ON A GENERALIZATION OF DEURING’S RESULTS

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Abstract. Using the Dieudonné theory we will study a reduction of an abelian variety with complex multiplication at a prime. Our results may be regarded as generalization of the classical theorem due to Deuring for CM-elliptic curves. We will also discuss a sufficient condition for a prime at which the reduction of a CM-curve is maximal.

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1. Introduction

Let $E$ be an elliptic curve over the field of rational numbers $\mathbb{Q}$ with complex multiplication (which will be abbreviated by CM) by the integer ring of an imaginary quadratic field $K$ that has a non-singular reduction $E_p$ at a prime $p$. The classical Deuring’s theorem states that it is ordinary or supersingular according to whether $p$ completely splits or remains prime in $K$, respectively. It is known that $E_p$ is ordinary (resp. supersingular) if and only if the $p$-divisible group $E_p[p^\infty]$ is $\mathbb{L}$ (resp. $G_{1,1}$) (see §2 for details). Suppose that $E_p$ is supersingular. If $p$ is greater than 3 the characteristic polynomial of the $p$-th power Frobenius on an $l$-adic Tate module $(l \neq p)$ is $t^2 + p$ and the number of $\mathbb{F}_{p^2}$-rational points attains the Hasse-Weil upper bound:

$$|E_p(\mathbb{F}_{p^2})| = 1 + p^2 + 2p,$$

where $|\cdot|$ stands for cardinality. In this report we will generalize these results to a proper smooth curve over $\mathbb{Q}$ of a higher genus with CM.

Let us first explain terminologies used in the paper. Let $X$ be an object defined over a field $F$. The base change over an extension $F'$ of $F$ is denoted by $X \otimes_F F'$. If $F' = \overline{F}$, the algebraic closure, it is simply described by $\overline{X}$. Let $V$ be a proper smooth variety defined over $\mathbb{F}_q (q = p^f)$. Then $\Phi_{V,q}(t)$ stands for the characteristic polynomial of the $q$-th power Frobenius on $H^1_{et}(\overline{V}, \mathbb{Q}_l)$ ($l \neq p$).

Let $k$ be a field of characteristic $p$. An abelian variety $A$ of dimension $g$ over $k$ will be called supersingular (resp. superspecial) if $\overline{A}$ is isogeneous (resp. isomorphic) to a
product of supersingular elliptic curves and ordinary if the group of $p$-torsion points $[\bar{A}]$ is isomorphic to $\left(\mathbb{Z}/p\mathbb{Z}\right)^9$. We mention that $A$ has CM (or sometimes CM by $\mathcal{O}_K$) if there is a finite extension $k'$ of $k$ such that the endomorphism ring of $A \otimes_k k'$ contains the integer ring $\mathcal{O}_K$ of a CM-field $K$ satisfying $[K : \mathbb{Q}] = 2g$. Let $K_0$ be the maximal totally real subfield and $[K_0 : \mathbb{Q}] = g$. We assume that $p$ is unramified in $K$ and let

$$(1) \quad p = \mathfrak{P}_1 \cdots \mathfrak{P}_t$$

be the prime factorization in $K_0$.

**Theorem 1.1.** Let $A$ be an abelian variety of dimension $g$ defined over a finite field $k$ of characteristic $p$ endowed with CM by $\mathcal{O}_K$. Suppose that every $\mathfrak{P}_i$ of (1) remains prime in $K$. Then $A$ is supersingular. If moreover $t = g$ (i.e. $p$ completely splits in $K_0$) it is superspecial.

If $p \geq 5$ this determines the characteristic polynomial of $A$. In fact suppose that the assumptions of **Theorem 1.1** are satisfied. Then there is a positive integer $m$ so that $A \otimes_k \mathbb{F}_{p^m}$ is isogeneous to a product of supersingular elliptic curves $\{E_i\}_{1 \leq i \leq g}$ defined over $\mathbb{F}_{p^m}$ and $\Phi_{A,p^m}(t) = \prod_{i=1}^g \Phi_{E_i,p^m}(t)$. Since $p \geq 5$, $\Phi_{E_i,p^m}(t)$ is one of the following (20)

$$(t^2 + p^m), \quad t^2 \pm p^m t + p^m, \quad t^2 \pm 2p^m t + p^m,$$

where the last two occur when $m$ is even.

**Theorem 1.2.** Let $A$ be an abelian variety of dimension $g$ defined over a finite field $k$ of characteristic $p$ endowed with CM by $\mathcal{O}_K$. Suppose that $p$ completely splits in $K$ then $A$ is ordinary.

**Theorem 1.3.** Let $A$ be an abelian variety over $\mathbb{F}_p$ of dimension $g$ with CM by $\mathcal{O}_K$ and we assume that the following conditions are satisfied:

1. $p$ completely splits in the maximal totally real subfield $K_0$:

$$p = \mathfrak{P}_1 \cdots \mathfrak{P}_g,$$

and that each prime $\mathfrak{P}_i$ remains prime in $K$.

2. The action of $\mathcal{O}_{K_0}$ is defined over $\mathbb{F}_p$.

Then $A$ is a product of supersingular elliptic curve $\{E_i\}_{1 \leq i \leq g}$,

$$A = E_1 \times \cdots \times E_g,$$

over $\mathbb{F}_p$. If moreover $p$ is greater than 3 we have $\Phi_{A,p}(t) = (t^2 + p)^g$.

A projective smooth curve $C$ of genus $g$ defined over a number field $F$ is called a CM-curve if the endomorphism ring of the Jacobian variety $\text{Jac}(C)$ contains $\mathcal{O}_K$, where $K$ is a CM-field satisfying $[K : \mathbb{Q}] = 2g$. We call a finite prime $v$ of $F$ is good if the reduction $C_v$ is nonsingular. A good prime $v$ is mentioned ordinary (resp. supersingular, superspecial) if so is $\text{Jac}(C_v)$. **Theorem 1.1, Theorem 1.2, Theorem 1.3** and the Hasse-Weil’s formula yield the following consequence, which generalizes the Deuring’s results.
Theorem 1.4. Let $C$ be a proper smooth curve of genus $g$ over $\mathbb{Q}$ with CM by $\mathcal{O}_K$ and $p$ a good prime.

1. If $p$ completely splits in $K$, $C_p$ is ordinary.
2. Let $p = \mathfrak{P}_1 \cdots \mathfrak{P}_t$ be the prime factorization in $K_0$. If every $\mathfrak{P}_i$ remains prime in $K$, $C_p$ is supersingular. If moreover $t = g$ (i.e. $p$ completely splits in $K_0$) it is superspecial.
3. Suppose that $t = g$ in (2) and that either the following (a) or (b) is satisfied.
   - (a) The action of $\mathcal{O}_K$ on $\text{Jac}(C)$ is defined over $K_0$.
   - (b) The action of $\mathcal{O}_K$ on $\text{Jac}(C)$ is defined over $K$.

   Then $\text{Jac}(C_p)$ is a product of supersingular elliptic curves over $\mathbb{F}_p$. If moreover $p \geq 5$, the number of $\mathbb{F}_p$-points attains the Hasse-Weil upper bound:
   $$|C(\mathbb{F}_p^2)| = 1 + p^2 + 2 gp.$$ 

The following corollaries are special cases of this theorem. Let $C$ be the curve in Theorem 1.4.

Corollary 1.1. Suppose that $K$ is a cyclotomic field $\mathbb{Q}(\zeta_N)$ where $\zeta_N$ is a primitive $N$-th root of unity that satisfies $\phi(N) = 2g$, where $\phi$ is the Euler function.

1. If $p \equiv 1(N)$, $C_p$ is ordinary.
2. Suppose that there is a positive integer $h$ with $p^h \equiv -1(N)$. Then $C_p$ is supersingular.
3. If $p \equiv -1(N)$, $\text{Jac}(C_p)$ is a product of supersingular elliptic curves over $\mathbb{F}_p$. If moreover $p \geq 5$,
   $$|C(\mathbb{F}_p^2)| = 1 + p^2 + 2 gp.$$ 

Corollary 1.2. Suppose that $K = \mathbb{Q}(\zeta_M + \zeta_M^{-1}, \zeta_d)$, where $M$ is a positive integer satisfying $\phi(M) = 2g$ and $d = 3$ or 4.

1. Assume that $p \equiv 1(d)$ and that $p \equiv \pm 1(M)$. Then $C_p$ is ordinary.
2. If $p \equiv -1(d)$, $C_p$ is supersingular.
3. Suppose that $p \equiv -1(d)$ and that $p \equiv \pm 1(M)$. Then $\text{Jac}(C_p)$ is a product of supersingular elliptic curves over $\mathbb{F}_p$ and if moreover $p \geq 5$,
   $$|C(\mathbb{F}_p^2)| = 1 + p^2 + 2 gp.$$ 

In the final section we will show several curves over $\mathbb{F}_p$ whose the number of $\mathbb{F}_p$-points attains the Hasse-Weil’s upper bound. We hope that our theorems may offer a new construction of such a maximal curve, i.e. the number of $\mathbb{F}_p$-points attains the Hasse-Weil’s upper bound.

Let us briefly explain how the theorems will be proved. We will reduce a problem of an abelian variety to one of the $p$-divisible group. For simplicity suppose that $p$ completely splits in $K_0$, $p = \mathfrak{P}_1 \cdots \mathfrak{P}_g$. Then completion $(\mathcal{O}_{K_0})_{\mathfrak{P}_i}$ is isomorphic to $\mathbb{Z}_p$ and

$$\mathcal{O}_{K_0} \otimes_{\mathbb{Z}} \mathbb{Z}_p \simeq (\mathcal{O}_{K_0})_{\mathfrak{P}_1} \times \cdots \times (\mathcal{O}_{K_0})_{\mathfrak{P}_g}, \quad (\mathcal{O}_{K_0})_{\mathfrak{P}_i} \simeq \mathbb{Z}_p.$$
Since $\mathcal{A}[p^\infty]$ is naturally acted by $O_{K_0} \otimes \mathbb{Z}_p$ this yields a decomposition,

$$\mathcal{A}[p^\infty] = G_{\mathfrak{P}_1} \times \cdots \times G_{\mathfrak{P}_g},$$

by the Dieudonné theory and it will proved that the height of $G_{\mathfrak{P}_i}$ is two for all $i$. In order to determine them we will study the action of the Frobenius and the Verschiebung on the Dieudonné module of each factor. This will connect a type of $G_{\mathfrak{P}_i}$ and a decomposing pattern of $\mathfrak{P}_i$ in $K$. In fact using the classification of $p$-divisible group we will show that $G_{\mathfrak{P}_i}$ is isomorphic to $G_{1,1}$ or $G_{1,0} \times G_{0,1}$ according to whether $\mathfrak{P}_i$ remains prime or completely splits in $K$, respectively (Lemma 3.1 and Lemma 3.2). Now Theorem 1.1 and Theorem 1.2 will follow from the facts that relate the type of $\mathcal{A}$ and the shape of $\mathcal{A}[p^\infty]$ (2, 14, 13, 17, 19, see Fact 2.2, Fact 2.3 and Fact 2.4 below). In order to prove Theorem 1.3 it is sufficient to show that the product in Theorem 1.1 is defined over $\mathbb{F}_p$. We will deduce it from the corresponding decomposition of the $p$-divisible group (Proposition 4.1) and Theorem 1.4 will be a consequence of preceding theorems. But a little care is needed to show (3) and here the assumption that $C_p$ is a reduction of a curve over $\mathbb{Q}$ is necessary. That is we have to check that the action of $O_{K_0}$ on $\text{Jac}(C_p)$ is defined over $\mathbb{F}_p$. Since a simple observation shows that (b) implies (a), we may assume that (a) is satisfied. Let $J$ be the Neron’s model of $\text{Jac}(C)$ over $\mathbb{Z}_p$, which is an abelian scheme. We claim the action of $O_{K_0}$ is defined over $\mathbb{Z}_p$. By the Neron’s mapping property our claim is true if the action on the generic fiber $\text{Jac}(C) \otimes \mathbb{Q}_p$ is defined over $\mathbb{Q}_p$. This will be checked by the faithful representation of $O_{K_0}$ on the cotangent space of $\text{Jac}(C) \otimes \mathbb{Q}_p$ at the origin, which is identified with $H^0(C, \Omega^1) \otimes \mathbb{Q}_p$. Now consider the special fiber and the rationality of the action will be obtained.

Let us mention precedent results that generalize Deuring’s results. Let $A$ be an abelian surface with CM by the integer ring of a cyclic quartic CM-field $K$ defined over a number field and $A_v$ its reduction at a good prime $v$ over $p$. Goren[5] has shown that if $p$ completely splits in $K_0$, $p = \mathfrak{P}_1\mathfrak{P}_2$ and if each $\mathfrak{P}_i$ remains prime in $K$, $\mathcal{A}_p$ is a product of supersingular elliptic curves. This coincides with Theorem 1.1. But he has also proved that $\mathcal{A}_p$ is simple and ordinary (resp. isogeneous but not isomorphic to a product of supersingular elliptic curves) if $p$ completely splits (resp. remains prime) in $K$. Using the Kraft’s diagrams, Zaytsev[21] has completely determined $p$-torsion group of an abelian variety over a finite field $k$ (Char $k = p$) with the dimension less than 4. Our Theorem 1.1 and Theorem 1.2 are contained in his results if the dimension of the abelian variety is less than 4.

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2. A review of $p$-divisible groups.

In this section we summarize facts of $p$-divisible groups and the Dieudonné functor which will be used later. The references are [3], [6], [9], [11], [12] and [15]. Throughout the section $k$ will be a field of characteristic $p$.

**Definition 2.1.** Let $h$ be a nonnegative integer. A $p$-divisible group $G$ of height $h(G) = h$ over $k$ is an inductive system of finite group schemes $G_i \to \text{Spec} k$, $(i \geq 1)$, satisfying

1. the dimension of the coordinate ring $k[G_i]$ over $k$ equals to $p^{h-i}$,
2. $p^i$ annihilates $G_i$,
3. there are inclusions $G_i \hookrightarrow G_{i+1}$ such that
   \[ G_{i+1}[p^i] = G_i, \]

and we denote
\[ G = \lim_{\to} G_i. \]

**Remark 2.1.** Let $\Gamma$ be a finite group scheme over $k$. The order is defined to be the dimension of the coordinate ring $k[\Gamma]$ and is described by $|\Gamma|$.

In the definition,
\[ X[f] = \text{Ker}[X \xrightarrow{f} X], \]
for an endomorphism $f$ of a group scheme $X$ and note that $G_{i+j}/G_i = G_j$. Let $Z$ be a connected commutative formal smooth group scheme of finite type over $k$. Then $\{Z[p^i]\}_i$ is a $p$-divisible group and the *dimension* is defined to be one of $Z$. It is equal to the order of the kernel of the Frobenius on $Z$ ([3] Chapter 3). Here are examples of $p$-divisible groups.

1. $\mathbb{G}_m[p^\infty] := \lim_{\to} \mathbb{G}_m[p^n]$, whose height and dimension are 1.
2. $\mathbb{Q}_p/\mathbb{Z}_p := \lim_{\to} \mathbb{Z}/p^n\mathbb{Z}$, is the étale $p$-divisible group of height one:
3. Let $A$ be an abelian variety over $k$ of dimension $g$. Then
   \[ A[p^\infty] := \lim_{\to} A[p^n], \]
is a $p$-divisible group of height $2g$.

Let $G = \lim_{\to} G_i$ be a $p$-divisible group. Then so is the collection of the Cartier dual $G' := \lim_{\to} G'_{i}$ and called the *Serre dual*. For example $\mathbb{Q}_p/\mathbb{Z}_p$ and $\mathbb{G}_m[p^\infty]$ are Serre dual to each other. Taking the Serre dual induces an involution of the category of $p$-divisible groups.
From now on we assume that $k$ is a perfect field of characteristic $p$. Let $W$ be the Witt group scheme defined over $k$ of $\infty$-length. It is isomorphic to a product of infinite affine lines as a scheme

$$\prod_{n \geq 0} \mathbb{A}^1 = \text{Spec}[x_0, x_1, \ldots],$$

and we denote a point $x$ by $x = (x_0, x_1, \ldots)$. The Frobenius $F$ and the Verschiebung $V$ which are endomorphism of $W$ are defined to be

$$F((x_0, x_1, \ldots)) = (x_0^p, x_1^p, \ldots), \quad V((x_0, x_1, \ldots)) = (0, x_0, x_1, \ldots),$$

and since $F V = V F = p$,

$$p((x_0, x_1, \ldots)) = (0, x_0^p, x_1^p, \ldots).$$

For positive integer $n$ let $W_n$ denote the additive group scheme of Witt vectors of length $n$. It is isomorphic to $W/V W$ and the collection of all $\{W_n\}$ forms a direct system by the natural inclusions.

Let $W(k)$ be the ring of Witt vectors whose coefficients are in $k$. The Frobenius induces the ring homomorphism and will be denoted by $\sigma$. Let $D(k)$ be a non-commutative algebra whose coefficient ring is $W(k)$, which is generated by semi-linear operators $F$ and $V$ with the relations

$$FV = VF = p, \quad F \lambda = \lambda^\sigma F, \quad \lambda V = V \lambda^\sigma, \quad \forall \lambda \in W(k).$$

Consider the torsion $W(k)$-module

$$T := W(k)[\frac{1}{p}]/W(k).$$

Then the functor

$$N \mapsto N^* := \text{Hom}_{W(k)}(N, T),$$

defines an anti-equivalence from the abelian category of finite length $W(k)$-modules to itself and

$$N \simeq (N^*)^*,$$

We define the actions of $F$ and $V$ on $N^*$ is defined as

$$(Fl)(n) := \sigma(l(Vn)), \quad (Vl)(n) := \sigma^{-1}(l(Fn)), \quad l \in N^*, n \in N.$$ Let $\Gamma$ be an affine unipotent group over $k$. The the Dieudonné module of $\Gamma$ is defined to be

$$M(\Gamma) := \lim_{n \to \infty} \text{Hom}(\Gamma, W_n).$$

Here "Hom" is taken in the category of affine unipotent group schemes over $k$. It is a contra-variant functor from the category of the affine unipotent group over $k$ to that of all $D(k)$-modules killed by a power of $V$ and induces an anti-equivalence between them. $\Gamma$ is algebraic (resp. finite) if and only if $M(\Gamma)$ is a finitely generated $D(k)$-module (resp. a $W(k)$-module of finite length). By restriction it induces an anti-equivalence between the category of finite unipotent étale (resp. infinitesimal) groups over $k$ and that of $D(k)$-modules which are $W(k)$-module of finite length, killed by a power of $V$ and on which
F is bijective (resp. killed by a power of F). Finally the Dieudonné module of a finite infinitesimal multiplicative group Γ is defined by
\[ M(\Gamma) := M(\Gamma^\vee)^*. \]
Then the functor M induces an anti-equivalence between the abelian category of finite commutative group schemes over k of a p-power order to that of left \( D(k) \)-modules of finite length. It is known that the length of \( M(\Gamma) \) is equal to \( \log_p |\Gamma| \). Here are some examples.

**Example 2.1.**

1. Let \( W_n^m \) be the kernel of \( F^m \) on \( W_n \). Then
   \[ M(W_n^m) = D(k)/(D(k)F^m + D(k)V^n). \]
2. \( M(\mathbb{Z}/p\mathbb{Z}) \) (resp. \( M(\mathbb{G}_m[p]) \) ) is isomorphic to \( k \) with \( F = 1, V = 0 \) (resp. \( F = 0, V = 1 \)).
3. Let \( E \) be a supersingular elliptic curve over \( k \), then
   \[ M(E[p]) = D(k) \otimes W(k) k/D(k) \otimes W(k) k(F - V). \]

We define the Dieudonné module of a \( p \)-divisible group \( G = \lim_{i \to \infty} G_i \) as
\[ M(G) := \lim_{i \to \infty} M(G_i), \]
which is a \( D(k) \)-module by definition. It is a free \( W(k) \)-module with rank \( h(G) \). In this way the Dieudonné functor \( M \) gives a contra-equivalence between the category of \( p \)-divisible groups defined over \( k \) and one of \( D(k) \)-modules that are free over \( W(k) \) with finite rank. For a pair of coprime integers \( (d, c) \) so that \( d > 0, c \geq 0 \) the \( p \)-divisible group \( G_{d,c} \) is defined to be
\[ G_{d,c} := \ker[F^c - V^d : W[p^\infty] \to W[p^\infty]], \]
where \( W[p^\infty] := \lim_{n \to \infty} W[p^n] \). Its Dieudonné module is
\[ M(G_{d,c}) = D(k)/D(k)(F^c - V^d), \]
and the dimension and the height are \( d \) and \( c+d \), respectively. One sees that \( G_{1,0} \simeq \mathbb{G}_m[p^\infty] \) and it is convenient to set \( G_{0,1} := \mathbb{Q}_p/\mathbb{Z}_p \). Then \( G_{c,d} = (G_{d,c})^\vee \) for every pair of coprime non-negative integer \( (d, c) \neq (0, 0) \). Temporally we assume that \( k \) is algebraically closed. Then \( G_{d,c} \) is characterized by the height \( h = c + d \) and the slope \( d/h = d/(c + d) \). A \( p \)-divisible group \( G \) over \( k \) is mentioned simple if any epimorphism from \( G \) is either an isogeny or the structure morphism (here an isogeny is a homomorphism whose kernel and cokernel are finite). Then \( G_{d,c} \) is simple and conversely any simple \( p \)-divisible group is isomorphic to a certain \( G_{d,c} \). Note that a simple \( p \)-divisible group \( G \) is isomorphic to \( G_{d,c} \) if and only if there is a pair of non-negative integer \( (m, n) \neq (0, 0) \) so that
\[ G[F^m] = G[V^n], \quad \frac{n}{m+n} = \frac{d}{c+d}. \]
In fact (2) shows that \( G \simeq G_{d,c} \) if and only if
\[ M(G)/V^nM(G) = M(G)/F^mM(G), \]
for a pair of non-negative integer \( (m, n) \neq (0, 0) \) satisfying \( d/(c + d) = n/(m + n) \) and this is equivalent to (3). Here are \( p \)-divisible groups whose height is less than 3 (3, p.93):
(1) \( h(G) = 0 \) iff \( G = 0 \).
(2) If \( h(G) = 1 \), then \( G = G_{1,0} \) or \( G_{0,1} \).
(3) If \( h(G) = 2 \), \( G \) is the one of the followings,
\[ G^2_{1,0}, \ G_{1,0} \times G_{0,1}, \ G^2_{0,1}, \ G_{1,1}. \]

Set \( \mathbb{L} = G_{1,0} \times G_{0,1} \). Then \( G_{1,1} \) (resp. \( \mathbb{L} \)) is isomorphic to the \( p \)-divisible group of a supersingular (resp. an ordinary) elliptic curve. Let \( X \) and \( Y \) be \( p \)-divisible groups. If there is an isogeny between them we say that they are isogeneous and describe as \( X \sim Y \).

This notion defines an equivalence relation on the set of \( p \)-divisible groups.

**Fact 2.1.** ([3], p.85 or [9], p.35) Let \( G \) be a \( p \)-divisible group over an algebraically closed field \( k \). Then there is an isogeny:
\[ G \sim G_{1,0}^f \times G_{0,1}^f \times \prod_i G_{d_i,c_i}, \]
where \( \{d_i, c_i\}_i \) are pairs of positive coprime integers.

Let \( A \) be an abelian variety over \( k \) of dimension \( g \) and set \( \overline{A} := A \otimes_k \overline{k} \). We define \( p \)-rank \( f(A) = f \) as an integer such that \( A[p](\overline{k}) \simeq (\mathbb{Z}/p\mathbb{Z})^f \), which equals to \( \dim_{\overline{k}} \text{Hom}(\mathbb{G}_m[p], \overline{A}[p]) \).

Let \( \alpha_p \) be a finite group scheme defined by \( \alpha_p := \text{Spec} \overline{k}[x]/(x^p) \). The \textit{a-number} is defined to be \( a(A) := \dim_{\overline{k}} \text{Hom}(\alpha_p, \overline{A}[p]) \). It is known that \( 0 \leq f(A) \leq g \) and that \( 0 \leq a(A) + f(A) \leq g \). We say \( A \) ordinary if \( f(A) = g \).

**Fact 2.2.** ([2]) Let \( A \) be an abelian variety over \( k \) of dimension \( g \). Then the following are equivalent:

1. \( A \) is ordinary,
2. \( A[p]\infty \simeq \mathbb{L}^g \).

This is well-known if \( g = 1 \). In fact let \( E \) be an elliptic curve defined over an algebraic closed field of characteristic \( p \). Since there is an exact sequence
\[ 0 \to \alpha_p \to G_{1,1}[p] \to \alpha_p \to 0, \]
the previous classification of \( p \)-divisible group of height 2 shows that \( a(E) = 1 \) if and only \( E \) is supersingular. This observation is generalized to a higher dimensional abelian varieties. The following fact is due to Deligne, Oort, Shioda and Tate.

**Fact 2.3.** ([14], [17], [19]) Let \( A \) an abelian variety over an algebraic closed field \( k \) with dimension \( g \). If
\[ A[p]\infty \simeq G_{1,1}^g, \]
then \( A \) is supersingular.

**Fact 2.4.** ([13] Theorem 2) Let \( A \) an abelian variety over an algebraic closed field \( k \) of with dimension \( g \). If
\[ a(A) = g, \]
then \( A \) is superspecial.
Let $A$ be an abelian variety over $k$ of dimension $g$. We denote the Dieudonné module of $A[p^{\infty}]$ by $T_p(A)$, which is a free $W(k)$-module of rank $2g$.

**Fact 2.5.** ([11] Theorem 6) Let $A$ and $B$ are abelian varieties over a finite field $k$. Then

$$\text{Hom}_k(A, B) \otimes_{\mathbb{Z}} \mathbb{Z}_p \simeq \text{Hom}_{D(k)}(T_p(B), T_p(A)).$$

### 3. The $p$-divisible group of an abelian variety with CM

Let $k$ be a finite field of characteristic $p$ with $[k : \mathbb{F}_p] = r$ and $A$ an abelian variety over $k$ of dimension $g$ endowed with CM by $\mathcal{O}_K$. We fix an imbedding of the CM-field $K$ into $\mathbb{C}$ and denote the restriction of the complex conjugation to $K$ by $"r"$. We assume that $\text{End}_k(A) = \text{End}_k(\overline{A})$ and denote it by $R$. We first note that $F^r$ and $V^r$ are contained in $\mathcal{O}_K$. In fact here is a proof after [21] Lemma 3.2. Set $\pi_F := F^r$ and $\pi_V := V^r$. The assumption is equivalent to that $\pi_F$ and $\pi_V$ are contained in the center $C$ of $R \otimes \mathbb{Z}$. Since the commutant of $K$ in $R \otimes \mathbb{Z}$ is itself ([10] S4 Corollary 1), $C$ is contained in $K$ and $\pi_F, \pi_V \in \mathcal{O}_K = K \cap R$. It is known that $\pi_F$ is a $q$-Weil number ($q = p^r$) and $\pi_F \cdot \pi_F' = p^r$ ([12] Theorem 3.2 and Proposition 2.2). On the other hand since $FV = VF = p$, $\pi_F \cdot \pi_V = F^rV^r = p^r$. Therefore $(\pi_F' - \pi_V)\pi_F = 0$ and because $\pi_F$ is surjective we conclude $\pi_F' = \pi_V$.

For simplicity we will denote $\pi_F$ and $\pi_V$ by $\pi$ and $\pi'$, respectively. As we have seen in (3) a simple $p$-divisible group is characterized by its slope. Since $\pi = F^r$ and $\pi' = V^r$ are contained in $\mathcal{O}_K$, we will obtain an information of the Dieudonné module of a simple component of $A[p^{\infty}]$ from the behavior of $\{\pi, \pi'\}$ in $\mathcal{O}_K \otimes \mathbb{Z} \mathbb{Z}_p$ with help of Fact 2.5. This is our strategy.

We assume that $p$ is unramified in $K$ and let

$$p = \mathfrak{P}_1 \cdots \mathfrak{P}_t,$$

be the prime factorization in $K_0$ so that $\mathfrak{P}_i$ remains prime (resp. completely splits) in $K$ for $1 \leq i \leq s$ (resp. $s + 1 \leq i \leq t$). Thus

$$p = \mathfrak{P}_1 \cdots \mathfrak{P}_s \mathfrak{P}_{s+1} \mathfrak{P}_{s+1}' \cdots \mathfrak{P}_t \mathfrak{P}_t',$$

in $K$. We set $\mathcal{P}_{\text{inert}} := \{\mathfrak{P}_1, \ldots, \mathfrak{P}_s\}$ and $\mathcal{P}_{\text{split}} := \{\mathfrak{P}_s+1, \ldots, \mathfrak{P}_t\}$, where $f(\mathfrak{P}/p)$ is the inertia degree. Similarly $\mathcal{P}_{\text{split}} := \{\mathfrak{P}_s+1, \ldots, \mathfrak{P}_t\}$ and $\mathcal{P}_{\text{split}} := \{\mathfrak{P}_s+1, \ldots, \mathfrak{P}_t\}$. By (5),

$$\mathcal{O}_K \otimes \mathbb{Z} \mathbb{Z}_p \simeq W(F_{\mathfrak{P}_1}) \times \cdots \times W(F_{\mathfrak{P}_s}) \times \{W(F_{\mathfrak{P}_{s+1}}) \times W(F_{\mathfrak{P}_{s+1}'})\} \times \cdots \times \{W(F_{\mathfrak{P}_t}) \times W(F_{\mathfrak{P}_t'})\},$$

where $F_{\mathfrak{P}}$ is the residue field. Using this we define $e_i$ to be an idempotent of $\mathcal{O}_K \otimes \mathbb{Z} \mathbb{Z}_p$ which corresponds to $0, 0, 1, 0, \ldots, 0$ in RHS, where "1" is placed at the $i$-th from the left. Remember that since the height of $A[p^{\infty}]$ is $2g$, $T_p(A)$ is a free $W(k)$-module of rank $2g$. Moreover by Fact 2.5 it has a faithful action of $\mathcal{O}_K \otimes \mathbb{Z} W(k)$. The following lemma is a consequence of these facts.
**Lemma 3.1.** $e_i T_p(A)$ is a $D(k)$-module that is free over $W(k)$ with rank $f(\hat{\mathfrak{p}}_i/p)$.

**Proof.** Note that, by Fact 2.5, $e_i T_p(A)$ is a $D(k)$-module which is free over $W(k)$ and we only have to identify the rank. By the decomposition we find that $e_i(\mathcal{O}_K \otimes_{\mathbb{Z}} W(k)) \simeq W(\mathbb{F}_{\hat{\mathfrak{p}}_i}) \otimes_{\mathbb{Z}_p} W(k)$, which is a product of complete discrete valuation rings

$$e_i(\mathcal{O}_K \otimes_{\mathbb{Z}} W(k)) \simeq R^{(1)}_i \times \cdots \times R^{(\nu(j))}_i,$$

so that every component is free and has a finite rank over $W(k)$. Let $e_j$ be an element of $e_i(\mathcal{O}_K \otimes_{\mathbb{Z}} W(k))$ that corresponds to $(0, \cdots, 0, 1, 0, \cdots, 0)$ in RHS as before and set

$$M_j^{(j)} = e_j(e_i T_p(A)).$$

It is a non-zero free $R^{(j)}_i$-module because the action of $\mathcal{O}_K \otimes_{\mathbb{Z}} W(k)$ on $T_p(A)$ is faithful. Let $\mu_i(j)$ be its rank and

$$T_p(A) \simeq \bigoplus_i \bigoplus_j (R^{(j)}_i)^{\mu_i(j)}, \quad \mu_i(j) \geq 1.$$

Since $\mathcal{O}_K \otimes_{\mathbb{Z}} W(k) \simeq \prod_i \prod_j R^{(j)}_i$ and since rank$_{W(k)} T_p(A) = \text{rank}_{W(k)} \mathcal{O}_K \otimes_{\mathbb{Z}} W(k) = 2g$ we find that $\mu_i(j) = 1$ for all $i$ and $j$. Therefore

$$\text{rank}_{W(k)} e_i T_p(A) = \sum_j \text{rank}_{W(k)} R^{(j)}_i = \text{rank}_{W(k)} W(\mathbb{F}_{\hat{\mathfrak{p}}_i}) \otimes_{\mathbb{Z}_p} W(k) = f(\hat{\mathfrak{p}}_i/p).$$

Since the Dieudonné functor $M$ gives the anti-equivalence, there is a $p$-divisible subgroup $G_i$ of $A[p^\infty]$ such that

$$M(G_i) = e_i T_p(A).$$

Set $\Gamma_j := G_{s+(2j-1)} \times G_{s+2j}$ $(1 \leq j \leq t-s)$ and

$$A[p^\infty] = G_1 \times \cdots \times G_s \times \Gamma_1 \times \cdots \times \Gamma_{t-s}. \quad (6)$$

For a prime factor $\mathfrak{p}$ of $p$ in $K$ we denote the corresponding factor of $A[p^\infty]$ by $G_{\mathfrak{p}}$. For a prime $\mathfrak{p}$ of $K_0$ dividing $p$ we define $p$-divisible subgroup $G_\mathfrak{p}$ of $A[p^\infty]$ as follows. If $\mathfrak{p}$ is contained in $\mathcal{P}_{\text{inert}}$ define $G_{\mathfrak{p}} := G_{\hat{\mathfrak{p}}}$ where $\hat{\mathfrak{p}}$ is the unique prime of $K$ over $\mathfrak{p}$. Since $\mathbb{F}_{\mathfrak{p}} \simeq \mathbb{F}_{p^f(\mathfrak{p}/p)}$ we see that by Lemma 3.1

$$h(G_\mathfrak{p}) = \text{rank}_{W(\mathfrak{p})} M(G_{\hat{\mathfrak{p}}}) = 2f(\mathfrak{p}/p). \quad (7)$$

On the other hand if it splits: $\mathfrak{p} = \hat{\mathfrak{p}} \times \mathfrak{p}'$, we define $G_{\mathfrak{p}} := G_{\hat{\mathfrak{p}}} \times G_{\mathfrak{p}'}$. Since $\mathbb{F}_{\hat{\mathfrak{p}}} \simeq \mathbb{F}_{\mathfrak{p}'} \simeq \mathbb{F}_{p^f(\mathfrak{p}/p)}$, a similar observation shows

$$h(G_{\mathfrak{p}}) = h(G_{\hat{\mathfrak{p}}}) = f(\mathfrak{p}/p), \quad h(G_{\mathfrak{p}'}) = h(G_{\hat{\mathfrak{p}}}) + h(G_{\mathfrak{p}'}) = 2f(\mathfrak{p}/p). \quad (8)$$

**Lemma 3.2.** For $\mathfrak{p} \in \mathcal{P}_{\text{split}}$, $G_{\mathfrak{p}}$ is isomorphic to $\mathbb{L}$. 
Proof. By (8), \( h(G_{\mathfrak{P}}) = h(G_{\mathfrak{P}'})) = 1 \) and the classification of \( p \)-divisible groups shows that \( G_{\mathfrak{P}} \) or \( G_{\mathfrak{P}'} \) is one of \( \{ G_{1,0}, G_{0,1} \} \). Remember that \( G_{1,0} \) (resp. \( G_{0,1} \)) is characterized by the fact \( V \) (resp. \( F \)) is an isomorphism on \( \mathbb{M}(G_{1,0}) \) (resp. \( \mathbb{M}(G_{0,1}) \)). Let

\[
(F') = (\pi) = \mathfrak{P}^a(\mathfrak{P}')^a \delta, \quad (\delta, \mathfrak{P}) = (\delta, \mathfrak{P}') = 1,
\]

be the factorization. Then

\[
(V') = (\pi') = \mathfrak{P}'^a(\mathfrak{P}')^a \delta',
\]

and

\[
(p') = (F'V') = (\pi \pi') = (\mathfrak{P} \mathfrak{P}')^a + a' \delta'.
\]

By (5), \( r = a + a' \). Suppose that \( G_{\mathfrak{P}} = G_{1,0} \). This implies that \( V \) is an isomorphism on \( \mathbb{M}(G_{\mathfrak{P}}) \) and so is \( \pi' \) (note that the slope of a \( p \)-divisible group is invariant under a base change). Since the action of \( \mathcal{O}_K \) on \( \mathbb{M}(G_{\mathfrak{P}}) \) (resp. \( \mathbb{M}(G_{\mathfrak{P}'}) \)) factors through an imbedding \( \mathcal{O}_K \hookrightarrow \mathcal{O}_{K,\mathfrak{P}} \simeq \mathbb{Z}_p \) (resp. \( \mathcal{O}_K \hookrightarrow \mathcal{O}_{K,\mathfrak{P}'} \simeq \mathbb{Z}_p \)), \( \pi' \) should be a unit in \( \mathcal{O}_{K,\mathfrak{P}} \) and so \( a' = 0 \). Therefore \( (V') = (\mathfrak{P}')^a \delta' \) and \( (F') = \mathfrak{P}' \delta' \), which implies that \( F' \) is an isomorphism on \( \mathbb{M}(G_{\mathfrak{P}'}) \). Hence \( \mathcal{G}_{\mathfrak{P}'_1} \simeq \mathbb{M}_{0,1} \) and \( \mathcal{G}_{\mathfrak{P}'_2} \simeq \mathbb{M}_{1,0} \times \mathbb{M}_{0,1} = \mathbb{L} \). In the case of \( \mathcal{G}_{\mathfrak{P}} = G_{0,1} \) the proof is similar.

\[ \square \]

Lemma 3.3. For \( \mathfrak{P} \in \mathcal{P}_{\text{inert}} \), \( \mathcal{G}_{\mathfrak{P}} \) is isogeneous to \( G_{1,1}^{f(\mathfrak{P}/p)} \).

Proof. By Fact 2.1 \( \mathcal{G}_{\mathfrak{P}} \) is isogeneous to \( \prod G_{d,c} \). Using (3) we will show all simple factors are isomorphic to \( G_{1,1} \). Let

\[
(F') = (\pi) = \mathfrak{P}^a \delta, \quad (\delta, \mathfrak{P}) = 1,
\]

be the factorization. Since \( \mathfrak{P}' = \mathfrak{P} \),

\[
(V') = (\pi') = \mathfrak{P}'^a \delta', \quad (\delta, \mathfrak{P}') = 1,
\]

and

\[
(9) \quad \mathcal{G}_{\mathfrak{P}}[F'] = \mathcal{G}_{\mathfrak{P}}[V'].
\]

Let \( G \) be a simple factor of \( \mathcal{G}_{\mathfrak{P}} \). Then (9) shows

\[
G[F'] = G[V'],
\]

and \( G = G_{1,1} \) by (3). Thus \( \mathcal{G}_{\mathfrak{P}} \) is isogeneous to \( G_{1,1}^{f_1} \) and

\[
h(G_{\mathfrak{P}}) = 2f,
\]

which implies the claim by (7).

\[ \square \]

Set \( \mathcal{P} = \mathcal{P}_{\text{inert}} \cup \mathcal{P}_{\text{split}} = \{ \mathfrak{P}_1, \cdots, \mathfrak{P}_t \} \).

Proposition 3.1. \( (1) \) Suppose \( \mathcal{P} = \mathcal{P}_{\text{inert}} \). Then \( A \) is supersingular.

\( (2) \) The \( a \)-number of \( A \) is greater than or equal to \( |\mathcal{P}_{\text{inert}}| \).
Suppose that $P = P_{\text{inert}}^f \cup P_{\text{split}}^f = 1$.

Then

$$a(A) = |P_{\text{inert}}^f|, \quad f(A) = |P_{\text{split}}^f|.$$ 

In particular if $P = P_{\text{inert}}^f$ (resp. $P = P_{\text{split}}^f$) $A$ is superspecial (resp. ordinary).

**Proof.** (1) is an immediate consequence of Lemma 3.3 and Fact 2.3. If necessary arranging the indices, $P_{\text{inert}}^f = \{P_1, \ldots, P_r\}$ ($r \leq s$) and we consider a subgroup $G_{\mathfrak{p}_i} \times \cdots \times G_{\mathfrak{p}_r}$ of $A[p^\infty]$. By (7) and Lemma 3.3 $\overline{G}_{\mathfrak{p}_i}$ is isomorphic to $G_{1,1}$ for $1 \leq \forall i \leq r$. Therefore $\overline{A}[p]$ contains $G_{1,1}[p]^s$ and by (4),

$$a(A) = \dim \overline{\mathbb{F}}_p \text{ Hom}(\alpha, \overline{A}[p]) \geq \dim \overline{\mathbb{F}}_p \text{ Hom}(\alpha_p, G_{1,1}[p]^s) = r.$$ 

Finally we prove (3). Set $s = |P_{\text{inert}}^f|$ and $g - s = |P_{\text{split}}^f|$. Lemma 3.2 and Lemma 3.3 show that the product (6) becomes

$$\overline{A}[p^\infty] \simeq G_{1,1}^1 \times \mathbb{L}^{g-s},$$

which implies $a(A) = |P_{\text{inert}}^f|$ and $f(A) = |P_{\text{split}}^f|$. The last statement follows from Fact 2.2 and Fact 2.4, respectively.

□

Now Theorem 1.1 and Theorem 1.2 are direct consequences of Proposition 3.1.

**Corollary 3.1.** Suppose that $K$ is a Galois extension of $\mathbb{Q}$.

1. If $P_{\text{inert}}^f$ is not empty, $A$ is supersingular.

2. Suppose that $p$ completely splits in $K_0$. If it completely splits even in $K$, $A$ is ordinary and otherwise $A$ is superspecial.

4. **Rationality**

Let $A$ be an abelian variety over $\mathbb{F}_p$ which satisfies the assumption of Theorem 1.3. The completion $(\mathcal{O}_K)_0 \otimes_{\mathbb{Z}} \mathbb{F}_p$ of $\mathcal{O}_K$ at $\mathfrak{p}_i$ is isomorphic to $\mathbb{Z}_p$ and

$$(10) \quad \mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p \simeq \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p, \quad \alpha = (\alpha_1, \ldots, \alpha_g).$$

Let $e_i^0$ be the idempotent in $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p$ which corresponds to $0, 1, 0, \ldots, 0, 1$ in RHS of (10) (where ”1” is placed at the $i$-th from the left, as before) and $G_i^0$ a $p$-divisible subgroup of $A[p^\infty]$ such that $\mathbb{M}(G_i^0) = e_i^0 T_p(A)$. Then

$$(11) \quad A[p^\infty] = G_1^0 \times \cdots \times G_g^0,$$

and since the action of $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p$ on $A[p^\infty]$ defined over $\mathbb{F}_p$, so is the product. The following lemma is clear from Lemma 3.3 (see also the proof of Proposition 3.1).

**Lemma 4.1.** $G_i^0$ is isomorphic to $G_{1,1}$ for all $i$. 

Proposition 4.1. Let $J$ be an abelian variety of dimension $g$ over $\mathbb{F}_p$ so that
\[ J[p^\infty] = G_1 \times \cdots \times G_g, \]
over $\mathbb{F}_p$, where $G_i$ is a $p$-divisible group defined over $\mathbb{F}_p$ with $\overline{G_i} \simeq G_{1,1}$ ($\forall i$). Then it is a product of supersingular elliptic curve \( \{E_i\}_{1 \leq i \leq g} \)
over $\mathbb{F}_p$. If moreover $p$ is greater than 3, $\Phi_{J,p}(t) = (t^2 + p)^g$.

**Proof.** By (4) the $a$-number of $J$ is $g$ and, by Fact 2.4,
\[ J \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p \simeq E_1 \times \cdots \times E_g, \]
where $E_i$ is a supersingular elliptic curve ($\forall i$). A simple consideration shows that, if necessary changing indices, we may assume that $G_i = E_i[p^\infty]$. Let $\phi$ be the $p$-th power Frobenius. Since $J$ is defined over $\mathbb{F}_p$ we have the diagram:
\[
\begin{array}{ccc}
E_i & \xrightarrow{\phi} & E_i^\phi \\
\nu_i \downarrow & & \nu_i^\phi \downarrow \\
J & \xrightarrow{\phi} & J,
\end{array}
\]
where $\nu_i$ is the imbedding. Take the $p$-divisible groups and
\[
\begin{array}{ccc}
E_i[p^\infty] & \xrightarrow{\phi} & E_i^\phi[p^\infty] \\
\nu_i[p^\infty] & \downarrow & \nu_i^\phi[p^\infty] \\
J[p^\infty] & \xrightarrow{\phi} & J[p^\infty].
\end{array}
\]
Since by the assumption all $\{E_i[p^\infty](= G_i)\}_i$ and $\{\nu_i[p^\infty]\}_i$ are defined over $\mathbb{F}_p$, $E_i[p^\infty] = E_i^\phi[p^\infty]$ and $\nu_i[p^\infty] = \nu_i^\phi[p^\infty]$ ($\forall i$). Thus $E_i = E_i^\phi$ and $\nu_i = \nu_i^\phi$, which shows that each component $\{E_i\}_i$ and the product are defined over $\mathbb{F}_p$. The last claim follows from the well-known fact that the characteristic polynomial of $p$-power Frobenius of a supersingular elliptic curve over $\mathbb{F}_p$ is $t^2 + p$ if $p \geq 5$ ([18]).

\[ \square \]

**Theorem 1.3** follows from (11), Lemma 4.1 and Proposition 4.1.

Let $F$ be a field of characteristic 0 and $A$ an abelian variety over $F$ of dimension $g$. Fix a base $\{\omega_1, \cdots, \omega_g\}$ of $H^0(A, \Omega^1)$ over $F$ and we consider the faithful representation,
\[ \rho : \text{End}_{\overline{\mathbb{F}}}(A) \to M_g(\overline{F}), \]
deﬁned by
\[ \alpha^* \omega = \omega \cdot \rho(\alpha), \quad \omega = (\omega_1, \cdots, \omega_g). \]
This is compatible with the action of $\text{Gal}(\overline{F}/F)$ and the faithfulness of $\rho$ yields,
\[ \text{End}_F(A) = \{ \alpha \in \text{End}_{\overline{\mathbb{F}}}(A) : \rho(\alpha) \in M_g(F) \}. \]
Remark 4.1. \( \rho \) may be identified with the representation on the cotangent space of \( A \) at the origin.

Proof of Theorem 1.4. (1) and (2) are consequences of Theorem 1.2 and Theorem 1.1, respectively. Let us show (3). We first claim that (b) implies (a). Take a base \( \{ \omega_1, \cdots, \omega_g \} \) of \( H^0(\text{Jac}(C), \Omega^1) \) over \( \mathbb{Q} \) and consider the representation \( \rho \). The assumption that the action of \( \mathcal{O}_K \) is defined over \( K \) implies \( \rho(\mathcal{O}_K) \subset M_g(K) \) by (12). Since \( \rho \) is compatible with action of Galois group, take the invariant part of the complex conjugation and \( \rho(\mathcal{O}_{K_0}) \subset M_g(K_0) \). Thus, by (12), \( \mathcal{O}_{K_0} \) is contained in \( \text{End}_{K_0}(\text{Jac}(C) \otimes_{\mathbb{Q}} K_0) \). Now we show that (a) implies the claim. Let \( \mathcal{J} \) be the Neron’s model of \( \text{Jac}(C) \) over \( \mathbb{Z}_p \), which is an abelian scheme. Since \( p \) completely splits in \( K_0 \), \( Z_\rho \simeq (\mathcal{O}_{K_0})_{\mathfrak{P}_i} \) (here we have used the notation of Theorem 1.3). Together with (12) this implies that the action of \( \mathcal{O}_{K_0} \) on \( \mathcal{J} \) is defined over \( \mathbb{Z}_p \). In fact by the Neron’s mapping property of \( \mathcal{J} \) ([I], §1 Proposition 8) it is sufficient to show that the action of \( \mathcal{O}_{K_0} \) on the generic fiber \( \text{Jac}(C) \otimes_{\mathbb{Q}} \mathbb{Q}_p \) is defined over \( \mathbb{Q}_p \). Since \( (K_0)_{\mathfrak{P}_i} \simeq \mathbb{Q}_p \), the image \( \rho(\mathcal{O}_{K_0}) \) is contained in \( M_g((K_0)_{\mathfrak{P}_i}) \simeq M_g(\mathbb{Q}_p) \) and (12) shows that \( \mathcal{O}_{K_0} \subset \text{End}_{\mathbb{Q}_p}(\text{Jac}(C) \otimes_{\mathbb{Q}} \mathbb{Q}_p) \). Take the special fiber and \( \text{Jac}(C_p) \) satisfies the assumption of Theorem 1.3. Therefore \( \text{Jac}(C_p) \) is a product of supersingular elliptic curves defined over \( \mathbb{F}_p \). Suppose that \( p \) is greater than 3. Since \( \Phi_{\text{Jac}(C_p), p}(t) = (t^2 + p)^g \) the eigenvalues of the action of the \( p \)-th power Frobenius on \( H^1_{et}(\overline{\mathcal{C}_p}, \mathbb{Q}_l) \) is \( \{ \sqrt{-p}, -\sqrt{-p} \} \). Use the Grothendieck-Lefschetz trace formula ([I10], Theorem 12.3) and

\[
|C_p(\mathbb{F}_p^2)| = 1 + p^2 - \text{Tr}[\text{Fr}_{p^2} : H^1_{et}(\overline{\mathcal{C}_p}, \mathbb{Q}_l)] = 1 + p^2 + 2gp.
\]

□

Proof of Corollaries. We first show Corollary 1.1. \( K_0 \) is \( \mathbb{Q}(\zeta_N + \zeta_N^{-1}) \) and the sequence

\[
1 \to \text{Gal}(K/K_0) \simeq \{ \pm 1 \} \to \text{Gal}(K/\mathbb{Q}) \simeq (\mathbb{Z}/(N))^{\times} \to \text{Gal}(K_0/\mathbb{Q}) \to 1,
\]

shows that \( p \equiv -1(N) \) iff \( p \) completely splits in \( K_0 \) and every prime factor of \( p \) in \( K_0 \) remains prime in \( K \). On the other hand \( p \equiv 1(N) \) iff \( p \) completely splits in \( K \). An existence of a positive integer \( h \) with \( p^h \equiv -1(N) \) implies that every prime factor of \( p \) in \( K_0 \) remains prime in \( K \). Now the desired claims follow from Theorem 1.4. In the case of Corollary 1.2 observe \( K_0 = \mathbb{Q}(\zeta_M + \zeta_M^{-1}) \) and

\[
\text{Gal}(K/\mathbb{Q}) \simeq \text{Gal}(\mathbb{Q}(\zeta_M + \zeta_M^{-1})/\mathbb{Q}) \times \text{Gal}(\mathbb{Q}(\zeta_d)/\mathbb{Q}).
\]

This is isomorphic to \( ((\mathbb{Z}/(M))^{\times}/\{ \pm 1 \}) \times \{ \pm 1 \} \) and the proof is similar.

\[\square\]

5. Examples

Example 5.1. ([I]) Let \( l \) be an odd prime and we define a curve \( C(l) \) to be the smooth projective model of

\[
y^l = x(1-x),
\]

over \( \mathbb{Q} \). The genus is \( (l-1)/2 \) and an \( l \)-th primitive root of unity \( \zeta_l \) acts by

\[
\zeta_l(x) = x, \quad \zeta_l(y) = \zeta_l y.
\]
Since it is defined over \( K := \mathbb{Q}(\zeta_l) \), so is the action of \( \mathbb{Z}[\zeta_l] \) on \( \text{Jac}(C(l)) \). Thus \( C(l) \) satisfies the assumption **Theorem 1.4**. It is also known that there is a \( \mathbb{Q} \)-rational base \( \{ \omega_1, \ldots, \omega_{(l-1)/l} \} \) of \( H^0(C(l), \Omega^1) \) satisfying
\[
(\zeta_l)^*(\omega_i) = \zeta_l^{\mu_i}, \quad 0 \leq \mu_i \leq l - 1,
\]
(\text{\[7\] \text{\$1 \text{ Theorem 7.1} \]}). Let \( p \) be a good prime so that \( p^h \equiv -1(l) \) for a certain positive integer \( h \). Set \( q = p^h \) and let \( F_{q+1} \) denote the Fermat curve, \( X^{q+1} + Y^{q+1} = 1 \).

By
\[
x = X^{q+1}, \quad y = (XY)^{q+1},
\]
we have a surjective morphism defined over \( \mathbb{Q} \),
\[
\pi : F_{q+1} \to C(l),
\]
and \( H^1_{et}(C(l)_p \otimes_{\mathbb{F}_p} \mathbb{F}_p, \mathbb{Q}_l) \) is a Galois submodule of \( H^1_{et}((F_{q+1})_p \otimes_{\mathbb{F}_p} \mathbb{F}_p, \mathbb{Q}_l) \). On the other hand \( |F_{q+1}(\mathbb{F}_{p^2})| \) attains the Hasse-Weil upper bound \( 1 + p^2 + 2gp \) and
\[
\Phi_{F_{q+1}, p}(t) = (t^2 + p)^g
\]
where \( g = q(q - 1)/2 \) is the genus of \( F_{q+1} \) (\text{\[8\] \text{Example 6.3.6} \]). Thus \( \Phi_{C(l), p}(t) = (t^2 + p)^{(l-1)/2} \) and
\[
|C(l)_p(\mathbb{F}_{p^2})| = 1 + p^2 + (l - 1)p,
\]
if there is a positive integer \( h \) such that \( p^h \equiv -1(l) \). **Corollary 1.1** recovers this observation if \( h = 1 \) but otherwise it only states that \( C(l) \) has a supersingular reduction. Therefore **Corollary 1.1** is only a sufficient condition for a prime \( p \) at which the reduction of a CM-curve is \( \mathbb{F}_{p^2} \)-maximal, i.e. the number of \( \mathbb{F}_{p^2} \)-points attains the Hasse-Weil upper bound.

**Example 5.2.** (\text{\[9\] \text{Example 2.3} \}) Let us consider a curve
\[
C : y^3 = x(x^7 + 1),
\]
that has automorphisms
\[
(x, y) \xrightarrow{\zeta_7} (\zeta_7^3 x, \zeta_7 y), \quad (x, y) \xrightarrow{\zeta_3} (x, \zeta_3 y),
\]
and the involution,
\[
\tau : (x, y) \mapsto \left( \frac{1}{x}, \frac{y}{x^3} \right).
\]
The quotient \( C/\langle \tau \rangle \) has a smooth model
\[
X : 2y^3 = x^4 - 2 \cdot 7^2 x^2 + 2 \cdot 7^2 x - 7^3,
\]
of genus 3 and the endomorphism ring of \( \text{Jac}(X) \) is the integer ring \( \mathcal{O}_K \) of \( K := \mathbb{Q}(\zeta_7 + \zeta_7^{-1}, \zeta_3) \). Therefore \( X \) is a CM-curve. A base of \( H^0(X, \Omega^1) \) is given by
\[
\frac{(1 - x)dx}{y}, \quad \frac{(1 - x^4)dx}{y^2}, \quad \frac{(x - x^3)dx}{y^2},
\]
which are eigenvectors of the action of $\zeta_7 + \zeta_7^{-1}$ whose eigenvalues are contained in $K_0 = \mathbb{Q}(\zeta_7 + \zeta_7^{-1})$. Hence by (12) the action of $O_{K_0}$ on Jac$(X)$ is defined over $K_0$ and $X$ satisfies the assumption of Theorem 1.4. By Corollary 1.2 we see that a good prime $p$ is supersingular if $p \equiv -1(3)$. Moreover it is superspecial if $p \equiv \pm 1(7)$ and $p \equiv -1(3)$. On the other hand, using [4], one can determine $\Phi_{X,p}(t)$. In fact they have shown that Jac$(C)$ is isogeneous to Jac$(X)^2 \times E$ where $E$ is a CM-elliptic curve whose defining equation is $x^3 = y(y + 1)$.

**Fact 5.1.** ([4] Corollary 3.1 and Corollary 3.2)

1. For $p \equiv 2, 5, 11, 17 (21)$,
   \[ \Phi_{C,p}(t) = (t^4 - pt^2 + p^2)^2(t^2 + p)^3. \]
2. For $p \equiv 8, 20 (21)$,
   \[ \Phi_{C,p}(t) = (t^2 + p)^7. \]

Since $X$ has genus 3 the degree of $\Phi_{X,p}(t)$ should be six and therefore

- If $p \equiv 2, 5, 11, 17 (21)$,
  \[ \Phi_{X,p}(t) = (t^4 - pt^2 + p^2)(t^2 + p), \]

- If $p \equiv 8, 20 (21)$,
  \[ \Phi_{X,p}(t) = (t^2 + p)^3. \]

These coincide with the above results of Corollary 1.2 (see the discussion after Theorem 1.1). Moreover in the second case Jac$(X_p)$ is a product of supersingular elliptic curves over $\mathbb{F}_p$ and $|X_p(\mathbb{F}_p^2)| = 1 + p^2 + 6p$.

**Example 5.3.**([4] Example 2.1) Remember that the $n$-th Chebyshev polynomial $U_n$ is defined by the recursive relation,

\[ U_{n+1}(x) = xU_n(x) - U_{n-1}(x), \quad U_0(x) = 2, \quad U_1(x) = x. \]

For a prime $l \geq 5$ we define a curve $X_l$ as

\[ X(l) : y^2 = U_l(x). \]

The genus is $(l - 1)/2$ and it is the quotient of a hyperelliptic curve

\[ Y(l) : y^2 = x(x^{2l} + 1), \]

by the involution

\[ \tau : (x, y) \mapsto \left( \frac{1}{x}, \frac{y}{x^{l+1}} \right). \]

Set $K = \mathbb{Q}(\zeta_l + \zeta_l^{-1}, \zeta_4)$ and $K_0 = \mathbb{Q}(\zeta_l + \zeta_l^{-1})$. The automorphisms of $Y(l)$:

\[ (x, y) \overset{\zeta_l}{\mapsto} (\zeta_l^2 x, \zeta_l y), \quad (x, y) \overset{\zeta_4}{\mapsto} (-x, \zeta_4 y), \]
induce an action of $O_K$ on $\text{Jac}(X(l))$. The authors have shown that $H^0(X(l), \Omega^1)$ has a $\mathbb{Q}$-rational base $\{\omega_1, \ldots, \omega_{(l-1)/2}\}$ satisfying

$$(\zeta_l + \zeta_l^{-1})^{i} \omega_i = (\zeta_l^i + \zeta_l^{-i})\omega_1, \quad 1 \leq i \leq \frac{l-1}{2}.$$  

By (12) this implies that the action of $O_{K_0}$ on the Jacobian is defined over $K_0$. Thus $X(l)$ satisfies the assumption of Theorem 1.4. Corollary 1.2 shows that a good prime $p$ satisfying $p \equiv -1(4)$ is supersingular. If moreover $p \equiv -1(4)$ and $p \equiv \pm 1(l)$, $\text{Jac}(X(l)_p)$ is a product of supersingular elliptic curves over $\mathbb{F}_p$ and $|X(l)_p(\mathbb{F}_{p^2})|$ attains the Hasse-Weil upper bound $1 + p^2 + (l - 1)p$.

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