CONVEX ANALYTIC METHOD REVISITED: FURTHER OPTIMALITY RESULTS AND PERFORMANCE OF DETERMINISTIC CONTROL POLICIES*

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Abstract. The convex analytic method (generalized by Borkar) has proved to be a very versatile method for the study of infinite horizon average cost optimal stochastic control problems. In this paper, we revisit the convex analytic method and make three primary contributions: (i) We present an existence result for controlled Markov models that lack weak continuity of the transition kernel but are strongly continuous in the action variable for every fixed state variable. (ii) For average cost stochastic control problems in standard Borel spaces, while existing results establish the optimality of stationary (possibly randomized) policies, few results are available on the optimality of deterministic policies, and these are under rather restrictive hypotheses. We provide mild conditions under which an average cost optimal stochastic control problem admits optimal solutions that are deterministic and Markov, building upon a study of strategic measures. We also review existing results establishing the optimality of stationary deterministic policies. (iii) We establish conditions under which the performance under stationary deterministic policies is dense in the set of performance values under randomized stationary policies.

Key words. ergodic control, optimality of deterministic policies

AMS subject classifications. 90C40, 93E20

1. Introduction. We start by reviewing the usual model in the literature for controlled Markov chains, otherwise referred to as Markov decision processes (MDPs). In general, for a topological space $\mathcal{X}$, we denote by $\mathcal{B}(\mathcal{X})$ its Borel $\sigma$-field and by $\mathcal{P}(\mathcal{X})$ the set of probability measures on $\mathcal{B}(\mathcal{X})$.

A controlled Markov chain consists of the tuple $(\mathcal{X}, \mathcal{U}, \mathcal{T}, c)$, whose elements can be described as follows.

(a) The state space $\mathcal{X}$ and the action or control space $\mathcal{U}$ are Borel subsets of complete, separable, metric (i.e., Polish) spaces.

(b) The map $\mathcal{U}: \mathcal{X} \rightarrow \mathcal{B}(\mathcal{U})$ is a strict, measurable multifunction. The set of admissible state/action pairs is

$$ \mathcal{K} := \{(x, u) : x \in \mathcal{X}, u \in \mathcal{U}(x)\}, $$

endowed with the subspace topology corresponding to $\mathcal{B}(\mathcal{X} \times \mathcal{U})$.

(c) The map $\mathcal{T}: \mathcal{K} \rightarrow \mathcal{P}(\mathcal{X})$ is a stochastic kernel on $\mathcal{K} \times \mathcal{B}(\mathcal{X})$, that is, $\mathcal{T}(\cdot | x, u)$ is a probability measure on $\mathcal{B}(\mathcal{X})$ for each $(x, u) \in \mathcal{K}$, and $(x, u) \mapsto \mathcal{T}(A | x, u)$ is measurable for each $A \in \mathcal{B}(\mathcal{X})$.

(d) The map $c: \mathcal{K} \rightarrow \mathbb{R}_+$ is measurable, and is called the running cost or one stage cost. We assume that it is bounded from below in $\mathcal{K}$, so without loss of generality, it takes values in $[1, \infty)$.

The (admissible) history spaces are defined as

$$ \mathbb{H}_0 := \mathcal{X}, \quad \mathbb{H}_t := \mathbb{H}_{t-1} \times \mathcal{K}, \quad t \in \mathbb{N}, $$

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and the canonical sample space is defined as \( \Omega := (X \times U)^\infty \). These spaces are endowed with their respective product topologies and are therefore Borel spaces. The state, action (or control), and information processes, denoted by \( \{X_t\}_{t \in \mathbb{N}_0}, \{U_t\}_{t \in \mathbb{N}_0} \) and \( \{H_t\}_{t \in \mathbb{N}_0} \), respectively, are defined by the projections

\[
X_t(\omega) := x_t, \quad U_t(\omega) := u_t, \quad H_t(\omega) := (x_0, \ldots, u_{t-1}, x_t)
\]

for each \( \omega = (x_0, \ldots, u_{t-1}, x_t, u_t, \ldots) \in \Omega \). An admissible control policy, or policy, is a sequence \( \gamma = \{\gamma_t\}_{t \in \mathbb{N}_0} \) of stochastic kernels on \( \mathbb{H}_t \times \mathcal{B}(\mathbb{U}) \) satisfying the constraint

\[
\gamma_t(U(X_t) \mid h_t) = 1, \quad h_t \in \mathbb{H}_t.
\]

The set of all admissible policies is denoted by \( \Gamma_A \). It is well known (see [41, Prop. V.1.1, pp. 162–164]) that for any given \( \nu \in \mathcal{P}(X) \) and \( \gamma \in \Gamma_A \) there exists a unique probability measure \( \mathbb{P}_\nu^\gamma \) on \( (\Omega, \mathcal{B}(\Omega)) \) satisfying

\[
\mathbb{P}_\nu^\gamma(X_0 \in D) = \nu(D) \quad \forall D \in \mathcal{B}(X),
\]

\[
\mathbb{P}_\nu^\gamma(U_t \in C \mid H_t) = \gamma_t(C \mid H_t) \quad \mathbb{P}_\nu^\gamma\text{-a.s.}, \quad \forall C \in \mathcal{B}(U),
\]

\[
\mathbb{P}_\nu^\gamma(X_{t+1} \in D \mid H_t, U_t) = T(D \mid X_t, U_t) \quad \mathbb{P}_\nu^\gamma\text{-a.s.}, \quad \forall D \in \mathcal{B}(X).
\]

The expectation operator corresponding to \( \mathbb{P}_\nu^\gamma \) is denoted by \( \mathbb{E}_\nu^\gamma \). If \( \nu \) is a Dirac mass at \( x \in X \), we simply write these as \( \mathbb{P}_x^\gamma \) and \( \mathbb{E}_x^\gamma \).

A policy \( \gamma \) is called Markov if there exists a sequence of measurable maps \( \{v_t\}_{t \in \mathbb{N}_0} \), where \( v_t : X \to \mathcal{P}(U) \) for each \( t \in \mathbb{N}_0 \), such that

\[
\gamma_t(\cdot \mid H_t) = v_t(X_t)(\cdot) \quad \mathbb{P}_\nu^\gamma\text{-a.s.}
\]

With some abuse in notation, such a policy is identified with the sequence \( v = \{v_t\}_{t \in \mathbb{N}_0} \). Note then that \( \gamma_t \) may be written as a stochastic kernel \( \gamma_t(\cdot \mid x) \) on \( X \times \mathcal{B}(U) \) which satisfies \( \gamma_t(U(x) \mid x) = 1 \). Let \( \Gamma_M \) denote the set of all Markov policies.

We say that a Markov policy \( \gamma \) is deterministic, or simple, if \( \gamma_t \) is a Dirac mass, in which case \( \gamma_t \) is identified with a Borel measurable function \( \gamma_t : X \to U \). In other words, \( \gamma_t \) is a measurable selector from the set-valued map \( U(x) \) [20]. We let \( \Gamma_{MD} \) denote the set of deterministic Markov policies.

We add the adjective stationary to indicate that the Markov policy does not depend on \( t \in \mathbb{N}_0 \), that is, \( \gamma_t = \gamma \) for all \( t \in \mathbb{N}_0 \). We let \( \Gamma_S \) denote the class of stationary Markov policies, henceforth referred to simply as stationary policies, and let \( \Gamma_{SD} \subset \Gamma_S \) denote the subset of those that are deterministic.

In summary, under a policy \( \gamma \in \Gamma_S \), the process satisfies the following: for all Borel sets \( B \in \mathcal{B}(X) \), \( t \geq 0 \), and (\( \mathbb{P}_\nu^\gamma \) almost all) realizations \( X_{[0,t]}, U_{[0,t]} \), we have

\[
\mathbb{P}_\nu^\gamma(X_{t+1} \in B \mid X_{[0,t]} = x_{[0,t]}, U_{[0,t]} = u_{[0,t]}) = \mathbb{P}_\nu^\gamma(X_{t+1} \in B \mid X_t = x_t, U_t = u_t) := T(B \mid x_t, u_t).
\]

Using stochastic realization results (see [21, Lemma 1.2], or [11, Lemma 3.1]), stochastic processes that satisfy (1.1) admit a realization in the form

\[
X_{t+1} = f(X_t, U_t, W_t)
\]

almost surely, where \( f \) is measurable and \( w_t \) is i.i.d. \([0,1]–valued. Since a system of the form (1.2) satisfies (1.1), it follows that the representations in these equations are equivalent.
In this paper, we consider the problem of minimizing the average cost
\begin{equation}
J^*(x) := \inf_{\gamma \in \Gamma_x} J(x, \gamma) = \inf_{\gamma \in \Gamma_x} \limsup_{T \to \infty} \frac{1}{T} \mathbb{E}_x^\gamma \left[ \sum_{t=0}^{T-1} c(X_t, U_t) \right].
\end{equation}
We say that a policy $\gamma \in \Gamma_x$ is optimal if it attains the infimum in (1.3).

This is an important problem in applications where one is concerned about the long-term behaviour, unlike the discounted cost setup where the primary interest is in the short-term time stages.

For the study of the average cost problem, there are three commonly adopted approaches [3]: contraction or value iteration based methods (see e.g. [52, 25, 2]), the vanishing discount method (see e.g. [3, 19, 27, 28, 13, 22, 54] which have various conditions and relaxations), and the convex analytic method (to be reviewed further below).

The first two are based on the arrival at what is known as the average cost optimality equation (ACOE) (and its variation involving an inequality (ACOI)). Efforts under this method typically require some some recurrence/ergodicity/Dobrushin type geometric or at least subgeometric convergence conditions, which may be too strong for a large class of applications (e.g., for belief-MDP reduction of Partially Observable Markov Decision Processes).

The third approach, via the convex analytic method, is based on the properties of expected (or sample path) occupation measures and their limit behaviours, leading to a linear program involving the space of probability measures. The convex analytic approach, typically attributed to Manne [37] and Borkar [12] (see additionally [33, 24, 3, 27, 55]), is a versatile approach to the optimization of infinite-horizon problems, which leads to a linear program. This approach is particularly effective for constrained optimization problems and infinite horizon average cost optimization problems. It avoids the use of dynamic programming and can also be tailored towards obtaining results on sample-path optimality via martingale convergence theorems under mild continuity conditions [51, 36, 1, 55]. Most importantly perhaps, this approach generally requires less restrictive conditions on the existence of an optimal policy for average cost stochastic control.

These approaches are related through a duality analysis, as noted in [27, Chapter 6] (see also [24] for a direct argument under positive Harris recurrence assumptions). However the more general conditions leading to solutions under these approaches are not identical, therefore, the corresponding conditions of existence and structural results for optimal policies are somewhat different. That is, going from one approach to another one (e.g., from the convex analytic solution to an ACOE) still entails open problems.

For MDPs with weakly or strongly continuous transition kernels, if ACOE/ACOI can be established (under somewhat strong conditions as reviewed above), the existence of deterministic stationary optimal policies naturally follows. However, with the convex analytic approach, whether an optimal policy, which can be assumed to be stationary, can also be assumed to be deterministic is an incomplete problem: While the convex analytic method typically provides less conservative conditions for existence of optimal policies, whether the optimal policy can be taken to be deterministic and stationary is generally an open question with only few results reported in the literature. This question is a further primary motivation for this paper.

Contributions.

(i) In Theorem 2.6, we present an existence result for average cost controlled Markov models that are strongly continuous in the action for every fixed state.
variable. Prior results on the convex analytic method (in particular due to Borkar [12] and nearly all the papers cited above [33, 24, 3, 27, 51, 36, 1]) have assumed weak continuity of the kernel in both the state and action variables. Related to this contribution, recently H. Yu [55] established the existence of an optimal solution for countable action and Borel state spaces through majorization conditions via Lusin’s theorem. A careful study of the topology of $w$-$s$ convergence, which our existence analysis builds upon in this paper, reveals that Lusin’s theory is what establishes the connections between weak topology and the $w$-$s$ topology via majorization conditions. Accordingly, in this paper the direct use of $w$-$s$ topology makes the analysis here more direct and concise, and as opposed to the countable action space case (which makes functions continuous in the actions) in [55], here we consider general action spaces.

(ii) In Theorem 3.6, we provide conditions under which the solution to an optimal average cost stochastic control problem is a deterministic Markov policy. To our knowledge, there exists only two main such results employing the convex analytic method, which as noted above generally require more relaxed conditions compared with approaches directly utilizing the ACOE/ACOI. The first one is by Meyn [38, Proposition 9.2.5] and Borkar [12, Lemma 2.4] for the countable probability space setup, and the second one due to Borkar [12, Section 3.2] for the continuous space setup, with the latter under quite restrictive conditions needed for applying Schauder’s fixed point theorem. We also note that via a direct relationship between average cost optimality and ACOI and utilizing Blackwell [7, 9]; [24, Corollary 5.4(b)] establishes the optimality of stationary and deterministic policies under a positive Harris recurrence assumption (see Section 3.3).

(iii) Finally, in Theorem 4.2, we establish conditions for not only the optimality, but also the denseness of the attained performance values under deterministic stationary policies in those attained under randomized stationary policies. In other words, we show that, under mild conditions, the cost under any randomized stationary policy can be approximated arbitrarily closely by the cost under some deterministic stationary policy.

2. The Convex Analytic Approach and a Refined Existence Result on the Optimality of Stationary (Possibly Randomized) Policies. Recall that we are interested in the minimization

\begin{equation}
\inf_{\gamma \in \Gamma_S} \limsup_{T \to \infty} \frac{1}{T} E_{x_0}^{\gamma} \left[ \sum_{t=1}^{T} c(X_t, U_t) \right],
\end{equation}

where $E_{x_0}^{\gamma}$ denotes the expectation over all sample paths with initial state given by $x_0$ under the admissible policy $\gamma$.

We refer the reader to [44] for an example where an optimal policy may not be stationary under an average cost optimality criterion even for countable state/action spaces. Therefore, the conditions presented in the following are not superfluous.

2.1. Some definitions. We summarize here some definitions which we use frequently in the paper.

For $\gamma \in \Gamma_S$, we let

\begin{equation}
T^{\gamma}(A \mid x) := \int_{\mu(x)} T(A \mid x, u) \gamma(du \mid x).
\end{equation}
We let \( \mathcal{M}_b(X) \) denote the space of bounded Borel measurable (continuous) real-valued functions on \( X \). For \( \mu \in \mathcal{P}(\mathbb{R}) \) and \( f \in \mathcal{M}_b(X) \), we define \( \mu T \in \mathcal{P}(\mathbb{R}) \) and \( Tf: \mathbb{R} \to \mathbb{R} \) by

\[
\mu T(A) := \int_{X} \mu(dx, du) T(A | x, u), \quad A \in \mathcal{B}(X),
\]

and

\[
Tf(x, u) := \int_{X} f(y) T(dy | x, u), \quad (x, u) \in \mathbb{R},
\]

respectively.

We use the convenient notation for integrals of functions

\[
\mu(f) := \int_{X} f(x, u) \mu(dx, du),
\]

and similarly for \( f \in \mathcal{M}_b(X) \) and \( \mu \in \mathcal{P}(\mathbb{R}) \) if no ambiguity arises. Clearly then, we have

\[
\langle \mu T, f \rangle = \langle \mu, Tf \rangle \quad \text{for} \quad \mu \in \mathcal{P}(\mathbb{R}), \ f \in \mathcal{M}_b(X).
\]

The set of invariant occupation measures (or, as is used more commonly in the literature: ergodic occupation measures\(^1\)) is defined by

\[
\mathcal{G} := \left\{ \mu \in \mathcal{P}(\mathbb{R}) : \mu(B \times U) = \mu T(B), \ B \in \mathcal{B}(X) \right\}.
\]

We also let

\[
\mathcal{H} := \left\{ \pi \in \mathcal{P}(X) : \exists \gamma \in \Gamma_S \text{ such that } \pi(A) = \int_{X} T^\gamma(A | x) \pi(dx), \ A \in \mathcal{B}(X) \right\}
\]

denote the set of invariant probability measures of the controlled Markov chain.

Let \( \mu \in \mathcal{G} \). It is well known that \( \mu \) can be disintegrated into a stochastic kernel \( \phi \) on \( X \times \mathcal{B}(U) \) and \( \pi \in \mathcal{P}(X) \) such that

\[
\mu(dx, du) = \phi(du | x) \pi(dx),
\]

and \( \phi \) is \( \pi \)-a.e. uniquely defined on the support of \( \pi \). We denote this disintegration by \( \mu = \phi \otimes \pi \). Therefore, if \( \gamma \in \Gamma_S \) is any policy which agrees \( \pi \)-a.e. with \( \phi \), then we have \( \pi(A) = T^\gamma(A | x) \pi(dx) \) for \( A \in \mathcal{B}(X) \). Therefore, \( \pi \in \mathcal{H} \). Conversely, if \( \pi \in \mathcal{H} \) with an associated \( \gamma \in \Gamma_S \), then it is clear from the definitions that \( \gamma \otimes \pi \in \mathcal{G} \).

Define

\[
\delta^* := \inf_{\mu \in \mathcal{G}} \langle \mu, c \rangle.
\]

A measure \( \mu \in \mathcal{G} \) for which the infimum is attained is called optimal. subsections 2.2 and 2.3 concern the existence of optimal invariant occupation measures.

\(^1\)It is perhaps more appropriate to use the term invariant occupation measures, instead of ergodic occupation measures since clearly the measures in \( \mathcal{G} \) are not all ergodic: we say that an invariant measure \( \mu \) is ergodic if the support of \( \mu \) does not contain two disjoint absorbing sets. However, traditionally the latter term has been used in the literature, see e.g. \cite{3}.
2.2. Review: optimality under weakly continuous kernels. We first review the general proof method of some existing results, due to [3, 12, 33, 24, 3, 27], on the existence of an optimal $\mu \in \mathcal{G}$ under the hypothesis that the transition kernel $T$ is weakly continuous. This property is defined as follows.

(H1) The transition kernel $T$ is called weakly continuous if the map

\[ K \ni (x, u) \mapsto \int_{\mathcal{X}} f(z) T(dz \mid x, u) \]

is continuous for all $f \in C_b(\mathcal{X})$.

Continuing, for $T \geq 1$, we let

\[ v_T(D) = \frac{1}{T} \sum_{t=0}^{T-1} 1_D(X_t, U_t), \quad D \in \mathcal{B}(\mathcal{X} \times \mathcal{U}). \]

Consider any policy $\gamma$ in $\Gamma_A$, $X_0 \sim \nu$, and let for $T \geq 1$,

\[ \mu_T^\gamma(D) = \mathbb{E}_\nu^\gamma [v_T(D)] = \frac{1}{T} \mathbb{E}_\nu^\gamma \left[ \sum_{t=0}^{T-1} 1_D(X_t, U_t) \right], \quad D \in \mathcal{B}(\mathcal{X} \times \mathcal{U}). \]

We refer to $\{\mu_T^\gamma\}_{T>0}$ as the family of mean empirical occupation measures under the policy $\gamma \in \Gamma_A$, and with initial distribution $\nu$. Through what is often referred to as a Krylov-Bogoliubov-type argument, for every $A \in \mathcal{B}(\mathcal{X})$, we have

\[ |\mu_T^\gamma(A \times \mathcal{U}) - \mu_T^\gamma T(A)| = \frac{1}{T} \left| \mathbb{E}_\nu^\gamma \left[ \sum_{t=0}^{T-1} 1_{A \times \mathcal{U}}(X_t, U_t) - \sum_{t=1}^{T} 1_{A \times \mathcal{U}}(X_t, U_t) \right] \right| \leq \frac{1}{T} \to 0 \quad \text{as } T \to \infty. \]

Observe that (2.6) holds for any policy $\gamma \in \Gamma_A$.

Suppose that, along some subsequence $\{t_k\} \subset \mathbb{N}$, $\mu_{t_k}^\gamma$ converges weakly to some $\mu \in \mathcal{P}(K)$, which we denote as $\mu_{t_k}^\gamma \Rightarrow \mu$. Using (2.5), we write the triangle inequality

\[ |\mu(f) - \mu T(f)| \leq |\mu(f) - \mu_{t_k}^\gamma(f)| + |\mu_{t_k}^\gamma(f) - \mu_{t_k}^\gamma T(f)| + |\mu_{t_k}^\gamma T(f) - \mu T(f)| \]

for $f \in C_b(\mathcal{X})$. This notation is consistent since $f$ may be viewed also as an element of $C_b(\mathcal{X})$. Suppose that (H1) holds. The first term on the right hand side of (2.7) vanishes as $k \to \infty$ by weak convergence, while the second term does the same by (2.6).

Since

\[ \mu_{t_k}^\gamma T(f) = \mu_{t_k}^\gamma (T f), \]

and $T f \in C_b(\mathcal{X})$ by (H1), it follows that the third term also vanishes as $k \to \infty$ by the weak convergence $\mu_{t_k}^\gamma \Rightarrow \mu$. Since the class $C_b(\mathcal{X})$ distinguishes points in $\mathcal{P}(\mathcal{X})$, this shows that $\mu(A, \mathcal{U}) = \mu T(A)$ for all $A \in \mathcal{B}(\mathcal{X})$, which implies that $\mu \in \mathcal{G}$ by the definition of the latter. Thus we have shown the following.

**Lemma 2.1.** Under (H1), the limit of any weakly converging subsequence of mean empirical occupation measures is in $\mathcal{G}$. 


Recall (1.3). This expected cost can be equivalently written as

\[ J(x, \gamma) := \limsup_{T \to \infty} \langle \mu^T, c \rangle, \]

where \( \mu^T \) is the mean empirical occupation measure under \( \gamma \). Let \( \{t_k\} \subset \mathbb{N} \) be a subsequence along which \( \langle \mu_{t_k}^T, c \rangle \) converges to \( J(x, \gamma) \) and suppose that \( \mu_{t_k} \Rightarrow \mu \in \mathcal{G} \). Then

\[ J(x, \gamma) = \liminf_{t_k \to \infty} \langle \mu_{t_k}^T, c \rangle \geq \langle \mu, c \rangle = \delta^*, \]

where for the first inequality we use the fact that, since \( c \) is lower semi-continuous (l.s.c.) and bounded from below, the map \( \mu \rightarrow \langle \mu, c \rangle \) is lower semi-continuous. The above shows that \( J^*(x) \geq \delta^* \). We now establish conditions for which the above is indeed an equality.

**Assumption 2.2.**

(A) The state and action spaces \( X \) and \( U \) are Polish. The set-valued map \( U : X \rightarrow B(U) \) is upper semi-continuous and closed-valued.

(A') The state and action spaces \( X \) and \( U \) are compact. The set-valued map \( U : X \rightarrow B(U) \) is upper semi-continuous and closed-valued.

(B) The non-negative running cost function \( c(x, u) \) is l.s.c. and \( c : K \rightarrow \mathbb{R} \) is inf-compact, i.e. \( \{(x, u) \in K : c(x, u) \leq \alpha\} \) is compact for every \( \alpha \in \mathbb{R}_+ \).

(B') The cost function \( c \) is bounded and l.s.c.

(C) There exists a policy and an initial state leading to a finite cost \( \eta \in \mathbb{R}_+ \).

(D) \((H1)\) holds.

(E) Under every stationary policy, the induced Markov chain is Harris recurrent.

Before we present a theorem, we now review the following concerning ergodic properties of (control-free) Markov chains: Let \( c \in L_1(\mu) := \{ f : X \rightarrow \mathbb{R}, \int |f(x)| \mu(dx) < \infty \} \). Suppose that \( \mu \) is an invariant ergodic probability measure for an \( X \)-valued Markov chain. Then, it follows that for \( \mu \) almost everywhere \( x \in X \):

\[ \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} c(X_t) = \int c(x) \mu(dx), \]

\( P_x \) almost surely (that is conditioned on \( x_0 = x \), with probability one, the above holds). Furthermore, again with \( c \in L_1(\mu) \), for \( \mu \) almost everywhere \( x \in X \)

\[ \lim_{T \to \infty} \frac{1}{T} E_x \left[ \sum_{t=1}^{T} c(X_t) \right] = \int c(x) \mu(dx), \]

On the other hand, the positive Harris recurrence property allows the almost sure convergence to take place for every initial condition: If \( \mu \) is the invariant probability measure for a positive Harris recurrent Markov chain, it follows that for all \( x \in X \) and for every \( c \in L_1(\mu) \) [39, Theorem 17.1.7] or [29, Theorem 4.2.13]

\[ \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} c(X_t) = \int c(x) \mu(dx), \]

almost surely. However, for every \( c \in L_1(\mu) \), while (2.11) holds for all \( x \in X \), it is not generally true that [39, Chapter 14] (see e.g. [55, Example 3.1]) that

\[ \lim_{T \to \infty} \frac{1}{T} E_x [\sum_{t=1}^{T} c(X_t)] = \int c(x) \mu(dx), \]

where for the first inequality we use the fact that, since \( c \) is lower semi-continuous (l.s.c.) and bounded from below, the map \( \mu \rightarrow \langle \mu, c \rangle \) is lower semi-continuous. The above shows that \( J^*(x) \geq \delta^* \). We now establish conditions for which the above is indeed an equality.

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\[ \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} c(X_t) = \int c(x) \mu(dx), \]

\( P_x \) almost surely (that is conditioned on \( x_0 = x \), with probability one, the above holds). Furthermore, again with \( c \in L_1(\mu) \), for \( \mu \) almost everywhere \( x \in X \)

\[ \lim_{T \to \infty} \frac{1}{T} E_x \left[ \sum_{t=1}^{T} c(X_t) \right] = \int c(x) \mu(dx), \]

On the other hand, the positive Harris recurrence property allows the almost sure convergence to take place for every initial condition: If \( \mu \) is the invariant probability measure for a positive Harris recurrent Markov chain, it follows that for all \( x \in X \) and for every \( c \in L_1(\mu) \) [39, Theorem 17.1.7] or [29, Theorem 4.2.13]

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\[ \lim_{T \to \infty} \frac{1}{T} E_x [\sum_{t=1}^{T} c(X_t)] = \int c(x) \mu(dx), \]
(2.12) \[ \lim_{T \to \infty} \frac{1}{T} E_x \left[ \sum_{i=1}^{T} c(X_t) \right] = \int c(x) \mu(dx) \]

This follows as a consequence of Fatou’s lemma and (2.11). Further refinements are possible via return properties to small sets and f-regularity of cost functions [2, 39] (e.g. this convergence holds if [39, Theorem 14.0.1] holds and \( X_0 = x \) with \( x \in \{ z : V(z) < \infty \} \)). We refer the reader to [39, Chapters 14 and 17] or [29, Chapters 2 and 4] for additional discussions.

**Theorem 2.3.** a) Under Assumption 2.2 (A, B, C, D) there exists an optimal measure in \( \mathcal{G} \). b) Under Assumption 2.2 (A’, B’, D, E), there exists a policy in \( \Gamma_S \) which is optimal for the control problem given in (2.1) for every initial condition.

**Proof.** a) Consider Assumption 2.2 (A, B, C, D). By (B, C) we have that the set of policies \( \gamma \) which lead to a finite cost is so that \( \langle \mu_T^\gamma, c \rangle < \infty \) for all \( T \), which implies that \( \{ \mu_T^\gamma, T > 0 \} \) is tight. Thus along some subsequence \( \mu_k \to \mu \in \mathcal{P} (\mathcal{K}) \). As shown in the paragraph preceding Lemma 2.1, \( \mu \in \mathcal{G} \).

Furthermore, under hypothesis (A), the set \( \mathcal{K} = \cup_{x \in \mathcal{X}} \{ (x, u), u \in \mathcal{U}(x) \} \) is closed by [27, Lemma D.3]. Thus, by the Portmanteau theorem that every weak limit of a converging sequence of probability measures on \( \mathcal{K} \) is also supported on \( \mathcal{K} \).

Consider a sequence \( \{ \mu_k \}_{k \in \mathbb{N}} \subset \mathcal{G} \) such that \( \langle \mu_k^\gamma, c \rangle \to \delta^* \) as \( k \to \infty \), the sequence \( \mu_k \) is tight by inf-compactness, and any limit point \( \mu_* \) of this sequence is in \( \mathcal{G} \) with \( \mu_*(\mathcal{K}) = 1 \). Thus, by [27, Prop. D.8] we have an optimal control policy \( \phi \). Taking limits as in (2.9), we obtain \( \langle \mu_*, c \rangle = \delta^* \). This establishes the first part of the theorem.

Define a stationary policy \( \gamma \) via the disintegration

\[ \mu_*(dx, du) = \gamma_*(du | x) \pi_* (dx) \]

\( \mu_* \) almost surely. Note that via this disintegration the control \( \gamma_* \) is defined \( \pi_* \)-a.e. Let \( \phi \in \Gamma_S \) be any policy that agrees with \( \gamma_* \) on the support of \( \pi_* \).

b) Under (A’, B’, D), via (2.8) and that \( Tf \in C_b(\mathcal{K}) \) by (H1), we have that \( \mathcal{G} \) is compact; we also have that Portmanteau theorem applies as in part a). By hypothesis (E), since the chain under an optimal \( \phi \), is Harris recurrent, \( \pi_* \) is its unique invariant probability measure. Optimality of \( \phi \), for every initial condition, then follows by positive Harris recurrence given that \( c \) is bounded via (2.12) under hypothesis (B’). Thus, \( J(x, \phi) = \langle \mu_*, c \rangle \) and \( \mu_* \) is optimal.

**Theorem 2.3** can be stated under weaker assumptions. See, for example, [2, Theorem 2.1] among other references in the literature. We have chosen to state it under somewhat stronger hypotheses in order to present a simple and short proof that conveys the essential arguments.

In general, in the absence of Assumption 2.2 (E), there is a consideration of reachability. Suppose that the chain under the policy \( \phi \) as defined in the proof of Theorem 2.3 is a \( T \) model (see [49]). Then, as asserted in [49, Theorem 6.1], the Doeblin decomposition of the state space contains, in general, a countable collection of maximal Harris sets. In particular, we have a decomposition into the disjoint union \( \mathcal{K} = \bigcup_{i \in \mathbb{N}} H_i \cup E \), where each \( H_i \) is a maximal Harris set with invariant measure \( \pi_i \), and \( E \) is transient. Now, by part (ii) of Theorem 6.1 in [49], only a finite number of the sets \( H_i \) may have a nonempty intersection with any given compact set. This
implies that $\pi_*$ can always be expressed as a convex combination of finitely many ergodic invariant measures. Thus, if the Markov Chain is not recurrent, the stationary policy defined above, in general, is only optimal in a restricted set of initial conditions. On implications related to insensitivity to such initial state dependence, the reader is referred to [36] and [28, Prop. 11.4.4(c) and Lemma 11.4.5(a)], among other references, for further results on sample path average cost optimality and expected average cost optimality.

2.3. New conditions: optimality under setwise convergence and strong continuity in actions for each state. There are many important applications where the kernel $T$ is not weakly continuous. For example, consider dynamics described by a stochastic difference equation on $\mathbb{R}^d$ of the form

$$X_{n+1} = F(X_n, U_n) + W_n, \quad n = 0, 1, 2, \ldots,$$

where $X = \mathbb{R}^n$ and the $W_n$’s are independent and identically distributed (i.i.d.) random vectors whose distribution has a bounded and continuous density function. We assume that $F$ is bounded and $u \mapsto F(x, u)$ is continuous for all $x \in X$. It is clear that the transition kernel $T$ is not, in general, weakly continuous. However, it satisfies the following hypothesis.

(H2) The transition kernel $T$ satisfies the following:

(a) For any $x \in X$, the map $u \mapsto \int f(z)T(dz | x, u)$ is continuous for every bounded measurable function $f$.

(b) There exists a finite measure $\nu$ majorizing $T$, that is

$$T(dy | x, u) \leq \nu(dy), \quad x \in X, \ u \in U.$$

If in addition, the distribution of $W_n$ has a continuous, bounded, and a strictly positive probability density function (a non-degenerate Gaussian distribution satisfies this condition), then positive Harris recurrence can be established by Lebesgue-irreducibility and a uniform countable additivity condition for compact sets following [50, Condition A], which leads to the presence of accessible compact petite sets (where one can take $V(x) = x^2$ as the Lyapunov function). For more details see [45, Example 3.1].

Assumption 2.4. The following hold:

(A) The state and action spaces $X$ and $U$ are Polish. The set $K = \bigcup_{x \in X} \{(x, u), u \in U(x)\}$ is measurable (see [27, Lemma D.3] for conditions) and the set-valued map $U: X \to \mathcal{B}(U)$ is compact-valued.

(A') The state and action spaces $X$ and $U$ are compact. The set $K$ is measurable and set-valued map $U: X \to \mathcal{B}(U)$ is compact-valued.

(B) The non-negative running cost function $c(x, u)$ is continuous in $u \in U(x)$ for every $x \in X$ and $c: K \to \mathbb{R}$ is inf-compact.

(B') The cost function $c$ is bounded, and continuous in $u \in U(x)$ for every $x \in X$.

(C) There exists a policy and an initial state leading to a finite cost $\eta \in \mathbb{R}_+^d$.

(D) (H2) holds.

(E) Under every stationary policy, the induced Markov chain is Harris recurrent.

Let us recall the $w$-$s$ topology studied by Schäi [48] (see Balder [5] for further properties).

Definition 2.5. The $w$-$s$ topology on the set of probability measures $\mathcal{P}(X \times U)$ is the coarsest topology under which $\int f \, d\nu: \mathcal{P}(X \times U) \to \mathbb{R}$ is continuous for every
measurable and bounded $f(x,u)$ which is continuous in $u$ for every $x$ (but unlike the weak topology, $f$ does not need to be continuous in $x$).

It is a consequence of [48, Theorem 3.10] or [5, Theorem 2.5] that (2.14), by implying setwise sequential pre-compactness of marginal measures on the state, ensures that every weakly converging sequence of mean empirical occupation measures also converges in the w-s sense. Equation (2.14) implies setwise sequential pre-compactness by [45, Proposition 3.2], which in turn builds on [29, Corollary 1.4.5]; see also [23, Theorem 4.17].

**Theorem 2.6.** a) Under Assumption 2.4 $(A, B, C, D)$, there exists an optimal measure in $\mathcal{G}$. b) Under Assumption 2.4 $(A', B', D, E)$, there exists a policy in $\Gamma_S$ which is optimal for the control problem given in (2.1) for every initial condition.

First, note the following counterpart to Lemma 2.1.

**Lemma 2.7.** Under (H2), the limit of any w-s converging subsequence of mean empirical occupation measures is in $\mathcal{G}$.

**Proof.** We follow the notation used in the discussion leading to Lemma 2.1. Suppose that, along some subsequence $\{t_k\} \subset \mathbb{N}$, $\mu_{tk}^\gamma$ converges to some $\mu \in \mathcal{P}(\mathcal{K})$ in the w-s sense, which we denote as $\mu_{tk}^\gamma \Rightarrow \mu$. As in (2.7) we have the triangle inequality

\[
|\mu(f) - \mu \mathcal{T}(f)| \leq |\mu(f) - \mu_{tk}^\gamma(f)| + |\mu_{tk}^\gamma(f) - \mu_{tk}^\gamma \mathcal{T}(f)| + |\mu_{tk}^\gamma \mathcal{T}(f) - \mu \mathcal{T}(f)|
\]

for $f \in \mathcal{M}_b(\mathcal{X})$. If (H2) holds, the first term on the right hand side of (2.15) vanishes as $k \to \infty$ by w-s convergence, while the second term does so by (2.6). We have

\[
(2.16) \quad \mu_{tk}^\gamma \mathcal{T}(f) = \mu_{tk}^\gamma (\mathcal{T}f),
\]

where $\mathcal{T}f$ is as defined in (2.4). Since $\mathcal{T}f$ is continuous in $u$ for every fixed $x$, by (H2), it follows that the third term also vanishes as $k \to \infty$ by the w-s convergence $\mu_{tk}^\gamma \Rightarrow \mu$. This shows that $\mu(A, U) = \mu \mathcal{T}(A)$ for all $A \in \mathcal{B}(\mathcal{X})$, which implies that $\mu \in \mathcal{G}$. \[\square\]

**Proof of Theorem 2.6.** The proof follows along the lines of Theorem 2.3, but instead of weak convergence, we work with w-s convergence.

As noted earlier, (2.14) ensures that every weakly converging sequence also does so under the w-s sense (see [45, Proposition 3.2], which in turn builds on [29, Corollary 1.4.5] or [23, Theorem 4.17]).

a) Accordingly, Assumption 2.4 $(A, B, C, D)$ ensures that each mean empirical occupation measure leading to a finite cost has a weakly converging subsequence, and which then is a w-s converging subsequence. Lemma 2.7 then implies that the limit of this sequence $\mu$ is in $\mathcal{G}$.

Furthermore, under hypothesis $(A')$ or $(A')$, the set $\mathcal{K}$ is measurable. Thus, by the generalized Portmanteau theorem [5, Proposition 3.2] for w-s convergence that every w-s limit of a converging sequence of probability measures $\mu_n$ with $\mu(\mathcal{K}) = 1$ is also supported on $\mathcal{K}$.

Now, Equation (2.14) implies also that the set of measures in $\mathcal{G}$ leading to a cost less than $\eta$ is w-s pre-compact, that is, for every sequence $\mu_n \in \mathcal{G}$ with $\langle \mu_n, c \rangle \leq \eta$, there exists a subsequence which converges (in the w-s sense) to a limit. Now, let $\mu_n$ be a sequence in $\mathcal{G}$ such that $\mu_n \overset{w-s}{\longrightarrow} \mu$. We show that $\mu \in \mathcal{G}$ and this also leads to a cost less than $\eta$.\[10]
Using the definition in (2.3), we note first that
\[ G = \left\{ \mu \in \mathcal{P}(X \times U) : \int_{X \times U} f(x) \mu(dx, du) = \int_X f(y) \mu(dy), \forall f \in \mathcal{M}_b(X) \right\}, \]
where as defined in subsection 2.1, \( \mathcal{M}_b(X) \) denotes the set of bounded Borel measurable functions on \( X \). Thus, for every \( f \in \mathcal{M}_b(X) \), we have
\[
\lim_{n \to \infty} \int_{X \times U} f(x) \mu_n(dx, du) = \lim_{n \to \infty} \int_X f(y) \mu_n(dy) = \lim_{n \to \infty} \int_X T f(x, u) \mu_n(dx, du),
\]
where \( T f \) is as defined in (2.4). Since \( T f \) is continuous in \( u \) for every fixed \( x \), by Assumption 2.4 (D), and \( \mu_n \xrightarrow{w-s} \mu \), we obtain
\[
\lim_{n \to \infty} \langle \mu_n, T f \rangle = \langle \mu, T f \rangle = \langle \mu, f \rangle,
\]
and
\[
\lim_{n \to \infty} \langle \mu_n, f \rangle = \langle \mu, f \rangle.
\]
Since the terms on the left hand side are equal by (2.17), we have equality of the terms on the right hand side, which implies that \( \mu \in G \).

Note that the integral \( \langle \mu, c \rangle \) is lower semi-continuous in \( \mu \). This follows by truncating \( c \) as \( c^N(x, u) = \min(N, c(x, u)) \), and then taking the limit \( N \to \infty \) noting that for every finite \( N \), \( \langle \mu, c^N \rangle \) is continuous in \( \mu \) by the \( w-s \) convergence. Thus, we also have that \( \langle \mu, c \rangle \leq \eta \). As a result, there exists an optimal measure \( \mu_* \in G \) with \( \mu_*(\mathcal{K}) = 1 \), and by, e.g., [27, Prop. D.8], we have an optimal control policy \( \phi \).

b) Now, under (A') \( w-s \) compactness follows from the existence of a \( w-s \) converging subsequence and (2.16) and the discussion following it. Then, under Assumption 2.4(A', B', D) and (E), as in Theorem 2.3, optimality of \( \phi \) for every initial condition follows by positive Harris recurrence given that \( c \) is bounded via (2.12). \( \square \)

**Remark 2.8.** In the above, we can relax continuity with a lower semi-continuity type condition: The non-negative cost function \( c \) is lower semi-continuous in \( u \in U(x) \) for every \( x \in X \) with the additional condition that \( \liminf_{u_n \to u} c(x, u_n) = c(x, u) \) for every \( x \in X \) and \( u_n \to u \) (with values in \( U(x) \)). This follows from [47, Remark 11.5] that every function satisfying this condition can be expressed as a monotone pointwise limit of measurable functions \( c_n \) continuous in \( u \) for every \( x \), and the lower semi-continuity argument in the proof applies directly, leading to existence.

### 3. Optimality of Deterministic (Markov or Stationary) Policies

In this section, we provide conditions under which an optimal average cost stochastic control problem is a deterministic stationary policy.

#### 3.1. Preliminaries

**Definition 3.1.** A policy \( \gamma \in \Gamma_S \) under which the chain has an invariant probability measure \( \pi_\gamma \), is called \( \pi_\gamma \)-deterministic (or simply, deterministic), if
\[
\pi_\gamma \left( \{ x \in X : \gamma(\cdot | x) \ is\ Dirac \} \right) = 1.
\]
If the policy is not $\pi_{\gamma}$-deterministic, we say that it is $\pi_{\gamma}$-randomized (or simply, randomized).

Here, $\mu_\phi$ denotes the invariant occupation measure of the chain under a stationary Markov policy $\phi$.

3.1.1. Convexity of the set of invariant occupation measures. Under the conditions noted above, the space $\mathcal{G}$ is closed under either the weak convergence or the $w$-$s$ topologies.

We now discuss convexity of $\mathcal{G}$. Let $\kappa \in (0, 1)$ and consider two invariant occupation measures $\mu^1, \mu^2 \in \mathcal{G}$. Let

\begin{equation}
\mu^i(dx, du) = \phi^i(du \mid x)\pi_{\phi^i}(dx) \text{ for } i = 1, 2,
\end{equation}

denote their disintegration into invariant probability measures $\pi_{\phi^i}$, and Markov policies $\phi^i$, $i = 1, 2$, respectively. Define

\begin{equation}
\pi(dx) := \kappa\pi_{\phi^1}(dx) + (1 - \kappa)\pi_{\phi^2}(dx).
\end{equation}

Note that $\pi(dx) = 0 \implies \pi_{\phi^i}(dx) = 0$ for $i = 1, 2$. As a consequence, the Radon-Nikodym derivative of $\pi_{\phi^i}$ with respect to $\pi$ exists. Let $f^i(x) := \frac{d\pi_{\phi^i}}{d\pi}(x)$, $i = 1, 2$, and

\begin{equation}
\phi(du \mid x) := \kappa f^1(x) \phi^1(du \mid x) + (1 - \kappa) f^2(x) \phi^2(du \mid x) \text{ } \pi\text{-a.e.}
\end{equation}

Then

\begin{equation}
\mu(dx, du) := \phi(du \mid x)\pi(dx) = \kappa\mu^1(dx, du) + (1 - \kappa)\mu^2(dx, du),
\end{equation}

and it follows by applying the definition that $\mu \in \mathcal{G}$. Therefore, $\mathcal{G}$ is convex.

In the following we let $\mathcal{G}_e$ denote the set of extreme points of $\mathcal{G}$.

3.1.2. A partial characterization of $\mathcal{G}_e$.

Lemma 3.2. If a measure $\mu$ is not in $\mathcal{G}_e$, then one of the following conditions are satisfied: (i) The control policy inducing it is randomized, or (ii) under this policy the Markov chain has multiple invariant probability measures.

Proof. Let $\mu$ be an invariant occupation measure in $\mathcal{G}$ which is not extreme. This means that there exist $\kappa \in (0, 1)$ and distinct invariant occupation measures $\mu^1, \mu^2 \in \mathcal{G}$ such that (3.1)–(3.4) hold.

Suppose $f^1 f^2 = 0$ $\pi$-a.e. Then the invariant measures $\pi_{\phi^i}$, $i = 1, 2$, are singular with respect to each other, so under the policy $\phi$ the Markov chain has two distinct invariant probability measures. On the other hand, if $f^1 f^2 \neq 0$ on a set of positive $\pi_{\phi^i}$ measure, then by (3.3), the policy is randomized on that set.

However, the converse direction is more consequential for optimization purposes, as we wish to show the optimality of deterministic policies. Towards this end, in what follows, we characterize the extreme points of the convex set $\mathcal{G}$. Since an optimal solution can, without any loss of generality, be searched over the extreme points of this set due to the linear programming formulation, this characterization provides insights on the structure of optimal policies. In particular, we establish the optimality of deterministic stationary policies.
3.1.3. Revisiting the countable state/action space setup: Optimality of deterministic policies. As noted earlier, the countable setup has been studied in [12, 24] and [38, Proposition 9.2.5]. We provide a different proof for Lemma 3.3 which may also be utilized in the continuous space setup, see subsection 3.5.

Following Definition 3.1, if \( \phi \) is a non-deterministic policy, we can select \( \alpha \in \mathcal{X} \) and lying on the support of \( \mu_\phi \), such that \( \phi(du|\alpha) \) can be expressed as a non-trivial convex combination of two different probability measures \( \gamma_1 \) and \( \gamma_2 \) on \( U \)

\[
(3.5) \quad \phi(du|\alpha) = \theta\gamma_1(du) + (1 - \theta)\gamma_2(du),
\]

and \( \theta \in (0,1) \).

**Lemma 3.3.** We assume that the chain is controlled by some \( \phi \in \Gamma_S \) has an invariant probability measure \( \pi_\phi \). Suppose that \( \phi \) is non-deterministic on some set that has positive \( \pi_\phi \) measure. Then the corresponding invariant occupation measure \( \mu_\phi \) cannot lie in \( \mathcal{G}_e \).

**Proof.** Let \( \phi \) be a non-deterministic policy so that (3.5) holds. Let \( \phi^i, i = 1, 2 \), denote the Markov policy which at \( \alpha \) (with \( \pi_\phi(\alpha) > 0 \)) selects an action under \( \gamma_i \) and agrees with \( \phi \) everywhere else. It is clear that, with \( \tau_\alpha = \min(k > 0 : x_k = \alpha) \) denoting the first return time to \( \alpha \), we have the stochastic representations

\[
(3.6) \quad \pi^i_\phi(x) = \frac{E^\phi^i_\alpha[\sum_{k=0}^{\tau_\alpha-1} 1\{X_k = x\}]}{E^\phi^i_\alpha[\tau_\alpha]}, \quad i = 1, 2,
\]

and

\[
\pi_\phi(x) = \frac{\theta E^\phi_\alpha[\sum_{k=0}^{\tau_\alpha-1} 1\{X_k = x\}] + (1 - \theta) E^{\phi^2}_\alpha[\sum_{k=0}^{\tau_\alpha-1} 1\{X_k = x\}]}{\theta E^\phi_\alpha[\tau_\alpha] + (1 - \theta) E^{\phi^2}_\alpha[\tau_\alpha]},
\]

where in the second equality we use (3.6), and the constant \( \kappa \in (0,1) \) defined by

\[
\kappa := \frac{\theta E^\phi_\alpha[\tau_\alpha]}{\theta E^\phi_\alpha[\tau_\alpha] + (1 - \theta) E^{\phi^2}_\alpha[\tau_\alpha]}.
\]

It follows from (3.4) that

\[
\mu_\phi = \kappa \phi^1 \circ \pi^1_\phi + (1 - \kappa) \phi^2 \circ \pi^2_\phi.
\]

It is clear that \( \pi^{\phi^i}(\alpha) > 0 \) for \( i = 1, 2 \). Thus \( \phi^1 \circ \pi^1_\phi \neq \phi^2 \circ \pi^2_\phi \) since the \( \gamma_i \)'s are not identical. This shows that \( \mu_\phi \notin \mathcal{G}_e \).

As a result, we can deduce that for such countable state and action spaces an optimal policy is stationary and deterministic, provided that the convex analytic method is applicable.

3.2. Uncountable standard Borel setup: Optimality of deterministic Markov policies. For an uncountable setup, the optimality of deterministic policies under the convex analytic approach has been an open problem. We present our results first and then compare these with the only available results in the literature, which to our knowledge, are due to Borkar.
Our approach will build on an analysis via strategic measures: For stochastic control problems, strategic measures are defined (see [47], [15] and [18]) as the set of probability measures induced on the product spaces of the state and action pairs by measurable control policies. As noted earlier, given an initial distribution on the state, and a policy, one can uniquely define a probability measure on the product space. Topological properties, such as measurability and compactness, of sets of strategic measures are studied in [47], [15], [18] and [8].

For the following result, the reader is referred to [17, Theorem 1.2] and [21, Lemma 1.2] (see also [57, Theorem 2.3] and [16, Theorem 3.2 or Corollary 3.3]).

**Theorem 3.4.** Let \( L_A(\mu) \) be the set of strategic measures induced by (possibly randomized) policies in \( \Gamma_A \) with \( X_0 \sim \mu \). Then, for any \( P \in L_A(\mu) \), there exists an augmented space \( \Omega' \) and a probability measure \( \eta \) on \( B(\Omega') \) such that

\[
P(B) = \int_{\Omega'} \eta(d\omega) P_\mu^{\gamma(\omega)}(B), \quad B \in B((X \times U)^T),
\]

for every finite horizon \( T \), where \( \gamma(\omega) \in \Gamma_{AD} \) for all \( \omega \in \Omega' \).

**Proof.** Here, we build on Lemma 1.2 in Gikhman and Shorodhod [21] and Theorem 1 in [17]. Any stochastic kernel \( P(du|x) \) can be realized by some measurable function \( u = f(x, v) \) where \( v \) is a uniformly distributed random variable on \([0, 1]\) and \( f \) is measurable (see also [11] for a related argument). One can define a new random variable \( (\omega = (v_0, v_1, \ldots, v_{T-1})) \). In particular, \( \eta \) can be taken to be the probability measure constructed on the product space \([0, 1]^T \) by the independent variables \( v_k, k \in \{0, 1, \ldots, T-1\} \).

One implication of this theorem is that if one relaxes the measure \( \eta \) to be arbitrary, a convex representation would be possible. That is, the set

\[
P(B) = \int_{\Omega'} \eta(d\omega) P_\mu^{\gamma(\omega)}(B), \quad B \in B((X \times U)^N), \eta \in P(\Omega)
\]

is convex, when one does not restrict \( \eta \) to be a fixed measure. Furthermore, the extreme points of these convex sets consist of policies which are deterministic. A further implication then is that, since the expected cost function is linear in the strategic measures, one can without any loss consider the extreme points while searching for optimal policies. In particular,

\[
\inf_{\gamma \in \Gamma_{AR}} J(x, \gamma) = \inf_{\gamma \in \Gamma_A} J(x, \gamma).
\]

In other words, if \( L_D(\mu) \) and \( L_A(\mu) \) denote the the set of strategic measures induced by deterministic and admissible (possibly randomized) policies, respectively, then the extreme points of \( L_A(\mu) \) are in \( L_D(\mu) \). A related celebrated result in economics theory, known as Kuhn’s theorem [32], asserts that the convex hull of admissible (i.e. those in \( L_A(\mu) \)) strategic measures is equivalent to \( L_A(\mu) \) in classical stochastic control. For a comprehensive discussion on the geometry of strategic measures, we refer the reader to [46].

In the following, we say that an invariant measure \( \mu \) is ergodic if the support of \( \mu \) does not contain two disjoint absorbing sets.

**Lemma 3.5.** Suppose that the conditions of either Theorem 2.6 or Theorem 2.3 hold so that there exists an optimal occupation measure. Without any loss of generality, such an optimal invariant occupation measure can be assumed to be ergodic.
Proof. Let \( \nu \) be an optimal occupation measure, which leads to a policy \( \gamma \) by (2.13). By an ergodic decomposition theorem [29, Lemma 5.2.4] (see [53] for more general results on Polish spaces and a review), every stationary measure can be expressed as a convex combination of ergodic invariant measures: let \( \nu \) be expressed as a convex combination of ergodic invariant measures. Then, the infimum over all these ergodic invariant measures is at least as good as the performance of \( \langle \nu, c \rangle \) (for otherwise there would be a contradiction), and each of these measures also belongs to \( G \) by definition, since the control policy is defined on the support of \( \nu \), and thus is defined also for each of the invariant measures (in the ergodic decomposition). Thus, without any loss, we can take \( \nu \) to be an ergodic invariant measure.

Theorem 3.6. a) Let \( c \) be bounded. Under the conditions of either Theorem 2.6 or Theorem 2.3, if the induced Markov chain under an optimal stationary (possibly randomized) policy \( \gamma \) satisfies one of the following:

(i) is positive Harris recurrent,

(ii) is uniquely ergodic with invariant probability measure \( \mu \), and with \( X_0 \sim \nu \), for some \( k \in \mathbb{Z}_+ \), we have \( P^\gamma(X_k \in \cdot) \ll \mu(\cdot) \), with \( \mu \) being an invariant probability measure arising from an optimal invariant occupation measure, then there exists an optimal policy which is Markov and deterministic.

b) Under the conditions of either Theorem 2.6 or Theorem 2.3, if the cost is not bounded, there exists an optimal ergodic measure \( \nu \) so that for \( \nu \) almost all \( x \), with \( X_0 = x \), there exists an optimal policy which is Markov and deterministic.

Proof. We first prove a).

(step 1.) Feinberg [16, Section 4.3] establishes that the criterion (where limit superior in (1.3) is replaced by limit inferior)

\[
\lim_{T \to \infty} \frac{1}{T} \mathbb{E}^{\gamma}_{x_0} \left[ \sum_{t=1}^{T} c(x_t, u_t) \right],
\]

is a concave function on the set of strategic measures. Accordingly, the infimum

\[
\inf_{\gamma \in \Gamma} \lim_{T \to \infty} \frac{1}{T} \mathbb{E}^{\gamma}_{x_0} \left[ \sum_{t=1}^{T} c(x_t, u_t) \right]
\]

can be taken to be the infimum over the extreme points of the set strategic measures, which implies that the infimum can be taken over deterministic admissible, history-dependent policies [16, Theorem 3.2 or Corollary 3.3].

Furthermore, if one follows the construction in the proof of Theorem 3.4, via expressing the strategic measure induced by any \( \gamma = \{f, f, f, \cdots\} \in \Gamma_S \) as a mixture of \( \gamma(\omega) = \{f(\cdots, v_0), f(\cdots, v_1), f(\cdots, v_2), \cdots\} \) with \( \omega = (v_0, v_1, v_2, \cdots) \).

Thus, we have that the infimum

\[
\inf_{\gamma \in \Gamma_S} \lim_{T \to \infty} \frac{1}{T} \mathbb{E}^{\gamma}_{x_0} \left[ \sum_{t=1}^{T} c(x_t, u_t) \right]
\]

can be taken to be the infimum over the extreme points of the set strategic measures induced by randomized stationary policies, which implies that the infimum can be taken over deterministic Markov policies \( \Gamma_{MD} \). Thus, deterministic policies Markov policies are as good as any randomized stationary policy.

(step 2.) The convex analytic method studied for the criterion (1.3), in fact establishes the optimality of stationary policies under the liminf criterion (3.8) also, via an
identical argument as that presented for the inequalities in the proof of Theorem 2.3: in particular, (2.9) can be seen to also be applicable for this criterion leading to the optimality of stationary (possibly randomized) policies.

**step 3.** Thus, provided that the convex analytic method holds for (3.7), by steps (1) and (2), an optimal policy lies in \( \Gamma_{MD} \). Hence, the criterion (3.8) admits an optimal solution which is Markov and deterministic. Furthermore, this criterion serves as a lower bound to (1.3). If it can be shown that this lower bound is attained for the limit superior criterion (1.3), the proof will be complete.

**step 4.** Under positive Harris recurrence with the induced optimal policy, the optimality of a deterministic Markov policy is then established with no further conditions, since the limit infimum and limit superior lead to the same performance.

Now, recall the following result \([53, \text{Prop. 5.4}]\): Let \( \bar{P} \) be an invariant probability measure for a Markov process. For all \( \mu \in \mathcal{P}(X) \) which satisfies that \( \mu \ll \bar{P} \) (that is, \( \mu \) is absolutely continuous with respect to \( \bar{P} \)), there exists an invariant probability measure \( v^* \) such that

\[
\lim_{N \to \infty} \frac{1}{N} E_\mu \left[ \sum_{t=0}^{N-1} 1_{\{X_t \in \cdot\}} \right] - v^*(\cdot) \xrightarrow{TV} 0.
\]

By this result, we have that if under the optimal policy the chain is uniquely ergodic, then \( v^* = \bar{P} \) and the ‘lim inf’ and ‘lim sup’ of the expected average cost sequences are identical. This also applies when \( \bar{P} \) is not the unique invariant measure, but \( \bar{P} \) is just an invariant measure with \( \nu \ll \bar{P} \).

Accordingly, for item (ii), by Lemma 3.5, an optimal occupation measure can be assumed ergodic. Then, if we have \( X_0 \sim \mu(\cdot, U) \) so that for some \( k \in \mathbb{Z}_+ \), the measure \( P^k_\nu(X_k \in \cdot) \) satisfies the absolute continuity condition, the optimal cost will be attained noting that

\[
\lim_{N \to \infty} \frac{1}{N} E^\gamma_\mu \left[ \sum_{n=0}^{N-1} 1_{\{X_n \in \cdot\}} \right] - v^*(\cdot) \xrightarrow{TV} \lim_{N \to \infty} \frac{1}{N} E^\gamma_\mu \left[ \sum_{n=k}^{N-1} 1_{\{X_n \in \cdot\}} \right] - v^*(\cdot)
\]

for any \( k \in \mathbb{Z}_+ \). The result a) then follows.

For b), let \( \mu^* \) be an optimal occupation measure (clearly leading to a finite cost). By Lemma 3.5, we may assume this measure to be ergodic (say for some \( \nu \) in the ergodic decomposition of \( \mu^* \)). By (2.10), the results above apply so that limit inferior and superior are identical for the integrable cost function for \( \nu \) almost all initial conditions.

We do not, however, show the optimality of deterministic and stationary policies in the above. This is studied next.

### 3.3. Hernández-Lerma’s approach via arriving at ACOE/ACOI from the convex analytic method.

In general, establishing conditions for the existence of a solution to ACOI is an unfinished problem. Our findings through the convex analytic method, through the duality analysis below, provides further conditions. One may express the linear program

\[
(3.10) \quad \min_{\nu \in U} \langle \nu, c \rangle
\]

as an infinite dimensional linear program, present its convex dual formulation and arrive at the ACOI. This then leads to an existence result.
A direct argument along this direction is presented by Hernandez-Lerma [24, Theorem 5.3]. This result shows that an average cost optimal randomized policy \( \phi \), with invariant measure \( \pi_\phi \) satisfies the ACOI \( \pi_\phi \) almost everywhere:

\[
g + h(x) \geq c(x, \phi(x)) + \int h(x') T(dx'|x, \phi(x))
\]

where \( h \) is bounded from below. [24, Prop. 5.2] shows that under this condition on \( h \), (3.11) implies that such a policy is indeed optimal. Now, if one can ensure that the above holds for all \( x \in X \) (and not just \( \pi \) almost everywhere), by utilizing Blackwell’s theorem of optimality of deterministic policies (also called irrelevant information theorem) [7, 9], it turns out that we can replace \( \phi \) with a deterministic \( f \in \Gamma_{SD} \), which will then be optimal [24, Corollary 5.4(b)].

3.4. Borkar’s approach via Schauder’s Fixed Point Theorem. To the best of our knowledge, the only result on the optimality of deterministic policies directly via the convex analytic method that is available for uncountable state spaces is Borkar [12, Section 3.2], which imposes the following.

**Assumption 3.7.** There exists a \( \sigma \)-finite non-negative measure \( \lambda \) on \( X \) such that

\[
T(dy \mid x, u) = f(x, u, y)\lambda(dy), \quad x \in X, \quad u \in U,
\]

\( f \) is continuous in all its variables, and \( f(x, u, \cdot) \) is bounded and equicontinuous (over \( x \in X, \ u \in U \)) and bounded away from zero uniformly over all compact sets. The state and control variables are finite dimensional real valued. Furthermore \( G \) is compact and every stationary and randomized policy leads to a Markov chain which admits an invariant probability measure.

**Theorem 3.8** (Lemma 11.16 in [12]). Under Assumption 3.7, suppose that with \( a \in (0,1) \) and \( \phi^1, \phi^2 \) two stationary control policies

\[
\phi(du \mid x) = a\phi^1(du \mid x) + (1-a)\phi^2(du \mid x),
\]

where \( \phi^1(du \mid x) \neq \phi^2(du \mid x) \) for some \( x \in B_R \) for some ball of radius \( R \). Then, the invariant probability measure induced by \( \phi \) cannot be an extreme point.

The conditions above are needed in order to employ a version of Schauder’s fixed point theorem on maps on the space of probability measures under the total variation distance. Lemma 3.7 in [12] also requires randomizations to be constant over a non-trivial set.

3.5. An approach via the small/petite set theory. For reader’s knowledge, we present an approach via the theory of small/petite sets to arrive at complementary conditions for the optimality of stationary and randomized policies. The approach is to follow the proof method utilized in Lemma 3.3. The results boil down to a stochastic realization condition, which however does not appear to be lenient. This is reported in the appendix: the realization condition itself is likely a useful property for further applications and for this reason the analysis is reported in the appendix.

4. Denseness of performance of stationary deterministic policies. In some applications it may be useful to know not only that optimal policies are deterministic, but that deterministic policies are dense in the sense of approximability of the costs induced under randomized and stationary policies.
We have the following supporting *denseness* result involving measurable policies over randomized ones.

**Theorem 4.1.** Let \((X,U)\) be finite dimensional real valued state and control action random variables, where the compact \(U\) valued \(U\) is generated by a randomized stationary policy. Suppose further that \(X\) admits a non-atomic probability measure. Then we have the following:

(i) There exists a collection of measurable policies \(U_n = \gamma_n(X)\) so that \((X,U_n)\) converges weakly to \((X,U)\).

(ii) \((X,U_n) \to (X,U)\) in the \(w\)-s (setwise-weak) topology also.

(iii) If \(X_n\) is a sequence of random variables whose associated probability measure converges in total variation to that of \(X\), then the joint random variable \((X_n,\gamma_n(X_n))\) converges weakly to \((X,U)\) as well as in the \(w\)-s sense (setwise in \(x\) and weakly in \(u\)).

**Proof.** (i) is due to [40, Theorem 3], though there exist other related results, e.g. [6, Proposition 2.2], [35], [40, Theorem 3], but also many texts in optimal stochastic control where denseness of deterministic controls have been established inside the set of relaxed controls [10].

(ii) The marginal on \(X\) is fixed along the sequence. The result then follows from [48, Theorem 3.10] (or [5, Theorem 2.5]).

(iii) Let \(\rho\) denote the Prohorov metric on the joint state-action random variables.

\[
\rho((X_n,\gamma_n(X_n)),(X,U)) \leq \rho((X_n,\gamma_n(X_n)),(X,\gamma_n(X))) + \rho((X,\gamma_n(X)),(X,U)).
\]

The first term on the right converges to zero due to total variation convergence of \(X_n\) to \(X\) (since we apply the same deterministic measurable policy \(\gamma_n\), and convergence is uniform over all measurable functions as in the proof of [31, Lemma 1.1(iii)]). The second term converges to zero is by (i).

As in the proof of Theorem 2.6, by [48, Theorem 3.10] or [5, Theorem 2.5], since the measure converges weakly and the marginal in \(X\) converges setwise, the convergence is also in the \(w\)-s sense.

**Theorem 4.1** helps us in establishing the following.

**Theorem 4.2.** Suppose that

(i) \(G\) is weakly compact. Furthermore \(X = \mathbb{R}^n\) for some finite \(n\), and for all \(x \in \mathbb{R}, \cup(x) = U\) is compact.

(ii) For some \(\alpha \in [0,1)\), under every stationary policy \(\gamma\) the induced kernel \(P^\gamma\) of the Markov chain given by

\[
P^\gamma(\pi)(\cdot) := (\pi \mathcal{T}^\gamma)(\cdot) = \int \pi(dx)\gamma(du \mid x) \int \mathcal{T}(\cdot \mid x,u)
\]

satisfies

\[
(4.1) \quad \|P^\gamma(\pi) - P^\gamma(\bar{\pi})\|_{TV} \leq \alpha \|\pi - \bar{\pi}\|_{TV}
\]

for any pair of probability measures \((\pi, \bar{\pi})\). This condition implies, naturally, that every stationary policy leads to a unique invariant probability measure.

(iii) The kernel \(\mathcal{T}(dy \mid x,u)\) is such that, the family of conditional probability measures \(\{\mathcal{T}(dy \mid x,u), x \in X, u \in U\}\) admit densities \(f_{x,u}(y)\) with respect to
a reference measure and all such densities are bounded and equicontinuous
(over \(x \in \mathcal{X}, u \in \mathcal{U}\)).

(iv) One of the following holds: (H1) holds and the bounded cost function \(c(x, u)\)
is continuous; or (H2)(a) holds and the bounded cost function \(c(x, u)\) is continuous
in \(u\) for every \(x\).

Then, deterministic and stationary policies are dense among those that are randomized
and stationary, in the sense that the cost under any randomized stationary policy can
be approximated arbitrarily well by deterministic and stationary policies.

Before presenting the proof, we note that a list of sufficient conditions for \((4.1)\) are
presented in \([26, Theorem 3.2]\) and these all have a relationship with the Dobrushin’s
ergodicity coefficient \([14]\).

Proof. Observe that the family of densities \(f_{x,u}(\cdot)\) being equicontinuous over \(x \in \mathcal{X}, u \in \mathcal{U}\)
implies that \(\{ f_{x,u}(y)\mu(dx)\gamma(du|x), \mu \in \mathcal{P}(\mathcal{X}), \gamma \in \Gamma_S \}\) is also equicontinuous.
Thus, the family of invariant probability measures under any stationary policy
admit densities (with respect to some reference measure \(\psi\)) which are equicontinuous
and uniformly bounded and if \(\mu_n(dy) = f_{n}(y)\psi(dy) \rightarrow \mu(dy) = f(y)\psi(dy)\) weakly,
then as a consequence of the Arzelà-Ascoli theorem (applied to \(\sigma\)-compact spaces)
\(f_n \rightarrow f\) pointwise and by Scheffé’s theorem, \(\mu_n \rightarrow \mu\) in total variation.

Let \(\gamma\) be any randomized policy. Suppose that this policy gives rise to an
invariant probability measure \(\pi_{\gamma}(dx, du)\). Now, consider a sequence of deterministic
policies \(f_n\) so that under this sequence of policies \(\pi_{\gamma}(dx)\delta_{f_n(x)}(du)\) converges weakly
to \(\pi_{\gamma}(dx, du)\) by Theorem 4.1(i).

Now, let us apply the same measurable policy sequence to the random variable
\(X_n\) which has the probability measure \(\pi_{f_n}(dx)\) equal to the marginal of the invariant
measure under policy \(U = f_n(X)\). Then, for every continuous and bounded \(g \in C_b(\mathcal{X})\)
\[
\int \pi_{f_n}(dx)\delta_{f_n(x)}(du) \left( \int g(y)T(dy|x,u) \right) = \int \pi_{f_n}(dx)g(x).
\]

Let \(\pi_{f_{n_k}}(dx)\) be a weakly converging subsequence with limit \(\eta\) (by the compactness
assumption on \(\mathcal{G}\)). By hypothesis, this convergence is also in total variation.

Define for any stationary policy \(f\)
\[
P^f(\pi) = \int \pi(dx)f(du|x)\int T(dy|x,u),
\]
and by hypothesis note that \(\|P^f(\pi) - P^f(\bar{\pi})\|_{TV} \leq \alpha\|\pi - \bar{\pi}\|\). Then,
\[
\|\pi_{f_{n_k}} - \pi_\gamma\|_{TV} = \|P^{f_{n_k}}(\pi_{f_{n_k}}) - P^{\gamma}(\pi_\gamma)\|_{TV}
\]
\[
= \|P^{f_{n_k}}(\pi_{f_{n_k}}) - P^{f_{n_k}}(\pi_\gamma)\|_{TV} + \|P^{f_{n_k}}(\pi_\gamma) - P^{\gamma}(\pi_\gamma)\|_{TV}
\]
\[
\leq \alpha\|\pi_{f_{n_k}} - \pi_\gamma\|_{TV} + \|P^{f_{n_k}}(\pi_\gamma) - P^{\gamma}(\pi_\gamma)\|_{TV},
\]
and thus
\[
\|\pi_{f_{n_k}} - \pi_\gamma\|_{TV} \leq \frac{\|P^{f_{n_k}}(\pi_\gamma) - P^{\gamma}(\pi_\gamma)\|_{TV}}{1 - \alpha}
\]
Now, for the right hand side, we have that \(P^{f_{n_k}}(\pi_\gamma) \rightarrow P^{\gamma}(\pi_\gamma)\) weakly, since \(\pi_\gamma(dx)f_n(du|x)\)
Theorem 4.1

(iii), as (ii) and Assumption is that the follow-

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the weak convergence should be supported by pointwise convergence of densities, and therefore, the weak convergence should be supported by pointwise convergence of densities, and thus by Scheffé’s lemma, the convergence is also in total variation.

Since the right hand side converges to zero, we can conclude that indeed \( \pi_{f_{\nu_k}}(dx) \to \pi_\gamma(dx) \).

Finally, by Theorem 4.1(iii), as \( \pi_{f_{\nu_k}}(dx)\delta_{f_{\nu_k}(x)}(du) \to \pi_\gamma(dx)\gamma(du \mid x) \) in the setwise-weak (setwise in the state, weakly in the control action), the result follows.

5. Conclusion.

In this paper, we established conditions for the optimality of deterministic policies in average cost optimal stochastic control. We also presented a new result on the existence of optimal policies in average cost optimal stochastic control with kernels that do not satisfy weak kernel continuity in both state and actions, but with strong kernel continuity in the actions for every fixed state variable. We finally presented a denseness result of costs induced under deterministic and stationary policies among those that are attained by randomized and stationary policies.

Appendix A. An approach based on the theory of small sets and an open realizability question. In the following we present a sufficient condition to establish the desired optimality result on deterministic policies through the theory of small sets.

**Definition A.1.** A set \( A \in \mathcal{B}(\mathbb{X}) \) is \( n \)-small on \( (\mathbb{X}, \mathcal{B}(\mathbb{X})) \) if for some positive measure \( \mu_n \)

\[
P^n(x, B) \geq \mu_n(B), \quad \forall x \in A \text{ and } B \in \mathcal{B}(\mathbb{X}).
\]

**Definition A.2.** [39] A set \( A \in \mathcal{B}(\mathbb{X}) \) is \( \nu_T \)-petite on \( (\mathbb{X}, \mathcal{B}(\mathbb{X})) \), if for some distribution \( K \) on \( \mathbb{N} \) (set of natural numbers), and some positive measure \( \nu_T \),

\[
\sum_{n=0}^{\infty} P^n(x, B)K(n) \geq \nu_K(B), \quad \forall x \in A \text{ and } B \in \mathcal{B}(\mathbb{X}).
\]

By [39, Proposition 5.5.6], if a Markov chain is \( \psi \)-irreducible, and if a set \( C \) is \( \nu \)-petite, then \( K \) can be taken to be a geometric distribution \( \alpha_n(i) = (1 - \epsilon)^{i+1}, \quad i \in \mathbb{N} \) (with the randomly sampled chain also known as the resolvent kernel).

**The 1-small case.** We impose the following assumption.

**Assumption A.3.** For any two policies \( \gamma^1 \) and \( \gamma^2 \) in \( \Gamma_5 \) (possibly randomized), and every Borel set \( B \) that satisfies \( \psi_{\gamma^i}(B) > 0, \quad i = 1, 2 \), where \( \psi_{\gamma^i} \) denotes the maximal \( \psi \)-irreducibility measure under policy \( \gamma^i \), there exists a measurable \( C \subset B \) that is a 1-small set with positive maximal irreducibility measure under either of the transition probabilities \( P_{\gamma^1} \) and \( P_{\gamma^2} \).

**Proposition A.4.** A sufficient condition for Assumption A.3 is that the following hold:

(i) The transition kernel \( T \) is bounded from below by a conditional probability measure that admits a density with respect to some positive measure \( \phi \). In other words there exist a measurable \( f : \mathbb{X} \times U \times \mathbb{X} \to \mathbb{R}_+ \), such that

\[
T(D \mid x, u) \geq \int_D f(x, u, y)\phi(dy)
\]
for every $D \in \mathcal{B}(\mathcal{X})$.

(ii) The function $f(x,u,y)$ in (i) is continuous in $x,u$ for every fixed $y$, and $\mathbb{U}$ is compact.

(iii) It holds that

$$\int_{\mathcal{X}} \inf_{x \in A, u \in \mathbb{U}} f(x,u,y) \phi(dy) > 0$$

for every nonempty compact set $A \subset \mathcal{X}$.

Proof. The measurable selection results in [47, 34] and [30, Theorem 2] show that, for any compact $A \subset \mathcal{X}$, there exist measurable functions $g$ and $F$ such that

$$\inf_{x \in A, u \in \mathbb{U}} f(x,u,y) = \min_{x \in A, u \in \mathbb{U}} f(x,u,y) =: F(g(y),y)$$

Thus, using the notation in (2.2), we have

$$P^\gamma(x,D) \geq \int_D \inf_{x \in A, u \in \mathbb{U}} f(x,u,y) \phi(dy) = \int_D F(g(y),y) \phi(dy) =: \nu(D)$$

for some finite (sub-probability) measure $\nu$. Thus, every compact set is 1-small under a given policy.

Theorem A.5. Under Assumption A.3, and the realizability condition given in (A.2) and (A.3) a randomized policy cannot lead to an extreme measure in $\mathcal{G}$, that is, all measures in $\mathcal{G}_c$ are induced by deterministic policies.

Proof. The proof is divided into four steps.

Step 1. Let there be a policy $\phi$ which is randomizing between two policies $\phi^1$ and $\phi^2$ on some measurable set $B$, so that for some $\kappa(x) \in (0,1)$ with $x \in B$, we have that

$$\phi(du \mid x) = \kappa(x)\phi^1(du \mid x) + (1 - \kappa(x))\phi^2(du \mid x), \quad x \in B.$$ 

By Assumption A.3, there exists a $C \subset B$ so that this set is small for either of the transition probabilities and on this set the above randomization also holds, and that the measure on $C$ is positive under either of the irreducibility measures under $P^{\phi^1}$ and $P^{\phi^2}$.

We can assume that the transition kernels admit small sets with measure $\nu^1$ and $\nu^2$, where we take $\nu^1(\mathcal{X}) = \nu^2(\mathcal{X})$, without any loss of generality, since we can always scale down the measure with the larger total mass to match the one with the smaller total mass.

Step 2. Define, for $K \in (0,\frac{1}{4})$,

$$C^K := \{x \in C : 1 - K \geq \kappa(x) \geq K\}.$$ 

Thus, we have

$$\phi(du \mid x) = \kappa(x)\phi^1(du \mid x) + (1 - \kappa(x))\phi^2(du \mid x), \quad x \in C^K,$$

and $C^K$ is also small (since it is a subset of a small set). Furthermore, we can take $C^K$ be so that it has positive measure under the irreducibility measures (by a continuity of measures argument, as $K \to 0$, $\psi(C^K) \to \psi(C)$ for any measure $\psi$). Now, by the Nummelin-Athreya-Ney split chain argument [43, 42, 4], we split $C^K$ into $C^K \times \{0\}$.
and \( C^K \times \{1\} =: \alpha \), where \( \alpha \) is a pseudo-atom, in the sense that the transition kernels are independent of the particular \( x \in \alpha \), as we make more explicit below. To motivate this construction, we note that for \( x \in C^K \), for any Borel \( A \), we can write

\[
\mathcal{T}^\phi(A \mid x) = \left( 1 - K\nu^1(\mathcal{X}) \right) \frac{\kappa(x)\mathcal{T}^{\phi^1}(A \mid x) - K\nu^1(A)}{1 - K\nu^1(\mathcal{X})} + K\nu^1(\mathcal{X}) \frac{K\nu^1(A)}{K\nu^1(\mathcal{X})} \\
+ \left( 1 - K\nu^2(\mathcal{X}) \right) \frac{(1 - \kappa(x))\mathcal{T}^{\phi^2}(A \mid x) - K\nu^2(A)}{1 - K\nu^2(\mathcal{X})} + K\nu^2(\mathcal{X}) \frac{K\nu^2(A)}{K\nu^2(\mathcal{X})}.
\]

Write the above as

\[
\mathcal{T}^\phi(A \mid x) = \frac{1}{2} \left( 1 - 2K\nu^1(\mathcal{X}) \right) \frac{\kappa(x)\mathcal{T}^{\phi^1}(A \mid x) + (1 - \kappa(x))\mathcal{T}^{\phi^2}(A \mid x) - K(\nu^1(A) + \nu^2(A))}{1 - 2K\nu^1(\mathcal{X})} \\
+ \frac{1}{2} \left( 1 - 2K\nu^2(\mathcal{X}) \right) \frac{\kappa(x)\mathcal{T}^{\phi^1}(A \mid x) + (1 - \kappa(x))\mathcal{T}^{\phi^2}(A \mid x) - K(\nu^1(A) + \nu^2(A))}{1 - 2K\nu^2(\mathcal{X})} + 2K\nu^1(\mathcal{X}) \frac{2K\nu^1(A)}{2K\nu^1(\mathcal{X})} \\
+ 2K\nu^2(\mathcal{X}) \frac{2K\nu^2(A)}{2K\nu^2(\mathcal{X})}
\]

**Step 3 (The realizability step).** We now realize (i.e., construct) two control policies, called \( \tilde{\phi}^1 \) and \( \tilde{\phi}^2 \), so that these policies agree with \( \phi \) outside \( C^K \), and inside \( C^K \) they admit a split chain where the transitions outside the atom \( \alpha \), that is on \( C^K \times \{0\} \), are also in agreement: the only difference is on the atom itself, therefore, the policies act as if they are in agreement everywhere except on the atom. That is, for \( x \notin C^K \), we have

\[
\tilde{\phi}^1(du \mid x) = \tilde{\phi}^2(du \mid x) = \phi(du \mid x)
\]

But on \( C^K \), we have that \( \mathcal{T}^\phi(\cdot \mid x) \) is attained by randomizing between \( \tilde{\phi}^1 \) and \( \tilde{\phi}^2 \) according to:

\[
(A.1) \quad \phi(du \mid x) = \frac{1}{2} \tilde{\phi}^1(du \mid x) + \frac{1}{2} \tilde{\phi}^2(du \mid x), \quad x \in C^K
\]

where in the split chain, for \( x \in C^K \times \{0\} \), \( \tilde{\phi}^i(du \mid x) \) leads to the one step transition kernel

\[
\mathcal{T}^{\tilde{\phi}^i}(\cdot \mid x) = \kappa(x)\mathcal{T}^{\phi^i}(\cdot \mid x) + (1 - \kappa(x))\mathcal{T}^{\phi^i}(\cdot \mid x) - K\nu^1(\cdot) - K\nu^2(\cdot)
\]

for \( i = 1, 2 \). And on \( x \in \alpha \), \( \tilde{\phi}^i(du \mid x) \) leads to the one step transition kernel

\[
\mathcal{T}^{\tilde{\phi}^i}(dy \mid x) = \frac{\nu^i(dy)}{\nu^i(\mathcal{X})}
\]

for \( i = 1, 2 \). Note that the above lead to the following virtual aggregate transition kernels under \( \tilde{\phi}^i \), \( i = 1, 2 \) for \( x \in C^K \):

\[
(A.2) \quad \mathcal{T}^{\tilde{\phi}^i}(\cdot \mid x) = \kappa(x)\mathcal{T}^{\phi^i}(\cdot \mid x) + (1 - \kappa(x))\mathcal{T}^{\phi^i}(\cdot \mid x) + K\nu^1(\cdot) - K\nu^2(\cdot),
\]
\[
\mathcal{T}^\phi (\cdot | x) = \kappa(x)\mathcal{T}^\phi (\cdot | x) + (1 - \kappa(x))\mathcal{T}^\tilde{\phi} (\cdot | x) - KV^1 (\cdot) + KV^2 (\cdot)
\]
so that (A.1) holds.

**The Realizability Condition:** There exist stationary control policies \(\psi^1\) and \(\psi^2\), such that, \(\psi^1\) realizes (A.2) and \(\psi^2\) realizes (A.3).

As a result, the only difference in the transition kernels, as seen from the split chain/atom is that, in the atom \(\alpha\) randomization occurs; outside the atom the transition probabilities are identical.

**Step 4.** \(\alpha\) is the accessible atom of interest: \(E_\alpha^\phi [\tau_\alpha] < \infty\) and \(P^\phi (x, B) = P^\psi (y, B)\)
for all \(x, y \in \alpha\), where \(\tau_\alpha = \min(k > 0 : x_k = \alpha)\) is the return time to \(\alpha\). Since \(\phi\) is randomizing between the two policies \(\phi^1\) and \(\phi^2\), let \(v, v^1, v^2\) be corresponding invariant measures to \(\phi, \phi^1\) (only different at the atom), \(\phi^2\) (only different at the atom); as noted above, for states other than those in the atom the transition kernels are identical.

In this case, the invariant measures computed through the mean empirical occupation measures normalized with the expected return times is obtained through the following analysis: For every Borel \(A \in \mathcal{X}, B \in \mathcal{U}\),

\[
v(A, B) = \frac{E_\alpha^\phi [\sum_{k=0}^{\tau_\alpha - 1} 1_{A \times B}(X_k, U_k)]}{E_\alpha^\phi [\tau_\alpha]}
= \frac{1}{2} E_\alpha^\phi [\sum_{k=0}^{\tau_\alpha - 1} 1_{A \times B}(X_k, U_k)] + \frac{1}{2} E_\alpha^\tilde{\phi} [\sum_{k=0}^{\tau_\alpha - 1} 1_{A \times B}(X_k, U_k)]
= \frac{E_\alpha^\phi [\tau_\alpha] E_\alpha^{\phi^2} [\tau_\alpha] + E_\alpha^{\phi^1} [\tau_\alpha] E_\alpha^{\tilde{\phi}^2} [\tau_\alpha]}{E_\alpha^{\phi^1} [\tau_\alpha] + E_\alpha^{\tilde{\phi}^2} [\tau_\alpha]}
\]

which is a convex combination of \(v^1\) and \(v^2\). In the above, the second equality is critical for the validity of the convex combination: \(E_\alpha^\phi [\tau_\alpha] = \frac{1}{2} E_\alpha^{\phi^1} [\tau_\alpha] + \frac{1}{2} E_\alpha^{\tilde{\phi}^2} [\tau_\alpha]\), which holds due to the construction in **Step 3**.

We note also that the similar program applies for a construction building on both \(m\)-small sets and petite sets. These sets exist under much less stringent conditions than those required on \(1\)-small sets.

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