Inflation Free, Stringy Generation of Scale-Invariant Cosmological Fluctuations in $D = 3 + 1$ Dimensions

Ali Nayeri

Jefferson Physical Laboratory, Harvard University, Cambridge, MA 02138, USA
(Dated: March 27, 2022)

We propose an alternative scenario to cosmic inflation for producing the initial seeds of cosmic structures. The cosmological fluctuations are generated by thermal fluctuation of the energy density of the ideal string gas in three compact spatial dimensions. Statistical mechanics of the strings reveals that scalar power spectrum of the cosmological fluctuations on cosmic scales is scale-invariant for closed strings and inclines towards red for open strings in three compact spatial dimensions. This generation of thermal fluctuations happens during the Hagedorn era of string gas cosmology and without invoking an inflationary epoch the perturbations enter the radiation-dominated era. The amplitude of the fluctuations is proportional to the ratio of the two length scales in the theory, i.e., the Planck length over the string length, $(\ell_P/\ell_s)$. Since modes with the shorter wavelengths exit the Hubble radius at the end of the Hagedorn phase at later times compare to the modes with long wavelengths, the scalar fluctuations gain mild tilt towards red.

PACS numbers: 98.80.Cq

If the string theory is claimed to be the theory of every fundamental things in the universe then it is better to describe the cosmos itself. There have been many attempts to combine string theory with cosmology. One such approach is the string gas cosmology (SGC)\[1, 2\]—also \[3, 4\] for an overview of the subject, and \[5\] for a critical review—which is based specifically on new symmetries (T-duality) and new degrees of freedom (string winding modes) of string theory\[1\]. Based on considerations of string thermodynamics, it was argued that string theory could provide a nonsingular cosmology. Going backwards in time, the universe contracts and the temperature grows. However, the temperature will not exceed the Hagedorn temperature for weakly interacting closed strings. As the radius of space approaches the self-dual radius (the string length), the pressure of the string gas will almost vanishes because of T-duality (the positive contribution to the pressure from momentum modes will cancel against the negative pressure from string winding modes). Using the background equations of motion from dilaton gravity, it follows that the evolution of the scale factor near the self-dual radius will be quasi-static\[2\]. Once the radius of space decreases below the self-dual radius, the string gas temperature will decrease, demonstrating that string gas cosmology will be non-singular. There has recently been quite a lot of work on further developing string gas cosmology (see e.g. \[6, 7, 8, 9, 10\] and \[11, 12, 13, 14\] for recent reviews and comprehensive lists of references).

In string gas cosmology, it is assumed that the universe starts in a Hagedorn phase, a phase in which the universe is quasi-static and in thermal equilibrium at a temperature close to the Hagedorn temperature\[15\], the limiting temperature of perturbative string theory. As the universe slowly expands, heavy degrees of freedom gradually fall out of equilibrium. String winding modes keep all but $d$ spatial dimensions compact\[1\] (see, however, \[16, 17\] for a critical view of this aspect of the scenario). In this sense, one can view SGC as being a stringy generalization of the big bang cosmology.

Hence SGC appears to offer a dynamical explanation for the dimensionality of macroscopic spacetime in the context of a non-singular cosmology. Independent of whether or not the mechanism of\[1\] is realized, SGC has more recently found a new application as a possible solution of the moduli problem. In the context of heterotic or bosonic string theory, it has been demonstrated that massless string states which appear at the self dual radius can stabilize all shape and radial moduli corresponding to a toroidal compactification at the string scale\[18]\[19\], in a way that is consistent with observational bounds and late time cosmology\[3, 10\].

On the other hand, one of the biggest triumph of the cosmic inflation is explaining the origin of the structures by generating quantum fluctuations in the very early universe\[20, 21, 22, 23, 24\]. Despite many theoretical and observational successes of the inflationary models, perhaps the most unsatisfactory feature of all the inflationary models is how to embed inflation in a more fundamental theory. For that, there have been lots of attempts to find inflation in the context of string theory in recent years\[25, 26\]. Alternatively, one can ask whether a fundamental theory like string theory can provide an alternative model of generating the initial seeds for formation of structures\[27, 28, 29\]. Recently, a mechanism to generate an initial scale-invariant spectrum of scalar metric fluctuations in the context of SGC and without invoking an inflationary period, has been proposed\[28\]. It was demonstrated that, during the quasi-static Hagedorn phase of SGC, thermal fluctuations of a closed string gas generate an almost scale-invariant spectrum of metric fluctuations for the gravitational potential, which is

*Electronic address: nayeri@feynman.harvard.edu
the quantity which generates the observed anisotropies in the cosmic microwave background (CMB). In this paper, I will elaborate more on the result of \( \mathcal{A} \) and compare the results with the ones for open strings and massless relativistic particles.

In what follows, we shall assume that there are only \( d \) already sufficiently large spatial dimensions and will work in natural units \( (c = \hbar = k_B = 1) \).

I. STRING GAS COSMOLOGY WITH DILATON: DYNAMICS

String theory predicts the presence of a scalar field known as dilaton which along with the graviton and the antisymmetric tensor couples to matter. The low energy string theory effective action in the ‘string’ frame is

\[
\mathcal{A} = \frac{-1}{2\kappa^2_D} \int \sqrt{-G} d^D x \, e^{-2\phi} \left[ (D)R + 4\nabla_{\mu} \phi \nabla^{\mu} \phi \right. \\
- \frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \bigg] + \mathcal{A}_m, \tag{1}
\]

where \((D)R\) is the \( D \)-dimensional Ricci scalar, \( \phi \) the dilaton, \( H_{\mu\nu\lambda} = \partial_{[\mu} B_{\nu\lambda]} \), the field strength of the antisymmetric field tensor \( B_{\mu\nu} \), \( F_{\mu\nu} = \partial_{[\mu} A_{\nu]} \), the field strength of the \( U(1) \) gauge field \( A_\mu \) and \( \mathcal{A}_m \) is the action for string matter which will be discussed later.

If one rescales the metric in the following way

\[
G_{\mu\nu} \rightarrow e^{2\phi/(D-2)} g_{\mu\nu}, \tag{2}
\]

the low energy effective action becomes a \( D \) dimensional modified Einstein action. This transformation of the metric defines the Einstein frame in which the corresponding action to \( \mathcal{A} \) takes the following form in this frame

\[
\mathcal{A}_E = \frac{-1}{2\kappa^2_D} \int \sqrt{-\bar{g}} d^D x \left[ (D)R + \frac{2}{D-2} \nabla_{\mu} \phi \nabla^{\mu} \phi \right. \\
- \frac{1}{12} e^{2(\phi/\sqrt{\bar{g}})} H_{\mu\nu\lambda} H^{\mu\nu\lambda} \\
- \frac{1}{4} e^{-2\phi/2(D-2)} F_{\mu\nu} F^{\mu\nu} \bigg] + \mathcal{A}_m. \tag{3}
\]

If we now consider bosonic strings in \( D = 10 \) critical dimensions by ignoring the antisymmetric tensor \( B_{\mu\nu} \) for simplicity, the action for the gravitational degrees of freedom interacting with string matter then, in string frame, takes the form

\[
\mathcal{A} = \frac{-1}{2\kappa^2_{10}} \int \sqrt{-G} d^{10} x \, e^{-2\phi} \left[ (10)R + 4(\nabla \phi)^2 + \ldots \right] \\
+ \mathcal{A}_m, \tag{4}
\]

where \( \kappa^2_{10} = \frac{1}{2} (2\pi)^7 \ell_s^2 \), and

\[
\mathcal{A}_m = \int dt \, F(a, \beta), \tag{5}
\]

is the action for the stringy matter which consists of a gas of almost “free” string modes, by assuming small effective string coupling constant, in thermal equilibrium at the temperature \( \beta^{-1} \). Here \( F \) is the (one-loop) free energy which can be expressed in terms of the one-loop string partition function in a torus background of radii \( a \) and periodic Euclidean time of perimeter \( \beta \).

For the spatial rectangular torus background with two isotropic sets of dimensions of the form

\[
ds^2 = -dt^2 + \ell_s^2 \left[ \sum_{i=1}^{(d)} a_i^2(t) d\theta_i^2 + \sum_{i=9-d} a_i^2(t) d\theta_i^2 \right], \tag{6}
\]

following equations of motion are found

\[
-(d)\ddot{\mu}^2 - (9-d)\ddot{\nu}^2 + \ddot{\phi}^2 = 2 \frac{\kappa^2_{10}}{(2\pi \sqrt{\alpha'})^9} e^\phi F_{d}, \tag{7}
\]

\[
\ddot{\mu} - \dot{\phi} \dot{\mu} = \frac{\kappa^2_{10}}{(2\pi \sqrt{\alpha'})^9} e^\phi P_d, \tag{8}
\]

\[
\ddot{\nu} - \dot{\phi} \dot{\nu} = \frac{\kappa^2_{10}}{(2\pi \sqrt{\alpha'})^9} e^\phi P_{9-d}, \tag{9}
\]

\[
\ddot{\phi} - (d)\ddot{\mu}^2 - (9-d)\ddot{\nu}^2 = \frac{\kappa^2_{10}}{(2\pi \sqrt{\alpha'})^9} e^\phi E, \tag{10}
\]

or alternatively, in terms of \( \phi \)

\[
-(d)\ddot{\mu}^2 - (9-d)\ddot{\nu}^2 + [\dot{\phi} - (d)\dot{\mu} - (9-d)\dot{\nu}]^2 = 2 \kappa^2_{10} e^{2\phi} \rho, \tag{11}
\]

\[
\ddot{\mu} - [\dot{\phi} - (d)\dot{\mu} - (9-d)\dot{\nu}] \dot{\mu} = \kappa^2_{10} e^{2\phi} p_d, \tag{12}
\]

\[
\ddot{\nu} - [\dot{\phi} - (d)\dot{\mu} - (9-d)\dot{\nu}] \dot{\nu} = \kappa^2_{10} e^{2\phi} p_{9-d}, \tag{13}
\]

\[
[\ddot{\phi} - (d)\ddot{\mu}^2 - (9-d)\ddot{\nu}^2 - (d)\ddot{\nu}^2 - (9-d)\ddot{\mu}^2 = \kappa^2_{10} e^{2\phi} \rho. \tag{14}
\]

Here \( (10)R \) is the ten-dimensional Ricci scalar, \( \phi \) is the 10-dimensional dilaton of the string theory which is related to the shifted dilaton through

\[
\sqrt{G} e^{-2\phi} = e^{-\varphi}, \tag{15}
\]

or

\[
\varphi \equiv 2\phi - (d)\mu - (9-d)\nu, \tag{16}
\]

where

\[
a(t) = e^{\varphi(t)}, \quad a_s(t) = e^{\varphi(t)}, \tag{17}
\]
are the homogenous scalar factors for \(d\) expanding dimensions and \(9-d\) dimensions remaining in string scale, respectively. \(\nu\) is normally taken to be zero.

The total energy of the system is represented by \(E \equiv F + \beta (\partial F / \partial \beta)\). The variables \(P_d \equiv - (\partial F / \partial \mu)\) and \(P_{9-d} \equiv - (\partial F / \partial \nu)\) are related to the pressures \(p_d\), \(p_{9-d}\) in the respective directions by a volume rescaling, \(P_X = p_X V\), with \(V\) being the total volume of the system given by

\[
V = (2\pi \sqrt{\alpha'})^d a^d a_s^{(9-d)} \equiv (2\pi \sqrt{\alpha'})^d e^{(d-1)\mu} e^{(9-d)\nu}. \tag{18}
\]

Therefore we can find the modified conservation law of energy-momentum

\[
\dot{E} + (d) \dot{\mu} P_d + (9-d) \dot{\nu} P_{9-d} = 0 = \frac{\dot{S}}{\beta}, \tag{19}
\]
or

\[
\dot{\mu} + (d) \frac{\dot{a}}{a} (\rho + p_d) + (9-d) \frac{\dot{a}}{a} (\rho + p_{9-d}) = 0, \tag{20}
\]

where \(S \equiv \beta^2 (\partial F / \partial \beta)\), the total entropy, is conserved due to the fact that \(E = F[\mu(t), \nu(t), \beta(t)]\). The details of this free energy will be discussed later. The adiabaticity condition tells us that the temperature \(\beta^{-1}\) adjusts itself such that the total entropy, \(S\), remains constant for a given radii determined by \(\mu\) and \(\nu\).

The above equations of motion are duality invariant since \(F\) is invariant under \(\mu \rightarrow -\mu, \nu \rightarrow -\nu\) and \(\varphi \rightarrow -\varphi\) (or \(\phi \rightarrow \phi - \mu - \nu\) for a given temperature). Note that while \(E(\mu, \nu)\) and \(\beta(\mu, \nu)\) are invariant under the duality transformations, the pressures \(P_d\) and \(P_{9-d}\) change sign under these transformations.

The system of equations (11)-(14) or (15)-(18) has a mechanical interpretation of describing a motion of a particle either in the potential \(U_\phi(\mu, \nu, \phi) = e^{\varphi} E\) or \(U_\phi(\mu, \nu, \phi) = e^{\phi} \rho\). Based on the behavior of \(\varphi(\phi)\) and \(E(\rho)\) one can distinguished two distinct regime for the system of a very weakly coupled string gas. I will assume hereafter the expansion/contraction only exists in \(d\) dimensions and that the \((9-d)\) dimensions have stretched (contracted) to their maximum (minimum) length, i.e., string (Planck) length and thus they do not take part in expansion (contraction). Hence, \(\nu = 0\) or \(\dot{a}_s = 0\).

**Hagedorn phase.** For weakly coupled strings, \(g_s \ll 1\), there is a limiting temperature known as the Hagedorn temperature \(T_H\). In this regime, because of weak interaction among the strings, to a good approximation the free energy of the string gas, \(F = E - T_H S\), vanishes (since for the string gas \(S \approx \beta_H E\) and so does the total pressure \(P = P_d = - (\partial F / \partial \ln a) \approx 0\). The equations of motion (11), (14) and (5) for this vacuum reduce to

\[
\ddot{\varphi} = \frac{1}{2} [\varphi^2 + (d) \mu^2], \tag{21}
\]
\[
\ddot{\mu} - \dot{\varphi} \dot{\mu} = 0. \tag{22}
\]

**Radiation dominated era.** As the universe slowly expands, \(\mu(t)\) reaches the intermediate region in which \(E\) drops and the system moves towards the direction of large \(\mu\). In this regime the dilaton \(\phi\) is a constant and thus one can easily find out that

\[
P = \left(\frac{1}{d}\right) E, \tag{25}
\]

with \(E = E_0 e^{-\mu}\) and thus

\[
\rho = \rho_0 e^{-(d+1)\mu}. \tag{26}
\]
This is a description of a matter with traceless energy-momentum tensor, i.e., a gas of massless particles in a thermodynamical equilibrium. In this regime, a power law solution for $\mu$ and $\varphi$ exists in the form of

$$\mu(t) = \mu_0 + \left(\frac{2}{d + 1}\right) \ln t, \quad (27)$$

$$\varphi(t) = \varphi_0 - \left(\frac{2d}{d + 1}\right) \ln t, \quad (28)$$

which implies that

$$\phi = \frac{1}{2} [\varphi + (d) \mu] = \text{constant}, \quad (29)$$

and

$$a(t) \sim t^{2/(d+1)} \rightarrow H(t) \sim \frac{2}{d + 1} \frac{1}{t}. \quad (30)$$

To summarize, the universe, in the expanding sector, starts out in a quasi-static phase known as the Hagedorn phase in which the thermodynamical equilibrium can be established over the entire universe. In this regime the universe is large enough so that after shrinking to a microscopic scale ($H^{-1} \sim \ell_s^2/\ell_{Pl}^2$), in the follow up radiation dominated era, can grow to the present size without undergoing any inflationary phase. The phase transition to the radiation dominated FRW universe occurs when the dilaton gets fixed at the end of the Hagedorn phase.

We want to study the statistical mechanics of the string gas in the Hagedorn regime in which the coordinate radii are all near the string scale, $\sqrt{\alpha'} = \ell_s$. I first give a quick review of the fundamentals of statistical mechanics that we need later.

## II. BRIEF REVIEW OF FUNDAMENTAL OF STATISTICAL MECHANICS

For any thermodynamical system there are two almost identical statistical descriptions. The microcanonical ensemble is the most basic description of statistical mechanics and is valid for a closed system with fixed total energy $E$ (and volume $V$). The fundamental quantity in the ensemble is the total density of states $\Omega(E)$ which is defined as

$$\Omega(E) \equiv \sum_i \delta(E - E_i), \quad (30)$$

where the sum is over all states $i$ of the system and $E_i$ denotes the energy of the state $i$. The quantity $\Omega(E)$ is related to the thermodynamic entropy $S$ of the system, up to an additive constant, as

$$S(E) \equiv \ln \Omega(E). \quad (31)$$

Once the entropy is determined, other thermodynamical quantities like temperature, $T$, pressure, $P$ and the specific heat $C_V$ can be defined through the first law of thermodynamics $dE = TdS - PdV$. Thus

$$\frac{1}{T(E)} = \beta(E) \equiv \left(\frac{\partial S}{\partial E}\right)_V = \left(\frac{\partial \ln \Omega}{\partial E}\right)_V, \quad (32)$$

$$P \equiv T \left(\frac{\partial S}{\partial V}\right)_E = T \left(\frac{\partial \ln \Omega}{\partial V}\right)_E, \quad (33)$$

and

$$C_V \equiv \left(\frac{\partial E}{\partial T}\right)_V = -\beta^2 \left(\frac{\partial E}{\partial \beta}\right)_V = - \left[T^2 \left(\frac{\partial^2 S}{\partial E^2}\right)_V \right]^{-1}. \quad (34)$$

On the other hand, the canonical ensemble can be used to describe the properties of any large subsystem of a closed system, providing the energy is an extensive variable. The basic quantity in this ensemble is the canonical partition function

$$Z(\beta) = \sum_i e^{-\beta E_i},$$

which is the Laplace transformation of the total energy density $\Omega(E)$ for any system whose energy is bounded from below (say, unlike the gravitating system),

$$Z(\beta) = \int_0^\infty dE e^{-\beta E} \Omega(E), \quad (35)$$

or inversely, if $Z(\beta)$ exists and is known as a function of $\beta$ then $\Omega(E)$ can be obtained as a function of $E$ from the inverse Laplace transformation,

$$\Omega(E) = \int_{L - i\infty}^{L + i\infty} \frac{d\beta}{2\pi i} e^{\beta E} Z(\beta), \quad (36)$$

where $L (= \Re \beta)$ is the contour which is chosen to be right of all the singularities of $Z(\beta)$ in the complex $\beta$ plane.

Note that while (35) always exists, the transformation (36) is subject to the existence of the saddle point.

Both canonical and microcanonical descriptions of statistical mechanics can be used in an interchangeable manner for describing the gaseous systems.

While for a system that is described by microcanonical distribution, the total energy $E$ is fixed, the canonical distribution describes the same system in a way that the total energy is not a fixed quantity but rather fluctuates among the canonical ensemble. If we should be able to use either of the two descriptions, in an interchangeable manner, then to describe the system one must satisfies the following two conditions: (a) The mean energy $\langle E \rangle$ of the canonical ensemble must be the same as the fixed energy of the microcanonical ensemble and (b) the root-mean-square fluctuations in the energy, in
the canonical distribution, \( \Delta E/(E) \equiv \sqrt{(E^2)/(E)^2 - 1} \), must be negligible for sufficiently large systems, i.e., \( \Delta E/(E) \propto 1/\sqrt{E} \). Since in the canonical distribution, \( (E) \equiv -Z^{-1}(\partial Z/\partial \beta) = -(Z'/Z) \) and \( (E^2) \equiv Z^{-1}(\partial^2 Z/\partial \beta^2) = (Z''/Z) \), where the primes denote the differentiation with respect to \( \beta \), it follows that

\[
C_V = \beta^2 ((E^2) - (E)^2) = \beta^2 \left( \frac{Z''}{Z} - \frac{Z'^2}{Z^2} \right) = \beta^2 \frac{\partial}{\partial \beta} \left( \frac{Z'}{Z} \right) = -\beta^2 \frac{\partial (E)}{\partial \beta}.
\]

(37)

The specific heat defined in the canonical distribution must, therefore, be positive definite. As long as the two distributions are equivalent the specific heat in microcanonical description is negative and thus the equivalence between the two prescriptions breaks down. One example of this situation is near the phase transition, where the fluctuations may become very large.

### III. DENSITY OF STATES FOR THE IDEAL STRING GAS

When the string coupling is sufficiently small, \( g_s \ll 1 \), and the local spacetime geometry is close to flat \( \mathbb{R}^{d+1} \) over the length scale of the finite size box of volume \( V = R^d \), there are two distinct regimes that characterize the statistical mechanics of string thermodynamics: the massless modes with field theoretic entropy, \( S \propto E^{d/(d+1)} \), and the highly excited strings with \( S \propto E \). One simple realization of this setup for a string background is a spatial toroidal compactification with \( d \) dimensions of size \( R \) and \( 9 - d \) dimensions of string scale size. Small string coupling ensures us that we can measure energies with respect to the flat time coordinate.

To review the features of string thermodynamics we use the intuitive geometrical picture for a highly excited string as a random walk in target space \( \mathbb{R}^d \). The large entropy factor corresponding to the shape of the random walk in space explains why highly energetic strings dominate the thermodynamics despite of their large energy.

**Closed strings.** For a highly excited closed string represented as a random walk in a target space, the energy \( \varepsilon \) of the string is proportional to the length of the random walk. Thus the number of the strings with a fixed starting point grows as \( \exp(\beta_H \varepsilon) \), with \( T_H = 1/\beta_H \) being the Hagedorn temperature. This explains the bulk of the entropy of highly energetic strings. Closed strings correspond to random walks that must close on themselves. This overcounts by a factor of roughly the volume of the walk, denoted \( V_{\text{walk}}(\varepsilon) \). The global translation of the random walk in volume \( V = R^d \) and \( 1/\varepsilon \) due the fact that any point in the string can be considered as a new starting point are other factors that contribute to the number of closed string. Therefore, the final result is

\[
\omega_{\text{closed}}(\varepsilon) \sim V \frac{1}{\varepsilon} V_{\text{walk}}(\varepsilon).
\]

(38)

There are two limiting case here:

**(a)** Volume of the random walk is well-contained in \( d \) spatial dimensions (i.e., \( R \gg \sqrt{\varepsilon} \)) which corresponds to a string in \( d \) non-compact dimensions. In this case the \( V_{\text{walk}}(\varepsilon) \sim \varepsilon^{d/2} \), the density of states per unit volume is

\[
\omega_{\text{closed}}(\varepsilon) \sim R^d e^{\beta_H \varepsilon} \varepsilon^{d/2 + 1}.
\]

(39)

**(b)** Volume of the random walk is space-filling \( (R \ll \sqrt{\varepsilon}) \) and saturates at order \( V \) which corresponds to \( d \) compact dimensions that contains the highly excited string states. Hence,

\[
\omega_{\text{closed}}(\varepsilon) = \frac{e^{\beta_H \varepsilon}}{\varepsilon}.
\]

(40)

This is an exact leading result. Note that here the density of states are almost independent of the topology of the spacetime and depends only on the volume of the random walk. One can combine the two regimes in one equation

\[
\omega_{\text{closed}}(\varepsilon) = \beta_H R^d e^{\beta_H \varepsilon} \gamma_{c+1}, \quad \text{with} \quad \gamma_c = \frac{d}{2},
\]

(41)

where \( d_c \) is the number of dimensions in which closed strings have no windings and again \( R^d \) is the volume of this space. In other words, at low energies the winding modes are frozen and thus corresponds to large radius, while at high energies the the winding modes can be excited and thus all the radii are compact. We want to find the total density of states when the radius of the compact space expands and thus one can see the effect of interplay between the winding modes and the momentum modes of the closed strings.

The total density of states, \( \Omega(E) \), can be obtained through \( \omega_{\text{closed}}(\varepsilon) \). The partition function \( Z(\beta) \) can be evaluated explicitly in the one loop approximation \( Z(\beta) = \sum_{\alpha} e^{-\beta E_{\alpha}(R)} \),

(42)

where \( \alpha = (N, q_1, \ldots, q_N) \) labels a state for \( N \) strings and \( g_k \) are the quantum numbers of the \( k^{th} \) string and stand for the whole set of momentum, winding and oscillatory modes. Here \( E_{\alpha} \) is the energy of the multi-string state \( \alpha \) in a universe of radius \( R \). In the ideal gas approximation, \( E_{\alpha} \) is given by the sum all over the single-state energies, i.e., \( E_{\alpha}(R) = \sum_{k=1}^{N} \varepsilon_{q_k}(R) \), where \( \varepsilon \), say, for the closed bosonic strings is \( \varepsilon = \frac{L^2}{R^2} + \frac{w^2 R^2}{\ell_s^4} + \frac{2}{\ell_s^4} \left[ -2 + \sum_{I=1}^{d-1} \sum_{m=1}^{\infty} m(N'I_m + \tilde{N}'m) \right] \),

(43)
which maps to itself under the duality transformations

$$R \leftrightarrow \tilde{R} = (\ell_s^2/R), \quad 1 \leftrightarrow w.$$  

In [43], $l$, $w$, $N^l_m$ and $\tilde{N}^l_m$ are momentum, winding and oscillatory quantum numbers, respectively.

Near the Hagedorn temperature, we can assume Maxwell-Boltzman statistics and thus treat the system quasiclassically. Then we can write $Z(\beta) = \exp[z(\beta)]$, where $z(\beta)$ is the single-string partition function (free thermal energy)

$$z(\beta) = \int_0^\infty d\varepsilon \omega(\varepsilon) e^{-\beta\varepsilon} = \sum_q e^{-\beta\varepsilon_q}. \quad (44)$$

According to [43, 44, 57], the singular part of the partition function at finite volume for closed strings is given by a set of poles of even multiplicity $g_i = 2k_i = 2d$

$$Z^\text{singular}_{\text{closed}} \sim \left( \frac{\beta_s}{\beta - \beta_s} \right)^{k_i}, \quad (45)$$

with $k_i = k_0 = 1$ for the leading Hagedorn singularity

$$\beta_s = \beta_0 = \beta_H = \sqrt{2\pi \ell_s} \left( \frac{\omega_l + \omega_r}{2} \right)^{1/2}, \quad (46)$$

for $i = 1, 2, \ldots, O(R/\ell_s)$ or

$$\beta_i = \sqrt{2\pi \ell_s} \left( \frac{\omega_l - i \ell_s^2/2R^2}{2} \right)^{1/2} + \frac{\omega_r - i \ell_s^2/2R^2}{2}, \quad (47)$$

for $i = 1, 2, \ldots, O(R/\ell_s)$, depending on which one has a larger real part. In other words, while [43] shows the location of the singularities for an expanding universe, the singularities of its dual contracting universe are given by [43] and vise versa.

The regular part of the free energy to the leading order in energy is

$$Z^\text{regular} \sim n_H V - \rho_H V (\beta - \beta_i) + O(V(\beta - \beta_i)^2). \quad (48)$$

The leading singularity at very high and finite volume is always a simple pole of the partition function at the Hagedorn singularity. Considering the subleading singularities that are located to the left of $\beta_0 = \beta_H$, e.g., $\beta_1$, one can parameterize $Z(\beta)$ in the region in which there are two singularities,

$$Z_{\text{closed}}(\beta) \sim \frac{\beta_H}{(\beta - \beta_H)} \left( \frac{\beta_H - \beta_1}{\beta - \beta_1} \right)^{2d} Z^\text{regular}(\beta). \quad (49)$$

The total density of states of closed strings when all the dimensions are compact and large is [33]

$$\Omega(E, R) \approx \Omega_0 + \Omega_1 \approx \beta_H e^{\beta_H E + n_H V} [1 + \delta\Omega(1)(E, R)], \quad (50)$$

with $\Omega_0$ and $\Omega_1$ being, respectively, the contributions to the density of states from $\beta_0 = \beta_H = (1/T_H)$ and the closest singularity to $\beta_H$, i.e., $\beta_1 < \beta_H$ (which is real). The subleading contributions are encoded in $\delta\Omega(1)$ which is

$$\delta\Omega(1)(E, R) = -\frac{(\beta_H E)^{2d-1}}{(2d-1)!} e^{-(\beta_H - \beta_1)(E - \rho_H V)} \quad (51)$$

Here, the density of states has specifically written for $d$ large compact dimensions, i.e., $R \gg \ell_s$. Also, $n_H$ is a constant number density of order $\ell_s^{d-1}$ and $\rho_H$ is the ‘Hagedorn Energy density’ of the order $\ell_s^{\nu(d+1)}$ while according to [16] and [17]

$$\beta_H - \beta_1 \sim \left\{ \begin{array}{ll}
(\ell_s^2/R^2), & \text{for } R \gg \ell_s, \\
(R^2/\ell_s), & \text{for } R \ll \ell_s.
\end{array} \right.$$  

To ensure the validity of Eq. (51) we demand that $-\delta\Omega(1) \ll 1$ by assuming $\rho \equiv (E/V) \gg \rho_H$. So the entropy of the string gas in the Hagedorn phase is given by

$$S(E, R) \approx \beta_H E + n_H V + \ln [1 + \delta\Omega(1)], \quad (52)$$

and therefore the temperature $T \equiv \left[ (\partial S/\partial E) V \right]^{-1}$ will be

$$T(E, R) \approx \left( \beta_H + \frac{\delta\Omega(1)}{E + \delta\Omega(1)} \right)^{-1} \approx T_H \left( 1 + \frac{\beta_H - \beta_1}{\beta_H} \delta\Omega(1) \right). \quad (53)$$

Open Strings. For the open strings, however, the geometrical picture is a bit more involved. Let’s consider a highly excited string between $Dp$- and $Dq$-branes. In addition to the leading exponential degeneracy for random walks with a fix starting point on the $Dp$-brane, there would be a degeneracy factor due to the fixing of the endpoints on each brane

$$V_{NN} V_{ND} \cdots V_{NN} V_{DD}, \quad (V_{NN} V_{ND}) (V_{NN} V_{DN}) (V_{NN} V_{DD}) e^{\beta_H \varepsilon}, \quad (54)$$

where $N$ and $D$ refer to Neumann and Dirichlet boundary conditions. Finally there would be an overall factor due to the translation of the walk in the excluded $NN$ volume which is $(V_{NN} V_{ND})$. Thus for the open string, the density of the states looks like

$$\omega_{\text{open}}(\varepsilon) \sim \frac{V_{NN} V_{ND} (V_{NN} V_{DN}) (V_{NN} V_{DD}) e^{\beta_H \varepsilon}}{V_{\text{walk}}^{\text{open}}} \sim \frac{V_{NN} e^{\beta_H \varepsilon}}{V_{DD}^{\text{walk}}}, \quad (54)$$

where $V_{\text{walk}}^{\text{open}} = V_{\text{walk}}^{\text{walk}} V_{\text{ND}} V_{\text{DN}} V_{\text{DD}}$ is the total volume of the random walk. Like the close string there are two limiting case here (a) The random walk is well-contained in the $d_{DD} = d_-$
directions with \(DD\) boundary conditions \(R_{DD} = R_i \gg \sqrt{\epsilon}\) we have \(V_{DD}^{walk} \sim \epsilon^{d_{\perp}/2}\) and thus
\[
\frac{\omega_{open}}{V_{NN}} \sim \frac{e^{\beta_H \epsilon}}{\epsilon^{d_{\perp}/2}}. \tag{55}
\]

(b) If the random walk is filling the \(DD\) volume then
\[
\frac{\omega_{open}}{V_{NN}} \sim \int_0^\infty e^{\beta_H \epsilon} \frac{\epsilon^{d_{\perp}/2}}{\epsilon^{d_{\perp}/2}}. \tag{56}
\]

In summary the density of states for a single open string can be written as \[30, 36, 37\]
\[
\omega_{open}(\epsilon) = \beta H V_o \frac{\epsilon^{d_{\perp}/2}}{e^{\beta_H \epsilon}} \left( \beta H \epsilon \right)^{\gamma_0 + 1}, \tag{57}
\]
where \(V_{\parallel} = V_{NN}\) and \(V_{\perp} = V_{DD}\) are the volumes transverse and perpendicular to the \(D\)-brane. \(0 \leq d_o \leq d_{\perp} = d_{DD}\) is the number of dimensions transverse to the brane with no windings and \(V_o\) is the volume of this space (which is \(O(1)\) in string units when there are windings in all directions). Both \(\gamma_0\) and \(\gamma_c\) are \(\epsilon\)-dependent critical exponent. The ‘effective’ number of large spacetime dimensions (i.e., the total number of \(NN + DD\) dimensions) as a function of \(\epsilon\) is
\[
d_{eff}(\epsilon) = d_{NN} + d_o(\epsilon), \tag{58}
\]
with \(d_{NN} = p + 1\) being the \(p\) spatial non-compact Neumann directions of the \(Dp\)-brane. In particular, if we consider \(3 + 1\) directions which are much larger than the string scale, then there are \(d_{comp}\) compactified dimensions of the string scale and \(d_{DD} = 6 - d_{comp}\) compactified directions much smaller than the string scale. For the large internal energies open strings can move freely in the entire space and we have \(\gamma_0 + 1 = 0\). In the low energies limit, on the other hand, \(\gamma_0 + 1 = d_{DD}/2 = 3 - d_{comp}/2\).

The behavior of \(z_{\text{singular}}(\beta)\) for open strings near the singularity \(\beta \approx \beta_i\), by direct substitution, is given by \[30\]
\[
z_{\text{singular}}(x_{\text{open}}(\beta)) \approx \begin{cases} \frac{1}{2} \frac{(-1)^{\gamma_0+1}}{\Gamma(\gamma_0+1)} f(\beta - \beta_i)^{\gamma_0} \log(\beta - \beta_i), & \text{for } \gamma_0 \in \mathbb{Z}^+ \cup \{0\}, \\ \frac{1}{2} \frac{(-\gamma_0)}{\Gamma(-\gamma_0)} f(\beta - \beta_i)^{\gamma_0}, & \text{for } \gamma_0 \notin \mathbb{Z}^+ \cup \{0\}; \end{cases} \tag{59}
\]

where \(f = V_{\parallel}/V_{\perp} = V_{NN}/V_{DD}\) is the volume factor. The critical exponent for the compact \(DD\) directions \(\gamma_0 = -1\). If a number \(d_{\infty}\) of \(DD\) directions are strictly non-compact, then \(\gamma_0 \rightarrow \gamma_0 + d_{\infty}/2\) and \(f \rightarrow f_{V_{\infty}}\).

Eq. (48) is a generic result for both closed and open strings. For open strings \(V_o = V_{\parallel}\).

Whenever the specific heat is positive (and large), there is a correspondence between the canonical and microcanonical ensembles and thus the saddle point approximation is applicable. A necessary condition for this is that \(\gamma < 1\), ensuring the canonical internal energy \(E(\beta) \sim \beta z(\beta)\) diverges at the Hagedorn singularity. In other words, these systems are unable to reach the Hagedorn temperature since their require an infinite amount of energy to do so. For these systems the Hagedorn temperature is limiting, and this is true for all the open strings with \(d_{NN} > d_{e,eff} - 4\). For the closed strings the Hagedorn temperature is non-limiting for any model in which \(d_c > 3\). In other words, stable canonical (i.e., no phase transition) can be achieved for the closed strings in low dimensional thermodynamical limits \(d_c \leq 2\), or open strings with \(d_{\perp} \leq 4\) non-compact \(DD\) dimensions, i.e., \(Dp\)-brane with \(p \geq 5\) and non-compact transverse dimensions.

Systems with close-packing of random walks (high energy in a fixed volume) have \(\gamma_0 = 0\) for closes strings and \(\gamma_0 = -1\) for open strings. As it was mentioned above in these cases we can use the saddle point approximation. For a gas of open strings we have \[30\]
\[
\Omega_{\text{open}}(\gamma_0 = -1) \simeq \beta H f^{-1} 1_{I_1}(2x) e^{\beta_H E + a_H V_{\parallel}} \left[ 1 + O\left( \frac{x^2}{V_{\parallel}(\rho - \rho_H)^2} \right) + O\left( e^{-\beta_H (\beta_H - \beta_i)(E - \rho_H V_{\parallel})} \right) \right], \tag{60}
\]
where \(x = \sqrt{f(E - \rho_H V_{\parallel})}\) is the control parameter for saddle point approximation and \(I_1\) is the modified Bessel function of the first kind. For \(x \gg 1\) the Hagedorn temperature is limiting and thus:
\[
\Omega_{\text{open}}(E) \sim \exp\left( \beta_H E + 2\sqrt{fE} \right), \tag{61}
\]
for \(\gamma_o = -1\) and hence \(d_{e,eff} = d_{NN}\).
The validity of the results in Eqs. (61) and (50) depends on the condition
\[
\log \left( \frac{\Omega_0}{\Omega_1} \right) \gg 1, \quad (62)
\]
where \(\Omega_1\) is the contribution to the density of states from the closest singularity to \(\beta_H\), i.e., \(\beta_1 < \beta_H\) which is real. Thus the necessary condition is
\[
\beta_H E + a_H V_\parallel \gg \beta_1 E + \text{Re}(a_1)V_\parallel \quad \text{(63)}
\]
or,
\[
(\beta_H - \beta_1)(E - \rho_H V_\parallel) \gg 1 \Rightarrow (\rho - \rho_H)R^{D-3} \gg 1, \quad (64)
\]
which is satisfied for large \(R\) and high \(E\), provided \(D > 3\).

IV. POWER SPECTRUM OF THE ENERGY FLUCTUATIONS

Now, having the total density of states, we are in the position to evaluate the power spectrum of our ideal gas in this 9-torus.

The entropy of the open and closed string system in the Hagedorn temperature limiting case for large \(R\) and \(\beta_H E \gg 1\) is
\[
S_{\text{closed}}(E) \approx \beta_H E + n_H V_{D-1} + \ln \left[ 1 + \delta \Omega(1) \right], \quad (65)
\]
and
\[
S_{\text{open}}(E) \approx \beta_H E + n_H V_\parallel + 2\sqrt{f E}. \quad (66)
\]
This clearly shows that the asymptotic entropy of the system of open strings dominates over the closed strings in the finite volume (compact dimensions):
\[
S_{\text{open}}(E) \gg S_{\text{closed}}(E).
\]
It is worth mentioning that the entropy for an ideal gas of relativistic point particles with energy density \(\rho\) in a large volume \(V\) in \(d\) spatial dimensions is
\[
S_{\text{particle}} \approx V \rho^{d/(d+1)}. \quad (67)
\]
Note that unlike the string case the coefficient of \(V\) is not just a constant, \(n_H\), but the energy density, \(\rho\).

By using the microcanonical density of states we can find the temperature \((\beta = \partial_E S(E))\),
\[
\frac{1}{T} \approx \frac{1}{T_H} + \left\{ \begin{array}{ll}
\sqrt{\frac{E}{\beta_H E}} & \text{for open strings}, \\
\frac{\beta_H E^2 (2^{d-1})}{(2d-1)!} \rho^2 e^{-E/R^2} & \text{for closed strings},
\end{array} \right. \quad (68)
\]
or invert them to get the energy of the system
\[
E \approx \left\{ \begin{array}{ll}
\frac{T T_H^2}{R^2 \ell_s^2 \ln \left( \frac{\ell_s}{R^2 (1-T/T_H)} \right)} & \text{for open strings}, \\
\frac{T^2}{2} \ln \left( \frac{T}{R^2 (1-T/T_H)} \right) & \text{for closed strings}.
\end{array} \right. \quad (69)
\]
The specific heat \(C_V = \partial E/\partial T\), therefore, is
\[
C_V^{\text{closed}} \approx \frac{R^2 / \ell_s^2}{T (1 - T/T_H)}, \quad (70)
\]
for closed strings and
\[
C_V^{\text{open}} \approx \frac{2f T}{(1 - T/T_H)^3}, \quad (71)
\]
for open strings with \(f = (R^d / \ell_s^{d-1})\). In both cases the specific heat is positive and approach the Hagedorn temperature from below unlike the nonlimiting cases. For instance, the specific heat for closed strings in an infinite volume (i.e., non-compact dimensions) is negative and approach the Hagedorn temperature from above \(\Omega(1)\).

\[
C_V^{\text{nc}} = -\frac{d + 2}{2} \left( \frac{T_H}{T - T_H} \right)^2,
\]
which could be a sign of phase transition or no thermal equilibrium. For comparison, note that the specific heat of an ideal gas of relativistic point particles in a large volume \(V = R^d\) is always positive
\[
C_V^{\text{particle}} \approx (d + 1) \left( \frac{d}{d + 1} \right)^d R^d T^d. \quad (72)
\]
Note that the specific heat for open strings and for massless relativistic point particles scales similarly like \(R^d\), while for closed strings scales as \(R^2\) in any number of spatial dimensions larger than two. This is a crucial point and later on will play an important role to get the scale-invariant spectrum in \(d = 3\).

V. STRING GAS COSMOLOGY: THERMAL FLUCTUATIONS

Now, let’s calculate the fluctuations of the energy-momentum tensor during the Hagedorn phase.

For a warm up let’s recall that the average of a thermodynamical quantity, \(\mathcal{H}\) can be derived from the partition function \(Z\) through
\[
\langle \mathcal{H} \rangle = -\frac{\partial \ln Z}{\partial \beta}. \quad (73)
\]
Taking another derivative of \(Z\) will give us
\[
\langle \mathcal{H}^2 \rangle = \langle \mathcal{H} \rangle^2 + \frac{\partial^2 \ln Z}{\partial \beta^2}. \quad (74)
\]
The fluctuation in \(\mathcal{H}\) is
\[
\delta \mathcal{H} = \mathcal{H} - \langle \mathcal{H} \rangle, \quad (75)
\]
and thus the mean square is
\[
\langle \delta \mathcal{H}^2 \rangle = \langle \mathcal{H}^2 \rangle - \langle \mathcal{H} \rangle^2 = \frac{\partial^2 \ln Z}{\partial \beta^2}. \quad (76)
\]
By the same fiat, I can find the average of an operator $\mathcal{O}^\alpha$ by

$$\langle \mathcal{O}^\alpha \rangle = \frac{\partial \ln Z}{\partial \mathcal{O}^\alpha},$$

(77)

where $\mathcal{Q}^\alpha$ is the conjugate to $\mathcal{O}^\alpha$. Taking another derivative with respect to $\mathcal{Q}^\alpha$ yields

$$\langle \mathcal{O}^\alpha \mathcal{O}^\beta \rangle = \langle \mathcal{O}^\alpha \rangle \langle \mathcal{O}^\beta \rangle + \frac{\partial^2 \ln Z}{\partial \mathcal{Q}^\alpha \partial \mathcal{Q}^\beta}.$$  (78)

The mean square fluctuation is, hence

$$\langle \delta \mathcal{O}^\alpha \delta \mathcal{O}^\beta \rangle = \langle \mathcal{O}^\alpha \mathcal{O}^\beta \rangle - \langle \mathcal{O}^\alpha \rangle \langle \mathcal{O}^\beta \rangle = \frac{\partial^2 \ln Z}{\partial \mathcal{Q}^\alpha \partial \mathcal{Q}^\beta},$$  (79)

where $\delta \mathcal{O}^\alpha = \mathcal{O}^\alpha - \langle \mathcal{O}^\alpha \rangle$ is the fluctuation from the mean.

The mean energy–momentum tensor $\langle T^\mu_\nu \rangle$ can be defined by

$$\langle T^\mu_\nu \rangle = 2 \frac{G_\mu^\alpha}{\sqrt{-G}} \frac{\partial \ln Z}{\partial \mathcal{Q}^\alpha \partial \mathcal{Q}^\nu},$$  (80)

where $G_{\mu\nu}$ is the metric. Taking another derivative of $Z$ gives,

$$\langle T^\mu_\nu T^\sigma_\lambda \rangle = \langle T^\mu_\nu \rangle \langle T^\sigma_\lambda \rangle + 2 \frac{G_\mu^\alpha}{\sqrt{-G}} \frac{\partial}{\partial \mathcal{Q}^\alpha} \left( \frac{G_\sigma^\delta}{\sqrt{-G}} \frac{\partial \ln Z}{\partial \mathcal{Q}^\delta} \right) + 2 \frac{G_\sigma^\gamma}{\sqrt{-G}} \frac{\partial}{\partial \mathcal{Q}^\gamma} \left( \frac{G_\mu^\beta}{\sqrt{-G}} \frac{\partial \ln Z}{\partial \mathcal{Q}^\beta} \right),$$  (81)

and thus the mean square fluctuation is

$$C_\mu^\nu_\sigma_\lambda = \langle \delta T^\mu_\nu \delta T^\sigma_\lambda \rangle = \langle T^\mu_\nu T^\sigma_\lambda \rangle - \langle T^\mu_\nu \rangle \langle T^\sigma_\lambda \rangle,$$

(82)

with $\delta T^\mu_\nu = T^\mu_\nu - \langle T^\mu_\nu \rangle$.

In order to make our thermodynamical approach sensible, we divide the spacetime inside the Hubble radius (Kayhaanistan [14], $H^{-1}$) to small blocks (Kayhaanaks) of size $\ell_s \ll R \ll H^{-1}$, where $R$ is almost independent of time during the Hagedorn phase (see Figure 2). Now in each of these Kayhaanaks the energy (mass) fluctuations can be calculated. Mass fluctuation inside of each Kayhaanak is responsible for the observed fluctuations in the cosmic microwave background radiation (CMBR). The partition function $Z = \exp \left( -\beta F \right)$, where $F = F(\beta \sqrt{G_{00}} R)$ is the string free energy with $\beta \sqrt{-G_{00}} = T^{-1} \sqrt{-G_{00}}$ acting as the Euclidean time. Therefore $C_0^0_0_0$, becomes

$$C_0^0_0_0 = \langle \delta \rho^2 \rangle = \langle \rho^2 \rangle - \langle \rho \rangle^2 = -\frac{1}{R^{2d} \partial \beta} \left( F + \beta \frac{\partial F}{\partial \beta} \right) = -\frac{1}{R^{2d} \partial \beta} \equiv \frac{T^2}{R^{2d} C_V}.$$  (83)

Now using the result that we obtained in (71) and (74) give us the following results for the mean energy density fluctuations squared

$$\langle \delta \rho^2 \rangle_{\text{closed}} \approx \frac{R^{-2(d-1)}}{\ell_s^3} \frac{T}{(1 - T/T_H)},$$  (84)

for closed strings and

$$\langle \delta \rho^2 \rangle_{\text{open}} \approx \frac{2R^{-d}}{\ell_s^{(d-1)}} \frac{T^3}{(1 - T/T_H)^3},$$  (85)

for open strings. As far as the length scaling is concerned, $\langle \delta \rho^2 \rangle$ for open strings is same as the one for relativistic massless point particles according to (12)

$$\langle \delta \rho^2 \rangle_{\text{particle}} \sim R^{-d} T^d,$$  (86)

VI. THE POWER SPECTRUM OF THE GENERATED PERTURBATIONS

In order to compute the power spectrum of these fluctuations, we will use the theory of linear cosmological...
perturbations around a four-dimensional homogeneous and isotropic cosmology [38, 39, 40, 41, 42].

The key point here is the fact that the thermal fluctuations generated during the Hagedorn phase are well inside the Hubble radius and exit the Hubble radius at a time very close to the transition time from Hagedorn era to the radiation phase. These fluctuations then will reenter the Hubble radius at some later time well within radiation dominated epoch.

In the absence of anisotropic stress, there is only one physical degree of freedom, namely the relativistic generalization of the Newtonian gravitational potential. In a flat universe in the conformal Newtonian gauge, the metric takes the form

\[ ds^2 = -(1 + 2\Phi)dt^2 + a(t)^2(1 - 2\Phi)d\mathbf{x}^2, \]  

(87)

where \( t \) is physical time, \( \mathbf{x} \) are the comoving spatial coordinates of the three large spatial dimensions, \( a(t) \) is the cosmological scale factor and \( \Phi(\mathbf{x}, t) \ll 1 \) represents the scalar fluctuation mode in the gravitational potential. Here we discount the tensor modes (see [29] for the discussion of the tensor modes in the Hagedorn phase).

For the metric (87) with small perturbations, the Einstein and energy-momentum tensor can be split to unperturbed and linear terms in fluctuations. The linearized equations for the perturbations are

\[ \delta G^\alpha_\beta = \kappa_{d+1}^2 \delta T^\alpha_\beta, \]  

(88)

where \( \kappa_{d+1}^2 \) is the Einstein gravitational coupling constant in an arbitrary spatial dimension \( d \) [43].

The short-wavelength perturbations are very important as they are the reason for the acoustic peaks in the cosmic microwave background (CMB) spectrum. On scales smaller than the Hubble radius, the gravitational potential \( \Phi \) is determined by the matter fluctuations via the Einstein constraint equation \( \delta G^0_0 = \kappa_{d+1}^2 \delta T^0_0 \) which is the relativistic generalization of the Poisson equation of Newtonian gravitational perturbation theory:

\[ \nabla^2 \Phi = \frac{2\pi^{d/2}G_{d+1}}{\Gamma[d/2]} \delta \rho, \]  

(89)

where \( G_{d+1} \) is the Newton’s Gravitational coupling constant in \( d \) compact but yet large spatial dimensions and \( \Gamma[x] \equiv (x - 1)! \). Using Fourier transform of (89) we find that at the Hubble radius crossing time

\[ |\Phi_k|^2 = \frac{4\pi^d G_{d+1}^2}{\Gamma^2[d/2]} k^{-4} \langle \delta \rho_k^2 \rangle. \]  

(90)
By (90) one can calculate the power spectrum of the metric fluctuations $\Phi_k$ at the end of the Hagedorn phase or to be more precise, at the time $t_s(k)$ when the fluctuate mode labeled by $k$ exits the Hubble radius (see Figure 4). In the context of standard cosmological perturbation theory, for modes with $k/a(t) \ll H(t)$, one can find a conserved quantity $\mathcal{R}$ which is related to the spatial curvature in comoving slicing of spacetime and in Newtonian gauge (in the absence of anisotropic pressure) is given by

$$\mathcal{R} = \zeta + \left( \frac{k^2}{3a^2 H} \right) \Phi,$$

where

$$\zeta = -\Phi + \frac{\delta \rho}{3(\rho + p)} = -\Phi + \frac{\delta \rho}{3\rho(1+w)}$$

is the spatial curvature on the constant energy density spacelike surface and $w = p/\rho$ is the equation of state. Note that for the super-Hubble radius fluctuations, i.e., in the limit of $k \to 0$, $\mathcal{R} \to \zeta$. The conservation of $\mathcal{R}$ (or $\zeta$) means conservation of $\Phi$ on super-Hubble scale as far as the equation of state of the background does not change drastically from one phase to another. For string gas cosmology the equation of state at the end of the Hagedorn phase changes from $w = 0$ to $w = 1/3$ at the beginning of the radiation dominated era which is much milder than a change compare to the cosmic inflationary scenario. As long as the equations of four spacetime dimensional general relativistic cosmological perturbation theory apply, then $\Phi$ is conserved on super-Hubble scales as long as the equation of state of the background does not change significantly. The use of ordinary general relativistic cosmological perturbation theory is certainly justified after the end of the Hagedorn phase, but not necessarily in the time interval between mode exiting from the Hagedorn phase and radiation dominated epoch.

Assuming the validity of the arguments of the previous paragraphs by ignoring the contribution of the dilaton fluctuation, then the scalar spectral index $n_s$ of the cosmological perturbations can be determined.

The dimensionless power spectrum, in $d$ spatial dimensions is given by

$$\Delta^2(k) = \frac{V}{(2\pi)^d} \frac{\pi^{d/2}}{d!} k^d P(k),$$

where $P_{\delta \rho} = \langle \delta \rho_k^2 \rangle$ and $P_{\Phi_k} = \langle \Phi_k^2 \rangle$ are the power spectra of the energy density fluctuations and the metric perturbation, respectively. Thus the dimensionless power spectrum of the metric fluctuation is

$$\Delta_{\Phi_k}^2(k) = \frac{4\pi^2 G_5^2}{\ell_s^2} k^{-4} \Delta_{\delta \rho_k}^2(k),$$

which has the following forms at the end of Hagedorn regime

$$\Delta_{\Phi_{k,cl}}^2(k) \simeq \frac{\pi(d/2) G_5^{d+1}}{2(d-3)\Gamma(d/2)\ell_s^3} \frac{T^3}{(1 - T/T_H)^3} k^{2(d-3)},$$

for closed strings and

$$\Delta_{\Phi_{k,op}}^2(k) \simeq \frac{\pi(d/2) G_5^{d+1}}{2(d-4)\Gamma(d/2)\ell_s^3} \frac{T^3}{(1 - T/T_H)^3} k^{d-4},$$

for open strings.

Obviously, as one can see from (96) and (99), the power spectrum is independent of the length of fluctuations in $d = 3$ and $d = 4$ for closed and open strings, respectively. Therefore, whereas only closed strings provide us a scale-invariant spectrum in $D = 3 + 1$ dimensions,

$$\Delta_{\Phi_{k,cl}}^2(k) \simeq \frac{4G_5^2}{\ell_s^3} \frac{T^3}{k(1 - T/T_H)^3},$$

the spectrum of open strings in $d = 3$ slopes towards red (more power on large scales)

$$\Delta_{\Phi_{k,op}}^2(k) \simeq \frac{4G_5^2}{\ell_s^3} \frac{T^3}{k(1 - T/T_H)^3}.$$

Note that near the Hagedorn temperature $T_H = 1/4\pi \ell_s$ -- $T \approx 3 \times 10^{-3}$ for observed amplitude of $\delta \rho_k^2$ corresponding to a hierarchy of $(\ell_s/x)_s \sim 10^{-4} - 10^{-3}$ for an observed amplitude of $10^{-5}$. On the other hand, the ratio $(\ell_s/\ell) = 10^{-2}$ and $g_s$ is proportional to the string coupling constant, $g_s$. Our assumption from the beginning was that $g_s \ll 1$. It seems that our result is consistent with that assumption.

Finally, notice that the spectrum (97) looks scale-invariant to the first approximation (assuming $T$ is independent of $k$), but since $T(t_{exit}(k))$ at time $t_{exit}(k)$ depends on $k$, the power spectrum gains a slight red tilt due to the $1 - T(t_{exit}(k))/T_H$ in the denominator. Since the exact form of $k$ dependency of $T$ is hard to find due to our lack of knowledge about phase transition at the end of Hagedorn epoch, one possible ansatz for the factor in the denominator, noticing that $T(t_{exit}(k))$ is a slowly decreasing function of $k$ (shorter modes or larger $k$ modes exit Hubble radius later), is

$$1 - \frac{T(t_{exit}(k))}{T_H} \approx \alpha \left( \frac{k}{k_0} \right)^\epsilon,$$

where both $\alpha$ and $\epsilon$ are positive numbers much smaller than unity for some constant $k_0$. Therefore the scalar spectral index $n_s$ is

$$n_s - 1 \approx -\epsilon,$$

which yields $n_s \approx 0.95$ if we assume $\epsilon \approx 0.05$ to be compatible with the third year WMAP result [10]. So the spectrum has a mild red tilt depending on the value of $\epsilon$.

VII. CONCLUSION

In this paper, we have studied the generation and evolution of cosmological fluctuations in a model of string gas cosmology in which an early quasi-static Hagedorn
phase is followed by the radiation-dominated phase of standard cosmology, without an intervening period of inflation. Due to the fact that the Hubble radius during the Hagedorn phase is cosmological, it is possible to produce fluctuations using causal physics. Assuming thermal equilibrium on scales smaller than the Hubble radius, we have used string thermodynamics to study the amplitude of density fluctuations during the Hagedorn phase. The mean square energy fluctuations are determined by the specific heat of the string gas. To compute the perturbations on a physical length scale $R$, we apply string thermodynamics to a box of size $R$. Working under the assumption that all spatial dimensions are compact (but sufficiently large), the specific heat turns out to scale as $R^2$ for closed strings. This is an intrinsically stringy effect: in the case of point particle thermodynamics, the specific heat would scale as $R^d$. The $R^2$ scaling of the specific heat leads to a scale-invariant spectrum of metric fluctuations. The compactness of the spatial dimensions are very crucial here. If we were dealing with noncompact dimensions, instead, the specific heat would have been negative. The positiveness of the specific heat for closed strings in compact spatial dimensions, among other things, can assure us the absence of formation of any primordial black holes in the Hagedorn phase, for instance.

Although our cosmological scenario provides a new mechanism for generating a scale-invariant spectrum of cosmological perturbations, it does not solve all of the problems which inflation solves. In particular, it does not solve the flatness problem. Without assuming that the there are some large spatial dimensions are much larger than the string scale, we do not obtain a universe which is sufficiently large today. The longevity of the Hagedorn phase, however, could help us to explain the homogeneity and the absence of $U(1)$ monopoles. During the Hagedorn phase, closed strings would have enough time to travel the entire universe and communicate with strings of other parts of the universe so that the whole universe becomes almost homogenous. On the other hand, if monopoles form during the Hagedorn phase, perhaps there would be enough time for them to annihilate each other. These issues, of course, need more detailed studies and will be discussed in future publications.

Our scenario may well be testable observationally. Taking into account the fact that the temperature $T$ evaluated at the time $t_i(k)$ when the scale $k$ exits the Hubble radius depends slightly on $k$, the formula (9) leads to a calculable deviation of the spectrum from exact scale-invariance. Since $T(t_i(k))$ is decreasing as $k$ increases, a slightly red spectrum is predicted. Since the equation of state does not change by orders of magnitude during the transition between the initial phase and the radiation-dominated phase as it does in inflationary cosmology, the spectrum of tensor modes is not expected to be suppressed compared to that of scalar modes.

Finally, it would interesting to study the fluctuation of the dilatonic field and its contribution to the metric fluctuation during the Hagedorn phase in more details. We hope we can address this and other related issues in future publications.

Acknowledgements

I really am thankful to R. Brandenberger, C. Vafa and L. Motl without whom this work could no be accomplished. I also would like thank R. Allahverdi, A. Guth, J. Khury, L. Kofman, A. Linde, V. F. Mukhanov, S. Patil, A. Peat, P. Ramond, C. Thorn and S. Watson for many useful discussions. The work of A.N. is supported in part by NSF grant PHY-0244821 and DMS-0244464.

\[ \text{References} \]

[1] R. H. Brandenberger and C. Vafa, Nucl. Phys. B 316, 391 (1988).
[2] A. A. Tseytlin and C. Vafa, Nucl. Phys. B 372, 443 (1992).
[3] R. H. Brandenberger, “Moduli stabilization in string gas cosmology,” arXiv:hep-th/0509159.
[4] R. H. Brandenberger, “Challenges for string gas cosmology,” arXiv:hep-th/0500009.
[5] T. Battefeld and S. Watson, “String gas cosmology,” arXiv:hep-th/0510022.
[6] J. Kripfanz and H. Perl, Class. Quant. Grav. 5, 453 (1988).
[7] S. Alexander, R. H. Brandenberger and D. Easson, Phys. Rev. D 62, 103509 (2000) arXiv:hep-th/0005212.
[8] S. Watson and R. Brandenberger, JCAP 0311, 008 (2003) arXiv:hep-th/0307044.
[9] S. P. Patil and R. Brandenberger, Phys. Rev. D 71, 103522 (2005) arXiv:hep-th/0401037.
[10] S. P. Patil and R. H. Brandenberger, arXiv:hep-th/0502069.
[11] R. Brandenberger, Y. K. Cheung and S. Watson, arXiv:hep-th/0501032.
[12] R. Brandenberger, arXiv:hep-th/0509099, “Challenges for String Gas Cosmology” to appear in the proceedings of the 59th Yamada Conference “Inflating Horizon of Particle Astrophysics and Cosmology” (Univ. of Tokyo, Tokyo, Japan, June 20 - June 24, 2005).
[13] T. Battefeld and S. Watson, arXiv:hep-th/0510022.
[14] R. Brandenberger, “Moduli Stabilization in String Gas Cosmology” to appear in the proceedings of YKIS 2005 (Yukawa Institute for Theoretical Physics, Kyoto, Japan, June 27 - July 1, 2005). arXiv:hep-th/0509159.
[15] R. Hagedorn, Nuovo Cim. Suppl. 3, 147 (1965).
[16] R. Easther, B. R. Greene, M. G. Jackson and D. Kabat, JCAP 0502, 009 (2005) arXiv:hep-th/0409121.
[17] R. Danos, A. R. Frey and A. Mazumdar, Phys. Rev. D 70, 106010 (2004) arXiv:hep-th/0409162.
[18] S. Kanno and J. Soda, “Moduli stabilization in string gas.
compactification," arXiv:hep-th/0509074.

[19] A. H. Guth, “The Inflationary Universe: A Possible Solution To The Horizon And Flatness Problems,” Phys. Rev. D 23, 347 (1981).

[20] A. D. Linde, “A New Inflationary Universe Scenario: A Possible Solution Of The Horizon, Flatness, Homogeneity, Isotropy And Primordial Monopole Problems,” Phys. Lett. B 108, 389 (1982).

[21] A. Albrecht and P. J. Steinhardt, “Cosmology For Grand Unified Theories With Radiatively Induced Symmetry Breaking,” Phys. Rev. Lett. 48, 1220 (1982).

[22] A. H. Guth and S. Y. Pi, “Fluctuations In The New Inflationary Universe,” Phys. Rev. Lett. 49, 1110 (1982).

[23] A. D. Linde, “Scalar Field Fluctuations In Expanding Universe And The New Inflationary Universe Scenario,” Phys. Lett. B 116, 335 (1982).

[24] A. D. Linde, “Chaotic Inflation,” Phys. Lett. B 129, 177 (1983).

[25] S. Kachru, R. Kallosh, A. Linde and S. P. Trivedi, “De Sitter vacua in string theory,” Phys. Rev. D 68, 046005 (2003) [arXiv:hep-th/0301240].

[26] S. Kachru, R. Kallosh, A. Linde, J. M. Maldacena, L. McAllister and S. P. Trivedi, “Towards inflation in string theory,” JCAP 0310, 013 (2003) [arXiv:hep-th/0308055].

[27] P. J. Steinhardt and N. Turok, “A cyclic model of the universe,” Science 296 (2002) 1436.

[28] A. Nayeri, R. H. Brandenberger and C. Vafa, “Producing a scale-invariant spectrum of perturbations in a Hagedorn phase of string cosmology,” arXiv:hep-th/0511140.

[29] R. H. Brandenberger, A. Nayeri, S. P. Patil and C. Vafa, “Tensor modes from a primordial Hagedorn phase of string cosmology,” [arXiv:hep-th/0604126].

[30] S. A. Abel, J. L. F. Barbón, I. I. Kogan and E. Rabinovici, JHEP 04, 015 (1999) [hep-th/9902058].

[31] J. Polchinski, Commun. Math. Phys. 104, 37 (1986).

[32] M. B. Green, J. H. Schwarz and L. Brink, Nucl. Phys. B 198, 474 (1982).

[33] N. Deo, S. Jain and C.-I. Tan, Phys. Lett. B 220, 125 (1989).

[34] N. Deo, S. Jain and C.-I. Tan, Phys. Rev. D 40, 2626 (1989).

[35] N. Deo, S. Jain, O. Narayan and C.-I. Tan, Phys. Rev. D 45, 3641 (1992).

[36] D. A. Lowe and L. Thorlacius, Phys. Rev. D 51, 665 (1995), hep-th/9408134.

[37] S. Lee and L. Thorlacius, Phys. Lett. B 413, 303 (1997), hep-th/9707167.

[38] V. F. Mukhanov, H. A. Feldman and R. H. Brandenberger, Phys. Rept. 215, 203 (1992).

[39] R. H. Brandenberger, Lect. Notes Phys. 646, 127 (2004) [arXiv:hep-th/0306071].

[40] J. M. Bardeen, P. J. Steinhardt and M. S. Turner, Phys. Rev. D 28, 679 (1983).

[41] R. H. Brandenberger and R. Kahn, Phys. Rev. D 29, 2172 (1984).

[42] R. H. Brandenberger, Nucl. Phys. B 245, 328 (1984).

[43] R. Mansouri and A. Nayeri, “Gravitational coupling constant in arbitrary dimension,” Grav. Cosmol. 4, 142 (1998) [arXiv:gr-qc/9609061].

[44] S. Weinberg, “Adiabatic modes in cosmology,” Phys. Rev. D 67, 123504 (2003) [arXiv:astro-ph/0302326].

[45] J. A. Peacock, *Cosmological Physics*, (Cambridge University Press, Cambridge 1999).

[46] D. N. Spergel et al., “Wilkinson Microwave Anisotropy Probe (WMAP) three year results: Implications for cosmology,” [arXiv:astro-ph/0603449].

[47] Kayhaanistan is a Persian word and means a place for many mini cosmoses or *Kayhaanaks*.

[48] See [23] for the calculation of other components of the correlation function.