SPARSE EXPANDERS HAVE NEGATIVE CURVATURE

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Abstract. We prove that bounded-degree expanders with non-negative Ollivier–Ricci curvature do not exist, thereby solving a long-standing open problem suggested by A. Naor and E. Milman and publicized by Y. Ollivier (2010). In fact, this remains true even if we allow for a vanishing proportion of large degrees, large eigenvalues, and negatively-curved edges. Moreover, the same conclusion applies to the Bakry–Émery curvature condition $CD(0, \infty)$, thereby settling a recent conjecture of D. Cushing, S. Liu and N. Peyerimhoff (2019). To establish those results, we work directly at the level of Benjamini–Schramm limits, and exploit the entropic characterization of the Liouville property on stationary random graphs to show that non-negative curvature and uniform expansion are incompatible “at infinity”. We then transfer this conclusion to finite graphs via local weak convergence. Our approach also shows that the class of finite graphs with non-negative curvature and degrees at most $d$ is hyperfinite, for any fixed $d \in \mathbb{N}$.

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1 Introduction

1.1 Non-negative curvature. Ricci curvature is a fundamental concept in Riemannian geometry, and considerable efforts have been made to develop
satisfactory analogues in discrete settings; see the recent survey [Maa17] and the many references therein. In particular, Y. Ollivier [Oll07, Oll09] proposed a notion of curvature based on optimal transport which applies to arbitrary metric spaces, hence in particular to graphs. Specifically, let $G = (V_G, E_G)$ be a locally finite connected graph. Write $\text{deg}_G(x)$ for the degree of a vertex $x$, and $d_G(x, y)$ for the length of a minimal path from $x$ to $y$ in $G$. Let also $P_G: V_G \times V_G \to [0, 1]$ denote the transition matrix of the lazy simple random walk on $G$, i.e.

$$P_G(x, y) := \begin{cases} \frac{1}{2\text{deg}_G(x)} & \text{if } \{x, y\} \in E_G; \\ \frac{1}{2} & \text{if } x = y; \\ 0 & \text{else}. \end{cases}$$

The Ollivier–Ricci curvature at an edge $\{x, y\} \in E_G$ is defined as

$$\kappa_G(x, y) := 1 - W_1(P_G(x, \cdot), P_G(y, \cdot)),$$

where $W_1$ denotes the $L^1$–Wasserstein distance on $P_1(V_G, d_G)$, see (20) below. The Ollivier–Ricci curvature of the whole graph is then defined as

$$\kappa(G) := \inf_{\{x, y\} \in E_G} \kappa_G(x, y).$$

This fundamental geometric quantity measures how distances are contracted, on average, under the action of $P_G$. When $\kappa(G) \geq 0$, the graph $G$ is called non-negatively curved. This is the case, for example, when $G$ is a Cayley graph of an abelian group, as witnessed by the obvious coupling that uses the same increments for both trajectories. Non-negative curvature is equivalent to the requirement that $P_G$ is a contraction under the Wasserstein metric $W_1$, and constitutes the essence of the powerful path coupling method for bounding mixing times [BD07]. Consequences in terms of geometry, mixing, and concentration of measure have been massively investigated, and quantified by a variety of functional inequalities. The literature is too vast for an exhaustive account, and we refer the reader to the seminal papers [Jou07, Oll09, LLY11, JO10], the survey [Oll10], and the more recent works [ELL17, MW19, CKKLMP20, KLM20, Mun19] for details, variations, references, and open problems. In particular, the present work was motivated by the following long-standing question, due to A. Naor and E. Milman, and publicized by Y. Ollivier [Oll10, Problem T]. Recall that an expander sequence is a sequence of finite graphs with uniformly bounded degrees, diverging sizes, and spectral gap bounded away from 0.

**Question 1.** (Problem T in [Oll10]). Is there an expander sequence with non-negative curvature?

An instructive special class of graphs for which non-negative curvature is completely understood is that of cubic graphs. Specifically, it was shown in [CKLLS19] that prism graphs and Möbius ladders are the only cubic graphs with non-negative
Ollivier–Ricci curvature. Since these are not expanders, the answer to Question 1 is negative for cubic graphs. To the best of our knowledge, this is the only result in the direction of Question 1, despite the rich body of works on non-negative curvature.

1.2 Main result. In the present paper, we answer Question 1 negatively in full generality, as well as its CD(0,∞) analogue raised by D. Cushing, S. Liu and N. Peyerimhoff (see Remark 1 below). Moreover, we show that the answer to Question 1 remains negative even if we significantly relax the required properties. Specifically, let $\Delta(G)$ denote the maximum degree of a finite graph $G$, and write

$$1 = \lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_N(G) \geq 0,$$

for the $N = |V_G|$ ordered eigenvalues of its transition matrix $P_G$. With these notations, Question 1 simply asks whether there exist constants $d \geq 1$, $\rho < 1$ and arbitrary large graphs satisfying

(A) sparsity: $\Delta(G) \leq d$;
(B) spectral expansion: $\lambda_2(G) \leq \rho$;
(C) non-negative curvature: $\kappa(G) \geq 0$.

Our main result says that no large graph can even come close to satisfying these three requirements.

**Theorem 2.** (Main result). Fix $d \geq 1$ and $\rho < 1$. Then, there is a constant $\varepsilon = \varepsilon_{d,\rho} > 0$ such that every finite graph $G$ must satisfy one of the following conditions:

- either $G$ is far from satisfying the sparsity requirement (A), in the following sense:

  $$\sum_{x \in V_G} \deg_G(x) \log \deg_G(x) > (d \log d)|V_G|;$$

- or $G$ is far from satisfying the expansion requirement (B), in the following sense:

  $$\text{card}\{i: \lambda_i(G) > \rho\} \geq \varepsilon|V_G|;$$

- or $G$ is far from satisfying the curvature requirement (C), in the following sense:

  $$\text{card}\{e \in E_G: \kappa_G(e) < -\varepsilon\} \geq \varepsilon|E_G|.$$

Note that the conclusion is only meaningful for large graphs, since the second condition is trivially satisfied when $|V_G| \leq \frac{1}{\varepsilon}$. Here is an equivalent—but perhaps more intuitive—formulation.
Theorem 3. (Rephrasing). Let $G_n = (V_n, E_n), n \geq 1$ be finite graphs with the sparsity property

$$
\sup_{n \geq 1} \left\{ \frac{1}{|V_n|} \sum_{x \in V_n} \deg_{G_n}(x) \log \deg_{G_n}(x) \right\} < \infty. \quad (1)
$$

Suppose in addition that the Ollivier–Ricci curvature is almost non-negative on most edges, i.e.

$$
\forall \varepsilon > 0, \quad \frac{1}{|E_n|} \text{card}\{e \in E_n: \kappa_{G_n}(e) < -\varepsilon\} \xrightarrow{n \to \infty} 0. \quad (2)
$$

Then, a macroscopic proportion of eigenvalues of the transition matrix must accumulate near 1:

$$
\forall \rho < 1, \quad \liminf_{n \to \infty} \left\{ \frac{1}{|V_n|} \text{card}\{i: \lambda_i(G_n) \geq \rho\} \right\} > 0. \quad (3)
$$

Here again, the theorem is only meaningful in the large-size limit $|V_n| \to \infty$, since the conclusion (3) trivially holds otherwise. The high-level message is that on large sparse graphs, non-negative curvature induces extremely poor spectral expansion. This stands in stark contrast with the traditional idea—quantified by a broad variety of functional inequalities over the past decade—that non-negative curvature is associated with good mixing behavior. As pointed out to us by M. Miklós and G. Pete (whom we warmly thank), our approach actually also implies poor expansion in the strong sense of hyperfiniteness introduced by G. Elek [Ele07]. Specifically, a collection $G$ of finite graphs is called hyperfinite if for each $\varepsilon > 0$, there is $k \in \mathbb{N}$ such that every $G \in G$ can be broken into connected components of size at most $k$ by removing at most $\varepsilon|V_G|$ edges. Such a poor connectivity is easily seen to be incompatible with the definition of an expander sequence. Thus, the following statement provides another striking negative answer to Question 1.

Theorem 4. (Non-negative curvature and hyperfiniteness). For any fixed $d \in \mathbb{N}$, the collection of all non-negatively curved finite graphs with degrees at most $d$ is hyperfinite.

Remark 1. (Bakry-Émery curvature). D. Bakry and M. Émery developed a different notion of non-negative curvature based on $\Gamma$–calculus and known as the $\text{CD}(0, \infty)$ condition [BE85, Bak87, BGL14] (see also [KKRT16, FS18]). Since this notion is local, our proof also applies, with the role of Theorem 12 being played by a recent result of B. Hua [Hua19, Theorem 2]. Consequently, no expander sequence can satisfy $\text{CD}(0, \infty)$. This settles a conjecture of D. Cushing, S. Liu and N. Peyerimhoff [CLP20, Conjecture 9.11]. The weaker statement obtained by replacing $\text{CD}(0, \infty)$ with $\text{CD}(0, n)$ was recently established by F. Münch [Mun19]. We warmly thank D. Cushing, S. Liu and F. Münch for pointing this out.
Remark 2. (Laziness). The literature actually contains a whole family of variants \((\kappa_\alpha)_{\alpha \in [0,1)}\) of the Ollivier–Ricci curvature \(\kappa\), obtained by replacing the matrix \(P_G\) with its \(\alpha\)--idle version:

\[
P_G^{(\alpha)} := (2 - 2\alpha)P_G + (2\alpha - 1)\text{Id}.
\]

There is even a continuous-time version \(\kappa_* := \lim_{\alpha \to 1} \frac{\kappa_\alpha}{1-\alpha}\), proposed in [LLY11] and largely adopted since then. In fact, it was later shown (see [BCLMP18, Remark 5.4]) that \(\frac{\kappa_*}{1-\alpha} \leq \kappa_* = 2\kappa\), where \(\kappa = \kappa_{1/2}\) is the version considered in the present paper. Consequently, our result is stated in the strongest possible form, and applies to all versions of the Ollivier–Ricci curvature.

Remark 3. (Eigenvectors). Our proof will actually reveal more than (3): not only are there many eigenvalues near 1, but the corresponding eigenvectors furthermore charge most vertices significantly. In other words, the poor spectral expansion of non-negatively curved graphs is not restricted to any specific region: it applies everywhere. See Remark 6 for a precise statement.

1.3 Proof outline.

Proof outline. The most natural route towards Question 1 would consist in looking for a quantitative upper-bound on the spectral gap of a finite non-negatively curved graph, in terms of its size and maximum degree. Interestingly, we do not pursue this approach here. Neither do we try to obtain asymptotic estimates along sequences of non-negatively curved graphs. Instead, we work directly at the elegant level of local weak limits of finite graphs, and exploit their built-in stationarity to prove that non-negative curvature and expansion are incompatible “at infinity”. This relies on the central concept of asymptotic entropy, and its classical relations with the Liouville property, the speed of random walk, and the spectral radius. We then transfer this incompatibility result to finite graphs via a relative-compactness argument. As far as we know, the idea of using local weak limits as a tool to deduce generic bounds on the mixing parameters of sparse Markov chains have not received much attention. We firmly believe that this viewpoint will find other applications.

Further questions. The surprising “deg log deg” requirement (1) is used to define the asymptotic entropy on which our whole argument relies. We do not know whether it is necessary for the conclusion (3) to hold, or whether it can be further relaxed. Note that one can not simply replace “deg log deg” with “deg”, since the \(n\)--star \(G = K_{1,n}\) satisfies \(\kappa(G) = 1/n\) and \(\lambda_2(G) = 1/2\). Also, a drawback of our approach—as of any limit argument—is its non-quantitative nature. It would be interesting to find an explicit upper-bound (vanishing as \(n \to \infty\)) on the spectral gap of a non-negatively curved graph with \(n\) vertices and maximum degree \(d\), i.e. to estimate

\[
\gamma_d(n) := \max\{1 - \lambda_2(G) : \vert V_G \vert = n, \Delta(G) \leq d, \kappa(G) \geq 0\}.
\]
Organization of the paper. The remainder of the paper is organized as follows: Section 2 offers a brief, self-contained introduction to the framework of random rooted graphs. In particular, we recall the definition of local weak convergence (Section 2.1), introduce the key notions of unimodularity, stationarity and tightness (Section 2.2), and gather important results on the asymptotic entropy of random walks on stationary graphs (Section 2.3). Section 3 is devoted to the proof of the main results, which is reduced (in Section 3.1) to the following three steps:

1. Proving that non-negative curvature implies zero entropy (Section 3.2).
2. Proving that zero entropy implies poor spectral expansion (Section 3.3).
3. Proving that zero entropy implies hyperfiniteness (Section 3.4).

2 Random Rooted Graphs

In this section, we provide a self-contained introduction to the framework of local weak convergence. This limit theory for sparse graphs was introduced by I. Benjamini and O. Schramm [BS01] and developed further by D. Aldous and M. Steele [AS04] and D. Aldous and R. Lyons [AL07]. The limit points are random rooted graphs enjoying a powerful form of stationarity. They describe the “internal” geometry of large graphs, as seen from a uniformly chosen vertex. Local weak limits are often more convenient to work with than the finite-graph sequences that they approximate, and have been shown to capture the asymptotic behavior of a number of graph parameters [Lyo05, BLS13, BLS11, AS16]. The present paper can be viewed as another illustration of the strength of this modern viewpoint.

2.1 Local weak convergence.

The space of rooted graphs. All graphs considered in this paper will be simple, undirected, countable, and locally finite. A rooted graph is a pair \((G, o)\), where \(G\) is a graph and \(o\) is a distinguished vertex, called the root. Two rooted graphs \((G, o)\) and \((G', o')\) are isomorphic, written \(G \simeq G'\), if there is a bijection \(\phi: V_G \to V_{G'}\) which preserves the root \((\phi(o) = o')\) and the edges:

\[
\forall x, y \in V_G, \quad \{x, y\} \in E_G \iff \{\phi(x), \phi(y)\} \in E_{G'}.
\]

We let \(\mathcal{G}_\bullet\) denote the set of connected rooted graphs, considered up to the isomorphism relation \(\simeq\). To lighten the exposition, we will use the same notation \((G, o)\) for the rooted graph and its equivalence class. We write \(\mathcal{B}_t(G, o)\) for the ball of radius \(t\) around the root in \(G\), i.e. the (finite) rooted subgraph of \(G\) induced by the set \(\{x \in V_G: d_G(o, x) \leq t\}\). We equip \(\mathcal{G}_\bullet\) with the local metric \(d_{\text{loc}}: \mathcal{G}_\bullet \times \mathcal{G}_\bullet \to [0, 1]\), defined by

\[
d_{\text{loc}}((G, o), (G', o')) := \frac{1}{1 + r}, \quad \text{with} \quad r = \sup\{t \geq 0: \mathcal{B}_t(G, o) \simeq \mathcal{B}_t(G', o')\}.
\]
In words, two elements of \( \mathcal{G} \) are “close” to each other if one has to look “far away” from the root to distinguish them apart. It can be shown that \((\mathcal{G}, d_{\text{loc}})\) is a complete separable metric space. We equip it with its Borel \( \sigma \)-algebra, and call \( \mathcal{G} \)-valued random variables random rooted graphs.

**Local weak convergence.** Write \( \mathcal{P}(\mathcal{G}) \) for the space of Borel probability measures on \( \mathcal{G} \), equipped with the usual topology of weak convergence. If \( G \) is an arbitrary finite graph, define its local profile \( L_G \in \mathcal{P}(\mathcal{G}) \) to be the empirical distribution of all possible rootings of \( G \), i.e.

\[
L_G := \frac{1}{|V_G|} \sum_{x \in V_G} \delta_{(G,x)},
\]

(4)

where \((G,x)\) is here implicitly restricted to the connected component of \( x \) if \( G \) is not connected. Finally, if \( G_n = (V_n, E_n), n \geq 1 \) are finite graphs whose local profiles \((L_{G_n})_{n \geq 1}\) admit a limit \( L \) in \( \mathcal{P}(\mathcal{G}) \), we call \( L \) the local weak limit of the sequence \((G_n)_{n \geq 1}\), and write simply

\[
G_n \xrightarrow{n \to \infty} L.
\]

Thus, the local weak limit \( L \) is the law of a certain random rooted graph \((G,o)\) which describes how \( G_n \) asymptotically looks when seen from a uniformly chosen root. More formally,

\[
\frac{1}{|V_n|} \sum_{x \in V_n} f(G_n,x) \xrightarrow{n \to \infty} \mathbb{E}[f(G,o)] = \int f \, dL,
\]

(5)

for each continuous, bounded observable \( f: \mathcal{G} \to \mathbb{R} \). The left-hand side can be thought of as a spatial average of “local contributions” from the various vertices of \( G_n \). In short, local weak convergence allows one to conveniently replace the asymptotic analysis of such averages with the direct computation of an expectation at the root of a limiting random graph.

**Local observables.** The class of continuous functions on \( \mathcal{G} \) clearly contains all \( t \)-local observables \((t \geq 0)\), where \( f: \mathcal{G} \to \mathbb{R} \) is called \( t \)-local if the value \( f(G,o) \) is determined by the (isomorphic class of the) finite ball \( B_t(G,o) \). Here is a list of examples, which will be used later without notice:

- The root degree \( \text{deg}: (G,o) \mapsto \text{deg}_G(o) \) is 1-local.
- The minimum curvature at \( o \), \( (G,o) \mapsto \min_{x \sim o} \kappa_G(o,x) \) is 2-local.
- For each \( t \geq 0 \), the return probability \( (G,o) \mapsto P^t_G(o,o) \) is \( t \)-local (in fact, \((\lfloor t/2 \rfloor + 1)\)-local).
- For each \( t \geq 0 \), the \( t \)-step entropy \( (G,o) \mapsto -\sum_{x \in V_G} P^t_G(o,x) \log P^t_G(o,x) \) is \( t \)-local.
2.2 Tightness, unimodularity and stationarity.

Tightness. One of the many reasons for the success of the local weak convergence framework (compared to other limit theories for sparse graphs) is the fact that every “reasonable” sequence of sparse graphs admits a local weak limit. The following tightness criterion, due to I. Benjamini, R. Lyons and O. Schramm, gives an honest mathematical content to this vague claim. Note, of course, that passing to subsequences is unavoidable.

**Theorem 5.** (Tightness, see Theorem 3.1 in [BLS15]). Let $G_n = (V_n, E_n), n \geq 1$ be finite graphs so that

$$\sup_{n \geq 1} \left\{ \frac{1}{|V_n|} \sum_{x \in V_n} f(\deg_{G_n}(x)) \right\} < \infty,$$

for some function $f: \mathbb{Z}_+ \to \mathbb{R}_+$ satisfying $f(d) \gg d$ as $d \to \infty$. Then, $(G_n)_{n \geq 1}$ has a subsequence which admits a local weak limit.

In particular, this criterion applies to the sequence $(G_n)_{n \geq 1}$ in Theorem 3, with $f(d) = d \log d$. This will ensure that we can “pass to the limit” and study the question of existence of non-negatively curved expanders directly at the level of local weak limits.

Unimodularity. Local weak limits of finite graphs happen to enjoy a powerful distributional invariance, which is directly inherited from the fact that the root is equally likely to be any vertex under the local profile (4). More precisely, a random rooted graph $(G, o)$ (or more accurately, its law $\mathcal{L}$) is called unimodular if it satisfies

$$E \left[ \sum_{x \in V_G} f(G, o, x) \right] = E \left[ \sum_{x \in V_G} f(G, x, o) \right],$$

(6)

for every Borel function $f: \mathcal{G}_{\bullet \bullet} \to [0, \infty]$, where $\mathcal{G}_{\bullet \bullet}$ denotes the analogue of the space $\mathcal{G}_{\bullet}$ with two distinguished roots instead of one. Thinking of $f(G, o, x)$ as an amount of mass sent from $o$ to $x$, the identity (6) expresses the fact that the expected masses received and sent by the root coincide. This Mass Transport Principle is clearly satisfied when $\mathcal{L}$ is the local profile of a finite graph, and is preserved under weak convergence. Thus, we obtain the following fundamental result.

**Theorem 6.** (Inherited unimodularity). All local weak limits of finite graphs are unimodular.

Whether the converse holds is a notoriously hard open problem with deep implications; see [AL07, Ele10, BLS15]. Let us here record a first simple consequence of unimodularity, which will be useful.
Lemma 7. (Everything shows at the root, see Lemma 2.3 in [AL07]). If \((G, o)\) is a unimodular random graph and \(B \subseteq \mathcal{G}\), a Borel set such that \(\mathbb{P}((G, o) \in B) = 1\), then, we actually have
\[
\mathbb{P}((G, x) \in B \text{ for all } x \in V_G) = 1.
\]

Proof. Just apply the Mass Transport Principle with \(f(G, o, x) = 1_{(G, o) \notin B}\). \(\square\)

Stationarity. Under a mild integrability condition and a trivial change of measure, unimodularity can be rephrased as reversibility under a natural Markov chain on \(\mathcal{G}\). We will here only need the weaker notion of stationarity: a random rooted graph \((G, o)\) (or more accurately, its law \(\mathcal{L}\)) is called stationary if its law is invariant for the Markov chain on \(\mathcal{G}\), which, at each step, keeps the underlying graph as it is and moves the root according to the transition matrix \(P_G\). More formally,
\[
\mathbb{E} \left[ \sum_{x \in V_G} P^t_G(o, x) h(G, x) \right] = \mathbb{E} [h(G, o)],
\]
for every Borel function \(h : \mathcal{G} \to [0, \infty]\) and every \(t \geq 0\) (equivalently, for \(t = 1\)). The relation with unimodularity is summed up in the following classical lemma (see, e.g., [BC12]).

Lemma 8. (Degree-biasing). Let \(\mathcal{L} \in \mathcal{P}((\mathcal{G})_{\bullet})\) be unimodular with \(\int \deg d\mathcal{L} < \infty\). Then, the law \(\mathcal{L}^\ast = \mathcal{P}((\mathcal{G})_{\bullet})\) defined by the following change of measure is stationary:
\[
d\mathcal{L}^\ast(G, o) := \frac{\deg_G(o)}{\int \deg d\mathcal{L}(G, o)} d\mathcal{L}(G, o).
\]

Proof. Apply the Mass Transport Principle to \(\mathcal{L}\) with \(f(G, o, x) = h(G, o)1_{\{x, o\} \in E_G}\). \(\square\)

Remark 4. (Mutual absolute continuity). It follows from (8) that the original law \(\mathcal{L}\) and its degree-biased version \(\mathcal{L}^\ast\) are mutually absolutely continuous. In other words, we have
\[
\mathcal{L}(B) = 1 \iff \mathcal{L}^\ast(B) = 1,
\]
for any Borel set \(B \subseteq \mathcal{G}\), allowing us to transfer results from one law to the other.

2.3 Asymptotic entropy Stationarity is a powerful property, because it enables the development of an ergodic theory of random rooted graphs. See the inspiring works [LPP95] on Galton–Watson trees, [BC12] on random rooted graphs, and [BDKY15] on general random environments. In particular, a classical application of Kingman’s sub-additive ergodic theorem allows one to define the (asymptotic) entropy of random walks on stationary random graphs, as recalled in the following lemma.
Lemma 9. (Entropy). If \((G, o)\) is a stationary random graph with \(\mathbb{E}[\log \deg_G(o)] < \infty\), the limit

\[
\mathcal{H}(G, o) := \lim_{t \to \infty} \frac{1}{t} \sum_{x \in V_G} P_t^G(o, x) \log \frac{1}{P_t^G(o, x)},
\]

exists almost-surely and in \(L^1\), and does not depend on the choice of the root \(o\).

We will henceforth simply write \(\mathcal{H}(G)\) instead of \(\mathcal{H}(G, o)\), and call it the entropy of \(G\).

Proof. Conditionally on \((G, o)\), let \(X = (X_t)_{t \geq 0}\) be a lazy simple random walk on \(G\) starting from \(X_0 = o\). For \(0 \leq s \leq t\), define a non-negative random variable \(Z_{s,t}\) by

\[
Z_{s,t} := \log \frac{1}{P_{t-s}^G(X_s, X_t)}.
\]

Note that \(Z_{t,s} \overset{d}{=} Z_{0, t-s}\). Indeed, for any Borel function \(f : \mathbb{R}_+ \to \mathbb{R}_+\), we have by definition

\[
\mathbb{E}[f(Z_{s,t})] = \mathbb{E} \left[ \sum_{x,y \in V_G} P_s^G(o, x) P_{t-s}^G(x, y) f \left( \log \frac{1}{P_{t-s}^G(x, y)} \right) \right]
= \mathbb{E} \left[ \sum_{y \in V_G} P_{t-s}^G(o, y) f \left( \log \frac{1}{P_{t-s}^G(o, y)} \right) \right]
= \mathbb{E}[f(Z_{0, t-s})],
\]

where the second line uses the stationarity (7) with \(h(G, o) = \sum_y P_{t-s}^G(o, y) f \left( \log \frac{1}{P_{t-s}^G(o, y)} \right)\). Moreover, the trivial inequality \(P_{t-s}^G(o, y) \geq P_s^G(o, x) P_{t-s}^G(x, y)\) readily implies the sub-additive property

\[
Z_{0,t} \leq Z_{0,s} + Z_{s,t}. \tag{9}
\]

Finally, the assumption \(\mathbb{E}[\log \deg_G(o)] < \infty\) ensures that \(\mathbb{E}[Z_{0,1}] < \infty\). Consequently, Kingman’s sub-additive ergodic theorem (see, e.g. [LP16, Theorem 14.44]) guarantees the existence of a non-negative, integrable random variable \(Z_\infty\) such that almost-surely and in \(L^1\),

\[
\frac{Z_{0,t}}{t} \xrightarrow{t \to \infty} Z_\infty.
\]

Averaging this convergence over the random walk \(X\) (i.e., taking conditional expectation given \((G, o)\)) yields the existence of the limit \(\mathcal{H}(G, o)\). By Lemma 7, the same is true if \(o\) is replaced by any \(x \in V_G\). Moreover, the sub-additive property (9) with \(s = 1\) shows that almost-surely,

\[
\mathcal{H}(G, o) \leq \sum_{x \in V_G} P_G(o, x) \mathcal{H}(G, x).
\]
For any fixed $a \in \mathbb{R}$, the function $\theta \mapsto (\theta - a)_+$ is monotone and convex, so we have almost-surely
\[
(\mathcal{H}(G, o) - a)_+ \leq \sum_{x \in V_G} P_G(o, x) (\mathcal{H}(G, x) - a)_+.
\]

But the two sides have the same law by stationarity, so they must coincide almost-surely. The fact that this is true for all $a \in \mathbb{R}$ forces the almost-sure equality $\mathcal{H}(G, x) = \mathcal{H}(G, o)$ for all neighbours $x$ of $o$, and hence for all $x \in V_G$ by Lemma 7.

\[\square\]

The Liouville property. One of the interests of asymptotic entropy lies in its relation with the Liouville property. A function $f : V_G \rightarrow \mathbb{R}$ is called harmonic on $G$ if $P_G f = f$, where
\[
\forall x \in V_G, \quad (P_G f)(x) := \sum_{y \in V_G} P_G(x, y) f(y).
\]

This is trivially the case, in particular, when $f$ is constant. The graph $G$ has the Liouville property if it admits no non-constant bounded harmonic function. For stationary random graphs, this functional-analytic property turns out to admit the following simple entropic characterization.

**Theorem 10.** (Entropic characterization of the Liouville property). The equivalence
\[
\mathcal{H}(G) = 0 \iff G \text{ has the Liouville property},
\]
holds almost-surely for any stationary random graph $(G, o)$ with $\mathbb{E}[\log \deg_G(o)] < \infty$.

This result has a long history: it originates with the pioneering works of A. Avez [Ave72, Ave74, Ave76] on groups, and was then made famous in a celebrated paper of V. Kaimanovich and A. Vershik [KV83]. In the present setting of stationary random graphs, the implication $\implies$ was established by I. Benjamini and N. Curien [BC12], and refined by I. Benjamini, H. Duminil-Copin, G. Kozma and A. Yadin [BDKY15]. The converse $\iff$ was proved by M. Carrasco Piaggio and P. Lessa [PL16] (see also [BPP18]), but under an additional growth assumption. Since this is the implication that we are going to use, we need to give more details.

**Proof of Theorem 10.** Fix a connected graph $G$, and let $X = (X_t)_{t \geq 0}$ denote a lazy simple random walk on $G$ starting at some fixed vertex $o \in V_G$. Write $P^G$ for its law, which is a probability measure on the product space $V_G^{\mathbb{Z}_+}$. On this space, let $\mathcal{I}$ denote the $\sigma$-field of all events which are invariant under the natural shift $(x_t)_{t \geq 0} \mapsto (x_{t+1})_{t \geq 0}$. Then [LP16, Proposition 14.12] states that
\[
G \text{ has the Liouville property } \iff \mathcal{I} \text{ is } P^G \text{-trivial}.
\]
On the other hand, writing $T = \bigcap_{t=0}^{\infty} \sigma(x_t, x_{t+1}, \ldots)$ for the tail $\sigma$-field on $V^\mathbb{Z}_G$, we have

$$T \text{ is } \mathbb{P}^G \text{- trivial } \iff T \text{ is } \mathbb{P}^G \text{- trivial},$$

by Theorem [LP16, Theorem 14.18] and because $X$ is lazy. Now, for a stationary random graph $(G, o)$ with $\mathbb{E}[\log \deg_G(o)] < \infty$, the equivalence

$$\mathbb{P}(T \text{ is } \mathbb{P}^G\text{-trivial}) = 1 \iff \mathbb{P}(\mathcal{H}(G) = 0) = 1,$$

was established in [BC12, Theorem 3.2]. Putting things together, we obtain

$$\mathbb{P}(G \text{ has the Liouville property}) = 1 \iff \mathbb{P}(\mathcal{H}(G) = 0) = 1,$$

and this annealed statement will actually suffice for the present paper. However, deducing the quenched claim is easy, as we now explain. Define the events $A := \{G \text{ has the Liouville property}\}$ and $B := \{\mathcal{H}(G) = 0\}$, and let $A \Delta B$ denote their symmetric difference. We want to show that

$$\mathcal{L}(A \Delta B) = 0.$$  \hspace{1cm} (12)

for any stationary law $\mathcal{L}$ with $\int (\log \deg) \, d\mathcal{L} < \infty$. We already know this if $A, B$ are $\mathcal{L}\text{-trivial}$, thanks to (11). Moreover, the events $A, B$ are clearly root-invariant, i.e.

$$(G, o) \in A \implies \{\forall x \in V_G, (G, x) \in A\}.$$  \hspace{1cm} (11)

Consequently, (12) holds under the assumption that root-invariant events are $\mathcal{L}\text{-trivial}$. But this is known as ergodicity, and any stationary law can be decomposed as a mixture of ergodic laws, by [AL07, Theorem 4.7]. Thus, (12) extends to all stationary laws $\mathcal{L}$ with $\int (\log \deg) \, d\mathcal{L} < \infty$. \hfill \Box

**Speed and spectral radius.** The entropy $\mathcal{H}(G)$ is classically related to two other fundamental graph-theoretical quantities. The first is the *speed* $\mathcal{S}(G)$: if $(G, o)$ is a stationary random graph, then the same argument as the one used in the proof of Lemma 9 shows that the limit

$$\mathcal{S}(G, o) := \lim_{t \to \infty} \frac{1}{t} \sum_{x \in V_G} P^t_G(o, x) d_G(o, x),$$

exists almost-surely and in $L^1$, and does not depend on the choice of the root $o \in V_G$. We call it the *speed* of $G$, and write simply $\mathcal{S}(G)$.

The second related quantity is the *spectral radius* $\varrho(G)$, defined as follows. Fix a rooted graph $(G, o) \in \mathcal{G}$. For any $s, t \geq 0$, we trivially have $P^{t+s}_G(o, o) \geq P^t_G(o, o) P^s_G(o, o)$. By Fekete's lemma, we deduce that the limit

$$\varrho(G, o) := \lim_{t \to \infty} \left(P^t_G(o, o)\right)^{\frac{1}{t}},$$

exists and is finite. We call it the *spectral radius* of $G$, and write simply $\varrho(G)$.
exists in \((0, 1]\). Moreover, the connectivity of \(G\) together with the trivial inequality
\[
P_{G}^{t+2s}(o, o) \geq P_{G}^{s}(o, x)P_{G}^{t}(x, x)P_{G}^{s}(x, o),
\]
shows that \(\varrho(G, o)\) does not depend on the choice of the root \(o\). Thus, we will henceforth simply write \(\varrho(G)\), and call this quantity the \textit{spectral radius} of \(G\). The speed and spectral radius provide simple lower-bounds on the entropy, as stated below.

**Lemma 11.** (Lower bounds). If \((G, o)\) is stationary with \(E[\log \deg_{G}(o)] < \infty\), then almost-surely,
\[
\mathcal{H}(G) \geq \frac{\mathcal{J}^{2}(G)}{2} \quad \text{and} \quad \mathcal{H}(G) \geq 2 \log \frac{1}{\varrho(G)}.
\]

**Proof.** For any \(t \geq 0\), we have by concavity
\[
\log \left( P_{G}^{2t}(o, o) \right) = \log \left( \sum_{x \in V_{G}} P_{G}^{t}(o, x)P_{G}^{t}(x, o) \right)
\geq \sum_{x \in V_{G}} P_{G}^{t}(o, x) \log P_{G}^{t}(x, o)
= \sum_{x \in V_{G}} P_{G}^{t}(o, x) \log P_{G}^{t}(o, x) + \sum_{x \in V_{G}} P_{G}^{t}(o, x) \log \left( \frac{\deg_{G}(o)}{\deg_{G}(x)} \right),
\]
where the last line uses the reversibility \(\deg_{G}(o)P_{G}^{t}(o, x) = \deg_{G}(x)P_{G}^{t}(x, o)\). Dividing by \(-2t\) and taking the \(t \to \infty\) limit in \(L^{1}\) will yield the first claim, provided we can show that
\[
\frac{1}{t} \sum_{x \in V_{G}} P_{G}^{t}(o, x) \log \left( \frac{\deg_{G}(o)}{\deg_{G}(x)} \right) \xrightarrow{L^{1}} 0.
\]

But this follows from the crude bound
\[
E \left[ \left| \sum_{x \in V_{G}} P_{G}^{t}(o, x) \log \left( \frac{\deg_{G}(o)}{\deg_{G}(x)} \right) \right| \right] \leq E \left[ \sum_{x \in V_{G}} P_{G}^{t}(o, x) \left( \log \deg_{G}(o) + \log \deg_{G}(x) \right) \right]
= 2E \left[ \log \deg_{G}(o) \right],
\]
where the second line uses the stationarity property (7) with \(h(G, o) = \log \deg_{G}(o)\). For the second claim, we use the Varopoulos–Carne bound [Var85, Car85] (see also [LP16, Theorem 13.4]), which asserts that
\[
P_{G}^{t}(o, x) \leq 2 \sqrt{\frac{\deg_{G}(x)}{\deg_{G}(o)}} e^{-\frac{\varrho^{2}(o, x)}{2t}}.
\]
Taking logarithms, and then averaging over all $x \in V_G$, we obtain

$$
\sum_{x \in V_G} P_G^t(o,x) \log \frac{1}{P_G(o,x)} \geq \frac{1}{2t} \sum_{x \in V_G} P_G^t(o,x) \log \frac{\deg_G(o)}{\deg_G(x)} + \frac{1}{2} \sum_{x \in V_G} P_G^t(o,x) \log \frac{\deg_G(o)}{4 \deg_G(x)}.
$$

As above, dividing by $t$ and taking the $t \to \infty$ limit in $L^1$ yields the desired conclusion. \hfill \Box

**Remark 5.** (Unimodular analogues). By Lemma 8 and Remark 4, all results in this section also apply to any unimodular random graph $(G, o)$ with $\mathbb{E}[\deg_G(o) \log \deg_G(o)] < \infty$.

### 3 Proofs

We are now ready to prove our main result. We work with the formulation given in Theorem 3. Section 3.1 below reduces it to two key results, which are then proved in Sections 3.2 and 3.3. Finally, Section 3.4 adapts the whole argument to prove the hyperfiniteness stated in Theorem 4.

**3.1 Setting the stage**

Let $G_n = (V_n, E_n)$, $n \geq 1$ be finite graphs satisfying the assumptions of Theorem 3, i.e.

$$
\sup_{n \geq 1} \left\{ \frac{1}{|V_n|} \sum_{x \in V_n} \deg_{G_n}(x) \log \deg_{G_n}(x) \right\} < \infty; \quad (16)
$$

$$
\forall \varepsilon > 0, \quad \frac{1}{|E_n|} \text{card}\{e \in E_n: \kappa_{G_n}(e) < -\varepsilon\} \xrightarrow{n \to \infty} 0. \quad (17)
$$

Recall that our goal is to establish

$$
\forall \rho \in (0,1), \quad \liminf_{n \to \infty} \left\{ \frac{1}{|V_n|} \text{card}\{i: \lambda_i(G_n) > \rho\} \right\} > 0. \quad (18)
$$

By (16) and Theorem 5, we may assume, upon extracting a subsequence if necessary, that

$$
G_n \xrightarrow{n \to \infty} \mathcal{L}, \quad (19)
$$

for some $\mathcal{L} \in \mathcal{P}(\mathcal{G}_\bullet)$. Note that $\mathcal{L}$ is unimodular by Theorem 6, and that $\int (\deg \log \deg) \, d\mathcal{L} < \infty$ because of (16). We claim that the condition (17) also passes to the limit, in the following sense.

**Claim 1.** The limit $\mathcal{L}$ in (19) is supported on non-negatively curved graphs.
Proof. By its very definition, the observable \( f : (G, o) \mapsto \min_{x \sim o} \kappa_G(o, x) \) is 2-local, hence continuous on \( \mathcal{G} \). By the Portmanteau Theorem, we deduce that for any \( \varepsilon > 0 \),

\[
\mathcal{L} (f < -\varepsilon) \leq \liminf_{n \to \infty} \mathcal{L}_{G_n} (f < -\varepsilon) = \liminf_{n \to \infty} \left\{ \frac{1}{|V_n|} \text{card}\{o \in V_n : f(G_n, o) < -\varepsilon\} \right\} \leq \liminf_{n \to \infty} \left\{ \frac{2}{|V_n|} \text{card}\{e \in E_n : \kappa_{G_n}(e) < -\varepsilon\} \right\} = \left( \int \text{deg} \, d\mathcal{L} \right) \liminf_{n \to \infty} \left\{ \frac{1}{|E_n|} \text{card}\{e \in E_n : \kappa_{G_n}(e) < -\varepsilon\} \right\}.
\]

The last inequality follows from the observation that \( \frac{2|E_n|}{|V_n|} \to \int \text{deg} \, d\mathcal{L} \), by the continuity and uniform integrability of the observable \( \text{deg} : (G, o) \mapsto \text{deg}_G(o) \). Sending \( \varepsilon \to 0 \) yields \( \mathcal{L}(f < 0) = 0 \), by (17). To conclude, we simply apply Lemma 7 to the event \( B = \{ f \geq 0 \} \). \( \square \)

The main step will consist in deducing from this that \( \mathcal{L} \) is supported on graphs with zero entropy. This is the content of the following theorem, which will be proved in Section 3.2.

**Theorem 12.** (Non-negative curvature implies zero-entropy). If \( (G, o) \) is a stationary random graph with \( \mathbb{E} [\log \text{deg}_G(o)] < \infty \), then we have almost-surely

\[
\kappa(G) \geq 0 \implies \mathcal{H}(G) = 0.
\]

In view of Remark 4, this result also applies to any unimodular random graph \( (G, o) \) satisfying \( \mathbb{E} [\text{deg}_G(o) \log \text{deg}_G(o)] < \infty \), hence in particular to the one whose law \( \mathcal{L} \) appears in (19). Combining this with Lemma 11, we immediately deduce that \( \mathcal{L} \) is supported on graphs whose spectral radius is 1. It turns out that this simple condition suffices to guarantee (18). This is the content of the following result, established in Section 3.3 below, and which completes the proof of Theorem 3.

**Theorem 13.** (Zero-entropy implies poor spectral expansion). Let \( G_n = (V_n, E_n), n \geq 1 \) be finite graphs having a local weak limit supported on graphs with spectral radius 1. Then, for any \( \rho < 1 \),

\[
\liminf_{n \to \infty} \left\{ \frac{1}{|V_n|} \text{card}\{i : \lambda_i(G_n) > \rho\} \right\} > 0.
\]

In fact, a stronger statement about eigenvectors will be derived, as claimed in Remark 3.
3.2 Non-negative curvature implies zero entropy

Consider a connected graph \( G \) and two vertices \( x, y \in V_G \). The proof of Theorem 12 relies on the following intuitive idea: if \( G \) has non-negative curvature and bounded degrees, then it takes time \( O(d_G^2(x,y)) \) for two random walks starting at \( x \) and \( y \) to meet. This classical observation constitutes the very essence of the path coupling method of M. Borewicz and M. Dyer [BD07]. It was later re-discovered and further developed by F. Münch [Mun19]. We will here prove a refinement that does not require bounded degrees, see Corollary 17 below.

Write \( B_x, B_y \) for the balls of radius 1 around \( x \) and \( y \), and recall that the Wasserstein distance \( W_1(P_G(x,\cdot), P_G(y,\cdot)) \) is defined as

\[
W_1(P_G(x,\cdot), P_G(y,\cdot)) = \inf_{\pi} \left\{ \sum_{u \in B_x} \sum_{v \in B_y} \pi(u,v) d_G(u,v) \right\},
\]

where the infimum runs over all probability distributions \( \pi \in \mathcal{P}(B_x \times B_y) \) with marginals \( P_G(x,\cdot) \) and \( P_G(y,\cdot) \). By compactness, the above infimum is actually achieved, and the minimizers will be called optimal couplings. As in [BD07, Mun19], our first task consists in showing that an optimal coupling can always be chosen so as to assign a “decent” probability to the “good” set

\[
\Gamma := \{(u,v) \in B_x \times B_y : d_G(u,v) < d_G(x,y)\}.
\]

The argument only uses the laziness of \( P_G \). In particular, non-negative curvature is not required.

**Lemma 14.** (Good optimal couplings). If \( x \neq y \), then there is an optimal coupling \( \pi \) such that

\[
\pi(\Gamma) \geq \frac{1}{2} \max \left\{ \frac{1}{\deg_G(x)}, \frac{1}{\deg_G(y)} \right\}.
\]

**Proof.** By compactness, we can find an optimal coupling \( \pi \) which, among all optimal couplings, maximizes \( \pi(\Gamma) \). Suppose for a contradiction that this “doubly optimal” coupling satisfies

\[
\pi(\Gamma) < \frac{1}{2 \deg_G(x)}.
\]

The set \( A := \{ u \in B_x : (u,y) \in \Gamma \} \) is not empty, as it contains the first vertex on a geodesic from \( x \) to \( y \). Thus, \( \pi(A \times B_y) = P_G(x,A) \geq 1/(2 \deg_G(x)) \). By (21), this forces \( \pi((A \times B_y) \setminus \Gamma) > 0 \), i.e.

\[
\exists (x_0, y_0) \in (A \times B_y) \setminus \Gamma, \quad \pi(x_0, y_0) \geq \varepsilon,
\]

for some \( \varepsilon > 0 \). On the other hand, we have \( \pi(A \times \{y\}) + \pi(A^c \times \{y\}) = P_G(y,y) = \frac{1}{2} \). This forces \( \pi(A^c \times \{y\}) > 0 \), because \( \pi(A \times \{y\}) \leq \pi(\Gamma) < \frac{1}{2} \). In other words,

\[
\exists x_1 \in A^c, \quad \pi(x_1,y) \geq \varepsilon,
\]
provided $\varepsilon > 0$ is chosen small enough. We now use the vertices $x_0, y_0, x_1$ found at (22)–(23) to construct a new coupling $\hat{\pi}$ which contradicts the optimality of $\pi$. For all $(u, v) \in B_x \times B_y$, we set

$$\hat{\pi}(u, v) := \begin{cases} 
\pi(u, v) & \text{if } u \notin \{x_0, x_1\} \text{ or } v \notin \{y_0, y\}; \\
\pi(u, v) - \varepsilon & \text{if } (u, v) = (x_0, y_0) \text{ or } (u, v) = (x_1, y); \\
\pi(u, v) + \varepsilon & \text{if } (u, v) = (x_0, y) \text{ or } (u, v) = (x_1, y_0).
\end{cases}$$

By construction, $\hat{\pi}$ is non-negative on $B_x \times B_y$ and has the same marginals as $\pi$. Thus, it is a coupling of $P_G(x, \cdot), P_G(y, \cdot)$. This coupling is moreover optimal, since

$$\sum_{u \in B_x} \sum_{v \in B_y} d_G(u, v) (\hat{\pi}(u, v) - \pi(u, v)) = \varepsilon (d_G(x_0, y_0) + d_G(x_1, y) - d_G(x_0, y_0) - d_G(x_1, y))$$

$$\leq \varepsilon (d_G(x, y) - 1 + d_G(x_1, y_0) - d_G(x, y) - d_G(x_1, y))$$

$$\leq 0,$$

where the first inequality uses $x_0 \in A$ and $(x_0, y_0) \notin \Gamma$, while the second uses the triangle inequality $d_G(x_1, y_0) \leq d_G(x_1, y) + d_G(y, y_0)$. Finally, since $\Gamma$ contains $(x_1, y)$ but not $(x_0, y_0), (x_1, y)$, we have

$$\hat{\pi}(\Gamma) \geq \pi(\Gamma) + \varepsilon,$$

contradicting the definition of $\pi$. Thus, (21) can not be true, and the claim follows by symmetry.

We will also need the following technical lemma, which quantifies the intuition that non-negative super-martingales that “move a lot” must “quickly” hit zero.

**Lemma 15.** (Non-negative super-martingales hit zero quickly). Let $\tau := \inf\{t \geq 0 : Z_t = 0\}$ be the hitting time of zero by a discrete-time non-negative super-martingale $Z = (Z_t)_{t \geq 0}$. Suppose that $Z_0 = z$, and that all increments $(Z_{t+1} - Z_t)_{t \geq 0}$ are upper-bounded by a constant $K$. Then,

$$\mathbb{P}(\tau \geq t) \leq \frac{z (4a + K - z)}{a^2} + \mathbb{P} \left( \tau \geq t, \sum_{s=0}^{t-1} W_s < a^2 \right),$$

for all $t \in \mathbb{Z}_+, a > 0$, where $W_s = \mathbb{E} \left[ (Z_{s+1} - Z_s)^2 | \mathcal{F}_s \right]$ and $(\mathcal{F}_s)_{s \geq 0}$ is the underlying filtration.

**Proof.** First note that the process $Z$ is trivially square-integrable, because $Z_t \in [0, z + Kt]$ for each $t \geq 0$. Now fix $t \geq 0$ and $a > 0$, and consider the bounded stopping time

$$\sigma := \inf\{s \geq 0 : Z_s \geq a\} \wedge t.$$

Using the Optional Stopping Theorem, the non-negativity of $Z$ and the definition of $\sigma$, we have
\[ z \geq \mathbb{E}[Z_{\sigma \land \tau}] \]
\[ \geq \mathbb{E}[Z_{\sigma \land \tau} \mathbf{1}_{(\sigma < \tau \land \tau)}] \]
\[ \geq a \mathbb{P}(\sigma < \tau \land t) \].

On the other hand, observe that for all \( s \geq 0 \), we may rewrite \( W_s \) as
\[ W_s = \mathbb{E}[Z_{s+1}^2 - Z_s^2 | \mathcal{F}_s] + 2Z_s \mathbb{E}[Z_s - Z_{s+1} | \mathcal{F}_s]. \]
Note that the second conditional expectation is non-negative by assumption. Moreover, we have
\[ Z_s \leq a \text{ on the event } \{ \sigma > s \} \text{, which is in } \mathcal{F}_s. \]
Thus,
\[ W_s \mathbf{1}_{\sigma > s} \leq \mathbb{E}[Z_{s+1}^2 - Z_s^2] + 2a \mathbb{E}[(Z_s - Z_{s+1}) \mathbf{1}_{\sigma > s} | \mathcal{F}_s]. \]
Taking expectations and summing over all \( s \geq 0 \), we obtain
\[ \mathbb{E} \left[ \sum_{s=0}^{\sigma-1} W_s \right] \leq \mathbb{E}[Z_{\sigma}^2] - 2a \mathbb{E}[Z_{\sigma}] - z^2 + 2az \]
\[ \leq (K + 3a - z)z, \]
where the second inequality follows from the observations that \( Z_{\sigma} \leq K + a \) and \( \mathbb{E}[Z_{\sigma}] \leq z \). Let us now use these two estimates to conclude. By union bound, we have
\[ \mathbb{P}(\tau \geq t) \leq \mathbb{P}(\sigma < \tau \land t) + \mathbb{P}(\sigma \land \tau \geq t) \]
\[ \leq \mathbb{P}(\sigma < \tau \land t) + \mathbb{P}\left( \tau \geq t, \sum_{s=0}^{\sigma-1} W_s \geq \sum_{s=0}^{t-1} W_s \right) \]
\[ \leq \mathbb{P}(\sigma < \tau \land t) + \mathbb{P}\left( \sum_{s=0}^{\sigma-1} W_s \geq a^2 \right) + \mathbb{P}\left( \tau \geq t, \sum_{s=0}^{t-1} W_s < a^2 \right) \]
\[ \leq \frac{z}{a} + \frac{(K + 3a - z)z}{a^2} + \mathbb{P}\left( \tau \geq t, \sum_{s=0}^{t-1} W_s < a^2 \right). \]
This is exactly the claimed bound. \( \square \)

Combining these two lemmas, we may now deduce the following estimate, which exploits non-negative curvature to control the action of \( P_G \) on the variations of bounded observables.

**Proposition 16.** (Variational estimate via non-negative curvature) Let \( G \) be a connected graph with \( \kappa(G) \geq 0 \). Then, for any \( f : V_G \to [-1, 1] \), any vertices \( x, y \in V_G \), and any \( a > 0, t \in \mathbb{Z}_+ \),
\[ |P_G^t f(x) - P_G^t f(y)| \leq \frac{10 d_G(x, y)}{a} + 2\mathbb{P}\left( \sum_{s=0}^{t-1} \frac{1}{\deg_G(X_s)} < 2a^2 \right), \]
where \( X \) denotes a lazy random walk on \( G \) starting from \( x \).
Proof. Let \((X, Y)\) be the Markov chain on \(V_G \times V_G\) which, from any state \((x, y) \in V_G \times V_G\), draws the next state according to the “good” optimal coupling of \(P_G(x, \cdot), P_G(y, \cdot)\) described in Lemma 14. We use the standard notations \(\mathbb{P}_{(x,y)}(\cdot), \mathbb{E}_{(x,y)}[\cdot]\) to specify the choice of the initial state. Since the two coordinates \(X, Y\) are marginally distributed as lazy random walks on \(G\), we have

\[
|P^t_G f(x) - P^t_G f(y)| = \left| \mathbb{E}_{x,y} [f(X_t)] - \mathbb{E}_{x,y} [f(Y_t)] \right|
\]

\[
\leq \mathbb{E}_{x,y} [|f(X_t) - f(Y_t)|]
\]

\[
\leq 2 \mathbb{P}_{x,y} (X_t \neq Y_t)
\]

\[
\leq 2 \mathbb{P}_{x,y} (\tau > t),
\]

where \(\tau = \inf\{t \geq 0 : X_t = Y_t\}\) denotes the meeting time of the two walkers. Note that \(\tau\) is also the hitting time of zero by the non-negative process \(Z = (Z_t)_{t \geq 0}\) defined as follows:

\[
\forall t \geq 0, \quad Z_t := d_G(X_t, Y_t).
\]

We claim that \(Z\) is a super-martingale w.r.t. the natural filtration \((\mathcal{F}_t)_{t \geq 0}\) associated with \((X, Y)\). Indeed, by the Markov property and the optimality of the chosen couplings, this claim reduces to

\[
\mathcal{W}_1(P_G(x, \cdot), P_G(y, \cdot)) \leq d_G(x, y),
\]

for all \(x, y \in V_G\). But this inequality readily follows from the assumption \(\kappa_G(x, y) \geq 0\) in the case \(\{x, y\} \in E_G\), and it then automatically extends to all \(x, y \in V_G\) by the triangle inequality of \(\mathcal{W}_1(\cdot, \cdot)\) (see, e.g., [Vil03]). On the other hand, Lemma 14 ensures that on the event \(\{\tau > t\}\),

\[
\mathbb{E}_{x,y} [(Z_{t+1} - Z_t)^2 | \mathcal{F}_t] \geq \frac{1}{2 \deg_G(X_t)}.
\]

Finally, note that the distance between the two walkers can not increase by more than 2 at each step. Thus, we may invoke Lemma 15 to conclude that

\[
\mathbb{P}_{x,y} (\tau \geq t) \leq d_G(x, y) \left( \frac{4a + 1}{a^2} \right) + \mathbb{P}_{x,y} \left( \sum_{s=0}^{t-1} \frac{1}{\deg_G(X_s)} < 2a^2 \right)
\]

\[
\leq \frac{5d_G(x, y)}{a} + \mathbb{P}_{x,y} \left( \sum_{s=0}^{t-1} \frac{1}{\deg_G(X_s)} < 2a^2 \right),
\]

where the second line follows from the first if \(a \geq 1\), and is trivial otherwise. \(\square\)

In particular, this applies to any bounded harmonic function \(f\), after a trivial normalization. Since \(P^t_G f = f\) for all \(t \geq 0\), we may send \(t \to \infty\) and then \(a \to \infty\) in the resulting estimate to obtain the following key result, which ensures that non-negatively curved graphs satisfy the Liouville property, provided they have a “decent proportion” of vertices with “reasonable” degree.
Corollary 17. (Liouville property and non-negative curvature). Let $G$ be a connected graph with $\kappa(G) \geq 0$. Fix $o \in V_G$ and suppose that the simple random walk $X$ on $G$ starting from $o$ satisfies

$$\mathbb{P} \left( \sum_{t=0}^{\infty} \frac{1}{\deg_G(X_t)} = \infty \right) = 1. \quad (24)$$

Then, $G$ has the Liouville property.

A simple situation where the above condition trivially holds is that where $G$ has bounded degrees. In that case, the Liouville property was recently established by J. Jost, F. M"unch, and C. Rose [JMR19]. Our relaxation allows for arbitrarily large degrees, as long as the random walk can “avoid them from times to times”. This property turns out to hold almost-surely on any stationary random graph, allowing us to finally establish Theorem 12.

Proof of Theorem 12. Let $(G, o)$ be a stationary random graph with $\mathbb{E}[\log \deg_G(o)] < \infty$ and conditionally on $(G, o)$, let $X$ be a lazy random walk on $G$ starting from $o$. Then $(\deg_G(X_t))_{t \geq 0}$ is a stationary $N$-valued sequence, so the Poincaré Recurrence Theorem (see, e.g. [LP16, Exercise 2.3]) ensures that the set \( \{ t \in \mathbb{N} : \deg_G(X_t) = \deg_G(X_0) \} \) is almost-surely infinite. In particular, the random rooted graph $(G, o)$ satisfies (24) almost-surely. By the above corollary, this implies that $G$ has the Liouville property almost-surely on the event \( \{ \kappa(G) \geq 0 \} \). Finally, by Theorem 10, we conclude that \( \mathcal{H}(G) = 0 \) almost-surely on the same event. \( \square \)

3.3 Zero entropy implies poor spectral expansion. This section is devoted to proving Theorem 13, which relates the eigenvalues of finite graphs to the spectral radius of their local weak limits. If $G$ is a finite graph, the $N = |V_G|$ eigenvalues $\lambda_1(G) \geq \cdots \geq \lambda_N(G)$ of its transition matrix $P_G$ can be conveniently encoded into a probability measure $\mu_G \in \mathcal{P}([0,1])$, called the empirical eigenvalue distribution of the matrix $P_G$:

$$\mu_G := \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i(G)}. $$

It turns out that the large-size asymptotics of this fundamental object can be understood directly at the level of local weak limits. When $P_G$ is replaced with the more standard adjacency matrix, this classical observation is the starting point of a rich and well-established theory [Bor17].

Local spectral measures. The transition kernel $P_G$ of a graph $G$ can be viewed as a linear operator acting via (10) on the Hilbert space

$$\ell^2(G) := \left\{ f \in \mathbb{C}^{V_G} : \sum_{o \in V_G} \deg_G(o) |f(o)|^2 < \infty \right\},$$
with inner product \( \langle f, g \rangle = \sum_{o \in V_G} \deg_G(o) \overline{f(o)} g(o) \). The stochasticity, laziness and reversibility

\[
\sum_{y \in V_G} P_G(x, y) = 1, \quad P_G(x, x) \geq 1/2, \quad \deg_G(x) P_G(x, y) = \deg_G(y) P_G(y, x),
\]
easily (and classically) imply that \( P_G \) is a positive contraction on \( \ell^2(G) \), i.e.

\[
\forall f \in \ell^2(G), \quad 0 \leq \langle f, P_G f \rangle \leq \langle f, f \rangle.
\]

In particular, for each \( o \in V_G \), the spectral theorem for self-adjoint operators ensures the existence of a \textit{local spectral measure} \( \mu_{(G, o)} \in \mathcal{P}([0, 1]) \), characterized by the moment identity

\[
\forall t \geq 0, \quad \int_0^1 \lambda^t \mu_{(G, o)}(d\lambda) = P_G^t(o, o).
\]

As we will now see, \( \mu_{(G, o)} \) can be interpreted as the local contribution of \( o \) to the spectrum of \( P_G \). Local spectral measures are a powerful tool to investigate the mixing properties of graphs, see [LG18].

\textbf{The finite case.} When \( G \) is finite with \( N \) vertices, there is an orthonormal basis \( (\phi_1, \ldots, \phi_N) \) of \( \ell^2(G) \) consisting of eigenvectors of \( P_G \) with eigenvalues \( \lambda_1(G), \ldots, \lambda_N(G) \), and we easily find

\[
\mu_{(G, o)} = \sum_{i=1}^N \deg_G(o) |\phi_i(o)|^2 \delta_{\lambda_i(G)}.
\]

Thus, the local spectral measure \( \mu_{(G, o)} \) is a mixture of Dirac masses located at the various eigenvalues of \( P_G \), and weighted by the squared amplitudes of the corresponding eigenvectors at \( o \). Moreover, thanks to the orthonormality of \( (\phi_1, \ldots, \phi_N) \), the identity (26) readily implies

\[
\mu_G = \frac{1}{|V_G|} \sum_{o \in V_G} \mu_{(G, o)}.
\]

In other words, the empirical eigenvalue distribution of a finite graph \( G \) coincides with the spatial average of its local spectral measures.

\textit{Spectral continuity.} In light of (5), it is tempting to pass to the limit in the formula (27) along a convergent sequence of finite graphs \( (G_n)_{n \geq 1} \). This is made rigorous by the following continuity principle. As usual, \( \mathcal{P}([0, 1]) \) is here equipped with the topology of weak convergence.
Lemma 18. (Spectral continuity). The map \((G, o) \mapsto \mu_{(G, o)}\) is continuous on \(\mathcal{G}_\bullet\). In particular, if a sequence of graphs \((G_n)_{n \geq 1}\) admits a local weak limit \(L = \text{Law}(G, o)\), then

\[ \mu_{G_n}(d\lambda) \xrightarrow{n \to \infty} \mu_L(d\lambda) := \mathbb{E}[\mu_{(G, o)}(d\lambda)]. \]

Proof. For each fixed \(t \geq 0\), the observable \((G, o) \mapsto P_t^G(o, o)\) is clearly \(t\)-local, hence continuous. In particular, via the identity (25), the convergence \((G_n, o_n) \to (G, o)\) in \(\mathcal{G}_\bullet\) implies

\[ \forall t \geq 0, \int_0^1 \lambda^t \mu_{G_n, o_n}(d\lambda) \xrightarrow{n \to \infty} \int_0^1 \lambda^t \mu_{(G, o)}(d\lambda). \] (28)

Since convergence in \(\mathcal{P}([0, 1])\) is equivalent to the convergence of moments, we conclude that \(\mu_{(G_n, o_n)} \xrightarrow{n \to \infty} \mu_{(G, o)}\), and the continuity is proved. Similarly, the second claim is obtained by applying (5) to the \(t\)-local observable \(f : (G, o) \mapsto P_t^G(o, o)\), for each \(t \geq 1\).

Corollary 19. (Unit spectral radius implies poor spectral expansion). Let \((G_n)_{n \geq 1}\) be finite graphs with a local weak limit supported on graphs with spectral radius 1. Then, for any \(\rho < 1\),

\[ \liminf_{n \to \infty} \mu_{G_n}([\rho, 1]) > 0. \] (29)

Moreover, we have the refinement

\[ \sup_{n \geq 1} \left| \left\{ x \in V_n : \mu_{(G_n, o_n)}([\rho, 1]) \leq \varepsilon \right\} \right| \frac{1}{|V_n|} \xrightarrow{\varepsilon \to 0} 0. \] (30)

Proof. First observe that by (25), the supremum of the support of the local spectral measure \(\mu_{(G, o)}\) of any rooted graph \((G, o) \in \mathcal{G}_\bullet\) is precisely the spectral radius \(\varrho(G)\):

\[ \sup \left\{ \rho : \mu_{(G, o)}([\rho, 1]) > 0 \right\} = \varrho(G). \]

Now fix \(\rho < 1\). By the second part of Lemma 18, the convergence \(G_n \to L\) implies

\[ \liminf_{n \to \infty} \mu_{G_n}([\rho, 1]) \geq \mu_{L}((\rho, 1)), \]

and the right-hand side is positive because \(L\) is supported on graphs with \(\rho(G) = 1\). To prove the second claim, fix \(\varepsilon \in [0, 1]\) and note that the continuity of \((G, o) \mapsto \mu_{(G, o)}\) implies that the event

\[ F_\varepsilon := \left\{ (G, o) \in \mathcal{G}_\bullet : \mu_{(G, o)}([\rho, 1]) \leq \varepsilon \right\}, \]

is closed in \(\mathcal{G}_\bullet\). Consequently, the convergence \(G_n \to L\) implies

\[ \limsup_{n \to \infty} \mathcal{L}_{G_n}(F_\varepsilon) \leq \mathcal{L}(F_\varepsilon), \]

and the right-hand side tends to \(\mathcal{L}(F_0) = 0\) as \(\varepsilon \to 0\), because \(\mathcal{L}\) is supported on graphs with \(\varrho(G) = 1\). The limsup can then be replaced with a sup, since for each \(n \geq 1\), \(\mathcal{L}_{G_n}(F_\varepsilon)\) decreases monotonically to 0 with \(\varepsilon\). \(\square\)
Remark 6. (Corollary 19 vs Theorem 13). The statement (29) asserts that a macroscopic proportion of eigenvalues of $G_n$ accumulate in $[\rho, 1]$, which is exactly the conclusion of Theorem 13. The refinement (30), on the other hand, constitutes a rigorous formalization of the “delocalization” announced in Remark 3. To see this, recall that for any graph $G$ with $N$ vertices, we have by (26),

$$
\mu(G,x)([\rho, 1]) = \sum_{i=1}^{N} \deg_G(x)|\phi_i(x)|^2 1_{\lambda_i(G) \geq \rho}.
$$

In words, the number $\mu(G,x)([\rho, 1]) \in [0, 1]$ measures the cumulative squared amplitude at $x$ of all the basis eigenvectors corresponding to “bad” eigenvalues (those in $[\rho, 1]$). In particular, the set $\{x \in V_G: \mu(G,x)([\rho, 1]) \leq \epsilon\}$ represents the region where those “bad” eigenvectors have a small cumulative squared amplitude. The statement (30) asserts that the relative size of this region can be made arbitrarily small by choosing $\epsilon$ small, uniformly in $n$. Thus, bad eigenvectors have their cumulative mass “spread out” across most vertices.

3.4 Zero-entropy implies hyperfiniteness In this final section, we establish the hyperfiniteness result stated in Theorem 4. The approach is similar to the one used for Theorem 3. Fix $d \in \mathbb{N}$, and suppose for a contradiction that the class of all non-negatively curved finite graphs with degrees at most $d$ is not hyperfinite. Unwrapping the definition, this means that there is $\epsilon > 0$ and a sequence of non-negatively curved finite graphs $(G_n)_{n \geq 1}$ with degrees at most $d$, such that

$$M_{\epsilon}(G_n) \xrightarrow{n \to \infty} +\infty,$$

where $M_{\epsilon}(G)$ denotes the minimum possible size of the largest connected component, over all subgraphs that can be formed by removing at most $\epsilon|V_G|$ edges from $G$. Now, the property (31) is clearly preserved under taking subsequences, so by virtue of Theorem 5, we may further assume that the sequence $(G_n)_{n \geq 1}$ admits a local weak limit $\mathcal{L}$. The latter is automatically unimodular, and supported on non-negatively curved graphs with degrees at most $d$. By Theorem 12, it must then be supported on graphs with zero entropy. To obtain a contradiction, it now only remains to prove the following hyperfinite analogue of Theorem 13.

Theorem 20. (Zero-entropy implies hyperfiniteness). Any sequence of finite graphs $(G_n)_{n \geq 1}$ having bounded degrees and a local weak limit supported on graphs with zero entropy is hyperfinite.

Proof. Let $(G_n)_{n \geq 1}$ be finite graphs with degrees at most $d$ and whose local weak limit $\mathcal{L}$ is supported on graphs with zero entropy. By Lemma 11, $\mathcal{L}$ is then also supported on graphs with zero speed. Now, a remarkable result established by I. Benjamini, R. Lyons and O. Schramm [BLS99, Theorem 4.3] in the special case of Cayley graphs, and later extended to unimodular random graphs by D. Aldous and
R. Lyons [AL07, Theorem 8.15], asserts that any unimodular law $\mathcal{L}$ supported on bounded-degree graphs with zero speed is amenable in the sense that

$$\inf \left\{ \mathbb{E} \left[ \frac{|\partial K(o)|}{|K(o)|} \right] : K \text{ is a finitary percolation of } \mathcal{L} \right\} = 0. \quad (32)$$

This definition requires a bit of explanations (more details can be found in [AL07]): the notion of unimodularity naturally extends to networks (i.e., graphs with marks on the edges), and a percolation $K$ of $\mathcal{L}$ is a unimodular random rooted network with $\{0, 1\}$—valued marks, such that the random rooted graph obtained by forgetting the marks has law $\mathcal{L}$. Deleting from this random rooted graph all edges whose mark was 0 gives rise to a random subgraph; we let $K(o)$ denote the connected component of the root in that subgraph, and $\partial K(o)$ its edge-boundary, i.e., the set of zero-marked edges incident to $K(o)$. Finally, we call the percolation finitary if $\mathbb{P}(|K(o)| < \infty) = 1$, and bounded if $\mathbb{P}(|K(o)| \leq k) = 1$ for some $k \in \mathbb{N}$. Now, the unimodularity of a finitary percolation $K$ implies

$$\mathbb{E} \left[ \frac{|\partial K(o)|}{|K(o)|} \right] = \mathbb{E}[\deg_{K^c}(o)],$$

where $\deg_{K^c}(o)$ denotes the number of zero-marked edges incident to the root (this is just the Mass Transport Principle with $f(K, o, x) = \frac{\deg_{K^c}(o)}{|K(o)|} 1_{x \in K(o)}$). Moreover, if we start with a finitary percolation $K$ of $\mathcal{L}$, and then change to zero all marks in connected components of size more than a given threshold $k \in \mathbb{N}$, we obtain a new percolation $\tilde{K}$ of $\mathcal{L}$ which is now bounded, and such that

$$\mathbb{E}[\deg_{\tilde{K}^c}(o)] \leq \mathbb{E}[\deg_{K^c}(o)] + d\mathbb{P}(|K(o)| > k).$$

Since the second term on the right-hand side can be made arbitrarily small by choosing $k$ large, we conclude that the amenability property (32) is in fact equivalent to the following one:

$$\inf \{ \mathbb{E}[\deg_{K^c}(o)] : K \text{ is a bounded percolation of } \mathcal{L} \} = 0. \quad (33)$$

To conclude, we now invoke the following result of O. Schramm [Sch08]: if $(G_n)_{n \geq 1}$ are finite graphs with degrees at most $d$ and local weak limit $\mathcal{L}$, then (33) holds if and only if $(G_n)_{n \geq 1}$ is hyperfinite.

\[ \square \]

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