Gravitational energy of rotating black holes

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Abstract

In the teleparallel equivalent of general relativity the energy density of asymptotically flat gravitational fields can be naturally defined as a scalar density restricted to a three dimensional spacelike hypersurface Σ. Integration over the whole Σ yields the standard ADM energy. Here we obtain the formal expression of the localized energy for a Kerr black hole. The expression of the energy inside a surface of constant radius can be explicitly calculated in the limit of small $a$, the specific angular momentum. Such expression turns out to be exactly the same as the one obtained by means of the method proposed recently by Brown and York. We also calculate the energy contained within the outer horizon of the black hole, for any value of $a$. The result is practically indistinguishable from $E = 2M_{ir}$, where $M_{ir}$ is the irreducible mass of the black hole.

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I. Introduction

Although it is widely believed that Einstein’s equations describe the dynamics of the gravitational field, it has not been possible so far to arrive at a definite expression for the gravitational energy in the context of Einstein’s general relativity. Attempts based on the Hilbert-Einstein action integral fail to yield an expression for the gravitational energy density\(^1\), \(^2\). The total gravitational energy is normally obtained from surface terms in the action or in the Hamiltonian\(^3\), \(^4\), or from pseudotensor methods which make use of coordinate dependent expressions.

Recently an expression for quasi-local energy has been proposed by Brown and York\(^5\). Such expression is derived directly from the action functional \(A_{cl}\). The latter is identified as Hamilton’s principal function and, in similarity with the classical Hamilton-Jacobi equation, which expresses the energy of a classical solution as minus the time rate of the change of the action, the quasilocal gravitational energy is identified as minus the proper time rate of change of the Hilbert-Einstein action (with surface terms included). Expressions for the quasilocal energy have been obtained for the Schwarzschild solution\(^5\) and for the Kerr solution\(^5\).

Einstein’s equations can also be obtained from the teleparallel equivalent of general relativity (TEGR). The Lagrangian formulation of the TEGR is established by means of the tetrad field \(e^a_\mu\) and the spin affine connection \(\omega_{\mu ab}\), which are taken to be completely independent field variables, even at the level of field equations. This formulation has been investigated in the past in the context of Poincaré gauge theories\(^7\), \(^8\). However, as we will explain ahead, this is not an alternative theory of gravity. This is just an alternative formulation of general relativity, in which the curvature tensor constructed out of \(\omega_{\mu ab}\) vanishes, but the torsion tensor is non-vanishing. The physical content of the
theory is dictated by Einstein’s equations. In this alternative geometrical formulation the gravitational energy density can be naturally defined.

The expression for the gravitational energy density arises in the framework of the Hamiltonian formulation of the TEGR. It has been demonstrated that under a suitable gauge fixing of $\omega_{\mu ab}$, already at the Lagrangian level, the Hamiltonian formulation of the TEGR is well defined. The resulting constraints are first class constraints. The Hamiltonian formulation turns out to be very much similar to the usual ADM formulation. However there are crucial differences. The integral form of the Hamiltonian constraint equation $C = 0$ in the TEGR can be written in the form $C = H - E_{\text{ADM}} = 0$, when we restrict considerations to asymptotically flat spacetimes. The quantity $\varepsilon(x)$ which appears in the expression of $C$ and which under integration yields $E_{\text{ADM}}$ is recognized as the gravitational energy density.

We have calculated the energy inside a sphere of radius $r_o$ in a Schwarzschild spacetime by means of $\varepsilon(x)$. The expression turns out to be exactly the same as the one obtained by means of the procedure of ref. (expression (6.14) of ). In this paper we consider the Kerr black-hole. We obtain the formal expression for the energy contained in any space volume in terms of non-trivial integrals. In the limit of slow rotation (small specific angular momentum) the energy contained within a surface of constant radius $r_o$ can be calculated. Again the result obtained here is exactly the same as that obtained by Martinez who adopted Brown and York’s procedure. The advantage of our procedure rests on the fact that the localized energy associated with a Kerr spacetime can be calculated in the general case, without recourse to particular limits, at least by means of numerical integration, whereas in Brown and York’s procedure one has to calculate the subtraction term $\varepsilon^0$ and for this purpose it is necessary to embed an arbitrary two dimensional boundary surface of the Kerr space $\Sigma$ in the appropriate reference space ($E^3$, say), which is not always
We have also calculated the energy contained within the outer horizon of the black hole. Such a quantity has been obtained by Martinez in the limit of small $a$, and reads $E = 2M_{ir}$ (plus corrections of order $O(a^4 M_{ir}^4)$), where $M_{ir}$ is the irreducible mass of the black hole. The concept of irreducible mass was introduced by Christodoulou. He showed that the mass of a rotating black hole cannot be decreased to values below $M_{ir}$ by means of Penrose’s process of extraction of energy. One would thus consider $E = 2M_{ir}$ to be the energy that cannot escape from the black hole. Here we obtain the expression of the energy contained within the horizon for any value of $a$. The result is striking. The numerical values of this expression are practically coincident with $2M_{ir}$ in the whole range $0 \leq a \leq m$, although the expression is algebraically different from $2M_{ir}$.

In section II we present the mathematical preliminaries of the TEGR, its Hamiltonian formulation and the expression of the energy for an arbitrary asymptotically flat spacetime. In section III we carry out the construction of triads for a three dimensional spacelike hypersurface of the Kerr type, obtain the general expression of the energy contained in a volume $V$ of space and provide the exact expression of the latter in the limit of slow rotation. Comments and conclusions are presented on section IV.

Notation: spacetime indices $\mu, \nu, ...$ and local Lorentz indices $a, b, ...$ run from 0 to 3. In the 3+1 decomposition latin indices from the middle of the alphabet indicate space indices according to $\mu = 0, i, \quad a = (0), (i)$. The tetrad field $e^a_\mu$ and the spin connection $\omega_{\mu ab}$ yield the usual definitions of the torsion and curvature tensors: $R^a_{\;b\mu\nu} = \partial_\mu \omega^a_{\nu b} + \omega^a_{\mu c} \omega^c_{\nu b} - ...$, $T^a_{\;\mu\nu} = \partial_\mu e^a_{\;\nu} + \omega^a_{\mu b} e^b_{\;\nu} - ...$. The flat spacetime metric is fixed by $\eta_{(0)(0)} = -1$. 

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II. The TEGR in Hamiltonian form

In the TEGR the tetrad field $e^{a \mu}$ and the spin connection $\omega_{\mu ab}$ are completely independent field variables. The latter is enforced to satisfy the condition of zero curvature. The Lagrangian density in empty spacetime is given by

$$L(e, \omega, \lambda) = -ke\left(\frac{1}{4}T^{abc}T_{abc} + \frac{1}{2}T^{abc}T_{bac} - T^{a}T_{a}\right) + e\lambda^{ab\mu}R_{ab\mu\nu}(\omega).$$  \hspace{1cm} (1)

where $k = \frac{1}{16\pi G}$, $G$ is the gravitational constant; $e = \det(e^{a \mu})$, $\lambda^{ab\mu}$ are Lagrange multipliers and $T_{a}$ is the trace of the torsion tensor defined by $T_{a} = T^{b}_{\ b a}$.

The equivalence of the TEGR with Einstein’s general relativity is based on the identity

$$eR(e, \omega) = eR(e) + e\left(\frac{1}{4}T^{abc}T_{abc} + T^{abc}T_{acb} - T^{a}T_{a}\right) - 2\partial_{\mu}(eT^{\mu}),$$  \hspace{1cm} (2)

which is obtained by just substituting the arbitrary spin connection $\omega_{\mu ab} = \omega_{\mu ab}(e) + K_{\mu ab}$ in the scalar curvature tensor $R(e, \omega)$ in the left hand side; $\omega_{\mu ab}(e)$ is the Levi-Civita connection and $K_{\mu ab} = \frac{1}{2}e^{a \lambda}e^{b \nu}(T_{\lambda\mu\nu} + T_{\nu\lambda\mu} - T_{\mu\nu\lambda})$ is the contorsion tensor. The vanishing of $R^{a \ b}_{\ mu}(\omega)$, which is one of the field equations derived from (1), implies the equivalence of the scalar curvature $R(e)$, constructed out of $e^{a \mu}$ only, and the quadratic combination of the torsion tensor. It also ensures that the field equation arising from the variation of $L$ with respect to $e^{a \mu}$ is strictly equivalent to Einstein’s equations in tetrad form. Let $\frac{\delta L}{\delta e^{a \mu}} = 0$ denote the field equations satisfied by $e^{a \mu}$. It can be shown by explicit calculations that

$$\frac{\delta L}{\delta e^{a \mu}} = \frac{1}{2}\{R_{a\mu}(e) - \frac{1}{2}e_{a \mu}R(e)\}.$$  \hspace{1cm} (3)

(we refer the reader to ref.[9] for additional details).
It is important to notice that for asymptotically flat spacetimes the total divergence in (2) does not contribute to the action integral. This term is a scalar density that falls off as $\frac{1}{r^3}$ when $r \to \infty$. In this limit we should consider variations in $g_{\mu\nu}$ or in $e_{a\mu}$ that preserve the asymptotic structure of the flat spacetime metric; the allowed coordinate transformations must be of the Poincaré type. The variation of $\partial_{\mu}(eT^\mu)$ at infinity under such variations of $e_{a\mu}$ vanishes. Moreover all surface integrals arising from partial integration in the variation of the action integral vanish as well. Therefore the action does not require additional surface terms, as it is invariant under transformations that preserve the asymptotic structure of the field quantities. This property fixes the action integral, together with the requirement that the variation of the latter must yield Einstein’s equations (the Hilbert-Einstein Lagrangian requires the addition of a surface term for the variation of the action to be well defined; a clear discussion of this point is given in ref. [12]). In what follows we will be interested in asymptotically flat spacetimes.

The Hamiltonian formulation of the TEGR can be successfully implemented if we fix the gauge $\omega_{0ab} = 0$ from the outset, since in this case the constraints (to be shown below) constitute a first class set [9]. The condition $\omega_{0ab} = 0$ is achieved by breaking the local Lorentz symmetry of (1). We still make use of the residual time independent gauge symmetry to fix the usual time gauge condition $e_{(k)}^0 = e_{(0)k} = 0$. Because of $\omega_{0ab} = 0$, $H$ does not depend on $P^{kab}$, the momentum canonically conjugated to $\omega_{kab}$. Therefore arbitrary variations of $L = p\dot{q} - H$ with respect to $P^{kab}$ yields $\dot{\omega}_{kab} = 0$. Thus in view of $\omega_{0ab} = 0$, $\omega_{kab}$ drops out from our considerations. The above gauge fixing can be understood as the fixation of a global reference frame.

Under the above gauge fixing the canonical action integral obtained from (1) becomes [3]

$$A_{TL} = \int d^4x \{ \Pi^{(j)k} \dot{e}_{(j)k} - H \}, \quad (4)$$
\[ H = NC + N^i C_i + \Sigma_{mn} \Pi^{mn} + \frac{1}{8\pi G} \partial_k (Ne^{T_k}) + \partial_k (\Pi^{jk} N_j) . \]  

(5)

\[ N \text{ and } N^i \text{ are the lapse and shift functions, and } \Sigma_{mn} = -\Sigma_{nm} \text{ are Lagrange multipliers.} \]

The constraints are defined by

\[ C = \partial_j (2ke^{T_j}) - ke^{kij} T_{kij} - \frac{1}{4ke} (\Pi^{ij} \Pi_{ji} - \frac{1}{2} \Pi^2) , \]  

(6)

\[ C_k = -e_{(j)k} \partial_i \Pi^{(j)i} - \Pi^{(j)k} T_{(j)ik} , \]  

(7)

with \( e = det(e_{(j)k}) \) and \( T^i = g^{ik} e^{(j)l} T_{(j)lk} \). We remark that (4) and (5) are invariant under global SO(3) and general coordinate transformations.

We assume the asymptotic behaviour \( e_{(j)k} \approx \eta_{jk} + \frac{1}{2} h_{jk} \left( \frac{1}{r} \right) \) for \( r \rightarrow \infty \). In view of the relation

\[ \frac{1}{8\pi G} \int d^3 x \partial_j (e^{T_j}) = \frac{1}{16\pi G} \int_S dS_k (\partial_i h_{ik} - \partial_k h_{ii}) \equiv E_{ADM} \]  

(8)

where the surface integral is evaluated for \( r \rightarrow \infty \), the integral form of the Hamiltonian constraint \( C = 0 \) may be rewritten as

\[ \int d^3 x \left\{ ke^{kij} T_{kij} + \frac{1}{4ke} (\Pi^{ij} \Pi_{ji} - \frac{1}{2} \Pi^2) \right\} = E_{ADM} . \]  

(9)

The integration is over the whole three dimensional space. Given that \( \partial_j (e^{T_j}) \) is a scalar density, from (7) and (8) we define the gravitational energy density enclosed by a volume \( V \) of the space as

\[ E_g = \frac{1}{8\pi G} \int_V d^3 x \partial_j (e^{T_j}) . \]  

(10)
It must be noted that $E_g$ depends only on the triads $e_{(k)i}$ restricted to a three-dimensional spacelike hypersurface; the inverse quantities $e^{(k)i}$ can be written in terms of $e_{(k)i}$. From the identity (3) we observe that the dynamics of the triads does not depend on $\omega_{\mu ab}$. Therefore $E_g$ given above does not depend on the fixation of any gauge for $\omega_{\mu ab}$.

### III. Energy of the Kerr geometry

The Kerr solution\[13\] describes the field of a rotating black hole. In terms of Boyer and Lindquist coordinates\[14\] ($t, r, \theta, \phi$) it is described by the metric

$$
\begin{align*}
\text{ds}^2 &= - \frac{\Delta}{\rho^2} [dt - a \sin^2 \theta d\phi]^2 + \frac{\sin^2 \theta}{\rho^2} [(r^2 + a^2) d\phi - a dt]^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2, \\
\Delta &= r^2 - 2mr + a^2, \\
\rho^2 &= r^2 + a^2 \cos^2 \theta;
\end{align*}
$$

(11)

where $a$ is the specific angular momentum defined by $a = \frac{J}{m}$. The components of the metric restricted to the three dimensional spacelike hypersurface are given by $g_{11} = \frac{\Delta}{\rho^2}$, $g_{22} = \rho^2$ and $g_{33} = \frac{\Sigma^2}{\rho^2} \sin^2 \theta$, where $\Sigma$ is defined by

$$
\Sigma^2 = (r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta.
$$

We define the triads $e_{(k)i}$ as

$$
e_{(k)i} = 
\begin{pmatrix}
\frac{\rho}{\sqrt{\Delta}} \sin \theta \cos \phi & \rho \cos \theta \cos \phi & -\frac{\Sigma}{\rho} \sin \theta \sin \phi \\
\frac{\rho}{\sqrt{\Delta}} \sin \theta \sin \phi & \rho \cos \theta \sin \phi & \frac{\Sigma}{\rho} \sin \theta \cos \phi \\
\frac{\rho}{\sqrt{\Delta}} \cos \theta & -\rho \sin \theta & 0
\end{pmatrix}
$$

(12)
(k) is the line index and i is the column index. The one form \( e^{(k)} \) is defined by

\[
e^{(k)} = e^{(k)}_r dr + e^{(k)}_\theta d\theta + e^{(k)}_\phi d\phi,
\]

from what follows

\[
e^{(k)}e_{(k)} = \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{\Sigma^2}{\rho^2} \sin^2 \theta d\phi^2.
\]

We also obtain \( e = \det(e_{(k)i}) = \frac{\rho \Sigma}{\sqrt{\Delta}} \sin \theta \). Therefore the triads given by (12) describe the components of the Kerr solution restricted to the three dimensional spacelike hypersurface.

One readily notices that there is another set of triads that yields the Kerr solution, namely, the set which is diagonal and whose entries are given by the square roots of \( g_{ii} \). This set is not appropriate for our purposes, and the reason can be understood even in the simple class of flat spacetime. In the limit when both \( a \) and \( m \) go to zero (12) describes flat space: the curvature tensor and the torsion tensor vanish in this case. However, for the diagonal set of triads (again requiring \( a \to 0 \) and \( m \to 0 \)),

\[
e^{(r)} = dr \ , \ e^{(\theta)} = r \ d\theta \ , \ e^{(\phi)} = r \sin \theta \ d\phi,
\]

some components of the torsion tensor do not vanish, \( T_{(2)12} = 1 \), \( T_{(3)13} = \sin \theta \), and \( E_g \) calculated out of the diagonal set above diverges when integrated over the whole space. Therefore the use of (12) is mandatory in the present context.

The components of the torsion tensor can be calculated in a straightforward way from (12). Only \( T_{(3)13} \) and \( T_{(3)23} \) are vanishing. The others are given by:

\[
T_{(1)12} = \cos \theta \cos \phi \left( \frac{r}{\rho} + \frac{a^2}{\rho \sqrt{\Delta}} \sin^2 \theta - \frac{\rho}{\sqrt{\Delta}} \right)
\]
\[ T_{(1)13} = \sin\theta \sin\phi \left\{ -\frac{1}{\rho \Sigma} [2r(r^2 + a^2) - a^2 \sin^2 \theta (r - m)] + \frac{r \Sigma}{\rho^3} + \frac{\rho}{\sqrt{\Delta}} \right\} \]

\[ T_{(1)23} = \cos\theta \sin\phi \left\{ \rho - \frac{\Sigma}{\rho} + a^2 \sin^2 \theta \left( \frac{\Delta}{\rho \Sigma} - \frac{\Sigma}{\rho^3} \right) \right\} \]

\[ T_{(2)12} = \cos\theta \sin\phi \left( \frac{r}{\rho} + \frac{a^2}{\rho \sqrt{\Delta}} \sin^2 \theta - \frac{\rho}{\sqrt{\Delta}} \right) \]

\[ T_{(2)13} = -\sin\theta \cos\phi \left\{ -\frac{1}{\rho \Sigma} [2r(r^2 + a^2) - a^2 \sin^2 \theta (r - m)] + \frac{r \Sigma}{\rho^3} + \frac{\rho}{\sqrt{\Delta}} \right\} \]

\[ T_{(2)23} = -\cos\theta \cos\phi \left\{ \rho - \frac{\Sigma}{\rho} + a^2 \sin^2 \theta \left( \frac{\Delta}{\rho \Sigma} - \frac{\Sigma}{\rho^3} \right) \right\} \]

\[ T_{(3)12} = \sin\theta \left[ -\frac{r}{\rho} + \frac{\rho}{\sqrt{\Delta}} + \frac{a^2}{\rho \sqrt{\Delta}} \cos^2 \theta \right] \]

In order to evaluate (9) we need to obtain \( T^i \). After a long calculation we arrive at

\[ T^1 = \frac{\sqrt{\Delta}}{\rho^2} + \frac{\sqrt{\Delta}}{\Sigma} - \frac{\Delta}{\rho^2 \Sigma^2} [2r(r^2 + a^2) - a^2 \sin^2 \theta (r - m)] , \]

\[ T^2 = \sin\theta \cos\theta \frac{a^2}{\rho^4} + \frac{1}{\rho \Sigma} \frac{\cos\theta}{\sin \theta} \left[ \rho - \frac{\Sigma}{\rho} + a^2 \sin^2 \theta \left( \frac{\Delta}{\rho \Sigma} - \frac{\Sigma}{\rho^3} \right) \right] , \]

\[ T^3 = 0 . \]

The gravitational energy density inside a volume \( V \) of a three dimensional spacelike hypersurface of the Kerr solution can now be easily calculated. It is given by

\[ E_g = \frac{1}{8\pi} \int_V d\rho \, d\theta \, d\phi \left\{ \frac{\partial}{\partial r} \left[ \sin \theta \left[ \frac{\Sigma}{\rho} + \frac{\sqrt{\Delta}}{\rho \Sigma} \left( 2r(r^2 + a^2) - a^2 \sin^2 \theta (r - m) \right) \right] \right] \]
\[ + \frac{\partial}{\partial \theta} \left[ \frac{\Sigma}{\sqrt{\Delta \rho^3}} \sin^2 \theta \cos \theta + \frac{\cos \theta}{\sqrt{\Delta}} \left( \rho - \frac{\Sigma}{\rho} + a^2 \sin^2 \theta \left( \frac{\Delta}{\rho \Sigma} - \frac{\Sigma}{\rho^3} \right) \right) \right] \]  \tag{13}

Next we specialize \( E_g \) to the case when the volume \( V \) is contained within a surface with constant radius \( r = r_o \) assuming \( r_o \geq r_+ \), where \( r_+ = m + \sqrt{m^2 - a^2} \) is the outer horizon of the black hole. The integrations in \( \phi \) and \( r \) are trivial. Also, because we integrate \( \theta \) between 0 and \( \pi \), the second line of the expression above vanishes. We then obtain

\[
E_g = \frac{1}{4} \int_0^\pi d\theta \sin \theta \left\{ \rho + \frac{\Sigma}{\rho} - \frac{\sqrt{\Delta}}{\rho \Sigma} \left( 2r(r^2 + a^2) - a^2 \sin^2 \theta (r - m) \right) \right\}_{r=r_o}. \tag{14}
\]

We have not managed to evaluate exactly the integral above. However, in the limit of slow rotation, namely, when \( \frac{a}{r_o} \ll 1 \) all integrals have a simple structure and we can obtain the approximate expression of \( E_g \). It reads

\[
E_g = r_o \left( 1 - \sqrt{1 - \frac{2m}{r_o} + \frac{a^2}{r_o^2}} \right) + \frac{a^2}{6r_o} \left[ 2 + \frac{2m}{r_o} + \left( 1 + \frac{2m}{r_o} \right) \sqrt{1 - \frac{2m}{r_o} + \frac{a^2}{r_o^2}} \right]. \tag{15}
\]

This is exactly the expression found by Martinez\cite{Martinez} for the energy inside the surface of constant radius \( r_o \) in a spacelike hypersurface of a Kerr black hole, in the limit of small specific angular momentum. As in ref.\cite{Martinez}, we have not expanded the square root which appears in (15) in powers in \( \frac{a^2}{r_o} \).

We remark that the expansion of \( \rho + \frac{\Sigma}{\rho} \) in the integrand of (14) yields \( -\varepsilon_0 \), whereas the remaining term corresponds exactly to \( \varepsilon \), expressions (3.17) and (3.1) respectively of \cite{Martinez}. It does not seem to be possible, however, to split \( \partial_i (eT^i) \) into two terms such that their integrals arise in the form \( \varepsilon - \varepsilon_0 \).

As a very interesting application of (14), let us calculate the energy contained within the outer horizon, i.e., we will calculate (14) when the surface of constant radius is defined
by $r_o = r_+$. This surface is characterized by $\Delta = 0$. The integral can be calculated exactly for any value of $a$. The latter is parametrized in terms of the black hole mass $m$ according to

$$a = km, \quad 0 \leq k \leq 1.$$  

After a number of integrations we arrive at

$$E_g = m\left[\frac{\sqrt{2p}}{4} + \frac{6p - k^2}{4k} \ln\left(\frac{\sqrt{2p} + k}{p}\right)\right], \quad (16)$$

where $p$ is defined by

$$p = 1 + \sqrt{1 - k^2}.$$  

This is the amount of energy expected not to escape from the black hole by any process in which the black hole interacts with external particles. It is then important to compare (16) with $E = 2M_{ir}$.

We recall that a rotating black hole can have its mass decreased by means of Penrose’s process of extraction of energy [15]. The idea is the following. We consider a particle that is emitted towards the black hole and penetrates into the ergosphere. Suppose we arrange the particle to break up into two fragments, in such a way that one of the fragments has total negative energy. This is possible in principle, since the energy need not be positive in the ergosphere. By conservation of energy, the fragment with positive energy has an energy greater than that of the incident particle. Thus energy will be extracted from the black hole if the positive energy particle escapes to infinity, while the black hole absorbs the negative energy one. As a consequence, the mass of the black hole is decreased. We expect, however, that not the whole energy of the black hole can be extracted in this
manner. The existence of the horizon certainly prevents one from exhausting the total energy. Christodoulou\cite{11} has given an argument to determine how much energy can be extracted from the black hole by Penrose’s process. He concluded that at the end of this process (when the ergosphere disappears and the black hole becomes static) the final (irreducible) mass of the black hole is given by

\[ M_{ir} = \frac{1}{2} \sqrt{r_+^2 + a^2}. \]

Martinez\cite{6} has calculated the energy inside the horizon of the Kerr black hole in the limit of small \( a \). He arrived at \( E = 2M_{ir}[1 + O(\frac{a^4}{M_{ir}^4})] \). A similar result (in the same approximation) has been obtained by Zaslavskii\cite{16} in the analysis of a generic axially-symmetric spacetime. The question immediately arises as to whether this relationship holds for any value of \( a \). This is in fact the conjecture made in ref.\cite{6}.

Since expressions (14) and (16) are valid for any value of \( a \) in the appropriate range, it is worth comparing (16) with \( 2M_{ir} \). In our parametrization we have

\[ E = 2M_{ir} = m \sqrt{2p}. \] (17)

The expression above certainly looks different from (16). However in the range \( 0 \leq a \leq m \) expressions (16) and (17) as functions of \( k \) are strikingly indistinguishable, as we can see in Fig.1. In the latter we have plotted \( \frac{E}{m} \) against \( k \). The upper curve represents (16), the lower one (17). We see that for small values of the parameter \( k \) the two curves are essentially coincident. A tiny deviation occurs for values of \( k \) near 1. Inspite of this deviation, this is a remarkable result in favour of (14).

Unfortunately we have not been able to explain such small deviation between (16) and (17) for values of \( k \) near 1, although we expect such explanation to be of fundamental
importance. It might be related to some physical property of the Kerr black hole which we do not understand yet.

Comments

The gravitational energy \( E_g \) defined by (14) can be evaluated for an arbitrary value of \( a \) by means of numerical integration. This is the major advantage of our procedure as compared to that of Brown and York\([5]\). By means of the latter one cannot construct expressions like (13) and (14), which may be useful in the study of astrophysical problems, since in a general situation Brown and York's procedure requires the embedding of an arbitrary two dimensional boundary surface of the Kerr space in the reference space \( E^3 \), a construction which is not possible in general\([6]\) (the evaluation of \( \varepsilon_0 \) in \([6]\) is only possible in the limit \( \frac{a}{\varepsilon_0} <<< 1 \)). Therefore the present approach is more general than that of ref.\([6]\). Finally we remark that we expect expression (10) to be useful in the study of the thermodynamics of self-gravitating systems, where the gravitational energy plays the role of the thermodynamical internal energy that is conjugate to the inverse temperature. We hope to come to this issue in the future.

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**Figure Captions**

Figure 1: $\frac{E}{m}$ against $k$. The upper curve represents the energy expression (16), the lower one (17).
