A note on finite-time Lyapunov dimension of the Rossler attractor

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For the Rössler system we verify Eden’s conjecture on the maximum of local Lyapunov dimension. We compute numerically finite-time local Lyapunov dimensions on the Rössler attractor and embedded unstable periodic orbits. The UPO computation is done by Pyragas time-delay feedback control technique.

I. RÖSSLER ATTRACTOR AND PYRAGAS STABILIZATION OF EMBEDDED UNSTABLE PERIODIC ORBITS

Consider the following Rössler system [1]

\begin{align}
\dot{x} &= -y - z, \\
\dot{y} &= x + ay, \\
\dot{z} &= b - cz + xz,
\end{align}

(1)

with arbitrary real parameters \(a, b, c \in \mathbb{R}\). If \(c^2 \geq 4ab\), then system (1) has the following equilibria:

\[ O^\pm = (ap^\pm, -p^\pm, p^\pm), \quad \text{where} \quad p^\pm = \pm \sqrt{\frac{c^2 - 4ab}{2a}}. \]

(2)

For some values of parameters system (1) exhibits chaotic behavior. To get a visualization of chaotic attractor one needs to choose an initial point in the basin of attraction of the attractor and observe how the trajectory, starting from this initial point, after a transient process visualizes the attractor: an attractor is called a self-excited attractor if its basin of attraction intersects with any open neighborhood of an equilibrium, otherwise, it is called a hidden attractor \([2–5]\). It was discovered numerically by Rössler that in the phase space of system (1) with parameters \(a = 0.2, b = 0.2, c = 5.7\) there exist a chaotic attractor of spiral shape, which is self-excited with respect to both equilibria \(O^\pm\).

One of the building blocks of chaotic attractor are embedded unstable periodic orbits (UPOs). An effective method for the computation of UPOs is the time-delay feedback control (TDFC) approach, suggested by K. Pyragas [6] (see also discussions in \([7–10]\)). Let \(u^{\text{upo}}(t)\) be an UPO with period \(\tau > 0\), \(u^{\text{upo}}(t + \tau) = u^{\text{upo}}(t)\), satisfying a differential equation

\[ \dot{u} = f(u). \]

(3)

To compute the UPO, we add the TDFC:

\[ \dot{u} = f(u) + kBC^*(u(t - T) - u(t)), \]

(4)

where \(B, C\) are vectors and \(k\) is a real gain. If \(T = \tau\), then \(kBC^*(u(t - T) - u(t)) = 0\) along the UPO, and periodic solution of system (4) coincides with periodic solution of system (3).

For the Rössler system (1) we solved numerically system (4) and stabilized a period-1 UPO \(u^{\text{upo}}_1(t, u_0)\) with period \(\tau = 5.8811\) (see Fig. 1).

Then for the initial point \(u_0^{\text{upo}}\), chosen on the UPO \(u^{\text{upo}} = \{u^{\text{upo}}(t), t \in [0, \tau]\}\), we numerically compute the trajectory \(\tilde{u}(t, u_0^{\text{upo}})\) of system (4) without the stabilization (i.e. with \(k = 0\)) on sufficiently large time interval \([0, T = 500]\) (see Fig. 1b). One can see that on the initial small time interval \([0, T_1 \approx 60]\), even without the control, the obtained trajectory \(\tilde{u}(t, u_0^{\text{upo}})\) traces approximately the ”true” periodic orbit \(u^{\text{upo}}(t, u_0^{\text{upo}})\). But for \(t > T_1\) without control the trajectory \(\tilde{u}(t, u_0^{\text{upo}})\) diverge from \(u^{\text{upo}}\) and wind on the attractor \(A\).

II. FINITE-TIME LYAPUNOV DIMENSION AND EDEN CONJECTURE

For an attractor, an interesting question [11, p.98] (known as Eden conjecture) is whether the supremum of the local Lyapunov dimensions is achieved on a stationary point or an unstable periodic orbit embedded in the strange attractor. In general, a conjecture on the Lyapunov dimension of self-excited attractor \([12, 13]\) is that for a typical system the Lyapunov dimension of a self-excited attractor does not exceed the Lyapunov dimension of one of unstable equilibria, the unstable manifold of which intersects with the basin of attraction and visualize the attractor.

Below we follow the concept of the finite-time Lyapunov dimension \([12, 13]\), which is convenient for carrying out numerical experiments with finite time. The finite-time local Lyapunov dimension \([12, 13]\) can be defined via an analog of the Kaplan-Yorke formula with respect to the set of finite-time Lyapunov exponents:

\[ \dim_l(t, u) = d_l^{\text{KY}}(\{E_i(t, u)\}_{i=1}^3) = j(t, u) + \frac{\sum_{i=1}^{m} \text{LE}_i(t, u) \geq 0}{\text{LE}_i(t, u) \geq 0}, \]

(5)

where \(j(t, u) = \max\{m : \sum_{i=1}^{m} \text{LE}_i(t, u) \geq 0\}\). Then the finite-time Lyapunov dimension (of dynamical system generated by (3) on compact invariant set \(A\)) is de-
The period-1 UPO \( u_{\text{upo}} \) with period \( \tau = 5.8811 \) has the following multipliers: \( \rho_1 = -2.40398, \rho_2 = 1, \rho_3 = -1.2946 \cdot 10^{-14} \). Thus, for the local Lyapunov dimension of the UPO \( u_{\text{upo}}(t) \) we obtain \( \dim_L u_{\text{upo}} = d_L^{KY}(\{\frac{1}{\tau} \log \rho_j\}_{j=1}^3) = 2.0274 \leq 2.0283 = \dim_L(500, u_{\text{upo}}) \).

III. CONCLUSION

In this note we have confirmed the Eden conjecture for the Rössler system (1) and obtained the following relations between the Lyapunov dimensions:

\[
3 = \dim_L O_+ > 2.0341 = \dim_L O_- > 2.0274 = \dim_L u_{\text{upo}} > 2.0160 = \max_{u \in C_{\text{grid}}} \dim_L(500, u) \geq \dim_L A \geq \dim_H A.
\]

Concerning the time of integration, remark that while the time series obtained from a physical experiment are assumed to be reliable on the whole considered time interval, the time series produced by the integration of a mathematical dynamical model can be reliable on a limited time interval only due to computational errors (caused by finite precision arithmetic and numerical integration of ODE). Thus, in general, the closeness of the real trajectory \( u(t, u_0) \) and the corresponding pseudo-trajectory \( \tilde{u}(t, u_0) \) calculated numerically can be guaranteed on a limited short time interval only. However, for two different long-time pseudo-trajectories \( \tilde{u}(t, u_0^1) \) and \( \tilde{u}(t, u_0^2) \) visualizing the same attractor, the corresponding finite-time LEs can be, within the considered error, similar due to averaging over time and similar sets of points \( \{\tilde{u}(t, u_0^1)\}_{t \geq 0} \) and \( \{\tilde{u}(t, u_0^2)\}_{t \geq 0} \). At the same time, the corresponding real trajectories \( u(t, u_0^1) \) may have different LEs, e.g. \( u_0 \) may correspond to an unstable periodic trajectory \( u(t, u_0) \) which is embedded in the attractor and does not allow one to visualize it.
Figure 2: $\text{LE}_1(t, u_0^{\text{upo}})$ and $\dim(t, u_0^{\text{upo}})$ on the time interval $t \in [0, 500]$ along the UPO $u^{\text{upo}}(t)$ (red) and trajectory integrated without stabilization (blue). Both trajectories start from the point $u_0^{\text{upo}} = (6.491, -7.0078, 0.1155)$.

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