Convergence time estimate and tuning of twisting algorithm

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Abstract

Gain tuning is given for the twisting controller to ensure that the closed-loop trajectories of the perturbed double integrator, initialized within a bounded domain and affected by uniformly bounded disturbances, settle at the origin in prescribed time.

1. Introduction

In [1], a settling time estimate of the perturbed double integrator

\begin{align}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= u(x_1, x_2) + \omega(t),
\end{align}

was obtained in terms of the positive constant gains $\mu_1, \mu_2$, of the upper magnitude bound $N > 0$ on the external disturbance $\omega(\cdot)$ such that

$$|\omega(t)| \leq N \quad \text{for all } t,$$

was obtained in terms of the positive constant gains $\mu_1, \mu_2$, of the upper magnitude bound $N > 0$ on the external disturbance $\omega(\cdot)$ such that

$$|\omega(t)| \leq N \quad \text{for all } t,$$
and of the size $R > 0$ of the initial domain
\[ \Gamma_R = \{ x : V(x_1, x_2) \leq R \}, \tag{4} \]
where $V(x_1, x_2)$ is the positive definite function
\[ V(x_1, x_2) = \mu_2 |x_1| + \frac{1}{2} x_2^2. \tag{5} \]
The aim of this note is to further simplify the afore-mentioned estimate to suit it to control applications of the twisting algorithm in the closed-loop form. For convenience of the reader, the nomenclature of [1] is used throughout.

2. Settling time estimate

Given the parameters
\[ R > 0, \quad \beta > 1, \quad \rho \in (0, 1), \quad \delta > \frac{\sqrt{2R(\beta + 1)}}{\beta - 1} \tag{6} \]
let us choose
\[ \mu_1 > \frac{2\delta}{T_s \sqrt{1 - \beta^{-2}}} + N, \tag{7} \]
\[ \mu_2 > \max \left\{ \sqrt{\frac{R}{2}}, \rho \sqrt{\frac{R}{2(1 - \rho)}}, \rho, \beta \mu_1, \mu_1 + N \right\}, \tag{8} \]
where $T_s > 0$ is a prescribed convergence time.

Setting the initial domain boundary
\[ \partial \Gamma_R = \{ x : V(x_1, x_2) = R \}, \tag{9} \]
the following result is in order.

**Theorem 1.** Consider the perturbed double integrator [1], driven by the twisting algorithm [2] and affected by an external disturbance [3]. Let $x(\cdot) = (x_1(\cdot), x_2(\cdot))^T \in \mathbb{R}^2$ be an arbitrary solution of the closed-loop system [1] - [2], initialized on the domain boundary [9] of the size $R > 0$, and let the gains
\(\mu_1, \mu_2\) be chosen according to (7)–(8), specified with \(N, T_s > 0\) and the parameters \(R, \beta, \rho, \delta\), satisfying (6). Then \(x(t) \equiv 0\) for all \(t \geq T_s\), regardless of whichever external disturbance (3) affects the closed-loop system.

3. Proof of Theorem 1

The proof follows the line of reasoning used in [1]. All relations, which are subsequently invoked from [1], are accordingly referenced.

For later use, let us introduce embedded balls

\[ B_{r_1} = \{ x_1^2 + x_2^2 \leq r_1^2 \} \quad \text{and} \quad B_{r_2} = \{ x_1^2 + x_2^2 \leq r_2^2 \} \]

such that \( B_{r_2} \subset \Gamma_R \subset B_{r_1} \).

The balls, thus introduced, are depicted in Figure 1, reproduced from [1].

![Figure 1: Sets \(\Gamma_R, B_{r_1}, B_{r_2}\) and \(B_\delta\).](image)

The radiuses \(r_1, r_2\) of such balls are established in [1] relations (4.18) and (4.25) to be

\[ r_1 = \max \left\{ \frac{R}{\mu_2}, \sqrt{2R} \right\} \]  \hspace{1cm} (10)

\[ r_2 = \min \left\{ \frac{\rho R}{\mu_2}, \sqrt{2R(1-\rho)} \right\} \]  \hspace{1cm} (11)
with \( \rho \in (0, 1) \). Since \( \mu_2 > \sqrt{\frac{R}{2}} \) due to [8], relation [10] is simplified to [11] relation (6.6)]

\[
r_1 = \sqrt{2R} \tag{12}
\]

In addition, \( \mu_2 > \rho \sqrt{\frac{R}{2(1-\rho)}} \) by its choice [8], and [11] is therefore simplified to [11] relation (4.28)]

\[
r_2 = \frac{\rho R}{\mu_2} \tag{13}
\]

Taking into account that [8] results in \( \mu_2 > \rho \), it follows that \( r_2 < R \).

The convergence time \( T_2 \) of the closed-loop system (1)–(2), initialized on the domain boundary [9], is upper estimated by that of initialized at the intersection point \( O_4 \) of the vertical axis and the circle \( \partial B_\delta = \left\{ x_1^2 + x_2^2 = \delta^2 \right\} \) of a (properly chosen) radius \( \delta \), satisfying [6]; see Figure 1. As shown in [11] p.480], such a value of \( \delta \) ensures that the resulting convergence time estimate is conservative regardless of whichever initial conditions are chosen on the domain boundary \( \partial \Gamma_R \).

Thus, the convergence time \( T_2 \) of the closed-loop system (1)–(2), initialized at \( O_4 \), is bigger than the convergence time \( T_2 \) of (1)–(2), initialized on [9], i.e.,

\[
T_2 \geq T_2, \tag{14}
\]

and it is given by [11] relation (5.37)]

\[
T_2 = \frac{\delta \left( \sqrt{1 - \eta^2} + 1 \right)}{(\mu_1 - N)\sqrt{1 - \eta^2}} \tag{15}
\]

where

\[
\eta = \frac{\mu_1 - N}{\mu_2} < \frac{1}{\beta}. \tag{16}
\]

Relations (15) and (16), coupled together, ensure [11] cf. relation (5.40)] that
\[ T_2 = \frac{\delta \left( \sqrt{1 - \eta^2} + 1 \right)}{(\mu_1 - N) \sqrt{1 - \eta^2}} \leq \frac{2\delta}{(\mu_1 - N) \sqrt{1 - \beta^2}}. \]  
(17)

Employing (14) and taking into account that under condition (7), imposed on \( \mu_1 \), one has

\[ \frac{2\delta}{(\mu_1 - N) \sqrt{1 - \beta^2}} \leq T_s, \]  
(18)

it follows that

\[ T_2 \leq T_s. \]  
(19)

Hence, the convergence time \( T_2 \) of the closed-loop system (1)–(2), initialized on the domain boundary (9), is smaller than the prescribed time instant \( T_s \). The proof of Theorem 1 is thus completed.

4. Supporting simulations

Capabilities of the tuning of the twisting controller gains, resulting from Theorem 1, are illustrated in the numerical study of the fixed-time regulation problem of a simple pendulum, initialized in a prespecified domain. Consider a simple pendulum governed by

\[ \ddot{q} = b [\tau(q, \dot{q}, t) - f_v \dot{q} - mgl \sin(q) + d(t)] \]  
(20)

where \( b = 1/(m_p l^2 + J) \) and \( q \) is the angular position. Hereinafter, \( m_p \) is the mass of the pendulum, \( l \) is the distance from its rotation axis to its center of mass, \( J \) is the moment of inertia of the pendulum with respect to its center of mass, \( f_v \) is the viscous friction coefficient, \( g \) is the gravity acceleration, \( \tau \in \mathbb{R} \) is the torque produced by the actuator, the unknown term \( d(t) \) stands to account for external uniformly bounded disturbances. The parameters of the pendulum (20) are given in Table 1. The pendulum model (20), represented in terms of
Table 1: Parametric values of the pendulum driven by actuator

| parameter | value    | unit          |
|-----------|----------|---------------|
| m         | 0.0474   | kg            |
| l         | 0.11     | m             |
| J         | 3.11×10^{-3} | kg m^2       |
| g         | 9.81     | m/s^2         |
| f_v       | 2.43×10^{-4} | N s/rad |

the state errors $x_1(t) = q(t) - r$, $x_2(t) = \dot{q}(t)$, takes the form

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= b [\tau - f_v x_2 - mgl \sin(x_1 + r)] + \omega,
\end{align*}
\]

where $x = (x_1, x_2)^T \in \mathbb{R}^2$, $r \in \mathbb{R}$ is the desired position for the pendulum such that the initial error state vector $x^0 = (x_1^0, x_2^0) \in \mathbb{R}^2$ with $x_1^0 = q(t_0) - r$ and $x_2^0 = \dot{q}(t_0)$ satisfies $x^0 \in \Gamma_R$ and

\[
\omega(t) = bd(t).
\]

Setting

\[
\tau = \frac{1}{b} u + f_v x_2 + mgl \sin(x_1 + r),
\]

the control input [23] is then composed of the friction-gravitation compensator $f_v x_2 + mgl \sin(x_1 + r)$ and the twisting algorithm $u(x_1, x_2)$ to be designed as in [2]. Substituting [23] in [21] for $\tau$ yields

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= u(x_1, x_2) + \omega(t).
\end{align*}
\]

The resulting system [24] is consistent with the underlying system [1] provided that $\omega(t)$ is a uniformly bounded disturbance entering the system.
The control objective is that the closed–loop trajectories \( (20), (23) \) reach the origin of the state space in a prescribed time \( T_s \) for any initial error \( x^0 \in \Gamma_R \), regardless of whichever uniformly bounded disturbance \( \omega(t) \) affects the system.

The simulations are carried out with several initial conditions for the pendulum as \( q(0) = \{0, \frac{0.9R}{\mu_2}\} \) and \( \dot{q}(0) = \{0.8\sqrt{2R}, 0\} \). With this initial conditions we guaranties that \( x^0 \in \Gamma_R \) in both cases. The desired value for the reference \( r \) is set to zero, i.e., \( r = 0 \). The unknown function \( d(t) \) in (22) is modeled as

\[
d(t) = A \sin(wt)
\]  

where \( A = 7 \times 10^{-4} \) and \( w = 2 \text{ rad/s} \), thus the disturbance \( \omega(t) \) given in (22) is uniformly bounded as \( |\omega(t)| \leq 0.19 \).

The initial domain \( \Gamma_R \) in (4) is specified with \( R = 2 \), and the prescribed settling time is set to \( T_s = 1 \text{ s} \). Taking \( N = 0.2 \), and using the Theorem 1, the tuning variables (6) are selected as

\[
R = 2, \quad \beta = 5, \\
\rho = 0.5, \quad \delta = 3.1.
\]  

Then, using (7)–(8) together with (26), the twisting algorithm parameters are set to

\[
\mu_1 = 6.63, \quad \mu_2 = 33.24.
\]

As it can be seen in the Figure 2, the closed–loop system \( (2), (20), (23) \) escape to the origin \( (x_1, x_2) = 0 \) in a prescribed–time \( t \leq T_s \) irrespective of the initial conditions \( x^0 \in \Gamma_R \) and the uniformly bounded disturbances entering the system. It can additionally be concluded from the phase portrait of Figure 3 that once in \( \Gamma_R \), the trajectories of the closed-loop system \( (2), (20), (23) \) never leave the level set \( \Gamma_R \) of the Lyapunov function (5).
Figure 2: Simulation results: time responses of the pendulum position error, velocity error, and control input.
5. Conclusions

In the present note, a tuning procedure is formalized for the fixed time stabilization of the perturbed double integrator, initialized on an a priori given domain and driven by the twisting algorithm. The resulting tuning procedure is supported by numerical simulations enriching that of [1].

References

[1] H. B. Oza, Y. V. Orlov, S. K. Spurgeon, Lyapunov-based settling time estimate and tuning for twisting controller, IMA Journal of Mathematical Control and information 29 (4) (2012) 471–490.