Modulo 2 counting of Klein-bottle leaves in smooth taut foliations

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Abstract

This article proves that the parity of the number of Klein-bottle leaves in a smooth cooriented taut foliation is invariant under smooth deformations within taut foliations, provided that every Klein-bottle leaf involved in the counting has non-trivial linear holonomy.

1 Introduction

It was proved in [1] that for a cooriented foliation, a $C^0$-generic smooth perturbation destroys all closed leaves with genus greater than 1. This article explores the other side of the story. It shows that under certain conditions, one cannot get rid of Klein-bottle leaves of a taut foliation by smooth deformations.

Let $\mathcal{L}$ be a smooth cooriented 2-dimensional foliation on a smooth three manifold $Y$. The foliation $\mathcal{L}$ and the manifold $Y$ are allowed to be unorientable. By definition, the foliation $\mathcal{L}$ is called a taut foliation if for every point $p \in Y$ there exists an embedded circle in $Y$ passing through $p$ and being transverse to $\mathcal{L}$.

Definition 1.1. Let $K \subset Y$ be a closed leaf of $\mathcal{L}$. The leaf $K$ is called nondegenerate if it has non-trivial linear holonomy.

Consider a closed 2-dimensional submanifold $K$ of $Y$. If $K$ is cooriented, one can define an element $PD[K] \in \text{Hom}(H_1(Y;\mathbb{Z});\mathbb{Z})$ as follows. Let $[\gamma]$ be a homology class represented by a closed curve $\gamma$, then $PD[K]$ maps $[\gamma]$ to the oriented intersection number of $\gamma$ and $K$. Since $\text{Hom}(H_1(Y;\mathbb{Z});\mathbb{Z}) \cong H^1(Y;\mathbb{Z})$, the element $PD[K]$ can be considered as an element of $H^1(Y;\mathbb{Z})$. If both $Y$ and $K$ are oriented and if the orientations of $Y$ and $K$ are compatible with the coorientation of $K$, the element $PD[K]$ is equal to the Poincaré dual of the fundamental class of $K$. 

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Definition 1.2. Let $A \in H^1(Y; \mathbb{Z})$. A closed leaf $K$ of $\mathcal{L}$ is said to be in the class $A$ if $PD[K] = A$. The foliation $\mathcal{L}$ is called $A$-admissible if every Klein-bottle leaf of $\mathcal{L}$ in the class $A$ is nondegenerate.

The following result is the main theorem of this article.

Theorem 1.3. Let $A \in H^1(Y; \mathbb{Z})$. Let $\mathcal{L}_s$, $s \in [0, 1]$ be a smooth family of coorientable taut foliations on $Y$. Suppose $\mathcal{L}_0$ and $\mathcal{L}_1$ are both $A$-admissible. For $i = 0, 1$, let $n_i$ be the number of Klein-bottle leaves in the class $A$. Then $n_0$ and $n_1$ have the same parity.

Notice that if there is no Klein-bottle leaf of $\mathcal{L}$ in the homology class $A$, then $\mathcal{L}$ is automatically $A$-admissible. Therefore, the following result follows immediately.

Corollary 1.4. Let $A \in H^1(Y; \mathbb{Z})$, and let $\mathcal{L}$ be an $A$-admissible smooth coorientable taut foliation on $Y$. Assume that $\mathcal{L}$ has an odd number of Klein-bottle leaves in the class $A$. Then every smooth deformation of $\mathcal{L}$ through taut foliations has at least one Klein-bottle leaf in the class $A$.

It would be interesting to understand whether a similar result holds for torus leaves of taut foliations. For example, suppose $\mathcal{L}_0$ and $\mathcal{L}_1$ are two oriented and cooriented taut foliations on $Y$ that can be deformed to each other through taut foliations. Suppose every closed torus leaf in a homology class $e \in H_2(Y; \mathbb{Z})$ has non-trivial linear holonomy, is it always true that the numbers of torus leaves in the homology class $e$ in $\mathcal{L}_0$ and $\mathcal{L}_1$ have the same parity? The answer to this question is not clear to the author at the time of writing this article.

The article is organized as follows. Sections 2 and 3 build up necessary tools for the proof of theorem 1.3. Sections 4 and 5 prove the theorem. Section 6 gives an explicit example of corollary 1.4, constructing a foliation with a Klein-bottle leaf that cannot be removed by deformations.

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2 Moduli spaces of $J$-holomorphic tori

In [4], Taubes studied the behaviour of the moduli space of pseudo-holomorphic curves on a compact symplectic 4-manifold, and used it to define a version of Gromov invariant. This section recalls some results from [4] to prepare for the proof of theorem 1.3. The moduli space considered here is not exactly the moduli space used in the
definition of Taubes’s Gromov invariant, but essentially what is developed in this section is a special case of Taubes’s result. For a survey on different versions of Gromov invariants of symplectic 4-manifolds based on Taubes’s work, see \[3\].

Let \( X \) be a smooth 4-manifold. To avoid complications caused by exceptional spheres, assume throughout this section that \( \pi_2(X) = 0 \). This will be enough for the proof of theorem 1.3. Let \( J \) be a smooth almost complex structure on \( X \).

Consider an immersed closed \( J \)-holomorphic curve \( C \) in \( X \). Let \( N \) be the normal bundle of \( C \), the fiber of \( N \) then inherits an almost complex structure from \( J \). Let \( \pi : N \to C \) be the projection from \( N \) to \( C \). Choose a local diffeomorphism \( \varphi \) from a neighborhood of the zero section of \( N \) to a neighborhood of \( C \) in \( X \), which maps the zero section of \( N \) to \( C \). The map \( \varphi \) can be chosen in such a way that the tangent map is \( \mathbb{C} \)-linear on the zero section of \( N \). Every closed immersed \( J \)-holomorphic curve that is \( C^1 \)-close to \( C \) is the image of a section of \( N \). Fix an arbitrary connection \( \nabla_0 \) on \( N \) and let \( \bar{\partial}_0 \) be the \((0,1)\)-part of \( \nabla_0 \). If \( s \) is a section of \( N \) near the zero section, the equation for \( \varphi(s) \) to be a \( J \)-holomorphic curve in \( X \) can be schematically written as

\[
\bar{\partial}_0 s + \tau(s)(\nabla_0(s)) + Q(s)(\nabla_0(s), \nabla_0(s)) + T(s) = 0. \tag{2.1}
\]

Here \( \tau \) is a smooth section of \( \pi^*(\text{Hom}_\mathbb{R}(T^*C \otimes \mathbb{R} N, T^{0,1}C \otimes \mathbb{C} N)) \), and \( Q \) is a smooth section of \( \pi^*(\text{Hom}_\mathbb{R}(T^*C \otimes \mathbb{R} N, T^{0,1}C \otimes \mathbb{C} N)) \), and \( T \) is a smooth section of \( \pi^*(T^{0,1}C \otimes \mathbb{C} N) \). The values of \( \tau \), \( Q \), and \( T \) are defined pointwise by the values of \( J \) in an algebraic way, and \( \tau \), \( Q \), \( T \) are zero when \( s = 0 \). The linearized equation of (2.1) at \( s = 0 \) is \( \bar{\partial}_0(s) + \frac{\partial T}{\partial s}(s) = 0 \). Define

\[
L(s) := \bar{\partial}_0(s) + \frac{\partial T}{\partial s}(s). \tag{2.2}
\]

Notice that \( L \) is a real linear operator. The curve \( C \) is called nondegenerate if \( L \) is surjective as a map from \( L^2_k(N) \) to \( L^2(N) \). By elliptic regularity, if \( C \) is nondegenerate then the operator \( L \) is surjective as a map from \( L^2_k(N) \) to \( L^2_{k-1}(N) \) for every \( k \geq 1 \). The index of the operator \( L \) equals

\[
\text{ind } L = \langle c_1(N), [C] \rangle - \langle c_1(T^{0,1}X), [C] \rangle. \tag{2.3}
\]

It follows from the definition that nondegeneracy only depends on the 1-jet of \( J \) on \( C \). Namely, if there is another almost complex structure \( J' \) such that \( (J - J')|_C = 0 \) and \( (\nabla(J - J'))|_C = 0 \), then \( C \) is nondegenerate as a \( J \)-holomorphic curve if and only if it is nondegenerate as a \( J' \)-holomorphic curve.
For a homology class \( e \in H_2(X; \mathbb{Z}) \), define
\[
d(e) = e \cdot e - \langle c_1(T^{0,1}X), e \rangle.
\]
By equation (2.3), \( d(e) \) is the formal dimension of the moduli space of embedded pseudo-holomorphic curves in \( X \) in the homology class \( e \).

By the adjunction formula, the genus \( g \) of such a curve satisfies
\[
e \cdot e + 2 - 2g = -\langle c_1(T^{0,1}X), e \rangle.
\]
Therefore \( d(e) = 2(g - \langle c_1(T^{0,1}X), e \rangle - 1) \). In general, the formal dimension of the moduli space of \( J \)-holomorphic maps from a genus \( g \) curve to \( X \) in the homology class \( e \), modulo self-isomorphisms of the domain, is equal to \( 2(g - \langle c_1(T^{0,1}X), e \rangle - 1) \).

Now assume \( X \) is has a symplectic structure \( \omega \). Recall that an almost complex structure \( J \) is compatible with \( \omega \) if \( \omega(\cdot, J\cdot) \) defines a Riemannian metric. Let \( \mathcal{J}(X, \omega) \) be the set of smooth almost complex structures compatible with \( \omega \). For a closed surface \( \Sigma \) and a map \( \rho : \Sigma \to X \), define the topological energy of \( \rho \) to be \( \int_\Sigma \rho^*(\omega) \).

**Definition 2.1.** Let \((X, \omega)\) be a symplectic manifold. Let \( E > 0 \) be a constant. An almost complex structure \( J \in \mathcal{J}(X, \omega) \) is called \( E \)-admissible if the following conditions hold:

1. Every embedded \( J \)-holomorphic curve \( C \) with energy less than or equal to \( E \) and with \( d([C]) = 0 \) is nondegenerate.

2. For every homology class \( e \in H_2(X; \mathbb{Z}) \), if \( \langle [\omega], e \rangle \leq E \), and if \( \langle c_1(T^{0,1}X), e \rangle > 0 \) (namely, the formal dimension of the moduli space of \( J \)-holomorphic maps from a torus to \( X \) in the homology class \( e \), modulo self-isomorphisms of the domain, is negative), then there is no somewhere injective \( J \)-holomorphic map from a torus to \( X \) in the homology class \( e \).

The next lemma is a special case of proposition 7.1 in [5]. Recall that the \( C^\infty \) topology on \( \mathcal{J}(X, \omega) \) is defined as the Fréchet topology, namely it is induced by the distance function
\[
d(j_1, j_2) = \sum_{n=1}^{\infty} 2^{-n} \cdot \frac{\|j_1 - j_2\|_{C^n}}{1 + \|j_1 - j_2\|_{C^n}}.
\]

**Lemma 2.2.** Let \( E > 0 \) be a constant. If \((X, \omega)\) is a compact symplectic manifold, the set of \( E \)-admissible almost complex structures form an open and dense subset of \( \mathcal{J}(X, \omega) \) in the \( C^\infty \)-topology.

A homology class \( e \) is called primitive if \( e \neq n \cdot e' \) for every integer \( n > 1 \) and every \( e' \in H_2(X; \mathbb{Z}) \). If \( e \in H_2(X; \mathbb{Z}) \) is a primitive class,
define $\mathcal{M}(X, J, e)$ to be the set of embedded $J$-holomorphic tori in $X$ with fundamental class $e$.

Now consider smooth families of almost complex structures. Assume $\omega_s$ ($s \in [0, 1]$) is a smooth family of symplectic forms on $X$. For $i = 0, 1$, let $J_i \in \mathcal{J}(X, \omega_i)$. Define

$$\mathcal{J}(X, \{\omega_s\}, J_0, J_1)$$

to be the set of smooth families $\{J_s\}$ connecting $J_0$ and $J_1$, such that $J_s \in \mathcal{J}(X, \omega_s)$ for each $s \in [0, 1]$. The ideas of the following lemma can be found implicitly in [4].

**Lemma 2.3.** Let $X$ be a compact 4-manifold and let $\omega_s$ ($s \in [0, 1]$) be a smooth family of symplectic forms on $X$. Let $e \in H_2(X; \mathbb{Z})$ be a primitive class with $\langle c_1(T^{0,1}X), e \rangle = 0$ and $e \cdot e = 0$, and let $E > 0$ be a constant such that $E > \langle [\omega_s], e \rangle$ for every $s$. For $i \in \{0, 1\}$, let $J_i \in \mathcal{J}(X, \omega_i)$ be an $E$-admissible almost complex structure on $X$. Then there is an open and dense subset $U \subset \mathcal{J}(X, \{\omega_s\}, J_0, J_1)$ in the $C^\infty$-topology, such that for every element $\{J_s\} \in U$, the moduli space $\mathcal{M}(X, \{J_s\}, e) = \bigsqcup_{s \in [0, 1]} \mathcal{M}(X, J_s, e)$ has the structure of a compact smooth 1-manifold with boundary $\mathcal{M}(X, J_0, e) \cup \mathcal{M}(X, J_1, e)$.

**Proof.** The formal dimension of the moduli space of $J_s$-holomorphic maps from a genus $g$ curve to $X$ in homology class $e$, modulo self-isomorphisms of the domain, is equal to $2(g - \langle c_1(T^{0,1}X), e \rangle - 1)$, which always even. When the formal dimension is negative, it is less than or equal to $-2$. Therefore, there is an open and dense subset $U_\infty \subset \mathcal{J}(X, \{\omega_s\}, J_0, J_1)$, such that condition 2 of definition 2.1 holds for each $J_s$. The standard transversality argument shows that on an open and dense subset $U \subset U_0$, the space $\mathcal{M}(X, \{J_s\}, e)$ is a smooth 1-manifold with boundary $\mathcal{M}(X, J_0, e) \cup \mathcal{M}(X, J_1, e)$. For general $X$ and $e$ the space $\mathcal{M}(X, \{J_s\}, e)$ does not have to be compact. However, since it is assumed that $\pi_2(X) = 0$, there is a non-constant $J_s$-holomorphic maps from a sphere to $X$. By Gromov’s compactness theorem (see for example [6]), for every sequence $\{C_n\} \subset \mathcal{M}(X, \{J_s\}, e)$, there is a subsequence $\{C_{n_i}\}$ with $C_{n_i} \in \mathcal{M}(X, J_{n_i}, e)$ and $\lim_{i \to \infty} s_i = s_0$, such that the sequence $C_{n_i}$ is convergent to one of the following: (1) a branched multiple cover of a somewhere injective $J_{s_0}$-holomorphic map, (2) a somewhere injective $J_{s_0}$-holomorphic map with bubbles or nodal singularities or both, (3) a somewhere injective $J_{s_0}$-holomorphic torus. Case (1) is impossible since $e$ is assumed to be a primitive class. Case (2) is impossible because there is no non-constant $J_{s_0}$-holomorphic maps from a sphere to $X$. When case (3) happens, for the limit curve the adjunction formula states that $e \cdot e + 2 - 2g = -\langle c_1(T^{0,1}X), e \rangle + \kappa$, where $\kappa$ depends on the
behaviour of singularities and self-intersections of the curve, and \( \kappa \) is always positive if the curve is not embedded (see [2]). Since \( g = 1 \), \( e \cdot e = 0 \), \( \langle c_1(T^{0,1}X), e \rangle = 0 \), it follows that \( \kappa = 0 \), hence the limit curve is an embedded curve, namely it is an element of \( \mathcal{M}(X, J_{s_0}, e) \). Therefore the space \( \mathcal{M}(X, \{J_s\}, e) \) is compact.

With a little more effort one can generalize lemma [2, 3] to non-compact symplectic manifolds. To start, one needs the following definition.

**Definition 2.4.** Let \((X, \omega)\) be a symplectic manifold, not necessarily compact. Let \( J \in \mathcal{J}(X, \omega) \). The pair \((\omega, J)\) defines a Riemannian metric \( g \) on \( X \). The triple \((X, \omega, J)\) is said to have bounded geometry with bounding constant \( N \) if the following conditions hold:

1. The metric \( g \) is complete.
2. The norm of the curvature tensor of \( g \) is less than \( N \).
3. The injectivity radius of \((X, g)\) is greater than \( 1/N \).

One says that a path \( \{(X, \omega_s, J_s)\} \) has uniformly bounded geometry if each \((X, \omega_s, J_s)\) has bounded geometry, and the bounding constant \( N \) is independent of \( s \).

The following lemma is a well-known result.

**Lemma 2.5.** Let \((X, \omega, J)\) be a triple with bounded geometry, with bounding constant \( N \). Let \( e \in H_2(X; \mathbb{Z}) \), and let \( E > 0 \) be a constant such that \( E \geq \langle [\omega], e \rangle \). Then there is a constant \( M(N, E) \), depending only on \( N \) and \( E \), such that every connected \( J \)-holomorphic curve \( C \) with fundamental class \( e \) has diameter less than \( M(N, E) \) with respect to the metric defined by \( \omega(\cdot, J\cdot) \).

**Proof.** By the monotonicity of area, there is a constant \( \delta \) depending only on \( N \), such that for every point \( p \in C \) the area of \( B_p(1/N) \cap C \) is greater than \( \delta \). Since \( C \) is connected, this implies that the total area of \( C \) bounds its diameter. Notice that the area of \( C \) equals \( \langle [\omega], e \rangle \), which is bounded by \( E \), hence the the diameter is bounded by a function of \( N \) and \( E \). \( \square \)

In the noncompact case, one needs to be more careful about the topology of the spaces of almost complex structures. A topology on \( \mathcal{J}(X, \omega) \) can be defined as follows. Cover \( X \) by countably many compact sets \( \{A_i\}_{i \in \mathbb{Z}} \). For each \( A_i \) define the \( C^\infty \)-topology on \( \mathcal{J}(A_i, \omega) \). Endow the product space

\[
\prod_{i \in \mathbb{Z}} \mathcal{J}(A_i, \omega)
\]
with the box topology, and consider the map
\[ \mathcal{J}(X, \omega) \rightarrow \prod_{i \in \mathbb{Z}} \mathcal{J}(A_i, \omega) \]
defined by restrictions. The topology on \( \mathcal{J}(X, \omega) \) is then defined as the pull back of the box topology on the product space.

For \( N > 0 \), define \( \mathcal{J}(X, \omega, N) \) to be the set of almost complex structures \( J \in \mathcal{J}(X, \omega) \) such that \( (X, \omega, J) \) has bounded geometry with bounding constant \( N \). With the topology given above, the space \( \mathcal{J}(X, \omega, N) \) is an open subset of \( \mathcal{J}(X, \omega) \).

A topology on \( \mathcal{J}(X, \{\omega_i\}, J_0, J_1) \) can be defined in a similar way. Cover \( X \) by countably many compact sets \( \{A_i\}_{i \in \mathbb{Z}} \). For each \( A_i \), define the \( C^\infty \)-topology on \( \mathcal{J}(A_i, \{\omega_i\}, J_0, J_1) \). The topology on the space \( \mathcal{J}(X, \{\omega_i\}, J_0, J_1) \) is then defined as the pull back of the box topology on the product space.

For \( N > 0 \), define the set \( \mathcal{J}(X, \{\omega_i\}, J_0, J_1, N) \) to be the set of families \( \{J_i\} \in \mathcal{J}(X, \{\omega_i\}, J_0, J_1) \) such that \( \{(X, J_s, \omega_s)\} \) has uniformly bounded geometry with bounding constant \( N \). Then the set \( \mathcal{J}(X, \{\omega_i\}, J_0, J_1, N) \) is an open subset of \( \mathcal{J}(X, \{\omega_i\}, J_0, J_1) \).

The following lemma is essentially a diagonal argument. It explains why the topologies defined above are the correct topologies to accommodate the perturbation arguments for the rest of the article.

**Lemma 2.6.** Let \( \{A_n\}_{n \geq 1} \) be a countable, locally finite cover of \( X \) by compact subsets. Let \( \omega \) be a symplectic form on \( X \), let \( \omega_i \) be a smooth family of symplectic forms on \( X \). Let \( N > 0 \) be a constant.

Let \( J_i \in \mathcal{J}(X, \omega_i, N) \), where \( i = 0 \) or 1.

1. Let \( \varphi : \mathcal{J}(X, \omega) \rightarrow \prod_n \mathcal{J}(A_n, \omega) \) be the embedding map. For every \( n \), let \( U_n \) be an open and dense subset of \( \mathcal{J}(A_n, \omega) \), then \( \varphi^{-1}(\prod_n U_n) \) is an open and dense subset of \( \mathcal{J}(X, \omega) \).

2. Let \( \varphi : \mathcal{J}(X, \{\omega_i\}, J_0, J_1) \rightarrow \prod_n \mathcal{J}(A_i, \{\omega_i\}, J_0, J_1) \) be the embedding map. For every \( n \), let \( U_n \subset \mathcal{J}(A_n, \{\omega_i\}, J_0, J_1) \)

be an open and dense subset, then \( \varphi^{-1}(\prod_n U_n) \) is an open and dense subset of \( \mathcal{J}(X, \{\omega_i\}, J_0, J_1) \).

**Proof.** For part 1, the set \( \varphi^{-1}(\prod_n U_n) \) is open by the definition of box topology. To prove that \( \varphi^{-1}(\prod_n U_n) \) is dense, let \( J \) be an element of \( \mathcal{J}(X, \omega) \). Let \( J_n = J|_{A_n} \in \mathcal{J}(A_n, \omega) \). Let \( V_n \subset \mathcal{J}(A_n, \omega) \) be an arbitrary open neighborhood of \( J_n \). One needs to find an element \( J' \in \mathcal{J}(X, \omega) \) such that \( J'|_{A_n} \in V_n \cap U_n \). For each \( n \), let \( D_n \) be an open neighborhood of \( A_n \) such that the family \( \{D_n\} \) is still a locally finite
cover of \( X \). One obtains the desired \( J' \) by perturbing \( J \) on the open sets \( \{ D_n \} \) one by one. To start, perturb the section \( J \) on \( D_1 \) to obtain a section \( J_1 \). Since \( U_1 \) is dense it is possible to find a perturbation such that \( J_1|_{A_1} \in U_1 \cap V_1 \). Now assume that after perturbation on \( D_1, D_2, \cdots, D_k \), one obtains a section \( J_k \) such that \( J_k|_{A_i} \in U_j \cap V_j \) for \( j = 1, 2, \cdots, k \). Then a perturbation of \( J_k \) on \( D_{k+1} \) gives a section \( J_{k+1} \) such that \( J_{k+1}|_{A_{k+1}} \in U_{k+1} \cap V_{k+1} \). When the perturbation is small enough, it still has the property that \( J_{k+1}|_{A_i} \in U_j \cap V_j \) for \( j = 1, 2, \cdots, k \). Since \( \{ D_n \} \) is a locally finite cover of \( X \), on each compact subset of \( X \) the sequence \( \{ J_k \} \) stabilizes for sufficiently large \( k \). The limit \( \lim_{k \to \infty} J_k \) then gives the desired \( J' \).

The proofs for part 2 is exactly the same, one only needs to change the notation \( \mathcal{J}(\cdot, \omega) \) to \( \mathcal{J}(\cdot, \{ \omega_s \}, J_0, J_1) \).

**Lemma 2.7.** Let \( X \) be a 4-manifold, let \( e \in H_2(X; \mathbb{Z}) \) be a primitive class. Assume \( \omega_s \ (s \in [0, 1]) \) is a smooth family of symplectic forms on \( X \). Let \( E \) be a positive constant such that \( E > \langle \omega_s, e \rangle \) for every \( s \). For \( i = 0, 1 \), assume \( J_i \in \mathcal{J}(X, \omega_i, N) \) is \( E \)-admissible. If the set \( \mathcal{J}(X, \omega_s, J_0, J_1, N) \) is not empty, then there is an open and dense subset \( U \subset \mathcal{J}(X, \{ \omega_s \}, J_0, J_1, N) \), such that for each \( \{ J_s \} \in U \), the moduli space \( \mathcal{M}(X, \{ J_s \}, e) = \coprod_{s \in [0, 1]} \mathcal{M}(X, J_s, e) \) has the structure of a smooth 1-manifold with boundary \( \mathcal{M}(X, J_0, e) \cup \mathcal{M}(X, J_1, e) \).

Moreover, if \( f : X \to \mathbb{R} \) is a smooth proper function on \( X \), then the function defined as

\[
\mathcal{J} : \mathcal{M}(X, \{ J_s \}, e) \to \mathbb{R}
\]

\[
C \mapsto (\int_C f \, dA)/(\int_C 1 \, dA)
\]

is a smooth proper function on \( \mathcal{M}(X, \{ J_s \}, e) \), where \( dA \) is the area form of \( C \).

**Proof:** One first prove that there is an open and dense subset \( U \subset \mathcal{J}(X, \omega_s, J_0, J_1, N) \), such that for every \( \{ J_s \} \in U \), the moduli space \( \mathcal{M}(X, \{ J_s \}, e) \) is a smooth 1-dimensional manifold. Let \( g_s \) be the metric on \( X \) compatible with \( J_s \) and \( \omega_s \). Let \( g \) be a complete metric on \( X \) such that \( g_s \geq g \) for every \( s \). From now on, the distance function on \( X \) is defined by the metric \( g \). By lemma 2.5 there exists a constant \( M > 0 \) such that the diameter of every \( J_s \)-holomorphic curve with energy no greater than \( E \) is bounded by \( M \). Let \( \{ B_n \} \) be a countable locally finite cover of \( X \) by open balls of radius 1. For every \( n \), let \( A_n \) be the closed ball with the same center as \( B_n \) and with radius \((M + 1)\). The family \( \{ A_n \} \) is also a locally finite cover of \( X \). For each \( n \), let \( \mathcal{M}_n(X, \{ J_s \}, e) \) be the open subset of \( \mathcal{M}(X, \{ J_s \}, e) \) consisting of the curves \( C \in \mathcal{M}(X, \{ J_s \}, e) \) such that \( C \cap A_n \neq \emptyset \). By the diameter
bound of $J_s$-holomorphic curves and the results for the compact case, there is an open and dense subset $U_n \subset \mathcal{J}(A_n, \{\omega_s\}, J_0, J_1, N)$ such that if $\{J_s\}|_{A_n} \in U_n$, then the set $\mathcal{M}_n(X, \{J_s\}, e)$ is a smooth 1-dimensional manifold. It then follows from part 2 of lemma 2.6 that there is an open and dense subset $U \subset \mathcal{J}(X, \{\omega_s\}, J_0, J_1, N)$ such that for every element $\{J_s\} \in U$, the set $\mathcal{M}(X, \{J_s\}, e)$ is a smooth 1-manifold.

When set $\mathcal{M}(X, \{J_s\}, e)$ is a smooth 1-manifold, its boundary is $\mathcal{M}(X, J_0, e) \cup \mathcal{M}(X, J_1, e)$, and the function $f$ is a smooth function on $\mathcal{M}(X, \{J_s\}, e)$. It remains to prove that $f$ is a proper function. For any constant $z > 0$, take a sequence of curves $C_n \in \mathcal{M}(X, \{J_s\}, e)$ such that $|f(C_n)| < z$. By the definition of $f$, there exists a sequence of points $p_n \in C_n$ such that $|f(p_n)| < z$. Since $f$ is a proper function on $X$, the sequence $p_n$ is bounded on $X$. By lemma 2.3 this implies that the curves $C_n$ stay in a bounded subset of $X$. By the argument for the compact case (lemma 2.3), the sequence $\{C_n\}$ has a subsequence that converges to another point in $\mathcal{M}(X, \{J_s\}, e)$, hence function $f$ is proper.

3 Symplectization of taut foliations

This section discusses a symplectization of oriented and cooriented taut foliations. It is the main ingredient for the proof of theorem 1.3.

Let $M$ be a smooth 3-manifold, let $F$ be a smooth oriented and cooriented taut foliation on $M$. Since $F$ is cooriented, it can be written as $F = \ker \lambda$ where the positive normal direction of $F$ is positive on $\lambda$. Since $F$ is taut, there exists a closed 2-form $\omega$ such that $\omega \wedge \lambda > 0$ everywhere on $M$. Choose a metric $g_0$ on $M$ such that $*g_0 \lambda = \omega$. By Frobenius theorem, $d\lambda = \mu \wedge \lambda$ for a unique 1-form $\mu$ satisfying $\mu \perp \lambda$. Locally, write $\omega = e^1 \wedge e^2$ where $e^1$ and $e^2$ are orthonormal with respect to the metric $g_0$. Consider the 2-form $\Omega = \omega + d(t\lambda)$ on $M \times \mathbb{R}$ and the metric $g$ defined by

$$g = \frac{1}{1 + t^2} \cdot (dt + t\mu)^2 + (1 + t^2)\lambda^2 + (e^1)^2 + (e^2)^2$$

The 2-form $\Omega$ is a symplectic form on $M \times \mathbb{R}$, and the metric $g$ is independent of the choice of $\{e^1, e^2\}$ and is compatible with $\Omega$. Let $J$ be the almost complex structure given by $(\Omega, g)$. To simplify notations, let $X$ be the manifold $M \times \mathbb{R}$.

Lemma 3.1 ([7], lemma 2.1). The triple $(X, \Omega, J)$ has bounded geometry. □
Locally, let \( \{e_0, e_1, e_2\} \) be the basis of \( TM \) dual to \( \{\lambda, e^1, e^2\} \), and extend them to \( \mathbb{R} \)-translation invariant vector fields on \( M \times \mathbb{R} \). Let \( \hat{e}_1 = e_1 - t\mu(e_1) \frac{\partial}{\partial t} \), \( \hat{e}_2 = e_2 - t\mu(e_2) \frac{\partial}{\partial t} \). The almost complex structure \( J \) is then given by
\[
J \frac{\partial}{\partial t} = \frac{1}{1 + t^2} e_0,
J \hat{e}_1 = \hat{e}_2.
\]

Define \( \tilde{F} = \text{span}\{\hat{e}_1, \hat{e}_2\} \), it is a \( J \)-invariant plane field on \( X \).

**Lemma 3.2.** The plane field \( \tilde{F} \) is a foliation on \( X \). Under the projection \( M \times \mathbb{R} \to M \), the leaves of \( \tilde{F} \) projects to the leaves of \( F \).

**Proof.** Since \( d\mu \wedge \lambda = d(d\lambda) = 0 \), there is a \( \mu_1 \) such that \( d\mu = \mu_1 \wedge \lambda \). Therefore, one has \( d(dt + t\mu) = (dt + t\mu) \wedge \mu + t\mu_1 \wedge \lambda \), and \( d\lambda = \mu \wedge \lambda \).
By Frobenius theorem, the plane field \( \tilde{F} = \ker(dt + t\mu) \cap \ker \lambda \) is a foliation. The tangent planes of \( \tilde{F} \) projects isomorphically to the tangent planes of \( F \) pointwise, thus the leaves of \( \tilde{F} \) projects to the leaves of \( F \).

It turns out that every closed \( J \)-holomorphic curve in \( X \) is a closed leaf of \( \tilde{F} \).

**Lemma 3.3.** Let \( \rho : \Sigma \to X \) be a \( J \)-holomorphic map from a closed Riemann surface to \( X \). Then either \( \rho \) is a constant map, or it is a branched cover of a closed leaf of \( \tilde{F} \).

**Proof.** Since \( \rho \) is \( J \)-holomorphic, \( \rho^*((dt + t\mu) \wedge \lambda) \geq 0 \) pointwise on \( \Sigma \). On the other hand,
\[
\int_{\Sigma} \rho^* ((dt + t\mu) \wedge \lambda) = \int_{\Sigma} \rho^* (d(t\lambda)) = 0.
\]
Therefore \( \rho(\Sigma) \) is tangent to \( \ker(dt + t\mu) \cap \ker \lambda \), hence either \( \rho \) is a constant map, or it is a branched cover of a closed leaf of \( \tilde{F} \).

**Lemma 3.4.** Let \( L \) be a leaf of \( F \) and \( \gamma \) a closed curve on \( L \). Let \( \pi : M \times \mathbb{R} \to M \) be the projection map. The foliation \( \tilde{F} \) is then transverse to \( \pi^{-1}(\gamma) \) and gives a horizontal foliation on \( \pi^{-1}(\gamma) \cong \gamma \times \mathbb{R} \). The holonomy of this foliation along \( \gamma \) is given by multiplication of \( l(\gamma)^{-1} \), where \( l(\gamma) \) is the linear holonomy of \( F \) along \( \gamma \).

**Proof.** Suppose \( \gamma \) is parametrized by \( u \in [0,1] \). Let \( (\gamma(u), t(u)) \) be a curve in \( M \times \mathbb{R} \) that is a lift of \( \gamma \) and tangent to \( \tilde{F} \). Then the function \( t(u) \) satisfies \( \dot{t} + t\mu(\dot{\gamma}) = 0 \). Therefore
\[
t(1) = e^{-\int_0^1 \mu(\dot{\gamma}) du} t(0).
\]
Now let $U$ be a tubular neighborhood of $\gamma$ on the leaf $L$, and let $U \times (-\epsilon, \epsilon) \subset M$ be a tubular neighborhood of $U$ in $M$. Parametrize the second factor of $U \times (-\epsilon, \epsilon)$ by $z$, then on this neighborhood of $\gamma$ the 1-form $\lambda$ can be written as $f \cdot dz + \nu(z)$, where $f$ is a nowhere zero function on $U \times (-\epsilon, \epsilon)$, and $\nu(z)$ is a 1-form on $U$ depending on $z$ with $\nu(0) = 0$. The restriction of the 1-form $\mu$ on $U$ then has the form

$$\mu = f^{-1}df - f^{-1}\frac{\partial \nu}{\partial z}|_{z=0} + g \cdot dz$$

for some function $g$. If $(\gamma(u), z(u))$ is a curve in $U \times (-\epsilon, \epsilon)$ tangent to $F$, then

$$\dot{z} + f(\gamma, z)^{-1} \cdot \nu(z)(\dot{\gamma}) = 0.$$ (3.1)

If $z_s(u), s \in [0, \epsilon)$ is a smooth family of solutions to (3.1) with $z_0(u) = 0$, then the linearized part $l(u) = \frac{\partial z_s}{\partial s}|_{s=0}(u)$ satisfies

$$\dot{l} + l \cdot f^{-1}(\gamma, 0) \cdot \frac{\partial \nu}{\partial z}|_{z=0}(\dot{\gamma}) = 0.$$ 

Therefore the linear holonomy of $F$ along $\gamma$ is

$$e^{-\int_0^1 f^{-1}(\gamma, 0) \cdot \frac{\partial \nu}{\partial z}|_{z=0}(\dot{\gamma}) du},$$

which is equal to $e^{\int_0^1 \nu(\dot{\gamma}) du}$, hence the linear holonomy of $F$ along $\gamma$ is inverse to the holonomy on $\pi^{-1}(\gamma)$ given by $\tilde{F}$.

The following result follows immediately from lemmas 3.3 and 3.4.

**Corollary 3.5.** Let $C$ be a closed embedded $J$-holomorphic curve on $X$. Then either $C \subset M \times \{0\}$ and $C$ is a closed leaf of $F$, or $C$ does not intersect the slice $M \times \{0\}$ and it projects diffeomorphically to a closed leaf of $F$ with trivial linear holonomy.

The next lemma studies $J$-holomorphic tori on $X$.

**Lemma 3.6.** Suppose $T$ is a torus leaf of $F$ with non-trivial linear holonomy. Then $T \times \{0\}$ is a nondegenerate $J$-holomorphic curve in $X$.

**Proof.** Notice that $d([T]) = 0$, thus the index of the deformation operator is zero, and one only needs to prove that for $T$ the operator $L$ defined by equation (2.2) has a trivial kernel.

Let $T_0 = T \times \{0\}$ be the torus in $X$. Recall that locally $(\lambda, e_1, e_2)$ is an orthonormal basis of $T^*M$ and $(e_0, e_1, e_2)$ is its dual basis. Let $T \times (-\epsilon, \epsilon) \subset M$ be a tubular neighborhood of $T$ in $M$ such that the fibers of $(-\epsilon, \epsilon)$ are flow lines of $e_0$. Choose a parametrization $z$ for the second factor of $T \times (-\epsilon, \epsilon)$, such that $\lambda(\frac{\partial}{\partial z}) = 1$. Then on this neighborhood $e_0 = \frac{\partial}{\partial z}$, and $\lambda$ has the form $\lambda = dz + \nu(z)$ where $\nu(z)$
is a 1-form on $T$ depending on $z$ and $\nu(0) = 0$. The condition that ker $\lambda$ is a foliation is equivalent to

$$d\nu + \frac{\partial \nu}{\partial z} \wedge \nu = 0.$$ 

Let $\beta = \left. \frac{\partial \nu}{\partial z} \right|_{z=0}$. Apply $\frac{\partial}{\partial z}$ on the equation above at $z = 0$, one obtains $d\beta = 0$. Let $\lambda' = dz + z \cdot \beta$, then ker $\lambda'$ defines another foliation near $T$. Let $\mu' = -\beta$.

Let $e'_1, e'_2$ be vector fields on $T \times (-\epsilon, \epsilon)$ such that they are tangent to ker $\lambda'$, and their projections to $T$ form a positive orthonormal basis. Extend $e'_1, e'_2$ to a neighborhood of $T_0$ in $X$ by translation on the $t$-coordinate. Define an almost complex structure $J'$ on $T \times (-\epsilon, \epsilon)$ as

$$J' \frac{\partial}{\partial t} = \frac{\partial}{\partial z},$$

$$J'(e'_1 - t\mu'(e'_1) \frac{\partial}{\partial t}) = e'_2 - t\mu'(e'_2) \frac{\partial}{\partial t}.$$ 

Since $T$ has nontrivial linear holonomy, the same argument as in lemma 3.3 and lemma 3.4 shows that $T_0$ is the only embedded $J'$-holomorphic torus in a neighborhood of $T_0$. On the other hand, a straightforward calculation shows that the equation (2.1) for deformation of $J'$-holomorphic curves near $T_0$ is a linear equation, therefore $T_0$ is nondegenerate as a $J'$-holomorphic curve. Since $J'$ and $J$ agree up to first order derivatives along the curve $T_0$, this proves that $T_0$ is nondegenerate with respect to $J$. \hfill $\Box$

4 Proof of theorem 1.3

Now let $\mathcal{L}$ be a cooriented smooth taut foliation on a smooth 3-manifold $Y$. Consider its orientation double cover $\tilde{\mathcal{L}}$. It is an oriented and cooriented taut foliation on the orientation double cover of $Y$. Let $p : \tilde{Y} \to Y$ be the covering map. If $K$ is a Klein-bottle leaf of $\mathcal{L}$, then $p^{-1}(K)$ is a torus leaf of $\tilde{\mathcal{L}}$. Recall that in the beginning of section 1.1, a homology class $PD[K] \in H^1(Y; \mathbb{Z})$ was defined for every Klein-bottle leaf.

Lemma 4.1. Let $K$ be a Klein-bottle leaf of $\mathcal{L}$. Let $PD[p^{-1}(K)]$ be the Poincaré dual of the fundamental class of $p^{-1}(K)$. Then $p^*(PD[K]) = PD[p^{-1}(K)]$.

Proof. Let $\gamma$ be a closed curve in $\tilde{Y}$. Use $I(\cdot, \cdot)$ to denote the intersection number. Then

$$\langle PD[p^{-1}(K)], [\gamma] \rangle = I(p^{-1}(K), \gamma)$$

$$= I(K, p(\gamma)) = \langle PD[K], p_*[\gamma] \rangle = \langle p^*(PD[K]), [\gamma] \rangle.$$
Therefore \( p^*(PD[K]) = PD[p^{-1}(K)] \).

**Lemma 4.2.** The pull-back map \( p^* : H^1(Y; \mathbb{Z}) \to H^1(\tilde{Y}; \mathbb{Z}) \) is injective.

**Proof.** Every element in \( \ker p^* \) is represented by an element \( \alpha \in \text{Hom}(\pi_1(S), \mathbb{Z}) \) such that \( \alpha \) is zero on the image of \( p_* : \pi_1(\tilde{Y}) \to \pi_1(Y) \). Since \( \text{Im} p_* \) is a normal subgroup of \( \pi_1(Y) \) of index 2, the map \( \alpha \) is decomposed as

\[
\alpha : \pi_1(Y) \to \pi_1(Y)/\pi_1(\tilde{Y}) \cong \mathbb{Z}/2 \to \mathbb{Z},
\]

which has to be zero. Therefore \( p^* \) is injective.

By lemma 4.1 and 4.2, a Klein-bottle leaf \( K \) has \( PD[K] = A \) if and only if \( PD([p^{-1}(K)]) = p^*(A) \). The next lemma shows that for every Klein-bottle leaf \( K \) of \( L \) the fundamental class \( [p^{-1}(K)] \) is a primitive class.

**Lemma 4.3.** Let \( F \) be an oriented and cooriented taut foliation on a smooth three manifold \( M \), then the fundamental class of every closed leaf of \( F \) is a primitive class.

**Proof.** Let \( L \) be a closed leaf of \( F \). Take a point \( p \in L \). By the definition of tautness, there exists an embedded circle \( \gamma \) passing through \( p \) and transverse to the foliation. Let \( \gamma : [0, 1] \to M \) with \( \gamma(0) = \gamma(1) = p \) be a parametrization of \( \gamma \). By transversality, \( \gamma^{-1}(L) \) is a finite set. Let \( t_0 \) be the minimum value of \( t > 0 \) such that \( \gamma(t_0) \in L \). Then for \( \epsilon \) sufficiently small one can slide the part of \( \gamma \) on \( (t_0 - \epsilon, t_0 + \epsilon) \) along the foliation, such that the resulting curve is still transverse to \( F \), and such that \( \gamma(t_0) = p \). Now \( \gamma|_{[0,t_0]} \) defines a circle whose intersection number with \( L \) equals 1. The existence of such a curve implies that the fundamental class of \( L \) is primitive.

With the preparations above, one can now prove theorem 1.3.

**Proof of theorem 1.3.** Let \( A \in H^1(Y; \mathbb{Z}) \). Suppose \( L_0 \) and \( L_1 \) are two smooth \( A \)-admissible taut foliations on \( Y \), such that they can be deformed to each other by a smooth family of taut foliations \( L_s \), \( s \in [0, 1] \). Let \( \tilde{Y} \) be the orientation double cover of \( Y \). Then the orientation double covers \( \tilde{L}_s \) of \( L_s \) form a smooth family of oriented and cooriented taut foliations on \( \tilde{Y} \).

Let \( \tilde{\sigma} : \tilde{Y} \to \tilde{Y} \) be the deck transformation of the orientation double cover. Then the map \( \tilde{\sigma} \) preserves the coorientation of \( \tilde{L}_s \) and reverses its orientation for each \( s \).

There exists a smooth family of 1-forms \( \lambda_s \) and closed 2-forms \( \omega_s \) on \( \tilde{Y} \) such that \( \tilde{L}_s = \ker \lambda_s \) and \( \lambda_s \land \omega_s > 0 \). By changing \( \lambda_s \)
to \((\lambda_s + \bar{\sigma}^* \lambda_s)/2\) and changing \(\omega_s\) to \((\omega_s - \bar{\sigma}^* \omega_s)/2\), one can assume that \(\bar{\sigma}^* \lambda_s = \lambda_s\), and \(\bar{\sigma}^* \omega_s = -\omega_s\). Let \((\Omega_s, J_s)\) be the corresponding symplectic structures and almost complex structures on \(X = \tilde{Y} \times \mathbb{R}\). Define

\[
\sigma : X \rightarrow X \\
(x, t) \mapsto (\bar{\sigma}(x), -t).
\]

Then \(\sigma^*(\Omega_s) = -\Omega_s\), and \(\sigma^*(J_s) = -J_s\). The family \(\{(X, \Omega_s, J_s)\}\) has uniformly bounded geometry. This means that there is a constant \(N > 0\) such that \(J_s \in \mathcal{J}(X, \Omega_s, N)\) for each \(s\).

If neither \(\mathcal{L}_0\) nor \(\mathcal{L}_1\) has any Klein-bottle leaf in the class \(A\), the statement of theorem 1.3 obviously holds. From now on assume that \(\mathcal{L}_1\) implies that \(\pi\) is \(\sigma\)-admissible. Moreover, \(\sigma^*(\Omega_s) = -\Omega_s\) and \(\sigma^*(J_s) = -J_s\). Let \(e\) be the push forward of \(PD(p^*(A)) \in H_2(\tilde{Y}; \mathbb{Z})\) to \(H_2(X; \mathbb{Z})\) via the inclusion map \(\tilde{Y} \cong \tilde{Y} \times \{0\} \hookrightarrow X\). The class \(e\) then satisfies \(\sigma^*(e) = -e\). By lemma 4.3, \(e\) is a primitive class. Roussarie-Thurston theorem implies that \(\pi_2(X) = \pi_2(\tilde{Y}) = 0\).

Take a positive constant \(E\) such that \(E > \langle [\Omega_s], e \rangle\) for all \(s\). Let \(M(N, E)\) be the diameter upper bound from lemma 2.6 for the geometry bound \(N\) and the energy bound \(E\). Let \(T_0 > 0\) be sufficiently large such that the distance of \(\tilde{Y} \times \{T_0\}\) and \(\tilde{Y} \times \{-T_0\}\) is greater than \(M(N, E) + 1\) for every metric \(g_s\) induced from \((\Omega_s, J_s)\).

For \(i = 0, 1\), the union of torus leaves \(L\) in \(\mathcal{L}_i\) in the homology class \(p^*(A)\) such that \(\int_L \omega_i \leq E\) and \(L\) is not the lift of any Klein-bottle leaf form a compact set \(\tilde{B}_i\). The set \(\tilde{B}_i\) satisfies \(\tilde{\sigma}(\tilde{B}_i) = \tilde{B}_i\). Let \(\tilde{U}_i\) be a neighborhood of \(\tilde{B}_i\) such that \(\tilde{\sigma}(\tilde{U}_i) = \tilde{U}_i\) and the closure of \(\tilde{U}_i\) does not intersect the lift of any Klein-bottle leaf of \(\mathcal{L}_i\). Let

\[
V = \tilde{Y} \times ((-\infty, -T_0) \cup (T_0, \infty)),
\]

\[
U_i = (\tilde{U}_i \times \mathbb{R}) \bigcup \tilde{Y} \times ((-\infty, -T_0) \cup (T_0, \infty))
\]

which are open subsets of \(X\). The following two lemmas will be proved in section 5.

**Lemma 4.4.** The almost complex structure \(J_i\) can be perturbed to \(J_i' \in \mathcal{J}(X, \Omega_i, N)\), such that \(J_i' = J_i\) near Klein-bottle leaves, and \(J_i'\) is \(E\)-admissible. Moreover, \(\sigma^*(J_i') = -J_i'\) on \(U_i\), and every \(J_i'\)-holomorphic torus of \(X\) in the homology class \(e\) is either contained in \(U_i\) or is the lift of a Klein-bottle leaf in \(\mathcal{L}_i\) in the class \(A\). If \(C\) is a \(J_i'\)-holomorphic curve in the homology class \(e\) contained in \(U_i\), then \(\sigma(C) \neq C\).
Lemma 4.5. The almost complex structures $J'_0$ and $J'_1$ given by lemma 4.4 can be connected by a smooth family of almost complex structures $J'_s \in \mathcal{J}(X, \Omega_s, N)$, such that $\sigma^*(J'_s) = -J'_s$ on $V$, and the moduli space $\mathcal{M}(X, \{J'_s\}, e) = \bigsqcup_{s \in [0, 1]} \mathcal{M}(X, J'_s, e)$ has the structure of a smooth 1-manifold with boundary $\mathcal{M}(X, J'_0, e) \cup \mathcal{M}(X, J'_1, e)$. Moreover, let $t : X \to \mathbb{R}$ be the projection of $X = \tilde{Y} \times \mathbb{R}$ to $\mathbb{R}$, then the function defined as

$$f : \mathcal{M}(X, \{J'_s\}, e) \to \mathbb{R}$$

$$C \mapsto \left( \int_C t \, dA \right) / \left( \int_C 1 \, dA \right).$$

is a smooth proper function on $\mathcal{M}(X, \{J'_s\}, e)$, where $dA$ is the area form of $C$.

Let $\{J'_s\}$ be the family of almost complex structures given by the lemmas above, let $t_0 > 0$ be sufficiently large such that every $J'_s$-holomorphic torus $C$ in the homology class $e$ with $|f(C)| > t_0$ is contained in $V$. Take a constant $t_1 > t_0$ such that $t_1$ and $-t_1$ are regular values of $f$, and that $t_1 \notin \mathcal{M}(X, J'_0, e) \cup \mathcal{M}(X, J'_1, e)$. Let $S_i = \mathcal{M}(X, J'_i, e) \cap f^{-1}([-t_1, t_1])$. The set $f^{-1}(t_1) \cup f^{-1}(-t_1) \cup S_0 \cup S_1$ is the boundary of the compact 1-manifold $f^{-1}([-t_1, t_1])$, hence it has an even number of elements. On the other hand, the properties of $\{J'_s\}$ given by lemma 4.5 shows that $\sigma$ maps $f^{-1}(t_1)$ to $f^{-1}(-t_1)$, therefore the set $f^{-1}(t_1) \cup f^{-1}(-t_1)$ has an even number of elements. The properties given by lemma 4.4 implies that $\sigma$ acts on the set $S_i$, and the fixed point set of this action consists of the $J'_s$-holomorphic tori in $\tilde{Y} \times \{0\}$ which are lifts of Klein-bottle leaves. Let $K_i$ be the set of lifts of Klein-bottle leaves in $L_i$ in the class $A$, then the arguments above shows that the number of elements in $f^{-1}(t_1) \cup f^{-1}(-t_1) \cup S_0 \cup S_1$ has the same parity as the number of elements in $K_0 \cup K_1$. Therefore, the set $K_0 \cup K_1$ has an even number of elements, and the desired result is proved.

5 Technical lemmas

The purpose of this section is to prove lemma 4.4 and lemma 4.5. The proofs are routine and straightforward, they are given here for lack of a direct reference. Throughout this section $X$ will be a smooth 4-manifold with $\pi_2(X) = 0$.

Definition 5.1. Let $(X, \omega)$ be a symplectic manifold. Let $B \subset X$ be a closed subset. Let $E, N > 0$ be constants. An almost complex
structure $J \in \mathcal{J}(X, \omega, N)$ is called $(B, E)$-admissible if the following conditions hold:

1. Every embedded curve $C$ with energy less than or equal to $E$ and $d([C]) = 0$, and satisfies $C \cap B \neq \emptyset$ is nondegenerate.

2. For every homology class $e \in H_2(X; \mathbb{Z})$, if $\langle [\omega], e \rangle \leq E$, and if $\langle c_1(T^{0,1}X), e \rangle > 0$ (namely, the formal dimension of the moduli space of $J$-holomorphic maps from a torus to $X$ in the homology class $e$, modulo self-isomorphisms of the domain, is negative), then there is no somewhere injective $J$-holomorphic map $\rho$ from a torus to $X$ in the homology class $e$ such that $\text{Im}(\rho) \cap B \neq \emptyset$.

The next lemma follows immediately from Gromov’s compactness theorem and the diameter bound of lemma 2.5.

**Lemma 5.2.** Let $(X, \omega)$ be a symplectic manifold. Let $B \subset X$ be a closed subset, and $E, N > 0$ be constants. The elements of $\mathcal{J}(X, \omega, N)$ that are $(B, E)$-admissible form an open subset of $\mathcal{J}(X, \omega, N)$.

From now on assume that $\sigma : X \to X$ is a map that acts diffeomorphically on $X$, such that $\sigma^2 = \text{id}_X$ and the quotient map $X \to X/\sigma$ is a covering map.

**Definition 5.3.** Let $(X, \omega)$ be a symplectic manifold. Let $d, E, N > 0$ be constants. Let $B$ be a closed subset of $X$ such that $\sigma(B) = B$. An almost complex structure $J \in \mathcal{J}(X, \omega, N)$ is called $(d, E)$-regular with respect to $B$ if for every $J$-holomorphic map $\rho$ from a torus to $X$ with topological energy less than or equal to $E$, at least one of the following conditions hold:

1. The distance between the sets $\text{Im}(\rho)$ and $\sigma(\text{Im}(\rho))$ is greater than $d$.

2. The distance of $\text{Im}(\rho)$ and $B$ is greater than $d$.

Here the distance is defined by the metric $g_J = \omega(\cdot, J \cdot)$ on $X$.

Notice that since the map $\rho$ in the definition above can be a constant map, for a $(d, E)$-regular almost complex structure $J$ with respect to $B$, one has $\text{dist}(p, \sigma(p)) > d$ for every $p \in B$.

The following result is also a corollary of Gromov’s compactness theorem.

**Lemma 5.4.** Let $d, E, N > 0$ be constants, and $B$ is a closed subset of $X$ such that $\sigma(B) = B$. The elements of $\mathcal{J}(X, \omega, N)$ that are $(d, E)$-regular with respect to $B$ form an open subset of $\mathcal{J}(X, \omega, N)$.
Proof. First consider the case when $B$ is compact. Let $M(N, E)$ be the upper bound of diameter given by lemma 2.5. Let $A$ be a compact set containing $B$ such that the distance between $\partial A$ and $B$ is greater than $M(N, E) + d + 2$. Suppose $J$ is a $(d, E)$-regular almost complex structure with respect to $B$. Let $U$ be a sufficiently small open neighborhood of $J|_A \in \mathcal{J}(A, \omega)$, such that for every $J' \in \mathcal{J}(X, \omega, N)$, if $J'|_A \in U$ then the distance between $\partial A$ and $B$ is greater than $M(N, E) + d + 1$. One claims that there is a smaller neighborhood $V \subset U$ containing $J$, such that for every $J' \in \mathcal{J}(X, \omega, N)$, if $J'|_A \in V$ then $J'$ is $(d, E)$-regular with respect to $B$. In fact, assume the claim is not true, since $\mathcal{J}(A, \omega)$ is first countable, there is a sequence $\{J_n\} \subset \mathcal{J}(X, \omega, N)$, such that $J_n|_A \to J|_A$ in the $C^\infty$ topology, and that every $J_n$ is not $(d, E)$-regular with respect to $B$. By the definition of $(d, E)$-regularity, there is a sequence of $J_n$-holomorphic maps $\rho_n$ from torus to $X$ with topological energy less than or equal to $E$, such that the distance of $\text{Im}(\rho)$ to $B$ with respect to the metric given by $J_n$ is less than or equal to $d$, and the distance between $\text{Im}(\rho)$ and $\sigma(\text{Im}(\rho))$ with respect to the metric given by $J_n$ is less than or equal to $d$. By the diameter bound, every curve $C_n$ is contained in the set $A$. Gromov’s compactness theorem then implies that there is a subsequence of $\rho_n$ such that at least part of the map converges to a non-constant $J$-holomorphic map. Since is it assumed that $\pi_2(X) = 0$, the domain of the limit map is a torus. The limit map has topological energy less than or equal to $E$, and it violates the assumption that $J$ is $(d, E)$-regular with respect to $B$.

Now consider the case when $B$ is not necessarily compact. Let $J$ be a $(d, E)$-regular almost complex structure with respect to $B$. Cover $B$ by a locally finite family of compact subsets $B_n$ such that $\sigma(B_n) = B_n$ for each $n$. Let $A_n$ be the closed $(M(N, E) + d + 2)$-neighborhood of $B_n$. By the argument of the previous paragraph, for each $n$ there is an open neighborhood $V_n$ of $J|_{A_n}$ in $\mathcal{J}(A_n, \omega)$, such that for every $J' \in \mathcal{J}(X, \omega, N)$, if $J'|_{A_n} \in V_n$ then $J'$ is $(d, E)$-regular with respect to $B_n$. Notice that $J'$ is $(d, E)$-regular with respect to $B$ if and only if it is $(d, E)$-regular with respect to every $B_n$. The result of the lemma then follows from part 1 of lemma 2.6.

The following lemma is a 1-parametrized version of lemma 5.4.

**Lemma 5.5.** Let $d, E, N > 0$ be constants, and $B$ is a closed subset of $X$ such that $\sigma(B) = B$. Let $\omega_s$ ($s \in [0, 1]$) be a smooth family of symplectic forms on $X$, and let $J_i \in \mathcal{J}(X, \omega_i, N)$. Then the set of elements $\{J_s\} \in \mathcal{J}(X, \{\omega_s\}, J_0, J_1, N)$ such that every $J_s$ is $(d, E)$-regular with respect to $B$ form an open subset of $\mathcal{J}(X, \{\omega_s\}, J_0, J_1, N)$.
Proof. The proof is exactly the same as lemma 5.4. One only needs to change the notation $J$ to $\{ J_s \}$, and change the notation $J(X, \omega, N)$ to $J(X, \{ \omega_s \}, J_0, J_1, N)$. 

Lemma 5.6. Let $(X, \omega)$ be a symplectic manifold such that $\sigma^*(\omega) = -\omega$. Let $d, E, N > 0$ be constants. Let $B$ be a closed subset of $X$ such that $\sigma(B) = B$. Assume $J \in J(X, \omega, N)$ is $(d, E)$-regular with respect to $B$, and assume that $\sigma^*(J) = -J$ on $B$. Then for every open neighborhood $U$ of $J$ in $J(X, \omega, N)$, there is an element $J'$ such that $J'$ is $(d, E)$-regular with respect to $B$ and is $E$-admissible, and $\sigma^*(J') = -J'$ on $B$. Moreover, if there is a closed subset $H \subset X$ such that $\sigma(H) = H$ and $J$ is $(H, E)$-admissible, then $J'$ can be taken to be equal to $J$ on the set $H$.

Proof. By shrinking the open neighborhood $U$, one can assume that every element of $U$ is $(d, E)$-regular with respect to $B$, and that there is a complete metric $g_0$ on $X$ such that $g_0 \geq g_J$ for every $J' \in U$. For the rest of this proof, the distance function on $X$ is defined by $g_0$.

Cover $X$ by a locally finite family of closed balls with radius $d/10$. Say $$X = \bigcup_{i=1}^{N} B_i,$$
where $\{ B_i \}$ are closed balls with radius $d/10$. Let $D_i$ be the open $d/10$-neighborhood of $B_i$. Let $A_j = \bigcup_{i \leq j} B_j$, where $A_0 = \emptyset$. The construction of $J'$ follows from induction. Assume that $J_j$ is already $(A_j, E)$-admissible with $\sigma^*(J_j) = -J_j$ on $B$, the following paragraph will perturb $J_j$ to $J_{j+1}$ such that $J_{j+1}$ is $(A_{j+1}, E)$-admissible with $\sigma^*(J_{j+1}) = -J_{j+1}$ on $B$.

In fact, if $D_{j+1} \cap B = \emptyset$, then a generic perturbation on $D_{j+1}$ will do the job. If $D_{j+1} \cap B \neq \emptyset$, make a small perturbation on $D_{j+1}$ such that the resulting almost complex structure $J'_{j+1}$ is $(B_{j+1}, E)$-admissible. Now make a corresponding perturbation on $\sigma(D_{j+1})$ such that the resulting almost complex structure $J_{j+1}$ satisfies $\sigma(J_{j+1}) = -J_{j+1}$ on $B$. Since every element in $U$ is $(d, E)$-regular with respect to $B$, there is no $J_{j+1}$-holomorphic map with topological energy less than or equal to $E$ and with image passing through both $D_{j+1}$ and $\sigma(D_{j+1})$, therefore $J'_{j+1}$ being $(D_{j+1}, E)$-admissible implies that $J_{j+1}$ is $(D_{j+1}, E)$-admissible. Since being $(A_j, E)$-admissible is an open condition, when the perturbation is sufficiently small the almost complex structure $J_{j+1}$ is also $(A_j, E)$-admissible. Therefore $J_{j+1}$ is $(A_{j+1}, E)$-admissible. Since the family $\{ D_n \}$ is locally finite, on each compact set the sequence $\{ J_j \}$ stabilizes for sufficiently large $j$. The desired $J'$ can then be taken to be $\lim_{j \to \infty} J_j$. Moreover, if there is a
closed subset $H \subset X$ such that $\sigma(H) = H$ and $J$ is $(H, E)$-admissible, then each step of the perturbation can be taken to be outside of $H$. 

The following lemma is a 1-parametrized version of lemma 5.6 and the proof is essentially the same.

**Lemma 5.7.** Let $e \in H_2(X; \mathbb{Z})$ be a primitive class. Let $B$ be a closed subset of $X$ such that $\sigma(B) = B$. Assume $\omega_s (s \in [0, 1])$ is a smooth family of symplectic forms on $X$ such that $\sigma^*(\omega_s) = -\omega_s$ for each $s$. Let $d, N > 0$ be constants. Let $E$ be a positive constant such that $E > \langle [\omega_s], e \rangle$ for every $s$. For $i = 0, 1$, assume $J_i \in \mathcal{J}(X, \omega_i, N)$ is $E$-admissible and $(d, E)$-regular with respect to $B$. Assume $\{J_s\} \in \mathcal{J}(X, \{\omega_s\}, J_0, J_1, N)$, such that for each $s$, the almost complex structure $J_s$ is $(d, E)$-regular with respect to $B$, and $\sigma^*(J_s) = -J_s$ on $B$. Then for every open neighborhood $U$ of $\{J_s\}$ in $\mathcal{J}(X, \{\omega_s\}, J_0, J_1, N)$, there is an element $\{J'_s\}$ such that $\{J'_s\}$ is $(d, E)$-regular with respect to $B$ and is $E$-admissible, and $\sigma^*(J'_s) = -J'_s$ on $B$ for every $s$. Moreover, if there is a closed subset $H \subset X$ such that $\sigma(H) = H$ and $\{J_s\}$ is $(H, E)$-admissible, then $J'_s$ can be taken to be equal to $J_s$ on the set $H$.

**Proof.** The proof follows verbatim as the proof of lemma 5.6. One only needs to change the notation $J$ to $\{J_s\}$, and change $\mathcal{J}(X, \omega, N)$ to $\mathcal{J}(X, \{\omega_s\}, J_0, J_1, N)$.

Combining the results above, one obtains the following lemma.

**Lemma 5.8.** Let $e \in H_2(X; \mathbb{Z})$ be a primitive class. Let $B$ be a closed subset of $X$ such that $\sigma(B) = B$. Assume $\omega_s (s \in [0, 1])$ is a smooth family of symplectic forms on $X$ such that $\sigma^*(\omega_s) = -\omega_s$ for each $s$. Let $d, N > 0$ be constants. Let $E$ be a positive constant such that $E > \langle [\omega_s], e \rangle$ for every $s$. For $i = 0, 1$, assume $J_i \in \mathcal{J}(X, \omega_i, N)$ is $E$-admissible and $(d, E)$-regular with respect to $B$. Let $\mathcal{J}$ be the subset of elements $\{J_s\}$ of $\mathcal{J}(X, \{\omega_s\}, J_0, J_1, N)$ such that for each $s$, the almost complex structure $J_s$ is $(d, E)$-regular with respect to $B$, and $\sigma^*(J_s) = -J_s$ on $B$. If $\mathcal{J}$ is not empty, let $\mathcal{U} \subset \mathcal{J}$ be the subset of $\mathcal{J}$, such that for every $\{J_s\} \in \mathcal{U}$, the moduli space $\mathcal{M}(X, \{J_s\}, e) = \bigsqcup_{s \in [0, 1]} \mathcal{M}(X, J_s, e)$ has the structure of a smooth 1-manifold with boundary $\mathcal{M}(X, J_0, e) \cup \mathcal{M}(X, J_1, e)$. Then $\mathcal{U}$ is open and dense. Moreover, if $f : X \to \mathbb{R}$ is a smooth proper function on $X$, then the function defined as

$$f : \mathcal{M}(X, \{J_s\}, e) \to \mathbb{R}$$

$$C \mapsto \left( \int_C f \, dA \right) / \left( \int_C 1 \, dA \right).$$
is a smooth proper function on $\mathcal{M}(X, \{J_s\}, e)$, where $dA$ is the area form of $C$.

Proof. The openness of $U$ follows from lemma 5.5. The fact that $U$ is dense follows from lemma 5.7. The properness of the function $f$ was proved in lemma 2.7.

The following lemma controls the location of pseudo-holomorphic curves after perturbation of the almost complex structure.

**Lemma 5.9.** Let $(X, \omega)$ be a symplectic manifold, let $J \in \mathcal{J}(X, \omega, N)$. Let $E > 0$ be a positive constant, and let $B$ be a closed subset of $X$. Assume that there is no non-constant $J$-holomorphic map $\rho$ from a torus to $X$, such that $\text{Im}(\rho) \cap B$ is nonempty and the topological energy of $\rho$ is no greater than $E$. Then there is an open neighborhood $U$ of $J$ in $\mathcal{J}(X, \omega, N)$, such that for every $J' \in U$, there is no embedded $J'$-holomorphic torus in $X$ intersecting $B$ with energy less than or equal to $E$.

Proof. Cover the set $B$ by a locally finite family of compact subsets $B_n$. Let $M(N, E)$ be the upper bound given by lemma 2.5 for geometry bound $N$ and energy bound $E$. Let $A_n$ be the closed $M(N, E) + 1$-neighborhood of $B_n$. One claims that there is an open neighborhood $\mathcal{U}_n$ of $J|A_n \in J(A_n, \omega)$ such that for every $J' \in \mathcal{U}_n$, if $J'|A_n \in \mathcal{U}_n$, then there is no embedded $J'$-holomorphic torus in $X$ intersecting $B_n$ with topological energy less than or equal to $E$. Assume the result does not hold, then there is a sequence of $J_n \subset J(A_n, \omega, N)$ such that for each $n$ there exists a $J_n$-holomorphic map $\rho_n$ from a torus to $X$ which intersects $B$ and has topological energy less than or equal to $E$, and $J_n|A_n \to J|A_n$. For sufficiently large $n$, the distance between $\partial A_n$ and $B_n$ is greater than $M(N, E)$ with respect to the distance given by $J_n$, therefore the relevant $J_n$-holomorphic curve is contained in $A_n$. By Gromov’s compactness theorem, a subsequence of $\rho_n$ will give a non-constant $J$-holomorphic map from a torus to $A_n$, such that the intersection $\text{Im}(\rho) \cap B$ is nonempty, and the topological energy of $\rho$ is less than or equal to $E$, which is a contradiction. Therefore, the claim holds. The result of the lemma then follows from part 1 of 2.6.

With the preparations above, one can now give the proofs of lemma 4.4 and lemma 4.5.

**Proof of lemma 4.4.** By the definition of the set $U_i$, the almost complex structure $J_i$ is $(d, E)$-regular for some constant $d > 0$ with respect to $\overline{U_i}$. Apply lemma 5.6 for $B = \overline{U_i}$, there is a perturbation $J'_i \in \mathcal{J}(X, \Omega_i, N)$ of $J_i$, such that $J'_i$ is $E$-admissible and $\sigma^*(J'_i) = -J'_i$ on $\overline{U_i}$. Let $W_i$ be a small compact neighborhood of the union of
lifts of Klein-bottle leaves such that \( \sigma(W_i) = W_i \). The almost complex structure \( J'_i \) can be taken to be equal to \( J_i \) on \( W_i \) since \( J_i \) is already \((W_i, E)\)-admissible. By the definition of the set \( U_i \), every \( J'_i \)-holomorphic map from a torus to \( X \) is either a lift of Klein-bottle leaf or is mapped into the set \( U_i \). Therefore lemma 5.9 shows that when the perturbation is sufficiently small, every \( J'_i \)-holomorphic torus with homology class \( e \) is either contained in \( U_i \) or is contained in \( W_i \). In the latter case the curve is contained in \( \tilde{Y} \times \{0\} \) and it is a lift of a Klein-bottle leaf of \( L \) in class \( A \). Since \( J'_i \) is \((d, E)\)-regular with respect to \( U_i \), for every \( J'_i \) holomorphic torus \( C \) in \( U_i \) one has \( \sigma(C) \neq C \).

Proof of lemma 4.5. The almost complex structures \( J'_0 \) and \( J'_1 \) can be connected by a smooth family of almost complex structures \( J'_s \in \mathcal{J}(X, \Omega_s, J'_0, J'_1, N) \) such that \( \sigma^*(J'_s) = -J'_s \) on \( V \). Use lemma 5.8, the family \( J'_s \) can be further perturbed to satisfy the desired conditions.

6 An example

This section gives an example of a taut foliation with an odd number of Klein-bottle leaves such that every closed leaf is nondegenerate. By corollary 1.4, every deformation of such a foliation via taut foliations has at least one Klein-bottle leaf.

Think of the torus \( T_0 = S^1 \times S^1 \) as a trivial \( S^1 \)-bundle over \( S^1 \). Let \( z_1, z_2 \in \mathbb{R}/2\pi \) be the coordinates of the two \( S^1 \) factors, where \( z_1 \) is the coordinate for the fiber, and \( z_2 \) is the coordinate for the base. Let \( \gamma \) be a closed curve on the base that wraps the \( S^1 \) once in the positive direction. Take a horizontal foliation \( \hat{\mathcal{I}} \) on \( T_0 \) such that the holonomy along \( \gamma \) has two fixed points: \( z_1 = 0 \) and \( z_1 = \pi \), and that holonomy map has nontrivial linearization at these two points. Moreover, choose \( \hat{\mathcal{I}} \) so that it is invariant under the map \( (z_1, z_2) \mapsto (z_1 + \pi, -z_2) \) and the map \( (z_1, z_2) \mapsto (z_1, z_2 + \pi) \).

Consider the pull back of the foliation \( \hat{\mathcal{I}} \) to \( T_0 \times S^1 \). Let \( z_3 \in \mathbb{R}/2\pi \) be the coordinate for the \( S^1 \) factor, then \( \text{span}\{\hat{\mathcal{I}}, \frac{\partial}{\partial z_3}\} \) defines a foliation \( \mathcal{I} \) on \( T_0 \times S^1 \). The foliation \( \mathcal{I} \) is invariant under the maps

\[
\begin{align*}
\sigma_1 : (z_1, z_2, z_3) &\mapsto (z_1 + \pi, \pi - z_2, z_3) \\
\sigma_2 : (z_1, z_2, z_3) &\mapsto (z_1, z_2 + \pi, \pi - z_3) \\
\sigma_3 : (z_1, z_2, z_3) &\mapsto (z_1 + \pi, -\pi - z_2, \pi - z_3)
\end{align*}
\]

The set \( V = \{\text{id}, \sigma_1, \sigma_2, \sigma_3\} \) is a group acting freely and discontinuously on \( T_0 \times S^1 \) and it preserves the coorientation of \( \mathcal{I} \). The quotient foliation \( \mathcal{I}/V \) has exactly one Klein-bottle leaf and it is nondegenerate. Therefore, one has the following result.
Proposition 6.1. Every deformation of $\mathcal{I}/\mathcal{V}$ through taut foliations must have at least one Klein-bottle leaf.

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