ON $n$-CUBIC PYRAMID ALGEBRAS

Abstract. In this paper we study a class of algebras having $n$-dimensional pyramid shaped quiver with $n$-cubic cells, which we call $n$-cubic pyramid algebras. This class of algebras includes the quadratic dual of the basic $n$-Auslander absolutely $n$-complete algebras introduced by Iyama. We show that the projective resolution of the simples of $n$-cubic pyramid algebras can be characterized by $n$-cuboids, and prove that they are periodic. So these algebras are almost Koszul and $(n - 1)$-translation algebras.

1. Introduction

Quivers, especially translation quivers, are very important in the representation theory of algebras [1, 3, 21]. There are many algebras, such as Auslander algebras, preprojective algebras, related translation quivers [1, 11, 15]. They are widely used in many mathematical fields, such as Cohen-Macaulay modules, cluster algebras, Calabi-Yau algebras and categories, non-commutative algebraic geometry and mathematical physics [3, 5, 19, 20, 10, 5, 2]. Recently, Iyama has developed the higher representation theory [13, 17, 18], where a class of higher translation quivers also plays an important role. In [18], he characterizes a class of higher representation-finite algebras, $n$-Auslander absolutely $n$-complete algebras. We observed that the quivers of such algebras are $n$-dimensional pyramid shaped quivers with $n$-cubic cells. We study in this paper a class of algebras, call $n$-cubic pyramid algebras, defined on such quivers. This class of algebras includes quadratic dual of $n$-Auslander absolutely $n$-complete algebras introduced and studied by Iyama [18], which is a class of $n$-representation finite algebras.

Almost Koszul rings are introduced by Brenner, Butler and King in [7]. One of their main results in the paper is the periodicity of the trivial extensions of representation finite hereditary algebras of bipartite oriented quiver. In fact, they prove that such algebras are almost Koszul of type $(3, h - 1)$ for the Coxeter number $h$ of the quiver of the algebra. Our aim is to generalize a modified version of this result to $n$-cubic pyramid algebras. We proved that a stable $n$-cubic pyramid algebra of height $m \geq 3$ is almost Koszul of type $(n + 1, m - 1)$. Our proof is based on the combinatoric characterization of the projective resolutions of the simple modules of the stable $n$-cubic pyramid algebras, using the integral points on an $n + 1$-cuboid defined on the corresponding vertex in the quiver.

In [11], one of the authors introduce translation algebras as algebras with translation quivers as their quivers and the translation corresponds to an operation related to the Nakayama functor. Such algebras includes the quadratic dual of Auslander algebras, preprojective algebras. In this paper, we using 0-extension from [15], a method for constructing $(n + 1)$-translation algebra using trivial extension followed by a smash product with $\mathbb{Z}$, and a truncation called cuboid truncation to construct $n + 1$-cubic pyramid algebra from $n$-cubic pyramid algebra. By taken quadratic dual of a special case, we recover Iyama’s cone construction of an $(n + 1)$-Auslander

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absolutely \( (n+1)\)-complete algebra from an \( n \)-Auslander absolutely \( n \)-complete algebra in [15]. Observe that the class of trivial extensions of hereditary algebras of bipartite oriented quiver is the same as the trivial extensions of the quadratic dual of the hereditary algebras. Our result can be regarded as a higher representation theory version of the result of Brenner, Butler and King [7] in the case \( \Lambda_m \).

In Section 2 we introduce pyramid shaped \( n \)-cubic quivers and corresponding algebras.

2. Pyrimid Shaped \( n \)-cubic Quivers and Related Algebras

Let \( k \) be a field. In this paper, the algebra assumed to be a graded quotient algebra of the path algebra of a locally finite quiver \( Q \) over \( k \), that is, \( \Lambda = kQ/(\rho) = \Lambda_0 + \Lambda_1 + \cdots \), with \( \Lambda_0 \) a direct sum of (possibly, infinite) copies of \( k \) and \( \Lambda \) is generated by \( \Lambda_1 \) over \( \Lambda_0 \), with relation set \( \rho \) (see [15]). So we have a complete set of idempotents \( \{e_i | i \in Q_0 \} \) such that \( \Lambda_0 = \bigoplus_{i \in Q_0} \Lambda_0 e_i \) and the number of arrows from \( i \) to \( j \) is exactly \( \dim_k e_j \Lambda_1 e_i \) for any \( i, j \in Q_0 \).

To introduce pyramid shaped \( n \)-cubic quiver, we first make some convention on the non-negative integral vectors. Given a positive integer \( p \), let \( e_0 = 0 \), \( e^{(p)} = (1, \ldots, 1) \) be the \( p \)-dimensional vector with all the components 1 and let \( e^{(p)}_i = (0, \ldots, 0, 1, 0, \ldots, 0) \) be the \( p \)-dimensional vector with \( t \)-th component 1 and all the other components zero. Write \( e^{(p)}_{ij} = e^{(p)}_{i_1 \ldots i_s} = \sum_{h=1}^{p} e^{(p)}_{i_h} \) for a subset \( J = \{t_1, \ldots, t_s\} \) of \( \{1, \ldots, p\} \), set \( e^{(p)}_t = e^{(p)}_{[t,t+1]} \). Conventionally, we write them as \( e, e_t, e_J \) and \( e_i \) when no confusion will occur.

For a vector denoted by the boldface letter such like \( a_t \) write \( a_s \) for its \( s \)-th component, so \( a = (a_1, \ldots, a_p) \). For a vector \( i = (i_1, \ldots, i_p) \in \mathbb{Z}^p \), write \( i(1) = i + e_1 \), \( i(t) = i - e_{t-1} + e_t \) for \( 2 \leq t \leq p \), and write \( i(t_1, \ldots, t_s) = i(t_1) \cdots (t_s) \). Similarly, write \( (t)i = i - e_t \) and \( (t)i = i + e_{t-1} - e_t \) for \( 2 \leq t \leq p \), and write \( (t_1, \ldots, t_s)i = (t_1) \cdots (t_s)i \). Clearly \( (t)i(t) = i \).

Let \( \mathbb{Z}^+ \) be the set of non-negative integers. For \( a = (a_1, \cdots, a_p) \in \mathbb{Z}^+ \), write \( |a| = \sum_{t=1}^{p} a_t \). Let \( \mathbb{Z}^+_{l} = \{a = (a_1, \cdots, a_p) \in \mathbb{Z}^+ | |a| = l \} \). We need the following subsets of \( \mathbb{Z}^+ \). The vertex set

\[
U^{(p)} = \{u = (u_1, u_2, \cdots, u_p) | 0 \leq a_t \leq 1 \} \subset \mathbb{Z}^+,
\]

and of the unit \( p \)-cube and the vertex sets

\[
U_{l}^{(p)} = \{u = (u_1, u_2, \cdots, u_p) | |u| = l \},
\]

of \( l \)-nets inside the unit \( p \)-cube, for \( 0 \leq l \leq p \).

Fix \( m \geq 3 \) and \( i = (i_1, \ldots, i_p) \in \mathbb{Z}^+ \) satisfying

\[
1 \leq i_t \text{ and } |i| \leq m + s - 1 \text{ for } s = 1, \ldots, p.
\]

Set

\[
b_t(i) = m + p - 1 - |i|, \quad b_t(i) = i_{t-1} - 1, \quad 2 \leq t \leq p + 1,
\]

write \( b_t = b_t(i) \) and \( b = b(i) = (b_1, \ldots, b_{p+1}) \). Let

\[
C^{(p)}(i) = \{(a_1, a_2, \cdots, a_{p+1}) | 0 \leq a_t \leq b_t(i), 1 \leq t \leq p + 1 \} \subset \mathbb{Z}^{p+1}.
\]
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this is the set of integral vertices of a $(p + 1)$-cuboid with sides of length $b_t$ for $1 \leq t \leq p + 1$. We call it the $(p + 1)$-cuboid associated to the vertex $i$. The vertex $C_0^{(p)}(i) = (0, \ldots, 0)$ is called the initial vertex of the $C^{(p)}(i)$. Let

$$C_l^{(p)}(i) = \{a = (a_1, \ldots, a_{p+1}) \in C(i) \mid |a| = l\},$$

for $0 \leq l \leq m - 1$. $C_l^{(p)}(i)$ is a net inside the $(p + 1)$-cuboid formed by integral vertices which can be reached from the initial vertex in $l$ steps. We have that $C_m^{(p)}(i) = \{b = (b_1, \ldots, b_{p+1})\}$, and $C_l^{(p)}(i) = \emptyset$ for $l > m$.

For $i = (i_1, \ldots, i_p) \in \mathbb{Z}^p$ satisfying (1), for each $a = (a_1, \ldots, a_{p+1}) \in \mathbb{Z}^{p+1}$, define

$$\mathbf{v}^i(a) = (i_1 + a_1 - a_2, i_2 + a_2 - a_3, \ldots, i_{p-1} + a_{p-1} - a_p, i_p + a_p - a_{p+1}).$$

For $i = (i_1, \ldots, i_{p+1}) = (i', i_{p+1}) \in \mathbb{Z}^{p+1}$ with $(i_1, \ldots, i_p)$ satisfying (1), write $C^{(p)}(i) = C^{(p)}(i')$. For each $a = (a_1, \ldots, a_{p+1}) \in \mathbb{Z}^{p+1}$, define

$$\mathbf{v}^i(a) = (i_1 + a_1 - a_2, i_2 + a_2 - a_3, \ldots, i_p + a_p - a_{p+1}, i_{p+1} + a_{p+1}).$$

From now on, we fix integers $n \geq 1$ and $m \geq 3$. Define $n$-cubic pyramid quiver $Q(n)$ of height $m$ as the quiver with

vertex set: $Q(n)_0 = \{i = (i_1, \ldots, i_n) \mid 1 \leq i_t \leq s, \sum_{t=1}^{s} i_t \leq m + s - 1, 1 \leq s \leq n\}$

arrow set: $Q(n)_1 = \{\gamma^i_{(t)} : i \rightarrow i(t) \mid 1 \leq t \leq n, i, i(t) \in Q(n)_0\}$. (2)

Call $\gamma^i_{(t)}$ an arrow of type $t$ starting at $i$. In the case of $n = 3$, these quivers look like a pyramid of side length of $m$ built up with cubes:

These quivers are one of those defined in [18] inductively to describe the absolutely Auslander $n$-complete algebras. It is shown in [14] that such quivers are truncation of the McKay quivers of some finite Abelian subgroups of $GL_n(\mathbb{C})$.

For $i = (i_1, \ldots, i_n) \in Q(n)_0$, call the integral vector $a \in \mathbb{Z}^{n+1}$ an $i$-quiver vertex if $\mathbf{v}^i(a) \in Q(n)_0$. Clearly, $\mathbf{v}^i(a)$ is in $Q(n)_0$ if and only if $i_t > a_{t+1} - a_t$ and $a_{s+1} - a_1 \leq m + s - 1 - \sum_{t=1}^{s} i_t$. 
Let \( i \) be a vertex in \( Q(n) \). The \( n \)-cubic cell at \( i \) in \( Q(n) \) is the full subquiver \( H_i^1 \) of \( Q(n) \) with the vertex set the set of the \( i \)-quiver vertices in \( \{ \psi^i(a) \in Q(n) \mid a \in U^{(n)} \} \). The vertex \( \psi^i(e) \) is called the end vertex of \( H_i^1 \) and \( H_i^1 \) is called complete if \( \psi^i(e) \) is a vertex of \( H_i^1 \). \( H_i^1 \) can be regarded as formed by certain vertices and the edges in the unit \( n \)-cube directed from 0 to \( e \). Define \( n \)-hammock \( H_i \) at \( i \) as the full subquiver of \( Q(n) \) with the vertex set the set of the \( i \)-quiver vertices in \( \{ \psi^i(a) \in Q(n) \mid a \in C^{(n)}(i) \} \). The vertex \( \psi^i(b(i)) \) is called the end vertex of \( H_i \) and \( H_i \) is called complete if \( \psi^i(b(i)) \) is a vertex of \( H_i \).

Let

\[
g = (g_1 = 1, g_2, \ldots, g_n)
\]  

be a sequence of linear maps on \( kQ(n) \) such that the restriction on the vertex set of \( g_s \) is only defined on the vertices with \( s \) component \( i_s > 1 \), in this case \( g_s(i) = i - e_s \), or, \( g_s(e_1) = e_{1-e_s} \), and \( g_s(e_1) = 0 \) if \( i_s = 1 \). \( g_s \) restricts to a bijective linear map on \( \psi^i(\Lambda(n)_{t} e_j) \) if \( i_s \neq 1 \neq j_s \). Since there is at most one arrow between each pair of vertices, we have that \( g_s(\gamma^{(t)}_{i1}) = d_{s,t,1} \gamma^{(t)}_{g_s(i)} \) for some \( 0 \neq d_{s,t,1} \in k \) when \( i_s > 1 \) and \( t < s \).

Define a relation set:

\[
\rho^g(n) = \{ \gamma^{(t)}_{i1} \mid i, i(t) \in Q(n), 1 \leq t \leq n \} \cup \{ d_{s,t,1} \gamma^{(t)}_{i1} \gamma^{(s)}_{i1} - \gamma^{(s)}_{i1} \gamma^{(t)}_{i1} \mid i, i(s), i(t)(s) \in Q(n), 1 \leq t < s \leq n \}.
\]  

The relations are of two kind, one is zero relation consisting of arrows of same type, and the other is commutative relation consisting of arrows of two different types.

Let \( \Lambda^g(n) \) be the algebra with bound quiver \((Q(n), \rho^g(n)) \), and we call it a \( n \)-cubic pyramid algebra. \( 1 \)-cubic pyramid algebras are the quadratic duals of the hereditary algebras of quiver of type \( A_m \) with linear orientation.

A path \( p \) of a bound quiver \((Q, \rho)\) is called a bound path if its image in \( kQ/\rho \) is nonzero. Recall that a bound quiver \((Q, \rho)\) is called an \( n \)-translation quiver \([15]\) if there is a bijective map \( \tau : Q_0 \setminus \mathcal{P} \to Q_0 \setminus \mathcal{I} \) for two subsets \( \mathcal{P}, \mathcal{I} \subset Q_0 \) satisfying the following conditions:

1. Any maximal bound path is of length \( n+1 \) from \( ri \) in \( Q_0 \setminus \mathcal{I} \) to \( i \), for some vertex \( i \) in \( Q_0 \setminus \mathcal{P} \).
2. Two bound paths of length \( n+1 \) from \( ri \) to \( i \) are linearly dependent, for any \( i \in Q_0 \setminus \mathcal{P} \).
3. For each \( i \in Q_0 \setminus \mathcal{P} \) and \( j \in Q_0 \), any bound element \( u \) which is linear combination of paths of the same length \( t \leq n+1 \) from \( j \) to \( i \), there is a path \( q \) of length \( n+1-t \) from \( ri \) to \( j \) such that \( uq \) a bound element.
4. For each \( i \in Q_0 \setminus \mathcal{I} \) and \( j \in Q_0 \), any bound element \( u \) which is linear combination of paths of the same length \( t \leq n+1 \) from \( i \) to \( j \), there is a path \( p \) of length \( n+1-t \) from \( j \) to \( ri \) such that \( uq \) a bound element.

\( \tau \) is called the \( n \)-translation of \( Q \), the vertices in \( \mathcal{P} \) are called projective vertices and the vertices in \( \mathcal{I} \) are called injective vertices.

An algebra \( \Lambda \) with bound quiver an \( n \)-translation quiver \((Q, \rho)\) is called an \( n \)-translation algebra if there is an \( q \in \mathbb{N} \cup \{ \infty \} \) such that \( \Lambda \) is \((n+1, q)\)-Koszul in the following sense

1. \( \Lambda_t = 0 \) for \( t > n+1 \), and
2. for each \( i \in Q_0 \), let
The stable quiver of our early example is as follows:

\[ P^0(i) \xrightarrow{f_0} \cdots \xrightarrow{f_q} P^1(i) \xrightarrow{f_0} P^0(i) \xrightarrow{f_0} S(i) = \Lambda_0 e_i \rightarrow 0 \]  

be the first \( q+1 \) terms in a minimal projective resolution of the simple \( S(i) \simeq \Lambda_0 e_i \), then \( P^t(i) \) is generated by its component of degree \( t \) for \( 0 \leq t \leq q \) and the kernel of \( f_q \) is concentrated in degree \( n+1+q \).

**Theorem 2.1.** \((Q(n), \rho^S(n))\) is an admissible \((n-1)\)-translation quiver with \( n-1 \)-translation defined by \( \tau_n : i \rightarrow i - e_n \) if \( i_n > 1 \).

\( \Lambda^S(n) \) is an \((n-1)\)-translation algebra with admissible \((n-1)\)-translation quiver.

The theorem will be proven in the last Section inductively.

Choosing \( g = (g_1, \ldots, g_n) \) such that \( g_s(\gamma^{(t)}) = -\gamma^{(t)} \) for \( t < s \) and for \( i = (i_1, \ldots, i_n) \) with \( i_s > 1 \), the following proposition follows directly.

**Proposition 2.2.** The quadratic dual \( \Lambda^{g_1}(n) \) of \( \Lambda^S(n) \) is the \( n \)-Auslander absolutely \( n \)-complete algebra \( T_m(n)(k) \).

It is easy to see that the set of the projective vertices in \( Q(n) \) is \( P = \{i| i_n = 1\} \), and the set of the injective vertices is \( I = \{i| i = \sum_{t=1}^{n} i_t = m + n - 1\} \).

Let \( m \geq 3 \). Define **stable \( n \)-cubic pyramid quiver** \( \tilde{Q}(n) \) of height \( m \) as the quiver with

vertex set: \( \tilde{Q}(n)_0 = Q(n)_0 \)

arrow set: \( \tilde{Q}(n)_1 = Q(n)_1 \cup \{\gamma^{(n+1)} : i \rightarrow i - e_n | i_n > 1\} \).

The stable quiver of our early example is as follows:

![Diagram](chart.png)

It follows from [14] and [12] that such quiver is a truncation of the McKay quiver of some finite Abelian subgroup of \( SL_{n+1}(\mathbb{C}) \).

Let \( g^{n+1} = (g, g_n) \) for some graded endomorphism \( g' = g_n \) induced by the \( n-1 \)-translation defined in Theorem 2.1, then \( g'(\gamma^{(s)}_{i-e_n}) = d_{n+1,s,i-e_n} \gamma^{(s)}_{i-e_n} \), since there is at most one arrow of each type \( s \) from any vertex of \( Q(n) \). Define relation set

\[
\tilde{\rho}^S(n) = \rho^S \cup \{\gamma^{(n+1)}_{i-e_n} | i_n > 1\} \cup \{d_{n+1,s,i-e_n} \gamma^{(s)}_{i-e_n} | i \in \tilde{Q}(n)_0, i_n > 1, i_{s-1} > 1 \text{ and } \sum_{t=1}^n i_t < m + s - 1, 1 \leq s \leq n\}.
\]

(7)
Let $\hat{\Lambda}(n)$ be the algebra given by the stable $n$-cubic pyramid quiver $\hat{Q}(n)$ and the relation $\hat{p}(n)$. Our key results in this paper, Lemma 3.3 essentially proves the following theorem.

**Theorem 2.3.** $\hat{\Lambda}(n)$ is an $n$-translation algebra with stable $n$-translation quiver and trivial $n$-translation.

Set $\mathcal{R} = \{1, 2, \ldots, n\}$ and $\mathcal{R}^+ = \{1, 2, \ldots, n + 1\}$. Write
\[
\gamma_{w(a)}^{(t_1)} = \gamma_{w(a)}^{(t_1)} \cdots \gamma_{w(a)}^{(t_s)}
\]
for a path determined by a sequences $\{t_1, \ldots, t_s\} \subset \mathcal{R}^+$. For a subset $P = \{p_1, \ldots, p_s\}$ of $\mathcal{R}^+$ with $1 \leq p_1 < \cdots < p_s \leq n + 1$, write
\[
\gamma_{v(a)}^{(P)} = \gamma_{v(a)}^{(p_1 \cdots p_s)}.
\]

We obviously have the following result.

**Lemma 2.4.** Assume that $0 \leq l \leq n + 1$. For any two vertices $i, j \in \hat{Q}(n)_0$, we have that $\dim_{k e_j} \hat{\Lambda}(n)^i e_i \leq 1$, and $\dim_{k e_j} \hat{\Lambda}(n)^i e_i = 1$ if and only if any path of length $l$ from $i$ to $j$ consists of arrows of different types.

So we see that a bound path starting at a vertex $i$ in $\hat{Q}(n)$ is a path on $H^I$ starting from $i$. Since the $n$-translation is defined by $\tau_i = i - e_n$ for each non-projective vertex $i$ in $Q(n)_0$, that is, the ones with $i_n > 1$.

The following theorem follows from Theorem 2.3 and Proposition 4.2 of [15].

**Theorem 2.5.** $\hat{\Lambda}(n)$ is isomorphic to some twisted trivial extension of $\Lambda(n)$.

For any $i \in Q(n)_0$, let $S(i)$ be the simple $\Lambda(n)$-module at $i$ and let $P(i)$ and $I(i)$ be its projective cover and injective envelop as $\hat{\Lambda}(n)$ and $\Lambda(n)$-modules, respectively. Let $\hat{S}(i)$ be the simple $\hat{\Lambda}(n)$-module at $i$ and let $\hat{P}(i)$ and $\hat{I}(i)$ be its projective cover and injective envelop as $\hat{\Lambda}(n)$-modules, respectively. Conventionally, set $\hat{S}(i)$, $\hat{P}(i)$ and $\hat{I}(i)$ to be zero when $i \notin \hat{Q}(n)_0$.

The supports of $\hat{P}(i)$ and $\hat{I}(i)$ are described with $U^{(n+1)}$, the following result follows directly from Lemma 2.6.

**Lemma 2.6.**

\[
\tau^l \hat{P}(i)/\tau^{l+1} \hat{P}(i) \simeq \bigoplus_{a \in U^{(n+1)}_l} \hat{S}(v^l(a)).
\]

Since the components of $a$ are 0 or 1, it is easy to see that $v^l(a)$ is a quiver vertex if and only if the following hold for all $s$: (i). $a_{s+1} \leq a_s$ if $i_s = 1$; (ii). $a_{s+1} \geq a_s$ if $\sum_{t=1}^s i_t = m + s - 1$. A vertex $i$ with $i_1 = 1$ or $\sum_{t=1}^s i_t = m + s - 1$ for some $s$ is called a $\Lambda$ boundary vertex otherwise it is called an internal vertex. Now let $S(i) = \{s \in \mathcal{R} | i_s = 1\}$ and $W(i) = \{s \in \mathcal{R} | \sum_{t=1}^s i_t = m + s - 1\}$.

The set $U^{(n+1)}(i)$ of $i$-quiver vertices in $U^{(n+1)}$ is

\[
U^{(n+1)}(i) = \{a = (a_1, \ldots, a_{n+1}) | a_s \geq a_s + 1 \text{ if } s \in S(i), a_s \leq a_{s+1} \text{ if } s \in W(i)\}.
\]

This implies that if $i_1 = 1$ and $a_{s+1} = 1$, then $a_s = 1$, and if $s \in W(i)$ and $a_s = 1$, then $a_{s+1} = 1$. We can refine Lemma 2.6 as following.
Lemma 2.7. \[
\mathbf{r}^t \tilde{P}(i)/\mathbf{r}^{t+1} \tilde{P}(i) \simeq \bigoplus_{a \in U_t^{(n+1)}(i)} \tilde{S}(\mathbf{v}^i(a)).
\]

Now consider \( \overline{Q}(n) = Z_{(n-1)}Q(n) \) and 0-extension \( \overline{\Lambda}(n)^{\#} = \overline{\Lambda}(n)^{\#} \# \mathbb{Z}^* \), respectively. (See [15]). We have
\[
\begin{align*}
\overline{Q}(n)_0 &= Q(n)_0 \times \mathbb{Z} \\
\overline{Q}(n)_1 &= Q(n)_1 \times \mathbb{Z} \cup \{ \gamma_{(n+1)}^{(n)} : (i, v) \rightarrow (i - e_n, v + 1)(i, v) \in \overline{Q}(n)_0 \}. (8)
\end{align*}
\]

Set \( e_i^{(n+1)} = (e_i^{(n)}, 0) \) for \( 1 \leq t \leq n \) and \( e_i^{(n+1)} = (0, 1) \), then the arrows are \( \gamma_{(n+1)}^{(n)} : i \rightarrow i - e_{t-1} + e_i \) for \( i \in \overline{Q}(n)_0, 1 \leq t \leq n+1 \), and the relations for \( \overline{\Lambda}(n) \) is induced from \( \overline{p}(\mathbf{r}) \). Thus
\[
\overline{p}(\mathbf{r}) = \rho \times \mathbb{Z} \cup \{ \gamma_{i-e_n+e_{n+1}}^{(n+1)}| \gamma_{i}^{(n)} > 1 \} = \bigcup \{ d_{n+1}, s = e_{n-1}, e_i, e_{n+1} | \gamma_{i}^{(n)} > 1 \}
\]

\[
i \in \overline{Q}(n)_0, i_n > 1, i_{s-1} > 1 \text{ and } \sum_{i=1}^{n} i_t < m - 1, 1 \leq s \leq n \}. \]

The following theorem follows from Theorem [2], Theorem 2.3 and Theorem 5.3 of [15].

Theorem 2.8. \( \overline{\Lambda}(n) \) is an \( n \)-translation algebra with stable \( n \)-translation quiver \( \overline{Q}(n) \) and \( n \)-translation \( \tau_n : i = i - e_{n+1} \).

Let \((Q, \rho)\) be a bound quiver and \( Q' \) a full subquiver of \( Q \). Let \( \rho(Q') = \{ e_{j} \sum a_{p}e_{i} | \sum a_{p}p \in \rho, i, j \in Q'_0 \} \), we call \((Q', \rho')\) a full bound subquiver of \( Q \), if for any \( i, j \in Q'_0 \), and for any path \( p \) in \( Q \) with \( a_{p} \neq 0 \) for some \( \sum a_{p}p \in \rho \) and \( e_{j-p} \neq 0 \), then \( p \) is a path in \( Q' \). In this case, we say \( \rho' = \rho(Q') \) is a restriction of \( \rho \) on \( Q' \).

Obviously, we have the following lemma.

Lemma 2.9. Let \( \Lambda = kQ/\rho \) be the algebra with bound quiver \((Q, \rho)\) and let \((Q', \rho')\) be a full bound subquiver of \((Q, \rho)\). Let \( I \) be the ideal generated by the set of idempotents \( \{ e_{j} | j \in Q - Q'_0 \} \). Then
\[
kQ'/\rho' \simeq \Lambda/I.
\]

3. Minimal Projective Resolutions of the Simples of \( \overline{\Lambda}(n) \)

In this section, we study the projective resolution of the simples of the algebra \( \overline{\Lambda}(n) \). Fix \( i \in \overline{Q}(n)_0 \), the following lemma follows from the the definition of \( \mathbf{v}^i(a) \) for \( a \in \mathbb{Z}^{+n+1} \).

Lemma 3.1. For \( 0 \leq l \leq m \), if \( a, a' \in \mathbb{Z}^{+n+1} \), then \( \mathbf{v}^i(a) = \mathbf{v}^i(a') \) if and only if \( a = a' \).

Let \( O(a) = \{ l | a_l = 0 \}, T(a) = \{ 1 \leq t \leq n + 1 | a_t' > b_t \} \) and let \( R(a) = R^+ \setminus (O(a) \cup T(a)) \) for any \( a \in \mathbb{Z}^{+n+1} \). Then \( a \in C^{(n)}(i) \) if and only if \( T(a) = \emptyset \). Let \( \cdots \rightarrow \tilde{P}_2(\tilde{S}(i)) \xrightarrow{f_2} \tilde{P}_1(\tilde{S}(i)) \xrightarrow{f_1} \tilde{P}(\tilde{S}(i)) \xrightarrow{f} \tilde{S}(i) \rightarrow 0 \) (10)
Lemma 3.3. In the projective resolution $\tilde{S}(i)$ of corresponding to the vertex $i \in \tilde{Q}(n)_0$.

Our main aim of this section is characterizing the linear part of this projective resolution using $n+1$-cuboid $C^{(n)}(i)$ as in the following proposition. The technical detail of the proof will be given in Lemma 3.3, for the last assertion, note that $v^l(a + e) = v^l(a)$.

Proposition 3.2. For $l = 0, 1, \ldots, m - 1$

\[
\tilde{P}^l(\tilde{S}(i)) \simeq \bigoplus_{a \in C^{(n)}_l(i)} \tilde{P}(v^l(a)),
\]

and $\text{Ker } f_{m-1} \simeq \tilde{S}(v^l(b(i)))$.

Let \( \{ \epsilon^{(t)}_{v^l(a)} | a \in C^{(n)}_t(i) \} \) be the standard basis of this direct sum, then

\[
\epsilon_{v^l(a)} \epsilon_{v^l(a)}^{(l)} = \epsilon_{v^l(a)}^{(l)} \text{ and } \epsilon_{v^l(a)}^{(l)} = 0 \text{ if } l \neq v^l(a).
\]

We assume that $\epsilon_{v^l(a)}^{(l)} = 0$ when $a \not\in C^{(n)}_t(i)$ or $v^l(a) \not\in \tilde{Q}(n)_0$, for $l = 1, \ldots, m - 1$.

Conventionally, we assume that $\epsilon_{a'}^{(l)} = 0$ if $a' \not\in C^{(n)}(i)$. For each $a \in C^{(n)}_t(i)$, there are elements $\theta_{v^l(a)}^{(l)}$ $k$ for $t = 1, \ldots, n+1$, with $\epsilon_{a-e_t}^{(l)} \neq 0$ if $a - e_t \in C^{(n)}(i)$, let

\[
\theta_{v^l(a)}^{(l)} = \sum_{t=1}^{n+1} \epsilon_{v^l(a-e_t)}^{(l)} \epsilon_{v^l(a-e_t)}^{(l)} \epsilon_{v^l(a-e_t)}^{(l)}.
\]

Then

\[
\epsilon_{v^l(a)} \theta_{v^l(a)}^{(l)} = \theta_{v^l(a)}^{(l)}.
\]

Set

\[
K_i = \{ \theta_{v^l(a)}^{(l)} | a \in C^{(n)}_{l+1}(i) \}.
\]

Lemma 3.3. In the projective resolution $\tilde{P}(i)$, we have

(a) \( \text{Ker } f_l \) hold for $0 \leq l \leq m - 1$;

(b) for $1 \leq l \leq m - 1$, the map $f_l$ is defined by

\[
f_l(\epsilon_{v^l(a)}^{(l)}) = \theta_{v^l(a)}^{(l-1)},
\]

for each $a \in C^{(n)}_l(i, j)$;

(c) \( \text{Ker } f_l \) generates $\text{Ker } f_l$ for $0 \leq l \leq m - 2$;

(d) $C^{(n)}_{m-1}(i) = \{ b \}$, $b + e$ is a $i$-quiver vertex and $\text{Ker } f_{m-1} \simeq S(v^l(b + e))$.

Proof. We now prove (a), (b) and (c) using induction on $l$. The assertions for $\tilde{P}^0(S(i))$, $\text{Ker } f_0$ and $f_1$ are obvious.

Assume that $1 \leq l \leq m - 1$ and the assertions hold for $\tilde{P}^h(i)$, $\text{Ker } f_h$, and $f_h$ for $h < l$. By the inductive assumption, $\text{Ker } f_{l-1}$ is a graded module generated in degree $l$ and as $k$-spaces

\[
(\text{Ker } f_{l-1})_1 = \text{top } (\text{Ker } f_{l-1}) = L(\theta_{v^l(a)}^{(l-1)} | a \in C^{(n)}_l(i)) = \sum_{a \in C^{(n)}_l(i)} k \theta_{v^l(a)}^{(l-1)}.
\]

By (12), we have

\[
k \theta_{v^l(a)}^{(l-1)} \simeq \tilde{A}_0 \epsilon_{v^l(a)}.
\]
as \( \hat{A}_0 \)-modules. Since for each \( a \in C^i_1(i) \), we have exactly one nonzero element \( \theta^{(l-1)}_{v(a)} \) in \( K_{l-1} \). Thus

\[
\text{top}(\text{Ker } f_{l-1}) \simeq \bigoplus_{a \in C^i_1(i)} \hat{A}_0 e_{v(a)}.
\]

Thus

\[
\tilde{P}^l(S(i)) \simeq \bigoplus_{a \in C^i_1(i)} \tilde{P}(v^1(a)).
\]

This proves that (11) holds for \( l \).

Define the homomorphism \( f_l \) from \( \tilde{P}^l(S(i)) \) to \( \tilde{P}^{l-1}(S(i)) \) with

\[
f_l(\epsilon^{(l)}_{v^1(a)}) = \theta^{(l-1)}_{v^1(a)},
\]

Clearly \( f_l \) is an epimorphism from \( \tilde{P}^l(S(i)) \) to \( \text{Ker } f_{l-1} \), and \( f_l(\tilde{P}^l(S(i)))_p \subseteq (\tilde{P}^{l-1}(S(i)))_{p+1} \) for \( p \in \mathbb{Z} \). This proves (11) for \( 1 \leq l \leq m-1 \).

Now we compute generators of \( \text{Ker } f_1 \) for \( l = m-2 \). Clearly (\text{Ker } f_l)_{l-1} = 0 \) and \( \text{Ker } f_l \) for \( l = m-1 \) is \( \text{soc } \tilde{P}(S(i)) \), by the inductive assumption.

Assume that \( 0 \neq x_n \) is a homogeneous element in \( (\text{Ker } f_1)_p \), then we can write

\[
x_{l+1} = \sum_{a \in C^i_1(i)} (\sum_{t=1}^{n+1} \sum_{t'=1}^{n+1} z_{a,t} \gamma_{v^1(a)}^{(t)} \epsilon_{v^1(a)}^{(t)})
\]

Thus

\[
0 = f_l(x_{l+1}) = \sum_{a \in C^i_1(i)} (\sum_{t=1}^{n+1} \sum_{t'=1}^{n+1} z_{a,t} \gamma_{v^1(a)}^{(t)} \epsilon_{v^1(a)}^{(t)}) f_l(\epsilon_{v^1(a)}^{(l)})
\]

\[
= \sum_{a \in C^i_1(i)} (\sum_{t=1}^{n+1} \sum_{t'=1}^{n+1} z_{a,t} \gamma_{v^1(a)}^{(t)} \epsilon_{v^1(a)}^{(l)}) \theta^{(l-1)}_{v^1(a)}
\]

\[
= \sum_{a \in C^i_1(i)} (\sum_{t=1}^{n+1} \sum_{t'=1}^{n+1} z_{a,t} \gamma_{v^1(a)}^{(t)} \epsilon_{v^1(a+e_t)}^{(l)} \gamma_{v^1(a-e_t)}^{(l-1)} \epsilon_{v^1(a-e_t)}^{(l-1)})
\]

\[
= \sum_{a \in C^i_1(i)} (\sum_{t=1}^{n+1} \sum_{t'=1}^{n+1} z_{a,t} \gamma_{v^1(a-e_t)}^{(l)} \epsilon_{v^1(a+e_t)}^{(l-1)} \gamma_{v^1(a-e_t)}^{(l-1)} \epsilon_{v^1(a-e_t)}^{(l-1)})
\]

If \( a' \in \mathbb{Z}^{n+1}_{l+1} \) such that \( a' - e_i, a' - e_{i'} \in C^i_1(i) \) for \( t \neq t' \), then \( 0 \leq a'_t = (a' - e_i)_t \leq b_i, 0 \leq a'_{i'} = (a' - e_{i'}) \leq b_{i'}, \) and \( 0 \leq a_{i'} = (a' - e_i)^{i'} \leq b_{i'} \) for \( t'' \in \{t, t'\}. \) Thus \( a' \in C^i_1(i) \), and we have that \( a' - e_i, a' - e_{i'} \in C^i_1(i) \), then \( a' - e_t, a' - e_{i'} \in C^i_1(i) \), left multiple (16) with \( \epsilon_{v_1(a'_{i'})}^{(t')}, \) we get,

\[
0 = \sum_{1 \leq t < t' \leq n+1, a' - e_t, e_{i'} \in C^i_1(i)} (z_{a' - e_t, e_{i'}}^{(l-1)} \gamma_{v_1(a' - e_t, e_{i'})}^{(l-1)}) (z_{a' - e_t, e_{i'}}^{(l-1)} \epsilon_{v_1(a' - e_t, e_{i'})}^{(l-1)})
\]

Thus we get a system of linear equations

\[
z_{a' - e_t, e_{i'}}^{(l-1)} - z_{a' - e_t, e_{i'}}^{(l-1)} = 0
\]

for \( 1 \leq t < t' \leq n+1 \) with \( a' - e_i, e_{i'} \in C^i_1(i) \).

Let \( t' = l - n - 1 + |O(a')| \). The composition factor \( \tilde{S}(a') \) in the degree \( l+1 \) component of the projective resolution (10) is

\[
0 \rightarrow \epsilon_{v_1(a')}^{(t')} (\text{Ker } f_l)_{l+1} \rightarrow \bigoplus_{a \in C^i_1(i)} \epsilon_{v_1(a')}^{(t')} (\tilde{P}(v_1(a))[l])_{l+1} \rightarrow \cdots
\]

\[
\rightarrow \bigoplus_{a \in C^i_1(i)} \epsilon_{v_1(a')}^{(t')} (\tilde{P}(v_1(a))[l])_{l+1} \rightarrow 0.
\]
Note that
\[
\sum_{a \in C_{l+1}^{(n)}(i)} e_{v(a)}(P(v^l(a))[l'])_{l+1} = \sum_{a \in C_{l+1}^{(n)}(i)} e_{v(a)}P^{l+1-t}((n)/l^{t+2-t}A(n)e_{v(a)}
\]
is the space spanned by the bound paths of length \(l + 1 - t\) ending at \(v^l(a)\). For an path \(p\) ending at \(v^l(a')\), the types of the arrows in \(p\) are in
\[
\{1, \ldots, n, n + 1\} \setminus O(a').
\]
Since bound paths are formed by arrows of different types, and two bound paths are linearly independent if the set of the types of their arrows are different, a maximal linearly independent set of paths of length \(t\) ending at \(v^l(a')\) has \(C_{l+1}^{(n)}(O(v^l(a'))\) elements, that is
\[
\dim_k \bigoplus_{a \in C_{l+1}^{(n)}(i)} e_{v(a)}(P(v^l(a))[l'])_{l+1} = C_{l+1}^{(n)}(O(v^l(a')).
\]
So it follows from (19) that
\[
0 = \dim_k e_{v(a')}(\text{Ker } f_l)_{l+1}
\]
\[+ \sum_{t=1}^{n+1-|O(a')|} (-1)^t \dim_k \bigoplus_{a \in C_{l+1}^{(n)}(i)} e_{v(a')}(P(v^l(a))[l-t])_{l+1}
\]
\[= \dim_k e_{v(a')}(\text{Ker } f_l)_{l+1} + \sum_{t=1}^{n+1-|O(a')|} \left( n + 1 - O(a') \right)
\]
\[= \dim_k e_{v(a')}(\text{Ker } f_l)_{l+1} - 1 + (1 - 1)^{n+1-|O(a')|}.
\]
Thus \(\dim_k e_{v(a')}(\text{Ker } f_l)_{l+1} = 1\), and this is equivalent to that the solution space of (13) is one dimensional. Take a nonzero solution
\[
z_{a'-e_i,t} = \zeta(l)_{a'-e_i,t} \in k,
\]
for \(a' \in C_{l+1}^{(n)}(i)\) with \(a' - e_i - e_j \in C_{l+1}^{(n)}(i)\) for some \(1 \leq l' < l'' \leq n + 1\). Then each \(z_{a'-e_i,t} \neq 0\) if \(a'_t \neq 0\), and we have \(z_{a'-e_i,t} = 0\) for those \(t\) with \(a'_t = 0\).
\[
\theta(l)_{v^l(a')} = \sum_{t=1}^{n+1} \zeta(l)_{a'-e_i,t} \gamma(l)_{v^l(a'-e_i),v^l(a'-e_i)}
\]
(20)
is a nonzero element of \(e_{v(a')}(\text{Ker } f_l)_{l+1}\).

If \(a' \in \mathbb{Z}^{n+1}_{l+1}\) such that \(a' - e_i \in C_{l+1}^{(n)}(i)\) and \(a' - e_i \notin C_{l+1}^{(n)}(i)\) for \(t' \neq t\).

If \(a' \in C_{l+1}^{(n)}(i)\), then \(a'_t = 0\) for \(t' \neq t\). The exact sequence (19) of composition factor \(S(a')\) in the degree \(l+1\) component of the projective resolution (11) becomes
\[
0 \longrightarrow e_{v(a')}(\text{Ker } f_l)_{l+1} \longrightarrow e_{v(a')}(P(v^l(a'-e_i))[l])_{l+1} \longrightarrow 0
\]
Left multiple (13) with \(e_{v(a')}, we get a zero term:
\[
z_{a'-e_i,t} \gamma(l-1)_{v^l(a'),v^l(a'-e_i)} \gamma(l)_{v^l(a'-e_i),v^l(a'-e_i)} = 0.
\]
So we can take \(z_{a'-e_i,t} = 1\) and
\[
\theta(l)_{v^l(a')} = \gamma(l)_{v^l(a'-e_i),v^l(a'-e_i)}
\]
is a nonzero element of \(e_{v(a')(\text{Ker } f_l)}_{l+1}\).

Now we consider the case that \(a' \in \mathbb{Z}^{n+1}_{l+1} \setminus C_{l+1}^{(n)}(i)\) such that \(a' - e_i \in C_{l+1}^{(n)}(i)\) and \(a' - e_i \notin C_{l+1}^{(n)}(i)\) for \(t' \neq t\). Thus \(a'_t = b_i(i) + 1\). Now we show that \(\tilde{S}(v^l(a'))\) is not a composition factor in \(\text{Ker } f_l)_{l+1}\)
Assume that $\mathbf{v}(a') \in \hat{Q}(n)_0$ and consider the composition factor $S(a')$ in the degree $l + 1$ component of the projective resolution $[10]$. Since in this case, each bound path from a vertex $\mathbf{v}(a)$ with $a \in C^{(n)}(i)$ to $\mathbf{v}(a')$ in $\hat{Q}(n)$ is a multiple of a path of same length with the last arrow $\gamma_v^{(l)}(a' - e_{a'})$, so $[10]$ becomes

$$
0 \longrightarrow e_{\mathbf{v}(a')}(\text{Ker} f_l)_{l+1} \longrightarrow \bigoplus_{a \in C^{(n)}(i)} e_{\mathbf{v}(a')}(\gamma_v^{(l)}(a' - e_{a'}))\tilde{P}(\mathbf{v}(a))[l] \longrightarrow \cdots
$$

$$
\longrightarrow \bigoplus_{a \in C^{(n)}(i)} e_{\mathbf{v}(a')}(\gamma_v^{(l)}(a' - e_{a'}))\tilde{P}(\mathbf{v}(a))[l']_{l+1} \longrightarrow 0,
$$

for $l' = l - n - 1 + |O(a')|$. Similar to the argument after $[10]$ we get that $e_{\mathbf{v}(a')}(\text{Ker} f_i)_{l+1} = 0$.

This proves that the set $K_l$ spans the space $(\text{Ker} f_i)_{l+1}$.

Since $f_i$ is degree zero map, Ker $f_i$ is graded. Consider the degree $l + s$ component of $[10]$, for each $2 \leq s \leq n + 1$, we have for each $a' \in \mathbb{Z}^{l+s}$,

$$
0 \longrightarrow e_{\mathbf{v}(a')}(\text{Ker} f_i)_{l+s} \longrightarrow \bigoplus_{a \in C^{(n)}(i)} e_{\mathbf{v}(a')}(\tilde{P}(\mathbf{v}(a))[l])_{l+s} \longrightarrow \cdots
$$

$$
\longrightarrow \bigoplus_{a \in C^{(n)}(i)} e_{\mathbf{v}(a')}(\tilde{P}(\mathbf{v}(a))[l']_{l+s} \longrightarrow 0,
$$

with $l' = l + s - n - 1 + |O(a')|$. Note that $\bigoplus_{a \in C^{(n)}(i)} e_{\mathbf{v}(a')}(\gamma_v^{(l)}(a' - e_{a'}))\tilde{P}(\mathbf{v}(a))[l - s']_{l+s}$ is the space of the bound paths of length $s + s'$ to the vertex $e_{\mathbf{v}(a')}$ in $\hat{Q}(n)_0$. Thus paths formed by arrows of different types are nonzero and the representatives of those with the same set of types form a basis. On and other hand, $a'_t = b_t + 1$ since $a'_t \leq b_t$ for all $t$ and there is a bound path from $\mathbf{v}(a)$ to $\mathbf{v}(a')$, so $|T(a')| \leq s$. So each bound path from $\mathbf{v}(a)$ to $\mathbf{v}(a')$ passes the arrows of all the types in $T(a')$. This implies that a set of the linear independent path to $\mathbf{v}(a')$ is determined by the arrows of types in $R^+ \setminus (O(a') \cup T(a'))$. Thus

$$
\dim_k \bigoplus_{a \in C^{(n)}(i)} e_{\mathbf{v}(a')}(\tilde{P}(\mathbf{v}(a))[l - s'])_{l+s} = \binom{n + 1 - |O(a')| - |T(a')|}{s + s' - |T(a')|}
$$

and

$$
\dim_k e_{\mathbf{v}(a')}(\text{Ker} f_i)_{l+s}
$$

$$
= \sum_{s=0}^{n+1-|O(a')|-|T(a')|-s} (-1)^s \bigoplus_{a \in C^{(n)}(i)} \dim_k e_{\mathbf{v}(a')}(\tilde{P}(\mathbf{v}(a))[l])_{l+s}
$$

$$
= \sum_{s=0}^{n+1-|O(a')|-|T(a')|-s} (-1)^s \binom{n + 1 - |O(a')| - |T(a')|}{s + |T(a')| + s'}
$$

$$
= \binom{n - |O(a')| - |T(a')|}{s - |T(a')| - 1}
$$

For each subset $P$ of $s - |T(a')| - 1$ elements in $R^+ \setminus (O(a') \cup T(a'))$, let

$$
\kappa(P, a) = \gamma_{\mathbf{v}(a')}(P^{(a')}_{\mathbf{v}(a)})_{\mathbf{v}(a)} = \gamma_{\mathbf{v}(a')}(P^{(a')}_{\mathbf{v}(a)})_{\mathbf{v}(a)} \sum_{t \in T^+ \setminus (O(a') \cup T(a'))} \gamma_{\mathbf{v}(a')}(P^{(a')}_{\mathbf{v}(a)})_{\mathbf{v}(a')}
$$

$$
= \sum_{t \in T^+ \setminus (O(a') \cup T(a'))} \gamma_{\mathbf{v}(a')}(P^{(a')}_{\mathbf{v}(a)})_{\mathbf{v}(a')}
$$

for any $a = a' - e_{T(a')}: P \in C_{l+s-|P|-|T(a')|}(i)$.

Let $s'' = l + s - |P| - |T(a')|$, and let $\mathcal{P}(s'', t_0) = \{ P \subset R^+ \setminus (O(a') \cup T(a') \cup \{ t_0 \}) | |P| = s'' \}$. Fix a $t_0 \in R(a')$, assume that

$$
\sum_{P \in \mathcal{P}(s'', t_0), a \in C^{(n)}(i)} d_{P, a} \kappa(P, a) = 0
$$
for \( d_P, a \in k \). That is

\[
0 = \sum_{P \in \mathcal{P}(s', t_0), a \in C_{s'}(i)} d_P, a \kappa(P, a)
\]

\[
= \sum_{P \in \mathcal{P}(s', t_0), a \in C_{s'}(i), t \in \mathcal{R}(a') \setminus P} \sum_{a \in C_{s'-1}(i), t \in \mathcal{R}(a') \setminus \mathcal{R}(a) \setminus \{t_0\}] P \in \mathcal{P}(s', t_0), t \not\in P} d_P, a, e_t, t, \gamma_{(a')}(P) \gamma_{(a-e_t)}(t) \gamma_{(a-e_t)}(l) \gamma_{(a)}(v_{a-a_0}) \gamma_{v_{a-a_0}}(a)
\]

\[
= \sum_{\substack{a \in C_{s'-1}(i), t, t' \in \mathcal{R}(a') \setminus \{t_0\} \setminus P \subset \mathcal{R}(a') \setminus \{t_0\} \setminus P \in \mathcal{P}(s', t_0), t \not\in P}} d_P, a, e_t, t, \gamma_{(a')}(P) \gamma_{(a-e_t)}(t) \gamma_{(a-e_t)}(l) \gamma_{(a-e_t)}(l) \gamma_{(a-e_t)}(l) \gamma_{v_{a-a_0}}(a)
\]

\[
+ \sum_{\substack{a \in C_{s'-1}(i), P \subset \mathcal{P}(s', t_0)}} d_P, a, e_t, t, \gamma_{(a')}(P) \gamma_{(a-e_t)}(t) \gamma_{(a-e_t)}(l) \gamma_{(a-e_t)}(l) \gamma_{(a-e_t)}(l) \gamma_{v_{a-a_0}}(a)
\]

(22)

Thus \( d_P, a = 0 \) for all \( P \in \mathcal{P}(s', t_0) \) and \( a \in C_{s'}(i) \) with \( a - e_{t_0} \in C_{s'-1}(i) \).

Since \( a_{t_0} > 0 \), and if \( e_{v_{(a')}} \gamma_{v_{(a'-e_{t_0})}} \gamma_{v_{(a)}} \neq 0 \), then \( a = a' - e_{t_0} \in \mathcal{P}(s', t_0) \) and \( a_{t_0} = a'_{t_0} > 0 \), thus \( a - e_{t_0} \in C_{s'-1}(i) \).

This shows that \( \kappa(P, a) \mid P \in \mathcal{P}(s', t_0), a \in C_{s'}(i) \) is a linearly independent set and thus

\[
\dim_k(\Lambda(n)K_l)_{t+s'} \geq \left( n - |O(a')| - |T(a')| - 1 \right).
\]

Since \( (\Lambda(n)K_l)_{t+s'} \subset (\mathcal{K}f_l)_{t+s'} \), this implies that \( (\Lambda(n)K_l)_{t+s'} = (\mathcal{K}f_l)_{t+s'} \) for \( s' = 2, \ldots, n + 1 \).

This proves that \( \mathcal{K}f_l \) is generated by \( K_l \) for \( l \leq m - 1 \) and thus proves (a), (b) and (c), by induction.

Now we prove (d), consider the case of \( l = m \), note that \( C_{m}(i) = \{ b = (b_1, \ldots, b_{m+1}) \} \), with \( b_1 = m + n - 1 - \sum_{t=1}^{n} i_t \), and \( b_t = i_{t-1} - 1 \) for \( 2 \leq t \leq n + 1 \).

Clearly, we have that \( \text{soc} \, P(v_{(b')}) = \bar{P}(v_{(b)})_{m+1} = (\mathcal{K}f_{m})_{n+1} \subset \mathcal{K}f_{m} \).

For each \( a' \neq b + e \) such that \( \bar{S}(v_{(a')}) \) is a composition factor of \( \bar{P}(v_{(b)})_{m+1} \), for \( 1 \leq s \leq n \), \( e_{v_{(a')}} \bar{\Lambda} \bar{v}_{(b)} \neq 0 \) and \( a' \in \mathbb{Z}_{++}^{n+1} \setminus C_{m}(i) \). Thus \( |T(a')| = s \) and there is a bound path of \( \gamma_{(a')}^{(a)} \gamma_{(a)} \) type \( T(a') \) from \( v_{(b)} \) to \( v_{(a')} \). So paths of the form \( \gamma_{(a')}^{(a)}(P) \gamma_{(b)}^{(b-e_{t_0})} \) for \( P \subset \mathcal{R}(a') \) with \( |P| = s' \) form a basis of \( \bigoplus_{a \in C_{m-1}(i)} e_{v_{(a')}}^{(b)} \bar{P}(v_{(a)})_{m+s} \) for \( 0 \leq s' \leq n + 1 - |O(a')| - |T(a')| \).

Consider the composition factor \( \bar{S}(a') \) in the degree \( m + s \) component of the projective resolution \( \{11\} \), we get

\[
0 \longrightarrow e_{v_{(a')}}(\mathcal{K}f_{m})_{m+s} \longrightarrow e_{v_{(a')}}(\bar{P}(v_{(b)})_{[m]})_{m+s} \longrightarrow \bigoplus_{a \in C_{m-1}(i)} e_{v_{(a')}}(\bar{P}(v_{(a)})_{[m]})_{m+s} \longrightarrow \cdots \rightarrow 0,
\]

with \( s'' = n + 1 - |O(a')| - |T(a')| \). So we have that

\[
\dim_k e_{v_{(a')}}(\mathcal{K}f_{m})_{m+s} = \sum_{t=0}^{s''} (-1)^t \dim_k \bigoplus_{a \in C_{m-1}(i)} e_{v_{(a')}}(\bar{P}(v_{(a)})_{[m]})_{m+s}
\]

\[
= \sum_{t=0}^{s''} (-1)^t \left( \begin{array}{c}
}s'' \\
-t
\end{array} \right) = (1 - 1)^{s''} = 0.
\]
This proves that \((\text{Ker } f_m)_{m+s} = 0\) for \(1 \leq s \leq n\), and thus \(\text{Ker } f_m = \text{soc } \overline{P}(v^i(b))\). 

Let \(\tilde{A}_0\) be the subspace spanned by the idempotents of \(\tilde{A}(n)\), and let \(\tilde{A}_1\) be the subspace spanned by the arrows. Then \(\overline{\rho}(n) \subset \tilde{A}_1 \otimes_{\tilde{\rho}} \tilde{A}_1\), and is spans an subspace \(R^g \subset \tilde{A}_1 \otimes_{\tilde{\rho}} \tilde{A}_1\). Let \(T_{\tilde{\rho}}(\tilde{A}_1) = \tilde{A}_0 + \tilde{A}_1 \otimes_{\tilde{\rho}} \tilde{A}_1 + \cdots \tilde{A}_1 \otimes_{\tilde{\rho}} \tilde{A}_1 + \cdots\) be the tensor algebra and let \((R^g)\) be the ideal generated by \(R^g\), then \(\tilde{A}(n) \simeq T_{\tilde{\rho}}(\tilde{A}_1)/(R^g)\) is a quadratic algebra. The quadratic dual \(\tilde{A}(n)\) of \(\tilde{A}(n)\) is defined as the quotient \(T_{\tilde{\rho}}(\tilde{D}\tilde{A}_1)/(R^g)\), where \(\tilde{D}\tilde{A}_1\) is the dual space of \(\tilde{A}_1\) and \(R^g\) is the annihilator of \(R^g\) in \(D(\tilde{A}_1 \otimes_{\tilde{\rho}} \tilde{A}_1)\). From a different view point, Theorem 2.3 is restated as following theorem:

**Theorem 3.4.** Assume that \(m \geq 3\).

\[ \tilde{A}(n) \text{ is an almost Koszul algebra of type } (n+1, m-1), \text{ and its quadratical dual } \tilde{A}^g_1(n) \text{ is almost Koszul algebra of type } (m-1, n+1).\]

Both \(\tilde{A}(n)\) and \(\tilde{A}^g_1(n)\) are periodic algebras.

More precisely, the minimal periodicity of \(\tilde{A}(n)\) is \((n+1)m\) and if , the minimal periodicity of \(\tilde{A}^g_1(n)\) is \((n+1)(n+2)\).

**Proof.** If follows directly from Lemma 3.3 that \(\tilde{A}(n)\) is an almost Koszul algebra of type \((n+1, m-1)\). So \(\tilde{A}^g_1(n)\) is almost Koszul algebra of type \((m-1, n+1)\), by Proposition 3.11 of [7].

Note that for each \(i \in Q(n)_0\), by (d) of Lemma 3.3, we have that \(\Omega^n \tilde{S}(i) \simeq \tilde{S}(v^i(b(i))) = \tilde{S}(\omega(i)).\) If \(i = (i_1, \ldots, i_n)\), then

\[ b(i) = (m + n - 1 - \sum_{t=1}^{n} i_t, i_1 - 1, i_2 - 1, \ldots, i_n - 1) \]

and we have

\[ \omega(i) = v^i b(i) = (m + n - \sum_{t=1}^{n} i_t, i_1, \ldots, i_{n-1}). \]

Thus \(\omega(i) = (i_{n-t+2}, \ldots, i_n, m + n - \sum_{t=1}^{n} i_t, i_1, \ldots, i_{n-1})\), and \(\omega(i) + 1)(i) = i\). This proves that \(\Omega^{n(n+1)} \tilde{S}(i) = \tilde{S}(i)\), and \(n + 1\) is minimal such that this holds for all \(i \in Q(n)_0\). So the minimal periodicity of \(\tilde{A}(n)\) is \((n+1)m, (n+1)(n+2)\).

Since \(\tilde{A}^g_1(n)\) is \((m-1, n+1)\)-Koszul, the \(\Omega^{n+1}\) of a simple is the first simple among its syzygies. Clearly, the minimal periodicity is exactly the order of permutation matrix \(N(\tilde{A}(n), m-1)\) of \(\tilde{A}(n)\) in Proposition 3.14 of [7], thus the permutation matrix \(N(\tilde{A}^g_1(n), n+1)\) of \(\tilde{A}^g_1(n)\) have the same order as \(N(\tilde{A}(n), m-1)\). This proves that the minimal periodicity of \(\tilde{A}^g_1(n)\) is \((n+1)(n+2)\). 

4. \((n+1)\)-CUBOID TRUNCATIONS

Assume that \(\Lambda(n) = \Lambda^g(n)\) is an \(n\)-translation algebra with admissible \(n\)-translation quiver \(Q(n)\). Let \(g = (g', g)\) be as defined in Section 2, let \(\overline{\Lambda}(n) = \Lambda^g(n)\) be a twisted trivial extension of \(\Lambda(n)\). Now consider truncations on \(\overline{\Lambda}(n) = \Lambda^g(n) = \overline{\Lambda}(n) \neq \mathbb{Z}^*\) and we write \(\overline{\rho}(n)\) for \(\overline{\rho}(n)\). It follows from Theorem 2.3 that \(\overline{\Lambda}(n)\) is an \(n\)-translation algebra with stable \(n\)-translation quiver \(\overline{Q}(n)\) and \(n\)-translation \(\overline{\tau} = \overline{\tau}_n\) for the \(n\)-translation of \(\overline{Q}(n)\).
Let $\mathcal{S}(i)$ be the simple $\Lambda(n)$-module at $i \in Q(n)_0$ and let $\mathcal{P}(i)$ and $\mathcal{T}(i)$ be its projective cover and injective envelop as $\Lambda(n)$-modules, respectively. Conventionally, set $\mathcal{S}(i)$, $\mathcal{P}(i)$ and $\mathcal{T}(i)$ to be zero when $i \notin Q(n)_0$.

For each $i = (i_1, \ldots, i_n, i_{n+1}) \in Q(n)_0$, let $i' = (i_1, \ldots, i_n)$ be its $n$-truncation in $Q(n)_0 = Q(n)_{0}$. Define its $(n+1)$-cuboid $C^{(n)}(i) = C^{(n)}(i')$. We have characterization of the linear part of the projective of simple $\Lambda(n)$-modules using the $(n+1)$-cuboids $C^{(n)}(i)$ similar to Proposition 3.2. Let

$$\cdots \rightarrow \mathcal{P}^2(\mathcal{S}(i)) \xrightarrow{f_2} \mathcal{P}^1(\mathcal{S}(i)) \xrightarrow{f_1} \mathcal{P}^0(\mathcal{S}(i)) \xrightarrow{f_0} \mathcal{S}(i) \rightarrow 0 \ (23)$$

be a minimal projective resolution of the simple $\Lambda$-modules $\mathcal{S}(i)$ of corresponding to the vertex $i \in Q(n)_0$. If $i \in Q(n)_0$, call $a$ an $i$-quiver vertex if $\psi^i(a) \in Q(n)_0$.

Consider the linear part of this projective resolution. We have

**Proposition 4.1.** For $l = 0, 1, \ldots, m − 1$ and $i \in Q(n)_0$, we have that $\mathcal{P}^l(\mathcal{S}(i))$ is generated in degree $l$ and

$$\mathcal{P}^l(\mathcal{S}(i)) \simeq \bigoplus_{a \in C^{(n)}(i)} \mathcal{P}(\psi^i(a)). \ (24)$$

Further more, if $C^{(n)}_{m-1}(i) = \{b\}$, then $\ker f_{m-1} \simeq \soc \mathcal{P}(\psi^i(b)) = \mathcal{S}(\psi^i(b + e))$ is simple.

Recall that a full bound subquiver $Q'$ of $Q(n)$ is called a $\tau_n$-slice of $Q$ if it has the following property $[13]$:

(a) for each vertex $i$ of $Q(n)$, the intersection of the $\tau_n$-orbit of $v$ and the vertex set of $Q'$ is a single-point set;

A $\tau_n$-slice is called a complete $\tau_n$-slice, if it also has the following property:

(b) $Q'$ is path complete in the sense that for any path $p : v_0 \rightarrow v_1 \rightarrow \ldots \rightarrow v_l$ of $Q(n)$ with $v_0$ and $v_l$ in $Q'$, the whole path $p$ lies in $Q'$.

The algebra defined by a complete $\tau$-slice of $Q(n)$ is called a $\tau$-slice algebra of $Q(n)$ (or of $\Lambda(n)$).

Let $Q(n) \times \{t\}$ be the full subquiver of $Q(n)$ with vertex set $Q(n)_0 \times \{t\}$ for any $t \in \mathbb{Z}$. It is easy to check that we have the following result.

**Proposition 4.2.** $Q(n) \times \{t\}$ are complete $\tau$-slices of $Q(n)$ for all $t$.

Let $\Lambda(n,t)$ be the algebra defined by the bound quiver $Q(n) \times \{t\}$. $\Lambda(n,t)$ are isomorphic to $\Lambda^{(n)}(n)$ for all $t$.

Now we define the $(n+1)$-cuboid completion $Q(n+1)(t)$ of $Q(n) \times \{t\}$ as the full subquiver of $Q(n)$ with the vertex set:

$$Q(n+1)(t)_0 = \cup_{a \in Q(n)_0 \times \{t\}} \{\psi^a(a) | a \in C^{(n)}(i)\}.$$ 

Clearly, $Q(n+1)(t)_0$ are isomorphic quivers for all $t$.

Let $Q(n+1)(1)^{(k)}$ and $Q(n+1)(1)^{(k)}$ be the full subquivers of $Q(n+1)(1)$ and $Q(n+1)$ with vertex set $Q(n+1)(1)^{(k)} = \{i = (i_1, \ldots, i_k, 1, \ldots, 1) \in Q(n+1)(1)_0\}$ and $Q(n+1)(1)^{(k)} = \{i = (i_1, \ldots, i_k, 1, \ldots, 1) \in Q(n+1)_0\}$, respectively.

**Lemma 4.3.** For $1 \leq t \leq n + 1$, $(Q(t), \rho(t))$ can be identify with

$$(Q(n+1)(1)^{(t)}, \mathcal{P}(n)(Q(n+1)(1)^{(t)}))$$

as full bound subquivers.
Proof. We prove by using induction on $n$ and $t$. The assertion clearly holds for $n = 0$ since $Q(1)(1) = Q(1)(1)^{(1)}$.

Assume that $n > 0$ and lemma holds for $n' < n$. By inductive assumption $(Q(t), \rho(t))$ can be identify $Q(n)(1)^{(t)}$, which can be identify with $Q(n) \times \{1\} = Q(n + 1)(1)^{(n)}$, a full subquiver of $Q(n + 1)$ inside $Q(n + 1)(1)$, we only need to prove the case of $t = n + 1$.

For any $i = (i_1, \ldots, i_n, i_{n+1}) \in Q(n + 1)_0$, we have that $(i_1, \ldots, i_n) \in Q(n)_0$ and $i_{n+1} \leq m + n - 1 - \sum_{t=1}^{n} i_t$, by definition. It follows easily that $i_n + i_{n+1} - 1 \leq m + n - 1 - \sum_{t=1}^{n-1} i_t$, thus $i' = (i_1, \ldots, i_{n-1}, i'_n = i_n + i_{n+1} - 1, 1) \in Q(n)_0 \times \{1\}$.

Let $a = (0, \ldots, 0, i_{n+1} - 1)$ then $a \in C(n)(i')$, and we have that

$$\nu'(a) = (i_1, \ldots, i_n, i_{n+1}) = 1.$$ 

This proves that $Q(n + 1)_0 \subset Q(n + 1)(1)_0$.

On the other hand, for any $i = (i_1, \ldots, i_n, i_{n+1}) \in Q(n + 1)(1)_0$. Then there is a $i' = (i'_1, \ldots, i'_n, 1) \in Q(n + 1)_0 \times \{1\}$ such that $i = \nu'(a)$ for some $a \in C(n)(i')$. Thus $i_t = i'_t + a_t - a_{t+1} > 1$ for $t \leq n$ and $i_{n+1} = 1 + a_{n+1}$. For $1 \leq s \leq n$, we have

$$\sum_{t=1}^{s} i_t = \sum_{t=1}^{s} i'_t + a_s - a_{s+1} \leq m + n - 1 - \sum_{t=1}^{n} i'_t - a_{s+1} \leq m + s - 1,$$

and

$$\sum_{t=1}^{n+1} i_t = \sum_{t=1}^{n} i'_t + a_1 \leq m + n.$$

This proves that $i \in Q(n + 1)_0$ and thus $Q(n + 1)_0 = Q(n + 1)(1)_0$.

Now let $i = (i_1, \ldots, i_n, i_{n+1}) \in Q(n + 1)(1)_0$, and let $i' = i = (i_1, \ldots, i_n, 1)$. If $\nu'(e_t), \nu'(e_{t} + e_t) \in Q(n + 1)(1)_0$, and $\nu'(e_t) \in Q(n)_0$, then we have $i_{t-1} > 1$, and $i_{s+1} > 1$ when no of $s, t$ equals 1, and $i_1 < m$ when $s = 1$ or $t = 1$. So we have that if $b(i') = (b'_1, \ldots, b'_n, b'_{n+1})$, then $b'_t > 0$ and $b'_s > 0$ and thus $e_s \in C(i')$. Thus $\nu'(e_s) \in Q(n + 1)(1)_0$. This proves that $(Q(n + 1)(1), \nu'(n)(Q(n + 1)(1)))$ is a full bound quiver of $(\overline{Q}(n), \overline{\nu}(n))$, since elements of $\overline{\nu}(n)$ are either paths of length 2, or linear combinations of two paths of length 2 of the same type.

Write $\rho(n + 1)(1)^{(t)} = \overline{\nu}(n)(Q(n + 1)(1)^{(t)})$, and $\rho(n + 1)(t) = \overline{\nu}(n)(Q(n + 1)(t))$. As a corollary of Lemma 4.3 we have

**Corollary 4.4.** $(Q(n + 1)(1), \rho(n + 1)(1))$ are full bound subquivers of $(\overline{Q}(n), \overline{\nu}(n))$.

Let $\Lambda(n + 1)(t)$ be the algebra defined by the bound quiver $(Q(n + 1)(1), \rho(n + 1)(1))$ and they will be called $(n + 1)$-cuboid truncations of $\overline{\nu}(n)$.

We have the following theorem.

**Proposition 4.5.**

(a) $\Lambda(n + 1)(t)$ are all isomorphic to $\Lambda(n + 1)(1) = \Lambda(n + 1)$.

(b) Let $I(t)$ be the ideal of $\overline{\Lambda}(n)$ generated by the set $\{e_i | i \in \overline{Q}(n)_0 \setminus Q(n + 1)(1)_0\}$, then $\Lambda(n + 1) \simeq \overline{Q}(n)/I(t)$ for all $t$.

(c) $\Lambda(n + 1)$ is $n$-translation algebra.

*Proof. (a), (b) follows directly from Lemma 4.3.*
Now we prove (c). Identify $Q(n + 1)$ with $Q(n + 1)(1)$. As an truncation of an $n$-translation quiver, it follows from (a) and (b) that $Q(n + 1)$ is an $n$-translation quiver.

For each $i \in Q(n + 1)_0 = Q(n + 1)(1)_0 \subset \overline{Q(n)}_0$, we have a minimal projective resolution \((23)\) of simple $\Lambda(n)$-module $\overline{S}(i)$. Note that $\Lambda(n + 1) \cong \overline{\Lambda(n)} / I$, where $I$ is the ideal generated by $\{ e_j | j \in Q(n)_0 \setminus Q(n + 1)(1)_0 \}$. Since $Q(n + 1)(1)_0$ is finite, let $e = \sum_{i \in Q(n + 1)(1)_0} e_i$, then $\overline{e}$ is the unit of $\overline{\Lambda(n)} / I$. We have $S(i) = \overline{S}(i)$ for $i \in Q(n + 1)(1)_0$. Tensor $\overline{\Lambda(n)} / I$, we get a complexes of $\Lambda(n + 1)$-modules

$$\cdots \rightarrow \overline{\Lambda(n)} / I \otimes \overline{\Lambda(n)} \overline{P}^l(\overline{S}(i)) \xrightarrow{1 \otimes f_l} \overline{\Lambda(n)} / I \otimes \overline{\Lambda(n)} \overline{P}^l(\overline{S}(i)) \rightarrow 0$$

Note that $\overline{\Lambda(n)} / I \otimes \overline{\Lambda(n)} \overline{P}^l(\overline{S}(i)) \cong \overline{P}^l(\overline{S}(i)) / I \overline{P}^l(\overline{S}(i))$. For any $y = \sum_p \bar{a}_p \otimes y_p$, we have $\sum_p \bar{a}_p y_p \in \text{Ker} (1 \otimes f_l)$, and $x \in \overline{P}^{l + 1}(\overline{S}(i))$ such that $f_{l + 1}(x) = \sum_p \bar{a}_p y_p$. This proves that \((25)\) is exact and hence a projective resolution $S(n + 1)$ as $\Lambda(n + 1)$-modules.

Now fix $i \in Q(n)(1)_0 \subset Q(n + 1)(1)_0$, then $\overline{\Lambda(n)} / I \otimes \overline{\Lambda(n)} \overline{S}(i) \cong \overline{\Lambda(n)} / I \overline{S}(i)$ and hence is generated in degree $l$ and $\text{Ker} (1 \otimes f_{l + 1})$ is either simple or zero.

This proves that $\Lambda(n + 1)$ is almost Koszul of type $(n + 1, m - 1)$ and is $n$-translation algebra.

We call $\Lambda(n + 1)$ a $(n + 1)$-cuboidal completion of $\Lambda(n)$.

5. Pyramid $n$-cubic Algebras and $n$-almost Split Sequences

Now fix $m \geq 3$, and a integer $n \geq 1$. Let $Q(n)$, $\rho(n) = \rho\rho^\perp(n)$ and $\Lambda(n)$ be the data for a pyramid shaped $n$-cubic quiver $Q(n) = Q\rho(n)$ as defined in \(22\) and \(23\), and let $\Lambda(n)$ be the pyramid $n$-cubic algebra defined by the bound quiver $Q(n)$.

Recall that an $n$-translation quiver $Q$ is called admissible if it satisfies the following conditions:

(i) For each bound path $p$, there are paths $q'$ and $q''$ such that $q'pq''$ is a bound path of length $n + 1$.

(ii) Any bound path $p$ from a non-injective vertex $i$ to a non-projective vertex $j$ is shiftable.

(iii) Let $i$ be a non-projective vertex. Let $p$ be a bound path and let $q$ be a bound path starting at $ni$ with $l(p) + l(q) \leq n$. If $p$ passes a projective vertex and $q$ passes an injective vertex, the $p$ is either left stark with respect to $t(q)$, or $q$ is right stark with respect to $s(p)$, of length $n + 1 - (l(p) + l(q))$.

(iv) If $p, q$ are two bound paths such that $s(p) = s(q)$ and $t(p), t(q) \in P$ then there are paths $p', q'$ with $s(p') = t(q)$ such that $p, p', q, q'$ are bound paths with $t(p') = t(q') \in P$.
Theorem 5.1. main Pyramid n-cubic algebra \( \Lambda(n) \) is an extendible \( n-1 \)-translation algebra with admissible \( n-1 \)-translation quiver \( Q(n) \).

Proof. We prove using induction on \( n \). We have

\[
Q(1) = Q : \overset{1}{\circ} \rightarrow \overset{2}{\circ} \rightarrow \cdots \rightarrow \overset{m}{\circ},
\]

with the relation \( \rho(1) = \{ \alpha_{i+1}|i = 1, \ldots, m-2 \} \). Clearly, \( Q(1) = (Q(1), \rho(1)) \) is a admissible 0-translation quiver with 0-translation \( \tau_0 : i+1 \rightarrow i \) for \( i = 1, \ldots, m-1 \). \( \Lambda(1) \) is a Koszul algebra with radical squared zero. So it is a 0-translation algebra, and it is extendible by Theorem 2.1 of [7].

Assume that \( \Lambda(n) \) is an extendible \( n-1 \)-translation algebra and its bound quiver \( Q(n) \) is an admissible \( n-1 \)-translation quiver.

It follows from Proposition 1.5 that \( \Lambda(n+1) \) is an \( n \)-translation algebra with \( n \)-translation \( Q(n+1) \) and \( n \)-translation \( \tau_n : i \rightarrow e_{n+1} \). By Theorem 5.4 \( \Lambda(n+1) \) is extendible. We need only to prove that \( Q(n+1) \) is admissible.

Let \( \mathcal{P} \) and \( \mathcal{I} \) be the sets of projective vertices and injective vertices of \( Q(n+1) \), respectively. Then

\[
\mathcal{P} = \{ i \in Q(n+1)_0| i_{n+1} = 1 \} \quad \mathcal{I} = \{ i \in Q(n+1)_0| i = m+n \}.
\]

Let \( p = \gamma_{\ell(t_1)} \cdots \gamma_{\ell(t_1)} \) be a bound path in \( \Lambda(n+1) \). If \( t_h = 1 \), then

\[
i, i(t_1), \ldots, i(t_1, \ldots, t_{h-1})
\]

are not injective. If \( t_h = n+1 \), then

\[
i(t_1, \ldots, t_h), i(t_1, \ldots, t_h, t_{h+1}), \ldots, i(t_1, \ldots, t_s)
\]

are not projective.

Assume that \( p \) is a bound path from \( i \) to \( j \) in \( Q(n+1) \). If \( j \) is non-projective, then there is a path \( p' \) from \( \tau_n j \) to \( i \) such that \( pp' \) is a bound path of length \( n+1 \), take \( p'' = e_j \) be the trivial path. If \( i \) is non-injective then there is a path \( p'' \) from \( j \) to \( \tau_n^{-1}i \) such that \( pp'' \) is a bound path of length \( n+1 \), take \( p' = e_i \) be the trivial path. If \( i \) is injective and \( j \) is projective, then we have that \( \sum_{i=1}^{n+1} i = m+n \) and \( j_{n+1} = 1 \). Thus we have that \( j \in \{ v^l(a) | a \in U^{n+1} \} \) and \( i \in \{ v^j-e_{n+1}(a) | a \in U^{n+1} \} \). Thus we have that \( i_{n+1} = 1 \) and \( j = m+n \), and all the vertices on the path \( p \) are projective and injective, which is on the subquiver \( Q(n) \times \{ 1 \} \) of \( Q(n+1) \). So there are paths \( q' \) and \( p'' \) in \( Q(n) \times \{ 1 \} \), such that \( pp'' \) is a bound path from \( i' \) to \( j' \) of length \( n \). But \( t'_{l} \geq 1 \) for \( l = 1, \ldots, n+1 \) and \( t'_h \geq 1 \), \( q'p'' \) is formed by arrows of different type, thus \( j'_n > 1 \) and \( \gamma_{\ell(t_1)} \cdots \gamma_{\ell(t_1)} : j \rightarrow j(n+1) \) is an arrow of \( Q(n+1) \). Let \( p' = \gamma_{\ell(t_1)} \cdots \gamma_{\ell(t_1)} \) be a bound path in \( Q(n+1) \). This proves the Condition (i) of the admissibility.

Let \( p = \gamma_{\ell(t_1)} \cdots \gamma_{\ell(t_1)} \) be a bound path from a non-injective vertex \( i \) to a non-projective vertex \( j \). Then we have that \( j_{n+1} > 1 \) and \( |j| < m+n \). We prove that \( p \) is shiftable by using induction on \( s \).

If \( s = 1 \), then either \( t_1 = 1 \), then \( i_{n+1} = j_{n+1} \), both \( i \) and \( j \) are non-projective and \( p \) is left shiftable, or \( t_1 \neq 1 \) and \( |j| = |i| < n \), thus both \( i \) and \( j \) are non-injective and \( p \) is right shiftable.

Assume that \( s \geq 1 \) is an integer and any bound path of length \( l \) from a non-injective vertex to an non projective vertex is shiftable for \( s = l \).
Now let $s = l + 1$. Assume that $p$ is not left or right shiftable then $p$ passes through both projective and injective vertex. Since for any arrow $\alpha : i \to i'$, $i'$ is injective if $i$ is so, and $i'$ is projective if $i$' is so, thus $i$ is projective and $j$ is injective and we have $j_{n+1} = 1$ and $|j| = m + n$. This implies that $j_{n+1} = 2$ and $|i| = m + n - 1$ and there is one arrow of type 1 one arrow of type $n + 1$ in $p$.

If $t_i \neq 1$, then $p' = \gamma_i^{(t_i)} \ldots \gamma_i^{(t_2)}$ is a bound path from a non-injective vertex $i(t_i)$ to a non-projective vertex $j$ and thus it is shiftable by inductive assumption. So $p$ is also shiftable by definition.

If $t_i \neq n + 1$, then $p' = \gamma_i^{(t_i)} \ldots \gamma_i^{(t_2)}$ is a bound path from a non-injective vertex $i$ to a non-projective vertex $j$ and thus it is shiftable by inductive assumption. So $p$ is also shiftable by definition.

Let $q = \gamma_i^{(t_r)} \gamma_i^{(t_{r-1})} \ldots \gamma_i^{(t_2)}\gamma_i^{(t_1)}$ be a bound path. If $t_r \neq 1$ and $i_{r-1} > 1$ then $t_u \neq t_r$ if $u \neq r$, since $p$ is a bound path, let $j^{(u)} = i(1, \ldots, t_u - 1) = (i_1^{(u)}, \ldots, j_{n+1}^{(u)})$, then $j_{n+1}^{(u)} \geq i_{r-1} > 1$, and we have that $j^{(u)}(t_r)$ is a vertex in $Q(n + 1)$. Thus there are nonzero $d_2, \ldots, d_r \in k$ such

$$q = \gamma_i^{(t_r)} \gamma_i^{(t_{r-1})} \ldots \gamma_i^{(t_2)}\gamma_i^{(t_1)}$$

Similarly, if $j_{t_1} > 1$ or $t_1 = 1$ then we also have $j_{t_{r-1}}^{(u)} > 1$ for $u = 3, \ldots, n$. This implies $(t_1)j^{(u)}(1) \in Q(n + 1)$ and there are nonzero $d_2, \ldots, d_r \in k$ such that

$$q = \gamma_i^{(t_r)} \gamma_i^{(t_{r-1})} \ldots \gamma_i^{(t_2)}\gamma_i^{(t_1)}$$

Assume that $t_1 = 1$, $t_{r+1} = n + 1$, if $i_{r-1} > 1$ for some $1 < r \leq l + 1$, or $j_{r-1} > 1$ for some $1 \leq r \leq l$, then we have that $p$ is shiftable, similar to above. Otherwise $t_1 = 1$, $t_{r+1} = n + 1$, $i_{r-1} = 1$ for $1 < r \leq l + 1$ and $j_{r-1} = 1$ for $1 \leq r \leq l$. It follows that $t_r = r$ for $r = 1, \ldots, l < n$ and $t_{r-1} = n - r$ for $r = 0, \ldots, \min\{n - 1, l - 1\}$ and thus $n = l$. But $|i| = m + n - 1$ so $i_1 = m - 1$ and $|j| = n + m$ and $j_{n+1} = m$ and $m - 1 > 1$, there is no path from $i$ to $j$.

This proves that bound paths from a non-injective vertex to a non-projective vertex in $Q(n + 1)$ are shiftable and (ii) hold.

(iii) and (iv) holds since for any pair of vertex $i,j$ of $Q(n + 1)$, all the bound paths from $i$ to $j$ are linearly dependent, if there is one.

This proves that a pyramid $n$-cubic algebra is $n - 1$-translation algebra with admissible $n - 1$-translation quiver.

\[\square\]

**Lemma 5.2.** Let $Q(n)$ be a pyramid shaped $n$-cubic quiver and let $i$ be a vertex of $Q(n)$.

If $H^i$ is complete, then $H^i_{C^i(e)}$ is not complete.

If $H^i_{C^i}$ is complete, then $H^i_{C^i(b(i))}$ is not complete.
Proof. For $i = (i_1, \ldots, i_n) \in G(n)_0$, then $\psi^1(i(e)) = (i_1, \ldots, i_n + 1)$. Thus $b(\psi^1(i(e))) = (m + n - 1 - \sum_{t=1}^{n} i_t, i_2 - 1, \ldots, i_{n-1} - 1, i_n)$ and

$$\bar{v}^1(e)(b(\psi^1(i(e)))) = (i_1 + m + n - 1 - \sum_{t=1}^{n} i_t - i_2 + 1, 2i_2 - i_3, \ldots, 2i_{n-1} - i_n - 1, 2i_n + 1),$$

and we have

$$|\bar{v}^1(e)(b(\psi^1(i(e))))| = m + n - 1 + 1 = m + n.$$

This implies that $\bar{v}^1(e)(b(\psi^1(i(e)))) \notin Q(n)_0$ and $H^1_C^{v^1(e)}$ is not complete.

The other statement is proven similarly. \hfill \Box

Let $\Gamma(n + 1) = \Lambda(n + 1)^{op}$ be the Koszul dual of $\Lambda(n + 1)$. The have the same quiver with quadratic dual relations. Their projective are described by the hammocks (see Lemma 2.7 for $\Lambda(n)$):

**Lemma 5.3.** The following are equivalent for $\Lambda(n + 1)$.

i. $\Lambda(n + 1)e_1$ is projective injective.

ii. The Loewy length of $\Lambda(n + 1)e_1$ is $n + 2$.

iii. $H^1$ is complete.

It follows from Theorem 6.2 of [15], that $\hat{\Lambda}(n)^{op}$ and $\hat{\Lambda}(n)^{op}$ are self-injective of Loewy length $m$, so they are $m - 2$-translation algebras. As a truncation of $\hat{\Lambda}(n + 1)^{op}$ we also have the following result.

**Lemma 5.4.** The following are equivalent for $\Gamma(n)$.

i. $\Gamma(n)e_1$ is projective injective.

ii. The Loewy length of $\Gamma(n)e_1$ is $m$.

iii. $H^1_C$ is complete.

Then we have the following result:

**Theorem 5.5.** $\Gamma(n + 1)$ is an $m - 2$-translation algebra.

add $\Gamma(n + 1)$ has $n$-almost split sequence.

add $\Lambda(n + 1)$ has $m - 2$-almost split sequence.

Proof. Similar to the case for $\Lambda(n + 1)$, it is easy to see that

$$\Gamma(n + 1) \simeq \overline{\Gamma(n)}/I',$$

where $I'$ is the ideal of $\overline{\Gamma(n)}$ generated by the set $\{\epsilon j | j \in Q(n)_0 \setminus Q(n + 1)_0\}$ of idempotents. Thus $\Gamma(n + 1)$ is an $n + 1$-translation algebra.

Note that the $n$-translation of $\Lambda(n + 1)$ is defined by $\tau^{-1}_n 1 = i + e$ and the $m - 2$-translation of $\Gamma(n + 1)$ is defined by $\tau^{m-1}_{n-1} 1 = \psi^1(i)(b(\psi^1(i)))$. The last two assertion follows from Theorem 7.2 of [15], Lemma 5.3 and Lemma 5.3. \hfill \Box

We have the following version of Iyama’s cone construction of absolutely $(n + 1)$-complete algebras from an absolutely $n$-complete algebra.

**Theorem 5.6.** Let $\Lambda^n(n) = T^m_t^{(n)}(k)$ be the $n$-cubic pyramid algebra defined in Proposition 2.3, then there is an $(n + 1)$-cuboid completion $\Lambda^n(n + 1)$ of $\Lambda^n(n)$ such that $T^m_t^{(n+1)}(k) \simeq \Lambda^n(n + 1)$. 
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