Entangled superfluids: condensate dynamics of the entangled Bose-Einstein condensation

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We study the condensate dynamics of the so-called entangled Bose-Einstein condensation (EBEC), which is the ground state of a mixture of two species of pseudospin-$\frac{1}{2}$ atoms with interspecies spin-exchange scattering in certain parameter regimes. EBEC leads to four inter-dependent superfluid components, each corresponding to the orbital wave function associated with a spin component of a species. The four superflows have various counter-relations, and altogether lead to a conserved total supercurrent and a conserved total spin supercurrent. In the homogeneous case, we also obtain the elementary excitations due to variations of the single-particle orbital wave functions, by exactly solving the generalized time-dependent Bogoliubov equations. There are three gapless Bogoliubov modes and one Klein-Gordon-like gapped mode. The origin of these excitations are also discussed from the perspective of spontaneous breaking of the symmetries possessed by the system.

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I. INTRODUCTION

Bose-Einstein condensation (BEC) amplifies quantum mechanical behavior of individual particles into macroscopic quantum phases. In the simplest case, BEC occurs in a single-particle spatial or orbital state. Quantum features are more pronounced when there are additional degrees of freedom. For example, BEC occurring in a superposed single particle state leads to Josephson effect [1], while BEC of atoms with spin degree of freedom leads to a spinor condensate [2]. For spin $F = 1$, in a mean field state [3, 4], which is exact for a gas with ferromagnetic spin exchange, BEC occurs in a single-particle superposition of the three hyperfine states. On the other hand, in the exact ground state for a spin-1 gas with antiferromagnetic spin exchange [4–7], BEC occurs in a superposition of two-particle state. As a further development in this perspective, the so-called entangled BEC (EBEC), i.e. BEC occurring in an interspecies entangled two-particle state, amplifies entanglement of distinguishable particles into a macroscopic phase of a many-particle systems [8]. EBEC was found to be the ground state of a mixture of two species of pseudospin-$\frac{1}{2}$ atoms in a considerable parameter regime [9, 10]. It is entirely different from the two-component BEC, which occurs in a gas of two kinds of atoms distinguished by only one degree of freedom [4, 11], whose ground state is simply a direct product of the states of the two kinds of atoms, each separately undergoing BEC. Just like the simplest BEC may be a source of coherent atoms, EBEC could be a source of entangled atom pairs.

EBEC, in the case where the total number of atoms of each species is equal to $N$, refers to the many-body ground state

$$|G_0\rangle = \frac{1}{\sqrt{N+1!N!}}(a_1^\dagger a_1^\dagger - a_1^\dagger b_1^\dagger)^N|0\rangle,$$

where $a_\sigma$ and $b_\sigma$ are, respectively, the annihilation operators of the two species $a$ and $b$ for pseudospin $\sigma$ ($\sigma = \uparrow, \downarrow$). In $|G_0\rangle$, BEC occurs in a two-particle state of a maximally entangled interspecies pair

$$\eta(r_a, r_b) = \frac{1}{\sqrt{2}}[\phi_{a\uparrow}(r_a)|\uparrow\rangle_a\phi_{b\downarrow}(r_b)|\downarrow\rangle_b - \phi_{a\downarrow}(r_a)|\downarrow\rangle_a\phi_{b\uparrow}(r_b)|\uparrow\rangle_b],$$

and is thus called EBEC or BEC with an entangled order parameter $\eta(r_a, r_b)$. Here $\phi_{\alpha\sigma}$ ($\alpha = a, b$, $\sigma = \uparrow, \downarrow$) is the single-particle orbital wave function for each spin state of an atom of each species. In the most general case, spin dependence of the potential $U_{\alpha\sigma}$ and the scattering lengths, which determine the orbital wave functions, lead to a kind of spin-orbit coupling. Consequently, $\phi_{a\uparrow} \neq \phi_{a\downarrow}$, hence spin and orbital parts in $\eta(r_a, r_b)$ cannot be factorized as an orbital part and a spin part, i.e. there exists spin-orbit “entanglement” in $\eta$. If, however, $\phi_{a\uparrow} = \phi_{a\downarrow} = \phi_a$, as

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in the case where they are dominantly determined by the spin-independent part of the Hamiltonian, then \( \eta(r_a, r_b) = \phi_a(r_a)\phi_b(r_b) \) \( \frac{1}{\sqrt{2}}(|\uparrow\rangle_{ab} - |\downarrow\rangle_{ab}) \), i.e. spin and orbit become disentangled, as in the usual consideration in SU(2) model spin-1 model, consequently, the entanglement between \( a \)-atom and \( b \)-atom becomes entirely spin entanglement.

A state analogous to \( |G_0\rangle \) appears in a SU(2) symmetric model of a single species of pseudospin-\( \frac{1}{2} \) atoms [8, 9], with the role of the two different species played by the single-particle orbital ground state and the first excited state of a single species of atoms. But there are also differences: (i) In the SU(2) model, the number of atoms in the two single-particle orbital states are not conserved, but are fixed by a measurement or controlled by a microcanonical distribution, while in EBEC the number of atoms in the two species are strictly conserved. (ii) In the SU(2) model, in which there is only a single species, the occupation of two orbital modes, rather than a single mode, is due to the constraint of spin conservation in the cooling process, which could be compromised as the collision rate during the evaporative cooling might depend on hyperfine spins, and there could be atom loss. In EBEC, in contrast, two distinguishable species can have small total spin, as distinguishable atoms are not subject to the constraint of Bose symmetry, consequently all the atoms of each species can occupy the lowest orbital modes. (iii) In the SU(2) model, the analog of \( |G_0\rangle \) is not the true ground state of the system. In EBEC, \( |G_0\rangle \) is the true ground state. (iv) In the SU(2) model, the orbital identity of atoms, unlike different species, and the correlation between identical particles in the two orbital modes, unlike the entanglement between atoms of different species, are lost after the atoms are taken out of the trap.

An important open issue about EBEC is its condensate dynamics, i.e. superfluid behavior determined by the orbital wave functions and elementary excitations due to the fluctuations of the orbital wave functions. In this article, under the presumption that EBEC exists, we study the condensate dynamics of EBEC, based on a generalized version of the time-dependent Gross-Pitaevskii (GP) equations. EBEC leads to four inter-dependent components of the superfluid with a few counter-relations between each other. We also study the elementary excitations by exactly solving a set of generalized version of the Bogoliubov equations, as well as from the perspective of symmetry breaking.

II. HAMILTONIAN AND THE GENERALIZED GROSS-PITAEVSKII EQUATIONS

Consider a dilute gas of two species of atoms in a trap. Each atom has an internal degree of freedom represented as a pseudospin-\( \frac{1}{2} \). The field theoretic Hamiltonian is

\[
H = \sum_{\alpha, \sigma} \int d^3r \phi^\dagger_{\alpha \sigma} \hat{h}_{\alpha \sigma} \phi_{\alpha \sigma} + \frac{1}{2} \sum_{\alpha \sigma, \alpha' \sigma'} g^{(\alpha \sigma)} \int d^3r \phi^\dagger_{\alpha \sigma} \phi^\dagger_{\alpha' \sigma'} \phi_{\alpha' \sigma'} \phi_{\alpha \sigma} + H_{ab},
\]

with

\[
H_{ab} = \sum_{\sigma \sigma'} g^{(ab)} \int d^3r \phi^\dagger_{\alpha \sigma} \phi^\dagger_{\beta \sigma'} \phi_{\beta \sigma'} \phi_{\alpha \sigma} + g_c \int d^3r (\psi^\dagger_{a \uparrow} \phi^\dagger_{b \uparrow} \phi_{b \uparrow} \psi_{a \uparrow} + \psi^\dagger_{a \downarrow} \phi^\dagger_{b \downarrow} \phi_{b \downarrow} \psi_{a \downarrow}),
\]

where \( \alpha = a, b \) represents the two species, \( \sigma = \uparrow, \downarrow \) represents the two basis states of the pseudospin-\( \frac{1}{2} \), \( h_{\alpha \sigma} = -\hbar^2 \nabla^2_{\alpha} / 2m_{\alpha} + U_{\alpha \sigma} \) is the single particle Hamiltonian, \( \psi_{\alpha \sigma} \) is the field operator of species \( \alpha \). The coefficients \( g \)'s are shorthands for \( g_{\sigma_1 \sigma_2 \sigma_3 \sigma_4} = (2\pi\hbar^2 \xi^{(\sigma)}_{\sigma_1 \sigma_2 \sigma_3 \sigma_4} / \mu_{\alpha \beta}) \), where \( \xi^{(\sigma)}_{\sigma_1 \sigma_2 \sigma_3 \sigma_4} \) is the scattering length for the scattering in which an \( \alpha \)-atom flips from \( \sigma_1 \) to \( \sigma_1 \) while an \( \beta \)-atom flips from \( \sigma_3 \) to \( \sigma_2 \); \( \mu_{\alpha \beta} = m_{\alpha} m_{\beta} / (m_{\alpha} + m_{\beta}) \) is the effective mass.

For scattering lengths, we use the shorthands \( \xi^{(\alpha \beta)}_{\sigma_1 \sigma_2 \sigma_3 \sigma_4} = \xi^{(\alpha \beta)}_{\sigma_1 \sigma_2 \sigma_3 \sigma_4} \equiv \xi^{(\alpha \beta)}_{\sigma_1 \sigma_3 / \sigma_2 \sigma_4} \equiv \xi^{(\alpha \beta)}_{\sigma_1 \sigma_4 / \sigma_2 \sigma_3} = 2 \xi^{(\alpha \beta)}_{\sigma_1 \sigma_2 \sigma_3 \sigma_4} \) for \( \sigma \neq \tilde{\sigma} \). \( \xi^{(\alpha \beta)}_{\sigma_1 \sigma_2 / \sigma_3 \sigma_4} = \xi^{(\alpha \beta)}_{\sigma_1 \sigma_4 / \sigma_2 \sigma_3} \equiv \xi^{(\alpha \beta)}_{\sigma_1 \sigma_3 / \sigma_2 \sigma_4} \) for \( \sigma \neq \tilde{\sigma} \), \( g^{(\alpha \beta)}_{\sigma_1 \sigma_2 / \sigma_3 \sigma_4} = g^{(\alpha \beta)}_{\sigma_1 \sigma_4 / \sigma_3 \sigma_2} = g^{(\alpha \beta)}_{\sigma_1 \sigma_3 / \sigma_4 \sigma_2} \). Correspondingly \( g^{(\alpha \beta)}_{\sigma_1 \sigma_2 \sigma_3 / \sigma_4} = 2 \pi \hbar^2 \xi^{(\alpha \beta)}_{\sigma_1 \sigma_2 \sigma_3 \sigma_4} / \mu_{\alpha \beta} \), that is, \( g^{(\alpha \beta)}_{\sigma_1 \sigma_2 \sigma_3 / \sigma_4} = g^{(\alpha \beta)}_{\sigma_1 \sigma_3 \sigma_2 / \sigma_4} = g^{(\alpha \beta)}_{\sigma_1 \sigma_4 \sigma_2 / \sigma_3} = 2 g^{(\alpha \beta)}_{\sigma_1 \sigma_2 \sigma_3 \sigma_4} = 2 g^{(\alpha \beta)}_{\sigma_1 \sigma_2 \sigma_4 \sigma_3} = 2 g^{(\alpha \beta)}_{\sigma_1 \sigma_3 \sigma_2 \sigma_4} = 2 g^{(\alpha \beta)}_{\sigma_1 \sigma_3 \sigma_4 \sigma_2} = 2 g^{(\alpha \beta)}_{\sigma_1 \sigma_4 \sigma_2 \sigma_3} = 2 g^{(\alpha \beta)}_{\sigma_1 \sigma_4 \sigma_3 \sigma_2} \).

Under the single orbital mode approximation for each species, for each atom of species \( \alpha (\alpha = a, b) \) and pseudospin \( \sigma (\sigma = \uparrow, \downarrow) \), only the single-particle spatial ground state \( \phi_{\alpha \sigma}(r) \) is occupied. Therefore \( \psi_{\alpha \sigma} = \alpha \phi_{\alpha \sigma}, \) where \( \alpha \sigma \) is the annihilation operator corresponding to the single-particle orbital wave function \( \phi_{\alpha \sigma} \). Then the many-body Hamiltonian can be simplified as

\[
H = \sum_{\alpha, \sigma} f_{\alpha \sigma} N_{\alpha \sigma} + \frac{1}{2} \sum_{\alpha \sigma, \sigma', \sigma''} K^{(\alpha \beta)}_{\sigma_1 \sigma_2 \sigma_3 / \sigma_4} N_{\alpha \sigma} N_{\alpha \sigma'} + \sum_{\sigma} K^{(\alpha \beta)}_{\sigma_1 \sigma_2 \sigma_3 \sigma_4} N_{\alpha \sigma} N_{\beta \sigma'} + K_c (a_{\alpha \sigma} a_{\beta \sigma} b_{\sigma} + a_{\alpha \sigma}^\dagger a_{\beta \sigma}^\dagger b_{\sigma}^\dagger),
\]

where \( N_{\alpha \sigma} = a_{\alpha \sigma}^\dagger a_{\alpha \sigma} \). The total number of atoms of each species \( N_\alpha = N_{\alpha \uparrow} + N_{\alpha \downarrow} \) is conserved. The coefficients \( K \)’s are shorthands for

\[
K^{(\alpha \beta)}_{\sigma_1 \sigma_2 \sigma_3 / \sigma_4} = g^{(\alpha \beta)}_{\sigma_1 \sigma_2 \sigma_3 / \sigma_4} \int \phi^*_{\alpha \sigma_1}(r) \phi^*_{\beta \sigma_2}(r) \phi_{\beta \sigma_3}(r) \phi_{\alpha \sigma_4}(r) d^3r,
\]

where \( \eta(r_a, r_b) = \phi_a(r_a)\phi_b(r_b) \) \( \frac{1}{\sqrt{2}}(|\uparrow\rangle_{ab} - |\downarrow\rangle_{ab}) \), i.e. spin and orbit become disentangled, as in the usual consideration in SU(2) model spin-1 model, consequently, the entanglement between \( a \)-atom and \( b \)-atom becomes entirely spin entanglement.
that is, \( K_{\sigma \sigma}^{(a)} = K_{\sigma \sigma}^{(\alpha)} \), \( K_{\sigma \sigma}^{(a)} = K_{\sigma \sigma}^{(\alpha)} \) for \( \sigma \neq \sigma', K_{\sigma \sigma}^{(a)} = K_{\sigma \sigma}^{(\alpha)} \). 

\( f_{\sigma \sigma} = \epsilon_{\sigma \sigma} - K_{\sigma \sigma}^{(a)} / 2 \), where \( \epsilon_{\sigma \sigma} = \int \hat{\phi}_{\sigma \sigma}^{(a)} h_{\sigma \sigma} \hat{\phi}_{\sigma \sigma} d^3r \) is the single-particle energy.

When \( N_a = N_b \), the ground state is exactly \( |G_0 \rangle \), under the following conditions, which ensure consistency of simplifying the Hamiltonian to that of isotropic Heisenberg coupling between two giant spins representing the two species: (i) \( U_{\alpha \alpha}(r) = U_{\alpha \alpha}(r) = U_\alpha(r) \). (ii) The interspecies scattering lengths satisfy \( \xi_{\sigma \sigma}^{(a)} = \xi_{\sigma \sigma}^{(\alpha)} = \xi_\sigma \). (iii) The interspecies scattering lengths satisfy the relations \( \xi_{\uparrow \uparrow}^{(a)} = \xi_{\uparrow \uparrow}^{(\alpha)} \), denoted as \( \xi_{\uparrow \uparrow}^{(a)} \), and \( \xi_{\downarrow \downarrow}^{(a)} = \xi_{\downarrow \downarrow}^{(\alpha)} \), denoted as \( \xi_{\downarrow \downarrow}^{(a)} \), where the subscripts “s” and “d” stand for “same” and “different”, respectively. (iv) \( \xi_{\alpha \beta}^{(a)} = \xi_{\alpha \beta}^{(\alpha)} = \xi_{\alpha \beta}^{(\alpha)} \). Under these conditions, \( \mu_{\uparrow \uparrow} = \mu_{\downarrow \downarrow} = \phi_{\sigma \sigma} \) in the many-body ground state. The four wave functions \( \phi_{\alpha \sigma} \)'s, of which the condensate wave function \( \eta \) is built on, satisfy four generalized GP equations. It has also been shown that in a considerable magnetic parameter regime, the ground state approaches \( |G_0 \rangle \). The conditions (i) and (ii) are also satisfied in the SU(2) symmetric model of a single species of pseudospin-\( \pi \) atoms.

In the thermodynamic limit, the energetic advantage of BEC is lost to the stability of simple BEC, like other fragmented BEC. Hence BEC should be realized in a mesoscopic scale with a finite number of atoms and a finite volume. We can obtain upper bounds on the number of atoms as the following. According to previous discussions \([10]\), the effect of symmetry-breaking perturbation, which causes the ground state to deviate from BEC, tends to diminish when the volume remains finite while \( \Delta < 2K_\epsilon \), where \( \Delta \) is the energy gap of the perturbed Hamiltonian, and is given by \( \Delta = \sqrt{4K_{\epsilon} \alpha_\pi} \), where \( \alpha_\pi = \int [J_\pi - 2K_\epsilon - C_\pi - \hat{c}_\pi^N] + [\hat{b}_\pi - B_\pi N, \hat{c}_\pi = (N_a + N_b)/2] \), where \( J_\pi = K_{\uparrow \uparrow}^{(a)} + K_{\downarrow \downarrow}^{(a)} - K_{\uparrow \downarrow}^{(a)} - K_{\downarrow \uparrow}^{(a)} \), \( B_\pi = f_{\pi \pi} - f_{\alpha \alpha} + \frac{N_a}{2} [K_{\uparrow \uparrow}^{(a)} + K_{\uparrow \downarrow}^{(a)} - K_{\downarrow \uparrow}^{(a)} - K_{\downarrow \downarrow}^{(a)}] \). Under the conditions given in the above paragraph, \( \Delta \to 0 \), hence BEC is indeed the ground state. With deviation from these conditions, we may use the requirement \( \Delta < 2K_\epsilon \), i.e. \( \Delta < K_\epsilon \), to derive a constraint on \( N \) for the occurrence of BEC, which turns out to be \( N < \sqrt{[B_{\pi} - B_{\sigma}]/K_{\epsilon}(\Delta) - [C_{\pi} - C_{\sigma}]/[B_{\pi} - B_{\sigma}]}/(2) \). Under \( \Delta < 2K_\epsilon \) at \( \epsilon \), we find that \( E_\epsilon - E_0 \approx 2K_\epsilon \). According to \( [10] \), \( K_\epsilon = \gamma \int \hat{\phi}_{\alpha \sigma}(r)^* \hat{\phi}_{\alpha \sigma}(r) \hat{\phi}_{\alpha \sigma}(r) \hat{\phi}_{\alpha \sigma}(r) d^3r \).

To make a rough estimation using uniform wave functions, we have \( K_\epsilon = \epsilon_\pi / \Omega \), where \( \Omega \) is the volume of the gas. Hence we should have \( T < 2\epsilon_\pi / k_B \Omega \). Alternatively, for atoms in a trap, we may roughly assume \( \phi_{\alpha \sigma} \approx \frac{\sqrt{m_\pi}}{\sqrt{2\pi}} \exp(-\frac{m_\pi^2}{2\Omega}) \), where it is assumed that the atoms of both species have equal mass, and that the trap is isotropic with frequency \( \omega \). Then \( K_\epsilon = \gamma \sqrt{m_\pi / (2\pi \Omega)} \). Hence we have \( T < \sqrt{(\gamma k_B / \sqrt{2\pi m_\pi})/\Omega} \). BEC transition temperature can be roughly estimated using one component of the gas with particle number \( N/2 \). Hence \( T_c \approx 3.31 \hbar^2 (N/\Omega)^{2/3} / m \) for a uniform gas or \( T_c \approx 0.94 \hbar^2 (N/\Omega)^{1/3} / m \) for a trapped gas. Moreover, combining the estimation of \( T_c \) with the result derived from \( k_B T < E_1 - E_0 \), we obtain a further constraint on \( N \), namely \( N < (2k_B m_\pi)^{1/3} / [(3.31 \hbar^2 k_B)^{2/3} \Omega]^{1/2} \) for a uniform gas, or \( N < (2k_B m_\pi)^{1/3} / [\sqrt{3} k_B (2\pi m_\pi)^{1/2} \hbar^{1/2}] \) for a trapped gas.

EBEC might be experimentally realizable by using two species of spin-1 alkali atoms in an optical trap with hyperfine states constrained in a two-dimensional subspace of \( \mid \uparrow \rangle \equiv \mid F = 2, m_F = 2 \rangle \) and \( \mid \downarrow \rangle \equiv \mid F = 1, m_F = 1 \rangle \). In order that the spin-exchange scattering is energetically guaranteed, we should have \( \epsilon_{\uparrow \uparrow} - \epsilon_{\downarrow \downarrow} = \epsilon_{\uparrow \uparrow} - \epsilon_{\downarrow \downarrow} \), where \( \epsilon_{\alpha \alpha} \) (\( \alpha = a, b \)) is the single particle energy. However, for two different species of alkali atoms, the hyperfine splitting is different. A method of overcoming this difficulty is to apply a magnetic field such that \( \epsilon_{\uparrow \uparrow} - \epsilon_{\downarrow \downarrow} = \epsilon_{\uparrow \uparrow} - \epsilon_{\downarrow \downarrow} \), which is the sum of the hyperfine splitting \( A_\alpha \), plus the difference in Zeeman shift of the two hyperfine states \( (2g_\alpha F_{\alpha} - g_\alpha F_{\alpha} = 1) \mu_B B \), is the same for the two species. For an alkali atom, \( g_{\alpha \epsilon}(F J J + I J J - I_\alpha (I_\alpha + 1))/F(F + 1) \), where \( J = 1/2 \), \( I_\alpha \) is the nuclear spin of species \( \alpha \). Consider species \( a \) to be \( ^8 \text{Rb} \) while species \( b \) to be \( ^8 \text{Rb} \). Then \( A_a = 6835 \text{MHz} \), \( A_b = 3036 \text{MHz} \), \( I_a = 3/2 \), and \( I_b = 5/2 \). It can be estimated that \( B = 0.325 \text{T} \).

If BEC exists, then what about its physical properties? Here we focus on its orbital dynamics, which is determined by the generalized time-dependent GP equation

\[
\frac{i\hbar}{\partial \phi_{\alpha \sigma}(r)} = \left\{ -\frac{\hbar^2}{2m_\pi} \nabla^2 + U_\alpha(r) + \frac{2(N - 1)}{3} g_{\sigma \sigma}^{(a)} |\phi_{\alpha \sigma}(r)|^2 + \frac{N - 1}{3} g_{\sigma \sigma}^{(\alpha)} |\phi_{\alpha \sigma}(r)|^2 \right\} \phi_{\alpha \sigma}(r)
\]

\[
+ \frac{2}{3} \int \phi_{\alpha \sigma}(r) \phi_{\sigma \sigma}(r) \phi_{\alpha \sigma}(r) \phi_{\sigma \sigma}(r) d^3r,
\]

where \( \alpha \neq \alpha \) represents the species other than species \( \alpha \), and \( \sigma \neq \sigma \) represents the pseudospin opposite to \( \sigma \).

These time-dependent GP equations can be obtained from the static GP equations \([8]\), by replacing \( \mu_{\alpha \sigma} \) as \( i\hbar \partial / \partial t \), and can also be justified by using the action principle

\[
\delta \int_{t_1}^{t_2} L dt = 0,
\]
where

\[ L = \int dr \left\{ (i\hbar/2) \sum_{\alpha\sigma} (\phi_{\alpha\sigma}^* \partial \phi_{\alpha\sigma}/\partial t - \phi_{\alpha\sigma} \partial \phi_{\alpha\sigma}^*/\partial t) - \langle G_0 | \mathcal{H} | G_0 \rangle \right\} \]  

(9)

is the Lagrangian functional for \( |G_0\rangle \). The static GP-like equations have been obtained minimization of the energy functional under \( |G_0\rangle \). This self-consistent determination of the equation of motion of the orbital wave function actually underlies the derivation of the simplest GP equation, for which the many-body ground state is \((1/\sqrt{N})(a^\dagger)^N|0\rangle\). This methodology has also been used, e.g., by Ashhab and Leggett in studying the SU(2) model [7].

In studying the orbital dynamics of the simplest BEC, it is assumed that the system remains in BEC though the condensate wave function is time-dependent. Likewise, in the present case of EBEC, the generalized time-dependent GP equations (7) presumes that the system remains in the many-body ground states in the form of \( |G_0\rangle \), though the corresponding orbital wave functions \( \phi_{\alpha\sigma} \) is time dependent. In other words, the system remains as BEC though the orbital wave functions are time dependent. It is this kind of dynamics that corresponds to superfluidity and is considered here.

### III. SUPERCURRENTS AND SPIN SUPERCURRENTS

The number density of species \( \alpha \) with pseudospin \( \sigma \) is

\[ n_{\alpha\sigma} = (N/2)\phi_{\alpha\sigma}^* \phi_{\alpha\sigma} \]  

(10)

while the supercurrent is

\[ J_{\alpha\sigma} = \frac{\hbar}{2mi} \frac{N}{2} \left( \phi_{\alpha\sigma}^* \nabla \phi_{\alpha\sigma} - \nabla \phi_{\alpha\sigma}^* \phi_{\alpha\sigma} \right) = n_{\alpha\sigma} v_{\alpha\sigma} \]  

(11)

where \( v_{\alpha\sigma} \) is the superfluid velocity of species \( \alpha \) of pseudospin \( \sigma \).

From the generalized time-dependent GP equation (7), we obtain

\[ \frac{\partial n_{\alpha\sigma}(r,t)}{\partial t} + \nabla \cdot J_{\alpha\sigma}(r,t) = S_{\alpha\sigma} \]  

(12)

with

\[ S_{\alpha\sigma} = -S_{\beta\sigma} = -S_{\sigma\alpha} = S_{\sigma\beta} = -\frac{2(N + 2)\bar{g}_{e}}{3\hbar} \text{Im}(\phi_{\beta\sigma}^* \phi_{\alpha\sigma} \phi_{\alpha\sigma}^* \phi_{\beta\sigma}^*) \]  

(13)

which is due to interspecies spin-exchange and acts as a source. Thus the supercurrent is not conserved individually in each pseudospin component of each species. Equation (13) indicates the counter relations between the two pseudospin components of a same species, as well as those between two components with a same pseudospin and of two different species.

The total supercurrent of each species is conserved:

\[ \frac{\partial (n_{\alpha\uparrow} + n_{\alpha\downarrow})}{\partial t} + \nabla \cdot (J_{\alpha\uparrow} + J_{\alpha\downarrow}) = 0. \]  

(14)

So is the total supercurrent for each pseudospin,

\[ \frac{\partial (n_{\alpha\uparrow} + n_{\alpha\downarrow})}{\partial t} + \nabla \cdot (J_{\alpha\uparrow} + J_{\alpha\downarrow}) = 0. \]  

(15)

Of course, the total supercurrent of the four components is conserved:

\[ \frac{\partial}{\partial t} \sum_{\alpha,\sigma} n_{\alpha\sigma} + \nabla \cdot \sum_{\alpha,\sigma} J_{\alpha\sigma} = 0. \]  

(16)

Furthermore, we can also define spin density and spin supercurrent for each species,

\[ n_{\alpha}^s = m_F(\alpha, \uparrow)n_{\alpha\uparrow} + m_F(\alpha, \downarrow)n_{\alpha\downarrow} \]  

(17)
\[ \mathbf{J}_\alpha' = m_F(\alpha, \uparrow)\mathbf{J}_{\alpha\uparrow} + m_F(\alpha, \downarrow)\mathbf{J}_{\alpha\downarrow}, \]  

where \( m_F(\alpha, \sigma) \) denotes the hyperfine \( z \) component represented by pseudospin \( \sigma \) for species \( \alpha \).

The spin supercurrent of each species is not conserved:

\[ \frac{\partial n^s_\alpha}{\partial t} + \nabla \cdot \mathbf{J}^s_\alpha = [(m_F(\alpha, \uparrow) - m_F(\alpha, \downarrow))] S_{\alpha\uparrow}. \]  

If \( m_F(a, \sigma) = m_F(b, \sigma) \), then the total spin supercurrent of the two species is conserved,

\[ \frac{\partial(n^s_a + n^s_b)}{\partial t} + \nabla \cdot (J^s_a + J^s_b) = 0. \]

Thus we have a conserved total supercurrent as well as a conserved total spin supercurrent.

### IV. HYDRODYNAMICS

In parallel to some discussions in Ref. [4], here we derive some hydrodynamic equations from the generalized GP equations. Our discussion is restricted to zero temperature, hence does not require thermodynamic limit and also applies to a finite system in a trap.

With \( \phi_{\alpha\sigma} = f_{\alpha\sigma}e^{i\Phi_{\alpha\sigma}} \), where the phase \( \Phi_{\alpha\sigma} \) is not singular, Eq. (7) yields

\[ \frac{\partial f_{\alpha\sigma}}{\partial t} = -\hbar \frac{\hbar}{m_\alpha} \nabla \cdot ( f_{\alpha\sigma}^2 \nabla \Phi_{\alpha\sigma} - \frac{2}{3} f_{\alpha\sigma} f_{\beta\sigma} f_{\gamma\sigma} f_{\alpha\beta\sigma\gamma} \sin(\Phi_{\beta\sigma} + \Phi_{\alpha\sigma} - \Phi_{\alpha\beta\sigma}) ), \]  

which is just the continuity equation expressed in terms of amplitudes and phases, and

\[ -\hbar \frac{\partial \Phi_{\alpha\sigma}}{\partial t} = -\frac{\hbar^2}{2m_\alpha f_{\alpha\sigma}} \nabla^2 f_{\alpha\sigma} + \frac{m_\alpha}{2} \nabla^2 \Phi_{\alpha\sigma} + \frac{2(N-1)}{3} g^{(\alpha\alpha)} f_{\alpha\sigma}^2 + \frac{2N-1}{3} g^{(\alpha\beta)} f_{\alpha\sigma} f_{\beta\sigma} + \frac{2N+1}{3} g^{(\alpha\beta)} f_{\beta\sigma}^2 + \frac{3}{2} g_e f_{\alpha\sigma} f_{\beta\sigma} f_{\alpha\beta\sigma\gamma} \cos(\Phi_{\beta\sigma} + \Phi_{\alpha\sigma} - \Phi_{\alpha\beta\sigma} - \Phi_{\alpha\beta\sigma}). \]

According to (11),

\[ \mathbf{v}_{\alpha\sigma} = \frac{\hbar}{m} \nabla \Phi_{\alpha\sigma}, \]

hence the gradient of (22) becomes

\[ m_\alpha \frac{\partial \mathbf{v}_{\alpha\sigma}}{\partial t} = -\nabla (\tilde{\mu}_{\alpha\sigma} + \frac{1}{2} m_\alpha \mathbf{v}_{\alpha\sigma}^2), \]

where \( \tilde{\mu}_{\alpha\sigma} = -\frac{\hbar^2}{2m_\alpha n_{\alpha\sigma}} \nabla^2 \sqrt{n_{\alpha\sigma}} + U_{\alpha\sigma} + \frac{2(N-1)}{3} g^{(\alpha\alpha)} n_{\alpha\sigma} + \frac{N-1}{3} g^{(\alpha\alpha)} n_{\alpha\sigma} + \frac{N-1}{3} g^{(\alpha\beta)} n_{\alpha\sigma} + \frac{N+1}{3} g^{(\alpha\beta)} n_{\alpha\sigma} + V^{(\alpha)}), \]

where the third to sixth terms on right-hand side are mean-field interaction energies due to scattering without spin exchange, while

\[ V^{(\alpha)} = \frac{N+2}{3} g_e \sqrt{n_{\alpha\sigma} n_{\alpha\sigma} n_{\alpha\sigma}} \cos(\Phi_{\beta\sigma} + \Phi_{\alpha\sigma} - \Phi_{\alpha\beta\sigma} - \Phi_{\alpha\beta\sigma}) \]

is due to spin exchange scattering. \( \nabla V^{(\alpha)} \) = \( [(N+2)/3] g_e (\sqrt{n_{\alpha\sigma} n_{\alpha\sigma} n_{\alpha\sigma} / n_{\alpha\sigma}}) \cos(\Phi_{\beta\sigma} + \Phi_{\alpha\sigma} - \Phi_{\alpha\beta\sigma} - \Phi_{\alpha\beta\sigma}) - [(N+2)/(3\hbar)] g_e \sqrt{n_{\alpha\sigma} n_{\alpha\sigma} n_{\alpha\sigma} / n_{\alpha\sigma}} \sin(\Phi_{\beta\sigma} + \Phi_{\alpha\sigma} - \Phi_{\alpha\beta\sigma} - \Phi_{\alpha\beta\sigma}) \) \( m_{\alpha}(\mathbf{v}_{\alpha\sigma} - \mathbf{v}_{\alpha\sigma}) - m_{\alpha}(\mathbf{v}_{\alpha\sigma} - \mathbf{v}_{\alpha\sigma}) \), implying the inter-dependence between superfluid velocities of different components.

It is interesting that with the spin-exchange term, Eq. (22) can still be written as a generalized Josephson relation

\[ \frac{\partial \Phi_{\alpha\sigma}}{\partial t} = \frac{1}{\hbar} \frac{\delta E}{\delta n_{\alpha\sigma}}, \]

where

\[ E = E_\alpha + E_\beta + E_{\alpha\beta}. \]
with

\[ E_{\alpha} = \frac{N}{2} \int d\mathbf{r} \left\{ \frac{\hbar^2}{2m_\alpha} |\nabla \psi_{\alpha\sigma}|^2 + U_{\alpha\sigma} + \frac{N - 1}{3} g^{(\alpha\alpha)}_{\sigma\sigma}|\psi_{\alpha\sigma}|^4 + \frac{N - 1}{3} g^{(\alpha\alpha)}_{\sigma\sigma}|\psi_{\alpha\sigma}|^2 \right\} , \tag{28} \]

\[ E_{ab} = \frac{N}{2} \int d\mathbf{r} \left\{ \frac{N - 1}{3} g_{\sigma\sigma}^{(ab)}|\psi_{\alpha\sigma}|^2 |\psi_{\beta\sigma}|^2 + \frac{2N + 1}{3} g_{\sigma\sigma}^{(ab)}|\psi_{\alpha\sigma}|^2 |\psi_{\beta\sigma}|^2 \right\} - \frac{N + 2}{3} g_{e} (\phi_{b\sigma}^* \phi_{a\sigma}^* \phi_{a\Lambda} + \phi_{a\sigma}^* \phi_{a\sigma}^* \phi_{b\sigma}) \right\} , \tag{29} \]

In case \( \Phi_{\alpha\sigma} \) is independent of position or time, this leads to time-independent GP equations.

Like the usual condensate, Eq. (24) can be expressed in terms of pressure,

\[ \frac{\partial \nabla \alpha\sigma}{\partial t} = - \frac{1}{m_\alpha n_{\alpha\sigma}} \nabla P_{\alpha\sigma} - \nabla \left( \frac{\hbar^2}{2m_\alpha} \nabla^2 \sqrt{n_{\alpha\sigma}} \right) - \frac{1}{m_\alpha} \nabla U_{\alpha\sigma} , \tag{30} \]

where pressure \( P_{\alpha\sigma} \) is related to chemical potential through

\[ dP_{\alpha\sigma} = n_{\alpha\sigma} d\mu'_{\alpha\sigma} , \tag{31} \]

where \( \mu'_{\alpha\sigma} = \mu_{\alpha\sigma} - U_{\alpha\sigma} \). \( \mu'_{\alpha\sigma} \) and thus \( P_{\alpha\sigma} \) are contributed by interspecies spin-exchange scattering.

V. ELEMENTARY EXCITATIONS

The variation \( \delta \phi_{\alpha\sigma} \), away from \( \phi_{\alpha\sigma} \) in the ground state, satisfies the equation

\[ i\hbar \frac{\partial}{\partial t} \delta \phi_{\alpha\sigma} = \left( - \frac{\hbar^2}{2m_\alpha} \nabla^2 + U_{\alpha\sigma} \right) \delta \phi_{\alpha\sigma} + \frac{4(N - 1)}{3} g^{(\alpha\alpha)}_{\sigma\sigma} |\psi_{\alpha\sigma}|^2 \delta \phi_{\alpha\sigma} + \frac{2(N - 1)}{3} g^{(\alpha\alpha)}_{\sigma\sigma} \phi_{\alpha\sigma}^2 \delta \phi_{\alpha\sigma}^* + \frac{2}{3} g_{\sigma\sigma}^{(\alpha\alpha)} |\phi_{\alpha\sigma}|^2 |\phi_{\alpha\sigma}|^2 \delta \phi_{\alpha\sigma}^* + \frac{3}{3} g_{\sigma\sigma}^{(\alpha\alpha)} |\phi_{\alpha\sigma}|^2 |\phi_{\alpha\sigma}|^2 \delta \phi_{\alpha\sigma}^* + \frac{3}{3} g_{\sigma\sigma}^{(\alpha\alpha)} |\phi_{\alpha\sigma}|^2 |\phi_{\alpha\sigma}|^2 \delta \phi_{\alpha\sigma}^* \tag{32} \]

where \( \phi_{\alpha\sigma} \) refers to orbital wave functions in the ground state.

For simplicity, here we only consider a homogeneous system, i.e. \( U_{\alpha\sigma} = 0 \). It is reasonable to suppose that a trapping potential \( U_{\alpha\sigma} \) would not change the essential physics. Furthermore, we assume the above-mentioned conditions \( g^{(\alpha\alpha)}_{\sigma\sigma} = g^{(\alpha\alpha)}_{\sigma\sigma} = g_{\alpha}, g^{(\alpha\alpha)}_{\sigma\sigma} = g_{\sigma}, \) and \( g^{(\alpha\alpha)}_{\sigma\sigma} = g_{d}. \) Then the chemical potentials \( \mu_{\alpha\sigma} = \mu_{\alpha\sigma}, \) which is equal to

\[ \mu_{\alpha\sigma} = \frac{N - 1}{\Omega} g_{\alpha} + \frac{2N + 1}{3\Omega} g_{d} - \frac{N + 2}{3\Omega} g_{e} . \tag{33} \]

Thus single particle wave function in the ground state is \( \phi_{\alpha\sigma} = e^{-i\mu_{\alpha\sigma}t/\hbar} / \sqrt{\Omega} \), which is then substituted to Eq. (32).

Even though \( \phi_{\alpha\sigma} = \phi_{\alpha\sigma}, \) their variations should still be considered respectively. Since we are considering a uniform system, we may set

\[ \delta \phi_{\alpha\sigma} = \frac{e^{-i\mu_{\alpha}\tau/\hbar}}{\sqrt{\Omega}} \left[ u_{\alpha\sigma}(q)e^{i(q\cdot r - \omega t)} - v_{\alpha\sigma}^*(q)e^{-i(q\cdot r - \omega t)} \right] . \tag{34} \]

Then Eq. (32) yields

\[ \frac{\omega_{u_{\alpha\sigma}}}{3\Omega} = \frac{2(N - 1)}{3\Omega} g_{\alpha} \omega_{v_{\alpha\sigma}} + \frac{2N + 1}{3\Omega} g_{d} \omega_{v_{\alpha\sigma}} - \frac{N - 1}{2\Omega} g_{e} \omega_{v_{\alpha\sigma}} \tag{35} \]
and
\[
\begin{align*}
\omega \nu_{\alpha\sigma} &= \frac{2(N - 1)}{3\Omega} g_{\alpha} u_{\alpha\sigma} - W \nu_{\alpha\sigma} \\
&+ \frac{N - 1}{3\Omega} g_{\alpha} u_{\alpha\sigma} + [- \frac{N - 1}{3\Omega} g_{\alpha} + \frac{N + 2}{3\Omega} g_{\delta}] v_{\alpha\delta} \\
&+ \frac{N - 1}{3\Omega} g_{\sigma} u_{\alpha\sigma} + [- \frac{N - 1}{3\Omega} g_{\sigma} + \frac{N + 2}{3\Omega} g_{\delta}] v_{\alpha\delta} \\
&+ \frac{2N + 1}{3\Omega} g_d - \frac{N + 2}{3\Omega} g_e v_{\alpha\delta} - \frac{2N + 1}{3\Omega} g_d v_{\alpha\delta}.
\end{align*}
\]

where
\[
W = -\mu \alpha + \frac{\hbar^2 q^2}{2m_\alpha} + \frac{5(N - 1)}{3\Omega} g_{\alpha} + \frac{N - 1}{3\Omega} g_{\sigma} + \frac{2N + 1}{3\Omega} g_d.
\]

Therefore we have eight coupled equations, which can be written as a matrix
\[
AU = \omega U,
\]

where
\[
U \equiv (u_{\alpha\uparrow}, v_{\alpha\uparrow}, u_{\alpha\downarrow}, v_{\alpha\downarrow}, u_{\beta\uparrow}, v_{\beta\uparrow}, u_{\beta\downarrow}, v_{\beta\downarrow})^T,
\]

the matrix elements of \(A\) can be read from Eqs. (35) and (36).

We obtain four pairs of eigenvalues \(\omega = \pm E^{(j)}_q / \hbar\) \((j = 1, 2, 3, 4)\), \(E^{(j)}_q\) being the energy of the elementary excitations. As usual, a positive eigenvalue \(\omega\) corresponds to addition of a quasiparticle with momentum \(\hbar q\) and removal of a quasiparticle with zero-momentum, while a negative eigenvalue \(\omega\) corresponds to removal of a quasiparticle with momentum \(-\hbar q\) and addition of a quasiparticle with zero-momentum.

\(E^{(j)}_q\) \((j = 1, 2, 3, 4)\) can be obtained exactly. The first and second excitations are Bogoliubov-like gapless modes, and can be written together as
\[
E^{(j)}_q = \left(\frac{X_a + X_b}{2}\right) \pm \left(\frac{X_a - X_b}{2}\right)^2 + \frac{4D}{9\Omega^2} \epsilon^a_q \epsilon^b_q \frac{1}{2},
\]

where \(j = 1, 2\) correspond to + and \(-\) in \(\pm\) respectively, \(\epsilon^a_q = \hbar^2 q^2 / 2m_a\), \(X_a \equiv \epsilon^a_q / \hbar^2 q^2 / 2m_a = \epsilon^a_q / \hbar^2 q^2 / 2m_a / 2\Omega\), \(\epsilon^b_q = \hbar^2 q^2 / 2m_b\), \(D = [(N + 2)g_e - (N - 1)g_s - (2N + 1)g_d]^2\). In terms of \(q\),
\[
E^{(j)}_q = \hbar^2 q^2 / 2m_a + \frac{1}{\Omega} \left(\frac{X_a - X_b}{2}\right)^2 + \frac{4D}{9\Omega^2} \epsilon^a_q \epsilon^b_q \frac{1}{2},
\]

If \(m_a = m_b = m\), then these two energy spectra reduce to
\[
E^{(j)}_q = E^0_q [E^0_q + \Gamma^{(j)}] \frac{1}{2},
\]

where \(E^0_q = \hbar^2 q^2 / 2m\), \(\Gamma^{(j)} \equiv (N - 1)(g_a + g_b) / \Omega \pm \{(N - 1)^2(g_a - g_b)^2 + 4D / 9\Omega^2\}^{1/2} / \Omega\).

From (42), it can be seen that in the long wavelength limit \(q \rightarrow 0\), the excitation \(E^{(j)}_q\) \((j = 1, 2)\) becomes linear
\[
E^{(j)}_q \rightarrow s^{(j)} \hbar q,
\]

with the sound velocity
\[
s^{(j)} = \left(\frac{N - 1}{2\Omega} \frac{g_a}{m_a} + \frac{g_b}{m_b}\right) \pm \left(\frac{N - 1}{2\Omega} \frac{g_a}{m_a} - \frac{g_b}{m_b}\right)^2 + \frac{D}{9\Omega^2 m_a m_b} \frac{1}{2},
\]

The existence of \(E^{(2)}_q\) for \(q \rightarrow 0\) is subject to the condition \((N - 1)^2 g_a g_b \geq D / 9\), or \(g_a g_b \geq (g_e - g_s - 2g_d)^2\) for \(N \gg 1\).

In the short wavelength limit \(q \rightarrow \infty\),
\[
E^{(1)}_q \rightarrow \epsilon^a_q, \quad E^{(2)}_q \rightarrow \epsilon^b_q,
\]

(46)
that is, they reduce to single particle excitations.

The third and fourth excitations can be written as

$$E^{(j)}_q = [Z_q \mp (Z_q^2 - Y_q)^{1/2} + i],$$  \hspace{1cm} (47)$$

where \( j = 3, 4 \) correspond to \(-\) and \(+\) in \( \pm \) respectively, \( Z_q = e_q^g e_q^b/2 + (N - 1)g_a/(3\Omega) + 2(N + 2)g_c/(3\Omega) \) and \( Y_q = e_q^g e_q^b/2 + (N - 1)g_b/(3\Omega) + 2(N + 2)g_c/(3\Omega) \) + \( R \), where \( R = 2(N + 2)g_c[(N - 1)(g_a + g_b - 2g_s) + 2(N + 1)g_d + 2(N + 2)g_c]/(9\Omega^2) \), \( Y_q = e_q^g e_q^b/2 + F \), \( F = 2(N - 1)g_b + 2(N + 2)g_c e_q^b/(3\Omega) + 2[(N - 1)g_a + (N + 2)g_c] e_q^b/(9\Omega^2) + 4C/3\Omega \) + \( 4e_q^g e_q^b([(N - 1)g_a + (N + 2)g_c] e_q^b/(9\Omega^2) + 2C/3\Omega \), where \( C \equiv \{(N + 2)g_a + (N - 1)g_b, (N + 2)g_c + (N - 1)g_a, (N - 1)g_a - (N + 1)g_d - 2(N + 1)g_c\}/(9\Omega^2) \).

If \( q \to 0 \), \( Z_q \to R, Y_q \to 0 \), hence \( E^{(3)}_q \) is gapless while \( E^{(4)}_q \) is gapped with the gap \( E^{(4)}_q = \sqrt{2R} \), under the condition \( R > 0 \), that is, \((N - 1)(g_a + g_b - 2g_s) + 2(N + 1)g_d + 2(N + 2)g_c > 0 \), or \( g_a + g_b - 2g_s + 4g_d + 2g_c > 0 \) when \( N \gg 1 \).

In the long wavelength limit \( q \to 0 \),

$$E^{(3)}_q \to s^{(3)} q, \hspace{1cm} (48)$$

with the sound velocity

$$s^{(3)} = \frac{2(N + 2)g_c C}{3\Omega R} \left[ \frac{1}{m_a} + \frac{1}{m_b} \right]^{1/2}, \hspace{1cm} (49)$$

\( E^{(3)}_q \) being gapless is subject to the condition \( C/R \geq 0 \). When \( N \gg 1 \), \( s^{(3)} = |(g_c + g_a)(g_c + g_b) - (g_s + 2g_d)^2|/m_a + 1/m_b/[3(g_a + g_b - 2g_s + 4g_d + 2g_c)\Omega] \).

In the long wavelength limit \( q \to 0 \),

$$E^{(4)}_q \to \sqrt{\Delta^2 + c^2\hbar^2 q^2} \hspace{1cm} (50)$$

$$\approx \Delta + \frac{\hbar^2 q^2}{2m_{eff}}, \hspace{1cm} (51)$$

where

$$\Delta = \sqrt{2R} \hspace{1cm} (52)$$

is the energy gap at \( q = 0 \), similar to the energy due to the rest mass of a relativistic particle,

$$c^2 = \frac{N - 1}{3\Omega} \left( \frac{g_a}{m_a} + \frac{g_b}{m_b} \right) + \frac{2(N + 2)}{3\Omega} \left( \frac{g_c}{m_a} + \frac{g_e}{m_b} \right)(1 - \frac{C}{R}) \hspace{1cm} (53)$$

is a constant similar to the square of the speed of light for a relativistic particle,

$$m_{eff} = \frac{\Delta}{c^2} \hspace{1cm} (54)$$

is the effective rest mass of the particle-like excitation near \( q = 0 \). Therefore, in the long wavelength limit, while the three excitations \( E^{(1)}_q, E^{(2)}_q \) and \( E^{(3)}_q \) behave like massless particles, the fourth excitation \( E^{(4)}_q \) behaves like a massive Klein-Gordon particle, under the condition \( R > 0 \) and \( c^2 \geq 0 \).

In the short wavelength limit \( q \to \infty \),

$$E^{(3)}_q \to e^a_q, \hspace{1cm} E^{(4)}_q \to e^b_q \hspace{1cm} (55)$$

if \( m_a \geq m_b \), \( E^{(3)}_q \to e^b_q, \hspace{1cm} E^{(4)}_q \to e^a_q \) if \( m_b \geq m_a \).

It is always the case that when \( q \to \infty \), the excitations reduce to the free particle spectra \( e^a_q \) and \( e^b_q \), each being double degenerate as there are two pseudospin states for each species. The effect of interspecies spin exchange is manifested as \( q \to 0 \), in both a nonzero sound velocity of the third excitation and the nonzero gap of the fourth excitation.
VI. NATURE OF SYMMETRY BREAKING

From the point of view of symmetry breaking, Bogoliubov modes are Goldstone modes associated with \( U(1) \) gauge symmetry breaking. In our model, there are three conserved particle numbers \( N_a, N_b \) and \( N_{a\uparrow} + N_{b\uparrow} - N_{a\downarrow} - N_{b\downarrow} = 2S_z \), hence there are three \( U(1) \) symmetries, the breaking of which gives rise to the three Bogoliubov modes. \( U(1) \) group is isomorphic to \( \text{SO}(2) \). Indeed, the \( U(1) \) symmetry generated by \( N_{a\uparrow} + N_{b\uparrow} - N_{a\downarrow} - N_{b\downarrow} \) is just the \( \text{SO}(2) \) symmetry generated by \( S_z \).

It is not difficult to identify \( E_q^{(1)} \) and \( E_q^{(2)} \) as the Goldstone modes associated \( N_a \) and \( N_b \), respectively. In fact, they reduce to the spectra of \( a \)-atoms and \( b \)-atoms if there is no interspecies scattering. \( E_q^{(3)} \) is associated with \( N_{a\uparrow} + N_{b\uparrow} - N_{a\downarrow} - N_{b\downarrow} \). It is interesting that at the long-wavelength limit, \( E_q^{(3)} \) depends only on the spin-exchange scattering.

The gapped mode \( E_q^{(4)} \) is due to the spin exchange between the two species. In fact, the gap vanishes if \( g_e = 0 \). At the isotropic parameter point, the Hamiltonian can be rewritten as \( H = 2K_c S_a \cdot S_b \), where \( S_a \) is the total spin of \( \alpha \)-species. Hence

\[
H = 2K_c S_a S_b \cos \theta,
\]

where \( S_a = N_a/2 \), \( S_b = N_b/2 \), \( \theta \) is the angle between \( S_a \) and \( S_b \). For the ground state, and \( \theta \) is uniquely \( \pi \). Around \( \theta = \pi \), the Hamiltonian is

\[
H_e = 2K_c S_a S_b [-1 + (\delta \theta)^2],
\]

where \( \delta \theta \equiv \theta - \pi \). Therefore \( \delta \theta \) is massive. This leads to the gapped mode.

One can also consider this issue in terms of the phases \( \Phi_{a\sigma} \) of the four components. In the absence of the interspecies spin-exchange part, \( H_e \), of the Hamiltonian, \( N_{a\sigma} \) would be conserved. Then the spontaneous breaking of these four \( U(1) \) symmetries would give four phases \( \Phi_{a\sigma} \)'s, as well as four Goldstone modes, each corresponding to a combination of the four phases. With interspecies spin exchange, the Hamiltonian imposes an extra constraint on these four phases, as

\[
H = 2K_c (n_{a\uparrow} n_{a\downarrow} n_{b\uparrow} n_{b\downarrow})^{1/2} \cos (\Phi_{a\uparrow} + \Phi_{b\downarrow} - \Phi_{a\downarrow} - \Phi_{b\uparrow}),
\]

which fixes one combination of the four phases. By comparison of (56) and (58), it can be identified that

\[
\theta = \Phi_{a\uparrow} + \Phi_{b\downarrow} - \Phi_{a\downarrow} - \Phi_{b\uparrow}.
\]

Around the minimum of \( H_e \),

\[
H_e = 2K_c (n_{a\uparrow} n_{a\downarrow} n_{b\uparrow} n_{b\downarrow})^{1/2} [-1 + (\delta \Phi)^2],
\]

where \( \delta \Phi \equiv \Phi_{a\uparrow} + \Phi_{b\downarrow} - \Phi_{a\downarrow} - \Phi_{b\uparrow} - \pi \). Hence the mode corresponding to \( \delta \Phi \) becomes gapped. This gapped mode corresponds to the source term \( S_{a\sigma} \) in Eq. (12). Therefore there are only three Goldstone modes remained. The gapped mode is due to the fluctuation of \( \theta = \Phi_{a\uparrow} + \Phi_{b\downarrow} - \Phi_{a\downarrow} - \Phi_{b\uparrow} \).

In the number conserved ground state \( |G_0 \rangle \), \( \langle a^\dagger_{\alpha} a_{\beta} \rangle = \langle a^\dagger_{\alpha'} a_{\beta'} \rangle = N/2 \), implying off-diagonal long-range order in each pseudospin component of each species. Equivalently, in the language of gauge symmetry breaking, \( U(1) \) gauge symmetry for each pseudospin component of each species is broken in the symmetry breaking ground state, i.e. \( \langle a_{a\sigma} \rangle = \sqrt{N/2} e^{i\Phi_{a\sigma}} \neq 0 \).

On the other hand, EBEC means \( \langle a_{\uparrow} b_{\downarrow} - a_{\downarrow} b_{\uparrow} \rangle \neq 0 \), as can be justified as follows. In the symmetry breaking ground state, \( \langle a_{\uparrow} b_{\downarrow} - a_{\downarrow} b_{\uparrow} \rangle = (N/2) \{ \exp[i(\Phi_{a\uparrow} + \Phi_{b\downarrow})] - \exp[i(\Phi_{a\downarrow} + \Phi_{b\uparrow})] \} \), as \( \Phi_{a\uparrow} + \Phi_{b\downarrow} - \Phi_{a\downarrow} - \Phi_{b\uparrow} = \pi \) in the ground state, we have \( \exp[i(\Phi_{a\uparrow} + \Phi_{b\downarrow})] = - \exp[i(\Phi_{a\downarrow} + \Phi_{b\uparrow})] \). Therefore \( \langle a_{\uparrow} b_{\downarrow} - a_{\downarrow} b_{\uparrow} \rangle = N \exp[i(\Phi_{a\uparrow} + \Phi_{b\downarrow})] \). Therefore, differing from the case of molecular BEC, we have \( \langle a_{a\uparrow} \rangle \neq 0 \), \( \langle a_{a\downarrow} \rangle \neq 0 \), as well as \( \langle a_{\uparrow} b_{\downarrow} - a_{\downarrow} b_{\uparrow} \rangle \neq 0 \), i.e. both atoms and the nonlocal interspecies entangled pairs are Bose-condensed. These field theoretical discussions are all consistent with the exact results in the particle-number conserved ground state above.

VII. SUMMARY

To summarize, we have studied condensate dynamics of EBEC, which exhibits peculiar superfluidity. EBEC leads to various counter relations among the four superfluid components. The total supercurrent and the total spin supercurrent are conserved. For the homogeneous case, we have also studied the elementary excitations due
to fluctuations of the four orbital wave functions around those in the ground state of the system. There are four modes, three of which are Bogoliubov-like while another is gapped. The three Bogoliubov modes correspond to the spontaneous breaking of the $U(1)$ symmetries associated with three conserved particle numbers, while the gapped mode is associated with the spin exchange. Alternatively, the excitations can be understood as that the interspecies spin exchange gives mass to a certain combination of the four Bogoliubov modes corresponding to the four components of the system, hence there are only three gapless modes remaining.

The emergence of massless elementary excitations of superfluids have been regarded as a paradigm showing how effective theory emerges from physics above the “Planckian” scale \cite{14}. In this perspective, our model provides a way generating massive Klein-Gordon particles.

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