Overgroups of exterior powers of an elementary group. levels

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ABSTRACT
We prove a first part of the standard description of groups $H$ lying between an exterior power of an elementary group $\bigwedge^m E_n(R)$ and a general linear group $\operatorname{GL}_n^m(R)$ for a commutative ring $R, 2 \in R^*$ and $n \geq 3m$. The description uses the classical notion of a level: for every group $H$ we find a unique ideal $A$ of the ground ring $R$, which describes $H$.

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Introduction
The present paper is devoted to the solution of the following general problem.

Problem: Let $R$ be an arbitrary commutative associative ring with 1 and let $\Phi$ be a reduced irreducible root system. $G(\Phi, -)$ is a Chevalley–Demazure group scheme and $\rho : G(\Phi, -) \rightarrow \operatorname{GL}_N(-)$ is its arbitrary representation. Describe all overgroups $H$ of the elementary subgroup $E_G(\Phi, R)$ in the representation $\rho$:

$$E_{G, \rho}(\Phi, R) \leq H \leq \operatorname{GL}_N(R).$$

The conjectural answer, the standard description of overgroups, can be formulated as follows. For any overgroup $H$ of the elementary group there exists a net of ideals $\mathbb{A}$ of the ring $R$ such that

$$E_{G, \rho}(\Phi, R) \cdot E_N(R, \mathbb{A}) \leq H \leq N_{\operatorname{GL}_N(R)}\left( E_{G, \rho}(\Phi, R) \cdot E_N(R, \mathbb{A}) \right),$$

where $E_N(R, \mathbb{A})$ is a relative elementary subgroup for the net $\mathbb{A}$.

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In the special case of a trivial net, i.e. $\mathfrak{A} = \{A\}$ consists of one ideal $A$ of the initial ring $R$, overgroups of the group $E_G(\Phi, R)$ can be parametrized by the ideal $A$ of the ring $R$:

$$E_{G, \rho}(\Phi, R) \cdot E_N(R, A) \leq H \leq N_{GL_N(R)}(E_{G, \rho}(\Phi, R) \cdot E_N(R, A)), \quad (1)$$

where $E_N(R, A)$ equals $E_N(A)^{E_N(R)}$ by definition.

Based on the classification of finite simple groups in 1984, Michael Aschbacher proved the Subgroup Structure Theorem [1]. It states that every maximal subgroup of a finite classical group either falls into one of the eight explicitly described classes $C_1$–$C_8$ or is an ‘almost’ simple group in an irreducible representation (class $S$). In the recent past, many experts studied overgroups of groups from the Aschbacher classes for some special cases of fields. For finite fields and algebraically closed fields maximality of subgroups was obtained by Peter Kleidman and Martin Liebeck, see [2,3]. Oliver King, Roger Dye, Shang Zhi Li, and others proved maximality of groups from Aschbacher classes for arbitrary fields or described its overgroups in cases, where they are not maximal, see [4–12]. We recommend the surveys [13–15], which contain necessary preliminaries, complete history, and known results about the initial problem.

In the present paper, we consider the case of the $m$th fundamental representation of a simply connected group of type $A_{n-1}$, i.e. the scheme $G_\rho(\Phi, \_)$ equals a [Zariski] closure of the affine group scheme $SL_n(\_)$ in the representation with the highest weight $\sigma_m$. In our case the extended Chevalley group scheme coincides with the $m$th fundamental representation of the general linear group scheme $GL_n(\_)$.

Morally, the paper is a continuation of a series of papers by the St. Petersburg school on subgroups in classical groups over a commutative ring, see [14,16–28].

We deal only with the case of a trivial net, i.e. a net consisting of only one ideal $A$. As shown below (Propositions 1.1 and 3.13), it imposes the constraint $n \geq 3m$, so we proceed with this restriction. In this case the general answer has the following form. Let $N = \binom{n}{m}$ and let $H$ be a subgroup in $GL_N(R)$ containing $\wedge^m E_n(R)$. Then there exists a unique maximal ideal $A \trianglelefteq R$ such that

$$\wedge^m E_n(R) \cdot E_N(R, A) \leq H \leq N_{GL_N(R)}(\wedge^m E_n(R) \cdot E_N(R, A)). \quad (*)$$

The present paper is approximately one half of the total proof for this case. We construct levels and calculate the normalizer of connected (i.e. perfect) intermediate subgroups. Further, it is necessary to construct invariant forms for $\wedge^m SL_n(R)$ and calculate the normalizer of $\wedge^m E_n(R)$. Finally, we will extract an elementary transvection from an intermediate subgroup $H$. These steps are enough to solve the problem completely, see [29].

There are separate results for special cases of the ring $R$ such as a finite field $K$ or an algebraically closed field. For finite fields, Bruce Cooperstein proved maximality of the normalizer $N_{G}(\wedge^2 E_n(K))$ in $GL_N(K)$ [30]. For algebraically closed fields, description of overgroups of $\wedge^m E_n(K)$ follows from results of Gary Seitz on maximal subgroups of classical algebraic groups, for instance, see [31].

The present paper is organized as follows. In the next Section, we formulate main results of the paper. In Section 1 we set up the notation. Section 2 contains all complete proofs; for instance, in Section 3.1 we considered an important special case – a level computation for exterior squares of elementary groups. In Subsections 3.2–3.4 we develop a technique
for an arbitrary exterior power. Finally, a level reduction for exterior powers is proved in Section 3.5.

1. Main results

Fundamental representations of the general linear group $GL_n$, as well as of the special linear group $SL_n$, are the ones with the highest weights $\omega_m = (1, \ldots, 1)$ for $m = 1, \ldots, n$. The representation with the highest weight $\omega_n$ degenerates for the group $SL_n$. The explicit description of these representations uses exterior powers of the standard representation.

In detail, for a commutative ring $R$ by $\wedge^n R^n$ we denote an $m$th exterior power of the free module $R^n$. We consider the following natural transformation, an exterior power,

$$ \wedge^m : GL_n \to GL_{(n^m)}, $$

which extends the action of the group $GL_n(R)$ from $R^n$ to $\wedge^n R^n$.

An elementary group $E_n(R)$ is a subgroup of the group of points $GL_n(R)$, so its exterior power $\wedge^m E_n(R)$ is a well defined subgroup of $\wedge^n GL_n(R)$. A more user-friendly description of the elementary group $\wedge^m E_n(R)$ is presented in Subsections 3.1 and 3.2.

Let $H$ be an arbitrary overgroup of the elementary group $\wedge^m E_n(R)$:

$$ \wedge^m E_n(R) \leq H \leq GL_{(n^m)}(R). $$

For any unequal weights $I, J \in \wedge^m [n]$, which are indices for matrix entries of elements from $GL_{(n^m)}(R)$, by $A_{IJ}$ we denote the following set

$$ A_{IJ} := \{ \xi \in R | t_{IJ}(\xi) \in H \} \subseteq R. $$

It turns out these sets are ideals that coincide for any pair of unequal weights $I \neq J$.

**Proposition 1.1**: Sets $A_{IJ}$ coincide for $n \geq 3m$.

In the case $\frac{n}{3} \leq m \leq n$ description of overgroups cannot be done by a parametrization only by a single ideal. Moreover, as it could be seen from further calculations we need up to $m$ ideals for a complete parametrization of overgroups. There are a lot of nontrivial relationships between the ideals. So even the notion of a relative elementary group is far more complicated and depends on a Chevalley group (for instance, see [16]), let alone formulations of the main theorems. The authors work in this direction and hope this problem will be solved in the near future. In the general case, this [partially ordered] set of ideals forms a net of ideals (due to Zenon Borevich; for definitions see [32] and for further progress in the direction see [33,34]).

Back to the case $n \geq 3m$, the set $A := A_{IJ}$ is called a level of an overgroup $H$. The description of overgroups goes as follows.

**Theorem 1.2 (Level computation)**: Let $R$ be a commutative ring and $n \geq 3m$. For an arbitrary overgroup $H$ of the group $\wedge^m E_n(R)$, there exists a unique maximal ideal $A$ of the ring $R$ such that

$$ \wedge^m E_n(R) \cdot E_n(R, A) \leq H. $$

Namely, if a transvection $t_{IJ}(\xi)$ belongs to the group $H$, then $\xi \in A$. 

The left-hand side subgroup is denoted by $E \wedge^m E_n(R, A)$. We note that this group is perfect (Lemma 3.19). Motivated by the expected relations (*), we present an alternative description of the normalizer $N_{GL_n(R)}(E \wedge^m E_n(R, A))$.

For this purpose we introduce the canonical projection $\rho_A : R \longrightarrow R/A$ mapping $\lambda \in R$ to $\overline{\lambda} = \lambda + A \in R/A$. Applying the projection to all entries of a matrix, we get the reduction homomorphism

$$\rho_A : GL_n(R) \longrightarrow GL_n(R/A)$$

$$a \mapsto \overline{a} = (\overline{a}_{ij})$$

Eventually, we have the following explicit congruence description.

**Theorem 1.3 (Level reduction):** Let $n \geq 3m$. For any ideal $A \trianglelefteq R$, we have

$$N_{GL_n(R)}(E \wedge^m E_n(R, A)) = \rho_A^{-1}(\wedge^m GL_n(R/A)).$$

2. **Principal notation**

Our notation for the most part is fairly standard in Chevalley group theory. We recall all necessary notions below for the purpose of self-containment.

First let $G$ be a group. By a commutator of two elements we always mean the left-normed commutator $[x, y] = xyx^{-1}y^{-1}$, where $x, y \in G$. Multiple commutators are also left-normed; in particular, $[x, y, z] = [[x, y], z]$. By $^xy = xyx^{-1}$ we denote the left conjugates of $y$ by $x$. Similarly, by $^y = x^{-1}yx$ we denote the right conjugates of $y$ by $x$. In the sequel, we use the Hall–Witt identity:

$$[x, y^{-1}, z^{-1}]^x \cdot [z, x^{-1}, y^{-1}]^z \cdot [y, z^{-1}, x^{-1}]^y = e.$$ 

For a subset $X \subseteq G$, we denote by $\langle X \rangle$ a subgroup it generates. The notation $H \trianglelefteq G$ means that $H$ is a subgroup in $G$, while the notation $H \trianglelefteq G$ means that $H$ is a normal subgroup in $G$. For $H \trianglelefteq G$, we denote by $\langle X \rangle^H$ the smallest subgroup in $G$ containing $X$ and normalized by $H$. For two groups $F, H \trianglelefteq G$, we denote by $[F, H]$ their mutual commutator: $[F, H] = \langle [f, g] \rangle$ for $f \in F, h \in H$.

Also we need some elementary ring theory notation. Let $R$ be an associative ring with 1. By default, it is assumed to be commutative. By an ideal $I$ of a ring $R$, we understand the two-sided ideal and this is denoted by $I \trianglelefteq R$. As usual, $R^*$ denotes a multiplicative group of a ring $R$. A multiplicative group of matrices over a ring $R$ is called a general linear group and is denoted by $GL_n(R) = M_n(R)^*$. A special linear group $SL_n(R)$ is a subgroup of $GL_n(R)$ consisting of matrices of determinant 1. By $a_{ij}$ we denote an entry of a matrix $a$ at the position $(i, j)$, where $1 \leq i, j \leq n$. Further, $e$ denotes the identity matrix and $e_{ij}$ denotes the standard matrix unit, i.e. the matrix that has 1 at the position $(i, j)$ and zeros elsewhere. For entries of the inverse matrix we use the standard notation $a_{ij}^{-1} := (a^{-1})_{ij}$.

By $t_{ij}(\xi)$ we denote an elementary transvection, i.e. a matrix of the form $t_{ij}(\xi) = e + \xi e_{ij}$, $1 \leq i \neq j \leq n$, $\xi \in R$. Hereinafter, we use (without any references) standard relations [35] among elementary transvections such as

1. **Additivity:**

$$t_{ij}(\xi)t_{ij}(\zeta) = t_{ij}(\xi + \zeta).$$
(2) the Chevalley commutator formula:

$$[t_{ij}(\xi), t_{hk}(\zeta)] = \begin{cases} 
  e, & \text{if } j \neq h, i \neq k, \\
  t_{ik}(\xi\zeta), & \text{if } j = h, i \neq k, \\
  t_{hj}(-\zeta\xi), & \text{if } j \neq h, i = k.
\end{cases}$$

A subgroup $E_n(R) \leq \text{GL}_n(R)$ generated by all elementary transvections is called an (absolute) elementary group:

$$E_n(R) = \langle t_{ij}(\xi), 1 \leq i \neq j \leq n, \xi \in R \rangle.$$ 

Now define a normal subgroup of $E_n(R)$, which plays a crucial role in calculating levels of intermediate subgroups. Let $I$ be an ideal in $R$. Consider a subgroup $E_n(R, I)$ generated by all elementary transvections of level $I$, i.e. $E_n(R, I)$ is a normal closure of $E_n(I)$ in $E_n(R)$. This group is called an (relative) elementary group of level $I$:

$$E_n(R, I) = \langle t_{ij}(\xi), 1 \leq i \neq j \leq n, \xi \in I \rangle^{E_n(R)}.$$ 

It is well known (due to Andrei Suslin [36]) that the elementary group is normal in the general linear group $\text{GL}_n(R)$ for $n \geq 3$. Furthermore, the relative elementary group $E_n(R, I)$ is normal in $\text{GL}_n(R)$ if $n \geq 3$. This fact, first proved in [36], is cited as Suslin’s Lemma. Moreover, if $n \geq 3$, then the group $E_n(R, I)$ is generated by transvections of the form $z_{ij}(\xi, \zeta) = t_{ij}(\xi)t_{ij}(\zeta)t_{ij}(-\zeta), 1 \leq i \neq j \leq n, \xi, \zeta \in I, \zeta \in R$. This fact was proved by Leonid Vaserstein and Andrey Suslin [37] and, in the context of Chevalley groups, by Jacques Tits [38].

By $[n]$ we denote the set $\{1, 2, \ldots, n\}$ and by $\wedge^m [n]$ we denote an exterior power of the set $[n]$. Elements of $\wedge^m [n]$ are ordered subsets $I \subseteq [n]$ of cardinality $m$ without repeating entries:

$$\wedge^m [n] = \{(i_1, i_2, \ldots, i_m) \mid i_j \in [n], i_j \neq i_l\}.$$ 

We use the lexicographic order on $\wedge^m [n]$ by default: $1 \cdots (m - 1)m < 1 \cdots (m - 1)(m + 1) < \cdots$

Usually, we write an index $I = \{i_j\}_{j=1}^m$ in the ascending order, $i_1 < i_2 < \cdots < i_m$. Sign $\text{sgn}(I)$ of the index $I = (i_1, \ldots, i_m)$ equals the sign of the permutation mapping $(i_1, \ldots, i_m)$ to the same set in the ascending order. For example, $\text{sgn}(1234) = \text{sgn}(1342) = +1$, but $\text{sgn}(1324) = \text{sgn}(4123) = -1$.

Finally, let $n \geq 3$ and $m \leq n$. By $N$ we denote the binomial coefficient $\binom{n}{m}$. In the sequel, we denote an elementary transvection in $E_N(R)$ by $t_{I,J}(\xi)$ for $I, J \in \wedge^m [n]$ and $\xi \in R$. For instance, the transvection $t_{12,13}(\xi)$ equals the matrix with 1’s on the diagonal and $\xi$ in the position $(12, 13)$.

3. Proofs & computations

We consider a case of an exterior square of the group scheme $\text{GL}_n$ at first. We have two reasons for this way of presentation. First proofs of statements in the general case belong to the type of technically overloaded statements. At the same time, simpler proofs in the basic case present all ideas necessary for the general case. In particular, for $n = 4$ Nikolai
Vasilov and Victor Petrov completed the standard description of overgroups. Secondly for exterior squares there are several important results that cannot be obtained for the exterior cube or other powers, see [39–41]. For instance, in [40] the author constructs a transvection \( T \in \bigwedge^2 E_n(R) \) such that it stabilizes an arbitrary column of a matrix \( g \in \text{GL}_n(R) \). And there are no such transvections for other exterior powers.

### 3.1. Exterior square of elementary groups

Let \( R \) be a commutative ring with 1, let \( n \) be a natural number greater than 3, and let \( R^n \) be a right free \( R \)-module with the standard basis \( \{e_1, \ldots, e_n\} \). By \( \bigwedge^n R^n \), we denote the universal object in the category of alternating bilinear maps from \( R^n \) to \( R \)-modules. Concretely, take a free module of rank \( N = \binom{n}{2} \) with the basis \( e_i \land e_j, 1 \leq i \neq j \leq n \). The elements \( e_i \land e_j \) for arbitrary \( 1 \leq i, j \leq n \) are defined by the relation \( e_i \land e_j = -e_j \land e_i \).

An action of the group \( \text{GL}_n(R) \) on the module \( \bigwedge^n R^n \) is diagonal:

\[
\bigwedge^n(g)(e_i \land e_j) := (ge_i) \land (ge_j) \quad \text{for any } g \in \text{GL}_n(R) \text{ and } 1 \leq i \neq j \leq n.
\]

In the basis \( \{e_1, I \in \bigwedge^n[n]\} \) of the module \( \bigwedge^n R^n \), a matrix \( \bigwedge^n(g) \) consists of second order minors of the matrix \( g \) with lexicographically ordered columns and rows:

\[
(\bigwedge^n(g))_{i,j} = (\bigwedge^n(g))_{(i_1,i_2),\ldots,(j_1,j_2)} = M_{i_1,i_2}^{j_1,j_2}(x) = g_{i_1,j_1} \cdot g_{i_2,j_2} - g_{i_1,j_2} \cdot g_{i_2,j_1}.
\]

By the Cauchy–Binet Theorem, the map \( \pi : \text{GL}_n(R) \rightarrow \text{GL}_N(R), x \mapsto \bigwedge^n(x) \) is a homomorphism. Thus the map \( \pi \) is a representation of the group \( \text{GL}_n(R) \). It is called the bivector representation or the second fundamental representation (the representation with the highest weight \( \varpi_2 \)). The image of the latter action is called the exterior square of the group \( \text{GL}_n(R) \). \( E_n(R) \) is a subgroup of \( \text{GL}_n(R) \), therefore the exterior square of the elementary group is well defined. The following lemma is a corollary of Suslin’s theorem.

**Lemma 3.1:** The image of an elementary group is normal in the image of a general linear group under the exterior square homomorphism:

\[
\bigwedge^2(E_n(R)) \trianglelefteq \bigwedge^2(\text{GL}_n(R)).
\]

Note that \( \bigwedge^2(\text{GL}_n(R)) \) does not equal \( \bigwedge^2 \text{GL}_n(R) \) for arbitrary rings. For details, see the extended description in Section 3.2.

Let us consider a structure of the group \( \bigwedge^2 E_n(R) \) in detail. The following proposition can be extracted from the very definition of \( \bigwedge^2(\text{GL}_n(R)) \).

**Proposition 3.2:** Let \( t_{ij}(\xi) \) be an elementary transvection. For \( n \geq 3 \), \( \bigwedge t_{ij}(\xi) \) can be presented as the following product:

\[
\bigwedge^2 t_{ij}(\xi) = \prod_{k=1}^{i-1} t_{ki,kj}(\xi) \cdot \prod_{l=i+1}^{j-1} t_{il,jl}(-\xi) \cdot \prod_{m=j+1}^{n} t_{im,jm}(\xi)
\]

for any \( 1 \leq i < j \leq n \).
Remark: For $i > j$ a similar equality holds:

$$\bigwedge^2 t_{i,j}(\xi) = \prod_{k=1}^{j-1} t_{k_i,k_j}(\xi) \cdot \prod_{l=j+1}^{i-1} t_{l_i,l_j}(\xi) \cdot \prod_{m=i+1}^{n} t_{m_i,m_j}(\xi).$$

Likewise, one can get an explicit form of torus elements $h_{ei}(\xi)$ of the group $\bigwedge^2 \text{GL}_n(R)$.

**Proposition 3.3:** Let $d_i(\xi) = e + (\xi - 1)e_{i,i}$ be a torus generator, $1 \leq i \leq n$. Then the exterior square of $d_i(\xi)$ equals a diagonal matrix, with diagonal entries 1 everywhere except in $n-1$ positions:

$$\bigwedge^2 (d_i(\xi))_{I, I} = \begin{cases} \xi, & \text{if } i \in I, \\ 1, & \text{otherwise} \end{cases} \quad (3)$$

It follows from the propositions that $\bigwedge^2 t_{i,j}(\xi) \in \text{E}^{n-2}(N, R)$, where a set $\text{E}^n(N, R)$ consists of products of $M$ or less elementary transvections, e.g. $\bigwedge^2 t_{1,3}(\xi) = t_{12,23}(\xi) t_{14,24}(\xi) t_{15,25}(\xi) \in \bigwedge^2 \text{E}_5(R)$.

Let $H$ be an overgroup of the exterior square of the elementary group $\bigwedge^2 \text{E}_n(R)$:

$$\bigwedge^2 \text{E}_n(R) \leq H \leq \text{GL}_N(R).$$

We consider two indices $I, J \in \bigwedge^2 [n]$. By $A_{I, J}$ we denote the set

$$A_{I, J} := \{ \xi \in R | t_{I,J}(\xi) \in H \} \subseteq R.$$

By definition, diagonal sets $A_{I, J}$ equal whole ring $R$ for any index $I$. In the rest of the section, we prove that these sets are ideals, i.e. $A_{I, J}$ form a net of ideals. Moreover, we will get $D$-net in terms of Zenon Borevich [32] by the latter statement.

Let $t_{I,J}(\xi)$ be an elementary transvection. We define a distance between $I$ and $J$ as the cardinality of the intersection $I \cap J$:

$$d(I, J) = |I \cap J|.$$

This combinatorial characteristic plays the same role as the distance function $d(\lambda, \mu)$ for roots $\lambda$ and $\mu$ on the weight diagram of a root system.

The distance splits up all sets $A_{I, J}$ into two classes: the one with $d(I, J) = 0$ and the other with $d(I, J) = 1$. In fact, these classes are equal for $n \geq 6$ or $n \geq 4, 3 \in R^*$. The set $A := A_{I, J}$ is called a level of an overgroup $H$.

**Lemma 3.4:** Every set $A_{I, J}$ is an ideal of the ring $R$. Moreover, if $n \geq 6$, then for any $I \neq J$ and $K \neq L$ the ideals $A_{I, J}$ and $A_{K, L}$ coincide.

**Proof:** A complete proof is presented in Section 3.4, Proposition 3.13. Here we sketch calculations in the case $(n, m) = (4, 2)$ exclusively. These calculations present the general idea in a transparent way.
(1) First take any \( \xi \in A_{12,34} \), i.e. \( t_{12,34}(\xi) \in H \). Then
\[
[t_{12,34}(\xi), \wedge^2 t_{4,2}(\zeta)] = t_{14,23}(-\xi \zeta^2) t_{14,34}(-\zeta \xi) t_{12,23}(-\xi \zeta) \in H.
\]
Multiplying \( [t_{12,34}(\xi), \wedge^2 t_{4,2}(\zeta)] \) and \( [t_{12,34}(\xi), \wedge^2 t_{4,2}(-\zeta)] \), we obtain \( t_{14,23}(-2\xi \zeta^2) \in H \). By the condition \( 2 \in R^\times \) this means that \( A_{12,34} \subseteq A_{14,23} \). It follows that
\[
A_{I,J} \subseteq A_{K,L} \quad \text{for } I \cup J = K \cup L = \{1234\}.
\]
(2) Secondly take any \( \xi \in A_{12,34} \), then \( [t_{12,34}(\xi), \wedge^2 t_{4,5}(\zeta)] = t_{12,35}(\xi \zeta) \). Consequently,
\[
A_{I,J} \subseteq A_{K,L} \quad \text{for } d(I, J) = d(K, L) = 0.
\]
(3) Thirdly let \( \xi \in A_{12,13} \), then \( [t_{12,13}(\xi), \wedge^2 t_{1,4}(\zeta)] = t_{12,34}(\xi \zeta) \in H \). Consider two commutators of the latter transvection with \( \wedge^2 t_{4,1}(\zeta_1) \) and \( \wedge^2 t_{4,1}(-\zeta_1) \), respectively. We obtain that \( t_{24,13}(\zeta_1^2 \xi \zeta) \in H \) and also \( t_{12,13}(-\xi \zeta_1) t_{24,34}(\zeta_1 \xi \zeta) \in H \). Hence \( t_{24,34}(\zeta_1 \xi \zeta) \in H \). This means that
\[
A_{I,J} \subseteq A_{K,L} \quad \text{for any } d(I, J) = d(K, L) = 1.
\]
(4) Now take any \( \xi \in A_{12,23} \); then \( [t_{12,23}(\xi), \wedge^2 t_{4,2}(\zeta)] = t_{14,23}(-\zeta \xi) \). Thus
\[
A_{I,J} \subseteq A_{K,L} \quad \text{for } d(I, J) = 1, d(K, L) = 0.
\]
(5) Finally, let \( \xi \in A_{12,34} \). As in (1) consider the commutator \( t_{12,34}(\xi) \) with \( \wedge^2 t_{4,2}(\zeta) \). We obtain \( t_{14,23}(-2\xi \zeta^2) \in H \) and \( t_{14,34}(-\zeta \xi) t_{12,23}(-\xi \zeta) \in H \). By the same argument we provide these calculations with the transvection \( t_{45,16}(\xi) \) and \( \wedge^2 t_{6,4}(\zeta_1) \). We get that \( t_{56,14}(-2\xi_1^2 \zeta) \in H \) and \( t_{45,14}(\zeta_1 \xi) t_{56,16}(\zeta_1 \xi) \in H \). To finish the proof it remains to commutate the latter two products. Then \( t_{45,34}(-\xi_2^2 \zeta_1 \zeta) \in H \), or
\[
A_{I,J} \subseteq A_{K,L} \quad \text{for } d(I, J) = 0, d(K, L) = 1.
\]

The following lemma is crucial for the rest. It gives an alternative description of the relative elementary group.

**Lemma 3.5:** Let \( n \geq 4 \). For any ideal \( A \subseteq R \), we have
\[
E_N(A) \wedge^2 E_n(R) = E_N(R, A),
\]
where by definition \( E_N(R, A) = E_N(A)E_n(R) \).

**Proof:** The inclusion \( \leq \) is trivial. By Vaserstein–Suslin’s lemma \([37]\), the group \( E_N(R, A) \) is generated by elements of the form
\[
z_{ij,hk}(\xi, \zeta) = z_{IJ}(\xi, \zeta) = t_{ij,I}(\xi) t_{ij,J}(\xi) t_{ij,I}(\zeta), \quad \xi \in A, \zeta \in R.
\]
Hence to prove the reverse inclusion, it is sufficient to check the matrix \( z_{ij,hk}(\xi, \zeta) \) belongs to \( F := E_N(A) \wedge^2 E_n(R) \) for any \( \xi \in A, \zeta \in R \). Let us consider two cases:
• Suppose that there exists one pair of the same indices. Without loss of generality, we can assume that \( i = k \). Then this inclusion is obvious:

\[
\left. z_{ij,hi}(\xi, \zeta) = t_{hi,ij}(\zeta) t_{ij,hi}(\xi) = \Lambda^2_{\zeta} t_{i,jh}(\xi) t_{ij,hi}(\xi) \in F. \right.
\]

• Thus we are left with the inclusion \( z_{ij,hk}(\xi, \zeta) \in F \) with different indices \( i, j, h, k \). First we express \( t_{ij,hk}(\xi) \) as a commutator of elementary transvections:

\[
z_{ij,hk}(\xi, \zeta) = t_{hk,ij}(\zeta) t_{ij,hk}(\xi) = t_{hk,ij}(\zeta) [t_{ij,h}(\xi), t_{jh,hk}(1)].
\]

Conjugating arguments of the commutator by \( t_{hk,ij}(\zeta) \), we get

\[
z_{ij,hk}(\xi, \zeta) = [t_{ij,h}(\xi) t_{hk,jh}(\xi, \zeta), t_{jh,i}(-\zeta) t_{jh,hk}(1)] = [ab, cd].
\]

Next we decompose the right-hand side with a help of the formula

\[
[ab, cd] = a[b, c] \cdot ac[b, d] \cdot [a, c] \cdot c[a, d],
\]

and observe that the exponent \( a \) belongs to \( E_3(A) \), so it can be ignored. Now a direct calculation, based upon the Chevalley commutator formula, shows that

\[
[b, c] = [t_{hk,jh}(\xi, \zeta), t_{jh,i}(-\zeta)] = t_{hk,ij}(-\zeta^2 \xi) \in E_3(A);
\]

\[
c[b, d] = t_{jh,i}(-\zeta) [t_{hk,jh}(\xi, \zeta), t_{jh,hk}(1)]
\]

\[
= t_{hk,ik}(-\xi^2(1 + \xi \zeta)) t_{jh,ik}(-\xi \zeta^2) \cdot \Lambda^2_{\zeta} t_{i,jh}(\xi \zeta) \Lambda^2_{\xi} t_{j,hk}(1);
\]

\[
[a, c] = [t_{ij,h}(\xi), t_{jh,i}(-\zeta)] = [t_{ij,h}(\xi), \Lambda^2_{\xi} t_{j,h}(\zeta)];
\]

\[
c[a, d] = t_{jh,i}(-\zeta) [t_{ij,h}(\xi), t_{jh,hk}(1)]
\]

\[
= t_{jh,ik}(\xi \zeta^2) t_{ij,ik}(-\xi \zeta) \cdot \Lambda^2_{\zeta} t_{i,jh}(\xi) \Lambda^2_{\xi} t_{j,hk}(1),
\]

where all factors on the right-hand side belong to \( F \). □

**Remark:** The attentive reader can notice these calculations to almost completely coincide with the calculations for the orthogonal and symplectic cases [24,27,28]. In the special case \((n, m) = (4, 2)\) calculations are the same due to the isomorphism \( \Lambda^2 E_4(R) \cong EO_6(R) \). Amazingly, this argument proves a similar proposition for the general exterior power (see Section 3.4, Lemma 3.17).

**Corollary 3.6:** Let \( A \) be an arbitrary ideal of \( R \). Then

\[
\Lambda^2 E_n(R) \cdot E_N(R, A) = \Lambda^2 E_n(R) \cdot E_N(A).
\]

Summarizing above two lemmas, we get the main result of the paper for bivectors.
Theorem 3.7 (Level Computation): Let \( n \geq 6 \) and let \( H \) be a subgroup in \( \text{GL}_N(R) \) containing \( \wedge^2 E_n(R) \). Then there exists a unique maximal ideal \( A \trianglelefteq R \) such that

\[
\wedge^2 E_n(R) \cdot E_N(R, A) \leq H.
\]

Namely, if \( t_{i,j}(\xi) \in H \) for some \( I \) and \( J \), then \( \xi \in A \).

Lemma 3.5 asserts precisely \( \wedge^2 E_n(R) \cdot E_N(R, A) \) to be generated as a subgroup by exterior transvections \( \wedge^2 t_{i,j}(\zeta) \), \( \zeta \in R \) and by elementary transvections \( t_{ij,hk}(\xi) \), \( \xi \in A \). As usual, we assume that \( n \geq 6 \) and \( 2 \in R^* \).

We formulate a perfectness of the lower bound subgroup from the latter theorem. The proof follows from Lemma 3.19.

Lemma 3.8: Let \( n \geq 6 \). The group \( \wedge^2 E_n(R) \cdot E_N(R, A) \) is perfect for any ideal \( A \trianglelefteq R \).

3.2. Exterior powers of elementary groups

In this section, we lift the previous statements from the level of the exterior square to the case of an arbitrary exterior power.

Let us define an \( m \)th exterior power of an \( R \)-module \( R^n \) as follows. A basis of this module consists of exterior products \( e_{i_1} \wedge \cdots \wedge e_{i_m} \), where \( 1 \leq i_1 < \cdots < i_m \leq n \). Products \( e_{i_1} \wedge \cdots \wedge e_{i_m} \) are defined for any set \( i_1, \ldots, i_m \) as \( e_{\sigma(i_1)} \wedge \cdots \wedge e_{\sigma(i_m)} = \text{sgn}(\sigma) e_{i_1} \wedge \cdots \wedge e_{i_m} \) for any permutation \( \sigma \) in the permutation group \( S_m \). We denote the \( m \)th exterior power of \( R^n \) by \( \wedge^m R^n \).

For every \( m \), the group \( \text{GL}_n(R) \) acts diagonally on the module \( \wedge^m R^n \). Namely, an action of a matrix \( g \in \text{GL}_n(R) \) on decomposable \( m \)-vectors is set according to the rule

\[
\wedge^m(g)(e_{i_1} \wedge \cdots \wedge e_{i_m}) := (ge_{i_1}) \wedge \cdots \wedge (ge_{i_m})
\]

for every \( e_{i_1}, \ldots, e_{i_m} \in R^n \). In the basis \( e_I, I \in \wedge^m [n] \) a matrix \( \wedge^m(g) \) consists of \( m \)-order minors of the matrix \( g \) with lexicographically ordered columns and rows:

\[
(\wedge^m(g))_{I,J} = (\wedge^m(g))_{i_1, \ldots, i_m,j_1, \ldots, j_m} = M_{i_1, \ldots, i_m}^{j_1, \ldots, j_m}(g).
\]

By the Cauchy–Binet theorem, the map \( \pi: \text{GL}_n(R) \rightarrow \text{GL}_N(R), x \mapsto \wedge^m(x) \) is homomorphism. Thus the map \( \pi \) is a representation of the group \( \text{GL}_n(R) \). It is called the \( m \)th vector representation or the \( m \)th fundamental representation (the representation with the highest weight \( \sigma_m \)). The image of the latter action is called the \( m \)th exterior power of the group \( \text{GL}_n(R) \). \( E_n(R) \) is a subgroup of \( \text{GL}_n(R) \), and therefore the exterior power of the elementary group is well defined.

We cannot but emphasize the difference for arbitrary rings between the groups

\[
\wedge^m(\text{GL}_n(R)) < \wedge^m \text{GL}_n(R) < \text{GL}_m^n(R).
\]

The first group is a set-theoretic image of the [abstract] group \( \text{GL}_n(R) \) under the Cauchy–Binet homomorphism \( \wedge^m: \text{GL}_n(R) \rightarrow \text{GL}_m^n(R) \), while the second one is a group of \( R \)-points of the categorical image of the group scheme \( \text{GL}_n \). Since epimorphisms of algebraic groups on points are not surjective, we see that \( \wedge^m \text{GL}_n(R) \) is strictly larger than
\(\Lambda^m(\text{GL}_n(R))\). In fact, elements of \(\Lambda^m \text{GL}_n(R)\) are still images of matrices, but coefficients are not from the ring itself, but from its extension. This means that for any commutative ring \(R\) elements \(\tilde{g} \in \Lambda^m \text{GL}_n(R)\) can be represent in the form \(\tilde{g} = \Lambda^m g, g \in \text{GL}_n(S)\), where \(S\) is an extension of the ring \(R\). We refer the reader to [42] for more precise results about the difference between these groups.

As in Section 3.1, \(\Lambda^m E_n(R)\) is a normal subgroup of \(\Lambda^m (\text{GL}_n(R))\) by Suslin’s lemma. Moreover, \(\Lambda^m E_n(R)\) is normal in \(\Lambda^m \text{GL}_n(R)\). This fact follows from [43, Theorem 1].

**Theorem 3.9:** Let \(R\) be a commutative ring, \(n \geq 3\), then \(\Lambda^m E_n(R) \leq \Lambda^m \text{GL}_n(R)\).

For further computations we calculate the exterior power of an elementary transvection in the following proposition. The proof is straightforward by the very definition of the [classical] Binet–Cauchy homomorphism.

**Proposition 3.10:** Let \(t_{ij}(\xi)\) be an elementary transvection in \(E_n(R), n \geq 3\). Then \(\Lambda^m t_{ij}(\xi)\) equals

\[
\Lambda^m t_{ij}(\xi) = \prod_{L \in \Lambda^{m-1}([n \setminus \{ij\}])} t_{L \cup i, L \cup j}(\text{sgn}(L, i) \text{sgn}(L, j) \xi)
\]

for any \(1 \leq i \neq j \leq n\).

Similarly, one can get an explicit form of torus elements \(h_{\sigma_m}(\xi)\) of the group \(\Lambda^m \text{GL}_n(R)\).

**Proposition 3.11:** Let \(d_i(\xi) = e + (\xi - 1)e_{i,i}\) be a torus generator, \(1 \leq i \leq n\). Then the exterior power of \(d_i(\xi)\) equals a diagonal matrix, with diagonal entries \(1\) everywhere except in \(\binom{n-1}{i}\) positions:

\[
\Lambda^m (d_i(\xi))_{I,J} = \begin{cases} 
\xi, & \text{if } i \in I, \\
1, & \text{otherwise}.
\end{cases}
\]

As an example, consider \(\Lambda^3 t_{1,3}(\xi) = t_{124,234}(\xi) t_{125,235}(\xi) t_{145,345}(\xi) = \Lambda^3 E_5(R)\) and \(\Lambda^4 d_2(\xi) = \text{diag}(\xi, \xi, \xi, 1, \xi) \in \Lambda^4 E_5(R)\). It follows from the propositions \(\Lambda^m t_{ij}(\xi) \in E^{\binom{n-2}{m-1}}(N, R)\), where by definition every element of the set \(E^M(N, R)\) is a product of \(M\) or less elementary transvections. In other words, the residue of a transvection \(\text{res}(\Lambda^m t_{ij}(\xi))\) equals the binomial coefficient \(\binom{n-3}{m-1}\). Recall that a residue \(\text{res}(g)\) of a transformation \(g\) is called the rank of \(g - e\). Finally, there is a simple connection between the determinant of a matrix \(g \in \text{GL}_n(R)\) and the determinant of \(\Lambda^m g \in \Lambda^m \text{GL}_n(R)\), see [44, Proof of Theorem 4]:

\[
\det \Lambda^m g = (\det (g))^{\binom{n-1}{m-1}} = (\det (g))^{\binom{n-1}{m-1}}.
\]

### 3.3. Elementary calculations technique

For an arbitrary exterior power, calculations with elementary transvections are huge. In this section, we organize all possible calculations of a commutator of an elementary transvection with an exterior transvection.
Proposition 3.12: Up to the action of the permutation group there exist three types of commutators with a fixed transvection \( t_{I,J}(\xi) \in E_N(R) \):

1. \([t_{I,J}(\xi), \Lambda^m t_{i,j}(\xi)] = 1\) if both \( i \not\in I \) and \( j \not\in J \) hold;
2. \([t_{I,J}(\xi), \Lambda^m t_{i,j}(\xi)] = t_{I,J}(\pm \xi \xi)\) if either \( i \in I \) or \( j \in J \) and then \( \tilde{I} = I \setminus i \cup j \) or \( \tilde{J} = J \setminus j \cup i \) respectively;
3. If both \( i \in I \) and \( j \in J \) hold, then we have the equality:

\[
[t_{I,J}(\xi), \Lambda^m t_{i,j}(\xi)] = t_{I,J}(\pm \xi \xi) \cdot t_{I,J}(\pm \xi \xi) \cdot t_{I,J}(\pm \xi^2 \xi).
\]

Note that the latter case is true whenever \( I \setminus i \neq J \setminus j \), otherwise we obtain \([t_{I,J}(\xi), t_{I,J}(\pm \xi \xi)]\). This commutator cannot be presented in a simpler form than the very definition.

The rule for commutator calculations from the latter proposition can be translated into the language of weight diagrams:

Weight diagrams tutorial.

1. Let \( G(A_{n-1,-}) \) be a Chevalley–Demazure group scheme, and let \((I, J) \in \Lambda^m[n]^2\) be a pair of different weights for the \(m\)th exterior power of \(G(A_{n-1,-})\). Consider any unipotent \( x_\alpha(\xi) \) for a root \( \alpha \) of the root system \( A_{n-1} \), i.e. \( x_\alpha(\xi) \) equals an elementary transvection \( \Lambda^m t_{i,j}(\xi) \in \Lambda^m E_n(R) \);
2. By \( Ar(\alpha) \) denote all paths on the weight diagram \(^3\) of this representation corresponding to the root \( \alpha \);
3. Then there exist three different scenarios corresponding to the cases of Proposition 3.12:
   - sets of the initial and the terminal vertices of paths from \( Ar(\alpha) \) do not contain the vertex \((I, J)\);
   - the vertex \((I, J)\) is initial or terminal for one path from \( Ar(\alpha) \);
   - the vertex \((I, J)\) is simultaneously initial and terminal for some path \(^4\) from \( Ar(\alpha) \).
4. Finally, let us consider a commutator of the transvection \( t_{I,J}(\xi) \) and the element \( \Lambda^m t_{i,j}(\xi) \). It equals a product of transvections. These transvections correspond to the paths from the previous step. Arguments of the transvections are monomials in \( \xi \) and \( \zeta \). Namely, in the second case the argument equals \( \pm \xi \xi \); in the third case it equals \( \pm \xi^2 \).

In Figure 1(a) we present all three cases from step (3) for \( m = 2 \) and \( \alpha = \alpha_2 \):

- \((I, J) = (14, 15)\), then \([t_{14,15}(\xi), \Lambda^2 t_{2,3}(\xi)] = 1\);
- \((I, J) = (13, 35)\), then \([t_{13,35}(\xi), \Lambda^2 t_{2,3}(\xi)] = t_{12,35}(-\xi \xi)\);
- \((I, J) = (13, 24)\), then \([t_{13,24}(\xi), \Lambda^2 t_{2,3}(\xi)] = t_{12,24}(-\xi \xi) t_{12,34}(\xi^2) t_{13,34}(\xi \xi)\).

Similarly, for the case \( m = 3 \) the elementary calculations can be seen directly from Figure 1(b).
Figure 1. Weight diagrams for (a): \((A_4, \omega_2), \alpha = \alpha_2\) and (b): \((A_5, \omega_3), \alpha = \alpha_4\).

3.4. Level computation

We recall that our final goal is to parametrize overgroups of \(\wedge^m E_n(R)\) by ideals of the ring \(R\). In this section we compute such ideal for every overgroup, see Theorem 1.2.

We generalize the notion of ideals \(A_{IJ}\) to the case of the \(m\)th exterior power. Let \(H\) be an overgroup of the exterior power of the elementary group \(\wedge^m E_n(R)\):

\[\wedge^m E_n(R) \leq H \leq GL_N(R)\]

Let

\[A_{IJ} := \{\xi \in R | t_{IJ}(\xi) \in H\}\]

for any indices \(I, J \in \wedge^m [n]\). As usual, diagonal sets \(A_{IJ}\) equal the whole ring \(R\) for any index \(I \in \wedge^m [n]\). Thus we will construct \(D\)-net of ideals of the ring \(R\).

**Proposition 3.13:** If \(|I \cap J| = |K \cap L|\), then sets \(A_{IJ}\) and \(A_{KL}\) coincide. In fact, \(A_{IJ}\) are ideals of \(R\).

But first we prove a weaker statement.

**Lemma 3.14:** Let \(I, J, K, L\) be different elements of the set \(\wedge^m [n]\) such that \(|I \cap J| = |K \cap L| = 0\). If \(n \geq 2m\), then sets \(A_{IJ}\) and \(A_{KL}\) coincide.

**Proof of the lemma:** The sets \(A_{IJ}\) coincide when the set \(I \cup J\) is fixed. This fact can be proved by the third type commutation due to Proposition 3.12 with \(\zeta\) and \(-\zeta\). If \(\xi \in A_{IJ}\), we get a transvection \(t_{IJ}(\xi) \in H\). Then the following two products also belong to \(H\):

\[[t_{IJ}(\xi), \wedge^m t_{j,i}(\xi)] = t_{IJ}(\pm \zeta \xi) \cdot t_{IJ}(\pm \zeta \xi) \cdot t_{IJ}(\pm \zeta^2 \xi)\]
\[ [t_{i,j}(\xi), \wedge^m t_{i,j}(\xi)] = t_{i,j}(\mp \xi \xi) \cdot t_{i,j}(\mp \xi \xi) \cdot t_{i,j}(\pm \xi^2 \xi). \]

This implies that the products of two factors on the right-hand sides \( t_{i,j}(\pm 2 \xi^2 \xi) \) belong to \( H \).

It can be easily proved that the set \( I \cup J \) can be changed by the second type commutations. For example, the set \( I_1 \cup J_1 = \{1, 2, 3, 4, 5, 6\} \) can be replaced by the set \( I_2 \cup J_2 = \{1, 2, 3, 4, 5, 7\} \) as follows

\[ [t_{123,456}(\xi), \wedge^3 t_{6,7}(\xi)] = t_{123,457}(\xi \xi). \]

**Proof of Proposition 3.13:** Arguing as above, we see that the sets \( A_{i,j} \) and \( A_{K,L} \) coincide in the case \( I \cap J = K \cap L \), where \( n_1 = n - |I \cap J| \geq 2 \cdot m - 2 \cdot |I \cap J| = 2 \cdot m_1 \).

In the general case, we can prove the statement by both the second and the third type commutations. Let us give an example of this calculation with replacing the set \( I \cap J = \{1, 2\} \) by the set \( \{1, 5\} \).

Let \( t_{123,124}(\xi) \in H \). So we have \( [t_{123,124}(\xi), \wedge^3 t_{2,5}(\xi)] = t_{123,145}(\xi \xi) \in H \). We commute this transvection with the element \( \wedge^3 t_{5,2}(\xi) \). Then the transvection \( t_{123,124}(\xi^2 \xi \xi) \) belongs to \( H \) as well as the product \( t_{123,124}(\xi \xi \xi) \cdot t_{123,145}(\xi \xi \xi) \in H \). From the latter inclusion we see \( t_{123,145}(\xi \xi \xi) \in H \) and \( I \cap J = \{1, 5\} \).

To prove that all \( A_{i,j} \) are ideals in \( R \) it is sufficient to commute any elementary transvection with exterior transvections with \( \xi \) and \( 1 \):

\[ t_{i,j}(\xi \xi) = [t_{i,j}(\xi), \wedge^m t_{i,j}(\xi), \wedge^m t_{i,j}(\pm 1)] \in H. \]

Recall that the distance between any two indices \( I \) and \( J \) equals to \( |I \cap J| \). Now Proposition 3.13 can be rephrased as follows. Sets \( A_{i,j} \) and \( A_{K,L} \) coincide for the same distances: \( A_{i,j} = A_{K,L} = A_{|I \cap J|} \). Suppose that \( d(I, J) \) is larger than \( d(K, L) \); then using Proposition 3.12, we get \( A_{i,j} \subseteq A_{K,L} \).

Summarizing the above arguments, we have

\[ A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots \supseteq A_{m-2} \supseteq A_{m-1}. \]

The following result proves a coincidence of the sets \( \{A_k\}_{k=0}^{m-1} \).

**Proposition 3.15:** The ideals \( A_k \) coincide for \( n \geq 3m \). More accurately, the inverse inclusion \( A_k \subseteq A_{k+1} \) takes place if \( n \geq 3m - 2k \).

**Proof:** The statement can be proved by applying twice the third type commutation as follows. Let \( \xi \in A_k \), i.e. a transvection \( t_{i,j}(\xi) \in H \) and \( d(I, J) = k \). By the third type commutation with a transvection \( \wedge^m t_{i,j}(\xi) \), we have \( t_{i,j}(\pm \xi \xi) \cdot t_{i,j}(\xi \xi \xi) \in H \). We consider an analogous commutator with a specifically chosen transvection \( t_{i,j}(\xi) \in H \) and \( \wedge^m t_{i,j}(\xi) \). We get that \( t_{i,j}(\pm \xi \xi) \cdot t_{i,j}(\pm \xi \xi) \in H \). The final step is to commute two results of previous commutations.

The choice of transvections is made in a way that commutator in the final step (initially of the form \( [ab, cd] \)) equals to an elementary transvection. This choice is always possible due to the condition \( n \geq 3m - 2k \).

Let us give a particular example of such calculations for the case \( m = 4 \). This calculation could be easily generalized. The first three steps below correspond to the inclusions
$A_0 \leq A_1$, $A_1 \leq A_2$, and $A_2 \leq A_3$, respectively. We emphasize that the ideas of the proof of all three steps are completely identical. The difference has to do only with a choice of the appropriate indices. We replace the numbers 10, 11, 12 with the letters $\alpha$, $\beta$, $\gamma$ respectively.

(1) Let $\xi \in A_0$. Consider the mutual commutator

$$\left[[t_{1234, 5678}(\xi), t_{54, 123}(\xi)], [t_{49\alpha\beta, 123\gamma}(\xi), t_{\gamma, 4}(\xi_1)]\right] \in H.$$ 

It is equal to the commutator

$$[t_{1234, 4567}(\xi) \cdot t_{1238, 5678}(\xi), t_{123, 4567}(\xi_1) \cdot t_{9\alpha\beta, 123\gamma}(\xi_1)] \in H,$$

which is a transvection $t_{49\alpha\beta, 4567}(\xi_2) \in H$. As a result, $A_0 \leq A_1$.

(2) For $\xi \in A_1$, consider similar commutator

$$\left[[t_{1234, 1567}(\xi), t_{7, 4}(\xi)], [t_{1489, 123}(\xi), t_{\alpha, 7}(\xi_1)]\right] \in H.$$ 

Thus

$$[t_{1234, 1456}(\xi) \cdot t_{1237, 1567}(\xi), t_{1489, 123}(\xi_1) \cdot t_{189\alpha, 123\alpha}(\xi_1)] \in H.$$ 

Again this commutator is equal to $t_{1489, 1456}(\xi_2) \in H$, i.e. $A_1 \leq A_2$.

(3) Finally, let $\xi \in A_2$. Consider the commutator

$$\left[[t_{1234, 1256}(\xi), t_{6, 4}(\xi)], [t_{1248, 1237}(\xi), t_{7, 4}(\xi_1)]\right] \in H.$$ 

It is equal to the commutator

$$[t_{1234, 1245}(\xi) \cdot t_{1236, 1256}(\xi), t_{1248, 1237}(\xi_1) \cdot t_{1278, 1237}(\xi_1)] \in H,$$

which is an elementary transvection $t_{1248, 1245}(\xi_2) \in H$. Thus, $A_2 \leq A_3$.

We proved that all ideals $A_i$ coincide for a large enough $n$. However, the following proposition shows relations between the ideals without this restriction. Recall that the residue $\text{res}$ of an exterior transvection $\wedge^m t_{ij}(\xi)$ equals the binomial coefficient $\binom{n-2}{m-1}$.

**Proposition 3.16:** The ideals $\{A_0, \ldots, A_{m-1}\}$ are interrelated as follows:

$$A_k \leq A_{k+1}, \quad \text{for } n \geq 3m - 2k;$$

$$A_0 \geq A_1 \geq A_2 \geq \cdots \geq A_{m-2} \geq A_{m-1};$$

$$\text{res} \cdot A_{m-2} \leq A_{m-1}.$$ 

**Proof:** The first two series of relations is proved in previous calculations. Therefore, we must only prove that $\text{res} \cdot A_{m-2} \leq A_{m-1}$. Again, we use the third type commutation.
Let $\xi \in A_{m-2}$, i.e. for any indices $I, J$ with $d(I, J) = m - 2$ a transvection $t_{I,J}(\xi) \in H$. Note that if $i \in I, j \in J$, then in the commutator

$$[t_{I,J}(\xi), \wedge^m t_{i,j}(\xi)] = t_{I,J}(\pm \xi \xi) \cdot t_{I,J}(\pm \xi \xi) \cdot t_{I,J}(\pm \xi^2 \xi)$$

the transvection $t_{I,J}(\pm \xi^2 \xi)$ belongs to the group $H$. Indeed, the distance of indices $\tilde{I} = I \setminus i \cup j$ and $\tilde{J} = J \setminus j \cup i$ coincide with the distance of $I, J$. At the same time, the distance of $\tilde{I}, \tilde{J}$ and $I, J$ equals $m - 1$. Thus $t_{I,J}(\pm \xi \xi) \cdot t_{I,J}(\pm \xi \xi) \in H$ for all indices $I, J$ with $d(I, J) = m - 2$ and all different $i \in I, j \in J$.

Consider $\wedge^m t_{1,2}(\xi \xi) \in H$, where $\xi \in R$. By the definition of exterior transvections (4), we have $\wedge^m t_{1,2}(\xi \xi) = \prod_I t_{\cup 1, \cup 2}(\xi \xi)$. The proof is to consistently reduce the number of factors in the product by multiplication $\wedge^m t_{1,2}(\xi \xi)$ on suitable transvections $t_{I,J}(\pm \xi \xi) \cdot t_{I,J}(\pm \xi \xi) \in H$. Finally, we get an elementary transvection $t_{P \cup 1, P \cup 2}(\xi \xi)$, where the distance of indices equals $m - 1$ and the coefficient $c$ equals $\binom{n-1}{m-1}$.

Let us give an example of such an argument for the exterior cube of the elementary group of dimension 5. Take $\xi \in A_1, \xi \in R$, and $\wedge^3 t_{1,2}(\xi \xi) = t_{134,234}(\xi \xi)t_{135,235}(\xi \xi)t_{145,245}(\xi \xi)$.

First consider the commutator $[t_{134,234}(\xi \xi), \wedge^3 t_{5,3}(\xi \xi)] \in H$. As we mentioned above, the matrix $z_1 := t_{134,234}(-\xi \xi)t_{145,245}(\xi \xi) \in H$. Thus

$$\wedge^3 t_{1,2}(\xi \xi) \cdot z_1 = t_{135,235}(\xi \xi)t_{145,245}(2\xi \xi) \in H.$$ 

To get an elementary transvection, consider one more commutator $[t_{135,245}(\xi \xi), \wedge^3 t_{4,3}(-\xi \xi)] \in H$. Then the matrix $z_2 := t_{145,245}(\xi \xi)t_{135,235}(-\xi \xi) \in H$. It remains to multiply $\wedge^3 t_{1,2}(\xi \xi)$ and $z_1z_2$. We get the transvection $t_{145,245}(3\xi \xi) \in H$. Therefore, $3\xi \xi \in A_2$. 

If $n \geq 3m$, then the set $A = A_{1,J}$ is called a level of an overgroup $H$.

**Lemma 3.17:** For any ideal $A \trianglelefteq R$, we have

$$E_N(A) \wedge^m E_n(R) = E_N(R, A),$$

where by definition $E_N(R, A) = E_N(A)^{E_N(R)}$.

**Proof:** Clearly, the left-hand side is contained in the right-hand side. The proof of the inverse inclusion goes by induction on the distance of $(I, J)$. By Vaserstein–Suslin’s Lemma [37], it is sufficient to check that the matrix $z_{I,J}(\xi, \xi)$ belongs to $F := E_N(A) \wedge^m E_n(R)$ for any $\xi \in A, \xi \in R$.

In the base case $|I \cap J| = m - 1$, the inclusion is obvious:

$$z_{I,J}(\xi, \xi) \cdot t_{I,J}(-\xi) = [t_{I,J}(\xi), t_{I,J}(\xi)] = [\wedge^m t_{i,j}(\xi), t_{I,J}(\xi)] \in F.$$ 

Now let us consider the general case $|I \cap J| = p$, i.e. $I = k_1 \cdots k_p i_1 \cdots i_q$ and $J = k_1 \cdots k_p j_1 \cdots j_q$. For the following calculations we need two more sets $V := k_1 \cdots k_p i_1 \cdots i_{q-1}j_q$ and $W := k_1 \cdots k_p j_1 \cdots j_{q-1}i_q$. 


First we express \( t_{I,J}(\xi) \) as a commutator of elementary transvections,
\[
z_{I,J}(\xi, \zeta) =_{t_{I,J}(\zeta)} t_{I,J}(\xi) =_{t_{I,V}(\xi)} [t_{I,V}(\xi), t_{V,I}(1)].
\]
Conjugating the arguments of the commutator by \( t_{I,J}(\zeta) \), we get
\[
[t_{I,V}(\xi \zeta)t_{I,V}(\xi), t_{V,I}(-\zeta)t_{V,I}(1)] = [ab, cd].
\]
Next we decompose the right-hand side with help of the formula
\[
[ab, cd] = a[b, c] \cdot ac[b, d] \cdot [a, c] \cdot c[a, d],
\]
and observe the exponent \( a \) to belong to \( E_n(A) \), so can be ignored. Now a direct calculation, based upon the Chevalley commutator formula, shows that
\[
\begin{align*}
[b, c] &= [t_{I,V}(\xi), t_{V,I}(-\zeta)] \in F \quad \text{(by the induction step for the distance } m - 1); \\
c[b, d] &= _{t_{I,V}(-\zeta)} [t_{I,V}(\xi), t_{V,I}(1)] = t_{V,W}(\xi \zeta^2) t_{I,W}(-\xi \zeta) \cdot \land^m_{t_{I,J}(-\zeta)} t_{I,J}(\xi); \\
[a, c] &= [t_{I,V}(\xi \zeta), t_{V,I}(-\zeta)] = t_{I,J}(-\xi^2 \zeta); \\
c[a, d] &= _{t_{I,V}(-\zeta)}[t_{I,V}(\xi \zeta), t_{V,I}(1)] \\
&= t_{I,W}(-\xi \zeta^2 (1 + \xi \zeta)) t_{V,W}(-\xi \zeta^2) \cdot \land^m_{t_{I,J}(-\zeta)} t_{I,V}(\xi \zeta) \\
&\quad \cdot \land^m_{t_{I,J}(-\zeta)} z_{I,V}(-\xi \zeta, 1) \in F,
\end{align*}
\]
where all factors on the right-hand side belong to \( F \).

\[\blacksquare\]

**Remark:** Since we do not use the coincidental elements of \( I \) and \( J \), we also can prove this Lemma by induction on \(|I \setminus J| = |J \setminus I| = 1/2 \cdot |I \triangle J|\). Then we can assume that \( m \) is an arbitrarily large number (mentally, \( m = \infty \)).

**Corollary 3.18:** Suppose \( A \) be an arbitrary ideal of the ring \( R \); then
\[
\land^m E_n(R) \cdot E_N(R, A) = \land^m E_n(R) \cdot E_N(A).
\]

Summarizing Proposition 3.15 and Lemma 3.17, we get the main result of the paper for the general case.

**Theorem 1.2 (Level computation):** Let \( R \) be a commutative ring and \( n \geq 3m \). For an arbitrary overgroup \( H \) of the group \( \land^m E_n(R) \), there exists a unique maximal ideal \( A \) of the ring \( R \) such that
\[
\land^m E_n(R) \cdot E_N(R, A) \leq H.
\]
Namely, if a transvection \( t_{I,J}(\xi) \) belongs to the group \( H \), then \( \xi \in A \).

For \( n < 3m \) the level computation is formulated with a set of ideals. An \( m \)-tuple of ideals \( \mathcal{A} = (A_0, \ldots, A_{m-1}) \) of the ring \( R \) is called *admissible* if \( \mathcal{A} \) satisfies the relations in Proposition 3.16. Then every admissible \( m \)-tuple \( \mathcal{A} \) corresponds to the group \( E \land^m E_n(R, \mathcal{A}) := \).
Lemma 3.19: Let \( n \geq 4 \). In this section, we describe the normalizer of the lower bound for a group \( \mathbf{C} \) the scalar matrices \( \mathbf{A} \). Theorem 1.2: Let, as above, \( A = \mathbf{N}_n(R) \). Theorem 1.2': Let \( R \) be a commutative ring and let \( n \geq 4 \). For an arbitrary overgroup \( H \) of the group \( \wedge^m E_n(R) \), there exists a net of ideals \( \mathbf{A} \) of the ring \( R \) such that
\[
\wedge^m E_n(R) \cdot E_N(R, \mathbf{A}) \subseteq H.
\]
Namely, if a transvection \( t_{1, j}(\xi) \) belongs to the group \( H \), then \( \xi \in A_{1, j} \). □

3.5. Normalizer of \( \wedge^m E_n(R, A) \)

In this section, we describe the normalizer of the lower bound for a group \( H \).

Lemma 3.19: Let \( n \geq 3m \). The group \( E \wedge^m E_n(R, A) := \wedge^m E_n(R) \cdot E_N(R, A) \) is perfect for any ideal \( A \subset R \).

Proof: It is sufficient to verify all generators of the group \( \wedge^m E_n(R) \cdot E_N(R, A) \) to lie in its commutator subgroup, which is denoted by \( F \). The proof goes in two steps.

- For the transvections \( \wedge^m t_{1, j}(\xi) \) this follows from the Cauchy–Binet homomorphism:
\[
\wedge^m t_{1, j}(\xi) = \wedge^m ([t_{1, j}(\xi), t_{1, j}(1)]) = [\wedge^m t_{1, j}(\xi), \wedge^m t_{1, j}(1)].
\]

- For elementary transvections \( t_{1, j}(\xi) \) this can be done as follows. Suppose that \( I \cap J = K = k_1 \cdots k_p \), where \( 0 \leq p \leq m - 1 \), i.e. \( I = k_1 \cdots k_p i_1 \cdots i_q \) and \( J = k_1 \cdots k_p j_1 \cdots j_q \). As in Lemma 3.17, we define a set \( V = k_1 \cdots k_p j_1 \cdots j_q - 1 i_q \). Then
\[
t_{1, j}(\xi) = [t_{1, v}(\xi), t_{1, j}(1)] = [t_{1, v}(\xi), \wedge^m t_{i_q j_q}(\pm 1)],
\]
as required. □

Let, as above, \( A \subset R \) and let \( R/A \) be the factor-ring of \( R \) modulo \( A \). Denote by \( \rho_A : R \longrightarrow R/A \) the canonical projection sending \( \lambda \in R \) to \( \tilde{\lambda} = \lambda + A \in R/A \). Applying the projection to all entries of a matrix, we get the reduction homomorphism
\[
\rho_A : \text{GL}_n(R) \longrightarrow \text{GL}_n(R/A).
\]
\[
a \mapsto \bar{a} = (\bar{a}_{i,j})
\]
The kernel of the homomorphism \( \rho_A \), \( \text{GL}_n(R, A) \), is called the principal congruence subgroup in \( \text{GL}_n(R) \) of level \( A \). Now let \( C_n(R) \) be the centre of the group \( \text{GL}_n(R) \) consisting of the scalar matrices \( \lambda e, \lambda \in R^* \). The full preimage of the centre of \( \text{GL}_n(R/A) \), denoted by \( C_n(R, A) \), is called the full congruence subgroup of level \( A \). The group \( C_n(R, A) \) consists of
all matrices congruent to a scalar matrix modulo \( A \). We further concentrate on study of the full preimage of the group \( \wedge^n \text{GL}_n(R/A) \):

\[
C \wedge^n \text{GL}_n(R, A) = \rho_A^{-1}(\wedge^n \text{GL}_n(R/A)).
\]

A key point in a reduction modulo an ideal is the following standard commutator formula, proved by Leonid Vaserstein \([45]\), Zenon Borevich, and Nikolai Vavilov \([18]\).

\[
[E_n(R), C_n(R, A)] = E_n(R, A) \quad \text{for a commutative ring } R \text{ and } n \geq 3.
\]

Finally, we are ready to state the level reduction result.

**Theorem 1.3 (Level reduction):** Let \( n \geq 3m \). For any ideal \( A \trianglelefteq R \), we have

\[
N_{\text{GL}_N(R)}\left( E \wedge^n E_n(R, A) \right) = \rho_A^{-1}(\wedge^n \text{GL}_n(R/A)).
\]

**Proof:** In the proof, by \( N \) we mean \( N_{\text{GL}_N(R)} \).

Since \( E_N(R, A) \) and \( \text{GL}_N(R, A) \) are normal subgroups in \( \text{GL}_N(R) \), we see that

\[
N\left( E \wedge^n E_n(R, A) \right) \leq N\left( E \wedge^n E_n(R, A) \text{GL}_N(R, A) \right) = C \wedge^n \text{GL}_n(R, A). \tag{6}
\]

Note that the latter equality is due to the normalizer functoriality:

\[
N\left( E \wedge^n E_n(R, A) \text{GL}_N(R, A) \right) = N\left( \rho_A^{-1}(\wedge^n E_n(R/A)) \right) = \rho_A^{-1}\left( N\left( \wedge^n E_n(R/A) \right) \right) = \rho_A^{-1}\left( \wedge^n \text{GL}_n(R/A) \right).
\]

In particular, using (6), we get

\[
\left[ C \wedge^n \text{GL}_n(R, A), E \wedge^n E_n(R, A) \right] \leq E \wedge^n E_n(R, A) \text{GL}_N(R, A). \tag{7}
\]

On the other hand, it is completely clear \( E \wedge^n E_n(R, A) \) to be normal in the right-hand side subgroup. Indeed, it is easy to prove the following stronger inclusion:

\[
\left[ \wedge^n \text{GL}_n(R) \text{GL}_N(R, A), E \wedge^n E_n(R, A) \right] \leq E \wedge^n E_n(R, A). \tag{8}
\]

To check this, we consider a commutator of the form

\[
[x, h] \cdot [x, h] \cdot h[x, g], \quad x \in \wedge^n \text{GL}_n(R), \ y \in \text{GL}_N(R, A), \ h \in \wedge^n E_n(R), \ g \in E_N(R, A).
\]

Then \([xy, hg] = x[y, h] \cdot [x, h] \cdot h[x, g]. We need to prove that all factors on the right-hand side belong to \( E \wedge^n E_n(R, A) \). Right away, the second factor lies in the group \( E \wedge^n E_n(R, A) \). For the first commutator, we need to consider the following inclusions:

\[
\wedge^n \text{GL}_N(R) \left[ \text{GL}_N(R, A), \wedge^n E_n(R) \right] \leq \left[ \wedge^n \text{GL}_N(R) \text{GL}_N(R, A), \wedge^n \text{GL}_N(R) \wedge^n E_n(R) \right] = \wedge^n E_n(R).
\]
The element \(h \in \wedge^m E_n(R)\), so we ignore it in conjugation. The third commutator lies in \(E \wedge^m E_n(R, A)\) due to the following inclusion:

\[
[\wedge^m \text{GL}_n(R) \text{GL}_N(R, A), E_N(R, A)] \leq [\text{GL}_N(R), E_N(R, A)] = E_N(R, A).
\]

Now if we recall (7) and (8), we get

\[
[C \wedge^m \text{GL}_n(R, A), E \wedge^m E_n(R, A), E \wedge^m E_n(R, A)] \leq E \wedge^m E_n(R, A).
\]  
(9)

To invoke the Hall–Witt identity, we need a slightly more precise version of the latter inclusion:

\[
[[C \wedge^m \text{GL}_n(R, A), E \wedge^m E_n(R, A)], [C \wedge^m \text{GL}_n(R, A), E \wedge^m E_n(R, A)]] \leq E \wedge^m E_n(R, A).
\]  
(10)

Observe that by formula (7) we have already checked the left-hand side to be generated by the commutators of the form

\[
[uv, [z, y]], \quad \text{where } u, y \in E \wedge^m E_n(R, A), \; v \in \text{GL}_N(R, A), \; z \in C \wedge^m \text{GL}_n(R, A).
\]

However,

\[
[uv, [z, y]] = u[v, [z, y]] \cdot [u, [z, y]].
\]

By formula (9) the second commutator belongs to \(E \wedge^m E_n(R, A)\), whereas by (10) the first is an element of \([\text{GL}_N(R, A), E_N(R)] \leq E_N(R, A)\).

Now we are ready to finish the proof. By the previous lemma, the group \(E \wedge^m E_n(R, A)\) is perfect, and thus, it suffices to show \([z, [x, y]] \in E \wedge^m E_n(R, A)\) for all \(x, y \in E \wedge^m E_n(R, A), z \in C \wedge^m \text{GL}_n(R, A)\). Indeed, the Hall–Witt identity yields

\[
[z, [x, y]] = x^z[[z^{-1}, x^{-1}], y] \cdot y^z[[y^{-1}, z], x^{-1}],
\]

where the second commutator belongs to \(E \wedge^m E_n(R, A)\) by (4). Removing the conjugation by \(x \in E \wedge^m E_n(R, A)\) in the first commutator and carrying the conjugation by \(z\) inside the commutator, we see that it only remains to prove the relation \([[x^{-1}, z], [z, y]y] \in E \wedge^m E_n(R, A)\). Indeed,

\[
[[x^{-1}, z], [z, y]y] = [[x^{-1}, z], [z, y]] \cdot [z, y][x^{-1}, z], y],
\]

where both commutators on the right–hand side belong to \(E \wedge^m E_n(R, A)\) by formulas (9) and (10), and moreover, the conjugating element \([z, y]\) in the second commutator is an element of the group \(E \wedge^m E_n(R, A) \text{GL}_N(R, A)\), and thus by (8), normalizes \(E \wedge^m E_n(R, A)\).

\[\blacksquare\]

**Notes**

1. The restriction of the exterior square square map \(\wedge^2: \text{GL}_4(R) \rightarrow \text{GL}_6(R)\) to the group \(E_4(R)\) is an isomorphism onto the elementary orthogonal group \(EO_6(R)\) [24].
2. The same strict inclusions are still true with changing \(\text{GL}\) to \(\text{SL}\).
3. Recall that we consider the representation with the highest weight \(\varpi_m\).
4. From root systems geometry any vertex can be initial or terminal for at most one \(\alpha\)-path.
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