Direct Localization of Multiple Sources by Partly Calibrated Arrays

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Abstract—We present novel solutions to the problem of direct localization of multiple narrowband and arbitrarily correlated sources by partly calibrated arrays, i.e., arrays composed of fully calibrated subarrays yet lacking inter-array calibration. The solutions presented vary in their performance and computational complexity. We present first a relaxed maximum likelihood solution whose concentrated likelihood involves only the unknown locations of the sources and requires an eigen-decomposition of the array covariance matrix at every potential location. To reduce the computational load, we introduce an approximation which eliminates the need for such an eigen-decomposition at every potential location. To further reduce the computational load, novel MUSIC-like and MVDR-like solutions are presented which are computationally much simpler than the existing solutions. The performance of these solutions is evaluated and compared via simulations.

Index Terms—Partly calibrated arrays, direct localization, relaxed maximum likelihood, coherent signals, multipath, signal subspace.

I. INTRODUCTION

Partly calibrated arrays are arrays composed of fully calibrated subarrays, yet lacking inter-array calibration. Such arrays are common in large scale systems composed of small subarrays with large inter-array distances, as is the case in multi-site surveillance systems, multi-site communication systems, and multi-site radar systems.

In such arrays, the large inter-array distances make the calibration of the whole array problematic. Time and phase synchronization in such arrays may be problematic as well. If all the subarrays are time and phase synchronized, they are referred to as coherent. If each subarray is time and phase synchronized internally, but there is no time and phase synchronization across the whole array, they are referred to as noncoherent. If there is time synchronization across the whole array and each subarray is time and phase synchronized internally, but there are unknown phase offsets between the subarrays, they are referred to as phase offset.

A powerful model for partly calibrated arrays, referred to as the Partly Calibrated Array (PCA) model, was introduced by See and Gershman [] in the problem of direction finding. This model can cope with a variety of uncertainties in the direction finding problem, including unknown subarrays displacements and unknown phase offsets between the subarrays. It can be regarded as a generalization of a more limited model, introduced by Pesavento et. al [], addressing partly calibrated arrays composed of identically oriented subarrays with unknown subarray displacements.

Apart from introducing the PCA model, [] presented a MUSIC-like technique for estimating the direction-of-arrival of multiple narrowband sources and the Cramer-Rao bound (CRB) for this problem. This work was followed by Lie et al. [] and Mavrychev et al. [] who introduced MVDR-like techniques. Liao and Chan [] exploited the special structure of the uniform linear array to reduce the computational complexity. A sparse recovery approach for direction finding in partly calibrated arrays composed of subarrays with unknown displacements was introduced by Steffens and Pesavento [].

Independently of this work on direction finding, Weiss [] and Weiss and Amar [] introduced the PCA model in the direct localization problem, to cope with the unknown propagation to the subarrays. Direct localization, advocated first in [-], and further developed in []-[], is a localization scheme in which the location is estimated directly from the data in one-step, as opposed to the more conventional two-step scheme, where the directions-of-arrival to the subarrays are estimated in the first step and then, in the second step, the location is estimated using triangulation. Direct localization provides not only higher accuracy at low signal-to-noise and low signal-to-interference ratios, but not less importantly, reduced ambiguity. This is because the data association step, needed in the two-step procedure and prone to ambiguity errors, is eliminated.

Apart from introducing the PCA model, [] introduced the maximum likelihood solution for a single narrowband source, while [] extended this approach to wideband sources, introduced MUSIC-like solution for sources with unknown waveforms and a maximum likelihood solution for sources with known waveforms, as well as the CRB for these problems. This work was followed by Bosse et al. [], who introduced an alternative space-time approach for wideband sources, by Delestre et al. [] who extended this space-time approach to include the time delay information between the subarrays and by Tirer and Weiss [] who introduced MVDR-like technique for multiple wideband sources. There has been also extension to different types of signals and scattering. Reuven and Weiss [] to cyclostationary signals, Bar-Shalom and Weiss [] to environments with local scattering, Yin et al [] to noncircular sources, and Yi et al. [] to signals parametrized by a small number of parameters.

The PCA model has been used also in related localization problems. Closas et al. [] used it for direct localization in the Global Navigation Satellites System (GNSS), to cope with the unknown propagation coefficients to the satellites, and derived the maximum likelihood estimator of the location. Dmochowski et al. [] used it for direction finding of an acoustic source, to cope with uncertainties in the propagation environment and the acoustic array, and derived a modified version.
of the Steered Minimum Variance (STMV) [] method for a wideband source. Biakowski et al [], and subsequently Hack et al [], used it in passive multistatic radar with widely separated receiving subarrays, to cope with uncertainties regarding the propagation of the reference signal, and derived the maximum likelihood solution for a single moving target. Bar Shalom and Weiss [] used it for active radar involving multiple widely separated transmit antennas and multiple widely separated receiving arrays, to cope with uncertainties regarding the propagation of the transmuted signals, and derived maximum likelihood solutions for locating a stationary target.

In this paper we present novel solutions to the problem of direct localization of narrowband sources by partly calibrated arrays. We address the general case of arbitrarily correlated sources, including the case of fully correlated sources, happening in coherent multipath propagation. Note that since direction finding can be considered as a special case of direct localization, corresponding to the case that the sources are in the far-field of the array, our solutions apply to both problems.

First, we present a relaxed maximum likelihood solution which, by eliminating all the nuisance parameters in the partly calibrated array model, reduces the problem to a concentrated likelihood involving only the $Q$ unknown locations of the sources. The concentrated likelihood requires an eigen-decomposition of the array covariance matrix for every potential location. In the special case of a single source with no multipath, this solution coincides with the maximum likelihood solution []. Second, using the structure of the signal subspace, we introduce an approximation which eliminates the need for an eigen-decomposition of the covariance matrix at every potential location, thus reducing significantly the computational load. In the special case of a single source with no multipath, this solution is computationally much simpler than the existing maximum likelihood solution []. Third, to further reduce the computational complexity, we present MUSIC-like and MVDR-like solutions which, in contrast to the existing MUSIC-like and MVDR-like solutions [], [], [], avoid the need for an eigen-decomposition for every potential location.

The rest of the paper is organized as follows. The problem formulation is presented in section II. Section III presents the "relaxed" maximum likelihood solution, while section IV presents the reduced complexity solution and the MUSIC-like and MVDR-like solutions which trade off performance for computational load. The performance of the various solutions are compared using simulated data in section V. Finally, section VIII presents the conclusions.

II. PROBLEM FORMULATION

Consider an array composed of $L$ fully calibrated subarrays, each composed of $M_l$ antennas with arbitrary locations and arbitrary directional characteristics. Let $M = \sum_{l=1}^{L} M_l$ denote the total number of antennas. Assume that $Q$ sources, located at locations $\{p_q\}_{q=1}^{Q}$, with $p_q \in \mathbb{R}^{D \times 1}$, $D = 2, 3$, and emitting signals $\{s_q(t)\}_{q=1}^{Q}$, are impinging on the array.

To capture both the direct localization and the direction finding problems, we allow the dimension $D$ to be a parameter. If the sources are in the far-field of the array then either $D = 1$, if both the sources and the array are confined to a plane, or $D = 2$, if otherwise. In case the sources are in the near-field of the array, then either $D = 2$, if both the sources and the array are confined to a plane, or $D = 3$, if otherwise.

We further make the following assumptions regarding the emitted signals, the array and the noise:

A1: The number of sources $Q$ is known.
A2: The emitted signals are narrowband, i.e, their bandwidth is much smaller than the reciprocal of the propagation time across the array, and centered around angular frequency $\omega_c$.
A3: The emitted signals are unknown with zero mean and arbitrary correlation, including being fully correlated, as happens in coherent multipath propagation.
A4: The array is synchronized in time, but there is unknown phase offsets between the subarrays.
A5: The locations of the subarrays are known, but with an uncertainty of $\sigma^2$.
A6: The propagation model is spherical waves (it degenerates to plane waves if the sources are in the far-field of the array).
A7: The steering vectors of the subarrays toward any potential location $p_l$, given by $\{a_l(p)\}_{l=1}^{L}$, are known and have unit norm, i.e., $\|a_l(p)\| = 1$.
A8: The additive noises at the subarrays are independent of the signals and independent of each other, and distributed as complex Gaussian with zero mean and covariance $\sigma^2 I_M$.

Assumptions A1-A3 and A6-A8 are conventional and do not need further justification. Assumptions A4-A5 reflect the current limitation of the Global Positioning System (GPS). A4 reflects the current accuracy of the GPS time data - typically 10 ns - which is good enough for time synchronization in the case of narrowband signals, but not good enough for phase synchronization. A5 reflects the current accuracy of the GPS location data, which is typically 10 meters. Under these assumptions, the PCA model for the $M_l \times 1$ vector of the complex envelopes of the received signals at the $l$-th subarray is given by

$$x_l(t) = \sum_{q=1}^{Q} b_{l,q} a_l(p_q)s_q(t - \tau_l(p_q)) + n_l(t),$$

where $b_{l,q}$ is a complex coefficient associated with the propagation of the $q$-th signal to the $l$-th subarray, $a_l(p_q)$ is the steering vector of the $l$-th subarray toward location $p_q$, $j = \sqrt{-1}$, $\tau_l(p_q)$ is the delay from $p_q$ to the $l$-th subarray, and $n_l(t)$ is the noise at the $l$-th subarray.

The partly calibrated nature of the array is embodied by the set of $QL$ complex coefficient $\{b_{l,q}\}$, $q = 1, ..., Q$; $l = 1, ..., L$, assumed to be unknown parameters. In our problem these parameter capture the combined effect of the unknown propagation to the subarrays, the unknown subarrays displacement due to subarrays location error, and the unknown phase offset between subarrays. Though $\{b_{l,q}\}$ are assumed here to be fixed in time, letting them vary over time can serve as a good model for the quasi-stationarity nature of some propagation channel, resulting from small temporal changes due to movement of people, cars, trees, etc. [][].
The narrowband assumption A2 implies that the time delays are well approximated by phase shifts, which allow us to rewrite (1) as
\[
x_l(t) = \sum_{q=1}^{Q} b_{l,q} a_l(p_q) s_q(t) e^{-j \omega_c \tau_l(p_q)} + n_l(t),
\]
Assuming the array is sampled N times, we can express the received signals by the l-th subarray as
\[
X_l = A_l(P_0) B_l S + N_l,
\]
where \(X_l\) is the \(M_l \times N\) matrix
\[
X_l = [x_l(t_1), ..., x_l(t_N)],
\]
\(A_l(P_0)\) is the \(M_l \times Q\) matrix of the steering vectors towards the Q locations (to simplify the notation, the explicit dependence on the locations \(P_0 = \{p_1, ..., p_Q\}\) will be sometimes dropped)
\[
A_l(P_0) = A_l = [a_l(p_1) e^{-j \omega_c \tau_l(p_1)}, ..., a_l(p_Q) e^{-j \omega_c \tau_l(p_Q)}],
\]
\(B_l\) is a \(Q \times Q\) diagonal matrix
\[
B_l = \text{diag}(b_l),
\]
with
\[
b_l = [b_{l,1}, ..., b_{l,Q}]^T,
\]
\(S\) is the \(Q \times N\) signals matrix
\[
S = [s(t_1), ..., s(t_N)] = \begin{bmatrix} s_1 \\ \vdots \\ s_Q \end{bmatrix},
\]
with
\[
s(t) = [s_1(t), ..., s_Q(t)]^T,
\]
and \(N_l\) is the \(M_l \times N\) matrix of the noise
\[
N_l = [n_l(t_1), ..., n_l(t_N)],
\]
To equalize the contributions of the subarrays, we normalize their power, namely set
\[
\text{tr}(X_l X_l^H) = 1, \quad l = 1, ..., L
\]
where \(\text{tr}(\cdot)\) denotes the trace operator and \(H\) denotes the conjugate transpose.

We can now state the direct localization problem as follows:
**Given the received data \(\{X_l\}_{l=1}^{L}\), estimate the Q locations \(\{p_q\}_{q=1}^{Q}\).**

**III. RELAXED MAXIMUM LIKELIHOOD SOLUTION**

In this section we derive the Relaxed Maximum Likelihood (RML) solution.

To this end, regarding the signals matrix \(S\) and the coefficient matrices \(\{B_l\}\) as unknown parameters, it follows from (3) and the Gaussian noise assumption A8 that the maximum likelihood cost function is given by
\[
\hat{P} = \text{argmin}_{\{A_l(B_l)S\}} \sum_{l=1}^{L} \| X_l - A_l(P) B_l S \|_F^2
\]
Note that this cost function is a multidimensional nonlinear minimization with a total of \(DQ + 2QL + 2QN\) real unknown parameters, corresponding to \(P, \{B_l\}\), and \(S\), respectively. Out of this large number of unknowns, only the \(DQ\) unknowns corresponding to the locations \(P\) are of our interest, while the other are considered as nuisance parameters.

As we show in Appendix A, the exact solution of (12) yields a complicated expression which does not seem to enable the elimination of all the nuisance parameters. Consequently, we next present a relaxed maximum likelihood solution which enables the desired elimination and yields a concentrated likelihood involving only the unknown locations of the sources.

Our first step is to eliminate the unknown coefficients \(\{B_l\}\) by expressing them in terms of the other parameters \(P\) and \(S\). To this end, note that \(B_l\) appears only in the \(l\)-th term in (12), implying that it can be estimated by the following minimization problem:
\[
\hat{B}_l = \text{argmin}_{B_l} \| X_l - A_l B_l S \|_F^2
\]
where we hold \(A_l\) and \(S\) fixed. Denoting by \(J_l\) the cost function of (13), we have
\[
J_l = \text{tr}(X_l - A_l B_l S)^H(X_l - A_l B_l S)
\]
\[
= \text{tr}(X_l^H X_l) - \text{tr}(B_l^H A_l^H X_l S^H) - \text{tr}(A_l B_l S X_l^H)
\]
\[
+ \text{tr}(B_l^H A_l^H A_l B_l S S^H)
\]
Dropping the terms which do not contain \(B_l\), we can rewrite it as
\[
J_l = -\text{tr}(A_l S X_l^H \text{diag}(b_l)) - \text{tr}(A_l^H X_l S^H B_l)
\]
\[
+ \text{tr}(S S^H B_l^H A_l^H A_l \text{diag}(b_l)).
\]
Using, equating to zero the derivative with respect to \(b_l\), using the well known complex differentiation rules [7] and the following matrix differentiation rule [8],
\[
\frac{\partial}{\partial B_l} \text{tr}(A_l S X_l^H \text{diag}(b_l)) = \text{diag}(A_l),
\]
we get
\[
\text{diag}(S X_l^H A_l) = \text{diag}(S S^H B_l^H A_l^H A_l).
\]
To solve this equation for \(B_l\), we first relax the equality of the diagonals of the two matrices to an equality of the whole matrices, yielding
\[
S X_l^H A_l = S S^H B_l^H A_l^H A_l.
\]
Next, we relax the constraint that \(B_l\) is diagonal and allow it to be an arbitrary matrix, which enables us to straightforwardly solve this equation for \(B_l\), yielding
\[
\hat{B}_l = (A_l^H A_l)^{-1} A_l^H X_l S^H (S S^H)^{-1}.
\]
Now, multiplying from the left and right by \(A_l\) and \(S\), respectively, we get
\[
A_l \hat{B}_l S = A_l (A_l^H A_l)^{-1} A_l^H X_l S^H (S S^H)^{-1} S
\]
or alternatively,
\[
A_l \hat{B}_l S = P A_l X_l P S^H
\]
where $P_{A_l}$ is the projection matrix on column span of $A_l$
\[ P_{A_l} = A_l(A_l^H A_l)^{-1} A_l^H, \]  
and $P_{S^H}$ is the projection matrix on the column span of $S^H$
\[ P_{S^H} = S^H (S S^H)^{-1} S. \]
Substituting (21) into (12), yields
\[ \hat{P} = \arg\min_{P, S} \sum_{l=1}^L \| X_l - P_{A_l(P)} X_l P_{S^H} \|^2, \]
which, using the properties of the trace operator and the projection matrix, with some straightforward manipulations, reduces to
\[ \hat{P} = \arg\max_{P, S} \text{tr}(P_{S^H} \sum_{l=1}^L X_l^H P_{A_l(P)} X_l) \]  
This expression can be interpreted as a search for the locations $P$ and the signal matrix $S$ for which there is maximum correlation between the signal subspace defined by $P_{S^H}$ and the sum of projections of $X_l$ on the signal subspaces defined by $P_{A_l(P)}$, $l = 1, ..., L$.

To further eliminate the unknowns parameters of the matrix $S$, we next evaluate (25) separately for noncoherent and coherent signals.

A. Noncoherent Signals

In case the signals are noncoherent, the signal subspace defined by $P_{S^H}$ is $Q$-dimensional. This, in turn, implies that we can express $P_{S^H}$ as
\[ P_{S^H} = \tilde{S}^H \tilde{S}, \]
where $\tilde{S}$ obeys
\[ \tilde{S} \tilde{S}^H = I_Q \]  
where $I_Q$ is the $Q \times Q$ identity matrix. Substituting this expression in (25), using the properties of the trace operator, we get
\[ \hat{P} = \arg\max_{P, \tilde{S}} \text{tr}(\tilde{S}^H \sum_{l=1}^L X_l^H P_{A_l(P)} X_l \tilde{S}^H) \]
Maximizing this expression over $\tilde{S}$, holding $P$ constant, we get
\[ \tilde{S}^H(P) = \tilde{v}_S(P) = [\tilde{v}_1, ..., \tilde{v}_Q], \]
where $\tilde{v}_q$ denotes the $N \times 1$ eigenvector corresponding to the $q$-th eigenvalue of the $N \times N$ matrix $\sum_{l=1}^L X_l^H P_{A_l(P)} X_l$. Substituting this expression for $\tilde{S}(P)$ back into (28), we get
\[ \hat{P} = \arg\max_P \sum_{q=1}^Q \lambda_q \left( \sum_{l=1}^L X_l^H P_{A_l(P)} X_l \right) \]
where $\lambda_q()$ denotes the $q$-th eigenvalue of the bracketed matrix.

For large $N$, computing the eigenvalues of the $N \times N$ matrix $X^H P_{A_l(P)} X_l$ may be prohibitive. We next show how to reduce the dimensionality of this problem.

To this end, denote by $P_{A(P)}$ the $M \times M$ block-diagonal matrix
\[ P_{A(P)} = \begin{pmatrix} P_{A_1(P)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & P_{A_L(P)} \end{pmatrix}, \]
by $X$ is the $M \times N$ matrix of the sampled data
\[ X = (X_1^T, ..., X_L^T)^T, \]
and by $\hat{R} = XX^H$ the $M \times M$ sample-covariance matrix of the array
\[ \hat{R} = \begin{pmatrix} X_1 X_1^H & \cdots & X_1 X_L^H \\ \vdots & \ddots & \vdots \\ X_L X_1^H & \cdots & X_L X_L^H \end{pmatrix} = \begin{pmatrix} \hat{R}_{1,1} & \cdots & \hat{R}_{1,L} \\ \vdots & \ddots & \vdots \\ \hat{R}_{L,1} & \cdots & \hat{R}_{L,L} \end{pmatrix}. \]
Now, as we show in Appendix B,
\[ \lambda_q \left( \sum_{l=1}^L X_l^H P_{A_l(P)} X_l \right) = \lambda_q(P_{A(P)} \hat{R} P_{A(P)}), \]
which when substituted into (30) yields
\[ \hat{P} = \arg\max_P \sum_{q=1}^Q \lambda_q(P_{A(P)} \hat{R} P_{A(P)}). \]
To further simplify this expression, let $\tilde{A}$ denote the block-diagonal matrix
\[ \tilde{A} = \begin{pmatrix} \tilde{A}_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{A}_L \end{pmatrix}, \]
where $\tilde{A}_l$ is given by
\[ \tilde{A}_l = A_l(A_l^H A_l)^{-1/2}. \]
Using this notation we can rewrite $P_{A}$ as
\[ P_{A} = \tilde{A} \hat{A}^H = \begin{pmatrix} \tilde{A}_1 \hat{A}_1^H & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{A}_L \hat{A}_L^H \end{pmatrix}, \]
which implies, using the invariance of the eigenvalues of a product of matrices to their cyclic permutation $[,]$, that
\[ \lambda_q(P_{A} \hat{R} P_{A}) = \lambda_q(P_{A} P_{A} \hat{R}) = \lambda_q(P_{A} \hat{R}) = \lambda_q(\hat{A}^H \hat{R} \tilde{A}) \]
Substituting this result into (35), we get
\[ \hat{P} = \arg\max_P \sum_{q=1}^Q \lambda_q(\tilde{A}^H(P) \hat{R} \tilde{A}(P)) \]
Note that since the matrix $\hat{A}^H(P) \hat{R} \tilde{A}(P)$ is $LQ \times LQ$, and since typically $LQ \ll N$, the computational complexity of the solution (40) is significantly smaller than that of (30). Yet, the complexity of this solution is still high, as it involves the
computation of the $Q$ largest eigenvalues of this matrix for every potential location $P$, and a $Q$-dimensional search for the location $P$ for which this sum of eigenvalues is maximized.

To reduce the computational load of the $Q$-dimensional search over $P$, we can employ the Alternative Projection (AP) algorithm [], which transforms a $Q$-dimensional search into an iterative process involving only single source searches.

Denote the cost function by

$$g(P) = \sum_{q=1}^{Q} \lambda_q(\hat{A}^H(P)\hat{R}\hat{A}(P))$$

(41)

The AP algorithm involves two phases. In the first phase, referred to as initialization, the number of sources is increased from $q = 1$ to $q = Q$, with the $q$-th step involving a maximization over $p_q$, with the other $q - 1$ pre-computed locations held fixed:

$$\hat{P}_q = \arg\max_{P_q} g(P_q^{(0)})$$

(42)

where

$$P_q^{(0)} = (\hat{p}_1, ..., \hat{p}_{q-1}, p_q).$$

(43)

In the second phase, the algorithm involves multiple iterations till convergence, with the $k + 1$ iteration for the $q$-th source given by

$$\hat{P}_q^{(k+1)} = \arg\max_{P_q} g(P_q^{(k)})$$

(44)

where

$$\hat{P}_q^{(k)} = (\hat{p}_1^{(k)}, ..., \hat{p}_{q-1}^{(k)}, p_q^{(k)}, \hat{p}_{q+1}^{(k)}, ..., \hat{p}_Q^{(k)}).$$

(45)

B. Coherent Signals

When the signals are coherent, the rows of the signals matrix $S$ are identical hence we have,

$$s_1 = s_2 = ... = s_Q = s.$$ (46)

It then follows that $P_{S^H}$, the projection matrix on the signal subspace, is in this case rank-1 and given by

$$P_{S^H} = s^H (ss^H)^{-1} s = \hat{s}^H \hat{s},$$ (47)

which when substituted into (25), yields

$$\hat{P} = \arg\max_{P,\hat{s}} tr(\hat{s}(L \sum_{l=1}^{L} X_l^H P A_l(P) X_l) \hat{s}^H).$$ (48)

Maximizing this expression over $\hat{s}$, while holding $P$ constant, yields

$$\hat{s}^H(P) = \hat{v}_1,$$ (49)

Substituting this result back into (48) we get

$$\hat{P} = \arg\max_{P} \lambda_1(L \sum_{l=1}^{L} X_l^H P A_l(P) X_l).$$ (50)

Using (34), this becomes

$$\hat{P} = \arg\max_{P} \lambda_1(P A(P) \hat{R} P A(P)),$$ (51)

which can be rewritten as

$$\hat{P} = \arg\max_{P} \lambda_1(\hat{A}^H(P)\hat{R}\hat{A}(P)).$$ (52)

This expression is similar to that obtained for the noncoherent signals (35), with the difference that here only the first eigenvalue of $A^H(P) \hat{R} A(P)$ is involved. The complexity of this solution is still high, as it involves the computation of the largest eigenvalue of this matrix for every potential locations $P$, and a $Q$-dimensional search over $P$ for the $Q$ potential locations for which this sum of eigenvalues is maximized. To simplify the computational load of the $Q$-dimensional search over $P$, one can use the AP algorithm described in (43)-(47), with the difference being that $g(P) = \lambda_1(\hat{A}^H(P)\hat{R}\hat{A}(P)).$

The solution (52) admits a beamforming interpretation. To reveal it, first note that by the definition of the largest eigenvector, we can rewrite it as

$$\hat{p} = \arg\max_{P, w_1, w_2, \cdots, w_L} w^H \hat{A}^H(P)\hat{R}\hat{A}(P) w$$ (53)

where $w$ is the $LQ \times 1$ vector

$$w = [w_1^T, ..., w_L^T]^T.$$ (54)

with

$$w_l = [w_{1l}, ..., w_{Ql}]^T.$$ (55)

Now, it can be readily verified that

$$w^H \hat{A}^H\hat{R}A w = \sum_{l,k=1}^{L} w_l^H \hat{A}_l^H \hat{R}_{l,k} \hat{A}_k w_k$$

$$= \sum_{n=1}^{N} \sum_{l=1}^{L} w_l^H \hat{A}_l^H x_l(t_n) x_l^H(t_n),$$ (56)

which implies that

$$\hat{P} = \arg\max_{P, w_1, w_2, \cdots, w_L} \sum_{n=1}^{N} \sum_{l=1}^{L} w_l^H \hat{A}_l^H(P) x_l(t_n) x_l^H(t_n).$$ (57)

Note that $\hat{A}_l^H(P) x_l(t_n)$ can be interpreted as beamforming at the $l$-th subarray towards locations $P$, while $\sum_{l} w_l^H \hat{A}_l^H(P) x_l(t_n)$ can be interpreted as a second level of beamforming, aimed at combing coherently the outputs of the subarrays’ beamformers. The whole expression can therefore be interpreted as a search for the weights $w$ and locations $P$ for which the power output of this two-level beamforming is maximized.

C. Single Signal

In the case of a single signal, the matrix $A$ reduces to

$$A(p) = \hat{A}(p) = \begin{pmatrix} \hat{a}_1(p) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \hat{a}_L(p) \end{pmatrix},$$ (58)

where

$$\hat{a}_l(p) = a_l(p)e^{-j\omega_c \gamma_l(p)},$$ (59)

and (52) becomes

$$\hat{p} = \arg\max_{P} \lambda_1(\hat{A}^H(p)\hat{R}\hat{A}(p)),$$ (60)

which is identical to the expression derived by Weiss [].
As in the coherent signals case, this expression has a beamforming interpretation. Indeed, following the same steps leading from (53) to (57), we get
\[ \hat{p} = \arg\max_{p, w: w^Hw = 1} \sum_{n=1}^{N} \left| \sum_{l=1}^{L} w_l^H \tilde{a}_l^H(p) x_l(t_n) \right|^2 \]  
(61)

Note that \( \tilde{a}_l^H x_l(t_n) \) can be interpreted as beamforming at the \( l \)-th subarray towards location \( p \), while \( \sum_{l=1}^{L} w_l^H \tilde{a}_l^H(p) x_l(t_n) \) can be interpreted as a second level of beamforming, aimed at combining coherently the outputs of the subarrays’ beamformers. The whole expression can therefore be interpreted as a search for the weights \( w \) and location \( p \) for which the power output of this two-level beamforming is maximized.

IV. REduced COMPLEXITY SOLUTIONS

The RML solution derived above is computationally complex. In this section we present two reduced complexity solutions with different level of complexity.

A. Reduced Complexity Signal Subspace Solution

We first present a solution which simplifies considerably the computation but still requires a \( Q \)-dimensional maximization, based on exploiting the structure of the signal subspace.

To reveal the structure of the signal subspace, let us rewrite \( X \) as
\[ X = Y + N, \]  
(62)
where \( Y \) is the signal component and \( N \) is the noise. From (3), we can express \( Y \) as
\[ Y = \begin{bmatrix} Y_1 \\ \vdots \\ Y_L \end{bmatrix} = \begin{bmatrix} A_1 B_1 S \\ \vdots \\ A_L B_L S \end{bmatrix} = A \bar{S} \]  
(63)
where \( A \) is the \( M \times QL \) block-diagonal matrix
\[ A = \begin{pmatrix} A_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_L \end{pmatrix}, \]  
(64)
and \( \bar{S} \) is the \( QL \times N \) matrix given by
\[ \bar{S} = \begin{bmatrix} B_1 S \\ \vdots \\ B_L S \end{bmatrix}. \]  
(65)
Substituting (63) into (62) we get
\[ X = A \bar{S} + N. \]  
(66)

Now, denoting by \( R \) and \( R_\bar{S} \) the covariance matrices of \( X \) and \( \bar{S} \), respectively, it follows from (66) and from assumptions A4 and A8 regarding the properties of the signals and noise that
\[ R = A R_\bar{S} A^H + \sigma^2 I_M. \]  
(67)

Recalling that
\[ P_A A = A, \]  
(69)
this becomes
\[ P_A R P_A = A R_\bar{S} A^H + \sigma^2 P_A. \]  
(70)
Comparing (67) and (70), it is clear that both \( R \) and \( P_A R P_A \) have the same \( Q \)-dimensional signal subspace \( A R_\bar{S} A^H \). Since this subspace is spanned by the \( Q \) largest eigenvectors of \( R \), we have
\[ v_q(P_A R P_A) = v_q(R) = v_q, \quad q = 1, ..., Q. \]  
(71)

or more explicitly, inserting the explicit dependence on the true location \( p_0 \), we have
\[ v_q(P_{A(p_0)} R_{A(p_0)}) = v_q(R) = v_q, \quad q = 1, ..., Q. \]  
(72)

We will next exploit this relation to simplify the computational load of the RML solution.

To this end, from (35), using the well known properties of the eigenvectors, we can write
\[ \hat{p} = \arg\max_{p} tr(\tilde{V}_S^H(p) P_{A(p)} \bar{R} P_A(p) \tilde{V}_S(p)). \]  
(73)
where \( \tilde{V}_S(p) \) is the \( M \times Q \) matrix of the \( Q \) largest eigenvectors of the matrix \( P_{A(p)} \bar{R} P_A(p) \).

Now, since \( \hat{p} \), the maximizing value of (73), is close to true value \( p_0 \), and since \( \bar{R} \) is close to \( R \) - the error in these two approximations diminishes as the number of samples grows - it follows from (72) that
\[ v_q(P_{A(p)} \bar{R} P_A(p)) \approx v_q(R) = v_q, \quad q = 1, ..., Q, \]  
(74)
where \( v_q \) denotes the eigenvector corresponding to the \( q \)-th eigenvalue of the matrix \( R \). This implies that
\[ V_S(\hat{p}) \approx \tilde{V}_S, \]  
(75)
where \( V_S \) is the matrix of the \( Q \) largest eigenvectors of \( \bar{R} \),
\[ V_S = [v_1, ..., v_Q]. \]  
(76)

This in turn implies that the maximum value of (73) is unchanged if \( V_S(p) \) is replaced by \( V_S \), implying that the maximization problem (73) can be reformulated as
\[ \hat{p} = \arg\max_{p} tr(\tilde{V}_S^H(p) P_{A(p)} \bar{R} P_A(p) \tilde{V}_S(p)). \]  
(77)
In the coherent signals case, this becomes
\[ \hat{p} = \arg\max_{p} tr(v_1^H P_{A(p)} \bar{R} P_A(p) v_1). \]  
(78)
and in the single signal case, this reduces to
\[ \hat{p} = \arg\max_{p} tr(v_1^H P_{A(p)} \bar{R} P_A(p) v_1). \]  
(79)

Note that the computational complexity of (77), (78) and (79) is significantly lower than those of the corresponding solutions (35), (53) and (60), since here no eigen-decomposition is required for every potential location. This implies also a significant computational saving over the existing solution for the case of a single signal presented by Weiss [ ].
The expression (78) for coherent signals admits a beam-forming interpretation. To see it, let $\hat{v}_1$ be segmented into its $L$ subvectors corresponding to the $L$ subarrays
\[ \hat{v}_1 = [\hat{v}^T_1, \ldots, \hat{v}^T_L] \quad (80), \]
where $\hat{v}_{1,l}$ is the $l$-th segment of $\hat{v}_1$. Substituting it into (74) with some straightforward manipulation, yield
\[
\hat{P} = \arg \max_P \sum_{n=1}^N \sum_{l=1}^L \| (P_{A_l}(P)\hat{v}_1_l)^H x_l(t_n) \|^2, \quad (81)
\]
This expression can be interpreted as a two-step beamforming. In the first step, a set of beamformers are applied at the subarrays, which are based on the largest eigenvector $\hat{v}_1$ and given by $w_l = P_{A_l}(P)\hat{v}_1$. Then in the second step, the total beamformer power output is maximized over all potential location. As we show in Appendix C, the beamformer $w_l = P_{A_l}(P)\hat{v}_1$ compensates, in a suboptimal way, for the unknown $b_l$.

Expressions (77)-(79) can be further simplified. To this end, we first rewrite (77), using the properties of the trace operator, as
\[
\hat{P} = \arg \max_P \text{tr}(\hat{V}_S \hat{V}_S^H P_{A_l(P)} \hat{R} P_{A_l(P)}) \quad (82).
\]
Now, from the eigen-decomposition of $\hat{R}$ we have
\[
\hat{R} = \sum_{m=1}^M \hat{\lambda}_m \hat{v}_m \hat{v}_m^H, \quad (83),
\]
where $\hat{\lambda}_m$ denotes $m$-th eigenvalue of $\hat{R}$, and similarly
\[
\hat{V}_S \hat{V}_S^H = \sum_{q=1}^Q \hat{v}_q \hat{v}_q^H. \quad (84)
\]
Substituting these expression into (82), with some straightforward manipulations, we get
\[
\hat{P} = \arg \max_P \sum_{m=1}^M \sum_{q=1}^Q \hat{\lambda}_m |\hat{v}_m^H P_{A_l(P)} \hat{v}_q|^2, \quad (85),
\]
which can be interpreted as a weighted projection of the eigenvectors on the signal subspace defined by $P_{A_l(P)}$, with the weights given by the corresponding eigenvalues.

Using (31), (85) can be further simplified to
\[
\hat{P} = \arg \max_P \sum_{l=1}^L \sum_{m=1}^M \sum_{q=1}^Q \hat{\lambda}_m |\hat{v}_{1,l}^H P_{A_l(P)} \hat{v}_q|^2, \quad (86),
\]
where $\hat{v}_{1,l}$ is the $l$-th segment of $\hat{v}_1$.

For the case of coherent sources this reduces to
\[
\hat{P} = \arg \max_P \sum_{l=1}^L \sum_{m=1}^M \hat{\lambda}_m |\hat{v}_{1,l}^H P_{A_l(P)} \hat{v}_1_l|^2, \quad (87),
\]
and for a single signal this becomes
\[
\hat{p} = \arg \max_P \sum_{l=1}^L \sum_{m=1}^M \hat{\lambda}_m |\hat{v}_{1,l}^H P_{a_l(P)} \hat{v}_1_l|^2. \quad (88),
\]
As the maximization of (86) and (87) still involves a $Q$-dimensional maximization over $P$, further reduction in the computational load can be achieved by using the AP algorithm presented in (45)-(47), with the required modification of the cost function $g(P)$.

## B. Low Complexity MUSIC-like MVDR-like Solutions

We next present solutions which eliminate the $Q$-dimensional search, by resorting to MUSIC-like and MVDR-like techniques. These solutions are applicable only in case the signals are noncoherent.

Before we present these solutions, it would be instructive to present the basis of the existing MUSIC-like and MVDR-like solutions, so as to better understand the differences.

To this end, note that from (63) we can express the signal component $Y$ as
\[
Y = \begin{pmatrix} A_1B_1 \\ \vdots \\ A_LB_L \end{pmatrix} \mathbf{s} = \begin{pmatrix} \tilde{a}_1(p_1)b_{1,1} & \ldots & \tilde{a}_1(p_Q)b_{1,Q} \\ \vdots & & \vdots \\ \tilde{a}_L(p_1)b_{L,1} & \ldots & \tilde{a}_L(p_Q)b_{L,Q} \end{pmatrix} \mathbf{s}. \quad (89)
\]
As is evident from this structure, the signal subspace, i.e., the space spanned by the columns of $Y$, is spanned by $Q$ columns having the following form:
\[
\begin{pmatrix} \tilde{a}_1(p_1) \\ \vdots \\ \tilde{a}_L(p_L) \end{pmatrix}. \quad (90)
\]
Note that this characterization of the signal subspace is parametrized by the $L$ unknown parameters $b_{1,1}, \ldots, b_{L,1}$, which need to be estimated from the data. This estimation step complicates the existing MUSIC-like and MVDR-like solutions, presented in [],[],[],[], since it requires an eigen-decomposition of an $L \times L$ matrix for each potential location $p$.

Alternatively, our solution is based on a different parametrization of the signal subspace given by (63)-(64) and based on the block-diagonal matrix $A$, which we can rewrite as
\[
A = \begin{pmatrix} \tilde{a}_1(p_1) & \ldots & \tilde{a}_1(p_Q) & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \tilde{a}_L(p_1) & \tilde{a}_L(p_L) \end{pmatrix}. \quad (91)
\]
As is evident, $A$ is parametrized only by the unknown locations. For each location $p$, it contains a set of $L$ columns having the following form:
\[
\{\tilde{a}_1(p), \ldots, \tilde{a}_L(p)\}, \quad (92)
\]
where $\tilde{a}_l(p)$ is a block vector with all zeros except the $l$-th block, which value is $\tilde{a}_l(p)$:
\[
\tilde{a}_1(p) = \begin{pmatrix} \tilde{a}_1(p) \\ 0 \\ \vdots \\ 0 \end{pmatrix} ; \quad \tilde{a}_L(p) = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \quad (93)
\]
Now, if the signals are noncoherent, it follows from the well-known MUSIC technique [] that
\[
\text{span} A \approx \text{span}(\tilde{v}_1, \ldots, \tilde{v}_Q), \quad (94)
\]
A. Exact Maximum Likelihood Solution

Thus, for each location \( p \), the set of columns given by (85) are approximately orthogonal to the noise subspace. This, in turn, implies that the locations \( \{ p_i \}_{i=1}^{Q} \) can be obtained by searching for the \( Q \) highest maxima of the following function:

\[
f(p) = \frac{1}{\sum_{l=1}^{Q} \sum_{i=Q+1}^{M} |\hat{v}_l^H \hat{\Pi}(p)|^2}
\]

Using (93), this can be rewritten as

\[
f(p) = \frac{1}{\sum_{l=1}^{L} \sum_{i=Q+1}^{M} |\hat{v}_l^H \hat{\alpha}(p)|^2}
\]

Note that this solution is computationally much simpler than the MVDR-like solution of Marvichev et al [7] and Tirer and Weiss [8]. Similarly, a MVDR-like solution can be obtained by searching for the highest maxima of the following function:

\[
g(p) = \frac{1}{\sum_{i=1}^{L} \hat{\alpha}_l^H(p) \hat{R}^{-1} \hat{\alpha}_l(p)}
\]

Using (93), this can be rewritten as

\[
g(p) = \frac{1}{\sum_{l=1}^{L} \hat{\alpha}_l^H(p)(\hat{R}^{-1})_{ll} \hat{\alpha}_l(p)}
\]

where \((\hat{R}^{-1})_{ll}\) denotes the \( l, l \) block of \((\hat{R}^{-1})\). Note that this solution is computationally much simpler than the MVDR-like solutions of Marvichev et al [7] and Tirer and Weiss [8] since here no eigen-decomposition is needed for each potential location \( p \).

Similarly, a MVDR-like solution can be obtained by searching for the highest maxima of the following function:

\[
x(p) = \frac{1}{\sum_{i=1}^{L} \hat{x}_l^H(p) \hat{R}^{-1} \hat{x}_l(p)}
\]

where \((\hat{R}^{-1})_{ll}\) denotes the \( l, l \) block of \((\hat{R}^{-1})\). Note that this solution is computationally much simpler than the MVDR-like solutions of Marvichev et al [7] and Tirer and Weiss [8] since here no eigen-decomposition is needed for each potential location \( p \).

B. An Equality Regarding the Eigenvalues

In this Appendix we derive the exact maximum likelihood solution. Using the following identity [1],

\[
tr(D^H \hat{A} D \hat{B}^T) = y^H (A \circ B) z,
\]

where \( D_{\hat{z}} = \text{diag}(\hat{z}) \) and \( \circ \) denotes the Hadamard product, we can rewrite (14), dropping the terms which do not contain \( B_l \), as

\[
J_l = -1^H (A_l \circ (X_l^H)^H) B_l - b_l^H (A_l^H \circ (X_l^H)^H) 1
+ b_l^H (A_l^H A_l) \circ (SS^H)^T 1.
\]

Using this equality of the eigenvalues of \( \sum_{l=1}^{L} X_l^H P A_i X_l \) and \( P_{A_i(P)} \hat{R} P_{A_i(P)} \).

To this end, note first that using the properties of the projection matrix, we have

\[
\sum_{l=1}^{L} X_l^H P A_i X_l = \sum_{l=1}^{L} (P A_i X_l)^H P A_i X_l
\]

Now, since the eigenvalues of a product two matrices are unchanged by their permutation, we have

\[
\lambda_q\left(\sum_{l=1}^{L} D_l^H D_l\right) = \lambda_q\left(\begin{bmatrix} D_1 & \cdots & D_L \end{bmatrix}\begin{bmatrix} D_1^H & \cdots & D_L^H \end{bmatrix}\right)
\]

\[
= \lambda_q\left(\begin{bmatrix} D_1 & \cdots & D_L \end{bmatrix}\begin{bmatrix} D_1^H & \cdots & D_L^H \end{bmatrix}\right)
\]

\[
= \begin{bmatrix} \lambda_q(D_1^H D_1) & \cdots & \lambda_q(D_L^H D_L) \end{bmatrix}
\]

Using this identity, we have

\[
\lambda_q \sum_{l=1}^{L} P_{A_i X_l} X_l^H P_{A_i X_l} = \lambda_q\left(\begin{bmatrix} \hat{X}_1 \hat{X}_1^H & \cdots & \hat{X}_L \hat{X}_L^H \\ \vdots & \ddots & \vdots \\ \hat{X}_L \hat{X}_1^H & \cdots & \hat{X}_L \hat{X}_L^H \end{bmatrix}\right)
\]
where $\hat{X}_l$ is the $M_l \times N$ matrix

$$\hat{X}_l = P_A X_l.$$  

(112)

Now, as can readily be verified,

$$
\left( \begin{array}{ccc}
\hat{X}_1 \hat{X}_1^H & \cdots & \hat{X}_L \hat{X}_L^H \\
\vdots & \ddots & \vdots \\
\hat{X}_1 \hat{X}_L^H & \cdots & \hat{X}_L \hat{X}_L^H
\end{array} \right) = P_A XX^H P_A = P_A \hat{R} P_A
$$

(113)

Combining (109)-(113), we get

$$
\lambda_l (\sum_{i=1}^{L} X_i^H P_{A_i(P)} X_l) = \lambda_l (P_A(P) \hat{R} P_{A(P)}),
$$

(114)

which is (34).

C. Beamforming-Based Signal Subspace Solution

In this Appendix we present a beamforming-based derivation of the signal subspace solution for the coherent signal case. The single signal case is a special case of coherent signals corresponding to $Q = 1$.

Note that when the signals are coherent, (2) can be rewritten as

$$x_l(t) = A_l b_l s(t) + n_l(t),$$

(115)

Observing this expression it is clear that the optimal weight vector for beamforming at the $l$-th subarray, which will enable coherent summation across the subarrays, is given by

$$w_l = A_l b_l.$$  

(116)

Yet, since $b_l$ is unknown, we need to estimate it from the data. To this end, first note that in the absence of noise $x_l(t)$ spans a rank-1 subspace given by $A_l b_l$. This subspace is well approximated by the largest eigenvector of $\hat{R}_{l,l}$, denoted by $\hat{v}_l$. Now, as we show in Appendix D

$$\hat{v}_l = \lambda_{\hat{v}_l}.$$  

(117)

Thus, a natural way to estimate $b_l$ is by the following least squares criterion:

$$b_l = \arg \min_{b_l} \| \hat{v}_l - A_l b_l \|^2,$$  

(118)

whose solution is given by

$$b_l = (A_l^H A_l)^{-1} A_l^H \hat{v}_l.$$  

(119)

Substituting this into (116), we get

$$\hat{w}_l = A_l \hat{b}_l = A_l (A_l^H A_l)^{-1} A_l^H \hat{v}_l = P_{A_l} \hat{v}_l.$$  

(120)

Using this vector for beamforming at the $l$-th subarray, the location $P$ for which the sum of the beamformers’ power output is maximized is given by

$$\hat{P} = \arg \max_P \sum_{n=1}^{N} \left| \sum_{l=1}^{L} (P_{A_l(P)} \hat{v}_l)^H x_l(t_n) \right|^2$$

(121)

which is (81).

D. The Case of Rank-1 Covariance Matrix

In this Appendix we prove the relation (112) between the array and the subarrays covariance matrices in case the array covariance matrix is rank-1.

To this end, note that if the array covariance matrix is rank-1 then we have

$$R_l - \sigma^2 I_M = v_1 v_1^H = \left( \begin{array}{ccc}
v_{11} & \cdots & v_{1L} \\
\vdots & \ddots & \vdots \\
v_{1L} & \cdots & v_{11}
\end{array} \right),$$

(122)

where $v_1$ is the $l$-th subvector of $v_1$. Now, for the rank-1 case we also have

$$R_{l,l} - \sigma^2 I_M = \nabla \hat{v}_1,$$

(123)

Since the $l,l$ element of $R_l - \sigma^2 I_M$ should be equal to $R_{l,l} - \sigma^2 I_M$, it follows that

$$\nabla \hat{v}_1 = v_1,$$

(124)

which is (117).

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