Components and Exit Times of Brownian Motion in Two or More $p$-Adic Dimensions

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Abstract
The fundamental solution to a pseudo-differential equation for functions defined on the $d$-fold product of the $p$-adic numbers, $\mathbb{Q}_p$, induces an analogue of the Wiener process in $\mathbb{Q}_p^d$. As in the real setting, the components are 1-dimensional $p$-adic Brownian motions with the same diffusion constant and exponent as the original process. Asymptotic analysis of the conditional probabilities shows that the vector components are dependent for all time. Exit time probabilities for the higher dimensional processes reveal a concrete effect of the component dependency.

Keywords Fractional Brownian motion · $p$-Adic diffusion · Component dependence · Exit times

1 Introduction

Two main ideas initially motivated the study of non-Archimedean diffusion: the idea that non-Archimedean physical models describe the observed ultrametricity in certain complex systems, and the idea that the extremely small scale structure of spacetime could be non-Archimedean. Ultrametric structures in spin glasses were already implicit in Parisi’s investigations [29, 30]. Volovich proposed the idea that spacetime could have a non-Archimedean structure at small enough distance and time scales [34] and he initiated a program to study analogues of physical theories with $p$-adic state spaces for this reason. The study of diffusion in non-Archimedean local fields goes...
back more than 30 years, and the study of diffusion processes in vector spaces over such fields goes back nearly as far. Seminal works in this area include the work of Kochubei [25] and the work of Albeverio and Karwowski [1]. Kochubei gave the fundamental solution to the $p$-adic analogue of the diffusion equation, developed a theory of $p$-adic diffusion equations, and proved a Feynman–Kac formula for the operator semigroup with a $p$-adic Schrödinger operator as its infinitesimal generator [25]. Albeverio and Karwowski constructed a continuous time random walk on $Q_p$, computed its transition semigroup and infinitesimal generator, and showed among other things that the associated Dirichlet form is of jump type [1].

For any finite dimensional vector space $S$ with coefficients in a division algebra that is finite dimensional over a non-Archimedean local field of arbitrary characteristic, Varadarajan constructed [33] a general class of diffusion processes with sample paths in the Skorohod space $D([0, \infty) : S)$ of càdlàg paths that take values in $S$. The current work follows this approach of Varadarajan and takes it as a starting point, but specializes to the setting where, for any prime number $p$ and any natural number $d$, paths take values in the $d$-fold Cartesian product of the $p$-adic numbers, $Q_p^d$. The results in the more general setting [33] specialize to show that there is a triple $(D([0, \infty) : Q_p^d), P^d, \mathcal{X})$ so that $P^d$ is a probability measure on $D([0, \infty) : Q_p^d)$ and, for any $\omega$ in $D([0, \infty) : Q_p^d)$ and any positive $t$,

$$\mathcal{X}(t, \omega) = \omega(t).$$

Furthermore, the probability measure on $Q_p^d$ that for any Borel set $B$ of $Q_p^d$ is given by

$$B \mapsto P^d(\mathcal{X}(t, \omega) \in B)$$

has a density function that is a solution to a pseudo-differential equation that is analogous to the real diffusion equation. Refer to any process of this type as a Brownian motion in $Q_p^d$.

For any Brownian motion $\mathcal{X}$ in $Q_p^d$ with sample paths in $(D([0, \infty) : Q_p^d), P^d)$, the current paper establishes that the component processes of $\mathcal{X}$ are each Brownian motions in $Q_p$ with the same parameters (diffusion constant and exponent) as $\mathcal{X}$, and that for no positive real $t$ are the components of $\mathcal{X}_t$ independent. Section 2 briefly reviews some necessary results in $p$-adic analysis, discusses the max-norm process that is a special case of the more general process that Varadarajan discusses [33], and determines the infinitesimal generator of the product process in $Q_p^d$. Section 3 studies the component processes of the max-norm process. The main results are Theorems 3.1 and 3.5. Theorem 3.1 establishes that the component processes are Brownian motions in $Q_p$. Theorem 3.5 gives precise estimates on certain conditional probabilities that establish the dependency of the component processes. The effect of the dependency of the components becomes strikingly apparent in the calculation in Sect. 4 of the exit probabilities for the product and max-norm process that generalize Weisbart’s earlier result [36, Theorem 3.1].
Dragovich, Khrennikov, Kozyrev, Volovich, and Zelenov give a detailed review of the history of research in non-Archimedean mathematical physics [16] that updates the earlier review [15] by the first four authors. This review helps to put the current paper in context. In their book [23], Khrennikov, Kozyrev, and Zúñiga-Galindo discuss recent developments in and applications of the study of ultrametric pseudodifferential equations. Both their book and the references they provide present many areas of study where the current paper could be useful. The works [4–6] of Avetisov, Bikulov, Kozyrev, and Osipov that deal with $p$-adic models for complex systems seem to be particularly relevant to this current paper, as is the work [3] of Avetisov and Bikulov that involves biological applications. Ultrametricity also appears in data structures, as Bradley recently discussed [11]. The current paper may find application in the study of data structures and, in particular, in the recent work of Bradley, Keller, and Weinmann [12], Bradley [13], and Bradley and Jahn [14]. Khrennikov and Kochubei recently studied a $p$-adic analogue of the porous medium equation [24]. Their subsequent work with Antoniouk [2] more generally investigates pseudo-differential equations that involve radially symmetric operators of a type that Vladimirov, Volovich, and Zelenov first introduced [35]. These operators generalize the operator that gives rise to the max-norm process with state space $Q_{p}^{d}$. The interest of their work demonstrates that the current study merits generalization to a broader class of diffusion equations, even in the setting of a $Q_{p}^{d}$ state space.

2 The Norm and Product Processes

See Gouvêa’s book [21] for an accessible supplement to the cursory review of $p$-adic analysis that the current section presents. For more detail, see the book [31] of Ramakrishnan and Valenza, and Weil’s book [37]. Vladimirov, Volovich, and Zelenov give a self-contained introduction to $p$-adic analysis and mathematical physics in their now classic book [35]. Zúñiga-Galindo’s recent book [38] is an accessible and current reference that, among other things, investigates diffusion processes of a type that includes the max-norm processes that appear shortly.

2.1 Basic Analysis in $Q_{p}^{d}$

For any prime $p$, the field of $p$-adic numbers, $Q_{p}$, is the analytic completion of the rational numbers, $Q$, with respect to the metric on $Q$ that the $p$-adic absolute value induces. This metric is an ultrametric on $Q_{p}$ and gives $Q_{p}$, with its additive operation, the structure of a locally compact, totally disconnected, Abelian group. For any natural number $d$ and any $d$-tuple $(x_1, \ldots, x_d)$ in $Q_{p}^{d}$, write

$$\vec{x} = (x_1, \ldots, x_d).$$

Denote by $| \cdot |$ the $p$-adic absolute value on $Q_{p}$ and by $\| \cdot \|$ the max-norm on $Q_{p}$ that takes any $\vec{x}$ in $Q_{p}^{d}$ to $\|\vec{x}\|$, where

$$\|\vec{x}\| = \max_{i \in \{1, \ldots, d\}} |x_i|.$$
The max-norm induces an ultrametric on \( \mathbb{Q}_d^p \). The general linear group in \( d \) dimensions with coefficients in \( \mathbb{Z}/p\mathbb{Z} \), \( GL_d(\mathbb{Z}/p\mathbb{Z}) \), is the maximal compact subgroup of the \( p \)-adic general linear group in \( d \) dimensions, \( GL_d(\mathbb{Q}_p) \). Since the real orthogonal group in \( d \) dimensions, \( O_d(\mathbb{R}) \), is the maximal compact subgroup of the real general linear group \( GL_d(\mathbb{R}) \), the group \( GL_d(\mathbb{Z}/p\mathbb{Z}) \) is the natural analogue in the \( \mathbb{Q}_p^d \) setting of the real orthogonal group \( O_d(\mathbb{R}) \). The max-norm is \( GL_d(\mathbb{Z}/p\mathbb{Z}) \) invariant, and so the max-norm is a natural analogue in the \( \mathbb{Q}_p^d \) setting of the Euclidean norm on \( \mathbb{R}^d \).

For each \( \vec{x} \) in \( \mathbb{Q}_d^p \), take \( B_d(k, \vec{x}) \) and \( S_d(k, \vec{x}) \) to be the compact open sets
\[
B_d(k, \vec{x}) = \{ \vec{y} \in \mathbb{Q}_d^p : \| \vec{y} - \vec{x} \| \leq p^k \} \text{ and } S_d(k, \vec{x}) = \{ \vec{y} \in \mathbb{Q}_d^p : \| \vec{y} - \vec{x} \| = p^k \},
\]
the ball and the circle of radius \( p^k \) with center at \( \vec{x} \), respectively. The unit ball in \( \mathbb{Q}_p^d \) is the \( d \)-fold Cartesian product, \( \mathbb{Z}_p^d \), of the ring of integers in \( \mathbb{Q}_p \). Compress notation by denoting, respectively, by \( B_d(k) \) and \( S_d(k) \) the ball and circle in \( \mathbb{Q}_p^d \) with center \( \vec{0} \) and radius \( p^k \). Suppression of \( d \) in the notation for balls and circles indicates that \( d \) is equal to 1.

### 2.2 Structural Features of \( \mathbb{Q}_p^d \)

A categorical approach to the study of \( \mathbb{Q}_p \) vividly captures certain structural features that are significant to Fourier analysis in the \( \mathbb{Q}_p \) setting. It seems worthwhile to sketch the main ideas here without going into the proofs.

The set of non-decreasing ordered pairs \((i, j)\) of natural numbers, \( \mathbb{N}^{2}_{\text{ord}} \), is the set of morphisms in the category \( \text{NatOrd} \), whose objects are natural numbers. Identify, respectively, the first and second place of an ordered pair as the source and target of a morphism. Transitivity of \( \leq \) determines the composition of morphisms, namely,
\[
(i, k) \circ (j, k) = (i, j).
\]

Take \( \text{LCAb} \) to be the category of locally compact Abelian groups (LCA groups). Take \( \psi \) to be the contravariant functor from \( \text{NatOrd} \) to \( \text{LCAb} \) that takes each object \( i \) to the object \( \mathbb{Z}/p^i\mathbb{Z} \), and each morphism \((i, j)\) to the epimorphism \( \psi_{i,j} \) from \( \mathbb{Z}/p^i\mathbb{Z} \) to \( \mathbb{Z}/p^j\mathbb{Z} \) that is given by
\[
\psi_{i,j} : z \mod p^j \mapsto z \mod p^i.
\]

The projective limit \( \lim \longrightarrow \mathbb{Z}/p^i\mathbb{Z} \) of the projective system \( \psi \) ambiguously means both a universal cone over \( \psi \) and the LCA group that is the common source of all morphisms of the cone. The common source may be taken as the definition of \( \mathbb{Z}_p \), and then one may use the existence of the Haar measure on \( \mathbb{Z}_p \) and the scaling properties of the measure to define a metric on \( \mathbb{Z}_p \) that is compatible with its topology. This approach offers a construction of \( \mathbb{Z}_p \) that is, in a sense, independent of the analytical construction. The field \( \mathbb{Q}_p \) may then be defined to be \( \mathbb{Z}_p[p^{-1}] \).
Take $\phi$ to be the functor from $\text{NatOrd}$ to $\text{LCAb}$ that takes each object $i$ to the object $\mathbb{Z}/p^i\mathbb{Z}$ and each morphism $(i, j)$ to the monomorphism $\phi_{i, j}$ from $\mathbb{Z}/p^i\mathbb{Z}$ to $\mathbb{Z}/p^j\mathbb{Z}$ by

$$\phi: z \mod p^i \mapsto p^{j-i}z \mod p^j.$$ 

The direct limit $\lim\rightarrow \mathbb{Z}/p^i\mathbb{Z}$ of the directed system $\phi$ ambiguously means both a universal co-cone over $\phi$ and the LCA group that is the common target of all morphisms of the co-cone. The common target is isomorphic to the Prüfer $p$-group $\mathbb{Z}(p^\infty)$, the discrete group of all $p^n$th roots of unity, where $n$ varies in $\mathbb{N}$. It may be helpful to clarify that for each $i$ the image of each $\mathbb{Z}/p^i\mathbb{Z}$ in the direct limit is isomorphic to the group of $p^i$th roots of unity. Any element $x_i$ in $\mathbb{Z}/p^i\mathbb{Z}$ is given for some $z$ in $\mathbb{Z}$ by

$$x_i = z \mod p^i.$$ 

Use this representation of $x_i$ to identify $x_i$ with the root of unity $\exp(2\pi z p^{-i} \sqrt{-1})$. This identification is independent of representative and $\mathbb{Z}(p^\infty)$ is isomorphic in $\text{LCAb}$ to $\mathbb{Q}_p/\mathbb{Z}_p$.

Identify the circle group, $S^1$, as the multiplicative group of unit elements of $\mathbb{C}$. For any object $G$ in $\text{LCAb}$, compress notation by writing $\widehat{G}$ rather than $\text{Hom}(G, S^1)$, so that $\widehat{G}$ is the group of continuous, additive characters of $G$. Refer to each element of $\widehat{G}$ as a character of $G$. For each $i$, 

$$\widehat{\mathbb{Z}/p^i\mathbb{Z}} \cong \mathbb{Z}/p^i\mathbb{Z}, \text{ and } \psi_{i, j} = \text{Hom}(\phi_{i, j}, S^1),$$

so the functor $\text{Hom}(-, S^1)$ takes the directed system $\phi$ to the projective system $\psi$. The functor $\text{Hom}(-, S^1)$ takes the directed limit to the projective limit, so

$$\widehat{\mathbb{Z}_p} \cong \mathbb{Q}_p/\mathbb{Z}_p \quad \text{and} \quad \mathbb{Z}_p \cong \mathbb{Q}_p/\mathbb{Z}_p.$$

Take $i_0$ and $q_0$ to be, respectively, the inclusion and projection maps

$$i_0: \mathbb{Z}_p \to \mathbb{Q}_p \quad \text{and} \quad q_0: \mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p,$$

so that $\text{Hom}(-, S^1)$ induces these mappings between exact sequences:

$$0 \to \mathbb{Z}_p \xrightarrow{i_0} \mathbb{Q}_p \xrightarrow{q_0} \mathbb{Q}_p/\mathbb{Z}_p \to 0$$

The category $\text{LCAb}$ is not an Abelian category, which complicates an otherwise straightforward appeal to the Five Lemma to justify that the morphism $\alpha$ from $\mathbb{Q}_p$ to $\mathbb{Q}_p$ is an isomorphism. Much work has involved the study of the structure of LCA
groups [26] and use of homological algebra in the LCAb setting [27]. Among many other things, Borceux and Clementino show [10] that the Short Five Lemma holds in the setting of Abelian topological groups. The Short Five Lemma is all that is needed to justify that $\mathbb{Q}_p$ is Pontryagin self-dual.

For any non-trivial character $\chi$ of $\mathbb{Q}_p$ and any $x$ in $\mathbb{Q}_p$, the morphism

$$x \mapsto (\chi(x \cdot) : y \mapsto \chi(xy)) \quad (\forall y \in \mathbb{Q}_p)$$

is an isomorphism from $\mathbb{Q}_p$ to $\hat{\mathbb{Q}}_p$. Take $(\mathbb{Q}_p^d)^*$ to be the vector space dual to $\mathbb{Q}_p$. For any vector space isomorphism $\eta$ from $\mathbb{Q}_p^d$ to $(\mathbb{Q}_p^d)^*$ that takes an element $\bar{x}$ of $\mathbb{Q}_p^d$ to an element $\eta\bar{x}$ of $(\mathbb{Q}_p^d)^*$, take $\chi_\eta$ to be the morphism in LCAb from $\mathbb{Q}_p^d$ to $\hat{\mathbb{Q}}_p^d$ that is defined for each $\bar{x}$ in $\mathbb{Q}_p^d$ by

$$\chi_\eta(\bar{x}, \cdot) : \mathbb{Q}_p^d \to S^1 \quad \text{by} \quad \chi_\eta(\bar{x}, \bar{y}) = \chi(\eta\bar{x}(\bar{y})).$$

Linearity of $\eta$ and linearity for any $\bar{z}$ in $\mathbb{Q}_p^d$ of $\eta\bar{z}$ together with the additivity and nontriviality of $\chi$ imply that $\chi_\eta$ is an isomorphism, and so $\mathbb{Q}_p^d$ and $\hat{\mathbb{Q}}_p^d$ are isomorphic. The specific choices for $\eta$ and $\chi$ in the next subsection are not unique, but they simplify the calculations that follow.

### 2.3 Fourier Analysis in $\mathbb{Q}_p^d$

For any $x$ in $\mathbb{Q}_p$, there is a unique function

$$a_x : \mathbb{Z} \to \{0, 1, \ldots, p - 1\}$$

with the property that

$$x = \sum_{k \in \mathbb{Z}} a_x(k) p^k.$$

Denote by $\{x\}$ the fractional part of $x$, the sum

$$\{x\} = \sum_{k < 0} a_x(k) p^k. \quad (2)$$

For any $x$ in $\mathbb{Q}_p$, the support of $a_x$ is bounded below and so the sum that defines $\{x\}$ in (2) is a finite sum. Take $\chi$ to be the additive character of $\mathbb{Q}_p$ that is given by

$$\chi(x) = e^{2\pi \sqrt{-1} \{x\}}.$$

For each $\bar{x}$ in $\mathbb{Q}_p$, take $\eta$ to be the isomorphism from $\mathbb{Q}_p^d$ to $(\mathbb{Q}_p^d)^*$ that is given by

$$(\eta(\bar{x}))(\bar{y}) = \bar{x} \cdot \bar{y} = x_1 y_1 + \cdots + x_d y_d.$$
This choice of isomorphism between $\mathbb{Q}_p^d$ and its vector space dual is equivalent to a choice of basis for $\mathbb{Q}_p^d$. With these choices of $\chi$ and $\eta$, the function $\chi_\eta$ is a measure preserving isomorphism in $\text{LCA}_b$ from $\mathbb{Q}_p^d$ to $\hat{\mathbb{Q}}_p^d$. Take $\mu_d$ to be the unique Haar measure on $\mathbb{Q}_p^d$ for which $\mathbb{Z}_p^d$ has unit measure. For any integer $k$, the substitution formula for $p$-adic integrals implies that $\mu_1(B(k))$ is equal to $p^k$. The measure $\mu_d$ is a product measure and $B_d(k)$ is the $d$-fold Cartesian product of the ball $B(k)$. Furthermore, the circle $S(k)$ is the set $B(k) \setminus B(k-1)$. Translation invariance of $\mu_d$ implies that for any $\vec{x}$ in $\mathbb{Q}_p^d$,

$$\mu_d(B_d(k, \vec{x})) = p^{kd} \quad \text{and} \quad \mu_d(S_d(k, \vec{x})) = p^{kd} - p^{(k-1)d} = p^{kd} \left(1 - \frac{1}{p^d}\right).$$

(3)

Initially define $F_d$ and $F_d^{-1}$ on $L^1(\mathbb{Q}_p^d)$ by

$$(F_d f)(\vec{x}) = \int_{\mathbb{Q}_p^d} \chi(-\vec{x} \cdot \vec{y}) f(\vec{y}) d\mu_d(\vec{y}) \quad \text{and}$$

$$(F_d^{-1} f)(\vec{x}) = \int_{\mathbb{Q}_p^d} \chi(\vec{x} \cdot \vec{y}) f(\vec{y}) d\mu_d(\vec{y}).$$

These operators are unitary on $L^1(\mathbb{Q}_p^d) \cap L^2(\mathbb{Q}_p^d)$ and extend to unitary operators on $L^2(\mathbb{Q}_p^d)$. The extensions of these operators, again denoted by $F_d$ and $F_d^{-1}$, are the Fourier and inverse Fourier transforms on $L^2(\mathbb{Q}_p^d)$, respectively. To simplify notation, suppress the measure in the notation for integrals by writing $d\vec{x}$ to mean $d\mu_d(\vec{x})$. Furthermore, write $dx$ to mean $d\mu_1(x)$ in the case when $d$ is equal to 1 and the integral is taken over a subset of $\mathbb{Q}_p$.

The following lemma is helpful for performing calculations that involve the integration of characters.

**Lemma 2.1** For any $m$ and $n$ in $\mathbb{Z}$,

$$\int_{B_d(n)} \chi(\vec{x} \cdot \vec{y}) \, d\vec{x} \, d\vec{y} = p^d(n + \min(-n, m)).$$

**Proof** Since the character $\chi$ is identically equal to 1 on $\mathbb{Z}_p$ and the sum of the $p^{th}$ roots of unity is equal to 0,

$$\int_{B(n)} \chi(x) \, dx = \begin{cases} p^n & \text{if } p^n \leq 1 \\ 0 & \text{if } p^n > 1. \end{cases}$$

(4)

For any $y$ in $\mathbb{Q}_p$, the substitution formula for $p$-adic integration implies that

$$\int_{B(n)} \chi(xy) \, dx = \begin{cases} p^n & \text{if } p^n \leq \frac{1}{|y|} \\ 0 & \text{if } p^n > \frac{1}{|y|}. \end{cases}$$

(5)
For any subset $S$ of $\mathbb{Q}_p^d$, take $\mathbb{1}_S$ to be the indicator function on $S$. Equation 5 implies that

$$\int_{B_d(n)} \chi(\vec{x} \cdot \vec{y}) \, d\vec{x} = p^{dn} \mathbb{1}_{B_d(-n)}(\vec{y}),$$

and so

$$\int_{B_d(m)} \int_{B_d(n)} \chi(\vec{x} \cdot \vec{y}) \, d\vec{x} \, d\vec{y} = \int_{B_d(-m)} p^{dn} \mathbb{1}_{B_d(-n)}(\vec{y}) \, d\vec{y} = p^{d(n+\min(-n,m))}. \tag{6}$$

\[\square\]

### 2.4 A Space of Test Functions

Defining an unbounded self-adjoint operator $H$ on an LCA group $G$ with a Haar measure $\mu$ typically involves taking the graph closure of an operator $H'$ that is initially defined on a subspace of $L^2(G, \mu)$, a space of test functions on which $H'$ is easier to define and on which $H'$ is essentially self-adjoint. For the present work, it is enough to take the space of test functions to be the space $SB(\mathbb{Q}_p^d)$, the Schwartz–Bruhat space of complex valued, compactly supported, locally constant functions on $\mathbb{Q}_p^d$. In some ways, this is a $\mathbb{Q}_p^d$ analogue of the space of complex valued, compactly supported, smooth functions on $\mathbb{R}^d$, but unlike its real analogue, $SB(\mathbb{Q}_p^d)$ is closed under the Fourier transform. From this perspective, all distributions on $L^2(\mathbb{Q}_p^d)$ are tempered. The space of Schwartz–Bruhat functions features prominently in the discrete and finite approximation of diffusion processes [7, 9] and quantum systems [8], and is natural in these settings. Osborne introduced and studied a space of functions that includes the Schwartz–Bruhat functions but that is defined by a rapid decay property that is more similar to that of the Schwartz functions than the compact support of the Schwartz–Bruhat functions [28]. He also identified a condition that characterizes the Schwartz–Bruhat functions within this space.

There are drawbacks to identifying the space of Schwartz–Bruhat functions as the space of test functions, just as there are drawbacks to using the Schwartz space as a space of test functions in the study of distributions and distribution solutions to differential and pseudo-differential equations. Reiter deeply investigated the connection between the metaplectic group and algebras of functions known as Segal algebras [32]. He further discussed the importance of the special Segal algebra that Feichtinger introduced [17, 20]. This Feichtinger algebra is, in a particular sense, a minimal Segal Algebra [19] and has many interesting and important applications. Jakobsen’s excellent review [22] of the area covers many interesting developments. It is important to note that the Feichtinger algebra is a Banach space rather than just a Frechét space [18, 19], and so the tempered distributions that have the Feichtinger algebra as their test functions also form a Banach space.
2.5 The Max-Norm Process

For any positive real number $b$, take $\mathcal{M}_b$ to be the multiplication operator that acts on $SB(\mathbb{Q}_p^d)$ by

$$(\mathcal{M}_b f)(\bar{x}) = \|\bar{x}\|^b f(\bar{x}).$$

Take $\Delta_{b,d}$ to be the self-adjoint closure of the densely defined, essentially self-adjoint operator that acts on $SB(\mathbb{Q}_p^d)$ by

$$(\Delta_{b,d} f)(\bar{x}) = (\mathcal{F}_d^{-1} \mathcal{M}_b \mathcal{F}_d f)(\bar{x}). \quad (7)$$

Extend $\Delta_{b,d}$ to act on a complex valued function $f$ on $\mathbb{R}_+ \times \mathbb{Q}_p^d$ by currying variables, as in [36]. This is to say that if for each $t$ in $\mathbb{R}_+$ the function $f(t, \cdot)$ is in the domain of $\Delta_{b,d}$, then $f$ is in the domain of the extended operator, again denoted by $\Delta_{b,d}$, and that

$$(\Delta_{b,d} f)(t, \bar{x}) := (\Delta_{b,d} f(t, \cdot))(\bar{x}).$$

This extension is the Taibleson–Vladimirov operator with exponent $b$. To simplify notation, suppress $d$ to denote that $d$ is equal to 1. Again follow [36] by currying variables to extend the Fourier and inverse Fourier transforms to act on functions on $\mathbb{R}_+ \times \mathbb{Q}_p^d$. For any positive real number $\sigma$, the pseudo-differential equation

$$\frac{df(t, \bar{x})}{dt} = -\sigma \Delta_{b,d} f(t, \bar{x}) \quad (8)$$

is a $d$-dimensional $p$-adic diffusion equation and has fundamental solution $\rho_d$, where for each $t$ in $(0, \infty)$ and for each $\bar{x}$ in $\mathbb{Q}_p^d$,

$$\rho_d(t, \bar{x}) = (\mathcal{F}_d^{-1} e^{-\sigma t \|\cdot\|_p^b})(\bar{x}).$$

Once again, suppress $d$ in the notation for $\rho_d$ when $d$ is equal to 1.

With the necessary modifications to include the diffusion constant $\sigma$, follow the arguments in [33] to see that $\rho_d(t, \cdot)$ is a probability density function that gives rise to a probability measure $P^d$ on $D([0, \infty) : \mathbb{Q}_p^d)$ that is concentrated on the set of paths originating at 0. The inclusion of a diffusion constant that is not equal to 1 amounts to a rescaling of the real time parameter. The max-norm process is the stochastic process $\left(D([0, \infty), P^d, \bar{X})\right)$, where for any pair $(t, \omega)$ in $[0, \infty) \times D([0, \infty) : \mathbb{Q}_p^d),$

$$\bar{X}(t, \omega) = \omega(t).$$

This stochastic process specializes the process that Varadarajan constructed in [33] to the setting of $\mathbb{Q}_p^d$. To compress notation, refer to $\bar{X}$ ambiguously as both the process...
and the function $\tilde{X}$ that is defined on $(D((0, \infty), P^d))$. Furthermore, if $d$ is equal to 1, then write $X$ rather than $\tilde{X}$. The process $X$ is the Brownian motion in $Q_p$ with diffusion constant equal to $\sigma$ and diffusion exponent equal to $b$.

Denote by $\tilde{X}_t$ the random variable $\tilde{X}(t, \cdot)$, whose density function $\rho_d(t, \cdot)$ satisfies the equality

$$\rho_d(t, \tilde{x}) = \int_{Q_p} \chi(\tilde{x} \cdot \tilde{y}) e^{-\sigma t \|\tilde{y}\|^b} \, d\tilde{y}$$

$$= \sum_{r \in \mathbb{Z}} e^{-\sigma tp^{rb}} \int_{S_d(r)} \chi(\tilde{x} \cdot \tilde{y}) \, d\tilde{y}$$

$$= \sum_{r \in \mathbb{Z}} \left( e^{-\sigma tp^{rb}} - e^{-\sigma tp^{(r+1)b}} \right) p_d^{dr} \mathbb{1}_{B_d(-r)}(\tilde{x}).$$

$$\tau(t, \tilde{x}) = \int_{Q_p} \chi(\tilde{x} \cdot \tilde{y}) e^{-\sigma t \|\tilde{y}\|^b} \, d\tilde{y}$$

$$= \sum_{r \in \mathbb{Z}} e^{-\sigma tp^{rb}} \int_{S_d(r)} \chi(\tilde{x} \cdot \tilde{y}) \, d\tilde{y}$$

$$= \sum_{r \in \mathbb{Z}} \left( e^{-\sigma tp^{rb}} - e^{-\sigma tp^{(r+1)b}} \right) p_d^{dr} \mathbb{1}_{B_d(-r)}(\tilde{x}).$$

\[ (9) \]

### 2.6 The Product Process

A function $f$ is a Schwartz–Bruhat monomial with domain $Q_p^d$ if for every $i$ in $\{1, \ldots, d\}$ there is a Schwartz–Bruhat function $f_i$ on $Q_p$ so that for every $\tilde{x}$ in $Q_p^d$,

$$\tilde{x} = (x_1, \ldots, x_d) \text{ implies that } f(\tilde{x}) = \prod_{i=1}^d f_i(x_i). \quad (10)$$

The space $SB_0(Q_p^d)$ of simple Schwartz–Bruhat functions on $Q_p^d$ is the $d$-fold algebraic tensor product of the space of functions $SB(Q_p)$—It is the complex vector space of functions that are finite linear combinations of Schwartz–Bruhat monomials. Each simple Schwartz–Bruhat function is a function in $L^2(Q_p^d)$, and $L^2(Q_p^d)$ is the analytic completion of $SB_0(Q_p^d)$ under the $L^2$-norm on $L^2(Q_p^d)$.

For each $i$ in $\{1, \ldots, d\}$, take $H_i$ to be the linear extension to $SB_0(Q_p^d)$ of the operator that is initially defined for any Schwartz–Bruhat monomial $f$ by

$$H_i(f) = f_1 \otimes \cdots \otimes f_{i-1} \otimes \sigma \Delta_b f_i \otimes f_{i+1} \otimes \cdots \otimes f_d.$$

Take $H_0$ to be the sum

$$H_0 = H_1 + \cdots + H_d.$$ 

The operator $H_0$ is the Fourier transform of a real valued multiplication operator on a dense subset of $L^2(Q_p^d)$, and so it is essentially self-adjoint on $SB_0(Q_p^d)$. Take $H$ to be the self-adjoint closure of $H_0$.

The max-norm process $\tilde{X}$ has a radially symmetric law with respect to the max-norm, but it is not the only possible choice for a Brownian motion in $Q_p^d$. Denote...
by \( P \) the probability measure \( P^d \) on \( D([0, \infty) : Q_p) \) with \( d \) equal to 1. For each positive \( t \), the probability density function \( \rho(t, \cdot) \) for \( X_t \) satisfies (8). Take \( \otimes_{i=1}^d P \) to be the measure on \( D([0, \infty) : Q_p^d) \) that is induced by the product measure on \( D([0, \infty) : Q_p)^d \). Simplify notation by henceforth writing \( \otimes^d P \) rather than \( \otimes_{i=1}^d P \).

For each positive \( t \), take \( \vec{Y}_t \) to be the random variable that acts on any sample path \( \omega \) in the probability space \( (D([0, \infty) : Q_p^d), \otimes^d P) \) by

\[
\vec{Y}_t(\omega) = \omega(t) = (\omega_1(t), \ldots, \omega_d(t)).
\]

Take \( \vec{Y} \) to be the function that is defined for all \( t \) in \((0, \infty)\) by

\[
\vec{Y}(t) = \vec{Y}_t.
\]

The process that is given by the triple \((D([0, \infty) : Q_p^d), \otimes^d P, \vec{Y})\) is the *product process*, which is radially symmetric if and only if \( d \) is equal to 1. For any positive real \( t \), take \( g_d(t, \cdot) \) to be the probability density function for \( \vec{Y}_t \). Independence of the components of \( \vec{Y} \) implies that

\[
g_d(t, \vec{x}) = \rho(t, x_1)\rho(t, x_2) \cdots \rho(t, x_d).
\]

Differentiate to obtain for any positive \( t \) and \( \vec{x} \) in \( Q_p^d \) the equality

\[
\frac{d}{dt} g_d(t, \vec{x}) = \frac{d}{dt} \left( \rho(t, x_1)\rho(t, x_2) \cdots \rho(t, x_d) \right) = \sigma H g_d(t, \vec{x}),
\]

which implies that \( H \) is the infinitesimal generator of the product process on \( Q_p \).

In the setting of Brownian motion in \( \mathbb{R}^d \), where the norm is the usual Euclidean \( \ell^2 \)-norm, the analogous norm process is that which is given by the product of \( d \) independent Brownian motions in \( \mathbb{R} \) that have the same diffusion constant. The next section will show that this is not the case in the \( Q_p \) setting, and the goal of the present work is to precisely understand some differences between the stochastic processes \( \vec{X} \) and \( \vec{Y} \) when \( d \) is greater than 1.

### 3 The Component Processes and Their Dependence

Section 3.1 establishes that the processes \( \vec{X} \) and \( \vec{Y} \) are similar in that the components of each are themselves Brownian motions with the same diffusion constant, \( \sigma \), and the same diffusion exponent, \( b \). Section 3.3 establishes that the max-norm process, \( \vec{X} \), has dependent components, where the product process, \( \vec{Y} \), has independent components, and so \( \vec{X} \) and \( \vec{Y} \) are qualitatively different.
3.1 Calculation of the Marginals

For any natural number $i$ in $\{1, \dots, d\}$, take $X^{(i)}$ to be the stochastic process that is given by the $i^{th}$ component of the max-norm process $\mathbf{X}$. The goal of this subsection is to prove Theorem 3.1.

Some additional notation facilitates the presentation. For each $\mathbf{x}$ in $Q^d_p$, denote by $\mathbf{x}_i$ the vector in $Q^{d-1}_p$ that is given by

$$
\mathbf{x}_i = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_d).
$$

To simplify notation, identify any ordered pair $(x_1, (x_2, \dots, x_d))$ in $Q^d_p \times Q^{d-1}_p$ with the $d$-tuple $(x_1, \dots, x_d)$ in $Q^d_p$. For each positive real number $t$, take $\rho^{(i)}(t, \cdot)$ to be the probability density function for the $i^{th}$ component function of $\mathbf{X}_t$, the random variable $X_t^{(i)}$. For each $x$ in $Q^d_p$, the marginal $\rho^{(i)}(t, x)$ is given by the integral

$$
\rho^{(i)}(t, x) = \int_{Q^{d-1}_p} \rho_d(t, (x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_d)) \, d\mathbf{x}_i
$$

where the change of variables formula and the symmetry of the integrand together imply the penultimate equality.

Switching the order of integration in (11) that determines the marginals facilitates calculation of the law for the component processes. The failure of the integrand to be absolutely integrable with respect to the product measure on $Q^{d-1}_p \times Q^d_p$ precludes a naive application of the Fubini-Tonelli theorem that would quickly verify Theorem 3.1. The technical complication necessitates the following more involved argument and implicitly makes use of the fact that the given integral is an oscillatory integral.

**Theorem 3.1** Each $X^{(i)}$ is a Brownian motion in $Q^d_p$ with diffusion constant equal to $\sigma$ and diffusion exponent equal to $b$.

**Proof** For any $i$ in $\{1, \dots, d\}$, the radial symmetry of $\rho_d$ and (11) together imply that $X^{(i)}$ and $X^{(1)}$ have the same law. For any positive $t$, the characteristic function $\phi_d(t, \cdot)$ of $\mathbf{X}_t$ is given for each $\mathbf{y}$ in $Q^d_p$ by

$$
\phi_d(t, \mathbf{y}) = \mathbb{E}e^{-\sigma t \|\mathbf{y}\|^b}.
$$

The function $\phi_d(t, \cdot)$ is bounded and integrable, and so the Fubini-Tonelli theorem guarantees that
\[
\rho^{(1)}(t, x_1) = \int_{Q_{p}^{d-1}} \rho_d(t, (x_1, \bar{x}_1)) \, d\bar{x}_1 \\
= \int_{Q_{p}^{d-1}} \left\{ \int_{Q_{p}^{d}} \chi((x_1, \bar{x}_1) \cdot \bar{y}) e^{-\sigma \|\bar{y}\|^b} \, d\bar{y} \right\} \, d\bar{x}_1 \\
= \int_{Q_{p}^{d-1}} \left\{ \int_{Q_{p}^{d-1}} \chi(x_1y_1) \chi(\bar{x}_1 \cdot \bar{y}_1) e^{-\sigma \|y_1, \bar{y}_1\|^b} \, d\bar{y}_1 \, dy_1 \right\} \, d\bar{x}_1.
\]

For each \(y_1\), decompose the innermost integral into a sum of integrals over the ball of radius \(|y_1|\) and its complement to obtain the equalities

\[
\rho^{(1)}(t, x_1) = \int_{Q_{p}^{d-1}} \left\{ \int_{Q_{p}^{d-1}} \chi(x_1y_1) \chi(\bar{x}_1 \cdot \bar{y}_1) e^{-\sigma \|y_1, \bar{y}_1\|^b} \, d\bar{y}_1 \, dy_1 \right\} \, d\bar{x}_1 \\
+ \int_{Q_{p}^{d-1}} \left\{ \int_{Q_{p}^{d-1}} \chi(x_1y_1) \chi(\bar{x}_1 \cdot \bar{y}_1) e^{-\sigma \|y_1, \bar{y}_1\|^b} \, d\bar{y}_1 \, dy_1 \right\} \, d\bar{x}_1 \\
= \int_{Q_{p}^{d-1}} \chi(x_1y_1) \left[ \int_{Q_{p}^{d-1}} e^{-\sigma \|y_1\|^b} \, dy_1 \right] \chi(\bar{x}_1 \cdot \bar{y}_1) \, d\bar{y}_1 \\
+ \int_{Q_{p}^{d-1}} \chi(\bar{x}_1 \cdot \bar{y}_1) e^{-\sigma \|\bar{y}_1\|^b} \, d\bar{y}_1 \chi(x_1y_1) \, dy_1 \}
\]

Since \(\rho_d(t, \cdot)\) is integrable,

\[
\rho^{(1)}(t, x_1) = \lim_{n \to \infty} \int_{B_{d-1}(n)} \rho_d(t, (x_1, \bar{x}_1)) \, d\bar{x}_1. \tag{12}
\]

Take \(I_1(x_1)\) and \(I_2(x_1)\) to be the quantities

\[
I_1(x_1) = \lim_{n \to \infty} \int_{B_{d-1}(n)} \left\{ \int_{Q_{p}^{d-1}} \chi(x_1y_1) e^{-\sigma \|y_1\|^b} \, dy_1 \int_{Q_{p}^{d-1}} \chi(\bar{x}_1 \cdot \bar{y}_1) \, d\bar{y}_1 \right\} \, d\bar{x}_1 \tag{13}
\]

and

\[
I_2(x_1) = \lim_{n \to \infty} \int_{B_{d-1}(n)} \left\{ \int_{Q_{p}^{d-1}} \chi(x_1y_1) \int_{Q_{p}^{d-1}} \chi(\bar{x}_1 \cdot \bar{y}_1) e^{-\sigma \|\bar{y}_1\|^b} \, d\bar{y}_1 \, dy_1 \right\} \, d\bar{x}_1, \tag{14}
\]

so that if both limits exist, then

\[
\rho^{(1)}(t, x_1) = I_1(x_1) + I_2(x_1). \tag{15}
\]
The domain of integration of the outermost integral of both (13) and (14) is the $\mu_{d-1}$ finite measure space $B_{d-1}(n)$. The integrand

$$\chi(x_1 y_1) \int_{\|\tilde{y}_1\| > |y_1|} \chi(\tilde{x}_1 \cdot \tilde{y}_1) e^{-\sigma t \|\tilde{y}_1\|^b} \, d\tilde{y}_1$$

$$= \chi(x_1 y_1) \int_{Q_p \setminus B_{d-1}(n)} \chi(\tilde{x}_1 \cdot \tilde{y}_1) e^{-\sigma t \|\tilde{y}_1\|^b} \mathbf{1}_{\|\tilde{y}_1\| > |y_1|} \, d\tilde{y}_1$$

is bounded with compact support in $Q_p$, so both

$$\chi(x_1 y_1) e^{-\sigma t |y_1|^b} \int_{\|\tilde{y}_1\| \leq |y_1|} \chi(\tilde{x}_1 \cdot \tilde{y}_1) \, d\tilde{y}_1$$

and

$$\chi(x_1 y_1) \int_{\|\tilde{y}_1\| > |y_1|} \chi(\tilde{x}_1 \cdot \tilde{y}_1) e^{-\sigma t \|\tilde{y}_1\|^b} \, d\tilde{y}_1$$

are $L^1(B_{d-1}(n) \times Q_p)$. The Fubini-Tonelli theorem implies that

$$I_1(x_1) = \lim_{n \to \infty} \int_{Q_p} \left\{ \int_{B_{d-1}(n)} \chi(x_1 y_1) e^{-\sigma t |y_1|^b} \int_{\|\tilde{y}_1\| \leq |y_1|} \chi(\tilde{x}_1 \cdot \tilde{y}_1) \, d\tilde{y}_1 \right\} \, dy_1$$

(16)

and

$$I_2(x_1) = \lim_{n \to \infty} \int_{Q_p} \left\{ \int_{B_{d-1}(n)} \chi(x_1 y_1) \int_{\|\tilde{y}_1\| > |y_1|} \chi(\tilde{x}_1 \cdot \tilde{y}_1) e^{-\sigma t \|\tilde{y}_1\|^b} \, d\tilde{y}_1 \right\} \, dy_1.$$  

(17)

Lemma 2.1 and (16) together imply that

$$I_1(x_1) = \lim_{n \to \infty} \int_{Q_p} \chi(x_1 y_1) e^{-\sigma t |y_1|^b} p^{\log_p |y_1| + \min(- \log_p |y_1|, n)} \, dy_1.$$  

(18)

For any natural number $M$, decompose the integral over $Q_p$ into a sum of integrals over $B(-M)$ and $B(-M)^c$ to obtain the equalities

$$I_1(x_1) = \lim_{n \to \infty} \left\{ \int_{B(-M)^c} \chi(x_1 y_1) e^{-\sigma t |y_1|^b} p^{\log_p |y_1| + \min(- \log_p |y_1|, n)} \, dy_1 \right\}$$

$$+ \int_{B(-M)} \chi(x_1 y_1) e^{-\sigma t |y_1|^b} p^{\log_p |y_1| + \min(- \log_p |y_1|, n)} \, dy_1$$

$$= \int_{B(-M)^c} \chi(x_1 y_1) e^{-\sigma t |y_1|^b} \, dy_1$$

$$+ \lim_{n \to \infty} \int_{B(-M)} \chi(x_1 y_1) e^{-\sigma t |y_1|^b} p^{\log_p |y_1| + \min(- \log_p |y_1|, n)} \, dy_1$$

$$= \int_{Q_p} \chi(x_1 y_1) e^{-\sigma t |y_1|^b} \, dy_1 + E_M(x_1),$$

(18)
where

\[ E_M(x_1) = -\int_{B(-M)} \chi(x_1 y_1) e^{-\sigma |y_1|^b} \, dy_1 + \lim_{n \to \infty} \int_{B(-M)} \chi(x_1 y_1) e^{-\sigma |y_1|^b} p \log_p |y_1| + \min(-\log_p |y_1|, n) \, dy_1. \]

The inequality

\[ |E_M(x_1)| \leq 2p^{-M} \]

and (18) together imply that

\[ I_1(x_1) = \int_{Q_p} \chi(x_1 y_1) e^{-\sigma |y_1|^b} \, dy_1. \]  

(19)

Take \( F \) to be the function that is given for each pair \((y_1, \vec{y}_1)\) in \( Q_p \times Q_{p-1}^d \) by

\[ F(y_1, \vec{y}_1) = e^{-\sigma \|\vec{y}_1\|^b} \mathbb{1}_{B_{d-1}((\log_p |y_1|)^\epsilon)}(\vec{y}_1) \]  

(20)

and for each \( y_1 \), take \( \tilde{F}(y_1, \cdot) \) to be the Fourier transform, taken over \( Q_{p-1}^d \), of \( F(y_1, \cdot) \). Use (17) and (20) to rewrite \( I_2(x_1) \) in terms of \( F \) and obtain the equalities

\[ I_2(x_1) = \lim_{n \to \infty} \int_{B_{d-1}(n)} \left\{ \int_{Q_p} \chi(x_1 y_1) \int_{Q_{p-1}^d} \chi(\vec{x}_1, \vec{y}_1) F(y_1, \vec{y}_1) \, d\vec{y}_1 \, dy_1 \right\} d\vec{x}_1 \]

\[ = \lim_{n \to \infty} \int_{B_{d-1}(n)} \left\{ \int_{Q_p} \chi(x_1 y_1) \tilde{F}(y_1, \vec{x}_1) \, d\vec{x}_1 \right\} d\vec{y}_1. \]

Since \( B_{d-1}(n) \) has finite measure, the Fubini-Tonelli theorem together with the equality

\[ F(y_1, \vec{0}) = 0 \]

implies that

\[ I_2(x_1) = \lim_{n \to \infty} \int_{Q_p} \left\{ \int_{B_{d-1}(n)} \chi(x_1 y_1) \tilde{F}(y_1, \vec{x}_1) \, d\vec{x}_1 \right\} dy_1 \]

\[ = \lim_{n \to \infty} \int_{Q_p} \left\{ \int_{Q_{p-1}^d} \chi(x_1 y_1) \tilde{F}(y_1, \vec{x}_1) \, d\vec{x}_1 \right\} dy_1 \]

\[ - \int_{B_{d-1}(n)^c} \chi(x_1 y_1) \tilde{F}(y_1, \vec{x}_1) \, d\vec{x}_1 \]
\[ \lim_{n \to \infty} \int_{Q_p} \left\{ F(y_1, \tilde{0}) - \int_{B_{d-1}(n)}^c \chi(x_1 y_1) \tilde{F}(y_1, \tilde{x}_1) \, d\tilde{x}_1 \right\} \, dy_1 = - \lim_{n \to \infty} \int_{Q_p} \left\{ \int_{B_{d-1}(n)}^c \chi(x_1 y_1) \tilde{F}(y_1, \tilde{x}_1) \, d\tilde{x}_1 \right\} \, dy_1. \]  
\[ (21) \]

For any natural number \( M \), decompose the integral over \( Q_p \) in (21) into a sum of integrals over \( B(-M) \) and \( B(-M)^c \) to obtain the equality

\[ I_2(x_1) = \lim_{n \to \infty} \int_{Q_p} \left\{ \int_{B(-M)} \int_{B_{d-1}(n)}^c \chi(x_1 y_1) \tilde{F}(y_1, \tilde{x}_1) \, d\tilde{x}_1 \right\} \, dy_1 + \int_{B(-M)^c} \int_{B_{d-1}(n)}^c \chi(x_1 y_1) \tilde{F}(y_1, \tilde{x}_1) \, d\tilde{x}_1 \, dy_1. \]  
\[ (22) \]

For any non-zero \( y_1 \) in \( Q_p \), \( F(y_1, \cdot) \) is locally constant with a radius of local constancy equal to \( |y_1| \). The function \( \tilde{F}(y_1, \cdot) \) is therefore supported on the ball of radius \( \frac{1}{|y_1|} \) that is centered at the origin, and so the second summand in (22) is the integral of the zero function for any \( n \) that is greater than \( M \). Since the innermost integral of the first term is bounded, there is a constant \( C \) so that for any natural number \( M \),

\[ |I_2(x_1)| \leq \frac{C}{p^M}, \]

and so

\[ I_2(x_1) = 0. \]  
\[ (23) \]

Equations (15), (19), and (23) together imply that

\[ \rho^{(1)}(t, x_1) = \int_{Q_p} \chi(\tilde{x}_1 \cdot \tilde{y}_1) e^{-\sigma t |y_1|^b} \, dy_1. \]

\[ \square \]

### 3.2 Probabilities for the Conditioned Components

For each \( i \) in \( \{1, \ldots, d\} \), denote by \( \tilde{X}_{t,i} \) the \( d-1 \) tuple

\[ \tilde{X}_{t,i} = \left( X_t^{(1)}, \ldots, X_t^{(i-1)}, X_t^{(i+1)}, \ldots, X_t^{(d)} \right), \]

where \( \tilde{X}_t = \left( X_t^{(1)}, \ldots, X_t^{(d)} \right) \).

The components of the max-norm process fail to be independent because the spatial dependence of the law for the max-norm process involves only the component with the largest \( p \)-adic absolute value. For any integers \( r \) and \( R \), for any \( i \) in \( \{1, \ldots, d\} \), and for any \( a \) in \( B(R) \), if \( r \) is less than \( R \), then

\[ P^d \left( X_t^{(i)} \in B(r, a) \, \bigg| \, \tilde{X}_{t,i} \in S_{d-1}(R) \right) \leq p^{r-R} < \frac{1}{p} \]
because this conditional probability is independent of \(a\). However, as long as \(t\) is small enough,

\[
P^d\left(X^{(i)}_t \in B(r, 0)\right) > \frac{1}{p},
\]

and so \(\tilde{X}_t\) does not have independent components as long as \(t\) is small enough. Lemma 3.2 provides an explicit calculation of certain conditional probabilities that lead not only to a proof that the components of \(\tilde{X}_t\) are dependent for any positive \(t\), but also to an explicit description of certain local (small time) behaviors of the conditioned component processes.

**Lemma 3.2** For any integers \(r\) and \(R\), for any \(i\) in \(\{1, \ldots, d\}\), and for any \(a\) in \(B(R)\), if \(r\) is less than or equal to \(R\), then

\[
P^d\left(X^{(i)}_t \in B(r, a) \mid \tilde{X}_{t,i} \in S_{d-1}(R)\right) = \frac{p^r \sum_{j \leq -R} \left(e^{-\sigma tp^{(j+1)b}} - e^{-\sigma tp^{(j)b}}\right) p^{d-j}}{\sum_{j \leq -R} \left(e^{-\sigma tp^{(j+1)b}} - e^{-\sigma tp^{(j)b}}\right) p^{(d-1)j}}.
\]

**Proof** Without loss in generality, take \(i\) to be equal to 1. For any \(\tilde{x}\) in \(B(r, a) \times S_{d-1}(R)\) and for any \(\tilde{x}\) in \(B(R, a) \times S_{d-1}(R)\),

\[
\|\tilde{x}\| = p^R,
\]

and so if \(-j\) is less than \(R\), then

\[
\int_{B(r, a) \times S_{d-1}(R)} \mathbb{1}_{B_d(-j)}(\tilde{x}) \, d\tilde{x} = \int_{B(R, a) \times S_{d-1}(R)} \mathbb{1}_{B_d(-j)}(\tilde{x}) \, d\tilde{x} = 0. \quad (24)
\]

For any \(\tilde{x}\) in \(B(R, a)^c \times S_{d-1}(R)\),

\[
\|\tilde{x}\| > p^R,
\]

and so if \(-j\) is less than or equal to \(R\), then

\[
\int_{B(R, a)^c \times S_{d-1}(R)} \mathbb{1}_{B_d(-j)}(\tilde{x}) \, d\tilde{x} = 0. \quad (25)
\]

The Fubini–Tonelli theorem and (3) together imply that if

\[
j \leq -R,
\]

then

\[
\int_{B(r, a) \times S_{d-1}(R)} \mathbb{1}_{B_d(-j)}(\tilde{x}) \, d\tilde{x} = \int_{B(r, a)} \mathbb{1}_{B(-j)}(x_1) \, dx_1 \int_{S_{d-1}(R)} \mathbb{1}_{B_{d-1}(-j)}(\tilde{x}_1) \, d\tilde{x}_1 \\
= p^r p^{(d-1)R} \left(1 - \frac{1}{p^{d-1}}\right), \quad (26)
\]
\[
\int_{B(R,a) \times S_{d-1}(R)} 1_{B_d(-j)}(\vec{x}) \, d\vec{x} = \int_{B(R,a)} 1_{B(-j)}(x_1) \, dx_1 \int_{S_{d-1}(R)} 1_{B_{d-1}(-j)}(\vec{x}_1) \, d\vec{x}_1 \\
= p^{dR} \left( 1 - \frac{1}{p^{d-1}} \right), \quad (27)
\]
and if
\[
j < -R,
\]
then
\[
\int_{B(R,a)^c \times S_{d-1}(R)} 1_{B_d(-j)}(\vec{x}) \, d\vec{x} = \int_{B(R,a)^c} 1_{B(-j)}(x_1) \, dx_1 \int_{S_{d-1}(R)} 1_{B_{d-1}(-j)}(\vec{x}_1) \, d\vec{x}_1 \\
= \left( p^{-j} - p^R \right) p^{(d-1)R} \left( 1 - \frac{1}{p^{d-1}} \right). \quad (28)
\]

The equality (9) implies that for any positive real \( t \) and any Borel subset \( U \) of \( Q_p^d \),
\[
P^d(X_t \in U) = \sum_{j \in \mathbb{Z}} \left( e^{-\sigma tp^{j+b}} - e^{-\sigma tp^{(j+1)b}} \right) p^{dj} \int_U 1_{B_d(-j)}(\vec{x}) \, d\vec{x}. \quad (29)
\]
Together with (24), (25), (26), (27), and (28), (29) implies that
\[
P^d(\vec{X}_t \in B(r, a) \times S_{d-1}(R)) = p^r p^{(d-1)R} \left( 1 - \frac{1}{p^{d-1}} \right) \\
\sum_{j \leq -R} \left( e^{-\sigma tp^{j+b}} - e^{-\sigma tp^{(j+1)b}} \right) p^{dj}, \quad (30)
\]
\[
P^d(\vec{X}_t \in B(R, a) \times S_{d-1}(R)) = p^{dR} \left( 1 - \frac{1}{p^{d-1}} \right) \\
\sum_{j \leq -R} \left( e^{-\sigma tp^{j+b}} - e^{-\sigma tp^{(j+1)b}} \right) p^{dj}, \quad (31)
\]
and
\[
P^d(\vec{X}_t \in B(R, a)^c \times S_{d-1}(R)) = p^{(d-1)R} \left( 1 - \frac{1}{p^{d-1}} \right) \sum_{j < -R} \left( e^{-\sigma tp^{j+b}} - e^{-\sigma tp^{(j+1)b}} \right) p^{dj} \left( p^{-j} - p^R \right). \quad (32)
\]
The equality
\[
P \left( \vec{X}_{t,1} \in S_{d-1}(R) \right) = P \left( (X_t^{(1)} \in B(R)) \cap (\vec{X}_{t,1} \in S_{d-1}(R)) \right) \\
+ P \left( (X_t^{(1)} \in B(R)^c) \cap (\vec{X}_{t,1} \in S_{d-1}(R)) \right)
\]
implies that
\[
P^d \left( X_t^{(i)} \in B(r, a) \mid \bar{X}_{t,1} \in S_{d-1}(R) \right)
= \frac{P^d \left( (X_t^{(i)} \in B(r, a)) \cap (\bar{X}_{t,1} \in S_{d-1}(R)) \right)}{P^d \left( (X_t^{(i)} \in B(R)) \cap (\bar{X}_{t,1} \in S_{d-1}(R)) \right) + P^d \left( (X_t^{(i)} \in B(R^c)) \cap (\bar{X}_{t,1} \in S_{d-1}(R)) \right)},
\]
and so (30), (31), and (32) together imply that
\[
P^d \left( X_t^{(i)} \in B(r, a) \mid \bar{X}_{t,1} \in S_{d-1}(R) \right)
= \frac{p^r \sum_{j \leq -R} \left( e^{-\sigma t p^j b} - e^{-\sigma t p^{(j+1)b}} \right) p^d j}{p^R \sum_{j \leq -R} \left( e^{-\sigma t p^j b} - e^{-\sigma t p^{(j+1)b}} \right) p^d j + \sum_{j < -R} \left( e^{-\sigma t p^j b} - e^{-\sigma t p^{(j+1)b}} \right) p^d j \left( p^{-j} - p^R \right)}
= \frac{p^R \left( e^{-\sigma t p^{-R} b} - e^{-\sigma t p^{(-R+1)b}} \right) p^d R + \sum_{j < -R} \left( e^{-\sigma t p^j b} - e^{-\sigma t p^{(j+1)b}} \right) p^d (d-1) j}{\sum_{j \leq -R} \left( e^{-\sigma t p^j b} - e^{-\sigma t p^{(j+1)b}} \right) p^d (d-1) j}.
\]

\[
\square
\]

### 3.3 Component Dependency

The formula that Lemma 3.2 provides for the conditional probabilities is rather complicated. Lemma 3.3 gives a description of the local behavior of these conditional probabilities that is especially useful for understanding the effect of conditioning on the component processes. Denote by \( \Gamma(p, b, d) \) the quantity
\[
\Gamma(p, b, d) = \frac{p^{b+d} - p}{p^{b+d} - 1}.
\]

Use the standard “big oh” notation to simplify the statements and proofs of the statements below.

**Lemma 3.3** For any integers \( r \) and \( R \), for any \( i \) in \( \{1, \ldots, d\} \), and for any \( a \) in \( B(R) \), if \( r \) is less than or equal to \( R \), then
\[
P^d \left( X_t^{(i)} \in B(r, a) \mid \bar{X}_{t,1} \in S_{d-1}(R) \right) = \left( \Gamma(p, b, d) p^{-R} + O(t) \right) p^r.
\]

**Proof** Take \( G(R, d, \cdot) \) to be the function that is given for any \( t \) in \([0, 1]\) by
\[
G(R, d, t) = \sum_{j \leq -R} \left( e^{-\sigma t p^j b} - e^{-\sigma t p^{(j+1)b}} \right) p^d j.
\]
Differentiate $G(R,d,\cdot)$ to obtain the equality
\[
G'(R,d,0) = \sum_{j \leq -R} (\sigma p^{(j+1)b} - \sigma p^{jb}) p^{dj} = \frac{\sigma (p^b - 1) p^{(b+d)(1-R)}}{p^{b+d} - 1}.
\] (35)

The twice continuous differentiability of $G(R,d,\cdot)$ implies that
\[
G(R,d,t) = G'(R,d,0)t + O(t^2) \quad \text{and} \quad G(R,d-1,t) = G'(R,d-1,0)t + O(t^2),
\] (36)
and so if $t$ is positive, then
\[
\frac{G(R,d,t)}{G(R,d-1,t)} = \frac{G'(R,d,0)}{G'(R,d-1,0)} + O(t)
= p^{-R} \frac{p^{b+d} - p}{p^{b+d} - 1} + O(t).
\] (37)

Use (37) to rewrite the righthand side of the equality that is given by Lemma 3.2 and obtain the desired description of the local behavior of the conditional probabilities. □

For any positive real number $t$ and any integer $R$, take $U(R)_t$ to be the random variable with the following law: For any Borel subset $V$ of $\mathbb{Q}_p$,
\[
\text{Prob}(U(R)_t \in V) = P^d \left( X_i^{(1)} \in V \mid \bar{X}_{t,1} \in S_{d-1}(R) \right).
\] (38)

**Proposition 3.4** The random variable $U(R)_t$ is asymptotically uniformly distributed in $B(R)$. Furthermore, for any Borel subset $V$ of $B(R)$,
\[
\lim_{t \to 0^+} \text{Prob}(U(R)_t \in V) = \mu(V) p^{-R} \Gamma(p, b, d).
\]

**Proof** For any Borel subset $V$ of $B(R)$, there is an at most countable index set $J$ and sequences $(a_j)$ in $B(R)$ and $(r_j)$ in $\mathbb{Z} \cap (-\infty, R]$, both indexed in $J$, so that $(B(r_j, a_j))$ is a sequence of disjoint balls whose union is $V$. Lemma 3.2 and (37) together imply that for each $j$,
\[
\text{Prob}(U(R)_t \in B(r_j, a_j)) = p^{r_j} \frac{G(R, d, t)}{G(R, d-1, t)}
= p^{r_j} \left( p^{-R} \Gamma(p, b, d) + O(t) \right).
\] (39)
The countable additivity of the conditioned measure implies that
\[
\text{Prob}(U(R), t \in V) = \sum_{j \in J} p^{r_j} \left( p^{-R} \Gamma(p, b, d) + O(t) \right) \\
= \left( \Gamma(p, b, d) p^{-R} + O(t) \right) \sum_{j \in J} \mu(B(r_j, a_j)) \\
= \left( \Gamma(p, b, d) p^{-R} + O(t) \right) \mu(V).
\]

Lemma 3.3 and (40) together imply that
\[
\text{Prob}(U(R), t \in V) = \left( \Gamma(p, b, d) p^{-R} + O(t) \right) \mu(V) \to \mu(V) p^{-R} \Gamma(p, b, d)
\]

as \( t \) tends to 0 from the right, and so \( U(R), t \) is asymptotically uniformly distributed in \( B(R) \).

The symmetry between spatial and temporal scalings for the law for a \( p \)-adic Brownian motion suggests the following significant extension of Proposition 3.4.

**Theorem 3.5** For any positive real numbers \( t \) and \( \epsilon \), there is an integer \( M \) so that for any integer \( N \) and for any Borel subset \( V \) of \( B(N) \),

\[
N \geq M \quad \text{implies that} \quad \left| \text{Prob}(U(N), t \in V) - \mu(V) \Gamma(p, b, d) p^{-N} \right| < \epsilon.
\]

**Proof** For any integer \( R \) and any natural number \( K \), take \( G(R + K, d, t) \) and \( G(R + K, d - 1, t) \) to be given by (34) so that
\[
\frac{G(R + K, d, t)}{G(R + K, d - 1, t)} = \frac{\sum_{j \leq -R - K} \left( e^{-\sigma t p^{j+b}} - e^{-\sigma t p^{(j+1)b}} \right) p^{dj}}{\sum_{j \leq -R - K} \left( e^{-\sigma t p^{j+b}} - e^{-\sigma t p^{(j+1)b}} \right) p^{(d-1)j}}.
\]

Reindex the sums in (42) to obtain the equalities
\[
\frac{G(R + K, d, t)}{G(R + K, d - 1, t)} = \frac{\sum_{j \leq -R} \left( e^{-\sigma t p^{(j-K)b}} - e^{-\sigma t p^{(j-K+1)b}} \right) p^{d(j-K)}}{\sum_{j \leq -R} \left( e^{-\sigma t p^{(j-K)b}} - e^{-\sigma t p^{(j-K+1)b}} \right) p^{(d-1)(j-K)}} \\
= \frac{\sum_{j \leq -R} \left( e^{-\sigma t (p^{-K} p^{j})b} - e^{-\sigma t (p^{-K} p^{(j+1)b})} \right) p^{dj} p^{-Kd}}{\sum_{j \leq -R} \left( e^{-\sigma t (p^{-K} p^{j})b} - e^{-\sigma t (p^{-K} p^{(j+1)b})} \right) p^{(d-1)j} p^{-K(d-1)}} \\
= p^{-K} \frac{G(R, d, t p^{-K} b)}{G(R, d - 1, t p^{-K} b)}.
\]
For any positive real number $t$, Lemma 3.3 implies that

$$\frac{G(R, d, tp^{-Kb})}{G(R, d-1, tp^{-Kb})} = \left( \Gamma(p, b, d)p^{-R} + O(tp^{-Kb}) \right),$$

and so (43) implies that

$$\frac{G(R + K, d, t)}{G(R + K, d-1, t)} = p^{-K} \left( \Gamma(p, b, d)p^{-R} + O(tp^{-Kb}) \right). \quad (44)$$

Lemma 3.3, (38), and (44) together imply that for any $r$ that is less than $R + K$ and any $a$ in $B(R + K)$,

$$\text{Prob}(U(R + K), t \in B(r, a)) = p^{-K} \left( \Gamma(p, b, d)p^{-R} + O(tp^{-Kb}) \right).$$

Follow the proof of Proposition 3.4 to generalize to the case where $V$ is a Borel set that is not a ball and obtain for any Borel set $V$ in $B(R + K)$ the equality

$$\text{Prob}(U(R + K), t \in V) = \mu(V) \Gamma(p, b, d)p^{-R-K} + \mu(V)p^{-K}O(tp^{-Kb}).$$

Since $V$ is a subset of $B(R + K)$, $\mu(V)$ is no greater than $p^{R+K}$, and so Proposition 3.4 implies that

$$\left| \mu(V)p^{-K}O(tp^{-Kb}) \right| \leq p^R \left| O(tp^{-Kb}) \right| < \varepsilon$$

as long as $K$ is large enough. \hfill $\square$

**Corollary** For any positive real number $t$, the components of $\tilde{X}_t$ are not independent.

**Proof** For any $i$ in $\{1, \ldots, d\}$ and for any positive real number $t$, Theorem 3.1 implies that

$$\lim_{r \rightarrow \infty} P^d\left(X_t^{(i)} \in B(r)\right) = \lim_{r \rightarrow \infty} P(X_t \in B(r)) = 1,$$

and so for any positive real number $\varepsilon$ there is an integer $R$ so that for any $r$ that is greater than or equal to $R$,

$$P^d\left(X_t^{(i)} \in B(r)\right) > 1 - \varepsilon.$$

Since $\Gamma(p, b, d)$ is less than 1, Theorem 3.5 and radial symmetry together imply that there is a natural number $K$ and a real number $e$ in $(-\varepsilon, \varepsilon)$ so that

$$P^d\left(X_t^{(i)} \in B(R + K) \mid \tilde{X}_{t,i} \in S_{d-1}(R + 1 + K)\right) = \mu(B(R + K)) p^{-R-K-1}\Gamma(p, b, d) + e = \frac{1}{p} \Gamma(p, b, d) + e < \frac{1}{p} + \varepsilon.$$
As long as $\varepsilon$ is small enough,

$$
P^d\left(\bar{X}_t^{(i)} \in B(R + K) \mid \bar{X}_{t,i} \in S_{d-1}(R + 1 + K)\right) < P^d\left(\bar{X}_t^{(i)} \in B(R + K)\right).
$$

\[\square\]

## 4 First Exit Probabilities

Each component process of the product and max-norm processes is a $p$-adic Brownian motion with diffusion constant $\sigma$ and diffusion exponent $b$. The dependency of the components of these processes impacts the first exit times from balls. Namely, dimension has a large impact on the first exit time probabilities for the product process, but it has a rather small effect on these probabilities for the max-norm process.

### 4.1 Exit Times for the Components

Take $\alpha$ to be the positive real number that is given by

$$
\alpha = 1 - \frac{p^{b-1}}{p^{b+1} - 1},
$$

and take $X$ to be a one dimensional $p$-adic Brownian motion with diffusion constant $\sigma$ and diffusion exponent $b$. For sake of clarity, slightly modify the prior notation in [36] and for any positive real number $T$ denote by $\|X\|_T$ the quantity

$$
\|X\|_T = \sup_{0 \leq t \leq T} |X_t|.
$$

The probability that a sample path for $X$ remains in $B(R)$ until time $T$, $P\left(\|X\|_T \leq p^R\right)$, is a survival probability for $X$. The complement of this probability is a first exit probability, the probability that a sample path for $X$ has first exit from $B(R)$ before time $T$. Since every point in $B(R)$ is the center of $B(R)$, the exit times for $X$ from $B(R)$ do not depend on starting points, and so

$$
P\left(\|X\|_{T+S} \leq p^R \mid X_S \leq p^R\right) = P\left(\|X\|_T \leq p^R\right).
$$

The survival probabilities for $X$ are continuous from the right at 0 and satisfy Cauchy’s multiplicative functional equation, and so the first exit time for $X$ is an exponentially distributed random variable. Theorem 3.1 of [36] determines the parameter of this exponential distribution by establishing the equality

$$
P\left(\|X\|_T \leq p^R\right) = e^{-\sigma \alpha T p^{-Rb}}.
$$

\[45\]
4.2 Exit Times for the Processes

Take $\alpha_d$ to be the quantity

$$\alpha_d = 1 - \frac{p^b - 1}{p^{b+d} - 1},$$

and for any $Q_d^p$-valued stochastic process $\tilde{Z}$ and any positive real number $T$, take $\|\tilde{Z}\|_T$ to be the quantity

$$\|\tilde{Z}\|_T = \sup_{0 \leq t \leq T} \|\tilde{Z}_t\|.$$

**Theorem 4.1** For any integer $R$,\[\bigotimes_d P\left(\|\tilde{Y}\|_T \leq p^R\right) = e^{-d\sigma \alpha_1 T p^{-Rb}}.\]

**Proof** Since $\tilde{Y}_t$ lies outside $B_d(R)$ if and only if at least one component of $\tilde{Y}_t$ lies outside $B_d(R)$,

$$\left(\|\tilde{Y}\|_T \leq p^R\right) = \bigcap_{i \in \{1, \ldots, d\}} \left(\|Y^{(i)}\|_T \leq p^R\right).$$

Independence of the components of $\tilde{Y}$ implies that

$$\bigotimes_d P\left(\|\tilde{Y}\|_T \leq p^R\right) = \prod_{i \in \{1, \ldots, d\}} P\left(\|Y^{(i)}\|_T \leq p^R\right) = e^{-d\sigma \alpha T p^{-Rb}}.$$

$\square$

With only minor modification, the arguments of [36] extend to the more general max-norm setting and determine the survival probabilities for the max-norm process $\tilde{X}$. For this reason, the proof below for Theorem 4.2 will omit certain details that the proof of Theorem 3.1 in [36] includes.

**Theorem 4.2** For any integer $R$,

$$P_d\left(\|\tilde{X}\|_T \leq p^R\right) = e^{-\sigma \alpha_d T p^{-Rb}}.$$

**Proof** Take $U$ in (29) to be $B(R)$ in order to obtain the equality

$$P_d\left(\tilde{X}_t \in B_d(R)\right) = \sum_{j \in \mathbb{Z}} \left( e^{-\sigma tp^b} - e^{-\sigma tp^{(j+1)b}} \right) p^d \int_{B_d(R)} \mathbb{1}_{B_d(-j)}(\tilde{x}) \, d\tilde{x}$$

$$= p^d R \sum_{j \leq -R} \left( e^{-\sigma tp^b} - e^{-\sigma tp^{(j+1)b}} \right) p^d \quad \quad \quad + \sum_{j > -R} \left( e^{-\sigma tp^b} - e^{-\sigma tp^{(j+1)b}} \right). \quad (46)$$

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The summands of the second sum in (46) telescope, and so

\[
P^d\left(\tilde{X}_t \in B_d(R)\right) = e^{-\sigma_t p^{(-R+1)b}} + p^R \sum_{j \leq -R} \left( e^{-\sigma_t p^{jb}} - e^{-\sigma_t p^{(j+1)b}} \right) p^{dj}.
\]

(47)

For any natural numbers \(N\) and \(j\), where \(j\) is less than or equal to \(N\), take \(t_j\) to be given by

\[t_j = \frac{iT}{N}.
\]

Since the max-norm on \(Q^d_p\) satisfies the ultra-metric inequality,

\[
P^d\left(\max_{t_j} (\|\tilde{X}_{t_1}\|, \ldots, \|\tilde{X}_{t_n}\|) \leq p^R\right) = P^d\left(\max_{t_j} (\|\tilde{X}_{t_1}\|, |\tilde{X}_{t_1} - \tilde{X}_{t_2}|, \ldots, |\tilde{X}_{t_n} - \tilde{X}_{t_{n-1}}|) \leq p^R\right).
\]

(48)

Take the random variable \(X_0\) to be the zero function. The independence of the set of increments \(\{\tilde{X}_{t_j} - \tilde{X}_{t_{j-1}} : i \in \{1, \ldots, N\}\}\) implies that

\[
P^d\left(\max_{t_j} (\|\tilde{X}_{t_1}\|, \ldots, \|\tilde{X}_{t_n}\|) \leq p^R\right) = P^d\left(\|\tilde{X}_{t_1}\| \leq p^R \cap \cdots \cap \|\tilde{X}_{t_n}\| \leq p^R\right) = \prod_{1 \leq j \leq N} P^d\left(\|\tilde{X}_{t_j} - \tilde{X}_{t_{j-1}}\| \leq p^R\right).
\]

(49)

The increments are identically distributed, and so

\[
P^d\left(\max_{t_j} (\|\tilde{X}_{t_1}\|, \ldots, \|\tilde{X}_{t_n}\|) \leq p^R\right) = P^d\left(\|\tilde{X}_T\| \leq p^R\right)^N.
\]

(50)

Take \(B\) to be the twice continuously differentiable function that is defined for any \(t\) in [0, \(\infty\)] by

\[B(t) = P^d\left(\|\tilde{X}_t\| \leq p^R\right).
\]

The twice continuous differentiability of \(B\) implies that

\[B\left(\frac{T}{n}\right) = 1 + \frac{B'(0)T}{n} + O\left(\frac{1}{n}\right).
\]

The right continuity of the sample paths of \(\tilde{X}\) implies that
\[ P^d \left( \| \tilde{X} \|_T \leq p^R \right) = \lim_{n \to \infty} P^d \left( \max_j \left( \| \tilde{X}^T_n \|, \| \tilde{Y}^T_n \|, \ldots, \| \tilde{X}^T_n \|, \| \tilde{X}^T_n \| \right) \leq p^R \right) = \lim_{n \to \infty} B^d \left( \frac{T}{n} \right)^n = e^{TB'(0)}. \] (51)

Rewrite (47) to obtain the equality
\[ B(t) = e^{-\sigma tp^{-Rb}} + p^dR \sum_{j < -R} \left( e^{-\sigma tp_jb} - e^{-\sigma tp_{(j+1)b}} \right) p^d_j. \] (52)

Differentiate both sides of (52) to obtain the equality
\[ B'(0) = -\sigma p^{-Rb} + \sigma \sum_{j \leq -R-1} p^d(R+j) \left( p^{(j+1)b} - p^{jb} \right). \] (53)

Simplify the righthand side of (53) to obtain the equality
\[ B'(0) = -\sigma \alpha_d p^{-Rb} \] (54)

which, together with (51), implies that
\[ P^d \left( \| \tilde{X} \|_T \leq p^R \right) = e^{-\sigma \alpha_d Tp^{-Rb}}. \]

Together with Theorem 4.2, the equality
\[ \lim_{d \to \infty} \alpha_d = \lim_{d \to \infty} \left( 1 - \frac{p^{R-1}}{p^{R+d-1}} \right) = 1 \]
implies that for any integer \( R \),
\[ \lim_{d \to \infty} P^d \left( \| \tilde{X} \|_T \leq p^R \right) = e^{-\sigma Tp^{-Rb}}. \] (55)

In contrast, Theorem 4.1 implies that
\[ \lim_{d \to \infty} \otimes^d P \left( \| \tilde{Y} \|_T \leq p^R \right) = \lim_{d \to \infty} e^{-d\sigma \alpha_1 Tp^{-Rb}} = 0. \] (56)

This marked contrast between (55) and (56) demonstrates that exit probabilities from a fixed ball depend only to a small degree on dimension for the max-norm process, but are very sensitive to changes in dimension for the product process.

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