Lomax-Rayleigh distribution: traditional and heuristic methods of estimation

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Abstract. We introduce a continuous distribution called the Lomax-Rayleigh (L-R) distribution that extends the Lomax distribution. The generalization of the probability density function, cumulative distribution function of this distribution and the expression for moment generating function was established. We considered a traditional methods of estimation such as the maximum likelihood method and nonlinear least square estimation method to estimate the parameters and utilizing the Artificial Intelligence Algorithms such as Genetics Algorithm and pattern search method in estimation process. A comparison study among that methods is carried out through simulation experiments. We concluded that pattern search method is more efficient than other methods depending upon mean square error criteria.

1. Introduction:
Probability distribution has many applications in describing real world situations. There are lot of researches have shown that some real life data that cannot be modeled adequately by traditional statistical distributions, because the complexity found in it. Recently there is rapid grown direction toward generalization, mixing, Transmuting and exponentiation of existing distributions, so some families and new general formulas of distributions is appeared in papers that dealing with skewed data and data drawn from non-homogenous populations. Some of the earlier works include those Gupta and Kundu, 1999[12], Eugene et al., 2004[9], Famoye et al.,2005[10], Akinsete et al., 2008[4], Miroslav and Balakrishnan., 2012[14], Alzaatreh et al.,2013[3], Adeleke et al., 2013[5]; Akarawak et al., 2013[6] and Akarawak et al., 2015[6], Ghosh and Hamedani 2015[11] and Khaleel et al.,2016[13].

The main idea was that any parametric family of distributions can be incorporated into larger families through an application of the probability integral transform. Then the number of parameters and complexity of new families is increased, so almost in most cases numerical techniques to get estimates of parameters are used, since closed form of estimators of that parameters are difficult to derive it.

Recently Venegas et al. 2019[17] introduced the two parameter Lomax-Rayleigh distribution as compound between Lomax and Rayleigh distribution. Özsoy, 2020[16] used the heuristic optimization approaches such as Genetic Algorithms, Differential Evolution, Particle Swarm Optimization, and Simulated Annealing to estimate the parameter of generalized gamma distribution: comparison among maximum likelihood method and heuristic optimization approaches are proved that later have nice properties in estimation of parameters.

The rest of the paper is organized as follows. In Section 2 we present the probability density function and cumulative distribution function, reliability and hazard function of the Lomax-Rayleigh model with its general formula of moments. In Section 3 we discuss traditional methods such as: maximum likelihood estimation and non-linear least square estimation method and we introduce heuristic optimization approaches such as Genetic Algorithms and pattern search method. In section 4 we present a comparison...
among different estimation methods via simulation experiments. Finally, in Section 5,6 we report the discussion and final conclusions.

2. Lomax-Rayleigh Distribution (L-R)

\[ F(x) = \int f(u) du \]

(1)

Let \( x = -\log (1 - G) \)

Where \( G \) is a cumulative distribution function of another distribution

\[ F(x) = \int_{0}^{\log (1-G)} f(u) du \]

(2)

By differentiating the equation (2) for the base distribution, we get:

\[ f(x) = f(-\log (1 - G)) \frac{g}{(1-G)} \]

(3)

Equation (3) represents a transformation for mixing two distributions through inserting the cumulative distribution function of any distribution for the base distribution so we call it Lomax-X family.

2.1. Derivation of Lomax-Rayleigh Distribution (L-R)

2.2.1. Derivation of the pdf and cdf of R-L

The pdf and cdf of the Lomax-Rayleigh distribution is derived in this section as a class of Lomax-X family of generalized distributions.

**Theorem 2.1.** Let the pdf of a Rayleigh distribution which is a base variable be:

\[ f(x) = 2\lambda x e^{-\lambda x^2}, \ x > 0, \lambda > 0 \]

(4)

And the pdf of a Lomax distributed random variable be:

\[ f(y) = \frac{\alpha}{\beta} \left( 1 + \frac{y}{\beta} \right)^{-(\alpha+1)}, \ y > 0, \alpha, \beta > 0 \]

(5)

Then the pdf of the Rayleigh-Lomax distribution is given by:

\[ f(x) = 2\alpha \theta x(1 + \theta x^2)^{-(\alpha+1)} \ x > 0, \alpha, \theta > 0 \]

(6)

**Proof**

The pdf of the Lomax-X family of distribution is given by:

\[ f(x) = f(-\log (1 - G)) \frac{g}{(1-G)} \]

(7)

Where \( g \) and \( G \) are the pdf and cdf of Rayleigh distribution.

By substituting the pdf and cdf of Rayleigh distribution in equation (4), we get:

\[ f(x) = 2\alpha \theta x \left( 1 + \frac{\lambda x^2}{\beta} \right)^{-(\alpha+1)} \ x > 0, \alpha, \beta, \lambda > 0 \]

(8)

Let: \( \frac{\lambda}{\beta} = \theta \), then the Lomax-Rayleigh distribution be:

\[ f(x) = 2\alpha \theta x(1 + \theta x^2)^{-(\alpha+1)} \ x > 0, \alpha, \theta > 0 \]

Where \( \alpha \) is the shape parameter and \( \theta \) is the scale parameter. The Lomax-Rayleigh distribution is right skewed as shown in Figure 1.
Corollary 2.1. The function of Lomax-Rayleigh is pdf function.

Proof

The integral of function must equal one. That is:

\[ \int_{0}^{\infty} f(x) \, dx = 1 \]

Then

\[ \int_{0}^{\infty} f(x) \, dx = \int_{0}^{\infty} 2\alpha \theta x (1 + \theta x^2)^{-\alpha/2} \, dx \]
\[ = 2\alpha \theta \int_{0}^{\infty} xe^{-\alpha x \log (1 + \theta x^2)} \, dx \]

Let:

\[ y = (\alpha + 1) \log (1 + \theta x^2) \]

Then:

\[ x = \left( \frac{1}{\theta} \right)^{\frac{1}{2}} \left( 1 - e^{\frac{y}{\alpha+1}} \right)^{\frac{1}{2}} \]
\[ dx = \left( \frac{1}{\theta} \right)^{\frac{1}{2}} \left( 1 - e^{\frac{y}{\alpha+1}} \right)^{-\frac{1}{2}} \left( -\frac{y}{\alpha+1} \right) \, dy \]

Then

\[ \int_{0}^{\infty} f(x) \, dx = 2\alpha \theta \int_{0}^{\infty} \left( \frac{1}{\theta} \right)^{\frac{1}{2}} \left( 1 - e^{\frac{y}{\alpha+1}} \right)^{\frac{1}{2}} e^{-y} \left( \frac{1}{\theta} \right)^{\frac{1}{2}} \left( 1 - e^{\frac{y}{\alpha+1}} \right)^{-\frac{1}{2}} \left( -\frac{y}{\alpha+1} \right) \, dy \]
\[ = \frac{\alpha}{\alpha+1} \int_{0}^{\infty} e^{-\gamma \left( \frac{1}{\alpha+1} \right)} \, dy \]

Let:

\[ z = \frac{\alpha}{\alpha+1} y \]
\[ y = \frac{\alpha}{\alpha+1} z \]
\[ dy = \frac{\alpha}{\alpha+1} \, dz \]

\[ \int_{0}^{\infty} f(x) \, dx = \frac{\alpha}{\alpha+1} \int_{0}^{\infty} e^{-\gamma \left( \frac{1}{\alpha+1} \right)} \, dy = \frac{\alpha}{\alpha+1} \int_{0}^{\infty} e^{-\frac{\gamma}{\alpha+1} \left( \frac{1}{\alpha+1} \right)} \, dy = 1 \]

Corollary 2.2. The cdf of Lomax-Rayleigh is:

\[ F(x) = 1 - e^{-\alpha \log (1 + \theta x^2)} \]
Proof

\[ F(x) = \Pr(X \leq x) = \int_{0}^{x} f(u) \, du \]

Then:

\[ F(x) = \int_{0}^{x} 2\alpha \theta u e^{-\left(\alpha + 1\right) \log \left(1 + \theta u^2\right)} \, du \]

Let:

\[ y = (\alpha + 1) \log \left(1 + \theta u^2\right) \]

Then:

\[
\begin{align*}
\frac{1}{\theta} \left(1 - e^{(\alpha + 1) y} \right)^{1/2} & = \frac{1}{\alpha} \left(1 - e^{(\alpha + 1) y} \right)^{1/2} \\
\frac{1}{\alpha} \left(1 - e^{(\alpha + 1) y} \right)^{1/2} & = -\frac{y}{(\alpha + 1)} \\
F(x) & = \frac{\alpha}{(\alpha + 1)} \int_{0}^{(\alpha + 1) \log \left(1 + \theta x^2\right)} e^{-y \left(1 - \frac{1}{\alpha + 1} y^2\right)} \, dy
\end{align*}
\]

Let:

\[
\begin{align*}
z & = \frac{\alpha}{(\alpha + 1)} y \\
y & = \frac{\alpha}{(\alpha + 1)} z \\
dy & = \frac{\alpha}{(\alpha + 1)} dz
\end{align*}
\]

\[ F(x) = \int_{0}^{(\alpha + 1) \log \left(1 + \theta x^2\right)} e^{-z \left(\frac{\alpha + 1}{\alpha}\right)} \, dz = 1 - e^{-\alpha \log \left(1 + \theta x^2\right)} \quad \text{(9)} \]

The survival function and hazard function are as follow:

\[ S(x) = 1 - F(x) = e^{-\alpha \log \left(1 + \theta x^2\right)} \quad \text{(10)} \]

\[ h(x) = \frac{f(x)}{S(x)} = \frac{2 \theta}{1 + \theta x^2} \quad \text{(11)} \]

2.2.2. Moment generating function:

\[ \mu_x(t) = \mathbb{E}(e^{xt}) = \int_{0}^{\infty} e^{xt} 2\alpha \theta x (1 + \theta x^2)^{-(\alpha + 1)} \, dx \]

\[ = 2\alpha \theta \int_{0}^{\infty} x e^{xt} e^{-\left(\alpha + 1\right) \log \left(1 + \theta x^2\right)} \, dx \]

Since: \( e^y = \sum_{l=0}^{\infty} \frac{y^l}{l!} \)

\[ = 2\alpha \theta \int_{0}^{\infty} x \sum_{j=0}^{\infty} \frac{(xt)^j}{j!} e^{-\left(\alpha + 1\right) \log \left(1 + \theta x^2\right)} \, dx \]

\[ = 2\alpha \theta \sum_{j=0}^{\infty} \frac{(t)^j}{j!} \int_{0}^{\infty} x^{j+1} e^{-\left(\alpha + 1\right) \log \left(1 + \theta x^2\right)} \, dx \]

Let:

\[ y = (\alpha + 1) \log \left(1 + \theta x^2\right) \]

Then:

\[ x = \left(\frac{y}{e^{\alpha + 1} - 1}\right)^{1/2} \left(\frac{1}{\theta}\right)^{1/2} \]
\[
dx = \frac{1}{2} \left( \frac{y}{e^\alpha - 1} \right)^{\frac{1}{2}} \left( \frac{y}{e^\alpha - 1} - \frac{1}{\alpha + 1} \right) \left( \frac{1}{\theta} \right)^{1/2}
\]

Then:
\[
\mu_x(t) = \frac{\alpha \theta}{\alpha + 1} \int_0^\infty \left( \frac{y}{e^\alpha - 1} - 1 \right)^{1/2} e^{-y (\alpha + 1)} dy
\]
\[
= \frac{\alpha \theta}{\alpha + 1} \sum_{i=0}^{\infty} c_i^{1/2} (-1)^{1/2} \int_0^\infty e^{-\left(\frac{\alpha - i}{\alpha + 1}\right)} dy
\]

Let:
\[
z = y \left( \frac{\alpha - i}{\alpha + 1} \right)
\]

Then:
\[
y = z \left( \frac{\alpha + 1}{\alpha - i} \right)
\]
\[
dy = \left( \frac{\alpha + 1}{\alpha - i} \right) dz
\]
\[
\mu_x(t) = \alpha \sqrt{\theta} \sum_{i=0}^{\infty} \frac{1}{1} \sum_{i=0}^{\infty} c_i^{1/2} (-1)^{1/2} \frac{1}{(\alpha - i)}
\]

3. Estimation Methods
In this section we will derive the estimators of the unknown parameters by using traditional methods (Maximum likelihood and Nonlinear Least Square) and heuristic method (Genetic algorithm and pattern search)

3.1. Maximum Likelihood Method MLE
The principle of this method is to find the values of the parameters that maximize the likelihood function, where the likelihood function is a function of data and unknown parameters only, so it represent the information of sample.

Then:
\[
\text{Likelihood Function} = L(x, \alpha, \theta) = 2^\alpha \theta^\theta \prod_{i=1}^n x_i e^{-\left(\alpha + 1\right) \sum_{i=1}^n \log (1 + \theta x_i^2)}
\]

Equation (13) is monotonic, so the values of parameters that maximize it is same as that maximize log of it. Then:
\[
\text{LogL}(\alpha, \theta) = n \log(2) + n \log(\alpha) + n \log(\theta) + \sum_{i=1}^n \log(1 + \theta x_i^2) - (\alpha + 1) \sum_{i=1}^n \log (1 + \theta x_i^2)
\]

By taking partial derivatives of equation (14) and equating them to zero, we get normal equation. Then:
\[
\frac{n}{\alpha} - \sum_{i=1}^n \log \left(1 + \theta x_i^2\right) = 0
\]
\[
\frac{n}{\theta} - (\alpha + 1) \sum_{i=1}^n \frac{x_i}{1 + \theta x_i^2} = 0
\]

The solution of equation (15) and (16), represent the estimate of parameters, by substitute equation (15) in equation (16), we get
\[
\frac{n}{\theta} - \left( \frac{\alpha}{\sum_{i=1}^n \log (1 + \theta x_i^2)} + 1 \right) \sum_{i=1}^n \frac{x_i^2}{1 + \theta x_i^2} = 0
\]
The equation (17) is highly nonlinear, so we use Newton Raphson method to get the solution of the parameter $\theta$. Let the $\hat{\theta}_{\text{mle}}$ be the maximum likelihood estimate of $\theta$ then from equation (17) the estimator of parameters $\alpha$ will be:

$$
\hat{\alpha}_{\text{mle}} = \frac{n}{\sum_{i=1}^{n} \log(1 + \theta x_i^2)}
$$

(18)

3.2. Non-Linear Least Square Method NLLSM

The principle of this method is to find the values of the parameters that minimize the square sum of errors. Let:

$$
\hat{F} = \frac{i}{n+1}, \quad \text{be non-parametric estimator of cdf, where}
$$

$n$: sample size.
$i = 1, 2, ..., n$

Then by equating the non-parametric estimator by cdf of distribution, we get:

$$
\hat{F} = 1 - e^{-\alpha \log(1 + \theta x_i^2)}
$$

And

$$
(1 - \hat{F}) = e^{-\alpha \log(1 + \theta x_i^2)}
$$

If we let: $y = (1 - \hat{F})$, then:

$$
y_i = e^{-\alpha \log(1 + \theta x_i^2)} + \epsilon_i
$$

(19)

Is nonlinear model of parameters $\alpha$ and. We can use nonlinear least square method that minimize the squared sum square of errors to get estimate of parameters. Where sum of square of error is:

$$
Q = \sum_{i=1}^{n} \left(y_i - e^{-\alpha \log(1 + \theta x_i^2)}\right)^2
$$

(20)

3.3. Genetic Algorithm GA\textsuperscript{[8]}

Genetic Algorithm is one of the most powerful stochastic optimization technique, its base idea from Charles Darwin’s theory of natural evolution “survival of the fittest”. It is very useful in estimation of nonlinear models, particularly in cases where the function cannot be solved in more traditional ways so it is more efficient of obtaining global optimum solution which is represent of parameter estimates. This algorithm reflects the method of natural selection where the fittest individuals are selected for reproduction in order to select better offspring from the parent population.

GA has its basic steps from genetics artificially to construct search algorithms that are robust and require minimal problem information with small overall computational time. It has three main operators which is selection, crossover and mutation, that making it an important tool for optimization.

The process of natural selection starts with the selection of fittest individuals from a population. So it works with a population of solutions instead of a single solution. They produce offspring which inherit the characteristics of the parents and will be added to the next generation. If parents have better fitness, their offspring will be better than parents and have a better chance at surviving. This process continually will find a generation with the best individuals.

GA is a population-based algorithm, where an individual in the population, representing a solution, which is called a chromosome. It is basically a binary vector, where each item in the vector is called a gene. Since the individuals are represented in binary, it is important to choose a proper encoding of the solution. The fitness is assigned to each chromosome and new generation of chromosomes is created in the reproduction process. Parent chromosomes, from which new chromosomes are created, are chosen quasi-randomly, so the better the fitness will have higher probability for the chromosome to be chosen. Next, the genetic operators are used to create descendant of parents. These include:

1. Initial population or start Generate random population of $n$ chromosomes which is more suitable solutions for the problem.
2. Fitness function Evaluate the fitness $f(x)$ of each chromosome $x$ in the population
3. Selection of two parent chromosomes from a population according to their fitness so the best fitness will be with higher probability to be selected.

4. Crossover: with a crossover probability cross over the parents to form a new offspring (children). If no crossover was performed, offspring is an exact copy of parents.

5. Mutation: with a mutation probability mutate new offspring at each local which is position in chromosome.

6. Place new offspring in a new population.

7. Replace: use new generated population for a further run of algorithm.

8. Testing: if the end condition is satisfied, stop, and return the best solution in current population.

9. Loop: Go to step 2.

GA works on a population consisting of some solutions where the population size is the number of solutions. Each solution is called individual. Each individual solution has a chromosome. The chromosome is represented as a set of parameters (features) that defines the individual. Each chromosome has a set of genes. Each gene is represented by somehow such as being represented as a string of 0 and 1, the string which represent parameter solution is evaluated in terms of its fitness or its objective Function which is often represent sum of square of residual.

The important part in this Algorithm is formulation fitness or objective function to be minimized, which will take the form:

$$\sum_{i=1}^{n} \left( y_i - e^{-\alpha \log (1 + \theta x_i^2)} \right)^2$$

(21)

Where:

$$y_i = 1 - \frac{1}{n+1} \quad i = 1, 2, ..., n$$

And by applying the GA Algorithm function in MatLab program, that required the x and y values, fitness function, population size, crossover probability, mutation Probability for minimization of equation (21) and number of generation, we will get the estimate of the parameters.

3.4. Pattern Search Method PSM

Pattern search (PS) algorithm is one class of direct search evolutionary algorithms used to solve constrained optimization problems. While it was first formally proposed in early 1960, it has popularity with users due to their simplicity and their practical success on a wide range of optimization problems. This method do not require any information about the gradient of the objective function at hand, while searching for an optimum solution, so it is directional method that make use of a finite number of directions with appropriate descent properties. It is suitable for situation where the first and second derivatives of fitness or objective function are not exist, so that do not make explicit use of derivatives. It need only some values of objective function for some values of variable to run, therefore for this reason it named derivative free algorithm.

This algorithm calculates objective or fitness function values of the pattern and then try to find a minimizer. If it finds a new minimum, then it changes the center of pattern and iterates. This search continues until the search step gets sufficiently small, thus ensuring convergence to a local minimum.

The algorithm required:

1. Starting points or Initialization.
2. Value of acceleration factor.
3. Initial perturbation factor.
4. Perturbation tolerance factor.

The important part in this Algorithm is fitness function that take the form:

$$\sum_{i=1}^{n} \left( y_i - e^{-\alpha \log (1 + \theta x_i^2)} \right)^2$$

(22)
Where:

\[ y_i = 1 - \frac{1}{n+1}, \quad i = 1,2,...,n \]

And by applying the PSM algorithm in MatLab, that required the \( x \) and \( y \) values and fitness function, we will get the estimate of the parameters.

### 4. Simulation Experiments:

In order to compare among methods, a simulation experiments were carried. A range of sample sizes that represent small, moderate and high are used. A simulated data are generated according to inverse cdf method as in formula (23):

\[
x = \left(\theta\right)^{\frac{1}{\alpha}} \left(e^{\frac{-\theta}{\alpha^{\frac{1}{\alpha}}}} - 1\right)^{\frac{1}{\alpha}}
\]

Where \( U \) is a uniform random variant.

The GA parameters is set as: population size equal to 600, crossover probability equal to 0.9, mutation probability for minimization equal to 0.01 and number of generations as 100. The terminated with accuracy level is equal to 0.001.

For different values of parameters that represent small and large range of values of parameters the results of mean square error of estimates are listed in tables 1 to table 6.

| Table 1. Mean Square Error Values for \((\alpha = 1, \theta = 1)\). |
|---------------------------------------------------------------|
| \( n \) | Parameters | MLE | NLLSM | GA | PSM | best |
| 10  | \( \alpha \) | 0.667988 | 0.056365 | 0.195805 | 0.056506 | NLLSM |
|  & \( \theta \) | 18.544000 | 8.020789 | 8.193958 | 5.09695 | NLLSM |
| 15  | \( \alpha \) | 0.649208 | 0.048823 | 0.127806 | 0.088396 | NLLSM |
|  & \( \theta \) | 16.393910 | 0.161808 | 0.493110 | 0.161498 | NLLSM |
| 25  | \( \alpha \) | 0.595117 | 0.020371 | 0.125151 | 0.017380 | NLLSM |
|  & \( \theta \) | 0.008116 | 0.169578 | 1.656273 | 0.156868 | MLE |
| 50  | \( \alpha \) | 0.442140 | 0.012686 | 0.118044 | 0.012252 | NLLSM |
|  & \( \theta \) | 12.221690 | 0.140998 | 1.434298 | 1.005031 | NLLSM |
| 100 | \( \alpha \) | 0.263562 | 0.012388 | 0.102130 | 0.011835 | PSM |
|  & \( \theta \) | 11.71317 | 0.148881 | 0.250246 | 0.131913 | PSM |
| 150 | \( \alpha \) | 0.165628 | 0.011762 | 0.064163 | 0.010663 | PSM |
|  & \( \theta \) | 9.160459 | 0.126539 | 0.474766 | 0.244446 | NLLSM |
| 200 | \( \alpha \) | 0.141976 | 0.011443 | 0.085911 | 0.011381 | PSM |
|  & \( \theta \) | 4.539029 | 0.118452 | 0.134299 | 0.109151 | PSM |

| Table 2. Mean Square Error Values for \((\alpha = 1, \theta = 2)\). |
|---------------------------------------------------------------|
| \( n \) | parameters | MLE | NLLSM | GA | PSM | best |
| 10  | \( \alpha \) | 0.157305 | 0.194045 | 0.189704 | 0.148948 | PSM |
|  & \( \theta \) | 0.057850 | 14.27066 | 13.973860 | 7.980625 | MLE |
| 15  | \( \alpha \) | 0.179088 | 0.094797 | 0.050026 | 0.071181 | NLLSM |
|  & \( \theta \) | 0.045906 | 7.769966 | 3.461991 | 5.297190 | MLE |
| 25  | \( \alpha \) | 0.119167 | 0.017216 | 0.030205 | 0.014241 | PSM |
|  & \( \theta \) | 0.043018 | 3.082504 | 0.089634 | 2.797779 | MLE |
Table 3. Mean Square Error Values for \((\alpha = 1, \theta = 3)\).

| n  | \(\alpha\) Parameters | MLE | NLLSM | GA | PSM | best |
|----|------------------------|-----|-------|----|-----|------|
|    |                        |     |       |    |     |      |
| 10 | \(\alpha\) 0.377612    | 3.085701 | 13.22014 | 0.372275 | PSM  |
|    | \(\theta\) 2.056876    | 19.663020 | 16.64826 | 12.731780 | MLE  |
| 15 | \(\alpha\) 0.365532    | 3.228973 | 1.72176 | 0.168565 | PSM  |
|    | \(\theta\) 2.363185    | 17.814360 | 12.37500 | 11.278790 | MLE  |
| 25 | \(\alpha\) 0.357080    | 2.052149 | 0.06541 | 0.034932 | PSM  |
|    | \(\theta\) 1.363700    | 4.172735 | 8.30707 | 3.679630 | MLE  |
| 50 | \(\alpha\) 0.334262    | 1.674159 | 0.63487 | 0.071647 | PSM  |
|    | \(\theta\) 1.063835    | 3.691743 | 10.82640 | 3.452363 | MLE  |
| 100| \(\alpha\) 0.012047    | 0.104377 | 0.68987 | 0.094123 | MLE  |
|    | \(\theta\) 0.060599    | 0.658005 | 5.12502 | 0.589990 | MLE  |
| 150| \(\alpha\) 0.012510    | 0.006508 | 0.01276 | 0.006175 | PSM  |
|    | \(\theta\) 0.046318    | 0.44234  | 0.07274 | 0.040933 | MLE  |
| 200| \(\alpha\) 0.049992    | 0.003836 | 0.01142 | 0.003654 | PSM  |
|    | \(\theta\) 0.032683    | 0.035722 | 0.02958 | 0.034731 | GA   |

Table 4. Mean Square Error Values for \((\alpha = 2, \theta = 1)\).

| n  | Parameters | MLE | NLLSM | GA | PSM | best |
|----|------------|-----|-------|----|-----|------|
|    |            |     |       |    |     |      |
| 10 | \(\alpha\) 3.914358 | 9.866198 | 1.505866 | 0.711264 | PSM  |
|    | \(\theta\) 3.615759 | 0.618972 | 8.545414 | 1.248075 | NLLSM |
| 15 | \(\alpha\) 3.267255 | 0.192999 | 1.206978 | 0.093107 | PSM  |
|    | \(\theta\) 2.165781 | 0.313614 | 0.510557 | 1.214682 | NLLSM |
| 25 | \(\alpha\) 1.710191 | 0.778096 | 1.051513 | 0.054885 | PSM  |
|    | \(\theta\) 1.241398 | 0.105136 | 0.029204 | 1.591178 | GA   |
| 50 | \(\alpha\) 0.491340 | 0.333224 | 0.040609 | 0.091582 | GA   |
|    | \(\theta\) 1.112081 | 0.938523 | 0.936618 | 2.354406 | GA   |
| 100| \(\alpha\) 0.536402 | 0.030253 | 0.017792 | 0.044540 | GA   |
|    | \(\theta\) 0.907246 | 0.450256 | 0.492739 | 0.490337 | NLLSM |
| 150| \(\alpha\) 0.536402 | 0.030253 | 0.017792 | 0.044540 | GA   |
|    | \(\theta\) 0.907246 | 0.450256 | 0.492739 | 0.490337 | NLLSM |
| 200| \(\alpha\) 0.002725 | 0.008118 | 0.005135 | 0.040473 | MLE  |
|    | \(\theta\) 0.059195 | 0.545362 | 0.006545 | 0.167839 | GA   |
Table 5. Mean Square Error Values for \( (\alpha = 3, \theta = 1) \).

| n  | parameters | MLE  | NLLSM  | GA  | PSM  | best    |
|----|------------|------|--------|-----|------|---------|
| 10 | \( \alpha \) | 33.47505 | 6.299863 | 30.25490 | 2.668031 | PSM    |
|    | \( \theta \) | 10.42354 | 107.50470 | 73.35895 | 43.725440 | MLE    |
| 15 | \( \alpha \) | 5.22474 | 4.663007 | 29.33486 | 1.550078 | PSM    |
|    | \( \theta \) | 10.94425 | 2.892324 | 4.25160 | 1.337027 | PSM    |
| 25 | \( \alpha \) | 6.14677 | 3.44570 | 17.20558 | 0.913376 | PSM    |
|    | \( \theta \) | 6.24812 | 1.513890 | 9.35474 | 0.900419 | PSM    |
| 50 | \( \alpha \) | 2.32009 | 3.660520 | 17.20558 | 0.913376 | PSM    |
|    | \( \theta \) | 4.88894 | 1.043221 | 9.35474 | 0.900419 | PSM    |
| 100| \( \alpha \) | 1.46685 | 0.150727 | 0.39147 | 0.077133 | PSM    |
|    | \( \theta \) | 0.01284 | 0.068221 | 0.07955 | 0.497864 | MLE    |
| 150| \( \alpha \) | 1.94818 | 0.06009 | 0.20943 | 0.065627 | NLLSM  |
|    | \( \theta \) | 0.01148 | 0.052933 | 0.06477 | 0.604872 | MLE    |
| 250| \( \alpha \) | 0.92491 | 0.019210 | 0.49432 | 0.061260 | NLLSM  |
|    | \( \theta \) | 0.01136 | 0.03581 | 0.01029 | 0.054012 | GA     |

Table 6. Mean Square Error Values for \( (\alpha = 3, \theta = 3) \).

| n  | Parameters | MLE  | NLLSM  | GA  | PSM  | best    |
|----|------------|------|--------|-----|------|---------|
| 10 | \( \alpha \) | 3.652754 | 73.15335 | 58.539880 | 1.743710 | PSM    |
|    | \( \theta \) | 41.212920 | 352.97470 | 4.264320 | 11.168230 | GA     |
| 15 | \( \alpha \) | 3.332197 | 23.20017 | 4.481708 | 0.242188 | PSM    |
|    | \( \theta \) | 24.972810 | 6.58103 | 2.744208 | 3.437160 | GA     |
| 25 | \( \alpha \) | 1.805887 | 1.22346 | 2.358017 | 0.252733 | PSM    |
|    | \( \theta \) | 21.419360 | 6.79508 | 2.962349 | 3.270942 | GA     |
| 50 | \( \alpha \) | 1.468550 | 1.38174 | 2.765233 | 0.073792 | PSM    |
|    | \( \theta \) | 16.025720 | 5.68290 | 1.060771 | 2.964387 | GA     |
| 100| \( \alpha \) | 0.112281 | 1.29696 | 1.350315 | 0.087335 | PSM    |
|    | \( \theta \) | 0.045603 | 1.38915 | 1.218220 | 0.387443 | MLE    |
| 150| \( \alpha \) | 0.117031 | 1.46919 | 1.104577 | 0.096361 | PSM    |
|    | \( \theta \) | 0.012876 | 1.44618 | 0.341098 | 0.081406 | MLE    |
| 200| \( \alpha \) | 0.062984 | 1.45436 | 0.959521 | 0.029787 | PSM    |
|    | \( \theta \) | 0.011285 | 0.86290 | 0.108012 | 0.053908 | MLE    |

5. Discussion
The minimum mean square error of parameters for the four method are marked in last column for each case. It is shown that PSM method attained the first, since it reach minimum in 44% of cases. The MLE attained the second, since it get 25% of cases, NLLSM 17% and GA 14%, as illustrated in table(7).

Table 7. Mean Square Error Values for \( (\alpha = 3, \theta = 1) \)

| method | counts | Percentage |
|--------|--------|------------|
| MLE    | 21     | 25%        |
| NLLSM  | 14     | 17%        |
6. Conclusion
This paper introduce new probability distribution that is mixed between Lomax distribution and Rayleigh distribution, we get closed form for the pdf and cdf, the since the theoretical mean square error was difficult to find for estimation method, so we used simulation experiments. We concluded that pattern search method is more efficient than the rest methods.

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