Multiplying unitary random matrices – universality and spectral properties

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Abstract
In this paper we calculate, in the large $N$ limit, the eigenvalue density of an infinite product of random unitary matrices, each of them generated by a random hermitian matrix. This is equivalent to solving unitary diffusion generated by a hamiltonian random in time. We find that the result is universal and depends only on the second moment of the generator of the stochastic evolution. We find indications of critical behavior (eigenvalue spacing scaling like $1/N^{3/4}$) close to $\theta = \pi$ for a specific critical evolution time $t_c$.

1 Introduction

A key feature of random matrix theory is that many properties of random matrix models do not depend on the fine details of these models but only on some very general symmetry properties and a very limited number of numerical coefficients (usually just a single coefficient is enough). These universal properties facilitate the widespread applications of random matrix

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models in various fields since one can use the models to learn something about
the behavior of complex systems without knowing all the precise microscopic
details of these systems.

The simplest example of such a behavior is the eigenvalue density of a
matrix with entries independently distributed according to some probabil-
ity distribution. Then the eigenvalue density in the $N \to \infty$ limit follows
Wigner’s semicircle law with the scale set just by the second moment of the
distribution. All the dependence on other properties of the initial probabil-
ity distribution disappears. In this work we will uncover a similar type of
universality in a different context.

The aim of this paper is study infinite products of unitary random mat-
rices, and derive their properties in the large $N$ limit. This can be interpreted
as a multiplicative diffusion process on the unitary group\textsuperscript{1}. A natural physi-
cal interpretation would be of a quantum-mechanical evolution governed by
a hamiltonian which changes randomly in time. Another possibility would
be the modelling of Wilson loops in lattice gauge theory. In this paper we
will not examine further these possible application but rather concentrate on
mathematically solving the model.

We show that the eigenvalue density exhibits universality properties i.e.
it only depends on the second moment of the random hamiltonian which
generates the stochastic evolution. But of course the resulting eigenvalue
density is much more complex than the semicircle law. We derive equations
for the eigenvalue density, give explicit expression for the lowest moments
and study the properties of the model close to a phase transition where the
eigenvalue support begins to cover the whole unit circle.

## 2 Multiplicative unitary diffusion

We consider the product of $M$ unitary $N \times N$ random matrices $U_k$ in the
$M, N \to \infty$ limit:

$$U = \lim_{M \to \infty} \lim_{N \to \infty} \prod_{k=1}^{M} U_k,$$

(1)

where the $U_k$’s are generated by

$$U_k = e^{i\varepsilon H_k},$$

(2)

\textsuperscript{1}Recently matrix valued multiplicative diffusion has been considered for $2 \times 2$ real
matrices in [1] and for infinite hermitian and complex matrices in [2].
and where $\varepsilon = \sqrt{t/M}$. Such a scaling is standard for diffusive processes and works also very well for matrix-valued diffusion processes studied in [1] and [2]. $t$ is then a real parameter corresponding to ‘diffusive’ evolution time and the continuum limit $M \to \infty$ exists. The generators of the evolution $H_k$ are $N \times N$ hermitian matrices drawn from a probability distribution

$$P(H) \sim e^{-N \text{tr} V(H)}$$

where we assume that the first moment $m_1 = \langle \frac{1}{N} \text{tr} H \rangle$ vanishes $m_1 = 0$. We will show below that the spectral properties of (1) depend only on the second moment $m_2$ of the distribution (3)

$$m_2 = \langle \frac{1}{N} \text{tr} H^2 \rangle.$$  

The main aim of this paper is to find the eigenvalue distribution of the product (1)

$$\rho(\theta, t) = \langle \frac{1}{N} \sum_{j=1}^{N} i e^{i\theta} \delta(e^{i\theta} - e^{i\theta_j}) \rangle,$$

where the $e^{i\theta_j}$ are the eigenvalues of $U$ defined through (1).

In the next section we will use free random variable methods to derive an equation from which one can get $\rho(\theta, t)$. Sometimes we will omit the second argument but of course the dependence on $t$ will be there.

3 The S-transform method

The main difficulty encountered in [2] when considering products of random matrices was the necessity to deal with nonhermitian matrices and eigenvalues covering two dimensional regions of the complex plane. Here fortunately, since the product matrices are always unitary, the eigenvalues lie on the unit circle and hence they can be uniquely reconstructed from just the knowledge of the moments. Therefore all information is encoded in the asymptotic expansion of the Green’s function

$$G(z) = \int_{0}^{2\pi} \frac{\rho(\theta)}{z - e^{i\theta}} d\theta.$$  

Our aim now is to obtain the spectral density $\rho_{\text{PROD}}(\theta)$ of product $U = \prod_{k=1}^{M} U_k$, equivalently the corresponding Green’s function $G_{\text{PROD}}(z)$, from the spectral density $\rho_H(\theta)$ for the generator $H$. 

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To do that we use $S$-transforms introduced in [3]. Firstly we define an auxiliary function $\chi(z)$ through:
\[
\frac{1}{\chi} G \left( \frac{1}{\chi} \right) - 1 = z; \tag{7}
\]
Then the $S$-transform is:
\[
S(z) = \frac{1 + z}{z} \chi(z). \tag{8}
\]
Their main property is that the $S$-transform of a product of random matrices is a product of $S$-transforms of the individual factors. Before we apply this setup to (1) let us note that putting together the two previous equations we arrive at a functional relation satisfied by $S$ and $G$:
\[
\frac{1}{zS} G \left( \frac{1 + z}{z} \frac{1}{S} \right) = 1. \tag{9}
\]
In our case all single matrix Green’s functions are the same since they come from the same distribution so we can write:
\[
S_{\text{PROD}}(z) = \lim_{M \to \infty} \prod_{i=1}^{M} S_i(z) = \lim_{M \to \infty} (S_1(z))^M, \tag{10}
\]
Let us now find $S_1(z)$. The eigenvalue density of $U_1 = e^{i\theta H}$ is given by
\[
\rho_1(\theta, \varepsilon) = \frac{1}{\varepsilon} \rho_H \left( \frac{\theta}{\varepsilon} \right), \tag{11}
\]
where $\varepsilon = \sqrt{t/M}$. Inserting (11) and the definition (6) into (9) leads to an equation for $S_1$:
\[
\int \frac{1}{\varepsilon} \frac{\rho_H \left( \frac{u}{\varepsilon} \right)}{1 + z - e^{i\theta} z S_1} d\theta = 1. \tag{12}
\]
From the form of (10) we see that we need to calculate $S_1$ only to the order $O(\varepsilon^2)$. Substituting $u = \theta/\varepsilon$ and expanding in Taylor series in $\varepsilon$ we obtain:
\[
\int \rho_H(u) \left( 1 + i\varepsilon uz + (-u^2 z^2 - \frac{1}{2}u^2 z + sz)e^2 + O(\varepsilon^3) \right) du = 1, \tag{13}
\]
where $s = s(z)$ comes from Taylor expansion of $S_1(z) = 1 + s(z)\varepsilon^2 + O(\varepsilon^3)$.

\[\text{Here we used the assumption that the first moment of } H \text{ vanishes.}\]
We may now calculate \( s(z) \)
\[
s(z) = (z + 1/2) \langle u^2 \rangle \equiv (z + 1/2)m_2. \tag{14}
\]
From (10) we may now obtain the \( S \)-transform for product:
\[
S_{\text{PROD}} = \lim_{M \to \infty} (S_1)^M = \lim_{M \to \infty} \left( 1 + \frac{t}{M} s(z) \right)^M = e^{t(z + \frac{1}{2})m_2}. \tag{15}
\]
This result shows that \( S_{\text{PROD}} \) depends only on the second moment \( m_2 \) of \( H \). So in the limit \( M \to \infty \) we obtain universal behavior of the system independent of the spectral density of generator of the stochastic evolution \( H \), as long as the first moment vanishes (no drift) and the second moment is finite. It will be interesting to consider cases where these assumptions are violated which would lead to anomalous diffusion. We leave these problems for future investigation.

The final step is to come back from \( S_{\text{PROD}} \) to the Green’s function \( G_{\text{PROD}}(z,t) \) (from now on we drop the subscript). It is convenient to introduce auxiliary function \( f(z,t) \) as:
\[
G(z,t) = \frac{1 + f(t,z)}{z}. \tag{16}
\]
It is easy to check that \( f \) fulfills an equation: \( f \left( \frac{1}{\chi(z)} \right) = z \). This means that \( f \) and \( 1/\chi \) are functional inverses of each other so the following relation \( 1/\chi(f) = z \) is also true. This observation, together with the result (15) and the definition (8) leads us to the final equation:
\[
z f = (1 + f) e^{-t(f + \frac{1}{2})m_2}. \tag{17}
\]
This equation encodes all the spectral properties of the unitary diffusion process (1). In the next section we will proceed to investigate some of its properties.

4 The dynamical properties of the unitary diffusion

In this section we analyze the dynamical behavior of unitary matrix diffusion. For small times \( t \) the eigenvalues will be concentrated only in a small neighbourhood of \( \theta = 0 \). For longer times the support of the eigenvalue density \( \rho \)
Figure 1: The time evolution of the spectral function $\rho(\theta, t)$ (the dots represent numerical simulation). The figures show $\rho(\theta, t)$ after a time $t = 1, 250, 500$ up to 2750 (with $m_2 = 1/500$). The solid lines represent the eigenvalue density (29) obtained by the numerical solution of (17).

will expand and when some critical time $t_c$ is reached the eigenvalues will fill the whole circle. But of course the eigenvalue density $\rho$ will be nonuniform. In fact we expect a critical behavior close to $\theta = \pi$ with nonstandard fractional eigenvalue spacing. Only later for $t \to \infty$ the eigenvalues will become uniformly spread over the whole unit circle. In fig. 1 we show numerical results for the eigenvalue density obtained by generating unitary matrices and compare it to the one extracted from (17) (see below).

In this section we will quantitatively analyze this behavior.
The support of the eigenvalue distribution and the critical time $t_c$

Although one cannot find an analytical formula for the eigenvalue density one can analytically find the edges of the eigenvalue support. These occur when the Green’s function has an infinite derivative $\partial_z G = \infty$. Differentiating (17) with respect to $z$ gives

$$1 = -\frac{\partial_z f}{f^2} \left(1 + m_2 tf + m_2 t f^2\right) e^{-t(f+\frac{i}{2})m_2}.$$  

(18)

So the end points are determined through the solutions of equation $1 + m_2 tf + m_2 t f^2 = 0$. Once we know $f$ we can reconstruct the end-points $z$ using (17). The result is

$$z_{\text{edge}} = \frac{\sqrt{4 - m_2 t} + i\sqrt{m_2 t}}{\sqrt{4 - m_2 t} - i\sqrt{m_2 t}} e^{i\sqrt{m_2 t}\sqrt{4 - m_2 t}}.$$  

(19)

and its complex conjugate $z_{\text{edge}}^*$. When these two solution are equal ($z_{\text{edge}} = z_{\text{edge}}^* = -1$), the eigenvalues will cover the whole circle. This will happen for the critical time

$$t_c = \frac{4}{m_2}.$$  

(20)

The moments of $U$

Another quantity which can be analytically calculated are the moments of $U$. The coefficients of the auxiliary function $f$ around $z = \infty$

$$f(z) = \sum_{k=1}^{\infty} a_k z^k,$$  

(21)

are indeed directly linked to the moments:

$$a_k = \left\langle \frac{1}{N} \text{tr} U^k \right\rangle.$$  

(22)

So inserting (21) into (17) allows us to find the moments iteratively. The expressions for the lowest ones are:

$$a_1 = e^{-\frac{1}{2}m_2 t}.$$  

(23)
Figure 2: The comparison of numerical simulations (dots) with analytical results (lines), for \( n = 1, 2, 3, 4 \) and \( m_2 = 1/500 \).

\[
\begin{align*}
a_2 &= e^{-m_2 t} (-1 + m_2 t) \quad (24) \\
a_3 &= \frac{1}{2} e^{-\frac{3 m_2 t}{2}} (2 - 6 m_2 t + 3 m_2^2 t^2) \quad (25) \\
a_4 &= -\frac{1}{3} e^{-2 m_2 t} (-3 + 18 m_2 t - 24 m_2^2 t^2 + 8 m_2^3 t^3) \quad (26) \\
a_5 &= \frac{1}{24} e^{-\frac{5 m_2 t}{2}} (24 - 240 m_2 t + 600 m_2^2 t^2 - 500 m_2^3 t^3 + 125 m_2^4 t^4) \quad (27)
\end{align*}
\]

In fig. 2 we compare the formulae for the lowest 4 moments with numerical simulations of the unitary matrix diffusion and find complete agreement.

The eigenvalue density

The equation (17) allows us to directly reconstruct the Green’s function. However it is very simple to recover also the eigenvalue density. This follows from the observation that the moments of \( U \) are just the Fourier coefficients.
of the eigenvalue density \( \rho(\theta) \)

\[
\left< \frac{1}{N} \text{tr} U^k \right> = \int_0^{2\pi} \rho(\theta)e^{ik\theta}.
\]  

(28)

Using the relation of \( f \) to moments derived earlier, and the symmetry \( \rho(\theta) = \rho(-\theta) \) one finds finally

\[
\rho(\theta) = -\frac{1}{2\pi} \text{Re} \left( \frac{1}{2} + f \right).
\]  

(29)

Equation (17) may be easily solved numerically. In fig. 1, we show the resulting eigenvalue density together with numerical simulations for various times \( t \).

**Critical behavior at \( t = t_c \) and level spacing**

At \( t = t_c \) the edges of the eigenvalue support touch at \( z = -1 \). Typically in such cases we expect new critical type of behavior and nonstandard scaling of eigenvalue spacing with \( N \). Let us analyze now this behavior. Inserting \( t = t_c \) and \( f = -1/2 + F \) to (17) we obtain:

\[
z = \frac{F + 1/2}{F - 1/2} e^{-4F},
\]  

(30)

To find the behavior close to \( z = -1 \) (equivalent to \( \theta = \pi \)) we expand the left hand side of (30) in \( F \) and put \( z = -1 + iy \) to get

\[-1 + iy \approx -1 - \frac{16F^3}{3} .\]  

(31)

Using the relation between \( f \) and the eigenvalue density (29) we thus find the behavior close to \( \theta = \pi \):

\[
\rho(\theta) \sim \left\{ \frac{1}{2\pi} \left( \frac{3}{16} \right)^{\frac{1}{2}} \cos \frac{\pi}{6} \right\} \cdot |\theta - \pi|^\frac{1}{2}.
\]  

(32)

Such behavior of the eigenvalue density leads to nonstandard eigenvalue spacing and signifies the appearance of new universal regime on the scale of eigenvalue spacing (analogous to Airy universality and \( 1/N^{2/3} \) spacing on the edges of the eigenvalue distribution of a generic hermitian random
matrix [4, 5, 6, 7] in contrast to the standard $1/N$ spacing in the classical Wigner-Dyson regime [8]).

In our case, the number of eigenvalues between $\pi$ and $\Lambda$ is approximately equal to $n \sim N(\Lambda - \pi)^{4/3}$. Reexpressing $\Lambda$ in terms of $n$ shows that the eigenvalue spacing in the vicinity of $\theta = \pi$ scales like $1/N^{3/4}$. A similar scaling appeared in a certain class of chiral random matrix models at finite temperature [9]. It would be interesting to compare these regimes and/or try to apply the methods of [10] to the case at hand. We leave this problem for further investigation.

5 Discussion

In this paper we considered multiplicative unitary matrix diffusion generated by random hermitian matrices. We found the eigenvalue density as a function of evolution time in the large $N$ limit using $S$-transform methods. The eigenvalue distribution turns out to be universal and depends only on the second moment of the random hermitian matrix which generates the diffusion process.

We found that at a critical time of evolution $t = t_c$ the eigenvalues start to fill the whole unit circle, and close to $\theta = \pi$ a nonstandard eigenvalue spacing $\sim 1/N^{3/4}$ sets in which signifies the appearance of a new critical regime.

There are various further issues that one could investigate. Firstly, relaxing the assumption of the existence of the second moment might lead to defining anomalous diffusion processes. Secondly it would be interesting to study microscopic properties of these unitary matrices, however in order to do that new methods have to be developed. Thirdly a more detailed investigation of the critical behavior at $t = t_c$ close to $\theta = \pi$ would be interesting and last but not least the application of these results to some physical situations.

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