MEAN CURVATURE FLOW OF KILLING GRAPHS

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Abstract. We study a Neumann problem related to the evolution of graphs under mean curvature flow in Riemannian manifolds endowed with a Killing vector field. We prove that in a particular case these graphs converge to a trivial minimal graph which contacts the cylinder over the domain orthogonally along its boundary.

1. Introduction

Let \( M \) be an \((n + 1)\)-dimensional Riemannian manifold endowed with a Killing vector field \( Y \). Suppose that the distribution orthogonal to \( Y \) is of constant rank and integrable. Given an integral leaf \( P \) of that distribution, let \( \Omega \subset P \) be a bounded domain with regular boundary \( \Gamma = \partial \Omega \). Let \( \vartheta : \mathbb{I} \times \Omega \rightarrow M \) be the flow generated by \( Y \) with initial values in \( M \), where \( \mathbb{I} \) is a maximal interval of definition. In geometric terms, the ambient manifold is a warped product \( M = P \times 1/\sqrt{\gamma} \mathbb{I} \) where \( \gamma = 1/|Y|^2 \).

Given \( T \in [0, +\infty) \), let \( u : \bar{\Omega} \times [0, T) \rightarrow \mathbb{I} \) be a smooth function. Fixing this notation, the Killing graph of \( u(\cdot, t), t \in [0, T) \), is the hypersurface \( \Sigma_t \subset M \) parametrized by the map
\[
X(t, x) = \vartheta(u(x, t), x), \quad x \in \bar{\Omega}.
\]
Notice that this definition could be slightly more general if we suppose that the coordinates of \( x \in \bar{\Omega} \) change with the parameter \( t \in [0, T) \). To abolish this possibility is equivalent to ruling out tangential diffeomorphisms of \( \Omega \).

The Killing cylinder \( K \) over \( \Gamma \) is in turn defined by
\[
(1.1) \quad K = \{ \vartheta(s, x) : s \in \mathbb{I}, x \in \Gamma \}.
\]

Let \( N \) be a unit normal vector field along \( \Sigma_t \). In what follows, we denote by \( H \) the mean curvature of \( \Sigma_t \) with respect to the orientation given by \( N \). We are then concerned with establishing conditions for longtime existence of a prescribed mean curvature flow of the form
\[
(1.2) \quad \frac{\partial X}{\partial t} = (nH - H)N,
\]
\[
(1.3) \quad X(0, \cdot) = \vartheta(u_0(\cdot), \cdot),
\]
for given functions \( u_0 : \bar{\Omega} \rightarrow \mathbb{R} \) and \( H : \bar{\Omega} \rightarrow \mathbb{R} \). In order to define boundary conditions for the evolution problem \((1.2)\) we consider a function \( \phi \in C^\infty(\Gamma) \) such that \( |\phi| \leq \phi_0 < 1 \) for some positive constant \( \phi_0 \). Let \( \nu \) be the inward unit normal
vector field along $K$. We impose the following Neumann condition associated to (1.2):

$$(1.4) \quad \langle N, \nu \rangle |_{\partial \Sigma_t} = \phi,$$

where $\langle \cdot, \cdot \rangle$ denotes the Riemannian metric in $M$.

The main result in this paper may be stated as follows.

**Theorem 1.** There exists a unique solution $u : \bar{\Omega} \times [0, \infty) \to \mathbb{I}$ to the problem (1.2)-(1.4). Moreover, if $\phi = 0$ and $\mathcal{H} = 0$, the functions $u(\cdot, t)$ converge to a constant function as $t \to \infty$.

Theorem 1 extends Theorem 1.1 in [3] as well as Theorem 2.4 in [2] and Theorem 2.4 in [1] in a twofold way. The corresponding theorems in [3] and [2] concern evolution of graphs in Euclidean space whereas [1] deals with the case of graphs in Riemannian product spaces of the form $P \times \mathbb{R}$. Moreover those earlier results hold only for the case when the prescribed mean curvature is $\mathcal{H} = 0$. An existence result for evolution of graphs in Euclidean space by the Gauss-Kronecker curvature under Neumann boundary conditions is proved in [8]. We also mention that the Dirichlet problem for the evolution of graphs in warped spaces is extensively studied in [7].

The paper is organized as follows. Section 2 describes the evolution problem in nonparametric terms. Height and boundary gradient a priori estimates for (1.2)-(1.4) are presented respectively in Sections 3 and 4. Interior gradient estimates are obtained in Section 5. Some technical computations needed in the body of the proofs are compiled in an appendix. In Section 6 we prove the convergence of the mean curvature flow (1.2) for $\phi = 0$ and $\mathcal{H} = 0$ to slices of the form $\vartheta(\{s\} \times \Omega)$ for some $s \in \mathbb{I}$. This generalizes the corresponding convergence statement of Theorem 1.1 in [3].

## 2. Fundamental Equations

Since we will consider the mean curvature flow in nonparametric terms it seems adequate to describe all geometric invariants as well as their evolution equations in terms of graphical coordinates.

Let $x^1, \ldots, x^n$ be local coordinates in $P$. This system is augmented to be a coordinate system in $M$ by setting $x^0 = s$, the flow parameter of $Y$. The tangent space of $\Sigma_t$ at a point $X(t, x), x \in \Omega$, is spanned by the coordinate vector fields

$$(2.1) \quad X_s \frac{\partial}{\partial x^i} = \partial_s \frac{\partial}{\partial x^i} + u_i \partial_s \frac{\partial}{\partial x^0} = \frac{\partial}{\partial x^i} |_X + u_i \frac{\partial}{\partial x^0} |_X.$$

In terms of these coordinates the induced metric in $\Sigma_t$ is expressed in local components by

$$(2.2) \quad g_{ij} = \sigma_{ij} + \frac{1}{\gamma} u_i u_j,$$

where $\gamma = \frac{1}{|Y|^2}$ and $\sigma_{ij}$ are the local components of the metric in $P$.

In order to compute the mean curvature of $\Sigma_t$, we fix $N$ as the vector field

$$(2.3) \quad N = \frac{1}{W}(\gamma Y - \partial_s \nabla u),$$

where $\nabla u$ is the gradient of $u$ in $P$ and

$$(2.4) \quad W = \sqrt{\gamma + |\nabla u|^2}.$$
The second fundamental form of $\Sigma_t$ calculated with respect to this choice of normal vector field has local components

\begin{equation}
(2.5) \quad a_{ij} = \langle \nabla_{X_i} \frac{\partial}{\partial x^i}, X_j \frac{\partial}{\partial x^j}, N \rangle,
\end{equation}

where $\nabla$ denotes the covariant derivative in $M$. We then compute

\begin{equation}
a_{ij} = \langle \nabla_{X_i} \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, N \rangle + \langle \nabla_{X_i} \frac{\partial}{\partial x^i}, u_j \frac{\partial}{\partial x^j}, N \rangle
= \langle \nabla_{\partial, \frac{\partial}{\partial x^i}}, \frac{\partial}{\partial x^j}, N \rangle + u_i \langle \nabla_{\partial, \frac{\partial}{\partial x^i}}, \frac{\partial}{\partial x^j}, N \rangle + u_j \langle \nabla_{\partial, \frac{\partial}{\partial x^i}}, \frac{\partial}{\partial x^j}, N \rangle
\end{equation}

\begin{equation}
+ u_{i,j} \langle \theta^s \frac{\partial}{\partial x^0}, N \rangle + u_i u_j \langle \nabla_{Y,Y}, \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, N \rangle.
\end{equation}

Hence, using the fact that the maps $x \mapsto \theta(s,x)$ are isometries and that the hypersurfaces defined by $\{\theta(s,x) : x \in P\}$, $s \in \mathbb{I}$, are totally geodesic, one concludes that

\begin{equation}
a_{ij} = \langle \nabla_{\partial, \frac{\partial}{\partial x^i}}, \frac{\partial}{\partial x^j}, \frac{1}{W} \nabla u \rangle + u_i \langle \nabla_{\partial, \frac{\partial}{\partial x^i}}, \frac{1}{W} \gamma Y, \frac{1}{W} \gamma Y \rangle + u_j \langle \nabla_{\partial, \frac{\partial}{\partial x^j}}, \frac{1}{W} \gamma Y, \frac{1}{W} \gamma Y \rangle
\end{equation}

\begin{equation}
+ u_{i,j} \langle \nabla_{Y,Y}, \frac{1}{W} \gamma Y, \frac{1}{W} \nabla u \rangle.
\end{equation}

It follows from Killing’s equation that

\begin{equation}
a_{ij} = \frac{u_{ij}}{W} - \frac{u_i \gamma_j}{W 2\gamma} - \frac{u_j \gamma_i}{W 2\gamma} - \frac{u_i u_j}{2W} \frac{1}{\gamma^2}.
\end{equation}

It turns out that $a_{ij}$ could also be expressed by

\begin{equation}
a_{ij} = \frac{u_{ij}}{W} - \frac{u_i}{W} \gamma \langle \nabla_{Y,Y}, \frac{\partial}{\partial x^j} \rangle - \frac{u_j}{W} \gamma \langle \nabla_{Y,Y}, \frac{\partial}{\partial x^j} \rangle - \frac{u_i u_j}{W} \langle \nabla_{Y,Y}, \nabla u \rangle.
\end{equation}

Taking traces with respect to the induced metric, one obtains the following expression for the mean curvature $H$ of the hypersurface $\Sigma_t$:

\begin{equation}
(2.8) \quad nH = \left( \sigma^{ij} - \frac{u^i u^j}{W} \right) \frac{u_{ij}}{W} - \frac{2\gamma + |\nabla u|^2}{W^3} \frac{\langle \nabla \gamma, \nabla u \rangle}{2\gamma^2}.
\end{equation}

Alternatively one has

\begin{equation}
(2.9) \quad nH = \left( \sigma^{ij} - \frac{u^i u^j}{W} \right) \frac{u_{ij}}{W} - \frac{2\gamma + |\nabla u|^2}{W^3} \gamma \langle \nabla_{Y,Y}, \nabla u \rangle.
\end{equation}

At this point we recall that

\begin{equation}
(2.10) \quad \nabla_{\frac{\partial}{\partial x^i}} Y = - \frac{1}{2 \gamma} \gamma Y
\end{equation}

and

\begin{equation}
(2.11) \quad \nabla_{Y,Y} = \frac{1}{2 \gamma^2} \gamma,
\end{equation}

which implies that

\begin{equation}
(2.12) \quad \langle \nabla_{Y,Y}, \nabla u \rangle = - \langle \nabla \nabla_{Y} Y, Y \rangle = \frac{1}{2\gamma^2} \langle \nabla \gamma, \nabla u \rangle.
\end{equation}

Using this one easily verifies that (2.8) may be written in divergence form as

\begin{equation}
(2.13) \quad \text{div} \left( \frac{\nabla u}{W} - \frac{1}{2\gamma W} \frac{\nabla \gamma}{\nabla u} \right) = nH.
\end{equation}
In fact we have
\[
\left( \frac{u^i}{W} \right)_i = \frac{1}{W} u^i - \frac{1}{W^3} u^i u^j u_{ij} - \frac{1}{2W^3} u^i \gamma_i.
\]

It is worth pointing out that (2.13) is equivalent to
\[
\text{div} \left( \frac{\gamma}{W} \nabla Y, \nabla u \right) = n H.
\]

We conclude that (1.2) may be written nonparametrically as
\[
\frac{\partial u}{\partial t} = W \text{div} \left( \frac{\nabla u}{W} - W H - \frac{\gamma}{W} \langle \bar{\nabla} Y, \nabla u \rangle \right).
\]

Indeed it holds that
\[
\frac{1}{W} u^i W_i = \frac{1}{W} (\sigma_{ij} - u^i u^j) u_{ij} - \frac{1}{2} \gamma + \frac{1}{2W^2} \langle \nabla \gamma, \nabla u \rangle - W H.
\]

We conclude that the Neumann problem (1.2)-(1.4) has the nonparametric form
\[
\frac{\partial u}{\partial t} = \left( \sigma_{ij} - u^i u^j \right) u_{ij} - \frac{1}{2} \gamma u_i - \frac{1}{2W^2} \langle \nabla \gamma, \nabla u \rangle - W H \quad \text{in } \Omega \times [0,T),
\]
\[
u_t = \nu(0, \cdot) \quad \text{in } \Omega \times \{0\},
\]
with boundary condition
\[
\langle N, \nu \rangle = \phi \quad \text{on } \partial \Omega \times [0,T).
\]

This boundary value problem describes the evolution of the Killing graph of the function \(u(\cdot, t)\) by its mean curvature in the direction of the unit normal \(N\) with prescribed contact angle at the boundary.

The standard theory for quasilinear parabolic equations [5] guarantees that the problem of solving (1.2)-(1.4) is reduced to obtaining a priori height and gradient estimates for solutions to (2.17)-(2.19).

3. Height estimates

From now on, we consider the parabolic linear operator given by
\[
\mathcal{L} v = g^{ij} v_{ij} - \left( \frac{1}{2} + \frac{1}{2W^2} \right) \gamma^i v_i - H \frac{u^i}{W} v_i - v_t,
\]
where \(v \in C^\infty(\Omega \times [0,T))\).

**Proposition 1.** For a solution \(u \in C^\infty(\Omega \times [0,T^*))\), \(T^* < T\), of (2.17)-(2.19), it holds that
\[
\max_{\Omega \times [0,T^*]} |u_t| = \max_{\Omega} |u_t(0, \cdot)|.
\]

Then it follows that
\[
\max_{\Omega \times [0,T^*]} |u| \leq CT^*
\]
for a given constant \(C > 0\) which depends on \(T^*\).
Proof. First of all we verify that $u_t$ is a solution for a linear parabolic equation. Indeed one has
\begin{align*}
\mathcal{L}u_t &= g^{ij}u_{tt;i} - \left(\frac{1}{2\gamma} + \frac{1}{2W^2}\right)\langle\nabla\gamma, \nabla u_t\rangle - u_{tt} \\
&= (g^{ij}u_{t;i})_t - g^{ij}u_{t;i} - \left(\frac{1}{2\gamma} + \frac{1}{2W^2}\right)\langle\nabla\gamma, \nabla u_t\rangle - u_{tt} \\
&= -g^{ij}u_{t;i} + \left(\frac{1}{2\gamma} + \frac{1}{2W^2}\right)(\nabla\gamma_t, \nabla u_t) + \left(\frac{1}{2\gamma} + \frac{1}{2W^2}\right)(\nabla\gamma_t, \nabla u_t) + W_t\mathcal{H}.
\end{align*}

However since $\gamma = \gamma(x)$ in (2.16) and $x$ is independent of $t$ it follows that
\begin{equation}
\left(\frac{1}{2\gamma} + \frac{1}{2W^2}\right)_t = \left(\frac{1}{2\gamma}\right)_t = -\frac{1}{W^4}\left(\gamma_t + 2u^ku_{k,t}\right) = -\frac{1}{W^4}u^ku_{t,k}.
\end{equation}

In the same way we have
\begin{equation}
W_t = \frac{1}{2W}\left(\gamma_t + 2u^ku_{k,t}\right) = \frac{1}{W}u^ku_{t,k}.
\end{equation}

We conclude that
\begin{align*}
\mathcal{L}u_t &= -g^{ij}u_{t;i} - \frac{1}{W^4}\langle\nabla\gamma, \nabla u_t\rangle u^k(u_t)_k + \frac{1}{W}\mathcal{H}u^k(u_t)_k.
\end{align*}

Now using the fact that $\sigma^{ij}_{:t} = 0$ and $\gamma_t = 0$ we have
\begin{align*}
\mathcal{L}u_t &= \frac{2}{W}\left(\frac{u^iu^j}{W} - \frac{u^i}{W}\frac{u^j}{W}W_t\right)u_{t;i} - \frac{1}{W^4}\langle\nabla\gamma, \nabla u_t\rangle u^k(u_t)_k + \frac{1}{W}\mathcal{H}u^k(u_t)_k \\
&= \frac{2}{W}\left((W_i - \frac{\gamma_i}{2W})u^i_t - (W_i - \frac{\gamma_i}{2W})\frac{u^i}{W}\frac{u^k}{W}u_{t;k}\right) \\
&\quad - \frac{1}{W^4}\langle\nabla\gamma, \nabla u_t\rangle u^k(u_t)_k + \frac{1}{W}\mathcal{H}u^k(u_t)_k \\
&= \frac{2}{W}(W_i - \frac{\gamma_i}{2W})(\sigma^{ik} - \frac{u^i}{W}\frac{u^k}{W})u_{t;k} - \frac{1}{W^4}\langle\nabla\gamma, \nabla u_t\rangle u^k(u_t)_k + \frac{1}{W}\mathcal{H}u^k(u_t)_k.
\end{align*}

Hence it follows that
\begin{equation}
\mathcal{L}u_t - \frac{2}{W}g^{ik}(W_i - \frac{\gamma_i}{2W})(u_t)_k + \frac{1}{W^4}\langle\nabla\gamma, \nabla u_t\rangle u^k(u_t)_k - \frac{1}{W}\mathcal{H}u^k(u_t)_k = 0.
\end{equation}

Thus fix $T^* \in [0, T)$ and let $(x_0, t_0)$ be a point in $\bar{\Omega} \times [0, T^*)$ such that
\begin{equation}
\max_{\Omega \times [0, T^*)} |u_t| = u_t(x_0, t_0) = \max_{\Omega \times [0, T^*)} |u_t|.
\end{equation}

Hence we choose a coordinate system adapted to the boundary $\Gamma$ in such a way that $\frac{\partial}{\partial x^i} = \nu$ at $x_0$. Then, at the point $(x_0, t_0)$ we have
\begin{equation}
u_{i;t} = u_{t;i} = 0
\end{equation}

for $1 \leq i < n$ which implies that
\begin{equation}
W_t = \frac{1}{W}u^nu_{n;t} = -\phi(x_0)u_{n;t},
\end{equation}

where we used (2.19) and (3.2). On the other hand, (2.19) implies that
\begin{equation}
\max_{\Omega \times [0, T^*)} |\phi W_t| = \max_{\Omega \times [0, T^*)} |\phi W_t|.
\end{equation}

Thus we conclude that
\begin{equation}
(1 - \phi^2(x_0))u_{n;t} = 0.
\end{equation}
However since \(|\phi| < 1\), it follows that \(u_{t:n} = 0\) which contradicts the parabolic Hopf Lemma \([5]\).

From this contradiction we conclude that \(t_0 = 0\). Since \(T^*\) is arbitrary, the conclusion follows. \(\square\)

4. Boundary gradient estimates

Now we will prove a gradient bound for a solution of (2.17)-(2.19) by applying a modification of Korevaar’s technique \([4]\) which formerly appeared in \([2]\).

From now on, we consider a nonnegative extension \(d : \bar{\Omega} \to \mathbb{R}\) of the distance function \(\text{dist}_P(\cdot, \Gamma)\) satisfying \(|\nabla d| \leq 1\) in \(\Omega\). In the same way, we consider a \(C^\infty\) extension of the boundary data \(\phi\) to the domain \(\tilde{\Omega}\) which we also denote by \(\phi\). Then we define

\[
\eta = e^{ Ku } h,
\]

where

\[
h = 1 + \alpha d - \phi \langle \nabla d, N \rangle,
\]

where \(K\) and \(\alpha\) are positive numbers to be fixed later.

**Proposition 2.** For \(\alpha > 0\) sufficiently large independent of \(K\) and \(t\), if for some \(t \geq 0\) fixed, \(\eta W(\cdot, t)\) attains a local maximum value at a point \(x_0 \in \partial \Omega\), then \(W(x_0, t) \leq K\).

**Proof.** Let \(t \geq 0\) be such that

\[
\max_{\Omega} \eta W(t, \cdot) = \eta W(t, x_0)
\]

for a point \(x_0 \in \Gamma\). Hence we choose a coordinate system adapted to \(\Gamma\) such that \(\partial_{x:n} = \nu\) at \(x_0\) and

\[
u_1(x_0) \geq 0 \quad \text{and} \quad u_i(x_0) = 0, \quad \text{for} \quad 2 \leq i \leq n - 1.
\]

We have at \(x_0\)

\[
0 = (\eta W)_1 = \eta_1 W + \eta W_1 = e^{ Ku } \left(W Ku_1 (1 - \phi^2) - 2W \phi \phi_1 + W_1 (1 - \phi^2)\right)
\]

from which follows that

\[
W_1 = -Ku_1 W + \frac{ 2\phi \phi_1 }{ (1 - \phi^2) W }.
\]

On the other hand, at \(x_0\) we have

\[
\eta_n = e^{ Ku } \left(Ku_n (1 - \phi^2) + \alpha - \phi \phi_n - \phi \langle \nabla \text{dist}_d N, \nabla d \rangle + \langle N, \nabla d \rangle \right)
\]

\[
= e^{ Ku } \left(Ku_n (1 - \phi^2) + \alpha - \phi \phi_n - \phi \left(\frac{1}{W} \gamma Y - \nabla u, \partial_n\right)\right)
\]

\[
+ \left\langle \frac{1}{W} \nabla \partial_n (\gamma Y - \nabla u), \partial_n\right\rangle
\]

\[
= e^{ Ku } \left(Ku_n (1 - \phi^2) + \alpha - \phi \phi_n - \frac{1}{W^2} \phi u_n W_n + \frac{1}{W} \phi u_{n:n}\right).
\]

Since \((\eta W)_n \leq 0\) at \(x_0\) it holds that

\[
0 \geq WKu_n (1 - \phi^2) + \alpha W - W \phi \phi_n = \frac{1}{W} \phi u_n W_n + \phi u_{n:n} + (1 - \phi^2) W_n
\]

\[
= WKu_n (1 - \phi^2) + \alpha W + W_n + \phi u_{n:n} + u_n \phi_n
\]

\[
= WKu_n (1 - \phi^2) + \alpha W + W_n + \phi u_{n:n} - W \phi \phi_n.
\]
On the other hand,

\[(4.6) \quad W_n = \frac{\gamma_n}{2W} + \frac{1}{W} (u_1 u_{1;n} + u_n u_{n;n}) = \frac{\gamma_n}{2W} - \frac{1}{W} \phi u_1 W_1 - \phi_1 u_1 - \phi u_{n;n},\]

which implies that

\[
W_n = \frac{\gamma_n}{2W} - \frac{1}{W} \phi u_1 \left( \frac{2\phi \phi_1 W}{1 - \phi^2} - Ku_1 W \right) - \phi_1 u_1 - \phi u_{n;n} \\
= \frac{\gamma_n}{2W} - \frac{1 + \phi^2}{1 - \phi^2} u_1 \phi_1 + K \phi u_1^2 - \phi u_{n;n}.
\]

Therefore since

\[
u_1^2 = |\nabla u|^2 - u_n^2 = W^2 - \gamma - \phi^2 W^2 = W^2 (1 - \phi^2) - \gamma,
\]

we conclude that

\[
0 \geq \alpha + \frac{\gamma_n}{2W^2} - \frac{1 + \phi^2}{1 - \phi^2} u_1 \phi + \frac{K \phi u_1^2}{W} - \phi \phi_n + Ku_n (1 - \phi^2) \\
= \alpha + \frac{\gamma_n}{2W^2} + \frac{1 + \phi^2}{1 - \phi^2} N_1 \phi_1 + K \phi \left( W (1 - \phi^2) - \frac{\gamma}{W} \right) - \phi \phi_n - K \phi W (1 - \phi^2) \\
= \alpha + \frac{\gamma_n}{2W^2} + \frac{1 + \phi^2}{1 - \phi^2} N_1 \phi_1 - \frac{K \phi \gamma}{W} - \phi \phi_n \\
\geq \alpha + C - \frac{K \gamma}{W}
\]

for a given constant \(C\) depending solely on \(\gamma\) and \(\phi\). It follows that \(W(x_0, t) \leq K\) if \(\alpha\) is chosen large enough and independent of \(K\) and \(t\).

\[\square\]

5. Interior gradient estimates

In this section we deduce a global gradient bound using the techniques in [1] and [2]. However the more general context of warped product gives rise to a long list of additional terms which require careful tracking along the calculations.

In the sequel, we consider the parabolic linear operator given by

\[(5.1) \quad L v = g^{ij} v_{i;j} - \left( \frac{1}{2\gamma} + \frac{1}{2W^2} \right) \gamma^i v_i - v_t,
\]

where \(v \in C^\infty(\Omega \times [0, T])\).

**Proposition 3.** For fixed \(T^* < T\) there exists \(K > 0\) sufficiently large so that if

\[\eta W(x_0, t_0) = \max_{\Omega \times [0, T^*]} \eta W\]

for some \((x_0, t_0) \in \Omega \times [0, T^*]\), then \(W(x_0, t_0) \leq C\) for some constant \(C\).

**Proof.** We can assume \(x_0 \in \Omega\) and \(t_0 > 0\). At a point \((x_0, t_0)\) where \(\eta W\) attains maximum value we have

\[\eta W + \eta W_i = 0\]

and

\[\frac{1}{\eta} L \eta + \frac{1}{W} \left( LW - \frac{2}{W} g^{ij} W_i W_j \right) \leq 0.\]
We conclude that
\[
\frac{1}{\eta} L \eta = KL \eta + \frac{1}{\hbar} L h + K^2 g_{ij} u_i u_j + 2 K g^{ij} u_i \frac{h_j}{h} \\
= K \mathcal{H}(W + \frac{1}{\hbar} L h + K^2 \frac{\gamma |\nabla u|^2}{W^2} + 2 K g^{ij} u_i \frac{h_j}{h}).
\]

Now we have
\[
g^{ij} u_i h_j = \frac{\gamma}{W^2} u^j h_j = - \frac{\gamma}{W} (\alpha \langle N, \nabla d \rangle - \langle N, \nabla \phi \rangle \theta - \phi(N, \nabla \theta)).
\]

However
\[
\langle N, \nabla \theta \rangle = W \langle A Y^T, \nabla^{\Sigma} d \rangle + \langle \nabla \Sigma \nabla d, \nabla u \frac{W}{W} \rangle - \kappa |\nabla u|^2 W^2.
\]

Therefore
\[
g^{ij} u_i h_j = - \alpha \frac{\gamma}{W} \langle N, \nabla d \rangle + \frac{\gamma}{W} \langle N, \nabla \phi \rangle \theta + \gamma \phi \langle A Y^T, \nabla^{\Sigma} d \rangle \\
+ \frac{\gamma}{W} \phi \langle \nabla \Sigma \nabla d, \nabla u \frac{W}{W} \rangle - \gamma \phi \kappa \frac{|\nabla u|^2}{W^3}.
\]

Thus the expression for \( L h \) in the Appendix allows us to conclude that
\[
\frac{1}{\eta} L \eta = K \mathcal{H}(W + K^2 \frac{\gamma |\nabla u|^2}{W^2}) \\
+ \frac{2 K}{h} ( - \alpha \frac{\gamma}{W} \langle N, \nabla d \rangle + \frac{\gamma}{W} \langle N, \nabla \phi \rangle \theta + \gamma \phi \langle A Y^T, \nabla^{\Sigma} d \rangle + \frac{\gamma}{W} \phi \langle \nabla \Sigma \nabla d, \nabla u \frac{W}{W} \rangle - \gamma \phi \kappa \frac{|\nabla u|^2}{W^3}) \\
- \frac{1}{h} |A|^2 \phi \theta + \frac{1}{h} \phi H W \langle A Y^T, \nabla^{\Sigma} d \rangle + \frac{1}{h} (\kappa \gamma - \frac{1}{2} \langle \nabla d, \nabla \gamma \rangle) \phi \langle A Y^T, Y^T \rangle \\
+ \frac{1}{2} \langle A \nabla^{\Sigma} d, \nabla^{\Sigma} \phi \rangle + \frac{1}{h} \langle A, \nabla^{\Sigma} d \rangle \Sigma \phi - \frac{1}{h W^2} \phi \langle A \nabla^{\Sigma} d, X, \nabla \gamma \rangle \\
+ \frac{1}{h} (\phi - 3 \dot{\gamma}) (\langle N, \nabla \phi \rangle \theta - \alpha \theta - \frac{1}{2W^2} \langle \nabla \gamma, \nabla d \rangle \phi + \langle \nabla \Sigma \nabla d, \nabla u \frac{W}{W} \rangle) \phi - \frac{\kappa}{h} H \phi \\
- \frac{\kappa}{h} H d + \frac{1}{h} (2 \langle N, \nabla \phi \rangle - \alpha) \langle \nabla \Sigma \nabla d, \nabla u \frac{W}{W} \rangle + \frac{1}{h} \langle \nabla \Sigma \nabla d, \nabla \phi \rangle \\
- \frac{1}{\gamma} \phi \langle \nabla \Sigma \nabla d, \nabla \gamma \frac{W}{2} \rangle + \frac{1}{h} \phi \langle \nabla^{\Sigma} \Sigma, \nabla d \rangle + \frac{1}{h} \langle \nabla H d, N \rangle \phi - \frac{1}{h} \phi \gamma^3 d \left( \frac{\nabla u W}{W^2}, \frac{\nabla u W}{W^2}, \frac{\nabla u W}{W^2} \right) \\
+ \frac{1}{h} \text{Ric}(\nabla d, \nabla u W) \phi + \frac{\gamma}{h W^2} \langle N, \nabla \kappa \rangle \phi - \frac{1}{h} \left( \frac{1}{2 \gamma} + \frac{1}{2 W^2} \right) \alpha \langle \nabla d, \nabla \gamma \rangle \\
+ \frac{1}{h} \left( \frac{1}{2 \gamma} + \frac{1}{2 W^2} \right) \langle \nabla \phi, \nabla \gamma \rangle \theta \\
- \frac{\kappa}{h} \phi \langle N, \nabla \gamma \rangle + \frac{2}{h} \frac{\gamma}{W^2} \langle \nabla \phi, N \rangle - \frac{1}{h} \left( \Delta \phi - \langle \nabla \Sigma \nabla \phi, \nabla u \frac{W}{W} \rangle \right) \phi \theta.
\]
On the other hand Lemma 3 yields

\[
\frac{1}{W} \left( LW - \frac{2}{W} g^{ij} W_i W_j \right) = |A|^2 + nHW^2 \langle AY^T, Y^T \rangle - nHW^2 \langle \nabla \gamma, N \rangle
\]

\[
-3 \frac{1}{W} \gamma \langle AY^T, X_s \nabla \gamma \rangle + g^{ij} \frac{\gamma_{ij}}{2} - \frac{3}{4} \frac{\langle AY \rangle^2}{4\gamma^2} - \frac{1}{4} \frac{\langle \nabla \gamma, N \rangle^2}{4\gamma^2} + \gamma \langle \nabla N, \nabla \gamma \rangle
\]

\[
- \langle \nabla \Sigma \Phi, N \rangle - \frac{1}{4\gamma} \frac{\langle \nabla \gamma \rangle^2}{W^2} - \frac{W}{W_t}.
\]

Now we use the fact that \( x_0 \) is a critical point to \( \eta W \). We have

\[
e^{Ku} (Ku h_i + h_i) W = -e^{Ku} h W_i,
\]

which implies that

\[
-KW^2 h N_i N^i + W h_i N^i = -h W_i N_i
\]

and then

\[
-K h |\nabla u|^2 + W h_i N^i = -h W_i N_i.
\]

However

\[
W_i N^i = \frac{\gamma_i}{2\gamma} N^i W + N_i N^i W^3 \langle \frac{\nabla \gamma}{2\gamma^2}, N \rangle + W^2 \langle AY^T, N_i X_s \frac{\partial}{\partial x_i} \rangle
\]

\[
= \frac{1}{2\gamma} \langle \nabla \gamma, N \rangle W + |\nabla u|^2 W \langle \frac{\nabla \gamma}{2\gamma^2}, N \rangle - W^3 \langle AY^T, Y^T \rangle
\]

and

\[
h_i N^i = \alpha \theta - \langle \nabla \phi, N \rangle \theta + \phi a^j N^i d_j - \phi (d_{ij} N^i N^j - \kappa \sigma_{ij}) N^i N^j
\]

\[
= \alpha \theta - \langle \nabla \phi, N \rangle \theta - \phi W \langle AY^T, \nabla^2 d \rangle - \phi \langle \nabla \nabla \phi, \nabla d \rangle + \phi \kappa \frac{|\nabla u|^2}{W^2}.
\]

We then conclude that

\[
- K \frac{|\nabla u|^2}{W} + \alpha \theta - \frac{1}{h} \langle \nabla \phi, N \rangle \theta - \phi W \langle AY^T, \nabla^2 d \rangle - \phi \langle \nabla \nabla \phi, \nabla d \rangle + \frac{\phi \kappa}{h} \frac{|\nabla u|^2}{W^2}
\]

\[
= - \frac{1}{2\gamma} \langle \nabla \gamma, N \rangle - |\nabla u|^2 \langle \frac{\nabla \gamma}{2\gamma^2}, N \rangle + W^2 \langle AY^T, Y^T \rangle.
\]

Moreover

\[
- \frac{W_t}{W} = \frac{\eta}{\eta} = K u_t + \frac{h_t}{h} = W K (nH - \mathcal{H}) + \frac{h_t}{h}
\]

\[
= n H K W - K W \mathcal{H} - \frac{1}{h} (nH - \mathcal{H}) \left( \langle \nabla \phi, N \rangle \theta - \alpha \theta - \frac{\phi}{2W^2} \langle \nabla \gamma, \nabla d \rangle \right)
\]

\[
+ \frac{\phi}{h} \langle \nabla u, \nabla \nabla \phi, \nabla d \rangle.
\]
Then we have

\[
\frac{1}{W} \left( LW - \frac{2}{W} g^{ij} W_i W_j \right) \\
= |A|^2 + KnH \gamma W + \frac{\alpha \theta}{h} nH - \frac{1}{h} nH \langle \nabla \phi, N \rangle \theta - \frac{\phi}{h} nHW \langle AY^T, \nabla \Sigma d \rangle \\
- \frac{1}{h} \left( nH - \mathcal{H} \right) \left( \langle \nabla \phi, N \rangle \theta - \alpha \theta - \frac{\phi}{2W^2} \langle \nabla \gamma, \nabla d \rangle \right) - \frac{\phi}{h} nHW \langle \nabla \Sigma, \nabla d, \nabla u \rangle \\
+ \frac{\phi}{h} nHk \frac{\nabla |u|^2}{W^2} - 3 \frac{1}{W} \gamma \langle AY^T, X_s \nabla \gamma \rangle + g^{ij} \gamma_{ij} \frac{\nabla |\nabla \gamma|^2}{4\gamma} - 3 \frac{1}{4} \langle \nabla \gamma, N \rangle^2 \\
+ \gamma \langle \nabla_{\Sigma} \nabla \gamma, N \rangle - \langle \nabla \Sigma \mathcal{H}, N \rangle - \frac{\nabla |\nabla \gamma|^2}{4\gamma} \frac{1}{W^2} - KW \mathcal{H} + \frac{\phi}{h} \langle \nabla u, \nabla \Sigma, \nabla d \rangle.
\]

We conclude that

\[
\frac{1}{\eta} L \eta + \frac{1}{W} \left( LW - \frac{2}{W} g^{ij} W_i W_j \right) = K^2 \frac{\nabla |u|^2}{W^2} + A + B,
\]

where

\[
A = \left( 1 + \frac{\phi \theta}{h} \right) |A|^2 + \frac{2K}{h} \gamma \phi \langle AY^T, \nabla \Sigma d \rangle + \frac{h}{h} \left( \kappa \gamma - \frac{1}{2} \langle \nabla d, \nabla \gamma \rangle \right) \langle AY^T, Y^T \rangle \\
+ \frac{2}{h} \langle A \nabla \Sigma d, \nabla \Sigma \phi \rangle + \frac{2}{h} \langle A, \nabla^2 d \rangle \Sigma \phi - \frac{1}{hW^2} \phi \langle A \nabla \Sigma d, X_s \nabla \gamma \rangle \\
+ KnH \frac{\gamma}{W} + \frac{\alpha \theta}{h} nH - \frac{1}{h} nH \langle \nabla \phi, N \rangle \theta - \frac{\phi}{h} nHk \frac{\gamma}{W^2} - 3 \frac{1}{W} \gamma \langle AY^T, X_s \nabla \gamma \rangle
\]

and

\[
B = \frac{2K}{h} \left( - \alpha \gamma \langle N, \nabla d \rangle - \frac{\gamma}{W} \langle N, \nabla \phi \rangle \theta - \frac{\gamma}{W} \phi \langle \nabla \Sigma, \nabla d, \nabla u \rangle - \frac{\gamma \phi \kappa}{W^3} \frac{\nabla |u|^2}{W^2} \right) \\
- \mathcal{H} \langle \nabla \Sigma, \nabla d, \nabla u \rangle \phi - n \frac{\alpha}{h} H_d + \frac{1}{h} \left( 2 \langle N, \nabla \phi \rangle - \alpha \right) \langle \nabla \Sigma, \nabla d, \nabla u \rangle \phi \\
+ \frac{1}{h} \phi \langle \nabla \Sigma, \nabla d, \nabla \gamma \rangle - \frac{1}{h} \left( \nabla \Sigma, \nabla \phi, \nabla \gamma \right) + \frac{1}{h} \phi \langle \nabla \Sigma, \nabla d, N \rangle \phi \\
- \frac{1}{h} \phi \nabla^3 d \left( \nabla u \frac{W}{W}, \nabla u \frac{W}{W}, \nabla u \frac{W}{W} \right) + \frac{1}{h} \left( \nabla u \frac{W}{W}, \nabla u \frac{W}{W} \right) \phi + \frac{\gamma}{hW^2} \langle N, \nabla_k \phi \rangle \\
\left( \frac{1}{2\gamma} + \frac{1}{2W^2} \right) \alpha \langle \nabla d, \nabla \gamma \rangle + \frac{1}{h} \left( \frac{1}{2\gamma} + \frac{1}{2W^2} \right) \langle \nabla \phi, \nabla \gamma \rangle \theta - \frac{1}{h} \phi \langle N, \nabla \gamma \rangle \\
+ \frac{2}{h} \kappa \frac{\gamma}{W^2} \langle \nabla \phi, N \rangle - \frac{1}{h} \left( \Delta \phi - \langle \nabla \Sigma, \nabla \phi, \nabla u \rangle \right) \langle \nabla \phi, \nabla \gamma \rangle \theta + g^{ij} \gamma_{ij} \frac{\nabla |\nabla \gamma|^2}{2\gamma} - \frac{3}{4} \frac{\nabla |\nabla \gamma|^2}{4\gamma^2} \\
- \frac{1}{4} \left( \frac{\nabla \phi, N \rangle^2}{2\gamma} + \gamma \langle \nabla N \nabla \gamma, \nabla \mathcal{H}, N \rangle - \langle \nabla \Sigma \mathcal{H}, N \rangle - \frac{\nabla |\nabla \gamma|^2}{4\gamma} \right) + \frac{\phi}{h} \langle \nabla u, \nabla \Sigma, \nabla d \rangle.
\]

However, using some standard inequalities, we obtain

\[
A \geq \left( 1 + \frac{\phi \theta}{h} \right) |A|^2 - \left( \frac{2K \gamma}{h\sqrt{\gamma}} + \frac{\kappa}{h} + \frac{1}{2h\gamma} |\nabla \gamma| + \frac{2}{h} |\nabla \phi| + \frac{2}{h} |\nabla^2 d|_{\Sigma} \\
+ \frac{1}{hW^2} |X_s \nabla \gamma| + \frac{K \sqrt{n}}{W} + \frac{\alpha \theta \sqrt{n}}{h} + \theta \sqrt{n} |\nabla \phi| + \frac{\gamma \sqrt{n} \kappa}{hW^2} + \frac{3}{\sqrt{n}} |\nabla \mathcal{H}, X_s \nabla \gamma \rangle \right) |A|.
\]
Using that \( W^2 \geq \gamma \) and choosing \( \alpha \) sufficiently large and depending only on \( n, \gamma, \phi \) and \( \kappa \), we have
\[
\mathcal{A} \geq \frac{1}{2} |A|^2 - \left( \epsilon + 2\sqrt{\gamma} \frac{K}{h} + \frac{K\gamma\sqrt{n}}{W} + \frac{3\sqrt{\gamma}}{W} |X_{2\gamma}^*| \right) |A| \\
\geq -\left( \epsilon + 2\sqrt{\gamma} \frac{K}{h} + \frac{K\gamma\sqrt{n}}{W} + \frac{3\sqrt{\gamma}}{W} |X_{2\gamma}^*| \right)^2.
\]
Moreover
\[
\mathcal{B} \geq -C \left( 1 + \frac{\alpha}{h} + \frac{\alpha}{hW^2} + \frac{1}{h} + \frac{1}{W^2} + \frac{1}{hW^2} + K\frac{\alpha}{h} + \frac{K}{h} \right),
\]
where \( C \) is a constant depending on \( n, \gamma, \phi, d, \kappa \) and \( \mathcal{H} \).

Hence we obtain
\[
\frac{1}{\eta L} + \frac{1}{W} \left( \mathcal{L}W - \frac{2}{W} g^{ij} W_i W_j \right) \geq K^2 \frac{|\nabla u|^2}{W^2} - C(\epsilon) - \frac{K}{W} C(\epsilon, \gamma, n) - \frac{K^2}{W^2} C(\gamma, n) \\
- \frac{1}{W} C(\gamma, \epsilon) - \frac{K^2}{W^2} C(\gamma, n) - \frac{K^2}{h^2} C(\gamma) - \frac{K^2}{hW^2} C(\gamma, n) - \frac{1}{W^2} C(\gamma) - \frac{K}{hW} C(\gamma) \\
- K\frac{\alpha}{h} C - \frac{K}{h} C(\epsilon, \gamma) - C - \frac{\alpha}{hW^2} C - \frac{1}{h} C - \frac{1}{W^2} C - \frac{1}{hW^2} C.
\]

Then
\[
-K^2 \frac{|\nabla u|^2}{W^2} \geq -C \left( \frac{K}{h^2} + \frac{K}{h} + \frac{1}{h} + 1 \right) C \leq \left( K^2 + K + \frac{1}{\alpha} + 1 \right) C \\
+ (K + \frac{K}{h} + 1) CW.
\]

Now suppose that \( W(x_0, t_0) \geq 1 \). Otherwise we are done. In this case we have \( W \leq W^2 \) and absorbing the terms with \( W \) into the one with \( W^2 \) transforms the inequality above into
\[
\left( K^2 \gamma - \left( \frac{K}{h^2} + \frac{K}{h} + \frac{1}{h} + 1 \right) C \right) W^2 \leq \left( K^2 + K + \frac{1}{h} + 1 \right) C \\
+ (K + \frac{K}{h} + 1) CW.
\]

If \( d_0 = d(x_0) \), then choosing \( \alpha \geq 1/(C(d_0)d_0 - 1) \) for some constant \( C(d_0) > 1/d_0 \), we obtain \( (1 + \alpha)/h \leq C(d_0) \) which implies that
\[
\left( K^2 \gamma - \left( \frac{K}{h^2} C - \frac{K}{h} C - KC(d_0) - C(d_0) \right) \right) W^2 \leq (K^2 + K + C(d_0)) C.
\]

Then for \( \alpha > \frac{1}{d_0} \max\{1, \sqrt{2C} / \gamma \} \) we have
\[
\left( K^2 \gamma \frac{\sqrt{2C}}{\gamma} - KC(d_0) - C(d_0) \right) W^2 \leq (K^2 + K + C(d_0)) C.
\]
It follows that for \( K > \frac{C(d_0) + \sqrt{C(d_0)^2 + 2\gamma C(d_0)}}{\gamma} \) we have \( K^2 \frac{2}{\gamma} - KC(d_0) - C(d_0) > 0 \) and
\[
W^2 \leq \frac{C(K^2 + K + C(d_0))}{K^2 \frac{2}{\gamma} - KC(d_0) - C(d_0)}.
\]
This finishes the proof of the proposition.

**Theorem 2.** There exists a unique solution \( u : \bar{\Omega} \times [0, \infty) \rightarrow \mathbb{I} \) to the problem \((1.2)-(1.4)\).

**Proof.** Propositions 1, 2 and 3 yield the global gradient bound
\[
W(x, t) \leq W(x_0, t_0) \eta(x, t) \eta(x_0, t_0) \leq C_1 e^{C_2 MT^*}
\]
for \((x, t) \in \bar{\Omega} \times [0, T^*] \), where \( C_1 \) and \( C_2 \) are positive constants and
\[
M = \max_{\Omega \times [0, T^*]} |u - u_0|.
\]
It results that \((2.17)\) is uniformly parabolic and then the standard theory of quasi-linear parabolic PDEs may be applied to assure the existence of a unique smooth solution to \((2.17)-(2.19)\).

6. **Asymptotic behavior**

Suppose from now on that \( \mathcal{H} = 0 \) and \( \phi = 0 \). In the particular case when the evolving functions have the form \( u(x, t) = v(x) + Ct \), \((x, t) \in \bar{\Omega} \times [0, T)\), the initial value problem \((2.17)-(2.19)\) becomes
\[
\text{div} \frac{\nabla v}{W} - \gamma \langle \nabla Y Y, \nabla v \rangle = \frac{C}{W} \text{ in } \Omega,
\]
\[
\langle \nu, N \rangle = 0 \text{ on } \partial \Omega.
\]
Conversely, notice that if \( v(x) \) is a solution of \((6.1)-(6.2)\), then \( u = v + Ct \) is a solution of \((2.17)\) which is translating along the flow lines of \( Y \) with speed \( C \).

Now observe that
\[
\text{div} \frac{\nabla v}{W} - \gamma \langle \nabla Y Y, \nabla v \rangle = \text{div} \frac{\nabla v}{W} + \gamma \langle \nabla v \nabla Y Y \rangle = \text{div} \frac{\nabla v}{W} + \gamma \langle \nabla Y \nabla v Y \rangle = \text{div} M \frac{\nabla v}{W}.
\]
Therefore it follows from the divergence theorem that
\[
\int_{\partial([0, s] \times \Omega)} \frac{C}{W} + \mathcal{H} = - \int_{\partial([0, s] \times \Gamma)} \langle \nabla v \rangle \langle W, \nu \rangle = \int_{\partial([0, s] \times \Gamma)} \langle \nu, N \rangle = \int_{\partial([0, s] \times \Gamma)} \phi.
\]
Since the integrands do not depend on \( s \) we have
\[
\int_{\Omega} C \frac{1}{\sqrt{\gamma W}} = \int_{\Gamma} \frac{1}{\sqrt{\gamma}} \phi,
\]
from which results that
\[
C = 0.
\]
We then obtain the following height estimate.

**Proposition 4.** Given a solution \( u(x, t) \) of \((2.17)\) there exists a constant \( M \) such that
\[
|u(x, t)| \leq M
\]
for \((x, t) \in \Omega \times [0, +\infty)\).
Proof. We observe that since $C$ is necessarily zero, $v = \text{cte}$ is a solution to (6.1). In particular the constant functions $v_1 = \inf_{\Omega} u_0$ and $v_2 = \sup_{\Omega} u_0$ are solutions of (6.1) with $v_1 \leq u_0 \leq v_2$. Hence the parabolic maximum principle implies that

$$v_1 \leq u(\cdot, t) \leq v_2$$

for $t \in [0, T)$ from which we obtain (6.6). □

Now, proceeding as in [3], we prove the following convergence result.

**Theorem 3.** Suppose that $H = 0$ and $\phi = 0$. Then $\lim_{t \to \infty} u_t = 0$. In particular the mean curvature flow converges to a slice of the form $\vartheta(\{s\} \times \Omega)$ for some $s \in \mathbb{R}$.

**Proof.** It is immediate that $v = s$ is a trivial solution to (6.1) with (necessarily) $C = 0$. We also have

$$\frac{d}{dt} \int_{\Omega} W = \int_{\Omega} \frac{u_t^2}{W} - \int_{\Omega} \frac{1}{2W^3} \langle \nabla u, \nabla \gamma \rangle - \int_{\Omega} \frac{1}{2\gamma W^2} \langle \nabla u, \nabla \gamma \rangle.$$

Therefore

$$(6.7) - \int_{\Omega} u_t^2 W = \frac{d}{dt} \left( \int_{\Omega} W \right) + \int_{\Omega} \frac{1}{2W^3} \langle \nabla u, \nabla \gamma \rangle + \int_{\Omega} \frac{1}{2\gamma W^2} \langle \nabla u, \nabla \gamma \rangle.$$

It follows that

$$\int_0^T \int_{\Omega} \frac{u_t^2}{W} = - \int_{\Omega} W(x, T) + \int_{\Omega} W(x, 0)$$

$$+ \int_0^T \int_{\Omega} \frac{1}{2W^3} \langle \nabla u, \nabla \gamma \rangle + \int_0^T \int_{\Omega} \frac{1}{2\gamma W^2} \langle \nabla u, \nabla \gamma \rangle \leq \tilde{C}$$

for some positive constant $\tilde{C}$. It also follows that $\lim_{t \to \infty} \frac{u_t^2}{W} = 0$. Since $W$ is bounded, then $\lim_{t \to \infty} u_t = 0$. This finishes the proof of the theorem. □

7. Appendix

In what follows, $II$ and $A$ denote respectively the second fundamental form and the Weingarten map of $\Sigma_t$. Their components are given by

$$a_{ij} = II(X_* \frac{\partial}{\partial x^i}, X_* \frac{\partial}{\partial x^j}) := (AX_* \frac{\partial}{\partial x^i}, X_* \frac{\partial}{\partial x^j}).$$

Some lemmata will be needed in the sequel. Their content could also be of independent interest for other applications.

**Lemma 1.** Denote $\theta = \langle \nabla d, N \rangle$. The differentials of the functions $\theta$ and $h$ have components given by

$$\theta_i = -a_{ij}d_j + (d_{ij} - \kappa \sigma_{ij})N_j$$

and

$$h_i = (\alpha \delta^j_i + \phi a_{ij})d_j - (\phi(d_{ij} - \kappa \sigma_{ij}) + \phi_i d_j)N_j,$$

respectively, where $\kappa = \langle \gamma \nabla_Y Y, \nabla d \rangle$. 

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Proof. We have
\[
\frac{\partial \theta}{\partial x^i} = X_\tau \frac{\partial}{\partial x^i} \langle N, \nabla d \rangle = \langle \nabla_{X_\tau} \frac{\partial}{\partial x^i} N, \nabla d \rangle + \langle N, \nabla_{X_\tau} \frac{\partial}{\partial x^i} \nabla d \rangle
\]
\[
= -\langle AX_\tau \frac{\partial}{\partial x^i}, \nabla d \rangle + \langle N, \nabla \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial x^i} \nabla d \rangle
\]
\[
= -\langle AX_\tau \frac{\partial}{\partial x^i}, \nabla d \rangle + \nabla \left( \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^0} \nabla d \right) - \nabla \left( \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^0} \nabla d \right)
\]
\[
+ u_i \left( \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^0} \nabla d \right) - u_i \left( \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^0} \nabla d \right).
\]

Since \( P \) is totally geodesic we have
\[
\langle \frac{\partial}{\partial x^i}, \nabla_{\frac{\partial}{\partial x^i}} \nabla d \rangle = \langle \frac{\partial}{\partial x^0}, \nabla_{\frac{\partial}{\partial x^0}} \nabla d \rangle = 0.
\]
Moreover we compute
\[
\langle \frac{\partial}{\partial x^0}, \nabla_{\frac{\partial}{\partial x^0}} \nabla d \rangle = |Y|^2 \langle \frac{\partial}{\partial x^0}, u \frac{\partial}{\partial x^0} \nabla d \rangle = |Y|^2 \kappa = \frac{1}{\gamma}
\]
and
\[
\langle \frac{\partial}{\partial x^0}, \nabla_{\frac{\partial}{\partial x^0}} \nabla d \rangle = \nabla \left( \frac{\partial}{\partial x^0} \nabla d \right) + \nabla \left( \frac{\partial}{\partial x^0} \nabla d \right) = 0,
\]
where we used the fact that \( \left[ \frac{\partial}{\partial x^0}, \nabla d \right] = 0 \) and that \( P \) is totally geodesic.

Thus we conclude that
\[
\frac{\partial \theta}{\partial x^i} = -\langle AX_\tau \frac{\partial}{\partial x^i}, \nabla d \rangle - \langle \frac{\partial}{\partial x^0}, \nabla_{\frac{\partial}{\partial x^0}} \nabla d \rangle + \kappa u_i \frac{\partial}{\partial x^i} \nabla d
\]

However
\[
\langle AX_\tau \frac{\partial}{\partial x^i}, \nabla d \rangle = a^i_j \langle X_\tau \frac{\partial}{\partial x^j}, \nabla d \rangle = a^i_j \langle \frac{\partial}{\partial x^j} + u_j Y, \nabla d \rangle = a^i_j d_j = g^{jk} a_{ik} d_j.
\]
Therefore we write
\[
(7.4) \quad \theta_i = -g^{jk} a_{ik} d_j + (d_{i;j} - \kappa s_{ij}) N^j.
\]
This finishes the proof of the proposition. \( \square \)

We denote the components of the tensor \( X^* II \) in \( P \) by
\[
(7.5) \quad b_{ij} = X^* II \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) := \langle AX_\tau \frac{\partial}{\partial x^i}, X_\tau \frac{\partial}{\partial x^j} \rangle.
\]
Notice that the covariant derivatives of \( X^* II \) and \( II \) are related by
\[
\nabla_k b_{ij} = \langle (\nabla X_\tau \frac{\partial}{\partial x^k} A) X_\tau \frac{\partial}{\partial x^i}, X_\tau \frac{\partial}{\partial x^j} \rangle + \langle AX_\tau \frac{\partial}{\partial x^j}, \nabla X_\tau \frac{\partial}{\partial x^i} X_\tau \frac{\partial}{\partial x^j} - X_\tau \nabla \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} \rangle
\]
\[
+ \langle AX_\tau \frac{\partial}{\partial x^i}, \nabla X_\tau \frac{\partial}{\partial x^j} - X_\tau \nabla \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} \rangle.
\]

However since \( X_\tau \frac{\partial}{\partial x^i} = \frac{\partial}{\partial x^i} + u_i Y \) we compute
\[
\nabla X_\tau \frac{\partial}{\partial x^i} - X_\tau \nabla \frac{\partial}{\partial x^i} = \nabla \frac{\partial}{\partial x^i} + u_i Y + u_i \nabla Y
\]
\[
+ u_k \nabla Y \frac{\partial}{\partial x^i} + u_i u_k \nabla Y - \nabla \frac{\partial}{\partial x^i} - \left( \nabla u, \nabla \frac{\partial}{\partial x^i} \right) Y.
\]
Therefore
\[
\nabla X_\tau \frac{\partial}{\partial x^i} - X_\tau \nabla \frac{\partial}{\partial x^i} = u_{i;k} Y + u_i \nabla Y + u_k \nabla Y + u_i u_k \nabla Y.
\]
Hence using (2.6), (2.10) and (2.11) we obtain

\[ \nabla_X \frac{\partial}{\partial x^i} X - X \nabla \frac{\partial}{\partial x^i} = (W a_{ik} + u_i u_k \frac{\gamma}{2 \gamma^2}) Y + \frac{1}{2} u_i u_k \nabla \gamma \]

\[ = W a_{ik} Y + \frac{1}{2 \gamma^2} u_i u_k ((\nabla u, \nabla \gamma) Y + \nabla \gamma) = W a_{ik} Y + \frac{1}{2 \gamma^2} u_i u_k X \nabla \gamma. \]

Hence it follows that

\[ \langle AX, \frac{\partial}{\partial x^i} \rangle, \nabla_X \frac{\partial}{\partial x^i} - X \nabla \frac{\partial}{\partial x^i} = \langle AX, \frac{\partial}{\partial x^i} \rangle, u_i u_k - \frac{1}{2 \gamma^2} a_{ij} \gamma^{l} \]

We conclude that

\[ \nabla_k b_{ij} = ((\nabla \Sigma, X) X, X) \frac{\partial}{\partial x^i} + \frac{1}{\gamma} W a_{ik} a_i^l u_l + \frac{u_i u_k}{2 \gamma^2} a_{il} \gamma^{l}, \]

that is,

\[ (7.6) \quad \nabla_k b_{ij} = \nabla_k^\Sigma a_{ij} + \frac{1}{\gamma} W a_{ik} a_i^l u_l + \frac{1}{\gamma} W a_{jk} a_i^l u_l + \frac{u_i u_k}{2 \gamma^2} a_{il} \gamma^{l} + \frac{u_i u_k}{2 \gamma^2} a_{il} \gamma^{l}. \]

Now we use (7.6) for computing the Hessian of the function \( \theta \).

**Lemma 2.** The trace of the Hessian of \( \theta \) in \( \Omega \) calculated with respect to the metric in \( \Sigma \) is given by

\[ g^{ik} \theta_{i;k} = -|A|^2 \theta - 2 \langle \nabla^2 d, X^* H \rangle_S - n \langle \nabla^2 H, \nabla^2 d \rangle - n H W \langle AY^T, \nabla^2 d \rangle \]

\[ - \text{Ric}(\nabla d, \frac{\nabla u}{W}) - \text{Tr} \nabla^2 d \nabla^2 d - \frac{|\nabla u|^2}{W^2} \langle A \nabla H, \frac{\nabla \gamma}{\gamma} \rangle + \frac{1}{2} \langle AY^T, Y^T \rangle \langle \nabla d, \nabla \gamma \rangle \]

\[ - \frac{1}{2 W^2} \langle \nabla d, \nabla \gamma \rangle - \frac{\gamma}{W^2} \langle N, \nabla \kappa \rangle + \kappa (n H - \gamma \langle AY^T, Y^T \rangle) - \kappa \frac{1}{2 W^2} \langle N, \nabla \gamma \rangle. \]

**Proof.** Notice that we may write (7.4) as

\[ (7.7) \quad \theta_i = -g^{il} b_{il} d_j + (d_{ij} - \kappa \sigma_{ij}) N^j. \]

Hence we have

\[ n g^{ik} \theta_{i;k} = -g^{ik}(g^{jl} b_{ij} d_j)_k + g^{ik}(d_{ij;k} - \kappa \sigma_{ij}) N^j + g^{ik}(d_{ij} - \kappa \sigma_{ij}) N^j \]

\[ = -g^{ik}(g^{jl} b_{ij} d_j)_k + g^{ik}(d_{ij;k} + R^l_{jkli} d_l - \kappa \sigma_{ij}) N^j - g^{ik}(d_{ij} - \kappa \sigma_{ij})(a^j_k - N_k \frac{\gamma^j}{2 \gamma}). \]

However

\[ g^{ik}(g^{jl} b_{ij} d_j)_k = g^{ij} g^{ik} b_{ij} d_j + g^{ik} g^{jl} b_{ij} d_j + g^{ik} g^{jl} b_{ij} d_j \]

\[ = g^{ij} g^{ik} (\nabla^k_\Sigma a_{il} + \frac{1}{\gamma} W a_{ik} a^m_i u_m + \frac{1}{\gamma} W a_{ik} a^m_i u_m + u_{i k} a_{im} \frac{\gamma_m}{2 \gamma^2} + u_{i k} a_{im} \frac{\gamma_m}{2 \gamma^2}) d_j \]

\[ + g^{ij} g^{jl} b_{ij} d_j + g^{ik} g^{jl} b_{ij} d_j. \]
Hence using Codazzi’s equation we obtain

\[ g^{ik}(g^{jl}b_{il}d_{j})_{;k} = g^{ij}(nH_{l} + \frac{1}{\gamma} WHa_{m}^{l}u_{m} + \frac{1}{\gamma} W^{2}a_{m}^{l}a_{m}^{n}u_{m} + \frac{|\nabla u|^{2}}{W^{2}}a_{m}^{l}N_{m}^{k} \]

\[ + u_{t}u_{k}a_{m}^{l}N_{m}^{k} + g^{ij}g^{kl}(\dot{R}(X_{*}, \frac{\partial}{\partial x^{l}}, X_{*}, \frac{\partial}{\partial x^{k}})N, X_{*}, \frac{\partial}{\partial x^{j}})d_{j} + g^{ik}g^{jl}b_{il}d_{j} \]

\[ + g^{ik}g^{jl}b_{il}d_{j;k}. \]

Using that \( g^{jl}u_{l} = \frac{\omega_{j}}{\omega_{y}}u_{j} \) we conclude that

\[ g^{ik}(g^{jl}b_{il}d_{j})_{;k} = ng^{jl}H_{l}d_{j} - \frac{1}{\gamma} W^{2}Hg^{il}a_{m}^{l}N_{m}d_{j} - \frac{1}{\gamma} W^{2}a_{m}^{l}a_{m}^{n}N_{m}d_{j} + \frac{|\nabla u|^{2}}{W^{2}}a_{m}^{l}N_{m}^{k} \]

\[ + N^{j}N_{k}a_{m}^{k}N_{m}^{k}d_{j} + g^{ik}g^{jl}b_{il}d_{j} + g^{ik}g^{jl}b_{il}d_{j;k}. \]

However we have

\[ g^{ik} = (\sigma^{ij} - N^{j}N_{i})_{;k} = -N^{j}_{;k}N^{l} - N^{j}N_{i;k} = (a_{k}^{l} - N_{k}^{j}a_{m}^{l})N^{l} + N^{j}(a_{k}^{i} - N_{k}^{j}a_{m}^{i})N^{l} \]

and

\[ \tilde{\nabla}_{\tilde{\omega}x}N = \tilde{\nabla}_{x}N - \tilde{\nabla}_{u}YN = -AX_{*} \frac{\partial}{\partial x^{k}} - u_{k}\tilde{\nabla}_{y}Y(\gamma Y_{y} - \nabla u_{y}) \]

\[ = -AX_{*} \frac{\partial}{\partial x^{k}} - \frac{u_{k}}{2W}(\frac{\nabla Y}{\gamma} + \langle \nabla u, \nabla Y \rangle), \]

from which follows that

\[ g^{ik}(g^{jl}b_{il}d_{j})_{;k} = ng^{jl}H_{l}d_{j} - \frac{1}{\gamma} W^{2}Hg^{il}a_{m}^{l}N_{m}g^{jl}d_{j} - \frac{1}{\gamma} W^{2}a_{m}^{l}a_{m}^{n}N_{m}g^{jl}d_{j} \]

\[ + \frac{|\nabla u|^{2}}{W^{2}}a_{m}^{l}N_{m}^{k} \]

\[ + N^{j}N_{k}a_{m}^{k}N_{m}^{k}d_{j} + g^{ik}g^{jl}b_{il}d_{j} + \frac{|\nabla u|^{2}}{W^{2}}a_{m}^{l}N_{m}^{k} \]

\[ + g^{ik}(d_{i;k} + R_{j;ki}^{l}d_{l} - \kappa_{k}a_{ij})N^{j} - g^{ik}(d_{i;k} - \kappa_{ij})(a_{l}^{k} - N_{k}^{j})N^{l} + g^{ik}d_{i;k}. \]

Therefore

\[ g^{ik}\theta_{i;k} = -ng^{jl}H_{l}d_{j} + \frac{1}{\gamma} W^{2}Hg^{il}a_{m}^{l}N_{m}g^{jl}d_{j} + \frac{1}{\gamma} W^{2}a_{m}^{l}a_{m}^{n}N_{m}g^{jl}d_{j} \]

\[ - \frac{|\nabla u|^{2}}{W^{2}}a_{m}^{l}N_{m}^{k} \]

\[ - a_{k}^{l}a_{k}^{j}N^{l}d_{j} + a_{k}^{l}N_{k}N^{j}d_{j} - a_{k}^{l}N_{k}^{j}N^{l}d_{j} + g^{ik}(d_{i;k} + R_{j;ki}^{l}d_{l} - \kappa_{k}a_{ij})N^{j} - g^{ik}(d_{i;k} - \kappa_{ij})(a_{l}^{k} - N_{k}^{j})N^{l} + g^{ik}d_{i;k}. \]

Now using the fact that \( g^{ij}u_{j} = \frac{\omega_{j}}{\omega_{y}}u_{j} \) and therefore \( g^{ij}N_{j} = \frac{\omega_{y}}{\omega_{y}}N_{i} \) we obtain

\[ a_{m}^{m}N_{m} = g^{km}a_{ik}N_{m} = \frac{\gamma}{W^{2}}a_{ik}N^{k} = \frac{\gamma}{W^{2}}\langle AX_{*}\frac{\partial}{\partial x^{l}}, N^{k}X_{*}\frac{\partial}{\partial x^{k}} \rangle \]

\[ = \frac{\gamma}{W^{2}}\langle AX_{*}\frac{\partial}{\partial x^{l}}, N^{k}\frac{\partial}{\partial x^{k}} + \langle N^{k}\frac{\partial}{\partial x^{k}}, \nabla u \rangle Y \rangle \]

\[ = \frac{\gamma}{W^{2}}\langle AX_{*}\frac{\partial}{\partial x^{l}}, N - \frac{\gamma}{W}Y \rangle + \langle N, \nabla u \rangle Y \rangle \]

\[ = -\frac{\gamma}{W^{2}}\langle AX_{*}\frac{\partial}{\partial x^{l}}, Y \rangle \langle \gamma Y + \frac{|\nabla u|^{2}}{W^{2}} \rangle - \frac{\gamma}{W^{2}}\langle AX_{*}\frac{\partial}{\partial x^{l}}, Y \rangle = -\frac{\gamma}{W^{2}}\langle MY^{T}, X_{*}\frac{\partial}{\partial x^{l}} \rangle. \]
Therefore
\[ a^m_i N_m g^{jl} d_j = -\frac{\gamma}{W} \langle AY^T, g^{jl} d_j X_s \frac{\partial}{\partial x^l} \rangle = -\frac{\gamma}{W} \langle AY^T, \nabla d \rangle. \]

Moreover notice that
\[ a^i_k N^l = g^{km} a_{ml} N^l = -g^{km} W \langle AY^T, X_s \frac{\partial}{\partial x^m} \rangle \]
and
\[ a_{ik} N^k = -W \langle AY^T, X_s \frac{\partial}{\partial x^i} \rangle. \]

Similarly we have
\[ a^j_k d_j = g^{jm} d_j \langle AX_s \frac{\partial}{\partial x^j}, X_s \frac{\partial}{\partial x^m} \rangle = \langle AX_s \frac{\partial}{\partial x^j}, \nabla \Sigma d \rangle = \langle AX_s \frac{\partial}{\partial x^j}, A \nabla d \rangle. \]

Replacing the above we obtain
\[ g^{ik} \theta_{i;k} = -n \langle \nabla^\Sigma H, \nabla d \rangle - n HW \langle AY^T, \nabla d \rangle - W \langle AY^T, A \nabla^\Sigma d \rangle - \frac{|\nabla u|^2}{W^2} \langle A \nabla^\Sigma d, X_s \nabla_{2\gamma} \rangle + W \langle AY^T, A \nabla d \rangle + \gamma \langle AY^T, Y^T \rangle \langle \nabla \Sigma d, \nabla_{2\gamma} \rangle - |A|^2 \theta - g^{il} a_{i;j} d_{j;k} + g^{ik} (d_{i;j} + R_{l;ki} d_l - \kappa \kappa \sigma_{ij}) N^j - g^{ik} (d_{i;j} - \kappa \kappa \sigma_{ij}) (a^l_k - N^l_k \gamma^j_{2\gamma}). \]

Therefore
\[ g^{ik} \theta_{i;k} = -n \langle \nabla^\Sigma H, \nabla d \rangle - n HW \langle AY^T, \nabla d \rangle - \frac{|\nabla u|^2}{W^2} \langle A \nabla^\Sigma d, X_s \nabla_{2\gamma} \rangle + \gamma \langle AY^T, Y^T \rangle \langle \nabla \Sigma d, \nabla_{2\gamma} \rangle - |A|^2 \theta - g^{il} a_{i;j} d_{j;k} + g^{ik} (d_{i;j} + R_{l;ki} d_l - \kappa \kappa \sigma_{ij}) N^j - g^{ik} (d_{i;j} - \kappa \kappa \sigma_{ij}) (a^l_k - N^l_k \gamma^j_{2\gamma}). \]

However
\[ g^{ik} \sigma_{ij} = g^{ik} (g_{ij} - \frac{u_i u_j}{\gamma}) = \delta^k_j - \frac{1}{W^2} u^k u_j = \delta^k_j - N^k N^j. \]

Hence we have
\[ g^{ik} \theta_{i;k} = -n \langle \nabla^\Sigma H, \nabla d \rangle - n HW \langle AY^T, \nabla d \rangle - \frac{|\nabla u|^2}{W^2} \langle A \nabla^\Sigma d, X_s \nabla_{2\gamma} \rangle + \frac{1}{2} \langle AY^T, Y^T \rangle \langle \nabla d, \nabla \gamma \rangle - |A|^2 \theta - 2 g^{ik} g^{jl} d_{i;j} a_{kl} + \frac{1}{2W^2} d_{i;j} N^i \gamma^j + g^{ik} d_{i;j} N^j - \text{Ric} \langle \nabla d, \nabla \gamma \rangle - \frac{\gamma}{W^2} \langle N, \nabla \kappa \rangle + \kappa (nH - \gamma \langle AY^T, Y^T \rangle) - \kappa \frac{1}{2W^2} \langle N, \nabla \gamma \rangle. \]

This finishes the proof of the lemma. \( \square \)

Using Lemma 2 we will obtain an expression for \( Lh \). Notice that
\[ h_{i;k} = \alpha d_{i;k} - \phi_i \theta_k - \phi_k \theta_i - \phi_{i;k} \theta - \phi_{i;k}. \]
Moreover it holds that
\[ 2g^i{}^k \phi_i \theta_k = 2g^i{}^k \phi_i \langle A \nabla^\Sigma d, X^t \partial_x^k \rangle - 2g^i{}^k d_{k ; l} \phi_i N^l + 2\kappa g^i{}^k \sigma_{k l} \phi_i N^l \]
\[ = 2 \langle A \nabla^\Sigma d, \nabla^\Sigma \phi \rangle - 2g^i{}^k d_{k ; l} \phi_i N^l + 2\kappa \frac{\gamma}{W^2} \langle \nabla \phi, N \rangle. \]

We conclude that
\[ g^i{}^k h_{i ; k} = \alpha g^i{}^k d_{i ; k} + 2 \langle A \nabla^\Sigma d, \nabla^\Sigma \phi \rangle - 2g^i{}^k d_{k ; l} \phi_i N^l + 2\kappa \frac{\gamma}{W^2} \langle \nabla \phi, N \rangle - g^i{}^k \phi_i \theta \]
\[ + n \phi \nabla^\Sigma H, \nabla^\Sigma d \rangle + n \phi \phi (A Y^T, \nabla^\Sigma d) + \frac{|\nabla u|^2}{W^2} \phi \langle A \nabla^\Sigma d, X^t \nabla \gamma \rangle \]
\[ - \frac{1}{2} \phi (A Y^T, Y^T) \langle \nabla d, \nabla \gamma \rangle + |A|^2 \phi \theta + 2g^i{}^k g^{j i} d_{i ; j} a_{k l} \phi - \frac{1}{2W^2} \phi d_{i ; j} N^i \gamma^j \]
\[ - g^i{}^k d_{i ; j} N^j \phi + \text{Ric}(\nabla d, \nabla u) \phi + \frac{\gamma}{W^2} \langle N, \nabla \kappa \rangle \phi - \kappa (\nabla \gamma) \phi - \frac{\kappa}{2W^2} \langle N, \nabla \gamma \phi \rangle. \]

Now we compute the derivatives with respect to \( t \). We have
\[ \theta_t = X^t \partial_x^i \langle N, \tilde{\nabla} d \rangle = \langle \tilde{\nabla} X^t, \partial_x^i \tilde{\nabla} d \rangle + \langle N, \tilde{\nabla} X^t, \partial_x^i \tilde{\nabla} d \rangle \]
\[ = - \langle \nabla^\Sigma (n H - \kappa), \tilde{\nabla} d \rangle + (n H - \kappa) \langle N, \tilde{\nabla} N \tilde{\nabla} d \rangle. \]

Hence we have
\[ \theta_t = - \langle \nabla^\Sigma (n H - \kappa), \tilde{\nabla} d \rangle + (n H - \kappa) \langle N, \tilde{\nabla} d \rangle + \langle \nabla \frac{u}{W}, \tilde{\nabla} \frac{u}{W} \tilde{\nabla} d \rangle \]

Moreover we have
\[ (7.8) \quad d_t = \langle X^t \partial_x^i, \tilde{\nabla} d \rangle = (n H - \kappa) \langle N, \tilde{\nabla} d \rangle = (n H - \kappa) \theta. \]

Therefore
\[ h_t = \alpha (n H - \kappa) \theta - (n H - \kappa) \langle N, \nabla \phi \rangle \theta + \phi \langle \nabla^\Sigma (n H - \kappa), \tilde{\nabla} d \rangle \]
\[ - \phi (n H - \kappa) \langle - \frac{1}{2W^2} \langle \nabla \gamma, \tilde{\nabla} d \rangle + \langle \nabla \frac{u}{W}, \tilde{\nabla} \frac{u}{W} \tilde{\nabla} d \rangle \rangle. \]

We also compute
\[ \langle \nabla \gamma, \nabla h \rangle = \alpha \langle \nabla d, \nabla \gamma \rangle + \phi \langle A \nabla^\Sigma d, X^t \nabla \gamma \rangle - \langle \nabla \phi, \nabla \gamma \rangle \theta - \phi d_{i ; j} \gamma^i N^j + \kappa \phi \langle N, \nabla \gamma \rangle. \]

Now we obtain
\[ g^i{}^k d_{i ; k} = \Delta d - \langle \nabla \frac{u}{W} \tilde{\nabla} d, \frac{u}{W} \rangle = -n H d - \langle \nabla \frac{u}{W} \tilde{\nabla} d, \nabla \frac{u}{W} \rangle \]
and
\[ g^i{}^k d_{k ; l} \phi_i N^l = d_{k ; l} \phi_k N^l - d_{k ; l} N^k N^l \phi_i \]
\[ = - \langle \nabla \frac{u}{W} \tilde{\nabla} d, \nabla \phi \rangle - \langle \nabla \frac{u}{W} \tilde{\nabla} d, \frac{u}{W} \rangle \langle N, \nabla \phi \rangle. \]

Moreover we have
\[ g^{ij} \phi_{i ; j} = \Delta \phi - \langle \nabla \frac{u}{W} \tilde{\nabla} \phi, \frac{u}{W} \rangle \]
and
\[ g^{ij}d_{i;k}N^j = (\sigma^{ik}d_{i;k})_jN^j - d_{i;kj}N^iN^kN^j \]
\[ = -n(Hd)_jN^j + \nabla^3d\left(\frac{\nabla u}{W}, \frac{\nabla u}{W}, \frac{\nabla u}{W}\right). \]

Therefore grouping and rearranging these expressions we obtain
\[ Lh = |A|^2\phi\theta + n\phi HW\langle AY^T, \nabla\Sigma d\rangle + (\kappa\gamma - \frac{1}{2}\langle \nabla d, \nabla\gamma \rangle)\phi\langle AY^T, Y^T \rangle \]
\[ + 2\langle A\nabla\Sigma d, \nabla\Sigma \phi \rangle + 2\langle A, \nabla^2d \Sigma \phi - \frac{1}{W^2}\phi(A\nabla\Sigma d, X_\ast \nabla\gamma) \rangle \]
\[ (nH - \mathcal{H})(\langle N, \nabla\phi \rangle - \alpha\theta - \frac{1}{2W^2}\langle \nabla\gamma, \nabla d \rangle\phi + \langle \nabla\Sigma, \nabla d, \frac{\nabla u}{W} \rangle\phi) - n\kappa H\phi \]
\[ - n\alpha H_d + (2\langle N, \nabla\phi \rangle - \alpha)\langle \nabla\Sigma, \nabla d, \frac{\nabla u}{W} \rangle + 2\langle \nabla\Sigma, \nabla d, \nabla\phi \rangle \]
\[ - \phi\langle \nabla\Sigma, \nabla d, \frac{\nabla\gamma}{2\gamma} \rangle + \phi(\nabla\Sigma\mathcal{H}, \nabla d) + n\langle \nabla H_d, N \rangle \phi \]
\[ - \phi\nabla^3d(\frac{\nabla u}{W}, \frac{\nabla u}{W}, \frac{\nabla u}{W}) + \text{Ric}(\nabla d, \frac{\nabla u}{W})\phi \]
\[ + \frac{\gamma}{2W^2}(\langle N, \nabla\kappa \rangle\phi - \frac{1}{2\gamma} + \frac{1}{2W^2})\alpha(\langle \nabla d, \nabla\gamma \rangle + \frac{1}{1\gamma} + \frac{1}{2W^2})\langle \nabla\phi, \nabla\gamma \rangle \theta \]
\[ - \kappa\phi\langle N, \frac{\nabla\gamma}{2\gamma} \rangle + 2\kappa\frac{\gamma}{W^2}\langle \nabla\phi, N \rangle - (\Delta\phi - \langle \nabla\Sigma, \nabla\phi, \frac{\nabla u}{W} \rangle)\theta. \]

**Lemma 3.** We have
\[ LW - \frac{2}{W}g^{ij}W_iW_j \]
\[ = |A|^2W + nHW^3\langle AY^T, Y^T \rangle - nHW^3\langle \frac{\nabla\gamma}{2\gamma^2}, N \rangle - 3\gamma\langle AY^T, X_\ast \frac{\nabla\gamma}{2\gamma} \rangle \]
\[ + g^{ij}\frac{\gamma^i_j}{2\gamma}W^3 - \frac{3}{4}\frac{\nabla\gamma^2}{4\gamma^2}W - \frac{1}{2}\frac{\nabla\gamma^2}{2\gamma}W + \gamma W\langle \nabla N, \frac{\nabla\gamma}{2\gamma^2}, N \rangle - W\langle \nabla\Sigma\mathcal{H}, N \rangle \]
\[ - \frac{\nabla\gamma^2}{4\gamma} - \frac{1}{W}W_t. \]

**Proof.** Notice that
\[ W_i = -W^2\left(\langle \nabla X_\ast \frac{\partial}{\partial x^i} Y, N \rangle + \langle Y, \nabla X_\ast \frac{\partial}{\partial x^i} N \rangle\right) \]
\[ = -W^2\left(\langle \nabla \frac{\partial}{\partial x^i} Y, N \rangle + u_i(\langle \nabla Y, N \rangle - \langle Y, AX_\ast \frac{\partial}{\partial x^i} \rangle)\right) \]
\[ = -W^2\left(\langle Y, N \rangle - u_i\langle \nabla \frac{\partial}{\partial x^i} Y, N \rangle - \langle Y, AX_\ast \frac{\partial}{\partial x^i} \rangle\right). \]

Therefore
\[ W_i = \frac{\gamma_i}{2\gamma}W + N_iW^3\langle \frac{\nabla\gamma}{2\gamma^2}, N \rangle + W^2\langle AY^T, X_\ast \frac{\partial}{\partial x^i} \rangle. \]

However
\[ \langle AY^T, X_\ast \frac{\partial}{\partial x^i} \rangle = g^{kl}\langle Y, X_\ast \frac{\partial}{\partial x^k} \rangle\langle X_\ast \frac{\partial}{\partial x^l}, AX_\ast \frac{\partial}{\partial x^i} \rangle = g^{kl}\langle Y, u_k Y \rangle b_{il} = \frac{1}{W^2}u^l b_{il}. \]
Hence it follows that

\[ W_i = \frac{\gamma_i}{2\gamma} W + N_i W^3 \langle \frac{\nabla \gamma}{2\gamma^2}, N \rangle - WN^i b_{it}. \]

Hence we obtain

\[
\frac{1}{W} g^{ij} W_i W_j = \frac{\left| \nabla \Sigma \gamma \right|^2}{4\gamma^2} W + \langle \frac{\nabla \gamma}{2\gamma}, N \rangle^2 W + \langle AY^T, \frac{\nabla \Sigma \gamma}{2\gamma} \rangle W^2
\]

\[ + \gamma |\nabla u|^2 \langle \frac{\nabla \gamma}{2\gamma^2}, N \rangle^2 W - \langle AY^T, Y^T \rangle \langle \frac{\nabla \gamma}{\gamma}, N \rangle W^3 + \langle AY^T, AY^T \rangle W^3. \]

Now we compute

\[ W_{ij} = \left( \frac{\gamma_i}{2\gamma} - \frac{\gamma_j}{2\gamma} \right) W + \frac{\gamma_i}{2\gamma} W_j + N_{ij} W^3 \langle \frac{\nabla \gamma}{2\gamma^2}, N \rangle + 3N_i W^2 W_j \langle \frac{\nabla \gamma}{2\gamma^2}, N \rangle \]

\[ + N_i W^3 \left( \langle \nabla X, \frac{\nabla \gamma}{2\gamma^2}, N \rangle - \langle \frac{\nabla \gamma}{2\gamma^2}, AX_\nu \frac{\partial}{\partial x^j} \rangle \right) - W_j N^i b_{it} - WN_{ij} b_{it} - WN_{ij} b_{it,j}. \]

However we have

\[ g^{ij} \frac{\gamma_i}{2\gamma} W_j = \frac{\left| \nabla \Sigma \gamma \right|^2}{4\gamma^2} W + \langle \frac{\nabla \gamma}{2\gamma}, N \rangle^2 W + W^2 \langle AY^T, \frac{\nabla \Sigma \gamma}{2\gamma} \rangle \]

and

\[ g^{ij} N_{ij} = g^{ij} \sigma_{ik} N_{jk} = -\delta_j^i - N_j N_k (a_k^i - N_j \frac{\gamma_k}{2\gamma}) \]

\[ = -nH + \frac{\gamma}{W^2} \langle N, \frac{\nabla \gamma}{2\gamma} \rangle + \gamma \langle AY^T, Y^T \rangle. \]

Moreover we compute

\[ g^{ij} N_i W_j = \frac{\gamma}{W} \langle N, \frac{\nabla \gamma}{2\gamma} \rangle + \gamma \frac{|\nabla u|^2}{W} \langle \frac{\nabla \gamma}{2\gamma^2}, N \rangle - \gamma W \langle AY^T, Y^T \rangle \]

and

\[ g^{ij} N_i W^3 \left( \langle \nabla X, \frac{\nabla \gamma}{2\gamma^2}, N \rangle - \langle \frac{\nabla \gamma}{2\gamma^2}, AX_\nu \frac{\partial}{\partial x^j} \rangle \right) \]

\[ = \gamma W \left( \langle \nabla N - WY, \frac{\nabla \gamma}{2\gamma^2}, N \rangle - \langle A \frac{\nabla \Sigma \gamma}{2\gamma^2}, -WY \rangle \right) \]

\[ = \gamma W \langle \nabla N, \frac{\nabla \gamma}{2\gamma^2}, N \rangle + W \frac{|\nabla \gamma|^2}{4\gamma^2} + \gamma W^2 \langle A \frac{\nabla \Sigma \gamma}{2\gamma^2}, Y^T \rangle. \]

We also have

\[ 2W g^{ij} W_j \langle AY^T, X_\nu \frac{\partial}{\partial x^i} \rangle = 2W^2 \langle AY^T, \frac{\nabla \Sigma \gamma}{2\gamma} \rangle - W^3 \langle \frac{\nabla \gamma}{\gamma}, N \rangle \langle AY^T, Y^T \rangle \]

\[ + 2W^3 \langle AY^T, AY^T \rangle. \]

Now we compute

\[ g^{ij} WN^i b_{it,j} = WN^i g^{ij} \nabla \Sigma \gamma a_{it} + \frac{1}{\gamma} W^2 g^{ij} a_{ij} a_{im} N^i u_m + \frac{1}{\gamma} W^2 g^{ij} a_{ij} N^i a_{im} u_m \]

\[ + W g^{ij} \frac{u_{it} u_{jm}}{2\gamma^2} a_{im} \gamma^m. \]
Hence we have

\[ g^{ij} W N^l b_{it,j} = WN^l (n \nabla_l^\Sigma H + g^{ij} \langle R(X_s \frac{\partial \Sigma}{\partial x^i}, X_s \frac{\partial}{\partial x^j}) N, X_s \frac{\partial}{\partial x^l} \rangle) \]

\[ + nHW^2 \langle AY^T, N^k X_s \frac{\partial}{\partial x^k} \rangle \]

\[ + W^2 g^{ij} \langle AY^T, X_s \frac{\partial}{\partial x^j} \rangle (-W \langle AY^T, X_s \frac{\partial}{\partial x^i} \rangle) \]

\[ - |\nabla u|^2 \langle AY^T, X_s \frac{\nabla}{\gamma} \rangle + \frac{|\nabla u|^2}{2\gamma} W (-W \langle AY^T, X_s \frac{\partial}{\partial x^m} \rangle) \gamma^m. \]

Therefore

\[ g^{ij} W N^l b_{it;j} = nWN^l \nabla_l^\Sigma H - nHW^3 \langle AY^T, Y^T \rangle - W^3 \langle AY^T, AY^T \rangle \]

\[ - |\nabla u|^2 \langle AY^T, X_s \frac{\nabla}{\gamma} \rangle. \]

Moreover

\[ g^{ij} W_j N^l b_{il} = -W^2 \langle AY^T, \frac{\nabla_l^\Sigma}{2\gamma} \rangle + W^3 \langle \frac{\nabla}{2\gamma}, N \rangle \langle AY^T, Y^T \rangle - W^3 \langle AY^T, AY^T \rangle \]

and

\[ W g^{ij} N_{j}^l b_{il} = -W g^{ij} (a_j^l - N_j \frac{\gamma}{2\gamma}) a_{il} = -|A|^2 W - \frac{1}{2} \langle AY^T, X_s \nabla \gamma \rangle. \]

We conclude that

\[ g^{ij} W_{i,j} = |A|^2 W + 2W^3 \langle AY^T, AY^T \rangle + (nH - 3\langle \frac{\nabla}{2\gamma}, N \rangle) W^3 \langle AY^T, Y^T \rangle \]

\[ + 3W^2 \langle AY^T, \frac{\nabla_l^\Sigma}{2\gamma} \rangle + |\nabla u|^2 \langle AY^T, X_s \frac{\nabla}{\gamma} \rangle + \frac{1}{2} \langle AY^T, X_s \nabla \gamma \rangle \]

\[ + g^{ij} \frac{\gamma_x}{2\gamma} W - \frac{|\nabla^2_l}{4\gamma^2} W + \frac{|\nabla^2}{4\gamma^2} W + (5W + 3W |\nabla u|^2) \langle \frac{\nabla}{2\gamma}, N \rangle^2 \]

\[ - nHW^3 \langle \frac{\nabla}{2\gamma}, N \rangle + \gamma W \langle \nabla N \frac{\nabla}{2\gamma}, N \rangle - nWN^l \nabla_l^\Sigma H. \]

Now

\[ \langle \nabla, \nabla \rangle = \frac{|\nabla|^2}{2\gamma} W + \frac{1}{2\gamma^2} \langle \nabla, N \rangle^2 W^3 + W^2 \langle AY^T, X_s \nabla \gamma \rangle. \]
Hence

\[ L W - \frac{2}{W} g^{ij} W_i W_j = |A|^2 W + (n H + \left< \frac{\nabla \gamma}{2 \gamma}, N \right>) W^2 \left< A Y^T, Y^T \right> \]
\[- W^2 \left< A Y^T, \frac{\nabla \Sigma \gamma}{2 \gamma} \right> + |\nabla u|^2 \left< A Y^T, X_* \frac{\nabla \gamma}{\gamma} \right> + \frac{1}{2} \left< A Y^T, X_* \nabla \gamma \right> \]
\[- \left( \frac{1}{2 \gamma} + \frac{1}{2 W^2} \right) W^2 \left< A Y^T, X_* \nabla \gamma \right> \]
\[+ g^{ij} \frac{\gamma_{ij}}{2 \gamma} W - \frac{3}{4} \frac{|\nabla \Sigma \gamma|^2}{4 \gamma^2} W + \frac{|\nabla \gamma|^2}{4 \gamma^2} W + (5W + 3 \frac{W}{\gamma} |\nabla u|^2) \left< \frac{\nabla \gamma}{2 \gamma}, N \right>^2 \]
\[- n H W^3 \left< \frac{\nabla \gamma}{2 \gamma}, N \right> + \gamma W \left< \nabla N \frac{\nabla \gamma}{2 \gamma}, N \right> - n W N^I \nabla^I H \]
\[- \left( \frac{1}{2 \gamma} + \frac{1}{2 W^2} \right) \left( \frac{|\nabla \gamma|^2}{2 \gamma} W + \frac{1}{2 \gamma^2} \left< \nabla \gamma, N \right|^2 W^3 \right) - \frac{1}{\gamma^2} \left< \nabla \gamma, N \right|^2 W \]
\[- 2 \gamma |\nabla u|^2 \left< \frac{\nabla \gamma}{2 \gamma}, N \right|^2 W - W_t. \]

However

\[ 3 \frac{W}{\gamma} |\nabla u|^2 \left< \frac{\nabla \gamma}{2 \gamma}, N \right|^2 W - 2 \gamma |\nabla u|^2 \left< \frac{\nabla \gamma}{2 \gamma}, N \right|^2 W = \frac{W}{\gamma} |\nabla u|^2 \left< \frac{\nabla \gamma}{2 \gamma}, N \right|^2, \]

\[ 5W \left< \frac{\nabla \gamma}{2 \gamma}, N \right|^2 W - \frac{1}{\gamma^2} \left< \nabla \gamma, N \right|^2 W = \left< \frac{\nabla \gamma}{2 \gamma}, N \right|^2 W \]

and

\[ - W^2 \left< A Y^T, \frac{\nabla \Sigma \gamma}{2 \gamma} \right> + |\nabla u|^2 \left< A Y^T, X_* \frac{\nabla \gamma}{\gamma} \right> + \frac{1}{2} \left< A Y^T, X_* \nabla \gamma \right> \]
\[- \left( \frac{1}{2 \gamma} + \frac{1}{2 W^2} \right) W^2 \left< A Y^T, X_* \nabla \gamma \right> \]
\[= -3 \gamma \left< A Y^T, X_* \frac{\nabla \gamma}{2 \gamma} \right> - W^3 \left< \frac{\nabla \gamma}{2 \gamma}, N \right> \left< A Y^T, Y^T \right>. \]

Moreover we compute

\[ \left( \frac{1}{2 \gamma} + \frac{1}{2 W^2} \right) \left( \frac{|\nabla \gamma|^2}{2 \gamma} W + \frac{1}{2 \gamma^2} \left< \nabla \gamma, N \right|^2 W^3 \right) \]
\[= \frac{|\nabla \gamma|^2}{4 \gamma^2} W + \gamma W^3 \left< \frac{\nabla \gamma}{2 \gamma}, N \right|^2 + \frac{|\nabla \gamma|^2}{4 \gamma} \frac{1}{W} + \left< \frac{\nabla \gamma}{2 \gamma}, N \right|^2 W \]

and

\[ - n W N^I \nabla^I H = - n W \left< \nabla \Sigma H, N \right> = - W \left< \nabla \Sigma H, N \right>. \]
We conclude that
\[
LW - \frac{2}{W} g^{ij} W_i W_j = |A|^2 W + nHW^3 \langle AY^T, Y^T \rangle - 3\gamma \langle AY^T, X, \nabla \gamma \rangle \\
+ g^{ij} \gamma_{ij} W - \frac{3}{4} \frac{|
abla \Sigma|^2}{4\gamma^2} W + \frac{|
abla u|^2}{\gamma} \langle \frac{\nabla \gamma}{2\gamma}, N \rangle^2 W \\
-nHW^3 \langle \frac{\nabla \gamma}{2\gamma}, N \rangle + \gamma W \langle \nabla N, \frac{\nabla \gamma}{2\gamma}, N \rangle - W \langle \nabla \Sigma \mathcal{H}, N \rangle \\
- \frac{1}{\gamma} W^3 \langle \frac{\nabla \gamma}{2\gamma}, N \rangle^2 - \frac{|
abla \gamma|^2}{4\gamma} \frac{1}{W} - W_t.
\]

However
\[
\frac{|
abla u|^2}{\gamma} \langle \frac{\nabla \gamma}{2\gamma}, N \rangle^2 W - \frac{1}{\gamma} W^3 \langle \frac{\nabla \gamma}{2\gamma}, N \rangle^2 = -\langle \frac{\nabla \gamma}{2\gamma}, N \rangle^2 W
\]

and
\[
- \frac{3}{4} \frac{|
abla \Sigma|^2}{4\gamma^2} W - \langle \frac{\nabla \gamma}{2\gamma}, N \rangle^2 W = - \frac{3}{4} \frac{|
abla \gamma|^2}{4\gamma^2} W + \frac{3}{4} \langle \frac{\nabla \gamma}{2\gamma}, N \rangle^2 W - \langle \frac{\nabla \gamma}{2\gamma}, N \rangle^2 W
\]
\[
= - \frac{3}{4} \frac{|
abla \gamma|^2}{4\gamma^2} W - \frac{1}{4} \langle \frac{\nabla \gamma}{2\gamma}, N \rangle^2 W.
\]

Hence we obtain
\[
LW - \frac{2}{W} g^{ij} W_i W_j = |A|^2 W + nHW^3 \langle AY^T, Y^T \rangle - nHW^3 \langle \frac{\nabla \gamma}{2\gamma}, N \rangle \\
-3\gamma \langle AY^T, X, \frac{\nabla \gamma}{2\gamma} \rangle + g^{ij} \gamma_{ij} W - \frac{3}{4} \frac{|
abla \gamma|^2}{4\gamma^2} W - \frac{1}{4} \langle \frac{\nabla \gamma}{2\gamma}, N \rangle^2 W \\
+ \gamma W \langle \nabla N, \frac{\nabla \gamma}{2\gamma}, N \rangle - W \langle \nabla \Sigma \mathcal{H}, N \rangle - \frac{|
abla \gamma|^2}{4\gamma} \frac{1}{W} - W_t.
\]

This finishes the proof of the lemma. \(\square\)

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