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Erratum to: Model-checking continuous-time Markov chains by Aziz et al.

David N. Jansen*

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Abstract

This note corrects a discrepancy between the semantics and the algorithm of the multiple until operator of CSL, like in Pr_{t_0} > 0.0025(a U_{[1,2]} b U_{[3,4]} c), of the article: Model-checking continuous-time Markov chains by Aziz, Sanwal, Singhal and Brayton, TOCL 1(1), July 2000, pp. 162–170.

1 Introduction

The widely cited article [1] defines continuous stochastic logic (CSL), a logic to reason about continuous-time Markov chains, with a multiple until operator to write formulas (with atomic propositions $a$, $b$, and $c$) like:

\[ a U_{[1,2]} b U_{[3,4]} c. \]

The semantics given in the article is:

A path $\pi$ satisfies $f_1 U_{[a_1,b_1]} f_2 U_{[a_2,b_2]} \cdots U_{[a_{n-1},b_{n-1}]} f_n$ "if and only if there exist real numbers $t_1, \ldots, t_{n-1}$ such that for each integer in $[1,n]$ we have $(a_i \leq t_i \leq b_i) \land (\forall t' \in [t_{i-1}, t_i)](\pi(t')$ satisfies $f_i$), where $t_{-1}$ is defined to be 0 for notational convenience."

This definition uses the undefined variables $t$, $t_0$, $i$, $a_n$, $b_n$, and $t_n$ (while it defines the unused variable $t_{-1}$), and it seems to require that $\pi(t_0)$ satisfy $f_1 \land f_2$. Obviously, the authors meant something like:

A path $\pi$ satisfies $f_1 U_{[a_1,b_1]} f_2 U_{[a_2,b_2]} \cdots U_{[a_{n-1},b_{n-1}]} f_n$ if and only if there exist real numbers $0 < t_1 < t_2 < \cdots < t_{n-1}$ such that for each integer $i$ in $[1,n-1]$ we have $(a_i \leq t_i \leq b_i) \land (\forall t' \in [t_{i-1}, t_i)](\pi(t')$ satisfies $f_i)$, where $t_0$ is defined to be 0 for notational convenience, and additionally $\pi(t_{n-1})$ satisfies $f_n$.

However, the implementation, i.e., the algorithm that estimates the probability of this until operator, uses another semantics implicitly, namely the following:

A path $\pi$ satisfies $f_1 U_{[a_1,b_1]} f_2 U_{[a_2,b_2]} \cdots U_{[a_{n-1},b_{n-1}]} f_n$ if and only if for each integer $i$ in $[1,n-1]$ we have $(\forall t' \in [b_{i-1}, a_i])(\pi(t')$ satisfies $f_i) \land (\forall t' \in (a_i, b_i))(\pi(t')$ satisfies $f_i \lor f_{i+1})$, where $b_0$ is defined to be 0 for notational convenience, and additionally $\pi(b_{n-1})$ satisfies $f_n$.

The implementation allows to switch back and forth between states satisfying $f_i \land \neg f_{i+1}$ and states satisfying $\neg f_i \land f_{i+1}$, and it requires to stay in a $f_n$-state longer than the semantics.

The present article exhibits the error and shows how it can be corrected. In the remainder of the article, we will assume that the intervals do not overlap, i.e., that $b_i < a_{i+1}$ for all $i = 1, 2, \ldots, n-2$.

*Institute for Computing and Information Sciences, Radboud Universiteit, P. O. Box 9010, 6500 GL Nijmegen, The Netherlands; e-mail: D.Jansen@cs.ru.nl.
2 Example

For example, consider the Markov chain drawn in Figure 1. The probability that a path satisfies the formula \( g = a U_{[1,2]} b U_{[3,4]} c \) can be calculated as the product of a few Poisson probabilities:

\[
\begin{align*}
\Pr(0 \text{ transitions during time } [0, 1]) & \cdot \Pr(1 \text{ transition during time } [1, 2]) \cdot \frac{1}{2}, \\
\Pr(0 \text{ transitions during time } (2, 3)) & \cdot \Pr(> 0 \text{ transition(s) during time } [3, 4]) = \\
&= \frac{2^6 e^{-2}}{0!} \cdot \frac{2^1 e^{-2}}{1!} \cdot \frac{1}{2} \cdot \frac{2^0 e^{-2}}{0!} \cdot \left(1 - \frac{2^0 e^{-2}}{0!}\right) = \frac{2}{2} e^{-6} (1 - e^{-2}) \approx 0.00214
\end{align*}
\]

However, [1]'s algorithm does the following calculations: The probability of the formula \( g \) is

\[
\mu^1(g) = (1, 0, 0, 0) P_a(1) I_a P_a \lor b(2 - 1) I_b P_b(3 - 2) I_b P_{b \lor c}(4 - 3) I_c \\
\begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix}
\]

where \( P_f(t) \) is the transition probability matrix of the Markov chain with all \( \neg f \)-states changed to absorbing states, for time interval \( t \); and \( I_f \) is the diagonal matrix with entries 1 for \( f \)-states and 0 for \( \neg f \)-states. For example,

\[
P_a(t) = \exp \begin{pmatrix}
-2t & 2t & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad I_b = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

Multiplying all these matrices as indicated in Formula (1) produces the outcome:

\[
\mu^1(g) = \frac{1}{2} e^{-8} (e^{\sqrt{2}} - e^{-\sqrt{2}}) \sqrt{2} \approx 0.000918
\]

which is less than half the actual value.

3 First problem: final transition

A problem arises with paths that enter state 4 during the time interval \((3, 4]\). These paths are counted as non-satisfying by the algorithm (Formula (1) only counts the paths that are in a \( c \)-state at time 4), while they have passed through a \( c \)-state timely and actually may satisfy \( g \).

To solve this problem, \( P_{b \lor c}(t) \) should be replaced by a matrix based on a Markov chain where additionally all \( c \)-states have been made absorbing, so that a path entering an \( c \)-state stays there until time 4. This is basically the same transformation as described by [2] for simple until formulas \( f_1 U_{[a_1, b_1]} f_2 \): Make all states except the \( f_1 \land \neg f_2 \)-states absorbing. In our example, we have to replace the factor \( P_{b \lor c}(t) \) of Formula (1) by:

\[
P_{b \land \neg c}(t) = \exp \begin{pmatrix}
0 & 0 & 0 & 0 \\
t & -2t & t & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]
and so, the calculated probability becomes

\[
\mu^1(g) = (1, 0, 0, 0) P_a(1) I_a P_{a \lor b}(2 - 1) I_b P_b(3 - 2) I_b P_{b \land \neg c}(4 - 3) I_c \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \\
\frac{1}{4}e^{-6}(1 - e^{-2})(e\sqrt{2} - e^{-\sqrt{2}})\sqrt{2} \approx 0.00293 \quad (2)
\]

which unfortunately is still wrong.

4 Second problem: intermediary transitions

The remaining discrepancy after the first correction reveals another problem. In the example, there is a transition from \(b\)-state 2 back to \(a\)-state 1. According to Formula (1), the path \(1 \xrightarrow{t=1} 2 \xrightarrow{t=1.5} 1 \xrightarrow{t=1.8} 2 \xrightarrow{t=3.2} 3\) is counted as a path satisfying \(g\), as it continuously satisfies \(a \lor b\) during the interval \((1, 2)\). However, the semantics requires that one choose when \(t_1 \in [1, 2]\) has come. This must happen upon entering state 2 (a \(b \land \neg a\)-state) at the latest, so \(t_1 = 1\). After \(t_1\), one is no longer allowed to enter \(a \land \neg b\)-states, so the transition \(2 \xrightarrow{t=1.5} 1\) is forbidden.

4.1 Wrong correction

We could try to correct this problem by deleting transitions from \(b\)-states to \(a\)-states in the example. This, however, gives rise to two new problems:

1. The exit rate of state 2 (in the Markov chain of Figure 1) would change.

2. What about \(a \land b\)-states? These states are allowed both before and after \(t_1\). If \(t_1\) has passed, switching back and forth between \(a \land b\)-states and \(b \land \neg a\)-states should be allowed, while it should be counted as an error to enter an \(a \land \neg b\)-state. Before \(t_1\), the opposite condition holds. It is impossible to make a subset of the states absorbing in a way that satisfies both conditions.

4.2 Better correction

To solve the problem mentioned above, I propose to add extra states to the Markov chain for \(P_{a \lor b}\): Introduce a second copy \(s'\) of each \(a\)-state \(s\) that has a \(b\)-predecessor state. One copy \((s)\) stands for “\(t_1\) (possibly) has not yet passed” and the second \((s')\) for “\(t_1\) has passed definitely”. So, transitions from \(b \land \neg a\)-states to \(s\) are deflected to \(s'\), and if \(s\) satisfies \(a \land \neg b\), then \(s'\) is an error state and is rendered absorbing.

In our example, we have to replace the factor \(P_{a \lor b}(t)\) of Formula (2) by the transition probability matrix of the Markov chain shown in Figure 2 denoted by \(P'_{a \lor b}(1)\) (I treat state 1’ as fifth
The following table shows which transitions the extended Markov chain contains, i.e., the Markov chain to base $P_s$ chain with an additional copy $P_{a_i}$. When we extend the above corrections to general until formulas, we get the following basis for an algorithm to compute the probability of until formulas:

$$
\mu^f(g) = (1, 0, 0, 0) P_a(1) I'_a P_{a_i}^f (2 - 1) I''_b P_b (3 - 2) I_b P_{b \land -c} (4 - 3) I_c \left( \begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array} \right) = e^{-6}(1 - e^{-2}) \approx 0.00214
$$

which is the correct answer.

In this formula, I also modified $I'_a$ and $I''_b$ to include the transformation between the four- and five-state-Markov chains:

$$
I'_a = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad I''_b = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
$$

### 5 General formulation of the corrected semantics

When we extend the above corrections to general until formulas, we get the following basis for an algorithm to compute the probability of until formulas:

$$
g := f_1 U_{[a_1, b_1]} f_2 U_{[a_2, b_2]} f_3 \cdots f_{n-1} U_{[a_{n-1}, b_{n-1}]} f_n \quad ,
$$

where the intervals do not overlap, is given by

$$
\mu^s(g) = \pi_s \cdot P_{f_1}(a_1) \cdot I'_{f_1} \cdot P'_{f_1 \lor f_2} (b_1 - a_1) \cdot I''_{f_2} \cdot P_{f_2}(a_2 - b_1) \cdot I'_{f_2} \cdot P'_{f_2 \lor f_3} (b_2 - a_2) \cdot I''_{f_3} \cdot P_{f_3}(a_3 - b_2) \cdots I'_{f_{n-2}} \cdot P'_{f_{n-2} \lor f_{n-3}} (b_{n-2} - a_{n-2}) \cdot I''_{f_{n-1}} \cdot P_{f_{n-1}}(a_{n-1} - b_{n-2}) \cdot I'_{f_{n-1}} \cdot P'_{f_{n-1} \lor f_n} (b_{n-1} - a_{n-1}) \cdot I''_{f_n} \cdot \left( \begin{array}{c} 1 \\ 1 \\ \vdots \\ 1 \end{array} \right)
$$

where $P_f(t)$ is the transition probability matrix for time $t$ corresponding to the Markov chain where all $\neg f$-states are made absorbing; $\pi_s$ is the starting probability distribution (which in our case has unity for state $s$ and zeroes otherwise); $P'_{f_i \lor f_{i+1}}(t)$ is the transition probability matrix for time $t$ based on the extended Markov chain (details follow); and $I'_{f_i}$ and $I''_{f_i}$ are the matrices that map $f_i$-states and $f_{i+1}$-states, respectively, to and from the extended Markov chain (details follow).

The following table shows which transitions the extended Markov chain contains, i.e., the Markov chain to base $P'_{f_i \lor f_{i+1}}(t)$ upon. As mentioned above, the states are the same as the original Markov chain with an additional copy $s'$ of each $f_i$-state $s$ that has a $f_{i+1}$-predecessor.
If the original Markov chain contains a transition \( s \xrightarrow{\lambda} t \), then the extended Markov chain contains the following transition(s):

| \( s \) | \( s' \) is added | \( t' \) is added | \( t' \) is not added | \( s' \) is not added | \( t' \) is added | \( t' \) is not added |
|-----|----------------|----------------|----------------|----------------|----------------|----------------|
| \( s \models f_i \land \neg f_{i+1} \) | \( s \xrightarrow{\lambda} t; s' \) is absorbing | \( s \xrightarrow{\lambda} t \) | \( s \xrightarrow{\lambda} t \) | \( s' \xrightarrow{\lambda} t \) | \( s \xrightarrow{\lambda} t \) | \( s \xrightarrow{\lambda} t \) |
| \( s \models f_i \land f_{i+1} \) | \( s \xrightarrow{\lambda} t \) and \( s' \xrightarrow{\lambda} t' \) | \( s \xrightarrow{\lambda} t \) and \( s' \xrightarrow{\lambda} t \) | \( s \xrightarrow{\lambda} t \) | \( s \xrightarrow{\lambda} t \) |
| \( s \models \neg f_i \land f_{i+1} \) | impossible | \( s \xrightarrow{\lambda} t' \) | \( s \xrightarrow{\lambda} t \) |
| \( s \models \neg f_i \land \neg f_{i+1} \) | impossible | \( s \xrightarrow{\lambda} t \) | \( s \) is absorbing |

\( I'_f \) and \( I'_{f_{i+1}} \) can be used to convert the probability vectors to and from the extended Markov chain. At time \( a_i \), all probability mass should go into the first copy \( s \) of a state which has a second copy \( s' \), so

\[
(I'_{f_i})_{st} = \begin{cases} 
1 & \text{if } s = t \text{ and } s \text{ satisfies } f_i \\
0 & \text{otherwise}
\end{cases}
\]

Both copies of \( f_i \land f_{i+1} \)-states are allowed at time \( b_i \) (the latest possible \( t_i \)), so their probabilities should be added in the modified \( I'_{f_{i+1}} \):

\[
(I'_{f_{i+1}})_{st} = \begin{cases} 
1 & \text{if } s = t \text{ or } s = t' \text{ and } s \text{ satisfies } f_{i+1} \\
0 & \text{otherwise}
\end{cases}
\]

5.1 Correctness

To convince ourselves that the product in Formula [3] corresponds closely to the semantics given at the beginning of the article, let us look at some of its factors.

- \( P_{a_n-1} := I_{f_{a_n-1}} \cdot P_{f_{a_n-1} \land f_{a_{n-1}}} \cdot P_{f_{a_n-1}} \cdot 1 \) can be seen as a vector of probabilities: \( P_{a_n-1}(s) \) is the probability that one gets a path for which there exists a \( t_{a_n-1} \in (a_{n-1}, b_{n-1}] \) such that the path is in \( f_{a_n-1} \)-states during the time interval \([a_{n-1}, t_{n-1})\) and it is in a \( f_{a_n} \)-state at time \( t_{n-1} \) under the condition that it is in state \( s \) at time \( a_{n-1} \), as shown by [2]. (Note that \( t_{n-1} \not\in a_{n-1} \) as the path has to be in a \( f_{a_n-1} \)-state at time \( a_{n-1} \). This is not a relevant difference as the paths with \( t_{n-1} = a_{n-1} \) form a set that has probability 0.)

- Let \( M_{b_{n-2}} := I_{f_{a_n-1}} \cdot P_{f_{a_n-1} \land f_{a_{n-1}}} \cdot P_{f_{a_n-1}} \cdot P_{b_{n-2}} \cdot 1 \). Then, \( M_{b_{n-2}}(s, t) \) is the probability that a path is in \( f_{a_n-1} \)-states during the time interval \([b_{n-2}, a_{n-1}]\) and ends in state \( t \) at time \( a_{n-1} \), under the condition that it is in state \( s \) at time \( b_{n-2} \).

Let \( P_{b_{n-2}} := M_{b_{n-2}} \cdot P_{a_{n-1}} \cdot P_{f_{a_n-1}} \cdot P_{f_{a_n-1} \land f_{a_{n-1}}} \cdot P_{b_{n-2}} \cdot 1 \). \( P_{b_{n-2}}(s) \) is the probability that one gets a path for which there exists a \( t_{n-1} \in (a_{n-1}, b_{n-1}] \) such that the path is in \( f_{a_n-1} \)-states during the time interval \([b_{n-2}, t_{n-1})\) and it is in a \( f_{a_n} \)-state at time \( t_{n-1} \), under the condition that it is in state \( s \) at time \( b_{n-2} \).

- Let \( M_{a_n-2} := I'_{f_{a_n-2}} \cdot P_{f_{a_n-2} \land f_{a_{n-2}}} \cdot P_{f_{a_n-2}} \cdot P_{a_{n-2}} \cdot 1 \). Then, \( M_{a_n-2}(s, t) \) is the probability that one gets a path for which there exists a \( t_{n-2} \in (a_{n-2}, b_{n-2}] \) such that the path is in \( f_{a_n-2} \)-states during the time interval \([a_{n-2}, t_{n-2})\), it is in \( f_{a_n-1} \)-states during the time interval \([t_{n-2}, b_{n-2}]\) and it is in state \( t \) at time \( b_{n-2} \), under the condition that the path is in state \( s \) at time \( a_{n-2} \).

Let \( P_{a_{n-2}} := M_{a_n-2} \cdot P_{b_{n-2}} \). (Note that \( I'_{f_{a_n-1}} \cdot I_{f_{a_n-1}} = I'_{f_{a_n-1}} \)) \( P_{a_{n-2}}(s) \) is the probability that one gets a path for which there exist \( t_{n-2} \in (a_{n-2}, b_{n-2}] \) and \( t_{n-1} \in (a_{n-1}, b_{n-1}] \) such that the path is in \( f_{a_n-2} \)-states during the time interval \([a_{n-2}, t_{n-2})\), it is in \( f_{a_n-1} \)-states during the time interval \([t_{n-2}, t_{n-1})\), and it is in a \( f_{a_n} \)-state at time \( t_{n-1} \), under the condition that it is in state \( s \) at time \( a_{n-2} \).
• etc.
• Finally, let $P_0 := M_0 \cdot P_{a_1}$, and $P_0(s)$ is the probability that one gets a path for which there exist $t_1 \in (a_1, b_1], \ldots, t_{n-1} \in (a_{n-1}, b_{n-1}]$ such that the path is in $f_1$-states during the time interval $[0, t_1)$, it is in $f_2$-states during the time interval $[t_1, t_2)$, \ldots, it is in $f_{n-1}$-states during the time interval $[t_{n-2}, t_{n-1})$ and it is in a $f_n$-state at time $t_n$, under the condition that it is in state $s$ at time 0.

The first factor in the product, $\pi_s$, serves to uncondition on the initial state. So, overall, the product $\pi_s \cdot P_0$ calculates the probability that a path satisfies the semantics given.

6 Concluding remarks

The correction presented above provides a calculation principle to find the probability that a path satisfies an until formula, corresponding closely to the intended semantics as given in the introduction. It does no longer require to stay in $f_n$-states overly long; it does no longer allow to switch back and forth between $f_{i-1}$- and $f_i$-states too often.

The present note passes over a choice that one has if intervals overlap: Would $a \ U_{[1,3]} b \ U_{[2,4]} c$ be satisfied by a path that jumped from an $a \land \neg b$-state to a $c \land \neg b$-state in the interval $[2,3]$, i.e. a path with $t_1 = t_2$? [1]’s remarks about overlapping intervals suggest they choose to forbid such paths, and my formulation of the semantics is aligned thereto; however, in some cases it may be more intuitive to allow them. It is possible to solve this discrepancy by adding the probability of $a \ U_{[2,3]} c$ if desired. [2] choose the other way for simple until formulas: they consider $f_2$-states as satisfying $f_1 \ U_{[0,b]} f_2$.

The main goal of [1] was to prove decidability of CSL model checking. This erratum does not invalidate their proof idea; it only requires to fill in slightly different matrices in some proof parts, but the main argument – namely, that basic operations on (matrices containing) algebraic numbers produce (matrices containing) algebraic numbers again – remains valid for the modified matrices. So, it still holds up that CSL model checking is decidable.

Acknowledgement. Most of the above matrix calculations have been performed using Maple.

References

[1] Adnan Aziz, Kumud Sanwal, Vigyan Singhal, and Robert Brayton. Model-checking continuous-time Markov chains. *ACM transactions on computational logic*, 1(1):162–170, July 2000.

[2] Christel Baier, Boudewijn Haverkort, Holger Hermans, and Joost-Pieter Katoen. Model-checking algorithms for continuous-time Markov chains. *IEEE transactions on software engineering*, 29(6):524–541, 2003.