On the Mean Residence Time in Stochastic Lattice-Gas Models

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Abstract A heuristic law widely used in fluid dynamics for steady flows states that the amount of a fluid in a control volume is the product of the fluid influx and the mean time that the particles of the fluid spend in the volume, or mean residence time. We rigorously prove that if the mean residence time is introduced in terms of sample-path averages, then stochastic lattice-gas models with general injection, diffusion, and extraction dynamics verify this law. Only mild assumptions are needed in order to make the particles distinguishable so that their residence time can be unambiguously defined. We use our general result to obtain explicit expressions of the mean residence time for the Ising model on a ring with Glauber + Kawasaki dynamics and for the totally asymmetric simple exclusion process with open boundaries.

Keywords Residence time · Interacting particle systems · Sample-path averages · Strong law of large numbers

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1 Introduction

Residence time is the amount of time that the particles of a fluid spend in a control volume. Residence time is a ubiquitous concept involved for instance in the water cycle in hydrology [1], in the water and wastewater treatment in environmental engineering [2], in continuous flow reactions in chemistry [3], and in drug kinetics in pharmaceutics [4]. Beyond fluid dynamics and flow chemistry, the concept of residence time can be applied to the flow of generic resources from biology up to economic and social sciences. Recently, we have used the mean residence time of proteins on lipid membranes in eukaryotic cells to evaluate the efficiency of the molecular sorting process [5], whereby specific proteins and lipids are concentrated and distilled into lipid vesicles.

The mean residence time $\tau$ of a fluid in a fixed control volume is commonly determined for steady flows through the law $\rho = \phi \tau$ [1,2,3], where $\rho$ is the total amount of fluid in the volume and $\phi$ is the fluid influx. This law has been justified on the basis of heuristic arguments, but it has never been formally expressed and rigorously proven in a microscopic framework accounting for single fluid particles. A companion principle was proposed in queuing theory but, unlike the case of fluid dynamics, it was formulated and demonstrated in a rigorous setting based on sample-path averages of stochastic queuing processes. This principle, which is widely known as Little’s law, states that $l = \lambda w$ [6], where $l$ is the mean number of units in the system, $\lambda$ is their arrival rate, and $w$ is the mean time spent by a unit in the system. In this work, we resort to a similar sample-path formulation to show that the law $\rho = \phi \tau$ for fluids is rigorously verified in the microscopic framework of stochastic lattice-gas models with rather general mechanisms of injection, diffusion, and extraction of particles. Precisely, $\rho$ and $\phi$ are here the mean number of particles in the system and the influx of particles in the stationary state. Mild conditions making particles distinguishable and trackable must be imposed in order to unambiguously define their residence time.

Stochastic lattice-gas models are continuous-time Markov processes describing systems of particles moving in a lattice and interacting with each other. Since Spitzer’s pioneering studies in the late 1960’s on spatially distributed stochastic systems [7], stochastic lattice-gas models have become a main subject of research both in physics, for the deep insight they provide on non-equilibrium statistical mechanics [8,9], and in mathematics, for the new problems they pose in probability theory [10]. Nevertheless, although issues of existence and uniqueness have long been settled [10], proving anything nontrivial about the properties of such models is surprisingly difficult. With a few exceptions [11,12], explicit calculations are not feasible and one has to be satisfied with Monte Carlo simulations, qualitative statements based on mean-field theories, and some explicit bounds. Complicating the situation is the fact that the variety of non-equilibrium phenomena one can conceive, combined with the major role played by the details of the microscopic dynamics, makes it arduous to define general classes of systems for which a unified analysis is possible [9]. For comparison, the macroscopic behavior of systems at thermodynamic equilibrium is to a considerable extent independent of the microscopic details, so that different systems exhibit qualitatively the same phenomenology at large scales. In this scenario, our proof of the universal law $\rho = \phi \tau$ is a breakthrough, in that it provides an exact connection between distinct dynamical observables in stochastic lattice-gas models. It is worth observing
here that knowledge of exact relations is precious in checking the validity of general polynomial-time approximation schemes, such as mean-field theories where correlations are neglected.

In order to demonstrate the practical usefulness of the law $\rho = \phi \tau$, we compute the mean residence time for two well-known stochastic lattice-gas models: the Ising model on a ring with Glauber + Kawasaki dynamics and the totally asymmetric simple exclusion process with open boundaries. The Ising model is proposed as an example of a system that is time-reversible at equilibrium, whereas the totally asymmetric simple exclusion process violates time-reversal symmetry. The mean residence time in stochastic lattice-gas models has been the subject of two recent works, which however ignore, and therefore do not take advantage of, the exact law $\rho = \phi \tau$. The first work [13] deals with the mean residence time of particles undergoing an asymmetric simple exclusion dynamics on a two-dimensional vertical strip whose top and bottom sides are in contact with infinite particle reservoirs. In that work, the mean residence time is approximated numerically and analytically by means of a mean-field theory and of an analogy with a first-passage-time problem for a birth-and-death process. The second work [14] focuses on the totally asymmetric simple exclusion process with open boundaries and some of its variants. In that work, the on-site mean residence time, defined as the mean time a particle spends on a given site before moving on to the next site, is approximated analytically using a mean-field theory and domain-wall theory at the coexistence of the low-density phase with the high-density phase. In the case of the standard totally asymmetric simple exclusion process, Ref. [14] provides an approximate analytical expression for the mean residence time, which can be easily computed as the sum of on-site mean residence times over the entire lattice. Comparison with Monte Carlo simulations suggests that this approximate expression is exact in the large system-size limit [14]. Here we show how the law $\rho = \phi \tau$ applied to the totally asymmetric simple exclusion process allows to compute the mean residence time exactly for any system size, obtaining a result that is perfectly consistent with the findings of Ref. [14] when the system size is sent to infinity.

The paper is organized as follows. In Sect. 1.1 we introduce the class of stochastic lattice-gas models on which the work is focused. Sect. 1.2 is devoted to define the mean residence time for such stochastic lattice-gas models in terms of sample-path averages and to state the law $\rho = \phi \tau$ as a limit theorem for these sample-path averages. In Sect. 1.3 we apply this law to the Ising model on a ring with Glauber + Kawasaki dynamics and to the totally asymmetric simple exclusion process with open boundaries. Finally, Sect. 2 addresses the proof of the law.

1.1 The Stochastic Lattice-Gas Model

Let $\Lambda$ be a finite set and let $S$ be the collection of all functions $\eta : \Lambda \rightarrow \{0, 1\}$. We will refer to the set $\Lambda$ as the lattice, to $x \in \Lambda$ as a site of the lattice, and to $\eta \in S$ as a microscopic configuration of the lattice. Given a site $x \in \Lambda$ and a microscopic configuration $\eta \in S$, the binary number $\eta(x)$ will be interpreted as the number of particles of a fluid at $x$ once an exclusion principle is imposed. The stochastic lattice-gas model we consider is a homogeneous continuous-time Markov chain $(\eta_t)_{t \geq 0}$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with state space $S$ and right-continuous sample paths. The microscopic dynamics is specified by an infinitesimal
generator $\mathcal{L}$ (see [15], page 94) that acts on any observable $f : S \to \mathbb{R}$ providing its evolution in time in the sense that

$$\frac{d}{dt} \mathbb{E}[f(\eta_t)] = \mathbb{E}[\mathcal{L}f(\eta_t)],$$

where $\mathbb{E}$ denotes expectation with respect to the probability measure $\mathbb{P}$. We write the infinitesimal generator as the superposition $\mathcal{L} := \mathcal{L}_I + \mathcal{L}_D + \mathcal{L}_E$ of a generator $\mathcal{L}_I$ accounting for injection of particles in the lattice, a generator $\mathcal{L}_D$ describing diffusion on the system, and a generator $\mathcal{L}_E$ governing extraction of particles from the system. Our prescriptions for the generators $\mathcal{L}_I$, $\mathcal{L}_D$, and $\mathcal{L}_E$ are provided below. Hereafter, given a state $\eta \in S$ and a subset $v \subseteq \Lambda$, we denote by $\eta^v$ the microscopic configuration defined by $\eta^v(x) := 1 - \eta(x)$ if $x \in v$ and $\eta^v(x) := \eta(x)$ otherwise.

**Assumption 1** When the system is in the state $\eta \in S$, then

(I) a particle can be injected at a site $x \in \Lambda$ with injection rate $i_x(\eta) \geq 0$ if $x$ is not occupied. Thus, $i_x(\eta) = 0$ if $\eta(x) = 1$. The action of the generator $\mathcal{L}_I$ on the observable $f$ reads

$$\mathcal{L}_I f(\eta) := \sum_{x \in \Lambda} i_x(\eta)[f(\eta^{(x)}) - f(\eta)];$$

(D) a particle occupying a site $x$ can diffuse on the system through a jump on an empty site $y$ with diffusion rate $d_{x,y}(\eta) \geq 0$. Thus, $d_{x,y}(\eta) = 0$ if either $\eta(x) = 0$ or $\eta(y) = 1$. The action of $\mathcal{L}_D$ on $f$ is

$$\mathcal{L}_D f(\eta) := \sum_{(x,y) \in \Lambda^2} d_{x,y}(\eta)[f(\eta^{(x,y)}) - f(\eta)];$$

(E) all particles in an arbitrary subset $v \subseteq \Lambda$ are simultaneously removed from the system with extraction rate $e_v(\eta) \geq 0$ if $v$ is completely filled. Thus, $e_v(\eta) = 0$ if there exists $x \in v$ such that $\eta(x) = 0$. The action of $\mathcal{L}_E$ on $f$ reads

$$\mathcal{L}_E f(\eta) := \sum_{v \subseteq \Lambda} e_v(\eta)[f(\eta^v) - f(\eta)].$$

Some remarks are in order. The injection mechanism described by $\mathcal{L}_I$ entails that particles are injected in the system one at a time. This hypothesis is not necessary to distinguish particles but largely simplifies the presentation, covering at the same time most of the interesting physical systems and basically all models found in the literature. The diffusion mechanism identified by $\mathcal{L}_D$ accounts for only one particle jump at a time. This hypothesis is necessary to distinguish and to track particles, which are unavoidable operations when one needs to link the particles that leave the system with the particles that have previously entered in order to define residence times. Finally, it is worth observing here that the generator $\mathcal{L}_E$ allows the extraction of any possible subset of the lattice, thus providing in principle the most general extraction mechanism.

The process $\{\eta_t\}_{t \geq 0}$ can be conveniently represented in terms of the associated jump chain and holding times. It will be important from now on not to confuse jumps of the system between microscopic configurations with jumps of the particles on the lattice. Let $J_0 := 0$ and $J_i := \inf\{t > J_{i-1} : \eta_t \neq \eta_{J_{i-1}}\}$ for each $i \geq 1$.
be the jump times at which the system moves to a new state. We point out that
\( \lim_{t \to \infty} J_t = \infty \) \( \mathbb{P} \)-a.s.\(^1\) because the state space \( S \) is finite (see [15], page 90). Set \( \zeta_i := \eta_{J_i} \) for all \( i \geq 0 \), so that \( \eta_t = \zeta_i \) whenever \( t \) satisfies \( J_i \leq t < J_{i+1} \). The sequence \( \{\zeta_i\}_{i \geq 0} \) collecting the states that the system progressively visits is called the jump chain and results in a homogeneous discrete-time Markov chain (see [15], page 88). Denote by \( q(\eta) \) the real number defined for each \( \eta \in S \) by

\[
q(\eta) := \sum_{x \in A} i_x(\eta) + \sum_{(x,y) \in A^2} d_{x,y}(\eta) + \sum_{v \subseteq A} e_v(\eta).
\]

Transition probabilities of the jump chain are given for each \( \eta, \eta' \) in \( S \) by the formula \( \mathbb{P}[\zeta_{i+1} = \eta' | \zeta_i = \eta] = 1(\eta' = \eta) \) or the formula

\[
\mathbb{P}[\zeta_{i+1} = \eta' | \zeta_i = \eta] = \sum_{x \in A} \frac{i_x(\eta)}{q(\eta)} 1(\eta' = \eta^{(x)}) + \sum_{(x,y) \in A^2} \frac{d_{x,y}(\eta)}{q(\eta)} 1(\eta' = \eta^{(x,y)}) + \sum_{v \subseteq A} \frac{e_v(\eta)}{q(\eta)} 1(\eta' = \eta^v)
\]

depending on whether \( q(\eta) = 0 \) or \( q(\eta) > 0 \) (see [15], page 87). The time \( H_i \) that the process \( \{\eta_i\}_{i \geq 0} \) spends in the state \( \zeta_i \) is \( H_i := J_{i+1} - J_i \) and is called holding time. For each \( i \geq 0 \), conditional on \( \zeta_0, \ldots, \zeta_i \), the holding times \( H_0, \ldots, H_i \) are independent exponential random variables of parameters \( q(\zeta_0), \ldots, q(\zeta_i) \) respectively (see [15], page 88).

The process \( \{\eta_i\}_{i \geq 0} \) is said to be irreducible if for each microscopic configurations \( \eta \) and \( \eta' \) there exists an integer \( i \geq 0 \) such that \( \mathbb{P}[\zeta_i = \eta' | \zeta_0 = \eta] > 0 \). Most of the physical phenomena that can be described in terms of stochastic lattice-gas models originate processes that do not become trapped in proper subsets of the state space, thus resulting irreducible [9]. Irreducibility is assumed here.

**Assumption 2** The process \( \{\eta_i\}_{i \geq 0} \) is irreducible.

Irreducibility combined with the fact that the state space \( S \) is finite due to the finiteness of \( A \) has a number of consequences. First of all, no state is absorbing, meaning that \( q(\eta) > 0 \) for all \( \eta \in S \). This gives in particular that the jump chain \( \{\zeta_i\}_{i \geq 0} \) cannot stay at rest, satisfying for each \( i \geq 1 \) one of the following alternatives \( \mathbb{P} \)-a.s.:

(I) there exists a site \( x \in A \) such that \( \zeta_{i-1}(x) = 0 \) and \( \zeta_i = \zeta_i^{(x)} \);

(D) there exist \( x \) and \( y \) in \( A \) such that \( \zeta_{i-1}(x) = 1, \zeta_{i-1}(y) = 0 \), and \( \zeta_i = \zeta_i^{(x,y)} \);

(E) there exists a cluster \( v \subseteq A \) such that \( \zeta_{i-1}(x) = 1 \) for all \( x \in v \) and \( \zeta_i = \zeta_i^v \).

Secondly, the jump chain is recurrent (see [15], page 27), so that for every \( \eta \in S \) there exist \( \mathbb{P} \)-a.s. infinitely many \( i \) with the property that \( \zeta_i = \eta \). Third, there exists a unique invariant distribution \( \pi \) (see [15], page 118). We recall that a distribution \( \pi \) on \( S \) is invariant if \( \sum_{\eta \in S} \mathcal{L}f(\eta) \pi(\eta) = 0 \) for all observables \( f : S \to \mathbb{R} \). Last, strong laws of large numbers hold for functionals of the process \( \{\eta_i\}_{i \geq 0} \) (see [15], page 126) and the jump chain \( \{\zeta_i\}_{i \geq 0} \) (see [16], page 267).

\(^1\) As usual, we say that a property holds \( \mathbb{P} \)-almost surely (\( \mathbb{P} \)-a.s. for short) if it holds for all \( \omega \in \Omega_0 \in \mathcal{F} \) with \( \mathbb{P}(\Omega_0) = 1 \).
1.2 The Mean Residence Time Law

Let \( \theta_i(x, y) \) be the binary random variable defined for each integer \( i \geq 1 \) and sites \( x \) and \( y \) in \( A \) by

\[
\theta_i(x, y) := \begin{cases} 
\zeta_i(x) & \text{if } x = y; \\
\zeta_i(x)[1 - \zeta_i(y)][1 - \zeta_i(x)] & \text{if } x \neq y,
\end{cases}
\]

where \( \{\zeta_i\}_{i=0}^{\infty} \) is the jump chain. Considering separately the alternatives (I), (D), and (E) for the \( i \)th configuration jump it is not difficult to verify that \( \theta_i(x, x) = 1 \) if and only if \( x \) host a particle that stays at rest during this change of configuration. Similarly, \( \theta_i(x, y) = 1 \) with \( x \neq y \) if and only if there is a particle at \( x \) that moves to \( y \) in the \( i \)th configuration jump. Tracking particles on the lattice is now possible. Given a couple of integers \( \theta \) and \( \eta \), we have that \( x_{i-1} \) hosts a particle that moves progressively to \( x_k \) in the \( k \)th configuration jump with \( k \) running from \( i \) to \( j \) and only if \( \prod_{k=i}^{j} \theta_k(x_{k-1}, x_k) = 1 \). It follows in particular that a particle located at \( x_{i-1} \) before the \( i \)th configuration jump is still in the system after the \( j \)th change of configuration if and only if there exist \( x_i, \ldots, x_j \) in \( A \) such that \( \prod_{k=i}^{j} \theta_k(x_{k-1}, x_k) = 1 \).

This condition is tantamount to \( \Theta_{i,j}(x_{i-1}) = 1 \), where \( \Theta_{i,j}(x_{i-1}) \) is the binary random variable defined for each \( j \geq i \geq 1 \) and \( x_{i-1} \in A \) by

\[
\Theta_{i,j}(x_{i-1}) := \sum_{x_j \in A} \cdots \sum_{x_i \in A} \prod_{k=i}^{j} \theta_k(x_{k-1}, x_k).
\]

Let \( |\eta| := \sum_{x \in A} \eta(x) \) denote the number of particles in the system under the state \( \eta \in S \). Given an integer \( i \geq 1 \), a particle enters the system in the \( i \)th configuration jump if and only if the condition \( |\zeta_i| > |\zeta_{i-1}| \) that excludes alternatives (D) and (E) is fulfilled. If \( |\zeta_i| > |\zeta_{i-1}| \), then the particle is injected at that unique site \( x \) such that \( \zeta_{i-1}(x) = 0 \) and \( \zeta_i(x) = 1 \). This way, for each \( j \geq i \) we can state that a particle enters the system in the \( i \)th configuration jump and is still in the system after the \( j \)th change of configuration if and only if \( U_{i,j} = 1 \), where \( U_{i,j} \) is the binary random variable defined by

\[
U_{i,j} := \begin{cases} 
1(|\zeta_i| > |\zeta_{i-1}|) & \text{if } j = i; \\
1(|\zeta_i| > |\zeta_{i-1}|) \sum_{x \in A}[1 - \zeta_i(x)]\Theta_{i+1,j}(x) & \text{if } j > i.
\end{cases}
\]

For any \( j \geq i \geq 1 \), we have that \( U_{i,j-1} \geq U_{i,j} \) and that a particle enters the system in the \( i \)th configuration jump and leaves it exactly in the \( j \)th change of configuration if and only if \( U_{i,j-1} - U_{i,j} = 1 \). The random variables (3) and (4) satisfy \( \lim_{j \to \infty} \Theta_{i,j}(x) = 0 \) \( \mathbb{P} \)-a.s. and \( \lim_{j \to \infty} U_{i,j} = 0 \) \( \mathbb{P} \)-a.s. for any \( i \geq 1 \) and \( x \in A \), meaning that every particle eventually leaves the system. The simplest way to prove these limits is to observe that recurrence of the jump chain implies that there exist \( \mathbb{P} \)-a.s. infinitely many \( k \) such that \( \zeta_k(x) = 0 \) for all \( x \). For such \( k \) it holds that \( \theta_k(x, y) = 0 \) for all \( x \) and \( y \) in \( A \).

We are now able to define the mean residence time in terms of sample-path averages. A particle that enters the system in the \( i \)th configuration jump and leaves it exactly in the \( j \)th change of configuration spends in the lattice the time \( J_j - J_i \). Thus, denoting by \( N_t := \sup\{i \geq 0 : J_i \leq t\} \) the number of configuration
on the mean residence time in stochastic lattice-gas models

jumps up to a certain time \( t \geq 0 \), we introduce the mean residence time \( T_t \) of the particles that have been injected by the time \( t \) as

\[
T_t := \frac{\sum_{i=1}^{N_t} \sum_{j=i+1}^{\infty} (J_{i,j} - J_{j,i})(U_{i,j-1} - U_{i,j})}{\sum_{i=1}^{N_t} \sum_{j=i+1}^{\infty} (U_{i,j-1} - U_{i,j})}. \tag{5}
\]

Hereafter we assume that a sum with upper limit smaller than the lower one is equal to zero and that \( 0/0 := 0 \). The following theorem stating the law \( \rho = \phi \tau \) for stochastic lattice-gas models is our main result.

**Theorem** Let \( \pi \) be the invariant distribution of \( L \) and set \( \rho := \sum_{\eta \in S} |\eta| \pi(\eta) \) and \( \phi := \sum_{\eta \in S} \sum_{x \in A} t_x(\eta) \pi(\eta) \). Then, the limit \( \lim_{t \to \infty} T_t =: \tau \) exists \( \mathbb{P} \)-a.s. and satisfies \( \rho = \phi \tau \).

The real number \( \rho \) is the mean number of particles in the system with respect to \( \pi \). The real number \( \phi \) is the rate at which particles enter the system measured as follows. The number of particles that are injected in the lattice by the time \( t \geq 0 \) is \( \sum_{i=1}^{N_t} U_{i,j} = \sum_{i=1}^{N_t} \mathbb{1}(|\zeta_i | > \zeta_{i-1}) \). Thus, appealing to the strong law of large numbers for functionals of the jump chain (see [16], page 267) first and to the explicit expression (2) of its transition probabilities later we get

\[
\lim_{t \to \infty} \frac{1}{t} \sum_{i=1}^{N_t} U_{i,i} = \sum_{\eta \in S} \sum_{\eta' \in S} \mathbb{1}(|\eta'| > |\eta|) \mathbb{P}[\zeta_1 = \eta' | \zeta_0 = \eta] q(\eta) \pi(\eta) = \phi \quad \mathbb{P}\text{-a.s.} \tag{6}
\]

1.3 Applications

In this section we make use of the law \( \rho = \phi \tau \) to compute explicitly the mean residence time for two well-known stochastic lattice-gas models. The first model is the Ising model on a ring with Glauber + Kawasaki dynamics, which is proposed as an example of a system that is time-reversible at equilibrium. The second model is the totally asymmetric simple exclusion process with open boundaries, which violates time-reversal symmetry. We recall that the process \( \{\eta_t\}_{t \geq 0} \) is said to be time-reversible if \( \{\eta_t\}_{t \leq T} \) and \( \{\eta_{T-t}\}_{t \leq T} \) share the same finite-dimensional marginal distributions for any number \( T > 0 \). The irreducible homogeneous continuous-time Markov chain \( \{\eta_t\}_{t \geq 0} \) with invariant distribution \( \pi \) is time-reversible if and only if \( \mathbb{P}[\eta_0 = \eta] = \pi(\eta) \) for all \( \eta \in S \), so that \( \{\eta_t\}_{t \geq 0} \) is stationary, and the infinitesimal generator \( L \) satisfies detailed balance with respect to \( \pi \) (see [15], page 125). The generator \( L \) is said to satisfy detailed balance with respect to a probability distribution \( \lambda \) on \( S \) if \( \sum_{\eta \in S} (f \mathcal{L}g - g \mathcal{L}f)(\eta) \lambda(\eta) = 0 \) for every two observables \( f : S \to \mathbb{R} \) and \( g : S \to \mathbb{R} \). The distribution \( \lambda \) is invariant if \( L \) satisfies detailed balance with respect to \( \lambda \) (see [15], page 125).

Conditions on the rates providing reversibility can be easily obtained as follows. Assume that \( L \) satisfies detailed balance with respect to \( \pi \) and observe that irreducibility and finiteness of the state space entail \( \pi(\eta) > 0 \) for all \( \eta \in S \) (see [15], page 118). Given a site \( \bar{x} \) and a state \( \bar{\eta} \) such that \( \bar{\eta}(\bar{x}) = 1 \), the condition \( \sum_{\eta \in S} (f \mathcal{L}g - g \mathcal{L}f)(\eta) \lambda(\eta) = 0 \) results in \( \epsilon_{\{\bar{x}\}}(\bar{\eta}) \pi(\bar{\eta}) - t_{\bar{x}}(\bar{\eta}(\bar{x})) \pi(\bar{\eta}(\bar{x})) = 0 \) when
shows that 0 = v for all \( \eta \). The arbitrariness of \( \bar{x} \) and \( \bar{\eta} \), combined with the fact that \( e_{\{x\}}(\eta) = e_{\{x\}}(\bar{\eta}) = 0 \) by construction if \( \eta(\bar{x}) = 0 \), yields \( e_{\{x\}}(\eta)\pi(\eta) = e_{\{x\}}(\bar{\eta})\pi(\bar{\eta}) \) for every \( x \in A \) and \( \eta \in S \). Furthermore, if \( \bar{v} \) is a set of at least two sites and \( \bar{\eta} \) is a microscopic configuration such that \( \bar{\eta}(x) = 1 \) for each \( x \in \bar{v} \), then the choice \( f(\eta) := 1(\eta = \bar{\eta}) \) and \( g(\eta) := 1(\eta = \bar{\eta}) \) for any \( \eta \) shows that 0 = \( \sum_{\eta \in S} (f \mathcal{L} g - g \mathcal{L} f)(\eta)\pi(\eta) = e_0(\bar{\eta})\pi(\bar{\eta}) \). The arbitrariness of \( \bar{v} \) and \( \bar{\eta} \) and the fact that \( e_0(\eta) = 0 \) by construction if \( \eta(x) = 0 \) for some \( x \in \bar{v} \) imply that \( e_0(\bar{\eta}) = 0 \) for all \( v \subseteq A \) containing more than one site and all \( \eta \in S \). In conclusion, we find that for each \( v \subseteq A \) and \( \eta \in S \)

\[
e_v(\eta) = \begin{cases} 
    i_{\{x\}}(\eta(x))\pi(\eta(x)) & \text{if } v = \{x\} \text{ for some } x \in A; \\
    0 & \text{otherwise}.
\end{cases}
\]

(7)

Such extraction rates make the dynamics generated by \( \mathcal{L}_I + \mathcal{L}_E \) a Glauber dynamics [17], whereby only the update of one site at a time is involved and the detailed balance condition is fulfilled. Given now two distinct sites \( \bar{x} \) and \( \bar{y} \) and a microscopic configuration \( \bar{\eta} \) such that \( \bar{\eta}(\bar{x}) = 1 \) and \( \bar{\eta}(\bar{y}) = 0 \), the condition \( \sum_{\eta \in S} (f \mathcal{L} g - g \mathcal{L} f)(\eta)\pi(\eta) = 0 \) with \( f(\eta) := 1(\eta = \bar{\eta}) \) and \( g(\eta) := 1(\eta = \bar{\eta}(x,\bar{y})) \) for all \( \eta \) becomes \( d_{x,y}(\bar{\eta})\pi(\bar{\eta}) - d_{y,x}(\bar{\eta}(x,\bar{y}))\pi(\bar{\eta}(x,\bar{y})) = 0 \). The arbitrariness of \( \bar{x} \), \( \bar{y} \), and \( \bar{\eta} \), combined with the fact that \( d_{x,y}(\eta) = d_{y,x}(\eta(x,\bar{y})) = 0 \) by construction if either \( \eta(\bar{x}) = 0 \) or \( \eta(\bar{y}) = 1 \), leads to the relationship

\[
d_{x,y}(\eta)\pi(\eta) = d_{y,x}(\eta(x,\bar{y}))\pi(\eta(x,\bar{y}))
\]

(8)

to be satisfied for all \( x \) and \( y \) in \( A \) and \( \eta \in S \). The dynamics generated by \( \mathcal{L}_D \) is called a Kawasaki dynamics [17] if the set \( A \) is endowed with a graph structure and if the diffusion rates satisfy both (8) and the property that \( d_{x,y}(\eta) = 0 \) for all \( \eta \) whenever \( x \) and \( y \) are not nearest-neighbor sites.

Conditions (7) and (8) are necessary conditions for the generator \( \mathcal{L} \) to satisfy detailed balance with respect to the distribution \( \pi \). Simple algebra shows that they also are sufficient conditions to give \( \sum_{\eta \in S} (f \mathcal{L} g - g \mathcal{L} f)(\eta)\lambda(\eta) = 0 \) for all \( f : S \rightarrow \mathbb{R} \) and \( g : S \rightarrow \mathbb{R} \). Thus, \( \mathcal{L} \) satisfies detailed balance with respect to \( \pi \) if and only if (7) and (8) hold.

1.3.1 The Ising Model on a Ring with Glauber + Kawasaki Dynamics

Let \( A := \mathbb{Z}/L\mathbb{Z} \) be the one-dimensional discrete torus of size \( L \geq 2 \) and let the function \( \mathcal{H} : S \rightarrow \mathbb{R} \) be the Ising Hamiltonian defined for each \( \eta \in S \) by

\[
\mathcal{H}(\eta) := V \sum_{x \in A} \eta(x)\eta(x + 1) - \mu \sum_{x \in A} \eta(x),
\]

(9)

where \( V \in \mathbb{R} \) is the interaction parameter and \( \mu \in \mathbb{R} \) is the chemical potential. The Gibbs state associated to \( \mathcal{H} \) is the distribution \( \pi_G := (1/Z)\exp(-\mathcal{H}) \), \( Z \) being the partition function. In this section we consider a stochastic lattice-gas model whose generator \( \mathcal{L} \) satisfies detailed balance with respect to the Gibbs state \( \pi_G \), so that extraction and diffusion rates fulfill the conditions (7) and (8) respectively with \( \pi = \pi_G \). For simplicity, we focus here on the local and translationally invariant injection rates defined for all \( x \in A \) and \( \eta \in S \) by the formula

\[
i_{x}(\eta) := [1 - \eta(x)]\alpha_{\eta(x-1),\eta(x+1)},
\]
where the parameters \( \alpha_{0,0}, \alpha_{1,0}, \alpha_{0,1}, \text{ and } \alpha_{1,1} \) are assumed to be strictly positive. Non-vanishing extraction rates inherit the same local and translationally invariant structure, since combining (7) with (9) we get \( \epsilon_{\{x\}}(\eta) = \eta(x) \beta_{\eta(x-1), \eta(x+1)} \) for any \( x \) and \( \eta \) with the strictly positive parameters \( \beta_{0,0} := \alpha_{0,0} e^{-\mu}, \beta_{1,0} := \alpha_{1,0} e^{V-\mu}, \beta_{0,1} := \alpha_{0,1} e^{V-\mu}, \text{ and } \beta_{1,1} := \alpha_{1,1} e^{2V-\mu} \). Since \( \epsilon_{\{x\}}(\eta) > 0 \) if \( \eta(x) = 0 \) and \( \epsilon_{\{x\}}(\eta) > 0 \) if \( \eta(x) = 1 \) for every \( x \) and \( \eta \), the process \( \{\eta_t\}_{t \geq 0} \) turns out to be irreducible irrespective of the features of the generator \( \mathcal{L}_D \). Although the mean residence time does not depend on the details of diffusion rates as long as condition (8) holds, to fix the ideas we consider here the Kawasaki dynamics where particles can only jump to nearest-neighbor sites. We will refer to this model as the Ising model on a ring with Glauber + Kawasaki dynamics.

The mean residence time \( \tau \) can be computed explicitly as follows. The translational symmetry of the invariant distribution and of the injection rates yields \( \rho = L \sum_{\eta \in S} \eta(1) \pi(\eta) \) and \( \phi = L \sum_{\eta \in S} i_1(\eta) \pi(\eta) \). Consequently, we have

\[
\tau = \frac{\rho}{\phi} = \frac{\sum_{\eta \in S} \eta(1) \exp[-H(\eta)]}{\sum_{\eta \in S} i_1(\eta) \exp[-H(\eta)]}. \tag{10}
\]

The sums over \( \eta \) that appear in (10) can be carried out by means of the transfer matrix method. Let \( \mathcal{T} \in \mathbb{R}^{2 \times 2} \) be the symmetric matrix with entries \( T_{0,0} := 1, T_{1,0} = T_{0,1} := e^{\mu/2}, \text{ and } T_{1,1} := e^{\mu-V} \). The matrix \( \mathcal{T} \) allows us to recast the weight \( \exp[-H(\eta)] \) as \( \prod_{x \in A} T_{\eta(x), \eta(x+1)} \) for each \( \eta \in S \). This way, we get

\[
\sum_{\eta \in S} \eta(1) \exp[-H(\eta)] = \sum_{\eta \in S} \eta(1) \prod_{x \in A} T_{\eta(x), \eta(x+1)} = (T^L)_{1,1} \tag{11}
\]

and

\[
\sum_{\eta \in S} i_1(\eta) \exp[-H(\eta)] = \sum_{\eta \in S} [1 - \eta(1)] \alpha_{\eta(0), \eta(2)} \prod_{x \in A} T_{\eta(x), \eta(x+1)} = \alpha_{0,0} (T^{L-2})_{0,0} + \alpha_{1,0} e^{\mu/2} (T^{L-2})_{1,0} + \alpha_{0,1} e^{\mu/2} (T^{L-2})_{0,1} + \alpha_{1,1} e^{\mu} (T^{L-2})_{1,1}. \tag{12}
\]

We now use the fact that \( \mathcal{T} \) is symmetric to write down for any \( n \geq 0 \) the spectral decomposition \( \mathcal{T} = t_-^n \mathcal{P}_- + t_+^n \mathcal{P}_+ \), where \( t_- < t_+ \) are the eigenvalues of \( \mathcal{T} \) and \( \mathcal{P}_- \) and \( \mathcal{P}_+ \) are the orthogonal projections onto the corresponding eigenspaces. The eigenvalues and the projections are given by the formulas

\[
t_{\pm} = \frac{1 + e^{\mu-V} \pm \sqrt{[1 - e^{\mu-V}]^2 + 4e^{\mu}}}{2}
\]

and \( \mathcal{P}_\pm = (\mathcal{T} - t_\pm \mathcal{I})/(t_\pm - t_\pm) \), \( \mathcal{I} \) being the identity matrix. Thus, combining (10) with (11) and (12) first and making use of this spectral decomposition later we reach the result

\[
\tau = \frac{r_+ t_+^L + r_- t_-^L}{a_+ t_+^{L-2} + a_- t_-^{L-2}}, \tag{13}
\]

where

\[
r_{\pm} := (\mathcal{P}_\pm)_{1,1} = \frac{e^{\mu-V} - t_\pm}{t_\pm - t_\pm} = \frac{(t_\pm - 1)^2}{(t_\pm - 1)^2 + e^{\mu}}.
\]
and

\[ a_\pm := a_{0,0}(P_{\pm})_{0,0} + a_{1,0}\varepsilon^{\mu/2}(P_{\pm})_{1,0} + a_{0,1}\varepsilon^{\mu/2}(P_{\pm})_{0,1} + a_{1,1}\varepsilon^{\mu}(P_{\pm})_{1,1} \]

\[ = \frac{a_{0,0}(1 - t_{\mp}) + (a_{1,0} + a_{0,1})e^{\mu} + a_{1,1}e^{\mu - V - t_{\mp}}}{t_{\pm} - t_{\mp}} \]

\[ = \frac{a_{0,0} + (a_{1,0} + a_{0,1})(t_{\pm} - 1) + a_{1,1}(t_{\pm} - 1)^2}{1 + e^{-\alpha}(t_{\pm} - 1)^2} \]

The explicit expression of the mean residence time \( \tau \) for the Ising model on a ring with Glauber + Kawasaki dynamics is thus provided by (13). The time \( \tau \) is bounded with respect to the system size \( L \) because particles can leave the system at each site. A different situation is observed in the totally asymmetric simple exclusion process, where particles have to travel a macroscopic distance before being allowed to leave the system.

### 1.3.2 The Totally Asymmetric Simple Exclusion Process

Let \( A \) be the set \( \{1,\ldots,L\} \) for some integer \( L \geq 2 \). The totally asymmetric simple exclusion process with open boundaries is the irreducible stochastic lattice-gas model associated with the lattice \( A \) and the following rates, where \( \alpha > 0 \) and \( \beta > 0 \) are model parameters: \( i_{\pm}(\eta) := \alpha[1 - \eta(1)] \) and \( i_{x}(\eta) := 0 \) if \( x > 1 \) as far as injection rates are concerned, \( d_{x,x+1}(\eta) := \eta(x)[1 - \eta(x + 1)] \) if \( x < L \) and \( d_{x,y}(\eta) := 0 \) if \( y \neq x + 1 \) for diffusion rates, \( e_{(L)}(\eta) := \beta\eta(L) \) and \( e_{x}(\eta) := 0 \) if \( v \neq \{L\} \) for extraction rates. Thus, particles enter the lattice \( A \) at the left boundary with rate \( \alpha \), can move rightwards, and leave the system at the right boundary with rate \( \beta \). The invariant distribution \( \pi \) is known \([11]\) and an explicit expression for the probability with respect to \( \pi \) that a certain site is occupied can be obtained \([11]\). We need this expression in order to compute the mean residence time. Let \( Z_x \) be the real number defined for each integer \( x \geq 0 \) by

\[ Z_x := \begin{cases} 1 & \text{if } x = 0; \\ \sum_{k=1}^{x} B_{x,k} \sum_{l=0}^{k} \frac{\alpha}{\beta} & \text{if } x \geq 1, \end{cases} \]

where \( B_{x,k} \) is the combinatorial coefficient given for all \( x \geq 1 \) and \( k \geq 1 \) by the formula

\[ B_{x,k} := \frac{k(2x - k - 1)!}{x!(x-k)!}. \]

The probability \( \sum_{\eta \in S} \eta(x)\pi(\eta) \) with respect to \( \pi \) that a generic site \( x \in \Lambda \) is occupied is \([11]\)

\[ \sum_{\eta \in S} \eta(x)\pi(\eta) = \frac{1}{Z_L} \left( \sum_{k=1}^{L-x} \left[ Z_{L-k} B_{k,1} + \frac{Z_{L-k} B_{L-k-1}}{\beta} \right] \right) \quad \text{if } x < L; \]

\[ \frac{Z_{L-k} B_{k,1}}{\beta} \quad \text{if } x = L. \quad (14) \]

The mean residence time of the totally asymmetric simple exclusion process with open boundaries can be immediately determined by combining the \( \rho = \phi \tau \) law with (14). To get at a more compact expression, we notice that the influx \( \phi \) defined as \( \phi := \alpha \sum_{\eta \in \eta \in S} [1 - \eta(1)]\pi(\eta) \) equals \( \beta \sum_{\eta \in \eta \in S} \eta(L)\pi(\eta) \). Indeed, from
the definition of \( \pi \) we have that 0 = \( \sum_{\eta \in S} \mathcal{L} f(\eta) \pi(\eta) = \phi - \beta \sum_{\eta \in S} \eta(L) \pi(\eta) \) if \( f(\eta) := |\eta| \) for each \( \eta \in S \). This way, we can write

\[
\tau = \frac{\rho}{\phi} = \frac{\sum_{x=1}^{L} \sum_{\eta \in S} \eta(x) \pi(\eta)}{\beta \sum_{\eta \in S} \eta(L) \pi(\eta)} = \frac{1}{\beta} + \frac{1}{Z_{L-1}} \sum_{x=1}^{L-1} \left[ (L-x)Z_{L-x}B_{x,1} + \sum_{k=1}^{x} \frac{Z_{x-1}B_{L-x,k}}{\beta^{k+1}} \right].
\]

(15)

This formula provides the exact mean residence time \( \tau \) for any system size \( L \geq 2 \). Even though (15) is slightly cumbersome to deal with, asymptotic analysis shows that \( \tau \) is proportional to \( L \) in the large \( L \) limit with the simple coefficient of proportionality \( r \) given by

\[
\begin{aligned}
\rho :=
\begin{cases}
2 & \text{if } \alpha \geq 1/2 \text{ and } \beta \geq 1/2; \\
\frac{2\alpha(1-\alpha)}{\beta} & \text{if } \alpha = \beta < 1/2; \\
\frac{1}{\beta} & \text{if } \alpha < 1/2 \text{ and } \alpha < \beta; \\
\frac{1}{\alpha} & \text{if } \beta < 1/2 \text{ and } \beta < \alpha.
\end{cases}
\end{aligned}
\]

Indeed, the following proposition holds, confirming that \( \tau \) is proportional to the distance \( L \) that particles have to travel before leaving the system.

**Proposition** For each \( \alpha > 0 \) and \( \beta > 0 \) there exists a positive constant \( c < \infty \) independent of \( L \) such that

\[
\left| \frac{\tau}{L} - r \right| \leq \frac{c}{\sqrt{L}}.
\]

The proof of this proposition goes through the asymptotic analysis of the number \( Z_i \) in the large \( x \) limit, which can be performed by means of Laplace’s method for sums as in Ref. [11]. We omit the details because they are easily imaginable and not very informative. We point out that the coefficient \( r \) has been previously determined in Ref. [14], where the mean time that a particle spends on a given site before moving on to the next site has been investigated by means of mean-field theory. In particular, it has been shown there by comparison with Monte Carlo simulations that a mean-field theory neglecting time correlations in the local density of particles provides the exact value of \( r \) for all \( \alpha > 0 \) and \( \beta > 0 \), except for the case \( \alpha = \beta < 1/2 \) where it fails. The coefficient of \( r \) in the case \( \alpha = \beta < 1/2 \), corresponding to coexistence between a low-density phase and a high-density phase, has been found in Ref. [14] by combining mean-field estimations with domain-wall theory.

### 2 Proof of the Mean Residence Time Law

In this section we prove that \( \lim_{t \to \infty} T_i = \rho/\phi \) P.a.s., thus demonstrating the \( \rho = \phi \tau \) law. We first observe that the denominator of (5) divided by \( t \) tends to \( \phi \) in the large \( t \) limit since the number of particles that enter the system equals the number of particles that enter the system and eventually leave it. Formally, \( \lim_{t \to \infty} (1/t) \sum_{i=1}^{N_t} \sum_{j=i+1}^{\infty} (U_{i,j-1} - U_{i,j}) = \phi \) P.a.s. follows from (6) since \( \sum_{j=i+1}^{\infty} (U_{i,j-1} - U_{i,j}) = U_{i,i} \) P.a.s. for each \( i \geq 1 \) due to the fact that
\begin{align}
\lim_{t \to \infty} U_{i,j} &= 0 \quad \mathbb{P}\text{-a.s.}. \quad \text{This way, in order to prove that } \lim_{t \to \infty} T_t = \rho/\phi \quad \mathbb{P}\text{-a.s. it suffices to show that } \lim_{t \to \infty} (1/t) \sum_{i=1}^{N_i} \sum_{j=i+1}^{\infty} (J_j - J_i) (U_{i,j-1} - U_{i,j}) = \rho \quad \mathbb{P}\text{-a.s.}.
\end{align}

The latter limit is verified if

\begin{align}
\lim_{t \to \infty} \frac{1}{t} \sum_{i=1}^{N_i} \sum_{j=i}^{\infty} H_j U_{i,j} = \rho \quad \mathbb{P}\text{-a.s.}. \tag{16}
\end{align}

Indeed, we have \( \sum_{j=i+1}^{\infty} (J_j - J_i) (U_{i,j-1} - U_{i,j}) = \sum_{j=i}^{\infty} H_j U_{i,j} \quad \mathbb{P}\text{-a.s.} \) for every \( i \geq 1 \), thanks to the identity \( J_j - J_i = \sum_{k=i}^{j-1} H_k \) and the fact that \( U_{i,j} = 0 \) for all sufficiently large \( j \) \( \mathbb{P}\text{-a.s.} \), since on the one hand \( \lim_{t \to \infty} U_{i,j} = 0 \quad \mathbb{P}\text{-a.s.} \), and on the other hand \( U_{i,j} \) can take only two values. We shall therefore concentrate on proving (16) starting from the strong law of large numbers for functionals of the process \( \{ \eta_t \}_{t \geq 0} \) (see [15], page 126), which in particular gives

\begin{align}
\lim_{t \to \infty} \frac{1}{t} \int_0^t |\eta_t| \, d\tau = \sum_{\eta \in \mathcal{S}} |\eta| \pi(\eta) = \rho \quad \mathbb{P}\text{-a.s.}. \tag{17}
\end{align}

To begin with, we notice that the particles still in the system after the \( j \)th configuration jump are those that were present at the beginning or that have been injected up to the \( j \)th change of configuration and have not yet left the lattice. The following lemma concerning the number of particles holds.

**Lemma 1** \( |\zeta_j| = \sum_{x \in A} \Theta_{1,j}(x) + \sum_j U_{i,j} \) for each \( j \geq 1 \).

**Proof** For brevity, set \( \psi_i(x) := 1(|\zeta_i| > |\zeta_{i-1}|)[1 - \zeta_{i-1}(x)] \zeta_i(x) \) for each \( i \geq 1 \) and \( x \in A \). For every \( i \geq 1 \) and \( y \in A \) we have

\begin{align}
\zeta_i(y) &= \sum_{x \in A} \theta_i(x,y) + \psi_i(y). \tag{18}
\end{align}

This identity can be easily verified considering separately the alternatives (I), (D), and (E) for the \( i \)th configuration jump. It simply states that a particle in the system either was already present before the last configuration jump or it has been injected during this change of configuration. Making use of (18) we show by induction that for all integers \( j \geq 1 \) and \( i \) running from \( j \) to 1

\begin{align}
|\zeta_j| &= \sum_{x \in A} \Theta_{i,j}(x) + U_{i,j} + \cdots + U_{j,j}. \tag{19}
\end{align}

The lemma follows from this last formula when \( i = 1 \). In order to demonstrate (19), pick \( j \geq 1 \) and notice that \( \sum_{y \in A} \psi_j(y) = U_{j,j} \) since \( \sum_{y \in A}[1 - \zeta_{j-1}(y)]\zeta_j(y) = 1 \) if \( |\zeta_j| > |\zeta_{j-1}| \), which corresponds to alternative (I). Then, identity (18) yields

\begin{align}
|\zeta_j| &= \sum_{x \in A} \sum_{y \in A} \theta_j(x,y) + \sum_{y \in A} \psi_j(y) = \sum_{x \in A} \Theta_{j,j}(x) + U_{j,j}.
\end{align}

This proves (19) when \( i = j \). Suppose now that (19) holds with \( i + 1 \leq j \) in the place of \( i \). From definition (3) we have that \( \sum_{y \in A} \theta_i(x,y) \theta_{i+1,j}(y) = \Theta_{i,j}(x) \) and that \( \Theta_{i+1,j}(y) \) is proportional to \( \zeta_i(y) \), so that in particular (4) can be recast as
\[ \sum_{y \in A} \psi_i(y) \Theta_{i+1,j}(y) = U_{i,j}. \]

Then, we get from the inductive hypothesis first and (18) later that

\[
|\zeta_i| = \sum_{y \in A} \Theta_{i+1,j}(y) + U_{i+1,j} + \ldots + U_{j,j} \\
= \sum_{y \in A} \zeta_i(y) \Theta_{i+1,j}(y) + U_{i+1,j} + \ldots + U_{j,j} \\
= \sum_{x \in A} \sum_{y \in A} \theta_i(x,y) \Theta_{i+1,j}(y) + \sum_{y \in A} \psi_i(y) \Theta_{i+1,j}(y) + U_{i+1,j} + \ldots + U_{j,j} \\
= \sum_{x \in A} \Theta_{i,j}(x) + U_{i,j} + \ldots + U_{j,j}.
\]

This proves (19) when \( i < j \).

The fact that \( \eta_i = \zeta_i \) if \( J_j \leq t < J_{j+1} = J_j + H_j \) and that \( J_{N_i} \leq t < J_{N_i+1} \) allows to show that

\[
\int_0^t |\eta_i| \, dt = \sum_{j=0}^{N_i-1} H_j |\zeta_j| + (t - J_{N_i}) |\zeta_{N_i}| = \sum_{j=0}^{N_i} H_j |\zeta_j| + (t - J_{N_i+1}) |\zeta_{N_i}|.
\]

Using for all \( j \geq 1 \) the identity \( |\zeta_j| = \sum_{x \in A} \Theta_{1,j}(x) + \sum_{i=1}^t U_{i,j} \) provided by Lemma 1 we then find

\[
\int_0^t |\eta_i| \, dt = H_0 |\zeta_0| + (J_{N_i+1} - t) |\zeta_{N_i}| + \sum_{j=1}^{N_i} \sum_{x \in A} H_j \Theta_{1,j}(x) + \sum_{i=1}^{N_i} \sum_{j=N_i+1}^{N_i+1} H_j U_{i,j}.
\]

This way, noticing that \( 0 \leq J_{N_i+1} - t \leq J_{N_i+1} - J_{N_i} = H_{N_i} \) we obtain the bound

\[
\left| \sum_{i=1}^{N_i} \sum_{j=1}^{\infty} H_j U_{i,j} - \int_0^t |\eta_i| \, dt \right| \leq H_0 |\zeta_0| + (J_{N_i+1} - t) |\zeta_{N_i}| + \sum_{j=1}^{N_i} \sum_{x \in A} H_j \Theta_{1,j}(x) + \sum_{i=1}^{N_i} \sum_{j=N_i+1}^{\infty} H_j U_{i,j} \\
\leq |A| H_0 + |A| H_{N_i} + \sum_{j=1}^{N_i} \sum_{x \in A} H_j \Theta_{1,j}(x) + \sum_{i=1}^{N_i} \sum_{j=N_i+1}^{\infty} H_j U_{i,j}. \tag{20}
\]

The limit (16) follows from (17) if we prove that the r.h.s. of (20) divided by \( t \) goes to zero \( \mathbb{P} \)-a.s. when \( t \) is sent to infinity. It is clear that \( \lim_{t \to \infty} (1/t) H_0 = 0 \) \( \mathbb{P} \)-a.s., since \( H_0 < \infty \) \( \mathbb{P} \)-a.s.. Then, we must show that \( \lim_{t \to \infty} (1/t) V_{N_i} = 0 \) \( \mathbb{P} \)-a.s. with \( V_{N_i} \) once equal to \( H_{N_i} \) once equal to \( \sum_{x \in A} \sum_{j=1}^\infty H_j \Theta_{1,j}(x) \), and once equal to \( \sum_{i=1}^{N_i} \sum_{j=n+1}^{\infty} H_j U_{i,j} \). The average number of configuration jumps per unit time is \( \lim_{t \to \infty} (1/t) N_t = \sum_{q \in S} q(\sigma) \pi(\eta) \) \( \mathbb{P} \)-a.s. with \( q(\sigma) \) as in (1) (see [16], page 265). As \( \sum_{q \in S} q(\sigma) \pi(\eta) < \infty \), we obtain \( \lim_{t \to \infty} (1/t) V_{N_i} = 0 \) \( \mathbb{P} \)-a.s. if we demonstrate that \( \lim_{n \to \infty} (1/n) V_n = 0 \) \( \mathbb{P} \)-a.s.. The Borel-Cantelli lemma states that \( \lim_{n \to \infty} (1/n) V_n = 0 \) \( \mathbb{P} \)-a.s. if \( \sum_{n=1}^{\infty} \mathbb{P}[V_n > \epsilon n] < \infty \) for all \( \epsilon > 0 \) and the Markov’s inequality yields \( \mathbb{P}[V_n > \epsilon n] \leq (1/\epsilon n)^2 \mathbb{E}[V_n^2] \) for every \( n \geq 1 \) and \( \epsilon > 0 \).
This way, we conclude that $\lim_{t \to \infty} (1/t)V_{N_t} = 0$ a.s. if there exists a positive constant $C < \infty$ such that $\mathbb{E}[V_n^2] \leq C$ for all $n \geq 1$. Let us show that such a constant exists. We recall that $H_0, \ldots, H_\pi$ are independent exponential random variables of parameters $q(\zeta_0), \ldots, q(\zeta_\pi)$ conditional on $\zeta_0, \ldots, \zeta_\pi$. We set $\delta := \min_{\eta \in S} \{q(\eta)\}$ and we observe that $\delta > 0$ since $q(\eta) > 0$ for all $\eta$ belonging to the finite set $S$.

We have that for all $n \geq 1$

$$
\mathbb{E}[H_n^2] = \sum_{\eta_0 \in S} \sum_{\eta_n \in S} \mathbb{E}[H_n^2|\eta_0 = \eta_0 \land \ldots \land \eta_n = \eta_n] \mathbb{P}[\zeta_0 = \eta_0 \land \ldots \land \zeta_n = \eta_n] = \sum_{\eta_0 \in S} \sum_{\eta_n \in S} \frac{2}{\delta^2} \mathbb{P}[\zeta_0 = \eta_0 \land \ldots \land \zeta_n = \eta_n] \leq \frac{2}{\delta^2}.
$$

Thus, there exists $C < \infty$ such that $\mathbb{E}[V_n^2] \leq C$ for each $n \geq 1$ when $V_n := H_\pi$.

The cases $V_n := \sum_{x \in \Lambda} \sum_{j=1}^n H_j \Theta_{1,j}(x)$ and $V_n := \sum_{i=1}^n \sum_{j=n+1}^\infty H_j \Theta_{i,j}$ are more involved and require the use of the following lemma.

**Lemma 2** There exist positive constants $c < \infty$ and $r < 1$ with the property that $\mathbb{E}[\Theta_{i,j}(x)] \leq cr^{j-i}$ for all $j \geq i \geq 1$ and $x \in \Lambda$.

**Proof** Let $E$ be the vector space of the functions $f : \Lambda \times S \to \mathbb{C}$ endowed with the norm $\|f\| := \max_{(x,\eta) \in \Lambda \times S}|f(x,\eta)|$ and let $W : E \to E$ be the linear operator defined for each $f \in E$ by

$$
Wf(x,\eta) := \sum_{\eta' \in S} \eta(x') \eta'(x) \mathbb{P}[\zeta_1 = \eta'|\zeta_0 = \eta]f(x,\eta') + \sum_{y \in \Lambda} \sum_{\eta' \in S} \eta(x)[1 - \eta(y)] \mathbb{P}[\zeta_1 = \eta'|\zeta_0 = \eta]f(y,\eta').
$$

Denoting by $\|W\| := \sup_{f \in E} \|f\| \{\|Wf\|\}$ the norm of $W$ induced by the vector norm on $E$ and by $\sigma(W)$ the spectrum of $W$, Gelfand’s formula for the spectral radius states that $\lim_{n \to \infty} \|W^n\|^{1/n} = \max_{\xi \in \sigma(W)} \{\xi\}$. The powers of $W$ are related to certain expected values, as we shall see in a moment.

Pick a function $f \in E$ and consider the random variable $\Theta_{i,j}(x_{i-1})$ defined for each couple of integers $j \geq i \geq 1$ and site $x_{i-1} \in \Lambda$ by

$$
\Theta_{i,j}(x_{i-1}) := \sum_{x_i \in \Lambda} \cdots \sum_{x_j \in \Lambda} \prod_{k=1}^j \theta_k(x_{k-1},x_k)f(x_j,\zeta_j).
$$

The variable $\Theta_{i,j}(x_{i-1})$ reduces to $\Theta_{i,j}(x_{i-1})$ given by (3) when $f$ is identically equal to one. The Markov property of $\{\zeta_i\}_{i \geq 0}$ in combination with the fact that $\theta_k(x,y)$ is a deterministic function of only $\zeta_k$ and $\zeta_k$ yields for each $x_{i-1} \in \Lambda$ and $\eta_{i-1} \in S$ the relationship

$$
\mathbb{E}[\Theta_{i,j}(x_{i-1})|\zeta_{i-1} = \eta_{i-1}] = \sum_{x_i \in \Lambda} \cdots \sum_{x_j \in \Lambda} \sum_{\eta_j \in S} \sum_{\eta_{i-1} \in S} \prod_{k=1}^j \mathbb{E} \left[\theta_k(x_{k-1},x_k)\mathbb{1}(\zeta_k = \eta_k) \big| \zeta_{k-1} = \eta_{k-1}\right] f(x_j,\eta_j).
$$
It follows from here that $E[\Theta_{i,j}^{\ell}(x_{i-1})\mid \zeta_{i-1} = \eta_{i-1}] = W^{j-i+1} f(x_{i-1}, \eta_{i-1})$, with $W$ the above linear operator, since the homogeneity of the jump chain implies that for each $k \geq 1$ and function $f$

\[
\sum_{y,\eta' \in S} \sum_{\eta'' \in S} E[\theta_k(x, y)E(\zeta_k = \eta')\mid \zeta_{k-1} = \eta] f(y, \eta') = \\
\sum_{\eta' \in S} \eta(x)\eta'(x)P[\zeta_k = \eta'\mid \zeta_{k-1} = \eta] f(x, \eta')
\]

\[
\sum_{y,\eta' \in S} \sum_{\eta'' \in S} \eta(x)[1 - \eta(y)]\eta'(x)\eta(y)P[\zeta_k = \eta'\mid \zeta_{k-1} = \eta] f(y, \eta')
\]

$= Wf(x, \eta)$.

In conclusion, for every $j \geq i \geq 1$ and $x \in A$ we find

\[
E[\Theta_{i,j}^{\ell}(x)] = \sum_{\eta' \in S} P[\zeta_{i-1} = \eta] W^{j-i+1} f(x, \eta).
\]  (21)

We now prove that $|\xi| < 1$ for each eigenvalue $\xi \in \sigma(W)$. To this aim, we recall that there exist $P$-a.s. infinitely many $k$ such that $\theta_k(x, y) = 0$ for all $x$ and $y$ due to recurrence of $\{\zeta_i\}_{i \geq 0}$. Consequently, $\lim_{j \to \infty} \Theta_{i,j}^{\ell}(x) = 0$ $P$-a.s.. In its turn, Lebesgue’s dominated convergence theorem gives $\lim_{j \to \infty} E[\Theta_{i,j}^{\ell}(x)] = 0$ since $|\Theta_{i,j}^{\ell}(x)| \leq \|f\|$. This is true for every $i \geq 1$, $x \in A$, and $f \in E$ irrespective of the distribution of $\zeta_0 = \eta_0$. Pick $\xi \in \sigma(W)$ and let $f \in E$ be a corresponding eigenfunction, so that $Wf = \xi f$. There exists a pair $(x, \bar{\eta}) \in A \times S$ such that $f(x, \bar{\eta}) \neq 0$ and we can assume without loss of generality that $f(x, \bar{\eta}) = 1$. For the process $\{\eta_i\}_{i \geq 0}$ defined by the initial condition $\eta_0 = \bar{\eta}$ $P$-a.s. the application of (21) with $i = 1$ and $f$ the previously introduced eigenfunction yields $E[\Theta_{1,j}^{\ell} (\bar{x})] = \xi^j$.

This way, the bound $|\xi| < 1$ follows from $\lim_{j \to \infty} E[\Theta_{1,j}^{\ell} (\bar{x})] = 0$.

The fact that $\max_{\xi \in \sigma(W)} |\xi| < 1$ in combination with Gelfand’s spectral radius formula proves that there exist positive constants $c < \infty$ and $r < 1$ such that $\|W^{n+1}\| \leq cr^n$ for all $n \geq 0$. Expression (21) with $f$ identically equal to one, which has norm $\|f\| = 1$, shows that for all $j \geq i \geq 1$ and $x \in A$

\[
E[\Theta_{i,j}^{\ell}(x)] \leq \sum_{\eta' \in S} P[\zeta_{i-1} = \eta] \|W^{j-i} f\| \leq cr^{j-i}
\]

This concludes the proof.  

Let us now set $V_n := \sum_{x \in A} \sum_{j=1}^n H_j \Theta_{1,j}(x)$ for all $n \geq 1$. Since for any couple of integers $i \geq 1$ and $j \geq 1$ the binary random variables $\Theta_{1,i}(x)$ and $\Theta_{1,j}(y)$ are
deterministic functions of $\zeta_0, \ldots, \zeta_k$ with $k := \max\{i, j\}$, we have the bound

$$E\left[H_i H_j \Theta_{i,j}(x) \Theta_{i,j}(y)\right] = \sum_{\eta_0 \in S} \cdots \sum_{\eta_k \in S} E\left[H_i H_j \Theta_{i,j}(x) \Theta_{i,j}(y) \prod_{l=0}^{k} 1(\zeta_l = \eta_l)\right]$$

$$= \sum_{\eta_0 \in S} \cdots \sum_{\eta_k \in S} E\left[H_i H_j \right] 1(\xi_0 = \eta_0 \wedge \ldots \wedge \xi_k = \eta_k] E\left[\Theta_{i,j}(x) \Theta_{i,j}(y) \prod_{l=0}^{k} 1(\zeta_l = \eta_l)\right]$$

$$\leq \sum_{\eta_0 \in S} \cdots \sum_{\eta_k \in S} 2 \sum_{\eta_0 \in S} \sum_{\eta_k \in S} q(\eta_0) q(\eta_k) E\left[\Theta_{i,j}(x) \Theta_{i,j}(y) \prod_{l=0}^{k} 1(\zeta_l = \eta_l)\right]$$

$$\leq \frac{2}{\delta^2} E\left[\Theta_{i,j}(x) \Theta_{i,j}(y)\right] \leq \frac{2}{\delta^2} \sqrt{E\left[\Theta_{i,j}(x)\right] E\left[\Theta_{i,j}(y)\right]}.$$  (22)

where the Cauchy-Schwarz inequality has been exploited to obtain the last inequality. Thus, combining (22) with lemma 2 we arrive at the result

$$E\left[\left(\sum_{x \in A} \sum_{j=1}^{n} H_j U_{i,j}\right)^2\right] = \sum_{x \in A} \sum_{i=1}^{n} \sum_{j=1}^{n} E\left[H_i H_j \Theta_{i,j}(x) \Theta_{i,j}(y)\right]$$

$$\leq \frac{2}{\delta^2} \left(\sum_{x \in A} \sum_{j=1}^{n} \sqrt{E\left[\Theta_{i,j}(x)\right]}\right)^2 \leq \frac{2c|A|^2}{\delta^2(1 - \sqrt{\eta})^2}.$$

This proves that there exists $C < \infty$ such that $E[V_n^2] \leq C$ for each $n \geq 1$.

To conclude, set $V_n := \sum_{n=1}^{\infty} \sum_{j=1}^{n} H_i U_{i,j}$ for all $n \geq 1$. The same arguments that have led to (22) show that $\delta^2 E[H_i H_k U_{i,j} U_{h,k}] \leq 2$ for all $i \geq j \geq 1$ and $k \geq h \geq 1$ since the binary random variables $U_{i,j}$ and $U_{h,k}$ are deterministic functions of $\min_{i=1,j=1}, \ldots, \max_{i=1,j=1}$. Consequently, we can write

$$E\left[\left(\sum_{i=1}^{n} \sum_{j=n+1}^{\infty} H_j U_{i,j}\right)^2\right] = \sum_{i=1}^{n} \sum_{j=n+1}^{\infty} \sum_{h=1}^{n} \sum_{k=n+1}^{\infty} E\left[H_j H_k U_{i,j} U_{h,k}\right]$$

$$\leq \frac{2}{\delta^2} \sum_{i=1}^{n} \sum_{j=n+1}^{\infty} \sum_{h=1}^{n} \sum_{k=n+1}^{\infty} E[U_{i,j} U_{h,k}]$$

$$\leq \frac{2}{\delta^2} \sum_{i=1}^{n} \sum_{j=n+1}^{\infty} \sum_{h=1}^{n} \sum_{k=n+1}^{\infty} \sqrt{E[U_{i,j}] E[U_{h,k}]}$$

$$= \frac{2}{\delta^2} \left(\sum_{i=1}^{n} \sum_{j=n+1}^{\infty} \sqrt{E[U_{i,j}]}\right)^2.$$  (23)

On the other hand, from (4) we have $U_{i,j} \leq \sum_{x \in A} \Theta_{i+1,j}(x)$ for all $j > i$, giving $E[U_{i,j}] \leq |A| c r^{j-i-1}$ thanks to lemma 2. Combining this bound with (23) we get

$$E\left[\left(\sum_{i=1}^{n} \sum_{j=n+1}^{\infty} H_j U_{i,j}\right)^2\right] \leq \frac{2c|A|}{\delta^2(1 - \sqrt{\eta})^2}.$$

Thus, there exists $C < \infty$ with the property that $E[V_n^2] \leq C$ for any $n \geq 1$. 
References

1. van der Ent, R.J., Tuinenburg, O.A.: The residence time of water in the atmosphere revisited. Hydrol. Earth Syst. Sci. 21, 779-790 (2017)
2. Sincero, A.P., Sincero, G.A.: Physical-chemical treatment of water and wastewater, CRC Press, Boca Raton (2003)
3. Nauman, E.B.: Residence time theory. Ind. Eng. Chem. Res. 47, 3752-3766 (2008)
4. Weiss, M.: The relevance of residence time theory to pharmacokinetics. Eur. J. Clin. Pharmacol. 43, 571-579 (1992)
5. Zamparo, M., Valdenbri, D., Serini, G., Kolokolov, I.V., Lebedev, V.V., DallAsta, L., Gamba, A.: Optimality in self-organized molecular sorting. In preparation
6. Little, J.D.C.: Little’s law as viewed on its 50th anniversary. Oper. Res. 59, 536-549 (2011)
7. Griffeth, D.: Frank Spitzer’s pioneering work on interacting particle systems. Ann. Probab. 21, 608-621 (1993)
8. Kipnis, C., Landim, C.: Scaling limits of interacting particle systems, Springer, Berlin (1999)
9. Bertini, L., De Sole, A., Gabrielli, D., Jona-Lasinio, G., Landim, C.: Stochastic interacting particle systems out of equilibrium. J. Stat. Mech. 2007(07), P07014 (2007)
10. Liggett, T.M.: Interacting particle systems, Springer, New York (1985)
11. Derrida, B., Evans, M.R., Hakim, V., Pasquier, V.: Exact solution of a 1D asymmetric exclusion model using a matrix formulation. J. Phys. A: Math. Gen. 26, 1493-1517 (1993)
12. Schütz, G.M.: Exactly solvable models for many-body systems far from equilibrium. In: Domb, C., Lebowitz, J. (eds.) Phase Transitions and Critical Phenomena, vol. 19, pp. 1-251. Academic Press, San Diego (2001)
13. Cirillo, E.N.M., Krekel, O., Muntean, A., van Santen, R., Sengar, A.: Residence time estimates for asymmetric simple exclusion dynamics on strips. Physica A 442, 436-457 (2016)
14. Messelink, J., Rens, R., Vahabi, M., MacKintosh, F.C., Sharma, A.: On-site residence time in a driven diffusive system: violation and recovery of a mean-field description. Phys. Rev. E 93, 012119 (2016)
15. Norris, J.R.: Markov chains, reprinted ed. Cambridge University Press, Cambridge (1998)
16. Serfozo, R.: Basics of applied stochastic processes, Springer, Berlin (2009)
17. Presutti, E.: Scaling limits in statistical mechanics and microstructures in continuum mechanics, Springer, Berlin (2009)