Tight Kernel Bounds for Problems on Graphs with Small Degeneracy

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Abstract. Kernelization is a strong and widely-applied technique in parameterized complexity. In a nutshell, a kernelization algorithm for a parameterized problem transforms a given instance of the problem into an equivalent instance whose size depends solely on the parameter. Recent years have seen major advances in the study of both upper and lower bound techniques for kernelization, and by now this area has become one of the major research threads in parameterized complexity.

In this paper we consider kernelization for problems on \( d \)-degenerate graphs, i.e. graphs such that any subgraph contains a vertex of degree at most \( d \). This graph class generalizes many classes of graphs for which effective kernelization is known to exist, e.g. planar graphs, \( H \)-minor free graphs, and \( H \)-topological-minor free graphs. We show that for several natural problems on \( d \)-degenerate graphs the best known kernelization upper bounds are essentially tight. In particular, using intricate constructions of weak compositions, we prove that unless coNP \( \subseteq \) NP/poly:

- **Dominating Set** has no kernels of size \( O(k^{(d-1)(d-3)-\varepsilon}) \) for any \( \varepsilon > 0 \). The current best upper bound is \( O(k^{(d+1)^2}) \).
- **Independent Dominating Set** has no kernels of size \( O(k^{d-4-\varepsilon}) \) for any \( \varepsilon > 0 \). The current best upper bound is \( O(k^{d+1}) \).
- **Induced Matching** has no kernels of size \( O(k^{d-3-\varepsilon}) \) for any \( \varepsilon > 0 \). The current best upper bound is \( O(k^d) \).

To the best of our knowledge, the result on **Dominating Set** is the first example of a lower bound with a super-linear dependence on \( d \) in the exponent.

In the last section of the paper, we also give simple kernels for **Connected Vertex Cover** and **Capacitated Vertex Cover** of size \( O(k^d) \) and \( O(k^{d+1}) \) respectively. We show that the latter problem has no kernels of size \( O(k^{d+\varepsilon}) \) unless coNP \( \subseteq \) NP/poly by a simple reduction from \( d \)-**Set Cover** (a similar lower bound for **Connected Vertex Cover** is already known).

1 Introduction

Parameterized complexity is a two-dimensional refinement of classical complexity theory introduced by Downey and Fellows [13] where one takes into account not only the total input length \( n \), but also other aspects of the problem quantified in a numerical parameter \( k \in \mathbb{N} \). The main goal of the field is to determine which problems have algorithms whose exponential running time is confined strictly to the parameter. In this way, algorithms running in \( f(k) \cdot n^{O(1)} \) time for some computable function \( f() \) are considered feasible, and parameterized problems that admit feasible algorithms are said to be **fixed-parameter tractable**. This notion has proven extremely useful in identifying tractable instances for generally hard problems, and in explaining why some theoretically hard problems are solved routinely in practice.

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A closely related notion to fixed-parameter tractability is that of kernelization. A kernelization algorithm (or kernel) for a parameterized problem $L \subseteq \{0,1\}^* \times \mathbb{N}$ is a polynomial time algorithm that transforms a given instance $(x,k)$ to an instance $(x',k')$ such that: (i) $(x,k) \in L \iff (x',k') \in L$, and (ii) $|x'| + k' \leq f(k)$ for some computable function $f$. In other words, a kernelization algorithm is a polynomial-time reduction from a problem to itself that shrinks the problem instance to an instance with size depending only on the parameter. Appropriately, the function $f$ above is called the size of the kernel.

Kernelization is a notion that was developed in parameterized complexity, but it is also useful in other areas of computer science such as cryptography [22] and approximation algorithms [28]. In parameterized complexity, not only is it one of the most successful techniques for showing positive results, it also provides an equivalent way of defining fixed-parameter tractability: A decidable parameterized problem is solvable in $f(k) \cdot n^{O(1)}$ time iff it has a kernel [6]. From practical point of view, compression algorithms often lead to efficient preprocessing rules which can significantly simplify real life instances [16, 20]. For these reasons, the study of kernelization is one of the leading research frontiers in parameterized complexity. This research endeavor has been fueled by recent tools for showing lower bounds on kernel sizes [2, 4, 5, 7, 9, 10, 12, 23, 27] which rely on the standard complexity-theoretic assumption of $\text{coNP} \not\subseteq \text{NP/poly}$.

Since a parameterized problem is fixed-parameter tractable iff it is kernelizable, it is natural to ask which fixed-parameter problems admit kernels of reasonably small size. In recent years there has been significant advances in this area. One particularly prominent line of research in this context is the development of meta-kernelization algorithms for problems on sparse graphs. Such algorithms typically provide compressions of either linear or quadratic size to a wide range of problems at once, by identifying certain generic problem properties that allow for good compressions. The first work in this line of research is due to Guo and Niedermeier [21], which extended the ideas used in the classical linear kernel for DS in planar graphs [1] to linear kernels for several other planar graph problems. This result was subsumed by the seminal paper of Bodlaender et al. [3], which provided meta-kernelization algorithms for problems on graphs of bounded genus, a generalization of planar graphs. Later Fomin et al. [18] provided a meta-kernel for problems on $H$-minor free graphs which include all bounded genus graphs. Finally, a recent manuscript by Langer et al. [26] provides a meta-kernelization algorithm for problems on $H$-topological-minor free graphs. All meta-kernelizations above have either linear or quadratic size.

How far can these meta-kernelization results be extended? A natural class of sparse graphs which generalizes all graph classes handled by the meta-kernelizations discussed above is the class of $d$-degenerate graphs. A graph is called $d$-degenerate if each of its subgraphs has a vertex of degree at most $d$. This is equivalent to requiring that the vertices of the graph can be linearly ordered such that each vertex has at most $d$ neighbors to its right in this ordering. For example, any planar graph is 5-degenerate, and for any $H$-minor (resp. $H$-topological-minor) free graph class there exists a constant $d(H)$ such that all graphs in this class are $d(H)$-degenerate (see e.g. [11]). Note that the Independent Set problem has a trivial linear kernel in $d$-degenerate graphs, which gives some hope that a meta-kernelization result yielding small degree polynomial kernels might be attainable for this graph class.

Arguably the most important kernelization result in $d$-degenerate graphs is due to Philip et al. [29] who showed a $O(k(d+1)^2)$ size kernel for Dominating Set, and an $O(k^{d+1})$ size kernel for Independent Dominating Set. Erman et al. [14] and Kanj et al. [24] independently gave a $O(k^d)$ kernel for the Induced Matching problem, while Cygan et al. [8] showed a $O(k^{d+1})$
kernel is for Connected Vertex Cover. While all these results give polynomial kernels, the exponent of the polynomial depends on $d$, leaving open the question of kernels of polynomial size with a fixed constant degree. This question was answered negatively for ConVC in [8] using the standard reduction from $d$-Set Cover. It is also shown in [8] that other problems such as Connected Dominating Set and Connected Feedback Vertex Set do not admit a kernel of any polynomial size unless $\text{coNP} \subseteq \text{NP/poly}$. Furthermore, the results in [9, 23] can be easily used to show exclude a $O(k^{d-\varepsilon})$-size kernel for Dominating Set, for some small positive constant $\varepsilon$.

Our results: In this paper, we show that all remaining kernelization upper bounds for $d$-degenerate graphs mentioned above have matching lower bounds up to some small additive constant. Perhaps the most surprising result we obtain is the exclusion of $O(k^{(d-3)(d-1)-\varepsilon})$ size kernels for Dominating Set for any $\varepsilon > 0$, under the assumption of $\text{coNP} \not\subseteq \text{NP/poly}$. This result is obtained by an intricate application of weak compositions which were introduced by [10], and further applied in [9, 23]. What makes this result surprising is that it implies that Independent Dominating Set is fundamentally easier than Dominating Set in $d$-degenerate graphs. We also show a $O(k^{d-4-\varepsilon})$ lower bound for Independent Dominating Set, and an $O(k^{d-3-\varepsilon})$ lower bound for Induced Matching. The latter result is also somewhat surprising when one considers the trivial linear kernel for the closely related Independent Set problem. Finally, we slightly improve the $O(k^{d+1})$ kernel for Connected Vertex Cover of [8] to $O(k^{d})$, and show that the related Capacitated Vertex Cover problem has a kernel of size $O(k^{d+1})$, but no kernel of size $O(k^{d-\varepsilon})$ unless $\text{coNP} \subseteq \text{NP/poly}$. Table 1 summarizes the currently known state of the art of kernel sizes for the problems considered in this paper.

| Problem                        | Lower Bound       | Upper Bound       |
|--------------------------------|-------------------|-------------------|
| Dominating Set                 | $(d - 3)(d - 1) - \varepsilon$ | $(d + 1)^* [29]$ |
| Independent Dominating Set     | $d - 4 - \varepsilon$ | $d + 1 [29]$     |
| Induced Matching               | $d - 3 - \varepsilon$ | $d [14, 24]$     |
| Connected Vertex Cover         | $d - 1 - \varepsilon [8]$ | $d$              |
| Capacitated Vertex Cover       | $d - \varepsilon$  | $d + 1$           |

Table 1. Lower and upper bounds for kernel sizes for problems in $d$-degenerate graphs. Only the exponent of the polynomial in $k$ is given. Results without a citation are obtained in this paper.

2 Kernelization Lower Bounds

In the following section we quickly review the main tool that we will be using for showing our kernelization lower bounds, namely compositions. A composition algorithm is typically a transformation from a classical NP-hard problem $L_1$ to a parameterized problem $L_2$. It takes as input a sequence of $T$ instances of $L_1$, each of size $n$, and outputs in polynomial time an instance of $L_2$ such that (i) the output is a YES-instance iff one of the inputs is a YES-instance, and (ii) the parameter of the output is polynomially bounded by $n$ and has only “small” dependency on $T$. Thus, a composition may intuitively be thought of as an “OR-gate” with a guarantee bound on the parameter of the output. More formally, for an integer $d \geq 1$, a weak $d$-composition is defined as follows:
**Definition 1 (weak d-composition).** Let \( d \geq 1 \) be an integer constant, let \( L_1 \subseteq \{0,1\}^* \) be a classical (non-parameterized) problem, and let \( L_2 \subseteq \{0,1\}^* \times \mathbb{N} \) be a parameterized problem. A weak d-composition from \( L_1 \) to \( L_2 \) is a polynomial time algorithm that on input \( x_1, \ldots, x_d \in \{0,1\}^n \) outputs an instance \((y,k') \in \{0,1\}^* \times \mathbb{N}\) such that:

- \((y,k') \in L_2 \iff x_i \in L_1 \) for some \( i \), and
- \( k' \leq t \cdot n^{O(1)} \).

The connection between compositions and kernelization lower bounds was discovered by [2] using ideas from [22] and a complexity theoretic lemma of [19]. The following particular connection was first observed in [10].

**Lemma 1 ([10]).** Let \( d \geq 1 \) be an integer, let \( L_1 \subseteq \{0,1\}^* \) be a classical NP-hard problem, and let \( L_2 \subseteq \{0,1\}^* \times \mathbb{N} \) be a parameterized problem. A weak-d-composition from \( L_1 \) to \( L_2 \) implies that \( L_2 \) has no kernel of size \( O(k^{d-\varepsilon}) \) for any \( \varepsilon > 0 \), unless coNP \( \subseteq \text{NP/poly} \).

**Remark 1.** Lemma 1 also holds for compressions, a stronger notion of kernelization, in which the reduction is not necessarily from the problem to itself, but rather from the problem to some arbitrary set.

3 Dominating Set

We begin by considering the Dominating Set (DS) problem. In this problem, we are given an undirected graph \( G = (V,E) \) together with an integer \( k \), and we are asked whether there exists a set \( S \) of at most \( k \) vertices such that each vertex of \( G \) either belongs to \( S \) or has a neighbor in \( S \) (i.e. \( N[S] = V \)). The main result of this section is stated in Theorem 1 below.

**Theorem 1.** Let \( d \geq 4 \). The **Dominating Set** problem in \( d \)-degenerate graphs has no kernel of size \( O(k^{(d-1)(d-3)-\varepsilon}) \) for any constant \( \varepsilon > 0 \) unless NP \( \subseteq \text{coNP/poly} \).

In order to prove Theorem 1, we show a lower bound for a similar problem called the Red Blue Dominating Set problem (RBDS): Given a bipartite graph \( G = (R \cup B, E) \) and an integer \( k \), where \( R \) is the set of red vertices and \( B \) is the set of blue vertices, determine whether there exists a set \( D \subseteq R \) of at most \( k \) red vertices which dominate all the blue vertices (i.e. \( N(D) = B \)). According to Remark 1, the lemma below shows that focusing on RBDS is sufficient for proving Theorem 1 above.

**Lemma 2.** There is a polynomial time algorithm, which given a \( d \)-degenerate instance \( I = (G = (R \cup B, E), k) \) of RBDS creates a \((d+1)\)-degenerate instance \( I' = (G',k') \) of DS, such that \( k' = k + 1 \) and \( I \) is YES-instance iff \( I' \) is a YES-instance.

**Proof.** As the graph \( G' \) we initially take \( G = (R \cup B, E) \) and then we add two vertices \( r, r' \) and make \( r \) adjacent to all the vertices in \( R \cup \{r'\} \). Clearly \( G' \) is \((d+1)\)-degenerate. Note that if \( S \subseteq R \) is a solution in \( I \), then \( S \cup \{r\} \) is a dominating set in \( I' \). In the reverse direction, observe that w.l.o.g. we may assume that a solution \( S' \) for \( I' \) contains \( r \) and moreover contains no vertex of \( B \). Therefore \( I \) is a YES-instance iff \( I' \) is a YES-instance. \( \qed \)
We next describe a weak $d(d + 2)$-composition from MULTICOLORED PERFECT MATCHING to RBDS in $(d + 2)$-degenerate graphs. The MULTICOLORED PERFECT MATCHING problem (MPM) is as follows: Given an undirected graph $G = (V, E)$ with even number $n$ of vertices, together with a color function $\text{col} : E \rightarrow \{0, \ldots, n/2 - 1\}$, determine whether there exists a perfect matching in $G$ with all the edges having distinct colors. A simple reduction from 3-DIMENSIONAL PERFECT MATCHING, which is NP-complete due to Karp [25], where we encode one coordinate using colors, shows that MPM is NP-complete when we consider multigraphs. In the following lemma we show that MPM is NP-complete even for simple graphs.

**Lemma 3.** The MPM problem is NP-complete.

**Proof.** We show a polynomial time reduction from MPM in multigraphs, where several parallel edges are allowed. Let $(G = (V, E), \text{col} : V \rightarrow \{0, \ldots, |V|/2 - 1\})$ be an instance of MPM where $G$ is a multigraph, $|V|$ is even and $\pi : E \rightarrow \{0, \ldots, |E| - 1\}$ is an arbitrary bijection between the multiset of edges and integers from 0 to $|E| - 1$. Moreover we assume that there is some fixed linear order on $V$. We create a graph $G'$ as follows. The set of vertices of $G'$ is $V' = \{v : v \in V\} \cup \{y_{v,e} : e \in E, v \in V\}$, that is we create a vertex in $G'$ for each vertex of $G$ and for each endpoint of an edge of $G$. The set of edges of $G'$ is $E' = \{y_{u,e}y_{v,e} : e = uv \in E\} \cup \{x_{v,e}y_{v,e} : v \in E, e \in E, v \in e\}$. Finally, we define the color function $\text{col}' : E' \rightarrow \{0, \ldots, n'/2 - 1\}$, where $n' = 2|E| + |V|$. For $y_{u,e}y_{v,e} \in E'$ we set $\text{col}'(y_{u,e}y_{v,e}) = \pi(e)$, for $x_{v,e}y_{v,e} \in E'$, where $e = uv$ and $u < v$ we set $\text{col}'(x_{v,e}y_{v,e}) = \pi(e)$, while for $x_{v}y_{v,e} \in E'$, where $e = uv$ and $u > v$ we set $\text{col}'(x_{v}y_{v,e}) = |E| + \text{col}(e)$.

Clearly the construction can be performed in polynomial time, and the graph $G'$ is simple, hence it suffices to show that the instance $(G', \text{col}')$ is a YES-instance iff $(G, \text{col})$ is a YES-instance. Let $M \subseteq E$ be a solution for $(G, \text{col})$. Observe that the set $M' = \{y_{u,e}y_{v,e} : e = uv \in M\} \cup \{x_{v,e}y_{v,e} : e \in E \setminus M, v \in e\}$ is a solution for $(G', \text{col}')$.

In the other direction, assume that $M' \subseteq E'$ is a solution for $(G', \text{col}')$. Note that for each $e = uv \in E$ either we have $y_{u,e}y_{v,e} \in M'$ or both $x_{u}y_{u,e}$ and $x_{v}y_{v,e}$ belong to $M'$. Consequently we define $M \subseteq E$ to be the set of edges $e = uv$ of $E$ such that $y_{u,e}y_{v,e} \notin M'$. It is easy to verify that $M$ is a solution for $(G, \text{col})$.

The construction of the weak composition is rather involved. We construct an instance graph $H_{\text{inst}}$ which maps feasible solutions of each MPM instance into feasible solutions of the RBDS instance. Then we add an enforcement gadget $(H_{\text{enf}}, E_{\text{conn}})$ which prevents partial solutions of two or more MPM instances to form altogether a solution for the RBDS instance. The overall RBDS instance will be denoted by $(H, k)$, where $H$ is the union of $H_{\text{inst}}$ and $H_{\text{enf}}$ along with the edges $E_{\text{conn}}$ that connect between these graphs. The construction of the instance graph is relatively simple, while the enforcement gadget is rather complex. In the next subsection we describe $H_{\text{enf}}$ and its crucial properties. In the following subsection we describe the rest of the construction, and prove the claimed lower bound on RBDS (and hence DS). Both $H_{\text{enf}}$ and $H_{\text{inst}}$ contain red and blue nodes. We will use the convention that $R$ and $B$ denote sets of red and blue nodes, respectively. We will use $r$ and $b$ to indicate red and blue nodes, respectively. A color is indicated by $\ell$.

### 3.1 The Enforcement Graph

The enforcement graph $H_{\text{enf}} = (R_{\text{enf}} \cup B_{\text{enf}}, E_{\text{enf}})$ is a combination of 3 different gadgets: the encoding gadget, the choice gadget, and the fillin gadget (see also Fig. 1), i.e. $R_{\text{enf}} = R_{\text{code}} \cup R_{\text{fill}}$.
and \( B_{enf} = B_{code} \cup B_{choice} \cup B_{fill} \) (\( R_{choice} \) is empty).

**Encoding gadget:** The role of this gadget is to encode the indices of all the instances by different partial solutions. It consists of nodes \( R_{code} \cup B_{code} \), plus the edges among them. The set \( R_{code} \) contains one node \( r_{\delta,\lambda,\gamma} \) for all integers \( 0 \leq \delta < d+2, 0 \leq \lambda < d, \) and \( 0 \leq \gamma < t \). In particular, \(|R_{code}| = (d + 2)dt\). The set \( B_{code} \) is the union of sets \( B_{\ell}^{\ell} \) for each color \( 0 \leq \ell < n/2 \). In turn, \( B_{\ell}^{\ell} \) contains a node \( b_{a}^{\ell} \) for each integer \( 0 \leq a < (dt)^{d+2} \). We connect nodes \( r_{\delta,\lambda,\gamma} \) and \( b_{a}^{\ell} \) iff \( a_{\delta} = \lambda \cdot t + \gamma \), where \((a_{0}, \ldots, a_{d+1})\) is the expansion of \( a \) in base \( dt \), i.e. \( a = \sum_{0 \leq \delta < d+2} a_{\delta} (dt)^{\delta} \).

There is a subtle reason behind this connection scheme, which hopefully will be clearer soon. Note that since \( 0 \leq \gamma < t \), pairs \((\lambda, \gamma)\) are in one to one correspondence with possible values of digits \( a_{\delta} \).

**Choice gadget:** The role of the choice gadget is to guarantee the following choice property: Any feasible solution to the overall RBDS instance \((H, k)\) contains all nodes \( R_{code} \) except possibly one node \( r_{\delta,\lambda,\gamma,\delta,\lambda} \) for each pair \((\delta, \lambda)\) (hence at least \((d+2)d(t-1)\) nodes of \( R_{code} \) altogether are taken). Intuitively, the \( \gamma_{\delta,\lambda} \)'s will be used to identify the index of one MPM input instance. In order to do that, we introduce a set of nodes \( B_{choice} \), containing a node \( b_{\delta,\lambda,\gamma,\gamma,\gamma} \) for every pair \((\delta, \lambda)\) and for every \( 0 \leq \gamma_1 < \gamma_2 < t \). We connect \( b_{\delta,\lambda,\gamma,\gamma,\gamma} \) to both \( r_{\delta,\lambda,\gamma,\gamma} \) and \( r_{\delta,\lambda,\gamma,\gamma} \). It is not hard to see that, in order to dominate \( B_{choice} \), it is necessary and sufficient to select from \( R_{code} \) a subset of nodes with the choice property.

**Fillin gadget:** We will guarantee that, in any feasible solution, precisely \((d+2)d(t-1)\) nodes from \( R_{code} \) are selected. Given that, for each pair \((\delta, \lambda)\), there will be precisely one node \( r_{\delta,\lambda,\gamma,\delta,\lambda} \) which is not included in the solution. Consequently, as we will prove, for each \( 0 \leq \ell < n/2 \) in \( B_{\ell}^{\ell} \) there will be exactly \( d^{d+2} \) uncovered nodes, namely the nodes \( b_{a}^{\ell} = b_{(a_{0}, \ldots, a_{d+1})}^{\ell} \) such that for each \( 0 \leq \delta < d+2 \) and \( \lambda \leq a_{\delta} < (\lambda+1)t \) one has \( a_{\delta} = \lambda t + \gamma_{\delta,\lambda} \). Ideally, we would like to cover such nodes by means of red nodes in the instance graph \( H_{inst} \) (to be defined later), which encode a feasible solution to some MPM instance. However, the degeneracy of the overall graph would be too large. The role of the fillin gadget is to circumvent this problem, by leaving at most \( d \) uncovered nodes in each \( B_{\ell}^{\ell} \). The fillin gadget consists of nodes \( R_{fill} \cup B_{fill} \), with some edges incident to them. The set \( R_{fill} \) is the union of sets \( R_{\ell}^{\ell} \) for each color \( \ell \). In turn \( R_{\ell}^{\ell} \) contains one node \( r_{a,j}^{\ell} \) for each \( 1 \leq j \leq d^{d+2} - d \) and \( 0 \leq a < (dt)^{d+2} \). We connect each \( r_{a,j}^{\ell} \) to \( b_{a}^{\ell} \). The set \( B_{fill} \) contains one node \( b_{j}^{\ell} \), for each color \( \ell \) and for all \( 1 \leq j \leq d^{d+2} - d \). We connect \( b_{j}^{\ell} \) to all nodes \( \{r_{a,j}^{\ell} : 0 \leq a < (dt)^{d+2}\} \). Observe that, in order to cover \( B_{fill} \), it is necessary and sufficient to select one node \( r_{a,j}^{\ell} \) for each \( \ell \) and \( j \). Furthermore, there is a way to do that such that each selected \( r_{a,j}^{\ell} \) covers one extra node in \( B_{\ell}^{\ell} \) w.r.t. selected nodes in \( R_{code} \). Note that we somewhat abuse notation as we denote by \( b_{j}^{\ell} \) vertices of \( B_{fill} \), while we use \( b_{a}^{\ell} \) for vertices of \( B_{code} \), hence the only distinction is by the variable name.

**Lemma 4.** For any matrix \((\gamma_{\delta,\lambda})_{0 \leq \delta < d+2, 0 \leq \lambda < d}\) of size \((d+2) \times d\) with entries from \(\{0, \ldots, t-1\}\), there exists a set \( \tilde{R}_{enf} \subseteq R_{enf} \) of size \( k' := \frac{n}{2}(d^{d+2} - d) + (d+2)d(t-1) \), such that:

- each vertex in \( B_{choice} \cup B_{fill} \) has a neighbor in \( \tilde{R}_{enf} \), and
- for every \( 0 \leq \ell < n/2 \) we have \( B_{\ell}^{\ell} \cup N(\tilde{R}_{enf}) = \{b_{a}^{\ell} : 0 \leq \lambda < d, a = \sum_{0 \leq \delta < d+2} (\lambda t + \gamma_{\delta,\lambda}) (dt)^{\delta}\} \).

**Proof.** For each \( 0 \leq \delta < d+2 \) and \( 0 \leq \lambda < d \), add to \( \tilde{R}_{enf} \) the set \( \{r_{\delta,\lambda,\gamma} : 0 \leq \gamma < t, \gamma \neq \gamma_{\delta,\lambda}\} \) containing \( t-1 \) vertices. Note that by construction \( \tilde{R}_{enf} \) dominates the whole set \( B_{choice} \). Consider
a vertex $b'_a \in B^\ell_{\text{code}} \setminus N(\tilde{R}_{enf})$ and observe that for each coordinate $0 \leq \delta < d + 2$, there are exactly $d$ values that $a_\delta$ can have, where $(a_0, \ldots, a_{d+1})$ is the $(dt)$-ary representation of $a$. Indeed, for any $0 \leq \delta < d + 2$, we have $a_\delta \in X_\delta = \{\lambda t + \gamma_\delta, \lambda : 0 \leq \lambda < d\}$, since otherwise $b'_a$ would be covered by $\tilde{R}_{enf}$ due to the $\delta$-th coordinate. Moreover if we consider any $b'_a(a_0, \ldots, a_{d+1}) \in B^\ell_{\text{code}}$ such that $a_\delta \in X_\delta$ for $0 \leq \delta < d + 2$, then $b'_a(a_0, \ldots, a_{d+1})$ is not dominated by the vertices added to $\tilde{R}_{enf}$ so far.

Next, for each $\ell$ define $M^\ell := \{b'_a : 0 \leq \lambda < d, a = \sum_{0 \leq \delta \leq d+2}(\lambda t + \gamma_\delta)(dt)^\delta\}$ and observe that $M^\ell$ are not dominated by $\tilde{R}_{enf}$. For each $0 \leq \ell < n/2$, let $Z^\ell$ be the vertices of $B^\ell_{\text{code}}$ not yet covered by $\tilde{R}_{enf}$ and for each $1 \leq j \leq d^{d+2} - d$ select exactly one distinct vertex $v_j \in Z^\ell \setminus M^\ell$, where $v_j = b'_a$, and add to $\tilde{R}_{enf}$ the vertex $r'_{a,j}$. Observe that after this operation $\tilde{R}_{enf}$ covers $B_{\text{fill}}$ and moreover the only vertices of $B_{\text{code}}$ not covered by $\tilde{R}_{enf}$ are the vertices of $\bigcup_{0 \leq \ell < n/2} M^\ell$. Since the total size of $\tilde{R}_{enf}$ equals $d(d + 2)(t - 1) + \frac{2}{t}(d^{d+2} - d)$, the lemma follows.

\begin{lemma}
Consider an RBDS instance $(H = (R \cup B,E), k)$ containing $G_{enf} = (R_{enf} \cup B_{enf}, E_{enf})$ as an induced subgraph, with $R_{enf} \subseteq R$ and $B_{enf} \subseteq B$, such that no vertex of $B_{\text{choice}} \cup B_{\text{fill}}$ has a neighbor outside of $R_{enf}$. Then any feasible solution $\tilde{R}$ to $(H,k)$ contains at least $k' := \frac{2}{t}(d^{d+2} - d) + (d + 2)(t - 1)$ nodes $\tilde{R}_{enf}$ of $R_{enf}$. Furthermore, for any feasible solution $\tilde{R}$ to $(H,k)$ containing exactly $k'$ vertices of $R_{enf}$, there exists a matrix $(\gamma_{\delta,\lambda})_{0 \leq \delta < d + 2, 0 \leq \lambda < d}$ of size $(d + 2) \times d$ with entries from $\{0, \ldots, t - 1\}$, such that for each $0 \leq \ell < n/2$:

(a) there are at least $d$ vertices in $U^\ell = B^\ell_{\text{code}} \setminus N(\tilde{R} \cap R_{enf})$, and
(b) $U^\ell$ is a subset of the $d^{d+2}$ nodes $b'_a(a_0, \ldots, a_{d+1})$ such that for each $\delta \in \{0, \ldots, d + 1\}$ there exists $\lambda \in \{0, \ldots, d - 1\}$ with $a_\delta = \lambda + \gamma_{\delta,\lambda}$.

\end{lemma}

\begin{proof}
Let $\tilde{R}$ be any feasible solution to $(H,k)$. Observe that since $\tilde{R}$ dominates $B_{\text{choice}}$, for each $0 \leq \delta < d + 2$ and $0 \leq \lambda < d$ we have $|\tilde{R} \setminus \{r_{\delta,\lambda,\gamma} : 0 \leq \gamma < t\}| \geq t - 1$. Moreover in order to dominate vertices of $B_{\text{fill}}'$, the set $\tilde{R}$ has to contain at least $n/2(d^{d+2} - d)$ vertices of $R_{\text{fill}}$. Consequently, if $\tilde{R}$ contains exactly $k'$ vertices of $R_{enf}$, then for each $0 \leq \delta < d + 2$ and $0 \leq \lambda < d$, there is exactly one $\gamma_{\delta,\lambda}$ such that $r_{\delta,\lambda,\gamma_{\delta,\lambda}} \notin \tilde{R}$. By the same argument as in the proof of Lemma 4, we infer that for each $\ell$, the set $B^\ell_{\text{code}} \setminus N(\tilde{R} \cap R_{\text{code}})$ contains exactly $d^{d+2}$ vertices, and we denote them as $U^\ell$. Observe that the set $\tilde{R} \setminus R_{\text{code}}$ dominates at most $d^{d+2} - d$ vertices of $U^\ell$, for each $0 \leq \ell < n/2$, which proves properties (a) and (b) of the lemma.

\end{proof}

### 3.2 The Overall Graph

The construction of $H_{\text{inst}} = (R_{\text{inst}} \cup B_{\text{inst}}, E_{\text{inst}})$ is rather simple. Let $(G_i = (V, E_i), \text{col}_i)$ be the input MPM instances, with $0 \leq i < T = t(d+2)$. By standard padding arguments we may assume that all the graphs $G_i$ are defined over the same set $V$ of even size $n$, i.e. $G_i = (V, E_i)$. For each $v \in V$, we create a blue node $b_v \in B_{\text{inst}}$. For each $e_i = \{u, v\} \in E_i$, we create a red node $r_{e,i} \in R^i_{\text{inst}}$ and connect it to both $b_u$ and $b_v$. We let $R_{\text{inst}} := \bigcup_{0 \leq i < T} R^i_{\text{inst}}$. Intuitively, we desire that a RBDS solution, if any, selects exactly $n/2$ nodes from one set $R^i_{\text{inst}}$, corresponding to edges of different colors, which together dominate all nodes $B_{\text{inst}}$. This induces a feasible solution to MPM for the $i$-th instance.

It remains to describe the edges $E_{\text{con}}$ which connect $H_{enf}$ with $H_{\text{inst}}$. This is the most delicate part of the entire construction. We map each index $i$, $0 \leq i < T$, into a distinct $(d+2) \times d$ matrix $M_i$ with entries $M_i[\delta, \lambda] \in \{0, \ldots, t - 1\}$, for all possible values of $\delta$ and $\lambda$. Consider an instance $G_i$. We
connect $r_{e,i}$ to $b^i_\ell$ iff $\ell = \text{col}_i(e_i)$ and there exists $0 \leq \lambda < d$ such that the expansion $(a_0, \ldots, a_{d+1})$ of $a$ in base $dt$ satisfies $a_\delta = M_i[\delta, \lambda] + \lambda \cdot t$ on each coordinate $0 \leq \delta < d + 2$. The final graph $H := (R \cup B, E)$ we construct for our instance RBDS is then given by $R := R_{\text{inst}} \cup R_{\text{enf}}$ and $E := E_{\text{inst}} \cup E_{\text{enf}} \cup E_{\text{conn}}$. See Fig. 1.

Fig. 1. Construction of the graph $H$. For simplicity the figure does not include sets $R'_{\text{fill}}$ and $B'_{\text{code}}$ for $\ell' \neq \ell$.

Lemma 6. $H$ is $(d + 2)$-degenerate.

Proof. Observe that each vertex of $\bigcup_{0 \leq i < T} R^i_{\text{inst}}$ is of degree exactly $d + 2$ in $H$, since it is adjacent to exactly two vertices of $B_{\text{inst}}$ and exactly $d$ vertices of the enforcement gadget, so we put all those vertices first to our ordering. Next, we take vertices of $B_{\text{inst}}$, as those have all neighbors already put into the ordering. Therefore it is enough to argue about the $(d + 2)$-degeneracy of the enforcement gadget. We order vertices of $R_{\text{fill}} \cup B_{\text{choice}}$, since those are of degree exactly two in $H$. In $H \setminus R_{\text{fill}}$ the vertices of $B_{\text{fill}}$ become isolated, so we put them next to our ordering. We are left with the vertices of the encoding gadget. Observe, that each blue vertex of the encoding gadget has exactly $d + 2$ neighbors in $R_{\text{code}}$, one due to each coordinate, hence we put the vertices of $B_{\text{code}}$ next and finish the ordering with vertices of $R_{\text{code}}$. \hfill $\square$

Lemma 7. Let $k := (d + 2)d(t - 1) + n/2(d^{d+2} - d) + n/2 = k' + n/2$. Then $(H, k)$ is a YES-instance of RBDS iff $(G_i, \text{col}_i)$ is a YES-instance of MPM for some $i \in \{0, \ldots, T - 1\}$.

Proof. Let us assume that for some $i_0$ the instance $(G_{i_0}, \text{col}_{i_0})$ is a YES-instance and $E' \subseteq E_{i_0}$ is the corresponding solution. We use Lemma 4 with the matrix $M_{i_0}$ assigned to the instance $i_0$ to obtain the set $\tilde{R}_{\text{enf}}$ of size $(d + 2)d(t - 1) + n/2(d^{d+2} - d)$. As the set $\tilde{R}$ we take $\tilde{R}_{\text{enf}} \cup \{r_{e,i_0} : e \in E'\}$. 

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Clearly $|\tilde{R}| = k$. Since $E'$ is a perfect matching, $\tilde{R}$ dominates $B_{\text{inst}}$. By Lemma 4, $\tilde{R}$ dominates $B_{\text{fill}} \cup B_{\text{choice}}$ and all but $d$ vertices of each $B_{\text{code}}$, so denote those $d$ vertices by $M$. Consider each $0 \leq \ell < n/2$, and observe that since $E'$ is multicolored and by the construction of $H$, the set of neighbors of $r_{e,i}$ in $B_{\text{code}}$ is exactly $M_{\text{col}}(e)$; and hence $\tilde{R}$ is a solution for $(H,k)$.

In the other direction, assume that $(H,k)$ is a YES-instance and let $\tilde{R}$ be a solution of size at most $k$. By Lemma 5, the set $\tilde{R}$ contains at least $k' = \frac{n}{2} (d^\ell + d) + (d + 2) (d - 1)$ vertices of $R_{\text{enf}}$ and since $\tilde{R}$ needs to dominate also $B_{\text{inst}}$ it contains at least $\frac{n}{2} \ell$ vertices of $\bigcup_{0 \leq i < T} R_{\text{inst}}^i$, since no vertex of $H$ dominates more than two vertices of $B_{\text{inst}}$. Consequently $|\bigcup_{0 \leq i < T} R_{\text{inst}}^i \cap \tilde{R}| = n/2$ and $|R_{\text{enf}} \cap \tilde{R}| = k'$. We use Lemma 5 to obtain a matrix $M = (\gamma_{\delta,\lambda})$ of size $(d + 2) \times d$. Moreover, by property (a) of Lemma 5, there are at least $d$ vertices in $U^\ell$, and consequently for each color $\ell$ the set $\tilde{R}$ contains exactly one vertex of the set $\{ r_{e,i} : 0 \leq i < T, \text{col}_i(e) = \ell \}$. Our goal is to show that for each $0 \leq i < T$, such that a matrix different than $M$ is assigned to the $i$-th instance, we have $\tilde{R} \cap R_{\text{inst}}^i = \emptyset$, which is enough to finish the proof of the lemma. Consider any such $i$ and assume that the matrices $M_i$ and $M$ differ in the entry $M_i[\delta',\lambda'] \neq \gamma_{\delta',\lambda}$. Let $\ell$ be a color such that $r_{e,i} \in \tilde{R}$ and col$_i(e) = \ell$. By property (b) of Lemma 5, the set of at least $d$ vertices of $B_{\text{code}}$ not dominated by $\tilde{R} \cap R_{\text{enf}}$ is contained in $U_0^\ell = \{ b_{(a_0,\ldots,a_{d+1})}^\ell : \forall 0 \leq \delta < d+2 : \text{if } \lambda t \leq a_\delta < (\lambda + 1) t \text{ then } a_\delta = \lambda t + \gamma_{\delta,\lambda} \}$. However, by our construction of edges of $H$ between $R_{\text{inst}}^i$ and $B_{\text{code}}$, we have $(N_H(r_{e,i}) \cap B_{\text{code}}) \subsetneq U_0^\ell$ since the vertex $b_{(a_0,\ldots,a_{d+1})}^\ell \in N_H(r_{e,i}) \cap B_{\text{code}}$ with $\lambda t = \lambda t + M_i[\delta',\lambda']$ does not belong to $U_0^\ell$ and consequently does not belong to $U^\ell$, which leaves at least one vertex of $B_{\text{code}}$ not dominated by $\tilde{R}$; a contradiction. □

Lemmas 6 and 7 imply that, for any $d \geq 1$, there exists a weak $d(d + 2)$-composition from MPM to RBDS in $(d + 2)$-degenerate graphs. The proof of Theorem 1 thus follows from Lemmas 1 and 2.

4 Independent Dominating Set

The Independent Dominating Set (IDS) problem is the variant of DS where we require the dominating set $S$ to induce an independent set (i.e., nodes in $S$ have to be pairwise non-adjacent). We next describe a weak $d$-composition from 3-Exact Set Cover, to IDS in $(d + 4)$-degenerate graphs.

The input of 3-Exact Set Cover is a set system $(U,F)$, where each set in $F$ contains exactly three elements and the question is whether there is a collection $S \subseteq F$ of disjoint sets which partition $U$, i.e. $\bigcup S = U$. The 3-Exact Set Cover problem is NP-complete by a reduction from 3-Dimensional Perfect Matching, which is NP-complete due to Karp [25].

Consider a fixed value of $d \geq 1$ and let $(F_0,U_0),\ldots,(F_{T-1},U_{T-1})$ be $T := d^4$ instances of 3-Exact Set Cover. Without loss of generality we can assume that in each instance the same universe $U$ of size $n$ is used, such that $n \equiv 0 \pmod{3}$, since if for some $i$ we have $|U_i| \neq 0 \pmod{3}$, then $(F_i,U_i)$ is clearly a NO-instance, and moreover we can pad each universe with additional triples to make sure that each $U_i$ has the same cardinality. Therefore we assume that for each $i = 0,\ldots,T - 1$ we have $U_i = U$ and $|U| = n$.

We construct an instance $(H = (V,E),k)$ of IDS for a properly chosen parameter $k$. Similar to construction in Section 3, the graph $H$ will contain two graphs, the enforcement graph $H_{\text{enf}} = (V_{\text{enf}},E_{\text{enf}})$ and the instance graph $H_{\text{inst}} = (V_{\text{inst}},E_{\text{inst}})$, plus some edges $E_{\text{conn}}$ connecting them (see Fig. 2). The graph $H_{\text{inst}}$ contains the node set $V_{\text{univ}} := \{ v_u : u \in U \}$ (i.e., one node per element of the universe). Furthermore, it contains a node set $V_{\text{triple}}$ which includes one node $v_{i,S}$ for every instance index $0 \leq i < T$ and every triple of elements of the universe $S \in \binom{U}{3}$. We connect nodes
Fig. 2. The construction of the graph $H$ in the composition for INDEPENDENT DOMINATING SET.

$v_u$ and $v_{i,S}$ iff $S \in \mathcal{F}_i$ and $u \in S$. Observe that there might be some isolated nodes in $V_{\text{triple}}$. The graph $H_{enf}$ consists of two induced matchings. It contains an encoding matching $X_{\text{code}}$ with edges $\{x_{\gamma,\delta}, x'_{\gamma,\delta}\}$ for all integers $0 \leq \gamma < t$ and $0 \leq \delta < d$. Furthermore, it contains a choice matching $Y_{\text{choice}}$ with edges $\{y_S, y'_S\}$ for all triples $S \in \binom{U_3}{3}$. It remains to describe $E_{\text{conn}}$. First of all, we connect each $y_S$ to every $v_{i,S}$. Second, we connect $x_{\gamma,\delta}$ with $v_{i,S}$ iff $i_{\delta} = \gamma$ where $(i_0, \ldots, i_{d-1})$ is the $t$-ary expansion of index $i$, i.e. $i = \sum_{\delta=0}^{d-1} i_{\delta} \cdot t^\delta$ with $0 \leq i_{\delta} < t$.

**Lemma 8.** The graph $H$ is $d+4$ degenerate.

*Proof.* Consider any ordering of vertices where we put vertices of $V_{\text{triple}}$ first, and the remaining vertices. Observe, that each vertex of $V_{\text{triple}}$ is of degree exactly $d+4$ in $H$, since it has exactly $d$ neighbors in $X_{\text{code}}$, exactly one neighbor in $Y_{\text{choice}}$ and exactly three neighbors in $V_{\text{univ}}$. After removing $V_{\text{triple}}$, all vertices have degree at most 1. The claim follows. $\Box$

**Lemma 9.** Let $k := dt + \binom{n}{3} + n/3$. Then $(H, k)$ is a YES-instance of IDS iff there exists $0 \leq j < T$, such that $(\mathcal{F}_j, U)$ is a YES-instance.

*Proof.* For the if part, let us assume that for some $0 \leq j < T$ the instance $(\mathcal{F}_j, U)$ of 3-EXACT SET COVER is a YES-instance and $\mathcal{S} \subseteq \mathcal{F}_j$ be a solution, i.e. a collection of $n/3$ disjoint sets. Construct an independent dominating set $D$ of exactly $k$ vertices as follows. Let $(j_0, \ldots, j_{d-1})$ be the $t$-ary expansion of $j$. For $0 \leq \gamma < t, 0 \leq \delta < d$ if $j_{\delta} = \gamma$, then add $x'_{\gamma,\delta}$ to $D$ and otherwise add to $D$ the vertex $x_{\gamma,\delta}$. For $S \in \binom{U_3}{3}$, add $y'_S$ to $D$ if $S \in \mathcal{S}$ and add $y_S$ to $D$ otherwise. Finally, add to $D$ the $n/3$ vertices $v_{j,S}$ with $S \in \mathcal{S}$. Clearly $|D| = k$. Moreover $D$ is an independent set. In fact, $H[X_{\text{code}} \cup Y_{\text{choice}}]$ is a matching and we have taken exactly one endpoint of each one of its edges. Moreover for each $v_{j,S} \in D$ by construction there is no edge between $v_{j,S}$ and the remaining vertices of $D$. To prove that $D$ is a dominating set observe that all the vertices of $V_{\text{univ}}$ are dominated because $\mathcal{S}$ is a solution for $(\mathcal{F}_j, U)$, and all the vertices of $X_{\text{code}} \cup Y_{\text{choice}}$ are dominated because $D$ contains exactly one endpoint of each edge of the matching $H[X_{\text{code}} \cup Y_{\text{choice}}]$. Finally, each vertex
than $V$

Therefore, all the three mentioned inequalities are tight. Moreover an independent set, it means that no vertex of $V$

we have $|S|\geq n/$. Enough to prove that $(v_0, v_1, v_2, v_3)$ in $D$

Theorem 2. Let $d \geq 4$. Then $IDS$ has no kernel of size $O(k^{d-4-\varepsilon})$ for any constant $\varepsilon > 0$ unless coNP $\subseteq$ NP/poly.

Proof. From Lemmas 8 and 9, there exists a weak $d$-composition from 3-EXACT SET COVER to IDS in $(d + 4)$-degenerate graphs for any $d \geq 1$. The claim thus follows from Lemma 1.

5 Induced Matching

In this section we show a kernelization lower bound for the INDUCED MATCHING (IM) problem in $d$-degenerate graphs. In IM, the input is a graph $G$ and an integer $k$, and the goal is to determine whether there exists a set of $k$ edges $e_1, \ldots, e_k$ in $G$ such that there is no edge in $G$ connecting two endpoints of $e_i$ and $e_j$ for all $i \neq j \in \{1, \ldots, k\}$. The main result of this section is given by the following theorem.

Theorem 3. Let $d \geq 3$. Then $IM$ has no kernel of size $O(k^{d-3-\varepsilon})$ for any constant $\varepsilon > 0$ unless coNP $\subseteq$ NP/poly.

For our kernel lower bound on IM, we present a weak $d$-composition from the MULTICOLORED CLIQUE problem in which the input is a graph $G := (V, E)$ and a vertex-coloring $col : V \rightarrow \{1, \ldots, k\}$, and the goal is to determine whether there exists a multicolored clique of size $k$ in $G$, that is, whether there exists a set of pairwise adjacent vertices $v_1, \ldots, v_k$ in $G$ with $col(v_i) \neq col(v_j)$ for all $i \neq j \in \{1, \ldots, k\}$. It is well known that MULTICOLORED CLIQUE is NP-hard [17].

Let $(G_i = (V_i, E_i), col_i), 0 \leq i < T := t^{d}$, be the input instances of MULTICOLORED CLIQUE. By standard padding and vertex-renaming arguments, we can assume that all graphs $G_i$ are defined over the same vertex set $V$ of size $n$, and that each vertex $v \in V$ is assigned the same color $col(v) \in \{1, \ldots, k\}$ by all coloring functions $col_i$’s (note that this can be done even if the number of colors in each graph is different). We can further assume that for each $(u, v) \in E_i$ we have $col(u) \neq col(v)$, since all edges between vertices of the same color can never appear in any multicolored clique. Finally, we also assume that $(\binom{k}{2} - k > d$, since otherwise a weak $d$-composition can trivially be constructed by solving each instance separately in polynomial time.

We next construct and instance $(H = (W, F), k')$ of IM for a proper parameter $k'$. As in previous sections, $H$ consists of an instance graph $H_{inst} = (W_{inst}, F_{inst})$ and an enforcement gadget
Let holds iff the graph with vertex set \( V \), where \( \{1, ..., n\} \) is not matched in \( M \), is not contained in any edge of \( E \).

**Proof.**

\( IM \) solution to the \( (M, M') \) instance induce feasible solutions to the \( IM \) instance, and the enforcement gadget ensures that we cannot combine partial solutions of \( IM \) instances to obtain a feasible solution to the \( IM \) instance.

Graph \( H_{\text{inst}} = (A \cup B, F_{\text{inst}}) \) is bipartite, with \( A = A_{\text{nodes}} \cup A_{\text{edges}} \) and \( B = B_{\text{nodes}} \cup B_{\text{col-pairs}} \). Sets \( A_{\text{nodes}} \) and \( B_{\text{nodes}} \) contain a node \( a_v \) and \( b_v \), respectively, for each \( v \in V \). We have \( A_{\text{edges}} = \bigcup_{0 \leq i < T} A_{\text{edges}}^i \), where the set \( A_{\text{edges}}^i \) contains a node \( a_{i,e} \) for each instance \( i \) and \( e_i \in E_i \). Set \( B_{\text{col-pairs}} \) contains a node \( b_{\alpha,\beta} \in B \) for every pair of colors \( 1 \leq \alpha < \beta \leq k \). Set \( F_{\text{inst}} \) contains all the edges of type \( \{a_v, b_v\} \), plus edges between each \( a_{i,e} = a_{i,uv} \) and nodes \( b_u, b_v \) and \( b_{\text{col}(u), \text{col}(v)} \).

**Lemma 10.** Let \( M' \) be an induced matching in \( H_{\text{inst}} \). Then \( |M'| \leq \binom{k}{2} + n - k \). Moreover, equality holds iff the graph with vertex set \( V' = \{ v \in V : b_v \in V(M') \} \) and edge set \( E' := \{ e : a_{i,e} \in V(M') \text{ for some } i \in \{0, \ldots, T-1\} \} \) is a multicolored clique of the graph \( (V, \bigcup_i E_i) \).

**Proof.** Let \( M' \) be a maximum induced matching in \( H_{\text{inst}} \). Clearly \( n \leq |M'| \leq n + \binom{k}{2} \) (the lower bound comes from the matching induced by pairs \( \{a_v, b_v\} \) ). If \( |M'| = n \) then we are done, so assume \( |M'| > n \). Since \( \binom{k}{2} - k > d > 0 \), we have \( \binom{k}{2} + n - k > n \), and so some vertex \( b_{\alpha,\beta} \) must be matched in \( M' \). Consider some edge \( m \in M' \) which includes some vertex \( b_{\alpha,\beta} \). Then by construction, \( m = \{b_{\alpha,\beta}, a_{i,e}\} \) where \( a_{i,e} \) corresponds to an edge \( e = \{u, v\} \in E_i \), for some \( i \in \{0, \ldots, T-1\} \), where \( \text{col}(u) = \alpha \) and \( \text{col}(v) = \beta \). As \( \{b_u, a_{i,e}\}, \{b_v, a_{i,e}\} \in F_{\text{inst}} \), the vertices \( b_u \) and \( b_v \) cannot be matched in \( M' \). Thus, for each vertex \( b_{\alpha,\beta} \) that is matched in \( M' \), a vertex of \( \{b_u : u \in V, \text{col}(u) = \alpha\} \) is not matched in \( M' \), and a vertex of \( \{b_v : v \in V, \text{col}(v) = \beta\} \) is not matched in \( M' \). Since a vertex can only appear in at most \( k \) color pairs, we have \( M' \leq \binom{k}{2} + n - k \). Furthermore, if a vertex \( v \in V \) is not contained in any edge of \( E' \), then \( M' \cup \{a_v, b_v\} \) is also an induced matching. It is now not
difficult to see that \( |M'| = \binom{k}{2} + n - k \) iff \( |V \setminus V'| = n - k \), and \((V', E')\) is a multicolored clique of size \( k \).

By Lemma 10 suggests, to ensure that maximum induced matchings in \( H_{\text{inst}} \) are meaningful to us, we need to guarantee that only nodes \( c_{i,e} \) for a unique index \( i \) are matched. To this aim, we introduce the following enforcement gadget \((H_{\text{enf}}, E_{\text{conn}})\). Graph \( H_{\text{enf}} = (X \cup Y, E_{\text{enf}}) \) is bipartite. Sets \( X \) and \( Y \) contain a node \( x_{\gamma, \delta} \) and \( y_{\gamma, \delta} \), respectively, for all integers \( 0 \leq \gamma < t \) and \( 0 \leq \delta < d \).

We add edges between all pairs \( \{x_{\gamma, \delta}, y_{\gamma, \delta}\} \). Finally, we connect \( x_{\gamma, \delta} \) with \( a_{i,e} \) iff \( i_\delta = \gamma \) where \((i_0, i_1, \ldots, i_{d-1})\) is the \( t \)-ary expansion of index \( i \). Observe that each \( a_{i,e} \) is adjacent to exactly \( d \) distinct vertices \( x_{\gamma, \delta} \).

**Lemma 11.** \( H \) is \((d + 3)\)-degenerate.

**Proof.** Consider any ordering of the vertices which places first nodes \( A_{\text{nodes}} \) and \( Y \), then node \( A_{\text{edges}} \), then nodes \( B \), and finally nodes \( X \). It is not difficult to check that each vertex is adjacent to at most \( d + 3 \) vertices appearing to its right in this ordering.

**Lemma 12.** Let \( k' := t(d - 1) + \binom{k}{2} + n - k \). Then \((H, k')\) is a YES-instance of IM iff \((G_i, \text{col}_i)\) is a YES-instance of MULTICLORED CLIQUE for some index \( i \in \{0, \ldots, T - 1\} \).

**Proof.** Suppose some \( G_i \) has a multicolored clique of size \( k \). Then by Lemma 10 we can find an induced matching \( M' \) of size \( \binom{k}{2} + n - k \) in \( H_{\text{inst}} \) which matches only vertices of type \( a_{i,e} \). Let us add all the edges \( \{x_{\gamma, \delta}, y_{\gamma, \delta}\} \) such that \( i_\delta \neq \gamma \). There are precisely \( t(d - 1) \) such edges, which together with the edges of \( M' \), form an induced matching of size \( k' \) in \( H \).

For the converse direction, suppose \( M \) is an induced matching of size \( k' \) in \( H \). First observe that we can assume \( M \) does not contain any edges of \( F(A_{\text{edge}}, X) \) since any such edge \( \{a_{i,e}, x_{\gamma, \delta}\} \) can be safely replaced with the edge \( \{x_{\gamma, \delta}, y_{\gamma, \delta}\} \). Now as \( \binom{k}{2} - k > d \) w.l.o.g., the matching \( M \) must include some edge of \( F(B, A_{\text{edge}}) \) since there are \( n + td < k' \) edges altogether in \( F(A_{\text{node}}, B) \cup F(X, Y) \).

So let \( a_{i,e} \) be a vertex of \( A_{\text{edge}} \) which is matched in \( M \). Then as \( a_{i,e} \) is matched in \( M \), this means that \( d \) vertices of \( X \) cannot be matched in \( M \), precisely those vertices \( x_{\gamma, \delta} \) with \( i_\delta = \gamma \). Thus, \( |F(X, Y) \cap M| \leq t(d - 1) \). Since \( M \setminus F(X, Y) \) is a subset of edges in \( H_{\text{inst}} \), and since the maximum induced matching in \( H_{\text{inst}} \) is at most \( \binom{k}{2} + n - k \) by Lemma 10, this implies that \( M \) contains exactly \( \binom{k}{2} + n - k \) edges of \( H_{\text{inst}} \) and \( t(d - 1) \) edges of \( F(X, Y) \). By construction of \( F(A_{\text{edge}}, X) \), the latter assertion implies that there exists some \( i \) such that each vertex of \( A_{\text{edge}} \) that is matched by \( M \) is of the form \( a_{i,e} \) for some \( e \in E_i \). By Lemma 10, the second assertion implies that \( \{v \in V : \{b_v, a_{i,e}\} \in M \text{ for some } e \in E_i \} \) is a multicolored clique of size \( k \) in \( G_i \).

Theorem 3 now directly follows from Lemmas 11, 12, and 1.

## 6 Upper Bounds

Both upper bounds that we present rely on the following easy lemma.

**Lemma 13.** Let \( G := (A \cup B, E) \) be a bipartite \( d \)-degenerate graph where all vertices in \( B \) have degree greater than \( d \). Then \(|B| \leq d|A|\).

**Proof.** By the \( d \)-degeneracy of \( G \), we know that \(|E| \leq d(|A| + |B|)\). Since each vertex of \( B \) has degree greater than \( d \), we also know that \((d + 1)|B| \leq |E|\). Subtracting \( d|B| \) from both inequalities gives \(|B| \leq d|A|\). □
6.1 Dominating and independent dominating set

6.2 Connected and capacitated vertex cover

Let us begin with the kernel for ConVC. In ConVC, our goal is to determine whether a given graph $G$ has a set of vertices $A$ such that $G[V(G) \setminus A]$ is edgeless ($A$ is a vertex cover) and $G[A]$ is connected ($A$ is connected). Our kernelization algorithm uses two simple reduction rules which are given below, the second of which is a variant of the well-known crown reduction rule [15]. We say that a set of vertices $S \subseteq V(G)$ is a set of twins in $G$ if $N(v) = N(u)$ for all $u, v \in S$ (note that this implies that $S$ is an independent set in $G$). We let $N(S)$ denote the common set of neighbors of $S$.

**Rule 1.** If $G$ has an isolated vertex remove it.

**Rule 2.** If $S \subseteq V(G)$ is a subset of at most $d + 1$ twin vertices with $|N(S)| < |S|$, remove an arbitrary vertex of $S$ from $G$.

**Lemma 14.** Let $G$ be a graph, and let $G'$ be a graph resulting from applying either Rule 1 or Rule 2 to $G$. Then for any integer $k$, the graph $G$ has a connected vertex cover of size $k$ if and only if $G'$ has a connected vertex cover of size $k$.

**Proof.** The lemma is obvious for Rule 1, so let us focus on Rule 2. Observe that there exists a minimal size connected vertex cover $A$ in $G$ with $N(S) \subseteq A$. Furthermore, $A$ contains at most one vertex of $S$, and this vertex can be replaced by any other vertex of $S$ to obtain another connected vertex cover for $G$ of equal size. Replacing this vertex (if it exists) with a vertex of $S \cap V(G')$, we obtain a connected vertex cover $A'$ for $G'$ with $|A'| = |A|$. Conversely, using similar arguments one can transform a minimum size connected vertex cover for $G'$ to an equal size connected vertex cover for $G$. □

**Theorem 4.** ConVC in $d$-degenerate graphs has a kernel of size $O(k^d)$.

**Proof.** Our kernelization algorithm for Connected Vertex Cover in $d$-degenerate graphs exhaustively applies Rule 1 and Rule 2 until they no longer can be applied. Since Rule 1 can be implemented in linear time, and Rule 2 can be done in $n^{O(d)}$ time, this algorithm runs in polynomial time. Let $G'$ be the graph resulting from the kernelization. Observe that both reduction rules that were used do not increase the degeneracy of the graph, and so $G'$ is $d$-degenerate as well. Furthermore, due to Lemma 14, we know that $G$ has a connected vertex cover of size $k$ if $G'$ has one as well. We next show that $|V(G')| = O(k^d)$, or otherwise $G'$ has no connected vertex cover of size $k$.

Suppose that $G'$ has a connected vertex cover $A$ of size $k$. Then as $A$ is a vertex cover, the set $B := V(G) \setminus A$ is an independent set in $G$. For $i := 0, \ldots, d$, define $B_i \subseteq B$ to be set of all vertices in $B$ with degree $i$ in $G$, and let $B_{>d} \subseteq B$ be the vertices in $B$ with degree greater than $d$ in $G$. Then $B := B_0 \cup \cdots \cup B_d \cup B_{>d}$, and $|B_0| = 0$ since Rule 1 cannot be applied. Due to Rule 2, for each subset of $i$ vertices $A' \subseteq A$, $1 \leq i \leq d$, there are at most $i$ vertices $B' \subseteq B_d$ with $N(B') = A'$. We therefore have $|B_i| \leq i\binom{k}{i}$ for each $i \in \{1, \ldots, d\}$, and $\sum_{i=0}^{d} |B_i| \leq dk^d$. Furthermore, we also have $|B_{>d}| \leq dk$ by applying Lemma 13 to the bipartite graph on $A$ and $B_{>d}$. Accounting also for $A$, we get

$$|V(G')| = |A| + |B| \leq k + \sum_{i=0}^{d} |B_i| + |B_{>d}| \leq k + dk^d + dk = O(k^d),$$

and the theorem is proved. □
Next we consider CapVC. In this problem, we are given a graph $G$, an integer $k$, and a vertex capacity function $cap : V(G) \rightarrow \mathbb{N}$, and the goal is to determine whether there exists a vertex cover of size $k$ where each vertex covers no more than its capacity. That is, whether there is a vertex cover $C$ of size $k$ and an injective mapping $\alpha$ mapping each edge of $E(G)$ to one of its endpoints such that $|\alpha^{-1}(v)| \leq cap(v)$ for every $v \in V(G)$. We may assume that $k + 1 > d$, since otherwise a kernel is trivially obtained by solving the problem in polynomial time.

**Rule 3.** If $S \subseteq V(G)$ is a subset of twin vertices with a common neighborhood $N(S)$ such that $|S| = k + 2 > d \geq |N(S)|$, remove a vertex with minimum capacity in $S$ from $G$, and decrease all the capacities of vertices in $N(S)$ by one.

**Lemma 15.** Let $k \geq 1$ be an arbitrary integer, let $G$ be a vertex capacitated graph, and let $G'$ be a vertex capacitated graph resulting from applying either Rule 1 or Rule 3 to $G$. Then $G$ has a capacitated vertex cover of size $k$ iff $G'$ has a capacitated vertex cover of size $k$.

**Proof.** The lemma is trivial for Rule 1. For Rule 3, let $A$ be a capacitated vertex cover of size $k$ in $G$, and let $u$ be a vertex of minimum capacity in $S$. As $|S| > k$, there is some $v \in S \setminus A$, and moreover it must be that $N(S) \subseteq A$. Thus, if $u \notin A$, then $A$ is also a capacitated vertex cover of $G'$. Otherwise, if $u \in A$, we can replace $u$ with $v$ in $A$. As $cap(u) \leq cap(v)$, this would result in another capacitated vertex cover for $G$ which is also a solution for $G'$. Conversely, any capacitated vertex cover of size $k$ for $G'$ must also include all vertices of $N(S)$, and thus by increasing the capacities of all these vertices by one we obtain a solution of size $k$ for $G$. \hfill $\Box$

**Theorem 5.** CapVC has a kernel of size $O(k^{d+1})$ in $d$-degenerate graphs.

**Proof (sketch).** The argument is similar to the one used in Theorem 4. The only difference is that now the size of sets $B_i$ is bounded by $(k+1)\binom{k}{i}$ instead of $d\binom{k}{i}$, which yields a kernel size of $O(k^{d+1})$ instead of $O(k^d)$. \hfill $\Box$

The following complimenting lower bound follows from a simple reduction $d$-Set Cover. In this problem we are given a $d$-regular hypergraph $(V,E)$ and an integer $k$ and the question is whether there are $E_1, \ldots, E_k \in E$ with $V = \cup_i E_i$. Clearly we can assume that $|V| \leq dk$ since otherwise there is no solution, and the problem remains hard also when $|V| = dk$ (and the solution is a partition of $V$). Dell and Marx show that unless $\text{coNP} \subseteq \text{NP/poly}$, $d$-Set Cover has no kernel of size $O(k^{d-\varepsilon})$ for any $\varepsilon > 0$ [9]. Note that this lower bound holds even if the output of the kernel is an instance of another decidable problem; in particular, even if the output is an instance of CapVC.

**Theorem 6.** Let $d \geq 3$. Unless $\text{coNP} \subseteq \text{NP/poly}$, CapVC in $d$-degenerate graphs has no kernel of size $O(k^{d-\varepsilon})$ for any $\varepsilon > 0$.

**Proof.** Given an instance $(V,E,k)$ of $d$-Set Cover with $|V| = kd$, we construct a graph $H := (U,F)$ by initially taking the incidence bipartite graph on $V \cup \mathcal{E}$, and then connecting each $v \in V$ to new a leaf-vertex $v'$ which is adjacent only to $v$. In this way, $U := V' \cup V \cup \mathcal{E}$, where $V' := \{v' : v \in V\}$ is a set of copies of $V$, and $F := \{(v,E) : v \in V, e \in \mathcal{E}, \text{ and } v \in e\} \cup \{\{v,v'\} : v \in V\}$. To complete our construction, we set the capacity of each vertex $u \in U$ to be its degree in $H$ minus one. Note that $H$ is $d$-degenerate.

It is not difficult to see that $(V,E,k)$ has a solution iff $H$ has a capacitated vertex cover of size $k + |V| = (d + 1)k$. Indeed if $E_1, \ldots, E_k$ is a solution for $(V,E,k)$, then $\{E_1, \ldots, E_k\} \cup V$ is a
capacitated vertex cover of $H$. Conversely, any minimal capacitated vertex cover of $H$ must include all vertices of $V$ and none of $V'$, and hence if it is of size $|V| + k$, it must include $k$ vertices which correspond to $k$ edges of $E$ that cover $V$. Thus, combining the above construction with a $O(k^{d-\varepsilon})$ kernel for CapVC in $d$-degenerate graphs shows that coNP $\subseteq$ NP/poly according to [9].

\[\square\]

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