Gap sequences of 1-Weierstrass points on non-hyperelliptic curves of genus 10

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Abstract

In this paper, we compute the 1-gap sequences of 1-Weierstrass points of non-hyperelliptic smooth projective curves of genus 10. Furthermore, the geometry of such points is classified as flexes, sextactic and tentactic points. Also, an upper bounds for their numbers are estimated.

MSC 2010: 14H55, 14Q05

Keywords: 1-Weierstrass points, \( q \)-gap sequence, flexes, sextactic points, tentactic points, kanonical linear system, Kuribayashi sextic curve.
0 Introduction

Weierstrass points on curves have been extensively studied, in connection with many problems. For example, the moduli space $M_g$ has been stratified with subvarieties whose points are isomorphism classes of curves with particular Weierstrass points. For more details, we refer for example [3], [6].

At first, the theory of the Weierstrass points was developed only for smooth curves, and for their canonical divisors. In the last years, starting from some papers by R. Lax and C. Widland (see [10], [11], [12], [13], [14], [18]), the theory has been reformulated for Gorenstein curves, where the invertible dualizing sheaf substitutes the canonical sheaf. In this context, the singular points of a Gorenstein curve are always Weierstrass points. In [16], R. Notari developed a technique to compute the Weierstrass gap sequence at a given point, no matter if it is simple or singular, on a plane curve, with respect to any linear system $V \subseteq H^0(C, O_C(n))$. This technique can be useful to construct examples of curves with Weierstrass points of given weight, or to look for conditions for a sequence to be a Weierstrass gap Sequence. He used this technique to compute the Weierstrass gap sequence at a point of particular curves and of families of quintic curves.

In this paper, we compute the 1-gap sequence of the 1-Weierstrass points on smooth non-hyperelliptic algebraic curves of genus 10 which can be embedded holomorphically and effectively in algebraic curves of degree 6. Furthermore, the geometry of Weierstrass points is classified as flexes, sextactic and tentactic points. On the other hand, we show that a smooth non-hyperelliptic curve of genus 10 has no 4-flex, 1,2-sextactic or 9-tentactic points. Also, an upper bound of the numbers of flexes, sextactic and tentactic points on such curves is estimated.

1 Preliminaries

Notations. We assume the following notations throughout the present paper.

$I(C_1, C_2; p)$; the intersection number of the curves $C_1$ and $C_2$ at the point $p$ [4].

$G_p^{(q)}(Q)$; the $q$-gap sequence of the point $p$ with respect to the linear system $Q$ [1, 2].

$\omega^{(q)}(p)$; the $q$-weight of the point $p$ [15].

$N^{(q)}(C)$; the number of $q$-Weierstrass points on $C$ [8].
Lemma 1.1. \[9 \ 15\] Let $X$ be a smooth projective plane curve of genus $g$. The number of $q$-Weierstrass points $N^{(q)}(C)$, counted with their $q$-weights, is given by

$$N^{(q)}(C) = \begin{cases} g(g^2 - 1), & \text{if } q = 1 \\ (2q - 1)^2(g - 1)^2g, & \text{if } q \geq 2. \end{cases}$$

In particular, for smooth projective plane sextic (i.e. $g = 10$), the number of 1-Weierstrass points is 990 counted with their weights.

Theorem 1.2. \[15\] Let $X$ be a non-hyperelliptic curve of genus $\geq 3$. Write $G^{(1)}_p(Q) = \{n_1 < n_2 < \ldots < n_g\}$, then

1. $n_1 = 1$,
2. $n_r \leq 2r - 2$ for every $r \geq 2$,
3. $\omega^{(1)}(p) \leq \frac{(g - 1)(g - 2)}{2}$.
4. There are at least $2g + 6$ 1-Weierstrass points on $X$.

Remark 1.3. For more details on $q$-Weierstrass points on Riemann surfaces, we refer for example to \[15 \ 5\].

2 Main results

Let $X$ be a smooth projective plane curve of genus 10, and let $Q := |K|$ be its canonical linear system.

Proposition 2.1. The linear system $Q$ is $g^{9}_{18}$.

Proof. The result is an immediate consequence, since

$$\dim Q := \dim |K| = g - 1 = 9,$$

and

$$\deg (Q) := \deg (K) = 2(g - 1) = 18.$$

Corollary 2.2. Let $p \in X$, then $\sharp G^{(1)}_p(Q) = 10$ and $G^{(1)}_p(Q) \subset \{1, 2, 3, \ldots, 19\}$.

Lemma 2.3. The set of cubic divisors on $X$ form a linear system which is $g^{9}_{18}$. 


2.1 Flexes

Definition 2.4. [17] A point \( p \) on a smooth plane curve \( C \) is said to be a flex point if the tangent line \( L_p \) meets \( C \) at \( p \) with contact order \( I_p(C, L_p; p) \) at least three. We say that \( p \) is \( i \)-flex, if \( I_p(C, L_p) - 2 = i \). The positive integer \( i \) is called the flex order of \( p \).

Our main results for this part are the following.

Theorem 2.5. Let \( p \) be a flex point on a smooth projective non-hyperelliptic plane curve \( C \) of \( g = 10 \). Let \( L_p \) be the tangent line to \( C \) at \( p \) such that \( I(C, E_p) = \mu_f \). Then \( G_p^{(1)}(Q) = \{1, 2, 3, 1 + \mu_f, 2 + \mu_f, 3 + \mu_f, 2\mu_f + 1, 2\mu_f + 2, 3\mu_f + 1, 3\mu_f + 2\} \). Moreover, the geometry of such points is given by the following table:

| \( \omega_p^{(1)}(Q) \) | \( G_p(Q) \) | Geometry |
|--------------------------|----------------|----------|
| 2                        | \{1,2,3,...,8,10,11\} | 1-flex   |
| 15                       | \{1,2,3,5,6,7,9,10,13,14\} | 2-flex   |
| 28                       | \{1,2,3,6,7,8,11,12,16,17\} | 3-flex   |

Proof. The dimensions of \( Q(-1.p) \) and \( Q(-2.p) \) do not depend on whether \( p \) is 1-Weierstrass point or not, i.e.,

\[ 1, 2 \in G_p^{(1)}(Q). \]

The spaces \( Q(-3.p) = ... = Q(-\mu_f.p) \), consists of divisor of cubic curves of the form \( L_pR \), where \( R \) is an arbitrary conic. Hence, \( dim Q(-\ell.p) = 6 \) for \( \ell = 3, ..., \mu_f \). That is,

\[ 3 \in G_p^{(1)}(Q). \]

The space \( Q(-(1 + \mu_f).p) \) consists of divisor of cubic curves of the form \( L_pR \), where \( R \) is a conic passing through \( p \). Hence, \( dim Q(-(1 + \mu_f).p) = 5 \). That is,

\[ 1 + \mu_f \in G_p^{(1)}(Q). \]

The space \( Q(-(2 + \mu_f).p) \) consists of divisor of cubic curves of the form \( L_pR \), where \( R \) is a conic passing through \( p \) with contact order at least 2. Hence, \( dim Q(-(2 + \mu_f).p) = 4 \). That is,

\[ 2 + \mu_f \in G_p^{(1)}(Q). \]

The spaces \( Q(-(3 + \mu_f).p) = ... = Q(-2\mu_f.p) \) consists of divisor of cubic curves of the form \( L_pH \), where \( H \) is an arbitrary hyperplane. Hence, \( dim Q(-\ell.p) = 3 \) for \( \ell = 3 + \mu_f, ..., 2\mu_f \). That is,

\[ 3 + \mu_f \in G_p^{(1)}(Q). \]
The space $Q(-(2\mu_f + 1).p)$ consists of divisor of cubic curves of the form $L_p^2 H$, where $H$ is a hyperplane through $p$. Hence, $\dim Q(-(2\mu_f + 1).p) = 2$. That is,

$$2\mu_f + 1 \in G_p^{(1)}(Q).$$

The spaces $Q(-(2\mu_f + 2).p) = \ldots = Q(-(3\mu_f).p)$ consists of divisor of cubic curves of the form $L_p^2 H$, where $H$ is a hyperplane through $p$ with contact order at least 2. Hence, $\dim Q(-(2\mu_f + 2).p) = 1$. That is,

$$2\mu_f + 2 \in G_p^{(1)}(Q).$$

The space $Q(-(3\mu_f + 1).p)$ consists of divisor of the cubed tangent line $L_p^3$. Hence, $\dim Q(-(3\mu_f + 1).p) = 0$. That is,

$$3\mu_f + 1 \in G_p^{(1)}(Q).$$

The spaces $Q(-\ell.p) = \phi$ for $\ell \geq 3\mu_f + 2$. Consequently,

$$3\mu_f + 2 \in G_p^{(1)}(Q).$$

Consequently,

$$G_p^{(1)}(Q) = \{1, 2, 3, 1 + \mu_f, 2 + \mu_f, 3 + \mu_f, 2\mu_f + 1, 2\mu_f + 2, 3\mu_f + 1, 3\mu_f + 2\}.$$ 

Finally, it is well known that curves of genus 10 can be embedded into algebraic curves of degree 6. Hence, by the famous Bezout’s theorem, the tangent line meets $C$ at the flex point $p$ with $3 \leq \mu_f \leq 6$. On the other hand, by Theorem 1.2, it follows that, $\mu_f \neq 6$.

Which completes the proof.

**Corollary 2.6.** If a smooth projective curve $X$ of $g = 10$ has 4-flex points, then $X$ is hyperelliptic.

**Corollary 2.7.** On a smooth non-hyperelliptics projective plane curve $C$ of $g = 10$, the 1-weight of a flex point is given by

$$\omega_p^{(1)}(Q) = 13\mu_f - 37,$$

where $\mu_f$ is the multiplicity of the tangent line $L_p$ to $C$ at $p$.

**Proof.** Let $p$ be a flex point on $C$, then by Theorem 2.5

$$G_p^{(1)}(Q) = \{1, 2, 3, 1 + \mu_f, 2 + \mu_f, 3 + \mu_f, 2\mu_f + 1, 2\mu_f + 2, 3\mu_f + 1, 3\mu_f + 2\}.$$
Consequently,

\[ \omega_{p}^{(1)}(Q) := \sum_{r=0}^{g}(n_{r} - r) \]

\[ = (1 + \mu_{f} - 4) + (2 + \mu_{f} - 5) + (3 + \mu_{f} - 6) + (2\mu_{f} + 1 - 7) + (2\mu_{f} + 2 - 8) \]

\[ + (3\mu_{f} + 1 - 9) + (3\mu_{f} + 2 - 10) \]

\[ = 13\mu_{f} - 37. \]

\[ \square \]

**Notation.** Let \( F_{i}^{(q)}(C) \) be the set of \( i \)-flex points which are \( q \)-Weierstrass points on \( C \). Also, let \( NF_{i}^{(q)}(C) \) denotes the cardinality of \( F_{i}^{(q)}(C) \).

**Corollary 2.8.** For a smooth non-hyperelliptic projective plane curve \( C \) of \( g = 10 \), the maximal cardinality of \( F_{i}^{(1)} \) is given by the following table:

| \( i \) | Maximum \( NF_{i}^{(1)}(C) \) |
|-------|-------------------|
| 1     | 495               |
| 2     | 66                |
| 3     | 35                |
| 4     | 0                 |

**Proof.** The number of the 1-Weierstrass points on \( C \) is 990 counted with their 1-weight. Hence,

\[ NF_{i}^{(1)} \leq \left\lceil \frac{990}{13(i + 2) - 37} \right\rceil, \]

where \( i = 1, 2, 3 \).

\[ \square \]

### 2.2 Sextactic Points

In analogy with tangent lines and flexes of projective plane curves, one can consider *osculating conics and sextactic points in the following way:*

**Lemma 2.9.** \([2]\) Let \( p \) be a non-flex point on a smooth projective plane curve \( X \) of degree \( d \geq 3 \). Then there is an unique irreducible conic \( D_{p} \) with \( I_{p}(X, D_{p}; p) \geq 5 \). This unique irreducible conic \( D_{p} \) is called the *osculating conic of \( X \) at \( p \).*

**Definition 2.10.** \([1]\) A non-flex point \( p \) on a smooth projective plane curve \( X \) is said to be a **sextactic point** if the osculating conic \( D_{p} \) meets \( X \) at \( p \) with contact order at least six. A sextactic point \( p \) is said to be *\( i \)-sextactic*, if \( I_{p}(X, D_{p}; p) - 5 = i \). The positive integer \( i \) is called the sextactic order.

Now, the main results for this part are the following.
Theorem 2.11. Let \( p \) be a sextactic point on a smooth projective non-hyperelliptic curve \( C \) of \( g = 10 \). Let \( D_p \) be the osculating conic to \( C \) at \( p \) such that \( I(C, D_p; p) = \mu_s \). Then \( G_p^{(1)}(Q) = \{1, 2, 3, \ldots, 7, 1 + \mu_s, 2 + \mu_s, 3 + \mu_s\} \).

Proof. Again, the dimensions of \( Q(-1.p) \) and \( Q(-2.p) \) do not depend on whether \( p \) is 1-Weierstrass point or not, i.e.,

\[
1, 2 \in G_p^{(1)}(Q).
\]

Moreover, let \( L_p \) be the tangent line to \( C \) at \( p \), then

\[
div(L_pR) \in Q(-2.p) - Q(-3.p),
\]

where, \( R_0 \) is a conic not through \( p \). That is,

\[
3 \in G_p^{(1)}(Q).
\]

Furthermore,

\[
div(L_pR_1) \in Q(-3.p) - Q(-4.p),
\]

where, \( R_1 \) is a conic passing through \( p \) with multiplicity 1. That is,

\[
4 \in G_p^{(1)}(Q).
\]

Also,

\[
div(L_p^2H_0) \in Q(-4.p) - Q(-5.p),
\]

where, \( H_0 \) is a hyperplane not through \( p \). That is,

\[
5 \in G_p^{(1)}(Q).
\]

Similarly,

\[
div(L_p^2H_1) \in Q(-5.p) - Q(-6.p),
\]

where, \( H_1 \) is a hyperplane passing through \( p \) with multiplicity 1. So that,

\[
6 \in G_p^{(1)}(Q).
\]

Now, the spaces \( Q(-7.p) = \ldots = Q(-\mu_s.p) \) consists of divisors of cubic curves of the form \( D_pH \), where \( H \) is an arbitrary line. Hence,

\[
7 \in G_p^{(1)}(Q).
\]

On the other hand, the space \( Q(-(1 + \mu_s).p) \) consists of divisors of cubic curves of the form \( D_pH \), where \( H \) is a hyperplane through \( p \). Consequently,

\[
1 + \mu_s \in G_p^{(1)}(Q).
\]
Also, the space $Q(- (2 + \mu_s).p)$ contains only the cubic divisor $D_p L_p$. Then,

$$2 + \mu_s \in G_p^{(1)}(Q).$$

Finally, $Q(- \ell.p) = \phi$, for $\ell \geq 3 + \mu_s$. That is,

$$3 + \mu_s \in G_p^{(1)}(Q).$$

Which completes the proof. \hfill \Box

**Corollary 2.12.** Let $p$ be a sextactic point on a smooth projective non-hyperelliptic curve $C$ of $g = 10$. Then, the geometry of such points is given by the following table:

| $\omega_p^{(1)}(Q)$ | $G_p(Q)$ | Geometry |
|----------------------|----------|----------|
| 3                    | $\{1,2,3,...,7,9,10,11\}$ | 3-sextactic |
| 6                    | $\{1,2,3,...,7,10,11,12\}$ | 4-sextactic |
| 9                    | $\{1,2,3,...,7,11,12,13\}$ | 5-sextactic |
| 12                   | $\{1,2,3,...,7,12,13,14\}$ | 6-sextactic |
| 15                   | $\{1,2,3,...,7,13,14,15\}$ | 7-sextactic |

**Proof.** It follows by Theorem 2.11 and Bezout’s theorem, that $D_p$ meets $C$ at $p$ with $8 \leq \mu_s \leq 12$. Hence, varying $\mu_s$ produces the last table. \hfill \Box

**Corollary 2.13.** If a smooth projective curve $X$ of $g = 10$ has 1-sextactic or 2-sextactic points, then $X$ is hyperelliptic.

**Notation.** Let $S_i^{(q)}(C)$ be the set of $i$-sextactic points which are $q$-Weierstrass points on $C$. Also, let $NS_i^{(q)}(C)$ denotes the cardinality of the set $S_i^{(q)}(C)$.

**Corollary 2.14.** For a smooth non-hyperelliptic projective curve $C$ of $g = 10$, the maximal cardinality of $S_i^{(1)}$ is given by the following table:

| $i$ | Maximum $NS_i^{(1)}(C)$ |
|-----|------------------------|
| 1   | 0                      |
| 2   | 0                      |
| 3   | 330                    |
| 4   | 165                    |
| 5   | 110                    |
| 6   | 82                     |
| 7   | 66                     |
Proof. The number of the 1-Weierstrass points of such curves is 990. Hence,
\[ NS^{(1)} \leq \left[ \frac{990}{3(i + 5) - 21} \right], \]
where \( i = 3, 4, 5, 6, 7 \).

**Corollary 2.15.** On a smooth non-hyperelliptic projective plane curve \( C \) of \( g = 10 \), the 1-weight of a sextactic point is given by
\[ \omega^{(1)}_p(Q) = 3\mu_s - 21, \]
where \( \mu_s \) is the multiplicity of the osculating conic \( D_p \) at \( p \).

**Proof.** If \( p \) is a sextactic point on \( C \), then
\[ G^{(1)}_p(Q) = \{1, 2, 3, ..., 7, 1 + \mu_s, 2 + \mu_s, 3 + \mu_s\}. \]
Consequently,
\[ \omega^{(1)}_p(Q) := \sum_{r=0}^{g} (n_r - r) \]
\[ = (1 + \mu_s - 8) + (2 + \mu_s - 9) + (3 + \mu_s - 10) \]
\[ = 3\mu_s - 21. \]

2.3 Tentactic points

**Definition 2.16.** A point \( p \) on a smooth plane curve \( C \) of genus \( g \geq 2 \), which is neither flex nor sextactic point, is said to be a tentactic point, if there a cubic \( E_p \) which meets \( C \) at \( p \) with contact order at least 10. The positive integer \( t \) such that \( i := I(C, E_p; p) - 9 \) is called the tentactic order of \( p \). Moreover, the point \( p \) is said to be \( i \)-tentactic.

**Theorem 2.17.** Let \( p \) be a tentactic point on a smooth projective non-hyperelliptic curve \( C \) of \( g = 10 \) and let \( E_p \) be its osculating cubic curve such that \( I(C, E_p; p) = \mu_t \). Then \( G^{(1)}_p(Q) = \{1, 2, 3, ..., 9, 1 + \mu_t\} \). Moreover, the geometry of such points is given by the following table:

| \( \omega^{(1)}_p(Q) \) | \( G^{(1)}_p(Q) \) | Geometry |
|------------------------|-----------------|----------|
| 1                      | \{1,2,3,...,9,11\} | 1-tentactic |
| 2                      | \{1,2,3,...,9,12\} | 2-tentactic |
| 3                      | \{1,2,3,...,9,13\} | 3-tentactic |
| 4                      | \{1,2,3,...,9,14\} | 4-tentactic |
| 5                      | \{1,2,3,...,9,15\} | 5-tentactic |
| 6                      | \{1,2,3,...,9,16\} | 6-tentactic |
| 7                      | \{1,2,3,...,9,17\} | 7-tentactic |
| 8                      | \{1,2,3,...,9,18\} | 8-tentactic |
Proof. Since, the point $p$ is neither flex nor sextactic, then
\[
\dim Q(-\ell.p) = 9 - \ell \text{ for } \ell = 1, 2, 3, ..., 9.
\]
Hence,
\[1, 2, 3, ..., 9 \in G_p^{(1)}(Q).
\]
Moreover, assuming that $I(C, E_p) = \mu_t$, then
\[
div(E_p) \in Q(-\mu_t.p) - Q(-(1 + \mu_t).p).
\]
Therefore, $1 + \mu_t \in G_p^{(1)}(Q)$. Consequently,
\[G_p(Q) = \{1, 2, 3, ..., 9, 1 + \mu_t\}.
\]
Finally, $19 \notin G_p^{(1)}(Q)$ as $n_{10} \leq 18$.

Corollary 2.18. If a smooth projective curve $X$ of $g = 10$ has $9$-tentactic points, then $X$ is hyperelliptic.

Notation. Let $T_i^{(q)}(C)$ be the set of $i$-tentactic points which are $q$-Weierstrass points on $C$. Also, let $NT_i^{(q)}(C)$ denotes the cardinality of the set $T_i^{(q)}(C)$.

Corollary 2.19. For a smooth projective non-hyperelliptic curve $C$ of $g = 10$, the maximal cardinality of $T_i^{(1)}$ is given by the following table:

| $i$ | Maximum $NT_i^{(1)}(C)$ |
|-----|-------------------------|
| 1   | 990                     |
| 2   | 495                     |
| 3   | 330                     |
| 4   | 247                     |
| 5   | 198                     |
| 6   | 165                     |
| 7   | 141                     |
| 8   | 123                     |
| 9   | 0                       |

Proof. The number of the 1-Weierstrass points of a smooth projective non-hyperelliptic curve of genus $g$ is $(g - 1)g(g + 1)$. Hence, in particular,
\[NT_i^{(1)} \leq \left\lfloor \frac{990}{i} \right\rfloor,
\]
where $i = 1, 2, 3, ..., 8$. □
Concluding remarks.
We conclude the paper by the following remarks and comments.

- The 1-Weierstrass points of an algebraic curve of \( g = 10 \) are classified as flexes, sextactic and tentactic points. In particular, the present authors computed the 1-gap sequences of such points on non-hyperelliptic curves of genus 10. Furthermore the geometry of these points is investigated and an upper bound for their number is estimated.

- The main theorems constitute a motivation to solve other problems. One of these problems is the investigation of the geometry of the 1-Weierstrass points of Kuribayashi sextic curve with three parameters \( a, b \) and \( c \) defined by the equation:

\[
KI_{a,b,c} : X^6 + Y^6 + Z^6 + aX^3Y^3 + bX^3Z^3 + cY^3Z^3.
\]

However, this problem will be the object of a forthcoming work.

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