ASYMPTOTIC STABILIZATION OF A FLEXIBLE BEAM WITH ATTACHED MASS

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We consider a mathematical model of a simply supported Euler–Bernoulli beam with an attached spring-mass system. The model is controlled by distributed piezo-actuators and a lumped force. We address the issue of asymptotic behavior of the solutions of this system driven by a linear feedback law. The precompactness of trajectories is established for the operator formulation of the closed-loop dynamics. Sufficient conditions for the strong asymptotic stability of the trivial equilibrium are obtained.

1. Introduction

Mechanical structures with flexible beams are widely used in modern engineering, in particular, in the fields of spacecraft manufacturing, large-scale robotics, wind-turbine industry, and offshore drilling technology. Parallel with rapid ongoing technological progress, the line between the control engineering and the mathematical control theory is blurring. Inspired by industrial challenges, the control theory of elastic systems with distributed parameter has been refined over the several last decades. The overview of some important results in the field of stabilization of mechanical systems with flexible beams is presented in Section 2. The present paper is devoted to the problem of stabilization of a model of flexible beam with \( k \) distributed piezo-actuators and attached mass (shaker).

Our study is motivated by the needs of rigorous theoretical treatment of an experimental setup considered in [4]. The current paper continues our previous works on the nonasymptotic stability analysis [26] and the distribution of eigenvalues [5] for this class of systems.

The considered model consists of a flexible beam of length \( l \) and a controlled spring-mass system (shaker) attached at a point with coordinate \( l_0 \in (0, l) \). The state of this system is described by the cross-sectional deflection \( w(x, t) \) of the beam centerline at a point \( x \in (0, l) \) and time \( t \); \( E(x) > 0 \) and \( I(x) > 0 \) are Young’s modulus and the cross-sectional moment of inertia, respectively, and \( \rho(x) > 0 \) is the mass per unit length of the beam. A schematic diagram of the analyzed mechatronic model is shown in Fig. 1.

The distributed control is provided by \( k \) piezoelectric actuators. The action of the \( j \)th actuator is described by a torque density \( M_j \) and a shape function \( \psi_j(x) \) satisfying the assumptions

\[
\text{supp } \psi_j \cap \{0, l_0, l\} = \emptyset \quad \text{and} \quad \psi_j''(x) \in C^2[0, l], \quad j = 1, \ldots, k.
\]

The lumped control is a force \( F \) implemented by the electromagnetic shaker at \( l_0 \in (0, l) \) and \( m > 0 \) and \( \kappa > 0 \) are, respectively, the mass of the moving part of the shaker and its stiffness. The dynamics of the system is
A feedback control guaranteeing the stability of the trivial equilibrium of the system described above in Lyapunov’s sense was proposed in [26]. The question about the asymptotic stability of this flexible system with attached mass remained open up to now. The positive answer to this question is the main result of the present paper established in Theorem 5.1.

The paper is organized as follows: A survey of related results available from the literature is summarized in Section 2. A special emphasis is made on the systems with distributed parameters described by the wave and beam equations. The operator representation of the control system (1)–(3) is presented in Section 3 together with the necessary auxiliary results. The maximal invariant set is also investigated. Our study of the asymptotic stability is based on the infinite-dimensional version of LaSalle’s invariance principle [10] (cf. [25]) in which the precompactness of trajectories plays a crucial role. Thus, we present a detailed precompactness analysis in Section 4.
The following contributions distinguish our results from the results known from the literature:

- explicit design of feedback controllers guaranteeing strong asymptotic stability of the closed-loop;
- asymptotic stability conditions with arbitrarily many actuators;
- sufficient conditions for the precompactness of the trajectories (see Theorem 4.1) allowing the localization of limit sets that are applicable to the case of actuators at the nodes of eigenfunctions.

2. Related Works

The stability and control theory of dynamical systems described by hyperbolic differential equations was developed by means of different approaches in various works since the second half of the 20th century.

In [18], the nonharmonic Fourier theory was extended to the case of second-order differential equations, and the properties of eigenvalues of the moment problem were established under basic assumptions for the control function.

The optimal control problems for the vibrations of elastic beams were discussed in detail in [6] for all principal kinds of boundary conditions derived from the mechanical statement. Some general types of control, including impulse, were discussed for symmetric hyperbolic systems, while the case of distributed load control was described for the model of vibrating beam.

The generalization of the finite-dimensional linear control problem to an abstract linear control problem in the Hilbert space can be found in [8]. The conditions of controllability of vibrating strings and beams with distributed and boundary controls are presented in the general form.

Since it is suitable to investigate the trajectories of complex mechanical systems in the form of properties of abstract differential equations in infinite-dimensional spaces, numerous results cannot be obtained without the developed theory of $\mathcal{C}_0$-semigroups, which was comprehensively covered in [14, 17] with applications to stability problems.

Strong stability, stabilizability, and detectability were studied by the spectral methods in [16] for the control systems with bounded input and output operators that are not necessarily exponentially stable.

Some applications of the semigroups theory to the observability and controllability problems for abstract wave equations were reviewed in [7].

The authors of [3] brought universal approaches employing the nonharmonic Fourier series to the investigation of controllability of the networks of strings, thus giving useful tools also for the analysis of the beam equations.

The inequalities of controllability of the linear partial differential equations were proved in [1].

The semigroup theory and its application to the investigation of controllability, observability, stabilizability, exponential stabilizability, and detectability of linear control systems with bounded input and output operators in the infinite-dimensional spaces was covered in the integrated way in [2].

The properties of asymptotic stability of the boundary control Euler–Bernoulli beam connected to a nonlinear mass spring damper system were discussed in [11] under weak assumptions imposed on the nonlinear spring and damper.

A series of works by M. Shubov, et al. [19–21] was devoted to the controllability and spectral analysis of the Euler–Bernoulli beam models with different types of boundary conditions.

In recent years, more and more pronounced mathematical interest is attracted by control problems arising in modeling the offshore drilling structures [9, 12] and different kinds of robotic manipulators [15, 24].

The problem of stabilization for flexible-link manipulators with payload was solved in [27] for the case of multiple passive joints and in [28] for a manipulator represented by the Timoshenko beam model.

New results in the field of stabilization of infinite-dimensional dynamical systems in abstract spaces were presented in [22].
Despite the advanced general approaches to the theory of stabilization and control of oscillating systems, numerous special cases exhibit their peculiarities affecting the final stability results. Thus, the elastic structures consisting of flexible beams with distributed or boundary controls and multilink networks of joint beams are extensively studied in the context of controllability and stabilizability, while the stability and asymptotic stability for the flexible beam with attached rigid body and distributed control form the framework for this investigation.

3. Auxiliary Results

In order to formally treat the property of asymptotic stability, we introduce an operator representation of the control system (1)–(3). For this purpose, we consider a Hilbert space

\[ X = H^2(0, l) \times L^2(0, l) \times \mathbb{R}^2 \]

equipped with the inner product

\[
\left\langle \begin{pmatrix} u_1 \\ v_1 \\ p_1 \\ q_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ v_2 \\ p_2 \\ q_2 \end{pmatrix} \right\rangle_X = \int_0^l \left( E(x)I(x)u''_1(x)v''_2(x) + \rho(x)v_1(x)v_2(x) \right) dx + \kappa p_1 \bar{p}_2 + m q_1 \bar{q}_2.
\]

Consider a differential operator \( \tilde{A} : D(\tilde{A}) \to X \) with the domain

\[
D(\tilde{A}) = \left\{ \xi = \begin{pmatrix} u \\ v \\ p \\ q \end{pmatrix} \in X : \begin{array}{l}
u(x) \in H^4(0, l) \cap H^4(l_0, l) \\
u''(0) = u''(l) = 0 \\
|u''|_{x=l_0-0} = |u''|_{x=l_0+0} \\
v(x) \in \mathcal{H}^2(0, l) \\
p = u(l_0), \quad q = v(l_0) \end{array} \right\} \subset X
\]

given by the formula:

\[
\tilde{A} : \xi = \begin{pmatrix} u \\ v \\ p \\ q \end{pmatrix} \mapsto \tilde{A} \xi = \begin{pmatrix} v \\ -\frac{1}{\rho(x)}(E(x)I(x)u''\)'' + \frac{1}{\rho(x)} \sum_{j=1}^k \psi_j''(x)M_j \\ q \\ \frac{1}{m}(L - \kappa p + F) \end{pmatrix},
\]

where \( y = (M_1, \ldots, M_k, F)^T \) is a control and

\[
L = E(x)I(x)\left( u''\big|_{x=l_0-0} - u''\big|_{x=l_0+0} \right).
\]
A feedback control that ensures the property of stability of zero equilibrium was proposed in [26] in the form of the following functionals:

\[ M_j = -\alpha_j \int_0^l \psi_j''(x)v(x)dx, \quad \alpha_j > 0, \quad j = \overline{1,k}, \]

\[ F = -\alpha_0 q, \quad \alpha_0 > 0. \tag{5} \]

In [26], it was shown that the equations of motion (1)–(3) can be represented as the following differential equation in the operator form:

\[ \frac{d}{dt} \xi(t) = \tilde{A}\xi(t), \quad \xi(t) \in X. \tag{6} \]

Consider an operator \( A \) with domain \( D(A) = D(\tilde{A}) \) that acts exactly like \( \tilde{A} \) with vanishing controls, i.e., \( M_j = 0, \ j = \overline{1,k}, \) and \( F = 0. \)

In this section, we construct the inverse operators \( A^{-1} \) and \( \tilde{A}^{-1} \) and focus our attention on some important features of the operators \( A \) and \( \tilde{A}. \)

Here and in what follows, for the sake of simplicity, we assume that \( E, I, \) and \( \rho \) are positive constants.

**Lemma 3.1.** The operator \( \tilde{A} : D(\tilde{A}) \to X \) is closed.

**Proof.** Solving the equation

\[ \tilde{A}\xi = \dot{\xi} \tag{7} \]

with respect to \( \xi \in D(A) \) for \( \dot{\xi} = (\dot{u}, \dot{v}, \dot{p}, \dot{q})^T \in X, \) we obtain the inverse operator of the following form:

\[ \tilde{A}^{-1} : \dot{\xi} = \dot{\tilde{A}}^{-1}\dot{\xi} = \begin{bmatrix} B_1x + B_2x^3 - \frac{1}{6} \int_0^x (s - x)^3 \Gamma(\tilde{\xi}(s))ds, \ x \leq l_0 \\ B_3(x - l) + B_4(x - l)^3 + \frac{1}{6} \int_x^l (s - x)^3 \Gamma(\tilde{\xi}(s))ds, \ x > l_0 \\ \hat{u}(x) \\ \frac{EI}{\kappa} \left( \int_0^l \Gamma(\tilde{\xi}(s))ds + 6(C_2 - C_4) \right) - \frac{1}{\kappa}(\alpha_0\hat{p} + m\hat{q}) \end{bmatrix} \tag{8} \]

where

\[ \Gamma(\tilde{\xi}(s)) = -\frac{1}{EI} \left( \rho\dot{v}(x) + \sum_{j=1}^k \psi_j''(x) \int_0^l \psi_j''(x)\dot{u}(x)dx \right), \]

the bounded linear functionals \( B_1(\tilde{\xi}), \ldots, B_4(\tilde{\xi}) \) are solutions of the algebraic system.
with the matrix

\[
K = \begin{pmatrix}
  l_0 & l_0^3 & l - l_0 & (l - l_0)^3 \\
  1 & 3l_0^2 & -1 & -3(l - l_0)^2 \\
  0 & 1 & 0 & l - l_0 \\
-\kappa l_0^2 & 6EI - \kappa l_0^3 & 0 & -6EI
\end{pmatrix},
\]

and

\[
a_n(\hat{\xi}) = \int_0^l (l_0 - s)^{4-n} \hat{\Gamma}(\hat{\xi}(s)) ds, \quad n = 1, 4.
\]

As

\[
\det K = -EI(l - l_0 + 1) - \frac{\kappa}{3} l_0(l - l_0)^2(l - l_0 + l_0^2) < 0,
\]

there exist constants \(\bar{B}_j^m > 0\) such that \(B_1(\hat{\xi}), \ldots, B_4(\hat{\xi})\) can be expressed in the form

\[
B_j(\hat{\xi}) = \sum_{n=0}^3 \bar{B}_j^m \int_0^l (l_0 - x)^n \hat{v}(x) dx + \bar{B}_j^4 \int_0^l \hat{v}(x) dx + \bar{B}_j^5 \hat{q}, \quad j = 1, 4.
\]

Note that \(\bar{B}_j^m > 0\) are determined solely by the mechanical parameters of the considered system.

Mapping (8) is obviously linear. Assume that the sequence \(\{\xi_n\}_{n=1}^\infty\) is bounded, i.e., \(\|\xi_n\| \leq \bar{C}\). We now show that the sequence \(\{A^{-1}\hat{\xi}_n\}_{n=1}^\infty\) is bounded in \(X\).

For \(\hat{\xi} \in X\), the boundary conditions \(\hat{u}(0) = \hat{u}(l) = 0\) are satisfied. According to the Poincaré inequality,

\[
\int_0^l |\hat{u}|^2 dx \leq c \int_0^l \left| \frac{d\hat{u}}{dx} \right|^2 dx, \quad c > 0.
\]

For \(\hat{u}(x) \in H^2(0, l)\) and \(H^2(0, l) \subset C^1(0, l)\), according to the Lagrange mean-value theorem, there exists a point \(\xi \in (0, l)\) such that

\[
\hat{u}'(\xi) = \frac{\hat{u}(l) - \hat{u}(0)}{l} = 0.
\]
Hence,

\[
\dot{u}' = \int_0^x \dot{u}''(z) dz + \dot{u}'(\zeta) = \int_0^x \dot{u}''(z) dz.
\]

Applying the Cauchy–Schwarz inequality, we obtain

\[
\int_0^l |\dot{u}'(x)|^2 dx = \int_0^\zeta |\dot{u}'(x)|^2 dx + \int_\zeta^l |\dot{u}'(x)|^2 dx
\]

\[
= \int_0^\zeta \left( \int_0^x |\dot{u}''(z)| dz \right)^2 dx + \int_\zeta^l \left( \int_\zeta^x |\dot{u}''(z)| dz \right)^2 dx
\]

\[
\leq \int_0^\zeta \left( (x - \zeta) \int_\zeta^x |\dot{u}''(z)|^2 dz \right) dx + \int_\zeta^l \left( (x - \zeta) \int_\zeta^x |\dot{u}''(z)|^2 dz \right) dx
\]

\[
\leq \frac{(\zeta, x) \subset (0, l)}{l^2} \frac{l^2}{2} \int_0^l |\dddot{u}(x)|^2 dx.
\]

This yields the existence of a positive definite quadratic form \( \tilde{\mathcal{A}} = \tilde{\mathcal{A}}(C_1(\bar{\xi}), \ldots, C_4(\bar{\xi})) \) such that the following estimate is true:

\[
\| \tilde{\mathcal{A}}^{-1} \bar{\xi} \|_X^2 \leq \tilde{\Lambda} \| \bar{\xi} \|_X^2.
\]

Hence, the operator \( \tilde{\mathcal{A}}^{-1} : X \to X \) is bounded. Since \( D(\tilde{\mathcal{A}}^{-1}) = X \), it follows from [23, p. 162] that the operator \( \tilde{\mathcal{A}} \) is closed in \( X \).

Lemma 3.1 is proved.

In Theorem 4.1, it is shown that the operator \( \tilde{\mathcal{A}} \) is maximal in a sense that the range of \( (I - \lambda \tilde{\mathcal{A}}) \) coincides with \( X \) for some \( \lambda > 0 \). In addition, the direct substitution implies that

\[
\langle \tilde{\mathcal{A}} \xi, \bar{\xi} \rangle_X \leq 0 \quad \forall \xi \in D(\tilde{\mathcal{A}}),
\]

where \( \bar{\xi} \) is the complex conjugate of \( \xi \). According to Sobolev’s embedding theorems, \( H^4 \subseteq H^2 \) and \( H^2 \subseteq L^2 \) (\( \subseteq \) stands for the compact embedding), which means that the operator \( \tilde{\mathcal{A}} \) is densely defined in \( X \).

Summarizing the arguments presented above, we conclude that the operator \( \mathcal{A} \), which is densely defined, \( m \)-dissipative, and closed in \( X \), satisfies the conditions of the Lume–Phillips theorem [13]. Thus, we have the following corollary:

**Corollary 3.1.** The operator \( \mathcal{A} : X \to X \) is the infinitesimal generator of a \( C_0 \)-semigroup of operators on \( X \).
Consequently, the Cauchy problem for (6) with the initial value from $X$ is well posed.

**Lemma 3.2.** The operator $A^{-1} : X \rightarrow X$ is compact.

**Proof.** Exploiting the same technique as in Lemma 3.1, we can obtain the following norm estimate for the inverse operator $A^{-1}$ with the domain $D(A^{-1}) = X$:

$$\|A^{-1}\hat{\xi}\|^2_{X'} \leq \Lambda \|\hat{\xi}\|^2_X$$

with some positive-definite quadratic form $\Lambda$. Here, $X' = H^3 \times H^1 \times \mathbb{C}^2$. Thus, the operator $A^{-1} : X \rightarrow X'$ is closed. According to Sobolev’s embedding theorems, $X' \subseteq X$. Therefore, the operator $A^{-1} : X \rightarrow X$ is compact.

Lemma 3.2 is proved.

Note that $A^{-1}$ is well defined and $D(A^{-1}) = X$ (because equation (7) is solvable for any $\hat{\xi} \in X$). It is easy to see that the operator $A^{-1}$ is skew-symmetric, i.e.,

$$\langle A^{-1}\xi_1, \xi_2 \rangle_X = -\langle \xi_1, A^{-1}\xi_2 \rangle_X.$$ 

Thus, the operator $A^{-1}$ satisfies the conditions of the Hilbert–Schmidt theorem and, consequently, the eigenvectors of $A^{-1}$ form a basis in $X$. Taking into account the match of the eigenvectors of $A$ and $A^{-1}$, we arrive at the following corollary:

**Corollary 3.2.** Let $\{\xi_n\}_{n=1}^\infty$ be eigenvectors of the operator $A$. Then the system $\{\xi_n\}_{n=1}^\infty$ forms a basis in $X$.

In the present work, we concentrate on the asymptotic behavior of trajectories of the closed-loop system. The asymptotic behavior is determined by the invariant subsets of the set $Z = \{\xi \in D(A) \mid \hat{V}(\xi) = 0\}$, where $V(\xi)$ is the weak Lyapunov functional constructed in [26] in the form

$$2V = \int_0^l \left(\rho v^2(x) + EI(u''(x))^2\right)dx + m q^2 + \kappa p^2 = \|\xi\|^2_X.$$

The required spectral properties of the infinitesimal generator were obtained in [5]. It was shown there that the eigenvalues $\lambda_j$, $j = 1, 2, \ldots$, of the corresponding spectral problem can be obtained as the roots of the following simplified frequency equation:

$$\Phi_0(\mu) = 0, \quad (9)$$

where

$$\Phi_0(\mu) = 2 \sin \mu (l - l_0) \sin \mu l_0 - \sin \mu l, \quad \mu = \left(\frac{\rho}{EI} \omega^2\right)^{1/4}, \quad \omega = \text{Im} \lambda.$$

In fact, the frequency equation was deduced in a much more complicated form, and its equivalence to (9) in a sense of limit behavior of the roots was also proved in the work cited above.

**Lemma 3.3.** Assume that $\mu_j$ are the roots of the truncated frequency equation (9) and that there are no multiple roots among $\mu_j$. If $\frac{l_0}{l}$ is rational, then there exists a $\tau > 0$ such that the system of functions $\{e^{\lambda_j t}\}_{j=1}^\infty$ is minimal in $L^2(0, \tau)$. 

Proof. The function $\Phi_0$ is analytic. Since $\Phi_0 \not\equiv \text{const}$, the set
\[ \{ \mu \in [0, +\infty) \mid \Phi_0(\mu) = 0 \} \]
is totally disconnected. The condition
\[ \frac{l_0}{l} = \frac{p_1}{p_2} \in \mathbb{Q} \]
guarantees that the function $\Phi_0$ is periodic. Its period $P$ can be found by using the following formula:
\[ P = \frac{2\pi}{|2l_0 - l|} \frac{|2p_1 - p_2|}{\text{GCD}(p_2, 2p_1 - p_2)}, \quad p_1, p_2 \in \mathbb{N}. \]

If $\mu_0, \ldots, \mu_{k-1}$ are the roots of the function $\Phi_0$ on $[0; P)$, then
\[ \mu_n = \left[ \frac{n}{k} \right] P + \mu\left\{ \frac{n}{k} \right\} k, \quad n = 1, 2, \ldots. \]

As shown in [5], the eigenvalues
\[ \lambda_j = i \sqrt{\frac{EI}{p}} \mu_j^2 \]
quadratically grow with respect to $j$. Consider a function $Q'(x) = \max\{ n \in \mathbb{N} \mid \mu_n^2 < x \}$. The following estimates are true:
\[ \frac{k}{P} \left( \sqrt{x} - \mu\left\{ \frac{n}{k} \right\} k \right) - 1 \leq Q'(x) \leq \frac{k}{P} \left( \sqrt{x} - \mu\left\{ \frac{n}{k} \right\} k \right). \]

The number of eigenvalues $\lambda_j$ on $[y, y + z)$ is determined by the function $Q(y + z) = Q'(y + z) - Q'(y)$ satisfying the following estimate:
\[ Q(y, y + z) \leq \left[ \frac{k}{P} \sqrt{y + z} \right] - \left[ \frac{k}{P} \sqrt{y - P} \right] + 1 \leq \frac{k}{P} \left( \sqrt{y + z} - \sqrt{y} + P \right) + 2. \]
Thus, we get
\[ \limsup_{y \to +\infty} \limsup_{z \to +\infty} \frac{Q(y, y + z)}{z} = \limsup_{y \to +\infty} \limsup_{z \to +\infty} \frac{1}{z} \left( \frac{k}{P} \left( \sqrt{y + z} - \sqrt{y} + P \right) + 2 \right) = 0. \]

According to Theorem 1.2.17 [8], the system of functions $\{ e^{\lambda_j t} \}_{j=1}^{\infty}$ is minimal in $L^2(0, \tau)$ for all $\tau > 0$.

Lemma 3.3 is proved.

4. Precompactness of the Trajectories

In this section, we show that the trajectories of the closed-loop system are precompact. For this purpose, we construct the resolvent of $\tilde{A}$ and prove its compactness. First, we construct the resolvent for the operator $\tilde{A}_M$ with constant parameters in the place of controls $M_j$, and then we substitute feedback (5) in the obtained equations.
Theorem 4.1. Let the operator $\tilde{A}$ be defined above. Then the resolvent

$$R_\lambda(\tilde{A}) = (I - \lambda \tilde{A})^{-1} : X \to X$$

of $\tilde{A}$ is compact.

Proof. Let $M_1, \ldots, M_k$ in (4) be constants and let $F = -\alpha_0 q$. Denote

$$\eta = \left(\frac{\rho}{\lambda^2 EI}\right)^{1/4}, \quad \Gamma(\hat{\xi}(s)) = \frac{1}{EI} \left(\sum_{j=1}^{k} \psi_j'' M_j + \frac{\rho}{\lambda} \left(\hat{\upsilon} + \hat{\bar{\upsilon}}\right)\right),$$

$$z_1(x) = \sin \eta x \cosh \eta x, \quad z_2(x) = \cos \eta x \sin \eta x, \quad z_3(x) = z_1(x - s) - z_2(x - s).$$

Then the equation

$$(I - \lambda \tilde{A}_M) \xi = \hat{\xi} \tag{10}$$

is solvable with respect to $\xi \in D(\tilde{A})$ for any vector $\hat{\xi} \in X$, and it solution can be represented for some $\lambda > 0$ in the following form:

$$\xi = \begin{pmatrix} R_1(\hat{\xi}) \\ R_2(\hat{\xi}) \\ R_3(\hat{\xi}) \\ R_4(\hat{\xi}) \end{pmatrix},$$

where

$$R_1(\hat{\xi}) = \begin{cases} B_1 z_1(x) + B_2 z_2(x) + \frac{1}{4\eta^3} \int_{0}^{x} z_3(s) \Gamma(\hat{\xi}(s)) ds, & x \leq l_0, \\ B_3 z_1(x - l) + B_4 z_2(x - l) - \frac{1}{4\eta^3} \int_{0}^{x} z_3(s) \Gamma(\hat{\xi}(s)) ds, & x > l_0, \end{cases}$$

$$R_2(\hat{\xi}) = \begin{cases} \left(\frac{EI}{\rho}\right)^{1/2} \left(2\eta^2 (B_1 z_1(x) + B_2 z_2(x) - \hat{\upsilon}) + \frac{1}{2\eta} \int_{0}^{x} z_3(s) \Gamma(\hat{\xi}(s)) ds\right), & x \leq l_0, \\ \left(\frac{EI}{\rho}\right)^{1/2} \left(2\eta^2 (B_3 z_1(x - l) + B_4 z_2(x - l) - \hat{\upsilon}) - \frac{1}{2\eta} \int_{0}^{x} z_3(s) \Gamma(\hat{\xi}(s)) ds\right), & x > l_0, \tag{11} \end{cases}$$

$$R_3(\hat{\xi}) = B_1 z_1(l_0) + B_2 z_2(l_0) + \frac{1}{4\eta^3} \int_{0}^{l_0} z_3(l_0) \Gamma(\hat{\xi}(s)) ds,$$

$$R_4(\hat{\xi}) = \left(\frac{EI}{\rho}\right)^{1/2} \left(2\eta^2 (B_1 z_1(l_0) + B_2 z_2(l_0) - \hat{\bar{\upsilon}}) + \frac{1}{2\eta} \int_{0}^{l_0} z_3(l_0) \Gamma(\hat{\xi}(s)) ds\right).$$
By using the interface conditions $u^{(j)}(l_0 - 0) - u^{(j)}(l_0 + 0) = 0, \ j = 0, 2,$ the values of bounded linear functionals $B_1(\hat{x}), \ldots, B_4(\hat{x})$ can be obtained as the solution of the following linear algebraic system:

$$
\begin{pmatrix}
B_1 \\
B_2 \\
B_3 \\
B_4
\end{pmatrix} = \frac{1}{4\eta^3} 
\begin{pmatrix}
\int_0^l z_3(l_0 - s)\Gamma(\hat{x}(s))ds \\
2\eta \int_0^l \sin \eta(l_0 - s) \sinh \eta(l_0 - s)\Gamma(\hat{x}(s))ds \\
2\eta^2 \int_0^l (z_1(l_0 - s) + z_2(l_0 - s))\Gamma(\hat{x}(s))ds \\
\int_0^l z_3(l_0 - s)\Gamma(\hat{x}(s))ds
\end{pmatrix} + \gamma,
$$

where

$$\gamma = \left(0, 0, 0, \int_0^l \cos \eta(l_0 - s) \cosh \eta(l_0 - s)\Gamma(\hat{x}(s))ds + \frac{2\eta^2}{\rho} \left(2\eta^2(m + \alpha_0)\bar{p} + \sqrt{\frac{\rho}{EI}(m\bar{q} + \bar{p})}\right)\right)^T.$$

The determinant of $M$ for the system presented above can be expanded in Taylor’s series with respect to $\eta$ as $\eta \to 0$:

$$\det(M) = \frac{8l}{3m} \left(\frac{\rho}{EI}\right)^{1/2} \left(k\lambda_0^2(l - l_0)^2 + 3EI\eta^6 + O(\eta^{10})\right).$$

Hence, it is possible to choose sufficiently small $\eta$ such that $\det(M) \neq 0$.

The parameters $M_j$ in (11) can be excluded by substituting $v = R_2(\hat{x})$ in the distributed control formula (5). The determinant of the resulting matrix of algebraic system can be decomposed in Taylor’s series with respect to $\mu$

$$\left(\mu = \sqrt{\frac{\sigma}{4\lambda^2EI}}\right):$$

$$\det(M_{ij}) = 1 + O(\mu).$$

Hence, we can select sufficiently small $\mu$ such that $\det(M_{ij}) \neq 0$. Thus, the resolvent of $\tilde{A}$ is constructed.

Consider the space $X' = H^2 \times H^2 \times C^2$. Relations (11) define a linear continuous mapping $X \mapsto X'$, and there exists a positive definite quadratic form $\Lambda(B_1(\hat{x}), \ldots, B_4(\hat{x}))$ such that the inequality

$$\| (I - \lambda\tilde{A})^{-1}\hat{x}\|_{X'}^2 \leq \Lambda\|\hat{x}\|_X^2$$

holds for each $\hat{x} \in X$.

As $X' \subseteq X$, the mapping $R_\lambda(\tilde{A}) : X \to X$ is a compact operator.

Theorem 4.1 is proved.

**Remark 4.1.** The solvability of equation (10) proves that the operator $\tilde{A}$ is maximal. This fact was used as an assumption in our previous work on nonasymptotic stability [26].
5. Asymptotic Stability

Sufficient conditions for the asymptotic stability of the considered closed-loop system are formulated in the following theorem:

**Theorem 5.1.** Let \( \{ \xi_i \}_{i \in \mathbb{N}} \) be eigenvectors of the operator \( A \) and let, for each \( i \in \mathbb{N} \), either \( v_i(l_0) \neq 0 \) or
\[
\int_0^l \psi''_j(x)v_i(x)dx \neq 0
\]
for some \( j \in 1, \ldots, k \). Then the solution \( \xi = 0 \) of system (6) is strongly asymptotically stable.

Basically, the assumption that
\[
\int_0^l \psi''_j(x)v_i(x)dx \neq 0
\]
for some \( j \in \{1, \ldots, k\} \) means that the \( j \)th piezoactuator is not located at the node of eigenfunction of the beam.

**Proof.** It suffices to show that the set \( Z = \{ \dot{V} = 0 \} \) does not contain any nontrivial trajectory of the closed-loop system.

Let \( \xi \) be a solution of (6) and let \( \xi \in Z, \ t \geq 0 \). The control \( y \) vanishes on the set \( Z \). This means that the solution \( \xi \) satisfies the following equation in \( Z \):
\[
\dot{\xi} = A\xi. \tag{12}
\]

We now expand the vector \( \xi \) in the series in eigenfunctions of \( A \):
\[
\xi(t) = \sum_{i=1}^{\infty} r_i(t)\xi_i. \tag{13}
\]

Here, we refer to the system \( \{ \xi_i \}_{i=1}^{\infty} \) as a basis of \( X \) according to Corollary 3.2.

Substituting expansion (13) in (12) and taking into account the fact that \( A\xi_i = \lambda_i\xi_i \), where \( \lambda_i \) are the eigenvalues of \( A \), \( i = 1, 2, \ldots, \), we obtain
\[
\sum_{i=1}^{\infty} \dot{r}_i(t)\xi_i = \sum_{i=1}^{\infty} \lambda_i r_i(t)\xi_i.
\]

In view of the uniqueness of expansion in a basis, we get
\[
\dot{r}_i = \lambda_i r_i.
\]

This yields
\[
r_i(t) = r_i^0 e^{\lambda_i t}.
\]

Since \( \xi \in M \), we find
\[
\int_0^l \psi''_j(x)v(x)dx = 0, \quad j = 1, k. \tag{14}
\]
Let us write the decomposition of $v(x)$ in the basis $\{\xi_i\}_{i=1}^{\infty}$:

$$v(x) = \sum_{i=1}^{\infty} r_i(t)v_i(x) = \sum_{i=1}^{\infty} r_i^0 e^{\lambda_i t}v_i(x). \quad (15)$$

Substituting (15) in (14), we conclude that the linear combination of functions $e^{\lambda_i t}$ vanishes. According to Lemma 3.3, the functions $e^{\lambda_i t}$ are linearly independent and, hence, all coefficients of the linear combination must be zero:

$$r_i^0 \int_0^l \psi''_j(x)v_i(x)dx = 0, \quad i = 1, 2, \ldots.$$  

Since

$$\int_0^l \psi''_j(x)v_i(x)dx \neq 0,$$

we conclude that $r_i^0 = 0$ for all $i \in \mathbb{N}$ and, hence, $\xi \equiv 0$.

Thus, the set $Z$ does not contain any nontrivial trajectory of system (6). According to LaSalle’s invariance principle [10], the trivial solution of the considered system is asymptotically stable.

6. Conclusions

The present work concludes the investigations of asymptotic stability of the vibrating simply-supported flexible beam with distributed control and an attached point mass. The answer to the question about asymptotic stability is positive, while the system is not necessarily exponentially stable.

In the present paper, we do not take into account dissipation with an aim to investigate the applicability of the Lyapunov method in the case where there is no natural damping. The proposed state feedback law is of mathematical interest, while the problems of observer design and observer-based stabilization remain for further studies.

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