THE OPLAX LIMIT OF AN ENRICHED CATEGORY

In memory of our colleague Marta Bunge

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Abstract. We show that 2-categories of the form \( \mathcal{B}\text{-Cat} \) are closed under slicing, provided that we allow \( \mathcal{B} \) to range over bicategories (rather than, say, monoidal categories). That is, for any \( \mathcal{B}\text{-category} \ X \), we define a bicategory \( \mathcal{B}/X \) such that \( \mathcal{B}\text{-Cat}/X \cong (\mathcal{B}/X)\text{-Cat} \). The bicategory \( \mathcal{B}/X \) is characterized as the oplax limit of \( X \), regarded as a lax functor from a chaotic category to \( \mathcal{B} \), in the 2-category \( \text{BICAT} \) of bicategories, lax functors and icons. We prove this conceptually, through limit-preservation properties of the 2-functor \( \text{BICAT} \to \text{2-CAT} \) which maps each bicategory \( \mathcal{B} \) to the 2-category \( \mathcal{B}\text{-Cat} \). When \( \mathcal{B} \) satisfies a mild local completeness condition, we also show that the isomorphism \( \mathcal{B}\text{-Cat}/X \cong (\mathcal{B}/X)\text{-Cat} \) restricts to a correspondence between fibrations in \( \mathcal{B}\text{-Cat} \) over \( X \) on the one hand, and \( \mathcal{B}/X \)-categories admitting certain powers on the other.

1. Introduction

It is well-known that for any monoidal category \( \mathcal{V} \) and monoid \( M = (M, e: I \to M, m: M \otimes M \to M) \) therein, the slice category \( \mathcal{V}/M \) has a canonical monoidal structure; the unit is \( e \) and the monoidal product of objects \( (s: S \to M) \) and \( (t: T \to M) \) is

\[
S \otimes T \xrightarrow{s \otimes t} M \otimes M \xrightarrow{m} M.
\]

Moreover, there is a canonical isomorphism of categories

\[
\text{Mon}(\mathcal{V}/M) \cong \text{Mon}(\mathcal{V})/M.
\]

This paper originated from a natural generalization of this, replacing the notion of monoid in \( \mathcal{V} \) by that of \( \mathcal{V} \)-category. That is, for any \( \mathcal{V} \)-category \( X \), there is an appropriate “base” \( \mathcal{V}/X \) admitting a canonical isomorphism of 2-categories

\[
(\mathcal{V}/X)\text{-Cat} \cong \mathcal{V}\text{-Cat}/X.
\] (1)

Here, the “base” \( \mathcal{V}/X \) is in general not a monoidal category but a bicategory. Enriched category theory over bicategories is developed in, e.g., \([\text{BCSW83}, \text{Str83}]\). We recall that, for a bicategory \( \mathcal{B} \), a \( \mathcal{B} \)-category \( X \) is given by

\[
\begin{align*}
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\end{align*}
\]
• a set \( \text{ob}(X) \);

• a function \(|-| : \text{ob}(X) \to \text{ob}(\mathcal{B}) \) (\(|x| \) is called the *extent* of \( x \));

• for all \( x, x' \in \text{ob}(X) \), a 1-cell \( \mathcal{X}(x, x') : |x| \to |x'| \) in \( \mathcal{B} \);

• for all \( x \in \text{ob}(X) \), a 2-cell

\[
\begin{array}{c}
|x| \\
\downarrow \mathcal{X}(x, x) \\
|x|
\end{array}
\]

in \( \mathcal{B} \), where \( 1_{|x|} \) is the identity 1-cell on \(|x|\); and

• for all \( x, x', x'' \in \text{ob}(X) \), a 2-cell

\[
\begin{array}{c}
|x| \\
\downarrow \mathcal{X}(x, x') \\
|x'| \\
\downarrow \mathcal{X}(x', x'') \\
|X(x, x'')| \\
\downarrow \mathcal{X}(x, x''') \\
|x''|
\end{array}
\]

in \( \mathcal{B} \), subject to the associativity and identity laws, generalizing the usual axioms for a category.

Since the isomorphism (1) already forces us to consider enrichment over bicategories, it is natural to wonder whether there is a generalization of the isomorphism involving a bicategory \( \mathcal{B} \) in place of the monoidal category \( \mathcal{V} \). Indeed this turns out to be the case: for any bicategory \( \mathcal{B} \) and \( \mathcal{B} \)-category \( \mathcal{X} \), there is a bicategory \( \mathcal{B}/\mathcal{X} \) with a canonical isomorphism of 2-categories \((\mathcal{B}/\mathcal{X})\text{-Cat} \cong \mathcal{B}\text{-Cat}/\mathcal{X}\). Thus 2-categories of the form \( \mathcal{B}\text{-Cat} \) are closed under slicing, provided that we allow \( \mathcal{B} \) to range over bicategories.

The construction of \( \mathcal{B}/\mathcal{X} \) is simple enough to carry out at this point; see also Remark 4.8 for a more abstract point of view. We set \( \text{ob}(\mathcal{B}/\mathcal{X}) = \text{ob}(\mathcal{X}) \) and, for all \( x, x' \in \text{ob}(\mathcal{B}/\mathcal{X}) \), the hom-category \( (\mathcal{B}/\mathcal{X})(x, x') \) is the slice category \( \mathcal{B}(|x|, |x'|)/\mathcal{X}(x, x') \). The identity 1-cell at \( x \) is \( j_x \), and the composite of 1-cells \((s : S \to \mathcal{X}(x, x')) : x \to x'\) and \((t : T \to \mathcal{X}(x', x'')) : x' \to x''\) is the pasting composite

\[
\begin{array}{c}
|X| \\
\downarrow \mathcal{X}(x, x') \\
|x'| \\
\downarrow \mathcal{X}(x', x'') \\
|x''|
\end{array}
\]

Of course, when both \( \mathcal{B} \) and \( \mathcal{X} \) have only one object, the construction of \( \mathcal{B}/\mathcal{X} \) reduces to that of the slice of a monoidal category over a monoid.

This observation allows one to view (enriched) functors as (enriched) categories, and suggests new perspectives even on notions which are not directly related to enrichment.
For example, for any \((\text{Set})\)-category \(X\), there is a bicategory \(\text{Set} / X\) with an isomorphism \((\text{Set} / X)-\text{Cat} \cong \text{Cat} / X\). Thus we can view functors into \(X\) as enriched categories (see Example 4.6 below and [Gar14] for a related construction), and we may potentially interpret properties of functors via enriched categorical terms. Indeed, we shall show that a functor \(Y \to X\) is a Grothendieck fibration if and only if the corresponding \(\text{Set} / X\)-category \(Y\) has powers by a certain class of 1-cells in \(\text{Set} / X\), as well as a \(\mathcal{B}\)-enriched version of this result.

The notation \(\mathcal{B} / X\) is justified by its characterization as the oplax limit of a 1-cell in a suitable 2-category. To explain this, recall that a \(\mathcal{B}\)-category \(X\) can be given equivalently as a lax functor \(X: X_c \to \mathcal{B}\), where \(X_c\) is the chaotic category with the same set of objects as \(X\).\(^1\) Thus we can view the \(\mathcal{B}\)-category \(X\) as a 1-cell in the 2-category \(\text{BICAT}\) of bicategories, lax functors and icons [Lac10]. The bicategory \(\mathcal{B} / X\) is the oplax limit of this 1-cell in \(\text{BICAT}\):

\[
\begin{tikzcd}
X_c & \text{BICAT} \\
\mathcal{B} / X \\
\mathcal{B}
\end{tikzcd}
\]

(Although \(\text{BICAT}\) is not complete, it does have oplax limits of 1-cells [Lac05, LS12].) This generalizes the characterization of the slice monoidal category \(\mathcal{V} / M\) as the oplax limit of the monoid \(M\) in \(\mathcal{V}\), regarded as a lax monoidal functor from the terminal monoidal category to \(\mathcal{V}\), in the 2-category of monoidal categories, lax monoidal functors and monoidal natural transformations.

In this paper, we study properties of the 2-functor \(\text{Enr}: \text{BICAT} \to \text{2-CAT}\) mapping each bicategory \(\mathcal{B}\) to the 2-category \(\mathcal{B}\)-\text{Cat}, in order to understand the isomorphism \((\mathcal{B} / X)-\text{Cat} \cong \mathcal{B}\)-\text{Cat} / X\) conceptually, as well as to establish further closure properties of 2-categories of the form \(\mathcal{B}\)-\text{Cat}. To this end, it is useful to factorize \(\text{Enr}\) as

\[
\begin{tikzcd}
\text{BICAT} & \text{2-CAT} \\
\text{2-CAT}/\text{Enr}(1) \ar[swap]{u}{\text{Enr}} \ar{u}{\text{Enr}_1}
\end{tikzcd}
\]

where \(1\) is the terminal bicategory. The 2-functor \(\text{Enr}_1\) maps each bicategory \(\mathcal{B}\) to \(\mathcal{B}\)-\text{Cat} equipped with the 2-functor \(\text{Enr}(!): \mathcal{B}\)-\text{Cat} \to \text{Enr}(1)\) induced from the unique lax functor \(!: \mathcal{B} \to 1\). The underlying category of \(\text{Enr}(1)\) is \(\text{Set}\), and \(\text{Enr}(!)\) can be regarded as \(\text{ob}(-)\), mapping each \(\mathcal{B}\)-category \(X\) to its set of objects \(\text{ob}(X)\). (Although \(\text{Enr}\) is usually denoted simply as \((-)\)-\text{Cat}, we adopted the current notation in order to avoid the potentially misleading expression \(1\)-\text{Cat}.)

In our main theorem (Theorem 2.1), we show that \(\text{Enr}_1: \text{BICAT} \to \text{2-CAT}/\text{Enr}(1)\) preserves any limit which happens to exist in \(\text{BICAT}\). This implies that \(\text{Enr}\) preserves

\(^1\)Lax functors of this form were studied by Bénabou [Bén67] under the name \textit{polyad}; for the connection with enriched categories see [Str83].
any limit which happens to exist in \textbf{BICAT} and is preserved by the forgetful 2-functor \textbf{2-CAT}/Enr\((1)\) \to \textbf{2-CAT}; the latter condition is satisfied whenever the limit in question is small enough to exist in \textbf{2-CAT} and is created by the forgetful 2-functor. In ordinary category theory, the limits created by the forgetful functors from slice categories are precisely the connected limits. In Section 3 we generalize this to 2-categories (or in fact to \(\mathcal{V}\)-categories where \(\mathcal{V}\) is any complete and cocomplete cartesian closed category), introducing the class of \textbf{Cat}-connected limits with several characterizations. Thus \(\text{Enr}: \textbf{BICAT} \to \textbf{2-CAT}\) preserves any \textbf{Cat}-connected limit which happens to exist in \textbf{BICAT}. This includes Eilenberg–Moore objects of comonads, for example. Although oplax limits of 1-cells are not \textbf{Cat}-connected, the isomorphism \((\mathcal{B}/\mathcal{X})\text{-Cat} \cong \mathcal{B}\text{-Cat}/\mathcal{X}\) is explained via the limit-preservation property of \(\text{Enr}\) and a 2-categorical argument in Section 4.

Finally, in Section 5, we investigate (internal) fibrations in the 2-category \(\mathcal{B}\text{-Cat}\) of \(\mathcal{B}\)-categories. Specifically, we show that (assuming a mild local completeness condition on \(\mathcal{B}\)) a \(\mathcal{B}\)-functor \(\mathcal{Y} \to \mathcal{X}\) is a fibration in \(\mathcal{B}\text{-Cat}\) if and only if the corresponding \(\mathcal{B}/\mathcal{X}\)-category \(\overline{\mathcal{Y}}\) admits certain powers.

We intend to revisit the results of this paper in the future, in the context of enrichment over pseudo double categories.

2. The limit-preservation theorem

Size does not play a significant role in this paper; nonetheless we make a few comments here about the issues which arise and our approach to them. The typical \textit{monoidal} categories over which one enriches, such as \textbf{Set}, \textbf{Cat}, or \textbf{Ab}, have small hom-sets but are not themselves small. Thus the corresponding bicategories will not even have small hom-categories. We do still need some control of the size of these bicategories, and therefore fix two Grothendieck universes \(\mathcal{U}_0\) and \(\mathcal{U}_1\) with \(\mathcal{U}_0 \in \mathcal{U}_1\). Sets, categories, etc. in \(\mathcal{U}_0\) and \(\mathcal{U}_1\) are called \textit{small} and \textit{large} respectively.

Let \textbf{BICAT} be the 2-category of large bicategories, lax functors and icons [\textit{Lac10}, Theorem 3.2], and \textbf{2-CAT} be the 2-category of large 2-categories, 2-functors and 2-natural transformations. We have a 2-functor \(\text{Enr}: \textbf{BICAT} \to \textbf{2-CAT}\) sending each bicategory \(\mathcal{B}\) to the 2-category \(\mathcal{B}\text{-Cat}\) of all small \(\mathcal{B}\)-categories, \(\mathcal{B}\)-functors and \(\mathcal{B}\)-natural transformations. It is the limit-preservation properties of this 2-functor \(\text{Enr}\) that is our main focus. The limits in question will be 2-limits weighted by 2-functors of the form \(F: \mathcal{D} \to \textbf{CAT}\), where \(\mathcal{D}\) is a large 2-category and \textbf{CAT} is the 2-category of large categories.

The bicategory \(\mathbf{1}\) with a single 2-cell is the terminal object of \textbf{BICAT}, and hence \(\text{Enr}\) induces the 2-functor \(\text{Enr}_1: \textbf{BICAT} \to \textbf{2-CAT}/\text{Enr}(1)\), where \(\textbf{2-CAT}/\text{Enr}(1)\) denotes the (strict) slice 2-category of \textbf{2-CAT} over \(\text{Enr}(1) \in \textbf{2-CAT}\). The 2-category \(\text{Enr}(1)\) is the locally chaotic 2-category whose underlying category is \textbf{Set}. More precisely, the objects of \(\text{Enr}(1)\) can be identified with the small sets, and for each pair of small sets \(X\) and \(Y\)
we have $\text{Enr}(1)(X, Y) = \mathbf{Set}(X, Y)_c$, where $(-)_c$ appears in the string of adjunctions

\[
\begin{array}{ccc}
\pi_0 & \Downarrow & \mathbf{CAT}_0. \\
\downarrow & \downarrow & \downarrow \\
\downarrow & \downarrow & \downarrow \\
\mathbf{SET} & \xleftarrow{(-)_d} & \mathbf{CAT}. \\
\downarrow & \downarrow & \downarrow \\
\downarrow & \downarrow & \downarrow \\
\downarrow & \downarrow & \downarrow \\
\mathbf{SET} & \xleftarrow{(-)_l} & \mathbf{CAT}. \\
\end{array}
\]

Here, $\mathbf{SET}$ and $\mathbf{CAT}_0$ denote the categories of large sets and of large categories respectively. The (finite-product-preserving) functors in (2) induce 2-adjunctions

\[
\begin{array}{ccc}
\pi_0 & \Downarrow & \mathbf{CAT}_2. \\
\downarrow & \downarrow & \downarrow \\
\downarrow & \downarrow & \downarrow \\
\mathbf{CAT} & \xleftarrow{(-)_d} & \mathbf{2-CAT}. \\
\downarrow & \downarrow & \downarrow \\
\downarrow & \downarrow & \downarrow \\
\downarrow & \downarrow & \downarrow \\
\mathbf{2-CAT} & \xleftarrow{(-)_l} & \mathbf{SET}. \\
\end{array}
\]

Thus we shall write the 2-category $\text{Enr}(1)$ as $\mathbf{Set}_c$.

Explicitly, the 2-functor $\text{Enr}_1 : \mathbf{BICAT} \to \mathbf{2-CAT}/\mathbf{Set}_c$ maps each bicategory $B$ to the 2-category $B\text{-Cat}$ equipped with the 2-functor $\text{ob}(-) : B\text{-Cat} \to \mathbf{Set}_c$ which extracts the set of objects of a $B$-category.

2.1. Theorem. The 2-functor $\text{Enr}_1 : \mathbf{BICAT} \to \mathbf{2-CAT}/\mathbf{Set}_c$ preserves all weighted limits which happen to exist in $\mathbf{BICAT}$.

Proof. We shall show the following.

(a) The set $G$ of all objects of $\mathbf{2-CAT}/\mathbf{Set}_c$ of the form $(\mathbf{2}_2 \to \mathbf{Set}_c)$, where $\mathbf{2}_2$ denotes the free 2-category on a single 2-cell, is a strong generator of the 2-category $\mathbf{2-CAT}/\mathbf{Set}_c$.

(b) For each object $A \in G$, the 2-functor $\mathbf{2-CAT}/\mathbf{Set}_c(A, \text{Enr}_1(-)) : \mathbf{BICAT} \to \mathbf{CAT}$ is a 2-limit of representable 2-functors, and hence preserves all weighted limits which happen to exist in $\mathbf{BICAT}$.

From these, the main claim follows. Indeed, let $\mathbf{D}$ be a large 2-category, $F : \mathbf{D} \to \mathbf{CAT}$ be a 2-functor (the weight) and $S : \mathbf{D} \to \mathbf{BICAT}$ be a 2-functor such that the weighted limit $\{F, S\}$ exists in $\mathbf{BICAT}$. Then the weighted limit $\{F, \text{Enr}_1 \circ S\}$ exists in $\mathbf{2-CAT}/\mathbf{Set}_c$, because $\mathbf{2-CAT}/\mathbf{Set}_c$ has all (large) weighted limits. We have a comparison 1-cell $M : \text{Enr}_1\{F, S\} \to \{F, \text{Enr}_1 \circ S\}$ in $\mathbf{2-CAT}/\mathbf{Set}_c$. Now for each $A \in G$, the functor

\[
\mathbf{2-CAT}/\mathbf{Set}_c(A, M) : \mathbf{2-CAT}/\mathbf{Set}_c(A, \text{Enr}_1\{F, S\}) \to \mathbf{2-CAT}/\mathbf{Set}_c(A, \{F, \text{Enr}_1 \circ S\})
\]

is an isomorphism by (b), from which we conclude that $M$ is an isomorphism by (a).
$\mathcal{G}$ is a strong generator of $2\text{-}\text{CAT}/\text{Set}_{lc}$ because, given any 1-cell $T: (\mathcal{X} \to \text{Set}_{lc}) \to \mathcal{Y}$, i.e., a 2-functor $T: \mathcal{X} \to \mathcal{Y}$ between 2-categories $\mathcal{X}$ and $\mathcal{Y}$ over $\text{Set}_{lc}$, the condition that $2\text{-}\text{CAT}/\text{Set}_{lc}(A,T)$ be an isomorphism for all $A \in \mathcal{G}$ means that $T$ is bijective on 2-cells.

To show (b), observe that a 2-functor $2_2 \to \text{Set}_{lc}$ corresponds to a parallel pair of functions $f_0, f_1: X \to Y$. Such a 2-functor can be seen as an object of $2\text{-}\text{CAT}/\text{Set}_{lc}$. Given $((f_0, f_1): 2 \to \text{Set}_{lc})$ where $f_0, f_1: X \to Y$, first consider the category $2 \times X_c$ where $2 = \{0 < 1\}$ is the two-element chain. We regard $2 \times X_c$ as a bicategory as well. We have the projection functor $\pi: 2 \times X_c \to X_c$ and the functor $[(f_0, f_1): 2 \times X_c \to Y_c$ defined by $[f_0, f_1](i, x) = f_i(x)$; these can also be regarded as lax functors, i.e., morphisms in $\text{BICAT}$. The 2-functor

$$2\text{-}\text{CAT}/\text{Set}_{lc}((f_0, f_1), \text{Enr}_1(-)): \text{BICAT} \to \text{CAT}$$

is the comma object (in $[\text{BICAT}, \text{CAT}]$) as in

$$\xymatrix{ 2\text{-}\text{CAT}/\text{Set}_{lc}((f_0, f_1), \text{Enr}_1(-)) \ar[d] \ar[r] & \text{BICAT}(X_c, -) \ar[d] \ar[dr] \ar[dd] & \text{BICAT}(\pi, -) \ar[d] \ar[dr] \ar[dd] \\
\text{BICAT}(Y_c, -) & & \text{BICAT}(2 \times X_c, -).}
$$

Indeed, for any bicategory $\mathcal{B} \in \text{BICAT}$, an object of the comma category of the functors $\text{BICAT}([f_0, f_1], \mathcal{B})$ and $\text{BICAT}(\pi, \mathcal{B})$ consists of lax functors $\mathcal{C}: X_c \to \mathcal{B}$ and $\mathcal{D}: Y_c \to \mathcal{B}$ together with an icon

$$
\xymatrix{ 2 \times X_c \ar[d]^{[f_0, f_1]} \ar[r]^\pi & X_c \ar[d]^\mathcal{C} \\
Y_c & \mathcal{B} \ar[l]_\mathcal{D} }
$$

This corresponds to $\mathcal{B}$-categories $\mathcal{C}$ and $\mathcal{D}$ with $\text{ob}(\mathcal{C}) = X$ and $\text{ob}(\mathcal{D}) = Y$ such that $|x|_\mathcal{C} = |f_i(x)|_\mathcal{D}$ for all $x \in X$ and $i \in \{0, 1\}$, together with a 2-cell $\alpha_{(i, x), (i', x')} : \mathcal{C}(x, x') \to \mathcal{D}(f_i(x), f_i'(x'))$ in $\mathcal{B}$ for all $(i, x), (i', x') \in 2 \times X_c$ with $i \leq i'$, satisfying some equations. These latter data in turn correspond to $\mathcal{B}$-functors $F_0: \mathcal{C} \to \mathcal{D}$ and $F_1: \mathcal{C} \to \mathcal{D}$ (with $\text{ob}(F_i) = f_i$) together with a $\mathcal{B}$-natural transformation $\alpha: F_0 \to F_1$. (We record in Lemma 2.2 below an observation which is useful for the verification.)

This gives a bijective correspondence on objects of $2\text{-}\text{CAT}/\text{Set}_{lc}((f_0, f_1), \text{Enr}_1(\mathcal{B}))$ and the comma category of $\text{BICAT}([f_0, f_1], \mathcal{B})$ and $\text{BICAT}(\pi, \mathcal{B})$, which routinely extends to an isomorphism of categories natural in $\mathcal{B}$. $\blacksquare$
2.2. Lemma. Let \( \mathcal{B} \) be a bicategory, \( \mathcal{C}, \mathcal{D} \) be \( \mathcal{B} \)-categories and \( T, S : \mathcal{C} \to \mathcal{D} \) be \( \mathcal{B} \)-functors. To give a \( \mathcal{B} \)-natural transformation \( \alpha : T \to S \), i.e., a family of 2-cells

\[
\begin{array}{c}
\xymatrix{
1_{|x|} \\
\mathbb{D}(T x, S x) \ar@{|->}[rr]^\alpha_x & & |x| \\
|x| \ar@{|->}[rr]_{1_{|x|}} & & |x|
}
\end{array}
\]

in \( \mathcal{B} \) for all \( x \in \mathcal{C} \), satisfying the naturality axiom saying that for all \( x, x' \in \mathcal{C} \),

\[
\begin{align}
1_{|x'|}.\mathbb{C}(x, x') & \xrightarrow{\alpha_{x', T x, x'}} \mathbb{D}(T x', S x').\mathbb{D}(T x, T x') \\
\cong & \\
\mathbb{C}(x, x') & \xrightarrow{\alpha_{x, S x, S x}} \mathbb{D}(T x, S x') \quad (3)
\end{align}
\]

commutes, is equivalent to giving a family of 2-cells

\[
\begin{array}{c}
\xymatrix{
\mathbb{C}(x, x') \\
\mathbb{D}(T x, S x') \ar@{|->}[rr]^\alpha_{x, x'} & & |x'| \\
|x| \ar@{|->}[rr]_{1_{|x'|}} & & |x'|
}
\end{array}
\]

in \( \mathcal{B} \) for all \( x, x' \in \mathcal{C} \), such that for all \( x, x', x'' \in \mathcal{C} \),

\[
\begin{align}
\mathbb{C}(x', x'').\mathbb{C}(x, x') & \xrightarrow{\alpha_{x', x'', x'}} \mathbb{C}(x, x'') \\
S_{x', x'', x'}, \alpha_{x, x'} & \\
\mathbb{D}(S x', S x'').\mathbb{D}(T x, S x') & \xrightarrow{\alpha_{x, x''}} \mathbb{D}(T x, S x'') \\
\cong & \\
\mathbb{C}(x', x'').\mathbb{C}(x, x') & \xrightarrow{\alpha_{x', x'', x'}} \mathbb{C}(x, x'') \\
\alpha_{x', x'', T x, x'} & \\
\mathbb{D}(T x', S x'').\mathbb{D}(T x, T x') & \xrightarrow{\alpha_{x, x''}} \mathbb{D}(T x, S x'')
\end{align}
\]

commute; the correspondence is given by mapping \( (\alpha_x) \) to \( (\alpha_{x, x'}) \) whose component at \( (x, x') \) is the composite (3).

As observed in [Lac10, Section 6.2], the 2-category \( \text{BICAT} \) can be seen as the 2-category of strict algebras, lax morphisms, and algebra 2-cells for a 2-monad \( T \) on a
certain locally presentable 2-category of \textsc{cat}-enriched graphs, and so by [Lac05] has oplax limits, Eilenberg–Moore objects of comonads, and limits of diagrams containing only strict morphisms; this last class includes in particular products and powers. It also has various other sorts of limits where certain parts of the diagram are required to be pseudofunctors. For a more precise characterization see [LS12].

The case of oplax limits of 1-cells is our motivating example, and is formalized in Section 4, specifically in Theorem 4.5. The case of Eilenberg–Moore objects of comonads is treated in Example 3.9. As a final example, we consider products. In this case, Theorem 2.1 says that, for bicategories \( \mathcal{B} \) and \( \mathcal{C} \), the diagram

\[
\begin{array}{ccc}
\mathcal{B} \times \mathcal{C} \text{-Cat} & \longrightarrow & \mathcal{C} \text{-Cat} \\
\downarrow & & \downarrow \text{ob} \\
\mathcal{B} \text{-Cat} & \longrightarrow & \text{Set}_{lc}
\end{array}
\]

is a pullback of 2-categories. In particular, to give a \( \mathcal{B} \times \mathcal{C} \)-category is equivalent to giving a \( \mathcal{B} \)-category and a \( \mathcal{C} \)-category with the same set of objects.

2.3. Remark. It is possible to remove any size-related conditions on the notion of weighted limit in Theorem 2.1. That is, for any (possibly larger than “large”) 2-category \( \mathcal{D} \) and a weight \( F: \mathcal{D} \rightarrow \text{cat}' \), where \( \text{cat}' \) is a 2-category of categories in a universe containing \( \mathcal{U}_1 \), \( \text{Enr}_1 \) preserves all \( F \)-weighted limits which happen to exist in \( \text{bicat} \). Indeed, let \( S: \mathcal{D} \rightarrow \text{bicat} \) be a 2-functor such that \( \{F, S\} \) exists in \( \text{bicat} \). Then, although a priori we do not know if \( \{F, \text{Enr}_1 \circ S\} \) exists in \( \text{bicat}'/\text{set}_{lc} \) or not, we can certainly consider a large enough variant \( 2\text{-cat}'/\text{set}_{lc} \) in which it does. Then by the above discussion we have \( \text{Enr}_1 \{F, S\} \cong \{F, \text{Enr}_1 \circ S\} \) in \( 2\text{-cat}'/\text{set}_{lc} \). Since the fully faithful 2-functor \( 2\text{-cat}'/\text{set}_{lc} \rightarrow 2\text{-cat}'/\text{set}_{lc} \) reflects limits, and \( \text{Enr}_1 \) does land in \( 2\text{-cat}'/\text{set}_{lc} \), we see that the limit \( \{F, \text{Enr}_1 \circ S\} \) actually exists in \( 2\text{-cat}'/\text{set}_{lc} \).

3. Weighted limits created by forgetful 2-functors \( \mathcal{K}/A \rightarrow \mathcal{K} \)

Theorem 2.1 implies that the 2-functor \( \text{Enr}: \text{bicat} \rightarrow \text{2-cat} \) preserves all weighted limits preserved by the forgetful 2-functor \( \text{2-cat}'/\text{set}_{lc} \rightarrow \text{2-cat} \). We now investigate these.

A large part of this section (until the end of Example 3.7) is devoted to the study of this class of limits, which we shall call \text{cat}-connected. Since this notion does not require two separate universes, and since it may be of interest in other contexts, we work with a single universe \( \mathcal{U} \), whose elements we call \textit{small} sets. (We temporarily ignore \( \mathcal{U}_0 \) introduced at the beginning of Section 2.) When we later return to the study of \( \text{bicat} \) and \( \text{2-cat}'/\text{set}_{lc} \), we apply our results in the case \( \mathcal{U} = \mathcal{U}_1 \), and so speak of \text{cat}-connected limits.
In the literature there are (at least) two definitions of creation of limit. Given 2-functors $F: \mathcal{D} \to \mathbf{Cat}$, $S: \mathcal{D} \to \mathcal{A}$, and $G: \mathcal{A} \to \mathcal{B}$, the phrase “$G$ creates the $F$-weighted limit of $S$” could mean either of the following.

- For any $F$-weighted limit $\{F, GS\}, \mu: F \to \mathcal{B}(\{F, GS\}, GS_\cdot)$ of $GS$, there exists a unique $F$-cylinder $(L, \nu: F \to \mathcal{A}(L, S_\cdot))$ over $S$ in $\mathcal{A}$ such that $\{F, GS\} = GL$ and $\mu = G_{L,S_\cdot} \circ \nu$ hold. Moreover, $(L, \nu)$ is an $F$-weighted limit of $S$.

- For any $F$-weighted limit $\{F, GS\}, \mu$ of $GS$, there exists an $F$-cylinder $(L, \nu)$ over $S$ in $\mathcal{A}$ such that the mediating 1-cell $GL \to \{F, GS\}$ is an isomorphism. Moreover, such an $F$-cylinder $(L, \nu)$ is always an $F$-weighted limit of $S$.

These two conditions are equivalent when $G$ is the forgetful 2-functor $\mathcal{K}/A \to \mathcal{K}$ from a slice 2-category, since such 2-functors reflect identities and lift invertible 1-cells.

In the following, 1 and $\mathbf{1}$ denote the terminal category and the terminal 2-category, respectively.

**3.1. Theorem.** Let $\mathcal{D}$ be a small 2-category and $F: \mathcal{D} \to \mathbf{Cat}$ be a 2-functor. Then the following are equivalent.

1. All $F$-weighted limits are created by the forgetful 2-functor $\mathcal{K}/A \to \mathcal{K}$ for any locally small 2-category $\mathcal{K}$ and $A \in \mathcal{K}$.

2. All $F$-weighted limits commute with copowers in $\mathbf{Cat}$. In other words, $F$-weighted limits are preserved by the 2-functor $X \times (-): \mathbf{Cat} \to \mathbf{Cat}$ for any $X \in \mathbf{Cat}$.

3. The $F$-weighted limit of the unique 2-functor $\mathcal{D} \to 1$ is preserved by any 2-functor $1 \to \mathbf{Cat}$: that is, $X \cong [\mathcal{D}, \mathbf{Cat}](F, \Delta X)$ for any $X \in \mathbf{Cat}$.

4. The $F$-weighted limit of the unique 2-functor $\mathcal{D} \to 1$ is preserved by any 2-functor $1 \to \mathcal{K}$: that is, $A \cong \{F, \Delta A\}$ for any locally small 2-category $\mathcal{K}$ and $A \in \mathcal{K}$.

5. $F \ast (-)[\mathcal{D}^{op}, \mathbf{Cat}] \to \mathbf{Cat}$ preserves the terminal object. In other words, the $F$-weighted colimit of $\Delta 1: \mathcal{D}^{op} \to \mathbf{Cat}$ is the terminal category: $F \ast \Delta 1 \cong 1$.

6. The (conical) colimit of $F$ is the terminal category: $\Delta 1 \ast F \cong 1$.

**Proof.**

1. $(1 \implies 2)$ For any $X \in \mathbf{Cat}$, copowers by $X$ are given by $X \times (-): \mathbf{Cat} \to \mathbf{Cat}$, which is the composite of the right adjoint 2-functor $X \times (-): \mathbf{Cat} \to \mathbf{Cat}/X$ and the forgetful 2-functor $\mathbf{Cat}/X \to \mathbf{Cat}$.

2. $(2 \implies 3)$ Note that we have $1 \cong \{F, \Delta 1\}$ in $\mathbf{Cat}$. Since $X \times (-): \mathbf{Cat} \to \mathbf{Cat}$ preserves the $F$-weighted limit $\{F, \Delta 1\}$, we have $X \cong \{F, \Delta X\}$.

3. $(3 \implies 4)$ For any $B \in \mathcal{K}$ we have $\mathcal{K}(B, A) \cong [\mathcal{D}, \mathbf{Cat}](F, \Delta \mathcal{K}(B, A))$. This shows that $A \in \mathcal{K}$ is the weighted limit $\{F, \Delta A\}$. 


[(4) \implies (1)] Let \( T : \mathcal{D} \to \mathcal{K}/A \) be a 2-functor, with the corresponding oplax cone

\[
\begin{array}{c}
1 \\
\downarrow^{\gamma} \\
A \\
\downarrow \\
\mathcal{K}.
\end{array}
\]

In particular, \( S \) is the composite of \( T \) and the forgetful 2-functor \( \mathcal{K}/A \to \mathcal{K} \). Suppose that the weighted limit \( \{ F, S \} \) exists in \( \mathcal{K} \). We have a 1-cell \( \{ F, \gamma \} : \{ F, S \} \to \{ F, \Delta A \} \approx A \) in \( \mathcal{K} \). We claim that the object \( (\{ F, \gamma \} : \{ F, S \} \to A) \) is the limit \( \{ F, T \} \) in \( \mathcal{K}/A \).

[(4) \implies (5)] Applying (4) to \( 1 : \mathbf{1} \to \mathbf{Cat}^{op} \), we obtain \( F \ast \Delta 1 \cong 1 \) in \( \mathbf{Cat} \).

[(5) \implies (3)] For any \( X \in \mathbf{Cat} \), we have

\[
X \cong [1, X] \cong [F \ast \Delta 1, X] \cong [\mathcal{D}, \mathbf{Cat}](F, \Delta 1(-), X) \cong [\mathcal{D}, \mathbf{Cat}](F, \Delta X).
\]

[(5) \iff (6)] By \( F \ast \Delta 1 \cong \Delta 1 \ast F \).

A 2-functor \( F : \mathcal{D} \to \mathbf{Cat} \) is called \textit{Cat-connected} if \( F \) satisfies the equivalent conditions of Theorem 3.1. Similarly, a weighted limit is \textit{Cat-connected} if its weight is so. Note that \( F : \mathcal{D} \to \mathbf{Cat} \) is connected (in the sense that \( [\mathcal{D}, \mathbf{Cat}](F, -) : [\mathcal{D}, \mathbf{Cat}] \to \mathbf{Cat} \) preserves small coproducts) if and only if \( [\mathcal{D}, \mathbf{Cat}](F, \Delta X) \cong X \) for any small discrete category \( X \), or equivalently just for \( X = 1 + 1 \); on the other hand it is \textit{Cat-connected} if this holds for all small categories \( X \).

3.2. Remark. Theorem 3.1 can be proved more generally for categories enriched over a complete and cocomplete cartesian closed category \( \mathcal{V} \) in place of \( \mathbf{Cat} \), indeed the proof carries over essentially word-for-word upon replacing each instance of \( \mathbf{Cat} \) by \( \mathcal{V} \).

We now give a few simple results about \textit{Cat-connected} weights in order to clarify the scope of the notion.

3.3. Proposition. If \( \mathcal{D} \) has a terminal object, then \( F : \mathcal{D} \to \mathbf{Cat} \) is \textit{Cat-connected} if and only if \( F \) preserves the terminal object.

Proof. If \( \mathcal{D} \) has a terminal object \( 1 \) then the colimit of \( F \) is \( F(1) \).

3.4. Proposition. Let \( \mathcal{C} \) be a small ordinary category, and \( G : \mathcal{C} \to \mathbf{Set} \) a functor. This determines a 2-functor \( G_d : \mathcal{C} \times \mathbf{1} \to \mathbf{Cat} \), where now \( \mathcal{C} \times \mathbf{1} \) is regarded as a locally discrete 2-category. This \( G_d \) sends an object \( C \) to the discrete category \( G(C)_d \) with object-set \( G(C) \).

Then \( G_d \) is \textit{Cat-connected} if and only if the corresponding \( G \) is connected.

Proof. Since the functor \( (-)_d : \mathbf{Set} \to \mathbf{Cat}_0 \) preserves colimits, \( \text{colim}(G_d) = \text{colim}(G)_d \).
3.5. Proposition. $\Delta 1: \mathcal{D} \to \textbf{Cat}$ is $\textbf{Cat}$-connected if and only if $\mathcal{D}_0$ is connected.

Proof. The colimit of $\Delta 1: \mathcal{D} \to \textbf{Cat}$ is the discrete category corresponding to the set of connected components of $\mathcal{D}_0$. □

3.6. Example. Equifiers are $\textbf{Cat}$-connected: here it is easiest to verify directly that equifiers in $\textbf{Cat}$ commute with copowers. Similarly, one verifies that Eilenberg–Moore objects of monads and of comonads are $\textbf{Cat}$-connected. Equalizers and pullbacks are $\textbf{Cat}$-connected by Proposition 3.5.

3.7. Example. Non-trivial products are not $\textbf{Cat}$-connected: they are not even connected. Powers by a category $X$ are limits weighted by $X: 1 \to \textbf{Cat}$; since the colimit of such a weight is just $X$, powers by $X$ are $\textbf{Cat}$-connected if and only if $X = 1$. Inserters, comma objects and oplax limits of 1-cells are not $\textbf{Cat}$-connected: in particular they are not preserved by the 2-functor $\mathbb{N}: 1 \to \textbf{Cat}$ which picks out the additive monoid $\mathbb{N}$ of natural numbers. More generally, inserters are not preserved by $X: 1 \to \textbf{Cat}$ if $X$ has a non-identity endomorphism, while comma objects and oplax limits of 1-cells are not preserved by $X: 1 \to \textbf{Cat}$ unless $X$ is discrete.

As anticipated at the beginning of the section, we now take $U$ to be $U_1$, and use the resulting notion of $\textbf{Cat}$-connected limit, involving a large 2-category $\mathcal{D}$ and a 2-functor $F: \mathcal{D} \to \textbf{CAT}$ as weight. Since the inclusion $\textbf{Cat} \to \textbf{CAT}$ preserves small limits and small colimits, $\textbf{Cat}$-connected limits are also $\textbf{CAT}$-connected. As an immediate consequence of Theorems 2.1 and 3.1, we have:

3.8. Corollary. The 2-functor $\text{Enr}: \textbf{BICAT} \to \textbf{2CART}$ preserves all $\textbf{CAT}$-connected limits which happen to exist in $\textbf{BICAT}$.

3.9. Example. Eilenberg–Moore objects of comonads are $\textbf{Cat}$-connected (as well as $\textbf{CAT}$-connected), and exist in $\textbf{BICAT}$ by the results of [Lac05, LS12], thus they are preserved by $\text{Enr}$. In more detail, a comonad $G$ in $\textbf{BICAT}$ on a bicategory $\mathcal{B}$ consists of a comonad $G = G_{a,b}$ on each hom-category $\mathcal{B}(a,b)$, together with 2-cells $G_2: GgGf \to G(gf)$ for all $f: a \to b$ and $g: b \to c$, and 2-cells $G_0: 1_{Ga} \to G1_a$ for all objects $a$, subject to various conditions, which say that the $G_{a,b}$, the $G_2$ and the $G_0$ can be assembled into an identity-on-objects lax functor $\mathcal{B} \to \mathcal{B}$, in such a way that the counits and comultiplications for the comonads become icons. The Eilenberg–Moore object $\mathcal{B}^G$ is the bicategory with the same objects as $\mathcal{B}$, and with hom-category $\mathcal{B}^G(a,b)$ given by the Eilenberg–Moore category $\mathcal{B}(a,b)^{G_{a,b}}$ of $G_{a,b}$. Corollary 3.8 then says that $\mathcal{B}^G\text{-Cat}$ is the Eilenberg–Moore 2-category for the induced 2-comonad on $\mathcal{B}\text{-Cat}$. 
4. Oplax limits and fibrations

A 1-cell \( f: A \to B \) in a 2-category \( \mathcal{K} \) is called a \textit{fibration}, when \( \mathcal{K}(C, f): \mathcal{K}(C, A) \to \mathcal{K}(C, B) \) is a Grothendieck fibration for each \( C \in \mathcal{K} \), and

\[
\begin{array}{c}
\mathcal{K}(C, A) \xrightarrow{\mathcal{K}(c, A)} \mathcal{K}(D, A) \\
\downarrow \quad \downarrow \\
\mathcal{K}(C, B) \xrightarrow{\mathcal{K}(c, B)} \mathcal{K}(D, B)
\end{array}
\]

is a morphism of fibrations for each \( c: D \to C \) in \( \mathcal{K} \), in the sense that \( \mathcal{K}(c, A) \) sends cartesian morphisms (with respect to \( \mathcal{K}(C, f) \)) to cartesian morphisms (with respect to \( \mathcal{K}(D, f) \)). If \( q: F \to B \) and \( p: E \to B \) are fibrations in \( \mathcal{K} \) with the common codomain \( B \), then a 1-cell \( r: (F, q) \to (E, p) \) in \( \mathcal{K}/B \) is a \textit{morphism of fibrations} if for each \( C \in \mathcal{K} \), \( \mathcal{K}(C, r) \) is a morphism of fibrations, i.e., preserves cartesian morphisms.

As explained by Street [Str74], these notions can be reformulated if the 2-category \( \mathcal{K} \) has oplax limits of 1-cells, as we shall henceforth suppose. Recall that the oplax limit of a 1-cell \( f: A \to B \) in \( \mathcal{K} \) is the universal diagram

\[
\begin{array}{c}
B/f \xrightarrow{u_f} A \\
\downarrow \quad \downarrow \\
B \xrightarrow{v_f} B
\end{array}
\]

wherein we often drop the subscripts \( f \) unless multiple oplax limits are being used.

If \( \mathcal{K} = \text{Cat} \), then these oplax limits are comma categories, as the notation suggests. On the other hand, we have:

4.1. \textbf{Example}. Let \( X \) be a small set, seen as a chaotic bicategory \( X_c \) (that is, \( (X_c)_{id} \) or equivalently \( (X_c)_{lc} \)). To give an \( X_c \)-enriched category is just to give a set of objects with a map into \( X \). Similar calculations involving \( X_c \)-enriched functors and natural transformations show that the diagram

\[
\begin{array}{c}
X_c-\text{Cat} \xrightarrow{\land} 1 \\
\downarrow \quad \downarrow x \\
\text{Set}_{lc} \xrightarrow{\text{ob}}
\end{array}
\]

is an oplax limit in \( 2\text{-CAT} \); in other words, the 2-category \( X_c-\text{Cat} \) is isomorphic to the slice 2-category \( \text{Set}_{lc}/X \); this in turn is isomorphic to \( (\text{Set}/X)_{lc} \).
The fibrations in $\mathcal{K}$ with codomain $B$ can be understood in terms of a 2-monad $T_B$ on $\mathcal{K}/B$ whose underlying 2-functor maps $f: A \to B$ to $v_f: B/f \to B$; the component at $f: A \to B$ of its unit is the unique map $d = d_f: A \to B/f$ with $ud = 1_A$, $vd = f$, and $\lambda d$ equal to the identity 2-cell on $f$. This 2-monad is colax idempotent (has the dual of the “Kock–Zöberlein property”), and so an object $f: A \to B$ of $\mathcal{K}/B$ admits the structure of a pseudo $T_B$-algebra if and only if $d: (A, f) \to (B/f, v_f)$ has a right adjoint in $\mathcal{K}/B$; and this in turn is the case if and only if $f$ is a fibration. See for example [Str74, Proposition 3(a)] and [Web07, Theorem 2.7].

Also, if $q: F \to B$ and $p: E \to B$ are fibrations in $\mathcal{K}$, then a 1-cell $r: (F, q) \to (E, p)$ in $\mathcal{K}/B$ admits the structure of a (pseudo) morphism of pseudo $T_B$-algebras if and only if the mate of the identity 2-cell

$$
\begin{array}{ccc}
(F, q) & \xrightarrow{r} & (E, p) \\
\downarrow d_q & & \downarrow d_p \\
(B/q, v_q) & \xrightarrow{T_B r} & (B/p, v_p)
\end{array}
$$

is invertible; and this in turn is the case if and only if $r$ is a morphism of fibrations.

Likewise, the strict $T_B$-algebras are the split fibrations in $\mathcal{K}$: those $f: A \to B$ for which each $\mathcal{K}(C, f): \mathcal{K}(C, A) \to \mathcal{K}(C, B)$ is a split fibration, and each $\mathcal{K}(c, A): \mathcal{K}(C, A) \to \mathcal{K}(D, A)$ preserves the chosen cartesian lifts.

In particular, $v: B/f \to B$ is a split fibration for any $f: A \to B$, and $d$ exhibits $v: B/f \to B$ as the free (split) fibration on $f$. Thus if $p: E \to B$ is a fibration, and $g: A \to E$ defines a morphism $(A, f) \to (E, p)$ in $\mathcal{K}/B$, there is an essentially unique morphism of fibrations $r: (B/f, v) \to (E, p)$ extending $g$.

4.2. Proposition. The 2-functor $\text{Enr}_1: \text{BICAT} \to \text{2-CAT}/\text{Set}_{lc}$ factors through the locally full sub-2-category of $\text{2-CAT}/\text{Set}_{lc}$ having

- the fibrations in $\text{2-CAT}$ to $\text{Set}_{lc}$ as objects, and
- the fibration morphisms as 1-cells.

Proof. First we describe fibrations in $\text{2-CAT}$ explicitly. Given a 2-functor $F: \mathcal{Y} \to \mathcal{X}$ between 2-categories, a 1-cell $h: y' \to y$ in $\mathcal{Y}$ is called cartesian (with respect to $F$) if

$$
\begin{array}{ccc}
\mathcal{Y}(z, y') & \xrightarrow{\mathcal{Y}(z, h)} & \mathcal{Y}(z, y) \\
F_{z, y'} \downarrow & & \downarrow F_{z, y} \\
\mathcal{X}(Fz, Fy') & \xrightarrow{\mathcal{X}(Fz, Fh)} & \mathcal{X}(Fz, Fy)
\end{array}
$$

is a pullback in $\text{CAT}$ for each $z \in \mathcal{V}$. Then $F$ is a fibration if and only if, for each object $y \in \mathcal{V}$ and each 1-cell $g: x \to Fy$ in $\mathcal{X}$, there is a cartesian morphism $\overline{g}: g^*y \to y$ in $\mathcal{V}$ with $F\overline{g} = g$; such a $\overline{g}$ is called a cartesian lifting of $g$ to $y$. Moreover, given fibrations $F: \mathcal{V} \to \mathcal{X}$ and $G: \mathcal{Z} \to \mathcal{X}$ over $\mathcal{X}$, a 2-functor $H: \mathcal{V} \to \mathcal{Z}$ satisfying $F = G \circ H$ is a morphism of fibrations if and only if $H$ preserves cartesian 1-cells. (This is a special case of Proposition 5.3 below, whose proof does not depend on the current proposition.)

For any $\mathcal{B} \in \text{BICAT}$, a $\mathcal{B}$-functor $S: \mathcal{V}' \to \mathcal{V}$ is called fully faithful when the 2-cell $S_{y,y'}: \mathcal{V}(y, y') \to \mathcal{V}'(Sy, Sy')$ in $\mathcal{B}$ is invertible for all $y, y' \in \mathcal{V}$. It is easy to see that a $\mathcal{B}$-functor is cartesian with respect to $\text{ob}(-): \mathcal{B}\text{-Cat} \to \text{Set}_{llc}$ if it is fully faithful, and indeed by essential uniqueness of cartesian lifts the reverse implication also holds. The claim follows at once. \hfill $\blacksquare$

4.3. REMARK. In the above proposition, we used fibrations in the 2-category $\text{2-CAT}$, called 2-categorical fibrations in [Gra74, I.2.9]. These were also called 2-fibrations in [Gra74], but for the purposes of this remark we shall save that name for the more restrictive notion studied by Hermida [Her99]; see also [Bak, Buc14]. In general, $\text{ob}(-): \mathcal{B}\text{-Cat} \to \text{Set}_{llc}$ is not a 2-fibration in the sense of [Her99]. Indeed, a 2-fibration is a 2-functor which among other things is locally a fibration, but the forgetful functor $\mathcal{B}\text{-Cat}(\mathcal{X}, \mathcal{V}) \to \text{Set}_{llc}(\text{ob}(\mathcal{X}), \text{ob}(\mathcal{V}))$ induced by $\text{ob}(-)$ is rarely a fibration of categories.

In general, oplax limits of 1-cells are not preserved by the projection $\mathcal{K}/\mathcal{B} \to \mathcal{K}$, but to some extent fibrations can be used to remedy this, as the following result shows.

4.4. PROPOSITION. Let $p: A \to B$ be a fibration in $\mathcal{K}$, and consider a morphism $g$ in $\mathcal{K}/\mathcal{B}$ into $p$, and the (essentially unique) induced morphism $r$ of fibrations, as below

```
    C  B
  p|    |
 A/ \ vpg
  p
    A
```

Then the oplax limit of $g$ in $\mathcal{K}$ is the oplax limit of $r$ in $\mathcal{K}/\mathcal{B}$.

PROOF. As usual we write $A/g$ for the oplax limit of $g$ in $\mathcal{K}$. We also write $(A,p)/r$ for the oplax limit of $r$ in $\mathcal{K}/\mathcal{B}$.

A morphism $D \to A/g$ consists of morphisms $a: D \to A$, $c: D \to C$, and a 2-cell $\alpha: a \to gc$.

A morphism $D \to B/pg$ consists of morphisms $b: D \to B$, $c: D \to C$, and a 2-cell $\beta: b \to pgc$, and composing with $r$ gives the domain of the cartesian lifting $\overline{\beta}: \beta^*gc \to gc$ of $\beta$. A morphism $(D,b) \to (A,p)/r$ in $\mathcal{K}/\mathcal{B}$ consists of $(b,c,\beta): D \to B/pg$, $a: D \to A$, and a 2-cell $\alpha': a \to \beta^*gc$ with $pa' = b$. But by the fibration property of $p$, to give such an $\alpha'$ is equivalent to give $\alpha: a \to gc$ with $pa = \beta$.

This shows that the one-dimensional aspect of the universal properties of $A/g$ and $(A,p)/r$ agree, and similarly the two-dimensional aspects also agree. \hfill $\blacksquare$
4.5. **Theorem.** Let $\mathcal{B}$ be a bicategory and $\mathcal{X}$ a $\mathcal{B}$-category. Then the slice 2-category $\mathcal{B}-\text{Cat}/\mathcal{X}$ is isomorphic to $(\mathcal{B}/\mathcal{X})-\text{Cat}$ for a bicategory $\mathcal{B}/\mathcal{X}$.

**Proof.** If we regard $\mathcal{X}$ as a lax functor $\mathcal{X}: \mathcal{X}_c \to \mathcal{B}$, where $\mathcal{X}_c = \text{ob}(\mathcal{X})$, we may take its oplax limit $\mathcal{B}/\mathcal{X}$ in $\text{BICAT}$. Explicitly, $\text{ob}(\mathcal{B}/\mathcal{X}) = \text{ob}(\mathcal{X}) = \mathcal{X}_c$, while the hom-category $(\mathcal{B}/\mathcal{X})(x, x')$ is given by the slice category $\mathcal{B}(|x|, |x'|)/\mathcal{X}(x, x')$ for all $x, x' \in \mathcal{X}_c$.

It follows by Theorem 2.1 that $(\mathcal{B}/\mathcal{X})-\text{Cat}$ is the oplax limit

$$
\begin{array}{c}
\mathcal{B}/\mathcal{X} \\
\searrow \downarrow \mathcal{X} \\
\mathcal{B}
\end{array}
$$

in $\text{BICAT}$. Explicitly, $\text{ob}(\mathcal{B}/\mathcal{X}) = \text{ob}(\mathcal{X}) = \mathcal{X}_c$, while the hom-category $(\mathcal{B}/\mathcal{X})(x, x')$ is given by the slice category $\mathcal{B}(|x|, |x'|)/\mathcal{X}(x, x')$ for all $x, x' \in \mathcal{X}_c$.

It follows by Theorem 2.1 that $(\mathcal{B}/\mathcal{X})-\text{Cat}$ is the oplax limit

$$
\begin{array}{c}
\mathcal{X}_c-\text{Cat} \\
\downarrow \mathcal{B}/\mathcal{X} \quad \text{ob} \\
\mathcal{B}-\text{Cat}
\end{array}
$$

in $\text{2-CAT}/\text{Set}_{lc}$.

Now $\text{ob}(\mathcal{B}/\mathcal{X}): X_{\mathcal{X}}-\text{Cat} \to \text{Set}_{lc}$ is the free fibration on $X: 1 \to \text{Set}_{lc}$ by Example 4.1, while $\text{Enr}(\mathcal{X})$ is the morphism of fibrations induced by $\mathcal{X}: 1 \to \mathcal{B}-\text{Cat}$ by Proposition 4.2, and so by Proposition 4.4 the diagram

$$
\begin{array}{c}
\mathcal{B}/\mathcal{X} \\
\downarrow \mathcal{X} \\
\mathcal{B}-\text{Cat}
\end{array}
$$

is an oplax limit in $\text{2-CAT}$. But this says precisely that $(\mathcal{B}/\mathcal{X})-\text{Cat} \cong \mathcal{B}-\text{Cat}/\mathcal{X}$.

4.6. **Example.** In particular, when $\mathcal{B}$ is the cartesian monoidal category $\text{Set}$ regarded as a one-object bicategory, we have for each $(\text{Set})$-category $\mathcal{X}$ the bicategory $\text{Set}/\mathcal{X}$ whose set of objects is $\text{ob}(\mathcal{X})$ and whose hom-category $(\text{Set}/\mathcal{X})(x, x')$ is the slice category $\text{Set}/\mathcal{X}(x, x')$. Each functor $F: \mathcal{Y} \to \mathcal{X}$ corresponds to a $\text{Set}/\mathcal{X}$-category $\mathcal{Y}$ given as follows: the objects of $\mathcal{Y}$ are the same as those of $\mathcal{Y}$, the extent of $y$ in $\mathcal{Y}$ is $Fy$, and the hom $\mathcal{Y}(y, y')$ is $F_{y, y'}$: $\mathcal{Y}(y, y') \to \mathcal{X}(Fy, Fy')$. Note that since $\text{Set}/\mathcal{X}(x, x') \cong \text{Set}^{\mathcal{X}(x, x')}$, $\text{Set}/\mathcal{X}$ is (biequivalent to) the free local cocompletion of $\mathcal{X}$ regarded as a locally discrete bicategory, as pointed out to us by Ross Street.
A variant of \( \text{Set/}X \) is the free quantaloid \( \mathcal{P}X \) over \( X \). Specifically, \( \mathcal{P}X \) is also a bicategory with the same objects as \( X \), but whose hom-category \( (\mathcal{P}X)(x, x') \) is the powerset lattice \( \mathcal{P}(X(x, x')) \), which is equivalent to the full subcategory of the slice category \( \text{Set/}X(x, x') \) consisting of the injections to \( X(x, x') \). Accordingly, the \( \mathcal{P}X \)-categories correspond to the faithful functors \( Y \to X \) [Gar14, Proposition 3.5].

4.7. Example. Let \( \mathcal{B} \) be a bicategory with all right liftings. Then for each \( b \in \mathcal{B} \), we have a \( \mathcal{B} \)-category \( \mathcal{B}_b \) whose objects are the 1-cells \( f: x \to b \) in \( \mathcal{B} \) with codomain \( b \), with extent \( |(x, f)| = x \), and whose hom \( \mathcal{B}_b((x, f), (y, g)) : x \to y \) is the right lifting of \( f \) along \( g \):

\[
\begin{array}{ccc}
x & \xrightarrow{\mathcal{B}_b((x, f), (y, g))} & y \\
f \downarrow & < & g \\
x & \downarrow & y \\
\end{array}
\]

(See [GP97, Section 2] for the dual construction.) Given a \( \mathcal{B} \)-category \( X \), the \( \mathcal{B} \)-functors \( X \to \mathcal{B}_b \) correspond to the \( \mathcal{B} \)-presheaves on \( X \) with extent \( b \). Hence if we consider the bicategory \( \mathcal{B}/\mathcal{B}_b \), then a \( \mathcal{B}/\mathcal{B}_b \)-category can be identified with a \( \mathcal{B} \)-category equipped with a \( \mathcal{B} \)-presheaf with extent \( b \).

By the universality of right liftings, the bicategory \( \mathcal{B}/\mathcal{B}_b \) is canonically isomorphic to the lax slice bicategory \( \mathcal{B}/b \): this has 1-cells with codomain \( b \) as objects, and diagrams of the form

\[
\begin{array}{ccc}
x & \xrightarrow{\mathcal{B}/((x, f), (y, g))} & y \\
f \downarrow & < & g \\
\end{array}
\]

as 1-cells from \( f \) to \( g \). Unlike \( \mathcal{B}_b \), this lax slice bicategory \( \mathcal{B}/b \) can be defined even when \( \mathcal{B} \) does not have right liftings, and it is true in general that a \( \mathcal{B}/b \)-category corresponds to a \( \mathcal{B} \)-category equipped with a \( \mathcal{B} \)-presheaf with extent \( b \). (For a general bicategory \( \mathcal{B} \), the notion of \( \mathcal{B} \)-presheaf can be defined in terms of actions; see [Str83] for a definition of the more general notion of module.)

4.8. Remark. The bicategory \( \mathcal{B}/X \) can be obtained from \( X \) via a change-of-base process for bicategories enriched in a tricategory. Since the theory of tricategory-enriched bicategories, let alone change-of-base for them, has not really been developed in detail, we merely sketch the details. (See [GS16, Section 13] for change-of-base for bicategories enriched over monoidal bicategories.)

We regard \( \mathcal{B} \) as a tricategory with no non-identity 3-cells, and we regard the cartesian monoidal 2-category \( \text{Cat} \) as a one-object tricategory \( \Sigma(\text{Cat}) \). There is a lax morphism of tricategories \( \Theta: \mathcal{B} \to \Sigma(\text{Cat}) \) sending each object \( b \in \mathcal{B} \) to the unique object of \( \Sigma(\text{Cat}) \), and sending a 1-cell \( f: b \to b' \) in \( \mathcal{B} \) to the category \( \mathcal{B}(b, b')/f \). Composition with \( \Theta \)
then sends each $\mathcal{B}$-enriched bicategory to a $\Sigma(\mathbf{Cat})$-enriched bicategory. Since $\mathcal{B}$ has no non-identity 3-cells, a $\mathcal{B}$-enriched bicategory is just a $\mathcal{B}$-enriched category; on the other hand, a $\Sigma(\mathbf{Cat})$-enriched bicategory is just a bicategory in the ordinary sense. Applying this to the $\mathcal{B}$-category $\mathcal{X}$ gives the bicategory $\mathcal{B}/\mathcal{X}$.

5. Variation through enrichment

In the paper [BCSW83], the authors showed how fibrations with codomain $\mathcal{X}$ can be seen as certain categories enriched over a bicategory $\mathcal{W}(\mathcal{X})$ depending on the category $\mathcal{X}$. In this section we give a result of the same type, although it differs in several important respects. The bicategory we use is $\mathbf{Set}/\mathcal{X}$ (see Example 4.6), which is like $\mathcal{W}(\mathcal{X})$ in having as objects the objects of $\mathcal{X}$: see Remark 5.1 below for the relationship between the two bicategories. Then we show that fibrations over $\mathcal{X}$ can be identified with $\mathbf{Set}/\mathcal{X}$-categories which have certain powers.

5.1. Remark. Given objects $x, x' \in \mathcal{X}$, a 1-cell in $\mathcal{W}(\mathcal{X})$ from $x$ to $x'$ consists of a presheaf $E$ on $\mathcal{X}$ equipped with maps to $\mathcal{X}(-, x)$ and $\mathcal{X}(-, x')$; in other words, it consists of a span of presheaves from $\mathcal{X}(-, x)$ to $\mathcal{X}(-, x')$. Now a 1-cell $S \to \mathcal{X}(x, x')$ in $\mathbf{Set}/\mathcal{X}$ from $x$ to $x'$ determines, via Yoneda, a map $S \cdot \mathcal{X}(-, x) \to \mathcal{X}(-, x')$ of presheaves, where $S \cdot \mathcal{X}(-, x)$ denotes the copower of $\mathcal{X}(-, x)$ by $S$: the coproduct of $S$ copies of $\mathcal{X}(-, x)$. On the other hand there is the codiagonal $S \cdot \mathcal{X}(-, x) \to \mathcal{X}(-, x)$, and so we obtain a span

$$
\mathcal{X}(-, x) \xleftarrow{\text{copower}} S \cdot \mathcal{X}(-, x) \xrightarrow{\text{coproduct}} \mathcal{X}(-, x')
$$

of presheaves; that is, a 1-cell in $\mathcal{W}(\mathcal{X})$ from $x$ to $x'$. This defines the 1-cell part of a homomorphism of bicategories $\mathbf{Set}/\mathcal{X} \to \mathcal{W}(\mathcal{X})$ which is the identity on objects and locally fully faithful. Just as we characterize fibrations over $\mathcal{X}$ as $\mathbf{Set}/\mathcal{X}$-categories with certain limits, so in [BCSW83] these fibrations are seen as $\mathcal{W}(\mathcal{X})$-categories with certain limits; one key difference is that in the case of $\mathcal{W}(\mathcal{X})$ the limits in question are absolute.

In fact we work not just with fibrations of ordinary categories, but rather fibrations in the 2-category $\mathcal{B}\mathbf{-Cat}$ of $\mathcal{B}$-enriched categories, as in Section 4. One recovers the case of ordinary categories upon taking $\mathcal{B}$ to be the one-object bicategory $\Sigma(\mathbf{Set})$. We have seen in Theorem 4.5 that, for a $\mathcal{B}$-category $\mathcal{X}$, $\mathcal{B}$-functors with codomain $\mathcal{X}$ correspond to $\mathcal{B}/\mathcal{X}$-enriched categories. We shall see in this section that a $\mathcal{B}$-functor $F: \mathcal{Y} \to \mathcal{X}$ is a fibration in $\mathcal{B}\mathbf{-Cat}$ if and only if the corresponding $\mathcal{B}/\mathcal{X}$-category $\mathcal{Y}$ has certain powers.

First, however, we give an elementary characterization of fibrations in $\mathcal{B}\mathbf{-Cat}$. To do this, we start with the fact that every $\mathcal{B}$-category $\mathcal{X}$ has an underlying ordinary category $\mathcal{X}_0$ with the same objects; a morphism $x \to x'$ in $\mathcal{X}_0$ can exist only if $x$ and $x'$ have the same extent ($|x| = |x'|$), in which case it amounts to a 2-cell $1_{|x|} \to \mathcal{X}(x, x')$ in $\mathcal{B}$.

2The assignment $\mathcal{X} \mapsto \mathcal{X}_0$ is the object-part of a 2-functor $\mathcal{B}\mathbf{-Cat} \to \mathbf{Cat}$, arising via change-of-base with respect to a lax functor from $\mathcal{B}$ to the cartesian monoidal category $\mathbf{Set}$, seen as a one-object bicategory. The lax functor sends each object $b$ to this unique object; it sends a 1-cell $f: b \to c$ to the set $\mathcal{B}(b, c)(1_b, f)$ if $b = c$ and the empty set otherwise; with the evident action on 2-cells.
shall sometimes refer to such morphisms in \( X_0 \) simply as morphisms in \( X \). If \( f: x' \to x'' \) is a morphism in \( X \) and \( x \) is an object, there is an induced 2-cell \( \Xi(x, f): \Xi(x, x') \to \Xi(x, x'') \) defined by pasting \( f: 1_{|x'|} \to \Xi(x', x'') \) together with the composition 2-cell \( M_{x, x', x''}: \Xi(x', x'') \Xi(x, x') \to \Xi(x, x'') \).

5.2. Definition. Let \( F: \mathcal{Y} \to \mathcal{X} \) be a \( \mathcal{B} \)-functor. A morphism \( h: y' \to y \) in \( \mathcal{Y}_0 \) is said to be cartesian with respect to \( F \) if the square

\[
\begin{array}{ccc}
\mathcal{Y}(z, y') & \xrightarrow{\mathcal{Y}(z, h)} & \mathcal{Y}(z, y) \\
F_{z, y'} \downarrow & & \downarrow F_{z, y} \\
\Xi(Fz, Fy') & \xrightarrow{\Xi(Fz, Fh)} & \Xi(Fz, Fy)
\end{array}
\]

is a pullback in \( \mathcal{B}(|z|, |y|) \) for all objects \( z \) in \( \mathcal{Y} \).

This implies in particular that \( h \) is cartesian with respect to the ordinary functor \( F_0: \mathcal{Y}_0 \to \mathcal{X}_0 \), but in general is stronger than this.

5.3. Proposition. Suppose that the bicategory \( \mathcal{B} \) has pullbacks in each hom-category \( \mathcal{B}(a, b) \). A \( \mathcal{B} \)-functor \( F: \mathcal{Y} \to \mathcal{X} \) is a fibration in \( \mathcal{B} \)-Cat if and only if, for each object \( y \in \mathcal{Y} \) and each morphism \( g: x \to Fy \) in \( \mathcal{X} \) there is a cartesian morphism \( \overline{g}: g^*y \to y \) in \( \mathcal{Y} \) with \( F\overline{g} = g \). Given fibrations \( F: \mathcal{Y} \to \mathcal{X} \) and \( G: \mathcal{Z} \to \mathcal{X} \), a \( \mathcal{B} \)-functor \( H: \mathcal{Y} \to \mathcal{Z} \) with \( F = G \circ H \) is a morphism of fibrations if and only if \( H: \mathcal{Y} \to \mathcal{Z} \) preserves cartesian morphisms.

Proof. The pullbacks in the hom-categories of \( \mathcal{B} \) can be used to construct oplax limits in \( \mathcal{B} \)-Cat, as we shall now show. Let \( F: \mathcal{Y} \to \mathcal{X} \) be a \( \mathcal{B} \)-functor; then the oplax limit \( \mathbb{L} = \mathcal{X}/F \) has:

- objects given by pairs \((g, y)\), with \( y \in \mathcal{Y} \) and \( g: x \to Fy \) in \( \mathcal{X}_0 \)
- the extent of \((g, y)\) equal to the extent of \( y \) (which is also the extent of \( x \))
- homs given by pullbacks as in

\[
\begin{array}{ccc}
\mathbb{L}(((g', y'), (g, y))) & \xrightarrow{U((g', y'), (g, y))} & \mathcal{Y}(y', y) \\
V((g', y'), (g, y)) \downarrow & \downarrow F_{y', y} & \downarrow \Xi(Fy', Fy) \\
\mathcal{X}(x', x) & \xrightarrow{\Xi(g', Fy)} & \mathcal{X}(x', Fy)
\end{array}
\]
\* projections $V : \mathbb{L} \to \mathbb{X}$ and $U : \mathbb{L} \to \mathbb{Y}$ sending an object $(g, y)$ to $x$ and to $y$, and defined on homs as in the diagram above.

The diagonal $\mathcal{B}$-functor $D : \mathbb{Y} \to \mathbb{L}$ sends an object $z \in \mathbb{Y}$ to $(1_{Fz}, z) \in \mathbb{L}$. Taking $(g', y') = Dz$ in the above diagram gives a pullback

$$
\begin{array}{ccc}
\mathbb{L}(Dz, (g, y)) & \xrightarrow{U} & \mathbb{Y}(z, y) \\
V \downarrow & & \downarrow F_{z, y} \\
\mathbb{X}(Fz, x) & \xrightarrow{\mathbb{X}(Fz, g)} & \mathbb{X}(Fz, Fy).
\end{array}
$$

Now $F$ is a fibration just when $D$ has a right adjoint in $\mathcal{B}$-$\mathbf{Cat}/\mathbb{X}$. Such an adjoint is given on objects by a lifting of $g : Fx \to y$ to some $g^*y \to y$, and the universal property says that this lifting is cartesian. \hfill \blacksquare

We now turn to the characterization of fibrations of $\mathcal{B}$-categories in terms of $\mathcal{B}/\mathbb{X}$-categories. First recall that if $\mathcal{W}$ is a bicategory and $\mathcal{Z}$ is a $\mathcal{W}$-category then powers in $\mathcal{Z}$ involve an object $y$ of $\mathcal{Z}$ and a 1-cell $v : x \to \|y\|$ in $\mathcal{W}$ with codomain the extent of $y$. The power of $y$ by $v$ consists of an object $v \pitchfork y$ of $\mathcal{Z}$ with extent $\|v \pitchfork y\| = x$, together with a 2-cell

$$
\begin{array}{ccc}
\|v \pitchfork y\| & \xrightarrow{v} & \|y\| \\
\downarrow \eta & & \downarrow \\
\mathcal{Z}(v \pitchfork y, y)
\end{array}
$$

such that for all $z \in \mathcal{Z}$ and all

$$
\begin{array}{ccc}
\|v \pitchfork y\| & \xrightarrow{v} & \|y\| \\
\downarrow \alpha & & \downarrow \\
\|z\| & \xrightarrow{\mathcal{Z}(z, y)} & \|y\|
\end{array}
$$

there exists a unique $\gamma$ making the pasting composite
equal to $\alpha$. (In other words, the pasting of $\eta$ and $M$ exhibits $Z(z, v \pitchfork y)$ as the right lifting of $Z(z, y)$ along $v$.)

We consider this in the case where $W = B/X$ and $Z$ is the $B/X$-category $\mathcal{Y}$ corresponding to a $B$-functor $F: \mathcal{Y} \to X$. Then an object $y$ of $\mathcal{Y}$ is just an object of $\mathcal{Y}$, and the extent of $y$, as an object of $B/X$, is the object $Fy$ of $X$. A general 1-cell $x \to Fy$ in $B/X$ has the form

$$
\begin{array}{c}
v \\
\downarrow \\
\mathcal{X}(x, Fy)
\end{array}
\quad
\begin{array}{c}
|Fy| \\
\downarrow \\
\mathcal{X}(Fz, Fy)
\end{array}
\quad
\begin{array}{c}
|x| \\
\downarrow \\
\mathcal{X}(Fz, Fy)
\end{array}
$$

but we shall only consider the special case where $|x| = |Fy|$ and $v = 1_{|x|}$, so that in fact we are dealing with a morphism $w: x \to Fy$ in $X_0$. In general, we call a 1-cell $(w: v \to \mathcal{X}(x, x')): x \to x'$ in $B/X$ a singleton 1-cell if $|x| = |x'|$ and $v = 1_{|x|}$. Note that the category $X_0$ can be regarded as a sub-bicategory of $B/X$ whose 1-cells are the singleton 1-cells. When $B = \text{Set}$, a 1-cell $x \to x'$ in $\text{Set}/X$ corresponds to a set $v$ equipped with a function $w: v \to \mathcal{X}(x, x')$; in this case, the singleton 1-cells in $\text{Set}/X$ can be identified with those 1-cells with $v$ a singleton, whence the name singleton.

A power of $y$ by $w: 1 \to \mathcal{X}(x, Fy)$ then consists of an object $w \pitchfork y$ of $\mathcal{Y}$ with $F(w \pitchfork y) = x$ together with a morphism $\overline{w}: w \pitchfork y \to y$ in $Y_0$ with $F\overline{w} = w$ — that is, a lifting $\overline{w}$ of $w$ — subject to the universal property stating that for all $z \in Y$, $b: |z| \to |y|$, $\alpha$, and $\beta$ making

$$
\begin{array}{c}
|z| \\
\downarrow \\
\mathcal{X}(Fz, Fy)
\end{array}
\quad
\begin{array}{c}
|w \pitchfork y| \\
\downarrow \\
\mathcal{X}(Fz, x)
\end{array}
\quad
\begin{array}{c}
|z| \\
\downarrow \\
\mathcal{X}(Fz, Fy)
\end{array}
\quad
\begin{array}{c}
|w \pitchfork y| \\
\downarrow \\
\mathcal{X}(Fz, x)
\end{array}
$$

there exists a unique $\gamma$ making the pasting composites

$$
\begin{array}{c}
|z| \\
\downarrow \\
\mathcal{X}(Fz, Fy)
\end{array}
\quad
\begin{array}{c}
|w \pitchfork y| \\
\downarrow \\
\mathcal{X}(Fz, x)
\end{array}
\quad
\begin{array}{c}
|z| \\
\downarrow \\
\mathcal{X}(Fz, Fy)
\end{array}
\quad
\begin{array}{c}
|w \pitchfork y| \\
\downarrow \\
\mathcal{X}(Fz, x)
\end{array}
$$
equal respectively to $\beta$ and $\alpha$. But this says exactly that if the exterior of the diagram

\[ \begin{array}{ccc}
\beta & \vdash & \alpha \\
Y(z, w \vdash y) & \xrightarrow{\gamma} & Y(z, y) \\
F_{y, w \vdash y} & \downarrow & F_{y, y} \\
X(F_z, x) & \xrightarrow{\chi(F_z, w)} & X(F_z, F_y)
\end{array} \]

in $\mathcal{B}(|z|, |y|)$ commutes, then there is a unique $\gamma$ making the triangular regions commute; in other words, that the internal square is a pullback. This in turn says that $\overline{w}$ is a cartesian lifting of $w$. This now proves:

5.4. Proposition. Let $\mathcal{B}$ be a bicategory in which each hom-category has pullbacks. A $\mathcal{B}$-functor $F : \mathcal{Y} \to \mathcal{X}$ is a fibration if and only if the corresponding $\mathcal{B}/\mathcal{X}$-category $\overline{\mathcal{Y}}$ has powers by morphisms in $\mathcal{X}_0$; that is, powers by singleton 1-cells.

We conclude by strengthening this correspondence to an isomorphism between suitable 2-categories. Let $\mathcal{W}$ be a bicategory and $H : \mathcal{Z} \to \mathcal{Z}'$ a $\mathcal{W}$-functor. Suppose that the power $v \vdash y$ of $y \in \mathcal{Z}$ by $v : x \to |y|$ exists in $\mathcal{Z}$, with the associated 2-cell $\eta : v \to \mathcal{Z}(v \vdash y, y)$. Then $H$ is said to preserve the power $v \vdash y$ if the 2-cell $H_{v\vdash y, y} \circ \eta : v \to \mathcal{Z}'(H(v \vdash y), Hy)$ exhibits $H(v \vdash y)$ as the power $v \vdash Hy$ in $\mathcal{Z}'$.

5.5. Theorem. Let $\mathcal{B}$ be a bicategory in which each hom-category has pullbacks. The canonical isomorphism of 2-categories $(\mathcal{B}/\mathcal{X})$-$\text{Cat} \cong \mathcal{B}$-$\text{Cat}$/\mathcal{X}$ restricts to an isomorphism between the locally full sub-2-category of $(\mathcal{B}/\mathcal{X})$-$\text{Cat}$ having

- the $\mathcal{B}/\mathcal{X}$-categories with powers by singleton 1-cells as objects, and
- the $\mathcal{B}/\mathcal{X}$-functors preserving these powers as 1-cells,

and the locally full sub-2-category of $\mathcal{B}$-$\text{Cat}$/\mathcal{X}$ having

- the fibrations to $\mathcal{X}$ as objects, and
- the fibration morphisms as 1-cells.

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