Asymmetric extension of Pascal-Dellanoy triangles

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Abstract
We give a generalization of the Pascal triangle called the quasi s-Pascal triangle where the sum of the elements crossing the diagonal rays produce the s-bonacci sequence. For this, consider a lattice path in the plane whose step set is \( \{ L = (1, 0), L_1 = (1, 1), L_2 = (2, 1), \ldots, L_s = (s, 1) \} \); an explicit formula is given. Thereby linking the elements of the quasi s-Pascal triangle with the bi

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1 Introduction
A lattice path in the plane-\((x, y)\) is a set of edges \( \{ p_0, p_1, \ldots, p_n \} \) in \( \mathbb{Z}^2 \), such that two edges are related by one vertex, the set of vertices connecting \( p_0 \) to \( p_n \) is called a lattice path. Several authors studied and enumerated the lattice path, Mohanty and Handa \[18\] enumerate the unrestricted lattice paths from \((0, 0)\) to \((n, k)\) where \( u \) diagonal steps are allowed at each position, Dziewianczuk \[16\] count the lattice path with four steps horizontal \( H = (1, 0) \), vertical \( V = (0, 1) \), diagonal \( D = (1, 1) \), and sloping \( L = (-1, 1) \), Fray and Roselle \[17\] determine the number of unrestricted weighted lattice paths from \((0, 0)\) to \((n, k)\) with horizontal, vertical, and diagonal steps, Rohatgi \[22\] enumerate the paths which must remain below the line \( y = ax + b \) where the diagonal steps are allowed in addition to the usual horizontal and vertical steps. In a Pascal triangle, the binomial coefficients \( \binom{n}{k} \) count the number of lattice paths from \((0, 0)\) to \((n, k)\) using the steps \( \{ H = (1, 0) \rightarrow, D = (1, 1) \} \), see sloane \[23\] A007318.
It is well known that the terms of Fibonacci sequence \((F_n)_n\) are obtained by summing the elements crossing the principal diagonal rays in the Pascal triangle,

\[ F_{n+1} = \sum_k \binom{n-k}{k}, \]

where \( \binom{n}{k} = \frac{n!}{k!(n-k)!} \) for \( n \geq k \geq 0 \) and \( \binom{n}{k} = 0 \) otherwise.
The generating function of the binomial coefficients is given by

\[ \sum_{n \geq 0} \binom{n}{k} x^n = \frac{x^k}{(1-x)^{k+1}}. \]
Alladi and Hoggat \cite{AlladiHoggat} extended the Pascal triangle. They established the Tribonacci triangle and proved that the sum of elements lying over the principal diagonal rays in the Tribonacci triangle gives the Tribonacci sequence

\[ T_{n+1} = T_n + T_{n-1} + T_{n-2}, \]

with \( T_0 = 0, T_1 = 1, T_2 = 1 \).

Denote by \( \binom{n}{k}[2] \) the element in the \( n^{th} \) row and \( k^{th} \) column of the Tribonacci triangle, the triangle is produced by the recurrence relation

\[ \binom{n}{k}[2] = \binom{n-1}{k}[2] + \binom{n-1}{k-1}[2] + \binom{n-2}{k-1}[2], \]

where \( \binom{n}{0}[2] = \binom{n}{n}[2] = 1 \). We use the convention \( \binom{n}{k}[2] = 0 \) for \( k \notin \{0, \ldots, n\} \).

Moreover, Barry \cite{Barry} has shown that for \( 0 \leq k \leq n \) these coefficients satisfy the relation

\[ \binom{n}{k}[2] = \sum_{j=0}^{k} \binom{k}{j} \binom{n-j}{k}. \] (3)

![Table 1. Tribonacci triangle.](image)

We find in \textsc{sloane} \cite{Sloane} A008288 that \( \binom{n}{k}[2] \) counts the number of lattice paths from \((0,0)\) to \((n,k)\) using the steps \( \{H = (1,0), D = (1,1), L = (2,1)\} \).

In what follows \( s \) is a positive integer.

1.1 The \( s \)-Pascal triangle

Let \( k \in \{0, 1, \ldots, sn\} \), the bi\(^*\)nomial coefficient \( \binom{n}{k}_s \) is defined as the \( k^{th} \) coefficient in the expansion

\[ (1 + x + x^2 + \cdots + x^s)^n = \sum_{k \in \mathbb{Z}} \binom{n}{k}_s x^k, \]

with \( \binom{n}{k}_s = 0 \) for \( k > sn \) or \( k < 0 \).

Using the classical binomial coefficient, see \cite{AlladiHoggat, Sloane, Barry}, one has

\[ \binom{n}{k}_s = \sum_{j_1+j_2+\cdots+j_s=k} \binom{n}{j_1} \binom{j_1}{j_2} \cdots \binom{j_{s-1}}{j_s}. \] (4)

Some other readily well known established properties are:

the symmetry relation

\[ \binom{n}{k}_s = \binom{n}{sn-k}_s, \] (5)
the longitudinal recurrence relation
\[ \binom{n}{k}_s = \sum_{j=0}^{s} \binom{n-1}{k-j}_s, \] (6)
the diagonal recurrence relation
\[ \binom{n}{k}_s = \sum_{j=0}^{n} \binom{n}{j}_s \binom{j}{k-j}_{s-1}, \] (7)
and de Moivre expression see [19, 20]
\[ \binom{n}{k}_s = \sum_{j=0}^{n} (-1)^j \binom{n}{j}_s \binom{k-(s+1)+n-1}{n-1}. \] (8)

Belbachir and Benmezai [6] gave the following relation
\[ \binom{n}{k}_s = (-1)^k \sum_{j_1+j_2+\cdots+j_s=k} \binom{n}{j_1}_s \binom{n}{j_2}_s \cdots \binom{n}{j_s}_s a^{-\sum_{r=1}^{s} r j_r}, \] (9)
where \( a = \exp(2i\pi/(s+1)) \).

These coefficients, as for the usual binomial coefficients, are built, as for the Pascal triangle, the called s-Pascal triangle. One can find the first values of the s-Pascal triangle in sloane A027907 for \( s = 2 \), A008287 for \( s = 3 \) and A035343 for \( s = 4 \).

| \( n \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 |   |   |   |   |   |   |   |   |   |   |   |   |
| 1 | 1 | 1 | 1 | 1 |   |   |   |   |   |   |   |   |   |
| 2 | 1 | 2 | 3 | 4 | 3 | 2 | 1 |   |   |   |   |   |   |
| 3 | 1 | 3 | 6 | 10 | 12 | 12 | 10 | 6 | 3 | 1 |   |   |   |
| 4 | 1 | 4 | 10 | 20 | 31 | 40 | 44 | 40 | 31 | 20 | 10 | 4 | 1 |
| 5 | 1 | 5 | 15 | 35 | 65 | 101 | 135 | 155 | 155 | 135 | 101 | 65 | 35 | 15 |

Table 2. Biquadranomial triangle \((s = 3)\).

1.2 The \( q \)-binomial coefficient

The \( q \)-analogue of binomial coefficient or the \( q \)-binomial coefficient \([n]_k\) generalizes the binomial coefficient \( \binom{n}{k} \) [10] [14]. It is defined as follows
\[ \binom{n}{k} = \frac{[n]_q!}{[k]_q! [n-k]_q!} q^{\binom{k}{2}}, \]
with \( [n]_q = 1 + q + q^2 + \cdots + q^{n-1} \) and \( [n]_q! = [1]_q [2]_q \cdots [n]_q \), we use the convention \( \binom{n}{k} = 0 \) for \( k \notin \{0, \ldots, n\} \).

The \( q \)-binomial coefficient satisfies the following recurrence relations
\[ \binom{n}{k} = \binom{n-1}{k} + q^{n-1} \binom{n-1}{k-1}, \] (10)
and
\[ \binom{n}{k} = q^k \binom{n-1}{k} + q^{k-1} \binom{n-1}{k-1}. \] (11)
And the generating functions are

\[
\sum_{k=0}^{n} \binom{n}{k} x^k = (1 + x)(1 + qx)(1 + q^2x) \cdots (1 + q^{n-1}x),
\]

(12)

and

\[
\sum_{n \geq 0} \binom{n}{k} x^n = \frac{x^k q^{\binom{k}{2}}}{(1 - x)(1 - qx) \cdots (1 - q^kx)}.
\]

(13)

Belbachir and Benmezai \[6\] proposed the q-bi\(^s\)nomial coefficient, denoted by \(\binom{n}{k}\)\(^{(s)}\), as follows

\[
\binom{n}{k}\(^{(s)}\) := (-1)^k \sum_{j_1 + j_2 + \cdots + j_s = k} \binom{n}{j_1} \binom{n}{j_2} \cdots \binom{n}{j_s} a^{-\sum_{r=1}^{s} r j_r}.
\]

(14)

The q-bi\(^s\)nomial coefficient satisfies the following recurrence relations

\[
\binom{n}{k}\(^{(s)}\) = \sum_{j=0}^{s} q^{k-j} \binom{n-1}{k-j},
\]

(15)

\[
\binom{n}{k}\(^{(s)}\) = \sum_{j=0}^{s} q^{(n-1)j} \binom{n-1}{k-j}.
\]

(16)

According to (12) the generating function is given by

\[
\sum_{k=0}^{n} \binom{n}{k}\(^{(s)}\) x^k = \prod_{j=0}^{n-1} (1 + q^j x + (q^j x)^2 + \cdots + (q^j x)^s).
\]

(17)

In the first section we introduce the quasi s-Pascal triangle by using a family of lattice paths; we establish an explicit formula for the elements of the quasi s-Pascal triangle, and we prove that the sums of the elements crossing the diagonal rays yield the terms of s-bonacci sequence; we close this section by giving a relation between s-Pascal triangle and quasi s-Pascal triangle. The second section is devoted to the linear recurrence relation obtained by summing the elements lying over any finite rays of the quasi s-Pascal triangle and we give the corresponding generating function. In the third section we give the de Moivre like summation with some other identities. In section four, we establish the q-analogue of the elements of the quasi s-Pascal triangle.

2 The quasi s-Pascal triangle

In this section we define the quasi s-Pascal triangle, we denote by \(\binom{n}{k}\)\([s]\) the coefficient in the \(n^{th}\) row and \(k^{th}\) column of the quasi s-Pascal triangle.

**Definition 1.** The quasi-bi\(^s\)nomial coefficient \(\binom{n}{k}\)\([s]\) is defined by the number of lattice path from \((0, 0)\) to \((n, k)\) with steps in \(\{L = (1, 0), L_1 = (1, 1), L_2 = (2, 1), \ldots, L_s = (s, 1)\}\). With \(\binom{n}{0}\)\([s]\) = \(\binom{n}{n}\)\([s]\) = 1 and the convention \(\binom{n}{k}\)\([s]\) = 0 for \(k > n\) or \(k < 0\).
Lemma 2. The quasi-binomial coefficient \( \binom{n}{k}_{[s]} \) satisfies the following recurrence relation

\[
\binom{n}{k}_{[s]} = \binom{n-1}{k}_{[s]} + \binom{n-1}{k-1}_{[s]} + \binom{n-2}{k-1}_{[s]} + \cdots + \binom{n-s}{k-1}_{[s]},
\]

(18)

Proof. By Definition [1] the last step of any path is one of \( \{L = (1, 0), L_1 = (1, 1), L_2 = (2, 1), \ldots, L_s = (s, 1)\} \), then if the last step is \( L = (1, 0) \), it remains to enumerate the number of lattice paths from \((0, 0)\) to \((n - 1, k)\) which is \( \binom{n-1}{k}_{[s]} \), or if the last one is \( L_1 = (1, 1) \) it remains to enumerate the number of lattice paths from \((0, 0)\) to \((n - 1, k - 1)\) which is \( \binom{n-1}{k-1}_{[s]} \), . . . , if the last step is \( L_s = (s, 1) \) it remains to enumerate the number of lattice paths from \((0, 0)\) to \((n - s, k - 1)\) which is \( \binom{n-s}{k-1}_{[s]} \), considering all possibilities we construct our recurrence. \( \square \)

2.1 Generating function

Here is given the generating function of \( \{ \binom{n}{k}_{[s]} \}_n \):

Theorem 3. Let \( F_k(x) := \sum_{n \geq 0} \binom{n}{k}_{[s]} x^n \), then

\[
F_k(x) = (1 + x + x^2 + \cdots + x^{s - 1})^k \frac{x^k}{(1 - x)^{k+1}}.
\]

Proof. It follows from Relation (18) that

\[
F_k(x) = x F_k(x) + x F_{k-1}(x) + x^2 F_{k-1}(x) + \cdots + x^s F_{k-1}(x),
\]

repeated applications of this recurrence give the result. \( \square \)

2.2 Binomial coefficients explicit formula

The following result gives an explicit formula for the coefficients of the quasi s-Pascal triangle in terms of binomial coefficients and a variant with multinomial coefficients.

Theorem 4. The explicit formula for the quasi-binomial coefficient is given by

\[
\binom{n}{k}_{[s]} = \sum_{j_1} \sum_{j_2} \cdots \sum_{j_{s-1}} \binom{k}{j_1} \binom{j_1}{j_2} \cdots \binom{j_{s-2}}{j_{s-1}} \binom{n - \sum_{i=1}^{s-1} j_i}{k},
\]

(19)

the multinomial version is

\[
\binom{n}{k}_{[s]} = \sum_{k_1, k_2, \ldots, k_s} \binom{k}{k_{1, k_2, \ldots, k_s}} \binom{n + k - \sum_{i=1}^{s} i k_i}{k},
\]

(20)

where \( \binom{k}{k_{1, k_2, \ldots, k_s}} = \frac{k!}{k_1! k_2! \cdots k_s!} \) for \( k_1 + k_2 + \cdots + k_s = k \) and \( \binom{k}{k_{1, k_2, \ldots, k_s}} = 0 \), else.
Proof. For Relation (19) we need to prove that

\[
\sum_{n=0}^{\infty} \sum_{j_1} \sum_{j_2} \cdots \sum_{j_{s-1}} \left( \frac{k}{j_1} \right) \left( \frac{j_1}{j_2} \right) \cdots \left( \frac{j_{s-2}}{j_{s-1}} \right) \left( n - \sum_{i=1}^{s-1} j_i \right) x^n = \left( \frac{x + x^2 + \cdots + x^s}{1 - x} \right)^k \frac{1}{1 - x}.
\]

So

\[
\sum_{n=0}^{\infty} \sum_{j_1} \sum_{j_2} \cdots \sum_{j_{s-1}} \left( \frac{k}{j_1} \right) \left( \frac{j_1}{j_2} \right) \cdots \left( \frac{j_{s-2}}{j_{s-1}} \right) \left( n - \sum_{i=1}^{s-1} j_i \right) x^n
\]

\[
= \sum_{j_1} \left( \frac{k}{j_1} \right) x^{j_1} \sum_{j_2} \left( \frac{j_1}{j_2} \right) x^{j_2} \cdots \sum_{j_{s-2}} \left( \frac{j_{s-2}}{j_{s-2}} \right) x^{j_{s-2}} \sum_{n=0}^{\infty} \left( n - \sum_{i=1}^{s-1} j_i \right) x^{n-\sum_{i=1}^{s-1} j_i}
\]

\[
= \frac{x^k}{(1 - x)^{k+1}} \sum_{j_1} \left( \frac{k}{j_1} \right) x^{j_1} \sum_{j_2} \left( \frac{j_1}{j_2} \right) x^{j_2} \cdots \sum_{j_{s-2}} \left( \frac{j_{s-2}}{j_{s-2}} \right) x^{j_{s-2}} \sum_{j_{s-1}} \left( \frac{j_{s-2}}{j_{s-1}} \right) x^{j_{s-2}}
\]

\[
= \frac{x^k}{(1 - x)^{k+1}} \sum_{j_1} \left( \frac{k}{j_1} \right) x^{j_1} \sum_{j_2} \left( \frac{j_1}{j_2} \right) x^{j_2} \cdots \sum_{j_{s-2}} \left( \frac{j_{s-2}}{j_{s-2}} \right) x^{j_{s-2}} (x + x^2)^{j_{s-2}}
\]

\[
\vdots
\]

\[
= \frac{x^k}{(1 - x)^{k+1}} \sum_{j_1} \left( \frac{k}{j_1} \right) (x + x^2 + \cdots + x^{s-1})^{j_1} = \left( \frac{x + x^2 + \cdots + x^{s-1}}{1 - x} \right)^k \frac{1}{1 - x}.
\]

For Relation (20) we have

\[
\sum_{n \geq 0} \sum_{k_1, k_2, \ldots, k_s} \left( \frac{k}{k_1, k_2, \ldots, k_s} \right) \left( n - \sum_{i=1}^{s-1} ik_i + k \right) x^n
\]

\[
= \frac{1}{x^s} \sum_{k_1, k_2, \ldots, k_s} \left( \frac{k}{k_1, k_2, \ldots, k_s} \right) x^{\sum_{i=1}^{s} ik_i} \sum_{n \geq 0} \left( n - \sum_{i=1}^{s-1} ik_i + k \right) x^{n-\sum_{i=1}^{s} ik_i}
\]

\[
= \frac{1}{x^s} \sum_{k_1, k_2, \ldots, k_s} \left( \frac{k}{k_1, k_2, \ldots, k_s} \right) x^{\sum_{i=1}^{s} ik_i}
\]

\[
= \left( \frac{x + x^2 + \cdots + x^s}{1 - x} \right)^k \frac{1}{1 - x}.
\]

\[
\square
\]

2.3 Link with generalized Dellanoy matrix

Ramirez and Sirvent [21] propose a generalization of Dellanoy and Pascal Riordan arrays, they denoted by \( D_m(n, k) \) the element in the \( i^{th} \) row and \( k^{th} \) column of the generalized Dellanoy matrix, such that \( D_m(n, k) \) satisfy the following recurrence relation

\[
D_m(n + 1, k) = aD_m(n + 1, k - 1) + \sum_{i=0}^{m-1} a_{i+1} D_m(n, k - i),
\]

with \( k \geq m - 1, n \geq 1 \) and initial conditions \( D_m(0, k) = a^k \) and \( D_m(n, 0) = a^n \).

The coefficient \( D_m(n, k) \) is given by the following explicit formula

\[
D_m(n, k) = \sum_{j_1, j_2, \ldots, j_{m-1}} \binom{n}{j_1} \binom{n-j_1}{j_2} \cdots \binom{n-j_1-j_{m-2}}{j_{m-1}} \binom{n+k-u}{n} \times
\]

\[
\times a_1^{j_1} a_2^{j_2} \cdots a_{m-1}^{j_{m-1}} \alpha_m \sum_{i=1}^{m-1} j_i \alpha_{k-u},
\]
where \( u = (m - 1)(n - j_1) + \sum_{i=2}^{m-1} (i - m)j_i \).

The following result gives a relation between \( D_m(n, k) \) and \( \binom{n}{k} \).

**Theorem 5.** For \( m = s, a = 1 \) and \( a_i = 1, i \in \{1, \ldots, s\} \)
\[
D_s(k, n-k) = \binom{n}{k}_{[s]}.
\]

**Proof.** For \( a = 1 \) and \( a_i = 1, i \in \{1, \ldots, s\} \) we have
\[
D_s(n, k) = \sum_{j_1, j_2, \ldots, j_{s-1}} \binom{n}{j_1} \binom{n-j_1}{j_2} \cdots \binom{n-j_1-\cdots-j_{s-2}}{j_{s-1}} \binom{n+k-u}{n},
\]
where \( u = (s - 1)(n - j_1) + \sum_{i=2}^{s-1} (i - s)j_i \), then
\[
D_s(n, k) = \sum_{j_1, j_2, \ldots, j_{s-1}} \binom{n}{n-j_1} \binom{n-j_1}{n-j_1-j_2} \cdots \binom{n-j_1-\cdots-j_{s-2}}{n-j_1-\cdots-j_{s-2}-j_{s-1}} \binom{n+k-u}{n},
\]
we put \( j'_v \rightarrow n - \sum_{i=1}^{v} j_i, v \in \{1, \ldots, s-1\} \) then \( j'_1 + j'_2 + \cdots + j'_{s-1} = u \),
\[
D_s(n, k) = \sum_{j'_1, j'_2, \ldots, j'_{s-1}} \binom{n}{j'_1} \binom{j'_1}{j'_2} \cdots \binom{j'_{s-2}}{j'_{s-1}} \binom{n+k-s-1}{n}.
\]
Thus \( D_s(k, n-k) = \binom{n}{k}_{[s]} \).

\[\Box\]

### 2.4 Recurrence relation or \( s \)-bonacci sequence

Now we establish the recurrence relation for the \( s \)-bonacci sequence, which is a generalization of Fibonacci sequence. Let \( (T_{n,s})_n \) be the terms of the \( s \)-bonacci sequence obtained by summing the elements lying over the principal diagonal rays in the quasi \( s \)-Pascal triangle.

Let be the sequence
\[
T_{n+1,s} := \sum_k \binom{n-k}{k}_{[s]},
\]
with \( T_{0,s} = 0 \).

**Theorem 6.** For \( n \geq 0 \), \( (T_{n,s})_n \) satisfies the following recurrence relation
\[
T_{n+1,s} = T_{n,s} + T_{n-1,s} + \cdots + T_{n-s,s},
\]
with \( T_{1,s} = 1, T_{i,s} = 0 \) for \( i \in \{0, -1, \ldots, -(s-1)\} \).

It is not else \( s \)-bonacci sequence.

**Proof.** We have \( T_{n+1,s} = \sum_k \binom{n-k}{k}_{[s]} \) and by Relation 118 we obtain
\[
T_{n+1,s} = \sum_k \binom{n-k-1}{k}_{[s]} + \sum_k \binom{n-k-1}{k-1}_{[s]} + \cdots + \sum_k \binom{n-k-s}{k-1}_{[s]},
\]
\[
= \sum_k \binom{n-k-1}{k}_{[s]} + \sum_{k'} \binom{n-k'-2}{k'-1}_{[s]} + \cdots + \sum_{k'} \binom{n-k'-s-1}{k'-1}_{[s]},
\]
\[
= T_{n,s} + T_{n-1,s} + \cdots + T_{n-s,s}.
\]
\[\Box\]
For $s = 1$ and $s = 2$ we obtain the terms of Fibonacci and Tribonacci sequences respectively.

**Example 7.** For $s = 3$ we have the quadrabonacci triangle

| $n$, $k$ | 0   | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   |
|----------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0        | 1   |     |     |     |     |     |     |     |     |     |
| 1        | 1   | 1   |     |     |     |     |     |     |     |     |
| 2        | 1   | 3   | 1   |     |     |     |     |     |     |     |
| 3        | 1   | 6   | 5   | 1   |     |     |     |     |     |     |
| 4        | 1   | 9   | 15  | 7   | 1   |     |     |     |     |     |
| 5        | 1   | 12  | 33  | 28  | 9   | 1   |     |     |     |     |
| 6        | 1   | 15  | 60  | 81  | 45  | 11  | 1   |     |     |     |
| 7        | 1   | 18  | 96  | 189 | 66  | 33  | 13  | 1   |     |     |
| 8        | 1   | 21  | 141 | 378 | 459 | 281 | 91  | 15  | 1   |     |
| 9        | 1   | 24  | 195 | 675 | 1107| 946 | 449 | 120 | 17  | 1   |

Table 2. The quadrabonacci triangle.

By Relation (18) the elements of the quadrabonacci triangle ($s = 3$) are given by $(\binom{n}{k})_3 = (\binom{n}{n})_3 = 1$, and

$$
\binom{n}{k}_3 = \binom{n-1}{k}_3 + \binom{n-1}{k-1}_3 + \binom{n-2}{k-1}_3 + \binom{n-3}{k-1}_3.
$$

For $s = 3$, $(\binom{n}{k})_3$ counts the number of lattice paths with steps in $(1,0)$, $(1,1)$, $(2,1)$, $(3,1)$ from $(0,0)$ to $(n,k)$, for example for the value 6 in the quadrabonacci triangle we have the lattice path

![Illustration of possible paths from (0,0) to (3,1) using the steps (1,0), (1,1), (2,1), (3,1).](image)

Notice that for $s = 1$ and $s = 2$ the obtained triangles are symmetric, unlike the cases where $s > 2$.

### 2.5 $s$-Pascal triangle versus quasi-\(s\)-Pascal triangle

The following result establishes the relation between the quasi $s$-Pascal triangle and $s$-Pascal triangle

**Theorem 8.** For fixed non negative integers $n$, $k$ and $s$, we have

$$
\binom{n}{k}_s = \sum_i \binom{n-i}{k}_s \binom{k}{i}_{s-1}.
$$
Proof. We have

\[
\binom{n}{k}_{[s]} = \sum_{j_1} \sum_{j_2} \cdots \sum_{j_{s-1}} \binom{k}{j_1} \binom{j_1}{j_2} \cdots \binom{j_{s-2}}{j_{s-1}} \binom{n - j_1 - j_2 - \cdots - j_{s-1}}{k},
\]

considering the summations by blocks \( j_1 + j_2 + \cdots + j_{s-1} = i \) we get

\[
\binom{n}{k}_{[s]} = \sum_i \left( \binom{n - i}{k} \sum_{j_1 + j_2 + \cdots + j_{s-1} = i} \binom{k}{j_1} \binom{j_1}{j_2} \cdots \binom{j_{s-2}}{j_{s-1}} \right) = \sum_i \left( \binom{n - i}{k} \binom{k}{i} \right)_{s-1}.
\]

3 Linear recurrence relation and generating function associated to finite transversals of the quasi s-Pascal triangle

This section is devoted to establish a recurrence relation associated to the sums of the elements lying over the transversals of direction \((\alpha, r)\) in the quasi \(s\)-Pascal triangle. In [8] we find the details about the concept of direction in Pascal triangle. The study was extended for the arithmetic triangle, see [3, 9]. We generalize the concept to our triangle as an extension of Theorem 6 to the case where \(r \in \mathbb{Z}, \alpha \in \mathbb{N}, \beta \in \mathbb{Z}^+ \) with \(0 \leq \beta < \alpha \) and \(r + \alpha > 0\). This corresponds to the finite sequences lying over finite transversals of the quasi \(s\)-Pascal triangle.

Let be the sequence

\[
T_{n+1,s}^{(\alpha,\beta,r)} := \sum_k \binom{n - rk}{\beta + ak}_{[s]}, \text{ with } T_{0,s}^{(\alpha,\beta,r)} = 0.
\]

The following figure illustrate the direction \((\alpha, r) = (2, 1)\) and \(\beta = 0\) on the Tribonacci triangle.

![Tribonacci triangle](image)

**Figure 1:** Tribonacci triangle.

**Theorem 9.** For \(n \geq \alpha s + r\), \(T_{n+1,s}^{(\alpha,\beta,r)}\) satisfies the following linear recurrence relation

\[
\sum_{i=0}^{\alpha} (-1)^i \binom{\alpha}{i} T_{n-i,s}^{(\alpha,\beta,r)} = \sum_{i=0}^{\alpha} \binom{\alpha}{i} T_{n-s-i,r-s}^{(\alpha,\beta,r)}, \quad (21)
\]

we can recover the initial conditions \(T_{1,s}^{(\alpha,\beta,r)}, \ldots, T_{\alpha s+r-s,s}^{(\alpha,\beta,r)}\) by \(\sum_k \binom{n-rk}{\beta+ak}_{[s]}\).

For the proof we need the following Lemma.
Lemma 10. Let $a,b$ and $\alpha$ be non-negative integers satisfying the conditions $\alpha \leq a$, then

$$\sum_{i=0}^{\alpha} (-1)^i \binom{\alpha}{i} \binom{a-i}{b} = \binom{a-\alpha}{b-\alpha}.$$  \hspace{1cm} (22)$$

Proof. Of Theorem 9 from Relations 13 and 22, we get

$$\sum_{i=0}^{\alpha} (-1)^i \binom{\alpha}{i} T_{n-i,s}^{(\alpha,\beta,r)}$$

$$= \sum_{i=0}^{\alpha} (-1)^i \binom{\alpha}{i} \sum_k \binom{n-rk-i-1}{\beta+ak}$$

$$= \sum_{i=0}^{\alpha} (-1)^i \binom{\alpha}{i} \sum_{j_1,j_2,\ldots,j_{s-1}} \binom{\beta+ak}{j_1} \binom{j_1}{j_2} \cdots \binom{j_{s-2}}{j_{s-1}} \binom{n-rk-i-\sum_{i=1}^{s-1} j_i-1}{\beta+ak}$$

$$= \sum_{j_1,j_2,\ldots,j_{s-1}} \binom{\beta+ak}{j_1} \binom{j_1}{j_2} \cdots \binom{j_{s-2}}{j_{s-1}} \binom{n-rk-\alpha-\sum_{i=1}^{s-1} j_i-1}{\beta+\alpha(k-1)}$$

$$(k' \to k-1)$$

$$= \sum_{k'} \sum_{j_1,j_2,\ldots,j_{s-1}} \binom{\beta+ak'}{j_1} \binom{j_1}{j_2} \cdots \binom{j_{s-2}}{j_{s-1}} \binom{n-rk'-r-\alpha-\sum_{i=1}^{s-1} j_i-1}{\beta+ak'}$$

$$= \sum_{k'} \sum_{j_1,j_2,\ldots,j_{s-1}} \binom{\beta+ak'}{j_1} \binom{j_1-i_1+i_1}{j_2} \cdots \binom{j_{s-2}-i_{s-2}+i_{s-2}}{j_{s-1}} \times \frac{(n-rk'-r-\alpha-\sum_{i=1}^{s-1} j_i-1)}{\beta+ak'}$$

by Vandermonde Formula

$$\implies \sum_{k'} \sum_{j_1,j_2,\ldots,j_{s-1}} \binom{\alpha}{i_1} \binom{\beta+ak'}{j_1-i_1} \binom{j_1-i_1}{i_2} \binom{j_2-i_2}{i_2} \cdots \binom{j_{s-2}-i_{s-2}+i_{s-2}}{i_{s-2}} \times \frac{(n-rk'-r-\alpha-\sum_{i=1}^{s-1} j_i-1)}{\beta+ak'}$$

$$(l_v \to j_v - l_v)$$

$$= \sum_{k'} \sum_{l_1,l_2,\ldots,l_{s-1}} \binom{\alpha}{i_1} \binom{\beta+ak'}{l_1} \binom{l_1}{l_2} \cdots \binom{l_{s-2}}{l_{s-2}} \times \frac{(n-rk'-r-\alpha-\sum_{j=1}^{s-1} i_j-\sum_{j=1}^{s-1} l_j-1)}{\beta+ak'}$$

we take the summation as by block $i_1+i_2+\cdots+i_{s-1} = i$
\[= \sum_{k=1}^{\infty} \sum_{l_1, l_2, \ldots, l_{s-1}, l_1+l_2+\cdots+l_{s-1}=i} \left( \alpha \sum_{i=1}^{l_1} (\beta + \alpha k') (l_1) \left( \frac{i_2}{l_2} \right) \ldots \left( \frac{i_{s-2}}{l_{s-1}} \right) \left( \frac{l_{s-2}}{l_{s-1}} \right) \times \left( n - r k' - r - \alpha - i - \sum_{j=1}^{s-1} l_j - 1 \right) \beta + \alpha k' \right) \times \sum_{i=0}^{s-1} \binom{\alpha}{i} s_{n-\alpha-r-i,s}^{(\alpha, \beta, r)} \]

For \( \alpha = 1, r = 1, \beta = 0 \) we obtain the terms of \( s \)-bonacci sequence, for \( s = 2 \), we obtain Theorem 7 of [2].

**Example 11.** For \( \alpha = 2, r = 1, \beta = 0 \) and \( n \geq 2s + 1 \) we have the following recurrence relation

\[
T_{n,s}^{(2,0,1)} = 2 \sum_{j=0}^{2} \binom{2}{j} T_{n-j,s}^{(2,0,1)} + 2 \sum_{j=0}^{2} \binom{2}{j} T_{n-j-3,s}^{(2,0,1)}
\]

\[
= 2T_{n-1,s}^{(2,0,1)} - T_{n-2,s}^{(2,0,1)} + T_{n-3,s}^{(2,0,1)} + 2T_{n-4,s}^{(2,0,1)} + \cdots + sT_{n-s-2,s}^{(2,0,1)} + \cdots + 2T_{n-2s,s}^{(2,0,1)} + T_{n-2s-1,s}^{(2,0,1)}
\]

as \( \binom{2}{s-1} = s \).

The following result establishes the generating function for the sequence \( (T_{n,s}^{(\alpha, \beta, r)})_n \) for quasi \( s \)-Pascal triangles.

**Theorem 12.** The generating function of the sequence \( \{T_{n,s}^{(\alpha, \beta, r)}\}_{n \geq 0} \) is given by

\[
\sum_{n \geq 0} T_{n+1,s}^{(\alpha, \beta, r)} x^n = \frac{(1 - x)^{\alpha - r - 1} (x + x^2 + \cdots + x^s)^{\beta}}{(1 - x)^{\alpha - x^{r+\alpha}} (1 + x + \cdots + x^{s-1})^s}
\]

**Proof.** We have

\[
\sum_{n \geq 0} T_{n+1,s}^{(\alpha, \beta, r)} x^n = \sum_{n \geq 0} \sum_{k} \binom{n - rk}{\beta + ak} x^n
\]

\[
= \sum_{n \geq qk} \sum_{k} \binom{n - rk}{\beta + ak} x^{n-rk} x^k
\]

\[
= \sum_{k} \frac{(x + x^2 + \cdots + x^s)^{\beta + ak} x^k}{(1 - x)^{\beta + ak + 1}}
\]
4 The de Moivre summation and other nested sums

Butler and Karasik see [12] showed how the binomial coefficient can be written as nested sums, in this section we establish an identity for the quasi binomial coefficients \( \binom{n}{k}_{[s]} \) equivalent to the de Moivre summation for binomial coefficient and we give some other nested sums for the coefficient \( \binom{n}{k}_{[s]} \). The following identity is important in the sense that it gives a simple summation with a product of two binomials.

**Theorem 13.** The following identity holds true

\[
\binom{n}{k}_{[s]} = \sum_{j} (-1)^j \binom{k}{j} \left( \frac{n - sj}{2k} \right).
\] (23)

**Proof.** By Theorem 3 we have

\[
\sum_{n \geq 0} \binom{n}{k}_{[s]} x^n = \frac{x^k (1 + x + x^2 + \cdots + x^{s-1})^k}{(1 - x)^{k+1}}
\]

\[
= x^k \frac{(1 - x)^k}{(1 - x)^{2k+1}}
\]

\[
= x^k \sum_{j} (-1)^j \binom{k}{j} x^j \sum_{i} \binom{i + 2k}{2k} x^i
\]

\[
= \sum_{n} \sum_{i = sj} (-1)^j \binom{k}{j} \binom{i + 2k}{2k} x^{n+k}
\]

\[
= \sum_{n} \sum_{j} (-1)^j \binom{k}{j} \left( \frac{n - sj + k}{2k} \right) x^n.
\]

Identity (23) is a dual version of Relation (8).

Now we give an identity for \( \binom{n}{k}_{[s]} \) dual to Relation (9).

**Theorem 14.** For \( w = \exp \left( 2\pi i / s \right) \) we have

\[
\binom{n}{k}_{[s]} = \sum_{j} \binom{n - j}{k}_{[s]} (-1)^j \sum_{k_1 + k_2 + \cdots + k_{s-1} = j} \binom{k}{k_1} \binom{k}{k_2} \cdots \binom{k}{k_{s-1}} \times w^{-\sum_{r=1}^{s-1} r k_r}.
\]

**Proof.** By Theorem 3 we have

\[
\sum_{n \geq 0} \binom{n}{k}_{[s]} x^n
\]

\[
= \frac{x^k}{(1 - x)^{k+1}} \prod_{j=1}^{s-1} (x - w^j)^k
\]

\[
= \sum_{n \geq 0} \binom{n}{k}_{[s]} x^n \sum_{k_1} (-1)^{k-k_1} \binom{k}{k_1} \binom{w^{k-k_1} x^{k_1}}{k_2} \sum_{k_2} (-1)^{k-k_2} \binom{k}{k_2} \binom{w^{2(k-k_2)} x^{k_2}}{k_{s-1}} \cdots
\]

\[
\sum_{k_{s-1}} (-1)^{k-k_{s-1}} \binom{k}{k_{s-1}} \binom{w^{(s-1)(k-k_{s-1})} x^{k_{s-1}}}{1}
\]
Proof.

Using the generating function we obtain the following beautiful nested relation.

\[ \binom{n}{k} x^n \sum_{j \geq 0} \sum_{k_1 + k_2 + \cdots + k_{s-1} = j} \binom{k}{k_1} \binom{k}{k_2} \cdots \binom{k}{k_{s-1}} \times (-1)^{(s-1)k-j} u^{k_{s-1}} r(k_{s-1}) x^j \]

\[ = \sum_{n \geq 0} \sum_{j} (-1)^{(s-1)j} u^{\sum_{i=1}^{s-1} r(k_{i})} x^n. \]

This yields the result.

\[ \square \]

Remark 15. We can also deduce Theorem 14 using Theorem 8 and Identity (9).

Using the generating function we obtain the following beautiful nested relation.

**Theorem 16.** The terms of the s-Pascal triangle satisfy the following identity

\[ \binom{n}{k} = \sum_{j_1, j_2, \ldots, j_{s-1}} \binom{k}{j_1} \binom{j_1}{j_2} \cdots \binom{j_{s-2}}{j_{s-1}} \left( n - k - \sum_{i=1}^{s-2} j_i \right) x \times (2)^{j_1} (3/2)^{j_2} \cdots (s/s - 1)^{j_{s-1}} \]

Proof. We have

\[ \sum_{n \geq 0} \sum_{j_1, j_2, \ldots, j_{s-1}} \binom{k}{j_1} \binom{j_1}{j_2} \cdots \binom{j_{s-2}}{j_{s-1}} \left( n - k - \sum_{i=1}^{s-2} j_i \right) x \times (2)^{j_1} (3/2)^{j_2} \cdots (s/s - 1)^{j_{s-1}} \]

\[ = \frac{x}{1-x} \sum_{j_1} \binom{k}{j_1} \left( (2x)^{j_1} \sum_{j_2} \binom{j_1}{j_2} \left( (3x/2)^{j_2} \sum_{j_{s-2}} \binom{j_{s-2}}{j_{s-2}} \left( x(s-1)/(s-2) \right)^{j_{s-2}} \right) \right) \]

\[ \times \left( \frac{(s/s-1)^{j_{s-1}}}{1-x} \right) \]

\[ = \frac{x}{1-x} \sum_{j_1} \binom{k}{j_1} \left( (2x)^{j_1} \sum_{j_2} \binom{j_1}{j_2} \left( (3x/2)^{j_2} \right) \right) \]

\[ \cdots \sum_{j_{s-2}} \binom{j_{s-2}}{j_{s-2}} \left( \frac{x(s-1)/(s-2) + x^2/(s-2)}{1-x} \right) \]

\[ = \frac{x}{1-x} \sum_{j_1} \binom{k}{j_1} \left( (2x)^{j_1} \sum_{j_2} \binom{j_1}{j_2} \left( (3x/2)^{j_2} \right) \right) \]

\[ \cdots \sum_{j_{s-3}} \binom{j_{s-3}}{j_{s-3}} \left( \frac{(x(s-2) + x^2 + x^3)/(s-3)}{1-x} \right) \]

\[ \vdots \]

\[ = \frac{x}{1-x} \sum_{j_1} \binom{k}{j_1} \left( \frac{2x + x^2 + \cdots + x^{s-1}}{1-x} \right)^{j_1} \]

\[ = \frac{x^k(1+x+x^2+\cdots+x^{s-1})^k}{(1-x)^{k+1}}. \]
5 The $q$-analogue of the quasi $s$-Pascal triangle

In this section we define the $q$-analogue of the quasi $s$-Pascal triangle, we denote by $\left[ \begin{array}{c} n \\ k \\ s \end{array} \right]$ these coefficients; for that, we give an explicit formula and generating function of $\left[ \begin{array}{c} n \\ k \\ s \end{array} \right]$, and finally we propose a $q$-deformation for the $s$-bonacci sequence.

**Definition 17.** We define the $q$-quasi-bi$^s$nomial coefficient, according to Relation (24), as

$$
\left[ \begin{array}{c} n \\ k \\ s \end{array} \right] = \left[ \begin{array}{c} n-1 \\ k \\ s \end{array} \right] + \sum_{j=1}^{s} q^{n-j} \left[ \begin{array}{c} n-j \\ k-1 \\ s \end{array} \right],
$$

or equivalently

$$
\left[ \begin{array}{c} n \\ k \\ s \end{array} \right] = q^k \left[ \begin{array}{c} n-1 \\ k \\ s \end{array} \right] + \sum_{j=1}^{s} q^{(k-1)j} \left[ \begin{array}{c} n-j \\ k-1 \\ s \end{array} \right].
$$

We use the convention $\left[ \begin{array}{c} 0 \\ k \\ s \end{array} \right] = 1$ and $\left[ \begin{array}{c} n \\ k \\ s \end{array} \right] = 0$ for $k \notin \{0, \ldots, n\}$.

**Remark 18.** For $s = 1$ we obtain Relations (10) and (11) respectively.

The generating function of $\left[ \begin{array}{c} n \\ k \\ s \end{array} \right]$ is given by

**Theorem 19.** Let $F_k(x) := \sum_{n \geq 0} \left[ \begin{array}{c} n \\ k \\ s \end{array} \right] x^n$ the generating function of the $q$-quasi bi$^s$nomial coefficient, then

$$
F_k(x) = x^{k} q^{\binom{k}{2}} \prod_{j=0}^{k-1} (1 + q^j x + (q^j x)^2 + \cdots + (q^j x)^{s-1}) \prod_{j=0}^{k} (1 - q^j x).
$$

**Proof.** We have $F_k(x) = \sum_{n \geq 0} \left[ \begin{array}{c} n \\ k \\ s \end{array} \right] x^n$, then using Relation (24)

$$
F_k(x) = \sum_{n \geq 0} \left[ \begin{array}{c} n-1 \\ k \\ s \end{array} \right] x^n + \sum_{n \geq 0} \sum_{j=1}^{s} q^{n-j} \left[ \begin{array}{c} n-j \\ k-1 \\ s \end{array} \right] x^n

= x \sum_{n \geq 0} \left[ \begin{array}{c} n \\ k \\ s \end{array} \right] x^n + (x + x^2 + \cdots + x^s) \sum_{n \geq 0} \left[ \begin{array}{c} n \\ k-1 \\ s \end{array} \right] (qx)^n,
$$

thus

$$
\frac{(1-x)F_k(x)}{(x + x^2 + \cdots + x^s)} = \sum_{n \geq 0} \left[ \begin{array}{c} n \\ k-1 \\ s \end{array} \right] (qx)^n,
$$

on the other side, using Relation (24) again

$$
\frac{(1-x)F_k(x)}{(x + x^2 + \cdots + x^s)}

= \sum_{n \geq 0} \left[ \begin{array}{c} n-1 \\ k-1 \\ s \end{array} \right] (qx)^n + \sum_{n \geq 0} q^{n-1} \left[ \begin{array}{c} n-1 \\ k-2 \\ s \end{array} \right] (qx)^n + \cdots + \sum_{n \geq 0} q^{n-s} \left[ \begin{array}{c} n-s \\ k-2 \\ s \end{array} \right] (qx)^n

= qx \sum_{n \geq 0} \left[ \begin{array}{c} n \\ k-1 \\ s \end{array} \right] (qx)^n + (qx + (qx)^2 + \cdots + (qx)^s) + \sum_{n \geq 0} \left[ \begin{array}{c} n \\ k-2 \\ s \end{array} \right] (q^2 x)^n,
$$

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and by Relation (26) we obtain
\[
\frac{(1-x)(1-qx)F_k(x)}{(x + x^2 + \cdots + x^s)(qx + (qx)^2 + \cdots + (qx)^s)} = \sum_{n \geq 0} \left[ \frac{n}{k-2} \right]_{[s]} (q^2x)^n, \tag{27}
\]
we repeat the process and get
\[
\frac{(1-x)(1-qx)\cdots(1-q^{k-1}x)F_k(x)}{\prod_{j=0}^{k-1}(q^jx + (q^jx)^2 + \cdots + (q^jx)^s)} = \sum_{n \geq 0} \left[ \frac{n}{0} \right]_{[s]} (q^kx)^n
= \frac{1}{1 - q^kx},
\]
and finally we conclude to the result.

**Remark 20.** With the same method we can also prove Theorem[19] using Relation (25). The following result establishes the explicit formula of the $q$-quasi-bi\textsuperscript*nomial coefficient

**Theorem 21.** The coefficient $\left[ \frac{n}{k} \right]_{[s]}$ satisfy
\[
\left[ \frac{n}{k} \right]_{[s]} = \sum_{j} \left[ \frac{n-j}{k} \right]_{[s]} \left[ \frac{s-1}{j} \right]. \tag{28}
\]

**Proof.** We have
\[
\sum_{n \geq 0} \sum_{j} \left[ \frac{n-j}{k} \right]_{[s]} \left[ \frac{s-1}{j} \right] x^n = \sum_{j} \left[ \frac{k}{j} \right]_{[s]} \left[ \frac{s-1}{j} \right] x^j \sum_{n \geq 0} \left[ \frac{n-j}{k} \right] x^{n-j}
= x^k q^j \prod_{j=0}^{k-1} (1 + q^jx + (q^jx)^2 + \cdots + (q^jx)^{s-1}) / \prod_{j=0}^{k-1} (1 - q^jx),
\]
this last equality comes from Relation (13) and Relation (17).

Cigler and Carlitz propose the $q$-analogue of the Fibonacci sequence see[13, 15], the following result establish the recurrence relation for the $q$–analogue of the $s$–bonacci sequence, this generalize $s$ Theorem[6]

**Theorem 22.** Let $T^{(s)}_{n+1}(x) := \sum_k \left[ \frac{n-k}{k} \right]_{[s]} x^k$ for $n \geq 0$ and $T^{(s)}_0(x) = 0$ then
\[
T^{(s)}_{n+1}(x) = T^{(s)}_n(x) + x \sum_{j=1}^s q^{n-j-1} T^{(s)}_{n-j}(x/q), \tag{29}
\]
and
\[
T^{(s)}_{n+1}(x) = T^{(s)}_n(xq) + x \sum_{j=1}^s T^{(s)}_{n-j}(xq^j). \tag{30}
\]

**Proof.** For Relation (29), we have
\[
T^{(s)}_{n+1}(x) = \sum_k \left[ \frac{n-k}{k} \right]_{[s]} x^k, \]
then by Relation (24), we have

\[ T_{n+1}^{(s)}(x) = \sum_k \left[ \begin{array}{c} n-k-1 \\ k \end{array} \right] x^k + \sum_k \left[ \begin{array}{c} n-k-1 \\ k-1 \end{array} \right] x^k q^{n-k-1} + \ldots \]

\[ \ldots + \sum_k \left[ \begin{array}{c} n-k-s \\ k-1 \end{array} \right] x^k q^{n-k-s} \]

\[ \ldots \]

\[ = \sum_k \left[ \begin{array}{c} n-k-1 \\ k \end{array} \right] x^k + x \sum_{k'} \left[ \begin{array}{c} n-k'-2 \\ k' \end{array} \right] x^{k'} q^{n-k'-2} + \ldots \]

\[ \ldots \]

\[ \ldots \]

\[ = T_n^{(s)}(x) + x \sum_{j=1}^s q^{n-j-1} T_{n-j}^{(s)}(x/q). \]

The proof is the same for the Relation (30), we use Relation (25).

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