Phase transitions for excited states in $^{16}O + \alpha \rightarrow ^{20}Ne$ within the SACM

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Abstract. We present the cranking of the Semimicroscopic Algebraic Cluster Model (SACM) for two spherical clusters. A geometrical mapping is applied and a discussion on phase transition as a function of the cranking parameter is given. This parameter can be related to the average angular momentum of the nucleus. The particular cluster system considered is $^{16}O + \alpha \rightarrow ^{20}Ne$.

We also investigate the phase transition property when the Pauli exclusion principle is not observed. We show that phase transitions may occur within the same dynamical symmetry limit.

1. Introduction
Cluster models play an important role in describing structural properties of nuclei at low energy. They contributed in the understanding of decay properties, like the $\alpha$-particle decay or the emission of larger particles [1] up to the fission [2, 3]. In the usual approach, one has to antisymmetrize a two cluster system explicitly, which leads to great practical difficulties. An alternative approach was presented in [4, 5] (the above mentioned SACM), where the Pauli-principle was taken into account by the construction of a model space, which has the same content as the shell model space. While this space is microscopic, the Hamiltonian is phenomenological and contains interactions which are related to the generators of a unitary or orthogonal groups. The model was applied to many different cluster systems with great success. Also some decay properties could be understood through this model [6].

In order to study the geometrical structure, the mapping to geometrical coordinates is given in [7]. The method is based on the use of coherent states [8, 9, 10]. It is general and can be applied to any algebraic model which involves bosons. For more recent literature and excellent reviews, please consult [11, 12, 13, 14]. These considerations helped to understand the properties of a cluster system under phase transitions [15, 16]. In [15, 16] it was shown that only three effective parameters appear in the SACM for the description of phase transitions, while only two effective parameters appear, when Pauli principle is not taken into account. A model which does not observe the Pauli exclusion principle, we denote as a Phenomenological Algebraic Cluster Model (PACM). As a further result we showed that the PACM can, in principle, reproduce the results of the SACM when highly non-polynomial terms are included, making it more involved to apply [15].
The present contribution is related to the question, if a phase transition can appear when the total system is rotated. Phase transitions may occur when the angular momentum is increased and the phenomena may be related to the concept of backbending [17] or other phenomena, like the change of deformation of the united nucleus. The method in question is called cranking [17]. Cranking applied to cluster systems can be found in [18]. It was first applied to an algebraic model, though not a cluster model, in [19, 20]. A more recent application to an algebraic model can be found in [21]. In [22] the cranking formalism was for the first time applied to the SACM. Also the system \(^{16}\text{O}+\alpha \rightarrow ^{20}\text{Ne}\) was considered. The reason for choosing this system is that no additional contribution from the individual clusters enter and all structure comes from the relative motion of the clusters only. However, the general structure of phase transitions was not considered and the discussion was restricted from the beginning to the \(\text{SU}(3)\) dynamical symmetry.

In this contribution we will retake the question of how to crank the SACM. We again restrict to two spherical clusters, the system \(^{16}\text{O}+\alpha \rightarrow ^{20}\text{Ne}\), but the general treatment will already become obvious. An overview of the formalism, for both SACM and PACM, will be presented. No details will be shown and we refer to a later publication. We finally compare the results for both models and show that in the SACM a second order phase transition occurs when the angular momentum reaches a critical value, while in the PACM no phase transition occurs. We show that a phase transition in the SACM may occur within the same dynamical symmetry limit.

2. The Model and the Geometric Mapping
The basic degrees of freedom are the relative oscillations, described by the \(\pi^\dagger_m\) creation operators and by the corresponding annihilation operators \(\pi^m\) \((m = \pm 1, 0)\). The products of a creation with an annihilation operator form the \(U_R(3)\) group, where \(R\) refers to the relative motion. A cutoff is introduced by adding a scalar boson, whose creation and annihilation operator are \(\sigma^\dagger\) and \(\sigma\) respectively. These bosons do not have a physical meaning but are merely introduced to create a cut-off. \(N = n_\pi + n_\sigma\). The products of all types of boson creation with boson annihilation operators form a \(U_R(4)\) group. For a detailed description of the SACM, please consult [4, 5].

The model space is constructed by first calculating the product \((\lambda_1, \mu_1) \otimes (\lambda_2, \mu_2) \otimes (n_\pi, 0)\), where the \((\lambda_k, \mu_k)\) \((k = 1, 2)\) describe the structure of the clusters 1 and 2 and \(n_\pi\) is the relative oscillation number. This results in a sum of many \(\text{SU}(3)\) irreducible representations (irreps) \((\lambda, \mu)\) with a multiplicity \(m_{(\lambda, \mu)}\). This list has to be compared with the content of the shell model, which can be easily obtained by computer routines, which are available to us and can be obtained on request. When a \(\text{SU}(3)\) irrep in the former mentioned list does not appear in the shell model, this irrep is excluded, while when it appears it is included into the model space of the SACM. In this way, the model space of the SACM is microscopic. For the case of two spherical clusters, this procedure is equivalent to the Wildermuth condition [23], where the number of \(\pi\) bosons is restricted from below, in order to satisfy the Pauli exclusion principle.

The Hamiltonians, however, is phenomenologic and is written as a sum of Casimir operators of subgroups of the \(U_R(4)\). When we restrict to spherical clusters, i.e. \((\lambda_k, \mu_k) = (0, 0)\), there is no contributions from the individual clusters. In \(\text{SU}(3)\) notation, they are in a scalar \((0, 0)\) irrep of \(\text{SU}(3)\). There are two dynamical symmetry group chains of importance, called the \(\text{SU}_R(3)\) and the \(\text{SO}_R(4)\) dynamical symmetries. For the case discussed here, the Hamiltonian has the form

\[
H = xH_{\text{SU}(3)} + (1 - x)H_{\text{SO}(4)} - \Omega H_{\text{crank}}
\] (1)

with
\begin{align}
H_{\text{SU(3)}} &= \hbar \omega n_\pi + (a - b \Delta n_\pi) C_2(n_\pi, 0) + \xi L^2 \\
H_{\text{SO(4)}} &= \frac{c}{4} [\pi^\dagger \cdot \pi^\dagger] - (\sigma^\dagger)^2 [[\pi \cdot \pi] - (\sigma)^2] + \xi L^2 \\
H_{\text{crank}} &= L_x,
\end{align}

where the \( \Delta \hat{n}_\pi = \hat{n}_\pi - n_0 \) is the excitation number of quanta and \( n_0 \) is the minimal number of relative oscillation quanta [23], a necessary condition to satisfy the Pauli exclusion principle. The \( L_x \) is the x-component of the angular momentum operator and \( \Omega \) is the frequency parameter which fixes the amount of the angular momentum. The structure of the Hamiltonian is equal to introduce a Lagrange multiplier [17].

The semi-classical potential is obtained by first defining a coherent state

\[ |\alpha\rangle = \frac{N!}{(N+n_0)!} \mathcal{N}_{N,n_0} \frac{d^{n_0}}{d\gamma^{n_0}} \left[ \sigma^\dagger + (\alpha^* \cdot \pi^\dagger) \right]^{N+n_0} |0\rangle, \tag{3} \]

where

\[ \mathcal{N}_{N,n_0}^{-1} = \frac{(N!)^2}{(N+n_0)!} \frac{d^{n_0}}{d\gamma^{n_0}} \frac{d^{n_0}}{d\gamma^{n_0}} \left[ 1 + \gamma_1 \gamma_2 (\alpha^* \cdot \alpha) \right]^{N+n_0} \tag{4} \]

is the normalization constant. The \( \alpha_m \ (m = \pm 1, 0) \) are the coherent state parameters. The \( \gamma \) is set to zero after having applied the derivatives. The corresponding coherent state for the PACM is obtained by setting \( n_0 = 0 \), leading to a more known form of the coherent state in algebraic models. For notational reasons, the total number of quanta is now \( N+n_0 \), i.e., \( n_\pi + n_\sigma = N+n_0 \).

Finally, calculating the expectation value of the total Hamiltonian, i.e., \( \langle \alpha | H | \alpha \rangle \), the semi-classical potential is obtained. When \( \Omega = 0 \), then the only relevant parameter is \( \alpha = \alpha_0 \), describing the relative distance of the two clusters [7, 15, 16]. However, in general we need all three components when cranking is included. For reason of simplicity, we use polar coordinates, i.e.

\[ \alpha_{\pm 1} = \frac{\alpha}{\sqrt{2}} e^{\pm i \phi} \sin(\theta) \]
\[ \alpha_0 = \alpha \cos(\theta) \tag{5} \]

The next steps are: First, determining the minimum in the variables \( \phi \) and \( \theta \). The minimization with respect to \( \phi \) is easy with the result that the minimum is always at \( \phi = 0 \). The minimization with respect to \( \theta \) is not so easy, it results in a fourth order equation, which can be expressed as \( \sin 2\theta = F(\alpha) \), which is a function of \( \alpha \), \( N \), \( n_0 \) and all the interaction parameters. In a second step one has to substitute the value of \( \theta \) into the potential giving an effective potential \( V_{\text{eff}} \) depending only on \( \alpha \). Results get much easier, when only the \( SU(3) \) dynamical symmetry limit is considered [22].

The \( \Omega \) parameter is determined by the following steps: The expectation value of \( L_x \) is set to \( L \), the angular momentum, which here is rather an average value. The expectation value of \( L_x \), within the SACM, is given by

\[ \langle L_x \rangle = (N+n_0) \cos \phi \sin 2\theta \alpha^2 \frac{F_{22}(\alpha)}{F_{00}(\alpha)} \tag{6} \]

where \( F_{pq}(\alpha) \) are polynomial functions in \( \alpha \) [15]. The expressions get particular simple for small \( \alpha \). For small \( \alpha \), the minimum of the potential is given by

\[ \phi = 0 \quad \text{and} \quad \sin 2\theta = \frac{\Omega}{4 \xi n_0} \tag{7} \]
For large $\alpha$ the factor $\frac{\Omega}{\xi n_0}$ is multiplied by $[1 + G(\alpha)]$, with $G(\alpha)$ being a sum of ratios of polynomials in $\alpha$ [15, 16]. With this, the expectation value is

$$\langle L_x \rangle = \frac{\Omega}{4\xi} \frac{(n_0 - 1)}{(N + n_0 - 1)} = L,$$

(8)

Which allows to deduce an approximate value for $\Omega$, for a given $L$. In general, there will be a dependence on $\alpha$, where one has to substitute for $\alpha$ the value at the minimum of the potential.

In order to study the phase transition properties, one first has to determine the minima of the potential. It turns out that $\bar{\alpha}_1 = 0$ is always an extremum, which can either be a minimum or maximum. We call this a spherical solution, though in the SACM the distance between the two clusters has a minimal value which is proportional to $\sqrt{n_0}$ [7], i.e., even in the intrinsic ground state a deformation may exist. There is a deformed solution with $\bar{\alpha}_2 > 0$. This solution not always exists. In the next step we have to find points in the phase space diagram [15, 16], where the values of the potential in both minima coincide, i.e. $V(\bar{\alpha}_1) = V(\bar{\alpha}_2)$. This describes a surface in the phase space diagram. In order to decide which order the transition is, one has to calculate on both sides the first and second derivatives with respect to the parameters. The order of the phase transition is one when the first derivatives in both minima are not equal, while it is of second order when the first derivatives are equal but the second are not. As shown in [15, 16] there is at most a second order phase transition in the SACM and cranking the model does not change the structure. Calculating the first and second derivatives is very involved because of the complex structure of the potential. The considerations can be greatly simplified when one considers the special structure of the potential within the SACM. As shown in [15, 16], when at the point of a phase transition the spherical and the deformed minima cross at different values of $\alpha$, the phase transition is of first order, while when they merge at the point of phase transition it is of second order. For later discussion, note that the $L_x$ is a generator of the $SO(3)$ subgroup of $SU(3)$ and thus within the $SU(3)$ dynamical symmetry limit.

As shown in this section, a general treatment is feasible but involved. This is still work in progress and the full results will be published elsewhere.

3. An application: The system $^{16}O+\alpha \rightarrow ^{20}Ne$

The method presented is applied to the system $^{16}O+\alpha \rightarrow ^{20}Ne$. In this system, the minimal number $\pi$-bosons is 8 and we choose 20 as the total number of quanta $N + n_0$, i.e., $N = 12$. The interaction parameters where fitted to experimental data, with good agreement. We will not present details here but merely report the parameter values obtained, which are $\hbar \omega = 13.2$MeV, $a = -0.5$MeV, $b = -0.009$MeV and $\xi = 0.208$MeV. It is not a surprise that the $SU(3)$ dynamical symmetry fits well the spectrum, i.e. $x = 1$ [24]. This fit was applied for $\Omega = 0$. The resulting potential shows a minimum at $\alpha = 0$, raises for larger $\alpha$ and finally equilibrating at a constant value for very large $\alpha$, which is an effect of the finite space ($N$ is finite) used [15, 16].

In order to simplify the consideration we will consider $\alpha << 1$. This is justified because for $\Omega = 0$ the potential of the system has a spherical minimum, so increasing $\Omega$ we may reach a point when this minimum flips over and the global minimum will be a deformed one. In other words, we expect a second order phase transition, if it exists. Note, that this implies that even when we are in a determined dynamical symmetry limit (here $SU(3)$), still phase transition may occur ($L_x$ is within the $SU(3)$ dynamical symmetry), rendering the association of a dynamical symmetry to a determined phase questionable.

In the limit of $\alpha << 1$ the solution of $\sin \theta$ is particular easy, namely $\sin(2\theta) \approx \frac{\Omega}{\pi n_0}$. The
potential has, up to second order on $\alpha$, the approximate form

\[ V(\alpha) \approx (N + n_0)(N + n_0 - 1)(N + n_0 - 2)(-b) \left( A - \Omega^2 B \right) \alpha^2 \]

\[ = 61.56 \left( 8.0491 - 0.1318 \Omega^2 \right) \alpha^2 \ [\text{MeV}] \ . \tag{9} \]

In the last line the fitted parameter values for the system $^{16}\text{O}+\alpha$ were substituted. The $A$ is positive, using the adjusted parameters and also $B$ is positive. The last equation gives a very intuitive picture: When $\Omega^2 < A/B$, the global minimum is spherical and for $\Omega^2 > A/B$ it flips over, i.e. the new global minimum is a deformed one. The fact that the two minima (spherical and deformed) merge at the point of phase transition, we conclude that it is a second order phase transition [15, 16].

With the parameter values used, the limiting value of $\Omega$ is given by 7.81. Using (8), leads to the angular momentum of approximately $9 - 10$ (9.39). Of course, this does not mean that the phase transition can be seen exactly from $L = 9 - 10$, probably slightly above. This phase transition happens within the $SU(3)$ dynamical symmetry. However, if this is related to backbending or another phenomenon can not be determined yet.

We also considered the PACM, when the Pauli exclusion principle is not taken into account. The equation for $\theta$ changes such that $\sin \theta$ is proportional to $(1 + \alpha^2)/\alpha^2$. As a result a mere constant term, proportional to $\Omega$ appears, whose effect is nothing but moving the potential up and down, by changing $\Omega$. Therefore, in the PACM no phase transition exists when the nucleus is excited towards larger angular momentum.

4. Conclusions

In this contribution we presented the general formalism on how to introduce cranking in the SACM. We restricted to the case of two spherical clusters and the particular system considered was $^{16}\text{O}+\alpha \rightarrow ^{20}\text{Ne}$. The extension to systems with one or two clusters deformed is direct.

We showed that a phase transition can occur in the SACM when the angular momentum increases sufficiently. However, in the PACM no phase transition occurs, which is related to not taking into account the Pauli exclusion principle. The result also demonstrates that different phases may exist within the same dynamical symmetry, rendering the association of a dynamical symmetry to a single phase questionable.

The results presented here are preliminary and a list of questions still remain open. In a future work we will add the internal structure of the clusters, i.e. we take into account deformed nuclei.

Acknowledgements

We acknowledge financial help from DGAPA-PAPIIT (IN103212) and from the National Research Council of Mexico (CONACyT).

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