Homology of systemic modules

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Abstract. In this paper, we develop the rudiments of a tropical homology theory. We use the language of “triples” and “systems” to simultaneously treat structures from various approaches to tropical mathematics, including semirings, hyperfields, and super tropical algebra. We enrich the algebraic structures with a negation map where it does not exist naturally. We obtain an analogue to Schanuel’s lemma which allows us to talk about projective dimension of modules in this setting. We define two different versions of homology and exactness, and study their properties. We also prove a weak Snake lemma type result.

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1. Introduction

In this paper we explore homology theory for tropical mathematics. This has been initiated in a strong paper of Connes and Consani [11], which we utilize, but we work more algebraically in the theory of triples and systems which was espoused in [34,45], as surveyed in [46], (and applied in [3,18,31]) in order to unify classical algebra with the algebraic theories of supertropical algebra, symmetrized semirings, hyperfields, and fuzzy rings.

We briefly recall that we start with a set $T$ (often a multiplicative monoid), whose intrinsic additive structure is not rich enough to utilize established algebraic techniques. Two such examples related to tropical mathematics are the max-plus algebra, where we get into difficulties unless we eliminate the idempotent rule $a + a = a$, and the hyperfield, whose addition requires passing to the power set; see [46, §1.2] for more examples. We embed $T$ into a fuller algebraic structure $A$ on which $T$ acts, which possesses a formal negation map $(-)$. $(A, T, (-))$ is called a triple. ([11] utilizes a categorical “involution” which is much like the negation map here.)

The negation map is essential for developing a viable homology theory. Semirings (such as idempotent semirings) often do not have a classical negation. In this paper we look at two natural ways to introduce a negation map in the context of tropical mathematics. The first is simply taking the negation map to be the identity, as done in supertropical math [29] and idempotent mathematics. The second is to construct a negation map by a symmetrization functor, which recovers much of [11], and moreover provides a decomposition into positive and negative parts.
We consider a more general setting (allowing us to treat rings and semirings at the same time), by replacing equality with a “surpassing relation” $\preceq$, given below in Definition 2.11. This relation restricts to equality on $T$. It is reflexive and transitive but not symmetric on $A$. A triple with such relation we call a system. The surpassing relation $\preceq$ plays a key structural role.\footnote{There is a related weaker relation $\nabla$, where $a_1 \nabla a_2$ if and only if $0 \leq a_1 (-) a_2$, which also restricts to equality on $T$. The relation $\nabla$ is reflexive and symmetric, but not transitive. According to experience, $\nabla$ has been more effective in linear algebra [2,3], but $\preceq$ serves better in developing representation theory, and features in [31] as well as this paper.}

In [34] we have considered both “ground systems” which often are semirings, and systemic modules over ground systems. In this paper we investigate the rudiments of homology theory of systemic modules over a given ground system. Although there is justification in limiting the class of modules, as in [27], here we consider all the modules. [11] already has a categorical theory, with the emphasis on homological categories. Although one can just consider their homomorphisms, the major role of the surpassing map $\preceq$ naturally leads us to study “$\leq$-morphisms,” in which $f(a + a') \preceq f(a) + f(a')$. These have already shown up in the hyperfield literature, and have a natural structure theory which parallels the classical theory of homomorphisms.

In this paper we study chains of modules and long exact sequences, in terms of the surpassing relation $\preceq$, and provide a systemic version of [1].

One major obstacle in obtaining a meaningful homology theory for semirings is the lack of correspondence between ideals (modules) and kernels of homomorphisms (congruences). Classically, homology is defined to be “the kernel modulo the image” in a chain. In the semiring case however, the kernel is a congruence and the image does not necessarily correspond to the kernel of a homomorphism (i.e. is not a normal object), which makes the factor object hard to define at best. For this reason we need to work with congruences which are considerably more complicated, and cannot be viewed as morphisms in the original module category. Moreover, we often have to generalize congruences to precongruences, in order to overcome some of the difficulties in handling images and cokernels.

The discrepancy between the category of modules already has been addressed in the literature [11,17,23]. One approach to homology is the use of the double arrow chain complex introduced by Patchkoria [42] or Flores [17]. However, Connes and Consani [11] provide some troublesome examples [11, Examples 4.9,4.14] concerning this approach.

A categorical approach is taken by Connes and Consani [11, Theorem 6.12] who obtain as decisive a theorem as possible for modules over the Boolean semifield. In § 6 we obtain part of [11, Theorem 6.12] in the systemic setting of $\circ$-idempotent modules.

In this paper, we search for the precise algebraic notions which will permit us to obtain explicit versions of standard homological results such as Schanuel’s lemma and the Snake Lemma. It does not seem possible to get the full analog of these results, but there are convincing partial results, which provide insight not available in the categorical approach.
1.1. Main Results

In [31] we studied versions of projective modules and Schanuel’s Lemma. Here we continue with projective dimension via a version of Schanuel’s Lemma which is more technical but which fits in well with the systemic environment, i.e., by means of negation maps and symmetrization as in [3, 45], Sect. 2.7. Then we study homology semigroups of chains of modules, in terms of congruences and $\preceq$-precongruences, avoiding the pitfalls of Remark 3.26. We conclude by bringing in the categorical viewpoint in Sect. Grand.

In order to state the main results we need the following definitions. Let the map $f : M \rightarrow M'$ be a $\preceq$-morphism of systemic modules over a semiring system $A$.

1. The null-module kernel denoted $\ker_{\text{Mod}, M} f$ is defined as the preimage of the set $\{a \in M' : a \succeq 0\}$.
2. The $\succeq$-module image of $f$ is the set $f(M)_{\succeq} := \{b' \in M' : f(b) \succeq b', \text{ for some } b \in M\}$.
3. The systemic cokernel of $f : M \rightarrow M'$, denoted $\text{coker}_{\text{sys}, f}$, is the set $\{[b] : b \in M' \land f(c) \preceq b'(-)b, \text{ for some } c \in M\}$.

**Theorem A.** (Theorems 2.41, 2.43) For any $T$-module $A$, we can embed $A$ into $\hat{A} := A \oplus A$ via $b \mapsto (b, 0)$, thereby obtaining a faithful functor from the category of semirings into the category of semirings with a negation map (and preserving additive idempotence). This yields a faithful functor from ordered semigroups to signed ($-$)-bipotent systems. Any $A$-module $M$ yields a $\hat{A}$-module $\hat{M} = M \oplus M$, which has a signed decomposition where $M^+$ is the first component.

**Theorem B.** (Theorem 4.5) (Semi-Schanuel, homomorphic-version) Let $P \xrightarrow{f} M$ and $P' \xrightarrow{f'} M'$ be $\preceq$-onto homomorphisms, where $P$ is $h$-projective and $P'$ is $(\preceq, h)$-projective, and let $K' = \ker_{\text{Mod}, P'} f'$. There is a $\preceq$-onto $\preceq$-splitting homomorphism $g : K' \oplus P \rightarrow P'$, with a $\preceq$-isomorphism (i.e., $\preceq$-monic and $\preceq$-onto) $\Phi : K \rightarrow K''$, where $K'' = \{(b, b') : b \preceq \bar{\mu}(b')\}$.

Next, we prove that a “weak version” of the Snake Lemma holds in our setting. Consider the following commutative diagram of $\preceq$-morphisms:

\[
\begin{array}{ccc}
M' & \xrightarrow{q} & N' & \xrightarrow{p} & L' \\
\downarrow f & & \downarrow g & & \downarrow h \\
M & \xrightarrow{l} & N & \xrightarrow{r} & L
\end{array}
\]

**Theorem C.** (Theorem 5.11) Under technical conditions for (1.1), there exists a natural $T_A$-equivariant map $d : \ker_{\text{Mod}, L'} h \rightarrow \text{coker}_{\text{sys}, f}$ (given in the proof). An analogous systemic version provides a natural $\preceq$-morphism $d : \ker_{\text{Mod}, L'} h \rightarrow M$, if the $f(M')$-minimality condition of Definition 3.28 holds.
Theorem D. (Weak Snake Lemma) [Lemma 5.14, Theorem 5.15] Under technical conditions for (1.1), we have the sequence of $T_A$-modules with $T_A$-equivariant maps

$$
\ker_{\text{Mod}, \mathcal{M}} f \xrightarrow{q} \ker_{\text{Mod}, \mathcal{N}} g \xrightarrow{p} \ker_{\text{Mod}, \mathcal{L}} h \xrightarrow{d} \text{coker}(f)_{\text{sys}}
$$

$$
\xrightarrow{\bar{i}} \text{coker}(g)_{\text{sys}} \xrightarrow{\bar{r}} \text{coker}(h)_{\text{sys}},
$$
satisfying the following properties:

(i) If the top row of (1.1) is exact, then

$$
\bar{q}(\ker_{\text{Mod}, \mathcal{M}} f) = \ker_{\text{Mod}, \mathcal{H}} \tilde{p},
$$

where $\mathcal{H} = \ker_{\text{Mod}, \mathcal{N}} g$.

(ii) $\bar{p}(\ker_{\text{Mod}, \mathcal{N}} g) \subseteq \{ b \in \ker_{\text{Mod}, \mathcal{L}} h : d(b) = [0] \}$,

where $[0]$ is the equivalence class of $0$ in $\text{coker}(f)_{\text{sys}}$.

(iii) $d(\ker_{\text{Mod}, \mathcal{L}} h) \subseteq \{ [b] \in \text{coker}(f)_{\text{sys}} : \bar{i}([b]) = [0] \}$,

where $[0]$ is the equivalence class of $0$ in $\text{coker}(g)$.

(iv) If the bottom row of (1.1) is exact, then

$$
\bar{i}(\text{coker}(f)_{\text{sys}}) \subseteq \{ b' \in \text{coker}(g)_{\text{sys}} : \bar{r}(b') = [0] \},
$$

where $[0]$ is the equivalence class of $0$ in $\text{coker}(h)_{\text{sys}}$.

Theorem F. (Theorem 6.9) Suppose $\mathcal{C}$ is a pointed category. Define $\hat{\mathcal{C}}$ to have

1. Objects: the pairs $(A, A)$ for each object $A$ of $\mathcal{C}$.
2. Morphisms: for $(A, A), (B, B) \in \text{Obj}(\hat{\mathcal{C}}),$

$$
\text{Hom}_{\hat{\mathcal{C}}}(A, A, (B, B)) = \{ (f, g) \mid f, g \in \text{Hom}_{\mathcal{C}}(A, B) \}.
$$

The composition of two morphisms are given by the twist product, that is, for $(f_1, f_2) : (B, B) \to (C, C)$ and $(g_1, g_2) : (A, A) \to (B, B),$

$$
(f_1, f_2)(g_1, g_2) = (f_1 g_1 + f_2 g_2, f_2 g_1 + f_1 g_2)
$$

Define the switch $(-)_{sw}(f_1, f_2) = (f_2, f_1)$. Then $\hat{\mathcal{C}}$ is a category with negation functor $(-)_{sw}$, and there is a faithful functor $F : \mathcal{C} \to \hat{\mathcal{C}}$ such that $F(A) = (A, A)$ for an object $A$, and $F(f) = (f, 0_A)$ for $f \in \text{Hom}_{\mathcal{C}}(A, B)$.

Theorem G. (Theorem 6.16) Let $(A, T, (-))$ be a triple, and $\text{Mod}_{A, h}$ be the subcategory of $\text{Mod}_A$ with:

1. Objects: The $A$-modules $\mathcal{M}$ for which $(b^o)^o = b^o$ for each $b \in \mathcal{M}$.
2. Morphisms: homomorphisms of $A$-modules.

With $N$ as in Example 6.11, $\text{Mod}_{A, h}$ is a semiexact category.
2. Basic notions

See [46] for a relatively brief introduction; more details are given in [31,34], and [45]. Throughout the paper, we let \( \mathbb{N} \) be the additive monoid of nonnegative integers. Similarly, we view \( \mathbb{Q} \) (resp. \( \mathbb{R} \)) as the additive monoid of rational numbers (resp. real numbers).

A semiring (cf. [12,22]) \( (A, +, \cdot, 1) \) is an additive commutative semigroup \( (A, +, 0) \) and multiplicative monoid \( (A, \cdot, 1) \) satisfying \( 0b = b0 = 0 \) for all \( b \in A \), as well as the usual distributive laws.

The semiring predominantly used in tropical mathematics has been the max-plus algebra, where \( \oplus \) designates max, and \( \odot \) designates +. However, we proceed with the familiar algebraic notation of addition and multiplication in whichever setting under consideration. A magma \( T \) is a set with binary multiplication. We need not assume that multiplication in \( A \) is associative, so that we may treat “Lie algebra homology” in later work.

2.1. \( T \text{-modules} \)

We review the definitions throughout the next three subsections for the reader’s convenience. We assume that \( T \) is a magma.

Definition 2.1. A (left) \( T \text{-module} \) is a set \( (S, +, 0) \) with a zero element (denoted by \( 0 \)) and scalar multiplication \( T \times S \to S \) satisfying the following axioms, for all \( a_i \in T \) and \( b, b_j \in S \):

(i) \( (a_1a_2)b = a_1(a_2b) \).
(ii) \( a0 = 0a = 0 \).
(iii) \( a(\sum_{j=1}^u b_j) = \sum_{j=1}^u (ab_j) \), \( a \in A \),
(iv) Let \( S_1 \) and \( S_2 \) be \( T \text{-modules} \). A homomorphism \( f : S_1 \to S_2 \) is a function such that \( f(ta) = tf(a) \), \( f(a+b) = f(a)+f(b) \) for all \( t \in T \) and \( a, b \in S_1 \).

Remark 2.2. One example treated in the literature is the property that \( b + b' = 0 \) implies \( b = b' = 0 \), called zero sum free in [22] and lacking zero sums in [28], also treated in [11]. This property holds in tropical mathematics, as well as numerous other situations, as indicated in [28, Examples 1.9]. Using this property one can recover some analogues to classical results such as all strongly projective modules being direct sums of cyclic strongly projective modules; However, this assumption is too strong to result in a useful homology theory, and will not be pursued in this paper.

\footnote{In order not to get caught up in later complications, but not following the convention in [8, Definition 1.1], we adjoin a formal absorbing element \( 0 \) to \( T \), i.e., \( a0 = 0a = 0 \), for all \( a \in T \).}
2.2. Triples

Definition 2.3. A negation map on a $T$-module $A$ is a semigroup isomorphism $(-)_A : A \to A$ of order $\leq 2$, written $a \mapsto (-)a$, together with a map $(-)_T$ of order $\leq 2$ on $T$ which also respects the $T$-action in the sense that

$((-)_T a)b = (-)(ab) = a((-)b)$

for $a \in T, b \in A$. When the context is clear, we drop the subscripts and simply denote the negation map by $(-)$.

Lemma 2.4. Let $A$ be a $T$-module with a negation map. Then $(-)0 = 0$.

Proof. Pick any $a \in T$. Then $(-)0 = (-)(a0) = ((-)(a))0 = 0$. \(\square\)

We write $b_1(-)b_2$ for $b_1 + ((-))b_2$, and $b^\circ$ for $b(-)b$, called a quasi-zero. $(-)b$ is called the quasi-negative of $b$. An assortment of negation maps is given in [18,34,45]. When $1 \in T$, the negation map is given simply by $(-)b = ((-)(1))Tb$ for $b \in A$.

Remark 2.5. Let $A$ be a $T$-module with a negation map. Then

$(-)b^\circ = (-)(b(-)b) = ((-)(b)) + b = b^\circ$.

In other words, quasi-zeros are fixed under any negation map.

The set $A^\circ$ of quasi-zeros is a $T$-submodule of $A$ that plays an important role. When $A$ is a semiring, $A^\circ$ is an ideal.

Note. In this paper, we always assume that $A$ is a $T$-module, and also assume that $T$ is a multiplicative monoid, leaving the Lie theory for later. When $A$ is a semiring, we essentially have Lorscheid’s blueprints, [40,41]. We can make a $T$-module with associative multiplication into a semiring by means of [45, Theorem 2.9].

In general, $T$ need not be a subset of $A$. For example, let $T = \mathbb{N}$ and $A$ be any monoid. In this case, $A$ is equipped with a natural $T$-module structure, although $T$ is not a subset of $A$. In what follows, we will further assume that $T$ is indeed a subset of $A$, starting from the following definition.

Definition 2.6. ([45]) A pseudo-triple $(A, T, (-))$ is a $T$-module $A$, called the set of tangible elements, and a negation map $(-)$ on $A$. We write $T_0$ for $T \cup \{0\}$.

A triple is a $T$-pseudo-triple, in which $T \cap A^\circ = \emptyset$ and $T_0$ generates $(A, +)$.

Definition 2.7. (i) Recall from [45, Definition 1.28] that $T$-triple $(A, T, (-))$ is metatangible if $a + b \in T$ whenever $a, b \in T$ with $b \neq (-)a$. The main special case: we call a triple $(-)$-bipotent if $a + a' \in \{a, a'\}$ whenever $a' \neq (-)a$.

(ii) A $(-)$-bipotent triple $(A, T, (-))$ has height 2 if $A = T \cup A^\circ$. 

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Both the supertropical semiring [29] and the symmetrization of an ordered
group (viewed as an idempotent semiring), cf. Sect. 2.7 below, are (−)-bipotent of
height 2.

**Lemma 2.8.** For any (−)-bipotent triple \((A, T, (−))\) of height 2, \(b^0 + b = b^0\) for
all \(b \in A\), and thus also in any free \(A\)-module.

**Proof.** If \(b \neq (−)b\) then \(b^0 + b = b(−)b + b = b(−)b = b^0\). The last assertion
is by checking components. □

### 2.2.2. Signed decompositions

We will need a further refinement for triples of the
second kind, taken from [3].

#### 2.2.2.1. Triples of the first and second kind

The triple is of the **first kind** if \((−)\) is
the identity, of the **second kind** if \((−)\) is not the identity.

When a given \(T\)-module \(A\) does not come equipped with a negation map, there
are two main ways of providing one: Either take \((−)\) to be the identity (first kind),
as is done in supertropical algebra (and implicitly in much of the tropical literature),
or we “symmetrize” \(A\) as in Sect. 2.7 below (second kind), as elaborated in [3] and
[19].

#### 2.2.2.2. Signed decompositions

We will need a further refinement for triples of the
second kind, taken from [3].

**Definition 2.9.** A triple \((A, T, (−))\) of the second kind has a **signed decomposition**
if there is a submonoid \(T^+\) of \(T\) and a \(T^+\)-submodule \(A^+\) spanned by \(T^+_0 := T^+ \cup \{0\}\),
together with injective homomorphisms \(\mu_i : A^+ \to A\) with \(\mu_i = (−)\mu_j\), for
\(i, j \in \{0, 1\}, i \neq j\), and \(T^+\)-module homomorphisms \(\pi_i : A \to A^+\) for \(i = 0, 1\)
satisfying \(\pi_0 \mu_0 = 1_{A^+}, \pi_1 \mu_0 = \pi_0 \mu_1 = 0\), and \(\mu_0 \pi_0 (−)\mu_0 \pi_1 = 1_A\).

We identify \(A^+\) with \(\mu_0(A^+)\) and define \(A^- := \mu_1(A^+)\), both as \(T^+\)
submodules of \(A\).

**Lemma 2.10.** Let \((A, T, (−))\) be a triple with a signed decomposition.

(i) \(A^+ + A^- = A\).

(ii) Any element of \(A\) is uniquely represented as \(b_0(−)b_1\) for \(b_0, b_1 \in A^+\). In
particular, \(A^0 = \{a(−)a : a \in A^+\}\).

(iii) Defining \(T^- := (−)T^+\), we have \(T^+ \cap T^- = \emptyset\).

**Proof.** (i) Any \(b \in A\) satisfies \(b = \mu_0 \pi_0(b)(−)\mu_0 \pi_1(b) \in A^+ + A^-\).

(ii) From (i), any element \(b \in A\) can be written as \(b = b_0(−)b_1\) for \(b_0, b_1 \in A^+\).

We have

\[
 b_0 = \pi_0 \mu_0(b_0(−)b_1) = \pi_0 \mu_0 b, \quad b_1 = \pi_1 \mu_1(b_0(−)b_1) = \pi_1 \mu_1 b,
\]

showing that the decomposition is unique. The second assertion is clear.

(iii) If \(a \in T^+ \cap T^-\), then \(a(−)0 = 0(−)a\), so \(a = 0\). □
2.3. Systems

We round out the structure with a **surpassing relation** $\preceq$ ([45, Definition 1.31] and also described in [34, Definition 2.11]).

**Definition 2.11.** A **surpassing relation** on a triple $(\mathcal{A}, \mathcal{T}, (-))$, denoted $\preceq$, is a partial preorder satisfying the following, for elements $a \in \mathcal{T}$ and $b_i \in \mathcal{A}$:

(i) $0 \preceq c^\circ$ for any $c \in \mathcal{A}$.
(ii) If $b_1 \preceq b_2$ then $(-)b_1 \preceq (-)b_2$.
(iii) If $b_1 \preceq b_2$ and $b'_1 \preceq b'_2$ for $i = 1, 2$ then $b_1 + b'_1 \preceq b_2 + b'_2$.
(iv) If $a \in \mathcal{T}$ and $b_1 \preceq b_2$ then $ab_1 \preceq ab_2$.
(v) If $a \preceq b$ for $a, b \in \mathcal{T}$, then $a = b$.

The justification for these definitions is given in [45, Remark 1.34].

**Remark 2.12.** Any surpassing relation $\preceq$ induces a partial preorder $\leq$ on $\mathcal{A}$ given by $a_0 \leq a_1$ iff $a_0^\circ \preceq a_1^\circ$. This restricts to a partial preorder on $\mathcal{A}^\circ$, which ties in with the modulus, as seen in [31, Example 2.16].

**Definition 2.13.** For a triple $\mathcal{A}$ with a surpassing relation $\preceq$, we let $\mathcal{A}_{\text{Null}} := \{ b \in \mathcal{A} : b \geq 0 \}$.

**Remark 2.14.** $\mathcal{A}_{\text{Null}}$ is a $\mathcal{T}$-submodule of $\mathcal{A}$ containing $\mathcal{A}^\circ$.

**Lemma 2.15.** Let $\mathcal{A}$ be a triple with a surpassing relation $\preceq$. Then we have

$$ b \preceq b + c, \quad \forall b \in \mathcal{A}, \forall c \geq 0. $$

**Proof.** This directly follows from (iii) of Definition 2.11. \qed

**Proposition 2.16.** ([31, Lemma 2.11]) If $b_1 \preceq b_2$, then $b_2(-)b_1 \geq 0$ and $b_1(-)b_2 \geq 0$. In particular, if $b \in \mathcal{A}_{\text{Null}}$, then $(-)b \in \mathcal{A}_{\text{Null}}$.

Some main cases of triples and systems are defined as follows:

**Example 2.17.** ([34, Definition 2.17], [45, Definition 1.70])

(i) Given a triple $(\mathcal{A}, \mathcal{T}, (-))$, define $a \preceq c$ if $a + b^\circ = c$ for some $b \in \mathcal{A}$. Here the surpassing relation $\preceq$ is $\preceq_\circ$, and $\mathcal{A}_{\text{Null}} = \mathcal{A}^\circ$.

(ii) The symmetrized triple (to be considered in Sect. 2.7, in particular Definition 2.40) is a special case of (i).

(iii) More generally, given a surpassing relation $\preceq$, define its restriction $\preceq_{\text{Null}}$ by $a \preceq_{\text{Null}} c$ if $a + b = c$ for some $b \in \mathcal{A}_{\text{Null}}$.

(iv) (Even more generally, taken from [3]) Given a triple $(\mathcal{A}, \mathcal{T}, (-))$ and a $\mathcal{T}$-submodule $\mathcal{I} \supseteq \mathcal{A}^\circ$ of $\mathcal{A}$ closed under $(-)$, we define the $\mathcal{I}$-relation $\preceq_{\mathcal{I}}$ by $b_1 \preceq_{\mathcal{I}} b_2$ if $b_2 = b_1 + c$ for some $c \in \mathcal{I}$. This need not be a surpassing relation, but is relevant to Sect. 6.

---

3 One could also require $b^\circ = b$. This has categorical advantages seen in Sect. 6 but has algebraic drawbacks.
Take $\leq$ to be set inclusion when $\mathcal{A}$ is obtained from the power set of a hyperfield, see [45, §3.6, Definition 4.23], [34, §10]; we denote it here as $\leq_{\text{hyp}}$. $\mathcal{A}_{\text{Null}}$ consists of those sets containing 0, which is the version usually considered in the hypergroup literature, for instance, [5] and [21].

**Definition 2.18.** (i) A **system** (resp. **pseudo-system**) is a quadruple $(\mathcal{A}, \mathcal{T}_A, (-), \preceq)$, where $\preceq$ is a surpassing relation on the triple (resp. pseudo-triple) $(\mathcal{A}, \mathcal{T}_A, (-))$, which is **uniquely negated** in the sense that for any $a \in \mathcal{T}_A$, there is a unique element $b$ of $\mathcal{T}_A$ for which $0 \preceq a + b$ (namely $b = (-)a$).

(ii) The system $(\mathcal{A}, \mathcal{T}_A, (-), \preceq)$ is called a **semiring system** when $\mathcal{A}$ is a semiring.

We want to view triples and their systems as the ground structure over which we build our representation theory. We call this a **ground system**.

A range of examples of ground systems is given in [31, Example 2.16], including “supertropical mathematics.”

Here is a weakened version of $\preceq$, in view of Proposition 2.16:

**Definition 2.19.** For any $b, b' \in \mathcal{A}$, We say that $b$ **balances** $b'$ written $b \nabla b'$, if $b(-)b' \in \mathcal{A}_{\text{Null}}$.

**Proposition 2.20.** $\nabla$ is transitive for any $(-)$-bipotent triple of height 2.

*Proof.* We want to show that $b_0 \nabla b_1$ and $b_1 \nabla b_2$ imply $b_0 \nabla b_2$. This is obvious if all $b_0, b_1$ are in $\mathcal{T}$, since then $b_0 = b_1$. If $b_0$ or $b_1$ is in $\mathcal{A}^\circ$, say $b_0$, then $b_1 \preceq b_0$. We can easily see that transitivity holds in this case. Similarly, the result follows if both $b_0$ and $b_1$ are in $\mathcal{A}^\circ$. \(\square\)

### 2.4. Systemic modules

We fix a ground triple $(\mathcal{A}, \mathcal{T}, (-))$ or ground system $(\mathcal{A}, \mathcal{T}, (-), \preceq)$, according to context, which we simply denote by $\mathcal{A}$ as long as there is no possible confusion.

**Definition 2.21.** A **module with a negation map** (or just called **module with negation**) over the triple $(\mathcal{A}, \mathcal{T}, (-), \preceq)$ is an $\mathcal{A}$-module with a negation map $(-)$ satisfying

(a) $((-)a)m = (-(am)) = a((-)m)$ for $a \in \mathcal{A}$, $m \in \mathcal{M}$.

(b) $\mathcal{M}$ is **uniquely negated** in the sense that for any $a \in \mathcal{T}_\mathcal{M}$, there is a unique element $b$ of $\mathcal{T}_\mathcal{M}$ for which $0 \preceq a + b$ (namely $b = (-)a$).

A **systemic module** $\mathcal{M} := (\mathcal{M}, \mathcal{T}_\mathcal{M}, (-), \preceq)$ over $(\mathcal{A}, \mathcal{T}, (-), \preceq)$ is an $(\mathcal{A}, \mathcal{T}, (-), \preceq)$-module $\mathcal{M}$ together with:

(i) a subset $\mathcal{T}_\mathcal{M}$ spanning $\mathcal{M}$, and satisfying $\mathcal{T}_\mathcal{M} \subseteq \mathcal{T}_\mathcal{M}$;

(ii) a surpassing relation $\preceq_\mathcal{M}$ also denoted as $\preceq$, satisfying $am \preceq a'm'$ for $a \preceq a' \in \mathcal{A}$, $m \preceq m' \in \mathcal{M}$.

---

4 This slightly strengthens the version of “uniquely negated,” for triples, used in [45], which says that there is a unique element $b$ of $\mathcal{T}$ for which $a + b \in \mathcal{A}^\circ$. 
Definition 2.22. A module $\mathcal{M}$ with negation is $\circ$-idempotent if $(b^\circ)^\circ = b^\circ$ for each $b \in \mathcal{M}$.

Lemma 2.23. Any $(-)$-bipotent module of second kind is $\circ$-idempotent.

Proof. If $(-)$ is of second kind then $b^\circ + b^\circ = (b + b)(-)(b + b) = b(-)b = b^\circ$. □

For $(-)$ of first second kind, $b + b + b = b + b$ for supertropical algebra, but various “layered semialgebras” of [45, Example 1.50] are counterexamples. One can extract a “largest” $\circ$-idempotent submodule.

Lemma 2.24. For any module $\mathcal{M}$ with negation, $\{b \in \mathcal{M} : (b^\circ)^\circ = b^\circ\}$ is a $\circ$-idempotent submodule with negation.

Proof. If $(b^\circ)^\circ = b^\circ$ and $(c^\circ)^\circ = c^\circ$, then $((b + c)^\circ)^\circ = (b^\circ)^\circ + (c^\circ)^\circ = b^\circ + c^\circ = (b + c)^\circ$, and $((-b)^\circ)^\circ = (b^\circ)^\circ = b^\circ = (-b)^\circ$. □

In Sect. 6 $\circ$-idempotence will play a special role. Following [34], we consider the category of systemic modules $(\mathcal{M}, T_\mathcal{M}, (-), \preceq)$ over a semiring system $(\mathcal{A}, T_\mathcal{A}, (-), \preceq)$.

Example 2.25. (i) For any module $\mathcal{M}$ with negation, $a\mathcal{M}$ is a submodule of $\mathcal{M}$ with negation, for any $a \in T_\mathcal{A}$.

(ii) For a system $(\mathcal{A}, T_\mathcal{A}, (-), \preceq)$, and an index set $I$, $(\mathcal{A}^{(I)}, \cup_{i \in I} T e_i, (-), \preceq)$ is a systemic module, where $(-)$ and $\preceq$ are defined componentwise, and the $e_i$ are the usual vectors with $1$ in the $i$ position, comprising a base $\{e_i : i \in I\}$ of $\mathcal{A}^{(I)}$. It is uniquely negated, seen componentwise.

(iii) This is in the spirit of hyperfields. Suppose $(\mathcal{A}, T, (-), \preceq)$ is a system. Then the operations extend elementwise to the power set $P(\mathcal{A})$ and $P(T)$, which can be viewed as a systemic module over $(\mathcal{A}, T, (-), \preceq)$. But now we define $\preceq$ to be set inclusion.

(iv) More generally, for any systemic module $\mathcal{M} := (\mathcal{M}, T_\mathcal{M}, (-), \preceq)$ we define the power set (systemic) hypermodule $(P(\mathcal{M}), T_{P(\mathcal{M})}, (-), \preceq)$, where $T_{P(\mathcal{M})} = P(T_\mathcal{M})$, with elementwise operations, elementwise $(-)$, and with surpassing relation $\subseteq$ on $P(\mathcal{M})$.

Definition 2.26. $\preceq$-submodules $\mathcal{N}$ of a systemic module $\mathcal{M}$ are defined in the usual way, with the extra assumption that if $b \in \mathcal{N}$ then $b + c \in \mathcal{N}$ for all $c \succeq 0$ in $\mathcal{M}$.

In particular, any $\preceq$-submodule contains $\mathcal{M}_{\text{Null}}$ since $0 \in \mathcal{N}$.

Definition 2.27. An element $b \in \mathcal{A}$ is null-regular over a $\preceq$-submodule $\mathcal{N} \subseteq \mathcal{M}$, if the following property:

“For each $y \in \mathcal{M}$, there exists $z \in \mathcal{N}$ such that $by \succeq z$,”

implies $b \succeq 0$.

The element $b \in \mathcal{A}$ is null-regular if $b$ is null-regular for $\mathcal{N} = 0$, i.e., if $b\mathcal{M} \succeq 0$ then $b \succeq 0$. 
Example 2.28. (i) Let $\mathcal{A}$ be a system and $\mathcal{M}$ be a systemic module. For any subset $S \subseteq \mathcal{A}$, define the (module) annihilator

$$\text{Ann}_\mathcal{M} S = \{ b \in \mathcal{M} : sb \in \mathcal{M}_{\text{Null}}, \forall s \in S \}.$$ 

Then $\text{Ann}_\mathcal{M} S$ is a $\preceq$-submodule of $\mathcal{M}$. In fact, since $\text{Ann}_\mathcal{M}$ is clearly a $T$-module, the only nontrivial part is to show that for $c \in \text{Ann}_\mathcal{M} S$, we have that $c + d \in \text{Ann}_\mathcal{M} S$ for any $d \geq 0$. But, we have that $sc \in \mathcal{M}_{\text{Null}}$ for any $s \in S$, and hence

$$s(c + d) = sc + sd \geq 0 + 0 = 0,$$

showing that $c + d \in \text{Ann}_\mathcal{M} S$.

(ii) In particular we have the submodule $\text{Ann}_\mathcal{M} a = \{ b \in \mathcal{M} : ab \in \mathcal{M}_{\text{Null}} \}$, for any $a \in T$.

Definition 2.29. For subsets $S_0, S_1 \subseteq \mathcal{M}$, we write $S_0 \preceq S_1$ if for any $b_0 \in S_0$, there exists $b_1 \in S_1$ such that $b_0 \preceq b_1$.

An alternate definition to $\preceq$ on sets could be that for any $b_1 \in S_1$, there exists $b_0 \in S_0$ such that $b_0 \preceq b_1$. However, this would mean $S_0 \preceq \mathcal{M}^\circ$ for any set $S_0$ containing $\emptyset$, which is too inclusive to be a useful criterion.

Definition 2.30. A systemic module $\mathcal{M}$ with a signed decomposition, is an $\mathcal{A}$-module over a system $(\mathcal{A}, \mathcal{T}_\mathcal{A}, (-), \preceq)$ which itself has a signed decomposition, defined to be a union $\mathcal{M} = \mathcal{M}^+ \cup \mathcal{M}^- \cup \mathcal{M}^\circ$ of pairwise disjoint submodules, where $\mathcal{M}^+ = \mathcal{A}^+ \mathcal{M}^+ + \mathcal{A}^- \mathcal{M}^-$ and $\mathcal{M}^- = \mathcal{A}^+ \mathcal{M}^- + \mathcal{A}^- \mathcal{M}^+$.

We have the analog of Lemma 2.10:

Lemma 2.31. Let $(\mathcal{M}, \mathcal{T}_\mathcal{M}, (-), \preceq)$ be a systemic module with a signed decomposition.

(i) $\mathcal{M}^+$ and $\mathcal{M}^-$ are $\mathcal{A}^+$-submodules of $\mathcal{M}$, with $\mathcal{M}^+ + \mathcal{M}^- = \mathcal{M}$.

(ii) Any element of $\mathcal{M}$ is uniquely represented as $b_0(-)b_1$ for $b_0, b_1 \in \mathcal{M}^+$. In particular, $\mathcal{M}^\circ = \{ b(-)b : b \in \mathcal{M}^+ \}$.

(iii) $(\mathcal{M}^+, \mathcal{T}_\mathcal{M}^+, (-), \preceq)$ is a systemic module over the system $(\mathcal{A}^+, \mathcal{T}_{\mathcal{A}^+}, (-), \preceq)$.

Proof. We see by Definition 2.30 that $\mathcal{M}^+$ and $\mathcal{M}^-$ are closed under multiplication by $\mathcal{A}^+$. □

This is to be made relevant via symmetrization, in Theorem 2.43.

2.5. $\preceq$-Morphisms

A key philosophical point here is that $\preceq$, which plays such a prominent structural role, most also enter into the other aspects of the theory. But then we are confronted with the thorny question, when do we use $\preceq$ and when $\succeq$? Our pragmatic response is always to take that direction which provides the best theorem.

We work over a ground system $\mathcal{A} = (\mathcal{A}, \mathcal{T}_\mathcal{A}, (-), \preceq)$ and consider systemic modules $(\mathcal{M}, \mathcal{T}_\mathcal{M}, (-), \preceq)$.
Definition 2.32. Let \( \mathcal{A} = (A, \mathcal{T}_A, (-), \leq) \) be a system.

(i) [34, Definition 2.37] An ordered map of pre-ordered modules

\[ f : (\mathcal{M}, \mathcal{T}_\mathcal{M}, (-), \leq) \to (\mathcal{M}', \mathcal{T}_\mathcal{M}', (-)', \leq') \]

is a function \( f : \mathcal{M} \to \mathcal{M}' \) satisfying the following properties for \( a \in \mathcal{T} \) and \( b \leq b' \), \( b, b' \in \mathcal{M} \):

(a) \( f(0) = 0 \).

(b) \( f(ab) = af(b) \).

(c) \( f((-)b) = (-f(b)) \) (automatic from (b) since we are assuming that \((-)1 \in \mathcal{T})\);

(d) \( f(b) \leq' f(b') \).

(ii) [34, Definition 2.37] A \( \leq \)-morphism of systemic modules

\[ f : (\mathcal{M}, \mathcal{T}_\mathcal{M}, (-), \leq) \to (\mathcal{M}', \mathcal{T}_\mathcal{M}', (-)', \leq') \]

is an ordered map \( f : \mathcal{M} \to \mathcal{M}' \) (taking the preorder to be \( \leq \)) also satisfying

\[ f(b + b') \leq' f(b) + f(b'), \; \forall b, b' \in \mathcal{M} \]

(iii) [34, Definition 2.37] A \( \leq \)-morphism

\[ f : (\mathcal{M}, \mathcal{T}_\mathcal{M}, (-), \leq) \to (\mathcal{M}', \mathcal{T}_\mathcal{M}', (-)', \leq') \]

of systemic modules is systemic if \( f(\mathcal{M}) \) is a systemic submodule of \( \mathcal{M}' \) satisfying the property that \( f^{-1}(\mathcal{T}_\mathcal{M}') \cap \mathcal{T}_\mathcal{M} \neq \emptyset \); \( f \) is tangible if \( f(\mathcal{T}_\mathcal{M}) \subseteq \mathcal{T}_\mathcal{M}' \).

(iv) A \( \geq \)-morphism of systemic \( \mathcal{A} \)-modules is an ordered map \( f : \mathcal{M} \to \mathcal{M}' \) also satisfying

\[ f(b + b') \geq' f(b) + f(b'), \; \forall b, b' \in \mathcal{M} \]

Note for ordered maps that if \( b \geq 0 \) then \( f(b) \geq 0 \), by (a) and (d). Furthermore, ordered maps obviously satisfy the convexity condition that if \( b \leq b' \leq b'' \) then \( f(b) \leq f(b') \leq f(b'') \).

By a homomorphism we mean that equality holds instead of \( \leq' \) in (ii). (This would correspond to a “strict homomorphism” of hyperrings in [32, Definition 2.3].) A homomorphism of semiring systems is also required to satisfy \( f(bb') = f(b)f(b') \).

Example 2.33. (i) Let \( \mathcal{A} \) be a system and \( \mathcal{M} \) be a systemic module over \( \mathcal{A} \). The null homomorphism \( f_{\text{Null}} : \mathcal{M} \to \mathcal{M} \) is given by \( f_{\text{Null}}(b) = b^0 \).

(ii) The zero homomorphism \( 0 : \mathcal{M} \to \mathcal{M} \) is given by \( f_{\text{Null}}(b) = 0 \), \( \forall b \).

(iii) An example of a \( \leq \)-morphism which is not a homomorphism. Let \( (\mathcal{A}, \mathcal{T}, (-)), \leq \) be a \((-)\)-bipotent system of height 2, i.e., \( \mathcal{A} = \mathcal{T} \cup \mathcal{A}^0 \). Define \( \varphi : \mathcal{A} \to \mathcal{A} \) by \( \varphi(a) = a^0 \) for \( a \in \mathcal{T} \) and \( \varphi(b) = 0 \) for \( b \in \mathcal{A}^0 \). We claim that \( \varphi \) is a \( \leq \)-morphism which is not a homomorphism. Indeed, for \( a' \neq (-)a \) in \( \mathcal{T} \), \( \varphi(a + a') = (a + a')^0 = a^0 + (a')^0 = \varphi(a) + \varphi(a') \), but \( \varphi(a(-)a) = 0 \neq \varphi(a^0) = \varphi(a) + \varphi((-)a) \); for \( a + b^0 = b^0 \), \( \varphi(a + b^0) = 0 \leq a^0 = \varphi(a) \) whereas for \( a + b^0 = a \), \( \varphi(a + b^0) = a^0 = \varphi(a) + \varphi(b^0) \).
Example 2.46 below gives instances where we would prefer to use \( \preceq \)-morphisms rather than homomorphisms. On the other hand, as with \([10, 32]\), to facilitate results, we often make the stronger assumption that \( f \) is a homomorphism. Ironically, although the zero homomorphism is null, we prefer to bypass it because it is too special. For example, the zero homomorphism is not tangible. The following observation helps.

Remark 2.34. In the case of signed systemic modules, any \( \preceq \)-morphism

\[
f : (\mathcal{M}, T_{\mathcal{M}}, (-), \preceq) \rightarrow (\mathcal{M}', T_{\mathcal{M}'}, (-)', \preceq')
\]

can be decomposed as \( f = (f_0, f_1) \), where \( f(b) = f_0(b)(-)f_1(b) \) with \( f_i(b) \in \mathcal{M}^+ \). In other words, \( f|_{\mathcal{M}^+} = f_0 \) and \( f|_{\mathcal{M}^-} = f_1 \). We can continue this decomposition by defining \( f_0, i(b) = \pi_0(f_i(b)) \) and \( f_1, i(b) = \pi_1(f_i(b)) \) for \( i = 0, 1 \). This is all well-defined, in view of Lemma 2.31.

Conversely, given \( f_i : \mathcal{M} \rightarrow \mathcal{M}'^+ \), we define \( f = (f_0, f_1) : \mathcal{M} \rightarrow \mathcal{M}' \) by \( f(b) = f_0(b)(-)f_1(b) \), and we continue as before. Thus any \( \preceq \)-morphism \( f \) corresponds to positive \( \preceq \)-morphisms \( (f_i, j : 0 \leq i, j \leq 1) \).

2.5.1. Images

Definition 2.35. Let \( \mathcal{M} \) and \( \mathcal{M}' \) be systemic modules over a system \( \mathcal{A} \), and

\[
f : (\mathcal{M}, T_{\mathcal{M}}, (-), \preceq) \rightarrow (\mathcal{M}', T_{\mathcal{M}'}, (-), \preceq)
\]

be a \( \preceq \)-morphism of systemic modules.

(i) We define the **module image** \( f(\mathcal{M}) \) of \( f \) in the usual way as \( \{f(b) : b \in \mathcal{M}\} \).

(ii) The **\( \preceq \)-module image** of \( f \) is

\[
f(\mathcal{M})_{\preceq} := \{b' \in \mathcal{M}' : f(b) \preceq b', \text{ for some } b \in \mathcal{M}\}.
\]

(Thus \( f(\mathcal{M}) \preceq f(\mathcal{M})_{\preceq} \)). The **\( \geq \)-module image** of \( f \) is

\[
f(\mathcal{M})_{\geq} := \{b' \in \mathcal{M}' : f(b) \geq b', \text{ for some } b \in \mathcal{M}\}.
\]

(iii) \( f \) is **null** if \( f(\mathcal{M}) \subseteq \mathcal{M}'_{\text{Null}} \).

(iv) \( f \) is **\( \preceq \)-onto** if \( f(\mathcal{M})_{\preceq} = \mathcal{M}' \) i.e., if for all \( b' \in \mathcal{M}' \) there is \( b \in \mathcal{M} \) for which \( f(b) \preceq b' \).

(v) \( f \) is **\( \geq \)-onto** if \( f(\mathcal{M})_{\geq} = \mathcal{M}' \) i.e., if for all \( b' \in \mathcal{M}' \) there is \( b \in \mathcal{M} \) for which \( f(b) \geq b' \).

The set \( f(\mathcal{M})_{\geq} \) is a systemic submodule of \( \mathcal{M}' \) for any \( \geq \)-morphism \( f \). In fact, in this case, \( f(a) + f(b) \geq f(a + b) \). In particular, \( f(\mathcal{M})_{\geq} \) is closed under addition. One can easily check the remaining conditions.
2.6. The role of universal algebra and model theory

Although we do not want to get bogged down in formalism, it is appropriate to see how all of this ties in with model theory and the venerable theory of universal algebra, invented for general algebraic theories. Universal algebra is defined by a “signature” comprised of various sets \( A_1, \ldots, A_r \), called “carriers,” operations \( \omega_{j,m} : A_1 \times \cdots \times A_{im} \to A_{im+1} \) on the the sets, and “universal relations,” also called “identities,” which equate evaluations of operators, i.e.,

\[
p(x_1, \ldots, x_t) = q(x_1, \ldots, x_t),
\]

where \( p \) and \( q \) involve composites of various operators \( \omega_{j,m} \). In the theory of triples, we take carriers \( A_1 = A, A_2 = T \), operations \( \omega_{0,1} = \emptyset, \omega_{0,2} = 1, \omega_{1,1} = (\cdot) \), \( \omega_{2,1} : A_1 \otimes A_1 \to A_1 \) to be addition, \( \omega_{2,2} : A_2 \otimes A_1 \to A_1 \) to be the \( T \)-action, and \( \omega_{2,2} : A_1 \otimes A_1 \to A_1 \) to be multiplication when \( A \) is a semiring, and in case when \( T \subseteq A, \omega_{2,1} : A_2 \otimes A_2 \to A_1 \) to be addition on \( T \). The universal relations we have are associativity, distributivity (when it is considered part of the structure), and the properties of the negation map. For example \((\cdot)(\cdot \cdot b) = b\) can be written as \( \omega_{1,1}(\omega_{1,1}(b)) = b \). We can also describe a module over a triple as a universal algebra.

As in [45, §2], universal algebra provides a guide for our definitions, especially with regard to the roles of possible multiplication on \( T \) and the negation map.

So far we are missing one critical ingredient, \( \leq \). For this purpose we incorporate a surpassing relation into the signature, and stipulate that if \( x_i \leq x'_i \) then

\[
\omega_{j,m}(x_1, \ldots, x_m) \leq \omega_{j,m}(x'_1, \ldots, x'_m).
\]

A common example of such a modification needed in universal algebra is the theory of ordered groups. One must be careful. The ordered group \((\mathbb{Q}, +)\) fits into this approach, but the ordered group \((\mathbb{Q}, \cdot)\) does not, since \(-2 < -1\) and \(-5 < 1\) but \(10 > 1\).

Note. Surpassing relations, not being identities, play a role in modifying universal algebra. Having \( \leq \) at our disposal, we want it to replace equality in the basic definitions, especially since it is relevant to recent work in hyperfields in [10, 24, 32–34].

Let \( A = (A_1, \ldots, A_t) \) and \( B = (A'_1, \ldots, A'_t) \) be carriers of a given signature with surpassing relations \( \leq \) and \( \leq' \), respectively.

Let us formalize \( \leq \)-morphisms described above, in terms of universal algebra with a surpassing relation. \( A \leq \)-morphism \( f : A \to B \) is a set of maps \( f_j : A_j \to A'_j, 1 \leq j \leq m \), satisfying the properties:

(i) \( f((\omega(b_1, \ldots, b_m)) \leq' \omega(f_1(b_1), \ldots, f_m(b_m))) \), for every operator \( \omega : A_{i_1} \times \cdots \times A_{im} \to A_{im+1} \), with \( b_j \in A_{ij} \).

(ii) For every operator \( \omega : A_{i_1} \times \cdots \times A_{im} \to A_{im+1} \), if \( b_j \leq c_j \) in \( A_{ij} \) for each \( 1 \leq j \leq m \), then

\[
\omega(f_1(b_1), \ldots, f_m(b_m)) \leq' \omega(f_1(c_1), \ldots, f_m(c_m)).
\]
2.7. Symmetrization

We introduce the symmetrization, which is a way to obtain a negation map, when there is not a natural one. It is an important technique, introduced by Gaubert [19] to study vector spaces over semifields; Here we put it in the context of triples and systems.

A $\mathbb{Z}_2$-graded semigroup $\hat{A}$ is also called a super-semigroup (not to be confused with “supertropical”), i.e., $\hat{A} = A_0 \oplus A_1$ as semigroups. A super-semiring $\mathcal{A}$ is a super-semigroup $\mathcal{A} = A_0 \oplus A_1$ that is a semiring satisfying $A_0^2, A_1^2 \subseteq A_0$ and $A_0A_1, A_1A_0 \subseteq A_1$. A major example is a signed decomposition, where $A_0 = \hat{A}^+$ and $A_1 = (-)\hat{A}^+$. We view $\hat{A} := \mathcal{A} \oplus \mathcal{A}$ as a $\mathcal{T}$-module via the diagonal action, and as a super-semigroup, where $A_0, A_1$ each is a copy of $\mathcal{A}$. Following [34, §2.2] we impose a canonical $\hat{\mathcal{T}}$-module structure on $\hat{A}$ in the following way.

**Definition 2.36.** For any $\mathcal{T}$-module $\mathcal{A}$, the twist action on $\hat{\mathcal{A}}$ over $\hat{T}_0 := \mathcal{T}_0 \oplus \mathcal{T}_0$ is given by the super-action, namely

$$(a_0, a_1) \cdot_{tw} (b_0, b_1) = (a_0b_0 + a_1b_1, a_0b_1 + a_1b_0), \quad a_i \in \mathcal{T}_0, \quad b_i \in \mathcal{A}. \quad (2.1)$$

The switch map $(-)_{sw}$ on $\hat{\mathcal{A}}$ is given by $(-)_{sw}(b_0, b_1) = (b_1, b_0)$.

**Example 2.37.** The symmetrized $\mathcal{T}$-module $\hat{\mathcal{A}}$ of a $\mathcal{T}$-module $\mathcal{A}$ containing $\mathcal{T}$ is defined as the submodule of $\hat{\mathcal{A}}$ generated by $\{(0, 0)\}$ and $\hat{T}_{\hat{\mathcal{A}}} := (\mathcal{T} \oplus 0) \cup (0 \oplus \mathcal{T})$.

**Remark 2.38.** The switch map $(-)$ on $\hat{\mathcal{A}}$ is a negation map, and all quasi-zeros have the form $(b, b)$ since $(b_0, b_1)(-)(b_0, b_1) = (b_0 + b_1, b_0 + b_1)$. Also notice that $T_{\hat{\mathcal{A}}} \cap \hat{\mathcal{A}}^0 = \emptyset$. Hence, $(\hat{\mathcal{A}}, \hat{T}_{\hat{\mathcal{A}}}, (-))$ is a triple for any $\mathcal{T}$-module $\mathcal{A}$ generated by $\mathcal{T}$. Thinking of $(b_0, b_1)$ as a replacement for as $b_0 - b_1$, we see that $(b_1, b_0)$ corresponds to $b_1 - b_0 = -(b_0 - b_1)$. When $\mathcal{A}$ is additively idempotent, so is $\hat{\mathcal{A}}$.

In this situation, Example 2.17 (i) becomes:

**Definition 2.39.** $(b_0, b_1) \leq_{\text{sym}} (b'_0, b'_1)$ if there is $c \in \mathcal{A}$ such that $b_i + c = b'_i$ for $i = 0, 1$.

$\leq_{\text{sym}}$, the main surpassing relation used in this paper, is $\preceq_\circ$ with respect to the switch map (viewed as a negation map).

**Definition 2.40.** Let $\mathcal{A}$ be a system. The symmetrized system (of $\mathcal{A}$) is $(\hat{\mathcal{A}}, \hat{T}_{\hat{\mathcal{A}}}, (-), \leq_{\text{sym}})$.

**Theorem 2.41.** (i) For any $\mathcal{T}$-module $\mathcal{A}$, we can embed $\mathcal{A}$ into $\hat{\mathcal{A}}$ via $b \mapsto (b, 0)$, thereby obtaining a faithful functor from the category of semirings into the category of semirings with a negation map (and preserving additive idempotence).

$(\hat{\mathcal{A}}, \hat{T}_{\hat{\mathcal{A}}}, (-), \leq_{\text{sym}})$ is a system when $\mathcal{T}$ generates $\mathcal{A}$.

(ii) The symmetrized system $(\hat{\mathcal{A}}, \hat{T}_{\hat{\mathcal{A}}}, (-), \leq_{\text{sym}})$ has a signed decomposition with $\hat{\mathcal{A}}^+ = \mathcal{A} \oplus \{0\}$, the image of $\hat{\mathcal{A}}$ in (i), and $\hat{\mathcal{A}}^- = \{0\} \oplus \mathcal{A}$. Thus the functor in (i) goes to the category of triples with a signed decomposition, and the projection to the positive part is a retract.
(iii) Any \( A \)-module \( M \) yields a \( \hat{A} \)-module \( \hat{M} = M \oplus M \), which has a signed decomposition where \( M^+ \) is the first component.

(iv) For any \( b_0, b_1 \in M \) and \( S \subseteq A \), and \( b = (b_0, b_1) \), the set
\[
[b : S] := \{ c \in \hat{M} : Sc \geq b_0(-)b_1 \}
\]
is a \( \preceq \)-submodule of \( \hat{M} \).

**Proof.** (i) The twist multiplication matches the usual multiplication on the first component (where the multiplicative unit is \((1, 0)\)), and then any \( \preceq \)-morphism \( f \) is sent to the \( \preceq \)-morphism \((f, 0)\) on \( \hat{A} \).

The proofs of (ii), (iii), and (iv) are straightforward. \( \square \)

Remark 2.34 also applies in the symmetrized case \( \hat{M} \) of an arbitrary module \( M \), and the positive part of \( \hat{M} \), which is the original module \( M \).

The one downside here is that the symmetrized triple of \( A \) is not \((-)\)-bipotent. The following modification, which is \((-)\)-bipotent and thus more amenable to the systemic theory and tropical mathematics, was introduced in Gaubert’s dissertation [19] and [2, Proposition 5.1], and explored further under the name of “symmetrized max-plus semiring” in [2, Proposition-Definition 2.12]; also see [3, Example 1.42].

**Example 2.42.** One starts with an ordered semigroup \( T \) and a \( T \)-module \( A := T_0 = T \cup \{0\} \), and defines
\[
\tilde{T} := (T \oplus \{0\}) \cup (\{0\} \oplus T),
\]
\[
T_{\text{sym}} := \tilde{T} \cup D \text{ where } D = \{(a, a) : a \in T_0\} \subset T_0 \oplus T_0.
\]
Thus, viewing \( T_0 \) as a bipotent semiring, addition on \( T_{\text{sym}} \) also is according to components on \( T_0 \oplus \{0\} \), \( \{0\} \oplus T_0 \), and \( (a, a) : a \in T_0 \}, whereas “mixed” addition satisfies:
\[
(a_0, 0) + (\{0\}, a_1) = \begin{cases} (a_0, 0) & \text{if } a_0 > a_1; \\ (0, a_1) & \text{if } a_0 < a_1; \\ (a_1, a_1) & \text{if } a_0 = a_1. \end{cases}
\]
\[
(a_0, \{0\}) + (a_1, a_1) = \begin{cases} (a_0, 0) & \text{if } a_0 > a_1; \\ (a_1, a_1) & \text{if } a_0 \leq a_1; \end{cases}
\]
\[
(\{0\}, a_0) + (a_1, a_1) = \begin{cases} (0, a_0) & \text{if } a_0 > a_1; \\ (a_1, a_1) & \text{if } a_0 \leq a_1. \end{cases}
\]

Multiplication in \( T_{\text{sym}} \) is the twist action as in Definition 2.36, but taken with respect to this addition. Then \((T_{\text{sym}}, \tilde{T}, (-))\) is a \((-)\)-bipotent triple.

\( \tilde{T} \) is ordered, when we take the elements of the first component to be positive, and thus greater than the elements of the second component. (But this is only preserved under multiplication by positive elements.)

**Theorem 2.43.** The assertion of Theorem 2.41 also holds for Example 2.42, providing a faithful functor from ordered semigroups to \((-)\)-bipotent triples with signed decompositions.

**Proof.** The same verifications as in Theorem 2.41, with the obvious adjustment for addition. \( \square \)
2.7.1. Application to [11]  Symmetrization provides the link to [11] in obtaining homology for modules over semirings.

Remark 2.44. Connes and Consani [11] work with the category $\mathbb{B}$-Mod of modules over the Boolean semifield $\mathbb{B} = \{0, 1\}$ (where $1 + 1 = 1$). Its symmetrization then is $\hat{\mathbb{B}}(-) = \{(0, 0), (1, 0), (0, 1), (1, 1)\}$, where $(-)(1, 0) = (0, 1)$, and the quasi-zeros are the diagonal elements. Symmetrization decomposes into the first and second components as modules over the Boolean semifield.

There is a homomorphism from $\hat{\mathbb{B}}(-)$ onto the supertropical extension $\mathcal{A} = \mathbb{B} \cup \mathbb{B}^\circ$ of $\mathbb{B}$, where $(a, 0), (0, a) \mapsto a$ and $(a, a)$ is fixed. The twist action (Definition 2.36) on $\hat{\mathcal{A}}$ over $\hat{T}$ is studied in [11, §4] over $\hat{\mathbb{B}}(-)$ under the name of the Kleisli category. Thus [11], which studies modules over $\hat{\mathbb{B}}(-)$, can be viewed in this context. The approach in [11] is more categorical, and the reader is welcome to interpret the results of [11, §3] concerning coequalizers and comonads for systemic modules. Our negation map is similar to the Eilenberg-Moore notion of involution treated in [11, §5].

2.8. Major examples

Let us summarize the discussion of triples and systems via examples. More details are provided in [4].

Example 2.45. (i) Given a triple $(\mathcal{A}, T, (-))$, take the surpassing relation $\leq$ to be $\leq_o$ of Example 2.17.

(ii) The $I$-relation of Example 2.17(iv) provides a rather wide ranging generalization of (i); sometimes it is better to define $b \preceq b'$ when $b' = b^o$ or $b' = b + c$ for some $c \in I$, in order to take quasi-zeros into account.

(iii) The set-up of supertropical mathematics [26,29] is a special case of (i), where $\mathcal{A} = T_0 \cup G$ is the supertropical semiring, $(-)$ is the identity, $o$ is the “ghost map,” $G = A^o$, and $\leq$ is “ghost surpasses.” Another way of saying this is that $a + a' \in \{a, a'\}$ for $a \neq a' \in T$, and $a + a = a^o$. Tropical mathematics is encoded in $G$, which (excluding $0$) often is an ordered group, and can be viewed for example as the target of the Puiseux valuation on the field of Puiseux series (tropicalization).

(iv) The symmetrized triple can be made into a system viewed as special case of (i), which includes idempotent mathematics, as explained in Theorem 2.41.

(v) For a hypergroup $T$, $\mathcal{A}$ is the subset of the power set $\mathcal{P}(T)$ generated by $T$, where $(-)$ is induced from hypergroup negation, and $\preceq$ is set inclusion. We call this a hypersystem. $\mathcal{A}_{\text{Null}}$ consists of those sets containing $0$, which in the hypergroup literature is the set of hyperzeros. The “tropical” hypergroup and sign hypergroup actually are isomorphic to supertropical algebras, so (ii) is applicable in these cases.

(vi) The fuzzy ring of [14] is a special case of (i), and provides valuated matroids as explained in [5] and [3, §1.3.6].

(vii) Tracts, introduced recently in [5], are mostly special cases of systems, where $T$ is the given Abelian group $G$, $\mathcal{A} = \mathbb{N}[G]$, $\varepsilon = (-)\mathbb{I}$, and $NG$ is $\mathcal{A}_{\text{Null}}$, usually taken to be $\mathcal{A}^o$. 


(viii) (The connection to idempotent algebra) In any semiring, one has the Green relation given by \( a \leq b \) iff \( a + b = b \), [45, Example 2.60(i)]. Conversely, any ordered monoid with 0 gives rise to an idempotent (in fact, bipotent) semiring by putting \( a + b = b \) whenever \( a \leq b \).

The only natural negation map would be the identity, and one gets a pseudo-triple by taking \( T \) to be a generating set of \( A \). But then every element \( a = a + a \) is a quasi-zero, and \( A_{\text{Null}} = A \), so one needs to apply symmetrization in order to obtain structure theory along the lines of systems.

Semple and Whittle [47] implemented the notion of partial fields, which could be seen as a weakened definition of hypergroup, where addition \( T \times T \to T \) is only defined on certain sets of elements. Thus \( A \) is a \( T \)-set, but not necessarily a \( T \)-module. Following an idea of [6, §2.2], we can define partial addition on \( T \) in a triple \((A, T, (-))\) by accepting the operation \( a_1 + a_2 = a_3 \) only when \( a_3 \in A^0 \). \( T \) together with its partial addition is called a partial semigroup. In a metatangible triple the only defined sums in the partial addition are for \( a_2 = (-)a_1 \), so the partial semigroup is trivial.

**Example 2.46.** We describe \( \succeq \)-morphisms for the systems from Example 2.45. These are often preferable to homomorphisms.

(i) In supertropical mathematics, a \( \succeq \)-morphism \( f \) satisfies

\[
\text{ iff } f(a_1 + a_2) = f(a_1) + f(a_2);
\]

(2.2) implies that either \( f(a_1 + a_2) = f(a_1) + f(a_2) \), or \( f(a_1) + f(a_2) \) is ghost.

(ii) For hypersystems, a \( \succeq \)-morphism \( f \) satisfies

\[
f(b_1 \odot b_2) \subseteq f(b_1) \odot f(b_2),
\]

the definition used in [10, Definition 2.1] and [21, Definition 2.4]. This is intuitive when \( f \) maps the hyperring \( T \) into itself.

Given a hypersystem \((A \subseteq \mathcal{P}(T), T, (-), \subseteq)\) and a hypergroup morphism \( f \) over \( T \), it is natural to extend \( f \) to \( A \) via

\[
f(b) = \{ f(a) : a \in b \}.
\]

In this case, if \( f(b)(-f(b')) \succeq 0 \), there is some hypergroup element \( a \in f(b) \cap f(b') \).

(iii) For fuzzy rings, in [14, §1], also see [21, Definition 2.17], a homomorphism

\[
f : (K; +, \cdot, \varepsilon_K, K_0) \to (L; +, \cdot, \varepsilon_L, L_0)
\]

of fuzzy rings is defined as satisfying: For any \( \{a_1, \ldots, a_n\} \in K^x \) if \( \sum_{i=1}^n a_i \in K_0 \) then \( \sum_{i=1}^n f(a_i) \in L_0 \). Any \( \succeq \)-morphism in our setting is a fuzzy homomorphism since \( L_0 \) is an ideal, and thus \( \sum_{i=1}^n f(a_i) \in f(\sum_{i=1}^n a_i) + L_0 = L_0 \). The other direction might not hold. The same reasoning holds for tracts of [5].

One subtle point about fuzzy rings is that, as pointed out in [45, Theorem 11.8], although any fuzzy ring gives rise to a triple and thus a system, conceivably the
set \(A_0\) could properly contain \(A^\circ\), cf. [45, Remark 11.3], and thus we could not define \(\leq\) according to \(A_0\). (One could define \(\leq\) according to Example 2.17, but \(A_0\) might not match the set of quasi-zeros.) We refer the readers to [21] for more details on the functorial relation between hyperrings and fuzzy rings.

(iv) Another interesting example comes from valuation theory. In [45, Definition 8.8(ii)], valuations are displayed as \(\leq\)-morphisms of semirings, writing the target of the valuation as a semiring (using multiplicative notation instead of additive notation) via Green’s relation of Example 2.45(viii). Here \(\varphi(b_1b_2) = \varphi(b_1)\varphi(b_2)\). If we instead wrote \(\varphi(b_1b_2) \leq \varphi(b_1)\varphi(b_2)\), we would have a “quasi-valuation.” This pertains in particular to the Puiseux valuation.

3. Semiring analogs of classical module theory

In this section we introduce the analogs of classical module concepts, for modules over semiring systems. We consider the category of systemic modules \((\mathcal{M}, T_\mathcal{M}, (-), \leq)\) over a semiring system \((A, T_A, (-), \leq)\).

3.1. Congruences, precongruences, and weak congruences

One of the the big differences between homomorphisms in semiring theory in comparison to ring theory is the lack of a bijection between modules and kernels of homomorphisms. This makes necessary the use of congruences, which are equivalence relations that respect the given operations (in our case multiplication, addition, and negation). Congruences are subsets of \(\hat{A}\) which also are subalgebras. Moreover, failure of the surpassing relation to be an identity hampers the definition of quotient (factor) objects. Here we impose some restrictive conditions in order to overcome this problem.

**Definition 3.1.** A congruence \(\Phi\) on a system is **convex** if whenever \(b \leq b'\) and \((b, c) \in \Phi\) and \((b', c') \in \Phi\) then \(c \leq c'\).

**Lemma 3.2.** (i) If \(\mathcal{M}\) is a module with negation \((-)\) over a triple and \(\Phi\) is a congruence relation on \(\mathcal{M}\), then the quotient \(\mathcal{M}/\Phi\) also has a negation given by \((-)[b] = [(-)b]\).

(ii) Let \(\mathcal{M}\) be a systemic module and \(\Phi\) be a congruence relation on \(\mathcal{M}\); then the quotient \(\mathcal{M}/\Phi\) is also a systemic module with the surpassing relation given for \([b], [b'] \in \mathcal{M}/\Phi\) by:

\[
[b] \leq [b'] \iff c \leq c' \text{ in } \mathcal{M} \text{ for some } c \in [b], c' \in [b'],
\]

where \([b]\) is the equivalence class of \(b \in \mathcal{M}\) in \(\mathcal{M}/\Phi\).

**Proof.** (i) If \([b'] = [b]\) then \((b, b') \in \Phi\) so \((-)(b, b') = ((-b, (-(b') \in \Phi).\n
(ii) It is clear that \(\mathcal{M}/\Phi\) is a module. Furthermore, it is well-known that the relation on a quotient of a poset defined as in (3.1) is again a partial order. Now, one can easily check that \(\mathcal{M}/\Phi\) is indeed a systemic module.
Lemma 3.2 points to an important difference between triples and systems. Whereas \( \preceq \) plays a critical model-theoretic role, it hampers the category theory since it restricts the use of quotient objects. (The quotient of a system need not be a system, cf. Example 3.6 below.) This will be reflected in Sect. 6.

We generalize congruences slightly, since reflexivity often fails.

**Definition 3.3.** Let \((A, T, (-), \preceq)\) be a system.

(i) The **diagonal congruence** \(\text{Diag}_M\) of a systemic \(A\)-module \(M\) is \(\{(b, b) : b \in M\}\).

(ii) A **precongruence** of a systemic module \(M\) is a \(\preceq\)-submodule \(\Phi\) of \(\hat{M}\), which also is symmetric. In other words, precongruences satisfy all the properties of congruences except perhaps reflexivity and transitivity, and preserve the \(T\)-action.

(iii) \(\Phi'\) is a **weak congruence** if \(\Phi' = \Phi \cup \text{Diag}_A\), where \(\Phi\) is a transitive precongruence.

**Lemma 3.4.** Any weak congruence is a transitive precongruence.

**Proof.** The adjunction of \(\text{Diag}_A\) does not affect symmetry or transitivity. \(\square\)

**Remark 3.5.** For any subset \(S\) of \(A \times A\), the intersection of all congruences containing \(S\) is the “smallest” congruence containing \(S\). But this can be much larger than \(S\), even when \(S\) is a weak congruence.

Let us see some examples.

**Example 3.6.** Let \(M\) be a module over \(A\).

(i) For any subset \(S \subseteq A\), the **congruence annihilator**

\[
\text{Ann}_{M, \text{cong}} S = \{(b, b') \in \hat{M} : sb = sb', \forall s \in S\}
\]

is a congruence of \(M\). In particular,

\[
\text{Ann}_{M, \text{cong}} a = \{(b, b') \in \hat{M} : ab = ab'\}.
\]

However, the congruence \(\text{Ann}_{M, \text{cong}}\) need not be convex.

(ii) Suppose that \(M\) has a signed decomposition over a triple \(A\). For any \(b_0, b_1 \in M^+\), define

\[
\hat{A}(b_0, b_1) = \{(c_0b_0 + c_1b_1, c_0b_1 + c_1b_0) : c_0, c_1 \in A^+\}.
\]

Then the set \(\hat{A}(b_0, b_1)\) is a precongruence.

(iii) The **\(\preceq\)-diagonal precongruence** is \(\{(b, b') \in \hat{M} : b'(-)b \geq 0\}\).

**Definition 3.7.** If \(\Phi\) is a precongruence of \(M\), and \(\Phi'\) is a congruence, then the **quotient precongruence** \(\Phi'/\Phi\) is the precongruence on \(M/\Phi'\) consisting of all pairs of equivalence classes \([(b_0), [b_1])\) with respect to \(\Phi'\), with \(b_i \in M\) and \((b_0, b_1) \in \Phi\).

We can act directly on the precongruences.

**Remark 3.8.** (i) If \(\Phi' \subseteq \Phi\) are congruences, the quotient \(\Phi/\Phi'\) is a congruence.

When \(\Phi = \Phi'\) and \([b_0] = [b_1]\) with respect to \(\Phi\), then \([b_0] = [b_1]\) with respect to \(\Phi'\), so \(\Phi/\Phi'\) is the diagonal congruence.

(ii) When \(\Phi\) is merely a precongruence, the quotient \(\Phi/\Phi'\) is a precongruence.
3.1.1. Morphisms of lex-precongruences and congruences We consider the category of systemic modules \((\mathcal{M}, T_\mathcal{M}, (\cdot), \leq)\) over a semiring system \((\mathcal{A}, T_\mathcal{A}, (\cdot), \leq)\).

**Definition 3.9.** Let \(\Phi\) (resp. \(\Phi'\)) be a precongruence on \(\mathcal{M}\) (resp. \(\mathcal{N}\)).

(i) A **precongruence \(\leq\)-morphism** \(\hat{f} = (f_0, f_1) : \Phi \to \Phi'\) is given by

\[
f(b, b') = (f_0(b) + f_1(b'), f_1(b) + f_1(b')) \quad \forall b, b' \in \mathcal{M}
\]

where \(f_i : b \mapsto (f_i(b), f_i(b))\) for \(\leq\)-morphisms \(f_{ij} : \mathcal{M} \to \mathcal{N}\).

When \(\hat{f}\) also is a homomorphism of modules, we call it a precongruence homomorphism.

Of course we can delete “pre” when \(\Phi, \Phi'\) are congruences. Clearly the composition \(\hat{f} \hat{g}\) of precongruence \(\leq\)-morphisms (resp. homomorphisms) is the precongruence \(\leq\)-morphism (resp. homomorphism) \(\hat{f} \hat{g}\).

**Definition 3.10.** (i) A precongruence \(\leq\)-morphism \(\hat{f} = (f_0, f_1) : \Phi \to \Phi'\) is \((\leq_{\Phi''})\)-trivial for a sub-\(\leq\)-precongruence \(\Phi''\) of \(\Phi'\), if \(\hat{f}(\Phi) \geq \Phi''\).

(ii) A precongruence \(\leq\)-morphism \(\hat{f} : \Phi \to \Phi'\) is \(\leq_{\text{sym}}\)-trivial if \(\hat{f}\) is \(\Phi''\)-trivial for \(\Phi'' = \text{Diag}_{\Phi'}\), i.e., \(\hat{f}(\Phi) \geq \text{Diag}_{\Phi'}\).

**Example 3.11.** The **left \(\leq\)-trivial precongruence homomorphism** \(\hat{f} : \Phi \to \text{Diag}_\mathcal{M}\) is given by \(\hat{f}(a_0, a_1) = (a_0, a_0)\). It is a precongruence homomorphism since

\[
\hat{f}((a_0, a_1) + (a'_0, a'_1)) = \hat{f}(a_0 + a'_0, a_1 + a'_1) = (a_0 + a'_0, a_0 + a'_0) = (a_0, a_0) + (a'_0, a'_0) = \hat{f}(a_0, a_1) + \hat{f}(a'_0, a'_1).
\]

3.2. Monics, kernels, cokernels, and images

**Definition 3.12.** Let \(\mathcal{M}\) and \(\mathcal{N}\) be systemic modules over a semiring system \(\mathcal{A}\), and \(f : \mathcal{M} \to \mathcal{N}\) be a \(\leq\)-morphism.

(i) We say that \(f\) is \(\leq\)-**monic** if \(f(b_0) \leq f(b_1)\) implies \(b_0 \leq b_1\).

(ii) We say that \(f\) is **\(N\)-monic** if the symmetrization of \(f\), that is \(\hat{f} = (f, f)\), is \(\leq\)-monic; i.e., \(f(b_0) + c = f(b_1)\) and \(f(b'_0) + c = f(b'_1)\) for \(c \in \mathcal{N}\) implies \(b_0 + c' = b_1\) and \(b'_0 + c' = b'_1\) for some \(c' \in \mathcal{M}\).

(iii) \(f\) is an \(\leq\)-**isomorphism** if there is a \(\leq\)-morphism \(g : \mathcal{N} \to \mathcal{M}\) for which \(gf = 1_\mathcal{M}\) and \(1_\mathcal{N} \leq fg\) (elementwise).

This has been done in general in universal algebra by means of “equalizers” [7, § 9.1] and “kernel pairs” [7, Exercise 9.4.4(5)], but the theory is rather intricate, so we just bypass it.
3.2.1. Kernels

Kernels are tricky. Let us start with a naive definition,

**Definition 3.13.** Let $\mathcal{M}$ and $\mathcal{N}$ be systemic modules over $\mathcal{A}$, and $f : \mathcal{M} \to \mathcal{N}$ a $\preceq$-morphism.

(i) A $\preceq$-morphism $f : \mathcal{M} \to \mathcal{N}$ is **null** if $f(\mathcal{M}) \subseteq \mathcal{N}_{\text{Null}}$.
(ii) A $\preceq$-submodule $\mathcal{M}'$ of $\mathcal{M}$ is $f$-**null** if $f(a) \in \mathcal{N}_{\text{Null}}$ for all $a \in \mathcal{M}'$. The **null-module kernel** $\ker_{\text{Mod}, \mathcal{M}}f$ of $f$ is the sum of all $f$-null $\preceq$-submodules of $\mathcal{M}$.
(iii) Suppose further that $f$ is a homomorphism. The **classical kernel** is $\{ b \in \mathcal{M} : f(b) = 0 \}$.

We bypass the classical kernel in tropical homology, since 0 is too special, cf. [11, Example 3.4].

**Example 3.14.** Define the **left multiplication map** $\ell_a : \mathcal{M} \to a\mathcal{M}$ by $\ell_a(b) = ab$. $\ell_a$ is a homomorphism, and its kernel is $\text{Ann}_{\mathcal{M}, \text{cong}}a$.

**Remark 3.15.** (i) The composition of a null $\preceq$-morphism with any $\preceq$-morphism is null. Thus null $\preceq$-morphisms take the place of the 0 morphism.
(ii) Let $f : \mathcal{M} \to \mathcal{N}$ be a $\preceq$-morphism. If $b \in \mathcal{M}_{\text{Null}}$ then $f(b) \notin T_N$ since $T_N \cap \mathcal{N}_{\text{Null}} = \emptyset$.

3.2.2. Congruence kernels and congruence cokernels

Here are other candidates for kernel, which may be more natural when working directly within the module category. Since morphisms are defined in terms of congruences, we are led to bring congruences into the picture.

**Definition 3.16.** Let $f : \mathcal{M} \to \mathcal{M}'$ be a $\preceq$-morphism of systemic modules.

(i) The **precongruence kernel** $\text{preker}_{\text{cong}}f$ of $f$ is
$$\{(b_0, b_1) \in \mathcal{M} \oplus \mathcal{M} : f(b_0) \nabla f(b_1)\}.$$ 
(ii) The $\preceq$-**precongruence kernel** $\text{preker}_{\preceq, \text{cong}}f$ of $f$ is
$$\{(b_0, b_1) \in \mathcal{M} \oplus \mathcal{M} : f(b_0) \preceq f(b_1)\}.$$ 

The **N-congruence kernel** $\ker_N f$ of $f$ is
$$\{(b_0, b_1) \in \mathcal{M} \oplus \mathcal{M} : f(b_0) = f(b_1)\}.$$ 

**Lemma 3.17.** The precongruence kernel of a homomorphism $f$ is a reflexive and symmetric precongruence.

**Proof.** By the definition, it is clear that the precongruence kernel is symmetric and closed under coordinate-wise negation map and scalar multiplication. If $f(b_0)$ balances $f(b_1)$ and $f(b'_0)$ balances $f(b'_1)$, then since $f$ is a homomorphism, $f(b_0 + b'_0)$ balances $f(b_1 + b'_1)$. This shows that the precongruence kernel is closed under addition and hence it is a precongruence. \[\square\]

\[\text{Note that this is just an equalizer of } f.\]
For any homomorphism \( f : \mathcal{M} \to \mathcal{N} \), there is an injection \( \bar{f} : \mathcal{M}/\ker N f \to \mathcal{N} \), given by \( \bar{f}(b) = f(b) \).

**Proof.** If \((b, b') \in \ker N f\), then \( f(b) = f(b')\), showing that \( \bar{f} \) is well-defined. Clearly \( \bar{f} \) is a homomorphism since \( \ker N(f) \) is a congruence. Finally, one can easily check that \( \bar{f} \) is an injection. \( \square \)

**Lemma 3.20.** Any homomorphism \( f : \mathcal{M} \to \mathcal{N} \) is composed as \( \mathcal{M} \to \mathcal{M}/\ker N f \to \mathcal{N} \), where the first map is the canonical homomorphism, and the second map is the injection given in Lemma 3.19.

**Proof.** The first map sends \( b \in \mathcal{M} \) to \( \bar{b} \), which is clearly a homomorphism since \( \ker N f \) is a congruence. The second map follows from Lemma 3.19. \( \square \)

### 3.2.3. Precongruences to ≤-submodules

Since the theory mixes congruences and ≤-submodules, the next observation could be useful.

**Definition 3.21.** For a ≤-precongruence \( \Phi \) on a systemic module \( \mathcal{M} \), define \( \mathcal{N}_\Phi = \{b_0(-)b_1 : (b_0, b_1) \in \Phi\} \).

**Lemma 3.22.** \( \mathcal{N}_\Phi \) is a ≤-submodule of \( \mathcal{M} \), and gives rise to a homomorphism \( \psi_\Phi : \Phi \to \mathcal{N}_\Phi \) given by \( \psi_\Phi(b_0, b_1) = b_0(-)b_1 \).

**Proof.** If \( b_0(-)b_1, b'_0(-)b'_1 \in \mathcal{N}_\Phi \), where we have \( (b_0, b_1), (b'_0, b'_1) \in \Phi \), then \( (b_0 + b'_0, b_1 + b'_1) \in \Phi \), so

\[
(b_0(-)b_1) + (b'_0(-)b'_1) = (b_0 + b'_0)(-)(b_1 + b'_1) \in \mathcal{N}_\Phi.
\]

Also \( a(b_0, b_1) = ab_0(-)ab_1 \), whereas \( (ab_0, ab_1) \in \Phi \). This shows that \( \mathcal{N}_\Phi \) is closed under addition and scalar multiplication. Furthermore, since \( (-(b_0, (-)b_1) \in \Phi \), we have that

\[
(-(b_0(-))(-(b_1)) = (-)b_0 + b_1 = b_1(-)b_0 \in \mathcal{N}_\Phi,
\]

showing that \( \mathcal{N}_\Phi \) is closed under negation. Next, suppose that \( c \) is any element of \( \mathcal{M} \) such that \( 0 \leq c \). Since \( \Phi \) is a ≤-precongruence, we have that \( (0, c) \in \Phi \), and hence \( c \in \mathcal{N}_\Phi \). It now follows from the fact that \( \mathcal{N}_\Phi \) is closed under addition, we have \( b + c \in \mathcal{N}_\Phi \) for any \( b \in \mathcal{N}_\Phi \) and \( c \in \mathcal{M}_{\text{Null}} \). The last assertion is an easy verification, noting that \( \psi_\Phi(a(b_0, b_1)) = ab_0(-)ab_1 = a\psi_\Phi(b_0, b_1) \) and \((3.2)\). \( \square \)
Lemma 3.23. With the same notation as above, for any homomorphism $f$,

$$
\psi_\Phi(\text{preker}_{\leq,\text{cong}} f) = \ker_{\text{Mod},\mathcal{M}} f = \psi_\Phi(\text{preker}_{\text{cong}} f).
$$

Proof. Suppose that $(b, b') \in \text{preker}_{\leq,\text{cong}} f$, i.e., $f(b) \leq f(b')$. Since $f$ is a homomorphism, we have that

$$
0 \leq f(b')(-)f(b) = f(b'(-)b),
$$

showing that $\psi_\Phi(b, b') = b'(-)b \in \ker_{\text{Mod},\mathcal{M}} f$. Hence $\psi_\Phi(\ker_{\leq,\text{cong}} f) \subseteq \ker_{\text{Mod},\mathcal{M}} f$.

Conversely, if $f(b) \geq 0$, then we have that $(0, b) \in \text{preker}_{\leq,\text{cong}} f$ since $\psi_\Phi(0, b) = (-)b$. It follows that $\psi_\Phi(\ker_{\leq,\text{cong}} f) \supseteq \ker_{\text{Mod},\mathcal{M}} f$.

The same argument works for $\text{preker}_{\text{cong}} f$, since $f(b) \triangleright f(b')$ implies $0 \leq f(b'(-)f(b) = f(b'(-)b)$. \qed

3.2.4. Images We have the difficulty that, for any reasonable definition, the congruence $\leq$-image of a homomorphism might not contain the diagonal, so modding it out can destroy well-definedness of addition.

Definition 3.24. (i) The precongruence image of a $\leq$-morphism $f : \mathcal{M} \to \mathcal{M}'$ is

$$
f(\mathcal{M})_{\text{pcong}} := \{(b, b') \in \mathcal{M}' \times \mathcal{M}' : f(c) \geq b, \ f(c') \geq b' \text{ for some } c, c' \in \mathcal{M}\}.
$$

The weak precongruence image $f(\mathcal{M})_{\text{wpcong}}$ of $f : \mathcal{M} \to \mathcal{M}'$ is Diag$_{\mathcal{M}'} \cup f(\mathcal{M})_{\text{pcong}}$.

The strict precongruence image of a $\leq$-morphism $f : \mathcal{M} \to \mathcal{M}'$ is

$$
f(\mathcal{M})_{\text{stpcong}} := \{(b, b') \in \mathcal{M}' \times \mathcal{M}' : f(c) = b, \ f(c') = b' \text{ for some } c, c' \in \mathcal{M}\}
$$

(ii) Alternatively we can take the systemic congruence image

$$
f(\mathcal{M})_{\text{cong,sys}} = \text{Diag}_{\mathcal{M}'} \cup \{(b, b') : f(c) \geq b'(-)b \text{ for some } c \in \mathcal{M}\}.
$$

Proposition 3.25. (i) Let $f : \mathcal{M} \to \mathcal{M}'$ be a $\leq$-morphism of systemic modules. Then the weak precongruence image $f(\mathcal{M})_{\text{wpcong}}$ is an equivalence relation on $\mathcal{M}'$.

(ii) Let $f : \mathcal{M} \to \mathcal{M}'$ be a homomorphism of systemic modules. The systemic congruence image $f(\mathcal{M})_{\text{cong,sys}}$ is a congruence on $\mathcal{M}'$.

6 It might be more natural to use $\leq$ instead of $\geq$; for example, when $f = 0$ we would get $b'(-)b$ a quasi-zero. But then we would lose the important Proposition 3.25.
Proof. (i) Clearly, \( f(M)_{wpcong} \) is reflexive. Suppose that \((a, b) \in f(M)_{pcong}\). In other words, there exist \(c, c' \in M\) such that \(f(c) \geq a\) and \(f(c') \geq b\). In particular, obviously we have that \((b, a) \in f(M)_{pcong}\), and hence \((b, a) \in f(M)_{wpcong}\). This shows that \( f(M)_{wpcong} \) is symmetric. Finally, suppose that \((a, b), (b, c) \in f(M)_{wpcong}\). If \((a, b) \in \text{Diag}_{M'}\) or \((b, c) \in \text{Diag}_{M'}\), then clearly \((a, c) \in f(M)_{wpcong}\). Hence, we may assume that \((a, b), (b, c) \in f(M)_{pcong}\). That is, there exist \(d, d', e, e' \in M\) such that

\[
f(d) \geq a, \quad f(d') \geq b, \quad f(e) \geq b, \quad f(e') \geq c.
\]

This implies that \((a, c) \in f(M)_{pcong}\), showing that \( f(M)_{wpcong} \) is transitive.

(ii) The only nontrivial verification is transitivity, i.e., if \((b, b'), (b', b'') \in f(M)_{cong, sys}\) then \((b, b'') \in f(M)_{cong, sys}\). This is obvious if \(b = b'\) or \(b' = b''\), so we can write \(f(c) \geq b(-)b'\) and \(f(c') \geq b''(-)b'\), implying

\[
f(c + c') = f(c) + f(c') \geq b(-)b' + b''(-)b' = b''(-)b + (b'(-)b') \geq b''(-)b.
\]

\(\Box\)

Remark 3.26. (i) Unfortunately \( \text{Diag}_{M'} + f(M)_{pcong} \) is not a congruence, since it is reflexive, but need not be symmetric nor transitive. Connes and Consani [11, before Lemma 3.2] take the smallest congruence containing this, which is difficult to compute and could even be everything. Transitivity is a major obstruction to the theory. For instance, suppose we are given \(f(a) + b = b\) and \(b + f(c) = f(c)\). Then we have

\[
(b, b) + (f(a), f(c)) = (b + f(a), b + f(c)) = (b, f(c)),
\]

which means we are identifying \(b\) with \(f(c)\) in any ordered group with \(c\) large enough. If transitivity holds and for any \(b \in M'\) there is \(c \in M\) for which \(f(c) \geq b\) then the congruence identifies any two elements of \(M'\) and degenerates.

Alternatively one may be tempted to require that \(\text{Diag}_{M'} + f(M)_{pcong}\) is already a congruence; i.e., if \(b \leq f(c), b' \leq f(c'), b'' \leq f(c'')\), and \((b + d, b' + d), (b' + d', b'' + d') \in \Phi\), with \(b' + d = b' + d'\), then \((b + d, b'' + d') \in \Phi\); i.e., transitivity requires some extra cancelation or convexity property.

Although this approach is used in the literature, it seriously restricts the congruences we may use.

(ii) In many instances, we might prefer the weak precongruence image \( f(C)_{wpcong} \). We lose addition, but have a congruence of sets on which \(T\) acts.

(iii) The systemic congruence image \( f(M)_{cong, sys} \) is the best behaved theoretically, in view of Proposition 3.25(ii), but it could be prone to blow up to all of \(M'\) in the tropical case (when \(f(M)\) has elements that are arbitrarily large.)
3.2.5. Cokernels  Here are alternatives to the categorical definition to be given in Sect. 6.2, which seem more promising for specific results.

**Definition 3.27.** Let $f : \mathcal{M} \to \mathcal{M}'$ be a $\preceq$-morphism of systemic modules over a semiring system $\mathcal{A}$. Given $b \in \mathcal{M}'$, we define $[b]_f = \{b' \in \mathcal{M}' : f(c) \preceq b'(-)b$ for some $c \in \mathcal{M}\}$. The cokernel of $f : \mathcal{M} \to \mathcal{M}'$, denoted $\text{coker}_f$, is the set $\mathcal{M}'/f(\mathcal{M})_{\text{wpcong}}$, i.e., we mod out by the weak precongruence image.

The systemic cokernel of $f : \mathcal{M} \to \mathcal{M}'$, denoted $\text{coker}_f^{\text{sys}}$, is the set $\{[b]_f : b \in \mathcal{M}'\}$. $\text{coker}_f$ is not a systemic module since it is not closed under addition, but is a $T$-module. $\text{coker}_f^{\text{sys}}$ provides a rather explicit definition of cokernel, although it is viewed in the power set hypermodule of Example 2.25, a possible drawback.

we can progress further by introducing extra conditions, such as the following.

**Definition 3.28.** The $f$-minimality condition is that any element of $[b]_f$ has a unique minimal element $b_f$.

In the presence of the $f$-minimality condition we define the minimal $f$-cokernel denoted $\text{coker}_{\text{min}} f$, as $\{b_f : b \in \mathcal{M}'\}$.

The $f$-minimality condition is motivated by the following observations.

**Remark 3.29.** (i) If $b \succeq f(c) \in f(\mathcal{M})$ then $0 \in [b]_f$ since we could take $b' = 0$, implying $b_f = 0$ since 0 is minimal.

(ii) If $b$ is tangible and $b \not\preceq f(c)$ and $\mathcal{M}$ is $(-)$-bipotent then $b' = b$.

(iii) From (i) and (ii) we obtain the $f$-minimality condition for $\mathcal{A}^{(n)}$ when $\mathcal{A}$ is $(-)$-bipotent.

(iv) $(b + b')_f \preceq b_f + b'_f$ since at worst one would descend further.

**Note.** Each method has its advantages and limitations. $\text{coker}_f$ is the closest to the classical version, although it loses the property of being a module, and also becomes rather weak in the absence of negation. $\text{coker}_f^{\text{sys}}$ is more in line with the systemic theory, but it is rather strong. $\text{coker}_{\text{min}} f$ provides workable results in tropical situations, but is quite restrictive.

3.3. Chain complexes and exact sequences of systemic modules

Now, we introduce the notion of chain complexes and exact chains of systemic modules. As before, we will consider various versions involving surpassing relation $\preceq$ to encapsulate as many as generalized algebraic structures by one unified framework of systemic modules.

**Definition 3.30.** Let

$$
\cdots \to \mathcal{K} \xrightarrow{k} \mathcal{M} \xrightarrow{f} \mathcal{N} \to \cdots
$$

be a sequence of $\mathcal{A}$-modules with $\preceq$-morphisms. \footnote{Even if we use the term $\succeq$-chain and $\succeq$-exact in this definition, all morphisms are assumed to be $\preceq$-morphisms unless otherwise stated.}
(i) (3.3) is a chain at $\mathcal{M}$ if $k(\mathcal{K}) \subseteq \ker_{\text{Mod},\mathcal{M}} f$. A sequence is said to be a chain complex if it is a chain at each $\mathcal{A}$-module appearing in a sequence.

(ii) (3.3) is a chain at $\mathcal{M}$ if $k(\mathcal{K}) \subseteq \ker_{\text{Mod},\mathcal{M}} f$. A sequence is said to be a chain complex if it is a chain at each $\mathcal{A}$-module appearing in a sequence.

(iii) (3.3) is a chain at $\mathcal{M}$ if $k(\mathcal{K}) \subseteq \ker_{\text{Mod},\mathcal{M}} f$. A sequence is said to be a chain complex if it is a chain at each $\mathcal{A}$-module appearing in a sequence.

(iv) A sequence is said to be a chain complex if it is a chain at each $\mathcal{A}$-module appearing in a sequence.

(v) A sequence is said to be a chain complex if it is a chain at each $\mathcal{A}$-module appearing in a sequence.

(vi) A sequence is said to be a chain complex if it is a chain at each $\mathcal{A}$-module appearing in a sequence.

(vii) A sequence is said to be a chain complex if it is a chain at each $\mathcal{A}$-module appearing in a sequence.

(viii) A sequence

$$\cdots \longrightarrow \mathcal{M}_n \xrightarrow{d_n} \mathcal{M}_{n-1} \xrightarrow{d_{n-1}} \mathcal{M}_{n-2} \longrightarrow \cdots$$

(3.4)

is exact if the sequence

$$\mathcal{M}_{n+1} \xrightarrow{d_{n+1}} \mathcal{M}_n \xrightarrow{d_n} \mathcal{M}_{n-1}$$

is exact for each $n$.

(ix) A chain complex

$$\cdots \longrightarrow \mathcal{M}_n \xrightarrow{d_n} \mathcal{M}_{n-1} \xrightarrow{d_{n-1}} \mathcal{M}_{n-2} \longrightarrow \cdots$$

(3.5)

is exact if the sequence

$$\mathcal{M}_{n+1} \xrightarrow{d_{n+1}} \mathcal{M}_n \xrightarrow{d_n} \mathcal{M}_{n-1}$$

is exact for each $n$.

The following is a special case of Definition 3.30 with symmetrization.

**Definition 3.31.** With the same notation as in Definition 3.30, suppose that the sequences (3.3) and (3.4) are chain complexes.

(i) The chain (3.3) is N-exact at $\mathcal{M}$ if the symmetrized sequence

$$\cdots \longrightarrow \hat{\mathcal{K}} \xrightarrow{\hat{k}} \hat{\mathcal{M}} \xrightarrow{\hat{f}} \hat{\mathcal{N}} \longrightarrow \cdots$$

(3.6)

is exact.

(ii) The chain complex (3.4) is N-exact if the sequence

$$\text{\overline{\mathcal{M}}}_{n+1} \xrightarrow{d_{n+1}} \text{\overline{\mathcal{M}}}_n \xrightarrow{d_n} \text{\overline{\mathcal{M}}}_{n-1}$$

is N-exact for each $n$.

**Remark 3.32.** One can easily see from Definition 3.30 that a chain complex is a chain complex, and exactness implies $\leq$-exactness since $k(\mathcal{K}) \subseteq k(\mathcal{K}) \subseteq \ker_{\text{Mod},\mathcal{M}} f$. 
Exactness is more limited in the systemic context.

**Example 3.33.** (Compare with [16, p. 620]) The chain

\[
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{pmatrix} \xrightarrow{A^o} \begin{pmatrix}
  x_1^o & x_3 & (-)x_2 \\
  (-)x_3 & x_2^o & x_1 \\
  x_2 & (-)x_1 & x_3^o
\end{pmatrix} \xrightarrow{A^{(3)}} \begin{pmatrix}
  x_1 & x_2 & x_3
\end{pmatrix}
\]

is a free chain but exactness fails in the middle.

Indeed, a vector \((a_1, a_2, a_3)\) is in the kernel, i.e., \(a_1 x_1^o (-) a_2 x_3 + a_3 x_2 \geq \emptyset\) for \(a_2 x_3, a_3 x_2 \leq a_1 x_1; a_2 x_3^o (-) a_3 x_1 + a_1 x_3 \geq \emptyset\) for \(a_3 x_1 = a_1 x_3 > a_2 x_2\) and \(a_3 x_3^o (-) a_1 x_2 + a_2 x_1 \geq \emptyset\) for \(a_2 x_1 = a_1 x_2 > a_3 x_3\), so we take \(x_1 = a_1\) large and \(x_2 = a_2, x_3 = a_3\). This vector \((a_1, a_2, a_3)\) is not in the image.

Let us see how Definition 3.31 relates to the literature.

**Remark 3.34.**

(i) Any chain complex

\[
\cdots \to \mathcal{M}_n \xrightarrow{d_n} \mathcal{M}_{n-1} \xrightarrow{d_{n-1}} \mathcal{M}_{n-2} \to \cdots
\]

thereby induces a chain complex

\[
\cdots \to \left[\mathcal{M}_n \xrightarrow{\bar{d}_n} \mathcal{M}_{n-1} \xrightarrow{\bar{d}_{n-1}} \mathcal{M}_{n-2}\right] \to \cdots ,
\]

and thus yields a special case of the double arrow chain complex introduced by Patchkoria [42] and Flores [17].

4. (h, \(\preceq\))-Projective dimension

We recall the various definitions of “projective” from [31]. See [13,25,38,39] for comparison.

**Definition 4.1.** Let \(\mathcal{P} := (\mathcal{P}, \mathcal{T}_{\mathcal{P}}, (-), \preceq)\) be a systemic module.

(i) \(\mathcal{P}\) is **projective** if for any onto homomorphism of systemic modules \(h : \mathcal{M} \to \mathcal{M}'\), every homomorphism \(f : \mathcal{P} \to \mathcal{M}'\) **lifts** to a homomorphism \(\tilde{f} : \mathcal{P} \to \mathcal{M}\), in the sense that \(h \tilde{f} = f\).

(ii) \(\mathcal{P}\) is **\(\preceq\)-projective** if for any \(\preceq\)-onto \(\preceq\)-morphism \(h : \mathcal{M} \to \mathcal{M}'\), every \(\preceq\)-morphism \(\tilde{f} : \mathcal{P} \to \mathcal{M}'\) **\(\preceq\)-lifts** to a \(\preceq\)-morphism \(\tilde{f} : \mathcal{P} \to \mathcal{M}\), in the sense that \(\tilde{f} \preceq h \tilde{f}\).

(iii) \(\mathcal{P}\) is **(\(\preceq\), h)-projective** if for any \(\preceq\)-onto homomorphism \(h : \mathcal{M} \to \mathcal{M}'\), every \(\preceq\)-morphism \(f : \mathcal{P} \to \mathcal{M}'\) **\(\preceq\)-lifts** to a \(\preceq\)-morphism \(\tilde{f} : \mathcal{P} \to \mathcal{M}\), in the sense that \(\tilde{f} \preceq h \tilde{f}\).

(iv) \(\mathcal{P}\) is **h-projective** if for any \(\preceq\)-onto homomorphism \(h : \mathcal{M} \to \mathcal{M}'\), every homomorphism \(f : \mathcal{P} \to \mathcal{M}'\) **\(\preceq\)-lifts** to a homomorphism \(\tilde{f} : \mathcal{P} \to \mathcal{M}\), in the sense that \(f \preceq h \tilde{f}\).

(v) \(\mathcal{P}\) is **N-projective** if the symmetrization \(\mathcal{P}'\) is h-projective with regard to the symmetrized system of Definition 2.40.
4.1. Resolutions

Definition 4.2. Let $\mathcal{M}$ be a systemic module over a semiring system $\mathcal{A}$.

(i) (The classical case) A resolution of $\mathcal{M}$ is an exact chain complex

$$
\cdots \to N_2 \xrightarrow{d_2} N_1 \xrightarrow{d_1} N_0 \xrightarrow{\varepsilon := d_0} \mathcal{M} \tag{4.1}
$$

of homomorphisms, such that $\varepsilon$ is onto.

(ii) A $\leq$-resolution of $\mathcal{M}$ is an $\leq$-exact chain complex

$$
\cdots \to N_2 \xrightarrow{d_2} N_1 \xrightarrow{d_1} N_0 \xrightarrow{\varepsilon := d_0} \mathcal{M} \tag{4.2}
$$

of homomorphisms, such that $\varepsilon$ is onto.

(iii) (The classical case) A projective resolution of $\mathcal{M}$ is a resolution where each $N_i$ is projective.

(iv) An $h$-projective resolution of $\mathcal{M}$ is a $\leq$-resolution where each $N_i$ is $h$-projective. The null-module kernel $\ker_{\text{Mod},_P} d_n$ is called the $n$-th $\leq$-syzygy.

(v) An N-projective resolution of $\mathcal{M}$ is a $\leq$-resolution of $\hat{\mathcal{M}}$ where each $N_i$ is N-projective.

(vi) A free resolution of $\mathcal{M}$ is an h-projective resolution, where each $N_i$ is free.

The definitions of $h$-projective resolution and $N$-projective resolution match, when $\leq$ is taken in the context of symmetrization. But there are other cases of interest, such as in Definition 3.28.

Projective resolutions lead naturally to the notion of projective dimension.

Definition 4.3. Let $\mathcal{M}$ be a systemic module over a semiring system $\mathcal{A}$.

(i) The projective dimension $\text{proj. dim}(\mathcal{M})$ of $\mathcal{M}$ is the smallest possible $n$ (if it exists) among all projective resolutions, such that the $n$-th syzygy $\ker_{\text{Mod},_P} d_n$ is projective. We say $\text{proj. dim}(\mathcal{M}) = 0$ when $\mathcal{M}$ is projective.

(ii) The $h$-projective dimension $\text{h-proj. dim}(\mathcal{M})$ of $\mathcal{M}$ is the smallest possible $n$ (if it exists) among all $\leq$-projective resolutions, such that the $n$-th $\leq$-syzygy $\ker_{\text{Mod},_P} d_n$ is $h$-projective. We say $\text{h-proj. dim}(\mathcal{M}) = 0$ when $\mathcal{M}$ is $h$-projective.

(iii) The $N$-projective dimension $\text{N-proj. dim}(\mathcal{M})$ of $\mathcal{M}$ is the smallest possible $n$ (if it exists) among all $N$-projective resolutions of $\hat{\mathcal{M}}$, such that the $n$-th $\leq$-syzygy $\ker_{\text{Mod},_P} d_n$ is $h$-projective. We say $\text{N-proj. dim}(\mathcal{M}) = 0$ when $\mathcal{M}$ is $N$-projective.

Since a free module over a system is projective and $h$-projective, and $N$-projective (see, [31, §4]), one can show that the category of $\mathcal{T}$-modules has enough projectives, $h$-projectives, and $N$-projectives; any $\mathcal{T}$-module $\mathcal{M}$ has a projective resolution, simply by taking $\mathcal{P}_n$ to be the free $\mathcal{T}$-module mapping onto $\ker_{\text{Mod},_P} d_n$. This raises the question of which projective $\mathcal{T}$-modules are necessarily free.

On the other hand, many chains which are resolutions in the classical sense now fail to be resolutions because of failure of exactness, as in Example 3.33.

Example 4.4. One can easily see that a module over a system $F[\lambda]$ has an infinite projective dimension.
4.1.1. Schanuel’s Lemma revisited

The $h$-projective dimension could be less than the projective dimension; the projective dimension could even be infinite while the $h$-projective dimension were finite. In classical algebra, the projective dimension does not depend on the choice of projective resolution in the following sense. Let $\mathcal{M}$ be a module over a commutative ring $A$ of the projective dimension $n$. One can truncate any projective resolution of $\mathcal{M}$ in the following sense: If $\mathcal{M}$ has a projective resolution

$$0 \to P_m \xrightarrow{d_m} P_{m-1} \to \cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} \mathcal{M},$$

with $m \geq n$, then there exists a projective module $P'$ such that the following is a projective resolution of $\mathcal{M}$:

$$0 \to P' \to P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} \mathcal{M}.$$ 

This elementary result is called Schanuel’s lemma.

We do not yet have the full analog of Schanuel’s lemma for systems and $\preceq$-projective dimension, but we still can attain some reasonable results. For example, [31, §5] contains various versions of Schanuel’s Lemma. Here is a slightly more technical version, whose essence is in [44, Theorem 2.8.26].

Let $P \xrightarrow{f} \mathcal{M}$ and $P' \xrightarrow{f'} \mathcal{M}'$ be $\preceq$-onto homomorphisms, where $P$ is $h$-projective and $P'$ is $(\preceq, h)$-projective. Suppose that we have a given $\preceq$-onto homomorphism $\mu : \mathcal{M} \to \mathcal{M}'$. By the definition of $h$-projective, the $\preceq$-onto homomorphism $\mu f$ lifts to a homomorphism $\tilde{\mu} : P \to P'$ such that

$$\mu f \preceq f' \tilde{\mu}.$$ 

We let $K = \ker_{\text{Mod}, P} f$ and $K' = \ker_{\text{Mod}, P'} f'$.

We refer the readers to [31, §3] for the definition and basic properties of $\preceq$-splitting, a key systemic notion. The following theorem concerns $\preceq$-splitting of homomorphisms.

**Theorem 4.5.** (Semi-Schanuel, homomorphic-version) With the same notation as above, there is a $\preceq$-onto (Definition 3.12) $\preceq$-splitting homomorphism $g : K' \oplus P \to P'$, with an $\preceq$-isomorphism $\Phi : K \to K''$, where $K'' = \{(b', b) \in \ker_{\text{Mod}, K' \oplus P} g : b' \preceq \tilde{\mu}(b)\}$.

**Proof.** Define the map

$$g : K' \oplus P \to P', \quad g(b', b) = \tilde{\mu}(b)(-)b'.$$

We first claim that $g$ is a homomorphism which is $\preceq$-onto. In fact, clearly $g$ is a homomorphism since $\tilde{\mu}$ is a homomorphism. We first claim that for any $b' \in P'$, there exists $b \in P$ such that

$$f'(b') \preceq \mu f(b).$$ 

(4.4)
Indeed, let \( c = f'(b') \). Since \( \mu \) is a \( \preceq \)-onto homomorphism, there exists \( x \in M \) such that \( c \preceq \mu(x) \). Moreover, since \( f \) is \( \preceq \)-onto, we have \( b \in P \) such that \( x \preceq f(b) \); in particular,

\[
f'(b') = c \preceq \mu(x) \preceq \mu f(b).
\]

Since \( f' \) is a homomorphism,

\[
f'(\tilde{\mu}(b)(-b')) = f'(\tilde{\mu}(b))(-f'(b')) \succeq \mu f(b)(-f'(b')) \succeq 0.
\]

Therefore \( \tilde{\mu}(b)(-b') \in K' \). Furthermore, \( g(\tilde{\mu}(b)(-b'), b) = \tilde{\mu}(b)(-)(\tilde{\mu}(b)(-b')) \succeq b' \), implying \( g \) is \( \preceq \)-onto (since \( b' \) was taken arbitrarily). Since \( P' \) is \( (\preceq, h) \)-projective, \( g \preceq \)-splits (by [31, Proposition 4.4]).

For the last assertion take the map

\[
\Phi : K \to \ker_{\text{Mod, } K' \oplus P} g, \quad b \mapsto (\tilde{\mu}(b), b).
\]

\( \Phi \) is well-defined since \( b \in K \) implies that

\[
f'\tilde{\mu}(b) \succeq \mu f(b) \succeq \mu(\emptyset) = 0,
\]

showing that \( \tilde{\mu}(b) \in K' \). Also, we have

\[
g(\tilde{\mu}(b), b) = \tilde{\mu}(b)(-\tilde{\mu}(b)) \succeq 0,
\]

showing that \( (\tilde{\mu}(b), b) \in \ker_{\text{Mod, } K' \oplus P} g \). The retract \( \Phi' \) of \( \Phi \) is given by projection onto the second coordinate, i.e., \( (b', b) \mapsto b \) when \( b \preceq \tilde{\mu}(b') \). Then \( \Phi' \Phi(b) = b \) whereas, when \( b' \preceq \tilde{\mu}(b), \Phi \Phi'(b', b) = \Phi(b) = (\tilde{\mu}(b), b) \succeq (b', b), \) implying \( \Phi \Phi' \succeq 1_{K''} \).

\[\square\]

5. Homological theory of systems

Homology in general should be defined using congruences, not modules, cf. Connes and Consani [10,11]. We take a similar approach, but often bypassing congruences in favor of precongruences in order to broaden the range of applications.

5.1. Homology of precongruences

Consider the following sequence of systemic modules.

\[
\cdots \to M_{n+1} \xrightarrow{d_{n+1}} M_n \xrightarrow{d_n} M_{n-1} \xrightarrow{d_{n-1}} \cdots \tag{5.1}
\]

We assume that (5.1) is a \( \succeq \)-chain complex as in Definition 3.30, that is, \( d_n \) is a \( \preceq \)-morphism and \( d_{n+1}(M_{n+1}) \succeq \ker_{\text{Mod, } M_n} d_n \) for each \( n \).

There are four versions of homology, based on the three versions of image in Remark 3.26 (Definition 5.1) and a direct generalization from the classical definition (Definition 5.2). We use the weak precongruence image in Definition 3.24 to define homology, which is going be a set with \( T \)-action.
To be precise, for each $n \in \mathbb{N}$, we have

$$d_{n+1}(\mathcal{M}_{n+1})_{\text{wpcong}} = \text{Diag}_{\mathcal{M}_n} \cup d_{n+1}(\mathcal{M}_{n+1})_{\text{pcong}},$$

where

$$d_{n+1}(\mathcal{M}_{n+1})_{\text{pcong}} = \{(b, b') \in \widehat{\mathcal{M}}_n : d_{n+1}(c) \geq b, \ d_{n+1}(c') \geq b', \text{ for some } c, c' \in \mathcal{M}_{n+1}\}.$$

From Proposition 3.25(i), $d_{n+1}(\mathcal{M}_{n+1})_{\text{wpcong}}$ is an equivalence relation on $\mathcal{M}_n$.

**Definition 5.1.** Letting $C$ be a $\geq$-chain complex (5.1), we define the following. $Z_n(C) = \ker_{\text{Mod}, \mathcal{M}_n} d_n$.

(i) $H_n(C)_{\text{sys}} = Z_n(C)/d_{n+1}(\mathcal{M}_{n+1})_{\text{cong,sys}}$.
(ii) $H_n(C) = Z_n(C)/d_{n+1}(\mathcal{M}_{n+1})_{\text{wpcong}}$. In other words, $H_n(C)$ is the set of equivalence classes of elements in $Z_n(C)$ under the equivalence relation $d_{n+1}(\mathcal{M}_{n+1})_{\text{wpcong}}$.
(iii) (in line with [11]) $H_n(C)_{\text{CC}} = Z_n(C)/d_{n+1}(\mathcal{M}_{n+1})_{\Phi}$, where $\Phi$ is the smallest congruence containing $\text{Diag}_{\mathcal{M}} + d_{n+1}(\mathcal{M}_{n+1})_{\text{pcong}}$.

Here is our last version directly generalizing the classical definition.

**Definition 5.2.** Suppose that a sequence of systemic modules (5.1) is a $\leq$-chain complex, and let this be $C$. We define the following. $Z_n(C) = \ker_{\text{Mod}, \mathcal{M}_n} d_n$, and

$$H_n(C)_{\text{str}} = Z_n(C)/\langle d_{n+1}(\mathcal{M}_{n+1}) \rangle,$$

where $\langle d_{n+1}(\mathcal{M}_{n+1}) \rangle$ is the smallest congruence in $\mathcal{M}_n$ generated by $d_{n+1}(\mathcal{M}_{n+1})$.

If the context is clear, we write $H_n$ for $H_n(C)$, $H_n(\mathcal{M})_{\text{sys}}$ for $H_n(C)_{\text{sys}}$, $H_n(\mathcal{M})_{\text{str}}$ for $H_n(C)_{\text{str}}$, and $Z_n$ for $Z_n(C)$. We say that the homology is **trivial** if it consists of a single element.

**Lemma 5.3.** With the same notation as above, an action of $T$ on $\mathcal{M}_n$ induces an action on $H_n$.

**Proof.** The induced action is defined as $a \cdot [b] := [ab]$ since $d_{n+1}(ab') = a \cdot d_{n+1}(b')$.

**Proposition 5.4.** Suppose that (5.1) is a $\geq$-chain complex $C$, where $d_n$ is a homomorphism. The following are equivalent:

(i) $C$ is $\geq$-exact at $\mathcal{M}_n$ in the sense of Definition 3.30,
(ii) $H_n, \text{sys}$ is trivial.

**Proof.** (i) $\Rightarrow$ (ii) Suppose that $C$ is $\geq$-exact at $\mathcal{M}_n$, that is, $Z_n = d_{n+1}(\mathcal{M}_{n+1})_{\geq}$. Let $b, b' \in Z_n$. Then there exist $c, c' \in \mathcal{M}_{n+1}$ such that $d_{n+1}(c) \geq b$ and $d_{n+1}(c') \geq b'$ from the $\geq$-exactness. Since $d_n$ is a homomorphism, we have

$$d_{n+1}(c)(-c') = d_{n+1}(c)(-d_{n+1}(c') \geq b(-b').$$
and hence \([b] = [b']\) in \(H_{n,\text{sys}} = Z_n(C)/d_{n+1}(\mathcal{M}_{n+1})_{\text{cong,sys}}\), showing that the equivalence class of \(b\) is the same as the equivalence class of \(b'\) in \(H_{n,\text{sys}}\). Therefore, \(H_{n,\text{sys}}\) is trivial.

(ii) \(\Rightarrow\) (i) Suppose that \(H_{n,\text{sys}}\) is trivial, i.e. for any \(b, b' \in Z_n(C)\), we have \([b] = [b']\) in \(H_{n,\text{sys}}\). We claim that \(Z_n = d_{n+1}(\mathcal{M}_{n+1})_{\geq}\). In fact, we only have to prove that \(Z_n \subseteq d_{n+1}(\mathcal{M}_{n+1})_{\geq}\). Let \(x \in Z_n\). Since \(H_{n,\text{sys}}\) is trivial, we have that \([x] = [0]\) in \(H_{n,\text{sys}}\). It follows that there exists \(c \in \mathcal{M}_{n+1}\) such that

\[
d_{n+1}(c) \geq x(-)0 = x,
\]

showing that \(x \in d_{n+1}(\mathcal{M}_{n+1})_{\geq}\). \(\square\)

**Proposition 5.5.** The following are equivalent for a \(\geq\)-chain complex \(C\) of (5.1):

(i) \(C\) is \(\geq\)-exact at \(M_n\) in the sense of Definition 3.30.

(ii) \(H_n\) is trivial.

**Proof.** (i) \(\Rightarrow\) (ii) Suppose that \(C\) is \(\geq\)-exact at \(M_n\), that is, \(Z_n = d_{n+1}(\mathcal{M}_{n+1})_{\geq}\). Let \(b, b' \in d_{n+1}(\mathcal{M}_{n+1})_{\geq}\). Then there exist \(c, c' \in \mathcal{M}_{n+1}\) such that \(d_{n+1}(c) \geq b\) and \(d_{n+1}(c') \geq b'\). Then clearly, \((b, b') \in d_{n+1}(\mathcal{M}_{n+1})_{\wp\cong}\), showing that the equivalence class of \(b\) is same as the equivalence class of \(b'\) in \(H_n\). Therefore, \(H_n\) is trivial.

(ii) \(\Rightarrow\) (i) Suppose that \(H_n\) is trivial, i.e, \(Z_n \times Z_n = d_{n+1}(\mathcal{M}_{n+1})_{\wp\cong}\). From the condition of \(\geq\)-chains we have that \(d_{n+1}(\mathcal{M}_{n+1})_{\geq} \subseteq Z_n\), so we only have to show the other inclusion. Let \(b \in Z_n\), and assume that there exists a pair \((a, a) \in \text{Diag}_{\mathcal{M}_n} \setminus d_{n+1}(\text{Diag}_{\mathcal{M}_{n+1}})\). Since \(d_{n+1}(\mathcal{M}_{n+1})_{\wp\cong}\) is an equivalence relation and \(H_n = \{1\}\), the pair \((a, b) \in d_{n+1}(\mathcal{M}_{n+1})_{\wp\cong}\). By assumption \(b \neq a\), that is, \((a, b) \notin \text{Diag}_{\mathcal{M}_n}\). This means that the pair \((a, b)\) must be an element of \(d_{n+1}(\mathcal{M}_{n+1})_{\pcong}\), more precisely, there exists \(c\) such that \(a \leq d_{n+1}(c)\). Hence \(a \in d_{n+1}(\mathcal{M}_{n+1})_{\geq}\), for every \((a, a)\) in \(\text{Diag}_{\mathcal{M}_n}\). This implies that \(\text{Diag}_{\mathcal{M}_n} \subseteq d_{n+1}(\mathcal{M}_{n+1})_{\pcong}\) and \(d_{n+1}(\mathcal{M}_{n+1})_{\wp\cong} = d_{n+1}(\mathcal{M}_{n+1})_{\pcong}\). \(\square\)

Note that in order for Definition 5.1(i) to make sense we require \(d_{n+1}(\mathcal{M}_{n+1})_{\cong,\text{sys}}\) to be a congruence, which happens when the maps \(d_n\) are homomorphisms, as shown in Proposition 3.25.

For \(H_n(C)_{\text{CC}}\), the analog of Lemma 5.3 goes through with the same proof. A statement similar to Proposition 5.4 holds in this new setting as well.

5.1.1. Specific examples

Here are some examples of systemic modules defined in terms of chains, and their homologies.

**Example 5.6.** As one can see in Definitions 5.1 and 5.2, there are several possible homologies which one can define. This is due to the generality of systems. Here are some examples paralleling classical examples.

(i) Let \(\ell_a\) be the left multiplication map from \(M\) to \(M\). Consider the following \(\geq\)-chain complex:

\[
K(a) : 0 \rightarrow M \xrightarrow{\ell_a} M
\]
In this case, one obtains

\[ H_1(K(a))_{\text{str}} = \text{Ann} \mathcal{M}a / \langle 0 \rangle = \text{Ann} \mathcal{M}a. \]

(ii) Generalizing the above, given \( a, a' \in T \) we have the commuting diagram

\[
\begin{array}{ccc}
0 & \rightarrow & \mathcal{M} \\
\downarrow & & \downarrow \ell_{a'} \\
0 & \rightarrow & \mathcal{M}
\end{array}
\]

which can be rewritten as the chain complex

\[ K(a, a') : 0 \rightarrow \mathcal{M} \xrightarrow{\ell} \bigwedge^2 \mathcal{M} \rightarrow \cdots \rightarrow \bigwedge^n \mathcal{M} \xrightarrow{d_n} \cdots \]

where \( d_n(v) = v \wedge b \). The homology of the Koszul complex casts light on regular sequences (Sect. 5.1.2).

Many classical examples, say from D. Rogalski [43], can be put in this systemic context.

5.1.2. Regular sequences and homology Homology ties in with regular sequences, which classically lead in to Koszul complexes, cf. [16, §17].

**Definition 5.7.** A **regular sequence** on a \( \preceq \)-module \( \mathcal{M} \) is a subset \( S = \{b_1, \ldots, b_t\} \subseteq A \) for which \( S \mathcal{M} \neq \mathcal{M} \) but \( b_j \) is null-regular for \( S_j \mathcal{M} \subseteq \mathcal{M} \) for all \( 0 \leq j \leq t - 1 \), where \( S_j = \{b_1, \ldots, b_j\} \).

**Lemma 5.8.** (As in [16, p. 425])

(i) \( \{\lambda_1, \ldots, \lambda_r\} \) is a regular sequence in the polynomial system \( A[\lambda_1, \ldots, \lambda_r] \).

(iii) (suggested by Lorscheid) The truncated systemic module \( \mathcal{A}[[\lambda]] \theta \Phi_1 \), where \( \Phi_1 = \{ (f + \lambda m + 1 g, f + \lambda m + 1 h) : \deg f = m \} \).

This mods out all polynomials of degree \( > m \) since we can take \( f = g = 0 \) or \( f = h = 0 \). Then \( H_{m+1, \text{str}} \) is trivial.

(iv) The Grassmann (exterior) semialgebra system of a module \( \mathcal{M} \) (over a commutative semiring) is treated (and exploited) in [18] where we formally define (−) on the elements of the tensor algebra of degree \( \geq 2 \) by \( (−)(v \otimes v') = v' \otimes v \). In particular all elements \( v \otimes v \) are quasi-zeros (but nonzero), but we can further mod out \( v \otimes v \) to \( 0 \). In case \( \mathcal{M} \) is free with base \( \{b_i : i \in I\} \) where \( I \) is an ordered set, the tangible elements now are spanned by \( \{(\pm) b_{i_1} \cdots b_{i_m} : i_1 < \cdots < i_n\} \).

This enables us to define, for any \( b \in \mathcal{M} \), the **Koszul complex**

\[ K(b) : \mathcal{M} \rightarrow \bigwedge^2 \mathcal{M} \rightarrow \cdots \rightarrow \bigwedge^n \mathcal{M} \rightarrow \bigwedge^{n+1} \mathcal{M} \]

where \( d_n(v) = v \wedge b \). The homology of the Koszul complex casts light on regular sequences (Sect. 5.1.2).

Many classical examples, say from D. Rogalski [43], can be put in this systemic context.
(ii) For \( t = 1 \), the sequence \( S = \{ b \} \) is regular if and only if the \( H_1(K(b)) \) of the complex
\[
\emptyset \to A \xrightarrow{b} A
\]
is trivial.

(iii) For \( t = 2 \), and \( a_1 \) null-regular, the sequence \( S = \{ a_1, a_2 \} \) is regular if and only if the systemic homology of the complex
\[
K(a_1, a_2) : \emptyset \to M \xrightarrow{(\overline{a_1}, a_2)} \widehat{M} \xrightarrow{\overline{a_1}, a_2} M
\]
of Example 5.6(ii) is trivial.

Proof. (i) The regularity condition is clear for \( t = 1 \). In general modding out \( \{ \lambda_1, \ldots, \lambda_j \} \) still leaves the polynomial system \( A[\lambda_{j+1}, \ldots, \lambda_t] \), which satisfies the condition by induction.

(ii) By Proposition 5.5 and Example 5.6, since each assertion says \( \text{Ann}_M a = \emptyset \).

(iii) We follow the fourth paragraph of [16, p. 425]. An element is in the \( \leq \)-image of (3) iff it surpasses \( (ca', ca) \), so \( [a : a'] \) corresponding to the elements of the image are the elements surpassing \( Aa \). The assertion follows when \( a_1 \) is null-regular. \( \square \)

5.1.3. The differential approach
We can define a specific chain complex from a given module \( M \) with negation, applying symmetrization to the classical theory given in [9, 35, 48]. We define \( (-)^1 b = (-)b \) and inductively \( (-)^k b = (-)^{k-1}b \).

Tensor products over semirings parallel tensor products over rings, and are well studied in the literature [36–38]; the systemic version is given in [45, § 6.4], where it is stipulated that \( (-) (b \otimes b') = ((-) b) \otimes b' \) for all \( b, b' \). Given a commutative semiring system \( (A, T, (-), \leq) \) and a systemic module \( M \) we define \( M^{\otimes n} = M \otimes M^{\otimes n-1} \).

One needs to use homomorphisms for the theory to contain tensor products in the context of monoidal categories, as illustrated in [34, Example 7.7] and even more extreme examples of Gaubert [19].

Example 5.9. Here is the systemic version of [9, 48]. Given a semiring \( M \) which is a module with negation over a commutative semiring system \( (A, T, (-), \leq) \), define \( M_{ab} = \{ bb' (-) b' : b, b' \in M \} \). (\( M_{ab} \) is null when \( M \) is commutative.)
Define the differential map \( d_n : M^{\otimes n} \to M^{\otimes n-1} \) by
\[
d_n(b_1 \otimes \cdots \otimes b_n) = b_n b_1 \otimes \cdots \otimes b_{n-1} + \sum_{k=1}^{n-1} (-)^k b_1 \otimes \cdots \otimes b_{k-1} \otimes b_k b_{k+1} \otimes b_{k+2} \otimes \cdots \otimes b_n.
\]

Then each term in \( d_k d_{k-1} \) appears with its quasi-negative, implying \( d_k d_{k-1} \) is null for each \( k \), and we have a chain \( M^{\otimes n} \xrightarrow{d_n} M^{\otimes n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} M_{ab} \). (since \( d_1(b_1 \otimes b_2) = b_2 b_1 (-) b_1 b_2 \).
By modding out the congruence spanned by all \((b_1 \otimes \cdots \otimes b_n, b_nb_1 \otimes \cdots \otimes b_{n-1})\), one obtains the systemic version of cyclic homology. Instead of homomorphisms of tensor products, one could consider \(n\)-linear maps, and work in the semigroup algebra of the cartesian power \(M^{(n)}\).

The version \(d_n : \hat{M}^{\otimes n} \rightarrow \hat{M}^{\otimes n-1}\) under symmetrization for an arbitrary semi-algebra over a semiring is to symmetrize the differential map, i.e., put the positive parts in the first component and the negative parts in the second component, taking

\[
d_n((b_1, b_1') \otimes \cdots \otimes (b_n, b_n')) = (c, c')
\]

where

\[
c = b_nb_1 + b_nb_1' \otimes (b_1, b_1') \otimes \cdots \otimes (b_{n-1}, b_{n-1}') + \sum_{k \text{ even}} (b_1, b_1') \otimes \cdots \otimes (b_k, b_k') \otimes (b_k b_{k+1} + b_1' b_{k+1}') \otimes (b_{k+2}, b_{k+2}') \otimes \cdots \otimes (b_n, b_n') + \sum_{k \text{ odd}} (b_1, b_1') \otimes \cdots \otimes (b_k, b_k') \otimes (b_k b_{k+1} + b_1' b_{k+1}').
\]

\[
c' = b_nb_1' + b_nb_1 \otimes (b_1, b_1') \otimes \cdots \otimes (b_{n-1}, b_{n-1}') + \sum_{k \text{ odd}} (b_1, b_1') \otimes \cdots \otimes (b_k, b_k') \otimes (b_k b_{k+1} + b_1' b_{k+1}') \otimes (b_{k+2}, b_{k+2}') \otimes \cdots \otimes (b_n, b_n').
\]

5.2. Chains of systemic modules

**Lemma 5.10.** Consider the following commutative diagram of systemic modules over a semiring system \(A\):

\[
\begin{array}{c}
M' \xrightarrow{q} N' \\
\downarrow f \quad \downarrow g \\
M \xrightarrow{l} N
\end{array}
\]

with \(\leq\)-morphisms \(f\), \(g\), \(q\), and \(l\). The commutative diagram (5.4) induces the following natural maps:

(i) A \(\leq\)-morphism \(\ker_{\Mod, M'} f \rightarrow \ker_{\Mod, N'} g\) given by \(x_0 \mapsto q(x_0)\).

(ii) A \(T_A\)-equivariant map \([l]\) : \(\coker f \rightarrow \coker g\) given by \([y]\) \(\mapsto [l(y)]\), where \([y]\) is the equivalence class of \(y \in M\) in \(\coker f\).

(iii) A precongruence \(\leq\)-morphism \(\preker_{\cong, f} \rightarrow \preker_{\cong, g}\) given by \((x_0, x_1) \mapsto (q(x_0), q(x_1))\).
(iv) A precongruence $\preceq$-morphism $\text{preker}_{\preceq,\text{cong}} f \rightarrow \text{preker}_{\preceq,\text{cong}} g$ given by $(x_0, x_1) \mapsto (q(x_0), q(x_1))$.

Proof. (i) Suppose $x_0 \in \ker_{\text{Mod},\mathcal{M}'} f$. By definition, $f(x_0) \succeq 0$, and hence

$$gq(x_0) = lf(x_0) \succeq l(0) \succeq 0,$$

proving $q(x_0) \in \ker_{\text{Mod},\mathcal{N}'} g$. This map is clearly a $\preceq$-morphism.

(ii) We claim that $[l]$ is well-defined. In fact, if $[y] = [y'] \in \text{coker } f$, then there exist $c, c' \in \mathcal{M}'$ such that $f(c) \succeq y$ and $f(c') \succeq y'$. It follows that

$$g(q(c)) = l(f(c)) \succeq l(y) \text{ and } g(q(c')) = l(f(c')) \succeq l(y'),$$

showing that $[l(y)] = [l(y')]$. This map is clearly $T$-equivariant since for any $t \in T$, we have

$$[l](t \cdot [y]) = [l](ty) = [l(ty)] = [tl(y)] = t \cdot [l(y)].$$

(iii) Let $(x_0, x_1) \in \text{preker}_{\text{cong}} f$, that is $0 \preceq f(x_0)(-)f(x_1)$. It follows that

$$gq(x_0)(-)gq(x_1) \succeq gq(x_0(-)x_1) = lf(x_0(-)x_1) \succeq l(0) \succeq 0,$$

showing that $(q(x_0), q(x_1)) \in \text{preker}_{\text{cong}} g$. This map is clearly a precongruence $\preceq$-morphism.

(iv) Let $(x_0, x_1) \in \text{preker}_{\preceq,\text{cong}} f$, that is $f(x_0) \preceq f(x_1)$. From the commutative diagram (5.4), we have that

$$g(q(x_0)) = l(f(x_0)) \preceq l(f(x_1)) = g(q(x_1)),$$

and hence $(q(x_0), q(x_1)) \in \text{preker}_{\preceq,\text{cong}} g$. In particular, the map $\text{preker}_{\preceq,\text{cong}} f \rightarrow \text{preker}_{\preceq,\text{cong}} g$ sending $(x_0, x_1)$ to $(q(x_0), q(x_1))$ is well defined. Finally, it is clear that this map is a precongruence $\preceq$-morphism. \hfill $\square$

Consider the following commutative diagram of $\preceq$-morphisms:

$$
\begin{array}{ccc}
\mathcal{M}' & \xrightarrow{q} & \mathcal{N}' \\
\downarrow f & & \downarrow g \\
\mathcal{M} & \xrightarrow{l} & \mathcal{N} \\
\end{array}
\rightarrow
\begin{array}{ccc}
\mathcal{L}' & \xrightarrow{p} & \mathcal{L} \\
\downarrow h & & \downarrow r \\
\mathcal{L} & \xrightarrow{r} & \mathcal{L} \\
\end{array}
$$

(5.5)

where $p$ is a $\preceq$-onto homomorphism, $g$ is a homomorphism, and $l$ is a $\preceq$-monic homomorphism.

We want a usable result that says there exists a natural $T$-equivariant map $d$ sending $\ker_{\text{Mod},\mathcal{L}'} h$ to some cokernel. Unfortunately well-definedness does not hold in general, due to the difficulties with transitivity noted in Remark 3.26. We see two ways to overcome this, given the set-up of (5.5).

Theorem 5.11. (Systemic connecting map) In addition to the hypotheses above, suppose that the top row is $\preceq$-exact and the bottom row is $\succeq$-exact (cf. Definition 3.30). Then there exists a natural $T_{\mathcal{A}}$-equivariant map

$$d : \ker_{\text{Mod},\mathcal{L}'} h \rightarrow \text{coker } f_{\text{sys}}$$

(given in the proof). If the $f(\mathcal{M}')$-minimality condition of Definition 3.28 holds, there exists a natural $\preceq$-morphism $d : \ker_{\text{Mod},\mathcal{L}'} h \rightarrow \mathcal{M}$. 

Proof. The construction of \( d \) is similar to the classical case. For each \( x_0 \in \ker_{\text{Mod}, L'} h \), we want to define an element \( d(x_0) \) in \( \ker f_{\text{sys}} \). Since \( p \) is \( \geq \)-onto, we have \( v_0 \in N' \) such that \( p(v_0) \geq x_0 \). Hence,

\[
rg(v_0) = hp(v_0) \geq h(x_0) \in L_{\text{null}}, \tag{5.6}
\]

so \( g(v_0) \in \ker_{\text{Mod}, N'} r \). From the \( \geq \)-exactness of the bottom row we have \( u_0 \in M \) such that

\[
l(u_0) \geq g(v_0).
\]

We then define \( d(x_0) \) to be the image of \( u_0 \) in \( \ker f_{\text{sys}} \).

We claim that \( d \) is well-defined. In fact, suppose that we have \( p(v_0') \geq x_0 \) as well as \( p(v_0) \geq x_0 \). Then

\[
p(v_0'(-)v_0) = p(v_0'(-))p(v_0) \geq x_0(-)x_0 \geq 0 \tag{5.7}
\]

since \( p \) is a homomorphism. In particular, \( v_0'(-)v_0 \in \ker_{\text{Mod}, N'} p \). Now, the \( \leq \)-exactness of the top row shows that there exists \( y \in M' \) such that

\[
q(y) \leq v_0'(-)v_0.
\]

Since \( g \) and \( l \) are homomorphisms,

\[
lf(y) = gq(y) \leq g(v_0'(-)v_0) = g(v_0')(-)g(v_0) \leq l(u_0'(-))l(u_0) = l(u_0'(-)u_0). \tag{5.8}
\]

Now, since \( l \) is \( \leq \)-monic we get \( f(y) \leq u_0'(-)u_0 \), so \( u_0' \) and \( u_0 \) map to the same element in \( \ker f_{\text{sys}} \).

Clearly this map \( d \) is \( T_A \)-equivariant since for each relation in the proof, multiplying an element of \( \ker f_{\text{sys}} \) does not change anything.

If the \( f(M')\)-minimality condition holds, take such \( u_0 \) \( f \)-minimal (using the minimality condition). In proving uniqueness, the minimality condition implies \( u_0' = u_0 \). Furthermore, if \( d(x_0 + x_1) = u_2 \), the minimality condition implies that \( u_2 \leq u_0 + u_1 \), showing that \( d \) is a \( \leq \)-morphism in this case. \( \square \)

Here is an alternate version that holds in tropical math. In the case that \((A, T, (-))\) is metatangible, we have a natural algebraic semilattice structure on \( T \) by putting \( a \wedge a = a \vee a = a \), and \( a \vee a' = a + a' \) for \( a \neq a' \). We say that a semilattice is complete if any set has a sup. A systemic module \( M \) is complete if its underlying tangible set is complete. Thus, when \( M \) is \((-)\)-bipotent we can take infinite sums.

Write \( T_{\ker, L'} \) for the tangible elements of \( \ker_{\text{Mod}, L'} h \), and analogously for \( T_{\ker, M'} \) and \( T_{\ker, N'} \).

**Theorem 5.12.** (Tropical Connecting map) Suppose that \( M \) is a free module over a complete metatangible triple \((A, T, (-))\). Then, given the set-up of (5.5), with \( g \) tangible (Definition 2.32) and the bottom row tangibly exact, there exists a natural \( T_A \)-equivariant map \( d \) from \( T_{\ker, L'} \) to \( T_M \) given in the proof.
Proof. The construction of $d$ is similar to the previous case. We may assume that $\mathcal{M} = \mathcal{A}$ since $\mathcal{M}$ is free. For each $x_0 \in \ker_{\mathcal{L}'},$ we want to define an element $d(x_0)$ in coker $f$. Since $p$ is $\geq$-onto, we have $v \in \mathcal{N}'$ such that $x_0 \leq p(v)$, and $v$ must be tangible. (This follows from [45, Theorem 4.31].) Take $v_0$ to be the sup of all such $v$. Hence,

$$rg(v_0) = hp(v_0) \geq h(x_0) \in \mathcal{L}_{\text{Null}},$$

(5.9)

so $g(v_0) \in \ker_{\mathcal{M}, \mathcal{N}' r}$. It follows from the tangibly exactness of the bottom row that

$$\exists u_0 \in \mathcal{T}_M \text{ such that } g(v_0) \leq l(u_0).$$

(5.10)

We define $d(x_0)$ to be the image of $u_0$ in coker $f$; $d$ is well-defined since there were no choices involved in its definition. (If we had instead $u_0'$ with $g(v_0) \leq l(u_0')$ then $0 \leq g(v_0)(-)g(v_0) \leq l(u_0')(-)l(u_0)$, implying $0 \leq u_0'(--)u_0$, so $u_0' = u_0$ since both are tangible.

Finally, one can easily see that this map is indeed $\mathcal{T}_A$-equivariant. $\square$

Remark 5.13. When $\mathcal{A}$ is $(\cdot)$-bipotent of height 2, $d$ provides a map $\ker_{\mathcal{M}, \mathcal{L}'} h \to \mathcal{T}_M$. Thus Theorem 5.12 has useful content.

Lemma 5.14. Suppose that we have the same respective set-ups as in Theorems 5.11 and 5.12, with all maps tangible in the latter situation. We have the following respective sequences of $\mathcal{T}_A$-modules with $\mathcal{T}_A$-equivariant maps:

(a)$$\begin{align*}
\ker_{\mathcal{M}, \mathcal{M}'} f \xrightarrow{\tilde{q}} \ker_{\mathcal{M}, \mathcal{N}'} g \xrightarrow{\tilde{p}} \ker_{\mathcal{M}, \mathcal{L}} h \xrightarrow{d} \coker(f)_{\text{sys}} \\
\xrightarrow{\tilde{r}} \coker(g)_{\text{sys}} \xrightarrow{\tilde{r}} \coker(h)_{\text{sys}},
\end{align*}$$

(5.11)

(b)$$\begin{align*}
\mathcal{T}_{\ker, \mathcal{M}'} f \xrightarrow{\tilde{q}} \mathcal{T}_{\ker, \mathcal{N}'} g \xrightarrow{\tilde{p}} \mathcal{T}_{\ker, \mathcal{L}} h \xrightarrow{d} \mathcal{T}_M \xrightarrow{\tilde{r}} \mathcal{T}_N \xrightarrow{\tilde{r}} \mathcal{T}_L,
\end{align*}$$

(5.12)

where:

1. $\tilde{q}$ and $\tilde{p}$ are the respective restrictions of $q$ and $p$,
2. $d$ is the connecting map in Theorem 5.11, 5.12 respectively, and
3. $\tilde{l}$ and $\tilde{r}$ are the respective restrictions of $l$ and $r$.

Proof. (a): We have to prove that $\tilde{q}, \tilde{p}, \tilde{l}, \tilde{r}$ are well defined. First, we claim that $\tilde{q}$ is well-defined. In fact, take any $x \in \ker_{\mathcal{M}, \mathcal{M}'} f$. From the commutative diagram (5.5), we have that

$$g(q(x)) = l(f(x)) \geq 0,$$

showing that $q(x) \geq 0$, or equivalently, $\tilde{q}(x) \in \ker_{\mathcal{M}, \mathcal{M}'} g$, showing that $\tilde{q}$ is well defined. One can easily see that $\tilde{p}$ is well-defined by the same argument. Next, we claim that $\tilde{l}$ and $\tilde{r}$ are well-defined. In fact, if $[b] = [b']$ in $\ker_{\text{sys}}(f)$, then we
have $c \in \mathcal{M}'$ such that $f(c) \preceq b'(-)b$. From the commutative diagram (5.5), we have that

$$g(q(c)) = l(f(c)) \preceq l(b')(--)l(b),$$

showing that $[l(b')] = [l(b)]$ in $\text{coker}(g)$. Similarly, $\bar{r}$ is well-defined.

(b) One can easily see that $\bar{q}, \bar{p}, \bar{l}$ and $\bar{r}$ are well-defined from the exact same argument as in (a) with the fact that the maps send tangible elements to tangible elements. $\square$

Then, we have the following version of the Snake Lemma. The hypothesis in (i) is needed since otherwise its proof would reverse the direction.

**Theorem 5.15. (Weak Systemic Snake Lemma)** Suppose that we have the same respective set-up as in Theorem 5.11. Notation as above, the sequence (5.11) satisfies the following:

(i) If the top row of (5.5) is exact, then

$$\tilde{q}(\ker_{\text{Mod}, M'} f)_\preceq = \ker_{\text{Mod}, \mathcal{N}'} \tilde{p},$$

where $\mathcal{H} = \ker_{\text{Mod}, \mathcal{N}'} g$.

(ii) $\tilde{p}(\ker_{\text{Mod}, \mathcal{N}'} g) \subseteq \{b \in \ker_{\text{Mod}, \mathcal{L}'} h : d(b) = [0]\}$,

where $[0]$ is the equivalence class of $0$ in $\text{coker}(f)_{\text{sys}}$.

(iii) $d(\ker_{\text{Mod}, \mathcal{L}'} h) \subseteq \{[b] \in \text{coker}(f)_{\text{sys}} : \bar{l}([b]) = [0]\}$,

where $[0]$ is the equivalence class of $0$ in $\text{coker}(h)_{\text{sys}}$.

(iv) If the bottom row of (5.5) is exact, then

$$\bar{l}(\text{coker}(f)_{\text{sys}}) \subseteq \{b' \in \text{coker}(g)_{\text{sys}} : \bar{r}(b') = [0]\},$$

where $[0]$ is the equivalence class of $0$ in $\text{coker}(h)_{\text{sys}}$.

**Proof.** (i) For $b \in \tilde{q}(\ker_{\text{Mod}, M'} f)_\preceq$, clearly $p(b) \succeq 0$ from the exactness of the top row of (5.5). Furthermore $g(b) \succeq 0$. Indeed, from our assumption, we have $u \in \ker_{\text{Mod}, M'} f$ such that $q(u) = b$. It follows from the commutative diagram (5.5) that

$$g(b) = g(q(u)) = l(f(u)) \succeq l(0) \succeq 0,$$

showing that $g(b) \succeq 0$. This shows that $\tilde{q}(\ker_{\text{Mod}, M'} f)_\preceq \subseteq \ker_{\text{Mod}, \mathcal{H}} \tilde{p}$. Conversely, let $a \in \ker_{\text{Mod}, \mathcal{H}} \tilde{p}$, that is, $p(a) \succeq 0$ and $g(a) \succeq 0$. But, since $q(M') = \ker_{\text{Mod}, \mathcal{N}} p$ from the exactness assumption, there exists $x \in \mathcal{M}'$ such that $q(x) = a$. Furthermore we have

$$l(f(x)) = g(q(x)) = g(a) \succeq 0,$$
showing that \( f(x) \succeq 0 \) since \( l \) is \( \preceq \)-monic. In particular, \( x \in \ker_{\text{Mod}, \mathcal{M}' f} \). This implies that \( a \in \tilde{g}(\ker_{\text{Mod}, \mathcal{M}' f}) \).

(ii) Let \( x \in \ker_{\text{Mod}, \mathcal{N} g} \). We know from Lemma 5.14 that \( b := p(x) \in \ker_{\text{Mod}, \mathcal{L}' h} \). We claim that \( d(p(x)) = [0] \). In fact, from the definition of \( d \) in Theorem 5.11, we have the following:

\[
\begin{align*}
\xrightarrow{p} \quad g(x) & \quad \text{such that } p(x) = b, \quad l(u) \succeq g(x) \succeq 0, \quad d(b) = [u].
\end{align*}
\]

(5.13)

Since \( l \) is \( \preceq \)-monic, (5.13) implies that \( u \succeq 0 \), showing that \( [u] = [0] \) in \( \text{coker}(f)_{\text{sys}} \) since, in this case, we have that \( f(0) = 0 \preceq u(-)0 \). This shows that \( \tilde{p}(\ker_{\text{Mod}, \mathcal{N} g}) \subseteq \{ b \in \ker_{\text{Mod}, \mathcal{L}' h} : d(b) = [0] \} \).

(iii) Suppose \( x_0 \in \ker_{\text{Mod}, \mathcal{L}' h} \). We can find \( u_0 \) and \( v_0 \) as in the proof of Theorem 5.11 such that \( d(x_0) = [u_0] \), \( p(v_0) \succeq x_0 \), \( r(g(v_0)) \succeq 0 \), and \( l(u_0) \succeq g(v_0) \). But, the last condition that \( l(u_0) \succeq g(v_0) \) directly implies

\[
\tilde{l}(u_0) = [l(u_0)] = [0]
\]

in \( \text{coker}(g)_{\text{sys}} \), showing that

\[
d(\ker_{\text{Mod}, \mathcal{L}' h}) \subseteq \{ [b] \in \text{coker}(f)_{\text{sys}} : \tilde{l}([b]) = [0] \}.
\]

(iv) Suppose that \( [b] \in \text{coker}(f)_{\text{sys}} \). Then we have, from the exactness of the bottom row, \( r(l(b)) \succeq 0 \). In particular, we have

\[
\tilde{r}(\tilde{l}(b)) = \tilde{r}([l(b)]) = [r(l(b))] = [0],
\]

showing that \( \tilde{l}(\text{coker}(f)_{\text{sys}}) \subseteq \{ b' \in \text{coker}(g)_{\text{sys}} : \tilde{r}(b') = [0] \} \). \( \square \)

Remark 5.16. (Minimal Weak Snake Lemma) Assume the \( f \)-minimality condition holds, with the set-up of 5.12, where all maps are tangible. With the same set-up as in Theorem 5.15, one can easily prove a similar result (“Minimal Weak Snake Lemma”) about the sequence (5.12), where \( \sim \) denotes the restriction to the kernel and \( \tilde{\sim} \) denotes the induced map on the image.

6. Categorical aspects in the perspective of homological categories

The objective of this section is to show how the spirit of this paper, via triples, is consistent with work in the more abstract categorical-theoretic literature, thereby leading towards a categorical description of systems. We only take the first step in this direction, laying out the guidelines for future research. All of the categories considered here are locally small.

Grandis in [23] introduced the notion of homological categories to investigate homological properties of various categories such as the category of modules over
a semiring or the category of pairs of topological spaces. See [23, pp. 9] for a comprehensive list of examples.

We conjecture that the category of systemic $\mathcal{A}$-modules yields a homological category in the sense of Grandis [23]. The idea is to view everything in terms of morphisms, so $\mathcal{A}$ would be replaced by $\text{Hom}$ sets, and $\mathcal{T}$ would be replaced by distinguished subsets. In this treatment the null morphisms play a prominent role.

6.1. Categories with negation

We start by considering the negation map categorically. We assume, generalizing “preadditive,” that $\text{Hom}(A, B)$, the set of morphisms from $A$ to $B$, is an additive semigroup for all objects $A, B$.

**Definition 6.1.** Let $\mathcal{C}$ be a category. A **categorical ideal of morphisms** of $\mathcal{C}$ is a set $N$ of morphisms such that, for $f \in N$, the composite $hfg$ (when it is defined) is also in $N$ for any morphisms $h, g$.

**Definition 6.2.** Let $\mathcal{C}$ be a category with a categorical ideal $N$. A **negation functor** on $\mathcal{C}$ is an endofunctor $(-) : \mathcal{C} \to \mathcal{C}$ satisfying $(-)A = A$ for each object $A$, and, for all morphisms $f, g$:

(i) $(-)((-)f) = f$.
(ii) $(-)(fg) = ((-)f)g = f((-)g)$, i.e., any composite of morphisms (if defined) commutes with $(-)$.
(iii) $f(-)f \in N$ for any $f$.

**Unique negation** means $f(-)g \in N$ implies $g = f$.

**Remark 6.3.** Note that being a negation functor depends on a choice of a categorical ideal $N$ because of Definition 6.2 (iii).

When lacking a negation functor, we obtain one via the categorical version of symmetrization, performing Example 2.37 at the level of $\text{Hom}$, and thus also obtaining a symmetrization functor.

**Definition 6.4.** Let $\mathcal{C}$ be a pointed category. The symmetrized category $\widehat{\mathcal{C}}$ of $\mathcal{C}$ consists of the following.

(1) Objects; the pairs $(A, A)$ for each object $A$ of $\mathcal{C}$.
(2) Morphisms; for $(A, A), (B, B) \in \text{Obj}(\mathcal{C})$,

\[ \text{Hom}_{\widehat{\mathcal{C}}}((A, A), (B, B)) = \{(f, g) \mid f, g \in \text{Hom}_\mathcal{C}(A, B)\} \]

---

8 [23, § 1.3.1] calls this an “ideal” but we prefer to reserve this terminology for semirings.

9 By this notation, we mean $f + (-)f$. This makes sense since we assume that $\text{Hom}(A, B)$ is an additive semigroup. Alternatively, more in line with [11], one could take $N$ to be \{ $f : f = (-)f$ \}. This is more inclusive since $f^\circ = (-)f^\circ$. On the other hand when $(-)$ is the identity, $f = (-)f$ always.
The composition of two morphisms are given by the twist product, that is, for $(f_1, f_2) : (B, B) \to (C, C)$ and $(g_1, g_2) : (A, A) \to (B, B)$,

\[
(f_1, f_2)(g_1, g_2) = (f_1g_1 + f_2g_2, f_2g_1 + f_1g_2)
\]

(6.1)

Proposition 6.8. Let $C$ be a pointed category. Then, $\hat{C}$ is a category.

Proof. For each object $(A, A)$, one can easily check that the morphism $(1_A, 0_A) : (A, A) \to (A, A)$ is the identity morphism. The composition defined by (6.1) is associative since

\[
((f_1, f_2)(g_1, g_2))(h_1, h_2)
= ((f_1g_1 + f_2g_2)h_1 + (f_2g_1 + f_1g_2)h_2, (f_1g_1 + f_2g_2)h_2 + (f_2g_1 + f_1g_2)h_1)
= (f_1(g_1h_1 + g_2h_2) + f_2(g_1h_2 + g_2h_1), f_1(g_1h_2 + g_2h_1) + f_2(g_1h_1 + g_2h_2))
= (f_1, f_2)((g_1, g_2)(h_1, h_2))
\]

$\square$

Definition 6.6. Let $C$ be a pointed category and $\hat{C}$ be the symmetrized category of $C$. We define the following endofunctor, called the switch, on $\hat{C}$; for any $(f_1, f_2) \in \text{Hom}_{\hat{C}}((A, A), (B, B))$,

\[
(\neg)_{\text{sw}}(f_1, f_2) = (f_2, f_1).
\]

Remark 6.7. As we mentioned above, we assume that the set $\text{Hom}_C(A, B)$ is an additive semigroup for any objects $A, B$ of $C$. This will induce additive semigroup structure on $\text{Hom}_{\hat{C}}((A, A), (B, B))$ (in a coordinate-wise way).

Proposition 6.8. With the same notation as above, $(\neg)_{\text{sw}} : \hat{C} \to \hat{C}$ is a negation functor with the categorical ideal:

\[
N_{\hat{C}} = \{(f, f) : f \in \text{Hom}_C(A, B), \forall A, B \in \text{Obj}(C)\}.
\]

Proof. We first claim that $N_{\hat{C}}$ is a categorical ideal of $\hat{C}$. Indeed, we have

\[
(f_1, f_2)(g, g) = (f_1g + f_2g, f_1g + f_2g) \in N_{\hat{C}},
\]

and

\[
(g, g)(f_1, f_2) = (gf_1 + gf_2, gf_1 + gf_2) \in N_{\hat{C}},
\]

proving $N_{\hat{C}}$ is a categorical ideal of morphisms.

Next, we prove that the functor $(\neg)_{\text{sw}}$ satisfies the conditions in Definition 6.2. Clearly we have

\[
(\neg)_{\text{sw}}((\neg)_{\text{sw}}(f, g)) = (\neg)_{\text{sw}}(g, f) = (f, g).
\]

---

10 See [11, Equation (22)]. This is a categorical version of the twist product defined for modules in Definition 2.36. For application to tropical geometry, see [30].
Now, consider \((f_1, f_2), (g_1, g_2)\) such that \(f_i : B \to C\) and \(g_i : A \to B\) for \(i = 1, 2\). Then, we have

\[
\begin{align*}
(\mathcal{-sw})(f_1, f_2)(g_1, g_2) &= (\mathcal{-sw})(f_1g_1 + f_2g_2, f_2g_1 + f_1g_2) \\
&= (f_2g_1 + f_1g_2, f_1g_1 + f_2g_2) \\
&= (f_2, f_1)(g_1, g_2) = ((\mathcal{-sw})(f_1, f_2))(g_1, g_2).
\end{align*}
\]

Similarly, one can show that

\[
(\mathcal{-sw})(f_1, f_2)(g_1, g_2) = (f_1, f_2)((\mathcal{-sw})(g_1, g_2)),
\]

showing the second condition.

Finally, we have

\[
(f, f)(\mathcal{-sw}(f, f)) = (f, f) + (f, f) = (f + f, f + f) \in N_{\mathcal{C}}.
\]

By combining the above results, we conclude the following.

**Theorem 6.9.** Let \(C\) be a pointed category. Suppose that \(C\) is equipped with a negation functor \((-)\) with respect to a categorical ideal \(N\). Then, there is a faithful functor \(F : C \to \mathcal{C}\) such that \(F(A) = (A, A)\) for an object \(A\), and \(F(f) = (f, 0_A)\) for \(f \in \text{Hom}_C(A, B)\).

**Proof.** We first prove that \(F\) is a functor. In fact, \(F(1_A) = (1_A, 0_A) = 1_{(A,A)}\). Furthermore,

\[
F(gf) = (gf, 0) = (g, 0)(f, 0) = F(g)F(f),
\]

showing that \(F\) is a functor. One can easily see that \(F\) is faithful. \qed

### 6.2. Towards semiexact categories and homological categories

In this subsection, we briefly explore how our framework of systems is related to more categorical framework of Grandis [23] as well as Connes and Consani’s work [11]. In particular, we prove that the category of \(A\)-modules is semiexact (under certain condition), which we expect to be a homological category. We further investigate the category of systemic modules in this context. To make this section self-contained, we briefly recall the definition of semiexact category and homological category.

**Definition 6.10.** Let \(C\) be a category.

(i) A categorical ideal \(N\) is said to be **closed** if there exists a set \(\mathcal{O}\) of objects in \(C\) such that

\[
N = \{ f \in \text{Mor}(C) \mid f \text{ factors through some object in } \mathcal{O}\}. \tag{6.2}
\]
(ii) An **N-category** is a category with a specified categorical closed ideal $N$ of morphisms.

From the systemic point of view, the set $\mathcal{O}$ is comprised of objects whose elements are quasi-zeros as the following example illustrates.

**Example 6.11.** Let $(\mathcal{A}, \mathcal{T}, (-))$ be a triple, and $\text{Mod}_\mathcal{A}$ be the category of systemic $\mathcal{A}$-modules. For each $\mathcal{M} \in \text{Mod}_\mathcal{A}$, recall that we have a submodule:

$$\mathcal{M}^o = \{a(-)a \mid a \in \mathcal{M}\}.$$  

Consider the following set

$$\mathcal{O} = \{\mathcal{M}^o \mid \mathcal{M} \in \text{Mod}_\mathcal{A}\}.$$  

It is clear that $N$ defined as in (6.2) is a closed categorical ideal of $\text{Mod}_\mathcal{A}$. Hence, $\text{Mod}_\mathcal{A}$ is an $N$-category.

One ingredient of homological category is to replace the zero object (or zero morphism) of an abelian category with the set of zero objects (or zero morphisms) through a closed categorical ideal $N$. To be precise, Gradis introduced the following:

**Definition 6.12.** Let $(\mathcal{C}, N)$ be an $N$-category. Let $f : X \to Y$ be a morphism of $\mathcal{C}$.

(i) The **N-kernel** $\text{ker}_N f$ of $f$ is a morphism $\text{ker}_N f : \text{Ker} f \to X$ satisfying the usual universal property with respect to $N$, that is, $\text{ker}_N f$ satisfies the following conditions:

\begin{enumerate}
  \item The composite $f(\text{ker}_N f)$ is null, i.e., $f(\text{ker}_N f) \in N$.
  \item For any morphism $h$, if $fh$ is defined and null, then $h$ uniquely factors through $\text{ker}_N f$.
\end{enumerate}

(ii) The **N-cokernel** of $f$ is a morphism $\text{coker}_N f : Y \to \text{Coker}(f)$ (suitably defined) satisfying the usual universal property with respect to $N$: The composite $f(\text{coker}_N f) \in N$, and for any morphism $h$, if $hf$ is defined and null, then $h$ uniquely factors through $\text{coker}_N f$.

(iii) A morphism $f$ is said to be **N-monic** (resp. **N-epic**) if $fh$ (resp. $kf$) is null, then $h$ is null (resp. $k$ is null) for any morphism $h$ (resp. $k$) where $fh$ (resp. $kf$) is defined.

(iv) A **normal** N-monic (resp. **normal** N-epic) is an N-monic (resp. N-epic) which is an N-kernel (resp. N-cokernel) of some morphism.

We now recall the definition of semiexact category and homological category. We then prove that the category of $\mathcal{A}$-modules (satisfying some mild condition) is semiexact. We leave open the problem of proving (or disproving) that the category of $\mathcal{A}$ modules (with some suitable condition) is homological.

---

11 In [23], this is called the kernel with respect to $N$. 
Let \((C, N)\) be an \(N\)-category. Suppose that a morphism \(f : X \to Y\) has an \(N\)-kernel and \(N\)-cokernel. Then \(f\) uniquely factors through its normal coimage \(N\text{c}m\ f := \text{coker}(\ker f)\) and normal image \(N\text{im}\ f := \ker(\text{coker} f)\) as follows:

\[
\begin{array}{ccc}
\text{Ker} f & \longrightarrow & X \\
\downarrow p & & \downarrow f \\
N\text{c}m f & \longrightarrow & \text{Coker} f \\
\end{array}
\]

\[
\begin{array}{ccc}
\downarrow m & & \downarrow \text{im} f \\
N\text{im} f & \longrightarrow & \end{array}
\]

\[
(6.3)
\]

**Definition 6.13.** A morphism \(f : X \to Y\) is **exact** (in the sense of Grandis) if there exists an isomorphism \(g\) which makes (6.3) commute.\(^{12}\)

Now, we recall the definition of a homological category from [23].

**Definition 6.14.** [23, §1] Let \((C, N)\) be an \(N\)-category.

1. \((C, N)\) is said to be a **semiexact** category if every morphism \(f : X \to Y\) has an \(N\)-kernel and \(N\)-cokernel.
2. \((C, N)\) is said to be **homological** if the following conditions hold:
   1. \((C, N)\) is semiexact.
   2. Normal monics and normal epics are closed under composition.
   3. For any normal monic \(m : M \to X\) and normal epic \(q : X \to Q\) such that \(m \geq \ker q\) (i.e., \(m\) factors through \(\ker q\)), the morphism \(qm\) is exact in the sense of Definition 6.13.

**Example 6.15.** ([23, §1.4.2]) Let \(\text{Set}_2\) be the category of pairs of sets, i.e., an object is a pair \((A, B)\) of a set \(A\) and a subset \(B\) of \(A\). A morphism from \((A, B)\) to \((A', B')\) is a function \(f : A \to A'\) such that \(f(B) \subseteq B'\). A morphism \(f : (A, B) \to (A', B')\) is null if and only if \(f(A) \subseteq B'\). Then \(\text{Set}_2\) becomes a homological category.

Now, we prove that the category of \(\circ\)-idempotent \(A\)-modules is semiexact.

**Theorem 6.16.** Let \((A, T, (-))\) be a triple, and \(\text{Mod}_{A, h}\) be the subcategory of \(\text{Mod}_A\) with:

1. **Objects:** The \(\circ\)-idempotent \(A\)-modules, cf., Definition 2.22.
2. **Morphisms:** homomorphisms of \(A\)-modules.

With \(N\) as in Example 6.11, \(\text{Mod}_{A, h}\) is a semiexact category.

**Proof.** Let \(f : X \to Y\) be a homomorphism of \(A\)-modules. As explained in Example 2.17, we may impose a surpassing relation \(\preceq_\circ\) on \(X\) and \(Y\) as follows:

\[
c \preceq_\circ c' \iff c + b^\circ = c'.
\]

For notational convenience, we will just write \(\preceq_\circ = \preceq\). We will consider all \(A\)-modules in this way.

---

\(^{12}\) Once it exists, \(g\) as in the above diagram is unique. See [23, §1.5.5].
With (6.4), we first claim that the following is an $N$-kernel of $f$:

$$\ker f : \ker_{\text{Mod},X} f \longrightarrow X.$$  \hfill (6.5)

Note that (6.5) makes sense since $\ker f$ is a homomorphism and $\ker_{\text{Mod},X} f$ is a submodule of $X$. To prove our claim, for any $x \in \ker_{\text{Mod},X} f$, we have $f(x) \geq 0$, that is, $f(x) = b(-)b$ for some $b \in Y$. In particular, $f(x) \in Y^\circ$. Therefore, we have the following commutative diagram:

$$
\begin{array}{ccc}
\ker_{\text{Mod},X} f & \xrightarrow{\ker f} & X \\
\downarrow f & & \downarrow f \\
Y^\circ & \xrightarrow{\psi} & Y
\end{array}
$$

\hfill (6.6)

showing that $f(\ker f)$ is null.

Next, suppose that $h : Z \rightarrow X$ is a homomorphism such that $fh$ is null. Since $fh$ is null, there exists an $A$-module $N'$ such that the following commutes:

$$
\begin{array}{ccc}
Z & \xrightarrow{h} & X \\
\varphi & \downarrow & \psi \\
N^\circ & \xrightarrow{\psi} & Y
\end{array}
$$

for some homomorphisms $\varphi$ and $\psi$. Indeed, one can observe, for each $z \in Z$, $f(h(z)) \in Y^\circ$ since $fh = \psi \varphi$ and $\varphi(z) \in N^\circ$. In particular, $h(z) \in \ker_{\text{Mod},X} f$, and hence $h$ uniquely factors through $\ker f$. This proves that $\ker f : \ker_{\text{Mod},X} f \longrightarrow X$ is an $N$-kernel of $f$.

Now, we prove that the $N$-cokernel exists. We use the idea of Connes and Consani in [11] to define the $N$-cokernel. For a given homomorphism $f : X \rightarrow Y$, we define

$$\text{Coker} f := Y/\Phi,$$

where for $y, y' \in Y$,

$$(y, y') \in \Phi \iff g(y) = g(y') \forall Z \forall g \in \text{Hom}(Y, Z) \text{ such that } f(X) \subseteq \ker_{\text{Mod},X} g.$$  \hfill (6.7)

We let $\text{coker} f : Y \rightarrow \text{Coker} f$ be the projection map. This definition makes sense from the following two observations:

1. $\Phi$ is a congruence relation since $g$ is a homomorphism. Furthermore, for any $[y] \in Y/\Phi$, where $[y]$ is the equivalence class of $y \in Y$ in $Y/\Phi$, we have that $([y]^\circ)^\circ = [(y^\circ)^\circ] = [y^\circ] = [y]^\circ$. Hence, $\text{Coker} f = Y/\Phi$ is $\circ$-idempotent.

2. The projection $\text{coker} f : Y \rightarrow Y/\Phi$ is a homomorphism.
Now, we claim that for any \( c \in X \), we have
\[
[f(c)] = [f(c)^\circ].
\]
In fact, suppose that \( g \in \text{Hom}(Y, Z) \) such that \( f(X) \subseteq \ker_{\text{Mod}, X} g \). This implies that \( gf(X) \subseteq Z^\circ \). Hence, \( gf(c) = b^\circ \) for some \( b \in Z^\circ \). It follows that
\[
g(f(c)^\circ) = gf(c)^\circ = (b^\circ)^\circ = b^\circ = gf(c).
\]
In particular, we have \( [f(c)] = [f(c)^\circ] \). This shows the following diagram commutes, proving that \((\text{coker } f) f \in N:\)

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{f^\circ} & & \downarrow{(\text{coker } f)|_{Y^\circ}} \\
Y^\circ & \xrightarrow{\text{coker } f} & Y/\Phi
\end{array}
\quad (6.8)
\]

Next, suppose that \( hf \in N \) for some \( h : Y \to Z \). Then, there exists a module \( \mathcal{M} \) and the following commutative diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{\varphi} & & \downarrow{\psi} \\
\mathcal{M}^\circ & \xrightarrow{h} & Z
\end{array}
\quad (6.9)
\]

Since \( hf \) factors through \( \mathcal{M}^\circ \), we have that \( f(X) \subseteq \ker_{\text{Mod}, X} h \). In particular, from (6.7), the following map is well-defined:
\[
\tilde{\varphi} : \text{Coker } f \longrightarrow Z, \quad [y] \mapsto h(y).
\]
Furthermore, in this case, we have that \( h = \tilde{\varphi}(\text{coker } f) \), showing that \( h \) factors through \( \text{Coker } f \). One can easily see that this is unique.

This completes the proof that \( \text{Mod}_{A,h} \) is semiexact. \( \square \)

**Remark 6.17.** In the proof of Theorem 6.16, one may observe that for the \( N \)-kernel, we do not need the assumption that morphisms are homomorphisms or that our modules are \( \circ \)-idempotent. On the other hand, the \( N \)-cokernel is trickier than the \( N \)-kernel. For instance, as we mentioned before, for a \( \leq \)-morphism \( f : X \to Y \), the image \( f(X) \) does not have to be a submodule of \( Y \) since we only have \( f(x + y) \leq f(x) + f(y) \) which is not enough to ensure that \( f(x) + f(y) \in f(X) \). Towards this end, we have to restrict ourselves exclusively to homomorphisms. Furthermore, \( \circ \)-idempotence of modules is essential in our proof to show that \((\text{coker } f) f \in N \).

**Remark 6.18.** In [11, Theorem 6.12], Connes and Consani proved that the category of \( \mathbb{B} \)-modules endowed with an involution is a homological category. Much of their theory can be restated for modules over triples. We expect that the category of \( A \)-modules in Example 6.11 may be homological. Although we do not pursue this in this paper, one may be able to prove this by closely following the proofs in [11].
Since $\preceq_\circ$ is a special case of surpassing relation, one may ask whether or not the category of systemic modules is semiexact. To some extent, this seems to be doable, for example for convex images.

Let $\mathcal{A}$ be a system, and $\mathcal{C}$ be the category of systemic $\mathcal{A}$-modules. For each $\mathcal{M} \in \mathcal{C}$, recall that we have

$$\mathcal{M}_{\text{Null}} = \{ a \in \mathcal{M} \mid a \succeq 0 \}.$$  

Consider the following set

$$\mathcal{O} = \{ \mathcal{M}_{\text{Null}} \mid \mathcal{M} \in \mathcal{C} \}.$$  

It is clear that $N$ defined as in (6.2) is a closed categorical ideal of $\mathcal{C}$. Hence, $\mathcal{C}$ is an $N$-category.

**Proposition 6.19.** In $\mathcal{C}$, any $\preceq$-morphism $f : X \to Y$ has an $N$-kernel given as follows:

$$\ker f : \ker \text{Mod}_{\mathcal{X}} f \to X.$$  

**Proof.** The same proof as in Theorem 6.16 works.  

**Proposition 6.20.** Let $\mathcal{C}_h$ be the category of systemic modules with homomorphisms, viewed as an $N$-category with the same $N$. Let $f : X \to Y$ be a homomorphism. Then, with $\text{coker } f : \text{Coker } f \to Y/\Phi$ as in Theorem 6.16, if $hf \in N$ for some $h : Y \to Z$, then $h$ uniquely factors through $\text{Coker } f$.

**Proof.** The same proof as in Theorem 6.16 works.

An affirmative answer to the following question could provide a more general context for Connes and Consani’s result:

**Open Problem** Let $(\mathcal{A}, T, (−))$ be a triple, or $(\mathcal{A}, T, (−), \preceq)$ a system. Find general conditions for the category of $\mathcal{A}$-modules to be homological.

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