Chiral Symmetry Breaking via Multi-fermion Green Functions

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Abstract

Previous results on fermion chirality-flipping four-point functions are extended to $SU(N)$ gauge theories. The problem is purely non-perturbative, and it is approached by truncating the Schwinger-Dyson hierarchy. The large-$N$ limit also simplifies the problem substantially. The resulting equation is solved numerically by relaxation techniques and an estimate of the critical coupling and momentum behavior is obtained. We also comment on the behavior of chirality-flipping $2n$-point functions for general $n$.

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I Introduction

In this paper we study the dynamical generation of momentum dependent fermion four-point functions within the framework of non-abelian gauge theories. These will be purely non-perturbative quantities associated with breakdown of chiral symmetries, and would for example imply the existence of the corresponding four-fermion condensates. Our analysis is based on the Schwinger-Dyson (SD) formalism. The resulting equations are analytically intractable and thus we seek numerical solutions.

We have two main goals in this study. One is to estimate the critical value of the gauge coupling necessary for the formation of the four-point functions. We would like to study this in a case where four-point functions develop on scales higher than the scale of mass formation. This would make it consistent to treat the four-point function problem in isolation, independently of the mass generation problem. This is not the case for QCD, but even in that case it is useful to at least consider the existence of the effects we study here.

In extensions of the standard model the hierarchical symmetry breaking pattern we envisage can be a natural consequence of the fact that a dynamical four-point function may break fewer continuous symmetries than a dynamical mass. In the case that some of these symmetries are gauged, then these additional interactions could resist the formation of mass. If so, dynamical masses may form on a scale lower than the scale of the four-point functions, or not form at all. With these possibilities in mind, we shall simply ignore fermion mass in the present work.

A multiflavor case will be discussed elsewhere in which it is possible to add abelian gauge interactions such as to make the theory chiral, so that any mass will
break some gauge symmetry. In that theory some four-point functions do not break any gauge symmetry; this provides a dynamical reason for why four-point functions form and masses do not. There are also four-point functions for which the effects of the extra $U(1)$’s essentially cancel out, leaving only the attractive non-abelian interactions. The study of such four-point functions would be very similar to the study of four-point functions in our simplified single-flavor case, where we assume the mass vanishes.

Our other goal is to extract the momentum dependence of the dynamical fermion four-point function. For example if such an object is to play a role in generating a fermion or a gauge-boson mass, then two or four lines of the four-point function may be closed off into a loop or into another four-point function. It proves then interesting to study the details of the momentum dependence, such as the relative size of the four-point function when different pairs of momenta are large. The factorization hypothesis, according to which the four-point function can be treated as a product of two two-point functions, is of no use here since we neglect two-fermion condensates.

In the present study we use a one gauge-boson exchange approximation. Since the gauge boson can attach to any pair of the four legs, the SD equation sums up a much more complicated set of diagrams than the set of ladder graphs. The present work constitutes a clear progress with respect to our previous study [2], since, apart from considering a general non-abelian group, it includes a treatment of non-linearities, it does not neglect terms proportional to external momenta while at the same time exploring the full available momentum space, and in a certain limit allows us to comment on the behavior of certain $2n$-point functions.
In the next section we consider the SD equation for the fermion chirality-flipping four-point function in a non-abelian theory, and then compute the form it takes in the large-$N$ limit of an $SU(N)$ theory in section 3. In section 4 we discuss our numerical treatment and results, and in section 5 we generalize our results to $2n$-point functions.

II The equations in the non-abelian case

We focus our attention on fermion operators which are purely chirality changing of the form $\bar{\psi}_L \Gamma \psi_R \bar{\psi}_L \Gamma' \psi_R + \text{h.c.}$, where $\psi_L \equiv \frac{(1-\gamma_5)}{2} \psi$ and $\psi_R \equiv \frac{(1+\gamma_5)}{2} \psi$. We constrain our study to just one fermion flavor in a representation of a gauged simple Lie group. The four independent operators which have this property and respect parity are:

\[
\frac{1}{2} (\bar{\psi}_L \Gamma \psi_R \bar{\psi}_L \Gamma' \psi_R + \text{h.c.}) = \bar{\psi}_L \gamma_\mu \psi_R \bar{\psi}_L \gamma_\mu \psi_R + \text{h.c.}
\]

where $\psi \lambda^a \psi^A$ is anomalous, but its discrete subgroup $e^{i(n/2)\pi\gamma_5}$ where $n$ is an integer is still present and would have to be broken before fermion mass could

\[
\frac{1}{2} (\bar{\psi}_L \lambda^a \psi_R \bar{\psi}_L \lambda^a \psi_R + \text{h.c.}) = \bar{\psi}_L \lambda^a \gamma_\mu \psi_R \bar{\psi}_L \lambda^a \gamma_\mu \psi_R + \text{h.c.}
\]

\[
\psi_R \lambda^a \sigma^{\mu
u} \psi \bar{\psi}_L \lambda^a \sigma^{\mu
u} \psi + \text{h.c.}, \quad (1)
\]

The chiral $U(1)_A$ is anomalous, but its discrete subgroup $e^{i(n/2)\pi\gamma_5}$ where $n$ is an integer is still present and would have to be broken before fermion mass could

\[
\psi \Gamma \psi \bar{\psi} \Gamma' \psi, \quad (1)
\]

\[
\Gamma, \Gamma' \text{ are matrices with possibly non-trivial spinor and color structure, by } \Gamma \otimes \Gamma'.
\]

\[
\text{We choose to work in Euclidean space, so there is no difference between upper and lower Lorentz indices.}
\]
form. As mentioned in the introduction, other effects in a more realistic theory could resist the breaking of such a symmetry. Here we shall simply assume that fermion masses are smaller than the typical scale of our problem, and can thus be ignored.

Note that we have omitted chirality flipping operators with derivatives, having structures like $\partial^\mu_i \partial^\nu_j \sigma^{\mu \nu} \otimes (1$ or $\partial^\rho_k \partial^\tau_l \sigma^{\rho \tau})$ and $\partial^\mu_i \sigma^{\mu \nu} \otimes \partial^\rho_j \sigma^{\rho \nu}$, where each of the derivatives with indices $i, j, k, l = 1, ..., 4$ acts on one of the four fermion fields. We will present arguments supporting their omission later.

The SD formalism relevant to these four-point functions was discussed in Ref. [2] and led to an equation shown diagrammatically in Fig. 1. It results from a simple truncation of the SD hierarchy achieved by approximating the five-point function involving four fermions and a gluon by the sum of tree graphs involving the four-point function. (The equation has been symmetrized to include diagrams with gluons connecting all possible pairs of fermions, so the right-hand side is multiplied by a factor of $1/2$.) The analogous procedure used in the SD equation for the two-point function yields the popular ladder approximation. In our case the set of diagrams being resummed has a more complicated structure than sets of ladder graphs. In both cases gauge self-interactions no longer appear in the SD equation. In the two-point case it is then true that there is little difference between abelian and non-abelian interactions. On the contrary in the four-point case we show that there is still a big difference; for non-abelian interactions the space of four point functions is enlarged and the mixing between the four-point functions is completely different.

We are considering a four-point function associated with the Green function
\[ \langle 0 \mid T \{ \bar{\psi}_\alpha \psi_\beta \bar{\psi}_\rho \psi_\tau \} \mid 0 \rangle. \] In momentum space, the four-point function involving scalar functions of momentum only and receiving exclusively non-perturbative contributions is

\[ O_{\alpha\beta\rho\tau} = O_{S+P}(I_{\alpha\beta} \otimes I_{\rho\tau} + \gamma^5_{\alpha\beta} \otimes \gamma^5_{\rho\tau}) + O_T \sigma^\mu_{\alpha\beta} \otimes \sigma^\mu_{\rho\tau} \]

\[ + O_{S+P}^{\text{color}}(\lambda^a I_{\alpha\beta} \otimes \lambda^a I_{\rho\tau} + \lambda^a \gamma^5_{\alpha\beta} \otimes \lambda^a \gamma^5_{\rho\tau}) + O_T^{\text{color}} \lambda^a \sigma^\mu_{\alpha\beta} \otimes \lambda^a \sigma^\mu_{\rho\tau} \]

where the four scalar functions \( O_{S+P}, O_T, O_{S+P}^{\text{color}} \) and \( O_T^{\text{color}} \) depend on 6 variables, which are all the independent and Lorentz-invariant combinations of the external 4-momenta \( p_1, \ldots, p_4 \), corresponding to the fermions with spinor indices \( \alpha, \beta, \rho, \tau \) respectively. We wish to develop the SD equations for these scalar functions.

Similarly to our previous work [2], we begin by defining the following functional operators \( \Gamma^i \):

\[
\begin{align*}
\Gamma^A[K] & \equiv \frac{\alpha}{8\pi^3} \int d^4k \frac{K}{(p_1 - k)^2(p_2 + k)^2} \\
\Gamma^B[K] & \equiv \frac{\alpha}{8\pi^3} \int d^4k \frac{K}{(p_3 + k)^2(p_4 - k)^2} \\
\Gamma^C[K] & \equiv \frac{\alpha}{8\pi^3} \int d^4k \frac{K}{(p_1 - k)^2(p_4 - k)^2} \\
\Gamma^D[K] & \equiv \frac{\alpha}{8\pi^3} \int d^4k \frac{K}{(p_2 + k)^2(p_3 + k)^2} \\
\Gamma^E[K] & \equiv \frac{\alpha}{8\pi^3} \int d^4k \frac{K}{(p_1 - k)^2(p_3 - k)^2} \\
\Gamma^F[K] & \equiv \frac{\alpha}{8\pi^3} \int d^4k \frac{K}{(p_2 + k)^2(p_4 + k)^2},
\end{align*}
\] (3)

where \( K \) is a function of the loop and external momenta, with a possibly non-trivial spinor structure, the letters \( A, \ldots, F \) correspond to the diagrams shown in Fig. 1 with the gauge boson having four-momentum \( k \), and \( \alpha \) is the momentum-independent coupling. We work in the Landau gauge which is popular in studies of SD equations; the gauge boson propagator reads \( D^\mu_\nu = \frac{1}{k^2} \left( \delta^\mu_\nu - \frac{k^\mu k^\nu}{k^2} \right) \). The choice of this gauge
will be further justified later, when we consider the large-$N$ limit of an $SU(N)$ gauge theory.

By combining the results of the abelian case [2] with the study on the color structure given in Appendix A, and considering fermions in the fundamental representation of $SU(N)$, we have

$$O_{S+P} = \frac{3(N^2 - 1)}{2N}(\Gamma^A + \Gamma^B)[O_{S+P}] + 6(\Gamma^C + \Gamma^D - \Gamma^E - \Gamma^F)[O_{\text{color}}]$$

$$O_T = -\frac{N^2 - 1}{2N}(\Gamma^A + \Gamma^B)[O_T] + 2(\Gamma^C + \Gamma^D + \Gamma^E + \Gamma^F)[O_{\text{color}}] + \frac{1}{2}(\Gamma^C + \Gamma^D - \Gamma^E - \Gamma^F)[O_{S+P}]

O_{\text{color}}^{S+P} = -\frac{3}{2N}(\Gamma^A + \Gamma^B)[O_{\text{color}}^{S+P}] + \frac{3(N^2 - 1)}{2N^2}(\Gamma^C + \Gamma^D - \Gamma^E - \Gamma^F)[O_T] + 6\left(-\frac{N^2 + 2}{2N}(\Gamma^C + \Gamma^D) + \frac{1}{N}(\Gamma^E + \Gamma^F)\right)[O_{\text{color}}]

O_{\text{color}}^T = \left(\frac{1}{2N}(\Gamma^A + \Gamma^B) + \left(-\frac{N^2 + 2}{N}(\Gamma^C + \Gamma^D) - \frac{2}{N}(\Gamma^E + \Gamma^F)\right)\right)[O_{\text{color}}] + \frac{N^2 - 1}{2N^2}(\Gamma^C + \Gamma^D + \Gamma^E + \Gamma^F)[O_T] + \frac{N^2 - 1}{8N^2}(\Gamma^C + \Gamma^D - \Gamma^E - \Gamma^F)[O_{S+P}] + \frac{1}{2}\left(-\frac{N^2 + 2}{2N}(\Gamma^C + \Gamma^D) + \frac{1}{N}(\Gamma^E + \Gamma^F)\right)[O_{S+P}^\text{color}], \quad (4)

where $\Gamma^i[O] \equiv \Gamma^i(O^i)$ and $O^i$ are the form factors with loop-dependent arguments corresponding to diagrams $i = A, ..., F$ [2].
We are faced with a system of four coupled 4-dimensional integral equations involving functions of 6 variables. As it stands, the problem is analytically intractable, and even a numerical solution proves to be beyond our present means. In the next section we present the simplification that the large-$N$ limit of $SU(N)$ provides.

### III The large-$N$ limit

By taking the large-$N$ limit, (4) reduces to

\[
\begin{pmatrix}
O_{S+P} \\ O_T \\ O_{color}^{S+P} \\ O_{color}^T
\end{pmatrix}
= N
\begin{pmatrix}
\frac{3}{2}(\Gamma^A + \Gamma^B) & 0 & 0 & 0 \\
0 & -\frac{1}{2}(\Gamma^A + \Gamma^B) & 0 & 0 \\
0 & 0 & 0 & -3(\Gamma^C + \Gamma^D) \\
0 & 0 & -\frac{1}{2}(\Gamma^C + \Gamma^D) & -(\Gamma^C + \Gamma^D)
\end{pmatrix}
\begin{pmatrix}
O_{S+P} \\ O_T \\ O_{color}^{S+P} \\ O_{color}^T
\end{pmatrix}
\]  

(5)

The form factors $O_{S+P}$ and $O_T$ decouple from the others and from each other. The $O_{S+P}$ entry is positive, which means that there is an attractive interaction in this channel necessary for the formation of a non-zero four-point function. In contrast $O_T$ is not expected to be non-zero. It is also apparent that $O_{color}^{S+P}$ can be non-zero only if $O_{color}^T$ is somehow non-zero, but that seems unlikely because of the negative entry coming from $O_{color}^T$ itself. Even if the $O_{color}^{S+P,T}$ functions were somehow non-zero, we would expect for them a critical coupling larger than the one required for

\footnote{Note that we apply the large-$N$ limit only within the context of the truncated SD equation, and we do not claim that the large-$N$ limit in any way justifies the original truncation of the SD hierarchy.}
The large-$N$ limit also allows us to argue for the omission of contributions coming from operators proportional to external momenta. With regards to the operators with structure $p_i^\mu \sigma^{\mu\nu} \otimes p_j^\lambda \sigma^{\nu\lambda}$ and $p_i^\mu p_j^\nu \sigma^{\mu\nu} \otimes p_k^\lambda p_\ell^\tau \sigma^{\lambda\tau}$, we can see that the contributions they receive from the dominant diagrams having the same functions in their vertices (diagrams $A$ and $B$ for $1 \otimes 1$ color structures, diagrams $C$ and $D$ for $\lambda^a \otimes \lambda^a$ color structures) are negative, so we do not expect the corresponding four-point functions to be non-zero. This happens because, in the large-$N$ limit, tensor insertions give negative contributions. Even if other terms made them non-zero, they would require a larger gauge coupling than the one needed for $O_{S+P}$.

Interesting also are the operators with structure $O_1 \equiv p_i^\mu p_j^\nu \sigma^{\mu\nu} \otimes 1$ and $O_2 \equiv 1 \otimes p_i^\mu p_j^\nu \sigma^{\mu\nu}$ (and trivial color structure), which can obviously mix with the form factor $O_{S+P}$. If we take the operator $O_1$ for instance, we see that it receives a negative contribution from diagram $A$ and a positive one from diagram $B$. Therefore, it might not develop a non-zero value, and if it did, it would require a larger value for the gauge coupling than the critical value we are seeking. These operators will also be neglected.

We now write down the integral equation for $O_{S+P}$. We go to a reference frame where $\vec{p}_1 = -\vec{p}_2$, where the form of the kernel of the integral equation indicates that $O_{S+P}$ is a function of five variables, i.e.

$$O_{S+P}(p_1^0, |\vec{p}_1|, p_4^0, |\vec{p}_4|, q^0) = \frac{N\alpha}{16\pi^3} \int d^4k \gamma^\mu \frac{1}{k^2 - \vec{k} \cdot \vec{q}} \gamma^\nu \times \left( \frac{O_{S+P}(k^0, |k|, p_1^0, |\vec{p}_1|, q^0) D_1^{\mu\nu}}{(k - p_1)^2} + \frac{O_{S+P}(p_4^0, |\vec{p}_4|, k^0, |k|, q^0) D_4^{\mu\nu}}{(k - p_4)^2} \right)$$

where the two terms on the right-hand side come from diagrams $A$ and $B$ respectively.
tively, \( q \equiv p_1 + p_2 \), \( D_i^{\mu\nu} \equiv \delta^{\mu\nu} - \frac{(k-p_i)^\mu(k-p_i)^\nu}{(k-p_i)^2} \), and the two fermions in the loop have four-momenta \( k \) and \( k - q \) respectively.

We find that a solution to the above integral equation takes the factorized form \( \mathcal{O}_{S+P} \sim \tilde{B}(p_1^0, |\vec{p}_1|, q^0) \tilde{B}(p_4^0, |\vec{p}_4|, q^0) \). This just reflects the fact that the loop integral in diagram \( A \) does not depend on \( p_4^0 \) and \( |\vec{p}_4| \) for a given \( q^0 \), and similarly for diagram \( B \). By momentarily setting \( p_1^0 = p_4^0 \equiv p^0 \) and \( |\vec{p}_1| = |\vec{p}_4| \equiv |\vec{p}| \) we obtain an equation for \( \tilde{B} \).

\[
\tilde{B}(p^0, |\vec{p}|; q^0) = \frac{N\alpha}{8\pi^3} \int d^4 k \tilde{B}(k^0, |\vec{k}|; q^0) \gamma^\mu \frac{1}{k(k-q)} \gamma^\nu \frac{D^{\mu\nu}}{(k-p_1)^2} \tag{7}
\]

We have separated \( q^0 \) from the other two variables with a semi-colon since it is an argument of \( \tilde{B} \) that is not affected by loop momenta, and it can thus be treated as if it were a parameter in the kernel.

Note that for \( q^0 = 0 \) this reduces to the linearized SD equation for the fermion two-point function. For non-zero \( q^0 \), the problem reduces to the one of a three-point function, which has been studied before \cite{3}. After some Dirac algebra, and omitting terms with structure \( p_i^\mu p_j^\nu \sigma^{\mu\nu} \otimes 1 \) according to our previous discussion, the equation becomes

\[
\tilde{B}(p^0, |\vec{p}|; q^0) = \frac{3N\alpha}{4\pi^2} \int d^4 k d|k| \frac{\tilde{B}(k^0, |\vec{k}|; q^0)|\vec{k}|}{((q^0 - k^0)^2 + |\vec{k}|^2) 2|\vec{p}|} \times
\]

\[
\ln \left( \frac{(|k| + |\vec{p}|)^2 + (k^0 - p^0)^2}{(|k| - |\vec{p}|)^2 + (k^0 - p^0)^2} \right) \left( 1 - \frac{k^0 q^0}{k^2} \right) \tag{8}
\]

In the following we will change our variables from \( p^0, k^0, |\vec{p}|, \) and \( |\vec{k}|, \) to \( p = \sqrt{p^0^2 + |\vec{p}|^2}, \ k = \sqrt{k^0^2 + |\vec{k}|^2}, \phi = \arctan(|\vec{p}|/p^0)\) and \( \tilde{\phi} = \arctan(|\vec{k}|/k^0)\). Without loss of generality we will write \( \tilde{B}(p^0, |\vec{p}|; q^0) \equiv R(q^0)B(p, \phi; q^0) \) with \( R(0) \equiv 1 \) and \( \tilde{B}(0, 0; q^0) \equiv 1 \).
At this stage our equation is linear, and the function $R(q^0)$ remains undetermined. Clearly some essential physics, containing the non-linearities, has been omitted by the original truncation. In the two-point function case it is known that the main effect of the non-linearities can be modeled by introducing an infrared cutoff, which is determined in a self-consistent manner by identifying it with the dynamical mass evaluated at the infrared cutoff. We will follow a similar procedure in the four-point function case by identifying the infrared cutoff with $\Lambda R(q^0)$, where $\Lambda$ is the infrared cutoff when $q^0 = 0$, and then determining $R(q^0)$ in a self-consistent manner.

Non-linear effects will enter through diagrams involving an odd number of four-point functions, an example of which is provided by the diagram in Fig.2. For large gluon momentum this diagram is much more suppressed than the diagrams we have been considering; the diagram only becomes important for small gluon momentum. This diagram thus provides an example of how non-linear effects could introduce a natural infrared cutoff into our problem. Unfortunately, it is not easy to resum such diagrams, as is possible in the two-point function case.

Our result for the 4-point function will take the form

$$O_{S+P} = \left(\frac{r}{\Lambda}\right)^2 R(q^0)^2 B(p_1, \phi_1; q^0) B(p_4, \phi_4; q^0),$$

where $r$ is a dimensionless constant. It is interesting that $r$ may have to be significantly larger than unity, in order to compensate for the loop-phase-space suppression factors appearing in diagrams such as Fig.2.

In place of an explicit infrared cutoff we choose to multiply the kernel of our
equation by the function \( k^2/(k^2 + \Lambda^2 R^2(q^0)) \). The final equation becomes

\[
pB(p, \phi; q^0) = \frac{3N\alpha}{4\pi^2} \int dk d\tilde{\phi} \frac{kB(k, \tilde{\phi}; q^0)k \sin \tilde{\phi}}{(q^0 - k \cos \tilde{\phi})^2 + k^2 \sin^2 \tilde{\phi} \sin \phi} \times \\
\ln \left( \frac{(k \sin \tilde{\phi} + p \sin \phi)^2 + (k \cos \tilde{\phi} - p \cos \phi)^2}{(k \sin \phi - p \sin \phi)^2 + (k \cos \phi - p \cos \phi)^2} \right) \times \\
\left( 1 - \frac{q^0 k \cos \tilde{\phi}}{k^2} \right) \times \frac{k^2}{k^2 + \Lambda^2 R^2(q^0)},
\]

where \( 0 \leq \phi, \tilde{\phi} \leq \pi \). We solve for the function \( pB \) in order to increase the accuracy of the numerical solution that follows, since we expect \( B \) to decrease with increasing \( p \). We note that our crude representation of the infrared effects does not affect the functional form of \( B \) when \( p \) is large.

With regards to the gauge dependence of our results we note that the use of the Landau gauge, along with a bare massless fermion propagator and a bare gluon-fermion vertex, is consistent with the Ward-Takahashi identity at one-loop. Moreover, in the large-\( N \) limit only diagrams \( A \) and \( B \) are important and the situation is then expected to be similar to that of the two-point function. In particular in that case the use of the Landau gauge in the ladder approximation yields results resembling those found in gauge-invariant treatments [4].

### IV Numerical results

The form of the integral equation allows us to use the same discretization lattice for the arguments of the function \( B \) inside and outside the integral. The angles are discretized according to \( \phi(i), \tilde{\phi}(i) = \frac{i\pi}{(n+1)}, i = 1, \ldots, n \). We avoid the endpoints where the angles are equal to 0 or \( \pi \) since our integral has an integrable singularity.
The momenta are discretized according to \( \log_{10} p(i), \log_{10} k(i) = \log_{10}(\Lambda_{IR} + \frac{i-1}{n-1} \log_{10}(\Lambda_{UV}/\Lambda_{IR}) \), where \( \Lambda_{IR,UV} \) are the infrared and ultraviolet cut-offs. \( \Lambda_{UV} \) does not have to be much larger than the energy scale where our function is negligibly small. On the other hand, \( \Lambda_{IR} \) has to be smaller than \( \Lambda \) which provides an effective infrared cutoff to our theory. We choose to work with the ratios \( \Lambda/\Lambda_{IR} = 10 \) and \( \Lambda_{UV}/\Lambda = 100 \). We choose the number of points \( n \) in each dimension to be equal to 40. Our results change little when this number is increased.

The integral equation is solved via a relaxation method. We first insert an initial configuration for our function, and then iterate the equation until it is satisfied to a reasonable accuracy. We start by setting \( q^0 = 0 \) and determine the critical coupling necessary for a solution. We then keep this coupling fixed, and for different values of \( q^0 \) we compute \( B(p, \phi; q^0) \) and \( R(q^0) \). Our critical coupling is \( N\alpha_c = 2.7 \pm 0.3 \) which, from our previous discussion, is the same as the one for chiral symmetry breaking via the two-point function with the same cut-offs. In the large-\( N \) limit and for infinite \( \Lambda_{UV} \) this coupling is given by \( N\alpha_c = \frac{2\pi}{3} \approx 2.1 \).

The function \( B(p, \phi; q^0) \) has only a weak dependence on \( q^0 \), and in Fig. we plot \( \frac{B}{\Lambda}B(p, \phi; 0) \). We see that \( \frac{B}{\Lambda}B(p, \phi; 0) \) exhibits the \( \cos (\gamma \log (p/\Lambda)) \) behavior (with \( \gamma \) a coupling-dependent constant) that is well known in two-point function studies. \( B(p, \phi; 0) \) is independent of \( \phi \) as expected, and even for non-zero \( q^0 \) the dependence on \( \phi \) remains very weak.

The main \( q^0 \) dependence enters in the function \( R(q^0) \) which we plot in Fig. R\( (q^0) \) falls rapidly with increasing \( q^0 \); we find that \( q^0 \) cannot exceed \( \Lambda \) by much
before the whole solution collapses to zero.

V 2n-point Green functions

The form that the integral equation takes for large $N$ suggests a possible generalization of the above results. In particular, the diagrams that dominate are the ones in which a gluon is attached between two fermion fields with spinor indices contracted with each other. Therefore, one might expect only condensates of the form $<\bar{\psi}_L\psi_R...\bar{\psi}_L\psi_R + h.c.>$, where the ellipsis stands for $2n - 4$ fermion fields paired with each other. They should form at a critical value of the gauge coupling similar to that required by the two- and four-point functions.

Moreover, we expect the corresponding scalar 2n-point function $O_{S+P}^n$ to factorize as $O_{S+P}^n \sim R^n B_1...B_n$. Each $B_i$ depends on the momenta of the respective fermion pair. In order for each of the $B_i$ to satisfy the same final integral equation as before, however, we would have to assume that the three-momenta of the fermions in each pair are equal and opposite, so that $\vec{q}_i = 0$. We expect, though, that our previous functions $B$ and $R$ should describe the qualitative momentum behavior of the 2n-point functions if we are able to go to an optimal reference frame where the various $\vec{q}_i$ are reasonably small.

VI Conclusions

In this work we considered chirality-changing fermion four-point functions in a non-abelian gauge theory which receive non-perturbative contributions exclusively. We have tackled the problem by writing down an equation derived from a truncation
of the infinite SD hierarchy. The solution to this equation is expected to illustrate some generic properties of four-point functions. The large-$N$ limit of $SU(N)$ renders the four-point problem similar to that of the three-point function. We find that a particular four-point function is much more likely to form than other four-point functions, and that its critical coupling is similar to the one required for a two-fermion condensate. The large-$N$ limit also allows us to discuss the critical and momentum behavior of fermion chirality-flipping $2n$-point functions for $n$ higher than two.

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**Appendix A**

The purpose of this appendix is to investigate the effects that the gluons in diagrams $A,...,F$ have on the color structure of the fermion vertices. We consider the case of fermions in a general representation of a compact simple Lie group. The group is then defined by the commutation relations $[\lambda^a, \lambda^b] = t^{abc} \lambda^c$, where $\lambda^a$ are the (traceless) generators of the corresponding algebra.

The Dynkin index $T_f$ is defined by the relation $Tr(\lambda^a \lambda^b) = T_f \delta^{ab}$. By $N_f \equiv Tr1$ we denote the dimension of the fermion representation, and by $C_f$ and $C_g$ the quadratic Casimirs of the fermion and adjoint representations. These are defined by $\lambda^a \lambda^a = C_f 1$ and $t^{abc} t^{abc'} = C_g \delta^{cc'}$. For fermions in the fundamental representation of $SU(N)$, $N_f = C_g = N$, $T_f = \frac{1}{2}$, and $C_f = \frac{N^2-1}{2N}$.
There are two color structures in the problem. One is $1 \otimes 1$, which is associated with the functions $O_{S+P,T}$, and the other is $\lambda^a \otimes \lambda^a$, which is associated with the functions $O_{S+P,T}^{\text{color}}$.

I. VERTEX COLOR STRUCTURE: $1 \otimes 1$

i. Diagrams $A, B$

We have $\lambda^a \lambda^a \otimes 1 = C_f 1 \otimes 1$. Diagrams multiplied by $C_f$, no change in color structure.

ii. Diagrams $C, D, E, F$

Color structure becomes $\lambda^a \otimes \lambda^a$.

II. VERTEX COLOR STRUCTURE: $\lambda^a \otimes \lambda^a$

i. Diagrams $A, B$

$\lambda^b \lambda^a \lambda^b \otimes \lambda^a = (C_f - C_g/2)\lambda^a \otimes \lambda^a$. Diagrams multiplied by $C_f - C_g/2$, no change in color structure.

ii. Diagrams $E, F$

Color structure becomes $O_{EF} \equiv \lambda^a \lambda^b \otimes \lambda^a \lambda^b$. This has to be reduced down to the two initial color structures.

We first expand the matrix $\lambda^a \lambda^b$ with respect to the basis of the unity matrix $1$ and the group generators $\lambda^c$.

$$\lambda^a \lambda^b = f^{ab} 1 + h^{abc} \lambda^c$$  \hspace{1cm} (11)

By taking the trace of both sides, we find that $f^{ab} = \frac{T_r}{N_f} \delta^{ab}$. Moreover, considering the case $a = b$ proves that $h^{aac} = 0$. 

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The above considerations allow us to write

\[ O_{EF} = \frac{T_f}{N_f} C_f \mathbf{1} \otimes \mathbf{1} + h^{abc} h^{abc'} \lambda^c \otimes \lambda^{c'} \]  

(12)

We now have to calculate the quantity \( h^{abc} h^{abc'} \). By virtue of (11), we consider the tensor

\[
h^{abc} h^{abc'} \lambda^c \lambda^{c'} = \left( \lambda^a \lambda^b - \frac{T_f}{N_f} \delta^{ab} \mathbf{1} \right) \left( \lambda^a \lambda^b - \frac{T_f}{N_f} \delta^{ab} \mathbf{1} \right) = \left( C_f - \frac{C_g}{2} - \frac{T_f}{N_f} \right) C_f \mathbf{1}
\]

(13)

But the unique quadratic tensor proportional to \( \mathbf{1} \) is the quadratic Casimir of the fermion representation. This means that \( h^{abc} h^{abc'} = h \delta^{cc'} \), in which case we find

\[ h = C_f - \frac{C_g}{2} - \frac{T_f}{N_f} \]  

(14)

So finally we have a new structure \( O_{EF} = \frac{T_f}{N_f} C_f \mathbf{1} \otimes \mathbf{1} + h \lambda^c \otimes \lambda^c \), with \( h \) given above. For fermions in the fundamental representation of \( SU(N) \), \( h = -1/N \).

iii. Diagrams \( C, D \)

Color structure becomes \( O_{CD} \equiv \lambda^a \lambda^b \otimes \lambda^b \lambda^a \). By using the commutation relations of the group, we find that the color structure changes to \( O_{CD} = O_{EF} - \frac{C_a}{2} \lambda^a \otimes \lambda^a \).

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Figure 1: The schematic form of the SD equation. We have labeled the four fermions by their spinor indices. We label the diagrams by the capital letters $A, ..., F$, and we omit the factor of $1/2$ multiplying the right-hand side.

Figure 2: An example of a diagram which introduces non-linearities in the infrared.

Figure 3: The function $\frac{2}{\Lambda} B(p, \phi; 0)$

Figure 4: The function $R(q^0)$. 

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Figure 1: The schematic form of the SD equation. We have labeled the four fermions by their spinor indices. We label the diagrams by the capital letters $A, ..., F$, and we omit the factor of $1/2$ multiplying the right-hand side.
Figure 2: An example of the non-linear terms that act as an IR cut-off in our theory.

Figure 3: The function $\frac{p}{\Lambda} B(p, \phi; 0)$
Figure 4: The function $R(q^0)$. 

\[R(q^0)\]