GLOBAL EXISTENCE AND BLOW-UP OF SOLUTIONS FOR INFINITELY DEGENERATE SEMILINEAR PSEUDO-PARABOLIC EQUATIONS WITH LOGARITHMIC NONLINEARITY

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Abstract. In this paper, we study the initial-boundary value problem for infinitely degenerate semilinear pseudo-parabolic equations with logarithmic nonlinearity

\[ u_t - \Delta_X u_t - \Delta_X u = u \log |u|, \quad x \in \Omega, \quad t > 0, \]

where \( \Omega \) is a bounded open domain and \( \Omega \subset \subset \Omega' \). Here, we pose the following hypotheses:

1. Introduction. Let \( \Omega' \) be an open domain in \( \mathbb{R}^n \) (\( n \geq 2 \)), and \( X = (X_1, \ldots, X_m) \) be a system of real smooth vector fields defined in \( \Omega' \). We assume that each \( X_i \) is formally skew-adjoint operator (i.e. \( X_i = -X_i^* \)). Denote that \( I = (i_1, i_2, \ldots, i_k) \), and \( X_I = [X_{i_1}, [X_{i_2}, \ldots, [X_{i_{k-1}}, X_{i_k}] \ldots]] \) as the \( k \)-th repeated commutators of \( (X_1, X_2, \ldots, X_m) \), here \( 1 \leq i_j \leq m \), and \( |I| = k \) is called the length of the commutators. We say that \( X \) satisfies Hörmander’s condition in \( \Omega' \) with Hörmander’s index \( Q < +\infty \), if there is an integer \( Q \geq 1 \) such that the vector fields \( \{X_1, X_2, \ldots, X_m\} \) together with their commutators of length at most \( Q \) can span the tangent space at each point \( x \in \Omega' \). If \( X \) satisfies the Hörmander’s condition then we say that \( X \) is finite degenerate, otherwise \( X \) is infinitely degenerate and the operator \( \Delta_X := \sum_{j=1}^{m} X_j^2 \) is called an infinitely degenerate elliptic operator.

In this paper, we consider the following initial-boundary value problem of infinitely degenerate semilinear pseudo-parabolic equation with logarithmic nonlinear term

\[
\begin{cases}
    u_t - \Delta_X u_t - \Delta_X u = u \log |u|, & x \in \Omega, \quad t > 0, \\
    u(x, t) = 0, & x \in \partial \Omega, \quad t > 0, \\
    u(x, 0) = u_0(x), & x \in \Omega,
\end{cases}
\]

where \( \Omega \) is a bounded open domain and \( \Omega \subset \subset \Omega' \). Here, we pose the following hypotheses:

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\( \partial \Omega \) is \( C^\infty \) and non-characteristic for the system of vector fields \( X \).

(H-2) \( X \) satisfies the finite type of Hörmander’s condition with Hörmander index \( Q \geq 1 \) on \( \Omega \) except an union of smooth surfaces \( \Gamma \) which are also non-characteristic for \( X \).

(H-3) There exists a constant \( C > 0 \), such that \( X \) satisfies the following logarithmic regularity estimate for \( s \geq \frac{3}{2} \):

\[
\| (\log A)^s u \|_{L^2(\Omega)}^2 \leq C \left[ \int_\Omega | Xu |^2 dx + \| u \|_{L^2(\Omega)}^2 \right],
\]

for any \( u \in C^\infty_0(\Omega) \), (2)

where \( \Lambda = (e + |D|^2)^{1/2} \).

The infinitely degenerate elliptic operators have been studied by Morimoto [25, 26, 27], Morimoto-Morioka [28, 29], Christ [12], Kohn [17, 18] and Koike [19]. In particular, Morimoto [25] and Christ [12] proved that, if the infinitely degenerate vector fields \( X \) satisfies the logarithmic regularity estimate (2) with \( s > 1 \), then \( \Delta_X \) is hypo-elliptic. On the other hand, for any \( \mu > 2 \) and weighted Sobolev space \( H^{1,0}_X(\Omega) \) (see (5) below), the embedding \( H^{1,0}_X(\Omega) \hookrightarrow L^\mu(\Omega) \) will not hold, which means, if the vector fields \( X \) are infinitely degenerate, the critical index of the embedding mapping is at most 2. That means that for non-linear infinitely degenerate elliptic equation \(-\Delta_X u = f(u)\), if the non-linear term \( f(u) \) is the power-nonlinearity such as \( |u|^{p-1}u \) with \( p > 1 \), we can not ensure the existence of nontrivial weak solution (see [30]). Therefore in this paper we only consider the non-linear term is logarithmic nonlinear, i.e. \( f(u) = u \log |u| \).

Pseudo-parabolic equations describe a variety of important physical processes, such as the seepage of homogeneous fluids through a fissured rock [2], the unidirectional propagation of nonlinear, dispersive, long waves [4, 37], and the aggregation of populations [31]. The pseudo-parabolic equation

\[
u_t - k \Delta u_t - \Delta u = f(u), \quad x \in \Omega, \quad t > 0,
\]

where \( k \) is a positive constant, \( \Omega \) is a bounded domain of \( \mathbb{R}^n \) with smooth boundary \( \partial \Omega \) and \( \Delta \) is the standard Laplace operator, can be used in the analysis of nonstationary processes in semiconductors in the presence of sources [21, 22]. Furthermore, Eq. (3) can be regarded as a Sobolev type equation or a Sobolev-Galpern type equation [36].

Ting, Showalter and Gopala Rao [14, 35, 38] investigated the initial-boundary value problem and the Cauchy problem for the linear pseudo-parabolic equation, and established the existence and uniqueness of solutions. From then on, considerable attentions have been paid to the study of nonlinear pseudo-parabolic equations, even including singular pseudo-parabolic equations and degenerate pseudo-parabolic equations, see [5, 13, 33] and the references therein for details. For the nonlinear case, besides the existence and uniqueness results, the properties of solutions, such as asymptotic behavior and regularity, were also investigated.

On the one hand, if \( f(u) \) in (3) is a polynomial such as \( u^p \) with \( p > 0 \), Benedetto and Pierre [3] established the maximum principle of Eq. (3). Cao, Yin, and Wang [6] studied the Cauchy problem of Eq. (3). In fact, by the integral representation and the contraction-mapping principle, they not only proved the existence, uniqueness and comparison principle for mild solutions, but also got large time behavior of solutions, the critical global existence exponent and the critical Fujita exponent for Eq. (3). Later, Xu and Su [39] considered the initial-boundary value problem (3) with \( p > 1 \) and showed the invariance of some sets, global existence, blow-up in finite time and asymptotic behavior of solutions. Here, they used the so-called
potential well method, which was found and developed by Sattinger [34] in 1968 and Payne and Sattinger [32], respectively. Later, the method was greatly improved by Liu and Zhao [24].

On the other hand, in case of \( f(u) = u \log |u| \) in (3), Chen et al. [8, 11] first studied this kind of initial boundary value problem (3). They used the generalized potential well method and the following logarithmic Sobolev inequality for any \( u \in H^1_0(\Omega) \) and \( a > 0 \),

\[
2 \int_{\Omega} |u(x)|^2 \log \left( \frac{|u(x)|}{\|u\|_{L^2(\Omega)}} \right) dx + n(1 + \log a)\|u\|_{L^2(\Omega)}^2 \leq \frac{a^2}{\pi} \int_{\Omega} |\nabla u(x)|^2 dx,
\]

to prove that the global existence, blow-up at \(+\infty\) and asymptotic behaviour for the solutions. However, for the problem (1), such kind of “good” logarithmic Sobolev inequality (4) is not satisfied, and we have only the logarithmic regularity estimate (2). Thus it is more difficult to deal with the nonlinear term \( f(u) = u \log |u| \) for the infinitely degenerate problem (1). Ji, Yin and Cao [16] recently study the periodic problem for semilinear heat equation and pseudo-parabolic equation (3) with logarithmic source, and show the existence and instability of positive periodic solutions.

Associated with the system of vector fields \( X = (X_1, X_2, \ldots, X_m) \), we introduced the following Sobolev space (cf. [30]):

\[
H^1_{X_1}(\Omega') = \{ u \in L^2(\Omega') \mid X_j u \in L^2(\Omega'), \ j = 1, \ldots, m \},
\]

which is a Hilbert space with norm

\[
\|u\|_{H^1_{X_1}(\Omega')}^2 = \|u\|_{L^2(\Omega')}^2 + \|X u\|_{L^2(\Omega')}^2,
\]

where \( \|X u\|_{L^2(\Omega')}^2 = \sum_{j=1}^m \|X_j u\|_{L^2(\Omega')}^2 \). The space \( H^1_{X,0}(\Omega) \) is defined by the closure of \( C^\infty_0(\Omega) \) in \( H^1_{X,0}(\Omega) \), which is also a Hilbert space.

In this paper, under the hypotheses (H-1), (H-2) and (H-3), our goal is to prove the global existence and blow-up at \(+\infty\) of the solutions for the problem (1) in \( H^1_{X,0}(\Omega) \).

Now, for our purpose, we introduce the following definitions.

**Definition 1.1** (Weak solution). A function \( u = u(x, t) \) is called a weak solution of problem (1) on \( \Omega \times [0, T) \), if \( u \in L^\infty(0, T; H^1_{X,0}(\Omega)) \) with \( u_t \in L^2(0, T; H^1_{X,0}(\Omega)) \) satisfies the problem (1) in the distribution sense, i.e.

\[
(u_t, v) + (X u_t, X v) + (X u, X v) = (u \log |u|, v), \quad \forall \ v \in H^1_{X,0}(\Omega), \ t \in (0, T),
\]

where \( u(0, x) = u_0(x) \in H^1_{X,0}(\Omega) \), and \((\cdot, \cdot)_{2}\) means the inner product \((\cdot, \cdot)_{L^2(\Omega)}\).

**Definition 1.2** (Maximal existence time). Let \( u(x, t) \) be a weak solution of (1).

We define the maximal existence time \( T \) of \( u(x, t) \) as follows:

(i) If \( u(x, t) \) exists for all \( 0 \leq t < \infty \), then \( T = +\infty \).

(ii) If there exists \( t_0 \in (0, \infty) \) such that \( u(x, t) \) exists for \( 0 \leq t < t_0 \), but doesn’t exist at \( t = t_0 \), then \( T = t_0 \).

Define two functionals on \( H^1_{X,0}(\Omega) \) as follows:

\[
J(u) = \frac{1}{2} \|X u\|_{L^2(\Omega)}^2 - \frac{1}{2} \int_{\Omega} u^2 \log |u| dx + \frac{1}{4} \int_{\Omega} u^2 dx,
\]

\[
I(u) = \|X u\|_{L^2(\Omega)}^2 - \int_{\Omega} u^2 \log |u| dx.
\]
Then, it is obvious that

\[ J(u) = \frac{1}{2} I(u) + \frac{1}{4} \|u\|_{L^2(\Omega)}^2. \]  

(8)

The mountain pass level \( d \), also known as potential well depth, is defined as

\[ d = \inf \left\{ \sup_{\lambda \geq 0} J(\lambda u) \mid u \in H^1_{X,0}(\Omega), \int_\Omega |Xu|^2 \, dx \neq 0 \right\}, \]

\[ L := \frac{1}{4} e^{\lambda_1 - 1 - C_X}, \]

\[ C_X := \sup_{u \in H^1_{X,0}(\Omega) \setminus \{0\}} \frac{\int_\Omega |u|^2 \log\left(\frac{|u|}{\|u\|_{L^2(\Omega)}}\right)^2 \, dx}{\|u\|_{H^1_{X,0}(\Omega)}^2}, \]

(9)

where \( \lambda_1 \) is the first eigenvalue of the operator \(-\Delta_X\), \( C_X \) can be got by the hypothesis (H-3) and the logarithmic Sobolev inequality (13). (in Lemma 3.3 below we can show that \( d \geq L \).)

Our main results can be stated as follows.

**Theorem 1.3.** Under the hypotheses (H-1), (H-2) and (H-3), if \( u_0 \in H^1_{X,0}(\Omega) \) such that \( J(u_0) \leq d \) and \( I(u_0) \geq 0 \), then the problem (1) has a global weak solution \( u = u(x,t) \in L^\infty(0, +\infty; H^1_{X,0}(\Omega)) \) with \( u_t \in L^2(0, +\infty; H^1_{X,0}(\Omega)) \). Furthermore,

- if \( J(u_0) < L \) and \( I(u_0) \geq 0 \), then

\[ \|u(\cdot, t)\|_{H^1_{X,0}(\Omega)} \leq \|u_0\|_{H^1_{X,0}(\Omega)} e^{\frac{1}{2} \alpha t}, \quad \forall \ t \geq 0, \]

where

\[ \alpha = \min\{1 - a, a\lambda_1 - 1 - C_X - \log(4J(u_0))\} > 0, \quad \forall \ a \in \left(\frac{\log(4J(u_0)) + 1 + C_X}{\lambda_1}, 1\right); \]

- if \( L \leq J(u_0) \leq d \) and \( I(u_0) > 0 \), then for any given sufficiently small positive number \( \xi \in (0, L) \), there exists \( t_\xi > 0 \) such that

\[ \|u(\cdot, t)\|_{H^1_{X,0}(\Omega)} \leq \|u_0\|_{H^1_{X,0}(\Omega)} e^{\frac{1}{2} \beta t}, \quad \forall \ t \geq t_\xi, \]

where

\[ \beta = \min\{1 - a, a\lambda_1 - 1 - C_X - \log(4(L - \xi))\} > 0, \quad \forall \ a \in \left(\frac{\log(4(L - \xi)) + 1 + C_X}{\lambda_1}, 1\right). \]

**Theorem 1.4.** Under the hypotheses (H-1), (H-2) and (H-3), if \( u_0 \in H^1_{X,0}(\Omega) \) such that \( J(u_0) \leq d \) and \( I(u_0) < 0 \), then the weak solution \( u = u(x,t) \) of the problem (1) blows up at \(+\infty\), i.e.

\[ \lim_{t \to +\infty} \|u(\cdot, t)\|_{H^1_{X,0}(\Omega)} = +\infty. \]

Moreover,

- \( \|u(\cdot, t)\|_{H^1_{X,0}(\Omega)} \leq e^{C_1 e^t}, \quad \forall \ t \geq 0 \), where the positive constant \( C_1 \) is determined by \( C_X \) and \( u_0 \); 
- for any \( \eta \in (0, 1) \), there exists \( t_\eta > 0 \) such that

\[ \|u(\cdot, t)\|_{H^1_{X,0}(\Omega)}^2 \geq C_2 (t - t_\eta)^{\frac{1}{1-\eta}} t^{-1}, \quad \forall \ t \geq t_\eta, \]

(12)

where the positive constant \( C_2 \) is dependent on \( \eta \) and \( t_\eta \).
We organize this paper as follows. After recalling the basic properties for the infinitely degenerate system of vector fields \( X \) in Section 2, we introduce a family of potential wells relative to the logarithmic nonlinear term \( u \log |u| \) and discuss the invariance of some sets under the flow of (1) and vacuum isolating behavior of solutions for problem (1) in Section 3. Finally, we pose the proof of Theorem 1.3 and 1.4 in Section 4, respectively.

Throughout this paper, for simplicity, we set \( u = u(t) = u(x, t), I(u) = I(u(t)) = I(u(x, t)), J(u) = J(u(t)) = J(u(x, t)), \) and \( T \) the maximal existence time of \( u(x, t) \).

2. The basic properties for the infinitely degenerate system of vector fields.

Proposition 1. (Logarithmic Sobolev inequality, cf. [30]) Suppose that the system of vector fields \( X = (X_1, \cdots, X_m) \) verifies the estimate (2) for some \( s > \frac{1}{2} \). Then there exists \( C_0 > 0 \) such that

\[
\int_{\Omega} |u|^2 \log \left( \frac{|u|}{\|u\|_{L^2(\Omega)}} \right) 2s-1 \, dx \leq C_0 \left[ \int_{\Omega} |Xu|^2 \, dx + \|u\|_{L^2(\Omega)}^2 \right], \quad \forall \, u \in H^1_{X,0}(\Omega). \quad (13)
\]

Proposition 2. (Poincaré inequality, cf. [30]) Assume that the system of vector fields \( X \) verifies the logarithmic regularity estimate (2) for \( s > 1 \), \( \partial \Omega \) is \( C^\infty \) and non-characteristic. Then the first eigenvalue \( \lambda_1 \) of the operator \( -\Delta_X \) is strictly positive and there holds

\[
\lambda_1 \|u\|_{L^2(\Omega)}^2 \leq \|Xu\|_{L^2(\Omega)}^2, \quad \forall \, u \in H^1_{X,0}(\Omega). \quad (14)
\]

By Proposition 2, we can use \( \|Xu\|_{L^2(\Omega)} = \left( \sum_{j=1}^{m} |X_ju|^2_{L^2(\Omega)} \right)^{1/2} \) as an equivalent norm of the space \( H^1_{X,0}(\Omega) \).

Proposition 3 (See [10]). Assume that the system of vector fields \( X \) satisfies the logarithmic regularity estimate (2) with \( s > 1 \), \( \partial \Omega \) is \( C^\infty \) and non-characteristic. Then the operator \( -\Delta_X \) has a sequence of discrete eigenvalues \( 0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k \), and \( \lambda_k \to +\infty \) as \( k \to +\infty \), such that for each \( k \geq 1 \), the Dirichlet problem

\[
\begin{aligned}
-\Delta_X \varphi_k &= \lambda_k \varphi_k, \quad x \in \Omega, \\
\varphi_k &= 0, \quad x \in \partial \Omega
\end{aligned} \quad (15)
\]

admits a non-trivial solution \( \varphi_k \in H^1_{X,0}(\Omega) \). Moreover, the corresponding eigenfunctions \( \{\varphi_k\}_{k \geq 1} \) constitute an orthonormal basis of the Sobolev space \( H^1_{X,0}(\Omega) \).

Next, we give two typical examples of the system of vector fields \( X \), which satisfy the logarithmic regularity estimate (2).

Example 2.1. [See [9, 10, 30]] Let \( X = (\partial_{x_1}, \cdots, \partial_{x_{n-1}}, \varphi(x_1)\partial_{x_n}) \) be a system of vector fields, where

\[
\varphi(x_1) = \begin{cases} 
\frac{1}{|x_1|^{s-1}}, & x_1 \neq 0, \\
0, & x_1 = 0
\end{cases}
\]

for \( s > 0 \). Then \( X \) is infinitely degenerate on the hypersurface \( \Gamma = \{x_1 = 0\} \), and the logarithmic regularity estimate (2) holds.
Example 2.2. [See [9, 10, 30]] Let

\[ X = (\partial_{x_1}, \ldots, \partial_{x_{n-1}}, \varphi(x_1)\partial_{x_n}) \]

be a system of vector fields, where

\[ \varphi(x_1) = \begin{cases} 
  e^{-\frac{1}{|x_1|^{\alpha}}} & , \quad x_1 \neq 0, \\
  0, & , \quad x_1 = 0
\end{cases} \]

for \( s > 0 \). Then \( X \) is infinitely degenerate on \( \Gamma = \bigcup_{j \in \mathbb{Z}^+} \Gamma_j \), where \( \Gamma_j = \{ x_1 = \frac{1}{j} \} \) for \( j \in \mathbb{Z}^+ \), \( \Gamma_0 = \{ x_1 = 0 \} \), and \( X \) satisfies the finite type of Hörmander’s condition in \( \mathbb{R}^n \setminus \Gamma \) with Hörmander index \( Q = 1 \). Moreover, the logarithmic regularity estimate (2) holds.

3. Potential wells. In this section, under the hypotheses (H-1), (H-2), and (H-3), we first introduce a family of potential wells for problem (1) and give a series of their properties.

Lemma 3.1. For any \( u \in H^1_{\mathcal{X},0}(\Omega) \), \( \|u\|_{L^2(\Omega)} \neq 0 \), there hold

1. \( \lim_{\lambda \to 0} J(\lambda u) = 0 \), \( \lim_{\lambda \to +\infty} J(\lambda u) = -\infty \);
2. \( J(\lambda u) \) is strictly increasing on \( \lambda \in [0, \lambda_X] \), strictly decreasing on \( \lambda \in [\lambda_X, +\infty) \) and takes the maximum at \( \lambda = \lambda_X \);
3. \( I(\lambda u) = \lambda \frac{dJ(\lambda u)}{d\lambda} \begin{cases} > 0, & \lambda \in (0, \lambda_X), \\
  0, & \lambda = \lambda_X, \\
  < 0, & \lambda \in (\lambda_X, +\infty), \end{cases} \)

where

\[ \lambda_X = \exp \left( \frac{\| Xu \|^2_{L^2(\Omega)} - \int_{\Omega} u^2 \log |u| dx}{\|u\|^2_{L^2(\Omega)}} \right). \]

Proof. By (7) we get

\[ J(\lambda u) = \frac{1}{2} \lambda^2 \| Xu \|^2_{L^2(\Omega)} - \frac{1}{2} \lambda^2 \int_{\Omega} u^2 \log |u| dx + \frac{1}{4} \lambda^2 \int_{\Omega} u^2 dx - \frac{1}{2} \lambda^2 \log \lambda \int_{\Omega} u^2 dx, \]

which implies Lemma 3.1 (1) and

\[ \frac{dJ(\lambda u)}{d\lambda} = \lambda \| Xu \|^2_{L^2(\Omega)} - \lambda \int_{\Omega} u^2 \log |u| dx - \lambda \log \lambda \int_{\Omega} u^2 dx. \]

Together with (7) we obtain Lemma 3.1 (2) and (3). \( \square \)

By introducing the so-called Nehari manifold

\[ \mathcal{N} = \{ u \in H^1_{\mathcal{X},0}(\Omega) \mid I(u) = 0, \int_{\Omega} |Xu|^2 dx \neq 0 \}, \]

we see from Lemma 3.1 that \( d > 0 \), and the potential well depth \( d \) is also characterized by

\[ d = \inf_{u \in \mathcal{N}} J(u). \] (16)

Thus, we can define the potential well

\[ W = \{ u \in H^1_{\mathcal{X},0}(\Omega) \mid I(u) > 0, J(u) < d \} \cup \{0\}, \]

and the set

\[ V = \{ u \in H^1_{\mathcal{X},0}(\Omega) \mid I(u) < 0, J(u) < d \}. \]
For $\delta > 0$, we introduce

$$I_\delta(u) = \delta \| Xu \|_{L^2(\Omega)}^2 - \int_\Omega u^2 \log |u| dx,$$

$$\mathcal{N}_\delta = \{ u \in H^1_{X,0}(\Omega) \mid I_\delta(u) = 0, \int_\Omega |Xu|^2 dx \neq 0 \},$$

$$d(\delta) = \inf_{u \in \mathcal{N}_\delta} J(u), \quad r(\delta) = \lambda_1^2 e^{(\delta - \frac{1}{2}) \lambda_1 - \frac{1 + C_X}{2}}, \quad (17)$$

where $\lambda_1$ is the first eigenvalue of the operator $-\triangle_X$. Moreover, for $\delta \in (0, 1 + \frac{1}{2\lambda_1})$, we define

$$W_\delta = \{ u \in H^1_{X,0}(\Omega) \mid I_\delta(u) > 0, J(u) < d(\delta) \} \cup \{ 0 \},$$

$$V_\delta = \{ u \in H^1_{X,0}(\Omega) \mid I_\delta(u) < 0, J(u) < d(\delta) \}.$$

Now, we are ready to prove the following

**Lemma 3.2.** For $u \in H^1_{X,0}(\Omega)$ and $r(\delta)$ as defined by (17), there hold

1. If $0 < \| Xu \|_{L^2(\Omega)} < r(\delta)$, then $I_\delta(u) > 0$;
2. if $I_\delta(u) < 0$, then $\| Xu \|_{L^2(\Omega)} > r(\delta)$;
3. if $I_\delta(u) = 0$, then $\| u \|_{H^1_{X,0}(\Omega)} = 0$ or $\| Xu \|_{L^2(\Omega)} \geq r(\delta)$.

**Proof.** (1) Hypothesis (H-3) and the logarithmic Sobolev inequality (13) give that

$$\int_\Omega |u|^2 \log \left( \frac{|u|}{\| u \|_{L^2(\Omega)}} \right) dx \leq C_X \left[ \int_\Omega |Xu|^2 dx + \| u \|_{L^2(\Omega)}^2 \right], \forall u \in H^1_{X,0}(\Omega), \quad (18)$$

where $C_X$ is defined in (9). By Young’s inequality, (18) and Poincaré inequality (14), we see that

$$\int_\Omega u^2 \log |u| dx = \int_\Omega u^2 \left( \log \frac{|u|}{\| u \|_{L^2(\Omega)}} + \log \| u \|_{L^2(\Omega)} \right) dx$$

$$\leq \frac{1}{2} \| Xu \|_{L^2(\Omega)}^2 + \frac{1 + C_X}{2} \| u \|_{L^2(\Omega)}^2 + \| u \|_{L^2(\Omega)}^2 \log \| u \|_{L^2(\Omega)}$$

$$\leq \left( \frac{1}{2} + \frac{1 + C_X}{2\lambda_1} + \frac{1}{\lambda_1} \log \| u \|_{L^2(\Omega)} \right) \| Xu \|_{L^2(\Omega)}^2$$

$$\leq \left( \frac{1}{2} + \frac{1 + C_X}{2\lambda_1} + \frac{1}{\lambda_1} \log \frac{\| Xu \|_{L^2(\Omega)}}{\lambda_1^{1/2}} \right) \| Xu \|_{L^2(\Omega)}^2$$

$$< \delta \| Xu \|_{L^2(\Omega)}^2.$$

By the definitions of $r(\delta)$ and $I_\delta(u)$, Lemma 3.2 (1) follows.

(2) By $I_\delta(u) < 0$ we have $\| u \|_{H^1_{X,0}(\Omega)} \neq 0$, which together with (19) shows that

$$\delta \| Xu \|_{L^2(\Omega)}^2 < \int_\Omega u^2 \log |u| dx$$

$$\leq \left( \frac{1}{2} + \frac{1 + C_X}{2\lambda_1} + \frac{1}{\lambda_1} \log \frac{\| Xu \|_{L^2(\Omega)}}{\lambda_1^{1/2}} \right) \| Xu \|_{L^2(\Omega)}^2$$

$$\leq \left( \frac{1}{2} + \frac{1 + C_X}{2\lambda_1} + \frac{1}{\lambda_1} \log \frac{\| Xu \|_{L^2(\Omega)}}{\lambda_1^{1/2}} \right) \| Xu \|_{L^2(\Omega)}^2.$$

Hence $\| Xu \|_{L^2(\Omega)} \geq \lambda_1^{1/2} \| u \|_{L^2(\Omega)} > r(\delta)$. 

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(3) On the other hand, if \( \|u\|_{H^{1,0}_X(\Omega)} = 0 \), then \( I_\delta(u) = 0 \). On the other hand, if \( I_\delta(u) = 0 \) and \( \|u\|_{H^{1,0}_X(\Omega)} \neq 0 \), by (19) there holds

\[
\delta \|Xu\|_{L^2(\Omega)}^2 = \int_\Omega u^2 \log |u| dx \\
\leq \left( \frac{1}{2} + \frac{1 + CX}{2\lambda_1} + \frac{1}{\lambda_1} \log \|u\|_{L^2(\Omega)} \right) \|Xu\|_{L^2(\Omega)}^2 \\
\leq \left[ \frac{1}{2} + \frac{1 + CX}{2\lambda_1} + \frac{1}{\lambda_1} \log \|Xu\|_{L^2(\Omega)} \right] \|Xu\|_{L^2(\Omega)}^2.
\]

Then

\[
\|Xu\|_{L^2(\Omega)} \geq \lambda_1^{\frac{1}{2}} \|u\|_{L^2(\Omega)} \geq \lambda_1^{\frac{1}{2}} e^{(\delta - \frac{1}{2})\lambda_1 - \frac{1 + CX}{2\lambda_1}} = r(\delta).
\]  

(20)

Lemma 3.2 (3) follows.

\[ \square \]

**Lemma 3.3.** The continuous function \( d(\delta) \) of \( \delta \) satisfies the following properties

1. \( d(\delta) \geq \frac{1}{2} (1 - \delta + \frac{1}{2\lambda_1}) r^2(\delta) \) for \( \delta > 0 \). In particular, \( d \geq \frac{1}{4} e^{\lambda_1 - 1 - CX} = L \);
2. \( d_0 := \lim_{\delta \to 0^+} d(\delta) > 0, d(1 + \frac{1}{2\lambda_1}) = 0 \) and \( d(\delta) < 0 \) for \( \delta > 1 + \frac{1}{2\lambda_1} \);
3. \( d(\delta) \) is strictly increasing on \((0, 1]\), strictly decreasing on \([1, 1 + \frac{1}{2\lambda_1}]\) and takes the maximum \( d = d(1) \) at \( \delta = 1 \).

**Proof.** (1) If \( u \in \mathcal{N}_\delta \), i.e. \( I_\delta(u) = 0 \) and \( \|Xu\|_{L^2(\Omega)} \neq 0 \), then by Lemma 3.2 there holds

\[
\|Xu\|_{L^2(\Omega)} \geq r(\delta),
\]

which together with (20) implies that

\[
J(u) = \frac{1}{2} (1 - \delta) \|Xu\|_{L^2(\Omega)}^2 + \frac{1}{2} I_\delta(u) + \frac{1}{4} \int_\Omega u^2 dx \\
\geq \frac{1}{2} (1 - \delta + \frac{1}{2\lambda_1}) r^2(\delta), \quad \forall \delta > 0.
\]

It follows from (17) that

\[
d(\delta) \geq \frac{1}{2} (1 - \delta + \frac{1}{2\lambda_1}) r^2(\delta), \quad \forall \delta > 0.
\]

(2) For any \( u \in H^{1,0}_X(\Omega), \|u\|_{L^2(\Omega)} \neq 0, \delta > 0 \), we can define

\[
\lambda = \lambda(\delta) = \exp \left( \frac{\delta \|Xu\|_{L^2(\Omega)}^2 - \int_\Omega u^2 \log |u| dx}{\|u\|_{L^2(\Omega)}^2} \right),
\]  

(21)

such that \( I_\delta(\lambda u) = 0 \). Then, by (17) and Poincaré inequality (15) we have \( \lambda u \in \mathcal{N}_\delta \),

\[
d(\delta) \leq J(\lambda u) = \lambda^2 \frac{1}{2} (1 - \delta) \|Xu\|_{L^2(\Omega)}^2 + \frac{1}{4} \|u\|_{L^2(\Omega)}^2 \\
\leq \lambda^2 \frac{1}{2} (1 - \delta + \frac{1}{2\lambda_1}) \|Xu\|_{L^2(\Omega)}^2.
\]

This implies that \( d(1 + \frac{1}{2\lambda_1}) \leq 0 \) and \( d(\delta) < 0 \) for \( \delta > 1 + \frac{1}{2\lambda_1} \).

In addition, by Lemma 3.3 (1) we see that \( d_0 := \lim_{\delta \to 0^+} d(\delta) > 0 \) and \( d(1 + \frac{1}{2\lambda_1}) \geq 0 \), thus \( d(1 + \frac{1}{2\lambda_1}) = 0 \).

(3) Obviously, we only need to prove that for any \( 0 < \delta_1 < \delta_2 < 1 \) or \( 1 < \delta_2 < \delta_1 < 1 + \frac{1}{2\lambda_1} \) and any \( u \in H^{1,0}_X(\Omega) \) such that \( I_\delta(u) = 0 \) and \( \|Xu\|_{L^2(\Omega)} \neq 0 \), there
exists \( v \in H_{X,0}^1(\Omega) \) and a constant \( \varepsilon(\delta_1, \delta_2) > 0 \) such that \( I_{\delta_1}(v) = 0, \|Xv\|_{L^2(\Omega)} \neq 0 \) and \( J(\varepsilon(\delta_1, \delta_2)) > J(u) - \varepsilon(\delta_1, \delta_2) \).

In fact, we can define \( \lambda(\delta) \) by (21) such that \( I_{\delta}(\lambda(\delta)u) = 0 \) and \( \lambda(\delta_2) = 1 \). Set \( g(\lambda) = J(\lambda(\delta)u) \), by Lemma 3.1 (3) and above there holds

\[
\frac{d}{d\lambda} g(\lambda) = \frac{1}{\lambda} I(\lambda u) = \frac{1}{\lambda} [(1 - \delta)\|X(\lambda u)\|_{L^2(\Omega)}^2 + I_{\delta}(\lambda u)] = (1 - \delta)\|Xu\|_{L^2(\Omega)}^2.
\]

Taking \( \lambda = \lambda(\delta_1) \) we see by (21) that \( I_{\delta_1}(v) = 0 \) and \( \|Xv\|_{L^2(\Omega)} \neq 0 \). If \( 0 < \delta_1 < \delta_2 < 1 \), as \( \lambda(\delta) \) is increasing in \( \delta \), then

\[
J(u) - J(\varepsilon) = g(1) - g(\lambda(\delta_1)) = g(\lambda(\delta_2)) - g(\lambda(\delta_1))
\]

\[
= \int_{\lambda(\delta_1)}^{\lambda(\delta_2)} \frac{d}{d\lambda} g(\lambda) d\lambda
\]

\[
= \int_{\lambda(\delta_1)}^{\lambda(\delta_2)} (1 - \delta)\|Xu\|_{L^2(\Omega)}^2 d\lambda
\]

\[
\geq (1 - \delta_2)\lambda(\delta_1)(\lambda(\delta_2) - \lambda(\delta_1))\|Xu\|_{L^2(\Omega)}^2 := \varepsilon(\delta_1, \delta_2) > 0.
\]

If \( 1 < \delta_2 < \delta_1 < 1 + \frac{1}{d(\delta_2)} \), we get

\[
J(u) - J(\varepsilon) = \int_{\lambda(\delta_1)}^{\lambda(\delta_2)} (1 - \delta)\|Xu\|_{L^2(\Omega)}^2 d\lambda
\]

\[
\geq (\delta_2 - 1)\lambda(\delta_1)(\lambda(\delta_2) - \lambda(\delta_1))\|Xu\|_{L^2(\Omega)}^2 := \varepsilon(\delta_1, \delta_2) > 0.
\]

Lemma 3.3 (3) follows.

**Lemma 3.4.** Assume that \( J(u) \in (d_0, d) \) for some \( u \in H_{X,0}^1(\Omega) \), and \( \delta_1, \delta_2 \) are two roots of the equation \( d(\delta) = J(u) \) with \( \delta_1 < 1 < \delta_2 \). Then the sign of \( I_{\delta}(u) \) is unchangeable on \( \delta \in (\delta_1, \delta_2) \).

**Proof.** First, the inequality \( J(u) > d_0 > 0 \) implies \( \|Xu\|_{L^2(\Omega)} \neq 0 \). If the sign of \( I_{\delta}(u) \) is changed when \( \delta \in (\delta_1, \delta_2) \), then there exists \( \delta' \in (\delta_1, \delta_2) \) such that \( I_{\delta'}(u) = 0 \). Thus by the definition of \( d(\delta) \) we have \( J(u) \geq d(\delta') \), which is contradictive with \( J(u) = d(\delta_1) = d(\delta_2) < d(\delta') \).

**Proposition 4.** Suppose that \( u_0 \in H_{X,0}^1(\Omega), 0 < \mu < d \), then there exists \( \delta_2 \in (1, 1 + \frac{1}{d(\delta_2)}) \) such that \( d(\delta_2) = \mu \). Furthermore, let \( u \) be a weak solution of problem (1) with \( J(u_0) = \mu \), then, for any \( \delta \in [1, \delta_2) \) and \( t \in [0, T) \), there hold

1. if \( I(t) > 0 \), we have \( u_0 \in W_{\delta} \),
2. if \( I(t) < 0 \), we have \( u \in V_{\delta} \).
Proof. Since $0 < \mu < d$ and Lemma 3.3, there exists $\delta_2 \in (1, 1 + \frac{1}{2\lambda_1})$ such that $d(\delta_2) = \mu$.

(1) For any $\delta \in [1, \delta_2)$, we have
\[ I_\delta(u_0) = (\delta - 1)||Xu_0||^2_{L^2(\Omega)} + I(u_0) > 0, \quad J(u_0) = \mu = d(\delta_2) < d(\delta), \]
which imply that $u_0 \in W_\delta$.

Now we prove $u(x,t) \in W_\delta$ for $\delta \in [1, \delta_2)$ and $0 < t < T$. If it is false, then there exists $t_0 \in (0,T)$ such that $u(x,t_0) \in \partial W_\delta$ for some $\delta \in [1, \delta_2)$, which means that either $I_\delta(u(t_0)) = 0$, $\|Xu(t_0)\|_{L^2(\Omega)} \neq 0$ or $J(u(t_0)) = d(\delta)$ occurs. It follows from (22) that
\[
\int_0^t ||u_r||^2_{L^2(\Omega)} + \|Xu_r\|^2_{L^2(\Omega)}d\tau + J(u) = J(u_0) < d(\delta), \quad \forall t \in [0,T), \quad \delta \in [1, \delta_2),
\]
which implies $J(u(t_0)) \neq d(\delta)$, thus $I_\delta(u(t_0)) = 0$ and $\|Xu(t_0)\|_{L^2(\Omega)} \neq 0$. Then by (17) we get $J(u(t_0)) \geq d(\delta)$, which is contradictory with (23).

(2) First, we prove $u_0 \in \mathcal{V}_\delta$ for any $\delta \in [1, \delta_2)$. If it is false, let $\tilde{\delta} \in [1, \delta_2)$ be the first number such that $u_0 \in \mathcal{V}_\delta$ for $\delta \in [1, \tilde{\delta})$ and $u_0 \in \partial \mathcal{V}_\delta$, which implies that
\[ I_\tilde{\delta}(u_0) = 0 \quad \text{or} \quad J(u_0) = d(\tilde{\delta}). \]
By Lemma 3.3 we obtain $J(u_0) = d(\delta_2) < d(\tilde{\delta})$. This implies that $J(u_0) = d(\tilde{\delta})$, which is impossible. Thus $I_\tilde{\delta}(u_0) = 0$ and $I_\delta(u_0) < 0$ for $\delta \in [1, \tilde{\delta})$. Then by Lemma 3.2 we see that $\|Xu_0\|_{L^2(\Omega)} \geq r(\tilde{\delta}) \neq 0$. Together with (17), we can obtain $d(\delta_2) = J(u_0) \geq d(\tilde{\delta})$, which is also a contradiction.

Next, we prove $u(x,t) \in \mathcal{V}_\delta$, for any $t \in (0,T)$ and $\delta \in [1, \delta_2)$, if it is false, then there exists $t_0 \in (0,T)$ such that $u(x,t) \in \mathcal{V}_\delta$ for $0 < t < t_0$ and $u(x,t_0) \in \partial \mathcal{V}_\delta$, namely,
\[ I_\delta(u(t_0)) = 0 \quad \text{or} \quad J(u(t_0)) = d(\tilde{\delta}), \quad \text{for some} \ \delta \in [1, \delta_2). \]
Together with (23), $J(u(t_0)) = d(\tilde{\delta})$ is impossible. Thus $I_\delta(u(t_0)) = 0$ and $I_\delta(u(t)) < 0$ for $0 < t < t_0$. Then by Lemma 3.2 there holds $\|Xu(t)\|_{L^2(\Omega)} \geq r(\tilde{\delta})$ for $0 < t \leq t_0$. Therefore, by (17) we have $J(u(t_0)) \geq d(\delta)$, which is also contradictory with (23).

Corollary 1. Assume that $u_0 \in H^1_{\lambda,0}(\Omega)$, $d_0 < \mu < d$, and $\delta_1, \delta_2$ are two roots of the equation $d(\delta) = \mu$ with $\delta_1 < 1 < \delta_2$. Let $u$ be a weak solution of problem (1) with $d_0 < J(u_0) \leq \mu$, then, for any $\delta \in (\delta_1, \delta_2)$ and $t \in [0,T)$, there hold

(1) if $I(u_0) > 0$, we have $u \in W_\delta$,
(2) if $I(u_0) < 0$, we have $u \in \mathcal{V}_\delta$.

Proof. (1) As $J(u_0) \leq \mu$ and $I(u_0) > 0$, then by Lemma 3.4 we have $J(u_0) < d(\delta)$ and $I_\delta(u_0) > 0$ for $\delta \in (\delta_1, \delta_2)$. This implies that $u_0 \in W_\delta$ for $\delta_1 < \delta < \delta_2$. The proof left is similar to that in Proposition 4 (1).

(2) From $J(u_0) \leq \mu$, $I(u_0) < 0$ and Lemma 3.4 we see that $J(u_0) < d(\delta)$ and $I_\delta(u_0) < 0$, i.e. $u_0 \in \mathcal{V}_\delta$ for $\delta \in (\delta_1, \delta_2)$. The proof left is similar to that in Proposition 4 (2).

From Corollary 1 and Lemma 3.3 we get the following result.

Corollary 2. Suppose that $u_0 \in H^1_{\chi,0}(\Omega)$, $d_0 < J(u_0) \leq \mu < d$, and $\delta_1, \delta_2$ are two roots of the equation $d(\delta) = \mu$ with $\delta_1 < 1 < \delta_2$. Then, for any $\delta \in (\delta_1, \delta_2)$, both
sets $W_\delta$ and $V_\delta$ are invariant, thus sets

$$W_{\delta_1\delta_2} = \bigcup_{\delta_1 < \delta < \delta_2} W_\delta, \quad V_{\delta_1\delta_2} = \bigcup_{\delta_1 < \delta < \delta_2} V_\delta$$

are also invariant under the solution flow of (1).

Corollary 1 show that for the set of all solutions of the problem (1) with $d_0 < J(u_0) \leq \mu < d$, there exists a vacuum region

$$U_\mu = N_{\delta_{1,\delta_2}} = \bigcup_{\delta_1 < \delta < \delta_2} N_\delta = \{ u \in H^1_{X,0}(\Omega) \mid \int_\Omega |Xu|^2 \, dx \neq 0, \; I_\delta(u) = 0, \; \delta_1 < \delta < \delta_2 \}$$

such that no solution of the problem (1) belongs to $U_\mu$. The vacuum region $U_\mu$ becomes bigger and bigger when $\mu$ is decreasing. As the limit case we obtain

$$U_0 = \{ u \in H^1_{X,0}(\Omega) \mid \int_\Omega |Xu|^2 \, dx \neq 0, \; I_\delta(u) = 0, \; 0 < \delta < \delta_0 \}.$$ 

Here, by Lemma 3.3 we have chosen $\delta_0 \in (1, 1 + \frac{1}{2A'})$ such that $d(\delta_0) = d_0$.

**Proposition 5.** Assume that $u_0 \in H^1_{X,0}(\Omega), \; J(u_0) \leq 0, \; I(u_0) < 0$, and $u$ is a weak solution of the problem (1), then $I(u) < 0$ for all $0 \leq t < T$.

**Proof.** Arguing by contradiction, we suppose that there exists $t' \in (0, T)$ such that

$$I(u(t')) = 0, \; I(u(t)) < 0, \; \forall \; t \in [0, t').$$

From Lemma 3.2 (2) we have $\|Xu\|_{L^2(\Omega)} > r(1)$ for $t \in [0, t')$. Thus $\|Xu(t')\|_{L^2(\Omega)} \geq r(1)$. Then by (16) we have $J(u(t')) \geq d$, which is contradictory with (22).

To deal with the critical case, we have the following proposition.

**Proposition 6.** Suppose that $u_0 \in H^1_{X,0}(\Omega), \; J(u_0) = d$, and $u$ is a weak solution of the problem (1). Then, for any $t \in [0, T)$ there hold

1. if $I(u_0) > 0$, we have $I(u) > 0$,
2. if $I(u_0) < 0$, we have $I(u) < 0$.

**Proof.** (1) If the result is false, then there exists $t_1 \in (0, T)$ such that

$$I(u(t_1)) = 0, \; I(u(t)) > 0, \; \forall \; t \in (0, t_1),$$

which together with $I(u) = -(u_t, u) - \langle Xu_t, Xu \rangle$ mean that

$$\|u(t)\|_{H^1_{X,0}(\Omega)} > 0, \; \|u_t\|_{H^1_{X,0}(\Omega)} > 0, \; \forall \; t \in (0, t_1).$$

Then $\int_0^t \|u_t\|^2_{H^1_{X,0}(\Omega)} \, d\tau$ is increasing for $t$ on $(0, t_1)$ and

$$J(u(t)) = J(u_0) - \int_0^t \|u_t\|^2_{H^1_{X,0}(\Omega)} \, d\tau < d, \; \forall \; t \in (0, t_1].$$

By Lemma 3.2 and (24) there holds $\|Xu(t_1)\|_{L^2(\Omega)} \geq r(1) \neq 0$. Then we see from the definition of $d$ that $J(u(t_1)) \geq d$, which is contradictory with (25). Proposition 6 (1) follows.

(2) If the result is false, then there exists $t_1 \in (0, T)$ such that

$$I(u(t_1)) = 0, \; I(u(t)) < 0, \; \forall \; t \in (0, t_1).$$

Similar to Proposition 6 (1), a contradiction appears. Proposition 6 (2) follows. □
4. Proofs of Theorems 1.3 and 1.4.

4.1. Global existence with exponential decay. In this subsection, under the hypotheses (H-1), (H-2) and (H-3), we shall use the Galerkin approximation technique and the potential well method to prove the global existence of weak solutions for problem (1). Meanwhile, we shall obtain the asymptotic stability of the global solutions.

In order to get the asymptotic behavior of the global solutions, we need the following well-known estimate

**Lemma 4.1** (See [20]). Let \( y(t) : \mathbb{R}^+ \to \mathbb{R}^+ \) be a nonincreasing function. Assume that there is a constant \( A > 0 \) such that

\[
\int_s^{+\infty} y(t) dt \leq Ay(s), \quad 0 \leq s < +\infty.
\]

Then \( y(t) \leq y(0)e^{1-t/A} \), for all \( t > 0 \).

**Proof of Theorem 1.3.** We divide the proof into four steps.

**Step 1. Global existence in the case of \( J(u_0) < d \).**

First, we can exclude some special cases as follows.

(i) \( 0 < J(u_0) < d, I(u_0) = 0 \). This is contradictory with the definition of \( d \).

(ii) \( J(u_0) = 0, I(u_0) = 0 \). It follows from (8) that \( u_0 = 0 \), which is a trivial case.

(iii) \( J(u_0) = 0, I(u_0) > 0 \). This is contradictory with (8).

(iv) \( J(u_0) < 0, I(u_0) \geq 0 \). This is contradictory with (8).

It remains for us to consider the case \( 0 < J(u_0) < d \) and \( I(u_0) > 0 \). By Proposition 3 we see that a sequence of eigenfunctions \( \{\varphi_k(x)\}_{k \geq 1} \) is an orthogonal basis of the Sobolev space \( H^1_{X,0}(\Omega) \). We construct the approximate solutions \( u_m(x,t) \) of the problem (1)

\[
u_m(x,t) = \sum_{k=1}^{m} g_{km}(t) \varphi_k(x), \quad m = 1, 2, \ldots,
\]

which satisfy

\[
(u_m, \varphi_j)_2 + (Xu_m, X\varphi_j)_2 + (Xu_m, X\varphi_j)_2 = (u_m \log |u_m|, \varphi_j)_2, \quad j = 1, 2, \ldots, m, \tag{26}
\]

and as \( m \to \infty \),

\[
u_m(x,0) = \sum_{k=1}^{m} a_{km} \varphi_k(x) \to u_0(x) \text{ in } H^1_{X,0}(\Omega). \tag{27}
\]

Multiplying (26) by \( g'_{jm}(t) \), summing for \( j \), and integrating with respect to \( t \) from 0 to \( t \), we get

\[
\int_0^t \left[ \|u_{m \tau}\|_{L^2(\Omega)}^2 \, d\tau + \|Xu_{m \tau}\|_{L^2(\Omega)}^2 \, d\tau \right] + J(u_m(t)) = J(u_m(0)), \quad 0 \leq t < \infty. \tag{28}
\]

By (27) we have \( J(u_m(0)) \to J(u_0) \), which together with (28) implies that

\[
\int_0^t \left[ \|u_{m \tau}\|_{L^2(\Omega)}^2 \, d\tau + \|Xu_{m \tau}\|_{L^2(\Omega)}^2 \, d\tau \right] + J(u_m(t)) = J(u_m(0)) < d, \quad 0 \leq t < \infty
\]

for sufficiently large \( m \).
From (29) and an argument similar to that in the proof of Proposition 4 (1) we can show that \( u_m(x, t) \in W \) for \( 0 \leq t < \infty \) and sufficiently large \( m \). Then we see from (7), (8), (29) and (19) that

\[
\int_0^t \| u_m \|_{H_{X,0}^1(\Omega)}^2 \, dt < d, \quad \| u_m \|_{L^2(\Omega)}^2 < 4d.
\]

Moreover, by Young’s inequality, (18) and (30) we deduce that

\[
\| X u_m \|_{L^2(\Omega)}^2 \leq 2 J(u_m) + (2 \log \| u_m \|_{L^2(\Omega)} + 1 + C_X) \| u_m \|_{L^2(\Omega)}^2
\]

\[
= 4 J(u_m) + (2 \log \| u_m \|_{L^2(\Omega)} + C_X) \| u_m \|_{L^2(\Omega)}^2
\]

\[
< C_d.
\]

Denote by \( \overset{*}{\rightharpoonup} \) the weakly star convergence. It follows from (30) and (31) that there exist \( u \) and a subsequence, still denoted by \( \{u_m\} \), such that as \( m \to \infty \),

\[
u_m \overset{*}{\rightharpoonup} u \text{ in } L^\infty(0, \infty; H_{X,0}^1(\Omega)) \text{ and a.e. in } \Omega \times [0, \infty),
\]

\[
\mu_m \overset{*}{\rightharpoonup} \mu_t \text{ in } L^2(0, \infty; H_{X,0}^1(\Omega)),
\]

\[
\mu_m \log |u_m| \overset{*}{\rightharpoonup} u \log |u| \text{ in } L^\infty(0, \infty; L^2(\Omega)) \text{ and a.e. in } \Omega \times [0, \infty).
\]

Hence in (26), for \( j \) fixed, letting \( m \to \infty \), there holds

\[
(u_t, \varphi_j)_2 + (X u_t, X \varphi_j)_2 + (X u, X \varphi_j)_2 = (u \log |u|, \varphi_j)_2, \quad \forall \ j \geq 1.
\]

Furthermore,

\[
(u_t, v)_2 + (X u_t, X v)_2 + (X u, X v)_2 = (u \log |u|, v)_2, \quad \forall \ v \in H_{X,0}^1(\Omega), \ t > 0.
\]

Meanwhile, from (27) we obtain \( u(x, 0) = u_0(x) \) in \( H_{X,0}^1(\Omega) \). It follows from Proposition 4 that \( u(x, t) \in W \) for \( 0 \leq t < \infty \). Therefore, \( u(x, t) \) is a global weak solution of the problem (1).

**Step 2. Global existence in the critical case of** \( J(u_0) = d \).

Set \( \mu_0 = \mu m u_0 \), where \( \mu_m = 1 - \frac{1}{m}, \ m \geq 2 \). We consider the initial-boundary value problem

\[
\begin{cases}
\frac{\partial u}{\partial t} - \Delta X u_t - \Delta X u = u \log |u|, & x \in \Omega, \ t > 0, \\
u(x, t) = 0, & x \in \partial \Omega, \ t > 0, \\
u(x, 0) = \mu_0(m)(x), & x \in \Omega.
\end{cases}
\]

\[ (32) \]
Suppose that \( \|u_0\|_{L^2(\Omega)} = 0 \) and \( \|Xu_0\|_{L^2(\Omega)} \neq 0 \), then we see from \( J(u_0) = d \) and \( (8) \) that

\[
I(u_0) = 2d, \quad I(u_0m) = \mu^2_m I(u_0) = 2\mu^2_m d > 0, \\
J(u_0m) = \mu^2_m J(u_0) = \mu^2_m d < d.
\]

If \( \|u_0\|_{L^2(\Omega)} \neq 0 \), then by \( I(u_0) \geq 0 \) and Lemma 3.1 there hold

\[
\lambda_X = \lambda_X(u_0) \geq 1, \\
I(u_0m) = I(\mu_m u_0) > 0, \\
J(u_0m) = J(\mu_m u_0) < J(u_0) = d.
\]

Thus it follows from the step 1 that, for each \( u \), the problem \((32)\) admits a global weak solution \( u_m(x,t) \in L^\infty(0,\infty; H^1_{X,0}(\Omega)) \) with \( u_m(x,t) \in L^2(0,\infty; H^1_{X,0}(\Omega)) \) and \( u_m(x,t) \in W \) for \( 0 \leq t < \infty \), satisfying

\[
(u_m,v) + (Xu_m,v)_2 + (Xu_m,v)_2 = (u_m \log |u_m|,v)_2, \quad \forall v \in H^1_{X,0}(\Omega), \ t > 0,
\]

\[
\int^t_0 \|u_m\|^2_{L^2(\Omega)} d\tau + \|Xu_m\|^2_{L^2(\Omega)} d\tau + J(u_m(t)) = J(u_0m) < J(u_0) = d, \ 0 \leq t < \infty.
\]

The proof left is similar to that in the step 1.

**Step 3. Decay estimate in the case of \( J(u_0) < L \).**

As before we only need to consider \( 0 < J(u_0m) < L \) and \( I(u_0m) > 0 \). Then, by Proposition 4, we have \( u \in W^\delta \) for \( 1 \leq \delta < \delta_2, \ 0 \leq \tau < \infty \), which implies that \( I(u) > 0 \) for \( 0 \leq \tau < \infty \). Then \( (8) \) and the energy equality \( (22) \) imply that

\[
\|u\|^2_{L^2(\Omega)} < 4J(u) \leq 4J(u_0) < 4L.
\]

Inserting \((34)\) into \((19)\), for any \( a \in \left( \frac{\log[4J(u_0)] + 1 + C_X}{\lambda_1}, 1 \right) \), by Poincaré inequality \((14)\) we see that

\[
I(u) = \|Xu\|^2_{L^2(\Omega)} - \int_\Omega u^2 \log |u| dx \\
\geq \frac{1}{2} \|Xu\|^2_{L^2(\Omega)} - \frac{1}{2} \|u\|^2_{L^2(\Omega)} - \|u\|^2_{L^2(\Omega)} \log \|u\|_{L^2(\Omega)} \\
\geq \frac{1-a}{2} \|Xu\|^2_{L^2(\Omega)} + \left( \frac{a}{2} \lambda_1 - \frac{1}{2} - \frac{C_X}{2} \right) \log \|u\|^2_{L^2(\Omega)} \\
\geq \frac{1-a}{2} \|Xu\|^2_{L^2(\Omega)} + \frac{1}{2} \left( a \lambda_1 - 1 - C_X - \log[4J(u_0)] \right) \|u\|^2_{L^2(\Omega)} \\
\geq \frac{\alpha}{2} \|Xu\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)},
\]

where

\[
\alpha = \min\{1-a, a \lambda_1 - 1 - C_X - \log[4J(u_0)]\} > 0.
\]

Furthermore, we get from Definition 1.1 that

\[
\frac{1}{2} \frac{d}{dt} \|u\|^2_{L^2(\Omega)} + \|Xu\|^2_{L^2(\Omega)} + I(u) = 0, \ 0 \leq \tau < \infty,
\]

which implies that

\[
\int^T_t I(u(\tau)) d\tau = \frac{1}{2} \|u(\cdot, t)\|^2_{H^1_{X,0}(\Omega)} - \frac{1}{2} \|u(\cdot, T)\|^2_{H^1_{X,0}(\Omega)} \\
\leq \frac{1}{2} \|u(\cdot, t)\|^2_{H^1_{X,0}(\Omega)}, \ 0 \leq t < T.
\]
Step 4. Decay estimate in the case of $L \leq J(u_0) \leq d$, $I(u_0) > 0$.

Proof of Theorem 1.4. We divide the proof into three steps.

(1) when $J \leq 0$.

Step 1. Nonblow-up in finite time.

Applying Young’s inequality and (18), we obtain

$$\|u(t)\|_{H^1_{x,0}(\Omega)}^2 \leq \frac{1}{\alpha} \|u(t)\|_{H^1_{x,0}(\Omega)}^2.$$ \hfill (37)

Letting $T \to +\infty$ and applying Lemma 4.1, we have the decay estimate (10).

Step 4. Decay estimate in the case of potential wells, we can prove blow-up at $+\infty$.

If we take $t = t_\xi$ as the initial time, then similar to step 3, we have the decay estimate (11).

4.2. Blow-up at $+\infty$ of solution. In this subsection, by using the properties of a family of potential wells, we can prove blow-up at $+\infty$ of solutions for problem (1) when $J(u_0) \leq d$ and $I(u_0) < 0$.

Proof of Theorem 1.4. We divide the proof into three steps.

Step 1. Nonblow-up in finite time.

Let $u(x,t)$ be any weak solution of problem (1) with $J(u_0) \leq d$ and $I(u_0) < 0$. Then, we get from Proposition 4, Proposition 5 and Proposition 6 that $u \in V_\delta$ for any $\delta \in [1, \delta_2]$, which implies that $I(u(t)) < 0$ for all $t \in [0,T)$. Thus $\|u(t)\|_{H^1_{x,0}(\Omega)} > 0$.

Set

$$\Phi(t) = \int_0^t \|u(t)\|_{H^1_{x,0}(\Omega)}^2 + \|Xu\|_{L^2(\Omega)}^2 d\tau = \int_0^t \|u(t)\|_{H^1_{x,0}(\Omega)}^2 d\tau,$$ \hfill (38)

there hold

$$\dot{\Phi}(t) = \|u(t)\|_{H^1_{x,0}(\Omega)}^2 + \|Xu(t)\|_{L^2(\Omega)}^2,$$

$$\dot{\Phi}(t) = 2\|u(t)\|_{H^1_{x,0}(\Omega)}^2 + 2\|Xu(t)\|_{L^2(\Omega)}^2 = 2(u \cdot \nabla u) + 2X u = -2I(t).$$ \hfill (39)

Applying Young’s inequality and (18), we obtain

$$\dot{\Phi}(t) = 2\int_\Omega u^2 \log |u| dx - \|Xu\|_{L^2(\Omega)}^2$$

$$\leq 2(1 + \frac{C X}{4} + \log \|u(t)\|_{L^2(\Omega)}) \|u(t)\|_{L^2(\Omega)}^2$$

$$\leq 2(C_1 + \log \|u(t)\|_{L^2(\Omega)}) \|u(t)\|_{L^2(\Omega)}^2$$

$$\leq (2C_1 + \log \Phi(t)) \Phi(t),$$

where the positive constant $C_1$ is dependent on $C_X$ and satisfies $C_1 + \log \|u_0\|_{H^1_{x,0}(\Omega)} > 0$. Thus, we can deduce that

$$(\log \Phi(t))' \leq \log \Phi(t) + 2C_1.$$ 

Integrating this inequality over $(0,t)$, we have

$$\log \Phi(t) + 2C_1 \leq (\log \Phi(0) + 2C_1) e^t, \forall t \geq 0.$$
This implies that
\[ \|u(\cdot,t)\|_{H^1_{X,0}(\Omega)} \leq e^{C_1 t}, \quad \forall \, t \geq 0, \]  
where \( C_1 = \bar{C}_1 + \log \|u_0\|_{H^1_{X,0}(\Omega)} > 0 \). Therefore, \( u(x,t) \) does not blow up in finite time.

**Step 2. Blow-up at \( +\infty \) in the case of \( J(u_0) < L \).**

By Proposition 4 and Proposition 5 we have \( I(u(t)) < 0 \) for all \( t \geq 0 \), which together with Lemma 3.2 shows that
\[ \|u(\cdot,t)\|_{L^2(\Omega)}^2 > e^{\lambda_1 t - C_X}, \quad \forall \, t \geq 0. \]  
That is,
\[ \dot{\Phi}(t) = \|u(\cdot,t)\|_{H^1_{X,0}(\Omega)}^2 > 0, \quad \ddot{\Phi}(t) = -2I(u(\cdot,t)) > 0, \quad \forall \, t \geq 0. \]  
Then
\[ \Phi(t) \geq \Phi(0) + t\dot{\Phi}(0) = t\|u_0\|_{H^1_{X,0}(\Omega)}^2, \quad \forall \, t \geq 0. \]  
In addition, from (8) and (22) we obtain
\[ \ddot{\Phi}(t) = -4J(u) + \|u(t)\|_{L^2(\Omega)}^2 
= -4J(u_0) + 2t \int_0^t \|u_\tau\|^2_{H^1_{X,0}(\Omega)} d\tau + \|u(t)\|_{L^2(\Omega)}^2 
> 4(L - J(u_0)) > 0. \]  
Therefore,
\[ \dot{\Phi}(t) = \dot{\Phi}(0) + \int_0^t \ddot{\Phi}(\tau) d\tau > 4(L - J(u_0)), \quad \forall \, t \in [0, +\infty). \]  
This implies that \( u(x,t) \) blows up at \( +\infty \).

Note that
\[ \left( \int_0^t |(u_\tau, u_2) + (Xu_\tau, Xu_2)| d\tau \right)^2 = \left( \frac{1}{2} \int_0^t \frac{d}{d\tau} \|u\|_{H^1_{X,0}(\Omega)}^2 d\tau \right)^2 \]
\[ = \frac{1}{4} (\ddot{\Phi}(t) - 2\|u_0\|_{H^1_{X,0}(\Omega)}^2 \dot{\Phi}(t) + \|u_0\|_{L^2(\Omega)}^4), \]  
which together with (44) and Hölder’s inequality we deduce that
\[ \Phi(t)\dot{\Phi}(t) - \ddot{\Phi}(t) = -4J(u_0) + 4 \int_0^t \|u_\tau\|^2_{H^1_{X,0}(\Omega)} d\tau + \|u\|_{L^2(\Omega)}^2 \Phi(t) \]
\[ - \left[ (\int_0^t \frac{d}{d\tau} \|u\|_{H^1_{X,0}(\Omega)}^2 d\tau)^2 + 2\|u_0\|_{H^1_{X,0}(\Omega)}^2 \dot{\Phi}(t) - \|u_0\|_{L^2(\Omega)}^4 \right] \]
\[ = 4 \int_0^t \|u\|^2_{H^1_{X,0}(\Omega)} d\tau \int_0^t \|u_\tau\|^2_{H^1_{X,0}(\Omega)} d\tau + \|u\|_{L^2(\Omega)}^2 \Phi(t) \]
\[ - 2\|u_0\|^2_{H^1_{X,0}(\Omega)} \Phi(t) - 4 \left( \int_0^t |(u_\tau, u_2) + (Xu_\tau, Xu_2)| d\tau \right)^2 \]
\[ - 4J(u_0)\Phi(t) + \|u_0\|_{H^1_{X,0}(\Omega)}^4 \]
\[ > 4(L - J(u_0))\Phi(t) - 2\|u_0\|^2_{H^1_{X,0}(\Omega)} \Phi(t) \]
\[ > -2\|u_0\|^2_{H^1_{X,0}(\Omega)} \Phi(t). \]  
(46)
Therefore, for any $\eta \in (0, 1)$ there holds
\[ \Phi(t)\dot{\Phi}(t) - \eta \ddot{\Phi}(t) \geq \Phi(t)\dot{\Phi}(t) - \dot{\Phi}(t)^2 - \eta \Phi^2(t) \]
\[ > (1 - \eta)\dot{\Phi}(t)^2 - 2\|u_0\|^2_{H^1_{1,0}(\Omega)} \dot{\Phi}(t). \]

By (45) we see that there exists $t_0 = \frac{\|u_0\|^2_{H^1_{1,0}(\Omega)}}{2(1 - \eta)(L - J(u_0))} > 0$ such that
\[ \Phi(t)\dot{\Phi}(t) - \eta \ddot{\Phi}(t) > 0, \quad \forall t \geq t_0, \quad (47) \]
which, together with
\[ (\Phi(t)^{1-\eta})' = (1 - \eta)\Phi(t)^{-\eta}\Phi(t), \quad (48) \]
implies that
\[ (\Phi(t)^{1-\eta})'' = (1 - \eta)\Phi(t)^{-\eta-1}[\Phi(t)\dot{\Phi}(t) - \eta \Phi^2(t)] > 0, \quad \forall t \geq t_0. \quad (49) \]

Then we get from (48) and (49) that
\[ \Phi(t) = [\Phi(t_0)^{1-\eta}]^{1+\frac{1}{\eta-1}} \geq \left[ \Phi(t_0)^{1-\eta} + (1 - \eta)(t - t_0)\Phi(t_0)^{-\eta}\dot{\Phi}(t_0) \right]^{1+\frac{1}{\eta-1}}, \quad \forall t \geq t_0, \]
which together with (38) means that
\[ \int_0^t \|u(\cdot, \tau)\|^2_{H^1_{1,0}(\Omega)} d\tau \geq \tilde{C}_0(t - t_0)^{\frac{1}{\eta-1}}, \quad \forall t \geq t_0, \quad (50) \]
where $\tilde{C}_0 = [(1 - \eta)\Phi(t_0)^{-\eta}\dot{\Phi}(t_0)]^{1+\frac{1}{\eta-1}}$. By $\dot{\Phi}(t) > 0$ there holds $t\|u(\cdot, t)\|^2_{H^1_{1,0}(\Omega)} \geq \tilde{C}_0(t - t_0)^{\frac{1}{\eta-1}t^{-1}}$, $\forall t \geq t_0$. \(\text{(51)}\)

**Step 3. Blow-up at } +\infty \text{ in the case of } L \leq J(u_0) \leq d.\)**

In this case, by Proposition 4, Proposition 5 and Proposition 6 we have $I(u(t)) < 0$ for all $0 \leq t < +\infty$, which together with $I(u) = -(u_t, u_2) - (Xu, Xu)_2 \neq 0$ shows that $\|u_1\|^2_{H^1_{1,0}(\Omega)} > 0$ and $\int_0^t \|u(\cdot, \tau)\|^2_{H^1_{1,0}(\Omega)} d\tau$ is increasing for $t$ on $[0, +\infty)$. Then by (22), we can choose $t_1 > 0$ such that
\[ J(u(t_1)) = J(u_0) - \int_0^{t_1} \|u_2\|^2_{L^2(\Omega)} d\tau < L. \]

If we take $t_1$ as the initial time, then similarly to step 2 in the proof of Theorem 1.4, we can show that the weak solution $u$ of the problem (1) with $L \leq J(u_0) \leq d$ and $I(u_0) < 0$ blows up at $+\infty$ and, for any $\eta \in (0, 1)$, there exists
\[ t_1 = \frac{\|u(t_1)\|^2_{H^1_{1,0}(\Omega)}}{2(1 - \eta)(L - J(u_0))} > 0 \]
such that
\[ \|u(\cdot, t)\|^2_{H^1_{1,0}(\Omega)} \geq \tilde{C}_1(t - t_1)^{\frac{1}{\eta}t^{-1}}, \quad \forall t \geq t_1, \quad (52) \]
where $\tilde{C}_1 = \left[ (1 - \eta)\int_0^{t_1} \|u(\cdot, \tau)\|^2_{H^1_{1,0}(\Omega)} d\tau \right]^{\frac{1}{\eta}}$. \(\text{Taking } t_\eta = \max\{t_0, t_1\} \text{ and } C_2 = \min\{\tilde{C}_0, \tilde{C}_1\}, \text{ by (51) and (52) we have the estimate (12).} \)

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REFERENCES

[1] J. M. Ball, Remarks on blow-up and nonexistence theorems for nonlinear evolution equations, *Quart. J. Math. Oxford Ser.*, 28 (1977), 473–486.

[2] G. Barenblat, I. Zheltov and I. Kochiva, Basic concepts in the theory of seepage of homogeneous liquids in fissured rocks, *J. Appl. Math. Mech.*, 24 (1960), 1286–1303.

[3] E. D. Benedetto and M. Pierre, On the maximum principle for pseudoparabolic equations, *Indiana Univ. Math. J.*, 30 (1981), 821–854.

[4] T. B. Benjamin, J. L. Bona and J. J. Mahony, Model equations for long waves in nonlinear dispersive systems, *Philos. Trans. R. Soc. Lond. Ser. A*, 272 (1972), 47–78.

[5] H. Brill, A semilinear Sobolev evolution equation in a Banach space, *J. Differential Equations*, 24 (1977), 412–425.

[6] Y. Cao, J. Yin and C. Wang, Cauchy problems of semilinear pseudo-parabolic equations, *J. Differential Equations*, 246 (2009), 4568–4590.

[7] H. Chen and K. Li, The existence and regularity of multiple solutions for a class of infinitely degenerate elliptic equations, *Math Nach.*, 282 (2009), 368–385.

[8] H. Chen, P. Luo and G. Liu, Global solution and blow up of a semilinear heat equation with logarithmic nonlinearity, *J. Math. Anal. Appl.*, 422 (2015), 84–98.

[9] H. Chen, P. Luo and S. Tian, Existence and regularity of multiple solutions for infinitely degenerate nonlinear elliptic equations with singular potential, *J. Differ Equ.*, 257 (2014), 3300–3333.

[10] H. Chen, P. Luo and S. Tian, Multiplicity and regularity of solutions for infinitely degenerate elliptic equations with a free perturbation, *J. Math. Pures Appl.*, 103 (2015), 849–867.

[11] H. Chen and S. Tian, Initial boundary value problem for a class of semilinear pseudo-parabolic equations with logarithmic nonlinearity, *J. Differential Equations*, 261 (2016), 5446–5464.

[12] J. J. Kohn, Hypoellipticity of some degenerate subelliptic operators, *J. Funct. Anal.*, 159 (1998), 203–216.

[13] J. J. Kohn, Hypoellipticity at points of infinite type, Proceedings of the International Conference on Analysis, Geometry, Number Theory in honor of Leon Ehrenpreis (Philadelphia, 1998), *Contemp. Math.*, 250 (2000), 393–398.

[14] M. Koike, A note on hypoellipticity for degenerate elliptic operators, *Publ. Res. Inst. Math. Sci.*, 27 (1991), 995–1000.

[15] V. Komornik, *Exact Controllability and Stabilization*, The Multiplier Method, Mason-John Wiley, Paris, 1994.

[16] M. O. Korpusov and A. G. Sveshnikov, Three-dimensional nonlinear evolution equations of pseudoparabolic type in problems of mathematical physics, Zh. Vychisl. Mat. Mat. Fiz., 43 (2003), 1835–1869 (in Russian); transl. in: *Comput. Math. Math. Phys.*, 43 (2003), 1765–1797.

[17] H. A. Levine, Instability and nonexistence of global solutions to nonlinear wave equations of the form $Pu_{tt} = -Au + F(u)$, *Trans. Amer. Math. Soc.*, 192 (1974), 1–21.

[18] Y. Liu and J. Zhao, On potential wells and applications to semilinear hyperbolic equations and parabolic equations, *Nonlinear Anal.*, 64 (2006), 2665–2687.

[19] Y. Morimoto, *On the hypoellipticity for infinitely degenerate semi-elliptic operators*, *J. Math. Soc. Jpn.*, 30 (1978), 327–358.
[26] Y. Morimoto, A criterion for hypoellipticity of second order differential operators, Osaka J. Math., 24 (1987), 651–675.
[27] Y. Morimoto, Hypoellipticity for infinitely degenerate elliptic operators, Osaka J. Math., 24 (1987), 13–35.
[28] Y. Morimoto and T. Morioka, The positivity of Schrödinger operators and the hypoellipticity of second order degenerate elliptic operators, Bull. Sci. Math., 121 (1997), 507–547.
[29] Y. Morimoto and T. Morioka, Hypoellipticity for elliptic operators with infinite degeneracy, Partial Differential Equations and Their Applications (H. Chen and L. Rodino, eds.), World Sci. Publishing, River Edge, NJ, (1999), 240–259.
[30] Y. Morimoto and C. Xu, Logarithmic Sobolev inequality and semi-linear Dirichlet problems for infinitely degenerate elliptic operators, Astérisque, 284 (2003), 245–264.
[31] V. Padrón, Effect of aggregation on population recovery modeled by a forward-backward pseudoparabolic equation, Trans. Amer. Math. Soc., 356 (2004), 2739–2756.
[32] L. E. Payne and D. H. Sattinger, Saddle points and instability of nonlinear hyperbolic equations, Isr. J. Math., 22 (1975), 273–303.
[33] M. Ptashnyk, Degenerate quasilinear pseudoparabolic equations with memory terms and variational inequalities, Nonlinear Anal., 66 (2007), 2653–2675.
[34] D. H. Sattinger, On global solution of nonlinear hyperbolic equations, Arch. Ration. Mech. Anal., 30 (1968), 148–172.
[35] R. E. Showalter and T. W. Ting, Pseudoparabolic partial differential equations, SIAM J. Math. Anal., 1 (1970), 1–26.
[36] S. L. Sobolev, On a new problem of mathematical physics, Izv. Akad. Nauk SSSR Ser. Math., 18 (1954), 3–50.
[37] T. W. Ting, Certain non-steady flows of second-order fluids, Arch. Ration. Mech. Anal., 14 (1963), 1–26.
[38] T. W. Ting, Parabolic and pseudo-parabolic partial differential equations, J. Math. Soc. Japan, 21 (1969), 440–453.
[39] R. Xu and J. Su, Global existence and finite time blow-up for a class of semilinear pseudo-parabolic equations, J. Funct. Anal., 264 (2013), 2732–2763.

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