PARAMETER DETERMINATION FOR ENERGY BALANCE MODELS WITH MEMORY

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Abstract. In this paper, we study two Energy Balance Models with Memory arising in climatology, which consist in a 1D degenerate nonlinear parabolic equation involving a memory term, and possibly a set-valued reaction term (of Sellers type and of Budyko type, in the usual terminology). We provide existence and regularity results, and obtain uniqueness and stability estimates that are useful for the determination of the insolation function in Sellers’ model with memory.

1. Introduction

1.1. Energy balance models and the problems we consider.

We are interested in a problem arising in climatology, coming more specifically from the classical Energy Balance models introduced independently by Budyko [7] and Sellers [46]. These models, which describe the evolution of temperature as the effect of the balance between the amount of energy received from the Sun and radiated from the Earth, were developed in order to understand the past and future climate and its sensitivity to some relevant parameters on large times scales (centuries). After averaging the surface temperature over longitude, they take the form of the following one-dimensional nonlinear parabolic equation with degenerate diffusion:

\[ u_t - (\rho_0 (1 - x^2) u_x)_x = R_a - R_e \]

where

- \( u(t, x) \) is the surface temperature averaged over longitude,
- the space variable \( x = \sin \phi \in (-1, 1) \) (here \( \phi \) denotes the latitude),
- \( R_a \) represents the fraction of solar energy absorbed by the Earth,
- \( R_e \) represents the energy emitted by the Earth,
- \( \rho_0 \) is a positive parameter.

A crucial role in the analysis will be played by the absorbed energy \( R_a \), which is a fraction of the incoming solar flux \( Q(t, x) \), that is,

\[ R_a = Q(t, x) \beta, \]

where \( \beta \) is the coalbedo function. Additionally, as is customary in seasonally averaged models, we will assume that

\[ Q(t, x) = r(t)q(x), \]

where \( r \) is positive and \( q \) is the so-called ”insolation function”.

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It was noted (see Bhattacharya-Ghil-Vulis [4]) that, in order to take into account the long response times that cryosphere exhibits (for instance, the expansion or retreat of huge continental ice sheets occurs with response times of thousands of years), it is useful to let the albedo function depend not only on $u$, but also on the history function, which can be represented by the integral term

$$H(t, x, u) := \int_{-\tau}^{0} k(s, x) u(t + s, x) ds \quad \forall t > 0, x \in I,$$

where $k$ is the memory kernel (and $\tau \sim 10^4$ years, in real problems). As in Roques-Checkroun-Cristofol-Soubeyrand-Ghil [45], we will assume a nonlinear response to memory in the form $f(H(t, x, u))$.

Hence, we are interested in the following Energy Balance Model with Memory (EBBM) problem, set in the space domain $I := (-1, 1)$:

$$\begin{cases}
  u_t - (\rho_0(1 - x^2)u_x)_x = Q(t, x) + f(H(t, x, u)) - R_e(u), & t > 0, x \in I, \\
  \rho_0(1 - x^2)u_x(t, x) = 0, & t > 0, x \in \partial I \\
  u(s, x) = u_0(s, x), & s \in [-\tau, 0].
\end{cases}$$

Concerning the function $\beta$, we will assume, as it is classical for such problems, that

- either $\beta$ is positive and at least Lipschitz continuous (the classical assumption for Sellers type models),
- or $\beta$ is positive, monotone and discontinuous (the classical assumption for Budyko type models).

1.2. Relation to literature and presentation of our main results.

The mathematical analysis of quasilinear EBBM problems of the form

$$\partial_t u - \text{div} (\rho(x)|\nabla u|^{p-2}\nabla u) = f(t, x, u, H(t, x, u))$$

has been the subject of many deep works for a long time. Questions such as well-posedness, uniqueness, asymptotic behavior, existence of periodic solutions, bifurcation, free boundary, numerical approximation were investigated for:

- 1-D models without memory by Ghil in the seminal paper [28],
- 0-D models in Fraedrich [25, 26],
- 1-D models with memory in Bhattacharya-Ghil-Vulis [4] and Diaz [17, 18],
- 2-D models (on a manifold without boundary, typically representing the Earth’s surface) in Hetzer [30], Diaz-Tello [22], Diaz-Hetzer [20], Hetzer [31], Diaz [19], Diaz-Hetzer-Tello [21], and Hetzer [32].

In this paper, we are interested in the following inverse problem: is it possible to recover the insolation function (which is a part of the incoming solar flux in $Q(t, x)$) from measurements of the solution, for our EBBM model? Our motivation comes from the fact that, with suitable tuning of their parameters, EBMs have shown to mimic the observed zonal temperatures for the observed present climate [40], and can be used to estimate the temporal response patterns to various forcing scenarios, which is of interest in particular in the detection of climate change. Unfortunately, in practice, the model coefficients cannot be measured directly, but are quantified through the measures of the solution [45]. Hence, results proving that measuring the solution in some specific (small) part of the space and time domain is sufficient to recover a specified coefficient are of practical interest.

Several earlier papers are related to this question, in particular the ones that we recall below.

- In Tort-Vancostenoble [49], the question of determining the insolation function was studied for a 1D Sellers type model without memory, combining:
– the method introduced by Imanuvilov-Yamamoto in the seminal paper [33] (based on the use of Carleman estimates to obtain stability results for the determination of source terms for parabolic equations),
– the Carleman estimates from Cannarsa-Martinez-Vancostenoble [10] for degenerate parabolic equations,
– suitable maximum principles to deal with nonlinear terms.

In the same paper, the authors proved stability estimates measuring the solution on an open subset of the space domain. Similar questions were studied in Martinez-Tort-Vancostenoble [39] on manifolds without boundary.

• In Roques-Checkroun-Cristofol-Soubeyrand-Ghil [45], the question of determining the insolation function was studied for a 1D Sellers type model including memory effects, but for a nondegerate diffusion. These authors extended a method due to Roques-Cristofol [44, 15] which, based on analyticity, allows for measurements only at a single point $x_0$ (under a rather strong assumption on the kernel appearing in the history function).

In this paper we study, first, the 1D Sellers type problem with degenerate diffusion and memory effects. More precisely, we prove regularity results and use them to study the determination of the insolation function, obtaining

• a uniqueness result, under pointwise observation,
• a Lipschitz stability result, under localized observation,

in the spirit of the above mentioned references. Then, we address 1D Budyko type problems with degenerate diffusion and memory effects, for which we obtain precise existence results as in Diaz-Hetzer [20]. For this, we need to regularize the coalbedo and use the existence results obtained in the first part of the paper.

Let us note that our existence results for Sellers and Budyko type problems can be regarded as a consequence of the ones by Diaz-Hetzer [20] for manifolds. However, here we give a direct proof of such results in zonally averaged 1D settings. For this reason, we need to use the properties of degenerate diffusion operators.

Finally, to give a more complete overview of the literature on these questions, let us also mention the papers by:

• Pandolfi [41], for a similar question but on a different equation (the history function depending on the second-order derivative in space),
• Guerrero-Imanuvilov [29], that proves that null controllability does not hold for the linear heat equation perturbed by $\int_0^t u$ (hence with a memory term which takes into account all the history from time 0 to $t$),
• Tao-Gao [47], that gives positive null controllability results for a similar heat equation under additional assumptions on the kernel appearing in the history function (notice however that these assumptions are incompatible with our settings as they would force the kernel to depend also on $t$ and to uniformly vanish at some time $T$, which is unnatural in climate modelling).

2. MATHEMATICAL ASSUMPTIONS FOR THESE CLIMATE MODELS

We are interested in a class of EBMM:

$$
\begin{align*}
 u_t - (\rho(x)u_x)_x &= R_a(t, x, u, H) - R_e(t, x, u, H), & t > 0, x \in I, \\
 \rho(x)u_x &= 0, & x \in \partial I, \\
 u(s, x) &= u_0(s, x), & s \in [-\tau, 0], x \in I,
\end{align*}
$$

(2.1)

where $I = (-1, 1)$. We are going to precise our assumptions concerning Budyko type problems and Sellers ones.
2.1. Budyko type models with memory.

We make the following assumptions:

- concerning the diffusion coefficient: we assume that there exists $\rho_0 > 0$ such that
  \begin{equation}
  \forall x \in (-1, 1), \quad \rho(x) := \rho_0(1 - x^2);
  \end{equation}

- concerning $R_a$: we assume that
  \begin{equation}
  R_a(t, x, u, H) = Q(t, x)\beta(u) + f(H(t, x, u)),
  \end{equation}
  where
  - $Q(t, x)$ is the incoming solar flux; we assume that $Q(t, x) = r(t)q(x)$, where $q$, the insolation function, and $r$ are such that:
    \begin{equation}
    \begin{cases}
    q \in L^\infty(I), \\
    r \in C^1(\mathbb{R}_+) \text{ and } r, r' \in L^\infty(\mathbb{R}_+);
    \end{cases}
    \end{equation}
  - $\beta$ is the classical Budyko type coalbedo function: it is an highly variable quantity which depends on many local factors such as the cloud cover and the composition of the Earth’s atmosphere, moreover it is used as an indicator for ice and snow cover; usually it is considered roughly constant for temperatures far enough from the ice-line, that is a circle of constant latitude that separates the polar ice caps from the lower ice-free latitudes; the classical Budyko type coalbedo is:
    \begin{equation}
    \beta(u) = \begin{cases}
    a_i, & u < \bar{u}, \\
    [a_i, a_f], & u = \bar{u}, \\
    a_f, & u > \bar{u},
    \end{cases}
    \end{equation}
    where $a_i < a_f$ (and the threshold temperature $\bar{u} := -10^\circ$);
  - $H$ is the history function; it is assumed to be given by
    \begin{equation}
    H(t, x, u) = \int_{-\tau}^0 k(s, x) u(t + s, x) \, ds
    \end{equation}
    where the kernel $k$ is such that:
    \begin{equation}
    k \in C^1([-\tau, 0] \times [-1, 1]; \mathbb{R});
    \end{equation}
  - $f$: the nonlinearity that describes the memory effects; we assume that $f : \mathbb{R} \to \mathbb{R}$ is $C^1$ and such that
    \begin{equation}
    \begin{cases}
    f, f' \in L^\infty(\mathbb{R}) \\
    f, f' \text{ are } L - \text{Lipschitz};
    \end{cases}
    \end{equation}

- concerning $R_c$: the classical Budyko type assumption is
  \begin{equation}
  R_c(t, x, u, H) = a + bu,
  \end{equation}
  where $a$, $b$ are constants;

- the initial condition: since we define $H$ over a past temperature, the initial condition in such models has to be of the form
  \begin{equation}
  u(s, x) = u_0(s, x) \quad \forall s \in [-\tau, 0], \quad x \in I
  \end{equation}
  for some $u_0(s, x)$ defined on $[-\tau, 0] \times I$, for which we will precise our assumptions in our different results.

Sometimes we will only add positivity assumptions on $q$ and $r$; these assumptions are natural with respect to the model, but only useful in the inverse problems results.
2.2. Sellers type models with memory.

The differences concern the assumptions on the coalbedo and on the emitted energy:

- \( \beta \): in Sellers type models, we assume that

\[
\beta \in C^2(\mathbb{R}), \quad \beta, \beta', \beta'' \in L^\infty(\mathbb{R})
\]

(typically, \( \beta \) is \( C^2 \) and takes values between the lower value for the coalbedo \( a_i \) and higher value \( a_f \) (even if there is a sharp transition between these two values around the threshold temperature \( \bar{u} \))).

- \( R_e \) is assumed to follow a Stefan-Boltzmann type law (assuming that the Earth radiates as a black body):

\[
R_e = \varepsilon(u)|u|^3 u,
\]

where the function \( \varepsilon \) represents the emissivity; we assume that

\[
\exists \varepsilon_1 > 0, \text{ s.t. } \forall u, \quad \varepsilon(u) \geq \varepsilon_1 > 0.
\]

2.3. Plan of the paper.

- section 3 contains the statement of our results concerning Sellers type models:
  - concerning well-posedness questions: see Theorem 3.2 in section 3.3.
  - concerning inverse problems questions:
    * Theorem 3.3: uniqueness of the insolation function under point-wise measurements (in section 3.4.a).
    * Theorem 3.4: Lipschitz stability under localized measurements (in section 3.4.b).

- section 4 contains the statement of our well-posedness result concerning Budyko type models, see Theorem 4.1.

- section 5 is devoted to mention some open questions;

- section 6 contains the proof of Theorem 3.2;

- section 7 contains the proof of Theorem 3.3;

- section 8 contains the proof of Theorem 3.4;

- section 9 contains the proof of Theorem 4.1.

3. Main results for the Sellers type model

First we show the local and global existence of a regular solution to the following problem:

\[
\begin{cases}
  u_t - (\rho(x)u_x)_x = r(t)q(x)\beta(u) - \varepsilon(u)|u|^3 u + f(H), & t > 0, x \in I, \\
  \rho(x)u_x = 0, & t > 0, x \in \partial I, \\
  u(s,x) = u_0(s,x), & s \in [-\tau,0], x \in I,
\end{cases}
\]

(3.1)

In this section, we assume \( (2.2), (2.4), (2.6)-(2.8), (2.10)-(2.13) \).

In the following, we recast (3.1) into a semilinear evolution equation governed by an analytic semigroup.

3.1. Functional framework.

Since the diffusion coefficient has a degeneracy at the boundary, it is necessary to introduce the weighted Sobolev space \( V \) below in order to deal the well-posedness of problem (2.1). To know more about this functional framework for one-dimensional degenerate parabolic equations, the reader may also refer to [8, 9, 12, 49].

\[
V := \{ w \in L^2(I) : w \in AC_{loc}(I), \sqrt{\rho}w_x \in L^2(I) \}
\]
endowed with the inner product
\[(u, v)_V := (u, v)_{L^2(I)} + \langle \sqrt{\rho}u_x, \sqrt{\rho}v_x \rangle_{L^2(I)} \quad \forall u, v \in V\]
and then with the associated norm
\[||u||_V := \sqrt{(u, u)_V} = ||u||_{L^2(I)} + ||\sqrt{\rho}u_x||_{L^2(I)} \quad \forall u \in V.\]

We recall that \(\rho(x) = \rho_0(1-x^2)\) for all \(x \in I\) by definition. Let us remark that 
\((V, (\cdot, \cdot)_V)\) is a Hilbert space and that \(V \subset H^1_{loc}(I)\) and \(V \subset L^2(I) \subset V^*\). Moreover,

- the space \(C^0_0(I)\) is dense in \(V\), in particular \(V\) is dense in \(L^2(I)\) [8, Lemma 2.6];
- for all \(p \in [1, +\infty)\), the inclusion
\[(3.2) \quad V \hookrightarrow L^p(I)\]
holds and is continuous; moreover, the inclusion \(V \hookrightarrow L^2(I)\) is compact [15, Lemma 1].

In order to obtain our semilinear evolution equation let us define the operator
\[A : D(A) \subset L^2(I) \rightarrow L^2(I)\]
in the following way:
\[(3.3) \begin{cases} D(A) := \{u \in V : \rho u_x \in H^1(I)\} \\ Au := (\rho u_x)_x \quad u \in D(A) \end{cases}\]
(Note that the boundary condition appearing in (2.1) is contained in the definition of the unbounded operator \(A\) given in (3.3): indeed, if \(u \in D(A)\), then \(\rho u_x \in H^1(I)\), hence \(\rho u_x \in C^0(I)\), which implies that \(\rho u_x \rightarrow L\) as \(x \rightarrow 1^-\); but if \(L \neq 0\), then \(\sqrt{\rho}u_x \notin L^2(I)\), therefore \(L = 0\) and \((\rho u_x)(1) = 0\). And the case \(x = -1\) is analogous.)

We denote \(L(L^2(I))\) the space of linear continuous applications from \(L^2(I)\) into itself, endowed with the natural norm \(|||| \cdot ||||_{L(L^2(I))}\). We recall the following

**Theorem 3.1.** [8] \(A, D(A)\) is a self-adjoint operator and it is the infinitesimal generator of an analytic and compact semigroup \(\{e^{tA}\}_{t \geq 0}\) in \(L^2(I)\) that satisfies
\[|||e^{tA}|||_{L(L^2(I))} \leq 1.\]

(We give elements of its proof in section 6.1). Finally, we recall also the following

**Proposition 3.1.** [37]. Proposition 2.1 The real interpolation space constructed by the trace method \([D(A), L^2(I)]_{\frac{1}{2}}\) is the space \(V\).

### 3.2. The concept of mild solution for the Sellers type model (5.3).

Consider the problem (2.1) of the Sellers type. In order to recast it into an abstract form, we introduce the following notations:

- to manage the nonlinear term, we consider the following function
\[(3.4) \quad G : [0, T] \times V \rightarrow L^2(I), \quad G(t, u)(x) = Q(t, x)\beta(u(x)) - \varepsilon(u(x))u(x)^3u(x),\]
where we recall that \(Q(t, x) = r(t)q(x)\); (note that since \(V \hookrightarrow L^p(I)\) for all \(p \geq 1\), it is clear that \(G([0, T] \times V) \subset L^2(I)\));
- to manage the shifted memory term:
  - given \(u \in C([-\tau, T]; L^2(I))\), given \(t \in [0, T]\), we consider the right translation \(u^{(t)} \in C([-\tau, 0]; L^2(I))\) by the formula
\[(3.5) \quad u^{(t)} : [-\tau, 0] \rightarrow L^2(I), \quad u^{(t)}(s) := u(t + s),\]
  - and we define the following function
\[(3.6) \quad F : C([-\tau, 0]; L^2(I)) \rightarrow L^2(I), \quad F(v)(x) = f \left( \int_{-\tau}^0 k(s, x) (v(s)(x)) \, ds \right),\]
in such a way that the memory term can be written $F(u(t))$.

And then, given $T > 0$, (3.1) on $[0, T]$ can be recast into:

\[
\begin{aligned}
&\dot{u}(t) = Au(t) + G(t, u) + F(u(t)) \quad t \in [0, T] \\
u(s) = u_0(s) \quad s \in [-\tau, 0].
\end{aligned}
\] (3.7)

Before defining the concept of mild solution for (3.7), we precise the concept of mild solution for the following linear nonhomogeneous problem

\[
\begin{aligned}
&\dot{u}(t) = Au(t) + g(t) \quad t \in [0, T] \\
u(0) = u_0.
\end{aligned}
\] (3.8)

We consider the following

**Definition 3.1.** Let $g \in L^2(0, T; L^2(I))$ and let $u_0 \in L^2(I)$. The function $u \in C([0, T]; L^2(I))$ defined by

\[
\forall t \in [0, T], \quad u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}g(s)ds
\] (3.9)

is called the mild solution of (3.8).

We recall that $u$ defined by (3.9) has the following additional regularity:

\[
u \in H^1(0, T; L^2(I)) \cap L^2(0, T; D(A)).
\]

Now we are ready to define the concept of mild solution for (3.7):

**Definition 3.2.** Given $u_0 \in C([-\tau, 0]; V)$, a function

\[
u \in H^1(0, T; L^2(I)) \cap L^2(0, T; D(A)) \cap C([-\tau, T]; V)
\]

is called a mild solution of (3.7) on $[0, T]$ if

(i) $u(s) = u_0(s)$ for all $s \in [-\tau, 0]$;

(ii) for all $t \in [0, T]$, we have

\[
u(t) = e^{tA}u_0(0) + \int_0^t e^{(t-s)A}(G(s, u) + F(u(s))) ds.
\] (3.10)

3.3. Global existence and uniqueness result for the Sellers model (3.7).

Now we are ready to prove the global existence result of the integrodifferential problem.

**Theorem 3.2.** Consider $u_0$ such that

\[
u_0 \in C([-\tau, 0]; V) \quad \text{and} \quad u_0(0) \in D(A) \cap L^\infty(I).
\]

Then, for all $T > 0$, the problem (3.7) has a unique mild solution $\nu$ on $[0, T]$.

(Note that

- existence and uniqueness of a global regular solution to (2.1) without the memory term has been proved in [19];
- the local existence of our model without the boundary degeneracy has been studied in [45];
- the global existence of a similar 2D-model with memory (hence on a manifold but without the boundary degeneracy), has been investigated in [20].)
3.4. Inverse problem results: determination of the insolation function.

Here we prove that the insolation function \( q(x) \) can be determined in the whole space domain \( I \) by using only local information about the temperature.

To achieve this goal, we add the following extra assumptions, as in [45]: the very recent past temperatures are not taken into account in the history function:

\[
\exists \delta > 0 \text{ s.t. } k(s, \cdot) \equiv 0 \quad \forall s \in [-\delta, 0]
\]

where \( \delta < \tau \). (We will discuss about this assumption in section 5.)

Hence, we have the following situation: consider two insolation functions \( q \) and \( \tilde{q} \), two initial conditions \( u_0 \) and \( \tilde{u}_0 \), and the associated solutions: \( u \) satisfying (3.1) and \( \tilde{u} \) satisfying

\[
\begin{cases}
\tilde{u}_t - (\rho(x)\tilde{u}_x)_x = r(t)\tilde{q}(x)\beta(\tilde{u}) - \varepsilon(\tilde{u})|\tilde{u}|^3\tilde{u} + f(\tilde{H}), & t > 0, x \in I, \\
\rho(x)\tilde{u}_x = 0, & x \in \partial I, \\
\tilde{u}(s,x) = \tilde{u}_0(s,x), & s \in [-\tau,0], x \in I,
\end{cases}
\]

where we denote

\[
\tilde{H} := H(t,x,\tilde{u}) = \int_{-\tau}^{-\delta} k(s,x)\tilde{u}(t + s,x) \, ds.
\]

In the following, we state two inverse problems results, according to different assumptions on the control region.

3.4.a. Pointwise observation and uniqueness result.

Let us choose suitable regularity assumptions on the initial conditions and on the insolation functions, in order to have sufficient regularity on the time derivative of the associated solutions: we consider

- the set of admissible initial conditions: we consider

\[
U^{(pt)} = C^{1,2}([-\tau, 0] \times [-1,1]),
\]

- and the set of admissible coefficients: we consider

\[
Q^{(pt)} := \{ q \text{ is Lipschitz-continuous and piecewise analytic on } I \},
\]

where we recall the following

**Definition 3.3.** A continuous function \( \psi \) is called piecewise analytic if there exist \( n \geq 1 \) and an increasing sequence \( (p_j)_{1 \leq j \leq n} \) such that \( p_1 = -1, p_n = 1 \), and

\[
\psi(x) = \sum_{j=1}^{n-1} \chi_{[p_j, p_{j+1}]}(x) \varphi_j(x) \quad \forall x \in I,
\]

where \( \varphi_j \) are analytic functions defined on the intervals \( [p_j, p_{j+1}] \) and \( \chi_{[p_j, p_{j+1}]} \) is the characteristic function of the interval \( [p_j, p_{j+1}] \) for \( j = 1, \ldots, n - 1 \).

Then we prove the following uniqueness result:

**Theorem 3.3.** Consider

- two insolation functions \( q, \tilde{q} \in Q^{(pt)} \) (defined in 3.14),
- an initial condition \( u_0 = \tilde{u}_0 \in U^{(pt)} \) (defined in 3.13),

and let \( u \) be the solution of (3.1) and \( \tilde{u} \) the solution of (3.12).

Assume that

- the memory kernel satisfies (3.11),
- \( r \) and \( \beta \) are positive,
there exists $x_0 \in I$ and $T > 0$ such that

\begin{equation}
\forall t \in (0, T), \left\{ u(t, x_0) = \tilde{u}(t, x_0), \\
u_x(t, x_0) = \tilde{u}_x(t, x_0) \right. \right. \end{equation}

Then $q \equiv \tilde{q}$ on $I$.

This result means that the insolation function $q(x)$ is uniquely determined on $I$ by any measurement of $u$ and $u_x$ at a single point $x_0$ during the time period $(0, T)$.

Theorem 3.3 is a natural extension of [45] to the degenerate problem.

3.4.b. **Localized observation and stability result.**

Let us choose suitable regularity assumptions on the initial conditions and on the insolation functions, in order to have sufficient regularity on the time derivative of the associated solutions: we consider

- the set of admissible initial conditions: given $M > 0$, we consider $U_M^{(loc)}$:

\begin{equation}
U_M^{(loc)} := \{ u_0 \in C([-\tau, 0]: V \cap L^\infty(-1, 1)), u_0(0) \in D(A), Au_0(0) \in L^\infty(I), \\
\sup_{t \in [-\tau, 0]} (\|u_0(t)\|_V + \|u_0(t)\|_{L^\infty}) + \|Au_0(0)\|_{L^\infty(I)} \leq M \}, \end{equation}

- and the set of admissible coefficients: given $M' > 0$, we consider $Q_{M'}^{(loc)}$:

\begin{equation}
Q_{M'}^{(loc)} := \{ q \in L^\infty(I) : \|q\|_{L^\infty(I)} \leq M' \}.
\end{equation}

Now we are ready to state our Lipschitz stability result:

**Theorem 3.4.** Assume that

- the memory kernel satisfies (3.11),
- $r$ and $\beta$ are positive.

Consider

- $0 < T' < \delta$,
- $t_0 \in [0, T')$, $T > T'$,
- $M, M' > 0$.

Then there exists $C(t_0, T', T, M, M') > 0$ such that, for all $u_0, \tilde{u}_0 \in U_M^{(loc)}$ (defined in (3.16)), for all $q, \tilde{q} \in Q_{M'}^{(loc)}$ (defined in (3.17)), the solution $u$ of (3.1) and the solution $\tilde{u}$ of (3.12) satisfy

\begin{equation}
\|q - \tilde{q}\|_{L^2(I)}^2 \leq C \left( \|u(T') - \tilde{u}(T')\|_{D(A)}^2 + \|u_t - \tilde{u}_t\|_{L^2((t_0, T') \times (a, b))}^2 + \|u_0 - \tilde{u}_0\|_{C([-\tau, 0]: V)}^2 \right). \end{equation}

Theorem 3.3 is a natural extension of [19].

4. **Main result for the Budyko type model**

Now we treat the global existence of regular solutions for the Budyko model. In a classical way (see, e.g. Diaz [18]), we study the set valued problem

- first regularizing the coalbedo, hence transforming the Budyko type problem into a Sellers one, for which we have a (unique) regular solution,
- and then passing to the limit with respect to the regularization parameter.
Since $\beta$ is the graph given in (2.3), the Budyko type problem has to be understood as the following differential inclusion problem:

$$\begin{cases}
u_t - (\rho(x)u_x)_x \in r(t)q(x)\beta(u) - (a + bu) + f(H(u)), & t > 0, x \in I, \\
\rho(x)u_x = 0, & x = \pm 1, \\
u(s, x) = u_0(s, x), & s \in [-\tau, 0], x \in I.
\end{cases} \tag{4.1}$$

In this section, we assume (2.2)-(2.10).

4.1. The notion of mild solutions for the Budyko model (4.1).

Let us define a mild solution for this kind of problem.

**Definition 4.1.** Given $u_0 \in C([-\tau, 0]; V)$, a function

$$u \in H^1(0, T; L^2(I)) \cap L^2(0, T; D(A)) \cap C([-\tau, T]; V)$$

is called a mild solution of (4.1) on $[-\tau, T]$ iff

- $u(s) = u_0(s)$ for all $s \in [-\tau, 0]$;
- there exists $g \in L^2([0, T]; L^2(I))$ such that
  - $u$ satisfies
    $$\forall t \in [0, T], \quad u(t) = e^{tA}u_0(0) + \int_0^t e^{(t-s)A}g(s) \, ds,$$
  - and $g$ satisfies the inclusion
    $$g(t, x) \in r(t)q(x)\beta(u(t, x)) - (a + bu(t, x)) + f(H(t, x, u)), \quad \text{a.e. } (t, x) \in (0, T) \times I. \tag{4.2}$$

4.2. Global existence for the Budyko model (4.1).

**Theorem 4.1.** Assume that

$$u_0 \in C([-\tau, 0], V) \quad \text{and} \quad u_0(0) \in D(A) \cap L^\infty(I).$$

Then (4.1) has a mild solution $u$, which is global in time (i.e. defined in $[0, +\infty)$ and mild on $[0, T]$ for all $T > 0$).

5. Open questions

Let us mention some open questions related to this work.

- Concerning the Sellers type models and the inverse problems results given in Theorems 3.3 and 3.4: the assumption on the support of the kernel $k$ is crucial, since it allows us to easily get rid of the memory term, and we do not know what can be done without this assumption; hence
  - it would be interesting to weaken this support assumption; it seems reasonable to think that uniqueness and stability results could be obtained in a more general context, and on the other hand, memory terms can sometimes generate problems (see, in particular, [29]):
  - even under the support assumption: from a numerical point of view, it would be interesting to weaken some assumptions (in particular on $T$, since our proof is based on $T < \delta$) in order to have better estimates even if $\delta$ is small.

- Concerning the Budyko type models: a solution is obtained by regularization and passage to the limit; (note that, at least for an EBM without memory term, uniqueness of the solution depends on the initial condition, see Diaz [15]); several questions are mathematically challenging, in particular to obtain inverse problems results in that setting: one way could be to obtain suitable estimates from the regularized problem, succeeding in avoiding to use the $C^1$ norm of the regularized coalbedo (unfortunately, the $C^1$ norm of the regularized coalbedo appears in our estimates in the Sellers model).
Finally, it would be interesting to obtain inverse problems results for other problems, in particular of the quasilinear type mentioned in the beginning of section 1.2.

6. Proof of Theorem 3.2

6.1. Elements for Theorem 3.1
To prove that $(A, D(A))$ is self-adjoint and it generates a strongly continuous semigroup of contractions in $L^2(I)$, one can refer to Theorem 2.8 in [8] (where in our case $b \equiv 0$).

Moreover,
\[
\int_0^x \frac{ds}{\rho_0(1-s^2)} = \frac{1}{2\rho_0}(\log(1+x) - \log(1-x))
\]
which implies that $\int_0^x \frac{ds}{\rho(s)} \in L^1(I)$, and then one can show that the semigroup generated by $A$ is compact (see Theorem 3.3 in [8]).

For the analyticity of the semigroup see Theorem 2.12 in [3] or Theorem 3.6.1 in [50].

6.2. Proof of Theorem 3.2: local existence of mild solutions.
In this section, we prove the following

Proposition 6.1. Consider $u_0$ such that
\[
u_0 \in C([-\tau,0]; V) \quad \text{and} \quad u_0(0) \in D(A).
\]
Then there exists $t^* > 0$ such that the problem (3.7) has a unique mild solution on $[0, t^*]$.

6.2.a. The functional setting and main tools.
It will be more practical to have strictly dissipative operators, so let us consider
\[
\tilde{A} := A - I,
\]
that satisfies:
\[
\begin{cases}
D(\tilde{A}) = D(A) \quad &\tilde{A}u := (\rho u_x)_x - u, \quad u \in D(\tilde{A}) ; \\
\end{cases}
\]
integrating by parts:
\[
\forall u \in D(\tilde{A}), \quad (\tilde{A}u, u)_{L^2(I)} = -\int_I(\rho u^2_x + u^2)dx \leq -\|u\|_{L^2(I)}^2,
\]
hence $\tilde{A}$ is strictly dissipative. We will use the following estimates: using Pazy [42] (Theorem 6.13, p. 74) with $\alpha = 1/2$ and $\alpha = 3/4$, there exists $c > 0$ such that
\[
\forall t > 0, \quad \|(-\tilde{A})^{1/2}e^{t\tilde{A}}\|_{L^2(I)} \leq \frac{4}{\sqrt{t}},
\]
and
\[
\forall t > 0, \quad \|(-\tilde{A})^{3/4}e^{t\tilde{A}}\|_{L^2(I)} \leq \frac{c}{t^{3/4}}.
\]

Now, consider also $\tilde{G}(t, u) := u + G(t, u)$ so that, adding and subtracting $u$ to the equation, the problem (3.7) is equivalent to
\[
\begin{cases}
\dot{u}(t) = \tilde{A}u(t) + \tilde{G}(t, u(t)) + F(u(t)) \\
u(0) = u_0(0).
\end{cases}
\]
By the definitions above, we know that a mild solution of this problem is a function such that
\[
u(t) = \begin{cases}
e^{t\tilde{A}}u_0(0) + \int_0^t e^{(t-s)\tilde{A}} \left[\tilde{G}(s, u(s)) + F(u(s))\right]ds, & t > 0 \\
u_0(t), & t \in [-\tau, 0]
\end{cases}
\]
Then we consider the suitable functional setting:

- the space of functions
  \[ X_R := \{ v \in C([-\tau, t^*]; V) \mid \begin{cases} \|v(t)\|_V \leq R \forall t \in [-\tau, t^*], \\
v(t) = u_0(t) \forall t \in [-\tau, 0], \end{cases} \] 
  defined by

  \[ \Gamma : X_R \subset C([-\tau, t^*]; V) \to C([-\tau, t^*]; V) \]

  with values into \( L^2 \).

We will use the following properties of the function \( G \):

**Lemma 6.1.** ([49], Lemma 3.4) Take \( R > 0 \). Then \( G \) is well defined on \([0, T] \times V\) with values into \( L^2(I) \). Moreover, we have the following estimates:

(i) there exists \( C_R > 0 \) such that

\[ \forall t \in [0,T], \forall u \in V \text{ s.t. } \|u\|_V \leq R, \]

and

\[ \|G(t, u)\|_{L^2(I)} \leq C_R(1 + \|u\|_V), \]

\[ \forall t \in [0, T], \forall u, v \in V \text{ s.t. } \|u\|_V, \|v\|_V \leq R; \]

(ii) there exists \( C > 0 \) such that

\[ \|G(t, u) - G(t, v)\|_{L^2(I)} \leq C|t - t'|, \]

\[ \forall t, t' \in [0, T], \forall u \in V \text{ s.t. } \|u\|_V \leq R. \]

These results come directly from [52].

Concerning the memory term, we have a similar result:

**Lemma 6.2.** Take \( T > 0 \). Then

\[ F : [0, T] \times C([-\tau, T]; L^2(I)) \to L^2(I), \quad F(t, u)(x) := f \left( \int_{-\tau}^{0} k(s, x) u(t + s)(x) \, ds \right) \]

is well defined. Moreover, we have the following estimates:

(i) there exists \( C > 0 \) such that

\[ \forall t \in [0, T], \forall u \in C([-\tau, T], L^2(I)), \]

and

\[ \|F(t, u)\|_{L^2(I)} \leq C\|f\|_{\infty}, \]

\[ \forall t \in [0, T], \forall u, v \in C([-\tau, T], L^2(I)); \]

(ii) there exists \( C > 0 \) such that

\[ \|F(t, u) - F(t, v)\|_{L^2(I)} \leq C|u - v|_{C([-\tau, T], L^2(I))}, \]

\[ \forall t \in [0, T], \forall u, v \in C([-\tau, T], L^2(I)); \]

\[ \|F(t, u) - F(t', u)\|_{L^2(I)} \leq C|u(t) - u(t')|_{C([-\tau, 0], L^2(I))}, \]

\[ \forall t, t' \in [0, T], \forall u \in C([-\tau, T], L^2(I)) \]
6.2.b. Step 1: $\Gamma$ maps $\mathcal{X}_R$ into itself if $t^* > 0$ is sufficiently small.

We recall that

$$\forall t \in [0, t^*], \quad \Gamma(u)(t) = e^{t\tilde{A}}u_0(0) + \int_0^t e^{(t-s)\tilde{A}} \left[ \tilde{G}(s, u(s)) + \mathcal{F}(s, u) \right] ds,$$

$$= e^{t\tilde{A}}u_0(0) + \int_0^t e^{(t-s)\tilde{A}} \left[ \tilde{G}(s, u(s)) + \mathcal{F}(s, u) \right] ds.$$

Denote

$$U_1(t) := e^{t\tilde{A}}u_0(0),$$

$$U_2(t) := \int_0^t e^{(t-s)\tilde{A}}\tilde{G}(s, u(s)) ds,$$

$$U_3(t) := \int_0^t e^{(t-s)\tilde{A}}\mathcal{F}(s, u) ds.$$

Then we claim that

$$(6.10) \quad U_1, U_2, U_3 \in C([0, t^*]; V), \quad U_1(0) = u_0(0), \quad U_2(0) = 0 = U_3(0).$$

Indeed: from standard regularity arguments, it is clear that $U_1(t), U_2(t), U_3(t) \in V$ for all $t \in (0, t^*)$. Moreover, $U_1(0) = u_0(0), U_2(0) = 0 = U_3(0)$. It remains to show the continuity. If $t, t + h \in [0, t^*]$, then

- first

$$\|U_1(t + h) - U_1(t)\|_V = \left\| (-\tilde{A})^{1/2}(e^{(t+h)\tilde{A}} - e^{t\tilde{A}})u_0(0) \right\|_{L^2(I)}$$

$$= \left\| \left( \int_t^{t+h} e^{\sigma \tilde{A}} \frac{d\sigma}{\sigma} \right) (-\tilde{A})^{1/2}u_0(0) \right\|_{L^2(I)}$$

$$\leq \left\| \int_t^{t+h} e^{\sigma \tilde{A}} \frac{d\sigma}{\sigma} \right\| \|(-\tilde{A})^{3/4}u_0(0)\|_{L^2(I)}$$

which gives that $U_1 \in C([0, t^*]; V)$;

- next, assume that $h > 0$, to simplify; then we have

$$\|U_2(t + h) - U_2(t)\|_V$$

$$= \left\| (-\tilde{A})^{1/2} \left( \int_0^{t+h} e^{(t+h-s)\tilde{A}}\tilde{G}(s, u(s)) ds - \int_0^t e^{(t-s)\tilde{A}}\tilde{G}(s, u(s)) ds \right) \right\|_{L^2(I)}$$

$$= \left\| (-\tilde{A})^{1/2} \left( e^{h\tilde{A}}I - I \right) \int_0^t e^{(t-s)\tilde{A}}\tilde{G}(s, u(s)) ds \right\|_{L^2(I)}$$

$$\quad + \left\| (-\tilde{A})^{1/2} \int_t^{t+h} e^{(t+h-s)\tilde{A}}\tilde{G}(s, u(s)) ds \right\|_{L^2(I)}$$

$$\quad + \left\| \int_t^{t+h} (-\tilde{A})^{1/2} e^{(t+h-s)\tilde{A}}\tilde{G}(s, u(s)) ds \right\|_{L^2(I)}.$$
then, using once again (6.2), we have
\[
\left\| (e^{h\hat{A}} - Id) \int_0^t (-\hat{A})^{1/2} e^{(t-s)\hat{A}} \tilde{G}(s, u(s)) \, ds \right\|_{L^2(t)}
\]
\[
= \left\| \left( \int_0^h \hat{A} e^{\tau \hat{A}} \, d\tau \right) \int_0^t (-\hat{A})^{1/2} e^{(t-s)\hat{A}} \tilde{G}(s, u(s)) \, ds \right\|_{L^2(t)}
\]
\[
= \left\| \left( \int_0^h (-\hat{A})^{3/4} e^{\tau \hat{A}} \, d\tau \right) \left( \int_0^t (-\hat{A})^{1/4} e^{(t-s)\hat{A}} \tilde{G}(s, u(s)) \, ds \right) \right\|_{L^2(t)}
\]
\[
\leq C \left( \int_0^h \frac{1}{\tau^{3/4}} \, d\tau \right) \left( \int_0^t \frac{1}{(t-s)^{3/4}} C_R(1 + R) \, ds \right)
\]
\[
= 16CC_R(1 + R)h^{1/4}t^{1/4};
\]
in the same way, using (6.1) we have
\[
\left\| \int_t^{t+h} (-\hat{A})^{1/2} e^{(t+h-s)\hat{A}} \tilde{G}(s, u(s)) \, ds \right\|_{L^2(t)}
\]
\[
\leq \int_t^{t+h} \frac{c}{\sqrt{t + h - s}} \| \tilde{G}(s, u(s)) \|_{L^2(t)} \, ds
\]
\[
\leq C_R(1 + R) \int_t^{t+h} \frac{c}{\sqrt{t + h - s}} \, ds = C_R(1 + R)O(\sqrt{|h|});
\]
hence
\[
\| U_2(t + h) - U_2(t) \|_V \leq C_R(1 + R)O(|h|^{1/4}),
\]
which gives that \( u_2 \in C([0, t^*]; V) \);

• finally, using (6.1), (6.2) and (6.7) we have (still assuming that \( h > 0 \) in order to simplify)
\[
\left\| U_3(t + h) - U_3(t) \right\|_V
\]
\[
= \left\| \left( e^{h\hat{A}} - Id \right) \int_0^t e^{(t+s)\hat{A}} F(s, u) \, ds - \int_0^t e^{(t-s)\hat{A}} F(s, u) \, ds \right\|_{L^2(t)}
\]
\[
\leq \left\| (e^{h\hat{A}} - Id) \int_0^t (-\hat{A})^{1/2} e^{(t-s)\hat{A}} F(s, u) \, ds \right\|_{L^2(t)}
\]
\[
+ \left\| \int_t^{t+h} (-\hat{A})^{1/2} e^{(t+h-s)\hat{A}} F(s, u) \, ds \right\|_{L^2(t)}
\]
\[
\leq \| f \|_{\infty}O(|h|^{1/4}),
\]
which gives that \( U_3 \in C([0, t^*]; V) \).

We conclude that \( \Gamma(u) \in C([0, t^*]; V) \), and since \( \Gamma(u)(0^+) = \Gamma(u)(0^-) \), we have that \( \Gamma(u) \in C([-\tau, t^*]; V) \).

From the previous study, we also have that
\[
\forall t \in [0, t^*], \quad \| \Gamma u(t) \|_V \leq \| U_1(t) \|_V + \| U_2(t) \|_V + \| U_3(t) \|_V
\]
\[
\leq \| u_0(0) \|_V + \int_0^t \frac{c}{\sqrt{t - s}} (C_R(1 + R) + \| f \|_{\infty}) \, ds
\]
\[
= \| u_0(0) \|_V + (C_R(1 + R) + \| f \|_{\infty})O(\sqrt{t}).
\]

Choose, e.g.,
\[
R = \| u_0 \|_{C([-\tau, 0]; V)} + 1;
\]
then choosing \( t^* \) small enough, we have that
\[
\forall t \in [-\tau, t^*], \quad \| \Gamma u(t) \|_V \leq R,
\]
6.2.c. **Step 2:** \( \Gamma \) is a contraction if \( t^* \in (0,1) \) is sufficiently small.

Of course,

\[
\forall u, v \in X_R, \forall t \in [-\tau, 0], \quad \Gamma(u)(t) = \Gamma(v)(t).
\]

So we study the difference \( \Gamma(u)(t) - \Gamma(v)(t) \) for \( t \in [0, t^*] \). As we did previously, we have

\[
\Gamma(u)(t) - \Gamma(v)(t) = \int_0^t e^{(t-s)\tilde{A}} \left[ \tilde{G}(s, u(s)) - \tilde{G}(s, v(s)) \right] ds
+ \int_0^t e^{(t-s)\tilde{A}} \left[ F(s, u) - F(s, v) \right] ds.
\]

Then, using (6.1) and (6.5), we have

\[
\left\| \int_0^t e^{(t-s)\tilde{A}} \left[ \tilde{G}(s, u(s)) - \tilde{G}(s, v(s)) \right] ds \right\|_V
= \left\| \int_0^t (-\tilde{A})^{1/2} e^{(t-s)\tilde{A}} \left[ \tilde{G}(s, u(s)) - \tilde{G}(s, v(s)) \right] ds \right\|_{L^2(I)}
\leq \int_0^t \frac{C}{\sqrt{t-s}} \left\| u(s) - v(s) \right\|_V ds
\leq C_R \left\| u - v \right\|_{C([-\tau, T]; V)} \int_0^t \frac{e}{\sqrt{t-s}} ds
\leq 2cC_R \sqrt{t^*} \left\| u - v \right\|_{C([-\tau, T]; V)}.
\]

And using (6.1) and (6.8), we have

\[
\left\| \int_0^t e^{(t-s)\tilde{A}} \left[ F(s, u) - F(s, v) \right] ds \right\|_V
= \left\| \int_0^t (-\tilde{A})^{1/2} e^{(t-s)\tilde{A}} \left[ F(s, u) - F(s, v) \right] ds \right\|_{L^2(I)}
\leq \int_0^t \frac{C}{\sqrt{t-s}} \left\| u - v \right\|_{C([-\tau, T]; V)} ds
\leq 2cC_R \sqrt{t^*} \left\| u - v \right\|_{C([-\tau, T]; V)}.
\]

We obtain that

\[
\forall t \in [0, t^*], \quad \left\| \Gamma(u)(t) - \Gamma(v)(t) \right\|_V \leq 4cC_R \sqrt{t^*} \left\| u - v \right\|_{C([-\tau, T]; V)},
\]

hence \( \Gamma \) is a contraction if \( t^* \) is small enough.

6.2.d. **Step 3:** Additional regularity of the solution and conclusion of the proof of Proposition 6.1.

Let us recall that \( u \in C([-\tau, t^*]; V) \) and

\[
u(t) = e^{t\tilde{A}} u_0(0) + \int_0^t e^{(t-s)\tilde{A}} \tilde{G}(s, u(s)) ds + \int_0^t e^{(t-s)\tilde{A}} F(s, u) ds.
\]

Let us remark that

\[
t \mapsto \int_0^t e^{(t-s)\tilde{A}} \tilde{G}(s, u(s)) ds \in H^1(0, t^*; L^2(I)) \cap L^2(0, t^*; D(A))
\]

and

\[
t \mapsto \int_0^t e^{(t-s)\tilde{A}} F(s, u) ds \in H^1(0, t^*; L^2(I)) \cap L^2(0, t^*; D(A))
\]
Let us introduce
\[ u \in H^1(0, t^*; L^2(I)) \cap L^2(0, t^*; D(A)) \cap C([-\tau, t^*]; V). \]

This concludes the proof of Proposition 6.1. \(\square\)

6.3. Proof of Theorem 3.2 uniqueness.

In this section we prove the following

**Proposition 6.2.** Given \(T_0 > 0\) and \(u_0 \in C([\tau, 0]; V)\), assume that \(u\) and \(\tilde{u}\) are mild solutions of the problem (3.7) on \([0, T_0]\). Then \(u = \tilde{u}\) on \([0, T_0]\).

**Proof.** Consider \(w = u - \tilde{u}\). Then \(w\) solves
\[
\begin{cases}
  w_t - (\rho w_x)_x = G(t, u) - G(t, \tilde{u}) + f(H) - f(\tilde{H}), & t \in (0, T_0), x \in (-1, 1), \\
  \rho w_x = 0, & t \in (0, T_0), x = \pm 1, \\
  w(s, x) = 0, & s \in [-\tau, 0], x \in (-1, 1).
\end{cases}
\]

Take \(T \in (0, T_0)\). Multiplying the first equation by \(w\) and integrating on \((0, T) \times (-1, 1)\), we obtain
\[
\int_0^T \int_{-1}^1 w w_t \, dx \, dt = \frac{1}{2} \|w(T)\|_{L^2(-1,1)}^2,
\]
\[
\int_0^T \int_{-1}^1 w(-\rho w_x)_x \, dx \, dt \geq 0,
\]
\[
\int_0^T \int_{-1}^1 w(G(t, u) - G(t, \tilde{u})) \, dx \, dt \leq c \int_0^T \int_{-1}^1 w(u - \tilde{u}) \, dx \, dt = c \int_0^T \|w(t)\|_{L^2(-1,1)}^2 \, dt,
\]
\[
\int_0^T \int_{-1}^1 w(f(H) - f(\tilde{H})) \, dx \, dt \leq c \int_0^T \int_{-1}^1 |w| |H - \tilde{H}| \, dx \, dt
\]
\[
\leq c \int_0^T \|w(t)\|_{L^2(-1,1)}^2 \, dt + c \int_0^T \int_{-1}^1 |H - \tilde{H}|^2 \, dx \, dt.
\]

Let us introduce
\[
W(T) := \int_0^T \|w(T)\|_{L^2(-1,1)}^2 \, dT.
\]

Using the previous estimates, we have
\[
\frac{1}{2} W'(T) \leq 2c W(T) + c \int_0^T \int_{-1}^1 |H - \tilde{H}|^2 \, dx \, dt.
\]

Concerning the last term:
\[
\int_0^T \int_{-1}^1 |H - \tilde{H}|^2 \, dx \, dt = \int_0^T \int_{-1}^1 \left( \int_{-\tau}^0 k(s, x)(u(t + s, x) - \tilde{u}(t + s, x)) \, ds \right)^2 \, dx \, dt
\]
\[
\leq c \int_0^T \int_{-1}^1 \left( \int_{-\tau}^0 (u(t + s, x) - \tilde{u}(t + s, x))^2 \, ds \right) \, dx \, dt.
\]
but
\[
\int_0^T \int_{-\tau}^t \left( \int_0^s \left( u(t + s, x) - \tilde{u}(t + s, x) \right)^2 \, ds \right) \, dx \, dt
= \int_0^T \int_{-\tau}^0 \| w(t + s) \|_{L^2(\mathbb{R}^1)}^2 \, ds \, dt = \int_0^T \int_{-\tau}^t \| w(\sigma) \|_{L^2(\mathbb{R}^1)}^2 \, d\sigma \, dt.
\]

Note that the initial condition of \( w \) gives us that
\[
t - \tau \leq 0 \implies \int_{t-\tau}^t \| w(\sigma) \|_{L^2(\mathbb{R}^1)}^2 \, d\sigma = \int_0^t \| w(\sigma) \|_{L^2(\mathbb{R}^1)}^2 \, d\sigma,
\]
and of course,
\[
t - \tau \geq 0 \implies \int_{t-\tau}^t \| w(\sigma) \|_{L^2(\mathbb{R}^1)}^2 \, d\sigma \leq \int_0^t \| w(\sigma) \|_{L^2(\mathbb{R}^1)}^2 \, d\sigma.
\]

Hence
\[
\int_0^T \int_{-\tau}^t \| w(\sigma) \|_{L^2(\mathbb{R}^1)}^2 \, d\sigma \, dt \leq \int_0^T \int_0^t \| w(\sigma) \|_{L^2(\mathbb{R}^1)}^2 \, d\sigma \, dt,
\]
which gives that
\[
\int_0^T \int_{-\tau}^1 \| H - \tilde{H} \|_1^2 \, dx \, dt \leq c \int_0^T \int_0^t \| w(\sigma) \|_{L^2(\mathbb{R}^1)}^2 \, d\sigma \, dt = c \int_0^T W(t) \, dt.
\]

Since \( W \) is nondecreasing, we obtain that
\[
(6.12) \quad \int_0^T \int_{-\tau}^1 \| H - \tilde{H} \|_1^2 \, dx \, dt \leq c TW(T),
\]
and then we are in position to conclude: we deduce from (6.11) and (6.12) that
\[
\frac{1}{2} W'(T) \leq (2c + c'T)W(T).
\]

Finally, integrating with respect to \( T \in (0, T') \), and using that \( W(0) = 0 \), we obtain that
\[
W(T') \leq 2 \int_0^{T'} (2c + c'T)W(T) \, dT.
\]

Then Gronwall’s lemma tells us that \( W = 0 \), and then \( W' = 0 \), which gives that \( w = 0 \). This concludes the proof of Proposition 6.2.

6.4. Proof of Theorem 3.2: the maximal solution is global in time.

Proposition 6.2 is a standard uniqueness result. As a consequence, combining it with the local existence result given in Proposition 6.1, we are able to define the maximal existence time:
\[
(6.13) \quad T^*(u_0) \coloneqq \sup \{ T \geq 0 \text{ s.t. } (6.7) \text{ has a mild solution on } [0, T] \}.
\]

Then, for all \( T < T^*(u_0) \), we have a mild solution \( u_T \) on \([0, T]\), and
\[
0 < T < T' < T^*(u_0) \implies u_T = u_{T'} \text{ on } [0, T],
\]
and this allows us to define the associated maximal solution, defined exactly on \([0, T^*(u_0)]\). It remains to prove the global existence of the maximal solution:

Proposition 6.3. Consider \( u_0 \in C([-\tau, 0]; V) \) and such that \( u_0(0) \in D(A) \cap L^\infty(\Omega) \), and the associated maximal mild solution, defined in \([0, T^*(u_0)]\). Then \( T^*(u_0) = +\infty \).

Proposition 6.3 allows us to conclude the proof of Theorem 3.2. So it remains to prove Proposition 6.3 and for this we begin by proving a boundedness property.
6.4a. \( L^\infty \) bound of the solution.

First of all, let us prove a general and useful boundedness property. Following the proofs developed by [49], without the memory term, we first state a known preliminary result (see Lemma 6.1 in [49]):

**Lemma 6.3.** Let \( u \in V \). Then, for all \( M \), \((u - M)^+ := \sup(u - M, 0) \in V \) and \((u + M)^- := \sup(-(u + M), 0) \in V \). Moreover for a.e. \( x \in I \)

\[
(u - M)^+(x) = \begin{cases} 
  u_x(x) & (u - M)(x) > 0 \\
  0 & (u - M)(x) \leq 0 
\end{cases}
\]

and for a.e. \( x \in I \)

\[
((u + M)^-)(x) = \begin{cases} 
  0 & (u + M)(x) > 0 \\
  -u_x(x) & (u + M)(x) \leq 0 
\end{cases}
\]

Then we can prove the theorem below, where we just add the memory term to the proof of Theorem 3.3 in [49]:

**Theorem 6.1.** Consider \( u_0 \in C([-\tau, 0]; V) \) and \( u_0(0) \in D(A) \cap L^\infty(I) \), \( T > 0 \) and \( u \) a mild solution of (3.7) defined on \([0, T]\). Let us denote

\[
M_1 := \left( \|Q\|_{L^\infty(I)} \|\gamma\|_{L^\infty(\mathbb{R})} \|\beta\|_{L^\infty(\mathbb{R})} + \|f\|_{L^\infty(\mathbb{R})} \right)^+ / \varepsilon_1
\]

and

\[
M := \max\{\|u_0(0)\|_{L^\infty(I)}, M_1\}
\]

Then \( u \) satisfies

\[
\|u\|_{L^\infty((0,T) \times I)} \leq M.
\]

**Proof.** Let us set \( B := \{x \in I : u(t,x) > M\} \) and multiply the equation satisfied by \( u \) by \((u - M)^+\), then we get the equation below using the previous Lemma and the boundary conditions satisfied by \( u \).

\[
\int_I u_t(u - M)^+ dx + \int_I \rho((u - M)^+)^2_x dx = \int_I [Q\beta(u) - R_e(u) + F(u(t))] (u - M)^+ dx
\]

Moreover, for \( x \in B \),

\[
Q\beta(u) - \varepsilon(u)u|u|^3 + F(u(t)) \leq \|Q\|_{\infty}\|\beta\|_{\infty} - \varepsilon_1 M^4 + \|f\|_{\infty} \leq 0
\]

thanks to our definition of \( M \) and to the assumptions on \( F \). Then

\[
\frac{1}{2} \frac{d}{dt} \int_I (u - M)^+^2 dx = \int_I u_t(u - M)^+ dx \leq 0.
\]

for all \( t \in [0, T] \). Therefore \( t \mapsto ||(u - M)^+(t)||^2_{L^2(I)} \) is nonincreasing on \([0, T]\). Since \((u_0(0) - M)^+ \equiv 0\), we obtain that \( u(t,x) \leq M \) for all \( t \in [0, T] \) and for a.e. \( x \in I \).

In the same way, we can multiply the equation (2.4) by \((u + M)^-\) and then we obtain

\[
\frac{d}{dt} \int_I (u + M)^-^2 dx \leq 0.
\]

Finally, since \((u_0(0) + M)^- \equiv 0\) we have that \( u(t,x) \geq -M \) for all \( t \in [0, T] \) and a.e. \( x \in I \). \(\Box\)
6.4.b. Proof of Proposition 6.3

From the previous theorem one may deduce that, for any \( u_0(0) \in D(\Lambda) \cap L^\infty(I) \), the \( L^\infty \)-norm of the solution remains bounded on \([0, T^*(u_0))\). To ensure the global existence of the mild solution for the Sellers-type model, we argue by contradiction: we are going to prove that, if \( T^*(u_0) < +\infty \), then \( t \mapsto u(t) \) can be extended up to \( T^*(u_0) \) (and then further), which will be in contradiction with the maximality of \( T^*(u_0) \).

Let us assume that \( T^*(u_0) < +\infty \). Note that since \( u_0(0) \in D(\Lambda) \cap L^\infty(I) \), Theorem 6.1 implies that \( \|u\|_{L^\infty((0,T^*(u_0)) \times I)} \leq M \). It follows that

\[
R_* = \|u\|_{L^\infty((0,T^*(u_0)) \times I)} \leq C
\]

It remains to prove that there exists \( \lim_{t \uparrow T^*(u_0)} u(t) \) in \( V \). To prove this, we prove that the function \( t \mapsto u(t) \) satisfies the Cauchy criterion. By definition of mild solution, we have

\[
u(t) = U_1(t) + U_2(t) + U_3(t)
\]

with

\[
U_1(t) := e^{t\Lambda} u_0(0),
\]
\[
U_2(t) := \int_0^t e^{(t-s)\Lambda} G(s, u(s)) \, ds,
\]
\[
U_3(t) := \int_0^t e^{(t-s)\Lambda} F(s, u) \, ds.
\]

Now, first, \( U_1 \) has a limit in \( V \): \( U_1(t) \to U_1(t^*) \) as \( t \to t^* \) since it is a semigroup applied to the initial value.

Next, let us prove that \( U_2 \) has also a limit in \( V \) as \( t \to t^* \): if \( t' \leq t < t^* \), we have

\[
\|U_2(t) - U_2(t')\|_V = \left\| \int_0^t e^{(t-s)\Lambda} G(s, u(s)) \, ds - \int_0^{t'} e^{(t-s)\Lambda} G(s, u(s)) \, ds \right\|_V
\]
\[
\leq \left\| \int_0^{t'} e^{(t-s)\Lambda} G(s, u(s)) - e^{(t'-s)\Lambda} G(s, u(s)) \, ds \right\|_V + \left\| \int_{t'}^t e^{(t-s)\Lambda} G(s, u(s)) \, ds \right\|_V.
\]

We study these two last terms, proving that they satisfy the Cauchy criterion, hence both will have a limit in \( V \):

- first the last one: using Theorem 6.1, we have for all \( s \in [0, T^*(u_0)) \):

\[
\|G(s, u(s))\|_{L^2(I)} = \|r(s)q(x)\beta(u) - \varepsilon(u)|u|^3\|_{L^2(I)}
\]
\[
\leq 2 \left( \|r\|_\infty \|q\|_\infty \|\beta\|_\infty + \|\varepsilon\|_\infty M^4 \right),
\]

hence

\[
\left\| \int_{t'}^t e^{(t-s)\Lambda} G(s, u(s)) \, ds \right\|_V \leq \left\| (\tilde{A})^{1/2} \int_{t'}^t e^{(t-s)\Lambda} G(s, u(s)) \, ds \right\|_{L^2(I)}
\]
\[
\leq \int_{t'}^t \left\| (\tilde{A})^{1/2} e^{(t-s)\Lambda} G(s, u(s)) \right\|_{L^2(I)} \, ds \leq \int_{t'}^t \frac{C}{\sqrt{t-s}} \, ds = 2C \sqrt{t-t'};
\]
for the other term:

\[
\| \int_0^t e^{(t-s)A}G(s, u(s)) - e^{(t'-s)A}G(s, u(s)) \|_V \\
= \left\| (e^{(t-t')A} - Id) \int_0^t e^{(t'-s)A}G(s, u(s)) \right\|_V \\
= \left\| (-A)^{1/2}(e^{(t-t')A} - Id) \int_0^t e^{(t'-s)A}G(s, u(s)) \right\|_{L^2(I)} \\
= \left\| (\int_0^{t'-t'} Ae^{-A} d\tau) \left( \int_0^t (-A)^{1/2}e^{(t'-s)A}G(s, u(s)) ds \right) \right\|_{L^2(I)} \\
= \left\| \left( \int_0^{t'-t'} (-A)^{3/4}e^{-A} d\tau \right) \left( \int_0^t (-A)^{3/4}e^{(t'-s)A}G(s, u(s)) ds \right) \right\|_{L^2(I)}.
\]

Using (6.2) and the fact that \( G \) is bounded, we obtain that

\[
\left\| \left( \int_0^{t'-t'} (-A)^{3/4}e^{-A} d\tau \right) \left( \int_0^t (-A)^{3/4}e^{(t'-s)A}G(s, u(s)) ds \right) \right\|_{L^2(I)} \\
\leq C \left( \int_0^{t'-t'} \frac{1}{\tau^{3/4}} d\tau \right) \left( \int_0^t \frac{1}{(t'-s)^{3/4}} ds \right) = C'(t-t')^{3/4}(t')^{1/4}.
\]

From these two estimates, we deduce that \( t \mapsto U_2(t) \) satisfies the Cauchy criterion and has a limit as \( t \rightarrow T^*(u_0) \) if \( T^*(u_0) < +\infty \).

In the same way, \( t \mapsto U_3(t) \) satisfies the Cauchy criterion as \( t \rightarrow T^*(u_0) \). It follows that \( t \mapsto u(t) \) has a limit as \( t \rightarrow T^*(u_0) \) if \( T^*(u_0) < +\infty \), which contradicts the maximality of \( T^*(u_0) \), hence \( T^*(u_0) = +\infty \).

**Remark 6.1.** If we assume that \( R_c \) is linear as in the Budyko models, i.e. \( R_c(u) = A + Bu \), everything remains true, and we can apply the fixed point theorem and extend the solution to a global one.

### 7. Proof of Theorem 3.3

Without loss of generality, we can assume that

\[ 0 < T < \delta. \]

Using the extra assumption (3.11), we see that the history term satisfies

\[
\forall t \in (0, T), \quad H(t, x, u) = \int_{-\tau}^0 k(s, x)u(t+s, x)ds = \int_{-\tau}^{t-\delta} k(s, x)u(t+s, x)ds \\
= \int_{t-\tau}^{t-\delta} k(\sigma - t, x)u(\sigma, x)d\sigma = \int_{t-\tau}^{t-\delta} k(\sigma - t, x)u_0(\sigma, x)d\sigma = H(t, x, u_0),
\]

where we used (2.10) and that \([t-\tau, t-\delta] \subset [-\tau, 0]\) since \( t \leq T < \delta \).

Hence during this small interval of time, the memory term depends only on the initial condition, hence

\[
H(t, x, u) = H(t, x, u_0) = H(t, x, \bar{u}).
\]

Let us set \( v := u - \bar{u} \).
7.1. Step 1: the linear problem satisfied by \( v \).

Subtracting the problem (7.1) satisfied by \( u \) with the one satisfied by \( \tilde{u} \), we obtain that the function \( v \) verifies

\[
(7.2) \quad v_t - (\rho(x)v_x)_x = r(t)q(x)\beta(u) - r(t)\tilde{q}(x)\beta(\tilde{u}) - (\varepsilon(u))u^3u - \varepsilon(\tilde{u})u^3\tilde{u}
\]

for all \( t \in (0,T) \) and \( x \in (-1,1) \). We linearize (7.2) thanks to the regularity of \( \beta \) (in Sellers type models) and of \( \varepsilon \), defining

\[
\mu_1(u, \tilde{u}) := \begin{cases} 
\frac{\varepsilon(u)|u|^3u - \varepsilon(\tilde{u})|\tilde{u}|^3\tilde{u}}{u - \tilde{u}} & u \neq \tilde{u} \\
\frac{\partial}{\partial u}(\varepsilon(u)|u|^3u) & u = \tilde{u}
\end{cases}
\]

and

\[
\mu_2(u, \tilde{u}) := \begin{cases} 
\frac{\beta(u) - \beta(\tilde{u})}{u - \tilde{u}} & u \neq \tilde{u} \\
\frac{\partial}{\partial u}(\beta(u)) & u = \tilde{u}
\end{cases}
\]

Let us add and substract \( r(t)\tilde{q}(x)\beta(u) \) and then replace \( \mu_1 \) and \( \mu_2 \) in (7.2), so we obtain the following linear equation with respect to \( v \):

\[
(7.3) \quad v_t - (\rho(x)v_x)_x = r(t)\tilde{q}(x)(\mu_2(u, \tilde{u})v - \mu_1(u, \tilde{u})v + r(t)\beta(u)(q(x) - \tilde{q}(x))).
\]

7.2. Step 2: \( q = \tilde{q} \) on \((-1,x_0)\).

We define the largest interval \([y_1, x_0]\) where \( q \equiv \tilde{q} \) and we want to prove that \( y_1 = -1 \).

Let us set

\[
A^- := \{ x \leq x_0 : (q - \tilde{q})(y) \equiv 0 \quad \forall y \in [x, x_0] \}
\]

If \( A^- \neq \emptyset \), we consider

\[
y_1 := \inf A^-,
\]

and if \( A^- = \emptyset \), we consider

\[
y_1 := x_0,
\]

so that in any case, we know that if \( y_1 > -1 \), and if \( \eta > 0 \) is such that \( y_1 - \eta > -1 \), then there exists \( y_2 \in (y_1 - \eta, y_1) \) such that \( q(y_2) \neq \tilde{q}(y_2) \).

To show that \( y_1 = -1 \), we argue by contradiction, so let us assume that \( y_1 > -1 \).

STEP 2.1 First of all we want to prove that there exists \( t_1 \in [0,T] \) and \( y_2 \in (-1,y_1) \) such that \( v(t, y_2) \) never vanishes on \((0,t_1)\). Since \( q, \tilde{q} \in \mathcal{M} \), we have that \( q - \tilde{q} \in \mathcal{M} \). It follows that there exists \( y'_1 < y_1 \) such that \( q - \tilde{q} \) is analytic on \([y'_1, y_1]\), hence is constantly equal to zero or has only a finite number of zeros. Since we already noted that the definition of \( y_1 \) implies that \( q - \tilde{q} \) cannot be constantly equal to zero on some interval \([y_1 - \eta, y_1]\) (with \( \eta > 0 \)), then \( q - \tilde{q} \) has only a finite number of zeros in \([y'_1, y_1]\), and this implies that

\[
(7.4) \quad \exists y_2 \in (y'_1, y_1) \text{ s.t } (q - \tilde{q})(x) \neq 0 \text{ for all } x \in [y_2, y_1].
\]

Without loss of generality, we can assume that

\[
(7.5) \quad (q - \tilde{q})(x) > 0 \quad \forall x \in [y_2, y_1].
\]

Let us notice that since \( u \) and \( \tilde{u} \) have the same initial condition, we have that \( v(0,x) = 0 \) for all \( x \in [-1,1] \) and that \( v_x(0,x) = 0 \). Using this remark and computing (7.4) at \( t = 0 \) and \( x = y_2 \), we obtain

\[
v_t(0,y_2) = r(0)\beta(u_0)(q - \tilde{q})(y_2),
\]

and then (7.5) implies that \( v_t(0,y_2) > 0 \). Therefore, since \( v(0,y_2) = 0 \), we have that there exists some time \( t_1 \in (0,T) \) such that

\[
v(t, y_2) > 0 \quad \forall t \in (0,t_1).
\]

Moreover, the assumption (3.15) implies that \( v(t, x_0) = 0 \) for all \( t \in (0,T) \).
STEP 2.2 Using the strong maximum principle and the Hopf’s Lemma, we are going to prove that the assumption \( y_1 > -1 \) leads to a contradiction. Consider

\[
K := \max_{t \in [0, t_1], x \in [y_2, x_0]} -\mu_1(u(t, x), \bar{u}(t, x)) + r(t)\bar{q}(x)\mu_2(u(t, x), \bar{u}(t, x)):
\]

\( K \) is chosen so that

\[
R(t, x) := -\mu_1(u(t, x), \bar{u}(t, x)) + r(t)\bar{q}(x)\mu_2(u(t, x), \bar{u}(t, x)) - K \leq 0.
\]

Let also define

\[
w(t, x) := v(t, x)e^{-Kt}.
\]

Using (7.3), we observe that

\[
w_1 - (\rho(x)w_2)_x - R(t, x)w = r(t)\beta(u)(q(x) - \bar{q}(x))e^{-Kt}.
\]

Since

\[
\forall x \in [y_2, x_0], \quad q(x) - \bar{q}(x) \geq 0,
\]

we obtain that \( w \) satisfies

\[
\begin{cases}
w_1 - (\rho(x)w_2)_x - R(t, x)w \geq 0 & (t, x) \in (0, t_1) \times (y_2, x_0), \\
w(0, x) = 0 & x \in [y_2, x_0], \\
w(t, x_0) = 0 & t \in (0, t_1), \\
w(t, y_2) > 0 & t \in (0, t_1)
\end{cases}
\]

where the second condition follows from the initial conditions of (2.1), the third from the assumption (3.15), and the last from Step 2.1.

Let us notice that, since \([y_2, x_0] \subset (\bar{y}, 1)\), we can apply the strong maximum principle (Chapter 3 of [43]). It implies that

\[
w(t, x) > 0 \quad \forall (t, x) \in (0, t_1) \times (y_2, x_0).
\]

Moreover, since \( w(t, x_0) = 0 \) and \( x_0 \neq 1 \), we can apply the Hopf’s Lemma which implies that

\[
w_x(t, x_0) < 0 \quad \forall t \in (0, t_1)
\]

It follows that \( u_x(t, x_0) < \bar{u}_x(t, x_0) \) for all \( t \in (0, t_1) \) which contradicts the second assumption in (3.15).

As a consequence, the assumption \( y_1 > -1 \) is false and this implies that \( y_1 = -1 \), therefore \( q \equiv \bar{q} \) on \((-1, x_0]\).

7.3. Conclusion.

The proof is equivalent for \([x_0, 1]\), and one can show that \( q \equiv \bar{q} \) on \((-1, 1)\).

The uniqueness result of Theorem 3.2 implies that \( u \equiv \bar{u} \) on \([0, +\infty) \times (-1, 1] \). \( \square \)

8. Proof of Theorem 3.4

We follow the strategy used in [49, 39] (that was adapted from the method introduced in Imanuvilov-Yamamoto, to study the Sellers case), focusing on the changes brought by the memory term. The proof of the stability result is decomposed in several steps, we give the main intermediate results and we will refer to [49, 39] for some details.

Remember that \( u \) and \( \bar{u} \) are the solutions of (3.11) and (3.12). Thanks to the assumptions, we can assume that \( T < \delta \) without loss of generality.
8.1. Step 1: the problem solved by the difference $w := u - \tilde{u}$.

Clearly the difference

$$w := u - \tilde{u}$$

satisfies the problem

$$
\begin{cases}
    w_t - (\rho w_x)_x = K^* + K + \tilde{K} + K^h, & t > 0, x \in (-1, 1), \\
    \rho w_x = 0, & x = \pm 1, \\
    w(s, x) = u_0(s, x) - \tilde{u}_0(s, x), & s \in [-\tau, 0], x \in (-1, 1),
\end{cases}
$$

where the source terms $K^*, K, \tilde{K}$ and $K^h$ are defined by

$$
K^* := r(t)(q(x) - \tilde{q}(x))\beta(u),
$$

$$
K := r(t)\tilde{q}(x)(\beta(u) - \beta(\tilde{u})),
$$

$$
\tilde{K} := -\varepsilon(u)|u|^3 u + \varepsilon(\tilde{u})|\tilde{u}|^3 \tilde{u},
$$

$$
K^h := f(H) - f(\tilde{H}).
$$

Note that compared to [49, 39], our goal is similarly to estimate from above $K^*$ and the only difference lies in the presence of $K^h$. However, since we assumed that the memory kernel $k$ is supported in $[-\tau, -\delta]$, and that $T < \delta$, it is clear (as in the previous section) that the memory terms $\tilde{H}$ and $\tilde{H}$ are directly determined from the initial conditions $u_0$ and $\tilde{u}_0$.

8.2. Step 2: a useful property of the source term $K^*$.

We claim that $K^*$ satisfies the following property (classical in that question of determining a source term):

$$
\exists C_0 > 0 \quad \text{s.t.} \quad \forall t \in (0, T), \forall x \in (-1, 1), \quad \left| \frac{\partial K^*}{\partial t}(t, x) \right| \leq C_0 K^*(T', x).
$$

Since

$$
K^*_t := r'(t)(q(x) - \tilde{q}(x))\beta(u) + r(t)(q(x) - \tilde{q}(x))\beta'(u)u_t,
$$

(8.7) is an easy consequence of the following regularity result:

**Lemma 8.1.** Under the regularity assumptions of Theorem 3.4, the solution $u$ of (5.1) satisfies: $u \in L^\infty((0, T) \times I)$, and more precisely, there exists $C(T, M, M') > 0$ such that, for all $u_0 \in U^{(loc)}_M$, for all $q \in Q^{(loc)}_{M'}$, we have

$$
\|u_t\|_{L^\infty((0, T) \times I)} \leq C(T, M, M').
$$

Lemma 8.1 can be proved as Theorem 3.4 and Corollary 3.1 of [49], noting that the additive memory term satisfies

$$
\forall t \in (0, T), \quad H(t, x) = \int_{t-\tau}^{t-\delta} k(s - t, x)u(s, x) \, ds = \int_{t-\tau}^{t-\delta} k(s - t, x)u_0(s, x) \, ds,
$$

hence

$$
H_t = k(-\tau, x)u_0(t - \tau, x) - k(-\delta, x)u_0(t - \delta, x) - \int_{t-\tau}^{t-\delta} k_t(s - t, x)u_0(s, x) \, ds,
$$

which is bounded from the regularity assumptions of Theorem 3.4.
8.3. Step 3: a Carleman estimate on the problem solved by \( z := w \).

Consider \( z := w_t \). It is solution of the following problem

\[
\begin{aligned}
&z_t - \langle \rho z \rangle_x = K_t^\ast + K_t + \tilde{K}_t + K^h_t, \\
&\rho z_x = 0, \quad x = \pm 1,
\end{aligned}
\tag{8.8}
\]

and we can apply standard Carleman estimates for such degenerate operator (see [10, 49]): choosing

- \( \theta : (t_0, T) \to \mathbb{R}_+^* \) smooth, strictly convex, such that
  \( \theta(t) \to +\infty \) as \( t \to t_0^+ \) and as \( t \to T^- \),
  and \( \theta'(T') = 0 \) such that \( T' \) is the point of global minimum,
- and \( p : (-1, 1) \to [1, +\infty) \) well designed with respect to the degeneracy
  (see [10] in the typical degenerate case, and [49] for an explicit construction
  in the case of the Sellers model),

and considering

\[ \sigma(t, x) = \theta(t)p(x), \]

and \( R > 0 \) large enough, the following Carleman estimate holds true, see Theorem 4.2 in [49]:

\[
\begin{aligned}
\int_{t_0}^{T} \int_{-1}^{1} \left( R^3 \theta^3 (1 - x^2) z^2 + R \theta (1 - x^2) z_x^2 + \frac{1}{R \theta} z_t^2 \right) e^{-2 R \sigma} \\
\leq C \left( \int_{t_0}^{T} \int_{-1}^{1} (K_t^\ast + K_t + \tilde{K}_t + K^h_t)^2 e^{-2 R \sigma} + \int_{t_0}^{T} \int_{a}^{b} R^3 \theta^3 z^2 e^{-2 R \sigma} \right).
\end{aligned}
\tag{8.9}
\]

Then,

- since \( u_t \) is bounded, we immediately obtain
  \[ |K_t| + |\tilde{K}_t| \leq c(|w| + |z|), \]

and this allows to show that

\[
\int_{t_0}^{T} \int_{-1}^{1} (K_t^2 + \tilde{K}_t^2) e^{-2 R \sigma} \leq c \left( \|w(T')\|_{L^2(1)} + \int_{t_0}^{T} \int_{-1}^{1} z^2 e^{-2 R \sigma} \right); \tag{8.10}
\]

- for the memory term: using the explicit form given by the initial conditions, we immediately have
  \[ K^h_t = f'(H) H_t - f'((\tilde{H})) \tilde{H}_t = f'(H) (H_t - \tilde{H}_t) + (f'(H) - f'((\tilde{H}))) \tilde{H}_t, \]

hence

\[ |K^h_t| \leq c \left( |H_t - \tilde{H}_t| + |H - \tilde{H}| \right) \]

and then

\[
\int_{t_0}^{T} \int_{-1}^{1} K^h_t e^{-2 R \sigma} \leq c \|u_0 - \tilde{u}_0\|_{C([-\tau, 0], V)}^2. \tag{8.11}
\]

Using (8.10) and (8.11) in the Carleman estimate (8.9), we obtain

\[
\begin{aligned}
\int_{t_0}^{T} \int_{-1}^{1} \left( R^3 \theta^3 (1 - x^2) z^2 + R \theta (1 - x^2) z_x^2 + \frac{1}{R \theta} z_t^2 \right) e^{-2 R \sigma} \\
\leq C \left( \int_{t_0}^{T} \int_{-1}^{1} (K_t^\ast)^2 e^{-2 R \sigma} + \int_{t_0}^{T} \int_{a}^{b} R^3 \theta^3 z^2 e^{-2 R \sigma} \right) \\
+ C \left( \|w(T')\|_{L^2(1)} + \int_{t_0}^{T} \int_{-1}^{1} z^2 e^{-2 R \sigma} \right) + c \|u_0 - \tilde{u}_0\|_{C([-\tau, 0], V)}^2.
\end{aligned}
\]

Absorbing the term \( \int_{t_0}^{T} \int_{-1}^{1} z^2 e^{-2 R \sigma} \) in the left-hand side (this is classical, using Hardy type inequalities, see [10, 49]), we finally obtain
Step 4: an estimate from above.

and it is classical (see Imanuvilov-Yamamoto [33], equation (3.17)) that

\( \leq C \int_{t_0}^{T} \int_{-1}^{1} (K^*_t)^2 e^{-2R\sigma} \, dx \, dt \]

\[ + C\|w(T')\|_{L^2(I)}^2 + C\|u_0 - \tilde{u}_0\|_{C([-\tau,0],V)}^2. \]

8.4. Step 4: an estimate from above.

Using Step 2, we see that

\[ \int_{t_0}^{T} \int_{-1}^{1} (K^*_t)^2 e^{-2R\sigma} \, dx \, dt \leq C_0^2 \int_{t_0}^{T} \int_{-1}^{1} K^*(T')^2 e^{-2R\sigma} \, dx \, dt \]

and it is classical (see Imanuvilov-Yamamoto [33], equation (3.17)) that

\[ \int_{t_0}^{T} e^{-2R\sigma} \, dt = o(e^{-2R\sigma(T')}) \quad \text{as } R \to +\infty. \]

(This is due to the convexity of the function \( \theta \), that attains its minimum at \( T' \).)

Hence

\[ \int_{t_0}^{T} \int_{-1}^{1} (K^*_t)^2 e^{-2R\sigma} \, dx \, dt = o\left( \int_{-1}^{1} K^*(T')^2 e^{-2R\sigma(T')} \, dx \right). \]

8.5. Step 5: an estimate from below.

As in [49, 39], we have

\[ \int_{-1}^{1} z(T')^2 e^{-2R\sigma(T')} \leq cI_0. \]

8.6. Step 6: conclusion.

Now we are in position to conclude: using the equation in \( w \):

\[ K^*(T') = z(T') - (\rho w_x)_x(T') - K(T') - \tilde{K}(T') - K^h(T'), \]

hence

\[ \int_{-1}^{1} K^*(T')^2 e^{-2R\sigma(T')} \]

\[ \leq C\left( \int_{-1}^{1} (z(T')^2 + (\rho w_x)_x(T')^2) + K(T')^2 + \tilde{K}(T')^2 + K^h(T')^2 e^{-2R\sigma(T')} \right); \]

now note that

\[ |K(T')| \leq c|w(T')|, \quad |\tilde{K}(T')| \leq c|w(T')|, \]

and

\[ \int_{-1}^{1} K^h(T')^2 e^{-2R\sigma(T')} \leq C\|u_0 - \tilde{u}_0\|_{C([-\tau,0],V)}^2; \]

then using (8.11), (8.12) and (8.13), we obtain that

\[ \int_{-1}^{1} K^*(T')^2 e^{-2R\sigma(T')} \leq C\left( \|w(T')\|_{D(A)}^2 \right. \]

\[ \left. + \|w_1\|_{L^2([t_0,T] \times (a,b))]^2 + u_0 - \tilde{u}_0\|_{C([-\tau,0],V)}^2 \right). \]

Looking to the form of \( K^* \), we obtain [33,18]. \( \Box \)
9. Proof of Theorem 4.1

9.1. The strategy to prove Theorem 4.1

The method is usual:

- first we approximate the control β by a sequence of smooth functions βj,
- then we consider the approximate problem associated to βj, and we denote uj its (unique) solution,
- we obtain suitable assumptions on uj, and we pass to the limit.

Lemma 9.1. The family (uj)j is relatively compact in C([0, T]; L2(Γ)).

Lemma 9.2. The family (γj)j is weakly relatively compact in L2(0, T; L2(Γ)).

Assume that Lemmas 9.1 and 9.2 hold true. Then, we can extract from (uj, γj)j a subsequence (uj', γj')j' such that

uj' → u∞ in C([0, T]; L2(Γ)) and γj' → γ∞ in L2(0, T; L2(Γ)).

Then we prove that

Lemma 9.3. The functions u∞ and γ∞ satisfy the following formula

∀t ∈ [0, T], u∞(t) = e^tA u0(0) + \int_0^t e^{(t-s)A} γ∞(s) ds.

Moreover,
u∞ ∈ H^1(0, T; L^2(Γ)) ∩ L^2(0, T; D(A)) ∩ C(]-τ, T[; V),

and

∀s ∈ []-τ, 0[; u∞(s) = u0(s).

Finally, it remains to prove that u∞ is solution of the original Byudyko problem, and so we have to prove that γ∞ satisfies the set inclusion. This is the object of the last lemma:
Lemma 9.4. The limit function $\gamma_\infty$ satisfies the set inclusion:

$$ (9.4) \quad \gamma_\infty(t)(x) + (a + bu_\infty(t, x)) - f(\int_0^\infty k(s, x)u_\infty(t + s, x) \, ds) \in \gamma(t)q(x)\beta(u_\infty(t, x)) \text{ a.e.} (t, x) \in (0, T) \times I. $$

Therefore the function $u_\infty$ is solution of the original Byudyko problem (4.1).

Finally, note that using the Cantor diagonal process, we can extract subsequences that converge in the same way in all compact subsets of $[0, +\infty)$, hence $u_\infty$ is well defined on $[0, +\infty)$.

9.2. Proof of Lemma 9.1

First note that from Theorem 6.1 we already know that the family $(u_j)_j$ belongs to $L^\infty([0, +\infty) \times (-1, 1))$ and is uniformly bounded in this space.

To prove that the sequence $(u_j)_j$ is relatively compact in $C([0, T]; L^2(I))$, we are going to apply the Ascoli-Arzela theorem (we refer, e.g., to [51] Theorem 1.3.1 p. 10): we have to prove that

- $(u_j)_j$ is equicontinuous, that is:
  $$ \sup_j \sup_{t \in [0, T]} \|u_j(t + h) - u_j(t)\|_{L^2(I)} \to 0 \quad \text{as} \ h \to 0, $$
- and that, for all $t \in [0, T]$, the set of traces $\{u_j(t), j \geq 1\}$ is relatively compact in $L^2(I)$.

The result on the set of traces $\{u_j(t), j \geq 1\}$ follows from a regularity result: we know from 9.2 that $u_j(t) \in V$. Moreover, it follows from the proof of Proposition 6.1 that there is some $M^*$ independent of $j$ such that

$$ \sup_{t \in [0, T]} \|u_j(t)\|_V \leq M^*. $$

Since the injection of $V$ into $L^2(I)$ is compact, we obtain that the set of traces $\{u_j(t), j \geq 1\}$ is relatively compact in $L^2(I)$.

Next, from the problem satisfied by $u_j$, we have the following integral representation formula

$$ u_j(t) = e^{tA}u_0(0) + \int_0^t e^{(t-s)A} \gamma_j(s) \, ds, $$

hence (if $h > 0$)

$$ u_j(t + h) - u_j(t) = \left( e^{(t+h)A}u_0(0) - e^{tA}u_0(0) \right) + \left( \int_0^{t+h} e^{(t+h-s)A} \gamma_j(s) \, ds - \int_0^t e^{(t-s)A} \gamma_j(s) \, ds \right) $$

$$ = \left( e^{(t+h)A} - e^{tA} \right) u_0(0) + \left( \int_t^{t+h} e^{(t+h-s)A} \gamma_j(s) \, ds \right) $$

$$ + \left( \int_0^t \left( e^{(t+h-s)A} - e^{(t-s)A} \right) \gamma_j(s) \, ds \right). $$

We estimate these three terms:

- first, by usual estimates
  $$ \left\| e^{(t+h)A}u_0(0) - e^{tA}u_0(0) \right\|_{L^2(I)} \leq c|h\|u_0(0)\|_{D(A)}; $$
- next,
  $$ \left\| \int_t^{t+h} e^{(t+h-s)A} \gamma_j(s) \, ds \right\| \leq \int_t^{t+h} \|\gamma_j(s)\|_{L^2(I)} \, ds \leq c|h|; $$

and

$$ \left\| \int_0^t \left( e^{(t+h-s)A} - e^{(t-s)A} \right) \gamma_j(s) \, ds \right\| \leq c|h|. $$
These three estimates prove that

\[ \left\| \int_0^t (e^{(t+h)-s} - e^{(t-s)}A)\gamma_j(s) \, ds \right\|_{L^2(I)} = \left\| (e^{hA} - I) \int_0^t e^{(t-s)}A\gamma_j(s) \, ds \right\|_{L^2(I)} \]

\[ = \left\| \left( \int_0^h A e^{sA} \, ds \right) \left( \int_0^t e^{(t-s)}A\gamma_j(s) \, ds \right) \right\|_{L^2(I)} \]
\[ = \left\| \left( \int_0^h (-A)^{1/2} e^{sA} \, ds \right) \left( \int_0^t (-A)^{1/2} e^{(t-s)}A\gamma_j(s) \, ds \right) \right\|_{L^2(I)} \]
\[ \leq \left( \int_0^h \left\| (-A)^{1/2} e^{sA} \right\|_{L(L^2(I))} \, ds \right) \left( \int_0^t \left\| (-A)^{1/2} e^{(t-s)}A\gamma_j(s) \right\|_{L^2(I)} \, ds \right) \]
\[ \leq c \left( \int_0^h \frac{1}{\sqrt{\sigma}} \, d\sigma \right) \left( \int_0^t \frac{1}{\sqrt{t-s}} \, ds \right) \leq c' \sqrt{h}. \]

These three estimates prove that

\[ \sup_j \sup_{[0,T]} \| u_j(t+h) - u_j(t) \|_{L^2(I)} = O(\sqrt{|h|}) \quad \text{as } h \to 0. \]

Hence the family \((u_j)_j\) is equicontinuous in \(C([0,T]; L^2(I))\). Therefore the Ascoli-Arzelà theorem allows us to say that the family \((u_j)_j\) is relatively compact in \(C([0,T]; L^2(I))\).

\[ \square \]

9.3. Proof of Lemma 9.2

Since the family \((u_j)_j\) is uniformly bounded in \(L^\infty((0,T), \times (-1,1))\), we deduce that the same property holds true for the family \((\gamma_j)_j\), hence also in the space \(L^2(0,T; L^2(I))\). Hence the family \((\gamma_j)_j\) is weakly relatively compact in \(L^2(0,T; L^2(I))\).

\[ \square \]

9.4. Proof of Lemma 9.3

We start from the integral formula

\[ u_j(t) = e^{tA}u_0(0) + \int_0^t e^{(t-s)A}\gamma_j(s) \, ds. \]

Choose \(\xi \in L^2(I)\), and fix \(t \in [0,T]\). Then

\[ \langle \xi, \int_0^t e^{(t-s)A}\gamma_j(s) \, ds \rangle_{L^2(I)} = \int_0^t \langle \xi, e^{(t-s)A}\gamma_j(s) \rangle_{L^2(I)} \, ds \]
\[ = \int_0^t \langle e^{(t-s)A}\xi, \gamma_j(s) \rangle_{L^2(I)} \, ds = \langle z, \gamma_j \rangle_{L^2(0,T; L^2(I))} \]

where \(z \in L^2(0,T; L^2(I))\) is defined by

\[ z(s) := \begin{cases} e^{(t-s)A}\xi, & 0 \leq s \leq t, \\ 0, & t \leq s \leq T. \end{cases} \]

Since \(\gamma_j \rightharpoonup \gamma_\infty\) in \(L^2(0,T; L^2(I))\), we obtain that

\[ \langle z, \gamma_j \rangle_{L^2(0,T; L^2(I))} \rightharpoonup \langle z, \gamma_\infty \rangle_{L^2(0,T; L^2(I))} \quad \text{as } j' \to \infty. \]

But

\[ \langle z, \gamma_\infty \rangle_{L^2(0,T; L^2(I))} = \int_0^t \langle z(s), \gamma_\infty(s) \rangle_{L^2(I)} \, ds = \int_0^t \langle e^{(t-s)A}\xi, \gamma_\infty(s) \rangle_{L^2(I)} \, ds \]
\[ = \int_0^t \langle \xi, e^{(t-s)A}\gamma_\infty(s) \rangle_{L^2(I)} \, ds = \langle \xi, \int_0^t e^{(t-s)A}\gamma_\infty(s) \, ds \rangle_{L^2(I)}. \]
From the fact that $u_{j'} \to u_\infty$ in $C([0,T];L^2(I))$, we deduce from (9.5) that

$$u_\infty(t) = e^{tA}u_0(0) + \int_0^t e^{(t-s)A}\gamma_\infty(s) \, ds.$$ 

Finally, since $\gamma_\infty \in L^2(0,T;L^2(I))$, and $u_0(0) \in D(A)$, $u_\infty$ has the regularity claimed in Lemma 9.3. □

9.5. Proof of Lemma 9.4.

It remains to identify the weak limit $\gamma_\infty$. In order to do this, fix $\kappa > 0$ and let us introduce

$$Q_\kappa := \{(t,x) \in (0,T) \times (-1,1) \mid |r(t)q(x)| \geq \kappa\},$$

and then we define on $Q_\kappa$ the function

$$B(t,x) := \frac{1}{r(t)q(x)} \left(\gamma_\infty(t,x) + (a + bu_\infty) - f(\int_{-\tau}^0 k(s,x)u_\infty(t+s,x) \, ds)\right).$$

Of course this definition is motivated by the fact that, also on $Q_\kappa$, we have

$$\beta_j(u_j(t,x)) = \frac{1}{r(t)q(x)} \left(\gamma_j(t,x) + (a + bu_j) - f(\int_{-\tau}^0 k(s,x)u_j(t+s,x) \, ds)\right).$$

Hence we immediately have that

$$\beta_{j'}(u_{j'}) \to B \quad \text{in } L^2(Q_\kappa) \quad \text{as } j' \to \infty.$$ 

This already tells us that

$$a_i \leq B(t,x) \leq a_f \quad \text{a.e.} \quad (t,x) \in Q_\kappa.$$ 

Indeed, choose $E$ any measurable part of $Q_\kappa$ and $\chi_E$ its characteristic function: then

$$\int_E (B - a_i) = \int_{Q_\kappa} (B - a_i) \chi_E = \lim_{j' \to \infty} \int_{Q_\kappa} (\beta_{j'}(u_{j'}) - a_i) \chi_E.$$ 

Since $\beta_{j'}(u_{j'}) - a_i \geq 0$, the last quantity is nonnegative, and hence

$$\int_E (B - a_i) \geq 0.$$ 

Since this holds true for all $E$, we obtain that $B - a_i \geq 0$ on $Q_\kappa$. In the same way, $B - a_f \leq 0$ on $Q_\kappa$.

Now we conclude by proving that

$$\begin{cases}
(t,x) \in Q_\kappa, \\
u_\infty(t,x) < \bar{u} 
\end{cases} \implies B(t,x) = a_i.$$ 

For $\eta > 0$, we introduce

$$D_{\kappa,\eta} := \{(t,x) \in Q_\kappa \mid u_\infty(t,x) \leq \bar{u} - \eta\}.$$ 

Up to some subsequence, we can assume that

$$u_{j'} \to u_\infty \quad \text{a.e.} \quad (t,x) \in (0,T) \times (-1,1).$$

If $(t,x) \in D_{\kappa,\eta}$, there exists $j'(t,x)$ large enough such that

$$\forall j' \geq j'(t,x), \quad u_{j'}(t,x) \leq \bar{u} - \eta - \frac{\eta}{2}.$$ 

And then, the construction of $\beta_{j'}$ implies that, if $j'$ is large enough, we have

$$\beta_{j'}(u_{j'}(t,x)) = a_i.$$ 

Hence

$$\beta_{j'}(u_{j'}) \to a_i \quad \text{a.e.} \quad (t,x) \in D_{\kappa,\eta}.$$
Now, consider $\psi \in L^2(Q_\kappa)$. We have

$$\iint_{Q_\kappa} (\beta_{j'}(u_{j'}) - B) \psi \to 0 \quad \text{as } j' \to +\infty.$$ 

Moreover, if $\psi$ is supported in $D_{\kappa,\eta}$, then, since $\beta_{j'}(u_{j'}) \to a_i$ a.e. $D_{\kappa,\eta}$, we deduce from the using the dominated pointwise convergence theorem that

$$\iint_{Q_\kappa} (\beta_{j'}(u_{j'}) - B) \psi \to \iint_{Q_\kappa} (a_i - B) \psi \quad \text{as } j' \to +\infty.$$ 

Hence

$$\iint_{Q_\kappa} (a_i - B) \psi = 0$$

and this holds true for all $\psi$ supported in $D_{\kappa,\eta}$, therefore

$$a_i = B \quad \text{on } D_{\kappa,\eta}.$$ 

Letting $\eta \to 0$, we obtain that

$$B = a_i \quad \text{on } \{(t,x) \in Q_\kappa \mid u_\infty(t,x) < \bar{u}\}.$$ 

We can proceed in the same way to prove that

$$B = a_f \quad \text{on } \{(t,x) \in Q_\kappa \mid u_\infty(t,x) > \bar{u}\}.$$ 

And finally, letting $\kappa \to 0$, we obtain that, if $r(t)q(x) \neq 0$, we have

$$\frac{1}{r(t)q(x)} \left(\gamma_\infty(t,x) + (a + bu_\infty) - f \left(\int_{-\tau}^{0} k(s,x)u_\infty(t+s,x) \, ds\right)\right) \begin{cases} 
= a_i & \text{if } u_\infty(t,x) < \bar{u}, \\
= a_f & \text{if } u_\infty(t,x) > \bar{u}, \\
\in [a_i, a_f] & \text{in any case,}
\end{cases}$$

Of course, if $r(t)q(x) = 0$, then

$$\gamma_{j'}(t,x) = -(a + bu_{j'}(t,x)) + f(H_{j'}),$$

and we deduce that

$$\gamma_\infty(t,x) = -(a + bu_\infty(t,x)) + f \left(\int_{-\tau}^{0} k(s,x)u_\infty(t+s,x) \, ds\right).$$

Hence

$$\gamma_\infty(t,x) = r(t)q(x)B(t,x) - (a + bu_\infty(t,x)) + f(H_\infty(t,x)) \quad \in r(t)q(x)\beta(u_\infty(t,x)) - (a + bu_\infty(t,x)) + f(H_\infty(t,x)) \quad \text{a.e.}(t,x) \in (0,T) \times I$$

hence the set inclusion (9.4) is satisfied. \qed

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