COMPUTER PROGRAMS FOR KNOT TABULATION

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Abstract

While the problem of knot classification is far from solved, it is possible to create computer programs that can be used to tabulate knots up to a desired degree of complexity. Here we discuss the main ideas on which such programs can be based. We also present the actual results obtained after running a computer program on which knots are denoted through regular projections.

1. INTRODUCTION

In order to develop an algorithm to tabulate knots, one first needs to introduce an appropriate notation, so that knots may be presented through sequences of numbers. Such a notation may only be valid if each acceptable presentation uniquely determines a knot up to ambient isotopy. One then continues by obtaining the conditions that have to be satisfied by some sequence of numbers, in order for the sequence to be acceptable and thus denote an actual knot. This problem is far from trivial, as shall be made clear later in the paper.

The next step is to identify the sequences that denote ambient isotopic knots, through the introduction of appropriate equivalence moves. If two or more number sequences are related through such moves, all sequences except the one appearing first are eliminated and do not appear at the output, and thus the corresponding knot type is tabulated exactly once.

There is no upper bound however to the number of equivalence moves that may be needed to connect two sequences denoting ambient isotopic knots. Therefore if two sequences are not connected after some finite number of moves, this does not imply that the corresponding knots are inequivalent. In order to show inequivalence one needs to establish appropriate invariants. Each such invariant is a function on the set of notations, defined so that any sequences connected through equivalence moves are mapped to identical values. Therefore if one or more invariants map two sequences to distinct values, the corresponding knots are inequivalent.

Ideally one may run the program long enough, so that each knot up to some desired complexity has either been shown ambient isotopic to a preceding knot, or shown inequivalent to all preceding knots. In the former case the knot is eliminated, while in the latter it is tabulated. Due however to memory and time limitations, there are certain limits up to which one may carry such a process.
In Section 2 we discuss two methods of denoting knots. One is based on the braid word, and the other on presenting knots as closed Self-Avoiding Walks (SAW’s) on a cubic lattice. In Section 3 we present and discuss in detail the Dowker notation, which is based on studying knots through their regular projections. Finally in Section 4 we present the results we obtained after running a computer program based on the Dowker notation, and compare them to results obtained by others (Ref. 1).

2. EXAMPLES OF KNOT NOTATIONS

The Braid Word

According to the Alexander Theorem, every knot and link may be obtained through the closure of appropriate braids (Ref. 2,3). Let a knot $K$ be the closure of a braid where $b = \prod_{i=1}^{k} \sigma_i^{a_i}$ and $\sigma_i \in \{1, 2, ..., n - 1\}$. One may thus denote $K$ by the sequence $a_0, a_1, a_2, ..., a_k$ where $a_0 = n$ and $a_i = \sigma_i \epsilon_i$. Once this sequence is given, one may uniquely determine the braid $b$ and its closure $K$. In general such sequences yield multicomponent links, and one thus needs a criterion to distinguish the sequences that denote knots from the ones that denote links. In order to do so, one replaces the braid generators $\sigma_i$ with permutations $p_i = (i, i+1) \in S_k$. The permutation $P = \prod_{i=1}^{k} p_i$ belongs to a conjugacy class of $S_k$ characterized by a partition $\rho_1 \geq \rho_2 \geq ... \geq \rho_k > 0$ of $k$. The braid closure of $\prod_{i=1}^{k} \sigma_i^{a_i}$ is a $q$-component link. If $q = 1$, such a link is a knot.

Two distinct sequences denote the same knot if and only if they either correspond to identical braids, or to braids connected through equivalence moves. The first possibility occurs by inserting or removing trivial factors $\sigma_i^a \sigma_i^{-a}$, by replacing $(\sigma_i \sigma_j)^{\epsilon}$ with $(\sigma_j \sigma_i)^{\epsilon}$ for $|i - j| > 1$, or by replacing $(\sigma_i \sigma_{i+1} \sigma_i)^{\epsilon}$ with $(\sigma_{i+1} \sigma_i \sigma_{i+1})^{\epsilon}$.

Two distinct braids yield ambient isotopic knots if they are connected by a series of the following moves (Ref. 4).

\begin{align*}
\alpha) & \quad b \leftrightarrow b'bb'^{-1} \text{ where } b, b' \in B_k \\
\beta) & \quad b \leftrightarrow b \sigma_k^{\epsilon} \text{ where } b \in B_k, \quad \epsilon = \pm 1 \text{ and } b \sigma_k^{\epsilon} \in B_{k+1}
\end{align*}

Recently (Ref. 3) it has been shown that these two kinds of moves, known as the Markov moves, can be generalized and replaced by the $L$-moves

$$b = b_1 b_2 \leftrightarrow \sigma b_1^\prime \sigma \sigma_k^{\pm 1}(\sigma')^{-1} b_2^\prime \sigma^{-1}$$

where $\sigma = \sigma_i^\epsilon ... \sigma_k^\epsilon$, $\sigma' = \sigma_{i-1}^{\epsilon} ... \sigma_{k-1}^{\epsilon}$ and $b_1^\prime, b_2^\prime$ are derived from $b_1, b_2$ by replacing $\sigma_j$ with $\sigma_{j+1}$ for all $j \geq i$.

Since there is no upper bound in the number of the Markov or $L$-moves that may be needed, one has to establish appropriate invariants in order to distinguish braids yielding inequivalent knots. Such invariants may be obtained through the skein relations

$$Af(\hskip 0.1cm \hskip 0.1cm) + Bf(\hskip 0.1cm \hskip 0.1cm) = Cf(\hskip 0.1cm \hskip 0.1cm)$$


from which one deduces that $Af(b_1\sigma \sigma_2) + Bf(b_1 \sigma^{-1}b_2) = cf(b_1b_2)$. Examples of such invariants are the Alexander-Conway, the Jones and the homflypt polynomials (Ref. 5).

Such invariants however do not always distinguish inequivalent knots. In addition, knots expressed as braid closures may not appear in a minimum crossing number projection, as the knot 5-2 in Rolfsen’s tables demonstrates (Ref. 6). Therefore the braid word notation becomes inefficient when the crossing number is large.

**Closed SAW’s on a Cubic Lattice**

Let $\{(m, n, l)/m, n, l \in Z\}$ define a cubic lattice in $R^3$. By connecting neighboring points on the lattice, one obtains a walk. If the initial and final points coincide, and no other point appears more than once, the walk is called closed Self-Avoiding, and defines a knot. A tame knot is defined as a knot that is ambient isotopic to a closed Self-Avoiding Walk. Therefore the problem of tabulating (tame) knots can be reduced to tabulating closed SAW’s.

Let $\vec{r}_0, \vec{r}_1, \vec{r}_2, ..., \vec{r}_n = \vec{r}_0$ the points on the cubic lattice defining the walk, such that $|\vec{r}_i - \vec{r}_{i+1}| = (1\forall i \in \{1, 2, ..., n\}$). The unit vectors $\vec{v}_i = \vec{r}_i - \vec{r}_{i+1}$ take values in $\{\pm \vec{i}, \pm \vec{j}, \pm \vec{k}\}$, which may be identified with the numbers $\{1, 2, 3, -1, -2, -3\}$. One may thus denote a closed SAW and its ambient isotopy class through a sequence of $n$ numbers taking values in $\{1, 2, 3, -1, -2, -3\}$ and indicating the vectors $\vec{v}_i$. In order for some random sequence $a_1, a_2, ..., a_n$, where $a_i \in \{\pm 1, \pm 2, \pm 3\}$, to denote a closed SAW, two conditions have to be satisfied. First, the sum of the corresponding unit vectors $\sum_{i=1}^{n} \vec{v}_i = 0$ in order for the walk to be closed. Second, there should exist no two numbers $i, j$, where $i < j$ and $\{i, j\} \neq \{0, n\}$, such that $\sum_{k=i+1}^{j} \vec{v}_k = 0$, in order for the walk to be Self-Avoiding.

Distinct notations $a_i, b_i$ yield identical walks if there is a number $c \in \{1, 2, ..., n-1\}$ such that $\forall i \in \{1, 2, ..., n-c\}$, $b_{i+c} = a_i$ and $\forall i \in \{n-c+1, ..., n-1, n\}$, $b_{i+c-n} = a_i$, since a switch from to merely relocates the end points of the walk, and is thus a reparametrization of the walk.

Distinct walks yield ambient isotopic knots if they are related through the equivalence moves

I) $\begin{array}{c} \uparrow \downarrow \leftrightarrow \begin{array}{c} \uparrow \downarrow \uparrow \downarrow \begin{array}{c} \downarrow \uparrow \end{array} \\ b_i = a_{i+1}, \quad b_{i+1} = a_i, \quad j \neq i, i + 1 \Rightarrow b_j = a_j \end{array} \end{array}$

II) $\begin{array}{c} \begin{array}{c} \uparrow \downarrow \uparrow \downarrow \end{array} \leftrightarrow \begin{array}{c} \uparrow \downarrow \end{array} \\ j \leq i - 1 \Rightarrow b_j = a_j, \quad b_i = a_{i+1}, \quad j \geq i \Rightarrow b_j = a_{j+2} \end{array}$

(Possible only if $a_{i+2} = -a_i$)

While under such moves a closed SAW remains closed, it may not remain Self-Avoiding. One thus has to ensure that any new lattice points to be occupied after a move, were unoccupied before, in order for the move to be allowed (Ref. 7).

As was the case with the braid word notation, there is no upper bound in the number of moves needed to connect two ambient isotopic walks, and one thus needs to establish invariants. In order to do so, one first performs the necessary equivalence moves so that all segments along one of the six directions belong to the surface of a rectangular parallelepiped that encloses the
walk. One then removes them and converts the closed walk to a collection of open walks, which are considered as the strings of a braid whose closure is the knot. It is possible from these open walks to obtain the braid word, and continue as before by calculating the skein invariants (Ref. 8).

In addition however to the problem of inequivalent knots with identical skein invariants, running a program based on the cubic lattice notation is extremely timing consuming. The simplest non-trivial knot, for example, the trefoil, needs a minimum length of 24 in order to be represented as a closed SAW on a cubic lattice; therefore the program will spend an enormous amount of time merely showing that all walks of length shorter than 24 are trivial (Ref. 9). In contrast, one may obtain it very fast through other notations, since its braid word is $\sigma_1 \in B_2$ and its crossing number is 3.

### 3. THE DOWKER NOTATION

**Denoting Regular Projections**

Let a knot $K$ and $P(K)$ a two-dimensional projection of $K$. $P(K)$ is called a regular projection of $K$ if none of its multiple points has a neighborhood that looks as follows.

I) \hspace{1cm} \hspace{1cm} II) \hspace{1cm} III) \hspace{1cm}

In other words, let $\vec{f}(s)$ be the continuous one-to-one function from $S^1$ to $S^3$ that defines the knot, and let $\vec{t}$ be a vector normal to the projection surface. In order for the projection to be normal, the following conditions must be met.

I) $\forall s \in S^1 \exists \vec{f}(s) = \frac{\partial \vec{f}(s)}{\partial s} : \vec{f}'(s) \times \vec{t} \neq \vec{0}$

II) $\forall s_1 \neq s_2 \in S^1 : (\vec{f}(s_2) - \vec{f}(s_1)) \times \vec{t} = \vec{0}, \quad (\vec{f}'(s_2) - \vec{f}'(s_1)) \times \vec{t} \neq \vec{0}$

III) $s_1, s_2, s_3 \in S^1, s_1 \neq s_2 \neq s_3 \neq s_1 \Rightarrow |(\vec{f}(s_2) - \vec{f}(s_1)) \times \vec{t} + |(\vec{f}(s_3) - \vec{f}(s_1)) \times \vec{t} > \vec{0}$

In addition, in regular projections one distinguishes overcrossings from undercrossings.

Regular projections are denoted as follows. Starting from some point and moving along the knot, one assigns successive natural numbers 1, 2, ..., $2n$ to the crossing points according to the order these points are met. Each crossing is thus assigned two such numbers, $a_0$ for the overcrossing point and $a_u$ for the undercrossing. The set of the ordered pairs $(a_0, a_u)$ denotes the projection.

If one considers mirror symmetric knots equivalent and disregards connected sums, given an appropriate set of numbers one can uniquely determine the projection.

**Drawable – Undrawable Sets**

Let $n$ be the crossing number. Each number in $\{1, 2, ..., 2n\}$ appears exactly once in one of the $n$ pairs. Not all such sets however denote actual projections, a simple counterexample being $\{(1, 3), (2, 4)\}$. 


This set, as one notices from the figure above, yields no actual projection and may thus
be considered “undrawable”. The reason is the following. Since (1, 3) belongs to the set, the
segments 1−2−3 and 3−4−1 are loops. If \{(1, 3), (2, 4)\} were drawable, the loop 1−2−3 would
have divided the plane into two disjoint segments according to the Jordan Curve Theorem
(Ref. 10). Therefore the loop 3−4−1 would either be tangent to 1−2−3, or would have to
intersect it at an even number of points, in order to “enter” as many times as it “exits” (vertices
do not count). These two loops are nowhere tangent, since they share no common segment,
and they intersect at just one point, namely at \{(2, 4)\}. Therefore the set is undrawable.

The general condition in order for a set to be drawable is the following. Any two loops
defined by such a set must either share one or more segments, or intersect at an even number
of points, vertices not counting.

A corollary of this condition is that odd numbers must always be paired to even. If that
were not the case, there would be some pair \((i, j)\) belonging to the set such that \(|j − i|\) would
be even. Therefore the loops \(i, i + 1, ..., j − 1, j\) and \(j, j + 1, ..., 2n − 1, 2n, 1, ..., i − 1, i\) which
share no common segment, would intersect at an even number of points and thus violate the
condition. This corollary, however, while being necessary, is not sufficient, as the counterexam-
ples \{\(1, 4\), \(3, 6\), \(5, 8\), \(7, 10\), \(9, 2\)\} shown below demonstrates. In this case, one notice that
the loops 1−2−3−4 and 5−6−7−8 violate the drawability condition.

In order to check whether one set is drawable, one has to consider all possible loops, which
are \(3^n\), since each crossing may or may not be a vertex of the loop. In addition, if it is a vertex,
the loop may turn clockwise or counterclockwise. The number of pairs of loops is thus \(3^{2n}\). If
one was to check all such pairs of loops for each possible set, the CPU time needed would be
enormous and grow exponentially. This is not necessary, however, since the sets may be divided
into classes of equal drawability. Sets belonging to one class are either all drawable, or all
undrawable, and thus one has to check the pairs of loops in merely one set of each class. For
$n = 14$, for example, there are $14!$ (about one hundred billion) possible sets. (While checking drawability, one need not consider the ordering in each pair of numbers. When this is also taken into account, the number of possible sets becomes $2^{14} = 16,384$ times larger). These however can be divided into less than a thousand classes of equal drawability, and thus the time needed is significantly reduced.

Two projections are “equally drawable” and belong to the same class, if they can be connected through the following moves.

I) \[ \begin{align*}
&\begin{pmatrix}
\infty & \cdots & \infty \\
\cdots & \infty & \cdots \\
\infty & \cdots & \infty
\end{pmatrix} \\
\leftrightarrow \\
&\begin{pmatrix}
\infty & \cdots & \infty \\
\cdots & \infty & \cdots \\
\infty & \cdots & \infty
\end{pmatrix}
\end{align*} \]
Add or remove a pair $(i, i + 1)$, and increase or decrease numbers larger or equal to $i$ by 2.

II) \[ \begin{align*}
&\begin{pmatrix}
\infty & \cdots & \infty \\
\cdots & \infty & \cdots \\
\infty & \cdots & \infty
\end{pmatrix} \\
\leftrightarrow \\
&\begin{pmatrix}
\infty & \cdots & \infty \\
\cdots & \infty & \cdots \\
\infty & \cdots & \infty
\end{pmatrix}
\end{align*} \]
If a set contains pairs $(i, j), (i \pm 1, j \pm 1), (i \pm 2, j \pm 2)$ remove two consecutive pairs, decrease numbers between $i$ and $j$ by 2, numbers larger than $j$ by 4.

III) \[ \begin{align*}
&\begin{pmatrix}
\infty & \cdots & \infty \\
\cdots & \infty & \cdots \\
\infty & \cdots & \infty
\end{pmatrix} \\
\leftrightarrow \\
&\begin{pmatrix}
\infty & \cdots & \infty \\
\cdots & \infty & \cdots \\
\infty & \cdots & \infty
\end{pmatrix}
\end{align*} \]
Substitute pairs $(i, j), (i', k), (j', k')$ with pairs $(i, k'), (i', j'), (j, k)$ where $|i' - i| = |j' - j| = |k' - k| = 1$.

**Equivalence Moves**

According to the **Reidemeister Theorem** (Ref. 11), two regular knot projections belong to ambient isotopic knots if and only if they are connected through a series of the following moves.
These moves affect the set that denotes the projection as follows.

**First move:** A pair \((i, i + 1)\) or \((i + 1, i)\) is added or removed, while any number \(j\) not less than \(i\) is replaced by \(j + 2\).

**Second move:** Two pairs \((i, j), (i + 1, j + 1)\) or \((i + 1, j), (i, j + 1)\) are added or removed. Any number \(k\) not less than \(\min(i, j)\) and less than \(\max(i, j)\) is replaced by \(k + 2\), while any number \(k\) not less than \(\max(i, j)\) is replaced by \(k + 4\).

**Third move:** Pairs \((i, j), (i', k), (j', k')\) where \(|i' - i| = |j' - j| = |k' - k| = 1\), are replaced by \((i, k'), (i', j'), (j, k)\).

Sets connected through such moves denote ambient isotopic knots and are thus identified. In addition, two sets \(S_1\) and \(S_2\) are identified if \(\exists \epsilon \in \{-1, 1\}\) and \(c \in \{1, 2, ..., 2n\}\) such that \((i, j) \in S_1 \Rightarrow (c + \epsilon i \mod(2n), c + \epsilon \mod(2n)) \in S_2\), since these two sets denote the same projection. The parameter \(c\) indicates a change in the origin, while \(\epsilon = -1\) indicates an orientation reversal. Finally, in order to identify mirror symmetric knots, we consider equivalent two sets if \((i, j) \in S_1 \Rightarrow (j, i) \in S_2\).

**Knot Invariants**

Given a regular knot projection, one may inductively calculate any skein invariant by repeatedly applying the formula
Af(\includegraphics{crossing}) + Bf(\includegraphics{crossing}) = Cf(\includegraphics{unlink})

Eventually one ends up with a sum of multiples of the invariant of the trivial knot. This invariant is usually set to be equal to 1. When working on the set notation however, the following complication arises. While switching a crossing merely changes the type of the knot, eliminating a crossing converts a knot into a two-component link. During the inductive process one obtains a number of multicomponent links, which cannot be denoted through the Dowker notation as defined earlier. A way to overcome this problem is to use the unordered pair \(\{i, j\}\) each time a crossing \((i, j)\) or \((j, i)\) is removed. Therefore when an unordered pair appears in some set, it indicates the removal of a crossing from the original knot. It is thus possible to keep track of the emerging links, until one ends up with unlinks consisting of trivial knots. Then one continues by noticing that

\[
 f(n \text{ unlinked circles}) = \left(\frac{A + B}{c}\right)^{n-1}
\]

In contrast to the previous notations, one need not stop here. Inequivalent knots with identical skein invariants may be distinguished through “colorization” invariants. These invariants count the number of distinct mappings from the projection strands (segments joining undercrossings) to the set \(\{1, 2, \ldots, k\}\), such that at each crossing the three strands involved satisfy certain constraints (Ref. 12). The constraints are defined in such a way so that if projection becomes through the application of a Reidemeister move, to each acceptable mapping of \(P_1\) corresponds exactly one acceptable mapping of \(P_2\). This condition is necessary in order for the number of mappings to be an invariant. In particular, let a strand \(j\) separate the strand \(i\) from the strand \(i + 1\) as shown below, and let these strands be mapped to the “colors” \(c_i, c_{i+1}\) and \(c_j\). In order for the mapping to be allowed, the strands must satisfy a relation \(c_{i+1} = m_{c_ic_j}\), where \(M_{ab}\) is a \(k \times k\) matrix that defines the colorization invariant.

In order for the number of acceptable mappings to be a true invariant, the matrix has to fulfill the following conditions.

1) \(M_{rr} = r \quad \forall \quad r\)
2) \(M_{sr} = M_{ts} \quad \iff \quad s = t\)
3) \(M_{\alpha\beta} = \gamma, M_{\delta\alpha} = \epsilon, M_{\gamma\beta} = \eta \quad \Rightarrow \quad M_{\eta\gamma} = M_{\epsilon\beta}\)

Due to the enormous number of possible such matrices, it is not feasible to check them one after the other in order to obtain the necessary invariants. In practice we have been able to
obtain all matrices up to only 11 “colors”, which are insufficient to distinguish knots with more than 7 crossings. Instead, one uses special matrices that are guaranteed to obey the above relations. One such category consists of matrices \( M_{\alpha \beta} = t\alpha + (1 - t)\beta \mod q \) where the integer \( q \) is prime with respect to both \( t \) and \( t - 1 \). (Corollary: \( q \) is odd)

In order for the number of acceptable mappings to exceed the number of “colors”, one obtains a homogeneous linear system with \( n - 1 \) equations and \( n - 1 \) unknowns, where \( n \) is the crossing number. This system has non-zero solutions if and only if its determinant is 0; this determinant is in fact the \textit{Alexander} polynomial, and thus one may start by calculating this polynomial before actually obtaining the colorization invariants.

A second category consists of matrices \( M_{g\rho g\beta} = g\beta \circ g\alpha \circ g\beta^{-1} \), where the “colors” \( g\rho \) are elements of a conjugacy class of some group. In particular, matrices derived from permutation groups are extremely helpful.

One may easily check that matrices of both these categories obey all necessary conditions to define invariants.

**Connected Sums**

When compiling tables of knots, one usually avoids \textit{connected sums} and thus tabulates only \textbf{prime} knots. There is a number of reasons for this. First, once prime knots have been tabulated, one may obtain connected sums by combining prime knots. It is therefore unnecessary to consider connected sums in the computer program before the end, and one thus saves substantially in running time and computer memory. Second, connected sums of the same prime knots possess identical knot groups and cannot be distinguished through colorization invariants, even if the connected sums are distinct. Third, if one uses the Dowker notation, a set denoting the connected sum of \( k \) prime knots may yield up to \( 2^{k-1} \) distinct knots, and thus the notation becomes ambiguous.

When one uses the braid word notation, or eventually arrives to the braid word after using the closed lattice notation, one may check whether a knot is prime or a connected sum as follows. If some braid generator \( \sigma_i \in B_n \), where \( 1 \leq i \leq n - 1 \), appears exactly once in the braid word, the corresponding knot is the connected sum of the closures of the two braids that are formed from the products of the factors \( \sigma_j \) for \( j < i \) and \( \sigma_k \) for \( k > i \). If both of these closures are non-trivial knots, the initial knot is indeed a connected sum. The problem however with this criterion is that if a generator appears exactly once in the braid word, the knot is not presented in a minimal form. Therefore one has to add at least one crossing and also increase the braid index by one, in order to show that a knot is a connected sum.

This problem does not arise for the Dowker notation, which is one additional reason that makes such a notation preferrable. If there is some number \( k \) between 1 and \( 2n \) such that for any pair \((i, j)\) belonging to the set that represents the knot, \((k - 1) \cdot (k - j) \geq 0\), the knot is the connected sum of two sets, one consisting of pairs with numbers smaller than \( k \), the other of pairs with numbers larger than \( k \). If both sets denote non-trivial knots, the initial knot is a connected sum. One thus does not need to increase the crossing number in order to show that a knot is a connected sum.
4. COMPUTER PROGRAM RESULTS

Using the Dowker notation we developed and ran a computer program in order to tabulate knots. The main ideas of the algorithm were the following.

We first obtained all regular projections whose crossing number does not exceed a maximum value \( n \). To do so, we started by considering all permutations in \( S_k \) and all \( k \)-digit binary numbers, for \( 1 \leq k \leq n \). Let \( S \) be the set denoting a projection. \( p_i = j, b_i = 0 \) indicates that \((2i - 1, 2j) \in S\), while \( p_i = j, b_i = 1 \) indicates that \((2j, 2i - 1) \in S\). We then eliminated the \( \text{“undrawable”} \) sets and the sets denoting projections that had already been recorded.

Once the regular projections were obtained, we applied on each one of them all Reidemeister moves that do not increase the crossing number. If a projection was not connected to one checked before, the projection was stored in the computer’s memory. If it was connected to two or more projections already stored in the memory, these projections would be identified. When the process was completed, the computer recorded all projections stored in the memory that were not identified later to any previous projections.

When \( n \) was set 14, the number of recorded projections as a function of the crossing number came out as follows. (Connected sums are not included; mirror symmetric and orientation reversed knots are listed exactly once, even if they are not ambient isotopic)

| Number of Crossings | Recorded projections |
|---------------------|----------------------|
| 0                   | 1                    |
| 1                   | 0                    |
| 2                   | 0                    |
| 3                   | 1                    |
| 4                   | 1                    |
| 5                   | 2                    |
| 6                   | 3                    |
| 7                   | 7                    |
| 8                   | 21                   |
| 9                   | 49                   |
| 10                  | 165                  |
| 11                  | 552                  |
| 12                  | 2191                 |
| 13                  | 29781                |

All other projections are equivalent to one or more of the recorded projections. It is also possible that some of the projections of the table above are ambient isotopic to each other, but may only be connected through Reidemeister moves involving 15 or more crossings. In order to obtain the distinct knot types we calculated first the Alexander polynomials of these projections. For projections having identical Alexander polynomials we calculated colorization invariants for permutation groups \( S_p \), where \( 1 \leq p \leq m \) and \( m \) is a fixed parameter. For \( m = 7 \) we were able to show that all knots up to 11 crossings are inequivalent, while (at least) 2176 out of the 2191 knots with crossing number 12 are inequivalent to each other and to knots with smaller crossing numbers (for detailed tables and explicit proofs, see Ref. 13). These results are consistent to results recently reported by J. Hoste, M. Thistlethwaite and J. Weeks in Ref. 1.

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One observes that the whole program needs merely two input numbers \( n \) and \( m \), which indicate the maximum crossing number of the projections considered, and the largest permutation group used to obtain colorisation invariants. As stated earlier, we eventually set these parameters equal to \( n = 14 \) and \( m = 7 \).

While running the program for smaller parameters, we also noticed the following. First, in order to establish all ambient isotopies for knots up to \( c \) crossings, one had to set \( n \geq \frac{3c}{2} - 3 \) for \( 5 \leq c \leq 11 \). To be exact, the table obtained was the following.

| \( c \) | \( n \) |
|-------|-------|
| 0     | 0     |
| 1     | 0     |
| 2     | 0     |
| 3     | 3     |
| 4     | 4     |
| 5     | 5     |
| 6     | 6     |
| 7     | 8     |
| 8     | 9     |
| 9     | 11    |
| 10    | 12    |
| 11    | 14    |

Second, in order to prove that the tabulated knots were distinct, for \( c \leq 8 \) the Alexander polynomials were sufficient. For \( c = 9 \) or 10 one had to set \( m \geq 5 \), while for \( c = 11 \) one had to set \( m \geq 7 \).

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