Research Article

Reliability Estimation for the Remained Stress-Strength Model under the Generalized Exponential Lifetime Distribution

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A stress-strength reliability model compares the strength and stresses on a certain system; it is used not only primarily in reliability engineering and quality control but also in economics, psychology, and medicine. In this paper, a novel extension of stress-strength models is presented. The new model is applied under the generalized exponential distribution. The maximum likelihood estimator, asymptotic distribution, and Bayesian estimation are obtained. A comprehensive simulation study along with real data analysis is performed for illustrating the importance of the new stress-strength model.

1. Introduction

Stress-strength reliability analysis is a statistical analysis of the interference of the strength of the component and the stresses placed on the component. The stress-strength reliability analysis is a statistical tool used in reliability engineering.

In a stress-strength reliability model, both strength and stresses are considered as separate random variables. Stress experienced by a certain component is usually presented by the random variable X and the strength of the same component is presented by the random variable Y. A situation in which X > Y is one in which the stress is greater than the strength, and then, the component fails.

The abovementioned probability model can be expressed as \( R = \Pr(X > Y) \) and then called as the stress-strength quantity. The stress-strength reliability model has various applications in many fields such as reliability, quality control, and engineering. For more details in this matter, see the work of Kotz et al. [1] and Ventura and Racugno [2]. Rezaei et al. [3] presented a list of probability distributions used under the stress-strength reliability model. Recently, Rasekhi et al. [4] presented a Bayesian and the classical inference of reliability in multicomponent stress-strength under the generalized logistic model. Saber and Yousof [5] investigated the Bayesian and the classical inference for the generalized stress-strength parameter under generalized logistic distribution.

We suppose that we know these two components have been worked till a known time, and then, we are going to have some inferences on R. For this case, Saber and Khorsidian [6] introduce the conditional stress-strength model \( R^{a,b} \):

\[
R^{a,b} = \Pr(X > Y | X > a, Y > b).
\]

When independent random variables X and Y are continues, \( R^{a,b} \) can be computed by the following equation:
f_X(x) = \alpha \lambda e^{-\lambda x} \left(1 - e^{-\lambda x}\right)^{\alpha - 1}, \quad (3)

where \(x \geq 0\) and \(\alpha, \lambda > 0\) are the shape parameters and \(\lambda > 0\) is the scale parameter.

The rest of the paper is organized as follows. We devote Section 2 to study \(R_{ab}^{ia}\) in case of GE distribution. In Section 3, the ML estimator of quantity \(R_{ab}^{ia}\) and its corresponding asymptotic distribution and confidence interval are provided. A simulation study is presented in Section 4, and Section 5 has been devoted to applying a real dataset to the recommended model.

2. Conditional Stress-Strength Model for GE Distribution

In this section, quantity (2) is computed when distribution of components is GE.

**Theorem 1.** We suppose random variables \(X\) and \(Y\) are independent and \(X \sim \text{GE}(\alpha_1, \lambda)\) and \(Y \sim \text{GE}(\alpha_2, \lambda);\) then,

\[
R_{ab}^{ia} = \frac{1 - \left(1 - e^{-\lambda b}\right)^{\alpha_1}}{1 - \left(1 - e^{-\lambda b}\right)^{\alpha_1} - \left(1 - e^{-\lambda b}\right)^{\alpha_2}} \left[1 - \left(1 - e^{-\lambda b}\right)^{\alpha_2}\right]_{a < b}.
\]  

**Proof.** Let

\[
\begin{align*}
f_X(x) &= \alpha_1 \lambda e^{-\lambda x} \left(1 - e^{-\lambda x}\right)^{\alpha_1 - 1}, \\
f_Y(y) &= \alpha_2 \lambda e^{-\lambda y} \left(1 - e^{-\lambda y}\right)^{\alpha_2 - 1}, \\
F_X(x) &= \left(1 - e^{-\lambda x}\right)^{\alpha_1}, \\
F_Y(y) &= \left(1 - e^{-\lambda y}\right)^{\alpha_2}, \\
F_X(a) &= \left(1 - e^{-\lambda a}\right)^{\alpha_1}, \\
F_Y(b) &= \left(1 - e^{-\lambda b}\right)^{\alpha_2}.
\end{align*}
\]  

Substitute the last six equations in equation (2). Let \(a = b\); then,

\[
R_{ab}^{ia} = \frac{1 - F_Y(a) - \int_a^\infty F_X(x) f_Y(y) dy}{[1 - F_X(a)] [1 - F_Y(b)].}
\]  

Then,

\[
\Rightarrow R_{ab}^{ia} = \frac{1 - F_Y(a) - \int_a^\infty \alpha_2 \lambda e^{-\lambda y} \left(1 - e^{-\lambda y}\right)^{\alpha_2 - 1} dy}{(1 - \left(1 - e^{-\lambda b}\right)^{\alpha_1})(1 - \left(1 - e^{-\lambda b}\right)^{\alpha_2})},
\]
\[ R^{a,b} = \frac{1 - F_Y(a) - \int_a^\infty a_1 e^{-\lambda y} (1 - e^{-\lambda y})^{a_2 + a_3 - 1} dy}{\left(1 - \left(1 - e^{-\lambda b}ight)^{a_1}\right)\left(1 - \left(1 - e^{-\lambda b}ight)^{a_2}\right)} \]

\[ = \frac{1 - F_Y(a) - (a_3/(a_1 + a_2)) \int_a^\infty (a_1 + a_2)e^{-\lambda y} (1 - e^{-\lambda y})^{a_1 + a_3 - 1} dy}{\left(1 - \left(1 - e^{-\lambda b}ight)^{a_1}\right)\left(1 - \left(1 - e^{-\lambda b}ight)^{a_2}\right)} \]

\[ = \frac{1 - F_Y(a) - (a_2/(a_1 + a_2))(1 - F_W(a))}{\left(1 - \left(1 - e^{-\lambda b}ight)^{a_1}\right)\left(1 - \left(1 - e^{-\lambda b}ight)^{a_2}\right)} \]

In the abovementioned result, it is noted that \( W = GE(a_1 + a_2, \lambda) \); thus,

\[ F_W(a) = \left(1 - e^{-\lambda a}\right)^{a_1 + a_2}. \]  

(8)

Since \( a = b \), the proof of Theorem 1 is completed. 

If \( a < b \), then

\[ R^{a,b} = \frac{1 - F_Y(b) - \int_b^\infty F_X(y)f_Y(y) dy}{\left[1 - F_X(a)\right]\left[1 - F_Y(b)\right]} \]

(9)

The dominator of \( R^{a,b} \) in this case is exactly the same as in the first case with substitution of \( b \) instead of \( a \). Therefore, 

\[ R^{a,b} = \frac{1 - F_Y(b) - (a_3/(a_1 + a_2))\left[1 - F_W(b)\right]}{\left(1 - \left(1 - e^{-\lambda a}\right)^{a_1}\right)\left(1 - \left(1 - e^{-\lambda a}\right)^{a_2}\right)} \]

(10)

\[ R^{a,b} = \int_a^\infty \left(1 - e^{-\lambda y}\right)^{a_1} a_1 e^{-\lambda y} (1 - e^{-\lambda y})^{a_2 - 1} dy - F_Y(b)(1 - F_X(a)) \]

\[ \left[1 - F_X(a)\right]\left[1 - F_Y(b)\right] \]

(13)

which can be expressed as

\[ R^{a,b} = \frac{(a_3/(a_1 + a_2))\left[1 - F_W(a)\right] - F_Y(b)(1 - F_X(a))}{\left[1 - F_X(a)\right]\left[1 - F_Y(b)\right]} \]  

(14)

Below, we derive the maximum likelihood estimation (MLE) of the \( R^{a,b} \) model; and hence, the asymptotic distribution of those is presented in order to constructing the corresponding confidence interval.

Let \( X_1, X_2, \ldots, X_m \) be a random sample of size \( m \) of \( GE(a_1, \lambda) \) and \( Y_1, Y_2, \ldots, Y_n \) be a random sample of size \( n \) of \( GE(a_2, \lambda) \) such that \( X \) and \( Y \) are independent. Then, the likelihood function can be expressed as

\[ L = \alpha_1^m \alpha_2^n \alpha_3^{m+n} e^{-\lambda \left( \sum_{i=1}^m x_i + \sum_{j=1}^n y_j \right)} \]

\[ \times \left[ \prod_{i=1}^m \left(1 - e^{-\lambda x_i}\right) \right]^{a_1 - 1} \left[ \prod_{j=1}^n \left(1 - e^{-\lambda y_j}\right) \right]^{a_2 - 1}. \]  

(15)

Then, the log-likelihood function is given by

\[ \frac{\partial L}{\partial \alpha_1} = \frac{m}{\alpha_1} + \sum_{i=1}^m \ln\left(1 - e^{-\lambda x_i}\right), \]

\[ \frac{\partial L}{\partial \alpha_2} = \frac{n}{\alpha_2} + \sum_{j=1}^n \ln\left(1 - e^{-\lambda y_j}\right), \]

\[ \frac{\partial L}{\partial \lambda} = \frac{m + n}{\lambda} - \left( \sum_{i=1}^m x_i + \sum_{j=1}^n y_j \right) + (\alpha_1 - 1) \sum_{i=1}^m x_i e^{-\lambda x_i} \]

\[ + (\alpha_2 - 1) \sum_{j=1}^n y_j e^{-\lambda y_j}, \]

\[ \frac{\partial^2 L}{\partial \alpha_1^2} = -\sum_{i=1}^m \frac{x_i}{\alpha_1^2} \ln\left(1 - e^{-\lambda x_i}\right), \]

\[ \frac{\partial^2 L}{\partial \alpha_2^2} = -\sum_{j=1}^n \frac{y_j}{\alpha_2^2} \ln\left(1 - e^{-\lambda y_j}\right), \]

\[ \frac{\partial^2 L}{\partial \alpha_1 \partial \alpha_2} = \frac{m + n}{\alpha_1 \alpha_2} + \sum_{i=1}^m \frac{x_i}{1 - e^{-\lambda x_i}} + \sum_{j=1}^n \frac{y_j}{1 - e^{-\lambda y_j}}. \]

\[ \frac{\partial^2 L}{\partial \lambda^2} = -\frac{m + n}{\lambda^2} + \sum_{i=1}^m \frac{x_i}{\left(1 - e^{-\lambda x_i}\right)^2} + \sum_{j=1}^n \frac{y_j}{\left(1 - e^{-\lambda y_j}\right)^2}. \]

Therefore, the maximum likelihood estimator of parameters can be obtained by solving \( \partial L_{\alpha_1} / \partial \alpha_1 = 0, \) \( \partial L_{\alpha_2} / \partial \alpha_2 = 0 \) and \( \partial L_{\lambda} / \partial \lambda = 0. \) A simple computation shows

\[ \frac{\partial^2 L_{\alpha_1}}{\partial \alpha_1^2} = \frac{m}{\alpha_1^2} - \sum_{i=1}^m \frac{x_i}{\alpha_1^2 \left(1 - e^{-\lambda x_i}\right)} \ln\left(1 - e^{-\lambda x_i}\right), \]

\[ \frac{\partial^2 L_{\alpha_2}}{\partial \alpha_2^2} = \frac{n}{\alpha_2^2} - \sum_{j=1}^n \frac{y_j}{\alpha_2^2 \left(1 - e^{-\lambda y_j}\right)} \ln\left(1 - e^{-\lambda y_j}\right), \]

\[ \frac{\partial^2 L_{\lambda}}{\partial \lambda^2} = -\frac{m + n}{\lambda^2} + \sum_{i=1}^m \frac{x_i}{\left(1 - e^{-\lambda x_i}\right)^2} + \sum_{j=1}^n \frac{y_j}{\left(1 - e^{-\lambda y_j}\right)^2}. \]

Therefore, we have
\[ \alpha_1 = \frac{m}{\sum_{i=1}^{m} \ln(1 - e^{-\lambda x_i})}, \quad (18) \]
\[ \alpha_2 = \frac{n}{\sum_{j=1}^{n} \ln(1 - e^{-\lambda y_j})}, \quad (19) \]

Equations (18) and (19) depend on unknown parameter \( \lambda \). We substitute equations (18) and (19) in \( \partial L_{n,j}^{m,a,b}/\partial \lambda \), and we can find \( \hat{\lambda} \) by solving the following nonlinear equation:

\[
\frac{\partial L_{n,j}^{m,n}}{\partial \lambda} = m + n \left( \sum_{i=1}^{m} x_i + \sum_{j=1}^{n} y_j \right) - \left[ \sum_{i=1}^{m} x_i e^{-\lambda x_i} \right] + 1 \left[ \sum_{i=1}^{m} x_i e^{-\lambda x_i} \right] - \left[ \sum_{j=1}^{n} y_j e^{-\lambda y_j} \right] + 1 \left[ \sum_{j=1}^{n} y_j e^{-\lambda y_j} \right] = 0. \quad (20)
\]

Then, two other parameters are earned by substitution \( \hat{\lambda} \) in equations (18) and (19) as

\[
\hat{\alpha}_1 = \frac{m}{\sum_{i=1}^{m} \ln(1 - e^{-\hat{\lambda} x_i})}, \quad (21)
\]
\[ \hat{\alpha}_2 = \frac{n}{\sum_{j=1}^{n} \ln(1 - e^{-\hat{\lambda} y_j})}. \]

Therefore, the maximum likelihood estimator of \( R_{a,b} \) becomes

\[
R_{a,b} = \frac{1 - \left( 1 - e^{-\lambda a} \right) \alpha_2 - \left( \alpha_1 \alpha_2 \right) \lambda_{a,b}}{1 - \left( 1 - e^{-\lambda b} \right) \alpha_2} \quad \text{denoted by} \quad I(\theta) = E[I(\theta)], \quad \text{where} \quad I(\theta) = [I_{i,j}]_{i,j=1,2,3} \text{ is the observed information matrix; i.e.,}
\]

\[
I(\theta) = \begin{pmatrix}
\frac{\partial^2 L_{i,j}^{m,a,b}}{\partial \lambda^2} & \frac{\partial^2 L_{i,j}^{m,a,b}}{\partial \lambda \partial \alpha_1} & \frac{\partial^2 L_{i,j}^{m,a,b}}{\partial \lambda \partial \alpha_2} \\
\frac{\partial^2 L_{i,j}^{m,a,b}}{\partial \alpha_1 \partial \lambda} & \frac{\partial^2 L_{i,j}^{m,a,b}}{\partial \alpha_1^2} & \frac{\partial^2 L_{i,j}^{m,a,b}}{\partial \alpha_1 \partial \alpha_2} \\
\frac{\partial^2 L_{i,j}^{m,a,b}}{\partial \alpha_2 \partial \lambda} & \frac{\partial^2 L_{i,j}^{m,a,b}}{\partial \alpha_2 \partial \alpha_1} & \frac{\partial^2 L_{i,j}^{m,a,b}}{\partial \alpha_2^2}
\end{pmatrix} \quad (23)
\]
Now, the elements of \( J(\theta) \) follow by
\[
J_{11} = -E \left[ \frac{\partial^2 \nu_{m,n}}{\partial \alpha^2} \right], \quad \text{and then,}
\]
\[
J_{11} = -E \left( \frac{m + n}{\lambda^2} + \frac{m(\alpha_1 - 1)}{\lambda^2} \sum_{i=1}^{m} x_i e^{-\lambda x_i} + \frac{n(\alpha_2 - 1)}{\lambda^2} \sum_{j=1}^{n} y_j e^{-\lambda y_j} \right), \quad (25)
\]
It can finally be derived as
\[
J_{11} = \frac{m + n}{\lambda^2} + m(\alpha_1 - 1) \frac{\partial^2 \text{Beta}(t + 1, \alpha_1 - 2)}{\partial t} \bigg|_{t=2} + n(\alpha_2 - 1) \frac{\partial^2 \text{Beta}(t + 1, \alpha_2 - 2)}{\partial t} \bigg|_{t=2},
\]
\[
J_{12} = J_{21} = -E \left( \frac{\partial^2 L}{\partial \alpha_1} \right) = -E \left( \frac{\sum_{i=1}^{m} x_i e^{-\lambda x_i}}{1 - e^{-\lambda x_i}} \right),
\]
\[
J_{13} = J_{31} = -E \left( \frac{\partial^2 L}{\partial \alpha_2} \right) = -E \left( \frac{\sum_{j=1}^{n} y_j e^{-\lambda y_j}}{1 - e^{-\lambda y_j}} \right) = \frac{m\alpha_2}{\lambda} \frac{\partial \text{Beta}(t + 1, \alpha_2 - 1)}{\partial t} \bigg|_{t=1},
\]
\[
J_{22} = -E \left( \frac{\partial^2 \nu_{m,n}}{\partial \alpha_1^2} \right) = -E \left( \frac{m}{\alpha_1^2} \right) = \frac{m}{\alpha_1^2},
\]
\[
J_{23} = J_{32} = -E \left( \frac{\partial^2 \nu_{m,n}}{\partial \alpha_1 \partial \alpha_2} \right) = 0,
\]
\[
J_{33} = -E \left( \frac{\partial^2 \nu_{m,n}}{\partial \alpha_2^2} \right) = -E \left( \frac{n}{\alpha_2^2} \right) = \frac{n}{\alpha_2^2},
\]
By characteristics of MLEs, we have \( \hat{\theta} \approx N_3(\theta, \Sigma) \) for a large number of \( n \) and \( m \), where \( \Sigma \) is the inverse of the Fisher information matrix \( I(\theta) \).

Lemma 1. Let \( \{X_n\}_{n=1}^\infty \) be a sequence of random vector, where \( X_n \rightarrow N(\mu, \Sigma) \) in distribution. If the function \( g \) is continuous in the first partial derivatives and \( \tau = (\nabla g(\mu))^T \Sigma (\nabla g(\mu)) > 0 \), then \( g(X_n) \rightarrow N(g(\mu), \tau) \) in distribution. Lemma 1 is related to the the multivariate Delta method.

Here, with the help of a theorem well known as the multivariate Delta method, we find the asymptotic distribution of \( R_{a,b}^{ab} \). As \( n \rightarrow \infty \) and \( m \rightarrow \infty \); then,
\[
\begin{align*}
\frac{1}{\sigma_1} (R^a_{\alpha,b} - R^a_{\alpha,b}) &\rightarrow N(0,1)|_{a\neq b}, \\
\frac{1}{\sigma_2} (R^a_{\alpha,b} - R^a_{\alpha,b}) &\rightarrow N(0,1)|_{a\neq b}, \\
\frac{1}{\sigma_3} (R^a_{\alpha,b} - R^a_{\alpha,b}) &\rightarrow N(0,1)|_{a\neq b},
\end{align*}
\]

where \(\sigma_1^2, \sigma_2^2, \) and \(\sigma_3^2\) are obtained from the following equations. In all three cases,

\[
\begin{align*}
\frac{\partial R^a_{\alpha,b}}{\partial \lambda} &= \left( \frac{\alpha_2 b e^{-\lambda b} (1 - e^{-\lambda b})^{a_2 - 1} - \alpha_2 b e^{-\lambda b} (1 - e^{-\lambda b})^{a_2 + a_1 - 1}}{((1 - (1 - e^{-\lambda b})^{a_1}) (1 - (1 - e^{-\lambda b})^{a_2}))^2} \right) \\
\times \left( \frac{1 - (1 - e^{-\lambda b})^{a_2} - (\alpha_2/(\alpha_1 + \alpha_2)) (1 - (1 - e^{-\lambda b})^{a_2 + a_1})}{((1 - (1 - e^{-\lambda b})^{a_1}) (1 - (1 - e^{-\lambda b})^{a_2}))^2} \right),
\end{align*}
\]

\[
\begin{align*}
\frac{\partial R^a_{\alpha,b}}{\partial \alpha_1} &= \left( \frac{-(\alpha_2/(\alpha_1 + \alpha_2)) (1 - (1 - e^{-\lambda b})^{a_2 + a_1}) + (1 - e^{-\lambda b})^{a_2 + a_1} \ln(1 - e^{-\lambda b}) (\alpha_2/(\alpha_1 + \alpha_2))}{((1 - (1 - e^{-\lambda b})^{a_1}) (1 - (1 - e^{-\lambda b})^{a_2}))^2} \right) \\
\times \left( \frac{1 - (1 - e^{-\lambda b})^{a_1} - (\alpha_2/(\alpha_1 + \alpha_2)) (1 - (1 - e^{-\lambda b})^{a_2 + a_1})}{((1 - (1 - e^{-\lambda b})^{a_1}) (1 - (1 - e^{-\lambda b})^{a_2}))^2} \right),
\end{align*}
\]

\[
\begin{align*}
\frac{\partial R^a_{\alpha,b}}{\partial \alpha_2} &= \left( \frac{(1 - e^{-\lambda b})^{a_2} \ln(1 - e^{-\lambda b}) + (\alpha_2/(\alpha_1 + \alpha_2)) (1 - e^{-\lambda b})^{a_2 + a_1} + (\alpha_2/(\alpha_1 + \alpha_2)) \ln(1 - e^{-\lambda b}) (1 - e^{-\lambda b})^{a_2 + a_1}}{(1 - (1 - e^{-\lambda b})^{a_1}) (1 - (1 - e^{-\lambda b})^{a_2}))^2} \right) \\
\times \left( \frac{1 - (1 - e^{-\lambda b})^{a_2} - (\alpha_2/(\alpha_1 + \alpha_2)) (1 - (1 - e^{-\lambda b})^{a_2 + a_1})}{((1 - (1 - e^{-\lambda b})^{a_1}) (1 - (1 - e^{-\lambda b})^{a_2}))^2} \right).
\end{align*}
\]

\[\sigma^2(\lambda, \alpha_1, \alpha_2) = \nabla R^a_{\alpha,b} \Gamma^{-1}(\lambda, \alpha_1, \alpha_2)(\nabla R^a_{\alpha,b})^T,\]

\[\nabla R^a_{\alpha,b} = \left( \frac{\partial R^a_{\alpha,b}}{\partial \lambda}, \frac{\partial R^a_{\alpha,b}}{\partial \alpha_1}, \frac{\partial R^a_{\alpha,b}}{\partial \alpha_2} \right).\]

For all cases, \(\Gamma^{-1}(\lambda, \alpha_1, \alpha_2)\) is equal and the only difference is in \(\nabla R^a_{\alpha,b}\).

In the following, this quantity has been computed for cases \(a = b, a < b,\) and \(a > b,\) respectively.

**Case 1.** \(a = b.\)

**Case 2.** \(a < b.\)
\begin{align*}
\frac{\partial R_{i,b}}{\partial \lambda} &= \left[ \frac{\alpha_2 b e^{-\lambda b}(1-e^{-\lambda b})^{a_2-1} - \alpha_2 b e^{-\lambda b}(1-e^{-\lambda b})^{a_1+a_2-1}}{((1 - (1 - e^{-\lambda a})^{a_1})(1 - (1 - e^{-\lambda b})^{a_2}))^2} \right] \\
&\quad - \frac{\alpha_2 a e^{-\lambda a}(1-e^{-\lambda a})^{a_1-1}(1-(1-e^{-\lambda b})^{a_2}) + \alpha_2 b e^{-\lambda b}(1-e^{-\lambda b})^{a_1-1}(1-(1-e^{-\lambda a})^{a_1})}{((1 - (1 - e^{-\lambda a})^{a_1})(1 - (1 - e^{-\lambda b})^{a_2}))^2} \\
&\quad \left[ 1-(1-e^{-\lambda b})^{a_2} - (a_2/(a_1+a_2)) \left( 1-(1-e^{-\lambda b})^{a_2+a_1} \right) \right] \\
\frac{\partial R_{i,b}}{\partial a_1} &= \left[ \frac{-(a_2/(a_1+a_2))\left( 1-(1-e^{-\lambda b})^{a_2+a_1} \right) + (1-e^{-\lambda b})^{a_2+a_1}\ln(1-e^{-\lambda b}) (a_2/(a_1+a_2))}{((1 - (1 - e^{-\lambda a})^{a_1})(1 - (1 - e^{-\lambda b})^{a_2}))^2} \right] \\
&\quad \left[ 1-(1-e^{-\lambda a})^{a_1} - (1-e^{-\lambda b})^{a_2} \right] - \left[ (1-e^{-\lambda a})^{a_1}\ln(1-e^{-\lambda a})(1-(1-e^{-\lambda b})^{a_2}) \right] \\
&\quad \left[ ((1 - (1 - e^{-\lambda a})^{a_1})(1 - (1 - e^{-\lambda b})^{a_2}))^2 \right] \\
\frac{\partial R_{i,b}}{\partial a_2} &= \left[ \frac{(1-e^{-\lambda b})^{a_2}\ln(1-e^{-\lambda b}) + (a_2/(a_1+a_2))\left( 1-(1-e^{-\lambda b})^{a_2+a_1} \right) + (a_2/(a_1+a_2))^2\ln(1-e^{-\lambda b})(1-e^{-\lambda b})^{a_2+a_1}}{((1 - (1 - e^{-\lambda a})^{a_1})(1 - (1 - e^{-\lambda b})^{a_2}))^2} \right] \\
&\quad \times \left[ \frac{-\left( 1-(1-e^{-\lambda b})^{a_2} - (a_2/(a_1+a_2))\left( 1-(1-e^{-\lambda b})^{a_2+a_1} \right) \right]}{((1 - (1 - e^{-\lambda b})^{a_2})(1 - (1 - e^{-\lambda b})^{a_2+a_1}))^2} \right] \\
&\quad \times \left[ \frac{-\left( 1-(1-e^{-\lambda b})^{a_2} - (a_2/(a_1+a_2))\left( 1-(1-e^{-\lambda b})^{a_2+a_1} \right) \right]}{((1 - (1 - e^{-\lambda b})^{a_2})(1 - (1 - e^{-\lambda b})^{a_2+a_1}))^2} \right].
\end{align*}

Case 3. \( a > b. \)
Bayesian Method for Estimation

In this section, we provide a Bayes estimator for $R_{a,b}$. All priors for $\alpha_1$, $\alpha_2$, and $\lambda$ are considered Gamma distribution. We more exactly denote

Equation (28) can be used for finding the confidence interval of $R_{a,b}$, however, by using of estimates of asymptotic variance. From this theorem, the $(1 - \alpha) \times 100$ percentage confidence interval of $R_{a,b}$ is given by

In the abovementioned equations, $\bar{\sigma}_i^2 i = 1, 2, 3$ are similar to $\bar{\sigma}_i^2 i = 1, 2, 3$ in equation (28) with substitution $\bar{\lambda}$, $\bar{\alpha}_1$, and $\bar{\alpha}_2$ instead of $\lambda$, $\alpha_1$, and $\alpha_2$.

3. Bayesian Method for Estimation

In this section, we provide a Bayes estimator for $R_{a,b}$. All priors for $\alpha_1$, $\alpha_2$, and $\lambda$ are considered Gamma distribution. We more exactly denote

Equation (32) is complicated, and it does not belong to a known distribution. Therefore, we use Gibbs sampler to
generate samples from (35). By (35), the full posterior density functions are

\[
L(\alpha_1 \mid \alpha_2, \lambda, x, y) \propto \alpha_1^{m+\xi_1-1} e^{-\alpha_1 \left( d_1 + \sum_{i=1}^{m} \ln(1 - e^{-\lambda x_i}) \right)} \]

\[
L(\alpha_2 \mid \alpha_1, \lambda, x, y) \propto \alpha_2^{n+\xi_2-1} e^{-\alpha_2 \left( d_2 + \sum_{j=1}^{n} \ln(1 - e^{-\lambda y_j}) \right)} \]

\[
L(\lambda \mid \alpha_1, \alpha_2, x, y) \propto \lambda^{m+n+\xi_3} e^{-\lambda \left( \sum_{i=1}^{m} x_i + \sum_{j=1}^{n} y_j + d_3 \right)}
\]

\[
\times \left[ \prod_{i=1}^{m} (1 - e^{-\lambda x_i}) \right]^{\alpha_1-1} \left[ \prod_{j=1}^{n} (1 - e^{-\lambda y_j}) \right]^{\alpha_2-1}.
\]

(36)

(37)

It is clear that the posteriors of $\alpha_1$ and $\alpha_2$ are Gamma distribution while (37) does not have a known distribution. Henceforth, we use Metropolis–Hastings (M–H) algorithm to generate data from (37). The proposal distribution for M–H algorithm is considered Gamma with the shape parameter $m + n + \xi_3$ and scale parameter $\sum_{i=1}^{m} x_i + \sum_{j=1}^{n} y_j + d_3$.

4. Simulation Study

In this section, we conduct a simulation study in order to survey quality and efficiency of the introduced model and its estimator. All results are the mean of 10000 iteration. To put it more clearly, note that we have iterated our simulation 10000 times. In the $i^{th}$ iteration, two random samples with size $n$ and $m$ are generated and $R^{a,b}$ is computed. The values of $R^{a,b}$ demonstrated in the tables are the mean of these 10000 computed estimates as follows:

| $n$   | 15   | 25   | 100  | 40   | 25   | 30   | 10   | 45   | 60   | 80   |
|-------|------|------|------|------|------|------|------|------|------|------|
| $m$   | 15   | 25   | 100  | 40   | 25   | 30   | 10   | 45   | 60   | 80   |
| $\overline{R}^{a,b}$ | 0.500656 | 0.5006194 | 0.5006128 | 0.500839 | 0.500552 | 0.50056 | 0.50056 | 0.50056 |
| Bias  | $-0.00117$ | $-6 \times 10^{-5}$ | $-0.0007875$ | $0.000149$ | $-0.00067$ | 0.00082 | 0.000148 |
| MSE   | 1.37E−06 | 3.6 \times 10^{-9} | 6.2 \times 10^{-7} | 2.2E−08 | 4.4 \times 10^{-7} | 6.7 \times 10^{-7} | 2 \times 10^{-8} |
| CP    | 0.00472 | 0.0033517 | 0.0016044 | 0.004563 | 0.002777 | 0.002506 | 0.002723 |
| LCI   | 0.924  | 0.92  | 0.9478 | 0.9254 | 0.936 | 0.9439 | 0.9577 |

| $n$   | 15   | 25   | 100  | 40   | 25   | 30   | 10   | 45   | 60   | 80   |
|-------|------|------|------|------|------|------|------|------|------|------|
| $m$   | 15   | 25   | 100  | 40   | 25   | 30   | 10   | 45   | 60   | 80   |
| $\overline{R}^{a,b}$ | 0.500656 | 0.5006194 | 0.5006128 | 0.500839 | 0.500552 | 0.50056 | 0.50056 | 0.50056 |
| Bias  | $-0.00117$ | $-6 \times 10^{-5}$ | $-0.0007875$ | $0.000149$ | $-0.00067$ | 0.00082 | 0.000148 |
| MSE   | 1.4 \times 10^{-6} | 3.6 \times 10^{-9} | 6.2 \times 10^{-7} | 1.4 \times 10^{-6} | 3.6 \times 10^{-9} | 6.2 \times 10^{-7} |
| CP    | 0.00472 | 0.0033517 | 0.0016044 | 0.004563 | 0.002777 | 0.002506 | 0.002723 |
| LCI   | 0.924  | 0.92  | 0.9478 | 0.9254 | 0.936 | 0.9439 | 0.9577 |

$R^{a,b} = \frac{\sum_{i=1}^{10000} \overline{R}^{a,b}}{10000}$ (38)

Four criteria containing bias, Mean Square Error (MSE), Coverage Probability (CP), and Length of Confidence Interval (LCI) are used in order to investigate the effectiveness and potentialities of the method. The results are demonstrated in Table 1 for values of parameters $a_1 = 2$, $a_2 = 1.6$, $\lambda = 2$, $a = 2$, and $b = 2$. Also, the results are represented in Table 2 for values of parameters $a_1 = 3$, $a_2 = 3$, $\lambda = 1.5$, $a = 1$, and $b = 3$.

As these tables show two criteria MSE and bias are very small; therefore, our estimation method is appropriate. Also, values of CP and LCI as cover of probability and length of confidence interval for $R^{i,j}$ represent the same findings.

5. Application

In this section, two real datasets reported by Lawless [9] (data A) and Linhardt and Zucchini [10] (data B), respectively, are analyzed. We fit the GE distribution to the two datasets separately. The first dataset arose in tests on endure of deep groove ball bearings and is the number of million revolutions before failure for each of the 23 ball bearings in the life test. The other dataset denotes the failure times of the air-conditioning system of an airplane (in hours).

Gupta and Kundu [7] and Gupta and Kundu [11] studied the validity of GE distribution for these two datasets, respectively. In Table 3, the Kolmogorov–Smirnov distance and its corresponding $P$ value are provided for these data which confirm that the generalized exponential model fits quite well to both the datasets where data I: 17.88, 28.92, 33, 41.52, 42.12, 45.60, 48.80, 51.84, 51.96, 54.12, 55.56, 67.8, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, and 173.40 and data II: 23, 261, 87.7, 120.14, 62, 47, 225, 71, 246, 21, 42, 20.5, 12, 120, 11, 3, 14, 71, 11, 14, 11, 16, 90, 1, 16, and 52, 95.
Using the results previously presented in Section 2, we can obtain the MLE of parameters $\alpha_1$, $\alpha_2$, and $\lambda$. Our computations show that $\hat{\alpha}_1 = 2.8082$, $\hat{\alpha}_2 = 1.0067$, and $\hat{\lambda} = 0.5036$. For these estimated parameters, $\hat{R}$ and its corresponding confidence interval have been computed for some values of $a$ and $b$, as in Table 4.

### 6. Conclusions

In this paper, a new extension of the stress-strength model is defined, studied, and applied under many particular cases. The novel stress-strength model is applied under the generalized exponential distribution. The maximum likelihood estimator, asymptotic distribution, and Bayesian estimation are obtained with details. A simulation study along with analysis of two real datasets is also performed for illustrative purposes. We hope that the proposed method would enable engineers and system designers to design better products.

### Data Availability

All data are available on request.

### Conflicts of Interest

The authors have no conflicts of interest.

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