LINEAR ADJOINT RESTRICTION ESTIMATES FOR PARABOLOID

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Abstract. We prove a class of modified paraboloid restriction estimates with a loss of angular derivatives for the full set of paraboloid restriction conjecture indices. This result generalizes the paraboloid restriction estimate in radial case from [Shao, Rev. Mat. Iberoam. 25(2009), 1127-1168], as well as the result from [Miao et al. Proc. AMS 140(2012), 2091-2102]. As an application, we show a local smoothing estimate for a solution of the linear Schrödinger equation under the assumption that the initial datum has additional angular regularity.

1. Introduction

Let $S$ be a non-empty smooth compact subset of the paraboloid,

$$\{ (\tau, \xi) \in \mathbb{R} \times \mathbb{R}^n : \tau = |\xi|^2 \},$$

where $n \geq 1$. We denote by $d\sigma$ the pull-back of the $n$-dimensional Lebesgue measure $d\xi$ under the projection map $(\tau, \xi) \mapsto \xi$. Let $f$ be a Schwartz function and define the inverse space-time Fourier transform of the measure $f d\sigma$

$$\langle f d\sigma \rangle^\lor(t, x) = \int_S f(\tau, \xi) e^{2\pi i (x \cdot \xi + \tau t)} d\sigma(\xi).$$

The classical linear adjoint restriction estimate for the paraboloid reads

$$\|\langle f d\sigma \rangle^\lor\|_{L^q_t L^p_x(\mathbb{R} \times \mathbb{R}^n)} \leq C_{p, q, n, S} \|f\|_{L^p(S; d\sigma)},$$

where $1 \leq p, q \leq \infty$. The famous restriction problem is to find the optimal range of $p$ and $q$ such that the estimate (1.2) holds. It is known that the condition

$$q > \frac{2(n + 1)}{n} \quad \text{and} \quad \frac{n + 2}{q} \leq \frac{n}{p'},$$

is necessary for (1.2), see [21, 22]. Here $p'$ denotes the conjugate exponent of $p$. The adjoint restriction estimate conjecture on paraboloid reads as follows.

Conjecture 1.1. The inequality (1.2) holds true if and only if inequalities (1.3) are valid.
There is a large amount of literature on this problem. For $n = 1$, Conjecture \[1.1\] was proved by Fefferman-Stein \[11\] for the non-endpoint case and by Zygmund \[36\] for the endpoint case. Conjecture \[1.3\] in high dimension case becomes much more difficult. For $n \geq 2$, Tomas \[33\] showed \[1.2\] for $q > 2(n + 2)/n$ and Stein \[22\] fixed the limit case $q = 2(n + 2)/n$. Bourgain \[1\] further proved estimate \[1.2\] for $q > 2(n + 2)/n - \epsilon_n$ with some $\epsilon_n > 0$; in particular, $\epsilon_n = \frac{1}{2n}$ when $n = 2$. Further improvements were made by Moyua-Vargas-Vega \[16\] and Wolff \[34\]. Tao \[31\] used the bilinear argument to show that estimate \[1.2\] holds true for $q > 2(n + 3)/(n + 1)$ with $n \geq 2$. This result was improved by Bourgain-Guth \[2\] when $n \geq 4$. This conjecture is so difficult that it remains open up to now. For more details, we refer the reader to \[2, 29–32, 34\].

On the other hand, the restriction conjecture becomes simpler (but not trivial) when a test function has some angular regularity. For example, Conjecture \[1.1\] is proved by Carbery-Romera-Soria \[4\]. Müller-Seeger \[15\] established some sharp mixed space-time estimates, Bochner-Riesz estimate etc.) have easier counterparts when the corresponding operators act on radial functions. Let $L^q_{\rad}$ act on radial functions. Let $S^{n-1}$ denote the unit sphere in $\mathbb{R}^n$ and $L^q_{\sph} := L^q_0(S^{n-1})$, the intermediate situation is to replace the $L^q(\mathbb{R}^n)$ by $L^q_{\rad} L^2_{\sph}$ in \[1.2\]. This intermediate case has been settled for adjoint restriction estimates for a cone by the authors of \[17\]. More precisely, if $S$ is a non-empty smooth compact subset of the cone of

$$S = \{(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^n : \tau = |\xi|, M \leq |\xi| \leq 2M\},$$

then for $q > 2n/(n-1)$ and $(n+1)/q \leq (n-1)/p'$ we have

\[1.4\]
$$\|(f d\sigma)^\vee\|_{L^p(S; L^q_{\rad} L^2_{\sph})} \leq C_{p,q,n,S} \|f\|_{L^p(S; d\sigma)}.$$

The $L^2_{\sph}$-norm allows us to use spherical harmonic expanding, so the problem is converted to $L^q(\ell^2)$-bounds for sequences of operators \{\(H_k\)\} where each $H_k$ is an operator acting on radial functions. The pioneering paper using such intermediate space is the Mockenhaupt Diploma in which he proved weighted $L^p$ inequalities and then sharp $L^p_{\rad} L^2_{\sph}$ estimates for the disc multiplier operator, see either Mockenhaupt \[14\] or Córdoba \[3\]. Sharp endpoint bounds for the disk multiplier were obtained by Carbery-Romera-Soria \[4\]. Müller-Seeger \[15\] established some sharp mixed space-time $L^p_{\rad} L^2_{\sph}$ estimates in order to study a local smoothing of solutions for the linear wave equation. Córdoba-Latorre \[9\] revisited some classical conjecture including restriction estimate in harmonic analysis in this kind of mixed space-time. Gigante-Soria \[12\] studied a related mixed norm problem for Schrödinger maximal operators. Concerning the sphere restriction conjecture, Carli-Grafakos \[7\] also treated the same problem for spherically-symmetric functions and Cho-Guo-Lee \[8\] showed a restriction estimate for $q > 2(n+1)/n$ and $s \geq (n+2)/q - n/2$

\[1.5\]
$$\left\|\int_{S^n} e^{2\pi i x \cdot \xi} f(\xi) d\sigma(\xi)\right\|_{L^q(\mathbb{R}^{n+1})} \leq C \|f\|_{H^s(S^n)}, \quad x \in \mathbb{R}^{n+1},$$

where $H^s(S^n)$ is the Sobolev space of order $s$ on the sphere.
where $d\sigma$ is the induced Lebesgue measure on $S^n$ and $L^2(S^n)$ denote the $L^2$-Sobolev space of order $s$ on the sphere. An advantage of the proof consists in a fact that inequality (1.3) is based on $L^2$-spaces. The advantage of using the $L^2$-based Hilbert space also allows us to use effective the $TT^*$ arguments to obtain Strichartz estimate with a wider range of admissible indexes by compensating with extra regularity in angular direction; see Sterbenz [21] for wave equation, Cho-Lee [9] for general dispersive equations and the authors [18] for wave equation with an inverse-square potential. Concerning other results in this direction, Cho-Hwang-Kwon-Lee [10] studied profile decompositions of fractional Schrödinger equations under the angular regularity assumption.

In this paper, we prove that estimate (1.2) holds for all $p, q$ in (1.3) by compensating with some loss of angular derivatives. Our strategy is to use a spherical harmonic expanding as well as localized restriction estimates. In contrast to the radial case, e.g. [7][22], the main difficulty comes from the asymptotic behavior of the Bessel function $J_{\nu}(r)$ when $\nu \gg 1$. It is worth to point out that the method of treating cone restriction [17] is not valid since it can not be used to exploit the curvature property of paraboloid multiplier $e^{it|\xi|^2}$. We note that the bilinear argument used in [22], which is in spirit of Carleson-Sjölin argument or equivalently the $TT^*$ argument, can be used to deal with the oscillation of the paraboloid multiplier. To use this argument, one needs to write the Bessel function $J_{\nu}(r) \sim c_{\nu}r^{-1/2}e^{ir}$ when $r \gg 1$. This expression works well for small $\nu$ (corresponding to the radial case) but it seems complicate to write the Bessel function in that form when $\nu \gg 1$. Indeed, as in [37], one can do this when $\nu^2 \ll r$, but it will cause more loss of derivative for the case $\nu \lesssim r \lesssim \nu^2$, since it is difficult to capture simultaneously the oscillation and decay behavior of $J_{\nu}(r)$. Our new idea here is to establish a $L^4_{t,x}$-localized restriction estimate by directly analyzing the kernel associated with the Bessel function. The key ingredient is to explore the decay and oscillation property of $J_{\nu}(r)$ for $r \gg \nu$, and resonant property of paraboloid multiplier. We also have to overcome low decay shortage of $J_{\nu}(r)$ (when $\nu \sim r \gg 1$) by compensating a loss of angular regularity.

Before stating the main theorem, we introduce some notation. Incorporating the angular regularity, we set the infinitesimal generators of the rotations on Euclidean space:

$$\Omega_{j,k} := x_j \partial_k - x_k \partial_j$$

and define for $s \in \mathbb{R}$

$$\Delta_\theta := \sum_{j<k} \Omega_{2,j,k}, \quad |\Omega|^s = (-\Delta_\theta)^{s/2}.$$

Hence $\Delta_\theta$ is the Laplace-Beltrami operator on $S^{n-1}$. Define the Sobolev norm $\| \cdot \|_{H^s_{\text{sph}}(\mathbb{R}^n)}$ by setting

$$\|g\|_{H^s_{\text{sph}}(\mathbb{R}^n)}^p = \int_0^\infty \int_{S^{n-1}} |(1-\Delta_\theta)^{s/2}g(r\theta)|^p d\theta r^{n-1} dr.$$  

Given a constant $A$, we briefly write $A + \epsilon$ as $A_+$ or $A - \epsilon$ as $A_-$ for $0 < \epsilon \ll 1$.

Our main result is the following one.

**Theorem 1.1.** Let $n \geq 2$. The following estimates hold for all Schwartz functions $f$
to denote the statement

\[ \| f \sigma \|^\alpha \| L^\infty_{t,x}(\mathbb{R} \times \mathbb{R}^n) \leq C_p, q, n, s \| f (|\xi|^2, \xi) \|_{H^0_{sp}(\mathbb{R}^n)} , \]

where \( \sigma = (n - 2)(1/2 - 1/q_0) + 2/q_0, \)
- if \( 1 \leq q, p \leq \infty \) satisfy (1.3), then

\[ \| (f \sigma)^\alpha \|^\beta \| L^1_{t,x}(\mathbb{R} \times \mathbb{R}^n) \leq C_{p, q, n, S}(1 + |\Omega|)^s \| f \|_{L^p(S; \sigma)} , \]

where \( s = s(q, n) = \sigma_0 \alpha \) and \( 0 \leq \alpha \leq 1 \) satisfying \( 1/q = \alpha/q_0 + (1 - \alpha)/q_1. \)

Remark 1.1. Estimate (1.8) is an interpolation consequence of (1.7) and \( L^p \)-estimates in Bourgain-Guth [2]. Inequality (1.8) leads to the linear adjoint restriction estimate when \( q < 2(n+1)/n, q(n) \) with some loss of angular derivatives.

Remark 1.2. Since the sphere \( \mathbb{S}^n = \{ (\tau, \xi) : |\tau|^2 + |\xi|^2 = 1 \} \) is closely related to the paraboloid in sense of Taylor expansion \( \sqrt{1 - \rho^2} = 1 - 1/2 \rho^2 + O(\rho^4) \) near \( \rho = 0, \) it seems to be possible to show some modified version of (1.5) with \( H^{s, p}(\mathbb{S}^n) \)-norm on right hand side.

As an application of the modified restriction estimate, we show a result on the local smoothing estimate for the Schödinger equation for initial data with additional conditions angular regularity by Rogers’s argument in [20]. Our result here extend [20, Theorem 1] from \( q > 2(n + 3)/(n + 1) \) to \( q > 2(n + 1)/n \) under the assumption that initial data has additional angular regularity.

More precisely, we have the following local smoothing result.

Corollary 1.1. Let \( n \geq 2, q > 2(n+1)/n \) and \( s \) be as in Theorem (1.7). Then

\[ \| e^{i t \Delta} u_0 \|^\alpha \| L^1_{t,x}(0,1] \times \mathbb{R}^n \) \leq C \| 1 + \| \Omega \|^s u_0 \|_{W^{n, q}(\mathbb{R}^n)} , \]

where \( \alpha > 2n(1/2 - 1/q) - 2/q \) and \( W^{n, q}(\mathbb{R}^n) \) is the Sobolev space.

This paper is organized as follows: In Section 2, we introduce notation and present some basic facts about spherical harmonics and Bessel functions. Furthermore, we use the stationary phase argument to prove some properties of Bessel functions. Section 3 is devoted to the proof of Theorem 1.1. In Section 4, we prove the key Proposition 3.1. We prove Corollary 1.1 in the final section.

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2. Preliminaries

2.1. Notation. We use \( A \lesssim B \) to denote the statement that \( A \leq CB \) for some large constant \( C \) which may vary from line to line and depend on various parameters, and similarly employ \( A \sim B \) to denote the statement that \( A \lesssim B \lesssim A \). We also use \( A \ll B \) to denote the statement \( A \leq C^{-1} B \). If a constant \( C \) depends on a special parameter other than the above, we shall write it explicitly by subscripts. For instance, \( C_s \) should
be understood as a positive constant not only depending on \( p, q, n \) and \( S \), but also on \( \epsilon \).
Throughout this paper, pairs of conjugate indices are written as \( p, p' \), where \( \frac{1}{p} + \frac{1}{p'} = 1 \) with \( 1 \leq p \leq \infty \). Let \( R > 0 \) be a dyadic number, we define the dyadic annulus in \( \mathbb{R}^n \) by
\[
A_R := \{ x \in \mathbb{R}^n : \frac{R}{2} \leq |x| \leq R \}, \quad S_R := [R/2, R].
\]
For each \( M \in 2\mathbb{Z} \), we define \( L_M \) to be the class of Schwartz functions supported on a dyadic subset of the paraboloid in the form of
\[
(2.1) \quad \{ (\tau, \xi) \in \mathbb{R} \times \mathbb{R}^n : M \leq |\xi| \leq 2M, \tau = |\xi|^2 \}.
\]

### 2.2. Spherical harmonics expansions and Bessel function.

We recall an expansion formula with respect to the spherical harmonics. Let
\[
\xi = \rho \omega \quad \text{and} \quad x = r \theta \quad \text{with} \quad \omega, \theta \in \mathbb{S}^{n-1}.
\]
For every \( g \in L^2(\mathbb{R}^n) \), we have the expansion formula
\[
g(\xi) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} a_{k,\ell}(\rho) Y_{k,\ell}(\omega),
\]
where
\[
\{Y_{k,1}, \ldots, Y_{k,d(k)}\}
\]
is the orthogonal basis of the spherical harmonics space of degree \( k \) on \( \mathbb{S}^{n-1} \). This space is recorded by \( \mathcal{H}^k \) and it has the dimension
\[
d(k) = \frac{2k + n - 2}{k} \binom{k-1}{n+k-3} \sim \langle k \rangle^{n-2}.
\]
It is clear that we have the orthogonal decomposition of \( L^2(\mathbb{S}^{n-1}) \)
\[
L^2(\mathbb{S}^{n-1}) = \bigoplus_{k=0}^{\infty} \mathcal{H}^k.
\]

It follows that
\[
(2.3) \quad \|g(\xi)\|_{L^2} = \|a_{k,\ell}(\rho)\|_{L^2_{\rho,\ell}}.
\]

Using the spherical harmonic expansion, as well as \[19, 28\], we define the action of \((1 - \Delta_\omega)^{s/2}\) on \( g \) as follows
\[
(2.4) \quad (1 - \Delta_\omega)^{s/2} g = \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} (1 + k(k + n - 2))^{s/2} a_{k,\ell}(\rho) Y_{k,\ell}(\omega).
\]
Given \( s, s' \geq 0 \) and \( p, q \geq 1 \), define
\[
\norm{g}_{H^s_{\mu} \cap H^{s'}_{\mu} : p \to q} := \norm{(1 - \Delta)^{s/2} ((1 - \Delta_\omega)^{s/2} g)}_{L^q_{\mu}(\mathbb{R}^n) \cap L^p_{\mu}(\mathbb{S}^{n-1})},
\]
where \( \mu(\rho) = \rho^{n-1} d\rho \).
For our purpose, we need the inverse Fourier transform of \(a_{k,\ell}(\rho)Y_{k,\ell}(\omega)\). We recall the Bochner-Hecke formula, see [13] and [26, Theorem 3.10]

\[
\hat{g}(r\theta) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} 2\pi i^k Y_{k,\ell}(\theta) r^{-\frac{n-2}{2}} \int_{0}^{\infty} J_{\nu(k)}(2\pi r\rho) a_{k,\ell}(\rho) \rho^\frac{n}{2} d\rho.
\]

Here \(\nu(k) = k + \frac{n-2}{2}\) and the Bessel function \(J_{\nu}(r)\) of order \(\nu\) is defined by

\[
J_{\nu}(r) = \frac{(r/2)^\nu}{\Gamma(\nu + 1/2)\Gamma(1/2)} \int_{-1}^{1} e^{isr}(1 - s^2)^{(2\nu-1)/2} ds,
\]

where \(\nu > -1/2\) and \(r > 0\). It is easy to verify that there exists a constant \(C\) independent of \(\nu\) such that

\[
|J_{\nu}(r)| \leq \frac{Cr^\nu}{2\nu\Gamma(\nu + 1/2)\Gamma(1/2)} \left(1 + \frac{1}{\nu + 1/2}\right).
\]

To investigate a behavior of asymptotic bound on \(\nu\) and \(r\), we recall the Schl"afli integral representation \([35]\) of the Bessel function: for \(r \in \mathbb{R}^+\) and \(\nu > -\frac{1}{2}\)

\[
J_{\nu}(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ir\theta} \sin(\nu\theta) d\theta - \frac{\sin(\nu\pi)}{\pi} \int_{0}^{\infty} e^{-(r \sinh s + \nu s)} ds
\]

\(=: \tilde{J}_{\nu}(r) - E_{\nu}(r)\).

Clearly, \(E_{\nu}(r) = 0\) when \(\nu \in \mathbb{Z}^+\). An easy computation shows that

\[
|E_{\nu}(r)| = \left|\frac{\sin(\nu\pi)}{\pi} \int_{0}^{\infty} e^{-(r \sinh s + \nu s)} ds\right| \leq C(r + \nu)^{-1}.
\]

There is a number of references for the asymptotic behavior of a Bessel function, see e.g. [9, 23, 25, 35]. We recall some properties of a Bessel function for a convenience.

**Lemma 2.1** (Asymptotics of Bessel functions). Let \(\nu \gg 1\) and let \(J_{\nu}(r)\) be the Bessel function of order \(\nu\) defined as above. Then there exists a large constant \(C\) and small constant \(c\) independent of \(\nu\) and \(r\) such that:

- When \(r \leq \frac{\nu}{2}\), we have

\[
|J_{\nu}(r)| \leq Ce^{c(\nu + r)};
\]

- When \(\frac{\nu}{2} \leq r \leq 2\nu\), we have

\[
|J_{\nu}(r)| \leq C\nu^{-\frac{1}{2}}(\nu^{-\frac{1}{2}}|r - \nu| + 1)^{-\frac{1}{2}};
\]

- When \(r \geq 2\nu\), we have

\[
|J_{\nu}(r)| = r^{-\frac{1}{2}} \sum_{\pm} a_{\pm}(\nu, r)e^{\pm ir} + E(\nu, r),
\]

where \(|a_{\pm}(\nu, r)| \leq C\) and \(|E(\nu, r)| \leq Cr^{-1}\.\]
In this section, we prove Theorem 1.1 by using some localized linear estimates whose proof are postpone to the next section. Since inequality (1.8) is a special case of (1.8), we aim to prove (1.8). Since (1.8) is a direct consequence of the Stein-Tomas inequality [25] for the case \( p \leq 2 \), it suffices to prove (1.8) for the case \( p \geq 2 \). More precisely, we will only establish the estimate for \( q > 2(n+1)/n, (n+2)/q = n/p' \) with \( p \geq 2 \)

\[
\| (f d\sigma)^\vee \|_{L^q_t L^2_x(\mathbb{R}^n)} \lesssim C_{p,q,n,S} \| (1 + |\Omega|)^{\varepsilon} f \|_{L^p(S; d\sigma)}.
\]

Recall the notation \( \mathbb{L}_M \) and \( A_R \) in the subsection 2.1. We decompose \( f \) into a sum of dyadic supported functions

\[
f = \sum_M f_M,
\]

where \( f_M = f \chi_{\{ (r, \xi) : r|\xi|^2, M \leq |\xi| \leq 2M \} } \in \mathbb{L}_M \). It follows that

\[
\| (f d\sigma)^\vee \|_{L^q_t L^2_x(\mathbb{R}^n)} \leq \left( \sum_R \left( \sum_M \| (f_M d\sigma)^\vee \|_{L^q_t L^2_x(\mathbb{R}^n)} \right)^q \right)^{1/q}.
\]

(3.2)

To prove (3.1), we need localized linear restriction estimates.

**Proposition 3.1.** Assume \( f \in \mathbb{L}_1 \) and \( R > 0 \) is a dyadic number. Then the following linear restriction estimates hold true.

- Let \( q = 2 \), then

\[
\| (f d\sigma)^\vee \|_{L^q_t L^2_x(\mathbb{R}^n)} \lesssim \min \left\{ R^{\frac{1}{q}}, R^{\frac{1}{2}} \right\} \| f \|_{L^2(S; d\sigma)}.
\]

- Let \( q = 3p' \) with \( 2 \leq p \leq 4 \) and \( \varepsilon = (n-2)(\frac{1}{q} - \frac{1}{2}) + \frac{2}{q}, 0 < \varepsilon \ll 1 \), then

\[
\| (f d\sigma)^\vee \|_{L^q_t L^2_x(\mathbb{R}^n)} \lesssim \min \left\{ R^{(n-1)(\frac{1}{p} - \frac{1}{2}) + \varepsilon}, R^{\frac{1}{2}} \right\} \| (1 + |\Omega|)^{\varepsilon} f \|_{L^p(S; d\sigma)}.
\]

We postpone the proof of Proposition 3.1 to the next section, and we complete the proof of Theorem 1.1 by this proposition. By a scaling argument, we conclude from (3.3) that

\[
\| (f_M d\sigma)^\vee \|_{L^q_t L^2_x(\mathbb{R}^n)} \lesssim \min \left\{ (RM)^{\frac{1}{2}}, (RM)^{\frac{1}{3}} \right\} M^{n-\frac{n+2}{2} - \frac{n}{q}} \| f_M \|_{L^2(S; d\sigma)}.
\]

For any \((q, p)\) satisfying

\[
q > 2(n+1)/n, \ (n+2)/q = n/p' \quad \text{with} \quad p \geq 2,
\]

let \( \alpha = 2 - \frac{2}{q} - \frac{1}{p} \), then we choose \( \bar{q} = 3p' \) such that

\[
\frac{1}{q} = \frac{1}{2} + \frac{\alpha}{\bar{q}}, \quad \frac{1}{p} = \frac{1}{2} + \frac{\alpha}{\bar{p}}.
\]
From (3.4), we have that for $q = 3\tilde{p}'$ with $2 \leq \tilde{p}' \leq 4$ and $\tilde{\sigma} = (n - 2)(\frac{1}{2} - \frac{1}{q}) + \frac{2}{q}$

$$\|f M d\sigma\|_{L^q_{1,t}(\mathbb{R} \times A_R)} \lesssim \min\left\{\left( (RM)^{(n-1)(\frac{1}{2} - \frac{1}{q}) + \tilde{\epsilon}} \right), \left( RM \right)^{\frac{n}{q}} \right\} M^{n - \frac{n+2}{q} - \frac{n}{2} - q} \| (1 + |\Omega|) \tilde{\sigma} f_M \|_{L^{p}(S; d\sigma)},$$

where $0 < \tilde{\epsilon} \ll 1$. Therefore we obtain by an interpolation theorem

$$\|f M d\sigma\|_{L^q_{1,t}(\mathbb{R} \times A_R)} \lesssim \min\left\{\left( (RM)^{\frac{n}{q}}, (RM)^{- \frac{n+1}{2}[1 - \frac{2(n+1)}{qn}]+\epsilon} \right) \right\} \| (1 + |\Omega|) \tilde{\sigma} f_M \|_{L^{p}(S; d\sigma)},$$

Here $0 < \epsilon := \tilde{\epsilon} \alpha \ll 1$. According to (3.2), we obtain

$$\|f d\sigma\|_{L^q_{1,t}(\mathbb{R} \times C^n)} \lesssim \left( \sum_{R} \left( \sum_{M} \min\left\{\left( RM \right)^{\frac{n}{q}}, (RM)^{- \frac{n+1}{2}[1 - \frac{2(n+1)}{qn}]+\epsilon} \right) \right) \right) \left( \sum_{M} \left( (1 + |\Omega|) \tilde{\sigma} f_M \right) \right)^{\frac{q}{q'}}.$$

Since $q > 2(n+1)/n$, $\epsilon \ll 1$, and $R, M$ are both dyadic number, we have

$$\sup_{R > 0} \left( \sum_{M} \min\left\{\left( RM \right)^{\frac{n}{q}}, (RM)^{- \frac{n+1}{2}[1 - \frac{2(n+1)}{qn}]+\epsilon} \right) \right) < \infty,$$

and

$$\sup_{M > 0} \left( \sum_{R} \min\left\{\left( RM \right)^{\frac{n}{q}}, (RM)^{- \frac{n+1}{2}[1 - \frac{2(n+1)}{qn}]+\epsilon} \right) \right) < \infty.$$

Note that for $q > 2(n+1)/n > p \geq 2$, we have by the Schur lemma and embedding inequality

$$\|f d\sigma\|_{L^q_{1,t}(\mathbb{R} \times C^n)} \lesssim \left( \sum_{M} \left( (1 + |\Omega|) \tilde{\sigma} f_M \right)^p \right)^{\frac{1}{p}} \| (1 + |\Omega|) \tilde{\sigma} f \|_{L^{p}(S; d\sigma)}.$$

Choosing $q = q_0 = (2(n+1)/n)_+$ and $(n+2)/q_0 = n/p_0'$, we have

$$\|f d\sigma\|_{L^q_{1,t}(\mathbb{R} \times C^n)} \lesssim \| (1 + |\Omega|) \tilde{\sigma} f \|_{L^{p_0}(S; d\sigma)}.$$

This implies (1.7). Interpolating this inequality with the restriction estimate by Bourgain-Guth [2, Theorem 1], we prove (3.1). Hence, the proof of estimate (1.8) is completed.

4. Localized restriction estimate

In this section we prove Proposition 3.1. We start our proof by recalling

$$(f(\tau, \xi)d\sigma)^\vee(t, x) = \int_{\mathbb{R}^n} g(\xi)e^{2\pi i(x \cdot \xi + t|\xi|^2)} d\xi,$$
where \( g(\xi) = f(|\xi|^2, \xi) \in \mathcal{S}(\mathbb{R}^n) \) with \( \text{supp} \, g \subset \{ \xi : |\xi| \in [1, 2] \} \). We apply the spherical harmonic expansion to \( g \) to obtain
\[
g(\xi) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} a_{k,\ell}(\rho) Y_{k,\ell}(\omega).
\]
Recalling \( \nu(k) = k + (n - 2)/2 \), we have by (4.2)
\[
(fd\sigma)^y(t, x) = 2\pi r^{-n/2} \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} i^k Y_{k,\ell}(\theta) \int_0^\infty e^{-2\pi it\rho^2} J_{\nu(k)}(2\pi t\rho) a_{k,\ell}(\rho) \rho^2 \varphi(\rho) d\rho.
\]
Here we insert a harmless smooth bump function \( \varphi \) supported on the interval \((1/2, 4)\) into the above integral, since \( a_{k,\ell}(\rho) \) is supported on \([1, 2]\). Now we estimate the quantity \( \|(fd\sigma)^y\|_{L^q_y(L^\infty_{\mu(r)}(S_R))} \). To this end, we first prove the following lemma.

**Lemma 4.1.** Let \( \mu(r) = r^{n-1}dr \) and \( \omega(k) \) be a weight specified below. For \( q \geq 2 \), we have
\[
\left\| r^{-n/2} \left( \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \omega(k) \right) \int_0^\infty e^{it\rho^2} J_{\nu(k)}(r\rho) a_{k,\ell}(\rho) \rho^{n/2} \varphi(\rho) \rho^{n/2+1/q} d\rho \right\|_{L^q_y(L^\infty_{\mu(r)}(S_R))} \lesssim \left\| r^{-n/2} \left( \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \omega(k) \right) \int_0^\infty e^{it\rho^2} J_{\nu(k)}(r\rho) a_{k,\ell}(\rho) \rho^{n/2} \varphi(\rho) \rho^{n/2+1/q} d\rho \right\|_{L^q_y(L^\infty_{\mu(r)}(S_R))}.
\]

**Proof.** Since \( q \geq 2 \), the Minkowski inequality and the Fubini theorem show that the left hand side of (4.3) is bounded by
\[
\left\| r^{-n/2} \left( \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \omega(k) \right) \int_0^\infty e^{it\rho^2} J_{\nu(k)}(r\rho) a_{k,\ell}(\rho) \rho^{n/2} \rho d\rho \right\|_{L^q_y(L^\infty_{\mu(r)}(S_R))} \]
We rewrite this by making the variable change \( \rho^2 \rightarrow \rho \)
\[
\left\| r^{-n/2} \left( \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \omega(k) \right) \int_0^\infty e^{it\rho^2} J_{\nu(k)}(\rho) a_{k,\ell}(\rho) \rho^{n/2} \rho d\rho \right\|_{L^q_y(L^\infty_{\mu(r)}(S_R))} \]
We use the Hausdorff-Young inequality with respect to \( t \) and we change variables back to obtain
\[
\left\| r^{-n/2} \left( \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \omega(k) \right) \int_0^\infty e^{it\rho^2} J_{\nu(k)}(\rho) a_{k,\ell}(\rho) \rho^{n/2} \rho d\rho \right\|_{L^q_y(L^\infty_{\mu(r)}(S_R))} \]

Now we prove that the inequalities (5.3) and (5.4) with \( R \leq 1 \). For doing this, we need
Lemma 4.2. Let $q \geq 2$ and $R \lesssim 1$, we have the following estimate

$$
(4.5) \quad \| (f \, d\sigma)^\vee \|_{L^q_{t,x} (\mathbb{R} \times A_R)} \lesssim R^{\frac{3}{4}} \left( \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \omega(k) \| a_{k,\ell} (\rho) \varphi(\rho) \|_{L^q_{\rho}}^2 \right)^{\frac{1}{2}},
$$

where $\omega(k) = (1 + k)^{2(n-1)(1/2 - 1/q)}$.

We postpone the proof of this lemma for a moment. Note that for $q' \leq 2 \leq p$, we use (4.5), (2.4), the Minkowski inequality and the Hölder inequality to obtain

$$
\| (f \, d\sigma)^\vee \|_{L^q_{t,x} (\mathbb{R} \times A_R)} \lesssim R^{\frac{3}{4}} \left( \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \omega(k) |a_{k,\ell} (\rho)|^2 \right)^{\frac{1}{2}} \varphi(\rho) \|_{L^p_\rho},
$$

$$
\lesssim R^{\frac{3}{4}} \| g \|_{L^p_\rho H^{m'}(S^{n-1})} \lesssim R^{\frac{3}{4}} \| g \|_{L^p_\rho\mu^{m'}(S^{n-1})},
$$

where $m = (n-1)\left(\frac{1}{2} - \frac{1}{q} \right)$. In particular, for $q = 2$ and $4 \leq q \leq 6$, this proves (3.3) and (3.4) when $R \lesssim 1$. Hence it suffices to consider the case $R \gg 1$ once we prove Lemma 4.2.

Proof of Lemma 4.2. By scaling argument in variables $t, x$ and (4.2), we obtain

$$
(4.6) \quad \| (f \, d\sigma)^\vee \|_{L^q_{t,x} (\mathbb{R} \times A_R)} \lesssim R^{-\frac{n-2}{2}} \left( \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \omega(k) \int_0^\infty e^{-it\rho^2} J_{\nu(k)}(r\rho)a_{k,\ell}(\rho)\rho^{\frac{n-2}{2}} \varphi(\rho) \, d\rho \right) \|_{L^q_{t,x} (\mathbb{R} \times A_R)}. 
$$

By Sobolev’s embedding, (2.3) and (2.4), we have

$$
\| (f \, d\sigma)^\vee \|_{L^q_{t,x} (\mathbb{R} \times A_R)} \lesssim R^{-\frac{n-2}{2}} \left( \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \omega(k) \int_0^\infty e^{it\rho^2} J_{\nu(k)}(r\rho)a_{k,\ell}(\rho)\rho^{\frac{n-2}{2}} \varphi(\rho) \rho \, d\rho \right)^{\frac{1}{2}} \|_{L^q_{t,x} (\mathbb{R} \times A_R)}. 
$$

By Lemma 1.1, it is enough to show

$$
\| R^{-\frac{n-2}{2}} \left( \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \omega(k) \| J_{\nu(k)}(r\rho)a_{k,\ell}(\rho)\rho^{(n-2)/2 + 1/q} \|_{L^q_{\rho}}^2 \right)^{\frac{1}{2}} \|_{L^q_{t,x} (\mathbb{R} \times A_R)} \lesssim R^{\frac{3}{4}} \left( \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \omega(k) \| a_{k,\ell}(\rho) \varphi(\rho) \|_{L^q_{\rho}}^2 \right)^{\frac{1}{2}},
$$

$$
\| R^{-\frac{n-2}{2}} \left( \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \omega(k) \| a_{k,\ell}(\rho) \varphi(\rho) \|_{L^q_{\rho}}^2 \right)^{\frac{1}{2}} \|_{L^q_{t,x} (\mathbb{R} \times A_R)} \lesssim R^{\frac{3}{4}} \left( \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \omega(k) \| a_{k,\ell}(\rho) \varphi(\rho) \|_{L^q_{\rho}}^2 \right)^{\frac{1}{2}}.
$$
Writing briefly ν = ν(k), and noting that R > r < 2R and 1 < ρ < 2, we have by (2.6)
\[ \left\| r^{-\frac{n-2}{2}} \sum_{k=0}^{\infty} \omega(k) \| J_{\nu(k)}(r \rho) a_{k,\ell}(\rho) \varphi(\rho) \| L^q_{\nu} \right\| L^q_{\nu}(\mathbb{R}^n) \leq \left( \int_R^{2R} \frac{r^{n-2}}{r} \left| \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \omega(k) \| J_{\nu(k)}(r \rho) a_{k,\ell}(\rho) \varphi(\rho) \| L^q_{\nu} \right|^2 r^{n-1} dr \right)^{\frac{1}{2}}. \]

In the last inequality, we use the Stirling formula Γ(ν + 1) ∼ \sqrt{\nu/\epsilon} and the fact that R ≲ 1 and ν ∼ (n - 2)/2.

Now we are in a position to prove Proposition 3.1 when R ≫ 1. We first prove (3.3) by making use of (4.11). Since supp g ⊆ \{ξ : |ξ| \in [1,2]\}, we may assume |\xi_n| ∼ 1. Then we freeze one spatial variable, say x_n, with |x_n| ≤ R and free other spatial variables x' = (x_1, ..., x_{n-1}). After making the change of variables η_j = ξ_j, η_n = |ξ|^2 with j = 1, ..., n - 1, we use the Plancherel theorem on the spacetime Fourier transform in (t, x') to obtain (3.3).

When R ≫ 1, inequality (3.4) is a consequence of the interpolation theorem and the following proposition.

**Proposition 4.1.** Assume f ∈ L_1 and R ≫ 1 is a dyadic number. For every small constant 0 < \epsilon ≪ 1, we have the following inequalities

- For q = 4, we have
  \[ \| (f \, d\sigma)^4 \|_{L^4_{\infty}(\mathbb{R} \times A_R)} \lesssim R^{-\frac{n-1}{4} + \epsilon} \| (1 + |\Omega|)^{\frac{n}{2}} f \|_{L^1(S; d\sigma)}. \]

- For q = 6, we have
  \[ \| (f \, d\sigma)^6 \|_{L^6_{\infty}(\mathbb{R} \times A_R)} \lesssim R^{-\frac{n-1}{6} + \epsilon} \| (1 + |\Omega|)^{\frac{n}{3}} f \|_{L^2(S; d\sigma)}. \]

**Remark 4.1.** It seems to be possible to remove the \epsilon-loss in (4.3), but we do not purchase this option here because we do not need it in this paper.

To prove this proposition, we first show

**Lemma 4.3.** Assume f ∈ L_1 and R ≫ 1. We have the following estimate

\[ \| (f \, d\sigma)^\nu \|_{L^\nu_{\infty}(\mathbb{R} \times A_R)} \lesssim R^{-\frac{n-1}{\nu} + \epsilon} \| g \|_{L^\nu H^{\frac{n}{2} + 1}(\mathbb{R}^{n-1})}, \]

where 0 < \epsilon ≪ 1, and g(\xi) = f(|\xi|^2, \xi).

**Proof.** By the scaling argument and (1.2), it suffices to estimate the quantity

\[ \left\| r^{-\frac{n-2}{2}} \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} Y_k,\ell(\theta) \int_0^\infty e^{-\nu r^2} J_{\nu(k)}(r \rho) a_{k,\ell}(\rho) \rho^{\frac{n}{2}} \varphi(\rho) \, d\rho \right\|_{L^1_{\infty}(\mathbb{R} \times A_R)}. \]
In the following, we consider the three cases. For the first two cases, we establish the estimates for general \( q \geq 4 \) so that we can use them directly for \( q = 6 \) later.

\( \bullet \) Case 1: \( k \in \Omega_1 := \{ k : R \ll \nu(k) \} \). Let \( \omega(k) = (1 + k)^{2(n-1)/(2 - 1/q)} \) again. We have by a similar argument as in the proof of Lemma 1.2

\[
\left\| \frac{-n-2}{2} \sum_{k \in \Omega_1} \sum_{\ell = 1}^{d(k)} e^{it\rho^2} J_{\nu(k)}(r\rho)a_{k,\ell}(\rho)\rho^{\frac{n}{2}} \varphi(\rho) \right\|_{L^p_{t,x}(\mathbb{R} \times A_R)} \\
\lesssim \left\| \frac{-n-2}{2} \left( \sum_{k \in \Omega_1} \sum_{\ell = 1}^{d(k)} \omega(k) \int_0^\infty e^{it\rho^2} J_{\nu(k)}(r\rho)a_{k,\ell}(\rho)\varphi(\rho)\rho^{\frac{n}{2}} \right)^{\frac{1}{2}} \right\|_{L^p_{t,x}(\mathbb{R} \times A_R)} \\
\lesssim \left\| \frac{-n-2}{2} \left( \sum_{k \in \Omega_1} \sum_{\ell = 1}^{d(k)} \omega(k) \left| J_{\nu(k)}(r\rho)a_{k,\ell}(\rho)\varphi(\rho)\right|^{\frac{n-2}{2} + 1/q} \right)^{\frac{1}{2}} \right\|_{L^p_{t,x}(\mathbb{R} \times A_R)}. 
\]

Recall that for \( R \gg 1 \) and \( k \in \Omega_1 \), we have \( |J_{\nu(k)}(r)| \lesssim e^{-c(r)} \) by (2.9). Using \( R < r < 2R \) and \( 1 < \rho < 2 \), we obtain

\[
\left\| \frac{-n-2}{2} \left( \sum_{k \in \Omega_1} \sum_{\ell = 1}^{d(k)} \omega(k) \left| J_{\nu(k)}(r\rho)a_{k,\ell}(\rho)\varphi(\rho)\right|^{\frac{n-2}{2} + 1/q} \right)^{\frac{1}{2}} \right\|_{L^p_{t,x}(\mathbb{R} \times A_R)} \\
\lesssim \left( \int_R^{2R} r^{-\frac{n-2}{2}} \left( \sum_{k \in \Omega_1} \sum_{\ell = 1}^{d(k)} \omega(k) e^{-c(r)} \left| a_{k,\ell}(\rho)\varphi(\rho)\right|^{\frac{n-2}{2} + 1/q} \right)^{\frac{1}{2}} dr \right)^{\frac{1}{2}} \\
\lesssim e^{-cR} \left( \sum_{k \in \Omega_1} \sum_{\ell = 1}^{d(k)} \omega(k) \left| a_{k,\ell}(\rho)\varphi(\rho)\right|^{\frac{n-2}{2} + 1/q} \right)^{\frac{1}{2}}. 
\]

By Minkowski’s inequality and Hölder’s inequality, we obtain

\[
\left\| \frac{-n-2}{2} \sum_{k \in \Omega_1} \sum_{\ell = 1}^{d(k)} e^{it\rho^2} J_{\nu(k)}(r\rho)a_{k,\ell}(\rho)\rho^{\frac{n}{2}} \varphi(\rho) \right\|_{L^p_{t,x}(\mathbb{R} \times A_R)} \\
\lesssim e^{-cR} \left( \sum_{k = 0}^{\infty} \sum_{\ell = 1}^{d(k)} \omega(k) \left| a_{k,\ell}(\rho)\varphi(\rho)\right|^{\frac{n-2}{2} + 1/q} \right)^{\frac{1}{2}}. 
\]

By (4.11)
we have for $q$ Sobolev embedding, the Strichartz estimate and the fact $\sup p$

\begin{equation}
\|(f \, d\sigma)^\vee\|_{L^q_t L^r_x(\mathbb{R} \times \mathbb{R}^n)} \lesssim \|f \, d\sigma\|^\vee_{L^q(\mathbb{R}; H^m_u(\mathbb{R}^n))} \lesssim \|g\|_{H^m_u(\mathbb{R}^n)} \lesssim \|g\|_{L^2(\mathbb{R}^n)}
\end{equation}

where $m = \frac{(q-2)n-4}{2q} \geq 0$ since $n \geq 2$. If $g = \bigoplus_{k \in \Omega_2} \mathcal{H}^k$, then

\begin{equation}
\|g\|_{L^2_S(S^{n-1})}^2 = \sum_{k \in \Omega_2} \sum_{\ell = 1}^{d(k)} |a_{k,\ell}|^2
\end{equation}

\begin{equation}
\lesssim R^{-2(n-1)/2} \sum_{k \in \Omega_2} \sum_{\ell = 1}^{d(k)} (1 + k)^{2(1-n)/2} |a_{k,\ell}|^2
\end{equation}

\begin{equation}
\lesssim R^{-2(n-1)/2} \|g\|^2_{H^m_u(n-1)/2} \|g\|_{H^m_u(n-1)/2} \|g\|_{L^2_{S}(S^{n-1})}.
\end{equation}

Since $\text{supp} g \subset \{\xi \in \mathbb{R}^n : |\xi| \in [1,2]\}$ and $p \geq 2$, we have by Hölder’s inequality and

\begin{equation}
\left\| e^{-it\rho^2} J_{\nu(k)}(r \rho) \phi(\rho) \right\|_{L^2_{t,\rho}(\mathbb{R} \times \mathbb{R}^n)}
\end{equation}

In particular, when $q = p = 4$, inequality \(4.14\) implies that

\begin{equation}
\left\| e^{-it\rho^2} J_{\nu(k)}(r \rho) \phi(\rho) \right\|_{L^4_{t,\rho}(\mathbb{R} \times \mathbb{R}^n)}
\end{equation}

\begin{equation}
\lesssim R^{-2(n-1)/4} \|g\|^2_{L^4_{t}H^m_u(n-1)/4} \|g\|_{L^4_{t}H^m_u(n-1)/4} \|g\|_{L^2_{S}(S^{n-1})}.
\end{equation}

• Case 3: $k \in \Omega_3 := \{k : \nu(k) \ll R\}$. We need the following lemma about the oscillation and decay property of a Bessel function. This lemma was proved by Barcelo-Cordoba \cite{[3]}.

**Lemma 4.4** (Oscillation and asymptotic property, \cite{[3]}). *Let $\nu > 1/2$ and $r > \nu + \nu^{1/3}$. There exists a constant number $C$ independent of $r$ and $\nu$ such that*

\begin{equation}
J_{\nu}(r) = \sqrt{\frac{2}{\pi}} \frac{\cos \theta(r)}{(r^2 - \nu^2)^{1/4}} + h_\nu(r),
\end{equation}

where $\theta(r) = (r^2 - \nu^2)^{1/2} - \nu \arccos \frac{2}{r}$ and

\begin{equation}
|h_\nu(r)| \leq C \left( \left( \frac{\nu^2}{(r^2 - \nu^2)^{1/4}} + \frac{1}{r} \right) 1_{[\nu + \nu^{1/3}, 2\nu]}(r) + \frac{1}{r} 1_{[2\nu, \infty]}(r) \right).
\end{equation}

Note that $\nu(k) = k + (n-2)/2$ and $k \in \Omega_3$, we can write

\begin{equation}
J_{\nu}(r) = I_{\nu}(r) + \bar{I}_{\nu}(r) + h_\nu(r), \quad \text{where } |h_\nu(r)| \lesssim r^{-1},
\end{equation}
and

\[ I_\nu(r) = \frac{\sqrt{2/\pi} e^{i\theta(r)}}{(r^2 - \nu^2)^{1/4}}. \]

A simple computation yields to

\[
\begin{aligned}
\theta'(r) &= (r^2 - \nu^2)^{1/2} - 1, \\
\theta''(r) &= (r^2 - \nu^2)^{-1/2} - (r^2 - \nu^2)^{1/2} - (r^2 - \nu^2)^{-1/2} - \nu^2 r^{-2}, \\
\theta'''(r) &= \frac{\nu^2}{r} (r^2 - \nu^2)^{-3/2} - 2 (3 + \frac{2\nu^2}{r^2}).
\end{aligned}
\]

(4.18)

Using Sobolev embedding on sphere and Minkowski’s inequality, we estimate

\[
\left\| r^{-\frac{n-2}{2}} \sum_{k \in \Omega_3} \sum_{\ell = 1}^{d(k)} \int_0^\infty e^{-it\rho^2} J_{\nu(k)}(r\rho) a_{k,\ell}(\rho) \rho^{\frac{n}{2}} \varphi(\rho) \, d\rho \right\|_{L^p_t(L^1_x(\mathbb{R} \times \Omega_3))} \lesssim R^{-\frac{n-2}{4}} \left\| \left( \sum_{k \in \Omega_3} \sum_{\ell = 1}^{d(k)} (1 + k)^{(n-1)/2} \left\| \int_0^\infty e^{-it\rho^2} J_{\nu(k)}(r\rho) a_{k,\ell}(\rho) \rho^{\frac{n}{2}} \varphi(\rho) \, d\rho \right\|_2^2 \right\|_{L^1_t(L^4_x(S_R))}^{1/2}
\]

\[
\lesssim R^{-\frac{n-2}{4}} \left\| \left( \sum_{k \in \Omega_3} \sum_{\ell = 1}^{d(k)} (1 + k)^{(n-1)/2} \left\| \int_0^\infty e^{-it\rho^2} J_{\nu(k)}(r\rho) a_{k,\ell}(\rho) \rho^{\frac{n}{2}} \varphi(\rho) \, d\rho \right\|_2^2 \right\|_{L^1_t(L^4_x(S_R))}^{1/2}
\]

Since \( J_\nu(r) = I_\nu(r) + I_{\nu}(r) + h_\nu(r) \), it suffices to estimate two terms

\[
\left( \sum_{k \in \Omega_3} \sum_{\ell = 1}^{d(k)} (1 + k)^{(n-1)/2} \left\| \int_0^\infty e^{-it\rho^2} h_{\nu(k)}(r\rho) a_{k,\ell}(\rho) \rho^{\frac{n}{2}} \varphi(\rho) \, d\rho \right\|_2^2 \right)^{1/2}
\]

(4.19)

\[
\lesssim R^{-3/4} \| g \|_{L^p_{t}H^{\frac{n-1}{4}}(\mathbb{S}^{n-1})} \]

and

\[
\left( \sum_{k \in \Omega_3} \sum_{\ell = 1}^{d(k)} (1 + k)^{(n-1)/2} \left\| \int_0^\infty e^{-it\rho^2} I_{\nu(k)}(r\rho) a_{k,\ell}(\rho) \rho^{\frac{n}{2}} \varphi(\rho) \, d\rho \right\|_2^2 \right)^{1/2}
\]

(4.20)

\[
\lesssim R^{-1/2 + \epsilon} \| g \|_{L^p_{t}H^{\frac{n-1}{4}}(\mathbb{S}^{n-1})}^{1/2}.
\]

For the first purpose, we consider the operator

\[ T_\nu(a)(t, r) = \chi \left( \frac{r}{R} \right) \int_0^\infty e^{-it\rho^2} h_\nu(r\rho) a(\rho) \rho^{\frac{n}{2}} \varphi(\rho) \, d\rho \]

where \( |h_\nu(r)| \leq C/r \). By a similar argument as in the proof of Lemma 4.1, it is easy to see

\[
||T_\nu(a)(t, r)||_{L^q_t} \leq R^{-1/4} ||a\varphi||_{L^q_x}. \]

(4.21)
Hence we have
\[
\left( \sum_{k \in \Omega_3} \sum_{\ell = 1}^{d(k)} (1 + k)^{(n-1)/2} \left\| \int_0^\infty e^{-it \rho^2} h_{\nu(k)}(r \rho) a_{k, \ell}(\rho) \rho^{2 \alpha} \varphi(\rho) \, d\rho \right\|_{L^2_t(\mathbb{R}^4)} \right)^{1/2} \\
\lesssim R^{-3/4} \left( \sum_{k \in \Omega_3} \sum_{\ell = 1}^{d(k)} (1 + k)^{(n-1)/2} \left\| a_{k, \ell}(\rho) \varphi(\rho) \right\|_{L^{4/3}} \right)^{1/2} \\
\lesssim R^{-3/4} \left( \sum_{k \in \Omega_3} \sum_{\ell = 1}^{d(k)} (1 + k)^{(n-1)/2} \left| a_{k, \ell}(\rho) \right|^2 \varphi \right)_{L^{4/3}}^{1/2} \\
\lesssim R^{-3/4} \left\| g \right\|_{L^\infty_t H^{\frac{n-1}{4}}(S^{n-1})}
\]
which implies (4.19).

Next we prove (4.20). To this end, let \( \beta(\rho) = \rho^{2 \alpha} \varphi(\rho) \), we see that
\[
\left( \sum_{k \in \Omega_3} \sum_{\ell = 1}^{d(k)} (1 + k)^{(n-1)/2} \left\| \int_0^\infty e^{-it \rho^2} I_{\nu(k)}(r \rho) a_{k, \ell}(\rho) \rho^{2 \alpha} \varphi(\rho) \, d\rho \right\|_{L^2_t(\mathbb{R}^4)} \right)^{1/2} 
\]
\[
= \left( \sum_{k \in \Omega_3} \sum_{\ell = 1}^{d(k)} (1 + k)^{(n-1)/2} \left\| \int_{\mathbb{R}^2} e^{-it(\rho_1^2 - \rho_2^2)} I_{\nu(k)}(r \rho_1) I_{\nu(k)}(r \rho_2) \\
\times a_{k, \ell}(\rho_1) a_{k, \ell}(\rho_2) \beta(\rho_1) \beta(\rho_2) \, d\rho_1 d\rho_2 \right\|_{L^2_t(\mathbb{R}^4)} \right)^{1/2} 
\]
\[
\leq \left( \sum_{k \in \Omega_3} (1 + k)^{(n-1)/2} \left\| \int_{\mathbb{R}^2} e^{-it(\rho_1^2 - \rho_2^2)} I_{\nu(k)}(r \rho_1) I_{\nu(k)}(r \rho_2) \\
\times \sum_{\ell = 1}^{d(k)} a_{k, \ell}(\rho_1) a_{k, \ell}(\rho_2) \beta(\rho_1) \beta(\rho_2) \, d\rho_1 d\rho_2 \right\|_{L^2_t(\mathbb{R}^4)} \right)^{2} 
\]
\[
= \left( \sum_{k \in \Omega_3} (1 + k)^{(n-1)/2} \left( \int_{\mathbb{R}^4} \sum_{\ell = 1}^{d(k)} \sum_{\ell' = 1}^{d(k)} a_{k, \ell}(\rho_1) a_{k, \ell}(\rho_2) \beta(\rho_1) \beta(\rho_2) \beta(\rho_3) \beta(\rho_4) \right) \\
\int_{\mathbb{R}^4} e^{-it(\rho_1^2 - \rho_2^2 + \rho_3^2 - \rho_4^2)} dt K(R, \nu; \rho_1, \rho_2, \rho_3, \rho_4) \, d\rho_1 d\rho_2 d\rho_3 d\rho_4 \right)^{1/2} \right)^{2}
\]
where the kernel
\[
K(R, \nu; \rho_1, \rho_2, \rho_3, \rho_4) = \int_0^\infty \chi(\frac{r}{R}) e^{i(\theta(\rho_1 r) - \theta(\rho_2 r) + \theta(\rho_3 r) - \theta(\rho_4 r))} \frac{1}{(r \rho_1^2 - \nu^2)^{1/4} (r \rho_2^2 - \nu^2)^{1/4} (r \rho_3^2 - \nu^2)^{1/4} (r \rho_4^2 - \nu^2)^{1/4} dr.
\]
Now we analyze the kernel \( K \). Let
\[
\phi(r; \rho_1, \rho_2, \rho_3, \rho_4) = \theta(\rho_1 r) - \theta(\rho_2 r) + \theta(\rho_3 r) - \theta(\rho_4 r).
\]
Hence if $\rho_1^2 - \rho_2^2 = \rho_4^2 - \rho_3^2$, we have by \(4.18\)

\[
\phi'_r = (\rho_1^2 - \rho_2^2)r' \left( \frac{1}{\sqrt{(r\rho_1)^2 - \nu^2 + (r\rho_2)^2 - \nu^2}} - \frac{1}{\sqrt{(r\rho_3)^2 - \nu^2 + (r\rho_4)^2 - \nu^2}} \right)
\]

\[
= \frac{(\rho_1^2 - \rho_2^2)(\rho_3^2 - \rho_4^2)r^3}{\sqrt{(r\rho_1)^2 - \nu^2 + (r\rho_2)^2 - \nu^2} \left( \sqrt{(r\rho_3)^2 - \nu^2 + (r\rho_4)^2 - \nu^2} \right)}.
\]

Since $k \in \Omega_3$, one has $r \gg \nu(k)$. Therefore we have

\[
|\phi'_r| \geq |\rho_1^2 - \rho_2^2| \cdot |\rho_3^2 - \rho_4^2|.
\]

Applying integration by parts with respect to $r$ to \(4.23\), we have for any $N \geq 0$

\[
K(R, \nu; \rho_1, \rho_2, \rho_3, \rho_4) \lesssim R^{-1} \left( 1 + R|\rho_1^2 - \rho_2^2| \cdot |\rho_3^2 - \rho_4^2| \right)^{-N},
\]

when $\rho_1^2 - \rho_2^2 = \rho_4^2 - \rho_3^2$. Let $b_{k,\ell}(\rho) = 2a_{k,\ell}(\sqrt{\rho})\beta(\sqrt{\rho})/\sqrt{\rho}$, from \(4.22\) and \(4.24\), it suffices to estimate

\[
\left( \sum_{k \in \Omega_3} (1 + k)^{(n-1)/2} \left( \int_{\mathbb{R}^4} \delta(\rho_1 - \rho_2 + \rho_3 - \rho_4)K(R, \nu(k); \sqrt{\rho_1}, \sqrt{\rho_2}, \sqrt{\rho_3}, \sqrt{\rho_4}) \right) \right)^{1/2}
\]

\[
\times \sum_{k \in \Omega_3} k_{k,\ell}(\rho_1) b_{k,\ell}(\rho_2) \sum_{\ell' = 1}^{d(k)} b_{k,\ell'}(\rho_3) b_{k,\ell'}(\rho_4) d\rho_1 d\rho_2 d\rho_3 d\rho_4 \right)^{1/2}
\]

\[
\leq R^{-1} \left( \sum_{k \in \Omega_3} (1 + k)^{(n-1)/2} \left( \int_{\mathbb{R}^3} (1 + R|\rho_1 - \rho_2| |\rho_3 - \rho_2|) \right)^{-N} \right.
\]

\[
\times \sum_{k \in \Omega_3} b_{k,\ell}(\rho_1) b_{k,\ell}(\rho_2) \sum_{\ell' = 1}^{d(k)} b_{k,\ell'}(\rho_3) b_{k,\ell'}(\rho_1 - \rho_2 + \rho_3) d\rho_1 d\rho_2 d\rho_3 \right)^{1/2}
\]

\[
\leq R^{-1} \left( \sum_{k \in \Omega_3} (1 + k)^{(n-1)/2} \left( \int_{\mathbb{R}^3} (1 + R|\rho_1 - \rho_2| |\rho_3 - \rho_2|) \right)^{-N} \right.
\]

\[
\times b_k(\rho_1) b(\rho_2) b(\rho_3) b(\rho_1 - \rho_2 + \rho_3) d\rho_1 d\rho_2 d\rho_3 \right)^{1/2}
\]

where $b_k(\rho) = \left( \sum_{\ell = 1}^{d(k)} \left| b_{k,\ell}(\rho) \right|^2 \right)^{1/2}$. Then we aim to estimate

\[
\int_{\mathbb{R}^3} b(\rho_1) b(\rho_2) b(\rho_3) b(\rho_1 - \rho_2 + \rho_3) (1 + R|\rho_1 - \rho_2| |\rho_3 - \rho_2|)^N d\rho_1 d\rho_2 d\rho_3 \lesssim R^{-1+\epsilon} ||b||_{L^2}^4.
\]

\[
\int_{\mathbb{R}^3} b(\rho_1) b(\rho_2) b(\rho_3) b(\rho_1 - \rho_2 + \rho_3) (1 + R|\rho_1 - \rho_2| |\rho_3 - \rho_2|)^N d\rho_1 d\rho_2 d\rho_3 \lesssim R^{-1+\epsilon} ||b||_{L^2}^4.
\]
Indeed once we have proved (4.25), we show

\[
\left\| \sum_{k \in \Omega_3} \sum_{\ell=1}^{d(k)} (1 + k)^{(n-1)/2} \left| \int_0^\infty e^{-it\rho^2} I_{\ell(k)}(r\rho)a_{k,\ell}(\rho)\rho^{d/2} \varphi(\rho) d\rho \right|^2 \right\|_{L^4_t(R; L^4_x(S_R))}^{1/2} \]

\[
\lesssim R^{-1+\epsilon} \sum_{k \in \Omega_3} (1 + k)^{(n-1)/2 + \frac{1}{2} \cdot \epsilon} (1 + k)^{-\frac{1}{2} - \epsilon} \|b_k\|_{L^4_x}^2
\]

\[
\lesssim R^{-1+2\epsilon} \left( \sum_{k \in \Omega_3} (1 + k)^n \right)^{d(k)} \left( \sum_{\ell=1}^{d(k)} \|b_{k,\ell}(\rho)\|_{L^4_x}^2 \right)^{1/2} \|
\]

\[
\lesssim R^{-1+2\epsilon} \left( \sum_{k \in \Omega_3} (1 + k)^n \right)^{d(k)} \left( \sum_{\ell=1}^{d(k)} \|b_{k,\ell}(\rho)\|_{L^4_x}^2 \right)^{1/2} \|
\]

which implies (4.20). Therefore, it remains to prove

\[
(4.26) \quad \left( \sum_{i,j} \left\| \int \frac{b(\rho_1) b(\rho_2) b(\rho_3) b(\rho_1 - \rho_2 + \rho_3)}{\left( 1 + R|\rho_1 - \rho_2||\rho_3 - \rho_2| \right)^N} d\rho_1 d\rho_2 d\rho_3 \right\|_{L^4_x}^4 \right)^{1/2} \lesssim R^{-1+\epsilon} \|b\|_{L^4_x}^4.
\]

For \( R = 2^{k_0} \gg 1 \), we decompose the integral into

\[
(4.27) \quad \left( \sum_{(i,j) \in \mathbb{N}^2 : i+j \geq k_0} + \sum_{(i,j) \in \mathbb{N}^2 : i+j \leq k_0} \right) R^{-N(2^{N(i+j)+1})} \left( \sum_{|\rho_1 - \rho_2| \geq 2^{-i}} b(\rho_1) d\rho_1 \right) \left( \sum_{|\rho_3 - \rho_2| \geq 2^{-j}} b(\rho_3) d\rho_3 \right) \]

To estimate it, we need the following lemma.

**Lemma 4.5.** We have the following estimate for the integral

\[
(4.28) \quad \left( \sum_{|\rho_1 - \rho_2| \geq 2^{-i}} b(\rho_1) d\rho_1 \right) \left( \sum_{|\rho_3 - \rho_2| \geq 2^{-j}} b(\rho_3) b(\rho_1 - \rho_2 + \rho_3) d\rho_3 \right) \lesssim 2^{-(i+j)} \|b\|_{L^4_x}^4.
\]

**Proof.** We first have by Hölder’s inequality

\[
(4.29) \quad \left( \sum_{|\rho_1 - \rho_2| \geq 2^{-i}} b(\rho_3) d\rho_3 \right) \left( \sum_{|\rho_3 - \rho_2| \geq 2^{-j}} b(\rho_1 - \rho_2 + \rho_3) d\rho_3 \right)^{1/2}
\]

\[
\lesssim \left( \sum_{|\rho_1 - \rho_2| \geq 2^{-i}} b(\rho_3) d\rho_3 \right) \left( \sum_{|\rho_1 - \rho_2| \geq 2^{-j}} b(\rho_1 - \rho_2 + \rho_3) d\rho_3 \right)^{1/2}
\]

\[
\lesssim \left( \sum_{|\rho_1 - \rho_2| \geq 2^{-i}} b(\rho_3) d\rho_3 \right) \left( \sum_{|\rho_1| \geq 2^{-j}} b(\rho_1 + \rho) d\rho \right)^{1/2}
\]

\[
\lesssim \left( \sum_{|\rho_1 - \rho_2| \geq 2^{-i}} b(\rho_3) d\rho_3 \right) \left( \sum_{|\rho - \rho_1| \geq 2^{-j}} b(\rho) d\rho \right)^{1/2}.
\]
Let $I$ be the left hand side of (4.28). We estimate $I$ by (4.29) and Hölder’s inequality
\[
\int b(\rho_2) \int |b(\rho_1)| \int |b(\rho_3)| \int b(\rho_1) |b(\rho_2)| b(\rho_3) d\rho_1 d\rho_2 d\rho_3 d\rho_4 \lesssim \left\| \rho \right\|_{L^4} \left\| \int b(\rho_1) |b(\rho_2)| b(\rho_3) d\rho_1 \right\|_{L^2} \left\| \int b(\rho_2) b(\rho_3) d\rho_2 d\rho_3 \right\|_{L^4} 
\]

where $\chi = \chi(2^j \rho)$ and $\chi \in C_0^\infty([1/2, 4])$. It is easy to see by the Young inequality
\[
\| \chi \|_{L^{1/2}} \lesssim \| \chi \|_{L^1} \| b \|_{L^4} \lesssim 2^{-j/2} \| b \|_{L^4},
\]
and
\[
\left\| \chi_i * (\chi_j * |b|^2)^{3/2} |b| \right\|_{L^2} \lesssim \left\| \chi_i \right\|_{L^1} \left\| (\chi_j * |b|^2)^{3/2} |b| \right\|_{L^2} \lesssim \| \chi_j \|_{L^1} \| \chi_j * |b|^2 \|_{L^2} \lesssim 2^{-i-2j/2} |b|^4_{L^4}.
\]

Collecting the above estimates, we obtain
\[
I \lesssim 2^{-(i+j)} |b|^4_{L^4}.
\]
This completes the proof of Lemma 4.5.

Now we return to prove (4.26). Applying Lemma 4.5 to (4.27), we have
\[
\int b(\rho_1) b(\rho_2) b(\rho_3) b(\rho_1 - \rho_2 + \rho_3) \sum_{\{i,j,k \leq k_0 \}} 2^{-(i+j+k)} + R^{-N} \sum_{\{i,j \in \mathbb{N}, i+j \leq k_0 \}} 2^{(N-1)(i+j)} \left\| b \right\|_{L^4}^4 \lesssim R^{-1+\epsilon} |b|^4_{L^4}.
\]

Hence we prove (4.26), and so, we finish the proof of 4.47.

We next prove (4.8) in Proposition 4.1. We need to prove the following lemma.

**Lemma 4.6.** Let $R \gg 1$ and $f \in L_1$, we have the following estimate for every $0 < \epsilon \ll 1$
\[
\| (f \sigma)^\nu \|_{L^{6,\epsilon}(\mathbb{R} \times A_R)} \lesssim R^{-\frac{n-1}{2} + \epsilon} \| g \|_{L^2 H^\frac{n-1}{2} -1(\mathbb{R}^{n-1})},
\]
where $g(\xi) = f(|\xi|^2, \xi)$.

**Proof.** It suffices to estimate, by a scaling argument, the following quantity
\[
\left| \right| \int_0^\infty e^{-it\rho^2} J_{\nu(k)}(r \rho) a_{k,\ell}(\rho) \varphi(\rho) d\rho \left| \right|_{L^6_{r,x}(\mathbb{R} \times A_R)}
\]

We divide the above integral into three cases.
• Case 1: \( k \in \Omega_1 := \{ k : R \ll \nu(k) \} \). Using (4.11) with \( q = 6 \), we prove
  \[
  \left\| r^{-\frac{n-2}{2}} \sum_{k \in \Omega_1} \sum_{\ell=1}^{d(k)} i^{k} Y_{k,\ell}(\theta) \int_{0}^{\infty} e^{-i t \rho^2} J_{\nu(k)}(r \rho) a_{k,\ell}(\rho) \rho^{\frac{n}{2}} \varphi(\rho) \, d\rho \right\|_{L_{t,\rho}^{6}(\mathbb{R} \times A R)} \lesssim e^{-c R} \left\| \left( \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} (1 + k)^{2(n-1)/3} |a_{k,\ell}(\rho)|^2 \right)^{\frac{1}{2}} \varphi(\rho) \right\|_{L_{\rho}^{2}} \lesssim e^{-c R} \| g \|_{L_{t}^{2} H_{x}^{\frac{n-1}{2}}(\mathbb{S}^{n-1})}.
  \]

• Case 2: \( k \in \Omega_2 := \{ k : \nu(k) \sim R \} \). Applying (4.13) with \( q = 6 \) and \( p = 2 \), we show
  \[
  \left\| r^{-\frac{n-2}{2}} \sum_{k \in \Omega_2} \sum_{\ell=1}^{d(k)} i^{k} Y_{k,\ell}(\theta) \int_{0}^{\infty} e^{-i t \rho^2} J_{\nu(k)}(r \rho) a_{k,\ell}(\rho) \rho^{\frac{n}{2}} \varphi(\rho) \, d\rho \right\|_{L_{t,\rho}^{6}(\mathbb{R} \times A R)} \lesssim R^{-(n-1)/3} \| g \|_{L_{t}^{2} H_{x}^{\frac{n-1}{2}}(\mathbb{S}^{n-1})}.
  \]

• Case 3: \( k \in \Omega_3 := \{ k : \nu(k) \ll R \} \). We introduce the operator
  \[
  T_\nu(a)(t, r) = \chi \left( \frac{r}{R} \right) \int_{0}^{\infty} e^{-i t \rho^2} h_\nu(r \rho) a(\rho) \rho^{\frac{n}{2}} \varphi(\rho) \, d\rho
  \]
  where \( |h_\nu(r)| \leq C/r \) and the operator
  \[
  H_\nu(a)(t, r) = \chi \left( \frac{r}{R} \right) \int_{0}^{\infty} e^{-i t \rho^2} I_\nu(r \rho) a(\rho) \rho^{\frac{n}{2}} \varphi(\rho) \, d\rho,
  \]
  where \( \nu = \nu(k) = k + (n-2)/2 \). Since
  \[
  J_\nu(r) = I_\nu(r) + I_\nu(r) + h_\nu(r),
  \]
  our aim here is to estimate
  \[
  \left\| r^{-\frac{n-2}{2}} \sum_{k \in \Omega_3} \sum_{\ell=1}^{d(k)} i^{k} Y_{k,\ell}(\theta) \int_{0}^{\infty} e^{-i t \rho^2} J_{\nu(k)}(r \rho) a_{k,\ell}(\rho) \rho^{\frac{n}{2}} \varphi(\rho) \, d\rho \right\|_{L_{t,\rho}^{6}(\mathbb{R} \times A R)} \lesssim R^{-\frac{n-1}{2} + \frac{1}{2}} \left( \sum_{k \in \Omega_3} \sum_{\ell=1}^{d(k)} (1 + k)^{2(n-1)/3} \left( \| T_\nu(a_{k,\ell})(t, r) \|_{L_{t}^{6}(\mathbb{R} ; L_{x}^{6} (\mathbb{S}^{n}))} \right)^{2} \right)^{1/2}.
  \]
  By making use of (4.21) with \( q = 6 \), we have
  \[
  \| T_\nu(a)(t, r) \|_{L_{t,x}^{6}} \leq R^{-5/6} \| a \|_{L_{t,x}^{6/5}}.
  \]
This implies that
\[
\left( \sum_{k \in \Omega_3} \sum_{\ell = 1}^{d(k)} (1 + k)^{2(n-1)/3} \left\| T_{\nu(k)}(a_{k,\ell})(t, r) \right\|_{L^6_t(R; L^5(S^1_r))}^2 \right)^{1/2} \lesssim R^{-5/6} \left\| \left( \sum_{k \in \Omega_3} \sum_{\ell = 1}^{d(k)} (1 + k)^{2(n-1)/3} |a_{k,\ell}(\rho)|^2 \right)^{1/2} \varphi \right\|_{L^{5/2}}.
\]
(4.34)

On the other hand, by (2.11), one has \(|I_\nu(r)| \lesssim r^{-1/2}\) when \(k \in \Omega_3\). Consider the operator
\[
H_\nu(a)(t, r) = \chi \left( \frac{r}{R} \right) \int_0^\infty e^{-it\rho^2} I_\nu(r\rho)a(\rho)\rho^{2+\epsilon} \varphi(\rho) d\rho,
\]
where \(\nu = \nu(k) = k + (n-2)/2\) with \(k \in \Omega_3\).

On the one hand, it is easy to see
\[
\|H_\nu(a)(t, r)\|_{L^4_t \cap L^4_r(R \times \mathbb{R}^n)} \lesssim R^{-1/2} \|a\varphi\|_{L^1}.\]

On the other hand, we have the claim that for any \(\epsilon > 0\)
\[
\|H_\nu(a)(t, r)\|_{L^4_t \cap L^4_r(R \times \mathbb{R}^n)} \lesssim R^{-1/2+\epsilon} \|a\varphi\|_{L^6_r}.
\]
(4.35)

We postpone the proof of this claim to the end of this section. Hence, by the interpolation of the above two estimates, for any \(\epsilon > 0\), we obtain that
\[
\|H_\nu(a)(t, r)\|_{L^6_t \cap L^6_r(R \times \mathbb{R}^n)} \lesssim R^{-1/2+\epsilon} \|a\varphi\|_{L^6_r}.
\]

This shows
\[
\left( \sum_{k \in \Omega_3} \sum_{\ell = 1}^{d(k)} (1 + k)^{2(n-1)/3} \left\| H_{\nu(k)}(a_{k,\ell})(t, r) \right\|_{L^6_t(R; L^5(S^1_r))}^2 \right)^{1/2} \lesssim R^{-1/2+\epsilon} \left( \sum_{k \in \Omega_3} \sum_{\ell = 1}^{d(k)} (1 + k)^{2(n-1)/3} \|a_{k,\ell}(\rho)\varphi(\rho)\|_{L^2_r}^2 \right)^{1/2} \lesssim R^{-1/2+\epsilon} \|g\|_{L^2_t \cap L^2_r \cap L^{n-1}_r(S^{n-1})}.
\]
(4.36)

Collecting (4.34) and (4.36) yields
\[
\left\| r^{-n/2} \sum_{k \in \Omega_3} \sum_{\ell = 1}^{d(k)} s^k Y_{k,\ell}(\theta) \int_0^\infty e^{-it\rho} I_{\nu(k)}(r\rho) a_{k,\ell}(\rho)\varphi(\rho) d\rho \right\|_{L^6_t \cap L^6_r(R \times A_R)} \lesssim R^{-n/2+\epsilon} \|g\|_{L^2_t \cap L^2_r \cap L^{n-1}_r(S^{n-1})}.
\]

This implies (4.31), which completes the proof of Lemma 4.6.

The proof of claim \((4.35)\). The same argument in the proof the \((4.20)\) shows the claim \((4.35)\). Recall the kernel \((4.23)\), it is enough to estimate the integral
\[
\|H_\nu(a)(t,r)\|^{4}_{L^4_{t,r}(\mathbb{R} \times \mathbb{R}^n)}
= \int_{\mathbb{R}^4} \int_{\mathbb{R}} e^{-it(\rho_1^2-\rho_3^2+\rho_3^2-\rho_2^2)} K(R,\nu;\rho_1,\rho_2,\rho_3,\rho_4) a(\rho_1) a(\rho_2) a(\rho_3) a(\rho_4) \\
\times \beta(\rho_1) \beta(\rho_2) \beta(\rho_3) \beta(\rho_4) dt d\rho_1 d\rho_2 d\rho_3 d\rho_4,
\]
where \(\beta(\rho) = \rho^2\varphi(\rho)\). For \(b(\rho) = 2a(\sqrt{\rho}) \beta(\sqrt{\rho}) / \sqrt{\rho}\), therefore we obtain
\[
\|H_\nu(a)(t,r)\|^{4}_{L^4_{t,r}(\mathbb{R} \times \mathbb{R}^n)}
= \int_{\mathbb{R}^4} \delta(\rho_1 - \rho_2 + \rho_3 - \rho_4) K(R,\nu;\sqrt{\rho_1},\sqrt{\rho_2},\sqrt{\rho_3},\sqrt{\rho_4}) b(\rho_1) b(\rho_2) b(\rho_3) b(\rho_4) d\rho_1 d\rho_2 d\rho_3 d\rho_4
= \int_{\mathbb{R}^4} K(R,\nu;\sqrt{\rho_1},\sqrt{\rho_2},\sqrt{\rho_3},\sqrt{\rho_4}) b(\rho_1) b(\rho_2) b(\rho_3) b(\rho_4) d\rho_1 d\rho_2 d\rho_3 d\rho_4
\leq R^{-2+\epsilon} \|b\|^{4}_{L^4} \leq R^{-2+\epsilon} \|a \varphi\|^{4}_{L^4},
\]
where we use the kernel estimate \((4.24)\) and \((4.26)\) in the first inequality. \(\square\)

5. Local smoothing estimate

K. M. Rogers \([20]\) developed an argument showing that a restriction estimate implies a local smoothing estimate under some suitable conditions. For the sake of convenience, we closely follow this argument to prove Corollary \((4.11)\). In fact, by making use of the standard Littlewood-Paley argument, it can be reduced to prove the claim
\[
\|e^{it\Delta} (1 - \Delta_\theta)^{-s/2} u_0\|_{L^\infty_{t,x}(0,1] \times \mathbb{R}^n)} \lesssim N^{(2n(1/2-1/2) - 2/q) +} \|u_0\|_{L^2_{t,x}}, \quad \forall \ N \gg 1
\]
where
\[
\text{supp } \mathcal{F}((1 - \Delta_\theta)^{-s/2} u_0) \subset \{ \xi : |\xi| \leq N \}.
\]
Here we denote by \(\mathcal{F}\) the Fourier transform. We also use the notation \(\hat{h}\) to express the Fourier transform of \(h\). Let \(h = (1 - \Delta_\theta)^{-s/2} u_0\). Denote by \(P_N\) the Littlewood-Paley projector, i.e.
\[
P_N h = \mathcal{F}^{-1} (\chi \left( \frac{|\xi|}{N} \right) \hat{h}), \quad \chi \in \mathbb{C}_c^\infty([1/2, 1]).
\]
By the Littlewood-Paley theory and the claim \((5.1)\), one has for \(\alpha > 2n(1/2 - 1/q) - 2/q\)
\[
\|e^{it\Delta} h\|^{2}_{L^\infty_{t,x}((0,1] \times \mathbb{R}^n)} \lesssim \|e^{it\Delta} P_{\leq 1} h\|^{2}_{L^\infty_{t,x}((0,1] \times \mathbb{R}^n)} + \sum_{N \gg 1} \|e^{it\Delta} P_N h\|^{2}_{L^\infty_{t,x}((0,1] \times \mathbb{R}^n)}
\leq \|u_0\|^{2}_{L^2_{t,x}((0,1] \times \mathbb{R}^n)} + \sum_{N \gg 1} N^{2n(1/2 - 1/q) - 2/q +} \|P_N u_0\|^{2}_{L^2_{t,x}}
\lesssim \|u_0\|^{2}_{L^2_{t,x}((0,1] \times \mathbb{R}^n)} + \left( \sum_{N \gg 1} N^{2\alpha} |P_N u_0|^2 \right)^{1/2} \|P_N u_0\|^{2}_{L^2_{t,x}}
\lesssim \|u_0\|^{2}_{W^{2,q}(\mathbb{R}^n)}.
\]
Here we use Hölder’s inequality for the third inequality, Sobolev imbedding for the fourth one. Hence we have
\[ \|e^{it\Delta}u_0\|_{L^q_{t,x}([0,1] \times \mathbb{R}^n)} \lesssim \|(1 - \Delta_\theta)^{s/2}u_0\|_{W^{s,q}(\mathbb{R}^n)}. \]

Now we are left to prove claim (5.1). Assume \( \text{supp} \hat{f} \subset [0,1] \). Note that
\[ e^{it\Delta}f = \frac{1}{(it)^{n/2}} \int_{\mathbb{R}^n} e^{i|x-y|^2/t} f(y)dy, \quad \forall \ t \in \mathbb{R} \setminus \{0\}. \]

On the other hand, we have for \( t \neq 0 \)
\[ e^{it\Delta}f = \int_{\mathbb{R}^n} e^{i|\xi|^2+ix\cdot \xi} \hat{f}(\xi)d\xi = e^{-\frac{|x|^2}{4t}} \int_{\mathbb{R}^n} e^{i\xi + \frac{|x|^2}{4t}} \hat{f}(\xi)d\xi \]
\[ = \frac{1}{(it)^{n/2}} e^{-\frac{|x|^2}{4t}} \left( e^{\frac{\Delta}{t}} f \right) \left( -\frac{x}{2t} \right). \]

So we have for every dyadic number \( N \)
\[ \|e^{it\Delta}f\|_{L^q_{t,x}([t-2N^2; \xi \leq 2N^2])} \lesssim N^{-n} \left\| \left( e^{\frac{\Delta}{t}} f \right) \left( -\frac{x}{2t} \right) \right\|_{L^q_{t,x}([t-2N^2; \xi \leq 2N^2])} \]
\[ \lesssim N^{-n+\frac{2n+4}{q}} \left\| e^{it\Delta}f \right\|_{L^q_{t,x}([t-2N^2; \xi \leq 1])}. \]

By making use of Theorem 1.1 we obtain for \( q > 2(n+1)/n \) and \( \frac{n+2}{q} = \frac{n}{p'} \)
\[ \|e^{it\Delta}f\|_{L^q_{t,x}([t-2N^2; \xi \leq 2N^2])} \lesssim \|f\|_{L^p_{\mu(e)}(\mathbb{R}^+; H^{s,p}(\mathbb{S}^{n-1}))}. \]

This yields
\[ \|e^{it\Delta}f\|_{L^q_{t,x}([t-2N^2; \xi \leq 2N^2])} \lesssim N^{-n+\frac{2n+4}{q}} \|f\|_{L^p_{\mu(e)}(\mathbb{R}^+; H^{s,p}(\mathbb{S}^{n-1}))}. \]

This implies that
\[ \|e^{it\Delta}(1 - \Delta_\theta)^{-s/2}f\|_{L^q_{t,x}([t-2N^2; \xi \leq 2N^2])} \lesssim N^{-n+\frac{2n+4}{q}} \|f\|_{L^q_{\mu(e)}(\mathbb{R}^+; H^{s,p}(\mathbb{S}^{n-1}))}. \]

For the sake of convenience, we recall [20] Lemma 8

**Lemma 5.1.** Let \( q \geq p_1 \geq p_0 \), \( r \geq 1 \) and \( I \subset [0, R^2] \). If one has
\[ \|e^{it\Delta}f\|_{L^q_{I}}(B_{R^2};L^p_{I}) \leq CR^s \|f\|_{L^p_{R^1}(\mathbb{R}^n)} \]
where \( R \gg 1 \), and \( f \) is frequency supported in unite ball \( \mathbb{B}^n \). Then for all \( \epsilon > 0 \)
\[ \|e^{it\Delta}f\|_{L^q_{I}(\mathbb{R}^n;L^1_{I}(I))} \leq C \epsilon R^{s+2n(\frac{1}{p_0} - \frac{1}{p_0})+\epsilon} \|f\|_{L^p_{I}(\mathbb{R}^n)}. \]

Since \( q > p \) when \( q > 2(n+1)/n \), for any \( 0 < \epsilon \ll 1 \), we have by this lemma
\[ \|e^{it\Delta}(1 - \Delta_\theta)^{-s/2}f\|_{L^q_{t,x}([t-2N^2; \xi \in \mathbb{R}^n])} \]
\[ \lesssim N^{-n+\frac{2n+4}{q}+2n(\frac{1}{p_0} - \frac{1}{p_0})+\epsilon} \|f\|_{L^q_{\mu(e)}(\mathbb{R}^+; H^{s,p}(\mathbb{S}^{n-1}))}. \]

Using the scaling argument, if
\[ \text{supp} \hat{f}_{k,N} \subset B_{2^{k/2}N} := \{ \xi : |\xi| \in [0, 2^{k/2}N] \}, \quad \forall \ k \geq 0, \]
then
\[
\|e^{it\Delta} (1 - \Delta_\theta)^{-\frac{s}{2}} f_{k,N} \|_{L^q_t, L^q_x((2^{-k}, 2^{-k+1}) \times \mathbb{R}^n)} \lesssim N^{n(1 - \frac{1}{2}) + \epsilon} \left( \frac{2^\frac{s}{2} N}{2} \right)^{-\frac{2}{q}} \|f_{k,N}\|_{L^q_x}.
\]
(5.4)

Since \( \text{supp} \hat{h} \subset \{ \xi : |\xi| \in [N/2, N] \} \subset B_{2k/2N}, \forall k \geq 2, \) we replace \( (1 - \Delta_\theta)^{-s/2} f_{k,N} \) by \( h \) to obtain
\[
\|e^{it\Delta} h \|_{L^q_t, L^q_x((0,1] \times \mathbb{R}^n)} = \left( \sum_{k \geq 0} \left\| e^{it\Delta} (1 - \Delta_\theta)^{-s/2} u_0 \right\|_{L^q_t, L^q_x((2^{-k}, 2^{-k+1}) \times \mathbb{R}^n)}^q \right)^{1/q} \\
< \left( \sum_{k \geq 0} 2^{-k} \right)^{1/q} N^{(2n(1/2 - 1/q) - 2)/q} \|u_0\|_{L^q_x}.
\]
(5.5)

This proves inequality (5.1).

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