Quantum tunneling through two sequential barriers: A simple derivation

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Abstract. In this work, based on the relevant solutions to the Schrodinger equation, we present a simple derivation of one-dimensional quantum tunneling through two successive potential barriers, separated by an intermediate free region. In particular, we hypothesized that the transmission coefficient depends strictly on the number (n) of barriers. This finding has never been discussed previously and need to be confirmed by future experiments.

1. Introduction

Quantum tunneling (QT), at first approximation, is a microscopic phenomenon where a particle can penetrate and pass through a potential barrier, which is assumed to be higher than the kinetic energy of the particle. This effect, non-predicted by the laws of classical mechanics, plays an essential role in different physical phenomena, such as nuclear fusion in stars \([1]\), radioactive decay \([2]\), quantum biology \([3]\), cold emission \([4]\), tunnel junction \([5]\), quantum-dot cellular automata \([6]\), tunnel diode \([7]\), tunnel field-effect transistor \([8]\), quantum conductivity\([9]\), kinetic isotope effect \([10]\), scanning tunneling microscope \([11]\), and more recently, water quantum tunneling \([12]\).

From the point of view of classical mechanics, after the interaction of a particle coming from the left towards a potential barrier, it is expected the particle is going to bounce back from the potential barrier since it has insufficient energy to get over it. Surprisingly, from the point of view of quantum mechanics, there is a very small but finite probability that the particle could be found to the right of the potential barrier which is described by a transmission coefficient \(T\). Although, QT is widely observed and applied in many areas of microscopic science and technology, the understanding of this phenomenon to date does not seem complete yet.

To our knowledge, the one-dimensional QT continues being an old and controversial problem, and furthermore, a simple derivation considering successive potential barriers, still lacking. In this work we treat such a problem, for one-dimensional tunneling through two successive and rectangular potential barriers, which is compared to the conventional one-dimensional tunneling through one rectangular potential barrier. More importantly, our main result is a generalized solution for \(n\) potential barriers, not previously described.

2. Derivation

Let us consider the quantum mechanical stationary solution for the one dimensional tunneling of a non-relativistic particle, with mass \(m\) and kinetic energy:
thought one barrier with height $V$ ($V > E$) and width $a$, then, the Schrödinger equation is:

$$\left[-\hbar^2 \frac{\partial^2}{\partial x^2} + V(x)\right] \Psi(x) = E \Psi(x)$$

(2.2)

where $V(x)$ is zero outside the potential barrier, while $V(x) = V$ inside the potential barrier. This approach is valid to treat both cases addressed here.

2.1. One rectangular potential barrier

In Figure 1., we illustrate the first case, i.e., the tunneling through one potential barrier. In the three regions I ($x \leq 0$), II ($0 \leq x \leq a$), and III ($x \geq a$), the stationary solutions to equation (2.2) are the following:

$$\Psi_I(x) = A_I e^{ipx/\hbar} + B_I e^{-ipx/\hbar}$$

(2.1.1)

$$\Psi_{II}(x) = A_{II} e^{-kx/\hbar} + B_{II} e^{kx/\hbar}$$

(2.1.2)

$$\Psi_{III}(x) = A_{III} e^{ipx/\hbar} + B_{III} e^{-ipx/\hbar}$$

(2.1.3)

where $p = \sqrt{2m(E/E)}$ and $k = \sqrt{2m(V-E)/\hbar}$, and the quantities $A_I, A_{III}, B_I, B_{III}$ are the transition and reflection amplitudes, and $A_{II}, B_{II}$ are the coefficients of the evanescent and anti-evanescent waves inside the barrier, respectively. These quantities can be obtained from the matching (continuity) conditions:

$$\Psi_I = \Psi_{II} ; x = 0$$

(2.1.4)

$$\frac{\partial \Psi_I}{\partial x} = \frac{\partial \Psi_{II}}{\partial x} ; x = 0$$

(2.1.5)

$$\Psi_{II} = \Psi_{III} ; x = a$$

(2.1.6)
\[ \frac{\partial \Psi_I}{\partial x} = \frac{\partial \Psi_{II}}{\partial x}; x = a \]  

(2.1.7)

then, at \( x = 0 \), we have:

\[ A_I + B_I = A_{II} + B_{II} \]  

(2.1.8)

\[ A_I - B_I = \frac{ik}{p} A_{II} - \frac{ik}{p} B_{II} \]  

(2.1.9)

and, at \( x = a \):

\[ A_{II} e^{-\frac{ka}{\hbar}} + B_{II} e^{\frac{ka}{\hbar}} = A_{III} e^{\frac{ipa}{\hbar}} \]  

(2.1.10)

\[ A_{II} e^{-\frac{ka}{\hbar}} - B_{II} e^{\frac{ka}{\hbar}} = - \frac{ip}{k} A_{III} e^{\frac{ipa}{\hbar}} - \frac{ip}{k} B_{III} e^{\frac{ipa}{\hbar}} \]  

(2.1.11)

now, by canceling the reflection amplitudes and the coefficient of the anti-evanescent waves, and using the ratio \( \frac{p}{k} = \frac{\sqrt{\frac{E}{V}}}{\sqrt{\frac{V}{E}}} \ll 1; \frac{k}{p} = \frac{\sqrt{V}}{\sqrt{E}} \gg 1 \), we have:

\[ A_{II} = -2 \frac{ip}{k} A_I \]  

(2.1.12)

\[ A_{III} = 2A_{II} e^{-\frac{ka}{\hbar}} e^{-\frac{ipa}{\hbar}} \]  

(2.1.13)

thus, \( A_{III} \) can be expressed in terms of \( A_I \)

\[ A_{III} = 2 \left( -2 \frac{ip}{k} A_I \right) e^{-\frac{ka}{\hbar}} e^{-\frac{ipa}{\hbar}} \]  

(2.1.14)

at this point, by applying the well-known definition of the transmission coefficient \( T = \frac{|A_{III}|^2}{|A_I|^2} \), we obtain:

\[ T = 16 \frac{E}{V} e^{-\frac{2a\sqrt{2mV}}{k^2}} \]  

(2.1.15)

where the reflection coefficient \( (R) \) is:

\[ R = 1 - T \]  

(2.1.16)
2.2. Two rectangular potential barriers

![Two rectangular potential barriers](image)

Figure 2. Schematic representation of the tunneling process through two successive potential barriers.

In Figure 2., we illustrate the focus of this work, i.e., the tunneling through two equal successive potential barriers, with width $a$, and $L - a \geq 0$ begin the distance between them. In the five regions I ($x \leq 0$), II ($0 \leq x \leq a$), III ($a \leq x \leq L$), IV ($L \leq x \leq L + a$), and V ($x \geq L + a$) the stationary solutions to equation (2.2) are the following:

\[
\Psi_I(x) = A_Ie^{\frac{ipx}{h}} + B_Ie^{-\frac{ipx}{h}} \tag{2.2.1}
\]

\[
\Psi_{II}(x) = A_{II}e^{\frac{-kx}{h}} + B_{II}e^{\frac{kx}{h}} \tag{2.2.2}
\]

\[
\Psi_{III}(x, t) = T_1 \left[ A_{III}e^{\frac{ipx}{h}} + B_{III}e^{-\frac{ipx}{h}} \right] \tag{2.2.3}
\]

\[
\Psi_{IV}(x, t) = T_1 \left[ A_{IV}e^{\frac{-k(x-L)}{h}} + B_{IV}e^{\frac{k(x-L)}{h}} \right] \tag{2.2.4}
\]

\[
\Psi_V(x, t) = T_1T_2 \left[ A_Ve^{\frac{ipx}{h}} + B_{III}e^{-\frac{ipx}{h}} \right] \tag{2.2.5}
\]

as in the first case, the quantities $A_I, A_{III}, A_V, B_I, B_{III}, B_V$ are the transition and reflection amplitudes, and $A_{II}, A_{IV}, B_{II}, B_{IV}$ are the coefficients of the evanescent and anti-evanescent waves inside the barrier, respectively. These quantities can be obtained as follow:

\[
\Psi_I = \Psi_{II} ; x = 0 \tag{2.2.6}
\]

\[
\frac{\partial \Psi_I}{\partial x} = \frac{\partial \Psi_{II}}{\partial x} ; x = 0 \tag{2.2.7}
\]

\[
\Psi_{II} = \Psi_{III} ; x = a \tag{2.2.8}
\]

\[
\frac{\partial \Psi_{II}}{\partial x} = \frac{\partial \Psi_{III}}{\partial x} ; x = a \tag{2.2.9}
\]

\[
\Psi_{III} = \Psi_{IV} ; x = L \tag{2.2.10}
\]

\[
\frac{\partial \Psi_{III}}{\partial x} = \frac{\partial \Psi_{IV}}{\partial x} ; x = L \tag{2.2.11}
\]
\[ \Psi_{IV} = \Psi_{V} \quad ; x = L + a \quad \text{(2.2.12)} \]
\[ \frac{\partial \Psi_{IV}}{\partial x} = \frac{\partial \Psi_{V}}{\partial x} \quad ; x = L + a \quad \text{(2.2.13)} \]

at \( x = 0 \):
\[ A_I + B_I = A_{II} + B_{II} \quad \text{(2.2.14)} \]
\[ A_I - B_I = \frac{ik}{p} A_{II} - \frac{ik}{p} B_{II} \quad \text{(2.2.15)} \]

at \( x = a \):
\[ A_{II} e^{-\frac{ka}{\hbar}} + B_{II} e^{\frac{ka}{\hbar}} = A_{III} e^{\frac{ipa}{\hbar}} + B_{III} e^{-\frac{ipa}{\hbar}} \quad \text{(2.2.16)} \]
\[ A_{II} e^{-\frac{ka}{\hbar}} - B_{II} e^{\frac{ka}{\hbar}} = -\frac{ip}{k} A_{III} e^{\frac{ipa}{\hbar}} + \frac{ip}{k} B_{III} e^{-\frac{ipa}{\hbar}} \quad \text{(2.2.17)} \]

at \( x = L \):
\[ A_{III} e^{\frac{ipl}{\hbar}} + B_{III} e^{-\frac{ipl}{\hbar}} = A_{IV} e^{-\frac{kL}{\hbar}} + B_{IV} e^{\frac{kL}{\hbar}} \quad \text{(2.2.18)} \]
\[ A_{III} e^{\frac{ipl}{\hbar}} - B_{III} e^{-\frac{ipl}{\hbar}} = \frac{ik}{p} A_{IV} e^{-\frac{kL}{\hbar}} - \frac{ik}{p} B_{IV} e^{\frac{kL}{\hbar}} \quad \text{(2.2.19)} \]

finally, at \( x = L + a \):
\[ A_{IV} e^{-\frac{k(L+a)}{\hbar}} + B_{IV} e^{\frac{k(L+a)}{\hbar}} = A_{V} e^{\frac{ipl(L+a)}{\hbar}} \quad \text{(2.2.20)} \]
\[ A_{IV} e^{-\frac{k(L+a)}{\hbar}} - B_{IV} e^{\frac{k(L+a)}{\hbar}} = -\frac{ip}{k} A_{V} e^{\frac{ipl(L+a)}{\hbar}} \quad \text{(2.2.21)} \]

as is adopted, canceling the reflection amplitudes and the coefficients of the anti-evanescent waves, and by the ratio \( p = \sqrt{\frac{E}{V}} \ll 1 ; \frac{k}{p} = \sqrt{\frac{V}{E}} \gg 1 \):
\[ A_{II} = -2 \frac{ip}{k} A_I \quad \text{(2.2.22)} \]
\[ A_{III} = 2A_{II} e^{-\frac{ka}{\hbar}} e^{-\frac{ipa}{\hbar}} \quad \text{(2.2.23)} \]
\[ A_{IV} = -2 \frac{ip}{k} A_{III} e^{\frac{ipl}{\hbar}} e^{\frac{kl}{\hbar}} \quad \text{(2.2.24)} \]
\[ A_{V} = 2A_{IV} e^{-\frac{k(L+a)}{\hbar}} e^{-\frac{ipl(L+a)}{\hbar}} \quad \text{(2.2.25)} \]
thus, $A_V$ can be written in terms of $A_{IV}, A_{III}, A_{II},$ and $A_I$ up to obtain:

$$A_V = -4 \frac{i p}{k} \left( -4 \frac{i p}{k} A_I e^{\frac{-k a}{\hbar}} e^{\frac{i p a}{\hbar}} \right) e^{\frac{i p L}{\hbar}} e^{\frac{k (L+a)}{\hbar}} e^{\frac{i p (L+a)}{\hbar}}$$  \hspace{1cm} (2.2.24)

this strategy allows to compute the transmission coefficient after the first and second barrier denoted as $T = \left| \frac{A_{IV}}{A_I} \right|^2$:

$$T = \left( \frac{E}{V} e^{-\frac{2a \sqrt{2mV}}{\hbar^2}} \right)^2$$  \hspace{1cm} (2.2.25)

where the reflection coefficient ($R$) is:

$$R = 1 - T.$$  \hspace{1cm} (2.2.26)

We can note that, equation (2.2.25) is similar to equation (2.1.15), based on this result, it is possible to postulate that the transmission coefficient for $n$ barriers can be defined as:

$$T = \left( \frac{E}{V} e^{-\frac{2a \sqrt{2mV}}{\hbar^2}} \right)^n$$  \hspace{1cm} (2.2.25)

where $n$, since now, is the number of potential barriers.

3. Conclusions

Based on the solutions to the Schrodinger equation, we have derived and discussed the one-dimensional quantum tunneling problem considering one and two potential barriers. Our main result allows to hypothesize that the quantum transmission coefficient can be generalized which depends on the number potential barriers and furthermore, it is not affected by the intermediate free region between barriers. This finding needs to be corroborated by future experiments.

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