Space-time chaos in the nonlinear Schrödinger equation

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Abstract. The paper provides an analytical and numerical analysis of the transition to space-time chaos in the generalized nonlinear Schrödinger equation with complex, in the general case, parameters. Equation describes, in particular, the wave amplitude of a surface plasmon polariton propagating over the contact surface of a metal with a dielectric. It is proved that considered equation has an infinite number of different stable wave solutions running along the spatial axis with arbitrary velocities, as well as an infinite number of different modes of space-time chaos in full accordance with the universal Feigenbaum-Sharkovsky-Magnitskii bifurcation theory. In this case, the bifurcation parameter is the value of the speed of propagation of traveling waves along the spatial axis, which is clearly not included in the original equation.

1. Introduction
In papers [1-2], a nonlinear equation was derived that describes the dynamics of propagation of surface plasmon-polaritons on the metal-dielectric surface

\[ i \frac{\partial \psi}{\partial z} + \frac{1}{2\beta} \frac{\partial^2 \psi}{\partial y^2} + \left[i(l - g) + \Delta \beta\right] \psi + f \varepsilon \left| \psi \right|^2 \psi = 0. \]  

(1.1)

where \( \psi \) is a complex-valued function of the surface variables \( y \) and \( z \), which describes the amplitude of the surface plasmon-polariton wave propagating over the metal-dielectric contact surface. The \( \beta \) parameter is dispersion constant; dimensionless parameters \( g, l \) denote gain and loss factors, respectively; \( \Delta \beta \) is the shift between the wave propagation constants of the surface plasmon polariton, \( f \) is the scaling factor. All five parameters \( g, l, f, \beta, \Delta \beta \) are real constants. Finally, Kerr’s law is represented by the complex parameter \( \varepsilon = \varepsilon' + i \varepsilon'' \), which is responsible for the nonlinear part of the equation.

Surface plasmon polaritons provide one of the preferred approaches to implementing in-process photonic devices and are a well-established tool for sensor applications. In this regard, it is very important to study the dynamics of solutions of equation (1.1) and its various generalizations. Strictly speaking, in equation (1.1) there is no time derivative and thus it formally differs from the nonlinear Schrödinger equation with nonlinearity according to Kerr’s law. However, if the variable \( z \) plays the role of time, then equation (1.1) can be considered as a special case of the equation.
which is a generalized nonlinear Schrödinger equation with complex, in the general case, parameters $c_1, c_2, c_3$. In this paper, we consider the generalized nonlinear Schrödinger equation (1.2) in the case $c_2 = -c_3$. It is shown that, for certain values of its parameters, equation (1.2) has an infinite number of different stable wave solutions traveling along the spatial axis with arbitrary velocities, and an infinite number of different regimes of space-time chaos in full accordance with the universal bifurcation theory of Feigenbaum-Sharkovsky-Magnitskii. Moreover, the bifurcation parameter is the value of the propagation velocity of traveling waves along the spatial axis, which is clearly not included in the original equation.

2. Reduction to an ODE system using self-similar change of variables

We represent $\psi(y, t) = u(y, t) + i v(y, t)$, $c_1 = a + ib$, $c_2 = d + ie$, $c_3 = -d - ie$, and consider equation (1.2) on the whole number axis: $u(y, t): \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $v(y, t): \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. We analyze the regular and chaotic solutions of equation (1.2) using a self-similar change of independent variables $\xi = y - ct$, thereby reducing the original equation (1.2) with partial derivatives to the system of nonlinear ordinary differential equations

\[ i \frac{\partial \psi}{\partial t} + c_1 \frac{\partial^2 \psi}{\partial y^2} + c_2 \psi + c_3 |\psi|^2 \psi = 0, \quad (1.2) \]

\[ + \frac{\partial \varphi}{\partial \xi} = 0, \]

\[ \frac{\partial \zeta}{\partial \xi} = 0, \]

where the derivative is taken over $\xi$. Solving system (2.1) with respect to the second derivatives $\ddot{u}$ and $\ddot{v}$ we obtain the system of equation

\[ \ddot{u} = \left( bc \dddot{u} - ac \ddot{v} + ((ad + be)u - (ae - bd)v)((u^2 + v^2) - 1) \right)/((a^2 + b^2)), \]

\[ \ddot{v} = \left( bc \dddot{v} + ac \ddot{u} - ((bd - ae)u - (ad + be)v)((u^2 + v^2) - 1) \right)/((a^2 + b^2)). \]

We reduce the system of equations (2.2) to a nonlinear system of four ordinary differential equations

\[ \dot{z} = \left( bcz - acw + ((ad + be)u - (ae - bd)v)((u^2 + v^2) - 1) \right)/((a^2 + b^2)), \]

\[ \dot{\psi} = \left( \right)/((a^2 + b^2)). \]

\[ \dot{w} = \left( bcw + acz - ((bd - ae)u - (ad + be)v)((u^2 + v^2) - 1) \right)/((a^2 + b^2)). \]

If $(u(\xi), z(\xi), v(\xi), w(\xi))$ is a solution to the ODE system (2.3), then $\psi(y, t) = u(y - ct) + iv(y - ct)$ will be a solution of the partial differential system (1.2). In this case, since the dimension of the system of equations (2.3) is equal to four, the traveling waves in equation (1.2) can be described either by limit cycles or by two-dimensional tori of system (2.3), the traveling impulses in equation (1.2) can be described by the saddle-foci separatrix loops of the system (2.3) or homoclinic separatrix structures that are Cartesian products of the saddle-foci separatrix loops of the system (2.3) and limit cycles. And the space-time chaos in the equation (1.2) is a cyclic or toroidal singular attractor of the ODE system (2.3) in the sense of the universal bifurcation theory of Feigenbaum-Sharkovsky-Magnitskii [3-10]. A necessary condition for the existence of an FShM cascade of bifurcations of two-dimensional tori in system (2.3) is the birth of an initial two-dimensional torus from a cycle as a result of the Andronov-Hopf bifurcation.
3. Analytical study of the ODE system

For nonlinear systems of ordinary differential equations, it is usually possible to analyse analytically their dissipativity or conservativity, the presence of singular points and their stability regions and, in rare cases, the conditions for the birth of a limit cycle as a result of the Andronov-Hopf bifurcation.

3.1. Dissipative region and singular points

Let us investigate the region of dissipativity of the system (2.3):

$$\frac{\partial F_1}{\partial u} + \frac{\partial F_2}{\partial z} + \frac{\partial F_3}{\partial v} + \frac{\partial F_4}{\partial w} = \frac{2bc}{a^2 + b^2} < 0$$

for $bc < 0$, where $F_i$ is the right-hand side of the $i$-th equation of the system (2.3), $i = 1, 2, 3, 4$; $F = (F_1, F_2, F_3, F_4)^T$. Since $c > 0$ by definition, it follows that the system of equations (2.3) is dissipative everywhere for $b < 0$. We find the singular points $(u_*, z_*, v_*, w_*)$ of the system (2.3), equating to zero the right-hand sides of its equations:

$$z_* = w_* = 0, ((ad + be)u_* + (bd - ae)v_*) = 0, ((bd - ae)u_* - (ad + be)v_*) = 0.$$

Consequently, system (2.3) has a single zero singular point $O(0, 0, 0, 0)$. To find the stability region of the singular point O, it is necessary to calculate the linearization matrix of system (2.3) at this singular point:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{(ad + be)}{(a^2 + b^2)} & \frac{bc}{(a^2 + b^2)} & \frac{-(bd - ae)}{(a^2 + b^2)} & \frac{-(ad + be)}{(a^2 + b^2)} \\ 0 & 0 & 0 & \frac{1}{(a^2 + b^2)} \\ \frac{(bd - ae)}{(a^2 + b^2)} & \frac{ac}{(a^2 + b^2)} & \frac{-(ad + be)}{(a^2 + b^2)} & \frac{-(bd - ae)}{(a^2 + b^2)} \end{pmatrix}.$$  

The characteristic equation corresponding to the matrix $A$ has the form:

$$\det(A - \lambda I) = \lambda^4 - \frac{2bc}{(a^2 + b^2)} \lambda^3 + \frac{2(ad + be) + c^2}{(a^2 + b^2)} \lambda^2 - \frac{2ec}{(a^2 + b^2)} \lambda + \frac{d^2 + e^2}{(a^2 + b^2)} = 0. \quad (3.1)$$

**Theorem 1.** A necessary and sufficient condition for the stability of the singular point $O$ of the system of equations (2.3) is the condition: $b < 0$, $e < 0$, $c > \sqrt{(bd - ae)^2/(be)}$.

**Proof.** The necessity of the condition: $b < 0$, $e < 0$ follows from the necessary condition for all coefficients of the characteristic polynomial (3.1) to be positive. If the condition $b < 0$, $e < 0$ is satisfied, then, by virtue of the Routh-Hurwitz criterion, a necessary and sufficient condition for the stability of the singular point $O$ is the fulfillment of two inequalities:

$$\Delta_2 = \frac{2bc}{(a^2 + b^2)} \frac{2(ad + be) + c^2}{(a^2 + b^2)} + \frac{2ec}{(a^2 + b^2)} \lambda > 0,$$

$$\Delta_3 = -\frac{2ec}{(a^2 + b^2)} \Delta_2 - \left(\frac{2bc}{(a^2 + b^2)} \lambda^2 \frac{d^2 + e^2}{(a^2 + b^2)} > 0. \right.$$
It follows from the first inequality that
\[ \Delta_2 = \frac{2c}{(a^2 + b^2)^2} (1 - 2a - b) > 0, \quad c > \frac{(b^2 - a^2)e + 2abd}{-b}, \]
and from the second inequality it follows that
\[ \Delta_3 = \frac{4c^2}{(a^2 + b^2)^2} (be - (bd - ae)^2) > 0, \quad c > \frac{(bd - ae)^2}{be}. \]
But since \((bd - ae)^2 > -e((b^2 - a^2)e + 2abd)\), then a necessary and sufficient condition for the stability of the singular point \(O\) is the condition
\[
b < 0, \quad e < 0, \quad c > \frac{(bd - ae)^2}{be} \tag{3.2}\]
The theorem is proved.

3.2. Andronov-Hopf bifurcation of birth of a cycle

It follows from Theorem 1 that of greatest interest is the study of possible cascades of bifurcations of a stable singular point \(O\) for \(b < 0, e < 0, c < \sqrt{(bd - ae)^2/(be)}\) in the case, when its first bifurcation is the Andronov-Hopf bifurcation of the birth of a stable limit cycle. It is in this case in system (2.3) that there may be a second Andronov-Hopf bifurcation, the birth of a stable two-dimensional torus, a cascade of bifurcations of two-dimensional tori, and the existence of an infinite number of chaotic singular toroidal attractors in accordance with the universal Feigenbaum-Sharkovsky-Magnitskii bifurcation theory. In this case, the generalized nonlinear Schrödinger equation (1.2) can have an infinite family of the most complex periodic and non-periodic (chaotic) traveling waves up to traveling pulses. Let us prove that under the condition \(b < 0, e < 0, c < \sqrt{(bd - ae)^2/(be)}\) the Andronov-Hopf bifurcation actually occurs in system (2.3), that is, the soft generation of a stable limit cycle from the singular point \(O\).

**Theorem 2.** For fixed negative values of the parameters \(b < 0, e < 0\), the birth of a stable limit cycle from a stable focus \(O\) as a result of Andronov-Hopf bifurcation occurs when \(c < \sqrt{(bd - ae)^2/(be)}\)

**Proof.** We will seek a periodic solution of system (2.3) in the form
\[
u = r\cos(\omega t), \quad \nu = r\sin(\omega t), \quad r = \text{const.} \tag{3.3}\]

Substituting the assumed form of the solution into the system of equations (2.2), we obtain
\[
\begin{align*}
\rho^2 \cos(\omega t) &= (bc\rho \cos(\omega t) + ac\rho \cos(\omega t) - ((ad + be)\rho \cos(\omega t) + (bd - ae)\rho \sin(\omega t))(r^2 - 1)) \\
\rho^2 \sin(\omega t) &= (ac\rho \sin(\omega t) - bcr \cos(\omega t) - ((ad + be)\rho \sin(\omega t) + (bd - ae)\rho \cos(\omega t))(r^2 - 1)) \\
\end{align*}
\]
Equating the coefficients of \(\cos(\omega t)\) and \(\sin(\omega t)\), we obtain a system of equations for \(r\) and \(\omega\)
\[
r\omega^2 = \frac{(ac\rho - (ad + be)r(r^2 - 1))}{(a^2 + b^2)}, \quad bcr - (bd - ae)r(r^2 - 1) = 0,
\]
Expressing from the second equation \((r^2 - 1) = bc\omega/(bd - ae)\), we obtain from the first equation

\[
\omega^2 = \frac{\left(a^2r\omega - (ad + be)bc\omega/(bd - ae)\right)}{(a^2 + b^2)} = \frac{-c\omega}{(bd - ae)},
\]

whence it follows that

\[
\omega = \frac{-ce}{(bd - ae)}, \quad r^2 = 1 - \frac{bec^2}{(bd - ae)^2}.
\]

Thus, for \(c^2 < (bd - ae)^2/(be)\) the system (2.3) has as its solution a stable limit cycle (3.3), softly generated from the zero singular point, stable for \(c^2 \geq (bd - ae)^2/(be)\). The theorem is proved.

It follows from Theorem 2 that the bifurcation parameter in the system of equations (2.3) is the parameter \(c\), which characterizes the velocity of propagation of disturbances along the \(y\) axis in the generalized nonlinear Schrödinger equation. The parameter \(c\) is not explicitly included in the original equation (1.2). Investigation of the following bifurcations in the equation (1.2) with decreasing values of the parameter \(c\) by analytical methods, starting with the bifurcation of the creation of a two-dimensional torus, doubling its period, and further bifurcations of a two-dimensional torus with a doubled period in accordance with the Feigenbaum-Sharkovsky-Magnitskii theory, is an extremely difficult task. To do this, it is necessary to find analytically the multipliers of the cycle, which is possible in very rare cases, and to determine such a value of the parameter \(c\), for which one of the multipliers is equal to +1, two complex conjugate multipliers lie on the unit circle, and the fourth real multiplier lies in the interval \((-1, +1)\). Therefore, further investigation of the complication of the dynamics of solutions of system (2.3) and equation (1.2) will be carried out by numerical methods.

4. Scenario of transition to space-time chaos
Let us carry out a numerical study of system (2.3) for fixed values of parameters: \(a = 2, b = -1, e = -1, d = 0\) and decreasing the values of the bifurcation parameter \(c\). For \(c > \sqrt{(bd - ae)^2/(be)} = 2\) the singular point \(O\) of the system (2.3) is a stable focus; for \(c = 2\), a stable (asymptotically orbitally stable) limit cycle is born from this point as a result of the Andronov-Hopf bifurcation. Cycle exists up to the value \(c \approx 1.367\), at which a stable two-dimensional torus is born from it as a result of the second Andronov-Hopf bifurcation. With a further decrease in the values of the parameter \(c\), the Feigenbaum cascade of period doubling bifurcations of stable two-dimensional tori is observed in system (2.3). A stable two-dimensional torus of period two is born at \(c \approx 1.243\), a two-dimensional torus of period 4 is born at \(c \approx 1.223\), a two-dimensional torus of period 8 is born at \(c \approx 1.218\) etc. For \(c \approx 1.215\) the system (2.3) contains the first simplest singular (chaotic) attractor - the Feigenbaum toroidal attractor - a non-periodic trajectory lying on a manifold that is the Cartesian product of a limit cycle and a cyclic Feigenbaum attractor. With the further decrease in the values of the parameter \(c\), a sequence of stable two-dimensional tori is found in accordance with the Sharkovsky subharmonic order and then in accordance with the Magnitskii homoclinic order. For example, a stable two-dimensional torus of period 6 of the subharmonic cascade is found at \(c = 1.2141\), a two-dimensional torus of period 5 is found at \(c = 1.2094\) (Fig. 1a), and a two-dimensional torus of period 3 is found at \(c = 1.2046\) (Fig. 1b). A stable two-dimensional torus \(T_6\) of the homoclinic cascade of bifurcations is found at \(c = 1.1883\) (Fig. 1c). As is known (see [4]), a sequence of two-dimensional tori of a complete homoclinic cascade of bifurcations must converge to a toroidal separatrix manifold, which is the Cartesian product of a limit cycle and a saddle-focus separatrix loop. However, for a given set of values of the system parameters, this separatrix manifold, apparently, does not exists. Its detection in the space of five parameters of the system \((a, b, c, d, e)\) is a separate difficult problem, since this separatrix manifold has codimension greater than one.
Fig. 1. Projections of Poincaré sections: two-dimensional tori of periods five (a) and three (b) of the subharmonic cascade, two-dimensional torus of period four of homoclinic cascade (c) and a complex toroidal singular attractor (d) of the ODE system (2.3).

Figure 1 shows the projections of the Poincaré sections $\nu = 0$ onto the plane $(z, u)$: two-dimensional tori of periods five (a) and three (b) of the subharmonic cascade of bifurcations, a two-dimensional torus of period four (c) of the homoclinic cascade of bifurcations and a complex toroidal singular attractor (d) of the system of ODE (2.3) lying in a neighborhood of a toroidal separatrix manifold.

Thus, it is numerically established that as the values of the parameter $c$ decrease after two Andronov-Hopf bifurcations, the following are realized in the system (2.3): Feigenbaum cascade of the period doubling bifurcations of stable two-dimensional tori, the complete subharmonic cascade of bifurcations of stable two-dimensional tori in accordance with the Sharkovsky order, and an incomplete homoclinic cascade of bifurcations of stable two-dimensional tori. The found two-dimensional tori of system (2.3) correspond to traveling waves in the generalized nonlinear Schrödinger equation (1.2). Figure 2 shows such waves corresponding to two-dimensional tori of period two at $c = 1.225$ and period three at $c = 1.2046$ of the subharmonic cascade of bifurcations.

Fig. 2. Traveling periodic waves in the generalized nonlinear Schrödinger equation (1.2) corresponding to two-dimensional tori of periods two (a) and three (b) of the subharmonic cascade of bifurcations.
5. Conclusion
The paper presents an analytical and numerical analysis of the dynamics of wave solutions in the generalized nonlinear Schrödinger equation. Solutions of the equation are obtained in the form of periodic and chaotic traveling waves. It is shown that the transition to space-time chaos in the generalized nonlinear Schrödinger equation is carried out in full accordance with the universal Feigenbaum-Sharkovsky-Magnitsii bifurcation theory through the subharmonic and homoclinic cascades of bifurcations of stable two-dimensional tori of a system of ordinary differential equations obtained from the original equation by a self-similar change of variables. It is proved that the bifurcation parameter is the value of the propagation velocity of traveling waves along the spatial axis, which is clearly not included in the original equation.

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