Abstract

Depending on the interpretation of the type of edges, a chain graph can represent different relations between variables and thereby independence models. Three interpretations, known by the acronyms LWF, MVR, and AMP, are prevalent. Multivariate regression chain graphs (MVR CGs) were introduced by Cox and Wermuth in 1993. We review Markov properties for MVR chain graphs and propose an alternative global and local Markov property for them. Except for pairwise Markov properties, we show that for MVR chain graphs all Markov properties in the literature are equivalent for semi-graphoids. We derive a new factorization formula for MVR chain graphs which is more explicit than and different from the proposed factorizations for MVR chain graphs in the literature. Finally, we provide a summary table comparing different features of LWF, AMP, and MVR chain graphs.

Keywords: multivariate regression chain graph; Markov property; graphical Markov models; factorization of probability distributions; conditional independence; marginalization of causal latent variable models; compositional graphoids.

1. Introduction

A probabilistic graphical model is a probabilistic model for which a graph represents the conditional dependence structure between random variables. There are several classes of graphical models; Bayesian networks (BN), Markov networks, chain graphs, and ancestral graphs are commonly used (Lauritzen [1996], Richardson and Spirtes [2002]). Chain graphs, which admit both directed and undirected edges, are a type of graphs in which there are no partially directed cycles. Chain graphs were introduced by Lauritzen, Wermuth and Frydenberg (Frydenberg [1990], Lauritzen and Wermuth [1989]) as a generalization of graphs based on undirected graphs and directed acyclic graphs (DAGs). Later Andersson, Madigan and Perlman introduced an alternative Markov property for chain graphs (Andersson et al. [1996]). In 1993 (Cox and Wermuth [1993]), Cox and Wermuth introduced multivariate regression (MVR) chain graphs as a combination of a (sequence of) multivariate regression graph(s) and covariance graph(s), where a multivariate regression graph is a two-box graph in which all edges are dashed. Edges in both boxes are dashed lines, and between boxes are dashed arrows. Note that if the right-hand box has two lines around it, the distribution of its components have been considered fixed, and a covariance graph is a single box graph in which all edges are undirected dashed lines. The 1993 paper does not provide a (global) Markov property for the newly introduced MVR chain graphs. In 1996, Cox and Wermuth introduced the joint-response chain graphs (Cox and Wermuth [1996]) as a graph in which the nodes can be arranged in a line of boxes from left to right; each box contains either a single node, or a full-line concentration graph (i.e. a single box graph in which all edges are undirected full lines), or a dashed-line covariance graph. The variables in each box are considered conditionally independent given variables in boxes to the right, in a way that is now going to be specified precisely. Arrows pointing to any one box are
either all dashed arrows or all full arrows. Dashed arrows to a node $i$ indicates that regressions of $Y_i$ (the variable corresponding to node $i$; as customary in graphical models, we will not distinguish between nodes and variables) on variables in boxes to the right of $i$ are being considered, whereas full-line arrows mean that the regression is taken both on variables in boxes to the right of $i$ and on the variables in the same box as $i$. For example, see Figure 1 for a simple illustration of the distinctions. Figure 1 (a) shows a joint-response chain graph with variables $U$, $V$ as explanatory to $X$, $Y$. $Y$ regressed on $X$, $U$, $V$ and $X$ regressed on $Y$, $U$, $V$. In this graph: $Y \perp \perp U|(X, V)$ and $X \perp \perp V|(Y, U)$. Figure 1 (b) shows a joint-response chain graph with $Y$ as a response to $V$ and $X$ as a response to $U$, which implies $Y \perp \perp U|V$ and $X \perp \perp V|U$.

Figure 1: (Cox and Wermuth 1996, p. 70 and 72) (a) a LWF chain graph, (b) a MVR chain graph

Since the paper appear, the two kinds of chain graphs illustrated in the previous example have received special names; full-edge joint-response chain graphs are known as chain graph under the Lauritzen-Wermuth-Frydenberg (LWF) interpretation, or simply LWF chain graphs, and dashed-edge joint-response chain graphs are known as multivariate regression (MVR) chain graphs.

It is worthwhile to mention that in (Cox and Wermuth 1996, p. 43) Cox and Wermuth claim that if we start with any dashed-line joint-response chain graph we can obtain a synthetic directed acyclic graph, i.e. one in which specified nodes represent variables over which marginalization occurs (We prove this claim in Corollary 15). For example, both graphs in Figure 2 and also, both graphs in Figure 3 have the same Markov property. Note that at that time the multivariate regression (MVR) Markov property had not been introduced.

Figure 2: Both graphs specify the same Markov property: $Y \perp \perp U|V$ and $X \perp \perp V|U$. 
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Figure 3: Both graphs specify the same Markov property: $Y \perp \perp U, X \perp \perp V$, and $V \perp \perp U$.

Furthermore, Cox and Wermuth claim that a directed acyclic graph (DAG) will result, implying the same independencies for the observed nodes unless in the original joint-response chain graph there is a subgraph with a sink-oriented U-configuration or there is a sink-oriented V-configuration from which a direction-preserving path of arrows leads to nodes joined by an undirected line (Cox and Wermuth, 1996, p. 44). However, we know that there is no (finite) DAG model which, under marginalizing and conditioning, gives the set of conditional independence relations implied by graph $G$ in Figure 4 (Lauritzen and Richardson, 2002), so the claim is false.

Figure 4: (Lauritzen and Richardson, 2002) There is no (finite) DAG model which, under marginalizing and conditioning, gives the set of conditional independence relations implied by graph $G$.

Acyclic directed mixed graphs (ADMGs), also known as semi-Markov(ian) (Pearl, 2009) models contain directed ($\to$) and bi-directed ($\leftrightarrow$) edges subject to the restriction that there are no directed cycles (Richardson, 2003; Evans and Richardson, 2014). An ADMG that has no partially directed cycle is called a multivariate regression chain graph. In the first decade of the 21st century, several Markov property (global, pairwise, block recursive, and so on) were introduced by authors and researchers (Richardson and Spirtes, 2002; Wermuth and Cox, 2004; Marchetti and Lupparelli, 2008, 2011; Drton, 2009). Lauritzen, Wermuth, and Sadeghi (Sadeghi and Lauritzen, 2014; Sadeghi and Wermuth, 2016) proved that the global and (four) pairwise Markov properties of a MVR chain graph are equivalent for any independence model that is a compositional graphoid.

In this paper we focus on the class of multivariate regression chain graphs and we discuss their Markov properties. The discussion preceding Theorem 14 provides strong motivation for the importance of MVR CGs. The major contributions of this paper may be summarized as follows:
• Proposed an alternative global and local Markov property for MVR chain graphs, which are equivalent with other Markov properties in the literature for semi-graphoids and compositional semi-graphoids, respectively.

• Compared different proposed Markov properties for MVR chain graphs in the literature and considered conditions under which they are equivalent.

• Derived an alternative explicit factorization criterion for MVR chain graphs based on the proposed factorization criterion for acyclic directed mixed graphs in [Evans and Richardson, 2014].

The rest of the paper is organized as follows: Section 2 reviews background and related works, and also more definitions and concepts for MVR chain graphs is introduced. In Section 3, we provide an alternative global and local Markov property for MVR chain graphs and prove that all of proposed Markov properties in the literature, other than pairwise Markov properties, for MVR chain graphs are equivalent for semi-graphoids. Section 4 presents an alternative factorization for MVR chain graphs according to the proposed factorization criterion for acyclic directed mixed graphs in [Evans and Richardson, 2014]. Many properties of the common interpretations of chain graphs are summarized in a table at the end of the paper.

2. Definitions and Concepts

In this section, we introduce some definitions and concepts that we need to review MVR chain graphs.

**Definition 1** A vertex \( \alpha \) is said to be an **ancestor** of a vertex \( \beta \) if either there is a directed path \( \alpha \to \ldots \to \beta \) from \( \alpha \) to \( \beta \), or \( \alpha = \beta \). A vertex \( \alpha \) is said to be **anterior** to a vertex \( \beta \) if there is a path \( \mu \) from \( \alpha \) to \( \beta \) on which every edge is either of the form \( \gamma \to \delta \) with \( \delta \) between \( \gamma \) and \( \beta \), or \( \alpha = \beta \); that is, there are no edges \( \gamma \leftrightarrow \delta \) and there are no edges \( \gamma \leftarrow \delta \) pointing toward \( \alpha \). Such a path is said to be an anterior path from \( \alpha \) to \( \beta \). A vertex \( \alpha \) is said to be **antecedent** to a vertex \( \beta \) if there is a path \( \mu \) from \( \alpha \) to \( \beta \) on which every edge is either of the form \( \gamma \leftrightarrow \delta \), or \( \gamma \to \delta \) with \( \delta \) between \( \gamma \) and \( \beta \), or \( \alpha = \beta \); that is, there are no edges of the form \( \gamma \to \delta \). Such a path is said to be an antecedent path from \( \alpha \) to \( \beta \). We apply these definitions disjunctively to sets:

\[
\begin{align*}
\text{an}(X) &= \{ \alpha | \alpha \text{ is an ancestor of } \beta \text{ for some } \beta \in X \}; \\
\text{ant}(X) &= \{ \alpha | \alpha \text{ is an anterior of } \beta \text{ for some } \beta \in X \}; \\
\text{antec}(X) &= \{ \alpha | \alpha \text{ is an antecedent of } \beta \text{ for some } \beta \in X \}.
\end{align*}
\]

If necessary we specify the graph by a subscript, as in \( \text{ant}_G(X) \). The usage of the terms “ancestor” and “anterior” differs from Lauritzen [Lauritzen, 1996], but follows Frydenberg [Frydenberg, 1990].

**Definition 2** If \( \text{antec}(a) \subseteq A \) for all \( a \in A \), we say that \( A \) is an antecedental set. The smallest antecedental set containing \( A \) is denoted by \( \text{Antec}(A) \).

**Remark 3** Note that when \( a \) is an ancestor of \( b \), it can be considered as an anterior or an antecedent of \( b \) but not vice versa. Also, note that the notions anterior and antecedent are not comparable in general.
Definition 4 A mixed graph is a graph containing three types of edges, undirected (−), directed (→) and bidirected (↔).

Definition 5 An ancestral graph G is a mixed graph in which the following conditions hold for all vertices α in G:

(i) if α and β are joined by an edge with an arrowhead at α, then α is not anterior to β.
(ii) there are no arrowheads present at a vertex which is an endpoint of an undirected edge.

Condition (i) implies that if α and β are joined by an edge with an arrowhead at α, then α is not an ancestor of β. This is the motivation for terming such graphs ”ancestral.” Examples of ancestral and nonancestral mixed graphs are shown in Figure 5.

Definition 6 A nonendpoint vertex ζ on a path is a collider on the path if the edges preceding and succeeding ζ on the path have an arrowhead at ζ, that is, → ζ ←, or ↔ ζ ↔, or ↔ ζ ←, or → ζ ↔. A nonendpoint vertex ζ on a path which is not a collider is a noncollider on the path. A path between vertices α and β in an ancestral graph G is said to be m-connecting given a set Z (possibly empty), with α, β / Z, if:

(i) every noncollider on the path is not in Z, and
(ii) every collider on the path is in \( \text{ant}_G(Z) \).

If there is no path m-connecting α and β given Z, then α and β are said to be m-separated given Z. Sets X and Y are m-separated given Z, if for every pair α, β, with α ∈ X and β ∈ Y, α and β are m-separated given Z (X, Y, and Z are disjoint sets; X, Y are nonempty). This criterion is referred to as a global Markov property. We denote the independence model resulting from applying the m-separation criterion to G, by \( \mathcal{I}_m(G) \). This is an extension of Pearl’s d-separation criterion to mixed graphs in that in a DAG D, a path is d-connecting if and only if it is m-connecting.

Definition 7 Let \( G_A \) denote the induced subgraph of G on the vertex set A, formed by removing from G all vertices that are not in A, and all edges that do not have both endpoints in A. Two vertices x and y in a MVR chain graph G are said to be collider connected if there is a path from x to y in G on which every non-endpoint vertex is a collider; such a path is called a collider path. (Note that a single edge trivially forms a collider path, so if x and y are adjacent in a MVR chain...
graph then they are collider connected.) The augmented graph derived from \( G \), denoted \((G)^a\), is an undirected graph with the same vertex set as \( G \) such that

\[
c - d \text{ in } (G)^a \iff c \text{ and } d \text{ are collider connected in } G.
\]

**Definition 8** Disjoint sets \( X, Y, \) and \( Z \neq \emptyset \) are said to be \( m^* \)-separated if \( X \) and \( Y \) are separated by \( Z \) in \((G_{\text{ant}}(X \cup Y \cup Z))^a\). Otherwise \( X \) and \( Y \) are said to be \( m^* \)-connected given \( Z \). The resulting independence model is denoted by \( \mathcal{I}_{m^*}(G) \).

Richardson and Spirtes in (Richardson and Spirtes, 2002, Theorem 3.18.) show that for an ancestral graph \( G \), \( \mathcal{I}_{m^*}(G) = \mathcal{I}_{m^*}(G) \). Note that in the case of ADMGs and MVR CGs, anterior sets in definitions 6, 8 can be replaced by ancestor sets, because in both cases anterior sets and ancestor sets are the same.

**Definition 9** An ancestral graph \( G \) is said to be maximal if for every pair of vertices \( \alpha, \beta \) if \( \alpha \) and \( \beta \) are not adjacent in \( G \) then there is a set \( Z (\alpha, \beta \notin Z) \), such that \( \langle \{\alpha\}, \{\beta\} | Z \rangle \in \mathcal{I}_{m^*}(G) \). Thus a graph is maximal if every missing edge corresponds to at least one independence in the corresponding independence model.

A simple example of a nonmaximal ancestral graph is shown in Figure 6: \( \gamma \) and \( \delta \) are not adjacent, but are \( m \)-connected given every subset of \( \{\alpha, \beta\} \), hence \( \mathcal{I}_{m}(G) = \emptyset \).

![Figure 6: A nonmaximal ancestral graph.](image)

If \( G \) is an undirected graph or a directed acyclic graph, then \( G \) is a maximal ancestral graph (Richardson and Spirtes, 2002, Proposition 3.19). Richardson and Spirtes (Richardson and Spirtes, 2002) propose the following notation:

**Notation 10** If \( \mathcal{I} \) contains the independence relations present in a distribution \( P \), then \( \mathcal{I}_{L} \) contains the subset of independence relations remaining after marginalizing out the latent variables in \( L \); \( \mathcal{I}_{S} \) and constitutes the subset of independencies holding among the remaining variables after conditioning on \( S \).

The absence of partially directed cycles in MVR CGs implies that the vertex set of a chain graph can be partitioned into so-called chain components such that edges within a chain component are bidirected whereas the edges between two chain components are directed and point in the same direction. So, any chain graph yields a directed acyclic graph \( D \) of its chain components having \( T \) as a node set and an edge \( T_1 \rightarrow T_2 \) whenever there exists in the chain graph \( G \) at least one edge \( u \rightarrow v \) connecting a node \( u \) in \( T_1 \) with a node \( v \) in \( T_2 \). In this directed graph, we may define for each \( T \) the set \( pa_D(T) \) as the union of all the chain components that are parents of \( T \) in the directed
graph $D$. This concept is distinct from the usual notion of the parents $pa_G(A)$ of a set of nodes $A$ in the chain graph, that is, the set of all the nodes $w$ outside $A$ such that $w \rightarrow v$ with $v \in A$ (Marchetti and Lupparelli [2011]). For instance, in the graph of Figure 7 for $T = \{1, 2\}$, the set of parent components is $pa_D(T) = \{3, 4, 5, 6\}$, whereas the set of parents of $T$ is $pa_G(T) = \{3, 6\}$.

Figure 7: (Marchetti and Lupparelli [2011]) A MVR chain graph.

Given a chain graph $G$ with chain components $(T \mid T \in T)$, we can always define a strict total order $\prec$ of the chain components that is consistent with the partial order induced by the chain graph, such that if $T \prec T'$ then $T \notin pa_D(T')$. For instance, in the chain graph of Figure 7 there are four chain components ordered in the graph of Figure 8 as $\{1, 2\} \prec \{3, 4\} \prec \{5, 6\} \prec \{7, 8\}$. Note that the chosen total order of the chain components is in general not unique and that another consistent ordering could be $\{1, 2\} \prec \{5, 6\} \prec \{3, 4\} \prec \{7, 8\}$.

Figure 8: (Marchetti and Lupparelli [2011]) One possible consistent ordering of the four chain components: $\{1, 2\} \prec \{5, 6\} \prec \{3, 4\} \prec \{7, 8\}$.

For each $T$, the set of all components preceding $T$ is known and we may define the cumulative set $\text{pre}(T) = \bigcup_{T \prec T'} T'$ of nodes contained in the predecessors of component $T$, which we sometimes call the past of $T$. The set $\text{pre}(T)$ captures the notion of all the potential explanatory variables of the response variables within $T$ (Marchetti and Lupparelli [2011]).

By definition, as the full ordering of the components is consistent with $G$, the set of predecessors $\text{pre}(T)$ of each chain component $T$ always includes the parent components $pa_D(T)$. 

7
3. Markov Properties for MVR Chain Graphs

In this section, first, we show, formally, that MVR chain graphs are a subclass of the maximal ancestral graphs of Richardson and Spirtes (Richardson and Spirtes, 2002) that include only observed and latent variables. Latent variables cause several complications (Colombo et al., 2012). First, causal inference based on structural learning algorithms such as the PC algorithm (Spirtes et al., 2000) may be incorrect. Second, if a distribution is faithful to a DAG, then the distribution obtained by marginalizing out on some of the variables may not be faithful to any DAG on the observed variables i.e., the space of DAGs is not closed under marginalization. These problems can be solved by exploiting MVR chain graphs. This motivates the development of studies on MVR CGs.

Remark 11  Sonntag (Sonntag, 2014) mentions that unlike the other CGs (Chain Graphs) interpretations, the bidirected [dashed] edge in a MVR CG has a strong intuitive meaning. It can be seen to represent one or more hidden common causes between the variables connected by it. In other words, in a MVR CG any bidirected edge $X \leftrightarrow Y$ can be replaced by $X \leftarrow H \rightarrow Y$ to obtain a BN representing the same independence model over the original variables, i.e. excluding the new variables $H$. These variables are called hidden, or latent, and have been marginalized away in the CG model.

Remark 12  Also, Peña claims that of the three main interpretations of chain graphs (LWF, MVR, and AMP (Andersson et al., 1996)), only MVR chain graphs have a convincing justification. In addition, he claims that since MVR chain graphs are a subset of maximal ancestral graphs without undirected edges, every MVR chain graph represents the independence model represented by a DAG under marginalization (Richardson and Spirtes, 2002, Theorem 6.4). That is, every MVR chain graph can be accounted for by a causal model that is partially observed (Peña, 2015).

Remark 13  Sadeghi and Lauritzen in (Sadeghi and Lauritzen, 2014) claim that when the chain components consist entirely of bi-directed edges, the multivariate regression property is identical to $m$-separation.

Theorem 14  If $G$ is a MVR chain graph, then $G$ is an ancestral graph.

Proof  Obviously, every MVR chain graph is a mixed graph without undirected edges. So, it is enough to show that condition (i) in Definition 5 is satisfied. For this purpose, consider that $\alpha$ and $\beta$ are joined by an edge with an arrowhead at $\alpha$ in MVR chain graph $G$. Two cases are possible. First, if $\alpha \leftrightarrow \beta$ is an edge in $G$, by definition of a MVR chain graph, both of them belong to the same chain component. Since all edges on a path between two nodes of a chain component are bidirected, then by definition $\alpha$ cannot be an anterior of $\beta$. Second, if $\alpha \leftarrow \beta$ is an edge in $G$, by definition of a MVR chain graph, $\alpha$ and $\beta$ belong to two different components ($\beta$ is in a chain component that is to the right side of the chain component that contains $\alpha$). We know that all directed edges in a MVR chain graph are arrows pointing from right to left, so there is no path from $\alpha$ to $\beta$ in $G$ i.e. $\alpha$ cannot be an anterior of $\beta$ in this case. We have shown that $\alpha$ cannot be an anterior of $\beta$ in both cases, and therefore condition (i) in Definition 5 is satisfied. In other words, every MVR chain graph is an ancestral graph.

The following corollary proves Cox and Wermuth’s claim (Peña and Sonntag’s claims in Remark 11 and Remark 12 and also Sadeghi and Lauritzen’s claim in Remark 13).
Corollary 15  Every MVR chain graph has the same independence model as a DAG under marginalization.

Proof  From Theorem 14, we know that every MVR chain graph is an ancestral graph. The result follows directly from (Richardson and Spirtes 2002, Theorem 6.3).

3.1 Global and Pairwise Markov Properties

The following properties have been defined for conditional independences of probability distributions. Let $A, B, C$ and $D$ be disjoint subsets of $V_G$, where $C$ may be the empty set.

1. Symmetry: $A \indep B \Rightarrow B \indep A$;
2. Decomposition: $A \indep BD | C \Rightarrow (A \indep B | C \text{ and } A \indep D | C)$;
3. Weak union: $A \indep BD | C \Rightarrow (A \indep B | DC \text{ and } A \indep D | BC)$;
4. Contraction: $(A \indep B | DC \text{ and } A \indep D | C) \Leftrightarrow A \indep BD | C$;
5. Intersection: $(A \indep B | DC \text{ and } A \indep D | BC) \Rightarrow A \indep BD | C$;
6. Composition: $(A \indep B | C \text{ and } A \indep D | C) \Rightarrow A \indep BD | C$.

An independence model is a semi-graphoid if it satisfies the first four independence properties listed above. Note that every probability distribution $p$ satisfies the semi-graphoid properties (Studený 1989). If a semi-graphoid further satisfies the intersection property, we say it is a graphoid (Pearl and Paz 1987, Studený 2005, 1989). A compositional graphoid further satisfies the composition property (Sadeghi and Wermuth 2016). If a semi-graphoid further satisfies the composition property, we say it is a compositional semi-graphoid.

For a node $i$ in the connected component $T$, its past, denoted by $pst(i)$, consists of all nodes in components having a higher order than $T$. To define pairwise Markov properties for MVR CGs, we use the following notation for parents, anteriors and the past of node pair $i, j$:

- $pa_G(i, j) = pa_G(i) \cup pa_G(j) \setminus \{i, j\}$,
- $ant(i, j) = ant(i) \cup ant(j) \setminus \{i, j\}$,
- $pst(i, j) = pst(i) \cup pst(j) \setminus \{i, j\}$.

The distribution $P$ of $(X_n)_{n \in V}$ satisfies a pairwise Markov property (Pm), for $m = 1, 2, 3, 4$, with respect to MVR CG($G$) if for every uncoupled pair of nodes $i$ and $j$ (i.e., there is no directed or bidirected edge between $i$ and $j$):

(P1): $i \indep j | pst(i, j)$,
(P2): $i \indep j | ant(i, j)$,
(P3): $i \indep j | pa_G(i, j)$,
(P4): $i \indep j | pa_G(i)$ if $i \prec j$.

Notice that in (P4), $pa_G(i)$ may be replaced by $pa_G(j)$ whenever the two nodes are in the same connected component. Sadeghi and Wermuth in (Sadeghi and Wermuth 2016) proved that all of above mentioned pairwise Markov properties are equivalent for compositional graphoids. Also, they show that each one of the above listed pairwise Markov properties is equivalent to the global Markov properties in Definitions 6, 8 (Sadeghi and Wermuth 2016, Corollary 1).
Remark 16 For equivalence of pairwise and global Markov properties, the six compositional graphoid axioms are sufficient. In fact, in general, for the mentioned equivalence, all six axioms are also necessary. In fact, in general, for the mentioned equivalence, all six axioms are also necessary. The necessity of intersection and compositional properties follows from \cite{SadeghiLauritzen2014} Section 6.3.

Remark 17 Two graphical models are Markov equivalent whenever their associated graphs capture the same independence structure, that is, the graphs lead to the same set of implied independence statements. Two MVR chain graphs are Markov equivalent if and only if they have the same skeleton (the skeleton of a graph results by replacing each edge present by a full line) and the same sets of (unshielded) collision versus \cite{WermuthSadeghi2012} Theorem 1.

3.2 Block-recursive, Multivariate Regression (MR), and Ordered Local Markov Properties

According to \cite{MarchettiLupparelli2008}, multivariate regression chain graphs represent situations in which the variables can be arranged in an ordered series of groups and all the variables within a group are considered to be on an equal footing, while the relation between two variables in different groups is considered asymmetrically. The associated graph is a special type of chain graph in which the edges between components are arrows and the subgraphs within chain components are covariance graphs.

According to \cite{MarchettiLupparelli2011}, in the general case, the interpretation of the undirected graphs within a chain component is that of a covariance graph, but conditional on all variables in preceding components. For example, the missing edge (a, c) in the graph of Figure 9 is interpreted as the independence statement $X_a \independent X_c | X_d, X_e$, compactly written in terms of nodes as $a \independent c | d, e$.

![Figure 9: A MVR chain graph with chain components: $T = \{\{a, b, c\}, \{d\}\}$](image)

The interpretation of the directed edges is that of multivariate regression models, with a missing edge denoting a conditional independence of the response on a variable given all the remaining potential explanatory variables. Thus, in the chain graph of Figure 10 the missing arrow (a, d) indicates the independence statement $a \independent d | c$.

Definition 18 Given a chain graph $G$, the set $N_{bg}(A)$ is the union of $A$ itself and the set of nodes $w$ that are neighbors of $A$, that is, coupled by a dashed-line (bi-directed edge) to some node $v$ in $A$. Moreover, the set of non-descendants $nd_D(T)$ of a chain component $T$, is the union of all
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Figure 10: A MVR chain graph with chain components: \( \mathcal{T} = \{\{a, b\}, \{c, d\}\} \).

components \( T' \) such that there is no directed path from \( T \) to \( T' \) in the directed graph of chain components \( D \).

The following definition explains the meaning of the multivariate regression interpretation of a chain graph.

**Definition 19** (multivariate regression (MR) Markov property for MVR CGs) Let \( G \) be a chain graph with chain components \( (T | T \in \mathcal{T}) \). A joint distribution \( P \) of the random vector \( X \) obeys the (local) multivariate regression Markov property for \( G \) if it satisfies the following independencies. For all \( T \in \mathcal{T} \) and for all \( A \subseteq T \):

**(MR1)** if \( A \) is connected: \( A \perp \perp [\text{pre}(T) \setminus \text{pa}_G(A)] | \text{pa}_G(A) \).

**(MR2)** if \( A \) is disconnected with connected components \( A_1, \ldots, A_r \): \( A_1 \perp \ldots \perp A_r | \text{pre}(T) \).

Assuming that the distribution \( P \) has a density \( p \) with respect to a product measure, the definition can be stated by the following two equivalent conditions:

\[
P_{A | \text{pre}(T)} = P_{A | \text{pa}_G(A)}
\]

for all \( T \) and for all connected subset \( A \subseteq T \).

\[
P_{A | \text{pre}(T)} = \prod_j P_{A_j | \text{pre}(T)}
\]

for all \( T \) and for all disconnected subset \( A \subseteq T \) with connected components \( A_j, j = 1, \ldots, r \). In other words, for any connected subset \( A \) of responses in a component \( T \), its conditional distribution given the variables in the past depends only on the parents of \( A \). On the other hand, if \( A \) is disconnected (i.e., the subgraph \( G_A \) is disconnected) the variables in its connected components \( A_1, \ldots, A_r \), are jointly independent given the variables in the past (not just the parents).

**Remark 20** (Marchetti and Lupparelli, 2011, Remark 2) One immediate consequence of Definition 19 is that if the probability density \( p(x) \) is strictly positive, then it factorizes according to the directed acyclic graph of the chain components:

\[
p(x) = \prod_{T \in \mathcal{T}} p(x_T | x_{\text{pa}_D(T)}) .
\]
Example 1 In Figure consider that \( T = \{ a, b, c \} \), so \( \text{pre}(T) = \{ d, e \} \). Thus, for each connected subset \( A \in T \), by (MR1), we have:

\[
\begin{align*}
& a \perp \perp d; \quad b \perp \perp d, e; \quad c \perp \perp d|e; \quad (a, b) \perp \perp e|d; \quad (b, c) \perp \perp d|e.
\end{align*}
\]

Also, for the disconnected set \( A = \{ a, c \} \) we obtain by (MR2) the independence \( a \perp \perp c|d, e \).

Finally, based on Remark 20 the corresponding factorization for the strictly positive probability density \( p(x) \) is:

\[
p = p_{a, b, c|d, e} p_{d, e}.
\]

Drton discussed four different block-recursive Markov properties for chain graphs, of which we discuss here those with the Markov property of type IV (Drton, 2009).

Definition 21 (Chain graph Markov property of type IV (Drton, 2009)) Let \( G \) be a chain graph with chain components \( (T|T \in T) \) and directed acyclic graph \( D \) of components. The joint probability distribution of \( X \) obeys the block-recursive Markov property of type IV if it satisfies the following independencies:

\[
\begin{align*}
&(IV0) \ T \perp [\text{nd}_D(T) \setminus \text{pa}_G(T)]|\text{pa}_D(T), \text{for all } T \in T; \\
&(IV1) \ A \perp [\text{pa}_D(T) \setminus \text{pa}_G(A)]|\text{pa}_G(A), \text{for all } T \in T, \text{ and for all } A \subseteq T; \\
&(IV2) \ A \perp [T \setminus \text{Nb}_G(A)]|\text{pa}_D(T), \text{for all } T \in T, \text{ and for all connected subsets } A \subseteq T.
\end{align*}
\]

Remark 22 In a multivariate normal distribution, two pairwise marginal independences \( v \perp \perp w \) and \( v \perp \perp u \) implies that \( v \perp \perp \{ u, w \} \). However, for the models of type IV a positive discrete distribution that obeys the pairwise Markov property will generally not obey the block-recursive Markov property. This follows from the fact that, in general, for positive joint distributions, \( v \perp \perp w \) and \( v \perp \perp u \), do not imply that \( v \) is independent of \( \{ u, w \} \) (Drton, 2009, Remark 5). In other words, the family of multinomial distributions does not satisfy the composition property in general.

The following example shows that independence models, in general, resulting from Definitions 19, 21 are different.

Example 2 Consider the following MVR chain graph \( G \):

![Diagram of a MVR chain graph with chain components: \( T = \{1, 2, 3, 4\}, \{5, 6\}, \{7\} \).]

Figure 11: A MVR chain graph with chain components: \( T = \{1, 2, 3, 4\}, \{5, 6\}, \{7\} \).

For the connected set \( A = \{1, 2\} \) the condition (MR1) implies that \( 1, 2 \perp \perp 6, 7|5 \) while the condition (IV2) implies that \( 1, 2 \perp \perp 6|5 \), which is not implied directly by (MR1) and (MR2). Also, the condition (MR2) implies that \( 1 \perp \perp 3, 4|5, 6, 7 \) while the condition (IV2) implies that \( 1 \perp \perp 3, 4|5, 6 \), which is not implied directly by (MR1) and (MR2).
Theorem 1 in (Marchetti and Lupparelli 2011) states that for a given chain graph \( G \), the multivariate regression Markov property is equivalent to the block-recursive Markov property of type IV. Also, Drton in (Drton 2009, Section 7 Discussion) claims that (without proof) the block-recursive Markov property of type IV can be shown to be equivalent to the global Markov property proposed in (Richardson and Spirtes 2002; Richardson 2003).

Now, we introduce a(n ordered) local Markov property for ADMGs proposed by Richardson in (Richardson 2003), which is an extension of the local well-numbering Markov property for DAGs introduced in (Lauritzen et al. 1990). For this purpose, we need to consider the following definitions and notations:

**Definition 23** For a given acyclic directed mixed graph (ADMG) \( G \), the induced bi-directed graph \( (G)_{\leftrightarrow} \) is the graph formed by removing all directed edges from \( G \). The district (aka c-component) for a vertex \( x \) in \( G \) is the connected component of \( x \) in \( (G)_{\leftrightarrow} \), or equivalently
\[
dis_{G}(x) = \{y | y \leftrightarrow \ldots \leftrightarrow x \text{ in } G, \text{ or } x = y\}.
\]

As usual we apply the definition disjunctively to sets:
\[
dis_{A}(B) = \bigcup_{x \in B} \dis_{A}(x).
\]

A set \( C \) is path-connected in \( (G)_{\leftrightarrow} \) if every pair of vertices in \( C \) are connected via a path in \( (G)_{\leftrightarrow} \); equivalently, every vertex in \( C \) has the same district in \( G \).

**Definition 24** In an ADMG, a set \( A \) is said to be ancestrally closed if \( x \rightarrow \ldots \rightarrow a \) in \( G \) with \( a \in A \) implies that \( x \in A \). The set of ancestrally closed sets is defined as follows:
\[
\mathcal{A}(G) = \{A | an_{G}(A) = A\}.
\]

If \( A \) is an ancestrally closed set in an ADMG \( (G) \), and \( x \) is a vertex in \( A \) that has no children in \( A \) then we define the Markov blanket of a vertex \( x \) with respect to the induced subgraph on \( A \) as
\[
mb(x, A) = pa_{G}(\dis_{A}(x)) \cup (\dis_{A}(x) \setminus \{x\}),
\]
where \( \dis_{A} \) is the district of \( x \) in the induced subgraph \( G_{A} \).

**Definition 25** Let \( G \) be an acyclic directed mixed graph. Specify a a total ordering \( (\prec) \) on the vertices of \( G \), such that \( x \prec y \Rightarrow y \not\in \an(x) \); such an ordering is said to be consistent with \( G \). Define \( \pre_{G, \prec}(x) = \{v | v \prec x \text{ or } v = x\} \).

**Definition 26 (Ordered local Markov property)** Let \( G \) be an acyclic directed mixed graph. An independence model \( \Im \) over the node set of \( G \) satisfies the ordered local Markov property for \( G \), with respect to the ordering \( \prec \), if for any \( x \), and ancestrally closed set \( A \) such that \( x \in A \subseteq \pre_{G, \prec}(x) \),
\[
\{x\} \perp \perp [A \setminus (mb(x, A) \cup \{x\})] | mb(x, A).
\]

Since MVR chain graphs are a subclass of ADMGs, the ordered local Markov property in Definition 26 can be used as a local Markov property for MVR chain graphs.
Theorem 27 Let $G$ be a MVR chain graph. For an independence model $\mathcal{I}$ over the node set of $G$, the following conditions are equivalent:

(i) $\mathcal{I}$ satisfies the global Markov property w.r.t. $G$ in Definition 8

(ii) $\mathcal{I}$ satisfies the global Markov property w.r.t. $G$ in Definition 10

(iii) $\mathcal{I}$ satisfies the block recursive Markov property w.r.t. $G$ in Definition 21

(iv) $\mathcal{I}$ satisfies the MR Markov property w.r.t. $G$ in Definition 19

(v) $\mathcal{I}$ satisfies the ordered local Markov property w.r.t. $G$ in Definition 26

Proof (i)$\Rightarrow$(ii): This has already been proved in [Richardson, 2003 Theorem 1].

(ii)$\Rightarrow$(iii): Assume that the independence model $\mathcal{I}$ over the node set of MVR $CG(G)$ satisfies the global Markov property w.r.t. $G$ in Definition 8. We have the following three cases:

Case 1: Let $X = \tau \subseteq \tau \in \mathcal{T}, Y = nd_D(\tau) \setminus pa_D(\tau)$, and $Z = pa_D(\tau)$. So, $an(X \cup Y \cup Z) = \tau \cup nd_D(\tau)$ is an ancestor set, and $pa_D(\tau)$ separates $\tau$ from $nd_D(\tau) \setminus pa_D(\tau)$ in $(G_{\tau \cup nd_D(\tau)})^a$; this shows that the global Markov property in Definition 8 implies (IV0) in Definition 21.

Case 2: Assume that $X = \sigma \subseteq \tau \in \mathcal{T}, Y = pa_D(\tau) \setminus pa_G(\sigma)$, and $Z = pa_G(\sigma)$. Consider that $W = an(X \cup Y \cup Z) = an(\sigma \cup pa_D(\tau))$. We know that there is no directed edge from $pa_D(\tau) \setminus pa_G(\sigma)$ to elements of $\sigma$, and also there is no collider path between nodes of $Y$ and $\sigma$ in $W$. So, every connecting path that connects $pa_D(\tau) \setminus pa_G(\sigma)$ to $\sigma$ in $(G_{W})^a$ has intersection with $pa_G(\sigma)$, which means $pa_G(\sigma)$ separates $pa_D(\tau) \setminus pa_G(\sigma)$ from $\sigma$ in $(G_{W})^a$; this shows that the global Markov property in Definition 8 implies (IV1) in Definition 21.

Case 3: Assume that $X = \sigma \subseteq \tau \in \mathcal{T}$ is a connected subset of $\tau$. Also, assume that $Y = \tau \setminus N_{bg}(\sigma)$, and $Z = pa_D(\tau)$. Obviously, $\sigma$ and $\tau \setminus N_{bg}(\sigma)$ are two subsets of $\tau$ such that there is no connection between their elements. Consider that $A$ is the ancestor set containing $\sigma$, $\tau \setminus N_{bg}(\sigma)$, and $pa_D(\tau)$. Clearly, $pa_D(\tau) \subseteq A$. Since $\sigma$ and $\tau \setminus N_{bg}(\sigma)$ are disconnected in $\tau$, so any connecting path between them in $A$ (if it exists) must pass through $pa_D(\tau)$ in $(G_A)^a$; this shows that the global Markov property in Definition 8 implies (IV2) in Definition 21.

(iii)$\Rightarrow$(iv): Assume that the independence model $\mathcal{I}$ over the node set of MVR $CG(G)$ satisfies the block recursive Markov property w.r.t. $G$ in Definition 21. We show that $\mathcal{I}$ satisfies the MR Markov property w.r.t. $G$ in Definition 19 by considering the following two cases:

Case 1 (IV0 and IV1 $\Rightarrow$ MR1): Assume that $A$ is a connected subset of $\tau$. From (IV1) we have:

$$A \perp (pa_D(\tau) \setminus pa_G(A))|pa_G(A) \tag{1}$$

Also, from (IV0) we have $\tau \perp (nd_D(\tau) \setminus pa_D(\tau))|pa_D(\tau)$, the decomposition property implies that

$$A \perp (nd_D(\tau) \setminus pa_D(\tau))|pa_D(\tau) \tag{2}$$

Using the contraction property for (1) and (2) gives: $A \perp [(nd_D(\tau) \setminus pa_G(A)) \cup (pa_D(\tau) \setminus pa_G(A))]|pa_G(A)$. Using the decomposition property for this independence relationship gives (MR1): $A \perp (pre(\tau) \setminus pa_G(A))|pa_G(A)$, because $(pre(\tau) \setminus pa_G(A)) \subseteq [(nd_D(\tau) \setminus pa_G(\tau)) \cup (pa_D(\tau) \setminus pa_G(A))]$.

Case 2 (IV0 and IV2 $\Rightarrow$ MR2): Consider that $A$ is a disconnected subset of $\tau$ that contains $r$ connected components $A_1, \ldots, A_r$, i.e., $A = A_1 \cup \ldots \cup A_r$. From (IV2) we have: $A_1 \perp \tau \setminus N_{bg}(A_1)|pa_D(\tau)$. Using the decomposition property gives:

$$A_1 \perp A_2|pa_D(\tau) \tag{3}$$
Also, using decomposition for (IV0) gives: \((A_1 \cup A_2) \perp (\text{pre}(\tau) \setminus \text{pa}_D(\tau))|\text{pa}_D(\tau)\). Applying the weak union property for this independence relation gives: \(A_1 \perp (\text{pre}(\tau) \setminus \text{pa}_D(\tau))|\text{pa}_D(\tau)\). Using the contraction property for this and \([3]\) gives: \(A_1 \perp [A_2 \cup (\text{pre}(\tau) \setminus \text{pa}_D(\tau))]|\text{pa}_D(\tau)\). Using the weak union property leads to \(A_1 \perp A_2|[\text{pa}_D(\tau) \cup (\text{pre}(\tau) \setminus \text{pa}_D(\tau)) |= \text{pre}(\tau)\). Similarly, we can prove that for every \(1 \leq i \neq j \leq r\): \(A_i \perp A_j|\text{pre}(\tau)\).

(iv)\Rightarrow(v): Assume that the independence model \(\mathcal{I}\) over the node set of MVR CG\(G\) satisfies the MR Markov property w.r.t. \(G\) in Definition \([\text{19}]\) and \(\prec\) is an ordering that is consistent with \(G\). Let \(x \in A \subseteq \text{pre}_{G,\prec}(x)\). We show that \(\mathcal{I}\) satisfies the ordered local Markov property w.r.t. \(G\) in Definition \([\text{26}]\) by considering the following two cases:

Case 1: There is a chain component \(T\) such that \(x \in T\). Consider that \(A \cap T\) is a connected subset of \(T\). From (MR1) we have: \(\text{dis}_{GA}(x) \subseteq \text{pa}_G(\text{dis}_{GA}(x))\). Using the weak union property gives: \(\{x\} \perp [\text{pre}(\tau) \setminus \text{pa}_G(\text{dis}_{GA}(x))]|\text{pa}_G(\text{dis}_{GA}(x))\). Since \([A \setminus \text{mb}(x, A) \cup \{x\}] \subseteq \text{dis}_{GA}(x)\), using the decomposition property leads to: \(\{x\} \perp [A \setminus \text{mb}(x, A) \cup \{x\}]|\text{mb}(x, A)\).

Case 2: There is a chain component \(T\) such that \(x \in T\), and \(A \cap T\) is a disconnected subset of \(T\) with connected components \(A_1, \ldots, A_k\) i.e., \(A \cap T = A_1 \cup \ldots \cup A_k\). It is clear that there is a \(1 \leq d \leq k\) such that \(A_d = \text{dis}_{GA}(x)\). We have the following two sub-cases:

Sub-case I): \(\sigma := T \setminus \text{Nb}_G(A_d)\) is a connected subset of \(T\).

\[
\begin{align*}
\{ & \text{From (MR2): } A_d \perp \sigma|\text{pre}(\tau) \\text{ From (MR1): } A_d \perp (\text{pre}(\tau) \setminus \text{pa}_G(A_d))|\text{pa}_G(A_d) \\
& \}
\end{align*}
\]

Using the contraction property for \([4]\) gives: \(A_d \perp [\sigma \cup (\text{pre}(\tau) \setminus \text{pa}_G(A_d))]|\text{pa}_G(A_d)\). Using the weak union property gives: \(\{x\} \perp [\text{pre}(\tau) \setminus \text{pa}_G(\text{dis}_{GA}(x))]|\text{pa}_G(\text{dis}_{GA}(x))\). Since \([A \setminus \text{mb}(x, A) \cup \{x\}] \subseteq \text{dis}_{GA}(x)\), using the decomposition property leads to: \(\{x\} \perp [A \setminus \text{mb}(x, A) \cup \{x\}]|\text{mb}(x, A)\).

Sub-case II): \(T \setminus \text{Nb}_G(A_d)\) is a disconnected subset of \(T\) with connected component \(\sigma_1, \sigma_2\) i.e., \(T \setminus \text{Nb}_G(A_d) = \sigma_1 \cup \sigma_2\). From (MR1) we have: \(\sigma_1 \perp (T \setminus \text{Nb}_G(\sigma_1))|\text{pre}(\tau)\). Since \(A_d \cup \sigma_2 \subseteq (T \setminus \text{Nb}_G(\sigma_1))\), using the decomposition and weak union property give: \(\sigma_1 \perp A_d|(\text{pre}(\tau) \cup \sigma_2)\). Using the symmetry property implies that \(A_d \perp \sigma_1|(\text{pre}(\tau) \cup \sigma_2)\).

\[
\begin{align*}
\{ & \text{From (MR2): } A_d \perp \sigma_1|\text{pre}(\tau) \cup \sigma_2 \\
& \}
\end{align*}
\]

Using the contraction property for \([5]\) gives: \(A_d \perp (\sigma_1 \cup \sigma_2)|\text{pre}(\tau)\).

\[
\begin{align*}
\{ & \text{From (MR1): } A_d \perp (\text{pre}(\tau) \setminus \text{pa}_G(A_d))|\text{pa}_G(A_d) \\
& \}
\end{align*}
\]

Using the contraction property for \([6]\) gives: \(A_d \perp [(\sigma_1 \cup \sigma_2) \cup (\text{pre}(\tau) \setminus \text{pa}_G(A_d))]|\text{pa}_G(A_d)\). Using the decomposition property gives: \(\{x\} \perp [(\sigma_1 \cup \sigma_2) \cup (\text{pre}(\tau) \setminus \text{pa}_G(A_d))]|\text{mb}(x, A)\). Since \([A \setminus \text{mb}(x, A) \cup \{x\}] \subseteq [(\sigma_1 \cup \sigma_2) \cup (\text{pre}(\tau) \setminus \text{pa}_G(A_d))]|\text{mb}(x, A)\), using the decomposition property leads to: \(\{x\} \perp [A \setminus \text{mb}(x, A) \cup \{x\}]|\text{mb}(x, A)\).

(v)\Rightarrow(i): This has already been proved in \([\text{Richardson, 2003}]\) Theorem 2).
3.3 An Alternative Global Markov Property for MVR Chain Graphs

In this subsection we formulate an alternative global Markov property for MVR chain graphs. This property is different from the global Markov property resulting from the m*-separation criterion proposed in (Richardson, 2003; Richardson and Spirtes, 2002). First, show that this global Markov property implies the block-recursive Markov property of type IV in (Drton, 2009; Marchetti and Lupparelli, 2008) and MR Markov property in (Marchetti and Lupparelli, 2011). Finally, we show that they are equivalent.

Definition 28 (Alternative global Markov property for MVR chain graphs) For any triple \((A, B, S)\) of disjoint subsets of \(V\) such that \(S\) separates \(A\) from \(B\) in \((G_{Antec(A\cup B\cup S)})^a\), in the augmented graph of the smallest antecedental set containing \(A\cup B\cup S\), we have \(A \perp \perp B|S\).

Proposition 29 The alternative global Markov property in Definition 28 implies the block-recursive Markov property in Definition 27.

Proof The proof contains the three following steps:

1. Since \(\tau \cup \text{nd}_D(\tau)\) is an antecedental set, and \(\text{pa}_D(\tau)\) separates \(\tau\) from \(\text{nd}_D(\tau)\) \(\text{pa}_D(\tau)\) in \((G_{}\text{ind}_D(\tau))^c\); this shows that the global Markov property in Definition 28 implies (IV0) in Definition 21.

2. Assume that \(\sigma \subseteq \tau, \tau \in T\). Consider that \(A\) is the smallest antecedental set containing \(\sigma\) and \(\text{pa}_D(\tau)\). We know that for each vertex \(v \in \text{pa}_D(\tau) \setminus \text{pa}_G(\sigma), \ v \in \text{antec}(\sigma)\) and \(\text{pa}_G(\sigma) \subseteq \text{pa}_D(\tau)\). Also, we know that there is no directed edge from \(\text{pa}_D(\tau) \setminus \text{pa}_G(\sigma)\) to elements of \(\sigma\). So, every connecting path that connects \(\text{pa}_D(\tau) \setminus \text{pa}_G(\sigma)\) to \(\sigma\) in \((G_{A})^a\) has intersection with \(\text{pa}_G(\sigma)\), which means \(\text{pa}_G(\sigma)\) separates \(\text{pa}_D(\tau) \setminus \text{pa}_G(\sigma)\) from \(\sigma\) in \((G_{A})^a\); this shows that the global Markov property in Definition 28 implies (IV1) in Definition 21.

3. Assume that \(\sigma \subseteq \tau, \tau \in T\). Also, assume that \(\sigma\) is a connected subset of \(\tau\). Obviously, \(\sigma\) and \(\tau \setminus \text{nb}_G(\sigma)\) are two subsets of \(\tau\) such that there is no connection between their elements. Consider that \(A\) is the smallest antecedental set containing \(\sigma\) and \(\tau \setminus \text{nb}_G(\sigma)\). Clearly, \(\text{pa}_D(\tau) \subseteq A\). Since \(\sigma\) and \(\tau \setminus \text{nb}_G(\sigma)\) are disconnected in \(\tau\), any connecting path between them (if it exists) must pass through \(\text{pa}_D(\tau)\) in \((G_{A})^a\); this shows that the global Markov property in Definition 28 implies (IV2) in Definition 21.

Proposition 30 The alternative global Markov property in Definition 28 implies the MR Markov property in Definition 19.

Proof The proof is very similar to that of Proposition 29 and is omitted.
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Corollary 31  In a DAG $D$, $A$ is $d$-separated from $B$ given $S$ if and only if $S$ separates $A$ and $B$ in $\left( D_{\text{Antec}(A \cup B \cup S)} \right)^a$.

The following example shows that the proposed alternative global Markov property in Definition 28 is different from the pathwise $m$-separation criterion and the augmentation separation criterion in (Richardson, 2003; Richardson and Spirtes, 2002), in general.

Example 3  Consider the MVR chain graph $G$ in Figure 12.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure12.png}
\caption{A MVR chain graph with chain components: $T = \{\{a, b, c\}, \{d, e, f\}\}$.}
\end{figure}

According to the pathwise $m$-separation criterion and the augmentation separation criterion in (Richardson, 2003; Richardson and Spirtes, 2002), $a$ and $f$ are marginally independent. However, according to Definition 28, $a \not \perp \! \! \! \! \perp f \mid d$. Also, according to the pathwise $m$-separation criterion and the augmentation separation criterion in (Richardson, 2003; Richardson and Spirtes, 2002), $\{a, b\}$ and $f$ are marginally independent. However, according to Definition 28 we cannot obtain directly that $\{a, b\}$ and $f$ are marginally independent.

Theorem 32  Let $G$ be a MVR chain graph. An independence model $\Im$ over the node set of $G$ satisfies the alternative global Markov property w.r.t. $G$ in Definition 28 if and only if it satisfies the global Markov property w.r.t. $G$ in Definition 8.

\begin{proof}
Assume that $S$ separates $A$ from $B$ in $\left( G_{\text{Antec}(A \cup B \cup S)} \right)^a$, where $A, B, S$ are disjoint subsets of $V_G$. Since there is no undirected edge in a MVR CG, by definition of anterior and antecedent $\text{ant}(A \cup B \cup S)$ is a subgraph of $\text{Antec}(A \cup B \cup S)$. Therefore, $S$ separates $A$ from $B$ in $\left( G_{\text{ant}(A \cup B \cup S)} \right)^a$. In other words, the independence model induced by the global Markov property w.r.t. $G$ in Definition 28 is a subset of the independence model induced by the global Markov property w.r.t. $G$ in Definition 8.

The result follows from Proposition 29 and Theorem 27.
\end{proof}

3.4 An Alternative Local Markov Property for MVR Chain Graphs

In this subsection we formulate an alternative local Markov property for MVR chain graphs. This property is different from and much more concise than the ordered Markov property proposed in (Richardson, 2003). The new local Markov property can be used to parameterize distributions
efficiently when MVR chain graphs are learned from data, as done, for example, in (Javidian and Valtorta [2018, Lemma 9). We show that this local Markov property is equivalent to the global and ordered local Markov property for MVR chain graphs (for compositional semi-graphoids).

**Definition 33** If there is a bidirected edge between vertices $u$ and $v$, $u$ and $v$ are said to be neighbors. The boundary $bd(u)$ of a vertex $u$ is the set of vertices in $V \setminus \{u\}$ that are parents or neighbors of vertex $u$. The descendants of vertex $u$ are $de(u) = \{v| u$ is an ancestor of $v\}$. The non-descendants of vertex $u$ are $nd(u) = V \setminus (de(u) \cup \{u\})$.

**Definition 34** The local Markov property for a MVR chain graph $G$ with vertex set $V$ holds if, for every $v \in V$:

$$v \perp \perp [nd(v) \setminus bd(v)]pa_G(v).$$

**Remark 35** In DAGs, $bd(v) = pa_G(v)$, and the local Markov property given above reduces to the directed local Markov property introduced by Lauritzen et al. in (Lauritzen et al. [1990]).

**Remark 36** In covariance graphs the local Markov property given above reduces to the dual local Markov property introduced by Lauritzen et al. in (Lauritzen et al. [1990]).

The following theorem shows that the set of distributions satisfying the local and global properties are identical for compositional semi-graphoids. The necessity of composition property follows from the fact that local and global Markov properties for bi-directed graphs, which are a subclass of MVR CGs, are equivalent only for compositional semi-graphoids (Kauermann [1996] Proposition 2.2).

**Theorem 37** Let $G$ be a MVR chain graph. If an independence model $\mathcal{I}$ over the node set of $G$ is a compositional semi-graphoid, then $\mathcal{I}$ satisfies the alternative local Markov property w.r.t. $G$ in Definition 34 if and only if it satisfies the global Markov property w.r.t. $G$ in Definition 8.

**Proof** (Global $\Rightarrow$ Local): Let $X = \{v\}, Y = nd(v) \setminus bd(v)$, and $Z = pa_G(v)$. So, $an(X \cup Y \cup S) = v \cup (nd(v) \setminus bd(v)) \cup pa_G(v)$ is an ancestor set, and $pa_G(v)$ separates $v$ from $nd(v) \setminus bd(v)$ in $(G_{v \cup nd(v) \setminus bd(v)} \cup pa_G(v))^\alpha$; this shows that the global Markov property in Definition 8 implies the local Markov property in Definition 34.

(Local $\Rightarrow$ MR): We prove this by considering the following two cases:

Case 1): Let $A \subseteq T$ is connected. Using the alternative local Markov property for each $x \in A$ implies that: $\{x\} \perp \perp [nd(x) \setminus bd(x)]pa_G(x)$. Since $(pre(T) \setminus pa_G(A)) \subseteq (nd(x) \setminus bd(x))$, using the decomposition and weak union property give: $\{x\} \perp \perp (pre(T) \setminus pa_G(A))|pa_G(A)$, for all $x \in A$. Using the composition property leads to (MR1): $A \perp \perp (pre(T) \setminus pa_G(A))|pa_G(A)$.

Case 2): Let $A \subseteq T$ is disconnected with connected components $A_1, \ldots, A_r$. For $1 \leq i \neq j \leq r$ we have: $\{x\} \perp \perp [nd(x) \setminus bd(x)]pa_G(x)$, for all $x \in A_i$. Since $[(pre(T) \setminus pa_G(A)) \cup A_j] \subseteq (nd(x) \setminus bd(x))$, using the decomposition and weak union property give: $\{x\} \perp \perp A_j|pre(T)$, for all $x \in A_i$. Using the composition property leads to (MR2): $A_i \perp \perp A_j|pre(T)$, for all $1 \leq i \neq j \leq r$.

(MR $\Rightarrow$ Global): The result follows from Theorem 27.
4. An Alternative Factorization for MVR Chain Graphs

According to the definition of MVR chain graphs, it is obvious that they are a subclass of acyclic directed mixed graphs (ADMGs). In this section, we derive an explicit factorization criterion for MVR chain graphs based on the proposed factorization criterion for acyclic directed mixed graphs in [Evans and Richardson, 2014]. For this purpose, we need to consider the following definition and notations:

Definition 38 An ordered pair of sets \((H, T)\) form the head and tail of a term associated with an ADMG \(G\) if and only if all of the following hold:
1. \(H = \text{barren}(H)\), where \(\text{barren}(H) = \{v \in H | \text{de}(v) \cap H = \{v\}\}\).
2. \(H\) contained within a single district of \((G_{\text{an}(H)})_{\leftrightarrow}\).
3. \(T = \text{tail}(H) = (\text{dis}_{\text{an}(H)}(H) \setminus H) \cup \text{pa}(\text{dis}_{\text{an}(H)}(H))\).

Evans and Richardson in [Evans and Richardson, 2014, Theorem 4.12] prove that a probability distribution \(P\) obeys the global Markov property for an ADMG \(G\) if and only if for every \(A \in \mathcal{A}(G)\),

\[
p(X_A) = \prod_{H \in [A]_G} p(X_H|\text{tail}(H)),
\]

where \([A]_G\) denotes a partition of \(A\) into sets \(\{H_1, \ldots, H_k\} \subseteq \mathcal{H}(G)\) (for a graph \(G\), the set of heads is denoted by \(\mathcal{H}(G)\), defined with \(\text{tail}(H)\), as above. The following theorem provides an alternative factorization criterion for MVR chain graphs based on the proposed factorization criterion for acyclic directed mixed graphs in [Evans and Richardson, 2014].

Theorem 39 Let \(G\) be a MVR chain graph with chain components \((T|T \in T)\). If a probability distribution \(P\) obeys the global Markov property for \(G\) then \(p(x) = \prod_{T \in T} p(x_T|x_{\text{pa}_G(T)})\).

Proof According to Theorem 4.12 in [Evans and Richardson, 2014], since \(G \in \mathcal{A}(G)\), it is enough to show that \(\mathcal{H}(G) = \{T|T \in T\}\) and \(\text{tail}(T) = \text{pa}_G(T)\), where \(T \in T\). In other words, it is enough to show that for every \(T \in T\), \((T, \text{pa}_G(T))\) satisfies the three conditions in Definition 38.

1. Let \(x, y \in T\) and \(T \in T\). Then \(y\) is not a descendant of \(x\). Also, we know that \(x \in \text{de}(x)\), by definition. Therefore, \(T = \text{barren}(T)\).
2. Let \(T \in T\), then from the definitions of a MVR chain graph and induced bi-directed graph, it is obvious that \(T\) is a single connected component of the forest \((G_{\text{an}(T)})_{\leftrightarrow}\). So, \(T\) contained within a single district of \((G_{\text{an}(T)})_{\leftrightarrow}\).
3. \(T \subseteq \text{an}(T)\) by definition. So, \(\forall x \in T: \text{dis}_{\text{an}(T)}(x) = \{y|y \leftrightarrow \ldots \leftrightarrow x \text{ in an}(T), or x = y\} = T\). Therefore, \(\text{dis}_{\text{an}(T)}(T) = T\) and \(\text{dis}_{\text{an}(T)}(T) \setminus T = \emptyset\). In other words, \(\text{tail}(T) = \text{pa}_G(T)\).

Example 4 Consider the MVR chain graph \(G\) in Example 11. Since \([G]_G = \{\{1, 2, 3, 4\}, \{5, 6\}, \{7\}\}\) so, \(\text{tail}\{1, 2, 3, 4\} = \{5\}\), \(\text{tail}\{5, 6\} = \{7\}\), and \(\text{tail}\{7\} = \emptyset\). Therefore, based on Theorem 39 we have: \(p = p_{1234|567}p_{56|7}p_7\). However, the corresponding factorization of \(G\) based on the formula in [Drton, 2009; Marchetti and Lupparelli, 2017] is: \(p = p_{1234|567}p_{56|7}p_7\).

The advantage of the new factorization is that it requires only graphical parents, rather than parent components in each factor, resulting in smaller variable sets for each factor, and therefore speeding up belief propagation.
Conclusion and Summary

Based on the interpretation of the type of edges in a chain graph, there are different conditional independence structures among random variables in the corresponding probabilistic model. Other than pairwise Markov properties, we showed that for MVR chain graphs all Markov properties in the literature are equivalent for semi-graphoids. We proposed an alternative global and local Markov property for MVR chain graphs, and we proved that they are equivalent with other Markov properties for semi-graphoids and compositional semi-graphoids, respectively. Also, we obtained an alternative formula for factorization of a MVR chain graph. Table I summarizes some of the most important attributes of different types of common interpretations of chain graphs.

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| Type of chain graph | Does it represent independence model of DAGs under marginalization? | Global Markov property | Factorization of $p(x)$ | Model selection (structural learning) algorithm(s) [constraint based method] |
|---------------------|---------------------------------------------------------------------|------------------------|------------------------|-------------------------------------------------------------------|
| MVR CGs: Cox & Wermuth (Cox and Wermuth, 1993, 1996), Wermuth and Cox (2004), Peña & Sonntag (Peña, 2015, Sonntag, 2014), Sadeghi & Lauritzen (Sadeghi and Lauritzen, 2014), Drton (type IV) (Drton, 2009), Marchetti & Lupparelli (Marchetti and Lupparelli, 2008, 2011) | Yes (claimed in Cox and Wermuth 1996, Wermuth and Sadeghi 2012, Sadeghi and Lauritzen 2014, Peña 2015, Sonntag 2014, proved in Corollary 15) | (1) $X \perp Y \mid Z$ if $X$ is separated from $Y$ by $Z$ in $(G_{ant}(X \cup Y \cup Z))^a$ or $(G_{an}(X \cup Y \cup Z))^a$ (Richardson 2003, Richardson and Spirtes 2002). (2) $X \perp Y \mid Z$ if $X$ is separated from $Y$ by $Z$ in $(G_{Antec}(X \cup Y \cup Z))^a$. (1) and (2) are equivalent for compositional graphoids. | (1) Theorem 39 \[ \prod_{T \in \mathcal{T}} p(x_T \mid x_{pa(T)}) \] (2) \[ \prod_{T \in \mathcal{T}} p(x_T \mid x_{pa_D(T)}) \] where $pa_D(T)$ is the union of all the chain components that are parents of $T$ in the directed graph $D$ (Drton, 2009, Marchetti and Lupparelli 2011). | PC like algorithm for MVR CGs in (Sonntag, 2014), Decomposition-based algorithm for MVR CGs in (Javidian and Valtorta, 2018) |
| LWF CGs (Frydenberg, 1990, Lauritzen and Wermuth, 1989), Drton (type I) (Drton, 2009) | No | $X \perp Y \mid Z$ if $X$ is separated from $Y$ by $Z$ in $(G_{An}(X \cup Y \cup Z))^m$ (Lauritzen 1996). | \[ \prod_{\tau \in \mathcal{T}} p(x_\tau \mid x_{pa(\tau)}) \] where $p(x_\tau \mid x_{pa(\tau)}) = Z^{-1}(x_{pa(\tau)}) \prod_{c \in C} \phi_c(x_c)$, where $C$ are the complete sets in the moral graph $(\tau \cup pa(\tau))^m$. | PC like algorithm in (Studený, 1997), LCD algorithm in (Ma et al., 2008), CKES algorithm in (Peña et al., 2014, Sonntag 2014) |
| AMP CGs (Andersson et al., 1996), Drton (type II) (Drton, 2009) | No | $X \perp Y \mid Z$ if $X$ is separated from $Y$ by $Z$ in the undirected graph $Aug[CG; X, Y, Z]$ (Richardson 1998). | $\prod_{\tau \in \mathcal{T}} p(x_\tau \mid x_{pa(\tau)})$, where no further factorization similar to LWF model appears to hold in general (Andersson et al., 1996). For the positive distribution $p$ see (Peña, 2018). | PC like algorithm in (Peña, 2014) |

Table 1: Properties of chain graphs under different interpretations