Economic Neutral Position: How to best replicate not fully replicable liabilities

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Version: April 28, 2017

Abstract

Financial undertakings often have to deal with liabilities of the form “non-hedgeable claim size times value of a tradeable asset”, e.g. foreign property insurance claims times fx rates. Which strategy to invest in the tradeable asset is risk minimal? We expand the capital requirements based on value-at-risk and expected shortfall using perturbation techniques. We derive a stable and fairly model independent approximation of the risk minimal asset allocation in terms of the claim size distribution and the first three moments of asset return. The results enable a correct and easy-to-implement modularization of capital requirements into a market risk and a non-hedgeable risk component: the paper provides a stable expression for the financial benchmark against which the company’s asset allocation must be measured to obtain the market risk component.

Keywords: risk measure; risk minimal asset allocation; incomplete markets; modular capital requirements; perturbation theory; quantos; Solvency II; standard formula; SCR; market risk; internal model; replicating portfolio;

JEL Classification: D81; G11; G22; G28;

1 Introduction

We consider a liability of product structure \[ \sum_i L_i \cdot X_i, \] where \( X_i \) are hedgeable risk factors and \( L_i \) represent stochastic notionals or claim sizes that are not replicable by financial instruments. It is well known that such liability is not perfectly replicable, since the number of risk drivers exceeds the number of involved hedgeable capital market factors.

This liability structure is of high practical relevance. Prominent examples stem from insurance: \( L_i \) denoting the claims from property insurance portfolios in foreign currencies and \( X_i \) denoting the exchange rates, or, \( L_i \) the benefit payments of pure endowment policies staggered by maturities (depending on realized mortality) and \( X_i \) the risk-free discount factors. Also for the banking industry such liability structure is relevant, in particular for measuring the credit value adjustment (CVA) risk for non-collateralized derivatives with counterparties for which no liquid credit default swaps exists: the CVA for a non-collateralized commodity forward contract can be written in the above structure with \( L_i \) denoting the default rate of the counterparty in the time interval \( t_i \) (multiplied by the loss-given-default ratio) and \( X_i \) denoting a commodity call option expiring at \( t_i \) which represents the commodity price dependent loss potential due to counterparty default at \( t_i \).

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1 hereby we assumed independence of the default rates from the credit exposure against the counterparty due to an increase of the commodity forward rates beyond the pre-agreed strike, refer e.g. to [3] for details.
To which extent can the risk from the above liability structure be mitigated by trading in the capital market factors $X_i$? The residual risk must be warehoused and backed with capital. The capital requirement for a financial institution is obtained in theory by applying a risk measure $\rho$ on the distribution of its surplus (i.e. excess of the value of assets over liabilities) in one year, which is the typical time horizon for risk measurement. Hence we aim to find the optimal strategy to invest in the assets $X_i$ that minimizes the capital requirements. Intuition tells us that investing more than the expected claim size into the respective hedgeable asset $X_i$ makes sense, since large liability losses are usually driven by events where both the claim sizes and the asset values develop adversely. As risk measures focus on tail events, the excess investments in $X_i$ mitigate that part of the liability losses that stems from an increase in $X_i$. The essential task now is to quantify this excess amount.

We restrict in this paper to the one-dimensional case where the liability reads $L \cdot X$; the general case will be treated in a forthcoming paper [5]. Without losing too much of generality we assume that $L$ and $X$ are independent and that there is no continuous increase in information concerning the state of $L$ during the risk measurement horizon. The latter assumption is almost tantamount to the assumption that $L$ is not hedgeable. As a consequence there is no need to adjust the holdings in $X$ dynamically within the year. If $L$ and $X$ were not independent, then in most practical applications $L$ could be expressed by regression techniques as a function of $X$ plus some residual $L'$ which then is independent of $X$ by construction.

Even if $X$ or $L$ is normally or log-normally distributed, the derivation of the risk minimal asset allocation is not straightforward, since products of log-normal variables are again log-normal but sums are not and vice versa for normal variables. On the other hand, it is well understood how to derive in a general incomplete market situation the risk minimal asset strategy in the one period case, refer to [2] and the references included therein. Our problem can also be seen as a special case of the utility indifference pricing approach in a one period setting, refer to [4]. According to our knowledge, no detailed results have been published yet that address the specific case of the above product structure with a non-hedgeable factor.

In this paper, we analyze the risk measures value-at-risk and expected shortfall. Our first results concern the exceptional initial holding in $X$ that equals to the value-at-risk from the non-hedgeable component, i.e. when $X$ becomes constant and the value-at-risk of the surplus is only driven by the non-hedgeable claim size $L$. We can show without additional assumptions on the distribution of $X$ and $L$ that for this exceptional asset allocation the capital requirements collapse to those in case of constant $X$. Moreover, this exceptional asset allocation is risk minimal in the expected shortfall case; the value-at-risk based capital requirements on the other hand are still decreasing when less than this exceptional amount is invested in $X$.

In a second step, we use perturbation techniques to expand the capital requirements and their dependence on the asset allocation in terms of the log-normal volatility of the hedgeable factor up to third order. For a typical insurance portfolio it is reasonable to assume that the log-normal volatility of the hedgeable factor is much smaller than one. The expansion only involves local properties of the distribution of $L$ at its quantile point as well as the second and third moment of the log-return of $X$. In the value-at-risk case the expansion of the capital requirements and its dependency on the investment in $X$ leads to an approximation of the risk minimal asset allocation. Numerical studies show that the derived expansion is stable even for large log-normal asset volatility levels.

To determine the asset allocation that minimizes capital requirements in a rather generic and model independent way is important for its own sake. This objective is even more relevant for the modularization of capital requirements into a capital market and a non-hedgeable risk component. This has become market standard since deriving capital requirements via a joint stochastic modeling of all (hedgeable and non-hedgeable) risk factors turned out to be too complex. The financial benchmark (Economic Neutral Position) against which the actual investment portfolio is measured to obtain the capital market risk component must obviously coincide with the risk minimal asset allocation. Our results show that the Economic Neutral Position replicates the financial risk factors of the liabilities.
on the basis of the expected claim size plus some safety margin. Solvency II, the new capital regime for European insurers, does not recognize this safety margin in the modularized Standard Formula approach, which can result in significant distortions of the total risk compared with the (correct) fully stochastic approach, refer to [1] for details. The results of these paper provide a simple and stable approximation of the required safety margin in the Economic Neutral Position, such that the modularized capital requirement approach keeps its easy-to-implement property; e.g. for non-hedgeable risks with normal tails the safety margin amounts to 85% of the insurance risk component in the Solvency II context.

2 Setup

Consider a financial undertaking whose capital requirement is determined by applying a risk measure \( \rho \) on its surplus \( S \) in one year. The value of the liabilities at year one shall have a product structure \( X \cdot L \), where the real-valued random variables \( L \) and \( X \) denote the claim size and the value of a tradeable asset, respectively, which live on a probability space with measure \( P \) together with a risk free numeraire investment (money market account). \( X \) is assumed strictly positive and independent of \( L \). All financial quantities are expressed in units of the numeraire.

The financial undertaking can invest its assets with initial value \( A_0 \geq 0 \) into the tradeable asset \( X \) with initial value \( X_0 \) or into the numeraire. We assume that all additional information concerning \( L \) becomes known only at year one, i.e. there is no continuous increase in information concerning the state of \( L \) during the year. Hence there is no need to adjust the holdings in \( X \) dynamically within the year. We denote by \( \phi \geq 0 \) the units of \( X \) that the financial undertaking invests statically as of today; the remaining asset value \( A_0 - \phi \cdot X_0 \) is invested into the numeraire. The value of the surplus at year one is a function of \( \phi \) and reads expressed in units of the numeraire

\[
S(\phi) := \phi \cdot X + A_0 - \phi \cdot X_0 - X \cdot L .
\]

(1)

We analyze the risk measures value-at-risk \( \text{VaR}_\alpha \) and expected shortfall \( \text{ES}_\alpha \) at tolerance level \( 1 - \alpha \) for some small \( \alpha > 0 \). Typically \( \alpha = 0.01 \) for banks and \( \alpha = 0.005 \) for European insurance companies. Refer to [2] for details of the definition of \( \text{VaR}_\alpha \) and \( \text{ES}_\alpha \). We use the notation \( \rho \) if the expression is valid for both analyzed risk measures, i.e. \( \rho \in \{ \text{VaR}_\alpha, \text{ES}_\alpha \} \).

We aim to find the optimal holding \( \phi^* \) in the tradeable assets that minimizes the risk of the surplus, i.e.

\[
\rho[S(\phi^*)] = \min_{\phi} \rho[S(\phi)] .
\]

We assume the following technical conditions:

\[
X, X^{-1}, L, \text{ and } X \cdot L \text{ are integrabel,}
\]

(2)

\[
L \text{ has a bounded density } f_L \text{ and } f_L > 0 \text{ on } \{ z : F_L(z) \geq 1 - \alpha \},
\]

(3)

where \( F_L \) denotes the cumulative distribution function of \( L \). Hence the quantile function \( F_L^{-1} \) is well defined in the upper tail of \( L \).

To simplify the minimization of \( \rho[S(\phi)] \) we assume without loss of generality

\[
\mathbb{E}[X] = X_0 = 1 \quad \text{and} \quad A_0 = \mathbb{E}[L] = 0 .
\]

(4)

The first assumption means in particular that \( X \) is fairly priced. The second assumption implies that \( S(\phi) \) has zero mean and hence reads

\[
S(\phi) = \phi \cdot (X - 1) - X \cdot L .
\]

(5)
We can justify these simplifying assumptions by making use of the positive homogeneity property of \( \rho \), that allows to divide the surplus by \( X_0 \), and of the cash invariance property, that enables us to separate the stochastics of \( S(\phi) \) from its mean \( A_0 - X_0 \cdot E[L] \). If \( X \) has non-zero excess return, i.e. \( E[X] \neq X_0 \), then the additional linear term “\( \phi \) times excess return” arises, which enters the minimization of the risk of the surplus with respect to \( \phi \) in a straight forward way. The details are transferred to the appendix together with the procedure to reduce \( S(\phi) \) from the original form \([1]\) to its simplified expression \([5]\).

3 Particular Value of \( \phi \)

In this section we identify a particular initial investment amount \( \phi \) into the tradeable asset \( X \) such that \( \rho[S(\phi)] \) becomes fairly independent of the distribution of \( X \). We also show some more preliminary results.

We rewrite the cumulative distribution function of the surplus: for any \( z \in \mathbb{R} \)

\[
\mathbb{P}(S(\phi) \leq z) = \mathbb{P}(\phi \cdot (X - 1) - X \cdot L \leq z) = \mathbb{P}(X \cdot L \geq \phi \cdot (X - 1) - z)
\]

\[
= \mathbb{E}_X [\mathbb{P}(L \geq \phi - (z + \phi)/X \mid X)]
\]

\[
= \mathbb{E}_X [\bar{F}_L(w(\phi, z, X))], \text{ with } w(\phi, z, X) := \phi - (z + \phi)/X, \tag{6}
\]

where \( \bar{F}_L(l) := \mathbb{P}(L > l) \) denotes the tail function of \( L \) and \( \mathbb{E}_X \) denotes the expectation with respect to the variable \( X \) only. The last but one equality follows from the strict positivity of \( X \) and the last equality from the independence of \( L \) and \( X \).

The following lemma shows that the \( \alpha \)-quantile of the surplus \( S(\phi) \) is well defined and states further preliminary results; we denote by \( 1_A \) the indicator function of some set \( A \).

**Lemma 1.** Assume \([2]\) and \([3]\). Then for every \( \phi \geq 0 \) and \( \alpha \in (0, 1) \)

a) \( \mathbb{P}(S(\phi) \leq z) = \alpha \) has a unique solution \( z = z_{\phi, \alpha} \), i.e. the \( \alpha \)-quantile of \( S(\phi) \) is well defined.

b) \( \text{VaR}_\alpha[S(\phi)] = -z_{\phi, \alpha} \) and \( \text{ES}_\alpha[S(\phi)] = -\alpha^{-1} \cdot \mathbb{E}[S(\phi) \cdot 1_{S(\phi) \leq z_{\phi, \alpha}}] \).

c) \( \phi \mapsto \rho[S(\phi)] \) is differentiable for both risk measures \( \rho \in \{ \text{VaR}_\alpha, \text{ES}_\alpha \} \).

d) \( \phi \mapsto \text{ES}_\alpha[S(\phi)] \) is convex.

We denote the quantile of \( S(\phi) \) by \( z_\phi \) omitting the subscript \( \alpha \) when there is no confusion about the risk tolerance. The lemma results basically from the implicit function theorem applied to expression \([6]\); (b) is a consequence of the continuous distribution of \( S(\phi) \), and (e) follows from the convexity of the expected shortfall. The details of the proofs are transferred to the appendix.

**Remark 2.** If \( L \) has atoms, i.e. does not admit a density, then the function \( \phi \mapsto \text{VaR}_\alpha[S(\phi)] \) might not be continuous but can have kinks at \( \phi \) equals to the singular values of \( L \).

Denote by \( q := F_L^{-1}(1 - \alpha) \) the \((1 - \alpha)\)-quantile of \( L \), which is well defined due to assumption \([3]\). If the financial undertaking invests initially \( q \) units into the asset \( X \), i.e. \( \phi = q \), the value-at-risk of the surplus becomes independent of \( X \) as the following sequence of equivalent events demonstrates:

\[
\{S(q) \leq -q\} = \{q \cdot (X - 1) - X \cdot L \leq -q\} = \{X \cdot (q - L) \leq 0\} = \{q - L \leq 0\} = \{L \geq q\}. \tag{7}
\]

Note that the last but one equality follows from the strict positivity of \( X \). Hence we derive that \( \mathbb{P}(S(q) \leq -q) = 1 - F_L(q) = \alpha \), which implies that \( z_q = -q \) or, equivalently, \( \text{VaR}_\alpha[S(q)] = q \).
Also for the expected shortfall, \( \phi = q \) is a special case: since \( \{ S(q) \leq z_q \} = \{ L \geq q \} \), which follows directly from (7), we conclude

\[
-\alpha \cdot \text{ES}_\alpha[S(q)] = \mathbb{E}[S(q) \cdot 1_{S(q) \leq z_q}] = \mathbb{E}[(q \cdot (X - 1) - X \cdot L) \cdot 1_{L \geq q}]
\]

where the third equality follows from the independence of \( X \) and \( L \) and the forth equality from the unit mean of \( X \).

Also the first derivative of the function \( \phi \mapsto \rho[S(\phi)] \) shows special properties at \( \phi = q \). We summarize the findings in the following theorem together with all other results concerning the particular value for \( \phi \).

**Theorem 3.** Assume (2) and (3) and set \( q := F_L^{-1}(1 - \alpha) = -F_{-L}^{-1}(\alpha) = \text{VaR}_\alpha[-L] \). If \( q \) units are initially invested in \( X \), i.e. if \( \phi = q \), then

a) \( \rho[S(q)] = \rho[-L] \) for \( \rho \in \{ \text{VaR}_\alpha, \text{ES}_\alpha \} \).

b) The differential of the risk of the surplus with respect to \( \phi \) evaluated at \( \phi = q \) reads

\[
(\partial_\phi \rho[S(\phi)])_{|\phi=q} = \begin{cases} 
(-1) \cdot \left( \mathbb{E}[X^{-1}] - 1 \right) & \text{if } \rho = \text{VaR}_\alpha, \\
0 & \text{if } \rho = \text{ES}_\alpha.
\end{cases}
\]

The above inequality becomes strict if \( X \) is not constant.

c) The function \( \phi \mapsto \text{ES}_\alpha[S(\phi)] \) attains its global minimum value \( \text{ES}_\alpha[-L] \) at \( \phi^* = q \). (\( \phi^* \) is not necessarily unique.)

Part (a) has already been shown above, the proof of (b) is transferred to the appendix, and (c) follows from (b) using the differentiability and convexity of \( \phi \mapsto \text{ES}_\alpha[S(\phi)] \), see lemma [1]

**Remark 4.**

a) \( \rho[-L] \) is the risk of the surplus if the volatility of \( X \) collapse to zero and \( X \) becomes constant (with value one).

b) The initial amount \( \phi^* \) invested in \( X \) that minimizes the risk \( \rho[S(\phi)] \) is less than \( \rho[-L] \) for both risk measures \( \rho \in \{ \text{VaR}_\alpha, \text{ES}_\alpha \} \). For \( \text{VaR}_\alpha \) this follows from part (b) of the theorem, for \( \text{ES}_\alpha \) the minimum is attained at \( \phi^* = \text{VaR}_\alpha[-L] < \text{ES}_\alpha[-L] \). This phenomenon is due to the diversification between \( X \) and \( L \). The probability of a synchronous realization of \( X \) and \( L \) beyond their respective \((1-\alpha)\)-quantiles amounts to \( \alpha^2 \ll \alpha \). Hence it makes sense to immunize against shocks in \( X \) based on a claim size notional below \( \rho[-L] \).

## 4 Perturbation Results

We apply perturbation methods and derive an expansion of \( \rho[S(\phi)] \) in terms of the log-normal volatility of the tradeable asset \( X \).

To construct a version of \( X \) indexed by its log-normal volatility we introduce the family of tradeable assets \( (X_\sigma)_{\sigma \geq 0} \) based on the centered and normalized logarithm of \( X \) as follows:

\[
X_\sigma := e^{\sigma Y} / \mathbb{E}[e^{\sigma Y}], \quad \text{where } Y := \mathbb{V}ar[\log(X)]^{-1/2} \cdot \left( \log(X) - \mathbb{E}[\log(X)] \right).
\]

Note that the log-normal standard deviation of \( X_\sigma \) equals \( \sigma \) and \( X_\sigma \) keeps the unit mean property due to the normalization. If \( X \) is log-normally distributed then \( X_\sigma = e^{\sigma Y - \sigma^2/2} \) where \( Y \) is a standard normal random variable.
We further denote by $z_\phi(\sigma)$ the $\alpha$-quantile of the surplus $S(\phi)$ with $X$ replaced by $X_\sigma$ and assume that $z_\phi(\sigma)$ is analytic in $\sigma$, i.e. allows for an expansion

$$z_\phi(\sigma) = z_0(\phi) + z_1(\phi) \cdot \sigma + z_2(\phi) \cdot \frac{\sigma^2}{2} + \cdots = \sum_{i=0}^\infty z_i(\phi) \cdot \frac{\sigma^i}{i!}.$$ 

Note that the functions $z_i$ are differentiable in $\phi$ by lemma 1.(e). Analogously to the derivation of relation (6), $z_\phi(\sigma)$ solves for every $\phi, \sigma \geq 0$

$$0 = \hat{G}(z; \sigma, \phi) := \mathbb{E}_X \left[ \hat{F}_L \circ w(\phi, z, X_\sigma) \right] - \alpha = \mathbb{E}_Y \left[ \hat{F}_L(\phi - (z_0 + \phi)) - \alpha = \hat{F}_L(-z_0) - \alpha, \right].$$

If $\hat{G}$ is analytic, then the $n$-th order total differentials $(D^n \hat{G}(z_\phi(\sigma); \sigma, \phi))_{\sigma=0}$ of $\hat{G}$ with respect to $\sigma$ evaluated at $\sigma = 0$ must vanish for every order $n = 0, 1, 2, \ldots$.

We start with the zero-order term in $\sigma$:

$$0 = \hat{G}(z_0; 0, \phi) = \mathbb{E}_Y \left[ \hat{F}_L(\phi - (z_0 + \phi)) \right] - \alpha = \hat{F}_L(-z_0) - \alpha,$$

hence $F_L(-z_0) = 1 - \alpha$ or, equivalently, $z_0 = -q$ constant. This is intuitive since for $\sigma = 0$ we have $X_0 \equiv 1$ and the distribution of the surplus becomes independent of $\phi$.

Using similar techniques, we can show that the first-order term $z_1$ vanishes and that the second and third order terms $z_2$ and $z_3$ can be expressed as polynomials in $(\phi - q)$. We state the expansion results for the value-at-risk in the following theorem. The details of the proofs are transferred to the appendix. We denote by $\mu_3 := \mathbb{E}[Y^3]$ the third normalized moment (skew) of log($X$) and we write $o(\sigma^n)$ for a term that vanishes when divided by $\sigma^n$ in the limit $\sigma \to 0$.

**Theorem 5.**  
a) The expansion of $\text{VaR}_\alpha[S(\phi)]$ in terms of the small log-normal volatility $\sigma$ of the financial asset $X$ up to third order is given by

$$\text{VaR}_\alpha[S(\phi)] = q + (\phi - q) \cdot \left(1 - \frac{1}{2} \cdot \frac{f'_L(q)}{f_L(q)} \cdot (\phi - q)\right) \cdot \sigma^2 + (\phi - q)^2 \cdot \left(\frac{1}{2} \cdot \frac{f'_L(q)}{f_L(q)} - \frac{1}{6} \cdot \frac{f''_L(q)}{f_L(q)} \cdot (\phi - q)\right) \cdot \mu_3 \cdot \sigma^3 + o(\sigma^3).$$

b) If $f'_L(q) \neq 0$, the minimum of the second order expansion of $\text{VaR}_\alpha[S(\phi)]$ is attained at

$$\phi^* = q + f_L(q) / f'_L(q),$$

which also minimizes the third order expansion in the case $\mu_3 = 0$.

c) If $\mu_3 \cdot f''_L(q) \neq 0$, the third order expansion attains its local minimum at

$$\phi^* = q + f''_L(q)^{-1} \cdot \left((1 - \delta) \cdot f'_L(q) - \sqrt{(1 - \delta)^2 \cdot f'_L(q)^2 + 2 \cdot \delta \cdot f''_L(q) \cdot f_L(q)}\right),$$

where $\delta := (\sigma \cdot \mu_3)^{-1}$.

**Remark 6.**  
a) The expansion of the value-at-risk only involves local properties of $L$ around its $(1-\alpha)$-quantile, i.e. (higher order) derivatives of $f_L$ at $q$.

b) If the skew of log($X$) vanishes and $L$ is normally distributed with volatility $\sigma_L$, then $q = \sigma_L \cdot u_{1-\alpha}$ with $u_{1-\alpha}$ equals the $(1-\alpha)$-quantile of the standard normal distribution and $f'_L(q) / f_L(q) = -q / \sigma^2_L = -u_{1-\alpha} / \sigma_L$. Part (b) of the theorem implies that $\phi^* / q = 1 - u_{2-\alpha}^2$, which amounts to 0.815 or 0.849 for the risk tolerance $1-\alpha = 0.99$ (Basel II) or $0.995$ (Solvency II), respectively. This means that the total Solvency II capital requirement of the financial institution (when evaluated via a fully stochastic model) is minimized, if in addition to the expected claim size also 84.9% of the non-hedgeable risk component, i.e. 99.5%-quantile of the centered claim size $L$, is initially invested in $X$. 

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c) The presence of a negative log-normal skew of the asset \( X \) (the usual case in practical applications) shifts the optimal asset allocation \( \phi^* \) nearer to the \( 1-\alpha \) quantile \( q \) of \( L \), refer to Fig. 2 and vice versa for a positive log-normal skew.

d) An expansion in terms of the volatility of \( X \) (instead of the volatility of log-return of \( X \)) is also possible. But this expansion has bad convergence properties.

Let us turn to the expected shortfall. Using the well-known representation of the expected shortfall 
\[
\text{ES}_\alpha[\cdot] = \alpha^{-1} \int_0^\alpha \text{VaR}_\beta[\cdot] \, d\beta,
\]
we obtain from theorem 5 the following expansion of \( \text{ES}_\alpha[S(\phi)] \) in terms of powers of the log-normal volatility of the tradeable asset \( X \).

**Theorem 7.** If \( f_L \) and its derivatives decays faster than polynomially for large value of \( L \), the expansion of the \( \text{ES}_\alpha[S(\phi)] \) up to third order in terms of the log-normal volatility \( \sigma \) of \( X \) around the minimum at \( \phi^* = q \) is given by

\[
\begin{align*}
\text{ES}_\alpha[S(\phi)] &= \text{ES}_\alpha[-L] \\
&\quad + \alpha^{-1} \cdot f_L(q) \cdot (\phi - q)^2 \cdot \sigma^2/2 \\
&\quad + \alpha^{-1} \cdot f'_L(q) \cdot (\phi - q)^3 \cdot \mu_3 \cdot \sigma^3/6 + o(\sigma^3).
\end{align*}
\]

Changing the integration variable \( \beta \to F_L^{-1}(1-\beta) \) is an essential ingredient for the proof; the details are transferred to the appendix.

**Remark 8.**

a) In contrast to the value-at-risk case, the minimal \( \phi^* \) does not depend on the skew of \( X \). As already stated in theorem 3 the minimal \( \phi^* \) for the expected shortfall is completely independent of the distribution of \( X \).

b) The result looks like a simultaneous expansion in terms of \( \phi \cdot \ln(X) \) of an integral of \( F_L \), but this seems to hold for the first three orders only.

5 Numerical Analysis

We now compare our perturbation results from section 4 with numerical analysis. For realistic choices of the log-normal volatility \( \sigma \) the optimum is hard to resolve based on Monte Carlo simulation. We therefore use numerical integration and sample the cumulative distribution function (6) around the \( \alpha \)-quantile of the surplus \( S(\phi) \) in order to obtain the inverse.

Fig. 1 shows the function \( \phi \mapsto \rho[S(\phi)] \) for the risk measures \( \rho \in \{\text{VaR}_\alpha, \text{ES}_\alpha\} \) with the Solvency II risk tolerance \( 1-\alpha = 99.5\% \). The claim size \( L \) is normally distributed such that \( q = 1 \) and log-normal volatility and skew of the asset \( X \) are calibrated to typical values of a 30 year discount factor (here based on US interest rates) which are rather at the upper end of realistic values for a typical insurance portfolio, refer to description of Fig. 1 for calibration details. It can be seen that our analytical results for the second and third order expansion (theorems 5 and 7) approximate the numerical behavior quite well. As predicted the risk minimal investment amount in \( X \) is around \( \phi^* \approx 0.85 \) for \( \rho = \text{VaR}_\alpha \) and \( \phi^* = 1 \) for \( \rho = \text{ES}_\alpha \), respectively. Investing in \( X \) according to the best-estimate value of \( L \) (i.e. \( \phi = 0 \)) would result in risk figures, which are substantially larger than the risk minimal value. Surprisingly, for the value-at-risk the minimum can even be below \( q = 1 \), i.e. below the VaR\( \alpha \) of the stand-alone distribution of \( L \).

[Figure 1 should be inserted here]
Next we analyze for the risk measure VaR$\alpha$ the location of the risk minimal investment amount $\phi^*$ in more detail, which depends on the characteristics of the hedgeable risk factor $X$. Fig. 2 shows the dependence of $\phi^*$ on the log-normal volatility $\sigma$ for various log-normal skew values $\mu_3$.

[Figure 2 should be inserted here]

In case of zero skew the third order expansion term vanishes. Higher order terms lead only to very small corrections to our theoretical prediction of $\phi^* \approx 0.85$. For realistic skew values of around $\mu_3 = -0.3$ the third order expansion is a good approximation up to $\sigma = 0.5$. In case of very high skew $\mu_3 = -1.0$ the approximation is only good up to $\sigma = 0.3$. To sum up, for realistic parametrizations of the hedgeable risk factor $X$ our perturbation results up to third order reflect the behavior of the risk minimal investment amount $\phi^*$ very well.

6 Application to Solvency II Market Risk Measurement

In general, there are two ways of how to set up an internal model for calculating the Solvency Capital Requirement (SCR) under Solvency II: The integrated risk model calculates the surplus (=excess assets over liabilities) distribution of the economic balance sheet, by simulating simultaneously the stochastics of all risk drivers (hedgeable and non-hedgeable). Although it is the more adequate approach, it is rarely used in practice both for operational and steering reasons. Market standard is a modular approach similar to the one used in the Solvency II standard formula. In the modular risk model the profit and loss distribution for each risk category is computed in a separate module and the different risk modules are subsequently aggregated to the total SCR of the company. For risk categories which affect only one side of the economic balance sheet this approach works fine. The market risk module is more problematic, because risk drivers like FX or interest rates affect both sides of the balance sheet. Therefore so-called replicating portfolios are introduced, which translate the capital market sensitivities of the liability side into a portfolio of financial instruments (e.g. zero coupon bonds). The key question is, how the notional value of the liabilities should be chosen for the replicating portfolio? Market standard is to take the best-estimate value, which implies that the capital backing the surplus is attributed to the risk-free investment, e.g. EUR cash. We will show that this can lead to significant distortions of the measured market risk SCR as compared to an integrated risk model. To avoid this we have introduced at Munich Re the concept of the Economic Neutral Position (ENP) which is defined as (virtual) asset portfolio, which minimizes the total SCR of the integrated model. The ENP is the risk-neutral reference point for Solvency II market risk measurement in Munich Re’s certified internal model. Fig. 3 illustrates how the ENP is embedded in the modular structure of the internal risk model.

[Figure 3 should be inserted here]

For liabilities exhibiting the product structure $L \cdot X$ defined in section 2, the ENP corresponds exactly to the solution of the optimization problem addressed in this paper. The present value of the liability (being represented by a zero coupon bond in the ENP) equals the best estimate value of $L \cdot X$ plus a safety margin corresponding to the risk minimal investment amount $\phi^*$ of theorem 5. If $L$ is normally distributed then this safety margin equals 85% of the SCR for non-hedgeable risks. Strictly speaking, this result only holds for the one-dimensional case, i.e. a liability portfolio being represented by a single currency and a single time to maturity. We will show in [5] that the same result also holds for...

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2Except for with-profit live insurance business which exhibits significant interaction between the asset and the liability side of the insurer’s balance sheet.
the multi-dimensional case with arbitrary number of liability cash flows. In that case the additional component in the ENP equals 85% of the total non-hedgeable SCR, which is fully diversified within all non-hedgeable risks of the insurance business. This component is allocated to the individual liability cash flows according to the covariance principle.

Let us now analyze the total SCR of a modular risk model, which uses the ENP as risk-neutral reference portfolio for market risk measurement, and compare it with the outcome of an integrated risk model. We assume that the surplus is of the form (5). The non-hedgeable SCR of $L$ is measured in the insurance risk module and can be set to one without loss of generality. The market risk SCR of $M$ is measured by the VaR99.5% of the mismatch portfolio of assets minus ENP, i.e. $S(\phi) = (\phi - \phi^*) \cdot X - \phi$, and is a function of the units $\phi$ of the financial asset $X$. For the sake of simplicity the total SCR is calculated by aggregating $SCR_L$ and $SCR_M$ based on the square root formula, which is also used in the Solvency II standard formula (remember that $L$ and $X$ are assumed to be independent):

$$SCR_T = \sqrt{SCR^2_L + SCR^2_M}.$$  

This aggregation method is only valid for a sum of normally distributed stochastic variables. Therefore we assume that both risk drivers $L$ and $X$ follow a normal distribution, i.e. we violate here the positivity assumption on $X$ for technical reasons. Otherwise the aggregation method needs to be adjusted accordingly.

Fig. 4 compares the total SCR of the modular risk model with the total SCR of the integrated model, which is simply the value-at-risk of $S(\phi)$ at risk tolerance $1 - \alpha = 99.5\%$ with joint stochastics of all risk drivers.

The integrated and the ENP-based modular approach yield in good approximation the same total SCR, as desired. Only if the asset value $\phi$ differs strongly from the risk minimal value $\phi^*$ deviations between the outcomes of the two models can be observed. This is due to the fact, that the square root formula used for aggregation only holds for a sum of normally distributed stochastic variables. Due to the product structure $L \cdot X$ the total distribution of the surplus is in general not normally distributed (even though both $L$ and $X$ are normally distributed). This effect can be healed to some extent by refining the aggregation method for the modular model.

For comparison we show in Fig. 4 also the industry standard, which measures market risk versus the replicating portfolio (RP). This corresponds to setting the notional of the liability $L$ equal to its best-estimate value, which is zero in our example. This can lead to substantial deviations from the true SCR as measured by the integrated model. Especially if the asset notional is below the expected claim size – a typical case for life insurers whose asset duration is generally lower than the duration of the liabilities due to the long-term nature of the business – the modular RP-based approach understates the “true” risk significantly.

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A Proofs

Justification of the simplifying assumptions \(4\): \(S(\phi)\) as defined by (11) has mean \(\bar{S}(\phi) := \phi \cdot (\bar{X} - X_0) + A_0 - \bar{X} \cdot \bar{L}\), where the bar superscript denotes expected value. We analyze the centered surplus

\[
S^*(\phi) := S(\phi) - \bar{S}(\phi) = \phi \cdot (X - X_0) + A_0 - X \cdot L - (\phi \cdot (X - X_0) + A_0 - \bar{X} \cdot \bar{L}) = \phi \cdot (X - \bar{X}) - (X \cdot L - \bar{X} \cdot \bar{L}) = (\phi - \bar{L}) \cdot (X - \bar{X}) - X \cdot (L - \bar{L})
\]

(9)

Using the cash invariance and positive homogeneity of the risk measures we can write

\[
\rho[S(\phi)] = \rho[S^*(\phi) + \bar{S}(\phi)] = \bar{X} \cdot \rho[\bar{X}^{-1} \cdot S^*(\phi)] - \bar{S}(\phi)
\]

\[
= \bar{X} \cdot \rho[\bar{S}(\phi - \bar{L})] - \phi \cdot (X - X_0) - (A_0 - \bar{X} \cdot \bar{L}),
\]

(10)

where

\[
\bar{S}(\phi) := \phi \cdot (X' - 1) - X' \cdot L', \quad \text{with } \bar{X} := X/\bar{X}, \quad \bar{L} := L - \bar{L}.
\]

The last equation follows from (9) which implies that \(\bar{X}^{-1} \cdot S^*(\phi) = \bar{S}(\phi - \bar{L})\). Note that the adjusted surplus function \(\bar{S}\) satisfies the simplifying conditions \(4\).

If \(\bar{X} = X_0\), relation (10) shows that the risk of the surplus in its original definition \(4\) is a linear transform of the risk of adjusted surplus \(5\) evaluated at \(\phi - \bar{L}\), i.e. the initial investment in \(X\) in excess of the best estimate of \(L\); otherwise, i.e. if \(\bar{X} \neq X_0\), the additional linear term \(\phi \cdot (\bar{X} - X_0)\) appears.

Proof of lemma \[7\]: Ad (a) and (b): since \(L\) has a density, \(\bar{F}_L\) is continuous. Hence also \((\phi, z) \mapsto G(\phi, z) := \mathbb{E}_X [\bar{F}_L(w(\phi, z, X))]\) is continuous. For every \(\phi \geq 0, z \mapsto G(\phi, z)\) is an increasing function and \(G(\phi, \mathbb{R}) = [0, 1]\). Hence for every \(\phi \geq 0\) and \(\alpha \in [0, 1]\) there exists a unique \(z_{\phi, \alpha} \in \mathbb{R}\) such that \(\mathbb{P}(S(\phi) \leq z_{\phi, \alpha}) = G(\phi, z_{\phi, \alpha}) = \alpha\), which proves (a). The latter also implies that \(S(\phi)\) has no atoms, and hence upper and lower quantile of \(S(\phi)\) coincide; the representation for the expected shortfalls follows from Corollary 4.49 of \[2\], hence (b) is proved.

Ad (c): since \(f_L\) is bounded and \(X^{-1}\) is integrable, partial differentiation and \(\mathbb{E}_X\) can be interchanged in the definition of \(G\) by dominated convergence and the partial derivatives of \(G\) are continuous. In particular, \(\partial_z G(\phi, z) = \mathbb{E}_X [-f_L(w) \cdot \partial_z w] = \mathbb{E}_X [f_L(w)/X] > 0\), since \(f_L > 0\) in the tail. Hence the implicit function theorem implies that \(\phi \mapsto z_{\phi, \alpha}\) is differentiable. For the expected shortfall the differentiability with respect to \(\phi\) follows from the representation \(\mathbb{E}_{S_{\alpha}}[S(\phi)] = \alpha^{-1} \cdot \int_0^\alpha \text{VaR}_{\beta}[S(\phi)] d\beta\), since the differential \(\partial_\phi\) and the integral \(\int_0^\alpha\) can be interchanged.

Ad (d): for \(\phi_1, \phi_2 \geq 0\) and \(\lambda \in [0, 1]\),

\[
S(\lambda \cdot \phi_1 + (1 - \lambda) \cdot \phi_2) = (\lambda \cdot \phi_1 + (1 - \lambda) \cdot \phi_2) \cdot (X - 1) - X \cdot L
\]

\[
= \lambda \cdot \phi_1 \cdot (X - 1) - X \cdot L + (1 - \lambda) \cdot \phi_2 \cdot (X - 1) - X \cdot L
\]

\[
= \lambda \cdot S(\phi_1) + (1 - \lambda) \cdot S(\phi_2).
\]

Hence the assertion follows from the convexity of the expected shortfall.

Proof of part (b) of theorem \[8\]: To determine \(\partial_\phi z_\phi\) at \(\phi = q\), we apply the implicit function theorem on the defining equation for \(z_\phi\) which by (6) reads \(\alpha = \mathbb{E}_X [\bar{F}_L(w(\phi, z, X))]\). We denote by \(D_\phi\) the total differential with respect to \(\phi\), i.e. \(D_\phi = \partial_\phi + (\partial_\phi z_\phi) \cdot \partial_z\). Applying \(D_\phi\) on the defining equation of \(z_\phi\) yields

\[
0 = D_\phi \mathbb{E}_X [\bar{F}_L(w(\phi, z_\phi, X))] = -\mathbb{E}_X [f_L(w(\phi, z_\phi, X)) \cdot D_\phi w(\phi, z_\phi, X)]
\]

(11)
From \( w(\phi, z, X) = \phi - (z + \phi)/X \) we derive \( \partial_\phi w = 1 - 1/X \) and \( \partial_z w = -1/X \). Hence
\[
\partial_\phi z_\phi = \frac{\mathbb{E}_X [f_L(w) \cdot (1 - 1/X)]}{\mathbb{E}_X [f_L(w) \cdot (1/X)]} = \frac{\mathbb{E}_X [f_L(w)]}{\mathbb{E}_X [f_L(w) \cdot (1/X)]} - 1,
\]
provided the denominator is not zero. Since \( z_q = -q \), \( w(q, z_q, X) = q - (q + z_q)/X = q \) becomes constant. Hence also \( f(w) \) becomes constant and the expression for \( \partial_\phi z_\phi \) above collapses to
\[
(\partial_\phi z_\phi)_{\phi=q} = \mathbb{E}[X^{-1}]^{-1} - 1 \leq 0,
\]
with < if \( X \) is non constant. The latter inequalities follow from the strict convexity of the inverse function and Jensen’s inequality, which implies \( \mathbb{E}[X^{-1}] > \mathbb{E}[X]^{-1} = 1 \) for non-constant \( X \). Multiplying (12) with \(-1\) yields the assertion of the theorem for the value-at-risk.

For the expected shortfall, we can show that at \( \phi = q \) the derivative with respect to \( \phi \) vanishes: from the last but one equation in the derivation of the relation (6) we find that \( \{S(\phi) \leq z_\phi\} = \{L \geq w(\phi, z_\phi, X)\} \). Similar to (8) we calculate
\[
\mathbb{E}[S(\phi) \cdot 1_{S(\phi) \leq z_\phi}] = \mathbb{E}_X \left[ (\phi \cdot (X - 1) - X \cdot L) \cdot 1_{L \geq w(\phi, z_\phi, X)} \right] = \phi \cdot \mathbb{E}_X \left[ (X - 1) \cdot \mathbb{P} (L \geq w(\phi, z_\phi, X)) \right] - \mathbb{E}_X \left[ X \cdot E_L[L \cdot 1_{L \geq w(\phi, z_\phi, X)}] \right] = \phi \cdot \mathbb{E}_X [(X - 1) \cdot \bar{F}_L(w)] - \mathbb{E}_X [X \cdot \bar{H}_L(w)],
\]
where \( \bar{H}_L(x) := \mathbb{E}[L \cdot 1_{L \geq x}] = \int_x^\infty l \cdot f_L(l) \, dl \). Differentiation with respect to \( \phi \) yields
\[
\partial_\phi \mathbb{E}[S(\phi) \cdot 1_{S(\phi) \leq z_\phi}] = \mathbb{E}_X [(X - 1) \cdot \bar{F}_L(w)] - \phi \cdot \mathbb{E}_X [(X - 1) \cdot f_L(w) \cdot D_\phi w] + \mathbb{E}_X [X \cdot w \cdot f_L(w) \cdot D_\phi w].
\]
Recall that at \( \phi = q \), the term \( w(q, z_q, X) = q \) becomes constant. Hence the above expression simplifies
\[
\partial_\phi \mathbb{E}[S(\phi) \cdot 1_{S(\phi) \leq z_\phi}]_{\phi=q} = \bar{F}_L(q) \cdot \mathbb{E}_X [X - 1] + q \cdot f_L(q) \cdot \mathbb{E}_X [(X - 1) \cdot D_\phi w] = q \cdot f_L(q) \cdot \mathbb{E}_X [(D_\phi w)(q, z_q, X)] = 0
\]
where the last equality follows from (11) evaluated at \( \phi = q \) where \( f(w) \) becomes a positive constant. This proves the assertion of the theorem for the expected shortfall.

Proof of theorem 3: Ad (a): Recall the definition \( \hat{G}(z_\phi(\sigma); \sigma, \phi) = \mathbb{E}_X \left[ \bar{F}_L(w(\phi, z_\phi(\sigma), X_\sigma)) \right] - \alpha \).

The total differential \( D_\sigma \hat{G} = \partial_\sigma + \partial_\sigma z_\phi \cdot \partial_z \) applied to \( \hat{G} \) up to third order reads
\[
D_\sigma \hat{G} = -\mathbb{E}_Y [f_L(w) \cdot D_\sigma w],
D_\sigma^2 \hat{G} = -\mathbb{E}_Y [f_L'(w) \cdot (D_\sigma w)^2 + f_L(w) \cdot D_\sigma^2 w],
D_\sigma^3 \hat{G} = -\mathbb{E}_Y [f_L''(w) \cdot (D_\sigma w)^3 + 3 \cdot f_L'(w) \cdot D_\sigma w \cdot D_\sigma^2 w + f_L(w) \cdot D_\sigma^3 w].
\]
Set \( \xi(\sigma) := X_\sigma^{-1} = e^{-\sigma Y} \cdot M_Y(\sigma) \), where \( M_Y(\sigma) = \mathbb{E}[e^{\sigma Y}] \) is the moment generating function of \( Y \). Then the function \( w \) reads
\[
w(\phi, z_\phi(\sigma), X_\sigma) = \phi - (z_\phi(\sigma) + \phi) \cdot \xi(\sigma).
\]
The n-the order total differential \( D_\sigma^n \) of \( z_\phi \cdot \xi \) is given by
\[
D_\sigma^n (z_\phi \cdot \xi) = \sum_{i=0}^n \binom{n}{i} z_\phi^{(i)} \cdot \xi^{(n-i)},
\]
where \( z_\phi^{(i)} \) denotes the i-the order derivative with respect to \( \sigma \). Hence the n-the order total differential of \( w \) reads
\[
D_\sigma^n w = -\binom{n}{i} z_\phi^{(i)} \cdot \xi^{(n-i)}),
\]
where
and for the n-th order derivatives $\xi^{(n)}$ of $\xi$ with respect to $\sigma$ read

$$\xi^{(n)}(\sigma) = e^{-\sigma Y} \cdot \sum_{i=0}^{n} \left(-1\right)^{i} \cdot \binom{n}{i} \cdot Y^{i} \cdot M_{Y}^{(n-i)}(\sigma).$$

Since $Y$ has vanishing first moment and unit second moment, the derivatives of $\xi$ at $\sigma = 0$ become

$$\xi(0) = 1, \quad \xi'(0) = -Y, \quad \xi''(0) = 1 + Y^{2}, \quad \xi'''(0) = \mu_{3} - 3Y - Y^{3}. \quad (15)$$

We show that $z_{1}$, i.e., the first-order term in the expansion of $z_{\phi}$ with respect to $\sigma$, vanishes: evaluating $w$ and $D_{\sigma}w$ at $\sigma = 0$, having in mind that $(D_{\sigma}^{n}z_{\phi})|_{\sigma=0} = z_{n}(\phi)$ and $z_{0} = -q$, yields

$$w|_{\sigma=0} = q, \quad (D_{\sigma}w)|_{\sigma=0} = (\phi - q) \cdot Y - z_{1}(\phi).$$

Plugging this into the requirement $(D_{\sigma}\hat{G})|_{\sigma=0} = 0$, we obtain with (13)

$$0 = f_{L}(q) \cdot \left((\phi - q) \cdot \mathbb{E}[Y] - z_{1}\right).$$

This implies $z_{1} \equiv 0$ since $Y$ has zero mean and $f_{L}(q) > 0$ by (3).

Evaluating the higher order total differentials of $w$ at $\sigma = 0$, we obtain from (14) and (15)

$$(D_{\sigma}^{2}w)|_{\sigma=0} = -\left(\phi - q\right) \cdot (1 + Y^{2}) - z_{2},$$

$$(D_{\sigma}^{3}w)|_{\sigma=0} = -\left(\phi - q\right) \cdot (\mu_{3} - 3Y - Y^{3}) + 3Yz_{2} - z_{3}.$$  

To determine the second order term $z_{2}$, we plug the above expressions for $(D_{\sigma}w)|_{\sigma=0}$ and $(D_{\sigma}^{2}w)|_{\sigma=0}$ in the expression (13) to obtain from the requirement

$$0 = (D_{\sigma}^{2}\hat{G})|_{\sigma=0} = f'_{L}(q) \cdot (\phi - q)^{2} \cdot \mathbb{E}[Y^{2}] - f_{L}(q) \cdot \left((\phi - q) \cdot (1 + \mathbb{E}[Y^{2}]) + z_{2}(\phi)\right).$$

Since $Y$ has unit variance, we have $f_{L}(q) \cdot z_{2}(\phi) = (\phi - q) \cdot [f'_{L}(q) \cdot (\phi - q) - 2 \cdot f_{L}(q)]$. Multiplying with the Taylor term $\sigma^{2}/2$, dividing by $f_{L}(q) > 0$ and multiplying with $-1$ (to flip from the quantile to VaR) yields the second order expansion term of part (a) of the theorem.

To determine the third order term $z_{3}$, we first note that $\mathbb{E}[(D_{\sigma}^{3}w)|_{\sigma=0}] = -z_{3}$ and plug this together with the expressions for $(D_{\sigma}w)|_{\sigma=0}$, $n = 0, 1, 2$ into the third order equation of (13) to obtain

$$0 = (D_{\sigma}^{3}\hat{G})|_{\sigma=0} = f''_{L}(q) \cdot (\phi - q)^{3} \cdot \mathbb{E}[Y^{3}] - 3 \cdot f'_{L}(q) \cdot (\phi - q) \cdot \mathbb{E} \left[ Y \cdot \left( (\phi - q) \cdot (1 + Y^{2}) + z_{2}(\phi) \right) \right] - f_{L}(q) \cdot z_{3}(\phi),$$

where the last equality follows from the fact that $Y$ has zero mean. Multiplying with the Taylor term $\sigma^{3}/3!$, dividing by $f_{L}(q) > 0$ and multiplying with $-1$ (to flip from the quantile to VaR) yields the third order expansion term of part (a) of the theorem.

Ad (b): Differentiating the second order expansion of $\text{VaR}_{\alpha}[S(\phi)]$ with respect to $\phi$ and setting it equal to zero yields $0 = \sigma_{2} \cdot \left[ 1 - f''_{L}/f_{L}(q) \right] \cdot (\phi - q)$. Solving for $\phi$ leads to the minimum $\phi_{2}^{*}$.

Ad (c): Setting $\psi = \phi - q$, then the value-at-risk of the surplus reads in the third order expansion $\text{VaR}_{\alpha}[S(\psi)] = (a/3) \cdot \psi^{3} + (b/2) \cdot \psi^{2} + c \cdot \psi + q$ with $a = -\mu_{3} \cdot 3/2 \cdot (f''_{L}/f_{L}(q))$, $b = (\mu_{3} \cdot \sigma - 1) \cdot \sigma^{2} \cdot (f'_{L}/f_{L}(q))$, and $c = \sigma^{2}$. Setting the differential with respect to $\psi$ equal to zero yields the quadratic formula which is solved by $\psi = \psi_{\pm} = (-b \pm \sqrt{b^{2} - 4ac})/(2a)$. Only $\psi_{\pm}$ constitutes a (local) minimum of the third order polynomial in $\psi$, since its second order derivative evaluated at $\psi_{\pm}$ reads $2a\psi_{\pm} + b = \pm \sqrt{b^{2} - 4ac}$ which is only positive for $\psi_{+}$. Hence the locally minimal $\phi$ is
given by \( \phi^* = q + \psi_+ \). Inserting the parameters \( a, b, \) and \( c \) and straightforward calculus leads the assertion.

**Proof of theorem**\(^7\). The expansion of \( \text{ES}_\alpha[S(\phi)] \) in the log-normal volatility \( \sigma \) of \( X \) is obtained by calculating \( \alpha^{-1} \int_0^\alpha \text{VaR}_\beta[S(\phi)] \, d\beta \) using the expansion of \( \text{VaR}_\alpha[S(\phi)] \) in \( \sigma \) of theorem 5. The only terms in this expansion that depend on the risk tolerance \( \alpha \) are those terms depending on \( q \), which is denoted by \( q_\alpha \) in the sequel.

To analyze the zero order term, we calculate

\[
\alpha^{-1} \int_0^\alpha q_\beta \, d\beta = \alpha^{-1} \int_0^\alpha F_L^{-1}(1 - \beta) \, d\beta = \alpha^{-1} \int_0^\alpha -F_L^{-1}(\beta) \, d\beta = \alpha^{-1} \int_0^\alpha \text{VaR}_\beta[-L] \, d\beta = \text{ES}_\alpha[-L].
\]

We apply expensively the change the integration variable \( \beta \to y := q_\beta = F_L^{-1}(1 - \beta) \) in the sequel which implies \( d\beta = -f_L(y) \, dy \). We can write for the zero order term

\[
\text{ES}_\alpha[-L] = \alpha^{-1} \int_0^\alpha q_\beta \, d\beta = \alpha^{-1} \int_{F_L^{-1}(1)}^{F_L^{-1}(1 - \alpha)} y \cdot (-f_L(y)) \, dy = \alpha^{-1} \int_{q_\alpha}^\infty y \cdot f_L(y) \, dy.
\]

To prepare for the second and third order terms, we analyze the \( \alpha^{-1} \int_0^\alpha \frac{f_L^{(n)}(q_\beta)}{f_L(q_\beta)} \cdot (\phi - q_\beta)^{n+1} \, d\beta \) for \( n = 1, 2 \), where \( f_L^{(n)} \) denotes the \( n \)-th order derivative, and change again the integration variable \( \beta \to y \). We hence obtain similarly to the above calculations

\[
\begin{align*}
\alpha^{-1} \int_0^\alpha \frac{f_L^{(n)}(q_\beta)}{f_L(q_\beta)} \cdot (\phi - q_\beta)^{n+1} \, d\beta &= \alpha^{-1} \int_{q_\alpha}^\infty f_L^{(n)}(y) \cdot (\phi - y)^{n+1} \, dy \\
&= \alpha^{-1} \left\{ \left[ f_L^{(n-1)}(y) \cdot (\phi - y)^{n+1} \right]_{y=q_\alpha}^{\infty} + (n + 1) \int_{q_\alpha}^\infty f_L^{(n-1)}(y) \cdot (\phi - y)^n \, dy \right\} \\
&= \alpha^{-1} \left\{ -f_L^{(n-1)}(q_\alpha) \cdot (\phi - q_\alpha)^{n+1} + (n + 1) \int_{q_\alpha}^\infty f_L^{(n-1)}(y) \cdot (\phi - y)^n \, dy \right\}
\end{align*}
\]

which equals for \( n = 1 \)

\[
\begin{align*}
&= \alpha^{-1} \left\{ -f_L(q_\alpha) \cdot (\phi - q_\alpha)^2 + 2 \cdot \left( \phi \cdot (1 - F_L(q_\alpha)) - \int_{q_\alpha}^\infty f_L(y) \cdot y \, dy \right) \right\} \\
&= -\alpha^{-1} f_L(q_\alpha) \cdot (\phi - q_\alpha)^2 + 2 \cdot (\phi - \text{ES}_\alpha(-L)),
\end{align*}
\]

and for \( n = 2 \) (note that the second term coincides with the \( n = 1 \) case)

\[
\begin{align*}
&= -\alpha^{-1} f_L'(q_\alpha) \cdot (\phi - q_\alpha)^3 + 3 \cdot \alpha^{-1} \int_0^\alpha \frac{f_L'(q_\beta)}{f_L(q_\beta)} \cdot (\phi - q_\beta)^2 \, d\beta.
\end{align*}
\]

We now integrate the \( \sigma^2 \)-order term of theorem 5(a) with respect to the tolerance level and obtain

\[
\begin{align*}
\alpha^{-1} \int_0^\alpha \left\{ (\phi - q_\beta) - \frac{1}{2} \cdot \frac{f_L'(q_\beta)}{f_L(q_\beta)} \cdot (\phi - q_\beta)^2 \right\} \, d\beta \\
&= (\phi - \text{ES}_\beta) - \frac{1}{2} \left( -\frac{f_L(q_\alpha)}{\alpha} \cdot (\phi - q_\alpha)^2 + (\phi - \text{ES}_\alpha(-L)) \right) = \frac{f_L(q_\alpha)}{2\alpha} \cdot (\phi - q_\alpha)^2,
\end{align*}
\]

which proves the \( \sigma^2 \)-order term of the theorem.
We now integrate the $\mu_3 \cdot \sigma^3$-order term of theorem 5(a) with respect to the tolerance level and obtain

\[
\alpha^{-1} \int_0^\alpha \left\{ \frac{1}{2} \cdot \frac{f'_L(q_{\beta})}{f_L(q_{\beta})} \cdot (\phi - q_{\beta})^2 - \frac{1}{6} \cdot \frac{f''_L(q_{\beta})}{f_L(q_{\beta})} \cdot (\phi - q_{\beta})^3 \right\} \, d\beta
\]

\[
= \frac{1}{2\alpha} \int_0^\alpha \frac{f'_L(q_{\beta})}{f_L(q_{\beta})} \cdot (\phi - q_{\beta})^2 \, d\beta - \frac{1}{6\alpha} \left( -f'_L(q_{\alpha}) \cdot (\phi - q_{\alpha})^3 + 3 \int_0^\alpha \frac{f'_L(q_{\beta})}{f_L(q_{\beta})} \cdot (\phi - q_{\beta})^2 \, d\beta \right)
\]

\[
= \frac{f'_L(q_{\alpha})}{6\alpha} \cdot (\phi - q_{\alpha})^3,
\]

which proves the $\mu_3 \cdot \sigma^2$-order term of the theorem.
B Figures

Figure 1: Value-at-risk $\text{VaR}_\alpha[S]$ (left) and expected shortfall $\text{ES}_\alpha[S]$ (right) as a function of the units $\phi$ of the financial asset $X$. The risk tolerance is set to $1 - \alpha = 99.5\%$, the non-hedgeable component $L$ is normally distributed with $\sigma_L = 0.388$ such that $q = \text{VaR}_\alpha(-L) = 1$, and $\log(X)$ is log-normally distributed such that $X$ has log-normal volatility $\sigma = 0.2$ and log-normal skew $\mu_3 = -0.3$.

Figure 2: Optimal investment amount $\phi^*$ minimizing the value-at-risk $\text{VaR}_\alpha[S(\phi)]$ as a function of the log-normal volatility $\sigma$ of the financial asset $X$ for various log-normal skews $\mu_3$. Refer to the description of Fig. 1 for further calibration details.
Figure 3: Illustration of the concept of the Economic Neutral Position (ENP) via the link between integrated and modular risk model. Market risk is measured on the mismatch portfolio of assets minus ENP.

Figure 4: Total SCR as a function of the units $\phi$ of the financial asset $X$ for an integrated risk model (red solid) in comparison with a modular risk model, where the market risk is measured either vs. ENP (blue dashed-dotted) or vs. RP (black dashed). $X$ is assumed to be normally distributed with a volatility of 15%.