The Coulomb Branch of Yang-Mills Theory from the Schrödinger Representation

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Abstract

The Coulomb branch of the potential between two static colored sources is calculated for the Yang-Mills theory using the electric Schrödinger representation.

I. INTRODUCTION

We shall study the \((3 + 1)\) dimensional Yang-Mills theory with static sources in the Electric Schrödinger representation. The static sources are placed in order to visualize the behavior of the non-Abelian field \[1\]. We aim to extract in an easy way the Coulomb branch of the potential of the non-Abelian \(SU(N)\) theory. The method adopted here is a generalization of the one used for the equivalent \((1 + 1)\)-dimensional case in \[2\] and \[3\]. The Coulomb branch can be derived also in the A-representation with the use of the Wilson loop, but the steps there are approximate and require assumptions. The method presented here is straightforward. It focuses on the kinetic part of the Hamiltonian, which will give, as its expectation value, the Coulomb potential between the static sources. Further study in this direction is needed to obtain the linear confining potential.

II. WAVE FUNCTIONAL AND ITS TRANSFORMATION

In \[4\] the functional

\[
\Psi[E] = \int \mathcal{D}u e^{-\frac{1}{4\pi} \int E \partial u u^{-1}} \Phi[E]
\]

is presented as a solution of the free Gauss’ law, where \(\Phi[E]\) is a gauge invariant functional of the electric field, \(E\). In the case we have two static sources (source - anti-source) placed at the points \(x_0\) and \(x_1\) the Gauss’ law becomes

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\[ G_a(x)\Psi[E] = \left( \partial_i E^i_a(x) - i f_{abc} E^i_b(x) \frac{\delta}{\delta E^i_c(x)} \right) \Psi[E] = \Psi[E] T_a \delta(x - x_0) - T_a \Psi[E] \delta(x - x_1) \]

(2)

and the wave functional satisfying it is

\[ \Psi[E] = \int \mathcal{D}u e^{-\frac{i}{\hbar} \int E^i \partial_i uu^{-1} u(x_1) u^{-1}(x_0)} \Phi[E] \]

(3)

There could be a constant matrix between \( u(x_1) \) and \( u^{-1}(x_0) \) and still satisfying (2), as \( u \) transforms like \( u \to U u \) under a \( U \in SU(N) \) transformation. Under the gauge transformation, \( E \to E^U \), this wave functional transforms as

\[ \Psi[E^U] = e^{-\frac{i}{\hbar} \int E^i \partial_i uu^{-1} u(x_1) \Phi[E^U] } \]

(4)

This is the generalization to the case with sources of the gauge transformation with the wave functional in the E-representation \([3][5]\).

### III. PROPERTIES OF THE WAVE FUNCTIONAL

As the gauge transform of \( E^i \) is \( E^{iU} = U E^i U^{-1} \), we can decompose the electric field as \( E^i = g K^i g^{-1}. \) \( g \in SU(N) \) and transforms as \( g \to U g, \) while \( K^i \) are matrices belonging in \( su(N) \) and they do not transform under \( U \). The algebraic components of \( K^i \) are given by \( K^i = \sum_{a=1}^{N^2-1} k^i_a T^a. \) The fixed direction of \( K^i \) in the \( SU(N) \) space makes the theory to reduce to an Abelian theory with respect to \( K^i. \) A specific direction of it can be chosen by taking the average vector \( \sum_i K^i \) to be along the Cartan sub-algebra.

With this decomposition the functional (3) becomes

\[ \Psi[E] = e^{-\frac{i}{\hbar} \int E^i \partial_i gg^{-1} g(x_1) \left[ \int \mathcal{D}u e^{-\frac{i}{\hbar} \int K^i \partial_i uu^{-1} u(x_1) u^{-1}(x_0)} \right] g^{-1}(x_0) \Phi[E] } . \]

(5)

We shall use the decomposition of the \( SU(N) \) group presented in [3] of the form \( u = h(\phi)\tilde{u}(\theta)h(\bar{\phi}) \), where the elements, \( h, \) belong in the Cartan subgroup. The parameterization in \( N - 1 \) angles \( \phi^k, \) \( N - 1 \) angles \( \tilde{\phi}^k \) and the rest \( (N - 1)^2 \) angles \( \theta, \) is such that

\[ \mathcal{J}_k u \equiv -i \frac{\partial}{\partial \phi^k} u = -T^k u \quad , \quad \mathcal{J}^R_k u \equiv -i \frac{\partial}{\partial \tilde{\phi}^k} u = u T^k , \]

(6)

where \( T^k \) are the \( N - 1 \) Cartan generators. They can be chosen to be diagonal, with \( f_k(s) \) their \( s \) element. With these “diagonalization” conditions the group integration measure \( \mathcal{D}u \) becomes \([\mathcal{D}\phi][\mathcal{D}\bar{\phi}][\mathcal{D}\theta] J(\theta) \), where \( J(\theta) \) is the Jacobian of the reparameterization and depends only on the angles, \( \theta. \) This allows to calculate relatively easy the integrations in relation (5), obtaining finally

\[ \Psi[E] = e^{-\frac{i}{\hbar} \int E^i \partial_i gg^{-1} \sum_{\rho} g(x_1) P^\rho g^{-1}(x_0)} \]

\[ \times \prod_{k=1}^{N-1} \int_{x} \delta \left( \partial_i k^i_k - f_k(\rho) \delta(x - x_0) + f_k(\rho) \delta(x - x_1) \right) \Phi[E] . \]

(7)
Finally, the following kinetic potential is obtained

$$V_{\text{kin}} = V_{\text{kin}}^0 + V_{\text{kin}}^1 - g^2C_2^{(N-1)} \frac{1}{4\pi} \frac{1}{|x_1 - x_0|}$$

where $V_{\text{kin}}^0$ and $V_{\text{kin}}^1$ are the Coulomb self-energies of the sources at the points $x_0$ and $x_1$, and $C_2^{(N-1)}$ is the quadratic Casimir operator restricted only on the Cartan components, i.e. $\sum_{k=1}^{N-1} T^k T^k = C_2^{(N-1)} 1 = C_2/(N + 1) 1$. A sum over the $N + 1$ directions the Cartan subalgebra could take will result to the complete Coulomb potential

$$V_{\text{kin}} = -g^2C_2 \frac{1}{4\pi} \frac{1}{|x_1 - x_0|}.$$
V. CONCLUSIONS

With straightforward steps we have calculated the Coulomb branch of the potential of two static non-Abelian sources. It is extracted from the kinetic term of the Hamiltonian which is proportional to $g^2$. The more interesting magnetic part is proportional to $1/g^2$ and is expected that after a proper regularization (see e.g. [1] [8] [9]) its calculation will produce the linear potential between the static sources.
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