, \quad x \in X.
\]

By the minimality of \( (\mathcal{E}; V) \) it follows that \( g = h \). Thus, \( U \) is a bijection.

Clearly, the bijective map \( U \) defined at (3.18) is linear, hence a linear isomorphism \( \mathcal{E} \to \mathcal{R} \) of complex vector spaces. On \( \mathcal{R} \) we introduce a \( Z \)-valued pairing
\[
[Uf, Ug]_\mathcal{R} = [f, g]_\mathcal{E}, \quad f, g \in \mathcal{E}.
\]

Since \( (\mathcal{E}; [\cdot, \cdot]_\mathcal{E}) \) is a VE-space over \( Z \), so is \( (\mathcal{R}; [\cdot, \cdot]_\mathcal{R}) \): indeed, by (3.20), we transported the \( Z \)-gramian from \( \mathcal{E} \) to \( \mathcal{R} \), which makes the linear isomorphism \( U \) a unitary operator between the VE-spaces \( \mathcal{E} \) and \( \mathcal{R} \).

We show that \( (\mathcal{R}; [\cdot, \cdot]_\mathcal{R}) \) is a weak \( Z \)-reproducing kernel VE-space with corresponding reproducing kernel \( k \). By definition, \( \mathcal{R} \subseteq Z^X \). On the other
hand, since

$$k_y(x) = k(y, x) = [V(y), V(x)]_E$$

for all \(x, y \in X\), taking into account that \(V(x) \in \mathcal{E}\), from (3.17) it follows that \(k_x \in \mathcal{R}\) for all \(x \in X\). Further, for all \(f \in \mathcal{R}\) and all \(x \in X\) we have

$$[k_x, f]_\mathcal{R} = [k_x, [V(\cdot), g]_\mathcal{E}] = [V(x), g]_\mathcal{E},$$

where \(g \in \mathcal{E}\) is the unique vector such that \([V(\cdot), g]_\mathcal{E} = f\), which shows that \(\mathcal{R}\) satisfies the reproducing axiom as well. Finally, from the minimality of \((\mathcal{E}; V)\) and (3.17) it follows that \(\text{Lin}\{k_x \mid x \in X\} = \mathcal{R}\). Thus, \((\mathcal{R}; [\cdot, \cdot]_\mathcal{R})\) is a minimal weak \(Z\)-reproducing kernel VE-space with reproducing kernel \(k\).

**Proposition 3.9.** Let \(k\) be a weakly positive semidefinite kernel on \(X\) with values in the admissible space \(Z\).

(a) Any weak reproducing kernel VH-space \(\mathcal{R}\) associated to \(k\) gives rise to a weak VH-space linearisation \((\mathcal{H}; V)\) of \(k\), where \(\mathcal{H} = \mathcal{R}\) and

$$(3.21) \quad V(x) = k_x, \quad x \in X.$$ 

If \(\mathcal{R}\) is topologically minimal then \((\mathcal{H}; V)\) is topologically minimal.

(b) Any topologically minimal weak VH-space linearisation \((\mathcal{H}; V)\) of \(k\) gives rise to the topologically minimal weak reproducing kernel VH-space

$$(3.22) \quad \mathcal{R} = \{[V(\cdot), h]_\mathcal{H} \mid h \in \mathcal{H}\},$$

which is a linear subspace of \(Z^X\).

**Proof.** (a) The argument is similar to that used to prove Proposition 3.8(a).

(b) Let \((\mathcal{H}; V)\) be a topologically minimal weak VH-space linearisation of \(k\) and let \(\mathcal{R}\) be defined as in (3.22). The correspondence

$$(3.23) \quad \mathcal{H} \ni h \mapsto U h = [V(\cdot), h]_\mathcal{H} \in \mathcal{R}$$

is a linear bijection \(U : \mathcal{H} \to \mathcal{R}\). The argument is now similar to that in the proof of Proposition 3.8(b), with the difference being that having (3.19) we use the topological minimality of \((\mathcal{H}; V)\) to conclude that \(g = h\). Thus, \(U\) is a bijection.

On \(\mathcal{R}\) we introduce a \(Z\)-valued pairing as in (3.20). Since \((\mathcal{H}; [\cdot, \cdot]_\mathcal{H})\) is a VH-space over \(Z\), so is \((\mathcal{R}; [\cdot, \cdot]_\mathcal{R})\): this follows from the observation that, by (3.20), we transported the \(Z\)-gramian from \(\mathcal{H}\) to \(\mathcal{R}\), which makes \(U\) a unitary operator between the VH-spaces \(\mathcal{H}\) and \(\mathcal{R}\).

Finally, \((\mathcal{R}; [\cdot, \cdot]_\mathcal{R})\) is the topologically minimal weak \(Z\)-reproducing kernel VH-space with corresponding reproducing kernel \(k\): the argument is again similar to that in the proof of Proposition 3.8(b), with the difference that here we use topological minimality.

The following theorem adds one more characterisation of positive semidefinite kernels, compared to Theorem 3.4, in terms of reproducing ker-
Invariant weakly positive semidefinite kernels

Theorem 3.10. (a) Let $Z$ be an ordered $*$-space, $X$ a nonempty set, and $k : X \times X \to Z$ a Hermitian kernel. The following assertions are equivalent:

(1) $k$ is weakly positive semidefinite.
(2) $k$ is the $Z$-valued reproducing kernel of a VE-space $R$ in $Z^X$.

(b) If $Z$ is a topologically ordered $*$-space then (1) and (2) are equivalent to

(3) $k$ is the $Z$-valued reproducing kernel of a topological VE-space $R$ in $Z^X$.

(c) If, in addition, $Z$ is an admissible space then (1) and (2) are equivalent to

(4) $k$ is the $Z$-valued reproducing kernel of a VH-space $R$ in $Z^X$.

In particular, any weakly positive semidefinite $Z$-valued kernel $k$ has a topologically minimal weak $Z$-reproducing kernel VH-space $R$, uniquely determined by $k$.

As a consequence of the last assertion of Theorem 3.10, given a positive semidefinite kernel $k : X \times X \to Z$ for an admissible space $Z$, we can denote, without any ambiguity, by $R_k$ the unique topologically minimal weak $Z$-reproducing kernel VH-space on $X$ associated to $k$.

4. Invariant weakly positive semidefinite kernels. Let $X$ be a nonempty set equipped with an action of a (multiplicative) semigroup $\Gamma$ denoted by $\xi \cdot x$, for all $\xi \in \Gamma$ and all $x \in X$. By definition

\[ \alpha \cdot (\beta \cdot x) = (\alpha \beta) \cdot x \quad \text{for all } \alpha, \beta \in \Gamma \text{ and all } x \in X. \]

This means that we have a semigroup morphism $\Gamma \ni \xi \mapsto \xi \cdot \in G(X)$, where $G(X)$ denotes the semigroup, with respect to composition, of all maps $X \to X$. In case $\Gamma$ has a unit $\epsilon$, the action is called unital if $\epsilon \cdot x = x$ for all $x \in X$, or equivalently $\epsilon \cdot = \text{Id}_X$.

We assume further that $\Gamma$ is a $*$-semigroup: there is an involution $*$ on $\Gamma$, that is, $(\xi \eta)^* = \eta^* \xi^*$ and $(\xi^*)^* = \xi$ for all $\xi, \eta \in \Gamma$. If $\Gamma$ has a unit $\epsilon$ then $\epsilon^* = \epsilon$.

4.1. Invariant weak VE-space linearisations. Given an ordered $*$-space $Z$, we are interested in those Hermitian kernels $k : X \times X \to Z$ that are $\Gamma$-invariant, that is,

\[ k(y, \xi \cdot x) = k(\xi^* \cdot y, x) \quad \text{for all } x, y \in X \text{ and } \xi \in \Gamma. \]

A triple $(\mathcal{E}; \pi; V)$ is called a $\Gamma$-invariant weak VE-space linearisation of $k$ if:
Since \((k; V)\) is a weak VE-space linearisation of \(k\).

(iivel2) \(\pi: \Gamma \rightarrow \mathcal{L}^*(\mathcal{E})\) is a \(*\)-representation, that is, a multiplicative \(*\)-morphism.

(iivel3) \(V(\xi \cdot x) = \pi(\xi)V(x)\) for all \(x \in X\) and \(\xi \in \Gamma\).

Let \((\mathcal{E}; \pi; V)\) be a \(\Gamma\)-invariant weak VE-space linearisation of the kernel \(k\).

Since \((\mathcal{E}; V)\) is a weak linearisation and by (ivel3), for all \(x, y \in X\) and all \(\xi \in \Gamma\) we have
\[
(4.3) \quad k(y, \xi \cdot x) = [V(y), V(\xi \cdot x)]_{\mathcal{E}} = [V(y), \pi(\xi)V(x)]_{\mathcal{E}}
\]
\[
= [\pi(\xi^*)V(y), V(x)]_{\mathcal{E}} = [V(\xi^* \cdot y), V(x)]_{\mathcal{E}} = k(\xi^* \cdot y, x),
\]
hence \(k\) is \(\Gamma\)-invariant.

If, in addition to (ivel1)–(ivel3), the triple \((\mathcal{E}; \pi; V)\) also has the property (ivel4) \(\text{Lin} V(X) = \mathcal{E}\),

that is, the linearisation \((\mathcal{E}; V)\) is minimal, then \((\mathcal{E}; \pi; V)\) is called minimal.

Remarks 4.1. (a) The minimality condition (ivel4) does not depend on the representation \(\pi\). Apparently, \(\text{Lin} \pi(\Gamma)V(X)\) looks like a suitable candidate to replace \(\text{Lin} V(X)\) in (ivel4). However, if the \(*\)-semigroup has a unit, then \(\text{Lin} \pi(\Gamma)V(X) = \text{Lin} V(X)\), but when \(\Gamma\) does not have a unit, only the inclusion \(\text{Lin} \pi(\Gamma)V(X) \subseteq \text{Lin} V(X)\) holds, and hence \(\text{Lin} \pi(\Gamma)V(X)\) may be too small to accommodate all \(V(x)\) for \(x \in X\).

(b) Let \((\mathcal{E}; \pi; V)\) be a \(\Gamma\)-invariant weak VE-space linearisation of \(k\). Then, for each \(\gamma \in \Gamma\), we have \(\pi(\gamma)V(x) = V(\gamma \cdot x)\) for all \(x \in X\), hence \(\pi(\gamma)\) leaves \(\text{Lin} V(X)\) invariant, and consequently if we let \(\mathcal{E}_0 = \text{Lin} V(X)\), define \(\pi_0: \Gamma \rightarrow \mathcal{L}^*(\mathcal{E}_0)\) by \(\pi_0(\gamma)f = \pi(\gamma)f\) for all \(f \in \text{Lin} V(X)\), and define \(V_0: X \rightarrow \mathcal{E}_0\) by \(V_0(x) = V(x)\), then \((\mathcal{E}_0; \pi_0; V_0)\) is a minimal \(\Gamma\)-invariant weak VE-space linearisation of \(k\).

As usual \([32]\), minimal invariant VE-space linearisations preserve linearity.

Proposition 4.2. Assume that, given an ordered \(*\)-space \(Z\) valued kernel \(k\), invariant under the action of the \(*\)-semigroup \(\Gamma\) on \(X\), for some fixed \(\alpha, \beta, \gamma \in \Gamma\) we have \(k(y, \alpha \cdot x) + k(y, \beta \cdot x) = k(y, \gamma \cdot x)\) for all \(x, y \in X\). Then for any minimal weak \(\Gamma\)-invariant VE-space linearisation \((\mathcal{E}; \pi; V)\) of \(k\), the representation satisfies \(\pi(\alpha) + \pi(\beta) = \pi(\gamma)\).

Proof. This follows by a standard argument. ■

Two \(\Gamma\)-invariant weak VE-space linearisations \((\mathcal{E}; \pi; V)\) and \((\mathcal{E}'; \pi'; V')\) of the same Hermitian kernel \(k\) are called unitarily equivalent if there exists a unitary operator \(U: \mathcal{E} \rightarrow \mathcal{E}'\) such that \(U\pi(\xi) = \pi'(\xi)U\) for all \(\xi \in \Gamma\), and \(UV(x) = V'(x)\) for all \(x \in X\). Note that if both these \(\Gamma\)-invariant
linearisations are minimal, then this is equivalent to the weak VE-space linearisations \((E; V)\) and \((E'; V')\) being unitarily equivalent.

Here is the first main theorem of this article in which invariant weakly positive semidefinite kernels are characterised by invariant weak (topological) VE-space linearisations and by certain \(*\)-representations on weak \(Z\)-reproducing kernel (topological) VE-spaces.

**Theorem 4.3.** Let \(\Gamma\) be a \(*\)-semigroup that acts on the nonempty set \(X\), and let \(k: X \times X \to Z\) be a \(Z\)-valued kernel for some [topological] ordered \(*\)-space \(Z\). The following assertions are equivalent:

(1) \(k\) satisfies the following conditions:

(a) \(k\) is weakly positive semidefinite.

(b) \(k\) is \(\Gamma\)-invariant, that is, \((4.2)\) holds.

(2) \(k\) has a \(\Gamma\)-invariant weak [topological] VE-space linearisation \((E; \pi; V)\).

(3) \(k\) admits a weak \(Z\)-reproducing kernel [topological] VE-space \(R\) and there exists a \(*\)-representation \(\rho: \Gamma \to \mathcal{L}^*(R)\) such that \(\rho(\xi)k_x = k_{\xi \cdot x}\) for all \(\xi \in \Gamma\) and \(x \in X\).

Moreover, if (1), (2), or (3) holds, then a minimal \(\Gamma\)-invariant weak [topological] VE-space linearisation of \(k\) can be constructed, and a minimal weak \(Z\)-reproducing kernel [topological] VE-space \(R\) as in (3) can be constructed as well.

**Proof.** (1)\(\Rightarrow\)(2). We consider the notation and the minimal weak VE-space linearisation \((E; V)\) constructed as in the proof of (1)\(\Rightarrow\)(2) in Theorem 3.4. For each \(\xi \in \Gamma\) we let \(\pi(\xi): Z^X \to Z^X\) be defined by

\[
(\pi(\xi)f)(y) = f(\xi^* \cdot y), \quad f \in Z^X, \ y \in X, \ \xi \in \Gamma.
\]

We claim that \(\pi(\xi)\) leaves \(Z^X_\Lambda\) invariant, where \(\Lambda\) is the convolution operator defined at (3.3) and \(Z^X_\Lambda \subseteq Z^X\) denotes its range. To see this, let \(f \in Z^X_\Lambda\), that is, \(f = Kg\) for some \(g \in C^0\) or, even more explicitly, by (3.10),

\[
f(y) = \sum_{x \in X} g(x)k(x, y), \quad y \in X.
\]

Then

\[
f(\xi^* \cdot y) = \sum_{x \in X} g(x)k(x, \xi^* \cdot y) = \sum_{x \in X} g(x)k(\xi \cdot x, y) = \sum_{z \in X} g_\xi(z)k(z, y),
\]

where

\[
g_\xi(z) = \begin{cases} 
0 & \text{if } \xi \cdot x = z \text{ has no solution } x \in X, \\
\sum_{\xi \cdot x = z} g(x) & \text{otherwise.}
\end{cases}
\]

Since clearly \(g_\xi\) has finite support, \(\pi(\xi)\) leaves \(Z^X_\Lambda\) invariant. In the following we also denote by \(\pi(\xi)\) the map \(\pi(\xi): Z^X_\Lambda \to Z^X_\Lambda\).
We now prove that \( \pi \) is a representation of the semigroup \( \Gamma \) on the complex vector space \( Z^X_K \), that is,

\[
\pi(\alpha\beta)f = \pi(\alpha)\pi(\beta)f, \quad \alpha, \beta \in \Gamma, \ f \in Z^X_K.
\]

To see this, let \( f \in Z^X_K \) and \( h = \pi(\beta)f \), that is, \( h(y) = f(\beta^*y) \) for all \( y \in X \). Then \( \pi(\alpha)\pi(\beta)f = \pi(\alpha)h \), that is, \( (\pi(\alpha)h)(y) = h(\alpha^*y) = f(\beta^*\alpha^*y) = f((\alpha\beta)^*y) = (\pi(\alpha\beta))(y) \) for all \( y \in X \), which proves (4.7).

Next we show that \( \pi \) is actually a *-representation, that is,

\[
[\pi(\xi)f, f']_\mathcal{E} = [f, \pi(\xi^*)f']_\mathcal{E}, \quad f, f' \in Z^X_K.
\]

To see this, let \( f = Kg \) and \( f' = Kg' \) for some \( g, g' \in \mathbb{C}_0^X \). Then, by (3.11) and (4.6),

\[
[\pi(\xi)f, f']_\mathcal{E} = \sum_{y \in X} g'(y)f(\xi^*y) = \sum_{x, y \in X} g'(y)\overline{g(x)}k(\xi^*y, x)
\]

\[
= \sum_{x, y \in X} g'(y)\overline{g(x)}k(y, \xi x) = \sum_{x \in X} g(x)f'(\xi x)^* = [f, \pi(\xi^*)f']_\mathcal{E}.
\]

To show that axiom (vel3) holds as well, we use (3.14) and (4.4). Thus, for all \( \xi \in \Gamma \), \( x, y \in X \) and since \( k \) is invariant under the action of \( \Gamma \) on \( X \), we have

\[
(4.9) \quad (V(\xi \cdot x))(y) = k(\xi \cdot x, y) = k(x, \xi^*y) = (V(x))(\xi^*y) = (\pi(\xi)V(x))(y),
\]

which proves (vel3). Thus, \( (\mathcal{E}; \pi; V) \) is a \( \Gamma \)-invariant weak VE-space linearisation of the Hermitian kernel \( k \). Note that \( (\mathcal{E}; \pi; V) \) is minimal, that is, axiom (vel4) holds, since \( (\mathcal{E}; V) \) is minimal by the proof of Theorem 3.4.

To prove the uniqueness of the minimal weak \( \Gamma \)-invariant linearisation, let \( (\mathcal{E}'; \pi'; V') \) be another such linearisation. We consider the unitary operator \( U : \mathcal{E} \to \mathcal{E}' \) defined as in (3.9); we already know that \( UV(x) = V'(x) \) for all \( x \in X \). Since, for any \( \xi \in \Gamma \) and \( x \in X \),

\[
U\pi(\xi)V(x) = UV(\xi x) = V'(\xi x) = \pi'(\xi)V'(x) = \pi'(\xi)UV(x),
\]

and by minimality, \( U\pi(\xi) = \pi'(\xi)U \) for all \( \xi \in \Gamma \).

If \( Z \) is a topologically ordered *-space, \( \mathcal{E} \) becomes a topological VE-space (see Subsection 2.3).

(2)\(\Rightarrow\)(1). Let \( (\mathcal{E}; \pi; V) \) be a \( \Gamma \)-invariant weak [topological] VE-space linearisation of \( k \). We already know from the proof of Theorem 3.4 that \( k \) is positive semidefinite and it was shown in (4.3) that \( k \) is \( \Gamma \)-invariant.

(2)\(\Rightarrow\)(3). Let \( (\mathcal{E}; \pi; V) \) be a \( \Gamma \)-invariant weak [topological] VE-space linearisation of \( k \). We can assume that it is minimal: since we have already proven (2)\(\Rightarrow\)(1), we observe that during the proof of (1)\(\Rightarrow\)(2) we have obtained a minimal \( \Gamma \)-invariant weak VE-space linearisation of \( k \).
We use the notation and the facts established in the proof of (2)⇒(3) in Theorem 3.10. Then, for all \( x, y \in X \),
\[
k_{\xi \cdot x}(y) = [V(y), \pi(\xi)V(x)] = [V(y), \pi(\xi)V(x)]_K,
\]

hence, letting \( \rho(\xi) = U\pi(\xi)U^{-1} \), where \( U : K \to \mathcal{R} \) is the unitary operator defined as in (3.18), we obtain a \( * \)-representation of \( \Gamma \) on \( \mathcal{R} \) such that \( k_{\xi \cdot x} = \rho(\xi)k_x \) for all \( \xi \in \Gamma \) and \( x \in X \).

(3)⇒(2). Let \( \mathcal{R} \) be a weak \( Z \)-reproducing kernel [topological] VE-space of \( k \) and \( \rho : \Gamma \to \mathcal{L}^*(\mathcal{R}) \) a \( * \)-representation such that \( k_{\xi \cdot x} = \rho(\xi)k_x \) for all \( \xi \in \Gamma \) and \( x \in X \). As in the proof of (3)⇒(2) in Theorem 3.10, we show that \( (\mathcal{R}; V) \), where \( V \) is defined as in (3.16), is a minimal linearisation of \( k \).

**Remark 4.4.** Consider Theorem 4.3 for \( Z \) a topologically ordered \( * \)-space. If in (1), we additionally assume (1)(c) of Theorem 4.7, namely, (c) for any \( \alpha \in \Gamma \) and any \( p \in S(Z) \), there exists \( q \in S(Z) \) and a constant \( c(\alpha) \geq 0 \) such that

\[
(4.10) \quad p\left(\sum_{j,k=1}^{n} t_j \overline{t}_k k(\alpha \cdot x_k, \alpha \cdot x_j)\right) \leq c(\alpha)^2 q\left(\sum_{j,k=1}^{n} t_j \overline{t}_k k(x_k, x_j)\right),
\]

for all \( n \in \mathbb{N}, x_1, \ldots, x_n \in X \), and \( t_1, \ldots, t_n \in \mathbb{C} \), then we obtain item (2) with the additional property that \( \pi(\xi) \) is continuous for all \( \xi \in \Gamma \), and similarly we obtain (3) with the continuity of \( \rho(\xi) \) for all \( \xi \in \Gamma \). For a proof, see the proof of Theorem 4.7.

**4.2. Boundedly adjointable invariant weak VH-space linearisations.** Assume now that \( Z \) is an admissible space and \( k : X \times X \to Z \) is a kernel. A triple \( (\mathcal{K}; \pi; V) \) is called a **boundedly adjointable \( \Gamma \)-invariant weak VH-space linearisation** of \( k \) if:

- **(ivhl1)** \( (\mathcal{K}; V) \) is a weak VH-space linearisation of \( k \).
- **(ivhl2)** \( \pi : \Gamma \to \mathcal{B}^*(\mathcal{K}) \) is a \( * \)-representation.
- **(ivhl3)** \( V(\xi \cdot x) = \pi(\xi)V(x) \) for all \( x \in X \) and \( \xi \in \Gamma \).

Let \( (\mathcal{K}; \pi; V) \) be a boundedly adjointable \( \Gamma \)-invariant weak VH-space linearisation of \( k \). As in (4.3), it follows that \( k \) is \( \Gamma \)-invariant.

If, in addition to (ivhl1)–(ivhl3), the triple \( (\mathcal{K}; \pi; V) \) also has the property that

- **(ivhl4)** \( \text{Lin} V(X) \) is dense in \( \mathcal{K} \),

that is, the weak VH-space linearisation \( (\mathcal{H}; V) \) is topologically minimal, then \( (\mathcal{K}; \pi; V) \) is called **topologically minimal**. A similar observation to Remark 4.3 can be made: if \( \Gamma \) has a unit then (ivhl4) is equivalent to
Lin $\pi(\Gamma)V(X)$ being dense in $\mathcal{K}$, but in general the apparent candidate Lin $\pi(\Gamma)V(X)$ is too small to provide a suitable topological minimality condition.

By definition, boundedly adjointable $\Gamma$-invariant weak VH-space linearisations $(\mathcal{K};\pi;V)$ and $(\mathcal{K}';\pi';V')$ of the same kernel $k$ are unitarily equivalent if there exists a unitary $U \in B^*(\mathcal{K},\mathcal{K}')$ such that $U \pi(\xi) = \pi'(\xi)U$ for all $\xi \in \Gamma$ and $UV(x) = V'(x)$ for all $x \in X$. Note that, if both these linearisations are topologically minimal then they are unitarily equivalent.

The analog of Proposition 4.2 for topologically minimal invariant weak VH-space linearisations holds as well.

**Proposition 4.5.** Assume that, given an admissible space $Z$ and a $Z$-valued kernel $k$, invariant under the action of the $*$-semigroup $\Gamma$ on $X$, for some fixed $\alpha, \beta, \gamma \in \Gamma$ we have $k(y, \alpha \cdot x) + k(y, \beta \cdot x) = k(y, \gamma \cdot x)$ for all $x, y \in X$. Then, for any topologically minimal boundedly adjointable $\Gamma$-invariant weak VH-space linearisation $(\mathcal{K};\pi;V)$ of $k$, the representation satisfies $\pi(\alpha) + \pi(\beta) = \pi(\gamma)$.

Here is the second main theorem of this article in which invariant weakly positive semidefinite kernels are characterised by boundedly adjointable invariant weak VE-space linearisations and by certain $*$-representations with boundedly adjointable operators on weak $Z$-reproducing kernel VE-spaces. This is the first topological analogue of Theorem 4.3.

**Theorem 4.6.** Let $\Gamma$ be a $*$-semigroup that acts on the nonempty set $X$, and let $k: X \times X \to Z$ be a $Z$-valued kernel for some admissible space $Z$. The following assertions are equivalent:

1. $k$ satisfies the following conditions:
   a. $k$ is weakly positive semidefinite.
   b. $k$ is $\Gamma$-invariant.
   c. For any $\alpha \in \Gamma$ there exists $c(\alpha) \geq 0$ such that
      \[
      \sum_{j,k=1}^{n} t_j t_k^* k(\alpha \cdot x_k, \alpha \cdot x_j) \leq c(\alpha)^2 \sum_{j,k=1}^{n} t_j t_k^* k(x_k, x_j)
      \]
      for all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$ and $t_1, \ldots, t_n \in \mathbb{C}$.

2. $k$ has a boundedly adjointable $\Gamma$-invariant weak VH-space linearisation $(\mathcal{K};\pi;V)$.

3. $k$ admits a weak $Z$-reproducing kernel VH-space $\mathcal{R}$ and there exists a $*$-representation $\rho: \Gamma \to B^*(\mathcal{R})$ such that $\rho(\xi)k_x = k_{\xi \cdot x}$ for all $\xi \in \Gamma$ and $x \in X$.

Moreover, if (1), (2), or (3) holds, then a topologically minimal boundedly adjointable $\Gamma$-invariant weak VH-space linearisation can be constructed, and
a topologically minimal weak $Z$-reproducing kernel VH-space $\mathcal{R}$ as in assertion (3) can be constructed as well.

Proof. (1) $\Rightarrow$ (2). We consider the notation and the minimal $\Gamma$-invariant weak VE-space linearisation $(\mathcal{E}; \pi; V)$ constructed as in the proof of (1) $\Rightarrow$ (2) of Theorem 4.3. Considering $Z_k^X$ as a VE-space with $Z$-gramian $[\cdot, \cdot]_\mathcal{E}$, we consider its natural topology as in Subsection 2.3 and we now prove that $\pi(\xi)$ is bounded for all $\xi \in \Gamma$. Indeed, let $f = Kg$ for some $g \in C_0^X$. Using the definition of $\pi(\xi)$ and (c), we have

$$
[\pi(\xi)f, \pi(\xi)f]_\mathcal{K} = [\pi(\xi^*)\pi(\xi)f, f]_\mathcal{K} = [\pi(\xi^*)f, f]_\mathcal{K}
$$

$$
= \sum_{x,y \in X} g(y)\overline{g(x)}k(\xi^* \cdot y, x) = \sum_{x,y \in X} g(y)\overline{g(x)}k(\xi \cdot y, \xi \cdot x)
$$

$$
\leq c(\xi)^2 \sum_{x,y \in X} g(y)\overline{g(x)}k(y, x) = c(\xi)^2[f, f]_\mathcal{K}.
$$

Let $\mathcal{K}$ be the VH-space completion of $\mathcal{E}$ (see (1) $\Rightarrow$ (3) in the proof of Theorem 3.4). Then $\pi(\xi)$ can be uniquely extended by continuity to an operator $\pi(\xi) \in \mathcal{B}(\mathcal{K})$. In addition, since $\pi(\xi^*)$ also extends by continuity to $\pi(\xi^*) \in \mathcal{B}(\mathcal{K})$ and by (4.8), it follows that $\pi(\xi)$ is adjointable and $\pi(\xi^*) = \pi(\xi)^*$. We conclude that $\pi$ is a $*$-representation of $\Gamma$ in $\mathcal{B}^*(\mathcal{K})$, that is, axiom (ivhl2) holds.

The uniqueness of the topologically minimal boundedly adjointable $\Gamma$-invariant weak VH-space linearisation follows as usual.

(2) $\Rightarrow$ (1). Let $(\mathcal{K}; \pi; V)$ be a boundedly adjointable $\Gamma$-invariant weak VH-space linearisation of $k$. We already know from the proof of Theorem 3.4 that $k$ is positive semidefinite and it was shown in (4.3) that $k$ is $\Gamma$-invariant. To prove (c), let $\alpha \in \Gamma$, $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$ and $t_1, \ldots, t_n \in \mathbb{C}$. Then

$$
\sum_{j,k=1}^n \overline{t_k}t_j k(\alpha \cdot x_k, \alpha \cdot x_j) = \sum_{j,k=1}^n \overline{t_k}t_j [\pi(\alpha^*)\pi(\alpha)V(x_k), V(x_j)]_\mathcal{K}
$$

$$
= \sum_{j,k=1}^n \overline{t_k}t_j [\pi(\alpha)V(x_k), \pi(\alpha)V(x_j)]_\mathcal{K}
$$

$$
= [\pi(\alpha) \sum_{k=1}^n t_k V(x_k), \pi(\alpha) \sum_{j=1}^n t_j V(x_j)]_\mathcal{K}
$$

$$
\leq ||\pi(\alpha)||^2 [\sum_{k=1}^n t_k V(x_k), \sum_{j=1}^n t_j V(x_j)]_\mathcal{K}
$$

$$
= ||\pi(\alpha)||^2 \sum_{j,k=1}^n \overline{t_k}t_j k(x_k, x_j).
$$
(2)⇒(3). Let \((\mathcal{K}; \pi; V)\) be a boundedly adjointable \(\Gamma\)-invariant weak VH-space linearisation of \(k\). Again we can assume that it is topologically minimal.

We use the notation and the facts established in the proof of (2)⇒(3) in Theorem 3.10. For all \(x, y \in X\) we have

\[
k_{\xi \cdot x}(y) = k(y, \xi \cdot x) = [V(y), V(\xi \cdot x)]_{\mathcal{K}} = [V(y), \pi(\xi)V(x)]_{\mathcal{K}},
\]

hence letting \(\rho(\xi) = U\pi(\xi)U^{-1}\), where \(U: \mathcal{K} \to \mathcal{R}\) is the unitary operator defined as in (3.18), we obtain a \(*\)-representation of \(\Gamma\) on the VH-space \(\mathcal{R}\) such that \(k_{\xi \cdot x} = \rho(\xi)k_x\) for all \(\xi \in \Gamma\) and \(x \in X\).

(3)⇒(2). Let \(\mathcal{R} = \mathcal{R}(k)\) be the weak reproducing kernel VH-space of \(k\) and \(\rho: \Gamma \to \mathcal{B}^*(\mathcal{R})\) a \(*\)-representation such that \(k_{\xi \cdot x} = \rho(\xi)k_x\) for all \(\xi \in \Gamma\) and \(x \in X\). As in the proof of (3)⇒(2) in Theorem 3.10 we show that \((\mathcal{R}; V)\), where \(V\) is defined as in (3.16), is a minimal weak linearisation of \(k\). Letting \(\pi = \rho\), it is then easy to see that \((\mathcal{R}; \pi; V)\) is a boundedly adjointable \(\Gamma\)-invariant weak VH-space linearisation of \(k\).

4.3. Continuously adjointable invariant weak VH-space linearisations. Let \(Z\) be an admissible space. A triple \((\mathcal{K}; \pi; V)\) is called a continuously adjointable \(\Gamma\)-invariant weak VH-space linearisation of the \(Z\)-valued kernel \(k\) and the action of \(\Gamma\) on \(X\) if the requirements (ivhl1) and (ivhl2) hold and, instead of (ihvl2),

(ivhl2)′ \quad \pi: \Gamma \to \mathcal{L}^*_c(\mathcal{K})\) is a \(*\)-representation.

Clearly, for any continuously adjointable \(\Gamma\)-invariant weak VH-space linearisation \((\mathcal{K}; \pi; V)\) of \(k\), it follows that \(k\) is \(\Gamma\)-invariant.

If, in addition to (ivhl1), (ivhl2)′ and (ivhl3), the triple \((\mathcal{K}; \pi; V)\) also satisfies (ivhl4), that is, the weak VH-space linearisation \((\mathcal{H}; V)\) is topologically minimal, then \((\mathcal{K}; \pi; V)\) is called a topologically minimal continuously adjointable \(\Gamma\)-invariant weak VH-space linearisation of \(k\).

The unitary equivalence of two continuously adjointable \(\Gamma\)-invariant weak VH-space linearisations \((\mathcal{K}; \pi; V)\) and \((\mathcal{K}'; \pi'; V')\) of the same kernel \(k\) is defined as in the case of boundedly adjointable \(\Gamma\)-invariant weak VH-space linearisations, and their topological minimality implies their unitary equivalence.

The analog of Proposition 4.2 for topologically minimal continuously adjointable \(\Gamma\)-invariant weak VH-space linearisations holds as well.

The next theorem is the analogue of Theorem 4.6 for continuously adjointable \(\Gamma\)-invariant weak VH-space linearisations in which the boundedness condition (1)(c) of Theorem 4.6 is replaced with a weaker one.

**Theorem 4.7.** Let \(\Gamma\) be a \(*\)-semigroup that acts on the nonempty set \(X\), and let \(k: X \times X \to Z\) be a \(Z\)-valued kernel for some admissible space \(Z\). The following assertions are equivalent:
(1) \( k \) satisfies the following conditions:

(a) \( k \) is weakly positive semidefinite.
(b) \( k \) is \( \Gamma \)-invariant.
(c) For any \( \alpha \in \Gamma \) and any \( p \in S(Z) \), there exist \( q \in S(Z) \) and a constant \( c(\alpha) \geq 0 \) such that

\[
 p \left( \sum_{j,k=1}^{n} t_j \overline{t}_k k(\alpha \cdot x_k, \alpha \cdot x_j) \right) \leq c(\alpha)^2 q \left( \sum_{j,k=1}^{n} t_j \overline{t}_k k(x_k, x_j) \right)
\]

for all \( n \in \mathbb{N} \), \( x_1, \ldots, x_n \in X \) and \( t_1, \ldots, t_n \in \mathbb{C} \).

(2) \( k \) has a continuously adjointable \( \Gamma \)-invariant weak VH-space linearisation \( (K; \pi; V) \).

(3) \( k \) admits a weak \( Z \)-reproducing kernel VH-space \( \mathcal{R} \) and there exists a \( * \)-representation \( \rho : \Gamma \to \mathcal{L}_c^*(\mathcal{R}) \) such that \( \rho(\xi)k_x = k_{\xi \cdot x} \) for all \( \xi \in \Gamma \) and \( x \in X \).

Moreover, if (1), (2), or (3) holds, then a topologically minimal continuously adjointable \( \Gamma \)-invariant VH-space linearisation can be constructed, and a topologically minimal weak \( Z \)-reproducing kernel VH-space \( \mathcal{R} \) as in (3) can be constructed as well.

Proof. \((1) \Rightarrow (2)\). We consider the notation and constructions as in the proof of \((1) \Rightarrow (2)\) of Theorem 4.3, and follow the same idea as in the proof of \((1) \Rightarrow (2)\) of Theorem 4.6, with the use of (1)(c). For any \( \xi \in \Gamma \), \( f = Kg \) and \( p \in S(Z) \) there exist \( q \in S(Z) \) and \( c(\xi) \geq 0 \) such that

\[
p([\pi(\xi)f, \pi(\xi)f]_K) = p \left( \sum_{x,y \in X} g(y)g(x) k(\xi \cdot y, \xi \cdot x) \right) \leq c(\xi)^2 q \left( \sum_{x,y \in X} g(y)g(x) k(y, x) \right) = c(\xi)^2 q([f, f]_K),
\]

hence \( \pi(\xi) \) is continuous. This implies that \( \pi(\xi) \) can be uniquely extended by continuity to an operator \( \pi(\xi) \in \mathcal{L}_c(K) \). In addition, since \( \pi(\xi^*) \) also extends by continuity to an operator \( \pi(\xi^*) \in \mathcal{L}_c(K) \) and by \( (4.8) \), it follows that \( \pi(\xi) \) is adjointable and \( \pi(\xi^*) = \pi(\xi)^* \). We conclude that \( \pi \) is a \( * \)-representation of \( \Gamma \) in \( \mathcal{L}_c^*(K) \).

The uniqueness of the topologically minimal continuously adjointable \( \Gamma \)-invariant VH-space linearisation follows as usual.

\((2) \Rightarrow (1)\). By the proof of \((2) \Rightarrow (1)\) of Theorem 4.6 we only have to show that (c) holds. Let \( \alpha \in \Gamma \), \( n \in \mathbb{N} \), \( x_1, \ldots, x_n \in X \) and \( t_1, \ldots, t_n \in \mathbb{C} \). Then, since \( \pi(\alpha) \) is continuous and \( S(Z) \) is directed, there exist \( q \in S(Z) \) and
c(α) ≥ 0 such that
\[ p\left( \sum_{j,k=1}^{n} \overline{t}_kt_jk(\alpha \cdot x_k, \alpha \cdot x_j) \right) = p\left( \left[ \sum_{k=1}^{n} t_kV(x_k), \sum_{j=1}^{n} t_jV(x_j) \right]_{\mathcal{K}} \right) \]
\[ \leq c(\alpha)^2q\left( \left[ \sum_{k=1}^{n} t_kV(x_k), \sum_{j=1}^{n} t_jV(x_j) \right]_{\mathcal{K}} \right) = c(\alpha)^2q\left( \sum_{j,k=1}^{n} \overline{t}_kt_jk(x_k, x_j) \right). \]

(2)⇒(3). Let \((K; V; \pi)\) be a continuously adjointable \(\Gamma\)-invariant weak \(VH\)-space linearisation of \(k\). Using the same ideas as in the proof of (2)⇒(1) of Theorem 4.6 we obtain a continuous \(*\)-representation of \(\Gamma\) on the \(VH\)-space \(\mathcal{R}\) defined by \(\rho(\xi) = U\pi(\xi)U^{-1}\), where \(U : K \rightarrow \mathcal{R}\) is defined in (3.18).

(3)⇒(2). Let \(\mathcal{R} = \mathcal{R}(k)\) be the weak reproducing kernel \(VH\)-space of \(k\) and \(\rho : \Gamma \rightarrow \mathcal{L}^*(\mathcal{R})\) a \(*\)-representation such that \(k_{\xi_0x} = \rho(\xi)k_x\) for all \(\xi \in \Gamma\) and \(x \in X\). As in the proof of (3)⇒(2) in Theorem 4.6, letting \(\pi = \rho\), it is easy to see that \((\mathcal{R}; \pi; V)\) is a weak \(VH\)-space linearisation of \(k\), and \(\pi\) has the required properties.

5. Unification of some dilation theorems. In this section we show how various dilation theorems can be obtained as special cases of Theorems 4.3, 4.6 or 4.7.

5.1. Invariant kernels with values in the space of adjointable operators. We show that Theorem 2.8 in [3] can be seen as a special case of Theorem 4.3. We first recall the necessary definitions from [3].

In this subsection we will consider a kernel on a nonempty set \(X\) and taking values in \(\mathcal{L}^*(\mathcal{H})\), for a \(VE\)-space \(\mathcal{H}\) over an ordered \(*\)-space \(Z\), that is, a map \(l : X \times X \rightarrow \mathcal{L}^*(\mathcal{H})\).

A kernel \(l : X \times X \rightarrow \mathcal{L}^*(\mathcal{H})\) is called positive semidefinite if for all \(n \in \mathbb{N}\), \(x_1, \ldots, x_n \in X\) and \(h_1, \ldots, h_n \in \mathcal{H}\), we have
\[
(5.1) \quad \sum_{i,j=1}^{n} [l(x_i, x_j)h_j, h_i]_{\mathcal{H}} \geq 0.
\]

Let \(\Gamma\) be \(*\)-semigroup acting on \(X\). A \(\Gamma\)-invariant \(\mathcal{L}^*(\mathcal{H})\)-valued \(VE\)-space linearisation of a kernel \(l\) is a triple \((\mathcal{E}; \tilde{\pi}; \tilde{V})\) such that:

(hvel1) \(\mathcal{E}\) is a \(VE\)-space over the same ordered \(*\)-space \(Z\).

(hvel2) \(\tilde{\pi} : \Gamma \rightarrow \mathcal{L}^*(\mathcal{E})\) is a \(*\)-representation.

(hvel3) \(\tilde{V} : X \rightarrow \mathcal{L}^*(\mathcal{H}, \mathcal{E})\) satisfies \(k(x, y) = \tilde{V}(x)^*\tilde{V}(y)\) for all \(x, y \in X\) and \(\tilde{V}(\xi \cdot x) = \tilde{\pi}(\xi)\tilde{V}(x)\) for all \(x \in X\) and \(\xi \in \Gamma\).

If a \(\Gamma\)-invariant \(\mathcal{L}^*(\mathcal{H})\)-valued \(VE\)-space linearisation has the property that \(\text{Lin} V(X)\mathcal{H} = \mathcal{E}\), then it is called minimal. Two \(\Gamma\)-invariant \(\mathcal{L}^*(\mathcal{H})\)-\(VE\)-space
linearisations \((\tilde{E}; \tilde{\pi}; \tilde{V})\) and \((\tilde{F}; \tilde{\rho}; \tilde{W})\) of the same kernel \(I\) are called \textit{unitarily equivalent} if there exists a unitary operator \(U: \tilde{E} \to \tilde{F}\) such that \(U \tilde{\pi}(\gamma) = \tilde{\rho}(\gamma)U\) for all \(\gamma \in \Gamma\) and \(UV(x) = \tilde{W}(x)\) for all \(x \in X\).

Let \(\mathcal{H}^X\) be the vector space of all maps \(f: X \to \mathcal{H}\), for a nonempty set \(X\) and a VE-space \(\mathcal{H}\) over the ordered \(*\)-space \(Z\). A VE-space \(\tilde{\mathcal{R}}\) over the same ordered \(*\)-space \(Z\) is called an \(\mathcal{L}^*(\mathcal{H})\)-reproducing kernel VE-space on \(X\) of the kernel \(I\) if:

\[(hrk1)\] \(\tilde{\mathcal{R}}\) is a vector subspace of \(\mathcal{H}^X\).
\[(hrk2)\] For all \(x \in X\) and \(h \in \mathcal{H}\), the \(\mathcal{H}\)-valued function \(I_x h := I(\cdot, x)h\) belongs to \(\tilde{\mathcal{R}}\).
\[(hrk3)\] For all \(f \in \tilde{\mathcal{R}}\) we have \([f(x), h]_{\mathcal{H}} = [f, I_x h]_{\tilde{\mathcal{R}}}\) for all \(x \in X\) and \(h \in \mathcal{H}\).

The space \(\tilde{\mathcal{R}}\) is \textit{minimal} if \(\tilde{\mathcal{R}} = \text{Lin}\{I_x h \mid x \in X, h \in \mathcal{H}\}\).

\textbf{Theorem 5.1 (\cite{3} Theorem 2.8).} Let \(\Gamma\) be a \(*\)-semigroup acting on a nonempty set \(X\), \(\mathcal{H}\) be a VE-space on an ordered \(*\)-space \(Z\), and \(I: X \times X \to \mathcal{L}^*(\mathcal{H})\) be a kernel. The following assertions are equivalent:

\begin{enumerate}
\item \(I\) satisfies the following properties:
\begin{enumerate}
\item \(I\) is positive semidefinite.
\item \(I\) is \(\Gamma\)-invariant.
\end{enumerate}
\item \(I\) has a \(\Gamma\)-invariant \(\mathcal{L}^*(\mathcal{H})\)-valued VE-space linearisation \((\tilde{E}; \tilde{\pi}; \tilde{V})\).
\item \(I\) admits an \(\mathcal{L}^*(\mathcal{H})\)-reproducing kernel VE-space \(\tilde{\mathcal{R}}\) and there exists a \(*\)-representation \(\tilde{\rho}: \Gamma \to \mathcal{L}^*(\tilde{\mathcal{R}})\) such that \(\tilde{\rho}(\xi)I_x h = I_{\xi \cdot x} h\) for all \(\xi \in \Gamma\), \(x \in X\) and \(h \in \mathcal{H}\).
\end{enumerate}

Moreover, if (1), (2) or (3) holds, a minimal \(\Gamma\)-invariant \(\mathcal{L}^*(\mathcal{H})\)-VE-space linearisation can be constructed, and a pair \((\tilde{\mathcal{R}}; \tilde{\rho})\) as in (3) with \(\tilde{\mathcal{R}}\) minimal can be obtained as well.

\textbf{Proof.} (1) \(\Rightarrow\) (2). Define a kernel \(k: (X \times \mathcal{H}) \times (X \times \mathcal{H}) \to Z\) by
\[
k((x, h), (y, g)) := [I(y, x)h, g]_{\mathcal{H}}, \quad x, y \in X, h, g \in \mathcal{H},
\]
and an action of \(\Gamma\) on \(X \times \mathcal{H}\) by \(\xi \cdot (x, h) = (\xi \cdot x, h)\) for all \(\xi \in \Gamma\), \(x \in X\) and \(h \in \mathcal{H}\). Since \(I\) is positive semidefinite in the sense of \([5,1]\), and \(\Gamma\)-invariant, it follows that \(k\) is weakly positive semidefinite and \(\Gamma\)-invariant.

By Theorem 4.3 there exists a minimal \(\Gamma\)-invariant weak VE-space linearisation \((\mathcal{E}; \pi; \mathcal{V})\) of \(k\). For each \(x \in X\) a linear operator of VE-spaces \(\mathcal{V}(x): \mathcal{H} \to \mathcal{E}\) can be defined by \(\mathcal{V}(x)h = \mathcal{V}(x, h)\).

We now have \([\mathcal{V}(x)h, \mathcal{V}(y)g]_{\mathcal{E}} = k((x, h), (y, g)) = [I(y, x)h, g]_{\mathcal{H}}\) for all \(x, y \in X\) and \(h, g \in \mathcal{H}\). By the minimality of \(\mathcal{E}\), it follows that \(\mathcal{V}(x)\) is an adjointable operator with \(\mathcal{V}(y)^* \mathcal{V}(x) = I(y, x)\) for all \(x, y \in X\).

Since \(\pi(\xi)\mathcal{V}(x) = \tilde{\mathcal{V}}(\xi \cdot x)\) for all \(\xi \in \Gamma\) and \(x \in X\), it follows that \((\mathcal{E}; \pi; \tilde{\mathcal{V}})\) is a minimal \(\Gamma\)-invariant \(\mathcal{L}^*(\mathcal{H})\)-valued VE-space linearisation of \(I\).
\[(2) \Rightarrow (3). \text{ Let } (\tilde{\mathcal{E}}; \tilde{\pi}; \tilde{V}) \text{ be a } \Gamma \text{-invariant } \mathcal{L}^\ast(\mathcal{H}) \text{-valued VE-space linearisation of } l, \text{ hence } l(x, y) = \tilde{V}(x)^*\tilde{V}(y) \text{ for all } x, y \in X. \text{ Define } V : X \times \mathcal{H} \to \tilde{\mathcal{E}} \text{ by } \]
\[
(5.2) \quad V(x, h) = \tilde{V}(x)h, \quad x \in X, h \in \mathcal{H}.
\]

We also have
\[
(5.3) \quad \tilde{\pi}(\xi)V(x, h) = \tilde{\pi}(\xi)\tilde{V}(x)h = \tilde{V}(\xi \cdot x)h, \quad \xi \in \Gamma, x \in X, h \in \mathcal{H},
\]
hence \(\tilde{\pi}(\xi)\) leaves \(\tilde{\mathcal{E}}_0 = \text{Lin} V(X, \mathcal{H})\) invariant for all \(\xi \in \Gamma\). Consequently, \((\tilde{\mathcal{E}}_0; \tilde{\pi}; V)\) is a minimal \(\Gamma\)-invariant weak VE-space linearisation for the kernel \(k : (X \times \mathcal{H}) \times (X \times \mathcal{H}) \to Z\) defined by
\[
k((x, h), (y, g)) = [h, l(x, y)g]_{\mathcal{H}}, \quad x, y \in X, h, g \in \mathcal{H},
\]
and the action of \(\Gamma\) on \(X \times \mathcal{H}\) given by
\[
(5.4) \quad \xi \cdot (x, h) = (\xi \cdot x, h), \quad \xi \in \Gamma, x \in X, h \in \mathcal{H}.
\]

By Theorem 4.3 there exists a minimal weak \(Z\)-reproducing kernel VE-space \(\mathcal{R} \subseteq Z^X \times \mathcal{H}\), with reproducing kernel \(k\), and a \(*\)-representation \(\rho : \Gamma \to \mathcal{L}^\ast(\mathcal{R})\) such that \(\rho(\xi)k_{(x, h)} = k_{(\xi \cdot x, h)}\) for all \(\xi \in \Gamma, x \in X\) and \(h \in \mathcal{H}\). As the proof of Theorem 4.3 shows, we can assume that \(\mathcal{R}\) is the collection of all maps \(X \times \mathcal{H} \to Z\) defined by \(X \times \mathcal{H} \ni (x, h) \mapsto [V(x)h, f]_{\tilde{\mathcal{E}}},\) where \(f \in \tilde{\mathcal{E}}_0\), which provides an identification of \(\mathcal{R}\) with \(\tilde{\mathcal{E}}_0\) via
\[
(5.5) \quad f(x, h) = [V(x, h), f]_{\mathcal{R}} = [\tilde{V}(x)h, f]_{\tilde{\mathcal{E}}} = [h, \tilde{V}(x)^*f]_{\mathcal{H}}, \quad h \in \mathcal{H}.
\]
Consequently, for each \(f \in \mathcal{R}\) and \(x \in X\), there exists a unique vector \(\tilde{f}(x) = \tilde{V}(x)^*f \in \mathcal{H}\) such that
\[
(5.6) \quad f(x, h) = [h, \tilde{f}(x)]_{\mathcal{H}}, \quad h \in \mathcal{H},
\]
which gives rise to a map \(\mathcal{R} \ni f \mapsto \tilde{f} \in \mathcal{H}^X\). Let \(\tilde{\mathcal{R}}\) be the vector space of all \(\tilde{f}\) for \(f \in \mathcal{R}\). From the reproducing property of \(l\), it follows that \(l_x h = k_{(x, h)} \in \tilde{\mathcal{R}}\) for all \(x \in X\) and \(h \in \mathcal{H}\).

It is easy to see that the map \(U : \mathcal{R} \ni f \mapsto \tilde{f} \in \tilde{\mathcal{R}}\) is linear, one-to-one and onto. Therefore, defining \([f, g]_{\tilde{\mathcal{R}}} := [f, g]_{\mathcal{R}}\) makes \(\tilde{\mathcal{R}}\) a VE-space, and \(U\) becomes a unitary operator of VE-spaces. If we define \(\tilde{\rho} := U \rho U^\ast\), the pair \((\tilde{\mathcal{R}}; \tilde{\rho})\) has all the required properties.

\((3) \Rightarrow (1). \text{ Assume that } (\tilde{\mathcal{R}}; \tilde{\rho}) \text{ is a pair consisting of an } \mathcal{L}^\ast(\mathcal{H}) \text{-reproducing kernel VE-space of } l \text{ and a } \ast\text{-representation } \tilde{\rho} : \Gamma \to \mathcal{L}^\ast(\tilde{\mathcal{R}})\) such that
\[ \rho(\xi)l_{x,h} = l_{\xi,x}h \text{ for all } \xi \in \Gamma, x \in X \text{ and } h \in \mathcal{H}. \] We have
\begin{align*}
\sum_{i,j=1}^{n} [l(x_i, x_j)h_j, h_i] &= \sum_{i,j=1}^{n} [l_{x_j}h_j(x_i), h_i] = \sum_{i,j=1}^{n} [l_{x_j}h_j, l_{x_i}h_i] \\
&= \left[ \sum_{j=1}^{n} l_{x_j}h_j, \sum_{i=1}^{n} l_{x_i}h_i \right] \geq 0
\end{align*}
for all \( n \in \mathbb{N} \), \( \{x_i\}_{i=1}^{n} \subset X \) and \( \{h_i\}_{i=1}^{n} \subset \mathcal{H} \). Therefore \( l \) is positive semidefinite in the sense of (5.1). Moreover, by (hrk3),
\[ [l(x, \xi \cdot y)h, g] = [l_{\xi,y}h(x), g] = [\tilde{\rho}(\xi)l_yh, l_xg] = [l_yh, \tilde{\rho}(\xi^*)l_xg] = [l(\xi^*x, y)h, g] \]
for all \( x, y \in X \) and \( g, h \in \mathcal{H} \), and the \( \Gamma \)-invariance of \( l \) is proven. \( \blacksquare \)

**Remark 5.2.** The crucial point in the proof of \((2) \Rightarrow (3)\) in Theorem 5.1 is the proof of (5.6) which we obtained as a consequence of the identification of \( \mathcal{R} \) with \( \mathcal{E}_0 \). Now we give a direct proof of (5.6), without using this identification.

By minimality, \( \mathcal{R} = \text{Lin}\{k_{(x,h)} \mid x \in X, h \in \mathcal{H}\} \), so let \( f = \sum_{j=1}^{n} \alpha_j k_{(y_j, g_j)} \), for some \( n \in \mathbb{N} \), \( y_1, \ldots, y_n \in X \), \( g_1, \ldots, g_n \in \mathcal{H} \) and \( \alpha_1, \ldots, \alpha_n \in \mathbb{C} \), be an arbitrary element of \( \mathcal{R} \). Then, for any \( x \in X \) and \( h \in \mathcal{H} \),
\[ f(x, h) = \sum_{j=1}^{n} \alpha_j k_{(y_j, g_j)}(x, h) = \sum_{j=1}^{n} \alpha_j k((x, h), (y_j, g_j)) \]
\[ = \sum_{j=1}^{n} \alpha_j [V(x, h), V(y_j, g_j)]_{\mathcal{E}} = \sum_{j=1}^{n} \alpha_j [\tilde{V}(x)h, \tilde{V}(y_j)g_j]_{\mathcal{E}} \]
\[ = \sum_{j=1}^{n} \alpha_j [h, \tilde{V}(x)^* \tilde{V}(y_j)g_j]_{\mathcal{H}} = [h, \tilde{V}(x)^* \sum_{j=1}^{n} \alpha_j \tilde{V}(y_j)g_j]_{\mathcal{H}}, \]
hence (5.6) holds with \( \tilde{f}(x) = \tilde{V}(x)^* \sum_{j=1}^{n} \alpha_j \tilde{V}(y_j)g_j \).

**5.2. Invariant kernels with values continuously adjointable operators.** In this subsection we show that Theorem 2.10 of [4] can be recovered as a special case of Theorem 4.7. We first review some definitions of [4].

Let \( X \) be a nonempty set and let \( \mathcal{H} \) be a VH-space over an admissible space \( Z \). In this subsection we will consider kernels \( k : X \times X \rightarrow \mathcal{L}_c^*(\mathcal{H}) \). Such a kernel \( k \) is called **positive semidefinite** if it is \( n \)-positive for all \( n \) in the sense of (5.1).
An $L_c^*(\mathcal{H})$-valued VH-space linearisation of $k$, or $L_c^*(\mathcal{H})$-valued VH-space Kolmogorov decomposition of $k$, is a pair $(\mathcal{K}; V)$, subject to the following conditions:

(vhl1) $\mathcal{K}$ is a VH-space over $Z$.

(vhl2) $V: X \to L_c^*(\mathcal{H},\mathcal{K})$ satisfies $k(x, y) = V(x)^*V(y)$ for all $x, y \in X$.

$(\mathcal{K}; V)$ is called topologically minimal if

(vhl3) $\text{Lin} V(X)\mathcal{H}$ is dense in $\mathcal{K}$.

We call $k$ $\Gamma$-invariant if

\[(5.7) \quad k(\xi \cdot x, y) = k(x, \xi^* \cdot y), \quad \xi \in \Gamma, x, y \in X.\]

A triple $(\mathcal{K}; \pi; V)$ is called a $\Gamma$-invariant $L_c^*(\mathcal{H})$-valued VH-space linearisation for $k$ if:

(ihl1) $(\mathcal{K}; V)$ is an $L_c^*(\mathcal{H})$-valued VH-space linearisation of $k$.

(ihl2) $\pi: \Gamma \to L_c^*(\mathcal{K})$ is a $\ast$-representation.

(ihl3) $V(\xi \cdot x) = \pi(\xi)V(x)$ for all $\xi \in \Gamma$ and all $x \in X$.

Also, $(\mathcal{K}; \pi; V)$ is topologically minimal if the $L_c^*(\mathcal{H})$-VH-space linearisation $(\mathcal{K}; V)$ is topologically minimal, that is, $\mathcal{K}$ is the closure of the linear span of $V(X)\mathcal{H}$.

A VH-space $\mathcal{R}$ over the ordered $\ast$-space $Z$ is called an $L_c^*(\mathcal{H})$-reproducing kernel VH-space on $X$ if there exists a Hermitian kernel $k: X \times X \to L_c^*(\mathcal{H})$ such that:

(rkh1) $\mathcal{R}$ is a linear subspace of $\mathcal{H}^X$.

(rkh2) For all $x \in X$ and all $h \in \mathcal{H}$, the $\mathcal{H}$-valued function $k_x h = k(\cdot, x)h$ is in $\mathcal{R}$.

(rkh3) For all $f \in \mathcal{R}$ we have $[f(x), h]_\mathcal{H} = [f, k_x h]_\mathcal{R}$ for all $x \in X$ and $h \in \mathcal{H}$.

(rkh4) For all $x \in X$ the evaluation operator $\mathcal{R} \ni f \mapsto f(x) \in \mathcal{H}$ is continuous.

In this operator valued setting, note the appearance of axiom (rkh4) which makes a difference with classical cases; see [4] for some results pointing out its significance.

**Theorem 5.3 ([4] Theorem 2.10).** Let $\Gamma$ be a $\ast$-semigroup that acts on a nonempty set $X$ and let $l: X \times X \to L_c^*(\mathcal{H})$ be a kernel, for some VH-space $\mathcal{H}$ over an admissible space $Z$. Then the following assertions are equivalent:

(1) $l$ has the following properties:

(a) $l$ is positive semidefinite in the sense of (5.1), and $\Gamma$-invariant.
In addition, if \( x, y \in X \) and any \( \ell \in S(Z) \) and \( c_\ell(x) \geq 0 \) such that for all \( n \in \mathbb{N} \), \( \{h_i\}_{i=1}^n \subset \mathcal{H} \) and \( \{x_i\}_{i=1}^n \subset X \)

\[
p\left( \sum_{i,j=1}^n [I(x, x_i)h_j, h_i]_{\mathcal{H}} \right) \leq c_\ell(x) q\left( \sum_{i,j=1}^n [I(x_i, x_j)h_j, h_i]_{\mathcal{H}} \right).
\]

(c) For any \( x \in X \) and any seminorm \( p \in S(Z) \), there exists a seminorm \( q \in S(Z) \) and a constant \( c_p(x) \geq 0 \) such that for all \( n \in \mathbb{N} \), \( \{y_i\}_{i=1}^n \subset X \) and \( \{h_i\}_{i=1}^n \subset \mathcal{H} \) we have

\[
p\left( \sum_{i,j=1}^n [I(x, y_i)h_i, I(x, y_j)h_j]_{\mathcal{H}} \right) \leq c_p(x) q\left( \sum_{i,j=1}^n [I(y_i, y_j)h_i, h_j]_{\mathcal{H}} \right).
\]

(2) \( 1 \) has a \( \Gamma \)-invariant \( \mathcal{L}_c^* (\mathcal{H}) \)-valued VH-space linearisation \((\mathcal{K}; \pi; V)\).

(3) \( 1 \) admits an \( \mathcal{L}_c^* (\mathcal{H}) \)-reproducing kernel VH-space \( \mathcal{R} \) and there exists a \( * \)-representation \( \rho : \Gamma \rightarrow \mathcal{L}_c^* (\mathcal{R}) \) such that \( \rho(\xi) \pi_x h = I_{\xi, x} h \) for all \( \xi \in \Gamma \), \( x \in X \) and \( h \in \mathcal{H} \).

In addition, if (1), (2), or (3) holds, then a minimal \( \Gamma \)-invariant \( \mathcal{L}_c^* (\mathcal{H}) \)-valued VH-space linearisation of 1 can be constructed, and the pair \((\mathcal{R}; \rho)\) as in (3) can be chosen with \( \mathcal{R} \) topologically minimal as well.

**Proof.** (1)\( \Rightarrow \) (2). Define \( k : (X \times \mathcal{H}) \times (X \times \mathcal{H}) \rightarrow \mathcal{H} \) by

\[
k((x, h), (y, g)) := [I(y, x)h, g]_{\mathcal{H}}, \quad x, y \in X, \ h, g \in \mathcal{H}.
\]

As in the proof of (1)\( \Rightarrow \) (2) of Theorem 5.1, \( k \) is weakly positive semidefinite and \( \Gamma \)-invariant. It can be directly checked that this kernel has property (1)(c) of Theorem 4.6.

By Theorem 4.6, there exists a minimal \( \Gamma \)-invariant weak VH-space linearisation \((\mathcal{K}; \pi; V)\) of \( k \). The same arguments as in the proof of (1)\( \Rightarrow \) (2) of Theorem 5.1 show that, for any \( x \in X \), there exists an adjointable operator \( \tilde{V}(x) : \mathcal{H} \rightarrow \mathcal{K}_0 \) between VE-spaces, given by \( \tilde{V}(x) h := V(x, h) \) for \( x \in X \) and \( h \in \mathcal{H} \), where \( \mathcal{K}_0 := \text{Lin} V(X) \mathcal{H} \), with \( \tilde{V}(x)^* \tilde{V}(y) = I(x, y) \) for all \( x, y \in X \). Arguing as in [4] proof of (1)\( \Rightarrow \) (2) of Theorem 2.10, we find that \( \tilde{V}(x) \in \mathcal{L}_c^* (\mathcal{K}_0, \mathcal{H}) \) for any \( x \in X \). Consequently, \( \tilde{V}(x)^* \) extends uniquely to \( \tilde{V}(x)^* \in \mathcal{L}_c^* (\mathcal{K}, \mathcal{H}) \) for each \( x \in X \). It follows that \((\mathcal{K}; \pi; V)\) is a \( \Gamma \)-invariant \( \mathcal{L}_c^* (\mathcal{H}) \)-valued VH-space linearisation of 1.

(2)\( \Rightarrow \) (3). The proof is similar to the proof of (2)\( \Rightarrow \) (3) of Theorem 5.1. Let \((\tilde{\mathcal{K}}; \tilde{\pi}; \tilde{V})\) be a \( \Gamma \)-invariant \( \mathcal{L}_c^* (\mathcal{H}) \)-valued VH-space linearisation of 1. Define \( \tilde{V} : X \times \mathcal{H} \rightarrow \tilde{\mathcal{K}} \) by \( \tilde{V}(x, h) = \tilde{V}(x) h \) for all \( x \in X \) and \( h \in \mathcal{H} \). Letting \( \tilde{\mathcal{K}}_0 = \text{Lin} \tilde{V}(X, \mathcal{H}) \subseteq \tilde{\mathcal{K}} \), we see that \((\tilde{\text{Lin}} V(X, \mathcal{H}); \tilde{\pi}_0; \tilde{V})\) is a topological minimal \( \Gamma \)-invariant weak VH-space linearisation for the kernel \( k : (X \times \mathcal{H}) \times (X \times \mathcal{H}) \rightarrow \mathcal{H} \) defined by \( k((x, h), (y, g)) = [V(x, h), V(y, g)] \) for all \( x, y \in X \) and \( h, g \in \mathcal{H} \) and the usual action of \( \Gamma \) on \( X \times \mathcal{H} \).
By Theorem 4.6 there exists a topologically minimal weak Z-reproducing kernel VH-space $\mathcal{R}$ and a $*$-representation $\rho: \Gamma \to \mathcal{L}_c^*(\mathcal{R})$ such that $\rho(\xi)k_{(x,h)} = k_{\xi,(x,h)}$ for all $\xi \in \Gamma$, $x \in X$ and $h \in \mathcal{H}$. As in the proof of (2)$\Rightarrow$(3) of Theorem 5.1, there is a map $\mathcal{R} \ni f \mapsto Uf = \tilde{f} \in \mathcal{H}^X$ linear and bijective between $\mathcal{R}$ and its range $\tilde{\mathcal{R}} \subseteq \mathcal{H}^X$. Let $[\tilde{f}, \tilde{g}]_{\tilde{\mathcal{R}}} = [f, g]_{\mathcal{R}}$ for all $f, g \in \mathcal{R}$. Then $\tilde{\mathcal{R}}$ becomes an $\mathcal{H}$-valued reproducing kernel VH-space with kernel $l$, and then with $\tilde{\rho} = U\rho U^*$, $(\tilde{\mathcal{R}}, \tilde{\rho})$ has all the required properties.

(3)$\Rightarrow$(1). Assume that $(\tilde{\mathcal{R}}; \tilde{\rho})$ consists of an $\mathcal{L}_c^*(\mathcal{H})$-valued reproducing kernel VH-space of $l$ and a $*$-representation $\tilde{\rho}$ of $\Gamma$ on $\mathcal{L}_c^*(\tilde{\mathcal{R}})$ such that $\rho(\xi)l_{x,h} = l_{\xi,x,h}$ for all $\xi \in \Gamma$, $x \in X$ and $h \in \mathcal{H}$. As in the proof of (3)$\Rightarrow$(1) of Theorem 5.1, the kernel $l$ is shown to be positive semidefinite and $\Gamma$-invariant. On the other hand, the inequalities (b) and (c) are obtained from the continuity of $\rho(\xi): \mathcal{R} \to \mathcal{R}$ for any $\xi \in \Gamma$ and, respectively, from the continuity of the evaluation operator $E_x: \mathcal{R} \to \mathcal{H}$ for any $x \in X$.

5.3. Invariant kernels with values boundedly adjointable operators. We show that Theorem 4.2 in [11] is a special case of Theorem 4.6. We recall the necessary definitions of [11].

Given a $\mathcal{B}^*(\mathcal{H})$-valued kernel $l$ on a nonempty set $X$, where $\mathcal{H}$ is a VH-space over the admissible space $Z$, a $\mathcal{B}^*(\mathcal{H})$-valued VH-space linearisation of $l$ is a pair $(\tilde{K}; V)$, where:

(hvhl1) $\tilde{K}$ is a VH-space over $Z$.
(hvhl2) $V: X \to \mathcal{B}^*(\mathcal{H}; \tilde{K})$ satisfies $l(x,y) = \tilde{V}(x)^*\tilde{V}(y)$ for all $x, y \in X$.

If $\Gamma$ is a $*$-semigroup acting on $X$, then $(\tilde{K}; \tilde{\pi}; \tilde{V})$ is called an $\Gamma$-invariant $\mathcal{B}^*(\mathcal{H})$-valued VH-space linearisation of $l$ if, in addition to (hvhl1) and (hvhl2):

(hvhl3) $\tilde{\pi}: \Gamma \to \mathcal{B}^*(\tilde{K})$ is a $*$-representation.
(hvhl4) $\tilde{V}(\xi \cdot x) = \tilde{\pi}(\xi)V(x)$ for all $\xi \in \Gamma$ and $x \in X$.

If moreover

(hvhl5) Lin$\tilde{V}(X)$ is dense in $\tilde{K}$,

then $(\tilde{K}; \tilde{\pi}; \tilde{V})$ is called topologically minimal.

Given a nonempty set $X$ and a VH-space $\mathcal{H}$ over the admissible space $Z$, a VH-space $\tilde{\mathcal{R}}$ over $\tilde{Z}$ is called a $\mathcal{B}^*(\mathcal{H})$-valued reproducing kernel VH-space on $X$ if there exists a kernel $l: X \times X \to \mathcal{B}^*(\mathcal{H})$ such that:

(hkr1) $\tilde{\mathcal{R}}$ is a linear subspace of $\mathcal{H}^X$.
(hkr2) $l_{x,h} = 1(\cdot, x)h \in \tilde{\mathcal{R}}$ for all $x \in X$ and $h \in \mathcal{H}$.
(hkr3) $[f(x), h]_{\mathcal{H}} = [f, l_{x,h}]_{\tilde{\mathcal{R}}}$ for all $f \in \tilde{\mathcal{R}}$, $x \in X$ and $h \in \mathcal{H}$. 

\( \tilde{R} \) is called topologically minimal if

(hkr4) \( \text{Lin}\{1_x h \mid x \in X, h \in \mathcal{H}\} \) is dense in \( \tilde{R} \),

and, in this case, \( \tilde{R} \) is uniquely determined by the kernel \( 1 \).

**Theorem 5.4** ([III Theorem 4.2]). Let \( \Gamma \) be a \(*\)-semigroup acting on a nonempty set \( X \), \( \mathcal{H} \) be a VH-space on an admissible space \( Z \), and \( 1 : X \times X \to \mathcal{B}^*(\mathcal{H}) \) be a kernel. Then the following are equivalent:

(1) \( 1 \) has the following properties:

(a) \( 1 \) is positive semidefinite.
(b) \( 1 \) is invariant under the action of \( \Gamma \) on \( X \).
(c) For any \( \alpha \in \Gamma \) there exists \( c(\alpha) \geq 0 \) such that, for all \( n \in \mathbb{N} \),

\[
\sum_{i,j=1}^{n} [1(\alpha \cdot x_i, \alpha \cdot x_j) h_j, h_i]_{\mathcal{H}} \leq c(\alpha)^2 \sum_{i,j=1}^{n} [1(x_i, x_j) h_j, h_i]_{\mathcal{H}}.
\]

(2) \( 1 \) has a \( \Gamma \)-invariant \( \mathcal{B}^*(\mathcal{H}) \)-valued VH-space linearisation \( (\tilde{\mathcal{E}}; \bar{\pi}; \tilde{V}) \).

(3) \( 1 \) admits a \( \mathcal{B}^*(\mathcal{H}) \)-reproducing kernel VH-space \( \tilde{\mathcal{R}} \) and there exists a \(*\)-representation \( \tilde{\rho} : \Gamma \to \mathcal{B}^*(\tilde{\mathcal{R}}) \) such that \( \tilde{\rho}(\xi)1_x h = 1_{\xi \cdot x} h \) for all \( \xi \in \Gamma \),

\( x \in X \) and \( h \in \mathcal{H} \).

Moreover, if (1), (2) or (3) holds, then a topologically minimal \( \Gamma \)-invariant \( \mathcal{B}^*(\mathcal{H}) \)-valued VH-space linearisation can be constructed.

**Proof.** (1)\( \Rightarrow \)(2). Define \( k : (X \times \mathcal{H}) \times (X \times \mathcal{H}) \to Z \) by

\[
k((x, h), (y, g)) := [1(y, x) h, g]_{\mathcal{H}}
\]

for all \( x, y \in X \) and \( h, g \in \mathcal{H} \). Then \( k \) is weakly positive semidefinite and \( \Gamma \)-invariant under the action of \( \Gamma \) on \( X \times \mathcal{H} \) given by \( \xi \cdot (x, h) = (\xi \cdot x, h) \) for all \( \xi \in \Gamma \), \( x \in X \) and \( h \in \mathcal{H} \), and it satisfies condition (1)(c) of Theorem 4.6.

By Theorem 4.6 there exists a minimal \( \Gamma \)-invariant weak VH-space linearisation \( (\tilde{\mathcal{K}}; V; \pi) \) of \( k \). The same arguments as in the proof of (1)\( \Rightarrow \)(2) of Theorem 5.3 give an adjointable operator \( \tilde{V}(x) : \mathcal{H} \to \mathcal{K}_0 \) between VH-spaces, given by \( \tilde{V}(x) h := V(x, h) \) for \( x \in X \) and \( h \in \mathcal{H} \), where \( \mathcal{K}_0 := \text{Lin} V(X) \mathcal{H} \), with the property that \( \tilde{V}(x)^* \tilde{V}(y) = 1(x, y) \) for all \( x, y \in X \). Arguing as in the proof of Theorem 3.3, we find that \( V(x) \in \mathcal{B}^*(\mathcal{H}, \mathcal{K}_0) \) and \( \tilde{V}(x)^* \in \mathcal{B}^*(\mathcal{K}_0, \mathcal{H}) \). Hence \( \tilde{V}(x)^* \) extends uniquely to \( V(x)^* \) in \( \mathcal{B}^*(\tilde{\mathcal{K}}, \mathcal{H}) \) for each \( x \in X \). It follows that \( (\tilde{\mathcal{K}}; \pi; \tilde{V}) \) is a topologically minimal \( \Gamma \)-invariant \( \mathcal{B}^*(\mathcal{H}) \)-valued VH-space linearisation of \( 1 \).

(2)\( \Rightarrow \)(3). Let \( (\tilde{\mathcal{K}}; \bar{\pi}; \tilde{V}) \) be a \( \Gamma \)-invariant \( \mathcal{B}^*(\mathcal{H}) \)-valued VH-space linearisation of \( 1 \). We essentially use Theorem 4.6 with details very close to the proof of (2)\( \Rightarrow \)(3) in Theorem 5.3 except that we obtain bounded adjointable operators instead of continuously adjointable ones. Define \( V : X \times \mathcal{H} \to \tilde{\mathcal{K}} \) by \( V(x, h) = \tilde{V}(x) h \) for all \( x \in X \) and \( h \in \mathcal{H} \). Then \( (\text{Lin}(V(X, \mathcal{H}); V; \bar{\pi})) \) is a
topologically minimal weak $\Gamma$-invariant VH-space linearisation for the kernel $k: (X \times H) \times (X \times H) \to Z$ defined by $k((x, h), (y, g)) = [V(x, h), V(y, g)]$ for all $x, y \in X$ and $h, g \in H$.

By Theorem 4.6 there exists a weak $Z$-reproducing kernel VH-space $\mathcal{R}$ and a $\ast$-representation $\rho: \Gamma \to \mathcal{B}(\mathcal{R})$ such that $\rho(\xi)k(x, h) = k(\xi, x, h)$ for all $\xi \in \Gamma$, $x \in X$ and $h \in H$.

With the same notations and definitions as in (2)$\Rightarrow$(3) of Theorem 5.3, defining $\tilde{f}, \tilde{g} := [f, g]_{\mathcal{R}}$ makes $\tilde{\mathcal{R}}$ a $\mathcal{B}(H)$-reproducing kernel VH-space with reproducing kernel $l$, and $U$ becomes a unitary operator of VH-spaces. With $\tilde{\rho} := U \rho U^*$, the pair $(\tilde{\mathcal{R}}, \tilde{\rho})$ has all the required properties.

(3)$\Rightarrow$(1). Assume that $(\tilde{\mathcal{R}}; \tilde{\rho})$ is a $\mathcal{B}(H)$-reproducing kernel VH-space of $l$ with a representation $\tilde{\rho}$ of $\Gamma$ on $\mathcal{B}(\tilde{\mathcal{R}})$ such that $\rho(\xi)l(x, h) = l(\xi, x, h)$ for all $\xi \in \Gamma$, $x \in X$ and $h \in H$. Similarly to the proof of (3)$\Rightarrow$(1) in Theorem 5.1, the kernel $l$ is shown to be positive semidefinite and $\Gamma$-invariant. On the other hand, as the linear operator $\tilde{\rho}(\xi): H \to H$ is bounded for all $\xi \in \Gamma$, it follows that $l$ has property (c).

5.4. Positive semidefinite $\mathcal{L}(\mathcal{X}, \mathcal{X}_Z')$-valued maps on $\ast$-semigroups.

In this subsection we obtain stronger versions of [26, Theorems 3.1 and 4.2] as applications of Theorems 4.3 and 4.6, respectively. We first reorganise some definitions from [26] and [13].

Let $\mathcal{X}$ be a vector space, and $Z$ an ordered $\ast$-space. We denote by $\mathcal{X}_Z'$ the space of all conjugate linear functions from $\mathcal{X}$ to $Z$ and call it the algebraic conjugate $Z$-dual space. Let $\mathcal{L}(\mathcal{X}, \mathcal{X}_Z')$ denote the vector space of all linear operators $T: \mathcal{X} \to \mathcal{X}_Z'$. For any VE-space $\mathcal{E}$ over $Z$ and any linear operator $A: \mathcal{X} \to \mathcal{E}$, we define a linear operator $A': \mathcal{E} \to \mathcal{X}_Z'$, called the algebraic $Z$-adjoint operator, by

$$ (A'f)(x) = [Ax, f]_{\mathcal{E}}, \quad f \in \mathcal{E}, \ x \in \mathcal{X}. $$

If $\Gamma$ is a $\ast$-semigroup, a map $T: \Gamma \to \mathcal{L}(\mathcal{X}, \mathcal{X}_Z')$ is called $\mathcal{L}(\mathcal{X}, \mathcal{X}_Z')$-valued $n$-positive if

$$ \sum_{i,j=1}^n (T_{s_i^* s_j} x_j)(x_i) \geq 0_Z \quad \text{for all } (s_i)_{i=1}^n \subset \Gamma \text{ and all } (x_j)_{j=1}^n \subset \mathcal{X}. $$

If $T$ is $n$-positive for all $n \in \mathbb{N}$ then it is called $\mathcal{L}(\mathcal{X}, \mathcal{X}_Z')$-valued positive semidefinite.

Remarks 5.5. With notation as before, let $T: \Gamma \to \mathcal{L}(\mathcal{X}, \mathcal{X}_Z')$.

1. We define a kernel $k: (\Gamma \times \mathcal{X}) \times (\Gamma \times \mathcal{X}) \to Z$ by

$$ k((s, x), (t, y)) = (T_{s^* t} y)x, \quad s, t \in \Gamma, \ x, y \in \mathcal{X}. $$
Then for all \( n \in \mathbb{N} \), all \( \alpha_1, \ldots, \alpha_n \in \mathbb{C} \), and all \( (s_i, x_i)_{i=1}^n \subset S \times X \),
\[
\sum_{i,j=1}^n \overline{\alpha}_i \alpha_j k((s_i, x_i), (s_j, x_j)) = \sum_{i,j=1}^n \overline{\alpha}_i \alpha_j (T_{s_i^* s_j} x_j) x_i = \sum_{i,j=1}^n (T_{s_i^* s_j} \alpha_j x_j)(\alpha_i x_i).
\]
This shows that, for \( n \in \mathbb{N} \), the map \( T \) is \( n \)-positive if and only if the kernel \( k \) is weakly \( n \)-positive. In particular, \( T \) is positive semidefinite if and only if \( k \) is weakly positive semidefinite.

(2) Recall from (3.2) that the kernel \( k \) is Hermitian if \( k((s, x), (t, y)) = k((t, y), (s, x))^* \) for all \( s, t \in \Gamma \) and all \( x, y \in \mathcal{X} \). From (5.11) it follows that \( k \) is Hermitian if and only if
\[
(T_{s^* t} y) x = ((T_{t^* s} x) y)^* \quad \text{for all } s, t \in \Gamma \text{ and } x, y \in \mathcal{X}.
\]
Consequently, Lemma 3.1 implies that if \( T \) is 2-positive, then (5.12) holds.

In addition, if \( \Gamma \) has a unit \( e = e^* \), then (5.12) is equivalent to
\[
(T_{s^*} y) x = ((T_s x) y)^* \quad \text{for all } s \in \Gamma \text{ and } x, y \in \mathcal{X}.
\]

(3) We define a left action of \( \Gamma \) on \( \Gamma \times \mathcal{X} \) by
\[
u \cdot (s, x) = (u s, x) \quad \text{for all } u, s \in \Gamma \text{ and } x \in \mathcal{X}.
\]
For all \( u \in \Gamma \) and \( (s, x) \in \Gamma \times \mathcal{X} \) we have
\[
k((s, x), u \cdot (t, y)) = (T_{s^* u t} y) x = (T_{(u^* s)^* t y}) x = k(u^*(s, x), (t, y)),
\]
hence the kernel \( k \) is \( \Gamma \)-invariant.

**Theorem 5.6.** Let \( Z \) be an ordered \( * \)-space, let \( \mathcal{X} \) be a complex vector space with algebraic conjugate \( Z \)-dual space \( \mathcal{X}_Z' \), and consider \( T: \Gamma \to \mathcal{L}(\mathcal{X}, \mathcal{X}_Z') \) for some \( * \)-semigroup \( \Gamma \) with unit. The following assertions are equivalent:

(i) \( T \) is positive semidefinite in the sense of (5.10).

(ii) There exist a VE-space \( \mathcal{E} \) over \( Z \), a unital \( * \)-representation \( \pi: \Gamma \to \mathcal{L}(\mathcal{E}) \) and an operator \( A \in \mathcal{L}(\mathcal{X}, \mathcal{E}) \) such that
\[
T_t = A^\dagger \pi(t) A, \quad t \in \Gamma.
\]
If (i) or (ii) holds, then \( \mathcal{E} \) can be chosen minimal in the sense that it coincides with the linear span of \( \pi(\Gamma) A X \) and, in this case, it is unique modulo unitary equivalence.

**Proof.** (i)\( \Rightarrow \) (ii). We consider the kernel \( k: \Gamma \times \mathcal{X} \to Z \) as in (5.11) and the left action of \( \Gamma \) on \( \Gamma \times X \) as in (5.14). By Remark 5.5(1, 2), \( k \) is an \( Z \)-valued weakly positive semidefinite \( \Gamma \)-invariant kernel, hence, by Theorem 4.3 there exists a minimal \( \Gamma \)-invariant weak VE-space linearisation \((\mathcal{E}, \pi, V)\) of \( k \). Since
\[
[V(s, x), V(t, y)]_\mathcal{E} = k((s, x), (t, y)) = (T_{s^* t} y) x, \quad s, t \in \Gamma, x, y \in \mathcal{X},
\]
it follows that \( \mathcal{X} \ni x \mapsto V(s, x) \in \mathcal{E} \) is linear for all \( s \in \Gamma \). This shows that we can define \( \tilde{V}: \Gamma \to \mathcal{L}(\mathcal{X}, \mathcal{E}) \) by \( \tilde{V}(s)x = V(s, x) \) for all \( s \in \Gamma \) and \( x \in \mathcal{X} \). From (5.9) it follows that
\[
(\tilde{V}(s)f)x = [\tilde{V}(s)x, f]_{\mathcal{E}}, \quad s \in \Gamma, \ x \in \mathcal{X}, \ f \in \mathcal{E},
\]
hence, letting \( A = \tilde{V}(e) \in \mathcal{L}(\mathcal{X}, \mathcal{E}) \) it follows that, for all \( s \in \Gamma \) and all \( x, y \in \mathcal{X} \),
\[
(A'\pi(s)Ax)y = (\tilde{V}(e)'\pi(s)\tilde{V}(e)x)y = [\tilde{V}(e)y, \pi(s)\tilde{V}(e)x]_{\mathcal{E}}
= [V(e, y), \pi(s)V(e, x)]_{\mathcal{E}} = [V(e, y), V(s, x)]_{\mathcal{E}}
= k((e, y), (s, x)) = (Ts y),
\]
and hence (5.15) is proven. The minimality and the uniqueness property follow by standard arguments that we omit.

(ii) \( \Rightarrow \) (i). This follows by a standard argument that we omit. \( \blacksquare \)

Theorem 5.6 is stronger than Theorem 3.1 in [26], since in addition to positive semidefiniteness of \( T \) the latter requires (5.13) above as well. As we have seen in Remark 5.5 (2), this condition is a consequence of the positive semidefiniteness of \( T \). Also, the ordered \( * \)-space \( Z \) need not be admissible, actually, the topology of \( Z \) does not play any role.

From now on we assume that \( Z \) is a topologically ordered \( * \)-space and that \( \mathcal{X} \) is a locally bounded topological vector space, that is, in \( \mathcal{X} \) there exists a bounded neighbourhood of 0. We denote by \( \mathcal{X}_Z^* \) the subspace of \( \mathcal{X}_Z' \) of all continuous conjugate linear functions from \( \mathcal{X} \) to \( Z \) and call it the topological conjugate \( Z \)-dual space. The space \( \mathcal{X}_Z^* \) is considered with the topology of uniform convergence on bounded sets, that is, a net \( (f_i)_{i \in \mathcal{I}} \subset \mathcal{X}_Z^* \) converges to 0 if for any bounded subset \( B \subset \mathcal{X} \) the \( Z \)-valued net \( (f_i(y))_{i \in \mathcal{I}} \) converges to 0 uniformly with respect to \( y \in B \), or equivalently, for any bounded set \( B \subset \mathcal{X} \), any \( p \in S(Z) \) and any \( \epsilon > 0 \), there exists \( i_0 \in \mathcal{I} \) such that \( i \geq i_0 \) implies \( p(f_i(y)) < \epsilon \) for all \( y \in B \). Let \( \mathcal{L}_c(\mathcal{X}, \mathcal{X}_Z^*) \) be the space of all continuous linear operators from \( \mathcal{X} \) to \( \mathcal{X}_Z^* \).

Let \( \mathcal{E} \) be a VE-space over \( Z \), with topology defined as in Subsection 2.3. Following [26] and [14], for any \( A \in \mathcal{L}_c(\mathcal{X}, \mathcal{E}) \), the topological \( Z \)-adjoint operator of \( A \) is, by definition, the operator \( A^*: \mathcal{E} \to \mathcal{X}_Z^* \) defined by
\[
(A^*f)x = [Ax, f]_{\mathcal{E}}, \quad f \in \mathcal{E}, \ x \in \mathcal{X}.
\]
By Lemma 2.2 the definition of \( A^* \) is correct.

**Theorem 5.7.** Let \( \Gamma \) be a \( * \)-semigroup with unit \( e \) and \( \mathcal{X} \) be a locally bounded topological vector space with topological conjugate \( Z \)-dual space \( \mathcal{X}_Z^* \) for an admissible space \( Z \). Let \( T: \Gamma \to \mathcal{L}_c(\mathcal{X}, \mathcal{X}_Z^*) \) have the following properties:

(a) \( T \) is an \( \mathcal{L}(\mathcal{X}, \mathcal{X}_Z^*) \)-valued positive semidefinite map.
(b) For all \( u \in \Gamma \), there is a constant \( c(u) \geq 0 \) such that for all \( n \in \mathbb{N} \), all \( s_1, \ldots, s_n \in \Gamma \) and all \( x_1, \ldots, x_n \in \mathcal{X} \), we have

\[
(5.17) \quad \sum_{i,j=1}^{n} (T_{s_i} u^* u s_j x_j)(x_i) \leq c(u)^2 \sum_{i,j=1}^{n} (T_{s_i^* s_j} x_j)(x_i).
\]

(c) \( T(e) \in \mathcal{L}_c(\mathcal{X}, \mathcal{X}_Z^*) \).

Then:

(i) There exist a VH-space \( \mathcal{K} \) over \( Z \), a \(*\)-representation \( \pi : \Gamma \to \mathcal{B}^*(\mathcal{K}) \) and an operator \( A \in \mathcal{L}_c(\mathcal{X}, \mathcal{K}) \) such that \( T_s = A^* \pi(s) A \) for any \( s \in \Gamma \).

(ii) \( T_s \in \mathcal{L}_c(\mathcal{X}, \mathcal{X}_Z^*) \) for all \( s \in \Gamma \).

(iii) If \( (u_l)_{l \in \mathcal{L}} \) is a net in \( \Gamma \) with \( \sup_{l \in \mathcal{L}} c(u_l) < \infty \) and \( (T_{su}t)_{l \in \mathcal{L}} \) converges to \( T_{su}t \) for some \( u \in \Gamma \) and any \( s, t \in \Gamma \), in the weak topology of \( \mathcal{L}_c(\mathcal{X}, \mathcal{X}_Z^*) \), then \( (\pi(u_l))_{l \in \mathcal{L}} \) converges to \( \pi(u) \) in the weak topology of \( \mathcal{B}^*(\mathcal{K}) \).

**Proof.** Define \( k : (\Gamma \times \mathcal{X}) \times (\Gamma \times \mathcal{X}) \to Z \) as in (5.11). By Remark 5.5, \( k \) is a \( Z \)-valued weakly positive semidefinite kernel. Next, consider the left action of \( \Gamma \) on \( \Gamma \times \mathcal{X} \) as in (5.14); by Remark 5.5, \( k \) is \( \Gamma \)-invariant under this action. Condition (1)(c) of Theorem 4.6 is shown to hold using (5.17).

By Theorem 4.6, there exists a topologically minimal \( \Gamma \)-invariant weak VH-space linearisation \( (\mathcal{K}, \pi; V) \) of \( k \). Since \( [V(s, x), V(t, y)]_{\mathcal{K}} = k((s, x), (t, y)) = (T_{st} y)(x) \) for all \( s, t \in \Gamma \) and \( x, y \in \mathcal{X} \), we observe that \( V(s, x) \) depends linearly on \( x \in \mathcal{X} \) for each \( s \in \Gamma \). As a consequence, letting \( \tilde{V}(s)x = V(s, x) \) for all \( x \in \mathcal{X} \), we obtain a linear operator \( \tilde{V}(s) : \mathcal{X} \to \mathcal{K} \) for each \( s \in \Gamma \). To see that \( \tilde{V}(s) \) is continuous for each \( s \in \Gamma \), let \( (x_l)_{l \in \mathcal{L}} \) be a net in \( \mathcal{X} \) converging to 0. Since \( \mathcal{X} \) is locally bounded, there exists a bounded neighbourhood \( B \subset \mathcal{X} \) of 0 and then there exists \( l_1 \in \mathcal{L} \) such that \( (x_l)_{l \geq l_1} \) is contained in \( B \). Since \( T_e \in \mathcal{L}_c(\mathcal{X}, \mathcal{X}_Z^*) \), taking into account the topology of \( \mathcal{X}_Z^* \), given any \( \varepsilon > 0 \) and any \( p \in S(Z) \) we can find \( l_2 \in \mathcal{L} \) such that \( \mathcal{L} \ni l \geq l_2 \) implies \( p((T_e x_l)y) < \varepsilon \) for all \( y \in B \). Since \( \mathcal{L} \) is directed, there exists \( l_0 \in \mathcal{L} \) with \( l_0 \geq l_1 \) and \( l_0 \geq l_2 \). Then, for any \( l \geq l_0 \), by (5.17) and the definition of the topology of \( \mathcal{K} \) (see Subsection 2.3), we have

\[
p([\tilde{V}(s)x_l, \tilde{V}(s)x_l]_{\mathcal{K}}) = p(k((s, x_l), (s, x_l)))
= p(k(s \cdot (e, x_l), s \cdot (e, x_l))) \leq c(s)^2 p(k((e, x_l), (e, x_l)))
= c(s)^2 p((T_e x_l)x_l) \leq c(s)^2 \sup_{y \in B} p((T_e x_l)y) \leq c(s)^2 \varepsilon,
\]

hence \( \tilde{V}(s) \in \mathcal{L}_c(\mathcal{X}, \mathcal{K}) \) for any \( s \in \Gamma \). In addition, for each \( s \in \Gamma \) the operator \( V(s)^* \in \mathcal{L}(\mathcal{K}, \mathcal{X}_Z^*) \) is defined as in (5.16).
Letting $A := \tilde{V}(e)$ we have
$$(A^*\pi(s)Ax)y = (\tilde{V}(e)^*\pi(s)\tilde{V}(e)x)(y) = [V(e, y), V(s, x)]_K$$
$$= k((e, y), (s, x)) = (T_s x)y \quad \text{for all } s \in \Gamma \text{ and } x, y \in \mathcal{X}.$$ Therefore $A^*\pi(s)A = T_s \in \mathcal{L}_c(\mathcal{X}, \mathcal{X}_Z^*)$ for all $s \in \Gamma$.

The rest of the proof, which shows that $\pi(u_l)_{l \in \mathcal{L}}$ converges to $\pi(u)$ in the weak topology of $\mathcal{B}^*(\mathcal{K})$, as in the second part of the conclusion, uses standard arguments together with the Schwarz-type inequality (2.2) (see the correction in [27]), and can be found in [26].

**Remarks 5.8.** (1) Theorem [5.7] is stronger than Theorem 4.2 of [26] (see also the correction in [27]), in two aspects: firstly, since [26] has the additional assumption that (5.13) holds, which is actually a consequence of positive semidefiniteness, as Remark 5.5(2) shows; and secondly since their assumption $T_s \in \mathcal{L}_c(\mathcal{X}, \mathcal{X}_Z^*)$ for all $s \in \Gamma$ is actually a consequence of the weaker one $T_e \in \mathcal{L}_c(\mathcal{X}, \mathcal{X}_Z^*)$, as the proof of Theorem [5.7] shows.

(2) It is easy to see that there is a “converse” to Theorem [5.7] in the sense that if assertion (i) is assumed, then (a), (b), (c) and (ii) are obtained as consequences.

**Acknowledgements.** The second named author acknowledges support from the grant PN-III-P4-PCE-2016-0823 Dynamics and Differentiable Ergodic Theory from UEFISCDI, Romania.

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