THE DIOPHANTINE PROBLEM FOR SYSTEMS OF ALGEBRAIC EQUATIONS WITH EXPONENTS

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Abstract. Consider the equation $q_1 \alpha x_1^{z_1} + \cdots + q_k \alpha x_k^{z_k} = q$, with constants $\alpha \in \mathbb{Q} \setminus \{0, 1\}$, $q_1, \ldots, q_k, q \in \mathbb{Q}$ and unknowns $x_1, \ldots, x_k$, referred to in this paper as an algebraic equation with exponents. We prove that the problem to decide if a given equation has an integer solution is NP-complete, and that the same holds for systems of equations (whether $\alpha$ is fixed or given as part of the input). Furthermore, we describe the set of all solutions for a given system of algebraic equations with exponents and prove that it is semilinear.

1. Introduction

A classical Diophantine equation is an equation with integer coefficients and one or more unknowns, for which only the integer solutions are of interest. Such equations have been studied since antiquity, motivating much fruitful work in diverse areas of mathematics (Fermat’s last theorem is a particularly famous example). For a class $\mathcal{C}$ of Diophantine equations (or of systems of Diophantine equations), one can study three related algorithmic problems:

(i) decide whether a given equation from $\mathcal{C}$ has a solution or not;
(ii) find a solution for a given equation from $\mathcal{C}$, assuming one exists;
(iii) describe the set of all solutions for a given equation from $\mathcal{C}$.

The first problem is called the Diophantine problem for $\mathcal{C}$, and is denoted $\text{DP}_\mathcal{C}$. In modern mathematics, the notion is extended to any algebraic structure (or a class of algebraic structures) $\mathcal{S}$, where the notation $\text{DP}_\mathcal{C}(\mathcal{S})$ may be used. For instance, we may speak of the Diophantine problem $\text{DP}_P(\mathbb{Q})$ for systems of polynomial equations over $\mathbb{Q}$, or the Diophantine problem $\text{DP}(\mathcal{N}_3)$ for systems of arbitrary equations in the class $\mathcal{N}_3$ of nilpotent groups of step 3.

The most well studied classes of Diophantine equations are the classes $\mathcal{L}$ of linear systems and $\mathcal{P}$ of polynomial systems. The problem $\text{DP}_\mathcal{L}$ has an efficient (polynomial time) solution: it may be solved by computing the Hermite (or Smith) normal form of the corresponding matrix (see [14]). On the other hand, $\text{DP}_P(\mathbb{Z})$ is Hilbert’s famous tenth problem, first proposed in 1900 and only shown to have no algorithmic solution in 1970, as a result of the combined work of Y. Matiyasevich, J. Robinson, M. Davis, and H. Putnam, spanning 21 years (with Matiyasevich completing the theorem—known as the MRDP theorem—in 1970 [12]). Notably, the decidability of $\text{DP}_P(\mathbb{Q})$ remains an open question.

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In this paper we study the class \( \mathcal{E} \) of systems of Diophantine equations of the following form:

\[
\begin{align*}
q_{11} \alpha_1^{x_1} + \cdots + q_{1k} \alpha_k^{x_k} &= q_{10} \\
& \\
q_{s1} \alpha_1^{x_1} + \cdots + q_{sk} \alpha_k^{x_k} &= q_{s0}
\end{align*}
\]

(1)

with constants \( \alpha_1, \ldots, \alpha_s \in \overline{\mathbb{Q}} \setminus \{0, 1\} \) (where \( \overline{\mathbb{Q}} \) denotes an algebraic closure of \( \mathbb{Q} \)), \( q_{ij} \in \mathbb{Q}(\alpha_i) \), unknowns \( x_1, \ldots, x_k \), and whose solutions must be in \( \mathbb{Z}^k \). We call these equations *algebraic equations with exponents*. The Diophantine problem for \( \mathcal{E} \) may be considered in two distinct forms: the *uniform problem*, where the \( \alpha_i \) are given as part of the input, and the *fixed base* problem, where the \( \alpha_i \) are fixed beforehand.

Notice that there is no loss of generality in requiring that the coefficients of the \( i \)th equation be contained in \( \mathbb{Q}(\alpha_i) \). In fact, the more general problem in which \( q_{i1}, \ldots, q_{ik} \in \overline{\mathbb{Q}} \) (with the input given in a natural form described below) is polynomial-time equivalent to \( \text{DP}_\mathcal{E} \). This may be seen from the example of a single equation

\[
q_1 \alpha^{x_1} + \cdots + q_k \alpha^{x_k} = q_0
\]

(2)

in the following manner. Let \( K/\mathbb{Q}(\alpha) \) be a proper finite degree extension such that \( q_0, q_1, \ldots, q_k \in K \), and let \( m = [K : \mathbb{Q}(\alpha)] \). Let \( \beta \) be a primitive element which generates \( K \) over \( \mathbb{Q}(\alpha) \), and suppose that each \( q_i \) is given as a vector \( (f_{i1}, \ldots, f_{im}) \in \mathbb{Q}(\alpha)^m \) such that \( q_i = \sum_{j=1}^m f_{ij} \beta^{j-1} \). Then (2) has a solution if and only if the system

\[
\begin{align*}
f_{11} \alpha^{x_1} + \cdots + f_{k1} \alpha^{x_k} &= f_{01} \\
& \\
f_{1m} \alpha^{x_1} + \cdots + f_{km} \alpha^{x_k} &= f_{0m}
\end{align*}
\]

(3)

has a solution. Observe that (3) is an instance of \( \text{DP}_\mathcal{E} \) whose input is roughly the same size as that of (2). Moreover, it is clear that we can transform one into the other in polynomial time.

1.1. **Encoding the input of \( \text{DP}_\mathcal{E} \).** The constants \( \alpha_1, \ldots, \alpha_s \) are algebraic numbers (other than 0 and 1) called the *bases* of the system; in what follows, we use the notation \( \deg(w) \) to denote the degree of the minimal polynomial of an algebraic number \( w \in \overline{\mathbb{Q}} \). An instance of \( \text{DP}_\mathcal{E} \) is given with the following data.

- The bases \( \alpha_i \) are specified (either as part of the input, or fixed in advance) as lists of the integer coefficients of their minimal polynomials \( p_{\alpha_i}(x) = \sum_{j=0}^{d_i} c_{ij} x^j \), where \( d_i = \deg(\alpha_i) \).
- The coefficients \( q_{ij} \in \mathbb{Q}(\alpha_i) \) are given as vectors \( (r_{ij}^0, \ldots, r_{ij}^{d_i-1}) \in \mathbb{Q}^{d_i} \) such that
  
  \[
  q_{ij} = \sum_{h=0}^{d_i-1} r_{ij}^h \alpha_i^h,
  \]
  
  with \( r_{ij}^h \) given as a quotient (in lowest terms) \( r_{ij}^h = \frac{a_{ij}^h}{b_{ij}^h} \).

This naturally defines the size of an instance \( E \) of the problem as

\[
\text{size}(E) = \sum_{i=1}^s \sum_{j=1}^k \sum_{h=0}^{d_i-1} \left( \log_2(|a_{ij}^h| + 1) + \log_2(|b_{ij}^h| + 1) \right) + \sum_{i=1}^s \sum_{j=0}^{d_i} \log_2(|c_{ij}| + 2),
\]

(4)
if $E$ is an instance of the uniform problem, and

$$\text{size}(E) = \sum_{i=1}^{s} \sum_{j=1}^{k} \sum_{h=0}^{d_i-1} \left( \log_2(|a_{ij}^h| + 1) + \log_2(|b_{ij}^h| + 1) \right)$$

if $E$ is an instance of the fixed base problem. This is roughly the number of bits required to encode $E$ (the terms $\log_2(|c_{ij}| + 2)$ are needed to count the zero coefficients of $p_{\alpha_i}$, and to ensure that $\deg(\alpha_i) < \text{size}(E)$).

The minimal polynomials of $\alpha_i$ are assumed to be irreducible, which means that $\text{DP}_E$, as defined above, is a promise problem (i.e. the input is “promised” to belong to a certain subset of all possible inputs). However, this is only a matter of convenience, since D. G. Cantor has shown that the problem of deciding whether a given polynomial is irreducible is in $\text{NP}$ (see [2]). Thus, our proof that $\text{DP}_E \in \text{NP}$ can easily be adapted to allow for arbitrary input by combining Cantor’s $\text{NP}$-certificate with our own.

We assume a basic background in complexity theory and $\text{NP}$-completeness, as may be found in [5].

1.2. **Main results.** We prove that the Diophantine problem for $E$ is $\text{NP}$-complete, and that it remains $\text{NP}$-complete for single equations with any fixed choice of $\alpha \in \mathbb{Q} \setminus \{0, 1\}$. Additionally, we describe the solution set for a given instance, and prove that it is semilinear. On a finer scale, we show that the Diophantine problem for a system $E \in \mathcal{E}$ can be solved in polynomial time as long as the number of equations is bounded by a fixed constant. In other words, it is a fixed-parameter tractable problem.

1.3. **Related results and applications.** Equations of a similar form were considered by A. Semenov in 1984, in the course of his investigation of the first-order theory of the algebraic structure $\langle \mathbb{N}; +, k^x \rangle$ (where $k \in \mathbb{N}$ is a fixed base for the exponential). Using methods involving quantifier elimination, Semenov proved the decidability of this theory in [13]. Notably, while it is a simple matter to solve (1) over the real or complex numbers, the first-order theory of $\langle \mathbb{C}; +, k^x \rangle$ is undecidable (since $\langle \mathbb{Z}; +, \cdot \rangle$ is interpretable in this structure), and the decidability of $\langle \mathbb{R}; +, k^x \rangle$ is an open question (but known to follow from Schanuel’s conjecture; see [11]).

Equations with exponents play a very important role in study of discrete optimization problems (such as the knapsack problem, the power word problem, see [4, 9]) and decision problems over certain groups. In fact, our present interest in the class of equations $\mathcal{E}$ arose from studying the Diophantine problem over the Baumslag-Solitar groups. There turns out to be a close connection between algebraic equations with exponents in base $\mathfrak{m}$ and equations in $\text{BS}(m, n)$, suggesting the importance of decidability and complexity results for $\text{DP}_E$. In a forthcoming paper [10], we employ some of the present results to prove, for instance, that the quadratic Diophantine problem for $\text{BS}(1, n)$ is $\text{NP}$-complete for $n \neq 1$, and we expect further applications to the study of equations (quadratic and otherwise) in $\text{BS}(m, n)$ and related groups. For background and recent results on the Diophantine problem in various classes of groups, see, for instance, [7] and [6], as well as [3] for the decidability of the Diophantine problem in groups that are virtually a direct product of hyperbolic groups (including the unimodular Baumslag-Solitar groups $\text{BS}(n, \pm n)$).
1.4. **Outline and general approach.** In Section 2, we prove that the Diophantine problem for a single equation with \( \alpha \in \mathbb{Q} \setminus \{0, 1\} \) is \( \text{NP} \)-hard. This is accomplished via a reduction of either the *partition problem* (in case \( \alpha \) is a root of unity) or the *3-partition problem* (both well-known \( \text{NP} \)-complete problems) to \( \text{DP} \). In Section 3, we prove that the Diophantine problem for a single equation with \( \alpha \in \mathbb{Q} \setminus \{0, 1\} \) belongs to \( \text{NP} \).

Sections 4 and 5 extend the above results to finite systems of equations, and in Section 6 we describe the solution set for a system of equations. Finally, Section 7 addresses the parameterized complexity of the problem.

2. **Complexity lower bound for a single equation**

Fix \( \alpha \in \mathbb{Q} \setminus \{0, 1\} \) and consider the equation

\[
q_1 \alpha^{x_1} + \cdots + q_k \alpha^{x_k} = q_0
\]

with \( q_1, \ldots, q_k, q_0 \in \mathbb{Q}(\alpha) \), unknowns \( x_1, \ldots, x_k \) and solutions sought in \( \mathbb{Z} \). In this section, we prove that the Diophantine problem for (6) is \( \text{NP} \)-hard. We first handle the special case where \( \alpha \) is a root of unity.

2.1. \( \alpha \) is a root of unity. Let \( \alpha = e^{2\pi i/n} \neq 1 \) for \( n \geq 2 \). Below, we prove \( \text{NP} \)-hardness by reducing the partition problem (which is known to be \( \text{NP} \)-complete) to the decidability of (6). Recall that the *partition problem* is the problem to decide if a given multiset \( \{q_1, \ldots, q_k\} \) of positive integers can be partitioned into two submultisets \( S_0 \) and \( S_1 \) such that \( \sum_{x \in S_0} x = \sum_{x \in S_1} x \), see [5].

For an instance \( Q = \{q_1, \ldots, q_k\} \) of the partition problem, define \( L = \frac{1}{2} \sum q_i \) (we may assume that \( L \in \mathbb{Z} \)), and let \( E_Q \) denote the following equation:

\[
\sum q_i \alpha^{x_i} = L + L\alpha.
\]

Note that size(\( E_Q \)) = \( \sum_{i=1}^{k} \log_2(q_i + 1) + 2 \log_2(L + 1) \), which is linear in the size of \( Q \) (with \( Q \) represented in binary).

**Proposition 2.1.** \( Q \) is a positive instance of the partition problem if and only if \( E_Q \) has a solution.

**Proof.** Consider two cases. Suppose that \( \alpha = -1 \). Then the right-hand side of (7) is equal to zero, \( \alpha^{x_i} = \pm 1 \), and (7) translates to the equation

\[
\sum \varepsilon_i q_i = 0,
\]

with unknowns \( \varepsilon_1, \ldots, \varepsilon_k = \pm 1 \), which is equivalent to the partition problem. The converse is also true.

Suppose that \( \alpha \neq -1 \). If \( Q \) is a positive instance, then it is clear that \( E_Q \) has a solution (with \( x_i \in \{0, 1\} \)). Conversely, assume that \( E_Q \) has a solution \( \pi = (x_1, \ldots, x_k) \). Consider the complex number \( u = 1 + \alpha \) as a vector in \( \mathbb{R}^2 \), and let \( \pi_u(z) \) denote the signed scalar projection of \( z \) onto \( u \), defined by

\[
\pi_u(z) = \frac{\Re(\pi z)}{|u|}.
\]

It is easily verified that

\[
\pi_u(1) = \pi_u(\alpha) = \frac{|u|}{2} > \max\{\pi_u(\alpha^2), \ldots, \pi_u(\alpha^{n-1})\}.
\]
Since the projection of the right-hand side of (7) onto \( u \) is equal to \( L|u| \), and since \( \sum_{i=1}^{k} q_i = 2L \), it follows that the \( x_i \) must all be either 0 or 1 (otherwise the left-hand side would have a projection strictly less than \( L|u| \)). Letting \( S_0 = \{ i \in \{1, \ldots, k\} | x_i = 0 \} \) and \( S_1 = \{ i \in \{1, \ldots, k\} | x_i = 1 \} \), the \( \mathbb{Q} \)-linear independence of \( \{1, \alpha\} \) implies that
\[
\sum_{i \in S_0} q_i = L = \sum_{i \in S_1} q_i,
\]
proving that \( Q \) is a positive instance. \( \square \)

**Corollary 2.2.** Then the (fixed base) Diophantine problem for equations (6) is \( \text{NP}-\text{complete}. \) \( \square \)

**Proof.** \( \text{NP} \)-hardness is proved above. To establish that the problem is in \( \text{NP} \), observe that if there is a solution, there must be one satisfying
\[
0 \leq x_1, \ldots, x_k < n,
\]
where \( n \) is the order of \( \alpha \), and it is easily seen that \( n \) is \( O(\text{size}^2(E)) \). It is shown in Lemma 3.1 that such a solution may be verified in time polynomial in \( \text{size}(E) \). \( \square \)

There exist polynomial-time algorithms (see, for instance [1]) to determine, based on its minimal polynomial, whether \( \alpha \in \mathbb{Q}^* \) is a root of unity (and if so, to find its order). Thus, it is valid to consider this case separately for the uniform Diophantine problem (when \( \alpha \) is a part of the input), and we may consequently assume that \( \alpha \) is not a root of unity in Section 3 (where it is proved that the Diophantine problem for a single equation is in \( \text{NP} \)).

2.2. \( \alpha \) is not a root of unity. The following proposition is the main technical result of this section. In what follows, \( \mathcal{U} \) always denotes the set of roots of unity and \( \overline{\mathbb{Q}} = \mathbb{Q} \setminus \{0\} \).

**Proposition 2.3.** Let \( \alpha \in \overline{\mathbb{Q}}^* \setminus \mathcal{U} \) and \( d = \deg(\alpha) \). For \( s \in \mathbb{N} \), define the positive integer
\[
(8) \quad c(\alpha, s) = \left\lceil \frac{3(\ln 2 + 2 \ln s)}{\kappa(d)} \right\rceil,
\]
where
\[
\kappa(d) = \begin{cases} \frac{\ln 2}{d(\ln(3d))^d} & \text{for } d = 1, \\ 2^d & \text{for } d \geq 2. \end{cases}
\]
Then for any integers \( 0 \leq p_1 < p_2 < \cdots < p_s \) satisfying \( p_{i+1} - p_i \geq c(\alpha, s) \), the equation
\[
(9) \quad \alpha^{p_1} + \cdots + \alpha^{p_s} = \alpha^{p_1} + \cdots + \alpha^{p_s}
\]
has (up to a permutation) the unique integer solution \( x_i = p_i \).

For the proof, we make use of the following lemma, proved by H. W. Lenstra, see [8, Proposition 2.3]. Note that Lenstra’s original result is more general, as it applies to polynomials in \( \overline{\mathbb{Q}}[x] \); we only require the slightly weaker version for polynomials in \( \mathbb{Z}[x] \) stated below. For a polynomial \( f(x) = \sum_{i=1}^{h} c_i x^i \in \mathbb{Z}[x] \) define
\[
\text{height}(f) = \max(|c_1|, \ldots, |c_h|).
\]
Lemma 2.4 (Lenstra). Let $\alpha, d$ and $\kappa(d)$ be as in Proposition 2.3, and suppose that a polynomial $P(x) = P_0(x) + x^r P_1(x) \in \mathbb{Z}[x]$ contains $k$ monomials and satisfies
\[
(10) \quad r - \deg(P_0) > \frac{\ln(k - 1) + \ln(\text{height}(P))}{\kappa(d)}.
\]
Then $P(\alpha) = 0$ if and only if $P_0(\alpha) = 0$ and $P_1(\alpha) = 0$. \hfill \Box

Proof of Proposition 2.3. By way of contradiction, let $0 \leq p_1 < p_2 < \cdots < p_s$ satisfy $p_{i+1} - p_i \geq c(\alpha, s)$, and suppose there is a solution $x_1 \leq \cdots \leq x_s \in \mathbb{Z}$ to (9) such that $\{x_1, \ldots, x_s\} \neq \{p_1, \ldots, p_s\}$. Multiplying by an appropriate power of $\alpha$ if necessary, we may assume that the $x_i$ are nonnegative. Eliminating terms that appear on both sides and reindexing, we obtain $x_1, \ldots, x_s'$ and $p_1, \ldots, p_{s'}$ such that $\{x_1, \ldots, x_s\} \cap \{p_1, \ldots, p_s\} = \emptyset$ and $s' \leq s$. Hence, the polynomial
\[
P(x) = \sum_{i=1}^s x^{p_i} - \sum_{i=1}^s x^{x_i},
\]
has $k = 2s'$ monomials, with $k \leq 2s$, and satisfies $P(\alpha) = 0$. Since the coefficients of $P$ are bounded in absolute value by $s$ (this bound is realized if $s = s'$ and $x_1 = \cdots = x_s$), we have $\text{height}(P) \leq s$. Hence, the right-hand side of the inequality in Lemma 2.4 corresponding to $P$ is less than
\[
K = \frac{\ln 2 + 2 \ln s}{\kappa(d)},
\]
and we notice that $c(\alpha, s) = \lceil 3K \rceil \geq 3K$. Now suppose that there is some $t \in \{1, \ldots, s\}$ such that none of the $x_i$ are contained in the interval $[p_t - K, p_t + K]$. Applying Lemma 2.4 twice, this implies that $\alpha^t = 0$, a contradiction. Thus, by a pigeonhole argument there is exactly one $x_i$ contained in each of the $s$ disjoint $K$-neighborhoods of $p_1, \ldots, p_s$. This means that $P(x)$ is a sum of $s$ polynomials of the form $B_i(x) = x^{p_i} - x^{x_i}$, such that
\[
\min(p_{i+1}, x_{i+1}) - \max(p_i, x_i) > K.
\]
Hence, Lemma 2.4 shows that $B_1(\alpha) = 0$, implying that $\alpha$ is a root of unity, which contradicts our assumption. \hfill \Box

Fix $\alpha \in \overline{\mathbb{Q}} \setminus \mathcal{U}$. Below we prove NP-hardness of the Diophantine problem for (6) by reducing the 3-partition problem to the decidability of (6). For a given multiset $S = \{a_1, \ldots, a_{3k}\}$ of $3k$ integers, define
\[
L = \frac{1}{k} \sum_{i=1}^{3k} a_i.
\]
The 3-partition problem (abbreviated as 3PART) is the problem of deciding whether an integer multiset $S = \{a_1, \ldots, a_{3k}\}$, where $L/4 < a_i < L/2$, can be partitioned into $k$ triples, each of which sums to $L$. This problem is known to be strongly NP-complete, which means that it remains NP-complete even when the input is represented in unary. A thorough treatment of this problem may be found in [6]. Below we reduce an instance of 3PART to an instance of the Diophantine problem for (6). Note that because of the restriction $L/4 < a_i < L/2$, we may assume that $S$ contains only positive integers.
Let $S = \{a_1, \ldots, a_{3k}\}$ be an instance of 3PART, with $L = \frac{1}{k} \sum_{i=1}^{3k} a_i$ the anticipated sum and $L/4 < a_i < L/2$ (we may assume that $L \in \mathbb{N}$). Let $c = c(\alpha, L_k)$ be defined as in Proposition 2.3 and define the numbers
\[
q_y = 1 + \alpha^c + \alpha^{2c} + \cdots + \alpha^{(y-1)c} \quad \text{for} \quad y \in \mathbb{N}
\]
\[
r = q_L \left(1 + \alpha^{2cL} + \alpha^{4cL} + \cdots + \alpha^{2(k-1)cL}\right)
\]
and the equation
\[
q_{a_1} \alpha^{x_1} + \cdots + q_{a_{3k}} \alpha^{x_{3k}} = r.
\]

**Proposition 2.5.** $S$ is a positive instance of 3PART if and only if (11) has a solution. Furthermore, a solution $x_1, \ldots, x_{3k}$ for (11), if it exists, is unique up to a permutation and satisfies
\[
0 \leq x_1, \ldots, x_{3k} \leq 2ckL.
\]

**Proof.** Suppose that $S$ is a positive instance of 3PART. Reindexing the $a_i$ and $x_i$ if necessary, we may assume that $\sum_{j=1}^{3} a_{3i+j} = L$ for $i = 0, 1, \ldots, k - 1$. It is now easily checked that
\[
x_{3i+1} = 2iL
\]
\[
x_{3i+2} = c(2iL + a_{3i+1})
\]
\[
x_{3i+3} = c(2iL + a_{3i+1} + a_{3i+2})
\]
for $i = 0, 1, \ldots, k - 1$ satisfies (11) and (12).

For the other direction, suppose that $x_1, \ldots, x_{3k}$ is a solution of (11). By construction, the left-hand side of (11) is a sum of $Lk$ powers of $\alpha$, while the right-hand side is a sum of $Lk$ distinct powers of $\alpha^c$. In particular, the sum on the right-hand side contains blocks of consecutive powers of $\alpha^c$, with gaps between $\alpha^{(2i-1)c(L-1)}$ and $\alpha^{2icL}$ for $i = 1, \ldots, k - 1$. Proposition 2.3 implies that the left-hand side consists of the same distinct powers of $\alpha$, and the proof follows from a careful comparison of the powers on each side. First of all, it is clear that we must have $x_{ii} = 0$ for exactly one $x_{ii}$, and $q_{a_{ii}} \alpha^{x_{ii}} = 1 + \alpha^c + \cdots + \alpha^{c(a_{ii}-1)}$. Since $ca_{ii} < cL$ by assumption, the right-hand side of (11) contains $\alpha^{ca_{ii}}$ and so we must have $x_{i2} = ca_{ii}$ for some (unique) $x_{i2}$. Similarly, the highest degree term of $q_{a_{i1}} \alpha^{x_{i1}} + q_{a_{i2}} \alpha^{x_{i2}}$ is $\alpha^{c(a_{i1}+a_{i2}-1)}$, and since $c(a_{i1} + a_{i2}) < cL$, we must have the next consecutive power of $\alpha^c$ on the left-hand side. Hence, there must be $x_{i3} = c(a_{i1} + a_{i2})$. Finally, the highest power of $q_{a_{i1}} \alpha^{x_{i1}} + q_{a_{i2}} \alpha^{x_{i2}} + q_{a_{i3}} \alpha^{x_{i3}}$ is $\alpha^{c(a_{i1}+a_{i2}+a_{i3}-1)}$, which must be the last term in the first block of consecutive powers of $\alpha^c$ (since for any $a_{ii}$ we have $cL < c(a_{i1} + a_{i2} + a_{i3} + a_{ii})$). This implies that $c(a_{i1} + a_{i2} + a_{i3}) = cL$, and the next largest $x_i$ is equal to $2cL$. It is clear that this process may be continued to show that $S$ is a positive instance, and that $x_1, \ldots, x_{3k}$ satisfies (12). □

**Corollary 2.6.** For $\alpha \in \overline{\mathbb{Q}} \setminus \{0, 1\}$, the Diophantine problem for equations (6) is NP-hard.

**Proof.** If $\alpha \in \overline{\mathbb{Q}} \setminus \{0, 1\}$ is not a root of unity, Proposition 2.5 provides a polynomial-time (Karp) reduction from 3PART to the Diophantine problem for (6). The case where $\alpha \in \mathbb{U}$ was proved in Proposition 2.2. □
In this section, we prove that the Diophantine problem for single algebraic equations with exponents is in \textbf{NP}. The following lemma is required to establish a polynomial time procedure for checking a candidate solution. Note that the proof extends easily to the case of a system of equations; however, for convenience we state it for a single equation.

**Lemma 3.1.** Consider an equation \( E \) of type (6). Let \( \vec{z} = (z_1, \ldots, z_k) \in \mathbb{Z}^k \), and set
\[
M = \max\{1, |z_1|, \ldots, |z_k|\},
\]
\[
\mathcal{M}(E, \vec{z}) = \max\{\text{size}(E), \log M\}.
\]

Then there is an algorithm that checks whether \( x_i = z_i \) is a solution of \( E \) in time polynomial in \( \mathcal{M}(E, \vec{z}) \).

**Proof.** Let \( d = \deg(\alpha) \), so that each \( q_i \) is equal to \( \sum_{j=0}^{d-1} r_{ij} \alpha^j \) for some \( r_{ij} \in \mathbb{Q} \). Multiplying \( E \) by the product of all denominators, which is less than \( 2^{\text{size}(E)} \), we can ensure that each \( r_{ij} \in \mathbb{Z} \). This can be done in time polynomial in size\((E)\), and the new equation \( E' \) satisfies size\((E') = O(\text{size}^2(E))\).

Set \( m = \min\{z_1, \ldots, z_k\} \) and \( M = \max\{z_1, \ldots, z_k\} \), and let \( z'_i = z_i - m \) (so that each \( z'_i \) is non-negative and \( \min z'_1 = 0 \)); note that \( z'_1, \ldots, z'_k \) may be computed in time polynomial in \( \log M \). Define \( f_i(x) = \sum_{j=0}^{d-1} r_{ij} x^j \) for \( i = 0, \ldots, k \), and
\[
F(x) = f_1(x)x^{z'_1} + \cdots + f_k(x)x^{z'_k} - f_0(x) \in \mathbb{Z}[x].
\]

To compute the coefficients of \( F(x) \), we must perform comparisons and addition on \( d(k+1) \) terms, with coefficients and exponents encoded as binary numbers of length bounded by size\((E)\) and \( d + \log(M - m + 1) \), respectively. That can be done in time polynomial in \( \mathcal{M}(E, \vec{z}) \), because \( d, k \) are bounded by size\((E)\). In this way, we end up with the sparse representation of \( F \), i.e., the list of pairs \( (c_i, n_i) \), where \( c_i \) is the coefficient of \( x^n \) in \( F(x) \) and \( c_n \neq 0 \), with \( c_n \) and \( n \) given in binary. Moreover, it is encoded in space that is polynomial in \( \mathcal{M}(E, \vec{z}) \).

By construction, \( z'_1, \ldots, z'_k \) satisfy (6) if and only if \( \alpha \) is a zero of \( F(x) \). By [9], Theorem 2.1], the problem to decide if \( F(\alpha) = 0 \) (where the input consists of \( F(x) \) and the minimal polynomial of \( \alpha \), encoded as above) belongs to \textbf{TC}^0. In particular, it is polynomial-time decidable, so the result follows. \( \square \)

Let \( \alpha \in \mathbb{Q} \setminus \mathcal{U} \), and let \( d = \deg(\alpha) \). Consider, instead of (6), the equation
\[
(13) \quad q_1 \alpha^{x_1} + \cdots + q_k \alpha^{x_k} = 0
\]
with coefficients \( q_1, \ldots, q_k \in \mathbb{Z}[\alpha] \setminus \{0\} \), and each coefficient given as
\[
q_i = r_{i0} + r_{i1} \alpha + \cdots + r_{i(d-1)} \alpha^{d-1}, \quad r_{ij} \in \mathbb{Z}.
\]

We call (13) a homogeneous equation with exponents. The Diophantine problem for (13) is easily seen to be polynomial-time equivalent (in terms of size\((E)\)) to the Diophantine problem for (6).
3.1. **Block structure of a solution.** Let $x_1, \ldots, x_k \in \mathbb{Z}$ be a solution for (13). Denote the vector of the solution $(x_1, \ldots, x_k) \in \mathbb{Z}^k$ by $x$. A nonempty set $I \subseteq \{1, \ldots, k\}$ is called a block for $x$ if the following conditions hold:

(B1) $\sum_{i \in I} q_i \alpha^{x_i} = 0$;
(B2) $I$ does not have a nonempty proper subset satisfying (B1).

For a given solution $x$, the set of indices $\{1, \ldots, k\}$ can be represented as a disjoint union $\bigsqcup_{j=1}^m I_j$ of blocks, perhaps in more than one way. Such a collection $I = \{I_1, \ldots, I_m\}$ is called a block structure for a solution $x$. For a block $I$, define

- $\Delta_I = (\delta_1, \ldots, \delta_k) \in \mathbb{Z}^k$, where
  $\delta_i = \begin{cases} 1 & \text{if } i \in I \\ 0 & \text{if } i \notin I \end{cases}$
- $\overline{x}_I = \{ x_i \mid i \in I \} \subseteq \mathbb{Z}$.
- $\text{span}_I(x) = \max(\overline{x}_I) - \min(\overline{x}_I)$, called the span of $I$.

The next lemma follows immediately from the definition of $\Delta_I$.

**Lemma 3.2.** If $\{I_1, \ldots, I_m\}$ is a block structure for a solution $x$ for (13), then

$$x + b_1 \Delta_{I_1} + \cdots + b_m \Delta_{I_m}$$

is a solution for (13) for any $b_1, \ldots, b_m \in \mathbb{Z}$.

3.2. **Gap and maximum span of an equation.** Consider an equation $E$ of type (13). Define $\text{gap}(E)$ to be the least $n \in \mathbb{N}$ such that for every solution $x_1, \ldots, x_k \in \mathbb{Z}$ for (13) and for every partition $\{1, \ldots, k\} = S_1 \sqcup S_2$ the following holds:

$$\min_{i \in S_2} x_i - \max_{i \in S_1} x_i > n \Rightarrow S_1 \text{ and } S_2 \text{ are unions of blocks.}$$

If $E$ has no solutions, then set $\text{gap}(E) = 0$. The following lemma shows that $\text{gap}(E)$ is well-defined.

**Lemma 3.3.** For equations of type (13), $\text{gap}(E) = O(\text{size}^3(E))$. If the base $\alpha$ is fixed, or if $\alpha$ is restricted to rational values, then $\text{gap}(E) = O(\text{size}(E))$.

**Proof.** If $E$ has no solutions, then $\text{gap}(E) = 0$ and the statement holds. Let $x = (x_1, \ldots, x_k) \in \mathbb{Z}^k$ be a solution for (13) and $I$ a block structure for $x$. As in the proof of Lemma 3.1, construct the polynomial $F(x) \in \mathbb{Z}[x]$ corresponding to $x$. The number of monomials in $F$ is not greater than $dk$ and

$$\text{height}(F) \leq \sum_{i=1}^k \sum_{j=0}^{d-1} |r_{ij}|,$$
which gives the following upper bound on the “gap bound” (i.e. the right-hand side of inequality (10)) of Lemma 2.4 for $F(x)$:

$$\frac{d \ln^3(3d)}{2} \left( \ln(kd) - 1 + \ln \left( \sum_{i=1}^{k} \sum_{j=0}^{d-1} |r_{ij}| \right) \right)$$

$$\leq \frac{d \ln^3(3d)}{2} \left( \ln(k) + \ln(d) + \ln \left( \prod_{i=1}^{k} \prod_{j=0}^{d-1} (|r_{ij}| + 1) \right) \right)$$

$$\leq \frac{d \ln^3(3d)}{2} \left( \ln(k) + \ln(d) + \sum_{i=1}^{k} \sum_{j=0}^{d-1} \ln(|r_{ij}| + 1) \right).$$

By construction, each monomial in $F(x)$ is of the form $cx_i^j$, where $0 \leq j < d$. Hence, for a partition $\{1, \ldots, k\} = S_1 \sqcup S_2$ satisfying

$$\min_{i \in S_2} x_i - \max_{i \in S_1} x_i > d + \frac{d \ln^3(3d)}{2} \left( \ln(k) + \ln(d) + \sum_{i=1}^{k} \sum_{j=0}^{d-1} \ln(|r_{ij}| + 1) \right),$$

we may write $F(x)$ as a sum $F_1(x) + F_2(x)$ which satisfies the assumptions of Lemma 2.4 where $F_1$ contains the terms of degree less than $d + \max_{i \in S_1} x_i$ and $F_2$ contains the terms of degree at least $\min_{i \in S_2} x_i$. Therefore, Lemma 2.4 implies that $F_1(\alpha) = F_2(\alpha) = 0$, so each block in $I$ belongs either to $S_1$ or to $S_2$. The right-hand side of (15) is $O(\text{size}^3(E))$, which completes the proof in the general case (where $\alpha$ is part of the input). In the case where $\alpha$ (hence, $d$) is fixed, the right-hand side of (15) is $O(\text{size}(E))$. If $\alpha \in \mathbb{Q}$, then $\frac{1}{\ln 2}$ replaces $\frac{d \ln^3(3d)}{2}$ in (15), so we obtain $O(\text{size}(E))$ in this case as well. \qed

For an equation $E$ of type (13) define

$$\maxspan(E) = \max \{ \text{span}_I(\overline{x}) \mid I \text{ is a block in a solution } \overline{x} \text{ for } E \},$$

called the maximum span of $E$. If $E$ has no solutions, then $\maxspan(E) = 0$. The following lemma shows that the maximum span is well-defined.

**Lemma 3.4.** $\maxspan(E) = O(\text{size}^4(E))$ for every equation $E$ of type (13). If the base $\alpha$ is fixed, then $\maxspan(E) = O(\text{size}^2(E))$, and if $\alpha$ is restricted to rational values then $\maxspan(E) = O(\text{size}(E))$.

**Proof.** If $I$ is a block in a solution $\overline{x} = (x_1, \ldots, x_k)$ of an equation $E$ of type (13), then it is clear from the definitions that $|x_{i_1} - x_{i_2}| \leq \text{gap}(E)$ for any $i_1, i_2 \in I$. Thus, we obtain

$$\text{span}_I(\overline{x}) \leq |I| \text{gap}(E) \leq (k - 1) \text{gap}(E).$$

Therefore,

$$\maxspan(E) \leq (k - 1) \text{gap}(E),$$

whence the first and second statements follow immediately from Lemma 3.3 (together with the fact that $k \leq \text{size}(E)$). For the case where $\alpha$ is restricted to rational values, we defer the proof to Section 3.4. \qed
3.3. An NP-certificate for (13). In this section, we establish a polynomial upper bound on a minimal solution of (13), and demonstrate that such a solution may be checked in polynomial time.

**Theorem 3.5.** If an equation $E$ of type (13) has a solution, then there is a solution $(x_1, \ldots, x_k) \in \mathbb{Z}^k$ satisfying

$$0 \leq x_1, \ldots, x_k \leq \text{maxspan}(E).$$

**Proof.** Consider any solution $\mathbf{x} = (x_1, \ldots, x_k) \in \mathbb{Z}^k$ and a block structure $\mathcal{I} = \{I_1, \ldots, I_m\}$ for $\mathbf{x}$. By Lemma 3.2,

$$(x'_1, \ldots, x'_k) = \mathbf{x} - \sum_{j=1}^m \min(\mathbf{x}_{I_j}) \cdot \Delta_{I_j}$$

is also a solution. By construction, it satisfies $0 \leq x'_1, \ldots, x'_k \leq \text{maxspan}(E)$, as claimed. \(\square\)

**Corollary 3.6.** The Diophantine problem for equations (13) belongs to NP.

**Proof.** By Lemma 3.1, a solution for (13) satisfying the conclusion of Theorem 3.5 constitutes an NP-certificate. \(\square\)

**Corollary 3.7.** The (uniform or fixed base) Diophantine problem for (13) is NP-complete for $\alpha \in \mathbb{Q} \setminus \{0, 1\}$.

**Proof.** This follows from Proposition 2.2 and Corollaries 2.6 and 3.6. \(\square\)

3.4. An improved bound on maximum span for $\alpha \in \mathbb{Q}$. It turns out that in the special case where $\alpha \in \mathbb{Q}$, one can obtain the following improved bound on maxspan($E$).

**Theorem 3.8.** Let $E$ be an equation of type (13), such that $\alpha \in \mathbb{Q} \setminus \{-1, 0, 1\}$. Then

(a) $\text{maxspan}(E) \leq |\log_{|\alpha|}(|q_i| + 1)|$

(b) $\text{maxspan}(E) = O(|\text{size}(E)|)$ for fixed $\alpha$.

This is very useful in certain applications in which the base $\alpha$ is fixed (e.g., the authors make use of this result in [10]). Note that statement (b) of Theorem 3.8 is an immediate consequence of (a), which we prove in the remainder of this section. However, we do not refer to this material anywhere else in the present work. For convenience, we sometimes assume that a solution $x_1, \ldots, x_k$ has a block structure consisting of a single block, and that $x_1 \geq x_2 \geq \cdots \geq x_k = 0$. The following technical lemma is required.

**Lemma 3.9.** Let $k \geq 2$, $\alpha > 1$, and $q_1, \ldots, q_k \in \mathbb{Z} \setminus \{0\}$. The maximum value of the sum $\delta_1 + \cdots + \delta_{k-1}$, for $\delta_1, \ldots, \delta_{k-1} \in \mathbb{R}$ constrained by

$$(17) \quad 0 \leq \delta_i \leq \log_{\alpha}(|q_i| + 1) + |q_{i+2}|\alpha^{-\delta_{i+2}} + \cdots + |q_k|\alpha^{-\sum_{j=i+1}^{k-1}\delta_j} \quad \text{for } i = 1, \ldots, k-2,$$

$$0 \leq \delta_{k-1} \leq \log_{\alpha}(|q_k|),$$

is attained at $\delta_1 = \log_{\alpha}(|q_2| + 1), \ldots, \delta_{k-2} = \log_{\alpha}(|q_{k-1}| + 1), \delta_{k-1} = \log_{\alpha}(|q_k|)$.

**Proof.** Since (17) defines a non-empty compact set, the sum attains the maximum value at some point $(\delta_1, \ldots, \delta_{k-1})$. Fix $\delta_3, \ldots, \delta_{k-1}$, define

$$q^* = |q_3| + |q_4|\alpha^{-\delta_3} + \cdots + |q_k|\alpha^{-\sum_{j=3}^{k-1}\delta_j},$$
and notice that \( \delta_1, \delta_2 \) must maximize \( \delta_1 + \delta_2 \) while satisfying

\[
0 \leq \delta_1 \leq \log_\alpha \left( |q_2| + |q^*|^\alpha^{-\delta_2} \right), \\
0 \leq \delta_2 \leq \log_\alpha \left( |q^*| \right).
\]

Obviously, \( \delta_1 = \log_\alpha \left( |q_2| + |q^*|^\alpha^{-\delta_2} \right) \). Thus, the value of \( \delta_2 \) must maximize the value of the function

\[
f(\delta_2) = \delta_2 + \log_\alpha \left( |q_2| + |q^*|^\alpha^{-\delta_2} \right)
\]

for \( \delta_2 \in [0, \log_\alpha (|q^*|)] \). It is easily seen that \( f \) is monotonically increasing, so the maximum is attained at \( \delta_2 = \log_\alpha (|q^*|) \), and \( \delta_1 = \log_\alpha (|q_2| + 1) \).

Once the optimal value of \( \delta_1 \) is found, we can eliminate \( \delta_1 \) from the sum and remove the bounds on \( \delta_1 \) from (17) (notice that \( \delta_1 \) is not involved in the other bounds). That produces an optimization problem of the same type with \( k - 1 \) variables. Hence, the result follows by induction on \( k \).

**Lemma 3.10.** Suppose that \( 1 < \alpha \in \mathbb{Q} \) and \( \overline{x} = (x_1, \ldots, x_k) \) is a solution for an equation \( E \) of type (13) with a single block \( I = \{1, \ldots, k\} \) satisfying \( x_1 \geq x_2 \geq \cdots \geq x_k = 0 \). Define \( \delta_i = x_i - x_{i+1} \) for \( i = 1, \ldots, k - 1 \). Then we have

(a) \( 0 \leq \delta_i \leq \log_\alpha \left( |q_{i+1}| + |q_{i+2}|\alpha^{-\delta_{i+1}} + \cdots + |q_k|\alpha^{-(\delta_{i+1} + \cdots + \delta_{k-1})} \right) \)

(b) \( \text{span}_I(\overline{x}) = x_1 \leq \sum_{i=1}^{k} \log_\alpha (|q_i| + 1) \).

**Proof.** Let \( \alpha = \frac{c}{d} \), where \( \gcd(c, d) = 1, c \geq 2 \) and \( c > d \). Multiplying (13) by \( d^{x_1} \) we get

(18) \[ q_1 c^{x_1} + q_2 c^{x_2} d^{x_1-x_2} + \cdots + q_{k-1} c^{x_{k-1}} d^{x_1-x_{k-1}} + q_k c^{x_k} d^{x_1-x_k} = 0. \]

Taking (18) modulo \( c^{x_{k-1}} \) we get

\[ q_k c^{x_k} d^{x_1-x_k} \equiv 0 \mod c^{x_{k-1}} \Rightarrow c^{x_{k-1}-x_k} | q_k \]

\[ \Rightarrow \delta_{k-1} = x_{k-1} - x_k \leq \log_c (|q_k|) \]

\[ \Rightarrow \delta_{k-1} \leq \log_\alpha (|q_k|), \]

which proves (a) for \( i = k - 1 \). Writing (18) as

(19) \[ q_1 c^{x_1} + q_2 c^{x_2} d^{x_1-x_2} + \cdots + (q_{k-1} + q_k \left( \frac{d}{c} \right)^{\delta_{k-1}}) c^{x_{k-1}} d^{x_1-x_{k-1}} = 0, \]

we obtain an expression of the same form as (18), with one fewer term and a new rightmost coefficient \( q'_{k-1} \in \mathbb{Z} \) that satisfies the following:

- \( q'_{k-1} = 0 \iff k = 2 \) (because, by our assumption, \( \overline{x} \) has a single block);
- \( |q'_{k-1}| \leq |q_{k-1}| + |q_k| \).

Assuming \( k > 2 \), we may apply the same argument to (19). That is, take (19) modulo \( c^{x_{k-2}} \), and follow the same steps as above to obtain \( \delta_{k-2} \leq \log_\alpha (|q'_{k-1}|) \), proving (a) for \( i = k - 2 \). This process can be continued to yield, at each iteration, a new rightmost (non-trivial unless \( i = 1 \)) coefficient

\[ q'_i = q_i + q_{i+1}^\alpha^{-\delta_i} = q_i + q_{i+1}^\alpha^{-\delta_i} + \cdots + q_k^\alpha^{-(\delta_i + \cdots + \delta_{k-1})} \]

satisfying (a) for each \( i = 1, \ldots, k - 1. \)
To prove (b), notice that $x_1$ can be expressed as

$$x_1 = \delta_1 + \cdots + \delta_{k-1}$$

and by (a) each $\delta_i$ satisfies the constraints of Lemma 3.9. Hence, it follows from Lemma 3.9 that $x_1 \leq \sum_{i=1}^{k} \log_{|q_i|}(|q_i| + 1)$. \hfill \Box

Lemma 3.10 was proved under the assumption that $\alpha > 1$. Below we show that it holds with minor modifications for any $\alpha \in \mathbb{Q} \setminus \{-1, 0, 1\}$.

**Proposition 3.11.** Suppose that $\alpha \in \mathbb{Q} \setminus \{-1, 0, 1\}$ and $\mathbf{x} = (x_1, \ldots, x_k)$ is a solution for an equation $E$ of type (13) with a single block $I = \{1, \ldots, k\}$ satisfying $x_1 \geq x_2 \geq \cdots \geq x_k = 0$. Define $\delta_i = x_i - x_{i+1}$ for $i = 1, \ldots, k - 1$. Then we have

$$\text{span}_I(\mathbf{x}) = x_1 \leq \sum_{i=1}^{k} |\log_{|q_i|}(|q_i| + 1)|.$$

**Proof.** We consider four cases.

**Case 1:** Suppose that $\alpha > 1$. Then the statement follows from Lemma 3.10.

**Case 2:** The case when $0 < \alpha < 1$ is reduced to Case 1 by replacing $x_i$ with $-x_i$ and $\alpha$ with $1/\alpha$. This transformation changes the order of $x_i$'s and $\delta_i$'s, which also modifies the inequalities in item (a) of Lemma 3.10 in the following way:

$$\delta_1 \leq \log_{1/\alpha}(|q_1|), \quad \delta_2 \leq \log_{1/\alpha}(|q_2| + |q_1|(|1/\alpha| - \delta_1)), \quad \text{etc.}$$

Since the obtained inequalities are of the same type, Lemma 3.9 remains applicable and gives the claimed bound

$$x_1 \leq \sum_{i=1}^{k} \log_{1/\alpha}(|q_i| + 1) = \sum_{i=1}^{k} |\log_{\alpha}(|q_i| + 1)|.$$

**Case 3:** Suppose that $\alpha < -1$. It is easy to see that Lemma 3.10 holds in that case with minor modifications (with $\alpha$ replaced with $|\alpha|$).

**Case 4:** The case when $-1 < \alpha < 0$ is reduced to Case 3 by replacing $x_i$ with $-x_i$ and $\alpha$ with $1/\alpha$. \hfill \Box

Theorem 3.8 now follows from Proposition 3.11.

### 4. Complexity upper bound for a system of homogeneous equations

Let $\alpha_1, \ldots, \alpha_s \in \overline{\mathbb{Q}} \setminus U$ and $d_i = \deg(\alpha_i)$. Consider a system of equations

$$
\begin{cases}
q_{11}\alpha_1^{x_1} + \cdots + q_{1k}\alpha_1^{x_k} = 0 \\
\vdots \\
q_{s1}\alpha_s^{x_1} + \cdots + q_{sk}\alpha_s^{x_k} = 0
\end{cases}
$$

with coefficients $q_{ij} \in \mathbb{Z}[\alpha_i]$ and each coefficient given as $q_{ij} = r_{ij}^0 + r_{ij}^1\alpha_i + \cdots + r_{ij}^{d_i-1}\alpha_i^{d_i-1}$. Let us also require that at least one of $q_{1j}, q_{2j}, \ldots, q_{sj}$ is nonzero for each $j$. Let $\mathbf{x} = (x_1, \ldots, x_k) \in \mathbb{Z}^k$ be a solution for (20). A nonempty $J \subseteq \{1, \ldots, k\}$ is called a cluster for the solution $\mathbf{x}$ if the following conditions hold:

1. $\sum_{j \in J} q_{ij}\alpha_i^{x_j} = 0$ for every $i = 1, \ldots, s$.  

(C2) $J$ does not have a nonempty proper subset $J$ satisfying (C1).

The set of indices $\{1, \ldots, k\}$ can be represented as a union of disjoint clusters, perhaps in more than one way; a choice $J = \{J_1, \ldots, J_m\}$ of one such union is called a cluster structure for a solution $\bar{x}$. For a cluster $J$, we define $\bar{x}_J$, $\operatorname{span}_J(\bar{x})$ and $\Delta_J$ analogously to the case of a block. The next lemma follows immediately from the definitions.

Lemma 4.1. If $\bar{x}$ is a solution for (20) with cluster structure $J_1, \ldots, J_m$, then

$$\bar{x} + \beta_1 \Delta_{J_1} + \cdots + \beta_m \Delta_{J_m}$$

is a solution for (20) with the same cluster structure for any $\beta_1, \ldots, \beta_m \in \mathbb{Z}$. \hfill \square

4.1. Max span of a system. For a system $E$ of type (20), the definition of the maximum span is extended in the natural way, as follows:

$$\maxspan(E) = \max \{ \operatorname{span}_J(\bar{x}) \mid J \text{ is a cluster in a solution } \bar{x} \text{ for } E \} .$$

If $E$ has no solutions, then maxspan($E$) = 0. We obtain an upper bound on maxspan($E$) below.

A collection $A_1, \ldots, A_m$ (where $m \geq 2$) of finite nonempty subsets of $\mathbb{Z}$ is non-separable if the union of closed intervals

$$\bigcup_{i=1}^{m} [\min(A_i), \max(A_i)]$$

is an interval. Otherwise it is separable. The following lemma follows easily from this definition.

Lemma 4.2. If finite nonempty subsets $A_1, \ldots, A_m \subseteq \mathbb{Z}$ are non-separable, then the following inequality holds:

$$\max \left( \bigcup_{i=1}^{m} A_i \right) - \min \left( \bigcup_{i=1}^{m} A_i \right) \leq \sum_{i=1}^{m} (\max(A_i) - \min(A_i)).$$

Let $\bar{x}$ be a solution for (20) and $J$ a cluster structure for $\bar{x}$. Then $\bar{x}$ is a solution for each individual equation in (20). Hence, each equation in (20) has a block structure $\mathcal{I}_i = \{I_{i1}, \ldots, I_{im}\}$ for $\bar{x}$. We say that $\mathcal{I}_1, \ldots, \mathcal{I}_s$ are compatible with $J$ if for any cluster $J \in J$ and any block $I_{ij} \in \mathcal{I}_i$ we have

$$I_{ij} \subseteq J \text{ or } I_{ij} \cap J = \emptyset .$$

For any cluster structure $J$, (C1) obviously implies the existence of a compatible block structure $\mathcal{I}_i$ for each individual equation in (20), and the identity

(21) $$J = \bigcup_{I_{ij} \subseteq J} I_{ij}$$

holds for every $J \in J$ because every index in $\{1, \ldots, k\}$ belongs to at least one block $I_{ij}$ (since we assume that there is at least one nonzero $a_{ij}$ corresponding to each $x_i$).

Lemma 4.3. If $J$ and $\mathcal{I}_1, \ldots, \mathcal{I}_s$ are compatible, then for every $J \in J$, the collection of sets

$$\{ \bar{x}_{I_{ij}} \mid I_{ij} \subseteq J \}$$

is non-separable.

Proof. Separability contradicts the property (C2) of $J$. \hfill \square
Proposition 4.4. Let $E$ be a system of type $(20)$, and let $E_i$ denote the $i$th equation in $E$. Then we have
\[
\max\text{span}(E) \leq sk \max_{1 \leq i \leq s} (\max\text{span}(E_i)).
\]

Proof. Let $J$ be a cluster structure for a solution $\bar{x}$ of $E$, and let $J \in J$. We have
\[
\max(\bar{x}_J) - \min(\bar{x}_J) = \max \left( \bigcup_{I \subseteq J} \bar{x}_{I_{ij}} \right) - \min \left( \bigcup_{I \subseteq J} \bar{x}_{I_{ij}} \right) \quad \text{(the identity (21))}
\]
\[
\leq \sum_{I_{ij} \subseteq J} \text{span}_{I_{ij}}(\bar{x}) \quad \text{(Lemmas 4.2 and 4.3)}
\]
\[
\leq \sum_{I_{ij} \subseteq J} \max\text{span}(E_i)
\]
\[
\leq sk \max_{1 \leq i \leq s} (\max\text{span}(E_i)).
\]
\[\square\]

Corollary 4.5. For systems $E$ of type $(20)$, $\max\text{span}(E)$ is bounded by a polynomial in $\text{size}(E)$.

Proof. Follows immediately from Proposition 4.4 and Lemma 3.4 \[\square\]

4.2. An NP-certificate for $(20)$.

Theorem 4.6. If a system $E$ of type $(20)$ has a solution, then it has a solution $x_1, \ldots, x_k \in \mathbb{Z}$ satisfying
\[
0 \leq x_1, \ldots, x_k \leq \max\text{span}(E).
\]

Proof. Same as the proof of Theorem 3.5 using Lemma 4.1 instead of Lemma 3.2 \[\square\]

Finally, from Lemma 3.1, Corollary 4.5 and Theorem 22 we obtain the following.

Corollary 4.7. The Diophantine problem for systems of type $(20)$ belongs to NP.

5. Complexity upper bound for general systems of equations

5.1. Non-homogeneous equations. Let $\alpha_1, \ldots, \alpha_s \in \mathbb{Q}^* \setminus \mathbb{U}$, and consider a system of non-homogeneous equations
\[
\begin{align*}
q_{11}\alpha_1^{x_1} + \cdots + q_{1k}\alpha_1^{x_k} &= q_{10}, \\
\vdots \\
q_{s1}\alpha_s^{x_1} + \cdots + q_{sk}\alpha_s^{x_k} &= q_{s0},
\end{align*}
\]
with $q_{ij} \in \mathbb{Z}[\alpha_i]$. It is easy to see that a system $E$ of this form has a solution if and only if the homogeneous system $E'$
\[
\begin{align*}
q_{11}\alpha_1^{x_1} + \cdots + q_{1k}\alpha_1^{x_k} - q_{10}\alpha_1^{x_0} &= 0 \\
\vdots \\
q_{s1}\alpha_s^{x_1} + \cdots + q_{sk}\alpha_s^{x_k} - q_{s0}\alpha_s^{x_0} &= 0
\end{align*}
\]
has a solution. Hence, an NP-certificate for $E'$ can be used as an NP-certificate for $E$. Since $\text{size}(E') = \text{size}(E)$, it follows from Corollary 4.7 that the Diophantine problem for non-homogeneous systems of type $(23)$ belongs to NP.
5.2. Systems of equations with roots of unity. Finally, we consider the most general systems of equations with exponents, where \( \alpha_1, \ldots, \alpha_s \) are allowed to be roots of unity. As above, the non-homogeneous case can be reduced to the homogeneous one, so in fact we need only consider homogeneous systems. Recall that there is a polynomial-time algorithm (see [1]) that takes as input the minimal polynomial of \( \alpha_i \) and determines whether \( \alpha_i \) is a root of unity. If it is a root of unity, this algorithm also provides the order of \( \alpha_i \) in the group \( U \). Hence, we consider systems of the form

\[
\begin{align*}
q_{11} \alpha_1^{x_1} + \cdots + q_{1k} \alpha_1^{x_k} &= 0 \\
\vdots & \quad \\
q_{t1} \alpha_t^{x_1} + \cdots + q_{tk} \alpha_t^{x_k} &= 0 \\
\vdots & \quad \\
q_{s1} \alpha_s^{x_1} + \cdots + q_{sk} \alpha_s^{x_k} &= 0
\end{align*}
\]  

(25)

with \( \alpha_1, \ldots, \alpha_t \notin U \) and \( \alpha_{t+1}, \ldots, \alpha_s \in U \) for some \( t \in \{0, \ldots, s\} \), and \( q_{ij} \in \mathbb{Z}[\alpha_i] \). We may assume that \( n_i \) is the order of \( \alpha_i \) in \( U \) for \( t < i \leq s \). From the fact that \( \deg(\alpha_i) = \varphi(n_i) \) (where \( \varphi \) denotes Euler’s totient function), and using the well-known lower bound \( \varphi(n) \geq \sqrt{n} \), we obtain the following bound on \( n_i \).

\[
n_i < 2(\deg(\alpha_i))^2 < 2 \text{size}^2(E).
\]  

(26)

We also assume that every unknown \( x_i \) is non-trivially involved in \( E \), i.e. for every \( j \in \{1, \ldots, k\} \) there exists \( i \in \{1, \ldots, s\} \) such that \( q_{ij} \neq 0 \).

First, assume that \( E \) is an equation of type (25) in which all of the bases are roots of unity, and let \( N = \text{lcm}(n_1, \ldots, n_s) \). Clearly, if \( \overline{\tau} = (x_1, \ldots, x_k) \) is a solution to \( E \), and \( x_i \equiv x_i' \mod N \), then \( \overline{\tau}' = (x_1', \ldots, x_k') \) is another solution to \( E \). Hence, if \( E \) has a solution, then it has a solution \( \overline{\tau} = (x_1, \ldots, x_k) \) such that \( 0 \leq x_1, \ldots, x_k < N \). From (26) we have

\[
N < 2^k \text{size}^2(E),
\]  

(27)

so that

\[
\log N = O(2k \log(\text{size}(E))) = O(\text{size}^2(E)).
\]  

(28)

By Lemma 3.1 \( \overline{\tau} \) is an NP-certificate for decidability of \( E \).

Now suppose that at least one, but not all, of the bases are roots of unity, and consider the subsystem \( E_{\leq t} \) of the first \( t \) equations (i.e. the equations where \( \alpha_i \notin U \)). Let \( \overline{\tau} = (x_1, \ldots, x_k) \) be a solution to \( E \). In particular, \( \overline{\tau} \) is a solution to \( E_{\leq t} \), and we may choose a cluster structure \( \mathcal{J} = \{J_1, \ldots, J_m\} \) corresponding to \( \overline{\tau} \) and \( E_{\leq t} \) (i.e. \( \mathcal{J} \) is not necessarily a cluster structure with respect to \( E \)). The following variation on Lemma 4.1 follows easily from the preceding discussion.

**Lemma 5.1.** Let \( E \) be a system of type (25) as described above (i.e. where \( 1 \leq t < s \)), and let \( \overline{\tau} \) be a solution of \( E \). Let \( \{J_1, \ldots, J_m\} \) be a cluster structure corresponding to \( \overline{\tau} \) and \( E_{\leq t} \), and let \( N = \text{lcm}(n_{t+1}, \ldots, n_s) \). Then

\[
\overline{\tau} + N \beta_1 \Delta_{J_1} + \cdots + N \beta_m \Delta_{J_m}
\]

is a solution for \( E \) with the same cluster structure for any \( \beta_1, \ldots, \beta_m \in \mathbb{Z} \).
Theorem 5.2. If a system $E$ of type (25) has a solution, then it has a solution $\bar{x}$ (with the same cluster structure) satisfying
\begin{equation}
0 \leq x_1, \ldots, x_k < N + \text{maxspan}(E_{\leq t})
\end{equation}
where $N = \text{lcm}(n_{t+1}, \ldots, n_s)$.

Proof. Suppose that $y$ is a solution to $E$, and set $\beta_i$ so that $0 \leq \min(y_J) + N\beta_i < N$ (i.e. reduce $\min(y_J)$ modulo $N$). Then it follows from the definition of maxspan($E_{\leq t}$) that
$$
\bar{x} = y + N\beta_1\Delta_{J_1} + \cdots + N\beta_m\Delta_{J_m}
$$
satisfies (29), and by Lemma 5.1 it is also a solution to $E$. □

By (28), Corollary 4.5 and Lemma 3.1, a solution $x_1, \ldots, x_k$ for (25) satisfying (29) is an NP-certificate; thus, we have proved the following.

Corollary 5.3. The Diophantine problem for systems (25) belongs to NP.

Finally, from Corollary 2.6 we obtain

Corollary 5.4. The Diophantine problem for systems (25) is NP-complete.

6. Structure of the solution set

We say that a set $S \subseteq \mathbb{Z}^k$ is semilinear if $S$ is a finite union of cosets, i.e.,
$$
S = \bigcup_{i=1}^n (\delta_i + A_i), \quad \text{for some } \delta_1, \ldots, \delta_n \in \mathbb{Z}^k \text{ and } A_1, \ldots, A_n \leq \mathbb{Z}^k.
$$

Let $\bar{x} = (x_1, \ldots, x_k)$ be a solution for a system $E$ of type (25). Following Section 5.2, let $J = \{J_1, \ldots, J_m\}$ be a cluster structure corresponding to $\bar{x}$ and $E_{\leq t}$, and let us define
$$
N_E = \begin{cases} 
1 & \text{if } t = s \\
\text{lcm}(n_{t+1}, \ldots, n_s) & \text{otherwise.}
\end{cases}
$$

The set $J$ defines the tuples $\Delta_{J_1}, \ldots, \Delta_{J_m} \in \{0, 1\}^k$, so by Lemma 5.1 the pair $(\bar{x}, J)$ defines the following set of solutions for $E$
$$
S_{\bar{x}, J} = \{ \bar{x} + N_E\beta_1\Delta_{J_1} + \cdots + N_E\beta_m\Delta_{J_m} \mid \beta_1, \ldots, \beta_m \in \mathbb{Z} \} \subseteq \mathbb{Z}^k
$$
(note that $S_{\bar{x}, J}$ is a coset of $\mathbb{Z}^k$). Define $T(E)$ to be the set of all pairs $(\bar{x}, J)$ such that
- $\bar{x}$ is a solution for $E$;
- $J$ is a cluster structure for $\bar{x}$ and $E_{\leq t}$;
- (boundedness) $\bar{x} = (x_1, \ldots, x_k)$ satisfies (29).

By construction, $T(E)$ is finite.

Proposition 6.1 (Completeness). The set of all solutions of a system $E$ of type (25) is equal to
$$
S(E) = \bigcup_{(\bar{x}, J) \in T(E)} S_{\bar{x}, J}.
$$
Proof. By Lemma 5.1, \( \bigcup_{(\overline{x}, J) \in \mathcal{T}} S_{\overline{x}, J} \) is a set of solutions for \( E \). Conversely, if \( \overline{y} \) is a solution for \( E \) and \( J \) is a cluster structure for \( \overline{x} \) and \( E_{\leq t} \), then (as in the proof of Theorem 5.2) the set \( S_{\overline{y}, J} \) contains a solution \( \overline{x} \) satisfying (29) with the same cluster structure \( J \). Hence, \((\overline{x}, J) \in \mathcal{T}(E)\) and \( \overline{y} \in S_{\overline{y}, J} = S_{\overline{x}, J} \), proving that \( \overline{y} \in S(E) \). □

**Corollary 6.2.** The set of all solutions of (25) is semilinear.

With minor adjustments, the foregoing arguments apply to non-homogeneous systems as well. Specifically, if \( E \) is a non-homogeneous system, then we form the associated homogeneous system \( E' \) with auxiliary variable \( x_0 \), as in (24). We now consider the set of \( \overline{x} \) which are solutions to both \( E \) and \( E' \), and cluster structures \( J \) associated to \( \overline{x} \) and \( E'_{\leq t} \), where each \( J \) is of the form \( \{J_1, \ldots, J_k, J_0\} \) and \( 0 \in J_0 \). We further stipulate that only the clusters \( J_1, \ldots, J_k \) can be shifted (but not \( J_0 \)). Hence, the definition of \( S_{\overline{x}, J} \) remains the same, and the proof of Proposition 6.1 goes through unchanged.

**7. Parameterized complexity of the Diophantine problem**

In this section, we show that the Diophantine problem for systems (25) can be solved in polynomial time if the number of variables \( k \) is bounded by a fixed constant. In other words, the Diophantine problem for (25) is **fixed-parameter tractable**.

Let \( q \) be a polynomial such that the time complexity for validating an NP-certificate \( \overline{x} = (x_1, \ldots, x_k) \) for a system \( E \) of type (25) (i.e. for checking that \( \overline{x} \) is a solution) is \( O(q(\text{size}(E))) \), and let \( p \) be a polynomial such that \( \maxspan(E) < p(\text{size}(E)) \).

By Corollaries 5.3 and 4.5, such polynomials exist.

**Proposition 7.1.** There exists an algorithm that decides if a given system \( E \) of type (25) has a solution in time

\[
O\left(2^{k^2}\text{size}^{2k^2}(E) \cdot p(\text{size}(E))^k \cdot q(\text{size}(E))\right).
\]

**Proof.** Using (27), we see that the total number of NP-certificates \( \overline{x} = (x_1, \ldots, x_k) \) satisfying (29) is

\[
(N_E + \maxspan(E_{\leq t}))^k < (2^k \text{size}^{2k}(E) + p(\text{size}(E)))^k,
\]

which is \( O\left(2^{k^2}\text{size}^{2k^2}(E) \cdot p(\text{size}(E))^k \right) \). It is straightforward to enumerate all such certificates, and the time required to check each one is \( O(q(\text{size}(E))) \), so the result follows. □

**Corollary 7.2.** Fix \( K \in \mathbb{N} \). The Diophantine problem for systems (25) with the number of variables bounded by \( K \) can be solved in polynomial time. Hence, the Diophantine problem for systems (25) is a fixed-parameter tractable problem.

**Proof.** The complexity bound (30) of Proposition 7.1 is polynomial if \( k \) is bounded by the given constant \( K \). □

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