THE FROG MODEL ON TREES WITH DRIFT

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Abstract. We provide a uniform upper bound on the minimal drift so that the one-per-site frog model on a $d$-ary tree is recurrent. To do this, we introduce a subprocess that couples across trees with different degrees. Finding couplings for frog models on nested sequences of graphs is known to be difficult. The upper bound comes from combining the coupling with a new, simpler proof that the frog model on a binary tree is recurrent when the drift is sufficiently strong. Additionally, we describe a coupling between frog models on trees for which the degree of the smaller tree divides that of the larger one. This implies that the critical drift has a limit as $d$ tends to infinity along certain subsequences.

1. Introduction

We study the one-per-site frog model with drift on the rooted $d$-ary tree $T_d$. Initially there is a single awake frog at the root and one sleeping frog at each non-root vertex. Awake frogs move towards the root with probability $p$, and otherwise move away from the root to a uniformly sampled child vertex. Frogs at the root always move to a uniformly sampled child vertex. Whenever an awake frog visits a site with a sleeping frog, the sleeping frog wakes up and begins its own independent $p$-biased random walk. Denote this process by $FM(d, p)$ and the total number of visits to the root by $V(d, p)$. The process is recurrent if $V(d, p)$ is infinite almost surely, and is otherwise transient. The almost sure requirement is not overly stringent, because the probability the root is visited infinitely often in $FM(d, p)$ satisfies a 0-1 law by Kosygnia and Zerner’s general result [KZ17, Theorem 1].

There is a history of investigating recurrence for the frog model with drift. It was first studied by Gantert and Schmidt with i.i.d $\eta$ frogs per site and a drift in the $e_1$ direction on $\mathbb{Z}$ [GS09]. They showed that the process is recurrent if and only if $E \log \eta = \infty$ regardless of the drift. A followup work by Ghosh et. al. studied the range of the frog model in the transient case [GNR17]. Similar observations were made by Rosenberg when the frog paths are Brownian motions in $\mathbb{R}$ [Ros17a, Ros17b]. The question is more subtle and challenging in higher dimensions. Döbler and Pfeifroth showed that the frog model is recurrent on $\mathbb{Z}^d$ for $d \geq 2$ so long as $E \log^{(d+1)/2} \eta = \infty$ [?]. It was open for some time whether, unlike the $d = 1$ case, there is a phase transition as the drift is varied. This was recently answered by Döbler et. al. [DGH+17]. With one sleeping frog at each site of $\mathbb{Z}^d$, they found that recurrence depends on the strength of the drift with notably different behavior in $d = 2$ and $d \geq 3$. We prove here that transience and recurrence of $FM(d, p)$ also depends on the drift.

Trees are a natural setting to study the frog model with drift, because the graph structure already induces one. Indeed, $FM_d = FM(d, 1/(d + 1))$ is the frog model with simple random walk paths. Hoffman et. al. proved that $FM_2$ is recurrent, but that $FM_d$ is transient for $d \geq 5$ [HJJ17b]. What happens when $d = 3$ and $d = 4$ for the one-per-site frog model is not currently known. However, followup work by Hoffman et. al. showed that the frog model with unbiased random walks can be made recurrent for any $d$ so long as $\Omega(d)$ sleeping frogs
are placed at each site [HJJ16, JJ16a, JJ16b]. So, there is a phase transition as we change
the degree of the tree, or the initial density of asleep frogs.

Since $FM_d$ is known to be transient for $d \geq 5$, it is natural to ask how much drift
\[ p_d = \inf\{p: FM(d, p) \text{ is recurrent}\} \]
is needed to make the process recurrent. From this perspective, the main theorem of
[HJJ17b] is that $p_2 = 1/3$. In general, we know that $p_d \leq 1/2$, because the initially awake
frog will return to the root infinitely often when $p \geq 1/2$. A simple argument shows that if
$p < 1/(d+1)$ then, even with all frogs initially awake, there are only finitely many expected
visits to the root. This immediately gives the bounds $1/(d+1) \leq p_d \leq 1/2$.

It is not much more difficult to establish a non-vanishing lower bound on $p_d$. The frog
model is dominated by the branching random walk (BRW) on $\mathbb{T}_d$ in which particles do not
branch when moving towards the root (with probability $p$), but split in two when moving
away. This corresponds to $FM(\infty, p)$. This BRW is a common tool for analyzing the frog
model. By replacing $1/(d+1)$ with $p$ in the calculation at [HJJ16, Proposition 15] it follows
that the BRW, and thus $FM(d, p)$, is transient for $p < q^* = (2 - \sqrt{2})/2 \approx .1464$. Thus,
$p_d \geq q^*$. Our main contribution is an upper bound.

**Theorem 1.** $p_d \leq .4155$ for all $d \geq 2$.

**Proof.** This follows from Lemma 5 combined with Proposition 3 and Proposition 4. \qed

It is interesting to ask how the frog model relates to the dominating BRW. The extra
drift the frog model needs to be recurrent, $p_d - q^*$, is one way to measure the difference.
By using a BRW that approximates two steps of the frog model, it is not overly taxing to
show that $p_d - q^* > 0$ for all $d$ (see [HJJ17b, Proposition 19] for an example of a more
refined BRW). Since the dominating BRW corresponds in a sense to $d = \infty$, it is natural to
ask if $p_d \to q^*$, and, if so, at what rate? The answer is not obvious, as sites near the root
gradually get visited. Thus, as time elapses, the frog model branches less. The region with
less branching may grow quickly. Hoffman et. al. proved in [HJJ17a] that, when the density
of frogs is $\Omega(d^2)$, the set of activated sites on the $d$-ary tree contains a linearly expanding
ball. Awake frogs in this region cause no branching. We are not sure if this prevents $p_d$
from converging to $q^*$. In fact, we are not sure whether $p_d$ converges at all. This question
of convergence is the second reason we are interested in $FM(d, p)$.

Coupling frog models on different graphs is known to be difficult. Past work by Fontes et.
al. established that the critical probability for the frog model with death is not monotonic in
the graph [FMS04]. However, Lebensztayn et. al. in [LMP05] conjectured that monotonicity
holds on regular trees. No coupling has ever been exhibited between the process on different
degree trees. This would be nice because it might help understand how the frog model
behaves on random trees and investigate the convergence of $p_d$.

We say that one frog model *dominates* another, denoted $FM(d, p) \preceq FM(d', p')$, if there
is a coupling so that every awake frog in $FM(d, p)$ is coupled to an awake frog in $FM(d', p')$
at the same distance from the root. Intuitively, this means that the set of awake frogs
in $FM(d, p)$ can be embedded in the set for $FM(d', p')$. An immediate consequence of
domination is that $V(d, p) \preceq V(d', p')$ in the usual sense of stochastic domination.

It ought to hold that $FM(d, p) \preceq FM(d+1, p)$. This is because the drift is the same,
but there are significantly more frogs in the higher degree tree. Despite considerable effort,
we were unable to construct such a coupling. It remains an open problem to prove that
$p_{d+1} \leq p_d$, and thus that $p_d$ has a limit. Additionally, there is no obvious coupling so that
$FM(d, p)$ visits the root less than $FM(d, p')$ does when $p < p'$. Although preposterous, we
cannot rule out the possibility FM($d,p$) switches between being transient and recurrent multiple times as we increase $p$.

The obvious coupling to try between FM($d,p$) and FM($d+1,p$) is to have paired frogs mimic one another’s displacement from the root, but to move to uniformly chosen vertices when moving away. One can readily find realizations where the frog on $T_d$ wakes a new frog, while the coupled frog on $T_{d+1}$ does not. This breaks the coupling. We tried several more sophisticated couplings with no luck.

One special case in which a coupling works is for trees in which the degree of the smaller tree divides that of the larger tree.

**Proposition 2.** FM($d,p$) \( \preceq \) FM($kd,p$) for all $k \geq 1$.

The argument relies on a natural way to embed copies of $T_k$ into $T_{kd}$. It does not appear to generalize to any other degrees. It follows that the subsequences $(p_{kd})_{k=1}^\infty$ converge to some $p^*(d)$ for each $d \geq 2$. So, there is a limit along certain subsequences, but we are not sure if $p^*(d) = q^*$.

We make more substantial progress coupling across different graphs with a subprocess of FM($d,p$) that we call the **recursive frog model** RFM($d,p$). It is obtained by trimming and halting the random walk paths of awake frogs. This ensures that RFM($d,p$) visits the root less than FM($d,p$). See Section 2 for the formal definition. A related, but slightly different process known as the **self-similar frog model** has been a useful tool for studying recurrence [HJJ16, JJ16a, HJJ18, HJJ17a, Ros17c].

Let $p_d' = \inf \{ p : \text{RFM}(d,p) \text{ is recurrent} \}$ be the critical drift for the recursive frog model on $T_d$. It follows from the dominance relation in Lemma 5 that $p_d \leq p_d'$. As mentioned above, the usual frog model is difficult to couple on two trees of different degrees. Finding a coupling for the self-similar frog model also appears challenging. The discussion just after the definition of RFM($d,p$) in Section 2 explains why in more detail. It is both useful for our main theorem and of independent interest that there is a coupling where RFM($d,p$) is dominated by RFM($d+1,p$). See Lemma 7 for the proof. We use this to show that $p_d'$ is decreasing.

**Proposition 3.** $p_{d+1}' \leq p_d'$ for all $d \geq 2$.

The recursive frog model is useful because a coupling is possible across trees of different degrees. However, the coupling comes at the cost of removing a lot of awake frogs. Because so many frogs are removed, it is not obvious whether RFM($d,p$) is ever recurrent. Old techniques do not apply easily here. We provide a new, simpler argument for recurrence for large enough $p$.

**Proposition 4.** $p_2' \leq .4155$.

All previous results that establish recurrence for the frog model on trees rely on bootstrapping a recursive distributional equation involving $V(d, 1/(d+1))$. See (4) for the equation. The recursive frog model is intuitively less recurrent than the self-similar frog model because more frogs are being removed. So, it is not clear that the bootstrapping approach will work. Fortunately, we find a simpler way to proceed. It starts with the usual recursive distributional equation, but uses the second moment method to finish. This is similar to an argument used to prove that the parking process visits the origin infinitely often [DGJ+17]. To finish we prove a 0-1 law for the recursive frog model. This is necessary because the recursive frog model is not covered by the 0-1 law in [KZ17].
2. THE RECURSIVE FROG MODEL

The recursive frog model RFM\((d,p)\) has awake frogs that move towards the root at each step with probability \(p = p/(1-p)\) when \(p < 1/2\), and with probability 1 for \(p \geq 1/2\). If a frog reaches the root, it is removed. Once a frog moves away from the root, it moves to a uniformly sampled child vertex (possibly the vertex from which it just came) and will thereafter continue to move away from the root to a uniformly sampled child. Frogs are removed if they move away from the root and land on an already-visited site. To make removing frogs well-defined, we at each step sample one frog from the set of awake frogs and have it perform one step. The order frogs move does not change the law, and since the set of awake frogs grows by at most one at each time step, it is easy to see that every awake frog will be sampled infinitely often. RFM\((d,p)\) earns its name because these modifications allow us to embed a recursive structure.

Due to these modifications, any frog path in RFM\((d,p)\) contains two stages. After being woken up, a frog in RFM\((d,p)\) first moves directly towards the root (stage 1). Then, the frog either hits the root and gets removed or starts taking steps away from the root and gets removed if it hits an already-visited site (stage 2). The number of steps in stage 1 might be zero, in which case upon waking up, the first step of the frog is to move away from the root. If the frog is killed at the root, then stage 2 has zero steps. The path an awake frog follows in RFM\((d,p)\) comes from the downward-loop-erased random walk of the corresponding frog in FM\((d,p)\). This is the subrange of a simple random walk that ignores any loops created by steps away from the root.

The self-similar frog model SFM\((d,p)\) has the same killing rules as RFM\((d,p)\). The difference is that awake frogs follow the loop-erased path from their corresponding frog in FM\((d,p)\). This path ignores all loops, not just downward ones. Unlike RFM\((d,p)\), the first step an awake frog in SFM\((d,p)\) takes away from the root cannot be to the one from which it just came. So an awake frog cannot cause its own death. Intuitively, this makes it so that SFM\((d,p)\) invokes the killing rule less frequently than RFM\((d,p)\). The difficulty with coupling SFM\((d,p)\) and SFM\((d+1,p)\) is that, even with the same drift, loop-erased random walk behaves differently on \(T_d\) than on \(T_{d+1}\). This is because a random walk on \(T_d\) is more likely to create loops. We could not find a natural way to have the displacement of a frog in SFM\((d,p)\) align with a frog in SFM\((d+1,p)\). Consequently, it is hard to see how to couple the two. The advantage of the recursive frog model is that there is a canonical way to align displacements. This is the starting point for our coupling.

We begin by deducing RFM\((d,p)\) visits the root no more than FM\((d,p)\).

**Lemma 5.** If RFM\((d,p)\) is recurrent, then FM\((d,p)\) is recurrent. Hence, \(p_d \leq p'_d\).

**Proof.** Consider another modification of the frog model, denoted by FM’\((d,p)\), in which we impose the rule that if a frog makes a loop \(v_0 \rightarrow v_1 \rightarrow \cdots v_n \rightarrow v_0\) with \(v_0 \rightarrow v_1\) being a step moving away from the root, then the frog does not wake up any sleeping frogs at the sites \(v_1, \ldots, v_n\). Call such ignored loops silent. Because the wake-up time for each sleeping frog in FM’\((d,p)\) is no earlier than the corresponding wake-up time in FM\((d,p)\), FM’\((d,p)\) is a subprocess of FM\((d,p)\), i.e., the set of awake frogs in FM’\((d,p)\) is a subset of that in FM\((d,p)\). Furthermore, RFM\((d,p)\) can be constructed from FM’\((d,p)\) by removing the silent loops and killing frogs whenever they step away from the root and hit a site that has already been visited. It is a simple exercise to prove that the probability a \(p\)-biased random walk reaches its parent vertex is \(\rho\). It follows that the number of visits to the root in RFM\((d,p)\) is no more than that in FM’\((d,p)\). \(\square\)
A key monotonicity property in FM(d, p) is that introducing more killing results in fewer visits to every site. It is not quite as obvious that RFM(d, p) also enjoys this property. This is because when we remove a frog f before it is killed via the killing rule of RFM(d, p), it will visit fewer sites. As a result, f generates fewer already-visited sites, so other frogs might take more steps and survive longer. Lemma 6 explains why RFM(d, p) does in fact have this monotonicity property. For the sake of precision, when we say that a frog is removed early, this means that the frog is killed before the killing rule in RFM(d, p) would have killed it (if ever).

**Lemma 6.** Any modification of RFM(d, p) with frogs removed early will visit each site of \(T_d\) no more than the usual RFM(d, p).

**Proof.** It is equivalent in law to view RFM(d, p) as a collection of infinite stacks of i.i.d. instructions at each vertex. Frogs move according to the instructions. Once used, the instruction is deleted from the stack. When frogs are removed early, at any time step after that, fewer instructions are used from each stack than in RFM(d, p) without removal. That is, the vertices along the path of the removed-early frog have one extra instruction in their stack. Either these sites remain visited one fewer time, or another frog comes along and continues the path of the removed-early frog. This does not result in more visited sites, only that, fewer instructions are used from each stack than in RFM(d, p). It is equivalent in law to view RFM(d, p) as a collection of infinite stacks of i.i.d. instructions at each vertex. Frogs move according to the instructions. Once used, the instruction is deleted from the stack. When frogs are removed early, at any time step after that, fewer instructions are used from each stack than in RFM(d, p) without removal. That is, the vertices along the path of the removed-early frog have one extra instruction in their stack. Either these sites remain visited one fewer time, or another frog comes along and continues the path of the removed-early frog. This does not result in more visited sites, only that, fewer instructions are used from each stack than in RFM(d, p).

### 3. Proof of Proposition 3

The result follows immediately from Lemma 6 and Lemma 7. In the latter, we will couple RFM(d, p) with a frog process on \(T_{d+1}\) so that the recurrence of one implies that of the other. The process on \(T_{d+1}\) is a modified version of RFM(d + 1, p) with the possibility that frogs may be removed early. By Lemma 6 it follows that RFM(d + 1, p) is also recurrent.

**Lemma 7.** RFM(d, p) can be coupled with a modified version of RFM(d + 1, p) with frogs removed early so that recurrence is equivalent in both models.

**Proof.** We call the modified process RFM’(d + 1, p). Frogs in RFM’(d + 1, p) will follow the movement of frogs in RFM(d, p). To do this, we couple each frog f in RFM(d, p) and its sleeping site \(v \in T_d\) with a unique frog \(f’\) in RFM’(d + 1, p) and its sleeping site \(v’ \in T_{d+1}\), using the notation \(f \sim f’, v \sim v’\) to denote such coupling. The coupling is constructed recursively.

At time 0, there is a sleeping frog at every vertex of \(T_d\) in RFM(d, p), and, respectively, \(T_{d+1}\) in RFM’(d + 1, p). The frog at the root vertex in each model is awake. These two awake frogs are coupled, and so are the two root vertices. At each step, \(t = 1, 2, \ldots\), a frog is picked uniformly at random from the set of awake frogs in RFM(d, p) and performs one step of random walk according to the recursive frog model.

Suppose an awake frog f at vertex v is selected. We will assume for now and justify later by induction that f is already coupled with a unique awake frog \(f’\) in RFM’(d + 1, p) at some vertex \(v’\). For any vertex \(v \in T_d\) or \(T_{d+1}\), let \(S(v, t)\) denote the number child vertices of v at which there is a sleeping frog after time step \(t\) has occurred. For a frog f we write \(S(f, t)\) for \(S(v(f, t), t)\), where \(v(f, t)\) is the vertex that frog f lands on at time t. The crux of our induction is that

\[
S(f, t) + 1 = S(f’, t), \text{ for all } t \geq 0 \text{ and all frogs } f \text{ and } f’ \text{ with } f \sim f’.\]

Taking for granted that every awake frog f in RFM(d, p) is coupled with an awake frog \(f’\) and that (1) holds, we can now define how \(f’\) follows f:
We mark this proof as complete, and devote the rest of this section to that task.

1. After that, the frog is either removed due to hitting the root or begins moving away whereas in RFM vertices and frogs on the way (by waking up sleeping frogs) until they are removed. Thus statement holds for all coupled frogs up to time \( T \). Two coupled frogs stay on coupled vertices when they take steps away from the root. Coupling, while two coupled frogs walk away from the root, they couple newly-discovered up a sleeping frog or hits an already-visited site and gets removed. This means that in the from the root (stage 2). In stage 2, when a frog moves away from the root, it either wakes up a sleeping frog or hits an unvisited site, is killed with probability \( p \). Note that the above probability is smaller than or equal to \( p(f', t) \) since (1) implies \( 1 \leq S(f', t) \leq d + 1 \). Therefore, marginally, RFM\((d + 1, p)\) is equivalent to a modified version of RFM\((d + 1, p)\) with the additional removal rule that each awake frog \( f' \), upon moving away from the root and hitting an unvisited site, is killed with probability

\[
0 \leq \frac{d + 1 - S(f', t)}{d(d + 1)} \leq \frac{1}{d + 1}.
\]

Moreover, so long as (1) holds, the coupled frogs in RFM\((d, p)\) and RFM\((d + 1, p)\) are always in bijection and at the same distance from the root. To see why, notice that whenever a frog wakes up in RFM\((d, p)\), a sleeping frog must also wake up in RFM\((d + 1, p)\). These two new frogs become coupled, which, together with the coupling rule (c), ensures that there exists a bijection between the set of the awake frogs (and their displacements from the root) in RFM\((d, p)\) and the set of awake frogs (and their displacements from the root) in RFM\((d + 1, p)\) at all times. Therefore, the lemma will be established once we prove (1). We mark this proof as complete, and devote the rest of this section to that task.

A key observation to establishing (1) is that coupled frogs stay on coupled vertices until they are removed, i.e.,

**Lemma 8.** If \( f \sim f' \), then \( v(f, t) \sim v(f', t) \) for all \( t \geq 0 \).

*Proof.* Recall that in RFM\((d, p)\) an awake frog starts by taking steps toward the root (stage 1). After that, the frog is either removed due to hitting the root or begins moving away from the root (stage 2). In stage 2, when a frog moves away from the root, it either wakes up a sleeping frog or hits an already-visited site and gets removed. This means that in the coupling, while two coupled frogs walk away from the root, they couple newly-discovered vertices and frogs on the way (by waking up sleeping frogs) until they are removed. Thus two coupled frogs stay on coupled vertices when they take steps away from the root.

The steps moving toward the root can be taken care of by induction. Suppose the statement holds for all coupled frogs up to time \( T \). At the \((T + 1)\)th step, we consider
any two coupled frogs $f$ in $\text{RFM}(d, p)$ and $f'$ in $\text{RFM}'(d + 1, p)$, which were initially woken up at some step $t_0 \leq T$ at vertices $v_0$ and $v'_0$ by two coupled frogs $g$ and $g'$, respectively. Upon waking up, vertices $v_0$ and $v'_0$ were immediately coupled. Moreover, since $g$ and $g'$ were at the parent vertices (denoted by $v_f$ and $v'_f$) of $v_0$ and $v'_0$ at $t_0 - 1$, then by the inductive hypothesis, $v_1$ and $v'_1$ must be coupled. Again, there must be two frogs $f_1$ and $f'_1$ initially sleeping at vertices $v_1$ and $v'_1$, because otherwise $v_1$ and $v'_1$ can not be coupled. Now consider the two frogs $f_1$ and $f'_1$ that are originally woken up at $v_1$ and $v'_1$ and repeat the argument above. It follows that the parent vertices of $v_1$ and $v'_1$ must also be coupled. Iterating this argument, we see that the unique path $v'_0 \rightarrow v'_1 \rightarrow \cdots \rightarrow \emptyset'$ from $v'_0$ to the root in $\text{RFM}(d, p)$ consists of vertices that are coupled with the vertices on the unique path $v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow \emptyset$ from $v_0$ to the root in $\text{RFM}'(d + 1, p)$, i.e.,

$$v_0 \sim v'_0, \ v_1 \sim v'_1, \ldots, \emptyset \sim \emptyset'. $$

If at the $(T + 1)$-th step, $f$ and $f'$ both move toward the root, then $f$ must still be in stage 1. Thus both frogs must move to some coupled vertices $v_k$ and $v'_k$.

**Lemma 9.** The relation (1) holds during the coupling described in Lemma 7.

**Proof.** With Lemma 8, it is easy to see that

$$(3) \quad S(v, t) + 1 = S(v', t)$$

for all $t \geq 0, v \in T_d, v' \in T_{d+1}$ and $v \sim v'$ implies (1). We now prove that (3) is preserved following the coupling rules by induction.

At time 0, only the two root vertices $\emptyset \in \text{RFM}(d, p)$ and $\emptyset' \in \text{RFM}'(d + 1, p)$ are coupled. We have

$$S(\emptyset, 0) = d, \quad S(\emptyset', 0) = d + 1,$$

and thus (3) is satisfied. Now suppose (3) is satisfied up to the $T$-th step for some $T > 0$. By the discussion above, this means there is a bijection between awakened, and thus coupled, frogs after step $T$.

At the $(T + 1)$-th step, suppose an awake frog $f$ at vertex $v$ is picked in $\text{RFM}(d, p)$, and denote its coupled frog in $\text{RFM}'(d + 1, p)$ sitting at vertex $v'$ by $f'$. We are done once we show that (3) continues to hold after any of the three possible moves of $f$:

(i) $f$ moves away from the root to an already-visited site. In this case, $f$ is removed and so is $f'$. At the end of step $T + 1$, the net change is to remove the pair of coupled and awake frogs $f$ and $f'$. The number of sleeping frogs beneath each vertex is unchanged. So (3) continues to hold;

(ii) $f$ moves away from the root to $v_0$ with a sleeping frog $g$ and wakes it up. If this occurs, by the inductive hypothesis, it must be the case $S(v', T) = S(v, T) + 1 \geq 2$; According to the coupling rule (b), $f'$ will wake up a sleeping frog $g'$ at a child vertex $v'_0$ of $v'$. We couple $v_0$ and $v'_0$. By waking up $g$ and $g'$, we reduce the two numbers $S(v, T + 1)$ and $S(v', T + 1)$ by one simultaneously and (3) continues to hold for $v$ and $v'$. In addition, the newly coupled vertices, $v_0$ and $v'_0$, have all their child vertices unvisited. Thus (3) is preserved.

(iii) $f$ moves to the parent vertex of $v$. If this occurs, according to coupling rule (a), $f'$ moves to the parent vertex too ($f$ and $f'$ may be removed if $f$ hits the root). The number of sleeping frogs at each vertex does not change and neither do the collection of pairs of coupled vertices.

Thus, (3) is preserved after one time step, which completes the induction. \qed
4. Proof of Proposition 4

Another advantage of RFM(2, p) is that the number of visits to the root satisfies a recursive distributional equation. A similar but more complicated equation also holds for larger \( d \). See Figure 1 for a visual representation of the following notation. Let \( \emptyset \) be the root of \( T_2 \). The frog initially awake at the root will move to one of the two children of the root and then it, or the frog it wakes there, may move down another level. Call these sites \( \emptyset \) and \( x \), respectively. Let \( y \) be the sibling vertex of \( x \).

Let \( V_t \) be the number of visits to the root in RFM(2, p) with frogs placed at all sites up to distance \( t \) from the root and the rest of the sites empty. By similar reasoning as Lemma 6 we have \( V_t \preceq V_{t+1} \) (in the usual sense of stochastic dominance) and thus there is a distributional limit \( V := V_\infty \). Let \( V^x_t \) and \( V^y_t \) be the number of visits to \( \emptyset \) from \( x, y \), respectively, given that \( x, y \), respectively, are visited. Let \( A_t \) be the event that a frog ever enters the subtree rooted at \( y \). Although the law for paths in RFM(2, p) is different than in the self-similar frog model from [HJJ17b], it enjoys the same recursive properties. This is because both processes (RFM(\( d, p \)) and the self-similar frog model) have frogs follow non-backtracking paths and get removed when moving away from the root to already-visited sites.

Using arguments similar to [HJJ17b, Proposition 6, Proposition 7] we have pairwise independence between \( 1\{A_t\} \) and \( V^y_t \), and also that \( V^x_t \) and \( V^y_t \) are i.i.d. with distribution \( V_{t-1} \). The law for random walk paths gives a slightly different recursion than at [HJJ17b, (2)]. We have

\[
V_t = \text{Bin}(V^x_t + 1, \rho) + 1\{A_t\} \text{Bin}(V^y_t, \rho).
\]

The ‘+1’ term in the first binomial variable comes from the frog initially sleeping at \( \emptyset \). Analyzing the first and second moments of this recursive distributional equation is enough to deduce \( V \) is infinite for \( \rho \) large enough.

**Proof of Proposition 4.** Let \( x_t = EV_t^2/(EV_t)^2 \). We will prove that \( \sup_t x_t = C < \infty \). It follows from the Paley-Zygmund inequality that

\[
P(V_t > EV_t/2) \geq (4x_t)^{-1} \geq (4C)^{-1} \quad \text{for all } t \geq 1.
\]

We will also show that \( EV_t \to \infty \) as \( t \to \infty \), and the above line implies \( P(V = \infty) > 0 \).

Taking expectation in (4) and using independence between \( 1\{A_{t+1}\} \) and \( V^y_{t+1} \) gives

\[
EV_{t+1} = \rho(1 + P(A_{t+1}))EV_t + \rho.
\]
It is easy to show that \( P(A_{t+1}) \geq 1 - \prod_{i=0}^{t}(1 - \rho^i(1 - \rho)/2) \). This is because there always exists a line segment \( L_{t+1} \) from \( \emptyset \) to a vertex at distance \( t + 1 \) from the root along which all the frogs have been woken up. For the frog on this ray at distance \( i + 1 \) from the root to visit \( y \) it must take \( i \) steps toward the root, then move to \( y \). This occurs with probability \( \rho^i(1 - \rho)/2 \). Since there are \( t \) frogs along this line segment, we obtain the claimed bound on \( P(A_{t+1}) \) by only considering these frogs guaranteed to be awake.

A computer can easily verify that \( \rho(1 + P(A_{511})) > 1 \) for \( \rho > .7107 \). Converting from \( \rho = p/(1 - p) \) back to \( p \) implies that this holds for \( p > .4155 \). Using (6), for such \( p \) we have \( \epsilon > 0 \) so that \( EV_{t+1} \geq (1 + \epsilon)EV_t + \rho \) for \( t \geq 50 \). It follows that \( EV_t \) diverges as \( t \to \infty \). This alone is not enough to conclude that the root is visited infinitely often almost surely.

To establish this, we need to control the second moment.

Let \( X_t = \text{Bin}(V_t^x + 1, \rho) \) and \( Y_t = \text{Bin}(V_t^y, \rho) \) so that \( V_{t+1} = X_t + 1\{A_{t+1}\}Y_t \) and thus
\[
V_{t+1}^2 = X_t^2 + 1\{A_{t+1}\}Y_t^2 + 21\{A_{t+1}\}X_tY_t.
\]
Taking expectations and using independence as well as the bound \( 1\{A_t\} \leq 1 \) we have
\[
EV_{t+1}^2 \leq EX_t^2 + P(A_{t+1})EY_t^2 + 2EX_tELY_t.
\]
Using the formula for the second moment of a random sum of i.i.d. \( Z_i \sim \text{Ber}(\rho) \)
\[
E \left( \sum_{i=1}^{N} Z_i \right)^2 = \rho(1 - \rho)EN + \rho^2EN^2,
\]
we have \( EX_t^2 = \rho(1 - \rho)E(V_t + 1) + \rho^2E(V_t + 1)^2 \), and similarly \( EY_t^2 = \rho(1 - \rho)EV_t + \rho^2EV_t^2 \).
Plugging these expressions into (8) and gathering smaller order terms yields
\[
EV_{t+1}^2 \leq \rho^2(1 + P(A_{t+1}))EV_t^2 + 2\rho^2(EV_t)^2 + O(EV_t).
\]
Squaring (6) and ignoring smaller order terms gives
\[
(EV_{t+1})^2 \geq (\rho(1 + P(A_{t+1}))EV_t)^2.
\]
Recall that \( x_t = EV_t^2/(EV_t)^2 \). Dividing (9) by (10) gives
\[
x_{t+1} \leq \frac{1}{1 + P(A_{t+1})}x_t + O(1).
\]
Since \( 1 + P(A_{t+1}) \geq 1 + \epsilon > 1 \) for all \( t \), the leading coefficient is less than 1. This ensures that \( \sup_t x_t = C < \infty \) which gives (5).

There is a quick way to go from \( P(V = \infty) > 0 \) to \( P(V = \infty) = 1 \). Recall the definition of \( L_{t+1} \) from just below (6). We can extend this to obtain a ray \( L \) from the root to \( \infty \) with an awake frog at each site. Let \( \emptyset \) be the site at distance \( t \) on this ray. The awake frog at \( \emptyset \) moves to the child \( y_t \not\in L \) beneath it with probability \( (1 - \rho)/2 \). When this occurs, an independent \( V \)-distributed number of frogs will visit \( \emptyset \). If this quantity is infinite, then \( \emptyset \) is visited infinitely often. Since \( P(V = \infty) > 0 \) and there are infinitely many independent trials along \( L \), we must have \( V \) is infinite almost surely.

\[\Box\]

5. PROOF OF PROPOSITION 2

Proof. We impose coordinates on \( T_d \) by writing a vertex \( v \) at distance \( n \) from the root as \( v = x_1 \cdots x_n \) with \( x_i \in \{1, 2, \ldots, d\} \). For \( 1 \leq i \leq d \) define the set-valued function
Each segment $L_v$ is associated to a $k$-ary tree in $T_{kd}$. The bolded line $L_{122} \subseteq T_2$ on the left corresponds to the binary tree in $T_4$ shown on the right. The color coding represents the coordinate entries of $v$.

$G(i) = \{k(i - 1) + 1, \ldots, ki\}$. Let $L_v$ be the set of vertices $\emptyset = v_0, v_1, \ldots, v_n = v$ on the shortest path from the root to $v$. For each $L_v$ we define a subgraph of $T_k$:

$$T_k(L_v) = \bigcup_{x_1, \ldots, x_i \in L_v} G(x_1) \times \cdots \times G(x_i).$$

If $v$ has distance $n$ from the root, then $T_k(L_v)$ is a $k$-ary tree of height $n$. Call vertices $\{v' \in T_k(L_v): |v'| = n\}$ the leaves. The embedding is such that for $v, v' \in T_d$ we have

$$T_k(L_v) \cap T_k(L_{v'}) = T_k(L_v \cap L_{v'}).$$

We will define a modified version of FM($kd, p$) that sometimes removes frogs. Awake frogs $f'$ in the modified version will be coupled to a unique frog $f$ in FM($d, p$). The rules for the coupling are that:

(i) If $f$ moves towards the root, then $f'$ moves towards the root.

(ii) Suppose that $f$ is at $x_1 \cdots x_n$ and $f'$ is at $x'_1 \cdots x'_n$. If $f$ moves away from the root to $x_1 \cdots x_n x_{n+1}$, then $f'$ moves to a uniformly random vertex in $x'_1 \cdots x'_n \times G(x_{n+1})$.

(iii) $f'$ only wakes a frog when $f$ does. Upon doing so these newly awakened frogs are also coupled.

These rules ensure that $f$ and $f'$ have the same displacement from the root so Rule (ii) always holds. Moreover, Rule (ii) combined with (11) ensure that the first visit to $v \in T_d$ corresponds to the first visit to a leaf of $T_k(L_v)$. So, when the frog at $v$ is woken by $f$, then there will be a sleeping frog at whatever leaf vertex of $T_k(L_v)$ that $f'$ moves to. It follows that Rule (iii) holds for all steps in the coupling. The resulting process is a restricted version of FM($kd, p$) that dominates FM($d, p$). This gives the claimed result.

Remark 10. This coupling is wasteful; only one of the $k^n$ frogs at the leaves of $T_k(L_v)$ can ever be woken. Nonetheless, it is the only one we could find with FM($d, p$) $\lesssim$ FM($d', p$) for some $d < d'$.

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