Non-convex Feedback Optimization with Input and Output Constraints

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Abstract—In this paper, we present a novel control scheme for feedback optimization. That is, we propose a discrete-time controller that can steer the steady state of a physical plant to the solution of a constrained optimization problem without numerically solving the problem. Our controller can be interpreted as a discretization of a continuous-time projected gradient flow. Compared to other schemes used for feedback optimization, such as saddle-point flows or inexact penalty methods, our algorithm combines several desirable properties: It asymptotically enforces constraints on the plant steady-state outputs, and temporary constraint violations can be easily quantified. Our algorithm requires only reduced model information in the form of steady-state input-output sensitivities of the plant. Further, as we prove in this paper, global convergence is guaranteed even for non-convex problems. Finally, our algorithm is straightforward to tune, since the step-size is the only tuning parameter.

I. INTRODUCTION

In recent years, the design of feedback controllers that steer the steady state of a physical plant to the solution of a constrained optimization problem has garnered significant interest both for its theoretical depth [1]–[4] and potential applications. In particular, while the historic roots of feedback optimization trace back to process control [5], [6] and communication networks [7], [8], recent efforts have centered around online optimization of power grids [9]–[12].

We adopt the perspective that feedback optimization emerges as the interconnection of an optimization algorithm such as gradient descent (formulated as an open system) and a physical plant with well-defined steady-state behavior. This is in contrast, for example, to [4] that adopts an output-regulation viewpoint or extremum-seeking which is completely model-free [13]. In particular, for our purposes, we assume that the plant is stable with fast-decaying dynamics and the steady-state input-to-output map \( y = h(u) \) is well-behaved (Fig. 1). This assumption is motivated by previous work on timescale separation in these setups [14], [15].

The critical aspect of feedback optimization is that, instead of relying on a full optimization model, the algorithms take advantage of measurements of the system output. This entails that the system model \( h \) does not need to be known explicitly, nor does it need to be evaluated numerically. Instead, as we will see, only information about the steady-state sensitivities \( \nabla h \) is required. This renders feedback optimization schemes inherently more robust against disturbances and uncertainties than “feedforward” numerical optimization.

A particular focus in feedback optimization is the incorporation of (unilateral) constraints on inputs and steady-state outputs. Constraints on the inputs can usually be enforced directly by means of projection or by exploiting physical saturation and using anti-windup control [16], [17]. In contrast, constraints on the steady-state outputs cannot be enforced directly, especially when the plant is subject to disturbances. Hence, previous works either considered inexact penalty methods [11], [18] or saddle-point algorithms [3], [9], [10]. The former treat the output constraints as soft constraints using penalty functions, while the latter ensure the constraints to be satisfied asymptotically. However, saddle-point flows exhibit oscillatory behavior, are difficult to tune, and do not come with strong guarantees for non-convex problems.

In this paper, we present a new way of enforcing output constraints in feedback optimization with a novel discretization of projected gradient flows. Our algorithm works by projecting gradient iterates onto a linearization of the feasible set around the current state and then applying them as set-points to the system. For the main result, the global convergence of our scheme, we take inspiration from numerical algorithms like sequential quadratic programming [19]–[21]. In our setup, however, we cannot rely on line-search techniques, and instead, have to establish convergence for fixed step-sizes. Furthermore, we need to ensure that our algorithm admits an implementation as a feedback controller.

Compared to inexact penalty approaches, our scheme guarantees that constraints on the steady-state outputs are satisfied asymptotically. This is similar to saddle-point schemes. However, in contrast to saddle-point flows, our algorithm exhibits a benign convergence behavior without oscillations (which we illustrate with numerical examples), temporary constraint violations are quantifiable, convergence is guaranteed for non-convex problems, and tuning is restricted to a single parameter.

The rest of the paper is structured as follows: In Section II we recall results on nonlinear optimization and projected gradient flows. Section III presents the problem of feedback optimization in continuous time. In Section IV we describe our novel discrete-time controller and state our main convergence result. The proof is laid out in Section V. Finally, we give a numerical example in Section VI and discuss open

Fig. 1. Block diagram of the feedback optimization setup.
questions in Section VII.

II. Preliminaries

A. Notation and Technical Results

For $\mathbb{R}^p$, $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product and $|| \cdot ||$ its induced 2-norm. The non-negative orthant of $\mathbb{R}^p$ is written as $\mathbb{R}^p_+$. Any $G \in \mathbb{S}^n_+$ induces a 2-norm defined as $||v||_G := \sqrt{v^T G v}$ for all $v \in \mathbb{R}^p$. Given a set $\mathcal{C} \subset \mathbb{R}^p$, a map $G : \mathcal{C} \to \mathbb{S}^n_+$ is called a metric on $\mathcal{C}$.

For a continuously differentiable function $f : \mathbb{R}^p \to \mathbb{R}$, $\nabla f(x) \in \mathbb{R}^{p \times p}$ denotes the Jacobian of $f$ at $x$. The map $f$ is globally $L$-Lipschitz continuous if for all $x, y \in \mathbb{R}^p$ and some $L > 0$, it holds that $||f(x) - f(y)|| \leq L ||x - y||$. We recall the so-called Descent Lemma [22, Prop A.24]:

**Lemma 1.** Given a continuously differentiable function $f : \mathbb{R}^p \to \mathbb{R}$ with $L$-Lipschitz derivative $\nabla f$, for all $x, z \in \mathbb{R}^p$ it holds that $f(z) \leq f(x) + \nabla f(x)(z - x) + \frac{L}{2}||z - x||^2$.

B. Nonlinear Optimization

In this paper, we often consider feasible sets of the form

$$\mathcal{X} := \{x \in \mathbb{R}^p | g(x) \leq 0\}, \quad (1)$$

where $g : \mathbb{R}^p \to \mathbb{R}^q$ is continuously differentiable. Let $\mathbf{1}_x^+ := \{i \mid g_i(x) = 0\}$ denote the index set of active inequality constraints of $\mathcal{X}$ at $x$ and let $\mathbf{1}_x^- := \{i \mid g_i(x) < 0\}$ be the index set of inactive inequality constraints at $x$.

**Definition 1 (LICQ).** Given $\mathcal{X} \subset \mathbb{R}^p$ as in (1), the linear independence constraint qualification (LICQ) is said to hold at $x \in \mathcal{X}$, if the matrix $\nabla g_{\mathbf{1}_x^+}(x)$ has rank $|\mathbf{1}_x^+|$.

The tangent cone $T_x\mathcal{X}$ of a set $\mathcal{X} \subset \mathbb{R}^p$ at a point $x \in \mathcal{X}$ is, intuitively speaking, defined as the set of directions from which one can approach $x$ from within $\mathcal{X}$. More precisely, $w \in T_x\mathcal{X}$ if there exist sequences $x_k \to x$ with $x_k \in \mathcal{X}$ for all $k$ and $\delta_k \searrow 0$ such that $\frac{x_k - x}{\delta_k} \to w$. For sets of the form (1) and if LICQ holds, $T_x\mathcal{X}$ takes an explicit form as captured by the following definition [23, Thm 6.31]:

**Definition 2.** Consider a set $\mathcal{X}$ as in (1) satisfying LICQ for all $x \in \mathcal{X}$. The tangent cone of $\mathcal{X}$ at $x \in \mathcal{X}$ is given by

$$T_x\mathcal{X} := \{w \in \mathbb{R}^p | \nabla g_{\mathbf{1}_x^+}(x)w \leq 0\}.$$  

Note that, under the given assumptions on $\mathcal{X}$, $T_x\mathcal{X}$ is closed and convex for all $x \in \mathcal{X}$ [23, Thm 6.26].

For a continuously differentiable function $\Psi : \mathbb{R}^p \to \mathbb{R}$ and $\mathcal{X} \subset \mathbb{R}^p$ as in (1), consider the constrained problem

$$\text{minimize } \Psi(x) \quad \text{subject to } x \in \mathcal{X}. \quad (2)$$

The Lagrangian of (2) is defined as $L(x, \mu) := \Psi(x) + \mu^T g(x)$ for all $x \in \mathbb{R}^p$ and all Lagrange multipliers $\mu \in \mathbb{R}^q_+$. Recall the first-order optimality (KKT) conditions for (2):

**Theorem 1.** [24, Ch. 11.8] If $x^* \in \mathcal{X}$ is a local solution of (2) and LICQ holds at $x^*$, there exists a unique $\mu^{\ast} \in \mathbb{R}^q_+$ such that $\nabla_x L(x^*, \mu^\ast) = 0$ and $\mu_i^{\ast} = 0$ hold for all $i \in \mathbf{1}_x^+$. In particular, the LICQ assumption guarantees the uniqueness (and boundedness) of the dual multipliers $\mu^\ast$ [25].

Next, consider the parametric optimization problem

$$\text{minimize } \Psi(x, \varepsilon) \quad \text{subject to } x \in \mathcal{X}(\varepsilon) := \{x \in \mathbb{R}^p | g(x, \varepsilon) \leq 0\}, \quad (3)$$

where $\Psi : \mathbb{R}^p \times \mathcal{T} \to \mathbb{R}$ and $g : \mathbb{R}^p \times \mathcal{T} \to \mathbb{R}^q$ are parametrized in $\varepsilon \in \mathcal{T} \subset \mathbb{R}^r$. The (parametrized) Lagrangian of (3) is defined as $L(x, \mu, \varepsilon) := \Psi(x, \varepsilon) + \mu^T g(x, \varepsilon)$ for all $x \in \mathbb{R}^p, \mu \in \mathbb{R}^q_+$, and all $\varepsilon \in \mathcal{T}$.

We are interested in solutions of (3) as a function of $\varepsilon$. In particular, for convex optimization problems with strongly convex objective, [26, Thm 2.3.2] simplifies to the following:

**Theorem 2.** Consider (3) and assume that $\Psi$ and $g$ are twice continuously differentiable in $x$, and that $\Psi, g, \nabla_x \Psi, \nabla_x g, \nabla^2_x \Psi, \text{ and } \nabla^2_{xx} g$ are continuous in $\varepsilon$. Furthermore, for all $\varepsilon \in \mathcal{T}$, let

- $\Psi$ be strongly convex in $x$,
- $\mathcal{X}(\varepsilon)$ be non-empty and convex, and
- LICQ be satisfied for all $x \in \mathcal{X}(\varepsilon)$.

Then, there exist continuous functions $x^* : \mathcal{T} \to \mathbb{R}^p$ and $\mu^* : \mathcal{T} \to \mathbb{R}^q_+$ such that $x^*(\varepsilon)$ is the unique global optimizer of (3) for all $\varepsilon \in \mathcal{T}$ and $\mu^*(\varepsilon)$ is its Lagrange multiplier.

**Proof.** We show that, under the given convexity assumptions, the requirements for [26, Thm 2.3.2] are met globally. Namely, by assumption, (3) is feasible for all $\varepsilon \in \mathcal{T}$ and LICQ holds for all $x \in \mathcal{X}(\varepsilon)$ and all $\varepsilon \in \mathcal{T}$. Hence, by strong convexity of $\Psi, (3)$ admits a unique (global) optimizer for all $\varepsilon \in \mathcal{T}$. Therefore, the solution map $\varepsilon \mapsto x^*(\varepsilon)$ and $\varepsilon \mapsto \mu^*(\varepsilon)$ are single-valued. Moreover, for all $\varepsilon \in \mathcal{T}$, the KKT conditions are satisfied and the so-called second order sufficiency conditions hold (trivially) by (strong) convexity. It then follows from [26, Thm 2.3.2] that $x^*$ and $\mu^*$ are continuous around every $\varepsilon \in \mathcal{T}$ and hence on all of $\mathcal{T}$.

C. Projected Gradient Flows

Given a set $\mathcal{X} \subset \mathbb{R}^p$ of the form (1) satisfying LICQ for all $x \in \mathcal{X}$, a metric $G : \mathcal{X} \to \mathbb{S}^n_+$, and a vector $v \in \mathbb{R}^p$, we define the projection operator

$$\Pi_{\mathcal{X}}^G[v](x) := \arg \min_{w \in T_x\mathcal{X}} ||w - v||^2_{G(x)}, \quad (4)$$

that is, $\Pi_{\mathcal{X}}^G$ projects $v$ onto the tangent cone of $\mathcal{X}$ at $x$ with respect to $G$. If $f : \mathcal{X} \to \mathbb{R}^p$ is a vector field, we write $\Pi_{\mathcal{X}}^G[f](x) = \Pi_{\mathcal{X}}^G[f(x)](x)$ for brevity. Since $T_x\mathcal{X}$ is closed, convex, and non-empty for all $x \in \mathcal{X}$, (4) is a convex problem with strongly convex objective, admitting a unique solution for any $x \in \mathcal{X}$ (in particular, $w = 0$ is feasible for all $x$). Considering a differentiable potential function $\Psi : \mathbb{R}^p \to \mathbb{R}$, a projected gradient flow [27]-[29] is defined by the constrained, discontinuous ODE

$$\dot{x} = \Pi_{\mathcal{X}}^G[-\nabla G(\Psi)(x), \ x \in \mathcal{X}], \quad (5)$$

where we have applied $\Pi_{\mathcal{X}}^G[\cdot]$ to the negative gradient of $\Psi$ in $G$, defined as $\nabla G(\Psi)(x) := (G^{-1})^T \nabla \Psi(x)^T$. An absolutely continuous function $x : [0, \infty) \to \mathcal{X}$ that satisfies $\dot{x} = \Pi_{\mathcal{X}}^G[f](x)$ almost everywhere, i.e., for almost all
\( t \in [0, \infty) \) is a complete (Carathéodory) solution of (5). Note that any solution of (5) has to be viable by definition, i.e., \( x(t) \in \mathcal{X} \) for all \( t \geq 0 \). The existence of solutions of (5), their convergence to critical points, and the asymptotic stability of strict local minimizers of (2) is guaranteed by results in [28], which, for our purposes, can be summarized as follows:

**Proposition 1.** [28, Prop 5.4, Thm 5.5] Consider a compact set \( \mathcal{X} \subset \mathbb{R}^p \) of the form (1) satisfying LICQ for all \( x \in \mathcal{X} \), a continuous metric \( G : \mathcal{X} \to \mathbb{R}_+^p \), and a continuously differentiable function \( \Psi : \mathbb{R}^p \to \mathbb{R} \). Then,

(i) for every \( x(0) \in \mathcal{X} \), (5) admits a complete solution,

(ii) every complete solution of (5) converges to the set of first-order optimal points of (2), and

(iii) if \( x^* \in \mathcal{X} \) is strongly asymptotically stable for (5), then \( x^* \) is a strict local minimum of (2).

**III. PROJECTED FEEDBACK GRADIENT FLOW**

We consider the problem of steering a physical plant to a steady state that solves a pre-specified constrained optimization problem. We assume that the plant is described by a nonlinear steady-state input-to-output map \( h : \mathbb{R}^p \to \mathbb{R}^n \).

For simplicity we consider separate constraints on the input \( u \in \mathbb{R}^p \) and the output \( y \in \mathbb{R}^n \) given by polyhedra

\[
\mathcal{U} := \{ u \in \mathbb{R}^p \mid Au \leq b \} \quad \text{and} \quad \mathcal{Y} := \{ y \in \mathbb{R}^n \mid Cy \leq d \},
\]

where \( A \in \mathbb{R}^{q \times p}, b \in \mathbb{R}^q, C \in \mathbb{R}^{r \times n}, \) and \( d \in \mathbb{R}^r \). For many applications, separable constraint sets are sufficiently expressive. Future work will also address generalizations to joint constraints on inputs and outputs.

Given \( \Phi : \mathbb{R}^p \times \mathbb{R}^n \to \mathbb{R} \), we hence consider the problem

\[
\begin{align*}
\text{minimize} & \quad \Phi(u, y) \\
\text{subject to} & \quad y = h(u), \\
& \quad u \in \mathcal{U}, \quad y \in \mathcal{Y}.
\end{align*}
\]

Solving (6) numerically relies on the accurate knowledge of \( h \), including the knowledge of plant parameters and disturbances, which are not always available. In feedback optimization, however, an explicit computation of \( h \) is not required, as \( y = h(u) \) is available as the measured system output.

To design a controller that drives the physical plant to the solution of (6), we first define the desired closed-loop behavior in the input coordinates. By substituting \( y \) with \( h(u) \) in problem (6), we arrive at the equivalent problem

\[
\begin{align*}
\text{minimize} & \quad \tilde{\Phi}(u) \\
\text{subject to} & \quad u \in \tilde{\mathcal{U}},
\end{align*}
\]

where \( \tilde{\Phi}(u) := \Phi(u, h(u)) \) and \( \tilde{\mathcal{U}} := \mathcal{U} \cap h^{-1}(\mathcal{Y}) \), which can be solved with the projected gradient flow

\[
\dot{u} = \Pi_{\tilde{\mathcal{U}}} \left[- \nabla \tilde{\Phi}(u)\right], \quad u \in \tilde{\mathcal{U}},
\]

where \( G : \tilde{\mathcal{U}} \to \mathbb{R}_+^p \) is a metric. Under the following assumption, **Proposition 1** applies and (8) is well-posed and converges to first-order optimal points of (7).

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1 Stability is understood in the sense of Lyapunov. In case of non-unique solutions, strong (as.) stability implies that every solution is (as.) stable.

**Assumption 1.** For (8) the set \( \tilde{\mathcal{U}} \) is compact, \( \tilde{\mathcal{U}} \) is non-empty and satisfies LICQ for all \( u \in \tilde{\mathcal{U}} \). Furthermore, \( \tilde{\Phi} \) and \( h \) are continuously differentiable and \( G \) is continuous.

The assumption that \( \tilde{\mathcal{U}} \) is compact is motivated by the fact that physical plants are generally characterized by some physical (actuation) constraints which usually take the form of (non-strict) inequality constraints. Further, it can be shown that the LICQ assumption holds generically [30].

The projected gradient flow (8) is the desired closed-loop behavior of our control loop according to Fig. 1. To arrive at a control law, we first reformulate \( \tilde{\Phi}, \tilde{\mathcal{U}} \) by applying the chain rule to \( \nabla (\tilde{\Phi}(u, h(u))) \). Consequently, (8) can be expressed equivalently as

\[
\dot{u} = \Pi_{\tilde{\mathcal{U}}} \left[-G^{-1}(u)H(u)^T \nabla \Phi(u, y)^T \right](u), \quad y = h(u),
\]

where \( y \) is the measured system output and \( H(u)^T := \Pi_p \nabla h(u)^T \). Note that the input-output sensitivity \( \nabla h \) (or an approximation thereof) is usually available in practice.

Since controllers need to be implemented digitally, we turn to find a proper discretization of the control law (9) in the next section. In doing so, specific care is needed to deal with the discontinuous nature of \( \Pi_{\tilde{\mathcal{U}}} \).

**IV. FEEDBACK-ENABLED DISCRETIZATION OF PROJECTED GRADIENT FLOWS**

A well-established approach to discretize (8) (but only for \( G \equiv \mathbb{I} \) and \( \tilde{\mathcal{U}} \) convex) is the projected Euler integration [29]. For (9), this scheme would take the form

\[
u^+ = P_{\tilde{\mathcal{U}}} (u - \alpha H(u)^T \nabla \Phi(u, y)^T) \quad y = h(u),
\]

where \( P_{\tilde{\mathcal{U}}} \) denotes the Euclidean projection on \( \tilde{\mathcal{U}} \) and \( \alpha > 0 \) is a step-size parameter. This approach, however, is not practical for feedback optimization, as it requires the set \( \tilde{\mathcal{U}} \) (and thus \( h \)) to be fully known.

Hence, as the main topic of this paper, we consider a special discretization of the projected gradient flow (8) that can be implemented as a feedback controller similar to (9).

In contrast to (10), our proposed numerical integration scheme for projected gradient flows applies to non-convex domains \( \mathcal{U} \), general metrics \( G \), and requires knowledge of only the steady-state sensitivity \( \nabla h \) instead of \( h \).

As a starting point, we define the map \( \sigma_\alpha : \mathbb{R}^p \to \mathbb{R}^p \) as

\[
\sigma_\alpha(u) := \arg \min_{w \in \mathbb{R}^p} \left\| \nabla G(u) \Phi(u) + w \right\|_{G(u)}^2 \quad \text{subject to} \quad A(u + \alpha w) \leq b, \quad C(h(u) + \alpha \nabla h(u) w) \leq d,
\]

where \( \alpha > 0 \) is a (fixed) step-size parameter. For (11) to be well-defined, we make the following assumption:

**Assumption 2.** For all \( u \in \mathcal{U} \), the feasible set of (11) defined as \( \mathcal{U}_w := \{ w \mid A(u + \alpha w) \leq b, C(h(u) + \alpha \nabla h(u) w) \leq d \} \) is non-empty and satisfies LICQ for all \( w \in \mathcal{U}_w \).

Assumption 2 implies the LICQ and non-emptyness conditions in Assumption 1 (this can be verified for \( u \in \tilde{\mathcal{U}} \) with \( w = 0 \)). In general, Assumption 2 is stronger than these conditions and is common in the study of sequential quadratic
programming [19]–[21]. Providing sufficient conditions for Assumption 2 to hold, are the subject of ongoing work.

We claim that, for small $\alpha$, the discrete-time system
\begin{equation}
  u^+ = u + \alpha \sigma_\alpha(u), \quad u \in U
\end{equation}
approximates the projected gradient flow in (8) on $\hat{U}$. In particular, (12) admits different interpretations: On the one hand, $\sigma_\alpha(u)$ is the projection of $u - \alpha \nabla \Phi(u)$ onto a linearization of $U$ around $u$. Thus, (12) can also be seen as an approximation of (10). On the other hand, one can show that $\lim_{\alpha \to 0} \sigma_\alpha(u) = \Pi_U \left[ - \nabla \Phi(u) \right]$, i.e., $\sigma_\alpha$ approximates a projected gradient.

Most importantly, however, our following main result guarantees global convergence of (12) to the set of first-order optimal points of (7) for a small enough, fixed step-size.

**Theorem 3.** Under Assumptions 1 and 2 consider (7) and assume that $\nabla \Phi$ and $\nabla h$ are globally Lipschitz on $U$. Then, there exists an $\alpha^* > 0$ such that for every $\alpha < \alpha^*$

(i) the trajectory of (12) for any $u^0 \in U$ converges to the set of first-order optimal points of (7), and

(ii) if $u^*$ is an asymptotically stable equilibrium point of (12), it is a strict local minimum of $\hat{\Phi}$ on $U$.

On top of global convergence, (12) lends itself to a feedback implementation analogously to [9]. Namely, we can write (12) equivalently as
\begin{equation}
  u^+ = u + \alpha \tilde{\sigma}_\alpha(u, y), \quad y = h(u),
\end{equation}
with $\tilde{\sigma}_\alpha$ as the measurement-based evaluation of (11), i.e.,
\[ \tilde{\sigma}_\alpha(u, y) := \arg \min_{w \in \mathbb{R}^p} \|w + G^{-1}(u)H(u)^T \nabla \Phi(u, y)^T \|_{G(u)}^2 \]
subject to
\[ A(u + \alpha w) \leq b \]
\[ C(y + \alpha \nabla h(u)w) \leq d, \]
where $y = h(u)$ is the measured system output. In particular, the evaluation $\tilde{\sigma}_\alpha(u, y)$ does not directly rely on $h$. Instead, it is enough to know $\nabla h$ (or an approximation thereof).

V. PROOF OF THEOREM 3

To prove Theorem 3 we apply the following invariance principle for discrete-time systems [31, Thm 6.3]:

**Theorem 4.** Consider a discrete-time dynamical system $u^+ = T(u)$, where $T : S \to S$ is well-defined and continuous, and $S \subset \mathbb{R}^p$ is closed. Further, let $V : S \to \mathbb{R}$ be a continuous function such that $V(T(u)) \leq V(u)$ for all $u \in S$. Let $u = \{u_0, u_1, u_2, \ldots\} \subset S$ be a bounded solution. Then, for some $r \in V(S)$, $u$ converges to the non-empty set that is the largest invariant subset of $V^{-1}(r) \cap S \cap \{u \mid V(T(u)) = V(u) = 0\}$.

We first need to establish that the map $T(u) := u + \alpha \sigma_\alpha(u)$ is continuous and $T(u) \in U$ for all $u \in U$.

**Lemma 2.** Under Assumptions 1 and 2 $u + \alpha \sigma_\alpha(u) \in U$ holds for all $u \in U$.

**Lemma 2** follows since $\sigma_\alpha(u)$ satisfies (11b) by definition.

**Lemma 3.** Under Assumptions 1 and 2 $\sigma_\alpha$ and the map of associated Lagrange multipliers are continuous in $u$.

**Proof.** Continuity in $u$ of $\sigma_\alpha$ and the map of Lagrange multipliers is a consequence of Theorem 2. Namely, for all $u \in U$, the functions defining (11) and their first and second derivatives with respect to $w$ are continuous in $u$, the objective in (11) is strongly convex in $w$, the set $U_a$ is non-empty and convex in $w$, and LICQ is satisfied for all $w \in U_a$ (by Assumption 2).

Moreover, the constraint violation committed at every iteration of (12) can be bounded.

**Lemma 4.** Assume that $C_i \nabla h$ is $\ell_i$-Lipschitz for all $i = 1, \ldots, l$ on $U$. Given the iteration (12) and any $u \in U$, we have $C_i h(u^+) - d_i \leq \frac{\ell_i}{2} ||\alpha \sigma_\alpha(u)||^2$.

**Proof.** Using the Descent Lemma (Lemma 1), the desired bound can obtained by inserting (11c) into
\[ C_i h(u^+) - C_i h(u) \leq \alpha C_i \nabla h(u)w + \frac{\ell_i}{2} ||\alpha w||^2. \]

**Lyapunov Function:** Given $u \in U$, let $\mu^*_i(u)$ be the Lagrange multiplier of (11) for the $i$th constraint of (11c) with $i = 1, \ldots, l$. Since $\mu^*_i(u)$ is continuous by Lemma 3 on $U$, which is compact by Assumption 1, there exists an upper bound $\xi \geq \sup_{a \in \Omega_i} \{\mu^*_i(u)\}$. Hence, we may consider the function $V : \mathbb{R}^p \to \mathbb{R}$ defined by
\[ V(u) = \hat{\Phi}(u) + \xi \left[ \sum_{i=1}^{l} \max\{0, C_i h(u) - d_i\} \right], \]
which we show to be non-increasing along solutions of (12).

To prove this claim, note that (11) is equivalent to solving
\begin{align}
  \arg \min_{w \in \mathbb{R}^p} & \quad \frac{1}{2} w^T G(u)w + \alpha \nabla \hat{\Phi}(u)w \\
  \text{subject to} & \quad \alpha w \leq b - Au \\
  & \quad \alpha C_i \nabla h(u)w \leq d - Ch(u),
\end{align}
where we have multiplied the objective with the constant factor $\alpha$ and ignored the constant term in the objective. Since (15) is convex, the KKT conditions are necessary and sufficient to certify optimality of a solution $w$ of (15). Namely, $w \in \mathbb{R}^p$ is a solution if (15a)-(15c) are satisfied and, for some dual multipliers $\nu \in \mathbb{R}^{2q}$ and $\mu \in \mathbb{R}^l$, stationarity
\[ \alpha \nabla^T G(u) + \alpha \nabla^T \hat{\Phi} + \alpha \mu^T A + \alpha \mu^T C \nabla h(u) = 0 \]
holds and complementary slackness is satisfied, i.e., for all $j = 1, \ldots, q$ and all $i = 1, \ldots, l$, we have
\[ \nu_j (\alpha A_j w - b_j + A_j u) = 0 \]
\[ \mu_i (\alpha C_i \nabla h(u)w - d_i + C_i h(u)) = 0. \]

**Lemma 5.** Under Assumptions 1 and 2 let $V$ be as in (14). Further, assume that $\nabla \hat{\Phi}$ is $L$-Lipschitz and $C_i \nabla h$ is $\ell_i$-Lipschitz for all $i = 1, \ldots, l$ on $U$. For (12), $V(u^+) \leq V(u)$ is satisfied for all $u \in U$, if
\[ \alpha < \alpha^* := \frac{2 \min(G(u))}{L + \xi \sum_{i=1}^{l} \ell_i}. \]
where $\xi$ is the upper bound of the Lagrange multipliers of the constraints in (11c) on $\Omega$.

**Proof.** In the following let $w := \sigma_\alpha(u)$. With the Descent Lemma (Lemma 1), we can establish
\[
\tilde{\Phi}(u^*) - \tilde{\Phi}(u) \leq \alpha \nabla \tilde{\Phi}(u) w + \frac{\alpha}{2}||w||^2.
\] (20)
Further, from Lemma 4 we can derive
\[
\max(0, C_i h(u^*) - d_i) \leq \frac{\alpha}{2}||w||^2.
\] (21)

The following is inspired by the proof of [19, Lemma 10.4.1]. We take the inner product of (16) and $w$, which results in $\alpha \nabla \tilde{\Phi}(u) w = -\alpha \omega^T G(u) w - \sum_{j=1}^l \alpha \nu_j A_i w - \sum_{i=1}^l \alpha \mu_i C_i \nabla h(u) w$. Using (17), we replace the summands, i.e., $\alpha \nabla \tilde{\Phi}(u) w = -\alpha \omega^T G(u) w + \sum_{i=1}^l \mu_i (C_i h(u) - d_i)$, which can be estimated as
\[
\alpha \nabla \tilde{\Phi}(u) w \leq -\alpha \omega^T G(u) w + \sum_{i=1}^l \mu_i (C_i h(u) - d_i)
\leq -\alpha \omega^T G(u) w + \sum_{i=1}^l \mu_i \max(0, C_i h(u) - d_i).
\] (22)

Using (20), (21) and (22), we obtain
\[
V(u^*) - V(u) \leq -\alpha \lambda_{\min} (G(u)) ||w||^2
+ \frac{\alpha}{2} \left[ L + \xi \sum_{i=1}^l \ell_i \right] ||w||^2
- \sum_{i=1}^l (\xi - \mu_i) \max(0, C_i h(u) - d_i).
\] (23)
Choosing $\alpha$ as in (19) guarantees $V(u^*) \leq V(u)$.

**Convergence to first-order optimal points:** To show (i) in Theorem 3 we can now simply apply Theorem 4. Namely, Lemmas 2 and 3 guarantee that $T(u) := u + \alpha \sigma_\alpha(u)$ is continuous in $u$ and $\mathcal{U}$ is invariant. By Lemma 5 for the given $\alpha$ satisfying (19), the continuous function $V : \mathcal{U} \to \mathbb{R}$ in (14) is non-increasing along the trajectory of (12) for all $u \in \mathcal{U}$. The set $\mathcal{U}$ being compact, we have $u = \{u_0, u_1, u_2, \ldots\} \subset \mathcal{U}$ being a bounded solution. Hence, for some $c \in \mathcal{U}(\mathcal{U})$, the trajectory $u(\mathcal{U})$ converges to the largest invariant subset of $V^{-1}(c) \cap \{u \in \mathcal{U} : V(u^*) - V(u) = 0\}$.

Now, we show that (12) converges to the set of equilibrium points in $\mathcal{U}$, that is $V(u^*) - V(u) = 0$ implies that $u^* = u \in \mathcal{U}$. For $V(u^*) - V(u) = 0$, (23) reduces to
\[
0 \leq \left( \frac{\alpha}{2} \left[ L + \xi \sum_{i=1}^l \ell_i \right] - \lambda_{\min}(G(u)) \right) ||w||^2
- \sum_{i=1}^l (\xi - \mu_i) \max(0, C_i h(u) - d_i).
\] (24)
For $\alpha$ as in (19), the right-hand side of (24) is negative for all $w \neq 0$. It follows that $w = 0$ and $C h(u) \leq d$, i.e., $u^* = u \in \mathcal{U}$ is an equilibrium point. Note that, by considering (15) for $w = 0$, an uncertainty in $\nabla h$ does not affect the feasibility of equilibrium points.

Further, if $w^* = 0$ solves (15) at $u^*$ and $\nu^*, \mu^*$ are the associated Lagrange multipliers, then the triplet $(u^*, \nu^*, \mu^*)$ satisfies the first-order optimality conditions of (7). In particular, $u^*$ is feasible for (7) and given LICQ, the KKT conditions in Theorem 1 are satisfied for (7).

**Strict optimality of asymptotically stable equilibria:** For (ii) in Theorem 3, the argumentation is similar to the proof of [28, Thm 5.5], albeit for a discrete-time system. Consider the neighborhood $\mathcal{N}(u^*) \subset \mathcal{U}$ of $u^* \in \mathcal{U}$, such that any solution $u$ to (12) starting at $u_0 \in \mathcal{N}(u^*)$ converges to $u^*$. For the given $\alpha$, by Lemma 5, we have $V(u_0) \geq V(u^*)$, which implies either $\Phi(u_0) \geq \Phi(u^*)$ or $u_0 \in \mathcal{U}$ or both. Therefore, if $u_0 \in \mathcal{U}$, we have $V(u_0) = \Phi(u_0)$ and $\Phi(u_0) \geq \Phi(u^*)$ follows. Since this reasoning applies to all $u_0$ in the region of attraction of $u^*$, it follows that $u^*$ is a local minimizer of $\Phi$ on $\mathcal{U}$. To see that $u^*$ is a strict local minimizer of $\Phi$ on $\mathcal{U}$, assume for the sake of contradiction that for an $u \neq u^*$ in the region of attraction $\mathcal{N}(u^*)$ of $u^*$, such that $u \in \mathcal{N}(u^*) \cap \mathcal{U}$, it holds $\Phi(u) = \Phi(u^*)$, and therefore (by feasibility) $V(u) = V(u^*)$. Nevertheless, the solution $u$ starting at $u$ converges to $u^*$ by assumption. Since for the given $\alpha$, by Lemma 5, $V$ is non-increasing along the trajectory of (12), it follows $V(u^*) = V(u)$ for all iterates of the solution $u$ starting at $u$. However, as shown in the proof of (i) in Theorem 3 $V(u^*) = V(u)$ implies that the point $u^* = u$ is an equilibrium point in $\mathcal{U}$. Consequently, $u^*$ cannot be asymptotically stable in $\mathcal{N}(u^*)$.

**VI. Numerical Example**

We illustrate the behavior of our feedback optimization scheme by means of a small numerical example. Namely, for $u \in \mathbb{R}^2$, $y \in \mathbb{R}$ and the map $y = h(u) = u_2^2 + u_1 - 2u_2 + 0.5$, we consider the minimization of $\Phi(y, y) = 1.5u_1^2 + u_2^2 - 2u_1u_2 - 3u_2 + 1.5 + y$ on $\mathcal{U} := [-1, 1]^2$ and $\gamma := [0, 1]$. To find a local solution, we use the feedback control system (13).

Tuning is simple: We start with a small fixed step-size $\alpha$ and gradually increase it. The left of Figure 2 shows that all constraints are satisfied asymptotically and, depending on $\alpha$, temporary constraint violations can be made arbitrarily small (Lemma 4). Note that, in our example, we can identify a trade-off between the magnitude of the temporary constraint violations and the convergence rate (in terms of iterations).

To highlight the benign convergence behavior of our scheme we compare it to a generic projected saddle-point scheme. Let the augmented Lagrangian of (7) be defined as
\[
L(u, \mu) = \tilde{\Phi}(u) + \mu^T (Ch(u) - d) + \frac{\rho}{2} \max(0, Ch(u) - d)^2
\] where $\rho \geq 0$ is a fixed augmentation parameter. We consider the projected primal-dual scheme of the form
\[
u^* = P_\mathcal{U}(u - \alpha \nabla u^T), \quad \mu^* = \max(0, \mu + \gamma \nabla u^T)
\] where $\alpha > 0$ and $\gamma > 0$ are separate primal and dual step-size parameters, respectively.

For the saddle-point simulations in the right panel of Figure 2 we have fixed $\alpha = 0.01$ and $\rho = 1$, and we only vary $\gamma$. The results indicate how controller tuning with three parameters ($\alpha$, $\gamma$, $\rho$) can be challenging to manage various performance trade-offs and obtain a proper convergence behavior in the limit. For example, the blue trajectory (γ3) in Figure 3 shows that with a large dual step-size convergence of the saddle-point scheme is not guaranteed within a reasonable interval.
We have proposed a feedback optimization scheme based on a novel discretization of projected gradient flows with the particular property that constraints on the plant steady-state outputs are enforced asymptotically. Our algorithm comes with global convergence guarantees for non-convex problems and fixed step-sizes. In contrast to saddle-point flows, our scheme exhibits a very benign convergence behavior without oscillations, as we illustrate in our numerical example. Furthermore, our algorithm is easy to tune by adapting the step-size.

The subject of ongoing research are the questions of whether the assumptions required for convergence can be relaxed, how much general input and output constraints can be considered, and how robust our scheme is with respect to uncertainties in the model-based input-output sensitivities.

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