ENERGY PRINCIPLES FOR SELF-GRAVITATING BAROTROPIC FLOWS:

II. THE STABILITY OF MACLAURIN FLOWS

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Abstract

We analyze stability conditions of "Maclaurin flows" (self-gravitating, barotropic, two dimensional, stationary streams moving in closed loops around a point) by minimizing their energy, subject to fixing all the constants of the motion including mass and circulations. Necessary and sufficient conditions of stability are obtained when gyroscopic terms in the perturbed Lagrangian are zero. To illustrate and check the properties of this new energy principle, we have calculated the stability limits of an ordinary Maclaurin disk whose dynamical stability limits are known. Perturbations are in the plane of the disk. We find all necessary and sufficient conditions of stability for single mode symmetrical or antisymmetrical perturbations. The limits of stability are identical with those given by a dynamical analysis. Regarding mixed types of perturbations the maximally constrained energy principle give for some the necessary and sufficient condition of stability, for others only sufficient conditions of stability. The application of the new energy principle to Maclaurin disks shows the method to be as powerful as the method of dynamical perturbations.

Key words: Energy variational principle; Self-gravitating systems; Stability of fluids.
1. Introduction

The main purpose of this work is to illustrate on a test case our principle of "maximally constrained" energy minimum (Katz, Inagaki and Yahalom 1993), and find stability limits of stationary motions of barotropic flows. Paper I dealt with three-dimensional flows and gave no application. Here we consider two-dimensional "Maclaurin Flows" and apply the method to Maclaurin disks. With "Maclaurin Flows" we mean some sort of generalized rotating streams, with closed loops around a point, but no special symmetry. The simplest and most symmetric example of a Maclaurin flow is a Maclaurin disk. Other two-dimensional fluid models for galaxies are Maclaurin flows as well. Stationary flows are "stationary points" of their energy. Flows are stable if their energy is minimum, compared to that of all slightly different stationary trial configurations. By constraining trial configurations to satisfy all the constraints that a dynamical flow would satisfy: linear momentum, angular momentum, mass conservation and circulation, we may greatly enhance the value of the energy principle. Energy principles give only sufficient conditions of stability. However, we know that if the "gyroscopic term" in the Lagrangian of a dynamically perturbed configuration is zero, a minimum of energy becomes a necessary and sufficient condition.

Our test model is a Maclaurin disk. This is one of those rare models (Binney & Tremaine 1987) were a detailed dynamical stability analysis is possible. It is thus the one valuable test case in which the details of our method and its results can be compared with a different, independent calculation. Both method use spherical harmonic decompositions of perturbations or \((l, m)\) - modes.
What we find are the *same stability limits* for all symmetrical and antisymmetrical single modes as those given by the dynamical analysis of Binney and Tremaine. Other models have been tested which give also necessary and sufficient conditions of stability. Thus, Yahalom (1993) found the stability limits of a uniform rotating sheet and of two dimensional Raleigh flows, by the present technique, to be the same as that of a dynamical analysis. The new energy principle appears thus as a powerful instrument that may complete and occasionally replace the method of linear perturbation analysis. This is the main point of this work.

2. Description of Stationary Maclaurin Flows, and of the Energy Principle for their Stability.

2.1. Lagrange Variables and Equations of Motion for Stationary Flows.

Equations of motion in three and two dimensions are similar, though in two dimensions angular velocity, angular-momentum and vorticity have only one component, in the $z$ direction, and there are less variables. We use the following notations: for the positions of fluid elements $\vec{R}$ or $(x^K) = (x, y); K, L, ... = 1, 2$, the density of matter per unit surface is $\sigma$; $\vec{W}$ is the velocity field in inertial coordinates and $\vec{U}$ the velocity relative to coordinates moving with uniform velocity $\vec{b}$ and rotating with constant angular velocity $\Omega_c$; these are the coordinates in which configurations appear stationary. Thus, by definition,

$$\vec{W} = \vec{U} + \vec{\eta}_c,$$  \hfill (2.1.a)

$$\vec{\eta}_c = \vec{b} + \vec{\Omega}_c \times \vec{R}$$ \hfill (2.1.b)

$$\vec{\Omega}_c = \mathbb{I}_z \Omega_c.$$ \hfill (2.1.c)
The mass is conserved in moving coordinates

\[ \nabla \cdot (\sigma \vec{U}) = 0 \]  \hspace{1cm} (2.2)

On the boundary of free self-gravitating fluids the density and the pressure are zero

\[ \sigma|_B = 0, \]  \hspace{1cm} (2.3.a)
\[ P|_B = 0 \]  \hspace{1cm} (2.3.b)

and in flows of astrophysical interest the velocity of sound is there zero also,

\[ \frac{\partial P}{\partial \sigma}|_{\sigma=0} = 0. \]  \hspace{1cm} (2.4)

Notice that (2.2) and (2.3) imply

\[ (\vec{U} \cdot \nabla \sigma)|_B = 0. \]  \hspace{1cm} (2.5)

Since \( \nabla \sigma|_B \) is normal to the boundary as follows from (2.3a), (2.5) means, as one expects, that \( \vec{U} \) is tangent to the boundary. Because of (2.3a) an integral of a divergence of the form \( \nabla \cdot (\sigma \vec{F}) \) must thus also be zero by virtue of Green’s identity:

\[ \int_S \nabla \cdot (\sigma \vec{F}) d^2x = \int_B \sigma \vec{F} \cdot d\vec{B} = 0 \]  \hspace{1cm} (2.6)

provided \( \vec{F} \) is single valued and continuous in \( S \) and on \( B \). Euler’s equations for stationary motions in moving coordinates are well known and can be written:

\[ \vec{U} \cdot \nabla \vec{W} + \vec{\Omega}_c \times \vec{W} + \nabla (h + \Phi) = 0 \]  \hspace{1cm} (2.7)

where the specific enthalpy

\[ h = \int \frac{dP}{\sigma} = \varepsilon(\sigma) - \frac{P}{\sigma}, \]  \hspace{1cm} (2.8)
ε is the specific internal energy. The gravitational potential Φ is, in the plane of the motion, given by

\[ \Phi(\vec{R}) = -G \int \frac{\sigma(\vec{R}')}{|\vec{R} - \vec{R}'|} d^2x. \]  

(2.9)

The vorticity of the flow

\[ \vec{\omega} = \text{rot } \vec{W} = \text{rot } \vec{U} + 2\vec{\Omega}_c \]  

(2.10.a)

\[ \vec{\omega} = \omega \vec{1}_z \]  

(2.10.b)

2.2. Lagrange Variables for Maclaurin Flows.

The labels of fluid elements are appropriate Lagrange variables in fluid mechanics (Lamb 1945). A continuous labelling of fluid elements depends on the particular topology of the flow. Here we describe a special labelling, and a representation of the velocity field for which mass conservation and circulation conservation hold automatically. We consider Maclaurin flows that move in closed loops around a fixed point. Suppose that, in some coordinates moving with constant velocity \( \vec{b} \) and rotating with constant angular velocity \( \Omega_c \), we are given the density \( \sigma(x^K) \) and the relative velocity of the flow \( \vec{U}(x^K) \). We do not suppose that \( \sigma \) and \( \vec{U} \) represents a solution of Euler’s equations (2.7). We rather regard \( \sigma \) and \( \vec{U} \) as a trial configuration of the flow. With \( \vec{U}, \vec{b} \) and \( \Omega_c \) we define a velocity \( \vec{W}(x^K) \) in inertial coordinates by equation (2.1). Given \( \sigma \) and \( \vec{W} \) we can construct a quantity

\[ \lambda = \frac{\sigma}{\omega} \]  

(2.11)

where \( \omega \) is the vorticity defined in equation (2.10). If \( \sigma|_B = 0 \),

\[ \lambda|_B = 0 \]  

(2.12)
In time dependant flows, the quantity $\lambda$ is conserved along the motion; $\lambda$ is the inverse of the potential vorticity; it is also the load of Lynden-Bell and Katz (1982). The conservation of $\lambda$ is equivalent to the conservation of circulation along closed contours (Katz and Lynden-Bell 1985). Now consider all the closed loops of constant $\lambda$ that can be drawn in our trial configuration and parametrize them with the value of the circulation along each loop $2\pi\alpha(\lambda)$; thus

$$\alpha(\lambda) = \frac{1}{2\pi} C(\lambda) = \frac{1}{2\pi} \oint_{\lambda} \vec{W} \cdot d\vec{R}$$

(2.13)

$\alpha(\lambda)$ equals zero at the fixed point $O$ of fig. 1 and along the boundary $\alpha(0) = \alpha_B$. For simplicity we assumed that $\alpha$ increases uniformly between $0$ and $\alpha_B$. Having defined $\alpha(x^K)$ at every point of our trial flow, we now define an "angle" $\beta$ by the condition that

$$\vec{\omega} = \vec{\nabla} \alpha \times \vec{\nabla} \beta$$

(2.14.a)

or

$$\omega = \frac{\partial (\alpha, \beta)}{\partial (x, y)}$$

(2.14.b)

This is a first order differential equation which defines $\beta$ up to an arbitrary function of $\alpha$, $B(\alpha)$ that depends on the choice of the line $\beta = 0$ (see figure 1). We shall define $\beta = 0$ as the positive $x$ axis. Following (2.13)

$$\alpha = \frac{1}{2\pi} \oint_{\lambda} \vec{W} \cdot d\vec{R} = \frac{1}{2\pi} \int \omega dx dy = \frac{1}{2\pi} \int d\alpha d\beta = \alpha(\lambda) \frac{1}{2\pi} \oint_{\lambda} d\beta$$

(2.15)

The domain of $\beta$ is thus $0 \leq \beta \leq 2\pi$. Having now an $\alpha(x^K)$ and a $\beta(x^K)$, it follows from (2.10) and (2.14.a) that the velocity field is of the form

$$\vec{W} = \alpha \vec{\nabla} \beta + \vec{\nabla} \nu$$

(2.16)
which is a Clebsch form (Lamb (1945)). It has been shown in paper I that the function \( \nu \) is single valued. The value of \( \nu \) is defined if we impose mass conservation, i.e. equation (2.2) or

\[
\nabla \cdot (\sigma \nabla \nu) = \nabla \cdot [\sigma (-\alpha \nabla \beta + \eta_c)]
\]

(2.17)

This is an elliptic equation for \( \nu \) whose solution depends on one arbitrary constant only because \( \sigma|_B = 0 \). We chose \( \nu(\alpha = 0) = 0 \) to make \( \nu \) unique.

Thus given a mass preserving trial configuration of a stationary Maclaurin flow in moving coordinates, there exists a labelling \( \alpha, \beta \) of the fluid elements with \( 0 \leq \alpha \leq \alpha_B \) and \( 0 \leq \beta \leq 2\pi \). The labelling is isocirculational in the following sense: the circulation along any contour \( l \) defined by \( \alpha(\beta) \) depend on \( \alpha(\beta) \) only:

\[
\oint_l \vec{W} \cdot d\vec{R} = \oint_l \alpha(\beta) d\beta
\]

(2.18)

and not on \( x^K(\alpha, \beta) \). Any deformed trial configuration in which fluid elements labeled \( \alpha, \beta \) with coordinates \( x^K(\alpha, \beta) \) go to points \( \tilde{x}^K(\alpha, \beta) \) will have the same circulation along the contour defined by the same function \( \alpha(\beta) \) in \( \tilde{x}^k \) coordinates.

Reciprocally let \( x^K(\alpha, \beta) \) define the positions of fluid elements in a trial configuration labelled by \( \alpha, \beta \) with the topology of fig.1, with \( \lambda(\alpha) \) given, \( 0 \leq \alpha \leq \alpha_B, 0 \leq \beta \leq 2\pi \), and \( \alpha_B \) given as well. Then the density of the configuration \( \sigma(x^K) \) is defined by (2.11) and (2.14.b) as

\[
\sigma = \lambda(\alpha) \frac{\partial (\alpha, \beta)}{\partial (x, y)}
\]

(2.19)

and a \( \tilde{W}(x^K) \) is defined by (2.16) in which \( \nu \) is given by equation (2.17) and \( \nu(\alpha = 0) = 0 \). The flow so defined \( \sigma(x^K), \tilde{W}(x^K) \) is both mass preserving and isocirculational. Notice
that $\vec{b}$ and $\Omega_c$ appear in $\vec{W}$ through $\nu$. The three numbers $\vec{b}$ and $\Omega_c$, are, in general, related to the linear momentum $\vec{P}$ and the angular momentum $J$ of the flow; $\vec{P}$ can always be taken equal to zero:

$$\vec{P} = \int_S \vec{W} \sigma d^2x = 0 \quad (2.20.a)$$

$$J = \vec{I}_x \cdot \int_S \vec{R} \times \vec{W} \sigma d^2x = J_0 \quad (2.20.b)$$

2.3. Fixation of Coordinates

For definiteness, the origin and orientation of coordinates have to be fixed. For this we have only the topology of the flow. We shall fix the relative position of our trial flow (see fig. 1) in the following way: The point of zero relative velocity, where $\alpha = 0$, will be chosen as the origin of coordinates, and the $x$ axis oriented along one of the principal directions of the boundary contour $\alpha = \alpha_B$. This fixes in general the coordinate system of any trial configuration. In these coordinates $\vec{R}(\alpha, \beta)$ must satisfy the following conditions:

$$\vec{R}(0, \beta) = 0 \quad (2.21.a)$$

$$\left( \frac{\partial x}{\partial \beta} \right)_{\alpha=\alpha_B,\beta=0} = 0 \quad (2.21.b)$$

In addition we must have

$$x(\alpha, 0) \geq 0, \quad y(\alpha, 0) = 0 \quad (2.21.c)$$

because we chose $\beta = 0$ to be the positive $x$-axis.

2.4. The Energy Criteria for Stability

The following criteria of stability refers to what we called in paper I the strong energy condition, in which mass conservation always holds. Among all mass preserving
and isocirculational trial configurations \( x^K(\alpha, \beta) \), that satisfy (2.21), only those for which the energy

\[
E = \int \left[ \frac{1}{2} \vec{W}^2 + \varepsilon(\sigma) + \frac{1}{2} \Phi \right] \sigma d^3 x \tag{2.22}
\]

under small displacements

\[
\vec{R}(\alpha, \beta) \rightarrow \vec{R}(\alpha, \beta) + \vec{\xi}(\alpha, \beta) \tag{2.23}
\]

is stationary, \( \Delta E = 0 \), for given \( \vec{P} \) and \( J \) (\( \Delta \vec{P} = \Delta J = 0 \)), are real physical flows in that they satisfy Euler’s equations. Physical flows, with minimum of energy are stable with respect to small perturbations. The condition of stability for flows with given linear and angular-momenta are thus

\[
\Delta^2 E > 0 \quad \text{at} \quad \Delta E = \Delta \vec{P} = \Delta J = 0 \quad \text{with} \quad \Delta^2 \vec{P} = \Delta^2 J = 0 \tag{2.24}
\]

As emphasized in paper I, (2.24) is a necessary and sufficient condition of stability if the gyroscopic term of the Lagrangian, the sum of those terms that are linear in time derivatives, is equal to zero. The explicit expression for the gyroscopic* term \( \Delta^2 G \) obtained in 3-D flows holds here with minor changes. Thus, the necessary and sufficient conditions of stability for flows with given linear and angular momentum are equation (2.24) with

\[
\Delta^2 G = 0 \tag{2.25}
\]

3. Application to Maclaurin Disks

* Here we deal only with the strong energy principle and drop the sub index ”strong” in (6.9) of paper I

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3.1. Stationary Configurations

The following is taken from Binney and Tremaine (1987) with some changes of notations. In Maclaurin disks the fluid rotates with uniform angular velocity $\vec{\Omega} = \vec{I}_z \Omega$ and the velocity field in inertial coordinates is thus

$$\vec{W}_0 = \vec{\Omega} \times \vec{R} = \Omega R^2 \vec{\nabla} \varphi$$

(3.1)

an index 0 refers to stationary quantities, $\varphi$ is the polar angle, $R$ the radial distance. Maclaurin disks appear the same in coordinates rotating with uniform angular velocity $\Omega_c$. In such coordinates, following equation (2.1), the relative velocity

$$\vec{U}_0 = (\vec{\Omega} - \vec{\Omega}_c) \times \vec{R}$$

(3.2a)

and

$$\vec{b}_0 = 0$$

(3.2b)

The vorticity

$$\omega_0 = \vec{I}_z \cdot \text{rot}\vec{W}_0 = 2\Omega$$

(3.3)

The mass preserving density

$$\sigma_0 = \sigma C \sqrt{1 - \frac{R^2}{a^2}} \quad R \leq a,$$

(3.4.a)

$$\sigma_0 = 0 \quad R \geq a.$$  

(3.4.b)

Pressure and specific enthalpy are given by the equation of state

$$P = \frac{1}{2} \kappa \sigma^3$$

(3.5.a)
\[ h = \frac{3}{2} \kappa \sigma^2 \]  

(3.5.b)

The gravitational field in the plane of the disk

\[ \Phi_0 = \frac{1}{2} \Omega_0^2 R^2, \quad \Omega_0^2 = \frac{\pi^2 G \sigma_C}{2a} \]  

(3.6)

\( \Omega_0 \) is the angular velocity of a test particle on a circular orbit. Euler’s equations (2.7) relate \( \Omega \) to \( \Omega_0, \sigma_C \) and \( a \):

\[ \Omega^2 = \Omega_0^2 - \frac{3 \kappa \sigma_C^2}{a^2} \]  

(3.7)

The mass \( M \) and the angular momentum \( J_0 \) define \( \sigma_C \) and \( \Omega \) for any given radius \( a \):

\[ M = \frac{2\pi}{3} \sigma_C a^2 \]  

(3.8.a)

\[ \vec{P}_0 = 0 \]  

(3.8.b)

\[ J_0 = \frac{4\pi}{15} \Omega \sigma_C a^4 = \frac{MC_B}{5\pi} \]  

(3.8.c)

where

\[ C_B = 2\pi a^2 \Omega \]  

(3.9)

is the circulation along the boundary.

3.2. \((\alpha, \beta)\) Labelling and the Load \( \lambda(\alpha) \) in Maclaurin Disks.

The labels defined in section 2 are here as follows. Lines of constant load \( \frac{\sigma_a}{\omega_0} = \text{const.} \) are circles. The circulation along lines of constant load

\[ C(R) = 2\pi \Omega R^2 \]  

(3.10)

Thus, \( \alpha(x^K) \) defined by equation (2.13) is proportional to \( R^2 \):

\[ 0 \leq \alpha = \frac{C}{2\pi} = \Omega R^2 \leq \alpha_B = \frac{C_B}{2\pi} \]  

(3.11)
The load $\lambda(\alpha)$ is correspondingly

$$\lambda(\alpha) = \frac{\sigma_0}{\omega_0} = \lambda_C \sqrt{1 - \frac{\alpha}{\alpha_B}} \quad (3.12.a)$$

where, with $\sigma_C$ given in terms of $M$ by (3.8.a) and $\Omega$ related to $C_B$ in (3.9), we have

$$\lambda_C = \frac{3M}{2C_B} \quad (3.12.b)$$

$\beta$ is defined by equation (2.14.b) which reduces here to

$$\omega_0 = 2\Omega = \frac{\partial (\alpha, \beta)}{\partial (x, y)} = \frac{1}{R} \frac{\partial (\alpha, \beta)}{\partial (R, \varphi)} = 2\Omega \frac{\partial \beta}{\partial \varphi} \quad (3.13)$$

If $\beta = 0$ is the $x$ axis, then

$$\beta = \varphi \quad (3.14)$$

Equations (3.1), (3.11) and (3.14) give

$$\vec{W} = \alpha \vec{\nabla} \beta \quad (3.15.a)$$

or

$$\vec{\nabla} \nu_0 = 0 \quad (3.15.b)$$

and with $\nu_0(\alpha = 0) = 0$

$$\nu_0 = 0 \quad (3.16)$$

3.3. Trial Configurations of Maclaurin Flows that Differ Slightly from a Maclaurin Disk

Consider now mass preserving and isocirculational trial configurations $\vec{R}(\alpha, \beta)$ that differ slightly from a Maclaurin disk $\vec{R}_0(\alpha, \beta)$

$$\vec{R}(\alpha, \beta) = \vec{R}_0(\alpha, \beta) + \vec{\xi}(\alpha, \beta) \quad (3.17)$$
with

\[ |\mathbf{\xi}| \ll a \quad (3.18) \]

Instead of \((\alpha, \beta)\), we may conveniently use \((R, \varphi)\) as labels. A perturbed configuration is drawn in figure 1 with coordinates taken in such a way that formulas (2.21) hold. \(\bar{R}\) is given by equation (3.17), and \(\bar{R}\) as \(\bar{R}_0\) satisfy both equation (2.21.a); we must then have also

\[ \bar{\xi}(R = 0, \varphi) = 0 \quad (3.19.a) \]

Equations (2.21.b and c) will be satisfied by removing from \(\bar{\xi}\) any displacements that amounts to a change of the cut \(\beta = 0\) by an arbitrary infinitesimal function of \(\alpha\) i.e. \(\delta \beta = \tau(\alpha)\). In terms of equation (3.11) and (3.23) [below where \(\delta\) is related to \(\bar{\xi}\) taking index \(K = 2\)], we may equivalently write that condition as

\[ R^2 \delta \beta = I_z \cdot \bar{\xi} \times \bar{R} = R^2 \tau(R^2) \quad (3.19.b) \]

Equation (3.19.b) includes arbitrary rigid rotations, \(\delta \beta = \text{const.}\).

The density \(\sigma\) of our Maclaurin flows change from \(\sigma_0\) given by equation (2.19) in which \(\lambda(\alpha)\) is the load of Maclaurin disks defined by equation (3.12) with the same \(\lambda_C\) and the same \(\alpha_B\). These conditions preserve indeed vortex fluxes. Thus the change in \(\sigma\)

\[ \Delta \sigma = \sigma - \sigma_0 = -\sigma_0 \nabla \cdot \bar{\xi} \quad (3.20) \]

The velocity field \(\bar{W}\) is given by (2.16) and therefore the change in \(\bar{W}\)

\[ \Delta \bar{W} = \bar{W} - \bar{W}_0 = -\nabla \bar{\xi} \cdot \bar{W}_0 + \nabla \Delta \nu \quad (3.21) \]

\(\Delta \nu\) itself is defined by the perturbed mass-preserving equation \(\Delta [\nabla \cdot (\sigma \bar{U})] = 0\). To obtain this equation it is convenient to introduce the often used \(\delta\)-variations appearing already in
(3.19.b), associated with changes of the functions $\alpha^K(x^L)$ that leave the lines of constant $\alpha^K$ fixed. Such changes are defined by

$$\alpha^K(\vec{R} + \vec{\xi}) + \delta \alpha^K(\vec{R} + \vec{\xi}) = \alpha^K(\vec{R})$$

(3.22)

To order 1,

$$\delta \alpha^K = -\vec{\xi} \cdot \vec{\nabla} \alpha^K$$

(3.23)

Just like $\Delta \alpha^K = 0$, now $\delta x^K = 0$. The greatest quality of $\delta$-variations is that they commute with $x^K$ derivatives

$$\delta \vec{\nabla} = \vec{\nabla} \delta$$

(3.24)

One easily find that

$$\delta = \Delta - \vec{\xi} \cdot \vec{\nabla}$$

(3.25)

We shall move from $\Delta$ to $\delta$ variations and back at our convenience. Thus $\Delta \sigma$ in equation (3.20) gives

$$\delta \sigma = -\vec{\nabla} \cdot (\sigma_0 \vec{\xi})$$

(3.26)

and $\Delta \vec{W}$ in eq (3.21) gives

$$\delta \vec{W} = -\vec{\nabla} \cdot \vec{W} + \vec{\nabla} \Delta \nu - \vec{\xi} \cdot \vec{\nabla} \vec{W}$$

(3.27)

Let us now go back to the equation for $\Delta \nu$ or $\delta \nu$. Following equation (2.2), we have

$$\delta [\vec{\nabla} \cdot (\sigma \vec{U})]_0 = \vec{\nabla} \cdot (\delta \sigma \vec{U}_0 + \sigma_0 \delta \vec{U}) = 0$$

(3.28)

With equations (2.1) and (3.2.a), one can see that (3.28) is an elliptic equation for $\delta \nu$:

$$\vec{\nabla} \cdot (\sigma_0 \vec{\nabla} \delta \nu) = -\vec{\nabla} \cdot [\delta \sigma \vec{W}_0 + \sigma_0 \delta (\alpha \vec{V} \beta)] + \delta b \cdot \vec{\nabla} \sigma_0 - \vec{I}_z \cdot \vec{R} \times \vec{\nabla} \delta \sigma \Omega_c$$

(3.29)
Equation (3.29) with $\delta \nu(\alpha = 0) = 0$ defines $\delta \nu$ uniquely. $\delta \nu$ is a non homogeneous linear expression in $\delta \vec{b}$ and $\Omega_c$. The latter will now be retrieved from the conditions that perturbed Maclaurin flows have same linear and angular momenta as the Maclaurin disks.

3.4. Global Constraints

a. Fixation of $\delta \vec{b}$

The equation (2.2) for mass conservation implies the following identity:

$$\vec{R} \vec{\nabla} \cdot (\sigma \vec{U}) = \vec{\nabla} \cdot (\sigma \vec{U} \vec{R}) - \sigma \vec{U} = 0$$

(3.30)

With (2.6) and (2.1), we have thus

$$\vec{P} = M(\vec{b} + \vec{\Omega}_c \times \vec{R}_{CM})$$

(3.31.a)

where

$$\vec{R}_{CM} \equiv \frac{1}{M} \int \vec{R} \sigma d^2x$$

(3.31.b)

defines the position of the center of mass. For our Maclaurin disks

$$\vec{P}_0 = \vec{b}_0 = \vec{R}_{CM0} = 0$$

(3.32)

If we want to keep $\vec{P} = 0$, we must take

$$\delta \vec{b} = -\vec{\Omega}_c \times \delta \vec{R}_{CM}$$

(3.33)

This equation defines $\delta \vec{b}$, that appears in (3.29), in terms of $\vec{\xi}$ and $\Omega_c$.

b. Fixation of $\Omega_c$

$\Omega_c$ is arbitrary in a Maclaurin disk but as soon as we destroy axial symmetry, we can see the flow rotating. $\Omega_c$ will be defined by the condition that Maclaurin flows have the
same angular momentum as the Maclaurin disks. Turning our attention to $J$, defined in (2.20.b), we obtain

$$\Delta J = \vec{I}_z \cdot \int (\vec{\xi} \times \vec{W} + \vec{R} \times \Delta \vec{W}) \sigma d^2x$$ \hspace{1cm} (3.34)

At the stationary point (Maclaurin disks) the first term on the right hand side

$$\int \vec{\xi} \times \vec{W}_0 \sigma_0 d^2x = \int \vec{\xi} \times (\vec{\Omega} \times \vec{R}) \sigma_0 d^2x = \Omega \int \vec{\xi} \cdot \vec{R} \sigma_0 d^2x$$ \hspace{1cm} (3.35)

while, using (3.1) and (2.6), we can write the second term as

$$\int \vec{R} \times \Delta \vec{W}|_0 \sigma_0 d^2x = \int \vec{R} \times (-\vec{\nabla} \vec{\xi} \cdot \vec{W}_0) \sigma_0 d^2x = -\Omega \int \vec{\xi} \cdot \vec{R} \sigma_0 d^2x$$ \hspace{1cm} (3.36)

Since (3.35) + (3.36) $\equiv 0$, we see that

$$\Delta J|_0 \equiv 0$$ \hspace{1cm} (3.37)

Thus $\Delta J|_0 = 0$ does not define $\theta_c$. However $\Delta^2 J|_0 = 0$ does. Indeed, following equation (3.34),

$$\Delta^2 J|_0 = \vec{I}_z \cdot \int (2\vec{\xi} \times \Delta \vec{W}|_0 + \vec{R} \times \Delta^2 \vec{W}|_0) \sigma_0 d^2x = 0$$ \hspace{1cm} (3.38)

which contains $\Delta \nu$ and $\Delta^2 \nu$ that depend both linearly on $\theta_c$. Notice, however that $\Delta^2 \nu$ appears in the following integral

$$\int \vec{R} \times \vec{\nabla} \Delta^2 \nu|_0 \sigma_0 d^2x = \int \vec{\nabla} \times (\vec{R} \Delta^2 \nu|_0 \sigma_0) d^2x = \vec{I}_z \oint \Delta^2 \nu|_0 \sigma_0 \vec{R} \cdot d\vec{R}$$ \hspace{1cm} (3.39)

The latter is zero because $\Delta^2 \nu$ must be small and $\sigma_0|_B = 0$. Therefore $\theta_c$ will only be defined in terms of $\Delta^2 J = 0$ through $\Delta \nu$ that appears in $\Delta \vec{W}|_0$ and $\Delta^2 \vec{W}|_0$. The explicit expression for $\theta_c$ is somewhat complicated and will be calculated in due place.

3.5. The Energy Principle
We may now apply our strongly constrained energy principle $\Delta^2 E > 0$ to find stability limits in Maclaurin disks. With $\Delta \nu$ defined by (3.29) in which $\Omega_c$ is given by (3.38) and $\delta \vec{b}$ by (3.33), we can evaluate $\delta \vec{W}$ given in (3.27) and $\delta \sigma$ in (3.26). Here is a differential equality convenient for calculating $\Delta^2 E$. It is obtained from (2.22) in appendix A

$$\Delta^2 E - \Omega \Delta^2 J = \delta^2 E - \Omega \delta^2 J = \int \{\sigma (\delta \vec{W})^2 + \delta \sigma \delta (h + \Phi)\} |_0 d^2x \quad (3.40)$$

4. Spherical Harmonic Decompositions of the Perturbations

4.1. Spherical Harmonic Decompositions of $\vec{\xi}$, $\delta \sigma$, $\delta h$ and $\delta \Phi$

a. $\vec{\xi}$ defined by scalar functions

It is a good thing to define $\vec{\xi}$ in terms of two independant non dimensional infinitesimal scalars $\eta$ and $\psi$ as follows:

$$\vec{\xi} = a^2[\vec{\nabla} \eta + \text{rot} \vec{\psi}], \quad \vec{\psi} = \vec{1}_z \psi \quad (4.1)$$

$\eta$ is thus defined in terms of $\vec{\xi}$ by

$$\Delta \eta = \frac{1}{a^2} \vec{\nabla} \cdot \vec{\xi} \quad (4.2)$$

Some boundary conditions are needed to make $\eta$ unique. If we take, say,

$$\eta|_B = 0 \quad (4.3.a)$$

equation (4.2) has a unique solution. Equation (4.2) represents also the condition of integrability of (4.1), considered as a set of two first order differential equations for $\psi$, given $\vec{\xi}$ and $\eta$. Thus the $\psi$ equations are integrable and define $\psi$ up to a constant. This constant we shall fix by asking:

$$\psi|_C = 0 \quad (4.3.b)$$
b. Spherical Harmonic Decomposition of $\eta$ and $\psi$

We now decompose $\eta$ and $\psi$ in normalised spherical harmonic functions of $\chi$ and $\varphi$ with

$$\chi = \frac{\sigma_0}{\sigma_C} = \sqrt{1 - \frac{R^2}{a^2}}, \quad 0 \leq \chi \leq 1. \quad (4.4)$$

We have

$$\eta = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \eta_{lm} P_l^m(\chi)e^{im\varphi} + c.c. \quad (4.5a)$$

$$\psi = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \psi_{lm} P_l^m(\chi)e^{im\varphi} + c.c. \quad (4.5b)$$

$\eta_{lm}$ and $\psi_{lm}$ are arbitrary, infinitesimal, complex numbers related by equations (4.3). Associated Legendre polynomials are defined in the range $-1 \leq \chi \leq 1$, but $\eta$ and $\psi$ are only defined in the range $0 \leq \chi \leq 1$. We may extend $\eta$, $\psi$ to the negative region of $\chi$ in any way we want. However, these functions have bounded gradients at $\chi = 0$. Indeed, since

$$|\partial_R \eta| < \infty, \quad |\partial_R \psi| < \infty \quad (4.6a)$$

$$\left|\frac{1}{R} \partial_\varphi \eta\right| < \infty, \quad \left|\frac{1}{R} \partial_\varphi \psi\right| < \infty \quad (4.6b)$$

hold and since

$$\partial_R = -\frac{1}{a\chi} \sqrt{1 - \chi^2} \partial_\chi, \quad (4.7)$$

then,

$$\partial_\chi \eta|_{\chi=0} = \partial_\chi \psi|_{\chi=0} = 0. \quad (4.8)$$

Equations (4.8) will be satisfied if we make symmetrical continuous extensions

$$\eta(\chi) = \eta(-\chi), \quad (4.9a)$$
\[ \psi(\chi) = \psi(-\chi), \quad (4.9.b) \]

whose expansions are given by (4.5) with \((l - m)\) even. The expansions in the domain \(0 \leq \chi \leq 1\), of arbitrary continuous \(\eta\) and \(\psi\) that satisfy (4.6) everywhere are then given by (4.5) with \((l - m)\) even [Arfken 1985]. The boundary conditions (4.3.b) gives the following relations among the \(\psi_{l0}\)'s:

\[ \sum_{l=0}^{\infty} \psi_{l0} = 0, \quad l \text{ even} \quad (4.10.a) \]

while equation (4.3.a) gives a relation among \(\eta_{lm}\)'s: for every \(m \geq 0:\)

\[ \sum_{l=m}^{\infty} \eta_{lm} P_l^m(0) = 0, \quad l - m \text{ even} \quad (4.10.b) \]

Equation (4.10.a) defines, say, \(\psi_{00}\) in terms of \(\psi_{l \neq 0,0}\); and equations (4.10.b) define, say, \(\eta_{ll}\) in terms of \(\eta_{l,m \neq l}\).

c. Spherical Harmonic Decomposition of \(\delta \sigma\), \(\delta h\) and \(\delta \Phi\)

We can now calculate the spherical harmonic decomposition of \(\delta \sigma\) with equation (3.26); from (3.4.a) and (4.4) we obtain

\[ \sigma_0 = \sigma_C \chi. \quad (4.11) \]

With equation (4.1) \(\delta \sigma\) can be written as:

\[ \delta \sigma = -a^2 \sigma_C [\vec{\nabla} \cdot (\chi \vec{\nabla} \eta) + \vec{\nabla} \chi \times \vec{\nabla} \psi]. \quad (4.12) \]

and from equation (4.12) and (4.5) we find that \(\delta \sigma\) has the following expansion

\[ \delta \sigma = \sigma_C \sum_{l=m}^{\infty} \sum_{m=0}^{\infty} \sigma_{lm} \frac{P_l^m(\chi)}{\chi} e^{im\phi} + c.c. \quad l - m \text{ even} \quad (4.13.a) \]

in which

\[ \sigma_{lm} = [l(l + 1) - m^2] \eta_{lm} + im\psi_{lm} \equiv k_{lm} \eta_{lm} + im\psi_{lm} \quad (4.13.b) \]
With the expansion of $\delta \sigma$ we obtain directly through (3.5.b) the spherical harmonic decomposition of $\delta h$:

$$
\delta h = 3\kappa \sigma_C^2 \sum_{l=m}^{\infty} \sum_{m=0}^{\infty} \sigma_{lm} P_l^m(\chi) e^{im\varphi} + c.c. \quad l - m \text{ even} \quad (4.14)
$$

and from the varied Poisson equation :

$$
\triangle \delta \Phi = 4\pi G \delta \sigma \delta_D(z) \quad (4.15)
$$

where $\delta_D(z)$ is the Dirac function, we find the solution $\delta \Phi$ which has been given by Hunter (1963):

$$
\delta \Phi = 2\Phi_B \sum_{l=m}^{\infty} \sum_{m=0}^{\infty} \Phi_{lm} P_l^m(\chi) e^{im\varphi} + c.c. \quad l - m \text{ even} \quad (4.16.a)
$$

in which

$$
\Phi_{lm} = -g_{lm} \sigma_{lm} \quad (4.16.b)
$$

where

$$
g_{lm} = \frac{(l+m)!}{2^{2l-1}[(l+m)!]^2} < 1 \quad l > 0, \quad l - m \text{ even} \quad (4.16.c)
$$

4.2. Spherical Harmonic Decomposition of $\delta \vec{W}$

Following equation (3.27) and equations (3.1) and (4.1), $\delta \vec{W}$ may be written as a gradient plus a rotational, always a convenient form for vector fields:

$$
\delta \vec{W} = a^2 \Omega [\vec{\nabla} \zeta + rot(2\eta \vec{I}_z)] \quad (4.17.a)
$$

in which

$$
\zeta = \frac{1}{a^2 \Omega} (\Delta \nu - \vec{\xi} \cdot \vec{W}_0) - 2\psi \quad (4.17.b)
$$
Notice that not only is $\zeta \ll 1$, $|\nabla \zeta|$ must be small with respect to $a^{-1}$. $\Delta \nu$ is defined by equation (3.29) and therefore, with (3.29), we obtain an elliptic equation for $\zeta$:

$$
\nabla \cdot (\sigma_0 \nabla \zeta) = \frac{1}{2a^2\Omega} [-\nabla \sigma_0 \cdot \text{rot}(2\eta \tilde{\Omega}) - (\tilde{\Omega} - \tilde{\Omega}_c) \times \tilde{R} \cdot \nabla \delta\sigma + \nabla \sigma_0 \cdot \delta \tilde{b}].
$$

(4.18)

We now expand $\zeta$ in spherical harmonics

$$
\zeta = \sum_{l=m}^{\infty} \sum_{m=0}^{\infty} \zeta_{lm} P_l^m(\chi) e^{im\phi} + c.c., \quad l - m \text{ even} \quad (4.19a)
$$

We also use equation (4.1) and (4.5) to find the expansion of $\delta \tilde{b}$. With equation (4.5.a) for $\eta$ and equation (4.13) for $\delta \sigma$, we obtain, after some straightforward but a little tedious calculations, that

$$
\zeta_{lm} = -\frac{im}{k_{lm}} [2\eta_{lm} - (1 - \frac{\Omega_c}{\Omega})\sigma_{lm}], \quad (l, m) \neq (1, 1) \quad (4.19b)
$$

and

$$
\zeta_{11} = i[\sigma_{11} - 2\eta_{11}] = -\psi_{11} - i\eta_{11}. \quad (4.19c)
$$

Equations (4.19) and (4.5.a) inserted into (4.17.a) give the spherical harmonic expansion of $\delta \tilde{W}$.

4.3. **Spherical Harmonic decomposition of $\Delta^2 E$ and $\Delta^2 J$**

$\Delta^2 E$ is given by (3.40) with $\Delta^2 J = 0$ that defines $\Omega_c$. Near the stationary point,

$$
\Delta^2 E = \int \{\sigma(\delta \tilde{W})^2 + \delta \sigma \delta (h + \Phi)\} |_0 d^2 x
$$

(4.20)

To obtain the spherical harmonic decomposition of $\Delta^2 E$, we replace $\delta \sigma$, $\delta h$, $\delta \Phi$ and $\delta \tilde{W}$ by their respective expansions (4.12) (4.14) (4.16) and (4.17), in $\Delta^2 E$ [detailed calculations,
are given in appendix B]. The result is as the follows:

\[
\Delta^2 E = 4\pi \sigma C \Omega^2 a^4 \sum_{(l,m)>1,m=0}^{\infty} 4(k_{lm} - \frac{m^2}{k_{lm}})|\eta_{lm}|^2 + \frac{m^2}{k_{lm}}(1 - \frac{\Omega_c}{\Omega})^2 - 1 + (1 - g_{lm}) \frac{\Omega^2}{\Omega^2} |\sigma_{lm}|^2 
\]

(4.21)

The parenthesis \((1 - \frac{\Omega_c}{\Omega})\) is given by \(\Delta^2 J = 0\) (the calculation is in appendix C):

\[
1 - \frac{\Omega_c}{\Omega} = \frac{\sum (k_{lm}^2 - m^2)|\eta_{lm}|^2 - \frac{\sigma_{lm}}{k_{lm}}|^2 + \frac{m^2}{k_{lm}}|\sigma_{lm}|^2}{\sum \frac{m^2}{k_{lm}}|\sigma_{lm}|^2} (m \neq 0), (l, m \neq 1, 1) 
\]

(4.22)

4.4. Fictitious Marginal Instabilities

Direct inspection of \(\Delta^2 E\) in equation (4.21) reveals that the following modes are absent: \(\eta_{l1}, \sigma_{11} = \eta_{11} + i\psi_{11}\) and \(\psi_{l0}\). Equation (4.10.a) defines \(\psi_{00}\), equations (4.10.b) define \(\eta_{l1}\). The other modes, \(\psi_{l0} l > 0\) and \(\sigma_{11}\), must thus represent symmetries. It is important to figure out what these are.

The absence of \(\sigma_{11} = \eta_{11} + i\psi_{11}\) in \(\Delta^2 E\) is related to \(\Delta^2 E\)’s invariance for uniform translations of the coordinate axis. Translational invariance is disposed off by imposing equation (3.19.a) whose spherical harmonic decomposition is

\[
\sum_{l} l(l+1)(\eta_{l1} + i\psi_{l1}) = 0 
\]

(4.23)

From which we see that \(\sigma_{11} = \eta_{11} + i\psi_{11}\) can be retrieved in terms of \(\eta_{l1} + i\psi_{l1}\) for \(l \neq 1\), Hence, the origin of coordinates is redefined, in accordance with condition (2.21.a).

The coefficients \(\psi_{l0}, l > 0\) are defined by equation (3.19.b). If we use (4.1) and the spherical harmonic decomposition (4.5), we obtain from (3.19.b)

\[
\frac{\tilde{1}_z \cdot \tilde{\xi} \times \tilde{R}}{R} = a^2 \sum_{l=0}^{\infty} \psi_{l0} \frac{dP_l(\chi)}{dR} = R \tau(R^2) 
\]

(4.24)
Terms dependent on \( \varphi \) have been removed from the left because \( \tau \) is independant of \( \varphi \). Equation (4.24) depend on \( \psi_{l0} \), \( l > 0 \) only; the latter can be retrieved explicitly with two obvious integrations:

\[
\psi_{l0} = \int_0^1 \mathcal{P}_l(\chi) \left[ \int_\chi^1 \tau(\chi') \chi' d\chi' \right] d\chi \quad l > 0
\]

(4.25)

The way to keep the cut \( \beta = 0 \) fixed on the \( x \) axis is simply to set \( \psi_{l0} = 0 \) for \( l > 0 \). Notice that \( \psi_{20} = 0 \) alone avoids rigid rotations.

5. Necessary and Sufficient Conditions for Stability of Maclaurin Disks

5.1. The Gyroscopic Term

Following section 2.4 the inequality \( \Delta^2 E > 0 \), calculable from (4.21) with (4.22), gives sufficient conditions of stability. \( \Delta^2 E \) has the form \( \frac{\Omega^2}{\Omega_0^2} A^2 - B > 0 \). If \( B < 0 \), the disk is stable for any \( \frac{\Omega^2}{\Omega_0^2} \). For instance, Maclaurin disks are stable to any perturbation that keeps the same densities at the displaced points (\( \delta \sigma = 0 \)). We are naturally interested in perturbations that might upset stability and for which \( B > 0 \). If \( B > 0 \) then \( \frac{\Omega^2}{\Omega_0^2} \) must satisfy the following inequality

\[
Q \equiv \frac{\Omega^2}{\Omega_0^2} < \frac{\sum_{l=m>0, m=0}^\infty (1 - g_{lm}) |\sigma_{lm}|^2}{\sum_{l=m>0, m=0}^\infty [1 - \frac{m^2}{k_{lm}} (1 - \frac{\Omega^2}{\Omega_0^2})^2] |\sigma_{lm}|^2 - 4 (k_{lm} - \frac{m^2}{k_{lm}}) |\eta_{lm}|^2 - 4 (k_{lm} - \frac{m^2}{k_{lm}}) |\eta_{lm}|^2}
\]

(5.1)

The condition (5.1) becomes sufficient and necessary when the gyroscopic term for dynamical perturbations \( \Delta^2 G \) is zero (see paper I). It is therefore important to calculate \( \Delta^2 G \). For dynamical perturbations, \( \ddot{\xi} \) is a function of the time \( t \) as well as of \( \alpha, \beta \). Gyroscopic terms are those bilinear functionals of \( \ddot{\xi} \) and \( \dddot{\xi} = \frac{\partial \xi}{\partial t} \) which appear in the Lagrangian of
the dynamically perturbed equations. We have calculated $\Delta^2 G$ in paper I. For perturbed Maclaurin Disks, $\Delta^2 G$ of paper I reduces here to a rather simple expression [see appendix D]

$$\Delta^2 G = 2 \int_{t_0}^{t} \int \tilde{\xi} \cdot \Delta \tilde{W} \sigma_0 dt^2 dx dt$$ (5.2)

We can make a spherical harmonic decomposition of $\Delta^2 G$, with $\tilde{\xi}$ given by (4.1) and (4.5) and $\Delta \tilde{W}$ given in appendix by (C.5). The only novelty in these calculations is that $\eta_{lm}$ and $\psi_{lm}$ are now functions of $t$. A straightforward substitution of $\tilde{\xi}$ and $\Delta \tilde{W}$ in equation (5.2) leads to the following expression for $\Delta^2 G$:

$$\Delta^2 G = 4i\pi \sigma_C \int_{t_0}^{t} dt \sum_{l, m \neq 1, 1} m \Omega \{ \left( \frac{1 - \frac{\Omega_c}{\Omega}}{k_{lm}} \right) - \frac{1}{m^2} \} \tilde{\eta}_{lm}^* \sigma_{lm} +$$

$$\left( 1 - \frac{k_{lm}^2}{m^2} \right) \tilde{\eta}_{lm} \eta_{lm} - 2 \left( \frac{1}{k_{lm}} - \frac{k_{lm}}{m^2} \right) \tilde{\eta}_{lm}^* \sigma_{lm} \} + \text{complex conjugate}$$ (5.3)

in which $\left( 1 - \frac{\Omega_c}{\Omega} \right)$ has to be replaced by the value given in equation (4.22). Necessary and sufficient conditions of stability are obtained from (5.1) when $\Delta^2 G = 0$.

5.2. Symmetric and Antisymmetric Single-Mode Perturbations

For symmetrical and antisymmetrical modes, we take either real or imaginary components of $\eta_{lm}$ and $\psi_{lm}$ in the spherical harmonic expansion, so the Fourier expansion contains either $\cos(m\varphi)$ or $\sin(m\varphi)$. For such perturbations, the written part of $\Delta^2 G$ is imaginary; adding the complex conjugate makes thus $\Delta^2 G = 0$. For one single mode $(l, m)$, equation (5.1) reduces to:

$$Q < Q_{lm} = \frac{(1 - g_{lm}) \sigma_{lm}^2}{\left[ 1 - m^2 \frac{1}{k_{lm}^2} (1 - \frac{\Omega_c}{\Omega})^2 \right] \sigma_{lm}^2 - 4 (k_{lm} - \frac{m^2}{k_{lm}}) \eta_{lm}^2}$$ (5.4)

25
in which
\[ 1 - \frac{\Omega_c}{\Omega} = \frac{(k_{lm}^2 - m^2)(\eta_{lm} - \frac{\sigma_{lm}}{k_{lm}})^2 + \frac{m^2}{k_{lm}^2}\sigma_{lm}^2}{m^2 \sigma_{lm}^2} \quad (m \neq 0), (l, m \neq 1, 1) \quad (5.5) \]

It is advantageous to introduce a single arbitrary variable \( z \) and a constant \( x \) to analyze the lower bound of \( Q_{lm} \):
\[ z \equiv -k_{lm} \frac{\eta_{lm}}{\sigma_{lm}} \quad -\infty < z < \infty \quad (5.6.a) \]

and
\[ x \equiv \frac{m^2}{k_{lm}^2} \quad 0 \leq x \leq 1 \quad (5.6.b) \]

In term of \( z \)
\[ Q_{lm} = \frac{(1 - g_{lm})}{P_4(z)} \quad (5.7.a) \]

where \( P_4(z) \) is a polynomial of order 4 in \( z \):
\[ P_4(z) = 1 - \frac{1}{k_{lm}}\{(\frac{1}{x} - 1)[(1 - x)z^4 + 4(1 - x)z^3 + 6z^2 + 4z] + \frac{1}{x}\} \quad (5.7.b) \]

\( P_4(z) \) has one and only one real maximum for any \( x \) (see figure 2). We are only interested in values of \( z \) for which \( P_4(z) > 0 \). For \( P_4(z) < 0, \Delta^2 E > 0 \) for any value of \( Q \). The maximum of \( P_4(z) \) is obtained for:
\[ z_{max}(x) = \frac{x^{\frac{1}{3}}}{\sqrt{1 - x}}[(\sqrt{1 - x} - 1)^{\frac{1}{3}} + (\sqrt{1 - x} + 1)^{\frac{1}{3}}] - 1 \quad (5.8.a) \]

for which
\[ P_4(z)_{max} = 1 - \frac{1}{k_{lm}}\{(\frac{1}{x} - 1)[(1 - x)z_{max}^4 + 4(1 - x)z_{max}^3 + 6z_{max}^2 + 4z_{max}] + \frac{1}{x}\} \equiv 1 - \frac{y(x)}{k_{lm}} \quad (5.8.b) \]
The function $y(x)$ appears in figure 3. To any pair of values $(l, m)$ corresponds a value $x$ defined by (5.6.b) and a point $(x, y)$ on the curve. Following (5.8.a) and (5.4), we must thus have

$$Q < (Q_{lm})_{\text{min}} = \frac{1-g_{lm}}{1-k_{lm}} y(x)$$

(5.9)

The smallest minimum of $Q_{lm}$ is obtained for $(l, m) = (2, 2)$ for which $(Q_{22})_{\text{min}} = \frac{1}{2}$.

Therefore the necessary and sufficient condition of stability with respect to symmetric or antisymmetric single mode perturbations of Maclaurin disks is

$$Q < \frac{1}{2}$$

(5.10)

5.3 Comments

a) Binney and Tremaine have given the following dispersion relation for the $\omega$ - modes of dynamical perturbations:

$$\omega_r^3 - \omega_r \{4\Omega^2 + k_{lm} [\Omega_0^2 (1 - g_{lm}) - \Omega^2] \} + 2m\Omega [\Omega_0^2 (1 - g_{lm}) - \Omega^2] = 0, \quad \omega_r = \omega - m\Omega$$

(5.11)

Stability holds if the non spurious $\omega_r$’s are real roots. Equation (5.11) is a third order polynome. The condition for a polynome of the form $x^3 + a_2 x + a_3 = 0$ to have only real roots is given in standard handbooks [for instance: Schaum’s Mathematical Handbook (1968)]

$$\left( \frac{a_2}{3} \right)^3 + \left( \frac{a_3}{2} \right)^2 < 0$$

(5.12)

Inequality (5.12) for equation (5.11) is exactly our inequality (5.9).

b) For $l = m$ modes, $\Delta^2 G = 0$ for asymmetrical modes, that is, for $\sigma_{ll}$ complex [ $\eta_{ll}$ does not appear in $\Delta^2 G$ nor in $\Delta^2 E$] . However, the stability limit is still given by equation
(5.9) with \( l = m \), for which \( y = 1 \) and \( k_{lm} = l \):

\[
Q < Q_{ll} = \frac{1 - g_{ll}}{1 - \frac{1}{r}}, \quad g_{ll} = \frac{(2l)!}{2^{2l-1}(l)!^2}
\]  

(5.13)

c) For radial modes, \( m = 0 \), \( \Delta^2E \) does not depend on \( \Omega_c \), \( \Delta^2J \equiv 0 \) and \( \Delta^2G = 0 \). Moreover, since [see equation (4.13.b)]

\[
\sigma_{l0} = l(l + 1)\eta_{l0}
\]  

(5.14)

all \( l \)-modes are decoupled, \( \Delta^2E \) is a sum of squares of independent modes. In this case one obtains stability limits for non single modes. The most unfavorable limit, however, is again that given by \( Q_{l0} \) for which \( y = 4 \) and \( k_{l0} = l(l + 1) \):

\[
Q < Q_{l0} = \frac{1 - g_{l0}}{1 - \frac{4}{l(l+1)}} \quad (l \geq 2), \quad g_{l0} = \frac{l^2}{2^{2l-1}[l(l+1)]^2}
\]  

(5.15)

d) For coupled \( l = m \) modes, \( \Delta^2G \neq 0 \) and we obtain only sufficient conditions of stability. In particular if we couple a pair of modes \( l = m \) and \( l' = m' \), sufficient conditions of stability go from \( Q < Q_{ll} \) for \( l' = l \) to

\[
Q < 1 - g_{l'l'}
\]  

(5.16)

for \( l \gg l' \). The smallest values of (5.16) is \( \frac{1}{4} \) which is also the secular limit of stability found by Binney and Tremaine (1987).

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Appendix A: $\Delta^2 E - \Omega \Delta^2 J$ Variational Identity (3.40)

We start from (2.22). Varying (2.22), using (2.8) for $\varepsilon$ and (2.9) for $\Phi$, and calculating at $\Delta E = 0$, we have

$$\Delta^2 E = \int \{ (\Delta \vec{W})^2 + \vec{W} \cdot \Delta^2 \vec{W} + \Delta[\vec{\xi} \cdot \vec{\nabla}(h + \Phi)] \} |_0 \sigma_0 d^2 x$$  \hspace{1cm} (A.1)

By substracting from $\Delta^2 E$, $\Omega \Delta^2 J$ where $\Delta^2 J$ is given in (3.38), we obtain

$$\Delta^2 E - \Omega \Delta^2 J = \int \{ (\Delta \vec{W})^2 + \vec{W} \cdot \Delta^2 \vec{W} - \vec{\Omega} \cdot (2\vec{\xi} \times \Delta \vec{W} + \vec{R} \times \Delta^2 \vec{W}) + \Delta[\vec{\xi} \cdot \vec{\nabla}(h + \Phi)] \} |_0 \sigma_0 d^2 x$$  \hspace{1cm} (A.2)

The $\Delta^2 \vec{W}$ drops out of equation (A.2) because of (3.1). The terms containing $\Delta \vec{W}$ can be written as a difference of squares:

$$(\Delta \vec{W})^2 - \vec{\Omega} \cdot (2\vec{\xi} \times \Delta \vec{W}) = (\Delta \vec{W} - \vec{\Omega} \times \vec{\xi})^2 - (\vec{\Omega} \times \vec{\xi})^2 = (\Delta \vec{W} - \vec{\xi} \cdot \vec{\nabla} \vec{W}_0)^2 - \Omega^2 \vec{\xi}^2 = (\delta \vec{W})^2 - \Omega^2 \vec{\xi}^2$$  \hspace{1cm} (A.3)

In the last equality we have used (3.25). So equation (A.2) with (A.3) can now be written:

$$\Delta^2 E - \Omega \Delta^2 J = \int \{ (\delta \vec{W})^2 - \Omega^2 \vec{\xi}^2 + \Delta[\vec{\xi} \cdot \vec{\nabla}(h + \Phi)] \} |_0 \sigma_0 d^2 x$$  \hspace{1cm} (A.4)

With the following operator identity

$$\Delta \vec{\nabla} = \vec{\nabla} \Delta - \vec{\nabla} \vec{\xi} \cdot \vec{\nabla}$$  \hspace{1cm} (A.5)

and with equation (3.25), we may rewrite the $h + \Phi$ term in (A.1) as follows:

$$\Delta[\vec{\xi} \cdot \vec{\nabla}(h + \Phi)] = \vec{\xi} \cdot \vec{\nabla} \Delta(h + \Phi) - \vec{\xi} \cdot \vec{\nabla} \vec{\xi} \cdot \vec{\nabla}(h + \Phi) = \vec{\xi} \cdot \vec{\nabla} \delta(h + \Phi) + \vec{\xi} \cdot \vec{\nabla} \vec{\nabla}(h + \Phi) \cdot \vec{\xi}$$  \hspace{1cm} (A.6)

But from Euler’s equations (2.7) we see that

$$\vec{\nabla}(h + \Phi)|_0 = -\vec{W}_0 \cdot \vec{\nabla} \vec{W}_0$$  \hspace{1cm} (A.7)
Substituting (A.7) into (A.6) and $\Delta[\vec{\xi} \cdot \vec{\nabla}(h + \Phi)]$ back in (A.4), we find that $\Omega^2 \vec{\xi}^2$ cancels out. Finally with $\delta \sigma$ given in (3.26), and with some integration by parts, we obtain the following much simpler form for $\Delta^2 E - \Omega \Delta^2 J$ which is that written in (3.40):

$$\Delta^2 E - \Omega \Delta^2 J = \int \{\sigma(\delta \vec{W})^2 + \delta \sigma \delta(h + \Phi)\}|_{0}d^2 x \quad (A.8)$$

Appendix B: Spherical Harmonic Decomposition of $\Delta^2 E$

Let us start from (A.8) and take $\Delta^2 J = 0$. We shall set

$$\Delta^2 E = \Delta^2 E_k + \Delta^2 E_p \quad (B.1.a)$$

in which

$$\Delta^2 E_k = \int [\sigma(\delta \vec{W})^2]|_{0}d^2 x \quad (B.1.b)$$

and

$$\Delta^2 E_p = \int [\delta \sigma(\delta h + \delta \Phi)]|_{0}d^2 x. \quad (B.1.c)$$

Inserting (4.17.a) in (B.1.b) gives:

$$\Delta^2 E_k = a^4 \Omega^2 \int \{ \vec{\nabla}^2 \vec{\zeta}^2 + 2 \vec{\nabla} \cdot [\vec{\zeta} \text{rot}(2\eta \vec{I}_z)] + \text{rot}(2\eta \vec{I}_z)^2\} \sigma_0 d^2 x \quad (B.2)$$

Using identity (2.6), we can integrate (B.2) by part, and with $\sigma_0|_B = 0$, obtain:

$$\Delta^2 E_k = a^4 \Omega^2 \int [-\vec{\zeta} \cdot (\sigma_0 \vec{\nabla} \vec{\zeta}) - 2\vec{\nabla} \sigma_0 \cdot \text{rot}(2\eta \vec{I}_z) - 4\eta \vec{\nabla} \cdot (\sigma_0 \vec{\nabla} \eta)]d^2 x. \quad (B.3)$$

The spherical harmonic decomposition of $\Delta^2 E_k$ is obtained by replacing in equation (B.3), $\vec{\zeta}$ by (4.19), $\sigma_0$ by $\sigma C \chi$ (equation (4.11)) and $\eta$ by (4.5.a). The result comes out as follows:

$$\Delta^2 E_k = 4\pi \sigma C \Omega^2 a^4 \sum_{l=m, m=0}^{\infty} \left[ 4(klm - \frac{m^2}{k_{lm}}) |\eta_{lm}|^2 + \frac{m^2}{k_{lm}} (1 - \frac{\Omega c}{\Omega})^2 |\sigma_{lm}|^2 \right] \quad (B.4)$$
Notice that these are already some of the terms of $\Delta^2 E$ given in (4.21)

The spherical harmonic decomposition of $\Delta^2 E_p$ follows by inserting in (B.1.c) the respective expansions of $\delta \sigma$ in (4.13), $\delta h$ in (4.14) and $\delta \Phi$ in (4.15). For $\Delta^2 E_p$, we obtain:

$$\Delta^2 E_p = 2\pi a^4 \Omega^2 \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \sigma_{lm} \Phi^*_{lm} + 3\kappa \sigma_C \sigma_{lm}^* + c.c. \quad (B.5)$$

With (4.16.b) and (3.7), $\Delta^2 E_p$ can also be written

$$\Delta^2 E_p = 4\pi \sigma_C a^4 \Omega^2 \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} [-1 + (1 - g_{lm}) \frac{\Omega^2}{\Omega^2_0}] |\sigma_{lm}|^2 \quad (B.6)$$

The sum of $\Delta^2 E_k$ given by (B.4) and $\Delta^2 E_p$ of (B.6) is the expression of $\Delta^2 E$ written in (4.21).

**Appendix C: Spherical Harmonic Decomposition of $\Delta^2 J$**

Following (3.38):

$$\Delta^2 J = \mathbf{1}_z \cdot \int (2\tilde{\xi} \times \Delta \tilde{W} + \tilde{R} \times \Delta^2 \tilde{W})|_0 \sigma_0 d^2x. \quad (C.1)$$

By varying $\Delta \tilde{W}$ in (3.21), we obtain $\Delta^2 \tilde{W}$:

$$\Delta^2 \tilde{W} = -2\tilde{\nabla} \tilde{\xi} \cdot \Delta \tilde{W} + \tilde{\nabla} \Delta^2 \nu \quad (C.2)$$

Inserting this into (C.1) gives:

$$\Delta^2 J = \mathbf{1}_z \cdot \int (2\tilde{\xi} \times \Delta \tilde{W} - 2\tilde{R} \times \tilde{\nabla} \tilde{\xi} \cdot \Delta \tilde{W})|_0 \sigma_0 d^2x. \quad (C.3)$$

Notice that $\tilde{\nabla} \Delta^2 \nu$ do not contribute to (C.3) [see equation (3.39)]. Following equation (3.25) we can write:

$$\Delta \tilde{W} = \delta \tilde{W} + \tilde{\xi} \cdot \tilde{\nabla} \tilde{W}_0 \quad (C.4)$$

Using (3.1) for $\tilde{W}_0$, (4.1) for $\tilde{\xi}$, and (4.17.a) for $\delta \tilde{W}$, we obtain for (C.4)

$$\Delta \tilde{W} = a^2 \Omega [\tilde{\nabla}(\zeta + \psi) + rot(\eta \mathbf{1}_z)] \quad (C.5)$$
Inserting (C.5) and (4.1) into (C.3) gives

\[
\Delta^2 J = 2a^4 \Omega I_z \cdot \int \{ [\vec{\nabla} \eta + \text{rot} \vec{\psi}] \times [\vec{\nabla} (\zeta + \psi) + \text{rot}(\eta I_z)] \\
- \vec{R} \times \vec{\nabla} [\vec{\nabla} \eta + \text{rot} \vec{\psi}] \cdot [\vec{\nabla} (\zeta + \psi) + \text{rot}(\eta I_z)] \} \sigma_0 d^2 x.
\] (C.6)

The spherical harmonic decomposition of \( \Delta^2 J \) is obtained by replacing in equation (C.6) \( \zeta \) by (4.19), \( \sigma_0 \) by \( \sigma C \chi \) (equation (4.11)) and \( \eta, \psi \) by (4.5). Using also (4.13.b) for \( \sigma_{lm} \), the result comes out as follows:

\[
\Delta^2 J = 4\pi \sigma C a^4 \Omega \sum [(k_{lm}^2 - m^2)|\eta_{lm} - \sigma_{lm}|^2 + \left( \frac{1}{k_{lm}} - \left( 1 - \frac{\Omega_c}{\Omega} \right) \frac{m^2}{k_{lm}} \right)|\sigma_{lm}|^2 + \text{c.c.}]
\]

\[
= 8\pi \sigma C a^4 \Omega \left( \sum [(k_{lm}^2 - m^2)|\eta_{lm} - \sigma_{lm}|^2 + \frac{m^2}{k_{lm}} |\sigma_{lm}|^2 - \left( 1 - \frac{\Omega_c}{\Omega} \right) \sum \frac{m^2}{k_{lm}} |\sigma_{lm}|^2 \right) (C.7)
\]

and demanding \( \Delta^2 J = 0 \) we see that:

\[
1 - \frac{\Omega_c}{\Omega} = \frac{\sum (k_{lm}^2 - m^2)|\eta_{lm} - \sigma_{lm}|^2 + \frac{m^2}{k_{lm}} |\sigma_{lm}|^2}{\sum \frac{m^2}{k_{lm}} |\sigma_{lm}|^2} \quad (m \neq 0), (l, m \neq 1, 1) \quad (C.8)
\]

which is equation (4.22).

**Appendix D: \( \Delta^2 G \) For Maclaurin Disks**

Equation (6.9) of paper I reads as follows (we drop the index strong):

\[
\Delta^2 G = 2 \int \int [\vec{\nabla} \Delta \nu_D \cdot \vec{\nabla} \Delta \nu_S - \vec{\xi} \cdot \vec{\nabla} \vec{\xi} \cdot \vec{W}_0] \sigma_0 d^2 x dt \\
- \int \int [\Delta \vec{\eta}_c D \cdot \vec{\nabla} \Delta \nu_S + \Delta \vec{\eta}_c S \cdot \vec{\nabla} \Delta \nu_D + \Delta \vec{\eta}_c D \cdot \vec{\nabla} \vec{\xi} \cdot \vec{W}_0 \\
+ (\Delta \vec{\Omega}_c D \times \vec{W}_0 + \vec{\Omega}_c \times \vec{\nabla} \Delta \nu_D) \cdot \vec{\xi}]] \sigma_0 d^2 x dt,
\] (D.1)

\( \nu \) and \( \Delta \nu \) are defined by the time dependent equation of mass conservation and it’s first variation. \( \Delta \nu \) may be decomposed into a ”dynamical contribution” \( \Delta \nu_D \) which is zero when
\( \dot{\xi} = 0 \) and a ”steady” part \( \Delta \nu_S \) which is the one we encountered in perturbed steady flows; \( \Delta \nu_S \) is the \( \Delta \nu = \delta \nu \) of this paper which appears for the first time in (3.21); here, however:

\[
(\Delta \nu)_{Paper} = \Delta \nu_D + \Delta \nu_S \tag{D.2}
\]

The equation of mass conservation is what defines equations for \( \Delta \nu_D \) and \( \Delta \nu_S \):

\[
\nabla \cdot (\sigma_0 \nabla \Delta \nu_D) = \nabla \cdot [\sigma_0 (\dot{\xi} + \Delta \eta_{cD})] \tag{D.3.a}
\]

\[
\nabla \cdot (\sigma_0 \nabla \Delta \nu_S) = \nabla \cdot [\sigma_0 (\tilde{U}_0 \cdot \nabla \xi + \nabla \xi \cdot \tilde{U}_0) + \sigma_0 \nabla \xi \cdot \Delta \eta_{c} + \Delta \eta_{cS}] \tag{D.3.b}
\]

Similarly, \( \eta_{c} \) and \( \Delta \eta_{c} \) (equation (2.1.b)) have a dynamical and steady contribution and we write

\[
\Delta \eta_{c} = \Delta \eta_{cD} + \Delta \eta_{cS} \tag{D.4.a}
\]

\[
\Delta \eta_{cD} = \Delta \tilde{b}_{D} + \Delta \tilde{\Omega}_{cD} \times \tilde{R} \tag{D.4.b}
\]

\[
\Delta \eta_{cS} = \Delta \tilde{b}_{S} + \tilde{\Omega}_{cS} \times \xi + \Delta \tilde{\Omega}_{cS} \times \tilde{R} \tag{D.4.c}
\]

The four last terms of (D.1) can be written as follows with the help of

\[
\Delta \tilde{P}_{D} = \int \nabla \Delta \nu_D \sigma_0 d^2x \tag{D.5.a}
\]

and

\[
\Delta J_D = \tilde{I}_z \cdot \int \tilde{R} \times \nabla \Delta \nu_D \sigma_0 d^2x \tag{D.5.b}
\]

and \( \Delta \tilde{P}_S \) and \( \Delta J_S \) which are the same as our \( \Delta \tilde{P} \) leading to (3.33) and \( \Delta J \) given in (3.34); now using the definitions in (D.4) we see that the four last terms of \( \Delta^2 G \) (see D.1)

\[
- \int \int [\Delta \tilde{\eta}_{cS} \cdot \nabla \Delta \nu_D + \Delta \tilde{\eta}_{cD} \cdot \nabla \xi \cdot \tilde{W}_0 + (\Delta \tilde{\Omega}_{cD} \times \tilde{W}_0 + \tilde{\Omega}_c \times \nabla \Delta \nu_D) \cdot \tilde{\xi}] \sigma_0 d^2x dt, \tag{D.6.a}
\]
Inserting (D.6.b) back into (D.1) gives thus:

$$\Delta^2 G = 2 \int \left[ \tilde{\nabla} \Delta \nu_D \cdot \tilde{\nabla} \Delta \nu_S - \dot{\tilde{\xi}} \cdot \tilde{\nabla} \tilde{\xi} \cdot \tilde{W}_0 \right] \sigma_0 d^2 x dt - \int \left\{ \left[ 2 \Delta \tilde{\eta}_{cD} \cdot \tilde{\nabla} \Delta \nu_S \sigma_0 d^2 x \right] + 
\Delta \tilde{b}_S \cdot \Delta \tilde{P}_D + \Delta \tilde{\Omega}_{cS} \cdot \Delta \tilde{J}_D - \Delta \tilde{b}_D \cdot \Delta \tilde{P}_S - \Delta \tilde{\Omega}_{cD} \cdot \Delta \tilde{J}_S \right\} dt, \quad (D.7)$$

Conservation of linear and angular momentum works now as follows. \( \Delta \tilde{J}_S \) and \( \Delta \tilde{J}_D \) vanish identically due to the symmetry of the problem. Define \( \Delta \tilde{b}_S \) and \( \Delta \tilde{b}_D \) by setting separately \( \Delta \tilde{P}_S = 0 \) and \( \Delta \tilde{P}_D = 0 \). With this, \( \Delta^2 G \) reduces to

$$\Delta^2 G = 2 \int \left[ \tilde{\nabla} \Delta \nu_D \cdot \tilde{\nabla} \Delta \nu_S - \dot{\tilde{\xi}} \cdot \tilde{\nabla} \tilde{\xi} \cdot \tilde{W}_0 - \Delta \tilde{\eta}_{cD} \cdot \tilde{\nabla} \delta \nu \right] \sigma_0 d^2 x dt \quad (D.8)$$

Integrating the first and the last term of (D.8) by parts: and replacing finally \( \Delta \nu_S \) by our present \( \Delta \nu = \delta \nu \) defined in (3.29), we obtain:

$$\Delta^2 G = 2 \int \left\{ \Delta \nu \tilde{\nabla} \cdot \left[ \sigma_0 (\Delta \tilde{\eta}_{cD} - \tilde{\nabla} \delta \nu) \right] - \dot{\tilde{\xi}} \cdot \tilde{\nabla} \tilde{\xi} \cdot \tilde{W}_0 \sigma_0 \right\} d^2 x dt \quad (D.9)$$

and with (D.3.a), (D.9) can also be written:

$$\Delta^2 G = 2 \int \left[ -\Delta \nu \tilde{\nabla} \cdot (\sigma_0 \tilde{\xi}) - \dot{\tilde{\xi}} \cdot \tilde{\nabla} \tilde{\xi} \cdot \tilde{W}_0 \sigma_0 \right] d^2 x dt \quad (D.10)$$

And again, integrating by parts the first term of the integrant we obtain:

$$\Delta^2 G = 2 \int \left[ \tilde{\dot{\tilde{\xi}}} \cdot (\tilde{\nabla} \Delta \nu - \tilde{\nabla} \tilde{\xi} \cdot \tilde{W}_0) \sigma_0 d^2 x dt \quad (D.11)$$

and finally using (3.21) in (D.12), we get the simple form of \( \Delta^2 G \) written in equation (5.2):

$$\Delta^2 G = 2 \int_{t_0}^{t} \tilde{\dot{\tilde{\xi}}} \cdot \tilde{W} \sigma_0 d^2 x dt \quad (D.12)$$
Figure Captions

Figure 1: Examples of a trial configuration. The drawing shows some $\alpha = const$ and $\beta = const$ lines and the positioning of coordinate axis.

Figure 2: This represents one $P_4(z) > 0$ for $x = \frac{1}{64}$ corresponding to $(l,m) = (4,2)$ and a $z_{max} = -0.734$. The hatched part of this curve corresponds to perturbations stable for any $Q$.

Figure 3: (No Caption).