REMARK ON A DIAMETER BOUND FOR COMPLETE MANIFOLDS WITH POSITIVE BAKRY-ÉMERY RICCI CURVATURE

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Abstract. In this note, we give a new upper diameter estimate for complete Riemannian manifolds in the case that the Bakry-Émery Ricci curvature has a positive lower bound and the norm of the potential function has an upper bound. Our diameter estimate improves previous results obtained by Wei-Wylie (J. Differential Geom. 83, 377-405, 2009) and Limoncu (Math. Z. 271, 715-722, 2012). As an application, we give an upper diameter bound for compact Ricci solitons in terms of the maximum value of the scalar curvature. Using such a diameter bound, we provide a sufficient condition for four-dimensional compact Ricci solitons to satisfy the Hitchin-Thorpe inequality.

1. Introduction

Let \((M, g)\) be a complete Riemannian manifold and \(f : M \to \mathbb{R}\) a smooth function. A \textit{Bakry-Émery Ricci curvature} \([1]\) is defined by \(\text{Ric}_g + \text{Hess} f\), where \(\text{Ric}_g\) stands the Ricci curvature of \((M, g)\) and \(\text{Hess} f\) denotes the Hessian of \(f\). Recently, the Bakry-Émery Ricci curvature has received much attention in various areas of mathematics, since it is a good substitute for the Ricci curvature allowing us to establish many interesting theorems in metric measure spaces, such as, comparison theorems \([12]\), eigenvalue estimates \([5]\), Li-Yau Harnack inequalities \([7]\). In particular, Wei and Wylie \([12]\) proved the following Myers type theorem via Bakry-Émery Ricci curvature.

\textbf{Theorem 1.1 (Wei-Wylie \([12]\))}. Let \((M, g)\) be an \(n\)-dimensional complete connected Riemannian manifold satisfying

\[\text{Ric}_g + \text{Hess} f \geq (n-1)Hg\]

for some \(H > 0\). If \(|f| \leq k\) for some \(k \geq 0\), then

\[\text{diam}(M, g) \leq \frac{\pi}{\sqrt{H}} + \frac{4k}{(n-1)\sqrt{H}}.\] (1.2)

On the other hand, Limoncu \([8]\) gave the following diameter estimate for complete Riemannian manifold under the same assumption as in the previous theorem.

\textbf{Theorem 1.3 (Limoncu \([8]\))}. Let \((M, g)\) be an \(n\)-dimensional complete connected Riemannian manifold satisfying

\[\text{Ric}_g + \text{Hess} f \geq (n-1)Hg\]

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for some $H > 0$. If $|f| \leq k$ for some $k \geq 0$, then
\[
\text{diam}(M,g) \leq \frac{\pi}{\sqrt{H}} \sqrt{1 + \frac{2\sqrt{2}k}{n-1}}.
\] (1.4)

In particular, if $k \geq \frac{(n-1)\pi}{8}(\sqrt{2}\pi - 4)$, then the estimate \(1.4\) is sharper than \(1.2\).

The aim of this note is to improve these two diameter estimates under the same assumption as in the two previous theorems by giving the following.

**Theorem 1.5.** Let $(M, g)$ be an $n$-dimensional complete connected Riemannian manifold satisfying
\[
\text{Ric}_g + \text{Hess} f \geq (n-1)Hg
\] (1.6)
for some $H > 0$. If $|f| \leq k$ for some $k \geq 0$, then
\[
\text{diam}(M, g) \leq \frac{\pi}{\sqrt{H}} \sqrt{1 + \frac{8k}{(n-1)\pi}}.
\] (1.7)

**Remark 1.8.** Since \(\frac{8}{\pi} \approx 2.54647\) and \(2\sqrt{2} \approx 2.82842\), our diameter estimate \(1.7\) is sharper than \(1.4\). Moreover, we can easily see that our estimate \(1.7\) is sharper than \(1.2\) without any assumptions on $k$.

Our Theorem 1.5 has applications to an upper diameter bound and the Hitchin-Thorpe inequality for compact Ricci solitons. A complete Riemannian manifold $(M, g)$ is called a **Ricci soliton** if there exists a vector field $X \in \mathfrak{X}(M)$ satisfying the equation
\[
\text{Ric}_g + \frac{1}{2} \mathcal{L}_X g = \lambda g
\] (1.9)
for some real number $\lambda \in \mathbb{R}$, where $\mathcal{L}_X$ denotes the Lie derivative by $X$. We say that the soliton $(M, g)$ is **shrinking**, **steady** and **expanding** described as $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$, respectively. Note that if $X$ is a Killing vector field, then the soliton is an Einstein manifold. In such a case, we say that the soliton is **trivial**. When $X$ may be replaced with a gradient vector field $\nabla f$ for some smooth function $f : M \to \mathbb{R}$, called a **potential function**, we call $(M, g)$ a **gradient Ricci soliton**. Then \(1.9\) becomes
\[
\text{Ric}_g + \frac{1}{2} \mathcal{L}_X g = \lambda g.
\] (1.10)

Thanks to Perelman [11], any compact Ricci soliton is gradient. It is known [2] that any non-trivial compact Ricci soliton $(M, g)$ is shrinking with $\text{dim} \, M \geq 4$. Moreover, it is also known [2] that the potential function $f$ of any gradient Ricci soliton $(M, g)$ satisfies $R + |\nabla f|^2 - 2\lambda f = C$ for some real constant $C$, where $R$ denotes the scalar curvature on the soliton. By adding some constant on $f$, we may normalize $f$ such that
\[
R + |\nabla f|^2 = 2\lambda f.
\] (1.11)

Fernández-López and García-Río [4] investigated a lower diameter bound for compact shrinking Ricci solitons depending on the scalar and Ricci curvatures. By using Theorem 1.5, we then give an upper diameter bound for compact shrinking Ricci solitons in terms of the maximum value of the scalar curvature.
Corollary 1.12. Let $(M, g)$ be an $n$-dimensional compact connected shrinking Ricci soliton satisfying (1.10). Suppose that the soliton is normalized in sense of (1.11). Then
\[
\text{diam}(M, g) \leq \frac{\pi}{\sqrt{\lambda}} \sqrt{n - 1 + \frac{4R_{\max}}{\pi \lambda}},
\]
(1.13)
where $R_{\max}$ denotes the maximum value of the scalar curvature $R$ on the soliton.

Since the Ricci soliton is a natural generalization of an Einstein manifold, we may expect some topological obstructions to the existence of compact Ricci solitons. The Hitchin-Thorpe inequality for compact shrinking Ricci solitons was proved by Ma [9] assuming some upper bounds on the $L^2$-norm of the scalar curvature, while Fernández-López and García-Río [4] investigated that assuming some upper diameter bounds in terms of the Ricci curvature. Using Corollary 1.12, we then provide the following sufficient condition for four-dimensional compact shrinking Ricci solitons to satisfy the Hitchin-Thorpe inequality.

Corollary 1.14. Let $(M, g)$ be a four-dimensional compact connected shrinking Ricci soliton satisfying (1.10). Suppose that the soliton is normalized in sense of (1.11). If
\[
\sqrt{\frac{R_{\max}}{\lambda^2}} \left(4\pi + \frac{\pi^2}{2}\right) \leq \text{diam}(M, g),
\]
(1.15)
then the soliton satisfies the Hitchin-Thorpe inequality $2\chi(M) \geq 3|\tau(M)|$.

This note is organized as follows: In Section 2, by introducing our notation, we will give a proof of Theorem 1.5. Ending with Section 3, proofs of Corollary 1.12 and 1.14 will be given.

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2. A proof of Theorem 1.5

Before giving a proof of Theorem 1.5, we recall our notation. Let $X, Y, Z \in \mathcal{X}(M)$ be three vector fields on $M$. For any smooth function $f \in C^\infty(M)$, the gradient vector field and Hessian of $f$ are defined by
\[
g(\nabla f, X) = df(X) \quad \text{and} \quad \text{Hess}(X, Y) = g(\nabla_X \nabla f, Y),
\]
respectively. The curvature tensor and Ricci tensor are defined by
\[
R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \quad \text{and} \quad \text{Ric}(X, Y) = \sum_{i=1}^n g(R(e_i, X)Y, e_i),
\]
respectively. Here, $\{e_i\}_{i=1}^n$ is an orthonormal frame of $(M, g)$. In order to prove Theorem 1.5, we will use the index form of a minimizing unit speed geodesic segment. We refer the reader to books [6, 10] for basic facts about this topic.

Proof of Theorem 1.5. Our proof of Theorem 1.5 is similar to that by Limoncu [8]. Take arbitrary two points $p, q \in M$. By the compactness of the manifold $(M, g)$, there exists the minimizing unit speed geodesic segment $\sigma$ from $p$ to $q$ of length $\ell$. Let $\{e_1 = \hat{\sigma}, e_2, \ldots, e_n\}$
be a parallel orthonormal frame along $\sigma$. Recall that, for any smooth function $\phi \in C^\infty([0, \ell])$ satisfying $\phi(0) = \phi(\ell) = 0$, we obtain

$$I(\phi e_i, \phi e_i) = \int_0^\ell \left( g(\dot{\phi} e_i, \dot{\phi} e_i) - g(R(\phi e_i, \dot{\phi}) \dot{\sigma}, \phi e_i) \right) dt, \quad (2.1)$$

where $I(\cdot, \cdot)$ denotes the index form of $\sigma$. From (2.1), we have

$$\sum_{i=2}^n I(\phi e_i, \phi e_i) = \int_0^\ell \left( (n-1)\dot{\phi}^2 - \phi^2 \text{Hess } f(\phi, \dot{\phi}) \right) dt,$$

where, the last equality follows from the parallelism of the metric $g$ and $\nabla_\phi \dot{\sigma} = 0$. On the geodesic segment $\sigma(t)$, we have

$$\phi^2 \dot{\sigma}(g(\nabla f, \dot{\phi})) = \phi^2 \frac{d}{dt}(g(\nabla f, \dot{\phi}))$$

$$= -2\dot{\phi} \phi g(\nabla f, \dot{\phi}) + \frac{d}{dt}(\phi^2 g(\nabla f, \dot{\phi}))$$

$$= 2f \frac{d}{dt}(\phi \dot{\phi}) - 2f \frac{d}{dt}(\phi \dot{\phi}) + \frac{d}{dt}(\phi^2 g(\nabla f, \dot{\phi})), \quad (2.4)$$

where, in the last equality, we have used $g(\nabla f, \dot{\phi}) = \frac{dt}{d\sigma}(\sigma(t))$. Hence, by integrating both sides of (2.4), we have

$$\int_0^\ell \phi^2 \dot{\sigma}(g(\nabla f, \dot{\phi})) dt = \int_0^\ell 2f \frac{d}{dt}(\phi \dot{\phi}) dt - 2 \left[ f \phi \dot{\phi} \right]_0^\ell + \left[ \phi^2 g(\nabla f, \dot{\phi}) \right]_0^\ell$$

$$= 2 \int_0^\ell f \frac{d}{dt}(\phi \dot{\phi}) dt,$$

where, the last equality follows from $\phi(0) = \phi(\ell) = 0$. By (2.5) and the assumption $|f| \leq k$ in Theorem 1.5, we obtain

$$\int_0^\ell \phi^2 \dot{\sigma}(g(\nabla f, \dot{\phi})) dt \leq 2k \int_0^\ell \left| \frac{d}{dt}(\phi \dot{\phi}) \right| dt. \quad (2.6)$$

From (2.3) and (2.6), we have

$$\sum_{i=2}^n I(\phi e_i, \phi e_i) \leq \int_0^\ell (n-1)\dot{\phi}^2 - H \phi^2) dt + 2k \int_0^\ell \left| \frac{d}{dt}(\phi \dot{\phi}) \right| dt. \quad (2.7)$$

If the function $\phi$ is taken to be $\phi(t) = \sin \left( \frac{\pi t}{\ell} \right)$, then we obtain $\dot{\phi}(t) = \frac{\pi}{\ell} \cos \left( \frac{\pi t}{\ell} \right)$ and

$$\dot{\phi} \phi = \frac{\pi}{\ell} \sin \left( \frac{\pi t}{\ell} \right) \cos \left( \frac{\pi t}{\ell} \right) = \frac{\pi}{2\ell} \sin \left( \frac{2\pi t}{\ell} \right).$$
Then, (2.7) becomes
\[
\sum_{i=2}^{n} I(\phi e_i, \phi e_i) \leq (n - 1) \int_0^{\ell} \left( \frac{\pi^2}{\ell^2} \cos^2 \left( \frac{\pi t}{\ell} \right) - H \sin^2 \left( \frac{\pi t}{\ell} \right) \right) dt \\
+ 2k \left( \frac{\pi}{\ell} \right)^2 \int_0^{\ell} \left| \cos \frac{2\pi t}{\ell} \right| dt,
\]
and consequently, we have
\[
\sum_{i=2}^{n} I(\phi e_i, \phi e_i) \leq -\frac{1}{2\ell} ((n - 1)H\ell^2 - (n - 1)^2 - 8k\pi).
\]
Since \( \sigma \) is a minimizing geodesic, we must obtain
\[
(n - 1)H\ell^2 - (n - 1)^2 - 8k\pi \leq 0.
\]
From this inequality, we have
\[
\ell \leq \frac{\pi}{\sqrt{H}} \sqrt{1 + \frac{8k}{(n - 1)\pi}}.
\]
This proves Theorem 1.5. \( \square \)

Remark 2.8. Using Cauchy-Schwarz inequality, Limoncu estimated (2.5) from above by
\[
\int_0^{\ell} \phi^2 \sigma(g(\nabla f, \dot{\sigma})) dt = 2 \int_0^{\ell} f \frac{d}{dt}(\phi \dot{\sigma}) dt \leq 2 \sqrt{\int_0^{\ell} f^2 dt} \sqrt{\int_0^{\ell} \left( \frac{d}{dt}(\phi \dot{\sigma}) \right)^2 dt},
\]
while we estimated (2.5) from above by an absolute value in (2.6) and obtained a better estimate (1.7) than (1.4).

3. Applications to Theorem 1.5

In this section, by using Theorem 1.5, we give proofs of Corollary 1.12 and 1.14. Throughout this section, we assume that \((M, g)\) is a compact connected normalized shrinking Ricci soliton satisfying (1.10) and (1.11).

3.1. A proof of Corollary 1.12. The following lemma is useful to prove Corollary 1.12.

**Lemma 3.1.** The potential function \( f \) on the soliton \((M, g)\) satisfies
\[
0 \leq 2\lambda f \leq R_{\text{max}},
\]
where \( R_{\text{max}} \) denotes the maximum value of the scalar curvature \( R \) on the soliton.

**Proof.** Thanks to Chen [3], the scalar curvature of any complete shrinking Ricci soliton is non-negative. Hence, by (1.11), we have \( 2\lambda f \geq 0 \). On the other hand, by compactness of the manifold \( M \), there exists some global maximum point \( p \in M \) of the potential function. Then, it follows from (1.11) that for any point \( x \in M \),
\[
2\lambda f(p) = R(p) \geq 2\lambda f(x) = R(x) + |\nabla f|^2(x),
\]
and hence, \( R(p) \geq R(x) \). Therefore, the scalar curvature also attains its maximum at \( p \), and we obtain the result. \( \square \)

Corollary 1.12 follows immediately from Theorem 1.5 and Lemma 3.1.
3.2. A proof of Corollary 1.14. We use the following theorem to prove Corollary 1.14.

**Theorem 3.2 (Ma [9]).** Let \((M, g)\) be a four-dimensional compact shrinking Ricci soliton satisfying (1.10). If the scalar curvature satisfies
\[
\int_M R^2 \leq 24\lambda^2 \text{vol}(M, g),
\]
then the soliton \((M, g)\) satisfies the Hitchin-Thorpe inequality \(2\chi(M) \geq 3|\tau(M)|\).

**Proof of Corollary 1.14.** By taking the trace of (1.10), we have
\[
R + \Delta f = 4\lambda.
\]
Thanks to Theorem 1.5, the diameter of \((M, g)\) has the upper bound
\[
\text{diam}(M, g) \leq \frac{\pi}{\sqrt{\lambda}} \sqrt{3 + \frac{4R_{\text{max}}}{\pi\lambda}}.
\]
Suppose that the inequality (1.15) holds. Then, from (3.4), we obtain
\[
\left(4\pi + \frac{\pi^2}{2}\right) \frac{R_{\text{max}}}{\lambda^2} \leq \text{diam}^2(M, g) \leq \frac{\pi^2}{\lambda} \left(3 + \frac{4R_{\text{max}}}{\pi\lambda}\right),
\]
from where we have \(R_{\text{max}} \leq 6\lambda\). Hence, by (3.3), we have
\[
\int_M R^2 \leq R_{\text{max}} \int_M R = 24\lambda^2 \text{vol}(M, g),
\]
and the result follows from Theorem 3.2. \(\Box\)

By using the same way as in the previous proof, we can easily show the following.

**Corollary 3.5.** Let \((M, g)\) be a four-dimensional compact connected shrinking Ricci soliton satisfying (1.10). Suppose that the soliton is normalized in sense of (1.11). If
\[
\frac{R_{\text{max}}}{6\lambda} \cdot \frac{\pi}{\sqrt{\lambda}} \sqrt{3 + \frac{4R_{\text{max}}}{\pi\lambda}} \leq \text{diam}(M, g),
\]
then the soliton satisfies the Hitchin-Thorpe inequality \(2\chi(M) \geq 3|\tau(M)|\).

**Remark 3.7.** The condition (3.6) may be a better estimate than (1.15) when the maximum value of the scalar curvature \(R_{\text{max}}\) is sufficiently small.

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