Dimensional dependence of the metal-insulator transition

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We study the dependence on the spatial dimensionality of different quantities relevant in the description of the Anderson transition by combining numerical calculations in a $3 \leq d \leq 6$ disordered tight binding model with theoretical arguments. Our results indicate that, in agreement with the one parameter scaling theory, the upper critical dimension for localization is infinity. Typical properties of the spectral correlations at the Anderson transition such as level repulsion or a linear number variance are still present in higher dimensions though eigenvalues correlations get weaker as the dimensionality of the space increases. It is argued that such a critical behavior can be traced back to the exponential decay of the two-level correlation function in a certain range of eigenvalue separations. We also discuss to what extent different effective random matrix models proposed in the literature to describe the Anderson transition provide an accurate picture of this phenomenon. Finally, we study the effect of a random flux in our results.

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I. INTRODUCTION

After almost fifty years of the landmark paper by Anderson about localization, the study of the properties of a quantum particle in a random potential is still one of the central problems of modern condensed matter physics.

In the early days of localization theory, research was largely focused on the determination of the critical disorder at which the metal-insulator transition—also referred as the Anderson transition—occurs as a function of the connectivity of the lattice. In the original Anderson’s paper this was achieved by looking at the limits of applicability of a localization theory provided with a similar answer. The self-consistent method is only exact in the case of Cayley tree but it is believed to be accurate if the spatial dimensionality is large enough. We note that in the quantum chaotic case the metal-insulator transition is induced by increasing the hopping amplitude of an initially localized particle.

In the seventies the application of ideas and techniques of the theory of phase transitions such as scaling and the renormalization group opened new ways to tackle the localization problem especially in low dimensional systems. These progress led eventually to the proposal of the one parameter scaling theory which, despite being still under debate, has become the “standard” theory of localization. In the one parameter scaling theory, localization in a given disordered sample is described by the dimensionless conductance $g$. This quantity is defined either as the sensitivity of a given quantum spectrum to a change of boundary conditions in units of the mean level spacing $\Delta$ or as $g = E_c/\Delta$ where $E_c$, the Thouless energy, is an energy scale related to the classical diffusion time to cross the sample. The dimensionless conductance $g$ is sensitive to localization effects. In a metal (insulator), it increases (decreases) monotonically with the system size, $L$.

Under the assumption that the dimensionless conductance is only a function of the system size and by using simple scaling arguments, the one parameter scaling theory predicts that the metal-insulator transition is characterized by a scale invariant dimensionless conductance $g = g_c$. The lowest dimension in which the metal-insulator transition occurs is $d > 2$. In two and lower dimensions destructive interference caused by backscattering produces exponential localization of the eigenstates in real space for any amount of disorder in the limit $L \to \infty$. In this picture, the Anderson transition is considered as a standard second order transition with critical exponents $s, \nu$ that control how the conductivity $\sigma \sim |W - W_c|^s$ vanishes or the localization length $\xi \sim |W_c - W|^{-\nu}$ diverges as the critical disorder $W_c$ is approached.

In $d = 2 + \epsilon$ ($\epsilon \ll 1$) the transition occurs in the weak disorder region and consequently an analytical treatment is possible. Diagrammatic perturbation theory and field theory techniques predict that $\nu \sim 1/\epsilon$ and $W_c \sim \epsilon$. By contrast, the critical exponent associated to the Cayley tree, which should be close to that of a disordered conductor in $d > 2$ dimensions, is $\nu = 1/2$. In the context of second order phase transitions this value corresponds with the upper critical dimension $d_u$, namely, for $d \geq d_u$ fluctuations are irrelevant and the mean field approximation becomes exact. For the localization problem different values $d_u = 4, 6, 8, \infty$ of the upper critical dimension have been reported. The results of this paper discard $d_u = 4, 6$ and indicate that $d_u \to \infty$ is the upper critical dimension. However we would like to point that the exact significance of the upper critical dimension for localization is unclear. It is not known what fluctuations are suppressed at the upper critical dimension and to what extent spectral or transport properties at criticality are affected.

The Anderson transition in a disordered conductor is a consequence of a non trivial interplay between quantum destructive interference effects and quantum tunneling. In low dimensions, $d \sim 2$, weak quantum destructive interfer-
ence effects induce the Anderson transition. Analytical results are available based on perturbation theory around the metallic state. In high dimensions, $d > 2$, quantum tunneling is dominant and the locator expansion or the self-consistent formalisms can be utilized to describe the transition. We note that in these papers corrections due to interference of different paths are neglected.

The progress in numerical calculations during the last twenty years has increased dramatically our knowledge of the metal-insulator transition specially in intermediate dimensions such as $d = 3, 4$ for which a rigorous analytical treatment is not available. Below we cite a few of its most relevant results.

It was verified that for a disorder strength below the critical one, the system has a mobility edge at a certain energy which separates localized from delocalized states. Its position moves away from the band center as the disorder is decreased. Delocalized eigenstates, typical of a metal, are extended through the sample and the level statistics agree with the random matrix predictions for the appropriate symmetry. The spectral correlations at the Anderson transition, usually referred to as critical statistics, are scale invariant and intermediate between the prediction for a metal and for an insulator. By scale invariant we mean any spectral correlator utilized to describe the spectral properties of the disordered Hamiltonian does not depend on the system size.

Eigenfunctions at the Anderson transition are multifractal (for a review see) namely, their moments present an anomalous scaling, $\langle \psi(r) \psi(r') \rangle^{2q} \sim L^{-D_q(q-1)}$ with respect to the sample size $L$, where $D_q$ is a set of exponents describing the Anderson transition.

The main features of the Anderson transition only depend on the dimensionality of the space and the universality class, namely, the presence or not of a magnetic field (or other time reversal breaking mechanism) or a spin-orbit interaction. The dependence with the universality class diminishes as the spatial dimension increases. It has also been reported that certain spectral correlators at the Anderson transition are sensitive to different boundary conditions.

All of these numerical findings are compatible with the one parameter scaling theory. The applicability of the $\epsilon$-expansion ($d = 2 + \epsilon$) is by contrast much more restricted. A naive extrapolation to $\epsilon = 1, 2$ yields $\nu_{3D} \sim 1/\epsilon = 1, \nu_{4D} = 1/2$ thus suggesting that the upper critical dimension is four. However numerical calculations show undoubtedly that $\nu_{3D} \sim 3/2$ and $\nu_{4D} \sim 1$. Similarly, up to $d = 4$, the self-consistent theory overestimate by more than a factor two the value of the critical disorder at which the Anderson transition occurs. This suggests none of the theories traditionally utilized to describe the metal-insulator transition can be really extrapolated to the physically relevant case of $d = 3$. In order to make progress in this difficult problem a new basis for the study of the Anderson transition in any dimension is necessary. In this paper we have a more modest goal: a detailed exploration of the dependence of different quantities defining the Anderson transition on the spatial dimensionality.

We propose simple relations that describe how the parameters defining the Anderson transition depend on the dimensionality of the space. It is argued that the upper critical dimension must be infinite. Our results are supported by numerical evidence from a disordered Anderson model in a hyper-cubic lattice in $3 \leq d \leq 6$. This is the first time the Anderson transition in $d = 5, 6$ is investigated numerically in the literature (for some recent results in a small asymmetric lattice in $d = 5$ we refer to Ref. 26).

The organization of the paper is as follows. In section II, we introduce the model to be studied, explain the technical details of the numerical simulation, locate the mobility edge in different dimensions and investigate how the critical exponent and the critical disorder depend on the spatial dimensionality. In Sec. III, spectral correlation at the Anderson transition are investigated, three different regions of the two level correlation function are distinguished, we also study the dependence of the slope of the number variance and the asymptotic decay of the level spacing distribution with the spatial dimensionality. It is also discussed the range of validity of certain phenomenological models commonly used in the literature to describe the spectral correlations at the Anderson transition. Finally, we examine the effect of a magnetic flux in our results.

II. CRITICAL DISORDER, CRITICAL EXPONENTS AND UPPER CRITICAL DIMENSION

In this section we determine the critical disorder and critical exponents for different dimensions and then discuss their dependence with the spatial dimensionality.

A. The model: Technical details

Our starting point is the standard tight-binding Anderson model on a hyper-cubic $L^d$ lattice with $d = 3, \ldots, 6$

$$H = \sum_i \epsilon_i a_i^\dagger a_i + \sum_{i,j} t_{ij} a_i^\dagger a_j ,$$

where the operator $a_i (a_i^\dagger)$ destroys (creates) an electron at the $i$th site of the lattice and $t_{ij}$ is the hopping integral between sites $i$ and $j$ which is non zero only for nearest neighbors. In the following we take $t_{ij} = 1$ for $i,j$ and the lattice constant equal to unity, which sets the energy and length units, respectively. The uncorrelated random energies $\epsilon_i$ are distributed with constant probability within the interval $(-W/2, W/2)$, where $W$ denotes the strength of the disorder and hard-wall boundary conditions are imposed in all directions.

In order to proceed we compute eigenvalues of the Hamiltonian Eq. (1) for different volumes and disorders by using techniques for large sparse matrices, in particular a Lanczos tridiagonalization without reorthogonalization method. We restrict ourselves to a small energy window $(-2, 2)$ around the center of the band. Calculations have been carried out in samples of sizes up to $L = 30$ for $d = 3, 12 (d = 4), 10 (d = 5)$ and $7 (d = 6)$. The number of random realizations is such that for a given triad of $\{d, L, W\} \frac{\text{number of eigenvalues obtained}}{\text{at least } 3 \times 10^5}$. In order to study the level
Our first task is to find out the critical disorder $W_c$ and the critical exponent $\nu$ for different dimensions in the small spectral window $(-2, 2)$ around the origin. In order to proceed we determine the location of the mobility edge close to the band center by using the finite size scaling method.\cite{foot5} First we evaluate a certain spectral correlator for different sizes $L$ and disorder strengths $W$. Then we locate the mobility edge by finding the disorder $W_c$ such that the spectral correlator analyzed becomes size-independent. In our case we investigate the level spacing distribution $P(s)$ (probability of finding two neighboring eigenvalues at a distance $s = (\epsilon_{i+1} - \epsilon_i)/\Delta$, with $\Delta$ being the local mean level spacing). The scaling behavior of $P(s)$ is examined through the following function of its variance:\cite{foot5}

$$\eta(L, W) = [\text{var}(s) - \text{var}_{WD}]/[\text{var}_P - \text{var}_{WD}],$$

which describes the relative deviation of $\text{var}(s)$ from the Wigner-Dyson (WD) limit. In Eq. (2) $\text{var}(s) = \langle s^2 \rangle - \langle s \rangle^2$, where $\langle \ldots \rangle$ denotes spectral and ensemble averaging, and $\text{var}_{WD} = 0.286$ and $\text{var}_P = 1$ are the variances of WD and Poisson statistics, respectively. Hence $\eta = 1(0)$ for an insulator (metal). Any other intermediate value of $\eta$ in the thermodynamic limit is an indication of a mobility edge.

In Fig. 1 we plot the $W$ dependence of $\eta$ for different system sizes in $d = 5$ (left panel) and $d = 6$ (right panel). The critical disorder $W = W_c$ signaling the Anderson transition corresponds with the point for which $\eta$ is independent of $L$. For a weaker (stronger) disorder, $\eta$ tends to the metallic (insulator) prediction. This is the first time that an Anderson transition is found in such a high dimensional disordered system. For a precise determination of the critical disorder $W_c$ and the critical exponent $\nu$ we look at the correlation length near $W_c$

$$\xi(W) = \xi_0 |W - W_c|^{-\nu},$$

where $\xi_0$ is a constant. The numerical values of $W_c$ and $\nu$ are obtained by expressing $\eta(L, W) = f[L/\xi(W)]$ and then performing an expansion around the critical point

$$\eta(L, W) = \eta_c + \sum_n C_n (W - W_c)^n L^{n/\nu}.$$ 

In practice, we have truncated the series at $n = 4$. For each dimension ($d = 5, 6$) we have performed a statistical analysis of the data in the windows shown in Fig. 1 with the Levenberg-Marquardt method for nonlinear least-squares models. The most likely fit is determined by minimizing the $\chi^2$ statistics of the fitting function.\cite{foot6} We found the following critical disorders $W_c = 51.4 \pm 0.4$ in $d = 5$ and $W_c = 74.5 \pm 0.7$ in $d = 6$, and the corresponding critical exponents are equal to $\nu = 0.84 \pm 0.06$ and $\nu = 0.78 \pm 0.06$, respectively. A similar analysis for the $d = 3$ and $d = 4$ systems results in $W_c = 15.22 \pm 0.08$ and $\nu = 1.52 \pm 0.06$ for $d = 3$, and $W_c = 29.8 \pm 0.2$ and $\nu = 1.03 \pm 0.07$ for $d = 4$. We note that in the $d = 3$ case, the deviation of $W_c$ from the accepted value $W_c \sim 16.5$ is due to the utilization of rigid boundary conditions. See Fig. 2 for a plot of $W_c$ and $\nu$ as a function of the spatial dimensionality.

In certain limiting cases $W_c$ and $\nu$ are known analytically. For instance, if effects related to interference among different paths are neglected, the standard tight binding Anderson model is effectively defined on a Cayley tree and $\nu = 1/2$. On the other hand if only interference corrections to the metallic limit are considered then $\nu = 1/(d - 2)$\cite{foot7} The former prediction is supposed to be approximately valid for $d \gg 2$ and the latter for $d = 2 + \epsilon$ and $\epsilon \ll 1$. From the above numerical results it is clear that none of these limits is appropriate in the range of intermediate dimensions of interest. Additionally, it is believed\cite{foot7} that corrections to the $\nu = 1/2$ result should go as $\sim 1/d$ since this is the dependence on the spatial dimensionality of the neglected diagrams describing interference effects. Combining these two facts we propose that

$$\nu = \frac{1}{d - 2} + \frac{1}{2},$$

for all dimensions. As is shown in Fig. 2 (right panel), this relation verifies all limiting cases and reproduce the numerical results accurately. According to the above relation, the upper critical dimension for localization is infinite. This result is fully supported by the analysis of the spectral correlations (see next section). Moreover, in a recent paper\cite{foot8} it has been proved rigorously that the level statistics of a disordered system in a Cayley tree ($\nu = 1/2$) is Poisson as for an insulator. As was mentioned previously, the Cayley tree represents to
a $d$-dimensional conductor where all interference effects between different paths are neglected. It is thus supposed to be an accurate description of a disordered conductor only in the limit $d \gg 2$.

On the other hand, if the one parameter scaling theory is valid, quantum diffusion never stops (see next section) for any finite dimension. Level repulsion typical of a metal will be present in any finite dimension so level statistics at criticality can be that of an insulator only in the $d \to \infty$ limit. But this precisely the result for the Cayley tree which corresponds with the upper critical dimension for localization. It is thus clear that the upper critical dimension must be infinity.

A similar analysis can be carried out for the critical disorder $W_c$. The original estimation of Anderson, $W_c = 4K \ln(W_c/2)$ (where $K$ is the connective constant which is a bit less than the number of nearest neighbors minus one) greatly overestimates $W_c$. This is hardly surprising since the Anderson’s calculation involves crude approximations and consequently should be considered as an order of magnitude estimation rather than an accurate prediction. For instance, deviations coming from interference effects are neglected in this scheme. Roughly speaking they tend to reduce $W_c$ by an amount of order $W_c/d$.

In the opposite limit, $d = 2 + \epsilon$, simple perturbation theory yields $W_c \propto d - 2$. The discrepancy observed with the analytical results in the limit of high dimensionality prevent us from proposing an interpolating relation as in the case of the critical exponent. However we have noticed that a much better agreement with the numerical results is achieved if an effective connective constant $K_{\text{eff}} = K/2$ is utilized (solid line in left panel of Fig. 2). Furthermore, the remaining deviation gets smaller as the spatial dimensionality increases thus suggesting that it may be produced by destructive interference effects ($\propto W_c/d$).

### III. LEVEL STATISTICS

In this section we investigate the level statistics at the Anderson transition. We shall mainly focus on its dependence on spatial dimensionality and the exact functional form of the two-level correlation function (TLCF).

#### A. Theoretical analysis of $R_2(s)$

Our starting point is the connected TLCF,

$$R_2(s) = \frac{1}{\langle \rho(\epsilon) \rangle^2} \langle \rho(\epsilon - \omega/2) \rho(\epsilon + \omega/2) \rangle,$$

where $\rho(\epsilon)$ is the density of states at energy $\epsilon$, $\langle \cdot \rangle$ denotes averaging over disorder realizations and $s = \omega/\Delta$ where $\Delta = 1/L^d(\rho(\epsilon))$ in the mean level spacing. Once the spectrum has been unfolded $R_2(s)$ can be simply written as

$$R_2(s) = \delta(s) + \sum_n p(n; s),$$

where $p(n; s)$ is the distribution of distances $s_n$ between $n$ other energy levels and $\delta(s)$ describes self-correlation of levels. In numerical computations we use Eq. (7) since it gives much more accurate results than Eq. (6).

According to the one parameter scaling theory, the spectral properties depend on the dimensionless conductance $g$ which is a function of the system size $L$ only. In a metal $g \to \infty$ for $L \to \infty$, the Hamiltonian can be accurately approximated by a random matrix with the appropriate symmetry and Wigner-Dyson statistics applies. For instance, for broken time reversal invariance, $R_2(s) = \delta(s) + 1 - \sin^2(\pi s)/2\pi^2$. In an insulator, eigenvalues uncorrelated, Poisson statistics applies and $R_2(s) = \delta(s)$.

Right at the Anderson transition, the dimensionless conductance $g = g_c$ is size independent and level statistics are supposed to be universal and intermediate between Wigner-Dyson and Poisson statistics. Unfortunately there are few analytical results for the TLCF at criticality. In the $d \geq 2$ regime, $g_c \sim 1/(d-2) \gg 1$. The TLCF, can only evaluated explicitly in the limit $d = 2$ and $g \gg 1$ where $R_2(s) \sim s$ (time-reversal), for $s \ll g_c$, as for a metal, and $R_2(s) \sim e^{-\Delta s/g_c}$ for $s \gg g_c$, with $\Delta$ being a factor of order unity. The Anderson transition is thus characterized by level repulsion combined with an exponential decay of the TLCF.

In higher dimensions the exact form of the TLCF is not known. However, we note that the scale invariance of the spectral correlations at the Anderson transition restricts the decay of the TLCF in the $s \gg g_c$ region to be power-law or exponential. Our numerical results (see Fig. 3) for $d \geq 3$ support also this picture.

The limit of long times and small energy differences $s \ll g_c$, $s \ll g_c$ is well understood in high dimensions as well. Level repulsion of neighboring eigenvalues $R_2(s) \propto s$, typical of a metal, should be a generic feature in any dimension. According to the one parameter scaling theory, the averaged moments of the particle position at the Anderson transition increase asymptotically $t \to \infty$ as $\langle r(t)^{2m} \rangle \sim t^{2m/d}$ where $m$ is a positive integer. As the spatial dimensionality $d$ increases the diffusion is slowed down but it never stops even
for long times. This is an indication that the spectral correlations for sufficiently small energy intervals are similar to those of a metal and, as a consequence, $R_2(s) \sim s$ for $s \ll g_c$ (see Fig. 3).

Finite size effects modify the TLCF in the critical region. In any finite system at criticality the localization length $\xi \propto |E_c - E|^{-\nu}$ ($E_c$ is the location in energies of the mobility edge) is finite and the dimensionless conductance is not. Strictly speaking, scale invariant, $g(L_\xi) = g_c [1 + (L_\xi/L)^{1/\nu}]$ where $L_\xi$ is the localization length for a given $E \sim E_c$. As a consequence, the TLCF develops a power-law tail, $R_2^{\text{tail}}(s) \propto s^{-\gamma}$ with $\gamma = 1 - 1/(\nu d)$ for $s > \Delta_c/\Delta$, where $\Delta_c$ is the mean level spacing in a localization volume $\xi^d$. This tail is not related with the properties of the critical point but rather with how the system approaches to it. In $d = 2 + \epsilon$, $\nu = 1/\epsilon \gg 1$ and $R_2^{\text{tail}}(s) \sim 1/s$.

As a summary, we can distinguish three different regions in the critical TLCF. For $s \ll g_c$, $R_2(s) \propto s$, for $s \gg g_c$, $R_2(s)$ decays exponentially. For $s > \Delta_c/\Delta$ decays as power-law due to finite size effects. In order to observe the exponential decay related to the critical point our system size must be such that $g_c > \Delta_c/\Delta$. Finally we note that the exact dependence of $g_c$ on the spatial dimensionality is not known. We are only aware of the prediction of Vollhardt and Wolke by using a self-consistent diagrammatic theory valid for $4 > d > 2$, $g_c(d) = c_d/(d - 2)$ with $c_d = (2/\pi)[S_4/(2\pi)^d]$ and $S_4$ the surface of a $d$-dimensional sphere of radius unity. In principle it should be accurate only for $d \gtrsim 2$ though it is unclear its exact range of validity.

B. Numerical analysis of $R_2(s)$

After the theoretical analysis we are now ready to present our numerical results for $R_2(s)$ at the Anderson transition in $d = 3 - 6$ dimensions. Our motivation is to study the existence and extension of the three regions introduced above: level repulsion, power-law and exponential decay. Indeed our numerical results clearly show these three regimes in all dimensions $d = 3 - 6$ investigated.

We have first verified (not shown) that for sufficiently large $s$, $R_2(s) \sim 1/s^7$. The numerical value of the exponent $\gamma$ was in full agreement with the theoretical prediction $\gamma = 1 - (\nu d)^{-1}$.

Then we investigate to what extent level repulsion typical of the Anderson transition in $d = 3, 4$ is still present in higher dimensions. As is observed in Fig. 3, left, for sufficiently small $s$, $R_2(s) \sim s$ for all dimensions studied. The solid lines lines are linear fits of the form $R_2(s) = C + Ds$ with fitting parameters $D = 6.6 \pm 0.8$ for $d = 3$, $15.0 \pm 1.2$ ($d = 4$), $101 \pm 5$ ($d = 5$), and $373 \pm 32$ ($d = 6$). The parameter $C$ is equal to zero within the error bars in all cases. This is consistent with the prediction of the one parameter scaling theory that quantum diffusion never stops. However the range in which level repulsion is observed decreases dramatically with the spatial dimensionality thus suggesting that the critical conductance $g_c$ also decreases rapidly with the dimension. It is hard to give a more quantitative prediction of $g_c$, as a function of the spatial dimensionality: the estimation of Vollhardt and Wolke mentioned previously fails for $d > 3$. Another option is to extrapolate the result in the diffusive regime, $R_2(s) \sim s^{1+\alpha_d/g^2}$ for $s \ll g_c$ to the critical one. However the geometrical coefficient $\alpha_d$ diverges also for $d > 3$.

Our numerical results (see Fig. 3, right) show that for larger spectral separations $s \geq g_c$, the linear repulsion is replaced by an exponential decay. The solid lines correspond to a linear fit $\ln[1 - R_2(s)] = C - Ds$ with fitting parameters $D = 3.7 \pm 0.1$ for $d = 3$, $4.7 \pm 0.1$ ($d = 4$), $5.6 \pm 0.5$ ($d = 5$), and $9.5 \pm 0.2$ ($d = 6$). The maximum value of $s$ plotted was chosen attending to technical criteria. For larger values of $s$, $1 - R_2(s)$ fluctuates around zero thus suggesting that the maximum precision of the computer has been reached.

We note that such exponential decay has already observed in certain one-dimensional disordered systems with long range hopping and in phenomenological short-range plasma models whose spectral properties are strikingly similar to those of a disordered system with short range hopping at the Anderson transition. In these one-dimensional systems it can be proved analytically that $R_c(s) \sim e^{\Lambda s}/g$ where $A$ is a constant of order unity. It is thus tempting to speculate that in our case $g_c \sim 1/D$ with $D$ the fitting parameter above. However a deeper analytical knowledge about the Anderson transition is needed to discard that additional geometrical factors (as $\alpha_d$ above) enter in the exponent of $R_2(s)$ making thus less evident the relation between $g_c$ and $D$. 

![Fig. 3: $R_2(s)$ for $d = 3 - 6$ at the Anderson transition for different $s$ ranges.](image-url)
C. Spectral correlators

Level statistics at the Anderson transition are usually investigated by computing certain spectral correlators from the TLCF or higher $n$-level correlation functions. The level spacing distribution $P(s)$ is a popular choice to study the correlations of eigenvalues separated short distances of order the mean level spacing. On the other hand, the number variance in an insulator but with a slope $\chi$ and $A$ with the spatial dimensionality. In $d = 2 + \epsilon$ the Anderson transition occurs in the weak disorder region, $g_c \gg 1$ and $\chi \sim 1/g_c \sim d-2 \ll 1$. On the other hand, the prediction for $d \to \infty$ (Cayley tree) is $A = \chi = 1_3^{22}$. In principle, corrections to the Cayley tree limit due to interference between different paths decay as $1/d$ or faster so it is tempting to conjecture that $\chi = 1 - C/(d-2)$ and $A = 1 + D/(d-2)$. The numerical results of Fig. 5 confirm this dependence especially for the parameter $A$. In the case of $\chi$ the situation is less clear. A reason for the discrepancy with the theoretical prediction could be that $d \sim 3$ is still far from the limit $d \gg 2$ in which $\chi = 1 - C/(d-2)$ holds. Indeed we have observed that our numerical data are better described (dotted line in Fig. 5) by $\chi = \tanh[C/(d-2)]$ with $C = 0.29 \sim 1/\pi$. Such a dependence of $\chi$ on hyperbolic functions has already been reported on the generalized random matrix models whose spectral correlators are strikingly similar to the ones at the Anderson transition.

The straight lines in Fig. 5 are linear fits to the conjectured relations with fitting parameters $A = 0.78 \pm 0.06$ and $D = 0.55 \pm 0.01$. From a physical point of view these numerical results are a further confirmation that analytical approaches to the Anderson transition starting from the metallic limit and adding interference corrections or starting from the insulator state and inducing the transition to a metal by increasing the tunneling amplitude fail to capture key features of the Anderson transition in intermediate dimensions where both mechanisms are at work.

D. Random matrix models and the Anderson transition

Typical signatures of critical statistics have also been found in both generalized random matrix models whose joint distribution of eigenvalues can be mapped onto the Calogero-Sutherland model at finite temperature and phenomenological short-range plasma models whose joint distribution of eigenvalues is given by the classical Dyson gas with the logarithmic pairwise interaction restricted to a finite number of nearest neighbors (the spectral correlators of this model are usually referred to as Semi-Poisson statistics though this name

FIG. 4: Number variance $\Sigma^2(t)$ (left panel) and $P(s)$ (right panel) at the Anderson transition for $d = 3 - 6$. The system tends to the Poisson limit (P) as the dimension is increased. WD denotes the Wigner-Dyson distribution.

FIG. 5: Dependence of the slope of the number variance $\chi$ (left panel) and the asymptotic decay of the level spacing distribution $A$ (right panel) with the dimensionality of the space.
IV. CONCLUSIONS

We have studied the dependence of the spatial dimensionality of different quantities relevant in the description of the Anderson transition. As a result we have concluded that the upper critical dimension for localization is infinity. The level statistics tend to Poisson statistics, typical of an insulator, as the upper critical dimensionality is approached. We have also proposed that the exponential decay of the TLCF observed in numerical calculations is a signature of an Anderson transition problem. Neither the self-consistent theory of localization exact in the Cayley tree nor the $\epsilon$-expansion formalism are accurate for intermediate dimensions. A new basis for the localization problem is thus called for. Finally, the effect of a magnetic flux and the validity of certain effective models to describe the spectral correlations at the Anderson transition have been discussed.

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