A Multistep Frank-Wolfe Method

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Abstract

The Frank-Wolfe algorithm has regained much interest in its use in structurally constrained machine learning applications. However, one major limitation of the Frank-Wolfe algorithm is the slow local convergence property due to the zig-zagging behavior. We observe the zig-zagging phenomenon in the Frank-Wolfe method as an artifact of discretization, and propose multistep Frank-Wolfe variants where the truncation errors decay as $O(\Delta^p)$, where $p$ is the method’s order. This strategy “stabilizes” the method, and allows tools like line search and momentum to have more benefit. However, our results suggest that the worst case convergence rate of Runge-Kutta-type discretization schemes cannot improve upon that of the vanilla Frank-Wolfe method for a rate depending on $k$. Still, we believe that this analysis adds to the growing knowledge of flow analysis for optimization methods, and is a cautionary tale on the ultimate usefulness of multistep methods.

1. Introduction

The Frank-Wolfe method attacks problems of form

$$\min_{x \in D} f(x)$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is an everywhere-differentiable function and $D$ is a convex compact constraint set, via the repeated iteration

$$
\begin{align*}
    s^{(k)} &= \arg \min_{s \in D} \nabla f(x^{(k)})^T s \\
    x^{(k+1)} &= \gamma^{(k)} s^{(k)} + (1 - \gamma^{(k)}) x^{(k)}.
\end{align*}
$$

The first operation is often referred to as the linear minimization oracle (LMO), and is the support function of $D$ at $-\nabla f(x)$. In particular, computing the LMO is often computationally cheap, especially when $D$ is the level set of a sparsifying norm, e.g. the 1-norm or the nuclear norm. In this regime, the advantage of such projection-free methods over methods like projected gradient descent is the cheap per-iteration cost. However, the tradeoff of the cheap per-iteration rate is that the convergence rate, in terms of number of iterations $k$, is often much slower than that of projected gradient descent (Lacoste-Julien and Jaggi, 2015; Freund and Grigas, 2016). While various acceleration schemes (Lacoste-Julien and Jaggi, 2015) have been proposed and several improved rates given under specific problem geometry (Garber and Hazan, 2015), by and large the “vanilla” Frank-Wolfe method, using the “well-studied step size” $\gamma^{(k)} = O(1/k)$, can only be shown to reach $O(1/k)$ convergence rate in terms of objective value decrease (Canon and Cullum, 1968; Jaggi, 2013; Freund and Grigas, 2016)

Continuous-time Frank-Wolfe. In this work, we view the method FW as an Euler’s discretization of the differential inclusion

$$
\begin{align*}
    \dot{x}(t) &= \gamma(t) (s(t) - x(t)), \\
    s(t) &\in \arg \min_{s \in D} \nabla f(x(t))^2 (s - x(t)).
\end{align*}
$$

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where $x(t)$, $s(t)$, and $\gamma(t)$ are continuations of the iterates $x^{(k)}$, $s^{(k)}$, and coefficients $\gamma^{(k)}$; i.e. $x^{(k)} = x(k\Delta)$ for some discretization unit $\Delta$. This was first studied in Jacimovic and Geary (1999), and is a part of the construct presented in Diakonikolas and Orecchia (2019). However, neither paper considered the effect of using advanced discretization schemes to better imitate the flow, as a way of improving the method. From analyzing this system, we reach the following conclusions through numerical experimentation:

- (Positive result.) We show that for a class of mixing parameters $\gamma(t)$, FWFLOW can have an arbitrarily fast convergence rate given aggressive enough mixing parameters.

- (Interesting result.) We show qualitatively that, on a number of machine learning tasks, unlike FW, the iterates $x(t)$ in FWFLOW usually do not zig-zag.

**Multistep methods.** While continuous time analyses offer improved intuition in idealized settings, it does not provide a usable method. We therefore investigate improved discretization schemes applied to Frank-Wolfe. Here, we make the following discoveries:

- (Negative result.) We show that over for a particular popular class of multistep methods (Runge Kutta methods) no acceleration can be made when the step size $\gamma_k = O(1/k)$.

- (Usefulness.) However, higher order multistep methods tend to have better search directions, which accounts for less zig-zagging. This leads to better performance when mixed with line search or momentum methods.

### 2. Continuous time Frank-Wolfe

**Proposition 2.1 (Continuous flow rate).** Suppose that $\gamma(t) = \frac{c}{1+t}$, for some constant $c \geq 1$. Then the flow rate of FWFLOW has an upper bound of

$$f(x(t)) - f^* \leq \left( \frac{c}{c+t} \right)^c = O\left( \frac{1}{tc} \right).$$

Note that this rate is arbitrarily fast, as long as we keep increasing $c$. This is in stark contrast to the usual convergence rate of the Frank-Wolfe method, which in general cannot improve beyond $O(1/k)$ for any $c$. Figure 1 shows this continuous rate as the limiting behavior of FW, where the discretization steps $\Delta \to 0$.

### 2.1. Continuous time Frank Wolfe does not zig-zag

Figure 2 quantifies this notion more concretely. We first propose to measure “zig-zagging energy” by averaging the deviation of each iterate’s direction across $k$-step directions, for $k = 1, \ldots, W$, for some measurement window $W$:

$$\mathcal{E}_{\text{zigzag}}(x^{(k+1)}, \ldots, x^{(k+W)}) = \frac{1}{W-1} \sum_{i=k+1}^{k+W-1} \left\| \left( I - \frac{1}{\|\bar{d}^{(k)}\|^2} \bar{d}^{(k)}(\bar{d}^{(k)})^T \right) d^{(i)} \right\|_2,$n

where $d^{(i)} = x^{(i+1)} - x^{(i)}$ is the current iterate direction and $\bar{d}^{(k)} = x^{(k+W)} - x^{(k)}$ a “smoothed” direction. The projection operator $Q$ removes the component of the current direction in the direction of the smoothed direction, and we measure this “average deviation energy.” We divide the trajectory into these window blocks, and report the average of these measurements $\mathcal{E}_{\text{zigzag}}$ over $T = 100$ time steps (total iteration = $T/\Delta$). Figure 2 (top table) exactly shows this behavior, where the problem...
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### Zigzagging in continuous flow

| Test set | $\Delta = 1$ | $\Delta = 0.1$ | $\Delta = 0.01$ |
|----------|--------------|----------------|-----------------|
| Sensing  | 105.90 / 140.29 | 10.49 / 13.86 | 1.05 / 1.39 |
| Madelon  | 0.11 / 0.23 | 0.021 / 0.028 | 0.0021 / 0.0028 |
| Gisette  | 1.08 / 1.74 | 0.21 / 0.28 | 0.021 / 0.028 |

Zigzagging in multistep methods

| Test set | FW | FW-MID | FW-RK4 |
|----------|-----|--------|--------|
| Sensing  | 105.90 / 140.29 | 0.57 / 1.00 | 0.015 / 0.033 |
| Madelon  | 0.031 / 0.040 | 0.025 / 0.029 | 0.025 / 0.029 |
| Gisette  | 0.30 / 0.40 | 0.25 / 0.11 | 0.22 / 0.20 |

Figure 2: **Zig-zagging on real datasets.** Average deviation of different discretizations of FWFLOW. Top table uses different $\Delta$s and uses the vanilla Euler’s discretization (FW). Bottom uses $\Delta = 1$ and different multistep methods. The two numbers in each box correspond to window sizes 5 / 20.

is sparse constrained logistic regression minimization over several machine learning classification datasets (Guyon et al., 2004) (Sensing (ours), Gisette \(^1\) and Madelon \(^2\)) are shown in Fig. 2.

From this experiment, we see that any Euler discretization of FWFLOW will always zig-zag, in that the directions often alternate. But, by measuring the deviation across a windowed average, we see two things: first, the deviations converge to 0 at a rate seemingly linear in $\Delta$, suggesting that in the limit as $\Delta \to 0$, the trajectory is smooth. Second, since these numbers are more-or-less robust to windowing size, it suggests that the smoothness of the continuous flow is on a macro level.

### 3. Runge-Kutta multistep methods

#### 3.1. The generalized Runge-Kutta family

We now consider Runge-Kutta (RK) methods, a generalized class of higher order methods ($p \geq 1$). These methods are fully parametrized by some choice of $A \in \mathbb{R}^{q \times q}$, $\beta \in \mathbb{R}^{q}$, and $\omega \in \mathbb{R}^{q}$ and at step $k$ can be expressed as (for $i = 1, \ldots, q$)

\[
\xi_i = \dot{x}(k + \omega_i, x^{(k)} + \sum_{j=1}^{q} A_{ij} \xi_j),
\]

\[
x^{(k+1)} = x^{(k)} + \sum_{i=1}^{q} \beta_i \xi_i.
\]

For consistency, $\sum_i \beta_i = 1$, and to maintain explicit implementations, $A$ is always strictly lower triangular. As a starting point, $\omega_1 = 0$. Specifically, we refer to the iteration scheme in (4) as a $q$-stage RK discretization method. Note that via our formulation, we capture not just all RK methods, but all explicit multistep methods satisfying this mild consistency constraint, which then implies method feasibility. A full list of the RK methods used in our experiments is described in the Appendix A.

Figure 3 (top row) compares these three implementations on the toy problem,and shows their rate of convergence. The closeness of the new curves with the continuous flow is apparent; however, while multistep methods are converging faster than vanilla FW, the rate does not seem to change. However, one thing that is visually apparent in Figure 3 (top row) is that higher order multistep methods establish better search directions. Additionally, we numerically quantify less zig-zagging behavior (lower table in Figure 2). This is still good news, as there are still several key advantages to such an improvement: namely, better uses of momentum and line search.

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\(^1\)Full dataset available at [https://archive.ics.uci.edu/ml/datasets/Gisette](https://archive.ics.uci.edu/ml/datasets/Gisette). We use a subsampling, as given in [https://github.com/cyrillewcombettes/boostfw](https://github.com/cyrillewcombettes/boostfw).

\(^2\)Dataset: [https://archive.ics.uci.edu/ml/datasets/madelon](https://archive.ics.uci.edu/ml/datasets/madelon)
3.2. RK convergence behavior

All proofs are in the appendix.

**Proposition 3.1** (Positive result). *All Runge-Kutta methods converge at worst with rate \( f(x^{(k)}) - f(x^*) \leq O(1/k) \).

**Proposition 3.2** (Negative result). *Under mild conditions, regardless of the order \( p \) and choice of \( A, \beta, \) and \( \omega \), the worst best case bound for FW-RK, for any RK method, is of order \( O(1/k) \).*

4. A better search direction

Though multistep methods do not seem to improve the rate of convergence, it does improve the quality of the search direction. We leverage this in two ways. First, we consider more aggressive line searches, e.g. replacing \( \gamma^{(k)} \) with \( \max\{\frac{2}{\max\{x_{k-1},1\}}, \gamma\} \) and

\[
\bar{\gamma} = \max_{0 \leq \gamma \leq 1} \{ \gamma : f(x^{(k)} + \gamma^{(k)}d^{(k)}) \leq f(x^{(k)}) \}.
\]

Note that this is not the typical line search as in (Lacoste-Julien and Jaggi, 2015), which forces \( \gamma^{(k)} \) to be upper bounded by \( O(1/k) \)—we are hoping for more aggressive, not less, step sizes. Second, we follow the scheme presented in Li et al. (2021) which generalizes the 3-variable Nesterov acceleration (Nesterov, 2003) from gradient descent to Frank-Wolfe. Fig. 3, rows 2 and 3, illustrate the benefits of multistep methods for line search (row 2) and momentum (row 3), over the toy triangle problem.

![Figure 3: Triangle toy problem. Top: Straight implementation. Middle: Line search. Bottom: Momentum acceleration.](image)

Finally, we evaluate the benefit of our multistep Frank-Wolfe methods on sparse logistic regression on Gisette (Guyon et al., 2004) (Figure 4, left) and Nuclear-norm constrained Huber regression on Movielens 100K dataset (Harper and Konstan, 2015) \(^3\) (Figure 4, right).

\(^3\)MovieLens dataset is available at [https://grouplens.org/datasets/movielens/100k/](https://grouplens.org/datasets/movielens/100k/)

![Figure 4: Left: Gisette. Right: Movielens.](image)
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A. Runge Kutta methods

- Midpoint method
  \[ A = \begin{bmatrix} 0 & 0 \\ 1/2 & 0 \end{bmatrix}, \quad \beta = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \omega = \begin{bmatrix} 0/2 \\ 1/2 \end{bmatrix}, \quad z^{(1)} \approx \begin{bmatrix} -0.3810 \\ 1.1429 \end{bmatrix}, \quad z^{(2)} \approx \begin{bmatrix} -0.2222 \\ 0.8889 \end{bmatrix} \]

- Runge Kutta 4th Order Tableau (44)
  \[ A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \beta = \begin{bmatrix} 1/6 \\ 1/3 \\ 1/3 \\ 1/6 \end{bmatrix}, \quad \omega = \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \\ 1 \end{bmatrix}, \quad z^{(1)} \approx \begin{bmatrix} 0.2449 \\ 0.5986 \\ 0.5714 \\ 0.3333 \end{bmatrix} \]

- Runge Kutta 3/8 Rule Tableau (4)
  \[ A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 0 \\ -1/3 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \end{bmatrix}, \quad \beta = \begin{bmatrix} 1/8 \\ 3/8 \\ 3/8 \\ 1/8 \end{bmatrix}, \quad \omega = \begin{bmatrix} 0 \\ 1/3 \\ 1/3 \\ 1 \end{bmatrix}, \quad z^{(1)} \approx \begin{bmatrix} 0.1758 \\ 0.6409 \\ 0.6818 \\ 0.2500 \end{bmatrix} \]

- Runge Kutta 5 Tableau
  \[ A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1/4 & 0 & 0 & 0 & 0 \\ 0 & 32/90 & 1 & 0 & 0 \\ 3/16 & 12/7 & 10/9 & 16 & 0 \\ -3/7 & 2/7 & 12/7 & -12/7 & 8/7 & 0 \end{bmatrix}, \quad \beta = \begin{bmatrix} 7/90 \\ 1/4 \\ 12/90 \\ 32/90 \\ 7/90 \end{bmatrix}, \quad \omega = \begin{bmatrix} 0 \\ 1/4 \\ 1/2 \\ 3/4 \\ 1 \end{bmatrix}, \quad z^{(1)} \approx \begin{bmatrix} 0.1821 \\ 0.0068 \\ 0.3657 \\ 0.9956 \\ 0.2333 \end{bmatrix} \]

In all examples, \( \|z^{(k)}\|_{\infty} \) monotonically decays with \( k \).

B. Continuous Time Frank Wolfe Convergence Rate

Proof of Prop. 2.1

Proof. \(^4\) Note that by construction of \( \nabla f(x)^T s = \min_{y \in D} \nabla f(x)^T y \), and since \( f \) is convex,

\[ f(x) - f(x^*) \leq \nabla f(x)^T (x - x^*) \leq \nabla f(x)^T (x - s). \]

Quantifying the objective value error as \( \mathcal{E}(t) = f(x(t)) - f^* \) (where \( f^* = \min_{x \in D} f(x) \) is attainable) then

\[ \dot{\mathcal{E}}(t) = \nabla f(x(t))^T \dot{x}(t) = \gamma(t) \nabla f(x(t))^T (s(t) - x(t)). \]

Therefore,

\[ \dot{\mathcal{E}}(t) = -\gamma(t) \nabla f(x(t))^T (x(t) - s(t)) \geq f(x) - f(x^*) \]

\[ \leq -\gamma(t) \mathcal{E}(t) \]

giving an upper rate of

\[ \mathcal{E}(t) \leq \mathcal{E}(0)e^{-\int_0^t \gamma(\tau)d\tau}. \]

In particular, picking the “usual step size sequence” \( \gamma(t) = \frac{c}{t+c} \) gives the proposed rate (2).

\(^4\) Much of this proof is standard analysis for continuous time Frank-Wolfe, and is also presented in (Jacimovic and Geary, 1999).
C. Feasibility

**Proposition C.1** (Feasibility). Consider a q-stage multistep FW method defined by \( A, \beta, \) and \( \omega \). For each given \( k \geq 1 \), define

\[
\gamma_i^{(k)} = \frac{c}{c + k + \omega_i}, \quad \Gamma^{(k)} = \text{diag}(\gamma_i^{(k)}), \]

\[
P^{(k)} = \Gamma^{(k)}(I + A^T \Gamma^{(k)})^{-1}, \quad z^{(k)} = qP^{(k)}\beta.
\]

Then if \( 0 \leq z^{(k)} \leq 1 \) for all \( k \geq 1 \), then

\[
x^{(0)} \in \mathcal{D} \Rightarrow x^{(k)} \in \mathcal{D}, \quad \forall k \geq 1.
\]

**Proof.** For a given \( k \), construct additionally

\[
Z = \begin{bmatrix} \xi_1 & \xi_2 & \cdots & \xi_q \end{bmatrix},
\]

\[
\bar{X} = \begin{bmatrix} \bar{x}_1 & \bar{x}_2 & \cdots & \bar{x}_q \end{bmatrix}, \quad \bar{S} = \begin{bmatrix} \bar{s}_1 & \bar{s}_2 & \cdots & \bar{s}_q \end{bmatrix}.
\]

where

\[
\bar{x}_i = x^{(k)} + \sum_{j=1}^{q} A_{ij} \xi_j,
\]

\[
\bar{s}_i = \text{LMO}(\bar{x}_i).
\]

Then we can rewrite (4) as

\[
Z = (\bar{S} - \bar{X})\Gamma = (\bar{S} - x^{(k)}1^T - ZA^T)\Gamma = (\bar{S} - x^{(k)}1^T)P
\]

for shorthand \( P = P^{(k)} \) and \( \Gamma = \Gamma^{(k)} \). Then

\[
x^{(k+1)} = x^{(k)}(1 - 1^T P \beta) + \bar{S} P \beta
\]

\[
= \frac{1}{q} \sum_{i=1}^{q} \left(1 - z_i^{(k)}\right) x^{(k)} + z_i^{(k)} \bar{s}_i
\]

where \( z_i^{(k)} \) is the \( i \)-th element of \( z^{(k)} \), and \( \beta = (\beta_1, ..., \beta_q) \). Then if \( 0 \leq z_i^{(k)} \leq 1 \), then \( \hat{\xi}_i \) is a convex combination of \( x^{(k)} \) and \( \bar{s}_i \), and \( \hat{\xi}_i \in \mathcal{D} \) if \( x^{(k)} \in \mathcal{D} \). Moreover, \( x^{(k+1)} \) is an average of \( \hat{\xi}_i \), and thus \( x^{(k+1)} \in \mathcal{D} \). Thus we have recursively shown that \( x^{(k)} \in \mathcal{D} \) for all \( k \).

\[\square\]

**D. Positive Runge-Kutta convergence result**

**Lemma D.1.** After one step, the generalized Runge-Kutta method satisfies

\[
h(x^{(k+1)}) - h(x^{(k)}) \leq -\gamma^{(k+1)} h(x^{(k)}) + D_4 (\gamma^{(k+1)})^2
\]

where \( h(x) = f(x) - f(x^*) \) and

\[
D_4 = \frac{LD_2^2 + 2LD_2D_3 + 2D_3}{2}, \quad D_2 = c_1 D, \quad D_3 = c_2 c_1 D, \quad c_1 = q p_{\max}, \quad c_2 = q \max_{ij} |A_{ij}|, \quad D = \text{diam} (\mathcal{D}).
\]
**Proof.** For ease of notation, we write $x = x^{(k)}$ and $x^+ = x^{(k+1)}$. We will use $\gamma = \gamma^{(k)} = \frac{c}{c+k}$, and $\bar{\gamma}_i = \frac{c}{c+k+i}$. Now consider the generalized RK method

$$
\dot{x}_i = x + \sum_{j=1}^{q} A_{ij} \xi_j \\
\xi_i = \frac{c}{c+k+i}(s_i - x_i) \\
x^+ = x + \sum_{i=1}^{q} \beta_i \xi_i
$$

where $s_i = \text{LMO}(\bar{x}_i)$.

Define $D = \text{diam}(D)$. We use the notation from section 3. Denote the 2,$\infty$-norm as

$$
\|A\|_{2,\infty} = \max_j \|a_j\|_2
$$

where $a_j$ is the $j$th column of $A$. Note that all the element-wise elements in $P^{(k)} = \Gamma^{(k)}(I + AT\Gamma^{(k)})^{-1}$

is a decaying function of $k$, and thus defining $p_{\text{max}} = \|P^{(1)}\|_{2,\infty}$ we see that

$$
\|Z\|_{2,\infty} = \|(S - x^{(k)} 1)P^{(k)}\|_{2,\infty} \leq q p_{\text{max}} D.
$$

Therefore, since $\bar{Z} = (\bar{S} - \bar{X})\Gamma$, and all the diagonal elements of $\Gamma$ are at most 1,

$$
\|s_i - \bar{x}_i\|_2 \leq q p_{\text{max}} D =: D_2
$$

and

$$
\|x - \bar{x}_i\|_2 = \|\sum_{j=1}^{q} A_{ij} \gamma_j (s_j - \bar{x}_j)\|_2 \leq q \max_{ij} |A_{ij}| \gamma D_2 =: D_3 \gamma.
$$

Then

$$
\begin{align*}
  f(x^+) - f(x) &\leq \nabla f(x)^T(x^+ - x) + \frac{L}{2} \|x^+ - x\|_2^2 \\
  &= \sum_{i} \beta_i \bar{\gamma}_i \nabla f(x)^T(s_i - \bar{x}_i) + \frac{L}{2} \left\| \sum_{i} \beta_i \bar{\gamma}_i (s_i - \bar{x}_i) \right\|_2^2 \\
  &\leq \sum_{i} \beta_i \bar{\gamma}_i \left( \nabla f(x) - \nabla f(\bar{x}_i) \right)^T(s_i - \bar{x}_i) + \sum_{i} \beta_i \bar{\gamma}_i \nabla f(\bar{x}_i)^T(s_i - \bar{x}_i) + \frac{L \gamma^2 D_2^2}{2} \\
  &\leq \sum_{i} \beta_i \bar{\gamma}_i \left( \frac{L\|x - \bar{x}_i\|_2}{\|x - \bar{x}_i\|_2} \right) \|s_i - \bar{x}_i\|_2 \leq D_2 \\
  &\leq - \frac{\gamma^+}{2} \sum_{i} \beta_i h(\bar{x}_i) + \frac{L \gamma^2 D_2^2}{2} + 2L \gamma^2 D_2 D_3
  \\
  &\leq - \gamma^+ \sum_{i} \beta_i h(\bar{x}_i) + \frac{L \gamma^2 D_2 (D_2 + 2D_3)}{2}
\end{align*}
$$

where $\gamma = \gamma_k$, and $\gamma^+ = \gamma_{k+1}$. Now assume $f$ is also $L_2$-continuous, e.g. $|f(x_1) - f(x_2)| \leq L_2 \|x_1 - x_2\|_2$. Then, taking $h(x) = f(x) - f(x^*)$,
h(x^+) - h(x) \leq -\gamma^+ \sum_i \beta_i (h(\bar{x}_i) - h(x)) - \gamma^+ \sum_i \beta_i h(x) + \frac{L \gamma^2 D_2(D_2 + 2D_3)}{2} \\
\leq \gamma \sum_i \beta_i L_2 \|\bar{x}_i - x\|_2 - \gamma^+ h(x) + \frac{L \gamma^2 D_2(D_2 + 2D_3)}{2} \\
\leq -\gamma^+ h(x) + \frac{\gamma^2 (LD_2^2 + 2LD_2 D_3 + 2D_3)}{2} \\
\leq -\gamma^+ h(x) + D_4(\gamma^+)^2
\]

where $D_4 = \frac{LD_2^2 + 2LD_2 D_3 + 2D_3}{2}$ and we use $2 \geq (\gamma / \gamma^+)^2$ for all $k \geq 1$.

\[\square\]

Proof of Prop. 3.1

Proof. After establishing Lemma D.1, the rest of the proof is a recursive argument, almost identical to that in (Jaggi, 2013).

At $k = 0$, we define $h_0 = \max\{h(x^{(0)}), \frac{D_4c^2}{k+1}\}$, and it is clear that $h(x^{(0)}) \leq h_0$.

Now suppose that for some $k$, $h(x^{(k)}) \leq \frac{h_0}{k+1}$. Then

\[
h(x_{k+1}) \leq h(x_0) - \gamma_{k+1}^+ h(x^{(k)}) + D_4 \gamma_{k+1}^2 \\
\leq \frac{h_0}{k+1} \cdot \frac{k+1}{c+k+1} + D_4 \frac{c^2}{(c+k+1)^2} \\
= \frac{h_0}{c+k+1} + D_4 \frac{c^2}{(c+k+1)^2} \\
= \left(h_0 + \frac{D_4c^2}{c+k+1}\right) \left(\frac{k+2}{c+k+1}\right) \frac{1}{k+2} \\
\leq h_0 \left(1 + \frac{c-1}{c+k+1}\right) \left(\frac{k+2}{c+k+1}\right) \frac{1}{k+2} \\
\leq h_0 \left(\frac{2c+k}{c+k+1}\right) \left(\frac{k+2}{c+k+1}\right) \frac{1}{k+2}.
\]

\[\square\]

E. Negative Runge-Kutta convergence result

This section gives the proof for Proposition 3.2.

Lemma E.1 ($O(1/k)$ rate). Start with $x^{(0)} = 1$. Then consider the sequence defined by

\[
x^{(k+1)} = |x^{(k)} - \frac{c_k}{k}|
\]

where, no matter how large $k$ is, there exist some constant where $C_1 < \max_{k' > k} c_{k'}$. (That is, although $c_k$ can be anything, the smallest upper bound of $c_k$ does not decay.) Then

\[
\sup_{k' \geq k} |x^{(k')}| = \Omega(1/k).
\]

That is, the smallest upper bound of $|x^{(k)}|$ at least of order $1/k$. 

Proof. We will show that the smallest upper bound of $|x^{(k)}|$ is larger than $C_1/(2k)$.

Proof by contradiction. Suppose that at some point $K$, for all $k \geq K$, $|x^{(k)}| < C_1/(2k)$. Then from that point forward,

$$\text{sign}(x^{(k)} - \frac{c_k}{k}) = -\text{sign}(x^{(k)})$$

and there exists some $k' > k$ where $c_{k'} > C_1$. Therefore, at that point,

$$|x^{(k'+1)}| = \frac{c_{k'}}{k'} - |x^{(k')}| \geq \frac{C_1}{2k'} > \frac{C_1}{2(k'+1)}.$$ 

This immediately establishes a contradiction. \hfill \square

Now define the operator

$$T(x^{(k)}) = x^{(k+1)} - x^{(k)}$$

and note that

$$|x^{(k+1)}| = |x^{(k)} + T(x^{(k)})| = |x_k| + |\text{sign}(x^{(k)})T(x^{(k)})|.$$ 

Thus, if we can show that there exist some $\epsilon$, agnostic to $k$ (but possibly related to Runge Kutta design parameters), and

$$\exists k' \geq k, \quad -\text{sign}(x^{(k')})T(x^{(k')}) > \frac{\epsilon}{k'}, \quad \forall k,$$ 

then based on the previous lemma, this shows $\sup_{k' > k} |x_k| = \Omega(1/k)$ as the smallest possible upper bound.

**Lemma E.2.** Assuming that $0 < qP^{(k)}/\beta < 1$ then there exists a finite point $\bar{k}$ where for all $k > \bar{k}$,

$$|x^{(k)}| \leq \frac{C_2}{k}$$

for some $C_2 \geq 0$.

Proof. We again use the block matrix notation

$$Z^{(k)} = (S - x^{(k)}1^T)\Gamma^{(k)}(I + A^T\Gamma^{(k)})^{-1}$$

where $\Gamma^{(k)} = \text{diag}(\gamma_i^{(k)})$ and each element $\gamma_i^{(k)} \leq \gamma^{(k)}$.

First, note that by construction, since

$$\|S - x^{(k)}1^T\|_{2,\infty} \leq D_4, \quad \|(I + A^T\Gamma^{(k)})^{-1}\|_2 \leq \|(I + A^T\Gamma^{(0)})^{-1}\|_2$$

are bounded above by constants, then

$$\|Z^{(k)}\|_\infty \leq \frac{c}{c + k}C_4$$

for $C_1 = D_4\|(I + A^T\Gamma^{(0)})^{-1}\|_2$.

First find constants $C_3$, $C_4$, and $\bar{k}$ such that

$$\frac{C_3}{k} \leq 1^TP^{(k)}\beta \leq \frac{C_4}{k}, \quad \forall k > \bar{k},$$

and such constants always exist, since by assumption, there exists some $a_{\min} > 0$, $a_{\max} < 1$ and some $k'$ where

$$a_{\min} < qP^{(k')}\beta < a_{\max} \Rightarrow \frac{a_{\min}}{q\gamma_{\max}} \leq (I + A^T\Gamma^{(k')})^{-1}\beta \leq \frac{a_{\max}}{q\gamma_{\min}}$$

where

$$\gamma_{\min} = \min_i \frac{c}{c + k' + \omega_i^{(k')}}, \quad \gamma_{\max} = \frac{c}{c + k'}.$$
Additionally, for all \( k > c + 1 \),
\[
\frac{c}{2k} \leq \frac{c}{c+k+1} \leq \Gamma^{(k)}_{ii} \leq \frac{c}{c+k} \leq \frac{c}{k}.
\]
Therefore taking
\[
C_3 = \frac{c_{\text{min}}}{2q\gamma_{\text{max}}} , \quad C_4 = \frac{c_{\text{max}}}{q\gamma_{\text{min}}} , \quad \tilde{k} = \max\{k', c + 1\}
\]
satisfies (6).

Now define
\[
C_2 = \max\{|x^{(1)}|, 4cqC_1\|A\|_\infty, 4C_3, 4C_4\}.
\]
We will now inductively show that \( |x^{(k)}| \leq \frac{C_2}{k} \). From the definition of \( C_2 \), we have the base case for \( k = 1 \):
\[
|x^{(1)}| \leq \frac{|x^{(1)}|}{1} \leq \frac{C_2}{k}.
\]

Now assume that \( |x^{(k)}| \leq \frac{C_2}{k} \). Recall that
\[
x^{(k+1)} = x^{(k)} (1 - T^k P^{(k)} \beta) + S P^{(k)} \beta, \quad S = [s_1, ..., s_q], \quad x_i = -\text{sign}(\tilde{x}_i)
\]
and we denote the composite mixing term \( \tilde{\gamma}^{(k)} = T^k P^{(k)} \beta \). We now look at two cases separately.

- Suppose first that \( S = -\text{sign}(x^{(k)} 1^T) \), e.g. \( \text{sign}(\tilde{x}_i) = \text{sign}(x^{(k)}) \) for all \( i \). Then
\[
\tilde{S} P^{(k)} \beta = -\text{sign}(x^{(k)}) \tilde{\gamma}^{(k)},
\]
and
\[
|x^{(k+1)}| = |x^{(k)} (1 - \tilde{\gamma}^{(k)}) + SP^{(k)} \beta| \\
= |x^{(k)} (1 - \tilde{\gamma}^{(k)}) - \text{sign}(x^{(k)}) \tilde{\gamma}^{(k)}| \\
= |\text{sign}(x^{(k)} x^{(k)} (1 - \tilde{\gamma}^{(k)}) - \text{sign}(x^{(k)}) \text{sign}(x^{(k)}) \tilde{\gamma}^{(k)}| \\
= \|x^{(k)}|(1 - \tilde{\gamma}^{(k)}) - \tilde{\gamma}^{(k)}| \\
\leq \max\{|x^{(k)}|(1 - \tilde{\gamma}^{(k)}) - \tilde{\gamma}^{(k)}, \tilde{\gamma}^{(k)} - |x^{(k)}|(1 - \tilde{\gamma}^{(k)})\} \\
\leq \max\left\{ \frac{C_2}{k} (1 - \frac{C_3}{k}) - \frac{C_3}{k}, \frac{C_4}{k} \right\}
\]
and when \( k \geq \frac{C_2}{C_3} \Leftrightarrow C_3 \geq \frac{C_2}{k} \),
\[
(*) \leq C_2 \left( \frac{1}{k} - \frac{1}{k^2} \right) \leq \frac{C_2}{k+1}.
\]
Taking also \( C_4 \leq \frac{C_2}{k+1} \),
\[
|x^{(k+1)}| \leq \max\left\{ \frac{C_2}{k+1}, \frac{C_2}{4k} \right\} \leq \frac{C_2}{k+1}
\]
for all \( k \geq 1 \).

- Now suppose that there is some \( i \) where \( \tilde{s}_i = \text{sign}(x^{(k)} 1^T) \). Now since
\[
S = -\text{sign}(x^{(k)} 1^T + Z A^T)
\]
then this implies that \( |x^{(k)}| < (Z A^T)_i \). But since
\[
|(Z A^T)_i| \leq \|Z\|_\infty \|A\|_\infty q \leq \frac{c}{c+k} (C_1 \|A\|_\infty q) \leq \frac{C_2}{4k},
\]
this implies that
\[
|x^{(k+1)}| \leq \frac{C_2}{4k} (1 - \frac{C_3}{k}) + \frac{C_2}{4k} \leq \frac{C_2}{2k} \leq \frac{C_2}{k+1}, \quad \forall k > 1.
\]
Thus we have shown the induction step, which completes the proof.

\textbf{Lemma E.3.} There exists a finite point \( \hat{k} \) where for all \( k > \hat{k} \),
\[
\frac{c}{c + k} - \frac{C_4}{k^2} < |\xi| < \frac{c}{c + k} + \frac{C_4}{k^2}
\]
for some constant \( C_4 > 0 \).

\textit{Proof.} Our goal is to show that
\[
\gamma^{(k)} - \frac{C_4}{k^2} \leq \|Z\|_\infty \leq \gamma^{(k)} + \frac{C_4}{k^2}
\]
for some \( C_4 \geq 0 \), and for all \( k \geq k' \) for some \( k' \geq 0 \). Using the Woodbury matrix identity,
\[
\Gamma(I + A^T\Gamma)^{-1} = \Gamma - A^T(I + \Gamma A^T)^{-1}\Gamma
\]
and thus
\[
Z^{(k)} = S\Gamma - (x^{(k)}1^T\Gamma + (\bar{S} - x^{(k)}1^T)\Gamma A^T(I + \Gamma A^T)^{-1}\Gamma)\cdot B
\]
and thus
\[
|\bar{s}_i\gamma_i| - \frac{C_4}{k^2} \leq |\xi^{(k)}| \leq |\bar{s}_i\gamma_i| + \frac{C_4}{k^2}
\]
where via triangle inequalities and norm decompositions,
\[
\frac{C_4}{k^2} = \max_i |B_i| \leq |x^{(k)}| \gamma_k + D_4\gamma^2_4 + A\|I + \Gamma^{(0)}A^T\|^{-1} = O(1/k^2).
\]
Finally, since \( \bar{s}_i \in \{-1, 1\} \), then \( |\bar{s}_i\gamma_i| = \gamma_i \), and in particular,
\[
\frac{c}{c + k + \omega_i} \leq \frac{c}{c + k}
\]
and
\[
\frac{c}{c + k + \omega_i} \geq \frac{c}{c + k + \omega_{\max}} = \frac{c}{c + k} - \frac{c}{c + k} \frac{\omega_{\max}}{c + k + \omega_{\max}} \geq \frac{c}{c + k} - \frac{c\omega_{\max}}{k^2}
\]
Therefore, taking \( C_4 = c\omega_{\max} + C_3 \) completes the proof.

\textbf{Lemma E.4.} There exists some large enough \( \tilde{k} \) where for all \( k > \tilde{k} \), it must be that
\[
\exists k' \geq k, \quad -\text{sign}(x^{(k')}^T x^{(k')}) > \frac{c}{k'}.
\]
\textit{Proof.} Define a partitioning \( S_1 \cup S_2 = \{1, \ldots, q\} \), where
\[
S_1 = \{i : \xi_i > 0\}, \quad S_2 = \{j : \xi_j \leq 0\}.
\]
Defining \( \bar{\xi} = \frac{c}{c + k} \),
\[
\left| \sum_{i=1}^{q} \beta_i \xi_i \right| = \left| \sum_{i \in S_1} \beta_i |\xi_i| - \sum_{j \in S_2} \beta_j |\xi_j| \right| \geq \left( \bar{\xi} - \frac{C_4}{k^2} \right) \cdot \left| \sum_{i \in S_1} \beta_i - \sum_{j \in S_2} \beta_j \right|.
\]
By assumption, there does not exist a combination of \( \beta_i \) where a specific linear combination could cancel them out; that is, suppose that there exists some constant \( \beta \), where for every partition of sets \( S_1, S_2 \),
\[
0 < \bar{\beta} := \min \sum_{i \in S_1} |\beta_i - \sum_{j \in S_2} \beta_j|.
\]
Then
\[
\left| \sum_{i=1}^{q} \beta_i \xi_i \right| \geq \left( \frac{c}{c + k} - \frac{C_4}{k^2} \right) \bar{\beta} \geq \bar{\beta} \frac{\max\{C_2, c\}}{k}.
\]
Picking \( \epsilon = \max\{C_2, c\} \) concludes the proof.
F. More Higher Order Discretization Methods

Figure 5 evaluates the performance of more multistep Frank-Wolfe methods, for a problem with $m = 500$, $n = 100$, and $\alpha = 1000$.

Figure 5: **Compressed sensing**, 500 samples, 100 features, 10% sparsity ground truth, $\alpha = 1000$. 