Topological Strings with Scaling Violation and Toda Lattice Hierarchy

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Abstract

We show that there is a series of topological string theories whose integrable structure is described by the Toda lattice hierarchy. The monodromy group of the Frobenius manifold for the matter sector is an extension of the affine Weyl group \( \tilde{W}(A_N^{(1)}) \) introduced by Dubrovin. These models are generalizations of the topological \( CP^1 \) string theory with scaling violation. The logarithmic Hamiltonians generate flows for the puncture operator and its descendants. We derive the string equation from the constraints on the Lax and the Orlov operators. The constraints are of different type from those for the \( c = 1 \) string theory. Higher genus expansion is obtained by considering the Lax operator in matrix form.
1 Introduction

Integrable structure of topological string theories has attracted much attention in recent years [1] [2]. Topological string theory is a kind of two dimensional cohomological field theory with topological matter coupled to topological gravity. As a cohomological field theory, topological string theory gives topological correlation functions among BRST cohomology classes. A generating function of these correlation functions is the free energy of the theory (the logarithm of the partition function). The integrability of topological string theory means that the partition function is a tau function of some integrable hierarchy. Furthermore, the partition function is uniquely characterized in terms of the so-called string equation. One of the motivations for investigating such a integrability of topological string theory is that the non-critical string theory has several common algebraic structures with topological string theory. Indeed, any string theory has more or less characteristic properties as topological field theory.

An important class of topological matter theory is topological conformal matter which can be obtained by twisting any $N = 2$ super conformal model. Taking the minimal $N = 2$ superconformal model of ADE type, which allows the Landau-Ginzburg description, one can construct topological string theory whose integrable structure is identified as the generalized KdV hierarchy, or the Drinfeld-Sokolov hierarchy of ADE type [3]. Topological matter theory usualy has a finite number of physical operators, called (chiral) primary fields. After coupling to topological gravity, an infinite series of gravitational descendants appears for each primary field. These descendants correspond to infinite number of commuting Hamiltonians of integrable hierarchy.

Recently it has been shown that there exist physically more interesting topological strings which are integrable. The $c = 1$ string theory is an important example [4] [5]. It is originally defined as a non-critical string theory and has physical scattering process among tachyons. Therefore, it is an astonishing fact that the $c = 1$ string theory is topological [6] [7] [8] [9] and integrable [10] [11]. Another example is the topological $CP^1$ model [12] [13] [14]. This is also quite intersting, since it opens a possibility of investigating quantum geometry of the target space, such as quantum cohomology ring, from the view point of integrable systems. The Toda lattice hierarchy is the integrable structure behind both
examples. In this paper we will show that there is a series of topological string theories whose integrable structure is described by the Toda lattice hierarchy. The difference of the models comes from the constraints imposed on the Lax operators, or in other words, the string equation. In [15], [16], it is shown that the string equation of the $c = 1$ string theory is given by the constraints:

$$L = \mathcal{M}, \quad \mathcal{L}^{-1} = \mathcal{N}^{-1}, \quad \text{(1.1)}$$

on the Lax operators $L, \mathcal{L}$ and the Orlov operators $M, \mathcal{M}$ of the (dispersionless) Toda lattice hierarchy. (For the definition of these objects, see section 4.) It has been established that the constraints (1.1) imply the $W_{1+\infty}$ constraints of the $c = 1$ theory which uniquely determine all the tachyon correlation functions at the self-dual radius [10], [11], [17]. The string equation in full genus is discussed in [18]. Quite recently the string equation of the deformed $c = 1$ string in the black hole background is also formulated in the framework of the Toda lattice hierarchy [19].

Topological string theories we will consider in this paper have a different type of the constraints:

$$\frac{\mathcal{L}}{N} = \mathcal{M}, \quad \mathcal{L}^{-1} = -\mathcal{M}, \quad \text{(N = 1, 2, \ldots)} \quad \text{(1.2)}$$

We note that similar constraints have been considered by Takasaki in connection with the topological strings of $D$ type [20]. Contrary to the constraint of the first type which allows infinitely many primary fields, the second type is quite restrictive on the number of the primaries which survive. After imposing the constraints (1.2), we have only $(N + 1)$ primary fields. The case $N = 1$ corresponds to the topological $CP^1$ model. Hence we have a series of topological string theories ($N \geq 2$) which generalize it. One of remarkable features in this type of topological string theory is that the flows for the puncture operator and its descendants are generated by the logarithmic Hamiltonians, which was first observed in [14]. The standard Hamiltonians in the Toda lattice hierarchy are of polynomial type. The logarithmic Hamiltonians involve the logarithm of the Lax operator. Such logarithmic Hamiltonians are well-defined only after the constrains of type (1.2) are imposed. Thus, they are peculiar to our models with the constraints (1.2).

In [21], it is shown that the constraints (1.1) arise quite naturally from the two matrix model without continuum limit. We easily see that the constraints (1.2) for $N = 1$ reduces
the two matrix model to the one matrix model [14]. Though we believe that there is an appropriate matrix model realization for $N \geq 2$, we have not found it at present.

In the following we will be mainly concerned with an example with three primaries for simplicity. The primary fields are the puncture operator $P$, one marginal operator $R$ and one relevant operator $Q$. (In general models, we will have several relevant operators but only one marginal operator.) Our model looks like a kind of “fusion” of the topological minimal model and the topological $CP^1$ model. The marginal operator comes from the $CP^1$ sector, while the relevant operator is similar to that of topological $A_N$ model. Like the topological $CP^1$ model, matter sector of our model does not have conformal invariance, or its “effective” first Chern class is non-vanishing. In fact the topological correlation functions have terms which may be interpreted as the instanton corrections, though their geometrical meaning is somewhat obscure.

The present paper is organized as follows. In the next section we take a topological matter theory first introduced by Dubrovin from his viewpoint of the Frobenius manifold and compute topological correlation functions using the topological recursion relation at tree level. The Lax formalism which implies all the result of tree level correlation functions is introduced in section 3. As we have seen in the topological $CP^1$ model, the flows corresponding to the descendants of the puncture operator are generated by the Hamiltonians including the logarithm of the Lax operator. In section 4, the integrable structure behind this Lax formalism is shown to be the Toda lattice hierarchy with a special constraint on the Lax operators. We will derive the string equation of the theory from the constraints on the Lax and the Orlov operators. This approach is similar to the one we have used for the $c = 1$ string theory. Section 5 is devoted to the higher genus structure. We will examine the genus expansion by using the Lax operator in matrix form.
2 Tree Level Correlation Functions

The topological string theory we will consider throughout the present paper is constructed from a topological matter introduced by Dubrovin. (See Exercise 4.3 and Example 5.5 in [2]) Before coupling to gravity the tree level free energy takes the form;

\[
F(t) = \frac{1}{2}(t_1)^2t_3 + \frac{1}{2}t_1(t_2)^2 - \frac{1}{24}(t_2)^4 + t_2e^{t_3}.
\]  

(2.1)

This model was discovered by considering the monodromy of the Frobenius manifold associated with topological matter. The monodromy of (2.1) is an extension of the affine Weyl group \(\tilde{W}(A^{(1)}_2)\). We can also find this model in the classification program of \(N = 2\) theories by Ceccoti and Vafa [22]. It is among three primary models\(^1\). The topological matter with the monodromy \(\tilde{W}(A^{(1)}_1)\) gives the same topological string theory as the topological \(CP^1\) model. There are a series of topological matter theories with the monodromy \(\tilde{W}(A^{(1)}_N)\), which generalizes the topological \(CP^1\) model. (Another more geometrical generalization is, of course, the topological \(CP^n\) model, or the topological Grassmannian model. But their integrable structure is still an open problem.) The free energy has an important property of generalized homogeneity;

\[
(3 - \mathcal{L}_E)F(t) = F(t),
\]  

(2.2)

where the Euler vector field \(E\) is given by

\[
E = t_1 \frac{\partial}{\partial t_1} + \frac{1}{2} t_2 \frac{\partial}{\partial t_2} + \frac{3}{2} \frac{\partial}{\partial t_3}.
\]  

(2.3)

We see that the dimension of this model is one. The parameter \(t_3\) is a marginal one. The coefficient \(3/2\) of \(\partial/\partial t_3\) can be regarded as the “effective” first Chern class in counting the ghost number anomaly.

The primary couplings \(t_i\) \((i = 1, 2, 3)\) are chosen to be flat coordinates. Let us introduce the primary operators \(P, Q, R\) conjugate to these couplings. Accordingly we will change the notation as \(t_{0,P} \equiv t_1, \ t_{0,Q} \equiv t_2, \ t_{0,R} \equiv t_3\). Then the basic two point functions, or the order parameters are

\[
\langle PP \rangle \equiv u(t) = t_{0,R}, \quad \langle PQ \rangle \equiv v(t) = t_{0,Q}, \quad \langle PR \rangle \equiv w(t) = t_{0,P}.
\]  

(2.4)

\(^1\)We thank S.-K. Yang for pointing it out.
On the small phase space the order parameters take the simple form; \( u(t) = t_{0,R}, v(t) = t_{0,Q}, w(t) = t_{0,P} \). After coupling to topological gravity, they do not have such simple forms any more. However, the topological recursion relation to be introduced shortly guarantees that any topological correlation functions are expressed in terms of these order parameters. That is, the dependence on the descendant couplings is only through the dependence of \( u(t), v(t), w(t) \). In [13], such relations for two point functions:

\[
\langle AB \rangle (t) = R_{AB}(u(t), v(t), w(t))
\]

are treated as the constitutive relations of the theory. Other primary two point functions, which are the first examples of the constitutive relations, are

\[
\langle QQ \rangle = w - \frac{1}{2}v^2, \quad \langle QR \rangle = e^u, \quad \langle RR \rangle = ve^u.
\]  

Topological correlation functions imply the following primary flow equations;

\[
\frac{\partial u}{\partial t_{0,Q}} = v', \quad \frac{\partial v}{\partial t_{0,Q}} = (w - \frac{1}{2}v^2)', \quad \frac{\partial w}{\partial t_{0,Q}} = (e^u)',
\]

\[
\frac{\partial u}{\partial t_{0,R}} = w', \quad \frac{\partial v}{\partial t_{0,R}} = (e^u)', \quad \frac{\partial w}{\partial t_{0,R}} = (ve^u)'.
\]

We will identify the primary field \( P \) as the puncture operator, which implies an identification of \( t_{0,P} \) as the space variable \( x \). The prime denotes the derivative with respect to \( t_{0,P} \equiv x \). As a consequence of the primary flows we recognize the dispersionless limit of the 2D Toda lattice equation;

\[
\frac{\partial^2 u}{\partial t_{0,Q} \partial t_{0,R}} = \frac{\partial^2}{\partial x^2} e^u.
\]  

After coupling to topological gravity, there appear hierarchies of the gravitational descendants in the non-trivial BRST cohomologies. For each primary \( \Phi_\alpha \), its \( n \)-th descendant is denoted by \( \sigma_n(\Phi_\alpha) \), \((n = 0, 1, \cdots)\). Using the topological recursion relation at genus zero,

\[
\langle \sigma_n(\Phi_\alpha) XY \rangle = n\langle \sigma_{n-1}(\Phi_\alpha) \Phi_\beta \rangle \eta^{\beta \gamma} \langle \Phi_\gamma XY \rangle,
\]  

we can compute the topological correlation functions including the descendants. The metric is \( \eta^{PR} = \eta^{QQ} = 1 \). As was shown by Dijkgraaf and Witten [13], the topological recursion
relation at genus zero is deeply connected to the integrable hierarchy in the dispersionless limit. We can also see this crucial relation from the viewpoint of the Gauss-Manin connection [2] [23] [24] [25].

Topological correlation functions for the first descendants are:

\[
\langle \sigma_1(P)P \rangle = uw + \frac{1}{2}v^2, \quad \langle \sigma_1(Q)P \rangle = vw - \frac{1}{6}v^3 + e^u, \quad \langle \sigma_1(R)P \rangle = \frac{1}{2}w^2 + ve^u. \tag{2.10}
\]

\[
\langle \sigma_1(P)Q \rangle = (u-1)e^u + vw - \frac{1}{3}v^3, \quad \langle \sigma_1(Q)Q \rangle = ve^u + \frac{1}{2}w^2 - \frac{1}{6}wv^2 + \frac{1}{8}v^4, \quad \langle \sigma_1(R)Q \rangle = ve^u. \tag{2.11}
\]

\[
\langle \sigma_1(P)R \rangle = uve^u + \frac{1}{2}w^2, \quad \langle \sigma_1(Q)R \rangle = (w + \frac{1}{2}v^2)e^u, \quad \langle \sigma_1(R)R \rangle = wve^u + \frac{1}{2}e^{2u}. \tag{2.12}
\]

To obtain the above result we have integrated once with respect to \( t_0, P \) assuming the absence of the integration constants. It is clear that we can continue this procedure of obtaining the constitutive relations recursively on the degree of the descendants. The results on the second descendants, for example, are:

\[
\langle \sigma_2(P)P \rangle = uw^2 + wv^2 - \frac{1}{6}v^4 + 2v(u-1)e^u, \\
\langle \sigma_2(P)Q \rangle = (2w(u-1) + v^2)e^u + vw^2 + \frac{2}{15}v^5 - \frac{2}{3}v^3w, \\
\langle \sigma_2(P)R \rangle = \frac{1}{3}w^3 + (2uvw + \frac{1}{3}v^3)e^u + (u - \frac{3}{2})e^{2u}. \tag{2.13}
\]

\[
\langle \sigma_2(Q)P \rangle = vw^2 - \frac{1}{3}v^3w + \frac{1}{20}v^5 + (2w + v^2)e^u, \\
\langle \sigma_2(Q)Q \rangle = \frac{1}{3}w^3 - \frac{1}{2}v^2w^2 + \frac{1}{4}v^4w + \frac{1}{24}v^6 + (2vw - \frac{1}{3}v^3)e^u + e^{2u}, \\
\langle \sigma_2(Q)R \rangle = (w^2 + v^2w - \frac{1}{12}v^4)e^u + 2ve^{2u}. \tag{2.14}
\]

\[
\langle \sigma_2(R)P \rangle = \frac{1}{3}w^3 + 2vww + \frac{1}{2}e^{2u}, \\
\langle \sigma_2(R)Q \rangle = w^2e^u + ve^{2u}, \\
\langle \sigma_2(R)R \rangle = w^2ve^u + (w + v^2)e^{2u}. \tag{2.15}
\]
3 Lax Formalism

Topological correlation functions are translated into the flow equations for \( u(t), v(t) \) and \( w(t) \). Let \( t_{n,P}, t_{n,Q}, t_{n,R} \) be the coupling parameters to \( \sigma_n(P), \sigma_n(Q), \sigma_n(R) \), respectively. Then the \( t_{1,P} \) flow equations are

\[
\frac{\partial u}{\partial t_{1,P}} = (uw + \frac{1}{2}v^2)', \quad \frac{\partial v}{\partial t_{1,P}} = ((u-1)e^u + vw - \frac{1}{3}v^3)', \quad \frac{\partial w}{\partial t_{1,P}} = (uve^u + \frac{1}{2}w^2)'). \quad (3.1)
\]

Similarly the \( t_{2,P} \) flows read

\[
\frac{\partial u}{\partial t_{2,P}} = (uw^2 + vw^2 - \frac{1}{6}v^4 + 2v(u-1)e^u)', \quad \frac{\partial v}{\partial t_{2,P}} = \left[(2w(u-1)e^u + v^2)e^u + vw^2 + \frac{2}{15}v^5 - \frac{2}{3}v^3w\right]', \quad \frac{\partial w}{\partial t_{2,P}} = \left[\frac{1}{3}w^3 + (2uvw + \frac{1}{3}v^3)e^u + (u - \frac{3}{2})e^{2u}\right]'. \quad (3.2)
\]

We can write down the \( t_{n,Q} \) and the \( t_{n,R} \) flow equations in the same manner.

Let us establish the Lax formalism for the above flow equations. Our choice of the Lax operator is,

\[
L = \frac{1}{2}p^2 + vp + w + e^u p^{-1}. \quad (3.3)
\]

This Lax operator is suggested by the Landau-Ginzburg potential proposed by Dubrovin. We employ the following Poisson bracket;

\[
\{A(p, x), B(p, x)\} = p\left(\frac{\partial A}{\partial p} \frac{\partial B}{\partial x} - \frac{\partial B}{\partial p} \frac{\partial A}{\partial x}\right). \quad (3.4)
\]

Given a Hamiltonian or a flow generator \( H(p, x) \), the flow equation for \( u(x, t), v(x, t) \) and \( w(x, t) \) is defined by

\[
\frac{\partial L}{\partial t} = \{H, L\}. \quad (3.5)
\]

It is natural to try \( (L^n)_+ \) first as flow generators. \( (\cdot)_+ \) means the non-negative power part. One can easily see that

\[
n \frac{\partial L}{\partial t_{n-1,R}} = \{(L^n)_+, L\}, \quad (n = 1, 2, \ldots) , \quad (3.6)
\]

coincides with the flow equations derived from the topological correlation functions. From the degree counting we expect that the \( t_{0,Q} \) flow is generated by the square root of the
Lax operator;

\[ L^{1/2} = \frac{1}{\sqrt{2}} \left( p + v + (w - \frac{v^2}{2})p^{-1} + (e^u - vw + \frac{1}{2}v^3)p^{-2} + \cdots \right). \tag{3.7} \]

Indeed the odd powers of it correspond to the \( t_{n-1,Q} \) flows. We have checked explicitly;

\[ \frac{1}{\sqrt{2}} \frac{\partial L}{\partial t_{0,Q}} = \left\{ (L^{1/2})_+, L \right\}, \]
\[ \frac{3}{2\sqrt{2}} \frac{\partial L}{\partial t_{1,Q}} = \left\{ (L^{3/2})_+, L \right\}. \tag{3.8} \]

From an examination of lower flow equations we conclude that the correct normalization of the Hamiltonians is

\[ (L^n)_+ \Rightarrow \left( \frac{1}{\sqrt{2}} \right)^{2n} \frac{\partial}{(n-1)! \partial t_{n-1,R}}, \quad (L^{(2n-1)/2})_+ \Rightarrow \left( \frac{1}{\sqrt{2}} \right)^{2n-1} \frac{\partial}{(n-1)! \partial t_{n-1,Q}}. \tag{3.9} \]

Now let us turn to the problem of the descendant flows of the puncture operator. Motivated by the result of the \( CP^1 \) model \([14]\), we will look at the logarithmic Hamiltonians. To define an expansion of \( \log L \) in \( p \), we use the following prescription to avoid an appearence of \( \log p \);

\[ \log L = \frac{1}{3} \log (1 + 2vp^{-1} + 2wp^{-2} + 2e^up^{-3}) \]
\[ + \frac{2}{3} \log \frac{1}{\sqrt{2}}(e^u + wp + vp^2 + \frac{1}{2}p^3). \tag{3.10} \]

For example,

\[ \left( 3\log L \right)_- = 2vp^{-1} + 2(w - v^2)p^{-2} + 2(e^u + \frac{4}{3}v^3 - 2vw)p^{-3} + \cdots. \tag{3.11} \]

Contrary to \( L^{1/2} \), \( \log L \) has an infinite series expansion in both directions of the positive and the negative powers. It is not appropriate to take a product with \( L^{(2n-1)/2} \) or \( \log L \), which is also an infinite series. Hence the next possible generator is

\[ \left( 3\log L \right)_- = (2u - \log 2)e^up^{-1} + (e^u + 2vw - \frac{2}{3}v^3)p^{-1} \]
\[ + \left( w^2 - 2v^2w + \frac{2}{3}v^4 + 2ve^u \right)p^{-2} \]
\[ + 2 \left( (w - v^2)e^u - vw^2 + \frac{4}{3}wv^3 - \frac{2}{5}v^5 \right)p^{-3} + \cdots, \tag{3.12} \]
We have found that the flow equations (3.1) and (3.2) for the descendants of the puncture operator are recovered from

\[
\frac{2}{3} \frac{\partial L}{\partial t_{1,P}} = \left\{ (L(\log L - 1 + \frac{\log 2}{3}))_+, L \right\} = -\left\{ (L(\log L - 1 + \frac{\log 2}{3}))_-, L \right\}, \\
\frac{2}{3} \frac{\partial L}{\partial t_{2,P}} = \left\{ (L^2(\log L - \frac{3}{2} + \frac{\log 2}{3}))_+, L \right\} = -\left\{ (L^2(\log L - \frac{3}{2} + \frac{\log 2}{3}))_-, L \right\}.
\]

(3.13)

Thus we have arrived at the following identification;

\[
\left( L^n(\log L - c_n) \right)_+ \implies \frac{2}{3} \frac{\partial}{\partial t_{n,P}},
\]

(3.14)

where the constants \( c_n \) are determined recursively by the relation \( c_n - c_{n-1} = 1/n \) with the initial condition \( c_0 = -(\log 2)/3 \). As a consequence of this recursion relation, we have a scaling relation;

\[
\frac{\partial}{\partial L} L^n(\log L - c_n) = nL^{n-1}(\log L - c_{n-1}),
\]

(3.15)

which is crucial in deriving the string equation. Finally we observe the relations

\[
\langle \sigma_n(P)P \rangle = \frac{3}{2} \left( L^n(\log L - c_n) \right)_0, \\
\langle \sigma_{n-1}(Q)P \rangle = (\sqrt{2})^{2n-1} \frac{(n-1)!}{(2n-1)!!} \left( L^{(2n-1)/2} \right)_0, \\
\langle \sigma_{n-1}(R)P \rangle = \frac{1}{n} \left( L^n \right)_0,
\]

(3.16)

where \( (\cdot)_0 \) means the degree zero part. These relations support the validity of the Landau-Ginzburg formalism with \( \log p \) as the Landau-Ginzburg variable.

4 Constraints and String Equation

The Lax formalism in section 3 is a special reduction of the Toda lattice hierarchy. We can reproduce the flow equations from the Toda lattice hierarchy by imposing constraints on the Lax operators and the associated Orlov operators. We should remark, however, that the flows corresponding to the descendants of the puncture operator are absent in the standard formulation of the Toda lattice hierarchy. We will show that the constraints imply the string equation. Though the type of the constraints is rather different from the \( c = 1 \) string theory, the method to derive the string equation is quite similar to it.
Let us first recall basic ingredients of the (dispersionless) Toda lattice hierarchy [26]. We consider two Lax operators $L, \overline{L}$ with the following expansion;

$$L = p + \sum_{n=0}^{\infty} u_{n+1}(t, \bar{t}, x)p^{-n},$$

$$\overline{L}^{-1} = \pi_0(t, \bar{t}, x)p^{-1} + \sum_{n=0}^{\infty} \pi_{n+1}(t, \bar{t}, x)p^n. \quad (4.1)$$

The commuting flows are defined by

$$\frac{\partial L}{\partial t_n} = \{H_n, L\}, \quad \frac{\partial L}{\partial \bar{t}_n} = \{\overline{H}_n, L\},$$

$$\frac{\partial \overline{L}}{\partial t_n} = \{H_n, \overline{L}\}, \quad \frac{\partial \overline{L}}{\partial \bar{t}_n} = \{\overline{H}_n, \overline{L}\}, \quad (n = 1, 2, \cdots) \quad (4.2)$$

where the Hamiltonians are

$$H_n = (L^n)_{+}, \quad \overline{H}_n = (\overline{L}^{-n})_{-}. \quad (4.3)$$

$X = (X)_+ + (X)_-$ is a decomposition into the non-negative and the negative power parts. The Poisson bracket $\{\cdot, \cdot\}$ has been introduced in section 3. It is convenient to consider the Orlov operators

$$M = \sum_{n=1}^{\infty} nt_n L^n + x + \sum_{n=1}^{\infty} v_n(t, \bar{t}, x) L^{-n},$$

$$\overline{M} = -\sum_{n=1}^{\infty} n\overline{t}_n \overline{L}^{-n} + x + \sum_{n=1}^{\infty} \overline{v}_n(t, \bar{t}, x) \overline{L}^n. \quad (4.4)$$

These operators satisfy

$$\{L, M\} = L, \quad \{\overline{L}, \overline{M}\} = \overline{L}. \quad (4.5)$$

In fact the above relations together with the following flow equations are the defining relations of the Orlov operators;

$$\frac{\partial M}{\partial t_n} = \{H_n, M\}, \quad \frac{\partial M}{\partial \bar{t}_n} = \{\overline{H}_n, M\},$$

$$\frac{\partial \overline{M}}{\partial t_n} = \{H_n, \overline{M}\}, \quad \frac{\partial \overline{M}}{\partial \bar{t}_n} = \{\overline{H}_n, \overline{M}\}, \quad (n = 1, 2, \cdots). \quad (4.6)$$

Now it is clear that the Lax operator $L$ introduced in section 3 is obtained by imposing the following constraint,

$$L \equiv \frac{1}{2} \overline{L}^2 = \overline{L}^{-1}. \quad (4.7)$$
As a consequence of this constraint we can eliminate the time variables $t_n$ by the identification:

$$t_n = -2^n t_{2n}. \quad (4.8)$$

According to the method of the Riemann-Hilbert problem \cite{27}, we should also impose the constraint for the conjugate operators:

$$-\mathcal{P}_L \equiv \mathcal{M}\mathcal{L}^{-2} = -\overline{\mathcal{M}} \overline{\mathcal{L}}. \quad (4.9)$$

It is the second constraint which leads to the string equation. But before proceeding to see it, we have to consider a modification of $\mathcal{M}$ due to additional flows. As we have seen in section 3, to incorporate the descendants of the puncture operator, it is necessary to introduce the flows generated by

$$K_n = \left( L^n (\log L - c_n) \right)_+. \quad (4.10)$$

The Hamiltonians $K_n$ can be properly defined only after imposing the constraint (4.7), since our prescription to define $\log L$ requires that the series expansion of $L$ is finite in both the positive and the negative directions. Let $s_n$ be the time variables for $K_n$. The Orlov operator is deformed by these new flows, because it should satisfy:

$$\frac{\partial \mathcal{M}}{\partial s_n} = \{ K_n, \mathcal{M} \}. \quad (4.11)$$

The expression of the modified Orlov operator is easily obtained by recalling how the defining relations (4.5) and (4.6) imply the expansion (4.4). The original Orlov operator can be reexpressed as

$$\mathcal{M} = W(x + \sum_n n t_n p^n)W^{-1}, \quad (4.12)$$

if one introduces a similarity transformation defined by

$$\mathcal{L} = WpW^{-1}. \quad (4.13)$$

(We have used a poor notation for the similarity transformation. It should be understood as the canonical transformation generated by $\phi$ with $W = e^{\phi}$. For a mathematically more precise definition, see \cite{27}.) Using the flow equation for $W$:

$$\frac{\partial W}{\partial t_n} = H_n W - Wp^n, \quad (4.14)$$

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one can check the defining relations are indeed satisfied. (We neglect \( \tilde{t}_n \) flows, since they have been eliminated by the constraint.) Defining the coefficient functions \( \{v_n\} \) by

\[
W x W^{-1} = x + \sum_{n=1}^{\infty} v_n \mathcal{L}^{-n},
\]

we get the expansion (4.4). Now the \( s_n \) flows for \( W \) are given by

\[
\frac{\partial W}{\partial s_n} = K_n W - W \left[ \left( \frac{p^2}{2} \right)^n (\log \frac{p^2}{2} - c_n) \right].
\]

We see that

\[
\mathcal{M}' = W \left( x + \sum_n n t_n p^n + \sum_n n s_n \frac{p^{2n}}{2^{n-1}} (\log \frac{p^2}{2} - c_{n-1}) \right) W^{-1},
\]

satisfies the additional flow equations. Here we have used the scaling property (3.15). Hence we obtain the following expansion of the modified Orlov operator

\[
\mathcal{M}' = x + \sum_n n t_n \mathcal{L}^n + \sum_n 2 n s_n L^n (\log L - c_{n-1}) + \sum_n v_n \mathcal{L}^{-n}.
\]

Let us return to a derivation of the string equation. The second constraint allows us to express the conjugate operator in a linear combination of the Hamiltonians. To see it, it is convenient to consider the non-negative power part and the negative power part separately. Since \( \mathcal{L}^{-n} (n \geq 1) \) does not have any positive power part and \( \mathcal{L}^{n} (n \geq 1) \) does not have any negative power part, we have

\[
(\mathcal{P}_L)_+ = (-\mathcal{M}' \mathcal{L}^{-2})_+ = - \sum_{n \geq 3} n t_n H_{n-2} - \sum_{n \geq 1} n s_n K_{n-1},
\]

\[
(\mathcal{P}_L)_- = (\mathcal{M} \mathcal{L})_- = - \sum_{n \geq 2} n \tilde{t}_n \bar{H}_{n-1}.
\]

Hence \( \mathcal{P}_L \) is expressed as

\[
- \mathcal{P}_L = 2 \sum_{n \geq 2} 2 n t_{2n} H_{2n-2} + \sum_{n \geq 1} (2n+1) t_{2n+1} H_{2n-1} + \sum_{n \geq 1} n s_n K_{n-1}.
\]

We have used the first constraint to eliminate \( \tilde{t}_n \) and \( \bar{H}_{n-1} \). Combined with the canonical commutation relation \( \{\mathcal{P}_L, L\} = 1 \), (4.20) implies,

\[
2 \sum_{n \geq 2} 2 n t_{2n} \frac{\partial L}{\partial t_{2n-2}} + \sum_{n \geq 1} (2n+1) t_{2n+1} \frac{\partial L}{\partial t_{2n-1}} + \sum_{n \geq 1} n s_n \frac{\partial L}{\partial s_{n-1}} = -1.
\]
After an appropriate identification of the flow parameters as the coupling parameters and the shift $s_1 \rightarrow s_1 - 1$ of the dilaton coupling, we obtain the string equation, or the puncture equation:

$$1 + \sum_{n \geq 1} nt_{n,R} \frac{\partial L}{\partial t_{n-1,R}} + \sum_{n \geq 1} nt_{n,Q} \frac{\partial L}{\partial t_{n-1,Q}} + \sum_{n \geq 1} nt_{n,P} \frac{\partial L}{\partial t_{n-1,P}} = \frac{\partial L}{\partial t_0,P}. \quad (4.22)$$

It is easy to generalize the above argument to the model with the constraint

$$L \equiv \frac{L^N}{N} = \mathcal{T}^{-1}, \quad (N = 3, 4, \ldots). \quad (4.23)$$

The Lax operator takes the form

$$L = \frac{1}{N} p^N + v_{N-1} p^{N-1} + \cdots + v_0 + e^u p^{-1}. \quad (4.24)$$

Since the Lax operator has a finite expansion in both directions, we can define an expansion of $\log L$ in a similar way to the case $N = 2$. The string equation will follow from the associated constraint;

$$\mathcal{P}_L = \mathcal{M} \mathcal{L}^{-N} = - \mathcal{M} \mathcal{T}. \quad (4.25)$$

### 5 Higher Genus Expansion

To obtain the genus expansion of the flow equations, we promote the dispersionless Lax operator into the $N \times N$ matrix Lax operator;

$$Q = \sum_n E_{n,n+1} + \sum_n \sum_{\ell \geq 0} a_n(\ell) E_{n,n-\ell}, \quad (5.1)$$

$$Q^2 = \sum_n E_{n,n+2} + \sum_n \sqrt{2} \tilde{v}_n E_{n,n+1} + \sum_n w_n E_{n,n} + \sum_n \frac{1}{\sqrt{2}} R_n E_{n,n-1}, \quad (5.2)$$

where $E_{i,j}$ stands for the matrix unit. The degree of $E_{i,j}$ is defined to be $(j - i)$. That is, the positive degree part is the upper triangular part of the matrix. The reason we add the tilde on $v_n$ will become clear shortly. Though we will not introduce an explicit matrix model realization, we believe the matrix Lax operator $Q$ would arise from an appropriate multi-matrix model developed, for example, in [28].
The commuting flows are defined by

\[
\frac{n}{N} \frac{\partial Q}{\partial t_{n-1,R}} = \left[ (Q^{2n})_+, Q \right], \tag{5.3}
\]

\[
\frac{1}{N} \left( \frac{1}{\sqrt{2}} \right)^{2n-1} \frac{(2n-1)!!}{(n-1)!} \frac{\partial Q}{\partial t_{n-1,Q}} = \left[ (Q^{2n-1})_+, Q \right], \tag{5.4}
\]

\[
\frac{2}{3N} \frac{\partial Q}{\partial t_{n,P}} = \left[ (Q^{2n}(\log Q^2 - c_n))_+, Q \right], \tag{5.5}
\]

where \([\cdot, \cdot]\) is the commutator of the matrices. The form of the flow equations is conformable to the genus expansion only when we make a good choice of (independent) dynamical variables. Since it is the free energy of the theory that should be expanded in genus by genus, two point functions are “good” variables. From the residue formula (3.16) in section 3 the diagonal matrix elements are identified with two point functions in full genus expansion. Thus we take \(w_n\) as one of the variables, but not \(\tilde{v}_n\). Instead we can choose \(a_n(0)\) as the second variable. Therefore, in the following we will use a notation \(v_n \equiv a_n(0)\). \(\tilde{v}_n\) coincides with \(v_n\) in the dispersionless limit. As the last variable, motivated by the matrix model method, we take \(\phi_n\) with \(R_n = \exp N(\phi_n - \phi_{n-1})\). The functions \(\log \phi_n\) correspond to the weight functions for the orthogonal polynomials in the matrix model.

The genus expansion can be obtained in the following way. We first write down the flow equations for the matrix elements. Then we make a Taylor expansion of \(w_{n+k}, v_{n+\ell}\) and \(R_{n+m}\) around \(w_n, v_n\) and \(R_n\), respectively, with an identification \(x \equiv n/N\). The continuum limit is defined to be the limit where the matrix size \(N\) goes to the infinity. We thus obtain the flow equations in terms of differential polynomials in \(w_n, v_n\) and \(\phi_n\). (As before the prime denotes the derivative with respect to \(x\).) The order of \(N^{-1}\) practically counts the number of the derivatives. Regarding \(N^{-1}\) as an expansion parameter or the cosmological constant, we can see the higher genus structure of the flow equations.

Let us illustrate the above procedure for the \(t_{n,R}\)-flows. The primary flow equations in terms of the matrix elements are

\[
\frac{\partial \phi_n}{\partial t_{0,R}} = w_n, \tag{5.6}
\]

\[
\frac{1}{N} \frac{\partial v_n}{\partial t_{0,R}} = R_{n+1} - R_n, \tag{5.7}
\]
\[ \frac{1}{N} \frac{\partial w_n}{\partial t_{0,R}} = \frac{1}{2} [(v_n + v_{n+1})R_{n+1} - (v_{n-1} + v_n)R_n] \] (5.8)

Similarly the \( t_{1,R} \)-flow equations are given by

\[ \frac{\partial \phi_n}{\partial t_{1,R}} = \frac{1}{2} w_n^2 + \frac{1}{4} [(v_n + v_{n+1})R_{n+1} + (v_{n-1} + v_n)R_n] \] (5.9)

\[ \frac{1}{N} \frac{\partial v_n}{\partial t_{1,R}} = \frac{1}{2} (w_n + w_{n+1})R_{n+1} - \frac{1}{2} (w_{n-1} + w_n)R_n \] (5.10)

\[ \frac{1}{N} \frac{\partial w_n}{\partial t_{1,R}} = \frac{1}{4} (w_n + w_{n+1})(v_n + v_{n+1})R_{n+1} - \frac{1}{4} (w_{n-1} + w_n)(v_{n-1} + v_n)R_n 
+ \frac{1}{4} (R_{n+1}R_{n+2} - R_{n-1}R_n). \] (5.11)

Making a Taylor expansion of \( w_n \pm 1, v_n \pm 1, R_n + 1 \) and \( R_n + 2 \), we get the genus expanded flow equations. Up to genus two the primary flows read

\[ \frac{\partial u_n}{\partial t_{0,R}} = w_n', \] (5.12)

\[ \frac{\partial v_n}{\partial t_{0,R}} = \left[ e^{u_n} + \frac{1}{24N^2}(u_n'' + 2u_n'')e^{u_n} + \frac{1}{N^4}\left( \frac{u_n'''}{360} + \frac{u_n' u_n''}{180} + \frac{7u_n' u_n''}{180} + \frac{u_n''}{1920} + \frac{u_n'}{240} \right)e^{u_n} \right]', \] (5.13)

\[ \frac{\partial w_n}{\partial t_{0,R}} = \left[ v e^{u_n} + \frac{1}{24N^2}(4v'' + 2u_n' v_n' + u_n'' v_n + 2u_n' v_n) e^{u_n} + \frac{1}{N^4}\left( \frac{v_n'''}{120} + \frac{v_n''' u_n''}{360} + \frac{u_n'' u_n'''}{80} + \frac{v_n' u_n'''}{80} + \frac{v_n u_n'''}{180} + \frac{v_n' u_n'''}{180} + \frac{11u_n''' v_n'}{720} + \frac{u_n^2 v_n''}{120} + \frac{v_n^2 u_n''}{240} + \frac{7v_n v_n u_n''}{1440} + \frac{u_n'' v_n'}{480} + \frac{v_n u_n' v_n'}{1920} + \frac{7v_n u_n' u_n''}{720} \right)e^{u_n} \right]' \] (5.14)

where we have introduced \( u_n = \phi_n' \). The \( t_{1,R} \)-flows are quite lengthy and we collect them in Appendix B. We think it is quite non-trivial that the odd-power terms in \( N^{-1} \) disappear from the flow equations in accord with our identification of \( N^{-1} \) as the genus expansion parameter. This fact supports our prescription of genus expansion and choice of dynamical variables.

Computation of \( t_{n,Q} \)-flows is more complicated, since the flow equations involve the matrix elements \( a_n(\ell) \). Identifying the square of (5.1) with (5.2), we have to eliminate these variables. We present only the final results up to genus two;

\[ \frac{\partial u_n}{\partial t_{0,Q}} = v_n' \] (5.15)
\[
\frac{\partial v_n}{\partial t_{0,Q}} = \left[ w_n - \frac{v_n^2}{2} - \frac{1}{12N^2}(w''_n - v_nv''_n - v^2_n) \right. \\
\left. + \frac{1}{120N^4}(w'''_n - v_nv'''_n + 4v'_n v'''_n - 7v''^2_n) \right]' \tag{5.16}
\]
\[
\frac{\partial w_n}{\partial t_{0,Q}} = \left[ e^{u_n} + \frac{1}{24N^2}(u'^2_n + 2u''_n)e^{u_n} \right. \\
\left. + \frac{1}{N^4}(u''_n + \frac{7v''_n^2}{1440} + \frac{u''_n^4}{1920} + \frac{u''_n^2}{240})e^{u_n} \right]' \tag{5.17}
\]

Note that \( \frac{\partial w_n}{\partial t_{0,Q}} = \frac{\partial v_n}{\partial t_{0,R}} \) as it should be. The \( t_{1,Q} \)-flows are again collected in Appendix B.

Finally let us look at the \( t_{1,P} \)-flow as an example of flow equations involving \( \log Q^2 \). In terms of the matrix element of \( \log Q^2 \), the flow equations are given by:

\[
\frac{2}{3} \frac{\partial \phi_n}{\partial t_{1,P}} = \frac{R_n}{\sqrt{2}} \left( b_{n-1} + w_n a_n + \sqrt{2} \tilde{v}_n d_{n+1} + e_{n+2} \right) \tag{5.18}
\]
\[
\frac{2}{3N} \frac{\partial \tilde{v}_n}{\partial t_{1,P}} = \tilde{v}_n \left( \frac{R_n b_{n-1} - R_{n+1} b_n}{\sqrt{2}} + w_n a_n - w_{n+1} a_{n+1} \right. \\
\left. + \sqrt{2} (\tilde{v}_n d_{n+1} - \tilde{v}_{n+1} d_{n+2}) + e_{n+2} - e_{n+3} \right) \\
+ (w_{n+1} - w_n) \left( \frac{R_n}{2} c_{n-1} + \frac{w_n b_n + \tilde{v}_n a_{n+1} + d_{n+2}}{\sqrt{2}} \right) \\
+ \frac{R_{n+2}}{2} \left( \frac{R_n}{\sqrt{2}} f_{n-1} + w_n c_n + \sqrt{2} \tilde{v}_n b_{n+1} + a_{n+2} \right) \\
- \frac{R_n}{2} \left( \frac{R_n}{\sqrt{2}} f_{n-2} + w_{n-1} c_{n-1} + \sqrt{2} \tilde{v}_{n-1} b_n + a_{n+1} \right) \tag{5.19}
\]
\[
\frac{2}{3N} \frac{\partial w_n}{\partial t_{1,P}} = R_{n+1} \left( \frac{R_n}{2} c_{n-1} + \frac{w_n b_n + \tilde{v}_n a_{n+1} + d_{n+2}}{\sqrt{2}} \right) \\
- R_n \left( \frac{R_n}{2} c_{n-2} + \frac{w_{n-1} b_{n-1} + \tilde{v}_{n-1} a_n + d_{n+1}}{\sqrt{2}} \right) \tag{5.20}
\]

where we have used the following notations for the matrix elements of \( \log Q^2 \):

\[
a_n := (\log Q^2)_{n,n}, \quad b_n := (\log Q^2)_{n,n+1}, \quad c_n := (\log Q^2)_{n,n+2} \tag{5.21}
\]
\[
f_n := (\log Q^2)_{n,n+3}, \quad d_n := (\log Q^2)_{n,n-1}, \quad e_n := (\log Q^2)_{n,n-2}.
\]

and omitted the counter terms proportional to \((3-\log 2)/3\). We have used \( \tilde{v}_n \) for simplicity of the expressions. The computation of the matrix elements of \( \log Q^2 \) is rather technical.
and given in Appendix C. Substituting it and making a Taylor expansion, we get the $t_{1,P}$-flow equations as follows;

$$\frac{\partial u_n}{\partial t_{1,P}} = \left[ u_n w_n + \frac{v_n^2}{2} + \frac{1}{24N^2}(6w_n'' - 3v_n'^2 - 2v_n'v_n'') \right]' , \quad (5.22)$$

$$\frac{\partial v_n}{\partial t_{1,P}} = \left[ (u_n - 1)e^{u_n} + v_n w_n - \frac{v_n^3}{3} + \frac{1}{24N^2}(5u_n'^2 + 6u_n'' + u_n'u_n'^2 + 2u_n'u_n'' + 2u_n'v_n'^2) e^{u_n} + 6v_n'v_n'' + 4v_n'' + 2v_n'u_n' + 6v_n'' + 4v_n'' + 2v_n'u_n' \right]' , \quad (5.23)$$

$$\frac{\partial w_n}{\partial t_{1,P}} = \left[ \frac{w_n^2}{2} + u_n v_n e^{u_n} + \frac{e^{u_n}}{24N^2}(4v_n'u_n'^2 + v_n'u_n'^2 + 6u_n'u_n'^2 + 6v_n'u_n'^2 + 2u_n'u_n'v_n') + 6v_n'' + 2v_n'u_n'u_n'' + 6v_n'' + 4v_n'' \right]' , \quad (5.24)$$

where we have calculated up to genus one. As we show in Appendix A, the formula of one loop free energy by Dijkgraaf and Witten implies the flow equations at one loop level. In this section we have computed several lower flow equations up to genus two (genus one for $t_{1,P}$) by making expansion of flow equations in matrix form. One loop order of our result enjoys a complete agreement with what is derived from the formula of one loop free energy.

The existence and consistency of higher genus expansion seems to impose very severe restrictions on possible models of topological string theory [29]. For example, the $A_2$ minimal model and the $CP^1$ model exhaust the two primary model with a consistent genus expansion. There are other topological matter theories, for instance, based on the Lie algebra $B_2 = C_2$. However, this model does not have a consistent higher genus expansion beyond genus one. It is believed that the generalized KdV hierarchy controls the genus expansion of the minimal topological string theory of ADE type in the sense that the genus expansion of the free energy comes from a single tau function of the integrable hierarchy. In this paper we have found another series of topological strings with a well-defined genus expansion, starting from the topological $CP^1$ model. The Toda lattice hierarchy plays the same role as the KP hierarchy for the minimal models. In this respect, to obtain more complete understanding of higher genus behavior, it is highly desirable to find a matrix model realization of the model, which will provide a closed expression of the genus expansion of the free energy.
We would like to thank T. Eguchi, K. Takasaki, Y. Yamada and S.-K. Yang for discussions. The work of H.K. is supported in part by the Grant-in-Aid for Scientific Research from the Ministry of Education, Science and Culture (No.06221255 and No.06740069).

Appendix A : One Loop Free Energy

There is a formula of Dijkgraaf-Witten for the one-loop free energy of topological string [12] [13]. We can derive the flow equations at one loop level from the one-loop free energy together with the tree level flow equations [4]. To see it in general model, let $O_\alpha, O_\beta, \cdots$ be the primary fields and $t_{n,\alpha}$ the flow parameter for $\sigma_n(O_\alpha)$. The flow equation for $u_\alpha = \langle PO_\alpha \rangle$ in full genus is assumed to be

$$\frac{\partial u_\alpha}{\partial t_{n,\beta}} = \left[ R_{\alpha,\beta,n}^{(0)} + \lambda^2 R_{\alpha,\beta,n}^{(1)} + \cdots \right]',$$  \hspace{1cm} (A.1)

where $R_{\alpha,\beta,n}^{(g)} = R_{\alpha,\beta,n}^{(g)}[u_\gamma]$ is a potential at genus $g$. It is important to remember that the order parameter $u_\gamma$ has to be expanded as

$$u_\gamma = u_\gamma^{(0)} + \lambda^2 u_\gamma^{(1)} + \cdots.$$  \hspace{1cm} (A.2)

What we need is the potential $R_{\alpha,\beta,n}^{(1)}[u_\gamma^{(0)}]$. The sub-leading part of the full genus flow equation gives

$$\frac{\partial u_{\alpha}^{(1)}}{\partial t_{\beta,n}} = \left[ \sum_\gamma \frac{\delta R_{\alpha,\beta,n}^{(0)}}{\delta u_\gamma} u_\gamma^{(1)} + R_{\alpha,\beta,n}^{(1)}[u_\gamma^{(0)}] \right]' .$$  \hspace{1cm} (A.3)

Now, in terms of the one-loop free energy;

$$F_1 = \frac{1}{24} \log \det \langle PO_\alpha O_\beta \rangle_0 ,$$  \hspace{1cm} (A.4)

we have

$$u_\gamma^{(1)} = \frac{\partial^2 F_1}{\partial t_{0,p} \partial t_{0,\gamma}} .$$  \hspace{1cm} (A.5)

Hence we obtain the potential

$$R_{\alpha,\beta,n}^{(1)}[u_\gamma^{(0)}] = \frac{\partial^2 F_1}{\partial t_{0,\alpha} \partial t_{n,\beta}} - \sum_\gamma \frac{\delta R_{\alpha,\beta,n}^{(0)}}{\delta u_\gamma} \frac{\partial^2 F_1}{\partial t_{0,p} \partial t_{0,\gamma}} .$$  \hspace{1cm} (A.6)

\footnote{We owe the material in this section to Y. Yamada.}
The existence of the second term is crucial for obtaining the potential in a form of differential polynomial in \( u_γ^{(0)} \). The first term gives a rational expression in general. We have computed one loop potentials for our model with three primaries using the above formula and found a complete agreement with the flow equations in section 5 (See also Appendix B).

**Appendix B : Flow Equations up to Genus Two**

\[
\begin{align*}
\frac{\partial u_n}{\partial t_{1,R}} &= \left[ v_n e^{u_n} + \frac{w_n^2}{2} + \frac{e^{u_n}}{24N^2} \left( 6v_n'' + 6u_n'v_n' + 4u_n''v_n + 3u_n'^2v_n \right) \\
&\quad + \frac{e^{u_n}}{5760N^4} \left( 15v_n u_n'^4 + 120v_n u_n'^2 u_n'' + 120v_n u_n' u_n'' + 120u_n'' v_n' + 120v_n''' \\
&\quad + 80v_n u_n'^2 + 48v_n u_n'' + 60u_n' v_n' + 240u_n' u_n'' + 240u_n'' v_n' + 240u_n'' v_n'' \right) \\
&\quad + 180u_n'^2 v_n' \right] ' \\
\frac{\partial v_n}{\partial t_{1,R}} &= \left[ w_n e^{u_n} + \frac{e^{u_n}}{24N^2} \left( 4w_n'' + 2u_n' w_n' + u_n'^2 w_n + 2u_n'' w_n \right) \\
&\quad + \frac{e^{u_n}}{5760N^4} \left( 48w_n''' + 16 w_n u_n'' + 32 w_n' u_n'' + 32w_n u_n'' + 8 u_n'' w_n' \right) \\
&\quad + 88u_n'' w_n' + 48u_n'' w_n' + 24 w_n u_n'' + 28 w_n u_n'' + 12 w_n' u_n'' + 3 w_n' u_n'' \right] ' \\
\frac{\partial w_n}{\partial t_{1,R}} &= \left[ v_n v_n e^{u_n} + \frac{1}{2} e^{2u_n} \\
&\quad + \frac{1}{24N^2} \left( 2v_n w_n u_n'' + 4v_n w_n'' + 4w_n v_n'' + w_n v_n' u_n + 2v_n' w_n' \right) \\
&\quad + 2v_n w_n' u_n' + 2w_n' v_n' u_n' \right] e^{u_n} + 8\left( u_n'' + u_n'^2 \right) e^{2u_n} \\
&\quad + \frac{1}{5760N^4} \left( 384u_n' u_n'' + 768u_n'' + 256u_n''' + 1024u_n' u_n'' + 1792u_n'^2 u_n'' \right) e^{2u_n} \\
&\quad + 96w_n' u_n' v_n'' + 96u_n' v_n' w_n'' + 72w_n' u_n' v_n'' + 72w_n' v_n'' + 72v_n' w_n'' \right] \\
&\quad + 48w_n u_n'' + 48v_n w_n'' + 48v_n u_n'' + 48v_n u_n'' + 32v_n w_n u_n'' \\
&\quad + 24v_n w_n u_n'' + 36v_n' w_n' u_n'' + 16v_n w_n u_n'' + 12v_n' w_n' u_n'' + 12v_n' w_n' u_n'' \right] \\
&\quad + 56w_n u_n' v_n' u_n'' + 56v_n' w_n' u_n'' + 56v_n' w_n' u_n'' + 88w_n u_n' v_n'' + 88v_n' w_n' u_n'' \\
&\quad + 28v_n w_n' u_n'' + 3v_n w_n' u_n'' \right] e^{u_n} \right] ' \\
\end{align*}
\]
\[
\frac{\partial u_n}{\partial t_{1,Q}} = \left[ e^{u_n} + v_n w_n - \frac{v_n^3}{6} + \frac{1}{24N^2}((5u_n^2 + 6u_n^\prime)v_n^2 + 4v_n v_n^2 - 2v_n^\prime w_n^\prime + 2v_n^2 v_n^\prime - 2v_n w_n^\prime) \\
+ \frac{1}{5760N^4}\left(5u_n^4 + 20u_n^2 u_n^\prime + 40u_n^\prime u_n^\prime + 80u_n u_n^\prime + 48u_n^\prime\right)^e^{u_n} \\
- 360v_n^\prime v_n^2 - 360v_n v_n^2 + 120v_n w_n^\prime - 600v_n v_n^\prime v_n^\prime + 120v_n w_n^\prime \\
- 120v_n^2 v_n^\prime\right]^\prime
\] (B.4)

\[
\frac{\partial v_n}{\partial t_{1,Q}} = \left[ v_n e^{u_n} + \frac{w_n^2}{2} - \frac{v_n^2 w_n^2}{2} + \frac{v_n^4}{8} + \frac{1}{24N^2}((6v_n^\prime + 4u_n^\prime v_n^\prime - v_n u_n^2)^e^{u_n} \\
- 6v_n^\prime v_n^2 + 2w_n v_n^2 - 2w_n v_n^2 - 2w_n w_n^\prime + 4v_n v_n^\prime w_n^\prime + 3v_n^2 w_n^\prime - 3v_n^2 v_n^\prime - w_n^2) \\
+ \frac{1}{5760N^4}\left(63v_n^4 + 72v_n^\prime u_n^2 + 368v_n u_n^2 u_n^2 + 112v_n^\prime u_n^2 + 256v_n^\prime u_n^2 \\
+ 184v_n u_n^2 + 88u_n^\prime v_n^\prime + 232v_n u_n^\prime u_n^\prime + 56v_n u_n^\prime + 28u_n^\prime v_n^\prime + 32u_n^\prime v_n^\prime + 48v_n^\prime\right)^e^{u_n} \\
+ 264v_n^4 + 2184v_n v_n^2 v_n^\prime - 456v_n^\prime v_n^\prime w_n^\prime + 1128v_n^\prime v_n^\prime v_n^\prime + 816v_n^2 v_n^\prime \\
- 48w_n v_n^\prime v_n^\prime + 144w_n w_n^\prime - 144w_n v_n^\prime - 144v_n^2 w_n^\prime + 144v_n^2 v_n^\prime - 368v_n^3 w_n^\prime \\
+ 112w_n^\prime v_n^\prime - 488v_n^\prime w_n^\prime + 104w_n^\prime - 472v_n^\prime w_n^\prime - 192w_n v_n^\prime \\
- 192v_n^\prime w_n^\prime\right]^\prime
\] (B.5)

\[
\frac{\partial w_n}{\partial t_{1,Q}} = \left[ (w_n + \frac{v_n^2}{2}) e^{u_n} \\
+ \frac{e^{u_n}}{24N^2}(2v_n v_n^\prime - 2v_n^\prime + 2w_n w_n^\prime + 6w_n^\prime + 2v_n v_n^\prime u_n^\prime + \frac{v_n^2 u_n^\prime}{2} + w_n u_n^2 \\
+ v_n^2 u_n^\prime + 2w_n u_n^\prime) + \frac{e^{u_n}}{5760N^4}(64v_n^2 - 64u_n^\prime v_n^\prime + 48v_n^\prime + 48w_n^\prime \\
+ 48v_n u_n^\prime + 32v_n u_n^\prime + 32v_n u_n^\prime u_n^\prime - 32v_n^2 u_n^\prime + 32v_n u_n^\prime u_n^\prime + 32w_n u_n^\prime \\
+ 32w_n u_n^\prime + 112v_n u_n^\prime + 16v_n^2 u_n^\prime + 16w_n u_n^\prime + 12v_n u_n^\prime + 12v_n^3 u_n^\prime - 12v_n^2 u_n^\prime \\
+ 12w_n^3 u_n^\prime + 12v_n^2 u_n^\prime + 28v_n^2 v_n^\prime u_n^\prime + 68w_n u_n^\prime + 28w_n u_n^\prime u_n^\prime + \frac{3v_n^2 u_n^\prime}{2} \\
+ 3w_n u_n^4 + 12w_n u_n^4 + 56w_n u_n^4 + 56v_n u_n^4 u_n^\prime + 8v_n^4 u_n^\prime + 14v_n^2 u_n^4 u_n^\prime + 24w_n u_n^4 \right]^\prime
\] (B.6)
Appendix C : Matrix Elements of log $Q^2$

To evaluate the matrix elements of $\log Q^2$, we use the following formula $^3$. The $(n, m)$ element is given by

$$(\log Q^2)_{n,m} = \frac{1}{2\pi i} \oint \frac{dz}{z} z^{n-m} \log(z^2 + \sqrt{2}z\bar{v}(n + z\partial) + w(n + z\partial) + \frac{1}{\sqrt{2}} z^{-1} R(n + z\partial)) \cdot 1 \,,$$

$$\equiv \frac{1}{2\pi i} \oint \frac{dz}{z} z^{n-m} \log Y(z).$$

(C.1)

Taylor expansion gives

$$Y(z) = Y_0(z) + \sum_{m=1}^{\infty} \frac{1}{m!} (\sqrt{2}z\bar{v}^{(m)} + w^{(m)} + \frac{1}{\sqrt{2}} z^{-1} R^{(m)}) \cdot (z\partial)^m \cdot 1,$$

(C.2)

where

$$Y_0(z) = z^2 + \sqrt{2} z \bar{v}_n + w_n + \frac{1}{\sqrt{2}} z^{-1} R_n,$$

(C.3)

and $f^{(m)}$ denotes the $m$-th derivative of $f$. Then $\log Y(z)$ may be expanded as

$$\log Y(z) = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} [Y_0 - 1]^{(m)} + \sum_{n=1}^{\infty} \frac{1}{n!} Y_1^{(n)} (z\partial)^n \cdot 1,$$

(C.4)

where

$$Y_1^{(m)} = \sqrt{2} z \bar{v}^{(m)} + w^{(m)} + \frac{1}{\sqrt{2}} z^{-1} R^{(m)}.$$

(C.5)

Let us organize the expansion by the number of the derivatives;

$$\log Y(z) = \log Y_0(z) + A_1 + A_2 + A_3 + \cdots,$$

(C.6)

where $A_n$ denotes the terms including $n$ derivatives. We need the contributions up to $A_3$ to obtain the one loop flow equations. The first order term, for example, is computed as follows. $[(Y_0 - 1) + Y_1^{(1)} z\partial]^{m} \cdot 1$ gives terms like

$$(Y_0 - 1)^{m-n} Y_1^{(1)} z\partial (Y_0 - 1)^{n-1} = (n-1)(Y_0 - 1)^{m-n} (Y_0 - 1)^{n-2} Y_1^{(1)} Y_0,$$

(C.7)

where the dot means $z(d/dz)$. Using the formulae

$$\sum_{m=1}^{n} m(m+1) \cdots (m+k) = \frac{1}{k+2} \frac{(n+k+1)!}{(n-1)!},$$

(C.8)

$$\sum_{n=1}^{\infty} (-1)^{n-1} (Y_0 - 1)^{n-1} = \frac{1}{Y_0},$$

(C.9)

$^3$We are grateful to S.-K. Yang for telling us a computation in the $CP^1$ model.
we obtain

\[
A_1 = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \sum_{n=1}^{m-1} (Y_0 - 1)^{m-n} \partial(Y_0 - 1)^{n-1}
\]

\[
= \frac{1}{2} \sum_{m=1}^{\infty} (-1)^{m-1} \frac{d}{dY_0} (Y_0 - 1)^m Y_1^{(1)} Y_0
\]

\[
= -\frac{1}{2Y_0} Y_1^{(1)} Y_0.
\]  

(C.10)

Similar computation gives

\[
A_2 = -\frac{1}{4Y_0^2} Y_1^{(2)} Y_0 + \frac{1}{3Y_0^3} (Y_1^{(2)} Y_0^2 + Y_1^{(1)} Y_0 Y_0 + (Y_1^{(1)})^2 Y_0) - \frac{3}{4Y_0^4} (Y_1^{(1)})^2 Y_0^2
\]  

(C.11)

\[
A_3 = -\frac{1}{12} Y_1^{(3)} Y_0 + \frac{1}{3} Y_1^{(3)} Y_1^{(2)} Y_0 + \frac{1}{6} Y_1^{(2)} \left( \frac{d}{dz} \right)^2 (Y_1^{(1)} Y_0) + \frac{1}{6} Y_1^{(1)} \left( \frac{d}{dz} \right) (Y_1^{(2)} Y_0)
\]

\[
-\frac{1}{4} Y_1 Y_0^3 - \frac{3}{8} Y_1^{(2)} Y_0 Y_0 - \frac{1}{4} Y_1^{(1)} \left( \frac{d}{dz} \right) (Y_1^{(2)} Y_0)
\]

\[
-\frac{3}{4} Y_1^{(2)} Y_0 \left( \frac{d}{dz} \right) (Y_1^{(1)} Y_0) - \frac{1}{4} Y_1^{(1)} \left( \frac{d}{dz} \right) (Y_1^{(1)} Y_0)
\]

\[
+ 2 \frac{Y_1 Y_1^{(2)}}{Y_0^5} Y_0^3 + 2 \frac{(Y_1^{(1)})^2 Y_0}{Y_0^5} \left( \frac{d}{dz} \right) (Y_1^{(1)} Y_0) - \frac{5}{2} \frac{Y_1^{(1)} Y_0}{Y_0^5}.
\]  

(C.12)

Now, we can evaluate the matrix elements \((\log Q^2)_{n,m}\). The residue integral picks up the coefficient of \(z^{m-n}\) in a Laurent expansion of \(\log Y(z)\). We have to be careful about the point where the expansion is made. In accord with our prescription of defining \(\log L\) for the dispersionless Lax operator, we add two contributions, one from the origin and the other from the infinity in an appropriate weight;

\[
(\log Q^2)_{n,m} = \frac{2}{3} c^{(0)}_{n,m} + \frac{1}{3} c^{(\infty)}_{n,m}.
\]  

(C.13)

With this prescription we obtain

\[
(\log Q^2)_{n,n} = \frac{2}{3} \log \frac{R_n}{\sqrt{2}} + \frac{R'_n}{3R_n N} + \frac{R''_n}{18R_n N^2} - \frac{R''_n}{18R_n^2 N^2} - \frac{R'_n R''_n}{4R_n^2 N^3}
\]  

(C.14)

\[
\frac{1}{\sqrt{2}} (\log Q^2)_{n,n+1} = \frac{2w_n R_n - 2w_n R'_n}{3R_n} + \frac{w_n R'_n}{3R_n^3 N} + \frac{1}{18R_n^3 N^2} (w''_n R^2_n - 6w_n R_n R''_n + 12w_n R''_n^2 - 6w'_n R'_n R_n)
\]
\begin{align*}
\frac{1}{2}(\log Q^2)_{n,n+2} &= \frac{2\tilde{v}_n - w_n}{3R_n} - \frac{w_n^2}{3R_n^2} + \frac{1}{3R_n N^3} \left[ 2w_n w'_n R_n - 3R_n^2 w'_n - 3w_n R'_n \right] + 12R_n R'_n w'_n - 15R_n^2 w'_n - 11R_n^2 R'_n w'_n
+ \frac{1}{18R_n^4 N^2} \left( 15w_n^2 R_n R''_n + 30\tilde{v}_n R_n R''_n + 36w_n w'_n R'_n R_n \right)
- 6w_n w''_n R_n^2 - 17\tilde{v}_n R_n^2 R''_n - 10\tilde{v}_n R'_n R_n^2 - 5w''_n R_n^2 + \tilde{v}'_n R_n^3 - 42w_n R_n^2 R''_n
+ \frac{1}{36R_n^5 N^3} \left[ 180w_n^2 R_n^3 + 36\tilde{v}_n R_n^2 R_n^2 + 30R_n^2 R'_n w''_n + 156\tilde{v}_n R_n R''_n R_n^2 \right]
+ 96w_n w''_n R'_n R_n R_n + 72w_n w''_n R'_n R_n + 18w_n R_n R''_n - 29\tilde{v}_n R_n R'_n R_n
- 13\tilde{v}_n R_n^2 R'_n - 17w'_n w''_n R_n - 24\tilde{v}_n R_n^2 R''_n - 4w_n w''_n R_n^3 - 108\tilde{v}_n R_n R_n R''_n
- 168w_n w''_n R_n R_n - 234w_n R_n R'_n R_n R''_n
\right] \tag{C.15}
\frac{1}{\sqrt{2}}(\log Q^2)_{n,n+3} &= \frac{2}{3R_n} - \frac{4\tilde{v}_n w_n}{3R_n^2} + \frac{4w_n^3}{9R_n^3}
+ \frac{2}{3R_n N^3} \left[ -2R_n^2 R'_n - 2\tilde{v}_n w_n R_n^2 - 3w'_n \tilde{v}_n R_n \right]
- 4w_n^2 R'_n + 3w_n^2 R'_n R_n + 8\tilde{v}_n w_n R_n R'_n
+ \frac{1}{9R_n^3 N^2} \left[ -14\tilde{v}_n w'_n R_n^3 - 17R_n^3 R'_n - 6\tilde{v}_n w'_n R_n^3 - 17w'_n \tilde{v}_n R_n^3 + 28R_n^2 R''_n \right]
+ 48w_n \tilde{v}_n R_n R_n^2 + 76\tilde{v}_n w'_n R_n^3 + 24w_n w'_n R_n^3 + 60\tilde{v}_n R_n R'_n R_n + 15w_n w''_n R_n^2
- 28w_n R_n R'_n R_n + 108w_n w'_n R'_n R_n - 156\tilde{v}_n w_n R_n R'_n + 100w_n w''_n R_n
+ \frac{1}{18R_n^6 N^3} \left[ -720w_n^3 R''_n + 960\tilde{v}_n w_n R_n R_n^3 + 900w_n^2 R_n^2 R_n^2 \right]
+ 660w_n^3 R'_n R''_n - 144R_n^2 R''_n + 112\tilde{v}_n w_n R_n R''_n + 18w_n w''_n R_n^3
- 312w_n \tilde{v}_n R_n R'n^2 R_n - 516\tilde{v}_n w_n R_n^2 R_n R''_n - 288w_n w''_n R_n R'_n R_n
- 996\tilde{v}_n w_n R_n^2 R'_n R_n - 342w_n^2 w'_n R_n^3 + 270w_n w''_n R_n R_n R''_n - 48w_n^3 R''_n R_n R''_n
+ 120\tilde{v}_n w_n R_n^3 R''_n + 18w_n^3 R_n^3 + 177R_n^3 R_n R'_n R_n + 156w_n \tilde{v}_n R_n R''_n R_n^2
+ 232\tilde{v}_n w_n R_n R''_n R_n^3 + 84w_n \tilde{v}_n R_n^2 R_n R''_n + 184\tilde{v}_n w''_n R_n R'_n R_n + 108w_n w'_n w''_n R_n
- 19w_n^2 \tilde{v}_n R_n^4 + 31w_n \tilde{v}_n R_n R''_n R_n - 36R_n^4 R''_n - 4\tilde{v}_n w_n R_n R_n + 24w_n \tilde{v}_n \tilde{v}_n R_n \right] \tag{C.16}
\frac{1}{\sqrt{2}}(\log Q^2)_{n,n-1} &= \frac{\tilde{v}_n - \tilde{v}'_n}{3N} + \frac{\tilde{v}''_n}{9N^2} \tag{C.17}
\end{align*}
\[(\log Q)_{n,n-2} = \frac{w_n}{3} - \frac{\bar{v}_n^2}{3} - \frac{w_n'}{3N} + \frac{\bar{v}_n v_n'}{N} + \frac{4w_n^{'''} - 11\bar{v}_n v_n'' - 12\bar{v}_n^2}{18N^2} + \frac{8}{3N^3} \bar{v}_n v_n'' + \frac{\bar{v}_n v_n'}{6N^3}. \]  
(C.19)

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