THE SINGULARLY CONTINUOUS SPECTRUM AND NON-CLOSED INVARIANT SUBSPACES

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Dedicated to Israel Gohberg on the occasion of his 75-th birthday

ABSTRACT. Let $A$ be a bounded self-adjoint operator on a separable Hilbert space $\mathcal{H}$ and $\mathcal{H}_0 \subset \mathcal{H}$ a closed invariant subspace of $A$. Assuming that $\mathcal{H}_0$ is of codimension 1, we study the variation of the invariant subspace $\mathcal{H}_0$ under bounded self-adjoint perturbations $V$ of $A$ that are off-diagonal with respect to the decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$. In particular, we prove the existence of a one-parameter family of dense non-closed invariant subspaces of the operator $A + V$ provided that this operator has a nonempty singularly continuous spectrum. We show that such subspaces are related to non-closable densely defined solutions of the operator Riccati equation associated with generalized eigenfunctions corresponding to the singularly continuous spectrum of $B$.

1. INTRODUCTION

In the present article we address the problem of a perturbation of invariant subspaces of self-adjoint operators on a separable Hilbert space $\mathcal{H}$ and related questions on the existence of solutions to the operator Riccati equation.

Given a self-adjoint operator $A$ and a closed invariant subspace $\mathcal{H}_0 \subset \mathcal{H}$ of $A$, we set $A_i = A|_{\mathcal{H}_i}$, $i = 0, 1$, with $\mathcal{H}_1 = \mathcal{H} \ominus \mathcal{H}_0$. Assuming that the perturbation $V$ is off-diagonal with respect to the orthogonal decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ consider the self-adjoint operator

$$B = A + V = \begin{pmatrix} A_0 & V \\ V^* & A_1 \end{pmatrix} \quad \text{with} \quad V = \begin{pmatrix} 0 \\ V^* \\ 0 \end{pmatrix},$$

where $V$ is a linear operator from $\mathcal{H}_1$ to $\mathcal{H}_0$. It is well known (see, e.g., \cite{7}) that the Riccati equation

$$A_1 X - X A_0 - X V X + V^* = 0$$

has a closed (possibly unbounded) solution $X : \mathcal{H}_0 \to \mathcal{H}_1$ if and only if its graph

$$G(\mathcal{H}_0, X) := \{x \in \mathcal{H} | x = x_0 \oplus X x_0, x_0 \in \text{Dom}(X) \subset \mathcal{H}_0\}$$

is an invariant closed subspace for the operator $B$.

Sufficient conditions guaranteeing the existence of a solution to equation \!(1\!) require in general the assumption that the spectra of the operators $A_0$ and $A_1$ are separated,

$$d := \text{dist}(\text{spec}(A_0), \text{spec}(A_1)) > 0,$$

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and hence \( \mathcal{H}_0 \) and \( \mathcal{H}_1 \) are necessarily spectral invariant subspaces of the operator \( A \). In particular (see [9]), if
\[
\|V\| < c_\pi d \quad \text{with} \quad c_\pi = \frac{3\pi - \sqrt{\pi^2 + 32}}{\pi^2 - 4} \approx 0.503288 \ldots,
\]
then the Riccati equation (1) has a bounded solution \( X \) satisfying the bound
\[
\frac{\|X\|}{\sqrt{1 + \|X\|^2}} \leq \frac{\pi}{2d} - \frac{\|V\|}{d - \delta_V} < 1
\]
with
\[
\delta_V = \|V\| \tan\left(\frac{1}{2} \arctan \frac{2\|V\|}{d}\right).
\]
It is plausible to conjecture that condition (4) can be relaxed by the weaker requirement \( \|V\| < \sqrt{3}d/2 \) (see [9] for details). However, no proof of that is available as yet.

In general, without additional assumptions, neither condition (3) nor a smallness assumption like (4) on the magnitude of the perturbation \( V \) can be dropped. However, if the spectra of \( A_0 \) and \( A_1 \) are subordinated in the sense that
\[
\sup \text{spec}(A_0) \leq \inf \text{spec}(A_1),
\]
then for any \( V \) with arbitrary large norm the Riccati equation (1) has a contractive solution [8] (see also [1]). Note that in this case the invariant subspaces \( \mathcal{H}_0 \) and \( \mathcal{H}_1 \) are not necessarily supposed to be spectral invariant subspaces of \( A \).

In the present work we prove new existence results for the Riccati equation under the assumption that the subspace \( \mathcal{H}_1 \) is one-dimensional. In particular, these results imply the existence of a one-parameter family of non-closed invariant subspaces of the self-adjoint operator \( B \), provided that \( B \) has nonempty singularly continuous spectrum.

The main result of our paper is presented by the following theorem.

**Theorem 1.** Assume that \( \dim \mathcal{H}_1 = 1 \) and suppose that \( \mathcal{H}_0 \) is a cyclic subspace for the operator \( A_0 \) generated by the one-dimensional subspace \( \text{Ran} \, V \). Let \( S_{pp} \) denote the set of all eigenvalues of the operator \( B \).

Then there exists a minimal support \( S_s \) of the singular part of the spectral measure of the operator \( B \) such that:

(i) For any \( \lambda \in S_{ac} = S_s \setminus S_{pp} \) the subspace \( \Psi(\lambda) = \mathcal{G}(\mathcal{H}_0, X_\lambda) \subset \mathcal{H} \) is a dense non-closed graph subspace with \( X_\lambda : \mathcal{H}_0 \rightarrow \mathcal{H}_1 \) a non-closed densely defined operator solving the Riccati equation (1) in the sense of Definition 2.3 below.

(ii) For any \( \lambda \in S_{pp} \subset S_s \) the subspace \( \Psi(\lambda) = \mathcal{G}(\mathcal{H}_0, X_\lambda) \subset \mathcal{H} \) is a closed graph subspace of codimension 1 with \( X_\lambda : \mathcal{H}_0 \rightarrow \mathcal{H}_1 \) a bounded operator solving the Riccati equation (1). Moreover, the operator \( X_\lambda \) is an isolated point (in the operator norm topology) of the set of all bounded solutions to the Riccati equation.

The mapping \( \Psi \) from \( S_s \) to the set \( \mathcal{M}(B) \) of all (not necessarily closed) subspaces of \( \mathcal{H} \) invariant with respect to the operator \( B \) is injective.

The article is organized as follows. In Section 2 we establish a link between non-closable densely defined solutions to the Riccati equation (1) and the associated non-closed invariant subspaces of the operator \( B \). In Section 3 accommodating the Simon-Wolff theory [10] to rank two off-diagonal perturbations we perform the spectral analysis of this operator under the assumption that \( \dim \mathcal{H}_1 = 1 \). The main result of this section is Theorem 3.4. Theorem 1 will be proven in Section 4.
Throughout the whole work the Hilbert space $H$ will assumed to be separable. The notation $B(H, H)$ is used for the set of bounded linear operators from the Hilbert space $H$ to the Hilbert space $H$. We will write $B(H)$ instead of $B(H, H)$.

2. **Non-Closed Graph Subspaces**

Let $H_0$ be a closed subspace of a Hilbert space $H$ and $X$ a densely defined (possibly unbounded and not necessarily closed) operator from $H_0$ to $H_1 = H_0^\perp := H \ominus H_0$ with domain $\text{Dom}(X)$. A linear subspace

$$G(H_0, X) := \{ x \in H | x = x_0 \oplus X x_0, x_0 \in \text{Dom}(X) \subset H_0 \}$$

is called the graph subspace of $H$ associated with the pair $(H_0, X)$ or, in short, the graph of $X$.

Recalling general facts on densely defined closable operators (see, e.g., [6]) we mention the following: If $X : H_0 \to H_1$ is a densely defined non-closable operator, then $G(H_0, X)$ is a non-closed subspace of $H$. Its closure is not a graph subspace, i.e., there is no closed operator $Y$ such that

$$G(H_0, X) = G(H_0, Y).$$

**Proposition 2.1.** Let $X : H_0 \to H_1$ be a densely defined non-closable operator. Then the closed subspace $\overline{G(H_0, X)}$ contains an element orthogonal to $H_0$.

**Proof:** First, for $X : H_0 \to H_1$ being a densely defined non-closable operator we prove the following alternative: either the closed subspace $\overline{G(H_0, X)}$ contains an element orthogonal to $H_0$ or the subspace $H_0$ contains an element orthogonal to $\overline{G(H_0, X)}$. Indeed, assume on the contrary that neither the closed subspace $\overline{G(H_0, X)}$ contains an element orthogonal to $H_0$ nor the subspace $H_0$ contains an element orthogonal to $\overline{G(H_0, X)}$. Then by Theorem 3.2 in [7] there is a closed densely defined operator $Y : H_0 \to H_1$ such that $\overline{G(H_0, X)} = G(H_0, Y)$, which is a contradiction.

Now assume that the subspace $H_0$ contains an element $x_0$ orthogonal to $\overline{G(H_0, X)}$. Obviously, this element is orthogonal to $G(H_0, X)$, that is, $\langle x_0 \oplus 0, x_0 \oplus X x_0 \rangle = 0$, and hence $x_0 = 0$. Then, by the alternative proven above the subspace $\overline{G(H_0, X)}$ contains an element orthogonal to $H_0$, completing the proof.

For notational setup assume the following hypothesis.

**Hypothesis 2.2.** Let $B$ be a self-adjoint operator represented with respect to the decomposition $H = H_0 \oplus H_1$ as a $2 \times 2$ operator block matrix

$$B = \begin{pmatrix} A_0 & V \\ V^* & A_1 \end{pmatrix},$$

where $A_i \in B(H_i), \ i = 0, 1$, are bounded self-adjoint operators in $H_i$ while $V \in B(H_1, H_0)$ is a bounded operator from $H_1$ to $H_0$. More explicitly, $B = A + V$, where $A$ is the bounded diagonal self-adjoint operator,

$$A = \begin{pmatrix} A_0 & 0 \\ 0 & A_1 \end{pmatrix},$$

and the operator $V = V^*$ is an off-diagonal bounded operator

$$V = \begin{pmatrix} 0 & V \\ V^* & 0 \end{pmatrix}.$$
Definition 2.3. A densely defined (possibly unbounded and not necessarily closable) operator $X$ from $H_0$ to $H_1$ with domain $\text{Dom}(X)$ is called a strong solution to the Riccati equation
\[ A_1X -XA_0 - XVX + V^* = 0 \]
if
\[ \text{Ran}(A_0 + VX)|_{\text{Dom}(X)} \subset \text{Dom}(X) \]
and
\[ A_1Xx - X(A_0 + VX)x + V^*x = 0 \quad \text{for any} \quad x \in \text{Dom}(X). \]

Theorem 2.4. Assume Hypothesis 2.2. A densely defined (possibly unbounded and not necessarily closed) operator $X$ from $H_0$ to $H_1$ with domain $\text{Dom}(X)$ is a strong solution to the Riccati equation (8) if and only if the graph subspace $G(H_0, X)$ is invariant for the operator $B$.

Proof. First, assume that $G(H_0, X)$ is invariant for $B$. Then
\[ B(x \oplus Xx) = (A_0x + VXx) \oplus (A_1Xx + V^*x) \in G(H_0, X) \]
for any $x \in \text{Dom}(X)$. In particular, $A_0x + VXx \in \text{Dom}(X)$ and
\[ A_1Xx + V^*x = X(A_0x + VXx) \]
for all $x \in \text{Dom}(X)$, which proves that $X$ is a strong solution to the Riccati equation (8).

To prove the converse statement assume that $X$ is a strong solution to the Riccati equation (8), that is,
\[ A_0x + VXx \in \text{Dom}(X) \]
and
\[ A_1Xx + V^*x = X(A_0x + VXx), \quad x \in \text{Dom}(X), \]
which proves that the graph subspace $G(H_0, X)$ is $B$-invariant. \hfill \Box

Remark 2.5. By Lemma 4.3 in [7] a closed densely defined operator $X : H_0 \to H_1$ is a strong solution to the Riccati equation (8) if and only if it is a weak solution to (8).

3. The Singular Spectrum of the Operator $B$

Assume the following hypothesis.

Hypothesis 3.1. Assume Hypothesis 2.2. Assume in addition that the Hilbert space $H_1$ is one-dimensional,
\[ H_1 = \mathbb{C}, \]
and the Hilbert space $H_0$ is the cyclic subspace generated by $\text{Ran} V$.

Note that under Hypothesis 3.1 the Hilbert space $H_0$ can be realized as a space of square integrable functions with respect to a Borel probability measure $m$ with compact support,
\[ H_0 = L^2(\mathbb{R}; m) \]
such that the bounded operator $A_0$ acts on $L^2(\mathbb{R}, m)$ as the multiplication operator
\[ (A_0x_0)(\lambda) = \lambda x_0(\lambda), \quad x_0 \in L^2(\mathbb{R}, m), \]
$A_1$ is the multiplication by a real number $a_1$ and, finally, the linear bounded map
\[ V^* : H_0 \to H_1 \]
is given by
\[ V^* x_0 = \langle v, x_0 \rangle_{\mathcal{H}_0}, \quad x_0 \in \mathcal{H}_0 \]
for some \( v \in \mathcal{H}_0 \).

**Lemma 3.2.** Assume Hypothesis 3.1. Then the element \( 0 \oplus 1 \in \mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \) is cyclic for the operator \( B \) given by \( (5) - (7) \) and, hence, \( B \) has a simple spectrum.

**Proof.** By hypothesis (in the above notations) the element \( v \in \mathcal{H}_0 \) is cyclic for the operator \( A_0 \). Therefore, the cyclic subspace with respect to the operator \( B \) generated by the elements \( v \oplus 0 \in \mathcal{H} \) and \( 0 \oplus 1 \in \mathcal{H} \) is the whole \( \mathcal{H} \). Without loss of generality we may assume that \( a_1 = 0 \). Observing that \( B(0 \oplus 1) = v \oplus 0 \) proves the claim. \( \Box \)

**Theorem 3.3.** Assume Hypothesis 3.1. Then the Herglotz function
\[ \phi(z) = \frac{1 + (a_1 - z) \langle v, (A_0 - z)^{-1} v \rangle_{\mathcal{H}_0}}{(a_1 - z) - \langle v, (A_0 - z)^{-1} v \rangle_{\mathcal{H}_0}} \]
admits the representation
\[ \phi(z) = \int d\omega(\lambda) \frac{\lambda - z}{\lambda - z}, \]
where \( \omega \) is a probability measure on \( \mathbb{R} \) with compact support. Moreover, the operator \( B \) is unitarily equivalent to the multiplication operator by the independent variable on \( L^2(\mathbb{R}, \omega) \).

**Proof.** Introduce the Borel measure \( \Omega \) with values in the set of non-negative operators on \( \mathcal{H}_1 \oplus \mathcal{H}_1 \) by
\[ \Omega(\delta) = \left( \begin{array}{cc} V & 0 \\ 0 & 1 \end{array} \right)^* \mathbb{E}_B(\delta) \left( \begin{array}{cc} V & 0 \\ 0 & 1 \end{array} \right), \]
where \( \left( \begin{array}{cc} V & 0 \\ 0 & 1 \end{array} \right) \) is the linear map from \( \mathcal{H}_1 \oplus \mathcal{H}_1 \) to \( \mathcal{H}_0 \oplus \mathcal{H}_1 \) and let
\[ \omega(\delta) = \text{tr} \Omega(\delta), \quad \delta \subset \mathbb{R} \text{ a Borel set.} \]
Clearly, the measure \( \omega \) vanishes on all Borel sets \( \delta \) such that \( \mathbb{E}_B(\delta) = 0 \). In fact, these measures have the same families of Borel sets, on which they vanish. Indeed, assuming \( \omega(\delta) = 0 \) yields
\[ \langle v \oplus 0, \mathbb{E}_B(\delta) v \oplus 0 \rangle_{\mathcal{H}} + \langle 0 \oplus 1, \mathbb{E}_B(\delta) 0 \oplus 1 \rangle_{\mathcal{H}} = 0 \]
and, hence, in particular,
\[ \langle 0 \oplus 1, \mathbb{E}_B(\delta) 0 \oplus 1 \rangle_{\mathcal{H}} = 0, \]
which implies \( \mathbb{E}_B(\delta) = 0 \).

Introducing the \( \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_1) \)-valued Herglotz function
\[ M(z) = \left( \begin{array}{cc} V & 0 \\ 0 & 1 \end{array} \right)^* (B - z)^{-1} \left( \begin{array}{cc} V & 0 \\ 0 & 1 \end{array} \right) \]
one concludes that the Herglotz function \( M(z) \) admits the representation
\[ M(z) = \int_R d\Omega(\lambda) \frac{\lambda - z}{\lambda - z}, \]
and hence
\[ \text{tr} M(z) = \int_R d\omega(\lambda) \frac{\lambda - z}{\lambda - z}. \]
Straightforward computations show that the operator-valued function \((11)\) with respect to the orthogonal decomposition \(\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1\) can be represented as the \(2 \times 2\) matrix
\[
M(z) = \begin{pmatrix}
M_{00}(z) & M_{01}(z) \\
M_{10}(z) & M_{11}(z)
\end{pmatrix}
\]
with the entries given by
\[
\begin{align*}
M_{00}(z) &= (a_1 - z)\langle v, (A_0 - z)^{-1}v \rangle [a_1 - z - \langle v, (A_0 - z)^{-1}v \rangle]^{-1}, \\
M_{11}(z) &= [a_1 - z - \langle v, (A_0 - z)^{-1}v \rangle]^{-1}, \\
M_{01}(z) &= -(a_1 - z)^{-1}M_{00}(z), \\
M_{10}(z) &= -(a_1 - z)^{-1}M_{00}(z).
\end{align*}
\]
Taking the trace of \(M(z)\) yields representation \((9)\).

Since by Lemma 3.2 the element \(0 \oplus 1\) is cyclic and the measure \(\omega\) and the spectral measure \(E_B\) have the same families of Borel sets, on which they vanish, one concludes (see, e.g., \([3]\)) that the operator \(B\) is unitarily equivalent to the multiplication operator by the independent variable on \(L^2(\mathbb{R}, \omega)\), completing the proof. \(\square\)

Recall that a measurable not necessarily closed set \(S \subset \mathbb{R}\) is a support of a measure \(\nu\) if \(\nu(\mathbb{R} \setminus S) = 0\). A support \(S\) is said to be minimal if any measurable subset \(S' \subset S\) with \(\nu(S') = 0\) has Lebesgue measure zero.

**Theorem 3.4.** The sets
\[(12)\quad S_s := \left\{ \lambda \in \mathbb{R} \mid a_1 - \lambda = \int \frac{|v(\mu)|^2 dm(\mu)}{\mu - \lambda - i0} \right\}
\]
and
\[(13)\quad S_{sc} := \left\{ \lambda \in \mathbb{R} \mid a_1 - \lambda = \int \frac{|v(\mu)|^2 dm(\mu)}{\mu - \lambda - i0}, \quad \int \frac{|v(\mu)|^2 dm(\mu)}{\mu - \lambda} = \infty \right\}
\]
are minimal supports of the singular part \(\omega_s\) and the singularly continuous part \(\omega_{sc}\) of the measure \(\omega\), respectively. The set
\[(14)\quad S_{pp} := \left\{ \lambda \in \mathbb{R} \mid a_1 - \lambda = \int \frac{|v(\mu)|^2 dm(\mu)}{\mu - \lambda}, \quad \int \frac{|v(\mu)|^2 dm(\mu)}{\mu - \lambda} < \infty \right\}
\]
coincides with the set of all atoms of the measure \(\omega\).

**Proof.** The fact that \((12)\) is a minimal support of \(\omega_s\) follows from Lemma 3.5 in \([4]\), where one sets \(m_+^s(z) = (a_1 - z)\) and
\[
m_+^s(z) = \langle v, (A_0 - z)^{-1}v \rangle_{\mathcal{H}_0} = \int \frac{|v(\mu)|^2 dm(\mu)}{\mu - z}, \quad \text{Im} \, z \neq 0.
\]

It is not hard to see (cf., e.g., Example 1 in \([2]\)) that the set \(S_{pp}\) coincides with the set of all eigenvalues of the operator \(B\). Hence, by Theorem 3.3 one proves that \(S_{pp}\) coincides with the set of all atoms of the measure \(\omega\). Therefore, to prove that \((13)\) is a minimal support of \(\omega_{sc}\) it suffices to check the inclusion
\[(15)\quad S_{pp} \subset S_s.
\]
Assume that \( \lambda \in S_{pp} \), that is,
\[
(16) \quad a_1 - \lambda = \int \frac{|v(\mu)|^2 d \mu}{\mu - \lambda} 
\]
and
\[
\int \frac{|v(\mu)|^2 d \mu}{|\mu - \lambda|^2} < \infty. 
\]
Since
\[
\int \frac{|v(\mu)|^2 d \mu}{|\mu - \lambda|} \leq \left( \int \frac{|v(\mu)|^2 d \mu}{|\mu - \lambda|^2} \right)^{1/2} \|v\|_{L^2(\mathbb{R}; m)},
\]
the dominated convergence theorem yields
\[
\int \frac{|v(\mu)|^2 d \mu}{\mu - \lambda - i \varepsilon} \equiv \lim_{\varepsilon \to 0^+} \int \frac{|v(\mu)|^2 d \mu}{\mu - \lambda - i \varepsilon} = \int \frac{|v(\mu)|^2 d \mu}{\mu - \lambda},
\]
which together with (16) proves inclusion (15). The proof is complete. \( \square \)

**Remark 3.5.** By Lemma 5 in [5] from Theorem 3.4 it follows that there exist minimal supports of the absolutely continuous part \( \omega_{ac} \), the singular part \( \omega_s \), and the singularly continuous part \( \omega_{sc} \) of the measure \( \omega \) such that their closures coincide with the absolute continuous part \( \text{spec}_{ac}(B) \), the singular part \( \text{spec}_s(B) \), and the singularly continuous part \( \text{spec}_{sc}(B) \) of the spectrum, respectively.

4. **Riccati Equation**

Given \( \lambda \in \mathbb{R} \), introduce the operator (linear functional)
\[
X_\lambda : L^2(\mathbb{R}; m) \to \mathcal{S}_1 = C 
\]
on
\[
\text{Dom}(X_\lambda) = \left\{ \varphi \in L^2(\mathbb{R}; m) \mid \lim_{\varepsilon \to 0^+} \int \frac{v(\mu) \varphi(\mu)}{\mu - \lambda - i \varepsilon} \, d \mu \text{ exists finitely} \right\} 
\]
by
\[
(17) \quad X_\lambda \varphi = \lim_{\varepsilon \to 0^+} \int \frac{v(\mu) \varphi(\mu)}{\mu - \lambda - i \varepsilon} \, d \mu, \quad \varphi \in \text{Dom}(X_\lambda). 
\]

**Lemma 4.1.** If \( \lambda \in S_{sa} \), then the operator \( X_\lambda \) is densely defined.

**Proof.** Since the element \( v \in L^2(\mathbb{R}; m) \) is generating for the operator \( A_0 \), the set
\[
D = \{ \varphi \mid \varphi(\mu) = v(\mu) \psi(\mu), \ \psi \text{ is continuously differentiable on } \mathbb{R} \}
\]
is dense in \( L^2(\mathbb{R}; m) \). For \( \varphi \in D \) and \( \varepsilon > 0 \) one obtains
\[
(18) \quad \int \frac{v(\mu) \varphi(\mu)}{\mu - \lambda - i \varepsilon} \, d \mu = \psi(\lambda) \int \frac{|v(\mu)|^2}{\mu - \lambda - i \varepsilon} \, d \mu 
\]
\[
(19) \quad + \int \frac{|v(\mu)|^2 (\psi(\mu) - \psi(\lambda))}{\mu - \lambda - i \varepsilon} \, d \mu.
\]
Since \( \lambda \in S_{sa} \), by Theorem 3.4 the limit
\[
\lim_{\varepsilon \to 0^+} \int \frac{|v(\mu)|^2 \, d \mu}{\mu - \lambda - i \varepsilon} = \int \frac{|v(\mu)|^2 \, d \mu}{\mu - \lambda - i 0}
\]
Moreover, if \( \parallel \frac{|v(\mu)|^2}{|\lambda - \mu|^2} d\mu \parallel < \infty \)
holds true. The converse is also true: If \( X_\lambda \) is bounded, then \( \parallel \frac{|v(\mu)|^2}{|\lambda - \mu|^2} d\mu \parallel \) holds. Indeed, by the uniform boundedness principle from definition \( \parallel \frac{|v(\mu)|^2}{(\lambda - \mu)^2 + \varepsilon^2} d\mu \parallel \)
proving \( \parallel \frac{|v(\mu)|^2}{|\lambda - \mu|^2} d\mu \parallel \) by the monotone convergence theorem.

\textbf{Theorem 4.3.} Let \( \lambda \in S_s \). Then the operator \( X_\lambda \) is a strong solution to the Riccati equation
\[ A_1 X - X A_0 - XV X + V^* = 0. \]
Moreover, if \( \lambda \in S_{pp} \), the solution \( X_\lambda \) is bounded and if \( \lambda \in S_{sc} = S_s \setminus S_{pp} \), the operator \( X_\lambda \) is non-closable.

\textbf{Proof.} Note that \( A_0 \operatorname{Dom}(X_\lambda) \subset \operatorname{Dom}(X_\lambda) \). If \( \lambda \in S_s \), then by Theorem \( 3.4 \)
\[ a_1 - \lambda = \int \frac{|v(\mu)|^2 d\mu}{\mu - \lambda - i0}. \]
In particular, \( v \in \operatorname{Dom}(X_\lambda) \) and
\[ X_\lambda V X_\lambda \varphi = \int \frac{|v(\mu)|^2 d\mu}{\mu - \lambda - i0} \varphi \in \operatorname{Dom}(X_\lambda). \]
Therefore, for an arbitrary \( \varphi \in \operatorname{Dom}(X_\lambda) \) one gets
\[ A_1 X_\lambda \varphi - X_\lambda A_0 \varphi = X_\lambda V X_\lambda \varphi \]
\[ = \int \frac{v(\mu) \varphi(\mu)(a_1 - \mu)}{\mu - \lambda - i0} d\mu - (a_1 - \lambda) \int \frac{v(\mu) \varphi(\mu)}{\mu - \lambda - i0} d\mu \]
\[ = \int \frac{v(\mu) \varphi(\mu)(\lambda - \mu)}{\mu - \lambda - i0} d\mu = - \int v(\mu) \varphi(\mu) d\mu = -V^* \varphi, \]
which proves that the operator \( X_\lambda \) is a strong solution to the Riccati equation \( \parallel \frac{|v(\mu)|^2}{\mu - \lambda - i0} d\mu \parallel \) holds. Indeed, by the monotone convergence theorem.

If \( \lambda \in S_{pp} \), then \( \parallel \frac{|v(\mu)|^2}{|\lambda - \mu|^2} d\mu \parallel \) holds, in which case \( X_\lambda \) is bounded. If \( \lambda \in S_{sc} = S_s \setminus S_{pp} \), then \( X_\lambda \) is an unbounded densely defined operator (functional) (cf. Remark \( 4.2 \)). Since every closed finite-rank operator is bounded \( \parallel \frac{|v(\mu)|^2}{|\lambda - \mu|^2} d\mu \parallel \), it follows that for \( \lambda \in S_{sc} \) the unbounded solution \( X_\lambda \) is non-closable.

\textbf{Proof of Theorem 4.3.} Introduce the mapping
\[ \Psi(\lambda) = \mathcal{G}(\mathcal{S}_0, X_\lambda), \quad \lambda \in S_s, \]
where \( X_\lambda \) is the strong solution to the Riccati equation referred to in Theorem 4.3. By Theorem 4.3 the subspace \( \Psi(\lambda), \lambda \in S_s \) is invariant with respect to \( B \).
To prove the injectivity of the mapping $\Psi$, assume that $\Psi(\lambda_1) = \Psi(\lambda_2)$ for some $\lambda_1, \lambda_2 \in S$. Due to (22), $X_{\lambda_1} = X_{\lambda_2}$, which by (17) implies $\lambda_1 = \lambda_2$.

(i). Let $\lambda \in S_{sc}$. By Theorem 4.3 the functional $X_\lambda$ is non-closable. Since $X_\lambda$ is densely defined, the closure $G(H_0, X_\lambda)$ of the subspace $G(H_0, X_\lambda)$ contains the subspace $H_0$. By Proposition 2.1, the subspace $G(H_0, X_\lambda)$ contains an element orthogonal to $H_0$. Since $H_0 \subset H$ is of codimension 1, one concludes that $G(H_0, X_\lambda) = H_0 \oplus H_1 = H$.

(ii). Let $\lambda \in S_{sp}$. By Theorem 5.3 in [7] the solution $X_\lambda$ is an isolated point (in the operator norm topology) of the set of all bounded solutions to the Riccati equation (21) if and only if the subspace $G(H_0, X_\lambda)$ is spectral, that is, there is a Borel set $\Delta \subset \mathbb{R}$ such that $G(H_0, X_\lambda) = \text{Ran} \, E_B(\Delta)$.

Observe that the one-dimensional graph subspace $G(H_1, -X_\lambda^*)$ is invariant with respect to the operator $B$. This subspace is spectral since by Lemma 3.2 $\lambda$ is a simple eigenvalue of the operator $B$. Thus, $G(H_0, X_\lambda) = G(H_1, -X_\lambda^*)^\perp$ is also a spectral subspace of the operator $B$. □

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