A Dichotomous Analysis of Unemployment Benefits†

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Abstract
Using equal employment opportunity as a fairness hypothesis, real-time balanced-budget rule as a constraint, and policy stability as an objective, we derive a scientific formula for taxation policy and formulate a fair allocation for unemployment benefits. The general setting is a coalitional game in which a random subset of the players is selected to take a task, resulting in a random payoff; we attempt to divide the payoff among the players fairly. The formula describes a fair, debt-free, and asymptotic risk-free payroll tax rate for given unemployment and spending levels. Also robust to the choice of other objectives, the tax rate stimulates employment and boosts productivity within this fair and sustainable framework. With additional assumptions on productivity, the tax rate results in equality of outcome, reducing poverty and economic inequality. The fair division rule and the valuation approach could be applied to similar profit- or cost-sharing situations, including voting rights, health insurance, road sharing, and machine learning.

Keywords: Payroll Tax, Unemployment Benefits, Equal Employment Opportunity, Fair Division, Equality of Outcome, Dichotomous Valuation

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1 Introduction
The problem we address relates to the following typical situation: an economy cannot employ all its labor force, and therefore, some people are employed, and others are

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not. As the employed receive wages and employment welfare (e.g., pension, health and life insurance, social security, and paid vacation), should the unemployed receive some unemployment benefits? If so, how much would be fair? In a specific jurisdiction system, the term “unemployment benefits” may also mean “unemployment insurance,” “unemployment payment,” or “unemployment compensation.” In an advanced economy, the answer to the first question is likely yes. This paper provides an affirmative answer to all individuals in the labor market, including those just graduating from school. It also responds to the second question — how much is fair — by justifying a fair share of unemployment benefits for the unemployed and deriving a fair payroll tax rate for the employed. The solution fairly distributes a random production among the two random labor market groups, and the justification may help reduce economic inequality and poverty.

The fair-division problem arises in various real-world settings. For a simple motivating example, consider a $k$-out-of-$n$ redundant system with $n$ identical components, any $k$ of which being in good condition makes the system work properly. When valuing the importance of each component (either working or idle), one may intuitively claim that these components should be equally valued. A similar situation occurs in a simple majority voting where only some voters support a ballot measure; thus, the measure could pass or fail. Nevertheless, voters are supposed to have the same voting power no matter what they support. For another example, not all policyholders in the health insurance industry are sick and use insurance to cover their medical expenses. The question is how to fairly share the total medical cost among sick and healthy policyholders. In a labor market, we have a similar but more complicated situation: on the one hand, the market could not hire the entire labor force even though everyone in the market would like to be employed; on the other hand, the participants in the market have heterogeneous performance in the production. In summary, there are four common features in these examples: a coalition of players with a cooperative nature, a random bipartition of the players, a payoff associated with the random partition, and an objective to share the random payoff with all the players. This paper derives a solution for situations with these features. In the $k$-out-of-$n$ redundant system or the simple majority voting, for example, we expect equality of outcome.

We face a few challenges when deriving a fair tax rate and a fair distribution of welfare and benefits, both implemented by the employed. First of all, fairness may be an abstract but vague concept. It could be defined in many ways. We argue that “fairness” is bound by the equality of employment opportunity, not by the equality of outcome or productivity. Secondly, despite limited opportunities, everyone in the labor market could contribute somehow. Thus, unemployment is neither a fault of the unemployed nor a flaw of the labor market but a self-adjustment mechanism toward the efficiency of the market. Next, we attempt to apply the taxation policy to the labor market, which operates in an ever-changing economy and with ever-changing productivity. For it to be helpful, the taxation and division rule should cope with the uncertainty and sustainability in generating and distributing the net production. Ideally, it should balance the account of value generated and that of value paid in each employment contingency. Lastly, in an objective world, a fair tax rate should depend only on observed data to avoid excessive political bargaining and
costly strategic voting. One significant data issue, however, is the non-observability of the heterogeneous-agent production function in all employment scenarios at all times. Another issue is the desynchronization between the unemployment and tax rates; the former has a high frequency while the latter has a low one. In a yearly time frame, policymakers often determine the tax rate after observing most monthly unemployment rates. We develop a fair rate for the high-frequency time frame but target a constant rate in the low-frequency time frame by minimizing its variability.

Much literature has studied fairness in taxation and unemployment payments from various aspects (e.g., Kornhauser 1995; Fleurbaey and Maniquet 2006). In particular, Shapley (1953) proposes an influential axiom of fairness to develop a fair-division method called the Shapley value, which is widely used in distributing employment compensation and welfare payments (for example, Moulin 2004; Devicienti 2010; Giorgi and Guandalini 2018; Krawczyk and Platkowski 2018). Beneath the pillars of the Shapley value and the Shapley axioms, however, are two underlying assumptions: players’ unanimous participation in the value-generating production and the distributor’s complete information about the production function. Recently, Hu (2006, 2020) relax the unanimity assumption and generalized the Shapley value, using some non-informative probability distributions for the dichotomy or bipartition of the players. In particular, Hu (2020) uses a beta-binomial distribution to address equal participation opportunities in production. This current research also capitalizes on the distribution. Furthermore, we assume no complete information about the production function. We only need its value at one observation, which does occur.

Our approach has two advantages. On the one hand, we provide a game-theoretic micro-foundation for a fiscal policy. In essence, the paper develops a fair-division solution of a random payoff in a coalitional game. One can apply the solution concept to many similar situations without substantive alternations, and one may also extend the framework using other identification schemes rather than those described in Section 4. On the other hand, the fair tax rate we provide is simple enough for practical use. It relies only on the unemployment rate and a reserved spending rate for non-personal use. The total unemployment compensation depends only on the tax rate and the realized production; equality of outcome is satisfied under an additional assumption. We attempt to immunize our solution from unnecessary randomness, hypotheticals, ambiguity, and latency. These include, but are not limited to, the competitive and cooperative features of the labor market, endogenous employment search behavior, non-linear schedules of tax rates, the exact sizes of the labor market, and the time-varying unemployment population. With this simplicity in hand, a certain level of abstraction is necessary, and any application of the fair solution should accommodate its specific reality.

We organize the remainder of the paper into six sections. Section 2 applies the framework of dichotomous valuation (or simply, “D-value”) in Hu (2006, 2020) to value each person in the labor market, assuming equal employment opportunity. The two sides of the D-value, conditional on two unknown parameters for a beta distribution, are aggregated separately for the unemployed and the employed labor. Section 3 formulates a set of fair divisions of the net production using the aggregate components of the D-value. In this section, we base the set of fair tax rates on two accounting
identities for a balanced budget. In Section 4, by maximizing the stability of the tax rate or minimizing the expected posterior unemployment rate, we identify a specific fair tax rate. This particular solution is robust to a few other criteria. The rate and an assumption on productivity result in equality of outcome in Section 5. Section 6 lists four other applications (voting power, health insurance, highway toll, and feature selection) of the division rule, and Section 7 concludes with several suggestions to extend this framework. The account is self-contained, and the proofs are in the Appendices.

2 Dichotomous Valuation

We introduce a few notations before formal discussions. For a general economy, its labor force consists of employed and unemployed labor, ignoring any part-timers. Let \( N = \{1, 2, \ldots, n\} \) denote the set of individuals in the labor force, indexed as 1, 2, ..., \( n \). The bold-cased \( S \subseteq N \) denotes the random subset of the employed labor in \( N \), which changes daily, and \( S \) denotes a realization of \( S \), which is observed. For any subset \( T \) of \( N \), let \(|T|\) denote its cardinality. For simplicity, we often use lower-cased \( n \) for \(|N|\), \( t \) for \(|T|\), \( z \) for \(|Z|\), and \( s \) for \(|S|\) and \(|S|\). We write the employment rate as \( \omega = \frac{s}{n} \), which is one minus the unemployment rate. Besides, we employ the vinculum (overbar) in naming a singleton set; for example, \( \bar{i} \) is for the singleton set \( \{i\} \). Also, \( \setminus \) is for set subtraction, \( \cup \) for set union, and \( \beta(\theta, \rho) \) for the beta function with parameters \( (\theta, \rho) \).

2.1 Equality of Employment Opportunity

Equal employment opportunity (EEO) is widely acknowledged and is the starting point or axiom for us to study fairness. In the United States, for example, EEO has been enacted to prohibit federal contractors from discriminating against employees by race, sex, creed, religion, color, or national origin since President Lyndon Johnson signed Executive Order 11246 in 1965. The Equality Act 2010 in Britain has similar prohibitions on all employers, service providers, and education providers. In the literature, there are abundant qualitative descriptions, formal and informal, about equal opportunity (e.g., Friedman and Friedman 1990; Roemer 1998; Rawls 1999). Thus, as there can be many versions of equality, many fair tax rates exist. Besides the justification from EEO, unemployment compensations could also be based on social protection, insurance of income flow, poverty prevention, and political considerations, among others (e.g., Sandmo 1998; Tzannatos and Roddis 1998; Vodopivec 2004).

We introduce a probabilistic version of EEO, whereby the employment opportunity is assumed equitable for all persons in the labor force, regardless of their professions and positions. Since both \( n \) and \( s \leq n \) are nonnegative integers, we assume three layers of uncertainty for the random subset \( S \). In the first layer, the employment size \(|S|\) follows a binomial distribution with parameters \((n, p)\). When independence is assumed, \( p \) is the probability of any given person being employed. In the second layer, the unknown parameter \( p \) has a prior beta distribution with hyperparameters \((\theta, \rho)\).
Thus, the joint probability density of \( p \) and \( |S| = t \) is

\[
\frac{p^{\rho-1}(1-p)^{\rho-1}}{\beta(\theta, \rho)} \binom{n}{t} p^t (1-p)^{n-t} = \frac{n!}{t!(n-t)!} \frac{p^{\rho+t-1}(1-p)^{\rho+n-t-1}}{\beta(\theta, \rho)}.
\] (1)

Eq. (1) implies that the marginal probability density for \( |S| = t \) is

\[
\mathbb{P}(|S| = t) = \int_0^1 \frac{n!}{t!(n-t)!} \frac{p^{\rho+t-1}(1-p)^{\rho+n-t-1}}{\beta(\theta, \rho)} \, dp = \frac{n!}{t!(n-t)!} \frac{\beta(\theta+t, \rho+n-t)}{\beta(\theta, \rho)}
\] (2)

for any \( t = 0, 1, \cdots, n \). In the third layer, for any given employment size \( t \), all subsets of size \( t \) have the same probability of being \( S \). As there are \( \frac{n!}{t!(n-t)!} \) subsets of size \( t \) in \( N \), the probability for the employment scenario \( S = T \) is

\[
\mathbb{P}(S = T) = \begin{cases} \frac{\beta(\theta+s, \rho+n-s)}{\beta(\theta, \rho)}, & \text{if } t = s; \\ 0, & \text{otherwise.} \end{cases}
\] (3)

The assumed triple-layered uncertainty implies that anyone in the labor market is equally likely to be employed. However, it never means that the workers are identical, as the production requires heterogeneous ones from the labor market. Also, this concept does not describe any particular job such that all applicants have an equal chance to win such a position. Instead, the hiring firm has the discretionary right to choose the best-fitted candidates from the market. Therefore, the EEO is more likely to occur when there is a high level of market depth and labor mobility and a lower level of structural unemployment and skills mismatch. Diversity and inclusiveness solidify the depth by bridging gaps between positions. Thus, opportunities abound in the inclusive market for everyone in \( N \), including inexperienced graduates just out of college. To achieve this equality, a government could create employment opportunities to balance the supply and demand from labor power across all industries. It could also encourage female labor participation, offer professional training programs in high-demand professions, and build efficient job-search platforms to match opportunities with job applicants, especially when opportunities shift from one industry to another. By taking taxation, the government is responsible for creating jobs, training the unemployed, and maintaining the EEO. Job-seekers may capitalize on their highly demanded skills in the shift to grow their career interests or leverage the market opportunities to jump to better-fit positions.

The hyperparameters \( \theta \) and \( \rho \) act as policy instruments — we place constraints and optimal objective through these parameters. Their effects are reflected in the posterior rate of employment. Based on the profile of the perspective posterior rate, we solve backward an optimal policy rule and the values of \( \theta \) and \( \rho \). By Eqs. (1) and (2), the posterior density function of \( p \) given \( |S| = s \) is

\[
\frac{n!}{s!(n-s)!} \frac{p^{\rho+s-1}(1-p)^{\rho+n-s-1}}{\beta(\theta, \rho)} = \frac{p^{\rho+s-1}(1-p)^{\rho+n-s-1}}{\beta(\theta+s, \rho+n-s)}.
\]
Thus, the posterior employment rate follows a beta distribution with parameters \((\theta + s, \rho + n - s)\). In the following, we use \(p_{n,\omega}\) to denote the posterior employment rate given the observance of \(|S| = n\omega\). In contrast, \(p\) is an unobservable parameter for the binomial random variable \(|S|\).

2.2 Aggregate Values of the Employed and the Unemployed

For any \(S \subseteq \mathbb{N}\), we use a heterogeneous-agent production function \(v(S)\) to measure the net aggregate productivity when \(S = S\). The net-profit production \(v(S)\) excludes the labor cost, which compensates for the time and efforts made by the employed labor in producing \(v(S)\). To isolate the added value by the labor force alone, \(v(S)\) also excludes the cost of consumed physical and financial resources. The labor and resource costs are exempt from taxation. Thus, without loss of generality, we assume that \(v(\emptyset) = 0\) for the empty set \(\emptyset\). However, \(v(S)\) does not necessarily increase with \(S\) or its size \(s\).

To retain its labor or to minimize its labor turnover, a firm would share part of its net profit with its employees. To keep things simple, we use the term “employment welfare” to denote the employees’ profit-sharing portion of \(v(S)\), in contrast to the term “unemployment benefits” for the unemployed. Welfare is crucial to employee retention and business continuation, significantly impacting productivity. The longer employees stay with their jobs, the more skilled they become and the fewer expensive mistakes they make. New hires generally take months to accommodate into a new workplace and years to become professional. Firm owners also require a share of \(v(S)\) for their risk-bearing investment, entrepreneurship, and financing costs.

Let us formally introduce the components of the D-value. For any \(i \in \mathbb{N}\), to analyze its marginal effect on the value-generating process, we consider two jointly exhaustive and mutually exclusive events:

- **Event 1**: \(i \in S\), who is currently employed. Then, \(i\)’s marginal effect is \(v(S) - v(S \setminus \{i\})\), called marginal gain, in that he or she contributes \(v(S) - v(S \setminus \{i\})\) to the production, due to his or her existence in \(S\). The expected marginal gain is
  \[
  \gamma_i[v] \overset{\text{def}}{=} \mathbb{E} \left[ v(S) - v(S \setminus \{i\}) \right] \tag{4}
  \]
  where “\(\text{def}\)” is for definition and “\(\mathbb{E}\)” for expectation with the probability density of \(S\).

- **Event 2**: \(i \notin S\), who is currently unemployed. He or she could just enter the market after school or be in a state of cyclical, structural, or frictional unemployment. Then, \(i\)’s marginal effect is \(v(S \cup \{i\}) - v(S)\) in that \(S\) faces a marginal loss \(v(S \cup \{i\}) - v(S)\) due to \(i\)’s absence from the employed labor force \(S\). In other words, the person would increase the production by \(v(S \cup \{i\}) - v(S)\) if the market included him or her in \(S\). The expected marginal loss is
  \[
  \lambda_i[v] \overset{\text{def}}{=} \mathbb{E} \left[ v(S \cup \{i\}) - v(S) \right] \tag{5}
  \]
  We let \(\gamma_i[v]\) be the employment welfare \(i\) receives when employed and \(\lambda_i[v]\) the unemployment benefits \(i\) receives when unemployed. Note that, even if \(i\) always
remains employed, both $S$ and $S \setminus i$ change in real-time; thus, $i$’s marginal gain is not a constant. Similarly, even if $i$ remains unemployed for a while, $S$, $S \cup i$, and $i$’s marginal loss are not constant. To account for this uncertainty, we take expectations in Eqs. (4) and (5) when defining $\gamma_i[v]$ and $\lambda_i[v]$.

There are a few implicit points in the profit-sharing strategy. First, in addition to receiving employment welfare, employed labor also receives reimbursement for labor costs, compensating for the human capital usage in generating $v(S)$. Human capital also accumulates in prior employment and pre-employment education. Unemployed $v_i$ also accumulates in prior employment and pre-employment education. Unemployed labor, however, only receives unemployment benefits. Secondly, if $i \in S$, $v(S \cup i)$ is not observable, but we observe $v(S)$. Similarly, when $j \notin S$, we cannot observe both $v(S \cup j)$ and $v(S)$ simultaneously. Thus, we must transform the aggregate marginals into observable forms, such as those in Theorem 1. Thirdly, the aggregate employment welfare $\sum_{i \in S} [v(S) - v(S \setminus i)]$ is not necessarily equal to $v(S)$. Thus, we distribute some of the surpluses $v(S) - \sum_{i \in S} [v(S) - v(S \setminus i)]$ to the unemployed labor $N \setminus S$. The distribution is not through personal giving but through government taxation and unemployment payment systems. This distribution channel also appeals to us for the aggregate welfare and aggregate benefits at the national level, as stated in Theorem 1.

**Theorem 1** The aggregate components of the $D$-value are

$$
\sum_{i \in N} \gamma_i[v] = \sum_{S \subseteq N} \left( v(S) - v(S \setminus i) \right) = \sum_{S \subseteq N} \frac{s(\theta + p - 1) - n\theta}{\theta + p + n - s - 1} \frac{\beta(\theta + s, \rho + n - s)}{\beta(\theta, \rho)} \cdot v(S),
$$

$$
\sum_{i \in N} \lambda_i[v] = \sum_{i \in N \setminus S} \left( v(S \cup i) - v(S) \right) = \sum_{S \subseteq N} \frac{s(\theta + p - 1) - n(\theta - 1)}{\theta + p + n - s} \frac{\beta(\theta + s, \rho + n - s)}{\beta(\theta, \rho)} \cdot v(S).
$$

The coalition $S$ realizes the value $v(S)$, however, it alone could not generate the value without the support from the country and the human supplies from the whole labor market, including $N \setminus S$. Thus, the government has the authority to control the aggregates in Theorem 1 by adjusting the hyper-parameters $\theta$ and $\rho$. This control aims at specific optimal fiscal policies subject to budget constraints. As shown in Theorem 11 of Hu (2020), player $i$’s Shapley value in the coalitional game $(N, v)$ equals the sum of $\gamma_i[v]$ and $\lambda_i[v]$ when $\theta = \rho = 1$, i.e., $p$ has a uniform distribution on $[0, 1]$. Also, if we enforce that $\sum_{i \in N} \gamma_i[v] = v(N)$, then $\gamma_i[v]$ is $i$’s Shapley value; similarly, if we restrict that $\sum_{i \in N} \lambda_i[v] = v(N)$, then $\lambda_i[v]$ is also $i$’s Shapley value (cf, Hu, 2020). Either the enforcement or the restriction identifies the values of the parameters $(\theta, \rho)$, which result in the Shapley value.

### 3 Accounting Identities for a Balanced Budget

By Eq. (3), the expected production is

$$
E[v(S)] = \sum_{S \subseteq N} \frac{\beta(\theta + s, \rho + n - s)}{\beta(\theta, \rho)} \cdot v(S).
$$
We observe only one out of 2ⁿ employment scenarios at a specific time. Let us consider the scenario at \( S = S \), which occurs with probability \( \frac{\beta(\theta, \rho + n - s)}{\beta(\theta, \rho)} \) and generates the value \( v(S) \). Our division rule should fully respect all entitlements to \( v(S) \). Each employed person receives their expected marginal gain \( \gamma_i[v] \), and each unemployed person receives their expected marginal loss \( \lambda_i[v] \). Additionally, we reserve a \( \delta \) portion of \( v(S) \) for the common good of the economy and the society, which provide a safe and regenerative environment for the value-generating process. In contrast, the Shapley value distributes \( v(N) \) to all players in \( N \), and Hu (2020) formulates solutions to divide \( \mathbb{E}[v(S)] \) to all players.

### 3.1 A Real-Time Balanced Budget Rule

We divide the realized net production \( v(S) \) into three components. The first is for the employment welfare. Comparing the coefficients of \( v(S) \) in Eq. (7) and in Eq. (6)’s \( \mathbb{E}\left[\sum_{i \in S} (v(S) - v(S \setminus i))\right] \), the employed labor together should retain \( \frac{n(\theta + \rho - 1) - n\theta}{\rho + n - s - 1} v(S) \) as their employment welfare. The rest, \( 1 - \frac{n(\theta + \rho - 1) - n\theta}{\rho + n - s - 1} v(S) \), is paid to the government as “payroll tax.” The second component goes to the unemployed. According to Eq. (6)’s \( \mathbb{E}\left[\sum_{i \in N \setminus S} (v(S \cup i) - v(S))\right] \), the unemployed labor \( N \setminus S \) should collectively claim \( \frac{n(\theta + \rho - 1) - n(\theta - 1)}{\theta + s - 1} v(S) \) as their unemployment benefits. The claim is not \( v(N) - v(S) \), in which we would never observe the value of \( v(N) \), and it is also possible that \( v(N) \leq v(S) \). Thirdly, we assume that a reserved proportion, \( \delta v(S) \), is not individually and not directly distributed to the labor force \( N \).

Thus, we define the tax rate \( \tau \) as

\[
\tau(\omega, \delta, n) \overset{\text{def}}{=} 1 - \frac{n\omega(\theta + \rho - 1) - n\theta}{\rho + n - n\omega - 1}.
\] (8)

The rate does not directly involve the production \( v(S) \) and \( (1 - \tau)v(S) \) is for the employment welfare. The tax is used for the unemployment benefits and reserve; consequently, the rate \( \tau \) includes both the reserve ratio \( \delta \) and the proportion of the unemployment benefits, i.e.,

\[
\tau(\omega, \delta, n) \equiv \delta + \frac{n\omega(\theta + \rho - 1) - n(\theta - 1)}{\theta + n\omega - 1} \] (9)

in which the second part is for the unemployment benefits.

Unemployment benefits could be managed separately or jointly with taxation in different countries. In Australia, for example, unemployment benefits, as part of social security benefits, are funded through the taxation system. In the United States, however, they are funded by a compulsory governmental insurance system, which manages the collection and payment accounts. Besides the Federal Unemployment Tax Act, the State Unemployment Insurance allows varying tax rates across years and states. The contribution and distribution may be de facto a type of payroll tax if new entrants
to the labor market are eligible to enjoy the benefits even though they pay no insurance premiums yet. Otherwise, the insurance may be unfair to those who are to retire soon. They pay the premiums until retirement but expect no unemployment benefits afterward.

The reserve \( \delta v(S) \) serves the general public’s interest and has broad appeal rather than meeting personal needs. More specifically, it includes but is not limited to, payments to people not in the labor force, social security, medicare, public administration and national defense, public services, and past tax deficits and their interests, if any. In the USA, Social Security and Medicare portions are managed separately and apply flat tax rates; they are not adjustable by gross household income, which uses progressive rates.

The real-time balanced budget rule forbids borrowing between different labor market scenarios and inter-temporal borrowing. Thus, this sustainable taxation policy meets the needs of the present market scenario while maintaining the ability of future market scenarios to meet their own needs. In practice, however, enforcing or enacting the balanced budget rule at the labor market scenario level or on a real-time basis is challenging. In the United States, for example, the employment rate \( \omega \) changes daily and is recorded monthly by the Bureau of Labor Statistics; as a policy variable, the tax rate \( \tau \) changes yearly. To balance the budget maximally, one could minimize the variance of the employment rate \( p_{n,\omega} \) within a yearly time frame. In an ideal situation, the employment rate follows a degenerate probability distribution and remains almost constant within the year.

A democratic government often has problems balancing a long-run budget. Each administration has no incentive to resolve the debt issue left by previous administrations. When the expenditure approach measures GDP, the administration tends to finance crises and wars with debt without compromising GDP growth. Competing political parties may also lower the income tax rate to an unsustainable level just to court a select group of voters. Consequently, a massive debt could accumulate over time; thus, taxation without balancing budgets plunders future generations if overseas plunder becomes impossible. In our real-time budget balance, the sitting administration takes accountability for the debt it creates.

Households could exhibit different behaviors from those of the government. In contrast to a government’s exhaustive distribution of the net production in the national accounts, a household could still maximize its utility through inter-temporal borrowing, saving, lending, and consumption.

### 3.2 The Set of Feasible Solutions

For a given triple of \((\omega, \delta, n)\), there are three indeterminates \((\theta, \rho, \tau)\) in the system of two equations, Eqs. (8) and (9). Let \( \Omega_{\omega,\delta,n} \) denote the set of all feasible combinations of \((\theta, \rho, \tau)\) which satisfy both budget constraints Eqs. (8) and (9):

\[
\Omega_{\omega,\delta,n} \overset{\text{def}}{=} \left\{ (\theta, \rho, \tau) \middle| \begin{array}{l}
\tau = 1 - \frac{n\omega(\theta + \rho - 1) - n\theta}{\rho + n\omega - 1 - n(\theta - 1)}, \\
\tau = \delta + \frac{n\omega(\theta + \rho - 1) - n\theta}{\theta + n\omega - 1 - n(\theta - 1)}, \\
0 \leq \tau \leq 1, \ \theta > 0, \ \rho > 0.
\end{array} \right\}.
\]
Uncertainty exists about the size of the labor market $n$. Even though the size is not random in our EEO model, it is a time-varying latent variable in that there is no clear dividing line between entry to and exit from the labor market, and many unemployed people may not be active job seekers. In practice, $n$ changes daily, whereas $\tau$ generally changes yearly. No matter how it changes or is latent, $n$ is a large number in the general economy. Thus, in contrast to Eqs. (8) and (9), we also seek a tax rule valid for all large $n$’s but not specific to a particular $n$. We define

$$\tau(\omega, \delta) \overset{\text{def}}{=} \lim_{n \to \infty} \tau(\omega, \delta, n)$$

if it exists for a series of $(\theta, \rho, \tau) \in \Omega_{\omega, \delta, n}$. For simplicity, we use “$\phi$-rate” for the limit of tax rates that satisfy Eqs. (8) and (9), both assuming the EEO of Eq. (3).

Eqs. (8) and (9) are indeed a system of linear functions for $\theta$ and $\rho$, in terms of $n$, $\omega$, $\tau$, and $\delta$. For notational simplicity, we introduce the following shorthands:

1. $\Delta \equiv \omega + \tau - \delta\omega - 1,$
2. $\Delta_1 \equiv \delta\omega - \omega - \tau + 2 \equiv 1 - \Delta > 0,$
3. $\Delta_2 \equiv \delta\omega - \omega - \tau^2 + 2\omega\tau + \tau - 1,$
4. $\Delta_3 \equiv (1 - \tau)(\delta - \tau) = \delta - \tau - \delta\tau + \tau^2 < 0,$
5. $\Delta_4 \equiv -\delta\omega\tau + \delta\omega + \delta\tau - 2\delta + \omega\tau^2 - 2\omega\tau + \omega - \tau^2 + 3\tau - 1.$

All these are bounded because $0 < \omega, \tau, \delta < 1$. Lemma 1, to be used in proving other theorems, extracts $\theta$ and $\rho$ out from Eqs. (8) and (9). Since $\Delta_1 > 0$ and $\Delta_3 < 0$ in Eq. (10), we need $\Delta > 0$ to ensure the positivity of $\theta$ and $\rho$ for a large but finite $n$. Consequently, Corollary 1 claims that a government should levy at least $1 - \omega + \delta\omega$ of the production as payroll tax.

**Lemma 1** In terms of $(n, \delta, \tau, \omega)$, we extract $(\theta, \rho)$ from Eq. (8) and Eq. (9) as

$$
\begin{align*}
\theta & = \frac{n^2\omega\Delta_1 + \rho\Delta_3 + \rho\Delta_4}{n\Delta_1 + \rho\Delta_3 + \rho\Delta_4}, \\
\rho & = \frac{n^2(1 - \omega)\Delta_1 + \rho\Delta_3 + \rho\Delta_4}{n\Delta_1 + \rho\Delta_3 + \rho\Delta_4}.
\end{align*}
$$

**Corollary 1** The minimal $\phi$-rate is $\tau(\omega, \delta) = 1 - \omega + \delta\omega$.

Furthermore, the posterior employment rate $p_{n, \omega}$ leverages between the prior and the observation. When the observation contains a large set of data, the effect from the prior becomes minimal. As a result, $p_{n, \omega}$ has a degenerated limit distribution as $n \to \infty$ if $\Delta > 0$, claimed in Theorem 2. Therefore, the limit tax rule does not affect the limit central tendency of $p_{n, \omega}$ as long as $\tau(\omega, \delta) > 1 - \omega + \delta\omega$. This leads us to examine its effect on other aspects of $p_{n, \omega}$, such as the variance and convergence speed.

**Theorem 2** For any $\phi$-rate $\tau \in (1 - \omega + \delta\omega, 1)$, as $n \to \infty$, $p_{n, \omega}$ converges in distribution to the degenerate probability distribution with mass at $\omega$.

We base our derivation on observed data to derive a unique solution from $\Omega_{n, \omega, \delta}$ or its limit. For a reasonable triple of $(\omega, \delta, n)$, there are infinitely many solutions in $\Omega_{\omega, \delta, n}$, demanding one more restriction to identify $(\theta, \rho, \tau)$ uniquely. One could capitalize on the statistical relation between $p$ and $(\theta, \rho)$: the mean and mode of $p$ are $\frac{\theta}{\theta + \rho - \tau}$, and $\frac{\theta - 1}{\theta + \rho - \tau}$, respectively (e.g., Johnson et al. 1995, Chapter 21). Alongside
this direction, for example, we could set $\theta \rho_{\pi \theta}$ or $\theta \rho_{\pi \theta - 2}$ to be the historical average of $\omega$ in the previous year. Alternatively, we could set $\theta \rho_{\pi \theta}$ or $\theta \rho_{\pi \theta - 2}$ to be a target employment rate or the natural employment rate. However, this identification scheme requires additional input, such as a historical average, target, or natural employment rate. The historical average, for example, may not capture the recent trend, and the natural employment rate is vulnerable to estimation errors.

When $\phi$-rate relies only on a realization of $p$, not on the uncertainty of $p$, a natural way to factor out this uncertainty is to study the posterior rate $p_{\omega, \omega}$, where both $\pi$ and $\omega$ are no longer random. As Theorem 2 indicates, $\omega$ is informative and indicative in revealing the central tendency of the posterior labor market, described by $p_{\omega, \omega}$. Asymptotic dispersion of $p_{\omega, \omega}$ and the market’s response to the $\phi$-rate $\tau(\omega, \delta)$ are among other essential ingredients in the complete profile of $p_{\omega, \omega}$. Thus, we could also set optimal criteria over $\Omega_{\omega, \delta, \infty}$ to minimize the dispersion measures or optimize the expected market response. The tax rule with the minimum dispersion, for example, makes the fastest convergence to the distribution in Theorem 2 and mitigates the frequency gap between the daily $\omega$ and the yearly $\tau$.

4 An Optimal Fair Tax Rate

In this section, we derive the $\phi$-rate $\tau(\omega, \delta) = 1 - \omega + \delta \omega$ from a few different angles. On the one hand, a proper distribution of benefits within the labor market should not compromise the efficiency of the market itself, and a good tax rule should not discourage the incentives of employment and productivity, such as those detailed in Theorems 5 and 7. On the other hand, we expect a good rule to be robust and optimal under multiple criteria; thus, we study five criteria in Theorems 3, 4, 5, 6, and 8, any of which uniquely identifies the solution. Together, they minimize the employment market risk or maximize the employment expectation under a given market capacity and a budget balance constraint.

To develop the tax rule, we heavily exploit the observed market behavior. While an efficient labor market stimulates productivity $v(S)$, a higher employment rate does not necessarily imply higher productivity and vice versa — the production function $v(S)$ does not necessarily increase with the employment size $s$. Acemoglu and Shimer (2000) find that a moderate level of unemployment could boost productivity by improving the quality of jobs. Indeed, it fosters peer pressure in producing $v(S)$, allows workers to move on from declining firms, and enables rising companies and the economy to respond optimally to external shocks. Therefore, our tax rule would not merely target a higher employment rate but grounds its assumptions in the observed market behavior and lets the market respond with a higher employment rate to the maximum extent permitted by the budget rule and the market capacity.

4.1 Asymptotic Risk-Free Tax Rate

Tax rate stability creates the right environment for a balance of payments, reduces the uncertainty of the labor market, and creates confidence in technology and human capital investment. As a function of $\omega$, $\tau$ transmits the risk from the unemployment rate to the tax rate. With the absence of other shocks, indeed, stability in the tax
rate is equivalent to that in the unemployment rate. However, targeting a stable $\omega$ would not prevent the variation of $\omega$, which is subject to many shocks. Instead, the policy could alleviate the original size of risks and mitigate the unfairness due to the inconsistent time frequencies between $\omega$ and $\tau$. For example, when $\omega$ is high in the first half of the year and low in the second half, the constant $\tau$ during the year still brings the same payment to an unemployed person in the first half and another in the second half. They may receive different unemployment benefits according to the $\phi$-rate $\tau(\omega, \delta) = 1 - \omega + \delta \omega$ or alike.

**Theorem 3** The $\phi$-rate $\tau = 1 - \omega + \delta \omega$ minimizes the limit variance of $\sqrt{n}p_{n,\omega}$, i.e.,

$$\arg\min_{\tau} \lim_{n \to \infty} \text{VAR} \left( \sqrt{n}p_{n,\omega} \mid (\theta, \rho, \tau) \in \Omega_{\omega,\delta,n} \right) = 1 - \omega + \delta \omega,$$

and the minimum limit variance is zero.

When the variance of $p_{n,\omega}$ measures its instability, Theorem 3 states that the $\phi$-rate $\tau(\omega, \delta) = 1 - \omega + \delta \omega$ minimizes the asymptotic variance of $p_{n,\omega}$. The limit may bring several potential misunderstandings about the theorem. First, a stable $\sqrt{n}p_{n,\omega}$ is achieved at $n = \infty$ where the limit variance is zero. However, $p_{n,\omega}$ is still exposed to exogenous shocks, such as those studied in Pissarides (1992) and Blanchard (2000). Secondly, $1 - \omega + \delta \omega$ is the limit tax rule as $n \to \infty$. For a large but finite $n$, a small positive number could be added to $1 - \omega + \delta \omega$ to ensure the positivity of $\theta$ and $\rho$. For example, $\tau(\omega, \delta, n) = 1 - \omega + \delta \omega + \frac{2\omega(1-\omega)(1-\delta)^2}{n}$ guarantees the positivity of $n\Delta + \Delta_3$ and, thus, $\theta > 0$ and $\rho > 0$. That small positive number, however, is negligible for a large $n$; thus, we can practically use the rule $\tau(\omega, \delta) = 1 - \omega + \delta \omega$ without any higher-order terms. Thirdly, with a zero or near zero variance in the unemployment rate, labor mobility means one layoff and one new hire almost coincide to ensure the total employment size $s$ remains nearly constant. It also means that the total number of the employed $s$ changes proportionally with the labor market size $n$ so that their ratio $s/n$ remains unchanged. Lastly, although the posterior distribution is skewed, the tax rule minimizes both the overall risk and either one-sided risk of $\sqrt{n}p_{n,\omega}$, as stated in Theorem 4. In particular, a policy-maker’s concern is on the downside risk.

**Theorem 4** As $n \to \infty$, the $\phi$-rate $\tau(\omega, \delta) = 1 - \omega + \delta \omega$ minimizes both the limit lower semivariance and the limit upper semivariance of $\sqrt{n}p_{n,\omega}$ where $(\theta, \rho, \tau) \in \Omega_{\omega,\delta,n}$.

### 4.2 Consistency and Robustness

The above $\phi$-rate captures several striking features of the labor market. Firstly, it is the policy-maker’s best response to the market to stimulate employment within Eq. (8) and Eq. (9). Secondly, we can also derive it by minimizing statistical dispersion measures more robust than the posterior variance or semivariance. Meanwhile, it helps mitigate income inequality. Lastly, the tax policy is predetermined before an economic scenario $S$ occurs. The predetermination reduces the size of the government, which could spend months deciding a tax rate schedule. It also avoids the government’s free-riding on favorite employment scenarios and manipulating bad ones into debt.

The rule $\tau(\omega, \delta) = 1 - \omega + \delta \omega$ is an effective taxation strategy to maximally boost the employment size without undermining equality of opportunity and balance.
of the budget. For an economic policymaker, one primary concern is the forward-looking employment profile \( p_{n,\omega} \). By Theorem 2, the mean of \( p_{n,\omega} \) converges to \( \omega \) as \( n \to \infty \) for any \( \phi \)-rate \( \tau(\omega, \delta) \in (1 - \omega + \delta \omega, 1) \). When \( n \) is finitely large, the mean responds adversely to an increasing \( \tau \) (cf. Theorem 5). Consequently, to maximize the posterior mean, we should minimize the tax rate \( \tau \), while still maintaining the conditions \( \tau(\omega, \delta) \in (1 - \omega + \delta \omega, 1), \theta > 0, \) and \( \rho > 0 \). Thus, the limit of fair tax rates that maximize the mean of \( p_{n,\omega} \) would be \( 1 - \omega + \delta \omega \). Also, the condition \( \omega > 0.5 \) in Theorem 5 is satisfied in a general economy.

Theorem 5 For any \( \omega \in (0.5, 1) \) and a finitely large \( n \), the mean of \( p_{n,\omega} \) reacts negatively to an increasing \( \phi \)-rate \( \tau(\omega, \delta) \in (1 - \omega + \delta \omega, 1) \). As a consequence, 

\[
\lim_{n \to \infty} \operatorname{argmax}_\tau \mathbb{E} [p_{n,\omega} \mid (\theta, \rho, \tau) \in \Omega(\omega, \delta, n)] = 1 - \omega + \delta \omega.
\]

Furthermore, the tax rule also minimizes the mean absolute deviation from the mean (thereafter, MAD) as \( n \to \infty \). For a beta distribution, especially with large parameters, the MAD is a more robust measure of statistical dispersion than the variance, while neither variance nor semivariance is a robust statistic. The MAD for the posterior \( p_{n,\omega} \) is (e.g., Gupta and Nadarajah 2004, page 37):

\[
\mathbb{E} [\left| p_{n,\omega} - \mathbb{E}(p_{n,\omega}) \right|] = \frac{2(\theta + s)^{\theta + s}(\rho + n - s)^{\rho + n - s}}{\beta(\theta + s, \rho + n - s)(\theta + \rho + n)^{\theta + \rho + n}}. \tag{11}
\]

In the next theorem, we identify the same \( \phi \)-rate by minimizing the asymptotic MAD.

Theorem 6 The \( \phi \)-rate \( \tau(\omega, \delta) = 1 - \omega + \delta \omega \) minimizes the limit MAD of \( np_{n,\omega} \), i.e.

\[
\operatorname{argmin}_\tau \lim_{n \to \infty} \mathbb{E} \left[ |np_{n,\omega} - \mathbb{E}(np_{n,\omega})| \mid (\theta, \rho, \tau) \in \Omega(\omega, \delta, n) \right] = 1 - \omega + \delta \omega.
\]

5 Labor Productivity and Equality of Outcome

Individual’s productivity changes with his or her workplace platform. For any \( i \in S \), his or her productivity in \( v(S) \) reflects how consistently and efficiently the individual completes tasks and accomplishes goals. However, the actual performance \( v(S) - v(S\setminus i) \) depends on the scenario of \( S \). When comparing two individuals, one may be more productive than another in some scenarios but not others. Given the size of \( s \), the production \( v(S) \) also varies with the platform \( S \). At a macro level, individual productivity only sometimes aligns with national productivity because \( v(S) - v(S\setminus j) \) does not perfectly correlate with \( v(S) \). However, with our distribution rule of \( v(S) \), the aggregate employment welfare always perfectly aligns with the production \( v(S) \) for a given tax rate \( \tau \). This collective alignment incentivizes the efficiency of the labor market, while the individual’s imperfect incentive compatibility promotes labor mobility and also strengthens the efficiency.

To connect personal productivity and incentives, we introduce a partial ordering in the labor market. For any \( i, j \in \mathbb{N} \) with \( i \neq j \), we say \( i \) uniformly outperforms \( j \) in \( v \) if
With these two inequality conditions, \( i \) has higher marginal productivity than \( j \) in all comparable employment contingencies — either both employed or both unemployed.

As productivity is highly valued in both Eqs. (4) and (5), \( j \) should receive less employment welfare and fewer unemployment benefits than \( i \) does. This is formally claimed in Theorem 7. Also, the theorem never requires the beta-binomial distribution in Eq. (3) as long as \( i \) and \( j \) alone have the same chance of being employed; other players in the labor market may have unequal employment opportunities. Additionally, the theorem is valid for all tax rates, including the special one \( \tau(\omega, \delta) = 1 - \omega + \delta \omega \).

**Theorem 7** If \( i \in \mathbb{N} \) uniformly outperforms \( j \in \mathbb{N} \) in \( v \) and they have EEO, then \( \gamma_i[v] \geq \gamma_j[v] \) and \( \lambda_i[v] \geq \lambda_j[v] \).

We say \( i, j \in \mathbb{N} \) are symmetric in the production function \( v \) if they uniformly outperform each other. By Theorem 7, \( \lambda_i[v] = \lambda_j[v] \) and \( \gamma_i[v] = \gamma_j[v] \) if \( i \) and \( j \) are symmetric and they have equal employment opportunity. In other words, they should receive the same amount of unemployment benefits if both are unemployed and the same amount of employment welfare if both are employed. However, the assumption of bilateral symmetry or uniform outperformance may be overly hypothetical. Indeed, \( j \) does not need to be symmetric with or outperform \( i \) as long as the probability of \( v(S \cup \overline{j} \setminus i) \geq v(S \cup \overline{i} \setminus j) \) is significantly large, or \( j \) is a quick learner, mastering the skill set soon. Education, interest, motivation, and experience contribute to this probability. An excellent job interviewer should be able to estimate the likelihood of these \( S \).

Without further analysis or prior knowledge about the production function, symmetry among the unemployed (or the employed) could be a reasonable a priori assumption to distribute the unemployment benefits (or employment welfare, respectively). For example, the Cobb-Douglas production function is symmetric among all employed persons in the labor market. In the USA, for example, the unemployed receive the same unemployment benefits no matter how much they pay the unemployment insurance premium. Additionally, if we assume symmetry among the employed labor and also assume symmetry among the unemployed labor, then the tax rule \( \tau(\omega, \delta) = 1 - \omega + \delta \omega \) eliminates the income inequality when the income is either employment welfare or unemployment benefits. Theorem 8 affirms this equality of outcome, though an employed individual and an unemployed one may not be symmetric in \( v \).

The theorem does not restrict the size of \( n \) and the specific probability distribution for the EEO. In the \( k \)-out-of-\( n \) redundant system in Section 1, the \( n \) components are thus equally important if they have equal quality. In addition, employed individuals and unemployed individuals may have different probabilities of being employed. This EEO discrepancy results from the market efficiency and helps the efficiency.

**Theorem 8** Assume EEO for the employed in \( \mathbb{N} \) and also EEO for the unemployed in \( \mathbb{N} \). If all employed individuals are symmetric in \( v \) and all unemployed individuals are also symmetric in \( v \), then \( \tau(\omega, \delta) = 1 - \omega + \delta \omega \) if and only if an employee’s employment welfare equals an unemployed person’s unemployment benefits.
6 Other Applications

In an abstract sense, the above account is a fair-division solution for the following game-theoretic setting:

- The players are randomly classified into two groups with equal opportunity.
- The random payoff to be divided comes from one group.

With its numerous players, this type of game finds practical applications in various scenarios. This section delves into four such applications, demonstrating the practicality of the formula derived in the previous sections. We also present a few reliable methods for estimating labor costs.

6.1 Voting Rights

In a voting game (e.g., Shapley 1962), \( v : 2^N \to \{0, 1\} \) is a monotonically increasing set function. Let \( S \) denote the random subset of voters who vote for the proposal. The proposal passes when \( v(S) = 1 \); otherwise, it blocks when \( v(S) = 0 \). However, \( v \) does not mean “production” or alike. No matter the outcome, Hu (2006) describes \( \gamma_i[v] \) as \( i \)’s probability of turning a blocked result into a passed one, and \( \lambda_i[v] \) as \( i \)’s probability of turning a likely passed result into a blocked one. Thus, the sum of \( \lambda_i[v] \) and \( \gamma_i[v] \) quantifies \( i \)’s power in the voting.

The ratio \( \delta \) plays a role in some circumstances. For example, 5% of voters petition to hold a referendum on a proposal, and the number of other supporting voters in the referendum follows a beta-binomial distribution. These petitioners are not in \( S \) because their votes are no longer random. They share \( \delta = .05 \) of the total voting power.

Many voting games are symmetric, for example, in the “One Person, One Vote” rule. In these cases, the equality of outcome becomes an egalitarian allocation of power.

6.2 Health Insurance

Health insurance has two types of policyholders: some are sick and use the insurance to cover their medical expenses; others are healthy and do not use it. Let \( S \) denote the random set of sick policyholders, \( v(S) \) be the total medical expenses with copays deducted, and \( \delta v(S) \) be the surcharge paid to the insurance company. Let \( \delta = -\delta \).

Then, the total expenses \((1 - \delta)v(S)\), except for the copays, are billed to all insurance policyholders.

If \( \tau = 1 - \omega + \delta \omega \) and \( v \) is symmetric among the two types of policyholders, respectively, then by the equality of outcome, the cost to buy the insurance policy would be \( \frac{(1-\delta)v(S)}{n} \) per policyholder. We take expectation on \( \frac{(1-\delta)v(S)}{n} \) because the policyholders pay it upfront. In contrast, unemployment benefits and employment welfare payments come after the production of \( v(S) \) is realized. In practice, the equality of outcome applies and, thus, so does the rule \( \tau = 1 - \omega + \delta \omega \).

In this example, patients also pay the predetermined copays. Since the same copay applies to any patient, no matter how serious their illness, we can set it by the minimum medical cost if anyone gets sick, i.e., \( \mathbb{E} \left[ \min_{i \in N} v(i) \right] \), from a humanitarian perspective.
6.3 Dynamic Highway Toll

The highway I-66 inside the Capital Beltway I-495 of the Washington metropolitan area has enforced a dynamic toll rule during rush hours: a carpool driver pays no toll, but a solo driver pays a dynamic toll \( \xi(n, s) \), say, where \( n \) and \( s \) are the numbers of total drivers and solo drivers, respectively, on the toll road. Also, let \( S \) denote the realized set of solo drivers. In this example, we use the equality of outcome to derive such payments as \( \xi(n, s) \), insurance copay, or labor cost.

Let \( g(n) \) be the average cost per driver on the toll road when the traffic volume is \( n \) cars. It is a nonlinear increasing function of \( n \). An excellent estimate of \( g(n) \) is the expected driving time in hours multiplied by the average hourly pay rate. Then, \( ng(n) \) is the total cost for all drivers, and \( (n-s)g(n-s) \) is the total cost for carpool drivers when there is no solo driver on the toll road. Thus,

\[
v(S) = ng(n) - (n-s)g(n-s) - s\xi(n, s)
\]

is the total over-traffic cost generated by the solo drivers, with tolls deducted.

The production function \( v \) is symmetric among all solo and carpool drivers. By the equality of outcome, each driver shares the same cost \( \frac{v(S)}{n} \). As a carpool driver pays no toll, their shared cost exactly offsets the extra cost caused by the solo drivers, which is \( g(n) - g(n-s) \). Finally, we solve the equation \( \frac{v(S)}{n} = g(n) - g(n-s) \) to get

\[
\xi(n, s) = g(n-s).
\] (12)

Moreover, an administration surcharge \( \delta \) may apply.

The toll in Eq. (12) is based on the equality of outcome per vehicle. We can extend the equality by including carpool passengers. Though each carpool passenger is a free-rider, they also bear the same cost \( \frac{v(S)}{n} \) as the carpool driver. Say, \( c \geq 1 \) is the expected number of passengers in a carpool vehicle, excluding the driver. Then the expected total cost of a carpool vehicle is \( \frac{(c+1)v(S)}{n} \), and the total cost of a solo vehicle is \( \frac{v(S)}{n} + \xi(n, s) \). In this break-even analysis,

\[
\frac{(c+1)v(S)}{n} = \frac{v(S)}{n} + \xi(n, s),
\]

which results in

\[
\xi(n, s) = \frac{c}{n+cs} \left[ ng(n) - (n-s)g(n-s) \right].
\]

6.4 Feature Selection in Machine Learning

In feature or variable selection, we select relevant features (variables, predictors) from a large set of candidate features to model and predict a dependent variable or feature. Say, we have \( n \) candidates, \( X_1, X_2, \ldots, X_n \in \mathbb{N} \), one dependent feature \( Y \), and a fitting or prediction measurement \( v: \mathbb{N} \rightarrow R \). The subset of relevant features \( S \subseteq \mathbb{N} \) is not observable, and we want to estimate it. Each relevant feature explains one or more characteristics of \( Y \); when other features largely account for its explanatory
power, it is unnecessary to be included in S. Without any prior information about these candidates, a selection method could assume that each candidate has an equal opportunity to be in S.

We apply Eqs. (8) and (9) to distribute v(S) for the finite n. First, X_i’s marginal gain is its explanatory or predictive power in Y. Since S is unobservable, we simulate it and use the sample mean of marginal gains, i.e., γ_i[v], to decide whether X_i is relevant. Secondly, we generally add a constant intercept to the model, accounting for v(∅) and the δ share of v(S). A linear regression, for example, uses the average of Y to predict or model Y without involving N. Lastly, irrelevant features should have no explanatory or predictive power and, thus, jointly receive nothing from v(S), i.e.,

\[
\frac{s(\theta + \rho - 1) - n(\theta - 1)}{\theta + s - 1} = 0
\]

from Eq. (9). Since τ is no longer of primary interest, Eqs. (8) and (9) jointly place another identification condition for the unknown hyperparameters (θ, ρ):

\[
1 - \frac{s(\theta + \rho - 1) - n\theta}{\rho + n - s - 1} = \delta + \frac{s(\theta + \rho - 1) - n(\theta - 1)}{\theta + s - 1} = \delta.
\]

Other identifications can also be placed on the beta distribution.

In a concrete selection process, for example, we admit relevant features to S one by one. Each time, we admit at most one candidate from the remaining candidates, and the admitted one should have a significant positive marginal contribution. This statistical significance is not due to the sampling errors in simulating S. Thus, s = 1 or E[|S|] = 1 since we expect to admit one but only take it if the selected feature passes the threshold of positive marginal contribution. Starting with the identified (θ, ρ), we simulate S and estimate γ_i[v] for all X_i ∈ N and admit the feature with the most significant estimated marginal gain. Next, we remove the admitted feature from the candidate set N and update n and (θ, ρ) accordingly. We then admit the second candidate with the most significant estimated marginal gain in the model, which already includes the first admitted feature. We continue this process until no more candidates are significant in the model, including all admitted features. Algorithm 1 details these steps in which ˆS and R contain the admitted and candidate features, respectively.

### 6.5 Labor Costs

Sections 6.3 and 6.2 use equality of outcome and E \[\min_{i \in N} v(i)\], respectively, to calibrate the labor costs or alike, excluded from v(S). We could also look into the demand and supply sides of the labor market. For example, let the labor cost for i ∈ S be the minimum wage requirement for all j ∈ N \ S, who is symmetric with i or uniformly outperforms i in v. That is, some j from N \ S can perform i’s work without compromising the production v. The minimum wage is called reservation wage, below which j is unwilling to work. At this minimum market replacement cost, S can switch i with someone else from N \ S without sacrificing its net profit. Also, the substitute person
Algorithm 1 Feature Selection by a Fair Division Rule

1: \( R \leftarrow \mathbb{N}, \quad \hat{S} \leftarrow \emptyset \)  
\( \triangleright \) admit at most one feature per loop
2: \( \text{while } R \neq \emptyset \text{ do} \)
3: \( n \leftarrow |R| \) and update \( (\theta, \rho) \) by the identification conditions
4: \( \gamma_i[v] \) in the game \((R, v)\) where \( v \) already includes regressors from \( \hat{S} \)
5: \( j \leftarrow \arg \max_{i \in R} \gamma_i[v] \)  
\( \triangleright j \) has largest marginal contribution
6: \( \text{if } \gamma_j[v] \text{ is significantly positive then} \)
7: \( \hat{S} \leftarrow \hat{S} \cup \{X_j\} \)
8: \( R \leftarrow \mathbb{N} \setminus \hat{S} \)  
\( \triangleright \text{bipartition of } \mathbb{N} \text{ into } R \text{ and } \hat{S} \)
9: \( \text{else} \)
10: \( \text{Return } \hat{S} \)
11: \( \text{end if} \)
12: \( \text{end while} \)

is willing to accept (e.g., Horowitz and McConnell 2003). Using the market prevailing minimal wage means that a raise or promotion is necessary whenever the market changes or player \( i \) gains more relevant skills and experience.

To avoid creating an undue disincentive to work, \( j \)'s unemployment benefits must be less than the reservation wage plus employment welfare. Moreover, to ensure every one of \( \mathbb{N} \) stays in the labor market, employment welfare must be bound to the incentive compatibility constraint. This incentive requirement places a lower bound or minimum wage on labor costs while the unemployment benefits cover some unpaid efforts in the job search. Both job-seeking and skill-learning promote market efficiency. Under the symmetry assumption in Theorem 8, the average employment welfare is no better than the average unemployment benefits, according to Corollary 2. With its many advantages mentioned above, the \( \phi \)-rate \( \tau = 1 - \omega + \delta\omega \) is most likely adopted. Thus, when the labor cost is included, an employee is still better off than an unemployed labor force participant.

**Corollary 2** Assume the EEO of Eq. (3). If all employed individuals are symmetric in \( v \) and all unemployed ones are symmetric in \( v \), then the average employment welfare is no more than the average unemployment benefits under any \( \phi \)-rate.

Lastly, labor costs may indirectly affect \( \tau \) when the reserve \( \delta v(S) \) is rigid. When labor costs increase, \( v(S) \) decreases. If \( \delta v(S) \) remains unchanged, then \( \delta \) increases and so does \( \tau = 1 - \omega + \delta\omega \). The copay in the health insurance example negatively affects the cost of buying the insurance policy, which decreases as the copay increases. In addition, the administration could implement a countercyclical fiscal policy by aligning the spending level \( \delta \) with the movement of \( \omega \) such that

\[
\frac{d\tau}{d\omega} = d(1 - \omega + \delta\omega) = -1 + \delta + \omega \frac{d\delta}{d\omega} > 0.
\]

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7 Conclusions

This paper derives a fair-division solution to allocate the unemployment benefits in an economy where the heterogeneous-agent production function is almost unknown, and the labor market consists of two random groups. We interpret “fairness” as equal employment opportunity and model it by a beta-binomial probability distribution. Our “sustainability” is meant to be free of debt and surplus in the taxation budget, holding the government accountable for the debt it creates. To justify the value of unemployed labor, we capitalize on the dichotomous valuation in Hu (2006, 2020).

The valuation approach specifies how much of the net production is retained with employed labor and what portion is distributed to the unemployed. Finally, we postulate that the labor market is static and identify a sustainable tax policy by minimizing the asymptotic variance of the posterior employment rate. The policy rule can also be uniquely determined by minimizing the asymptotic posterior mean of the unemployment rate, the downside risk of the posterior employment rate, or the posterior mean absolute deviation. Additionally, the tax rule is simple enough for practical use. It motivates the unemployed to seek employment and the employed to improve productivity – besides, additional symmetry assumption on productivity results in equality of outcome.

One could extend this framework in several ways. For example, we can re-specify the probability distribution of EEO by any of the following re-specifications. First, we can replace the two-parameter beta distribution with a four-parameter one or a beta rectangular distribution. Secondly, we can let $\theta$ and $\rho$ be some functions of other unknown parameters with sound economic meaning. Thirdly, we could substitute the beta-binomial distribution with a Dirichlet-multinomial or beta-geometric distribution. Alternatively, we could randomize $p$ without involving $(\theta, \rho)$ by generating two independent three-parameter Gamma random variables $X$ and $Y$. Then, the ratio $X/(X+Y)$ is a beta random variable. In any of the four cases, however, we need additional identification restrictions to precisely figure out a $\tau(\omega, \delta)$.

Similarly, we could apply other identification schemes or minimize other objective functions to find a unique $\phi$-rate. From a statistical viewpoint, one could try the maximum likelihood estimation using monthly, weekly, or daily data before determining the policy tax rate, minimize the ex-ante risk of $\omega$, or apply the statistical methods mentioned in Section 3.2. Alternatively, we could bind a good property to the skewness, kurtosis, or entropy of the posterior rate $p_{n, \omega}$. Economically, one could minimize the Gini coefficient of the beta distribution of $p_{n, \omega}$. Using a strategic game-theoretic approach, one could seek a bargaining solution from the feasible set $\Omega_{n, \omega, \delta}$; this solution is particularly useful when $n$ is small. Finally, a policymaker could treat the reserve ratio $\delta$ as endogenous, for example, letting it be an increasing function of $\omega$. They could also place a heavier weight on the marginal gain than the marginal loss to stimulate employment.

The simple static model, however, ignores several essential aspects of a real labor market. First, it better captures the dynamic features of income inequality and its rational response to tax rules. Secondly, while preventing the fungibility of borrowing funds from the future reduces the risk of a government administration piling up national debt, it impairs that administration’s ability (primarily monetary policy) to
intervene in the economy. The government, however, can still moderately stimulate the economy during a recession by adjusting the reserve ratio \( \delta \). Thirdly, the postulation of labor market efficiency neglects the recent development of the incomplete-market theory (e.g., Magill and Quinzii 1996). Also, a multi-criteria objective function may be a viable alternative to the minimum-variance one, especially when there is a high unemployment rate of \( 1 - \omega \) or a large \( \delta \). Lastly, a single tax rule \( \tau(\omega, \delta) \) could overly simplify the complexity of the taxation system, which is also affected by other determinants. These are just a few challenges our framework introduces that require further development.

Finally, the fair and sustainable tax policy studied here has solid theoretical underpinnings. It also has the added advantages of being simple for practical use, compatible with productivity and employment incentives, and robust to various objectives. When applying this framework to a real fair-division problem, one should also consider the benefits of alternative probability distributions for equal opportunity, alternative objective functions, alternative restrictions, and dynamic thinking.

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Appendix A  Proof of Theorem 1

In this proof, we use the following relations about beta functions:

\[
\begin{align*}
\beta(x-1, y+1) &= \frac{y}{x} \beta(x, y), \quad x > 1, \quad y > 0; \\
\beta(x+1, y-1) &= \frac{x}{y-1} \beta(x, y), \quad x > 0, \quad y > 1.
\end{align*}
\]

First, the expected aggregate marginal gain and loss are:

\[
\begin{align*}
E \left[ \sum_{i \in S} (v(S) - v(S \setminus i)) \right] &= \sum_{i \in S} \mathbb{P}(S = S) \sum_{i \in S} [v(S) - v(S \setminus i)] \\
&= \sum_{i \in S} \mathbb{P}(S = S) [v(S) - v(S \setminus i)] = \sum_{i \in S} \gamma_i[v], \\
E \left[ \sum_{i \in \mathbb{N} \setminus S} (v(S \cup i) - v(S)) \right] &= \sum_{i \in \mathbb{N} \setminus S} \mathbb{P}(S = S) \sum_{i \in \mathbb{N} \setminus S} [v(S \cup i) - v(S)] \\
&= \sum_{i \in \mathbb{N} \setminus S} \mathbb{P}(S = S) [v(S \cup i) - v(S)] = \sum_{i \in \mathbb{N}} \lambda_i[v].
\end{align*}
\]
Next, by Eqs. (3)–(5), we re-write the expected marginal gain and loss as:

\[
\gamma_i[v] = \sum_{S \subseteq N \cap S \ni i} \frac{\beta(\theta + s, \rho + n - s)}{\beta(\theta, \rho)} [v(S) - v(S \setminus i)] \\
Z = S \ni Z \quad (A.1)
\]

\[
\lambda_i[v] = \sum_{Z \subseteq N \setminus i} \frac{\beta(\theta + z, \rho + n - z)}{\beta(\theta, \rho)} [v(Z) - v(Z)]
\]

By Eq. (A.1), the aggregate value of the employed labor is

\[
\sum_{i \in N} \gamma_i[v] = \sum_{i \in N} \sum_{S \subseteq N \cap S \ni i} \frac{\beta(\theta + s, \rho + n - s)}{\beta(\theta, \rho)} v(S) - \sum_{i \in N} \sum_{Z \subseteq N \setminus i} \frac{\beta(\theta + z, \rho + n - z)}{\beta(\theta, \rho)} v(Z)
\]

Also by Eq. (A.1), the aggregate value of the unemployed labor is

\[
\sum_{i \in N} \lambda_i[v] = \sum_{i \in N} \sum_{S \subseteq N \cap S \ni i} \frac{\beta(\theta + s, \rho + n - s)}{\beta(\theta, \rho)} v(S) - \sum_{i \in N} \sum_{Z \subseteq N \setminus i} \frac{\beta(\theta + z, \rho + n - z)}{\beta(\theta, \rho)} v(Z)
\]
Appendix B  Proof of Lemma 1

For simplicity, we introduce the following shorthands:

\[
\begin{align*}
\Delta_5 &\equiv -\delta \omega + \delta \tau - 2\delta + \omega - \tau^2 + 4\tau - 2 \equiv \Delta_2 + \Delta_4, \\
\Delta_6 &\equiv -\omega \delta + \omega \tau + \tau - 1 \equiv \Delta_2 + \omega \Delta_3, \\
\Delta_7 &\equiv -\delta - \omega \tau + 2\tau + \omega - 1 \equiv \Delta_4 + (1 - \omega)\Delta_3, \\
\Delta_8 &\equiv -\delta \omega - \delta + \omega + 3\tau - 2 \equiv \Delta_3 + \Delta_5, \\
\Delta_9 &\equiv -2\delta \omega - \delta + 2\omega + 4\tau - 3 \equiv \Delta + \Delta_9.
\end{align*}
\]

All these are bounded because \(0 \leq \omega, \tau, \delta \leq 1\). With \(s = n\omega\), we re-write Eqs. (8) and (9) as a linear system of unknowns \((\theta, \rho)\):

\[
\begin{align*}
(1 - \tau)(\rho + n - s - 1) &= s(\theta + \rho - 1) - n\theta, \\
(\tau - \delta)(\theta + s - 1) &= s(\theta + \rho - 1) - n(\theta - 1).
\end{align*}
\]

As a consequence, the symbolic solution of \((\theta, \rho)\) is unique. We only need to verify that Eq. (10) satisfies the above linear system. Assuming Eq. (10), we have the following identities, also used in other proofs:

\[
\begin{align*}
\theta + s &= n^2\omega\Delta_1 + n\omega n(\Delta_3 + \Delta_5), \\
\theta + s - 1 &= n^2\omega n\Delta_1 + n^2\omega n\Delta_3 = n^2\omega n(\Delta_3 - \Delta_1), \\
\theta + \rho &= n^2\omega n\Delta_1 + n^2\omega n\Delta_3 + n\omega(1 - \omega)\Delta_1 - n^2\omega n\Delta_3, \\
\theta + \rho - 1 &= n^2\omega n\Delta_1 + n^2\omega n\Delta_3 + \omega^2(1 - \omega)\Delta_1 + n\omega(1 - \omega)\Delta_3, \\
\theta + \rho + n - 1 &= n^2\omega n\Delta_1 + n^2\omega n\Delta_3 - 2n\omega(1 - \omega)\Delta_3, \\
\theta + \rho + n - 2 &= n^2\omega n\Delta_1 + n^2\omega n\Delta_3 + n\omega(1 - \omega)\Delta_3, \\
\theta + \rho + n - 3 &= n^2\omega n\Delta_1 + n^2\omega n\Delta_3 + n\omega(1 - \omega)\Delta_3, \\
\theta + \rho + n - 4 &= n^2\omega n\Delta_1 + n^2\omega n\Delta_3 + n\omega(1 - \omega)\Delta_3.
\end{align*}
\]

Thus,

\[
\begin{align*}
s(\theta + \rho - 1) - n\theta &= n\omega(n^2\omega\Delta_1 + n\omega n(\Delta_3 + \Delta_5)), \\
&= n\omega(n^2\omega\Delta_1 + n\omega n(\Delta_3 + \Delta_5)), \\
&= n\omega(n^2\omega\Delta_1 + n\omega n(\Delta_3 + \Delta_5)), \\
&= n\omega(n^2\omega\Delta_1 + n\omega n(\Delta_3 + \Delta_5)), \\
&= n\omega(n^2\omega\Delta_1 + n\omega n(\Delta_3 + \Delta_5)), \\
&= n\omega(n^2\omega\Delta_1 + n\omega n(\Delta_3 + \Delta_5)).
\end{align*}
\]

Therefore,

\[
\begin{align*}
\frac{s(\theta + \rho - 1) - n\theta}{s(\theta + \rho - 1) - n(\theta - 1)} &= n^2\omega n(1 - \tau) + n\omega(1 - \omega)(\theta - \delta), \\
&= 1 - \tau, \\
\frac{n^2\omega n(1 - \tau) + n\omega(1 - \omega)(\theta - \delta)}{n^2\omega n(1 - \tau) + n\omega(1 - \omega)(\theta - \delta)} &= \tau - \delta.
\end{align*}
\]
Appendix C  Proof of Theorem 2

For any integer \( z \geq 0 \), by the proof of Lemma 1, as \( n \to \infty \),

\[
\theta + s + z = \frac{n^2 \omega + n \Delta_1 + \Delta_3}{n \Delta + \Delta_3} + \frac{z(n \Delta + \Delta_3)}{n \Delta + \Delta_3} = n^2 \omega + n(\Delta_6 + z \Delta) + (1 + z)\Delta_3 \to \omega.
\]

Section 2.1 shows that \( p_{n,\omega} \) has a beta distribution with parameters \( (\theta + s, \rho + n - s) \). Thus, the characteristic function of \( p_{n,\omega} \) (e.g., Johnson et al. 1995, Chapter 21) is

\[
E \left[ e^{i\eta p_{n,\omega}} \right] = 1 + \sum_{k=1}^{\infty} \frac{(i\eta)^k}{k!} \prod_{z=0}^{k-1} \frac{\theta + s + z}{\theta + \rho + n + z}
\]

where \( i \) is the unit imaginary number, i.e., \( i^2 = -1 \). We let \( n \to \infty \),

\[
\lim_{n \to \infty} E[e^{i\eta p_{n,\omega}}] = 1 + \sum_{k=1}^{\infty} \frac{(i\eta)^k}{k!} \lim_{n \to \infty} \prod_{z=0}^{k-1} \frac{\theta + s + z}{\theta + \rho + n + z} = 1 + \sum_{k=1}^{\infty} \frac{(i\eta\omega)^k}{k!} = \exp(i\eta\omega).
\]

Therefore, as \( n \to \infty \), \( p_{n,\omega} \) converges in distribution to the degenerate distribution with mass at \( \omega \), which has the characteristic function \( \exp(i\eta\omega) \).

Appendix D  Proof of Theorem 3

To analyze the asymptotic approximation between the functions \( f(n) \) and \( g(n) \) when \( n \) is large, we say \( f(n) = O \left( g(n) \right) \) if \( \limsup_{n \to \infty} \frac{|f(n)|}{g(n)} < \infty \), and \( f(n) \approx g(n) \) if \( \lim_{n \to \infty} \frac{f(n)}{g(n)} = 1 \).

Since \( p_{n,\omega} \) has a beta distribution with parameters \( (\theta + s, \rho + n - s) \), its variance is \( \frac{(\theta + s)(\rho + n - s)}{(\theta + \rho + n)^2(\rho + n + 1)} \) (e.g., Gupta and Nadarajah 2004, page 35). By the proof of Lemma 1, \( \text{VAR}(\sqrt{n} p_{n,\omega}) \) equals

\[
\frac{n^2 \omega + n \Delta_1 + \Delta_3}{n \Delta + \Delta_3} \frac{n^2 (1 - \omega) + n \Delta_2 + \Delta_3}{n \Delta + \Delta_3} = \frac{n(n \Delta + \Delta_3)(n^2 \omega + n \Delta_1 + \Delta_3)[n^2 (1 - \omega) + n \Delta_2 + \Delta_3]}{(n^2 + n \Delta_1 + 2 \Delta_3)(n^2 + n \Delta_2 + 3 \Delta_3)}
\]

\[
= \frac{(\Delta_1 + \Delta_3)}{\omega(1 + \frac{\Delta_1}{n^2} + \frac{\Delta_3}{n^2})} \left[ (1-\omega)(1 + \frac{\Delta_1}{n^2}) \right] \left( 1 + \frac{\Delta_1}{n^2} + \frac{\Delta_3}{n^2} \right)
\]

\[
= \omega(1 - \omega)(\Delta_1 + \Delta_3) \left[ 1 + \frac{\Delta_1}{n^2} + \frac{\Delta_3}{n^2} \right] - \frac{2 \Delta_1}{n^2} - \frac{\Delta_3}{n^2} + O \left( \frac{1}{n^2} \right)
\]

\[
= \omega(1 - \omega) \Delta + \frac{(1 - \omega)}{n} \left[ \Delta_3 + \Delta(\Delta_1 + \frac{\Delta_3}{\Delta_1} - 2 \Delta_2 - \Delta_3) \right] + O \left( \frac{1}{n^2} \right) \to \omega(1 - \omega) \Delta
\]

as \( n \to \infty \). To minimize \( \lim_{n \to \infty} \text{VAR}(\sqrt{n} p_{n,\omega}) = \omega(1 - \omega) \Delta \) while maintain

\( \lim_{n \to \infty} \text{VAR}(\sqrt{n} p_{n,\omega}) \geq 0 \), we have to set \( \Delta = 0 \). Thus, argmin \( \lim_{n \to \infty} \text{VAR}(\sqrt{n} p_{n,\omega}) = 1 - \omega + \delta \omega \).
Appendix E  Proof of Theorem 4

Let \( \mu_n = \frac{\theta + s}{\theta + \rho + n} \) and \( f(x) = \frac{e^{\theta + s - 1}(1-x)^{\theta + n - s - 1}}{\beta(\theta + s, \rho + n - s)} \), \( 0 \leq x \leq 1 \), be the mean and probability density function of \( p_{n,\omega} \), respectively. Also, let \( \sigma_n^2 \) be the variance of \( p_{n,\omega} \) and set \( \alpha_n = \sqrt{\frac{\omega (1 - \omega)}{n}} > 0 \). The lower semivariance of \( p_{n,\omega} \) is defined as

\[
\sigma_n^2 \equiv \int_0^{\mu_n} f(x)(x - \mu_n)^2 dx
\]

and \( n \sigma_n^2 \) is the lower semivariance of \( \sqrt{n}p_{n,\omega} \). We use it in the Chebychev inequality of

\[
P\left( p_{n,\omega} \leq \mu_n - \alpha_n \right) = \int_0^{\mu_n - \alpha_n} f(x) dx \leq \int_0^{\mu_n - \alpha_n} f(x) \left( \frac{x - \mu_n}{\alpha_n} \right)^2 dx = \frac{\sigma_n^2}{\alpha_n^2} = \frac{n \sigma_n^2}{\omega (1 - \omega) \Delta}.
\]

(A.2)

Let \( \kappa_n = \frac{\theta + s - 1}{\theta + \rho + n - 2} \) and \( \varepsilon_n \) be the mode and median of \( p_{n,\omega} \), respectively. The mode \( \kappa_n \) maximizes the density function \( f(x) \). As the median lies between the mean and mode,

\[
|\mu_n - \varepsilon_n| \leq |\mu_n - \kappa_n| = \left| \frac{\theta + s}{\theta + \rho + n} - \frac{\theta + s - 1}{\theta + \rho + n - 2} \right| = \left| \frac{n + p - \theta - 2s}{(\theta + \rho + n)(\theta + \rho + n - 2)} \right| = O(\frac{1}{n})
\]

by the identities in Appendix B. In the following lower-bound estimation, we use Gamma function \( \Gamma(-1) \) and its Stirling’s approximation \( \Gamma(z + 1) \approx \sqrt{2\pi} z^z e^{-z} \).

\[
P\left( p_{n,\omega} \leq \mu_n - \alpha_n \right) = \int_{\mu_n - \alpha_n}^{\varepsilon_n} f(x) dx \geq \frac{1}{2} - \left( \frac{\varepsilon_n - \alpha_n}{\alpha_n} \right) f(\kappa_n) = \frac{1}{2} - \left[ \alpha_n + O\left(\frac{1}{n}\right) \right] \left( \frac{(\theta + s - 1)(\theta + n - s - 1)}{(\theta + \rho + n)(\theta + \rho + n - 2)} \right)^{\theta + s - 1}(1 - \frac{\rho + \rho - 1}{\theta + \rho + n - 1})^{\rho + n - s - 1}
\]

\[
= \frac{1}{2} - \left[ \alpha_n + O\left(\frac{1}{n}\right) \right] \left( \frac{(\theta + s - 1)(\theta + n - s - 1)}{(\theta + \rho + n)(\theta + \rho + n - 2)} \right)^{\theta + s - 1}(1 - \frac{\rho + \rho - 1}{\theta + \rho + n - 1})^{\rho + n - s - 1}
\]

\[
= \frac{1}{2} - \left[ \alpha_n + O\left(\frac{1}{n}\right) \right] \left( \frac{\omega (1 - \omega)}{n} \right) + O\left(\frac{1}{n}\right) \left( \frac{\omega (1 - \omega)}{n} \right)^{\theta + s - 1}(1 - \frac{\rho + \rho - 1}{\theta + \rho + n - 1})^{\rho + n - s - 1}
\]

\[
\approx \frac{1}{2} - \left[ \alpha_n + O\left(\frac{1}{n}\right) \right] \left( \frac{\omega (1 - \omega)}{n} \right) + O\left(\frac{1}{n}\right) \left( \frac{\omega (1 - \omega)}{n} \right)^{\theta + s - 1}(1 - \frac{\rho + \rho - 1}{\theta + \rho + n - 1})^{\rho + n - s - 1}
\]

Finally, we re-write Eq. (A.2) as

\[
\left( \frac{1}{2} - \frac{1}{\sqrt{2\pi}} \right) \omega (1 - \omega) \Delta + O\left(\frac{1}{n}\right) \leq n \sigma_n^2 \leq n \sigma_n^2 = \omega (1 - \omega) \Delta + O\left(\frac{1}{n}\right).
\]

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Letting \( n \to \infty \), we get

\[
\left( \frac{1}{2} - \frac{1}{\sqrt{2\pi}} \right) \omega (1 - \omega) \Delta \leq \liminf_{n \to \infty} n \sigma^2_{\omega} \leq \limsup_{n \to \infty} n \sigma^2_{\omega} \leq \omega (1 - \omega) \Delta.
\]

Therefore, \( \Delta = 0 \) minimizes the limit of lower semivariance of \( \sqrt{n} p_{n,\omega} \).

We can apply similar arguments to the upper semivariance of \( \sqrt{n} p_{n,\omega} \).

**Appendix F  Proof of Theorem 5**

Note that \( \Delta_0 - \omega \Delta_8 = (1 - \tau)(2\omega - 1) + \omega^2 (\delta - 1) \). By the proof of Lemma 1,

\[
E \left[ p_{n,\omega} \mid (\theta, \rho, \tau) \in \Omega(\omega, \delta, n) \right] = \omega + n(\Delta_0 - \omega \Delta_8) \frac{+ (1 - 2\omega) \Delta_3}{n^2 + n \Delta_8 + 2 \Delta_3} = \omega + \omega\Delta + \frac{n^2 \omega + n \delta \omega + \Delta_3}{n^2 + n \Delta_8 + 2 \Delta_3} = \omega + \frac{1}{n^2} + O \left( \frac{1}{n^3} \right).
\]

In the above approximation, the mean reacts negatively with an increasing \( \tau \) when \( n \) is large and \( \omega > .5 \). To maximize the mean, we thus minimize \( \tau \in (1 - \omega + \delta \omega, 1) \) such that \( \theta > 0 \) and \( \rho > 0 \). It is easy to verify that \( \theta > 0 \) and \( \rho > 0 \) when \( \tau = 1 - \omega + \delta \omega + \frac{2\omega(1 - \omega)(1 - \delta)^2}{n} \) and \( n \) is large enough, i.e.

\[
1 - \omega + \delta \omega \leq \argmax_{\tau} E \left[ p_{n,\omega} \mid (\theta, \rho, \tau) \in \Omega(\omega, \delta, n) \right] \leq 1 - \omega + \delta \omega + \frac{2\omega(1 - \omega)(1 - \delta)^2}{n}.
\]

Finally, let \( n \to \infty \) in the above inequalities to get

\[
\lim_{n \to \infty} \argmax_{\tau} E \left[ p_{n,\omega} \mid (\theta, \rho, \tau) \in \Omega(\omega, \delta, n) \right] = 1 - \omega + \delta \omega.
\]

By the way, we could also use the derivative of \( n^2 \omega + n \Delta_3 + 2 \Delta_3 \) to get

\[
\frac{\partial E \left[ p_{n,\omega} \mid (\theta, \rho, \tau) \in \Omega(\omega, \delta, n) \right]}{\partial \tau} = \frac{(n \Delta_0 - (n^2 \omega)) (n \frac{\partial \Delta_8}{\partial \tau}) + O(n^2)}{(n^2 + n \Delta_8 + 2 \Delta_3)^2} = \frac{1 - 2\Delta_8}{n^2} + O \left( \frac{1}{n^3} \right)
\]

which is negative for \( \omega > .5 \) and any large \( n \).

**Appendix G  Proof of Theorem 6**

When \( n \) is large, \( (\theta, \rho, \tau) \in \Omega(\omega, \delta, n) \) implies \( \Delta > 0 \), based on Lemma 1, \( \Delta_1 > 0 \), and \( \Delta_3 < 0 \). Thus, by the proof of Lemma 1, both \( \theta + s \to \infty \) and \( \rho + n - s \to \infty \) as \( n \to \infty \). Applying Stirling’s formula, Johnson et al. (1995, page 219) derive the following approximation for the ratio of the variance and the squared MAD around the mean:

\[
\lim_{\theta + s \to \infty, \rho + n - s \to \infty} \frac{(E \left[ \left( \sqrt{\frac{n}{\pi}} \right) \right])^2}{\text{VAR} \left( p_{n,\omega} \right)} = \frac{2}{\pi}
\]

Thus, minimizing the MAD around the mean is equivalent to minimizing the variance of \( p_{n,\omega} \) when \( n \) is large. By Theorem 3, we have proved Theorem 6.
Appendix H  Proof of Theorem 7

Since \(i\) uniformly outperforms \(j\) and they have EEO, \(\gamma_j[v]\) equals

\[
\sum_{T \subseteq \mathbb{N}, i \notin T} \mathbb{P}(S = T)[v(T) - v(T \setminus j)]
\]

\[
= \sum_{T \subseteq \mathbb{N}, i \notin T} \mathbb{P}(S = T)[v(T - v(T \setminus j)] + \sum_{T \subseteq \mathbb{N}, j \notin T} \mathbb{P}(S = T)[v(T) - v(T \setminus j)]
\]

\[
= \mathbb{P}(S = Z \setminus j) \mathbb{E}(v(Z \setminus j)) + \mathbb{E}(v(Z))
\]

\[
= \mathbb{P}(S = Z \setminus j) \mathbb{E}(v(Z \setminus j)) + \mathbb{E}(v(Z))
\]

\[
= \mathbb{P}(S = Z \setminus j) \mathbb{E}(v(Z \setminus j)) + \mathbb{E}(v(Z))
\]

The beta-binomial distribution in Eq. (3) is not required in the above. Also, the EEO is not required for other players in \(\mathbb{N}\), except that \(\mathbb{P}(S = Z \setminus j) = \mathbb{P}(Z \setminus j)\) for all \(Z \subseteq \mathbb{N} \setminus j\), which implies that \(i\) and \(j\) have equal chance to be hired by \(Z\), when both are unemployed. Similarly, \(\lambda_j[v]\) equals

\[
= \sum_{T \subseteq \mathbb{N}, i \notin T} \mathbb{P}(S = T)[v(T \setminus j) - v(T)]
\]

\[
= \sum_{T \subseteq \mathbb{N}, j \notin T} \mathbb{P}(S = T)[v(T - v(T \setminus j)] + \sum_{T \subseteq \mathbb{N}, i \notin T} \mathbb{P}(S = T)[v(T) - v(T \setminus j)]
\]

\[
= \mathbb{P}(S = Z \setminus j) \mathbb{E}(v(Z \setminus j)) + \mathbb{E}(v(Z))
\]

\[
= \mathbb{P}(S = Z \setminus j) \mathbb{E}(v(Z \setminus j)) + \mathbb{E}(v(Z))
\]

\[
= \mathbb{P}(S = Z \setminus j) \mathbb{E}(v(Z \setminus j)) + \mathbb{E}(v(Z))
\]

\[
= \mathbb{P}(S = Z \setminus j) \mathbb{E}(v(Z \setminus j)) + \mathbb{E}(v(Z))
\]

In the above arguments, we use the identity \(\mathbb{P}(S = Z \setminus j) = \mathbb{P}(S = Z \setminus j)\) for any \(Z \subseteq \mathbb{N}\) such that \(i, j \in Z\). This identity states both \(i\) and \(j\) have equal opportunity to be laid off from \(Z\), when both are employed in \(Z\).

Appendix I  Proof of Theorem 8

We distribute \((1 - \tau)v(S)\) to \(S\) as employment welfare and \((\tau - \delta)v(S)\) to \(\mathbb{N} \setminus S\) as unemployment benefits. When the employed individuals are symmetric in \(v\) and they have EEO, each employed person receives \(\frac{(1 - \tau)v(S)}{s} = \frac{(1 - \tau)v(S)}{n}(\frac{n - s}{n})\) as his or her employment welfare, according to Theorem 7. Similarly, when all unemployed individuals are symmetric in \(v\) and they have EEO, each of them receives \(\frac{(\tau - \delta)v(S)}{n - s} = \frac{(\tau - \delta)v(S)}{n(1 - \omega)}\), as his
or her unemployment benefits. Therefore,

\[
\frac{(1 - \tau)v(S)}{s} = \frac{(\tau - \delta)v(S)}{n - s}
\]

is equivalent to \(\frac{1 - \tau}{\omega} = \frac{\tau - \delta}{1 - \omega}\), which itself is equivalent to \(\tau = 1 - \omega + \delta\omega\).

**Appendix J  Proof of Corollary 2**

By Corollary 1, any \(\phi\)-rate \(\tau \geq 1 - \omega + \delta\omega\). Thus, \(1 - \tau \leq \omega (1 - \delta)\) and

\[
(1 - \tau)(1 - \omega) \leq \omega(1 - \delta)(1 - \omega) = \omega(1 - \omega + \delta\omega - \delta) \leq \omega (\tau - \delta).
\]

Therefore, \(\frac{1 - \tau}{\omega} \leq \frac{1 - \delta}{1 - \omega}\) and \(\frac{(1 - \tau)v(S)}{s} \leq \frac{(\tau - \delta)v(S)}{n - s}\). The rest arguments follow the proof of Theorem 8 in Appendix I.

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