On weak-strong uniqueness property for full compressible magnetohydrodynamics flows

Weiping Yan$^{1,2,}$

1 College of Mathematics, Jilin University, 2699 Qianjin Street, Changchun 130012, China
2 Beijing International Center for Mathematical Research, Peking University, 5 Yiheyuan Road, Haidian District, Beijing 100871, China

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Abstract: This paper is devoted to the study of the weak-strong uniqueness property for full compressible magnetohydrodynamic flows. The governing equations for magnetohydrodynamic flows are expressed by the full Navier–Stokes system for compressible fluids enhanced by forces due to the presence of the magnetic field as well as the gravity and an additional equation which describes the evolution of the magnetic field. Using the relative entropy inequality, we prove that a weak solution coincides with the strong solution, emanating from the same initial data, as long as the latter exists.

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1. Introduction and main results

This paper studies the weak-strong uniqueness property of viscous compressible magnetohydrodynamic flows

\[ \frac{\partial}{\partial t} \rho + \nabla \cdot (\rho u) = 0, \]
\[ \frac{\partial}{\partial t} (\rho u) + \nabla \cdot (\rho u u) + \nabla P(\rho, \theta) + \nabla \times J \times H = \nabla \cdot S + J \times H, \]
\[ \frac{\partial}{\partial t} (\rho s(\rho, \theta)) + \nabla \cdot (\rho s(\rho, \theta) u) + \nabla \cdot \sigma = 0, \]
\[ \frac{\partial}{\partial t} H - \nabla \times (u \times H) + \nabla \times (v \nabla \times H) = 0, \]

* E-mail: yan8441@126.com
where $u$ is the vector field, $\rho$ is the density, $\theta$ is the temperature, $J$ is the electronic current, $e(\rho, \theta)$ is the (specific) internal energy and $H$ is the magnetic field. The electronic current satisfies Ampère’s law

$$J = \nabla \times H,$$

whereas the Lorentz force is given by

$$J \times H = \text{div}_s \left( \frac{1}{\mu} H \otimes H - \frac{1}{2\mu} |H|^2 I \right),$$

with $\mu$ being a permeability constant of free space, which here is assumed to be $\mu = 1$ for simplicity of the presentation. The electronic current $J$, electric field $E$ and magnetic field $H$ are related through Ohm’s law

$$J = \sigma (E + u \times H). \quad (5)$$

The interaction described by the theory of magnetohydrodynamics, “collective effects”, is governed by Faraday’s law,

$$\partial_t H + \nabla \times E = 0, \quad \text{div}_s H = 0. \quad (6)$$

Taking into account (5) we are able to write (6) in the form

$$\partial_t H + \nabla \times (H \times u) + \nabla \times (\nu \nabla \times H) = 0, \quad \nu = 1/\sigma.$$

Motivated by several recent studies devoted to the scale analysis as well as numerical experiments related to the proposed model (see Klein et al. [15]), we suppose that the viscous stress $S$ is a linear function of the velocity gradient, i.e. is described by Newton’s law

$$S(\theta, \nabla u) = \mu(\theta) \left( \nabla_s u + \nabla^\perp \nabla \theta - \frac{2}{3} \text{div}_s uI \right) + \eta(\theta) \text{div}_s uI, \quad (7)$$

while $q$ is the heat flux satisfying Fourier’s law

$$q = -\kappa(\theta) \nabla \theta, \quad (8)$$

and $\sigma$ stands for the entropy production rate which is non-negative measure given by

$$\sigma \geq \frac{1}{\theta} \left( S(\theta, \nabla u) : \nabla_s u - \frac{q(\theta, \nabla \theta) \cdot \nabla \theta}{\theta} \right). \quad (9)$$

We supplement compressible magnetohydrodynamic flows (1)–(4) with the conservation boundary condition

$$u|_{\partial \Omega} = q \cdot n|_{\partial \Omega} = 0, \quad (10)$$

and

$$H|_{\partial \Omega} = 0. \quad (11)$$

The concept of weak solution in fluid dynamics was introduced by Leray [18] in the context of incompressible, linearly viscous fluids. Later the original ideas of Leray have been put into the elegant framework of generalized derivatives (distributions) and the associated abstract function spaces of Sobolev type (for example, see Ladyzhenskaya [17] and Temam [22]). Lions [19] extended the theory to the class of barotropic flows (see also [6]). One of meaningful compressible
flow models is the compressible magnetohydrodynamics (MHD). It is a combination of the compressible Navier–Stokes equations of fluid dynamics and Maxwell’s equations of electromagnetism. Ducomet and Feireisl [4] proved existence of global in time weak solutions to a multi-dimensional nonisentropic MHD system for gaseous stars coupled with the Poisson equation with all the viscosity coefficients and the pressure depending on temperature and density asymptotically, respectively. Hu and Wang [13] studied the global variational weak solution to the three-dimensional full magnetohydrodynamic equations with large data by an approximation scheme and a weak convergence method. Jiang et al. [14] obtained convergence towards the strong solution of the ideal incompressible MHD system in the periodic domains. Recently, Kwon and Trivisa [16] established the incompressible limits of weak solutions to the compressible magnetohydrodynamics flows (1)–(4) on both bounded and unbounded domains.

The physical properties of the magnetohydrodynamics flows are reflected through various constitutive relations which are expressed as typically non-linear functions relating the pressure $P = P(\rho, \theta)$, internal energy $e(\rho, \theta)$, specific entropy $s = s(\rho, \theta)$ to the macroscopic variables $\rho$, $u$, and $\theta$. According to the fundamental principles of thermodynamics, the specific internal energy $e$ is related to the pressure $P$ and specific entropy $s$ through Gibbs’ relation

$$\theta Ds(\rho, \theta) = De(\rho, \theta) + P(\rho, \theta) D(\rho^{-1}),$$

where $D$ denotes the differential with respect to the state variables $\rho$ and $\theta$.

Due to the lack of information resulting from the inequality sign in (9), we need to supplement the resulting system with the energy inequality,

$$\frac{d}{dt} \int_{\Omega} \left( \frac{1}{2} \rho |u|^2 + P(\rho, \theta) + \frac{1}{2} |H|^2 \right) dx + \int_{\Omega} \left( |\nabla u|^2 + v |\nabla \times H|^2 \right) dx \leq 0.$$

Thus the total energy $E$ is given by $E = \rho |u|^2 / 2 + P(\rho, \theta) + |H|^2 / 2$. Under these circumstances, it can be shown [8, Chapter 2] that any weak solution to (1) which is sufficiently smooth satisfies instead of (9) the standard relation

$$s = \frac{1}{\theta} \left( \mathcal{S}(\theta, \nabla, u) : \nabla, u - \frac{q(\theta, \nabla, \theta) \cdot \nabla, \theta}{\theta} \right).$$

The pressure $P = P(\rho, \theta)$ is here expressed as

$$P = P_F + P_R, \quad P_R = \frac{a}{3} \theta^\beta, \quad a > 0,$$

where $P_R$ denotes the radiation pressure. Moreover, we shall assume that $P_F = P_M + P_E$, where $P_M$ is the classical molecular pressure obeying Boyle’s law, while $P_E$ is the pressure of electron gas constituent behaving like a Fermi gas in the degenerate regime of high densities and/or low temperatures (see Eliezer et al. [5, Chapters 1 & 15]). Thus necessarily $P_F$ takes the form

$$P_F = \theta^{\alpha^2} \left( \frac{\rho}{\theta^{\alpha^2}} \right),$$

where $\rho \in C^1[0, \infty)$ satisfies

$$\rho(0) = 0, \quad \rho'(Z) > 0, \quad Z \geq 0.$$

In agreement with Gibbs’ relation (12), the internal energy can be taken as

$$e = e_F + e_R \quad \text{with} \quad e_R = a \frac{\theta^\beta}{\rho},$$

where $e_F = e_F(\rho, \theta)$, $P_F(\rho, \theta)$ are interrelated through the following equation of state:

$$P_F(\rho, \theta) = \frac{2}{3} \rho e_F(\rho, \theta).$$
We also need the thermodynamic stability hypothesis, i.e.

\[
\frac{\partial P(p_\theta)}{\partial \rho} > 0, \quad \frac{\partial e(p_\theta)}{\partial \theta} > 0, \quad \rho, \theta > 0.
\] (19)

The second inequality in thermodynamic stability hypothesis (19) gives

\[
0 < \frac{5p(Z)/3 - p'(Z)Z}{Z} < c, \quad Z > 0,
\] (20)

which implies that the function \( Z \mapsto p(Z)/Z^{3/3} \) is decreasing and we suppose that

\[
\lim_{Z \to \infty} \frac{p(Z)}{Z^{3/3}} = p_\infty > 0.
\] (21)

In accordance with (12) and (17) we set the entropy as

\[
s = s_F + s_R \quad \text{with} \quad s_F = S\left(\frac{\rho}{Z^{1/2}}\right), \quad s_R = \frac{4a}{3p} \theta^3.
\] (22)

Furthermore, by the third law of thermodynamics,

\[
S'(Z) = -\frac{3}{2} \frac{5p(Z)/3 - p'(Z)Z}{Z^2} < 0, \quad \lim_{Z \to \infty} S(Z) = 0.
\] (23)

We choose the transport coefficients in the form

\[
\mu(\theta) = \mu_0 + \mu_1 \theta, \quad \kappa(\theta) = \kappa_0 + \kappa_1 \theta^2 + \kappa_2 \theta^3, \quad \mu_0, \mu_1 > 0, \quad \kappa_0 > 0, \quad i = 0, 2, 3.
\] (24)

A fundamental test of admissibility of a class of weak solutions to a given evolutionary problem is the property of weak-strong uniqueness. More specifically, the weak solution must coincide with a (hypothetical) strong solution emanating from the same initial data as long as the latter exists. This problem has been intensively studied for the incompressible Navier–Stokes system, for example, see [2, 11, 21]. It is a bit more delicate in compressible cases. The weak-strong uniqueness of compressible barotropic Navier–Stokes system and isentropic compressible Navier–Stokes system was established in [7, 10] and [12], respectively. Germain [12] provides only a partial and conditional answer to the weak-strong uniqueness problem for the compressible Navier–Stokes equations. The full solution is given in [7]. More recently, Feireisl and Novotný [9] extended the problem to compressible Navier–Stokes–Fourier system by the relative entropy inequality. The relative entropy in [9] is reminiscent to Dafermos [3] (who introduced the relatives entropies via the entropy flux pairs for the conservation laws), but is different from the Dafermos concept (in contrast to [3], it is based on the thermodynamic stability conditions).

Inspired by the work of Feireisl and Novotný [9], we prove weak-strong uniqueness of compressible three-dimensional magnetohydrodynamic equations. Our contribution is to construct suitable relative entropy inequality to (1)–(3). Then we overcome the presence of the magnetic field and its interaction with the hydrodynamic motion in the MHD flow of large oscillation.

We organize the rest of this paper as follows. In Section 2, we recall the definition of weak and strong solutions to the magnetohydrodynamic flows on bounded domains. Next, the relative entropy inequality to (1)–(4) is derived. In the last section, we give the rigorous proof of the weak-strong uniqueness property for the compressible magnetohydrodynamic flows on bounded domains in the spirit of Feireisl and Novotný [9].
2. Relative entropy and main result

Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. We recall definitions of weak and strong solutions to (1)–(4).

**Definition 2.1.**

We say that $(\rho, u, \theta, H)$ is a weak solution to the full magnetohydrodynamic flows (MHD) (1)–(4) supplemented with the initial data $(\rho_0, u_0, s(\rho_0, \theta_0), H_0)$, $\rho_0 \geq 0$, $\theta_0 > 0$, provided the following holds:

i) The density $\rho$ is a non-negative function, $\rho \in C_{\text{weak}}\left([0, T]; L^2(\Omega)\right)$, the velocity field $u \in L^2\left([0, T]; W^{1,2}_0(\Omega; \mathbb{R}^3)\right)$, $\rho u \in C_{\text{weak}}\left([0, T]; L^6(\Omega; \mathbb{R}^3)\right)$. Equation (1) is replaced by a family of integral identities

$$\int_\Omega \rho(\tau, \cdot) \varphi(\tau, \cdot) \, dx - \int_\Omega \rho_0 \varphi(0, \cdot) \, dx = \int_0^T \int_\Omega (\rho \varphi_t + \rho u \cdot \nabla \varphi) \, dx \, dt$$

(25)

for any $\varphi \in C^1([0, T] \times \overline{\Omega})$ and any $\tau \in [0, T]$.

ii) The balance of momentum holds in distributional sense, namely

$$\int_\Omega \rho u(\tau, \cdot) \cdot \varphi(\tau, \cdot) \, dx - \int_\Omega \rho u_0 \cdot \varphi(0, \cdot) \, dx = \int_0^T \int_\Omega \left( \rho u \cdot \varphi_t + \rho u \, \nabla \varphi + P \, \text{div} \, \varphi - S : \nabla \varphi + [(\nabla \times H) \times H] : \varphi \right) \, dx \, dt$$

(26)

for any $\varphi \in C^1([0, T] \times \overline{\Omega}; \mathbb{R}^3)$, $\varphi|_{\partial \Omega} = 0$ and any $\tau \in [0, T]$.

iii) The entropy balance (3) and (9) are replaced by a family of integral inequalities

$$\int_\Omega \rho s(\rho_0, \theta_0) \varphi(0, \cdot) \, dx - \int_\Omega \rho s(\rho, \theta)(\tau, \cdot) \varphi(\tau, \cdot) \, dx + \int_0^T \int_\Omega \frac{\varphi}{\theta} \left( S : \nabla \theta u - \frac{q}{\theta} \frac{\nabla \theta \cdot \nabla \varphi}{\theta} \right) \, dx \, dt$$

$$\leq - \int_0^T \int_\Omega \left( \rho s(\rho, \theta) \partial_t \varphi + \rho s(\rho, \theta) u \cdot \nabla \varphi + \frac{q}{\theta} \nabla \varphi \right) \, dx \, dt$$

(27)

for any $\varphi \in C^1([0, T] \times \overline{\Omega})$, $\varphi \geq 0$ and almost all $\tau \in [0, T]$. Here the quantities $S$ and $q$ are given through the constitutive equations (7) and (8). Moreover, similarly to the above, all quantities must be at least integrable on $(0, T) \times \Omega$. In particular, $\theta$ belongs to $L^\infty\left([0, T]; L^1(\Omega)\right)$ and $L^2\left([0, T]; W^{1,2}(\Omega)\right)$. In addition, we require $\theta$ to be positive for almost all $(t, x) \in (0, T) \times \Omega$.

iv) The total energy of the system satisfies the following inequality for almost all $\tau \in [0, T]$:

$$\int_\Omega \left( \frac{1}{2} \rho |u|^2 + \rho e(\rho, \theta) + \frac{1}{2} |H|^2 \right) \, dx + \int_0^T \int_\Omega \left( |\nabla u|^2 + v |\nabla \times H|^2 \right) \, dx \, dt$$

$$\leq \int_\Omega \left( \frac{1}{2} \rho_0 |u_0|^2 + \rho_0 e(\rho_0, \theta_0) + \frac{1}{2} |H_0|^2 \right) \, dx.$$  

(28)

v) The magnetic field $H \in L^2\left([0, T]; W^{1,2}(\Omega; \mathbb{R}^3)\right)$. The Maxwell equation (4) verifies

$$\int_\Omega \mathbf{H}(\tau, \cdot) \varphi(\tau, \cdot) \, dx - \int_\Omega \mathbf{H}_0 \varphi_0 \, dx = \int_0^T \int_\Omega \left( \mathbf{H} \, \partial_t \varphi - (\mathbf{H} \times \nabla \times \mathbf{u}) \cdot (\nabla \times \varphi) \right) \, dx \, dt,$$

(29)

where $\varphi \in C^1([0, T] \times \overline{\Omega}; \mathbb{R}^3)$, $\varphi|_{\partial \Omega} = 0$ and any $\tau \in [0, T]$. 

2009
Definition 2.2.
We say that \((\rho', u', \theta', H')\) is a classical (strong) solution to the full magnetohydrodynamic system (1)–(4) in \((0, T) \times \Omega\) if

\[
\begin{align*}
\rho' &\in C^1([0, T] \times \Omega), \\
\theta', \partial_t \theta', \nabla^2 \theta' &\in C([0, T] \times \Omega), \\
\nabla \cdot \rho' &\in C([0, T] \times \Omega; \mathbb{R}^3), \\
\rho' &\geq 0 \\
\theta' &\geq 0
\end{align*}
\]

\hspace{1cm} (30)

and \(\rho', u', \theta', H'\) satisfy equations (1)–(4), (13), together with the boundary conditions (10)–(11). Observe that hypothesis (30) implies the following regularity properties of the initial data:

\[
\begin{align*}
\rho(0) &= \rho_0 \in C^1(\overline{\Omega}), \\
\rho_0 &\geq 0 \\
\theta(0) &= \theta_0 \in C^2(\overline{\Omega}), \\
\theta_0 &\geq 0 \\
H(0) &= H_0 \in C^2(\overline{\Omega}).
\end{align*}
\]

Before presenting the main result, we deduce a relative entropy inequality which is satisfied by any weak solution to the full magnetohydrodynamic system (1)–(4).

Let \(\{A, B, C, D\}\) be a quantity of smooth function, \(A\) and \(C\) bounded below away from zero in \([0, T] \times \Omega\), and \(B|_{\partial \Omega} = D|_{\partial \Omega} = 0\). Moreover, assume that smooth functions \(B\) and \(D\) satisfy

\[
\partial_t D - \nabla \cdot (B \times D) + \nabla \times (\nu \nabla \times D) = 0.
\]

(32)

Taking \(\varphi = |B|^2/2, \varphi = B\) and \(\varphi = C > 0\) as a test function in (25), (26) and the entropy inequality (27), respectively, we get

\[
\begin{align*}
\int_0^T \frac{1}{2} \rho |B|^2(t, \cdot) dt - \int_0^T \frac{1}{2} \rho_0 |B|^2(0, \cdot) dt &= \int_0^T \int_\Omega (\rho B \cdot \partial_t B + \rho u \cdot \nabla B \cdot B) dx dt, \\
\int_0^T \int_\Omega \rho u \cdot B(t, \cdot) dt - \int_0^T \int_\Omega \rho_0 u_0 \cdot B(0, \cdot) dt &= \int_0^T \int_\Omega \left( \rho u \cdot \partial_t B + \rho u \cdot \nabla B + P(\rho, \theta) \right) \delta B, \\
\int_0^T \rho s(\rho, \theta) C(t, \cdot) dt - \int_0^T \rho s(\rho_0, \theta_0) C(0, \cdot) dt &\leq \int_0^T \int_\Omega \left( \frac{C}{\theta} \left( S(\theta, \nabla u) \cdot \nabla u - \frac{q(\theta, \nabla \theta) \cdot \nabla \theta}{\theta} \right) \right) dx dt.
\end{align*}
\]

(33)

(34)

(35)

It follows from (33), (34) and the energy inequality (28) that

\[
\begin{align*}
\int_0^T \left( \frac{1}{2} \rho |u - B|^2 + \rho e(\rho, \theta) + \frac{1}{2} |H|^2 \right) (t, \cdot) dt &+ \int_0^T \int_\Omega \left( |\nabla u|^2 + \nu |\nabla H|^2 \right) dx dt \\
&\leq \int_0^T \left( \frac{1}{2} \rho_0 |u_0 - B(0, \cdot)|^2 + \rho_0 e(\rho_0, \theta_0) + \frac{1}{2} |H_0|^2 \right) dt \\
&+ \int_0^T \int_\Omega \left( \left( \rho \partial_t B + \rho u \cdot \nabla B \right) \cdot (B - u) - P(\rho, \theta) \delta B + S(\theta, \nabla u) \cdot \nabla B - (\nabla \times H) \times H \cdot B \right) dx dt.
\end{align*}
\]

(36)

Then summing up (35) and (36), we deduce

\[
\begin{align*}
\int_0^T \left( \frac{1}{2} \rho |u - B|^2 + \rho e(\rho, \theta) + \frac{1}{2} |H|^2 - C \rho s(\rho, \theta) \right) (t, \cdot) dt &+ \int_0^T \int_\Omega \left( |\nabla u|^2 + \nu |\nabla H|^2 \right) dx dt \\
&\leq \int_0^T \left( \frac{1}{2} \rho_0 |u_0 - B(0, \cdot)|^2 + \rho_0 e(\rho_0, \theta_0) + \frac{1}{2} |H_0|^2 + C(0, \cdot) \rho_0 s(\rho_0, \theta_0) \right) dt \\
&+ \int_0^T \int_\Omega \left( \left( \rho \partial_t B + \rho u \cdot \nabla B \right) \cdot (B - u) - P(\rho, \theta) \delta B + S(\theta, \nabla u) \cdot \nabla B - (\nabla \times H) \times H \cdot B \right) dx dt.
\end{align*}
\]

(37)
Taking a test function $\varphi = D$ in (29) and $\varphi = \partial_\rho H_C(A, C)$ in (25), we have
\[
\int_{\Omega} H(\tau, \cdot) D(\tau, \cdot) \, dx - \int_{\Omega} H_0 D_0 \, dx = \int_{\Omega} \int \left[ \mathbf{H} \cdot \partial_\tau D - (\mathbf{H} \times \mathbf{u} + \nabla \times \mathbf{H}) \cdot (\nabla \times D) \right] \, dx \, dt,
\]
(38)
\[
\int_{\Omega} \rho \partial_\rho H_C(A, C) \tau \, dx - \int_{\Omega} \rho_0 \partial_\rho H_{\rho \mathbf{B}}(A, C) \left\{ A(0, \cdot), C(0, \cdot) \right\} \, dx
= \int_{\Omega} \int \left[ \rho \partial_\tau (\partial_\rho H_C(A, C)) + \rho \mathbf{u} \cdot \nabla_s (\partial_\rho H_C(A, C)) \right] \, dx \, dt.
\]
(39)

Multiplying (32) by $D$ and integrating over $(0, \tau) \times \Omega$, we get
\[
\int_{\Omega} \frac{1}{2} |D|^2(\tau, \cdot) \, dx - \int_{\Omega} \frac{1}{2} |D_0|^2 \, dx = \int_{\Omega} \int \left( (D \times B + \nabla \times D) \cdot (\nabla \times D) \right) \, dx \, dt.
\]
(40)

So by (37)–(40), we have
\[
\int_{\Omega} \left\{ \frac{1}{2} \rho |\mathbf{u} - B|^2 + \frac{1}{2} |\mathbf{H} - D|^2 + H_C(\rho, \theta) - \partial_\rho (H_C)(A, C)(\rho - A) - H_C(A, C) \right\} \, dx \, dt
+ \int_{\Omega} \int \left[ |\nabla u|^2 + \nu (|\nabla \times D|^2 + |\nabla \times D| |\nabla \times H| + |\nabla \times H|^2) \right] \, dx \, dt
+ \int_{\Omega} \int \frac{C}{\nu} \left( S(\theta, \nabla, \mathbf{u}) : \nabla \mathbf{u} - \frac{q(\theta, \nabla, \theta) \cdot \nabla \theta}{\nu} \right) \, dx \, dt
\leq \int_{\Omega} \left\{ \frac{1}{2} \rho_0 |\mathbf{u}_0 - B(0, \cdot)|^2 + \frac{1}{2} |H_0 - D_0|^2 + \left[ H_{\rho \mathbf{B}}(0, \cdot) \right] \right\} (\rho(0, \cdot), \theta(0, \cdot))
- \partial_\rho (H_{\rho \mathbf{B}})(A(0, \cdot), C(0, \cdot))(\rho_0 - A(0, \cdot)) - H_{\rho \mathbf{B}}(A(0, \cdot), C(0, \cdot)) \right\} \, dx \, dt
+ \int_{\Omega} \int \left[ (\rho \partial_\rho \mathbf{B} + \rho \mathbf{u} \cdot \nabla \mathbf{B}) \cdot (B - u) - P(\rho, \theta) \nabla \mathbf{u} + S(\theta, \nabla, \mathbf{u}) : \nabla \mathbf{B} \right] \, dx \, dt
+ \int_{\Omega} \int \left[ -\mathbf{H} \cdot \partial_\tau D - ((\nabla \times \mathbf{H}) \times \mathbf{H}) \cdot B - (D \times B) \cdot (\nabla \times D) + (\mathbf{H} \times \mathbf{u}) \cdot (\nabla \times D) \right] \, dx \, dt
- \int_{\Omega} \int \left[ \rho s(\rho, \theta) \partial_\tau \mathbf{C} + \rho s(\rho, \theta) \mathbf{u} \cdot \nabla_s \mathbf{C} + \frac{q(\theta, \nabla, \theta) \cdot \nabla \theta}{\nu} \right] \, dx \, dt
- \int_{\Omega} \int \left[ \rho \partial_\tau (\partial_\rho H_C(A, C)) + \rho \mathbf{u} \cdot \nabla_s (\partial_\rho H_C(A, C)) \right] \, dx \, dt
+ \int_{\Omega} \int \partial_\tau \left\{ A \partial_\rho (H_C)(A, C) - H_C(A, C) \right\} \, dx \, dt.
\]
(41)

Replacing $\partial_\tau D$ by (32) in (41) we get
\[
\int_{\Omega} \left\{ \frac{1}{2} \rho |\mathbf{u} - B|^2 + \frac{1}{2} |\mathbf{H} - D|^2 + H_C(\rho, \theta) - \partial_\rho (H_C)(A, C)(\rho - A) - H_C(A, C) \right\} \, dx \, dt
+ \int_{\Omega} \int \left[ |\nabla u|^2 + \nu (|\nabla \times D|^2 + |\nabla \times D| |\nabla \times H| + |\nabla \times H|^2) \right] \, dx \, dt
+ \int_{\Omega} \int \frac{C}{\nu} \left( S(\theta, \nabla, \mathbf{u}) : \nabla \mathbf{u} - \frac{q(\theta, \nabla, \theta) \cdot \nabla \theta}{\nu} \right) \, dx \, dt
\leq \int_{\Omega} \left\{ \frac{1}{2} \rho_0 |\mathbf{u}_0 - B(0, \cdot)|^2 + \frac{1}{2} |H_0 - D_0|^2 + \left[ H_{\rho \mathbf{B}}(0, \cdot) \right] \right\} (\rho(0, \cdot), \theta(0, \cdot))
- \partial_\rho (H_{\rho \mathbf{B}})(A(0, \cdot), C(0, \cdot))(\rho_0 - A(0, \cdot)) - H_{\rho \mathbf{B}}(A(0, \cdot), C(0, \cdot)) \right\} \, dx \, dt
+ \int_{\Omega} \int \left[ (\rho \partial_\rho \mathbf{B} + \rho \mathbf{u} \cdot \nabla \mathbf{B}) \cdot (B - u) - P(\rho, \theta) \nabla \mathbf{u} + S(\theta, \nabla, \mathbf{u}) : \nabla \mathbf{B} \right] \, dx \, dt
+ \int_{\Omega} \int \left[ -((\nabla \times \mathbf{H}) \times \mathbf{H}) \cdot B - (D \times B) \cdot (\nabla \times D) + (\mathbf{H} \times \mathbf{u}) \cdot (\nabla \times D) \right] \, dx \, dt
- \int_{\Omega} \int \left[ \rho s(\rho, \theta) \partial_\tau \mathbf{C} + \rho s(\rho, \theta) \mathbf{u} \cdot \nabla_s \mathbf{C} + \frac{q(\theta, \nabla, \theta) \cdot \nabla \theta}{\nu} \right] \, dx \, dt
- \int_{\Omega} \int \left[ \rho \partial_\tau (\partial_\rho H_C(A, C)) + \rho \mathbf{u} \cdot \nabla_s (\partial_\rho H_C(A, C)) \right] \, dx \, dt
+ \int_{\Omega} \int \partial_\tau \left\{ A \partial_\rho (H_C)(A, C) - H_C(A, C) \right\} \, dx \, dt.
\]
(42)
Note that
\[
\int_\Omega (\nabla \times H) \cdot B \, dx = - \int_\Omega \left( H^T B + \frac{1}{2} \nabla (|H|^2) \cdot B \right) \, dx, \tag{43}
\]
\[
\int_\Omega (\nabla \times (B \times H)) \cdot B \, dx = \int_\Omega \left( H^T B \mathbf{H} + \frac{1}{2} \nabla (|H|^2) \cdot B \right) \, dx.
\]

Thus it follows from (42) and (44) that
\[
\int_0^T \int_\Omega \left[ \left( (\nabla \times H) \cdot B - (D \times B) \cdot (\nabla \times D) \right) + (H \times u) \cdot (\nabla \times D) \right] \, dx \, dt - \int_0^T \int_\Omega (\nabla \times (B \times D)) \cdot H \, dx \, dt
\]
\[
= \int_0^T \int_\Omega \left[ (H \cdot - \frac{1}{4} \nabla (\nabla \times H) \cdot B \right) \right] \, dx \, dt
\]
\[
= \int_0^T \int_\Omega \left( (H \cdot - \frac{1}{4} \nabla (\nabla \times H) \cdot B \right) \right] \, dx \, dt
\]
\[
+ \int_0^T \int_\Omega \left( (D \cdot - \frac{1}{4} \nabla (\nabla \times D) \cdot B \right) \right] \, dx \, dt
\]
\[
- \int_0^T \int_\Omega \left( (D \cdot - \frac{1}{4} \nabla (\nabla \times D) \cdot B \right) \right] \, dx \, dt.
\]

Note that
\[
\partial_y (\partial_y H_c(A, C)) = -s(A, C) \partial_y C - A \partial_y s(A, C) \partial_y C + \partial^2_p H_c(A, C) \partial_y \rho + \partial^2_{\rho, \theta} H_c(A, C) \partial_y \theta \quad \text{for} \quad y = t, x.
\]

Thus it follows from (42) and (44) that
\[
\int_0^T \int_\Omega \left[ \left( \frac{1}{2} \rho |u - B|^2 + \frac{1}{2} |H - D|^2 + H_c(\rho, \theta) - \partial_p H_c(A, C) (\rho - A) - H_c(A, C) \right) (\tau, \cdot) \right] \, dx
\]
\[
+ \int_0^T \int_\Omega \left[ (\nabla |u|^2 + \nu \nabla \cdot - \nabla \times H) \right] \, dx \, dt + \int_0^T \int_\Omega \left( S(\theta, \nabla, u) \cdot \nabla u - q(\theta, \nabla, \theta) \cdot \nabla, \theta \right) \, dx \, dt
\]
\[
\leq \int_0^T \left[ \left( \frac{1}{2} \rho_0 |u_0 - B(0, \cdot)|^2 + \frac{1}{2} |H_0 - D(0, \cdot)|^2 + H_{C, 0, 1}(\rho(0, \cdot), \theta(0, \cdot)) - \partial_p (H_{C, 0, 1}(A(0, \cdot), C(0, \cdot)) \right) \right] \, dx
\]
\[
+ \int_0^T \int_\Omega \left( \rho \partial_t B + \rho u \cdot \nabla B \right) \cdot (B - u) - P(\rho, \theta) \div B + S(\theta, \nabla, u) \cdot \nabla B \right] \, dx \, dt
\]
\[
+ \int_0^T \int_\Omega \left( (H \cdot - \frac{1}{4} \nabla (\nabla \times H) \cdot B \right) \right] \, dx \, dt
\]
\[
+ \int_0^T \int_\Omega \left( (D \cdot - \frac{1}{4} \nabla (\nabla \times D) \cdot B \right) \right] \, dx \, dt
\]
\[
- \int_0^T \int_\Omega \left( (D \cdot - \frac{1}{4} \nabla (\nabla \times D) \cdot B \right) \right] \, dx \, dt
\]
\[
- \int_0^T \int_\Omega \left( \rho (s(\rho, \theta) - s(A, C)) \partial_t C + \rho (s(\rho, \theta) - s(A, C)) u \cdot \nabla C + \frac{q(\theta, \nabla, \theta) \cdot \nabla C} \theta \right) \, dx \, dt
\]
\[
+ \int_0^T \int_\Omega \left( A \partial_t s(A, C) + r \partial_p s(A, C) u \cdot \nabla C \right) \, dx \, dt
\]
\[
- \int_0^T \int_\Omega \left( \partial_p (H_c(A, C)) \partial_t A + \partial_p (H_c(A, C)) \partial_t C \right) \, dx \, dt
\]
\[
- \int_0^T \int_\Omega \left( \rho u (\partial_p (H_c(A, C)) \nabla A + \partial_p (H_c(A, C)) \nabla C \right) \, dx \, dt
\]
\[
+ \int_0^T \int_\Omega \left( \partial_t (A \partial_p (H_c(A, C)) - H_c(A, C) \right) \, dx \, dt.
\]
Following [1, 9, 20], we introduce the quantity

\[ \Gamma(\rho, \theta|C, C) = H_C(\rho, \theta) - \partial_\rho H_C(A, C)(\rho - A) - H_C(A, C), \]

where \( H_C(\rho, \theta) = \rho e(\rho, \theta) - C \rho s(\rho, \theta). \) Note that

\[
\begin{aligned}
\partial_{\rho, \rho}^2 H_C(A, C) &= -\frac{1}{A} \partial_\rho P(A, C), \\
A \partial_\rho s(A, C) &= \frac{1}{C} \partial_\rho P(A, C), \\
A \partial_\rho(H_C(A, C) - H_C(A, C)) &= P(A, C), \\
\partial_\rho^2 H_C(A, C) &= \partial_\rho(\rho(\theta - C) \partial_\rho s)(A, C) = (\theta - C) \partial_\rho(\rho \partial_\rho s(\rho, \theta))(A, C) = 0.
\end{aligned}
\]

Therefore, we can obtain a kind of relative entropy inequality by simplifying (45) as

\[
\begin{aligned}
\int_\Omega \left( \frac{1}{2} \rho|u - B|^2 + \frac{1}{2} |H - D|^2 + \Gamma(\rho, \theta|A, C) \right) (\tau, \cdot) \, dx + \int_0^T \int_\Omega \left| \nabla \times D - \nabla \times H \right|^2 \, dx \, dt \\
&+ \int_0^T \int_\Omega \left( \frac{C}{\theta} \cdot \left( \nabla, \nabla u \right) : \nabla u - \frac{q(\theta, \nabla \theta) \cdot \nabla \theta}{\theta} \right) \, dx \, dt \\
&\leq \int_\Omega \left( \frac{1}{2} \rho|u_0 - B(0, \cdot)|^2 + \frac{1}{2} |H_0 - D_0(0, \cdot)|^2 + \Gamma(\rho_0, \theta_0|A(0, \cdot), C(0, \cdot)) \right) \, dx \\
&- \int_0^T \int_\Omega \left( \rho(u - B) \cdot \nabla B \cdot (B - u) \right) \, dx \, dt + \int_0^T \int_\Omega \left( \rho(s(\rho, \theta) - s(A, C))(B - u) \cdot \nabla C \right) \, dx \, dt \\
&+ \int_0^T \int_\Omega \left( \rho \partial_\rho(1, \cdot) \right) \, dx \, dt + \int_0^T \int_\Omega \left( (H - B) \cdot (D - B) \right) \, dx \, dt \\
&+ \int_0^T \int_\Omega \left( (D - B) \cdot (u - B) \right) \, dx \, dt \\
&- \int_0^T \int_\Omega \left( \rho(s(\rho, \theta) - s(A, C)) \partial_\tau C + \rho(s(\rho, \theta) - s(A, C)) u \cdot \nabla C + \frac{q(\theta, \nabla \theta) \cdot \nabla C}{\theta} \right) \, dx \, dt \\
&+ \int_0^T \int_\Omega \left( -\frac{P}{A} \right) \partial_\rho(1, \cdot) \, dx \, dt.
\end{aligned}
\]

Now we state the weak-strong uniqueness property to the full magnetohydrodynamic system (1)–(4) on a bounded Lipschitz domains with Dirichlet boundary conditions.

**Theorem 2.3.**

Let \( \Omega \subset \mathbb{R}^3 \) be a bounded Lipschitz domain and \((\rho, u, \theta, H)\) be a weak solution to the full magnetohydrodynamic system (1)–(4) in \((0, T) \times \Omega \) and \((\rho', u', \theta', H')\) be a strong solution emanating from the same initial data (31). Assume that the thermodynamic functions \( P, e, s \) satisfy hypotheses (14)–(23) and that the transport coefficients \( \mu, \eta \) and \( \kappa \) satisfy (24). Then \( \rho \equiv \rho', u = u', \theta = \theta', H = H' \).

3. Proof of Theorem 2.3

In this section, we apply the relative entropy inequality to finish the proof of Theorem 2.3. Assume that \((\rho', u', \theta', H')\) is a classical (strong) solution to the full magnetohydrodynamic system in \((0, T) \times \Omega \) and it satisfies

\[
\begin{aligned}
\rho'(0, \cdot) = \rho_0, \\
u'(0, \cdot) = u_0, \\
\theta'(0, \cdot) = \theta_0, \\
H'(0, \cdot) = H_0.
\end{aligned}
\]
Following [8, 9], we introduce essential and residual components of each quantity appearing in (46). Thermodynamic stability hypothesis (19) implies that \( \rho \mapsto H_\rho(\rho, \theta) \) is strictly convex, while \( \theta \mapsto H_\rho(\rho, \theta) \) attains its global minimum at \( \theta = \theta^* \). Thus

\[
\Gamma(\rho, \theta; \rho', \theta') \geq c \left( |\rho - \rho'|^2 + |\theta - \theta'|^2 \right) \quad \text{if} \quad \rho, \theta \in [\rho_0, \rho_1] \times [\theta_0, \theta_1],
\]

where \( [\rho', \theta'] \in [\rho_0, \rho_1] \times [\theta_0, \theta_1] \), the constant \( c \) depends on positive constants \( \rho_0, \rho_1, \theta_0, \theta_1 \) and structural properties of thermodynamic function \( e, s \). The restriction of positive constants \( \rho_0, \rho_1, \theta_0, \theta_1 \) can be found in [9].

Thus we can write each measurable function as

\[
h_{\text{ess}} = \begin{cases} h(t, x) & \text{if } (\rho, \theta) \in [\rho_0', \rho_1'] \times [\theta_0', \theta_1'], \\ 0 & \text{otherwise}. \end{cases}
\]

Take \( (A, B, C, D) = (\rho', u', \theta', H') \) in (47). Due to the fact that the initial data coincide, we have

\[
\begin{align*}
\int_0^T \left( \frac{1}{2} \rho |u-u'|^2 + \frac{1}{2} |H-H'|^2 + \Gamma(\rho, \theta; \rho', \theta') \right) (t, \cdot) \, dx + \nu \int_0^T \left| \nabla \times H' - \nabla \times H \right|^2 \, dx \, dt \\
+ \int_0^T \int_\Omega \frac{\partial}{\partial \theta} \left( S(\theta, \nabla u) : \nabla u - \frac{q(\theta, \nabla \theta)}{\theta} \cdot \nabla \theta \right) \, dx \, dt \\
\leq \int_\Omega \left( \frac{1}{2} \rho_0 |u_0-u'(0, \cdot)|^2 + \frac{1}{2} |H_0-H'_0(0, \cdot)|^2 + \Gamma(\rho_0, \theta_0; \rho'(0, \cdot), \theta'(0, \cdot)) \right) \, dx \\
+ \int_0^T \int_\Omega \rho |u-u'|^2 |\nabla u'| \, dx \, dt + \int_0^T \int_\Omega (s(\rho, \theta) - s(\rho, \theta)) (u'-u) \cdot \nabla \theta' \, dx \, dt \\
+ \int_0^T \int_\Omega \left( (H-H')^\top \nabla u'(H-H') + \frac{1}{2} \nabla |(H-H')|^2 \right) u' \, dx \, dt \\
+ \int_0^T \int_\Omega \left( (H'-H)^\top \nabla (u-u') H + \frac{1}{2} \nabla |(H'-H)(u-u')| \right) \, dx \, dt \\
- \int_0^T \int_\Omega \left( H^\top \nabla (u-u') H + \frac{1}{2} \nabla |(H-u'|^2) \right) \, dx \, dt \\
- \int_0^T \int_\Omega \left( \rho (s(\rho, \theta) - s(\rho', \theta')) \partial_\theta \theta' + \rho (s(\rho, \theta) - s(\rho', \theta')) u' \cdot \nabla \theta' + \frac{q(\theta, \nabla \theta)}{\theta} \cdot \nabla \theta' \right) \, dx \, dt \\
+ \int_0^T \int_\Omega \left( 1 - \frac{\rho}{\rho'} \right) \partial_\rho P(\rho', \theta') - \frac{\rho}{\rho'} u' \cdot \nabla P(\rho', \theta') \right) \, dx \, dt.
\end{align*}
\]

In what follows, we estimate the right-hand side of (49). It is easy to see that

\[
\int_\Omega \rho |u-u'|^2 |\nabla u'| \, dx \leq \| \nabla u' \|_{L^2(\Omega; \mathbb{R})}^2 \int_\Omega \rho |u-u'|^2 \, dx.
\]

By virtue of (48), using interpolation inequality, for any \( \epsilon > 0 \), we derive

\[
\begin{align*}
\int_\Omega \rho (s(\rho, \theta) - s(\rho, \theta')) (u'-u) \cdot \nabla \theta' \, dx & \leq 2 \rho_1' \| \nabla \theta' \|_{L^2(\Omega; \mathbb{R}^2)}^2 + c(\epsilon) \int_0^T \Gamma(\rho, \theta; \rho', \theta') \, dx \\
+ \| \nabla \theta' \|_{L^2(\Omega; \mathbb{R}^2)} \left( \epsilon \| u'-u \|_{L^2(\Omega; \mathbb{R}^2)}^2 + c(\epsilon) \| (\rho(s(\rho, \theta) - s(\rho', \theta')) \|_{L^2(\Omega; \mathbb{R})}^2 \right) \\
\int_\Omega \rho (s(\rho, \theta) - s(\rho, \theta')) (u'-u) \cdot \nabla \theta' \, dx & \leq \epsilon \| u'-u \|_{L^2(\Omega; \mathbb{R}^2)}^2 + c(\epsilon) \| (\rho(s(\rho, \theta) - s(\rho', \theta')) \|_{L^2(\Omega; \mathbb{R})}^2 + c(\epsilon) \| \nabla \theta' \|_{L^2(\Omega; \mathbb{R}^2)}^2.
\end{align*}
\]

It follows from (21)–(22) that

\[
\left[ (\rho(s(\rho, \theta) - s(\rho', \theta'))) \right]_{\text{ess}} \leq c \rho \log \theta + c(\epsilon) \rho \log \theta + \theta^3.
\]
Using (17), (20)–(21),
\[ \rho e(\rho, \theta) \geq c (\rho^{5/3} + \theta^4), \]  
and (27)–(30) imply
\[ t \mapsto \int_\Omega \Gamma(\rho, \theta | \rho', \theta') \, dx \in L^\infty(0, T). \]
By (48), (52)–(54) and the Hölder inequality,
\[ ||[\rho(s(\rho, \theta) - s(\rho', \theta'))]||_{L^2(\Omega)}^2 \leq c \left( \int_\Omega \Gamma(\rho, \theta | \rho', \theta') \right)^{5/3}. \]
So by (51), for any \( \epsilon > 0 \), we obtain
\[ \int_\Omega \rho(s(\rho, \theta) - s(\rho, \theta))(u' - u) \cdot \nabla \theta' \, dx \leq \epsilon \|u' - u\|_{W^{1,2}(\Omega)}^2 + c'(\epsilon, \cdot) \int_\Omega \Gamma(\rho, \theta | \rho', \theta') \, dx, \]
where \( c'(\epsilon, \cdot) \) is a generic constant depending on \( \epsilon, \rho', u' \) and \( \theta' \) through the norms induced by (30)–(31), while \( c'(\cdot) \) is independent of \( \epsilon \) but depends on \( \rho', u', \theta', \rho_0 \) and \( \theta_0 \), through the norms induced by (30)–(31). Similarly to estimate (55), we get
\[ \int_\Omega \rho^{-1}(\rho - \rho')(u' - u) \cdot \left[ \text{div} S(\theta', \nabla u) - \nabla P(\rho', \theta') + \nabla \times H' \times H' \right] \, dx \]
\[ = \int_\Omega \left[ \rho^{-1}(\rho - \rho')(u' - u) \cdot \left[ \text{div} S(\theta', \nabla u) - \nabla P(\rho', \theta') + \nabla \times H' \times H' \right] \right]_{\text{ess}} \, dx \]
\[ + \int_\Omega \left[ \rho^{-1}(\rho - \rho')(u' - u) \cdot \left[ \nabla P(\rho', \theta') - \nabla \times H' \times H' \right] \right]_{\text{ess}} \, dx \]
\[ \leq c'(\epsilon, \cdot) ||\rho^{-1}(\rho - \rho')||_{L^2(\Omega)} \|u' - u\|_{L^2(\Omega)}^2 + c'(\epsilon, \cdot) \left( ||\rho||_{L^6(\Omega)} + ||\theta||_{L^6(\Omega)}^2 \right)^2 + \epsilon \|u' - u\|_{L^2(\Omega)}^2. \]
So using integration by parts, in virtue of (48), (54), (56) and \( W^{1,2}(\Omega) \hookrightarrow L^6(\Omega) \), we derive
\[ \left| \int_\Omega \rho(\theta, u' \cdot \nabla u) \cdot (u' - u) \, dx \right| = \left| \int_\Omega \frac{\rho - \rho'}{\rho'} (u' - u) \cdot \left[ \text{div} S(\theta', \nabla u) - \nabla P(\rho', \theta') + \nabla \times H' \times H' \right] \, dx \right| \]
\[ \leq \int_\Omega \left| \frac{\rho - \rho'}{\rho'} (u' - u) \cdot \left[ \text{div} S(\theta', \nabla u) - \nabla P(\rho', \theta') + \nabla \times H' \times H' \right] \right| \, dx \]
\[ + \int_\Omega (u' - u) \cdot \left[ \text{div} S(\theta', \nabla u) - \nabla P(\rho', \theta') + \nabla \times H' \times H' \right] \, dx \]
\[ \leq c'(\epsilon, \cdot) ||\rho^{-1}(\rho - \rho')||_{L^2(\Omega)} \|u' - u\|_{L^2(\Omega)}^2 + c'(\epsilon, \cdot) \left( ||\rho||_{L^6(\Omega)} + ||\theta||_{L^6(\Omega)}^2 \right)^2 + \epsilon \|u' - u\|_{L^2(\Omega)}^2 \]
\[ + \left| \int_\Omega \left[ \text{div} \theta', \nabla u' \right] (u' - u) + P(\rho', \theta') \text{div} (u' - u) + (\nabla \times H' \times H') \cdot (u' - u) \, dx \right| \]
\[ \leq \int_\Omega \left[ \text{div} \theta', \nabla u' \right] (u' - u) + P(\rho', \theta') \text{div} (u' - u) + (\nabla \times H' \times H') \cdot (u' - u) \, dx \]
\[ + \epsilon \|u' - u\|_{W^{1,2}(\Omega)}^2 + c(\epsilon) \int_\Omega \Gamma(\rho, \theta | \rho', \theta') \, dx. \]
By the Hölder inequality and (10)–(11), we derive

\[\int_0^T \left( |H-H'|^2 \nabla u' \cdot (H-H') \right) \, dx \, dt \leq \frac{1}{2} \int_0^T \left( \| \nabla u' \|_{L^\infty(\Omega; R^3)} \right) \left( |H-H'|^2 \right) \, dx \, dt.
\]

(58)

\[\frac{1}{2} \int_0^T \nabla \left( (H-H')^2 \right) \cdot u' \, dx \, dt \leq \frac{1}{2} \int_0^T \left( \| \nabla u' \|_{L^\infty(\Omega; R^3)} \right) \left( |H-H'|^2 \right) \, dx \, dt,
\]

(59)

\[\int_0^T \left( (H^2-H') \nabla u' \left( u-u' \right) \right) \, dx \, dt \leq \int_0^T c_\epsilon \| H' \|_{L^\infty(\Omega; R^3)} \left( |H-H'|^2 \right) \, dx \, dt
\]

(60)

\[+ \int_0^T e \| H' \|_{L^\infty(\Omega; R^3)} \| u-u' \|_{W^{1,2}(\Omega; R^3)} \, dt.
\]

Thus from (58)–(61) and (43), we have

\[\int_0^T \left( (H-H')^2 \nabla u' \cdot (H-H') \right) \, dx \, dt
\]

\[+ \int_0^T \left( \left( H^2-H' \right) \nabla \left( u-u' \right) H' \right) \, dx \, dt
\]

\[+ 1 \int_0^T \left( \nabla \left( u-u' \right) H' \right) \, dx \, dt
\]

\[\leq \int_0^T \left( c'' \| u-u' \|_{W^{1,2}(\Omega; R^3)}^2 \right) \, dx \, dt + \int_0^T \left( \nabla \times H' \right) \cdot \left( u-u' \right) \, dx \, dt,
\]

(62)

where \( c'' = c''\left( \| H' \|_{L^\infty(\Omega; R^3)}^2 \right) \) and \( c_\epsilon = c_\epsilon \left( \| \nabla u' \|_{L^\infty(\Omega; R^3)}, \| H' \|_{L^\infty(\Omega; R^3)} \right) \) denote constants, \( \epsilon > 0 \) is sufficiently small.

In what follows we estimate the next term. Using the Taylor–Lagrange formula, we derive

\[\int_0^T \rho (s(\rho, \theta) - s(\rho', \theta')) \partial \theta' \, dx \leq \int_0^T \rho' \left[ \partial s(\rho', \theta')(\rho-\rho') + \partial s(\rho', \theta')(\theta-\theta') \right] \partial \theta' \, dx + 4c(\cdot) \int_0^T \Gamma (\rho, \theta | \rho', \theta') \, dx.
\]

(63)

Similarly with getting (63) and (51), respectively, we have

\[- \int_0^T \rho (s(\rho, \theta) - s(\rho', \theta')) u' \cdot \nabla \theta' \, dx \leq - \int_0^T \rho' \left[ \partial s(\rho', \theta')(\rho-\rho') + \partial s(\rho', \theta')(\theta-\theta') \right] u' \cdot \nabla \theta' \, dx
\]

\[+ c(\cdot) \int_0^T \Gamma (\rho, \theta | \rho', \theta') \, dx,
\]

\[\left| \int_0^T \frac{\rho-\rho'}{\rho} u \cdot P(\rho', \theta') \cdot (u-u') \, dx \right| \leq c \left( \| \nabla \rho' \|_{W^{1,2}(\Omega; R^3)} \right) \left( \epsilon \| u-u' \|_{W^{1,2}(\Omega; R^3)}^2 \right) \left( \int_0^T \Gamma (\rho, \theta | \rho', \theta') \, dx \right).
\]

(64)

Now we estimate the last term of right-hand side of (49). By (65)

\[\left| \int_0^T \left( \frac{\rho-\rho'}{\rho} \partial P(\rho', \theta') - \frac{\rho}{\rho} u \cdot \nabla P(\rho', \theta') \right) \, dx \right| \leq \int_0^T \frac{\rho-\rho'}{\rho} \left( \partial P(\rho', \theta') + u \cdot \nabla P(\rho', \theta') \right) \, dx
\]

\[+ \int_0^T P(\rho', \theta') \, dx + c \left( \| \nabla \rho' \|_{W^{1,2}(\Omega; R^3)} \right) \left( \epsilon \| u-u' \|_{W^{1,2}(\Omega; R^3)}^2 \right) \left( \int_0^T \Gamma (\rho, \theta | \rho', \theta') \, dx \right).
\]

(65)
Thus by (49)–(50), (57)–(63) and (65), we obtain the following relative entropy inequality:

\[
\int_0^T \left( \frac{1}{2} \rho |u - u'|^2 + \frac{1}{2} |H - H'|^2 + \Gamma(p, \theta | \rho', \theta') \right) (\tau, \cdot) \, dx + \nu \int_0^T \int_\Omega |\nabla \times H' - \nabla \times H|^2 \, dx \, dt \\
+ \int_0^T \int_\Omega \left( \frac{\theta'}{\theta} S(\theta, \nabla u) \cdot \nabla u - S(\theta', \nabla u') \cdot \nabla u' \right) \, dx \, dt \\
+ \int_0^T \int_\Omega \left( \frac{q(\theta, \nabla \theta)}{\theta} \cdot \nabla \theta' - \frac{\theta'}{\theta} q(\theta, \nabla \theta) \cdot \nabla \theta + \frac{\theta - \theta'}{\theta^2} q(\theta', \nabla \theta') \cdot \nabla (\theta - \theta) \right) \, dx \, dt \\
\leq \int_0^T \left( c (\epsilon', \cdot) \int_\Omega \left( \Gamma(p, \theta | \rho', \theta') + \frac{1}{2} |u - u'|^2 \right) \, dx \right) \, dt \\
+ \int_0^T \left( c (\epsilon', \cdot) \int_\Omega \left( \Gamma(p, \theta | \rho', \theta') + \frac{1}{2} |u - u'|^2 \right) \, dx \right) \, dt \\
+ \int_0^T \left( c (\epsilon', \cdot) \int_\Omega \left( \Gamma(p, \theta | \rho', \theta') + \frac{1}{2} |u - u'|^2 \right) \, dx \right) \, dt.
\]

Next we simplify the relative entropy inequality (66). Using (46) and the fact that \( \rho' \) and \( u' \) satisfy the equation of continuity (1), we derive

\[
\int_0^T \rho' [\partial_\theta s(\rho', \theta') (\rho - \rho') + \partial_\rho s(\rho', \theta') (\theta - \theta')] \partial_\theta \theta' + u' \cdot \nabla \theta' \, dx + \int_0^T \frac{\rho - \rho'}{\rho'} \left( \partial_\theta P(\rho', \theta') + u \cdot \nabla P(\rho', \theta') \right) \, dx \\
= - \int_0^T \rho' (\theta - \theta') \left( \frac{1}{\theta} S(\theta', \nabla u') \cdot \nabla u' - q(\theta', \nabla \theta') \cdot \nabla \theta' \right) \, dx \\
+ \int_0^T \left( \Gamma(p, \theta | \rho', \theta') + \partial_\rho P(\rho', \theta') + (\rho - \rho') \partial_\rho P(\rho', \theta') \right) \, dx.
\]

Note that

\[
\left| \int_\Omega \left( \Gamma(p, \theta | \rho', \theta') - \partial_\rho P(\rho', \theta') (\rho - \rho') - \partial_\theta P(\rho', \theta') (\theta - \theta') - P(\rho, \theta) \right) \, dx \right| \leq c \| \nabla \theta' \| \int_\Omega \left( \Gamma(p, \theta | \rho', \theta') \right) \, dx.
\]

Thus by (66)–(67) and (68), for any \( \epsilon > 0 \), we obtain

\[
\int_0^T \left( \frac{1}{2} \rho |u - u'|^2 + \frac{1}{2} |H - H'|^2 + \Gamma(p, \theta | \rho', \theta') \right) (\tau, \cdot) \, dx + \nu \int_0^T \int_\Omega |\nabla \times H' - \nabla \times H|^2 \, dx \, dt \\
+ \int_0^T \int_\Omega \left( \frac{\theta'}{\theta} S(\theta, \nabla u) \cdot \nabla u - S(\theta', \nabla u') \cdot \nabla u' \right) \, dx \, dt \\
+ \int_0^T \int_\Omega \left( \frac{q(\theta, \nabla \theta)}{\theta} \cdot \nabla \theta' - \frac{\theta'}{\theta} q(\theta, \nabla \theta) \cdot \nabla \theta + \frac{\theta - \theta'}{\theta^2} q(\theta', \nabla \theta') \cdot \nabla (\theta - \theta) \right) \, dx \, dt \\
\leq \int_0^T \left( c (\epsilon', \cdot) \int_\Omega \left( \Gamma(p, \theta | \rho', \theta') \right) \, dx \right) \, dt \\
+ \int_0^T \left( c (\epsilon', \cdot) \int_\Omega \left( \Gamma(p, \theta | \rho', \theta') + \frac{1}{2} |u - u'|^2 \right) \, dx \right) \, dt \\
+ \int_0^T \left( c (\epsilon', \cdot) \int_\Omega \left( \Gamma(p, \theta | \rho', \theta') + \frac{1}{2} |u - u'|^2 \right) \, dx \right) \, dt \\
+ \int_0^T \left( c (\epsilon', \cdot) \int_\Omega \left( \Gamma(p, \theta | \rho', \theta') + \frac{1}{2} |u - u'|^2 \right) \, dx \right) \, dt.
\]
An estimate of the terms with $S$ and $q$ can be found in [9], so we omit it. Finally we get

\[
\int_0^T \left( \frac{1}{2} \rho |u - \bar{u}|^2 + \frac{1}{2} |H - H'|^2 + \Gamma(\rho, \theta) \right) (\tau, \cdot) \, dx + \nu \int_0^T \int_\Omega \left| \nabla \times H' - \nabla \times H \right|^2 \, dx \, dt \\
+ c_1 \int_0^T \int_\Omega \left[ |\nabla \cdot u' - \nabla u|^2 + |\nabla \theta' - \nabla \theta|^2 + |\nabla \log \theta' - \nabla \log \theta|^2 \right] \, dx \, dt \\
\leq c_2 \int_0^T \int_\Omega \left( \Gamma(\rho, \theta) \right) + \frac{1}{2} \rho |u - \bar{u}|^2 \right) \, dx \, dt \\
+ \int_0^T \left( c - c' \|u - u'\|_{W^{1,2}(\Omega, \mathbb{R}^3)}^2 + c_1 \int_\Omega |H' - H|^2 \right) \, dt \\
\text{for almost all } \tau \in (0, T).
\]

Furthermore, for sufficiently small $\epsilon > 0$, we obtain

\[
\int_0^T \left( \frac{1}{2} \rho |u - \bar{u}|^2 + \frac{1}{2} |H - H'|^2 + \Gamma(\rho, \theta) \right) (\tau, \cdot) \, dx + \nu \int_0^T \int_\Omega \left| \nabla \times H' - \nabla \times H \right|^2 \, dx \, dt \\
+ (c_1 - c' \epsilon) \int_0^T \int_\Omega \left[ |\nabla \cdot u' - \nabla u|^2 + |\nabla \theta' - \nabla \theta|^2 + |\nabla \log \theta' - \nabla \log \theta|^2 \right] \, dx \, dt \\
\leq c_2 \int_0^T \left( \int_\Omega \left( \Gamma(\rho, \theta) \right) + \frac{1}{2} \rho |u - \bar{u}|^2 + \frac{1}{2} |H' - H|^2 \right) \, dt
\]

which implies that $\rho \equiv \rho'$, $u \equiv u'$, $\theta \equiv \theta'$, $H \equiv H'$. This completes the proof.

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