DOMINATED SPLITTINGS FOR SEMI-INVERTIBLE OPERATOR COCYCLES ON HILBERT SPACE

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Abstract. A theorem of J. Bochi and N. Gourmelon states that an invertible linear cocycle admits a dominated splitting if and only if the singular values of its iterates become separated at a uniform exponential rate. It is not difficult to show that for cocycles of non-invertible linear maps over an invertible dynamical system – which we refer to as semi-invertible cocycles – this criterion fails to imply the existence of a dominated splitting. In this article we show that a simple modification of Bochi and Gourmelon’s singular value criterion is equivalent to the existence of a dominated splitting in both the invertible and the semi-invertible cases. This result extends to the more general context of semi-invertible cocycles of bounded linear operators acting on a Hilbert space, and generalises previous results due to J.-C. Yoccoz, J. Bochi and N. Gourmelon, and the present author.

1. Introduction

Let $T: X \to X$ be a continuous transformation of a nonempty topological space, $H$ a Hilbert space and $B(H)$ the set of all bounded linear operators on $H$, which we equip with the topology induced by the operator norm. We define a linear cocycle over the transformation $T$ to be a function $\mathcal{A}: X \times \mathbb{N} \to B(H)$ which satisfies the identity

$$\mathcal{A}(x, n + m) = \mathcal{A}(T^n x, m) \mathcal{A}(x, n)$$

for every $x \in X$ and $n, m \in \mathbb{N}$. Where no ambiguity results we shall use the notation $\mathcal{A}(x) := \mathcal{A}(x, 1)$ for all $x \in X$. Using this notation we observe that $\mathcal{A}: X \times \mathbb{N} \to B(H)$ is a cocycle if and only if

$$\mathcal{A}(x, n) = \mathcal{A}((T^{n-1} x) \cdots A(T x) A(x)$$

for every $x \in X$ and $n \in \mathbb{N}$. We define a continuous splitting of $H$ to be a pair of continuous functions $\mathcal{U}$, $\mathcal{V}$ from $X$ to the Grassmannian of $H$ such that $H = \mathcal{U}(x) \oplus \mathcal{V}(x)$ for all $x \in X$, and we express this relationship by writing $\mathcal{H} = \mathcal{U} \oplus \mathcal{V}$. We say that $\mathcal{U}$ and $\mathcal{V}$ are invariant with respect to $\mathcal{A}$ if $\mathcal{A}(x, n) \mathcal{U}(x) \subseteq \mathcal{U}(T^n x)$ and $\mathcal{A}(x, n) \mathcal{V}(x) \subseteq \mathcal{V}(T^n x)$ for every $x \in X$ and $n \in \mathbb{N}$, and when this is satisfied for a continuous splitting $\mathcal{H} = \mathcal{U} \oplus \mathcal{V}$ we call that splitting invariant with respect to $\mathcal{A}$.

Finally we define a dominated splitting of $\mathcal{H}$ with respect to the cocycle $\mathcal{A}$ to be a continuous invariant splitting $\mathcal{H} = \mathcal{U} \oplus \mathcal{V}$ for which there exist constants $C > 0$ and $\tau \in (0, 1)$ such that for all $x \in X$ and $n \in \mathbb{N}$

$$\sup_{\|v\| = 1} \|\mathcal{A}(x, n)v\| < C \tau^n \inf_{\|u\| = 1} \|\mathcal{A}(x, n)u\|.$$  

We will refer to the cocycle $\mathcal{A}$ as being $k$-dominated when it admits a dominated splitting such that dim$\mathcal{U}(x) = k$.  

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Dominated splittings originate in the work of V. A. Pliss [18], R. Mañé [14] and S. T. Liao [13] in smooth ergodic theory, in which context they continue to be of great significance and utility (see e.g. [5, 8, 19]). In this context $T: X \to X$ is a diffeomorphism of a compact Riemannian manifold and $A(x)$ is the derivative $D_x T$ acting on the tangent bundle of $X$. In particular, the cocycle $A$ here acts on a continuous bundle of vector spaces over $X$ rather than on a single space $\mathcal{H}$, but in order to simplify our presentation – and particularly because of the specific applications which we have in mind – we shall concentrate in this article on the case of a cocycle of operators acting on a single space as was defined previously.

Recently, dominated splittings and other related tools from multiplicative ergodic theory have been applied with considerable success to the study of joint spectral characteristics of sets of matrices or of linear operators, a topic in matrix analysis [4, 15, 16, 17]. In the original context of dominated splittings in cocycles of derivative maps over diffeomorphisms, the linear maps forming the cocycle were necessarily invertible and acted on finite-dimensional real vector spaces. These facts are reflected in subsequent work on dominated splittings in which the linear maps are always assumed to be invertible. In the context of the aforementioned applications to matrix analysis, on the other hand, the underlying dynamical system $T: X \to X$ is the shift on the infinite product $A^\mathbb{Z}$ where $A$ is a given compact set of matrices or bounded linear operators, and there is no a priori reason why the matrices or operators $A$ should be taken to be invertible, nor why the underlying vector space should always be defined over $\mathbb{R}$ in preference to $\mathbb{C}$. Applications of the theory of dominated splittings in matrix analysis have thus been hampered by a lack of results on dominated splittings for cocycles of non-invertible linear operators, and in these applications it has been necessary to either construct appropriate partial results on an ad-hoc basis [15, 17] or to restrict, somewhat arbitrarily, the scope of work to the context of invertible matrices only [4]. The purpose of this article is to give a general necessary and sufficient criterion for the existence of dominated splittings which on the one hand encompasses ad-hoc results established previously by the author for use on specific problems in matrix analysis [15], and on the other hand generalises an existing necessary and sufficient condition given in the invertible case by J. Bochi and N. Gourmelon [3]. In this article we will investigate dominated splittings in the situation where the underlying transformation $T: X \to X$ remains invertible, but where the values taken by the cocycle $A$ are allowed be non-invertible linear maps. We refer to such cocycles as semi-invertible.

The method which we will employ is sufficiently powerful to allow us to take the cocycle $A$ as acting on a Hilbert space which need not be finite-dimensional. Recent interest in multiplicative ergodic theorems for semi-invertible cocycles acting on finite and infinite-dimensional Banach spaces (see e.g. [6, 7, 9, 12, 16]) also serves to make natural the problem of constructing continuous invariant splittings for semi-invertible cocycles acting on finite or infinite-dimensional spaces.

Given a real (resp. complex) $d \times d$ matrix $A$ we recall that the singular values of $A$, which we denote by $\sigma_1(A), \ldots, \sigma_d(A)$, are defined to be the positive square roots of the eigenvalues of the positive semidefinite matrix $A^* A$ listed according to multiplicity in decreasing order. In particular we have $\sigma_1(A) \equiv \|A\|$. The singular values satisfy the characterisation

$$\sigma_k(A) = \sup_{\dim F = k} \inf_{\|u\| = 1} \|Au\|$$
where the supremum is taken over all \( k \)-dimensional linear subspaces of \( \mathbb{R}^d \) (resp. \( \mathbb{C}^d \)), see for example [10, Theorem 7.3.10]. By means of this characterisation the notion of singular value may be extended to bounded linear operators on infinite-dimensional Hilbert spaces, a topic to which we shall return in §2 below.

In this article we will prove a generalisation of the following theorem of J. Bochi and N. Gourmelon which characterises the existence of dominated splittings of invertible cocycles in terms of their singular values:

**Theorem 1** ([3]). Let \( T \colon X \to X \) be a homeomorphism of a compact Hausdorff space and \( A \colon X \times \mathbb{N} \to \text{GL}_d(\mathbb{R}) \) a continuous linear cocycle. Then \( A \) admits a dominated splitting \( \mathbb{R}^d = \mathcal{U} \oplus \mathcal{V} \), where each subspace \( U(x) \) has dimension \( k \), if and only if there exist \( C > 0 \) and \( \tau \in (0,1) \) such that for all \( x \in X \) and \( n \geq 1 \)

\[
\sigma_{k+1}(A(x,n)) < C \tau^n \sigma_k(A(x,n)).
\]

Let \( 1 \leq k \leq d \) and let \( A \) be a \( d \times d \) matrix, which we assume for the moment to be real-valued. We identify \( A \) with a linear transformation of \( \mathbb{R}^d \), and denote the set of all such transformations by \( \text{End}(\mathbb{R}^d) \). The transformation \( A \) induces a linear transformation \( \wedge^k A \) of the \( k \)-th exterior power \( \wedge^k \mathbb{R}^d \) of \( \mathbb{R}^d \). Given a cocycle \( A \colon X \times \mathbb{N} \to \text{End}(\mathbb{R}^d) \) and an integer \( k \) such that \( 1 \leq k \leq d \) we may therefore define an induced cocycle \( \wedge^k A \colon X \times \mathbb{N} \to \text{End}(\wedge^k \mathbb{R}^d) \) by taking \((\wedge^k A)(x,n) := \wedge^k(A(x,n)) \). Similar remarks apply to cocycles of linear transformations acting on \( \mathbb{C}^d \) or on an infinite-dimensional real or complex Hilbert space. When \( A \) is a linear transformation of a \( d \)-dimensional space and \( 1 \leq k < d \), the first and second singular values of the linear map \( \wedge^k A \) are given by

\[
\sigma_1 (\wedge^k A) = \prod_{i=1}^{k} \sigma_i(A), \quad \sigma_2 (\wedge^k A) = \left( \prod_{i=1}^{k-1} \sigma_i(A) \right) \sigma_{k+1}(A).
\]

It follows easily from Theorem 1 that an invertible cocycle \( A \) of linear maps acting on \( \mathbb{R}^d \) is \( k \)-dominated if and only if the cocycle \( \wedge^k A \) of linear maps acting on \( \wedge^k \mathbb{R}^d \) is 1-dominated. This equivalence was of central significance in the article [4], and in the context of semi-invertible cocycles we will return to the subject of this equivalence shortly.

J. Bochi and the present author observed in [4] that Theorem 1 does not extend directly to the case where the values of the cocycle \( A \) are general, not-necessarily-invertible matrices. We present the following simple example. Let \( X = \{0,1\}^\mathbb{Z} \) with the infinite product topology and let \( T \colon X \to X \) be given by \( T([x_i]_{i \in \mathbb{Z}}) := ([x_{i+1}]_{i \in \mathbb{Z}} \). We note that \( X \) is compact and metrisable and that \( T \) is a homeomorphism. Define

\[
A_0 := \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_1 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
\]

and let \( A \colon X \times \mathbb{N} \to M_2(\mathbb{R}) \) be the continuous linear cocycle given by

\[
A_0 \quad \text{and} \quad A_1 \quad \text{for} \quad n \in \mathbb{N}, \quad A := A_n \cdots A_1
\]

where \( M_2(\mathbb{R}) \) denotes the set of all \( 2 \times 2 \) real matrices. We note that \( A(x,n) \) is always equal either to \( A_0^n \) or to \( A_1^n \), and in particular we always have \( \sigma_2(A(x,n)) < 2^{1-n} \sigma_1(A(x,n)) \) so that (2) is satisfied with \( k := 1 \), \( C := 2 \) and \( \tau := \frac{1}{2} \). However we claim that \( A \) does not have the dominated splitting suggested by the na"ive generalisation of Theorem 1. Suppose for a contradiction that \( \mathbb{R}^2 = \mathcal{U} \oplus \mathcal{V} \) is an \( A \)-invariant dominated splitting with \( \mathcal{U}(x) \) and \( \mathcal{V}(x) \) being 1-dimensional for every \( x \in X \). Consider the sequence of points \( x^n = (x^n_i)_{i \in \mathbb{Z}} \) defined by \( x^n_i = 0 \)
if \( i \neq n \) and \( x_i^n = 1 \) otherwise, and let \( x = \lim_{n \to \infty} x^n \in X \) be the constant sequence of 0’s. In order for (1) to be satisfied for some suitable constants \( C \) and \( \tau \), together with the \( A \)-invariance of \( U \) and \( V \) being satisfied, it is necessary that \( U(x) \) should be the horizontal axis in \( \mathbb{R}^2 \) and \( V(x) \) the vertical axis, since these are the eigenspaces of \( A_0^n = A(x,n) \) corresponding to its larger and smaller eigenvalues respectively. On the other hand since \( A(x_n, n + 1) = A^{n+1}_0 \) the image of \( A(x_n, n + 1) \) is precisely the vertical axis, and therefore \( A(x_n, n + 1)U(x_n) \) must be either \( \{0\} \) or the vertical axis. If it were the former then we would have \( \|A(x_n, n + 1)u\| = 0 \) for all \( u \in U(x_n) \) which renders (1) impossible: we deduce that the latter holds, and therefore \( U(T^{n+1}x_n) \) must be the vertical axis since \( U(T^{n+1}x_n) \) is one-dimensional and contains \( A(x_n, n + 1)U(x_n) \). We find that \( \lim_{n \to \infty} T^{n+1}x_n = x \) yet \( \lim_{n \to \infty} U(T^{n+1}x_n) \neq U(x) \), contradicting the continuity of \( U \), and we conclude that \( A \) fails to have a dominated splitting as claimed.

The resolution of this obstruction to the extension of Theorem 1 is surprisingly easy to state: it transpires that for non-invertible cocycles the correct condition is simply

\[
\text{(3) } \max\{\sigma_{k+1}(A(x,n)), \sigma_{k+1}(A(Tx,n))\} < C\tau^n\sigma_k(A(x,n+1)).
\]

When \( A \) takes only invertible values it is not difficult to see that this is equivalent to (2) up to a change in the constant \( C \). However, in contrast to the condition (2) it is not a priori clear that a cocycle \( A \) will satisfy (3) for \( k := \ell \) if and only if the cocycle \( \wedge^kA \) satisfies (3) for \( k := 1 \). For this reason, and with a view to possible future applications of this property similar to those given in [14] in the invertible case, we make this issue explicit in our generalisation of Theorem 1. We prove the following:

**Theorem 2.** Let \( T : X \to X \) be a homeomorphism of a compact nonempty topological space, \( \mathcal{H} \) a real or complex Hilbert space and \( A : X \times \mathbb{N} \to B(\mathcal{H}) \) a continuous cocycle. Then the following are equivalent:

1. There exist constants \( C > 0 \) and \( \tau \in (0, 1) \) such that for every \( x \in X \) and \( n \geq 1 \)
   \[
   \max\{\sigma_{k+1}(A(x,n)), \sigma_{k+1}(A(Tx,n))\} < C\tau^n\sigma_k(A(x,n+1)).
   \]
2. There exists an \( A \)-invariant dominated splitting \( \mathcal{H} = U \oplus V \) such that \( \dim U = k \).
3. There exist an \( A \)-invariant splitting \( \mathcal{H} = U \oplus V \) and constants \( C, M, \delta > 0 \) and \( \tau \in (0, 1) \) such that for all \( x \in X \) and \( n \geq 1 \) we have
   \[
   \|A(x,n)u\| \geq \delta \sigma_k(A(x,n))\|u\|
   \]
   for all \( u \in U(x) \),
   \[
   \|A(x,n)v\| \leq M\sigma_{k+1}(A(x,n))\|v\|
   \]
   for all \( v \in V(x) \), and
   \[
   \sigma_{k+1}(A(x,n)) < C\tau^n\sigma_k(A(x,n)).
   \]
4. There exists a \( \wedge^kA \)-invariant dominated splitting \( \wedge^k\mathcal{H} = \hat{U} \oplus \hat{V} \) such that \( \dim \hat{U} = 1 \).

**Remark.** We note that by taking \( \mathcal{H} := \mathbb{R}^2 \) and making each \( A(x) \) take values in \( SL_2(\mathbb{R}) \) one obtains from Theorem 2 a result of J.-C. Yoccoz [24, Proposition 2]. With slightly more effort one may deduce from Theorem 2 the following result of the author (see [15, Theorem 2.1]): if \( T \) is a minimal homeomorphism of a compact topological space, \( A \) is a continuous linear cocycle of \( d \times d \) complex matrices, and
\(A\) is bounded in the sense that \(\sup_{x \in X} \sup_{n \geq 1} \| A(x, n) \| < \infty\), then there is a continuous \(A\)-invariant splitting \(C^d = U \oplus V\) such that \(\| A(x, n)u \|\) is uniformly bounded away from zero for every unit vector \(u \in U(x)\), and \(\| A(x, n)v \|\) converges to zero with uniform exponential speed for every unit vector \(v \in V(x)\). (Note that in some cases either \(U\) or \(V\) may be zero-dimensional.) To derive this result from Theorem 2 one must apply a lemma exploiting the minimality of \(T\) (see [15, Proposition 3.1]) to show that for every integer \(\ell\) in the range \(1 \leq \ell \leq d\), the products \(\sigma_1(A(x, n)) \cdots \sigma_\ell(A(x, n)) = \| \wedge^\ell A(x, n) \|\) either are uniformly bounded away from 0, or converge to 0 uniformly with exponential speed; taking \(k\) to be the largest such integer \(\ell\) for which the former holds yields the validity of the criterion (a) above. We leave the details to the reader.

The proof of Theorem 2 which we present here differs in many respects from the proof of Theorem 1 which was presented in [3]. In the latter, Bochi and Gourmelon begin by making use of Oseledets’ multiplicative ergodic theorem to show that to every \(T\)-invariant regular Borel measure on \(X\) one may associate a measurable invariant splitting, defined almost everywhere, which satisfies a property of uniform separation between Lyapunov exponents. They then show that these measurable splittings coincide almost everywhere with a continuous splitting. The domination property is proved for the continuous splitting by extending the dynamical system \(T : X \to X\) to a dynamical system on \(X \times \mathbb{RP}^d\) given by \((x, v) \mapsto (Tx, A(x)v)\), relating the Lyapunov exponents to the pointwise almost everywhere convergence of certain Birkhoff averages with respect to the extended dynamical system, and recovering uniform convergence of these Birkhoff averages (and hence domination of the splitting) by exploiting the weak-* compactness of the set of invariant measures of the extended dynamical system.

In our context such a strategy would encounter several obstacles. Firstly, if \(X\) is not taken to be Hausdorff then some or all parts of the above argument may fail due to a lack of inner regular measures on \(X\). Secondly, if the Hilbert space \(\mathcal{H}\) is not separable then the distributions of functions on \(\mathcal{H}\), or on the Grassmannian of \(\mathcal{H}\), may fail to be Radon measures, rendering problematic the construction of measurable splittings. Thirdly, since the values of \(A\) are not assumed to be invertible they do not induce a well-defined action on the projective unit sphere of \(\mathcal{H}\); and lastly, even when such an extended dynamical system exists, the noncompactness of the unit sphere of an infinite-dimensional Hilbert space means that the set of invariant measures of the extended dynamical system would fail to be weak-* compact and the final step of the argument would fail.

In proving Theorem 2 we therefore eschew ergodic theory entirely and construct the continuous invariant splitting from first principles from the singular spaces of \(A(x, n)\), which is to say the eigenspaces of the positive semidefinite operator \(A(x, n)^*A(x, n)\). The exponential gap between the \(k\)th and \((k+1)\)th singular values allows us to prove directly the existence of the lower invariant subspace \(V\) as a limit of orthogonal complements of singular spaces in a similar manner to the classic arguments of M. S. Raghunathan [20] and D. Ruelle [21, 22]. To deduce the existence of the upper invariant subspace \(U\) we define a new cocycle \(B\) over the inverse transformation \(T^{-1}\) by setting \(B(x, n) := A(T^{-n}x, n)^*\) and note that the orthogonal complement of the lower invariant subspace associated to \(B\) is an invariant subspace for the cocycle \(A\), a trick previously applied by the author in [15]. We define \(U\) to be this orthogonal complement. To show that \(U\) and \(V\)
together constitute a dominated splitting it must then be shown that vectors in $U$ are extended uniformly exponentially faster by $A$ than are vectors on $V$. This step is complicated by the fact that different vectors in $U$ may themselves be extended at different rates to one another. To overcome this problem we show first that the one-dimensional subspace $\wedge^k U$ of $\wedge^k H$ is the upper invariant subspace of a dominated splitting for $\wedge^k A$, and only upon having proved this do we go on to deduce that $U$ and $V$ together constitute a dominated splitting for $A$.

We remark that the proof of the implication $(a) \Rightarrow (b)$ in Theorem 2 uses the compactness of $X$ at only two stages: firstly in the rather trivial deduction that the set $\{A(x) : x \in X\}$ must be bounded, and secondly in passing from domination of $\wedge^k A$ to domination of $A$, where we require compactness in order to show that the angle between $U$ at different rates to one another. To overcome this problem we show first that dominated splitting for $\wedge^k A$ and only upon having proved this do we go on to show that $U$ to be assumed to be compact at all! In this case we are able to provide an effective estimate for general $k$ remains open.

In this introduction we have stepped lightly over the definition of a continuous function from a topological space to the set of subspaces of a Hilbert space, and we have also not stated explicitly the interpretation of exterior powers of a Hilbert space. A detailed exposition of these topics, together with a pair of preparatory lemmas concerning projection operators, is given in §2 below.

It is not difficult to show that Theorem 2 becomes false if the condition (3) is replaced with the weaker condition $\sigma_{k+1}(A(x, n)) < C r^n \sigma_k(A(x, n + 1))$, or if $A : X \to B(H)$ is assumed to be continuous only with respect to the strong operator topology on $B(H)$ instead of the topology induced by the operator norm. The examples which illustrate these facts are somewhat mundane to verify and we consequently relegate them to §4 below.

2. Preliminaries: projections, singular values and exterior powers

In this section we present some definitions and lemmas pertaining to Hilbert spaces which will be useful in the proof of Theorem 2.

Given a real or complex Hilbert space $H$ we recall that an operator $P \in B(H)$ is an orthogonal projection if and only $P^2 = P^* = P$. Since $\|A^* A\| = \|A\|^2$ for every $A \in B(H)$ it follows easily that if $P$ is a nonzero orthogonal projection then $\|P\|^2 = \|P^* P\| = \|P^2\| = \|P\|$ and therefore $\|P\| = 1$. For each integer $k \geq 1$ we let $Gr_k(H)$ denote the set of all $k$-dimensional subspaces of $H$, and we also let $Gr_k(H)^*$ denote the set of all $k$-codimensional subspaces. Given two elements $U_1, U_2$ of $Gr_k(H)$ we let $P_1, P_2 \in B(H)$ denote the operators of orthogonal projection onto $U_1$ and $U_2$ respectively, and we define $d(U_1, U_2) := \|P_1 - P_2\|$. It is clear that this defines a complete metric on $Gr_k(H)$. For $V_1, V_2 \in Gr_k(H)^*$ we define $d(V_1, V_2)$ to be the distance between the orthogonal complements $V_1^\perp, V_2^\perp \subseteq Gr_k(H)$, and since finite-codimensional subspaces of $H$ are completely characterised by their orthogonal complements it is clear that this defines a complete metric on $Gr_k(H)^*$.
The metric \(d\) on \(\text{Gr}_k(\mathcal{H})\) admits the alternative expression

\[
d(\mathcal{U}_1, \mathcal{U}_2) = \max \left\{ \sup_{u_1 \in \mathcal{U}_1} \inf_{u_2 \in \mathcal{U}_2} \| u_1 - u_2 \|, \sup_{u_2 \in \mathcal{U}_2} \inf_{u_1 \in \mathcal{U}_1} \| u_2 - u_1 \| \right\}
\]

which is more frequently used in the ergodic theory literature (when explicit metrics are used at all: see for example [2 §5.3]). For a proof of the equivalence between these two formulations of the metric \(d\) we direct the reader to [1, p.109–112].

Given a real or complex Hilbert space \(\mathcal{H}\) and a bounded operator \(A\) on \(\mathcal{H}\), we define the singular values or s-numbers of \(A\), which we denote by \(\sigma_k(A)\) for integers \(k \geq 1\), to be the quantities

\[
\sigma_k(A) := \sup_{F \in \text{Gr}_k(\mathcal{H})} \inf_{\|v\|=1} \|Av\|.
\]

(If \(\mathcal{H}\) is finite-dimensional then we adopt the convention that \(\sigma_n(A) = 0\) for every \(n > \dim \mathcal{H}\).) Clearly \(\sigma_{k+1}(A) \leq \sigma_k(A) \leq \sigma_1(A) = \|A\|\) for every integer \(k\), and each of the singular value functions \(\sigma_k: B(\mathcal{H}) \to \mathbb{R}\) is 1-Lipschitz continuous. If \(\sigma_{k+1}(A) < \sigma_k(A)\) for any integer \(k \geq 1\) then each of \(\sigma_1(A)^2, \ldots, \sigma_k(A)^2\) is an eigenvalue of the positive semidefinite operator \(A^*A\), see for example [23 §V.1.3].

Let \(\mathcal{H}\) be a real or complex Hilbert space and \(k \geq 1\) a fixed integer. For each \(k\)-tuple of elements \(v_1, \ldots, v_k \in \mathcal{H}\) we may form their exterior product \(v_1 \wedge \cdots \wedge v_k\), a formal expression which is subject to the rule \(v_1 \wedge \cdots \wedge v_k = \text{sign}(\pi) \cdot v_{\pi(1)} \wedge \cdots \wedge v_{\pi(k)}\) for every permutation \(\pi: \{1, \ldots, k\} \to \{1, \ldots, k\}\). We define addition and scalar multiplication between two exterior products according to the rule

\[
v_1 \wedge \cdots \wedge v_k + \lambda(v'_1 \wedge v_2 \wedge \cdots \wedge v_k) = (v_1 + \lambda v'_1) \wedge v_2 \wedge \cdots \wedge v_k,
\]

and define an inner product between two exterior products by

\[
\langle u_1 \wedge \cdots \wedge u_k, v_1 \wedge \cdots \wedge v_k \rangle := \det([\langle u_i, v'_j \rangle]_{i,j=1}^k).
\]

We extend this inner product by linearity to finite formal sums of exterior products \(v_1 \wedge \cdots \wedge v_k\) and define the Hilbert space \(\wedge^k \mathcal{H}\) to be the completion of the span of the set of all exterior products of \(k\) elements of \(\mathcal{H}\) with respect to this inner product. An exterior product \(v_1 \wedge \cdots \wedge v_k\) is zero if and only if the vectors \(v_1, \ldots, v_k\) are linearly dependent, and we have \(v_1 \wedge \cdots \wedge v_k = u_1 \wedge \cdots \wedge u_k\) if and only if the subspace of \(\mathcal{H}\) spanned by \(v_1, \ldots, v_k\) is equal to the subspace spanned by \(u_1, \ldots, u_k\); we also have \(\|v_1 \wedge \cdots \wedge v_k\| \leq \prod_{i=1}^k \|v_i\|\) with equality if the vectors \(v_1, \ldots, v_k\) are pairwise orthogonal. For a detailed exposition of this construction we refer the reader to [23 §V.1].

For each \(A \in B(\mathcal{H})\) we may define an induced operator \(\wedge^k A\) on \(\wedge^k \mathcal{H}\) by setting \(\wedge^k A(u_1 \wedge \cdots \wedge u_k) := Au_1 \wedge \cdots \wedge Au_k\) and by extending linearly to more general elements of \(\wedge^k \mathcal{H}\). The sequence of singular values of \(\wedge^k A\) is precisely the sequence, listed in decreasing order of value, of all possible products \(\sigma_{i_1}(A) \cdots \sigma_{i_k}(A)\) of singular values of \(A\) such that \(i_1 > i_2 > \ldots > i_k\). Thus in particular

\[
\|\wedge^k A\| = \sigma_1(\wedge^k A) = \prod_{i=1}^k \sigma_i(A) \leq \|A\|^k
\]
and
\[ \sigma_2(\land^k A) = \left( \prod_{i=1}^{k-1} \sigma_i(A) \right) \sigma_{k+1}(A), \]
see for example [23 §V.1.3]. For each \( A \in \mathcal{B}(\mathcal{H}) \) and every \( k \geq 1 \) we have \( \land^k (A^*) = (\land^k A)^* \) by direct calculation using the definition of the inner product on \( \land^k \mathcal{H} \), and we denote this operator simply by \( \land^k A^* \) in view of the lack of ambiguity. In particular it follows easily that if \( P \in \mathcal{B}(\mathcal{H}) \) is an orthogonal projection then so is \( \land^k P \in \mathcal{B}(\land^k \mathcal{H}) \).

Using (4) we observe that
\[ \prod_{i=1}^{k} \sigma_i(A) = \| \land^k A \| = \left\| (\land^k A)^* \right\| = \prod_{i=1}^{k} \sigma_i(A^*) \]
for every \( A \in \mathcal{B}(\mathcal{H}) \) and \( k \geq 1 \), and the identity \( \sigma_k(A) = \sigma_k(A^*) \) for all \( A \in \mathcal{B}(\mathcal{H}) \) and \( k \geq 1 \) follows easily using the above identity and induction on \( k \). The inequality \( \sigma_k(AB) \leq \|A\|\sigma_k(B) \), which is valid for all \( A, B \in \mathcal{B}(\mathcal{H}) \) and \( k \geq 1 \), may easily be obtained directly from the definition of the singular values; since this implies in particular \( \sigma_k(AB) = \sigma_k(B^*A^*) \leq \|B^*\|\sigma_k(A^*) \) we deduce that also \( \sigma_k(AB) \leq \sigma_k(A)\|B\| \). These identities will be used frequently without comment in the proof of Theorem 2.

We complete this section by proving the following two lemmas which will be used in the proof of Theorem 2.

**Lemma 2.1.** Let \( \mathcal{S}_k := \{ A \in \mathcal{B}(\mathcal{H}) : \sigma_{k+1}(A) < \sigma_k(A) \} \), and for each \( A \in \mathcal{S}_k \) let \( \mathcal{U}(A) \) denote the \( k \)-dimensional subspace of \( \mathcal{H} \) spanned by those eigenvectors of \( A^*A \) which correspond to eigenvalues of \( A^*A \) greater than or equal to \( \sigma_k(A)^2 \). Then \( \mathcal{U}: \mathcal{S}_k \to \mathfrak{g}_k(\mathcal{H}) \) is continuous.

**Proof.** Suppose first that \( \mathcal{H} \) is a complex Hilbert space. If \( A \) belongs to \( \mathcal{S}_k \) then the (not necessarily distinct) values \( \sigma_1(A)^2, \ldots, \sigma_k(A)^2 \) are eigenvalues of the operator \( A^*A \) with total multiplicity \( k \), and the remainder of the spectrum of \( A^*A \) is contained in the ball about the origin of radius \( \sigma_{k+1}(A)^2 \). Since \( \sigma_{k+1}(A) < \sigma_k(A) \) we may choose a counterclockwise-oriented circular curve \( \Gamma \subset \mathbb{C} \) which encloses the interval \( [\sigma_k(A)^2, \sigma_1(A)^2] \), does not enclose or intersect the closed ball about the origin of radius \( \sigma_{k+1}(A)^2 \), and is centred on the real axis. Since the singular value functions \( \sigma_i \) are all 1-Lipschitz continuous there exists an open set \( U \) of \( \mathcal{B}(\mathcal{H}) \) containing \( A \) such that \( U \subseteq \mathcal{S}_k \), and such that for every \( B \in U \) the curve \( \Gamma \) does not intersect the spectrum of \( B^*B \). By [11 Theorem IV.3.16] for every \( B \in U \) the integral
\[ P_B := \frac{1}{2\pi i} \int_{\Gamma} (\text{Id}_\mathcal{H} - zB^*B)^{-1} dz \]
is a well-defined projection operator on \( \mathcal{H} \) whose image is the span of the eigenvectors of \( B^*B \) corresponding to the eigenvalues \( \sigma_1(B)^2, \ldots, \sigma_k(B)^2 \), which is to say \( \mathcal{U}(B) \). Since \( B^*B \) is self-adjoint it follows easily that \( (\text{Id}_\mathcal{H} - zB^*B)^{-1} \) is self-adjoint and therefore \( P_B^* = P_B \), which implies that \( P_B \) is moreover an orthogonal projection. By [11 Theorem IV.3.16] the projection \( P_B \) varies continuously with respect to \( B \in U \), and this is precisely the statement which is required to prove the lemma in this case.

If on the other hand \( \mathcal{H} \) is a real Hilbert space then we may apply the above argument to the operator \( A^\mathbb{C} \) induced by \( A \) on the complexification \( \mathcal{H}^\mathbb{C} \). Since \( A^\mathbb{C} \)
commutes with the conjugation operator on \( \mathcal{H}^C \) and the contour \( \Gamma \) is symmetrical about the real axis, the integral \( \mathcal{H} \) commutes with the conjugation operator on \( \mathcal{H}^C \). It follows that \( P_{A^C} \) is precisely the complexification of the operator on \( \mathcal{H} \) given by orthogonal projection onto \( \mathcal{H}(A) \), and since by the above reasoning the former depends continuously on \( A \), the latter does also.

**Lemma 2.2.** Let \( \mathcal{H} \) be a Hilbert space and \( P_1, P_2 \in \mathcal{B}(\mathcal{H}) \) orthogonal projections with equal, finite rank. Then \( \| P_1 - P_2 \| = \| P_1(\text{Id}_{\mathcal{H}} - P_2) \| \).

**Proof.** Since \( P_1 \) and \( P_2 \) are orthogonal they satisfy \( P_1 = P_1^* \) and \( P_2 = P_2^* \), and therefore

\[
(6) \quad \| (\text{Id}_{\mathcal{H}} - P_2)P_1 \| = \| (\text{Id}_{\mathcal{H}} - P_2^*)P_1^* \| = \| (P_1(\text{Id}_{\mathcal{H}} - P_2))^* \| = \| P_1(\text{Id}_{\mathcal{H}} - P_2) \|.
\]

In particular if \( \| P_1(\text{Id}_{\mathcal{H}} - P_2) \| = 1 \) then \( \| (\text{Id}_{\mathcal{H}} - P_2)P_1 \| = 1 \). If \( (v_n) \) is a sequence of unit vectors such that \( \lim_{n \to \infty} \| (\text{Id}_{\mathcal{H}} - P_2)v_n \| = 1 \) then in particular \( \lim_{n \to \infty} \| P_1v_n \| = 1 \) so that \( (v_n) \) accumulates on the image of \( P_1 \). Since the image of \( P_1 \) is finite-dimensional it follows that there exists an accumulation point \( v \) of \( (v_n) \) such that \( \| (\text{Id}_{\mathcal{H}} - P_2)v \| = \| P_1v \| = \| v \| = 1 \). We have \( P_1v = v \) and therefore \( \| v - P_2v \| = 1 \), which implies that \( v \) is in the kernel of \( P_2 \). We therefore have \( \| P_1v - P_2v \| = \| P_1v \| = 1 \) and consequently \( \| P_1 - P_2 \| \geq 1 \). The converse inequality \( \| P_1 - P_2 \| \leq 1 \) holds for all pairs of orthogonal projections (see e.g. [1, p.109–110]) and the proof is complete in this case.

Now suppose instead that \( \| P_1(\text{Id}_{\mathcal{H}} - P_2) \| < 1 \), in which case \( \| (\text{Id}_{\mathcal{H}} - P_2)P_1 \| < 1 \). By applying [11 Theorem I.6.34] we arrive at one of two possible outcomes: either \( \| (\text{Id}_{\mathcal{H}} - P_2)P_1 \| = \| P_1 - P_2 \| \), or \( P_2 \) maps the image of \( P_1 \) bijectively onto a proper subspace of the image of \( P_2 \). Since by hypothesis \( P_1 \) and \( P_2 \) have equal, finite rank the latter outcome is impossible, and the first outcome together with (6) completes the proof. \( \square \)

### 3. Proof of Theorem 2

The technical core of Theorem 2 is the following result, the proof of which occupies the bulk of this section. The derivation of Theorem 2 from Theorem 3 is brief and is presented subsequently.

**Theorem 3.** Let \( T: X \to X \) be a homeomorphism of a nonempty topological space and \( A: X \times \mathbb{N} \to \mathcal{B}(\mathcal{H}) \) a continuous cocycle such that the quantity \( K := \sup_{x \in X} \| A(x) \| \) is finite. Suppose that there exist constants \( C_1 > 0 \) and \( \tau \in (0, 1) \) such that for every \( x \in X \) and \( n \geq 1 \),

\[
\max \{ \sigma_{k+1}(A(x, n)), \sigma_{k+1}(A(Tx, n)) \} < C_1 \tau^n \sigma_k(A(x, n + 1)).
\]

Then:

(i) There exists a continuous \( \wedge^k A \)-invariant splitting \( \wedge^k \mathcal{H} = \bar{\mathcal{U}} \oplus \bar{\mathcal{V}} \) with \( \text{dim} \bar{\mathcal{U}} = 1 \) such that for all \( x \in X \), if \( n \) is large enough that \( 2KC_1(\tau^{n+1} + 6k(1-\tau)^{-1}) \) is finite then \( 2KC_1(1+6k(1-\tau)^{-1}) \tau^n < 1 \);

(ii) \( \sup_{\tilde{v} \in \bar{\mathcal{V}}(x)} \| \wedge^k A(x, n)\tilde{v} \| \leq K \left( 2 + \frac{4k}{1-\tau} \right) \tau^n \inf_{\tilde{u} \in \bar{\mathcal{U}}(x)} \| \wedge^k A(x, n)\tilde{u} \| . \)

If additionally \( X \) is compact, then the splitting \( \wedge^k \mathcal{H} = \bar{\mathcal{U}} \oplus \bar{\mathcal{V}} \) is dominated.
(ii) There exists a continuous \( A \)-invariant splitting \( \mathcal{H} = \mathcal{U} \oplus \mathcal{V} \) with \( \dim \mathcal{U} = k \). If in addition \( X \) is compact, then there exist constants \( \delta, M > 0 \) such that for all \( x \in X \) and \( n \geq 1 \),

\[
\|A(x, n)u\| \geq \delta \sigma_k(A(x, n))\|u\|
\]

for all \( u \in \mathcal{U}(x) \), and

\[
\|A(x, v)v\| \leq M\sigma_{k+1}(A(x, n))\|v\|
\]

for all \( v \in \mathcal{V}(x) \).

Proof of Theorem \( \mathcal{H} \). Step 1: properties of singular spaces. Let \( n_0 \geq 1 \) be the smallest integer such that \( KC_1\tau^{n_0} < 1 \). For each \( n \geq n_0 \) and \( x \in X \) we have

\[
0 \leq \sigma_{k+1}(A(x, n)) < C_1\tau^n\sigma_k(A(x, n + 1)) \leq C_1\tau^n\|A(T^n x)\|\sigma_k(A(x, n)) \leq \sigma_k(A(x, n)) .
\]

The inequality \( \sigma_k(A(x, n)) > \sigma_{k+1}(A(x, n)) \) implies that the values \( \sigma_1(A(x, n))^2 \) to \( \sigma_k(A(x, n))^2 \) are eigenvalues of the positive semidefinite operator \( A(x, n)^*A(x, n) \). Let \( \mathcal{W}_n(x) \) be the \( k \)-dimensional subspace of \( \mathcal{H} \) which is spanned by the corresponding eigenvectors. Since \( A(x, n)^*A(x, n) \) is symmetric these eigenvectors may be chosen to be pairwise orthogonal. Let \( w_1, \ldots, w_k \) denote these eigenvectors (normalised to unit length) and note that for distinct \( i \) and \( j \)

\[
\langle A(x, n)w_i, A(x, n)w_j \rangle = \langle w_i, A(x, n)^*A(x, n)w_j \rangle = \sigma_j(A(x, n))^2 \langle w_i, w_j \rangle = 0.
\]

If \( w = \sum_{i=1}^k \alpha_i w_i \in \mathcal{W}_n(x) \) it follows that

\[
\|A(x, n)w\|^2 = \|A(x, n) \left( \sum_{i=1}^k \alpha_i w_i \right) \|^2
\]

\[
= \sum_{i=1}^k |\alpha_i|^2 \|A(x, n)w_i\|^2
\]

\[
= \sum_{i=1}^k |\alpha_i|^2 \langle A(x, n)w_i, A(x, n)w_i \rangle
\]

\[
= \sum_{i=1}^k |\alpha_i|^2 \langle w_i, A(x, n)^*A(x, n)w_i \rangle
\]

\[
\geq \sigma_k(A(x, n))^2 \left( \sum_{i=1}^k |\alpha_i|^2 \right) = \sigma_k(A(x, n))^2 \|w\|^2
\]

and we conclude that \( \|A(x, n)w\| \geq \sigma_k(A(x, n))\|w\| \) for every \( w \in \mathcal{W}_n(x) \). A similar but simpler calculation shows us that

\[
\| \wedge^k A(x, n)(w_1 \wedge \cdots \wedge w_k) \|^2 = \langle w_1 \wedge \cdots \wedge w_k, \wedge^k A(x, n)^*A(x, n)(w_1 \wedge \cdots \wedge w_k) \rangle
\]

\[
= \left( \prod_{i=1}^k \sigma_i(A(x, n))^2 \right) \langle w_1 \wedge \cdots \wedge w_k, w_1 \wedge \cdots \wedge w_k \rangle
\]

\[
= \| \wedge^k A(x, n) \|^2 \|w_1 \wedge \cdots \wedge w_k\|^2 .
\]
In particular since \( \sigma \) converges uniformly to a limit \((10)\)

\[
\|k\|_v \text{ of rank } v
\]

obtain the estimate \( P \) and since \( P \) sequence of Lemma 2.1. We wish to show that the sequence of continuous functions \( W \) with image \( W \) ment taking \( \sigma \) conclude that necessarily \( \|A\|_v \leq \sigma_{k+1}(A(x,n))\) then

\[
\langle A(x,n)v, A(x,n)w \rangle = \langle v, A(x,n)^*A(x,n)w \rangle = \sigma_i(A(x,n))^2\langle v, w \rangle = 0
\]

so that \( A(x,n)v \) and \( A(x,n)w \) are orthogonal to one another. It follows in a similar manner to the estimate on \( W_n(x) \) that if \( u \) is a unit vector which belongs to the \((k+1)\)-dimensional subspace spanned by \( W_n(x) \) together with \( v \), then \( \|A(x,n)u\| \geq \min\{\|A(x,n)v\|, \sigma_k(A(x,n))\} \). This minimum is clearly attained by taking \( u \) to be either \( v \) or a suitable eigenvector of \( A(x,n)^*A(x,n) \). By the definition of \( \sigma_{k+1}(A(x,n)) \) the value of this infimum is at most \( \sigma_{k+1}(A(x,n)) \), and we conclude that necessarily \( \|A(x,n)v\| \leq \sigma_{k+1}(A(x,n))\|v\| \)

as claimed.

**Step 2:** construction of the lower space \( V \) and its orthogonal complement \( W \). For each \( n \geq 1 \) let \( P_n(x) \in \mathcal{B}(\mathcal{H}) \) denote the orthogonal projection with image \( W_n(x) \) and kernel \( V_n(x) \), which depends continuously on \( x \) as a consequence of Lemma 2.1. We wish to show that the sequence of continuous functions \( P_n : X \to \mathcal{B}(\mathcal{H}) \) is uniformly convergent. Let \( n \geq n_0 \) and \( x \in X \), and let \( v \in V_n(x) \). We may write

\[
A(x,n+1)v = A(x,n+1)P_{n+1}(x)v + A(x,n+1)(\text{Id}_\mathcal{H} - P_{n+1}(x))v
\]

and since \( P_{n+1}(x)v \in W_{n+1}(x) \), \((\text{Id}_\mathcal{H} - P_{n+1}(x))v \in V_{n+1}(x) \) and \( v \in V_n(x) \) we may rearrange the above equation and apply the inequalities derived in step 1 to obtain the estimate

\[
\sigma_k(A(x,n+1))\|P_{n+1}(x)v\| \leq \|A(x,n+1)P_{n+1}(x)v\|
\]

\[
\leq \|A(x,n+1)v\| + \|A(x,n+1)(\text{Id}_\mathcal{H} - P_{n+1}(x))v\|
\]

\[
\leq \|A(T^n x)\| \cdot \|A(x,n)v\| + \sigma_k(A(x,n+1))\|v\|
\]

\[
\leq 2\|A(T^n x)\|\sigma_k(A(x,n))\|v\|
\]

In particular since \( \sigma_k(A(x,n)) > 0 \) by \((9)\) we obtain

\[
\|P_{n+1}(x)v\| \leq 2KC_1\tau^n\|v\|.
\]

Since \( v \in V_n(x) \) was arbitrary we deduce using Lemma 2.2

\[
\|P_n(x) - P_{n+1}(x)\| = \|P_{n+1}(x)(\text{Id}_\mathcal{H} - P_n(x))\| = \sup_{v \in V_n(x)} \|P_{n+1}(x)v\| \leq 2KC_1\tau^n.
\]

We conclude that the sequence of continuous functions \( P_n \) is Cauchy and hence converges uniformly to a limit \( P : X \to \mathcal{B}(\mathcal{H}) \) whose values are orthogonal projections of rank \( k \) and which satisfies

\[
(10) \quad \|P(x) - P_n(x)\| \leq \left(\frac{2C_1K}{1 - \tau}\right)\tau^n
\]

for all \( n \geq n_0 \) and \( x \in X \). For each \( x \in X \) let us define \( W(x) \) and \( V(x) \) to be respectively the image and the kernel of \( P(x) \).

**Step 3:** invariance of the lower space \( V \). We next claim that \( A(x)V(x) \subseteq V(Tx) \) for every \( x \in X \). Let \( x \in X \) and \( v \in V(x) \), and for every \( n \geq n_0 \) define \( v_n := (\text{Id}_\mathcal{H} - P_{n+1}(x))v \) so that \( v_n \in V_{n+1}(x) \) and \( \lim_{n \to \infty} v_n = v \). Since \( A(x)v \in V(Tx) \)
trivially when \( v = 0 \) we shall assume that \( v \) is nonzero. If \( n \geq n_0 \) is large enough that \( v_n \neq 0 \), then let us write

\[
\mathcal{A}(x, n + 1)v_n = \mathcal{A}(T x, n) P_n(T x) \mathcal{A}(x)v_n + \mathcal{A}(T x, n) (\text{Id}_H - P_n(T x)) \mathcal{A}(x)v_n.
\]

In a similar fashion to the previous step we obtain

\[
\sigma_k(\mathcal{A}(T x, n)) \| P_n(T x) \mathcal{A}(x)v_n \|
\leq \| \mathcal{A}(T x, n) P_n(T x) \mathcal{A}(x)v_n \|
\leq \| \mathcal{A}(T x, n) (\text{Id}_H - P_n(T x)) \mathcal{A}(x)v_n \|
\leq \sigma_{k+1}(\mathcal{A}(x, n + 1)) \| v_n \| + \sigma_{k+1}(\mathcal{A}(T x, n)) \| \mathcal{A}(x)v_n \|
\leq 2K \sigma_{k+1}(\mathcal{A}(T x, n)) \| v_n \|
< 2C_1 K \tau^n \sigma_k(\mathcal{A}(x, n + 1)) \| v_n \|
\leq 2C_1 K \tau^n \sigma_k(\mathcal{A}(T x, n)) \| v_n \|
\]

and since the last term is strictly greater than zero we may divide by \( \sigma_k(\mathcal{A}(T x, n)) \) to obtain

\[
\| P_n(T x) \mathcal{A}(x)v_n \| \leq 2C_1 K \tau^n \| v_n \|.
\]

We deduce that

\[
\| P(T x) \mathcal{A}(x)v \| = \lim_{n \to \infty} \| P_n(T x) \mathcal{A}(x)v_n \| = 0
\]

and thus \( \mathcal{A}(x)v \in \mathcal{V}(T x) \) as required to prove the claim.

**Step 4: lower growth estimates for elements of \( \wedge^k \mathcal{W} \)**. We next claim that for any \( k \) vectors \( w_1, \ldots, w_k \in \mathcal{W}(x) \) and any \( n \geq n_0 \) we have

\[
\| \wedge^k \mathcal{A}(x, n) (w_1 \wedge \cdots \wedge w_k) \| \geq (1 - 2C_2 \tau^n) \| \wedge^k \mathcal{A}(x, n) \| \cdot \| w_1 \wedge \cdots \wedge w_k \|
\]

where \( C_2 := 2kC_1 K/(1 - \tau) \). Let \( x \in X \) and \( n \geq n_0 \). Since \( \wedge^k \mathcal{W}(x) \) is one-dimensional it suffices to prove this inequality for a single linearly independent \( k \)-tuple \( w_1, \ldots, w_k \). Let us therefore assume that \( w_1, \ldots, w_k \) is an orthonormal basis for \( \mathcal{W}(x) \), in which case \( \| w_1 \wedge \cdots \wedge w_k \| = 1 \). By writing the difference

\[
w_1 \wedge \cdots \wedge w_k - P_n(x) w_1 \wedge \cdots \wedge P_n(x) w_k
\]

as the sum

\[
\sum_{j=1}^k w_1 \wedge \cdots \wedge w_{j-1} \wedge (w_j - P_n(x) w_j) \wedge P_n(x) w_{j+1} \wedge \cdots \wedge P_n(x) w_k
\]

we may obtain the estimate

\[
(11) \quad \| w_1 \wedge \cdots \wedge w_k - P_n(x) w_1 \wedge \cdots \wedge P_n(x) w_k \| \leq \sum_{j=1}^k \| w_j - P_n(x) w_j \| = \sum_{j=1}^k \| (P(x) - P_n(x)) w_j \| \leq k \| P(x) - P_n(x) \| \leq C_2 \tau^n,
\]

where \( C_2 := 2kC_1 K/(1 - \tau) \), using (10). It follows that the difference

\[
\| \wedge^k \mathcal{A}(x, n) (w_1 \wedge \cdots \wedge w_k) \| - \| \wedge^k (\mathcal{A}(x, n) P_n(x)) (w_1 \wedge \cdots \wedge w_k) \|
\]

is bounded by \( C_2 \tau^n \| \wedge^k \mathcal{A}(x, n) \| \). Now \( P_n(x) w_1 \wedge \cdots \wedge P_n(x) w_k \) belongs to the one-dimensional space \( \wedge^k \mathcal{W}_n(x) \), and hence is proportional to \( w'_1 \wedge \cdots \wedge w'_k \in \wedge^k \mathcal{W}_n(x) \)
Indeed, if as the linear combination (which in turn equals a finite linear combination of terms of the form $v \wedge (12)$ thus admits a dense linear subspace which is mapped to zero by $A$ where $w$ of terms of this form must also be sent to zero by $\wedge x$ complement step 4 by proving that for all $v$ and this clearly belongs to the right-hand side of (12). It follows that if $\hat{\omega}$ converse direction it is clear that if $v$ we have by the triangle inequality hence using (11)

$$\|\wedge^k (A(x,n)) P_n(x) (w_1 \wedge \cdots \wedge w_k)\| \geq (1 - C_2 x^n) \| \wedge^k A(x,n) \|$$

and therefore

$$\|\wedge^k A(x,n) (w_1 \wedge \cdots \wedge w_k)\| \geq (1 - 2C_2 x^n) \| \wedge^k A(x,n) \|$$

which proves the claim.

**Step 5: characterisation of the kernel of $\wedge^k P(x)$**. We assert that for all $x \in X$

$$\ker \wedge^k P(x) = \text{span}\{v_1 \wedge \cdots \wedge v_k : v_1, \ldots, v_{k-1} \in H \text{ and } v_k \in \mathcal{V}(x)\}.$$  

Indeed, if $v_1 \wedge \cdots \wedge v_k \in \wedge^k H$ then we may rewrite the vector

$$(\text{Id}_{\wedge^k H} - \wedge^k P(x))(v_1 \wedge \cdots \wedge v_k) = v_1 \wedge \cdots \wedge v_k - P(x)v_1 \wedge \cdots \wedge P(x)v_k$$

as the linear combination

$$\sum_{j=1}^{k} v_1 \wedge \cdots \wedge v_{j-1} \wedge (v_j - P(x)v_j) \wedge P(x)v_{j+1} \wedge \cdots \wedge P(x)v_k$$

which in turn equals

$$\sum_{j=1}^{k} (-1)^{k-j} v_1 \wedge \cdots \wedge v_{j-1} \wedge P(x)v_{j+1} \wedge \cdots \wedge P(x)v_k \wedge (\text{Id}_H - P(x))v_j,$$

and this clearly belongs to the right-hand side of (12). It follows that if $\hat{v} \in \wedge^k H$ is a finite linear combination of terms of the form $v_1 \wedge \cdots \wedge v_k$ then $(\text{Id}_{\wedge^k H} - \wedge^k P(x))\hat{v}$ belongs to the right-hand side of (12). Since every element of $\wedge^k H$ is a limit of a sequence of such linear combinations, and the right-hand side of (12) is closed, we conclude that the image of $\text{Id}_{\wedge^k H} - \wedge^k P(x)$ is contained in the right-hand side of (12), the former subspace being of course precisely the kernel of $\wedge^k P(x)$. In the converse direction it is clear that if $v_1, \ldots, v_{k-1} \in H$ and $v_k \in \mathcal{V}(x)$ then we have $P(x)v_1 \wedge \cdots \wedge P(x)v_k = 0$ since $P(x)v_k = 0$, and similarly any linear combination of terms of this form must also be sent to zero by $\wedge^k P(x)$. The right-hand side of (12) thus admits a dense linear subspace which is mapped to zero by $\wedge^k P(x)$, and by continuity it follows that the whole of the right-hand side of (12) is contained in the kernel of $\wedge^k P(x)$. This completes the proof of the identity (12).

**Step 6: upper growth estimates for elements of $\ker \wedge^k P(x)$**. We now complement step 4 by proving that for all $x \in X$ and $n \geq n_0$

$$\|\wedge^k A(x,n) (\text{Id}_{\wedge^k H} - \wedge^k P(x)) \| \leq (KC_1 + C_2) x^n \| \wedge^k A(x,n) \|.$$  

Let $v_1, \ldots, v_k \in H$ be orthonormal vectors with $v_k \in \mathcal{V}(x)$. Noting that $P(x)v_k = 0$, we have by the triangle inequality

$$\|\wedge^k A(x,n) (v_1 \wedge \cdots \wedge v_k)\|$$

$$\leq \left\| \wedge^k A(x,n) (v_1 \wedge \cdots \wedge v_{k-1} \wedge (P_n(x) - P(x))v_k) \right\|$$

$$+ \left\| \wedge^k A(x,n) (v_1 \wedge \cdots \wedge v_{k-1} \wedge (\text{Id}_H - P_n(x))v_k) \right\|.$$
The first of the two terms on the right-hand side admits the bound
\[ \| \Lambda^k A(x, n) (v_1 \wedge \cdots \wedge v_{k-1} \wedge (P_n(x) - P(x))v_k) \| \leq \| \Lambda^k A(x, n) \| \cdot \| P_n(x) - P(x) \| \leq C_2 \tau^n \| \Lambda^k A(x, n) \| \]
using (10). To estimate the second term we note that since
\[ \| A(x, n) (\text{Id}_H - P_n(x))v_k \| \leq \sigma_{k+1}(A(x, n)) \]
we may estimate
\[ \| \Lambda^k A(x, n) (v_1 \wedge \cdots \wedge v_{k-1} \wedge (\text{Id}_H - P_n(x))v_k) \| \leq \| \Lambda^{k-1} A(x, n) \| \cdot \sigma_{k+1}(A(x, n)) \]
\[ \leq C_1 \tau^n \| \Lambda^{k-1} A(x, n) \| \cdot \sigma_k(A(x, n)) \]
\[ \leq C_1 K \tau^n \| \Lambda^{k-1} A(x, n) \| \cdot \sigma_k(A(x, n)) \]
\[ = C_1 K \tau^n \| \Lambda^k A(x, n) \|. \]
Combining these estimates we obtain
\[ \| \Lambda^k A(x, n)(v_1 \wedge \cdots \wedge v_k) \| \leq (C_2 + KC_1) \tau^n \| \Lambda^k A(x, n) \| \cdot \| v_1 \wedge \cdots \wedge v_k \| \]
if \( v_1, \ldots, v_{k-1} \in H \) and \( v_k \in \mathcal{V}(x) \), and if these \( k \) vectors are orthonormal.

Now suppose that \( v_1, \ldots, v_{k-1} \in H \) and \( v_k \in \mathcal{V}(x) \) are arbitrary: by Gram-Schmidt orthogonalisation in \( H \) we may choose \( v_1', \ldots, v_{k-1}' \in H \) and \( v_k' \in \mathcal{V}(x) \) such that \( v_k' \) is proportional to \( v_k \), and such that \( v_1', \ldots, v_k' \) is an orthonormal basis for the space spanned by \( v_1, \ldots, v_k \). In particular the vectors \( v_1' \wedge \cdots \wedge v_k' \) and \( v_1 \wedge \cdots \wedge v_k \) are proportional to one another, and we conclude that (14) is valid for \( v_1, \ldots, v_{k-1} \in H \) and \( v_k \in \mathcal{V}(x) \) without assumption of orthonormality. More generally, suppose that \( \hat{v} \in \Lambda^k H \) may be written in the form \( \hat{v} = \sum_{i=1}^m \hat{v}_i \) where each \( \hat{v}_i \) has the form \( \hat{v}_i = v_1 \wedge \cdots \wedge v_k \) with \( v_1, \ldots, v_{k-1} \in H \) and \( v_k \in \mathcal{V}(x) \). By Gram-Schmidt orthogonalisation in \( \Lambda^k H \) we may assume without loss of generality that the vectors \( \hat{v}_i \in \Lambda^k H \) are orthogonal to one another. In this case we have
\[ \| \Lambda^k A(x, n) \hat{v} \| \leq \sum_{i=1}^m \| \Lambda^k A(x, n) \hat{v}_i \| \]
\[ \leq \left( \sum_{i=1}^m \| \Lambda^k A(x, n) \hat{v}_i \|^2 \right)^{\frac{1}{2}} \]
\[ \leq \left( \sum_{i=1}^m ((C_2 + KC_1) \tau^n \| \Lambda^k A(x, n) \| \| \hat{v}_i \|^2 \right)^{\frac{1}{2}} \]
\[ = (C_2 + KC_1) \tau^n \| \Lambda^k A(x, n) \| \cdot \| \hat{v} \|. \]
In view of the characterisation (12) of the kernel of \( \Lambda^k P(x) \), we have shown that the estimate
\[ \| \Lambda^k A(x, n) \hat{v} \| \leq (KC_1 + C_2) \tau^n \| \Lambda^k A(x, n) \| \cdot \| \hat{v} \| \]
is satisfied for all \( \hat{v} \) belonging to a dense subset of \( \ker \Lambda^k P(x) \), and by continuity it is satisfied on the entirety of \( \ker \Lambda^k P(x) \) which completes the proof of step 6.

**Step 7: construction of the dual space \( \mathcal{Y} \) and upper invariant space \( \mathcal{U} \).**
Now let us reapply the above chain of reasoning to the function \( \mathcal{B} : X \times \mathbb{N} \to \mathcal{B}(H) \)
defined by $B(x, n) := A(T^{-n}x, n)^*$, which is easily verified to be a cocycle over the homeomorphism $T^{-1}: X \to X$ and to satisfy

$$\max \{\sigma_{k+1}(B(x, n)), \sigma_{k+1}(B(T^{-1}x, n))\} < C_1 \tau^n \sigma_k(B(x, n + 1))$$

for every $x \in X$ and $n \geq 1$. Applying steps 1 through 4 to the cocycle $B$ over the transformation $T^{-1}$, we find in the same manner that there exists a continuous function $\mathcal{Y}: X \to \text{Gr}_k(\mathcal{H})^*$ such that $B(x)\mathcal{Y}(x) \subseteq \mathcal{Y}(T^{-1}x)$ for every $x \in X$, and such that for all $x \in X$, if $u_1, \ldots, u_k \in \mathcal{Y}(x)^\perp$ then for all $n \geq n_0$

$$(15) \quad \|\wedge^k B(x, n)(u_1 \wedge \cdots \wedge u_k)\| \geq (1 - 2C_2 \tau^n) \|\wedge^k B(x, n)\| \cdot \|u_1 \wedge \cdots \wedge u_k\|$$

for the same constant $C_2 > 0$. Let us define $\mathcal{U}(x) := \mathcal{Y}(x)^\perp$ for every $x \in X$. If $u \in \mathcal{U}(x)$ and $y \in \mathcal{Y}(T(x))$ then

$$(A(x)u, y) = \langle u, A(x)^*y \rangle = \langle u, B(Tx)y \rangle = 0$$

since $B(Tx)y \in \mathcal{Y}(x)$, so we have $A(x)u \in \mathcal{Y}(T(x)) = \mathcal{U}(T(x))$ and we conclude that $A(x)\mathcal{U}(x) \subseteq \mathcal{U}(T(x))$ for every $x \in X$. Clearly $\mathcal{U}$ is continuous and $\mathcal{U}(x)$ is $k$-dimensional for all $x \in X$.

**Step 8: lower growth for elements of $\wedge^k \mathcal{U}$.** We now claim that for every $x \in X$ and $n \geq n_0$, if $u_1, \ldots, u_k \in \mathcal{U}(x)$ then

$$(16) \quad \|\wedge^k A(x, n)(u_1 \wedge \cdots \wedge u_k)\| \geq (1 - C_3 \tau^n) \|\wedge^k A(x, n)\| \cdot \|u_1 \wedge \cdots \wedge u_k\|$$

where $C_3 := KC_1 + 3C_2$. Since $\wedge^k \mathcal{U}(x)$ is one-dimensional it is sufficient to prove this for a single nonzero vector $u_1 \wedge \cdots \wedge u_k \in \wedge^k \mathcal{U}(x)$. Given $x \in X$ and $n \geq n_0$, let $u_1, \ldots, u_k$ be an orthonormal basis for $\mathcal{U}(x)$ and $u_1', \ldots, u_k'$ an orthonormal basis for $\mathcal{U}(T^n x)$. Let $\hat{w} \in \wedge^k \mathcal{H}$ be a unit vector such that

$$\|\wedge^k B(T^n x, n)(u_1' \wedge \cdots \wedge u_k')\| = \|\langle \hat{w}, \wedge^k B(T^n x, n)(u_1' \wedge \cdots \wedge u_k') \rangle\|.$$

We may write

$$\langle \hat{w}, \wedge^k B(T^n x, n)(u_1' \wedge \cdots \wedge u_k') \rangle = \langle \langle \wedge^k P(x) \rangle \hat{w}, \wedge^k B(T^n x, n)(u_1' \wedge \cdots \wedge u_k') \rangle + \langle \langle \text{Id}_{\wedge^k \mathcal{H}} - \wedge^k P(x) \rangle \hat{w}, \wedge^k B(T^n x, n)(u_1' \wedge \cdots \wedge u_k') \rangle.$$

The absolute value of the second term on the right-hand side is equal to

$$|\langle \wedge^k A(x, n) \langle \text{Id}_{\wedge^k \mathcal{H}} - \wedge^k P(x) \rangle \hat{w}, u_1' \wedge \cdots \wedge u_k' \rangle| \leq (KC_1 + C_2)\tau^n \|\wedge^k A(x, n)\|$$

using (15), and since $\wedge^k P(x)\hat{w} = \|\wedge^k P(x)\hat{w}\|(u_1 \wedge \cdots \wedge u_k) \in \wedge^k \mathcal{U}(x)$, the absolute value of the first term is equal to

$$\|\wedge^k P(x)\hat{w}\| \cdot |\langle u_1 \wedge \cdots \wedge u_k, \wedge^k B(T^n x, n)(u_1' \wedge \cdots \wedge u_k') \rangle|.$$

Since $\|\wedge^k P(x)\hat{w}\| \leq \|\wedge^k P(x)\| = 1$ we deduce that

$$\|\langle u_1 \wedge \cdots \wedge u_k, \wedge^k B(T^n x, n)(u_1' \wedge \cdots \wedge u_k') \rangle\| \geq (1 - (KC_1 + 3C_2)\tau^n)\|\wedge^k A(x, n)\|$$
where we have used \([10]\) together with the identity \(\|\wedge^k A(x,n)\| = \|\wedge^k B(x,n)\|\). Thus

\[
\|\wedge^k A(x,n) (u_1 \wedge \cdots \wedge u_k)\| = \|\langle \wedge^k A(x,n) (u_1 \wedge \cdots \wedge u_k), u_1' \wedge \cdots \wedge u_k' \rangle\|
\]

\[
= \|\langle u_1 \wedge \cdots \wedge u_k, \wedge^k B(T^n x,n) (u_1' \wedge \cdots \wedge u_k') \rangle\|
\]

\[
\geq (1 - (KC_1 + 3C_2)\tau^n) \|\wedge^k A(x,n)\|
\]

since the vectors \(\wedge^k A(x,n) (u_1 \wedge \cdots \wedge u_k)\) and \(u_1' \wedge \cdots \wedge u_k'\) both belong to the one-dimensional space \(\wedge^k \mathcal{U}(T^n x)\) and hence are proportional to one another. This completes the proof of the claim.

**Step 9: completion of the proof of (i).** For each \(x \in X\) let \(Q(x) \in \mathcal{B}(\mathcal{H})\) denote the orthogonal projection with image \(\mathcal{U}(x)\). By construction \(Q: X \to \mathcal{B}(\mathcal{H})\) is continuous. For each \(x \in X\) let us define \(\hat{U}(x) := \wedge^k \mathcal{U}(x) = \text{im} \wedge^k Q(x)\) and \(\hat{V}(x) := \ker \wedge^k P(x)\). The operators \(\wedge^k P(x)\) and \(\wedge^k Q(x)\) are orthogonal projections which depend continuously on \(x\) and therefore \(\hat{U}\) and \(\hat{V}\) are continuous. Clearly \(\hat{U}(x)\) is 1-dimensional and \(\hat{V}(x)\) is 1-codimensional for every \(x \in X\). It follows from the \(A\)-invariance of \(\mathcal{U}\) that \(\wedge^k A(x,n) \mathcal{U}(x) \subseteq \mathcal{U}(T^n x)\) for every \(x \in X\) and \(n \geq 1\), and the characterisation

\[
\ker \wedge^k P(x) = \text{span} \{v_1 \wedge \cdots \wedge v_k: v_1, \ldots, v_k \in \mathcal{H} \text{ and } v_k \in \mathcal{V}(x)\}
\]

established in step 5 together with the \(A\)-invariance of \(\mathcal{V}\) yields the \(\wedge^k A\)-invariance of \(\hat{V}\).

We claim that \(\hat{U}(x) \cap \hat{V}(x) = \{0\}\) for every \(x \in X\). Indeed, if this is not the case for a given \(x \in X\) let \(\hat{u} \in \hat{U}(x) \cap \hat{V}(x)\) be a unit vector. For every \(n \geq n_0\) we have

\[
(1 - C_3\tau^n) \|\wedge^k A(x,n)\| \leq \|\wedge^k A(x,n)\hat{u}\| \leq (KC_1 + C_2)\tau^n \|\wedge^k A(x,n)\|
\]

by combining the results of steps 6 and 8, and this is obviously a contradiction when \(n\) is sufficiently large, so we conclude that \(\hat{U}(x) \cap \hat{V}(x) = \{0\}\) for every \(x \in X\) as claimed. In particular \(\wedge^k \mathcal{H} = \hat{U} \oplus \hat{V}\) is a continuous invariant splitting for the cocycle \(\wedge^k A\). If \(n\) is large enough that \(2C_3\tau^n = 2KC_1(1 + 6(1 - \tau)^{-1})\tau^n < 1\), then from step 8 we obtain

\[
\inf_{\hat{u} \in \hat{U}(x)} \frac{\|\wedge^k A(x,n)\hat{u}\|}{\|\hat{u}\|} \geq (1 - C_3\tau^n) \|\wedge^k A(x,n)\| > \frac{1}{2} \|\wedge^k A(x,n)\|
\]

and from step 6 and the definition of \(C_2\) we obtain

\[
\sup_{\hat{v} \in \hat{V}(x)} \frac{\|\wedge^k A(x,n)\hat{v}\|}{\|\hat{v}\|} \leq KC_1 \left(1 + \frac{2k}{1 - \tau}\right)\tau^n \|\wedge^k A(x,n)\|.
\]

It follows that \([7]\) holds as claimed, and this completes the proof of (i) in the case where \(X\) is not assumed to be compact.

To complete the proof of (i) in the case where \(X\) is compact we must show that the inequality \([7]\) can be extended to all \(n \geq 1\) with a suitable different constant in front of the term \(\tau^n\). Let us fix \(n_1 \geq n_0\) such that \(1 - C_3\tau^n \geq \frac{1}{2}\) for all \(n \geq 1\).

For each \(x \in X\) and integer \(m\) in the range \(1 \leq m < n_1\) we have

\[
\|\wedge^k A(x,m) (u_1 \wedge \cdots \wedge u_k)\| > 0
\]

when \(u_1 \wedge \cdots \wedge u_k \in \wedge^k \mathcal{U}(x)\) is a unit vector, since otherwise we could obtain \(\wedge^k A(x,n_1) (u_1 \wedge \cdots \wedge u_k) = 0\) which would contradict \([10]\) in view of the fact
that \( \| \wedge^k A(x, n_1) \| \geq \sigma_k(A(x, n_1))^k > 0 \). Since \( \wedge^k U(x) = \text{im} \wedge^k P(x) \) depends continuously on \( x \), and \( X \) is compact, it follows that there exists \( \delta \in (0, \frac{1}{2}) \) such that

\[
\| \wedge^k A(x, m) \hat{u} \| \geq \delta K^{kn_1} \| \hat{u} \| \geq \delta \| \wedge^k A(x, n) \| \cdot \| \hat{u} \|
\]

whenever \( x \in X \), \( 1 \leq m < n_1 \) and \( \hat{u} \in \wedge^k U(x) \). We deduce that for all \( x \in X \) and \( n \geq 1 \) we have

\[
\| \wedge^k A(x, n)(u_1 \wedge \cdots \wedge u_k) \| \geq \delta \| \wedge^k A(x, n) \| \cdot \| u_1 \wedge \cdots \wedge u_k \|
\]

for every \( u_1, \ldots, u_k \in U(x) \). On the other hand it is obvious that

\[
\sup_{\| \hat{v} \| = 1} \| \wedge^k A(x, n) \hat{v} \| \leq \| \wedge^k A(x, n) \| \leq \delta^{-n} \| \wedge^k A(x, n) \|
\]

for every \( n \) in the range \( 1 \leq n \leq n_1 \), and combining this with (17) and (18) we conclude that the splitting \( \wedge^k \mathcal{H} = \hat{U} \hat{\oplus} \hat{V} \) is dominated.

**Step 10: lower bound for the angle between \( \hat{U} \) and \( \hat{V} \).** We now claim that \( U(x) \cap V(x) = \{0\} \) for every \( x \in X \). Indeed, if this is false for some \( x \in X \) then there exists a nonzero vector \( v \in U(x) \cap V(x) \), which we may assume to have unit length. Let \( u_1, \ldots, u_{k-1}, \hat{v} \in U(x) \) such that \( u_1, \ldots, u_{k-1}, \hat{v} \) is an orthonormal basis for \( U(x) \). Since \( P(x)v = 0 \) we have \( u_1 \wedge \cdots \wedge u_{k-1} \wedge v \in \ker \wedge^k P(x) = \hat{V}(x) \). On the other hand, since \( u_1 \wedge \cdots \wedge u_{k-1} \wedge v \in \wedge^k U(x) = \hat{U}(x) \) we have \( \hat{U}(x) \cap \hat{V}(x) \neq \{0\} \), contradicting one of the results of step 9. We conclude that for every \( x \in X \) the subspace \( U(x) \cap V(x) \) contains only the zero vector, and in the case where \( X \) is not assumed to be compact this completes the proof of (ii).

It remains only to complete the proof of (ii) in the compact case, so we assume henceforth that \( X \) is compact. We assert that there exists a constant \( \kappa > 0 \) with the following property: if \( x \in X \), \( u_1, \ldots, u_k \in U(x) \), and \( v \in V(x) \), then

\[
\| u_1 \wedge \cdots \wedge u_k \wedge v \| \geq \kappa \| u_1 \wedge \cdots \wedge u_k \| \cdot \| v \|.
\]

Fix \( x \in X \). If \( (v_n) \) is a sequence of unit vectors in \( V(x) \) such that \( \| v_n - Q(x)v_n \| \) tends to zero then clearly \( (v_n) \) accumulates on the unit sphere of \( U(x) \), and since this sphere is compact such a sequence must have a nonzero accumulation point \( v \in U(x) \cap V(x) \), contradicting the previous paragraph. We deduce that for each \( x \in X \) there must exist \( \kappa_x > 0 \) depending on \( x \) such that \( \| v - Q(x)v \| \geq \kappa_x \| v \| \) for every \( v \in V(x) \), or equivalently \( \| (\text{Id}_H - Q(x))(\text{Id}_H - P(x))v \| \geq \kappa_x \| (\text{Id}_H - Q(x))v \| \) for every \( v \in \mathcal{H} \). By continuity of \( U \) and \( V \) it follows that for all \( z \) in a small open neighbourhood of \( x \) we have \( \| v - Q(z)v \| \geq (\kappa_x/2) \| v \| \) for all \( v \in V(z) \), and by compactness of \( X \) we deduce that there is \( \kappa > 0 \) such that for all \( x \in X \) we have \( \| v - Q(x)v \| \geq \kappa \| v \| \) for every \( v \in V(x) \). It follows in particular that if \( u_1, \ldots, u_k \in \hat{U}(x) \) and \( v \in V(x) \) then

\[
\| u_1 \wedge \cdots \wedge u_k \wedge v \| = \| u_1 \wedge \cdots \wedge u_k \wedge (v - Q(x)v) \|
\]

\[
= \| u_1 \wedge \cdots \wedge u_k \| \cdot \| v - Q(x)v \|
\]

\[
\geq \kappa \| u_1 \wedge \cdots \wedge u_k \| \cdot \| v \|
\]

using first the fact that \( Q(x)v \in U(x) \) and therefore \( u_1 \wedge \cdots \wedge u_k \wedge Q(x)v = 0 \), and second the fact that \( v - Q(x)v \) belongs to \( U(x)^\perp \) and is therefore orthogonal to every \( u_i \). This completes the proof of the claimed inequality (19).
Step 11: growth estimates for $\mathcal{U}$ and $\mathcal{V}$. We first prove the claimed estimate for the action of $\mathcal{A}$ on $\mathcal{U}$. Let $x \in X$ and $n \geq 1$, let $u_1 \in \mathcal{U}(x)$ be a unit vector, and choose $u_2, \ldots, u_k$ such that $u_1, \ldots, u_k$ is an orthonormal basis for $\mathcal{U}(x)$. Since

$$\|\wedge^k \mathcal{A}(x, n)(u_1 \wedge \cdots \wedge u_k)\| \leq \|\wedge^{k-1} \mathcal{A}(x, n)\| \cdot \|\mathcal{A}(x, n)u_1\|$$

$$= \left( \prod_{i=1}^{k-1} \sigma_i(\mathcal{A}(x, n)) \right) \|\mathcal{A}(x, n)u_1\|$$

and by (19)

$$\|\wedge^k \mathcal{A}(x, n)(u_1 \wedge \cdots \wedge u_k)\| \geq \delta \|\wedge^k \mathcal{A}(x, n)\| = \delta \prod_{i=1}^{k} \sigma_i(\mathcal{A}(x, n))$$

we find that $\|\mathcal{A}(x, n)u_1\| \geq \delta \sigma_k(\mathcal{A}(x, n))$ as was asserted in the statement of the theorem.

We now turn to the action of $\mathcal{A}$ on $\mathcal{V}$. Let $v \in \mathcal{V}(x)$, and let $u_1, \ldots, u_k$ be an orthonormal basis for $\mathcal{U}(x)$. Since $\mathcal{U}$ and $\mathcal{V}$ are invariant with respect to $\mathcal{A}$ we have $\mathcal{A}(x, n)u_i \in \mathcal{U}(T^i x)$ for each $i = 1, \ldots, k$, and $\mathcal{A}(x, n)v \in \mathcal{V}(T^n x)$. In particular it follows using (19) that

$$\|\wedge^{k+1} \mathcal{A}(x, n)(u_1 \wedge \cdots \wedge u_k \wedge v)\| \geq \kappa \|\wedge^k \mathcal{A}(x, n)(u_1 \wedge \cdots \wedge u_k)\| \cdot \|\mathcal{A}(x, n)v\|.$$ 

Using (18) we may therefore estimate

$$\kappa \delta \|\wedge^k \mathcal{A}(x, n)\| \cdot \|\mathcal{A}(x, n)v\| \leq \kappa \|\wedge^k \mathcal{A}(x, n)(u_1 \wedge \cdots \wedge u_k)\| \cdot \|\mathcal{A}(x, n)v\|$$

$$\leq \|\wedge^{k+1} \mathcal{A}(x, n)(u_1 \wedge \cdots \wedge u_k \wedge v)\|$$

$$\leq \|\wedge^{k+1} \mathcal{A}(x, n)\| \cdot \|v\|$$

and hence

$$\|\mathcal{A}(x, n)v\| \leq \left( \frac{\|\wedge^{k+1} \mathcal{A}(x, n)\|}{\kappa \delta \|\wedge^k \mathcal{A}(x, n)\|} \right) \|v\| = \kappa^{-1} \delta^{-1} \sigma_{k+1}(\mathcal{A}(x, n))\|v\|$$

as desired. The proof of Theorem 3 is complete. \(\square\)

Let us now deduce Theorem 2 from Theorem 3. The implications (a)⇒(c) and (a)⇒(d) follow directly from Theorem 3 and the implication (c)⇒(b) is trivial, so to establish Theorem 2 it suffices to prove both (b)⇒(a) and (d)⇒(a). We begin with the former. Given the validity of (b), let $C_1 > 0$ and $\tau \in (0, 1)$ such that for all $x \in X$ and $n \geq 1$ we have

$$\sup_{v \in \mathcal{V}(x)} \|\mathcal{A}(x, n)v\| < C_1 \tau^n \inf_{u \in \mathcal{U}(x)} \|\mathcal{A}(x, n)u\|.\quad (20)$$

For every $x \in X$ we in particular have $\mathcal{A}(x)u \neq 0$ for all nonzero $u \in \mathcal{U}(x)$, and the subspace $\mathcal{U}(x)$ is finite-dimensional. It follows by the continuity of $\mathcal{U}$ and $\mathcal{A}$ and the compactness of $X$ that there exists $\kappa > 0$ such that $\|\mathcal{A}(x)u\| \geq \kappa \|u\|$ whenever $x \in X$ and $u \in \mathcal{U}(x)$.

Fix $x \in X$ and $n \geq 1$. Since $\mathcal{U}(x) \in \text{Gr}_k(\mathcal{H})$ we have

$$\sigma_k(\mathcal{A}(x, n + 1)) \geq \inf_{u \in \mathcal{U}(x)} \|\mathcal{A}(x, n + 1)u\|.$$
Let $u \in \mathcal{U}(x)$. On the one hand we have
\[
\|A(x, n+1)u\| = \|A(T^nx)A(x, n)u\| \geq \kappa \|A(x, n)u\|
\]
since $A(x, n)u \in \mathcal{U}(T^nx)$, and thus by taking an infimum
\[
(21) \quad \sigma_k(A(x, n+1)) \geq \kappa \inf_{u \in \mathcal{U}(x) \atop \|u\|=1} \|A(x, n)u\|.
\]

On the other hand, for each $u \in \mathcal{U}(x)$ we have $A(x)u \in \mathcal{U}(Tx)$ and therefore
\[
\|A(x, n+1)u\| = \|A(Tx, n)A(x)u\| \geq \inf_{w \in \mathcal{U}(Tx) \atop \|w\|=\|A(x)u\|} \|A(Tx, n)w\|
\]
\[
= \|A(x)u\| \inf_{w \in \mathcal{U}(Tx) \atop \|w\|=1} \|A(Tx, n)w\|
\]
\[
\geq \kappa \inf_{w \in \mathcal{U}(Tx) \atop \|w\|=1} \|A(Tx, n)w\|
\]
so that we may similarly obtain
\[
(22) \quad \sigma_k(A(x, n+1)) \geq \kappa \inf_{w \in \mathcal{U}(Tx) \atop \|w\|=1} \|A(Tx, n)w\|.
\]

Now, if $F \in \text{Gr}_{k+1}(\mathcal{H})$ then there exists a unit vector $w \in \mathcal{V}(x) \cap F$, and thus
\[
\inf_{w \in F \atop \|w\|=1} \|A(x, n)u\| \leq \|A(x, n)w\| \leq \sup_{v \in \mathcal{V}(x) \atop \|v\|=1} \|A(x, n)v\|.
\]

Taking the supremum over all $F \in \text{Gr}_{k+1}(\mathcal{H})$ and invoking (20) we find that
\[
\sigma_{k+1}(A(x, n)) = \sup_{F \in \text{Gr}_{k+1}(\mathcal{H})} \inf_{u \in F \atop \|u\|=1} \|A(x, n)u\|
\]
\[
\leq \sup_{v \in \mathcal{V}(x) \atop \|v\|=1} \|A(x, n)v\| < C_n \inf_{u \in \mathcal{U}(x) \atop \|u\|=1} \|A(x, n)u\|
\]
\[
(24) \quad \sigma_{k+1}(A(x, n)) < C_n \sigma_k(A(x, n+1))
\]

Combining this with (24) and (22) yields
\[
\max \{\sigma_k(A(x, n)), \sigma_k(A(Tx, n))\} < C_1 \kappa^{-1} C_\tau^n \sigma_k(A(x, n+1))
\]
for all $x \in X$ and $n \geq 1$, and we have established (a).

To complete the proof we establish (d)⇒(a). Let $K := \sup_{x \in X} \|A(x)\|$. Applying the implication (b)⇒(a) to the cocycle $\wedge^k A$ we find that there are constants $C > 0$ and $\tau \in (0, 1)$ such that for every $x \in X$ and $n \geq 1$
\[
\prod_{i=1}^k \sigma_i(A(x, n)) = \sigma_1(\wedge^k A(x, n))
\]
\[
< C \tau^n \sigma_2(\wedge^k A(x, n+1))
\]
\[
= C \tau^n \left( \prod_{i=1}^{k-1} \sigma_i(A(x, n+1)) \right) \sigma_{k+1}(A(x, n+1))
\]
\[
\leq K^{k-1} C \tau^n \left( \prod_{i=1}^{k-1} \sigma_i(A(x, n)) \right) \sigma_{k+1}(A(x, n+1))
\]
and therefore
\[
\sigma_k(A(x, n)) < CK^{k-1} \tau^n \sigma_{k+1}(A(x, n+1)).
\]
In a similar manner we may obtain
\[ \sigma_k(A(Tx, n)) < CK^{k-1}r^n \sigma_{k+1}(A(x, n + 1)) \]
and therefore (a) is satisfied. The proof of Theorem 2 is complete.

4. Examples illustrating the sharpness of Theorem 2

In this section we present some examples to show that the hypotheses which yield the implication (a)⇒(b) in Theorem 2 cannot be substantially weakened. We begin with the observation that the topology on \( B(H) \) with respect to which \( A \) is assumed continuous cannot be significantly weakened:

**Proposition 4.1.** Let \( H \) be an infinite-dimensional real or complex Hilbert space. Then there exist a homeomorphism \( T \) of a compact metric space \( X \) and a cocycle \( A : X \times \mathbb{N} \to B(H) \) which satisfies
\[
\max \{ \sigma_2(A(x, n)), \sigma_2(A(Tx, n)) \} < C \tau^n \sigma_1(A(x, n + 1))
\]
with \( C = 2 \) and \( \tau = \frac{1}{2} \), such that \( A \) is continuous with respect to the strong operator topology on \( B(H) \) but does not admit a dominated splitting as in Theorem 2(b).

**Proof.** Let \( X \) be a metric space consisting of a sequence of distinct points \( z_n \) together with a sole additional point \( z = \lim_{n \to \infty} z_n \). Let \( \{e_n\}_{n=0}^{\infty} \) be a sequence of orthonormal vectors in \( H \), and let \( T : X \to X \) be the identity. Define \( A(x, n)e_0 = \frac{1}{n+1}e_0 \)
for all \( x \in X \) and \( n \geq 1 \), \( A(z_k, n)e_k = e_k \) for all \( x \in X \) and \( n, k \geq 1 \), \( A(x, n)e_1 = 0 \) otherwise, and \( A(x, n)v = 0 \) for all \( v \) which are perpendicular to the closed linear span of the sequence \( \langle e_n \rangle \). Clearly \( A : X \times \mathbb{N} \to B(H) \) is a cocycle and is continuous with respect to the strong operator topology. We have
\[
\max \{ \sigma_2(A(x, n)), \sigma_2(A(Tx, n)) \} = \frac{1}{2n} < 1 = \sigma_1(A(x, n + 1))
\]
if \( x \neq z \), and
\[
\max \{ \sigma_2(A(z, n)), \sigma_2(A(Tz, n)) \} = 0 < \frac{1}{2n} = \sigma_1(A(z, n + 1))
\]
so that (a) is satisfied with \( C_1 = 2 \) and \( \tau = \frac{1}{2} \). However, for Theorem 2(b) to be satisfied \( U(z_k) \) must be spanned by \( e_k \) for each \( k \geq 1 \) and \( U(z) \) must be spanned by \( e_0 \), but the resulting function \( U : X \to \text{Gr}_1(H) \) is discontinuous at \( z \) contrary to the requirements of Theorem 2(b). \( \square \)

The other observation which we make in this section is that the maximum of two terms on the left-hand side of the condition (3) cannot be replaced with just the first of the two terms. Note that by replacing \( A(x, n) \) with the dual cocycle \( B(x, n) := A(T^{-n}x, n)^* \) over \( T^{-1} \), this example also shows that the maximum cannot in general be replaced with only the second of the two terms.

**Proposition 4.2.** Let \( M_2(\mathbb{R}) \) denote the vector space of real \( 2 \times 2 \) matrices. Then there exist a homeomorphism \( T \) of a compact metric space \( X \) and a continuous cocycle \( A : X \times \mathbb{N} \to M_2(\mathbb{R}) \) such that
\[
\sigma_2(A(x, n)) < 2^{1-n} \sigma_1(A(x, n + 1))
\]
for all \( x \in X \) and \( n \geq 1 \), but which does not satisfy the hypothesis of Theorem 2(a) for any constants \( C_1 > 0 \) and \( \tau \in (0, 1) \).
Proof. Let $X$ be a compact metric space which may be partitioned into disjoint clopen subsets $X_1$ and $X_2$, and suppose that $T : X \to X$ has the property that for every $x \in X$ at most one element of the set $\{T^n x : n \in \mathbb{Z}\}$ belongs to $X_2$. For example, we could take $X$ to be the one-point compactification of $\mathbb{Z}$, let $T : X \to X$ be the transformation which maps $n$ to $n + 1$ and fixes the point at infinity, and take $X_2 := \{0\}$. In any case, given such a transformation $T$ define

$$A(x) = \begin{cases} 
    \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } x \in X_1 \\
    \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \text{if } x \in X_2
\end{cases}$$

which is a continuous function since the sets $X_1$ and $X_2$ are clopen. Now define $A(x, n) := A(T^{n-1} x) \cdots A(T x) A(x)$ for all $x \in X$ and $n \geq 1$. Clearly $A : X \times \mathbb{N} \to M_2(\mathbb{R})$ is a continuous linear cocycle.

The verification of (23) proceeds by a case-by-case analysis. We observe that the products $A(x, n)$ fall into several types. Let $x \in X$ and $n \geq 1$. If $T^i x \in X_1$ for all $i = 0, \ldots, n$ then we trivially have

$$A(x, n + 1) = \begin{pmatrix} 2^{n+1} & 0 \\ 0 & 1 \end{pmatrix}, \quad A(x, n) = \begin{pmatrix} 2^n & 0 \\ 0 & 1 \end{pmatrix}$$

so that obviously $\sigma_2(A(x, n)) = 1 < 2^{n+1} = \sigma_1(A(x, n + 1))$. In the second case let us suppose that $T^i x \in X_1$ for $i = 1, \ldots, n-1$, but $T^n x \in X_2$. In this case we obtain

$$A(x, n + 1) = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}^n \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2^n \\ 0 & 0 \end{pmatrix},$$

$$A(x, n) = \begin{pmatrix} 2^n & 0 \\ 0 & 1 \end{pmatrix}$$

and thus $\sigma_2(A(x, n)) = 1 < 2^n = \sigma_1(A(x, n + 1))$. In the third case we suppose that $T^k x \in X_2$ for some integer $k$ such that $0 \leq k < n$, and hence necessarily $T^i x \in X_1$ otherwise: in this case

$$A(x, n + 1) = \begin{pmatrix} 2^k & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2^{n-k} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2^k \\ 0 & 0 \end{pmatrix},$$

$$A(x, n) = \begin{pmatrix} 2^k & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2^{n-k-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2^k \\ 0 & 0 \end{pmatrix}$$

and therefore $\sigma_2(A(x, n)) = 0 < 1 = \sigma_1(A(x, n + 1))$. We have shown that in all cases $\sigma_2(A(x, n)) < 2^{1-n} \sigma_1(A(x, n + 1))$ as claimed. However, in the third and final case, if $k = 0$ then we have additionally

$$A(T x, n) = \begin{pmatrix} 2^n & 0 \\ 0 & 1 \end{pmatrix}$$

and thus $\sigma_2(A(T x, n)) = 1 = \sigma_1(A(x, n + 1))$. In particular there cannot exist constants $C > 0$ and $\tau \in (0, 1)$ such that $\sigma_2(A(x, n)) < C \tau^n \sigma_1(A(x, n + 1))$ for every $x \in X$ and $n \geq 1$. $\square$

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