Wigner Rotations and Iwasawa Decompositions in Polarization Optics

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Abstract

Wigner rotations and Iwasawa decompositions are manifestations of the internal space-time symmetries of massive and massless particles, respectively. It is shown to be possible to produce combinations of optical filters which exhibit transformations corresponding to Wigner rotations and Iwasawa decompositions. This is possible because the combined effects of rotation, phase-shift, and attenuation filters lead to transformation matrices of the six-parameter Lorentz group applicable to Jones vectors and Stokes parameters for polarized light waves. The symmetry transformations in special relativity lead to a set of experiments which can be performed in optics laboratories.

1 Introduction

In our earlier papers [1, 2], we have formulated the Jones vectors and Stokes parameters in terms of the two-by-two and four-by-four matrix representations of the six-parameter Lorentz group [3]. It was seen there that, to every two-by-two transformation matrix for the Jones vector, there is a corresponding four-by-four matrix for the Stokes parameters. It was found also that the Stokes parameters are like the components of Minkowskian four-vectors, and two-component Jones vectors are like two-component spinors in the relativistic world. This enhances our capacity to approach polarization optics in terms of the kinematics of special relativity.

Indeed, we can now design specific experiments which will test some of the consequences derivable from the principles of special relativity. The most widely known example is the Wigner rotation. This has been extensively discussed in

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the literature in connection with the Thomas effect [4], Berry’s phase [5, 6], and squeezed states of light [7].

In our earlier papers, we discussed an optical filter which will exhibit the matrix form of

\[
\begin{pmatrix}
1 & u \\
0 & 1
\end{pmatrix}
\]  \hspace{1cm} (1)

applicable to two transverse components of the light wave, where \( u \) is a controllable parameter. When applied to a two-component system, this matrix performs a superposition in the upper channel while leaving the low channel invariant. The question is whether it is possible to produce optical filters with this property.

In Ref. [1], we approached this problem in terms of the generators of the Lorentz group. It is very difficult, if not impossible, to manufacture optical devices performing the function of group generators. In the case of optical filters, this means an infinite number of layers of zero thickness. In the present paper, we deal with the same problem from an experimental point of view. We will present a specific design for optical filters performing this function. We will of course present our case in terms of a combination of three filters of finite thickness.

In order to achieve this goal, we use the fact that polarization optics and special relativity share the same mathematics. This aspect was already noted in the literature for the case of the Wigner rotation [6]. The concept of the Wigner rotation comes from the kinematics of special relativity, in which two successive non-collinear Lorentz boosts do not end up with a boost. The result is a boost followed or preceded by a rotation. Thus we can achieve a rotation from three non-collinear boosts starting from a particle at rest. Since each boost corresponds to an attenuation filter, it requires three attenuation filters to achieve a Wigner rotation in polarization optics.

While the Wigner rotation is based on Lorentz transformations of massive particles, there are similar transformations for massless particles. Here, two non-collinear Lorentz boosts do not result in one boost. They become one boost preceded or followed by a transformation which corresponds to a gauge transformation. In two-by-two formalism, the transformation takes the form of Eq. (1). We shall show in this paper that the filter possessing the property of Eq. (1) can be constructed from one rotation filter and one attenuation filter. In mathematics, this type of decomposition is called the Iwasawa decomposition [8, 9].

While the primary purpose of this paper is to discuss filters and their combinations in polarization optics, we provide also concrete illustrative examples of Wigner’s little group [10]. The little group is the maximal subgroup of the Lorentz group whose transformations leave the four-momentum of a given particle invariant, and has a long history [11]. The Wigner rotation and the Iwasawa decomposition are transformations of the little groups for massive and massless particles respectively. It is interesting to note that these transformations can also be achieved in optics laboratories.
In Sec. 2, we review the formalism for optical filters based on the Lorentz group and explain why filters are like Lorentz transformations. It is shown in Sec. 3, that a rotation can be achieved by three non-collinear Lorentz boosts. In Sec. 4, we spell out in detail how the Iwasawa decomposition can be achieved from the combination of two optical filters.

2 Formulation of the Problem

In studying polarized light propagating along the $z$ direction, the traditional approach is to consider the $x$ and $y$ components of the electric fields. Their amplitude ratio and the phase difference determine the degree of polarization. Thus, we can change the polarization either by adjusting the amplitudes, by changing the relative phases, or both. For convenience, we call the optical device which changes amplitudes an “attenuator” and the device which changes the relative phase a “phase shifter.”

Let us write these electric fields as

$$\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} A \exp \{ i(kz - \omega t + \phi_1) \} \\ B \exp \{ i(kz - \omega t + \phi_2) \} \end{pmatrix},$$

(2)

where $A$ and $B$ are amplitudes which are real and positive numbers, and $\phi_1$ and $\phi_2$ are the phases of the $x$ and $y$ components respectively. This column matrix is called the Jones vector. In dealing with light waves, we have to realize that the intensity is the quantity we measure. Then there arises the question of coherence and time average. We are thus led to consider the following parameters.

$$S_{11} = < E_x^* E_x >, \quad S_{22} = < E_y^* E_y >,$n
$$S_{12} = < E_x^* E_y >, \quad S_{21} = < E_y^* E_x >.$$

(3)

Then, we are naturally invited to write down the two-by-two matrix:

$$C = \begin{pmatrix}< E_x^* E_x >/ < E_y^* E_y > \\ < E_x^* E_y >/ < E_y^* E_y > \end{pmatrix},$$

(4)

where $< E_i^* E_j >$ is the time average of $E_i^* E_j$. The above form is called the coherency matrix [12].

It is sometimes more convenient to use the following combinations of parameters.

$$S_0 = S_{11} + S_{22},$$

$$S_1 = S_{11} - S_{22},$$

$$S_2 = S_{12} + S_{21},$$

$$S_3 = -i (S_{12} - S_{21}).$$

(5)

These four parameters are called the Stokes parameters in the literature [12].

We showed in our earlier papers that the Jones vectors and the Stokes parameters can be formulated in terms of the two-by-two spinor and four-by-four
vector representations of the Lorentz group. This group theoretical formalism allows to discuss three different sets of physical quantities using one mathematical device. In our earlier publications, we used the concept of Lie groups extensively and used their generators based on infinitesimal generators.

In this paper, we avoid the Lie groups and work only with explicit transformation matrices. For this purpose, we start with the following two matrices.

\[
B = \begin{pmatrix}
\cosh \chi & \sinh \chi & 0 & 0 \\
\sinh \chi & \cosh \chi & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

\[
R = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \phi & -\sin \phi & 0 \\
0 & \sin \phi & \cos \phi & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

(6)

If the above matrices are applied to the Minkowskian space of \((ct, z, x, y)\), the matrix \(B\) performs a Lorentz boost:

\[
t' = (\cosh \chi)t + (\sinh \chi)z,
\]

\[
z' = (\sinh \chi)t + (\cosh \chi)z,
\]

(7)

while \(R\) leads to a rotation:

\[
z' = (\cos \phi)z - (\sin \phi)x,
\]

\[
x' = (\sin \phi)z + (\cos \phi)x.
\]

(8)

In our previous paper, we discussed in detail what these matrices do when they are applied to the Stokes four-vectors.

In the two-component spinor space, the above transformation matrices take the form

\[
\begin{pmatrix}
e^{\chi/2} & 0 \\0 & e^{-\chi/2}
\end{pmatrix},
\]

\[
\begin{pmatrix}
\cos(\phi/2) & -\sin(\phi/2) \\\sin(\phi/2) & \cos(\phi/2)
\end{pmatrix}.
\]

(9)

We discussed the effect of these matrices on the Jones spinors in our earlier publications.

In this paper, we discuss some of nontrivial consequences derivable from the algebra generated by these two sets of matrices. We shall study Wigner rotations and Iwasawa decompositions. The Wigner rotation has been discussed in optical science in connection with Berry’s phase, but the Iwasawa decomposition is a relatively new word in optics. We would like to emphasize here that both the Wigner rotation and Iwasawa decomposition come from the concept of subgroup of the Lorentz groups whose transformations leave the momentum of a given particle invariant.
3 Wigner Rotations

There are many different versions of the Wigner rotation in the literature. Basically, this rotation is a product of two non-collinear Lorentz boosts. The result of these two boosts is not a boost, but a boost preceded or followed by a rotation. This rotation is called the Wigner rotation.

Figure 1: Closed Lorentz boosts. Initially, a massive particle is at rest with its four momentum \( P_a \). The first boost \( B_1 \) brings \( P_a \) to \( P_b \). The second boost \( B_2 \) transforms \( P_b \) to \( P_c \). The third boost \( B_3 \) brings \( P_c \) back to \( P_a \). The particle is again at rest. The net effect is a rotation around the axis perpendicular to the plane containing these three transformations. We may assume for convenience that \( P_b \) is along the \( z \) axis, and \( P_c \) in the \( zx \) plane. The rotation is then made around the \( y \) axis.

In this paper, we approach the problem by using three boosts described in Fig. 1. Let us start with a particle at rest, with its four momentum

\[
P_a = (m, 0, 0, 0),
\]

where we use the metric convention \((ct, z, x, y)\). Let us next boost this four-momentum along the \( z \) direction using the matrix

\[
B_1 = \begin{pmatrix}
\cosh \eta & \sinh \eta & 0 & 0 \\
\sinh \eta & \cosh \eta & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

resulting in the four-momentum

\[
P_b = m (\cosh \eta, \sinh \eta, 0, 0).
\]

Let us rotate this vector around the \( y \) axis by an angle \( \theta \). Then the resulting four-momentum is

\[
P_c = m \left( \cosh \eta, (\sinh \eta) \cos \theta, (\sinh \eta) \sin \theta, 0 \right).
\]
Instead of this rotation, we propose to obtain this four-vector by boosting the
four-momentum of Eq.(12). The boost matrix in this case is

\[
B_2 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \psi & -\sin \psi & 0 \\
0 & \sin \psi & \cos \psi & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
\cosh \lambda & \sinh \lambda & 0 & 0 \\
\sinh \lambda & \cosh \lambda & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

with

\[
\lambda = 2 \tanh^{-1} \{[\sin(\theta/2)] \tanh \eta\}, \quad \psi = \frac{\theta}{2} + \frac{\pi}{2}. \tag{15}
\]

If we carry out the matrix multiplication, the \(B_2\) matrix becomes

\[
\begin{pmatrix}
\cosh \lambda & -\sin(\theta/2) \sinh \lambda & \cos(\theta/2) \sinh \lambda & 0 \\
-\sin(\theta/2) \sinh \lambda & 1 + \sin^2(\theta/2)(\cosh \lambda - 1) & -\sin \theta \sinh^2(\lambda/2) & 0 \\
\cos(\theta/2) \sinh \lambda & -\sin \theta \sinh^2(\lambda/2) & 1 + \cos^2(\theta/2)(\cosh \lambda - 1) & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}. \tag{16}
\]

Next, we boost the four-momentum of Eq.(13) to that of Eq.(10). The particle
is again at rest. The boost matrix is

\[
B_3 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \theta & -\sin \theta & 0 \\
0 & \sin \theta & \cos \theta & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
\cosh \eta & -\sinh \eta & 0 & 0 \\
-\sinh \eta & \cosh \eta & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

with

\[
B_3 = \begin{pmatrix}
\cosh \eta & -\cos \theta \sinh \eta & -\sin \theta \sinh \eta & 0 \\
-\cos \theta \sinh \eta & 1 + \cos^2(\theta/2)(\cosh \eta - 1) & \sin \theta \cos(\cosh \eta - 1) & 0 \\
-\sin \theta \sinh \eta & \sin \theta \cos(\cosh \eta - 1) & 1 + \sin^2(\theta/2)(\cosh \eta - 1) & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}. \tag{18}
\]

The net result of these transformations is \(B_3B_2B_1\). This leaves the initial
four-momentum of Eq.(10) invariant. Is it going to be an identity matrix? The
answer is No. The result of the matrix multiplications is

\[
W = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \Omega & -\sin \Omega & 0 \\
0 & \sin \Omega & \cos \Omega & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \tag{19}
\]
with
\[
\Omega = 2 \sin^{-1} \left\{ \frac{(\sin \theta) \sinh^2(\eta/2)}{\sqrt{\cosh^2 \eta - \sinh^2 \eta \sin^2(\theta/2)}} \right\}.
\]  

This matrix performs a rotation around the $y$ axis and leaves the four-momentum of Eq.\,(10) invariant. This rotation is an element of Wigner’s little group whose transformations leave the four-momentum invariant. This is precisely the Wigner rotation.

This relativistic effect manifests itself in atomic spectra as the Thomas precession. Otherwise, the experiments on Wigner rotation in special relativity is largely academic. On the other hand, as noted in the literature, this effect could be tested in optics laboratories. As for the Stokes parameters, the above four-by-four matrices are directly applicable. Indeed, each four-by-four matrix corresponds to one optical filter applicable to polarized light.

In order to see this effect more clearly, let us use the Jones matrix formalism. The two-by-two squeeze matrix corresponding to the boost matrix $B_1$ of Eq.\,(11) is
\[
S_1 = \begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix}.
\]  

The two-by-two squeeze matrix corresponding to the boost matrix of Eq.\,(14) is now
\[
S_2 = \begin{pmatrix} \cos(\psi/2) & -\sin(\psi/2) \\ \sin(\psi/2) & \cos(\psi/2) \end{pmatrix} \begin{pmatrix} e^{\lambda/2} & 0 \\ 0 & e^{-\lambda/2} \end{pmatrix} \begin{pmatrix} \cos(\psi/2) & \sin(\psi/2) \\ -\sin(\psi/2) & \cos(\psi/2) \end{pmatrix},
\]  

where the parameters $\psi$ and $\lambda$ are given in Eq.\,(15). After the matrix multiplication, $S_2$ becomes
\[
S_2 = \begin{pmatrix} \cosh(\lambda/2) - \sin(\theta/2) \sinh(\lambda/2) & \cos(\theta/2) \sinh(\lambda/2) \\ \cos(\theta/2) \sinh(\lambda/2) & \cosh(\lambda/2) + \sin(\theta/2) \sinh(\lambda/2) \end{pmatrix}.
\]  

This is a matrix which squeezes along the direction which makes the angle $(\pi + \theta)/2$ with the $z$ axis. The two-by-two squeeze matrix corresponding to $B_3$ of Eq.\,(17) is
\[
S_3 = \begin{pmatrix} \cosh(\eta/2) - \cos \theta \sinh(\eta/2) & -\sin \theta \sinh(\eta/2) \\ -\sin \theta \sinh(\eta/2) & \cosh(\eta/2) + \cos \theta \sinh(\eta/2) \end{pmatrix}.
\]  

Now the matrix multiplication $S_3 S_2 S_1$ corresponds to the closure of the kinematical triangle given in Fig. 1. The result is
\[
S_3 S_2 S_1 = \begin{pmatrix} \cos(\Omega/2) & -\sin(\Omega/2) \\ \sin(\Omega/2) & \cos(\Omega/2) \end{pmatrix},
\]  

where $\Omega$ is given in Eq.\,(20).
4 Iwasawa Decompositions

In Sec. 3 the Lorentz kinematics was based on a massive particle at rest. If the particle is massless, there are no Lorentz frames in which the particle is at rest. Thus, we start with a massless particle whose momentum is in the $z$ direction:

$$K_a = (k, k, 0, 0), \quad (26)$$

where $k$ is the magnitude of the momentum. We can rotate this four-vector to

$$K_b = (k, -k \sin \alpha, k \cos \alpha, 0) \quad (27)$$

by applying to $K_a$ the rotation matrix

$$R_+ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \alpha_+ & -\sin \alpha_+ & 0 \\ 0 & \sin \alpha_+ & \cos \alpha_+ & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (28)$$

with $\alpha_+ = \alpha + \pi/2$.

If we rotate $K_b$ around the $y$ axis by $-2\alpha$, the resulting four-momentum will be

$$K_c = (k, k \sin \alpha, k \cos \alpha, 0). \quad (29)$$

It is possible to transform $K_b$ to $K_c$ by applying to $K_b$ the boost matrix

$$B = \begin{pmatrix} \cosh \gamma & \sinh \gamma & 0 & 0 \\ \cosh \gamma & \sinh \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (30)$$

with

$$\sinh \gamma = \frac{2 \sin \alpha}{\cos^2 \alpha}, \quad \cosh \gamma = \frac{1 + \sin^2 \alpha}{\cos^2 \alpha}. \quad (31)$$

We can transform $K_c$ to $K_a$ by rotating it around the $y$ axis by $(\alpha - \pi/2)$. The rotation matrix takes the form

$$R_- = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \alpha_- & -\sin \alpha_- & 0 \\ 0 & \sin \alpha_- & \cos \alpha_- & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (32)$$

with $\alpha_- = \alpha - \pi/2$. Thus, the multiplication of the three matrices, $R_- BR_+$, gives

$$T = \begin{pmatrix} 1 + u^2/2 & -u^2/2 & u & 0 \\ u^2/2 & 1 - u^2/2 & u & 0 \\ u & -u & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (33)$$

with

$$u = -2 \tan \alpha.$$
Figure 2: Two rotations and one Lorentz boost which preserve the four-momentum of a massless particle invariant. The four-momentum $K_a$ is rotated to $K_b$ by $R_+$. It is then boosted to $K_c$ by the boost matrix $B$. The rotation matrix $R_-$ brings back the four-momentum to $K_a$. The initial momentum is along the $z$ direction, and the boost $B$ is also made along the same direction. The rotations are performed around the $y$ axis.

This $T$ matrix plays an important role in studying space-time symmetries of massless particles. If this matrix is applied to the four-momentum $K_a$ given in Eq. (26), the four-momentum remains invariant. If this matrix is applied to the electromagnetic four-potential for the plane wave propagating along the $z$ direction with the frequency $k$, the result is a gauge transformation.

Again, the above four-by-four matrices are directly applicable to the Stokes parameters. On the other hand, if we are interested in designing optical filters, we need two-by-two representations corresponding to the four-by-four matrices given so far. The two-by-two squeeze matrix corresponding to the boost matrix $B$ of Eq. (30) is

$$S = \begin{pmatrix} e^{\gamma/2} & 0 \\ 0 & e^{-\gamma/2} \end{pmatrix}, \quad (34)$$

while the two-by-two matrices corresponding to $R_+$ of Eq. (28) and $R_-$ of Eq. (32) are

$$R_\pm = \begin{pmatrix} \cos(\alpha_\pm/2) & -\sin(\alpha_\pm/2) \\ \sin(\alpha_\pm/2) & \cos(\alpha_\pm/2) \end{pmatrix}, \quad (35)$$

where $\alpha_+$ and $\alpha_-$ are given in Eq. (28) and Eq. (32) respectively. They satisfy
the equations
\[ \alpha_+ + \alpha_- = 2\alpha, \quad \alpha_+ - \alpha_- = \pi. \]
The relation between \(\gamma\) and \(\alpha\) given in Eq. (31) can also be written as
\[ \cosh(\gamma/2) = 1/\cos\alpha, \]
which is more useful for carrying out the two-by-two matrix algebra.

The matrix multiplication \(R_- SR_+\) leads to
\[ T = R_- SR_+ = \begin{pmatrix} 1 & -2 \tan \alpha \\ 0 & 1 \end{pmatrix}. \] (36)

Conversely, we can write the
\[ \begin{pmatrix} 1 & -2 \tan \alpha \\ 0 & 1 \end{pmatrix} = R_- SR_+. \] (37)

The \(T\) matrix can be decomposed into rotation and squeeze matrices. This possibility is called the Iwasawa decomposition. In the present case, \(T\) of Eq. (36) can also be written as
\[ T = R_- S \left\{ (R_-)^{-1} R_- \right\} R_+ = \left\{ R_- S (R_-)^{-1} \right\} (R_- R_+). \] (38)

The matrix chain \(R_- S (R_-)^{-1}\) is one squeeze matrix whose squeeze axis is rotated by \(\alpha_-/2\), and the matrix product \(R_- R_+\) becomes one rotation matrix. The result is
\[ T = S(\alpha_-) R(2\alpha), \] (39)
with
\[ S(\alpha_-) = \begin{pmatrix} \cosh(\gamma/2) + \cos \alpha_- \sinh(\gamma/2) & \sin \alpha_- \sinh(\gamma/2) \\ \sin \alpha_- \sinh(\gamma/2) & \cosh(\gamma/2) - \cos \alpha_- \sinh(\gamma/2) \end{pmatrix}, \]
\[ R(2\alpha) = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}. \] (40)

It is indeed gratifying to note that the \(T\) matrix can be decomposed into one rotation and one squeeze matrix. The squeeze is made along the direction which makes an angle of \(\alpha_-/2\) or \(- (\pi/2 - \alpha)/2\) with the \(z\) axis. The angle \(\alpha\) is smaller than \(\pi/2\).

In our earlier papers [12], we have discussed optical filters with the property given in Eq. (36). We said there that filters with this property can be produced from an infinite number of infinitely thin filters. This argument was based on the theory of Lie groups where transformations are generated by infinitesimal generators. This may be possible these days, but the method presented in this paper is far more practical. We need only two filters [13].

We are able to achieve this improvement because we used here the analogy between polarization optics and Lorentz transformations which share the same mathematical framework.
Concluding Remarks

In this paper, we noted first that both the Wigner rotation and the Iwasawa decomposition come from Wigner’s little group whose transformations leave the four-momentum of a given particle invariant. Since the Lorentz group is applicable also to the Jones vector and the Stokes parameters, it is possible to construct corresponding transformations in polarization optics. We have shown that both the Wigner rotation and the Iwasawa decomposition can be realized in optics laboratories.

The matrix of Eq. (1) performs a shear transformation when applied to a two-dimensional object, and has a long history in physics and engineering. It also has a history in mathematics. The fact that a shear can be decomposed into a squeeze and rotations is known as the Iwasawa decomposition [8].

Among the many interesting applications of shear transformations, there is a special class of squeezed states of photons or phonons having the symmetry of shear [14, 15]. The wave-packet spread can be formulated in terms of shear transformations [10].

As we can see from this paper, a set of shear transformations can be formulated as a subset of Lorentz transformations. This set plays an important role in understanding internal space-time symmetry of massless particles, such as gauge transformation and neutrino polarizations [11, 17, 18].

Acknowledgments

We would like to thank A. E. Bak for bringing to our attention to his early works with C. S. Brown on applications of the Lorentz group to polarization optics. We are also grateful to S. Baskal for telling us about the recent paper by Simon and Mukunda on the Iwasawa decomposition [9].

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