MORE ELEMENTARY OPERATORS THAT ARE SPECTRALLY BOUNDED

NADIA BOUDI AND MARTIN MATHIEU

Dedicated to the memory of T.T. West.

Abstract. We discuss some necessary and some sufficient conditions for an elementary operator $x \mapsto \sum_{i=1}^{n} a_i x b_i$ on a Banach algebra $A$ to be spectrally bounded. In the case of length three, we obtain a complete characterisation when $A$ acts irreducibly on a Banach space of dimension greater than three.

1. Introduction

Let $A$ and $B$ be unital Banach algebras over the complex field $\mathbb{C}$. Let $r(x)$ denote the spectral radius of an element $x$ in $A$ or $B$. We say a linear mapping $T : A \to B$ is spectrally bounded if, for some constant $M \geq 0$ and all $a \in A$, the estimate $r(Ta) \leq M r(a)$ holds. This concept, together with its relatives spectrally isometric (i.e., $r(Ta) = r(a)$ for all $a \in A$) and spectrally infinitesimal (i.e., $r(Ta) = 0$ for all $a \in A$), was introduced in [17] in order to initiate a systematic investigation of mappings that had, on and off, been discussed in the literature; see, e.g., [2] or [25]. A number of fundamental properties of spectrally bounded operators can be found in [19] and [21] while [20] contains a structure theorem for such operators defined on properly infinite von Neumann algebras. Spectrally bounded operators also appear in connection with the noncommutative Singer–Wermer conjecture and with Kaplansky’s problem on invertibility-preserving operators; for details see [18].

An elementary operator on $A$ is a bounded linear operator $S : A \to A$ that can be written in the form $Sx = \sum_{i=1}^{n} a_i x b_i$, $x \in A$ for some $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$. These operators appear quite naturally in many contexts; for instance, if $A$ is finite dimensional and semisimple, every linear mapping is of this form. In general, additional assumptions on the algebra and on an operator $S$ may “force” the operator to be elementary: a typical example is the innerness of a derivation $d : A \to A$, that is, $dx = ax - xa$ for some $a \in A$. Properties of elementary operators have been studied under a vast variety of aspects; we refer the reader to [16] and [12] for an overview.

Date: December 23, 2013.

2000 Mathematics Subject Classification. 47B47; 46H99, 47A10, 47B48, 47L10.

Key words and phrases. Elementary operator; quasi-nilpotent; spectrally bounded.
Despite the rich literature on spectrally bounded operators in general, and spectral isometries in particular, see, e.g., [7, 9, 10, 18, 22, 26] and the references contained therein, the supply of examples is still somewhat limited. It is thus close at hand to ask which elementary operators are spectrally bounded, as these operators are given in a more concrete form. Continuing our work started in [4], we aim to provide further answers to this question in the present paper. We shall discuss a number of necessary conditions which, in the case of length three, turn out to be sufficient too. Our new approach exploits the relation with locally quasi-nilpotent elementary operators which, in the algebraic setting, were studied in [5]; in fact, that paper should be read in conjunction with the present one.

In order to illustrate the ideas, let us assume that the elementary operator $S: A \to A$ is spectrally infinitesimal. Let $\varphi$ be an irreducible representation of $A$ on a Banach space $E$. Since $S$ induces an elementary operator $S_\varphi: \varphi(A) \to \varphi(A)$ via $S_\varphi \circ \varphi = \varphi \circ S$, $S_\varphi$ is spectrally infinitesimal too. By Jacobson’s density theorem [2, Theorem 4.2.5], $\varphi(A)$ is a dense (i.e., $n$-transitive for all $n$) algebra on $E$, and we can apply the setting of [5]. Suppose $\zeta \in E$ and $x \in A$ are such that $\varphi(x)\varphi(b_i a_j)\zeta \subseteq \mathbb{C}\zeta$ for all $i, j$. It is easy to see that this implies that

$$S_\varphi \varphi(x) \left( \text{span}\{\varphi(a_1)\zeta, \ldots, \varphi(a_n)\zeta\} \right) \subseteq \text{span}\{\varphi(a_1)\zeta, \ldots, \varphi(a_n)\zeta\};$$

consequently, the restriction of $S_\varphi \varphi(x)$ to the finite-dimensional invariant subspace $\text{span}\{\varphi(a_1)\zeta, \ldots, \varphi(a_n)\zeta\}$ has to be nilpotent which then allows us to apply the theory developed in [5]. A first application of this method is presented in Proposition 3.1 and elaborations on this idea provide the main techniques for Section 3; see, in particular, Lemma 3.3.

The general questions that we pursue in this context are as follows. Let the elementary operator $Sx = \sum_{i=1}^n a_i x b_i$, $x \in A$ on $A$ be given.

(a) Suppose $S$ is spectrally bounded.

(i) What properties of the coefficients $a_i, b_i$ can we derive?

(ii) Can we find an “improved” representation of $S$ in the sense that the new coefficients have better properties?

(b) Which conditions on the coefficients $a_i, b_i$ ensure that $S$ is spectrally bounded?

After collecting a number of basic properties and tools in Section 2, we give several answers to question (a) above in Section 3 culminating in Theorem 3.6 which describes the size of various spaces associated to the coefficients of the induced elementary operator in an irreducible representation of $A$ in terms of the local dimension. Specialising to the case of length two elementary operators we derive further properties at the end of this section and also correct a small oversight in [4, Theorem 3.5] concerning an exceptional case that can appear in dimension two.
A full answer to both questions (a) and (b) is, at present, only available for elementary operators of short length. It was given in [4] for length two and is provided for length three under the assumption that $A$ acts irreducibly on a Banach space of dimension greater than three in Section 4 below (spaces with smaller dimension need to be treated separately). The formulation seems too technical to allow an extension to the general case so far.

2. Prerequisites

Throughout this paper, $A$ will denote a unital complex Banach algebra, and its group of invertible elements is written as $\text{Inv} A$. We let $\text{rad} A$ stand for the Jacobson radical of $A$, see [2, p. 34]. The algebra of all bounded linear operators on a Banach space $E$ will be designated by $L(E)$.

An elementary operator on $A$ is a bounded linear mapping $S: A \to A$ that can be written in the form

$$Sx = \sum_{i=1}^{n} a_{i}xb_{i} \quad (x \in A),$$

for some $a_{i}, b_{i} \in A$ and some $n \in \mathbb{N}$. Special cases are $L_{a}: x \mapsto ax$, $R_{b}: x \mapsto xb$ and $M_{a,b} = L_{a}R_{b}$. Clearly the representation of $S$ in a sum as in (2.1) is not unique. The smallest $n \in \mathbb{N}$ such that the non-zero elementary operator $S$ can be written as $S = \sum_{i=1}^{n} M_{a_{i},b_{i}}$ is called the length of $S$ and will be abbreviated as $\ell(S)$. We put $\ell(0) = 0$. If $S = \sum_{i=1}^{n} M_{a_{i},b_{i}}$ and $\ell(S) = n$ then, evidently, the sets $\{a_{1}, \ldots, a_{n}\}$ and $\{b_{1}, \ldots, b_{n}\}$ are linearly independent.

We will denote by $\mathcal{E}(A)$ and $\mathcal{E}_{n}(A)$, respectively, the algebra of all elementary operators on $A$ and the set of all elementary operators of length $n$, respectively.

Whenever convenient, we shall abbreviate an $n$-tuple $(a_{1}, \ldots, a_{n})$ of elements of $A$ by $a$ and indicate that $S \in \mathcal{E}(A)$ is written as $S = \sum_{i=1}^{n} M_{a_{i},b_{i}}$ by $S = S_{a,b}$. We shall further use the following notation for $S = S_{a,b}$:

$$L(S) = \text{span}\{a_{1}, \ldots, a_{n}\},$$

$$R(S) = \text{span}\{b_{1}, \ldots, b_{n}\},$$

$$V(S) = \text{span}\{b_{i}a_{j} : 1 \leq i, j \leq n\},$$

$$V'(S) = V(S) + \mathbb{C}I,$$

together with the abbreviations $S^{*}$ for $S_{b,a}$ and $ba$ for $\sum_{i=1}^{n} b_{i}a_{i}$.

The way representations of $S \in \mathcal{E}(A)$ as in (2.1) are related to each other can be rather intricate; however, we shall be content with representations arising from each other by linear combinations of the coefficients. In this case, we have the following result the argument for which is standard but we include a proof in order to illustrate how to work with different representations of the same elementary operator.
Lemma 2.1. Let $A$ be a unital Banach algebra, and let $S = \sum_{i=1}^n M_{a_i,b_i}$ be an elementary operator on $A$. Suppose that $S = \sum_{i=1}^m M_{c_i,d_i}$, where $c_i \in \mathcal{L}(S)$ and $d_i \in \mathcal{R}(S)$ for all $1 \leq i \leq m$. Then $\sum_{i=1}^n b_i a_i - \sum_{i=1}^m d_i c_i \in \text{rad}
abla$. 

Proof. As $S$ leaves each primitive ideal $P$ of $A$ invariant and we have to show that, for every $P$, we have 

$$\sum_{i=1}^n b_i a_i - \sum_{i=1}^m d_i c_i \in P,$$

we can assume that $A$ is primitive.

Without loss of generality, we may suppose that $S$ has length $n$ and that $\{c_1, \ldots, c_n\}$ is linearly independent. First assume that $n = m$. Write $c_i = \sum_{k=1}^n \alpha_{ik} a_k$, $1 \leq i \leq n$. Then it follows from [1, Theorem 5.1.7], e.g., that $b_k = \sum_{i=1}^n \alpha_{ik} d_i$, $1 \leq k \leq n$. Now it is easy to see that $\sum_{i=1}^n b_i a_i = \sum_{i=1}^n d_i c_i$.

Next suppose that $n < m$ and write 

$$c_j = \sum_{k=1}^n \alpha_{jk} c_k, \quad \text{for } n+1 \leq j \leq m.$$ 

Then

$$S = \sum_{i=1}^n M_{c_i,d_i} + \sum_{j=n+1}^m \sum_{k=1}^n \alpha_{jk} M_{c_k,d_j},$$

which entails that

$$S = \sum_{i=1}^n M_{c_i,d_i} + \sum_{k=1}^n \sum_{j=n+1}^m M_{c_k,\alpha_{jk} d_j} = \sum_{k=1}^n M_{c_k,d_k} + \sum_{j=n+1}^m \alpha_{jk} d_j,$$

Setting $d'_k = d_k + \sum_{j=n+1}^m \alpha_{jk} d_j$ we find, as shown above, that $\sum_{k=1}^n d'_k c_k = \sum_{k=1}^n b_k a_k$. Substituting the above expression for $d'_k$ back into this identity yields the desired conclusion. \qed

In the above argument we used implicitly that every primitive Banach algebra is centrally closed, that is, its so-called extended centroid is equal to $\mathbb{C}$; the important consequence for us is that we only have to work with linear combinations over the complex numbers.

We would like to use the fact that an elementary operator leaves every ideal of $A$ invariant to reduce the task of describing spectrally bounded elementary operators to the case of primitive Banach algebras. However, the induced elementary operator on the quotient of $A$ by a primitive ideal may not be spectrally bounded without further assumptions. Resulting from this we will have to work explicitly with irreducible representations in the following in order to apply the results obtained in [5] for locally quasi-nilpotent elementary operators on irreducible algebras of operators.
Nevertheless, there is a class of Banach algebras (which includes all $C^*$-algebras, for example) which allows us to reduce the problem fully to primitive quotients. Recall that a Banach algebra $A$ is said to be an $SR$-algebra if the spectral radius formula holds in every quotient; that is, whenever $I$ is a closed ideal of $A$, for each $x \in A$ we have

$$r(x + I) = \inf_{y \in I} r(x + y).$$

(2.2)

Whenever $T: A \to B$ is a linear mapping, $I$ is a closed ideal of $A$ and $J$ is a closed ideal of $B$ containing $TI$, we can define an induced linear mapping $\hat{T}: \hat{A} = A/I \to \hat{B} = B/J$ by $\hat{T}(x + I) = Tx + J$, $x \in A$. If $T$ is spectrally bounded, we define the spectral norm $\|T\|_\sigma$ of $T$ as the smallest $M \geq 0$ such that $r(Tx) \leq Mr(x)$ for all $x \in A$, see [19].

(Note that this is in general not a norm!)

The standard argument for induced bounded linear operators on quotient spaces enables us to prove the following result.

**Proposition 2.2.** Let $T: A \to B$ be a spectrally bounded operator from the unital $SR$-algebra $A$ into the unital Banach algebra $B$. Suppose that $I$ is a closed ideal of $A$ and $J$ is a closed ideal of $B$ containing $TI$. Then $\hat{T}: \hat{A} \to \hat{B}$ is spectrally bounded with $\|\hat{T}\|_\sigma \leq \|T\|_\sigma$.

**Proof.** Let $x \in A$ and, for given $\varepsilon > 0$, let $y \in I$ be such that $r(x + y) \leq r(x + I) + \varepsilon$. Then

$$r(\hat{T}(x + I)) = r(Tx + J) \leq \inf_{z \in J} r(Tx + z)$$

$$\leq r(Tx + Ty) \leq \|T\|_\sigma r(x + y)$$

$$\leq \|T\|_\sigma (r(x + I) + \varepsilon),$$

hence $r(\hat{T}(x + I)) \leq \|T\|_\sigma r(x + I)$ which yields the claim. \qed

As an illustration of how smooth the arguments become in the case of $C^*$-algebras, and to contrast the more elaborate techniques we have to use in the general situation, we formulate the following consequence which is a special case of Proposition 3.2 below.

**Corollary 2.3.** Let $S \in \mathcal{E}(A)$ for a unital $C^*$-algebra $A$ be spectrally bounded. If $S = S_{a,b}$ then $ba \in Z(A)$, the centre of $A$.

**Proof.** Let $\varrho$ be an irreducible representation of $A$ on a Hilbert space $H$ and put $P = \ker \varrho$. As $SP \subseteq P$ and since $A$ is an SR-algebra, by [24], see also [23], the induced elementary operator $S_{\varrho}$ defined by $S_{\varrho} \circ \varrho = \varrho \circ S$ is spectrally bounded with $\|S_{\varrho}\|_\sigma \leq \|S\|_\sigma$ by Proposition 2.2 above. As a result, we can assume that $A$ acts irreducibly on $H$ and aim to show that $\sum_{i=1}^n b_i a_i \in \mathbb{C}1$. Slightly simplified arguments like those used in the proof of Proposition 3.2 (which we do not spell out here to save space) allow us to arrive at the desired conclusion. \qed
For a unital Banach algebra $A$, let $\mathcal{Z}(A)$ denote the centre modulo the radical, that is, the inverse image of the centre of $A/\text{rad}A$ under the canonical epimorphism. The next proposition determines when the basic block of an elementary operator, the two-sided multiplication is a homomorphism.

**Proposition 2.4.** Let $A$ be a unital Banach algebra, let $u, v \in A$ and put $e = vu$. The two-sided multiplication $M_{u,v}$ is a homomorphism modulo $\text{rad}A$ if and only if $e$ is an idempotent in $\mathcal{Z}(A)$ and $M_{u,v}(1 - e)A \subseteq \text{rad}A$.

**Proof.** To establish the “only if”-part, let $\varrho$ be an irreducible representation of $A$ on a Banach space $X$. Suppose that there exists $\zeta \in X$ such that $\{ \varrho(vu)\zeta, \zeta \}$ is linearly independent. Choose $x, y \in A$ such that $\varrho(uxv)\zeta = 0$, $\varrho(yuv)\zeta = \zeta$.

Then
$$\varrho(M_{u,v}(xy)u)\zeta = \varrho(u)\zeta \text{ and } \varrho((M_{u,v}(x)M_{u,v}(y))u)\zeta = 0,$$
a contradiction. Consequently, $\{ \varrho(vu)\zeta, \zeta \}$ is linearly dependent for every $\zeta \in X$. As a result, $\varrho(vu) \in \mathbb{C}I$, say $\varrho(vu) = \lambda I$. In order to show that $\lambda \in \{0, 1\}$, let $x, y \in A$ be arbitrary. Then, by hypothesis,

$$\varrho(uxyv) = \varrho(uxvuyv) \implies \varrho((1 - \lambda)uxyv) = 0. \quad (2.3)$$

Specialising to $x = y = 1$ and multiplying on the left with $v$ and on the right with $u$ we find that $(1 - \lambda)\lambda^2 = 0$ which yields the claim. We conclude that $e$ is an idempotent in $\mathcal{Z}(A)$.

Moreover, by (2.3), $\varrho(u(1 - e)xv) = \varrho((1 - \lambda)uxv) = 0$ for all $x \in A$ which entails that $M_{u,v}(1 - e)A \subseteq \text{rad}A$.

Conversely, the hypotheses on $e$ and $M_{u,v}$ imply that

$$uxyv + \text{rad}A = uxeyv + \text{rad}A + ux(1 - e)yv + \text{rad}A = uxvuyv + \text{rad}A \quad (x, y \in A)$$

which is the desired assertion. \hfill $\square$

It follows from the above proposition that, if $u, v$ are elements of a unital semisimple Banach algebra $A$ and $e = vu$ is a central idempotent in $A$, then $M_{u,v}$ can be decomposed as

$$M_{u,v} = M_{u_1,v_1} + M_{u_2,v_2}$$

where $M_{u_1,v_1}$ is a homomorphism on $A$ and $M_{u_2,v_2}(eA) = 0$. This is easily verified by putting $u_1 = eu$, $v_1 = ev$, $u_2 = (1 - e)u$ and $v_2 = (1 - e)v$.

The next result shows how to build a new spectrally bounded operator out of a number of given ones; it generalises [4, Lemma 2.1].
Proposition 2.5. Let \( A \) be a unital Banach algebra, and let \( T_1, \ldots, T_n \) be linear mappings on \( A \) such that, for each \( i \), \( T_i \) is a homomorphism or \((T_i x)^2 \in \text{rad} A\) for every \( x \in A \). Suppose that \((T_i y)(T_j x) \in \text{rad} A\) for every \( x, y \in A \) and for all \( i > j \). Then, for all \( \lambda_1, \ldots, \lambda_n \in \mathbb{C} \), the mapping \( T = \sum_{i=1}^n \lambda_i T_i \) is spectrally bounded.

Proof. Suppose that for \( 1 \leq t \leq r \), \( T_t \) is a homomorphism and for \( r + 1 \leq t \leq n \), \((T_t x)^2 \in \text{rad} A\) for all \( x \in A \). Let \( \lambda_1, \ldots, \lambda_n \in \mathbb{C} \) and suppose, without loss of generality, that \( \lambda_i = 1 \) for \( r + 1 \leq t \leq n \). Put \( \lambda_t := \lambda_i \) for \( 1 \leq t \leq r \) and fix \( x \in A \). Choose a non-zero complex number \( \alpha \notin \bigcup_{t=1}^r \sigma(T_t x) \). Fix \( 1 \leq t \leq r \) and put \( y_t = \frac{\lambda_t}{\alpha} x \left( \frac{\lambda_t}{\alpha} x - 1 \right)^{-1} \). Then

\[
\frac{\lambda_t}{\alpha} x + y_t - \frac{\lambda_t}{\alpha} x = 0
\]

and thus

\[
\frac{\lambda_t}{\alpha} T_t x + T_t y_t - (T_t y_t) \frac{\lambda_t}{\alpha} T_t x = 0.
\]

We compute that

\[
(1 - \sum_{t=1}^r T_t y_t)(1 - \sum_{t=1}^r \frac{\lambda_t}{\alpha} T_t x - \sum_{t=r+1}^n \frac{1}{\alpha} T_t x) = 1 + q,
\]

where

\[
q = \sum_{t,s=1, t \neq s}^r \frac{\lambda_t}{\alpha} (T_t y_t)(T_s x) - \sum_{t=r+1}^n \frac{1}{\alpha} T_t x + \sum_{t=1}^r \sum_{s=r+1}^n \frac{1}{\alpha} (T_t y_t)(T_s x).
\]

As in the proof of Proposition 2.3 in [5], we show by induction on \( n \) that \( q \in \text{rad} A \). This implies that \( 1 + q \) is invertible which, together with identity (2.4), entails that \( \alpha - (\sum_{t=1}^r \lambda_t T_t + \sum_{t=r+1}^n T_t) x \) is left invertible. Since the boundary of the spectrum of \( T x \) is contained in the left approximate point spectrum, it follows that \( \alpha \notin \partial \sigma(T x) \) and thus

\[
\partial \sigma(T x) \subseteq \bigcup_{t=1}^r \sigma(T_t x).
\]

This implies that \( r(T x) \leq \max_{1 \leq i \leq n} \{|\lambda_i|\} r(x) \) for each \( x \in A \). The proof is complete. \( \square \)

As a consequence we obtain a sufficient criterion for spectral boundedness of an elementary operator.

Corollary 2.6. Let \( A \) be a unital Banach algebra, and let \( S \in \mathfrak{S}_n(A) \). If \( S \) can be written as \( S = \sum_{i=1}^n M_{v_i, v_i} \) with \( v_i v_j \in \text{rad} A \) for all \( i > j \) and \( v_i v_i \in \mathcal{Z}(A) \) for all \( i \) then \( S \) is spectrally bounded.

Proof. Let \( \varrho \) be an irreducible representation of \( A \); then \( S_\varrho \in \mathfrak{S}(\varrho(A)) \) is given by \( S_\varrho = \sum_{i=1}^n M_{\varrho(v_i), \varrho(v_i)} \). Since, for each \( i \), \( \varrho(v_i) \varrho(u_i) = \mu_i 1 \in \mathbb{C}1 \), the two-sided multiplication \( M_{\varrho(u_i), \varrho(v_i)} \) is spectrally bounded with \( \|M_{\varrho(u_i), \varrho(v_i)}\|_\sigma \leq |\mu_i| \) (see the first paragraph of
Section 3). If \( \mu_i = 0 \) then \( (M_{\varphi(u_i), \varphi(v_i)}\varphi(x))^2 = 0 \) for all \( x \in A \). Otherwise, \( \mu_i^{-1}M_{\varphi(u_i), \varphi(v_i)} \) is a homomorphism. Setting \( T_i = M_{\varphi(u_i), \varphi(v_i)}, \ 1 \leq i \leq n \) the hypothesis \( \varphi(u_i)\varphi(u_j) = 0, \ i > j \) yields \( T_i\varphi(y)T_j\varphi(x) = 0 \) for all \( x, y \in A \) and \( i > j \). Thus we can apply Proposition 2.5 to obtain that if
\[
S_{i > j} \geq 0, \quad \text{Section 3).}
\]

If \( S \) is a homomorphism. Setting \( T = a \) vector space \( S \) of \( V \) if \( \dim V < \) is called a separating vector of \( V \) and thus \( \dim V \leq \) is nilpotent. As \( A \) length two elementary operator \( M \) is spectrally bounded then \( ba + dc \in \mathcal{Z}(A) \), by [4, Theorem 3.5]; the latter theorem also gives a sufficient condition for spectral boundedness of \( M_{a,b} + M_{c,d} \). We will extend the necessary condition to arbitrary elementary operators in Proposition 3.2 below; it seems difficult to give concise necessary and sufficient conditions in general though.

We begin our discussion in this section by looking at spectrally infinitesimal elementary operators.

**Proposition 3.1.** Let \( A \) be a unital Banach algebra, and let \( S = S_{a,b} \) be an elementary operator on \( A \). Suppose that \( r(Sx) = 0 \) for every \( x \in A \). Then \( ba \in \text{rad} A \).

**Proof.** Let \( S_{\varphi} : \varphi(A) \to \varphi(A) \) denote the induced elementary operator, where \( \varphi \) is an irreducible representation of \( A \) on a Banach space \( E_{\varphi} \). Then \( r(S_{\varphi}\varphi(x)) = 0 \) for all \( x \in A \). As indicated in the Introduction, whenever \( \zeta \in E_{\varphi} \) and \( x \in A \) are such that \( \varphi(x)V(S_{\varphi})\zeta \subseteq C\zeta \) then \( S_{\varphi}L(S_{\varphi})\zeta \subseteq L(S_{\varphi})\zeta \) and thus \( S_{\varphi}|L(S_{\varphi})\zeta \) is nilpotent. As
Proposition 3.2. Let $\rho$ be a result, the assumptions in [5, Proposition 3.1] are satisfied and it thus follows that $\rho(\sum_{i=1}^{n} b_i a_i) = 0.$

In the spectrally bounded case, we have the following more general situation. Let us point out once again that, when $S \in \mathcal{E}(A)$ is spectrally bounded and $\rho$ is an irreducible representation of $A$, the elementary operator $S_{\rho} : \rho(A) \rightarrow \rho(A)$ may or may not be spectrally bounded; nevertheless the operator $\rho \circ S$ is spectrally bounded (with $\|\rho \circ S\|_\sigma \leq \|S\|_\sigma$).

Proposition 3.2. Let $A$ be a unital Banach algebra, and let $S = S_{a,b}$ be a spectrally bounded elementary operator on $A$. Then $ba \in \mathcal{Z}(A)$.

Proof. Let $\rho$ be an irreducible representation of $A$ on a Banach space $E$. Suppose that there exist $\zeta \in E$ and $t \in \{1, \ldots, n\}$ such that the set $\{\rho(b_t a_t) \zeta, \zeta\}$ is linearly independent (that is, $\rho(b_t a_t) \notin \mathbb{C}1$). Set $s = \dim \mathbf{L}(S_{\rho}) \zeta$. As in the proof of [5, Proposition 3.1], we assume without loss of generality that $\{\rho(a_1) \zeta, \ldots, \rho(a_r) \zeta\}$ is linearly independent and $\rho(a_t) \zeta = 0$ for $t > s$. Choose $i_1, \ldots, i_r$ such that $\Xi = \{\zeta, \rho(b_{i_1} a_{i_1}) \zeta, \ldots, \rho(b_{i_r} a_{i_r}) \zeta\}$ is linearly independent, $r$ being maximal (in particular, $\rho(b_{i_t} a_{i_t}) \notin \mathbb{C}1$ for all $1 \leq t \leq r$). Fix $j \in \{1, \ldots, r\}$. By Sinclair’s theorem [2, Corollary 4.2.6], for each $k \in \mathbb{N}$ there is $x_k \in \text{Inv} \ A$ such that

$$\rho(x_k) \zeta = \frac{1}{k} \rho(b_{i_j} a_{i_j}) \zeta, \quad \rho(x_k b_{i_j} a_{i_j}) \zeta = \zeta,$$

$$\text{and} \quad \rho(x_k b_{i_t} a_{i_t}) \zeta = \rho(b_{i_t} a_{i_t}) \zeta \quad (1 \leq t \leq r, t \neq j).$$

Choose $y \in A$ such that

$$\rho(y) \zeta = \rho(b_{i_j} a_{i_j}) \zeta \quad \text{and} \quad \rho(y b_{i_t} a_{i_t}) \zeta = 0 \quad (1 \leq t \leq r).$$

Then

$$\rho(x_k y x_k^{-1} b_{i_j} a_{i_j}) \zeta = k \zeta \quad \text{and} \quad \rho(x_k y x_k^{-1} b_{i_t} a_{i_t}) \zeta = 0 \quad (1 \leq t \leq r, t \neq j). \quad (3.1)$$

Replacing the above-chosen $x_k$ and $y$ by ones which, in addition, satisfy

$$\rho(x_k b_{i_t} a_{i_t}) \zeta = \rho(b_{i_t} a_{i_t}) \zeta \quad \text{and} \quad \rho(y b_{i_t} a_{i_t}) \zeta = 0 \quad (3.2)$$

for all $1 \leq i, \ell \leq n, i \neq \ell$ such that $\rho(b_{i_\ell}) \zeta \notin \text{span} \Xi$, we can also assume that

$$\rho(x_k y x_k^{-1}) \mathbf{V}(S_{\rho}) \zeta \subseteq \mathbb{C} \zeta. \quad (3.3)$$

Now let $J = \{1, \ldots, n\} \setminus \{i_1, \ldots, i_r\}$ and write, for all $i \in J$,

$$\rho(b_{i} a_{i}) \zeta = \alpha_i \zeta + \sum_{t=1}^{r} \alpha_{i_t} \rho(b_{i_t} a_{i_t}) \zeta.$$
Then, by (3.1), \(\varrho(x_ky_{x_k}^{-1}b_i)\zeta = \alpha_i^j k \zeta\) for all \(i \in J\). Since \(S_\varrho(x_ky_{x_k}^{-1})\) leaves the finite-dimensional subspace \(L(S_\varrho)\zeta\) invariant, by (3.3), we obtain

\[
\text{Tr}(S_\varrho(x_ky_{x_k}^{-1})|L(S_\varrho)\zeta) = k(1 + \sum_{i \in J} \alpha_i^j),
\]

where \(\text{Tr}\) denotes the trace on \(L(S_\varrho)\zeta\). Because the trace of a linear mapping \(u\) on \(L(S_\varrho)\zeta\) is dominated by \(s \cdot r(u)\) and thus

\[
k(1 + \sum_{i \in J} \alpha_i^j) \leq s \cdot r(S_\varrho(x_ky_{x_k}^{-1})) = s \cdot r(S(x_ky_{x_k}^{-1})) \leq s \cdot \|S\|_\sigma r(y) \quad (k \in \mathbb{N}),
\]

we must have \(1 + \sum_{i \in J} \alpha_i^j = 0\). As

\[
\sum_{i=1}^n \varrho(b_i a_i)\zeta = \sum_{i=1}^s \varrho(b_i a_i)\zeta = \sum_{i=1}^r \varrho(b_i a_i)\zeta + \sum_{i \in J} \varrho(b_i a_i)\zeta = \sum_{i=1}^r \varrho(b_i a_i)\zeta + \sum_{i \in J} (\alpha_i \zeta + \sum_{i=1}^r \alpha_i^j \varrho(b_i a_i)\zeta)
\]

we conclude that \(\sum_{i=1}^n \varrho(b_i a_i)\zeta \in \mathbb{C} \zeta\). We have thereby shown that \(\text{id}_E\) and \(\sum_{i=1}^n \varrho(b_i a_i)\zeta\) are locally linearly dependent. Thus \(\sum_{i=1}^n \varrho(b_i a_i)\zeta \in \mathbb{C} \zeta\), as desired.  

The following lemma is a key result enabling us to relate spectral boundedness to local quasi-nilpotency and thus to make use of the results in [5].

**Lemma 3.3.** Let \(A\) be a unital Banach algebra, and let \(S\) be a spectrally bounded elementary operator on \(A\). Let \(\varrho\) be an irreducible representation of \(A\) on a Banach space \(E\). Then, for every \(\zeta \in E\) and \(x \in A\) satisfying

\[
\varrho(x) V(S_\varrho)\zeta \subseteq \mathbb{C} \zeta \quad \text{and} \quad \varrho(x)\zeta = 0,
\]

the map \(S_\varrho(x)|_{L(S_\varrho)\zeta}\) is nilpotent.

**Proof.** Suppose that there exist \(\zeta \in E\) and \(x \in A\) satisfying \(\varrho(x) V(S_\varrho)\zeta \subseteq \mathbb{C} \zeta\), \(\varrho(x)\zeta = 0\) and \(S_\varrho(x)|_{L(S_\varrho)\zeta}\) is not nilpotent. Set \(s = \dim L(S_\varrho)\zeta\). Choose \(u \in L(S)\) such that \(\varrho(u)\zeta \neq 0\) and \(S_\varrho(x)\varrho(u)\zeta = \lambda \varrho(u)\zeta\), where \(\lambda\) is a non-zero eigenvalue of \(S_\varrho(x)|_{L(S_\varrho)\zeta}\). Write \(S = M_{u,v} + \sum_{i=1}^{n-1} M_{u,v_i}\). By the same argument as in the proof of Proposition 3.2, we can assume that \(\varrho(u_i)\zeta = 0\) for \(i > s - 1\). Therefore,

\[
\varrho(uxv)\zeta + \sum_{i=1}^{s-1} \varrho(u_i x v_i u)\zeta = \lambda \varrho(u)\zeta.
\]

Since \(\{\varrho(u)\zeta, \varrho(u_1)\zeta, \ldots, \varrho(u_{s-1})\zeta\}\) is linearly independent and

\[
\varrho(x) \{\varrho(u)\zeta, \varrho(v_1 u)\zeta, \ldots, \varrho(v_{s-1}u)\zeta\} \subseteq \mathbb{C} \zeta,
\]

we have a contradiction. Therefore, \(S_\varrho(x)|_{L(S_\varrho)\zeta}\) is nilpotent.
we get
\[ \varrho(xvu)\zeta = \lambda\zeta \quad \text{and} \quad \varrho(xv_iu)\zeta = 0 \quad (1 \leq i \leq s - 1). \quad (3.4) \]

Our assumption on \( \zeta \) and \( x \) implies that \( \{ \zeta, \varrho(vu)\zeta \} \) is linearly independent. Let \( r \leq s - 1 \) be maximal such that \( \{ \zeta, \varrho(vu)\zeta, \varrho(v_iu)\zeta, \ldots, \varrho(v_iu)\zeta \} \) is linearly independent. By the equations in (3.4), we infer that \( \varrho(v_iu)\zeta \in \text{span}\{ \zeta, \rho(v_iu)\zeta, \ldots, \rho(v_iu)\zeta \} \) for each \( 1 \leq i \leq s - 1 \). Let \( x_k \in \text{Inv}A, k \in \mathbb{N} \) be such that

\[ \varrho(x_k)\zeta = \frac{1}{k}\varrho(vu)\zeta, \quad \varrho(x_kvu)\zeta = \zeta \quad \text{and} \quad \varrho(x_kv_iu)\zeta = \varrho(v_iu)\zeta \quad (1 \leq t \leq r). \]

Choose \( y \in A \) such that

\[ \varrho(y)\zeta = \varrho(vu)\zeta \quad \text{and} \quad \varrho(yvu)\zeta = \varrho(yv_iu)\zeta = 0 \quad (1 \leq t \leq r). \]

Then

\[ \varrho(x_kyv_k^{-1}vu)\zeta = k\varrho(u)\zeta \quad \text{and} \quad \varrho(x_kyv_k^{-1}v_iu)\zeta = 0 \quad (1 \leq t \leq r) \]

and therefore \( \varrho(x_kyv_k^{-1}v_iu)\zeta = 0 \) for all \( 1 \leq i \leq s - 1 \). This entails that

\[ \varrho(S(x_kyv_k^{-1})u)\zeta = k\varrho(u)\zeta \quad (k \in \mathbb{N}), \]

which contradicts our assumption \( r(\varrho Sy) \leq \|S\|_\sigma r(y) \). \( \square \)

In the situation of the above lemma, the following notation is useful. For \( \zeta \in E \), denote by \( \pi : \mathbb{C}\zeta \to \mathbb{C}, \pi(\zeta) = 1 \) the canonical map. For a linear mapping \( y \) on \( E \) and a basis \( \{ \zeta_1, \ldots, \zeta_k \} \) of a \( y \)-invariant subspace of \( E \), let \( M(y, \{ \zeta_1, \ldots, \zeta_k \}) \) denote the corresponding matrix representation with respect to \( \{ \zeta_1, \ldots, \zeta_k \} \). For instance, if \( S = \sum_{i=1}^n M_{a_i,b_i} \in \mathcal{E}(A) \), \( \varrho \) is an irreducible representation of \( A \) on \( E \) and \( \{ \varrho(a_1)\zeta, \ldots, \varrho(a_n)\zeta \} \) is linearly independent, then

\[ M(S_\varrho\varrho(x), \{ \varrho(a_1)\zeta, \ldots, \varrho(a_n)\zeta \}) = (\pi(\varrho(xb_ia_j)\zeta))_{1 \leq i,j \leq n} \]

for every \( \zeta \in E \) and \( x \in A \) such that \( \varrho(x)\mathcal{V}(S_\varrho)\zeta \subseteq \mathbb{C}\zeta \).

Let \( S \) be an elementary operator of length \( n \) on an algebra \( A \). For each \( u \in L(S) \), we denote by \( S\langle u \rangle \) the set

\[ S\langle u \rangle = \{ v \in R(S) \mid \text{there exists } S_{n-1} \in \mathcal{E}_{n-1}(A) \text{ such that } S = M_{u,v} + S_{n-1} \}. \]

**Proposition 3.4.** Let \( A \) be a unital Banach algebra, and let \( S \) be a spectrally bounded elementary operator of length \( n \) on \( A \). Let \( \varrho \) be an irreducible representation of \( A \) on a Banach space \( E \). Suppose that \( \dim \mathcal{V}(S_\varrho) = \text{lindim} \mathcal{V}(S_\varrho) \). Then either for every \( \zeta \in E \) and \( x \in A \) satisfying \( \varrho(x)\mathcal{V}(S_\varrho)\zeta \subseteq \mathbb{C}\zeta \), we have \( S_\varrho\varrho(x)|_{L(S_\varrho)\zeta} \) is nilpotent or there exist \( S_1 \in \mathcal{E}_1(A), S_2 \in \mathcal{E}_{n-1}(A) \) such that \( \varrho S_1 \) is scalar multiple of a homomorphism and \( S = S_1 + S_2 \).
Proof. Suppose that there exist $\zeta \in E$ and $x \in A$ satisfying $g(x)V(S_{E})\zeta \subseteq \mathbb{C}\zeta$ and $S_{E}g(x)L(S_{E})\zeta$ is not nilpotent. Choose $u \in L(S)$ such that $g(u)\zeta \neq 0$ and $S_{E}g(xu)\zeta = \lambda g(u)\zeta$, where $\lambda$ is a non-zero eigenvalue of $S_{E}g(x)L(S_{E})\zeta$. We claim that we can assume that $\zeta$ is a separating vector of $V'(S_{E})$. Indeed, let $\zeta'$ be a separating vector of $V'(S_{E})$. If $\{g(v_{i}u_{ji})\zeta_{\zeta} : 1 \leq t \leq r\}$ is linearly independent, $r$ being maximal, then $\{g(v_{i}u_{ji})\zeta_{\zeta} : 1 \leq t \leq r\}$ is linearly independent too. Hence we can choose $y \in A$ so that $g(yv_{ij})\zeta' = (\pi g(xv_{ij})\zeta)\zeta'$, and consequently, $S_{E}g(uy)\zeta' = \lambda g(u)\zeta'$. The claim is proved. Next set $s = \dim L(S_{E})\zeta$ and write $S = M_{u,v} + M_{\sum i=1}^{n-1} M_{u_{i,v_{i}}}$. Then we may assume that $g(u_{i})\zeta = 0$ for $s \leq i \leq n - 1$. We have

$$g(uvxu + \sum_{i=1}^{s-1} u_{i}vxu_{i})\zeta = \lambda g(u)\zeta.$$ 

Suppose, for instance, that $\{g(v_{i}), g(v_{1}u), \ldots, g(v_{r}u)\}$ is linearly independent, $r$ being maximal and $r \leq s - 1$. Our assumption on $x$ and $\zeta$ implies that for each $1 \leq i \leq s - 1$, $g(v_{i}) \in \text{span}\{g(v_{1}), \ldots, g(v_{r})\}$. We write $I = \text{id}_{E}$ and distinguish two cases:

Case 1: $I \in \text{span}\{g(vu), g(v_{1}u), \ldots, g(v_{r}u)\}$.

Suppose for a moment that $I \in \text{span}\{g(v_{1}u), \ldots, g(v_{r}u)\}$. Since $g(xv_{1}u)\zeta = 0$, we must have $g(x)\zeta = 0$. Using Lemma 3.3, we get a contradiction. Thus, there exist a non-zero complex number $\gamma$ and complex numbers $\tau_{t}$ such that $I = g(\gamma vu + \sum_{t} \tau_{t}v_{t}u)$. Set $v' = v - \sum \tau_{t}v_{t}$. Then $v' \in S(u)$, $g(v'u) \in CI$, and clearly $g(v'u) \neq 0$.

Case 2: $I \not\in \text{span}\{g(vu), g(v_{1}u), \ldots, g(v_{r}u)\}$.

Since $\text{ldim} V'(S_{E}) = \text{dim} V'(S_{E})$, $\{\zeta, g(vu)\zeta, \ldots, g(v_{r}u)\zeta\}$ is linearly independent. As in the proof of Lemma 3.3, we get a contradiction using suitable $x_{k} \in \text{Inv}A$ and $y \in A$.

We have thereby shown that this case cannot occur. \hfill $\square$

Lemma 3.5. Let $A$ be a unital Banach algebra, and let $S$ be an elementary operator on $A$. Let $g$ be an irreducible representation of $A$ on a Banach space $E$. Suppose that $\ell(S_{E}) = n$ and $\text{ldim} V'(S_{E}) = 1$. Set $\text{ldim} L(S_{E}) = r$. Then $V(S_{E}) \subseteq CI$ and $S_{E}$ admits a representation of the form $S_{E} = \sum_{i=1}^{n} M_{u_{i,v_{i}}}$, where

$$(v_{i}u_{j})_{1 \leq i,j \leq n} = \begin{pmatrix} T & 0 \\ \ast & 0 \end{pmatrix}$$

and $T$ is a triangular matrix of order $r$. Moreover, $S_{E}^{*}S_{E} \in CI$.

Proof. Write $S_{E} = \sum_{i=1}^{n} M_{a_{i},b_{i}}$ for some $a_{i},b_{i} \in g(A)$. It follows from our assumption on $V'(S_{E})$ that $b_{ia_{j}} \in CI$ for each $i,j$. Hence $V(S_{E}) \subseteq CI$. Pick a vector $\zeta \in E$ such that $L(S_{E})\zeta$ has maximal dimension. With no loss of generality, we may suppose that $\{a_{1}\zeta, \ldots, a_{r}\zeta\}$ is linearly independent and $a_{i}\zeta = 0$ for $r + 1 \leq i \leq n$ (compare the proof of [5, Proposition 3.1]). Let $x \in A$ be such that $g(x)\zeta = \zeta$. Choose a basis $\{u_{1}, \ldots, u_{r}\}$
of \( \text{span}\{a_1, \ldots, a_r\} \) such that \( M(S_g \varrho(x), \{u_1 \zeta, \ldots, u_r \zeta\}) \) is triangular. Set \( u_k = a_k \) for \( k = r+1, \ldots, n \). Then \( R(S_g)u_k = 0 \) for all \( k > r \). It is clear that \( S_g^*S_g \in \mathbb{C}I \). The proof is complete.

Our next result is the analogue of Theorem 3.3 in [5].

**Theorem 3.6.** Let \( A \) be a unital Banach algebra, and let \( S \) be a spectrally bounded elementary operator on \( A \). Let \( \varrho \) be an irreducible representation of \( A \) on a Banach space \( E \). Suppose that \( \text{ldim} L(S_g) = \text{dim} L(S_g) = n \). Then \( \text{ldim} V'(S_g) \leq \frac{n(n-1)}{2} + 1 \). Moreover, if \( \text{ldim} V'(S_g) = \frac{n(n-1)}{2} + 1 \), then \( S_g \) admits a representation of the form \( S_g = \sum_{i=1}^n M_{\varrho(v_i), \varrho(v_i)} \), where \( \varrho(v_iu_j) = 0 \) for every \( i > j \) and \( \varrho(v_iu_i) \in \mathbb{C}I \). In particular, \( \text{dim} V'(S_g) = \frac{n(n-1)}{2} + 1 \).

**Proof.** Let \( S_g = \sum_{i=1}^n M_{\varrho(v_i), \varrho(v_i)} \) and \( \text{ldim} V'(S_g) = r + 1 \). Choose \( \zeta \in E \) such that \( \text{dim} V'(S_g)\zeta = r + 1 \) and \( \text{dim} L(S_g)\zeta = n \) (Lemma 2.7). The case \( r = 0 \) is trivial. Suppose that \( r \neq 0 \) and let \( \{\varrho(b_1, a_1), \ldots, \varrho(a_n)\zeta\} \) be a basis of \( V'(S_g)\zeta \). Pick \( x_1, \ldots, x_r \in A \) with

\[
\varrho(x_kb_ia_j)\zeta = \delta_{k\ell}\zeta \quad \text{and} \quad \varrho(x_k)\zeta = 0 \quad (1 \leq t, k \leq r).
\]

Let \( N \) be the vector subspace of \( M_n(\mathbb{C}) \) generated by \( M(S_g \varrho(x_i), \{\varrho(a_1)\zeta, \ldots, \varrho(a_n)\zeta\}) \). It follows from Lemma 3.3 that \( N \) is nilpotent. Since \( \text{dim} N = r \), applying Gerstenhaber’s theorem [14], we get \( r \leq \frac{n(n-1)}{2} \) and if \( r = \frac{n(n-1)}{2} \), there exists a basis \( B \) of \( L(S_g)\zeta \) such that \( M(S_g \varrho(x), B) \) is upper triangular for each \( x \in \text{span}\{x_1, \ldots, x_r\} \). Next suppose that \( r = \frac{n(n-1)}{2} \) and set \( B = \{\varrho(a_1)\zeta, \ldots, \varrho(a_n)\zeta\} \). Write \( gS = g(\sum_{i=1}^n M_{u_i,v_i}) \). Since \( M(S_g \varrho(x), B) = (\pi \varrho(xv_iu_j))_{1 \leq i, j \leq n} \), we have \( \pi \varrho(xv_iu_j)\zeta = 0 \) for all \( i \geq j \) and \( x \in \text{span}\{x_1, \ldots, x_r\} \). Since \( \text{dim} V'(S_g)\zeta = \frac{n(n-1)}{2} + 1 \), the set \( \{\varrho(v_iu_j)\zeta : i < j\} \cup \{\zeta\} \) is a basis of \( V'(S_g)\zeta \). Moreover, \( \varrho(v_iu_j)\zeta \in \mathbb{C}\zeta \) for all \( i \geq j \). Arguing similarly to the proof of [5, Theorem 3.3], we show that \( R(S_g)u_1 \subseteq \mathbb{C}I \) and deduce by induction that \( \varrho(v_iu_j) \in \mathbb{C}I \) for all \( i \geq j \). Thus \( \text{dim} V'(S_g) = \frac{n(n-1)}{2} + 1 \). Hence the set \( \{\varrho(v_iu_j) : i < j\} \cup \{I\} \) is a basis of \( V'(S_g) \), and using \( \zeta \), we infer that \( \varrho(v_iu_j) \in \mathbb{C}I \) for all \( i \geq j \).

Next suppose that \( \varrho(v_iu_l) \neq 0 \) for some \( t > l \) and let \( l \) be the smallest integer satisfying this property. With no loss of generality, we suppose that \( \varrho(v_iu_l) = I \). Suppose there is \( j > k \geq l \) with the property that \( \varrho(v_iu_j) \neq 0 \) and \( \varrho(v_ku_l) \neq 0 \). Write

\[
\varrho(S) = \varrho(M_{u_1,v_1} + \ldots + M_{u_k,v_k} - \lambda v_j + \ldots + M_{u_l+\lambda v_j,v_l} + \ldots + M_{u_n,v_n})
\]

for some complex number \( \lambda \) satisfying \( \varrho((v_k - \lambda v_j)u_l) = 0 \). Thus, by choosing the largest \( t \) with \( \varrho(v_iu_l) \neq 0 \), we may assume that \( \varrho(v_ku_l) = 0 \) for \( l \leq k \) and \( k \neq t \). Take \( x_k \in \text{Inv} A \), \( x \in A \) such that

\[
\varrho(x_k) = \frac{1}{k} \varrho(v_{i}u_{l})\zeta, \quad \varrho(x_kv_iu_{l})\zeta = \zeta \quad \text{and} \quad \varrho(x_kv_iu_k)\zeta = \varrho(v_iu_k)\zeta \quad (i < k, \ (i, k) \neq (l,t))
\]
and
\[ q(xv_l u_l) \zeta = q(x) \zeta = q(v_l u_l) \zeta, \quad q(xv_k u_k) \zeta = 0 \quad (i < k, (i, k) \neq (l, t)). \]

We have
\[ q(x_k x_k^{-1}) \zeta = \zeta, \quad q(x_k x_k^{-1} v_l u_l) \zeta = k \zeta \]
and
\[ q(x_k x_k^{-1} v_l u_k) \zeta = 0 \quad (i < k, (i, k) \neq (l, t)). \]

Let \( M \) be the matrix representation of \( S_q \sigma(y x y^{-1}) \) with respect to \( \{q(u_1) \zeta, \ldots, q(u_n) \zeta\} \). Then \( M \) has the form
\[
M = \begin{pmatrix}
D & 0 & 0 \\
0 & L & 0 \\
0 & * & *
\end{pmatrix},
\]
where \( D \) is a diagonal matrix of order \( l - 1 \), \( L \) has order \( t - l + 1 \) and is of the form
\[
S = \begin{pmatrix}
0 & 0 & \cdots & 0 & k \\
0 & * & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & 0 & 0 \\
0 & * & \cdots & 0 & 0 \\
1 & * & \cdots & * & *
\end{pmatrix}.
\]
Computing the characteristic polynomial along the row \( l \), and using the fact that \( v_k u_l = 0 \) for \( k > l \) with \( k \neq t \), we get
\[
P_M = Q + kR, \quad \deg Q = n, \quad \deg R = n - 2,
\]
and \( Q, R \) do not depend on \( k \). Thus, the set \( \{\sigma(S_q \sigma(y x y^{-1})) : y \in \text{Inv}A\} \) is not bounded, a contradiction. \( \square \)

**Corollary 3.7.** Let \( A \) be a unital Banach algebra, and let \( S \) be a spectrally bounded elementary operator on \( A \) of length 2. Let \( q \) be an irreducible representation of \( A \) on a Banach space \( E \). Then either \( S \) admits a representation of the form \( S = M_{a,b} + M_{c,d} \), where \( q(ba), q(dc) \in \mathbb{C}I \) and \( q(bc) = 0 \), or \( \dim E = 2 \) and \( S_q \) is similar to the map
\[
z \mapsto \begin{pmatrix}
\lambda \text{Tr}(z) & * \\
0 & 0
\end{pmatrix} \quad \text{for some } \lambda \in \mathbb{C}.
\]

**Proof.** If \( S_q \) has length 1, the desired conclusion follows from [11]; thus we can assume that \( \ell(S_q) = 2 \). Suppose that \( \text{ldim } L(S_q) = 2 \). The case \( \text{ldim } V'(S_q) = 1 \) follows from Lemma 3.5 and in the other case, the result follows from Theorem 3.6. Finally suppose that \( \text{ldim } L(S_q) = 1 \) and set \( S_q = q(M_{a_1,b_1} + M_{a_2,b_2}) \). Since \( \{q(a_1), q(a_2)\} \) is linearly independent, there exist \( \eta \in E \) and \( f_1, f_2 \in E^* \) such that \( q(a_i) = \eta \otimes f_i, \quad i = 1, 2; \) [6, Theorem 2.3]. Suppose first that \( \dim E \geq 3 \). We claim that \( V(S_q) = 0 \). Indeed, suppose
for instance that \( \rho(b_1) \eta \neq 0 \). Then there is \( \zeta \in E \) such that \( f_1(\zeta) = 1 \), \( f_2(\zeta) = 0 \), and \( \{ \zeta, \varrho(b_1) \eta \} \) is linearly independent. Choose \( x_k, x \in A \), \( k \in \mathbb{N} \) with \( x_k \) invertible satisfying

\[
\varrho(x_k) \zeta = \frac{1}{k} b_1 \eta, \quad \varrho(x_k x_1) \zeta = \zeta, \quad \varrho(x) \zeta = b_1 \eta, \quad \varrho(x b_1) \eta = 0 \quad \text{and} \quad \varrho(x_k x_k^{-1} b_2) \eta \in \mathbb{C} \zeta.
\]

Then \( S_{\varrho} \varrho(x_k x_k^{-1}) \eta = k \varrho(a_1) \zeta = k \eta \), a contradiction. Thus \( \varrho(b_i a_j) = 0 \) for all \( i, j \).

Next suppose that \( \dim E = 2 \) and thus \( \rho(A) = M_2(\mathbb{C}) \). Since \( \dim L(S_{\varrho}) = 1 \), we may assume that \( \rho(a_1) = e_{11} \) and \( \rho(a_2) = e_{12} \). Hence \( S_{\varrho} = M_{e_{11}, b} + M_{e_{12}, d} \) for some \( b, d \in M_2(\mathbb{C}) \). Using Proposition 3.2, we infer that the matrices \( b \) and \( d \) have the form

\[
b = \begin{pmatrix} \lambda & * \\ 0 & * \end{pmatrix}, \quad d = \begin{pmatrix} 0 & * \\ \lambda & * \end{pmatrix}
\]

for some \( \lambda \in \mathbb{C} \).

A straightforward computation yields the desired result and completes the proof. \( \square \)

**Corollary 3.8.** Let \( A \) be a unital Banach algebra, and let \( S \) be a spectrally bounded elementary operator on \( A \) of length 2. Let \( \varrho \) be an irreducible representation of \( A \) on a Banach space \( E \). If \( \dim E \neq 2 \) then \( S_{\varrho}^* S_{\varrho} \in \mathbb{C} I \); if \( \dim E = 2 \) then either \( S_{\varrho}^* S_{\varrho} \in \mathbb{C} I \) or \( S_{\varrho}^* S_{\varrho} \in \mathbb{C} (I \otimes \text{Tr}) \). In either case, \( [S^* S x, x] \in \text{rad} A \) for all \( x \in A \).

**Proof.** Suppose first that \( S_{\varrho} \) admits a representation of the form \( M_{a, b} + M_{c, d} \) such that \( \varrho(a b), \varrho(d c) \in \mathbb{C} I \) and \( \varrho(b c) = 0 \). Then, for every \( z \in \varrho(A) \), \( S_{\varrho}^* S_{\varrho} z \in \mathbb{C} z \) and hence \( S_{\varrho}^* S_{\varrho} \in \mathbb{C} I \). On account of Corollary 3.7, if \( S_{\varrho} \) does not have such a representation, then \( \dim E = 2 \) and \( S_{\varrho} \) can be written in the form \( S_{\varrho} = M_{e_{11}, b} + M_{e_{12}, d} \), using the same notation as in the proof of Corollary 3.7. It follows that

\[
S_{\varrho}^* S_{\varrho} z = \begin{pmatrix} \lambda^2 \text{Tr}(z) & 0 \\ 0 & \lambda^2 \text{Tr}(z) \end{pmatrix} = (\lambda^2 \otimes \text{Tr})(z) \quad (z \in \varrho(A))
\]

and hence \( S_{\varrho}^* S_{\varrho} \in \mathbb{C} I \otimes \text{Tr} \). In either case, \( [S^* S x, x] \in \text{rad} A \) for all \( x \in A \). \( \square \)

**Corollary 3.9.** Let \( S \) be a spectrally bounded elementary operator of length 2 on the matrix algebra \( A = M_n(\mathbb{C}) \), where \( n \geq 3 \). Then \( (Sx)^3 = 0 \) for every \( x \in A \) and \( S^* S = 0 \).

**Proof.** By Corollary 3.7, \( S \) has a representation of the form \( S = M_{a, b} + M_{c, d} \) such that \( b a, d c \in \mathbb{C} I \) and \( b c = 0 \). If \( b a \neq 0 \) then \( b \) is invertible, contradicting \( b c = 0 \). Analogously for \( d c \). Thus \( b a = d c = b c = 0 \). The first assertion hence follows from [5, Corollary 2.5] while the second is straightforward. \( \square \)

4. Spectrally Bounded Elementary Operators of Length 3

Like in the purely algebraic setting, [5], we have more precise information available for elementary operators of length 3. The first result is simply a translation of Corollary 4.2 in [5] into the context of Banach algebras.
Proposition 4.1. Let $A$ be a unital Banach algebra, and let $S$ be a spectrally infinitesimal elementary operator on $A$ of length 3. Then $S^*S(A) \subseteq \text{rad}A$ and $(Sx)^5 \in \text{rad}A$ for every $x \in A$.

In order to treat the more general situation of spectrally bounded elementary operators, we first need an auxiliary result.

Lemma 4.2. Let $E$ be a Banach space, and let $A$ be a closed irreducible algebra of bounded linear operators on $E$. Let $S = \sum_{i=1}^{3} M_{u_i, v_i}, u_i, v_i \in \mathcal{L}(E)$ be an elementary operator of length 3 on $A$ such that one of the following two cases occurs:

(i) $(v_i u_j)_{1 \leq i, j \leq 3} = \begin{pmatrix} \lambda I & \zeta_1 \otimes f & 0 \\ \zeta_0 \otimes f & \lambda I & \zeta_1 \otimes f \\ 0 & -\zeta_0 \otimes f & \lambda I \end{pmatrix} ; \lambda \in \mathbb{C}, \zeta_0, \zeta_1 \in E$ and $f \in E^*$;

(ii) $(v_i u_j)_{1 \leq i, j \leq 3} = \begin{pmatrix} \lambda I & \zeta_0 \otimes g & 0 \\ \zeta_0 \otimes f & \lambda I & \zeta_0 \otimes g \\ 0 & -\zeta_0 \otimes f & \lambda I \end{pmatrix} ; \lambda \in \mathbb{C}, \zeta_0 \in E$ and $f, g \in E^*$,

where $E^*$ denotes the dual of $E$. Then $S$ is spectrally bounded.

Proof. Case (i): If $\lambda = 0$, we are done by [5, Theorem 4.1]. Suppose next that $\lambda \neq 0$. A straightforward computation shows that $S^*1 = 3\lambda$ and $S^*S = 3\lambda^2 I$. Let $x \in A$ and let $\alpha \in \partial\sigma(Sx)$. Then

$$S^*(Sx - \alpha 1) = 3\lambda(\lambda x - \alpha 1).$$

Since $Sx - \alpha 1$ is a topological divisor of zero, we infer that $\alpha \in \sigma(\lambda x)$. It follows that $r(Sx) \leq |\lambda|r(x)$ for all $x \in A$, as desired.

Case (ii). This case is treated analogously. □

The following is the main result of this section.

Theorem 4.3. Let $E$ be a Banach space with $\dim E \geq 4$, and let $A$ be a closed irreducible algebra of bounded linear operators on $E$. Let $S: A \to \mathcal{L}(E)$ be an elementary operator of length 3. Then $S$ is spectrally bounded if and only if there exists a representation of $S$ of the form

$$S = \sum_{i=1}^{3} M_{u_i, v_i}, u_i, v_i \in \mathcal{L}(E)$$

such that one of the following three cases occurs:

(i) $v_i u_j = 0$ for all $i > j$ and $v_i u_i \in \mathbb{C}I$ for all $i$.

(ii) $(v_i u_j)_{1 \leq i, j \leq 3} =$ \begin{pmatrix} \lambda I & \zeta_1 \otimes f & 0 \\ \zeta_0 \otimes f & \lambda I & \zeta_1 \otimes f \\ 0 & -\zeta_0 \otimes f & \lambda I \end{pmatrix} ; \lambda \in \mathbb{C}, \zeta_0, \zeta_1 \in E$ and $f \in E^*$;
Proof. The “if part” of the statement follows directly from Lemma 4.2 and Corollary 2.6.

Now suppose that $S$ is spectrally bounded and $\text{ldim } L(S) = 3$. If $\text{ldim } V'(S) = 4$, we are done by Theorem 3.6. Next choose a separating vector $\zeta \in E$ of $L(S)$ such that $V'(S)\zeta$ has maximal dimension. Fix a basis $B$ of $L(S)\zeta$. It follows from Lemma 3.3 that the set

$$N = \{ M(Sx, B) : xV(S)\zeta \subseteq C\zeta, x\zeta = 0 \}$$

is a nilpotent subspace of $M_3(\mathbb{C})$. Suppose that $\text{ldim } V'(S) = 3$. In view of [13, Proposition 3], we distinguish two cases:

Case 1: There exists a representation of $S$ of the form $S = \sum_{i=1}^{3} M_{u_i, v_i}$ for some $u_i, v_i \in \mathcal{L}(E)$ such that $\{v_1u_2\zeta, v_1u_3\zeta, \zeta\}$ are linearly independent and $v_iu_j\zeta \in \mathbb{C}\zeta$ for all $i \geq j$. Suppose moreover that this fact is true for a dense set of separating vectors of $L(S)$ and $V'(S)$. Then, arguing as in the proof of Theorem 3.6, we show that $R(S)u_1 \subseteq CI$. We deduce that $v_ju_i \in CI$ for all $i \geq j$ and, finally, we show that $S = \sum M_{u_i', v_i'}$ where $u_i'v_j = 0$ for $i > j$ and $v_i'\zeta \in CI$.

Case 2: There exists a representation of $S$ of the form $S = \sum_{i=1}^{3} M_{u_i, v_i}$ for some $u_i, v_i \in \mathcal{L}(E)$ (depending on $\zeta$) such that

$$(v_iu_j\zeta)_{1 \leq i, j \leq 3} = \begin{pmatrix}
\lambda_1\zeta & v_2u_3\zeta & \lambda_3\zeta \\
v_2u_1\zeta & \lambda_2\zeta & v_2u_3\zeta \\
\lambda_3\zeta & -v_2u_1\zeta & \lambda_3\zeta \\
\end{pmatrix}$$

for some $\lambda_{ij} \in \mathbb{C}$.

Suppose that this assumption is true for every separating vector of $L(S)$, with $V'(S)\zeta$ of maximal dimension, in an open subset of $X$. For every $(i, j) \not\in \{(2, 1), (2, 3), (1, 2), (1, 3)\}$, the set of operators $\{v_2u_1, v_2u_3, I, v_iu_j\}$ is locally linearly dependent. Since $(v_iu_j - \lambda_{ij}I)\zeta = 0$, we must have $(v_iu_j - \lambda_{ij}I)E \subseteq \text{span}\{v_2u_1\zeta, v_2u_3\zeta, \zeta\}$. Set $v_2u_1\zeta = \eta_1$, $v_2u_3\zeta = \eta_2$ and $\zeta = \eta_0$. Pick $\eta_3 \in E \setminus \text{span}\{\zeta, \eta_1, \eta_2\}$. Write $E = \text{span}\{\zeta, \eta_1, \eta_2, \eta_3\} \oplus Y$, for some subspace $Y$ of $E$. Let $p: E \to Y$ be the natural projection. Write $v_su_t = \sum_{k=0}^{3} \eta_k \otimes f^k_{st} + pv_su_t$. Then for every $(i, j) \not\in \{(2, 1), (2, 3), (1, 2), (1, 3)\}$, we have $v_iu_j = \sum_{k=0}^{2} \eta_k \otimes f^k_{ij}$. Fix non-zero numbers $h_0, \ldots, h_3 \in \mathbb{C}$ and set $M(\zeta', h_0, \ldots, h_3) = (\sum_{k=0}^{3} h_k f^k_{ij}(\zeta'))_{1 \leq i, j \leq 3}$. Let $\zeta' \in E$. Then for all but finitely many $\lambda \in \mathbb{C}$, $\zeta' + \lambda\zeta$ is a separating vector of $L(S)$. Fix $\lambda \in \mathbb{C}$ and choose $x \in A$ satisfying

$$x\eta_k = h_k(\zeta' + \lambda\zeta), \quad xp_su_t(\zeta' + \lambda\zeta) = 0$$

and $x(\zeta' + \lambda\zeta) = 0$ ($0 \leq k \leq 3$). Then

$$M(Sx, \{u_1(\zeta' + \lambda\zeta), \ldots, u_3(\zeta' + \lambda\zeta)\}) = M(\zeta' + \lambda\zeta, h_0, \ldots, h_3)$$
which is nilpotent. Arguing as in the proof of [5, Theorem 4.1], we show that

\[(v_i u_j)_{1 \leq i,j \leq 3} = \begin{pmatrix}
\lambda_1 I & \zeta_1 \otimes f & \lambda_5 I \\
\zeta_0 \otimes f & \lambda_2 I & \zeta_1 \otimes f \\
\lambda_4 I & -\zeta_0 \otimes f & \lambda_3 I
\end{pmatrix},\]

where \(\lambda_i \in \mathbb{C}, \zeta_0, \zeta_1 \in E, f \in E^*\) and \(f(\zeta) = 1\). Choose \(x_k \in \text{Inv} A\) and \(x \in A\) such that

\[x_k \zeta = \frac{1}{k} \zeta_0, \quad x_k \zeta_0 = \zeta, \quad x_k \zeta_1 = \zeta_1 \text{ and } x \zeta_0 = x \zeta = \zeta_0, \quad x \zeta_1 = 0.\]

Then the matrix of \(S(x_k x_k^{-1})\) with respect to \(\{u_1 \zeta, u_2 \zeta, u_3 \zeta\}\) is:

\[\begin{pmatrix}
\lambda_1 & 0 & \lambda_5 \\
k & \lambda_2 & 0 \\
\lambda_4 & -k & \lambda_3
\end{pmatrix}.\]

This entails that \(\lambda_5 = 0\). Proceeding analogously, with \(\zeta_1\) instead of \(\zeta_0\), we show that \(\lambda_4 = 0\). Next choose \(x_k \in \text{Inv} A\) and \(x \in A\) such that

\[x_k \zeta = \frac{1}{k} \zeta_0, \quad x_k \zeta_0 = \zeta, \quad x_k \zeta_1 = \zeta_1 \text{ and } x \zeta_0 = x \zeta = x \zeta_1 = \zeta_0.\]

Then the matrix of \(S(x_k x_k^{-1})\) with respect to \(\{u_1 \zeta, u_2 \zeta, u_3 \zeta\}\) is:

\[\begin{pmatrix}
\lambda_1 & 1 & 0 \\
k & \lambda_2 & 1 \\
0 & -k & \lambda_3
\end{pmatrix}.\]

This implies that \(\lambda_1 = \lambda_3\). Next suppose there are \(\zeta, \zeta' \in E\) such that the sets \(\{\zeta_0, \zeta_1, \zeta, \zeta'\}\) and \(\{u_2 \zeta, u_3 \zeta, u_1 \zeta', u_2 \zeta', u_3 \zeta'\}\) are linearly independent. With no loss of generality, we may assume that \(f(\zeta') = 1\) and \(f(\zeta) = 0\). Pick \(x_k \in \text{Inv} A\) and \(x \in A\) such that

\[x_k \zeta' = \frac{1}{k} \zeta, \quad x_k \zeta_0 = \frac{1}{k} \zeta_0, \quad x_k \zeta_1 = \zeta_1, x_k \zeta = \zeta' \]

and

\[x \zeta' = x \zeta = x \zeta_1 = \zeta, \quad x \zeta_0 = \zeta'.\]

Then the matrix representation of \(S(x_k x_k^{-1})\) with respect to \(\{u_2 \zeta, u_3 \zeta, u_1 \zeta', u_2 \zeta', u_3 \zeta'\}\) is

\[\begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & \lambda_1 & 1 & 0 \\
\lambda_2 k & 0 & 0 & \lambda_2 & 1 \\
0 & \lambda_1 k & 0 & 0 & \lambda_1
\end{pmatrix}.\]

Its characteristic polynomial is

\[t^2(\lambda_1 - t)^2(\lambda_2 - t) + (\lambda_2 - \lambda_1)k(\lambda_1 - t)t\]
hence, we must have $\lambda_2 = \lambda_1$. We get the same conclusion if we swap $u_1$ and $u_3$.

Now suppose that for every $\zeta, \zeta' \in E$, the sets $\{u_2\zeta', u_k\zeta, u_1\zeta, u_2\zeta, u_3\zeta\}$, $k = 1, 3$ are linearly dependent. Then, for every separating vector $\zeta$ of $L(S)$ and $k \in \{1, 3\}$, either there are $\alpha_k, \beta_k$ such that $(\alpha_k u_k + \beta_k u_2)E \subseteq L(S)\zeta$ or $u_k, u_2$ are rank one modulo $L(S)\zeta$.

Suppose towards a contradiction that $\lambda_1 \neq \lambda_2$. Suppose that $E$ is finite dimensional. Since $v_1 u_3 = 0$, $v_1$ cannot be injective. Thus $\lambda_1 = 1$. Clearly, we can assume that $\lambda_2 = 1$. Since $v_2, u_2$ must be bijective, $u_1, u_3, v_1, v_3$ are rank 1. Set $u_k = \eta_k \otimes g_k$, $k = 1, 3$. Then, with no loss of generality, we may assume that $g_k = f$. Since $u_2$ is bijective and $v_2 u_2 = I$, $u_2 \zeta_0 = \eta_1$ and $u_2 \zeta_1 = \eta_3$. On the other hand,

$$v_1 u_2 \zeta_0 = f(\zeta_0) \zeta_1 = v_1 \eta_1 = 0.$$ Thus, $f(\zeta_0) = 0$. Analogously, we get $f(\zeta_1) = 0$. Pick $\eta \in E$ such that $f(\eta) = 1$ and $\{\eta_1, \eta_2, u_2 \eta\}$ is linearly independent. Choose $x_1 \in \text{Inv} A$, and $x \in A$ with

$$x_k \zeta_1 = \frac{1}{k} \zeta_0, x_k \zeta_0 = \zeta_1, x_k \eta = \eta, x \zeta_0 = \eta = x \eta, x \zeta_1 = \zeta_0.$$ Then

$$M(S(x_k x_k^{-1}), \{\eta_1, \eta_3, u_2 \eta\}) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ k & 0 & 0 \end{pmatrix}.$$ This yields a contradiction, as desired. Now suppose that $E$ has infinite dimension. Suppose first that $\lambda_2 \neq 0$. Then $u_2$ is injective, hence $u_1, u_3$ have rank one modulo $L(S)\zeta$, for each separating vector $\zeta$ of $L(S)$. This implies (using again the injectivity of $u_2$) that $\dim (u_1 E + u_3 E) \leq 2$. Applying $v_2$ to $u_1$ and $u_3$, we infer that $u_1, u_3$ are rank 1. Thus $\lambda_1 = 0$. Pick $\eta, \eta'$ such that the sets $\{u_1 \eta, u_2 \eta, u_3 \eta, u_2 \eta'\}$ and $\{\eta, \eta', \zeta_0, \zeta_1\}$ are linearly independent. With no loss of generality, we can suppose that $u_1 = u_1 \eta \otimes f$, $u_3 = u_3 \eta \otimes f$, $f(\eta) = 1$ and $f(\eta') = 0$. Choose $x_k \in \text{Inv} A$, $x \in A$ with

$$x_k \eta = \frac{1}{k} \zeta_0, x_k \zeta_0 = \eta, x_k \zeta_1 = \eta + \eta', x_k \eta' = \zeta_1$$ and

$$x \eta = \zeta_0, x \eta' = \zeta_1, x \zeta_0 = \zeta_0, x \zeta_1 = 2 \zeta_0.$$ Then

$$M(S(x_k x_k^{-1}), \{u_1 \eta, u_3 \eta, u_2 \eta, u_2 \eta'\}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -k & 0 \\ k & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$ Computing the characteristic polynomial, we get a contradiction. Thus, this case cannot occur.
Next suppose that $\lambda_1 \neq 0$, say $\lambda_1 = 1$. Then $u_1, u_3$ are injective. As above, we see that $u_2$ must have rank one. Set $u_2 = z \otimes g$, then we can assume that $g = f$, $v_1z = \zeta_1$ and $v_3z = -\zeta_0$. Choose $\eta, \eta'$ such that $\{u_1\eta, u_3\eta, u_2\eta', u_1\eta'\}$ is linearly independent. Assume that $u_2\eta = z$ and $f(\eta') = 0$. Choose $x, x_k$ as above to obtain

$$M(S(x_kx_k^{-1}), \{u_1\eta, u_3\eta, u_2\eta, u_1\eta'\}) = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -k & 0 \\ k & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$ 

Once again, we get a contradiction using the characteristic polynomial.

Now suppose that $\text{idim} \ V'(S) = 2$. Similar arguments as above yield that $S = \sum_{i=1}^{3} M_{u_i,v_i}$, $u_i, v_i \in \mathcal{L}(E)$ with either $v_iu_j = 0$ for all $i > j$ and $v_iu_i \in CI$ for all $i$, or

$$(v_iu_j)_{1 \leq i,j \leq 3} = \begin{pmatrix} \lambda_1I & \zeta_0 \otimes g & \lambda_4I \\ \zeta_0 \otimes f & \lambda_2I & \zeta_0 \otimes g \\ \lambda_5I & -\zeta_0 \otimes f & \lambda_3I \end{pmatrix};$$

where $\lambda_i \in \mathbb{C}$, $\zeta_0 \in \mathbb{E}$ and $f, g \in E^*$. This case is treated analogously.

Suppose now that $\text{idim} \ L(S) = 2$. Suppose first that there exists $b \in R(S)$ with $bL(S) = 0$, then we can write $S = M_{a,b} + \sum_{i=1}^{2} M_{u_i,v_i}$ for suitable $u_i, v_i$. Our arguments show that $\sum_{i=1}^{2} M_{u_i,v_i}$ must be spectrally bounded. Thus we get easily the desired conclusion.

Next suppose the contrary. Write $S = \sum_{i=1}^{3} M_{a_i,b_i}$. Using [8], we distinguish 3 cases.

Case 1: $\text{dim} \ L(S)E = 2$. Let $\zeta \in E$ be such that $L(S)\zeta$ has maximal dimension. Certainly we may assume that $a_3\zeta = 0$. Using the set

$$\{Sx | L(S)\zeta : xV(S)\zeta \subseteq \mathbb{C}\zeta, x\zeta = 0\}$$

we see that there exists $b \in R(S)$ such that $ba_iE \subseteq \mathbb{C}\zeta$. Consider $\zeta_1, \zeta_2, \zeta_3$ linearly independent separating vectors of $L(S)$. Then there exists $b'_1, b'_2, b'_3$ with $b'_1L(S)E \subseteq \mathbb{C}\zeta_i$. Therefore $\{b'_1, b'_2, b'_3\}$ is linearly independent. Thus, the dimension of $E$ cannot exceed 3. This case cannot occur.

Case 2: The vector space $L(S)$ is standard in the sense of [8]. Fix $u \in L(S)E$. We claim that either $\text{dim} \ R(S)u \leq 1$ or $\text{dim} \ R(S)u = 3$. Choose $\zeta \in E$ such that $u = a\zeta$ for some $a \in L(S)$. If $\zeta \not\in R(S)u$, choose $x \in A$ such that $x\zeta = 0$ and $xV(S)\zeta \subseteq \mathbb{C}\zeta$. Then $Sx$ must be nilpotent by Lemma 3.3. A straightforward argument shows that $\text{dim} \ R(S)u \leq 1$. Next suppose that $\text{dim} \ R(S)u \geq 2$. Then $\zeta \in R(S)u$ and $\ker a + \mathbb{C}\zeta \subseteq R(S)u$. Since $\text{dim} aE = 2$ and $\text{dim} R(S)u \leq 3$, we must have $\text{dim} E = 4$ and $\text{dim} R(S)u = 3$, as desired. Now write

$$a_1 = u_1 \otimes f_2 + u_2 \otimes f_3, \quad a_2 = -u_1 \otimes f_1 + u_3 \otimes f_3, \quad a_3 = u_2 \otimes f_1 + u_3 \otimes f_2;$$
where \( f_1, f_2, f_3 \) are linearly independent functionals and \( u_1, u_2, u_3 \) are linearly independent vectors of \( E \). Then, for every \( x \in A \), we have

\[
Sx = u_1 \otimes (f_2x_1 - f_1x_2) + u_2 \otimes (f_3x_1 + f_1x_3) + u_3 \otimes (f_3x_2 + f_2x_3).
\]

Choose \( e_j \in E \) with \( f_i(e_j) = \delta_{ij} \). Fix \( j \in \{1, 2, 3\} \). Let \( k, l \) be such that \( \{j, k, l\} = \{1, 2, 3\} \). The vector subspace

\[
\{M(Sx, \{a_ke_j, a_l e_j\}) : x \in A, xe_j = 0, xV(S)e_j \subseteq C e_j \}
\]

is nilpotent. Thus there exists \( z_j \in \text{span}\{a_ke_j, a_l e_j\} \) such that \( \text{span}\{b_k, b_l\}z_j \subseteq C e_j \). Since \( \dim R(S)z \neq 2 \) for every \( z \in L(S)E \), we must have \( \dim R(S)z \leq 1 \) for every \( z \in L(S)E \). Now using Proposition 3.2, we get the desired conclusion.

Case 3: The vector space \( L(S) \) is not minimal linearly dependent.

With no loss of generality, we may suppose that there exists \( \eta \in E \) such that \( a_k = \eta \otimes f_k \) for \( k = 1, 2 \). Then for all \( x \in A \) we have

\[
Sx = \eta \otimes \sum_{k=1}^{2} f_kx_k + a_3x_3.
\]

Suppose that \( b_3a_3 \notin CI \). Choose \( \zeta \in E \) such that the two sets \( \{b_3a_3\zeta, \zeta\} \) and \( \{a_1\zeta, a_3\zeta\} \) are linearly independent. The vector space

\[
\mathcal{N} = \{M(Sx, \{a_1\zeta, a_3\zeta\}) : x \in A, x\zeta = 0, x b_3a_3\zeta, xb_3\zeta \in C \zeta \}
\]

is nilpotent. Hence \( \dim \mathcal{N} \leq 1 \). Since for every \( x \in A \) with \( xb_3a_3\zeta, xb_3\eta \in C \zeta \), we have

\[
M(Sx, \{a_1\zeta, a_3\zeta\}) = \left( \sum_{k=1}^{2} f_k(xb_k\eta) \sum_{k=1}^{2} f_k(xb_k a_3\zeta) \right) \pi x b_3 a_3 \zeta
\]

(here we suppose that \( a_1\zeta = \eta \)). Thus \( b_k\eta \in \text{span}\{b_3a_3\zeta, \zeta\} \), for \( 1 \leq k \leq 3 \). Now assume that \( b_3\eta \notin C \zeta \). Then, replacing \( a_3 \) with \( a_3 + \lambda a_1 + \lambda' a_2 \) if needed, we see that we can assume that \( b_1\eta, b_2\eta \in C \zeta \). It follows from Proposition 3.2 that \( b_3a_3\zeta \in C \zeta \), a contradiction. Next suppose that \( b_3\eta \in C \zeta \). With no loss of generality, we can suppose that \( b_2\eta \in C \zeta \). It is easy to infer that \( b_1\eta = 0 \), and once again we get \( b_3a_3\zeta \in C \zeta \), which is not possible. Hence, we have \( b_3a_3 \in CI \). Since we can replace \( a_3 \) by \( a_3 + \lambda a_1 + \lambda' a_2 \), for every \( \lambda, \lambda' \in \mathbb{C} \), we get \( b_3a_k \in CI \) for each \( k \). But \( a_1, a_2 \) are rank one, therefore \( b_3a_1 = b_3a_2 = 0 \). Using again the set \( \mathcal{N} \) we see that \( b_1\eta, b_2\eta \in C \zeta \). Hence, \( b_1\eta = b_2\eta = 0 \). The proof of this case is complete.

Finally, assume that \( \dim L(S) = 1 \). Then there exists \( \eta \in E, f_i \in E^* \) such that \( a_i = \eta \otimes f_i \). Since \( \dim E \geq 4 \), we get easily the desired conclusion using Proposition 3.2. \( \square \)

The exceptional cases \( \dim E = 2 \) and \( \dim E = 3 \) do not seem to allow a concise description (such as in Corollary 3.7) and thus will not be discussed here.
Acknowledgement. Part of the research on this paper was carried out while the second-named author visited the Université Moulay Ismail in Morocco. He would like to express his gratitude for the generous hospitality and support received from his colleagues there. This visit was supported by the London Mathematical Society under their “Research in Pairs” programme.

References

[1] P. Ara and M. Mathieu, Local multipliers of C*-algebras, Springer Monographs in Mathematics, Springer-Verlag, London, 2003.
[2] B. Aupetit, A primer on spectral theory, Springer-Verlag, New York, 1991.
[3] N. Boudi, On the product of derivations in Banach algebras, Math. Proc. Royal Irish Academy 109 A (2009), 201–211.
[4] N. Boudi and M. Mathieu, Elementary operators that are spectrally bounded, Operator Theory: Advances and Applications 212 (2011), 1–15.
[5] N. Boudi and M. Mathieu, Locally quasi-nilpotent elementary operators, Operators and Matrices, to appear.
[6] M. Brešar and P. Šemrl, On locally linearly dependent operators and derivations, Trans. Amer. Math. Soc. 351 (1999), 1257–1275.
[7] M. Brešar and P. Šemrl, Linear maps preserving the spectral radius, J. Funct. Anal. 142 (1996), 360–368.
[8] M. A. Chebotar and P. Šemrl, Minimal locally linearly dependent spaces of operators, Linear Algebra Appl. 429 (4) (2008), 887–900.
[9] C. Costara, Commuting holomorphic maps on the spectral unit ball, Bull. London Math. Soc. 41 (2009), 57–62.
[10] C. Costara and D. Repovš, Spectral isometries onto algebras having a separating family of finite-dimensional irreducible representations, J. Math. Anal. Appl. 365 (2010), 605–608.
[11] R. E. Curto and M. Mathieu, Spectrally bounded generalized inner derivations, Proc. Amer. Math. Soc. 123 (1995), 2431–2434.
[12] R. E. Curto and M. Mathieu (eds.), Elementary operators and their applications, Proc. 3rd Int. Workshop, Belfast, 14–17 April 2009; Operator Theory: Advances and Applications 212, Springer, Basel, 2011.
[13] M. A. Fasoli, Classification of nilpotent linear spaces in M(4, C), Comm. in Algebra 25 (6) (1997), 1919–1932.
[14] M. Gerstenhaber, On nilalgebras and linear varieties of nilpotent matrices. I, Amer. J. Math. 80 (1958), 614–622.
[15] W. Gong, D. R. Larson and W. R. Wogen, Two results on separating vectors, Indiana Univ. Math. J. 43 (1994), 1159–1165.
[16] M. Mathieu (ed.), Elementary operators and applications, Proc. Int. Workshop, Blaubeuren, 9–12 June 1991; World Scientific, Singapore, 1992.
[17] M. Mathieu, Where to find the image of a derivation, Banach Center Publ. 30 (1994), 237–249.
[18] M. Mathieu, A collection of problems on spectrally bounded operators, Asian–Eur. J. Math. 2 (2009), 487–501.
[19] M. Mathieu and G. J. Schick, *First results on spectrally bounded operators*, Studia Math. **152** (2002), 187–199.

[20] M. Mathieu and G. J. Schick, *Spectrally bounded operators from von Neumann algebras*, J. Operator Theory **49** (2003), 285–293.

[21] M. Mathieu and A. R. Sourour, *Hereditary properties of spectral isometries*, Arch. Math. **82** (2004), 222–229.

[22] M. Mathieu and A. R. Sourour, *Spectral isometries on non-simple C*-algebras*, Proc. Amer. Math. Soc. **142** (2014), 129–145.

[23] G. J. Murphy and T. T. West, *Spectral radius formulae*, Proc. Edinburgh Math. Soc. **22** (1979), 271–275.

[24] G. K. Pedersen, *Spectral formulas in quotient C*-algebras*, Math. Z. **148** (1976), 299–300.

[25] V. Pták, *Derivations, commutators and the radical*, Manuscripta Math. **23** (1978), 355–362.

[26] P. Šemrl, *Spectrally bounded linear maps on B(H)*, Quart. J. Math. Oxford (2) **49** (1998), 87–92.

DÉPARTEMENT DE MATHEMATIQUES, UNIVERSITÉ MOULAY ISMAIL, FACULTÉ DES SCIENCES, MEKNÈS, MAROC

E-mail address: nadia.boudi@hotmail.com

PURE MATHEMATICS RESEARCH CENTRE, QUEEN’S UNIVERSITY BELFAST, UNIVERSITY ROAD, BELFAST BT7 1NN, NORTHERN IRELAND

E-mail address: m.m@qub.ac.uk