THE HOMOLOGY OF STRING ALGEBRAS I

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Abstract. We show that string algebras are ‘homologically tame’ in the following sense: First, the syzygies of arbitrary representations of a finite dimensional string algebra Λ are direct sums of cyclic representations, and the left finitistic dimensions, both little and big, of Λ can be computed from a finite set of cyclic left ideals contained in the Jacobson radical. Second, our main result shows that the functorial finiteness status of the full subcategory \( P^{<\infty}(\Lambda\text{-mod}) \) consisting of the finitely generated left Λ-modules of finite projective dimension is completely determined by a finite number of, possibly infinite dimensional, string modules – one for each simple Λ-module – which are algorithmically constructible from quiver and relations of Λ. Namely, \( P^{<\infty}(\Lambda\text{-mod}) \) is contravariantly finite in \( \Lambda\text{-mod} \) precisely when all of these string modules are finite dimensional, in which case they coincide with the minimal \( P^{<\infty}(\Lambda\text{-mod}) \)-approximations of the corresponding simple modules. Yet, even when \( P^{<\infty}(\Lambda\text{-mod}) \) fails to be contravariantly finite, these ‘characteristic’ string modules encode, in an accessible format, all desirable homological information about \( \Lambda\text{-mod} \).

1. Introduction

The representation theory of the Lorentz group is intimately linked to that of a certain string algebra Λ (for a definition of string algebras see Section 2), as was observed and exploited by Gelfand and Ponomarev in [17]. In particular, it was proved there that this algebra – along with a class of close relatives – has tame representation type; in fact, its finite dimensional indecomposable representations were explicitly pinned down. In a sequence of articles by Ringel [29], Bondarenko [6], Donovan-Freislich [12], Butler-Ringel [8], and others, the class of algebras amenable to techniques derived from the Gelfand-Ponomarev archetype was subsequently found to be much larger and, moreover, to be related to further classical scenarios, such as the representation theory of dihedral groups. This development ultimately led to a well-rounded representation-theoretic picture of the extended class of algebras on which we concentrate here, the class of string algebras. Among other tools, Auslander-Reiten methods were employed to place the finite dimensional indecomposable objects into a tightly knit categorical context. In tandem, certain portions of the infinite dimensional representation theory were rendered accessible. However, in spite of the availability of a full classification of the finitely generated indecomposable representations of string algebras, their homological properties, known to vary widely (see, e.g., [25]), remained far from understood.

Our goal is to supplement the structural information with equally precise homological data. For a more detailed preview of our results, let Λ be a finite dimensional string algebra over an algebraically closed field \( K \). We start by showing how the homological dimensions – global and finitistic – can be obtained from a finite collection of cyclic modules contained in the Jacobson radical \( J \) of Λ (Theorem 3). Then we turn to two far more deep-seated problems concerning the category \( P^{<\infty}(\Lambda\text{-mod}) \) that has as objects the finitely generated modules of finite projective dimension. These problems are as follows: (I) Can the internal structure of the objects in \( P^{<\infty}(\Lambda\text{-mod}) \) be characterized, so as to distinguish them from those in \( \Lambda\text{-mod}\setminus P^{<\infty}(\Lambda\text{-mod}) \)? – here \( \Lambda\text{-mod} \) stands for the category of all finitely generated left Λ-modules – and (II) How is the category \( P^{<\infty}(\Lambda\text{-mod}) \) embedded in \( \Lambda\text{-mod} \), in terms of maps entering or leaving \( P^{<\infty}(\Lambda\text{-mod}) \)? It is our answer to the second question which displays the homological mechanisms of Λ; in particular, it entails a solution to the first problem.

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To tackle Problem II, we establish a readily checkable characterization of contravariant finiteness of the category $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ in $\Lambda$-mod and describe the minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximations of the simple modules $S_1, \ldots, S_n$ in the positive case (cf. Section 4 for the relevant definitions); in fact, our description yields a procedure for constructing them (Theorem 5 and Proposition 14). Existence of such minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximations of the $S_i$ and – existence provided – their structure are known to have far-reaching consequences for the homology of $\Lambda$ (see, e.g. [3], [1], [24]); those which have direct impact on our present investigation are reviewed in Section 4.

To appreciate how contravariant finiteness of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ relates to our second problem, recall that this condition implies functorial finiteness of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$, i.e., dually defined left $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximations of the objects in $\Lambda$-mod come as free byproducts (see [24]). Suppose, for the moment, that $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is contravariantly finite, and fix an object $M \in \Lambda\text{-mod}$. The key roles played by the minimal right $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximation $\phi_M : A_M \to M$ and the minimal left approximation $\psi_M : M \to B_M$ of $M$ can be visualized as follows:

In other words, $A_M$ is minimal with the property that all homomorphisms from any object $A \in \mathcal{P}^{<\infty}(\Lambda\text{-mod})$ to $M$ pass through the way station $A_M$ via $\phi_M$, and consequently, the problem of controlling all maps in $\text{Hom}_\Lambda(\mathcal{P}^{<\infty}(\Lambda\text{-mod}), M)$ boils down to describing the approximation $\phi_M$ and understanding the internal maps of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$. The map $\psi_M$ plays a dual role. On the side, we mention that functorial finiteness of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ guarantees the existence of almost split sequences in that category (see [2] and [3]).

Whether or not $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is contravariantly finite in $\Lambda\text{-mod}$, one can associate with each simple $\Lambda$-module $S_i$ a representation $A_i$ in the category $\mathcal{P}^{<\infty}(\Lambda\text{-Mod})$ of all, not necessarily finite dimensional, left $\Lambda$-modules of finite projective dimension, together with a canonical map $\varphi_i : A_i \to S_i$, which is indicative of the map-theoretic ‘location’ of $S_i$ relative to $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ (Theorem 5). In fact, $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is contravariantly finite if and only if all of the $A_i$ are finite dimensional, in which case the maps $\varphi_i$ are the minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximations of the simple modules. Otherwise, the $A_i$ are still ‘phantoms’ of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximations of the $S_i$ in the sense of [18] (see Section 4 for a reminder). Roughly, this means that each $A_i$ exhibits, in the tightest possible format, the relations of those modules $M \in \mathcal{P}^{<\infty}(\Lambda\text{-mod})$ which map onto $S_i$; for instance, it is precisely when $S_i$ has finite projective dimension that $A_i \cong S_i$. All of these constructions are algorithmic. Indeed, even when the modules $A_i$ are infinite dimensional, they can be constructed in a predictable number of steps, growing polynomially with $\dim_K \Lambda$, due to their periodicity properties.

In the present work, we primarily focus on the homology of the ‘little’ category $\Lambda\text{-mod}$ – even though we need to resort to infinite dimensional modules to fully understand the latter – whereas part II will address additional phenomena arising in the homology of the ‘big’ category $\Lambda$-Mod.

As mentioned at the outset, the class of string algebras has developed into a showcase for representation-theoretic methods, thus attesting to the ‘state of the art’ on various fronts. Moreover, algebras degenerating to string algebras play a pivotal role in understanding more general classes of algebras. To tie the present investigation into the context of existing work, we add a relatively short, chronologically ordered list of further references providing historical background and samples of different lines of approach: [37], [15], [35], [39], [13], [10], [11], [30], [5], [16], [26], [33], [38], [34], [31], [32], [7].
2. Prerequisites and conventions

Consistently, $\Lambda = K\Gamma/I$ will denote a finite dimensional string algebra over an algebraically closed field $K$. This means that $\Lambda$ is a monomial relation algebra (that is, the admissible ideal $I$ of the path algebra $K\Gamma$ can be generated by paths), and $\Lambda$ is special biserial. The latter amounts to the combination of the following two conditions on $\Gamma$ and $I$: at most two arrows enter and at most two arrows leave any given vertex of $\Gamma$, and, for any arrow $\alpha$ in $\Gamma$, there is at most one arrow $\beta$ with $\alpha \beta \notin I$ and at most one arrow $\gamma$ with $\gamma \alpha \notin I$.

Our convention for multiplying paths is as follows: if $p$ and $q$ are paths in $\mathcal{K}$, then $pq$ stands for ‘$p$ after $q$’. Correspondingly, a right subpath of a path $p$ is a path $u$ such that $p = u'q$ for some path $u'$; left subpaths of $p$ are defined symmetrically. The set of vertices of $\Gamma$ will be identified with a full set of primitive idempotents. Given any left $\Lambda$-module $M$, a top element of $M$ is an element $x$ with the property that $ex = x$ for some primitive idempotent $e$, in which case we will also call $x$ a top element of type $e$ of $M$.

Throughout, $\Lambda$-mod and $\Lambda$-Mod will stand for the categories of finite dimensional and arbitrary left $\Lambda$-modules, respectively, $\mathcal{P}^{<\infty}(\Lambda$-mod) will denote the full subcategory of $\Lambda$-mod having as objects the modules of finite projective dimension, while $\mathcal{P}^{<\infty}(\Lambda$-Mod) will be the full subcategory of $\Lambda$-Mod consisting of all modules of finite projective dimension.

In essence, our conceptual and notational backdrop is that developed in successive steps in \cite{17, 29, 12, 8}, but some modifications to the presentation of the relevant data will be more convenient for our purposes. As is common, our notion of a ‘word’ is based on the fixed presentation $\Lambda = K\Gamma/I$ as follows: Syllables are elements of the set $\mathcal{P} \cup \mathcal{P}^{-1}$, where $\mathcal{P}$ is the set of all paths in $K\Gamma \setminus I$ and $\mathcal{P}^{-1}$ consists of the formal inverses of the elements of $\mathcal{P}$. The paths of length 0, i.e., the vertices of $\Gamma$, are included in $\mathcal{P}$ and will be called the trivial paths; both these trivial paths and their inverses are called trivial syllables. For $p \in \mathcal{P}$, let $(p^{-1})^{-1} = p$, so that $(\mathcal{P}^{-1})^{-1} = \mathcal{P}$.

(Generalized) words are $\mathbb{Z}$-indexed sequences of pairs of syllables $w = (p_i^{-1}q_i)_{i \in \mathbb{Z}}$ with $p_i, q_i \in \mathcal{P}$, which we also communicate as juxtapositions

\[ \cdots p_r^{-1}q_r \cdots p_{-1}^{-1}q_{-1}p_0^{-1}q_0p_1^{-1}q_1 \cdots p_s^{-1}q_s \cdots \]

(note that syllables from $\mathcal{P}$ alternate with syllables from $\mathcal{P}^{-1}$) subject to the following constraints:

- For each $i \in \mathbb{Z}$, the starting points of $p_i$ and $q_i$ coincide, but the first arrows of $p_i$ and $q_i$ are distinct whenever both $p_i$ and $q_i$ are nontrivial.
- For each $i \in \mathbb{Z}$, the end points of $q_i$ and $p_{i+1}$ coincide, but the last arrows of $q_i$ and $p_{i+1}$ are distinct whenever both $q_i$ and $p_{i+1}$ are nontrivial.
- No trivial syllable occurs between two nontrivial syllables (i.e., the nontrivial syllables form a ‘connected component’).

A word $w = (p_i^{-1}q_i)_{i \in \mathbb{Z}}$ will be called finite in case, for all $i \gg 0$ and all $i \ll 0$, the syllables $p_i^{-1}$ (and hence also the syllables $q_i$) are trivial; finite words are also communicated as finite juxtapositions $(p_i^{-1}q_i)$ in which all nontrivial syllables are preserved. More generally, we do not insist on recording trivial syllables; keep in mind that they can only occur at the left or right tail ends of a word. It is self-explanatory what we mean by a left or right finite word. Given a word $w = (p_i^{-1}q_i)_{i \in \mathbb{Z}}$, the inverse of $w$ is defined as $w^{-1} = (q_{-i}^{-1}p_{-i})_{i \in \mathbb{Z}}$ carrying the pair of syllables $q_{-i}^{-1}p_{-i}$ in position $i$. With each word, we associate a (not necessarily finite) directed graph which records the nontrivial syllables. Namely, if $w = (p_i^{-1}q_i)_{i \in \mathbb{Z}}$ with $p_i = a_{i_k} \ldots a_{i_1}$ and $q_i = b_{l(i)} \ldots b_{1}$, where the $a_{ij}$ and $b_{ij}$ are arrows, the graph of $w$ is
where the trivial syllables make no appearance, and the nodes are identified with the primitive idempotents occurring as the starting and end points of the arrows $a_{ij}$ and $b_{ij}$. When less detail is required, a simplified rendering of this graph will be preferred, namely:

```
... p_{i-1} \rightarrow q_{i-1} \rightarrow p_i \rightarrow q_i \rightarrow p_{i+1} \rightarrow q_{i+1} \rightarrow ... 
```

Graphs of words will only play a role in the proof of our main theorem, while graphs of string and pseudo-band modules, as given below, will be essential throughout our discussion.

To prepare for the upcoming definition, note that, for any nontrivial two-syllable word $w = p^{-1}q$, there exists at most one arrow $\alpha$ such that $(\alpha p)^{-1}q$ is again a word; analogously, there exists at most one arrow $\beta$ making $p^{-1}(\beta q)$ a word.

Each (generalized) word $w = (p_i^{-1}q_i)_{i \in \mathbb{Z}}$ gives rise to a (generalized) string module $\text{St}(w)$ defined as follows: If $w$ is trivial, say $w = e$, then $\text{St}(w)$ is the simple module $\Lambda e/Je$. Now suppose that $w$ is nontrivial, and let $\text{supp}(w)$ be the set of all those integers $j$ for which either $p_j$ or $q_j$ is nontrivial. If, moreover, the joint starting vertex of $p_i$ and $q_i$ is denoted by $e(i)$, then

$$
\text{St}(w) = \left( \bigoplus_{i \in \text{supp}(w)} \Lambda e(i) \right) / C,
$$

with cyclic correction terms $C_{\text{left}}$ and $C_{\text{right}}$ defined as follows: $C_{\text{left}} = 0$ if either $\text{supp}(w)$ is unbounded on the negative $\mathbb{Z}$-axis or $l = \inf \text{supp}(w)$ is an integer and $(\alpha p_l)^{-1}q_l$ fails to be a word for all arrows $\alpha$; in the remaining case, where $l \in \mathbb{Z}$ and there exists a (necessarily unique) arrow $\alpha$ with the property that $(\alpha p_l)^{-1}q_l$ is again a word, we set $C_{\text{left}} = \Lambda \alpha p_l e(l)$. The right-hand correction term $C_{\text{right}}$ is defined symmetrically.

Note that $\text{St}(w)$ is finite dimensional over $K$ precisely when $w$ is a finite word. Moreover, $\text{St}(w) \cong \text{St}(w^{-1})$, a fact which allows us to pass back and forth between the two presentations. Clearly, the family $(x_i)_{i \in \text{supp}(w)}$ consists of top elements which generate $\text{St}(w)$ and are $K$-linearly independent modulo $J \text{St}(w)$; by construction, they have the property that $q_i x_i = p_{i+1} x_{i+1}$, whenever $q_i$ and $p_{i+1}$ are both nontrivial. Any sequence of top elements of $\text{St}(w)$ with the listed properties is called a standardized sequence of top elements.

In the sense of [19] and [20], the module $\text{St}(w)$ has a layered graph relative to any standardized sequence of top elements: It is the undirected variant of the (directed) graph of $w$, layered in such a fashion that the vertices in the $i$-th row from the top correspond to the simple composition factors in $J^{-1} \text{St}(w)/J^i \text{St}(w)$. We will usually indicate the chosen sequence of top elements above the corresponding vertices in the first row of the graph as illustrated below.

```
... x_{i-1} \rightarrow x_i \rightarrow x_{i+1} \rightarrow ... 
```

The second class of representations of $\Lambda$ which will be pivotal in our discussion slightly generalizes the classical 'band modules' (a generalization which will prove convenient in the proof of the main theorem). This class consists entirely of finite dimensional representations, but this time, they need not be indecomposable. For a description following the classical road, suppose that $v = p_0^{-1}q_0 \ldots p_t^{-1}q_t$ is a finite word with $t \geq 0$ and $p_0$, $q_t$ both nontrivial; by our conventions, this amounts to the same as to require that all $p_i$ and $q_i$ be nontrivial. We call $v$ primitive if

- the juxtaposition $v^2 = vv$ is again a word (in which case all powers $v^r$ are words), and
- $v$ is not itself a power of a strictly shorter word.
In addition to the primitive word \( v \), let \( r \) be a positive integer and \( \phi : K^r \to K^r \) a cyclic automorphism (meaning that \( \phi \) turns \( K^r \) into a cyclic \( K[X] \)-module) with Frobenius companion matrix

\[
\begin{pmatrix}
0 & \cdots & 0 & c_1 \\
1 & \ddots & \vdots \\
\vdots & & \ddots & \vdots \\
0 & \cdots & 1 & c_r
\end{pmatrix}.
\]

Then the pseudo-band module \( \text{Bd}(v^r, \phi) \) is defined as follows: Let \( x_{10}, \ldots, x_{1t}, x_{20}, \ldots, x_{2t}, \ldots, x_{r0}, \ldots, x_{rt} \) be the standardized sequence of top elements of \( \text{St}(v^r) \) following the definition of a string module; in particular, \( q_j x_{ij} = p_{j+1} x_{i,j+1} \) for \( 0 \leq i \leq r \) and \( 1 \leq j < t \), and \( q_t x_{it} = p_0 x_{i+1,0} \) for \( i < r \). Then

\[
\text{Bd}(v^r, \phi) = \text{St}(v^r) / \Lambda \left( q_{r+1} x_{r0} - \sum_{i=1}^r c_i p_0 x_{i0} \right)
\]

by definition. If the residue classes of the \( x_{ij} \) in \( \text{Bd}(v^r, \phi) \) are denoted by \( y_{ij} \), then the latter are top elements of \( \text{Bd}(v^r, \phi) \) which are \( K \)-linearly independent modulo the radical, generate \( \text{Bd}(v^r, \phi) \), and satisfy the above equations, as well as the additional one

\[
q_{r+1} y_{rt} = \sum_{i=1}^r c_i p_0 y_{i0}.
\]

Any sequence of top elements with these properties is, in turn, called a standardized sequence of top elements of the pseudo-band module \( \text{Bd}(v^r, \phi) \). Relative to such a sequence, \( \text{Bd}(v^r, \phi) \) can be depicted in the form

![Diagram of pseudo-band module](image.png)

where the dotted line that encircles the vertices standing for the elements \( y_{0} y_{rt}, p_0 y_{i0}, \ldots, p_0 y_{r0} \) in the socle of \( \text{Bd}(v^r, \phi) \) indicates that the \( K \)-space spanned by these \( r + 1 \) elements has dimension \( r \) only.

In case the automorphism \( \phi \) of \( K^r \) is irreducible, the pseudo-band module \( \text{Bd}(v^r, \phi) \) is called a band module. It is readily seen that a pseudo-band module is a band module if and only if it is indecomposable (see, e.g., [8]). Note that, in contrast to the string case, the graph of a band module \( \text{Bd}(v^r, \phi) \) does not pin down the latter up to isomorphism, unless the scalars \( c_1, \ldots, c_r \) are recorded. Moreover, observe that, if we subject the pairs of syllables \( p_{ij}^{-1} q_i \) of the underlying primitive word \( v \) to a cyclical permutation resulting in \( \hat{v} \) say, then \( \text{Bd}(v^r, \phi) \cong \text{Bd}(\hat{v}^r, \phi) \).

The pivotal role played by the finite dimensional string and band modules is apparent from the following classification result, which will be used extensively in the sequel. In its present form, it was established by Butler and Ringel, but the ideas go back to Gelfand and Ponomarev who determined the finite dimensional representation theory of a somewhat more restricted class of algebras.

**Theorem 0.** (See [17, 29, 6, 12, 8]) The finitely generated string and band modules are precisely the indecomposable objects of \( \Lambda \)-mod. \( \square \)

3. Syzygies and the homological dimensions of string algebras

This short section provides a first installment of evidence that, not only from a representation-theoretic, but also from a homological viewpoint, string algebras show ‘tame behavior’. Not only can the global dimension of a string algebra be computed algorithmically from quiver and relations as we will shortly see, but this is true more generally for its finitistic dimensions.

Our first proposition determines the syzygies of string and band modules. All of these are direct sums of cyclic string modules which can be described in terms of the string and band data. Since we will repeatedly
Proof. and all second syzygies are direct sums of uniserial modules. Every submodule of a projective left $M$ obtained from $\subseteq M$ drop the requirement that $M$ of finitely generated modules and, by the preceding remark, the $M$ is nontrivial and $\Lambda$ is a $\Lambda$-module.

19 developed in \textquote[the proof of which is immediate from the definitions and the graphical methods for determining syzygies]{the proof of which is immediate from the definitions and the graphical methods for determining syzygies} distinct first arrows. Then, clearly, the string module $St(p^{-1}q)$ is a factor module of $\Lambda e$. Let $p$ and $q$ be the unique paths starting in the end points of $p$ and $q$, respectively, with the property that $St((pp^{-1}(qq)) \cong \Lambda e$. In other words, if $p$ is nontrivial, then $p$ is the longest path such that $p \subseteq KT \setminus I$, the path $q$ having an analogous description if $q$ is nontrivial; if on the other hand, $p$ is trivial, then $q$ is not, and $p$ is the longest path starting in $e$ which does not contain the first arrow of $q$ as a right subpath (in particular, $p = e$ in case $q$ is nontrivial and $\Lambda$ is uniserial). Finally, given any nontrivial path $u$, let $u^{(0)}$ be the path of length $\geq 0$ obtained from $u$ through deletion of the first arrow, and set $w^{(0)} = 0 \subseteq KT$ if $u$ is trivial.

We observe that the first syzygy of any cyclic string module $St(p^{-1}q)$ equals $St(p^{(0)}) \oplus St(q^{(0)})$ if $p$, $q$ are not both trivial, and equals $Je$ if $p = q = e$. Less obvious situations are addressed in the first proposition, the proof of which is immediate from the definitions and the graphical methods for determining syzygies developed in [19].

**Proposition 1.**

(1)(a) Suppose $w$ is a finite nontrivial word of the form $p_{0}^{-1}q_{0} \ldots p_{t}^{-1}q_{t}$ with $q_{0}$ and $p_{t}$ nontrivial. Then the first syzygy of the finite dimensional string module $St(w)$ is

$$St(p_{0}^{(0)}) \oplus \bigoplus_{i=0}^{t-1} St(p_{i+1}^{-1}q_{i}) \oplus St(q_{t}^{(0)}),$$

where the paths $p_{i}$, $q_{i}$, $q_{0}^{(0)}$, and $q_{t}^{(0)}$ are as introduced above.

(b) Now suppose that $w = \ldots p_{-1}^{-1}q_{-1}p_{0}^{-1}q_{1} \ldots$ is a word with $p_{i}$, $q_{i}$ nontrivial for all $i \in \mathbb{Z}$. The first syzygy of the infinite dimensional string module $St(w)$ equals

$$\bigoplus_{i \in \mathbb{Z}} St(p_{i+1}^{-1}q_{i}).$$

(2) If $v = p_{0}^{-1}q_{0} \ldots p_{t}^{-1}q_{t}$ is a primitive word with $p_{0}$ and $q_{t}$ nontrivial, $r$ a positive integer, and $\phi$ a cyclic automorphism of $K^{r}$, then the first syzygy of the pseudo-band module $Bd(v^{r}, \phi)$ is

$$\bigoplus_{i=0}^{t-1} (St(p_{i+1}^{-1}q_{i}))^{r} \oplus (St(p_{0}^{-1}q_{0}))^{r}.$$ 

Furthermore, the following statements are equivalent:

(i) For some positive integer $r$ and some cyclic automorphism $\phi$ of $K^{r}$, the pseudo-band module $Bd(v^{r}, \phi)$ belongs to $P^{<\infty}(\Lambda \text{-mod})$.

(ii) For any positive integer $r$ and any cyclic automorphism $\phi$ of $K^{r}$, the pseudo-band module $Bd(v^{r}, \phi)$ belongs to $P^{<\infty}(\Lambda \text{-mod})$.

(iii) The generalized string module $St(\ldots vvv \ldots)$ belongs to $P^{<\infty}(\Lambda \text{-Mod})$ \hfill $\square$

In view of Theorem 0 (Section 2), we glean from Proposition 1 that the syzygies of any finitely generated $\Lambda$-module $M$ are direct sums of cyclic string modules. Next we will see that this remains true even when we drop the requirement that $M$ be finitely generated.

**Proposition 2.** Every submodule of a projective left $\Lambda$-module is a direct sum of string modules of the form $St(p^{-1}q)$, where $p$ and $q$ are paths.

In particular, syzygies of arbitrary $\Lambda$-modules are direct sums of cyclic string modules which embed in $J$, and all second syzygies are direct sums of uniserial modules.

Proof. To verify the first assertion, let $P$ be a projective left $\Lambda$-module and $M \subseteq P$ a submodule. Since all of the indecomposable projective left $\Lambda$-modules are cyclic string modules, it is harmless to assume that $M \subseteq JP$.

Write $M$ as a directed union of finitely generated submodules, say $M = \bigcup_{i \in I} M_{i}$. Then all $M_{i}$ are syzygies of finitely generated modules and, by the preceding remark, the $M_{i}$ are direct sums of string modules with
simple tops. Since (up to isomorphism) there are only finitely many string modules of the form \( \text{St}(p^{-1}q) \), this entails that the direct sum \( \bigoplus_{i \in I} M_i \) is \( \Sigma \)-algebraically compact. Therefore, by [24, Observation 3.1] or [28], the category \( \text{Add}(\bigoplus_{i \in I} M_i) \) of arbitrary direct sums of direct summands of the \( M_i \) is closed under direct limits. In particular, \( M \) is in turn a direct sum of cyclic string modules, as claimed.

The final statements are immediate consequences. □

Recall that the left little and big finitistic dimensions of any finite dimensional algebra \( \Delta \) are defined as

\[
\text{l fin dim } \Delta = \sup \{ p \dim M \mid M \in \mathcal{P}^{<\infty}(\Delta \text{-mod}) \}
\]

\[
\text{l Fin dim } \Delta = \sup \{ p \dim M \mid M \in \mathcal{P}^{<\infty}(\Delta \text{-Mod}) \},
\]

respectively. According to [19] or [20], the finitistic dimensions of any monomial relation algebra can be computed up to an error of 1 by means of a simple graphical method. However, even for monomial relation algebras, the little finitistic dimension may be strictly smaller than the big finitistic dimension [20], and, for more general finite dimensional algebras, the difference \( \text{l Fin dim } \Delta - \text{l fin dim } \Delta \) attains arbitrarily high values in \( \mathbb{N} \) [36]. Our first theorem excludes such 'pathologies' for string algebras and pins down the finitistic dimensions in terms of the cyclic string modules of finite projective dimension; the latter are finite in number and easy to construct from \( \Gamma \) and \( I \). To that end, consider the set \( \mathcal{T} \) of those cyclic string modules in \( \mathcal{P}^{<\infty}(\Lambda \text{-mod}) \) which are contained in the radical \( J \) of \( \Lambda \), that is,

\[
\mathcal{T} = \{ \text{St}(p^{-1}q) \in \mathcal{P}^{<\infty}(\Lambda \text{-mod}) \mid \text{St}(p^{-1}q) \text{ embeds in } J \};
\]

here the paths \( p \) and \( q \) may be trivial, and so, in particular, \( \mathcal{T} \) includes all uniserial left modules of finite projective dimension contained in \( J \). The proof of Theorem 3 is an immediate consequence of Proposition 2.

**Theorem 3.** \( \text{l fin dim } \Lambda = \text{l Fin dim } \Lambda = t + 1 \), where \( t = \sup \{ p \dim M \mid M \in \mathcal{T} \} \) in case \( \mathcal{T} \) is nonempty, and \( t = -1 \) otherwise. □

The following example shows that shrinking \( \mathcal{T} \) to the set of all uniserial left modules from \( \mathcal{P}^{<\infty}(\Lambda \text{-mod}) \) contained in \( J \) does not leave the conclusion of Theorem 3 intact.

**Example 4.** Let \( \Lambda = K\Gamma/I \), where \( \Gamma \) is the quiver

```
1 \alpha \beta \gamma
\downarrow \downarrow \downarrow \downarrow
2 \gamma \delta \gamma
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and \( I \subseteq K\Gamma \) is the monomial ideal with the property that the indecomposable projective left \( \Lambda \)-modules have graphs

```
1 \alpha \beta
\downarrow \downarrow
2 \gamma \delta
```

Then \( \mathcal{T} \) is the singleton containing \( \Lambda e_2 \cong \Lambda(\alpha - \beta) \), whence, by Theorem 3, the left little and big finitistic dimensions of \( \Lambda \) are equal to 1.

4. Background on contravariant finiteness and phantoms

We give a brief summary of that part of the theory of contravariant finiteness of a full subcategory \( \mathcal{A} \subseteq \Lambda \text{-mod} \) which will be relevant here. The two categories on which we will focus in the sequel are \( \mathcal{P}^{<\infty}(\Lambda \text{-mod}) \) and \( \mathcal{S}^{<\infty}(\Lambda \text{-mod}) \), the full subcategory of \( \mathcal{P}^{<\infty}(\Lambda \text{-mod}) \) consisting of the finite direct sums
of string modules; note that the latter category depends, if only to a ‘minor’ degree, on the coordinatization $(\Gamma, I)$ of the string algebra $\Lambda$.

Since the background material we need is quite general, we let $\Delta$ be any finite dimensional algebra for the moment, and $A \subseteq \Delta$-mod a full subcategory which is closed under finite direct sums and direct summands. Recall from [2] that $A$ is said to be contravariantly finite in $\Delta$-mod in case each module $M \in \Delta$-mod has a (right) $A$-approximation, that is a homomorphism $\varphi : A \to M$ with $A \in A$ such that the induced sequence of functors

$$\text{Hom}_\Delta(-, A)|_A \to \text{Hom}_\Delta(-, M)|_A \to 0$$

is exact; in other words, the latter says that every map in $\text{Hom}_\Delta(B, M)$ with $B \in A$ factors through $\varphi$. In the sequel, we will suppress the qualifier ‘right’. Provided that $M$ has an $A$-approximation, there is a minimal such approximation which embeds, as a direct summand, into all other $A$-approximations of $M$. In particular, such a minimal $A$-approximation of $M$ is unique up to isomorphism; it is therefore only a mild abuse of language to refer to it as the minimal $A$-approximation of $M$. In case $A = \mathcal{P}^\infty(\Delta$-mod), the existence of approximations for the simple left $\Delta$-modules $S_i$, $1 \leq i \leq n$, already guarantees contravariant finiteness of $\mathcal{P}^\infty(\Delta$-mod). Moreover, in case of existence, the minimal $\mathcal{P}^\infty(\Delta$-mod)-approximations $A_i$ of the $S_i$ impinge on the structure of an arbitrary object in $\mathcal{P}^\infty(\Delta$-mod) as follows (see [1]): A finitely generated $\Delta$-module $M$ has finite projective dimension precisely when it is a direct summand of a module $X$ that has a finite filtration with successive factors in $\{A_1, \ldots, A_n\}$. This structure theory was extended to $\mathcal{P}^\infty(\Delta$-Mod), the category of all left $\Delta$-modules of finite projective dimension by the authors in [24]: If $\mathcal{P}^\infty(\Delta$-mod) is contravariantly finite, the objects of the ‘big’ category $\mathcal{P}^\infty(\Delta$-Mod) are precisely those which are direct limits of objects $X$ having filtrations $X = X_0 \supseteq X_1 \supseteq \cdots \supseteq X_m = 0$ with $X_i/X_{i+1} \cong A_{k(i)}$ for all $i < m$; in particular, the big and little left finitistic dimensions of $\Delta$ coincide in this situation and are attained on $\{A_1, \ldots, A_n\}$. This illustrates the pivotal role played by the minimal approximations of the simple modules in case $\mathcal{P}^\infty(\Delta$-mod) is contravariantly finite.

$A$-phantoms were introduced in [18], originally mainly with the aim of providing criteria for contravariant finiteness of $A$. Here they serve a dual purpose: the one just mentioned and that of replacing minimal $A$-approximations in case such approximations fail to exist. Our phantoms under (3) below are not to be confused with the phantom maps of algebraic topology, or the topology-inspired phantom maps in the modular representation theory of groups, as introduced by Benson and Gnacadja in [4].

**Definition.** Assume $A \subseteq \Delta$-mod to be closed under direct summands and finite direct sums. Moreover, fix $M \in \Delta$-mod.

1. Let $C$ be a subcategory of $A$. A relative $C$-approximation of $A$ in $A$ is a homomorphism $f : A \to M$ with $A \in A$ such that all maps in $\text{Hom}_\Delta(C, M)$ factor through $f$. In this situation, we will also refer to the module $A$ as a relative $C$-approximation of $M$ in $A$.

   If $C = \{C\}$, we write ‘relative $C$-approximation’ for short.

2. A finitely generated module $H$ is an $A$-phantom of $M$ in case there exists an object $C$ in $A$ with the property that $H$ occurs as a subfactor of every relative $C$-approximation of $M$ in $A$.

   More generally, a module $H \in \Delta$-Mod will be called an $A$-phantom of $M$ if each of its finitely generated submodules is a phantom in the sense just defined.

3. Given an $A$-phantom $H$ of $M$ and a subcategory $C$ of $A$, a homomorphism $\varphi : H \to M$ is called an effective $C$-phantom of $M$, provided that $H$ is a direct limit of objects in $C$, and every map in $\text{Hom}_\Delta(C, M)$ factors through $\varphi$. In that case, we will also say that $H$ is an effective $C$-phantom of $M$.

We add a few comments to set up an intuitive backdrop for the concept of a phantom. The terminology is to evoke a ‘phantom image’ of an object, assembled from witness reports, to aid a search effort. In that spirit, a finitely generated module $H$ is an $A$-phantom of $M$ precisely when there is a witness $C \in A$ testifying to the effect that the source of any homomorphism in $\text{Hom}_\Delta(A, M)$, which permits factorization of all maps in $\text{Hom}_\Delta(A, C, M)$, has an epimorphic image containing $H$.

In case $A$-approximations of $M$ exist, the minimal one, $A$ say, is the only effective $A$-phantom of $M$, and the class of all $A$-phantoms of $M$ coincides with the class of subfactors of $A$. So, in this situation, constructing $A$-phantoms of $M$ amounts to assembling ‘phantom images’ of the module $A$ that one would like to track down; existence is recognized in the process if one can argue that the $K$-dimensions of such phantoms need to be bounded (cf. the existence theorem below). Otherwise, namely when $M$ fails to have $A$-approximations, $A$-phantoms of $M$ still provide minimal building blocks of objects through which all maps
in \( \text{Hom}_A(A, M) \) can be factored. Of course, effective phantoms of \( M \) hold the highest structural interest also in this case: Effective \( C \)-phantoms, where \( C \subseteq A \), take over the role of minimal approximations, in that they carry full complements of information on how \( C \) relates to \( M \).

Note that, for any finite subcategory \( C \) of \( A \), relative \( C \)-approximations of \( M \) in \( A \) exist; to obtain candidates, we only have to add up a sufficient number of copies of the objects in \( C \). Moreover, observe that, given a subclass \( D \subseteq C \), every relative \( C \)-approximation of \( M \) in \( A \) is also a relative \( D \)-approximation. Hence calling for an object \( C \) in \( A \) such that \( H \) is a subfactor of \( \text{every} \) relative \( C \)-approximation of \( M \) in \( A \) – as we do in the definition of a finitely generated phantom – places strong ‘minimality pressure’ on \( H \). On the other hand, the class of \( A \)-phantoms of \( M \) is closed under subfactors and direct limits of directed systems, which often makes it enormous. This slack in the definition facilitates the search for phantoms and thus makes them an expedient tool in proving failure of contravariant finiteness: Indeed, the existence of \( A \)-phantoms of unbounded lengths of a given module \( M \) signals non-existence of an \( A \)-approximation of \( M \). We conclude this sketch with the following existence result.

**Theorem.** (see [18]) For \( M \) in \( \Delta \text{-mod} \) and \( A \subseteq \Delta \text{-mod} \) a subcategory as above, the following conditions are equivalent:

1. \( M \) fails to have an \( A \)-approximation.
2. \( M \) has \( A \)-phantoms of arbitrarily high finite \( K \)-dimensions.
3. \( M \) has an \( A \)-phantom of infinite \( K \)-dimension.
4. There exists a countable subclass \( C \subseteq A \) such that \( M \) has an effective \( C \)-phantom of infinite \( K \)-dimension. \( \square \)

For a more extensive overview of contravariant finiteness results, we refer the reader to [22].

5. **Statement of the main result, consequences, and examples**

Given a generalized word \( w = (p_i^{-1}q_i)_{i \in \mathbb{Z}} \), which is either trivial or has the property that \( \text{length}(p_0) + \text{length}(q_0) > 0 \), we will refer to the joint starting point of \( p_0 \) and \( q_0 \) as the center of \( w \); denote this center by \( e \). If we wish to emphasize the centered viewpoint, we will also refer to \( w \) as a **generalized word centered at** \( e \). Each generalized word \( w \) centered at \( e \) comes paired with an obvious homomorphism \( \varphi : \text{St}(w) \to \Lambda e/Je \): Indeed, consider the standard presentation

\[
\text{St}(w) = \left( \bigoplus_{i \in \text{supp}(w)} \Lambda e(i) \right) / C
\]

as specified in Section 2; here \( e(i) \) is again the starting vertex of \( p_i, q_i \) for \( i \in \mathbb{Z} \). Let \( x_i \in \text{St}(w) \) be the residue class of \( e(i) \). Then there exists a unique homomorphism \( \varphi : \text{St}(w) \to \Lambda e/Je \) with \( \varphi(x_0) = e + Je \) and \( \varphi(x_i) = 0 \) for \( |i| \geq 1 \); we will refer to it as the canonical map of the centered word \( w \).

Moreover, we call a generalized word \( w \) left periodic, resp. right periodic, in case \( w = \ldots uuww_1 \ldots \), resp. \( w = w_2ww_1 \ldots \), with \( u, v \) either trivial or primitive, and \( w_1, w_2 \) left, resp. right, finite. (On the side, we point out that the set of those left and right periodic words which are twosided infinite coincides with the union of the ‘periodic’ and ‘biperiodic Z-words’ introduced by Ringel in [30], while our left finite and right periodic words are ‘periodic’ or ‘almost periodic N-words’ in Ringel’s terminology.)

Finally, we recall that \( \mathcal{S}^{<\infty}(\Lambda \text{-mod}) \) denotes the full subcategory of \( \mathcal{P}^{<\infty}(\Lambda \text{-mod}) \) having as objects all finite direct sums of string modules of finite projective dimension.

We are now in a position to state our main theorem.

**Theorem 5.** As before, let \( \Lambda = \mathcal{K} \Gamma / I \) be a string algebra with simple left modules \( S_i = \Lambda e_i/Je_i, \ 1 \leq i \leq n \). Then there exist centered generalized words \( w_i = w(S_i) \), unique up to inversion, with the following properties:

(I) Each \( w_i \) is centered at \( e_i \), left and right periodic, and can be effectively constructed from \( \Gamma \) and \( I \) in a number of steps which grows polynomially with \( \dim_K \Lambda \).

(II) Each of the generalized string modules \( \text{St}(w_i) \) has finite projective dimension, and the canonical map \( \varphi_i : \text{St}(w_i) \to S_i \) is an effective \( \mathcal{S}^{<\infty}(\Lambda \text{-mod}) \)-phantom of \( S_i \).

(III) The category \( \mathcal{P}^{<\infty}(\Lambda \text{-mod}) \) is contravariantly finite in \( \Lambda \text{-mod} \) if and only if the words \( w_1, \ldots, w_n \) are all finite. In the positive case, the canonical maps \( \varphi_i : \text{St}(w_i) \to S_i \) are the minimal \( \mathcal{P}^{<\infty}(\Lambda \text{-mod}) \)-approximations of the simple modules. More precisely, for each \( i \in \{1, \ldots, n\} \), the following conditions are equivalent:

(i) \( w_i \) is finite.
(ii) $S_i$ has a $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximation.
(iii) $\varphi_i : \text{St}(w_i) \to S_i$ is the minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximation of $S_i$.
(iv) $S_i$ has an $S^{<\infty}(\Lambda\text{-mod})$-approximation.
(v) $\varphi_i : \text{St}(w_i) \to S_i$ is the minimal $S^{<\infty}(\Lambda\text{-mod})$-approximation of $S_i$.

Finally, if the equivalent conditions (i) – (v) are satisfied, the top of $\text{St}(w_i)$ has dimension at most $4n$.

A proof will be given in Section 7. As will be further substantiated when we describe and explore the generalized words $w_i$ of Theorem 5, they encode essentially all of the information required to understand the homology of $\Lambda$. Without providing an explicit algorithm, we mention that, in case of contravariant finiteness of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ in $\Lambda\text{-mod}$, the minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximations of arbitrary string and band modules can readily be constructed from the $\text{St}(w_i)$.

Theorem 5 remains unaffected if we replace the words $w_i$ by their inverses; indeed, $\text{St}(w) \cong \text{St}(w^{-1})$ for any generalized word $w$, and if $w$ is centered at $e$, the obvious isomorphism, namely the flip about the center, preserves the center and takes the canonical map $\text{St}(w) \to \Lambda e / Je$ to the canonical map $\text{St}(w^{-1}) \to \Lambda e / Je$. Consequently, we will invert whenever convenient. Note, moreover, that Theorem 5 guarantees uniqueness of any of the string modules $\text{St}(w_i)$ up to isomorphism, whenever $w_i$ is finite. On the other hand, the isomorphism type of $\text{St}(w_i)$ may depend on the coordinatization in general, a fact which reflects the dependence of $S^{<\infty}(\Lambda\text{-mod})$ on the coordinatization; see Section 8, Example 23. Yet, this failure of uniqueness is ‘minor’, in that the graph of $\text{St}(w_i)$, minus the labeling of the edges, is invariant up to a flip about the central axis; this is a consequence of Proposition 16 below. In categorical terms, if $w_1, \ldots, w_n$ and $w'_1, \ldots, w'_n$ are centered words having the properties described in Theorem 5 relative to two eligible coordinatizations of our string algebra $\Lambda$, there exists a Morita self-equivalence $F : \Lambda\text{-Mod} \to \Lambda\text{-Mod}$ such that $\text{St}(w'_i) \cong F(\text{St}(w_i))$ for all $i$, and $F$ carries the canonical epimorphisms $\varphi_i : \text{St}(w_i) \to S_i$ to those of the centered words $w'_i$.

We conclude the section with two immediate consequences of the main theorem, followed by examples showing that neither can be extended to arbitrary special biserial algebras.

**Corollary 6.** Suppose that $\Lambda$ is a string algebra. If $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is contravariantly finite, the minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximations of the simple left $\Lambda$-modules are string modules. In particular, they are indecomposable. □

While, under the hypothesis of the corollary, the category of all finite direct sums of string modules is always closed with respect to minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximations (we do not include a proof for this fact), indecomposability is not preserved in passing from a string module to its minimal approximation in general.

Actually, the property of being a string module may appear somewhat artificial, since it usually depends on the coordinatization of $\Lambda$. However, in light of Theorems 3 and 5, it becomes apparent that the concepts of ‘string’ and ‘band’ are more than devices permitting an explicit classification of the finitely generated indecomposable representations from quiver and relations of $\Lambda$. Indeed, we see that the class of all string modules determines (irrespective of the chosen coordinatization) the homological properties of the category $\Lambda\text{-mod}$.

Since the base field $K$ does not enter into the structure of string modules and their syzygies, this, in turn, guarantees that the homology of string algebras is a purely combinatorial game which is governed by the graphs of the indecomposable projective modules alone. In particular, the various homological dimensions of a string algebra $\Lambda = K\Gamma / I$ are completely determined by the quiver $\Gamma$ and any set of paths generating the ideal $I$. To see that this does not extend to arbitrary monomial relation algebras, compare, e.g., [21].

**Corollary 7.** The class of string modules determines the homological dimensions of a string algebra $\Lambda = K\Gamma / I$, as well as the contravariant finiteness status of the subcategory $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$. In particular, these data depend only on the quiver $\Gamma$ and the paths in $I$, not on the base field. □

Both corollaries fail for special biserial algebras in general.

**Example 8.** We present a finite dimensional special biserial algebra $\Delta$ and a simple $\Delta$-module $S_1$ whose $\mathcal{P}^{<\infty}(\Delta\text{-mod})$-approximation splits into three nontrivial summands. Suppose that $\Delta$ is a path algebra modulo relations over $K$ with $\dim_K \Delta / J(\Delta) = 8$ such that the indecomposable projective left modules have the following graphs:
Clearly, $\Delta$ is a special biserial algebra. Moreover, it is not difficult to check that the minimal $P^{<\infty}(\Delta\text{-mod})$-approximation of $S_1$ is the module with graph

![Diagram](image)

**Example 9.** For the following finite dimensional special biserial algebra $\Delta$, the category $S^{<\infty}(\Delta\text{-mod})$ has finite representation type and is thus contravariantly finite – in particular, all simple modules have $S^{<\infty}(\Lambda\text{-mod})$-approximations – whereas $P^{<\infty}(\Delta\text{-mod})$ fails to be contravariantly finite in $\Delta\text{-mod}$. Let $\Delta$ be a path algebra with relations such that the graphs of the indecomposable projective left $\Delta$-modules are

![Diagram](image)

It is straightforward (if somewhat tedious) to check that the only finitely generated string modules of finite projective dimension in $\Delta\text{-mod}$ are the indecomposable projective modules $\Delta e_6$ through $\Delta e_{13}$, and the module with graph

![Diagram](image). This shows that $S^{<\infty}(\Delta\text{-mod})$ has finite type. Note, in particular, that there are no string modules of finite projective dimension which have a copy of $S_1$ in their tops, whence the zero map $0 \to S_1$ is the minimal $S^{<\infty}(\Lambda\text{-mod})$-approximation of $S_1$. On the other hand, $S_1$ fails to have a $P^{<\infty}(\Delta\text{-mod})$-approximation; indeed, since each band module $\text{Bd}(v,\phi)$, where $v$ is the primitive word

![Diagram](image), belongs to $P^{<\infty}(\Delta\text{-mod})$, this amounts to a routine check with the aid of [26] for instance. □

6. **The characteristic words of the simple modules**

A *segment* of a generalized word $w = (p_i^{-1} q_i)_{i \in \mathbb{Z}}$ will be any subword of $w$, where we consider the syllables $p_i^{-1}$ and $q_i$ as indivisible building blocks of $w$. So, alternately expressed, the segments of $w$ are just the connected components of syllables of $w$. The segments $q_0 p_1^{-1} q_1 \ldots$ and $\ldots p_{-1}^{-1} q_{-1} p_{0}^{-1}$ will be called the *principal segments* of $w$; often, we will refer to the former as the *principal right segment* of $w$, and to the latter as the *principal left segment* for ease of optical reference, even though the distinction of sides is not substantive (cf. the remarks following Theorem 5).
Moreover, it will be convenient to say that a generalized word $w$ has finite projective dimension if the corresponding string module $St(w)$ has this property.

Given a simple module $S \in \Lambda$-mod, we will construct the (generalized) word $w = w(S)$ postulated in Theorem 5 as the limit of a sequence of successively growing segments of words of finite projective dimension. In doing so, we will observe that this process turns periodic after the construction of (at most) $4n + 1$ pairs of syllables, where $n$ is the $K$-dimension of $\Lambda/J$. Clearly, segments of words of finite projective dimension may fail to inherit this property. However, we do have

**Observations 10.** Let $w$ be a nontrivial finite word.

1. If $w$ is a segment of a generalized word of finite projective dimension, then $w$ is a segment of a finite word of finite projective dimension.

2. If $w$ is primitive and the band module $Bd(w^r, \phi)$ has finite projective dimension for some $r$ and $\phi$, then, again, $w$ is a segment of a finite word of finite projective dimension.

*Proof.* We prove (2) and leave the similar argument backing part (1) to the reader. So let $w = p_1^{-1}q_1 \cdots p_m^{-1}q_m$ be primitive with nontrivial flanking syllables $p_1$ and $q_m$. As in Proposition 1, we let $p_m$ (resp. $q_1$) be the longest paths in $\Gamma$ such that $p_mp_m$ is a path in $K\Gamma \setminus I$ (resp., such that $q_1q_1$ is a path in $K\Gamma \setminus I$). Invoking Proposition 1, we conclude that the word with graph

![Graph](attachment:image.png)

has finite projective dimension. □

In view of Observations 10, we will not run any risk of ambiguity if we henceforth simply refer to ‘segments of words of finite projective dimension’.

In order to recognize the algorithmic nature of our construction, the reader should be familiar with the computation of projective dimensions of modules with tree graphs over monomial relation algebras (see [19]). We precede the description of the procedure with an easy auxiliary statement spelling out the mechanism of the individual steps. In view of Proposition 1, the proof is straightforward and will be omitted.

The notationally somewhat involved second parts of statements (A) and (B) below, under the heading ‘more detail’, are only relevant for algorithmic purposes and do not impinge on the further development of the theory; the reader only interested in the latter is advised to skip them. Recall that, for any path $p$ of positive length in $K\Gamma \setminus I$, we denote by $p$ the unique longest path with the property that the concatenation $pp$ is still a path in $K\Gamma \setminus I$.

**Observations 11.**

(A) Suppose that $u = q_{-t}p_{-t-1}^{-1} \cdots q_{-1}p_t^{-1}$ with $t \geq 1$ is a segment of a centered word of finite projective dimension; so in particular, $u$ is nontrivial if and only if the segment $p_0^{-1}q_0$ is nontrivial.

Then there exist unique shortest paths $q_i$ and $p_{-i}$ such that $(p_{-i})^{-1}uq_i$ is again a segment of a word of finite projective dimension. In fact, the path $q_i$ depends only on the first arrow of $p_i$ in case the latter path has positive length, and is trivial otherwise; symmetrically, $p_{-i}$ depends only on the first arrow of $q_{-i}$ in case that path has positive length, and is trivial otherwise.

(Note that, in general, the choice $q_i = e(t)$ will be ruled out, since all syllables to the right of a trivial syllable $q_i$ with $i \geq 0$ are required to be trivial by the definition of a centered word. Analogous considerations apply to the left-hand side.)

More detail: If $p_t = p_te(t)$ has positive length and $\Lambda e(t)$ has graph

![Graph](attachment:image.png)
where \( p \) contains \( p_t \) as a right subpath and \( q \) is a path of length \( \geq 0 \), then \( q_t \) can be described as follows: It is the shortest right subpath of \( q \) such that, in writing \( q = q_0q_t \), we either have

- \( \text{St}(p^{-1}q_t) \in \mathcal{P}^\infty(\Lambda\text{-mod}), \) or else
- \( \text{length}(q_t) \geq 1, \) and there exists a path \( r \) of positive length with the property that \( qr^{-1} \) is a word and \( \text{St}(r^{-1}q_t) \) has finite projective dimension; here \( r \) and \( q \) relate to \( r \) and \( q \), respectively, as indicated ahead of the lemma.

The path \( p_{t-1} \) has an analogous description.

(B) Now suppose that \( u = p_{t-1}^{-1}q_{t-1}(t-1) \cdots p_{t-1}^{-1}q_{t-1} \) with \( t \geq 1 \) is a segment of a centered word of finite projective dimension.

Then there exist unique longest paths \( p_t \) and \( q_{t-1} \) such that \( q_{t-1}u(p_t)^{-1} \) is a segment of a word of finite projective dimension. In fact, the path \( p_t \) depends only on the last path of \( q_{t-1} \) in case the latter path has positive length, and is trivial otherwise; symmetrically, \( q_{t-1} \) depends only on the last arrow of \( p_{t-1} \) in case the latter path has positive length, and is trivial otherwise.

More detail: If \( q_{t-1} = \tilde{e}(t)q_{t-1} \) has positive length and the injective hull of \( \Lambda\tilde{e}(t)/\tilde{J}\tilde{e}(t) \) has graph

\[
\begin{array}{c}
\bullet \\
q \\
\downarrow \\
\bullet
\end{array}
\]

where \( q \) contains \( q_{t-1} \) as a left subpath, then \( p_t \) can be described as follows: it is the unique longest nontrivial subpath \( r \) of \( p \) such that

- \( \text{St}(r^{-1}q_{t-1}) \in \mathcal{P}^\infty(\Lambda\text{-mod}), \) if such a path \( r \) exists,
- and trivial otherwise.

Again, the path \( p_{t-1} \) can be described analogously. \( \square \)

In contrast to our usual practice, we will consistently record trivial syllables in the following construction.

12. Construction of the “characteristic word” of a simple module.

Start with a simple left module \( S = \Lambda e/Je \) and construct a generalized word \( w = w(S) \) centered at \( e \) as follows:

**Step 0:** Choose paths \( p_0 \) and \( q_0 \) starting in \( e \) and having minimal length with the property that \( w_0 = p_0^{-1}q_0 \) is a segment of a centered word of finite projective dimension. Note that, up to a swap of roles, \( p_0 \) and \( q_0 \) are uniquely determined by this requirement.

**Step \( t, t \geq 1 \):** Suppose that the centered word \( w_{t-1} = p_{t-1}^{-1}q_{t-1}(t-1) \cdots p_1^{-1}q_{t-1} \) has already been constructed. According to part B of Observations 11, we first find the unique longest paths \( q_{t-1} \) and \( p_t \) with the property that \( q_{t-1}w_{t-1}^{-1}p_1^{-1} \) is a segment of a word of finite projective dimension and, according to part A of Observations 11, we then choose \( p_{t-1} \) and \( q_t \) as the unique shortest paths such that \( w_t = p_1^{-1}q_{t-1}w_{t-1}^{-1}p_1^{-1}q_t \) is a segment of a word of finite projective dimension.

In Step \( 2n \), at the latest, we hit periodicity on both sides. We explain this for the principal right segment, the left-hand side behaving symmetrically. If \( q_{2n} \) has length zero, our claim is trivial, since in that case all further syllables on the right are paths of length zero. So suppose that length\( (q_{2n}) \geq 1 \). Then the principal right segment of \( w_{2n} \), that is, the word \( q_0p_1^{-1}q_1 \cdots p_2^{-1}q_2 \) consists of \( 4n + 1 \) nontrivial syllables and thus gives rise to a string module with socle dimension \( 2n + 1 \), meaning that some simple, say \( \Lambda \tilde{e}/\tilde{J}e \), occurs with multiplicity at least 3 in this socle. Hence at least two of the terminal arrows of the corresponding nontrivial paths \( q_t \) ending in \( \tilde{e} \) coincide, say \( q_e \) and \( q_{k} \) with \( k < l \leq 2n \). Thus Observations 11 guarantee that \( p_{k+1} = p_{l+1}, q_{k+1} = q_{l+1}, p_{k+2} = p_{l+2}, \) and so forth.

This completes the description of the construction.

Even though the left and right periodic centered word \( w = w(S) \) produced by this construction is only determined up to inversion, we will use the definite article in referring to it.

**Definition 13.** Given a simple left \( \Lambda \)-module \( S = \Lambda e/Je \), the centered word \( w = w(S) \) of Construction 12 will be called the characteristic word of \( S \), and the module \( \text{St}(w) \) the characteristic phantom of \( S \).

This terminology is justified by
**Proposition 14.** The characteristic words of the simple left \( \Lambda \)-modules are (up to inversion) the centered words \( w_1, \ldots, w_n \) postulated in Theorem 5.

Note that proving Proposition 14 will, at the same time, establish our main result, Theorem 5. This will be done in the next section. As is clear from their construction, the characteristic words of the simple modules are strongly interconnected. We record this fact in

**Remark 15.** Let \( w = (p_i^{-1}q_i)_{i \in \mathbb{Z}} \) be the characteristic word of a simple left \( \Lambda \)-module \( S \) and \( e(i) \) the coinciding starting point of the paths \( p_i \) and \( q_i \). If \( i \geq 1 \) and \( p_{i-1} \) is nontrivial, then the segment \( q_i p_{i+1} \ldots \) of \( w \) is, up to re-indexing, one of the principal segments of the characteristic word of \( \Lambda e(i) / Je(i) \). Analogously, if \( i \leq -1 \) and \( q_{i-1} \) is nontrivial, then \( \ldots p_{i-2} p_{i-1} q_{i-1} \) is, up to re-indexing, a principal segment of the characteristic word of \( \Lambda e(i) / Je(i) \).  

The paths arising as syllables of the characteristic words play a pivotal role in the structural makeup of arbitrary modules of finite projective dimension. This is reflected by the following facts.

**Proposition 16.** Let \( w = (p_i^{-1}q_i)_{i \in \mathbb{Z}} \) be the characteristic word of \( S = \Lambda e / Je \), and again denote the joint starting point of \( p_i \) and \( q_i \) by \( e(i) \). Moreover, let \( M \) be any object of \( \mathcal{P}^{<\infty}(\Lambda \text{-mod}) \).

(A) Suppose that \( x \) is a top element of \( M \). If \( x \) is of type \( e = e(0) \), then \( p_0 x \neq 0 \) and \( q_0 x \neq 0 \). If \( x \) is of type \( e(i) \) for some \( i \geq 1 \), then \( q_i x \neq 0 \), and if \( x \) is of type \( e(i) \) for some \( i \leq -1 \), then \( p_i x \neq 0 \).

(B) Now suppose that the path \( p_i \) is nontrivial and \( x = e(i) x \) for some \( i \geq 1 \) is an element of \( M \) with the property that \( p_i x \) is a nonzero element of \( \text{soc} M \cap q_{i-1} M \). Then \( x \) is a top element of \( M \). The same conclusion holds if \( i \leq -1 \), the path \( q_i \) is nontrivial, and \( q_i x \) is a nonzero element of \( \text{soc} M \cap p_{i+1} M \).

**Proof.** We may clearly restrict our attention to the case where \( M \) is a string or a band module. In the former case, our assertions are immediate consequences of our choices of the \( p_i \) and \( q_i \). So suppose that \( M \) is a band module based on a primitive word \( v \). By Observation 10(2), finite projective dimension of \( M \) forces \( v \) to be a segment of a word of finite projective dimension, whence our claims again follow from Observations 11 and Construction 12. \( \square \)

We conclude this section with examples demonstrating that all theoretically possible scenarios actually occur: left and right termination of a characteristic word \( w \), meaning that, for \( i \gg 0 \), the syllables \( p_i \), \( q_i \) and \( p_{-i} \), \( q_{-i} \) are primitive idempotents (recall that, in view of Theorem 5 and Proposition 14, this occurs precisely when the corresponding simple module \( S = \Lambda e / Je \) has a \( \mathcal{P}^{<\infty}(\Lambda \text{-mod}) \)-approximation); onesided termination of \( w \); non-trivial left and right periodicity;

\[
w = \ldots uu u \ast \ast \ast \ast \ast v v v \ldots,
\]

with primitive words \( u \) and \( v \) which are devoid of common syllables; and periodicity in the strongest possible sense, i.e., \( w = \ldots vv v \ldots \), where \( v \) is a primitive word.

**Example 17.** Let \( \Gamma \) be the quiver

![Quiver Diagram](image-url)

and \( \Lambda = K\Gamma / I \), where the ideal \( I \subseteq K\Gamma \) is chosen so that the indecomposable projective left \( \Lambda \)-modules have graphs
Then $\Lambda$ is a string algebra with simple left modules $S_i = \Lambda e_i / Je_i$, $1 \leq i \leq 12$. One readily finds that the characteristic phantoms (in the sense of the above definition – see also Theorem 5) of $S_1$, $S_7$, and $S_8$ are as follows:

In each case, the center is circled.

In particular, we see that the characteristic word $w_1$ of $S_1$ is two-sided infinite with distinct left and right periods, the characteristic word $w_7$ of $S_7$ is of the form $w_7 = \ldots vvv \ldots$, where $v$ is a primitive word, while the characteristic word $w_8$ of $S_8$ is infinite only on one side. Note moreover that the characteristic word of $S_7$ coincides with that of $S_{12}$, while that of $S_4$ results from that of $S_8$ through deletion of the first two syllables.

In view of Theorem 5, $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ fails to be contravariantly finite in $\Lambda\text{-mod}$. The theorem, in fact, supplies more precise information: Namely, the simple modules $S_2$, $S_3$, $S_6$, $S_9$, $S_{10}$, and $S_{11}$ are precisely those having $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximations and, for each of the listed indices $i$, the minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximation of $S_i$ coincides with $\Lambda e_i$. □

**Example 18.** Let $\Gamma'$ be the quiver with 12 vertices resulting from that of Example 17 through deletion of two arrows, those from 8 to 1 and from 12 to 11. We define the string algebra $\Lambda' = K\Gamma'/I'$, where $I' \subseteq K\Gamma'$ is chosen in such a way that the graphs of the indecomposable projective left $\Lambda'$-modules $\Lambda' e_i$, for
$i \in \{1, \ldots, 12\} \setminus \{8, 12\}$ coincide with those of the corresponding $\Lambda e_i$-modules of Example 17, while $\Lambda' e_8$ and $\Lambda' e_{12}$ have graphs

$$
\begin{array}{c}
\text{8} \\
\text{6} \\
\text{9} \\
\text{10} \\
\text{3} \\
\end{array} 
\quad \quad \quad 
\begin{array}{c}
\text{12} \\
\text{5} \\
\text{3} \\
\text{9} \\
\end{array}
$$

respectively. Then the characteristic phantoms of the simple left $\Lambda'$-modules are as follows, the centers being again highlighted:

Thus Theorem 5 tells us that $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is contravariantly finite in $\Lambda\text{-mod}$, and that the displayed characteristic phantoms are the minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximations of the simple modules corresponding to the centers. □

7. **Proof of the main result**

Our plan is to establish Theorem 5 by way of proving Proposition 14.

Throughout this section, we fix a simple left module $S = \Lambda e/J e$ with $e \in \{e_1, \ldots, e_n\}$ and let $w = w(S) = (p_i^{-1}q_i)_{i \in \mathbb{Z}}$ be the characteristic word of $S$ as described in Construction 12 of Section 6. To cope with the ambiguity arising from the fact that characteristic words are only unique up to inversion, we will, in the sequel, refer to the orientation of this fixed choice of $w$ as *normalized*. Since all parts of the theorem involving the word $w$ are true if the latter is trivial (recall that this occurs precisely when $S$ has finite projective dimension), we will henceforth assume that $w$ is nontrivial, i.e., that at least one of the paths $p_0$, $q_0$ is nontrivial. Relative to the standardized sequence of top elements $(x_i)_{i \in \text{supp}(w)}$ of $\text{St}(w)$, as introduced in Section 2, the string module $\text{St}(w)$ has graph
Here we denote by $e(i)$ the starting point of the path $q_i$ for $i \geq 0$, and by $\hat{e}(i + 1)$ its end point; then $e(i)$ and $\hat{e}(i)$ are also the starting and end points of the $p_i$ for $i \geq 1$, respectively; moreover, $e(0) = e$. The principal left segment of $w$ is labeled similarly, as shown in the graph of $St(w)$. Contrasting our usual convention, we have marked some of the edges by wiggly, as opposed to straight, lines to emphasize the following difference in roles: the straight edges indicate paths chosen as short as possible without forfeiting finite projective dimension of $St(w)$, whereas the paths represented by wiggly edges are chosen as long as possible under this restriction. As in the previous sections, $\varphi : St(w) \rightarrow S$ denotes the canonical map which sends $x_0$ to $e + Je$, while sending the other $x_i$ to zero.

We smooth the road towards a proof of Theorem 5 with a final definition and a few auxiliary facts.

### Definition 19
Two nontrivial centered words $\hat{w} = (\hat{p}_i^{-1}\hat{q}_i)_{i \in \mathbb{Z}}$ and $\hat{w} = (\hat{p}_i^{-1}\hat{q}_i)_{i \in \mathbb{Z}}$ are said to have the same orientation if they are centered in the same primitive idempotent and either $\hat{p}_0$ and $\hat{p}_0$ have the same first arrow, or else $\hat{q}_0$ and $\hat{q}_0$ have the same first arrow. (Note that, if all of the paths $\hat{p}_0$, $\hat{p}_0$, $\hat{q}_0$, $\hat{q}_0$ are nontrivial, the condition that $\hat{w}$ and $\hat{w}$ have the same orientation is equivalent to the requirement that $\hat{p}_0$ and $\hat{p}_0$, as well as $\hat{q}_0$ and $\hat{q}_0$, share first arrows.)

Let $S$ and $w = (\hat{p}_i^{-1}\hat{q}_i)$ be as fixed at the beginning of this section. If $\hat{w} = (\hat{p}_i^{-1}\hat{q}_i)$ has the same orientation as $w$, we call $\hat{q}_i$ (resp. $\hat{p}_i$) a right discontinuity of $\hat{w}$ relative to $w$ in case $i \geq 0$ and $\hat{q}_i \neq q_i$ (resp., $i \geq 1$ and $\hat{p}_i \neq p_i$); left discontinuities are defined symmetrically. Both types are also briefly called discontinuities of $\hat{w}$.

In the sequel, all discontinuities will be relative to the fixed characteristic word $w$ of $S = \Lambda e / Je$.

### Lemma 20
Let $\hat{v} = \hat{p}_0^{-1}\hat{q}_0 \ldots \hat{p}_i^{-1}\hat{q}_i$ be a primitive word with $t \geq 0$ and all of the listed syllables nontrivial; moreover, suppose that the joint starting point of $\hat{p}$ and $\hat{q}$ equals $e$. Expand $\hat{v}$ to a twosided infinite word $\hat{w} = \ldots \hat{v} \hat{v} \hat{v} \ldots$ centered in $e$ as illustrated by the following graph:

```
   e
   \hat{p}_{-(t+1)} \hat{q}_{-(t+1)} \hat{p}_{-1} \hat{q}_{-1} \hat{p}_0 \hat{q}_0 \hat{p}_1 \hat{q}_1 \hat{p}_t \hat{q}_t \hat{p}_{t+1} \hat{q}_{t+1} e
```

where $\hat{p}_i = \hat{p}_j$ and $\hat{q}_i = \hat{q}_j$ whenever $i$ is congruent to $j$ modulo $t + 1$. Finally, suppose that $\hat{w}$ has the same orientation as $w$.

If $\hat{w}$ has a right discontinuity, then the first right discontinuity of $\hat{w}$ is among the paths $\hat{q}_0, \ldots, \hat{q}_t, \hat{p}_1, \ldots, \hat{p}_{t+1}$. Similarly, if $\hat{w}$ has a left discontinuity, then the first such is among $\hat{p}_0, \ldots, \hat{p}_{-t}, \hat{q}_{-1}, \ldots, \hat{q}_{-(t+1)}$.

Now suppose that $\hat{w}$ is a word of finite projective dimension. If $\hat{w}$ has a right discontinuity, and the first right discontinuity is $\hat{q}_i$ for some $i$, then the path $\hat{q}_i$ contains the path $q_i$ as a proper right subpath; if, on the other hand, the first right discontinuity of $\hat{w}$ is $\hat{p}_i$, the latter path is contained in the path $p_i$ as a proper left subpath. In case of existence, the first left discontinuity of $\hat{w}$ is subject to mirror-symmetric conditions.

**Proof.** To verify the first assertion, suppose that $\hat{q}_i = q_i$ for $0 \leq i \leq t$ and $\hat{p}_i = p_i$ for $1 \leq i \leq t + 1$. Then the paths $q_0$ and $p_{t+1}$ starting in $e$ are both nontrivial, and the first arrow of $p_{t+1}$ differs from the first arrow of $q_0 = q_0 = q_{t+1}$. By Construction 12, this implies that $q_{t+1} = q_0$, i.e., $\hat{q}_{t+1} = q_{t+1}$, and further that...
where neither of the paths \( \hat{g} \geq \) discontinuities. and \( 1 \leq w \) relative to a standardized sequence \( \hat{w} \) as proper right subpaths, respectively; hence \( \hat{p} \) and \( \hat{q} \) are immediate consequence of the construction of \( w \) (Section 6, Construction 12). \( \square \)

In the proof of the next lemma, it will turn out handy that every homomorphism from a pseudo-band module \( \text{Bd}(v^r, \phi) \) to \( S \) factors through an ‘expanded’ pseudo-band module \( \text{Bd}(v^{2r}, \psi) \).

**Remark 21.** Since \( K \) is an infinite field, any pseudo-band module \( \text{Bd}(v^r, \phi) \) is contained as a direct summand in a pseudo-band module \( \text{Bd}(v^s, \psi) \), for any integer \( s \geq r \).

**Lemma 22.** Let \( \hat{v} \) and \( \hat{w} \) be as in the blanket hypothesis of Lemma 20 and retain all of the notation introduced there. Moreover, suppose that \( \hat{w} \) is a word of finite projective dimension having both right and left discontinuities.

Then, given any pseudo-band module \( B = \text{Bd}(\hat{v}^r, \phi) \), say with graph

```
\[
\begin{array}{c}
y_{10} & y_{11} & y_{1t} & y_{20} & y_{r-1, t} & y_{r0} & y_{rt} \\
e & \hat{e}(1) & \hat{e}(t) & e & \hat{e}(t) & e & \hat{e}(t) \\
p_0 & \hat{q}_0 & p_1 & \hat{q}_1 & p_0 & \hat{q}_1 & p_0 \hat{q}_1 p_t \hat{q}_t \\
\end{array}
\]
```

relative to a standardized sequence \( y_{10}, \ldots, y_{rt} \) of top elements, the homomorphism \( f : B \to S \) sending \( y_{10} \) to \( e + Je \) and the other \( y_{ij} \) to zero factors through the canonical map \( \varphi : \text{St}(w) \to S \).

**Proof.** We start by formalizing the graphical information provided: \( \hat{q}_j y_{ij} = \hat{p}_{j+1} y_{i,j+1} \) for \( j < t \), \( \hat{q}_i y_{it} = \hat{p}_{i+1} \hat{q}_{i+1} \) when \( i < r \), and \( \hat{q}_t y_{rt} = \sum_{i=1}^r c_i \hat{p}_0 \hat{q}_0 \), where

\[
\begin{pmatrix}
0 & \cdots & 0 & c_1 \\
1 & \ddots & \vdots \\
\vdots & \ddots & 0 & \vdots \\
0 & \cdots & 1 & c_r
\end{pmatrix}
\]

is the Frobenius companion matrix of the cyclic automorphism \( \phi \).

To facilitate visualization, we once more give the graph of the word \( \hat{w} = \ldots \hat{v} \hat{v} \hat{v} \ldots \),

where again \( \hat{p}_i = \hat{p}_j \) and \( \hat{q}_i = \hat{q}_j \) whenever \( i \equiv j \) \((\text{mod } t + 1)\). The word \( \hat{w} \) having the same orientation as \( w \), Lemma 20 guarantees that the first right discontinuity of \( \hat{w} \) is among the paths \( q_i, p_j \), with \( 0 \leq i \leq t \) and \( 1 \leq j \leq t + 1 \), and the first left discontinuity of \( \hat{w} \) is among the paths \( p_i, q_j \) with \( 0 \geq i \geq -t \) and \( 1 \geq j \geq -(t + 1) \). In factoring the homomorphism \( f \) through \( \varphi \), we will separately deal with the cases where neither of the paths \( \hat{p}_0, \hat{q}_0 \) is a discontinuity of \( \hat{w} \), where one of them is, and where both of them are discontinuities.

The last-mentioned case is immediate: Namely, we define \( g \in \text{Hom}_\Lambda(B, \text{St}(w)) \) by setting \( g(y_{10}) = x_0 \) and \( g(y_{ij}) = 0 \) for \( (i, j) \neq (1, 0) \). This is legitimate, since, by Lemma 20, the paths \( \hat{p}_0 \) and \( \hat{q}_0 \) contain \( p_0 \) and \( q_0 \) as proper right subpaths, respectively; hence \( \hat{p}_0 x_0 = \hat{q}_0 x_0 = 0 \), where \( (x_i)_{i \in \mathbb{Z}} \) is the standardized sequence of top elements of \( \text{St}(w) \) displayed at the beginning of the section.
Case A. One of \( \hat{p}_0 \), \( \hat{q}_0 \) is a discontinuity of \( \hat{w} \), but not the other; for symmetry reasons, it is harmless to assume that \( \hat{p}_0 \) is a (left) discontinuity. Lemma 20 tells us that \( \hat{p}_0 \) contains \( p_0 \) as a proper right subpath, and again we infer \( \hat{p}_0 x_0 = 0 \).

Subcase A.1. \( \hat{p}_{t+1} \) is the first right discontinuity of \( \hat{w} \).

In this case the segment of the word \( w \) relevant to our construction has the form

\[
\begin{align*}
p_{t+1} &= \hat{p}_{t+1} \nu = \hat{p}_0 \nu \quad \text{for a nontrivial path } \nu, \quad q_i = \hat{q}_i \quad \text{for } 0 \leq i < t \quad \text{and } p_i = \hat{p}_i \quad \text{for } 1 \leq i \leq t. \\
\end{align*}
\]

Define \( g \in \text{Hom}_A(B, \text{St}(w)) \) as follows, keeping in mind that \( c_1 \neq 0 \) because \( \phi \) is an automorphism of \( K^+ \): Namely, let \( g(y_{10}) = x_0 - (c_2/c_1)\nu x_{t+1} \), \( g(y_{1j}) = x_j \) for \( 1 \leq j \leq t \), \( g(y_{20}) = \nu x_{t+1} \), and \( g(y_{ij}) = 0 \) for \( 2 \leq i \leq r \) and \( 1 \leq j \leq t \).

Subcase A.2. The first right discontinuity of \( \hat{w} \) is \( \hat{p}_t \) or \( \hat{q}_t \) with \( 1 \leq t < l \).

In that case, we have \( p_t = \hat{p}_t \nu \), where \( \nu \) is a path of length \( \geq 0 \), and \( \hat{q}_t = \sigma q_t \) with length(\( \sigma \)) \( \geq 0 \), where either length(\( \nu \)) \( < 0 \) or length(\( \sigma \)) \( > 0 \), by Lemma 20. In either case we obtain \( \hat{q}_t \nu x_i = 0 \), and the following assignments give rise to a well-defined homomorphism \( g \in \text{Hom}_A(B, \text{St}(w)) \) with \( \varphi g = f \).

Lemma 20 guarantees that either \( \hat{p}_{t-1} = \hat{p}_{t-k} \) contains \( p_{t-1} \) as a proper right subpath, or else \( \hat{q}_{t-1} = \hat{q}_{t-k} \) is a proper left subpath of \( q_{t-1} \). In either case we can write the latter path in the form \( q_{t-1} = \hat{q}_{t-k} \nu \), where \( \nu \) is a path of length \( \geq 0 \) satisfying the equalities \( \hat{p}_{t-1} \nu x_{(t-1)} = \hat{p}_{t-k} \nu x_{(t-1)} = 0 \). Analogously, \( p_t = \hat{p}_t \nu \), where \( \nu \) is a path of length \( \geq 0 \) such that \( \hat{q}_t \nu x_i = 0 \). The segment of the characteristic word \( w \) which is decisive for our construction is depicted in the following diagram:

\[
\begin{align*}
x_{-t+1} &\quad x_{-k} &\quad x_0 &\quad x_{t-1} &\quad x_t \\
\end{align*}
\]

Here \( p_i = \hat{p}_i \) for \( -k \leq i \leq l - 1 \) and \( q_i = \hat{q}_i \) for \( -k \leq i \leq l - 1 \). Moreover, keep in mind that \( \hat{p}_t = \hat{p}_j \) and \( \hat{q}_t = \hat{q}_j \) whenever \( i \equiv j \pmod{t+1} \). Remark 21 permits us to assume \( r \geq 3 \). Referring to the above graph of \( B \) and distinguishing between the cases where \( l \leq t \) and \( l = t + 1 \), we can thus define a map \( g : \text{Bd}(w', \phi) \to \text{St}(w) \) as follows:

If \( l \leq t \), we set \( g(y_{10}) = x_0, g(y_{1j}) = x_j \) for \( 1 \leq j \leq t - 1, g(y_{1l}) = \nu x_1, g(y_{2j}) = 0 \) for \( l + 1 \leq j \leq t \), \( g(y_{2j}) = 0 \) for \( 2 \leq j \leq r - 1 \) and all \( j, g(y_{2r}) = 0 \) for \( 0 \leq j \leq t - k - 1 \) (note that this latter range is empty if \( k = t \), \( g(y_{t-k}) = c_1 \mu x_{(t-1)} \), and \( g(y_{t}) = c_1 x_{(t-1)-1} \) for \( t - k + 1 \leq j \leq t \) (this range being empty for \( k = 0 \)). The verification of the fact that \( g \) extends to a well-defined homomorphism \( B \to \text{St}(w) \) is a bit tedious – various possibilities for \( k \) need to be considered separately – and we leave it to the reader to fill in the pertinent computations.

In case \( l = t + 1 \), we can either play the situation back to one of the previously considered cases by relabeling, or else define \( g \) directly as follows: \( g(y_{10}) = x_0 - (c_2/c_1)\nu x_{t+1}, g(y_{1j}) = x_j \) for \( 1 \leq j \leq t \), and
\(g(y_{20}) = \nu x_{t+1}\); for the pairs \((i,j)\) lexicographically larger than \((2,0)\), we keep the above specifications of \(g(y_{ij})\). Again one checks that these definitions give rise to a map \(g \in \text{Hom}_A(B, \text{St}(w))\).

Since clearly \(\varphi g = f\) in either case, the proof of the lemma is complete. \(\square\)

**Lemma 23.** Let \(t \geq 0\), and suppose that the syllables \(q_0, \ldots, q_t, p_1^{-1}, \ldots, p_{t+1}^{-1}\) of the characteristic word \(w = w(S)\) are nontrivial, yielding a primitive word \(v = p_{t+1}^{-1}q_0p_1^{-1}q_1 \cdots p_{t}^{-1}q_t\) with graph

\[
\begin{array}{c}
e(0) \\
\bullet \\
p_{t+1} \\
\downarrow \downarrow \downarrow \downarrow \\
q_0 \bullet \\
p \bullet \\
q_1 \bullet \\
p \bullet \\
q_t \bullet \\
\vdots \bullet \\
\vdots \bullet \\
\vdots \bullet \\
en(t) \\
\bullet \\
\end{array}
\]

Moreover, suppose that \(e(0) = e\) and that the principal right segment \(w_{\text{right}}\) of \(w\) is periodic of the form \(w_{\text{right}} = (q_0p_1^{-1} \cdots q_{t}p_{t+1}^{-1})(q_0p_1^{-1} \cdots q_{t}p_{t+1}^{-1})\); in other words,

\[
p_{t+1}^{-1}w_{\text{right}} = vv\ldots.
\]

Then the direct sums

\[
\left( \bigoplus_{i=0}^{t} \Lambda e(i)/Je(i) \right)^{(N)}
\]

are \(P^{<\infty}(\Lambda\text{-mod})\)-phantoms and \(S^{<\infty}(\Lambda\text{-mod})\)-phantoms of \(S\). In particular, \(S\) has neither a \(P^{<\infty}(\Lambda\text{-mod})\)-nor an \(S^{<\infty}(\Lambda\text{-mod})\)-approximation.

Moreover, any effective \(P^{<\infty}(\Lambda\text{-mod})\)- or \(S^{<\infty}(\Lambda\text{-mod})\)-phantom of \(S\) has a copy of \(\left( \bigoplus_{i=0}^{t} \Lambda e(i)/Je(i) \right)^{(N)}\) in its top and a copy of \(\left( \bigoplus_{i=0}^{t} \Lambda \tilde{e}(i)/J\tilde{e}(i) \right)^{(N)}\) in its socle.

**Proof.** We will simultaneously verify that the above semisimple modules are \(P^{<\infty}(\Lambda\text{-mod})\)- and \(S^{<\infty}(\Lambda\text{-mod})\)-phantoms of \(S\), as all of our test maps will have sources in \(S^{<\infty}(\Lambda\text{-mod})\).

We set \(\tilde{p}_{t+1} = p_{t+1}p_{t+1}^{-1}\) and \(\tilde{q}_t = q_tq_t^{-1}\), where \(p_{t+1}\) and \(q_t\) are as in Proposition 1; the path \(p_{t+1}\) being nontrivial, this means that \(p_{t+1}\) is the longest path such that \(\tilde{p}_{t+1}\) is a path in \(KT\setminus I\); the path \(\tilde{q}_t\) has an analogous description. Moreover, we consider, for each positive integer \(m \geq 2\), the word \(u_m = \tilde{p}_{t+1}q_0 \cdots p_t^{-1}q_{t+m-2}^{-1}p_{t+1}q_0 \cdots p_t^{-1}q_t^{-1}\). Then \(u_m\) is a word of finite projective dimension and \(\text{St}(u_m)\) has graph relative to a standardized sequence of top elements \(y_{ij}\). We equip the set of pairs \((i,j)\) for \(1 \leq i \leq m\) and \(0 \leq j \leq t\) with the lexicographic order and let \(f : \text{St}(u_m) \rightarrow S\) be the homomorphism with \(f(y_{10}) = e + Je\) and \(f(y_{ij}) = 0\) for \((i,j) > (1,0)\). Next, we will check that any module \(A \in P^{<\infty}(\Lambda\text{-mod})\) with the property that \(f\) factors through a map \(\rho \in \text{Hom}_A(A, S)\) contains \((\Lambda \tilde{e}(j)/J\tilde{e}(j))^{m-1}\) in its socle and \((\Lambda e(j)/Je(j))^{m-1}\) in its top, for \(j \in \{0, \ldots, t\}\). This will prove all of our claims.

For that purpose, we let \(h \in \text{Hom}_A(\text{St}(u_m), A)\) be such that \(ph = f\). In a first step, we prove, more precisely, that the elements \(q_jh(y_{ij})\), \(1 \leq i \leq m-1\), of \(A\) are \(K\)-linearly independent for all \(j\). Clearly, these elements belong to the socle of \(A\) since the \(q_jy_{ij}\)'s belong to the socle of \(\text{St}(u_m)\) for \(i \leq m-1\). Assume, to the contrary of our claim, that the \(q_jh(y_{ij})'s\), \(i \leq m-1\), are linearly dependent, and let the pair \((k,l)\) be (lexicographically) minimal with the property that there exists an equality \(\sum_{i=k}^{m-1} a_iq_ih(y_{il}) = 0\) for scalars \(a_i \in K\) with \(a_k \neq 0\). Set \(y = \sum_{i=k}^{m-1} a_iy_{il} \in \text{St}(u_m)\). First we observe that \((k,l) \neq (1,0)\), for otherwise
we would have \( ph(y) = ak e + Je \neq 0 \), which would make \( h(y) \) a top element of \( A \), and hence guarantee that \( q_i h(y) \neq 0 \) by Proposition 16(A); but the latter is inconsistent with our choice of \( y \), and therefore we conclude \( (k, l) \neq (1, 0) \). This permits us to define an element \( z \in St(u_m) \) as follows: If \( l = 0 \), we have \( k \geq 2 \), which legitimizes the definition \( z = \sum_{i=k}^{m-1} a_i y_{i-1} \); if, on the other hand, \( l \geq 1 \), we set \( z = \sum_{i=k}^{m-1} a_i y_{i-1} \). We note that, in either case, the nonzero scalar \( a_k \) accompanies an element \( y_{ij} \) with \( (i, j) < (k, l) \). Hence, in case \( l = 0 \), the minimal choice of \( (k, l) \) ensures that \( p_{t+1} h(y) = q_i h(z) = \sum_{i=k}^{m-1} a_i q_i h(y_{i-1}) \neq 0 \), while, in case \( l \geq 1 \), that choice yields \( p_{i+1} h(y) = q_i h(z) = \sum_{i=k}^{m-1} a_i q_{i-1} h(y_{i-1}) \neq 0 \). In other words, \( p_{t+1} h(y) \) is a nonzero element in soc \( A \cap q_i A \) in the first case, and \( p_{i+1} h(y) \) is a nonzero element in soc \( A \cap q_{i-1} A \) in the second. In either case, our hypothesis combines with part (B) of Proposition 16 to show that \( h(y) \) is a top element of type \( e(l) \) of \( A \), whence our assumption that \( q_i h(y) \) be zero contradicts part (A) of Proposition 16. We thus conclude that \( (\Lambda e(j)/Je(j))^{m-1} \) is contained in the socle of \( A \) as claimed.

We have shown that, for each \( j \in \{0, \ldots, t\} \), all linear combinations \( \sum_{i=1}^{m-1} a_i h(y_{ij}) \) with \( (a_1, \ldots, a_{m-1}) \neq 0 \) are nonzero elements of the socle of \( A \), and invoking again part (B) of Proposition 16 on the model of the preceding paragraph, we infer that all linear combinations \( \sum_{i=1}^{m-1} a_i h(y_{ij}) \) with \( (a_1, \ldots, a_{m-1}) \neq 0 \) are top elements of \( A \). This means that the elements \( h(y_{j2k}), \ldots, h(y_{jnk}) \) are linearly independent modulo \( JA \) and thus yields the required containment of \( (\Lambda e(j)/Je(j))^{m-1} \) in \( A/JA \).

In conclusion, arbitrarily high finite powers of \( \bigoplus_{i=0}^{t} \Lambda e(i)/Je(i) \) and \( \bigoplus_{i=0}^{t} \Lambda e(i)/Je(i) \) are \( \mathcal{P}^{\infty}(\Lambda-\text{mod}) \)-phantoms, as well as \( \mathcal{S}^{\infty}(\Lambda-\text{mod}) \)-phantoms of \( A \), and hence so are their direct limits \( \left( \bigoplus_{i=0}^{t} \Lambda e(i)/Je(i) \right)^{(N)} \) and \( \left( \bigoplus_{i=0}^{t} \Lambda e(i)/Je(i) \right)^{(N)}. \)

Of course the mirror image of Lemma 23 relative to the central axis of \( w \) is also true, since replacing \( w \) by its inverse is harmless. The argument we gave for the lemma actually proves a little more than we stated in our conclusion: Namely, if \( u \) is a segment of \( v \) such that \( St(u) \) has square-free socle (this is for instance true for any syllable \( u \) of \( v \)), then \( (St(u))^{(N)} \) is a \( \mathcal{P}^{\infty}(\Lambda-\text{mod}) \)-phantom of \( S \). The part of the lemma that addresses \( \mathcal{S}^{\infty}(\Lambda-\text{mod}) \) will be superseded by the stronger assertion of Theorem 5 that \( St(w) \) is an \( \mathcal{S}^{\infty}(\Lambda-\text{mod}) \)-phantom of \( S \).

**Proof of Theorem 5 via Proposition 14.** Let \( e \) be one of the primitive idempotents \( e_1, \ldots, e_n \), and again denote the characteristic word of \( S = \Lambda e/Je \) by \( w = (p_i^{-1} q_i)_{i \in Z} \).

(I) The word \( w \) is centered at \( e \) by construction. For left and right periodicity of \( w \), we refer to the final paragraph of Construction 12. To justify the upper bound on the number of steps required to explicitly determine \( w \) from \( \Gamma \) and a set of paths generating \( I \), we recall that the principal right segment of \( w \) is completely determined, once we have constructed its first \( 4n + 1 \) syllables \( q_0, p_1^{-1}, q_1, p_2^{-1}, \ldots, q_{2n} \). The algorithm of [19] for determining the projective dimensions of \( \Lambda \)-modules which are either uniserial or have graphs of the form \( V \), moreover, allows us to find each successive pair \( q_i p_i^{-1} \) of syllables of \( w \) in \( \leq 2(\dim K \Lambda)^3 \) steps.

(II) In light of Proposition 1, finiteness of the projective dimension of \( St(w) \) is an immediate consequence of the construction.

To verify the second part of (II), let once more \( \varphi : St(w) \to S = \Lambda e/Je \) be the canonical map of the centered word \( w \). In reference to the graph of \( St(w) \) at the beginning of Section 7, we thus have \( \varphi(x_i) = \delta_{i0}(e + Je) \) for \( i \in Z \). We first show that every homomorphism \( M \to S \), where \( M \) is an object of \( \mathcal{S}^{\infty}(\Lambda-\text{mod}) \), factors through \( \varphi \); this will prove effectiveness, once we have shown that \( St(w) \) is an \( \mathcal{S}^{\infty}(\Lambda-\text{mod}) \)-phantom of \( S \). It is clearly harmless to assume \( M \) to be indecomposable; in other words, we focus on the situation where \( M \) is a string module of finite projective dimension, based on a finite word \( \hat{w} = p_0^{-1} q_0 \ldots p_n^{-1} \hat{q}_m \) with \( m \geq 0 \), such that the string module \( \hat{M} = St(\hat{w}) \) has a standardized sequence \( y_0, y_1, \ldots, y_m \) of \( m + 1 \) top elements; see Section 2 for our conventions. Then, clearly, those homomorphisms which send any top element \( y_i \) of type \( e \) to \( e + Je \) and all the other \( y_j \) to zero constitute a \( K \)-basis of \( \text{Hom}_\Lambda(St(\hat{w}), S) \), whence we may assume that our map \( f \in \text{Hom}_\Lambda(St(\hat{w}), S) \) is of the latter ilk; so suppose that there exists \( i \in \{1, \ldots, m\} \) such that \( y_i = ey_i \), and let \( f \in \text{Hom}_\Lambda(St(\hat{w}), S) \) be as described. We relabel the syllables of the word \( \hat{w} \) if necessary, to center it in the idempotent \( e \) corresponding to \( y_i \), and to ensure that \( w \) has the same orientation as \( w \). If the standardized sequence of top elements \( y_i \) is re-indexed accordingly, the graph of \( St(\hat{w}) \) takes on the form
for some \( t \geq 0 \). It is clearly innocuous to assume that \( \hat{p}_i \) is nontrivial.

We now construct a map \( g : \text{St}(\hat{w}) \to \text{St}(w) \) by starting with the assignment \( g(y_0) = x_0 \). Next, we inductively define the images \( g(y_j) \) for \( 1 \leq j \leq t \) in such a way that \( g(y_0), \ldots, g(y_t) \) satisfy all of the relations tying the \( \hat{p}_i y_j \) and \( \hat{q}_i y_j \) for \( 0 \leq j \leq t \) together; and finally, we do the same for \( t - m \leq j \leq -1 \). Since the relations of \( \text{St}(\hat{w}) \) can be generated by relations involving at most two consecutive \( y_i \), this will ensure that our assignments induce a well-defined homomorphism \( g \in \text{Hom}_\Lambda(\text{St}(\hat{w}), \text{St}(w)) \).

If none of the nontrivial paths \( \hat{q}_i, \hat{p}_{i+1} \) for \( i \geq 0 \) is a right discontinuity of \( \hat{w} \), the assignments \( g(y_i) = x_i \) for \( 1 \leq i \leq t \) satisfy our requirements. Next suppose that the first right discontinuity of \( \hat{w} \) is some nontrivial path \( \hat{q}_k \) with \( k \geq 0 \). Then the definitions \( g(y_i) = x_i \) for \( 1 \leq i \leq k \) and \( g(y_i) = 0 \) for \( i > k \) are as required, for Lemma 20 tells us that \( q_k \) is a proper right subpath of \( \hat{q}_k \), whence \( \hat{q}_k g(y_k) = 0 \). If, finally, \( \hat{w} \) has a first right discontinuity of the form \( \hat{p}_k \) for some \( k > 0 \), then \( p_k = p_k \nu \) for a nontrivial path \( \nu \), again by Lemma 20. In that case the assignments \( g(y_i) = x_i \) for \( i < k \), \( g(y_k) = \nu x_k \), and \( g(y_i) = 0 \) for \( i > k \) satisfy our demands; indeed, if \( \hat{q}_k \) is trivial, then \( k = t \), and otherwise \( g(y_k) = 0 \) in view of the nontriviality of \( \nu \).

Assignments \( g(y_j) \) for \( t - m \leq j \leq -1 \) which are compatible with the pertinent relations are made symmetrically. This procedure clearly leads to a homomorphism \( g \) with \( \varphi g = f \).

To show that \( \text{St}(w) \) is an \( S^{<\infty}(\Lambda, \text{-mod}) \)-phantom of \( S = \Lambda e/Je \), we consider, for each nonnegative integer \( k \), the centered word

\[
\begin{align*}
    u_k &= \hat{p}_{-(k+1)} q_{-(k+1)} \hat{p}_{-k} q_{-k} \cdots \hat{p}_0 q_0 \cdots \hat{p}_{-1} q_{k+1} \hat{p}_{k+1},
\end{align*}
\]

where \( \hat{p}_{-(k+1)} = p_{-(k+1)} \hat{p}_{-(k+1)} \) and \( \hat{q}_{k+1} = q_{k+1} \hat{q}_{k+1} \) with \( p_{-(k+1)} \) and \( q_{k+1} \) chosen as in Proposition 1, that is, \( p_{-(k+1)} \) and \( q_{k+1} \) are the longest paths in \( KT \setminus I \) containing \( p_{-(k+1)} \) and \( q_{k+1} \) as right subpaths. By construction of \( w = (u_k)_{k \in \mathbb{Z}} \), the assignments \( u_k \) are words of finite projective dimension. As usual, we fix a standardized sequence of top elements \( y_{-(k+1)} \), \( \ldots \), \( y_0 \), \( \ldots \), \( y_{k+1} \) of \( \text{St}(u_k) \), and focus on the canonical map \( \psi_k : \text{St}(u_k) \to S \), i.e., \( \psi_k(y_i) = \delta_{0i}(e + Je) \). We claim that any module \( A \in S^{<\infty}(\Lambda, \text{-mod}) \) with the property that \( \psi_k \) factors through some map in \( \text{Hom}_\Lambda(A, S) \) contains the string module \( \text{St}(p_{-k} q_{-k} \cdots p_{-1} q_{k+1}) \) as a submodule. Once established, this claim will entail that \( \text{St}(p_{-k} q_{-k} \cdots p_{-1} q_{k+1}) \) is an \( S^{<\infty}(\Lambda, \text{-mod}) \)-phantom of \( S \) for each \( k \), whence so is the obvious direct limit

\[
\lim_{k \in \mathbb{N}} \text{St}(p_{-k} q_{-k} \cdots p_{-1} q_{k+1}) = \text{St}(w).
\]

Thus, in order to prove our claim, we suppose that \( \psi_k \) factors through \( A \in S^{<\infty}(\Lambda, \text{-mod}) \) and write \( A = \bigoplus_{i=1} A_i \), where each \( A_i \) is a finite dimensional string module of finite projective dimension and \( \pi_i : A \to A_i \) is the corresponding canonical projection. That \( \psi_k \) factors through a map in \( \text{Hom}(A, S) \) clearly implies the existence of a map \( g \in \text{Hom}_\Lambda(\text{St}(u_k), A) \) and an index \( l \) such that \( \pi_l g(y_0) \) is a top element of \( A_l \). Suppose that \( A_l = \text{St}(\hat{w}) \), where \( \hat{w} = \hat{p}_0 q_0 \cdots \hat{p}_m q_m \) is a word of finite projective dimension such that \( m \geq 0 \) and \( \text{St}(\hat{w}) \) has a standardized sequence of \( m + 1 \) top elements, say \( z_0, \ldots, z_m \).

By [10, Theorem], \( \text{Hom}_\Lambda(\text{St}(u_k), \text{St}(\hat{w})) \) is generated, as a \( K \)-space, by maps

\[
h = h[\rho, u_k, \hat{w}]
\]

of the following ilk: \( u_k \) and \( \hat{w} \) are subgraphs of the graphs of the words \( u_k \) and \( \hat{w} \), respectively (see Section 2 for our conventions), the first closed under arrows whose endpoints belong to \( u_k \), the second closed under arrows whose starting points belong to \( \hat{w} \); moreover, \( \rho \) denotes an isomorphism \( u_k \to \hat{w} \) of directed graphs, sending any arrow in \( u_k \) to an arrow in \( \hat{w} \) that carries the same label, such that the homomorphism \( h \) is induced by \( \rho \) (in the only meaningful way). Note that the image of \( h \) is \( \text{St}(\hat{w}) \), the latter being a submodule of \( \text{St}(\hat{w}) \) by the closure condition imposed on \( \hat{w} \). So the fact that there exists a map in \( \text{Hom}_\Lambda(\text{St}(u_k), \text{St}(\hat{w})) \)
sending \(y_0\) to a top element of \(\text{St}(\hat{w})\) ensures the existence of a triple \([\rho, u_k, \hat{n}]\), as described, together with an index \(i \in \{0, \ldots, m\}\), satisfying the following requirements: \(\hat{n}\) includes that vertex in the graph of \(\hat{w}\) which corresponds to the top element \(z_i\) of \(\text{St}(\hat{w})\) — call that vertex \(z\) (it is the joint starting vertex of the paths \(\hat{p}_i\) and \(\hat{q}_i\) in the graph of \(\hat{w}\)); moreover, \(u_k\) includes that vertex in the graph of \(u_k\) which corresponds to the top element \(y_0\) of \(\text{St}(u_k)\) (namely, the joint starting vertex of \(p_0\) and \(q_0\) in the graph of \(u_k\)); and, finally, \(\rho\) sends this latter vertex to \(z\).

In light of the preceding discussion, it suffices to prove that \(\hat{n}\) contains a subgraph isomorphic to that of the word \(p_{-1}^{-1}q_k \ldots p_k^{-1}q_k\). As in the effectiveness proof above, we adjust the labeling of the syllables of \(\hat{w}\) and the standardized sequence of top elements of \(\text{St}(\hat{w})\) so that \(\hat{w}\) becomes a word which is centered at \(i = 0\), and the graph of \(\text{St}(w)\) has the form (\(\dagger\)), displayed at the outset of our proof of (II), for a suitable nonnegative integer \(t\). Moreover, it is clearly harmless to assume that \(\hat{w}\) has the same orientation as \(w\). Then \(\hat{p}_0\) contains \(p_0\) as a right subpath, and \(\hat{q}_0\) contains \(q_0\) as a right subpath by Proposition 16(A), because \(\hat{w}\) is a word of finite projective dimension. Since the graph \(\hat{n}\) contains the paths \(\hat{p}_0\) and \(\hat{q}_0\) due to its closure property, we deduce that \(\hat{p}_0 = p_0\) and \(\hat{q}_0 = q_0\). Part (B) of that same proposition now tells us that the paths \(\hat{p}_{-1}\) and \(\hat{p}_1\) are contained in \(p_{-1}\) and \(p_1\) as left subpaths, respectively. So we assume that \(\hat{p}_1\) is nontrivial and write \(\hat{p}_1 = \hat{p}_1\nu\) where \(\nu\) is a path of length \(\geq 0\). Then \(\hat{q}_0\hat{q}_0 = q_0\hat{q}_0 = \hat{p}_1\nu z_1\) is a nonzero element of \(\text{soc}\ \text{St}(\hat{w})\), whence Proposition 16(B) guarantees that \(\nu z_1\) is a top element of \(\text{St}(\hat{w})\). We infer that \(\nu\) is trivial and that the vertex of \(\hat{w}\) corresponding to \(z_1\) also belongs to \(\hat{n}\). Consequently, the preceding argument can be duplicated to show \(\hat{q}_1 = q_1\), and next, in case \(p_2\) is nontrivial, the equalities \(\hat{p}_2 = p_2\) and \(\hat{q}_2 = q_2\). An obvious induction on \(i \in \{1, \ldots, k\}\) yields \(\hat{p}_i = p_i\) and \(\hat{q}_i = q_i\), whenever \(p_i\) is nontrivial, and an analogous argument applies to the paths \(\hat{p}_i\) and \(\hat{q}_i\) with negative indices \(i \in \{-k, \ldots, -1\}\). This shows that the graph of the word \(p_{-1}^{-1}q_{-k} \ldots p_k^{-1}q_k\) is indeed a subgraph of \(\hat{n}\), thus proving our claim and finishing the proof of part (II) of Theorem 5.

(III) The first assertion is an immediate consequence of the equivalences. To prove the equivalences, we fix \(k\) and write \(S_k = S = \Lambda e / Je\). Moreover, we let \(w_k = w = (p_i^{-1}q_i)_{i \in \mathbb{Z}}\) be the characteristic word of \(S\) and \(\varphi : \text{St}(w) \to S\) the canonical homomorphism defined by \(\varphi(x_i) = \delta_{00}(e + Je)\).

‘(i) \implies (iii)’. Suppose that \(w\) is finite. We need to ascertain that every homomorphism \(f \in \text{Hom}_{\Lambda}(M, S)\), where \(M\) is an indecomposable object of \(\mathcal{P}^{<\infty}(\Lambda\text{-mod})\), factors through \(\varphi\). Once we know that \(\varphi\) is a \(\mathcal{P}^{<\infty}(\Lambda\text{-mod})\)-approximation of \(S\), minimality will be automatic, because \(\text{St}(w)\) is known to be indecomposable (see [8]). In case \(M\) is a string module of finite projective dimension, the required factorization property follows from the fact that \(\varphi\) is an effective \(\mathcal{S}^{<\infty}(\Lambda\text{-mod})\)-phantom of \(S\), which was established in part (II).

Now we focus on a band module \(M = \text{Bd}(\hat{v}, \phi)\) of finite projective dimension, where \(\hat{v} = \hat{p}_0^{-1}\hat{q}_0 \ldots \hat{p}_t^{-1}\hat{q}_t\) is a primitive word with all of the listed syllables nontrivial, and again denote by \(y_{01}, \ldots, y_{11}, y_{20}, \ldots, y_{22}, y_{30}, \ldots, y_{4t}\) a standardized sequence of top elements of \(M\). It clearly suffices to show that any map \(f \in \text{Hom}_{\Lambda}(M, S)\) which sends precisely one of the top elements \(y_{ij}\) of type \(e\) to \(e + Je\) and the others to zero factors through \(\varphi\) (if none of the \(y_{ij}\) is of type \(e\), our requirement is void). So let us assume that \(f : M \to S\) is a map of the described ilk. Moreover, it is harmless to adjust the setup so that \(f(y_{10}) = e + Je\) and \(f(y_{ij}) = 0\) for \((i, j) \neq (1, 0)\). In this situation, \(M\) has the same graph as the pseudo-band module \(B\) in the statement of Lemma 22.

As in Lemmas 20 and 22, we denote by \(\hat{w}\) the twosided infinite word \(\ldots \hat{v}\hat{v}\hat{v}\ldots\), which we again assume to be centered at \(e\) and have the same orientation as \(w\). Clearly, \(\hat{w}\) has right and left discontinuities, since the characteristic word \(w\) of \(S\) is finite by hypothesis, whereas \(\hat{w}\) is twosided infinite. In light of Proposition 1, moreover, \(\hat{w}\) is a word of finite projective dimension, and so Lemma 22 applies to guarantee that \(f\) factors through \(\varphi\) as required.

The implications ‘(iii) \implies (v) \implies (iv)’ and ‘(iii) \implies (ii)’ are obvious.

‘(iv) \implies (i)’ follows from the fact that \(\text{St}(w)\) is an \(\mathcal{S}^{<\infty}(\Lambda\text{-mod})\)-phantom of \(S\) by part (II).

‘(ii) \implies (i)’. Suppose \(w = (p_i^{-1}q_i)_{i \in \mathbb{Z}}\) is infinite; w.l.o.g., the principal right segment of \(w\) is infinite and so, in particular, the syllable \(q_0\) is nontrivial.

First assume that there exists an index \(j > 0\) such that \(q_j = q_0\), and let \(t \geq 1\) be minimal with the property that \(q_t = q_0\). Then the principal right segment \(w_{\text{right}}\) of \(w\) has the form

\[
w_{\text{right}} = (q_0p_1^{-1} \ldots q_tp_t^{-1}q_0p_1^{-1} \ldots q_tp_t^{-1}q_0p_1^{-1} \ldots q_tp_t^{-1}q_0p_1^{-1} \ldots)
\]

by Construction 12; in other words, if \(v = p_t^{-1}q_0 \ldots p_1^{-1}q_0\) is a primitive word such that \(p_{t+1}^{-1}w_{\text{right}} = \ldots vvv\ldots\), Lemma 23 tells us that \(S\) fails to have a \(\mathcal{P}^{<\infty}(\Lambda\text{-mod})\)-approximation. In case \(p_0\) is nontrivial
and there exists an index \( j < 0 \) with \( p_j = p_0 \), the situation is symmetric – just flip the characteristic word \( w \) about its central axis.

Now suppose that, for all \( j > 0 \), we have \( q_j \not= q_0 \), and \( p_0 \) is either trivial, or else \( p_j \not= p_0 \) for all \( j < 0 \). Assume that, to the contrary of our claim, \( S \) has a \( \mathcal{P}^{\leq \infty}(\Lambda \mbox{-mod}) \)-approximation. Our aim is to infer that then \( S \) has an \( \mathcal{S}^{\leq \infty}(\Lambda \mbox{-mod}) \)-approximation as well; but this is incompatible with the already established implication ‘(iv) \( \implies \) (i)’.

To that end, let \( A \) be any \( \mathcal{P}^{\leq \infty}(\Lambda \mbox{-mod}) \)-approximation of \( S \) and \( B \) a band module occurring as a direct summand of \( A \), say \( A = B \oplus C \). We will construct another \( \mathcal{P}^{\leq \infty}(\Lambda \mbox{-mod}) \)-approximation of \( S \) which has the form \( \bigoplus_{\text{finite}} \text{St}(u_j) \oplus C \), for suitable words \( u_j \) of finite projective dimension. In this way, we can eliminate all band module summands from \( A \) in favor of direct sums of string modules, to arrive at another \( \mathcal{P}^{\leq \infty}(\Lambda \mbox{-mod}) \)-approximation of \( S \), this one belonging to \( \mathcal{S}^{\leq \infty}(\Lambda \mbox{-mod}) \). This will then give us the desired contradiction.

In order to replace the summand \( B \) of \( A \) by a direct sum of string modules as indicated, write \( B = Bd(\hat{v}^r, \hat{\phi}) \), where \( \hat{v} = \hat{p}_0^{-1} \ldots \hat{p}_t^{-1} \hat{q}_t \) is a primitive word, \( r \geq 1 \). For a graph of \( B \), we refer to the graph of the pseudo-band module of the same name in the statement of Lemma 22; again, we let \( y_{10}, \ldots, y_{1t}, \ldots, y_{r0}, \ldots, y_{rt} \) be the corresponding standardized sequence of top elements of \( B \).

As in the argument for ‘(i) \( \implies \) (iii)’, it suffices to prove the following: Given any top element \( y \in \{y_{10}, \ldots, y_{1t}\} \) of type \( e \) of \( B \), the map \( f : B \to S \), sending \( y \) to the residue class \( \bar{e} + Je \) and the other top elements \( \bar{y}_{ij} \) to zero, factors through a map \( \text{St}(u) \to S \), where \( u \) is a suitable word of finite projective dimension. Moreover, it is harmless to assume that \( y \) is the top element corresponding to the joint starting vertex of the paths \( \hat{p} \) and \( \hat{q} \). Once more, we center the infinite periodic word \( \hat{w} = \ldots \hat{v} \hat{v} \ldots \) in some occurrence of this vertex, and assume (clearly still without losing generality) that \( \hat{w} \) has the same orientation as \( w \). Since \( B \) has finite projective dimension, so does the word \( \hat{w} \), by Proposition 1. The non-periodicity conditions we imposed on \( \hat{w} \), moreover, force left and right discontinuities on \( \hat{w} \). In light of Lemma 22, our test map \( f \) thus factors through the canonical homomorphism \( \varphi : \text{St}(w) \to S \). Say \( f = \varphi g \), for a suitable map \( g \in \text{Hom}_A(B, \text{St}(w)) \).

Let \( w' = (p_1^{-1}q_1) - m_{1 \leq i \leq m} \) be a finite segment of \( w \) such that \( \text{St}(w') \) is a submodule of \( \text{St}(w) \) containing the image of \( g \). Moreover, let \( u \) be a finite word of finite projective dimension which in turn contains \( w' \) as a segment; such a word \( u \) exists by Observations 10. Then \( f = \varphi h \), where \( h : B \to \text{St}(u) \) denotes the map resulting from \( g \) through restriction of the range, which finishes the argument we have laid out.

To prove the supplementary statement, let \( w \) be finite. Suppose that the \( K \)-dimension of \( \text{St}(w)/J \text{St}(w) \) exceeds \( 4n \), where \( n \) is the number of vertices of \( G \). Then either \( p_{2n} \) or \( q_{-2n} \) is nontrivial. We may assume the former path to be nontrivial, the other case leading to a symmetric situation. The string module \( \text{St}(q_0p_1^{-1} \ldots q_{2n-1}p_1^{-1}) \) then has \( 2n+1 \) top elements \( x_i \), which are \( K \)-linearly independent modulo the radical. At least three of these are normed by the same primitive idempotent. Hence at least two of the latter, say \( x_k \) and \( x_l \) for suitable \( k < l \), reside atop paths \( p_k, p_l \) starting in the same arrow. In this situation, Construction 12 yields \( q_k = q_l, p_{k+1} = p_{l+1}, \) etc. (consult Observations 11), and consequently all the \( p_i \) with positive index are nontrivial. But this makes the word \( w \) infinite, thus contradicting our hypothesis.

The proof of Theorem 5 is thus complete. \( \square \)

8. CONCLUDING REMARKS

Theorem 5 and Proposition 14 tell us that, if \( w = w(S) \) is the characteristic word of the simple module \( S \in \Lambda \mbox{-mod} \), the string module \( \text{St}(w) \) is an effective \( \mathcal{S}^{\leq \infty}(\Lambda \mbox{-mod}) \)-phantom of \( S \). On the other hand, \( \text{St}(w) \) is not an effective \( \mathcal{P}^{\leq \infty}(\Lambda \mbox{-mod}) \)-phantom of \( S \) in general. The first known example of a finite dimensional algebra \( \Lambda \) for which \( \mathcal{P}^{\leq \infty}(\Lambda \mbox{-mod}) \) fails to be contravariantly finite, presented in [25], already demonstrates this.

Example 23. Let \( \Lambda \) be the string algebra whose two indecomposable projective left modules have the following graphs:

\[
\begin{array}{c|c|c}
\alpha & 1 & 2 \\
2 & \beta & 2 \\
1 & & 1
\end{array}
\]

Using the techniques we developed, we see that the characteristic phantom \( \text{St}(w_1) \) of \( S_1 \) has graph
This infinite dimensional module is known to be both an $S^{<\infty}(\Lambda\text{-mod})$- and a $P^{<\infty}(\Lambda\text{-mod})$-phantom of $S_1$ (see [18]), but, while it is effective as an $S^{<\infty}(\Lambda\text{-mod})$-phantom by Theorem 5, it is not effective as a $P^{<\infty}(\Lambda\text{-mod})$-phantom. Indeed, for the band module $M = \Lambda e_1/\Lambda(\alpha - \beta) \in P^{<\infty}(\Lambda\text{-mod})$, the canonical epimorphism $f: M \to S_1$ clearly fails to factor through $\text{St}(w_1)$.

Furthermore, we note that the string module $\text{St}(w_1)$, unique relative to the given coordinatization of $\Lambda$, does depend on the coordinates. If we adopt the new arrows $\alpha' = \alpha - \beta$ and $\beta' = \beta$ from the vertex 1 to the vertex 2, the characteristic phantom $\text{St}(w')$ of $S_1$ with respect to the new coordinatization of $\Lambda$ has graph

It is readily seen that $\text{St}(w_1)$ is not isomorphic to $\text{St}(w'_1)$. □

In Part II, we will address the problem of supplementing the effective $S^{<\infty}(\Lambda\text{-mod})$-phantoms $\varphi_i: \text{St}(w_i) \to S_i$ to effective $P^{<\infty}(\Lambda\text{-mod})$-phantoms. It will turn out that all information required to construct such ‘completions’ of the $\text{St}(w_i)$ is stored in the characteristic words $w_i$.

Concerning the shape of the minimal $P^{<\infty}(\Lambda\text{-mod})$-approximations of the $S_i$ in case the word $w = w(S)$ is finite, Theorem 5 guarantees that the top of the minimal $P^{<\infty}(\Lambda\text{-mod})$-approximation $\text{St}(w)$ of $S$ has $K$-dimension at most $4n$, where $n$ is the number of distinct simple $\Lambda$-modules. This bound stems from the fact that the multiplicity of any simple module $T$ in $\text{St}(w)/J\text{St}(w)$ is bounded above by 4 (see Construction 12). While the bound on $\dim_K(\text{St}(w)/J\text{St}(w))$ can be tightened with some additional effort, the bound on multiplicities in $\text{St}(w)/J\text{St}(w)$ is sharp.

**Example 24.** Let $\Lambda$ be the string algebra whose indecomposable projective left modules have the following graphs

Then $P^{<\infty}(\Lambda\text{-mod})$ is contravariantly finite in $\Lambda\text{-mod}$, as can readily be ascertained with the aid of Theorem 5, and the graph of the minimal $P^{<\infty}(\Lambda\text{-mod})$-approximation of $S_1$ is

This final example also exhibits the necessity of recording the center of the characteristic word $w$ of $S$, if one aims at pinning down the correct homomorphism $\varphi: \text{St}(w) \to S$, through which all homomorphisms in $\text{Hom}_\Lambda(S^{<\infty}(\Lambda\text{-mod}), S)$ will factor. Indeed, the minimal $P^{<\infty}(\Lambda\text{-mod})$-approximation of $S_2$ is the string module with graph

□
where again the center is highlighted.

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