R-parity from string compactification

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The strategy for assigning $\mathbb{Z}_4$R parity in the string compactification is presented. For the visible sector, an anti-SU(5) (flipped-SU(5)) grand unification (GUT) model with three families is used to reduce the number of representations compared to the number in the minimal supersymmetric standard models (MSSMs). The SO(32) heterotic string is used to allow a large nonabelian gauge group SU($N$), $N \geq 9$, for the hidden sector such that the number of extra U(1) factors is small. A discrete subgroup of the gauge U(1)’s is defined as the $\mathbb{Z}_4$R parity. Spontaneous symmetry breaking of anti-SU(5) GUT is achieved by the vacuum expectation values of two index antisymmetric tensor Higgs fields $10_0^+ + 10^{-1}$ that led to our word ‘anti-SU(5)’. In the illustrated example, the multiplicity 3 in one twisted sector allows the permutation symmetry $S_3$ that leads us to select the third family members and one MSSM pair of the Higgs quintets.

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I. INTRODUCTION

Grand unified theories (GUTs) attracted a great deal of attention ethetically because they provided unification of gauge couplings and charge quantization $[1, 2]$. But there seems to be a fundamental reason leading to GUTs even at the standard model (SM) level. With the electromagnetic and charged currents (CCs), the leptons need representations which are a doublet or bigger. A left-handed (L-handed) lepton doublet ($\nu_e, e$) alone is not free of gauge anomalies because the observed electromagnetic charges are not $\pm \frac{1}{2}$. The anomalies from the fractional electromagnetic charges of the $u$ and $d$ quarks are needed to make the total anomaly to vanish $[4, 5]$. In view of this necessity for jointly using both leptons and quarks to cancel gauge anomalies even in the SM, we can view that unification of leptons and quarks is fundamentally needed in addition to the esthetic view.

In the SM, the largest number of parameters arises from the Yukawa couplings which form the bases of the family structure. Repetition of fermion families in 4-dimensional (4D) field theory or family-unified GUT (family-GUT) was formulated by Georgi $[6]$, requiring un-repeated chiral representations while not allowing gauge anomalies. Some interesting family-GUT models are the spinor representation of SO(14) $[7,8]$ and $84 \oplus 9 \cdot 9$ of SU(9) $[9]$. While Refs. $[7–9]$ do not provide interesting non-vanishing flavor quantum number, the SU(11) model $[6]$ allows a possibility for non-vanishing flavor quantum number such as U(1)$_{\mu-\tau}$ or U(1)$_{B-L}$ $[10]$.

On the other hand, the standard-like models from string have been the main focus of phenomenological activities for the ultraviolet completion of the SM toward the minimal supersymmetric standard model (MSSM) in the last several decades $[12–35]$. These models use the chiral spectrum from the level–1 construction which leads to unification of gauge couplings $[36]$. So, the standard-like models from string compactification achieved the goal of gauge coupling unification and GUT theories from string have not attracted much attention. Nevertheless, GUTs from strings $[37, 38]$ have been discussed sporadically for anti-SU(5) $[39]$ (or flipped SU(5) $[40]$), dynamical symmetry breaking $[41, 42]$, and family unification $[43]$. In fact, family-GUTs are much easier in discussing the family problem, in particular on the origin of the mixing between quarks/leptons because the number of representations in family-GUTs is generally much smaller than in their (standard-like model) subgroups.

In this paper we study the R-parity assignment for a family-GUT from string compactification. So far, most string compactification models used the $E_8 \times E_8$ heterotic string in which a GUT with rank greater than 8 is impossible. The group SU(11) has rank 10 which cannot arise from compactification of $E_8 \times E'_8$. Therefore, firstly we formulate the orbifold compactification $[43, 44]$ of SO(32) heterotic string $[11]$ whose rank is 16. From the compactification of the SO(32) heterotic string, however, we cannot derive the SU(11) model and the largest possible non-abelian gauge group we obtain is SU(9) $[44]$.

1 For more attempts of family-GUTs, see references in $[10]$. 


To discuss the R-parity from string compactification, we should consider a specific model. Among compactification schemes, we adopt the orbifold method. Among 13 possibilities listed in Ref. [45, 46], we employ $Z_{12-I}$ orbifold because it has the simplest twisted sectors. Twisted sectors are distinguished by Wilson lines [47]. The Wilson line in $Z_{12-I}$ distinguishes three fixed points at a twisted sector. Therefore, it suffices to consider only three cases at a twisted sector. In all the other orbifolds of Ref. [45], consideration of various possibilities of Wilson lines and the accompanying consistency conditions are much more involved.

In Sec. II, we recapitulate the orbifold methods used in this paper for an easy reference to Sec. III. In Sec. III, we construct a specific supersymmetric model possessing nonabelian groups SU(9) and SU(5). In Sec. IV, we assign $Z_4$ quantum numbers to the massless fields obtained in Sec. III. Section V is a conclusion.

II. ORBIFOLD COMPACTIFICATION

A. Tensor representations of SU(N) and SO(2N) spinors

All anti-symmetric representations $\Phi^A, \Phi^{AB} = \Phi^{[AB]}, \cdots \Phi^{[AB\cdots]}$, etc. of SU(N) contain only $3$ and $\bar{3}$ under its subgroup SU(3). Therefore, using only anti-symmetric representations for matter in an SU(N) GUT guarantees no high dimensional colored matter particles such as $6, \bar{6}$, etc. The SU(N) representations are either of vector-type or of spinor type. Let us define the vector representation of SU(N) as

\[
\text{SU(9)} : \begin{array}{c}
9 = (-1^0); \\
36 = (-1 - 1 0^7)
\end{array} \quad (1)
\]
\[
\text{SU(5)} : \begin{array}{c}
5 = (+1^0); \\
10 = (+1 + 1 0^3)
\end{array} \quad (2)
\]

where we defined differently for SU(9) and SU(5). It is a matter of convention to call what are $9$ and $5$. Here, the first (second) ones are called one-(two-)index tensors. The second ones carry two non-zero numbers. For spinor types, we also use this convention, but spinors fill all the $9$ (for SU(9)) and $5$ (for SU(5)) slots such as $(+ + \cdots)$. One index spinor is defined to have one sign and the rest the opposite sign. In string compactification, there appear only up to two index tensors. Define $36$ of the spinor type in SU(9) such that $36 \cdot 9 \cdot 9$ is allowed if two times the sum of entries is 0 modulo 9. Therefore, we have $36 = (+ + \cdots)$. Similarly, we have the spinor $10$ of SU(5) as $10 = (\pm 5 \cdot 5)$. Thus, one and two index spinors are

\[
\begin{array}{c}
\text{SU(9)} : \begin{array}{c}
\bar{9} = (+ \cdots); \\
\bar{36} = (- + \cdots)
\end{array} \\
\text{SU(5)} : \begin{array}{c}
\bar{5} = (\cdots); \\
\bar{10} = (+ \cdots)
\end{array}
\end{array}
\]

(3) (4)

One $36$ of (1) and two $\bar{9}$’s of (3) give $-4 - 7 - 7 = -18$ from $36 \cdot \bar{9} \cdot \bar{9}$.

B. Two dimensional orbifolds

In compactifying 10D string theory to 4D effective field theory, the small internal 6D is a compactified three two-tori. So, the basic is a two dimensional torus. A two dimensional orbifold is this two dimensional torus moded by discrete groups for which we adopt the discrete group $Z_{12-I}$. The shift vector we use here is similar to that of [44] but not the same, and the spectrum we obtain is a bit simpler. Since the strategy of compactifying SO(32) heterotic string is already presented there, we list key formulae used in the next chapter in Appendix A.

Most interesting spectra in the present paper are arising in the twisted sectors. Geometrically, twisted sectors correspond to fixed points. The moding vector of $Z_{12-I}$ is $\phi_s = \frac{1}{12}(5, 4, 1)$.

\[
\phi_s = \frac{1}{12}(5, 4, 1).
\]

2 The spinor $(++++)$ for $+\equiv \frac{1}{2}$ gives two times the sum as 9.
The multiplicities in the fixed points are three because \( \phi_s \) contains \( \frac{1}{3} \) in the second torus. This multiplicity 3 can be distinguished by Wilson lines \( a_3 = a_4 \) (3, 4 denoting two directions in the second torus) \[47\]. Entries of \( a_3 \) are some integer multiples of \( \frac{1}{3} \). But \( 3a_3 \) contains only integer entries, that leads to some conditions at twisted sectors \( T_3, T_6 \) and \( T_9 \).

There is one point to be noted for \( Z_{12} \). If \( N = \text{even} \), the \( k = 1, \cdots, N - 1 \) sectors provide the opposite chiralities in the \( k = N - 1, \cdots, N + 1 \) sectors,

\[
T_k \leftrightarrow T_{N-k}.
\]

Then, the corresponding phases of Eq. (A3) compare as

\[
e^{2\pi i \Theta_k} \leftrightarrow e^{2\pi i \Theta_{N-k}}
\]

whose difference is \( e^{2\pi i (N-2k)/12} \). Thus, if \( 2k = N \) then \( T_{N-k} \) do not provide the charge conjugated fields of \( T_k \), but they are identical. For \( T_3 \) or \( T_9 \), therefore, we must provide the additional charge conjugated fields with an extra phase \( e^{2\pi i (10/12)} = e^{2\pi i (-2/12)} \), the difference of \( \hat{s} \cdot \phi_s \) for \( \hat{s} = (+++) \) of \( R \) and \((-+-) \) of \( L \). We choose \( T_9 \) for this phase. In \( T_6 \), however, we do not need this since the charged conjugated fields also appear there.

C. Multiplicities in the twisted sectors

In the compactification of the SO(32) heterotic string, spinors in \( U \) are not appearing because it is not possible to have \( P^2 = 2 \) from sixteen \( \pm \frac{1}{2} \)'s. Only vector types are possible in \( U \). The \( T_k \) twisted sector has three possibilities

\[
T_k^{0,+,-} : kV_a = \begin{cases} 
  kV \equiv kV_0 & \text{if } k(V + a_3) \equiv kV_+ \\
  k(V - a_3) \equiv kV_- & \text{if } k(V - a_3) \equiv kV_-.
\end{cases}
\]

(7)

We select only the even lattices shifted from the untwisted lattices, therefore, we consider even numbers for the sum of absolute value of each element of \( P \). They should be even numbers if the absolute values are added. In Appendix A \( \Theta_k \) in Eq. (A3) is defined for the twisted sector \( T_k \). Since different \( P \)'s in the same twisted sector may lead to different gauge group representations, there is a need to distinguish them. So, we may use

\[
\Theta_{\text{Group}} = -\hat{s} \cdot \phi_s - k p_{\text{vec}}^{k \text{th}} \cdot \phi_s + k P \cdot V_0 + \frac{k}{2}(\phi_s^2 - V_0^2) + \Delta_N^N - \delta_k^N,
\]

(8)

where

\[
\delta_k^N = 2\delta_k.
\]

(9)

III. THE MODEL

The left-hand side (LHS) sector of the heterotic string is the gauge sector. The shift vector \( V_0 \) and Wilson line \( a_3 \) are restricted to satisfy the \( Z_{12-I} \) orbifold conditions,

\[
12(V_0^2 - \phi_s^2) = 0 \text{ mod even integer, } 12(V_0 \cdot a_3) = 0 \text{ mod even integer, } 12|a_3|^2 = 0 \text{ mod even integer.}
\]

(10)

Here, \( a_3 (= a_4) \) is chosen to allow and/or forbid some spectra, and is composed of fractional numbers with the integer multiples of \( \frac{1}{3} \) because the second torus has the \( Z_3 \) symmetry. The model is

\[
V_0 = \left( \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12} \right),
\]

\[
V_+ = \left( \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12} \right),
\]

\[
V_- = \left( \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12} \right),
\]

(11)

where

\[
a_3 = a_4 = \left( \frac{0^9}{3}, \frac{-2}{3}, 0; \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3} \right).
\]

(12)
The right-hand side (RHS) sector of the heterotic string is given by the spin lattice \( s = (\Theta \oplus \Theta; \bar{s}) \) with every entry being integer multiples of \( \frac{1}{2} \), satisfying \( s^2 = 2 \). The \( \hat{\phi}_s \) for \( \mathbb{Z}_{12-1} \) with three entries is Eq. 6,

\[
\hat{\phi}_s = \left( \frac{5}{12}, \frac{4}{12}, 1 \right), \quad \text{with } \phi_s^2 = \left( \frac{5}{12}, \frac{4}{12}, 1 \right)^2 = \frac{42}{144}.
\]

A. Untwisted sector \( U \)

In \( U \), we find the following nonvanishing roots of \( \text{SU}(5) \times \text{SU}(9) \times \text{U}(1)^4 \),

\[
\begin{align*}
\text{SU}(5) \text{ gauge multiplet: } & \quad P \cdot V = 0 \mod \text{ integer and } P \cdot a_3 = 0 \mod \text{ integer} \quad \text{SU}(5): \quad P = (0^9; 0^2; 12 \cdot 0,0,0). \\
\text{SU}(9)' \text{ gauge multiplet: } & \quad P \cdot V = 0 \mod \text{ integer} \quad \text{SU}(9)': \quad P = (12 \cdot 0,0,0) \mod 0,0,0).
\end{align*}
\]

In addition, there exists \( \text{U}(1)^4 \) symmetry. The non-singlet matter fields satisfy

\[
\text{SU}(5) \text{ and/or } \text{SU}(9)' \text{ matter multiplet: } \quad P \cdot V = \frac{1}{12} \cdot 5, P \cdot a_3 = 0 \mod \text{ integer}.
\]

The conditions allows the \( P^2 = 2 \) lattice shown in Table I.

| \( U_i \) | \( P \cdot V \) | Chirality | \( |p_{\text{spin}}| (p_{\text{spin}} \cdot \hat{\phi}) \) |
|---|---|---|---|
| \( U_1 (p \cdot V = \frac{5}{12}) \) | \((-1)^{a_3}; 0; 0^5)\) | L | \( [\Theta; -; -] \) \( \frac{12}{12} \) |
| \( U_2 (p \cdot V = \frac{1}{4}) \) | \(-; -; -)\) | L | \(-\) |
| \( U_3 (p \cdot V = \frac{1}{12}) \) | \(-; -; -)\) | L | \(-\) |

TABLE I: It is a \( 9_L \) in view of Eq. 19.

B. Twisted sectors

In \( T_9 \), for example, we have

\[
\begin{align*}
9V_0 & = \left( \left( \frac{9}{12} \right)^9; \left( \frac{27}{12} \right); \left( \frac{54}{12} \right); \left( \frac{54}{12} \right)^5 \right), \\
9V_+ & = \left( \left( \frac{9}{12} \right)^9; \left( \frac{27}{12} \right); \left( \frac{54}{12} \right); \left( \frac{54}{12} \right)^5 \right), \\
9V_- & = \left( \left( \frac{9}{12} \right)^9; \left( \frac{27}{12} \right); \left( \frac{54}{12} \right); \left( \frac{54}{12} \right)^5 \right).
\end{align*}
\]

We shift the lattice by adding \( \text{SU}(9)' \times \text{SU}(5) \) singlet vector; \( v : (v_1, \cdots, v_{16}) \) satisfying \( \sum_i |v_i| = 0 \modulo 2k \). In the following equations, we list all the shift vectors, \( \bar{k}V_{0,+,-} \), in all twisted sectors. Equations are in Eq. 24.

\[
\begin{align*}
T_0 & \quad (\left( \frac{9}{12} \right)^0; \left( \frac{27}{12} \right)^0; \left( \frac{54}{12} \right)^0), \\
T_1 & \quad (\left( \frac{9}{12} \right)^0; \left( \frac{27}{12} \right); \left( \frac{54}{12} \right); \left( \frac{54}{12} \right)^5), \\
T_2 & \quad (\left( \frac{9}{12} \right)^0; \left( \frac{27}{12} \right); \left( \frac{54}{12} \right)^5), \\
T_3 & \quad (\left( \frac{9}{12} \right)^0; \left( \frac{27}{12} \right); \left( \frac{54}{12} \right)^5), \\
T_4 & \quad (\left( \frac{9}{12} \right)^0; \left( \frac{27}{12} \right); \left( \frac{54}{12} \right)^5).
\end{align*}
\]
1. Twisted sector $T_9 (\delta_3 = \frac{9}{12})$

- Two indices spinor-form from $T_9^0$: with Eq. (17) the spinor forms satisfying the mass-shell condition $(P + 9V_0)^2 = \frac{234}{144} = \frac{14}{9}$ and the Wilson-line condition $12P(P + 9V_0) \cdot a_3 = 0$ are possible for SU(9):

$$P_9 = \left( \begin{array}{c} -3/2 \\ -3/2 \\ -7/2 \\ -9/2 \\ -9/2 \end{array} \right).$$

(26)

Instead of the above $P_9$, with Eq. (24) we have

$$P_9 = (- - + 7; + - ; - 5)$$

(27)

which is a shift from Eq. (26) by $(1^9; 3^4; 4^5)$. With Eqs. (24) and (27), we present the multiplicity in Table II.

| Chirality $\bar{s}$ | $-\bar{s} \cdot \phi_s$ | $-k P_v^{k_{th}} \cdot \phi_s$ | $k P_0 \cdot V_0^0$ | $(k/2) \phi_s^2$, $- (k/2) V_0^0$ | $\Delta_N^s$, $- \delta_9^s$ | $\Theta_9$, Mult. |
|---------------------|----------------------|-------------------------------|-------------------|-------------------------------|------------------|------------------|
| $\ominus = L$ (- - -) | $\frac{5}{12}$ | $\frac{3}{12}$ | $\frac{3}{12}$ | $\frac{3}{12}$ | $\frac{234}{144}$ | $\frac{144}{144}$ | $\frac{0}{12}$ | $\frac{0}{12}$ | $\frac{1}{12}$ | $\frac{0}{12}$ | $\frac{2}{12}$ | $2$ |
| $\oplus = R$ (+ + +) | $-\frac{5}{12}$ | $-\frac{3}{12}$ | $-\frac{3}{12}$ | $-\frac{3}{12}$ | $-\frac{189}{144}$ | $-\frac{144}{144}$ | $\frac{0}{12}$ | $\frac{0}{12}$ | $\frac{-8}{12}$ | $\frac{-2}{12}$ | $1$ |

TABLE II: Two index spinor-form from $T_9^0$: $(1, 36')_t$. This is the abbreviated one of Table XXIII presented in Appendix C. Note that we provided an extra phase $e^{2\pi i (-2/N)}$ as commented below Eq. (26).

2. Twisted sector $T_4 (\delta_4 = 0)$

- One index vector-form from $T_4^0$:

$$P_9 = (-1^8; 0^0; -1^5),$$

(28)

satisfies $(P_9 + 4V_0)^2 = \frac{192}{144}$. Thus, we obtain Table III.

| Chirality $\bar{s}$ | $-\bar{s} \cdot \phi_s$ | $-k P_v^{k_{th}} \cdot \phi_s$ | $k P_0 \cdot V_0^0$ | $(k/2) \phi_s^2$, $- (k/2) V_0^0$ | $\Delta_N^s$, $- \delta_9^s$ | $\Theta_9$, Mult. |
|---------------------|----------------------|-------------------------------|-------------------|-------------------------------|------------------|------------------|
| $\ominus = L$ (- - -) | $\frac{5}{12}$ | $\frac{5}{12}$ | $\frac{5}{12}$ | $\frac{5}{12}$ | $\frac{144}{144}$ | $\frac{144}{144}$ | $\frac{108}{12}$ | $\frac{108}{12}$ | $\frac{-8}{12}$ | $\frac{-2}{12}$ | $7$ |
| $\oplus = R$ (+ + +) | $\frac{5}{12}$ | $\frac{5}{12}$ | $\frac{5}{12}$ | $\frac{5}{12}$ | $\frac{144}{144}$ | $\frac{144}{144}$ | $\frac{108}{12}$ | $\frac{108}{12}$ | $\frac{-8}{12}$ | $\frac{-2}{12}$ | $7$ |

TABLE III: One index vector-form from $T_4^0$: Thus, we obtain $7 \cdot (1, 9')_R$.

- One index spinor-form for $T_4^{-}$: the spinor-form

$$P_9 = \left( \begin{array}{c} \pm \frac{8}{2} \\ - \frac{3}{2} \\ - \frac{5}{2} \end{array} \right),$$

(29)

satisfies $(P_9 + 4V_-)^2 = \frac{192}{144}$, which is shown in Table IV.

| Chirality $\bar{s}$ | $-\bar{s} \cdot \phi_s$ | $-k P_v^{k_{th}} \cdot \phi_s$ | $k P_0 \cdot V_0^0$ | $(k/2) \phi_s^2$, $- (k/2) V_0^0$ | $\Delta_N^s$, $- \delta_9^s$ | $\Theta_9$, Mult. |
|---------------------|----------------------|-------------------------------|-------------------|-------------------------------|------------------|------------------|
| $\ominus = L$ (- - -) | $\frac{5}{12}$ | $\frac{8}{12}$ | $\frac{4}{12}$ | $\frac{84}{144}$ | $-\frac{1272}{144}$ | $\frac{0}{12}$ | $\frac{0}{12}$ | $\frac{-4}{12}$ | $\frac{-9}{12}$ | $7$ |
| $\oplus = R$ (+ + +) | $\frac{5}{12}$ | $\frac{8}{12}$ | $\frac{4}{12}$ | $\frac{84}{144}$ | $-\frac{1272}{144}$ | $\frac{0}{12}$ | $\frac{0}{12}$ | $\frac{-4}{12}$ | $\frac{-9}{12}$ | $0$ |

TABLE IV: One index spinor-form from $T_4^{-}$: Thus, we obtain $7 \cdot (1, 9')_L$. 
3. Twisted sector $T_1 (\delta_1 = \frac{1}{12})$

In $T_1$, we use Eq. (11).

- One index vector-form for $T_1^+$: the vector

$$P_5 = (0^5; +1, 0; -1, 0, 0, 0),$$

satisfies $(P + V_+)^2 = \frac{186}{144}$ which is short by $\frac{24}{144} = \frac{2}{12}$ from the target value of $\frac{240}{144}$, and all states are shown in Table V.

| Chirality | $\hat{s}$ | $-\hat{s} \cdot \phi_s$ | $-k P_{vec}^{k_{th}} \cdot \phi_s$ | $k P_5 \cdot V_1$ | $(k/2)\phi_s^2$, $-(k/2)V_1^2$, $\Delta_{N_s}$, $-\delta_{N_s}^L$ | $\Theta_5$, Mult. |
|-----------|-----------|----------------|-----------------|---------------|-----------------|-------------|
| $\Theta = L$ (---) | $\frac{11}{17}$ | $\frac{15}{17}$ | $\frac{17}{17}$ | $\frac{21}{144}$ | $\frac{45}{144}$; $\frac{17}{17}$ | $\frac{4}{12}$ | 0 |
| $\Theta = R$ (+++) | $\frac{5}{17}$ | $\frac{5}{17}$ | $\frac{5}{17}$ | $\frac{21}{144}$ | $\frac{45}{144}$; $\frac{17}{17}$ | $\frac{2}{12}$ | 3 |

TABLE V: One index vector-form from $T_1^+$: Thus, we obtain $3 \cdot (5, 1)_R$ that is $\rho$ in Table XIX.

- One index vector-form for $T_1^+$: the vector

$$P_5 = (0^5; 0, -1, 0, 0, 0, 0),$$

satisfies $(P + V_+)^2 = \frac{186}{144}$ which is short by $\frac{24}{144} = \frac{2}{12}$ from the target value of $\frac{240}{144}$, and all states are shown in Table VI.

- One index vector-form for $T_1^-$: the vector

$$P_5 = (-1^5; -1, 0; 0^5),$$

satisfies $(P + V_-)^2 = \frac{186}{144}$ which is short by $\frac{24}{144} = \frac{2}{12}$ from the target value of $\frac{240}{144}$, and the spectrum is shown in Table VII.

| Chirality | $\hat{s}$ | $-\hat{s} \cdot \phi_s$ | $-k P_{vec}^{k_{th}} \cdot \phi_s$ | $k P_5 \cdot V_1$ | $(k/2)\phi_s^2$, $-(k/2)V_1^2$, $\Delta_{N_s}$, $-\delta_{N_s}^L$ | $\Theta_5$, Mult. |
|-----------|-----------|----------------|-----------------|---------------|-----------------|-------------|
| $\Theta = L$ (---) | $\frac{5}{17}$ | $\frac{15}{17}$ | $\frac{17}{17}$ | $\frac{21}{144}$ | $\frac{10}{144}$ | $\frac{4}{12}$ | 0 |
| $\Theta = R$ (+++) | $\frac{5}{17}$ | $\frac{15}{17}$ | $\frac{17}{17}$ | $\frac{21}{144}$ | $\frac{10}{144}$ | $\frac{4}{12}$ | 3 |

TABLE VII: One index vector-form from $T_1^-$: Thus, we obtain $3 \cdot (1, 9')_R$.

- One index vector-form for $T_1^-$: the vector

$$P_5 = (0^5; -1, 0; 1^4),$$

satisfies $(P + V_-)^2 = \frac{186}{144}$ which is short by $\frac{24}{144} = \frac{2}{12}$ from the target value of $\frac{240}{144}$, and we obtain Table VIII.
| Chirality | $\tilde{s}$ | $-\tilde{s} \cdot \phi_s$ | $-k P_{\text{vec}} ^{k_{\text{th}}} \cdot \phi_s \kappa P \cdot V$ | $(k/2) \phi_s^2$, $-(k/2)V_s^2$, $\Delta_s^N$, $-\delta_s^N$ | $\Theta_s$, Mult. |
|-----------|------------|----------------|---------------------------------|---------------------------------|----------------|
| $\varnothing = L$ (---) | $\frac{7}{12}$ | $\frac{1}{12}$ | $\frac{1}{12}$ | $\frac{21}{144}$ | $\frac{49}{144}$ | $\frac{5}{12}$ | 3 |
| $\oplus = R$ (+++ )| $\frac{5}{12}$ | $\frac{5}{12}$ | $\frac{7}{12}$ | $\frac{21}{144}$ | $\frac{93}{144}$ | $\frac{12}{12}$ | 3 |

TABLE VIII: One index vector-form from $T_1^-$: Thus, we obtain $3 \cdot (5, 1)_L + 3 \cdot (5, 1)_R$ that are $\beta$ and $\alpha$ in Table [XIX].

4. Twisted sector $T_2 (\delta_2 = \frac{9}{12})$

- One index vector-form from $T_2^0$: the vector

$$P_9 = (-10^8; 0, +1; 0^5),$$

which gives $(P + 2V_+)^2 = \frac{168}{144}$ which is short by $\frac{18}{144} = \frac{2}{12}$ from the target value of $\frac{216}{144}$, and massless fields are shown in Table IX.

| Chirality | $\tilde{s}$ | $-\tilde{s} \cdot \phi_s$ | $-k P_{\text{vec}} ^{k_{\text{th}}} \cdot \phi_s \kappa P \cdot V$ | $(k/2) \phi_s^2$, $-(k/2)V_s^2$, $\Delta_s^N$, $-\delta_s^N$ | $\Theta_s$, Mult. |
|-----------|------------|----------------|---------------------------------|---------------------------------|----------------|
| $\varnothing = L$ (---) | $\frac{7}{12}$ | $\frac{1}{12}$ | $\frac{1}{12}$ | $\frac{42}{144}$ | $\frac{84}{144}$ | $\frac{5}{12}$ | 0 |
| $\oplus = R$ (+++ )| $\frac{5}{12}$ | $\frac{5}{12}$ | $\frac{7}{12}$ | $\frac{42}{144}$ | $\frac{94}{144}$ | $\frac{12}{12}$ | 3 |

TABLE IX: One index vector-form from $T_2^0$: Thus, we obtain $3 \cdot (1, 9')_R$.

- One index spinor-form from $T_2^-$: the vector

$$P_5 = (-9; +, +; + + + +),$$

which gives $(P + 2V_-)^2 = \frac{192}{144}$ which is short by $\frac{2}{12}$ from the target value of $\frac{216}{144}$, and massless fields are shown in Table X.

| Chirality | $\tilde{s}$ | $-\tilde{s} \cdot \phi_s$ | $-k P_{\text{vec}} ^{k_{\text{th}}} \cdot \phi_s \kappa P \cdot V$ | $(k/2) \phi_s^2$, $-(k/2)V_s^2$, $\Delta_s^N$, $-\delta_s^N$ | $\Theta_s$, Mult. |
|-----------|------------|----------------|---------------------------------|---------------------------------|----------------|
| $\varnothing = L$ (---) | $\frac{5}{12}$ | $\frac{10}{12}$ | $\frac{1}{12}$ | $\frac{42}{144}$ | $\frac{90}{144}$ | $\frac{5}{12}$ | 3 |
| $\oplus = R$ (+++ )| $\frac{5}{12}$ | $\frac{5}{12}$ | $\frac{7}{12}$ | $\frac{42}{144}$ | $\frac{94}{144}$ | $\frac{12}{12}$ | 3 |

TABLE X: One index spinor-form from $T_2^-$. Thus, we obtain $3 \cdot (5, 1)_L + 3 \cdot (5, 1)_R$ that are $\eta$ and $\xi$ in Table [XIX].

5. Twisted sector $T_7 (\delta_7 = \frac{1}{12})$

- One index spinor-form from $T_7^0$:

$$P_9 = (- + 8; -, -; - 5),$$

gives $(P + 7V_0)^2 = \frac{138}{144}$, which is short of $\frac{4}{12}$ from $\frac{216}{144}$, and we obtain Table XI.

- Two index vector-form from $T_7^+$:

$$P_9 = (-1 0^8; +3, -4; 0^5),$$

gives $(P + 7V_+)^2 = \frac{186}{144}$ which is short of $\frac{2}{12}$ from the target value $\frac{210}{144}$, and massless fields are shown in Table XII.
Two index spinor-form from $g$ gives $(P_5)\phi^2$, $-(k/2)V_0^2$, $\Delta_0^N$, $-\delta_0^N$.

TABLE XI: One index spinor-form from $T^\sigma_\tau$: $3 \cdot (1, 9')_R$.

| Chirality $\tilde{s}$ | $-\tilde{s} \cdot \phi_5$ | $-k_p^{kth} \cdot \phi_s$ | $k P_5 \cdot V_0$ | $(k/2)\phi^2_s$, $-(k/2)V_0^2$, $\Delta_0^N$, $-\delta_0^N$ | $\Theta_5$, Mult. |
|------------------------|--------------------------|--------------------------|-------------------|---------------------------------|-----------------|
| $\oplus = L$ (- - -)   | $\frac{5}{12}$           | $\frac{1}{12}$          | $\frac{1}{12}$    | $\frac{147}{144}$, $\frac{4}{144}$, $\frac{449}{144}$, $\frac{1}{12}$ | $\frac{10}{12}$ 0 |
| $\oplus = R$ (+ + +)   | $\frac{5}{12}$           | $\frac{1}{12}$          | $\frac{1}{12}$    | $\frac{147}{144}$, $\frac{4}{144}$, $\frac{449}{144}$, $\frac{1}{12}$ | $\frac{10}{12}$ 3 |

TABLE XII: One index vector-form from $T^\sigma_\tau$: $3 \cdot (1, 9')_L$.

| Chirality $\tilde{s}$ | $-\tilde{s} \cdot \phi_5$ | $-k_p^{kth} \cdot \phi_s$ | $k P_5 \cdot V_1$ | $(k/2)\phi^2_s$, $-(k/2)V_0^2$, $\Delta_0^N$, $-\delta_0^N$ | $\Theta_5$, Mult. |
|------------------------|--------------------------|--------------------------|-------------------|---------------------------------|-----------------|
| $\oplus = L$ (- - -)   | $\frac{5}{12}$           | $\frac{1}{12}$          | $\frac{1}{12}$    | $\frac{147}{144}$, $\frac{4}{144}$, $\frac{449}{144}$, $\frac{1}{12}$ | $\frac{10}{12}$ 0 |
| $\oplus = R$ (+ + +)   | $\frac{5}{12}$           | $\frac{1}{12}$          | $\frac{1}{12}$    | $\frac{147}{144}$, $\frac{4}{144}$, $\frac{449}{144}$, $\frac{1}{12}$ | $\frac{10}{12}$ 3 |

TABLE XIII: Two index spinor-form from $T^\sigma_\tau$: $3 \cdot (10, 1)_R$ that is $T$ in Table XIX.

- Two index spinor-form from $T^\sigma_\tau$: The vector
  \[
  P_5 = (-9; -9; + + + -),
  \]
  gives $(P_5 + 7V_-)^2 = \frac{186}{144}$ which is short of $\frac{2}{12}$ from the target value $\frac{210}{144}$, and massless fields are shown in Table XIV.

- One index spinor-form from $T^\sigma_\tau$: The vector
  \[
  P_5 = (-9; -9; + + + +),
  \]
  gives $(P_5 + 7V_-)^2 = \frac{148}{144}$ which is short of $\frac{6}{12}$ from the target value $\frac{210}{144}$, and massless fields are shown in Table XIX.

| Chirality $\tilde{s}$ | $-\tilde{s} \cdot \phi_5$ | $-k_p^{kth} \cdot \phi_s$ | $k P_5 \cdot V_1$ | $(k/2)\phi^2_s$, $-(k/2)V_0^2$, $\Delta_0^N$, $-\delta_0^N$ | $\Theta_5$, Mult. |
|------------------------|--------------------------|--------------------------|-------------------|---------------------------------|-----------------|
| $\oplus = L$ (- - -)   | $\frac{5}{12}$           | $\frac{1}{12}$          | $\frac{1}{12}$    | $\frac{147}{144}$, $\frac{4}{144}$, $\frac{449}{144}$, $\frac{1}{12}$ | $\frac{10}{12}$ 3 |
| $\oplus = R$ (+ + +)   | $\frac{5}{12}$           | $\frac{1}{12}$          | $\frac{1}{12}$    | $\frac{147}{144}$, $\frac{4}{144}$, $\frac{449}{144}$, $\frac{1}{12}$ | $\frac{10}{12}$ 3 |

TABLE XIV: One index spinor-form from $T^\sigma_\tau$: $3 \cdot (10, 1)_L$ that is $\bar{T}$ in Table XIX.

6. Twisted sector $T_6 (\delta_6 = \frac{1}{12})$

Note that $\frac{k}{2}(V^2 - \phi^2)$ in Eq. (A3) is not distinguished by Wilson lines, and we just calculate the spectra from $6V_0$ with multiplicity 3 (for $T_6^{0, + -}$). With 3 and 0 in $2\delta$, we obtain only vectorlike representations. To clarify, we list all the possibilities for $5, \bar{5}$, 10, and 10. All the allowed ones are spinor forms.

- Spinor-form for 10,
  \[
  P_5 = (+9; \frac{-3}{2}, \frac{-5}{2}, \frac{-7}{2}; + + + -)
  \]  (40)
TABLE XV: \( T_6: \mathbf{10} \) appears 6 times for \( L \) and 6 times for \( R \). The same multiplicities occur also in \( T_6'^{-} \). So, in total, there appear 36 multiplicities of \( \mathbf{10} \) for \( L \) and also for \( R \).

which saturates the masslessness condition \( (P_5 + 6V_0)^2 = \frac{216}{144} \). The spectra are shown in Table XV. Considering Eq. (40) for \( \frac{-5}{2} \) and \( \frac{7}{2} \), we obtain 12 each for \( L \) and \( R \). Thus, we obtain 36 \( \cdot (\mathbf{10}_L \oplus \mathbf{10}_R) \).

- Spinor-form for \( \mathbf{10} \),

\[
P_5 = (+; \, \frac{-3}{2}, \, \frac{-5}{2}, \, \frac{-7}{2}; \, +, +, +)
\]

which saturates the masslessness condition \( (P_5 + 6V_0)^2 = \frac{216}{144} \). So, we obtain 36 \( \cdot (\mathbf{10}_L \oplus \mathbf{10}_R) \).

- Spinor-form for \( \mathbf{5} \),

\[
P_5 = (+; \, \frac{-3}{2}, \, \frac{-5}{2}, \, \frac{-7}{2}; \, +, +, +)
\]

which saturates the masslessness condition \( (P_5 + 6V_0)^2 = \frac{216}{144} \), and we obtain 36 \( \cdot (\mathbf{5}_L \oplus \mathbf{5}_R) \).

- Spinor-form for \( \mathbf{5} \),

\[
P_5 = (+; \, \frac{-3}{2}, \, \frac{-5}{2}, \, \frac{-7}{2}; \, +, +, +)
\]

which saturates the masslessness condition \( (P_5 + 6V_0)^2 = \frac{216}{144} \), and we obtain 36 \( \cdot (\mathbf{5}_L \oplus \mathbf{5}_R) \).

Summarizing the non-singlet chiral fields we obtained, \( 9f'_L(U) + 36f'_L(T^0_9) + (36f'_L(T^3_9) + 36f'_R(T^0_9)) + 7 \cdot 9f'_R(T^0_4) + 7 \cdot 9f'_L(T^-_9) + 3 \cdot 9f'_R(T^-_9) + 3 \cdot 9f'_R(T^0_7) + 3 \cdot 9f'_L(T^+_7) \) for SU(9). After removing vector-like representations, note that the SU(9) anomaly is absent with \( 36f'_R + 5 \cdot 9f'_R \).

Similarly after removing vector-like representations, for SU(5) we obtain

\[
3 \cdot 10_L(T^+_7) + 3 \cdot 5_L(T^-_7).
\]

These spectra in Eqs. (44) and (45) do not lead to non-Abelian gauge anomalies. For \( 5_L \) in Eq. (45), in fact it is a linear combination of \( 5_L(T^-_7) \) (\( F \) in Table XIX) and \( 5_L(T^-_2) \) (\( \eta \) in Table XIX). In addition, note that there appear 72 pairs of \( \mathbf{10} \oplus \overline{\mathbf{10}} \) from \( T_6' \), which are needed for breaking the anti-SU(5) GUT.

C. Singlets, quintets, and Higgs fields

We obtained the rank 16 group SU(5) × SU(9)’ × U(1)^4. The four U(1) charges are also shown in the following tables, where \( Q_X \) is the U(1)_X charge of anti-SU(5) GUT. The four U(1) charges in the gauge group SU(9)^’ × SU(5) × U(1)^4 are defined by

\[
Q_1 = \frac{1}{9} (-2, -2, -2, -2, -2, -2, -2, -2, 0, 0; 0^5)
\]

(46)

\[
Q_2 = (0^3; -2, 0; 0^5)
\]

(47)

\[
Q_3 = (0^3; 0, -2; 0^5)
\]

(48)

\[
Q_X = (0^3; 0, 0, -2, -2, -2, -2).
\]

(49)
We summarize the SU(5)×SU(9)' singlets in the following Tables XVII, XVII and XVIII with the format of Subsec. III.B In Table XVIII we tabulate the four U(1) charges given in Eqs. (49)–(49). For the SU(5) singlets, the electromagnetic charge is

$$Q_{em} = \frac{1}{5} Q_x.$$  

| Model | $-\hat{s} \cdot \phi_s$ | $-k p^\text{Vec} \cdot \phi_s$ | $k P_{\text{Group}} \cdot V_{0,+,-}$ | $(k/2)\phi_s^2$, $-(k/2)V_{0,+,-}$ | $\Delta_{\text{Group}} N$, $-\phi_s^N$ | $\Theta_{\text{Group}}$ |
|-------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $A(T_0^0)$ | $+3$ | $+6$ | $189$ | $+44$ | $0$ | $+3$ |
| $B(T_0^+$) | $+12$ | $-3$ | $189$ | $-144$ | $0$ | $-12$ |
| $C(T_0^-)$ | $+12$ | $0$ | $189$ | $-144$ | $0$ | $-12$ |
| $D(T_1^0)$ | $+3$ | $+6$ | $84$ | $+144$ | $+12$ | $+3$ |
| $E(T_1^-)$ | $+12$ | $0$ | $189$ | $+144$ | $0$ | $+12$ |
| $F(T_1^0)$ | $+3$ | $+6$ | $84$ | $+144$ | $+12$ | $+3$ |
| $G(T_2^0)$ | $+12$ | $0$ | $189$ | $+144$ | $0$ | $+12$ |
| $H(T_2^-)$ | $+12$ | $0$ | $189$ | $+144$ | $0$ | $+12$ |
| $I(T_2^0)$ | $+3$ | $+6$ | $147$ | $+189$ | $+12$ | $+3$ |
| $J(T_2^-)$ | $+12$ | $0$ | $189$ | $+144$ | $0$ | $+12$ |

**TABLE XVI:** Entries in calculating the multiplicities in the following Table XVII.

| Twisted Sector | $P : P + kv_1$ | $-\hat{s} \cdot \phi_s \Theta_{\phi}$ | Mult. of singlets (L or R) |
|----------------|----------------|----------------|----------------|
| $A(T_0^0)$ | $(+3, +; -,-)$ | $\frac{5}{12}$ | $1(L)$ |
| $B(T_0^+)$ | $(-3, -; +)$ | $\frac{5}{12}$ | $1(R)$ |
| $C(T_0^-)$ | $(-3, -; +)$ | $\frac{5}{12}$ | $1(R)$ |
| $D(T_1^0)$ | $(-3, +; \pm, +)$ | $\frac{5}{12}$ | $1(R)$ |
| $E(T_1^-)$ | $(-3, -; \pm, -)$ | $\frac{5}{12}$ | $1(R)$ |
| $F(T_1^0)$ | $(0^3, -1, -1; 0^6)$ | $\frac{5}{12}$ | $3(L) + 3(R)$, $3(R) + 3(R)$ |
| $G(T_2^0)$ | $(0^3, -1, -1; 0^6)$ | $\frac{5}{12}$ | $3(L) + 3(R)$ |
| $H(T_2^-)$ | $(0^3, -1, -1; 0^6)$ | $\frac{5}{12}$ | $3(L) + 3(R)$ |
| $I(T_2^0)$ | $(+3, \frac{1}{2}; -,-)$ | $\frac{5}{12}$ | $3(L)$ |
| $J(T_2^-)$ | $(-3, +, +; +)$ | $\frac{5}{12}$ | $3(L)$ |
| $K_1(T_0^0)$ | $(+3, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; +)$ | $\frac{5}{12}$ | $3 \times 7(L)$, $3 \times 7(R)$ |
| $K_2(T_0^+)$ | $(+3, -3, \frac{1}{2}; +)$ | $\frac{5}{12}$ | $3 \times 7(L)$, $3 \times 7(R)$ |
| $K_3(T_0^-)$ | $(+3, -3, -\frac{1}{2}; -)$ | $\frac{5}{12}$ | $3 \times 7(L)$, $3 \times 7(R)$ |
| $K_4(T_0^-)$ | $(+3, -3, -\frac{1}{2}; -)$ | $\frac{5}{12}$ | $3 \times 7(L)$, $3 \times 7(R)$ |

| $T_6(\mathbf{10})$ | $(+3, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; +)$ | $\frac{5}{12}$ | $36(L), 36(R)$ |
| $T_6(\mathbf{10})$ | $(+3, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; +)$ | $\frac{5}{12}$ | $36(L), 36(R)$ |

**TABLE XVII:** Summary of chiral singlets, 1’s. The value for $-\hat{s} \cdot \phi_s$ is for (0; 0; −), i.e. the ones in the top rows in the tables. We also listed the vectorlike (10 ⊕ 10, 1)_R obtained previously in Table XVI.

In Table XIX we collect all 5’s, 5’s, 10’s, and 10’s. A set of anti-SU(5) GUT representation, which is of course anomaly-free chiral set, is 10, 5, 5, and 10. Indeed, three 1’s (A, B, and C) in Table XVIII belong to the anti-SU(5) families. The remaining $Q_X = \pm 5$ singlets form vector-like representations under anti-SU(5). The neutral singlets, $F, G,$ and $H$ are the heavy neutrinos.
**Table XVIII**: U(1) charges of 1's and Higgs $\mathbf{10}$ and $\overline{\mathbf{10}}$. Here, $Q_R$ is calculated from Eq. (52).

| States          | $P : P + kV_i$            | $Q_1$ | $Q_2$ | $Q_3$ | $Q_X$ | $Q_R$ | Multiplicities of $\mathbf{5, 5, 10}$ and $\overline{\mathbf{10}}$ | $Q_R$ |
|-----------------|---------------------------|-------|-------|-------|-------|-------|-------------------------------------------------|-------|
| $\rho(\mathbf{5}_{R}(T_{1}^{+}))$ | $(0^{0};+1,0;+10^{2})$ | 0     | 0     | 2     | +2    | 3(R)  | +4                                             |       |
| $\sigma(\overline{\mathbf{5}}_{R}(T_{1}^{+}))$ | $(0^{0};0,0;+10^{2})$ | 0     | 0     | 0     | 2     | 3(L)  | +1                                             |       |
| $\xi(\mathbf{5}_{R}(T_{2}^{-}))$ | $(-9^{i};+,+,++;++++)$ | +1    | 1     | 1     | 3     | 3(R)  | +4                                             |       |
| $\eta(\overline{\mathbf{5}}_{R}(T_{2}^{-}))$ | $(0^{0};0,0;+10^{2})$ | 0     | 0     | 2     | 2     | 3(L)  | −4                                             |       |
| $\alpha(\overline{\mathbf{5}}_{R}(T_{1}^{+}))$ | $(0^{0};-1,0;+10^{2})$ | 0     | 2     | 0     | 2     | 3(R), Higgs $\mathbf{5}_{R}$ containing $H_u$ | −4                                             |
| $\beta(\mathbf{5}_{R}(T_{1}^{+}))$ | $(0^{0};0,0;+10^{2})$ | 0     | 0     | 2     | 2     | 3(L), Higgs $\mathbf{5}_{R}$ containing $H_d$ | +4                                             |
| $\overline{\mathbf{5}}(\mathbf{5}_{R}(T_{2}^{-}))$ | $(+9^{i};++,++;++++)$ | −1    | −1    | −1    | 3     | 3(L)  | +1                                             |       |
| $T(\overline{\mathbf{10}}_{R}(T_{7}^{+}))$ | $(-9^{i};-,--;++;++++)$ | +1    | 1     | 1     | 1     | 3(R)  | +1                                             |       |
| $H(\mathbf{10}_{R}(T_{b}))$ | $(-9^{i};-,--;++;++++)$ | +1    | 1     | 1     | +1    | $36(L), 36(R), VEV = 0$ | +3                                             |
| $\overline{H}(\overline{\mathbf{10}}_{R}(T_{b}))$ | $(-9^{i};-,--;++;++++)$ | +1    | 1     | 1     | 1     | $36(L), 36(R), VEV \neq 0$ | +4                                             |

**Table XIX**: U(1) charges of $\mathbf{5}$, $\overline{\mathbf{5}}$, $\mathbf{10}$ and $\overline{\mathbf{10}}$, where $Q_R$ is given in Eq. (53). The L fields are changed to R fields in the first column by taking charge conjugated quantum numbers presented in the text.

From Table XIX, the vector-like pairs $\{\rho, \sigma\}$ and $\{\xi, \eta\}$ are removed. The remaining ones constitute the anti-SU(5) spectra $T, \overline{T}, A, B, C$ and the MSSM Higgs fields $\alpha$ and $\beta$, containing $H_u$ and $H_d$. The vector-like pairs $H$ and $\overline{H}$ are needed to break the anti-SU(5) down to the SM.
IV. ASSIGNMENT OF R-PARITY

For a $\mathbb{Z}_{4R}$ symmetry, we may assign the following U(1)$_R$ charges to the MSSM plus $\sigma$ fields,

$$ q(T_{\mathbf{10}(-1, R)}), \, \ell(q(\mathbf{5}_{+3, R})), \, \tau^+(B(\mathbf{1}_{-5, R})), \, H_u(\alpha(\mathbf{5}_{-2, R})), \, H_d(\beta(\mathbf{5}_{+2, R})), \, \sigma \quad Q_R : \quad +1, \quad +1, \quad +1, \quad -4, \quad +4, \quad +1 \quad (51) $$

With the four gauged U(1) charges, let us choose $Q_R$ as

$$ Q_R = \frac{9}{2} Q_1 - Q_2 - \frac{3}{2} Q_3 + Q_X \quad (52) $$

which is tabulated in Table XIX and also in Table XVIII. To break U(1)$_R$ down to $\mathbb{Z}_{4R}$, we need the GUT breaking VEVs, $\langle \mathcal{H}(\mathbf{10}_R(T_0)) \rangle \neq 0$, but we demand $\langle H(\mathbf{10}_R(T_0)) \rangle$ should not develop a VEV such that $\mathbb{Z}_{4R}$ is not broken further by $\langle H(\mathbf{10}_R(T_0)) \rangle$. With $\mathbb{Z}_{4R}$, the Yukawa couplings are required to carry the $Q_R$ charge 2 modulo 4. With $\mathbb{Z}_{4R}$, the Higgs quintets carry $Q_R = 0$ modulo 4 and the Yukawa couplings $T \mathcal{F}_\alpha, TT \beta$ and $\mathcal{F}B\beta$ are obtained. So, $B$ is $\tau^+$.

A. One pair of Higgs quintets

The Higgs quintets are $\alpha$ and $\beta$. Twisted sector $T_1^-$ of $\alpha$ and $\beta$ gives multiplicity 3. Thus, three objects at these fixed points must have permutation symmetry $S_3$, the largest discrete symmetry of three objects. From three permuting elements of $S_3$, $\{x_1, x_2, x_3\}$, the $S_3$ singlet is formed as

$$ (x_1 + x_2 + x_3)/\sqrt{3}, \quad (53) $$

and the $S_3$ doublet becomes

$$ \left( \begin{array}{c} (x_1 + \omega x_2 + \omega^2 x_3)/\sqrt{3} \\ (x_1 + \omega^2 x_2 + \omega x_3)/\sqrt{3} \end{array} \right) \quad (54) $$

where $\omega$ is the complex cube root of unity. The tensor product of two $S_3$ doublets, $(x_1, x_2)$ and $(y_1, y_2)$, is $54^2$,

$$ 2 \otimes 2 \rightarrow 1 \oplus 1' \oplus 2 \quad (55) $$

where

$$ 1 = x_1 y_1 + x_2 y_2, \quad 1' = x_1 y_2 - x_2 y_1, \quad 2 = \left( \begin{array}{c} x_2 y_2 - x_1 y_1 \\ x_1 y_2 + x_2 y_1 \end{array} \right). \quad (56) $$

The Higgsino mass terms, constructed with the $\alpha$ and $\beta$ mass terms, are 54 symplectic components of 9'. Then, the $Q_R$ charge of $C \propto 36'_{9_0} \cdot 9'_{9_1} \cdot 9'_{9_1}$ is either 54 (both 9' from $T'_9$), 30 (both 9' from $T'_1$), or 42 (one 9' from $T'_9$ and the other from $T'_1$), i.e. $Q_R = 2$ modulo 4. Thus, the coupling $\alpha_1 \beta_1 C$ is allowed and $\alpha_1$ and $\beta_1$ are removed are removed at the scale where SU$(9')$ confines, (C). Note, however, that $\mathbb{Z}_{4R}$ is broken to $\mathbb{Z}_{2R}$ at the scale (C).

Next, let us consider doublets. Two doublets, one each from the $\alpha$ and $\beta$ sets, i.e. $\alpha_2$ and $\beta_2$ (the form given in Eq. 54), combine to form $1': \beta_2 \xi_2$, as noted in Eq. 55. As above, the coupling $\alpha_2 \beta_2 C$ is allowed and one of $\alpha_2$ and one
of $\beta_2$ are removed at the scale where SU(9)$'$ confines, $\langle C \rangle$, and there remains only one pair of $5_{hR}$ and $\overline{5}_{hR}$. This is the MSSM vector-like pair of $5'_{hR}$ and $\overline{5'_{hR}}$. Summarizing,

$$5_{hR} \oplus \overline{5}_{hR} : \text{one MSSM pair, two pairs at the scale } \langle C \rangle.$$  \hfill (58)

From two permuting elements $x_2$ and $x_3$, the $S_2$ singlet is $(x_2 + x_3)/\sqrt{2}$, and the orthogonal component to that is $(x_2 - x_3)/\sqrt{2}$. Thus, the MSSM doublets are

$$\frac{1}{\sqrt{2}} (5_{hR}(T^-_1)_2 - 5_{hR}(T^-_1)_3), \quad \frac{1}{\sqrt{2}} (\overline{5}_{hR}(T^-_1)_2 - \overline{5}_{hR}(T^-_1)_3).$$  \hfill (59)

B. The third family members

It is natural to choose the singlet combination $(x_1 + x_2 + x_3)/\sqrt{3}$ as the third family member, and the doublet as the light family members. The quark and lepton mass matrix can be constructed based on these bases.

V. CONCLUSION

We derived a $Z_{4R}$ parity in a family unification model with a GUT anti-SU(5) possessing three families and one pair of Higgs quintets. Spontaneous symmetry breaking of anti-SU(5) GUT is achieved by $\langle 10 + 1 \rangle \oplus \langle \overline{10} - 1 \rangle$. The $Z_{4R}$ parity together with the permutation symmetry $S_3$ are useful to select the third family members and one MSSM pair of the Higgs quintets.

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Appendix A: Fixed points in $Z_{12-I}$

In the $k$-th twisted sector of $Z_N$ orbifold, multiplicities $\mathcal{P}_k$ is

$$\mathcal{P}_k = \frac{1}{N} \sum_{l=0}^{N} \tilde{\chi}(k,l)e^{i2\pi l\Theta_k},$$  \hfill (A1)

where $\tilde{\chi}(k,l)$ in the $Z_{12-I}$ orbifold are listed in Table XX. In the calculation of multiplicity the information on $\Theta$ is the key. The right-hand side (RHS) mover is denoted by the spinor

$$s = (s_0; \tilde{s}) = (\ominus \text{ or } \oplus; \pm, \pm, \pm),$$  \hfill (A2)

| $k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|-----|---|---|---|---|---|---|---|---|---|---|----|----|
| $l$ | 1 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 2   | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 3   | 4 | 1 | 1 | 4 | 1 | 4 | 1 | 1 | 4 | 1 | 1 | 1 |
| 4   | 9 | 1 | 1 | 1 | 9 | 1 | 1 | 9 | 1 | 1 | 1 | 1 |
| 5   | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 6   | 16| 1 | 1 | 4 | 1 | 1 | 6 | 1 | 4 | 1 | 1 | 1 |

TABLE XX: $\tilde{\chi}(k,l)$ in the $Z_{12-I}$ orbifold [48].
where $s_0$ corresponds to L- or R- movers. In this paper, $+$ or $-$ in the spinor form as in Eqs. (27) and (A2), denote $+\frac{1}{2}$ and $-\frac{1}{2}$, respectively. The phase $\Theta_k$ in the $k$th twisted sector in is

$$\Theta_k = \sum_j (N^j_L - N^j_R) \phi^j - \frac{k}{2} (V^2_a - \phi^2_s) + (P + kV_a) \cdot V_a - (\hat{s} + k\phi_s) \cdot \phi_s \text{ + integer},$$

(A3)

where $\Delta_k$ is

$$\Delta_k = (P + kV_a) \cdot V_a - \frac{k}{2} (V^2_a - \phi^2_s) + \sum_j (N^j_L - N^j_R) \hat{\phi}^j$$

(A4)

$$\equiv \Delta^0_k + \Delta^N_k.$$  

(A5)

$V_a$ is the shift vector $V$ distinguished by Wilson lines $a, V_0, +, -, \text{ and } V_9$, and 

$$\Delta^0_k = P \cdot V_a + \frac{k}{2} (-V^2_a + \phi^2_s),$$

(A6)

$$\Delta^N_k = \sum_j (N^j_L - N^j_R) \hat{\phi}^j,$$

(A7)

and $\hat{\phi}^j$ is in the range $0 < \hat{\phi}^j \leq 1 \text{ mod integer}$. The oscillator contributions in $\Delta^N_k$ is for $N_{L,R} \geq 0$. We satisfy the massless spectrum condition for gauge sectors by

$$(P + kV_a)^2 + 2 \sum_j N^j_L \hat{\phi}^j = 2\tilde{c}_k,$$

(A8)

where $2\tilde{c}_k$ of $Z_{12-I}$ orbifold is

$$2\tilde{c}_k : \frac{210}{144} (k = 1), \frac{216}{144} (k = 2), \frac{234}{144} (k = 3), \frac{192}{144} (k = 4), \frac{210}{144} (k = 5), \frac{216}{144} (k = 6),$$

(A9)

which will lead to $\mathcal{N} = 1$ supersymmetry in 4D.

In the twisted sectors with with $3a_3 = 0 \text{ mod. integer}, \ i.e. \ at \ T_{3,6,9} \text{ and also } U^{38,3}$ the gauge invariance condition has to be taken into account explicitly,

$$\frac{\bar{a}_3}{V_0} \cdot \frac{V_0}{V_a} = 0.$$

(A10)

We distinguish the twisted sector $T_k$ by 0, + and −. The phase $\Delta^0_k$ of Eq. (A7) contains an extra $\frac{k}{2}$ factor, but at $T_6$ the factor $\frac{k}{2}$ is an integer. At $T_{1,2,4,5}$, we distinguish three fixed points just by $V_{0,+,−}$.

**Appendix B: Useful tables for model building**

Here we present multiplicities in Table XXI and $H$-momenta in Table XXII.

| $i$ | $P_k(0)$ | $P_k(\pm \frac{\pi}{3})$ | $P_k(\pm \frac{2\pi}{3})$ | $P_k(\pi)$ |
|-----|----------|--------------------------|---------------------------|-----------|
| 1   | 3        | 0                        | 0                         | 0         |
| 2   | 3        | 0                        | 0                         | 0         |
| 3   | 2        | 0                        | 1                         | 0         |
| 4   | 3        | 0                        | 0                         | 2         |
| 5   | 3        | 0                        | 0                         | 0         |
| 6   | 4        | 2                        | 3                         | 0         |

TABLE XXI: Multiplicities in the $k$-th twisted sectors of $Z_{12-I}$. $P_k$(angle) is calculated with angle $= \frac{2\pi}{12} \cdot l$ in Eq. (A1).  

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$^3$ $T_9$ contains the CTP conjugate states of $T_3$.  

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TABLE XXII: $H$ momenta, $p_{orb}$, in the twisted sectors of $\mathbb{Z}_{12-1}$, Table 10.1 of [18]. Requiring $(p_{\text{vec}} + k \hat{\phi})^2 = (\text{2nd line in Eq. (A1)})$, we have $p_{\text{vec}}$ in the 4th column. In the last column, $\delta_k$ is shown, from which we have the energy contribution from right movers $2\delta_k \geq 0$.

### Appendix C: Multiplicities for each $\hat{\phi}$

The multiplicities are calculated for each $\hat{\phi}$. In Table XXIII, we present one example for a complete list of the RHS spin vectors $\hat{\phi}$, deriving 36 of SU(9)$'$ in $T^9_y$.

| Chirality $\hat{\phi}$ | $\hat{\phi} \cdot \phi_s$ | $k P_{\text{th}} / \phi_s$ | $k P_{\text{th}} \cdot V_0$ | $(k/2)\phi_s^2$, $-(k/2)V_0^2$, $\Delta_g^N$, $-\delta_g^N$ | $\Theta_{9'}$, Mult. of SU(9) |
|------------------------|--------------------------|-----------------------------|-----------------------------|-------------------------------------------------|--------------------------|
| $\oplus = L$ ($(-+)$)  | $6 \over 12$             | $6 \over 12$                | $6 \over 12$                | $4 \over 12$                                   | $-2 \over 2$             |
| $\oplus = L$ ($(-+)$)  | $6 \over 12$             | $6 \over 12$                | $6 \over 12$                | $4 \over 12$                                   | $-2 \over 2$             |
| $\oplus = L$ $(++-)$    | $6 \over 12$             | $6 \over 12$                | $6 \over 12$                | $4 \over 12$                                   | $-2 \over 2$             |
| $\oplus = R$ $(++-)$    | $6 \over 12$             | $6 \over 12$                | $6 \over 12$                | $4 \over 12$                                   | $-2 \over 2$             |
| $\oplus = L$ $(++-)$    | $6 \over 12$             | $6 \over 12$                | $6 \over 12$                | $4 \over 12$                                   | $-2 \over 2$             |
| $\oplus = L$ $(+-)$     | $6 \over 12$             | $6 \over 12$                | $6 \over 12$                | $4 \over 12$                                   | $-2 \over 2$             |
| $\oplus = L$ $((++)$    | $6 \over 12$             | $6 \over 12$                | $6 \over 12$                | $4 \over 12$                                   | $-2 \over 2$             |

TABLE XXIII: The multiplicities for each $\hat{\phi}$ calculated with the row $i = 3$ in Table XXII
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