Cubical sets and the topological topos

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Abstract

Coquand’s cubical set model for homotopy type theory provides the basis for a computational interpretation of the univalence axiom and some higher inductive types, as implemented in the cubical proof assistant. This paper contributes to the understanding of this model. We make three contributions:

1. Johnstone’s topological topos was created to present the geometric realization of simplicial sets as a geometric morphism between toposes. Johnstone shows that simplicial sets classify strict linear orders with disjoint endpoints and that (classically) the unit interval is such an order. Here we show that it can also be a target for cubical realization by showing that Coquand’s cubical sets classify the geometric theory of flat distributive lattices. As a side result, we obtain a simplicial realization of a cubical set.

2. Using the internal ‘interval’ in the topos of cubical sets, we construct a Moore path model of identity types.

3. We construct a premodel structure internally in the cubical type theory and hence on the fibrant objects in cubical sets.

1 Introduction

Simplicial sets from a standard framework for homotopy theory. The topos of simplicial sets is the classifying topos of the theory of strict linear orders with endpoints. Cubical sets turn out to be more amenable to a constructive treatment of homotopy type theory. We consider the cubical set model in [CCHM15]. This consists of symmetric cubical sets with connections ($\wedge, \lor$), reversions ($\bar{-}$) and diagonals. In fact, to present the geometric realization clearly, we will leave out the reversions. We now give a precise definition.

2 Classifying topos and geometric realization

Cube category Consider the monad $DL$ on the category of finite sets with all maps which assigns to each finite set $F$ the finite set of the free distributive lattice
on $F$. That this set is finite can be seen using the disjunctive normal form. The *cube category* is the Kleisli category for the monad $DL$. The diagonals above refer to the use of all maps, as opposed to only the injective ones; cf [Pit14].

**Lawvere theory**  For each algebraic (=finite product) theory $T$, the Lawvere theory $\Theta_T^{op}$ is the opposite of the category of free finitely generated models. This is the classifying category for $T$: models of $T$ in any finite product category category $E$ correspond to product-preserving functors from the syntactic category $C_{fp}[T]$ to $E$. For a monad $T$ on finite sets, the Kleisli category $KL_T$ is precisely the opposite of the Lawvere theory: maps $I \to T(J)$ are equivalent to $T$-maps $T(I) \to T(J)$ since each such map is completely determined by its behavior on the atoms, as $T(I)$ is free.

**Cube category, contravariantly**  Let $2$ be the poset with two elements, one smaller than the other. Consider $\square$, the full subcategory of $\text{Cat}$ consisting of the powers of $2$. We have a Stone duality between finite posets and finite distributive lattices with $2$ as dualizing object. This duality is given by a functor $J = \text{hom}_{DL}(\cdot, 2)$ from finite distributive lattices to the opposite of the category of finite posets $\Theta DL$. This functor sends a distributive lattice to its join-irreducible elements. Its inverse is the functor $\text{hom}_{\text{poset}}(\cdot, 2)$ which sends a poset to its the distributive lattice of lower sets. This restricts to a duality between free finitely generated distributive lattices and powers of $2$ (the copowers of $DL$).

**De Morgan algebras**  There are two more specific dualities [CF77, CF79]:

Finite involutive posets and finite De Morgan-algebras, with dualizing object $2^2$.

Finite involutive posets such that $x \leq \neg x$ or $\neg x \leq x$ and finite Kleene algebras, with dualizing object $3$ (three points on a line).

The natural involution on $2$ provides us with an involutive poset and hence, dually, with a De Morgan algebra. Every free finitely generated De Morgan algebra on $n$ generators is a free distributive lattice on $2n$ generators. We obtain a duality with the category even powers of $2$ and maps preserving the De Morgan involution [CF77].

Although it is natural to consider De Morgan algebras or Kleene algebras in the implementation, we will focus on distributive lattices in what follows, mainly in view of the geometrical realization; see Section 2.1.1.

**Classifying topos**  Objects of the topos $\hat{\square} = \Theta DL$ are called cubical sets. The following theorem by Johnstone-Wraith [JW78, Thm 5.22] tells us what this topos classifies.

**Theorem 1.** Let $\Theta_T^{op}$ be a Lawvere theory. Then $\Theta_T$ classifies flat $T$-models.

Here a flat $T$-models is one that can be expressed as a filtered colimit of free models. Flatness is a geometric notion [JW78, Thm 5.22].

Below we will compute what this means for the algebraic theory of distributive lattices, but first we indicate how this theorem can be proved. To obtain the classifying topos for an algebraic theory $T$, we first need to complete with finite limits, i.e. to consider the classifying category $C_{fl}$ as the opposite of finitely presented $T$-algebras.
Then $C_{fl}^{op} \to \text{Set}$, i.e. functors on finitely presented $T$-algebras, is the classifying topos for the theory of $T$-algebras. This topos contains a generic $T$-algebra. $T$-algebras in any topos $\mathcal{F}$ correspond to left exact left adjoint functors from the classifying topos to $\mathcal{F}$.

Let $FG$ be the category of free finitely generated $T$-algebras and let $FP$ the category of finitely presented ones. We have a fully faithful functor $f : FG \to FP$. This gives a geometric morphism $\phi : \widehat{FG} \to \widehat{FP}$. Since $f$ is fully faithful, $\phi$ is an embedding [Joh02, A4.2.12(b)]. The subtopos $\widehat{FG}$ of the classifying topos for $T$-algebras is given by a quotient theory, the theory of the model $\phi^*M$. This model is given by pullback and thus is equivalent to the canonical $T$-algebra $\mathbb{I}(m) := m$ for each $m \in FG$.

**Flat distributive lattices** Let $D$ be a distributive lattice in a topos $\mathcal{X}$. Then by the standard construction in Lawvere theories, define $S_D : \Theta^{op}_{DL} \to \mathcal{X}$ on objects by $S_D(n) := D^n$ and for a map $\phi : n \to DLM$ define a map $S_D\phi : D^m \to D^n$. This is well-defined since a distributive lattice is an algebra for the DL-monad. It follows that $S_D : \square \to \mathcal{X}$ is a cocubical object in $\mathcal{X}$.

A Set-valued functor is $E : C \to \text{Set}$ is flat if it is filtering [MM12, VII.6]:

**inhabited** There is at least one object $c \in C$ such that $E(c)$ is an inhabited set.

**transitivity** For objects $c, d \in C$ and elements $y \in E(c)$, $z \in E(d)$, there exists an object $b \in C$, morphisms $\alpha : b \to c$, $\beta : b \to d$ and an element $w \in E(b)$ such that $E(\alpha)(w) = y$ and $E(\beta)(w) = z$.

**freeness** For two parallel morphisms $\alpha, \beta : c \to d$ and $y \in E(c)$ such that $E(\alpha)(y) = E(\beta)(y)$, there exists a morphism $\gamma : b \to c$ and an element $z \in E(b)$ such that $\alpha \circ \gamma = \beta \circ \gamma$ and $E(\gamma)(z) = y$.

Specializing the general definition of a flat model to $T$-algebras ($C = \Theta^{op}_T$), a $T$-algebra $D$, we observed that the first two conditions always hold:

**inhabited** $D^1$ is inhabited.

**transitivity** given $d \in D^n$ and $d' \in D^m$, then $d, d' \in D^{n+m}$ shows transitivity.

So, a $D$ is flat if for all $\alpha, \beta : n \to Tm$ and $d \in D^m$ st $\alpha d = \beta d$, there exists $\gamma : m \to Tk$ such that $\alpha \gamma = \beta \gamma$ and there exists a $d' \in D^k$ such that $\gamma d' = d$.

Put more simply, if we have two $n$-ary $T$-expressions (‘polynomials’) $\alpha, \beta$ which when applied to $d$ are equal, then there exists $\gamma$ such that $\gamma d' = d$ and $\alpha, \beta$ are both constructed from $\gamma$.

Flat functors generalizes the abstract definition of flat modules [MM12 VII]. The definition above is similar to the elementary definition of flat modules, with the difference that we lack subtraction, and hence equalities between terms cannot be replaced by being equal to 0.

Vickers [Vic07] observes that being flat is a geometric notion. The following is implicit in [JW78].

**Lemma 1.** A (free) finitely generated $T$-algebra is flat. Hence, the generic $T$-algebra is a flat model.
Proof. For readability, we fix $T$ to be the theory of distributive lattices. Every element $d$, e.g. $(x, x \lor y, y \land x)$, is the image under a map $\gamma$ of a list of generators $d'$, $(x, y)$ in the example. We also have $\alpha \gamma = \beta \gamma$, as $\gamma$ is completely determined by its behavior on the generators. This shows that finitely generated distributive lattices are free.

Since, being flat is a geometric statement it also holds for the generic element $I$.

Lemma 2. Flat distributive lattices have the disjunction property: If $a \lor b = 1$, then $a = 1$ or $b = 1$.

Proof. Consider $d = (a, b)$ and $\alpha = x \lor y$ and $\beta = 1$. There are essentially two possibilities for $\gamma$: $\gamma_d = p_{x,1}$ or $\gamma_d = p_{1,x}$. From $\gamma d = d$, it follows that $a = 1$ or $b = 1$.

This disjunction property crucial in CTT, as it is used to prove that we have a lattice homomorphism $I \to F$ [Coq15b, 3.1].

By Diaconescu’s theorem, flat functors correspond to geometric morphisms. In fact [Vic07], a presheaf topos is the classifying topos for flat functors. The generic element is the Yoneda flat functor.

Proposition 1. Every flat functor $\Box \to X$ is $S_D$ for some flat $D$ in $X$.

Proof. We first consider the generic flat functor: $y : \Box \to I$. Let $\Pi(n) := DL(n)$ be the generic free distributive lattice in $DL = \Box$. We have: $S_I(n) = \Pi^n = \text{hom}_{DL}(n, \Pi)$. The latter is the hom-set in the Kleisli category. Since products are geometric, $\Pi^n(m) = DL(m)^n = \text{hom}_{DL}(n, m)$ for all $m$. Note that $y(n) := \text{hom}_{\Box}(\Box, n) = \text{hom}_{DL}(n, -)$. So, $y = S_I!$.

As Johnstone [Joh79] observes (for simplicial sets) this suffices. Let $f$ be flat functor, then $f = F^*y$ for a geometric morphism $F$. Since the construction of $S$ is geometric, as it only uses natural numbers and free constructions, we have $f = F^*S_I = S_F*\Pi$. 

The previous proposition allows us to conclude a special case of Theorem 1.

Theorem 2. The topos $\hat{\Box}$ of cubical sets classifies flat distributive lattices.

2.1 Geometric realization

In Theorem 2 we will construct a cubical geometric realization.

Proposition 2. Every linear order $D$ defines a flat distributive lattice. Hence, we have a geometric morphism $\hat{\Delta} \to \hat{\Box}$.

Proof. Obviously $D$ defines a distributive lattice. To check freeness, take $d \in D^m$ and $\alpha, \beta : n \to DLm$ as above. Now choose $d' \in D^k$ which orders the elements of $d$ and removes duplicates. Define $\gamma_1 : D^k \to D^n$ such that $\gamma_1 d' = d$. This is possible because we have a (decidable) linear order. As $\alpha \gamma = \beta \gamma$, as $\gamma$ is completely determined by its behavior on the generators and so the functions $\alpha, \beta$ are equal to a list of projection functions on $m$. 

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Consider the map $\gamma_2 : k \to DLk$ defined by $\gamma_2(x_1, \ldots, x_k) = (x_1, x_1 \vee x_2, \ldots, \vee_{i=1}^{k} x_i)$. Then $\gamma_2(d') = d'$, as $d'$ is already ordered. Let $\gamma = \gamma_1 \gamma_2$. Each finite linear order is isomorphic to an interval in a free distributive lattice of the same size. In particular, $(x_1, x_1 \vee x_2, \ldots, \vee_{i=1}^{k} x_i)$ is isomorphic as a linear order to $d'$. Since $\alpha d = \alpha \gamma d' = \beta \gamma d'$, we have $\alpha \gamma = \beta \gamma$. It follows that $D$ is free.

Let $\mathcal{E}$ be Johnstone’s topological topos.

**Theorem 3 (Cubical geometric realization).** There is a geometric morphism $r : \mathcal{E} \to \hat{\Delta} \to \square$ defined using $I := [0, 1]$ in $\mathcal{E}$. Moreover, $r^*$ is the following weighted colimit

$$r^*K = K \otimes_\square I^* = \int^{n \in \square} K(n) \times I^n.$$

**Proof.** By analogy to the treatment by Johnstone [Joh79, Thm 8.1], we define the cubical realization by using that $I$ in Top is a flat distributive lattice, using proposition 2. By construction, $r^*K = S(I) \otimes_\square K$, where we use Proposition 1 that $S(I)$ is a cocubical set; see [MM12, VI.5]. Johnstone proves the preservation of colimits from Top to $\mathcal{E}$ and we know that $S_n(I) = I^n$. This gives the formula above, where the right hand side is the usual coend formula $(S \otimes_C T = \int S(c) \otimes D(c))$ of the tensor product.

For every $X$ in $\mathcal{E}$, $r_*X(n) = \text{hom}_{\mathcal{E}}(I^n, X)$. As the sequential spaces form a (reflective) subcategory of $\mathcal{E}$ [Joh79, Lem. 2.1], this reduces to the cubical singular complex in the case of such spaces. This is reminiscent of the cubical realization studied by Jardine [Jar02].

Concretely, by the computation above, this geometric morphism $r$ is represented by the flat functor $S_D(n) = D^n = \delta_0^n[1]$. From this we can compute the inverse image of the geometric morphism [Joh02, B3.2.7].

Johnstone uses classical logic to prove that the unit interval is a linear order. Without classical logic we can still define the geometric realization of simplicial sets and of cubical sets. This realization lands in the category $\mathcal{F}$ of so-called sequential spaces, a reflective subcategory of $\mathcal{E}$.

**Proposition 3.** The simplicial geometric realization $G$ is left exact iff $[0, 1]$ is a linear order. The cubical geometric realization is is left exact iff $[0, 1]$ is a flat distributive lattice.

**Proof.** If the geometric realization to spaces is left exact (i.e. preserves limits), then so is the realization in $\mathcal{E}$, because $\mathcal{F}$ is a reflective subcategory of $\mathcal{E}$ and hence the limits coincide. Being left exact this functor is part of a geometric morphism $\mathcal{E} \to \hat{\Delta}$. By the classifying property of $\hat{\Delta}$, this means that $[0, 1]$ is a linear order.

Conversely, if $[0, 1]$ is a linear order, then $r_* : \hat{\Delta} \to \mathcal{E}$ is left exact. Since it lands in $\mathcal{F}$ and the limits coincide, the geometric realization is left exact too.

A similar argument works for the cubical sets.

By our construction, the cubical realization factors via the simplicial realization. It is interesting to note that Jardine has geometrical realizations going both into $\hat{\Delta}$ and
into $\text{Top}$. His formulas for the simplicial realization \cite[p10]{Jar02} of a cubical set are analogous to the one in Theorem 3.

The cubical realization of a topological space in fact has compositions \cite{CCHM15}, so it is constructively fibrant.

### 2.1.1 Alternative combinatorial structures

#### Simplicial sets

Like the cube category, the simplex category can also be presented by a duality. The opposite of the simplex category (of ordinals with face and degeneracy maps) is the category $\nabla$ of intervals, finite linear orders with distinct $\top$ and $\bot$ and monotone functions preserving these. Here 2 is a dualizing object \cite{Wra93}. The Yoneda embedding of this element gives the universal order in sSets. It is called $V$ in \cite[p.458]{MM12}. To connect this with the cubical sets, observe that $\hom(\square, 2) = \hom_{DL}(DL(1), -) \cong DL(-)$. For the isomorphism in the previous sentence observe that such maps are determined by what happens to the generator.

#### Bi-pointed sets

Awodey uses cubes with diagonals (but without connections, or reversions). This is Grothendieck’s simplest test category. This cube category, $H$, is the free finite product category on an interval. Awodey observes that $H = \Theta_2^{op}$, the Lawvere theory of bi-pointed sets. As above, we also obtain a cubical realization for the topos $\tilde{H}$ which classifies strictly bi-pointed sets. Moreover, this can even be seen as a geometric morphism to the Giraud topos \cite{Joh79}, roughly sheaves over $\text{Top}$, as the topological interval is strictly bipointed, but not constructively a strict linear order.

We would like to relate Coquand’s and Awodey’s cubical sets. There is a free distributive lattice on a bi-pointed set. As both theories are Cartesian, the free distributive lattice over a bi-pointed set is geometric. We obtain a geometric morphism from $\tilde{H} \to \square$. This is the geometric morphism obtained from the functor $H^{op} = \Theta_2^{op} \to \Theta_{DL}^{op} = \square^{op}$ which is faithful, but not full. Hence the geometric morphism is not an embedding.

We obtain a geometric morphism in the opposite direction, $\square \to \tilde{H}$, by observing that every distributive lattice is bipointed. However, this is not the inverse of the former map.

#### De Morgan algebras and involutions

Since the cubical model uses De Morgan algebras, it is tempting to consider the geometric realization for these cubical sets.

Johnstone’s argument shows that $[0, 1]$ is also an involutive linear order with endpoints. Involutive means that we have extra rules: $a^{**} = a$, $a^* < b^*$ if $b < a$, and thus $a < b^*$ if $b < a^*$. An involutive linear order is a flat De Morgan algebra.

However, it seems that the classifying topos for involutive linear orders has not been used in homotopy theory, so we will not pursue this line further.

We would like to compare the toposes $\Theta_{DL}$ and $\Theta_{DM}$. However, the obvious maps are not embeddings: The functor $\Theta_{DL} \to \Theta_{DM}$ defined by $DL(n) \to DM(n)$ is not full: the map $DM(1) \to DM(1)$ generated by $x \mapsto x^*$ is not in the range. Likewise, the functor $\Theta_{DM} \to \Theta_{DL}$ defined by $DM(n) \to DL(2n)$ is not full as we have $f(a^*) = f(a)^*$, so the generators are related by the maps.

\[\text{https://ncatlab.org/homotopytypetheory/files/AwodeyDMVrev.pdf}\]
3 Moore paths and identity types

In this subsection, we focus on De Morgan algebras, instead of distributive lattices. It makes for a slightly smoother presentation, but these reversions are not strictly needed [Doc14]. Coquand’s presentation of the cubical model does not build on a general categorical framework for constructing models of type theory. Meanwhile a more abstract description has been given [BBP+16, OP16] motivated by Coq15a and the present work which was announced in Spi15.

Here we pursue Docherty’s model of identity types on cubical sets with connections [Doc14]. This uses the general theory of path object categories [vdBG12]. We present a slightly different construction from [Doc14] using similar tools, but simplified by the use of internal reasoning, starting from the observation that the generic De Morgan algebra $I$ represents the interval in cubical type theory. To obtain a model of identity types on a category $C$ it suffices to provide an involutive ‘Moore path’ category object on $C$ with certain properties. Now, category objects on cubical sets are categories in that topos. The Moore path category $MX$ consists of lists of composable paths $I \to X$ with the zero-length paths $e_x$ as left and right identity. This is an instance of the general path category on a directed graph. In this case, the graph with edges given by elements of $X^I$. To obtain a nice path object category ($M1 \cong 1$), we quotient by the relation which removes lists of constant paths $(\lambda i.x)$ from the list; cf [Doc14, Def 3.9]. Since $1^I \cong 1$ the only Moore path in 1 is in fact the zero-length path. Hence, $M1 \cong 1$.

The reversion $\to$ on $I$ allows us to reverse paths of length 1. This reversion extends to paths of any length. We obtain an involutive category: Moore paths provide strictly associative composition, but non-strict inverses.

A path contraction is a map $\text{con} : MX \to MEX$ which maps a path $p$ to a path from $p$ to the constant path on $tp$ ($t$ for target). Like Docherty [Doc14, Def 3.2.3], we use connections to first define the map $\text{con}_1$ from $X^I$ to $X^{I \times I}$ by $\lambda p.\lambda i.p(i \lor j)$ and then extended this to a contraction. For a Moore path of length three, this looks like:

All these constructions are algebraic and hence work functorially. We obtain a nice path object category.

We have obtained a model of identity types [vdBG12, Doc14] starting from the interval $I$ in the cubical model. The connection between this model and the one by Coquand is not entirely clear. We will explore what can be done when we restrict to the fibrant types, the types with composition structure.
3.1 Id, Path and Moore

In this section we study the relation between the three constructions of identity types. The Path type \(\text{Path}^I\) is the main identity type in cubical type theory. However, it does not satisfy the judgemental computation rule for the \(J\)-elimination. From \(\text{Path}\) one can define a new type \(\text{Id}\) which does satisfy the judgemental rule [CCHM15, 9.1].

\[
\Gamma, \phi \vdash a : A \quad \Gamma \vdash p : \text{Path}_A \ ab \[\phi \mapsto \langle - \rangle a\] \\
\Gamma \vdash (p, [\phi \mapsto a]) : \text{Id}_A \ ab
\]

**Id and Path** We start with some short observations about Path and Id. The section \(\lambda p.(p,0) : \text{Path} \rightarrow \text{Id}\) has \(\pi_1\) as a (judgemental) retraction. Moreover, it has been formally checked that they give rise a Path-quasi inverse\(^2\). This can be weakened to an Id-quasi-inverse.

The proofs that \(\Pi, \Sigma\) respect equivalence do not require univalence. It follows that all the predicates which are defined from an equality are equivalent with respect to either notion of equality. In particular, this holds for being contractible, being a proposition and being an equivalence.

**Moore paths** We will show that for each fibrant type \(A\), there is a new fibrant type \(M_0A\) of Moore paths. Cubical type theory supports inductive types and since it has a identity type, it is expected to support inductive families [Dyb94] too. However, this has not been formally checked yet. Instead, we can define the composition explicitly for one particular inductive family. This is a slight variation on the transitive reflexive closure of the Path relation. We make the following definition in cubical *sets* and will show that it has a composition operation.

Inductive \(M_0A(x : A) : A \rightarrow \text{Type} :=\)

\[
e : \Pi_{a:A}M_0Aa \\
\]

\(c : \Pi_{p:A}\Pi_{q:M_0A(p0)}a M_0A(p0)a\)

We will write \(M_0A\) for \(\Sigma_{ab}M_0A\ a\ b\). As we’ve seen above in the definition of contraction for Moore paths, paths between Moore paths will preserve the length, but may vary the points.

First we recall the definition of composition on lists. It is defined recursively by:

\[
\text{comp}^i(\text{list}\ A)[\phi \mapsto \text{nil}]\ \text{nil} \quad = \quad \text{nil} \\
\text{comp}^i(\text{list}\ A)[\phi \mapsto a :: l]\ a' :: l' \quad = \quad \text{comp}^i A[\phi \mapsto a] a' :: \text{comp}^i(\text{list}\ A)[\phi \mapsto l] l'
\]

Similarly,

\[
\text{comp}^i M_0A[\phi \mapsto e_a] e_{a'} \quad = \quad e_{\text{comp}^i A[\phi \mapsto a] a'} \\
\text{comp}^i M_0A[\phi \mapsto p :: l] p' :: l' \quad = \quad \text{comp}^i PA[\phi \mapsto p] p' :: \text{comp}^i M_0A[\phi \mapsto l] l'
\]

Where we have written \(p :: l\) for the concatenation \text{cpl}.

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\(^2\)\text{idToEqToId, eqToIdToEq in} \url{https://github.com/mortberg/cubicaltt/tree/defeq}
To obtain a nice path category, we would like to quotient Moore paths upto constant paths; as in [Doc14, Dfn 3.9]. Unfortunately, constancy of paths is not decidable and neither is it respected by comp. So, this approach seems stuck here. As Docherty [Doc14, Ch7] points out, it may be possible to relax the requirement that \( M1 \cong 1 \). So, it seems worth recording the facts above, even if they are not yet conclusive.

If such Moore paths can be defined on the fibrant types, they will have the judgemental computation rules for \( J \), and being equivalent type Path, will also satisfy univalence, since [CCHM15, Cor.11] works for any other notion of identity which is reflexive and satisfies the elimination rule for identity.

### 4 Pre-model-structure

The cubical model may be constructed in the internal language of the topos of cubical sets [Coq15a, BBC+16, OP16]. As recalled in the previous section, one can introduce a type \( \text{Id} \) which has a judgemental computation rule for the \( J \) eliminator. A factorization system can be defined in the cubical type theory [CCHM15, 9.1]. The Gambino-Garner factorization system [GG08] gives another factorization system. We will show that the two factorization systems together form a pre-model-structure. Here we follow Lumsdaine [Lum11], with the twist that he uses \( \text{Id} \)-types, rather than Path-types for the mapping cylinder.

To be precise, Gambino-Garner require identity contexts. Since the cubical model is only a category with families, we need to consider the associated contextual category. I.e. we need to restrict to the fibrant cubical sets.

**Definition 1.** A pre-model-structure on a category \( C \) consists of three classes \( (C, F, W) \) of maps of \( C \), such that \( W \) satisfies 3-for-2, and \((C, F \cap W)\) and \((C \cap W, F)\) are weak factorisation systems on \( C \).

**Theorem 4.** There is a pre-model-structure on fibrant cubical sets. The factorizations are defined in the cubical type theory, so they are uniform and functorial.

**Proof.** We define \( \mathcal{F}_0 \) to be the set of display maps \( \pi_1 : \Sigma AB \to A \). Let \( \mathcal{W} \) be the set of equivalences. Let \( \mathcal{T} \mathcal{F}_0 \) be the display maps which are also equivalences, i.e. every fiber is \( \text{Path} \)-contractible. The cofibrations \( \mathcal{C} \) are the maps with the lift lifting property (LLP) with respect to \( \mathcal{T} \mathcal{F}_0 \). This LLP is wrt diagrams which commute upto judgemental equality, equality in the topos, not just upto a \( \text{Path} \). We define the trivial cofibrations \( \mathcal{T} \mathcal{C} = \mathcal{W} \cap \mathcal{C} \).

Gambino-Garner factors a function \( f : A \to B \) through its graph \( \Sigma(y,x) \text{Id}_B(f(x),y) \). The maps \( tc_f(a) := (f(x); x; 1_{f(x)}) \) and \( \pi_1 \) give the factorization of \( f \). The map \( tc_f \) is a section with retraction \( \pi_1 \pi_2 \). This shows that \( tc_f \in \mathcal{T} \mathcal{C} \).

The other factorization is given by the type \( C_f := \Sigma_b T_f(b) \), where \( T_f(b) \) consists of partial sections \([\phi \mapsto a] \) such that \( f(a) = b \) on \( \phi \). The type \( C_f \) is reminiscent of the mapping cylinder, the homotopy pushout of \( 1 \) and \( f \), a higher inductive type (HIT) equivalent to \( B \); see [Uni13]. While there is no general theory of HITs in cubical yet, we can explicitly define the constructors of this HIT:

\[
\text{inbase } b := (b, [0 \mapsto a]) : C_f(b),
\]
intop \ a := (f(a), [1 \to a]) : C_f(fa) \text{ and}

\text{inyl} \ a := (\langle f(a), a_0 \rangle : \text{Path}_{C_f} \ \text{intopa}, \text{inbase}(fa)).

Coquand shows that we have a factorization \((C, T \mathcal{F}_0)\).

As Lumsdaine shows \(W\) has 3-for-2 and is closed under retracts. To be precise, these are retracts between fibrant objects, hence judgement equalities in the type theory.

We can close the classes \(\mathcal{F}_0\) and \(T \mathcal{F}_0\) by taking the double orthogonal, where being orthogonal is defined in the cubical type theory.

Lumsdaine shows that \(T \mathcal{F} = T \mathcal{F} \cap W\) and \(T \mathcal{C} = C \cap W\) by formal manipulations about retracts and orthogonality. The same arguments go through here.

\[\square\]

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