Quantum gravity in timeless configuration space

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Received 4 July 2017, revised 5 September 2017
Accepted for publication 15 September 2017
Published 10 November 2017

Abstract

On the path towards quantum gravity we find friction between temporal relations in quantum mechanics (QM) (where they are fixed and field-independent), and in general relativity (where they are field-dependent and dynamic). This paper aims to attenuate that friction, by encoding gravity in the timeless configuration space of spatial fields with dynamics given by a path integral. The framework demands that boundary conditions for this path integral be uniquely given, but unlike other approaches where they are prescribed—such as the no-boundary and the tunneling proposals—here I postulate basic principles to identify boundary conditions in a large class of theories. Uniqueness arises only if a reduced configuration space can be defined and if it has a profoundly asymmetric fundamental structure. These requirements place strong restrictions on the field and symmetry content of theories encompassed here; shape dynamics is one such theory. When these constraints are met, any emerging theory will have a Born rule given merely by a particular volume element built from the path integral in (reduced) configuration space. Also as in other boundary proposals, Time, including space-time, emerges as an effective concept; valid for certain curves in configuration space but not assumed from the start. When some such notion of time becomes available, conservation of (positive) probability currents ensues. I show that, in the appropriate limits, a Schrödinger equation dictates the evolution of weakly coupled source fields on a classical gravitational background. Due to the asymmetry of reduced configuration space, these probabilities and currents avoid a known difficulty of standard WKB approximations for Wheeler DeWitt in minisuperspace: the selection of a unique Hamilton–Jacobi solution to serve as background. I illustrate these constructions with a simple example of a full quantum gravitational theory (i.e. not in minisuperspace) for which the formalism is applicable, and give a formula for calculating gravitational semi-classical relative probabilities in it.
Keywords: shape dynamics, quantum cosmology, Hamiltonian GR, no boundary proposal, conditional probabilities in quantum mechanics, path integrals in field theory

(Some figures may appear in colour only in the online journal)

1. Introduction

1.1. Summary

Most formal schemes, for the full non-perturbative quantization of gravity, such as the Hartle–Hawking no-boundary proposal [1], do not contain time explicitly. From that specific proposal, one obtains a wavefunction $\psi(g)$, having instantaneous 3-dimensional metrics as arguments. Apart from issues of non-renormalizability, there are in such constructions other problems one must deal with; what is the meaning of probabilities and the experience of classical histories as derived from $\psi(g)$? Since $\psi(g)$ is explicitly constructed from path integrals in the space of 3-metrics, one must try to pose boundary conditions in this infinite-dimensional space in a manner that would be space-time covariant. However, there seems to be a good amount of arbitrariness in such choices. Moreover, 3 + 1 general relativity does not have a well-defined reduced configuration space—we do not know how to generically quotient out refoliation symmetries outside of minisuperspace. Indeed, a constructive ‘slice’ (or gauge-fixing) for the space of Lorentzian metrics exists only for the subspace of Einstein metrics which furthermore admit a constant-mean curvature foliation [2]. Thus such invariantly defined boundary conditions pose further challenges (again, outside of the minisuperspace approximations)\(^1\).

The present paper is the outcome of a long research project in which I deal with these issues in a context suited for shape dynamics [4] (see also [5] and references therein). It is the outcome of my attempt to find a consistent scheme for the quantization of theories such as shape dynamics: timeless and yet not fundamentally space-time covariant. Shape Dynamics is a theory of gravity which matches ADM gravity [6] for space-times that are globally foliable by surfaces of constant mean extrinsic curvature (CMC), but which can have global differences when they are not foliable in that manner. In shape dynamics, the local symmetries are given by spatial diffeomorphisms and Weyl transformations, and thus we can form a reduced configuration space in which to pose gauge-invariant boundary conditions for the path integral. This reduced configuration space has a rich geometric structure, and we can find preferred boundary conditions from variational principles of simple gauge-invariant quantities. In this paper, I will investigate what these are, and how they influence the resulting quantum theory, including the construction of probabilities and a semi-classical approximation. Although I will have little to say regarding the specifics of shape dynamics, it constitutes a first step towards a theory of quantum shape cosmology.

1.2. General motivation

Conventional approaches to quantum gravity suffer from serious conceptual and technical problems. On the technical side, most focus has been concentrated on the perturbative regime and on extending the theory to higher energy domains while maintaining some semblance of unitarity. Conceptual problems on the other hand, are starker when trying to make sense of the non-perturbative theory. There we witness a violent clash between the following pairs of quantum mechanical properties and classical ones:

\(^1\)See [3] for further discussion of this point.
These two pairs still leave out difficulties quantum cosmology faces arising directly from the measurement problem in the foundations of quantum mechanics, which the present work also aims to address. Of course, the non-perturbative regime poses infinitely more challenging obstacles to quantitative treatment. Nonetheless, resolving a conceptual incongruence between full quantum mechanics and general relativity might point us to different approaches, friendlier to quantum gravity. The aim of this paper is indeed to point out a framework in which such incongruences are resolved. The framework suggests the adoption of alternative formulations of gravity at a fundamental level, such as shape dynamics [4] (see also [5] and references therein).

The first clash, on the top row of (1), arises because, intuitively, a quantum superposition makes sense for space-like separated components of a system—i.e. belonging to a single constant-time hypersurface. But, to be meaningful, the label ‘space-like’ already requires a fixed, unique\(^2\), gravitational field.

Accounting for back-reaction, it becomes obvious a problem lurks here. We understand how non-back-reacting matter degrees of freedom can be in states of superposition in a background spacetime, e.g. what we usually mean when we write something of the sort \(|\psi\rangle = |\psi_1\rangle + |\psi_2\rangle\) in the position space representation. But due to the universality of the gravitational interactions, back-reaction would necessarily put the gravitational field in a state of superposition as well. How does a superposed state gravitate—and therefore affect the causal structure? The problem is also present in covariant axiomatic QFT language, where one demands that operators with space-like separated support, \(O_1, O_2\) commute, \([O_1, O_2] = 0\). Again, ‘space-like’ presumes a definite property for the very field we would like to be able to superpose\(^3\). What is the meaning of this demand when the operators are acting on states representing superpositions of gravitational fields? It is hard to tell, and it seems to me there are very few ways forward in thinking about these problems\(^4\).

Further tension lies on the bottom row of (1). It stems from distinct roles of (global) time in quantum theory and in general relativity. In quantum mechanics, all measurements are made at ‘instants of time’, and in this sense only quantities referring to the instantaneous state of a system have physical meaning. Dynamics is most naturally understood as an evolution from one instantaneous state to another, enacted by the quantum time evolution operator (\(\exp(-i\mathcal{H}t)\)). On the other hand, in general relativity, a global ‘time’ is an arbitrary label assigned to spacelike hypersurfaces, and physically meaningful quantities are required to be independent of such labels. In other words, in GR, only the spacetime geometry is measurable, and thus only the histories of the Universe have physical meaning. Here a transition between observables would have to be a transition between two complete spacetimes. The dynamics

\(^2\)Unique up to conformal transformations.

\(^3\)General relativity as an effective field theory works quite well [7]. However, standard effective field theory approaches are perturbative, i.e. they require a background, which gives, among other things, a preferred causal structure. One can try to move away from the perturbative domain by implementing a non-perturbative background-field method. In the case of gravity, this is mostly pursued by the asymptotic safety community [8]. However, also in that case you have manifestations of the problems I am mentioning here: firstly, the standard approaches use an Euclidean signature, which makes questions about causality murky at best (and Wick rotations are not well-defined for generally fluctuating space-times). Secondly, at a more abstract level, the preservation of unitarity along the renormalization group flow (even for the finite-truncations in types of operators) is completely unclear.

\(^4\)See [9] for a notable exception, using the formalism of partial and complete observables [10] (see [11, 12] for reviews).
in time of the gravitational field must be carved out of the theory by subjecting it to an initial value formulation, an unnatural procedure from the covariant standpoint.

When (naively) merging quantum theory and general relativity, it should cause no surprise that the only physically meaningful surviving quantities are those which are both: (1) instantaneously measurable (i.e. refer to quantities defined on a spacelike hypersurface) and (2) depend only on the spacetime geometry (i.e. are independent of a choice of spacelike hypersurface) (see e.g. [13, 14]). In a full geometrodynamical theory, this creates great constraints on what constitute reasonable quantum observables [15].

Lastly lies a further distinction (but not necessarily a clash) between the notions of local time, or duration, in the two theories. In classical GR, there exists one notion of local time inherent in the theory, independently of the space-time metric: proper time. Proper time gives an approximate notion of duration along a world-line, it is a dimensionful quantity that is not given in relation to anything else. It has this role whether a space-time satisfies the Einstein equations or not, and is therefore a kinematical, general proxy for elapsed time. On the other hand, in modern relational approaches to quantum mechanics [16, 17], one can parse two notions of time: time as measured by relations between subsystems and just an external evolution time. One of these subsystems is usually called the ‘clock’; its emerging notion of local time is akin to duration, but only becomes available through relational dynamics. The other, global evolution time, is indeed not dynamic, but it also has no meaningful parallel in GR.

These issues surface on the technical front as well; efforts to obtain a viable notion of quantum evolution in quantum gravity—even at the semi-classical level—founder for a variety of reasons [3]\textsuperscript{5}. My claim is that these reasons have a common birthplace. They arise from the distinction between unitary evolution and reduction processes in quantum mechanics, and from the confluence of dynamics and kinematics in general relativity.

1.3. A diplomatic resolution

This paper outlines a proposal to address such questions. Although it is still of largely qualitative character, the aim of this first paper is to discuss the mechanisms by which one could implement said proposal.

My present attempt to untangle both the problem of the superposition of causal structures and the problems of time consists in basing gravitational theories, at the kinematical level, on fields for which there is no notion of causality. Causality emerges, but only dynamically. In this way, physics should be encoded in a path integral operating in timeless configuration space, embodying only compatible spatial symmetries and with prescribed boundary conditions. Such symmetries allow us to define and exploit a reduced space of physical configurations (configuration space modulo gauge-symmetries). No such reduced, instantaneous configuration, space exists for theories with local refoliation invariance—such as GR—even abstractly.

The boundary conditions required for the constructions here are extremely special. They are based on a fundamental asymmetry of the reduced configuration space. Namely a reduced configuration space must not only exist, but also have a unique, most homogeneous element. This requirement places extreme constraints on the field content, on the symmetries, and on the topology of the manifold, but its satisfaction is what will allow most of the constructions here to be defined unambiguously. Anchoring the transition amplitude on this unique boundary\textsuperscript{6}, the present framework provides a ‘first principles’, non-covariant alternative to the Hartle–Hawking prescription of the global wavefunction of the Universe [1].

\textsuperscript{5} At least outside of drastic truncations of field space, such as minisuperspace. In many cases, such truncations do not commute with quantization [18].

\textsuperscript{6} Or more precisely, this unique ‘corner’, as we will see below.
It also addresses questions that need to be answered in any fully relational, timeless theory referring to instantaneous states, such as Hartle–Hawking and Vilenkin’s tunneling proposals. These proposals yield a wave-function whose argument is an instantaneous field content, e.g. \( \psi(g, \varphi) \), where \( g \) is the 3-metric and \( \varphi \) some spatial matter field. In this respect, such theories thus share many of the features of the framework constructed here. The difference, again, is that in such approaches, the wave-functions still need to correspond to covariant (i.e. space-time) quantities. Outside of minisuperspace approximations, the requirement of covariance (at a fundamental level) does not allow such theories to be describable in a physical configuration space, even abstractly, and thus they do not assuage the conflicts between QM and GR I have outlined in the previous section.

As I will show, the present setup yields a single static wavefunction and a corresponding volume-form in physical configuration space. The passage of time is then abstracted from a particular notion of ‘records’—subsets of configuration space where the volume form concentrates. Causality emerges thus as an approximate pattern encoded in the volume-form when it is well-described semi-classically. In this sense, no separate causal structures ‘superpose’—the frozen patterns we associate with them can at most be said to mesh and interfere in configuration space. Unlike previous work in which one attempts to emulate the effects of wave-function collapse within standard unitary evolution, here both evolution and reduction should be emergent from a fundamentally timeless description of the entire Universe.

Here, space-times do not exist \textit{a priori}, but require a relational construction of duration—or definitions of clocks—on classical solutions. i.e. in general one must define clocks and an experienced duration through the classical evolution of relational observables; I call such an emergent notion of time \textit{duration-time}. Indeed, the price to pay in the present, purely relational framework, is apparent in trying to recover, in the appropriate limits, the local, experienced time of GR. In general relativity, experienced time is directly related to proper time, but the gravitational theories compatible with the framework have no universal replacement for proper time. This feature might not be so damaging; it is also the case in relational approaches to quantum mechanics that one constructs clock-time from relations between subsystems [16], as discussed in the previous section.

The challenge to the sort of theories encompassed by the present work thus shifts, from the standard fundamental problems mentioned above—superposition of causal structures, untangling evolution and symmetries, and reduction versus unitary evolution—to one of recovering a smooth space-time description. While it may indeed be difficult to recognize classical relativity of simultaneity from the global evolution-time—i.e. the global time parametrization that will follow from the construction of records—it can be gleaned when using duration-time, as we will see.

In sum, the claim here is that, although unorthodox, this approach resolves issues both with the superposition of causal structures and the ‘evolution versus reduction’ dichotomy, while bringing the notion of duration in gravity closer to the non-unique versions of ‘clock systems’ in quantum mechanics. The constructions advocated in this paper can similarly work with other types of timeless configuration space; they do not limit themselves to the geometrodynamical setting in which they are mostly pursued here. Even the consequences of this approach to the construction of gravitational models will be only touched on here, being further pursued in an upcoming publication, as well as in [19].

1.3.1. Roadmap. I will now describe the main constructions of this paper and its relevance for quantum mechanics and quantum gravity. I will start by making clear what are the main assumptions, or axioms, of the work. This is done in section 2. The non-relativistic setting will in many ways resemble standard particle quantum mechanics. Leaving at most a single
reparametrization constraint, the assumptions are then perfectly compatible with past work on relational dynamics and quantum mechanics, which I very briefly review in section 3. At the end of this section, I include a proof that the principles single out the Born rule as a measure in configuration space. Finally, I discuss another problem that is not fully addressed in relational quantum mechanical approaches within the context of path integrals: a replacement for the role of the epistemological updating of probabilities in the timeless context, through the notion of ‘record-holding submanifolds’, in section 4. Lastly, I sketch a gravitational dynamical toy model based on strong gravity, to illustrate the structures introduced here.

2. Axioms

I would like to examine the conceptual picture that emerges from theories for which time plays no role before the implementation of the equations of motion. I will first clarify the axioms I am led to adopt in order to have a geometric theory of gravity without such kinematical causal relations.

The five given structures that the present work is based on are the following:

1. A closed topological manifold, \( M \). Prior to defining our fields, we require some weak notion of locality, which I will take to mean ‘open neighborhoods’. In the gravitational case, I will take this to be provided by the closed topological manifold \( M \) (i.e. compact without boundary), of dimension \( n = 3 \). If one wanted to apply the following constructions to a fundamentally discrete theory, \( M \) could be replaced by a lattice, or piecewise linear manifold, on top of which the physical degrees of freedom live.

2. The kinematic field space \( Q \). In the field theory case, this is the infinite-dimensional space of field configurations over \( M \). Each configuration—the ‘instantaneous’ field content of an entire Universe—is represented by a point of \( Q \), which we denote by \( \phi \). These should correspond to relational data, on which the dynamical laws act. For definiteness, I will take field configurations to be sections of tensor bundles over \( M \). The example for pure gravity would be \( Q = \{ g \in C^\infty(T^*M \otimes S^T^*M) \} \), the space of positive smooth sections on the symmetrized (0,2)-covariant tensor bundle. In principle the constructions here should apply for non-causally related observables of any kind, such as e.g. field values on a lattice.

3. ‘The Past Hypothesis’—boundary (or ‘initial’) conditions for the Universe: The requirement of such boundary conditions for the wave-function goes in line with the fact that, if time plays no fundamental role, the wave-function of the Universe must be given uniquely, and only once. Thus I require unique boundary conditions in order to anchor the construction of a unique transition amplitude. Unlike what is the case with the ‘no-boundary’ proposal [1], this should not be arbitrary, but given by a variational principle. The extrema should correspond to ‘the most homogeneous’ configuration, \( \phi^* \), with respect to which the wave-function is defined, \( \psi(\phi) = W(\phi^*, \phi) \), see equation (2), and [19], for a more extensive treatment of \( \phi^* \) below. As I will show, \( \phi^* \) furthermore plays a fundamental part in the definition of ‘records’, and for ‘records’ to function in the capacity their name suggests, the boundary conditions should correspond to field configurations which are as ‘structureless as possible’. Below, I give a criterion that can, in some circumstances, select a unique such boundary state.

4. An action functional on curves on \( Q \), i.e. \( S(\gamma) \), for \( \gamma : [0, 1] \rightarrow Q \), invariant wrt the gauge group \( G \) acting on \( Q, G \times Q \rightarrow Q \), and (up to boundary terms) depending only on first
derivatives in the time parametrization. The action should be invariant with respect to the
given gauge symmetry group \( G \), defining \( \tilde{S}(\gamma) \), for \( [\gamma] : [0, 1] \to Q/G \). For this definition,
the gauge-symmetry group needs to have a pointwise action on \( Q \). Such an action
functional will be used for the transition amplitude between two physical configurations,
for \([\phi] \in Q/G\), and \([\gamma](0) = [\phi^*], [\gamma](1) = [\phi_2]::

\[
\Psi([\phi]) \equiv W([\phi^*], [\phi]) := \int_{[\phi^*]} D\gamma \exp (i\tilde{S}[\gamma]) / h. \tag{2}
\]

To avoid cluttered notation, I will drop the square brackets in most of the paper, calling
attention when the distinction between full configuration space and the reduced one
becomes material.

5. A positive scalar function \( F : \mathbb{C} \to \mathbb{R}^+ \). I will call this function the pre-probability
density, \( F \), as it will give a measure in configuration space. To serve our purposes, \( F \) must
preserve the multiplicative group structure,

\[
F(z_1 z_2) = F(z_1) F(z_2). \tag{3}
\]

For future reasons I will term (3) the factorization property of the density. It is a required
condition for the definition of record that I introduce here to have physical significance\(^8\)
and represents a type of ‘Markov’ property of physical probabilities. It is the specific
form \( F(z) = |z|^2 \) of this \( F \) obeying (3) which encodes the Born rule.

These axioms are quite standard implicit choices in most work related to field theory, here
I am only making these choices explicit. Axiom 5 is implicit in the Born rule, and 3 is
usually explicitly stated in the literature, either as boundary conditions, or as some initial
condition.

The only structure required for doing physics, arising from the five premises above, is the
density over \( Q \), given by

\[
P(\phi) := F(W(\phi^*, \phi)) D\phi \tag{4}
\]
where \( F \) is restricted by axiom 4, \( \phi^* \) is given by axiom 5, and \( W(\phi^*, \phi) \) is defined from the
action functional given in axiom 3, and (16) and (18). This measure gives a volume form on
configuration space. It gives a way to ‘count’ configurations, \( P(\phi) \), and thereby the likelihood
of finding certain relational observables within \( Q \). It is assumed to be a positive functional
of the only non-trivial function we have defined pointwise on \( Q \), namely, \( W(\phi^*, \phi) \). I should
stress that roughly the same or similar assumptions are implicit in both Hartle–Hawking and
Vilenkin’s tunneling proposals.

Here, we have a pre-conceived notion of space, or rather, ‘of things which are not causally
related’, but not necessarily of time, in either its absolute or relativistic forms. We have the
space of relational objects on which dynamical laws should act.

At the end of the day, we have at our disposal the density over field space, \( P(\phi) \). But with-
out space-time, and without any notion of absolute time, can we still extract some physics
from the formalism? I would like to show that there can still be enough structure in the time-
less path integral in timeless configuration space for doing just that\(^9\). This is what I will focus
on for most of the paper.

\(^8\) And for it to have cluster decomposition properties (20).

\(^9\) Moreover, this formalism could apply to any relational system with a specified configuration space, action over
curves on it, and unique maximally homogeneous point. For example, certain types of tensor network models.
2.1. Relationalism and the symmetry group $\mathcal{G}$

Intimately related to our clash 2a and 2b is what is known as the ‘problem of time’. The GR Hamiltonian mingles local gauge symmetries and evolution [3, 13]$^{10}$. At least without the use of matter fields there is no preferred manner to split the Hamiltonian constraints such that all but one are fixed by a ‘definition of simultaneity’—a partial gauge-fixing of the Hamiltonian well defined everywhere in phase space.

Although the main constructions of the paper should be applicable in a more general setting, let us investigate the standard case for gravity in closer detail.

2.1.1. The standard ADM action and symmetries. Upon a Legendre transformation, the vacuum Einstein–Hilbert action yields primary and then secondary first-class constraints, $H^\alpha = (H^\perp, H^a)$. The action can then be put in the form

$$S[h_{ab}, \pi^{ab}, N^\alpha] = \int_{t_i}^{t_f} dt \int d^3x \left( \pi^{ab} \dot{h}_{ab} - H^\alpha N^\alpha \right)$$

where $N^\alpha$ are Lagrange multipliers and

$$H^\perp = \pi^{ab} \pi_{ab} - \frac{1}{2} \frac{\pi^2}{\sqrt{g}} - R \sqrt{g}$$

$$H^a = -\nabla_b \pi^{ab}$$

with some algebra given by

$$\{H^\alpha(x), H^\beta(y)\} = \int d^3x' U^\gamma_{\alpha(x)}(x') H^\gamma(x').$$

It has been shown that the local constraints above are essentially unique if the algebra they generate is required to mimic the commutation algebra of vector fields orthogonally decomposed along a hypersurface of space-time (the hypersurface deformation algebra). Thus the Hamiltonian constraint, and therefore the local Wheeler–DeWitt equation, are intimately tied to a covariant picture of space-time.

Under the transformation generated by the flow of the constraints,

$$\delta_\epsilon h_{ab}(x) := \{h_{ab}(x), \int d^3x H^\alpha \epsilon^\alpha\}$$

$$\delta_\epsilon \pi^{ab}(x) := \{\pi^{ab}(x), \int d^3x H^\alpha \epsilon^\alpha\}$$

and under the more ad hoc

$$\delta_\epsilon N^\alpha(x) = \dot{\epsilon}^\alpha(x) - \int d^3x' d^3y' U^\gamma_{\alpha(x)}(x') N^\gamma(x') \epsilon^\beta(y')$$

the action transforms by a boundary term:

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$^{10}$ I note that while a relativistic particle also has a Hamiltonian which generates a single reparametrization invariance, it does not mix this symmetry with local gauge transformations [3]. Moreover, as a subsystem, it is easily describable by relational observables. I give a broader analysis of the time problems in appendix H, and related ones in their formulation of a transition amplitude in superspace, in appendix H.2.
\[ \delta_s S = \left( \int d^3 x e^\alpha \left( \int d^3 y \pi_{ij}(y) \frac{\delta H_\alpha(x)}{\delta \pi_{ij}(y)} - H_\alpha(x) \right) \right) \bigg|_{t_i}^{t_f} \]  

which clearly vanishes for constraints linear in the momenta, or if the generator of refoliations, \( \epsilon^\perp \), vanishes at the initial and final surfaces.

Whereas,

\[ \delta \epsilon^\perp g_{ab} = L_\xi g_{ab} \]

has pointwise dependence in \( \mathcal{Q} \), the transformation

\[ \delta \epsilon^\perp g_{ab} = \frac{2\epsilon^\perp (\pi_{ab} - \frac{1}{2} \pi g_{ab})}{\sqrt{\pi}} \]

depends not only on the metric, but also on the momenta. We could also have obtained a similar result directly from the ADM decomposition (with no Legendre transformation): decomposing a vector field along the time direction and tangential components to the hypersurface, \( X^\mu(t, x) = (s(t, x), \xi^i(t, x)) \), one obtains (assuming zero shift):

\[ \delta_{X^\alpha(t, x)} g_{ab} = s g_{ab} + L_\xi g_{ab} \]

where \( L_\xi \) is the Lie derivative along the vector field tangential to the hypersurface. This part of the action is easy to make sense of: it is the infinitesimal action of a spatial diffeomorphism, acting on the spatial metric through pull-back: \( \text{Diff}(\mathcal{M}) \times \mathcal{Q} \ni (f, g_{ab}) \mapsto f^* g_{ab} \).

On the other hand, equation (11) shows us that refoliations act on the metric in a manner also depending on its velocity\(^ {11} \). It is the action of refoliations that makes it difficult to meaningfully define surfaces and points in superspace, the quotient space \( \mathcal{Q}/\text{Diff}(\mathcal{M}) \). For suppose one has two different curves intersecting at \( g_{ab}^{(0)} \), \( \gamma_{ab}^{(i)}(t) = g_{ab}^{(0)} + \pi^{(i)}_{ab} i = 1, 2 \), for \( \pi_{ab} \) a positive-definite symmetric \((0, 2)\) tensor (they intersect at \( t^{(i)} = 0 \)). Since the action of the spatial diffeomorphisms depends solely on the metric, the two curves will still intersect after the action of \( \xi^\alpha \) (or \( \delta_{\xi^\alpha} \)), as they will be jointly moved. But with the action of \( \epsilon^\perp \), the curves will be shifted: \( \gamma_{ab}^{(i)}(0) = g_{ab}^{(0)} + s(t, x) \pi_{ab}^{(i)} \), and will not intersect anymore. Taking the quotient by (spatial) diffeomorphisms will not help since it is easy to find \( \pi^{(i)} \) such that \( f^* (g_{ab}^{(0)} + s(t, x) \pi_{ab}^{(1)}) \neq (g_{ab}^{(0)} + s(t, x) \pi_{ab}^{(2)}) \) for all \( f \in \text{Diff}(\mathcal{M}) \)\(^ {12} \).

Even though \( \mathcal{Q} \) is an infinite-dimensional space, there is at least one situation in which such curves can still intersect at different \( t^{(i)} \). If the transformation corresponds merely to a reparametrization along each curve, \( s(t, x) = s(t^{(i)}) = s_0 + s_1 t^{(i)} \), the transformed curves merely shift their time-parameters\(^ {13} \): \( \gamma_{ab}^{(i)}(t) = \gamma_{ab}^{(i)}(s_0 + s_1 t^{(i)}) \), and thus the curves will still intersect, but now at \( t^{(i)} = -s_0 / s_1 \). However, if the refoliation is not spatially constant, generically the curves will miss each other, also in superspace (see figure 1).

This fact makes it doubtful that one can implement boundary conditions on superspace for the path integral that are physically meaningful from a covariant space-time perspective.

2.1.2. The present requirement on symmetries. The alternative explored here requires its fundamental symmetries to be generated by local constraints linear in the momenta. The treatment of such linear symmetries, including their quantization, is then much more straightforward.

\(^{11}\) In fact, the strict relationship between the Hamiltonian constraint and refoliations holds only on-shell.

\(^{12}\) That this must be so is easy to ascertain from a simple degree of freedom count: the space of symmetric \((0, 2)\) tensor has 6 degrees of freedom per space point, whereas the generators of spatial diffeomorphisms can only account for three.

\(^{13}\) This would happen order by order for higher contributions in powers of \( t \).
than for local constraints quadratic in the momenta. I will show how this non-fundamentally-covariant setting still allows the emergence of an on-shell refoliation invariance\(^{14}\).

Although the principle of having a pointwise action in configuration space is generically applicable, in the case of gravity in metric variables, i.e. \( Q = \text{Riem}(M) \), the symmetries which satisfy this criterion are found by requiring a phase space representation of first-class constraints linear in the momenta,

\[
\chi(g_{ab}, \pi^{ab}) \approx 0, \quad \chi(g_{ab}, \alpha \pi^{ab}) = \alpha \chi(g_{ab}, \pi^{ab}), \\
\{\chi(g_{ab}, \pi^{ab})(N_1), \chi(g_{ab}, \pi^{ab})(N_2)\} \approx 0
\]

where \( \alpha \in \mathbb{R} \) is a constant in \( M \), \( N_1 \) and \( N_2 \) are appropriate (tensor) smearings of the local constraint densities, \( \chi \), e.g.

\[
\chi(g_{ab}, \pi^{ab})(N) = \int d^3x \chi(g_{ab}, \pi^{ab}, x)c_{cdef}N^{cdef}(x)
\]

the conjugate momentum to the metric is \( \pi^{ab} \), and curly brackets denote the Poisson bracket wrt this canonical relation. Linearity in the momenta implies that the symmetries have an intrinsic action on configuration space, as required, i.e.

\[
\{\chi(g_{ab}, \pi^{ab})(N), g_{cd}(y)\} = \frac{\delta \chi(g_{ab}, \pi^{ab})(N)}{\delta g_{cd}(y)} = F[g_{ab}, \pi^{ab}; N; y] = F[g_{ab}, N; y]
\]

where we have used DeWitt notation for mixed ultra-local and functional dependence. What this tells us that the action of the symmetry on \( g_{ab} \) depends only on \( g_{ab} \) and the smearing. The first class property implies that these constraints correspond to symmetries. It turns out that, under some assumptions, the allowed non-trivial symmetries that act as a group in the metric configuration space—and thus fulfill our axioms—are diffeomorphisms and scale transformations (see [19] for more details).

\(^{14}\) I should note that in the ADM Hamiltonian formalism for GR [6], refoliations only generate a symmetry on-shell in any case [21].
In the field theoretic framework proposed here, relational observables are those that live in the quotient \( Q/\mathcal{G} \)—e.g.: those whose spatial position and scale are only defined relationally—and thus I will bypass a more complicated relational phrasing of the amplitudes by just assuming that all statements are suitably translated when necessary, from \( Q \) to \( Q/\mathcal{G} \). In the presence of this structure, boundary conditions for the path integral will have physical meaning.

The demands of our axioms thus severely restrict the form of the infinite-dimensional groups \( \mathcal{G} \), and imply that configuration space form a principal fiber bundle \( \mathcal{G} \hookrightarrow \mathcal{Q} \to \mathcal{Q}/\mathcal{G} \), making the quantum treatment of gauge-symmetries straightforward. For instance, unlike what is the case for ADM [6], in which one requires the more complicated use of the FBV formalism [23, 24], standard Fadeev–Popov is sufficient to give rise to a well-defined BRST charge; the Fadeev Popov determinant is a true determinant, which implements gauge-covariance of the gauge-fixed path integral. It is the principal fiber bundle structure that allows us to unambiguously consider individual points and dynamics in the quotient space \( \mathcal{Q}/\mathcal{G} \), an impossibility when gauge symmetries become tangled with dynamics (as is the case with ADM).

In the 3 + 1 path integral setting here, one can make use of the principal fiber bundle structure for a straightforward treatment of gauge-symmetries, with the use of a field-space connection 1-form, \( \varpi : T\mathcal{Q} \to T_{\mathcal{Q}}\mathcal{G} \), as described in appendix A. This field-space connection 1-form selects a way to horizontally lift to \( \mathcal{Q} \) a given curve in \( \mathcal{Q}/\mathcal{G} \) (see figure 2). For instance, take \( \mathcal{G} = \text{Diff}(M) \) and \( \mathcal{Q} = \text{Riem}(M) \), \( T_{\mathcal{Q}}\mathcal{G} = \mathcal{C}^\infty(TM) \), the vector fields with the algebra being just the commutator algebra, acting on the metric with the Lie derivative. The horizontal projection of a field-space vector, \( g_{ab} \), is given by \( g_{ab} : = g_{ab} - \mathcal{L}_{\varpi(\xi)}g_{ab} \), i.e. it acts as a field-dependent shift in the ADM formalism\(^{16}\). According to (A.10), under a time-dependent diffeomorphism, \( f(t) \in \text{Diff}(M) \), generated by the vector field \( \xi^a(t) \), we have the following transformation of \( \varpi^a \):

\[
\delta \varpi(\xi(\tilde{t}))^a = \tilde{\xi}^a - \varpi^a
\]

which is precisely the transformation required of the shift from (9) under time-dependent spatial diffeomorphisms. To compare to the standard gauge-fixing procedure in space-time, a horizontal lift with respect to the spatial diffeomorphisms would be equivalent to the choice of \( N^i = 0 \).

2.1.3. Summary of this section. In this section I have sketched how the local symmetry groups can be more or less uniquely selected in such a manner as to have a pointwise representation in configurational field space. In this manner, a well-defined reduced configuration space exists, and I guarantee that local symmetries will be distinguished from dynamics. Moreover, a gauge-invariant path integral may be constructed making use of the connection-form, possible from the emerging principal fiber bundle structure (see also [19, 25] for a more complete account).

\(^{15}\)This is not quite true, for the base space \( \mathcal{Q}/\mathcal{G} \) may not form a manifold, as it does not for the spatial diffeomorphisms. However, insofar as I will use this structure for a field-space connection 1-form—which I associate with an abstract observer—the group can be restricted by a further condition: only allow the diffeomorphisms that maintain a given (observer location) \( x_o \in M \) and a observer frame at \( x, e^a_o \), fixed. Calling this restricted set of diffeomorphisms \( \text{Diff}_o \), then the space \( \mathcal{Q}/\text{Diff}_o \) is indeed a manifold [22] and one can use its principal fiber bundle structure in the usual ways.

\(^{16}\)The connection 1-form itself can be related to a choice of equilocality and associated to abstract, non-backreacting, observers [25] (see also footnote 15). In a completely gauge-covariant way, it selects a manner in which an observer dynamically distinguishes physical transformations from pure gauge transformations, along time. This is a generalization of Barbour’s concept of “best-matching coordinates” [5].
2.2. Uniqueness of preferred boundary conditions

The variational principle I will use to select boundary conditions is based, roughly, on information content. I would like to set the anchor of the transition amplitude (2) to be the configuration (or reduced configuration) that carries the least amount of information. What I mean by that is that it carries the most amount of symmetries, under my group of transformations, i.e. it is the most ‘homogeneous’ configuration, and is in that sense an extremum, resulting from a variational principle.

As mentioned in footnote 15, it is not true, for groups which do not act freely on configuration space, that the quotient \( Q/G \) forms a manifold. Indeed, if there are elements of \( Q \) that remain fixed under the action of subgroups of \( G \), such as is the case with \( Q \) and \( \text{Diff}(M) \) for instance, then the quotient can be at most a stratified manifold [26]. A stratified manifold is basically a union of manifolds of different dimensions, with concatenated boundaries of boundaries. The standard example is a cube, with its bulk being of dimension 3, and its boundary being composed of a further union of manifolds; a face of dimension two, with its boundaries composed of a further union of manifolds; edges of dimension one, with its boundaries composed of a further union of (zero-dimensional) manifolds. The points with the highest isotropy group will correspond to the lowest dimensional boundaries (of all the other boundaries). In other words,

\[
U_m = \{ q \in Q | \dim(\text{Iso}_q(G)) = m \}
\]

we have:

\[
Q/G = \bigcup_{m=0}^{m_{\text{max}}} [U_m]
\]

where each \([U_m] := U_m/G\) is a manifold with boundaries, such that \([U_m] \subset \partial[U_{m'}]\) for \( m > m' \).

Given \( G \), let \( \Phi_o \subset Q \) be the set composed of all the most homogeneous field configurations, i.e. the subset of elements with the largest dimensional isotropy subgroup of \( G \). This will stand in for the configurations being ‘as structureless as possible’. For example, let us take \( Q = \text{Riem}(M) \) and \( G = \text{Diff}(M) \), acting through pull-back \( A_{f}(g_{ab}) \mapsto f^{*}g_{ab} \). Then let
\( \Phi_o = \{ g \in \text{Riem} \mid \text{Iso}_g \subset \text{Diff}(M) \} \text{ has maximum dimension}. \)

Since the dimensions of the isotropy group are discrete, this is a ‘discrete variational principle’. Allowing for degenerate metrics, we have a unique such point, \( \Phi_o = \{ \phi^* \} = \{ 0 \} \), the completely degenerate metric, since it has the full \( \text{Diff}(M) \) as an isotropy group.

In accordance with the symmetry principles of the theory, as described above, one could extend the group \( \text{Diff}(M) \) to the one given by the ‘maximum geometric group’, \( \text{Diff}(M) \ltimes C \), with \( C \) the Weyl group of conformal transformations (through pointwise multiplication of the metric by positive scalar functions, e.g.: for \( \alpha \in C^\infty(M) \), through \( g_{ab} \rightarrow e^{\alpha(g)} g_{ab} \), and a semi-direct product between the two groups) \([19, 27]\).

In fact, the case of \( \text{Diff}(M) \ltimes C \) does not require further specification of the field space boundary at all (as is required, for example in Hartle–Hawking \([1]\)). This property can be seen either by parametrizing physical space \( \mathcal{Q}/\mathcal{G} \) with unimodular metrics, \( \tilde{g}_{ab} := g^{-1/3} g_{ab} \) or using the horizontal lifts of the previous section. In the extended case, the orbit in \( \mathcal{Q}/\mathcal{G} \) corresponding to the completely degenerate metric is not continuously path-connected to the rest of \( \mathcal{Q}/\mathcal{G} \). For \( M = S^3 \), this leaves only \( \Phi_o = \{ \phi^* \} = \{ d\Omega^3 \} \), the round metric, as generating the unique allowed past orbit.

Using unimodular metrics as the conformal section of the principal fiber bundle \( C \leftrightarrow \mathcal{Q} \rightarrow \mathcal{Q}/\mathcal{G} \), for a curve of metrics in \( \mathcal{Q} \) to change signature, the determinant must become degenerate, which disconnects the physical spaces of positive definite signatures from those with other signatures. In other words, the ‘cone’ \([19]\) which makes up the boundary of Riem inside the affine space \( C^\infty(TM \otimes_S TM) \) becomes unreachable coming from \( \mathcal{Q}/\mathcal{G} \).

In the horizontal lift picture, the orbits are defined by vectors \( u_{ab} \in T_g \mathcal{Q} \) of the form \( \alpha g_{ab} \). For any ultralocal supermetric in configuration space of the form

\[
\langle u, v \rangle_g = \int d^3x \sqrt{\tilde{g}} F(g) G^{abcd}_{\alpha} u_{ab} v_{cd} \tag{13}
\]

where \( G^{abcd} = g^{ac} g^{bd} - \lambda g^{ab} g^{cd} \) where \( \lambda \neq 1/3 \), and \( F(g) \) is a function of the metric and its (spatial) derivatives, the orthogonal vectors to the orbits are going to be of traceless form. Thus the standard horizontal lifts orthogonal to the Weyl orbits (through any standard canonical supermetric in Riem) imply a traceless velocity, \( g_{ab} \dot{g}^{ab} = 0 \), which does not change the volume-form \( \sqrt{g} d^3x \). Thus such curves cannot reach metrics with different signatures than the initial ones, as this would require going through a zero in the determinant of the metric. We can thus formulate the path integral (see section below) in terms of horizontal lifts and without stipulating further boundary conditions in configuration space.

One could also obtain the same implementation of a variational principle for the preferred boundary conditions using a conformally invariant functional. These are not so easily constructed. Three examples are: the Yamabe functional \([29]\): \( Y[g] \rightarrow \mathbb{R} \), given by

\[
Y[g] := \inf_\theta \left( \frac{\int d^3x \sqrt{\tilde{g}} \left( (\nabla \theta)^2 + \frac{1}{2} R \theta^2 \right)^{1/3}}{\int d^3x \sqrt{\tilde{g}} \theta^3} \right)
\]

which is invariant under the simultaneous transformations \( g_{ab} \rightarrow e^{4\alpha} g_{ab} \), \( \theta \rightarrow \theta/\alpha \) and another is the Chern–Simons functional,

\[
CS[g] = \int d^3x (d\Gamma \wedge \Gamma + \frac{2}{3} \Gamma^3) \tag{14}
\]

\[\text{18} \text{ Note that since this group is Abelian, there is no Gribov problem } [28].\]

\[\text{19} \text{ Riem is a cone inside the affine space of sections of symmetric (0,2)-covariant tensors, } C^\infty(TM \otimes_S TM), \text{ in that the sum of two metrics is still inside Riem, but not their difference } [26].\]
where $\Gamma(g_{ab})$ is the Levi-Civita connection one form associated to $g_{ab}$. And yet another is the following integral \[19\]:
\[
\int d^3 \sqrt{g} \sum_{m} \beta_m (\mathcal{C}^1_{a_1} \cdots \mathcal{C}^m_{a_m})^{1/m}
\] (15)
where $\beta_m$ are constants, and the (undensitized) Cotton tensor is defined as: $\mathcal{C}^{ab} := \epsilon^{acd} \nabla_c (R^d_{\, \, \, \, b} - \frac{1}{2} g^d_b R)$ where here we are using the undensitized totally anti-symmetric pseudo-tensor $\epsilon^{abc}$. In all of these examples, the conformal equivalence class of the round sphere extremize the functional. In all but the last case, they are the unique extrema. This uniqueness is proven for the Yamabe functional in many places (e.g.\cite{30}), and for the Chern–Simons, one must merely know that its functional derivative wrt the metric gives the Cotton tensor, which vanishes only for conformally flat metrics. These all give converging arguments for selecting the round 3-sphere as giving the variationally selected boundary conditions for the wave-function of the Universe.

2.2.1. Summary of this section. In either the cases of $\text{Diff}(M)$ or $\text{Diff}(M) \ltimes \mathcal{C}$, the quotient space $Q/G$ is only a stratified manifold, and the orbits corresponding to $\Phi_o$ are the lowest, the ‘ultimate boundaries’, or the lowest dimensional corners, of $Q/G$. In this way we find simple variational principles to set up boundary conditions. When there is a unique least structured configuration, which is also the corner of corners in reduced configuration space, the boundary conditions can be uniquely specified. This is the case for $\text{Diff}(M) \ltimes \mathcal{C}$ on $M = S^3$. Moreover, no further boundary conditions for the path integral on field space need to be specified.

In other words, the specification of such a unique initial point in the amplitude kernel is sufficient to fully determine the entire wave-function of the Universe. Moreover, as mentioned above, the existence of such a preferred point will be fundamental in defining a ‘record’, which, in its turn, will replace standard notions of time.

3. Path integrals in configuration space

There are different ways of relating the standard relational setting for quantum mechanics to the path integral formalism. In the end, the constructed wave-function should obey some form of the reparametrization constraint. In appendix B I report on work of Chiou showing, for a relational particle model, how a rigorously defined timeless path integral regains this constraint, and how it reduces to the standard transition amplitude in the presence of a subsystems that behaves like a clock. The mechanisms used in this proof (e.g. the Riemann–Stieltjes integral in terms of a mesh) can be recycled for building the path integral with only global reparametrization constraints, as we have here.

Equation (B.6) guarantees that in the presence of a reliable clock in a given portion of configuration space, one can recover the standard quantum mechanics transition amplitude purely relationally. But it is silent in what regards the existence of such subsystems. As we will see, the asymmetry of reduced configuration space will give us enough structure to build such time functions in the appropriate approximations, even for the (infinite-dimensional) gravitational case.

3.1. The basic definitions

Given an action functional $S(\gamma)$ as above, a connection-form $\omega$, and a preferred ‘in’ configuration $\phi^*$ (see The Past Hypothesis, below), the (timeless) transition amplitude (or propagator)
to the orbit of the configuration $\phi$ is given by a timeless Feynman path integral in configuration space$^{20}$,

$$W(\phi^*, [\phi]) = A \int P \mathcal{D}\gamma \mathcal{D}g \exp \left[ i S_H(\phi^*, g : \phi) \right]/\hbar$$

(16)

where here $g \in \mathcal{G}$ acts on an arbitrary representative of the final point of the transition amplitude, $\phi$, and is integrated over with some measure. A Haar measure is not required here; unlike the standard case, we are not doing a group averaging procedure, each path on the base space corresponds to at most one group element. If the curvature of the field-space connection form is zero, there is no relative holonomy on the fiber for two paths $\gamma_1, \gamma_2$, interpolating between $\phi^*$ and the orbit of $[\phi]$. Thus all the lifts for the paths will end up in a single height of the orbit, let us say $\tilde{g} : \phi$. Then the path integral acquires a functional delta: $\delta(\tilde{g}, \phi)$, cancelling the integral over $\mathcal{D}g$.

In the case of gravity, we would have (where now $g$ stands for the metric field):

$$W(g_0, [g]) = A \int P \mathcal{D}\gamma \mathcal{D}f \exp \left[ i S_H(g_0, f^* g) \right]/\hbar.$$  

(17)

The class of paths $\mathcal{P}$ under which this is integrated over are the horizontal lifts of paths in $\mathcal{Q}/\mathcal{G}$, as explained in the previous section. i.e. $\gamma_H(g_0, f^* g)$ is a path that has a horizontal velocity (i.e. it is a horizontal lift through $g_0$), ending at $f^* g$. The implementation of horizontality will in general incur a Jacobian, which substitutes the standard Fadeev–Popov determinant.

For the purposes of this paper, the specification is not necessary$^{21}$. This procedure eliminates the degenerate directions of the action functional—and makes the projection of the Liouville measure non-degenerate—and does not suffer from Gribov ambiguities (see [19] for more details).

To give an explicit example of how this goes through for the spatial diffeomorphisms, as mentioned in the previous section, the horizontal lift condition is equivalent to $N^i = 0$ in the $3 + 1$ space-time picture. For a given path in configuration space, this fixing eliminates all gauge transformations with $\epsilon^i(\tau_1, x) = \epsilon^i(\tau_2, x)$. With no other symmetries present, the gauge-variation of the condition $N^i = 0$, i.e. (9), is given by $\delta_\epsilon N^i = [\epsilon, \tilde{N}]^i = \tilde{\epsilon}^i$. In the present context of the path integral with a fixed $g_0$, the gauge-transformations are fixed at the initial point of the paths, which means $\epsilon^i(x, \tau) = 0$. According to this gauge-fixing, the ‘coordinates’ for the metric are then completely fixed along any given path in configuration space$^{22}$. For $\eta_i$ being the anti-commuting ghost vector fields, the ghost action becomes $S_{\text{ghost}} = -i \int_0^1 d\tau \bar{\eta}_i \dot{\eta}^i$ and can be integrated out, as is done in [32]$^{23}$.

The measure for the spatial diffeomorphisms (connected to the identity), are given by first replacing the diffeomorphisms by the path-ordered exponential of vector fields, i.e. by vector

$^{20}$ Rigorously, I should start with a phase space action, and only if the momenta can be integrated out of the path integral—which up to the measure amounts to a Legendre transformation—move onto a configuration space action. Here I overlook these issues.

$^{21}$ In the interest of completeness, horizontality by being orthogonal to the fibers generated by the group of conformal diffeomorphisms, Diff(M) $\rtimes \mathcal{C}$, as discussed below, for the standard supermetric in $\mathcal{Q}$, will implement transverse and traceless conditions on the metric velocities, $\tilde{\epsilon}_{ab} = 0$ $^{22}$. The appropriate Jacobian is calculated in appendix E. We note that in the case of odd-dimensional Weyl symmetries, there is no conformal anomaly, and thus the measure can be suitably made Weyl-invariant in conjunction with the action. This is also true in the Hamiltonian setting if the anomalies have a local representation [31].

$^{22}$ Without a particular initial choice of $g_0$, one would have to keep coordinate choices at the initial/final point free. Then one must use a second integration interval, which incurs no dynamical evolution, to correct to any possible coordinate system. See Teitelboim [32], page 299 for a complete account.

$^{23}$ There for interval I, and, for comparison, set $N^\perp(x, \tau) = N^\perp(\tau), \epsilon = 0$.\[\]
fields $X^a$ such that $X^a(t,x) = \frac{df}{dt} f^a(t,x)$ (here on the rhs $f^a$ actually indicates the coordinates of the image of $x$ under $f$), and such that $f(0,x) = x$, i.e. $f^a(0,x) = x^a$. Calling again $U_{bc}^a$, the structure constants of the commutator algebra of diffeomorphisms, given implicitly in e.g. (9), and introducing the continuous matrix $\Omega_{bc}^a = X^c U_{ac}^b$, Teitelboim has shown that [32]:

$$Df = \det \left( 1 - \frac{\epsilon}{\Omega} \right) DX^a.$$ Finally, if the Weyl group $\mathcal{C}$ is part of $\mathcal{G}$, as it is in the example given, $\mathcal{G} = \text{Diff}(M) \ltimes \mathcal{C}$, the integration over the conformal factor cancels out with a functional delta, because the conformal field-space connection-form is flat; all lifted paths end up at the same height in the conformal orbit, as mentioned above (between equations (16)–(17)).

For more on the folding properties of the path integral, and other technical details, we can follow Teitelboim in the procedure given in [32], setting $N_\perp(x,\tau) = N_\perp(\tau)$ (and similarly with $\epsilon_\perp$). These details, for a specific conformally invariant model, are made explicit in [19].

Regarding the measure $D\gamma$, Barvinsky (see [33], section II) has shown how to split such functional measures into a lower dimensional functional integral over spatial fields, and then another integral over parametrizations. The first order part is just the projected Liouville measure:

$$D\gamma = \prod_{\tau, i} d\phi^i(t)|\text{det}(a)|^{1/2}(\tau) + O(\hbar)$$

where we are using DeWitt notation, so that $i$ here runs over both the tensor indexes and the spatial continuous ones, and

$$a = a_{ik} = \frac{\partial^2 \mathcal{L}}{\partial \phi^i \partial \phi^k}$$

and $\partial$ is the functional partial derivative. This is nothing but the invertibility matrix between velocity and momenta of course. It is only non-degenerate in the absence of gauge symmetries, and so its local symmetries must be excised.

Roughly, by using the horizontality conditions, here one would replace $\dot{\phi}^i \rightarrow \dot{\phi}^i - \omega(\dot{\phi}) \cdot \phi$, which would remove the gauge ambiguities. For example, suppose some gravitational action is given (again) by (13)

$$L[g] = \int d^3x \sqrt{\mathcal{F}(g)} G^{abcd}_{\mathcal{H}} \delta_{\mathcal{H}} + V(g)$$

with the group action of the spatial diffeomorphisms, where $V(g)$ does not depend on the velocities, and $\delta_{\mathcal{H}} = g_{ab} - L_{\mathcal{H}(g)} g_{ab}$, according to [25, 34]. Then

$$a \rightarrow a^{abcd}(x,y) = \sqrt{\mathcal{F}(g)} G^{abcd}_{\mathcal{X}} \delta(x,y)$$

which has non-zero determinant. As mentioned above, a choice of a connection form is largely equivalent to a choice of gauge, but more suited to the context we are applying here. For a very simple connection form for the diffeomorphisms, one would get that horizontal vectors are those for which $\nabla^a \xi_{ab} = 0$, i.e. the transverse ones [34].

Going back to the more general case (after dealing with the gauge-symmetries) when combining the measure with the one-loop determinant integrated over time parametrizations, one
finds the amplitude: \( \text{det} \left( -\frac{\delta^2 S(\gamma)}{\delta \phi^a \delta \phi^b} \right) \), which is only a determinant over the spatial fields, with no time integration, but for the on-shell action (equation (3.24) in [33]). The semi-classical approximation will be recapitulated in section 3.

3.1.1. Summary of this section. In this section, we have briefly and formally sketched how the field space path integral can be defined in the timeless context. We have explored the fact that the local gauge groups allow the construction of a reduced configuration space, to formulate a natural ‘particle-like’ path integral formally defining a gauge-invariant wave-function. We have largely used the example of gravity with spatial diffeomorphisms and Weyl transformations. From now, unless otherwise specified, I will keep a more abstract approach that ignores the presence of gauge-symmetry.

3.2. The semi-classical transition amplitude

Now we move back to the more general field configuration space specified above. First, I should note the ubiquity of interference experiments relying solely on multiple extremal paths; double slit and all sorts of interferometers rely on no dynamical information besides a semi-classical approximation with multiple extremal paths interpolating between initial and final configurations.

Indeed, here I will mostly study transition amplitudes between configurations that have at least one extremal (classical) path interpolating between them, and only briefly touch on more general cases in the accompanying [35]24. In the context of path integrals in configuration space, I will be in the semi-classical (or WKB, or saddle point), approximation (in the oscillatory domain).

Explicitly, the setting is given by a path integral in configuration space, (16), for (locally) extremal paths parametrized by the set \( I \subset \mathbb{N} \), \( \{ \gamma_{\alpha} \}_{\alpha \in I} \), between an initial and a final field configuration \( \phi_i, \phi_f \), where here \( a \) stands for both the tensorial and continuous indices. I will denote the on-shell action for these paths as \( S_{\gamma_{\alpha}} \). The expansion, which is accurate for \( 1 \ll S_{\gamma_{\alpha}} / \hbar \) in arc-length parametrization, is then:

\[
W_{\text{cl}}(\phi_i, \phi_f) = A \sum_{\alpha \in I} \left( \Delta_{\gamma_{\alpha}}^{\phi} \right)^{1/2} \exp \left( i S_{\gamma_{\alpha}} / \hbar \right)
\]

(20)

where \( A \) is field-independent, and the Van Vleck determinant has been defined as

\[
\Delta_{\gamma_{\alpha}}^{\phi}(\phi_i, \phi_f) := \text{det} \left( -\frac{\delta^2 S(\gamma_{\alpha})}{\delta \phi^a \delta \phi^b} \right) = \text{det} \left( \frac{\delta \pi^\gamma_{\alpha}(\phi_i)}{\delta \phi^a_f} \right)
\]

(21)

We here write the action as a functional of its initial and final points along \( \gamma_{\alpha} \). To clarify the notation, if we wanted to use continuous indices explicitly, we would have e.g.: the on-shell momenta defined as

\[
\pi^{\gamma}_{\alpha}(\phi_i; x) := \frac{\delta S_{\gamma_{\alpha}}[\phi_i, \phi_f]}{\delta \phi^a_f(x)}
\]

(22)

24 In the usual quantum mechanics setting, if the energy of the particle is lower than a potential barrier, then the fixed energy transition amplitude from one side to the other is exponentially decaying in the phenomenon known as tunneling. One can model the transition using imaginary time, or an Euclidean version of the path integral. Having said this, much progress has been made in explaining tunneling in the usual (or real time) path integral [36, 37]. Moreover, we can separate fields, as I will discuss in section 4.2. One field can be in a semi-classical approximation and serve as background, while the other undergoes evolution through the Schrödinger equation in this background.
where we used DeWitt’s mixed functional/local dependence notation \([\phi_i; x]\). For a proof of (20) in finite dimensions in this context, see [38], and in the Euclidean field theory setting, see [33], section III.

I do not concern myself with the normalization factor \(A\) for the moment, nor with the effect of the Maslov index (which emerges after focusing points only)\(^{25}\). The field determinant contained in the Van Vleck factor is the only element that would require regularization. As previously mentioned, this Van–Vleck determinant arises from the combination of the projected Liouville measure and the one-loop determinant. In more generality, these semi-classical prefactors can be translated into each other also in the field theory setting using what are known as ‘reduction methods for functional determinants’ [33], for which standard regularization methods can be applied.

It is fruitful to present this object in the context of Riemannian geometry, for later purposes. There, its given by a time integral of the expansion scalar along a geodesic congruence. i.e. let the Lagrangian be just the infinitesimal line element of a curve, in a \(d\)-dimensional (semi) Riemannian manifold, and the action is the length of the curve, \(S(\gamma) = \int ds\). Let \(\mu^\alpha\) then be a geodesic congruence, defining the expansion as \(\nabla_\mu \mu^\alpha = \theta\), the Van–Vleck can be written as:

\[
\Delta_{\gamma}(x, y) = s_{\gamma}(x, y)^d \exp \left(-\int_{\gamma} \theta ds\right)
\]

where \(s_{\gamma}(x, y)\) is the proper distance along \(\gamma\) between \(x\) and \(y\), and \(s\) is the arc-length parameter along \(\gamma\). The geometric version of the Van–Vleck can be seen as quantifying the focussing or defocussing of classical trajectories interpolating between \(x\) and \(y\) and around \(\gamma\). It provides a useful analogy to dynamical systems in configuration space, and a bridge between classical and quantum behavior, which we will explore later in section 5.

Indeed, the standard interpretation of (21) is as an indicator of the spread of classical trajectories in configuration space. That is, the Van Vleck determinant measures how the dynamics expand or contract extremal paths along configuration space, i.e. how densely the initial configurations are transported by the equations of motion to the final configurations\(^{26}\). As in the particle case, I assume that the classical field history maps an infinitesimal configuration space density \(\rho(\phi_i)\) to an infinitesimal configuration space density \(\rho_\alpha(\phi_f)\), and the Van Vleck determinant gives a ratio of these densities as propagated by paths around \(\gamma^\alpha_{cl}\):

\[
\Delta_{\gamma^\alpha_{cl}}(\phi_i, \phi_f) = \frac{\rho_\alpha(\phi_f)}{\rho(\phi_i)}.
\]

Under the assumption that there exists at least one classical (locally extremal) path between the two configurations, we find for the absolute value squared of the amplitude:

\[
|W_{cl}(\phi_i, \phi_f)|^2 = A \left(\sum_{\alpha \in I} \Delta_{\gamma^\alpha_{cl}} + 2 \sum_{\alpha \neq \alpha'} |\Delta_{\gamma^\alpha_{cl}}\Delta_{\gamma^{\alpha'}_{cl}}|^{1/2} \cos \left(\frac{S_{\gamma^\alpha_{cl}} - S_{\gamma^{\alpha'}_{cl}}}{\hbar}\right)\right).\]

\(^{25}\) Equation (20) is valid up to the point where the first eigenvalue of \(\delta^2 S/\delta \phi \delta \phi\) reaches an isolated zero, which is a focal point of the classical paths. At such points the approximation momentarily breaks down. However, it becomes again valid after the focal point, acquiring the phase factor known as the Maslov index \(\nu\), which is basically the Morse index (given by the signature of the Hessian) of the action.

\(^{26}\) The interpretation of this fact is quite ubiquitous, for instance, the Raychaudhuri equation—which describes the spreading of geodesic congruences—has a compact formulation in terms of the Van Vleck determinant [39]. The same is true of the Jacobi fields used to perform the semi-classical expansion of the path integral [40].
Here, to unclutter notation, I have omitted the dependence of the Van–Vleck on the configurations. Interference terms can be clearly identified from (24).

The most appropriate way of extending equations (20)–(24) to higher order of approximations was described in [40]. This is still based on extremal paths, and it can be incorporated with a piecewise approximation as well. Although I will not technically require or use these higher approximations, they provide—at least in principle—a way of extending the analysis done here to a more general context.

3.2.1. Summary of this section. I have here summarized necessary concepts regarding semi-classical approximations for oscillatory path integrals in configuration space. Most important is the appearance in second order of the Van–Vleck determinant; it has the interpretation of a ‘focussing’ of extremal trajectories.

3.3. The Born rule

In the absence of a separation into system and apparatus, the meaning of measurements and of the Born rule become more nebulous. Since my ultimate aim is to apply the constructions here to the whole Universe and to cosmology, I need to address the emergence and meaning of the Born amplitude in the present context.

I have left the form of the function $P(\phi) := F[W(\phi^*, \phi)]D\phi$, which ‘counts’ the number of configurations in small regions undetermined. We can associate the Born rule with a relative ‘density of observers’ if we interpret the likelihood of finding oneself e.g. in a region around configuration $\phi_1$ relative to configuration $\phi_2$ as the relative volume of these two regions. I now turn to this.

Decoherence is a form of dynamical diagonalization of the (reduced) density matrix. I will use a form of diagonalization appropriate to timeless configuration space, assuming there is no significant interference between different extremal trajectories. In this regime, using (24), we can compare the density function $P(\phi)$ to the classically propagated volume elements in configuration space.

From (23),

$$\rho(\phi_f) := \rho(\phi_i)\Delta_{\gamma_\alpha}(\phi_i, \phi_f).$$

Where there is only one extremal path connecting $\phi_i$ and $\phi_f$, namely $\gamma_\alpha$. More generally, one would have multiple extremal paths between $\phi_i$ and $\phi_f$. This fact forbids us to take $\Delta$ to define the densities, since it is path dependent, giving only $\rho_{\gamma_\alpha}(\phi_f)$, not $\rho(\phi_f)$—which needs to take into account interference from other paths. Nonetheless, for the semi-classical kernel, from the no-interference terms in (24), we also have

$$\Delta_{\gamma_\alpha}(\phi_i, \phi_f) \approx |W_{\text{cl}}(\phi_i, \phi_f)|^2.$$

Thus if we demand that our density functional gives the classically propagated densities in the no-interference limit, i.e.

$$F[W_{\text{cl}}(\phi_i, \phi_f)] \propto \rho(\phi_f) = \rho(\phi_i)\Delta_{\gamma_\alpha}(\phi_i, \phi_f)$$

we find that from (26)

$$F[W_{\text{cl}}(\phi_i, \phi)] := \rho(\phi_i)|W_{\text{cl}}(\phi_i, \phi)|^2.$$

Since we will have a unique choice of $\phi_i$, given by the preferred ‘vacuum’, or in-state $\phi^*$, we can absorb $\rho(\phi^*)$ into the normalization $A$. This proves that, semi-classically, the Born rule emerges from the classical propagation of volumes in configuration space.

---

27 This is the description of decoherence used extensively in the consistent histories formulation [41].
But even outside of the semi-classical regime, demanding positivity of $F$, it is easy to see that if one expands the function into polynomials of $z\bar{z}$, i.e. $F(z) = \sum_i a_i(z\bar{z})^i$, one would obtain:

$$F(z_1z_2) = \sum_i a_i(z_1z_2\bar{z}_1\bar{z}_2)^i \tag{28}$$

and, on the other hand

$$F(z_1)F(z_2) = \sum_i a_i(z_1\bar{z}_1)^i \sum_j a_j(z_2\bar{z}_2)^j. \tag{29}$$

Only the diagonal terms of (29) can match the polynomials of (28). Therefore, only one $a_i$, say for $i = k$ can survive, and only if $a_k = 1$. By the semi-classical limit above, (26), $k = 1$.

Alternatively, taking $F(z_1z_2) = F(z_1)F(z_2)$, by differentiating both sides wrt $z_1$ and setting $z_1 = 1, z_2 = z$ we obtain:

$$zF'(z) = F'(1)F(z)$$

and a similar equation for the complex conjugates, $\bar{z}$, which means that the function is homogeneous: $F(z) = z^\alpha\bar{z}^\beta$, and demanding positivity, $\alpha = \beta$. By the above classical limit, we must finally have that $\alpha = 1$.

Therefore, extending (27) to the full volume form, from the static wave-function over configuration space $\psi(\phi) := W(\phi^*, \phi)^{29}$, gives

$$P(\phi) = |W(\phi^*, \phi)|^2. \tag{30}$$

Equation (30)—i.e. the Born rule—is the extension of $F[W(\phi^*, \phi)]$ to arbitrary points $\phi$.

Note that no mention of ‘measurements’, or ‘observations’ need to be made. Volumes in configuration space suffice.

3.3.1. Summary of this section. The decomposition property of the function $F$ is necessary if we want to translate certain statements about the amplitude to statements about probability. Here I have shown that given just this property and implementing a semi-classical limit, one can uniquely derive the Born rule as giving the volume-form in configuration space.

4. Records

Given the past Hypothesis, even allowing for an objective meaning to a relational transition amplitude, $W(\phi_i, \phi_f)$, we should be able to do science from assumptions about relative number of configurations in $Q$. It is true that while examining the results of an experiment, all we have access to for comparison are the memories, or records, of the setup of the experiment.

Since we are in a context that cannot rely on absolute time, the meaning of measurements and the updating of probabilities needs to be significantly modified. Here, I will give a semi-classical definition of records. This definition should be seen as a mathematical pre-requisite for any functioning role for records; they do not, however, pinpoint the physical encoding of information in physical structures.

\[28\] I thank Wieland for these remarks.

\[29\] Alternatively, we could have defined a ‘vacuum state’ $|\phi^*\rangle$ existing over a trivial (one complex dimension) Hilbert space $\mathcal{H}_{\phi}$, with the usual complex space inner product, over $\phi^*$, and then taking $\hat{W}(\phi^*, \phi): \mathcal{H}_{\phi^*} \rightarrow \mathcal{H}_{\phi}$ as an operator such that $|\psi(\phi)\rangle := \hat{W}(\phi^*, \phi)|\phi^*\rangle$. 


4.1. Semi-classical records

I will denote a recorded configuration as $\phi_r$, and the manifold whose elements have $\phi_r$ as a record, as $Q(\phi)$. If these represent experiments, coexisting with the given record, each copy of ‘the experimenter’ will find itself in one specific configuration, $\phi \in Q(\phi)$. The manifold $Q(\phi)$, consisting of all those configurations with the same records, then represents the setup of the experiment. In the cosmological setting this could be, for instance, all of the configurations that contain a ‘record’ (as defined below) of the surface of last scattering. The post-selection is locating your configuration within $Q(\phi)$.

Loosely, a configuration $\phi$ will be defined to hold a record of (the recorded) configuration $\phi_r$ if, for a given ‘in’ configuration $\phi^*$, all extremal paths from $\phi^*$ to $\phi$ go through $\phi_r$. This definition is meant to embody the idea that $\phi_r$’s ‘happening’ is encoded in the state $\phi$. At least semi-classically, ‘branches’ of the wavefunction contributing to the amplitude of $\phi$ have ‘gone through’ $\phi_r$.

One note of caution: in the oscillatory semi-classical regime, when one speaks of the major contribution to the amplitude as coming from extremal paths, what is really meant is that there is a coarse-graining of paths, seeded by the extremal ones, in which the paths close to the extremal ones enjoy constructive interference, whereas ones that deviate much have their interference wash out. To be rigorous, in the companion paper [35] I have formulated the constructions here directly from specific types of coarse-grainings of paths. The results below go through without problems.

**Definition 1.** Given $\phi^*$, the action $S(\gamma)$ on curves in configuration space, and the collection of parametrized extremal paths $\{\gamma(\alpha)\}_{\alpha \in I} : [0, 1] \rightarrow Q$ such that $\gamma(0) = \phi^*, \gamma(1) = \phi$, $\forall \alpha \in I$, then $\phi$ is said to have a semi-classical record of $\phi_r$ if for each $\alpha \in I$ there exists a $t_\alpha \in [0, 1]$, such that $\gamma(\alpha)(t_\alpha) = \phi_r$ and $S[\gamma(\alpha)] \gg h$.

And with this definition we can prove the following

**Theorem 1.** Given a configuration $\phi$ with a semi-classical record of $\phi_r$, then

$$W_{cl}(\phi^*, \phi) = W_{cl}(\phi^*, \phi_r)W_{cl}(\phi_r, \phi) + O(h^2).$$

(31)

Combined with property (3) for the density functional, this means that the equation for the probability of $\phi$ automatically becomes an equation for conditional probability on $\phi_r$,

$$P(\phi) = P(\phi_r)P(\phi | \phi_r)$$

(32)

where $P(\phi | \phi_r) = |W(\phi_r, \phi)|^2$.

Using the semi-classical approximation (20),

$$W_{cl}(\phi^*, \phi) = \sum_{\gamma} (\Delta_{\gamma})^{1/2} \exp\left(iS_{\gamma}(\phi^*, \phi)/h\right) + O(h^2).$$

If the system is deparametrizable, this means that one of the configuration variables can be used as ‘time’, and extremal trajectories can be monotonically parametrized by this variable.

---

30 In that more rigorous setting, for the semi-classical record to be of order $h$, the length of the extremal paths that seed the coarse-graining, $\gamma(\gamma)$, need to obey $S_n := S[\gamma_n] \gg h$, and, up to this order of approximation, the preferred extremal coarse-graining defined in [35] cannot resolve if $\phi_r$ lies along the actual extremal paths, or just around them. Moreover, to extend this notion to that of piece-wise extremal paths, I need to use that notion of extremal coarse-grainings as well.
Then we could use (B.6), and the semi-classical composition law for the semi-classical transition amplitude (see [42])\(^{31}\) to write:

\[
W_{cl}(\phi^*, \phi) = \int D\phi_m W_{cl}((t_*, \phi^*), (t_m, \phi_m)) W_{cl}(t_m, \phi_m, (t, \phi))
\]

for a given intermediary time \(t_m\). Choosing \(t_m = t_r\), there is a unique configuration through which all of the paths go through, \(\phi_r = (\bar{\phi}_r, t_r)\). The integral gains a \(\delta(\bar{\phi}_r)\), since extremal paths pass only through that point at \(t_r\), and thus:

\[
W_{cl}(\phi^*, \phi) = W_{cl}((t_*, \phi^*), (t_r, \bar{\phi}_r)) W_{cl}((t_r, \bar{\phi}_r), (t, \phi)) \quad \Box.
\]

In the more general case however, there is more work to be done. Using (20), we first write the rhs of (31),

\[
\sum_{\alpha_1} \Delta_{\phi_1}^{1/2} \exp(iS_{\alpha_1}(\phi^*, \phi_r)/\hbar) \sum_{\alpha_2} \Delta_{\phi_2}^{1/2} \exp(iS_{\alpha_2}(\phi_r, \phi)/\hbar) + O(\hbar^2)
\]

where \(\alpha_1\) are the sets of extremal paths interpolating between \(\phi^*\) and \(\phi_r\) and \(\alpha_2\) those between \(\phi_r\) and \(\phi\). What we want to show is that to order \(O(\hbar^2)\), the contributing paths will be those that are continuous at \(\phi_r\). After this is done, we need to show that the Van Vleck determinants have the right composition law. This is done in appendix C.

4.1.1. Strings of records. For more than one recorded configurations, say \(\phi_1^*, \phi_2^*\), definition 1 demands that each extremal path \(\gamma_0 \in \Gamma(\phi^*, \phi)\) go through \(\phi_1^*, \phi_2^*\) in order or another. Let \(\phi\) contain multiple semi-classical records \(\phi_r\), i.e. \(\phi \in \cap \mathcal{Q}(r)\). Then an ordering of the records must exist for each extremal curve (see figure 3). For each \(\alpha\), the set \(\{\phi_i^*\}_{i=1}^n\) is ordered: \((\phi_1^*, \ldots, \phi_n^*)\).

In the presence of a single element \(\alpha\)\(^{32}\), we can concatenate \(P_{cl}(\phi^*, \phi_{i+1}^*) = P_{cl}(\phi^*, \phi_{i})P_{cl}(\phi_{i}, \phi_{i+1})\). The decomposition of the density follows:

\[
P_{cl}(\phi^*, \phi) \approx P_{cl}(\phi^*, \phi_i)P_{cl}(\phi_i, \phi_{i+1})P_{cl}(\phi_{i+1}, \phi).
\]

In other words, if there is only one single \(\alpha\) (no interference), and the semi-classical limit holds (i.e. the action spacing between the records is larger than \(\hbar\)), then the records in \(\phi \in \mathcal{Q}(r)\) yield a coarse-grained classical history of the field. In this case we recover a (granular) notion of classical time\(^{33}\).

We can still obtain a consistent string of records for many interfering branches if the ordering \((\phi_1^*, \ldots, \phi_n^*)\) given for each \(\alpha\), coincide, i.e. if for all \(i\) and any \(\alpha, \alpha'\), we have \(\phi_i^{\alpha} = \phi_i^{\alpha'} = \phi_i\). Then kernel still can be decomposed:

\[
P_{cl}(\phi^*, \phi) \approx P_{cl}(\phi^*, \phi_1)P_{cl}(\phi_1, \phi_2) \cdots P_{cl}(\phi_n, \phi)
\]

This matches the analysis performed by Halliwell, in which he recovers ‘time’ from a simple Hamiltonian Mott bubble chamber ansatz\(^{34}\), finding that the amplitude for \(n\)-bubbles to be excited, with the \(n\)th bubble configuration being \(q_n\), is given by:

\[
\langle q_n | \psi_n \rangle \propto \int d^N q_n \cdots d^N q_1 G(q_n, q_n) f_n(q_n) \cdots G(q_2, q_1) f_n(q_1)
\]

\(^{31}\) Here we are crucially assuming the ‘folding property’ for the path integral. See [32].

\(^{32}\) And remembering that the on-shell action of \(\gamma_0\) between \(\phi_r\) and \(\phi_{n+1}^*\) must be much larger than \(\hbar\) for records to be defined.

\(^{33}\) In a context of consistent histories, the separation of order greater than \(O(\hbar)\) between records will avoid the argument of Halliwell concerning the quantum Zeno effect [43].

\(^{34}\) Where spontaneous emission of \(\alpha\)-particles from a source ionize bubbles in a chamber of water vapor [44].
where $G$ are Green’s functions (for the free Hamiltonian), $f_i(q^i)$ are projections onto small regions of configuration space surrounding the $i$th configuration, and $N$ is the dimension of configuration space. For this derivation, Halliwell notes that it is essential that there is some asymmetry in configuration space, marked by the source of the $\alpha$-particles. It is the same here: we require axiom 5 of an ‘origin’ of configuration space (see also [35] for an extended version of this relationship).

The definition implies that a configuration can hold many records, and an ordering among these, with earlier recorded configurations being themselves recorded in later recorded configurations. Through this ordering, a semblance of global time emerges from a fundamentally timeless theory. Again, this is in line with Halliwell’s reconstruction of time in the Mott chamber context [45]. Configurations that are far from the given initial configuration $\phi^*$—but still connected to it by extremal paths—will in general have concentrated amplitude, and hold more records.

4.1.2. Relative probabilities of regions in $Q$ with the same records. The setup of an experiment implies the fixing of a submanifold in configuration space, $Q(r)$, characterized by its points all possessing the same records. In this abstract simple example, the submanifold is characterized by a single configuration $\phi_r$, which could represent for instance, ‘the whole laboratory setup of a double slit experiment and the firing of the electron gun’, or, ‘the cosmological surface of last scattering’.

We can compare transition amplitudes for configurations with the same records in the following way, given $\phi_1, \phi_2 \in Q(r)$, for any initial $\phi^*$:

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**Figure 3.** The types of sequence of records that can arise. The first gives rise to a notion of a granular history. The second and third represent a more general version of records of records. The fourth represents the sort of records which are not consistent, as their ordering disagrees.
\[ \frac{P(\phi^*, \phi_1)}{P(\phi^*, \phi_2)} \approx \frac{P(\phi^*, \phi_1)P(\phi_1, \phi_1)}{P(\phi^*, \phi_1)P(\phi_1, \phi_2)} = \frac{P(\phi_1, \phi_1)}{P(\phi_1, \phi_2)}. \] (35)

This ratio gets rid of a common quantity to both density functions.

In most experimental settings then, equation (35) justifies the practical use of the record configuration \( \phi_r \) as the effective initial point in the kernel. It is as if a measurement occurred, determining \( \phi_r \). While it is true that the theory is timeless, we can still give a meaning to active verbs such as ‘updating’ (e.g. of our confidence level): to the extent that humans are classical systems, extremal paths in configuration will reflect anything that the equations of motion predict, including rational (and irrational) ‘updating’ of our theories.

4.1.3. Summary of this section. I have here defined records. These are structures that can arise given our volume-form on \( \mathcal{Q} \), and which have many interesting properties. They yield conditional probabilities (in much the same way that the Mott bubbles do); as correlated volumes in configuration space that emulate a notion of ‘causation’. I have only discussed these structures in the semi-classical regime, whence one obtains at most a granular umes in configuration space that emulate a notion of causation. I have only discussed these structures in the semi-classical regime, whence one obtains at most a ‘granular’ history.\(^{35}\)

It is important to note here that our arguments from section 2.1, regarding the gauge-dependent character of intersecting curves in superspace if reparametrisations are allowed as a local symmetry, i.e. two curves might intersect before, but not after the action of a reparametrisation. This is not the case if one allows only the sort of local symmetries we have considered here, and/or global reparametrisations. This is the reason why the relativistic particle and minisuperspace models would not present such a problem. But it shows a disconnect, in so far as the concept of records explored here is concerned, between minisuperspace and the full theory with local reparametrisations.

4.2. Records and conservation of probability

The generic case should be one of redundancy of records in \( \mathcal{Q}_r \)\(^{36}\). In other words, within \( \mathcal{Q}_r \) one could have a polygamy of record relations; e.g.: \( \phi_1, \phi_2 \in \mathcal{Q}_r \) with \( \phi_1 \in \mathcal{Q}_2 \). This happens for example in the case of strings of records, discussed above. Ideally, when discussing conservation of probabilities, we would be able to discard such redundancy.

A choice of subset of \( \mathcal{Q}_r \) for which there is no such redundancy will be called a screen, written as \( \mathcal{S}_r \subset \mathcal{Q}_r \) (see figure 4). In other words,

\[ \mathcal{S}_r := \{ \phi_i \in \mathcal{Q}, i \in I \mid \phi_k \notin \mathcal{Q}_{(\phi)}, \forall k, j \in I \}. \] (36)

Screens are important when we want to discuss conservation of probability. As I will now show, we can expect probabilities to be conserved between records and the total amplitude of its screen.

According to (B.6), whenever we have a good ‘clock’ subsystem, we know that the propagator \( W(\varphi_1, t_1, \varphi_2, t_2) \) will obey a Schrödinger equation with respect to clock time. It follows that suitably defined currents and probabilities will obey conservation laws. However, part

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\(^{35}\) With records separated by action of order greater than \( \hbar \). In the arc-length parametrization of the following sections, this is a separation in superspace.

\(^{36}\) I mean this in two ways: first, that many different subsystems will have redundant records of the same ‘event’ (subset of a configuration, in the sense of [20]). For example, all the photons released by an electron hitting a fluorescent screen. By comparing the consistency of these records, one can formulate theories about the configuration space action and the probability amplitude. This ‘consistency of records’ approach, has been recently emphasized in the context of selecting the basis for branch selection in decoherence [46]. Since in this paper I am not explicitly dealing with subsystems—a topic I have dealt with in [20]—I will ignore this type of redundancy; although it is very important for doing science.
of the challenge of quantum cosmology is precisely to inform us of physics when such clock systems are not available. Therefore, here I will show how a stand in for conservation of probability can be extracted from records.

Now, in standard quantum gravity, there is no fixed causal structure, and thus there is difficulty in defining concepts such as conservation of probability. In our case it is clear that e.g., for actions that are of geodesic type in configuration space, at the semi-classical (WKB) level probability conservation laws should hold, since the form of the equations resemble that of a particle in a constant potential in a zero energy eigenstate.

For a screen as defined through (36), by definition 1, if extremal trajectories go through a small volume $D\phi_r$, the total semi-classical probability flux for any screen will not exceed that around $\phi_r$ (or of a previous record-screen). In other words:

$$P(\phi_r) \simeq \Delta(\phi^*, \phi_r) \geq \int_{S_{(r)}} D\phi \Delta(\phi^*, \phi) \simeq \sum_{\alpha \in I_S} \Delta_\alpha(\phi^*, \phi_\alpha) = P(S_{(r)})$$

where the extremal paths that leave $\phi_r$ and intersect the screen are parametrized by $I_S$. In this particular discussion, it is assumed that there is no interference at the screen; to each point on the screen corresponds a single $\alpha$. Moreover, the flux is equal to the Born volume of a (infinitesimally) thickened region. In particular, since

$$P(S_{(r)}) = \int_{S_{(r)}} D\phi |\Psi(\phi)|^2 \leq \int_{S_{(r)}} D\phi |W(\phi_r, \phi)|^2 \leq 1.$$  

If every extremal path that crosses $\phi_r$ also intersects the screen, then, up to higher orders of $\hbar$, the inequality of (38) is saturated.

4.2.1 Application to timeless configuration space. Suppose then that we were able to find a metric in configuration space for which extremal trajectories of the action are given by geodesics. From the Jacobi procedure, this does not imply that there is no potential for

\[\text{Figure 4. A schematic representation of a sequence of two screens, } S_{(r)}^1, S_{(r)}^2, \text{ for a given recorded configuration } \phi_r \text{ (the divergence between the rays after } \phi_r \text{ are exaggerated for illustration).}\]
the dynamical system, just that it can be reabsorbed into an effective metric in configuration space, with the use of specific conformal factors.

Using DeWitt functional notation, let the index $a$ represent the totality of both continuous and discrete indices of the spatial field in question (thus contraction includes spatial integration). Then, for a configuration-dependent supermetric $G^{ab}(\phi)$, and configuration space curves $\gamma_a$,

$$ S[\gamma] = \int dt \left( V(\phi)G^{ab}\dot{\gamma}_a\dot{\gamma}_b \right)^{1/2}. \quad (39) $$

Where $V(\phi)$ is the conformal Jacobi factor. Reparametrization invariance implies a reparametrization constraint, since

$$ p^a = \frac{\partial L}{\partial \dot{\gamma}_a} = \frac{V(\phi)^{1/2}G^{ab}\dot{\gamma}_b}{(G^{cd}\dot{\gamma}_c\dot{\gamma}_d)^{1/2}} \quad (40) $$

(where $\partial$ represents the functional partial derivative). We thus get:

$$ H \equiv p^a G_{ab}p^b - V(\phi) = 0. \quad (41) $$

Note that (41) is not a local equation like the usual scalar ADM constraint [6], since there is a hidden integration on all contracted indices. It is a bona fide global conservation law, and that makes all the difference. Although we are not here at a homogeneous (minisuperspace) approximation, we can still use many of its standard techniques and results. Of course, in the standard superspace approximation of homogeneous fields, equation (39) would no longer contain the local (integrated over) index, and only the actual indices appearing in (41).

For the reparametrizations generated by (41), the algebra is a bit more laborious, but in the end one obtains, in (10), for $\epsilon(t)$ a global (in space) parameter, just $\delta_s S = [2\epsilon(t)V(\phi(t))]\dot{\epsilon}$ and $\delta_s \phi_a = 2\epsilon(t)G_{ab}p^b$. Thus, from (H.8), it follows that the wave-function constructed from the path integral satisfies the constraint38:

$$ \hbar^2 \left( G^{ab}\frac{\delta^2 \Psi(\phi)}{\delta \phi^a \delta \phi^b} \right) - V(\phi)\Psi(\phi) = 0 \quad (42) $$

where I have chosen the non-self-adjoint factor ordering, with a possibly non-ultralocal supermetric $G^{ab}$ symmetric in $ab$.39 With the polar form for a wavefunction (assuming that there is at most one extremal solution connecting to $\phi$),

$$ \Psi(\phi) = A(\phi) \exp \left( iS(\phi)/\hbar \right) $$

with $S(\phi) := S(\phi^*, \phi)$ (the on-shell action from $\phi^*$ to $\phi$) and $A$ real functionals, we obtain

$$ \left( \hbar^2 G^{ab} \left( \nabla_a \nabla_b A - \frac{i}{\hbar} \left( 2\nabla_a A \nabla_b S + \nabla_a \nabla_b S - \frac{A}{\hbar^2} \nabla_a S \nabla_b S \right) - AV(\phi) \right) \exp \left( iS(\phi)/\hbar \right) = 0 \right) $$

38 We are here glossing over an important detail: in the construction of the wave-function $\Psi$ from a path integral, we are fixing the anchor point, $\phi^*$. This implies that the Hamiltonian constraint (41) is only valid away from $\phi^*$, or alternatively, that only positive reparametrizations are generated. In its turn this implies that instead of a zero on the rhs of (42) one would obtain a $\delta(\phi - \phi^*)$. Here we are considering the wavefunction for $\phi \neq \phi^*$, which reduces the case to the one displayed.

39 Non ultralocal, means that $G_{ab}p^a p^b = \int d^3x d^3y G_{ab}(x, y) p^a(x) p^b(y)$ and $G_{ab}(x, y)$ is not necessarily proportional to a Dirac delta. Such generalization is useful in order to facilitate point-splitting regularization methods, even if one takes the ultralocal limit later.
with \( \frac{\delta S(\phi)}{\delta \phi} = \nabla_b S \). To \( \mathcal{O}(\hbar) \), it implements the Hamilton–Jacobi equation (at order 0) and the standard current conservation law (at order \( \hbar \)),

\[
G^{ab} \nabla_a \left( A^2(\phi) \frac{\delta S}{\delta \phi} \right) = 0
\]

(44)

with \( A^2 \rightarrow \rho \), \( j_b \rightarrow A^2(\phi) \nabla_b S \). This will hold for any such Jacobi action, and without approximations other than the semi-classical one.

4.2.2. Recovering Schrödinger. But we would like to recover an approximation of the Schrödinger equation, for a weak coupling to gravitational degrees of freedom. We will now roughly go over a similar derivation by Banks [48], here adapted to our setting; namely a simple geometrodynamical model. The main difference to our case is that we are not in a minisuperspace approximation, and thus care must be taken to understand if local disturbances to the semi-classical approximation crop up. It is moreover instructive to go over the proofs for the proceeding discussion.

Suppose that one separates two kinds of fields, \( \phi = (g_{ab}, \varphi) \), i.e. \( \varphi \) should be seen as some sort of source matter field for the metric. Suppose moreover that the gravitational field is mostly unaffected by the source field, since the coupling is assumed to be very weak. In this approximation, for the analogue of Hamiltonian (41), we would have (reinstating \( \hbar \) and \( G \)):

\[
\left( -\frac{1}{2m_p^2} \nabla^2 + m_p^2 V[g] + H_{mat}(g, \varphi) \right) \Psi[g, \varphi] = 0
\]

(45)

where \( m_p \) is the Planck mass (which will give us an ordering), basically establishing the separation of scales between the gravitational and source Hamiltonians, and where, to avoid ambiguity now that we will explicitly refer to local and functional dependence, I used square brackets to denote functional dependence. The gravitational functional Laplacian is:

\[
\nabla^2 = \int d^3 x d^3 y \left( G_{abcd}(x, y) \frac{\delta^2}{\delta g_{ab}(x) \delta g_{cd}(y)} \right).
\]

(46)

where the simplest example of a supermetric is \( G_{abcd}(x, y) = g_{ac}(x) g_{bd}(y) \delta(x, y) \), and the potential is left completely arbitrary.

Assuming moreover that the WKB state now takes the form

\[
\Psi[g, \varphi] = \exp \left( \mathrm{i} m_p^2 S[g] \right) A[g] \psi[g, \varphi] + \mathcal{O}(m_p^{-2})
\]

(47)

Here I am assuming that the WKB part of the wave-function only holds for the gravitational field. In other words, we are at the no interference limit for gravity\(^{40}\). Note moreover that the split between \( A[g] \) and \( \psi[g, \varphi] \) is largely arbitrary. This allows us to set \( A[g] \) as satisfying the functional differential equation (implementing a conservation law along the direction of the momenta)\(^{41}\):

\[^{40}\text{In the path integral context, the perturbations due to the source fields should remain ‘small’, i.e. within the extremal coarse-graining (defined in [35]), which define a tubular bundle around the extremal paths. The two conditions can be shown to be identical.}\

\[^{41}\text{The difference between self-adjoint factor ordering and the one we chose here, amounts to a difference in the above definition; whether we put the supermetric inside or outside—our choice—of the functional derivatives. Had we chosen the self-adjoint one, with the supermetric inside the derivatives, this would have amounted to using a covariant divergence in superspace, instead of the simple one we used. I decided to use the non-covariant one for simplicity of the formulas, but everything here is translatable to the covariant (self-adjoint) context. Note moreover, that there are two covariances at work here. One is the one given by the principal fiber bundle structure; it ensures that the relevant structures live in the reduced configuration space. The other, which I have not given much attention to, ensures that quantities do not depend on the coordinates in reduced configuration space.}\

27
\[
\int d^3y \int d^3x \left( G_{abcd}(x,y) \frac{\delta}{\delta g_{ab}(x)} \left( \frac{\delta S[g]}{\delta g_{cd}(y)} A^2[g] \right) \right) = 0 \tag{48}
\]
with the components of this (bare) current being:
\[
J^{\alpha\beta}(x) = \left( \frac{\delta S[g]}{\delta g_{ab}(x)} A^2[g] \right) \tag{49}
\]
and still leaving the semi-classical wave-function general.

Expanding (45), at order \( m_p^2 \) we get the Hamilton–Jacobi equation, as expected:
\[
\int d^3x d^3y G_{abcd}(x,y) \frac{\delta S[g]}{\delta g_{ab}(x)} \frac{\delta S[g]}{\delta g_{cd}(y)} + V[g] = 0
\]
and next order (\( m_p^2 \)) we obtain:
\[
\int d^3x d^3y \left( G_{abcd}(x,y) \frac{\delta^2 S[g]}{\delta g_{ab}(x) \delta g_{cd}(y)} \right) A[g] \psi[g, \varphi] + 2 \int d^3y \int d^3x \left( G_{abcd}(x,y) \frac{\delta S[g]}{\delta g_{ab}(x)} \frac{\delta A[g]}{\delta g_{cd}(y)} \right) \psi[g, \varphi] \\
+ 2 \int d^3y \int d^3x \left( G_{abcd}(x,y) \frac{\delta S[g]}{\delta g_{ab}(x)} \frac{\delta \psi[g, \varphi]}{\delta g_{cd}(y)} \right) A[g] - i H_{\text{mat}}(g, \varphi) A[g] \psi[g, \varphi] = 0
\]
and finally, using (48), one obtains that the state functional satisfies the first order functional variational equation:
\[
i \int d^3x d^3y G_{abcd}(x,y) \frac{\delta S[g]}{\delta g_{ab}(x)} \frac{\delta \psi[g, \varphi]}{\delta g_{cd}(y)} = H_{\text{mat}}(g, \varphi) \psi[g, \varphi]. \tag{50}
\]

For a given region in \( Q \), one might choose a smooth screen by finding a functional time such that:
\[
\frac{\partial}{\partial T} = \int d^3x d^3y G_{abcd}(x,y) \frac{\delta S[g]}{\delta g_{ab}(x)} \frac{\delta}{\delta g_{cd}(y)} \tag{51}
\]
which is a standard choice for time functions for Wheeler–DeWitt in minisuperspace (see equation (10.25), [3]).

Using the choice of time (51) on (50), we obtain the Schrödinger equation (on a given classical background),
\[
i \frac{\partial}{\partial T} \psi[g, \varphi] = H_{\text{mat}}(g, \varphi) \psi[g, \varphi]. \tag{52}
\]

It is easy to show that the probability currents and densities of gravity and sources also factorize, and the Hamiltonian is Hermitean with respect to the usual Schrödinger inner product

\[\text{Footnote 42: This is almost the integrable arc-length parametrization (which need not exist for large regions of configuration space), but which in general should exist for given small region (around a record, for example). It is almost, but not quite, because we have not used the Jacobi metric. Had we done so, then it is trivially checked that, from the Hamilton–Jacobi equation, one would get } \frac{\partial^2 S}{\partial Q^2} = 1. \text{ In the Jacobi metric, the standard way of building such surfaces is the following: particular solutions to the Hamilton–Jacobi equations define a congruence of classical trajectories. By choosing an initial arbitrary surface and Lie dragging along the (arc-length parametrized) trajectories, one builds a foliation. In our case, we can Lie drag from the (small region around the) record configuration, and thus there is little arbitrary choice of the initial surface. The remaining directions would be the ones orthogonal to the gradient of the time function (they would span the screen).} \]

\[\text{Footnote 43: As made explicit in [35], records, and the coarse-grainings around extremal paths, are not infinitesimal, but form finite regions in configuration space, determined by the degree of distinguishability of said regions, a distinguishability which in its turn is determined by the accuracy of the semi-classical transition amplitude.} \]
for the source fields. I show this in appendix D. And thus we obtain the standard Schrödinger interpretation in the given timeless background satisfying the Hamilton–Jacobi equation\(^44\).

For example, if we want to find the partial infinitesimal flux of the bare probability current (49), through our smooth screen as defined by (51), we just need to take the inner product between the vectors \(\frac{d}{dT}\) and the respective current (i.e. \((dT)_a\frac{\partial}{\partial T}\)). In components, we have:

\[(dT)^{ab}(x) = \frac{1}{V[g]} \frac{\delta S[\phi]}{\delta g_{ab}(x)} \quad \text{and} \quad J_{cd}(x) = \int d^3y G_{abcd}(x, y) \left( \frac{\delta S[g]}{\delta g_{ab}(x)} A^2[g] \right) \tag{53}\]

since then, from the Hamilton–Jacobi equation, \(dT \cdot \frac{\partial}{\partial T} = 1\).

Infinitesimally, in this approximation, the flux of current is given by the probability:

\[dT \cdot \bar{J} = \int d^3x d^3y \frac{1}{V[g]} \frac{\delta S[\phi]}{\delta g_{ab}(x)} G_{abcd}(y, x) \frac{\delta S[\phi]}{\delta g_{cd}(y)} A^2[g] = A^2[g] \tag{54}\]

where we used the Hamilton–Jacobi equation. This equation is approximately conserved along the flow of \(\bar{J}\), as per (48), and it represents the infinitesimal Born volume of the region corresponding to the infinitesimal thickening of the screen area element.

### 4.2.3. Differences from Wheeler–DeWitt

This approach has profound advantages in comparison to the analogous Wheeler–DeWitt calculation—which has many significant flaws. Here I will list only two of these flaws, which I believe are the most relevant (for a complete list see [3], pp 54–62), and explain how the present approach overcomes them.

- **Superposition problem.** The previous calculations, but done in the standard—minisuperspace \(\text{WdW}\)—context, are bedeviled by a lack of uniqueness of the Hamilton–Jacobi functional. In our WKB ansatz, the only assumption we required was that we were in the no interference domain for the gravitational part (interference for the source terms was still allowed by our form of (47)). For most gravitational systems, this just means that records are sufficiently far away (in terms of arc-length distance) from the region being considered so that gravitational decoherence has ensued (see section 5 below)\(^45\). With our preferred initial configuration \(\phi^0\), there is no further choice that goes into determining the solutions to the Hamilton–Jacobi equation.

This is not the case in general, and in particular it is not the case for \(\text{WdW}\). Due to the linearity of the equations, one could in principle have a solution in the form:

\[\Psi(g, \varphi) \approx \exp \left( im_1^2 S_1[g] \right) A_1[g] \psi_1[g, \varphi] + \exp \left( im_2^2 S_2[g] \right) A_2[g] \psi_2[g, \varphi]. \tag{55}\]

The Hamilton–Jacobi equation is not linear in \(S\), and thus two solutions do not necessarily yield another solution\(^46\).

In the presence of more than one Hamilton–Jacobi solution, one loses almost all the nice features of the semi-classical approximation above. For instance, the derivation of the Schrödinger equation (52) no longer holds. Even if one assumes that \(S_1\) and \(S_2\) separately satisfy the Hamilton–Jacobi equation, and that \(A_1\) and \(A_2\) separately satisfy the conservation equation, one still only obtains a sum, for \(\psi_1[g, \varphi], \psi_2[g, \varphi]\):

\[\psi[g, \varphi] = \psi_1[g, \varphi] + \psi_2[g, \varphi].\]

\(^{44}\) We could also have records obeying the approximations for the different fields. This naturally allows for tunneling of the source fields on a given background.

\(^{45}\) For an instantiation of this fact, and a rationale based on chaos for which such non-linear classical interactions should quickly (in terms of arc-length time along extremal paths in reduced configuration space) should suffice for establishing decoherence (in the consistent histories sense). See also [35].

\(^{46}\) Not to mention they might have different boundary conditions.
\[
\left( i \frac{\partial}{\partial T} \psi_1 - H_{\text{mat}}(g, \varphi) \psi_1 \right) e^{i \omega_S[g]} A_1[g] + \left( i \frac{\partial}{\partial T} \psi_2 - H_{\text{mat}}(g, \varphi) \psi_2 \right) e^{i \omega_S[g]} A_2[g] = 0
\]

which provides many more solutions than just the individual Schrödinger equations. Moreover, such a superposition would allow for solutions with negative norm. In the WdW case, since the long-distance dynamics is silent about initial conditions (and their Hamilton–Jacobi associated solution), unless the short-distance dynamics somehow select a unique solution, the wave-function will not have a desired semi-classical interpretation. Moreover, it seems that, no matter what the short-distance dynamics is, if \( S[g] \) is a solution, so is \( -S[g] \). It is easy to find solutions of the form (55) which then have zero (Klein–Gordon) norm.

The reason we have avoided this fate is that due to our fundamental asymmetry of configuration space, we have naturally restricted the space of solutions to ones corresponding to a single Hamilton–Jacobi principal function, \( S[g] = S[g^*, g] \), and by choosing a smooth screen (corresponding to a time function) transverse to the classical trajectories and with the orientation coming from the classical trajectories themselves (away from the record). The point is that these are not choices in our framework, but consequences of the given approximation and our axioms.

• Many-fingered time problem. This is the problem of time, in the semi-classical approximation of minisuperspace. Outside of minisuperspace for standard Wheeler–deWitt, \( \partial_T \) is not a global time, it is space-dependent. Fluxes, conservation laws, and the like become ill defined concepts in the full superspace, partly because of the problems outlined in section 2.1 (after equation (11)): equal time surfaces in superspace will not be gauge-invariant quantities. Moving on to minisuperspace, consequences first appear in that the time function given in (51) ‘should correspond to a foliation in which geometry is homogeneous. It would be a consistent choice if we found that gravity had such homogeneity down to the Planck scale; which it does not’ [3]. Moreover, one should note that within gravity no space-time scalar can be built just from the spatial metric, as should be the case if (51) was to have meaning in a fully covariant theory. Thus an ontological barrier between the semi-classical approximation in a homogeneous (minisuperspace) approximation and the full theory is erected, isolating the two; the approximation becomes its own consistent theory, without a strong connection to the full theory [3].

As mentioned in the beginning of this section, the standard covariant interpretation of gravity has no place for (reduced) configuration space structures, such as records and screens. We do, because we have different symmetry principles and only global conservation laws associated to (global) time. In the context of timeless theories in configuration space as I have presented them here, the results of this section still of course arise from an approximation; but only one of convenience, supported by the hierarchy of the interactions and dynamical behavior of gravity.

4.2.4. Summary of this section. Although the Born rule gives a volume in configuration space, in the absence of external time we need to define what ‘conservation’ means. For this, I defined screens: sets of configurations with the same record, so that no element of this set is a record of another. That is, there is no redundancy of records within this set. I then showed that for smooth screens—defined by the flow of the extremal paths—one can indeed find conservation of probability (for the associated foliation, where it exists). Moreover, I showed that, for weak matter couplings, one can recover the Schrödinger equation for matter fields in the classical background (of each extremal path). Unlike what is the case for minisuperspace
WKB approximations of WdW, records and screens define the time functions and unique Hamilton–Jacobi functionals—each in its domain of existence—which allow the construction of positive currents associated to the probability densities, which can be at least formally extended to the full theory.

5. Applications to quantum gravity

I started the paper discussing fault lines between the foundations of quantum mechanics and those of gravity. I went on to explain some of the problems for defining a reduced configuration space in the presence of local refoliations. Most of these problems are not visible from minisuperspace. Thus, to highlight this aspect of the distinction between approaches, I now briefly present a simple gravitational toy model, corresponding to a form of strong gravity that is not a symmetry reduced model. Indeed, usually, one must resort to minisuperspace approximations to obtain information about quantum cosmology. Here I propose a different simplification (which might have even less to do with our Universe than standard minisuperspace cosmology [18]). The proposal is to use features of the infinite-dimensional geometry—specifically the relation between the Van–Vleck determinant and the geometric expansion scalar along a geodesic congruence [39]—to directly obtain first order quantum effects for gravitational theories. Here I will give only a very brief demonstration of this tool. The geometric analysis used below, however, is available for a much larger set of geometric actions than the one presented here (see [49]).

5.1. Gravitational toy model

First, a brief word on the use of the geometry of infinite-dimensional spaces in order to study quantum cosmology in the semi-classical limit.

5.1.1. Geometry and dynamics. Dynamical systems that are quadratic in the momenta and time-independent can have their flows associated to geodesics of a Riemannian metric (56). It turns out that even more general dynamical systems have similar geometric formulations, as shown in [47], albeit not necessarily through the use of Riemannian metrics.

This implies for example that the confluence or divergence of nearby trajectories are governed by the Jacobi equation (the geodesic deviation):

\[
\frac{D^2 J^\mu}{ds^2} = - R^\mu_{abcd} \hat{\gamma}^a \hat{\gamma}^b \hat{J}^c
\]

where \( R_{abcd} \) is the Riemann tensor associated with the Jacobi metric and \( D/ds = \dot{\gamma}^a \nabla_a \).

Equation (56) is easily obtained by defining a geodesic congruence \( \gamma(t,s) : [0,1] \times [0,1] \to \mathcal{Q} \),

\[
\frac{d}{dt} \gamma(t,s) = \dot{\gamma}^a(t,s) \quad \hat{\gamma}^a \nabla_a \hat{\gamma}^b = 0 \quad \text{and} \quad J^a(t,s) := \frac{d}{ds} \gamma(t,s)
\]

where we used abstract index notation. Definition through the map \( \gamma(t,s) \) implies that \( \hat{\gamma}^a \nabla_a J^h = J^a \nabla_a \hat{\gamma}^h \), and appropriately commuting the derivative on the lhs of (56) yields the rhs.

Many dynamical conclusions can be brought to bear here from geometry. For instance, the Hadamard–Cartan theorem states that if the sectional curvature is strictly negative along \( \gamma(t,0) \), then there are no conjugate points to \( \gamma(0,0) \); i.e. extremal paths do not ‘reconverge’

In \( N = 1 \) gauge, but with a strictly positive kinetic term, i.e. with no negative conformal modes.
or ‘recohere’. The other side of this is the Bonnet-Myers theorem (similar to the standard Focussing theorem), which states that for a bounded sectional curvature along the curve \( K(u, \gamma) = R_{abcd}u^adu^bd\gamma^c\gamma^d \geq \kappa > 0 \) then for \( \beta \geq \pi/\sqrt{\kappa} \), \( \gamma \) contains a conjugate point before reaching time \( \beta \). This is important in the context of the application of the semi-classical formula and the occurrence of interference (24). It also suggests that interference between these extremal paths will be suppressed for many types of chaotic behavior.

Moreover, in this geometric case, one can get many interesting relations between the Van–Vleck determinant (21), the Raychaudhuri equation, the expansion scalar, Jacobi fields and the Riemann curvature itself. Reference [39] gives a plethora of such relations, for congruences of different signatures.

5.1.2. The toy model. To recapitulate how my assumptions here fit the axioms of section 2: I will take \( M = S^3 \), the 3-sphere, \( Q \) to be given by Riem again, i.e. \( \text{Riem}(M) = C^\infty(T^*M \otimes \otimes T^*M) \) and the gauge-group to be the group of 3-dimensional diffeomorphisms, \( \text{Diff}(M) = \mathcal{G} \) acting through pull-back on the metric. The non-degenerate configurations with the highest isotropy subgroup are on the conformal class of \( g \rightarrow \mathcal{G} \). Moreover, in this geometric case, one can get many interesting relations between the Van–Vleck determinant (21), the Raychaudhuri equation, the expansion scalar, Jacobi fields and the Riemann curvature itself. Reference [39] gives a plethora of such relations, for congruences of different signatures.

For the action, given the space of 3-metrics, Riem (given in section 2), I will choose the simplest possible dynamical system, where the action is given by the length functional, for \( \gamma \rightarrow g_{ab}(t) \):

\[
S[g] = \int \sqrt{g} \sqrt{g^{cd}g^{ab}g^{cd}} \int dt \left( \int dx \sqrt{g_{ab}} \right)^{1/2}.
\] (57)

This action is globally (but not locally) reparametrization invariant. Here the path integral could be defined using the arc-length gauge-fixing of the parametrization, discussed in appendix E.2. Alternatively, I could use the Riemann–Stieltjes integration over parametrizations (see section IV, A in [38]). Note that we can do this here mainly because there is a single reparametrization constraint; there is no mixing of local gauge-symmetries with evolution, and the existence of a supermetric allows us to take limits of infinitesimal segments to zero, i.e. there is meaning in taking the size of the ‘mesh’ to zero, and no need to define the gauge-fixing before defining the path integral.48

However, the action (57) is still not invariant under configuration-space-dependent diffeomorphisms, i.e. under \( g_{ab}(t) \rightarrow f(t)^*g_{ab}(t) \) it acquires non-covariant terms depending on \( f(t) \). To covariantize, one has two alternatives, as discussed in section 3. The first, is to extend field space to include a further variable, the shift \( N^i \), which acts on the metric and transforms in such a way as to cancel the unwanted time derivatives of the coordinate transformations. i.e. \( g_{ab} \rightarrow \tilde{g}_{ab} = \mathcal{L}_\xi g_{ab} \), where \( N^i \) transforms according to (9) (or (12)).

The second, is to add a prescription for lifting curves from superspace, \( Q/\text{Diff} \) to \( Q \). This can be done by exploring the principal fiber bundle structure of \( Q \) and introducing a connection form, as explained in section 2.1. In this case, one can take advantage of the metric on \( Q \) to introduce a connection which is roughly a projection orthogonal to the orbits of \( \text{Diff}(M) \). In other words, configuration space vectors tangent to the orbits are of the form \( \nabla(a\xi^b) \) for \( \xi^b \)

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48 In the presence of refoliation invariance, one needs to define a physical slicing condition—so that small steps of the skeletonization make sense—even before defining the path integral. The challenge then is to determine how a change of gauge would affect this definition (see appendix H.2).
a vector field, and thus orthogonal vectors to the fibers wrt to the metric (57) are $v_{ab} \in T_gQ$ such that $\nabla v_{ab} = 0$. The connection itself is a type of Green’s function; the inverse of a second order differential operator, $\omega^a(\hat{g}) = (\delta^a_b \nabla^2 - R^a_{bc})^{-1}(\nabla b^c g^{ab})$. This implies that $\omega^a(\mathcal{L}_g g) = \xi^a$ and $\omega^a(\hat{v}) = 0$ iff $\nabla^a v_{ab} = 0$. Since the connection-form is defined by an orthogonality conditions for a $\mathcal{G}$-invariant metric in configuration space, it automatically satisfies both requirements, (A.9) and (A.10).

For our purposes here, in the semi-classical regime, it will be easier to deal with the second prescription. The action then becomes:

$$S[\hat{g}] = \int dt \left( \int d^3 x \left( \hat{g}_{ab} - \mathcal{L}_\omega(\hat{g})g_{ab} \right) \hat{g}^{ac} \hat{g}^{bd} (\hat{g}_{cd} - \mathcal{L}_\omega(\hat{g})g_{cd}) \sqrt{\hat{g}} \right)^{1/2}.$$  \hspace{1cm} (58)

Under a diffeomorphism, it is now easy to show that the integrand remains invariant\(^50\).

As I am mostly interested in a semi-classical approximation, I will take a short-cut, sufficient for my purposes in this analysis. Since the orbits of $\text{Diff}(M)$ are Killing directions of the metric given in (57), the inner product between the tangent to a geodesic and the orbit resulting expression is too involved to be written here (see theorem 4.7 in [51]). Thus assume here that all initial velocities are transverse, and thus we can use for all practical purposes just (57), not (58), since $\omega(x^b) = 0$ (see below for the appropriate Jacobian that would appear in the path integral).

The geodesic equations are:

$$\hat{g}_{ab} = \hat{g}_{ab} + \frac{1}{4} \hat{g}_{cd} \hat{v}^d \hat{g}_{ab} - \frac{1}{2} \hat{g}_{ab},$$

and one can, in fact, find explicit analytic solutions to the equations of motion [51]. Given $g_{ab} \in Q$ and $h_{ab} \in T_gQ$, let $h_{ab}^0 = h_{ab} - \frac{1}{2} h g_{0}^{0b}$, with $h = h_{ab}g_{0}^{0b}$ and $\nabla T h_{ab} = 0$. Then the geodesic starting out at $g_{ab}^0$ in the direction of $h_{ab}$ is given by:

$$g_{ab}(t) = g_{ab}^0 \left( e^{A(t)} (1 + B(t) h_{cd}) \right)_b^c$$  \hspace{1cm} (59)

where $h$ is the matrix given by $h_{bc} = h_{bc} g_{cd}^0$ of $\frac{4}{3} \sqrt{h_{cd}^2}$ and $A(t) = \frac{2}{3} \ln(1 + \frac{1}{4} h^2) + \frac{2}{8} h_{cd}^2 h_{cd}^2$ and $B(t) = \frac{4}{\sqrt{h_{cd}^2}} \arctan \left( \frac{\sqrt{3 h_{cd}^2}}{4 + \theta h} \right)$.

Note that the geodesics are ultralocal. Since the action has no spatial derivatives, the solution $g_{ab}(x, t)$ depends only on $g_{ab}(x, 0)$ and $\hat{g}_{ab}(x, 0)$. In arc-length units, the velocity $h_{ab}$ has units of $1/\text{sec}$. Let me briefly note that the Jacobi fields can also be explicitly solved for, but the resulting expression is too involved to be written here (see theorem 4.7 in [51]).

For standard homogeneous cosmological models, one would have $h_{ab}^T = 0$, greatly simplifying the solution, as $B = 0$ and $A = \frac{2}{3} \ln(1 + \frac{1}{4} h^2)$ where $h$ is then the instantaneous expansion. In this case, however, the volume of the manifold, which is dictated by $A(t)$ above, will reach a singular value in finite time for $h < 0$. If $h_{ab}^T \neq 0$, however, geodesics will run forever.

\(^{49}\) See also [25] for a relation between a choice of connection, locality, and the representation of abstract observers.

\(^{50}\) Note that the terms which take into account the metric variation of $\omega$ itself vanish by the equations defining $\omega$. The same sort of lift would ensure if the symmetry group included Weyl transformations, and the absence of Weyl anomalies in 3d extends, through the use of the horizontal lift, to the absence of Weyl anomaly in the lifted path integral. Note moreover that, unlike for refoiliations, in this case it is easy to show that the Fadeev–Popov determinant behaves like a true determinant, and so, taken together with the measure, is invariant under the given transformations.

\(^{51}\) $\frac{\partial}{\partial w} (g^{0b} u_{0a}) = \gamma^a w^b \nabla u_{0a} = 0$ for $u^a$ Killing. This elementary fact about Pseudo–Riemannian geometry remains true also in the infinite-dimensional case [50].
Endowed with such a metric, $\mathcal{Q} = \text{Riem}(M)$ has an associated Riemann super-curvature (of the Levi-Civita connection). Along pure trace directions (as would happen in a purely homogeneous expansion), it is zero. Otherwise, in index-free notation it is given by:

$$R(h_1, h_2) h_3 = -\frac{1}{4} [h_1, h_2], h_3] + \frac{3}{16} (tr_g(h_1, h_3) h_2 - tr_g(h_2, h_3) h_1)$$  \hspace{1cm} (60)

where $h$ are matrix vector fields, extending the sort of matrix vector given above for $T_g^0 \mathcal{Q}$ to $T \mathcal{Q}$. I have assumed that $h = h^f$, and the standard matrix commutator is used. The sectional super-curvature is then calculated to be52:

$$K(h_1, h_2) := \langle R(h_1, h_2) h_1, h_2 \rangle_g = - \int d^3 x \sqrt{g} \left( tr_g([h_1, h_2]^2) + \frac{3}{16} \left( (tr_g(h_1 h_2))^2 - tr_g(h_1^2) tr_g(h_2^2) \right) \right).$$

For $h_1, h_2$ orthogonal, i.e.

$$\int d^3 x \sqrt{g} \text{tr}(h_1 h_2) = \int d^3 x \sqrt{g} h_{ab}^1 g^{ac} h_{cd}^2 = 0$$

we immediately get from (61) that $K(h_1, h_2) \leq 0$. By the Hadamard–Cartan theorem, this means that there will be no conjugate points for these dynamical paths. It also means that interference of alternative paths will quickly subside, and semi-classical quantum effects will become predominantly due to other fields, present on a fixed metric background (see section 4.2).

For quite general cases, the geodesic equation, the exponential maps, Jacobi fields, curvature, etc, have been computed [49]. Quantities of interest become geometrical; for example, the Van–Vleck determinant:

$$\Delta(g, g') = \det (J(g', g')_{abcd}(x, y))^{-1} = \Delta(g, g').$$

(62)

Where $J$ is the Jacobi matrix associated to the geodesic variations53. The determinant here requires regularization, but in principle we can calculate everything needed for a semi-classical path-integral within the classical geometry of $\mathcal{Q}$, by using equation (20) and (21) together with (57) and the expression for the derivatives of the Riemann exponential map. Alternatively, we could use Barvinsky’s reduction methods for functional determinants [33], which reduce the 1-loop functional determinant to a spatial one, integrated over a given time parametrization. Here I will choose the first, simpler route.

There are two possible procedures to obtain closed expressions for the Van–Vleck determinant: (i) input equation (59) into the action (57), with the end-point $g_{ab}(1)$ as a function of $h_{ab}$ (for fixed $g_0$). Or, (ii) use the given closed expression for the Jacobi fields in theorem 4.7 in [51] to directly write the Jacobi matrix.

Both manners are calculable, but the resulting expression is not compact, or enlightening at face value. We pursue the first option in appendix F. Alternatively, we can use geometric approximations for the Van–Vleck which work, respectively, for weak field and small geodesic distance [39] (for finite-dimensions, with $\gamma$ the geodesics):

52 All the formulas in this section are generalizable to $n$ dimensions. For instance, the $3/16$ above in fact comes from $\frac{d}{d t} \langle 4 + n(n + 1) \rangle$ [51]. Moreover, one should note that there are two sorts of connections: one related to the internal gauge space, and the other the Levi-Civita. This is precisely the analogous case of Kaluza–Klein, and the two satisfy relations [51].

53 Jacobi fields–geodesic variations of $g_{ab}$ are uniquely defined by its value and first derivative at $\phi$, alternatively, it can be characterized by it values at $\phi_0$ and $\phi$, through the two-point tensor matrix $J(\phi_0, \phi)$. 
\[ \Delta^\text{weak}_g(x, y) = \exp \left( \frac{1}{t} \int_0^t (t - s) (R_{ab} \dot{\gamma}^a(s) \dot{\gamma}^b(s)) ds + \mathcal{O}(|\text{Riemann}|)^2 \right) \]  

\[ \Delta^\text{small}_g(x, y) = 1 + \frac{1}{6} (R_{ab} \dot{\gamma}^a \dot{\gamma}^b [g]_2^2 + \mathcal{O}(S[g])^3). \]

The Ricci super-curvature does not properly exist, since the mapping \( k \mapsto R(h, k) \) is just the push forward of the section by a certain tensor field, a differential operator of order 0. If this is not zero, it induces a topological linear isomorphism between certain infinite dimensional subspaces of \( T_g Q \), and is therefore never of trace class. In this point of this approximation we need to smuggle in some sort of regularization\(^{54}\).

Here, following [51], I take a shortcut by considering the pointwise trace of the local action of the Riemann tensor:

\[ \text{Ric}(h_1, h_2) = -\frac{3}{2} \int d^3 \chi \sqrt{g} h_{ab} g^{ac} g^{bd} h^{cd} \]  

which is almost proportional to the supermetric, except that one of the components gains a traceless projection. Due to this traceless projection, we do not have to worry about the extremal paths under consideration reaching the boundary of Riem (where we would have to set up boundary conditions). The problem with the super-curvature being proportional to the supermetric (and that inherent in (60)) is that it means in a sense we are in a constant curvature regime, and cannot really apply (63). We are better off applying (64). For points close to \( g_0 \) in arc-length, i.e. for \( S[\gamma] \sim \epsilon \), and initial velocity \( \dot{g}_{ab} = h_{ab} \), we get:

\[ \Delta_g(g_0, g(\epsilon)) = 1 - \frac{1}{4} \left( \int d^3 \chi \sqrt{g} h_{ab} g^{ac} g^{bd} h^{cd} \right) \epsilon^2 + \mathcal{O}(\epsilon)^3. \]

Clearly the pure trace directions (cosmologically homogeneous) have no contribution from \( \epsilon^2 \) terms, and thus, at least close to \( g_0 \), are of greater amplitude. Note however, that here we must strike a fine balance. That is because the semi-classical approximation is valid for \( S[g] \gg \hbar \), and thus we must have an expansion for small arc-length \( \epsilon \), but \( \epsilon \gg \hbar \), and there is no other dimensionful constant to compare to.

We must thus consider this Van Vleck as giving a one-loop functional determinant measure (combined with the projected Liouville measure), on the tangent space \( T_g Q \), i.e. on the space of initial cosmological directions at \( g_0 \).\(^{55}\)

According to (18), for the short distance approximation, we are in fact integrating over directions, \( h_{ab} \), and such that \( \nabla_a h_{ab} = 0 \). But because of the latter, ‘gauge-fixing’ condition, we gain a Jacobian, \( J_T(g^0) \). This is calculated in appendix E, equation (E.6). The determinant that emerges however, depends only on \( g^0 \), and thus is independent of the integration variable.

Finally, since (66) is obtained from an exponential, we obtain, for the relation between two total surface volumes (or screen-flux) for the screen regions parametrized by \( H_1, H_2 \in T_g Q \), at a geodesic distance of \( \epsilon \), and with the same nominal volume (i.e. same volume under the homogeneous measure \( \int_{H_1} D\ell = \int_{H_2} D\ell = V_H \)),

\(^{54}\) For an action with spatial derivatives, which would give Laplacians, and an expansion on eigenvalues of the tensor Laplacian would provide a cut-off for the trace inherent in the determinant. This will also be important to get some inhomogeneity in our probability for gravitational modes.

\(^{55}\) We should remind the reader that a more standard computation exists in terms of the integral of the one-loop determinant in time, which gives the Van–Vleck [33]. Because here we are exploring the geometrical analogy, we will not pursue that strategy.
\[
\int_{0}^{\Delta t} \mathcal{D}h \mathcal{D}r \left( g^{(\Delta t)} \right) P_{(\Delta t)}(\varepsilon) = \int_{0}^{\Delta t} \mathcal{D}h \left( 1 - \frac{1}{4} \left( \int \, d^{3}x \, \sqrt{g} \, h_{\text{Tab}} a_{\text{out}} b_{\text{out}} h_{\text{out}} d^{3}x \right) \right) + \mathcal{O}(\varepsilon)^{4} 
\]

This is the relative probability for these two screens defined by the same arc-length distance, in an expansion for arc-length distance from \( g^{0} \) larger than Planck. Explicit numbers can be computed by using an eigenbasis of transverse (0, 2) transverse traceless tensors on the round 3-sphere, \( g^{3} \). A convenient such basis is the tensor harmonic basis on \( S^{3} \), fully described in [52]. Since \( S^{3} \) is compact without boundaries, and the standard Laplacian is a self-adjoint operator with respect to the standard inner product (see (E.10)), by the spectral theorem the eigenfunctions with different eigenvalues are orthogonal. In terms of these specific eigenvalues, the measure becomes \( \prod_{i} \mathcal{D}h_{ij}(x) \rightarrow \prod_{n} d\lambda_{n}^{3} \), if the basis is orthonormal (see (E.11)). If it is not, another Jacobian appears for the transformation\(^{56}\). The value of (67) then becomes a difference for the integrals for different ranges of the prefactors of such a harmonic basis, which can be now calculated explicitly. But this is just a homogeneous measure\(^{57}\), and thus

\[
\int_{H_{1}} \mathcal{D}h = \int_{H_{2}} \mathcal{D}h \Rightarrow \int_{H_{1}} \mathcal{D}h^{2} \left( \int \, d^{3}x \, \sqrt{g} \, h_{\text{Tab}} a_{\text{out}} b_{\text{out}} h_{\text{out}} \right) = \int_{H_{1}} \mathcal{D}h^{4} \left( \int \, d^{3}x \, \sqrt{g} \, h_{\text{Tab}} a_{\text{out}} b_{\text{out}} h_{\text{out}} \right) .
\]

In other words, this measure does not care about the eigenvalues of the transverse–traceless modes. It is homogeneous, and thus would not probabilistically favor ‘flatter’ modes, as is expected of inflation.

However, preliminary calculations show that with coupling to a metric, i.e. having instead of (57),

\[
S[g] = \int \, dt \left( g^{-\frac{1}{2}} \right) \left( \int \, d^{3}x \, g^{\alpha\beta} \nabla_{\alpha} f(\mathcal{R}) \sqrt{g} \right) \left( \int \, d^{3}x \, g_{\alpha\beta} g^{\alpha\beta} \sqrt{g} \right)^{1/2}
\]

changes terms in (67) in the following way:

\[
h_{\text{Tab}} a_{\text{out}} b_{\text{out}} h_{\text{out}} \rightarrow h_{\text{Tab}} a_{\text{out}} b_{\text{out}} \nabla^{2} h_{}\text{cd}
\]

where \( n \) is the degree of \( f(\mathcal{R}) \). In this case, the higher the eigenvalues of the region \( H_{1} \) in comparison to \( H_{2} \), the smaller is its relative probability flux. This would mean indeed that more homogeneous modes would be favored. Although it should be noted that the eigenvalues of the basis \( \tau^{(n)}_{ij} \) in (E.11) are given by \( \nabla^{2} \tau^{(n)}_{ij} = -(n^{2} - 3) \tau^{(n)}_{ij} \), for \( n \geq 3 \). And thus, one would not have complete homogeneity represented in this space.

5.1.3. Comments on the cases where \( \varepsilon \) is not small. For the full expression of (62), we need replace the solution (59) into the action (57), between \( t = 0 \) and \( t = 1 \). Then the action as

\[
\frac{\delta^{2}S}{\delta g^{ab}_{\text{out}} \delta g_{\text{out}}(1)} = \frac{\delta}{\delta g^{ab}_{\text{out}}} \left( \frac{\delta S}{\delta h_{\text{ab}}} \frac{\delta g_{\text{out}}(1)}{\delta h_{\text{ab}}} \right) .
\]

\(^{56}\) These are the coefficients for the eigenfunctions described in (17)–(18c) of [52].

\(^{57}\) Had we chosen a basis which is not orthonormal, the Jacobian would have cancelled out with the emerging prefactors in any case.
This is a cumbersome procedure, requiring quite a bit of algebra. As the purpose of this exercise is a proof of principle for the method, we leave these calculations for the appendix F (see equation (F.11)).

In the general case it also happens that, since there will be at most one extremal path between \( g^0 \) and \( g \), the semi-classical probability density is given simply by

\[
P(g)Dg = F(\Delta^{1/2}(g^0, g)e^{iS[g, \delta]/\hbar})Dg = \Delta(g^0, g)Dg
\]

where \( g \) lies in a classical path from \( g^0 \) with a horizontal initial condition, i.e. \( \nabla_g g^0 = 0 \) and \( \Delta \) is defined by the determinant of the geometrical Jacobi matrix, in (62). It is clear from (59) that we could not have chosen the completely degenerate metric as \( g^* \).

There are two qualitative features for the gravitational quantum mechanics model that can be seen at this primitive stage: firstly, there is no gravitational interference, as the sectional super-curvature is strictly negative (61). Since generically interacting Hamiltonians in more than 2 dimensions are chaotic, it seems even in the more general geometric case we can use the Rauch comparison theorem in the other direction: generically, semi-classical interference effects will become suppressed after a short evolution time (in arc-length). Different cosmological histories will quickly decohere, with no chance of recohering. Thus, in exactly the same fashion as the Mott bubble chamber (a quantum mechanical decay process), metrics along the same geodesic (separated by more than \( \hbar \) in arc-length) can serve as records for ones coming later along the geodesic. At this level in \( \hbar \), with weak coupling to matter fields, the gravitational degrees of freedom would quickly become a background for the quantum effects of these other matter fields. Thus in principle we can fully determine the gravitational semi-classical theory, complete with records, interference (or lack thereof) and so on.

Secondly, geodesics can be extended indefinitely, but it is not true that any two points can be connected by a minimizing geodesic (thus the Hopf–Rinow theorem for finite-dimensional Riemannian geometry fails in this infinite-dimensional context). This already means that certain configurations have negligible volume from the start, without even taking into account the Van–Vleck determinant or the relevant records. It also counters some finite-dimensional intuition, that would lead one to believe that any configuration could be reached by a judicious choice of initial condition. There are, in the gravitational configuration space, deserts of very little volume.

On top of this structure, we could now proceed along the lines of section 4.2. This would allow us to discuss the quantum evolution of source fields on top of each one of these classical gravitational solutions, where each classical gravitational solutions is weighed by its own probability according to (67).

5.1.4. Summary of this section. This section was a proof of principle. Here I have described a simple gravitational toy model. With it I was able to implement the structures presented in this paper (e.g.: the field-space connection 1-form, the semi-classical approximation, etc) for building the semi-classical volume form. Extending an equivalence between the Van–Vleck determinant and geometric objects (depending on Ricci curvature), I proposed an equation for the semi-classical amplitude in the vicinity of the round 3-sphere. With it, I found a simple result for relative volumes of a geodesic screen close (in configuration space) to the round sphere. These represent relative transitions from a total amplitude which, according to the

58 Although, see [53] for arguments about the naturalness of certain reflecting boundary conditions for geodesics in superspace. Here too, one can impose certain artificial boundary conditions and obtain interference patterns directly from (24).
previous section, would be preserved in the geodesic arc-length time\(^{59}\). In this simple case, these amplitudes are just proportional to the respective ‘functional area’ encompassed by two sets of TT-modes (of the round \(S^3\)), on the tangent space \(T_{g_0}Q\). Nonetheless, I indicated how, including some coupling between points (through the inclusion of spatial derivatives in the action functional), the relative transition amplitude would naturally favor more homogeneous directions in field space (departing from the round sphere). It is important to note that this was all done without a minisuperspace assumption, and the results are not strictly classical, as they require the computation of the semi-classical effects through the Van–Vleck determinant.

5.2. Regaining space-time

As I mentioned in the introduction, it should be noted that even after having a good semi-classical quantum gravitational model of the sort presented here, one must still reconstruct an effective space-time from a curve of geometries. In the cases where we can use a classical approximation for a given amplitude, then we can just look at the solution curves of the action principle. Such a solution curve is of the form \(g_{ab}(t)\), i.e. a one-parameter family of 3-metrics (it does not have anything corresponding to \(g_{0\mu}\) yet).

In other words, although along a classical trajectory we can order records, this might not be relevant for what observers within these trajectories in Riem perceive as duration. This touches on the common question in quantum mechanics, surrounding what constitutes a relational clock. This section does not provide a definitive argument, but a sketch on different directions that could be pursued in this respect.

It should be said that all theories that refer fundamentally only to the 3-metric,—all standard interpretations of canonical quantum gravity, such as Hartle–Hawking—would require a reconstruction of space-time at a hypothetical classical limit. As with the necessity of records, these issues are less pressing in the minisuperspace context, and usually glossed over.

For shape dynamics [4], the dynamics of matter fields can reproduce those of GR (at least for some finite duration). As has been shown in different ways [54, 55], one can then reconstruct an effective slab of space-time which is indistinguishable from GR, and recover aspects of the equivalence principle [54]. But these solutions are more or less fine-tuned to match general relativity, and the way in which they acquire those features is not completely understood.

In the more general case pursued here—which gives a method and a paradigm, not a specific theory—we would like to understand the nuts and bolts about how any space-time can be recovered from a curve in (conformal) superspace, and what are its properties. This, we do not yet know how to do, but here is one attempt. I will first discuss the emergence of some limited, finite refoliation symmetry, based on scalar fields.

5.2.1. Refoliations. It is not really mysterious how regaining a space-time structure from matter fields can break refoliation invariance. Indeed, the useful matter fields for this role should be expected to break refoliation invariance; after all, they can define a rest-frame, or a surface in space-time where their gradient is purely timelike. For example, the CMB can be taken as one such reference. After one regains a space-time structure, what is the physical requirement for a fundamental refoliation invariance? It is not clear to me, but below I propose that it is encoded solely in the possibility of choosing different matter fields to describe the same space-time structure.

\(^{59}\) At least while these ‘geodesic spheres’ in field space are integrable. In the present case, by the Gauss lemma, their integrability could have a very large domain, since there are no focusing points.
It is also important to notice that the observability of a ‘preferred foliation’ is not obvious from the form of the action. For instance, the BSW action [56]:

$$S_{BSW} = \int d\tau \int d^3x \sqrt{g} \left( \dot{g}_{ab} G^{abcd} - R + \sum_i a_i (\dot{\psi}_i)^2 \right).$$  \hspace{1cm} (68)

where $\dot{g}_{cd} := \dot{g}_{cd} - L_{\xi} g_{cd}$ is the ‘dressed’ horizontal metric velocity, for $\xi^a$ a specific functional of the metric and its time metric. Equation (68) is not explicitly space-time covariant—it does not contain the lapse or the shift, components of the space-time metric in the covariant form—and yet it is exactly the Einstein–Hilbert action, written in 3 + 1 form and with eliminated lapse and shift. The important aspect of GR that can be seen in (68) is the fact that it contains the same amount of spatial and time derivatives.

Given these two considerations, I thus start with an action of the form:

$$S = \int d\tau \int d^3x \sqrt{g} \left( \dot{g}_{ab} G^{abcd} - R + \sum_i a_i (\dot{\psi}_i)^2 \right).$$  \hspace{1cm} (69)

(68)’

(69)’

(69)

\[
S = \int d\tau \int d^3x \sqrt{g} \left( \dot{g}_{ab} G^{abcd} - R + \sum_i a_i (\dot{\psi}_i)^2 \right).
\]

This is basically the standard gravitational Einstein action with $d/d\tau$ replaced by the derivative along the scalar field, and with the ‘lapse’ $N_i = \frac{d\psi_i}{d\tau}$. Now, one can shift from one $i$ to the other iteratively, i.e. forgetting about their original relation to $t$, obtaining the same results. i.e. by using $d\psi_i = \frac{d\psi_i}{d\tau} d\tau$ in the equations above, one gets precisely the same action, with $i \rightarrow k$, i.e. just described under evolution for a different scalar field. It is a ‘discrete’ sort of refoliation.

Moreover, the perturbation equations for the metric are hyperbolic, and would construct a universal gravitational light cone, i.e. in this extremely simple case, the equations of motion of the gravitational fields will be hyperbolic (for the unit lapse case):

$$\ddot{g}_{ab} = \dot{g}_{ij} \dot{g}_{kl} \Gamma^{ijkl}_{ab} + 2 (R_{ab} - \frac{1}{4} g_{ab} R)$$

where $\Gamma^{ijkl}_{ab}$ are the Christoffel symbols for the standard DeWitt supermetric. And thus, the propagation of gravitational perturbations around stationary solutions, such as $g_{ab} = \delta_{ab}$ (the flat Euclidean metric), will form null-cones. The requirement that these characteristics are indeed the null-cones of a metric, will determine the conformal class of the space-time metric (and thus a relation between the lapse and the spatial metric).

What is most unnatural about equation (69), however, is the fact that there are no spatial gradients of the scalar fields. This was motivated by the way one would use such a scalar field in space-time to define a foliation. Nonetheless, we can remove this condition by adding gradient terms—e.g. the standard one $\nabla_a \psi \nabla^a \psi$—and using shift vectors adapted to each scalar field, in the following way. Correcting the time velocities of the scalar fields: $\dot{\psi} \rightarrow \dot{\psi}_H = \dot{\psi} - \chi^a \nabla_a \psi$, we obtain the following correction to the kinetic terms:
\[ \dot{\psi}^2 - g^{ab} \nabla_a \psi \nabla_b \psi \rightarrow \psi^2 - 2 \dot{\psi} \chi a \nabla_a \psi + (\chi^2 \nabla_a \psi)^2 - g^{ab} \nabla_a \psi \nabla_b \psi. \] (71)

Now, choosing \( \chi^a = \chi g^{ab} \nabla_b \psi \), we get a quadratic equation on \( \chi \) so that
\[ \dot{\psi}^2 - g^{ab} \nabla_a \psi \nabla_b \psi \rightarrow \dot{\psi}^2 \]

namely,
\[ \chi^2 (\nabla_a \psi \nabla^a \psi) - 2 \chi \dot{\psi} - 1 = 0 \]

which can be solved for the fields,
\[ \chi = \frac{\dot{\psi} \pm \sqrt{\dot{\psi}^2 - \nabla_a \psi \nabla^a \psi}}{\nabla_a \psi \nabla^a \psi}. \]

It thus seems that one can choose the shift vectors to be physically given in such a way as to always ‘select a surface of spatially constant \( \psi \)’. In this way, one could in principle retain the spatial gradients of one \( \psi \) in a foliation, but get rid of another.

But there are many problems with this resolution however: when coupling to the metric, or with the other fields, it would imply a non-minimal coupling between \( \psi \) and \( g \), which would also reinstate the dependence of \( \nabla_a \psi \). This perhaps could be resolved by adding the terms from the beginning and solving a possibly more complicated equation, depending on all the fields. Another problem is that the solution for \( \chi \) does not seem to define a bona-fide connection-form, i.e. it does not have the right covariant transformation properties as in (A.10) or (9). Nonetheless, this is first attempt, a proof of principle that certain things can work, while others not at this point.

5.2.2. Null-cone and equivalence principle. As a simple example, given extremal paths of (57), we can attempt to reconstruct a space-time just with the use of scalar fields. The easiest manner to do this is to introduce a weakly back-reacting scalar field, with action in arc-length parametrization given by
\[ S_{\text{scalar}} = \epsilon \int dt \int d^3 x \sqrt{\gamma} (\dot{\psi}^2 - c^2 \gamma^{ab} \nabla_a \psi \nabla_b \psi + m^2 \psi^2) \]

for a very small \( \epsilon \). An objection here is that this action is largely inspired by a relativistic one—in that it contains the same number of spatial and temporal derivatives. Ideally, one would have more than one field, e.g.: \( \psi_1, \psi_2 \), and use the characteristics of one to define the propagation of the other. We leave this for further work. Using the characteristics of the field equations (on this background \( \gamma_{ab}(t) \)), we could have a metric given by:
\[ ds^2 = -c^2 dt^2 + \gamma_{ab}(t) dx^a dx^b \] (72)

where \( \gamma_{ab}(t) \) is given in (59). The first question one can ask is when (72) also obeys the Einstein equations for example.

Now, to discuss something such as the equivalence principle, one could proceed as in [54]; couple a point-like particle with trajectory \( z^a(t) \)—with the usual covariant action, parametrized by the superspace arc-length, and with an induced lapse given (in this simple case) by (72)—i.e.
\[ S_{\text{point}} = \int dt \int d^3 x \delta(x^a - z^a(t)) (c^2 - \dot{z}^a \dot{z}_a)^{1/2}. \]

The equations of motion are the usual geodesic ones for the metric (72). According to [54], to first order, one could use time-dependent spatial diffeomorphisms to correct the frame.
of reference centered on such a particle to be described by geodesics in Minkowski space, apart from transformations of the lapse\textsuperscript{60}. In the present simplified case, the physical lapse is already equal to one, so we regain one of the statements named ‘the equivalence principle’—namely, there is a gauge symmetry that will bring an arbitrary classical trajectory to a geodesic of Minkowski space, at least infinitesimally. This issue, however, needs to be further investigated in more general cases with more general matter fields; it is not clear to me that it must always hold.

5.2.3. Summary of this section. In this section I have sketched different ways one would go about reconstructing a space-time metric and regaining the equivalence principle (and other notions of refoliation invariance). This can be accomplished from the relative evolution of observables along an extremal path in configuration space. There, there can still be more than one way of defining duration, and perhaps these can replace different notions of refoliations (which is an inherently spacetime operation). Indeed, what do we physically do when we want to refoliate? We choose a different set of clocks spread out through space. Moreover, we still want a hyperbolic structure. Since we do observe that light has the same speed for all observers, and that can only happen if the metric has a null cone. In specific cases this translation seems to go through without much problems, but no generic statement was proven.

6. Conclusions

6.1. Problems between records and refoliation invariance

6.1.1. Differences from Hartle–Hawking and the tunneling proposals. As discussed in the main text, the formalism presented here has certain features in common with the Hartle–Hawking no-boundary [1], and with Vilenkin’s tunneling [57] proposals. The boundary conditions on Vilenkin’s model in mini-superspace coincides with our model, for the reduced ADM action, as I comment below.

There are, however, some crucial differences. The boundary conditions are based on completely different principles. In Hartle–Hawking, the path integral integrates over all 4-metrics interpolating between a given regular Euclidean 4-metric without boundary and a given 3-geometry with matter content. In the tunneling proposal, the boundary conditions should embody the notion that the Universe ‘tunnels’ from nothing. In both cases, the resulting wave-function (in terms of its one boundary 3-geometry and matter content) should satisfy the Wheeler–DeWitt equation.

Like the present work, these theories attempt to define a wave-function of the Universe from a path integral in some field configuration space. But there are a couple of important distinctions between such proposals and the one developed here:

(i) Both Hartle–Hawking and tunneling proposals are based on path integrals in superspace. But superspace is not the reduced configuration space of ADM, because one still has refoliations. In Hartle–Hawking, one usually stipulates boundary conditions such that $\Psi(g_{ab}) = 0$ for the degenerate determinant $g = 0$ [1], integrating only over metrics for which $g > 0$ and which can be ‘capped-off’ by part of a 4-sphere. In Vilenkin’s proposal (see also, Linde [58]), the definition of the wave-function is given by a transition function from the completely degenerate 3-metric, $g_{ab} = 0$. One also requires, however, further boundary conditions on superspace. For this, Vilenkin attempts to separate curvature

$^{60}$ Note here that the propagation of spatial coordinates are also ‘physical’, as they are given by a choice of $\varpi$. But different choices will be related by spatial diffeomorphisms, as is the case with different charts for the manifold.
singularities which would correspond to singularities of a 4-geometry—the singular boundary of superspace—and those which would correspond to singularities due to bad choices of slicings—the non-singular boundary of superspace. This is a difficult distinction to make in practice, at least away from minisuperspace. Assuming that it could be done, then the proposal requires the wave-function to have only out-going modes at the singular boundary. There is also a problem here, as in general there are no time-like Killing vector fields in superspace to define positive and negative frequencies. However, the proposal can be carried through in the WKB, minisuperspace context, yielding results similar to Hartle–Hawking.

In contrast, in our case, there must be a well-defined reduced configuration space for the volume form to have physical meaning. Records and screens required crossing extremal curves in superspace to have objective meaning (there cannot be one if refoliations are part of the local symmetries). In the case of $\text{Diff}(M) \ltimes C$ with $M = S^3$, our construction means no further boundary conditions need to be imposed apart from our single axiom. This can be seen by parametrizing physical space with unimodular metrics, $\tilde{g}_{ab} := g^{-1/3}g_{ab}$, which capture all shape degrees of freedom [27]. To change the signature, the determinant must become degenerate, which disconnects the physical space of positive definite signature from those with other signatures. In other words, the ‘cone’ which makes up the boundary of Riem inside the affine space $C^\infty(TM \otimes_{S} TM)$ becomes inaccessible from within reduced configuration space.

(ii) When one does not have access to the physical instantaneous configuration space, there seems to be a plethora of ways and choices to implement boundary conditions for the wave-function. Hartle and Hawking gave one; Vilenkin and Linde, another; but many others should be possible. In the present approach, the boundary conditions are selected uniquely by extremizing some gauge-invariant quantity, and are such that the arrow of time should point in the direction of greater inhomogeneity.

Said in another way, here the boundary (or initial) condition is motivated from an informational and timeless perspective. A reduced configuration space of observables does exist, and in it, one looks for the most homogeneous elements—i.e. the ones that encode the least irregularities, or ‘information’, in its configuration. Under certain symmetry groups, and with certain assumptions about the degrees of freedom of the theory, a unique such element can be singled out. In addition, due to the stratified nature of reduced configuration space—with concatenated boundaries—this point consists of a natural boundary. In a geometric sense it is the ‘boundary of all boundaries’ [26], and no further boundary conditions need to be given. Such a radical asymmetry of reduced configuration space is furthermore necessary for our notion of records (definition 1)—the crucial element which makes the recovery of time possible.

Needless to say, both Hartle–Hawking and Vilenkin’s proposals are more studied and developed theories of initial conditions than the one presented here, even if they face enduring questions about (i) the complex contour integration in which the path integral should be executed [59], (ii) the meaning of the Wick rotation (see e.g. [60] and references therein), (iii) the meaning of probabilities, and their conservation, (iv) the fact that for anisotropic minisuperspace, there seems to be no clear division between a purely imaginary and purely real solutions of the lapse for the saddle point approximation, but in fact a general solution should be complex [61], and lastly (iv) recent calculations show that the Hartle–Hawking calculation for the evolution of perturbations would be unbounded [62]. These are all points that could be made clearer with a new, first principles definition.
In the technical front, it should be noted that due to the algebra of constraints containing the metric, a rigorous proof of gauge-invariance requires the use of the FBV formalism [23], which for this case will include interaction terms of fourth order in ghosts, with no geometric meaning as a Fadeev–Popov determinant [24].

In the present work simple BRST symmetries with a geometric meaning as a FP determinant [63] can be derived because all constraints are linear in the momenta (see (10)). The wave-function built from the path integral satisfies the given functional constraint equations (even in the case of Weyl symmetry). A reduced (physical) configuration space exists, where we can make sense of a volume-form, including the records it creates, and decoherence in the sense of consistent histories. In the toy model we explored here, the action functional is positive-definite, with no requirements for choosing complex actions and complex contours of integration. More generally, the Wick rotation is well defined along each path. More work needs to be done in this direction to ascertain more precise properties of the semi-classical solution.

6.1.2. Incompatibility between records and fundamental refoliation invariance. It is not possible to apply the present formalism to the Wheeler–DeWitt equation in general—i.e. outside of the minisuperspace ‘approximation’—including its embodiment in the Hartle–Hawking no-boundary proposal. This comes about because of refoliation invariance, which makes it impossible to use the acquired notion of time through records, as I now comment on.

Firstly, there is no known form of reduced configuration space for the symmetries of ADM. This creates an insurmountable obstacle for the constructions of this paper. Even if one defines an initial configuration, records, screens, and so on, with respect to a given gauge-fixing of the foliations, these structures will not be preserved under a different choice of gauge-fixing, as I described in equations in section 2.1.

Moreover, to the extent that such foliations can be of physical character (i.e. some surface of homogeneity of a given field), they should be reproduced by the purely relational evolution of observables, which I described in section 6.1.3.

Lastly, note that if the theory possesses a single global reparametrization constraint, my notion of records remains fully functional. This makes it more akin to minisuperspace. A curve will go through a given point however it is parametrized. A profound asymmetry of reduced configuration space will then spontaneously form records and, consequently, an effective notion of time. This is how records untangle evolution and symmetries.

6.1.3. Non-causally related observables, and its problems. This paper was based on assigning a fundamental role to acausal relations, prior, and not defined by, the specific field content which one would like to superpose. Whereas in GR causality is inherited from the space-time metric—‘space-like’ is a statement about causal independence which depends on the metric—I would like to place local causally independent observables at the foundations of quantum gravity (QG). Space-time physics are to be recovered only effectively. This led me to a description of quantum mechanics on timeless configuration space.

This setup is perfectly suited for relational approaches to quantum mechanics [16, 17, 64], the results of which we are able to recycle. Importantly, foregoing the space-time picture off-shell (or before dynamics) does not imply that a preferred simultaneity surface is classically detectable (see [54] for a explicit example). In fact, models of gravity can be formulated where we get all the advantages of a preferred notion of simultaneity—compatibility with

61 In fact, due to this complexity, the BFV covariant version of (H.4) can only be shown to be gauge-fixing independent if the gauge-parameters are field independent, a result expected from [25].
QM collapse and superpositions—and none of the drawbacks—no detectable ‘ether’ for classical (relativistic) mechanics. The fact that such theories need not imply a preferred choice of simultaneity goes contrary to common wisdom. The misguided intuition comes from imagining such ‘space-like’ surfaces as being embedded in space-time, in which case indeed, they do define a notion of simultaneity. However, without the prior existence of space-time, one first requires a recipe to recover duration, using the dynamics of clocks and rods and fields throughout evolution in field-space.62

For instance, for a theory whose field space is that of spatial metrics, Riem, for a given action $S(\gamma)$, a solution to the equations of motion is a curve of 3-metrics $\gamma_{ab}(t)$, such that $S'_{\gamma} = 0$. However, to find out what a space-time corresponding to this curve is, one needs to ‘fill in’ the curve with a notion of duration (or a lapse), e.g.:

$$\text{d}s^2 = -N_{\text{eff}}[\gamma; x]^2 \text{d}t \otimes \text{d}t + \gamma_{ab}(x, t) \text{d}x^a \otimes \text{d}x^b + \varpi_a[\gamma; x](\text{d}t \otimes \text{d}x^a + \text{d}x^a \otimes \text{d}t)$$

(73)

where I used the mixed functional notation of DeWitt (square brackets signify functional dependence, and round brackets ultralocal dependence). In (73) both the ‘shift’ $\varpi$ and the lapse $N_{\text{eff}}$ are ‘effective’. The important point is that they only depend on $t$ through the curve $\gamma_{ab}(t)$ and are not inherited from space-time; quite the opposite: they build space-time.63 One example where this is explicitly realized, obtaining exactly the same empirical content of GR, is the BSW formulation of the Einstein–Hilbert action, (68), [56]64.

6.2. Summary

To summarize, in this paper I have described a theory existing in configuration space, $Q$, with no singled out time variable. In the presence of physical subsystems that act like clocks, the formalism is able to recover the transition amplitude of standard quantum mechanics (see section 3). However, this clock subsystem does not need to be defined over the entire configuration space for the theory to be consistent.

Given an action on curves in $Q$, and a preferred boundary condition which is part of the theory, I described how to construct a volume-form in $Q$ from the path integral. The volume-form must be constructed from the path integral. Under a simple further assumption required so that my notion of records (and a local decomposition property [20]) may later emerge, such a volume-form is uniquely given by the Born density.

What I termed ‘records’ is basically a sort of ‘information’ contained in the volume-form. Such records encode what we usually ascribe to the passage of time (in terms of conditional probabilities), and statements about conservation of probability using records were also shown to hold. It would be interesting to explicitly connect this notion of records with the one emergent in [66], and there claimed to resolve the issues of an arrow of time.

The preferred boundary condition here is an integral part of the theory, but only in the case of a topology $M \simeq S^3$ and with the group being $\text{Diff}(M) \ltimes C$ have I shown it to be unique. In this particular case, the boundary is the unique point which constitutes the least dimensional corner to the physical state space $Q/\text{Diff}(M) \ltimes C$, and no other boundary conditions need to be given.

This is the ultimate asymmetry of configuration space, and it is what makes most of our constructions possible. For example: the notion of conservation of probabilities, discussed in section 4.2; a notion of global time, and of a standard Schrödinger equation for source fields

62 See [14] for a related view.
63 This is what Wheeler used to call the struts and rods holding up space-time from its geometrodynamical skeleton [65].
64 Although, being essentially GR, the Hamiltonian version of the theory reverts back to the standard ADM.
propagating with this time; are made possible because there is such an asymmetry in configuration space. The unique $\phi^*$ acts in a similar fashion to the point of emission of alpha-particles in a Mott bubble chamber experiment. Here, as there, it is such an asymmetry that allows for the formation of ‘directional records of time’s passage’ [45].

Lastly, I proposed a simple gravitational toy model, similar to strong gravity, compatible with the conditions imposed by this setting, and showcasing the viability of its constructions [65]. Making use of its interpretation in terms of the geometry of superspace, there is a shot-cut way to calculate (and regularize) its emergent Van–Vleck determinant. I went on to compute the relative semi-classical quantum gravity probabilities (or volumes) for regions in configuration space, finally obtaining the approximate (67). This equation allows us to ask precise questions such as: what is the probability of finding a given range of perturbations of gravitational modes of the original round 3-sphere, with respect to some other range? It turns out that for this simple model any mode is just as likely as another. Note moreover that this is a semi-classical effect—depending on the Van–Vleck determinant—not a classical one.

6.2.1. Resolutions. Thus, summarizing the summary, we have the following resolutions of the questions posed at the start:

- In this framework there is no issue arising with the ‘superposition of causal structures’; causality can emerge semi-classically, and we can observe interference effects between alternative emergent space-times.
- Non-locality is a non-issue. Configuration space is the true arena of physics, and each point in it contains the entire physical space. Locality can and does emerge dynamically however [20].
- Neither is there a ‘problem of time’ (see [67]). Evolution is abstracted from records—made possible by the asymmetry of configuration space—and is not mixed with a local gauge symmetry.
- One can define positive probabilities and related probability currents in the semi-classical approximation. Unlike the corresponding approximation for WdW (which can only be done in minisuperspace), here there are no issues with positivity because time is defined around records, and at most a single solution of the Hamilton–Jacobi functional can exist. Source fields can be shown to propagate following the Schrödinger equation with respect to such an emergent time. The asymmetry of configuration space is what dictates and allows for these constructions and their consequent properties.
- There is no conflict of the sort ‘unitary versus reduction’ processes for the quantum wave-function. Relatedly, there is no ‘definite outcomes’ problem, since the set of all configurations exist, with a volume-form on top of it.

There are however, other challenges, concerned with regaining classical space-times.

6.3. Some future projects

There are standard formulations of gravity which would be amenable to this treatment. Deparametrized shape dynamics [55] is one such model. This approach would give a different take on old problems, both in cosmology and elsewhere. The use of a prescribed boundary condition will restrict quantum cosmological scenarios, in a very similar way that the Hartle–Hawking condition does. In the homogeneous, cosmological scenario, our prescription will

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65 There is in principle no requirement for the models to be geometrical; only that they satisfy the axioms 1–5 of section 2.
give a different boundary condition than the Hartle–Hawking one, but which can be uniquely
specified for certain types of field spaces and symmetries acting on them.

- Bianchi IX shape dynamics. Perhaps the ideal setting to test these ideas in the cosmological
scenario would be the Bianchi IX shape space, characterized in [68] (see also [5]).
There, we have an action, we have the explicit classical trajectories once again, and we
have a uniquely specified initial point—the round sphere. In such a reduced configuration
space, we get basically a Wheeler deWitt type equation, that should be solved with our
boundary conditions. This would give a first prescription of a shape quantum cosmology
model.

- Reconstructing space-time. The most straightforward project directly extending this
work however, is to flesh out the outline of the last paragraph of section 5: add a simple
matter scalar field to the action (57), perform the analogous calculations, and use the
prescription of [55] to find an effective space-time according to (73), below. Then use the
constructions of [54] to evaluate the standing of the equivalence principle. In this way we
can explicitly start to probe qualitative implications of quantum gravity for space-times,
which this approach suggests.

- Extending the analysis to other toy models. Lastly, as I mentioned, one of the disap-
pointing features of the present toy model is that it does not contain derivatives of the
metric, and thus it does not couple points. Thus, perhaps the most straightforward project,
would be to apply the formalism here—which exploits parallels between geometrical
quantities and the Van–Vleck determinant—to more complicated geometries in Q, explicit-
ly described in [49]. A particularly interesting geodesic action to try out is

\[
S[g] = \int dr (\dot{g}, \dot{\bar{g}})^{1/2}_{g(t)} = \int dr \left( \int d^3 x' \sqrt{g}(R^2 + \epsilon)^{1/2} \int d^3 x \sqrt{\bar{g}} g^{ac} g^{bd} \bar{g}_{cd} \right)^{1/2}
\]  

(74)

where \( g_{ab} \) is now unimodular, i.e. \( \sqrt{g(t)} = \sqrt{|g|} \), and with transverse–traceless veloc-
ities. For arbitrarily small \( \epsilon > 0 \), this is the Jacobi form of the Einstein–Hilbert action
with \( N = 1 \) (and transverse–traceless velocities), and thus its extremal trajectories should
coincide for positive Ricci scalar (but not its path integral). That is, the extremal paths of
(74) coincide with those of the spatially unimodular Einstein–Hilbert action with positive
spatial scalar curvature and small cosmological constant. In fact, much of the groundwork
for this analysis has already been set. In [49], general geodesic theories, with arbitrary
conformal factors depending on the Riemann tensor have been studied, and many similar
results to the ones presented here obtained.

Acknowledgments

I would like to thank Lee Smolin, Aldo Riello, Wolfgang Wieland, J Halliwell, Flavio
Mercati and Sean Gryb for comments. This research was supported by Perimeter Institute
for Theoretical Physics. Research at Perimeter Institute is supported by the Government
of Canada through Industry Canada and by the Province of Ontario through the Ministry of
Research and Innovation.
Appendix A. Connections on principal fiber bundles

The advantage of working directly with principal fiber bundles (PFB’s) is that structures are simpler. For example, it is easier to work out particular features of associated vector bundles, or features of gauge-fixed structures directly from the general formalism of PFB’s than vice-versa. The standard example of a PFB is the bundle of linear frames over a spacetime manifold $M$, with structure group $GL(n)$.

A principal fiber bundle $P$, is a smooth fiber bundle, for which a Lie group $G$ has an action from the right $G \times P \rightarrow P^g$, which we denote by $R_g P := p \cdot g$, for $g \in G$, $p \in P$. The action is assumed to be free, so that $p \cdot g = p$ iff $g = id_G$, the identity of the group. The quotient of $P$ by the equivalence relation given by the group, that is $p \sim p'$ iff $p' = p \cdot g$ for some $g \in G$, is usually identified with the spacetime manifold, i.e. $M = P/G$. We denote the projection map by $pr : P \rightarrow M$, where $R_g$ denotes the right action of $g$ on $P$ (which should be distinguished from the action of $g$ on the group $G$ itself). The set $O_p = \{ p \cdot g \mid g \in G \}$, is called the orbit through $p$, or the fiber of $[p] = pr(p)$. The action of $pr$ projects $O_p$ to $[p] \in M$.

The section then gives rise to a trivialization $G \times U \simeq pr^{-1}(U)$, given by the diffeomorphism $F_\sigma : U \times G \rightarrow pr^{-1}(U), ([p], g) \mapsto \sigma([p]) \cdot g$. Generically, there are no such $U = M$ and the bundle is said to be non-trivial. In the functional, i.e. field space, case, the base manifold is a modular space and the obstruction to triviality is known as the Gribov ambiguity.

A.1. PFB connections

Before going on to define a connection, we require the concept of a vertical vector in $P$. Let $\exp : \text{Lie}(G) = \mathfrak{g} \rightarrow G$ be the group exponential map. Then by dragging the point with the group action we can define a vertical vector at $p$, related to $X \in \mathfrak{g}$, as

$$X_p^\# := \frac{d}{dt}_{|t=0} \left( p \cdot \exp(tX) \right) \in T_p P.$$  \hfill (A.1)

The vector field $X^\# \in \Gamma(TP)$ is called the fundamental vector field associated to $X$. The vertical space $V_p$ is defined to be the span of the fundamental vectors at $p$.

From (A.1) one has that fundamental vector fields are ad–equivariant, in the following sense:

$$(R_g)_* X_p^\# = (\text{Ad}_{g^{-1}}X)_p^\#$$ \hfill (A.2)

where $R_g : TP \rightarrow TP$ denotes the pushforward tangential map associated to $R_g$ and $\text{Ad}_g : \mathfrak{g} \rightarrow \mathfrak{g}, X \mapsto gXg^{-1}$ is the adjoint action of the group on the algebra. Now, using the fact that the Lie derivative of a vector field $Y^\#$ along $X^\#$ is defined as the infinitesimal push-forward by the inverse of $\exp(tX)$ evaluated at $p$, by setting $g \mapsto \exp(-tX)$ and $p \mapsto p \cdot g^{-1}$ in (A.2) and deriving, we obtain

$$L_{X^\#} Y^\# = [X^\#, Y^\#]_{TP} = \frac{d}{dt}_{|t=0} \left( R_{\exp(-tX)} Y^\# \right) = \frac{d}{dt}_{|t=0} \left( \text{Ad}_{\exp(-tX)} Y \right)^\# = (\text{ad}_X Y)^\# = ([X, Y]^\#)^\#.$$  \hfill (A.3)

This shows that the vertical bundle, $V \subset TP$, is an integrable tangential distribution.

The definition of a connection amounts to the determination of an equivariant algebraic complement to $V$ in $TP$, i.e. an $H \subset TP$ such that

$$H_p \oplus V_p = T_p P$$ \hfill (A.4a)

66 This means that in the coordinate construction of $P$ the transition functions act from the left.
\[ R_{g^*} H_p = H_{p^*g}, \quad \forall p \quad (A.4b) \]

One can equally well define \( H \) to be the kernel of a \( g \)-valued one form \( \omega \) on \( P \), with the following properties:

\[ \omega(X^\#) = X \quad (A.5a) \]
\[ R_{g^*} \omega = \text{Ad}_{g^{-1}} \omega \quad (A.5b) \]

where \( R_{g^*} \omega \equiv \omega \circ R_{g^*} \). In other words, equation (A.5b) intertwines the action of the group \( G \) on \( P \) (lhs of the equation) and its action on \( g \) (rhs). Notice that these equations hold only for a global action of \( G \) on \( P \). For any \( v \in TP \), its vertical projection is given by \( \hat{V}(v) = \omega(v)^\# \).

The horizontal derivative is the generalization of a covariant derivative (in the associated vector bundle) to the PFB context. The role of a covariant derivative is to exactly cancel out the non-equivariant terms obtained for derivatives (either in field space or not). In the presence of a principal fiber bundle structure in field space, what plays the role of the Lie group before is now a diffeomorphism \( f : M \to M, \ x \mapsto f(x) \). It acts on the metric \( g \in Q \) via pullback,

\[ (Af^* g)_x := (f^* g)_x \quad (A.6) \]

where I have taken pains to distinguish the action (A) from the particular group element (f).

We can transplant the previous structures to the infinite-dimensional case:

\[ \frac{d}{dt} \overset{\sim}{\to} \frac{d}{dt} = \hat{H} \frac{d}{dt} = \frac{d}{dt} - \hat{V} \frac{d}{dt} \quad (A.7) \]

where \( \hat{V} \) is the inclusion operator in the field-theory context, \( \hat{V} \) is the vertical projection in \( Q \) and \( \hat{V} \) is a Lie-algebra-valued one-form, i.e. \( \hat{V} \in \Lambda^1(Q, \Gamma(TM)) \), which obeys the analogue of (A.5a), that is

\[ \hat{V}(X^\#) = X \quad (A.8a) \]
\[ A_{\hat{V}^*} \hat{V} = (\hat{V} \circ \hat{V} \circ \hat{V}^{-1}) \quad (A.8b) \]

As before, the horizontal projection \( \hat{H} \), is defined via the kernel of \( \hat{V} \). With the substitution (A.7) in the action, one automatically obtains general gauge-invariance for time-dependent gauge transformations. See [25] for a complete account on the relevance of this connection in the field space context, and how a choice of connection can embody a choice of abstract observer. Both for the functional case and in the finite-dimensional one, we have the infinitesimal form of the equations, generalized for possibly field-dependent gauge transformations:

\[ \mathcal{J}_\xi \hat{V} = \xi \quad (A.9) \]
\[ \mathcal{L}_\xi \hat{V} = [\hat{V}, \xi] + \delta \xi \quad (A.10) \]

where the commutator is the Lie algebra one. From these equations it is easy to see that a connection fulfills the purpose of canceling out terms that do not transform homogeneously under the group.
Appendix B. The timeless transition amplitude in quantum mechanics

B.1. Quantum mechanical timeless transition amplitude in configuration space

We start with a finite-dimensional system, whose configuration space, $Q$, is coordinitized by $q^a$, for $a = 1, \cdots, n$. Let us start by making it clear that no coordinate, or function of coordinates, singles itself out as a reference parameter of curves in $Q$. The systems we are considering are not necessarily ‘deparametrizable’—they do not necessarily possess a suitable notion of time variable.

Now let $\Omega = T^* Q$ be the cotangent bundle to configuration space, with coordinates $q^a$ and their momenta $p_a$. For a reparametrization invariant system the classical dynamics is fully determined once one fixes the Hamiltonian constraint surface in $\Omega$, given by $H = 0$. A curve $\gamma \in Q$ is a classical history connecting $q^a_1$ and $q^a_2$ if there exists an unparametrized curve $\bar{\gamma}$ in $T^* Q$ such that the following action is extremized:

$$S[\bar{\gamma}] = \int_{\bar{\gamma}} p_a dq^a$$

for curves lying on the constraint surface $H(q^a, p_a) = 0$, and such that the projection of $\bar{\gamma} \in T^* Q$ to $Q$ is $\gamma$, connecting $q^a_1$ and $q^a_2$. By parametrizing the curve with an innocent parameter $t$, we can get to the familiar form:

$$S[\gamma] = \int dt \left( p_a \dot{q}^a - N(t) H(q^a, p_a) \right)$$

where the $N$ is a Lagrange multiplier (and we are considering only one constraint).

One can now define the fundamental transition amplitudes between configuration eigenstates.

$$W(q, q') := \langle q | \hat{P} | q' \rangle$$

where the projector can be defined by:

$$\hat{P} := \lim_{t \to \infty} \frac{1}{2t} \int_{-t}^t e^{-iHt}$$

with the standard particle Hilbert space inner product, $\langle q | q' \rangle = \delta(q, q')$.

In [38], following previous studies of relational quantum mechanics (see [11] for a review), Chiou studied the relational timeless transition amplitude between configurations, $W(q_1, q_2)$ in the path integral formalism$^{67}$. The standard proof of equivalence between non-relativistic Schrödinger quantum mechanics and the path integral formulation relies on refining time slicings—partitioning paths into arbitrarily small segments. Without absolute time, a parametrized curve $\bar{\gamma} : [0, 1] \to \Omega$ need not be injective on its image (it may go back and forth). This issue is related to the one of integrating over only positive lapses, or over both positive and negative. One yields a propagator, and not a projection onto the constraint surface.

Chiou chooses the latter, and uses a Riemann–Stieltjes integral to make sense of the limiting procedure to infinite sub-divisions of the parametrization. Furthermore, one must then sum over all parametrizations, at which point one obtains $\delta[H]$, and the entire transition amplitude (B.3).

Crucially, Chiou uses a sub-division of the parametrization (available only when there is a single time direction). The mesh of the parameter sequence is then defined by the largest

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$^{67}$ Chiou calls it a ‘curve’ integral, as opposed to a path integral, because the integral only requires unparametrized paths in phase space. I will assume that there exists a metric in configuration space, for which I can then parametrize paths by arc-length.
interval encompassed by the parameter sequence. The parameter sequence is fine enough if its mesh is smaller than a given small number $\epsilon$.68

The transition amplitude can then be written as:

$$W(q_1, q_2) = A \int Dq^a \int Dp_a \delta[H] \exp \left[ \frac{i}{\hbar} \int p_a dq^a \right]$$  \hspace{1cm} (B.5)

where the path integral sums over those unparametrized curves $\gamma$ in phase space, whose projection to configuration space begins at $q_1$ and ends at $q_2$.69

I will not require this condition, but if configuration space can be decomposed wrt a ‘time’ variable, $q = (t, \vec{q})$, such that the total Hamiltonian can be written as $\hat{H} = \hat{p}_t + \hat{H}_o(\vec{q}, p_\gamma)$, with $H_o$ not explicitly dependent on $t$, then the timeless path integral amplitude kernel is equal to the standard non-relativistic one, up to an overall constant70:

$$W(q_1, q_2) = G((t_1, \vec{q}_1), (t_2, \vec{q}_2)) = \langle \vec{q}_1 | e^{-\frac{i}{\hbar} H_{o}(t_2-t_1)} | \vec{q}_2 \rangle.$$  \hspace{1cm} (B.6)

Note however, an important difference: in the lhs of (B.6), time is just part of the configuration space measure from the Liouville one, as in (18).

**Appendix C. Timeless semi-classical composition law**

Here I prove the theorem 1 in more generality. Assuming the action is additive, for a given path

$$S_{\gamma_2}(\phi^+, \phi) = S_{\gamma_2}(\phi^+, \phi_m) + S_{\gamma_1}(\phi_m, \phi).$$  \hspace{1cm} (C.1)

We need to show that

$$\det \left( \frac{\delta^2 S(\phi^+, \phi_m)}{\delta \phi^+ \delta \phi_m} \right) \det \left( \frac{\delta^2 S(\phi_m, \phi)}{\delta \phi_m \delta \phi} \right) \det \left( \frac{\delta^2 S(\phi^+, \phi_m)}{\delta \phi^+ \delta \phi_m} \right)^{-1} = \det \left( -\frac{\delta^2 S(\phi^+, \phi)}{\delta \phi^+ \delta \phi} \right)$$  \hspace{1cm} (C.2)

which is the required composition law of determinants to obtain (31) (see Kleinert’s textbook [42] for a proof). It is easiest to see this relation by transforming to the on-shell momenta and using the interpretation of focusing of the Van–Vleck determinant. Then,

$$\det \left( \frac{\delta^2 S(\phi^+, \phi_m)}{\delta \phi^+ \delta \phi_m} \right) + \frac{\delta^2 S(\phi_m, \phi)}{\delta \phi^+ \delta \phi_m} = \det \left( -\frac{\delta \pi_f(\phi^+, \phi_m)}{\delta \phi_m} \right) + \frac{\delta \pi_f(\phi_m, \phi)}{\delta \phi_m}$$

68 Still, here the integration still needs to be regularized: $W(q_1', q_2') \sim \lim_{\lambda \to 0} \int \frac{1}{\lambda^2} \delta^4(q_1'[e^{-\lambda \hat{H}}] q_2')$ with a cut-off $\lambda$, satisfying some hierarchical condition with respect to the mesh.

69 In the presence of gauge symmetries, if it is the case that these symmetries form a closed Lie algebra, one can in principle use the group averaging procedure, provided one uses a translation invariant measure of integration.

70 In [69], the conditions under which Briggs and Rost derive the time dependent Schrödinger equation from the time-independent one, corresponds to the system being deparametrizable. i.e. one can isolate degrees of freedom (the environment, in Briggs and Rosen) which are heavy enough to not suffer back reaction. The proposal here is not dependent on this condition.
becomes the relative amount of focusing that occurs at the intermediary point, which we are integrating out.

Using (C.1), from the stationarity condition at an intermediary field configuration, we have (dropping the subscripts for clarity):

$$\frac{\delta S(\phi^*, \phi_m)}{\delta \phi_m} + \frac{\delta S(\phi_m, \phi)}{\delta \phi_m} = -\pi_f(\phi^*, \phi_m) + -\pi_i(\phi_m, \phi) = 0$$

(C.3)

this requires the momenta to be continuous at $\phi_m$, setting $\phi_m$ to be along the extremal path $\phi_m = \phi_{\gamma}^*$. Thus, given a classical path $\gamma$, $\phi_m$ also depends on $\phi$ and $\phi^*$. Thus, deriving (C.3) by $\phi$:

$$-\frac{\delta \pi_f(\phi^*, \phi_m)}{\delta \phi_m} \cdot \frac{\delta \phi_m}{\delta \phi} + \frac{\delta \pi_i(\phi_m, \phi)}{\delta \phi_m} \cdot \frac{\delta \phi_m}{\delta \phi} + \frac{\delta \pi_i(\phi_m, \phi)}{\delta \phi_m} = 0$$

$$\Rightarrow -\frac{\delta \pi_f(\phi^*, \phi_m)}{\delta \phi_m} + \frac{\delta \pi_i(\phi_m, \phi)}{\delta \phi_m} = -\frac{\delta \pi_i(\phi_m, \phi)}{\delta \phi} \left( \frac{\delta \phi_m}{\delta \phi} \right)^{-1}.$$  

(C.4)

Writing equation (C.2) in terms of momenta we have:

$$\text{det} \left( -\frac{\delta \pi_i(\phi^*, \phi_m)}{\delta \phi_m} \right) \cdot \text{det} \left( -\frac{\delta \pi_i(\phi_m, \phi)}{\delta \phi_m} + \frac{\delta \pi_i(\phi_m, \phi)}{\delta \phi_m} \right) = \text{det} \left( -\frac{\delta \pi_i(\phi^*, \phi)}{\delta \phi} \right).$$  

(C.5)

Using the product rule for determinants we have that:

$$\text{det} \left( \frac{\delta \pi_i(\phi^*, \phi_m)}{\delta \phi_m} \cdot \frac{\delta \pi_i(\phi^*, \phi)}{\delta \phi} \right) = \text{det} \frac{\delta \phi_m}{\delta \phi}.$$  

Finally, substituting this and (C.4) into (C.5):

$$\text{det} \left( -\frac{\delta \pi_i(\phi^*, \phi_m)}{\delta \phi_m} \right) \cdot \text{det} \left( -\frac{\delta \pi_i(\phi_m, \phi)}{\delta \phi_m} \right) \cdot \text{det} \left( -\frac{\delta \pi_i(\phi_m, \phi)}{\delta \phi} \left( \frac{\delta \phi_m}{\delta \phi} \right)^{-1} \right)^{-1}$$

$$= \text{det} \left( -\frac{\delta \pi_i(\phi^*, \phi_m)}{\delta \phi_m} \right) \cdot \text{det} \left( -\frac{\delta \pi_i(\phi_m, \phi)}{\delta \phi_m} \right) \cdot \text{det} \left( -\frac{\delta \pi_i(\phi_m, \phi)}{\delta \phi} \left( \frac{\delta \pi_i(\phi^*, \phi_m)}{\delta \phi_m} \right)^{-1} \frac{\delta \pi_i(\phi^*, \phi)}{\delta \phi} \right)^{-1}$$

$$= \text{det} \left( -\frac{\delta \pi_i(\phi^*, \phi_m)}{\delta \phi_m} \right)$$  

(C.6)

where we note that an extra cancellation occurs since $\frac{\delta \pi_i(\phi_m, \phi)}{\delta \phi} \cdot \frac{\delta \phi_m}{\delta \phi} = \frac{\delta \pi_i(\phi_m, \phi)}{\delta \phi_m}$ because they are equal as second derivatives of the action.

Now, the continuity equation (C.3), turns the double sum of (33) into a single sum over the continuous extremal paths. Each overall extremal path decomposes into a unique composition of extremal paths, $\gamma_{\alpha'} = \gamma_{\alpha'_1} \circ \gamma_{\alpha'_2}$ with final (resp. initial) endpoints on $\phi_f$. Putting it all together we obtain:

$$\sum_{\alpha_1} \Delta_{\alpha_1}^{1/2} \exp \left( iS_{\alpha_1}(\phi^*, \phi_f)/\hbar \right) \cdot \sum_{\alpha_2} \Delta_{\alpha_2}^{1/2} \exp \left( iS_{\alpha_2}(\phi, \phi_f)/\hbar \right) = \sum_{\alpha} \Delta_{\alpha}^{1/2} \exp \left( iS_{\alpha}(\phi^*, \phi)/\hbar \right)$$

up to orders $O(\hbar^2)$.

(C.7)
Appendix D. Probability currents for weakly coupled systems

Following section 4.2, here I will briefly go over the derivation of the decomposition of currents and probabilities, under an inner product which is orthogonal in terms of gravity and its source fields.

\[ \Psi(g, \varphi) = \exp \left( i m^2 \frac{\delta^2}{\delta \varphi^n(x) \delta \varphi^b(x)} \right) \]  

(D.1)

I will assume the conservation law for \( A \), (48), and a specific form for the matter (or source) Hamiltonian given in (45),

\[ \hat{H}_{\text{mat}} = -\frac{1}{2m} \partial_x^2 + \frac{1}{2} m \nu(\varphi) \]  

(D.2)

where \( \nu(\varphi) \) is the potential for the sources, the quadratic momentum term in the Hamiltonian turns into

\[ \partial_x^2 \varphi = \int d^3x \left( g_{ab}(x) \frac{\delta^2}{\delta \varphi^a(x) \delta \varphi^b(x)} \right) \]  

(D.3)

following (46) and, again, I have taken a simple factor ordering\(^{71}\).

For the (block) diagonal metric, we decompose the current in components as:

\[ J_{ab}^{\text{grav}}(x) = \frac{1}{2i} m p \left( \Psi \frac{\delta \Psi}{\delta g_{ab}(x)} - \Psi \frac{\delta \Psi}{\delta g_{ab}(x)} \right) ; \quad J^\text{source}_a(x) = \frac{1}{2i} m \left( \Psi \partial_a \Psi - \Psi \partial_a \Psi \right) \]  

(D.4)

for \( \Psi \) given in (D.1). For a particle, we have

\[ J^\text{source}_a(x) \rightarrow \delta(x - x^\mu) \left( \nabla_a \Psi - \Psi \partial_a \nabla \right) \]

\[ \partial_x^2 \varphi \text{ turns into the standard spatial Laplacian. Let us look at this case for simplicity. In this case, the two currents of (D.5), reduce, with the approximation (D.1), to lowest order:} \]

\[ J_{ab}^{\text{grav}}(x) \rightarrow |A[g]|^2 \frac{\delta S[g]}{\delta g_{ab}(x)} \Psi^2; \quad J^\text{source}_a(x) \rightarrow \frac{1}{2i} m \left( \Psi \partial_a \Psi - \Psi \partial_a \Psi \right) \]  

(D.5)

and, using (48), the full continuity equation reduces to:

\[ |A[g]|^2 \left( \frac{\partial}{\partial T} |\Psi|^2 + \partial_x p^a \right) = 0 \]  

(D.6)

where we have used (51) as the arc-length parametrization (surfaces of equal on-shell action).

Now if we use the definition of the Klein–Gordon inner product, the normal to equal time surfaces is given by (51), i.e.

\[ n_{ab}(x) = \frac{1}{V[g]} \frac{\delta S}{\delta g_{ab}(x)} \]

and from the block diagonal form of the supermetric for gravity and sources, we get:

\[ \langle \Psi, \Psi \rangle_{\text{KG}} = \int_{S_{(r)}} Dg \int d^3x n_{ab}(x) J_{ab}^{\text{grav}}(x) = \int_{S_{(r)}} Dg |A[g]|^2 \int d^3x |\Psi|^2 \]  

(D.7)

where \( S_{(r)} \) is the screen, relating the Klein–Gordon inner product and the Schrödinger one.

\(^{71}\) This is the ordering ignoring Christoffel symbols, i.e. ignoring covariance issues, both in configuration space and the spatial manifold. But everything goes through with \( \frac{1}{\sqrt{|\det h|^{1/2} h^{ab} \partial_a \varphi}} \), which makes everything covariant. The same is true for the gravitational modes. See footnote 40.
Appendix E. Jacobian

E.1. The Jacobian for the TT decomposition

The spin decomposition wrt the background \( g_{ab} \):

\[
h_{ab} = h_{TT}^{ab} + \nabla_a X_b + \nabla_b X_a + 2 \nabla_a \nabla_b \theta - \frac{2}{3} g_{ab} \Delta \theta + \frac{1}{3} h g_{ab},
\]

(E.1)

where

\[
g^{ab} h_{TT}^{ab} = 0, \quad \nabla^b h_{TT}^{ab} = 0, \quad \nabla^a X_a = 0,
\]

(E.2)

we pick up a Jacobian

\[
1 = \int \mathcal{D}[h_{ab}] \exp \left[ - \frac{1}{2} \int d^3x \sqrt{g} h_{ab} h_{ab} \right] = J_{TT} \int \mathcal{D}[h_{ab}, X_a, \theta, h] \exp \left[ - \frac{1}{2} \int d^3x \sqrt{g} \left( h_{TT}^{ab} h_{TT}^{ab} + \frac{1}{3} h^2 - 2 X_a (R^{ab} + g^{ab} \Delta) X_b \right.ight.
\]

\[
+ \sigma \left( \frac{8}{3} \Delta + 4 R^{ab} \nabla^a + 4 R^{ab} \nabla^b \right) \sigma
\]

\[
+ 4 \sigma \left( R^{ab} \nabla_a - 4 X_a R^{ab} \nabla b \right)
\]

(E.3)

defining

\[
1 = \int \mathcal{D}[h] \exp \left[ - \frac{1}{2} \int d^3x \sqrt{g} h^2 \right] = \int \mathcal{D}[\theta] \exp \left[ - \frac{1}{2} \int d^3x \sqrt{g} \theta^2 \right]
\]

(E.4)

\[
1 = \int \mathcal{D}[X_a] \exp \left[ - \frac{1}{2} \int d^3x \sqrt{g} g^{ab} X_a X_b \right] = \int \mathcal{D}[h_{ab}] \exp \left[ - \frac{1}{2} \int d^3x \sqrt{g} g^{cd} h_{TT}^{ab} h_{TT}^{cd} \right]
\]

(E.5)

we get

\[
J_{TT} = \left| \det \left( \begin{array}{cc} -2 (R^{ab} + g^{ab} \Delta) & -4 R^{ab} \nabla_b \\ 4 (R^{ab} \nabla_a - 4 X_a R^{ab} \nabla b) & \frac{8}{3} \Delta + 4 R^{ab} \nabla^a + 4 R^{ab} \nabla^b \nabla^a \end{array} \right) \right|^{-1}
\]

(E.6)

E.2. Arc-length parametrization

If we gauge-fix the time-reparametrizations, we choose to be an arc-length one:

\[
G_t := \sqrt{\int_M \sqrt{g} \delta c^d(t) \delta c^d(t)} - 1 = 0.
\]

(E.7)

The infinitesimal version, i.e. for \( \delta t = \epsilon t + \mathcal{O}(\epsilon^2) \),

\[
G_{t} = |\epsilon| \sqrt{\int_M \sqrt{2} \delta c^d(t) \delta c^d(t)} - 1
\]

which gives the variation:
This gauge choice is responsible for the absence of the global square root. In other words, with this gauge choice one can express the action as an energy functional (in the Riemannian geometry sense) as opposed to a length functional:

$$\int ds \| \dot{\gamma} \|^2$$ as opposed to $$\int ds \| \dot{\gamma} \|.$$

The main feature however of the energy functional is that extremals wrt it automatically implement arc-length parametrization as one of the extremum conditions. In other words, since the length and energy functional coincide for arc-length parametrization, and this is implied by the extrema of the energy functional, in the semi-classical approximation one can directly use the energy functional as opposed to the length. In a way, this gauge fixing ‘trades’ global time reparametrization invariance for locality of the action functional.

### E.3. Eigenfunction basis

In terms of eigenfunctions of the Laplacian, using the notation of [52], we write

$$\nabla^2 G^{(n)}_{ij} = \lambda_n G^{(n)}_{ij}.$$

For the 3-sphere, one finds that \( \lambda_n = -(n^2 - 3) \), for \( n \geq 3 \). The solution to these equations, in standard spherical coordinates is given by a combination of \( n^2 - 4 \) terms:

$$G^{(n)}_{ij}(\chi, \theta, \phi) = \sum_{l=2}^{n-1} \sum_{m=-l}^{l} C^{n}_{lm}(G^{(n)}_{ij})_{lm}$$

(E.9)

where \( C^{n}_{lm} \) are the Fourier coefficients, and for each \( ij \), \( (G^{(n)}_{ij})_{lm} \) is a TT tensor harmonic, explicitly computed in [52] (equations (17)–(18c)). Since \( S^3 \) is compact without boundary, and \( \nabla^2 \) is self-adjoint with respect to the standard supermetric

$$\langle u | \nabla^2 v \rangle_g = \int d^3 x \ u_{ab} \ g^{ac} \ g^{bd} \ \nabla^2 v_{cd} \sqrt{|g|} = \langle \nabla^2 u | v \rangle_g$$

(E.10)

by the spectral theorem for bounded operators, eigenfunctions with distinct eigenvalues are orthogonal.

Now, for simplification, instead of (E.9), for a normalized basis (under the standard integral and sphere metric),

$$\int d^3 x \sqrt{g_0} \kappa_0^{\mu} g_0^\mu \kappa_0^{\nu} = \delta^{\nu \mu}$$

we simply write,

$$h_{ij}(x) = \sum_{n} \lambda_n^{ij} \kappa_n^{ij}(x) \prod dh_{ij}(x) \to \prod_{n=1}^{\infty} d\lambda_n^{ij}.$$  

(E.11)

If the basis is not normalized, a Jacobian appears for the transformation.
Appendix F. The full Van–Vleck determinant

I start by rewriting, for the reader’s convenience, (57):

\[ S[\gamma] = \int dt \left( \int d^3x \, \dot{\gamma}_{ab} g^{ac} g^{bd} \dot{\gamma}_{cd} \sqrt{g} \right)^{1/2} \]

and (59)

\[ \gamma_{ab}(t) = g_{ac}^0 (e^{(A(t) \text{Id} + B(t) \mathbf{h}_T) / b})_b \]

where \( \mathbf{h} \) is the matrix given by \( h_{ab} = h_{bc} g_{0ac} \), such that \( \nabla_a h_{ab} = 0 = h_{ab} g_{0b} \) (the traceless condition is taken for simplicity), and

\[ A(t) = \frac{2}{3} \ln(1 + \frac{3}{8} b t^2) \text{ and } B(t) = \frac{4}{\sqrt{3} b} \tan^{-1}\left( \frac{\sqrt{3} b t}{4} \right) \]

where \( b = h_{cd}^T h_{dc}^d \). The strategy will be to compute the Van–Vleck matrix with the following chain rule:

\[ \frac{\delta^2 S}{\delta g^{0c}(x) \delta g_{ab}(1)(y)} = \frac{\delta}{\delta g^{0c}(x)} \left( \int d^3x' \, \frac{\delta S}{\delta h_{ij}(x')} \cdot \frac{\delta \gamma_{ij}(1)(x')}{\delta h_{ab}(y)} \right) . \] (F.1)

I start by noticing that the usual trick to deal with the square root in (57) is to use instead the energy functional, i.e. the version without the square root. It is easy to show that the equations of motion then give two types of equations: one says that the paths should be arc-length parametrized, and the other that the curve will be geodesic. Moreover, I also note that the conservation arguments section 4.2 go through, slightly modified—the global Hamiltonian is conserved in time, thus giving a non-local conserved energy for each curve\(^72\).

To simplify the notation, I define the matrix \( \mathbf{M} := (A(t) \text{Id} + B(t) \mathbf{h}) \). Its derivative is:

\[ \dot{\mathbf{M}}_b^a = \frac{24 b t}{48 + 18 b t^2} \delta_b^a + \frac{16}{16 + 3 b t} h_b^d \] (F.2)

whose dimensions in arc-length units can be checked to be of 1/sec. From the geodesic equation (59), the inverse metric along it is

\[ g^{ab} = g^{0d}((e^\mathbf{M})^{-1})_d^b \] (F.3)

and the metric velocity is given by

\[ \dot{\gamma}_{ab} = g^{0c} \dot{\mathbf{M}}_d^c (e^\mathbf{M})_d^b \] (F.4)

thus:

\[ \dot{\gamma}_{ab} g^{ca} g^{bd} \dot{\gamma}_{cd} = g^{0b} \dot{\mathbf{M}}_d^f (e^\mathbf{M})_a^m g^{0f} ((e^\mathbf{M})^{-1})_d^j g^{0j} ((e^\mathbf{M})^{-1})_d^m \dot{\mathbf{M}}_a^f (e^\mathbf{M})_d^b = \dot{\mathbf{M}}_a^b \dot{\mathbf{M}}_b^a \]

and, since we are assuming the trace of \( h_{ab} \) vanishes:

\[ \dot{\mathbf{M}}_b^a \dot{\mathbf{M}}_a^b = 3 \left( \frac{24 b t}{48 + 18 b t^2} \right)^2 + \left( \frac{16}{16 + 3 b t} \right)^2 b \] (F.5)

where the traces are taken with respect to \( g^0 \).

\(^72\) Note however that in this case the Hamiltonian is conserved along each trajectory, but it is not a primary constraint, as it is in the case of the geodesic action.
The change in volume form (even for an initial traceless velocity) is
\[
\sqrt{g} d^3x(t) = \ln(1 + \frac{3}{8} b t^2) \sqrt{g} d^3x
\]  
(F.6)
and
\[
\frac{\delta S}{\delta h_{ij}(x)} = 2 \int d^3x' \frac{\delta S}{\delta b(x')} \frac{\delta b(x')}{\delta \delta h_{ij}(x)} = 2 \frac{\delta S}{\delta b(x)} h_i^j(x)
\]
since in these simple cases all quantities are ultra-local. Thus from (F.5) and (F.6) we calculate
\[
\frac{\delta S}{\delta h_{ij}(x)} = 2 \int_0^1 dt h_i^j(x) \frac{\delta}{\delta b(t)} \left[ \int d^3x' \sqrt{g'} \left( 3 \left( \frac{24 b t}{48 + 16 b t^2} \right)^2 + \left( \frac{16}{16 + 3 b t^2} \right)^2 b \right) \ln(1 + \frac{3}{8} b t^2) \right]
\]
and
\[
\frac{\delta S}{\delta h_{ij}(x)} = 96 \int_0^1 dt h_i^j(x) \left( b t^2 + \frac{48 (3 b t^3 + 8 b t^2)}{(3 b t^2 + 8 b t^2)^2} + \frac{b t^2 (b t^2 + 8)}{(3 b t^2 + 8)^2} - \frac{96}{(3 b t^2 + 16)^2} \ln \left( \frac{3 b t^2}{8} + 1 \right) \right)
\]  
(F.7)
For the other component of (F.1), we have
\[
\frac{\delta g_{ij}(1)(x)}{\delta h_{ab}(y)} = g^{0}_{ij} \frac{\delta M_{ab}^i}{\delta h_{ab}(y)} (e^{M_{ab}^j}(x))
\]  
(F.8)
where
\[
\frac{\delta M_{ab}^i}{\delta h_{ab}(y)} = 2 \frac{\delta M_{ab}^i}{\delta b(y)} h_{ab}(y) + \frac{16}{16 + 3 b t^2} g^{(\epsilon a'b)} h_{ab}(y) \delta(x, y)
\]  
(F.9)
and
\[
\frac{\delta M_{ab}^i}{\delta b(y)} = \left( \frac{2 \epsilon^2}{3 b t^2 + 8 \epsilon^2} + \frac{8 t}{3 b t^2 + 16 b} - \frac{2 \tan^{-1} \left( \frac{\sqrt{3} \sqrt{b t}}{\sqrt{b t^2 + 16}} \right)}{\sqrt{b t^2 \epsilon^2}} \right) h_i^j(y) \delta(x, y).
\]
Putting these two together we get:
\[
\frac{\delta g_{ij}(1)(x)}{\delta h_{ab}(y)} = \left( 4 \frac{\epsilon^2}{3 b t^2 + 8 \epsilon^2} + \frac{8 t}{3 b t^2 + 16 b} - \frac{2 \tan^{-1} \left( \frac{\sqrt{3} \sqrt{b t}}{\sqrt{b t^2 + 16}} \right)}{\sqrt{b t^2 \epsilon^2}} \right) h_{ab}(y) + \frac{32}{16 + 3 b t^2} g^{(\epsilon a'b)} h_{ab}(y) \right) (e^{M_{ab}^j}(x, y)).
\]  
(F.10)
Since there were no derivatives, the contraction between (F.10) and (F.7) required by (F.1) is a trivial multiplication. Finally,\[
\frac{\delta}{\delta g_{ij}(x)} \left( \int d^3x' \frac{\delta S}{\delta h_{ij}(x')} \frac{\delta g_{ij}(x')}{\delta h_{ab}(y)} \right)
\]
\[
= \frac{\delta}{\delta g_{ij}(x)} \left[ 96 \int_0^1 dt \left( b t^2 + \frac{48 (3 b t^3 + 8 b t^2)}{(3 b t^2 + 8 b t^2)^2} + \frac{b t^2 (b t^2 + 8)}{(3 b t^2 + 8)^2} - \frac{96}{(3 b t^2 + 16)^2} \ln \left( \frac{3 b t^2}{8} + 1 \right) \right) \right]
\]
\[
\times \left( 4 \frac{\epsilon^2}{3 b t^2 + 8 \epsilon^2} + \frac{8 t}{3 b t^2 + 16 b} - \frac{2 \tan^{-1} \left( \frac{\sqrt{3} \sqrt{b t}}{\sqrt{b t^2 + 16}} \right)}{\sqrt{b t^2 \epsilon^2}} \right) h_{ab}(y) + \frac{32}{16 + 3 b t^2} g^{(\epsilon a'b)} h_{ab}(y) \right) (e^{M_{ab}^j}(y)).
\]  
(F.11)
\[
\frac{\delta (e^M_{ij})}{\delta g^0_{ij}} = \frac{\delta M^a_{ij}}{\delta g^0_{ij}} (e^M_{ij})^a
\]
and implementing
\[
\frac{\delta}{\delta g^0_{ab}(x)} b(y) = 2 \hbar a_{\phi} \delta(x, y).
\]

One finds that even for this simple case, the result is an immensely complicated function of the initial velocity \( \hbar a_{\phi} \). Taking the determinant would once again require us to perform some regularization, as was implicitly done by taking the pointwise trace in the approximation in (65).

### Appendix G. Locality

There is a type of reciprocity between physical space and configuration space which can be described as follows. Whereas fixing the entire field configuration (non-locally on \( M \)) defines a point in the configuration space \( \mathcal{Q} \), fixing only a partial field configuration on a subset of \( M \) determines an entire submanifold of \( \mathcal{Q} \). Such submanifolds are formed by all of the configurations which have that same fixed field, let us say \( \phi_0 \) defined on \( O \subset M \), i.e. those fields \( \phi \) which coincide with \( \phi_0 \) on \( O \) but are arbitrary elsewhere:

\[
\mathcal{Q}_{\phi_0} := \{ \phi \in \mathcal{Q} \mid \phi|_O = \phi_0 \}.
\]

Extremal curves in the configuration space of the complete Universe will be in general non-local on \( M \), although of course local in \( \mathcal{Q} \). In the accompanying paper [20], we provide criteria for the sort of locality we are accustomed to, in the physical space \( M \).

Namely, suppose that the action admits a Jacobi-length representation, and let \( O \) be a region in \( M \), \( O \subset M \), defining an embedded submanifold \( \mathcal{Q}_O \) (with appropriate boundary conditions, in a specific way we spell out in [20]). We show that for semi-classical clustering—i.e. decoupling of \( O \) from the rest of \( M \)—to occur on a given region of configuration space, \( \mathcal{U} \subset \mathcal{Q} \), we must have that \( \mathcal{Q}_O \cap \mathcal{U} \) is a totally geodesic submanifold of \( \mathcal{U} \). We show that with this condition the semi-classical path integral kernel (in the arc-length gauge) decouples (here written for regions \( O_k \)), i.e. the kernel over the union of the regions turns into a product over each individual region:

\[
W^\mathcal{U} \cup \bigcup_k O_k (\phi_{|\bigcup_k O_k}^f, \phi_{|\bigcup_k O_k}^i) \approx \prod_k W^O_k (\phi_{|O_k}^f, \phi_{|O_k}^i)
\]

(g.1)
giving us our version of clustering decomposition.

I should also note that there is an inherent non-locality with the Jacobi form of reparametrization invariant actions (they possess a global square root). However, there is a preferred gauge in which they can become local: the arc-length gauge. In this gauge, the action becomes an energy functional (in Riemannian geometry terminology), and it is possible to show that the Fadeev Popov determinant is field independent. In other words, this particular gauge-fixing can trade time reparametrization invariance for locality.

\[73\text{To be more careful, I should only ascribe to them the title of ‘subsets’, not submanifolds. However, under reasonable assumptions, as shown in the accompanying paper [20] they indeed form submanifolds.}\]
Appendix H. The Wheeler DeWitt equation

H.1. Difficulties with the Wheeler DeWitt equation

Classical GR admits a Hamiltonian formulation, but it is a constrained theory. The configuration variable is taken to be the spatial Riemannian metric \( h_{ab} \) on the manifold \( M \), alluded to in (H.4). The space of all such metrics is called \( \text{Riem}(M) \). The emerging system (whose specific form is not relevant to our study here) is of a fully constrained Hamiltonian,

\[
\mathcal{H} = \int_M d^3x \left( N(x)H(x) + N'(x)H_i(x) \right). \tag{H.1}
\]

The standard canonical quantization rules would suggest that the states of the system be given by wave functions of the configuration variable, \( \Psi = \Psi(\tau, h_{ab}) \) where \( \tau \) denotes the time variable occurring in the Hamiltonian formulation. Replacing as usual \( h_{ab} \) by its action through multiplication, and of its conjugate momentum with functional derivation, \( \pi_{ab} \rightarrow \delta \delta h_{ab} \), we could have the evolution of \( \Psi \) given by the Schrödinger equation,

\[
i \frac{d\Psi}{d\tau} = \mathcal{H}\Psi. \tag{H.2}
\]

Since \( \mathcal{H} \) is pure constraint, \( \mathcal{H}\Psi = 0 \), but also \( \hat{H}\Psi = \hat{H_i}\psi = 0 \). The changes generated in phase space by the flow of \( \int_M d^3x N'(x)H_i(x) \) correspond simply to the action of spatial diffeomorphisms on \( h_{ab} \) and its conjugate momenta. Its quantization would simply mean that the value of \( \Psi(h_{ab}) \) should be unchanged upon the action of an infinitesimal diffeomorphism of \( M, f \in \text{Diff}(M) \), which acts pointwise on \( \text{Riem}(M) \) (through pull-backs) [72]. The space of metrics quotiented by this gauge symmetry, \( \text{Riem}/\text{Diff} \), is the space of geometries, also called superspace, and the action of \( \hat{H_i} \) as a quantum constraint operator implies the wavefunction lives over this superspace. This part is recovered in the present setting.

The action of \( \hat{H} \) on the wave-functions gives rise to the Wheeler–DeWitt equation [75]:

\[
\hat{H}\Psi = \left( G_{abcd} \frac{\delta^2}{\delta g_{ab}\delta g_{cd}} + \sqrt{g} \left( -R(g) + 2\Lambda \right) \right) \Psi[g_{ij}] = 0. \tag{H.3}
\]

The analogous operator interpretation to \( H_i \) here would be invariance wrt diffeomorphisms in space-time which move points orthogonally to \( M \). Although that interpretation is not as clear in this case, it is still compatible with the intuitive effect of the implementation of the constraint, namely \( \frac{d\Psi}{d\tau} = 0 \).

In spite of the indefinite signature in its kinetic term (coming from the DeWitt supermetric) giving rise to quadratic terms of negative sign, the WdW cannot be interpreted as a Klein–Gordon type of equation to be ‘second-quantized’, for (i) it is already quantized, and (ii) there are no straightforward ways to split solutions into positive and negative frequencies to construct the annihilation and creation operators, since there is no good time function in superspace. For a review, see [3]. This is not to mention finer points about regularization (point-splitting seems to be the more appropriate one here, due to the occurrence of the singular \( \delta(x, x) \) coming from quantization of the term quadratic in momenta [73]) or factor-ordering.

---

74 More formally, we should use the Radon–Nikodym derivatives in field space, which takes the measure into account [71].

75 It should be noted that to obtain (H.3) from (H.4) one needs to a priori fix refoliations at the initial and final configurations [24], in accordance with our expectations.
Moreover, the requirement that the Hamiltonian commute with all observables can be extremely restrictive. For ergodic systems, it will lead to the conclusion that the only observables (continuous in phase space, with the Liouville-induced topology) are polynomials of the Hamiltonian itself \[14–15\]; a defect which is at least camouflaged in minisuperspace approximations due to their simplicity.

**H.2. Issues with the path-integral definition of the wave-function**

\[
W(h_1, h_2) = \int_{h_1}^{h_2} \mathcal{D}g \exp \left[ i \int \mathcal{L}/\hbar \right]
\]  

(H.4)

where \( \mathcal{L} \) is the gravitational Lagrangian density, and I have omitted boundary terms, the required gauge-fixings, Fadeev–Popov determinant and regulators (since the theory is non-renormalizable, this action should only be taken as an effective theory at a given scale). Here \( h_i \) are initial and final spatial geometries, i.e. for the hypersurfaces \( M_i \) and the embedding \( i_i : M_i \hookrightarrow M \) then \( h = i^* g \), where here \( g \) is the abstract space-time metric field. The issue is that refoliations shift \( h_i \) (in a \( g \)-dependent manner), and change (H.4)\(^76\).

For a transition amplitude, one could define \( h_1 \) and \( h_2 \) in (H.4) as the inverse value of some scalar functional on the spatial metric (such as the value of a matter field, or the scalar curvature) \( f : Q \mapsto \mathbb{R} \), as \( h_i := f^{-1}(t_i) \), with restrictions on \( f \) so that \( h_i \) becomes invariant with respect to refoliations. Implicitly, this is what is done to obtain the invariance relations of the path integral with respect to refoliations, and it corresponds to the ‘embedding variables’ of Isham and Kuchar \[74\]. In this case, the embedding variables function largely as a ‘dust field’, which defines points on the space-time manifold.

Here adopting \( h_{ij} \) for the 3-metric and \( g_{\mu\nu} \) for the 4-metric, we fix one of the boundary conditions. From (H.4) the wavefunction of the universe on a hypersurface with intrinsic 3-metric \( h_{ij} \) and matter configuration, \( \varphi \), can be defined by

\[
\Psi(h, \varphi) = \int \mathcal{D}g\mathcal{D}\varphi \exp -I [g_{\mu\nu}, \varphi]
\]  

(H.5)

where the sum is over a class of 4-metrics, \( g_{\mu\nu} \), and matter configurations \( \varphi \), which when restricted to the boundary take values \( g_{ij} \) and \( \varphi \), and \( I \) is the spacetime covariant action functional.

To find the functional constraints related to the Ward–Takahashi identities such a wavefunction might satisfy, one must employ the more formal Hamiltonian definitions. In \[24\], Hartle and Halliwell attempt to formally clarify this. Instead of (H.5), one defines (dropping \( \varphi \)):

\[
\Psi(h) = \int \mathcal{D}z^A \delta(h_{ij}(t_f) - h_{ij}) \Delta_c[z^A] \delta[C(z^A)] \exp \{ iS[z^A] \}
\]  

(H.6)

where \( z^A = (\pi^\alpha, h_{ij}, N^\alpha) \) for the Hamiltonian path integral, or \( z^A = (h_{ij}, N^\alpha) \) if one can integrate away the momenta, \( \mathcal{D}z^A \) is the measure obtained from the Liouville one (for which BRST transformations are canonical), \( \Delta_c[z^A] \) is the Fadeev–Popov determinant, \( C(z^A) \) is the gauge-fixing condition, and \( N^\alpha \) are the Lagrange multipliers. Integration runs also over all the final variables, and there they are fixed by the Dirac delta.

\(^76\) As I comment on below, however, the issue is more subtle than it appears, if one implements the boundary conditions with a delta function on the boundary. Then gauge transformations do not act directly on the boundary field.
In the case of gravity, the very definition of the path integral through time slicing requires one to choose a notion of physical time already at that level, so that infinitesimal slicings are meaningful and one can take extremal approximations in between the slices. The very construction of the path integral in terms of infinitesimal slicings only makes sense in a gauge-fixing for gravity, which sets it apart from theories in which a background causal structure exists.

Clearly, from (10), if either the gauge transformations vanish at the boundaries, or if the Hamiltonian generators of the symmetries are linear in the momenta, then
\[ \delta \epsilon S = 0. \]
Moreover, in those cases, then we can resort to the standard BRST methods to show invariance of the wavefunction (H.6). This is the main difference between the constraints advocated here and the ones which shift the boundary itself.

By redefining the integration variables according to the \( \delta \epsilon \) transformation (and subtracting the untransformed wavefunction), one obtains
\[
0 = \int \mathcal{D}z^A \left[ - \int d^3 \delta \epsilon \left( \frac{\delta}{\delta h_{ij}(x')} \right) \delta(h_{ij}(t_f) - h_{ij}) \Delta_c[z^A] \delta[C(z')] \exp \left( iS[z^A] \right) \right].
\]

(Note that here the refoliations act on the paths (or on the \( z^A \) variables), and therefore on \( h_{ij}(t) \), but not on the boundary fields \( h_{ii} \).

Now, we must use their assumption 5 in section II B, which translates functions on the boundary variables in the path integral to functional equations on the resulting wavefunction (ignoring Lagrange multipliers, gauge-fixings and Fadeev–Popov determinants), i.e.:
\[
\int \mathcal{D}\phi \mathcal{D}\pi F(\phi(t''), \pi(t'')) \delta(\phi(t'') - \phi'') e^{iS(\phi, \pi)/\hbar} = F[\phi'', -h \frac{\delta}{\delta \phi''}] \psi[\phi''].
\]

And voila, one obtains the constraint equations applied to the wave-function, by taking into account the transformation of the boundary values together with the transformation of the action functional.

One of the issues here is that, unlike ordinary field theories such as Yang-Mills, the negative sign in front of the squared trace of the gravitational velocity (‘the conformal mode’), implies the covariant gravitational action is not positive-definite. Thus, restricting the sum to real Euclidean-signature metrics would not generically allow the the path integral to converge. Indeed, to obligate the path integral to converge in this way, it would be necessary to include complex metrics in the class of metrics to be integrated over. However, as a recent renewed debate surrounding this issue makes clear, there is no unique contour to integrate along in superspace. Moreover, the result obtained for the path integral would generically depend on the types of contours chosen. Such different alternatives are intimately connected with the boundary conditions chosen for the paths to be integrated over, and these choices should implicitly restrict the 4-manifolds included in the sum in (H.5).

However, the boundary conditions in the path integral should have a covariant meaning, and that makes them extremely averse to being embodied in superspace; where the individual paths of the path integral are explicitly realized. One objective consequence of this problem is that, although such contour integrations and boundary conditions for the Wheeler–DeWitt equation must be somehow related, it is nearly impossible to realize this connection in a precise manner.

77 While it is true the path integral for fermions is also not positive-definite, their anticommuting nature ensures that the path integral converges.
78 In practice, these problems are not seen directly, since working with the infinite dimensions of the full superspace is unfeasible, and the problems are not self-evident in minisuperspace.
H.2.1. Physically defining boundary states. This approach of physically defining the boundary states is also related to what is known as the approach of complete observables in canonical general relativity [11, 76], for which a large literature exists (see [12] for a review) and the use of ‘internal time’ (see e.g. [76]). In the case where only one reparametrization constraint exists, complete observables should be a one-parameter family of Dirac observables. Many technical problems arise, however, when an infinite amount of reparametrization constraints exist; as is the case of GR, and when one would like to extend this construction to the entire phase space.

Nonetheless, as shown in [78], even in this case, attempting to use an internal time for each different region in phase space, patching the transitions between them, actually works well only in the purely classical regime. From the point of view of state evolution, defining time for each finite range poses an enormous issue for unitary evolution of states. The major issue is then that non-unitary quantum evolution risks meaningless results—even away from the boundary of the patch where it should be valid—and it is not clear how to define quantum observables in such a situation.

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79 See [77, 78] for some candidate constructions.
80 Indeed, this happens even if classical evolution with respect to a local internal time can be made unproblematic in the transition region between one patch and another.
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