We construct the model of a quantum spherically symmetric plasma structure based on radial oscillations of ions. We suppose that ions are involved in ion-acoustic waves. We find the exact solution of the Schrödinger equation for an ion moving in the self-consistent oscillatory potential of an ion-acoustic wave. The system of ions is secondly quantized and its ground state is constructed. Then we consider the interaction between ions by the exchange of an acoustic wave. It is shown that this interaction can be attractive. We describe the formation of pairs of ions inside a plasma structure and demonstrate that such a plasmoid can exist in dense astrophysical medium. The application of our results for terrestrial plasmas is also discussed.

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I. INTRODUCTION

The Coulomb interaction is undoubtedly dominant for charged particles in plasma. For instance, various oscillatory processes, like Langmuir waves, are driven by the Coulomb interaction. However, in the last decades there is a growing interest for various effective interactions which can arise between charged particles in plasmas.

First we mention the wakefield interaction between particles of the same polarity moving in plasma [1]. Under certain conditions the wakefield interaction can be attractive. This kind of interaction can result in the formation of complex structures in dusty plasmas [2]. Recently the attractive wakefield interaction in the ions system was studied experimentally in Ref. [3].

Besides wakefield interaction which arises mainly in classical plasmas, there is an effective interaction between charged particles in dense strongly correlated plasmas which is responsible for the formation of Coulomb crystals [4]. Note that Coulomb crystals can exist in both classical and quantum plasmas. Coulomb crystals were observed in liquid helium [5], in a system of trapped laser cooled ions [6], and in dusty plasmas [7]. Magnetized Coulomb crystals were suggested in Ref. [8] to play an important role in the evolution of neutron stars.

Another type of the effective interaction between charged particles in plasma is based on the exchange of a virtual acoustic wave. It was first proposed in Ref. [9] to explain the stability of atmospheric plasmoids. This kind of effective interaction was recently studied in Ref. [10] to describe the pairing of ions performing spherically symmetric quantum oscillations. The weakness of the description of quantum plasmoids made in Ref. [10] was the assumption that ions participate in Langmuir oscillations which is valid only in the short waves limit.

In the present work we continue the analysis of spatially localized quantum plasma structures based on radial oscillations of charged particles. To describe the motion of charged particles we do not use quantum hydrodynamics since that approach is likely to be valid only when small perturbation of the plasma density are considered. In Sec. II we find the exact solution of the Schrödinger equation describing the motion of an ion in the collective potential of an ion-acoustic wave. Using the second quantization method we construct the ground state of a plasmoid corresponding to noninteracting ions. In Sec. III we consider the effective interaction between oscillating ions by the exchange of an acoustic wave. In particular, we show that this interaction can be attractive. In Sec. IV we discuss the pairing of ions due to this attractive interaction and demonstrate that a plasma structure, where bound states of ions are formed, can well exist in a dense matter of the inner crust of a neutron star (NS). Finally, in Sec. V we briefly summarize our results. The mathematical details of the analysis of ions wave functions in the coordinate representation are given in Appendix A.

II. QUANTUM STATES OF IONS IN A SPHERICAL PLASMOID

In this section we build the model of a quantum plasmoid based on radial oscillations of ions in plasma. We find the exact solution of the Schrödinger equation describing spherically symmetric motion of ions participating in ion-acoustic oscillations. Then this system is secondly quantized and its ground state is constructed. We formulate the criterion of the stability of quantum oscillations and consider the description of the plasmoid in the short waves limit.

We shall use the model of a quantum plasmoid proposed in Ref. [10]. Let us briefly remind the basic concepts of this model. We shall study the oscillatory motion of charged particles in dense plasma. Moreover we shall be interested in a spherically symmetric spatially confined system. Note that quantum hydrodynamics, which is a popular tool to study quantum plasmas [11], is not applicable for the analysis of such objects since it can account for only small perturbations on the plasma density [12]. Thus to describe the ground state of a plasmoid in question one has to use exact solutions of the Schrödinger equation for charged oscillating particles in plasma. Note that these wave functions should maximally account for the spatial symmetry and the dynamical features of the system. For this purpose we shall use wave functions of the 3D harmonic oscillator corresponding to the frequency of charged particles harmonic motion. Note that in the present work we will be mainly interested in the quantization of the ions motion.

As a rule, plasma oscillations are dominated by the motion an electron component of plasma. It happens since an electron is a very light particle compared to an ion. Despite the amplitude and the frequency of electrons oscillations are much higher than that of ions, in realistic situation we cannot neglect the ions motion. Typically ions have lower temperature. Thus, quantum effects can be more pronounced for them. Note that in Sec. IV we shall study the effective interaction of ions in a very dense matter. It justifies the application of the quantum approach for the description of the ions motion.

Let us suppose that ions in plasma oscillate with the frequency \( \omega \). Then the stationary Schrödinger equation for the ion wave function \( \psi \) has the form,

\[
E\psi = \hat{H}\psi, \quad \hat{H} = \frac{\hat{p}^2}{2m_i} + \frac{m_i\omega^2\hat{r}^2}{2},
\]  

(1)
where \( \hat{\mathbf{p}} \) and \( \hat{\mathbf{r}} \) are the operators of the momentum and the coordinate, \( E \) is the ion energy, and \( m_i \) is its mass.

In Ref. [10] we studied the quantization of the motion of charged particles participating in Langmuir oscillations, which is a rough model in case we deal with ions. Let us suppose that ions in plasma are involved in collective oscillations corresponding to ion-acoustic waves. Thus the frequency \( \omega \) in Eq. (1) obeys the dispersion relation,

\[
\omega^2 = \omega_i^2 \frac{\lambda^2 k^2}{1 + \lambda^2 k^2}.
\]

where \( \omega_i = \sqrt{4\pi e^2 n_0/m_i} \) is the Langmuir frequency for ions, \( \lambda_e = \sqrt{T_e/4\pi e^2 n_0} \) is the Debye length for electrons, \( n_0 \) is the background density of charged particles, \( T_e \) is the electron temperature, and \( k = |\mathbf{k}| \) is the absolute value of the wave vector.

We shall choose the momentum representation in Eq. (1), i.e. \( \hat{\mathbf{p}} = \mathbf{p} = \hbar \mathbf{k} \) and \( \hat{\mathbf{r}} = i\hbar \nabla_{\mathbf{p}} \), where \( \nabla_{\mathbf{p}} = \frac{\partial}{\partial \mathbf{p}} \). Using Eq. (2) we can rewrite Eq. (1) as

\[
\mathbf{E} = \mathbf{E}(\psi),
\]

where \( \mathbf{E} \) is the effective orbital quantum number. Note that \( -\lambda_e \leq \psi \leq \lambda_e \) and that \( \psi = 0 \) is the spherical harmonic corresponding to the orbital and magnetic quantum numbers: \( l = 0, 1, 2, \ldots \) and \( m = 0, \pm 1, \ldots, \pm l, \theta \) and \( \phi \) are the spherical angles fixing the momentum direction.

Let us introduce the new variable \( \rho = p^2/m_i \omega_i \) and the unknown function \( u \), as \( R = e^{-\rho/2} \rho^{l/2} u(\rho) \), where

\[
l' = \frac{1}{2} \left[ (2l+1)^2 - \frac{8E}{m_i \omega_i^2 \lambda^2} \right]^{1/2} - 1,
\]

is the effective orbital quantum number. Note that \(-1/2 \leq l' \leq l\). In these new variables Eq. (3) takes the form,

\[
\rho \frac{d^2 u}{d\rho^2} + \left( \frac{3}{2} + l' - \rho \right) \frac{du}{d\rho} + \left( \frac{E'}{2\hbar \omega_i} - \frac{3}{4} - \frac{l'}{2} \right) u = 0,
\]

where \( E' = E - \frac{\hbar^2}{2m_i \lambda^2} \) is the effective energy.

The solution of Eq. (5) can be expressed in the form (see Eq. (9.216) in Ref. [13]),

\[
u = 1 F_1 \left( -\frac{E'}{2m_i \omega_i} + \frac{l' + 3}{4}, \frac{3}{2}; \rho \right),
\]

where \( 1 F_1 (a, b, z) \) is the Kummer’s confluent hypergeometric function. The hypergeometric function in Eq. (6) should be finite at \( \rho \to \infty \). Thus the energy of an ion should satisfy the relation,

\[
E_{nl} = \hbar \omega_i \left\{ 2n + 1 - \frac{\hbar}{2m_i \omega_i \lambda^2} + \left[ \left( l + \frac{1}{2} \right)^2 - 4 \left( n + \frac{1}{2} \right) \frac{\hbar}{m_i \omega_i \lambda^2} \right]^{1/2} \right\},
\]

where \( n = 0, 1, 2, \ldots \) is the radial quantum number. It should be noted that, at big \( n \), ions oscillations become unstable since the energy in Eq. (6) acquires the imaginary part. Thus we should impose a restriction,

\[
n < n_{cr} = \left( l + \frac{1}{2} \right)^2 \frac{m_i \omega_i \lambda^2}{4\hbar} - \frac{1}{2},
\]

to guarantee the stability of oscillations.

In the following we shall study spherically symmetric plasma oscillations. Thus we should consider wave functions independent of \( \theta \) and \( \phi \). This case corresponds to \( l = 0 \). Using Eq. (6) one finds the properly normalized total wave function, which also includes the spin variables, in the following form:

\[
\psi_{n\sigma}(p) = \frac{2\pi}{1(l' + n + 1/2)} \left( \frac{n!}{\hbar} \right)^{1/2} \left( \frac{\hbar}{m_i \omega_i} \right)^{3/4} \left( \frac{p^2}{m_i \omega_i \hbar} \right)^{l'/2} \times \exp \left[ -\frac{p^2}{2m_i \omega_i \hbar} \right] \frac{l' + 1/2}{2} \left( \frac{p^2}{m_i \omega_i \hbar} \right)^{\chi},
\]
where $L_{n}^{\alpha}(z)$ is the associated Laguerre polynomial, $\Gamma(z)$ is the Euler gamma function, $\chi_{\sigma}$ is the spin wave function, and $\sigma$ is the spin variable. The new expression for $l'$ can be found on the basis of Eq. (4),

$$l' = \frac{1}{2} \left\{ \left[ \left( 1 - \frac{8E}{m_{i}\omega_{i}^{2}\lambda_{i}^{2}} \right)^{1/2} - 1 \right] \right\} .$$

(10)

Note that now $-1/2 \leq l' \leq 0$. The energy levels corresponding to the states described by $\psi_{n\sigma}$ in Eq. (9) have the form,

$$E_{n} = \hbar\omega_{i} \left\{ 2n + 1 - \frac{\hbar}{2m_{i}\omega_{i}\lambda_{i}^{2}} + \left[ \frac{1}{4} - 4 \left( n + \frac{1}{2} \right) \frac{\hbar}{m_{i}\lambda_{i}^{2}} \right]^{1/2} \right\} ,$$

(11)

which can be obtained directly from Eq. (7) by putting $l = 0$ there.

We shall assume that ions are singly ionized. Thus they should be fermions. Indeed, a typical neutral atom has an integer spin. Thus, if we remove one electron, an ion becomes a fermion. For simplicity we shall assume that ions have the lowest possible spin, i.e. they are 1/2-spin particles. It means that $\sigma = \pm 1$ in Eq. (9).

Now the ground state of the system can be constructed. Suppose that we have $N$ non-interacting ions performing ion-acoustic oscillations. On the basis of Eq. (9) we can introduce the operator valued wave function,

$$\hat{\psi} = \sum_{n\sigma} \psi_{n\sigma} \hat{a}_{n\sigma},$$

(12)

and the analogous expression for $\hat{\psi}^{\dagger}$, which contains $\hat{a}_{n\sigma}^{\dagger}$ and $\hat{a}_{n\sigma}$. Here $\hat{a}_{n\sigma}^{\dagger}$ and $\hat{a}_{n\sigma}$ are the creation and annihilation operators of the states corresponding to ions oscillations. Each state can be occupied by no more than two particles. These operators obey the usual anticommutation relation, $[\hat{a}_{n\sigma}, \hat{a}_{n'\sigma'}^{\dagger}]_+ = \delta_{n,n'}\delta_{\sigma,\sigma'}$, with other anticommutators being equal to zero.

The ground state constructed accounts for the dynamical features of the system and its geometric symmetry since all ions are involved in spherically symmetric oscillations. Moreover, this state does not contradict basic quantum mechanics principles. In particular, it does not violate the Pauli principle.

The energy of the ground state can be found as $E_{0} = \langle \hat{H}_{\text{ion}} \rangle$, where

$$\hat{H}_{\text{ion}} = \sum_{n\sigma} E_{n} \hat{a}_{n\sigma}^{\dagger} \hat{a}_{n\sigma},$$

is the Hamiltonian of non-interacting ions and the values of the energy are given in Eq. (11). Note that, since we have a great but limited number of ions involved in oscillations,

$$E_{0} \approx 2 \sum_{n < n_{\text{F}}} E_{n},$$

(14)

where $n_{\text{F}}$ is the number of occupied energy states. We shall call it the Fermi number.

We can define the “size” of a spherically symmetric plasmoid in the momentum space as the last maximum of the function $|\psi_{nF}(p)|^{2}$. This maximum is approximately achieved at the classical turn point.

Note that for the sufficiently short waves the frequency of ion oscillations is constant $\omega = \omega_{i}$, cf. Eq. (2). In the quantum description this limit is equivalent to $\lambda_{e} \gg \sqrt{\hbar/m_{i}\omega_{i}}$. In this case the energy spectrum in Eq. (11) coincides with that corresponding to Langmuir oscillations of ions [10]. Besides the coincidence of the spectra for big $\lambda_{e}$, the wave function in Eq. (9) is also consistent with the result of Ref. [11]. Indeed, taking into account that $l' \rightarrow 0$ at $\lambda_{e} \gg \sqrt{\hbar/m_{i}\omega_{i}}$, we get the following expression for the Fourier transform of the wave function in Eq. (9):

$$\psi_{n}(r) = \int \frac{d^{3}p}{(2\pi\hbar)^{3}} e^{i\vec{p}\cdot\vec{r}} \psi_{n}(p) = \frac{1}{\sqrt{4\pi(2n+1)!}} \left( \frac{m_{i}^{3}\omega_{i}^{3}}{\pi\hbar^{3}} \right)^{1/4} \times r_{0} \exp \left( -\frac{r^{2}}{2r_{0}^{2}} \right) H_{2n+1} \left( \frac{r}{r_{0}} \right) ,$$

(15)

where $H_{n}(z)$ is the Hermite polynomial and $r_{0} = \sqrt{\hbar/m_{i}\omega_{i}}$. One can notice that in Eq. (15) we reproduce the result of Ref. [10] up to the sign factor. The details of the derivation of Eq. (15) are provided in Appendix A (see Eqs. (A1)-(A4)).
III. THE EFFECTIVE INTERACTION OF IONS

In this section we study the effective interaction of ions in a spherically symmetric plasmoid owing to the exchange of virtual acoustic waves. It is demonstrated that this interaction can be attractive.

We consider the situation when the plasma temperature is not so high. It means that a neutral component can be present. In Sec. [X] we will study plasma structures in the NS crust, where a neutral component, consisting of neutrons, is always present. In this case rapidly oscillating ions will collide with neutral particles and generate acoustic waves. If an acoustic wave is coherently absorbed inside the system, it will result in the effective interaction between charged particles. One can expect that this effective interaction is more efficient for ions rather than for electrons.

We shall study oscillating ions in highly excited states with \( n \gg 1 \). Using Eqs. (A20) and (A35), we get that the asymptotic form of the wave functions of such ions in coordinate representation coincides with that in Eq. (15). Therefore, for \( n \gg 1 \), the effect of the spatial dispersion of ions-acoustic waves does not significantly contribute to the form of the ions wave functions.

We shall use the approximation of the contact interaction between ions and neutral particles. It means that the potential of interaction between an ion and a neutral particle has the form, \( K(r - r') = K_0 \delta^3(r - r') \), where \( K_0 \) is the phenomenological constant characterizing the strength of the interaction. In this case the Hamiltonian of the interaction between ions and acoustic waves has the form,

\[
\hat{H}_{\text{int}} = K_0 \int d^3r \hat{\psi}^\dagger(r) \hat{\psi}(r) \hat{n}_1(r),
\]

where \( \hat{n}_1(r) \) is the secondly quantized field corresponding to the perturbation of the neutral particles density. The expression for \( \hat{\psi} \) is given in Eq. (12).

The expression for \( \hat{n}_1(r) \) reads

\[
\hat{n}_1(r) = \int \frac{d^3k}{(2\pi)^{3/2}} \left( \frac{\hbar n_n(0)}{2m_n\omega_k} \right)^{1/2} k f_k \left( \hat{b}_k + \hat{b}_k^\dagger \right),
\]

where \( f_k = \sin(kr)/kr \) is the spherically symmetric solution of the wave equation for acoustic waves, \( \hat{b}_k^\dagger \) and \( \hat{b}_k \) are the creation and annihilation operators for phonons, \( \omega_k \) is the frequency of acoustic oscillations, \( n_n(0) \) is the unperturbed density of neutral particles, and \( m_n \) is the mass of a neutral particle.

We can rewrite Eq. (16) in the form,

\[
\hat{H}_{\text{int}} = \int \frac{d^3k}{(2\pi)^{3/2}} \sum_{n\sigma} D_{ns}(k) \hat{a}_{n\sigma}^\dagger \hat{a}_{s\sigma} \left( \hat{b}_k + \hat{b}_k^\dagger \right),
\]

where \( D_{ns} \) is the matrix element of this interaction. If we study the effective interaction between ions occupying the same energy level, we have for \( D_n = D_{nn} \).

\[
D_n(k) = K_0 \left( \frac{n_n(0) m_n\omega_i}{2m_n\omega_k} \right)^{1/2} \int d^3r \hat{\psi}_n^\dagger(r) \hat{\psi}_n(r) f_k(r)
\]

\[
= \frac{K_0}{4\sqrt{n}} \left( \frac{n_n(0) m_n\omega_i}{2m_n\omega_k} \right)^{1/2} \left[ 1 - \text{sgn}(\xi - 4\sqrt{n}) \right],
\]

where \( \xi = k \sqrt{\hbar/m\omega} \) and \( \text{sgn}(z) \) is the sign function.

After the standard elimination of the acoustic degrees of freedom with help of the canonical transformation,

\[
\hat{H}_{\text{int}} \to \exp \left( -\hat{S} \right) \hat{H}_{\text{int}} \exp \left( \hat{S} \right),
\]

\[
\hat{S} = \int \frac{d^3k}{(2\pi)^{3/2}} \sum_{n\sigma} D_{ns}(k) \hat{a}_{n\sigma}^\dagger \hat{a}_{s\sigma} \left( \frac{\hat{b}_k}{E_s - E_n - \hbar\omega_k} + \frac{\hat{b}_k^\dagger}{E_s - E_n + \hbar\omega_k} \right),
\]

the total Hamiltonian, which includes \( \hat{H}_{\text{ion}} \) given in Eq. (13), takes the form,

\[
\hat{H} = \sum_{n\sigma} E_n \hat{a}_{n\sigma}^\dagger \hat{a}_{n\sigma} - \sum_{nn'\sigma} F_{nn'} \hat{a}_{n\sigma}^\dagger \hat{a}_{n'\sigma}^\dagger \hat{a}_{n'\sigma} \hat{a}_{n\sigma}.
\]
To derive Eq. (21) it is important to assume that two interacting ions are at the same energy level. In Eq. (21) we also account for the fact that these ions must have oppositely directed spins because of the Pauli principle.

The amplitude of the effective interaction $F_{nn'}$ in Eq. (21) has the form,

$$F_{nn'} = \frac{K_0^2 n_n^{(0)} m_i \omega_i}{32 m_n \hbar \sqrt{n n'}} \int \frac{d^3 k}{(2\pi)^3} \frac{1}{\omega_k^3} [1 - \text{sgn}(\xi - 4\sqrt{n})][1 - \text{sgn}(\xi - 4\sqrt{n'})].$$  \hspace{1cm} (22)

Now we should take into account the fact that ions and neutral particles have practically the same mass. Thus the energy transfer in their collisions occurs rather effectively. Therefore we should take that the frequency of virtual acoustic waves in Eq. (22) is close to the frequency of ion-acoustic oscillations given in Eq. (2). After the integration one has the following expression for $F_{nn'}$:

$$F_{nn'} = \frac{4K_0^2 n_n^{(0)} m_i}{3\pi^2 \hbar \omega_i m_n} \left( \frac{m_i \omega_i}{\hbar} \right)^{3/2} \frac{\hat{n}^{3/2}}{\sqrt{n n'}} \left( 1 + \frac{3\hbar \omega_i}{16 m_i c_s^2 \hat{n}} \right),$$  \hspace{1cm} (23)

where $c_s = \lambda_\omega / m_i$ is the sound velocity and $\hat{n} = \min(n, n')$.

Note that $F_{nn'}$ in Eq. (23) is positive. It means that the effective interaction described by the Hamiltonian in Eq. (21) is attractive.

Now let us discuss the ions motion which corresponds to short waves. In this situation ion-acoustic waves are transformed into Langmuir oscillations of ions. As we have seen in Sec. III it happens in the limit $\lambda_\omega \gg \sqrt{\hbar / m_i \omega_i}$, that is equivalent to $c_s \gg \sqrt{\hbar \omega_i / m_i}$. Thus, using Eq. (23), we get that the matrix element of the effective interaction for Langmuir oscillations takes the form,

$$F_{nn'}^{(\text{Lang})} = \frac{4K_0^2 n_n^{(0)} m_i}{3\pi^2 \hbar \omega_i m_n} \left( \frac{m_i \omega_i}{\hbar} \right)^{3/2} \frac{\hat{n}^{3/2}}{\sqrt{n n'}}.$$

Note that Eq. (23) corrects our result obtained in Ref. [10]. The erroneous matrix element was derived in Ref. [10] since the incorrect dispersion relation for virtual acoustic waves was used in the calculation of the integral in Eq. (22).

To complete the analysis of the effective interaction we should express the constant $K_0$ in terms of the cross section $\sigma_s$ of the scattering of ions off neutral particles. We recall that we use the approximation of the contact interaction between ions and neutral particles. Thus, using the Born approximation for the description of the scattering, one has in this limit that $K_0^2 = \pi \hbar^2 \sigma_s / m_i^2$.

IV. PAIRING OF IONS

In this section we study the possibility for the formation of bound states of ions owing to the effective attraction described in Sec. III. Using this effective interaction, we discuss the pairing of protons in a dense matter of NS.

As in Sec. III we shall consider two ions occupying the same energy level. On the basis of Eq. (21) one can see that these ions can form a bound state if the amplitude of the effective interaction exceeds the kinetic energy of an ion,

$$E_n < F_{nn},$$  \hspace{1cm} (25)

where $E_n$ is given in Eq. (11) and $F_{nn}$ in Eq. (23). Introducing the properly defined Bogoliubov transformation, one can show that, if the condition in Eq. (25) is satisfied, the ground state of the system is transformed into the new one corresponding to the lower energy (see Ref. [10]).

Let us first examine the case of short ion-acoustic waves which is equivalent to Langmuir oscillations of ions. Using Eq. (24), one gets that the pairing of ions takes place, i.e. Eq. (25) is fulfilled, when

$$n < n_0 = \frac{4}{9\pi^2} \frac{\hbar m_i}{\omega_i} \left( \frac{n_n^{(0)} \sigma_s}{m_n} \right)^2 - \frac{3}{2}. $$  \hspace{1cm} (26)

Thus there is an upper limit on the number of occupied states.

In Sec. III we defined the Fermi number corresponding to the maximal possible number of occupied states. If one studies short ion-acoustic waves, we can neglect the spatial dispersion. Thus we can define ions wave functions in the coordinate space rather than in the momentum space, cf. Eq. (15). Therefore the Fermi number can be related to
and (27) and correspond to Langmuir oscillations of ions) are small but positive.

In this energy range the cross section of the proton-neutron scattering is approximately constant and equals to which is the upper bound on the plasmoid radius. Eq. (27), one obtains that a plasmoid. Thus the approximation of is mainly composed of neutrons and has proton superconductivity in the NS crust (see, e.g., Ref. [19]).

we shall assume that the proton density Ref. [17]) predict that a certain fraction of protons can be also present in the NS crust. We shall assume that the proton density is now determined by the nonlinear algebraic equation $F_{nn} = E_n$. After we find $n_n^{(s)}$, we can calculate $R_0^{(s)}$ taking that $n_F = n_0^{(s)}$. In Eq. (27) we show the parameters $\delta_n = (n_n^{(s)} - n_0) / n_0$ and $\delta_R = (R_0^{(s)} - R_0) / R_0$, where $n_n^{(s)}$ and $R_0^{(s)}$ are the critical occupation number and the plasmoid radius corresponding to nonzero $\kappa$, versus $\kappa$ for $n_0 \sim 10^3$. As one can see, if we account for the spatial dispersion of ion-acoustic waves, the corrections to $n_0$ and $R_0$ (we recall that $n_0$ and $R_0$ are defined in Eqs. (26) and (27) and correspond to Langmuir oscillations of ions) are small but positive.

We have found that, for $n \gg 1$, there is a very small enhancement of the plasmoid radius if we study long ion-acoustic waves. We obtained such a small effect because we used the ions wave functions in Eq. (15), which correspond to Langmuir oscillations, for the calculation of the matrix elements. Although we have shown that these wave functions are the correct asymptotics for $\psi_n$ at $n \gg 1$, if we consider great but limited values of $n$, we should use the exact $\psi_{n\sigma}(p)$.

FIG. 1. The dependence of the plasmoid parameters versus $\kappa = \hbar / m_\omega \lambda_c^2$ for $n_0 \sim 10^3$. (a) The relative change of the critical occupation number $\delta_n = (n_n^{(s)} - n_0) / n_0$. (b) The relative change of the critical plasmoid radius $\delta_R = (R_0^{(s)} - R_0) / R_0$. Which is the upper bound on the plasmoid radius.

We can expect that the described phenomenon of the ions pairing can happen in a dense matter. Let us consider a spherical plasma structure excited in the inner crust of NS. It should be noted that such a background matter is mainly composed of neutrons and has $n_n^{(0)} = 10^{38}$ cm$^{-3}$. Nevertheless some models of the NS matter (see, e.g., Ref. [17]) predict that a certain fraction of protons can be also present in the NS crust. We shall assume that the proton density $n_p^{(0)} = 10^{36}$ cm$^{-3}$, i.e. it is about 1% of the neutron density. In this case we get that $\omega_i = 10^{21}$ s$^{-1}$.

We shall be interested in the pairing of protons. Protons have energy in the MeV range inside the NS crust. In this energy range the cross section of the proton-neutron scattering is approximately constant and equals to $\sigma_\pi = 2 \times 10^{-23}$ cm$^2$ [18]. Thus the approximation of the contact interaction adopted in our work is valid. Using Eq. (27), one obtains that $R_0 \approx 5.3 \times 10^{-10}$ cm. Note that there are $N = \frac{3}{2}\pi R_0^3 n_p^{(0)} \approx 6.2 \times 10^8$ protons inside such a plasmoid. Thus the approximation of $n \gg 1$, used in our work, is valid.

Our estimate means that protons with oppositely directed spins can form bound states. This phenomenon is similar to the formation of Cooper pairs in metals. This result is in agreement with the fact that there can be a proton superconductivity in the NS crust (see, e.g., Ref. [19]).

Now we study the pairing of ions participating in long ion-acoustic waves. In this situation one should account for the effects of the spatial dispersion. In the quantum description it is equivalent to the consideration of the exact energy spectrum in Eq. (11). For long ion-acoustic waves, we have that $\omega < \omega_i$. Thus we can expect that the effective attraction will be stronger in this case. If we consider the case of slightly decreased frequency of ions oscillations, i.e. $\omega \lesssim \omega_i$, we can still assume that a plasmoid is localized in coordinate space and study the corrections to $n_0$ and $R_0$ defined above. We expect that the new values of $n_0$ and $R_0$ will increase if we account for the spatial dispersion of ion-acoustic waves.

If the ratio $\kappa = \hbar / m_\omega \lambda_c^2$ is small but nonzero, the value of $n_n^{(s)}$ is now determined by the nonlinear algebraic equation $F_{nn} = E_n$. After we find $n_n^{(s)}$, we can calculate $R_0^{(s)}$ taking that $n_F = n_0^{(s)}$. In Fig. 1 we show the parameters $\delta_n = (n_n^{(s)} - n_0) / n_0$ and $\delta_R = (R_0^{(s)} - R_0) / R_0$, where $n_n^{(s)}$ and $R_0^{(s)}$ are the critical occupation number and the plasmoid radius corresponding to nonzero $\kappa$, versus $\kappa$ for $n_0 \sim 10^3$. As one can see, if we account for the spatial dispersion of ion-acoustic waves, the corrections to $n_0$ and $R_0$ (we recall that $n_0$ and $R_0$ are defined in Eqs. (26) and (27) and correspond to Langmuir oscillations of ions) are small but positive.

We have found that, for $n \gg 1$, there is a very small enhancement of the plasmoid radius if we study long ion-acoustic waves. We obtained such a small effect because we used the ions wave functions in Eq. (15), which correspond to Langmuir oscillations, for the calculation of the matrix elements. Although we have shown that these wave functions are the correct asymptotics for $\psi_n$ at $n \gg 1$, if we consider great but limited values of $n$, we should use the exact $\psi_{n\sigma}(p)$.
given in Eq. (9). In this situation the effect of the nonzero spatial dispersion will be more significant. Unfortunately, this case can be analyzed only numerically.

V. CONCLUSION

In conclusion we mention that in the present work we have constructed the model of a spherical quantum plasmoid based on radial oscillations of ions. In our analysis we have suggested that the electron component of plasma is uniformly distributed in space and ions participate in ion-acoustic oscillations.

In Sec. II we have found the exact solution of the Schrödinger equation for an ion moving in the self-consistent field of an ion-acoustic wave. We have shown that in the limiting case of short waves the obtained ions wave functions transform into the previously found solution of the Schrödinger equation for a charged particle performing Langmuir oscillations. The we have secondly quantized our system. The creation and annihilation operators for oscillatory states of ions have been introduced and the ground state of the system has been constructed. We have defined the Fermi number as the maximal possible occupation number. Note that the quantization of the ions motion is justified since in Sec. IV we have considered plasma structures in the very dense matter of NS crust, where quantum effects are significant.

In Sec. III we have studied the effective interaction between oscillating ions by the exchange of a virtual acoustic wave. Discussing the situation of plasmoids containing a very great number of excited states $n \gg 1$, we have shown that two ions occupying the same energy level can attract each other.

The possibility of the formation of bound states of these ions have been analyzed in Sec. IV. Considering the short waves limit, we have derived the critical occupation number and the characteristic plamoid radius corresponding to a plasma structure in which all ions are in bound states. As an application of our results we have discussed the pairing of protons inside a plamoid in the inner crust of NS. We have shown that the existence of such a plasma structure is quite possible.

Since the spins of ions, which formed a bound state, should be antiparallel, this bound state is analogous to a Cooper pair of electrons in a metal. It is known that the formation of Cooper pairs underlies the phenomenon of superconductivity. Our result that protons can form bound states in the NS crust agrees with the hypothesis that the proton superconductivity should be present in this astrophysical environment.

It should be noted that in our work we have adopted rather simplified analysis of the ion’s motion in plasma. A more detailed description of the interaction between a charged particle and a plasma wave in frames of the classical electrodynamics was given in Ref. [21]. Various instabilities, which arise in this system, as well as the dynamics of the turbulence were also studied in Ref. [21] on the classical level.

In Sec. IV we have studied the contribution of the spatial dispersion of ion-acoustic waves to the dynamics of the system. We have obtained that, under the assumption of great number of excited states, this contribution to the critical occupation number and the effective plasmoid radius is small. However, if one studies a plasma structure with a significant but limited $n$, we expect that, e.g., the effective radius can be considerably enhanced.

Although we have examined plasma structures in a very dense medium of the inner crust of NS as a possible application of our results, we may expect that the phenomenon of pairing of charged particles can happen in a terrestrial plasma. Previously the formation of bound states of charged particles was studied in Ref. [20] to describe some properties of stable atmospheric plasma structures. The estimates given in Sec. IV show that our mechanism of pairing cannot be directly implemented inside plasmoids in low density atmospheric plasma since the radius of such a structure turns out to be quite small. Nevertheless, if one discusses a plasma structure a big but limited $n$, there is a possibility that the described phenomenon can take place in a terrestrial plasma as well.

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Appendix A: Wave functions in coordinate representation

In this Appendix we present the mathematical details required to express the ion’s wave function in the coordinate representation.
First we show that at $l' = 0$ the wave function coincides up to a sign factor with that found in Ref. [10]. To derive Eq. (A1) we rewrite the wave function in the coordinate representation as

\[
\psi_n(r) = \psi_n(r) = \left[ \frac{n!}{(l' + n + 3/2)!} \right]^{1/2} \left( \frac{m_i \omega_i}{\hbar} \right)^{3/4} \frac{1}{\pi \beta} \int_0^\infty dx' x'^{l'+1} \sin (\beta x') \exp \left( -\frac{x'^2}{2} \right) L_n^{l'+1/2}(x'),
\]

where account for its spherical symmetry. Here $\beta = r/r_0$. At $l' = 0$, the associated Laguerre polynomial in Eq. (A1) can be expressed through the Hermite polynomial (see Eq. (8.972.3) on page 1001 in Ref. [13])

\[
L_n^{1/2}(x^2) = \frac{(-1)^n}{2^{2n+1} n!} H_{2n+1}(x)/x.
\]

We also represent the gamma function of a half-integer argument in terms of factorials

\[
\Gamma \left( n + \frac{3}{2} \right) = \sqrt{\pi} \frac{[2(n + 1)]!}{4^n n!}.
\]

Using Eq. (A2) and Eq. (7.388.2) on page 806 in Ref. [13] which reads,

\[
\int_0^\infty dx \sin (\beta x) \exp \left( -\frac{x^2}{2} \right) H_{2n+1}(x) = (-1)^n \sqrt{\frac{\pi}{2}} \exp \left( -\frac{\beta^2}{2} \right) H_{2n+1}(\beta),
\]

one can calculate the integral in Eq. (A1). Finally, with help of Eq. (A3) one obtains Eq. (A1).

Now let us derive the asymptotics of the wave function in Eq. (A1) in case when $|l'| \ll 1$. Using the identity

\[
sin (\beta x) = \cos \left( \frac{\pi l'}{2} \right) \sin \left( \beta x - \frac{\pi l'}{2} \right) + \sin \left( \frac{\pi l'}{2} \right) \cos \left( \beta x - \frac{\pi l'}{2} \right),
\]

and the asymptotic of the Bessel function $J_{\nu}(z)$,

\[
J_{\nu+1/2}(z) \approx \sqrt{\frac{2}{\pi z}} \sin \left( z - \frac{\pi \nu}{2} \right) + \cdots,
\]

we can express the integral in Eq. (A1) in the following form:

\[
\int_0^\infty dx' x'^{l'+1} \sin (\beta x') \exp \left( -\frac{x'^2}{2} \right) L_n^{l'+1/2}(x'^2)
\approx \cos \left( \frac{\pi l'}{2} \right) (-1)^n \sqrt{\frac{\pi}{2}} \beta^{l'+1} \exp \left( -\frac{\beta^2}{2} \right) J_{l'+1/2}(\beta^2) + \sin \left( \frac{\pi l'}{2} \right) J,
\]

where

\[
J = \int_0^\infty dx x^{l'+1} \exp \left( -\frac{x^2}{2} \right) L_n^{l'+1/2}(x^2) \cos \left( \beta x - \frac{\pi l'}{2} \right).
\]

To derive Eq. (A7) we use the known value of the integral

\[
\int_0^\infty dx x^{l'+3/2} \exp \left( -\frac{x^2}{2} \right) L_n^{l'+1/2}(x^2) J_{l'+1/2}(\beta x)
= (-1)^n \beta^{l'+1/2} \exp \left( -\frac{\beta^2}{2} \right) L_n^{l'+1/2}(\beta^2),
\]

which is given in Eq. (7.421.4) on page 812 in Ref. [13].

Note that for $|l'| \ll 1$ we can set $l' = 0$ in the argument of cosine and in the upper index of the associated Laguerre polynomial in Eq. (A8) since $J$ is already multiplied by the small factor $\sin(\pi l'/2)$ in Eq. (A7). Thus we rewrite $J$ in the following way:

\[
J \approx \frac{(-1)^n}{2^{2n+1} n!} J', \quad J' = \int_0^\infty dx \exp \left( -\frac{x^2}{2} \right) H_{2n+1}(x) \cos (\beta x),
\]

(A10)
where we use Eq. (A2). To study the asymptotics of $J'$ at $n \gg 1$ we present the Hermite polynomial in Eq. (A10) in the explicit form,

$$H_{2n+1}(x) = (-1)^n 2(2n+1)! \sum_{k=0}^n \frac{(-1)^k 2^k}{(2k+1)! (n-k)!} x^{2k+1}.$$  \hfill (A11)

Then we use the known value of the integral,

$$\int_0^\infty dx \exp \left(-\frac{x^2}{2}\right) x^{2k+1} \cos(\beta x) = 2^k k! F_1 \left( k+1; \frac{1}{2}; -\frac{\beta^2}{2} \right),$$  \hfill (A12)

which can be found in Eq. (3.952.8) on page 503 in Ref. [13].

It should be noted that $1 F_1 \left( k+1; 1/2; -\beta^2/2 \right)$ is not a polynomial. Thus we can represent it in terms of the infinite series

$$1 F_1 \left( k+1; \frac{1}{2}; -\frac{\beta^2}{2} \right) = \sum_{s=0}^\infty (-1)^s \frac{(k+1)_s}{(1/2)_s s!} \left(\frac{\beta^2}{2}\right)^s,$$  \hfill (A13)

where $(z)_s = z(z+1)\cdots(z+s-1)$ is the Pochhammer symbol. Using the fact that

$$\sum_{k=0}^n \frac{(-1)^k k! 2^{3k}}{(2k+1)! (n-k)!} (k+1)_s = \frac{s!}{n!} \sum_{k=0}^\infty \frac{(-1)^k \beta^2}{2^k \Gamma(k+1/2)^2} F_1 \left( -n, s+1; \frac{3}{2}; 2 \right),$$  \hfill (A14)

where $2 F_1 \left( a, b; c; z \right)$ is the Gauss hypergeometric function and $(1/2)_s = \Gamma(s+1/2)/\sqrt{\pi}$, as well as Eqs. (A11)-(A14), we obtain $J'$ in the form,

$$J' = (-1)^n 2 \sqrt{\pi} \left(\frac{2n+1}{n!}\right) \sum_{k=0}^\infty \frac{(-1)^k \beta^2}{2^k \Gamma(k+1/2)^2} F_1 \left( -n, k+1; \frac{3}{2}; 2 \right).$$  \hfill (A15)

Note that one can study the limit $n \to \infty$ only in Eq. (A15).

To find the asymptotics of the Gauss hypergeometric function we use the following expression:

$$2 F_1 \left( a, b; c; z \right) = \frac{\Gamma(c) \Gamma(b-a)}{\Gamma(b) \Gamma(c-a)} (-1)^a z^{-a} 2 F_1 \left( a, a+1-c; a+1-b; \frac{1}{z} \right) + \frac{\Gamma(c) \Gamma(a-b)}{\Gamma(a) \Gamma(c-b)} (-1)^b z^{-b} 2 F_1 \left( b, b+1-c; b+1-a; \frac{1}{z} \right).$$  \hfill (A16)

One should mention that there is an ambiguity like $\Gamma(-n-k-1)/\Gamma(-n)$ in the second term in the rhs of Eq. (A16). Nevertheless we can treat it using the expression for the gamma function when its argument is close to a negative integer,

$$\Gamma(x) = \frac{(-1)^n}{n!} \frac{1}{x+n} + \cdots, \quad x \approx -n.$$  \hfill (A17)

With help of Eq. (A17) we obtain that

$$2 F_1 \left( -n, k+1; \frac{3}{2}; 2 \right) = \sqrt{\pi n!} \left(\frac{n+k+1}{k!} \Gamma(n+3/2)\right) (-1)^n 2^{n-1} 2 F_1 \left( -n, -n - \frac{1}{2}; -n-k; \frac{1}{2} \right)$$

$$- \frac{\sqrt{\pi n!}}{(n+k+1)! \Gamma(1/2-k)} (-1)^n 2^{-k-2}$$

$$\times 2 F_1 \left( k+1, k+\frac{1}{2}; n+k+2; \frac{1}{2} \right).$$  \hfill (A18)

Taking into account that at $n \to \infty$ one has that

$$2 F_1 \left( -n, -n - \frac{1}{2}; -n-k; \frac{1}{2} \right) \sim 2^{-n+k-1/2},$$

$$2 F_1 \left( k+1, k+\frac{1}{2}; n+k+2; \frac{1}{2} \right) \to 1,$$  \hfill (A19)
and

\[ \Gamma(n + k + 1) \sim \Gamma \left( n + \frac{3}{2} \right) \left( n + \frac{3}{2} \right)^{k-1/2} (n + k + 1)! \sim n! n^{k+2}, \quad \text{(A20)} \]

we obtain the following asymptotics:

\[ \binom{2}{F_1} \binom{-n, k + 1; 3}{2; \frac{3}{2}} \sim (-1)^n \frac{2^{k-1}}{k!} \sqrt{\frac{n}{2}} \left( n + \frac{3}{2} \right)^{k-1/2}, \quad \text{(A21)} \]

for the Gauss hypergeometric function.

It is interesting to compare Eq. (A21) with Eq. (2.7) in Ref. [22], where the asymptotics of the Gauss hypergeometric function was also studied. The result of Ref. [22] reads

\[ \binom{2}{F_1} \binom{-n, k + 1; 3}{2; \frac{3}{2}} \sim (-1)^n \frac{2^{k-1}}{k!} \sqrt{\frac{n}{2}} (n + 1)^{k-1/2}. \quad \text{(A22)} \]

We can see that besides the unessential difference, \( n + 3/2 \) vs. \( n + 1 \), which unimportant at big \( n \), our result coincides with that of Ref. [22]. However, it was claimed in Ref. [22] that the asymptotics in question is valid when the argument of the hypergeometric function \( z > 2 \). Here we demonstrate that the case \( z = 2 \) can be also described by Eq. (A21) or (A22) at least for the particular set of the parameters of the hypergeometric function.

The asymptotic behavior of the hypergeometric function in Eq. (A21) is shown in Fig. 2(a). We depict there the sequence

\[ y_n = (-1)^n \sqrt{\frac{2}{\pi}} \frac{k!}{2^{k-3/2}} \binom{2}{F_1} \binom{-n, k + 1; 3/2}{2; \frac{3}{2}} \frac{1}{(n + 3/2)^{k-1/2}}, \quad \text{(A23)} \]

for different values of \( k \). One can see in Fig. 2(a) that \( y_n \to 1 \). It proves the validity of the asymptotics in Eq. (A21).

On the basis of Eqs. (A11) - (A21) we obtain the behavior of \( J' \) as

\[ J' \sim \sqrt{2} 2^{2n} n! \cos \left( 2 \sqrt{n + 3/2} \beta \right), \quad \text{(A24)} \]

where we use the value of the sum of the series,

\[ \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(k + 1/2)} \left( \sqrt{n + 3/2} \beta \right)^{2k} = \frac{1}{\sqrt{\pi}} \cos \left( 2 \sqrt{n + 3/2} \beta \right), \quad \text{(A25)} \]

and the fact that at \( n \gg 1 \) one has \( (2n + 1)! = 2^{2n+1} n! \Gamma(n + 3/2) / \sqrt{\pi} \approx 2^{2n+1} n^2 \sqrt{n + 1} / \sqrt{\pi} \).
Using Eqs. (A7), (A10), and (A24), one gets the asymptotics of the wave function in Eq. (A1), which corresponds to the case \(|l'| \ll 1\) and \(n \gg 1\),

\[
\psi_n(r) \approx \frac{(-1)^n}{\sqrt{2\pi3n^{1/4}}} \left( \frac{m_i\omega_l}{\hbar} \right)^{3/4}
\times \left\{ \sin \left(2\sqrt{n}\beta \right) - \frac{\pi l'}{2} \left[ \cos \left(2\sqrt{n}\beta \right) - \cos \left(2\sqrt{n+3/2}\beta \right) \right] \right\}
\times \frac{(-1)^n}{\sqrt{2\pi3n^{1/4}}} \left( \frac{m_i\omega_l}{\hbar} \right)^{3/4} \sin \left(2\sqrt{n}\beta \right),
\]

(A26)

where we keep only the leading term in \(l'\) and use the Perron’s approximation for the associated Laguerre polynomial (see Eq. (8.978.3) on page 1003 in Ref. [13]),

\[
L_n^{l'+1/2}(\beta^2) \approx \frac{1}{\sqrt{\pi}} \beta^{-l'-1} n^{l'/2} \exp \left( \frac{\beta^2}{2} \right) \sin \left(2\sqrt{n}\beta - \frac{\pi l'}{2} \right),
\]

(A27)

One can see in Eq. (A26) that the correction for the wave function, linear in \(l'\), vanishes in the limit \(n \gg 1\).

Now let us consider another extreme situation which corresponds to \(l' = -1/2\). On the basis of Eq. (A1) one can see that in this case it is necessary to get the asymptotics at big \(n\) of the following integral:

\[
J = \int_0^{\infty} dx \sqrt{x} \sin (\beta x) \exp \left( -\frac{x^2}{2} \right) L_n(x^2),
\]

(A28)

where \(L_n(z) = L_0^n(z)\) is the Laguerre polynomial. We can analyze \(J\) in Eq. (A28) in the same manner as \(J'\) in Eq. (A10). We shall describe only the main steps of this analysis.

First, we represent the Laguerre polynomial in the explicit form,

\[
L_n(x^2) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{x^{2k}}{k!},
\]

(A29)

where \(\binom{n}{k} = \frac{n!}{(n-k)!k!}\) is the binomial coefficient. Then we use the known value of the integral (see Eq. (3.952.7) on page 503 in Ref. [13]),

\[
\int_0^{\infty} dx x^{2k+1/2} \sin (\beta x) \exp \left( -\frac{x^2}{2} \right) = \beta^{2k+1/4} \Gamma \left( k + \frac{5}{4} \right) \frac{\beta}{n} \left( \frac{3}{2} \right) F_2 \left( -n, s + \frac{9}{4}; 1, \frac{3}{2} \right),
\]

(A30)

and the sum of the series,

\[
\sum_{k=0}^{n} \frac{(-1)^k 2^k}{k!^2 (n-k)!} \left( k + \frac{9}{4} \right) s = \frac{4}{5} \Gamma \left( \frac{3}{2} \right) F_2 \left( -n, s + \frac{9}{4}; 1, \frac{3}{2} \right),
\]

(A31)

where \(\Gamma_a(b, c; z)\) is the generalized hypergeometric function. On the basis of Eqs. (A29)-(A31), we get the expression for \(J\) in the form,

\[
J = \frac{2^{5/4}}{5} \sqrt{\pi} \beta \sum_{k=0}^{n} \frac{(-1)^k \beta^{2k}}{2^{k}k!^2 \Gamma(k+3/2)} \frac{\Gamma(k+9/4)}{\Gamma(k+3/2)} \frac{\beta^{2k+1/4}}{n} \left( \frac{3}{2} \right) F_2 \left( -n, k + \frac{9}{4}; 1, \frac{3}{2} \right).\]

(A32)

Here we also use the analog of Eq. (A13) for \(\beta^{2k+1/4} \Gamma(k+9/4)\).

The hypergeometric function in Eq. (A32) has the following behavior at \(n \gg 1\):

\[
\left( -n, k + \frac{9}{4}, \frac{5}{4}; 1, \frac{3}{2} \right) \sim 5 (n+1)^{1/4} \left( n + \frac{9}{4} \right)^k \frac{(-1)^n 2^{k-7/4}}{\Gamma(k+9/4)}.\]

(A33)

To illustrate the asymptotics of the hypergeometric function in Eq. (A33), in Fig. (21b) we present the sequence

\[
y_n = \frac{(-1)^n \Gamma(k+9/4)}{2^{k-7/4} 5 (n+1)^{1/4} (n+9/4)^k} \left( -n, k + 9/4, 1, 9/4; 2 \right).\]

(A34)
for different values of \( k \). One can see in Fig. 2(b) that \( y_n \to 1 \), as it follows from Eq. \( \text{A33} \).

Finally, using Eqs. \( \text{A32} \) and \( \text{A33} \) one obtains the expression for the wave function at \( n \gg 1 \),

\[
\psi_n(r) \approx \frac{(-1)^n}{\sqrt{2\pi\beta}} \frac{(m_i \omega_i)}{h} \frac{3/4}{(n + 1)^{1/4}} \frac{(n + 9/4)^{1/2}}{\pi} \sin \left(2\sqrt{n\beta}\right)
\]

\[
\sim \frac{(-1)^n}{\sqrt{2\pi\beta n^{1/4}}} \frac{(m_i \omega_i)}{h} \sin \left(2\sqrt{n\beta}\right).
\]

To derive Eq. \( \text{A35} \) we use the known value for the sum of the series,

\[
\sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k + 3/2)} \left(\sqrt{n + 9/4}\beta\right)^{2k} = \frac{\sin(2\sqrt{n + 9/4}\beta)}{\sqrt{\pi\beta}\sqrt{n + 9/4}}.
\]

One can see in Eq. \( \text{A35} \) that, in the limit \( n \gg 1 \), the expression for the wave function corresponding to \( l' = -1/2 \) again coincides with the analogous expression for \( l' = 0 \).

To explicitly demonstrate that Eq. \( \text{A20} \) or \( \text{A35} \) corresponds to Eq. \( \text{15} \) one should use Perron’s approximation for the Hermite polynomial (see Eq. (8.955.2) on page 997 in Ref. \( \text{13} \)),

\[
H_{2n+1}(z) \sim (-1)^n 2^{n+1/2}(2n-1)!\sqrt{2n+1} \exp \left(\frac{z^2}{2}\right) \sin \left(2\sqrt{n}z\right),
\]

which is valid for big \( n \).

At the end of this section we mention that as a by-product have obtained the asymptotic expression for the Hilbert transform of the Hermite function with the odd index,

\[
\varphi_{2n+1}(z) = \frac{1}{\pi^{1/4}\sqrt{2^n n!}} \exp \left(\frac{-z^2}{2}\right) H_{2n+1}(z).
\]

The Hilbert transform \( H[f](\beta) \) of the function \( f(x) \) is defined as

\[
H[f](\beta) = \frac{1}{\pi} \text{V.P.} \int_{-\infty}^{+\infty} \frac{f(x)}{\beta - x} dx,
\]

where V.P. stays for the principle value of the integral.

Indeed, we can rewrite \( J' \) in Eq. \( \text{A10} \) as

\[
J' = \frac{1}{2} \int_{-\infty}^{+\infty} dx \exp \left(\frac{-x^2}{2}\right) \text{sgn}(x) H_{2n+1}(x) \cos(\beta x)
\]

\[
= \frac{(-1)^n}{2\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx \left[ \frac{1}{x + \beta - i0} + \frac{1}{x - \beta - i0} \right] \exp \left(\frac{-x^2}{2}\right) H_{2n+1}(x),
\]

where we use Eq. \( \text{A4} \) and the integral representation of the signum function,

\[
\text{sgn}(x) = \frac{1}{\pi} \int_{-\infty}^{+\infty} dt \frac{\sin(xt)}{t - i0}.
\]

Then, using the symbolic identity

\[
\frac{1}{x \pm \beta - i0} = \text{V.P.} \frac{1}{x \pm \beta} + i\pi \delta(x \pm \beta),
\]

we can express \( J' \) as

\[
J' = \frac{(-1)^n}{\sqrt{2\pi}} \text{V.P.} \int_{-\infty}^{+\infty} dt \exp \left(\frac{-x^2/2}{\beta - x}\right) H_{2n+1}(x).
\]

Note that the imaginary part, initially present in Eq. \( \text{A40} \), is washed out from Eq. \( \text{A43} \). Finally, using Eqs. \( \text{A24} \), \( \text{A38} \), \( \text{A39} \), and \( \text{A3} \), we obtain the following asymptotics of the Hilbert transform of the Hermite function:

\[
H[\varphi_{2n+1}](\beta) \sim (-1)^n \frac{2^{3n/2+1} \sqrt{n!}}{\pi^{3/4}} \cos \left(2\sqrt{n + 3/2}\beta\right).
\]
It should be noted that previously only the recurrence relation for \(H[\varphi_{2n+1}]\) was known (see, e.g., Ref. [23]), which can be used only for small \(n\). We have derived the explicit expression for the asymptotics of the Hilbert transform. Therefore, Eq. (A44) can be useful, for instance, in signal processing.

[1] M. Nambu, S. V. Vladimirov, and P. K. Shukla, Phys. Lett. A 203, 40 (1995).
[2] G. E. Morfill and A. V. Ivlev, Rev. Mod. Phys. 81, 1353 (2009).
[3] J. Carstensen, et al., Phys. Plasmas 15, 055704 (2012).
[4] M. Bonitz, et al., Phys. Plasmas 15, 055704 (2008), arXiv:0801.0754 [physics.plasm-ph].
[5] C. C. Grimes and G. Adams, Phys. Rev. Lett. 42, 795 (1979).
[6] G. Birk, S. Kassner, and H. Walther, Nature 357, 310 (1992).
[7] J. H. Chu and L. I, Phys. Rev. Lett. 72, 4009 (1994).
[8] D. A. Baiko, Phys. Rev. E 80, 046405 (2009), arXiv:0910.0171 [astro-ph.HE].
[9] A. A. Vlasov and M. A. Yakovlev, Theor. Math. Phys. 34, 124 (1978).
[10] M. Dvornikov, J. Phys. A: Math. Theor. 46, 045501 (2013), arXiv:1208.2208 [physics.plasm-ph].
[11] F. Haas, Quantum Plasmas: An Hydrodynamic Approach (New York, Springer, 2011).
[12] M. Bonitz, E. Pehlke, and T. Schoof, Phys. Rev. E 87, 033105 (2013), arXiv:1205.4922 [physics.plasm-ph].
[13] I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products, 7th ed. (Amsterdam, Elsevier, 2007).
[14] D. I. Blokhintsev, Quantum Mechanics (Dordrecht, Reidel, 1964), pp. 140–141.
[15] E. M. Lifshitz and L. P. Pitaevskii, Physical Kinetics (Burlington, MA, Elsevier, 2010) pp. 136–137.
[16] M. Dvornikov and S. Dvornikov, in Advances in Plasma Physics Research, vol. 5, ed. by F. Gerard (New York, NY, Nova Science Publishers, Inc., 2006), pp. 197–212, physics/0306157.
[17] M. Hempel, T. Fischer, J. Schaffner-Bielich, and M. Liebendorfer, Astrophys. J. 748, 70 (2012), arXiv:1108.0848 [astro-ph.HE].
[18] M. B. Chadwick, et al., Nucl. Data Sheets 107, 2931 (2006).
[19] N. Chamel and P. Haensel, Living Rev. Relativity 11, 10 (2008), arXiv:0812.3955 [astro-ph].
[20] G. C. Dijkhuis, Nature 284, 150 (1980); M. I. Zelikin, J. Math. Sci. 151, 3473 (2008); M. Dvornikov, Proc. R. Soc. A 468, 415 (2012), arXiv:1102.0944 [physics.plasm-ph]; A. V. Shavlov and V. A. Dzhumandzhii, Phys. Lett. A 377, 3131 (2013).
[21] R. Z. Sagdeev and A. A. Galeev, Nonlinear Plasma Theory (New York, NY, W. A. Benjamin, Inc., 1969) pp. 37–113.
[22] N. M. Temme, Constr. Approx. 2, 369 (1986).
[23] S. L. Hahn, Hilbert transform, in The Transforms and Applications Handbook, ed. by A. D. Poularikas, 2nd ed. (Boca Raton, FL, CRC Press), ch. 7.10.