WE WELL-POSEDNESS OF THE INITIAL-BOUNDARY VALUE
PROBLEM FOR THE FOURTH-ORDER NONLINEAR
SCHRÖDINGER EQUATION

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(Communicated by Yan Guo)

Abstract. The main purpose of this paper is to study local regularity properties of the fourth-order nonlinear Schrödinger equations on the half line. We prove the local existence, uniqueness, and continuous dependence on initial data in low regularity Sobolev spaces. We also obtain the nonlinear smoothing property: the nonlinear part of the solution on the half line is smoother than the initial data.

1. Introduction. In this work we are concerned with the following initial-boundary value problem (IBVP) of the fourth-order nonlinear Schrödinger (4NLS) equation posed on the right half line:

\begin{equation}
\begin{aligned}
&iu_t + u_{xx} + \gamma u_{xxxx} + F(u) = 0, \quad x > 0, \ t > 0, \\
&u(x, 0) = u_0(x), \\
&u(0, t) = f(t), \ u_x(0, t) = g(t),
\end{aligned}
\end{equation}

where \( F(u) = 2|u|^2u + \gamma \left(8|u|^2u_{xx} + 2u^2\overline{u}_{xx} + 6\overline{u}u_x^2 + 4|u_x|^2u + 6|u|^4u \right) \) with \( \gamma = \frac{\gamma_1^2}{12} \). \( \gamma_1 \) is a sufficiently small dimensionless parameter and also a non-zero real constant. \( \overline{u} \) denotes the conjugate complex quantity of \( u \). The data \((u_0, f, g)\) will be taken in the space \( H^s_x(\mathbb{R}^+) \times H^{\frac{s+3}{2}}(\mathbb{R}^+) \times H^s_x(\mathbb{R}^+) \) with the additional conditions \( u_0(0) = f(0) \) for \( \frac{1}{2} < s < \frac{3}{2} \) and \( u_0'(0) = g(0) \) for \( \frac{3}{2} < s < \frac{5}{2} \). In order for the solution to be continuous at space-time corner point \((x, t) = (0, 0)\), we must ensure that these compatibility conditions hold. The 4NLS equation appears in various physical applications, such as in a long-distance, high speed optical fiber transmission system, with the high-order nonlinear effects like the fourth-order dispersion, cubic-quintic nonlinearity, self-steepening and self-frequency shift, the ultrashort optical pulse propagation, see [10]; the nonlinear spin excitations in one-dimensional isotropic...
biquadratic Heisenberg ferromagnetic spin with the octupole-dipole interaction is modelled by the 4NLS equation, see [9].

The 4NLS equation with Kerr nonlinearity has been introduced by Karpman and Shagalov [20, 21] to describe the role of the small fourth order dispersion term in the propagation of intense laser beams in a bulk medium. There are many authors investigating the 4NLS equation with different nonlinearities, see [14, 17–19, 27–29]. Huo and Jia [17–19] studied the Cauchy problem of the 4NLS equation with nonlinear component containing the second order derivative nonlinearities on \( \mathbb{R} \). In [17], they showed the local well-posedness of the 4NLS equation for the initial data in \( H^s(\mathbb{R}) \) \( (s \geq \frac{1}{2}) \) with nonlinear part not containing the second order derivative term \( |u|^2 \partial^2_x u \). Huo and Jia overcame this difficulty of the second order derivative term, see [18]. The well-posedness of the 4NLS equation with the third order derivative nonlinearities for high-dimension \( n \geq 1 \) was also obtained by Huo and Jia, see [19]. For the 4NLS equation with the second order derivative nonlinearities on \( \mathbb{R}^d \) \( (d \geq 2) \), Chen and Zhang [6] established the local well-posedness and small data global existence for random initial data in \( H^s \).

The Cauchy problem of the biharmonic NLS equation \( iu_t + \gamma \Delta^2 u + |u|^p u = 0 \) is known to be global well-posedness and ill-posedness. In case \( \gamma = 1, p = 2 \) and \( 1 \leq n \leq 8 \), the global well-posedness for energy critical 4NLS with radial initial data was obtained in [27]. The cubic defocusing problem \( (\gamma = 1, p = 2) \) is ill-posedness when \( n \geq 9 \), while the scattering holds true in \( H^2 \) when \( 5 \leq n \leq 8 \). Fibich et al. [13] studied different properties of the 4NLS equation in the sub-critical regime. They proved that the solutions of the biharmonic NLS equation always be global if \( \gamma > 0 \), and the critical exponent for singularity formation is \( pn = 8 \) if \( \gamma < 0 \). Moreover, they also analyzed the NLS equation with mixed dispersion \( iu_t + \Delta u + \gamma \Delta^2 u + |u|^p u = 0 \) and found that whether the effect of biharmonic term be focused \( (\gamma < 0) \) or defocused \( (\gamma > 0) \) depends on its size compared with the Laplacian.

The IBVP of the biharmonic NLS equation was studied by several authors, cf. [5, 13, 26]. Özsari and Yolcu [26] investigated the local and global well-posedness of the IBVP for the biharmonic NLS equation, using the unified transform method (also known as Fokas method). Filho et al. [5] dealt with the IBVP of the biharmonic NLS equation by Fourier restriction norm method with the Duhamel boundary forcing operator. This tool was introduced by Colliander and Kenig [7] which solved the IBVP of the generalized Korteweg-de Vries (KdV) equation. Based on the idea, Holmer [15, 16] studied the NLS and KdV equation on the half line in the low regularity Sobolev spaces. The Fourier restriction norm method introduced by Bourgain [4] and mentioned by many authors, cf. [5, 7, 8, 12, 15–19, 22]. They were mainly concerned with the unitary groups and their relationship with the local and global smoothing properties of dispersive equations. Also, they applied the smoothing properties to nonlinear problems with the low regularity requirement.

In this paper we study the 4NLS equation on the half line by the tools initiated in [1, 3], which introduced a brand new approach by Laplace transform for the KdV equation. These tools contain two components: we extend data into the whole line and use the Laplace transform to construct an equivalent integral equation on \( \mathbb{R} \times \mathbb{R} \), see (7) below. Furthermore, we estimate the integral equation by Fourier restriction norm method (also known as \( X^{s,b} \) method) to close the fixed point argument. The tools have been applied to different dispersive equations on the half line, such as the cubic NLS, the derivative NLS, the KdV equation and the “good” Boussinesq
We are ready to state our main theorem of this paper. To this end, we start with a definition.

**Definition 1.1.** We say that (1) is locally well-posed in $H^s(\mathbb{R}^+)$ if for any $(u_0, f, g) \in H^s_x(\mathbb{R}^+) \times H^{2s+3}_{t,x}(\mathbb{R}^+) \times H^{2s+1}_{t,x}(\mathbb{R}^+)$, with the additional compatibility conditions $u_0(0) = f(0)$ for $\frac{1}{2} < s < \frac{3}{2}$ and $u_0'(0) = g(0)$ for $\frac{3}{2} < s < \frac{9}{2}$, the equation $\Phi u = u$, where $\Phi$ is defined in (7), has a unique solution in

$$X^{s,b}(\mathbb{R} \times [0,T]) \cap C^0_t H^s([0,T] \times \mathbb{R}) \cap C^0_t H^{2s+3}_t(\mathbb{R} \times [0,T]),$$

for any $b < \frac{1}{2}$ and some sufficiently small $T := T \left( \|u_0\|_{H^s(\mathbb{R}^+)}, \|f\|_{H^{2s+3}(\mathbb{R}^+)}, \|g\|_{H^{2s+1}(\mathbb{R}^+)} \right)$, where $X^{s,b}$ will be defined in Section 2.1. Furthermore, the solution depends continuously on the initial and boundary data.

Note that our main theorem establishes the sharp local well-posedness. We also prove that the nonlinear components of solution are smoother than the initial data. We now state our main theorem as follows.

**Theorem 1.2.** Fix $1 < s < \frac{9}{2}, s \neq \frac{3}{2}$. Then (1) is locally well-posed in $H^s(\mathbb{R}^+)$. Moreover, for $a < \min\{2s - 2, \frac{1}{2}, \frac{9}{2} - s\}$, the solution of (1) satisfies

$$u(x,t) - S^a_0(u_0, f, g) \in C^0_t H^{s+a}_x([0,T] \times \mathbb{R}^+),$$

where $S^a_0(u_0, f, g)$ is the solution of the corresponding linear problem (2) below.

The proof of the above theorem depends on a constructed integral formula which contains free Schrödinger group, the Duhamel term and the boundary operator. The form of the solution is the superposition of solutions to the linear equations which consist of the boundary and the initial data with the nonlinearity. For the two linear equations with initial and boundary data, we can use Fourier and Laplace transform, respectively. Before this, we shall extend the initial data to the whole line. With these in hand, it is easy to see that applying the restricted norm method in the integral formula. We note that the uniqueness of the constructed solution is not clear if different extensions of the initial data produce the same solution on $\mathbb{R}^+$. In other words, it is not clear that the restriction of the fixed point on the half line is independent of the extension of initial data.

The present work is organized as follows. In Section 2, we introduce some notations and function spaces. These can be used for the well-posedness of the IBVP (1). Moreover, we set up an integral representation which consists of a linear and a nonlinear evolution. In Section 3, we state some linear and nonlinear estimates which can be used to achieve the contraction argument. Section 4 exhibits the proof of our main theorem.

2. Preliminaries.

2.1. Notations. Define Fourier transform as

$$\hat{u}(\xi, \tau) = \mathcal{F}u(\xi, \tau) = \int \int_{\xi, \tau} e^{-ix\xi} e^{-it\tau} u(x,t) \, dx \, dt.$$

We define the homogeneous $L^2$-based Sobolev space $H^s = \dot{H}^s(\mathbb{R})$ by the norm

$$\|f\|_{H^s} = \|f\|_{\dot{H}^s(\mathbb{R})} = \left\| |\xi|^s \hat{f}(\xi) \right\|_{L^2_x}.$$
and the inhomogeneous $L^2$-based Sobolev space $H^s = H^s(\mathbb{R})$ by the norm
\[ \|f\|_{H^s} = \|f\|_{H^s(\mathbb{R})} = \left\| \langle \xi \rangle^s \hat{f}(\xi) \right\|_{L_t^2} \]
where $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$. In addition, for $g \in H^s(\mathbb{R}^+)$, if there exists $\tilde{g} \in H^s(\mathbb{R})$ such that $g(x) = \tilde{g}(x)$ for $x > 0$, then we set
\[ \|g\|_{H^s(\mathbb{R}^+)} = \inf \{ \|\tilde{g}\|_{H^s(\mathbb{R})} : \tilde{g} \chi_{(0,\infty)} = g \}, \]
where $\chi$ is the characteristic function. We also define the set $C_0^\infty(\mathbb{R}^+) = \{ f \in C^\infty(\mathbb{R}) ; \text{supp} f \subset [0, \infty) \}$.

Denote the fourth-order linear Schrödinger propagator (for any $g \in L^2(\mathbb{R})$) by
\[ S g(x,t) = e^{it(\partial_x^4 + \gamma \partial_x^2)} g(x) = F^{-1} \left( e^{it\phi(\cdot)} \hat{g}(\cdot) \right)(x), \]
where $\phi(\xi) = \gamma \xi^4 - \xi^2$. For $s,b \in \mathbb{R}$, we use the Bourgain space $X^{s,b}(\mathbb{R} \times \mathbb{R})$ corresponding to the 4NLS flow. The Bourgain space is defined on the full space by the norm
\[ \|u\|_{X^{s,b}} = \left( \int \int_{\xi,\tau} \langle \xi \rangle^{2s} \langle \tau - \phi(\xi) \rangle^{2b} |\hat{u}(\xi,\tau)|^2 d\xi d\tau \right)^{1/2}. \]

Let us define a smooth compactly supported function $\eta \in C_0^\infty(\mathbb{R})$ such that $\eta(t) = 1$ on $[-1,1]$, $\text{supp} \ \eta \subset [-2,2]$, and $\eta_T(t) = \eta(t/T^{-1})$. Let $\rho$ be a smooth function supported on $[-1,\infty)$, and $\rho(x) = 1$ for $x > 0$. For any function $f$, we set the notation
\[ D_0[f(x,t)] = f(0,t). \]

Finally, throughout this paper, the notation $a \lesssim b$ denotes that $a \leq C b$ for any constant $C$. The notation $a \gtrsim b$ has a similar definition: $a \geq C b$, and $a \approx b$ indicates that $a \lesssim b$ and $a \gtrsim b$. The notation $a + \epsilon$ means $a + \epsilon$, where $\epsilon$ is an arbitrarily small constant. The definition of $a - \epsilon$ is similar.

### 2.2. Notion of a solution

In this section we construct a solution of linear IBVP, and through this useful linear solution to reformulate the IBVP (1) as an integral equation on $\mathbb{R}$. To this end, we shall consider the linear IBVP:
\[
\begin{cases}
  iu_t + u_{xx} + \gamma u_{xxxx} = 0, \ (x, t) \in (0, +\infty) \times (0, \infty), \\
  u(x, 0) = u_0(x), \\
  u(0, t) = f(t), \ u_x(0, t) = g(t),
\end{cases}
\]  
(2)

where $u_0(x) \in H^s(\mathbb{R})$, $f(t) \in H^{2s+4}(\mathbb{R}^+)$, $g(t) \in H^{2s+1}(\mathbb{R}^+) + \text{additional compatibility conditions}$ $u_0(0) = f(0)$ for $\frac{1}{2} < s < \frac{3}{2}$ and $u_x(0) = g(0)$ for $\frac{3}{2} < s < \frac{5}{2}$. This idea of construction of solution introduced in [2] and the uniqueness of solution of (2) follows by considering the equation with $u_0 = f = 0$ with the method of odd extension. Denote the solution by $S_0^d(u_0, f, g)$. For extension $\tilde{u}_0$ to the full line $\mathbb{R}$ of the function $u_0$, we can write
\[ S_0^d(u_0, f, g) = S_0^d(0, f - p_1, g - p_2) + S(t)\tilde{u}_0, \]
where $\tilde{u}_0$ satisfies $\|\tilde{u}_0\|_{H^s(\mathbb{R})} \lesssim \|u_0\|_{H^s(\mathbb{R}^+)}$,
\[ p_1(t) = \eta(t)D_0 \left[ S(t)\tilde{u}_0 \right] = \eta(t) \left[ S(t)\tilde{u}_0 \right]_{x=0}, \]
\[ p_2(t) = \eta(t)D_0 \left[ S(t)\tilde{u}_0 \right]_x = \eta(t) \left[ S(t)\tilde{u}_0 \right]_{x=0}. \]
which are well-defined and are in $H^{\frac{2}{2}+\frac{3}{8}}(\mathbb{R}^+)$ and $H^{\frac{2}{2}+1}(\mathbb{R}^+)$, respectively. In other words, we decompose the solution operator as two parts: a modified boundary operator, which contains zero initial data, and the free Schrödinger propagator on the whole real line.

Namely, $S_0^j(0, f, g)$ is the solution of the following linear boundary value problem:

$$
\begin{align*}
\begin{cases}
iu_t + u_{xx} + \gamma u_{xxxx} = 0, \\
u(x, 0) = 0, \\
u(0, t) = f(t), \quad u_x(0, t) = g(t).
\end{cases}
\end{align*}
$$

(3)

Taking Laplace transform with respect to $t$ of (3), the IBVP is converted to a one-parameter family of boundary value problems:

$$
\begin{align*}
\begin{cases}
i\lambda \tilde{u}(x, \lambda) + \tilde{u}_{xx} + \gamma \tilde{u}_{xxxx} = 0, \\
\tilde{u}(0, \lambda) = \tilde{f}(\lambda), \quad \tilde{u}_x(0, \lambda) = \tilde{g}(\lambda), \\
\tilde{u}(+\infty, \lambda) = 0, \quad \tilde{u}_x(+\infty, \lambda) = 0.
\end{cases}
\end{align*}
$$

(4)

The characteristic equation of (4) is $i\lambda + \omega^2 + \gamma \omega^4 = 0$, which has roots satisfying

$$
\omega^2 = \frac{-1 \pm \sqrt{1 - 4i\gamma \lambda}}{2\gamma}.
$$

Let

$$
a = -\left( -\frac{1}{2\gamma} - \frac{\sqrt{1 - 4i\gamma \lambda}}{\gamma} \right)^{\frac{1}{2}}, \quad b = -\left( -\frac{1}{2\gamma} + \frac{\sqrt{1 - 4i\gamma \lambda}}{\gamma} \right)^{\frac{1}{2}}.
$$

Following the idea of construction of explicit linear solution formula in [8], we only care about the two roots above since we consider the solutions which decay at infinity. Thus, we have

$$
\tilde{u}(x, \lambda) = \frac{1}{a-b} \left[ (a \tilde{f} - \tilde{g}) e^{bx} - (b \tilde{f} - \tilde{g}) e^{ax} \right].
$$

By Mellin inversion, for any $c > 0$, we have

$$
u(x, t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{\lambda t}}{a-b} \left[ (a \tilde{f} - \tilde{g}) e^{bx} - (b \tilde{f} - \tilde{g}) e^{ax} \right] d\lambda.
$$

Letting $c \to 0$, we arrive at

$$
u(x, t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{\lambda t}}{a-b} \left[ (a \tilde{f} - \tilde{g}) e^{bx} - (b \tilde{f} - \tilde{g}) e^{ax} \right] d\lambda
$$

$$
= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{\lambda t}}{a^2 - b^2} (a + b) \left[ (a \tilde{f} - \tilde{g}) e^{bx} - (b \tilde{f} - \tilde{g}) e^{ax} \right] d\lambda.
$$

(5)

Substituting the change of variable $\lambda = i\mu^2(\gamma \mu^2 - 1)$ into the characteristic equation $i\lambda + \omega^2 + \gamma \omega^4 = 0$ yields

$$
a = -i\mu, \quad b = -\sqrt{\mu^2 - \frac{1}{\gamma}}.
$$
The representation (5) can be rewritten as $u(x,t) = \frac{1}{2\pi}(A_0 + B_0 + C_0 + D_0)$, where

$$A_0 = - \int e^{it(\gamma^4 - \beta^2) - i\beta x} \eta^\prime(t) \left( \gamma^4 - \beta^2 \right) \frac{\eta(t)}{\sqrt{\gamma^2 + 1}} d\beta,$$

$$B_0 = - \int e^{it(\gamma^4 - \beta^2) - i\beta x} \eta^\prime(t) \left( \gamma^4 - \beta^2 \right) \frac{\eta(t)}{\sqrt{\gamma^2 + 1}} d\beta,$$

$$C_0 = \int e^{it(\gamma^4 - \beta^2) - i\beta x} \eta^\prime(t) \left( \gamma^4 - \beta^2 \right) \frac{\eta(t)}{\sqrt{\gamma^2 + 1}} d\beta,$$

$$D_0 = \int e^{it(\gamma^4 - \beta^2) - i\beta x} \eta^\prime(t) \left( \gamma^4 - \beta^2 \right) \frac{\eta(t)}{\sqrt{\gamma^2 + 1}} d\beta.$$

Here $I = (-\infty, -1/\sqrt{\gamma}) \cup (1/\sqrt{\gamma}, +\infty)$. For $x \geq 0$, we can rewrite $A_0, B_0, C_0$ and $D_0$ as

$$A = \int_I \chi(t) e^{it(\gamma^4 - \beta^2) - i\beta x} \eta^\prime(t) \left( \gamma^4 - \beta^2 \right) \frac{\eta(t)}{\sqrt{\gamma^2 + 1}} d\beta,$$

$$B = \int_I \chi(t) e^{it(\gamma^4 - \beta^2) - i\beta x} \eta^\prime(t) \left( \gamma^4 - \beta^2 \right) \frac{\eta(t)}{\sqrt{\gamma^2 + 1}} d\beta,$$

$$C = \int_I \chi(t) e^{it(\gamma^4 - \beta^2) - i\beta x} \eta^\prime(t) \left( \gamma^4 - \beta^2 \right) \frac{\eta(t)}{\sqrt{\gamma^2 + 1}} d\beta,$$

$$D = \int_I \chi(t) e^{it(\gamma^4 - \beta^2) - i\beta x} \eta^\prime(t) \left( \gamma^4 - \beta^2 \right) \frac{\eta(t)}{\sqrt{\gamma^2 + 1}} d\beta,$$

where $\hat{f}(\xi) = F(\chi(t) f(\xi))$. We added the cut-off function $\eta$ in $A$ and $B$ so that the integrals converge for all $x$. Thus, the solution of (3) on $\mathbb{R}^+ \times \mathbb{R}^+$ can be written as $u(x,t) = \frac{1}{2\pi}(-A - B + C + D)$.

We set integral equation as follows,

$$\Phi(u(x,t)) = \eta^\prime(t) S(t) \bar{u}_0 + i \eta^\prime(t) \int_0^t S(t - t') G(u) dt' + \eta^\prime(t) S_0(0, f - p_1 - q_1, g - p_2 - q_2)(t),$$

where

$$G(u) = \eta^\prime(t) F(u),$$

$$p_1(t) = \eta(t) D_0 \left( S(t) \bar{u}_0 \right), \quad q_1(t) = \eta(t) D_0 \left( \int_0^t S(t - t') G(u) dt' \right),$$

$$p_2(t) = \eta(t) D_0 \left( S(t) \bar{u}_0 \right), \quad q_2(t) = \eta(t) D_0 \left( \int_0^t S(t - t') G(u) dt' \right).$$

Here $D_0 f(t) = f(0,t)$, $\bar{u}_0$ is an $H^s$ extension of $u_0$ to $\mathbb{R}$. For this integral equation, we will prove that it has a unique solution in $X^{s,b} \cap C^0\mathcal{H}_t^s \cap C^0\mathcal{H}_x^{2s+3}$ on $\mathbb{R} \times \mathbb{R}$ for sufficiently small $T$. By the definition of boundary operator, it is obvious that the restriction of $u$ to $\mathbb{R}^+ \times [0,T]$ satisfies (1) in the distribution sense and the smooth solution of (7) satisfies (1) in classical sense.

It is well known that the embedding $X^{s,b} \subset C^0\mathcal{H}_t^s$ for $b > \frac{1}{2}$. Since contraction theory work in $X^{s,b}$ space, and in order to bound the fourth-order linear Schrödinger
For any \( s \) and \( b \), we have
\[
\|\eta(t)S(t)g\|_{X^{s,b}} \lesssim \|g\|_{H^s}.
\] (11)

For \( s \in \mathbb{R} \), \( 0 \leq b_1 < \frac{1}{2} \), and \( 0 \leq b_2 < 1 - b_1 \), we have
\[
\left\| \eta(t) \int_0^t S(t - t')F(t')dt' \right\|_{X^{s,b_2}} \lesssim \|F\|_{X^{s,-b_1}}.
\] (12)

For sufficiently small \( T \), and \(-\frac{1}{2} < b_1 < b_2 < \frac{1}{2}\), we have
\[
\|\eta(t/T)F\|_{X^{s,b_1}} \lesssim T^{b_2-b_1} \|F\|_{X^{s,b_2}}.
\] (13)

Finally, we have the following Lemmas which are repeatedly useful throughout this paper.

**Lemma 2.1.** [12] Let \( h \in H^s(\mathbb{R}^+) \).

1. If \(-1/2 < s < 1/2\), then \( \|\chi_{(0,\infty)}h\|_{H^s(\mathbb{R})} \lesssim \|h\|_{H^s(\mathbb{R}^+)} \).
2. If \(1/2 < s < 3/2\), and \( h(0) = 0 \), then \( \|\chi_{(0,\infty)}h\|_{H^s(\mathbb{R})} \lesssim \|h\|_{H^s(\mathbb{R}^+)} \).

**Lemma 2.2.** [12] If \( \beta \geq \gamma \geq 0 \) and \( \beta + \gamma > 1 \), then we have
\[
\int \frac{1}{\|x-a\|^{\beta}(x-b)^{\gamma}} dx \lesssim (a-b)^{-\gamma}\varphi_\beta(a-b),
\]
where
\[
\varphi_\beta(c) = \begin{cases} 
1 & \beta > 1 \\
\log(1 + \langle c \rangle) & \beta = 1 \\
\langle c \rangle^{1-\beta} & \beta < 1.
\end{cases}
\]

3. **A priori estimates.** To obtain the contraction argument, we will deduce a priori estimates for the linear and nonlinear terms in (7).

3.1. **Linear estimates.** We start with the corresponding Kato smoothing estimates for fourth-order Schrödinger propagator.

**Lemma 3.1.** For fixed \( s \geq 0 \) and any \( g \in H^s(\mathbb{R}) \), we have \( \eta(t)S(t)g \in C^0_b \mathit{H}^{2+\frac{3}{s}}(\mathbb{R} \times \mathbb{R}) \), and
\[
\|\eta(t)S(t)g\|_{L_\infty^s \mathit{H}_t^{2+\frac{3}{s}}} \lesssim \|g\|_{H^s(\mathbb{R})}.
\]

In addition, for fixed \( s \geq 0 \) and any \( g \in H^s(\mathbb{R}) \), we have \( \eta(t)\partial_x S(t)g \in C^0_b \mathit{H}^{2+\frac{3}{s}}(\mathbb{R} \times \mathbb{R}) \), and
\[
\|\eta(t)\partial_x S(t)g\|_{L_\infty^s \mathit{H}_t^{2+\frac{3}{s}}} \lesssim \|g\|_{H^s(\mathbb{R})}.
\]

**Proof.** It is clear that
\[
\mathcal{F}_\hat{\eta}(\eta S(t)g) = \int \hat{\eta}(\tau - \phi(\xi))e^{ix\xi} \hat{g}(\xi) d\xi
\]
\[
= \int_{|\xi| < 2k} \hat{\eta}(\tau - \phi(\xi))e^{ix\xi} \hat{g}(\xi) d\xi + \int_{|\xi| \geq 2k} \hat{\eta}(\tau - \phi(\xi))e^{ix\xi} \hat{g}(\xi) d\xi,
\]
where \( k = \frac{1}{\sqrt{s}} \). We estimate the first term to \( \mathit{H}^{2+\frac{3}{s}} \) norm by
\[
\int_{|\xi| < 2k} \|\tau^{\frac{2+3}{s}} \hat{\eta}(\tau - \phi(\xi))\|_{L_\tau^2} \|\hat{g}(\xi)\| d\xi \lesssim \int_{|\xi| < 2k} \|\hat{g}(\xi)\| d\xi \lesssim \|g\|_{L^2} \lesssim \|g\|_{H^s}.
\]
For the second term, take a change of variable $\xi = \phi^{-1}$. If $|\xi| \geq 2k$, $\phi$ is invertible, then it is sufficient to prove that
\[
\left\| \int_{|\xi| \geq 2k} \tilde{\eta}(\tau - \phi(\xi))e^{ix\xi} \hat{g}(\xi) \, d\xi \right\|_{H^{2k+3}} \lesssim \|g\|_{H^{*}}.
\]
By a fact that $\langle a + b \rangle \lesssim \langle a \rangle \langle b \rangle$, we have
\[
\left\| \int_{|\phi^{-1}| \geq 2k} \langle \tau \rangle^{\frac{2k+3}{2}} |\tilde{\eta}(\tau - \phi)| \frac{|\hat{g}(\phi^{-1})|}{|\phi'|} \, d\phi \right\|_{L^2} \lesssim \left\| \int_{|\phi^{-1}| \geq 2k} \langle \tau - \phi \rangle^{\frac{2k+3}{2}} |\tilde{\eta}(\tau - \phi)| \langle \phi \rangle^{\frac{2k+3}{2}} \frac{|\hat{g}(\phi^{-1})|}{|\phi'|} \, d\phi \right\|_{L^2}.
\]
Since $\hat{\eta}$ is a Schwarz function, we can use Young’s inequality and then bound this by
\[
\left\| \langle \cdot \rangle^{\frac{2k+3}{2}} \hat{\eta} \right\|_{L^1} \left\| \langle \phi(\xi) \rangle^{\frac{2k+3}{2}} \frac{|\hat{g}(\xi)|}{|\phi'(\xi)|} \right\|_{L^2_{|\xi| \geq 2k}} \lesssim \|g\|_{H^*}(\mathbb{R}).
\]
The continuity argument follows from this result and the dominated convergence theorem.

For the derivative part, we have
\[
\mathcal{F}_t(\eta \partial_x S(t)g) = i \int \frac{\tilde{\eta}(\tau - \phi(\xi))e^{ix\xi} \hat{g}(\xi) \, d\xi}{|\xi| \geq 2k}
\]
\[
+ \int \frac{\tilde{\eta}(\tau - \phi(\xi))e^{ix\xi} \hat{g}(\xi) \, d\xi}{|\xi| < 2k}.
\]
We estimate the first term to $H^{2k+1}$ norm by
\[
\left\| \langle \tau \rangle^{\frac{2k+1}{2}} \tilde{\eta}(\tau - \phi) \hat{g}(\xi) \right\|_{L^2} \|g\|_{L^2} \lesssim \|g\|_{H^*}.
\]
The second term is bounded by
\[
\left\| \int_{|\phi| \geq k'} \langle \tau - \phi \rangle^{\frac{2k+1}{2}} |\tilde{\eta}(\tau - \phi)| |\hat{g}(\xi(\phi))| \left| \frac{\xi(\phi)}{4\gamma \xi^3 - 2\xi} \right| \, d\phi \right\|_{L^2} \lesssim \left\| \int_{|\phi| \geq k'} \langle \tau - \phi \rangle^{\frac{2k+1}{2}} |\tilde{\eta}(\tau - \phi)| |\hat{g}(\xi(\phi))| \left| \frac{1}{4\gamma \xi^2 - 2} \right| \, d\phi \right\|_{L^2},
\]
where $k' = \frac{12}{7}$. Since $\hat{\eta}$ is a Schwarz function, we also have
\[
\left\| \langle \cdot \rangle^{\frac{2k+1}{2}} \hat{\eta} \right\|_{L^1} \left\| \langle \phi \rangle^{\frac{2k+1}{2}} \frac{|\hat{g}(\phi^{-1})|}{4\gamma \xi^2 - 2} \right\|_{L^2_{|\xi| \geq k'}} \lesssim \|g\|_{H^*}(\mathbb{R}).
\]
Lemma 3.2. For any \( s \geq 0 \), and \((f,g)\) such that \((\chi_{(0,\infty)}f, \chi_{(0,\infty)}g) \in H^{\frac{2s+1}{4}}(\mathbb{R}) \times H^{\frac{2s+1}{4}}(\mathbb{R})\), we have
\[
S_0^t(0, f, g) \in C^0_t H^s_x(\mathbb{R} \times \mathbb{R})
\]
and
\[
\eta(t) S_0^t(0, f, g) \in C^0_t H^{\frac{2s+1}{4}}(\mathbb{R} \times \mathbb{R}).
\]

Proof. Recall that \(2\pi S_0^t(0, f, g) = -A - B + C + D\). We now show that \(A, B \in C^0_t H^s_x(\mathbb{R} \times \mathbb{R})\). Note that
\[
A = \int_{\mathbb{R}} \chi_I h \left( x \sqrt{\beta^2 - 1/\gamma} \right) \mathcal{F}(J \varphi_A)(\beta) \, d\beta,
\]
\[
B = \int_{\mathbb{R}} \chi_I h \left( x \sqrt{\beta^2 - 1/\gamma} \right) \mathcal{F}(J \varphi_B)(\beta) \, d\beta,
\]
where \(I = (-\infty, -1/\sqrt{\gamma}) \cup (1/\sqrt{\gamma}, +\infty)\),
\[
\varphi_A = 2i\gamma \beta \left( i\beta + \sqrt{\beta^2 - 1/\gamma} \right) \hat{f}(\gamma \beta^4 - \beta^2),
\]
\[
\varphi_B = 2\gamma \beta \left( i\beta + \sqrt{\beta^2 - 1/\gamma} \right) \hat{g}(\gamma \beta^4 - \beta^2),
\]
h\((x) = e^{-x} \rho(x)\) is a Schwarz function, \(J\) is the Fourier multiplier operator with multiplier \(e^{(\gamma \beta^4 - \beta^2)t}\). Now we have,
\[
\|\varphi_A\|_{H^s_x}^2
= \int_{\mathbb{R}} \chi_I \langle \beta \rangle^{2s} 4\gamma^2 \beta^4 (2\beta^2 - 1/\gamma) \left| \hat{f}(\gamma \beta^4 - \beta^2) \right|^2 \, d\beta
= \int_{\mathbb{R}} \chi_I 4\gamma^2 \beta^4 (2\beta^2 - 1/\gamma) \langle \beta \beta^2 - 2\beta \rangle^{2s+1} \left| \hat{f} \right|^2 (4\gamma \beta^3 - 2\beta) \, d\beta
\lesssim \int_{\mathbb{R}} \chi_I \langle \beta^2 (\beta^2 - 1) \rangle^{2s+1} \left| \hat{f} \right|^2 (4\gamma \beta^3 - 2\beta) \, d\beta
= \int_{\mathbb{R}} \chi_{(0,\infty)} (z)^{2s+1} \left| \hat{f}(z) \right|^2 \, dz \lesssim \|\chi f\|_{H^{\frac{2s+1}{4}}(\mathbb{R})}^2.
\]
By similar calculations, we obtain
\[
\|\varphi_B\|_{H^s_x}^2 \lesssim \|\chi g\|_{H^{\frac{2s+1}{4}}(\mathbb{R})}^2.
\]
Thus, using these and the continuity of \(J\) in \(H^s\), it suffices to prove that the map
\[
T g(x) := \int_{\mathbb{R}} \chi_I h \left( x \sqrt{\beta^2 - 1/\gamma} \right) \hat{g}(\beta) \, d\beta
\]
is bounded from \(H^s\) to \(H^s\). Considering \(s = 0\) and the change of variable \(z = x \sqrt{\beta^2 - 1/\gamma}\), we have
\[
T g(x) = \int_{\mathbb{R}^+} h(z) \left( \hat{g} \left( \sqrt{(z/x)^2 + 1/\gamma} \right) + \hat{g} \left( -\sqrt{(z/x)^2 + 1/\gamma} \right) \right) \frac{z/x^2}{\sqrt{(z/x)^2 + 1/\gamma}} \, dz.
\]
Then
\[ \|Tg\|_{L^2} \lesssim \int |h(z)| \left\| \hat{g} \left( \pm \sqrt{z/x}^2 + 1/\gamma \right) \frac{z/x^2}{\sqrt{(z/x)^2 + 1/\gamma}} \right\|_{L^2} \ dz, \]

and
\[ \int |\hat{g} \left( \pm \sqrt{z/x}^2 + 1/\gamma \right)|^2 \frac{z^2/x^4}{(z/x)^2 + 1/\gamma} \ dx \]
\[ = \frac{1}{|z|} \int_{\pm \frac{1}{\sqrt{\gamma}}}^{\pm \infty} |\hat{g}(y)|^2 \frac{\sqrt{y^2 - 1/\gamma}}{y} \ dy. \]

Notice that on the region \(|y| \geq 1/\sqrt{\gamma}\), we have
\[ \frac{1}{|z|} \int_{\pm \frac{1}{\sqrt{\gamma}}}^{\pm \infty} |\hat{g}(y)|^2 \frac{\sqrt{y^2 - 1/\gamma}}{y} \ dy \lesssim \frac{1}{|z|} \|g\|_{L^2}. \]

Since \(h \in \mathcal{S}, \frac{|h(z)|}{\sqrt{|z|}}\) is in \(L^1\), we have
\[ \|Tg\|_{L^2} \lesssim \|g\|_{L^2} \int_{\mathbb{R}} |h(z)| \frac{dz}{\sqrt{|z|}} \lesssim \|g\|_{L^2}. \]

This completes the proof that \(A, B \in C^0_\gamma H^s_x\) for \(s = 0\). For \(s > 0\) and any \(s \in \mathbb{N}\), we have
\[ \partial_x^s Tg(x) = \int_0^\infty \chi_x h(x) (\sqrt{\beta^2 - 1/\gamma})(\beta^2 - 1/\gamma)^{\frac{s}{2}} \hat{g}(\beta) \ d\beta. \]
By the result of \(s = 0\) and interpolation, we obtain the desired bounds for \(A, B\) in \(H^s_x\) for \(s > 0\).

Also recall that
\[ C = J\varphi_C, \quad \varphi_C = 2\gamma\beta \sqrt{\beta^2 - 1/\gamma} \left( i\beta + \sqrt{\beta^2 - 1/\gamma} \right) \hat{\varphi} \left( \gamma \beta^2 - \beta^2 \right), \]
\[ D = J\varphi_D, \quad \varphi_D = 2\gamma\beta \left( i\beta + \sqrt{\beta^2 - 1/\gamma} \right) \hat{\varphi} \left( \gamma \beta^4 - \beta^2 \right). \]

Following the continuity of linear operator \(J\) and the bounds for \(\varphi_C, \varphi_D\) which are proved by similar processes as \(A\) and \(B\), we obtain the \(C^0_\gamma H^s_x\) bounds for \(C\) and \(D\).

To prove that \(\eta(t)S^0_\gamma(0, f, g) \in C^0_\nu H^{2+\delta}_x(\mathbb{R} \times \mathbb{R})\), we notice the forms of \(C, D\) as the linear 4NLS group and apply the Kato smoothing estimates to obtain the bounds for \(C\) and \(D\). For \(A\) and \(B\), we have
\[ A = \int_{\mathbb{R}} \mathcal{F}_\beta \left( \chi_I h \left( x \sqrt{\beta^2 - 1/\gamma} \right) \right)(y) J\varphi_A \ dy, \]
\[ B = \int_{\mathbb{R}} \mathcal{F}_\beta \left( \chi_I h \left( x \sqrt{\beta^2 - 1/\gamma} \right) \right)(y) J\varphi_B \ dy, \]
where \(\varphi_A, \varphi_B\) are defined as before. Then we write
\[ A = \int_{\mathbb{R}} \frac{1}{x} \mathcal{F}_{x'} \left( h \left( \frac{\text{sgn}(x) \sqrt{z^2 - x^2}}{\gamma} \right) \right) \left( \frac{y}{x} \right) J\varphi_A(y) \ dy \]
\[ = \int_{\mathbb{R}} \mathcal{F}_{x'} \left( h \left( \frac{\text{sgn}(x) \sqrt{z^2 - x^2}}{\gamma} \right) \right)(xy) J\varphi_A(xy) \ dy, \]
where $z' = \beta x$. $B$ has a parallel result:

$$B = \int_{\mathbb{R}} \mathcal{F}_{z'} \left( h \left( \text{sgn}(x) \sqrt{z'^2 - x^2} \right) \right) (y) J \varphi_B (xy) \, dy.$$ 

By Lemma 3.1, it suffices to show that the function $\mathcal{F}_{z'} (h(\text{sgn}(x) \sqrt{z'^2 - x^2} / \gamma)) \in L_x^\infty L_y^1$. Since

$$\int |\hat{p}(y)| \, dy \lesssim \int \langle y \rangle^{-1} \, dy \lesssim \|p\|_{H^1} \|\langle \cdot \rangle^{-1}\|_{L^2},$$

we obtain that $h \left( \text{sgn}(x) \sqrt{z'^2 - x^2 / \gamma} \right) \in L_x^\infty H_y^1$. Then we complete the proof of the claim.

**Lemma 3.3.** Let $s \geq 0$, $b \leq \frac{1}{2}$. Then for any compactly supported smooth function $\eta$ and $(f, g)$ satisfying $(\chi_{(0, \infty)} f, \chi_{(0, \infty)} g) \in H^{2s+\frac{1}{4}}(\mathbb{R}) \times H^{2s+\frac{1}{4}}(\mathbb{R})$, we have

$$\|\eta(t) S^g_t (0, f, g)\|_{X^{s, b}} \lesssim \|\chi_{(0, \infty)} f\|_{H^{2s+\frac{1}{4}}(\mathbb{R})} + \|\chi_{(0, \infty)} g\|_{H^{2s+\frac{1}{4}}(\mathbb{R})}.$$  

**Proof.** Recall that

$$C = J \varphi_C, \quad \varphi_C = 2 \gamma \beta \sqrt{\frac{1}{\gamma^2} - 1} \left( i \beta + \sqrt{\frac{1}{\gamma^2} - 1} \right) \hat{f} \left( \gamma \beta^4 - \beta^2 \right),$$

$$D = J \varphi_D, \quad \varphi_D = 2 \gamma \beta \left( i \beta + \sqrt{\frac{1}{\gamma^2} - 1} \right) \hat{g} \left( \gamma \beta^4 - \beta^2 \right),$$

where $J$ is the Fourier multiplier operator with multiplier $e^{i (\gamma \beta^4 - \beta^2) t}$. Following the proof of (11), it is easy to obtain $\|\eta(t) C\|_{X^{s, b}} \lesssim \|\varphi_C\|_{H^s}$ and $\|\eta(t) D\|_{X^{s, b}} \lesssim \|\varphi_D\|_{H^s}$. Now we have,

$$\|\varphi_C\|_{H^s}^2 = \int_{\mathbb{R}} \chi_{(\frac{1}{2}, \infty)} (\gamma \beta^2 - \beta^2)( \gamma \beta^2 - 1/\gamma) (2 \gamma \beta^2 - 1/\gamma) \left| \hat{f} \left( \gamma \beta^4 - \beta^2 \right) \right|^2 \, d\beta$$

$$= \int_{\mathbb{R}} \chi_{(\frac{1}{2}, \infty)} \frac{4 \gamma^2 \beta^2 (\beta^2 - 1/\gamma) (2 \gamma \beta^2 - 1/\gamma) (\beta^2 (\gamma \beta^2 - 1))^{2s+\frac{1}{4}}}{4 \gamma \beta^4 - 2 \beta} \left( \beta^2 (\gamma \beta^2 - 1) \right)^{2s+\frac{1}{4}} \left( \beta^2 (\gamma \beta^2 - 1) \right)^{2s+\frac{1}{4}} \left| \hat{f} \left( \gamma \beta^4 - \beta^2 \right) \right|^2 \, d\beta$$

$$\lesssim \int_{\mathbb{R}} \chi_{(\frac{1}{2}, \infty)} \left( \beta^2 (\gamma \beta^2 - 1) \right)^{2s+\frac{1}{4}} \left( \beta^2 (\gamma \beta^2 - 1) \right)^{2s+\frac{1}{4}} \left| \hat{f} \left( \gamma \beta^4 - \beta^2 \right) \right|^2 \, d\beta$$

Using a similar calculations we obtain

$$\|\varphi_D\|_{H^s}^2 \lesssim \|\chi g\|_{H^{2s+\frac{1}{4}}(\mathbb{R})}^2.$$  

Thus we have the desired results for $C$ and $D$. Now we consider $A, B$. Let $h(x) = e^{-x} \rho(x)$, and assume that $s = 0$ and $b = 1/2$. Then we have

$$A = \int_{\mathbb{R}} \chi_{(\frac{1}{2}, \infty)} e^{i \gamma \beta^4 - \beta^2} (\gamma \beta + \sqrt{\gamma^2 - 1}) (2 \gamma \beta^2) (\gamma \beta - \beta) \hat{f} \left( \gamma \beta^4 - \beta^2 \right) \rho(x \sqrt{\gamma^2 - 1})$$

$$= \int_{\mathbb{R}} \chi_{(\frac{1}{2}, \infty)} e^{i \gamma \beta^4 - \beta^2} (2 \gamma \beta^2) (\gamma \beta + \sqrt{\gamma^2 - 1}) (\gamma \beta - \beta) \hat{f} \left( \gamma \beta^4 - \beta^2 \right) h \left( x \sqrt{\gamma^2 - 1} \right),$$
and

\[ \widehat{\eta A}(\xi, \tau) = \int_\mathbb{R} \chi \hat{\eta}(\tau - (\gamma \beta^4 - \beta^2)) \frac{2i\gamma \beta^2 (i\beta + \sqrt{\beta^2 - 1/\gamma})}{\sqrt{\beta^2 - 1/\gamma}} \hat{f} (\gamma \beta^4 - \beta^2) \cdot \mathcal{F}_x \left( h \left( x \sqrt{\beta^2 - 1/\gamma} \right) \right) (\xi) \ d\beta \]

\[ = \int_\mathbb{R} \chi \hat{\eta}(\tau - (\gamma \beta^4 - \beta^2)) \frac{2i\gamma \beta^2 (i\beta + \sqrt{\beta^2 - 1/\gamma})}{\sqrt{\beta^2 - 1/\gamma}} \hat{f} (\gamma \beta^4 - \beta^2) \cdot \hat{h} \left( \xi / \sqrt{\beta^2 - 1/\gamma} \right) \ d\beta. \]

Since \( h \) is a Schwarz function, we have

\[ \left| \hat{h} \left( \xi / \sqrt{\beta^2 - 1/\gamma} \right) \right| \lesssim \frac{1}{1 + \xi^2 / (\beta^2 - 1/\gamma)} = \frac{\beta^2 - 1/\gamma}{\xi^2 + \beta^2 - 1/\gamma}. \]

Note that \( \eta \) is also a Schwarz function, then

\[ |\hat{\eta}(\tau - (\gamma \beta^4 - \beta^2))| \lesssim (\tau - (\gamma \beta^4 - \beta^2))^{-9/2+} \lesssim (\tau - (\gamma \beta^4 - \beta^2))^{-4} (\tau - \phi(\xi))^{-1/2+} (\phi(\xi) - (\gamma \beta^4 - \beta^2))^{-1/2+}. \]

Therefore,

\[ \| \eta A \|_{\chi^0, \frac{1}{2}} \lesssim \left\| (\tau - \phi(\xi))^{\frac{1}{2}} \right\|_{\mathbb{R}} \int_\mathbb{R} \chi \left| \hat{\eta}(\tau - (\gamma \beta^4 - \beta^2)) \right| \frac{\beta^2 - 1/\gamma}{\xi^2 + \beta^2 - 1/\gamma} \frac{2\gamma \beta^2 \sqrt{2\beta^2 - 1/\gamma}}{\sqrt{\beta^2 - 1/\gamma}} \left| f \right| d\beta \|_{L^2_{\xi, \tau}} \]

\[ \lesssim \left\| \int_\mathbb{R} \chi \left( \tau - (\gamma \beta^4 - \beta^2) \right)^{-4} \frac{\sqrt{\beta^2 - 1/\gamma}}{\xi^2 + \beta^2 - 1/\gamma} 2\gamma \beta^2 \sqrt{2\beta^2 - 1/\gamma} \left| f \right| d\beta \right\|_{L^2_{\xi, \tau}} \]

\[ \lesssim \left\| \int_\mathbb{R} \chi \left( \tau - (\gamma \beta^4 - \beta^2) \right)^{-4} \frac{\sqrt{\beta^2 - 1/\gamma}}{(\beta^2 - 1/\gamma)^\tau} 2\gamma \beta^2 \sqrt{2\beta^2 - 1/\gamma} \left| f \right| d\beta \right\|_{L^2_{\tau}} \]

\[ \lesssim \left\| \left( \int_\chi \chi(\cdot, \infty)(\tau)^{\frac{1}{2}} \left| f(z) \right|^2 dz \right)^\frac{1}{2} \lesssim \| \chi f \|_{\mathcal{H}^{\frac{1}{2}}(\mathbb{R})}. \]

The last line follows from the Young’s inequality. The procedure of \( B \) is the same, and the bound of \( \| \eta B \|_{\chi^0, \frac{1}{2}} \) is \( \| \eta \|_{\mathcal{H}^{\frac{1}{2}}(\mathbb{R})} \). Notice that for any \( s \in \mathbb{N} \), we have

\[ \partial_x^s (\eta A) = \int_\mathbb{R} \chi \hat{f}(\gamma \beta^4 - \beta^2) e^{i(\gamma \beta^4 - \beta^2) t} \left( x \sqrt{\beta^2 - 1/\gamma} \right) \left( \beta - 1/\gamma \right)^{\frac{1}{2}} \]

\[ \cdot \hat{f} (\gamma \beta^4 - \beta^2) \ d\beta, \]

and a similar formula for \( \partial_x^s (\eta B) \). Then by interpolation, we obtain the desired result for \( s > 0 \). \( \square \)
Lemma 3.4. For fixed $b < \frac{1}{2}$ and any smooth compactly supported function $\eta(t)$, we have
\[
\left\| \eta(t) \int_0^t S(t - t') F(t') dt' \right\|_{\mathcal{C}_0^0 H_x^{\frac{2s+3}{8}} (\mathbb{R} \times \mathbb{R})} \lesssim \begin{cases} 
\|F\|_{X^{s,-b}}, & \text{for } 0 \leq s \leq \frac{1}{2}, \\
\|F\|_{X^{s,-b}} + \|F\|_{X^{\frac{1}{2} + \frac{2s-1-3b}{8}}}, & \text{for } \frac{1}{2} \leq s \leq \frac{3}{2}.
\end{cases}
\]

In addition, we also have
\[
\left\| \eta(t) \partial_x \int_0^t S(t - t') F(t') dt' \right\|_{\mathcal{C}_0^0 H_x^{\frac{2s+1}{8}} (\mathbb{R} \times \mathbb{R})} \lesssim \begin{cases} 
\|F\|_{X^{s,-b}}, & \text{for } 0 \leq s \leq \frac{3}{2}, \\
\|F\|_{X^{s,-b}} + \|F\|_{X^{\frac{1}{2} + \frac{2s-1-3b}{8}}}, & \text{for } \frac{3}{2} \leq s \leq \frac{5}{2}.
\end{cases}
\]

Proof. Since the $X^{s,b}$ norm is independent of space translation, it suffices to bound $\eta D_0 \left( \int_0^t S(t - t') F(t') dt' \right)$. First we consider the case $0 \leq s \leq \frac{1}{2}$. At $x = 0$, we have
\[
D_0 \left( \int_0^t S(t - t') F(t') dt' \right) = \int_{\mathbb{R}} \int_0^t e^{i(t-t')\phi(\xi)} F(\xi, t') dt' d\xi,
\]
where $\phi(\xi) = \gamma \xi^4 - \xi^2$. Using the facts that
\[
F(\xi, t') = \int_{\mathbb{R}} e^{i\xi t'} \tilde{F}(\xi, \lambda) d\lambda,
\]
and
\[
\int_0^t e^{i\xi(t-t') \phi(\xi)} dt' = \frac{e^{i\xi(\lambda - \phi(\xi))} - 1}{i(\lambda - \phi(\xi))},
\]
we obtain
\[
D_0 \left( \int_0^t S(t - t') F(t') dt' \right) = \int_{\mathbb{R}^2} \frac{e^{i\xi(\lambda - \phi(\xi))} - e^{i\xi\phi(\xi)}}{i(\lambda - \phi(\xi))} \tilde{F}(\xi, \lambda) d\lambda d\xi.
\]
Define a smooth cut-off function $\psi$ in $[-1, 1]$, and let $\psi^c = 1 - \psi$. We write
\[
\eta(t) D_0 \left( \int_0^t S(t - t') F(t') dt' \right) = \eta(t) \int_{\mathbb{R}^2} \frac{e^{i\xi\lambda} - e^{i\xi\phi(\xi)}}{i(\lambda - \phi(\xi))} \psi^c(\lambda - \phi(\xi)) \tilde{F}(\xi, \lambda) d\lambda d\xi
\]
\[
+ \eta(t) \int_{\mathbb{R}^2} \frac{e^{i\xi\lambda}}{i(\lambda - \phi(\xi))} \psi^c(\lambda - \phi(\xi)) \tilde{F}(\xi, \lambda) d\lambda d\xi
\]
\[
- \eta(t) \int_{\mathbb{R}^2} \frac{e^{i\xi\phi(\xi)}}{i(\lambda - \phi(\xi))} \psi^c(\lambda - \phi(\xi)) \tilde{F}(\xi, \lambda) d\lambda d\xi := I + II + III.
\]
By Taylor expansion, we obtain
\[
\frac{e^{i\xi\lambda} - e^{i\xi\phi(\xi)}}{i(\lambda - \phi(\xi))} = i e^{i\xi\lambda} \sum_{k=1}^{\infty} \frac{(-i \xi)^k}{k!} (\lambda - \phi(\xi))^k - 1,
\]
which leads to
\[
\|I\|_{\mathcal{H}^{2\nu+3}} \lesssim \sum_{k=1}^{\infty} \frac{\|\eta(t) t^k\|_{\mathcal{H}^\nu}}{k!} \left\| \int_{\mathbb{R}^2} e^{it\lambda (\lambda - \phi)^{k-1}} \psi(\lambda - \phi) \hat{F}(\xi, \lambda) \, d\lambda d\xi \right\|_{\mathcal{H}^{2\nu+3}}
\lesssim \sum_{k=1}^{\infty} \frac{1}{(k-1)!} \left\| \langle \lambda \rangle^{2\nu+3} \int_{\mathbb{R}} (\lambda - \phi)^{k-1} \psi(\lambda - \phi) \hat{F}(\xi, \lambda) \, d\xi \right\|_{L^2_{\lambda}}
\lesssim \left\| \langle \lambda \rangle^{2\nu+3} \int_{\mathbb{R}} \psi(\lambda - \phi) \left| \hat{F}(\xi, \lambda) \right| \, d\xi \right\|_{L^2_{\lambda}}.
\]
By Cauchy-Schwarz inequality in $\xi$, we have
\[
\left( \int \langle \lambda \rangle^{2\nu+3} \left( \int_{|\lambda-\phi|<1} \langle \xi \rangle^{-2s} \, d\xi \right) \left( \int_{|\lambda-\phi|<1} \langle \xi \rangle^{2s} \left| \hat{F}(\xi, \lambda) \right|^2 \, d\xi \right) \, d\lambda \right)^{\frac{1}{2}}
\lesssim \sup_{\lambda} \left( \langle \lambda \rangle^{2\nu+3} \int_{|\lambda-\phi|<1} \langle \xi \rangle^{-2s} \, d\xi \right)^{\frac{1}{2}} \left\| \hat{F} \right\|_{X^{s,-b}_\lambda} \lesssim \|F\|_{X^{s,-b}}.
\]
The above supremum is finite since
\[
\langle \lambda \rangle^{2\nu+3} \int_{|\lambda-\phi|<1} \langle \xi \rangle^{-2s} \, d\xi \lesssim \begin{cases} 1, & \text{if } |\lambda| \lesssim 1, \\ \langle \lambda \rangle^{2\nu+3} \int_{|\lambda-\phi|<1} (z)^{-\frac{s}{2}} \, dz, & \text{if } |\lambda| \gg 1, \end{cases}
\]
where the latter bound from the change of variable $z = \xi^4$. The right-hand side is finite since the integrand is of order $|\lambda|^{-\frac{s}{2} - \frac{3}{4}}$ over an interval of length $\approx 1$.

For the second term $II$ we have
\[
\|II\|_{\mathcal{H}^{2\nu+3}} \lesssim \|\eta(t)\|_{\mathcal{H}^1} \left\| \langle \lambda \rangle^{2\nu+3} \int_{\mathbb{R}} \frac{1}{\lambda - \phi} \psi(\lambda - \phi) \hat{F}(\xi, \lambda) \, d\lambda \right\|_{L^2_{\lambda}}
\lesssim \left\| \langle \lambda \rangle^{2\nu+3} \int_{\mathbb{R}} \frac{1}{\lambda - \phi} \left| \hat{F}(\xi, \lambda) \right| \, d\xi \right\|_{L^2_{\lambda}}.
\]
By Cauchy-Schwarz inequality in $\xi$, we have
\[
\left( \int \langle \lambda \rangle^{2\nu+3} \left( \int \frac{1}{(\lambda - \phi)^{2-2b} \langle \xi \rangle^{2s}} \, d\xi \right) \left( \int \frac{(\xi)^{2s}}{(\lambda - \phi)^{2b}} \left| \hat{F}(\xi, \lambda) \right|^2 \, d\xi \right) \, d\lambda \right)^{\frac{1}{2}}
\lesssim \sup_{\lambda} \left( \langle \lambda \rangle^{2\nu+3} \int \frac{1}{(\lambda - \phi)^{2-2b} \langle \xi \rangle^{2s}} \, d\xi \right)^{\frac{1}{2}} \left\| \hat{F} \right\|_{X^{s,-b}}
\lesssim \|F\|_{X^{s,-b}},
\]
which can be bounded by $\|F\|_{X^{s,-b}}$ as long as
\[
\sup_{\lambda} \left( \langle \lambda \rangle^{2\nu+3} \int \frac{1}{(\lambda - \phi)^{2-2b} \langle \xi \rangle^{2s}} \, d\xi \right)
\]
is finite. Consider $|\phi| \ll 1$. In this case, apply a change of variable $z = \xi^4$ and Lemma 2.2 to bound the supremum by
\[
\sup_{\lambda} \left( \langle \lambda \rangle^{2\nu+3} \int \frac{1}{(\lambda - \phi)^{2-2b} \langle \xi \rangle^{\frac{s}{2} + \frac{3}{4}}} \, d\xi \right) \lesssim \sup_{\lambda} \langle \lambda \rangle^{\frac{s}{2} + \frac{3}{4} - \min\{2-2b, \frac{3}{2} + \frac{3}{4}\}},
\]
which is finite for \( b < \frac{1}{2}, \  s \leq \frac{1}{2} \). If \( |\phi| \gtrsim 1 \), by setting \( z = \xi^4 \) and apply Lemma 2.2 to bound that

\[
\sup_\lambda (\lambda^{\frac{2}{3}} + \frac{3}{4} - 2 + 2b) < \infty,
\]

for \( b < \frac{1}{2}, \  s \leq \frac{1}{2} \).

For the third term III, when \( |\xi| \leq \sqrt{\frac{2}{7}} \), we have the bound that

\[
\|\text{III}\|_{H^s_x} \lesssim \int_R \int_{|\xi| \leq \sqrt{\frac{2}{7}}} \|\eta e^{it\phi}\|_{H^s_x} \frac{\psi''(\lambda - \phi)}{|\lambda - \phi|} |\hat{F}(\xi, \lambda)| d\xi d\lambda
\]

\[
\lesssim \int_R \frac{\chi(-\frac{1}{2}, \frac{3}{2}) (\phi)}{\langle \lambda - \phi \rangle} |\hat{F}(\xi, \lambda)| d\xi d\lambda
\]

\[
\lesssim ||F||_{X^{s,-b}} \left\| \frac{\chi(-\frac{1}{2}, \frac{3}{2}) (\phi)}{\langle \lambda - \phi \rangle^{1 - b}} \right\|_{L^2_{\xi,\lambda}}
\]

\[
\lesssim ||F||_{X^{s,-b}}.
\]

We estimate III for \( |\xi| \geq \sqrt{\frac{2}{7}} \), and setting \( z = \gamma \xi^4 \). Then we have

\[
\left\| \eta(t) \int_{R^2} e^{it\phi(\xi)} \frac{\psi''(\lambda - \phi(\xi)) \hat{F}(\xi, \lambda)}{i(\lambda - \phi)} d\lambda d\xi \right\|_{H^s_x} \lesssim \left\| \frac{\chi(-\frac{1}{2}, \frac{3}{2}) (\phi)}{\langle \lambda - \phi \rangle} \right\|_{L^2_{\xi,\lambda}}
\]

By Cauchy-Schwarz inequality in \( \lambda \), this is bounded by

\[
\left\| \frac{(z)^{\frac{2s+3}{2}} |\hat{F}(\xi(z), \lambda)|}{\langle \lambda - \phi \rangle^b} \right\|_{L^2_{\xi} L^2_{\lambda}}.
\]

Changing variables back to \( \xi \), we have the bound \( ||F||_{X^{s,-b}} \) for \( b < \frac{1}{2} \). This completes the proof for \( 0 \leq s \leq \frac{1}{2} \). If \( s = \frac{9}{2}, \ 2s + 3 = \frac{3}{2} \), we have the following inequality from [12]:

\[
||f||_{H^\frac{s}{2}} \lesssim ||f||_{L^2} + ||f'||_{H^{\frac{s}{2}}}
\]

Note that

\[
\frac{d}{dt} \left[ D_0 \left( \int_0^t S(t-t') F dt' \right) \right]
\]

\[
= \eta'(t) D_0 \left( \int_0^t S(t-t') F dt' \right) + i \eta(t) \int_R \frac{\lambda e^{i\lambda t} - \phi(\xi)e^{i\phi t}}{i(\lambda - \phi)} \hat{F}(\xi, \lambda) d\xi d\lambda
\]

\[
= \eta'(t) D_0 \left( \int_0^t S(t-t') F dt' \right) + i \eta(t) \int_R \frac{e^{i\lambda t} - e^{i\phi t}}{i(\lambda - \phi)} \phi(\xi) \hat{F}(\xi, \lambda) d\xi d\lambda
\]

\[
+ i \eta(t) \int_R \frac{e^{i\lambda t}}{i(\lambda - \phi)} (\lambda - \phi) \hat{F}(\xi, \lambda) d\xi d\lambda.
\]

Using the result of \( s = \frac{1}{2} \), we bound the second integral term in the last equality for \( \phi(\xi) \hat{F}(\xi, \lambda) \), and the last integral term by the proof of II for \( (\lambda - \phi) \hat{F}(\xi, \lambda) \). Then
we have
\[ \left\| \frac{d}{dt} \left[ D_0 \left( \int_0^t S(t-t') F \, dt' \right) \right] \right\|_{H^{s/2}} \lesssim \| F \|_{X^{s/2, -b}}, \]
for any \( b < \frac{1}{2} \). Therefore, we obtain
\[ \left\| D_0 \left( \int_0^t S(t-t') F \, dt' \right) \right\|_{H^{2s+3}} \lesssim \begin{cases} \| F \|_{X^{s, -b}}, & \text{if } 0 \leq s \leq \frac{1}{2}, \\ \| F \|_{X^{s/2, -b}} + \| F \|_{X^{s, -b/2}}, & \text{if } s = \frac{9}{2}. \end{cases} \]
The desired result for \( \frac{1}{2} < s < \frac{9}{2} \) is obtained by interpolation.

For the estimates of the derivative term, the procedure is similar. We can write the integral as three pieces:
\[ \begin{align*}
\eta(t)D_0 \left( \int_0^t S(t-t') \partial_x F \, dt' \right) &= \eta(t) \int_{\mathbb{R}^2} e^{it\lambda} - e^{it\phi(\xi)} \frac{e^{it\psi(\lambda - \phi(\xi))\xi}}{i(\lambda - \phi(\xi))} d\lambda d\xi \\
&\quad + \eta(t) \int_{\mathbb{R}^2} \frac{e^{it\lambda}}{i(\lambda - \phi(\xi))} \psi^c(\lambda - \phi(\xi)) \xi \hat{F}(\xi, \lambda) d\lambda d\xi \\
&\quad - \eta(t) \int_{\mathbb{R}^2} \frac{e^{it\phi(\xi)}}{i(\lambda - \phi(\xi))} \psi^c(\lambda - \phi(\xi)) \xi \hat{F}(\xi, \lambda) d\lambda d\xi := \hat{I} + \hat{II} + \hat{III}.
\end{align*} \]

For \( s \geq 1 \), the estimates of the derivative term follow from the first part since \( \partial_x \) commutes with \( S(t) \). For \( 0 \leq s \leq 1 \), the proof is based on the argument of the first part. Notice that each term has the additional \( |\xi| \) factor from the \( \partial_x \) and we should take the index of time \( \frac{2s+1}{s} \) instead of \( \frac{2s+3}{s} \).

To estimate \( \hat{I} \), we need to establish that \( \sup_{\lambda} \langle \lambda \rangle^{2s+1} \int \langle \xi \rangle^{2-2s} \, d\xi \) is finite. Then this brings us back to the estimate of \( I \).

To estimate \( \hat{II} \), considering \( |\phi| \gg 1 \), we have \( \sup_{\lambda} \langle \lambda \rangle^{2s+1} - \min(2s-2b, \frac{5}{2} - 4) \), which is finite for \( b < \frac{1}{2}, s \leq \frac{3}{2} \). When \( |\phi| \lesssim 1 \), we have \( \sup_{\lambda} \langle \lambda \rangle^{2s+1+2b} \), which is finite for \( b < \frac{1}{2}, s \leq \frac{3}{2} \).

For \( \hat{III} \), when \( |\xi| \leq \sqrt{2/\gamma} \), the bounds are the same as those on \( III \). When \( |\xi| \geq \sqrt{2/\gamma} \), we change variables as we made for \( III \), and we are again back to the estimate of \( III \).

3.2. Nonlinear estimates. In this section we establish nonlinear estimates of (7) for our contraction argument. We recall Kato smoothing inequality and Maximal function inequality, the details of these inequalities refer to [17,25,30]. Before these, we define the notations
\[ f(\xi, \tau) = (\xi)^s \langle \tau - \phi(\xi) \rangle^{b} |\hat{u}(\xi, \tau)|, \]
\[ \sigma_j = \tau_j - \phi(\xi), \quad j = 0, \ldots, 5, \]
\[ \mathcal{F} F_i^0 = \frac{g(\xi_0, \tau_0)}{\langle \sigma_0 \rangle^i}, \quad \mathcal{F} F_i^j = \frac{f(\xi_j, \tau_j)}{\langle \sigma_j \rangle^i}, \quad j = 1, 2, 3 \text{ or } j = 1, 2, 3, 4, 5. \]

Denote \( D_x^s \) for the derivative of fractional order \( s \) with respect to \( x \). Denote the notation \( J_x^s \) as the integrals
\[ \begin{align*}
\int_{\xi_0 - \xi_1 + \xi_2 + \xi_3 - \xi_4 - \xi_5 = 0; \tau_0 - \tau_1 + \tau_2 - \tau_3 + \tau_4 - \tau_5 = 0} d\tau_1 d\tau_2 d\tau_3 d\tau_4 d\tau_5 d\xi_1 d\xi_2 d\xi_3 d\xi_4 d\xi_5, \\
\int_{\xi_0 - \xi_1 + \xi_2 - \xi_3 = 0; \tau_0 - \tau_1 + \tau_2 - \tau_3 = 0} d\tau_1 d\tau_2 d\tau_3 d\xi_1 d\xi_2 d\xi_3.
\end{align*} \]
Then we have
\[ \|D_x^{\frac{3}{2}}F_{\frac{1}{2}+}\|_{L^\infty_t L^2_x} \lesssim \|f\|_{L^2_{t,x}}, \quad (14) \]
\[ \|D_x^{\frac{1}{2}}F_{\frac{1}{2}+}\|_{L^\infty_t L^2_x} \lesssim \|f\|_{L^2_{t,x}}. \quad (15) \]

Interpolation (14) and (15) with Plancherel identity we obtain
\[ \|D_x^{\frac{3}{2}}F_{\frac{1}{2}+}\|_{L^\infty_t L^2_x} \lesssim \|f\|_{L^2_{t,x}}, \quad (16) \]
\[ \|D_x^{\frac{1}{2}}F_{\frac{1}{2}+}\|_{L^\infty_t L^2_x} \lesssim \|f\|_{L^2_{t,x}}. \quad (17) \]

By Sobolev embedding we have
\[ \left\| \frac{D_x^{-\frac{1}{2}} + \frac{3}{2}}{\frac{1}{2}} F_{\frac{1}{2}+} \right\|_{L^2_t L^6_x} \lesssim \|f\|_{L^2_{t,x}}, \quad 2 \leq p < \infty. \quad (18) \]

Strichartz estimate is given by
\[ \left\| F_{\frac{3}{2}} \right\|_{L^2_t L^6_x} \lesssim \|f\|_{L^2_{t,x}}. \quad (19) \]

Interpolation (19) with Plancherel identity, we have
\[ \left\| F_{\frac{3}{2} - \frac{3}{2}} \right\|_{L^2_t L^6_x} \lesssim \|f\|_{L^2_{t,x}}, \quad 2 < p < 6. \quad (20) \]

Interpolation (19) with (18), we have
\[ \left\| D_x^{\frac{3}{2} + \frac{3}{2}} F_{\frac{1}{2}+} \right\|_{L^2_t L^6_x} \lesssim \|f\|_{L^2_{t,x}}, \quad 6 < p < \infty. \quad (21) \]

By (20), (21) and Hölder inequality, for \(0 \leq c \leq \frac{1}{2}\), we have
\[ \left\| \left( \frac{f(\xi, \tau)}{\langle \sigma \rangle^{c+}} \right)^{\frac{1}{2}} \left( \frac{f(\xi, \tau)}{\langle \sigma \rangle^{\frac{3}{2}-c} \langle \sigma \rangle^{\frac{3}{2}} - c} \right)^{\frac{1}{2}} \right\|_{L^2_{t,x}} \lesssim \|f\|_{L^2_{t,x}}^{\frac{3}{2}}. \quad (22) \]

By (16), (17) and Hölder inequality, we have
\[ \left\| \left( \frac{f(\xi, \tau)}{\langle \sigma \rangle^{\frac{3}{2}}} - f(\xi, \tau) \right)^{\frac{1}{2}} \right\|_{L^2_{t,x}} \lesssim \|f\|_{L^2_{t,x}}^{\frac{3}{2}}. \quad (23) \]

**Lemma 3.5.** For \(s > 0\) and \(a < \min\{4s + 1, \frac{3}{2}\}\), there exists \(\epsilon > 0\) such that \(\frac{1}{2} - \epsilon < b < \frac{1}{2}\), we have
\[ \|u\|_{X^{s+a,b}} \lesssim \|u\|_{X^{s,b}}^{\frac{3}{2}}. \]

**Proof.** Note that \(|u|^4u = u\bar{u}|u|u\) and write it as a convolution, then we have
\[ \hat{|u|^4u}(\xi_0, \tau_0) = \int_x \hat{u}(\xi_1, \tau_1)\hat{u}(\xi_2, \tau_2)\hat{u}(\xi_3, \tau_3)\hat{u}(\xi_4, \tau_4)\hat{u}(\xi_5, \tau_5). \]

By duality, it suffices to prove that
\[ I := \int_x \left( \frac{\langle \xi_0 \rangle^{\frac{3}{2}}}{\langle \xi_0 \rangle^{\frac{3}{2}}} \right)^{\frac{1}{4}} \left( \frac{\langle \xi_0 \rangle^{\frac{3}{2}}}{\langle \xi_0 \rangle^{\frac{3}{2}}} - \frac{\langle \phi(\xi) \rangle^{\frac{3}{2}}}{\langle \phi(\xi) \rangle^{\frac{3}{2}}} \right)^{\frac{1}{4}} \lesssim \|f\|_{L^2} \|g\|_{L^2}. \]
By symmetry, we consider the case \(|\xi_1| \geq |\xi_2| \geq |\xi_3| \geq |\xi_4| \geq |\xi_5|\), which implies that \(|\xi_1| \gtrsim |\xi_0|\). We have 
\[
I \lesssim \sup \left\{ \frac{(\xi_2)^{0+} (\xi_3)^{0+} (\xi_4)^{0+} (\xi_5)^{-1/2} - (\xi_5)^{-1} (\xi_0)^{s+a}}{(\xi_1)^{s-a}} \prod_{j=1}^{5} (\xi_j)^{s} \times \int_{\mathbb{R}^5} \left( \frac{g(\xi_0, \tau_0) f(\xi_2, \tau_2) f(\xi_3, \tau_3)}{(\xi_2)^{0^+} (\xi_3)^{0^+} (\sigma_0)^{b} (\sigma_2)^{b} (\sigma_3)^{b}} \frac{(\xi_4)^{s} (\xi_5)^{s-a} - (\theta_1)^{s-a} (\theta_4)^{b} (\theta_5)^{b}}{(\xi_1)^{s-a}} \frac{f(\xi_1, \tau_1) f(\xi_4, \tau_4) f(\xi_5, \tau_5)}{(\xi_1)^{s-a}} \right) \right\}.
\]

It is easy to bound the supremum by 
\[
\sup \frac{(\xi_2)^{0+} (\xi_3)^{0+} (\xi_4)^{0+} (\xi_5)^{-1/2} - (\xi_5)^{-1} (\xi_0)^{s+a}}{(\xi_1)^{s-a}} \prod_{j=1}^{5} (\xi_j)^{s} \lesssim \sup \left\{ \frac{(\xi_1)^{a-4s+1+}}{(\xi_1)^{a-4s+1+}} \right\} \text{ for } s \geq \frac{1}{4},
\]
which is finite for \(a < \min\{4s + 1, \frac{3}{2}\}\). Then we can achieve \(I \lesssim \|f\|_{L^2} \|g\|_{L^2}\) by the Plancherel identity, (22) with \(c = \frac{1}{2}\) and (23).

\[\text{Lemma 3.6. For } s \in (0, \frac{9}{2}) \text{ and } \frac{1}{2} - s < a < \min\{4s, \frac{9}{2}, \frac{9}{2} - s\}, \text{ there exists } c > 0 \text{ such that } \frac{1}{2} - c < b < \frac{1}{2}, \text{ we have } \]
\[
\|u\|_{X, \frac{1}{2}, 2a - 1 - nb}^4 \lesssim \|u\|_{X, s, b}^4.
\]

\[\text{Proof.} \text{ By duality, as in Lemma 3.5, it suffices to prove that }
\]
\[
I := \int_{\mathbb{R}^5} \frac{(\xi_0)^{1/2} (\tau_0 - \phi(\xi_0))^{2a+2a-1} - (\tau_5 - \phi(\xi_5))^b}{(\xi_1)^{s-a}} \prod_{j=1}^{5} (\xi_j)^{s} \prod_{j=1}^{5} (\tau_j - \phi(\xi_j))^b \lesssim \|f\|_{L^2} \|g\|_{L^2}.
\]

There are three cases for the sign of \(s + a\):

\[\text{Case 1. } \frac{5}{2} \leq s + a < \frac{9}{2}. \text{ Note that }
\]
\[
\langle \tau_0 - \phi(\xi_0) \rangle \lesssim \langle \xi_{\max} \rangle^4 \max_{j=1, \ldots, b} (\tau_j - \phi(\xi_j))
\]

Without loss of generality, we can set \(\max_{j=1, \ldots, b} (\tau_j - \phi(\xi_j)) = (\tau_5 - \phi(\xi_5))\). Then we have
\[
\frac{(\tau_0 - \phi(\xi_0))^{2a+2a-1} - (\tau_5 - \phi(\xi_5))^b}{(\xi_1)^{s-a}} \lesssim \langle \xi_{\max} \rangle^{s+a-4} \prod_{j=1}^{5} (\tau_j - \phi(\xi_j))^{2b}.
\]
Using (24) and the Cauchy-Schwarz inequality, \(I\) is bounded by
\[
\|f\|_{L^2} \|g\|_{L^2} \sup_{\xi_0} \int_{\mathbb{R}^5} \frac{(\xi_0)^{1/2} (\xi_{\max})^{2(s+a-1)} - (\xi_1)^{2s} \prod_{j=1}^{5} (\xi_j)^{2s}}{\prod_{j=1}^{5} (\xi_j)^{2s}}.
\]

The supremum is bounded by \(\sup \langle \xi_0 \rangle^{1/2} (\xi_{\max})^{2s+2a-1 - nb} \), which is finite for \(a \leq 4b\).

We now consider the case \( \frac{1}{2} < s + a < \frac{5}{2}\). By symmetry, we can restrict a case: \(|\xi_1| \geq |\xi_2| \geq |\xi_3| \geq |\xi_4| \geq |\xi_5|\), which implies that \(|\xi_1| \gtrsim |\xi_0|\). We separate the domain of \(s + a\) into two cases: \(s + a \leq \frac{3}{2}\) and \(s + a > \frac{3}{2}\).

\[\text{Case 2. } \frac{1}{2} < s + a \leq \frac{3}{2}. \text{ By (22) with } c = \frac{5-2(s+a)}{8} \in (\frac{1}{4}, \frac{1}{2}) \text{ for } f(\xi_1, \tau_1), f(\xi_2, \tau_2), \]
f(\xi_3, \tau_3) and (22) with \(c = \frac{5-2(s+a)}{8} \in (\frac{1}{4}, \frac{1}{2}) \) for \(g(\xi_0, \tau_0), f(\xi_4, \tau_4), f(\xi_5, \tau_5)\), we need to bound
\[
\sup \langle \xi_2 \rangle^{1/2-\epsilon} \langle \xi_3 \rangle^{1/2-\epsilon} \langle \xi_4 \rangle^{1/2-\epsilon} \langle \xi_5 \rangle^{1/2-\epsilon} \prod_{j=1}^{5} (\xi_j)^{s}.
\]
which is finite for $a < \min\{4s, \frac{3}{2}\}$.

**Case 3.** $\frac{3}{2} < s + a < \frac{5}{2}$. By (22) with $c = \frac{5 - 2(s+a)}{8} \in (0, \frac{1}{4})$ for $f(\xi_1, \tau_1)$, $f(\xi_2, \tau_2)$, $f(\xi_3, \tau_3)$ and (22) with $c = \frac{5 - 2(s+a)}{8} \in (0, \frac{1}{4})$ for $g(\xi_0, \tau_0)$, $f(\xi_4, \tau_4)$, $f(\xi_5, \tau_5)$, we need to bound

$$\sup(\xi_2)^{\frac{1}{2} - c^+} \langle \xi_3 \rangle^{\frac{1}{2} - c^+} \langle \xi_4 \rangle^{\frac{1}{2} - c^+} \langle \xi_5 \rangle^{\frac{1}{2} - c^+} \frac{\langle \xi_0 \rangle^{\frac{1}{2}}}{\prod_{j=1}^{5}} \langle \xi_j \rangle^{s}$$

which is finite for $a < \min\{4s, \frac{3}{2}\}$.

**Lemma 3.7.** For $s > 0$ and $a < \min\{2s + 1, \frac{3}{2}\}$, there exists $\epsilon > 0$ such that $\frac{1}{2} - \epsilon < b < \frac{1}{2}$, we have

$$\|u\|^2 \|u\|^3_{X^{s,a,-b}} \lesssim \|u\|^3_{X^{s,b}} .$$

In addition, for $s \in (0, \frac{2}{7})$ and $a < \min\{2s, \frac{1}{2}, \frac{9}{2} - s\}$, there exists $\epsilon > 0$ such that $\frac{1}{2} - \epsilon < b < \frac{1}{2}$, we have

$$\|u\|^2 \|u\|^3_{X^{\frac{1}{2} + 2s - \frac{1}{8}, -a}} \lesssim \|u\|^3_{X^{\frac{1}{2}, b}} .$$

**Proof.** For (25), as in Lemma 3.5, by duality it suffices to show that

$$I := \int \langle \xi_0 \rangle^{s+a} \langle \xi_0, \tau_0 \rangle \prod_{j=1}^{5} f(\xi_j, \tau_j) \prod_{j=1}^{5} \langle \xi_j \rangle^{s} \|f\|_{L^2} \|g\|_{L^2} .$$

By symmetry, we consider the case $|\xi_1| \geq |\xi_2| \geq |\xi_3|$, which implies that $|\xi_1| \geq |\xi_0|$. We have

$$I \lesssim \sup \left( \frac{\langle \xi_2 \rangle^{\frac{1}{2} - \langle \xi_3 \rangle^{\frac{1}{2} - \langle \xi_4 \rangle^{s+a}}}{\langle \xi_1 \rangle^{\frac{1}{2} - \prod_{j=1}^{5} \langle \xi_j \rangle^{s}}} \times \int \left( \langle \xi_1 \rangle^{\frac{1}{2} - \langle \xi_1, \tau_1 \rangle f(\xi_2, \tau_2) f(\xi_3, \tau_3) \langle \xi_0, \tau_0 \rangle \prod_{j=1}^{5} \langle \xi_j \rangle^{s} \langle \tau_j - \phi(\xi_j) \rangle^{b} \right) .

It is easy to obtain that the supremum is finite if $a < \min\{2s + 1, \frac{3}{2}\}$. Then we can achieve $I \lesssim \|f\|_{L^2} \|g\|_{L^2}$ by the Plancherel identity, (23) and (20). This completes the proof of (25).

For (26), as in Lemma 3.6, by duality it suffices to show that

$$I := \int \langle \xi_0 \rangle^{\frac{1}{2} - \langle \xi_0 - \phi(\xi_0) \rangle^{2s+2a-1-8b}} \langle \xi_0, \tau_0 \rangle \prod_{j=1}^{5} f(\xi_j, \tau_j) \prod_{j=1}^{5} \langle \xi_j \rangle^{s} \langle \tau_j - \phi(\xi_j) \rangle^{b} \lesssim \|f\|_{L^2} \|g\|_{L^2} .$$

There are three cases for the sign of $s + a$:

**Case 1.** $\frac{5}{2} \leq s + a < \frac{9}{2}$, Note that

$$\langle \tau_0 - \phi(\xi_0) \rangle \lesssim \langle \xi_{\text{max}} \rangle^{4} \max_{j=1, \ldots, 3} \langle \tau_j - \phi(\xi_j) \rangle .$$

Without loss of generality, we can set $\max_{j=1, \ldots, 3} \langle \tau_j - \phi(\xi_j) \rangle = \langle \tau_3 - \phi(\xi_3) \rangle$. Then $I$ is bounded by

$$\|f\|_{L^2}^{2s+2a-1-8b} \|g\|_{L^2}^{2s} \max_{\xi_0} \int \langle \xi_0 \rangle^{\langle \xi_{\text{max}} \rangle^{2s+2a-1-8b}} \prod_{j=1}^{5} \langle \xi_j \rangle^{2s} .$$

The supremum is bounded by $\max_{\xi_0} \langle \xi_{\text{max}} \rangle^{2s+2a-1-8b}$, which is finite for $a \leq 4b$. 


By symmetry, we can restrict a case: $|\xi_1| \geq |\xi_2| \geq |\xi_3|$, which implies that $|\xi_1| \gtrsim |\xi_0|$. We separate the domain of $s + a$ into two cases: $s + a < \frac{3}{2}$ and $s + a \geq \frac{3}{2}$.

**Case 2.** $\frac{1}{2} < s + a < \frac{3}{2}$. By (22) with $c = \frac{5 - 2(s + a)}{8} \in (\frac{1}{4}, \frac{1}{2})$ for $g(\xi_0, \tau_0)$, $f(\xi_2, \tau_2)$, $f(\xi_3, \tau_3)$, and (21) for $f(\xi_1, \tau_1)$, we need to bound that

$$\sup\langle\xi_1\rangle^{1-2c+}\langle\xi_2\rangle^{\frac{1}{2}-c+}\langle\xi_3\rangle^{\frac{1}{2}-c+} \frac{\langle\xi_0\rangle^{\frac{3}{2}}}{\prod_{j=1}^{3}\langle\xi_j\rangle^{s}},$$

which is finite for $a < \min\{2s, \frac{1}{2}\}$.

**Case 3.** $\frac{3}{2} < s + a < \frac{5}{2}$. By (22) with $c = \frac{5 - 2(s + a)}{8} \in (0, \frac{1}{4})$ for $g(\xi_0, \tau_0)$, $f(\xi_2, \tau_2)$, $f(\xi_3, \tau_3)$, and (21) for $f(\xi_1, \tau_1)$, we need to bound that

$$\sup\langle\xi_1\rangle^{\frac{1}{2}-2c+}\langle\xi_2\rangle^{\frac{1}{2}-c+}\langle\xi_3\rangle^{\frac{1}{2}-c+} \frac{\langle\xi_0\rangle^{\frac{3}{2}}}{\prod_{j=1}^{3}\langle\xi_j\rangle^{s}},$$

which is finite for $a < \min\{2s + \frac{1}{2}, 1\}$.

This completes the proof of (26). \hfill \Box

**Lemma 3.8.** For $s > \frac{1}{2}$ and $a < \min\{2s - 1, \frac{5}{4}\}$, there exists $\epsilon > 0$ such that $\frac{1}{2} - \epsilon < b < \frac{1}{2}$, we have

$$\|\bar{u}_x^2\|_{X^{s+a,-b}} \lesssim \|u\|^3_{X^{s,b}}. \quad (27)$$

In addition, for $s \in (\frac{1}{2}, \frac{3}{2})$ and $a < \min\{2s - 1, \frac{5}{4}, \frac{9}{4} - s\}$, there exists $\epsilon > 0$ such that $\frac{1}{2} - \epsilon < b < \frac{1}{2}$, we have

$$\|\bar{u}_x^2\|_{X^{\frac{1}{2}+\frac{2a+1-4b}{s},b}} \lesssim \|u\|^3_{X^{s,b}}. \quad (28)$$

**Proof.** For (27), by Fourier transform and duality as in Lemma 3.6, it suffices to prove that

$$I := \int \frac{\langle\xi_0\rangle^{s+a}\langle\xi_1\rangle\langle\xi_2\rangle g(\xi_0, \tau_0) \prod_{j=1}^{3} f(\xi_j, \tau_j)}{\prod_{j=1}^{3} \langle\xi_j\rangle^{s_b} \prod_{j=0}^{3} \langle\sigma_j\rangle^{b}} \lesssim \|f\|_{L^2} \|g\|_{L^2},$$

where $\sigma_j = \tau_j - \phi(\xi_j)$. Note that

$$\prod_{j=0}^{3} \langle\tau_j - \phi(\xi_j)\rangle^{b} \gtrsim \langle(\xi_0 - \xi_1)(\xi_0 - \xi_3)\rangle^{\frac{1}{2}} - \frac{\prod_{j=0}^{3} \langle\tau_j - \phi(\xi_j)\rangle^{\frac{1}{2}+}}{\max_{j=0,\ldots,3} \langle\tau_j - \phi(\xi_j)\rangle^{\frac{1}{2}+}},$$

where using a fact that $\max\{|\sigma_0|, |\sigma_1|, |\sigma_2|, |\sigma_3|\} \gtrsim |\xi_0 - \xi_1||\xi_0 - \xi_3|$ if $|\xi_0 - \xi_1| \geq k = 1/\sqrt{s}$, $|\xi_0 - \xi_3| \geq k$, $|\xi_1| \geq 2k$ and $|\xi_3| \geq 2k$ in [17].

Let

$$K(\xi_0, \xi_1, \xi_2, \xi_3) = \frac{\langle\xi_0\rangle^{s+a}\langle\xi_1\rangle\langle\xi_2\rangle}{\langle\xi_1\rangle^{s}\langle\xi_2\rangle^{s}\langle\xi_3\rangle^{s}}.$$

We split the integral domain into the following several cases.

**Case 1.** $|\xi_0 - \xi_1| \leq k$ or $|\xi_0 - \xi_3| \leq k$.

We can assume that $|\xi_0 - \xi_1| \leq k$. Then we have $\langle\xi_2\rangle \sim \langle\xi_3\rangle$ which follows that $\xi_0 = \xi_1 - \xi_2 + \xi_3$. Note that $\langle\xi_1\rangle\langle\xi_0 - \xi_1 + \xi_2\rangle \sim \langle\xi_0\rangle\langle\xi_2\rangle$.

**Subcase 1.** $\langle\xi_2\rangle \sim \langle\xi_3\rangle \leq 2k$. We have

$$K(\xi_0, \xi_1, \xi_2, \xi_3) \lesssim \sup_{\xi_0} \langle\xi_0\rangle^{1-2s+a} \lesssim 1,$$
which holds for $a < 2s - 1$. Then by Lemma 2.6 in [17], we can bound $I$ by
\[
I \lesssim \|F^0\|_{L^2_x L^7} \|F^2\|_{L^5_x L^9} \|F^3\|_{L^5_x L^9}
\lesssim \|f\|_{L^2_{x,t}} \|g\|_{L^2_{x,t}}.
\]

**Subcase 2.** $(\xi_2) \sim (\xi_3) \geq 2k$. We have
\[
K(\xi_0, \xi_1, \xi_2, \xi_3) \lesssim |\xi_1|.
\]
Using Lemma 2.7-2.10 in [17], we can bound $I$ by $\|f\|^3_{L^2_x} \|g\|_{L^2_x}$.

**Case 2.** $|\xi_0 - \xi_1| \geq k$ and $|\xi_0 - \xi_3| \geq k$.

**Subcase 1.** $|\xi_1| \leq 2k$ or $|\xi_3| \leq 2k$. By symmetry, without loss of generality, we can assume that $|\xi_1| \leq 2k$. If $|\xi_2| \leq 2k$, we obtain
\[
K(\xi_0, \xi_1, \xi_2, \xi_3) \lesssim \sup_{\xi_0} |\xi_0|^{-2s+a},
\]
which is finite for $a < 2s$. This leads to $I \lesssim \|f\|^3_{L^2_x} \|g\|_{L^2_x}$.

If $|\xi_2| \geq 2k$, we obtain
\[
K(\xi_0, \xi_1, \xi_2, \xi_3) \lesssim |\xi_1|.
\]
By the argument in [17], we follow the same line to bound $I$ by $\|f\|^3_{L^2_x} \|g\|_{L^2_x}$.

**Subcase 2.** $|\xi_1| \geq 2k$ and $|\xi_3| \geq 2k$. Using (14) and (15), we have
\[
\min \left( \frac{\langle \xi_1 \rangle^\frac{1}{4} \langle \xi_2 \rangle^\frac{1}{4} \langle \xi_3 \rangle^\frac{1}{4}}{\langle \xi \rangle^\frac{1}{2}}, \frac{\langle \xi_2 \rangle^\frac{1}{4} \langle \xi_3 \rangle^\frac{1}{4}}{\langle \xi \rangle^\frac{1}{2}}, \frac{\langle \xi_1 \rangle^\frac{1}{4} \langle \xi_3 \rangle^\frac{1}{4}}{\langle \xi \rangle^\frac{1}{2}} \right) \lesssim \frac{\langle \xi_1 \rangle^\frac{1}{4} \langle \xi_2 \rangle^\frac{1}{4} \langle \xi_3 \rangle^\frac{1}{4}}{\langle \xi \rangle^\frac{1}{2} + \langle \xi_2 \rangle^\frac{1}{4} + \langle \xi_3 \rangle^\frac{1}{4}}.
\]
Restricting the variables $\xi_0, \cdots, \xi_3$ by the size as $|\xi_{\min}| \leq |\xi_{\mid ID}| \leq |\xi_{\max}|$, we have
\[
\max_{0 \leq i, j, k \leq 3} \frac{\langle \xi_0 \rangle^\frac{1}{4} \langle \xi_1 \rangle^\frac{1}{4} \langle \xi_2 \rangle^\frac{1}{4} \langle \xi_3 \rangle^\frac{1}{4}}{\langle \xi \rangle^\frac{1}{2} + \langle \xi_2 \rangle^\frac{1}{4} + \langle \xi_3 \rangle^\frac{1}{4}} \lesssim \frac{\langle \xi_{\mid ID} \rangle^\frac{1}{4}}{\langle \xi_{\max} \rangle^\frac{1}{4}}.
\]
Then we need to bound that
\[
\sup_{\xi_0 - \xi_1 + \xi_2 - \xi_3 = 0} \frac{\langle \xi_0 \rangle^\frac{a + s}{4} \langle \xi_1 \rangle^{1-s} \langle \xi_2 \rangle^{1-s} \langle \xi_3 \rangle^{-s}}{\langle \xi_0 - \xi_1 \rangle^\frac{1}{2} - \langle \xi_0 - \xi_3 \rangle^\frac{1}{2} - \langle \xi_0 \rangle^{\frac{a+s}{4}}} \lesssim 1.
\]
By symmetry, when $\xi_0 - \xi_1 + \xi_2 - \xi_3 = 0$ and $|\xi_1| \leq |\xi_3|$, it suffices to show that
\[
\frac{\langle \xi_0 \rangle^\frac{a+s}{4} \langle \xi_1 \rangle^{1-s} \langle \xi_2 \rangle^{1-s} \langle \xi_3 \rangle^{-s}}{\langle \xi_0 - \xi_1 \rangle^\frac{1}{2} - \langle \xi_0 - \xi_3 \rangle^\frac{1}{2} - \langle \xi_0 \rangle^{\frac{a+s}{4}}} \lesssim 1.
\]
In the case $|\xi_1| \leq |\xi_3| \leq |\xi_2| \approx |\xi_0|$, we have
\[
(29) \lesssim \frac{\langle \xi_0 \rangle^{a-\frac{1}{4}}}{\langle \xi_1 \rangle^{a-1} \langle \xi_0 - \xi_1 \rangle^\frac{1}{2} - \langle \xi_0 - \xi_3 \rangle^\frac{1}{2} - \langle \xi_0 \rangle^{a-\frac{1}{4}}}.
\]
For $s > \frac{3}{4}$, this is bounded by $\langle \xi_0 \rangle^{a-\frac{1}{4}} \lesssim 1$ provided that $a < \frac{5}{4}$. For $\frac{1}{2} < s < \frac{3}{2}$, we have
\[
\frac{\langle \xi_0 \rangle^{a-\frac{1}{4}}}{\langle \xi_1 \rangle^{a-1} \langle \xi_0 - \xi_3 \rangle^\frac{1}{2} - \langle \xi_0 - \xi_3 \rangle^\frac{1}{2} - \langle \xi_0 \rangle^{a-\frac{1}{4}}} \lesssim \frac{\langle \xi_0 \rangle^{-\frac{1}{4} + a - s}}{\langle \xi_2 \rangle^{\min(\frac{s-\frac{3}{2}, \frac{s}{2})}}} \lesssim \langle \xi_0 \rangle^{-\frac{1}{4} + a - s - \min(\frac{s-\frac{3}{2}, \frac{s}{2}-\frac{3}{2})} \lesssim 1,
\]
which holds for $a < \min\{2s - 1, \frac{5}{4}\}$.  

In the case $|\xi_1| \leq |\xi_3| \approx |\xi_0|$, we have
\[ (29) \lesssim \frac{(\xi_0)^{a-\frac{4}{s}}(\xi_1)^{1-s}(\xi_2)^{1-s}(\xi_1)^{\frac{2}{s}} + (\xi_2)^{\frac{2}{s}})}{(\xi_0)^{\frac{a}{s}} - (\xi_1 - \xi_2)^{\frac{a}{s}}} \lesssim (\xi_0)^{a-2s} \lesssim 1, \]
which holds for $a < 2s$.
In the case $|\xi_1| \leq |\xi_3| \approx |\xi_0|$, we have
\[ (29) \lesssim \frac{(\xi_0)^{s+a}(\xi_2)^{-\frac{3}{s}} - 2s^+ (\xi_0)^{\frac{1}{s}} + (\xi_2)^{\frac{1}{s}})}{(\xi_1)^{s-1}(\xi_0 - \xi_1)^{\frac{a}{s}}} \lesssim (\xi_0)^{a+\frac{a}{s}} + (\xi_3)^{-\frac{3}{s} - 2s^+} \lesssim 1, \]
which holds for $a < 2s$.
In the case $|\xi_0|, |\xi_2| \ll |\xi_1| \approx |\xi_3|$, we have
\[ (29) \lesssim (\xi_0)^{a+|a|}(\xi_2)^{1-s}(\xi_3)^{-2s} - 2s^+ (\xi_0)^{\frac{1}{s}} + (\xi_2)^{\frac{1}{s}} \lesssim (\xi_3)^{a-2s} \lesssim 1, \]
which holds for $a < 2s$.

This completes the proof of (27).

For (28), as in Lemma 3.6, by duality and letting $b = \frac{1}{2} - \frac{1}{s}$, it suffices to show that
\[ I := \int (\xi_0)^{\frac{3}{2} + s} \langle \xi_1 \rangle \langle \xi_2 \rangle g(\xi_0, \tau_0) \prod_{j=1}^{3} f(\xi_j; \tau_j) \lesssim \|f\|_{L^2} \|g\|_{L^2}. \]

**Situation 1.** $\frac{1}{2} < s + a < \frac{3}{2}$. As in the proof of the first part, we have
\[ \langle \sigma_0 \rangle \lesssim \prod_{j=1}^{3} \langle \sigma_j \rangle \lesssim \langle (\xi_0 - \xi_1)(\xi_0 - \xi_3) \rangle^{\frac{s-2(s+a)}{s}} \lesssim \prod_{j=0}^{3} (\sigma_j + \frac{a}{2}) \lesssim \max_{j=0, \ldots, 3}(\sigma_j)^{s+a}. \]

Using (14) and (15), and taking the size of $\xi_{\text{max}}$ and $\xi_{\text{mid}}$ as in the first part, we need to bound that
\[ \sup_{\xi_0 - \xi_1 + \xi_2 - \xi_3 = 0} \frac{(\xi_0)^{\frac{3}{2} + s}}{(\xi_0 - \xi_1)(\xi_0 - \xi_3)} \lesssim \langle (\xi_0 - \xi_1)(\xi_0 - \xi_3) \rangle^{\frac{s-2(s+a)}{s}} \lesssim \langle \xi_{\text{max}} \rangle^s \lesssim 1. \]

The case $|\xi_0 - \xi_1| \leq k$ or $|\xi_0 - \xi_3| \leq k$: the case $|\xi_0 - \xi_1| \geq k$ and $|\xi_0 - \xi_4| \geq k$, $|\xi_1| \leq 2k$ or $|\xi_4| \leq 2k$ are similar to the first part. Now we consider the case: $|\xi_1| \geq 2k$ and $|\xi_3| \geq 2k$. By symmetry, when $\xi_0 - \xi_1 + \xi_2 - \xi_3 = 0$ and $|\xi_1| \leq |\xi_3|$, it suffices to show that
\[ \langle (\xi_0 - \xi_1)(\xi_0 - \xi_3) \rangle^{\frac{s-2(s+a)}{s}} \lesssim \langle \xi_{\text{mid}} \rangle^s \lesssim 1. \]

In the case $|\xi_1| \leq |\xi_3| \lesssim |\xi_2| \approx |\xi_0|$, we have
\[ (30) \lesssim (\xi_0)^{3-2s} (\xi_1)^{-\frac{3}{s} - 2s^+} \lesssim 1, \]
which holds for $a < 3s$.
In the case $|\xi_1| \leq |\xi_3| \approx |\xi_0|$, we have
\[ (30) \lesssim (\xi_0)^{-\frac{3}{s} - 2s^+} (\xi_1)^{1-s}(\xi_2)^{1-s}(\xi_1)^{\frac{2}{s}} + (\xi_2)^{\frac{2}{s}} \lesssim (\xi_0)^{-\frac{3}{s} - 2s^+ + \frac{3}{s} + 2a} (\xi_2)^{\frac{2}{s} + 3s^+ + 4a} \lesssim 1, \]
which holds for $a < 5s$. 

In the case $|\xi_1| \leq |\xi_3| \approx |\xi_2|$, $|\xi_3| \gg |\xi_0|$, we have
\[
(30) \lesssim \langle \xi_0 \rangle^{\frac{1}{2}} \langle \xi_1 \rangle^{1-s} \langle \xi_3 \rangle^{s-\frac{1}{2}s+\frac{1}{2}a+((\xi_1) \frac{1}{2} + (\xi_0) \frac{1}{2})} \langle \xi_0 - \xi_1 \rangle^{\frac{\frac{1}{2} - 2(s+a)}{8}} \\
\lesssim \langle \xi_0 \rangle^{-\frac{1}{2} + \frac{1}{2}s + \frac{1}{2}a + \langle \xi_2 \rangle^{-\frac{1}{2} - \frac{1}{2}s + \frac{1}{2}a}} \lesssim 1,
\]
which holds for $a < 3s + \frac{1}{2}$.

In the case $|\xi_0|, |\xi_2| \ll |\xi_1| \approx |\xi_3|$, we have
\[
(30) \lesssim \langle \xi_0 \rangle^{\frac{1}{2}} \langle \xi_2 \rangle^{1-s} \langle \xi_3 \rangle^{-\frac{1}{2} - \frac{1}{2}s + \frac{1}{2}a+((\xi_0) \frac{1}{2} + (\xi_2) \frac{1}{2})} \lesssim 1,
\]
which holds for $a < 2s - 1$.

**Situation 2.** $\frac{5}{2} \leq s + a < \frac{9}{2}$. Restricting the variables $\xi_1, \xi_2, \xi_3$ by the size as $|\xi_{\text{min}}| \leq |\xi_{\text{mid}}| \leq |\xi_{\text{max}}|$, we have
\[
\langle \sigma_0 \rangle \lesssim \max \{\langle \xi_{\text{max}} \rangle^4, \langle \sigma_1 \rangle, \langle \sigma_2 \rangle, \langle \sigma_3 \rangle\}.
\]
If the maximum is one of $\langle \sigma_j \rangle$, without loss of generality, we write that the maximum is $\langle \sigma_3 \rangle$. Then we obtain
\[
\langle \sigma_0 \rangle^{\frac{5 - 2(s + a)}{8} - \frac{3}{2}} \prod_{j=1}^{3} (\sigma_j)^{\frac{1}{2} - s} \gtrsim \langle \sigma_1 \rangle^{\frac{1}{2} - s} \langle \sigma_2 \rangle^{\frac{1}{2} - s} \langle \sigma_3 \rangle^{\frac{1}{2} + \frac{5 - 2(s + a)}{8}} \\
\gtrsim \langle \sigma_1 \rangle^{\frac{1}{2} + \sigma} \langle \sigma_2 \rangle^{\frac{1}{2} + \sigma} \langle \xi_{\text{max}} \rangle^{\frac{1}{2} - s(s + a)} + \langle \xi_{\text{max}} \rangle^{\frac{1}{2} - s(s + a)} + \langle \xi_{\text{max}} \rangle^{\frac{1}{2} - s(s + a)}.
\]

By the Cauchy-Schwarz inequality, it suffices to prove that
\[
\sup_{\xi} \int_{\mathbb{R}^3} \psi_{\max} \psi_0 \langle \xi_{\text{max}} \rangle^{9 - 2(s + a) - 8} \prod_{j=1}^{3} \langle \xi_j \rangle^{2s} \lesssim 1.
\]

Using Lemma 2.2 and $|\xi_{\text{max}}| \gtrsim |\xi_0|$, we obtain
\[
\int \psi_{\text{max}} \psi_0 \langle \xi_{\text{max}} \rangle^{2(s + a) - 8} \prod_{j=1}^{3} \langle \xi_j \rangle^{2s} d\xi_1 d\xi_3 \lesssim \langle \xi_0 \rangle^{2(s + a) - 8} \langle \xi_0 \rangle^{2s + 2} \lesssim 1
\]
provided that $a < 3$.

If the maximum is $\langle \xi_{\text{max}} \rangle^4$, we have
\[
\langle \sigma_0 \rangle^{\frac{5 - 2(s + a)}{8} - \frac{3}{2}} \prod_{j=1}^{3} (\sigma_j)^{\frac{1}{2} - s} \gtrsim \langle \xi_{\text{max}} \rangle^{\frac{5 - 2(s + a)}{8} - \frac{3}{2}} \prod_{j=1}^{3} (\sigma_j)^{\frac{1}{2} - s}.
\]
Thus, by (14) and (15), it suffices to prove that
\[
\langle \xi_0 \rangle^{\frac{1}{2}} \langle \xi_1 \rangle^{1-s} \langle \xi_2 \rangle^{1-s} \langle \xi_3 \rangle^{-s} \langle \xi_{\text{max}} \rangle^{s+a-\frac{1}{2}} \frac{\langle \xi_{\text{mid}} \rangle^{\frac{1}{2}}}{\langle \xi_{\text{max}} \rangle^{\frac{1}{2}}} \lesssim 1.
\]

We can bound this by
\[
\frac{\langle \xi_{\text{max}} \rangle^{s+a-\frac{1}{2}} \langle \xi_{\text{mid}} \rangle^{\frac{1}{2}}}{\prod_{j=1}^{3} \langle \xi_j \rangle^{s}} \lesssim \frac{\langle \xi_{\text{max}} \rangle^{s+a-\frac{1}{2}} \langle \xi_{\text{max}} \rangle^{\frac{1}{2}}}{\langle \xi_{\text{max}} \rangle^{s} \langle \xi_{\text{mid}} \rangle^{s}} \lesssim 1
\]
provided that $a < \frac{5}{4}$.

This completes the proof of (28). \(\square\)

**Lemma 3.9.** For $s > 1$ and $a < \min\{2s - 2, \frac{5}{4}\}$, there exists $\epsilon > 0$ such that $\frac{1}{2} - \epsilon < b < \frac{1}{2}$, we have
\[
\|u\|^2_{L^4} \lesssim \|u\|^2_{X^{s,b}}.
\]
In addition, for \( s \in (1, \frac{9}{2}) \) and \( a < \min\{2s - 2, \frac{5}{4}, \frac{9}{2} - s\} \), there exists \( \epsilon > 0 \) such that \( \frac{1}{2} - \epsilon < b < \frac{1}{2} \), we have
\[
\|u^2 u_{xx}\|_{X^{\frac{s-1}{2}, \frac{3}{2} + 2s-1 - b}_{\infty, b}} \lesssim \|u\|_{X^{s,b}}^3.
\]  
\[(32)\]

Proof. For \((31)\), by Fourier transform and duality as in Lemma 3.6, it suffices to prove that
\[
I := \int_{\mathbb{R}^3} \frac{\langle \xi_0 \rangle^{s+a}\xi_1^2 g(\xi_0, \tau_0) \prod_{j=1}^3 f(\xi_j, \tau_j)}{\prod_{j=1}^3 \langle \xi_j \rangle^s \prod_{j=0}^3 \langle \sigma_j \rangle^b} \, d\xi_0 \, d\tau_0 \lesssim \|f\|_{L^2} \|g\|_{L^2},
\]
where \( \sigma_j = \tau_j - \phi(\xi_j) \). Note that
\[
\prod_{j=0}^3 \langle \tau_j - \phi(\xi_j) \rangle^b \gtrsim \langle (\xi_0 - \xi_1)(\xi_0 - \xi_3) \rangle \frac{1}{\max_{j=0,\ldots,3} \langle \tau_j - \phi(\xi_j) \rangle \frac{1}{2} +}.\]

Let
\[
K(\xi_0, \xi_1, \xi_2, \xi_3) = \frac{\langle \xi_0 \rangle^{s+a}\xi_1^2}{\langle \xi_1 \rangle^s \langle \xi_2 \rangle^s \langle \xi_3 \rangle^s}.
\]
We split the integral domain into the following several cases.

**Case 1.** \( |\xi_0 - \xi_1| \leq k \) or \( |\xi_0 - \xi_3| \leq k \).
We can assume that \( |\xi_0 - \xi_1| \leq k \). Then we have
\[
K(\xi_0, \xi_1, \xi_2, \xi_3) \lesssim \sup_{\xi_0} \langle \xi_0 \rangle^{2s-2a},
\]
which is finite for \( a < 2s - 2 \). Then by Lemma 2.6 in [17], we can bound \( I \) by
\[
I \lesssim \|F_0\|_{L^2_t L^6_x} \|F_1\|_{L^3_t L^6_x} \|F_2\|_{L^3_t L^6_x} \|F_3\|_{L^3_t L^6_x} \lesssim \|f\|_{L^6_t L^3} \|g\|_{L^6_t}.
\]

**Case 2.** \( |\xi_0 - \xi_1| \geq k \) and \( |\xi_0 - \xi_3| \geq k \).

**Subcase 1.** \( |\xi_1| \leq 2k \) or \( |\xi_3| \leq 2k \). If \( |\xi_1| \leq 2k \), we obtain
\[
K(\xi_0, \xi_1, \xi_2, \xi_3) \lesssim \sup_{\xi_0} \langle \xi_0 \rangle^{a-2s},
\]
which is finite for \( a < 2s \). This leads to \( I \lesssim \|f\|_{L^2_t L^6_x} \|g\|_{L^2} \).

If \( |\xi_3| \leq 2k \). Then we have
\[
K(\xi_0, \xi_1, \xi_2, \xi_3) \lesssim \sup_{\xi_0} \langle \xi_0 \rangle^{2s-2a},
\]
which is finite for \( a < 2s - 2 \). Then we can bound \( I \) by \( \|f\|_{L^2_t L^6_x} \|g\|_{L^2} \).

**Subcase 2.** \( |\xi_1| \geq 2k \) and \( |\xi_3| \geq 2k \). Using \((14)\) and \((15)\), we have
\[
\min \left( \frac{\langle \xi_1 \rangle \frac{1}{2} \langle \xi_1 \rangle \frac{1}{2} \langle \xi_1 \rangle \frac{1}{2} \langle \xi_1 \rangle \frac{1}{2}}{\langle \xi_1 \rangle \frac{1}{2}}, \frac{\langle \xi_1 \rangle \frac{1}{2} \langle \xi_1 \rangle \frac{1}{2} \langle \xi_1 \rangle \frac{1}{2} \langle \xi_1 \rangle \frac{1}{2}}{\langle \xi_1 \rangle \frac{1}{2}} \right) \lesssim \frac{\langle \xi_1 \rangle \frac{1}{2} \langle \xi_1 \rangle \frac{1}{2} \langle \xi_1 \rangle \frac{1}{2} \langle \xi_1 \rangle \frac{1}{2}}{\langle \xi_1 \rangle \frac{1}{2} + \langle \xi_1 \rangle \frac{1}{2} + \langle \xi_1 \rangle \frac{1}{2} + \langle \xi_1 \rangle \frac{1}{2}}.
\]
Restricting the variables \( \xi_0, \cdots, \xi_4 \) by the size as \( |\xi_{\text{min}}| \leq |\xi_{\text{mid}}| \leq |\xi_{\text{mid}}| \approx |\xi_{\text{max}}| \),
we have
\[
\max_{0 \leq i,j,k \leq 3} \frac{\langle \xi_1 \rangle \frac{1}{2} \langle \xi_1 \rangle \frac{1}{2} \langle \xi_1 \rangle \frac{1}{2} \langle \xi_1 \rangle \frac{1}{2}}{\langle \xi_1 \rangle \frac{1}{2} + \langle \xi_1 \rangle \frac{1}{2} + \langle \xi_1 \rangle \frac{1}{2}} \approx \frac{\langle \xi_{\text{mid}} \rangle \frac{1}{2}}{\langle \xi_{\text{max}} \rangle \frac{1}{2}}.
\]
By symmetry, when $\xi_0 - \xi_1 + \xi_2 - \xi_3 = 0$ and $|\xi_1| \leq |\xi_3|$, it suffices to show that
\[
\frac{(\xi_0)^{s+a}(\xi_1)^{2-s}(\xi_2)^{-s}(\xi_3)^{-s}}{(\xi_0 - \xi_1)^{\frac{1}{2}} - (\xi_0 - \xi_3)^{\frac{1}{2}}} \lesssim 1.
\] (33)

In the case $|\xi_1| \leq |\xi_3| \lesssim |\xi_2| \approx |\xi_0|$, we have
\[
(33) \lesssim \frac{(\xi_0)^{a-\frac{s}{2}}}{(\xi_1)^{s-1}(\xi_0 - \xi_1)^{\frac{1}{2}} - (\xi_0 - \xi_3)^{\frac{1}{2}} - (\xi_3)^{a-\frac{s}{2}}}
\]
When $s \geq \frac{3}{2}$, we bound this by $(\xi_0)^{a-\frac{s}{2}} \lesssim 1$ provided that $a < \frac{5}{2}$. When $\frac{1}{2} < s < \frac{3}{2}$, we have
\[
\frac{(\xi_0)^{a-\frac{s}{2}}}{(\xi_1)^{s-a}(\xi_0 - \xi_3)^{\frac{1}{2}} - (\xi_0 - \xi_3)^{\frac{1}{2}} - (\xi_3)^{s-a}} \lesssim (\xi_0)^{a-s + \frac{s}{2}} - (\xi_0)^{a-\frac{s}{2} - \min(s - \frac{3}{2}, \frac{s}{2} - s)} \lesssim 1,
\]
which holds for $a < \min\{2s - 1, \frac{1}{2}\}$.

In the case $|\xi_1| \leq |\xi_3| \approx |\xi_2|$, we have
\[
(33) \lesssim \frac{(\xi_0)^{a-\frac{s}{2}}(\xi_1)^{2-s}(\xi_2)^{-s}(\xi_3)^{-s}}{(\xi_1)^{s-2}(\xi_0 - \xi_1)^{\frac{1}{2}} - (\xi_0)^{s-2}(\xi_3)^{-\frac{1}{2}}} \lesssim (\xi_0)^{a-2s+1} \approx 1,
\]
which holds for $a < 2s - 1$.

In the case $|\xi_1| \leq |\xi_3| \approx |\xi_2|$, we have
\[
(33) \lesssim \frac{(\xi_0)^{s+a}(\xi_3)^{-\frac{s}{2}} - 2(\xi_0)^{2}(\xi_1)^{-\frac{s}{2}} - (\xi_0)^{-\frac{s}{2}}}{} \lesssim (\xi_0)^{a+\frac{s}{2} + (\xi_0)^{-\frac{s}{2}} - 2s+1} \approx 1,
\]
which holds for $a < 2s - \frac{1}{2}$.

In the case $|\xi_2| < |\xi_1| \approx |\xi_3|$, we have
\[
(33) \lesssim (\xi_0)^{s+a}(\xi_2)^{-s}(\xi_3)^{-2s+\frac{s}{2}} + (\xi_0)^{-\frac{s}{2}} + (\xi_2)^{-\frac{s}{2}} \lesssim (\xi_3)^{a-2s+1} \approx 1,
\]
which holds for $a < 2s$.

This completes the proof of (31).

For (32), as in Lemma 3.6, by duality and letting $b = \frac{1}{2}$, it suffices to show that
\[
I := \int_\ast \frac{(\xi_0)^{\frac{s}{2}}(\xi_2)^{-s}(\xi_2)^{-s}(\xi_3)^{-s}}{(\xi_0 - \xi_1)^{\frac{1}{2}} - (\xi_0 - \xi_3)^{\frac{1}{2}}} \lesssim \frac{|| f ||_{L^2} || g ||_{L^2}}{\xi_0 - \xi_1 - \xi_3}.
\]

**Situation 1.** $\frac{1}{2} < s + a < \frac{5}{2}$. As in the proof of the first part, we have
\[
\frac{3}{s} \prod_{j=1}^{3} (\sigma_j)^{-\frac{1}{2}} \lesssim \frac{(\xi_0 - \xi_1)(\xi_0 - \xi_3)}{\xi_0 - \xi_1 - \xi_3} \frac{\prod_{j=0}^{3} (\sigma_j)^{\frac{1}{2}}}{\max_{j=0,\ldots,3}(\sigma_j)^{\frac{1}{2}}}.
\]

Using (14) and (15), and taking the size of $\xi_{\text{max}}$ and $\xi_{\text{mid}}$ as in Lemma 3.8, we need to bound that
\[
\sup_{\xi_0 - \xi_1 - \xi_2 - \xi_3 = 0} \frac{(\xi_0)^{\frac{1}{2}}(\xi_1)^{-s}(\xi_2)^{-s}(\xi_3)^{-s}}{(\xi_0 - \xi_1)(\xi_0 - \xi_3)} \frac{\prod_{j=0}^{3} (\sigma_j)^{\frac{1}{2}}}{\max_{j=0,\ldots,3}(\sigma_j)^{\frac{1}{2}}} \lesssim 1.
\]

The case $|\xi_0 - \xi_1| \leq k$ or $|\xi_0 - \xi_3| \leq k$; the case $|\xi_0 - \xi_1| \geq k$ and $|\xi_0 - \xi_3| \geq k$, $|\xi_1| \leq 2k$ or $|\xi_3| \leq 2k$ are similar to the first part. Now we consider the case:
\[ |\xi_1| \geq 2k \text{ and } |\xi_3| \geq 2k. \text{ By symmetry, when } \xi_0 - \xi_1 + \xi_2 - \xi_3 = 0 \text{ and } |\xi_1| \leq |\xi_3|, \text{ it suffices to show that} \]
\[
\frac{\langle \xi_0 \rangle^{\frac{1}{2}} \langle \xi_1 \rangle^{-s} \langle \xi_2 \rangle^{2-s} \langle \xi_3 \rangle^{-s} \langle \xi_{\text{mid}} \rangle^{\frac{1}{2}}}{\langle (\xi_0 - \xi_1)(\xi_0 - \xi_3) \rangle^{\frac{5-2(s+a)}{s}}} \left( \langle \xi_{\text{max}} \rangle^{\frac{1}{2}} \right)^{-2} \lesssim 1. \tag{34}
\]
In the case \(|\xi_1| \leq |\xi_3| \lesssim |\xi_2| \approx |\xi_0|\), we have
\[
(34) \lesssim \langle \xi_0 \rangle^{\frac{1}{2}} \langle \xi_1 \rangle^{-s} \langle \xi_2 \rangle^{2-s} \langle \xi_{\text{mid}} \rangle^{\frac{1}{2}} \left( \langle \xi_{\text{max}} \rangle^{\frac{1}{2}} \right)^{-2} \lesssim 1,
\]
which holds for \(a < 3s\).
In the case \(|\xi_1| \leq |\xi_3| \approx |\xi_0|, |\xi_3| \gg |\xi_2|\), we have
\[
(34) \lesssim \frac{\langle \xi_0 \rangle^{\frac{1}{2}} \langle \xi_1 \rangle^{-s} \langle \xi_2 \rangle^{2-s} (\langle \xi_1 \rangle^{\frac{1}{2}} + \langle \xi_2 \rangle^{\frac{1}{2}})}{\langle \xi_0 - \xi_1 \rangle^{\frac{5-2(s+a)}{s}}} \left( \langle \xi_{\text{mid}} \rangle \right)^{\frac{1}{2}} \left( \langle \xi_{\text{max}} \rangle \right)^{-2} \lesssim \langle \xi_0 \rangle^{\frac{1}{2}} \langle \xi_1 \rangle^{-s} \langle \xi_2 \rangle^{2-s} \langle \xi_{\text{max}} \rangle^{-2} \langle \xi_0 \rangle^{-s} \langle \xi_2 \rangle^{s-\frac{1}{2}a} \lesssim 1,
\]
which holds for \(a < 3s\).
In the case \(|\xi_0|, |\xi_2| \ll |\xi_1| \approx |\xi_3|\), we have
\[
(34) \lesssim \langle \xi_0 \rangle^{\frac{1}{2}} \langle \xi_2 \rangle^{2-s} \langle \xi_{\text{max}} \rangle^{-\frac{3}{2}} \langle \xi_0 \rangle^{\frac{1}{2}} + \langle \xi_2 \rangle^{\frac{1}{2}} \lesssim 1,
\]
which holds for \(a < 2s - 2\).

**Situation 2.** \(\frac{s}{2} \leq s + a < \frac{3}{2}\). Restricting the variables \(\xi_1, \xi_2, \xi_3\) by the size as \(|\xi_{\text{min}}| \leq |\xi_{\text{mid}}| \leq |\xi_{\text{max}}|\), we have
\[
\langle \sigma_0 \rangle^{\frac{5-2(s+a)}{s}} \prod_{j=1}^{3} \langle \sigma_j \rangle^{\frac{1}{2}} - \langle \sigma_1 \rangle^{\frac{1}{2}} - \langle \sigma_2 \rangle^{\frac{1}{2}} - \langle \sigma_3 \rangle^{\frac{1}{2}} \lesssim \langle \sigma_1 \rangle^{\frac{1}{2}} + \langle \sigma_2 \rangle^{\frac{1}{2}} + \langle \sigma_3 \rangle^{\frac{1}{2}} - \langle \sigma_{\text{max}} \rangle^{\frac{1}{2}} - \langle \sigma_{\text{max}} \rangle^{\frac{1}{2}} - \langle \xi_{\text{max}} \rangle^{\frac{1}{2}} - \langle \xi_{\text{max}} \rangle^{\frac{1}{2}} - \langle \xi_{\text{max}} \rangle^{\frac{1}{2}} - \langle \xi_{\text{max}} \rangle^{\frac{1}{2}} - \langle \xi_{\text{max}} \rangle^{\frac{1}{2}} ,
\]
where we use the fact that \(\langle \sigma_0 \rangle \lesssim \max \{\langle \xi_{\text{max}} \rangle^{\frac{1}{4}}, \langle \sigma_1 \rangle, \langle \sigma_2 \rangle, \langle \sigma_3 \rangle\}\), and assume that the maximum is one of \(\langle \sigma_j \rangle\).

By the Cauchy-Schwarz inequality, it suffices to prove that
\[
\sup_{\xi_0} \int_{\xi_0}^{\xi_{\text{max}}} \langle \xi_0 \rangle^{\frac{1}{2}} \langle \xi_2 \rangle^{\frac{1}{2}} \frac{d\xi_1}{\langle \xi_{\text{max}} \rangle^{9-2(s+a)} - \prod_{j=1}^{3} \langle \xi_j \rangle^{2s}} \lesssim 1.
\]
Using Lemma 2.2 and \(|\xi_{\text{max}}| \gtrsim |\xi_0|\), we obtain
\[
\int \frac{\langle \xi_0 \rangle^{2(s+a) - 8 + 2s} d\xi_1 d\xi_3}{\langle \xi_1 \rangle^{2s} \langle \xi_0 - \xi_1 - \xi_3 \rangle^{2s - 4} \langle \xi_{\text{max}} \rangle^{2s}} \lesssim \langle \xi_0 \rangle^{2(s+a) - 8 + 2s} \langle \xi_0 \rangle^{2s} \lesssim 1
\]
provided that \(a < 2\).

If the maximum is \(\langle \xi_{\text{max}} \rangle^{\frac{1}{4}}\), we have
\[
\langle \sigma_0 \rangle^{\frac{5-2(s+a)}{s}} \prod_{j=1}^{3} \langle \sigma_j \rangle^{\frac{1}{2}} \lesssim \langle \xi_{\text{max}} \rangle^{\frac{5-2(s+a)}{2}} - \prod_{j=1}^{3} \langle \sigma_j \rangle^{\frac{1}{2}}.
\]
Thus, by (14) and (15), it suffices to prove that
\[
\langle \xi_0 \rangle^{\frac{1}{2}} \langle \xi_1 \rangle^{-s} \langle \xi_2 \rangle^{2-s} \langle \xi_3 \rangle^{-s} \langle \xi_{\text{max}} \rangle^{s+a-\frac{3}{2}} + \frac{\langle \xi_{\text{mid}} \rangle^{\frac{1}{2}}}{\langle \xi_{\text{max}} \rangle^{\frac{3}{4}}} \lesssim 1.
\]
We can bound this by
\[
\frac{\langle \xi_{\text{max}} \rangle^{s+a-\frac{3}{2}} + \langle \xi_{\text{mid}} \rangle^{\frac{1}{2}}}{\prod_{j=1}^{3} \langle \xi_j \rangle^s} \lesssim \frac{\langle \xi_{\text{max}} \rangle^{s+a-\frac{3}{2}} + \langle \xi_{\text{mid}} \rangle^{\frac{1}{2}}}{\langle \xi_{\text{max}} \rangle^{s} \langle \xi_{\text{mid}} \rangle^{\frac{1}{2}}} \lesssim 1
\]
provided that \( a < \frac{5}{4} \).
This completes the proof of (32). \( \Box \)

In what follows, we state the nonlinear estimates for \( |u_x|^2 u \) and \( u^2 \bar{u}_{xx} \), which are similar to Lemma 3.8 and 3.9. Thus we omit the details.

**Lemma 3.10.** For \( s > \frac{1}{2} \) and \( a < \min \{2s - 1, \frac{5}{4} \} \), there exists \( \epsilon > 0 \) such that \( \frac{1}{2} - \epsilon < b < \frac{1}{2} \), we have
\[
\|u_x|^2 u\|_{X^{s,a,-b}} \lesssim \|u\|_{X^{s,b}}^3.
\]
In addition, for \( s \in (\frac{1}{2}, \frac{3}{4}) \) and \( a < \min \{2s - 1, \frac{5}{4}, \frac{9}{8} - s\} \), there exists \( \epsilon > 0 \) such that \( \frac{1}{2} - \epsilon < b < \frac{1}{2} \), we have
\[
\|u_x|^2 u\|_{X^{\frac{1}{2}+\frac{2s-1}{2}a,\frac{1}{2}-b}} \lesssim \|u\|_{X^{s,b}}^3.
\]

**Lemma 3.11.** For \( s > 1 \) and \( a < \min \{2s - 2, \frac{5}{4} \} \), there exists \( \epsilon > 0 \) such that \( \frac{1}{2} - \epsilon < b < \frac{1}{2} \), we have
\[
\|u^2 \bar{u}_{xx}\|_{X^{s,a,-b}} \lesssim \|u\|_{X^{s,b}}^3.
\]
In addition, for \( s \in (1, \frac{3}{2}) \) and \( a < \min \{2s - 2, \frac{5}{4}, \frac{9}{8} - s\} \), there exists \( \epsilon > 0 \) such that \( \frac{1}{2} - \epsilon < b < \frac{1}{2} \), we have
\[
\|u^2 \bar{u}_{xx}\|_{X^{\frac{1}{2}+\frac{2s-1}{2}a,\frac{1}{2}-b}} \lesssim \|u\|_{X^{s,b}}^3.
\]

4. **Proof of Theorem 1.2.** In this section, we will prove that the map \( \Phi \) defined in (7) has a fixed point in \( X^{s,b} \). Choose an extension \( \bar{u}_0 \in H^s(\mathbb{R}) \) of \( u_0 \) such that
\[
\|\bar{u}_0\|_{H^s(\mathbb{R})} \lesssim \|u_0\|_{H^s(\mathbb{R}^+)}.
\]
Recall that
\[
\Phi(u(x,t)) = \eta(t/T)S(t)\bar{u}_0 + i\eta(t/T) \int_0^t S(t-t')G(u) dt'
\]
\[
+ \eta(t/T)S'_0(0, f - p_1 - q_1, g - p_2 - q_2) (t),
\]
where \( G(u) \), \( p_i \) and \( q_i \) are defined in (8)–(10). To bound the first component of \( \Phi \), we use (11) to obtain
\[
\|\eta(t/T)S(t)\bar{u}_0\|_{X^{s,b}} \lesssim \|\bar{u}_0\|_{H^s(\mathbb{R})} \lesssim \|u_0\|_{H^s(\mathbb{R}^+)}.
\]
To bound the Duhamel term, we use (12), (13), (25), (27), (31), (35), (36) and Lemma 3.5 to obtain
\[
\left\| \eta(t/T) \int_0^t S(t-t')G(u) dt' \right\|_{X^{s,b}} \lesssim T^{\frac{1}{2} - b - \epsilon} \|G(u)\|_{X^{s,-b}} \lesssim T^{1 - 2b - \epsilon} \|F(u)\|_{X^{s,-b}}
\]
\[
\lesssim T^{1 - 2b - \epsilon} (\|u\|_{X^{s,b}}^5 + \|u\|_{X^{s,b}}^3).
\]
To bound the $S'_0$ term, we use Lemma 2.1 and Lemma 3.3 to obtain
\[ \|\eta(t/T)S'_0(0, f - p_1 - q_1, g - p_2 - q_2)(t)\|_{X^{s,b}} \]
\[ \lesssim \|\chi(0, \infty)(f - p_1 - q_1)\|_{H^s_t L^2_x(\mathbb{R})} + \|\chi(0, \infty)(g - p_2 - q_2)\|_{H^s_t L^2_x(\mathbb{R})} \]
\[ \lesssim \|f - p_1\|_{H^s_t L^2_x(\mathbb{R})} + \|q_1\|_{H^s_t L^2_x(\mathbb{R})} + \|g - p_2\|_{H^s_t L^2_x(\mathbb{R})} + \|q_2\|_{H^s_t L^2_x(\mathbb{R})}. \]  
(41)

Note that by Lemma 3.1, we have
\[ \| p_1 \|_{H^s_t L^2_x(\mathbb{R})} + \| p_2 \|_{H^s_t L^2_x(\mathbb{R})} \lesssim \| u_0 \|_{H^s(\mathbb{R})} \lesssim \| u_0 \|_{H^s(\mathbb{R})}. \]  
(42)

Moreover, by Lemma 3.4, (13) and Lemma 3.5, 3.6, 3.7, 3.8, 3.9, 3.10, 3.11, we have
\[ \| q_1 \|_{H^s_t L^2_x(\mathbb{R})} + \| q_2 \|_{H^s_t L^2_x(\mathbb{R})} \lesssim T^{\frac{2}{5} - \frac{1}{b}} (\| u \|_{X^{s,b}}^5 + \| u \|_{X^{s,b}}^3). \]  
(43)

Combining these estimates, we obtain
\[ \| \Phi u \|_{X^{s,b}} \lesssim \| u_0 \|_{H^s} + \| f \|_{H^s_t L^2_x(\mathbb{R})}^{2+3} + \| g \|_{H^s_t L^2_x(\mathbb{R})}^{2+1} + T^{\frac{2}{5} - b} (\| u \|_{X^{s,b}}^5 + \| u \|_{X^{s,b}}^3). \]

In addition, we also have the similar estimates for $\Phi u - \Phi v$. This yields the existence of a fixed point $u$ of $\Phi$ in $X^{s,b}$ for $T$ sufficiently small,
\[ T := T \left( \| u_0 \|_{H^s(\mathbb{R})}, \| f \|_{H^s_t L^2_x(\mathbb{R})}^{2+3}, \| g \|_{H^s_t L^2_x(\mathbb{R})}^{2+1} \right). \]

Now we verify the continuity that $u \in C^0_t H^s_x([0, T] \times \mathbb{R})$. The first term of $\Phi$, the Schrödinger group operator $S(t)u_0$ is continuous in $H^s$. The third term $S'_0(0, f, g)$ is continuous by Lemma 3.2 and (43). For the Duhamel term $\int_0^t S(t - t')G(u) \, dt'$, its continuity follows from (12), (25), (27), (31), (35), (37) and the embedding $X^{s,b} \subset C^0_t H^s_x$ for $b > \frac{1}{2}$. In addition, the fact that $u \in C^0_t H^{2+3}_x(\mathbb{R} \times [0, T])$ follows from Lemma 3.1, Lemma 3.2 and Lemma 3.4. The continuous dependence on initial and boundary data from the above contraction mapping argument and the a priori estimates. The solutions obtained in previous content also restricted to $\mathbb{R}^+$. However, it is not a priori clear if different extensions of the initial data produce the same solution on $\mathbb{R}^+$. We will discuss the uniqueness issue in Section 4.1 below. Before that, we prove that the nonlinear part of the solutions is smoother than the initial data. Recall that
\[ u - S'_0(u_0, f, g) = \eta(t/T) \int_0^t S(t - t')G(u) \, dt' - \eta(t/T)S'_0(0, q_1, q_2)(t). \]  
(44)

By embedding $X^{s+a, \frac{1}{2}+} \subset C^0_t H^{s+a}_x$ and (40), the first term is in $C^0_t H^{s+a}_x$. The second term also in $C^0_t H^{s+a}_x$ by Lemma 3.2 and (43). Thus, we have
\[ \| u - S'_0(u_0, f, g) \|_{C^0_t H^{s+a}_x} \lesssim \| G(u) \|_{X^{s+a,-b}} + \| q_1 \|_{H^{2+2a+3}_t L^2_x} \]
\[ \lesssim \| u \|_{X^{s,b}} + \| u \|_{X^{s,b}}^3, \]
which implies that $u(t) - S'_0(u_0, f, g)$ belongs to $C^0_t H^{s+a}_x$.

4.1. **Uniqueness.** Now we discuss the uniqueness argument of equations (1). The solution we obtained in the previous section is the unique fixed point of (7). Nevertheless, it is not a priori clear if different extensions of initial data produce the same solution on $\mathbb{R}^+$. We state an extension argument in [11]:
Lemma 4.1. Fix \( s \geq 0 \) and \( k \geq s \). Let \( u_0 \in H^s(\mathbb{R}^+) \), \( g \in H^k(\mathbb{R}^+) \), and let \( \tilde{u}_0 \) be an \( H^s \) extension of \( u_0 \) to \( \mathbb{R} \). Then there is an \( H^k \) extension \( \tilde{g} \) of \( g \) to \( \mathbb{R} \) such that

\[
\| \tilde{u}_0 - \tilde{g} \|_{H^s(\mathbb{R})} \lesssim \| u_0 - g \|_{H^s(\mathbb{R}^+)}.
\]

We start with the issue for \( s \geq 2 \). Taking two \( H^s \) local solutions \( u_1 \) and \( u_2 \), we have

\[
i((1 - u_1)_x + (u_1 - u_2)_x + \gamma_5 (u_1 - u_2)_{xxxx} + F(u_1) - F(u_2) = 0,
\]

with the initial-boundary value

\[
(u_1 - u_2)(x, 0) = 0, \quad (u_1 - u_2)(0, t) = 0, \quad (u_1 - u_2)_x(0, t) = 0.
\]

Multiplying (45) by \( u_1 - u_2 \) and integrating, we obtain

\[
\partial_t \| u_1 - u_2 \|^2_{L^2} \lesssim \| u_1 - u_2 \|^3_{L^2} (1 + \| u_1 \|_{H^2}^2 + \| u_2 \|_{H^2}^2) \lesssim \| u_1 - u_2 \|^2_{L^2},
\]

which implies uniqueness for \( s \geq 2 \).

We now discuss the uniqueness for \( s \in (1, 2) \), \( s \neq \frac{3}{2} \). Let \( u_0^1, u_0^2 \) be two extensions of \( u_0 \) in \( H^s(\mathbb{R}) \). Take a sequence \( \{ u_{n0} \} \subset H^2(\mathbb{R}) \) converging to \( u_0 \) in \( H^s(\mathbb{R}^+) \). Also take sequences \( \{ f_n \} \subset H^s(\mathbb{R}^+) \) converging to \( f \) in \( H^{s+1}(\mathbb{R}^+) \), and \( \{ g_n \} \subset H^s(\mathbb{R}^+) \) converging to \( g \) in \( H^{s+1}(\mathbb{R}^+) \). Let \( h_n^1, h_n^2 \in H^2(\mathbb{R}) \) be extensions of \( u_0^1, u_0^2 \) in \( H^s(\mathbb{R}) \). This is true from Lemma 4.1. Let \( u, v \) be two \( H^s(\mathbb{R}^+) \) solutions as in Definition 1.1. We know that by the uniqueness of \( H^2(\mathbb{R}^+) \) solutions, the corresponding solutions sequences \( u_n^1, u_n^2 \) on \( \mathbb{R}^+ \) are the same. By contraction argument, \( u = \lim u_n^1 \), \( v = \lim u_n^2 \) in \( H^s(\mathbb{R}) \) provided that the existence times \( T_n \) do not shrink to zero, and their restrictions on \( \mathbb{R}^+ \) are the same. We also note that the solution sequence \( u_n \) satisfies the fixed point argument, that is

\[
\| \Phi(u_n) \|_{X^{2,s}} \lesssim \| u_{n0} \|_{H^2} + \| f_n \|_{H^s} + \| g_n \|_{H^s} + T_n^{\frac{1}{2} - b} (\| u_n \|_{X^{2,s}}^5 + 1)
\]

\[
\lesssim \| u_{n0} \|_{H^2} + \| f_n \|_{H^s} + \| g_n \|_{H^s} + T_n^{\frac{1}{2} - b} (\| u_0 \|_{H^s} + \| f \|_{H^{s+1}} + \| g \|_{H^{s+1}} + 1)^5.
\]

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Received February 2021; revised June 2021; early access August 2021.

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