Problems of Real Scalar Klein-Gordon Field

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(Dated: March 27, 2022)

We examine the negative energy solution in Klein-Gordon equation in terms of the number of field components. A scalar field has only one component, and there is no freedom left for an anti-particle since the Klein-Gordon equation failed to take the negative energy solution into account. This is in contrast to the Dirac equation which has four components of fields. It is shown that the current density for a real scalar field is always zero if the field is classical, but infinite if the field is quantized. This suggests that the condition of a real field must be physically too strong.

PACS numbers: 03.50.-z, 11.10.-z

I. INTRODUCTION

When one treats a particle which moves with its velocity close to the velocity of light, then one should employ the relativistic kinematics. The Einstein relation for a particle with its mass $m$ is given as

$$E = \sqrt{m^2 + p^2}. \quad (1.1)$$

For the quantized equations of this particle, there are two equations, that is, Klein-Gordon and Dirac equations.

The equation for relativistic bosons can be described by the Klein-Gordon equation,

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)\phi(x) = 0 \quad (1.2)$$

where we denote $x = (t, r)$.

It should be important to note that the boson field $\phi$ has only one component, and therefore it is believed to describe the spin zero particle. In this case, however, it is interesting to ask where the freedom of the negative energy state may appear. Since the boson field $\phi$ has only one component, it is difficult to imagine that the effects of the negative energy solutions are properly taken into account. This means that we should find out where the freedom of the anti-particle should appear. In this respect, the boson field $\phi$ is essentially the same as the Schrödinger field $\psi$ in the non-relativistic quantum mechanics. At least, classically, the boson field $\phi$ which should be complex like the Schrödinger field $\psi$ should correspond to one boson state.

Therefore, if we could find an equation which has two components of field, then we should be able to understand where the degree of freedoms concerning the negative energy state appears. However, as long as we carry out the procedure of factoring out eq. (1.2) into equations with two component fields, we see that it is impossible.

As is well known, the equations with four component fields are only possible, which is the Dirac equation $\Psi$. There is an interesting attempt to construct field equations of bosons with two component spinor by Gross $\Psi$. However, the Hamiltonian becomes non-Hermite even though the right dispersion relation for bosons is obtained.

On the other hand, there is one example that has the proper two components for boson fields. That is, the gauge field in QED. The vector field $A$ has the two components and obeys coupled equations which have some similarities with the Klein-Gordon equation with some constraints. This indicates that there a vector field with a finite mass may have some difficulty since, in this case, the degree of freedom cannot be properly taken into account.

Here, several questions may arise. The first question is, what is the anti-particle of a real scalar field? Is it related to the negative energy solutions? The second question is connected to the current density of a real scalar field. Since it is easy to prove that the current density of a real scalar field vanishes, what is the physics of no current density of a scalar field? As the third question, what is the physical meaning of a real scalar field? In the Schrödinger field $\psi$, it is naturally complex. If the field $\psi$ is real, it does not depend on time and the energy eigenvalue must be zero, and it becomes unphysical. Therefore, if a real scalar field exists in Klein-Gordon equation, then how can one make a non-relativistic limit of the Klein-Gordon real scalar field?

Here, we discuss the above questions and examine whether a real scalar field can exist as a physical observable or not in Klein-Gordon equation. In most of the field theory textbooks, we find that pion with the positive charge is an anti-particle of pion with the negative charge. This can be understood easily if we look into the structure of the pion in terms of quarks. $\pi^\pm$ are indeed anti-particle to each other by changing quarks into anti-quarks.

Since pions are not an elementary particle, their dynamics must be governed by the complicated quark dynamics. Under some drastic approximation, the motion
of pion may be governed by the Klein-Gordon equation if one is only interested in the center of motion of pion.

II. NEGATIVE ENERGY STATE

It looks that eq.(1.2) contains the negative energy state. However, one sees that eq.(1.2) is only one component equation and, therefore the eigenvalue of $E^2$ can be obtained as a physical observable. There is no information from the Klein-Gordon equation for the energy $E$ itself, but only $E^2$ as we see it below,

$$(-\nabla^2 + m^2) \phi = E^2 \phi. \quad (2.1)$$

Thus, one obtains only one information from the Klein-Gordon equation. In this respect, there should be no negative energy physical state in the Klein-Gordon equation.

After one obtains the value of $E^2$, one may say that one finds two values for $E$. However, the way one obtains two solutions has nothing to do with physics. Physically, the positive solution of eq.(2.1) should be taken, and this is just the one found in eq.(2.2).

Mathematically, the wave function may include the solutions with $E = \pm \sqrt{m^2 + \mathbf{k}^2}$. The Klein-Gordon equation for the energy $E$ itself, but only $E^2$ as we see it below,

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A. Classical field

Eq. (2.1) has two independent solutions and can be given by requiring that the state should be the eigenstate of momentum

$$\phi(x) = C_+ e^{-i\omega_k t + i\mathbf{k} \cdot \mathbf{r}} \quad \text{or} \quad C_- e^{i\omega_k t - i\mathbf{k} \cdot \mathbf{r}} \quad (2.2)$$

where $\omega_k = \sqrt{k^2 + m^2}$. This shape of the solution is determined since the momentum $p^\mu = (\omega, \mathbf{k})$ and the coordinate $x^\mu = (t, \mathbf{r})$ can make a Lorentz scalar as

$$p^\mu x_\mu = \omega k + \mathbf{k} \cdot \mathbf{r}$$

which is just the one found in eq.(2.2).

Now, we discuss the current density of the Klein-Gordon field which is defined as

$$\rho(x) = i \left( \phi^\dagger(x) \frac{\partial \phi(x)}{\partial t} - \frac{\partial \phi^\dagger(x)}{\partial t} \phi(x) \right) \quad (2.3a)$$

$$\mathbf{j}(x) = -i \left[ \phi^\dagger(x) (\nabla \phi(x)) - (\nabla \phi^\dagger(x)) \phi(x) \right]. \quad (2.3b)$$

It should be noted that the current density must be hermitian and therefore the shape of eqs.(2.3) is uniquely determined. One cannot change the order between $\frac{\partial \phi^\dagger(x)}{\partial t}$ and $\phi(x)$ in the second term of eq.(2.3a).

Now, we come to an important observation that a real scalar field should have a serious problem. The real scalar field $\phi(x)$ can be written as

$$\phi(x) = \sum_k \frac{1}{\sqrt{2V_\omega k}} \left[ a_k e^{-i\omega_k t + i\mathbf{k} \cdot \mathbf{r}} + a_k^\dagger e^{i\omega_k t - i\mathbf{k} \cdot \mathbf{r}} \right] \quad (2.4)$$

where $V$ denotes the box. We assume that the $\phi(x)$ is still a classical field, that is, $a_k^\dagger$ and $a_k$ are not operators, but just the same as $C_\pm$ in eq.(2.2).

In this case, it is easy to prove that the current densities of $\rho(x)$ and $\mathbf{j}(x)$ which are constructed from the real scalar field $\phi(x)$ vanish to zero

$$\rho(x) = 0, \quad \mathbf{j}(x) = 0. \quad (2.5)$$

This means that there is no flow of the real scalar field, at least, classically. This is clear since a real wave function in the Schrödinger equation cannot propagate. Therefore, the condition that the scalar field should be a real field must be physically too strong. Even in the Schrödinger field, one cannot require that the field should be real as will be discussed in section IV.

In other words, the scalar field $\phi$ can be a real if it is a solution that satisfies the boundary conditions, but one cannot require that it should be a real field.

B. Quantized field

Now, we come to the current density when the field is quantized. Below it is shown that the current density of the real scalar field has some problem even if quantized, contrary to a common belief.

When we quantize the boson field of $\phi$, then $a_k^\dagger$ and $a_k$ become creation and annihilation operators

$$\hat{\phi}(x) = \sum_k \frac{1}{\sqrt{2V_\omega k}} \left[ a_k e^{-i\omega_k t + i\mathbf{k} \cdot \mathbf{r}} + a_k^\dagger e^{i\omega_k t - i\mathbf{k} \cdot \mathbf{r}} \right]. \quad (2.6)$$

In this case, the current density of eq.(2.4) becomes

$$\hat{\rho} = i \left( \hat{\phi}(x) \hat{\Pi}(x) - \hat{\Pi}(x) \hat{\phi}(x) \right) = i [\hat{\phi}(x), \hat{\Pi}(x)] \quad (2.7)$$

where $\hat{\Pi}(x)$ is a conjugate field of $\hat{\phi}(x)$, that is, $\Pi(x) = \hat{\phi}(x)$. However, the quantization condition of the boson fields with eq.(2.6) becomes

$$[\hat{\phi}(x), \hat{\Pi}(x')]_{t=t'} = i \delta(\mathbf{r} - \mathbf{r'}). \quad (2.8)$$

Therefore, eq.(2.7) becomes

$$\hat{\rho} = i [\hat{\phi}(x), \hat{\Pi}(x)] = -i \delta(\mathbf{r}). \quad (2.9)$$

Thus, the current density of the quantized real boson field becomes infinity after the quantization! In this sense, the current density of a real scalar field behaves...
completely in an unphysical manner. Therefore, it is by now obvious that the current density of the real scalar field has an improper physical meaning. This should be related to the fact that a real scalar field condition is too strong and one cannot impose this real condition on $\phi$ at the Lagrangian density level.

III. COMPLEX SCALAR FIELDS

Since a real scalar field has a difficulty not only in the classical case but also in the quantized case, it should be worth considering a complex scalar field. In this case, the complex scalar field can be written as

$$\phi(x) = \sum_k \frac{1}{\sqrt{2V\omega_k}} \left[ a_k e^{-i\omega_k t + i\mathbf{k} \cdot \mathbf{r}} + b_k^* e^{i\omega_k t - i\mathbf{k} \cdot \mathbf{r}} \right].$$

(3.1)

In this case, the current density is well defined and has no singularity when quantized.

According to a common belief, the complex scalar field should describe charged bosons, one which has a positive charge and the other which has a negative charge. But a question may arise as to where this degree of freedom comes from? By now, we realize that there is no negative energy solution in the Klein-Gordon equation. If one took into account the negative energy solution, then one should have had the field equations of two components. On the other hand, eq.(3.1) assumes a scalar field with two components, and not the result of the field equations. It is therefore most important to seek for the two component Klein-Gordon like equation which should be somewhat similar to the Dirac equation.

In any case, if it is a simply complex scalar field, then it should describe one boson state, at least, as a classical field. It is therefore surprising that the complex scalar field can describe charged particles when quantized.

IV. SCHRÖDINGER FIELD

It is worthwhile discussing the Schrödinger field $\psi(r, t)$ in the non-relativistic quantum mechanics. Apart from the kinematics, the Schrödinger field should have the same behavior as the scalar field in the classical field theory. The Lagrangian density of the Schrödinger field can be written

$$\mathcal{L} = i\psi^\dagger \frac{\partial \psi}{\partial t} - \frac{1}{2m} \nabla \psi^\dagger \nabla \psi - \psi^\dagger U \psi. \quad (4.1)$$

We note that the Lagrangian density of eq.(4.1) should have the $U(1)$ symmetry.

From eq.(4.1), we obtain the Schrödinger equation

$$i \frac{\partial \psi(r, t)}{\partial t} = \left( -\frac{1}{2m} \nabla^2 + U \right) \psi(r, t). \quad (4.2)$$

The solutions of eq.(4.2) in the $U = 0$ case can be obtained in a box with its volume $V$,

$$\psi_1 = \frac{C_1}{\sqrt{V}} e^{-iE_k t} e^{i\mathbf{k} \cdot \mathbf{r}}, \quad \psi_2 = \frac{C_2}{\sqrt{V}} e^{-iE_k t} e^{-i\mathbf{k} \cdot \mathbf{r}} \quad (4.3)$$

where $E_k = \frac{k^2}{2m}$ and $k$ denote quantum numbers which should correspond to the momentum of a particle.

A. Unphysical real field condition

It should be important to note that the complex Schrödinger field just corresponds to one particle state. In fact, if one imposes the condition that the field $\psi(r, t)$ should be a real function, that is

$$\psi(r, t) = \psi^\dagger(r, t) \quad (4.4)$$

then one obtains from the Schrödinger equation

$$\frac{\partial \psi(r, t)}{\partial t} = 0. \quad (4.5)$$

That is, the Schrödinger field becomes time-independent and in addition, the energy eigenvalue $E$ is zero

$$E = 0 \quad (4.6)$$

since

$$\left( -\frac{1}{2m} \nabla^2 + U \right) \psi(r) = 0. \quad (4.6)$$

Therefore, the field cannot propagate. It is by now clear that the real field condition of $\psi$ is not physically acceptable.

Since the Klein-Gordon field should be reduced to the Schrödinger field in the non-relativistic limit, the real field condition of the Klein-Gordon field must also be unphysical.

V. ELEMENTARY BOSONS

For practical problems, $\pi^\pm$ mesons are considered to be anti-particles to each other. However, they are not elementary bosons, and one sees that, in terms of the quark and anti-quark terminology, they are mutually anti-particles to each other.

As for elementary bosons, there are $W^\pm$ gauge bosons. However, they are not anti-particles to each other. Instead, they are different particles with different isospin components and they have nothing to do with negative energy solutions of the gauge field equations. This is just consistent with the present interpretation of the relativistic bosons.

Further, one often introduces complex scalar fields, and one says that $\phi$ and $\phi^*$ correspond to anti-particles to
each other. But this is completely different from the anti-particle of fermions where the degree of freedoms for the antiparticle exists as the negative energy solution in the Dirac equation.

Indeed, the Dirac equation is quite different from the Klein-Gordon equation. The Dirac equation becomes

\[ (-i \nabla \cdot \alpha + m\beta) \psi = E\psi. \tag{5.1} \]

Clearly, the Dirac equation is the eigenvalue equation for the energy \( E \), and therefore the \( E \) itself must be physical observables. This indicates that the negative energy states must be physically present, and indeed this is just the way to construct the physical vacuum state in fermion field theory models [1-4].

\section{VI. PHOTON}

There is one example which has the right degree of freedom of boson field. That is the electromagnetic field \( A \). The vector field \( A \) has only the transverse components ( \( \nabla \cdot A = 0 \) ) and has indeed two components as a boson field. In this respect, the gauge field takes into account the negative energy degree of freedom in a proper way, though it is realized as a spin degree of freedom.

There are massive gauge particles which are weak bosons. The weak gauge fields acquire their mass by the Higgs mechanism in which the gauge fixing should be done together with the spontaneous symmetry breaking for the Higgs fields. The problem of the Higgs mechanism is that the gauge fixing is done at the Lagrangian density level, and therefore, after the gauge fixing, the Lagrangian density has no gauge freedom any more. On the other hand, in the normal circumstance, the gauge fixing should be made when one wishes to solve the equation of motion since the gauge field has a redundancy and the number of freedom should be reduced.

\section{VII. SPONTANEOUS SYMMETRY BREAKING}

In the discussion of the spontaneous symmetry breaking in boson field theory model, Goldstone presented a complex scalar field model with a double well potential.

The Lagrangian density of the complex scalar field \( \phi(x) \) in the double well potential can be written as

\[ \mathcal{L} = \frac{1}{2} \partial_\mu \phi^\dagger \partial^\mu \phi - u_0 (|\phi|^2 - \lambda^2)^2 \tag{7.1} \]

where \( u_0 \) and \( \lambda \) are constant and the Lagrangian density has a \( U(1) \) symmetry.

Now, one rewrites the complex field as

\[ \phi(x) = (\lambda + \eta(x))e^{\frac{\xi(x)}{\lambda}} \tag{7.2} \]

where \( \eta \) is assumed to be much smaller than the \( \lambda \),

\[ |\eta(x)| << \lambda. \]

In this case, one can rewrite eq.\,(7.1) as

\[ \mathcal{L} = \frac{1}{2} \partial_\mu \phi^\dagger \partial^\mu \phi + \frac{1}{2} \partial_\mu \xi \partial^\mu \xi + U(|\lambda + \eta(x)|) + \ldots \tag{7.3} \]

Here, one finds the massless boson \( \xi \) which is associated with the degeneracy of the vacuum energy.

This is the spontaneous symmetry breaking which is indeed found by Goldstone, and he pointed out that there should appear a massless boson associated with the symmetry breaking \([1, 4, 5]\). The degeneracy of the potential vacuum is converted into a massless boson degree of freedom. This looks plausible, and at least approximately there is nothing wrong with this treatment of the spontaneous symmetry breaking phenomena. However, eq.\,(7.3) is written in terms of two real scalar fields, and physically as we discussed above, the real scalar fields cannot correspond to any physical particles.

In addition, if it is a realistic Lagrangian density for a physical massless boson, then its Lagrangian density must be always written as

\[ \mathcal{L} = \frac{1}{2} \partial_\mu \phi^\dagger \partial^\mu \phi \tag{7.4} \]

which is composed of the complex scalar field.

\section{VIII. SUMMARY}

In nature, all the elementary particles are composed of fermions, except gauge bosons. For this statement, people may claim that the Higgs particles must be elementary bosons. However, up to now, these particles are not yet discovered in nature, and it should be quite interesting to know whether there should exist elementary bosons apart from gauge bosons or not.

The present study shows that there is no anti-particle corresponding to the negative energy solution since its degree of freedom does not exist. Further, the current density of a real scalar field \( j_\mu \) is identically zero as a classical field theory and diverges as a quantum field theory. In the Schrödinger field, a real field condition leads to an unphysical state, and since the Klein-Gordon field should be reduced to the Schrödinger field in the non-relativistic limit, a real scalar field condition in the Klein-Gordon field cannot be justified, which is consistent with the vanishing current density.

The present study of the Klein-Gordon field strongly suggests that the Klein-Gordon field equation itself should not be physically acceptable. The first quantization condition of \( [x_i, p_j] = i\hbar \delta_{ij} \) is the basic assumption of deriving the Klein-Gordon equation, and therefore if the first quantization is not the fundamental principle, one cannot derive the Klein-Gordon equation any more. This point should be reexamined in future.

We would like to thank K. Fujikawa, M. Hiramato, T. Nihei and H. Takahashi for useful discussions.
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