We discuss the effect of disorder on the coherent propagation of the bound state of two attracting particles. It is shown that a result analogous to the Anderson theorem for dirty superconductors is also valid for the Cooper problem, namely, that the pair wave function is extended beyond the single-particle localization length if the latter is large. A physical justification is given in terms of the Thouless block-scaling picture of localization. These arguments are supplemented by numerical simulations. With increasing disorder we find a transition from a regime in which the interaction delocalizes the pair to a regime in which the interaction enhances localization.

I. INTRODUCTION

It has long been known that the wave function of an electron moving in a random potential becomes spatially localized. This effect was first predicted by Anderson and is termed Anderson localization. In one and two dimensions all quantum states are localized in the presence of any amount of disorder while in three dimensions localization occurs only above some critical disorder. While this phenomenon is now fairly well understood in a single-particle picture, the inclusion of interactions in the disordered many-body system is a non-trivial problem. Insight can be gained by studying the simpler case of just two interacting particles in a random potential. In this context, it has recently been claimed that the interaction can actually lead to a delocalization effect, in the sense that the spatial extent of the two-body wave function is larger than the single-particle localization length. This delocalization was found for both attractive and repulsive interactions, at least for some of the eigenstates in the continuous spectrum of energy eigenvalues. Some authors objected to this finding and this point is still being debated. Furthermore, the relevance of these results for the many-body problem has not yet been clarified.

In this paper our main concern is the case of an attractive interaction. The concept of a two-particle bound state (the Cooper pair) plays an important role in the theory of superconductivity. There is theoretical and experimental evidence for the existence of superconductor-insulator transitions, where localized states combine coherently into a superconducting condensate with a finite superfluid density. This finding motivates the question addressed in this paper: will the bound state of two attracting particles be extended over distances larger than the single-particle localization length? We find that a result analogous to the Anderson theorem for localized superconductors is valid for this problem in the limit of large single-particle localization lengths. By increasing the disorder we find a transition from a regime in which the interaction increases the localization length to a regime in which the interaction reduces the localization length.

II. LOCALIZED SUPERCONDUCTORS

The Hamiltonian for a disordered conventional (s-wave) superconductor is

$$H = H_0 - V \int \psi_{\uparrow}(r)\psi_{\downarrow}(r)\psi_{\uparrow}(r)\psi_{\downarrow}(r)dr$$

(1)

where $H_0$ represents the single-electron part including a spatially random external potential and the interaction parameter $V$ is taken as positive. The attractive interaction could be due to exchange of phonons or purely electronic mechanisms but its origin does not concern us here. Depending on the disorder, the eigenstates $\phi_n(r)$ of $H_0$ can be extended or localized (with localization length $L_c$ at the Fermi level).

At $T=0$ the system described by (1) is a superconductor with spatially constant order parameter $\Delta(r) = V < \psi_{\uparrow}(r)\psi_{\downarrow}(r) > = \Delta$ when the condition

$$\Delta N_p L_c^4 \gg 1$$

(2)

holds. $\Delta$ is then given by the same expression as that for a clean superconductor. This is the Anderson theorem. The important message of equation (2) is that if the disorder is strong enough to localize the single-particle states, the superconducting order parameter $\Delta$ and critical temperature $T_c$ will remain unaffected as long as the number of single-particle states in the energy range $\Delta$ contained in the localization volume $L_c^4$ is still large. (The superfluid density, on the other hand, is greatly reduced.) If the amount of disorder is further increased so that (2) is no longer valid the superconducting order parameter will fluctuate strongly in space and the critical temperature will be lowered.

Note that because $\Delta(r) \propto < \psi_{\uparrow}(r)\psi_{\downarrow}(r) >$ is the wave function of the condensate, the Anderson theorem tells us that the attractive interaction is delocalizing the Cooper
pairs. Keep in mind, however, that in this derivation the Cooper pairs are strongly interacting (and actually overlapping) with each other, forming a correlated liquid. In BCS theory the concept of Cooper pair only has the formal significance that strong pair correlations exist between the particles in phase space. In the regime of validity of the Anderson theorem $\Delta$ is given by

$$1 = V \sum_{\alpha} \frac{1}{2\Delta^2 + \xi_\alpha^2} = V \sum_{\alpha} \frac{1}{2\xi_\alpha}. \quad (3)$$

The denominator of this expression is the quasi-particle excitation energy. The binding energy of an "isolated" Cooper pair is given by an expression of the same form as this one (see eq.(8) below) but with a different denominator. This is because equation (3) takes into account the interactions between Cooper pairs.

Experimentally, it has been found that superconductivity persists up to the Anderson metal-insulator transition $\xi_{\text{loc}}$. These systems exhibit activated conductivity above $T_c$. The coexistence of superconductivity and localization has also been observed in underdoped high-$T_c$ superconductors [9–12].

### III. COOPER PROBLEM IN A RANDOM POTENTIAL

In what follows the problem of two interacting electrons in a random potential will be addressed for the specific case of an attractive interaction. The aim is to see how the interplay of disorder and interaction affects the coherent propagation of the electrons in the ground state. The attractive interaction will be assumed to be short-ranged. We can describe this system by an Anderson-Hubbard Hamiltonian

$$H = -t \sum_{<r,r'>,\sigma} \epsilon_{r,\sigma} c_{r,\sigma} \hat{c}_{r',\sigma} + U \sum_r n_{r,\uparrow} n_{r,\downarrow} + \sum_r \epsilon_r n_{r,\sigma} \quad (4)$$

with $\epsilon_r$ representing the site energies randomly distributed over a width $W$, and $U > 0$. The single-particle eigenfunctions $\phi_n$ are assumed to be localized by the disorder with a localization length $L_c$.

We search for the two-particle ground state

$$\hat{H} \Psi(r_1,r_2) = E_0 \Psi(r_1,r_2) \quad (5)$$

where $r_1$ and $r_2$ are the coordinates of the electrons. Because the bound state is a spin singlet, $\Psi(r_1,r_2)$ is a symmetric function of its arguments and has the general form

$$\Psi(r_1,r_2) = \sum_{n,m} g_{n,m} \phi_n(r_1) \phi^*_m(r_2) \quad (6)$$

where $\phi_n$ are normalized single-particle eigenfunctions of $H$ with energy eigenvalues $\xi_n$. In order to simulate the presence of a Fermi sea, it is assumed that the two electrons can only be paired in states $\phi_n$ which lie above the Fermi surface. According to the values of the parameters in (6) we recognize several different regimes. If $U \gg t$ (strong attraction) then the electrons are tightly bound and move together like a heavy boson with hopping amplitude $t^2/U$ in an environment with disorder $W$. This boson would then become easily localized. In the remainder of the paper we concentrate on the regime in which the interaction is not strong ($U \lesssim t$). In this case $\Psi$ is essentially the result of electron pairing in time-reversed single-particle eigenstates of (6) for not too strong disorder. This can be seen as follows.

The Schrödinger equation for the Cooper pair wave function is

$$\Psi(r_1,r_2) = \sum_{n,m} \frac{\phi_n(r_1) \phi^*_m(r_2)}{\sqrt{E_0 - \xi_n}} \phi_n(r_1) \phi^*_m(r_2). \quad (7)$$

It admits the solution $\Psi(r,r) = \text{const.}$ if the condition

$$1 = -U \sum_n \frac{|\phi_n(r)|^2}{E_0 - 2\xi_n}$$

$$= -U \int d\xi \frac{N(\xi,r)}{E_0 - 2\xi} \quad (8)$$

holds. (Here $N(\xi,r)$ denotes the local density of states at the point $r$.) Thus a result analogous to the Anderson theorem is valid for the function $\Psi(r,r)$. In such a case the solution to (6) is

$$\Psi(r_1,r_2) \propto \sum_n \frac{\phi_n(r_1) \phi^*_n(r_2)}{E_0 - 2\xi_n} \quad (9)$$

and $\Psi$ has no overlap with the state $\phi_n(r_1) \phi^*_m(r_2)$ if $n \neq m$. The condition (8) can be satisfied if the integral of the local density of states $d\xi N(\xi,r)$ over an energy interval of order $|E_0|$ does not depend on $r$. This is possible even if the disorder is strong enough to localize the single-particle states as long as the condition

$$|E_0| \sqrt{N_L L_c^d} \gg 1$$

holds. The binding energy $|E_0|$ will then be the same as that for a clean system. So we reach the conclusion that the attractive interaction can delocalize the pair or, at least, increase its localization length.

### IV. DELOCALIZATION BY INTERACTIONS

The delocalization by interactions due to the attractive interaction can be understood within the block-scaling...
picture of localization introduced by Thouless. In a recent paper, Imry has used the block-scaling picture in order to argue that interactions (attractive or repulsive) should, in some cases, delocalize some of the eigenstates of the continuous spectrum of the pair of electrons. In what follows we extend the argument to the case of a Cooper pair.

Suppose the system is divided into blocks of linear size $L_c$ (measured in units of the lattice spacing) so that the mean level spacing in a block is $\Delta_1 = B/L_c^d$ with $B$ equal to the band width. We can then solve the Cooper problem, as above, for each block by pairing the two electrons in time-reversed states localized inside the block. If the fluctuation of the binding energy is smaller than the effective scattering amplitude of the pair between blocks then the pair will be extended over many blocks. We denote the Cooper pair in the $j$-th block by $\Psi_j$. Next, we estimate the scattering amplitude $t_{eff}$, due to the interaction, of the Cooper pair between two neighboring blocks as

$$t_{eff} = - \langle \Psi_j | U \sum_r n_{r,\uparrow} n_{r,\downarrow} | \Psi_j' \rangle$$

$$= - U \sum_{n,n'} g_{nn} g_{n'n'} \sum_r | \phi_n(r) |^2 | \phi_{n'}(r) |^2$$

$$\approx - U \sum_{n,n'} g_{nn} g_{n'n'}.$$  \hspace{1cm} (10)

We now note that $g_{nn}$ depends smoothly on $n$ and has no nodes because it is the wave function of a ground state, namely

$$g_{nn} \propto \frac{1}{E_0 - 2\xi_n}.$$  

The normalization of $\Psi_j$ implies $g_{nn} \sim (| E_0 | / \Delta_1)^{\frac{1}{2}}$ and the number of terms in the sum in (10) is large, of the order of $(| E_0 | / \Delta_1)^2$. All those terms interfere constructively yielding

$$t_{eff} \sim \frac{| E_0 |}{\Delta_1} \frac{U}{L_c^d} \sim \frac{U}{B}.$$  

The fluctuation of the binding energy $| E_0 |$ due to the randomness in $\xi_n$ is of the order of $\delta E \sim \sqrt{| E_0 | \Delta_1}$ (see the Appendix). Thus the condition for delocalization is obtained as

$$\frac{t_{eff}}{\delta E} \sim \sqrt{\frac{| E_0 |}{\Delta_1}} \frac{U}{B} \ll 1.$$  

The ratio $U/B$ is the BCS product $N_F V$. The following points should be noted:

(i) There are no phase correlations to consider in the sum (10) because all the terms are real and negative;

(ii) The large effective hopping amplitude $t_{eff}$ resulted from $g_{nn}$ being a smooth function of $n$ with no nodes. In other words, because we have been considering the ground state.

(iii) If the sign of the interaction $U$ is reversed, to make it repulsive, we do not expect any delocalization to occur in the ground state. The reason for this is that $\Psi_j$ for each block (using time-reversed pairing) would now essentially involve only the single-particle eigenstate $\phi_0$ with the lowest energy: $\Psi_j(r_1, r_2) \approx \phi_0(r_1)\phi_0^*(r_2)$. Then $t_{eff} \sim U/L_c^d \sim \Delta_1$ if $U \approx B$. If we remove the constraint of time-reversed pairing then Imry’s argument would predict delocalization only if the pair has a certain excitation energy above the Fermi level $\Delta_1$. So it would not be in the ground state.

V. NUMERICAL RESULTS

We have also performed numerical calculations of the Cooper pair wave function for 1D systems of up to $L = 100$ sites using the Lanczos algorithm. In order to impose the constraint of a filled Fermi sea the single-particle eigenstates for a given realization of the disorder potential are calculated and the eigenenergies of the states below the Fermi energy are shifted by a large amount. Thus only the one-particle states above the Fermi surface are accessible for the Cooper pair. Before doing the Lanczos diagonalization of the two-particle problem the Hamiltonian is transformed back to real space representation where the number of nonzero matrix elements is much smaller than in the basis of the eigenstates of the non-interacting disordered system.

In order to determine the spatial extent of the Cooper pair wave function with respect to both the relative and the center-of-mass coordinates we have calculated two different quantities. The first one is the participation ratio

$$r = \frac{1}{\sum_r | \Psi(r, r) |^4}$$

where $\Psi(r, r)$ is normalized to unity. Since only the diagonal part $(r_1 = r_2)$ of the wave function is involved in the calculation the participation ratio can also be interpreted as the localization length of the Cooper pair.

The second quantity, which is related to the relative coordinate of the two electrons, is the average size $d$ of the Cooper pair defined as

$$d = \frac{1}{r_1, r_2} | \Psi(r_1, r_2) |^2 | r_1 - r_2 |.$$  

Our results are obtained for a system of 100 sites and each data point represents the average over 200 realizations of the disorder potential. Since the fluctuations of the participation ratio $r$ are very large we found it more convenient to average $\log r$ instead of $r$ itself.

Fig. 1 shows the behavior of $\log r$ as a function of $U/t$ for different values of the disorder strength $W/t$. While
in the case of strong disorder the interaction leads to a decrease of the participation ratio, for smaller values of $W/t$ an enhancement of log $r$ is observed, at least for not too large $U/t$. This is in qualitative agreement with our analytical arguments. One should however keep in mind that the calculations are done for a rather small system size due to finite size effects. In Fig. 2 we see that the size of the Cooper pair decreases rapidly with increasing interaction strength. The values for $U/t \to 0$ which should be of the order of the one-particle localization length are considerably reduced due to finite size effects.

VI. CONCLUSION

The interplay of disorder and interaction is a complex problem and therefore it is useful to start with the two-body problem in a disordered medium. The effect of a repulsive interaction seems to be very different from that of an attractive interaction since the latter can induce propagation of the bound state of two particles (their ground state). The binding energy and spatial extent of the bound state is insensitive to disorder beyond the point where single-particle states become localized. This result for the Cooper pair corresponds to the familiar Anderson theorem for the many-body problem of dirty superconductors and is valid in a regime where the attraction and the disorder are not too strong so that electrons are paired in time-reversed single-particle eigenstates. These localized states can then be combined coherently into an extended pair wave function (eq. (1)). In analogy with the case of dirty superconductors discussed in [4] where the superfluid density is greatly reduced by the disorder, we also expect the energy of the wave function (1) to be much less sensitive to the boundary conditions than that in a clean system. In other words, disorder has a stronger effect on the sensitivity to boundary conditions than on the localization length. It is possible to go away from this "Anderson regime" in two ways. If the interaction is increased then both the binding energy and the effective mass of the pair increase, so the pair becomes more localized. On the other hand, if the interaction is kept not too strong but the disorder is increased, the binding energy and the localization length are reduced.

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Appendix

We want to prove that $\delta E \sim \sqrt{E \Delta_1}$. The binding energy $E$ is obtained from the equation

$$\frac{1}{U} = \sum_n \frac{1}{E + 2\xi_n} = \int N(\xi)\frac{d\xi}{E + 2\xi}. \quad (11)$$

The density of states is $N(\xi) = N_0(\xi) + n(\xi)$ where $N_0(\xi)$ is the average density of states and $n(\xi)$ the noise due to disorder. Then $E = E_0 + \delta E$ where $E_0$ is the binding energy derived from $N_0(\xi)$. Expanding the right-hand side of the above equation for small $\delta E$ and taking into account the definition of $E_0$ we arrive at

$$\delta E \int \frac{N_0(\xi)}{(E_0 + 2\xi)^2}d\xi = \int \frac{n(\xi)}{E_0 + 2\xi}d\xi \quad (12)$$

We make the following assumptions about the moments (averages) of $n(\xi)$: $n(\xi) > 0$ and $n(\xi)n(\xi') > n_0^2 \exp(-\xi^2/\Delta_1^2)$. This latter condition is only intended to express the fact that the correlation persists over an energy of the order of the single-particle level.
spacing. Taking the square of (12) and averaging we obtain

\[ <\delta E^2> = \frac{(N_F)^2}{2E_0} = n_0^2 \int d\xi \int d\xi' e^{-\left(\xi-\xi'\right)^2/\Delta_1^2} \frac{1}{(E_0 + 2\xi)(E_0 + 2\xi')} \]

This gives \( \delta E \sim \frac{n_0}{N_F} \sqrt{\Delta_1 E_0} \). The quantity \( n_0/N_F \) should not be larger than unity.

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