Logic of temporal attribute implications

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Abstract

We study logic for reasoning with if-then formulas describing dependencies between attributes of objects which are observed in consecutive points in time. We introduce semantic entailment of the formulas, show its fixed-point characterization, investigate closure properties of model classes, present an axiomatization and prove its completeness, and investigate alternative axiomatizations and normalized proofs. We investigate decidability and complexity issues of the logic and prove that the entailment problem is NP-hard and belongs to EXPSPACE. We show that by restricting to predictive formulas, the entailment problem is decidable in pseudo-linear time.

Keywords: attribute implication, complete axiomatization, entailment problem, fixed point, functional dependency, temporal semantics

1 Introduction

Formulas describing if-then dependencies between attributes play fundamental role in reasoning about attributes in many disciplines including database systems [13, 42], formal concept analysis [26, 28], data mining [1, 55], logic programming [39, 48], and their applications. In these disciplines, the rules often appear under different names (e.g., attribute implications, functional dependencies, or

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simply “rules”) with semantics defined in various structures (e.g.,
transactional data, boolean matrices, or n-ary relations) but as it has been shown in [21],
the rules may be seen as propositional formulas with the semantic entailment
defined as in the propositional logic, possibly extended by additional measures
of interestingness. The rules are popular because of their easy readability for
non-expert users and tractability of the entailment problem which is decidable
in linear time [6]. Research on if-then rules is active and recent results include
new theoretical observations [5, 25, 31, 41, 49] on the rules and their general-
izations as well as applications in data analysis, formal languages, and related
areas [14, 19, 20, 22, 35, 38, 50, 53, 54].

In this paper, we introduce if-then formulas which express presence of attributes relatively in time and the formulas are evaluated in data where the presence or absence of attributes changes in time. In our approach, we adopt
the notion of a discrete time, i.e., the data are observed at distinct points in
time. We consider a formula valid in data changing over time if the if-then
dependency prescribed by the formula holds in all time points. We introduce
the formulas as expressions
\[
\{y_{i_1}^1, \ldots, y_{i_m}^m\} \Rightarrow \{z_{j_1}^1, \ldots, z_{j_n}^n\},
\]
where \(y_1, \ldots, y_m\) and \(z_1, \ldots, z_n\) are attributes which may be viewed as propositional variables, and \(i_1, \ldots, i_m, j_1, \ldots, j_n\) are integers annotating the attributes by relative time points with the following meaning: 0 denotes the present time
point, 1 is its immediate successor, −1 is the immediate predecessor of 0, 2 is
the immediate successor of 1, etc. With this interpretation of time points and
considering, for instance, the unit of time “a day”, formula \(\{x^{-1}, y^0\} \Rightarrow \{z^1\}\) prescribes the following dependency: “If \(x\) was present yesterday and \(y\) is present
today, then \(z\) will be present tomorrow.” From our perspective, a classic if-then
formula
\[
\{y_1, \ldots, y_m\} \Rightarrow \{z_1, \ldots, z_n\},
\]
may be seen as a particular case of (1), where all the relative time points
$i_1, \ldots, i_m, j_1, \ldots, j_n$ are equal to 0, and the data in which the formula is evaluated is constant in all time points.

We provide answers to several questions which emerge with formulas like (1). First, we define the notion of semantic entailment of the formulas, investigate closure structures of models of theories consisting of such formulas, and show that the problem of checking whether a formula is semantically entailed by a set of formulas can be reduced to checking its validity in a single model. Second, we prove that the semantic entailment has a complete axiomatization. That is, we show a notion of provability of formulas like (1) and show that it coincides with the semantic entailment. We discuss several possible axiomatizations, including ones that can be used to consider proofs in particular normal forms. Third, based on our insight into the properties of the semantic entailment and provability, we derive results on decidability and complexity of the entailment problem. Fourth, we include notes on the relationship of the formulas to formulas appearing in modal logics [9] and triadic formal concept analysis [34]. Similar rules as we consider in this paper appeared as inter-transaction association rules [52] inferred from time-changing transactional data. Despite the popularity of the rules in data mining, a logical analysis of the entailment of the rules and related properties is missing—providing the logical foundations is a goal of our paper.

Our paper is organized as follows. In Section 2 and Section 3 we present a survey of related work and short preliminaries. We introduce the formulas and present the results on their semantic entailment in Section 4. In Section 5 we give complete axiomatizations and in Section 6 we deal with the related computational issues. Finally, in Section 7 we present a conclusion.

2 Related Work

In database systems and knowledge engineering, there appeared isolated approaches which propose temporal semantics of if-then rules. We present here a short survey of the approaches and highlight the differences between our ap-
Formulas called temporal functional dependencies emerged in databases with time granularities [7]. In this approach, a time granularity is a general partition of time like seconds, weeks, years, etc., and a time granularity is associated to each relational schema. In addition, each tuple in a relation is associated with a part (so-called granule) of granularity. In this setting, temporal functional dependencies are like the ordinary functional dependencies [21] with a time granularity as an additional component. The concept of validity of temporal functional dependencies is defined in much the same way as its classic counterpart and includes an additional condition that granules of tuples need to be covered by any granule from granularity of the temporal functional dependency. Thus, [7] uses an ordinary notion of validity of functional dependencies which is restricted to some time segments. This is conceptually very different from the problem we deal with in this paper.

Several approaches to temporal if-then rules, which are conceptually similar to [7], appeared in the field of association rules [1] [55] as the so-called temporal association rules [2] [36] [46]. In these approaches, the input data is in the form of transactions (i.e., subsets of items) where each transaction occurred at some point in time and the interest of the papers lies in extracting association rules from data which occur during a specified time cycle. For instance, one may be interested in extracting rules which are valid in “every spring month of a year”, “every Monday in every year”, etc. As in the case of the temporal functional dependencies, the temporal association rules may be understood as classic association rules occurring during specified time cycles.

Other results motivated by temporal semantics of association rules includes the so-called inter-transaction association rules [23] [24] [33] [52], see [40] for a survey of approaches. The papers propose algorithms to extract, given an input transactional data and a measure of interestingness (based on levels of minimal support and confidence), if-then rules which are preserved over a given period of time. From this point of view, the rules can be seen as formulas studied in this paper restricted to so-called predictive rules (see Definition 33 in Sec-
tion 6) whose validity is considered with respect to the additional parameter of interestingness. As a consequence, the inter-transaction association rules are related to the rules in our approach in the same way as the ordinary association rules 11 are related to the ordinary attribute implications 26. The results in 23, 24, 33, 40, 52 are focused almost exclusively on algorithms for mining the inter-transaction association rules and are not concerned with problems of entailment of the rules and the underlying logic. In contrast, the problems of entailment of rules are central to this paper and we show there is reasonably strong logic for reasoning with such rules. Our observations may stimulate further development in the field of inter-transaction association rules and similar formulas and their applications in various domains 23, 30.

The formulas studied in this paper are also related to particular program rules which appear in Datalog extensions dealing with flow of time and related phenomena 10, 11, 12 such as Datalog\textsubscript{nS} (Datalog with \(n\) successors). The formulas we consider in our paper correspond to a fragment of rules which appear in such Datalog extensions. Despite the similar form of our formulas and the program rules, there does not seem to be a direct relationship (or a reduction) of the entailment problem of our formulas and the recognition problem of Datalog\textsubscript{nS} programs.

3 Preliminaries

In this section, we present the basic notions of closure systems (also known as Moore families) and closure operators which are used further in the paper. More details can be found in 8, 18.

If \(Y\) is a set, we denote by \(2^Y\) its power set. A closure operator on \(Y\) is a map \(c: 2^Y \rightarrow 2^Y\) such that

\[
A \subseteq c(A),
\]
\[
A \subseteq B \text{ implies } c(A) \subseteq c(B),
\]
\[
c(c(A)) \subseteq c(A),
\]

where
for all $A, B \subseteq Y$. The conditions (3)–(5) are called the extensivity, monotony, and idempotency of $c$, respectively. Note that (3) and (5) yield $c(A) = c(c(A))$ for all $A \subseteq Y$. A closure operator $c : 2^Y \to 2^Y$ is called an algebraic closure operator whenever

$$c(A) = \bigcup\{c(B) \mid B \subseteq A \text{ and } B \text{ is finite}\} \quad (6)$$

for all $A \subseteq Y$. Moreover, $A \subseteq Y$ is called a fixed point of $c$ whenever $c(A) = A$.

A system $S \subseteq 2^Y$ is called a closure system on $Y$ if it is closed under arbitrary intersections, i.e., $\bigcap A \in S$ for any $A \subseteq S$. In the paper we utilize the well-known correspondence between closure systems and closure operators on $Y$. In particular, if $c$ is an algebraic closure operator on $Y$, we call the closure system of all its fixed points the algebraic closure system induced by $c$.

### 4 Formulas, Models, and Semantic Entailment

In this section, we present a formalization of the formulas, their interpretation, and semantic entailment. Let us assume that $Y$ is a non-empty and finite set of symbols called attributes. Furthermore, we use integers in order to denote time points. We put

$$T_Y = \{y^i \mid y \in Y \text{ and } i \in \mathbb{Z}\} \quad (7)$$

and interpret each $y^i \in T_Y$ as “attribute $y$ observed in time $i$” (technically, $T_Y$ can be seen as the Cartesian product $Y \times \mathbb{Z}$). Under this notation, we may now formalize rules like (1) as follows:

**Definition 1.** An attribute implication over $Y$ annotated by time points in $\mathbb{Z}$ is a formula of the form $A \Rightarrow B$, where $A, B$ are finite subsets of $T_Y$.

As we have outlined in the introduction, the purpose of time points encoded by integers which appear in antecedents and consequents of the considered formulas is to express points in time relatively to a current time point. Hence, the intended meaning of (1) abbreviated by $A \Rightarrow B$ is the following: “For all time
points \( t \), if an object has all the attributes from \( A \) considering \( t \) as the current time point, then it must have all the attributes from \( B \) considering \( t \) as the current time point”. In what follows, we formalize the interpretation of \( A \Rightarrow B \) in this sense.

Since we wish to define formulas being true in all time points (we are interested in formulas preserved over time), we need to shift relative times expressed in antecedents and consequents in formulas with respect to a changing time point. For that purpose, for each \( M \subseteq T_Y \) and \( i \in \mathbb{Z} \), we may introduce a subset \( M + j \) of \( T_Y \) by

\[
M + j = \{ y^{i+j} \mid y^i \in M \}
\]  

(8)

and call it a time shift of \( M \) by \( j \) (shortly, a \( j \)-shift of \( M \)). In the paper, we utilize the following properties of time shifts.

**Proposition 2.** For all \( M, N \subseteq T_Y \), \( \{ N_k \subseteq T_Y \mid k \in K \} \), and \( i, j \in \mathbb{Z} \), we get

\[
\text{if } M \subseteq N \text{ then } M + i \subseteq N + i,
\]

(9)

\[
(M + i) + j = M + (i + j),
\]

(10)

\[
\bigcup_{k \in K} (N_k + i) = \bigcup_{k \in K} N_k + i,
\]

(11)

\[
\bigcap_{k \in K} (N_k + i) = \bigcap_{k \in K} N_k + i.
\]

(12)

**Proof.** All (9)–(12) follow directly from (8). \( \square \)

Based on (10), we may omit parentheses and write \( M + j + i \) instead of \( (M + i) + j \). Also, we write \( M - i \) to denote \( M + (-i) \).

Attribute implications annotated by time points are formulas, i.e., syntactic notions for which we define their semantics (interpretation) as follows.

**Definition 3.** A formula \( A \Rightarrow B \) is true in \( M \subseteq T_Y \) whenever, for each \( i \in \mathbb{Z} \),

\[
\text{if } A + i \subseteq M \text{, then } B + i \subseteq M
\]

(13)

and we denote the fact by \( M \models A \Rightarrow B \).
Figure 1: Daily weather observation from an airport station.

Remark 1. (a) The value of $i$ in the definition may be understood as a sliding time point. Moreover, $A + i$ and $B + i$ represent sets of attributes annotated by absolute time points considering $i$ as the current time point. Note that using (8), the condition (13) can be equivalently restated as “$A \subseteq M - i$ implies $B \subseteq M - i$,” i.e., instead of shifting the antecedents and consequents of the formula, we may shift the set $M$.

(b) Observe that $A \Rightarrow B$ is trivially true in $M$ whenever $B \subseteq A$ because in that case (13) trivially holds for any $i$. By definition, $A \Rightarrow B$ is not true in $M$, written $M \not\models A \Rightarrow B$ iff there is $i$ such that $A + i \subseteq M$ and $B + i \not\subseteq M$. In words, in the time point $i$, $M$ has all the attributes of $A$ but does not have an attribute in $B$, i.e., the time point $i$ serves as a counterexample.

Example 1. One particular example of a subset $M$ of $\mathcal{T}_Y$ can be a daily weather observation from an airport station. For instance, we can consider $Y$ as

$$Y = \{\text{rn, rl, rm, tv, tc, tm, wl, wm, ws}\},$$

where the attributes have the following meaning: “no rainfall” (denoted $\text{rn}$), “light rainfall” (denoted $\text{rl}$), “moderate rainfall” (denoted $\text{rm}$), “temperature
is very cold”, (denoted tv), “temperature is cold”, (denoted tc) “temperate is mild”, (denoted tm) “light wind” (denoted wl), “moderate wind” (denoted wm), and “strong wind” (denoted ws). A subset of $\mathcal{T}_Y$ may be depicted as a two-dimensional table with rows corresponding to time points, columns corresponding to attributes in $Y$, and crosses and blanks in the table, indicating whether attributes annotated by time points belong to the subset. For instance, if $M$ is given by the table in Figure 4, then $\text{rn}^{15} \in M$, $\text{rl}^{15} \notin M$, etc. In this case, we have $M \models \{\text{wl}^0, \text{wm}^1\} \Rightarrow \{\text{tc}^3\}$. On the other hand, $M \models \{\text{wm}^0, \text{wl}^1\} \Rightarrow \{\text{tc}^3, \text{rm}^3, \text{tc}^4\}$ because for $i = 22$, we have $\{\text{wm}^0, \text{wl}^1\} + 22 = \{\text{wm}^{22}, \text{wl}^{23}\} \subseteq M$ and $\{\text{tc}^3, \text{rm}^3, \text{tc}^4\} + 22 = \{\text{tc}^{25}, \text{rm}^{25}, \text{tc}^{26}\} \nsubseteq M$.

We consider the following notions of a theory and a model:

**Definition 4.** Let $\Sigma$ be a set of formulas (called a theory). A subset $M \subseteq \mathcal{T}_Y$ is called a model of $\Sigma$ if $M \models A \Rightarrow B$ for all $A \Rightarrow B \in \Sigma$. The system of all models of $\Sigma$ is denoted by $\text{Mod}(\Sigma)$, i.e.,

$$\text{Mod}(\Sigma) = \{M \subseteq \mathcal{T}_Y \mid M \models A \Rightarrow B \text{ for all } A \Rightarrow B \in \Sigma\}. \quad (14)$$

In general, $\text{Mod}(\Sigma)$ is infinite and there may be theories that do not have any finite model. For instance, consider a theory containing $\emptyset \Rightarrow \{y^0\}$.

We now turn our attention to the structure of systems of all models of attribute implications annotated by time points. In case of the ordinary attribute implications, it is well known that systems of their models are exactly closure systems in $Y$. Interestingly, the systems of models in our case are exactly the algebraic closure systems which are closed under time shifts. This additional closure property is introduced by the following definition.

**Definition 5.** A system $\mathcal{S} \subseteq 2^{\mathcal{T}_Y}$ of subsets of $\mathcal{T}_Y$ is called closed under time shifts whenever $M + i \in \mathcal{S}$ for all $M \in \mathcal{S}$ and $i \in \mathbb{Z}$.

We first show that $\text{Mod}(\Sigma)$ is a closure system closed under time shifts:

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1The data is based on discretization of real meteorological information for Aug 14 which can be found at [http://www.bom.gov.au/climate/dwo/IDCJDW0100.shtml](http://www.bom.gov.au/climate/dwo/IDCJDW0100.shtml)
Theorem 6. Let $\Sigma$ be a theory. Then, $\text{Mod}(\Sigma)$ is closed under arbitrary intersections and time shifts.

Proof. The fact that $\text{Mod}(\Sigma)$ is closed under arbitrary intersections follows by analogous arguments as in the case of ordinary attribute implications taking into account that (13) must hold for all $i \in \mathbb{Z}$. That is, for any $M \subseteq \text{Mod}(\Sigma)$ and arbitrary $A \Rightarrow B \in \Sigma$, we reason as follows. If $A + i \subseteq \bigcap \mathcal{M}$, then $A + i \subseteq M$ for all $M \in \mathcal{M}$ and thus $B + i \subseteq M$ for all $M \in \mathcal{M}$ because $\mathcal{M} \subseteq \text{Mod}(\Sigma)$. Therefore, $B + i \subseteq \bigcap \mathcal{M}$, proving $\bigcap \mathcal{M} | A \Rightarrow B$ which further gives $\bigcap \mathcal{M} \in \text{Mod}(\Sigma)$ since $A \Rightarrow B \in \Sigma$ was arbitrary.

In order to show that $\text{Mod}(\Sigma)$ is closed under time shifts, take $M \in \text{Mod}(\Sigma)$ and $j \in \mathbb{Z}$. It suffices to prove that $M + j \in \text{Mod}(\Sigma)$. In order to see that, take $A \Rightarrow B \in \Sigma$. If $A + i \subseteq M + j$, then $A + (i - j) \subseteq M$ and thus $B + (i - j) \subseteq M$ because $M \in \text{Mod}(\Sigma)$ and $A \Rightarrow B \in \Sigma$. Therefore, $B + i \subseteq M + j$, i.e., $M + j | A \Rightarrow B$ for arbitrary $A \Rightarrow B \in \Sigma$, showing $M + j \in \text{Mod}(\Sigma)$. 

Taking into account Theorem 6, for each theory $\Sigma$, we may consider a closure operator induced by $\text{Mod}(\Sigma)$ which maps each $M \subseteq T_Y$ to the least model of $\Sigma$ containing $M$.

Definition 7. Let $\Sigma$ be a theory. For each $M \subseteq T_Y$, we put

$$[M]_{\Sigma} = \bigcap \{ N \in \text{Mod}(\Sigma) | M \subseteq N \} \quad (15)$$

and call $[M]_{\Sigma}$ the **semantic closure** of $M$ under $\Sigma$.

Using the well-known relationship between closure operators and closure systems, $[\cdot \cdot]_{\Sigma}$ defined by (15) is indeed a closure operator. Note that in general, $[M]_{\Sigma}$ is infinite even if $Y$ and $M$ are finite. This is in contrast with the ordinary attribute implications using finite $Y$. Nevertheless, in our setting we can prove that even if $[M]_{\Sigma}$ is infinite, it can be obtained as a union of finitely generated elements of $\text{Mod}(\Sigma)$, showing that $\text{Mod}(\Sigma)$ is in fact an *algebraic* closure system.

Theorem 8. Let $\Sigma$ be a theory. For each $M \subseteq T_Y$, we have

$$[M]_{\Sigma} = \bigcup \{ [N]_{\Sigma} | N \text{ is finite subset of } M \}. \quad (16)$$
Proof. Observe that the monotony of $[\cdots]_{\Sigma}$ yields $[N]_{\Sigma} \subseteq [M]_{\Sigma}$ for any finite $N \subseteq M$ and thus the “⊇”-part of (16) is obvious.

For brevity, put $\mathcal{M} = \{[N]_{\Sigma} \mid N \text{ is finite subset of } M\}$. In order to prove the “⊆”-part of (16), it suffices to show that $\bigcup \mathcal{M}$ is a model of $\Sigma$ which contains $M$ because $[M]_{\Sigma}$ is the least model of $\Sigma$ containing $M$. For any $y^i \in M$, we have $\{y^i\}_{\Sigma} \in \mathcal{M}$ and thus $y^i \in \{y^i\}_{\Sigma} \subseteq \bigcup \mathcal{M}$ by the extensivity of $[\cdots]_{\Sigma}$ which proves $M \subseteq \bigcup \mathcal{M}$.

Now, take any $A \Rightarrow B \in \Sigma$ and suppose that $A + i \subseteq \bigcup \mathcal{M}$. Observe that for every $y^j \in A + i$ there is $[N_y]_{\Sigma} \in \mathcal{M}$ such that $y^j \in [N_y]_{\Sigma}$. Moreover, the fact that $A + i$ is finite yields that $\bigcup \{N_{y^j} \mid y^j \in A + i\}$ is finite and we thus have $\bigcup \{N_{y^j} \mid y^j \in A + i\}_{\Sigma} \in \mathcal{M}$. Clearly, $A + i \subseteq \bigcup \{N_{y^j} \mid y^j \in A + i\}_{\Sigma}$ and thus it follows that $B + i \subseteq \bigcup \{N_{y^j} \mid y^j \in A + i\}_{\Sigma} \subseteq \bigcup \mathcal{M}$ because $A \Rightarrow B \in \Sigma$. Altogether, $\bigcup \mathcal{M} \models A \Rightarrow B$ and so $\bigcup \mathcal{M} \in \text{Mod}(\Sigma)$. $\square$

Using Theorem 8, we may establish that each algebraic closure system closed under time shifts is a system of models of some theory consisting of attribute implications annotated by time points. Before we go to the proof, we show how the property of being closed under time shifts can be formulated in terms of closure operators.

Lemma 9. Let $\mathcal{S}$ be a closure system which is closed under arbitrary time shifts and let $C_{\mathcal{S}}$ be the induced closure operator. For each $M \subseteq T_Y$ and $i \in \mathbb{Z}$,

$$C_{\mathcal{S}}(M + i) = C_{\mathcal{S}}(M) + i. \quad (17)$$

Proof. “⊆”: Since $\mathcal{S}$ is closed under time shifts, we get $C_{\mathcal{S}}(M) + i \in \mathcal{S}$. In addition, $M + i \subseteq C_{\mathcal{S}}(M) + i$ on account of the extensivity of $C_{\mathcal{S}}$ and (16). Therefore, $C_{\mathcal{S}}(M + i) \subseteq C_{\mathcal{S}}(M) + i$ by monotony and idempotency of $C_{\mathcal{S}}$.

“⊇”: The extensivity of $C_{\mathcal{S}}$ gives $M + i \subseteq C_{\mathcal{S}}(M + i)$ and thus $M \subseteq C_{\mathcal{S}}(M + i) - i$. Moreover, $C_{\mathcal{S}}(M + i) - i \in \mathcal{S}$ because $\mathcal{S}$ is closed under time shifts and thus $C_{\mathcal{S}}(M) \subseteq C_{\mathcal{S}}(M + i) - i$ which gives $C_{\mathcal{S}}(M) + i \subseteq C_{\mathcal{S}}(M + i)$. $\square$

Lemma 10. Let $C$ be a closure operator satisfying $C(M + i) = C(M) + i$ for
each $M \subseteq T_Y$ and $i \in \mathbb{Z}$. Then, the system $S_C$ of all fixed points of $C$ is closed under arbitrary time shifts.

Proof. Take $M \in S_C$ and any $i \in \mathbb{Z}$, i.e., $M \subseteq T_Y$ such that $M = C(M)$. Clearly, $M + i = C(M) + i$ and since $C(M) + i = C(M + i)$, we get $M + i = C(M + i)$, proving that $M + i \in S_C$. \hfill \Box

The previous two lemmas give the following consequence.

Corollary 11. A closure system $S$ is closed under arbitrary time shifts iff the corresponding closure operator $C_S$ satisfies (17).

Based on our previous observations, we may now establish the connection between systems of models of attribute implications annotated by time points and algebraic closure systems closed under time shifts.

Theorem 12. Let $S \subseteq 2^{T_Y}$ be an algebraic closure system which is closed under time shifts. Then, there is a theory $\Sigma$ such that $S = \text{Mod}(\Sigma)$.

Proof. Assume that $C_S$ is the closure operator induced by $S$ and put

$$
\Sigma = \{ A \Rightarrow B \mid A \subseteq T_Y, B \subseteq C_S(A), \text{ and } A, B \text{ are finite}\}.
$$

We show that $S = \text{Mod}(\Sigma)$ by proving that both inclusions hold.

“\(\subseteq\)”: Take $M \in S$ and a finite $B \subseteq C_S(A)$ for a finite $A \subseteq T_Y$. We now check that $M \models A \Rightarrow B$. Assume that $A + i \subseteq M$. Then, $A \subseteq M - i$ and by the monotony of $C_S$ and utilizing (17), we have $C_S(A) \subseteq C_S(M - i) = C_S(M) - i = M - i$ which yields that $B \subseteq M - i$, i.e., $B + i \subseteq M$, showing $M \models A \Rightarrow B$. As a consequence, $S \subseteq \text{Mod}(\Sigma)$.

“\(\supseteq\)”: We let $M \in \text{Mod}(\Sigma)$ and prove that $M \in S$ which means to prove that $C_S(M) = M$. Since $S$ is an algebraic closure system, it suffices to check that $C_S(A) \subseteq M$ for each finite $A \subseteq M$. Assuming that $A \subseteq M$ and $A$ is finite, take any finite $B \subseteq C_S(A)$. By definition, $A \Rightarrow B \in \Sigma$ and since $M \in \text{Mod}(\Sigma)$, we get that for $i = 0$, $A + 0 \subseteq M$ implies $B + 0 \subseteq M$. Since $A + 0 = A$ and $A \subseteq M$, we therefore obtain $B = B + 0 \subseteq M$. Since $B$ was an arbitrary finite subset of $C_S(A)$, we conclude that $C_S(A) \subseteq M$. \hfill \Box
We now define semantic entailment of formulas and explore its properties. The notion is defined the usual way using the notion of a model introduced before.

**Definition 13.** Let \( \Sigma \) be a theory. Formula \( A \Rightarrow B \) is semantically entailed by \( \Sigma \) if \( M \models A \Rightarrow B \) for each \( M \in \text{Mod}(\Sigma) \).

The following lemma justifies the description of time points in attribute implications as relative time points. Namely, it states that each \( A \Rightarrow B \) semantically entails all formulas resulting by shifting the antecedent and consequent of \( A \Rightarrow B \) by a constant factor.

**Lemma 14.** \( \{ A \Rightarrow B \} \models A + i \Rightarrow B + i \).

*Proof.* Take \( M \in \text{Mod}(\{ A \Rightarrow B \}) \) and let \( (A + i) + j \subseteq M \). Then, \( A + i \subseteq M - j \) and by Theorem, we get \( M - j \in \text{Mod}(\{ A \Rightarrow B \}) \) which yields \( B + i \subseteq M - j \) and thus \( (B + i) + j \subseteq M \), proving \( M \models A + i \Rightarrow B + i \). \( \square \)

Analogously as for the classic attribute implications, the semantic entailment of \( A \Rightarrow B \) by a theory \( \Sigma \) can be checked using the least model of \( \Sigma \) generated by \( A \) as it is shown in the following theorem.

**Theorem 15.** For any \( \Sigma \) and \( A \Rightarrow B \), the following conditions are equivalent:

1. \( \Sigma \models A \Rightarrow B \),
2. \( [A]_\Sigma \models A \Rightarrow B \),
3. \( B \subseteq [A]_\Sigma \).

*Proof.* Clearly, (i) implies (ii) since \( [A]_\Sigma \in \text{Mod}(\Sigma) \); (ii) implies (iii) because \( A + 0 \subseteq [A]_\Sigma \). Assume that (iii) holds and take \( M \in \text{Mod}(\Sigma) \) and \( i \in \mathbb{Z} \) such that \( A + i \subseteq M \). Then, \( A \subseteq M - i \) and thus \( B \subseteq [A]_\Sigma \subseteq [M - i]_\Sigma = [M]_\Sigma - i \) by (17) from which it follows that \( B + i \subseteq [M]_\Sigma = M \), proving (i). \( \square \)

We conclude this section by notes on the propositional semantics of our formulas. The classic attribute implications on finite \( Y \) can be understood as
propositional formulas. Namely, an attribute implication of the from (2) can be seen as a propositional formula

\[(y_1 \& \cdots \& y_m) \Rightarrow (z_1 \& \cdots \& z_n),\]  

where & is the symbol for conjunction and \(y_1, \ldots, y_m, z_1, \ldots, z_n\) are propositional variables. Thus, (18) may be called a propositional counterpart of (2). Obviously, there are in general several propositional counterparts of (2) since formulas equivalent to (18) in sense of the propositional logic result, e.g., by re-arranging the propositional variables \(y_1, \ldots, y_m, z_1, \ldots, z_n\) in a different order. We neglect this aspect and always consider a fixed propositional counterpart of each attribute implication. It can be shown that if one takes the propositional counterparts of attribute implications, then their semantic entailment in sense of the propositional logic coincides with the semantic entailment as it is defined for attribute implications. We now show that an analogous correspondence can also be established in our case.

We start by considering the following notation. For any finite \(A, B \subseteq \mathcal{T}_Y\) and for any \(M \subseteq \mathcal{T}_Y\), we put \(M \models_{PL} A \Rightarrow B\) whenever \(A \not\subseteq M\) or \(B \subseteq M\). That is, \(M \models_{PL} A \Rightarrow B\) means that \(A \Rightarrow B\) is true in \(M\) as a classical attribute implication. Clearly, \(M \models_{PL} A \Rightarrow B\) does not imply that \(M \models A \Rightarrow B\) in sense of Definition 3. Moreover, we may introduce the set of models of \(\Sigma\) in the classic sense:

\[\text{Mod}_{PL}(\Sigma) = \{M \subseteq \mathcal{T}_Y \mid M \models_{PL} A \Rightarrow B \text{ for all } A \Rightarrow B \in \Sigma\}\]  

and put \(\Sigma \models_{PL} A \Rightarrow B\) whenever \(M \models_{PL} A \Rightarrow B\) for all \(M \in \text{Mod}_{PL}(\Sigma)\). Therefore, \(\models_{PL}\) denotes the semantic entailment of attribute implications in the classic sense. Again, \(\models_{PL}\) is in general different from \(\models\) introduced in Definition 13 but we can establish the following characterization:

**Theorem 16.** Let \(\Sigma\) be a theory and let

\[\Sigma^{PL} = \{A + i \Rightarrow B + i \mid A \Rightarrow B \in \Sigma \text{ and } i \in \mathbb{Z}\}.\]  

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Then $\text{Mod}(\Sigma) = \text{Mod}^{\text{PL}}(\Sigma^{\text{PL}})$. As a consequence, for each $A \Rightarrow B$, we have $\Sigma \models A \Rightarrow B$ iff $\Sigma^{\text{PL}} \models_{\text{PL}} A \Rightarrow B$.

Proof. The first part of the claim is easy to see. Indeed, for each $A \Rightarrow B$ we have $M \in \text{Mod}(\{A \Rightarrow B\})$ iff for each $i \in \mathbb{Z}$, we have $A + i \subseteq M$ implies $B + i \subseteq M$ which is true iff $M \in \text{Mod}^{\text{PL}}(\{A + i \Rightarrow B + i \mid i \in \mathbb{Z}\})$. Hence, it follows that $\text{Mod}(\Sigma) = \text{Mod}^{\text{PL}}(\Sigma^{\text{PL}})$.

Now, assume that $\Sigma \models A \Rightarrow B$ and take $M \in \text{Mod}^{\text{PL}}(\Sigma^{\text{PL}})$ such that $A \subseteq M$. Then $A + 0 \subseteq M$ and $M \in \text{Mod}(\Sigma)$ and thus $B = B + 0 \subseteq M$, proving that $\Sigma^{\text{PL}} \models_{\text{PL}} A \Rightarrow B$. Conversely, let $\Sigma^{\text{PL}} \models_{\text{PL}} A \Rightarrow B$ and $A + i \subseteq M$ for $M \in \text{Mod}(\Sigma)$. That is, we have $A \subseteq M - i$ and, owing to Theorem 6, $M - i \in \text{Mod}(\Sigma) = \text{Mod}^{\text{PL}}(\Sigma^{\text{PL}})$. As a consequence of $M - i \models_{\text{PL}} A \Rightarrow B$, we get $B \subseteq M - i$ and thus $B + i \subseteq M$, showing $\Sigma \models A \Rightarrow B$. Altogether, $\Sigma \models A \Rightarrow B$ iff $\Sigma^{\text{PL}} \models_{\text{PL}} A \Rightarrow B$. \qed

Now, based on Theorem 16, we may argue that for each $\Sigma$ there is a set of propositional formulas $\Sigma'$ such that the propositional counterpart of $A \Rightarrow B$ follows by $\Sigma'$ in sense of the propositional logic. Indeed, $\Sigma'$ can be taken as the set of propositional counterparts to all formulas in $\Sigma^{\text{PL}}$: Owing to Theorem 10, $A \Rightarrow B$ follows by $\Sigma^{\text{PL}}$ as a classic attribute implication over (a denumerable set of attributes) $\mathcal{T}_V$ and thus the propositional counterpart of $A \Rightarrow B$ follows by the propositional counterparts to all formulas in $\Sigma^{\text{PL}}$.

5 Deduction Systems and Complete Axiomatizations

In this section, we present a deduction system for our formulas and a related notion of provability which represents the syntactic entailment of formulas. The provability is based on an extension of the Armstrong axiomatic system [3] which is well known mainly in database systems [42]. The extension we propose accommodates the fact that time points in formulas are relative. The deductive system we use consists of the following deduction rules.

Definition 17. We introduce the following deduction rules:
(Ax) infer $A \cup B \Rightarrow A$,

(Cut) from $A \Rightarrow B$ and $B \cup C \Rightarrow D$ infer $A \cup C \Rightarrow D$,

(Shf) from $A \Rightarrow B$ infer $A + i \Rightarrow B + i$,

where $i \in \mathbb{Z}$ and $A, B, C, D$ are arbitrary finite subsets of $TY$.

Remark 2. (a) Note that there are several equivalent systems which are called the Armstrong systems \[42\]. In our presentation, the rule (Ax) can be seen as a nullary deduction rule which is an axiom scheme, i.e., each $A \cup B \Rightarrow A$ may be called an axiom. (Cut) and (Shf) are binary and unary deduction rules, respectively. In the classic case, (Ax) and (Cut) form a system which is equivalent to that from \[3\]. We call the additional rule (Shf) the rule of “time shifts.” Also note that in the database literature, (Cut) is also referred to as the rule of pseudo-transitivity \[42\].

(b) The rules in Definition 17 can be written as fractions with hypotheses (formulas preceding “infer”) above the conclusion (formula following “infer”) as

\[
\begin{align*}
\frac{A \cup B \Rightarrow A}{A \Rightarrow B}^{(Ax)}, & \quad \frac{A \Rightarrow B, B \cup C \Rightarrow D}{A \cup C \Rightarrow D}^{(Cut)}, & \quad \frac{A \Rightarrow B}{A + i \Rightarrow B + i}^{(Shf)}.
\end{align*}
\]

Although we are going to use (Ax), (Cut), and (Shf) as the basic deduction rules in our system, we define the notion of provability relatively to a collection of deduction rules because we later investigate systems consisting of other rules. Thus, a general deduction system is a set $R$ of $n$-ary rules of the form “from $\varphi_1, \ldots, \varphi_n$, infer $\psi$”.

Definition 18. Let $R$ be a deduction system. An $R$-proof of $A \Rightarrow B$ by $\Sigma$ is a finite sequence $\delta_1, \ldots, \delta_n$ such that $\delta_n$ equals $A \Rightarrow B$ and for each $i = 1, \ldots, n$ we have

(i) $\delta_i \in \Sigma$, or

(ii) $R$ contains a rule “from $\varphi_1, \ldots, \varphi_n$ infer $\psi$” such that $\psi$ is equal to $\delta_i$ and we have $\{\varphi_1, \ldots, \varphi_n\} \subseteq \{\delta_j \mid j < i\}$. 

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We say that \( A \Rightarrow B \) is \( \mathcal{R} \)-provable by \( \Sigma \), denoted \( \Sigma \vdash_{\mathcal{R}} A \Rightarrow B \), if there is an \( \mathcal{R} \)-proof of \( A \Rightarrow B \) by \( \Sigma \).

If \( \mathcal{R} \) consists solely of (Ax), (Cut), and (Shf), we write just \( \Sigma \vdash A \Rightarrow B \) and call \( A \Rightarrow B \) provable by \( \Sigma \). Analogously, we use the term “proof” instead of “\( \mathcal{R} \)-proof”. In the paper, we use the following properties of provability.

**Proposition 19.** For every finite \( A, B, C, D \subseteq \mathcal{T} \), we have

1. (Ref) \( \vdash A \Rightarrow A \),
2. (Wea) \( \{ A \Rightarrow C \} \vdash A \cup B \Rightarrow C \),
3. (Acc) \( \{ A \Rightarrow B \cup C, C \Rightarrow D \cup E \} \vdash A \Rightarrow B \cup C \cup D \),
4. (Add) \( \{ A \Rightarrow B, A \Rightarrow C \} \vdash A \Rightarrow B \cup C \),
5. (Aug) \( \{ B \Rightarrow C \} \vdash A \cup B \Rightarrow A \cup C \),
6. (Pro) \( \{ A \Rightarrow B \cup C \} \vdash A \Rightarrow B \),
7. (Tra) \( \{ A \Rightarrow B, B \Rightarrow C \} \vdash A \Rightarrow C \).

**Proof.** The laws hold because our system is an extension of the Armstrong system in which the laws hold as well, see [3, 42].

Our inference system is sound in the usual sense:

**Theorem 20** (soundness). If \( \Sigma \vdash A \Rightarrow B \) then \( \Sigma \models A \Rightarrow B \).

**Proof.** The proof goes by induction on the length of a proof, considering the facts that each axiom is true in all models, (Cut) is a sound deduction rule [42], and (Shf) is sound on account of Lemma 14. In a more detail, let \( \delta_1, \ldots, \delta_n \) be a proof by \( \Sigma \) and let \( \Sigma \models \delta_i \) for all \( i < j \). Then, if \( \delta_j \) results by \( \delta_i \) using (Shf) for some \( i < j \), then \( \Sigma \models \delta_i \) yields that \( M \models \delta_i \) for all \( M \in \text{Mod}(\Sigma) \) and thus, using Lemma 14, \( M \models \delta_j \) for all \( M \in \text{Mod}(\Sigma) \), showing \( \Sigma \models \delta_j \). The rest follows as in the classic case.

In the proof of completeness, we utilize the notion of a syntactic closure which is introduced as follows.
Definition 21. Let $\Sigma$ be a theory. For each $M \subseteq T_Y$, we put

$$M^0_\Sigma = M,$$

$$M^{n+1}_\Sigma = M^n_\Sigma \cup \bigcup \{ F + i \mid E \Rightarrow F \in \Sigma \text{ and } E + i \subseteq M^n_\Sigma \},$$

and call $M^\omega_\Sigma$ the syntactic closure of $M$ under $\Sigma$.

By the Tarski fixpoint theorem [51], the operator which maps $M$ to $M^\omega_\Sigma$ defined by (23) is indeed a closure operator, so the term “closure” in the name syntactic closure is appropriate. The following observation shows that the term “syntactic” is also appropriate since closures are directly related to provability.

Lemma 22. Let $A, B \subseteq T_Y$ be finite. Then, $B \subseteq A^n_\Sigma$ iff $\Sigma \vdash A \\Rightarrow B$.

Proof. Suppose that $B \subseteq A^n_\Sigma$. Since $B$ is finite, there is $m$ such that $B \subseteq A^m_\Sigma$. Thus, in order to show that $\Sigma \vdash A \Rightarrow B$, it suffices to check that for every $n$ and every finite $D \subseteq A^n_\Sigma$, we have $\Sigma \vdash A \Rightarrow D$ since then the claim readily follows for $D = B$ and $n = m$. By induction, assume the claim holds for $n$ and all finite $D \subseteq A^n_\Sigma$. Consider $n + 1$ and take a finite $D \subseteq A^{n+1}_\Sigma$. Now, consider a finite

$$D' = \{ (E \Rightarrow F, i) \mid E \Rightarrow F \in \Sigma \text{ and } E + i \subseteq A^n_\Sigma \}$$

such that

$$D \subseteq A^n_\Sigma \cup \bigcup \{ F + i \mid (E \Rightarrow F, i) \in D' \} \subseteq A^{n+1}_\Sigma.$$

Notice that since we assume $D$ finite, such finite $D'$ always exists. Now, by induction hypothesis, for each $(E \Rightarrow F, i) \in D'$, we have $\Sigma \vdash A \Rightarrow E + i$ owing to $E + i \subseteq A^n_\Sigma \subseteq A^{n+1}_\Sigma$. Furthermore, for $E \Rightarrow F \in \Sigma$, we have $\Sigma \vdash E + i \Rightarrow F + i$ using (Shf). Thus, (Tra) gives $\Sigma \vdash A \Rightarrow F + i$ for each $(E \Rightarrow F, i) \in D'$. In addition to that, $D \cap A^n_\Sigma \subseteq A^n_\Sigma$ and thus $\Sigma \vdash A \Rightarrow D \cap A^n_\Sigma$. Since $D'$ is finite and $D \subseteq (D \cap A^n_\Sigma) \cup \bigcup \{ F + i \mid (E \Rightarrow F, i) \in D' \}$, $\Sigma \vdash A \Rightarrow D$ follows by finitely many applications of (Add) and (Pro). As a consequence, $\Sigma \vdash A \Rightarrow B$.

Conversely, assume that $\Sigma \vdash A \Rightarrow B$. By Theorem [20] $\Sigma \models A \Rightarrow B$. We show that $A^n_\Sigma \in \text{Mod}(\Sigma)$. Take $E \Rightarrow F \in \Sigma$, $i \in \mathbb{Z}$ and let $E + i \subseteq A^n_\Sigma$. Since
$E+i$ is finite, there must be $n$ such that $E+i \subseteq A^n_\Sigma$ and thus $F+i \subseteq A^{n+1}_\Sigma \subseteq A^*_\Sigma$, proving that $A^*_\Sigma \in \text{Mod}(\Sigma)$. Now, $\Sigma \models A \Rightarrow B$ and $A + 0 = A \subseteq A^*_\Sigma$ yields that $B + 0 = B \subseteq A^*_\Sigma$.

Note that Lemma 22 is in fact a syntactic counterpart of Theorem 15. Now, using previous observations, we derive that our logic is complete:

**Theorem 23** (completeness). $\Sigma \vdash A \Rightarrow B$ iff $\Sigma \models A \Rightarrow B$.

**Proof.** If $\Sigma \not\vdash A \Rightarrow B$, we prove that there is $M \in \text{Mod}(\Sigma)$ such that $M \not\models A \Rightarrow B$. Indeed, we show that one can take $A^*_\Sigma$ for $M$. By Lemma 22, $\Sigma \not\vdash A \Rightarrow B$ yields $B \not\subseteq A^*_\Sigma$. So, for $i = 0$, we have that $A+i = A \subseteq A^*_\Sigma$ and $B+i = B \not\subseteq A^*_\Sigma$, i.e., $A^*_\Sigma \not\models A \Rightarrow B$. In addition to that, if $E+i \subseteq A^*_\Sigma$ for $E \Rightarrow F \in \Sigma$ and $i \in \mathbb{Z}$, then $\Sigma \vdash A \Rightarrow E+i$ by Lemma 22 and so $\Sigma \vdash A \Rightarrow F+i$ using (Shf) and (Tra). Using Lemma 22 again, $F+i \subseteq A^*_\Sigma$ which proves $A^*_\Sigma \in \text{Mod}(\Sigma)$. The rest is a consequence of Theorem 20.

As a corollary of the previous observations, we get the following assertion showing that both the syntactic and semantic closures coincide.

**Theorem 24.** For every $M \subseteq \mathcal{T}_Y$, we have $[M]_\Sigma = M^w_\Sigma$.

**Proof.** We get $[M]_\Sigma \subseteq M^w_\Sigma$ since $[M]_\Sigma$ is the least model of $\Sigma$ containing $M$. Conversely, observe that for any $N \in \text{Mod}(\Sigma)$ such that $M \subseteq N$, it follows that $M^w_\Sigma \subseteq N^w_\Sigma = N$. Hence, for $N$ being $[M]_\Sigma$, we get $M^w_\Sigma \subseteq [M]_\Sigma$.

**Remark 3.** Let us stress that the notions of semantic and syntactic entailment we have considered in our paper are different from their classic counterparts. Indeed, each attribute implication annotated by time points can also be seen as a classic attribute implication *per se* because the sets $A$ and $B$ in $A \Rightarrow B$ are subsets of $\mathcal{T}_Y$. Therefore, in addition to the semantic entailment from Definition 13, we may consider the ordinary one which disregards the special role of time points. The same applies to the provability—the classic notion is obtained by omitting the rule (Shf). For instance, $\Sigma = \{ \{x^1\} \Rightarrow \{y^2\}, \{y^5\} \Rightarrow \ldots \}$.
\{z^2\} proves \(\{x^4\} \Rightarrow \{y^5\}\) by (Shf) and thus \(\{x^4\} \Rightarrow \{z^2\}\) by (Tra). On the other hand, \(\Sigma\) does not prove \(\{x^4\} \Rightarrow \{z^2\}\) without (Shf).

**Remark 4.** (a) We can show that our system of deduction rules consisting of (Ax), (Cut), and (Shf) is non-redundant, i.e., all the rules in the system are independent. Indeed, no formulas are provable by \(\Sigma = \emptyset\) using only (Cut) and (Shf) and thus (Ax) is independent. Moreover, (Cut) is independent since all formulas provable by \(\Sigma = \emptyset\) using only (Ax) and (Shf) are exactly all instances of (Ax). The independence of (Shf) follows by Remark 3.

(b) Let us note that the deductive system in Definition 17 is not minimal in terms of the number of deduction rules. Indeed, we may replace (Cut) and (Shf) by a single deduction rule

\[
\frac{A \Rightarrow B + i, \ B \cup C \Rightarrow D}{A \cup (C + i) \Rightarrow D + i} \quad \text{(Cut)}. \tag{24}
\]

Indeed, observe that (Cut) is a particular case of (Cut) for \(i = 0\) and (Shf) results by (Cut) and (Ax) for \(A = B = \emptyset\). Conversely, \(\{A \Rightarrow B + i, \ B \cup C \Rightarrow D\} \vdash A \cup (C + i) \Rightarrow D + i\) because using (11), the sequence

\[
A \Rightarrow B + i, \ B \cup C \Rightarrow D, (B + i) \cup (C + i) \Rightarrow D + i, A \cup (C + i) \Rightarrow D + i
\]

is a proof of \(A \cup (C + i) \Rightarrow D + i\) using (Cut) and (Shf). As a consequence, the system consisting of (Ax), (Cut), and (Shf) is equivalent to (Ax) and (Cut).

(c) An alternative deduction system for our logic can be based on (Ref) instead of (Ax) and a single rule which is a modification of a simplification deduction rule [15]. First, it is easily seen that (Ax) and (Cut) may be equivalently replaced by the following rule and (Ref):

\[
\frac{A \Rightarrow B, \ C \Rightarrow D}{A \cup (C \setminus B) \Rightarrow D} \quad \text{(Sim)}. \tag{25}
\]

Indeed, (Sim) is a rule derivable by (Ax) and (Cut) because the sequence

\[
A \Rightarrow B, B \cup C \Rightarrow C, C \Rightarrow D, B \cup C \Rightarrow D, A \cup (C \setminus B) \Rightarrow D,
\]

is a proof of \(A \cup (C \setminus B) \Rightarrow D\) using (Ax) and (Cut).
is a proof of $A \cup (C \setminus B) \Rightarrow D$ by \{A $\Rightarrow$ B, C $\Rightarrow$ D\} using (Ax) and (Cut); apply the rule twice and observe that $B \cup C = B \cup (C \setminus B)$. Conversely, observe first that (Ax) is derivable by (Ref) and (Sim) because from $B \Rightarrow B$ and $A \Rightarrow A$ it follows that $B \cup (A \setminus B) \Rightarrow A$ that is, $A \cup B \Rightarrow A$. Moreover, (Cut) is derivable by (Ref) and (Sim) because the following sequence
\[ C \Rightarrow C, \emptyset \Rightarrow \emptyset, A \Rightarrow B, B \cup C \Rightarrow D, A \cup ((B \cup C) \setminus B) \Rightarrow D, A \cup C \Rightarrow D, \]
is a proof of $A \cup C \Rightarrow D$ by \{A $\Rightarrow$ B, $B \cup C \Rightarrow D$\} which we have used (Sim) three times and utilized the fact that $C \cup ((A \cup ((B \cup C) \setminus B)) \setminus \emptyset) = A \cup C$. Altogether, (Ax) and (Cut) can indeed be replaced by (Ref) and (Sim). Note that (Sim) may be perceived even more natural than (Cut) because it is applicable to any two input formulas. Note that a rule analogous to (Sim) with the inferred formula being $A \cup (C \setminus B) \Rightarrow B \cup D$ was first proposed by Darwen [16, page 140]. Now, we may consider an extension of (Sim) which involves time shifts:
\[ A \Rightarrow B + i, \; C \Rightarrow D \]
\[ A \cup ((C \setminus B) + i) \Rightarrow D + i \quad \text{(Sim$_i$).} \] (26)

Analogously as in the case of (Cut$_i$), (Sim) is a particular case of (Sim$_i$) for $i = 0$ and (Shf) results by (Sim$_i$) and (Ref) for $A = B = \emptyset$. Therefore, the deductive system in Definition L7 can be replaced by (Ref) and (Sim$_i$).

We now focus on the order in which the deduction rules may be applied in proofs. We show that each proof may be transformed into a normalized proof which involves applications of deduction rules in a special order. First, we show that (Shf) commutes with the other rules. Formally, we introduce the property for a general deduction rule $R$ as follows:

Let $R$ be a deduction rule of the form “from $\varphi_1, \ldots, \varphi_n$ infer $\psi$”. We say that (Shf) commutes with $R$ if for any formula $\chi$ which results by $\psi$ using (Shf) there are $\varphi'_1, \ldots, \varphi'_n$ which result by $\varphi_1, \ldots, \varphi_n$ using (Shf), respectively, such that $\chi$ is provable by $\{\varphi'_1, \ldots, \varphi'_n\}$ using $R$.

**Lemma 25.** (Shf) commutes with (Ax), (Cut), and (Shf).
Proof. Clearly, (Shf) commutes with (Ax) because the result of application of (Shf) to an instance of (Ax) is again an instance of (Ax). Moreover, (Shf) commutes with itself since \((A + i) + j = A + (i + j)\) for any \(A \subseteq T_Y\) and \(i, j \in \mathbb{Z}\). Therefore, it remains to check that (Shf) commutes with (Cut).

Consider formulas \(A \Rightarrow B\) and \(B \cup C \Rightarrow D\) and the formula \(A \cup C \Rightarrow D\) which results by (Cut) and formula \((A \cup C) + i \Rightarrow D + i\) which results by (Shf). Clearly, if we apply (Shf) to \(A \Rightarrow B\) and \(B \cup C \Rightarrow D\) for \(i\), we obtain \(A + i \Rightarrow B + i\) and \((B \cup C) + i \Rightarrow D + i\), respectively. The second formula equals \((B + i) \cup (C + i) \Rightarrow D + i\) which equals \((A \cup C) + i \Rightarrow D + i\), proving that (Shf) commutes with (Cut). \(
\)

**Theorem 26.** \(\Sigma \vdash A \Rightarrow B\) iff there is a finite \(\Sigma' \subseteq \Sigma_{PL}\) such that \(\Sigma' \vdash_R A \Rightarrow B\) for \(R\) containing (Ax) and (Cut).

Proof. In order to see the only-if part, assume that \(\Sigma \vdash A \Rightarrow B\) which means there is a proof of \(A \Rightarrow B\) by \(\Sigma\). The proofs contains only finitely many formulas in \(\Sigma\) and thus, we may consider a finite \(\Sigma'' \subseteq \Sigma\) such that \(\Sigma'' \vdash A \Rightarrow B\). Moreover, the proof contains only finitely many applications of (Shf) and, using Lemma 25 there is a proof of \(A \Rightarrow B\) by \(\Sigma''\) which starts by formulas in \(\Sigma''\), then continues with applications of (Shf), and terminates with formulas derived without using (Shf). Therefore, there is a finite \(\Sigma' \subseteq (\Sigma'')_{PL} \subseteq \Sigma_{PL}\) such that \(A \Rightarrow B\) is provable by \(\Sigma'\) using only (Ax) and (Cut). The if-part of the assertion is easy to see. \(
\)

The previous observation allows us to introduce special derivation sequences which represent proofs in a normalized form in that all utilized deduction rules are applied in a particular order. The proofs are constructed using deduction rules (Ref), (Shf), (Acc), and (Pro), see Proposition 19.

**Definition 27.** A finite sequence of formulas \(\varphi_1, \ldots, \varphi_n\) is called a normalized derivation sequence of \(A \Rightarrow B\) using formulas in \(\Sigma\) if the sequence

\(i\) starts with finitely many formulas in \(\Sigma\); \n
\(ii\) continues by formulas obtained using (Shf) applied to formulas in \(i\);
(iii) continues by \( A \Rightarrow A \);

(iv) continues by formulas obtained using (Acc) whose first argument is the preceding formula and the second argument is a formula in (i) or (ii);

(v) terminates with \( A \Rightarrow B \) which results by the preceding formula by (Pro).

Normalized derivation sequences are sufficient and adequate means for determining provability of formulas:

**Theorem 28.** \( \Sigma \vdash A \Rightarrow B \) iff there is a normalized derivation sequence of \( A \Rightarrow B \) using formulas in \( \Sigma \).

*Proof.* The if-part follows directly by the fact that a normalized derivation sequence of \( A \Rightarrow B \) using formulas in \( \Sigma \) is a proof of \( A \Rightarrow B \) by \( \Sigma \) using (Ref), (Shf), (Acc), and (Pro). Since all of them are rules derivable by (Ax), (Cut), and (Shf), see Proposition 19, we get \( \Sigma \vdash A \Rightarrow B \).

Conversely, by Theorem 26 we get that \( A \Rightarrow B \) is provable by a finite \( \Sigma' \subseteq \Sigma^{PL} \) using only (Ref) and (Cut). Therefore, we may form the (i) and (ii)-parts of the derivation sequence using the formulas in \( \Sigma' \) followed by \( A \Rightarrow A \).

Next, observe that there is a finite sequence \( A_0, \ldots, A_n \) of subsets of \( T_Y \) such that \( A_0 = A, A_i = A_{i-1} \cup F \) for some \( E \Rightarrow F \in \Sigma' \) satisfying \( E \subseteq A_{i-1} \), and \( A_n \supseteq B \). In order to see that, consider (22) and the fact that \( A \Rightarrow B \) is provable by \( \Sigma' \) without using (Shf). By moment’s reflection, we can see that the (iv)-part of the derivation sequence is formed of formulas \( A \Rightarrow A_i \ (i = 0, \ldots, n) \), and the sequence is terminated by a single application of (Pro) to obtain \( A \Rightarrow B \).  

We conclude the section by showing further properties of provability. The next assertion may be viewed as a type of a deduction theorem.

**Theorem 29.** Let \( \Sigma \) be a theory and \( A, B \subseteq T_Y \) be finite. Then the following statements are equivalent:

(i) \( \Sigma \cup \{\emptyset \Rightarrow A\} \vdash \emptyset \Rightarrow B \),

(ii) there are \( i_1, \ldots, i_n \in \mathbb{Z} \) such that \( \Sigma \vdash \bigcup_{m=1}^n (A + i_m) \Rightarrow B \).
Proof. “(i) ⇒ (ii)” Let $A_1 \Rightarrow B_1, \ldots, A_n \Rightarrow B_n$ be a proof of $\emptyset \Rightarrow B$ by $\Sigma \cup \{ \emptyset \Rightarrow A \}$. For each $p = 1, \ldots, n$, we show that there are $i_1, \ldots, i_{p_n} \in \mathbb{Z}$ for which $\Sigma \vdash A_p \cup \bigcup_{m=1}^{p_n} (A + i_m) \Rightarrow B_p$. The proof goes by induction on $p$. Thus, take $p = 1, \ldots, n$ and assume the claim holds for all $q < p$. We distinguish the following cases:

- $A_p \Rightarrow B_p$ is an instance of (Ax). Then, we let $p_n = 1$, $i_1 = 0$, and thus $A_p \cup \bigcup_{m=1}^{p_n} (A + i_m)$ equals $A_p \cup A$, i.e., $A_p \cup A \Rightarrow B_p$ follows using (Ax).

- $A_p \Rightarrow B_p \in \Sigma$. As in the previous case, for $p_n = 1$ and $i_1 = 0$ using (Wea) we infer $A_p \cup A \Rightarrow B_p$, showing $\Sigma \vdash A_p \cup A \Rightarrow B_p$.

- Let $A_p \Rightarrow B_p$ result by $A_q \Rightarrow B_q$ and $A_r \Rightarrow B_r$ using (Cut). In this case, there is $C$ such that $A_r = B_q \cup C$, $B_p = B_r$, and $A_p = A_q \cup C$. By induction hypothesis, there are $i_1, \ldots, i_{q_n} \in \mathbb{Z}$ and $i'_1, \ldots, i'_{q_r} \in \mathbb{Z}$ such that $\Sigma \vdash A_q \cup \bigcup_{m=1}^{q_n} (A + i_m) \Rightarrow B_q$ and $\Sigma \vdash B_q \cup C \cup \bigcup_{m=1}^{q_r} (A + i'_m) \Rightarrow B_r$. Therefore, using (Cut), $\Sigma \vdash A_q \cup \bigcup_{m=1}^{q_n} (A + i_m) \cup C \cup \bigcup_{m=1}^{q_r} (A + i'_m) \Rightarrow B_r$. Hence, for $i''_1 = i_1, \ldots, i''_{q_n} = i_{q_n}$, $i''_{q_n+1} = i'_1, \ldots, i''_{q_n+q_r} = i'_{q_r}$ it follows that $\Sigma \vdash A_q \cup C \cup \bigcup_{m=1}^{q_n+q_r} (A + i''_m) \Rightarrow B_p$, i.e., $\Sigma \vdash A_p \cup \bigcup_{m=1}^{q_n+q_r} (A + i''_m) \Rightarrow B_p$.

- Let $A_p \Rightarrow B_p$ result by $A_q \Rightarrow B_q$ using (Shf). Then, $A_p = A_q + i$ and $B_p = B_q + i$ for some $i \in \mathbb{Z}$. By induction hypotheses, there are $i_1, \ldots, i_{q_n}$ such that $\Sigma \vdash A_q \cup \bigcup_{m=1}^{q_n} (A + i_m) \Rightarrow B_q$. Using (Shf), we get $\Sigma \vdash (A_q \cup \bigcup_{m=1}^{q_n} (A + i_m)) + i \Rightarrow B_q + i$. Now, observe that $(A_q \cup \bigcup_{m=1}^{q_n} (A + i_m)) + i$ equals $(A_q + i) \cup \bigcup_{m=1}^{q_n} (A + i_m + i)$. Therefore, the claim holds for integers $i_1 + i, \ldots, i_{q_n} + i$.

As a special case for $p = n$, we get (ii) because $A_n = \emptyset$.

“(ii) ⇒ (i)” Let $\Sigma \vdash \bigcup_{m=1}^{p_n} (A + i_m) \Rightarrow B$ for some $i_1, \ldots, i_n \in \mathbb{Z}$. From the monotony of provability, we get that $\Sigma \cup \{ \emptyset \Rightarrow A \} \vdash \bigcup_{m=1}^{p_n} (A + i_m) \Rightarrow B$. Moreover, for each $m = 1, \ldots, n$ we get $\Sigma \cup \{ \emptyset \Rightarrow A \} \vdash \emptyset \Rightarrow A + i_m$ using (Shf). Hence, $\Sigma \cup \{ \emptyset \Rightarrow A \} \vdash \emptyset \Rightarrow \bigcup_{m=1}^{p_n} A + i_m$ by finitely many applications of (Add) and (Tra) gives $\Sigma \cup \{ \emptyset \Rightarrow A \} \vdash \emptyset \Rightarrow B$. □

Example 2. Let us observe that a direct counterpart of the classic deduction
theorem does not hold in our system. For instance, we may take a theory
\( \Sigma = \{ \emptyset \Rightarrow \{ x^1 \} \} \). Then, using (Shf) for \( i = 1 \), we easily see that \( \Sigma \vdash \emptyset \Rightarrow \{ x^2 \} \).

On the other hand, we have \( \not\vdash \{ x^1 \} \Rightarrow \{ x^2 \} \) and thus in general \( \Sigma \cup \{ \emptyset \Rightarrow A \} \vdash \emptyset \Rightarrow B \) does not imply that \( \Sigma \vdash A \Rightarrow B \) which holds in the classic case.

**Example 3.** One of the classic laws about provability that apply to attribute implications and can be formulated in terms of attribute implications as formulas with limited expressive power compared to general propositional formulas is the principle of the proof by cases. Formally, if \( R \) consists only of (Ax) and (Cut), then the following are equivalent:

- \( \Sigma \vdash_R A \Rightarrow B \);
- \( \Sigma \cup \{ C \Rightarrow D \} \vdash_R A \Rightarrow B \) and \( \Sigma \cup \{ D \Rightarrow C \} \vdash_R A \Rightarrow B \).

This follows immediately by the fact that in this case, \( \vdash_R \) becomes the classic propositional provability. The law does not apply in our system where \( R \) contains the additional rule (Shf). For instance, consider the following theory \( \Sigma = \{ \{ c^0 \} \Rightarrow \{ d^1 \}, \{ x^0 \} \Rightarrow \{ d^2 \}, \{ c^2 \} \Rightarrow \{ y^0 \}, \{ d^1 \} \Rightarrow \{ y^0 \} \} \).

Obviously, we have \( \Sigma \cup \{ \{ c^0 \} \Rightarrow \{ d^0 \} \} \vdash \{ x^0 \} \Rightarrow \{ y^0 \} \) using (Shf) and two applications of (Cut). Analogously, we get \( \Sigma \cup \{ \{ d^0 \} \Rightarrow \{ c^0 \} \} \vdash \{ x^0 \} \Rightarrow \{ y^0 \} \).

On the other hand, we can show that \( \Sigma \not\vdash \{ x^0 \} \Rightarrow \{ y^0 \} \), i.e., the principle of the proof by cases does not hold. In order to see that \( \Sigma \not\vdash \{ x^0 \} \Rightarrow \{ y^0 \} \), observe that \( [\{ x^0 \}]_{\Sigma} = \{ y^{-1}, x^0, c^1, y^1, d^2 \} \) for which \( [\{ x^0 \}]_{\Sigma} \neq \{ x^0 \} \Rightarrow \{ y^0 \} \). Thus, since our logic is sound and \( [\{ x^0 \}]_{\Sigma} \in \text{Mod}(\Sigma) \), we indeed have \( \Sigma \not\vdash \{ x^0 \} \Rightarrow \{ y^0 \} \).

**Remark 5.** We may say that \( \Sigma' \) is a completion of \( \Sigma \) if \( \Sigma \subseteq \Sigma' \) and for any finite \( C, D \subseteq T_Y \), we have either \( \Sigma' \vdash C \Rightarrow D \) or \( \Sigma' \vdash D \Rightarrow C \). Let us note that analogous notions of completions play an important role in completeness proofs of various logics, cf. [29]. Namely, if a given theory does not prove a formula it is often desirable to find its completion which does not prove the formula as well.

As a consequence of Example 3, we observe that this is not possible in our logic. Namely, the example shows a particular case where \( \Sigma \not\vdash \{ x^0 \} \Rightarrow \{ y^0 \} \) and there
is no completion $\Sigma'$ such that $\Sigma' \not\proves \{x^0\} \Rightarrow \{y^0\}$. Indeed, each completion $\Sigma'$ proves either $\{c^0\} \Rightarrow \{d^0\}$ or $\{d^0\} \Rightarrow \{c^0\}$ and thus it also proves $\{x^0\} \Rightarrow \{y^0\}$. Nevertheless, we were able to prove Theorem 23 without having this property.

6 Computational Issues

In this section, we show bounds on the computational complexity of deciding whether an attribute implication over attributes annotated by time points is provable by a finite set $\Sigma$ of other attribute implications. Then, we focus on a subproblem which typically appears in applications. For the subproblem we provide a pseudo-polynomial time \cite{27} decision algorithm.

We formalize the decision problem of entailment as a language of encodings of finitely many formulas, i.e., we put

$$L_{\text{ENT}} = \{ \langle \Sigma, A \Rightarrow B \rangle \mid \Sigma \text{ is a finite theory and } \Sigma \proves A \Rightarrow B \},$$

(27)

considering a fixed $T_Y$. In order to show the lower bound of the time complexity of $L_{\text{ENT}}$, we utilize a reduction of decision problems \cite{45} which involves the unbounded subset sum problem. The decision variant of the unbounded subset sum problem is formulated as follows: An instance of the problem is given by $n$ non-negative integers $j_1, \ldots, j_n$ and a target value $z$; the answer to the instance is “yes” iff there are non-negative integers $c_1, \ldots, c_n$ such that

$$\sum_{i=1}^{n} c_i j_i = z.$$  

(28)

The unbounded subset sum decision problem is NP-complete, see \cite[Proposition A.4.1]{32}.

Let us note that in the case of the ordinary attribute implications and functional dependencies, the problem of determining whether a given formula follows by a finite set of formulas is easy and there exist efficient linear time decision algorithms \cite{6}. In contrast, the corresponding decision problem in our setting is hard:

**Theorem 30** (lower bound). $L_{\text{ENT}}$ is NP-hard.
Proof. We prove the claim by showing that the unbounded subset sum problem (see Section 3) is polynomial time reducible to $L_{\text{ENT}}$. Consider an instance of the unbounded subset sum problem given by non-negative integers $j_1, \ldots, j_n$ and $z$. For the integers we consider

$$\Sigma = \{ \{y^0\} \Rightarrow \{y^i\} \mid i = 1, \ldots, n\}$$  \hspace{1cm} (29)$$

and put $A = \{y^0\}$, $B = \{y^z\}$. We now prove that $\sum_{i=1}^n c_i j_i = z$ holds true for some non-negative integers $c_1, \ldots, c_n$ iff $\Sigma \vdash \{y^0\} \Rightarrow \{y^z\}$ by proving both implications.

In order to prove the if-part, assume that $\Sigma \vdash \{y^0\} \Rightarrow \{y^z\}$. Using Theorem 28 it follows there is a normalized derivation sequence $\varphi_1, \ldots, \varphi_k$ of $\{y^0\} \Rightarrow \{y^z\}$ using formulas in $\Sigma$. In the proof, we utilize a part of the sequence which results by applications of (Acc), see Definition 27 (iv). All formulas in this part of the sequence can be written as

$$\{y^0\} \Rightarrow A_i, \{y^0\} \Rightarrow A_{i+1}, \ldots, \{y^0\} \Rightarrow A_{k-1},$$

where $A_1, \ldots, A_{k-1}$ are finite subsets of $\mathcal{T}_Y$, $A_1 = \{y^0\}$, and $y^z \in A_{k-1}$ because $\varphi_k$ results from $\varphi_{k-1}$ by (Pro), cf. Definition 27. By induction, we show for every $A_l$ ($i \leq l \leq k-1$) that the following property is satisfied:

If $y^w \in A_l$, then there are non-negative integers $c_1, \ldots, c_n$

such that $w = \sum_{i=1}^n c_i j_i$.

Notice the property is satisfied for $l = i$ since in that case we have $A_l = A_i = \{y^0\}$ and thus, we may put $c_1 = c_2 = \cdots = c_n = 0$. Assuming the claim holds for $l$, we prove it for $l + 1$ as follows. Inspecting Definition 27(iv), it follows that $\{y^0\} \Rightarrow A_{l+1}$ results from $\{y^0\} \Rightarrow A_l$ and $\{y^0\} + t \Rightarrow \{y^{j_m}\} + t$ using (Acc) where $t \in \mathbb{Z}$ and $1 \leq m \leq n$. As a consequence $\{y^0\} + t \subseteq A_l$ and thus, by induction hypothesis, there are non-negative integers $d_1, \ldots, d_n$ such that $t = 0 + t = \sum_{i=1}^n d_i j_i$. Then, $j_m + t = j_m + \sum_{i=1}^n d_i j_i$ and so $j_m + t = \sum_{i=1}^n c_i j_i$. 

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for non-negative integers \(c_1, \ldots, c_n\) defined by
\[
c_i = \begin{cases} 
d_i + 1, & \text{if } i = m, \\
d_i, & \text{otherwise.} 
\end{cases}
\]

Now, since we have \(A_{l+1} \subseteq A_l \cup \{y^{j_m+t}\}\), the property holds for \(A_{l+1}\). As a particular case, for \(\{y^x\} \subseteq A_{k-1}\) we conclude there are non-negative integers \(c_1, \ldots, c_n\) for which \(\sum_{i=1}^{n} c_i j_i = z\) which concludes the first part of the proof of Theorem 30.

Conversely, let \(\sum_{i=1}^{n} c_i j_i = z\) for some non-negative integers \(c_1, \ldots, c_n\). By induction, we show that \(\Sigma \vdash \{y^0\} \Rightarrow \{y^{z_k}\}\) for every \(z_k = \sum_{i=1}^{k} c_i j_i\) where \(k = 0, \ldots, n\). As a particular case for \(k = n\), we obtain the desired fact that \(\Sigma \vdash \{y^0\} \Rightarrow \{y^z\}\) because \(z_n = z\).

Observe that for \(k = 0\), the claim follows trivially by (Ax). Now, suppose the claim holds for \(k < n\). By induction hypothesis, \(\Sigma \vdash \{y^0\} \Rightarrow \{y^{z_k}\}\). Moreover, we have \(\Sigma \vdash \{y^0\} \Rightarrow \{y^{j_k+1}\}\) because \(\{y^0\} \Rightarrow \{y^{j_k+1}\} \in \Sigma\). Using (Shf), we also get \(\Sigma \vdash \{y^0\} + j_{k+1} \Rightarrow \{y^{j_k+1}\} + j_{k+1}\), i.e., using (Cut), it follows that \(\Sigma \vdash \{y^0\} \Rightarrow \{y^{2j_k+1}\}\). Repeating the last argument \(c_{k+1}\)-times, we obtain \(\Sigma \vdash \{y^0\} \Rightarrow \{y^{c_{k+1}j_{k+1}}\}\). Now, using (Shf), we get \(\Sigma \vdash \{y^0\} + z_k \Rightarrow \{y^{c_{k+1}j_{k+1}}\} + z_k\), i.e., \(\Sigma \vdash \{y^{z_k}\} \Rightarrow \{y^{c_{k+1}j_{k+1}+z_k}\}\). Hence, \(\Sigma \vdash \{y^0\} \Rightarrow \{y^{z_k}\}\) follows by (Cut) using the fact that \(z_{k+1} = z_k + c_{k+1} j_{k+1}\), which finishes the proof.

The reduction involved in Theorem 30 is illustrated in the following example.

**Example 4.** Let us show a particular instance of the unbounded subset sum problem and its reduction to \(L_{\text{ENT}}\). We consider integers 5, 7, 11, and a target number 31 as an instance of the problem. The answer to this instance is “yes” because for numbers 4, 0, and 1, the sum \(4 \cdot 5 + 0 \cdot 7 + 1 \cdot 11\) is equal to 31. The corresponding theory \(\Sigma\), see the proof of Theorem 30 is
\[
\Sigma = \{\{y^0\} \Rightarrow \{y^5\}, \{y^0\} \Rightarrow \{y^7\}, \{y^0\} \Rightarrow \{y^{11}\}\}.
\]
In this case, \(\{y^0\} \Rightarrow \{y^{11}\}\) is provable from \(\Sigma\) because we may chain four shifted instances of \(\{y^0\} \Rightarrow \{y^5\}\) and a single shifted instance of \(\{y^0\} \Rightarrow \{y^{11}\}\) by using...
(Cut). It corresponds with the sum \(4 \cdot 5 + 0 \cdot 7 + 1 \cdot 11\). In a more detail, the corresponding proof of \(\{y^0\} \Rightarrow \{y^{31}\}\) by \(\Sigma\) is the following sequence of formulas:

1. \(\{y^0\} \Rightarrow \{y^5\}\) formula in \(\Sigma\)
2. \(\{y^0\} + 5 \Rightarrow \{y^5\} + 5\) using (Shf) on 1.
3. \(\{y^0\} \Rightarrow \{y^{10}\}\) using (Cut) on 1. and 2.
4. \(\{y^0\} + 10 \Rightarrow \{y^{10}\} + 10\) using (Shf) on 1.
5. \(\{y^0\} \Rightarrow \{y^{15}\}\) using (Cut) on 3. and 4.
6. \(\{y^0\} + 15 \Rightarrow \{y^{15}\} + 15\) using (Shf) on 1.
7. \(\{y^0\} \Rightarrow \{y^{20}\}\) using (Cut) on 5. and 6.
8. \(\{y^0\} \Rightarrow \{y^{11}\}\) formula in \(\Sigma\)
9. \(\{y^0\} + 20 \Rightarrow \{y^{11}\} + 20\) using (Shf) on 8.
10. \(\{y^0\} \Rightarrow \{y^{31}\}\) using (Cut) on 7. and 9.

Remark 6. The entailment problem is closely related to the existence of non-negative solutions of linear Diophantine equations. Indeed, for a theory \(\Sigma\) which consists of formulas of the form \(\{y^0\} \Rightarrow \{y^{j_i}\}\) for \(i = 1, \ldots, n\), by inspecting the proof of Theorem 30 we can see that \(\Sigma \vdash \{y^0\} \Rightarrow \{y^z\}\) iff the linear Diophantine equation \(j_1x_1 + \cdots + j_nx_n = z\) has a non-negative solution.

Our observations on the upper bound of computational complexity involve additional classes of decision problems. In order to establish an upper bound, we utilize the fact that the satisfiability problem of temporal logic with “until” and “since” operators over a linear flow of time is decidable in polynomial space [47]. For the purpose of our proof, we use the linear temporal logic over \((\mathbb{Z}, <)\) with the unary temporal operators \(\square\) (always), \(\diamond_F\) (next time), and \(\diamond_P\) (previous time) because these operators are definable using operators “until” and “since”, see [4] for details.

From now on, we consider \(Y\) (the set of attributes) as (a subset of) the set of propositional variables. Recall that formulas of the temporal logic with the above-mentioned operators are defined as follows: Each \(y \in Y\) is a formula; if \(\varphi\)
and $\psi$ are formulas, then $\neg \varphi, \varphi \& \psi, \varphi \Rightarrow \psi, \boxdot \varphi, \circ_F \varphi,$ and $\circ_P \varphi$ are formulas.

In order to interpret the formulas we consider a standard structure $K = \langle W, e, r \rangle$ where $W = \mathbb{Z}$, $r$ is the genuine ordering $<$ on $\mathbb{Z}$, and $e$ is an evaluation such that $e(w, y) \in \{0, 1\}$ for all $w \in \mathbb{Z}$ and $y \in Y$. Given $K$ and $w \in \mathbb{Z}$, we interpret the formulas as usual: We put

(i) $K, w \models y$ whenever $e(w, y) = 1$;
(ii) $K, w \models \neg \varphi$ whenever $K, w \not\models \varphi$;
(iii) $K, w \models \varphi \& \psi$ whenever $K, w \models \varphi$ and $K, w \models \psi$;
(iv) $K, w \models \varphi \Rightarrow \psi$ whenever $K, w \not\models \varphi$ or $K, w \models \psi$;
(v) $K, w \models \boxdot \varphi$ whenever $K, w' \models \varphi$ for all $w' \in \mathbb{Z}$;
(vi) $K, w \models \circ_F \varphi$ whenever $K, w' \models \varphi$ for $w' \in \mathbb{Z}$ such that $w < w'$ and there does not exist $z \in \mathbb{Z}$ such that $w < z < w'$;
(vii) $K, w \models \circ_P \varphi$ whenever $K, w' \models \varphi$ for $w' \in \mathbb{Z}$ such that $w' < w$ and there does not exist $z \in \mathbb{Z}$ such that $w' < z < w$.

We say that $\varphi$ is true in $K$ whenever $K, w \models \varphi$ for all $w \in \mathbb{Z}$. Moreover, we say that $\varphi$ is satisfiable whenever there is a structure $K$ such that $K, 0 \models \varphi$.

Moreover for each formula of the form (1), we consider its counterpart in the considered temporal logic

\[ \boxdot \left( (\triangle^1 y_1 \& \cdots \& \triangle^m y_m) \Rightarrow (\triangle^1 z_1 \& \cdots \& \triangle^n z_n) \right), \tag{30} \]

where $\triangle^i$ is defined as follows:

\[ \triangle^i y = \begin{cases} y, & \text{if } i = 0, \\ \circ_F \triangle^{i-1} y, & \text{if } i > 0, \\ \circ_P \triangle^{i+1} y, & \text{if } i < 0. \end{cases} \tag{31} \]

Note that the construction of $\triangle^i y$ from $y^i$ requires space which is linear in (the absolute value of) $i \in \mathbb{Z}$, i.e., it is exponential in the length of the encoding of $i$.

**Theorem 31.** $\text{LENT}$ is reducible in exponential space to the satisfiability problem of the linear temporal logic over $(\mathbb{Z}, <)$ with unary temporal operators “always”, “next time”, and “previous time”.
Proof. First, observe that for each subset of $T_Y$ we may consider a corresponding structure which makes the same formulas true—any $A \Rightarrow B$ is true in the subset of $T_Y$ iff its counterpart given by (30) is true in the corresponding structure. Namely, for $M \subseteq T_Y$, we may consider $K_M = \langle W, e, r \rangle$, where $e(w, y) = 1$ if $y^w \in M$ and $e(w, y) = 0$ otherwise. Conversely, for $K = \langle W, e, r \rangle$, we put $M_K = \{y^w \mid e(w, y) = 1\}$. Now, for any $w \in W$, it is easy to see that $M | A \Rightarrow B$ iff $K_M, w | \varphi$ where $\varphi$ is the counterpart to $A \Rightarrow B$ given by (30). From now on, we tacitly identify attribute implications with their counterparts. Furthermore, we have $K, w | A \Rightarrow B$ iff $M_K | A \Rightarrow B$.

Now, for a given $\Sigma = \{A_1 \Rightarrow B_1, \ldots, A_m \Rightarrow B_m\}$ and $A \Rightarrow B$ we may consider formula $A_1 \Rightarrow B_1 \land \cdots \land A_m \Rightarrow B_m \land \neg(A \Rightarrow B)$ whose construction requires exponential space. From the previous observation, it is obvious that the formula is satisfiable iff $\Sigma \not| A \Rightarrow B$.

Corollary 32 (upper bound). $L_{\text{ENT}}$ belongs to EXPSPACE.

Proof. The decision procedure reduces the input of $L_{\text{ENT}}$ to the satisfiability problem of linear temporal logic over $\langle Z, < \rangle$ with unary temporal operators “always”, “next time”, and “previous time” in exponential space, see Theorem 31. Then, the input is reduced to the satisfiability problem of the linear temporal logic over $\langle Z, < \rangle$ with binary temporal operators “until” and “since” in linear space [4] which we can decide in polynomial space [47]. Altogether, the decision procedure decides $L_{\text{ENT}}$ in exponential space.

Remark 7. Note that the results of Theorem 31 and Corollary 32 can also be interpreted so that $L_{\text{ENT}}$ is decidable in a pseudo-polynomial space because we reduce an instance of $L_{\text{ENT}}$ to an instance (of the satisfiability problem of the above-mentioned temporal logic) the length of which is bounded from above by the numeric value encoded in the original input. With respect to the new instance, the decision procedure works in polynomial space.

We now turn our attention to issues of entailment of formulas which typically appear in applications in prediction. The restriction on particular formulas

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allows us to improve the complexity of the entailment problem. Based on the
time points present in antecedents and consequents of attribute implications,
we may consider formulas which describe presence of attributes in future time
points. That is, based on the presence of attributes in the past, the formulas
indicate which attributes are present in future time points. Technically, such
formulas can be seen as attribute implications where all the time points in
the antecedents are smaller (i.e., denote earlier time points) than all the time
points in the consequents which denote later time points. We call such formulas
predictive and define the notion as follows.

Definition 33. An attribute implication $A \Rightarrow B$ over $Y$ annotated by time
points in $Z$ is called predictive whenever $A \neq \emptyset$, $B \neq \emptyset$, and for each $x^i \in A$ and
$y^j \in B$, we have $i \leq j$. A theory $\Sigma$ is called predictive whenever all its formulas
are predictive.

Remark 8. Note that the deduction rules (Shf) and (Cut) preserve the property
of being predictive. That is, if $A \Rightarrow B$ is provable by a predictive theory $\Sigma$
without using (Ax), then $A \Rightarrow B$ is predictive. General instances of (Ax) are
not predictive formulas.

In the next assertion, we utilize lower and upper time bounds of finite non-
empty subsets of $\mathcal{T}_Y$: For a finite non-empty $M \subseteq \mathcal{T}_Y$, put

$$l(M) = \min \{i \in \mathbb{Z} \mid y^i \in M\},$$
\hspace{2cm} (32)

$$u(M) = \max \{i \in \mathbb{Z} \mid y^i \in M\}. \hspace{2cm} (33)$$

Thus, $l(M)$ and $u(M)$ are the lowest and greatest time points which appear
in $M$, respectively. Clearly, $A \Rightarrow B$ is predictive iff both $A$ and $B$ are non-
empty and $u(A) \leq l(B)$.

Theorem 34. Let $\Sigma$ and $A \Rightarrow B$ be predictive. Then, for

$$\Sigma^B_A = \{E + i \Rightarrow F + i \mid E \Rightarrow F \in \Sigma \text{ and } l(A) - l(E) \leq i \leq u(B) - l(F)\} \hspace{2cm} (34)$$

we have $\Sigma \vdash A \Rightarrow B$ iff $\Sigma^B_A \vdash_R A \Rightarrow B$ for $R$ containing (Ax) and (Cut).
Proof. Observe that the if-part of the claim is trivial. In order to prove the only-if part, assume that $\Sigma \vdash A \Rightarrow B$. That is, $B \subseteq [A]_\Sigma$ owing to Theorem 22 and Theorem 15. Note that $\Sigma_A^B \vdash_R A \Rightarrow B$ for $R$ containing (Ax) and (Cut) means that $A \Rightarrow B$ is provable by $\Sigma_A^B$ as an ordinary attribute implication. Let $A^\circ$ denote the least subset of $\mathcal{T}$ with the following properties:

(i) $A \subseteq A^\circ$, and

(ii) for each $E \Rightarrow F \in \Sigma_A^B$: if $E \subseteq A^\circ$ then $F \subseteq A^\circ$. 

Since $A^\circ$ is in fact the syntactic closure of $A$ with respect to $R$, $\Sigma_A^B \vdash_R A \Rightarrow B$ iff $B \subseteq A^\circ$. That is, in order to prove the desired claim, it suffices to show that $A^\circ \cap \mathcal{T} = [A]_\Sigma \cap \mathcal{T}$.

Trivially, we get that $A^\circ \cap \mathcal{T} \subseteq [A]_\Sigma \cap \mathcal{T}$. In order to prove the converse inclusion, according to Theorem 24, it suffices to check that $A^n \cap \mathcal{T} \subseteq A^\circ \cap \mathcal{T}$ for each non-negative integer $n$. By induction, assume that $A^n \cap \mathcal{T} \subseteq A^\circ \cap \mathcal{T}$ and take $y^j \in (A^{n+1}_\Sigma \cap \mathcal{T}) \setminus (A^n_\Sigma \cap \mathcal{T}) = (A^{n+1}_\Sigma \setminus A^n_\Sigma) \cap \mathcal{T}$. The fact $y^j \in A^{n+1}_\Sigma \setminus A^n_\Sigma$ yields there is $E \Rightarrow F \in \Sigma$ and $i \in \mathbb{Z}$ such that $E + i \subseteq A^n_\Sigma$ and $y^j \in F + i$. It can be shown that $E + i \Rightarrow F + i \in \Sigma_A^B$. Indeed, since $\Sigma$ is predictive, observe that $l(E) + i = l(E + i) \geq l(A^n_\Sigma) = l(A)$ and thus $i \geq l(A) - l(E)$. Moreover, $y^j \in F + i$ yields $l(F + i) = l(F) + i \leq j$ and thus $i \leq j - l(F)$ which gives $i \leq u(B) - l(F)$ on account of $j \leq u(B)$ since $y^j \in T$. As a consequence, $E + i \Rightarrow F + i \in \Sigma_A^B$. Furthermore, $E + i \subseteq A^n_\Sigma$ and the fact that $E \Rightarrow F$ is predictive give $E + i = (E + i) \cap \mathcal{T} \subseteq A^n_\Sigma \cap \mathcal{T}$. By induction hypothesis, $E + i \subseteq A^\circ$ and thus $F + i \subseteq A^\circ$ by (ii). Hence, $y^j \in A^\circ$ and so $A^{n+1}_\Sigma \cap \mathcal{T} \subseteq A^\circ \cap \mathcal{T}$.

Let $L_{\text{PRE}}$ be the language consisting of encodings of pairs of all finite predictive theories and predictive formulas, i.e.,

$$L_{\text{PRE}} = \{ \langle \Sigma, A \Rightarrow B \rangle \mid \Sigma \text{ finite and } \Sigma \text{ and } A \Rightarrow B \text{ are predictive}\}. \quad (35)$$

Based on Theorem 34, we establish the following observation on the time complexity of deciding whether a predictive formula is provable by a finite predictive
Theorem 35. \( L_{\text{ENT}} \cap L_{\text{PRE}} \) is decidable in a pseudo-polynomial time.

Proof. Take a finite predictive \( \Sigma \) and a predictive formula \( A \Rightarrow B \). The theory \( \Sigma_B^A \) given by (34) is finite. According to Theorem 34, the problem of deciding \( \Sigma \models A \Rightarrow B \) is reducible to the problem of deciding whether \( \Sigma_B^A \) entails \( A \Rightarrow B \) without using (Shf), i.e., in the sense of the entailment of ordinary attribute implications. Therefore, the problem is decidable in a time which is polynomial with respect to the size of \( \Sigma_B^A \) \( [6, 26, 42] \). Now, observe that the size of (the encoding of) \( \Sigma_B^A \) may be bounded from above by the size of (the encoding of) \( \Sigma \) multiplied by

\[
n = \max\{\max(0, u(B) + l(E) - l(A) - l(F) + 1) \mid E \Rightarrow F \in \Sigma\},
\]

i.e., the size of \( \Sigma_B^A \) is polynomial in the numeric value encoded in the input \( \Sigma \) and hence \( L_{\text{ENT}} \cap L_{\text{PRE}} \) is decidable in a pseudo-polynomial time. \( \square \)

Remark 9. (a) By considering only \( L_{\text{ENT}} \cap L_{\text{PRE}} \), we have improved the upper bound since pseudo-polynomial time algorithms belong to EXPTIME \( [27] \) which is believed to be better than EXPSPACE. Observe that \( L_{\text{ENT}} \cap L_{\text{PRE}} \) is also NP-hard because we can use the same reduction as in Theorem 34.

(b) Because of the complexity issues, in applications it is reasonable to consider attribute implications annotated by time points with small difference between lower and upper time bounds (maxspan \( [23] \)) since \( L_{\text{ENT}} \cap L_{\text{PRE}} \) is decidable in pseudo-linear time with respect to \( n \) given by (36).

An explicit procedure for deciding \( L_{\text{ENT}} \cap L_{\text{PRE}} \) in a pseudo-linear time is described in Algorithm 1. It is a generalization of LinClosure \( [6] \), cf. also \( [42] \), which incorporates applicable time shifts of formulas in \( \Sigma \). The algorithm accepts three arguments:

1. a finite predictive theory \( \Sigma \),
2. a finite \( A \subseteq T_Y \), and
Algorithm 1: PseudoLinClosure ($\Sigma, A, Max$)

1. forall the $E \Rightarrow F \in \Sigma$ do
2.     for $i$ from $l(A) - l(E)$ to $Max - l(F)$ do
3.         set $count[E \Rightarrow F, i]$ to $|E|$;
4.         forall the $y^i \in E$ do
5.             add $(E \Rightarrow F, i)$ to $list[y^{i+j}]$;
6.         end
7.     end
8. end
9. set $M$ to $A$;
10. set $update$ to $A$;
11. while $update \neq \emptyset$ do
12.     choose $y^i$ from $update$;
13.     set $update$ to $update \setminus \{y^i\}$;
14.     forall the $(E \Rightarrow F, j) \in list[y^i]$ do
15.         set $count[E \Rightarrow F, j]$ to $count[E \Rightarrow F, j] - 1$;
16.         if $count[E \Rightarrow F, j] = 0$ then
17.             set $new$ to $F + j \setminus M$;
18.             set $M$ to $M \cup new$;
19.             set $update$ to $update \cup new$;
20.         end
21.     end
22. end
23. return $M$
3. a non-negative number $Max \geq u(A)$,

and it returns a subset $M \subseteq \mathcal{A}_\Sigma$ such that $M \cap T = [\mathcal{A}_\Sigma] \cap T$ for

$$T = \{y^i \in \mathcal{T}_y \mid l(A) \leq i \leq Max\}.$$  \hspace{1cm} (37)

The soundness of the algorithm is justified by the following observation:

**Theorem 36.** Let $\Sigma$ and $A \Rightarrow B$ be predictive and let $\Sigma$ be finite. Then, Algorithm $\Sigma$ executed with arguments $\Sigma$, $A$, and $u(B)$, terminates after finitely many steps and for the returned value $M$ we have $\Sigma \vdash A \Rightarrow B$ iff $B \subseteq M$.

**Proof.** The arguments are fully analogous to those in case of the classic LIN-CLOSURE, so we present here comments on issues arising only in the context of attributes annotated by time points. Technical details can be found in [4].

Notice that Algorithm $\Sigma$ uses auxiliary structure $count$ and $list$ to store information about formulas. The structure $count$ can be seen as an associative array indexed by (pointers to) formulas in $\Sigma$ and integers $i$ representing time shifts. The value of $count[E \Rightarrow F, i]$ is initially set to the number of attributes in the antecedent of $E \Rightarrow F$ (shifted by $i$). During the computation, $count[E \Rightarrow F, i]$ represents the number of remaining attributes in $E + i$ which have not been “updated.” The structure $list$ is an array indexed by attributes annotated by time points and the value of $list[y^i]$ is a list of records $\langle E \Rightarrow F, j \rangle$ representing (pointers to) formulas in $\Sigma$ and their $j$-shifts such that $y^i$ appears in the antecedent of $E \Rightarrow F$ shifted by $j$. An additional variable $update$ is initialized at line 10 and maintains attributes annotated by time points which are waiting to be “updated.” An update of $y^i$, see lines 13–21, consists in decrementing the counter of occurrences of attributes in shifted antecedents in all formulas where $y^i$ appears. All such formulas (and their $j$-shifts) are found in $list[y^i]$, see line 14. If $count[E \Rightarrow F, j]$ reaches zero, see line 16, the antecedent of $E + j \Rightarrow F + j$ is already contained in $M$, and all new attributes in $F + j$ are prepared for update. Clearly, the procedure terminates after finitely many steps, and by Theorem 34 the attributes annotated by time points accumulated in $M$ represent a subset
of \([A]_{\Sigma}\). In addition, if \(u(B) \leq Max\), then \(B \subseteq M\) iff \(B \subseteq [A]_{\Sigma}\) iff \(\Sigma \vdash A \Rightarrow B\) as a consequence of our previous observations.

Remark 10. The procedure in Algorithm 1 is called \textsc{PseudoLinClosure} because for given parameters, \(\Sigma\), \(A\), and \(Max\), it computes a subset of the closure of \([A]_{\Sigma}\) in a linear time with respect to the numeric value of the encoding of its input arguments, i.e., its time complexity is \textit{pseudo-linear}. Indeed, this is a consequence of the fact that each \(y^i\) where \(l(A) \leq i \leq Max\) is updated during the computation at most once.

Example 5. Consider a set \(M\) given by the table in Figure 4. Since \(M\) can be regarded as transactional data over a set of items \(Y\) with a dimensional attribute \(d\) the domain of which is \(Z\), we can utilize the algorithm proposed in [10]. The parameters for the algorithm are numbers \(\text{maxspan}\), \(\text{minsupport}\), and \(\text{minconfidence}\) for which we obtain a set \(\Sigma\) of all predictive \(A \Rightarrow B\) where \(u(A \cup B) − l(A \cup B) \leq \text{maxspan}\), \(\text{minconfidence} \leq \text{confidence}(A \Rightarrow B)\), and \(\text{minsupport} \leq \text{support}(A \Rightarrow B)\). For this particular example we consider \(\text{maxspan} = 5\), \(\text{minconfidence} = 1\) since we are interested in formulas true in \(M\), and \(\text{support} = 5\). In this setting, we obtain

\[
\Sigma = \{\{w^0_m\} \Rightarrow \{tc^3\}, \{w^0_l\} \Rightarrow \{tc^3\}, \{w^1_l\} \Rightarrow \{w^1_m\}, \{w^0_l, tc^3\}, \{w^1_l, w^1_m\} \Rightarrow \{tc^3\}, \{r^0_n, w^2_l\} \Rightarrow \{tc^3\}, \{r^0_n, r^3_n\} \Rightarrow \{tc^5\}, \{tc^0, tc^3, r^5_n\} \Rightarrow \{tc^5\}, \{r^0_n, tc^0, r^3_n\} \Rightarrow \{tc^3\}, \{r^0_n, tc^0, w^2_m\} \Rightarrow \{tc^3\}\}.
\]

Now, we may successively reduce the set \(\Sigma\) by removing formulas \(A \Rightarrow B\) such that \(\Sigma \setminus \{A \Rightarrow B\} \vdash A \Rightarrow B\), i.e., without loss of information. Since \(\Sigma\) is
predictive we may use PSEUDO LIN CLOSURE and obtain the following set:

$$\Sigma' = \{ \{\text{wm}^0\} \Rightarrow \{\text{tc}^4\}, \{\text{wl}^0\} \Rightarrow \{\text{wm}^1, \text{tc}^3\}, \{\text{rn}^0, \text{rn}^3\} \Rightarrow \{\text{tc}^3\}, \{\text{rn}^0, \text{wm}^2\} \Rightarrow \{\text{tc}^3\}, \{\text{tc}^0, \text{rn}^5\} \Rightarrow \{\text{tc}^5\} \},$$

i.e., the equivalent non-redundant set contains less than half of the formulas in $\Sigma$. For maxspan = 5 and support = 2, the reduction is much more significant. From the total number of 34,440 generated formulas, PSEUDO LIN CLOSURE can be used to produce an equivalent set consisting of only 81 formulas.

7 Conclusion

We have presented logic for reasoning with if-then rules expressing dependencies between attributes changing in time. The logic extends the classic logic for dealing with if-then rules by considering discrete time points as an additional component. We have studied both the semantic entailment based on preserving validity in models in all time points and syntactic entailment represented by a provability relation. We have shown a characterization of the semantic entailment based on least models and syntactico-semantical completeness of the logic. We have shown the problem of entailment is NP-hard, decidable in exponential space, and its simplified variant which involves only predictive formulas is decidable in pseudo-linear time. Future research directions we consider interesting include utilization of generalized quantifiers to capture notions like “validity in all time points with possible exceptions”, connections to rules which may emerge in temporal databases [17], and further analysis of algorithms related to the entailment.

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