Novel PT-invariant Solutions For a Large Number of Real Nonlinear Equations

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Abstract:
For a large number of real nonlinear equations, either continuous or discrete, integrable or nonintegrable, we show that whenever a real nonlinear equation admits a solution in terms of \( \text{sech} x \), it also admits solutions in terms of the PT-invariant combinations \( \text{sech} x \pm i \text{tanh} x \). Further, for a number of real nonlinear equations we show that whenever a nonlinear equation admits a solution in terms of \( \text{sech}^2 x \), it also admits solutions in terms of the PT-invariant combinations \( \text{sech}^2 x \pm i \text{sech} x \text{tanh} x \). Besides, we show that similar results are also true in the periodic case involving Jacobi elliptic functions.
1 Introduction

Nonlinear equations are playing an increasingly important role in several areas of science in general and physics in particular. One of the major problems with these equations is the lack of a superposition principle. It is thus necessary to explicitly obtain more and more solutions of a given nonlinear equation. Thus if we can find some general results about the existence of solutions to a nonlinear equation, that would be invaluable. In this context it is worth recalling that some time ago we [1] had shown (through a number of examples) that if a nonlinear equation admits a periodic solution in terms of Jacobi elliptic functions $dn(x,m)$ and $cn(x,m)$, then it will also admit solutions in terms of $dn(x,m) \pm cn(x,m)$, where $m$ is the modulus of the elliptic function. Further, in the same paper [1], we also showed (again through several examples) that if a nonlinear equation admits a solution in terms of $dn^2(x,m)$, then it will also admit solutions in terms of $dn^2(x,m) \pm cn(x,m)dn(x,m)$.

The purpose of this paper is to propose general results about the existence of new solutions to real nonlinear equations, integrable or nonintegrable, continuous or discrete through the idea of PT-symmetry. It may be noted here that in the last 15-20 years the idea of PT symmetry [2] has given us new insight. In quantum mechanics it has been shown that even if Hamiltonian is not hermitian but if it is PT-invariant, then the energy eigenvalues are still real in case PT symmetry is not broken spontaneously. Further, there is tremendous growth in the number of studies of open systems which are specially balanced by PT symmetry [3, 4, 5] in several PT-invariant open systems bearing both loss and gain, one has obtained soliton solutions and they have been shown to be stable within certain parameter range [6, 7, 8].

In this paper we highlight one more novel aspect of PT-symmetry. In particular, we obtain new PT-invariant solutions through a general principle. We show, through several examples, that whenever a real nonlinear equation, either continuous or discrete, integrable or nonintegrable, admits a solution in terms of $sech x$, then it will necessarily also admit solutions in terms of the PT-invariant combinations $sech x \pm i \tanh x$. We also generalize these results to the periodic case and show that whenever a nonlinear equation admits a solution in terms of $dn(x,m)$ [or $cn(x,m)$], then it will necessarily also admit solutions in terms of the PT-invariant combinations $dn(x,m) \pm i\sqrt{m}sn(x,m)$ [or $cn(x,m) \pm isn(x,m)$].

Further, we show, through several examples, that whenever a real nonlinear equation admits a solution
in terms of \( \text{sech}^2 x \), then it will also admit solutions in terms of \( \text{sech}^2 x \pm i \text{sech} x \tanh x \). We also generalize these results to the periodic case and show that whenever a real nonlinear equation admits a solution in terms of \( \text{dn}^2(x, m) \), then it will necessarily also admit solutions in terms of \( \text{dn}^2(x, m) \pm i m \text{sn}(x, m) \text{cn}(x, m) \) as well as \( \text{dn}^2(x, m) \pm i \sqrt{m} \text{sn}(x, m) \text{dn}(x, m) \).

2 Solutions in Terms of \( \text{sech} x \pm i \tanh x \) as well as Their Periodic Generalization

We now discuss four examples, two from continuum field theories and two from the discrete case where \( \text{sech} x \) is a known solution and in all the four cases we obtain new PT-invariant solutions in terms of \( \text{sech} x \pm i \tanh x \) and also periodic PT-invariant solutions in terms of \( \text{dn}(x, m) \pm i \text{sn}(x, m) \) as well as \( \text{cn}(x, m) \pm i \text{sn}(x, m) \).

2.1 \( \phi^4 \) Field Theory

The \( \phi^4 \) field theory arises in several areas of physics including second order phase transitions. The field equation for the \( \phi^2 - \phi^4 \) field theory is given by

\[ \phi_{xx} = a\phi + b\phi^3, \]  

(1)

In case \( b < 0 \), one of the well known solution to this equation is

\[ \phi = A \text{sech}[\beta x], \]  

(2)

provided

\[ bA^2 = -2\beta^2, \quad a = \beta^2. \]  

(3)

Remarkably, even

\[ \phi = A \text{sech}(\beta x) \pm iB \tanh(\beta x) \]  

(4)

is an exact PT-invariant solution of Eq. (1) provided

\[ B = \pm A, \quad 2bA^2 = -\beta^2, \quad a = -(1/2)\beta^2. \]  

(5)
Further, as we now show, such PT-invariant solutions also exist in the periodic case. Let us first note that one of the exact, periodic solution to the $\phi^4$ Eq. (1) is

$$\phi = A \text{dn}(\beta x, m),$$  

provided

$$bA^2 = -2\beta^2, \quad a = (2 - m)\beta^2.$$  

Further, the same model (1) is known to admit another periodic solution

$$\phi = A \sqrt{m} \text{cn}(\beta x, m),$$  

provided

$$bA^2 = -2\beta^2, \quad a = (2m - 1)\beta^2.$$  

Remarkably, we find that the same model also admits the PT-invariant periodic solution

$$\phi = A \text{dn}(\beta x, m) + iB \sqrt{m} \text{sn}(\beta x, m),$$  

provided

$$B = \pm A, \quad 2bA^2 = -\beta^2, \quad a = -2m - 1 \beta^2.$$  

Further, the same model also admits another PT-invariant solution

$$\phi = A \sqrt{m} \text{cn}[\beta x, m] + iB \sqrt{m} \text{sn}[\beta x, m],$$  

provided

$$B = \pm A, \quad 2bA^2 = -\beta^2, \quad a = -2 - m \beta^2.$$  

### 2.2 mKdV Equation

We first discuss the celebrated mKdV equation

$$u_t + u_{xxx} + 6u^2u_x = 0,$$  

which is a well known integrable equation having application in several areas [9]. It is well known that

$$u = \text{Asech}[\beta(x - vt)].$$
is an exact solution of Eq. (14) provided
\[ A^2 = \beta^2, \ v = \beta^2. \] (16)

Remarkably, even
\[ u = A \text{sech}[\beta(x - vt)] \pm iB \tanh[\beta(x - vt)] \] (17)
is also an exact PT-invariant solution to the mKdV Eq. (14) provided
\[ B = \pm A, \ A^2 = 4\beta^2, \ v = -(1/2)\beta^2. \] (18)

Even more remarkable, such PT-invariant solutions also exist in the periodic case. For example, it is well known that one of the exact, periodic solution to the mKdV Eq. (14) is [10]
\[ u = A \text{dn}[\beta(x - vt), m], \] (19)
provided
\[ A^2 = \beta^2, \ v = (2 - m)\beta^2. \] (20)
Another periodic solution to the mKdV Eq. (14) is
\[ u = A\sqrt{m} \text{cn}[\beta(x - vt), m], \] (21)
provided
\[ A^2 = \beta^2, \ v = (2m - 1)\beta^2. \] (22)

Remarkably, even
\[ u = A \text{dn}[\beta(x - vt), m] + iB \sqrt{m} \text{sn}[\beta(x - vt), m] \] (23)
is an exact PT-invariant solution to the mKdV Eq. (14) provided
\[ B = \pm A, \ A^2 = 4\beta^2, \ v = -(2m - 1)\beta^2. \] (24)
We thus have two new periodic solutions of mKdV Eq. (14) depending on whether \( B = A \) or \( B = -A \).

Further, even
\[ u = A\sqrt{m} \text{cn}[\beta(x - vt), m] + iB \sqrt{m} \text{sn}[\beta(x - vt), m], \] (25)
is an exact PT-invariant solution of the mKdV Eq. (14) provided
\[ B = \pm A, \ A^2 = 4\beta^2, \ v = -(2 - m)\beta^2. \] (26)
2.3 Discrete $\phi^4$ Equation

We now discuss two discrete models and show that both these models also admit PT-invariant solutions. Let us first consider the discrete $\phi^4$ equation

$$\frac{1}{h^2}[\phi_{n+1} + \phi_{n-1} - 2\phi_n] + a\phi_n - \frac{\lambda}{2}\phi_n^2[\phi_{n+1} + \phi_{n-1}] = 0. \quad (27)$$

It is well known that the Eq. (27) admits an exact solution

$$\phi_n = A \text{sech}(\beta n), \quad (28)$$

provided

$$A^2 = -2 \frac{\sinh^2(\beta)}{h^2 \lambda}, \quad ah^2 = -4 \sinh^2(\beta/2). \quad (29)$$

Remarkably, the same model also admits a PT-invariant periodic solution

$$\phi_n = A \text{sech}(\beta n) \pm iB \tanh(\beta n), \quad (30)$$

provided

$$B = \pm A, \quad A^2 = -2 \frac{\tanh^2(\beta/2)}{h^2 \lambda}, \quad ah^2 = 2 \tanh^2(\beta/2). \quad (31)$$

Besides, the same model also has novel, PT-invariant periodic solutions. Let us first note that a well-known exact periodic solution to the Eq. (27) is

$$\phi_n = A \text{dn}(\beta n, m), \quad (32)$$

provided

$$A^2 \text{cs}^2(\beta, m) = -2 \frac{2}{h^2 \lambda}, \quad ah^2 = 2 \left[ 1 - \frac{\text{dn}(\beta, m)}{\text{cn}^2(\beta, m)} \right], \quad (33)$$

where $cs(x, m) = \text{cn}(x, m)/\text{sn}(x, m)$. Further, the same model (27) is known to admit another periodic solution

$$\phi_n = A \sqrt{m} \text{cn}(\beta n, m), \quad (34)$$

provided

$$A^2 \text{ds}^2(\beta, m) = -2 \frac{2}{h^2 \lambda}, \quad ah^2 = 2 \left[ 1 - \frac{\text{cn}(\beta, m)}{\text{dn}^2(\beta, m)} \right], \quad (35)$$
where \( ds(x, m) = \text{dn}(x, m)/\text{sn}(x, m) \).

We find that the same model also admits the PT-invariant periodic solution

\[
\phi_n = A\text{dn}(\beta n, m) + iB\sqrt{\text{msn}(\beta n, m)},
\]

provided

\[
B = \pm A, \quad A^2[\text{cs}(\beta, m) + \text{ns}(\beta, m)]^2 = -\frac{2}{h^2\lambda}, \quad ah^2 = 2 \left[ 1 - \frac{2\text{dn}(\beta, m)}{1 + \text{cn}(\beta, m)} \right].
\]

Further, the same model also admits another PT-invariant solution

\[
\phi_n = A\sqrt{m}\text{cn}(\beta n, m) + iB\sqrt{\text{msn}(\beta n, m)},
\]

provided

\[
B = \pm A, \quad A^2[ds(\beta, m) + \text{ns}(\beta, m)]^2 = -\frac{2}{h^2\lambda}, \quad ah^2 = 2 \left[ 1 - \frac{2\text{cn}(\beta, m)}{1 + \text{dn}(\beta, m)} \right].
\]

While deriving results in this and the next subsection, we have made use of several not so well known identities satisfied by the Jacobi elliptic functions \( [11] \).

### 2.4 Discrete mKdV Equation

Let us consider the discrete mKdV equation

\[
\frac{du_n}{dt} + \alpha(u_{n+1} - u_{n-1}) + \lambda u_n^2(u_{n+1} - u_{n-1}) = 0.
\]

(40)

It is well known that this model has an exact hyperbolic soliton solution

\[
u_n = \text{Asech}[\beta(n - vt)] ,
\]

provided

\[
\lambda A^2 = \alpha \sinh^2(\beta), \quad \beta v = 2\alpha \sinh(\beta).
\]

(42)

We find that this model also admits the PT-invariant solution

\[
u_n = \text{Asech}(\beta n) \pm iB\tanh(\beta n),
\]

provided

\[
B = \pm A, \quad \lambda A^2 = \alpha \tanh^2(\beta/2), \quad \beta v = 4\alpha \tanh(\beta/2).
\]

(44)
We find that this model also admits exact PT-invariant periodic solutions. Let us first note that the
well known periodic solution to Eq. (40) is
\[ u_n = A \text{dn}[\beta(n - vt), m], \] (45)
provided
\[ \lambda A^2 \text{cs}^2(\beta, m) = \alpha, \quad \beta v = \frac{2\alpha}{\text{cs}(\beta, m)}. \] (46)
Further, the same model (40) is known to admit another periodic solution
\[ u_n = A \sqrt{m} \text{cn}[\beta(n - vt), m], \] (47)
provided
\[ \lambda A^2 \text{ds}^2(\beta, m) = \alpha, \quad \beta v = \frac{2\alpha}{\text{ds}(\beta, m)}. \] (48)
We now show that the same model also admits a PT-invariant periodic solution
\[ u_n = A \text{dn}[\beta(n - vt), m] + iB \sqrt{m} \text{sn}[\beta(n - vt), m], \] (49)
provided
\[ B = \pm A, \quad \lambda A^2 [\text{cs}(\beta, m) + \text{ns}(\beta, m)]^2 = \alpha, \quad \beta v = \frac{4\text{sn}(\beta, m)}{1 + \text{cn}(\beta, m)}, \] (50)
where \( \text{ns}(x, m) = 1/\text{sn}(x, m). \) Further, the same model also admits another PT-invariant solution
\[ u_n = A \sqrt{m} \text{cn}(\beta n, m) + iB \sqrt{m} \text{sn}(\beta n, m), \] (51)
provided
\[ B = \pm A, \quad \lambda A^2 [\text{ds}(\beta, m) + \text{ns}(\beta, m)]^2 = \alpha, \quad \beta v = \frac{4\text{sn}(\beta, m)}{1 + \text{dn}(\beta, m)}. \] (52)

3 Solutions in Terms of sech\(^2\)x ± isech\(x\) tanh\(x\) as well as Their Periodic
Generalization

We now discuss two examples where sech\(^2\)x is a known solution and in both the cases we obtain new
PT-invariant solutions in terms of sech\(^2\)x ± isech\(x\) tanh\(x\) and also PT-invariant periodic solutions in terms
of \( \text{dn}^2(x, m) \pm i\text{msn}(x, m)\text{cn}(x, m) \) as well as \( \text{dn}^2(x, m) \pm i\sqrt{\text{msn}(x, m)\text{dn}(x, m)}. \)
3.1 KdV Equation

We first discuss the celebrated KdV equation

\[ u_t + u_{xxx} - 6uu_x = 0, \]  

(53)

which is a well known integrable equation having application in several areas including shallow water waves [9]. It is also well known that it admits the soliton solution

\[ u = A \text{sech}^2(x - vt), \]  

(54)

provided \( A = -2\beta^2, \ v = 4\beta^2. \) Remarkably, it also admits a PT-invariant solution

\[ u = A \text{sech}^2(x - vt) + iB \text{sech}(x - vt) \tanh(x - vt), \]  

(55)

provided

\[ B = \pm A, \ A = -\beta^2, \ v = \beta^2. \]  

(56)

We now show that KdV equation also admits periodic PT-invariant solutions. It is well known that one of the exact, periodic solution to the KdV Eq. (53) is

\[ u = A d_n^2[\beta(x - vt), m], \]  

(57)

provided

\[ A = -2\beta^2, \ v = 4(2 - m)\beta^2. \]  

(58)

Remarkably, even

\[ u = A d_n^2[\beta(x - vt), m] + iB m_s n[\beta(x - vt), m] c_n[\beta(x - vt), m], \]  

(59)

is an exact solution of the KdV Eq. (53) provided

\[ B = \pm A, \ A = -\beta^2, \ v = -(2 - m)\beta^2. \]  

(60)

We thus have two new periodic solutions of the KdV Eq. (53) depending on whether \( B = A \) or \( B = -A. \)

Remarkably, there is another PT-invariant solution to the same KdV equation

\[ u = A d_n^2[\beta(x - vt + \delta_1), m] + iB \sqrt{m} s_n[\beta(x - vt), m] d_n[\beta(x - vt), m], \]  

(61)
provided

\[ B = \pm A, \quad A = -\beta^2, \quad v = (5 - 4m)\beta^2. \]  \hspace{1cm} (62)

Few remarks are in order at this stage.

1. It is well known that the hyperbolic potential \(-2\beta^2\text{sech}^2(\beta x)\) which is a solution of the KdV equation, is a reflectionless potential. We then predict that the potentials \(-\beta^2\text{sech}^2(\beta x) \pm i\text{sech}(\beta x)\tanh(x)\) must also be reflectionless potentials.

2. It is well known that the periodic potential \(-2\beta^2\text{dn}^2(\beta x, m)\) which is a solution of the KdV equation, has precisely one band gap. We then predict that the potentials \(-\beta^2\text{dn}^2(\beta x, m) \pm im\beta^2\text{sn}(\beta x, m)\text{cn}(\beta x, m)\) as well as the potentials \(\beta^2\text{dn}^2(\beta x, m) \pm i\sqrt{m}\beta^2\text{sn}(\beta x, m)\text{dn}(\beta x, m)\) must also have precisely one band gap.

3.2 \(\phi^3\) Field Theory

This field theory arises in the context of third order phase transitions [12] and is also relevant to tachyon condensation [13]. The field equation for the \(\phi^2 - \phi^3\) field theory is given by

\[ \phi_{xx} = a\phi + b\phi^2, \] \hspace{1cm} (63)

which is known to admit an exact solution

\[ \phi = \text{Asech}^2(\beta x) + B, \] \hspace{1cm} (64)

provided

\[ A = \frac{3a}{2b}, \quad \beta^2 = \frac{a}{4}, \quad B = 0. \] \hspace{1cm} (65)

Remarkably, Eq. (63) also admits a PT-invariant solution

\[ \phi = \text{Asech}^2[\beta(x)] \pm iD\text{sech}[\beta(x)]\tanh[\beta(x)] + B, \] \hspace{1cm} (66)

provided

\[ D = \pm A, \quad A = -\frac{3a}{b}, \quad \beta^2 = a, \quad B = 0. \] \hspace{1cm} (67)
Further the model also admits PT-invariant periodic solutions. Let us first note that the model (63) also admits the periodic solution
\[ \phi = A \operatorname{dn}^2[\beta(x), m] + B, \] (68)
provided
\[ A = -\frac{3a}{2b\sqrt{1-m+m^2}}, \quad \beta^2 = \frac{a}{4\sqrt{1-m+m^2}}, \quad B = \frac{a[2-m-\sqrt{1-m+m^2}]}{2b\sqrt{1-m+m^2}}. \] (69)

It is easy to show that the same model also admits a PT-invariant periodic solution
\[ \phi = A \operatorname{dn}^2[\beta(x), m] + iD\sqrt{m}\operatorname{cn}[\beta(x), m]\operatorname{dn}[\beta(x), m] + B, \] (70)
provided
\[ D = \pm A, \quad A = -\frac{3a}{b\sqrt{16-16m+m^2}}, \quad \beta^2 = \frac{a}{\sqrt{16-16m+m^2}}, \quad B = \frac{a[2-m-\sqrt{16-16m+m^2}]}{2b\sqrt{16-16m+m^2}}. \] (71)

Further, the same model also admits another PT-invariant periodic solution
\[ \phi = A \operatorname{dn}^2[\beta(x), m] + iD\sqrt{m}\operatorname{sn}[\beta(x), m]\operatorname{dn}[\beta(x+c), m] + B, \] (72)
provided
\[ D = \pm A, \quad A = -\frac{3a}{b\sqrt{16-m+16m^2}}, \quad \beta^2 = \frac{a}{\sqrt{1-16m+16m^2}}, \quad B = \frac{a[5-4m-\sqrt{1-16m+16m^2}]}{2b\sqrt{1-16m+16m^2}}. \] (73)

4 PT-Invariant Solutions in Three Coupled models

We now consider three different coupled models and show that in all these cases one has PT-invariant solutions for all the coupled fields.

4.1 Coupled $\phi^4$ Model

We first consider a coupled $\phi^4$ model
\[ \phi_{xx} = 2a_1\phi + 4b_1\phi^3 + 2\gamma\phi\psi^2, \] (74)
\[
\psi_{xx} = 2a_2 \psi + 4b_1 \psi^3 + 2\gamma \psi \phi^2, \quad (75)
\]

and show that even in this case, PT-invariant solutions are allowed in both the fields.

It is well known that this coupled system admits the solution \[14\]

\[
\phi = A \text{sech}(\beta x), \quad \psi = D \text{sech}(\beta x), \quad (76)
\]

provided

\[
2b_1 A^2 + \gamma D^2 = -\beta^2 = 2b_2 D^2 + \gamma A^2, \quad a_1 = a_2 = \frac{\beta^2}{2}. \quad (77)
\]

Remarkably, the coupled model also admits the PT-invariant solution

\[
\phi = A \text{sech}(\beta x) + iB \tanh(\beta x), \quad \psi = D \text{sech}(\beta x) + iF \tanh(\beta x), \quad (78)
\]

provided

\[
B = \pm A, \quad F = \pm D, \quad a_1 = a_2 = -\frac{\beta^2}{4}, \quad \frac{4(2b_1 A^2 + \gamma D^2)}{4(2b_2 D^2 + \gamma A^2)}. \quad (79)
\]

Note that the signs of \( B = \pm A \) and \( F = \pm D \) are correlated.

This coupled model also admits PT-invariant periodic solutions. Let us first note that one of the well known periodic solution to the coupled Eq. (74) is

\[
\phi = A \text{dn}[\beta x, m], \quad \psi = D \text{dn}[\beta x, m], \quad (80)
\]

provided

\[
2b_1 A^2 + \gamma D^2 = -\beta^2 = 2b_2 D^2 + \gamma A^2, \quad a_1 = a_2 = \frac{(2m - 1)\beta^2}{2}. \quad (81)
\]

Further, the same coupled model is known to admit another periodic solution

\[
\phi = A \sqrt{m} \text{cn}[\beta x, m], \quad \psi = D \sqrt{m} \text{cn}[\beta x, m], \quad (82)
\]

provided

\[
2b_1 A^2 + \gamma D^2 = -\beta^2 = 2b_2 D^2 + \gamma A^2, \quad a_1 = a_2 = \frac{(2m - 1)\beta^2}{2}. \quad (83)
\]
Remarkably, we find that the same coupled model also admits a PT-invariant periodic solution

\[
\phi = A \text{dn}[\beta x, m] + iB \text{sn}[\beta x, m], \\
\psi = D \text{dn}[\beta x, m] + iF \text{sn}[\beta x, m],
\]

provided

\[
B = \pm A, \quad F = \pm D, \quad a_1 = a_2 = -\frac{(4m - 3)\beta^2}{4}, \\
4(2b_1A^2 + \gamma D^2) = -\beta^2 = 4(2b_2D^2 + \gamma A^2).
\]

(84)

Note that the signs of \(B = \pm A\) and \(F = \pm D\) are correlated.

Further, the same model also admits another PT-invariant periodic solution

\[
\phi = A \sqrt{m} \text{cn}[\beta x, m] + iB \sqrt{m} \text{sn}[\beta x, m], \\
\psi = D \sqrt{m} \text{cn}[\beta x, m] + iF \sqrt{m} \text{sn}[\beta x, m],
\]

provided

\[
B = \pm A, \quad F = \pm D, \quad a_1 = a_2 = -\frac{(4 - 3m)\beta^2}{4}, \\
4(2b_1A^2 + \gamma D^2) = -\beta^2 = 4(2b_2D^2 + \gamma A^2).
\]

(85)

Note that the signs of \(B = \pm A\) and \(F = \pm D\) are correlated.

### 4.2 Coupled KdV Equations

We now discuss the coupled KdV model which has also received some attention in the literature \[15\] and show that even in this case, PT-invariant solutions exist in both the coupled fields.

The coupled KdV equations are

\[
\begin{align*}
    u_t + \alpha uu_x + \eta vv_x + u_{xxx} &= 0, \\
    v_t + \delta uv_x + v_{xxx} &= 0.
\end{align*}
\]

(88)

One of the well known solution to the coupled Eqs. \[88\] is \[15\]

\[
\begin{align*}
    u &= A \text{sech}^2[\beta(x - ct)], \\
    v &= D \text{sech}^2[\beta(x - ct)].
\end{align*}
\]

(89)
provided
\[ \delta A = 12\beta^2, \quad \eta D^2 = (\delta - \alpha)A^2, \quad c = 4\beta^2. \] 
\hspace{1cm} (90)

Remarkably, the same coupled model also admits the hyperbolic PT-invariant solution
\[ u = A\text{sech}^2[\beta(x - ct)] + iB\tanh[\beta(x - ct)]\text{sech}[\beta(x - ct)], \]
\[ v = D\text{sech}^2[\beta(x - ct)] + iF\tanh[\beta(x - ct)]\text{sech}[\beta(x - ct)], \] 
\hspace{1cm} (91)

provided
\[ B = \pm A, \quad F = \pm D, \quad \delta A = 6\beta^2, \quad \eta D^2 = (\delta - \alpha)A^2, \quad c = \beta^2. \] 
\hspace{1cm} (92)

This discussion is easily generalized to the periodic case. In particular, it is easy to check that the coupled Eqs. (88) have the periodic solution
\[ u = A\text{dn}^2[\beta(x - ct), m], \quad v = D\text{dn}^2[\beta(x - ct), m], \] 
\hspace{1cm} (93)

provided
\[ \delta A = 12\beta^2, \quad \eta D^2 = (\delta - \alpha)A^2, \quad c = 4(2 - m)\beta^2. \] 
\hspace{1cm} (94)

Remarkably, the same model also admits a PT-invariant periodic solution
\[ u = A\text{dn}^2[\beta(x - ct), m] + iBm\sqrt{m}\text{sn}[\beta(x - ct), m]\text{dn}[\beta(x - ct), m], \]
\[ v = D\text{dn}^2[\beta(x - ct), m] + iFm\sqrt{m}\text{sn}[\beta(x - ct), m]\text{dn}[\beta(x - ct), m], \] 
\hspace{1cm} (95)

provided
\[ B = \pm A, \quad F = \pm D, \quad \delta A = 6\beta^2, \quad \eta D^2 = (\delta - \alpha)A^2, \quad c = (2 - m)\beta^2. \] 
\hspace{1cm} (96)

Note that the signs of \( B = \pm A \) and \( F = \pm D \) are correlated. Further, the same model also admits another PT-invariant periodic solution
\[ u = A\text{dn}^2[\beta(x - ct), m] + iB\sqrt{m}\text{sn}[\beta(x - ct), m]\text{dn}[\beta(x - ct), m], \]
\[ v = D\text{dn}^2[\beta(x - ct), m] + iF\sqrt{m}\text{sn}[\beta(x - ct), m]\text{dn}[\beta(x - ct), m], \] 
\hspace{1cm} (97)

provided
\[ B = \pm A, \quad F = \pm D, \quad \delta A = 6\beta^2, \quad \eta D^2 = (\delta - \alpha)A^2, \quad c = (2m - 1)\beta^2. \] 
\hspace{1cm} (98)

Note that the signs of \( B = \pm A \) and \( F = \pm D \) are correlated.
4.3 Coupled KdV-mKdV Model

Finally we consider a coupled KdV-mKdV model

\[
\begin{align*}
    u_t + u_{xxx} + 6uu_x + 2\alpha uvv_x &= 0, \\
    v_t + v_{xxx} + 6v^2v_x + \gamma vu_x &= 0,
\end{align*}
\]

and show that in this case too we have PT-invariant solutions of the form \( \text{sech}^2 x \pm i \text{sech} x \tanh x \) and \( \text{sech} x \pm i \tanh x \) in KdV and mKdV fields, \( u \) and \( v \), respectively.

It is easy to check that

\[
\begin{align*}
    u &= A \text{sech}^2 [\beta(x - ct)] + G, \\
    v &= D \text{sech} [\beta(x - ct)],
\end{align*}
\]

is an exact solution of the coupled Eqs. (99) provided

\[
12D^2 + 4\gamma A = 12\beta^2 = 6A + \alpha D^2, \quad c = \beta^2, \quad G = -\frac{A}{4}.
\]

Remarkably, the same model also admits a PT-invariant solution

\[
\begin{align*}
    u &= A \text{sech}^2 [\beta(x - ct)] + iB \tanh [\beta(x - ct)] \text{sech} [\beta(x - ct)], \\
    v &= D \text{sech} [\beta(x - ct)] + iF \tanh [\beta(x - ct)],
\end{align*}
\]

provided

\[
B = \pm A, \quad F = \pm D, \quad 12D^2 + 2\gamma A = 3\beta^2 = 3A + \alpha D^2, \quad c = -\frac{1}{2} \beta^2, \quad G = -\frac{A}{4}.
\]

We now show that the same model also has PT-invariant periodic solutions. Let us first note that

\[
\begin{align*}
    u &= A \text{dn}^2 [\beta(x - ct), m] + G, \\
    v &= D \text{dn} [\beta(x - ct), m],
\end{align*}
\]

is an exact solution of the coupled Eqs. (99) provided

\[
12D^2 + 4\gamma A = 12\beta^2 = 6A + \alpha D^2, \quad c = (2 - m)\beta^2, \quad G = -\frac{(2 - m)A}{4}.
\]

It is easy to check that the same model also admits a PT-invariant solution

\[
\begin{align*}
    u &= A \text{dn}^2 [\beta(x - ct), m] + iB \sqrt{m} \text{sn} [\beta(x - ct), m] \text{dn} [\beta(x - ct), m] + G, \\
    v &= D \text{dn} [\beta(x - ct), m] + iF \text{sn} [\beta(x - ct), m],
\end{align*}
\]
Note that the signs of $B = \pm A$ and $F = \pm D$ are correlated. Further, the same model also admits another PT-invariant solution

$$u = A\text{dn}^2[\beta(x - ct), m] + iB\text{msn}[\beta(x - ct), m]\text{cn}[\beta(x - ct), m],$$
$$v = D\sqrt{m}\text{cn}[\beta(x - ct), m] + iF\sqrt{m}\text{sn}[\beta(x - ct), m]\text{dn}[\beta(x - ct), m],$$

provided

$$B = \pm A, \quad F = \pm D, \quad 12D^2 + 2\gamma A = 3\beta^2 = 3A + \alpha D^2, \quad c = -\frac{(2m - 1)}{2}\beta^2, \quad G = -\frac{(3 - 2m)A}{4}.$$

Note that the signs of $B = \pm A$ and $F = \pm D$ are correlated.

5 Summary and Conclusions

In this paper we have shown through several examples that whenever a real nonlinear equation admits solution in terms of $\text{sech} x$ (or $\text{sech}^2 x$), then the same model also admits solutions in terms of $\text{sech} x \pm i\tanh x$ (or $\text{sech}^2 x \pm i\text{sech} x \tanh x$). Further, we have also shown that such PT-invariant solutions also exist in the corresponding periodic case involving Jacobi elliptic functions.

The obvious open question is whether these results are true in general. It would be nice if one can prove this in general, both in the hyperbolic as well as in the periodic case. In the absence of a general proof, it is worthwhile looking at more and more examples and see if this observation is true in general or if there are some exceptions. The other question is: What could be the deeper underlying reason because of which such solutions exist? Another question is about the significance of such solutions for a real nonlinear equation.

In this context we would like to remark that the symmetry of solutions of a nonlinear equation need not be the same as that of the nonlinear equation but could be less. Normally, the complex solutions of a real nonlinear equation are not of relevance. However, being $PT$ invariant complex solutions, we believe they could have some physical significance. One pointer in this direction is the fact that for both the KdV and
the mKdV equations, which are integrable equations, we have checked that the first 3 constants of motion for the PT-invariant complex solutions of both the KdV and the mKdV equations are in fact real but have different values than those for the usual hyperbolic solution (and we suspect that in fact all the constants of motion would be real and would be different than those for the real hyperbolic solution) thereby suggesting that such solutions could be physically interesting. Thus it would be worthwhile studying the stability of such PT-invariant solutions. That may shed some light on the possible significance of such solutions.

We hope to address some of these issues in the near future.

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