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To cite this article: Filipe C Mena and Paul Tod 2007 J. Phys.: Conf. Ser. 66 012019

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Lanczos potentials for linearly perturbed FLRW spacetimes

Filipe C Mena\textsuperscript{1} and Paul Tod\textsuperscript{2}
\textsuperscript{1}Dep. Matemática, Universidade do Minho, 4710 Braga, Portugal
\textsuperscript{2}Mathematical Institute, University of Oxford, 24-29 St. Giles', Oxford, U.K.
E-mail: fmena@math.uminho.pt, tod@maths.ox.ac.uk

Abstract. We study the problem of deriving the Lanczos potential and superpotential for linearly perturbed Friedman-Lemaitre-Robertson-Walker (FLRW) spacetimes.

1. Introduction
Penrose [17] conjectured that the gravitational entropy should be related to the clumping of matter and therefore associated with the Weyl or conformal curvature. Specifically, Penrose suggested that a measure of the gravitational entropy should involve an integral of a quantity derived from the Weyl tensor, and that the particle number operator for a linear spin-2 massless quantized free-field might provide some clues, since the entropy measure could be taken as an estimate of the ‘number of gravitons’ [17]. Since then, there have been several attempts to construct gravitational entropy measures using polynomial invariants of the Weyl tensor (see e.g. [8, 4, 16]) as well as density contrast functions [13, 10].

We have used Penrose’s conjecture and the particle number from linear theory in flat space to motivate a definition of gravitational entropy in curved space [14]. In order to do that we required a potential for the Weyl tensor which we took to be the Lanczos potential [12]. Illge [11] has shown that any spinor field with the symmetries of the Weyl spinor locally has a Lanczos potential which is determined by its value at a space-like hypersurface. Furthermore, for a vacuum spacetime there exists a potential for the Lanczos potential, i.e. a superpotential for the Weyl spinor [11] (see also [1]).

Apart from Illge’s result, which is difficult to apply, there is no general prescription for obtaining a Lanczos potential for a given spacetime. A general expression for a Lanczos potential in the case of perfect fluid spacetimes with zero shear and vorticity was given in [15]. More recently, this result has been extended by Holgersson [9] to Bianchi I perfect-fluid spacetimes. There are also several examples of Lanczos potentials for particular exact solutions, including Gödel, Schwarzschild, Taub and Kerr [3, 15, 5, 6].

In this short note, we consider the problem of deriving the Lanczos potential and superpotential for linearly perturbed Friedman-Lemaitre-Robertson-Walker (FLRW) spacetimes, which we then use to propose a new measure of the gravitational entropy in [14].
2. The perturbed FLRW model

We consider a spacetime with a distinguished time-like direction given by the velocity vector field \( u^a \) of the fluid, and use the formalism of [7, 18], with the projected metric
\[
h_{ab} = g_{ab} + u_a u_b,
\]
which is orthogonal to \( u^a \). The covariant derivative of \( u_a \) can be written as
\[
\nabla_b u_a = \frac{1}{3} \theta h_{ab} + \sigma_{ab} + \omega_{ab} - \dot{u}_a u_b
\]
where
\[
\sigma_{ab} = \sigma_{(ab)}; \quad \sigma^a_a = 0; \quad \sigma_{ab} u^b = 0; \quad \omega_{ab} = \omega_{[ab]}; \quad \omega_{ab} u^b = 0.
\]
Then \( \dot{u}^a \) is the acceleration (so that the overdot is \( u^a \nabla_a \)), \( \omega_{ab} \) is the vorticity tensor, \( \sigma_{ab} \) the shear, and \( \theta \) the expansion. The stress–energy tensor for perfect fluids is
\[
T_{ab} = \rho u_a u_b + p h_{ab},
\]
where \( \rho \) is the energy density and \( p \) the isotropic pressure of the fluid.

The Weyl tensor can be decomposed into its electric and magnetic parts, \( E_{ab} \) and \( H_{ab} \), relative to the velocity vector \( u^a \) as
\[
E_{ab} = C_{abcd} u^c u^d, \quad H_{ab} = C^*_{abcd} u^c u^d,
\]
where \( C^*_{abcd} = \frac{1}{2} \eta_{ac} \epsilon_{di} C_{stbd} \). An FLRW background is conformally-flat with the fluid-flow being geodesic, shear-free and twist-free so that \( u_a = \omega_{ab} = \sigma_{ab} = 0 = E_{ab} = H_{ab} \).

We shall now consider the FLRW metric \( g_{ab} \) with linear perturbations \( \delta g_{ab} = \Phi_{ab} \) such that
\[
\Phi_{ab} u^b = \Phi^a_a = \nabla^a \Phi_{ab} = 0.
\]
(1)

The perturbation is characterised as purely gravitational by requiring:
\[
\delta R^b_a = 0.
\]
(2)

This implies that \( \delta \rho = \delta p = 0 \), and with the gauge conditions (1) also \( \delta u^a = \delta u_a = 0 \), so that \( \delta T^b_a = 0 \) and \( \delta \theta = 0 = \delta \omega_{ab} = \delta \dot{u}_a \), while for the shear we introduce the notation:
\[
\Sigma_{ab} := \delta \sigma_{ab} = \frac{1}{2} \dot{\Phi}_{ab}.
\]
(3)

For the Weyl tensor, which is zero in the background, we find
\[
E^{ab} = -\Sigma^{ab} - \frac{2}{3} \theta \Sigma_{ab},
\]
(4)
\[
H^{ab} = \text{curl} \, \Sigma_{ab},
\]
(5)

with
\[
\text{curl} \, X^{ab} \equiv (\text{curl} \, X)^{ab} := \eta^{cd(a} D_c X^{b)}_{\, d},
\]
where \( D_c \) is the covariant derivative on hypersurfaces orthogonal to \( u^a \), \( \eta_{abc} = \eta_{abcd} u^d \) is the hypersurface volume form and \( \eta_{abcd} \) the space-time volume form. Now, the field equation (2) reduces to
\[
\Box \Phi_{ab} = \frac{2}{3} \rho \, \Phi_{ab}
\]
and from (3) and (2) we get
\[
\Box \Sigma_{ab} = \frac{2}{3} \theta \, \Sigma_{ab} + \left( \frac{1}{6} \rho - \frac{3}{2} \rho + \frac{1}{3} \theta^2 \right) \Sigma_{ab}.
\]
(6)
3. The Lanczos potential

The Lanczos potential is a tensor \( L_{abc} = -L_{bac} \) such that:

\[
C_{cd}^{ab} = -\nabla[c]L_{ab}^{d} - \nabla[a]L_{cd}^{b} - 2\delta^{[c}_{a} \nabla[e]L_{b]e}^{d]},
\]

in the Lanczos gauge:

\[
L_{ab}^{a} = 0 = \eta^{abcd}L_{abc} ; \quad \nabla[L_{ab}^{c}] = 0.
\]

Holgersson [9] gave a useful decomposition of the Lanczos potential into irreducible parts as:

\[
L_{abc} = 2u[a]A_{b]c} - 2u[a]C_{b]c} + \eta^{d}_{ab}S_{dc} + u[a]\eta^{b]cd}P^{d} - u[c]\eta^{abd}P^{d}, \quad (7)
\]

where \( A_{a} \) and \( P_{a} \) are orthogonal to \( u^{a} \) and \( S_{ab} \) and \( C_{ab} \) are trace-free, symmetric and orthogonal to \( u^{a} \).

Since the FLRW perturbation is trace-free, symmetric and orthogonal to \( u^{a} \), we seek a Lanczos potential as in (7) with \( A_{a} = P_{a} = 0 \). Then from (7) and (3) we find the following expressions for \( E_{ab} \) and \( H_{ab} \):

\[
E_{ab} = \frac{1}{2}(\text{curl} \, S_{ab} - \dot{C}_{ab}), \quad (8)
\]
\[
H_{ab} = \frac{1}{2}(\text{curl} \, C_{ab} + \dot{S}_{ab}), \quad (9)
\]

which equated to (4) and (5) give the expressions for \( C_{ab} \) and \( S_{ab} \).

Now, suppose a superpotential \( \phi_{ab} \) existed for all times with

\[
L_{abc} = \nabla[a]\phi_{b]c}, \quad (10)
\]

then from (7), we get expressions for \( C_{ab} \) and \( S_{ab} \) as:

\[
C_{ab} = \frac{1}{2}(\dot{\phi}_{ab} + \frac{\theta}{3}\phi_{ab}),
\]
\[
S_{ab} = \frac{1}{2}\text{curl} \, \phi_{ab},
\]

which turn out to be incompatible with the Bianchi identities [14] (as is to be expected, since the superpotential should not exist for non-vacuum). However, this procedure suggests the ansatz:

\[
C_{ab} = \frac{1}{2}(\psi_{ab} + \frac{\theta}{3}\phi_{ab}), \quad (11)
\]

in terms of another unknown tensor \( \psi_{ab} \). So, we find from (9) and (11)

\[
H_{ab} = \frac{1}{4}\text{curl} \, (\dot{\phi} + \psi)_{ab}.
\]

Comparing this equation with (5) we can choose

\[
\Sigma_{ab} = \frac{1}{4}(\dot{\phi}_{ab} + \psi_{ab}), \quad (12)
\]

so that \( \psi_{ab} \) is known once \( \phi_{ab} \) has been found. Then, from (8)

\[
E_{ab} = \frac{1}{4}(-\psi_{ab} - \frac{\theta}{3}\phi_{ab} - \frac{\theta}{3}\phi_{ab} + \text{curl} \, \phi_{ab}),
\]
and combining this with (4) and (12) we get

$$\Box \phi_{ab} + \frac{4}{3} \theta \dot{\phi}_{ab} + (\frac{\dot{\theta}}{3} + \frac{\theta^2}{9} - \rho) \phi_{ab} = \frac{8}{3} \theta \Sigma_{ab},$$

(13)

which is a wave equation for $\phi_{ab}$. We therefore have a complete prescription to determine a unique $L_{abc}$ for linearly perturbed FLRW, subject to choice of initial data. We can achieve (10), at a given instant $t_0$ by choosing the data for (13) to be

$$\phi_{ab}(x, t_0) = \Phi_{ab}(x, t_0),$$

$$\dot{\phi}_{ab}(x, t_0) = \dot{\Phi}_{ab}(x, t_0).$$

(14)

since then, by (12), $\psi_{ab}(x, t_0) = \dot{\phi}_{ab}(x, t_0)$.

We summarize our results in the following proposition:

**Proposition** Given a perturbed FLRW spacetime and a choice of time $t_0$, a Lanczos potential $L_{abc}$, in the Lanczos gauge, may be uniquely specified by (7) with (11), (12) and (13), subject to the data (14). We may define a superpotential $\phi_{ab}$ such that (10) holds at $t_0$ but this will not hold at other times.

**Acknowledgments**

FCM thanks FCT (Portugal) for grant SFRH/BPD/12137/2003 and Centre of Mathematics (CMAT), University of Minho, for support.

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