ON GRADED SECOND MODULES

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Abstract. This paper deals with some results concerning graded second modules.

1. Introduction

Throughout this paper, \( R \) will denote a commutative ring with identity.

A proper submodule \( N \) of an \( R \)-module \( M \) is said to be prime if for any \( r \in R \) and \( m \in M \) with \( rm \in N \), we have \( m \in N \) or \( r \in (N:_RM) \) [6].

In [7], I.G. Macdonald introduced the notion of secondary modules. A non-zero \( R \)-module \( M \) is said to be secondary if for each \( a \in R \) the endomorphism of \( M \) given by multiplication by \( a \) is either surjective or nilpotent [7].

In [11], S. Yassemi introduced the dual notion of prime submodules (i.e., second submodules) and investigated some properties of this class of modules. A non-zero submodule \( N \) of an \( R \)-module \( M \) is said to be second if for each \( a \in R \) the homomorphism \( N \rightarrow N \) is either surjective or zero. This implies that \( \text{Ann}_R(N) = P \) is a prime ideal of \( R \) and \( S \) is said to be \( P \)-second [11]. More information about of this class of modules can be found in [2] and [3].

Let \( G \) be a group with identity \( e \). The ring \( R \) graded by the group \( G \) will be denoted by \( R = \oplus_{g \in G} R_g \), where \( R_g \) is an additive subgroup of \( R \) and \( R_g.R_h \subseteq R_{gh} \) for every \( g, h \) in \( G \). If an element of \( R \) belongs to \( \cup_{g \in G} R_g = h(R) \), then it is called homogeneous and any \( x_g \in R_g \) is said to have degree \( g \). In the rest of this paper let \( R \) be a \( G \)-graded ring. An \( R \)-module \( M \) is said to be a graded module if \( M = \oplus_{g \in G} M_g \) for a family of subgroups \( \{M_g\}_{g \in G} \) of \( M \) such that \( R_g.M_h \subseteq M_{gh} \) for every \( g, h \) in \( G \). A graded submodule \( N \) of \( M \) is a submodule verifying \( N = \oplus_{g \in G}(N \cap M_g) \). Moreover, \( M/N \) becomes a graded \( R \)-module with \( (M/N)_g = (M_g + N)/N \). In this case, \( M/N \) is called a gr-quotient of \( M \). Also if an element of \( M \) belongs to \( \cup_{g \in G} M_g = h(M) \), then it is called homogeneous. Let \( M = \oplus_{g \in G} M_g \) and \( N = \oplus_{g \in G} N_g \) be graded \( R \)-modules. An \( R \)-homomorphism \( f : M \rightarrow N \) is said to be a gr-homomorphism of degree \( h \),
If \(I\) and \(J\) are graded ideals of \(R\) such that \(I \subseteq J\), then \(\text{Gr}(I) \subseteq \text{Gr}(J)\).
(c) If $P$ is a gr-prime ideal of $R$, then $Gr(P^n) = P$ for all $n > 0$.

A graded submodule $N$ of a graded $R$-module $M$ is said to be gr-minimal if it is minimal in the lattice of graded submodules of $M$ [8].

**Proposition 2.3.** Let $M$ be a graded $R$-module. Then the following hold.

(a) If $S$ is a gr-secondary submodule of $M$, then $S$ is gr-second if and only if $Ann_R(S)$ is a gr-prime ideal of $R$.

(b) Let $S$ be a graded submodule of a $P$-gr-second module $M$. Then $S$ is a $P$-gr-secondary submodule if and only if $S$ is a $P$-gr-second submodule.

(c) If $S$ is a gr-minimal submodule of $M$, then $S$ is a gr-second submodule of $M$.

**Proof.** (a) This is obvious.

(b) Assume $S$ is a $P$-gr-secondary submodule of $M$. Then $P = Ann_R(M) \subseteq Ann_R(S) \subseteq Gr(Ann_R(S)) = P$ by using Lemma 2.2 (a). Thus $P = Ann_R(S)$. Now the assertion follows from part (a). The reverse implication is clear.

(c) Let $S$ be a gr-minimal submodule of $M$. Since for each $r \in h(R)$, $rS$ is a graded submodule of $M$, by assumption, $rS = 0$ or $rS = S$ as desired.

**Proposition 2.4.** Let $P$ be a gr-prime ideal of $R$. Then the following hold.

(a) The sum of $P$-gr-second $R$-modules is a $P$-gr-second $R$-module.

(b) Every product of $P$-gr-second $R$-modules is a $P$-gr-second $R$-module.

(c) Every non-zero gr-quotient of a $P$-gr-second $R$-module is a $P$-gr-second $R$-module.

**Proof.** We only prove the part (a). The proofs of parts (b) and (c) are similar.

(a) Let $M_1, M_2, \ldots, M_n$ be $P$-gr-second $R$-modules. Then for each $1 \leq i \leq n$ we have $Ann_R(M_i) = P$ and hence $Ann_R(\sum_{i=1}^n M_i) = P$. If $r \in h(R) - P$, then $rM_i = M_i$. Hence $r(\sum_{i=1}^n M_i) = \sum_{i=1}^n M_i$, as desired.

**Lemma 2.5.** Let $P$ be a graded prime ideal of $R$ and let $S$ be a non-zero graded submodule of a graded $R$-module $M$. Then the following are equivalent.

(a) $S$ is a $P$-gr-second submodule of $M$.

(b) $W^{gr}(S) \subseteq Ann_R(S) = P$, where

$$W^{gr}(S) = \{a \in h(R) : \text{the homothety } S^a \rightarrow S \text{ is not surjective}\}.$$ 

**Proof.** It is straightforward.

A graded $R$-module $M$ is said to be gr-divisible if $ax = m$ with $a \in h(R)$ and $m \in h(M)$, has a solution in $M$ [8].
Theorem 2.6. Let $M$ be a graded $R$-module and let $S$ be a non-zero graded submodule of $M$ satisfying that $\text{Ann}_R(S) = P$ is a graded prime ideal of $R$. Then the following are equivalent.

(a) $S$ is a $P$-gr-second submodule of $M$.
(b) $S$ is a gr-divisible $R/P$-module.
(c) $rS = S$ for all $r \in h(R) - P$.
(d) $IS = S$ for all graded ideals $I$ with $I \not\subseteq P$.
(e) $W^{gr}(S) \subseteq P$.

Proof. $(a) \Rightarrow (b), (b) \Rightarrow (c), (c) \Rightarrow (d)$ and $(d) \Rightarrow (e)$ are straightforward.

$(e) \Rightarrow (a)$. By Lemma 2.5. □

Definition 2.1. Let $P$ be a graded prime ideal of $R$. A graded submodule $N$ of a graded $R$-module $M$ is called a minimal $P$-gr-secondary (resp. $P$-gr-second) submodule of $M$ if $N$ is a $P$-gr-secondary (resp. $P$-gr-second) submodule which contains no other $P$-gr-secondary (resp. $P$-gr-second) submodules of $M$.

Theorem 2.7. Let $M$ be a graded $R$-module. Then a submodule $N$ of $M$ is minimal $P$-gr-secondary if and only if $N$ is a minimal $P$-gr-second submodule of $M$.

Proof. $(\Leftarrow)$. By Proposition 2.3 (b).

$(\Rightarrow)$. Assume that $N$ is a minimal $P$-gr-secondary submodule of $M$. If $r \in W^{gr}(N)$, then $rN \neq N$. Since $rN$ is a graded quotient of $N$, we have that $rN$ is a $P$-gr-secondary submodule of $N$. As $N$ is a minimal $P$-gr-secondary submodule of $M$, $rN = 0$ so that $r \in \text{Ann}_R(N)$. Therefore, $W^{gr}(N) \subseteq \text{Ann}_R(N)$. Thus $N$ is a $P$-gr-second submodule of $M$ by using Lemma 2.5. Now the result follows from Proposition 2.3 (b). □

$R$ is said to be a gr-field if every nonzero homogeneous element of $R$ is invertible.

A graded $R$-module $M$ is said to be gr-injective if it is an injective object in the category of graded $R$-modules.

A graded $R$-module $M$ is said to be graded torsion-free if $a \in h(R)$ and $m \in M$ with $am = 0$ implies that either $m = 0$ or $a = 0$ [4].

Theorem 2.8. Let $M$ be a gr-prime module. Then the following are equivalent.

(a) $M$ is a gr-second module.
(b) $M$ is a gr-injective $R/\text{Ann}_R(M)$-module.
Proof. Since $M$ is a gr-primary module, we have that $P = \text{Ann}_R(M)$ is a gr-primary ideal of $R$ by [4, 2.7] and $M$ is a gr-torsion-free $R/P$-module by [4, 2.11]. Hence the graded $R/P$-homomorphism $\phi : M \to S^{-1}M$ given by $\phi(m) = m/1$, where $S = h(R/P) - 0$, is a monomorphism.

$(a) \Rightarrow (b)$. Since $M$ is a $P$-gr-second module, we have that $M$ is a gr-divisible $R/P$-module by Theorem 2.6. This implies that $\phi$ is an isomorphism. Hence $M$ is an $S^{-1}(R/P)$-module. As $S^{-1}(R/P)$ is a gr-field and $M$ is a gr-divisible $S^{-1}(R/P)$-module by [8, B.II.2], it is easy to see by a similar argument as the ungraded case that $M$ is a gr-injective $R/P$-module.

$(b) \Rightarrow (a)$. Since $M$ is a gr-injective $R/P$-module, we have that $M$ is a gr-divisible $R/P$-module. Thus we have that $M$ is gr-second by Theorem 2.6. □

Proposition 2.9. Let $M$ be a graded $R$-module and let $N$ be a graded submodule of $M$. Then we have the following.

(a) If $M$ is a gr-primary module and $N$ is a gr-second submodule of $M$, then $N$ is $\text{Ann}_R(N)$-gr-primary.

(b) If $M$ is a gr-primary module and $N$ is a gr-second submodule of $M$, then $rN = rM \cap N$ for each $r \in h(R)$.

(c) If $\text{Ann}_R(N)$ is a gr-primary ideal of $R$ and $N$ is a gr-minimal in the set of all graded submodules $K$ of $M$ such that $\text{Ann}_R(K) = \text{Ann}_R(N)$, then $N$ is a gr-second submodule of $M$.

Proof. (a) First we note that as $N$ is a gr-second submodule of $M$, $\text{Gr}(\text{Ann}_R(N)) = \text{Ann}_R(N)$ by Lemma 2.2 (c). Now let $rm \in N$, where $r \in h(R) - \text{Ann}_R(N)$ and $m \in h(M)$. Since $N$ is a gr-second submodule of $M$, we have $rN = N$. Thus $rm = rn$ for some $n \in N$. As $r \not\in \text{Gr}(\text{Ann}_R(N))$, we have $r \not\in \text{Gr}(\text{Ann}_R(M))$ by Lemma 2.2 (b). As $M$ is gr-primary, we have that $m \in N$ as required.

(b) Let $r \in h(R)$ and let $rm \in N$. Since $N$ is gr-second, $rN = 0$ or $rN = N$. If $rN = 0$, we have $r \in \text{Ann}_R(M)$ because $M$ is gr-prime. Hence $rN = rM \cap N = 0$. If $rN = N$, then $rm = rn$ for some $n \in N$. Since $M$ is gr-prime and $r \not\in \text{Ann}_R(N)$, we have $m = n$. Thus $rm \in rN$. Therefore $rM \cap N = N \subseteq rN$. Thus $rM \cap N = N = rN$ because the reverse inclusion is clear.

(c) As $\text{Ann}_R(N)$ is gr-prime, $N \neq 0$. Let $r \in h(R)$ and $rN \neq N$. Since $rN$ is a graded submodule of $M$, the claim is obviously true in the case that $\text{Ann}_R(rN) = \text{Ann}_R(N)$ by assumption. So we assume that $\text{Ann}_R(rN) \not\subseteq \text{Ann}_R(N)$. Then there exists $s \in h(\text{Ann}_R(rN))$ such that $s \not\in \text{Ann}_R(N)$. Hence $srN = 0$. Since $\text{Ann}_R(N)$ is gr-prime, it follows that $rN = 0$, as desired. □

A graded $R$-module $M$ is said to be graded injective cogenerator if it is injective cogenerator object in the category of graded $R$-modules.
Theorem 2.10. Let $E$ be a graded injective cogenerator of $R$ and let $N$ be a graded submodule of a graded $R$-module $M$. Then $N$ is a gr-prime submodule of $M$ if and only if $\text{Hom}_R(M/N, E)$ is a gr-second $R$-module.

Proof. Let $N$ be a gr-prime submodule of $M$ and let $r \in h(R)$. Then $M/N \neq 0$ if and only if $\text{Hom}_R(M/N, E) \neq 0$ by using similar arguments as the ungraded case. Further, $M/N \xrightarrow{r} M/N$ is either injective or zero if and only if

$$\text{Hom}_R(M/N, E) \xrightarrow{r} \text{Hom}_R(M/N, E)$$

is either surjective or zero by using similar arguments as the ungraded case. □

A graded submodule $N$ of a graded $R$-module $M$ is said to be gr-maximal if it is maximal in the lattice of graded submodules of $M$ [8].

Theorem 2.11. Let $R$ be an integral domain which is not a gr-field and $K$ the gr-field of quotients of $R$. Then the $R$-module $K$ has no gr-minimal submodule and $K$ is the only gr-second submodule of $K$.

Proof. Since $(0 :_K r) = 0$ for every non-zero element $r \in h(R)$, we have $\text{Ann}_R(N) = 0$ for every non-zero graded submodule $N$ of $M$. Consequently, $K$ has no gr-minimal submodule, for if $L$ is a gr-minimal submodule of $K$, then $\text{Ann}_R(L)$ is a gr-maximal ideal of $R$. But since $R$ is not a gr-field, $\text{Ann}_R(L) \neq 0$, which is a contradiction. Clearly $K$ is a 0-gr-second submodule of $K$.

To show that $K$ is the only gr-second submodule of $K$, we assume the contrary and let $S$ be a proper gr-second submodule of $K$. Since $S$ is proper, there exists $y/u \in h(K)$ and $y/u \notin S$. This implies that $1/u \notin S$. There exists $0 \neq x/t \in h(S)$ because $S$ is gr-second. Since $\text{Ann}_R(S) = 0$, we have $uS = S$. Thus $x/t = u(z/h)$ for some $z/h \in S$. It follows that $x/u = (tz)/h \in S$. Now $xS = S$ implies that $x/u = xw$ for some $w \in S$. Since $x \neq 0$, it follows that $1/u = w \in S$, which is a contradiction. □

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