Spatial log-Gaussian Cox processes in Hilbert spaces

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Abstract

A new class of spatial log-Gaussian Cox processes in function spaces is introduced under suitable conditions, and the corresponding least-squares spatial functional predictor is formulated. The special case where the log-intensity is a Gaussian first-order spatial autoregressive Hilbertian process is analysed, and its minimum-contrast componentwise functional parameter estimation is addressed. The asymptotic properties of these minimum contrast parameter estimators are studied. (See also the Supplementary Material). An application to respiratory disease mortality data illustrates the performance of the spatial functional estimation methodology proposed.

Keywords: Cox processes in Hilbert spaces, Minimum-contrast functional estimation, Spatial Autoregressive Hilbertian processes.

1. Introduction

Point pattern analysis constitutes an important branch of spatial and spatio-temporal statistics, having quite a long history (see, for example, \cite{23}; \cite{24}; \cite{27}; \cite{38}; \cite{43}; \cite{47}). A thorough treatment in a book-length format can be found in \cite{18} (see also \cite{19}; \cite{20}). The Cox process, also called the doubly stochastic Poisson process, generalises the Poisson process by allowing the underlying intensity to be a random function (\cite{15}). Doubly stochastic Poisson processes were already described in \cite{12}, applying martingale theory. The main probabilistic properties and the standard construction were addressed in \cite{34}. Different models have been proposed for the intensity process, such as the log-normal model (\cite{6}; \cite{28}; \cite{56}), the shot-noise type process (\cite{22}; \cite{15}), or a one-dimensional Feller process (see, e.g., \cite{65}). These approaches
overcome the non-negativity condition of the intensity. When the random intensity belongs to a class of affine diffusions, a flexible class of Cox processes is introduced in [21], useful, for example, in high frequency transaction data analysis.

Several statistical approaches, in the parametric (likelihood, pseudo-likelihood, composite likelihood), semi-parametric and non-parametric contexts, from a classical and Bayesian perspective, have been adopted for inference on spatial and spatio-temporal point processes (see, e.g., [4]; [3]; [14]; [26]; [32]; [33]; [36]; [40]; [48]).

Special attention has been paid to the log–Gaussian intensity model in the context of Cox processes. Indeed, log-Gaussian Cox processes define a flexible class of models for spatial and spatio-temporal point process data analysis ([28]; [33]). Log-Gaussian Cox processes with a constant intensity within square quadrants, and satisfying a conditional autoregressive structure are introduced in [46] (see also [7]). Gibbs sampling can be applied to explore these discretised models. Several properties of this class of processes, e.g., the complete characterisation of their distribution by the intensity, and the pair correlation function, were shown in [41] and [46]. These processes offer, in particular, a suitable framework for modelling and estimation in disease mapping due to their predictability properties (see, for example, [49], where a log-Gaussian Cox point process is considered to model the spatio-temporal distribution of substance abuse mortality in Iran). Additional areas of application are ecology ([55]; [63]), seismology ([42]) or texture recognition of visual scenes ([56]).

Functional Data Analysis (FDA) techniques applied to point process estimation is a relatively new branch in FDA. In the framework of doubly stochastic Poisson processes, [10] and [11] address the problem of modelling and estimation the intensity function from a FDA perspective. A pre-smoothing step, based on splines, is first applied. Smoothing techniques are also considered in the context of point process data classification, based on second-order statistics (see, e.g. [38], and the references therein). In [66], when the shapes of the intensity functions that generate the observed event times are not known, a functional approach is proposed to obtain the covariance structure of the associated random densities. Specifically, in the context of event times observed over a fixed time interval, a reconstruction formula is derived for the object-specific density functions, that reflect the distribution of such events in time.

Furthermore, up to our knowledge, the spatial functional statistical anal-
ysis of point patterns is an underdeveloped research area, where a number of problems remain open. Among the challenges posed in this area, the suitable definition of the process that generates the points plays a crucial role. An $\ell^2$-valued homogeneous Poisson process is introduced in [9], where its functional parameter estimation and prediction are addressed from both, a componentwise Bayesian and classical frameworks. The asymptotic efficiency and equivalence of both estimation approaches are also shown. In [58], sufficient conditions are derived for the existence and proper definition of an $\ell^2$-valued temporal log-Gaussian Cox process, with infinite-dimensional intensity given by a Hilbert-valued Ornstein-Uhlenbeck process. Its estimation is performed using a discrete ARH(1) approximation of such process in time.

This paper derives sufficient conditions for a suitable definition of a class of spatial log-Gaussian Cox processes in Hilbert spaces. Its $n$th-order product density can be characterised as the Laplace transform of a multivariate infinite-dimensional Gaussian distribution. The conditions derived on the covariance operator of the infinite-dimensional log-intensity ensure finite functional variance, and an almost surely finite Hilbert space norm of the realisations of the introduced log-Gaussian Cox process. The associated least-squares functional prediction problem is formulated as well. In the case where the log-intensity is a Gaussian spatial autoregressive Hilbertian process of order one (see [50]), its minimum contrast componentwise functional parameter estimation, based on Ibragimov functional (see [37]), is obtained. From Theorem 2 below, the weak–consistency of the derived minimum contrast componentwise functional parameter estimator follows, in the Hilbert-Schmidt and bounded linear operator norms under suitable conditions. See also [2] for the real-valued case, and the Supplementary Material, where a simulation study is undertaken to illustrate the asymptotic properties of these minimum contrast parameter estimators.

It is well-known that log-Gaussian Cox processes provide useful models to represent aggregated patterns ([24]; [28]). As commented in [28], Cox processes are natural models for point process phenomena that are environmentally driven, much less natural for phenomena driven primarily by interactions amongst the points. The present paper illustrates this fact in an infinite-dimensional random variable framework. Particularly, the real-data example analysed in Section 4, in the context of respiratory disease mapping, shows a good performance of the spatial functional log–Gaussian Cox process approach adopted here.

Summarizing, the outline of the paper is as follows. Section 2 introduces
some preliminary definitions and elements on $l^2$-valued homogeneous Poisson process estimation and prediction. The main ingredients used in the introduction of a new class of spatial log–Gaussian Cox processes in Hilbert spaces can be found in Section 2.2. Spatial functional prediction, in particular, plug-in prediction, is addressed in Section 2.3. The particular case where $H = L^2(S)$, $S \in B^n$, with $S$ being a bounded Borel set of $\mathbb{R}^n$, and $B^n$ being the Borel $\sigma$–algebra of $\mathbb{R}^n$, is contemplated in Section 2.4. A minimum contrast componentwise functional parameter estimator, and plug-in predictor of the Gaussian log–intensity is derived in Section 3, in the context of Spatial Autoregressive Hilbertian processes of order one (SARH(1) processes, see [50]). The estimation methodology adopted involves the periodogram operator. Spatial functional prediction of respiratory disease deaths, over a continuous time interval, is addressed in the real-data problem analysed in Section 4. A conclusion section ends the paper. The Supplementary Material illustrates the asymptotic properties (consistency, asymptotic efficiency and normality) of the minimum contrast estimators computed.

2. Functional Log-Gaussian Cox processes

In the remaining sections, we consider that all the stochastic processes are defined on the basic probability space $(\Omega, A, P)$. Let $\mathcal{H}$ be a real separable Hilbert space with the inner product $\langle \cdot, \cdot \rangle_\mathcal{H}$, and the associated norm $\| \cdot \|_\mathcal{H}$. We start introducing some preliminary elements involved in the definition of an $l^2$-valued homogeneous Poisson process in time (see [9]).

2.1. Homogeneous Poisson process in $l^2$ space

Let $\{M_t, t \geq 0\}$ be a continuous time process such that, for every $t \geq 0$, $M_t = \{N_{t,j}, j \geq 1\}$, with $\{N_{t,j}, t \in \mathbb{R}_+, j \geq 1\}$ being a sequence of independent homogeneous Poisson processes with respective intensities $\lambda_j > 0$, $j \geq 1$, satisfying $\sum_{j=1}^\infty \lambda_j < \infty$. Thus,

$$E\|M_t\|_{l^2}^2 = E\left(\sum_{j=1}^\infty N_{t,j}^2\right) = \sum_{j=1}^\infty E\left(N_{t,j}^2\right) = \sum_{j=1}^\infty \left\{\lambda_j t + (\lambda_j t)^2\right\} < \infty,$$

for every $t \geq 0$. Therefore, $\sum_{j=1}^\infty N_{t,j}^2 < \infty$, and $M_t \in l^2$ almost surely, for every $t \geq 0$. Thus, $\{M_t, t \geq 0\}$ defines an $l^2$-valued homogeneous Poisson process.
Let the sequences \((x_j) = \{x_j, j \geq 1\}\) be such that \(x_j\) is an integer for each \(j\), with \(x_j = 0\), for sufficiently large \(j\), and denote \(N_k = \{(x_j) : (x_1, \ldots, x_k) \in \mathbb{N}^k_k; x_j = 0, j > k\}\), with \(\mathbb{N}^k_k = \mathbb{N}^k - \{0\}\), for certain \(k \geq 1\). In addition, consider \(\mathcal{N} \subset \ell^2\) as the family of sequences \((x_j)\), then \(\mathcal{N} = \bigcup_k \mathcal{N}_k\). Note that \(\mathcal{N}_k\) is countable for every \(k \geq 1\). In addition, consider \(\mathcal{N} \subset \ell^2\) as the family of sequences \((x_j)\), then \(\mathcal{N} = \bigcup_k \mathcal{N}_k\).

One can define the values of the likelihood function, based on the observation \(M_T(\omega) \in \mathcal{N}, T > 0\), with respect to \(\mu\) by setting

\[
L(M_T(\omega), (\lambda)) = \prod_{j=1}^{j_0(\omega, \lambda, T)} \exp \left( -\lambda_j T \right) \frac{(\lambda_j T)^{N_{T,j}(\omega)}}{N_{T,j}(\omega)!},
\]

where \(j_0(\omega, \lambda, T)\) is such that \(N_{T,j}(\omega) = 0\), almost surely, for \(j > j_0\), and \(\omega \in \Omega_0\), with \(P(\Omega_0) = 1\), since \(\sum_j N_{T,j}^2 < \infty\) almost surely. The maximum likelihood estimator of \((\lambda) = \{\lambda_j, j \geq 1\}\) is computed as \((\hat{\lambda})_T = \left(\frac{N_{T,j}}{T}, j \geq 1\right)\). Then, \(\sum_{j=1}^{\infty} \hat{\lambda}_{T,j} < \infty\), almost surely, and \((\hat{\lambda})_T\) is an unbiased estimator of \((\lambda) \in \ell^1\). In particular, from (1),

\[
f(M_T) = \frac{T + h}{T} M_T = M_T + h(\hat{\lambda})_T\]

is an unbiased efficient estimator of \(E_{(\lambda)}(M_{T+h}|M_T), (\lambda) \in \ell^1\), providing a plug-in predictor of \(M_{T+h}\) (see [9]). In what follows we derive sufficient conditions for the reformulation of this functional plug-in predictor in the context of spatial doubly stochastic Poisson processes in Hilbert spaces.

### 2.2. Spatial log-Gaussian Cox processes in Hilbert spaces

Consider

\[
\Lambda = \{\Lambda_z = \exp(X_z), z \in \mathbb{R}^d\},
\]

where \(X = \{X_z, z \in \mathbb{R}^d\}\) is a spatial Gaussian process on \(\mathbb{R}^d\), with values in a separable Hilbert space \(\mathcal{H}\). That is, \(X_z : (\Omega, \mathcal{A}, P) \to \mathcal{H}\), for all \(z \in \mathbb{R}^d\).
We restrict our attention to the case where $X$ is weak-sense stationary, with zero mean, i.e.,

$$
E(X_z) = \mu(\cdot) \in \mathcal{H}, \quad \mu(\cdot) = 0
$$

$$
E\|X_z^2\|_\mathcal{H} = \sigma^2, \quad \forall z \in \mathbb{R}^d
$$

$$
E(X_z \otimes X_y) = \mathcal{R}_{z-y} \in \ell^1(\mathcal{H}), \quad z, y \in \mathbb{R}^d,
$$

(4)

where $\mathcal{R}_{z-y}$ is the covariance operator of $X$, defined on $\mathcal{H}$, given by

$$
\mathcal{R}_{z-y}(f)(g) = \langle E(X_z \otimes X_y)(f), g \rangle_\mathcal{H}, \quad \forall f, g \in \mathcal{H},
$$

(5)

and satisfying

$$
\|\mathcal{R}_{z-y}\|_{\ell^1(\mathcal{H})} = \sum_{j=1}^{\infty} \langle (\mathcal{R}_{z-y}^* \mathcal{R}_{z-y})^{1/2}(\varphi_j), \varphi_j \rangle_\mathcal{H} < \infty,
$$

for any orthonormal basis $\{\varphi_j\}_{j \geq 1}$ in $\mathcal{H}$. That is, $\mathcal{R}_{z-y} \in \ell^1(\mathcal{H})$, the class of trace operators on $\mathcal{H}$, having finite nuclear norm or Schatten norm $\|\cdot\|_{\ell^1(\mathcal{H})}$.

For $z = y \in \mathbb{R}^d$, $\mathcal{R}_0$ is a self-adjoint (symmetric) trace operator. The system $\{\phi_j, j \geq 1\}$ of eigenvectors of $\mathcal{R}_0$ defines an orthonormal basis of $\mathcal{H}$, satisfying

$$
\mathcal{R}_0(\phi_j) = \lambda_j(\mathcal{R}_0)\phi_j, \quad j \geq 1.
$$

Furthermore, for each $z \in \mathbb{R}^d$, $X_z$ admits the following orthogonal expansion in $L^2_{\mathcal{H}}(\Omega, A, P)$

$$
X_z = \sum_{j=1}^{\infty} \langle X_z, \phi_j \rangle_\mathcal{H} \phi_j.
$$

(6)

That is, $E\|X_z - \sum_{j=1}^{M} \langle X_z, \phi_j \rangle_\mathcal{H} \phi_j\|_\mathcal{H}^2 \to 0$, as $M \to \infty$, with

$$
E[\langle X_z, \phi_j \rangle_\mathcal{H} \langle X_z, \phi_p \rangle_\mathcal{H}] = \delta_{j,p}\lambda_j(\mathcal{R}_0), \quad j \geq 1,
$$

for each $z \in \mathbb{R}^d$. Here, $\delta_{j,p}$ denotes the Kronecker delta function. Thus, $\Lambda_z$ in (3) admits the following representation in $L^2_{\mathcal{H}}(\Omega, A, P)$

$$
\Lambda_z = \exp\left(\sum_{j=1}^{\infty} \langle X_z, \phi_j \rangle_\mathcal{H} \phi_j \right) = \prod_{j=1}^{\infty} [\Lambda_{z,j}]^{\phi_j}, \quad \forall z \in \mathbb{R}^d,
$$

6
where, for every \( j \geq 1 \), and \( \mathbf{z} \in \mathbb{R}^d \), \( \Lambda_{\mathbf{z}, j} = \exp \left( \langle X_{\mathbf{z}, j}, \phi_j \rangle_{\mathcal{H}} \right) = \exp \left( \langle X_{\mathbf{z}}, \phi_j \rangle \right) \).

Also, for each \( \mathbf{z} \in \mathbb{R}^d \), \( \{ \Lambda_{\mathbf{z}, j} \} \) is a sequence of independent log-Gaussian random variables, with \( \ln(\Lambda_{\mathbf{z}, j}) = \ln(\Lambda_{\mathbf{z}})(\phi_j) \), for every \( j \geq 1 \). Then, we also use the notation \( \{ \Lambda_{\mathbf{z}, j} = \tilde{\Lambda}(X_{\mathbf{z}}(\phi_j)), j \geq 1 \} \).

Let \( \mathcal{B}^d \) be the Borel \( \sigma \)-algebra on \( \mathbb{R}^d \). Assume that, for every \( j \geq 1 \), the realisations of \( \{ \Lambda_{\mathbf{z}, j}, \mathbf{z} \in \mathbb{R}^d \} \) are almost surely integrable on \( \mathbb{R}^d \), and, for every bounded Borel set \( A \in \mathcal{B}^d \), \( \int_A \Lambda_{\mathbf{z}, j} d\mathbf{z} < \infty \), almost surely. Let \( \{ d\mathbf{C}_\mathbf{z}, \mathbf{z} \in \mathbb{R}^d \} \) be an infinite-dimensional random measure with values in \( \mathcal{H} \) such that

\[
E[d\mathbf{C}_\mathbf{z}] = \Lambda_{\mathbf{z}}d\mathbf{z}, \quad \mathbf{z} \in \mathbb{R}^d \\
E[d\mathbf{C}_\mathbf{z} \otimes d\mathbf{C}_\mathbf{y}] = E[\Lambda_{\mathbf{z}} \otimes \Lambda_{\mathbf{y}}]d\mathbf{z}d\mathbf{y}, \quad \mathbf{z}, \mathbf{y} \in \mathbb{R}^d.
\]

For every bounded Borel set \( A \in \mathcal{B}^d \), and for every \( j \geq 1 \), assume that

\[
\mathbf{C}_A(\phi_j) := \int_A d\mathbf{C}_\mathbf{z}(\phi_j) < \infty,
\]

almost surely. Theorem \( \Box \) below derives sufficient conditions on the covariance operator \( \mathbf{C}_A(\phi_j) \) that ensure \( \| \mathbf{C}_A(\cdot) \|_{\mathcal{H}}^2 = \sum_{j=1}^{\infty} [\mathbf{C}_A(\phi_j)]^2 < \infty \), almost surely. Assume also that, for each bounded Borel set \( A \in \mathcal{B}^d \), and every \( j \geq 1 \), \( \Pr[\mathbf{C}_A(\phi_j) \in \mathbb{N}] = 1 \), and the conditional probability measure \( \mu_{\mathbf{C}_A(\phi_j)}(\tilde{\Lambda}(X_{\omega, \mathbf{z}}(\phi_j)), \mathbf{z} \in \mathbb{R}^d) \) of \( \mathbf{C}_A(\phi_j) \), given

\[
\left( \Lambda_{\mathbf{z}, j}(\omega) = \lambda_{\omega, \mathbf{z}, j} = \tilde{\Lambda}(X_{\omega, \mathbf{z}}(\phi_j)), \mathbf{z} \in \mathbb{R}^d \right),
\]

equivalently, given \( (X_{\omega, \mathbf{z}}(\phi_j)), \mathbf{z} \in \mathbb{R}^d \), for certain \( \omega \in \Omega \), is a Poisson measure with mean \( \int_A \Lambda_{\mathbf{z}, j}(\omega) d\mathbf{z} = \int_A \tilde{\Lambda}(X_{\omega, \mathbf{z}}(\phi_j)) d\mathbf{z} \), and Laplace transform

\[
\Gamma(\phi_j) = E_{\mu_{\mathbf{C}_A(\phi_j)}(\tilde{\Lambda}(X_{\omega, \mathbf{z}}(\phi_j)), \mathbf{z} \in \mathbb{R}^d)}[\exp(\mathbf{C}_A(\phi_j))]
\]

\[
= \exp \left( (e-1) \int_A \tilde{\Lambda}(X_{\omega, \mathbf{z}}(\phi_j)) d\mathbf{z} \right).
\]

Under the above setting of conditions (including the conditions assumed in Theorem \( \Box \) below), for every bounded Borel set \( A \in \mathcal{B}^d \), the probability measure \( \mu_{\mathbf{C}_A(|\mathbf{A}(\omega)|)} \) induced by \( \left( \mathbf{C}_A(\phi_j) \mid \left( \tilde{\Lambda}(X_{\omega, \mathbf{z}}(\phi_j)), \mathbf{z} \in \mathbb{R}^d \right), j \geq 1 \right) \) is defined on the countable set \( \mathcal{N} = \bigcup_k \mathcal{N}_k, \mathcal{N} \subset \ell^2 \), and extended to \( \ell^2 \) by setting \( \mu_{\mathbf{C}_A(|\mathbf{A}(\omega)|)}(\ell^2 - \mathcal{N}) = 0, \omega \in \Omega \). Here, as before, \( \mathcal{N}_k = \{(x_j) : (x_1, \ldots, x_k) \in \}

\( \mathbb{N}^\ast_k; x_j = 0, \ j > k \), with \( \mathbb{N}_k^\ast = \mathbb{N}_k - \{0\} \), for \( k \geq 1 \). In particular, for each \( j \geq 1 \), let \( C_j = C(\phi_j) \) be a locally finite random subset of \( \mathbb{R}^d \), such that

\[
C_A(\phi_j) = \text{Card} (C_j \cap A),
\]

for every bounded Borel set \( A \in \mathcal{B}^d \). Thus, \( C = \{C_j, \ j \geq 1\} \) defines an infinite-dimensional, or, in this case, spatial functional, point pattern.

**Lemma 1.** (See, for instance, Lemma 1.1.4, p.3, in [39], involved in the proof of Minlos-Sazanov Theorem) Let \( \mu \) be a finite Borel measure on \( \mathcal{H} \). Then, the following assertions are equivalent:

- (i) \( \int_\mathcal{H} \|x\|_H^2 \mu(dx) < \infty \).
- (ii) There exists a positive, symmetric, trace class operator \( Q \) such that \( \langle Qx, y \rangle_H = \int_\mathcal{H} \langle x, z \rangle_H \langle y, z \rangle_H \mu(dz) \), for \( x, y \in \mathcal{H} \).

If (ii) holds, then \( \|Q\|_{\ell^1(\mathcal{H})} = \int_\mathcal{H} \|x\|_H^2 \mu(dx) \).

**Lemma 2.** Let \( \{C_A(\phi_j), j \geq 1, A \in \mathcal{B}^d\} \) be introduced in equation (9), with \( \{dC_z, z \in \mathbb{R}^d\} \) satisfying equations (7)–(8). If \( E\|C_A(\cdot)\|_H^2 < \infty \), then, the following identity holds:

\[
E\|C_A(\cdot)\|^2_H = \left\| \int_\mathcal{A} \int_\mathcal{A} \exp \left( \frac{1}{2} \sum_{i,j \in \{1,2\}} \mathcal{R}_{z_i - z_j} \right) d\mathbf{z}_1 d\mathbf{z}_2 \right\|_{\ell^1(\mathcal{H})} , \tag{11}
\]

where \( \mathcal{R}_{z_i - z_j} = E[X_{z_i} \otimes X_{z_j}] \), \( z_i, z_j \in \mathbb{R}^d, i, j \in \{1, 2\} \).

**Proof.** From Lemma 1(i), if \( E\|C_A(\cdot)\|_H^2 < \infty \), then there exists a positive, symmetric, trace class operator \( Q_{C_A(\cdot)} \) such that \( \|Q_{C_A(\cdot)}\|_{\ell^1(\mathcal{H})} = E\|C_A(\cdot)\|_H^2 \). From equations (7)–(8),

\[
Q_{C_A(\cdot)}(f_1)(f_2) = \int_\mathcal{A} \int_\mathcal{A} E[\Lambda_{z_1}(f_1) \Lambda_{z_2}(f_2)] d\mathbf{z}_1 d\mathbf{z}_2, \quad \forall f_1, f_2 \in \mathcal{H}. \tag{12}
\]

We now obtain \( E[\Lambda_{z_1}(f_1) \Lambda_{z_2}(f_2)] \), for \( f_1, f_2 \in \mathcal{H} \), and \( z_1, z_2 \in \mathbb{R}^d \). It is well-known (see, for example, [16] and [17]) that, for every \( m \in \mathbb{N} \), the multivariate infinite-dimensional Gaussian measure associated with \( (X_{z_1}, \ldots, X_{z_m}) \),
\((z_1, \ldots, z_m) \in \mathbb{R}^{dm}\) has Fourier transform given by

\[
\phi_m(f) = \phi(f_1, \ldots, f_m) = \int_{\mathcal{H}^m} \exp \left( i f^*(x) \right) \mu(dx)
\]

\[
= \int_{\mathcal{H}^m} \exp \left( i \sum_{i=1}^m f^*_i(x_{z_i}) \right) \mu(dx)
\]

\[
= \exp \left( -\frac{1}{2} \langle \mathcal{R}_m f, f \rangle_{\mathcal{H}_m} \right), \quad \forall f = (f_1, \ldots, f_m) \in \mathcal{H}^m, \quad (13)
\]

where, for \(f^* = (f^*_1, \ldots, f^*_m) \in [\mathcal{H}^m]^*\), with \([\mathcal{H}^m]^*\) denoting the dual Hilbert space of \(\mathcal{H}^m\), and \(x = (x_{z_1}, \ldots, x_{z_m}) = (X_{z_1}(\omega), \ldots, X_{z_m}(\omega)) \in \mathcal{H}^m, \omega \in \Omega, \)

\[
f^*(x) = \sum_{i=1}^m f^*_i(x_{z_i}) = \sum_{i=1}^m \langle x_{z_i}, f_i \rangle_{\mathcal{H}} = \langle x, f \rangle_{\mathcal{H}_m}.
\]

Here,

\[
\mathcal{R}_m = \begin{pmatrix}
E[X_{z_1} \otimes X_{z_1}] & \ldots & E[X_{z_1} \otimes X_{z_m}] \\
\vdots & \ddots & \vdots \\
E[X_{z_m} \otimes X_{z_1}] & \ldots & E[X_{z_m} \otimes X_{z_m}]
\end{pmatrix} = \begin{pmatrix}
\mathcal{R}_0 & \ldots & \mathcal{R}_{z_1 - z_m} \\
\vdots & \ddots & \vdots \\
\mathcal{R}_{z_m - z_1} & \ldots & \mathcal{R}_0
\end{pmatrix}.
\]

From equations \((3)\) and \((13)\), for every \(f = (f_1, \ldots, f_m) \in \mathcal{H}^m, \)

\[
\mathbf{\Gamma}(f) = \mathbf{\Gamma}(f_1, \ldots, f_m) = E[\Lambda_m(f)] = E \left[ \prod_{i=1}^m \Lambda_{z_i}(f_i) \right]
\]

\[
= E \left[ \exp \left( \sum_{i=1}^m \langle X_{z_i}, f_i \rangle_{\mathcal{H}} \right) \right]
\]

\[
= \int_{\mathcal{H}^m} \exp \left( \sum_{i=1}^m \langle x_{z_i}, f_i \rangle_{\mathcal{H}} \right) \mu(dx) = \int_{\mathcal{H}^m} \exp \left( f^*(x) \right) \mu(dx)
\]

\[
= \exp \left( -\frac{1}{2} \langle \mathcal{R}_m f, f \rangle_{\mathcal{H}_m} \right), \quad (15)
\]

where \(\mu\) is the multivariate infinite-dimensional Gaussian measure with characteristic function \(\phi\) introduced in equation \((13)\).
Considering \( m = 2 \), in equation (15), for \( f = (f_1, f_2) \in \mathcal{H}^2 \),

\[
E[L_z(f_1)\Lambda_z(f_2)] = \exp\left(\frac{1}{2} \langle R_{22} f, f \rangle_{\mathcal{H}^2}\right) = \exp\left(\frac{1}{2} \sum_{i,j \in \{1,2\}} R_{z_i-z_j}(f_j)(f_i)\right).
\]

Thus, from equations (12) and (16), for \( f = (f_1, f_2) \in \mathcal{H}^2 \),

\[
Q_{C_A}(f_1)(f_2) = \int_A \int_A E[L_z(f_1)\Lambda_z(f_2)] \, dz_1 \, dz_2
= \int_A \int_A \exp\left(\frac{1}{2} \sum_{i,j \in \{1,2\}} R_{z_i-z_j}(f_j)(f_i)\right) \, dz_1 \, dz_2.
\]

(16)

Identity (11) then follows from Lemma 1(i), and equation (17).

Theorem 1. (i) For every \( z, y \in \mathbb{R}^d \),

\[
\|E[d\mathcal{C}_z \otimes d\mathcal{C}_y]\|_{l^1(\mathcal{H})} \leq \exp\left(\|R_0\|_{l^1(\mathcal{H})} + \|R_{z-y}\|_{l^1(\mathcal{H})} + \|R_{y-z}\|_{l^1(\mathcal{H})}\right).
\]

(ii) If, for each bounded Borel set \( A \in \mathcal{B}^d \),

\[
\int_A \int_A \exp\left(\|R_0\|_{l^1(\mathcal{H})} + \|R_{z-y}\|_{l^1(\mathcal{H})} + \|R_{y-z}\|_{l^1(\mathcal{H})}\right) \, dz \, dy < \infty,
\]

then, \( E\|\mathcal{C}_A(\cdot)\|_{\mathcal{H}}^2 < \infty \), thus, \( \|\mathcal{C}_A(\cdot)\|_{\mathcal{H}}^2 < \infty \), almost surely.

Proof. (i) From equations (7)–(8) and (16), for every \( z, y \in \mathbb{R}^d \),

\[
\|E[d\mathcal{C}_z \otimes d\mathcal{C}_y]\|_{l^1(\mathcal{H})} = \left\|\exp\left(\frac{1}{2} (2R_0 + R_{z-y} + R_{y-z})\right)\right\|_{l^1(\mathcal{H})}
\leq \sum_{k=0}^{\infty} \frac{(2)^{-k}}{k!} \sum_{h=0}^{k} \sum_{m=0}^{k-h} \frac{k!}{h!m!(k-h-m)!} 2^h \|R_0\|_{l^1(\mathcal{H})}\|R_{z-y}\|_{l^1(\mathcal{H})}\|R_{y-z}\|_{l^1(\mathcal{H})}^{k-h-m} = \exp\left(\|R_0\|_{l^1(\mathcal{H})} + \|R_{z-y}\|_{l^1(\mathcal{H})} + \|R_{y-z}\|_{l^1(\mathcal{H})}\right).
\]

(19)
(ii) This part of the present result follows from equations (7)–(8) and (17), applying inequality (19) and Lemma 1(ii). Specifically, for every bounded Borel set $A \in \mathcal{B}^d$,

$$\| QC_A(\cdot) \|_{\ell^1(\mathcal{H})} = \left\| \int_A \int_A \exp \left( \frac{1}{2} (2R_0 + R_{z-y} + R_{y-z}) \right) d\bar{y} \right\|_{\ell^1(\mathcal{H})}$$

$$\leq \int_A \int_A \left\| \exp \left( \frac{1}{2} (2R_0 + R_{z-y} + R_{y-z}) \right) \right\|_{\ell^1(\mathcal{H})} d\bar{y}$$

$$\leq \int_A \int_A \exp \left( \| R_0 \|_{\ell^1(\mathcal{H})} + \frac{\| R_{z-y} \|_{\ell^1(\mathcal{H})} + \| R_{y-z} \|_{\ell^1(\mathcal{H})}}{2} \right) d\bar{y} < \infty.$$ 

From Lemma 1(ii), we then obtain $E \| QC_A(\cdot) \|_{\ell^1(\mathcal{H})}^2 = \| QC_A(\cdot) \|_{\ell^1(\mathcal{H})} < \infty$. 

2.3. Spatial functional prediction

From equation (10), for each $\omega \in \Omega$, and for every bounded Borel sets $A, B \in \mathcal{B}^d$, and $j \geq 1$,

$$E \left( \tilde{\Lambda}(X_{\omega, z}(\phi_j)), z \in \mathbb{R}^d \right) \left( C_{A \cup B}(\phi_j) \big| C_A(\phi_j) \right) = \int_{A \cup B \setminus A} \tilde{\Lambda}(X_{\omega, z}(\phi_j)) d\bar{u} + C_A(\phi_j)$$

$$= \int_{A \cup B \setminus A} \exp \left( \langle X_{\bar{u}}(\omega), \phi_j \rangle_{\mathcal{H}} \right) d\bar{u} + C_A(\phi_j), \quad (20)$$

where $E \left( \tilde{\Lambda}(X_{\omega, z}(\phi_j)), z \in \mathbb{R}^d \right) [Z \big| Y]$ now denotes the conditional expectation of $Z \big| \left( \tilde{\Lambda}(X_{\omega, z}(\phi_j)), z \in \mathbb{R}^d \right)$ given $Y \big| \left( \tilde{\Lambda}(X_{\omega, z}(\phi_j)), z \in \mathbb{R}^d \right)$, for any real-valued random variables $Z$ and $Y$ on $(\Omega, \mathcal{A}, P)$.

Under the conditions assumed in Theorem 1(ii), for each bounded Borel set $A \in \mathcal{B}^d$, and for every $f \in \mathcal{H}$,

$$E \left[ \int_A \tilde{\Lambda}(X_{\omega, z}(f)) d\bar{z} \right]^2 = QC_A(f)(f) \leq C \| f \|_{\mathcal{H}}^2 < \infty.$$ 

Thus, $\int_A \tilde{\Lambda}(X_{\omega, z}(f)) d\bar{z} < \infty$, almost surely, for every $f \in \mathcal{H}$, and bounded Borel set $A \in \mathcal{B}^d$. From equation (10), for each bounded Borel set $A \in \mathcal{B}^d$, 


and for every $f \in \mathcal{H}$, we then have

$$
\Gamma(f) = \exp \left( (e - 1) \int_A \exp \left( X_{\omega,z}(f) \right) dz \right)
= \exp \left( (e - 1) \int_A \exp \left( \sum_{j=1}^{\infty} f(\phi_j) X_{\omega,z}(\phi_j) \right) dz \right)
= \exp \left( (e - 1) \int_A \prod_{j=1}^{\infty} \widetilde{\Lambda}(X_{\omega,z}(\phi_j)) f(\phi_j) dz \right),
$$

(21)

which can be interpreted as the infinite-dimensional Laplace transform of the measure $\mu_{C_{\Lambda}}(\cdot) \mid \Lambda(\omega)$ induced by $C_{\Lambda} \mid \Lambda(\omega)$, for a given $\omega \in \Omega$. From (20) and (21), the corresponding functional least-squares predictor, i.e., the orthogonal projector into the space $\ell^2_{\mu_{C_{\Lambda}}(\cdot) \mid \Lambda(\omega)}$ of square summable sequences with respect to the measure $\mu_{C_{\Lambda}}(\cdot) \mid \Lambda(\omega)$, for a given $\omega \in \Omega$ (see also Section 2.2), is obtained as

$$
\hat{C}_{A_{\Lambda}}(\omega), z \in \mathbb{R}^d \ (\cdot) := E_{\mu_{C_{A_{\Lambda}}(\cdot) \mid \Lambda(\omega)}} \left[ C_{A_{\Lambda}}(\cdot) \mid C_{A}(\cdot) \right]
\overset{D}{=} \int_{A_{\Lambda}} \exp \left( X_u(\omega, \cdot) \right) du + C_{A}(\cdot),
$$

(22)

where $\overset{D}{=}$ means the weak-sense identity, in distribution sense, and $E_{\mu_{C_{A_{\Lambda}}(\cdot) \mid \Lambda(\omega)}} [Z \mid Y]$ now denotes the conditional expectation, under the conditional infinite-dimensional probability measure $\mu_{C_{A_{\Lambda}}(\cdot) \mid \Lambda(\omega)}$, for any infinite-dimensional random variables $Z$ and $Y$ on $(\Omega, \mathcal{A}, P)$. Note that, from Theorem [1] $\hat{C}_{A_{\Lambda}}(\omega), z \in \mathbb{R}^d \ (\cdot) \in \mathcal{H}$ almost surely. Let $\hat{X}_z^A$ be a spatial functional predictor of $X_z$, $z \in B$, based on the functional observations $\left\{ X_y, \ y \in A \right\}$, for $A$ and $B$ being, as before, bounded Borel sets in $\mathcal{B}^d$. Denote also $\hat{\Lambda}_z^A = \exp \left( \hat{X}_z^A \right)$ the corresponding spatial functional predictor of $\Lambda_z$, for each $z \in B$. Equation (22) can then be approximated in terms of the following spatial functional plug-in predictor

$$
\hat{C}_{A_{\Lambda}}(\omega), z \in A, \hat{A}_z^A(\omega), z \in B \ (\cdot) = \int_{A_{\Lambda}} \hat{\Lambda}_u^A(\omega, \cdot) du + C_{A}(\cdot).
$$

(23)
2.4. The case of $H = L^2(S)$, $S \in \mathcal{B}^n$

For a bounded Borel set $S$ of $\mathbb{R}^n$, $n \geq 1$, let $X = \{X_z, z \in \mathbb{R}^d\}$, $d \geq 2$, be a zero-mean stationary Gaussian random field with values in $L^2(S)$, i.e., $X_z \in L^2(S)$, for every $z \in \mathbb{R}^d$. Consider $\mathcal{R}_0 = E[X_z \otimes X_z]$, $z \in \mathbb{R}^d$, its autocovariance operator, which defines a self-adjoint trace operator on $L^2(S)$. Denote, as before, $\{\phi_j, j \geq 1\}$ its corresponding system of eigenvectors that provides an orthonormal basis of $L^2(S)$. Let now $\{\Lambda_z, z \in \mathbb{R}^d\}$, with $\Lambda_z = \{\exp(X_{z,s}), s \in S\}$, for every $z \in \mathbb{R}^d$. Under the conditions assumed in Theorem 1(ii), from equation (22), since $Q_{\mathcal{A}}(\cdot)$ is a trace operator, applying Embedding Theorems between Besov spaces, in particular, from Embedding Theorem of fractional Sobolev spaces into Hölder spaces (see [59]), the following almost surely pointwise equality holds, defining the spatial functional predictor of $\mathcal{A} \cup \mathcal{B}(\cdot)$, given the functional (pointwise) observation of $\mathcal{A}(\cdot)$, for any $s \in S$:

$$\hat{\Lambda}_{\mathcal{A}}(s) = \int_{\mathcal{A} \cup \mathcal{B} \setminus \mathcal{A}} \exp(X_{\omega,u}(s))\,du + \mathcal{A}(s), \quad \omega \in \Omega.$$ 

Also, given a spatial functional predictor $\hat{\Lambda}_{\mathcal{A}}^A \hat{\Lambda}_{\mathcal{A}}(s) = \exp(\hat{X}_{\mathcal{A}}^A(s))$, $s \in S$, of $\Lambda_z$, for every $z \in B$,

$$\hat{\Lambda}_{\mathcal{A}}^A(s) = \int_{\mathcal{A} \cup \mathcal{B} \setminus \mathcal{A}} \hat{\Lambda}_{\mathcal{A}}^A(\omega)\,du + \mathcal{A}(s), \quad \forall s \in S.$$ 

In the particular case analysed in the real-data example in Section 4, $S \subset \mathbb{R}_+$ defines a continuous time interval. Thus, the number of random events (respiratory disease deaths) are predicted in space and any time. The same assertions hold for the case of $S$ being a sea depth interval, or a landscape elevation interval, which are also of interest in several fields of application (see Section 5, for more details).

The following section addresses the particular case where $X = \{X_z, z \in \mathbb{R}^2\}$ is a SARH(1) process (see [50]). For every $z \in B \in \mathcal{B}_2$, the predictor $\hat{X}_{\mathcal{A}}$, based on the functional observations $\{X_z, z \in A \in \mathcal{B}_2\}$, is computed from the minimum contrast componentwise estimation of the autocorrelation operators involved in the first-order spatial autoregressive Hilbertian state equation satisfied by $X$. 

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3. Parameter estimation

We restrict here our attention to the case of $d = 2$, assuming a constant spatial functional log-intensity within square quadrants, satisfying a first-order spatial autoregressive Hilbertian equation (see [50]), in the spirit of [7] and [46]. Specifically, $\{X_z, z \in \mathbb{Z}^2\}$ in (3) is assumed to satisfy the following state equation

$$X_{i,j} = R + L_1(X_{i-1,j}) + L_2(X_{i,j-1}) + L_3(X_{i-1,j-1}) + \epsilon_{i,j}, \quad (i, j) \in \mathbb{Z}^2,$$

(24)

where $R \in \mathcal{H}$ and $L_i \in \mathcal{L}(\mathcal{H})$, for $i = 1, 2, 3$, with $\mathcal{L}(\mathcal{H})$ denoting the space of bounded linear operators on $\mathcal{H}$. Here $\epsilon = \{\epsilon_{i,j}, (i, j) \in \mathbb{Z}^2\}$ is the spatial innovation process, which is assumed to be a two-parameter martingale difference, with $E|\epsilon|_H^2 = \sigma^2$, independently of the spatial location $(i, j)$, and $E(\epsilon_{i,j} \otimes \epsilon_{i,j}) = E(\epsilon_{0,0} \otimes \epsilon_{0,0})$ also independently of $(i, j)$. In the following, $\{\epsilon_{i,j}, (i, j) \in \mathbb{Z}^2\}$ will be Gaussian strong white noise. Assume the conditions derived in Propositions 3 and 4 in [50] for the existence of a unique stationary solution to equation (24). Assume also

**Assumption A1.** $\sum_{(i,j) \in \mathbb{Z}^2} \|R_{i,j}\|_{\mathcal{H}} < \infty$.

Under **Assumption A1**, the spectral density operator can be defined as

$$\mathcal{F}_{\omega_1,\omega_2} := \frac{1}{(2\pi)^2} \sum_{(j,m) \in \mathbb{Z}^2} R_{j,m} \exp \left(-i(j\omega_1 + m\omega_2)\right), \quad (\omega_1, \omega_2) \in [-\pi, \pi]^2,$$

(25)

which is a self-adjoint non-negative definite operator.

For a given functional sample of size $N = S_1 \times S_2$, $\{X_{i,j}, i = 1, \ldots, S_1, j = 1, \ldots, S_2\}$, the functional discrete Fourier transform is defined as

$$\tilde{X}_{\omega_1,\omega_2}^N(\cdot) := \frac{1}{2\pi \sqrt{N}} \sum_{j=1}^{S_1} \sum_{m=1}^{S_2} X_{j,m}(\cdot) \exp \left(-i(j\omega_1 + m\omega_2)\right).$$

(26)

This transform is linear, periodic and Hermitian. Under **Assumption A1**, the asymptotic covariance operator of (26) coincides with (25). Specifically, considering the periodogram operator being defined as

$$\mathcal{T}_{\omega_1,\omega_2}^N := \tilde{X}_{\omega_1,\omega_2}^N \otimes \tilde{X}_{\omega_1,\omega_2}^N,$$

(27)
for \(\omega_1, \omega_2 \in \left\{0, \frac{2\pi}{N}, \ldots, \frac{2\pi(N-1)}{N}, \pi \right\}\), we can write

\[
\mathcal{I}^N_{\omega_1, \omega_2}(\cdot, \cdot) := \sum_{j=1}^{S_1} \sum_{m=1}^{S_2} X_{j,m} \otimes X_{j',m'}(\cdot, \cdot) \exp \left(-i(j-j')\omega_1 + (m-m')\omega_2\right)
\]

\[
\frac{1}{(2\pi)^2 N^2} \mathcal{I}^N_{\omega_1, \omega_2}(\cdot, \cdot),
\]

where \([\cdot]^-\) denotes the integer part function (see Theorem 2.2 in [44]).

Let \(\{\phi_k, \ k \geq 1\}\) and \(\{\lambda_k(\mathcal{R}_0), \ k \geq 1\}\) be the respective orthonormal eigenfunction and eigenvalue systems of the autocovariance operator \(\mathcal{R}_0 = \mathbb{E} [X_{i,j} \otimes X_{i,j}'], (i,j) \in \mathbb{Z}^2\). Applying Parseval identity, the following identities hold:

\[
X_{j,m} \in \mathcal{L}^2(\Omega, A, P) \quad \sum_{k \geq 1} \sqrt{\lambda_k(\mathcal{R}_0)} \eta_k(j, m) \phi_k
\]

\[
\|\mathcal{I}^N_{\omega_1, \omega_2}\|_H = \left\|\tilde{X}^N_{\omega_1, \omega_2}\right\|_H^2
\]

\[
= \sum_{k \geq 1} \frac{1}{2\pi \sqrt{N}} \left\{ \sum_{j=1}^{S_1} \sum_{m=1}^{S_2} \langle X_{j,m}, \phi_k \rangle_H \exp \left(-i(j\omega_1 + m\omega_2)\right) \right\}^2
\]

\[
= \sum_{k \geq 1} \lambda_k(\mathcal{R}_0) \left\{ \frac{1}{2\pi \sqrt{N}} \sum_{j=1}^{S_1} \sum_{m=1}^{S_2} \eta_k(j, m) \exp \left(-i(j\omega_1 + m\omega_2)\right) \right\}^2
\]

\[
= \sum_{k \geq 1} \lambda_k(\mathcal{R}_0) \mathcal{I}^{N, \eta_k}_{\omega_1, \omega_2},
\]

where, for each \((i,j) \in \mathbb{Z}^2\), in the Gaussian case, \(\{\eta_k(i,j)\}_{k \geq 1}\) constitutes a sequence of independent standard Gaussian random variables. Note that, in the non-Gaussian case, they are uncorrelated. Here, the separable Hilbert space \(\tilde{H}\) is such that \(P \left[\tilde{X}^N_{\omega_1, \omega_2} \in \tilde{H}\right] = 1\). For each \(k \geq 1\), \(\mathcal{I}^{N, \eta_k}_{\omega_1, \omega_2}\) denotes the periodogram of the real-valued spatial process \(\eta_k\), based on a sample of size \(N\). Since \(\{X(i,j), (i,j) \in \mathbb{Z}^2\}\) satisfies equation (24), we obtain in (30), for each \(\omega_1, \omega_2 \in [-\pi, \pi]\),

\[
\lim_{N \to \infty} \mathbb{E} \left[\mathcal{I}^{N, \eta_k}_{\omega_1, \omega_2}\right] = \frac{1}{(2\pi)^2} \frac{1}{1 - \sum_{n=1}^{S^2} L_n(\phi_k)(\phi_k) \exp(-i\omega_n) - L_\lambda(\phi_k)(\phi_k) \exp(-i(\omega_1 + \omega_2))}
\]

\[
= \frac{\mathcal{F}_{\omega_1, \omega_2}(\phi_k)(\phi_k)}{\lambda_k(\mathcal{R}_0)}.
\]

(31)
Theorem 2. Let $X = \{X_{i,j}, (i, j) \in \mathbb{Z}^2\}$ be a SARH(1) process, satisfying equation (24), with $\{\varepsilon_{i,j}, (i, j) \in \mathbb{Z}^2\}$ being a Hilbert-valued spatial Gaussian strong white noise. Under Assumption A1, and since

$$\sum_{[(i_1,j_1),\ldots,(i_{k-1},j_{k-1})] \in \mathbb{Z}^{2(k-1)}} \|\text{cum}(X_{i_1,j_1}, \ldots, X_{i_{k-1},j_{k-1}}, X_0)\|_2 < \infty, \quad k \geq 2,$$

for each $(\omega_1, \omega_2) \in [0, \pi]^2$, the following limit holds:

$$\lim_{N \to \infty} E \left\| I_{\omega_1,\omega_2}^N - F_{\omega_1,\omega_2} \right\|_{l^1(\tilde{H})} = 0,$$

where, as before, $\| \cdot \|_{l^1(\tilde{H})}$ denotes the nuclear or Schatten norm.

Under Assumption A1 and (32), the conditions assumed in Theorem 2.2 in [44] are satisfied. Hence, for $N$ sufficiently large,

$$\left\| I_{\omega_1,\omega_2}^N \right\|_{\tilde{H} \otimes \tilde{H}} = \left\| T_{\omega_1,\omega_2}^N \right\|_{l^1(\tilde{H})}.$$

In particular, under those conditions, for $N = \text{Card}(\mathcal{N}_1 \times \mathcal{N}_2)$, sufficiently large, with $\mathcal{N}_1 \times \mathcal{N}_2 \subset \mathbb{Z}^2$,

$$E \left[ \left\| I_{\omega_1,\omega_2}^N - F_{\omega_1,\omega_2} \right\|_{l^1(\tilde{H})} \right]$$

$$= \frac{1}{(2\pi)^2} \sum_{k \geq 1} E \left[ \sum_{(j,m) \in \mathcal{N}_1 \times \mathcal{N}_2} \sum_{(j',m') \in \mathcal{N}_1 \times \mathcal{N}_2} \frac{X_{j,m} \otimes X_{j',m'}(\phi_k \otimes \phi_k)}{N} \right.$$

$$\times \exp \left( -i(j - j')\omega_1 + (m - m')\omega_2 \right)$$

$$- \sum_{(j,m) \in \mathbb{Z}^2} \left( \mathcal{R}_{j,m}(\phi_k), \phi_k \right)_H \exp \left( -i(j\omega_1 + m\omega_2) \right)$$

$$= \sum_{k \geq 1} E \left[ \sum_{(j,m) \in \mathcal{N}_1 \times \mathcal{N}_2} \frac{X_{j,m}(\phi_k)}{2\pi \sqrt{N}} \exp \left( -i(j\omega_1 + m\omega_2) \right) \right]^2$$

$$- \sum_{(j,m) \in \mathbb{Z}^2} \mathcal{R}_{j,m}(\phi_k)(\phi_k) \exp \left( -i(j\omega_1 + m\omega_2) \right).$$
\[
\begin{align*}
&= \sum_{k \geq 1} \lambda_k(R_0) E \left| T_{\omega_1,\omega_2}^{N,\eta_k} - \frac{F_{\omega_1,\omega_2}(\phi_k)(\phi_k)}{\lambda_k(R_0)} \right| \\
&\leq \sum_{k \geq 1} \lambda_k(R_0) \left[ E |T_{\omega_1,\omega_2}^{N,\eta_k}| + E \left| \frac{F_{\omega_1,\omega_2}(\phi_k)(\phi_k)}{\lambda_k(R_0)} \right| \right] \\
&\leq (M + 1) \sum_{k \geq 1} \lambda_k(R_0) \frac{F_{\omega_1,\omega_2}(\phi_k)(\phi_k)}{\lambda_k(R_0)} \\
&= (M + 1) \sum_{k \geq 1} F_{\omega_1,\omega_2}(\phi_k)(\phi_k) < \infty, \quad (34)
\end{align*}
\]

for certain \( M > 0 \), since \( F_{\omega_1,\omega_2} \) is a non-negative self-adjoint nuclear (or trace) operator. From equation (34), the sequence

\[ \left\{ \lambda_k(R_0) E \left| T_{\omega_1,\omega_2}^{N,\eta_k} - \frac{F_{\omega_1,\omega_2}(\phi_k)(\phi_k)}{\lambda_k(R_0)} \right| \right\}_{k \geq 1} \]

is absolutely upper bounded by the sequence \( \{(M + 1)F_{\omega_1,\omega_2}(\phi_k)(\phi_k)\}_{k \geq 1} \), which is absolutely summable. Thus, we can apply the Dominated Convergence Theorem, and under the conditions assumed in Theorem 2.2 in [44], from equation (31), we obtain

\[
\lim_{N \to \infty} E \left[ \left\| T_{\omega_1,\omega_2}^N - F_{\omega_1,\omega_2} \right\|_{\mu(\tilde{H})} \right] \\
= \frac{1}{(2\pi)^2} \sum_{k \geq 1} \lim_{N \to \infty} E \left[ \sum_{(j,m) \in N_1 \times N_2} \sum_{(j',m') \in N_1 \times N_2} X_{j,m} \otimes X_{j',m'}(\phi_k \otimes \phi_k) \right] \\
&\times \exp \left( -i(j - j')\omega_1 + (m - m')\omega_2 \right) \\
&- \sum_{(j,m) \in \mathbb{Z}^2} \langle R_{j,m}(\phi_k), \phi_k \rangle_{\mathcal{H}} \exp \left( -i(j\omega_1 + m\omega_2) \right) \\
&= \sum_{k \geq 1} \lambda_k(R_0) \lim_{N \to \infty} E \left| T_{\omega_1,\omega_2}^{N,\eta_k} - \frac{F_{\omega_1,\omega_2}(\phi_k)(\phi_k)}{\lambda_k(R_0)} \right| = 0. \quad (35)
\]

### 3.1. Minimum contrast componentwise parameter estimation

In the following, we write, for each \( p \geq 1 \), the vector \( \theta_p \) as

\[
\theta_p = (\theta_{p1}, \theta_{p2}, \theta_{p3}) = (L_1(\phi_p)(\phi_p), L_2(\phi_p)(\phi_p), L_3(\phi_p)(\phi_p)).
\]
Thus, for each \( p \geq 1 \), we compute the minimum contrast estimator

\[
\hat{\theta}_{N,p} = (\hat{\theta}_{N,p1}, \hat{\theta}_{N,p2}, \hat{\theta}_{N,p3})
\]

of the parameter vector \( \theta_{p} = (\theta_{p1}, \theta_{p2}, \theta_{p3}) \), from the observations

\[
\{(X_{i,j}, \phi_{p})_{H}, (i, j) \in \mathbb{Z}^{2}\}.
\]

Denote, for each \( p \geq 1 \),

\[
f_{p}(\varpi) = f_{p}(\omega_{1}, \omega_{2}) = F_{\omega_{1},\omega_{2}}(\phi_{p})(\phi_{p}), \quad \varpi = (\omega_{1}, \omega_{2}) \in [-\pi, \pi]^{2}.
\]

Define, for each \( p \geq 1 \), the projected contrast field by

\[
U_{p}(\theta_{p}) := -\int_{[-\pi,\pi]^{2}} f_{p}(\varpi, \theta_{0,p})\eta_{p}(\varpi) \log \Psi_{p}(\varpi, \theta_{p}) d\varpi,
\]

where \( \eta_{p}(\varpi), \varpi \in [-\pi, \pi]^{2} \), must be a nonnegative symmetric function such that \( \eta_{p}(\varpi)f_{p}(\varpi, \theta_{p}) \in L_{1}([-\pi, \pi]^{2}) \), for all \( \theta_{p} \in \Theta \). Here, for every \( p \geq 1 \),

\[
\sigma^{2}(\theta_{p}) = \int_{[-\pi,\pi]^{2}} f_{p}(\varpi, \theta_{p})\eta_{p}(\varpi) d\varpi, \quad f_{p}(\varpi, \theta_{p}) = \sigma^{2}(\theta_{p})\Psi_{p}(\varpi, \theta_{p}),
\]

with, for all \( \theta_{p} \in \Theta \),

\[
\int_{[-\pi,\pi]^{2}} \Psi_{p}(\varpi, \theta_{p})\eta_{p}(\varpi) d\varpi = 1
\]

(see, e.g., [2]). Note that, for each \( p \geq 1 \), \( \eta_{p} \) must satisfy suitable regularity and asymptotic conditions that ensure, under additional assumptions (see Theorem 2.1 in [2]) that

\[
K_{p}(\theta_{0,p}, \theta_{p}) := \int_{[-\pi,\pi]^{2}} f_{p}(\varpi, \theta_{0,p})\eta_{p}(\varpi) \log \frac{\Psi_{p}(\varpi, \theta_{0,p})}{\Psi_{p}(\varpi, \theta_{p})} d\varpi
\]

is the contrast function for the empirical contrast field

\[
\hat{U}_{N,p}(\theta_{p}) := -\int_{[-\pi,\pi]^{2}} I_{N,p}(\varpi)\eta_{p}(\varpi) \log \Psi_{p}(\varpi, \theta_{p}) d\varpi,
\]
where \( I_{N,p}(\varpi) = \lambda_p(\mathcal{R}_0)I_{\varpi}^{N,\eta_p} \) is the periodogram, based on the observations \( \{ (X_{i,j}, \phi_p)_{\mathcal{H}} , i = 1, \ldots, S_1, j = 1, \ldots, S_2 \} \), with \( I_{\varpi}^{N,\eta_p} \) as introduced after equation (30). Then \( I_{N,p}(\varpi) \) is given by

\[
I_{N,p}(\varpi) := \frac{1}{(2\pi)^2 N} \left| \sum_{s_1=1}^{S_1} \sum_{s_2=1}^{S_2} \exp(-i(s_1 \varpi_1 + s_2 \varpi_2)) \langle X_{s_1,s_2}, \phi_p \rangle_{\mathcal{H}} \right|^2, \quad p \geq 1.
\]

(39)

The contrast function satisfies \( K_p(\theta_{0,p}, \theta_p) \geq 0 \), for all \( \theta_p \in \Theta \), and has a unique minimum at \( \theta_p = \theta_{0,p} \), the true parameter value, for every \( p \geq 1 \). Furthermore, under the above-referred suitable conditions, for each \( p \geq 1 \),

\[
\hat{U}_{N,p}(\theta_{0,p}) - \hat{U}_{N,p}(\theta_p) \to_{P_0} K_p(\theta_{0,p}, \theta_p), \quad N \to \infty, \quad \forall \theta_p \in \Theta,
\]

(40)

where \( P_0 \) denotes the probability distribution with density function \( f_p(\varpi, \theta_{0,p}) \).

For \( p \geq 1 \), from equations (36)–(40), the minimum contrast estimator \( \hat{\theta}_{N,p} \) of \( \theta_p \) is given by

\[
\hat{\theta}_{N,p} = \arg \min_{\theta_p \in \Theta} \hat{U}_{N,p}(\theta_p).
\]

(41)

A similar procedure can be applied to estimate by minimum contrast the parameters \( \theta_{kpl} = L_l(\phi_k)(\phi_p), k \neq p, k, p \geq 1, l = 1, 2, 3 \), i.e., to estimate the parameter vectors \( \theta_{kp} = (\theta_{kp1}, \theta_{kp2}, \theta_{kp3}), k \neq p, k, p \geq 1 \). Note that the convergence, with respect to the mean nuclear norm in equation (35), also implies the convergence, with respect to the mean Hilbert-Schmidt norm. Thus, for each \( k, p \geq 1, k \neq p \), we arrive to the identity

\[
\hat{\theta}_{N,kp} = \arg \min_{\theta_{kp} \in \Theta} \hat{U}_{N,kp}(\theta_{kp}).
\]

(42)

The respective minimum contrast componentwise estimators \( \hat{L}^N_l, l = 1, 2, 3 \), of \( L_l, l = 1, 2, 3 \), based on a functional sample of size \( N \), are then defined as, for all \( f, g \in \mathcal{H} \),

\[
\hat{L}^N_l(f)(g) = \sum_{k,p,k \neq p} \hat{\theta}_{N,kpl} \langle g, \phi_k \rangle_{\mathcal{H}} \langle f, \phi_p \rangle_{\mathcal{H}} + \sum_{p=1}^{\infty} \hat{\theta}_{N,ppl} \langle g, \phi_p \rangle_{\mathcal{H}} \langle f, \phi_p \rangle_{\mathcal{H}}, \quad (43)
\]

where, for \( p \geq 1, \hat{\theta}_{N,ppl} \) denotes the minimum contrast estimator of \( L_l(\phi_p)(\phi_p) \), and, for \( k, p \geq 1, k \neq p, \hat{\theta}_{N,kpl} \) is the minimum contrast estimator of
L_l(\phi_k)(\phi_p), l = 1, 2, 3. From Theorem [2], applying
Dominated Convergence Theorem, the weak–consistency of \hat{L}_l^N, l = 1, 2, 3 follows, under the assumption that \( L_l, l = 1, 2, 3 \), are continuous, in the norm \( \| \cdot \|_{\mathcal{L}(\mathcal{H})} \) of the space of bounded linear operators on \( \mathcal{H} \). Note that this result also holds in the norm \( \| \cdot \|_{\mathcal{S}(\mathcal{H})} \), of Hilbert-Schmidt operators on \( \mathcal{H} \), under the assumption that \( L_l, l = 1, 2, 3 \), are Hilbert-Schmidt operators. In terms of the periodogram operator, the infinite-dimensional empirical contrast field is given by

\[
\hat{U}_N(\theta, \cdot, \cdot) := -\int_{[-\pi, \pi]^2} T^N_{\omega\omega}(\cdot, \cdot) \eta_{\omega\omega}(\cdot, \cdot) \log \Psi_{\omega\omega}(\cdot, \cdot) d\omega,
\]

where, for every \( p \geq 1 \), \( \eta_p = \langle \eta(\phi_p), \phi_p \rangle_{\mathcal{H}} \), and \( \Psi_p = \langle \Psi(\phi_p), \phi_p \rangle_{\mathcal{H}} \), and for
\( k, p \geq 1, k \neq p \), \( \eta_{kp} = \langle \eta(\phi_k), \phi_p \rangle_{\mathcal{H}} \), and \( \Psi_{kp} = \langle \Psi(\phi_k), \phi_p \rangle_{\mathcal{H}} \).

Thus, the componentwise infinite–dimensional vector solution \( \theta \), minimizing (44), constructed from equations (41)–(42), displays a minimum operator norm (in \( [\mathcal{S}(\mathcal{H})]^3 \), when \( L_l, l = 1, 2, 3 \), are Hilbert-Schmidt operators on \( \mathcal{H} \), or in \( [\mathcal{L}(\mathcal{H})]^3 \), when \( L_l, l = 1, 2, 3 \), are continuous operators on \( \mathcal{H} \)). That is,

\[
\hat{\theta}_N(\cdot, \cdot, \cdot) = \arg \min_{\theta \in \Theta; \| \theta \|_{[\mathcal{L}(\mathcal{H})]^3/[\mathcal{S}(\mathcal{H})]^3}} \hat{U}_N(\theta, \cdot, \cdot).
\]

Hence, the associated spatial functional plug-in predictor of \( X \) is defined as

\[
\hat{X}_{i,j}^N = \hat{L}_1^N X_{i-1,j} + \hat{L}_2^N X_{i,j-1} + \hat{L}_3^N X_{i-1,j-1}, \quad \forall (i, j) \in \mathcal{N}_B, \ B \in \mathcal{B}^2,
\]

where \( X_{i-k,j} \), \( X_{i,j-k} \) and \( X_{i-k,j-k} \), \( k \in \mathcal{N} \subset \mathbb{N} \), and \( \{ (i-k, j), (i, j-k), (i-k, j-k) \}, \ k \in \mathcal{N} \subset \mathbb{N} \} \subset \mathcal{N}_A \), with \( \text{Card}(\mathcal{N}_A) = N = S_1 \times S_2 \), are the functional observations over the nodes of the spatial regular grid \( \mathcal{N}_A \), covering the observed bounded spatial region \( A \in \mathcal{B}^2 \) (in the case of irregularly spatial distributed data, the original high-dimensional observations are spatially interpolated to the nodes of \( \mathcal{N}_A \)). Here, \( \mathcal{N}_B \) is the spatial regular grid covering the bounded spatial region of interest \( B \in \mathcal{B}^2 \). Thus,

\[
\hat{\Lambda}_{i,j}^N = \exp \left( \hat{X}_{i,j}^N \right), \quad \forall (i, j) \in \mathcal{N}_B.
\]

Under the conditions assumed in Theorem [11] the spatial functional plug-in predictor (23) of \( \mathcal{C}_{A \cup B}(\cdot) \) from the functional observation of \( \mathcal{C}_A(\cdot) \), and from \( \hat{X}_{i,j}^N \) in (46) is computed as follows:

\[
\mathcal{C}_A(\cdot) + \int_{A \cup B \setminus A} \hat{\Lambda}_{i,j}^N(\omega, \cdot) d\mu = \mathcal{C}_A(\cdot) + \sum_{(i,j) \in \mathcal{N}_{A \cup B \setminus A}} \exp \left( \hat{X}_{i,j}^N(\omega, \cdot) \right).
\]

20
4. Real-data example

The distribution of mortality patterns over a region are usually represented in maps. For estimating mortality risks, the statistical methodology used in the epidemiological literature in disease mapping is mainly based on extensions of the well-known autoregressive model studied in [8]. P-spline models are also being used for smoothing risks in space–time disease mapping (see, for instance, [62] and [61]). A comparative study between spatial Conditional Autoregressive and P-spline modeling in disease mapping is presented in [31].

In this section, a SARH(1) process based minimum contrast estimation methodology is adopted in the study of the distribution of respiratory disease mortality patterns over the Spanish regions, under the umbrella of the introduced framework of spatial infinite–dimensional log-Gaussian Cox processes. In the functional parameter estimation and plug-in prediction of the spatial log–intensity, a first–order Taylor expansion is applied, in the approximation of the logarithmic transformation of a realization of the introduced infinite–dimensional counting process, normalized by the number of expected cases. This expansion is centered at a given realization of the spatial functional log–intensity. A functional mixed-effect model approach can then be adopted to apply FDA techniques (see, e.g., [54]; [58]).

In our case, considering the observed number of respiratory disease deaths divided by the expected number of cases, after computing their logarithmic transformation, temporal interpolation and smoothing is applied to obtain our functional data set. The proposed SARH(1) based minimum contrast functional estimation methodology is implemented, after spatial interpolation to a regular grid of the observed spatial functional data over the Spanish regions. Detrending is previously applied to fit model (24) to our functional data set.

As commented before, we have worked under a constant functional-valued intensity assumption within square quadrants, in the spirit of the conditional autoregression strategy proposed in the pioneering work of [7] (see also [46]). The second step applied, in our implementation of the proposed estimation procedure, in this section, consists of the spatial functional plug-in prediction of the respiratory disease deaths, over a continuous time interval, by suitable integration of the estimated functional values of the spatial Gaussian log–intensity.

The Spanish National Statistical Institute provided the data on observed
and expected cases of respiratory disease deaths, consisting of 432 monthly records, in the period 1980–2015, collected at 48 Spanish provinces in the Iberian Peninsula. Temporal smoothing, based on robust regression, has been applied to the interpolated 1725 times at each Spanish region. Interpolation to a $20 \times 20$ spatial regular grid, based on the Inverse Distance Weighting Interpolation Method, is then performed. The real separable Hilbert space $\mathcal{H} = L^2([a, b])$, with $a = 1980$ and $b = 2015$, has been considered. After detrending the data, the spectral decomposition of the empirical autocovariance operator has been computed, to obtain the orthonormal system of eigenvectors for projection. The periodogram operator (27) is computed as well, from such functional data set. The projections of this operator on the tensorial product of the empirical eigenvectors lead to the desired minimum contrast estimates (41) and (42), from equations (36)–(38).

To assess the goodness-of-fit of the adopted spatial functional statistical modelling approach, the leave-one-out cross-validation technique is implemented. Specifically, by leaving aside the curves observed at the nodes located in the province defining the region of interest (the validation functional data set), equations (11)–(12) are computed from the remaining functional observations at the nodes located at the rest of the Spanish provinces (the training functional data set). The functional parameter estimation and prediction procedure were repeated 48 times. Thus, the leave-one-out cross-validation functional error is obtained as the mean of the absolute functional errors computed at each one of the 48 iterations. The annually averaged point values of the computed leave-one-out cross-validation functional error can be found in Table 1.

The annually averaged observed number of deaths, and the corresponding annually averaged functional estimates, at the 48 Spanish provinces, for each one of the years, are shown in the panels displayed in Figures 1 and 2.

Alternatively, the SARH(1) model in equation (1) of the Supplementary Material is also fitted to this data set. Figure 3 displays the observed risk curves (blue lines), and the functional estimated risk curves (pink lines), at each one of the Spanish provinces in the period analysed (since January, 1980, to December, 2015). This figure illustrates the relevance of the information stored on the diagonal projections of the periodogram operator, when the autocorrelation operators $L_l$, $l = 1, 2, 3$, in model (24) are sufficiently regular (see Theorem 2 and the Supplementary Material).
Figure 1: Annually averaged observed number of respiratory disease deaths at the 48 Spanish provinces in the period 1980–2015.
Figure 2: Annually averaged functional estimates of the number of respiratory disease deaths at the 48 Spanish provinces in the period 1980–2015.
Figure 3: Model (1) in the Supplementary Material is fitted. Here, the observed (blue lines) and estimated (pink lines) risk curves, since January, 1980, to December, 2015, are displayed.
| Year | ALOOCVE | Year | ALOOCVE | Year | ALOOCVE |
|------|---------|------|---------|------|---------|
| 1980 | 0.0247  | 1992 | 0.0118  | 2004 | 0.0132  |
| 1981 | 0.0144  | 1993 | 0.0130  | 2005 | 0.0117  |
| 1982 | 0.0112  | 1994 | 0.0163  | 2006 | 0.0135  |
| 1983 | 0.0125  | 1995 | 0.0159  | 2007 | 0.0140  |
| 1984 | 0.0144  | 1996 | 0.0111  | 2008 | 0.0118  |
| 1985 | 0.0122  | 1997 | 0.0099  | 2009 | 0.0113  |
| 1986 | 0.0126  | 1998 | 0.0108  | 2010 | 0.0143  |
| 1987 | 0.0155  | 1999 | 0.0141  | 2011 | 0.0131  |
| 1988 | 0.0161  | 2000 | 0.0167  | 2012 | 0.0122  |
| 1989 | 0.0144  | 2001 | 0.0161  | 2013 | 0.0115  |
| 1990 | 0.0125  | 2002 | 0.0143  | 2014 | 0.0145  |
| 1991 | 0.0118  | 2003 | 0.0140  | 2015 | 0.0221  |

Table 1: ALOOCVE. Annually averaged leave-one-out cross-validation functional error.

5. Conclusions

This paper introduces a new class of spatial log-Gaussian Cox processes in infinite dimensions. Its $n$th-order product density, characterising the $n$th-order moments of the functional random intensity, is defined from the Laplace transform of a multivariate infinite-dimensional Gaussian distribution. Under the conditions assumed, including the ones derived in Theorem 1, the Laplace transform functional of the conditional distribution of the introduced log-Gaussian Cox process, given a realisation of the random intensity, is obtained, defining an infinite dimensional version of the Laplace transform of a spatial dependent non–homogeneous Poisson measure (see Sections 2.2–2.4).

Recent developments in computer sciences have made possible the implementation of feasible probabilistic functional prediction techniques for the functional statistical analysis of high-dimensional data (see [3]; [1]; [51]; [52], among others). The spatial functional log-Gaussian Cox process setting adopted here covers, in particular, the second-order analysis of spatio-temporal log-Gaussian Cox processes (see, for instance, [13]; [57]; [25]; [24]; [30]). Specifically, the proposed SARH(1) minimum contrast estimation methodology, based on the periodogram operator (see also Theorem 2), allows to compute an approximation of the corresponding least-squares functional predictor of the introduced infinite-dimensional log-Gaussian Cox process.
cess in space.

Note that the approach presented is also useful for the analysis of high-dimensional data in additional fields of application, within the areas of Ecological and Environmental sciences. That is the case of control and risk assessment of disease (see, e.g., [60]), and wildfire occurrence mapping (see, e.g., [55]), as well as analysis of multi-species in ocean, and forest (see, e.g., [63]). In those cited problems, additionally to the usual spatial functional prediction of time supported curves, landscape elevation, and ocean depth supported curve prediction could also play a key role.

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Supplementary material

Supplementary material includes the simulation study undertaken, where the asymptotic efficiency, normality and weak-consistency properties of the minimum contrast estimators introduced in Section 3.1 are illustrated in a numerical example.

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