Gravitational effects in terms of paths in Minkowski space

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Abstract

An approach is developed which enables one to analyze gravitational effects without usage of any concrete model of geometry (or a class of models of geometry) of space-time and even without any coordinate system. Instead, the formalism of the group of paths is used. An element of this group (a class of curves in Minkowski space) is associated with each curve in the curved space-time and called its ‘flat model’. The analysis of observational data in terms of flat models of closed curves is interpreted as a formalization of the analysis which a ‘naive observer’ (knowing nothing about the space-time being curved) applies to his observations.

1 Introduction

Gravitational experiments and gravitational effects are usually described and analyzed in the framework of a certain assumption about the geometry of space-time, its belonging to a rather narrow class of geometries [1, 2]. The post-Newtonian approximation for the analysis of gravitational effects in the solar system and Friedmann model for the analysis of data about early Universe are examples of this situation. Sometimes the geometrical model used in the analysis enables one to ascribe a direct physical (geometrical) sense to a certain coordinate system.

However, such a model-dependent analysis is inconvenient if there is no sufficient ground for a reasonable choice of a geometrical model of the space-time region under investigation. In the extreme situation even topology of this region may be indefinite. Then one cannot introduce even structure of manifold, to say nothing about the curvature of the space-time. Even coordinate systems cannot be then correctly introduced. Such a situation may arise in astrophysics and cosmology if regions with strong gravitational field or early stages of Universe are under investigation.

Of course, one may use a number of radically different geometrical models in this case, trying to analyze the data in the framework of each of these models. Nevertheless, any a priori choice of the models may restrict the scope of research, introduce a subjective element into it. An ‘objective’ approach making use of no concrete model of geometry may be advantageous in this situation. We shall present here such an ‘objective’ description of gravitational effects which is independent of any concrete model of geometry. Particularly, no at all coordinate system (in a curved space-time) is used in this approach.

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At first sight it seems impossible to do without any model of geometry and even without any coordinate system. However, the following argument shows that the procedures of this type must exist. Indeed, an observer can objectively present his observations if he describes in detail the measuring devices which have been used, his actions with these devices and the readouts obtained in the course of the measurements. In other words, a sort of the measurement protocol presents the observational data in the objective way, with no reference to models of geometry and coordinate systems.

This way of objectification is of course inconvenient from theoretical point of view because it makes use of terms and concepts which are too different from those typical for geometry. We shall present the way of description of observational data which is adequate for geometry, yet quite similar to the measurement protocol. This description is based on the formalism of paths in Minkowski space. This formalism is closely connected with one used in the path-dependent field theory suggested by Mandelstam and Bialynicki-Birula [3, 4] and in the path-group approach to gauge theory and gravity suggested by the author [5, 6, 7, 8].

Although the formalism of paths is mathematically well elaborated, we shall only shortly concern here the precise mathematical concepts in Sect. 3. To avoid unessential mathematical details, we shall remain in most part of the paper on the ‘intuitive’ level. The resulting picture is in fact the point of view of a ‘naive’ observer who attempts to present the events, which are actually performed in the curved space-time, in terms of his local (and therefore flat) geometry. We shall call the corresponding procedure ‘flat modelling’ of a curved space-time. More precisely, the procedure provides flat models of curves which are given in a curved space-time. It will be especially important for us to construct flat models for those closed curves in a curved space-time which are characteristic for a gravitational experiment or observation. The result of flat modelling is in fact a kind of a protocol for a this observation.

Particularly, gravitational effects will manifest themselves as ‘discrepancies’ in the flat models of closed curves, or, in other words, as ‘discrepancies’ in observations as they are seen by a ‘naive’ local observer. For example, closed curves in the real (curved) space-time may be presented by non-closed flat models. Another kind of discrepancy which may arise is a Lorentz transformation of local frame after transporting it along the closed curve. In both cases the discrepancies in local (flat) presentation of observations have to be interpreted as evidences of non-zero curvature of the real space-time.

As examples illustrating the method we shall consider the classical effect of the deviation of light by Sun and the effect of gravitational lensing. It will be shown how these effects may be expressed in terms of flat models of the corresponding closed curves and compensating Lorentz transformations of local frames.

2 Flat modelling of a curved space

It is known that there is no natural mapping of a curved space onto the flat space (of the same dimension). However, a natural mapping exists of any curve in a curved space onto a curve in the flat space of the same dimension. In [5] and further publications of the author [6, 7, 8] this procedure was exploited to define ‘flat modelling’ of a curved space-time (actually of arbitrary curves in it). We shall apply this procedure for analysis of gravitational observations and gravitational effects.
2.1 Idea of flat modelling and Equivalence Principle

Consider all curves in a curved space-time $\mathcal{X}$ beginning at the same point $x_0 \in \mathcal{X}$. ‘Flat models’ of these curves are constructed in the tangent space to the point $x_0$. The tangent space has the structure of Minkowski space, thus the curve in the curved space-time has a curve in Minkowski space-time as its model (call it ‘flat model’). Later on (Sect. 3.1) we shall describe the process of flat modelling in precise terms, but now let us give a simple and obvious idea of this process.

The (tangent) Minkowski space-time will be naturally interpreted as a geometric image of the whole space-time as it is imagined by a ‘naive observer’ located at point $x_0$. We thus assume that the ‘naive observer’ thinks of the real space-time as a flat (Minkowski) one.

Having Minkowski space as a flat model of the space-time, the naive observer will try to describe gravitational experiments or observations in terms of this model. Particularly, he will attempt to present the characteristic curves arising in the course of the observation as curves in Minkowski space.

Evidently the local observer can do this. For example at a certain time moment he issues a photon (actually a number of photons) in a certain direction. Then he constructs the flat model of the world line of this photon as a light-like straight line in Minkowski space $\mathcal{M}$ starting in the point $\xi_0 \in \mathcal{M}$ which is identified with the location $x_0$ of the observer in the real space-time and pointing to the corresponding direction. The world line of the observer himself will be drawn in $\mathcal{M}$ as the straight line starting in $\xi_0$ and being a trajectory of a body at rest.

Let us assume that at a later time (corresponding, say, to the point $\xi_1$ at the flat model of his trajectory) the observer sees a photon (actually a number of photons) coming from a certain direction (determined by the direction of the telescope’s axis). Then he can construct the flat model of the world line of the absorbed photon. This will be a light-like straight line coming from the corresponding direction of the Minkowski space into the point $\xi_1$.

Analogously the observer may construct flat models of all curves crossing the point $x_0$ or any other point of his world line. If he can by some way or another obtain the information about lengths along any of these lines and directions of these lines in each point, then he will naturally construct the flat models of these lines. (Fig. 1). The process of flat modelling is similar to creating a topographical survey.

In terms of flat modelling, Equivalence Principle may be formulated in a simple and natural way [3, 4]: the flat model of the world line of any point-like body acted on by no but gravitational forces is a time-like straight line in Minkowski space.

![Figure 1: A curve in a curved space-time and its model in the (tangent) Minkowski space.](image-url)
2.2 Gravitational Effects

If an observer describes his observations by flat models of the corresponding curves, then gravitational effects manifest themselves as discrepancies in the resulting picture: the curves which are in reality closed may be modelled by non-closed curves, and the motion along these may be accompanied by a certain Lorentz transformation of the local frame.

This may be illustrated by the flat model of a curve consisting of pieces of geodesics on a sphere (Fig. 2). Imagine that the sphere is a globe. Let us start from a point on the equator, go to East along the quarter of the equator, then turn to the left (to North) and go along the quarter of the meridian (coming to North pole), then again turn to the left (in the direction of South) and go along the quarter of the meridian. As a result we return to the initial point at the equator passing along the closed curve on the globe.

However, the flat model of this curve will be non-closed (right diagram of Fig. 2). It consists of three sides of a square. Besides, the local frame, after being transported along the globe in a natural way (this corresponds to the procedure of parallel transport), will not coincide with the initial local frame. To achieve coincidence, the transported local frame has to be rotated by the angle $-\frac{\pi}{2}$. Thus, the flat model of a closed curve is non-closed and needs a compensating rotation of a local frame. These ‘discrepancies’ are consequences of non-zero curvature of the sphere. We may consider them to be manifestations of a ‘two-dimensional gravitational effect’.

If the flat models and the compensating transformations of local frames are known for all closed paths with the given initial point, the geometry may be completely reconstructed (see below Sect. 3.2). Real observations cannot of course give this complete information. Instead, the flat models and compensating transformations may be known only for restricted number of closed curves. This determines a class of geometries which lead to the effect. The complete Information about this class is contained in the description of those flat models (together with the corresponding compensating Lorentz transformations) which are obtained from the observations.

For example, if only a single flat model (with the accompanying compensating rotation), namely one of the right diagram of Fig. 2 is known, then the class of geometries which are compatible with this information consists of deformed spheres which however coincide with the ideal sphere in the narrow stripe along the path of the left diagram of Fig. 2. This may be illustrated in the following way. Let us cut out, from the right diagram of Fig. 2, a narrow stripe of paper along which the flat model of our curve is located. Let us glue up the end of this stripe with its beginning after rotating the end by right angle. Then the geometry corresponding to
the observation contains this stripe (glued up in this way) but is otherwise arbitrary.

3 Mathematical formalism of flat modelling

Here we shall shortly outline the rigorous mathematical formalism which is necessary for the construction and analysis of flat models. A more detailed exposition of this may be found in [6, 7, 8, 9].

3.1 Flat modelling of curves

There is no natural point-to-point mapping of a curved space-time onto the Minkowski space. However, a natural correspondence exists between the curves in the Minkowski space and the curves in the curved space-time provided the starting point of these curves and local reference frame in this point are fixed. We shall give here the main definitions assuming that the (pseudo-Riemannian) geometry is given.

Let \( T_x \) be a tangent space in the point \( x \) of the curved space-time \( \mathcal{X} \). Then a local frame \( b^\alpha \in T_x \), \( \alpha = 0, 1, 2, 3 \). Each of these vectors \( b_\alpha \) may be presented by its components (in any given coordinates) \( b_\alpha^\mu \), \( \mu = 0, 1, 2, 3 \). The set \( \mathcal{B} \) of all local frames (in all points \( x \in \mathcal{X} \)) form a fiber bundle over \( \mathcal{X} \) as a base. The structure group of \( \mathcal{B} \) is \( GL(4) \). The coordinates of points and components of vectors of local frames \( (x^\mu, b_\lambda^\beta) \) may serve as coordinates in \( \mathcal{B} \).

A local frame \( n \in \mathcal{B} \) is orthonormal if its components are orthonormalized, \( g_{\mu\nu} n_\alpha^\mu n_\beta^\nu = \eta_{\alpha\beta} \), with \( g_{\mu\nu} \) being metric in \( \mathcal{X} \) and \( \eta_{\alpha\beta} \) Minkowski tensor. The set \( \mathcal{N} \subset \mathcal{B} \) of all orthonormal local frames (in all points) is a fiber bundle over \( \mathcal{X} \) with the Lorentz group \( \Lambda = SO(1, 3) \) as a structure group, \( (n\lambda)_\alpha = n_\beta \lambda^\beta_\alpha \). Actually we need only orthonormal local frames, but the definitions will be simpler if all frames are introduced at intermediate steps (because it is simpler to introduce coordinates in \( \mathcal{B} \) than in \( \mathcal{N} \)).

Let us introduce now the so-called basis vector fields in the fiber bundle \( \mathcal{B} \) of local frames:

\[
B_\alpha = b_\mu^\alpha \frac{\partial}{\partial x^\mu} + b_\mu^\beta \Gamma_\mu^\lambda (x) \frac{\partial}{\partial b_\beta^\lambda}.
\]

Here \( \Gamma_{\mu\nu} (x) \) are coefficients of the connection which for simplicity (but not necessarily) may be taken to coincide with Christoffel symbols. The vector fields \( B_\alpha \) are horizontal in the fiber bundle \( \mathcal{B} \) (in respect to the given connection) and their restrictions on the fiber bundle \( \mathcal{N} \) of orthonormal local frames are horizontal in \( \mathcal{N} \).

1 Actually we need only these restrictions, but it is easier to introduce the field in \( \mathcal{B} \) first.

Let a curve \( [\xi] \) in the Minkowski space \( \mathcal{M} \) be given. The horizontal vector fields enable one to define, as an ordered exponential of an integral along this curve, the following operator acting in the space of functions on \( \mathcal{N} \):

\[
V[\xi] = P e^{\int B_\alpha d\xi^\alpha} = \lim_{N \to \infty} e^{B_\alpha \Delta \xi_1^\alpha} \cdots e^{B_\alpha \Delta \xi_N^\alpha}.
\]

In terms of these operators, the mapping \( n \to n[\xi] \) may be defined as follows:

\[
(V[\xi] \Psi)(n) = \Psi(n[\xi]).
\]
This mapping associate the end of the curve \([n]\) with its initial point. Applying the same procedure to an initial part of the curve \([\xi]\) (instead of the complete curve), we may obtain also an arbitrary intermediate point of the curve \([n]\). The projection of this curve onto the space-time \(X\) (the base of the fiber bundle \(N\)) gives the curve \([x]\) for which \([\xi]\) is a flat model.

This definition may be formulated also in terms of differential equations. The curve \([x]\) in the curved space \(X\), the corresponding curve \([n]\) in the fiber bundle \(N\) and the flat model \([\xi]\) of the curve \([x]\) (i.e. the corresponding curve in Minkowski space \(M\)) are connected by the following differential equations:

\[
\begin{align*}
\dot{x}^\mu(\tau) &= \dot{\xi}^\alpha(\tau)n^\mu_\alpha(\tau), \\
\dot{n}^\lambda_\beta(\tau) &= -\dot{x}^\mu(\tau)n^\mu_\beta(\tau)\Gamma^\lambda_\mu\nu(x(\tau)).
\end{align*}
\]

(4)

All local frames \(n(\tau)\) belonging to the curve \([n]\) are automatically orthonormal provided that the local frame in the initial point is chosen to be orthonormal.

### 3.2 Path Group, Holonomy Subgroup and reconstruction of the space-time geometry

Geometry of a (curved) space-time \(X\) can be reconstructed if all elements of the so-called Holonomy Subgroup of the Generalized Poincaré Group are known. In the simplified language used in this paper this means that the flat models and compensating Lorentz transformations are known for all closed curves beginning in an (arbitrary) fixed point \(x_0\). We shall present here the general scheme of the definitions, the details may be found in \([7, 8]\).

Generalized Poincaré Group \(Q\) is defined as a semidirect product of the Lorentz group \(\Lambda\) and the Path Group \(P\), the latter being a generalization of the translation group, consisting of classes of curves in Minkowski space (flat models in our context). A generic element \(q\) of \(Q\) may be presented as a product, \(q = p\lambda\), of a path \(p \in P\) and a Lorentz transformation \(\lambda \in \Lambda\). Products of any elements of \(Q\) are defined unambiguously if the following relation is accepted:

\[
\lambda[\xi]\lambda^{-1} = [\lambda\xi].
\]

The action of paths \(p \in P\) on the fiber bundle \(N\) of orthonormal local frames is defined above (Sect. 3.1), and Lorentz transformations act on \(N\) as a structure group. Therefore, the action of any element of \(Q\) on \(N\) is defined naturally as \(q : n \to nq = (np)\lambda\).

Let us choose an arbitrary orthonormal local frame \(n_0 \in N\) which will play the role of ‘origin’. Define the Holonomy Subgroup \(H \subset Q\) as the set of elements conserving \(n_0\):

\[
H = \{ h \in Q | n_0h = n_0 \}.
\]

Each element \(h \in H\) has the form \(h = p\lambda\) where \(p \in P\) is a flat model of some closed curve in the curved space-time \(X\) and \(\lambda \in \Lambda\) a corresponding compensating Lorentz transformation.

If one knows the whole subgroup \(H\) (i.e. flat models and compensating Lorentz transformations for all closed curves in \(X\) with the given initial point), then the geometry of \(X\) can be completely reconstructed. The procedure of reconstruction consists of the following steps.

- The fiber bundle of orthonormal local frames, \(N = H\backslash Q\), is a quotient space of the group \(Q\) in respect to the subgroup \(H\) so that a local frame is a coset, \(n = Hq = \{hq | h \in H\}\).

\(^2\)The same definition but with another choice of \(n_0\) would give an isomorphic Holonomy Subgroup.
The space-time $\mathcal{X}$ is a quotient space, $\mathcal{X} = \mathcal{N}/\Lambda$, of the fiber bundle $\mathcal{N}$ by its structure group $\Lambda$ so that a point of this space is a set (fiber) of local frames, $x = n\Lambda = \{n\lambda | \lambda \in \Lambda\}$. Therefore, the mapping $\pi : n \rightarrow x = n\Lambda$ is the canonical projection of the fiber bundle $\mathcal{N}$ onto its base $\mathcal{X}$.

In order to define the metric on the space $\mathcal{X}$, consider an arbitrary (orthonormal) local frame $n \in \mathcal{N}$ and the neighboring local frames $n p(a)$ which are obtained from $n$ by action of short straight paths $p(a) \in P$, coinciding with the four-vectors $a$. Then the interval between the point $x = \pi(n)$ and $x + dx = \pi(n p(da))$ is equal to $ds^2 = \eta_{\alpha\beta} da^\alpha da^\beta$.

4 Examples

Consider two examples of gravitational effects analyzed with the help of paths (flat models of curves). In each case we shall consider closed curves characteristic for the observation and construct their flat models (elements of holonomy). The first example is the well known experiment on light bending near Sun. The second example is gravitational lensing.

4.1 Light bending in terms of paths

The effect of light bending by Sun is easily analyzed by the conventional method. We shall speak about this effect only to illustrate our method. In the observation of light bending the direction of the light ray issued by a star turns out to be different in two cases: when this ray passes near Sun and when it passes far from Sun. It is shown at Fig. 3 how the observer construct flat models of the world lines of the two photons seen in the two points of his world line (flying correspondingly near Sun and far from it). For the analysis of the gravitational effect a closed curve is necessary. It may be formed by the world lines of the two photons together with the world lines of the observer and the star.

In order to obtain a flat model of this curve, one needs to assume that the only gravitational effect is caused by Sun deflecting the photon. If this hypothesis is accepted, then the flat model

Figure 3: Light bending by Sun and the corresponding element of holonomy.
of the closed curve may be reconstructed as it is shown in Fig. 3b. The flat model of the closed
curve (i.e. the path \( p \) in the expression \( h = p\lambda \) of the element of the Holonomy Subgroup) is
also a closed curve consisting of two light-like lines and two world lines of the bodies at rest.

The observed effect is a consequence of the fact that the local frame is rotated during its
parallel transport along the closed curve. To make the local frame identical to the initial one, it
has to be rotated after the parallel transport. This is a compensating Lorentz transformation
\( \lambda \) in the expression \( h = p\lambda \) of the element of holonomy.

### 4.2 Gravitational lensing

In case of gravitational lensing the observer sees light issued by some object (a star) as if coming
from two (or more) different directions (Fig. 4a). Trajectories of photons issued by the star are
curved because of being attracted by a heavy object located between the star and the observer.
As a result, the observer sees two separate light spots in different directions. The flat model of
the world lines of the two photons passing the attracting object from different sides will be as
in Fig. 4b: two light-like lines belonging to the past light cone of the observer.

However, with the help of the analysis of the radiation the observer knows that the two
photons are in fact issued simultaneously from the same point. The result of this analysis is
that the passage along one of these lines to the past leads to the event of radiation of the photon
by the star. Passing from this event along the world line of the second photon to the future
returns to the initial event of the perception of both photons by the observer.

The line consisting of two world lines of two photons is therefore closed. Its flat model is
shown in Fig. 4c and presents the path \( p \) in some element of holonomy \( h = p\lambda \). The above
arguments do not give the value of the compensating Lorentz transformation. One may assume
that \( \lambda = 1 \) so that the local frame parallely transported along the given closed curve will be
identical with the initial one. In principle astrophysical data may give the information about
\( \lambda \) too, for example if the source issues polarized photons.

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\(^3\)Actually the path \( p \) is non-closed, but this effect is very small and may be neglected because the distance
from the observer to the star is much longer than to Sun.
We shortly discussed here the way of objective presentation of observations of gravitational effects. The analysis was based on the mapping of curves in a curved space-time onto curves (paths) in Minkowski space. The most adequate development of such a formalism is presented in terms of Path Group, Generalized Poincaré group and Holonomy Subgroup [7, 8]. In correct terms of this formalism the observational data provide some elements of the Holonomy Subgroup, but we preferred to use also a simple image of ‘flat models’ of closed curves in a curved space-time (but a more rigorous terminology in Sect. 3).

It is important that the real observations give only some elements of holonomy (flat models of some closed curves) while the precise reconstruction of the geometry is possible only from the knowledge of the whole Holonomy Subgroup (flat models of all closed curves). Knowing a restricted set of elements of holonomy (as they are obtained from the observations), one may point out a class of geometries having these elements in its Holonomy Subgroup. The presentation of this class by paths and compensating Lorentz transformations (as in the examples considered here) is adequate because it is not based on any a priori choice of the class of geometries.

Let us remark that the formalism of Path Group and Generalized Poincaré Group has also other applications, among them the application to Quantum Equivalence Principle [9]. This gives an evidence that the formalism is natural in gravity and particularly in quantum gravity.

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