Multi-almost periodicity and invariant basins of general neural networks under almost periodic stimuli *

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SUMMARY
In this paper, we investigate convergence dynamics of $2^N$ almost periodic encoded patterns of general neural networks (GNNs) subjected to external almost periodic stimuli, including almost periodic delays. Invariant regions are established for the existence of $2^N$ almost periodic encoded patterns under two classes of activation functions. By employing the property of $\mathcal{M}$-cone and inequality technique, attracting basins are estimated and some criteria are derived for the networks to converge exponentially toward $2^N$ almost periodic encoded patterns. The obtained results are new, they extend and generalize the corresponding results existing in previous literature.

KEY WORDS: neural networks; almost periodic; encoded patterns; attracting basins; exponential stability

1. INTRODUCTION
In recent years, the dynamical behaviors of neural networks with delays have been widely investigated. Many important results on the existence and uniqueness of equilibrium point, global asymptotic (exponential) stability have been established and successfully applied to signal and image processing system, associative memories, pattern classification and so on. For corresponding results, we can refer to [1-9,17-18,34-36]. In the applications of neural networks to associative memory storage or pattern recognitions, the existence of multiple equilibria or multiple periodic orbits is an important feature [1-3,6-9]. It is worth noting that convergence analysis and coexistence of multiple equilibria or multiple periodic solutions have been investigated in [6-8] and these equilibria or periodic solutions are usually called encoded patterns [6-8,10].

As we know well, the nonautonomous phenomenon involved in periodic or almost periodic environment often occurs in many realistic systems [11-14,19]. Hence, in many applications,

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the property of periodic or almost periodic oscillatory solutions of neural networks is of great interest. Meanwhile, there often exist delays in artificial neural networks due to the finite switching speed of amplifiers and faults in the electrical circuit. They slow down the transmission rate and lead to some degree of instability. Therefore, complex dynamic behaviors of neural networks under periodic or almost periodic stimuli and delayed effects have been studied so far [10,15-16,20-31,37,39,41].

However, to the best of the authors' knowledge, few papers deal with general neural networks with both almost periodic coefficients and almost periodic delays. Furthermore, most of the results reported in the literature focus on the stability of unique almost periodic solution of neural networks. We can refer to [10,20-23,39] and the references cited therein. In this paper, similarly as [26,31], we consider the following nonautonomous general neural networks with transmission delays

\[
\frac{du^i(t)}{dt} = -c_i(t)u^i(t) + \sum_{l=1}^{M} \sum_{j=1}^{N} a_{ijl}(t)g_j\left(\sigma_j u^j(t - \kappa_{ij}(t))\right) + J_i(t),
\]

where \(i, j \in \mathcal{J} := \{1, 2, \ldots, N\}, l \in \mathcal{L} := \{1, 2, \ldots, M\}\); The main purpose of this paper is to study complex convergence dynamics of GNNs (1) in encoding external stimuli that vary almost periodically with time and recalling the encoded patterns associated with almost periodic delays. That is, we investigate exponential stability of \(2^N\) almost periodic encoded patterns (almost periodic solutions) of GNNs (1). The criteria we established are completely different from most of the existing results in [10,20-31]. Particularly, when GNNs (1) de-generates into the autonomous system, our results extend and generalize the related results existing in [6,8].

The rest of this paper is organized as follows. In Section 2, we shall make some preparations by giving some definitions and lemmas. Meanwhile, we establish \(2^N\) positively invariant basins for general neural networks under almost periodic stimuli. In Section 3, by using the property of \(\mathcal{M}\)-cone and inequality technique, attracting basins are determined and some new criteria for exponential stability of \(2^N\) almost periodic encoded patterns are obtained. In Section 4, we shall make some generalizations by considering activation functions with saturations and apply our obtained results to some special neural networks systems. It is shown that our results are general and improve the previously known results. Finally, numerical examples are presented to illustrate our results.

2. PRELIMINARIES AND \(2^N\) INVARIANT BASINS FOR GNNS

In GNNs (1), the integer \(N\) corresponds to the number of units in neural networks and \(M\) corresponds to the number of neural axons; that is, signals that emit from the \(i\)th unit have \(M\) pathways to the \(j\)th unit; \(u^i(t)\) corresponds to the membrane potential of the \(i\)th
neuron at time $t$; the dissipation coefficient $c_i(t) > 0$ represents the rate with which the $i$th neuron resets its potential when isolated from other neurons and inputs; $a_{ijl}(t)$ denotes synaptic connection weight of the $j$th neuron on the $i$th neuron at time $t - \kappa_{ijl}(t)$; $J_i(t)$ is an input to the $i$th neuron at time $t$ from outside the networks; $g_j(\cdot)$ denotes activation function; $\sigma_j$ denotes the amplifier gain; $\kappa_{ijl}(t)$ is the transmission delay of the $i$th unit along the $l$th axon of the $j$th unit at time $t$, it is a nonnegative bounded function with $0 < \kappa_{ijl}(t) \leq \kappa_{ijl} < \kappa := \max\{\kappa_{ijl}|i, j \in I, l \in L\}$.

As usual, we denote by $C([\sigma, 0], \mathbb{R}^N)$ the Banach space of all real-valued continuous mappings from $[-\sigma, 0]$ to $\mathbb{R}^N$ equipped with supremum norm defined by

$$||\phi||_\sigma = \max_{1 \leq i \leq N} ||\phi^i||_{\mathbb{R}}$$

and $\phi = (\phi^1, \phi^2, \cdots, \phi^N)^T \in C([-\sigma, 0], \mathbb{R}^N)$. Let $l > 0$. For any $u \in C([-\sigma, l], \mathbb{R}^N)$ and $t \in [0, l]$, we define $u_t(s) = u(t + s), s \in [-\sigma, 0]$. Then we have $u_t(\cdot) \in C([-\sigma, 0], \mathbb{R}^N)$. For any given $\phi \in C([-\sigma, 0], \mathbb{R}^N)$, we denote by $u(t; \phi)$ the solution of GNNs (1) with $u_0(s) = \phi(s)$ for all $s \in [-\sigma, 0]$.

Definition 1 (see [12-13])

A continuous function $f(t) : \mathbb{R} \rightarrow \mathbb{R}$ is called an almost periodic function if for any $\epsilon > 0$,

$$\mathcal{T}(f, \epsilon) = \left\{ T \in \mathbb{R} | |f(t + T) - f(t)| < \epsilon \text{ for all } t \in \mathbb{R} \right\}$$

is a relatively dense set in $\mathbb{R}$. That is, there exists a positive constant $l(\epsilon)$ such that any interval with length $l(\epsilon)$ contains at least one point of $\mathcal{T}(f, \epsilon)$. The number $T$ is called $\epsilon$-almost period of $f(t)$.

Let $(\mathcal{A}, || \cdot ||]$ be the space of all real-valued almost periodic functions defined on $\mathbb{R}$ with supremum norm defined by $||f|| = \sup_{t \in A} |f(t)|$ for any $f(t) \in \mathcal{A}$. It is easy for us to have the following basic properties (see [12-13]):

(\textbf{\textcircled{1}}) Let $f(t) \in \mathcal{A}$. Then $f(t)$ is bounded and uniformly continuous for all $t \in \mathbb{R}$.

(\textbf{\textcircled{2}}) Let $f_i(t) \in \mathcal{A}, i = 1, 2, \cdots, N$. Then for any $\epsilon > 0$, there exists a constant $l(\epsilon) > 0$ such that any interval with length $l(\epsilon)$ contains at least one common $\epsilon$-almost period of $f_i(t)$ for all $i = 1, 2, \cdots, N$. That is, $\bigcap_{i=1}^N \mathcal{T}(f_i, \epsilon)$ is relatively dense in $\mathbb{R}$.

(\textbf{\textcircled{3}}) Let $f(t), g(t) \in \mathcal{A}$. Then $f(t) - g(t) \in \mathcal{A}$.

Remark 1

Assume that $f(t), g(t) \in \mathcal{A}$. For any $\epsilon > 0$, by using (\textbf{\textcircled{1}}), there exists $\delta = \delta(\epsilon/2) > 0$ such that $|f(t_1) - f(t_2)| < \epsilon/2$ if $|t_1 - t_2| < \delta$. We only take $\overline{\epsilon} = \min\{\epsilon/2, \delta\}$. It is easy for us to check that $|f(t + T) - g(t + T)) - f(t - g(t))| < \epsilon$ for any $T \in \mathcal{T}(f, \overline{\epsilon}) \bigcap \mathcal{T}(g, \overline{\epsilon})$ and
all \( t \in \mathcal{R} \). From (\( \bullet_2 \)) and Definition 1, we get \( f(t - g(t)) \in \mathcal{A} \mathcal{R} \). That is, the property (\( \bullet_3 \)) holds.

Throughout this paper, we always assume the following assumptions hold:

\[ \bullet (H_1) \quad c_i(t) > 0, a_{ijl}(t), \kappa_{ijl}(t) > 0 \text{ and } J_i(t) \text{ are almost periodic functions defined on } \mathcal{R}, \sigma_j > 0 \text{ and } a_{ii}(t) > 0, \text{ where } i, j \in \mathcal{I} \text{ and } l \in \mathcal{L}. \]

Unless otherwise stated, we always use \( i, j = 1, 2, \cdots, N, l = 1, 2, \cdots, M \). The activation functions of first class we considered satisfy the following basic assumption:

\[ \text{Class } \mathcal{A} : \quad g_i \in C^2, \quad \begin{cases} |g_i(x)| \leq B_i, \quad g_i(0) = 0 \text{ and} \\ \dot{g}_i(x) > 0, \quad x \dot{g}_i(x) < 0, \text{ where } x \in \mathcal{R}. \end{cases} \]

**Lemma 1**

Assume that (\( H_1 \)) holds. For any \( \phi = (\phi^1, \phi^2, \cdots, \phi^N)^T \in C([-\kappa, 0], \mathcal{R}^N) \),

\[ \| \phi^i \|_\kappa \leq \left( \sum_{l=1}^{M} \sum_{j=1}^{N} \sup_{t \in \mathcal{R}} |a_{ijl}(t)|B_j + \sup_{t \in \mathcal{R}} |J_i(t)| \right) / \inf_{t \in \mathcal{R}} c_i(t) \]

implies that

\[ \| u^i(t; \phi) \|_\kappa \leq \left( \sum_{l=1}^{M} \sum_{j=1}^{N} \sup_{t \in \mathcal{R}} |a_{ijl}(t)|B_j + \sup_{t \in \mathcal{R}} |J_i(t)| \right) / \inf_{t \in \mathcal{R}} c_i(t) \]

for all \( t \geq 0 \), where \( u(t; \phi) \) is the solution of GNNs (1) with \( u_0(s) = \phi(s) \) for \( s \in [-\kappa, 0] \).

**Proof**

The proof is trivial, we omit it here. \( \square \)

We introduce the following auxiliary functions

\[ F_i(z) = - \sup_{t \in \mathcal{R}} c_i(t)z + \inf_{t \in \mathcal{R}} \sum_{l=1}^{M} a_{iil}(t)g_i(\sigma_i z). \]

**Lemma 2**

Suppose that the following assumption holds:

\[ \bullet (H_1^\mathcal{A}) \quad \inf_{\zeta \in \mathcal{R}} \dot{g}_i(\zeta) < \sup_{t \in \mathcal{R}} c_i(t) / \sigma_i \inf_{t \in \mathcal{R}} \sum_{l=1}^{M} a_{iil}(t) < \sup_{\zeta \in \mathcal{R}} \dot{g}_i(\zeta). \]

Then there exist two points \( z_{i1} \) and \( z_{i2} \) with \( z_{i1} < 0 < z_{i2} \) such that \( \dot{F}_i(z_{ik}) = 0 \) and \( \dot{F}_i(z) \cdot \text{sgn}\left\{ \frac{z - z_{i1}}{z - z_{i2}} \right\} < 0 \) (\( z \neq z_{ik}, k = 1, 2 \)).

**Proof**

We have \( \dot{F}_i(z) = 0 \) if and only if \( \dot{g}_i(\sigma_i z) = \sup_{t \in \mathcal{R}} c_i(t) / \sigma_i \inf_{t \in \mathcal{R}} \sum_{l=1}^{M} a_{iil}(t) \). For activation functions of class \( \mathcal{A} \), we know that the graph of positive function \( \dot{g}_i(z) \) concaves down and has its
maximal value at zero. By the continuity of $\dot{g}_i(z)$ and $(H^A_1)$, there exist two points $z_{i1}$ and $z_{i2}$ with $z_{i1} < 0 < z_{i2}$ such that $\dot{g}_i(\sigma_1 z_{ik}) = \sup_{t \in A} c_i(t)/\sigma_1 \inf_{t \in A} \sum_{l=1}^{M} a_{iil}(t)$; that is, $\dot{F}_i(z_{ik}) = 0$ $(k = 1, 2)$. Since $\dot{g}_i(z)$ is increasing on $(-\infty, z_{i1})$ and is decreasing on $[z_{i2}, +\infty)$, we get that

$$\left(-\sup_{t \in A} c_i(t) + \sigma_i \dot{g}_i(\sigma_i z) \inf_{t \in A} \sum_{l=1}^{M} a_{iil}(t)\right) \cdot \text{sgn}\left\{\frac{z - z_{i1}}{z - z_{i2}}\right\} < 0;$$

that is, $\dot{F}_i(z) \cdot \text{sgn}\left\{\frac{z - z_{i1}}{z - z_{i2}}\right\} < 0$ $(z \neq z_{ik})$. The proof is complete.

For the existence of $2^N$ positively invariant basins of GNNs (1), we consider the following assumption for activation functions of class $\mathcal{A}$:

$$(H^A_2) \quad (-1)^k \cdot \left\{F_i(z_{ik}) + J_i(t)\right\} > \sum_{l=1}^{M} \sum_{j \neq i}^{N} \sup_{t \in A} |a_{ijl}(t)| B_j$$

for all $t \in A$, where $k = 1, 2$.

Take $k = 1$ in $(H^A_2)$, it is easy for us to get that

$$F_i(z_{i1}) + \sum_{l=1}^{M} \sum_{j \neq i}^{N} \sup_{t \in A} |a_{ijl}(t)| B_j + \sup_{t \in A} J_i(t) < 0. \quad (2)$$

Noting that $F_i(z) \to +\infty$ as $z \to -\infty$, we know that there exists a $\tilde{z}_{i1}$ with $\tilde{z}_{i1} < z_{i1} < 0$ such that

$$F_i(\tilde{z}_{i1}) + \sum_{l=1}^{M} \sum_{j \neq i}^{N} \sup_{t \in A} |a_{ijl}(t)| B_j + \sup_{t \in A} J_i(t) = 0. \quad (3)$$

Take $k = 2$ in $(H^A_2)$, by the similar argument, we derive that there exists a $\tilde{z}_{i2}$ with $0 < z_{i2} < \tilde{z}_{i2}$ such that

$$F_i(\tilde{z}_{i2}) - \sum_{l=1}^{M} \sum_{j \neq i}^{N} \sup_{t \in A} |a_{ijl}(t)| B_j + \inf_{t \in A} J_i(t) = 0. \quad (4)$$

Next, we let

$$\begin{cases}
\alpha_{i1} := -\left(\sum_{l=1}^{M} \sum_{j=1}^{N} \sup_{t \in A} |a_{ijl}(t)| B_j + \sup_{t \in A} J_i(t)\right) \big/ \inf_{t \in A} c_i(t), & \beta_{i1} := \tilde{z}_{i1}, \\
\alpha_{i2} := \tilde{z}_{i2}, & \beta_{i2} := \left(\sum_{l=1}^{M} \sum_{j=1}^{N} \sup_{t \in A} |a_{ijl}(t)| B_j + \sup_{t \in A} J_i(t)\right) \big/ \inf_{t \in A} c_i(t).
\end{cases}$$

It is easy for us to check that $\alpha_{i1} < \beta_{i1} < 0 < \alpha_{i2} < \beta_{i2}$. Then we define $2^N$ subsets of $\mathcal{C}([-\kappa, 0], A)$ as follows:

$$\mathcal{K}_{i1} := \left\{\psi \in \mathcal{C}([-\kappa, 0], A) | \psi(s) \leq \beta_{i1} \text{ for all } s \in [-\kappa, 0]\right\},$$

$$\mathcal{K}_{i2} := \left\{\psi \in \mathcal{C}([-\kappa, 0], A) | \psi(s) \geq \alpha_{i2} \text{ for all } s \in [-\kappa, 0]\right\}.$$
Hence we have $2^N$ subsets $\mathcal{K}^\Sigma := \prod_{i=1}^N \mathcal{X}_i$ of $\mathcal{C}([-\kappa,0], \mathbb{R}^N)$, where $\Sigma = (\varsigma_1, \varsigma_2, \ldots, \varsigma_N)$ with $\varsigma_i = 1$ or 2, $i \in \mathcal{I}$. In what follows, we should prove that these $\mathcal{K}^\Sigma$ are $2^N$ positively invariant basins of GNNs (1).

**Theorem 1**
Under the assumptions $(H_1)$ and $(H_1^A)-(H_2^A)$, each $\mathcal{K}^\Sigma$ is a positively invariant basin with respect to the solution flow generated by GNNs (1).

**Proof**
For any initial condition $\phi \in \mathcal{K}^\Sigma$, we should prove that $u_*(\cdot; \phi) \in \mathcal{K}^\Sigma$ for all $t \geq 0$. For each $i \in \mathcal{I}$, we only consider the case $\varsigma_i = 2$, i.e., $\phi^i(s) \geq \alpha_{i2}$ for all $s \in [-\kappa,0]$. We assert that, for any sufficiently small $\epsilon > 0$ ($\epsilon \ll \alpha_{i2} - z_{i2}$), the solution $u^i(t; \phi) \geq \alpha_{i2} - \epsilon$ holds for all $t \geq 0$. If this is not true, there exists some $t^* > 0$ such that $u^i(t^*) = \alpha_{i2} - \epsilon$, $\dot{u}^i(t^*) \leq 0$ and $u^i(t) > \alpha_{i2} - \epsilon$ for $t \in [-\kappa, t^*]$. Due to $(H_1^A)$, $\inf_{t \in \mathcal{I}} \sum_{l=1}^M a_{iil}(t) > 0$ and the monotonicity of $g_i$, we derive from GNNs (1) that

$$
\frac{du^i(t^*)}{dt} = -c_i(t^*)u^i(t^*) + \sum_{l=1}^M \sum_{j=1}^N a_{ijl}(t^*) g_j \left( \sigma_j u^j \left( t^* - \kappa_{ijl}(t^*) \right) \right) + J_i(t^*)
\geq - \sup_{t \in \mathcal{I}} c_i(t)(\alpha_{i2} - \epsilon) + \inf_{t \in \mathcal{I}} \sum_{l=1}^M a_{iil}(t) g_i(\sigma_i(\alpha_{i2} - \epsilon))
- \sum_{l=1}^M \sum_{j \neq i} \sup_{t \in \mathcal{I}} |a_{ijl}(t)| B_j + \inf_{t \in \mathcal{I}} J_i(t)
\geq F_i(z_{i2} - \epsilon) - \sum_{l=1}^M \sum_{j \neq i} \sup_{t \in \mathcal{I}} |a_{ijl}(t)| B_j + \inf_{t \in \mathcal{I}} J_i(t). \tag{5}
$$

From Lemma 2, we know that $F_i(z)$ is strictly decreasing on $(z_{i2}, +\infty)$. By using (4) and (5), we get $\frac{du^i(t^*)}{dt} > 0$ which leads to a contradiction. Since the choice of $\epsilon$ is arbitrary, if $\phi^i(s) \geq \alpha_{i2}$ for all $s \in [-\kappa,0]$, we have $u^i(t; \phi) \geq \alpha_{i2}$ for all $t \geq 0$. When $\varsigma_i = 1$, similar argument can be performed to show that if $\phi^i(s) \leq \beta_{i1}$ for all $s \in [-\kappa,0]$, we have $u^i(t; \phi) \leq \beta_{i1}$ for all $t \geq 0$. Hence, for any initial condition $\phi \in \mathcal{K}^\Sigma$, we have that $u_*(\cdot; \phi) \in \mathcal{K}^\Sigma$ for all $t \geq 0$. That is, each $\mathcal{K}^\Sigma$ is a positively invariant basin with respect to the solution flow generated by GNNs (1). The proof is complete. \hfill \square

For convenience of discussing invariant regions for the existence of $2^N$ almost periodic encoded patterns of GNNs (1) in next section, we define $2N$ subsets $\mathcal{B}_{ik} \subset \mathcal{I}$ ($i \in \mathcal{I}$, $k = 1, 2$) which satisfy the following two basic properties:

$(\triangle_1)$ For any $\phi(t) \in \mathcal{B}_{ik}$, $\alpha_{ik} \leq \phi(t) \leq \beta_{ik}$ for all $t \in \mathcal{I}$.
(△₂) For any \( \epsilon > 0 \), \( \mathcal{I}(\mathcal{B}_{ik}, \epsilon) := \bigcap_{\phi \in \mathcal{B}_{ik}} \mathcal{I}(\phi, \epsilon) \) is relatively dense.

Let

\[ \mathcal{B}^\Sigma := \bigtimes_{\xi=1}^{N} \mathcal{B}_{\xi} \]

where \( \Sigma = (\xi_1, \xi_2, \cdots, \xi_N) \) with \( \xi_i = 1 \) or \( 2 \), \( i \in \mathcal{I} \). Then \( \mathcal{B}_{ik} \) \((i \in \mathcal{I}, k = 1, 2)\) are not only convex subsets of Banach space \( \mathcal{A} \mathcal{P} \), but also uniformly almost periodic families (see [12]). The compactness of \( \mathcal{B}_{ik} \) comes from the following lemma.

**Lemma 3** (see [12])

If \( \mathcal{B} \subset \mathcal{A} \mathcal{P} \) is a uniformly almost periodic family, then from every sequence in \( \mathcal{B} \) one can extract a subsequence which converges uniformly on \( \mathbb{R} \).

**Lemma 4** (see [26])

For any \( p > 1 \), \( x_k \geq 0 \), \( y \geq 0 \), the following inequality holds:

\[ y \prod_{k=1}^{m} x_k^p \leq \frac{1}{p} \sum_{k=1}^{m} p_k x_k^p + \frac{1}{p} y^p, \]

where \( p_k > 0 \) \((k = 1, 2, \cdots, m)\) are constants and \( \sum_{k=1}^{m} p_k = p - 1 \).

**Lemma 5** (see [32-34])

Let \( H = (h_{ij})_{N \times N} \in \mathcal{R}^{N \times N} \) with \( h_{ij} \leq 0 \) \((i \neq j)\). Then the following conditions are equivalent:

1. All the leading principal minors of \( H \) are positive;
2. \( H \) is quasi-dominant positive diagonal; that is, there exist positive numbers \( z_j \) \((j \in \mathcal{I})\) such that \( \sum_{j=1}^{N} z_j h_{ij} > 0 \) or \( \sum_{j=1}^{N} z_j h_{ji} > 0 \), \( i \in \mathcal{I} \).

We denote by \( \mathcal{M} \) the set of all matrices which satisfy one of the above properties. For any \( H \in \mathcal{M} \), let \( \Omega_{\mathcal{M}}(H) := \left\{ Z = (z_1, z_2, \cdots, z_N)^T \in \mathcal{R}^N | HZ > 0 \text{ and } z_i > 0, i \in \mathcal{I} \right\} \). It is obvious that \( \Omega_{\mathcal{M}}(H) \) is a cone without the vertex in \( \mathcal{R}^N \). Given any \( \bar{H} = (\bar{h}_{ij})_{N \times N} \leq 0 \), \( \bar{h}_{ij} \geq h_{ij} \) \((i, j \in \mathcal{I})\) and \( H \in \mathcal{M} \), then \( \bar{H} \in \mathcal{M} \).

3. ALMOST PERIODIC ENCODED PATTERNS FOR GNNS

In this section, by using the properties of almost periodicity and Schauder’s fixed point theorem, we should prove that each \( \mathcal{B}^\Sigma \) is an invariant region and there exist at least \( 2^N \) almost periodic encoded patterns of GNNs (1) in these \( \mathcal{B}^\Sigma \). Finally, attracting basins are estimated and some criteria are derived for the networks to converge exponentially toward \( 2^N \) almost periodic encoded patterns.
Theorem 2  
Under the basic assumptions (\(H_1\)) and (\(H_1^A\))-(\(H_2^A\)), for each \(\Sigma\), there exists at least one almost periodic encoded pattern of GNNs (1) in \(B^\Sigma\).

Proof  
For each \(\Sigma = (\varsigma_1, \varsigma_2, \cdots, \varsigma_N)\), we define a mapping \(\mathcal{F}^\Sigma = (\mathcal{F}_1^\Sigma, \mathcal{F}_2^\Sigma, \cdots, \mathcal{F}_N^\Sigma)\) by

\[
(\mathcal{F}_i^\Sigma \phi)(t) = \int_0^\infty \left[ \sum_{l=1}^M \sum_{j=1}^N a_{ijl}(t-s)g_j \left( \sigma_j \phi^i(t-s) - \kappa_{ijl}(t-s) \right) 
+ J_i(t-s) \right] \exp \left( - \int_0^s c_i(t-u)du \right) ds,
\]

where \(i \in \mathcal{I}\), \(\phi = (\phi^1, \phi^2, \cdots, \phi^N) \in B^\Sigma\). From (\(H_1\)) and the boundedness of activation functions, it is easy for us to check that each \(\mathcal{F}^\Sigma\) is well defined. Next we need three steps to complete our proof.

**Step 1:** For each \(i \in \mathcal{I}\), we should prove that \(\alpha_{i\varsigma_i} \leq (\mathcal{F}_i^\Sigma \phi)(t) \leq \beta_{i\varsigma_i}\) for all \(t \in \mathcal{R}\). Fix \(i \in \mathcal{I}\). From (\(H_1\)) and (6), one obtains that

\[
\left| (\mathcal{F}_i^\Sigma \phi)(t) \right| \leq \left( \sum_{l=1}^M \sum_{j=1}^N \sup_{t \in \mathcal{R}} |a_{ijl}(t)|B_j + \sup_{t \in \mathcal{R}} |J_i(t)| \right) \int_0^\infty \exp \left( - \int_0^s \inf_{t \in \mathcal{R}} c_i(t)du \right) ds
\]

\[
\leq \left( \sum_{l=1}^M \sum_{j=1}^N \sup_{t \in \mathcal{R}} |a_{ijl}(t)|B_j + \sup_{t \in \mathcal{R}} |J_i(t)| \right) \inf_{t \in \mathcal{R}} c_i(t) = \beta_{i2}.
\]

If \(\varsigma_i = 2\) (\(i \in \mathcal{I}\)), then \(\phi^i(t) \geq \alpha_{i2}\) for all \(t \in \mathcal{R}\). From (6) and (\(H_2^A\)), we get

\[
(\mathcal{F}_i^\Sigma \phi)(t) \geq \left( \inf_{t \in \mathcal{R}} \sum_{l=1}^M a_{iil}(t)g_l(\sigma_{i\alpha_{i2}}) - \sum_{l=1}^M \sum_{j \neq l}^N \sup_{t \in \mathcal{R}} |a_{ijl}(t)|B_j \right) \int_0^\infty \exp \left( - \int_0^s \sup_{t \in \mathcal{R}} c_i(t)du \right) ds
\]

\[
+ \inf_{t \in \mathcal{R}} J_i(t) \int_0^\infty \exp \left( - \int_0^s \sup_{t \in \mathcal{R}} c_i(t)du \right) ds
\]

\[
\geq \left( \inf_{t \in \mathcal{R}} \sum_{l=1}^M a_{iil}(t)g_l(\sigma_{i\alpha_{i2}}) - \sum_{l=1}^M \sum_{j \neq l}^N \sup_{t \in \mathcal{R}} |a_{ijl}(t)|B_j \right) \sup_{t \in \mathcal{R}} c_i(t)
\]

\[
+ \inf_{t \in \mathcal{R}} J_i(t) \sup_{t \in \mathcal{R}} c_i(t) = \alpha_{i2},
\]

for all \(t \in \mathcal{R}\). By (7) and (8), we have \(\alpha_{i2} \leq (\mathcal{F}_i^\Sigma \phi)(t) \leq \beta_{i2}\). From similar argument, if \(\varsigma_i = 1\), we can prove that \(\alpha_{i1} \leq (\mathcal{F}_i^\Sigma \phi)(t) \leq \beta_{i1}\) for all \(t \in \mathcal{R}\). Hence, we have \(\alpha_{i\varsigma_i} \leq (\mathcal{F}_i^\Sigma \phi)(t) \leq \beta_{i\varsigma_i}\) for each \(i \in \mathcal{I}\) and all \(t \in \mathcal{R}\). 

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Step 2: We should prove that \( \mathcal{F}^\Sigma : \mathcal{H}^\Sigma \to \mathcal{H}^\Sigma \). For any \( \epsilon > 0 \), we let

\[
\epsilon^* = \min \left\{ \frac{\epsilon}{5} \inf_{t \in \mathcal{T}} c_i(t) / \max \left[ \sum_{j=1}^{N} \sum_{l=1}^{M} \sup_{t \in \mathcal{T}} |a_{ij}(t)| B_j, \sup_{t \in \mathcal{T}} |J_i(t)| \right] , \right. \\
\epsilon \inf_{t \in \mathcal{T}} c_i(t) / \sum_{j=1}^{N} \sum_{l=1}^{M} \sup_{t \in \mathcal{T}} |a_{ij}(t)| \sup_{\zeta \in \mathcal{F}} |g_j(\zeta)| , \frac{\epsilon}{5} \inf_{t \in \mathcal{T}} c_i(t) / M \sum_{j=1}^{N} B_j , \left. \frac{\epsilon}{5} \inf_{t \in \mathcal{T}} c_i(t) \right\} .
\]

From basic properties (\( \mathcal{A}_2 \)) and (\( \mathcal{A}_3 \)), we know that there exists a positive constant \( l(\epsilon^*) \) such that any interval \([\zeta, \zeta + l] \) \( (\zeta \in \mathcal{T}) \) contains at least one common \( \epsilon^* \)-almost period \( T \), namely

\[
\left| \phi^j \left( t + T - s - \kappa_{ij}(t + T - s) \right) - \phi^j \left( t - s - \kappa_{ij}(t - s) \right) \right| 
\leq \epsilon^* < \frac{\epsilon}{5} \inf_{t \in \mathcal{T}} c_i(t) / \sum_{j=1}^{N} M B_j ,
\]

\[
\left| a_{ij}(t + T - s) - a_{ij}(t - s) \right| < \epsilon^* < \frac{\epsilon}{5} \inf_{t \in \mathcal{T}} c_i(t) ,
\]

\[
\left| J_i(t + T - u) - J_i(t - u) \right| < \epsilon^* < \frac{\epsilon}{5} \inf_{t \in \mathcal{T}} c_i(t) ,
\]

\[
\left| c_i(t + T - u) - c_i(t - u) \right| < \epsilon^* < \frac{\epsilon}{5} \inf_{t \in \mathcal{T}} c_i(t) / \max \left\{ \sum_{j=1}^{N} \sup_{t \in \mathcal{T}} |a_{ij}(t)| B_j, \sup_{t \in \mathcal{T}} |J_i(t)| \right\} ,
\]

For convenience, we define continuous functions \( \Xi_i(t, s) = \exp \left( - \int_{0}^{s} c_i(t - u) du \right) \). From (6), we get that

\[
\left| (\mathcal{F}^\Sigma \phi)(t + T) - (\mathcal{F}^\Sigma \phi)(t) \right|
= \int_{0}^{\infty} \left[ \sum_{l=1}^{M} \sum_{j=1}^{N} a_{ij}(t + T - s) g_j \left( \sigma_j \phi^j(t + T - s - \kappa_{ij}(t + T - s)) \right) \right] J_i(t + T - s) ds \\
\times \exp \left( - \int_{0}^{s} c_i(t + T - u) du \right) - \int_{0}^{\infty} \left[ \sum_{l=1}^{M} \sum_{j=1}^{N} a_{ij}(t - s) g_j \left( \sigma_j \phi^j(t - s - \kappa_{ij}(t - s)) \right) \right] \exp \left( - \int_{0}^{s} c_i(t - u) du \right) ds \\
+ \int_{0}^{\infty} \left[ \sum_{l=1}^{M} \sum_{j=1}^{N} \left( a_{ij}(t + T - s) - a_{ij}(t - s) \right) g_j \left( \sigma_j \phi^j(t + T - s - \kappa_{ij}(t + T - s)) \right) \right] \exp \left( - \int_{0}^{s} c_i(t + T - u) du \right) ds \\
\times \exp \left( - \int_{0}^{s} c_i(t + T - u) du \right) ds + a_{ij}(t - s) g_j \left( \sigma_j \phi^j(t + T - s - \kappa_{ij}(t + T - s)) \right) \\
\times \exp \left( - \int_{0}^{s} c_i(t + T - u) du \right) ds \right. \\
\left. + a_{ij}(t - s) g_j \left( \sigma_j \phi^j(t + T - s - \kappa_{ij}(t + T - s)) \right) \right].
\]
\[
-g_j \left( \sigma_j \phi^i (t - s - \kappa_i (t - s)) \right) \exp \left( - \int_0^s c_i(t + T - u) du \right) ds \\
+ a_{iij}(t-s)g_j \left( \sigma_j \phi^j (t - s - \kappa_i (t - s)) \right) \left[ \exp \left( - \int_0^s c_i(t + T - u) du \right) ds \right. \\
\left. - \exp \left( - \int_0^s c_i(t-u) du \right) ds \right] + \left[ J_i(t+T-s)-J_i(t-s) \right] \exp \left( - \int_0^s c_i(t+T-u) du \right) ds \\
+ J_i(t-s) \left[ \exp \left( - \int_0^s c_i(t+T-u) du \right) ds - \exp \left( - \int_0^s c_i(t-u) du \right) ds \right] \\
\leq \int_0^\infty \sum_{l=1}^M \sum_{j=1}^N \left| a_{iij}(t+T-s)-a_{iij}(t-s) \right| \left| g_j \left( \sigma_j \phi^j (t + T - s - \kappa_i (t + T - s)) \right) \right| \xi_i(t+T,s) ds \\
+ \int_0^\infty \sum_{l=1}^M \sum_{j=1}^N \left| a_{iij}(t-s) \right| \left| g_j \left( \sigma_j \phi^j (t + T - s - \kappa_i (t + T - s)) \right) - g_j \left( \sigma_j \phi^j (t - s - \kappa_i (t - s)) \right) \right| \\
\times \xi_i(t+T,s) ds + \int_0^\infty \sum_{l=1}^M \sum_{j=1}^N \left| a_{iij}(t-s) \right| \left| g_j \left( \sigma_j \phi^j (t - s - \kappa_i (t - s)) \right) \right| \left| \xi_i(t+T,s) - \xi_i(t,s) \right| ds \\
+ \int_0^\infty \left| J_i(t-s) \right| \left| \xi_i(t+T,s) - \xi_i(t,s) \right| ds + \int_0^\infty \left| J_i(t+T-s)-J_i(t-s) \right| \xi_i(t+T,s) ds. \quad (10)
\]

By using (\(H_1\)), (9)-(10) and mean value theorem of differential calculus, we obtain that

\[
\left| (\mathcal{F}_t^\Sigma \phi)(t+T) - (\mathcal{F}_t^\Sigma \phi)(t) \right| \leq \epsilon^* \sum_{l=1}^M \sum_{j=1}^N B_j \int_0^\infty \exp \left( - \int_0^s \inf_0^s c_i(t) du \right) ds + \sum_{l=1}^M \sum_{j=1}^N \sup_{t \in \mathbb{R}} \left| a_{iij}(t) \right| \\
\times \int_0^\infty \left| g_j(\theta) \right| \sigma^j \left( \phi^j (t + T - s - \kappa_i (t + T - s)) - \phi^j (t - s - \kappa_i (t - s)) \right) \exp \left( - \int_0^s \inf_0^s c_i(t) du \right) ds \\
+ \sum_{l=1}^M \sum_{j=1}^N \left( \sup_{t \in \mathbb{R}} \left| a_{iij}(t) \right| B_j + \sup_{t \in \mathbb{R}} \left| J_i(t) \right| \right) \int_0^s \int_0^\infty \exp(\bar{\theta}) \left| c_i(t+T-u) - c_i(t-u) \right| duds \\
+ \epsilon^* \int_0^\infty \exp \left( - \int_0^s \inf_0^s c_i(t) du \right) ds, \quad (11)
\]

where \( \theta \) lies between \( \sigma_j \phi^j (t+T-s-\kappa_i (t+T-s)) \) and \( \sigma_j \phi^j (t-s-\kappa_i (t-s)) \), \( \bar{\theta} \) lies between \( - \int_0^s c_i(t+T-u) du \) and \( - \int_0^s c_i(t-u) du \). Noting that \( - \int_0^s \sup_{t \in \mathbb{R}} c_i(t) du \leq \bar{\theta} \leq - \int_0^s \inf_{t \in \mathbb{R}} c_i(t) du \),
it follows that \( \exp(\hat{\theta}) \leq \exp(-s \inf_{t \in \mathcal{R}} c_i(t)) \). Therefore, by using (9) and (11), we have

\[
\begin{align*}
| (\mathcal{F}^{\Sigma}_i \phi)(t + T) - (\mathcal{F}^{\Sigma}_i \phi)(t) | & \leq e^* \sum_{j=1}^{N} MB_j / \inf_{t \in \mathcal{R}} c_i(t) + e^* \int_{0}^{\infty} \exp \left( - \int_{0}^{s} \inf_{t \in \mathcal{R}} c_i(t) du \right) ds \\
& + e^* \sum_{l=1}^{M} \sum_{j=1}^{N} (\sup_{t \in \mathcal{R}} |a_{ijl}(t)| B_j + \sup_{t \in \mathcal{R}} |J_i(t)|) \int_{0}^{\infty} \exp(-s \inf_{t \in \mathcal{R}} c_i(t)) s ds \\
& + e^* \sum_{l=1}^{M} \sum_{j=1}^{N} \sigma_j \sup_{t \in \mathcal{R}} |a_{ijl}(t)| \sup_{\zeta \in \mathcal{R}} \hat{g}_j(\zeta) / \inf_{t \in \mathcal{R}} c_i(t) \leq \epsilon,
\end{align*}
\]

which leads to the almost periodicity of \( \mathcal{F}^{\Sigma}(\phi) \). From Step 1, it follows that \( \mathcal{F}^{\Sigma} : \mathcal{B}^{\Sigma} \to \mathcal{B}^{\Sigma} \). That is, each \( \mathcal{B}^{\Sigma} \) is invariant region of \( \mathcal{F}^{\Sigma} \).

**Step 3:** We should prove that \( \mathcal{F}^{\Sigma} : \mathcal{B}^{\Sigma} \to \mathcal{B}^{\Sigma} \) is continuous. Take any two \( \phi_1, \phi_2 \in \mathcal{B}^{\Sigma} \).

From (6) and Lagrange’s mean value theorem, we have

\[
| (\mathcal{F}^{\Sigma}_i \phi_1)(t) - (\mathcal{F}^{\Sigma}_i \phi_2)(t) | \leq \int_{0}^{\infty} \sum_{l=1}^{M} \sum_{j=1}^{N} |a_{ijl}(t - s)|
\]

\[
\times | g_j(\sigma_j \phi_1(t - s - \kappa_{ijl}(t - s))) - g_j(\sigma_j \phi_2(t - s - \kappa_{ijl}(t - s)))| \exp \left( - \int_{0}^{s} c_i(t - u) du \right) ds
\]

\[
\leq \sum_{l=1}^{M} \sum_{j=1}^{N} \sup_{t \in \mathcal{R}} |a_{ijl}(t)| \sigma_j \sup_{\zeta \in \mathcal{R}} \hat{g}_j(\zeta) / \inf_{t \in \mathcal{R}} c_i(t) \| \phi_1 - \phi_2 \|,
\]

which leads to

\[
\| \mathcal{F}^{\Sigma} \phi_1 - \mathcal{F}^{\Sigma} \phi_2 \| \leq \max_{t \in \mathcal{R}} \left\{ \sum_{l=1}^{M} \sum_{j=1}^{N} \sup_{t \in \mathcal{R}} |a_{ijl}(t)| \sigma_j \sup_{\zeta \in \mathcal{R}} \hat{g}_j(\zeta) / \inf_{t \in \mathcal{R}} c_i(t) \right\} \| \phi_1 - \phi_2 \|.
\]

This implies that \( \mathcal{F}^{\Sigma} \) is continuous with respect to \( \phi \in \mathcal{B}^{\Sigma} \).

From Lemma 3, each \( \mathcal{B}^{\Sigma} \) is compact convex subset. Since \( \mathcal{F}^{\Sigma} : \mathcal{B}^{\Sigma} \to \mathcal{B}^{\Sigma} \) is continuous, by Schauder’s fixed point theorem, there exists at least one \( \tilde{u}_{\Sigma} \in \mathcal{B}^{\Sigma} \) such that \( \mathcal{F}^{\Sigma} \tilde{u}_{\Sigma} = \tilde{u}_{\Sigma} \).

It is easy for us to check that

\[
\frac{d\tilde{u}_{\Sigma}(t)}{dt} = \frac{d}{dt} \left[ \sum_{l=1}^{M} \sum_{j=1}^{N} a_{ijl}(s) g_j(\sigma_j \tilde{u}_{\Sigma}(s - \kappa_{ijl}(s))) + J_i(s) \right] \exp \left( - \int_{s}^{t} c_i(u) du \right) ds
\]

\[
= -c_i(t) \tilde{u}_{\Sigma}(t) + \sum_{l=1}^{M} \sum_{j=1}^{N} a_{ijl}(t) g_j(\sigma_j \tilde{u}_{\Sigma}(t - \kappa_{ijl}(t))) + J_i(t).
\]

Hence \( \tilde{u}_{\Sigma}(t) \) is an almost periodic solution of GNNs (1) in \( \mathcal{B}^{\Sigma} \). The proof is complete. □

From above theorem, we know that each \( \mathcal{B}^{\Sigma} \) is an invariant region of GNNs (1) and there exist at least \( 2^N \) almost periodic encoded patterns in these \( \mathcal{B}^{\Sigma} \). In what follows, we
should prove that each \( \mathcal{X}^\Sigma \) is an attracting basin for almost periodic encoded pattern in \( \mathcal{B}^\Sigma \) under our additional assumptions. For convenience, we take the following notations:

\[
C = \text{diag}\left( \inf_{t \in \mathcal{I}} c_1(t), \inf_{t \in \mathcal{I}} c_2(t), \cdots, \inf_{t \in \mathcal{I}} c_N(t) \right), \quad \mathcal{H} = \begin{pmatrix} h_{ij} \end{pmatrix}_{N \times N} \quad \text{with} \quad h_{ij} := \sum_{l=1}^{M} |a_{ijl}(t)|,
\]

\[
\mathcal{G} = \text{diag}\left( \sigma_1 \dot{y}_1(\zeta), \sigma_2 \dot{y}_2(\zeta), \cdots, \sigma_N \dot{y}_N(\zeta) \right) \quad \text{with} \quad \dot{y}_i(\zeta) = \max \left\{ \dot{y}_i(\zeta) \mid z = \sigma_i \beta_{i1}, \sigma_i \alpha_{i2} \right\}.
\]

Next, we should introduce two additional assumptions:

- \((H_3^A)\quad C - \mathcal{H} \mathcal{G} \in \mathcal{M}.
- \((H_4^A)\quad \text{There exist positive constants } p > 1, \ d_i, \ o_k, \ q_{jlk} \text{ and } p_{jlk} \text{ such that}
  \begin{align*}
  p d_i \inf_{t \in \mathcal{I}} c_i(t) &> \sum_{l=1}^{M} \sum_{j=1}^{N} \left[ d_j \left( \sup_{t \in \mathcal{I}} |a_{ijl}(t)| \right)^{pp_{jlk}+1} \left( \sigma_j \dot{y}_j(\zeta) \right)^{pp_{jlk}/o_k} \right] \\
  &+ \sum_{k=1}^{m} d_k o_k \left( \sup_{t \in \mathcal{I}} |a_{ijl}(t)| \right)^{pp_{jlk}/o_k} \left( \sigma_j \dot{y}_j(\zeta) \right)^{pp_{jlk}/o_k},
  \end{align*}
\]

where \( \dot{y}_i(\zeta) = \max \left\{ \dot{y}_i(\zeta) \mid z = \sigma_i \beta_{i1}, \sigma_i \alpha_{i2} \right\} \), \( \sum_{k=1}^{m} o_k = p - 1 \), \( \sum_{k=1}^{m+1} q_{jlk} = \sum_{k=1}^{m+1} p_{jlk} = 1 \) for each \( j \in \mathcal{I} \) and \( l \in \mathcal{L} \), \( m \) is a positive integer.

**Theorem 3**

Under the basic assumptions \((H_1)\) and \((H_1^A)-(H_3^A)\), for each \( \Sigma \), there exists a unique almost periodic encoded pattern of GNNs (1) which is exponentially stable in \( \mathcal{X}^\Sigma \).

**Proof**

From Theorem 2, there exists an almost periodic solution \( \tilde{u} \) of GNNs (1) in each \( \mathcal{B}^\Sigma \). For any initial condition \( \phi \in \mathcal{X}^\Sigma \), by Theorem 1, we know that \( x_t(\cdot; \phi) \in \mathcal{X}^\Sigma \) for all \( t \geq 0 \). Under translation \( y(t) = \tilde{u}(t) - x(t; \phi) \), we get that

\[
\frac{dy(t)}{dt} = -c_i(t)y_i(t) + \sum_{l=1}^{M} \sum_{j=1}^{N} a_{ijl}(t) \left[ g_j \left( \sigma_j \tilde{y}_j(t - \kappa_{ijl}(t)) \right) - g_j \left( \sigma_j x^j(t - \kappa_{ijl}(t)) \right) \right].
\]

From \((H_3^A)\) and Lemma 5, there exists a \( K = (K_1, K_2, \cdots, K_N)^T \in \Omega_{\mathcal{M}}(C - \mathcal{H} \mathcal{G}) \) such that \( \sup_{\theta \in [-\kappa, 0]} |\tilde{y}^i(\theta) - \phi^i(\theta)| \leq K_i \). Then we get that

\[
\inf_{t \in \mathcal{I}} c_i(t)K_i - \sum_{l=1}^{M} \sum_{j=1}^{N} \sup_{t \in \mathcal{I}} |a_{ijl}(t)| \sigma_j \dot{y}_j(\zeta)K_j > 0,
\]

where \( \dot{y}_j(\zeta) = \max \left\{ \dot{y}_j(\zeta) \mid z = \sigma_j \beta_{j1}, \sigma_j \alpha_{j2} \right\} \). We consider the single-variable functions \( W_i(\cdot) \) defined by

\[
W_i(\theta) = \left( \inf_{t \in \mathcal{I}} c_i(t) - \theta \right)K_i - \sum_{l=1}^{M} \sum_{j=1}^{N} \sup_{t \in \mathcal{I}} |a_{ijl}(t)| \sigma_j \dot{y}_j(\zeta)K_j e^{\theta \kappa_{ijl}}.
\]
Noting that \( W_i(0) > 0 \) and \( W_i(\theta) \to -\infty \) as \( \theta \to +\infty \), there exists a suitable \( \mu \) such that for all \( i \in \mathcal{F} \),

\[
\left( \inf_{t \in \mathcal{F}} c_i(t) - \mu I_i \right) K_i - \sum_{l=1}^{M} \sum_{j=1}^{N} \sup_{t \in \mathcal{F}} |a_{ijl}(t)| \sigma_j \dot{g}_j(\zeta) K_j e^{\mu \kappa^j,ij} > 0.
\]

(15)

Consider function \( Z_i(t) = e^{\mu^i} |y^i(t)| \) where \( t \in [-\kappa, \infty) \). Let \( \varrho > 1 \). It is obvious that \( Z_i(t) < \varrho K_i \) for all \( t \in [-\kappa, 0] \). Now we claim that \( Z_i(t) < \varrho K_i \) for all \( t > 0 \) and \( i \in \mathcal{F} \). Otherwise there is a first time \( t_0 > 0 \) and some \( i^* \in \mathcal{F} \) such that \( Z_{i^*}(t_0) = \varrho K_{i^*} \), \( \frac{d^+|Z_{i^*}(t_0)|}{dt} \geq 0 \) and \( Z_j(t) < \varrho K_j \) \((j \neq i^*)\) for all \( t \in [-\kappa, t_0] \). From (14), we derive that

\[
\frac{d^+ |y^i(t_0)|}{dt} \leq - \left( \inf_{t \in \mathcal{F}} c_{i^*}(t) - \mu \right) Z_{i^*}(t_0) + \sum_{l=1}^{M} \sum_{j=1}^{N} \sup_{t \in \mathcal{F}} |a_{ijl}(t)| \sigma_j \dot{g}_j(\xi) |y^j(t_0 - \kappa^j,ij(t_0))|,
\]

where \( \xi \) lies between \( \sigma_j \dot{u}^j(t_0 - \kappa^j,ij(t_0)) \) and \( \sigma_j x^j(t_0 - \kappa^j,ij(t_0)) \). From above inequality and (15), we get that

\[
\frac{d^+ |Z_{i^*}(t_0)|}{dt} \leq - \left( \inf_{t \in \mathcal{F}} c_{i^*}(t) - \mu \right) Z_{i^*}(t_0) + \sum_{l=1}^{M} \sum_{j=1}^{N} \sup_{t \in \mathcal{F}} |a_{ijl}(t)| \sigma_j \dot{g}_j(\zeta) e^{\mu \kappa^j,ij(t_0)} Z_j(t_0 - \kappa^j,ij(t_0))
\]

\[
\leq - \left( \inf_{t \in \mathcal{F}} c_{i^*}(t) - \mu \right) Z_{i^*}(t_0) + \sum_{l=1}^{M} \sum_{j=1}^{N} \sup_{t \in \mathcal{F}} |a_{ijl}(t)| \sigma_j \dot{g}_j(\zeta) e^{\mu \kappa^j,ij} \sup_{\theta \in [t_0 - \kappa, t_0]} Z_j(\theta)
\]

\[
\leq - \left( \inf_{t \in \mathcal{F}} c_{i^*}(t) - \mu \right) \varrho K_{i^*} + \sum_{l=1}^{M} \sum_{j=1}^{N} \sup_{t \in \mathcal{F}} |a_{ijl}(t)| \sigma_j \dot{g}_j(\zeta) e^{\mu \kappa^j,ij} \varrho K_j < 0,
\]

which leads to a contradiction. Hence \( Z_{i}(t) < \varrho K_{i} \) for all \( t > 0 \) and \( i \in \mathcal{F} \). That is, there exists a positive constant \( \tilde{\varrho} \) such that

\[
|\tilde{u}^i(t) - x^i(t; \phi)| \leq e^{-\mu \tilde{\varrho}} \sup_{\theta \in [-\kappa, 0]} |\tilde{u}^i(\theta) - \phi^i(\theta)|,
\]

for all \( t \geq 0 \) and \( i \in \mathcal{F} \). Therefore, for each \( \Sigma \), there exists a unique almost periodic encoded pattern \( \tilde{u}(t) \) which is exponentially stable in \( \mathcal{H}^\Sigma \). The proof is complete. \( \square \)

**Theorem 4**

Under the basic assumptions \((H_1), (H_1^A)-(H_2^A) \) and \((H_4^A)\), for each \( \Sigma \), there exists a unique almost periodic encoded pattern of GNNs \((1)\) which is exponentially stable in \( \mathcal{H}^\Sigma \).

**Proof**

From Theorem 2, there exists an almost periodic solution \( \tilde{u} \) of GNNs \((1)\) in each \( \mathcal{H}^\Sigma \). For any \( \phi \in \mathcal{H}^\Sigma \), by Theorem 1, we know that \( x_{i}(\cdot; \phi) \in \mathcal{H}^\Sigma \) for all \( t \geq 0 \). Let \( y(t) = \tilde{u}(t) - x(t; \phi) \).

We consider the Lyapunov functional \( V(y)(t) = \sum_{i=1}^{N} d_i |y^i(t)|^p \), where \( d_i > 0 \). From (14), we
can derive that
\[
\frac{d^+ V(y)(t)}{dt} \leq \sum_{i=1}^{N} p_d_i |y^i(t)|^{p-1} \left\{ -\inf_{t \in \mathbb{R}} c_i(t) |y^i(t)| + \sum_{l=1}^{M} \sum_{j=1}^{N} \sup_{t \in \mathbb{R}} \left| a_{ijl}(t) \sigma_j \dot{g}_j(\zeta) \right| |y^i(t - \kappa_{ijl}(t))| \right\}
\]
\[
\leq -p \sum_{i=1}^{N} \left\{ \inf_{t \in \mathbb{R}} c_i(t) d_i |y^i(t)|^p - \sum_{l=1}^{M} \sum_{j=1}^{N} d_i |y^i(t)|^{p-1} \sup_{t \in \mathbb{R}} \left| a_{ijl}(t) \sigma_j \dot{g}_j(\zeta) \right| |y^i(t - \kappa_{ijl}(t))| \right\}
\]
\[
\leq -p \sum_{i=1}^{N} \left\{ \inf_{t \in \mathbb{R}} c_i(t) d_i |y^i(t)|^p - \sum_{l=1}^{M} \sum_{j=1}^{N} d_i \prod_{k=1}^{m} \left[ \left( \sup_{t \in \mathbb{R}} |a_{ijl}(t)| \right)^{p_{jk}} \left( \sigma_j \dot{g}_j(\zeta) \right)^{q_{jk}} |y^i(t)| \right]^{\alpha_k} \right\} \times \left( \left( \sup_{t \in \mathbb{R}} |a_{ijl}(t)| \right)^{p_{ji}} \left( \sigma_j \dot{g}_j(\zeta) \right)^{q_{ji}} |y^i(t - \kappa_{ijl}(t))| \right)^{\alpha_k}
\]
\[
\leq -p \sum_{i=1}^{N} \left\{ \inf_{t \in \mathbb{R}} c_i(t) d_i |y^i(t)|^p - \sum_{l=1}^{M} \sum_{j=1}^{N} d_i \prod_{k=1}^{m} \left[ \left( \sup_{t \in \mathbb{R}} |a_{ijl}(t)| \right)^{p_{jk}} \left( \sigma_j \dot{g}_j(\zeta) \right)^{q_{jk}} |y^i(t)| \right]^{\alpha_k} \right\} \times \left( \left( \sup_{t \in \mathbb{R}} |a_{ijl}(t)| \right)^{p_{ji}} \left( \sigma_j \dot{g}_j(\zeta) \right)^{q_{ji}} |y^i(t - \kappa_{ijl}(t))| \right)^{\alpha_k}
\]
\[
+ \frac{d_i}{p} \left( \sup_{t \in \mathbb{R}} |a_{ijl}(t)| \right)^{p_{ji}} \left( \sigma_j \dot{g}_j(\zeta) \right)^{q_{ji}} |y^i(t - \kappa_{ijl}(t))|^p \right\}
\]
\[
\leq -p \sum_{i=1}^{N} \left\{ \inf_{t \in \mathbb{R}} c_i(t) d_i - \sum_{l=1}^{M} \sum_{j=1}^{N} d_i \prod_{k=1}^{m} \left[ \left( \sup_{t \in \mathbb{R}} |a_{ijl}(t)| \right)^{p_{jk}} \left( \sigma_j \dot{g}_j(\zeta) \right)^{q_{jk}} |y^i(t)| \right]^{\alpha_k} \right\} \times \left( \left( \sup_{t \in \mathbb{R}} |a_{ijl}(t)| \right)^{p_{ji}} \left( \sigma_j \dot{g}_j(\zeta) \right)^{q_{ji}} |y^i(t - \kappa_{ijl}(t))| \right)^{\alpha_k}
\]
\[
+ \sum_{i=1}^{N} \left\{ \sum_{l=1}^{M} \sum_{j=1}^{N} d_j \left( \sup_{t \in \mathbb{R}} |a_{ijl}(t)| \right)^{p_{jl}} \left( \sigma_j \dot{g}_j(\zeta) \right)^{q_{jl}} |y^i(t - \kappa_{ijl}(t))|^p \right\}
\]
\[
\leq -\alpha V(y)(t) + \beta \sup_{t - \kappa \leq s \leq t} V(y)(s), \quad (16)
\]
where
\[
\alpha = \min \left\{ p d_i \inf_{t \in \mathbb{R}} c_i(t) - \sum_{l=1}^{M} \sum_{j=1}^{N} d_i \prod_{k=1}^{m} \left[ \left( \sup_{t \in \mathbb{R}} |a_{ijl}(t)| \right)^{p_{jk}} \left( \sigma_j \dot{g}_j(\zeta) \right)^{q_{jk}} \right]^{\alpha_k} \right\},
\]
\[
\beta = \max \left\{ \sum_{l=1}^{M} \sum_{j=1}^{N} d_j \left( \sup_{t \in \mathbb{R}} |a_{ijl}(t)| \right)^{p_{jl}} \left( \sigma_j \dot{g}_j(\zeta) \right)^{q_{jl}} \right\}.
\]
From \((H^4)\), we have \(\alpha > \beta > 0\). By using Halanay inequality, we get for all \(t \in \mathbb{R}\),
\[
V(y)(t) \leq \left( \sup_{-\kappa \leq s \leq 0} V(y)(s) \right) \exp(-\gamma t), \quad (17)
\]
where \(\gamma = \alpha - \beta e^{\gamma \kappa}\). It follows that
\[
\sum_{i=1}^{N} \left| \bar{u}^i(t) - x^i(t; \phi) \right|^p \leq e^{-\gamma t} \frac{\max \{d_i\}}{\min \{d_i\}} \sum_{l=1}^{N} \sup_{t \in [-\kappa,0]} \left| \bar{u}^i(\theta) - \phi^i(\theta) \right|^p.
\]
Hence \(\bar{u}(t)\) is exponentially stable. The proof is complete.

\(\square\)

**Remark 2**
When \(c_i, a_{ijl}, \kappa_{ijl}, J_i : \mathbb{R} \rightarrow \mathbb{R}\) are \(\omega\)-periodic functions with \(\omega > 0\), we also obtain the existence and exponential stability of \(2^N\) periodic solutions of GNNs (1). Our results
in Theorem 3 and Theorem 4 are distinguished from the existing results on the following points: (i) Most of the previous results of neural networks only focus on the existence and stability of unique almost periodic (periodic) solution. Hence, we extend the related results [20-31, 37, 39, 41] to the convergence analysis of multiple almost periodic (periodic) solutions. (ii) We not only establish existing regions for almost periodic solutions, but also estimate attracting basins of these almost periodic solutions. (iii) Our sufficient conditions $(H_4^A)$-$(H_4^A)$ are dependent of system parameters and derivative of activation functions on boundary points which make our results new in the literature.

**Remark 3**

From $(H_4^A)$, we know that $F_1(z_{i2}) - \sum_{i=1}^{M} \sup_{t \in \mathcal{I}} |a_{ijl}(t)|B_j + \inf_{t \in \mathcal{I}} J_i(t) > 0$. It is obvious that $F_1(z_{i2}) + \sum_{i=1}^{M} \sup_{t \in \mathcal{I}} |a_{ijl}(t)|B_j + \sup_{t \in \mathcal{I}} J_i(t) > 0$. Together with (2), there exists a $\tilde{z}_{i1}$ with $\tilde{z}_{i1} > z_{i1}$ such that $F_1(\tilde{z}_{i1}) + \sum_{i=1}^{M} \sup_{t \in \mathcal{I}} |a_{ijl}(t)|B_j + \sup_{t \in \mathcal{I}} J_i(t) = 0$. Similarly, there exists a $\tilde{z}_{i2} < z_{i2}$ such that $F_1(\tilde{z}_{i2}) - \sum_{i=1}^{M} \sup_{t \in \mathcal{I}} |a_{ijl}(t)|B_j + \inf_{t \in \mathcal{I}} J_i(t) = 0$. For each $\Sigma = (\varsigma_1, \varsigma_2, \cdots, \varsigma_N)$, we denote

$$\Xi^\Sigma \coloneqq \left\{ z = (z_1, z_2, \cdots, z_N)^T \mid z_i \in [\tilde{z}_{i1}, \tilde{z}_{i2}], i = 1, 2, \cdots, N \text{ and } C - H \in \mathcal{M} \right\},$$

where $H = diag(\sigma_1g_1(\sigma_1 z_1), \sigma_2g_2(\sigma_2 z_2), \cdots, \sigma_N g_N(\sigma_N z_N))$. Under the basic assumptions of Theorem 3, we have $\Xi^\Sigma \neq \emptyset$. Assume that there exists a $\tau \in \Xi^\Sigma$ such that (i) $\varsigma_i = 1$, $\tau_i \geq \sup_{z \in \Xi^\Sigma} \{ z_i \}$ ($\tau_i \geq \tilde{z}_{i1}$); (ii) $\varsigma_i = 2$, $\tau_i \leq \inf_{z \in \Xi^\Sigma} \{ z_i \}$ ($\tau_i \leq \tilde{z}_{i2}$). For each $\Sigma$, we let (i) $\varsigma_1 = 1$, $\beta_{i1} := \tau_{i1}$; (ii) $\varsigma_i = 2$, $\alpha_{i2} := \tau_i$. From Theorem 1 and Theorem 3, we know that each $\mathcal{K}^\Sigma := \mathcal{K}_{i1} \times \mathcal{K}_{i2} \times \cdots \times \mathcal{K}_{N, i}$ is a **larger attracting basin**.

4. SOME GENERALIZATIONS AND IMPROVEMENTS

In this section, we should make some generalizations by considering the second class of saturated activation functions and make some improvements by applying our results to some special cases. The second class of activation functions we considered in this paper satisfies

$$\text{Class } \mathcal{B} : \quad g_j \in \mathcal{C}, \quad g_j(x) = \begin{cases} u_{j1} & \text{if } -\infty < x < \ell_{j1}, \\ \tilde{g}_j(x) & \text{if } \ell_{j1} \leq x \leq \ell_{j2}, \\ u_{j2} & \text{if } \ell_{j2} < x < +\infty, \end{cases}$$

where $\tilde{g}_j \in C^1$ is an increasing function with $\tilde{g}_j(0) = 0$, $\ell_{j1} < 0 < \ell_{j2}$ and $-\infty < u_{j1} < 0 < u_{j2} < +\infty$. Similarly as Lemma 1, it is easy for us to have the following lemma.

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Lemma 6
Assume that \((H_1)\) holds. For any \(\phi = (\phi^1, \phi^2, \ldots, \phi^N)^T \in C([-\kappa, 0], \mathbb{R}^N)\),
\[
\|\phi^i\|_\kappa \leq \left( \sum_{l=1}^M \sum_{j=1}^N \sup_{t \in \mathcal{S}} |a_{ijkl}(t)| \cdot \max_{k=1,2} \left\{ |u_{jk}| \right\} + \sup_{t \in \mathcal{S}} |J_i(t)| \right) / \inf_{t \in \mathcal{S}} c_i(t)
\]
implies that
\[
\|u^i(t; \phi)\|_\kappa \leq \left( \sum_{l=1}^M \sum_{j=1}^N \sup_{t \in \mathcal{S}} |a_{ijkl}(t)| \cdot \max_{k=1,2} \left\{ |u_{jk}| \right\} + \sup_{t \in \mathcal{S}} |J_i(t)| \right) / \inf_{t \in \mathcal{S}} c_i(t)
\]
for all \(t \geq 0\), where \(u(t; \phi)\) is the solution of GNNs (1) with \(u_0(s) = \phi(s)\) for \(s \in [-\kappa, 0]\).

Remark 4
Since each activation function of Class \(B\) is bounded with \(|g_j(x)| \leq \max_{k=1,2} \left\{ |u_{jk}| \right\}\), \(j \in \mathcal{J}\). As similar proof of Lemma 1, it is easy for us to get the result of Lemma 6.

For activation functions of class \(B\), we consider the following two parameter assumptions which are used to establish the existence and stability of \(2^N\) almost periodic encoded patterns of GNNs (1):
\[
\bullet (H^B_1) : \sup_{t \in \mathcal{S}} c_i(t) < \inf_{t \in \mathcal{S}} \sum_{l=1}^M a_{il} \sigma_i(t) \zeta, \ z \in [\ell_{i1}, \ell_{i2}].
\]
\[
\bullet (H^B_2) : (-1)^k \left\{ \sup_{t \in \mathcal{S}} c_i(t) \frac{\ell_{ik}}{\sigma_i} + \inf_{t \in \mathcal{S}} \sum_{l=1}^M a_{il} u_{jk} + J_i(t) \right\} > \sum_{l=1}^M \sum_{j \neq i} \sup_{t \in \mathcal{S}} |a_{ijkl}(t)| \cdot \max_{k=1,2} \left\{ |u_{jk}| \right\}
\]
for all \(t \in \mathcal{S}\), where \(i \in \mathcal{J}\) and \(k = 1, 2\).

Take \(k = 1\) in \((H^B_2)\), it is easy for us to derive that
\[
F_i(\frac{\ell_{i1}}{\sigma_i}) + \sum_{l=1}^M \sum_{j \neq i} \sup_{t \in \mathcal{S}} |a_{ijkl}(t)| \cdot \max_{k=1,2} \left\{ |u_{jk}| \right\} + \sup_{t \in \mathcal{S}} J_i(t) < 0.
\]
Noting that \(F_i(z) \to +\infty\) as \(z \to -\infty\), we know that there exists a \(\hat{\ell}_{i1}\) with \(\hat{\ell}_{i1} < \frac{\ell_{i1}}{\sigma_i} < 0\) such that
\[
F_i(\hat{\ell}_{i1}) + \sum_{l=1}^M \sum_{j \neq i} \sup_{t \in \mathcal{S}} |a_{ijkl}(t)| \cdot \max_{k=1,2} \left\{ |u_{jk}| \right\} + \sup_{t \in \mathcal{S}} J_i(t) = 0.
\]
Take \(k = 2\) in \((H^B_2)\), by the similar argument, there exists a \(\bar{\ell}_{i2}\) with \(0 < \frac{\ell_{i2}}{\sigma_i} < \bar{\ell}_{i2}\) such that
\[
F_i(\bar{\ell}_{i2}) - \sum_{l=1}^M \sum_{j \neq i} \sup_{t \in \mathcal{S}} |a_{ijkl}(t)| \cdot \max_{k=1,2} \left\{ |u_{jk}| \right\} + \inf_{t \in \mathcal{S}} J_i(t) = 0.
\]
For convenience, as Section 3, we also take the following denotations:

\[
\begin{align*}
\alpha_1 &= -\left( \sum_{l=1}^{M} \sum_{j=1}^{N} \sup_{t \in \mathcal{R}} |a_{ijl}(t)| \cdot \max_{k=1,2} \left\{ |u_{jk}| \right\} \right) / \inf_{t \in \mathcal{R}} c_i(t), \quad \beta_1 = \hat{\ell}_{i1}, \\
\alpha_2 &= \overline{\ell}_{i2}, \quad \beta_2 = \left( \sum_{l=1}^{M} \sum_{j=1}^{N} \sup_{t \in \mathcal{R}} |a_{ijl}(t)| \cdot \max_{k=1,2} \left\{ |u_{jk}| \right\} \right) / \inf_{t \in \mathcal{R}} c_i(t).
\end{align*}
\]

It is easy for us to check that \(\alpha_1 < \beta_1 < \alpha_2 < \beta_2\). The assumption \((H_1^B)\) implies that \(F_i(z)\) is increasing on \([\ell_{i1}/\sigma_i, \ell_{i2}/\sigma_i]\). Similarly as Theorem 1 and Theorem 2, we have the following two theorems.

**Theorem 5**
Under the assumptions \((H_1)\) and \((H_1^B)-(H_2^B)\), each \(\mathcal{X}^\Sigma\) is a positively invariant basin with respect to the solution flow generated by GNNs (1).

**Theorem 6**
Under the basic assumptions \((H_1)\) and \((H_1^B)-(H_2^B)\), for each \(\Sigma\), there exists at least one almost periodic encoded pattern of GNNs (1) in \(\mathcal{B}^\Sigma\).

**Remark 5**
From \((H_1^B)\) and piecewise linearity of activation functions in class \(\mathcal{B}\), we know that \(F_i(z)\) is strictly increasing on \([\ell_{i1}/\sigma_i, \ell_{i2}/\sigma_i]\) and is strictly decreasing on \((-\infty, \ell_{i1}/\sigma_i) \cup (\ell_{i2}/\sigma_i, +\infty)\).

By the definition of \(\hat{\ell}_{i1}, \hat{\ell}_{i2}\) and similar proof of Theorem 1, it is easy for us to know that each \(\mathcal{X}^\Sigma\) is a positively invariant basin with respect to the solution flow generated by GNNs (1). By Schauder’s fixed point theorem and positive invariancy of each \(\mathcal{X}^\Sigma\), similarly as Theorem 2, we can show that there exists at least one almost periodic encoded pattern of GNNs (1) in \(\mathcal{B}^\Sigma\).

Since each \(\mathcal{X}^\Sigma\) lies in the saturated parts to the activation functions of class \(\mathcal{B}\), we get that \(\hat{g}_i(z) = 0\) for all \(z \in [-\infty, \sigma_i\hat{\ell}_{i1}] \cup [\sigma_i\hat{\ell}_{i2}, +\infty]\), that is, \(\mathcal{C} - \mathcal{H}_G \in \mathcal{M}\) always holds. The exponential stability of almost periodic solutions of GNNs (1) follows as:

**Theorem 7**
Under the basic assumptions \((H_1)\) and \((H_1^B)-(H_2^B)\), for each \(\Sigma\), there exists a unique almost periodic encoded pattern of GNNs (1) which is exponentially stable in \(\mathcal{X}^\Sigma\).

**Remark 6**
When \(c_i, a_{ijl}, \kappa_{ijl}, J_i : \mathcal{R} \to \mathcal{R}\) are \(\omega\)-periodic functions with \(\omega > 0\), we also obtain the existence and exponential stability of \(2^n\) periodic solutions of GNNs (1). For activation functions of class \(\mathcal{B}\), we let \(\beta_{i1} := -\ell_{i1}/\sigma_i, \alpha_{i2} := \ell_{i2}/\sigma_i\). From Theorem 6 and Theorem 7, we can prove that each \(\mathcal{X}^\Sigma := \mathcal{X}_{151} \times \mathcal{X}_{232} \cdots \times \mathcal{X}_{N5N}\) is a larger attracting basin.
Now we should consider some special case of GNNs (1) and compare our results with the existing ones. When \( c_i(t) \equiv c_i, \ M = 2, \ \kappa_{ij1}(t) \equiv 0 \) and \( \sigma_j = 1 \), GNNs (1) reduces to the following GNNs considered by [20].

\[
\frac{du^j(t)}{dt} = -c_i u^j(t) + \sum_{j=1}^{N} a_{ij1}(t)g_j(u^j(t)) + \sum_{j=1}^{N} a_{ij2}(t)g_j(u^j(t - \kappa_{ij2}(t))) + J_i(t), \quad (18)
\]

where \( i \in \mathcal{J} = \{1, 2, \ldots, N\} \). From Theorem 2 to Theorem 4, it is easy for us to have the following two corollaries.

**Corollary 1**

For activation functions of class \( \mathcal{A} \), assume the following conditions hold:

\[
\begin{align*}
(A_1) & : \sup_{\zeta \in \mathcal{H}} \hat{g}_i(\zeta) < \frac{c_i}{\sigma_i \inf_{t \in \mathcal{J}} [a_{i11}(t) + a_{i12}(t)]} < \sup_{\zeta \in \mathcal{H}} \hat{g}_i(\zeta), \\
(A_2) & : (-1)^k \cdot \{ F_i(z_{ik}) + J_i(t) \} > \sum_{l=1}^{2} \sum_{j \neq l}^{N} \sup_{t \in \mathcal{J}} |a_{ijl}(t)| B_j \quad (k = 1, 2), \\
(A_3) & : \mathcal{C} - \mathcal{N} \mathcal{G} \in \mathcal{M} \quad \text{or}
\end{align*}
\]

\[
(A_3^*) : pd_i c_i > 2 \sum_{l=1}^{N} \sum_{j=1}^{N} \left[ d_j (\sup_{t \in \mathcal{J}} |a_{ijl}(t)|)^{pp_{ijl} \cdot \sigma_{ijl}/o_{ijl}} (\hat{g}_j(\xi))^q_{pp_{ijl} \cdot \sigma_{ijl}/o_{ijl}} \right] + \sum_{k=1}^{m} d_k o_k (\sup_{t \in \mathcal{J}} |a_{ijl}(t)|)^{p_{ijlk}/o_{ijl}},
\]

where \( \mathcal{C} = \text{diag}(c_1, c_2, \ldots, c_N) \), \( \hat{g}_j(\xi) = \max \{ \hat{g}_j(z) \mid z = \sigma_j \beta_{j1}, \sigma_j \alpha_{j2} \} \); \( z_{ik} \) are defined in Lemma 2; \( p > 1, d_i, a_k, q_{ijk}, \) and \( p_{ijl} \) are positive constants which satisfy with \( \sum_{k=1}^{m} a_k = p - 1, \)

\[
\sum_{k=1}^{m+1} q_{ijk} = \sum_{k=1}^{m+1} p_{ijlk} = 1 \quad \text{for each} \ j \in \mathcal{J}, \ l \in \mathcal{L}.
\]

Then there exist only \( 2^N \) almost periodic encoded patterns of GNNs (18) which are exponentially stable.

**Corollary 2**

For activation functions of class \( \mathcal{B} \), assume the following conditions hold:

\[
\begin{align*}
(B_1) & : \inf_{t \in \mathcal{J}} [a_{i11}(t) + a_{i12}(t)] > \sup_{\zeta \in \mathcal{H}} \hat{g}_i(\zeta), \quad \zeta \in [\ell_{i1}, \ell_{i2}] \\
(B_2) & : (-1)^k \cdot \left[ -c_i \frac{\ell_{ik}}{\sigma_i} + \inf_{t \in \mathcal{J}} [a_{i11}(t) + a_{i12}(t)] u_{ik} + J_i(t) \right] \\
& > \sum_{l=1}^{2} \sum_{j \neq l}^{N} \sup_{t \in \mathcal{J}} |a_{ijl}(t)| \max_{k=1,2} \{ |u_{jk}| \} \quad (k = 1, 2).
\end{align*}
\]

Then there exist only \( 2^N \) almost periodic encoded patterns of GNNs (18) which are exponentially stable.

**Remark 7**

For activation functions of class \( \mathcal{A} \), if we only assume that \( |g_i(x) - g_i(y)| \leq \tilde{L}_i|x - y| \) for all
Assume that $c_i(t) \equiv c_i$, $a_{ijl}(t) \equiv a_{ijl}$, $J_i(t) \equiv J_i$, $\sigma_j = 1$, $\kappa_{ijl}(t) \equiv 0$ and $\kappa_{ijl_2}(t) \equiv \kappa_{ijl_2}$, where $l_1 \in \mathcal{L}_1$, $l_2 \in \mathcal{L}_2$ and $\mathcal{L}_1 \cup \mathcal{L}_2 = \mathcal{L}$. Then GNNs (1) reduces to the following autonomous general neural networks including [6,8] as our special cases.

\[
\frac{du^i(t)}{dt} = -c_iu^i(t) + \sum_{l \in \mathcal{L}_1} \sum_{j=1}^{N} a_{ijl}g_j(u^j(t)) + \sum_{l \in \mathcal{L}_2} \sum_{j=1}^{N} a_{ijl}g_j(u^j(t - \kappa_{ijl})) + J_i, \quad (21)
\]

where $i \in \mathcal{I}$; From Theorem 2 to Theorem 4, it is easy for us to have the following two corollaries.

**Corollary 3**

For activation functions of class $\mathcal{A}$, assume the following conditions hold:

\[
\begin{cases}
\langle \bar{A}_1 \rangle : & \inf_{\zeta \in \mathcal{H}} \hat{g}_i(\zeta) < \frac{c_i}{\sum_{l=1}^{M} a_{ii}} < \sup_{\zeta \in \mathcal{H}} \hat{g}_i(\zeta), \\
\langle \bar{A}_2 \rangle : & (1)^k \cdot \{ F_i(z_{ik}) + J_i \} > \sum_{l=1}^{M} \sum_{j \neq i} |a_{ijl}|B_j \quad (k = 1, 2), \\
\langle \bar{A}_3 \rangle : & C - HG \in \mathcal{M} \quad \text{or} \quad \langle \bar{A}_3^* \rangle : & pd_i > \sum_{l=1}^{M} \sum_{j=1}^{N} \left[ d_j |a_{jil}|^{pp_{ijl} + 1}(\hat{g}_j(\xi))^{pp_{ijl} + 1} + \sum_{k=1}^{m} d_i |a_{ijl}|^{pp_{ijl}/\gamma_k}(\hat{g}_j(\xi))^{pp_{ijl}/\gamma_k} \right],
\end{cases}
\]

where $C = \text{diag}(c_1, c_2, \cdots, c_N)$, $\hat{g}_j(\xi) = \max \left\{ \hat{g}_j(z) \mid z = \sigma_j \beta_j, \sigma_j \alpha_j \right\}$; $p > 1$, $d_i$, $\gamma_k$, $q_{jk}$ and $p_{jk}$ are positive constants which satisfy with $\sum_{k=1}^{m} \gamma_k = p - 1$, $\sum_{k=1}^{m+1} q_{jk} = \sum_{k=1}^{m+1} p_{jk} = 1$ for each $j \in \mathcal{I}$, $l \in \mathcal{I}$. Then there exist only $2^N$ almost periodic encoded patterns of GNNs (21) which are exponentially stable.

**Corollary 4**

For the activation functions of class $\mathcal{B}$, assume the following conditions hold:

\[
\begin{cases}
\langle \bar{B}_1 \rangle : & c_i < \sum_{l=1}^{M} a_{ii} \sigma_i \hat{g}_i(\zeta), \quad \zeta \in [\ell_{i1}, \ell_{i2}] \\
\langle \bar{B}_2 \rangle : & (1)^k \cdot \left\{ -c_i \frac{\ell_{ik}}{\sigma_i} + \sum_{l=1}^{M} a_{ii} u_{ik} + J_i \right\} > \sum_{l=1}^{M} \sum_{j \neq i} a_{ijl} \cdot \max_{k=1,2} \left\{ |u_{jk}| \right\}
\end{cases}
\]

where $k = 1, 2$.

Then there exist only $2^N$ almost periodic encoded patterns of GNNs (21) which are exponentially stable.
Remark 8
In Corollary 3-4, $2^N$ almost periodic encoded patterns of GNNs (21) are indeed equilibria which are exponentially stable. We can replace activation functions of class $\mathcal{A}$ by
\[
\left\{ \begin{array}{l}
g_i \in C^2, \quad \eta_i \leq g_i(x) \leq \overline{\eta}_i, \quad \dot{g}_i(x) > 0, \\
(x - \vartheta_i)\dot{g}_i(x) < 0, \text{ for all } x \in \mathcal{R},
\end{array} \right.
\]
where $\eta_i$, $\overline{\eta}_i$ and $\vartheta_i$ are constants with $\eta_i < \overline{\eta}_i$, $i \in \mathcal{I}$. There exist $2^N$ equilibria of GNNs (21) which are exponentially stable. When $M = 2$, $\mathcal{L}_1 = \{1\}$ and $\mathcal{L}_2 = \{2\}$, the related results in [6,8] are the direct results of Corollary 3 and Corollary 4. It is obvious that our results are more general than corresponding results in [6,8].

Remark 9
Our approach can also be adapted to the following general neural networks:
\[
\frac{du^i(t)}{dt} = -c_i(t)u^i(t) + \sum_{l=1}^{M} \sum_{j=1}^{N} a_{ijl}(t)g_j\left(\sum_{l=1}^{N} K_{ijl}(s)u^j(t-s)ds\right) + J_i(t),
\]
or
\[
\frac{du^i(t)}{dt} = -c_i(t)u^i(t) + \sum_{l=1}^{M} \sum_{j=1}^{N} a_{ijl}(t)g_j\left(\sum_{l=1}^{N} \kappa_{ijl}(t)u^j(t)\right) + J_i(t),
\]
where $K_{ijl} : [0, \kappa_{ijl}] \rightarrow [0, +\infty]$ is assumed to be continuous and $0 < \int_{0}^{\kappa_{ijl}} K_{ijl}(s)ds < \infty$, $N_i(i) = \{i - l, \ldots, i + l\}$, $\kappa_{ijl} \leq +\infty$. The above general neural networks include [24,38-39,41] as special cases. Furthermore, our theory generalize stability and existence of multiple almost periodic (periodic) solutions to above general neural networks with delays. For more practical applications of multistability of neural networks, we can refer to [1-3,6-9,48-51].

5. NUMERICAL ILLUSTRATIONS

Example 1
Consider the following neural networks under almost periodic stimuli.
\[
\left\{ \begin{array}{l}
\frac{dx_1(t)}{dt} = -(1.2 + 0.2 \cos 2t)x_1(t) + 3g_1(x_1(t)) + \sin \sqrt{7}g_2(2x_2(t)) + (4 + \sin \sqrt{7})g_1(x_1(t)) + \cos \sqrt{3}g_2(2x_2(t - 9 - \sin t)) + 1.1458 \cos \sqrt{5}t, \\
\frac{dx_2(t)}{dt} = -(3 + 0.1 \sin 3t)x_2(t) + \cos \sqrt{3}g_1(x_1(t)) + 4g_2(2x_2(t)) + \sin \sqrt{5}g_1(x_1(t - 7 - 3 \cos t)) + (7 + \cos \sqrt{3}t)g_2(2x_2(t)) + 4.6679 \sin 2t,
\end{array} \right.
\]
where $g_1(\xi) = g_2(\xi) = \tanh(\xi)$, which belongs to class $\mathcal{A}$. It is easy for us to get that
\[
F_1(z) = -1.4z + 6g(z), \quad F_2(z) = -3.1z + 10g(2z).
\]
From some computations, we have $z_{11} = -1.3565$, $z_{12} = 1.3565$, $z_{21} = -0.792$, $z_{22} = 0.792$ such that $\dot{F}_1(z_{1k}) = 0$. Then $F_1(z_{1k}) = (-1)^k 3.3544$ and $F_2(z_{2k}) = (-1)^k 6.7370$ where $k = 1, 2$. From Lemma 1 and (3)-(4), we can have the following calculation result:

$$
\begin{align*}
\alpha_{11} &= -11.1458, \quad \beta_{11} = -1.8190, \quad \alpha_{12} = 1.8190, \quad \beta_{12} = 11.1458, \\
\alpha_{21} &= -6.4372, \quad \beta_{21} = -0.9095, \quad \alpha_{22} = 0.9095, \quad \beta_{22} = 6.4372.
\end{align*}
$$

It is easy for us to get

$$
\dot{g}_1(\zeta) = \max \left\{ \dot{g}_1(z) \bigg| z = \beta_{11}, \alpha_{12} \right\} = 0.1, \quad 2\dot{g}_2(\zeta) = \max \left\{ \dot{g}_2(z) \bigg| z = 2\beta_{21}, 2\alpha_{22} \right\} = 0.1.
$$

Therefore, the parameters satisfy our assumptions in Theorem 3:

Assumption $(H^A_1)$:

$$
0 < \sup_{t \in \mathbb{R}} c_1(t) \left/ \inf_{t \in \mathbb{R}} (a_{111}(t) + a_{112}(t)) \right. = 1.4/6 < 1,
$$

$$
0 < \sup_{t \in \mathbb{R}} c_2(t) \left/ \inf_{t \in \mathbb{R}} (a_{221}(t) + a_{222}(t)) \right. = 3.1/20 < 1.
$$

Assumption $(H^A_2)$:

$$
(-1)^k \cdot \{ F_1(z_{1k}) + J_1(t) \} = (-1)^k \cdot \{ (-1)^k 3.3544 + 1.1458 \cos \sqrt{t} \}
$$
$$
> 2 = \sup_{t \in \mathbb{R}} |a_{121}(t)|B_2 + \sup_{t \in \mathbb{R}} |a_{122}(t)|B_2,
$$

$$
(-1)^k \cdot \{ F_2(z_{2k}) + J_2(t) \} = (-1)^k \cdot \{ (-1)^k 6.7370 + 1.6679 \sin 2t \}
$$
$$
> 5 = \sup_{t \in \mathbb{R}} |a_{211}(t)|B_1 + \sup_{t \in \mathbb{R}} |a_{212}(t)|B_1.
$$

Assumption $(H^A_3)$:

$$
\mathcal{C} - \mathcal{H} \mathcal{G} = \begin{pmatrix} 0 & 0 \\ 0 & 2.9 \end{pmatrix} - \begin{pmatrix} 8 & 2 \\ 2 & 12 \end{pmatrix} \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix} = \begin{pmatrix} 0.2 & -0.2 \\ -0.2 & 1.7 \end{pmatrix} \in \mathcal{M}.
$$

Then there exist four almost periodic encoded patterns of (24) in $\mathcal{B}^\Sigma$ and their attracting basins are $\mathcal{X}^\Sigma$. We can compute the following regions:

Region I: $(-\infty, -1.77] \times [0.9, +\infty)$,  
Region II: $[1.77, +\infty] \times [0.9, +\infty)$,

Region III: $(-\infty, -1.77] \times (-\infty, -0.9]$,  
Region IV: $[1.77, +\infty] \times (-\infty, -0.9]$

which contain larger attracting basins mentioned in Remark 3. From Figure 1, we can see that four almost periodic encoded patterns of (24) lie in invariant regions $\mathcal{B}^{(1,2)}, \mathcal{B}^{(2,2)}, \mathcal{B}^{(1,1)}$ and $\mathcal{B}^{(2,1)}$ which borderlines are plotted in blue. Their larger attracting basins are denoted by Region I to Region IV which borderlines are plotted in black.
Example 2
Consider the following neural networks under periodic stimuli.

\[
\begin{aligned}
\frac{dx_1(t)}{dt} &= -1.4x_1(t) + 2g_1(x_1(t)) + 0.1g_2(2x_2(t)) \\
&\quad + 4g_1(x_1(t-10)) + 0.1g_2(2x_2(t-10)) + 3.1456\sin t, \\
\frac{dx_2(t)}{dt} &= -3.1x_2(t) + 0.1g_1(x_1(t)) + 3g_2(2x_2(t)) \\
&\quad + 0.1g_1(x_1(t-10)) + 7g_2(2x_2(t-10)) + 6.1705\cos t,
\end{aligned}
\]

where \(g_1(\xi) = g_2(\xi) = \tanh(\xi)\), which belongs to class \(\mathcal{A}\). Similarly as Example 1, we can check that \((H_{1A}^4)\) and \((H_{2A}^4)\) hold. From Lemma 1 and (3)-(4), we can have the following calculation result:

\[
\begin{aligned}
\alpha_{11} &= -6.6754, \quad \beta_{11} = -1.443, \quad \alpha_{12} = 1.443, \quad \beta_{12} = 6.6754, \\
\alpha_{21} &= -5.2801, \quad \beta_{21} = -1.0891, \quad \alpha_{22} = 1.0891, \quad \beta_{22} = 5.2801, \\
\dot{g}_1(\zeta) &= \max \left\{ \dot{g}_1(z) \bigg| z = \beta_{11}, \alpha_{12} \right\} = 0.2, \quad \dot{g}_2(\zeta) = \max \left\{ \dot{g}_2(z) \bigg| z = 2\beta_{21}, 2\alpha_{22} \right\} = 0.1.
\end{aligned}
\]

Let \(p = 4, \ m = 1, \ q_{11} = 3, \ q_{ik} = p_{ik} = 1/2 \ (i, l, k = 1, 2)\). From some calculations, we can check that \((H_{1A}^4)\) holds. Therefore, by Theorem 4, there exist four periodic encoded patterns of (25) in \(\mathcal{B}\). Their attracting basins follow as:
Figure 2: Convergence dynamics of four periodic encoded patterns of (25).

Region I: \((-\infty, -1.443] \times [1.0891, +\infty),\) \hspace{1cm} Region II: \([1.443, +\infty) \times [1.0891, +\infty),\)

Region III: \((-\infty, -1.443] \times (-\infty, -1.0891],\) \hspace{1cm} Region IV: \([1.443, +\infty) \times (-\infty, -1.0891].\)

We can refer to their convergence dynamics plotted in Figure 2.

Example 3
Consider the following neural networks under almost periodic stimuli.

\[
\begin{align*}
\frac{dx_1(t)}{dt} &= -(1.5 + 0.5 \cos \sqrt{7}t)x_1(t) + (4.5 + 0.5 \sin \sqrt{2}t)g_1(0.5x_1(t)) + 0.1 \sin t g_2(0.25x_2(t)) \\
&\quad + (2.5 + 0.5 \sin \sqrt{2}t)g_1(0.5x_1(t-10)) + 0.1 \cos t g_2(0.25x_2(t-10)) + 0.2 \sin 3t, \\
\frac{dx_2(t)}{dt} &= -(0.75 + 0.25 \sin t)x_2(t) + 0.2 \cos \sqrt{5}tg_1(0.5x_1(t)) + (8 + \cos \sqrt{3}tg_2(0.25x_2(t)) \\
&\quad + 0.2 \sin \sqrt{7}tg_1(0.5x_1(t-10)) + (2 + \cos \sqrt{3}tg_2(0.25x_2(t-10)) + 0.05 \cos 2t, \\
\end{align*}
\]

where \(g_1(\xi) = g_2(\xi) = \frac{1}{2}(\xi + 1) + |\xi - 1|),\) which belongs to class \(B.\) Similarly as Example 1, we can check that \((H^B_1)\) and \((H^B_2)\) hold. From Lemma 1 and (3)-(4), we can have the following calculation result:

\[
\begin{align*}
\alpha_{11} &= -8.4, \quad \beta_{11} = -2.8, \quad \alpha_{12} = 2.8, \quad \beta_{12} = 8.4, \\
\alpha_{21} &= -24.9, \quad \beta_{21} = -7.55, \quad \alpha_{22} = 7.55, \quad \beta_{22} = 24.9.
\end{align*}
\]
By Theorem 7, there exist four stable almost periodic encoded patterns of (26) in $B^\Sigma$. It is obvious that their larger attracting basins follow as:

Region I: $(-\infty, -2] \times [4, +\infty)$, Region II: $[2, +\infty] \times [4, +\infty)$,
Region III: $(-\infty, -2] \times (-\infty, -4]$, Region IV: $[2, +\infty] \times (-\infty, -4]$.

Their convergence dynamics are illustrated in Figure 3.

Example 4
Consider the following neural networks under periodic stimuli.

$$
\begin{cases}
\frac{dx_1(t)}{dt} = -(1.5 + 0.5 \cos t)x_1(t) + 10g_1(0.5x_1(t)) + 0.1g_2(0.25x_2(t)) \\
\quad + 3g_1(0.5x_1(t-10)) + 0.1g_2(0.25x_2(t-10)) + 0.2 \sin 4t, \\
\frac{dx_2(t)}{dt} = -(0.8 + 0.2 \sin t)x_2(t) + 0.2g_1(0.5x_1(t)) + 9g_2(0.25x_2(t)) \\
\quad + 0.2g_1(0.5x_1(t-10)) + 3g_2(0.25x_2(t-10)) + 0.05 \cos 8t,
\end{cases}
$$

(27)

where $g_1(\xi) = g_2(\xi) = \frac{1}{2}(|\xi + 1| + |\xi - 1|)$, which belongs to class $B$. Similarly as Example 1, we can check that $(H_1^B)$ and $(H_2^B)$ hold. By some calculations, we can have the following
result:
\[ \alpha_{11} = -13.4, \quad \beta_{11} = -6.3, \quad \alpha_{12} = 6.3, \quad \beta_{12} = 13.4, \]
\[ \alpha_{21} = -20.75, \quad \beta_{21} = -11.55, \quad \alpha_{22} = 11.55, \quad \beta_{22} = 20.75. \]

By Theorem 7, there exist four stable periodic encoded patterns of (27) in \( B^\Sigma \). Their larger attracting basins follow as:

Region I: \((-\infty, -2] \times [4, +\infty)\), Region II: \([2, +\infty] \times [4, +\infty)\),
Region III: \((-\infty, -2] \times (-\infty, -4]\), Region IV: \([2, +\infty] \times (-\infty, -4]\).

For their corresponding convergence dynamics, we can refer to Figure 4.

6. CONCLUDING REMARKS

In this paper, we investigate multi-almost periodicity of general neural networks under almost periodic stimuli. Invariant regions and attracting basins are established to investigate existence and exponential stability of \( 2^N \) almost periodic encoded patterns. Our results extend and generalize the related results reported in the literature [6,8,10,20-23].
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