Chameleon attractors in a turbulent flow

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Turbulent flows in geophysical systems often present rich dynamics originating from non-trivial
energy fluxes in scale space, non-stationary forcings and geometrical constraints. This complex-
ity appears via non-hyperbolic chaos, randomness, state-dependent persistence and predictability.
All these features have prevented a full characterization of the underlying turbulent (stochastic)
attractor, which will be the key object to unpin this complexity. Here we use a novel formalism
to map unstable fixed points to singularities of turbulent flows and to trace the evolution of their
structural characteristics when moving from small to large scales and vice versa, providing a full
characterization of the attractor. We demonstrate that the properties of the dynamically invariant
objects depend on the scale we are focusing on. Given the changing nature of such attractors in
time, space and scale spaces, we term them chameleon attractors.

A glass of water consists of about $10^{N_A}$ water
molecules (with $N_A = 6.022 \times 10^{23}$ mol$^{-1}$ being the Avo-
gadro number), which obey collisional dynamics that can
be derived from quantum mechanics equations. Yet, no-
body will attempt to compute the dynamics of the wa-
ter inside the glass by integrating the coupled equations
for the water molecules because it would mean taking
track of so many degrees of freedom that it would satu-
rate any computer memory, however large [1]. Instead,
one will rather use a coarse-grained set of equations, the
Navier-Stokes equations (NSE), derived as the macro-
scopic dynamic laws resulting from the mesoscopic Boltz-
mann equation [2]. The NSE describe accurately the dy-
namics of averaged quantities over spatial scales that are
much larger than the mean free path length of the wa-
ter molecules [3]. The number of Fourier modes needed
to simulate the NSE on a regular grid can roughly be
estimated as $(L/\eta)^3$, where $L$ is the largest scale of the
system, and $\eta$ the Kolmogorov scale, at which the input
energy is dissipated by viscosity [4]. In our glass of water
slowly stirred by a spoon, where laminar flow is realized,
the effective number of degrees of freedom then reduces to
just a few. By contrast, when we consider large-scale tur-
bulent motions, such as in most geophysical flows, even
the most powerful computer on Earth falls very short
from being able to describe accurately the behavior of
the NSE at all relevant scales of motions. Hence, there
is a need for deriving approximate equations able to de-
scribe accurately the dynamics on a reduced range of
spatial and temporal scales [5] and to develop accurate
yet efficient parametrizations for describing the impact

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of the unresolved scales of motions on those of interest [6, 7].

One way to approach the description of fluid flows containing a large number of scales is via a global statistical analysis. This was the main aim of the multifractal formalism developed by Parisi and Frisch [8]. It is basically framed in the large deviation theory [9, 10], contrasting the usual idea, coming from critical phenomena, that only a countable set of scaling exponents are relevant for a complete characterization of the statistical features of fluid flows. Thus, an infinite set of exponents, each belonging to a given fractal set, is required [11]. The probability of observing a singularity of scaling exponent $h$ depends on a function $C(h)$ which can be interpreted as the co-dimension of the fractal set [12]. The multifractal spectrum is then derived as $D(h) = D - C(h)$, with $D$ being the space dimension [13]. The multifractal theory assumes that the velocity has a local scale-invariance, i.e., $\partial u/\partial x \sim \xi^h$, to assess the statistical properties of fluids in a robust way with respect to noise and limited statistics [14]. Since its development experimental measurements of the velocity field in fluids have proved to be compatible with this multifractal picture, with the most probable scaling exponent $h \sim 1/3$, as predicted by the Kolmogorov theory [4, 15–19]. However, the multifractal theory only provides global information on the scale-dependent properties of fluids via the probability of occurrence of a given scaling exponent. Moreover, a direct computation of the multifractal spectrum from the NSE is not possible [20], although it would be helpful to explore the local statistics of velocity field fluctuations [1].

Since the pioneering work of Ed Lorenz [21], a lot of attention has been paid to dissipative chaotic dynamical systems, with a close connection between turbulence and chaos as proposed by Ruelle and Takens [22]. Indeed, three-dimensional viscous fluids, as described via the NSE, conform to this class of systems, being characterized by strange attractors, i.e., phase-space states toward which the system evolves for a wide range of initial conditions resulting from a series of bifurcations [22]. In this situation, we can theoretically implement a suitable coarse-graining procedure by projecting the motion onto the basis vectors of the phase-space containing the system’s attractor, resulting in new equations, or computer-accessible numerical simulations.

So far, the search for an attractor underlying turbulent flows in general, and geophysical turbulence in particular has however proved only partially successful. Since the 1990s several studies have suggested that the observed dynamical processes can be associated with the existence of non-hyperbolic strange and possibly stochastic attractors having a dimensionality much lower than the number of degrees of freedom of the system [23, 24]. Non-hyperbolicity manifests itself with the fact that the attractor is heterogeneous in terms of its local properties of persistence and predictability [24, 25]. When considering numerical models, this has important implications also in terms of error dynamics and efficiency of data assimilation [26].

In this letter, we consider a simple system derived from a turbulent flow by global averaging that removes all space dimensionality, and yet inherits intrinsic properties of the full turbulent system, such as intermittency, bistability and the existence of a low-dimensional stochastic attractor [25]. Upon application of new multiscale analysis tools to this system, we demonstrate that its attractor is in fact both scale- and time-dependent, being sensitive to the emergence of an intrinsic timescale solely determined by nonlinear interactions. Because the attractor adapts its geometric and statistical properties dynamically in time with respect to the intrinsic timescale, we call such attractor a “chameleon” attractor.

Data Our data originate from a turbulent von Kármán flow, obtained by stirring rapidly water in a vertical cylinder of length $L = 180$ mm and radius $R = 100$ mm. As a result of the forcing, turbulent motions of all temporal and spatial scales develop, and exert onto the two stirring counter-rotating impellers a back-reaction that can be measured through two torque-meters located along their common axis with resulting torques $C_1(t)$ and $C_2(t)$. Hence, these can be seen as large scale quantities that reflect the complex behaviour of the fluid contained in the vessel. Similarly, the instantaneous rotation frequencies $f_1(t)$ and $f_2(t)$ as a function of $C_1(t)$ and $C_2(t)$ therefore provides a global measure of the large scale circulation that develops under the action of the flux of angular momentum [27–29]. The study of the temporal properties of $f_1(t)$ and $f_2(t)$ as a function of $C_1(t)$ and $C_2(t)$ therefore provides a 1D (time-only) projection of the full 4D (space-time) dynamics of the turbulent flow. Despite this simplification, the resulting system still inherits complex and stochastic properties, and, for special forcing conditions, even develops large-scale circulation cells living on a stochastic attractor [25]. Such a situation is observed under conditions where the two torques $C_1$ and $C_2$ applied by the flow onto both impellers are constant, with a typical mean torque $C = ((C_1 + C_2)/2) = 1.68$ Nm. As a result, the two frequencies $f_1(t)$ and $f_2(t)$ fluctuate in time, with a typical mean frequency $f = (f_1 + f_2)/2$ between 4 and 7.5 Hz. The corresponding turbulent flow is then characterized by a Reynolds number $Re = 2\pi R^2 f/\nu \sim 3 \times 10^5$, where $\nu$ is the water viscosity. This value significantly exceeds the estimated critical Reynolds number for turbulence onset, $Re_T \approx 3500$.

The time fluctuations of $f_1(t)$ and $f_2(t)$ are not random, but follow an organized pattern that can be detected by using $\gamma = ((C_1(t) - C_2(t))/(C_1(t) + C_2(t)))$ as a control parameter, and $\Theta(t) = (f_1(t) - f_2(t))/(f_1(t) + f_2(t))$ as the order parameter: when $\gamma = 0$ then the top and the bottom impeller are exchangeable, and the turbulent state statistically follows this symmetry. As a result, the top and bottom rotation frequencies are statistically equal such that the variable $\Theta(t) = (f_1(t) - f_2(t))/(f_1(t) + f_2(t))$ fluctuates around zero and characterizes the symmetries of the turbu-
and a monotonic trend called Multivariate Intrinsic Mode Functions (MIMFs), peculiar dynamical feature that evolves on a typical mean.

Each multivariate signal $\Theta(t)$ moves from a univariate time series $\Theta(t)$ reported by red, green, blue, cyan, magenta, and black lines, respectively. The dashed gray lines refer to the value $\Theta(t) = 0$.

The dimension saturates at a value $d = 3$ [31]. Thus, the multivariate signal $\Theta(t)$ is interpreted as a superposition of scale-dependent fluctuations that can be individually investigated for their respective contribution to the collective properties of the whole measurements. A more detailed description is provided in the supplementary material.

Dynamic system metrics We next diagnose the dynamical properties of the instantaneous (in time) and local (in phase-space) states by means of two dynamical systems metrics, the instantaneous local dimension ($0 \leq d < \infty$) and the inverse persistence ($0 \leq \theta \leq 1$). The former is a measure of the active number of degrees of freedom [23], while the latter is a measure of the dynamical stability across the 3-D embedded phase-space [32]. Those instantaneous metrics are obtained by sampling the recurrences of a state $\zeta$ and observing that they are distributed according to extreme value theory [23, 33, 34]. Following [35], we can describe the dynamics at scales $\tau' < \tau$ as partial sums of MIMFs

$$\Theta(t) = \sum_{k|\tau_k<\tau} C_{\mu,k}(t) \quad (2)$$

such that we can define a scale-dependent local dimension $d_\tau$ and inverse persistence $\theta_\tau$ by diagnosing the dynamical properties of $\Theta(t)$. To do this, we compute both $d_\tau$ and $\theta_\tau$ for reconstructions based on the first $k$ MIMFs as in Eq. (1) until $k \to N$ for which $(d_\tau, \theta_\tau) \to (d, \theta)$. Further details can be found in the supplementary material.

Instantaneous scale-dependent dynamical features

Figure 2 shows the results of applying the instantaneous dynamical system metrics to (a) an approximately symmetric case with $\gamma = -0.0081$ and (b) a case with full symmetry breaking at $\gamma = -0.0682$. Other intermediate cases are presented in the supplementary material.

In the symmetric case, one observes that the local indicators are completely scale-invariant, suggesting that the properties of the system do not depend on the scale. The dimension saturates at a value $d = 3$, and the inverse persistence is maximal ($\theta = 1$) as expected for an unstructured stochastic system. By contrast, in the non-symmetric case, this scale-invariance is broken. As a result, one observes sudden bursts of local dimensions towards up to $d \approx 6$, localized temporally at frequencies above $f = 6$ Hz, the mean frequency of the system. The inverse persistence displays a markedly different behaviour between scales below and above the mean frequency, with a value close to one for the latter and close to 0 for the former. Since the 3D attractor that we are able to define using $\theta(t)$ is just a projection of a higher-dimensional attractor where the other degrees of freedom are lump in stochastic terms (i.e., at small scales), it is not surprising that we find dimensions larger than 3. As shown in [25], $d > 3$ points towards the existence of an unstable fixed point associated with abrupt changes and hints at the existence of an underlying stochastic attractor. Our scale-dependent results also suggest that, although the flow dynamics involves a wide range of scales, some of them can be described by stochastic theory [36].
FIG. 2. Temporal behavior of $\Theta(t)$ with $\gamma$ (upper panel) together with the behavior of the instantaneous dimension $d$ (middle panel) and instantaneous stability $\theta$ (lower panel) in the time-frequency domain. Left: $\gamma = -0.0081$, right: $\gamma = -0.0682$.

FIG. 3. Average dimension $\langle d_\tau \rangle$ (red circles) and stability indicator inverse persistence $\langle \theta_\tau \rangle$ (blue stars) as a function of the mean frequency $f = 1/\tau$. Error bars indicate the standard deviations of the instantaneous values of $d_\tau$ and $\theta_\tau$, respectively. The horizontal red line refers to the values $\langle d_\tau \rangle = 3$, while the two vertical black dashed lines correspond to $f_{s1} \sim 4$ Hz and $f_{s2} \sim 7$ Hz, respectively (see [25]). Left: $\gamma = -0.0081$, right: $\gamma = -0.0682$.

Chameleon attractor  The dynamical system metrics can finally be used to uncover the scale-specific properties of the topology of the chameleon attractor underlying the dynamics. To start with, we first reconstruct the attractor at a given scale using a Poincaré section obtained by plotting $\Theta^f(t)$ as a function of $\Theta^{l_1}(t)$ and $\Theta^{l_2}(t)$, where $\Theta^{l}(t)$ is the reconstruction of modes with mean frequencies $f_k > f$, with $f_k = 1/\tau_k$ as in Eq. (2). Figure 4 shows the attractor for the two limiting cases discussed previously, while the other cases are again reported in the supplementary material. We use a color scheme based on either the local dimension (upper two rows) or the instantaneous persistence (lower two rows).

In the symmetric case, $\gamma = -0.0081$, one observes that the local dimension and the inverse persistence are homogeneously distributed across the attractor, implying that its topology is very simple and can be associated with a noisy fixed point. By contrast, the non-symmetric case attractor displays scale-dependent features with a heterogeneous spatial distribution of the two metrics.

Discussion  Figure 2 (right) summarizes the strikingly different behavior of the dynamics above and below the timescale corresponding to $f = 6$ Hz, resulting in a change of the topological properties of the underlying attractor. This timescale corresponds to the natural characteristic forcing scale, thus suggesting that we observe a time behavior mirroring the well-known scaling behavior of a 3-D turbulent flow. At scales larger than the injection scale, the energy transfer is small, and the individual scales are in quasi-equilibrium; for scales smaller than the forcing scale, the mean energy transfer is positive, and there is an out-of-equilibrium energy cascade towards smaller scales, following a Kolmogorov spectrum with intermittency corrections.
In the present case, the low frequencies are associated to dynamics in lower dimensions, showing that the statistical equilibrium at large scales is driven by a few degrees of freedom, generating a well defined low-dimensional attractor. On the other hand, the dynamics at scales smaller than the forcing effectively plays the role of noise, which restores the broken symmetry and provides the "statistical temperature" for large scales, or the stochasticity of the attractor [27].

More generally, our results demonstrate that we cannot appropriately describe such attractors with averaged properties, and that we need refined analysis tools to detect their heterogeneity and the state-dependent properties of the system. Hence, it is apparent that the analysis of multiscale systems requires considering concepts allowing us to explore local and instantaneous properties of the system [25, 36]. Our analysis shows that the highly heterogeneous chameleon attractors discussed here could be common in high-dimensional dynamical systems as those encountered in climate sciences. We are confident that follow-up studies will further demonstrate their existence in such systems by exploiting the framework applied in the present work.

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