On maximum of Gaussian random field having unique maximum point of its variance*

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Abstract: Gaussian random fields on Euclidean spaces whose variances reach their maximum values at unique points are considered. Exact asymptotic behaviors of probabilities of large absolute maximum of theirs trajectories have been evaluated using Double Sum Method under the widest possible conditions.

Keywords: Non-stationary random field; Gaussian field; large excursion; Pickands’ method; Double sum method.

1 Introduction.

This contribution is a generalization of results of [7]. As discussed in [12], and then in [14], [15], [7], non-stationary Gaussian processes are more subtle to deal with since both the local properties of the variance function at its point of global maximum and those of the covariance function have to be carefully formulated. One can say the same about Gaussian fields, see [14]. Our aim is to show maximum capabilities of the Pickands’ Double Sum Method applying to Gaussian fields with unique point of global maximum; in case of processes this has been done in [7]. The Pickands’ method was developed originally for asymptotic behavior of the maximum tail distribution for Gaussian stationary processes in [11], with corrections in [13]. This method has been generalized to Gaussian random fields, [14], where stationary fields with power like behavior of the covariance function at zero are considered as well as fields with similar behavior of the covariance function at the unique maximum point of variance. However, while the power behavior of the covariance function, with possible slight generalization to regular variation of it, [13], is quite essential for the Pickand’s method, the required in [12], [14] power behavior of the variance, as it has been shown in [7], looks somewhat artificial. Thus the principal task of this contribution is to investigate the tail asymptotic behavior of supremum of non-stationary Gaussian fields by imposing a weak and natural assumption on their variance functions, see Conditions 5 and 6 below. In connection with, notice that in the recent article [2] it is proved that in the non-stationary case the behavior of variance does not need to be exactly power but may be just regularly varying. Here we do not assume even this.

Let $S \subset \mathbb{R}^d$ be the closure of a bounded open set containing zero, and let $X(t)$, $t \in S$, be a zero mean a.s. continuous Gaussian random field with covariance function

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\( R(s, t) = EX(s)X(t) \); denote by \( \sigma^2(t) = R(t, t) \) its variance function, which is continuous since \( X \) is a.s. continuous. We study the asymptotic behavior of the probability

\[
P(S; u) = P(\max_{t \in S} X(t) > u)
\]

as \( u \to \infty \). We need a slightly stronger condition than a.s. continuity of sample paths. Denote by \( \sigma^2(t) = R(t, t) \) its variance function, which is continuous since \( X \) is a.s. continuous. We study the asymptotic behavior of the probability

\[
P(S; u) = P(\max_{t \in S} X(t) > u)
\]

as \( u \to \infty \).

**Condition 1** \( X(t) \) is a.s. continuous. Moreover, there exists \( \varepsilon > 0 \) such that Dudley’s integral, \( [2], [12] \), for the standardized field \( \bar{X}(t) = X(t)/\sigma(t) \), \( t \in B_\varepsilon \) is finite.

Notice that for homogeneous Gaussian fields this condition is also necessary for existing of a.s. continuous version of the field (X. Fernique \[10\]). We need this condition in order to use V. A. Dmitrovsky’s inequality for estimating the exit probability from above. For reader’s convenience we give a corollary from the inequality adapted to our purposes, see Corollary 8.2.1, \[15\].

**Proposition 1** (V. Dmitrovsky, \[3\], \[4\], \[15\]) Let Condition 1 be hold. Then there exists \( \gamma(u) \), such that \( \gamma(u) \to 0 \) as \( u \to \infty \), and for any \( S_1 \subset B_\varepsilon \),

\[
P(S_1; u) \leq \exp \left( -\frac{u^2}{2\sigma^2(S_1)} + u\gamma(u) \right),
\]

where \( \sigma^2(S_1) = \sup_{t \in S_1} EX^2(t) \)

**Condition 2** \( \sigma(t) \) reaches its absolute maximum on \( S \) at only 0.

Without loss of generality assume that \( \sigma(0) = 1 \). Notice also that by time parameter shift the maximum point can be made arbitrary, with corresponding conditions on the parameter set. From Condition 2 it follows in particular that the normalized field \( \bar{X}(t) \), \( t \in B_\varepsilon \), see Condition 1 exists. Furthermore, it follows from it that \( r(s, t) \leq 1 \) and the equality holds only for \( s = t = 0 \).

Notice that we do not consider the case of boundary maximum point of variance. It can be considered with described here tools, with involving the structure of the boundary near the point. Such the consideration could not require any new ideas but makes the text longer and even more difficult to read. We would have to introduce a series new Pickands like constants. Only in the Talagrand case (see below) the asymptotic behavior reminds the same in this boundary maximum point case.

**Condition 3** (Local stationarity at 0). There exists a covariance function \( r(t) \) of a homogeneous random field with \( r(t) < 1 \) for all \( t \neq 0 \) such that

\[
\lim_{s,t \to 0, s \neq t} \frac{1 - R(s, t)}{1 - r(t - s)} = 1.
\]

**Remark 1** In contrast to \[12\], \[14\], \[13\], \[7\], and other works, we assume here local stationarity in terms of covariance function, not correlation (normed covariance) function. This is because we would like to impose minimal number of assumptions on the variance function. In particular, we do not assume Hölder condition in any neighborhood of zero, like we did this in \[7\].
For vectors \( \mathbf{a} = (a_1, ..., a_d) \), \( \mathbf{b} = (b_1, ..., b_d) \) define \( \mathbf{a} \mathbf{b} = (a_1 b_1, ..., a_d b_d) \). For a set \( T \) we write \( \mathbf{a} T = \{ \mathbf{a} t, t \in T \} \).

**Condition 4** There exists a basis in \( \mathbb{R}^d \), a vector function \( \mathbf{q}(u) = (q_1(u), ..., q_d(u)) \), \( u > 0 \), \( q_i(u) > 0 \), \( i = 1, ..., d \), and a positive for all \( \mathbf{t} \neq \mathbf{0} \) function \( h(t) \) such that for any \( \mathbf{t} \) written in these coordinates,

\[
\lim_{u \to \infty} u^2(1 - r(\mathbf{q}(u)\mathbf{t})) = h(t),
\]

uniformly in \( \mathbf{t} \) from any closed set.

In slightly other words, Condition 4 means that for some orthogonal matrix \( U \), (2) is fulfilled for \( \mathbf{t}' = U \mathbf{t} \) instead of \( \mathbf{t} \), that is

\[
\lim_{u \to \infty} u^2(1 - r(\mathbf{q}(u)U\mathbf{t})) = h(U\mathbf{t}).
\]

Remark that by definition of uniform convergence to a positive for all non-zero \( \mathbf{t} \) function with \( h(\mathbf{0}) = 0 \), from Condition 4 it follows that \( h(t) \) is continuous, and for any continuous function \( c_t \) with \( \lim_{\mathbf{t} \to \mathbf{0}} c_t = 1 \),

\[
\lim_{\mathbf{t} \to \mathbf{0}} \frac{1 - r(c_t \mathbf{t})}{1 - r(\mathbf{t})} = 1.
\]

Consider a simple example. Let \( d = 2 \) and

\[
1 - r(\mathbf{t}) = (|t_1 + t_2|^{\alpha_1} + |t_1 - t_2|^{\alpha_2})(1 + o(1))
\]
as \( \mathbf{t} \to \mathbf{0} \), where \( 2 \geq \alpha_1 > \alpha_2 > 0 \). For such covariance function one cannot find \( \mathbf{q}(u) \) satisfying (2), whereas rotating the basis turning at an angle of \( \pi/4 \), we have,

\[
1 - r(\mathbf{t}) = (2^{\alpha_1/2}|t_1|^{\alpha_1} + 2^{\alpha_2/2}|t_2|^{\alpha_2})(1 + o(1))
\]
as \( \mathbf{t} \to \mathbf{0} \), and \( \mathbf{q}(u) = (u^{-2/\alpha_1}, u^{-2/\alpha_2}) \) with \( h(t) = 2^{\alpha_1/2}|t_1|^{\alpha_1} + 2^{\alpha_2/2}|t_2|^{\alpha_2} \).

From now on we assume that the basis in \( \mathbb{R}^d \) satisfies Condition 4. From Conditions 3 and 4 it follows a regular variation property of \( r(\mathbf{t}) \). Indeed, remark first that since \( r(\mathbf{t}) \) is continuous at zero, we have \( \mathbf{q}(u) \to \mathbf{0} \) as \( u \to \infty \). Denote \( (\mathbf{e}_i, i = 1, ..., d) \), coordinate vectors in \( \mathbb{R}^d \). From Condition 4 since \( h(\mathbf{t}) \) is continuous, it also follows that

\[
\lim_{s \to 0} \frac{1 - r(s\mathbf{e}_i)}{1 - r(s\mathbf{e}_i)} = \frac{h(\mathbf{e}_i)}{h(\mathbf{e}_i)}
\]

for all \( t > 0 \) and \( i = 1, ..., d \),

and since the functions on the right are positive, continuous and cannot be equal identically to one \( (h(\mathbf{0}) = 0) \), \( 1 - r(s\mathbf{e}_i) \) regularly varies at zero with positive degree \( \alpha_i > 0 \), \( i = 1, ..., d \), and \( h(\mathbf{e}_i) = C_i |t|^{\alpha_i}, i = 1, ..., d \) for some \( C_i \). Notice that from properties of positive defined functions it follows that \( \alpha_i \leq 2 \) for all \( i \). That is, denoting by \( \ell_i(t), i = 1, ..., d \), the corresponding slowly varying functions, we write for all \( i \), that \( 1 - r(s\mathbf{e}_i) = \ell_i(t)|t|^{\alpha_i} \). Hence, using (2) and the definition of slowly variation,

\[
u^2(1 - r(q_i(u)s\mathbf{e}_i)) = u^2q_i(u)^{\alpha_i}|t|^{\alpha_i}\ell_i(tq_i(u)) = (1 + o(1))u^2q_i(u)^{\alpha_i}|t|^{\alpha_i}\ell_i(q_i(u))
\]
as \( u \to \infty \), and

\[
u^2q_i(u)^{\alpha_i}\ell_i(q_i(u)) \to 1
\]
as \( u \to \infty \). From here it follows that for some \( \ell_{1i}(u) \)

\[
q_i(u) = \ell_{1i}(u)u^{-2/\alpha_i}, \ell_{1i}(u)^{\alpha_i}\ell_i(u^{-2/\alpha_i}\ell_{1i}(u)) \to 1 \text{ as } u \to \infty.
\]
It is shown in [7] that
\[ \ell_{1i}(u) = (1 + o(1))\ell_i^\#(u-2)^{1/\alpha_i} \]  
(5)
as \( u \to \infty \), where \( \ell_i^\# \) is the de Bruijn conjugate of \( \ell_i \), see details below and also in [1], and \( \ell_{1i}(u) \) slowly varies as well. Moreover, in [7] it is shown that (2) holds for any \( \ell_i'(u) \), \( i = 1, \ldots, d \), such that \( \lim_{u \to \infty} \ell_i(u)/\ell_i'(u) = 1, \ i = 1, \ldots, d \). Consequently, without loss of generality we assume in the following that all \( \ell_i(u), \ i = 1, \ldots, d \) are monotone. Furthermore, using this argument and the fact that the ratio of two slowly varying positive functions, say \( L_1(u) \) and \( L_2(u) \), slowly varies as well, having in mind again its monotone equivalent, we may write that
\[ \lim_{u \to \infty} L_1(u)/L_2(u) = \kappa \in [0, \infty]. \]  
(6)

Let us say that \( L_1(u) \triangleright L_2(u) \) if \( \kappa = \infty \), \( L_1(u) \vartriangleleft L_2(u) \), if \( \kappa = 0 \), and \( L_1(u) \asymp L_2(u) \) if \( \kappa \in (0, \infty) \). We use these definitions in the proof of the following proposition.

**Proposition 2** Let Conditions 3 and 4 be fulfilled for a covariance function \( r(t) \). Then for any vector \( \mathbf{f} \) the function \( r(tf) \) regularly varies at zero with degree \( \alpha(\mathbf{f}) \in (0, 2] \). Moreover, if \( \alpha(\mathbf{f}) = 2 \), then \( \lim_{t \to 0} t^{-2}(1 - r(tf)) = 0 \).

**Proof:** Denote \( g(s) := 1 - r(sf) \), \( \mathbf{f} = (f_i, i = 1, \ldots, d) \),
\[ \alpha = \alpha(\mathbf{f}) := \min\{\alpha_i : f_i \neq 0\}, \]  
(7)
and \( J = J(\mathbf{f}) := \{i : \alpha_i = \alpha, f_i \neq 0\} \).

Introduce the “main” index set \( J_0 \subset J \) related to \( \mathbf{f} \), with property
\[ \forall i, j \in J_0, \ell_{1i}(u) \asymp \ell_{1j}(u), \ and \ \forall i, j \in J, j \in J \setminus J_0, \ \ell_{1i}(u) \triangleright \ell_{1j}(u). \]

Numerate indexes from \( J_0 = \{i_1, \ldots, i_{k_0}\} \) and denote
\[ \kappa_i = \lim_{u \to \infty} \ell_i(u)/\ell_{1i}(u) \in (0, \infty), \ i = i_1, \ldots, i_{k_0}, \]
where, as above, we mean monotone equivalent ratios, see (3). Put \( s = u^{-2/\alpha} \ell_{1i}(u) = q_{i_1}(u) \) and look at the behavior of \( g(q_{i_1}(u)) \) when \( u \to \infty \). We have by choice of \( J_0 \),
\[ \lim_{u \to \infty} \frac{q_{i_1}(u)}{q_i(u)} = \begin{cases} \kappa_i, & \text{if } i \in J_0; \\ 0, & \text{if } i \notin J_0. \end{cases} \]
Hence, by Condition 4
\[ \lim_{u \to \infty} u^2g(q_{i_1}(u)) = h(\kappa), \]  
(8)
where \( \kappa := (\kappa_i, i \in J_0; 0, i \notin J_0) \). Now put \( st = q_{i_1}(u_1) \); we have, as above,
\[ \lim_{u_1 \to \infty} u_1^2g(q_{i_1}(u_1)) = h(\kappa). \]  
(9)
Using Theorems 1.5.12, 1.5.13 (de Bruijn Lemma) and Proposition 1.5.15, [1], similarly to [7], we get that for some slowly varying function \( \tilde{\ell}(s) \),
\[ u = s^{-\alpha/2}\tilde{\ell}(s) \]  
and \( u_1 = (st)^{-\alpha/2}\tilde{\ell}(st) \),
which together with (8) gives
\[ \lim_{s \to 0} g(st)/g(s) = t^\alpha, \ \alpha = \alpha(\mathbf{f}), \]
that is, \( r(t) \) regularly varies at direction \( \mathbf{f} \). Further, if \( \alpha(\mathbf{f}) = 2 \) and \( s^{-2}(1 - r(sf)) \to 0 \) as \( s \to 0 \), then by properties of positive defined functions, \( r(tf) \equiv 1, t \in \mathbb{R} \), which contradicts Condition 3. Thus Proposition is established.
Remark 2 Remark that from Proposition 2 it follows that if \( \alpha_i = \alpha(e_i) = 2 \) then the corresponding slowly varying function \( \ell_i(t) \) is bounded from above. So, since \( u^2 q_i^2(u) = \ell_i^2(q_i(u)) = \ell_i^2(q_i(u)) \to 1 \) as \( u \to \infty \), the slowly varied function \( \ell_i(u) \) is bounded from below by a positive constant, that is,

\[
q_i(u) \geq q_0 u^{-1}, \quad q_0 > 0. \tag{10}
\]

Moreover, it is obvious that (I7) is valid for any \( q_i(u), i = 1, ..., d \). We shall use this below.

Remark 3 Observe that for any direction \( f \) we just choose appropriate \( q_i(u) \) from the collection of Condition 4.

Now assume a behavior of \( \sigma(t) \) near its point of absolute maximum. We shall see from the proof of Lemma 2 that the crucial point is the behavior of the ratio

\[
\frac{1 - \sigma^2(q(u)t)}{1 - r(q(u)t)}
\]

as \( u \to \infty \). In view of Condition 4 we assume the following.

Condition 5 For any \( t \) there exists the limit

\[
h_1(t) := \lim_{u \to \infty} u^2(1 - \sigma^2(q(u)t)) \in [0, \infty]. \tag{11}
\]

In case when the limit is equal to zero we speak about the stationary-like case. If the limit is equal to infinity, we refer to the Talagrand case, since M. Talagrand, [16], has shown that in most general conditions, for any closed set \( S \) and a Gaussian a. s. continuous function \( X(t), t \in S \), having unique point of maximum of variance, say, at \( t_0 \in S \),

\[
P(S; u) = P(X(t_0) > u)(1 + o(1)), \quad u \to \infty.
\]

In our conditions we show this below. At last, we say about the transition case if \( h_1(t) \) is neither zero nor infinity. Denote correspondingly

\[
\mathcal{K}_0 := \{ t \in S \setminus \{0\} : h_1(t) = 0 \}, \quad \mathcal{K}_c := \{ t \in S \setminus \{0\} : h_1(t) \in (0, \infty) \},
\]

\[
\mathcal{K}_\infty := \{ t \in S \setminus \{0\} : h_1(t) = \infty \}.
\]

We shall see that properties of these sets together with all above Conditions follow asymptotic behavior of the probability \( P(S; u) \). Consider one more simple example which shows that dimensions of \( \mathcal{K}_0, \mathcal{K}_c, \mathcal{K}_\infty \) may be arbitrary. Let \( d = 2 \),

\[
r(t_1, t_2) = 1 - |t_1|^\alpha_1 - |t_2|^\alpha_2 + o(|t|^\alpha), \quad t \to 0, \quad 2 \geq \alpha_1 \geq \alpha_2 > 0,
\]

\[
\sigma(t_1, t_2) = 1 - |t_1|^\beta_1 - |t_2|^\beta_2 + o(|t|^\beta), \quad t \to 0, \quad \beta_1 \geq \beta_2 > 0.
\]

Here one has not to change the basis, having obvious value of \( q(u) \). It is easy to calculate that

- If \( \beta_1 > \alpha_1 > \alpha_2 > \beta_2 \) then \( \dim \mathcal{K}_0 = 1, \dim \mathcal{K}_c = 0, \dim \mathcal{K}_\infty = 2 \).
- If \( \beta_2 > \alpha_1 \) then \( \dim \mathcal{K}_0 = 2, \dim \mathcal{K}_c = 0, \dim \mathcal{K}_\infty = 0 \).
- If \( \beta_1 = \alpha_1 > \alpha_2 > \beta_2 \) then \( \dim \mathcal{K}_0 = 0, \dim \mathcal{K}_c = 1, \dim \mathcal{K}_\infty = 2 \).
- If \( \beta_1 = \alpha_1 = \alpha_2 = \beta_2 \) then \( \dim \mathcal{K}_0 = 0, \dim \mathcal{K}_c = 2, \dim \mathcal{K}_\infty = 0 \).
- So on.
2 Homogeneous Gaussian fields

Let $S \subset \mathbb{R}^d$ satisfy the assumptions of the previous section, and $X(t), t \in S,$ be an a.s. continuous homogeneous zero-mean Gaussian field with covariance function satisfying Conditions 3 and 4.

**Lemma 1** In the above notations and conditions, for any bounded closed $T \subset \mathbb{R}^d$,

$$P\left( \max_{t \in T} X(t) > u \right) = (1 + o(1)) H_q(T) \Psi(u)$$

as $u \to \infty$, where

$$H_q(T) = \mathbb{E} \exp \left( \max_{t \in T} \chi(t) \right),$$

and $\chi(t)$ is a Gaussian a.s. continuous field with $\chi(0) = 0$,

$$\text{var}(\chi(t) - \chi(s)) = 2h(t - s), \quad \mathbb{E} \chi(t) = -h(t).$$

The proof of this lemma is a simple repetition of the proof of Lemma 6.1 of [14] with applying Condition 4 which implies among other that $\chi(t)$ exists.

**Theorem 1** In the assumptions of this section, for any closure $S_1 \subset S$ of an open set,

$$P\left( \max_{t \in S_1} X(t) > u \right) = (1 + o(1)) |S_1| H_q \prod_{i=1}^d q_i^{-1}(u) \Psi(u)$$

as $u \to \infty$, where $|\cdot|$ denotes the volume of $\{\cdot\}$, and

$$H_q = \lim_{T_i \to \infty, i=1,...,d} (T_1...T_d)^{-1} H_q \left( \bigotimes_{i=1}^d [0, T_i] \right) \in (0, \infty).$$

This assertion holds even if $S_1$ depends of $u$, $S_1 = S_1(u)$, provided there exist boxes $S_1^\pm(u) = \bigotimes_{i=1}^d [-S_1^i(u), S_1^i(u)]$ such that $S_1^+(u) \subset S_1(u) \subset S_1^-(u)$ with $S_1^i(u)q_i(u) \to \infty$ and for some $\delta < 1/2$, $S_1^+(u)e^{-6u^2} \to 0$ as $u \to \infty, i = 1, ..., d$.

The proof of the theorem follows step-by-step the proof of Theorem 7.1, [14].

3 Non-homogeneous Gaussian fields

Now we give two general results for all described above types of behavior of $\sigma(t)$. The first one is a standard local lemma of Double Sum Method, a generalization of Lemma 6 see [14], [15].

**Lemma 2** Under the Conditions 1 - 5, for any $T \in \mathbb{R}^d$,

$$P(q(u)T; u) = (1 + o(1)) P_q(T) \Psi(u)$$

as $u \to \infty$, where

$$P_q(T) = \mathbb{E} \max_{t \in T} e^{\chi_1(t)},$$

with $\chi_1(t) = \chi(t) - h_1(t)$ if $t \in K_0 \cup K_c$, and $\chi_1(t) \equiv 0$ if $t \in K_\infty$. 

The proof of this lemma is a repetition of the corresponding lemma proof in [14], using the Conditions 1 – 5. The only essential addition to the proof is careful consideration the points \( t \in K_\infty \), noticing that the weak convergence of the field

\[
\chi_u(t) := u(X(q(u)t) - u) + w
\]

conditioned on \( X(0) = u - w/\mu \in C(T) \) can be restricted to that in \( C(T \setminus K_\infty) \). That is, it can be proved that in the case of non-empty \( K_\infty \),

\[
\max_{t \in T} \chi(t) - h_1(t) = \max_{t \in T \setminus K_\infty} \chi(t) - h_1(t).
\]

The second general result is extraction of an informative parameter set. Denote

\[
\gamma_1(u) := \gamma(u) + u^{-1} \log^2 u,
\]

where \( \gamma(u) \) is taken from Proposition 1, and

\[
B_u := \{ t : 1 - \sigma^2(t) \leq 2u^{-1} \gamma_1(u) \}.
\]

**Lemma 3** In the above conditions and notations,

\[
P(S; u) = (1 + O(\exp(-\log^2 u))) \cdot P(S \cap B_u; u)
\]

as \( u \to \infty \).

**Proof:** Denote \( \sigma_\varepsilon := \sup_{t \in S \setminus B_\varepsilon} \sigma(t) \), where \( \varepsilon \) is taken from Condition 1. By Condition 2 we have, \( \sigma_\varepsilon < (1 + \sigma_\varepsilon)/2 < 1 \). By Borell-TIS inequality, see, for example, [14], Theorem D.1, for all sufficiently large \( u \),

\[
P(S \setminus B_\varepsilon; u) \leq 2\Psi \left( \frac{u}{(1 + \sigma_\varepsilon)/2} \right),
\]

and the right hand part is exponentially smaller than \( P(S; u) \). Indeed, \( 0 \in S \) by assumption, hence \( P(S; u) \geq P(X(0) > u) = \Psi(u)(1 + o(1)) \) as \( u \to 0 \). Further, we have for all sufficiently large \( u \),

\[
\max_{t \in S \setminus B_\varepsilon} \sigma^2(t) \leq 1 - 2u^{-1} \gamma_1(u),
\]

hence, by Proposition 1 we have,

\[
P(B_\varepsilon \setminus B_u; u) \leq \exp \left( -\frac{u^2}{2(1 - 2u^{-1} \gamma_1(u))} + u \gamma(u) \right) \leq \exp \left( -\frac{1}{2} u^2 - u \gamma_1(u) + u \gamma(u) \right) \]

\[
= \exp \left( -\frac{1}{2} u^2 - u (\gamma(u) + u^{-1} \log^2 u) + u \gamma(u) \right) = e^{-\log^2 u} e^{-u^2/2}.
\]

By the above trivial lower estimate, the right hand part is again infinitely smaller than \( P(S; u) \). Thus Lemma is established.

4 Gaussian fields with unique maximum point of variance.

We consider first the Gaussian field \( X(t) = \sigma(t)X_0(t) \), satisfying Conditions 1 – 5, where \( X_0(t) \) is a homogeneous centered Gaussian field with covariance function satisfying Condition 4 and \( \sigma(t) \) satisfies Condition 5. Then, using standard inequalities including Slepian Lemma, applied in \( B_u \), we pass to the general case.
4.1 Stationary like case.

Here we consider the stationary-like case, that is \( h_1(t) = 0 \) for all \( t \). Fix sufficiently large \( u = u_0 \) and denote

\[
f(t) = \frac{1}{2}(1 - \sigma^2(t)), \quad t \in B_{u_0}. \tag{14}\]

It will be convenient to extend \( f(t) \) to the unit cube \( K = \{ t : \max_{i=1,...,d} |t_i| \leq 1 \} \), having

\[
\max_{K \setminus B_u} f(t) \leq \max_{\partial B_{u_0}} f(t)
\]

with \( \partial B \), the boundary of \( B \). By Lemma 3, the only behavior of \( f(t) \) in any small neighborhood of 0 plays role for the desired asymptotic behavior. Hence we assume that \( f(t) \) is continuous in \( K \) and, taking in mind (14),

\[
f(t) \in (0, 1/2), \quad t \in K \setminus \{0\}. \tag{15}\]

Introduce the Laplace type integral,

\[
L_f(\lambda) := \int_K e^{-\lambda f(t)} dt, \quad \lambda > 0. \tag{16}\]

Notice that its asymptotic behavior as \( \lambda \to \infty \) depends only on behavior \( f(t) \) in a vicinity of zero, see, for example, [8].

**Proposition 3** Let Conditions 1–5 be fulfilled. If further \( h_1(t) = 0 \) for all \( t \), we have,

\[
P(S; u) = (1 + o(1)) H_q L_f(u^2) \prod_{i=1}^d q_i^{-1}(u) \Psi(u), \tag{17}\]

as \( u \to \infty \).

**Proof:** As it is mentioned above, we consider first a simplified model for \( X \), that is \( X(t) = X_0(t) \sigma(t), \quad t \in S \), so that \( X(t) \) satisfies Conditions 1–5. Recall that we consider the case

\[
\lim_{u \to \infty} \frac{1 - \sigma^2(q(u)t)}{1 - r(q(u)t)} = 0. \tag{18}\]

Denote

\[
T_i(u) = \sup\{|t_i| : t \in B_u\}, \quad i = 1, ..., d. \tag{19}\]

Obviously that for all \( i \), \( T_i(u) > 0 \) and it tends to zero as \( u \to \infty \). Moreover,

\[
\bigotimes_{i=1}^d [-T_i(u), T_i(u)] \supseteq B_u.
\]

By (18), (19) and the definition of \( B_u \),

\[
\lim_{u \to \infty} \frac{T_i(u)}{q_i(u)} = \infty, \quad i = 1, ..., d.
\]

Let an increasing \( \kappa(u) \) be such that

\[
\lim_{u \to \infty} \kappa(u) = \lim_{u \to \infty} \frac{T_i(u)}{q_i(u) \kappa(u)} = \infty, \quad i = 1, ..., d. \tag{20}\]
The box

$$\Delta_0 := \kappa(u) \prod_{i=1}^{d} [0, q_i(u)]$$  \hspace{1cm} (21)

satisfies conditions Theorem 1, hence,

$$P\left( \max_{t \in \Delta_0} X_0(t) > u \right) = (1 + \delta(u)) H_q |\Delta_0| \prod_{i=1}^{d} q_i^{-1}(u) \Psi(u),$$  \hspace{0.5cm} (22)

with $\delta(u) \to 0$ as $u \to \infty$. Denote

$$\Delta_k(u) = \kappa(u) k q(u) + \Delta_0(u), \ k \in \mathbb{Z}^d, \ u > 0.$$

For all $k$ with $\Delta_k(u) \cap B_u \neq \emptyset$ introduce events

$$A_k(u) = \left\{ \max_{t \in \Delta_k(u)} X_0(t) > u_k \right\} \text{ with } u_k = u/\sigma_k, \ \sigma_k = \max_{t \in \Delta_k(u)} \sigma(t),$$

and

$$A_k'(u) = \left\{ \max_{t \in \Delta_k(u)} X_0(t) > u_k' \right\} \text{ with } u_k' = u/\sigma_k', \ \sigma_k' = \min_{t \in \Delta_k(u)} \sigma(t).$$

By definition of $B_u$, after some easy calculations we have that

$$u \leq u_k, u_k' \leq u + \gamma_1(u), \ k \in K_u := \{k : \Delta_k(u) \cap B_u \neq \emptyset\}. \hspace{1cm} (23)$$

Hence all the boxes $\Delta_k(u)$ satisfy Theorem 1 conditions with $u_k$ instead of $u$. Therefore

$$P(A_k(u)) = (1 + \delta(u_k)) H_q |\Delta_k| \prod_{i=1}^{d} q_i^{-1}(u_k) \Psi(u_k),$$

and

$$P(A_k'(u)) = (1 + \delta(u_k')) H_q |\Delta_k| \prod_{i=1}^{d} q_i^{-1}(u_k') \Psi(u_k').$$

$k \in K_u$. By definition (23) of $K_u$, there exists a positive non-increasing $\delta_1(u)$ tending to zero as $u \to \infty$, such that for all $k \in K_u$,

$$1 - \delta_1(u) \leq \frac{\min(q_i(u_k), q_i(u_k'))}{q_i(u)} \leq \frac{\max(q_i(u_k), q_i(u_k'))}{q_i(u)} \leq 1 + \delta_1(u), \ \ i = 1, \ldots, d.$$  

Further, since $\delta_2(u) = \sup_{v \geq u} |\delta(v)| \to 0$ as $u \to \infty$ and $u_k, u_k' \geq u$, we get that for all $k \in K_u$,

$$P(A_k(u)), P(A_k'(u)) \leq (1 + \delta_1(u))(1 + \delta_2(u)) H_q |\Delta_k| \prod_{i=1}^{d} q_i^{-1}(u) \Psi(u_k). \hspace{1cm} (24)$$

By Bonferroni inequality

$$P(B_u; u) \leq \sum_{k: \Delta_k(u) \cap B_u \neq \emptyset} P(A_k(u)), \hspace{1cm} (25)$$

and

$$P(B_u; u) \geq \sum_{k: \Delta_k(u) \subset B_u} P(A_k'(u)) - \sum_{k, l: \Delta_k(u) \cap B_u \neq \emptyset, \Delta_l(u) \cap B_u \neq \emptyset, k \neq l} P(A_k(u)A_l(u)). \hspace{1cm} (26)$$
Write
\[ u_k^2 = u^2 + u^2(1 - \sigma_k^2) + \frac{u^2(1 - \sigma_k^2)^2}{\sigma_k^2}. \] (27)

Using
\[ \frac{u^2(1 - \sigma_k^2)^2}{2\sigma_k^2} \geq \frac{u^2u^2\gamma_k^2(u)}{2}, \] (28)
since \( \gamma_1(u) \to 0 \) as \( u \to \infty \), we have for some positive \( \delta_3(u) \) with \( \delta_3(u) \to 0, u \to \infty \), uniformly in \( k \in K_u \),
\[ \sigma_k \exp \left( -\frac{u^2(1 - \sigma_k^2)^2}{2\sigma_k^2} \right) \leq 1 \pm \delta_3(u). \] (29)

Hence, by (24)
\[ \sum_{k: \Delta_k(u) \cap B_u \neq \emptyset} P(A_k(u)) \leq (1 \pm \delta_4(u))H_a \prod_{i=1}^d q_i^{-1}(u)\Psi(u) \sum_{k: \Delta_k \cap B_u \neq \emptyset} |\Delta_k|e^{-u^2(1-\sigma_k^2)/2}, \] (30)
where
\[ 1 \pm \delta_4(u) = (1 \pm \delta_1(u))(1 \pm \delta_2(u))(1 \pm \delta_3(u)). \]

Remark that all the relations (27 – 29) are also valid for \( \sigma'_k \), with some other \( \delta'_\nu(u) \), \( \nu = 1, 2, 3, 4 \), having the same properties, say, for \( \delta'_\nu(u) \), \( \nu = 1, 2, 3, 4 \). Hence we also have (30) with \( A'_k(u) \) and \( \delta'_4(u) \) instead of \( A_k(u) \) and \( \delta_4(u) \). Denote
\[ \Sigma(u) := \sum_{k:1-\sigma_k^2 \leq 2u^{-1}\gamma_1(u)} |\Delta_k|e^{-u^2(1-\sigma_k^2)/2}, \text{ and } \Sigma'(u) := \sum_{k:1-\sigma_k^2 \leq 2u^{-1}\gamma_1(u)} |\Delta_k|e^{-u^2(1-\sigma_k^2)/2}. \] (31)
Notice that \( |\Delta_k| = |\Delta_0| = \kappa^d(u) \prod_{i=1}^d q_i(u) \), we used \( |\Delta_k| \) above for visibility. We have that \( \Sigma(u) \) and \( \Sigma'(u) \) are integral sums for the integral
\[ I(u) := \int_{f(t) \leq 2u^{-1}\gamma_1(u)} e^{-u^2f(t)}dt. \]

Besides,
\[ \Sigma'(u) \leq I(u) \leq \Sigma(u), \]
therefore from inequalities (30) and mentioned there their counterparts for \( A'_k(u) \) and \( \delta'_4(u) \) it follows that for some \( \tilde{\delta}(u) \) tending to zero as \( u \to \infty \),
\[ \sum_{k: \Delta_k(u) \cap B_u \neq \emptyset} P(A_k(u)) \leq (1 \pm \tilde{\delta}(u))H_a \prod_{i=1}^d q_i^{-1}(u)\Psi(u) \int_{f(t) \leq 2u^{-1}\gamma_1(u)} e^{-u^2f(t)}dt. \]

Now, using given by (14 – 16) definitions of \( f(t) \) and \( L(\lambda) \), we have,
\[ \int_{f(t) \leq 2u^{-1}\gamma_1(u)} e^{-u^2f(t)}dt = \left( 1 + O \left( e^{-2\log^2 u} \right) \right) L_f(u^2). \]
Indeed, it follows from (12) that \( \gamma_1(u) \geq u^{-1}\log^2 u \), so that for all sufficiently small \( t \) lying outside of the integration domain, \( u^2f(t) \geq 2u^2u^{-1}\gamma_1(u) \geq 2u^2u^{-2}\log^2 u. \)
Now estimate from above the double sum in (26). We have for non neighboring boxes,

\[
P(A_k A_l) \leq P \left( \max_{(s,t) \in \Delta_k \otimes \Delta_l} X(s) + X(t) > u_k + u_l \right)
\]

\[
\leq P \left( \max_{(s,t) \in \Delta_k \otimes \Delta_l} \frac{X_0(s) + X_0(t)}{\sqrt{2 + 2r(t-s)}} \geq \frac{u_k + u_l}{\sqrt{2(1 + r_{k,l}(u))}} \right),
\]

(32)

where \(2 + 2r(t-s)\) is the variance of \(X_0(s) + X_0(t)\) and

\[
r_{k,l}(u) = \max_{(s,t) \in \Delta_k \otimes \Delta_l} r(t-s).
\]

For the increments of the zero mean field

\[
Y(s, t) := \frac{X_0(s) + X_0(t)}{\sqrt{2 + 2r(t-s)}},
\]

with unit variance, one can get by simple algebra that for an absolute constant \(C\),

\[
E(Y(s, t) - Y(s', t')) \leq C( E(X_0(s) - X_0(s'))^2 + E(X_0(t) - X_0(t'))^2).
\]

From this inequality by standard Gaussian technique including Slepian inequality it follows that the right hand part of (32) is at most

\[
C_1 \kappa^{2d}(u) \Psi \left( \frac{u_k + u_l}{\sqrt{2(1 + r_{k,l}(u))}} \right),
\]

where the constant \(C_1\) does not depend of \(k, l\). Now write,

\[
(u_k + u_l)^2 = u^2 (\sigma_k^{-1} + \sigma_l^{-1})^2 = u^2 \left( 2 + 1 - \sigma_k + 1 - \sigma_l + \sum_{j=2}^{\infty}((1 - \sigma_k)^j + (1 - \sigma_l)^j) \right)^2.
\]

Using that \(r(t)\) regularly varies at zero in any direction \(f\), \(|f| = 1\), with indexes \(\alpha(f) \in (0, 2]\), see (7), we get for some positive \(\gamma \in [\max_{f} \alpha(f), 2]\) and \(\Gamma\) that \(1 - r(t) \geq \Gamma |t|^\gamma\). Remark that from (2) and followed then argument in case \(\alpha(f) = 2\) the corresponding slowly varying function is bounded at zero otherwise square mean derivative of \(X_0\) in this direction exists and is a constant, this contradicts the conditions on \(r\) and \(R\). Thus we have for non neighboring \(k, l\),

\[
u^2(1 - r_{k,l}(u)) \geq \Gamma \kappa^\gamma(u) u^{2-2/\gamma} \geq \Gamma \kappa^\gamma(u).
\]

Therefore, by analogy to (27, 28), we have,

\[
\frac{(u_k + u_l)^2}{2(1 + r_{k,l}(u))} \geq \frac{4u^2}{2(2 - (1 - r_{k,l}(u)))} + \frac{1}{2} u^2 (1 - \sigma_k) + \frac{1}{2} u^2 (1 - \sigma_l)
\]

\[
\geq u^2 + \frac{1}{2\Gamma} \kappa^\gamma(u) \geq \frac{1}{4} u^2 (1 - \sigma_k^2) + \frac{1}{4} u^2 (1 - \sigma_l^2),
\]

where we also used that \(r \leq 1\) and \(\sigma \leq 1\). Finally we have for non neighboring \(k, l\), that is, \(\max_{i=1, \ldots, d} |k_i - l_i| > 1\),

\[
P(A_k A_l) \leq C_2 \kappa^2(u) \exp \left( -\frac{1}{2\Gamma} |\kappa(u)|^\gamma \right) \Psi(u) \exp \left( -\frac{1}{4} u^2 (1 - \sigma_k^2) - \frac{1}{4} u^2 (1 - \sigma_l^2) \right),
\]

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Summing up, denoting $\mathcal{A}(k, l) := \{k, l : \Delta_k(u) \cap B_u \neq \emptyset, \Delta_l(u) \cap B_u \neq \emptyset\}$, we have for some $C_2 > 0$,

$$\sum_{k, l \in \mathcal{A}(k, l) : \max_{i=1, \ldots, d} |k_i - l_i| > 1} P(A_k A_l) \leq C_2 \kappa^2(u) \exp(-\Gamma|\kappa(u)|^\gamma) \Psi(u) \times \left( \sum_{k : \Delta_k(u) \cap B_u \neq \emptyset} e^{-\frac{1}{4} u^2 (1 - \sigma_k^2)} \right)^2 \tag{33}$$

By the above argument the last sum multiplied by $\kappa^d(u) \prod_{i=1}^d q_i(u)$ is an integral sum for the integral $I(u/\sqrt{2})$, with inessential changing of the integration domain. Obvious application of Schwarz inequality gives

$$I^2(u/\sqrt{2}) \leq (1 + \varepsilon) I(u),$$

for any $\varepsilon > 0$ and all sufficiently large $u$. The first exponent in the right hand part of (33) gives that the double sum over non-neighboring intervals is infinitely smaller than both the single sums in (23, 26).

Consider the double sum over neighboring intervals, that is over $k, l$ with $\max_{i=1, \ldots, d} |k_i - l_i| = 1$. This part is quite similar to the corresponding argument in [14], [15]. Let for definiteness $l_1 - k_1 = 1$. Denote

$$\Delta'_0 := \left[0, \frac{\kappa(u)q_1(u)}{\sqrt{\kappa(u)}}\right] \otimes \prod_{i=2}^d \left[0, \kappa(u)q_i(u)\right]$$

and write

$$P(A_k A_l) \leq P\left( \max_{t \in \kappa(u)q_i(u) + \Delta'_0} X_0(t) \geq u_i \right) + P\left( \max_{s \in \Delta_k, t \in \kappa(u)q_i(u) + \Delta'_0 \Delta_0} X_0(s) + X_0(t) \geq u_k + u_l \right).$$

The sum of the first probabilities on the right can be estimated using the same argument as the estimation of single sums above. Wherein the multiplier $\sqrt{\kappa(u)q_1(u)}$ appears which gives that the sum is infinitely smaller the single sum above. For the second probability on the right the argument of the double sum estimation over non-neighboring boxes can be applied because of the distance between boxes is not zero but $\sqrt{\kappa(u)q_1(u)}$. This also gives that the sum is infinitely smaller than the single sums.

Hence in view of already mentioned standard passage from the particular $X(t) = X_0(t)\sigma(t)$ to the general Gaussian process (by applying Slepian inequality), the proof follows. Thus Proposition is established.

**Remark 4** In the case when the fraction (18) tends to zero sufficiently fast, for example for $d = 1$,

$$\lim_{t \to 0} \frac{1 - \sigma^2(t)}{t^\varepsilon (1 - r(t))} = 0$$

with some $\varepsilon > 0$, the estimation of the double sum is quite similar to that in [14], [15]. But the fraction may tend to zero very slowly, so that for this situation the evaluations and estimations have to be more precise, what we have done here.
4.2 Talagrand case.

**Proposition 4** In the above conditions, if \( K_\infty = S \setminus \{0\} \), then

\[
P(S; u) = (1 + o(1)) \Psi(u),
\]

as \( u \to \infty \).

**Proof:** By Proposition 2 and Remark 2 in view of Lemma 3 we have that for \( \varepsilon \in (0, q_0) \), and all sufficiently large \( u \), \( B_u \subseteq \varepsilon \bigotimes_{i=1}^{d} [-q_i(u), q_i(u)] \). Further, similarly to the proof of Lemma 8.4, [14], we get that

\[
\limsup_{u \to \infty} \frac{P(B_u; u)}{\Psi(u)} \leq \limsup_{u \to \infty} \frac{P(\varepsilon \bigotimes_{i=1}^{d} [-q_i(u), q_i(u)]; u)}{\Psi(u)} \leq \mathbb{E} \max_{t \in [-\varepsilon, \varepsilon]^d} e^{\chi(t)}.
\]

Observe that in the latter inequality we again pass to a homogeneous field using monotonicity with respect to the variance and Slepian’s inequality, with following application of Lemma 1. Then, as in the proof of Lemma 8.4, we use Monotone Convergence Theorem to let \( \varepsilon \downarrow 0 \).

4.3 The transition case

Asymptotic evaluations for the exceeding probabilities in this case are quite similar to the corresponding evaluations in [12], [14], [15], with applying Lemma 2. Take in the Lemma

\[
T = T_0 = [-K, K]^d,
\]

and denote

\[
T_k = 2kK + T_0, \ k \in \mathbb{Z}^d.
\]

First we apply Lemma 2 for \( T_0 \), then we estimate from above the sum of the probabilities \( P(q(u)T_k \cap B_u; u) \) over \( k \neq 0 \) with \( q(u)T_k \cap B_u \neq \emptyset \) using the same Lemma and regular varying of \( r(t) \) in any direction. Then we let \( K \) tend to infinity. On this way we get the following.

**Proposition 5** In the above conditions, if \( K_c = S \setminus \{0\} \), then

\[
P(S; u) = (1 + o(1)) P_q \Psi(u)
\]

as \( u \to \infty \), with \( P_q = \lim_{K \to \infty} P_q([-K, K]^d) \in (0, \infty) \).

4.4 General case.

First formulate several simple generalizations of above propositions. The first one is a generalization of Proposition 4.

**Proposition 6** In the above conditions,

\[
P(S; u) = P(S \setminus K_\infty; u)(1 + o(1)) \text{ as } u \to \infty.
\]

The proof repeats the proof of Proposition 4 with Lemma 3 application. The second one is a simple reformulation of Proposition 5.
Proposition 7 In the above conditions, if \( \dim \mathcal{K}_c > 0 \), then
\[
P(\mathcal{K}_c; u) = (1 + o(1)) P_q(\mathcal{K}_c) \Psi(u),
\]
as \( u \to \infty \), where
\[
P_q(\mathcal{K}_c) = \lim_{K \to \infty} \mathbb{E} \max_{t \in [-K,K] \cap \mathcal{K}_c} e^{\chi_1(t)} \in (0, \infty).
\]
The proof starts with the set
\[
T = T_0 = [-K, K]^d \cap \mathcal{K}_c,
\]
with followed corresponding definition of \( T_k \).

Proposition 8 If \( \dim \mathcal{K}_0 = d \), then
\[
P(\mathcal{K}_0; u) = (1 + o(1)) H_q L_f(\mathcal{K}_0, u^2) \prod_{i=1}^{d} q_i^{-1}(u) \Psi(u),
\]
as \( u \to \infty \), where
\[
L_f(\mathcal{K}_0, \lambda) := \int_{\mathcal{K}_0 \cap \mathbb{K}} e^{-\lambda f(t)} dt, \quad \lambda > 0,
\]
and \( f(t) \) is given by \((14, 15)\).

Recall that \( \mathbb{K} \) is the unit cube, see \((16)\).

Notice that the case \( \dim \mathcal{K}_0 = 0 \) means \( \mathcal{K}_0 = \emptyset \), by definition. Assume now that \( \dim \mathcal{K}_0 > 0 \). Generally, since the behavior of \( \sigma(t) \) near its maximum point \( 0 \) can be various, \( \mathcal{K}_0 \) may consist of several non-intersecting connected manifolds of various dimensions. In order to avoid technical difficulties due to too exotic behavior of \( \sigma(t) \), assume the following.

Condition 6 The set \( \mathcal{K}_0 \) consists of finite number of smooth (two times continuously differentiable) disjoint manifolds, namely,
\[
\mathcal{K}_0 = \bigcup_{i=1}^{n} \mathcal{K}_{0i}, \quad \dim \mathcal{K}_{0i} = k_i, \quad 0 < k_1 \leq k_2 \leq \ldots \leq k_n \leq d.
\]
Assume that for any \( i \), \( k_i \)-dimensional volume of \( \mathcal{K}_{0i} \) is finite, \( |\mathcal{K}_{0i}| < \infty \).

Fix \( i \) with \( k_i < d \), and consider in \( \mathcal{K}_{0i} \) curvilinear coordinates. For \( t \in \mathcal{K}_{0i} \), using Proposition 2, choose coordinate vectors \( e_j(t), \ j = 1, \ldots, k_i \) of this curvilinear coordinates and complete them to a basis \( \{ e_j(t), \ j = 1, \ldots, k_i, \tilde{e}_j(t), \ j = k_i + 1, \ldots, d \} \) in \( \mathbb{R}^d \). and denote
\[
q^{i,t}(u) := (q_j^{i,t}(u), \ j = 1, \ldots, k_i, \tilde{q}_j^{i,t}(u), \ j = k_i + 1, \ldots, d),
\]
with corresponding positive limits
\[
\lim_{u \to \infty} u^2(1 - r(q^{i,t}(u)s)) =: h^{i,t}(s),
\]
where \( s = (s_1, \ldots, s_d) \) is written in these coordinates. Remark that by Proposition 2 functions \( q_j^{i,t}(u) \) are taken from the collection of Condition 4, but the choice of them can depend on \( t \) and \( i \), the index of the manifold. In fact, \( h^{i,t}(s) = h(U_{i,t}s) \), where \( U_{i,t} \) is an orthogonal
transition matrix to the curvilinear coordinates with the above orthogonal complement to a basis in \( \mathbb{R}^d \).

We have,

\[
\lim_{u \to \infty} \frac{1 - \sigma(q^{i,t}(u)s)}{1 - r(q^{i,t}(u)s)} = 0.
\] (41)

By analogy with relations (13 - 20) in Proposition 3 proof, we build a partition of \( K_{0i} \) similar to \( \Delta_0 \) with \( k_i \)-dimensional blocks, denote them by \( \Delta(t_\nu) \), where \( \{t_\nu, \nu = 1, ..., N\} \) is a grid satisfying (39 - 41) for all \( s \in \Delta(t_\nu), \nu = 1, ..., N \), correspondingly, that is,

\[
K_{0i} = \bigcup_{i=1}^N \Delta(t_\nu).
\] (42)

Similarly to the proof of Proposition 3 but for \( K_{0i} \cap B_u \) instead of \( S \cap B_u \), using Theorem 1 to get (22) for all \( \Delta(t_\nu) \) and thicken the grid unboundedly, we get, using Condition 6, the following Lemma.

**Lemma 4** For any \( K_{0i} \) from the partition (38) with \( k_i < d \),

\[
P(K_{0i}; u) = (1 + o(1)) \int_{K_{0i}} H_{q^{i,t}}(u) \prod_{i=1}^{k_i} (q_j^{i,t}(u) - 1) e^{-u^2 f(t) \nu_{k_i}(dt)} \Psi(u)
\]
as \( u \to \infty \), where \( \nu_{k_i}(dt) \) is an elementary \( k_i \)-dimensional volume of \( K_{0i} \), and \( f(t) \) is given by (14).

**Remark 5** Notice that if a manifold \( K_{0i} \) is a linear subspace, all \( q^{i,t}(u) \) do not depend of \( t \), so that

\[
P(K_{0i}; u) = (1 + o(1)) H_{q^{i,t}}(u) \prod_{i=1}^{k_i} (q_j^{i,t}(u) - 1) e^{-u^2 f(t) \nu_{k_i}(dt)} \Psi(u)
\]
as \( u \to \infty \).

Turning to (38), we have,

**Proposition 9** If \( \dim K_0 < d \),

\[
P(K_0; u) = (1 + o(1)) \sum_{i=1}^n \int_{K_{0i}} H_{q^{i,t}}(u) \prod_{i=1}^{k_i} (q_j^{i,t}(u) - 1) e^{-u^2 f(t) \nu_{k_i}(dt)} \Psi(u)
\]
as \( u \to \infty \).

Remark that the summands in (43) can have different orders in \( u \) depending on the dimension of the corresponding component \( K_{0i} \), on behavior of \( q_j^{i,t}(u)s \) and on behavior of \( \sigma^2(t) \). Hence only summands with slowest order play a role. Remark also that by Proposition 8 if for some \( i \), \( \dim K_{0i} = d \), no summands in (43) contribute to the asymptotics of \( P(S; u) \).

**Proof:** We have only to estimate the double probabilities \( P(A(\Delta(s_\nu))A(\Delta(t_\mu))) \), where \( A(\Delta(s_\nu)) \) and \( A(\Delta(t_\mu)) \) are events generated by corresponding partitions in different component manifolds. In view of Proposition 2 denote

\[
K_{\alpha/2} := u^{-4/\alpha} K = \{t : |t| \leq u^{-4/\alpha}\},
\]
where $\alpha$ is defined in (7), the minimal index of regular variation of $r(tf)$ at zero, $|f| = 1$. Then for the sets $\Delta(t)$ from partitions of the component manifolds containing in the manifold $K_0 \setminus K_{\alpha/2}$, we have that all the sets are not neighboring, and, as above, the sum of the corresponding double probabilities is negligibly small with respect to any single sum over the partitions of $K_{0i}$, $i = 1, \ldots, n$. Furthermore, the probability $P(K_0 \cap K_{\alpha/2}, u)$ is also infinitely smaller than the single sums. Finally, as it was mentioned above, the only summands with slowest order give contribution in the final asymptotic behavior, we may take the multipliers $1 + o(1)$ out of the sum. Thus Proposition is established.

4.5 Main result

Now we collect all obtained above asymptotic relations.

**Theorem 2** Let $S$ be a bounded open set in $\mathbb{R}^d$ containing zero, and $X(t), t \in S$, be an a.s. continuous zeromean Gaussian field satisfying Conditions 1 - 6. Then for the probability $P(S, u)$ given by (7) the following asymptotic relations take places as $u \to \infty$.

- If $\dim K_0 = d$, $P(S; u)$ satisfies the relation (17).
- If $\dim K_0 \in [1, d - 1]$, $P(S; u) = P(K_0; u)(1 + o(1))$, and $K_0$ satisfies the relation (43).
- If $\dim K_0 = 0$ and $\dim K_c > 0$, $P(S; u) = P(K_c; u)(1 + o(1))$, and $P(K_c; u)$ satisfies the relation (36).
- If $\dim K_0 = \dim K_c = 0$, $P(S; u)$ satisfies the relation (34).

**Proof:** Since $B_u \subset K_0 \cup K_c \cup K_\infty$ and both $P(K_c; u)$ and $P(K_\infty; u)$ have order $\Psi(u)$ which is infinitely smaller than $P(K_0; u)$ provided $K_0$ is not empty, the first two assertions follow from Propositions 3 and 9. The third assertion follows from Propositions 6 and 7. The last relation is given in Proposition 4.

Remark that the assertions of this Theorem agree with assertions of Theorem 3, [7], where $d = 1$. There the case $\dim K_0 = 1$ is described in the items one ($K_0$ consists of two manifolds) and two ($K_0$ consists of one manifold, with the list of corresponding cases for dimensions of $K_c$ and $K_\infty$). The case $\dim K_0 = 0$ is considered in the remaining items, with various relations between dimensions of $K_c$ and $K_\infty$.

5 Examples and discussion.

First give examples of covariance functions satisfying the above conditions.

5.1 Example 1. Covariance functions of Pickands type.

As we have seen, in one dimension case, $d = 1$, the only behavior of $1 - r(t)$ satisfying Condition 4 is as following,

$$1 - r(t) = |t|^{\alpha} \ell(|t|)(1 + o(1)), \quad t \to 0,$$  \hspace{1cm} (44)
up to time scaling, with $\alpha \in (0, 2]$ and slowly varying $\ell$. In two dimension case one can see two types of the behavior,

$$1 - r(t) = (1 + o(1)) \sum_{i=1}^{2} |t_i|^{\alpha_i} \ell_i(|t_i|), \quad \text{and}$$

$$1 - r(t) = |t|^\alpha \ell(|t|)(1 + o(1)), \quad t \to 0,$$

up to linear time transformation of $\mathbb{R}^2$ and with the same properties of $\alpha$ and $\ell$s. Observe that the first one can be a covariance function of two independent stationary processes $\xi_1(t_1) + \xi_2(t_2)$, with covariance functions as $[14]$. Going this way, that is, summing independent fields with different time parameters, one comes in $d$-dimension case to a “structured” covariance function,

$$1 - r(t) = (1 + o(1)) \sum_{j=1}^{n} |t_j|^{\alpha_j} \ell_j(|t_j|), \quad t \to 0,$$

where $t_j = (t_{j+1}, ..., t_{l_j+1})$, $j = 1, ..., n$, is a partition of coordinates of $t$, $0 = l_1 < ... < l_n = d$, $\alpha_j \in (0, 2]$, $j = 1, ..., n$. In the case without $\ell_j$s such functions are considered in $[14]$, the corresponding normalization $q(u)$ is also given there, namely

$$q_i(u) = u^{-2/\alpha_j}, \quad i = l_j + 1, ..., l_{j+1}, \quad j = 0, ..., n - 1,$$

with

$$h(t) = \sum_{i=1}^{n} |t_i|^{\alpha_i}.$$

One can continue considering sums of independent fields, even with different linear transforms of time parameters. A general case with $d = 2$ is considered in $[2]$, see the following example.

### 5.2 Example 2. The field from $[2]$.

In $[2]$ the covariance function satisfying

$$1 - r(t) = \rho_1^2(|a_{11}t_1 + a_{12}t_2|) + \rho_2^2(|a_{21}t_1 + a_{22}t_2|), \quad t \to 0,$$

is considered, where $\rho_i, i = 1, 2$, are regularly varying at zero functions with indexes $\alpha_i \in (0, 1]$. The rank of matrix $A = (a_{ij}, i, j = 1, 2)$ can be 2, 1, or 0. The first case is described here by Condition 4. When the rank is equal to 1, the standardized field in corresponding basis is equal to $X_0(t_1)\sigma(t_1, t_2)$, with a Gaussian stationary process $X_0(t_1)$, that is the field is degenerated along $t_2$. In case of zero rank, the standardized field is $X\sigma(t_1, t_2)$ with a Gaussian random variable $X$. Here such degenerated cases are not considered, but in a corresponding basis one can represent $\mathbb{R}^d$ as a product of two spaces, dimension of one of them should be equal to the rank of a matrix which generalized $A$ to $d$-dimension case, with subsequent application of given here results.

Notice that in $[2]$ the function $1 - \sigma(t)$ is also assumed to have a similar to $1 - r(t)$ form, with some other regularly varying functions. The corresponding matrix must be, of course, not degenerated, otherwise one has infinitely many maximum points. From results here it follows that such restriction on the variance is not necessary, in contrast of the above representation for the covariance function. We would like to mention that discussions with authors of $[2]$ helped us a lot in formulation of our Conditions.
5.3 On behavior of the variance.

An example of very gentle behavior of the variance function at zero is considered in [7], Example 1. Its trivial $d$-dimension generalization can be as following,

$$1 - \sigma^2(t) = \exp \left( -|t|\beta(t/|t|) \right) \left( 1 + o(1) \right)$$

as $t \to 0$, where $\beta(e)$ is a strictly positive function given on the unit sphere $S_{d-1}$. Such behavior is a subject of Proposition 3, the behavior of the integral $L_f(\lambda)$ as $\lambda \to \infty$ can be investigated similarly to that in [7].

A generalization of the model from [2] is

$$1 - \sigma^2(t) = |t|\beta(t/|t|)\ell_{t/|t|}(|t|)(1 + o(1)),$$

with similar $\beta(e)$ and a collection of slowly varying at zero functions $\ell_e(t)$, $e \in S_{d-1}$. Remark again that, in contrast of Pickands’ behavior of the covariance function at zero, the behavior of $\beta(e)$ can be very variable.

An example when in the stationary like case $1 - \sigma^2(t)$ is close to $1 - r(s, t)$, $s, t \to 0$, is also considered in [7] for the one dimension case. Let $\ell_e(t)$, $|e| = 1$, be a family of slowly varying at zero functions. Take $1 - \sigma^2(t) = (1 - r(t))\ell_{t/|t|}(|t|)$ and denote $t_u := q(u)t$. When $\ell_{t_u/|t_u|}(|t_u|) \to 0$ as $u \to \infty$, we have $t \in K_0$, when the limit is equal to infinity, $t \in K_\infty$, so on.

One should assume, following Condition 6, that the limit changes from zero to non-zero and to infinity at most finite number of times. In [7], $\ell(t) = \log(1/|t|)$ is taken, that is, only the stationary like case is considered.

It is possible to evaluate similarly to [7] asymptotic behavior of $P(S; u)$ for particular cases of $R(s, t)$ and described here behavior of $\sigma(t)$

References

[1] Bingham, N. H., Goldie C. M. and Teugels J. L.: Regular variation. Cambridge university press. Cambridge(1987)

[2] Dębicki, K. Hashorva, E. and Liu, P.: Extremes of Gaussian Random Fields with regularly varying dependence structure. Extremes, 20, 333–392 (2017)

[3] Dmitrovsky V. A.: Estimates for the distribution of the maximum of a Gaussian Field. Random Processes and Fields, Moscow State University Ed., Moscow, 22–31 (in Russian) (1979)

[4] Dmitrovsky V. A.: On the integrability of the maximum and the local properties of Gaussian fields. In: Probability Theory and Mathematical Statistics, Vol. I (B. Grigelionis, Yu. V. Prohorov, V. V. Sazonov, and V. Statulevičius, eds.), ”Mokslas“, Vilnius, 271–284 (1990).

[5] Dudley, R.M.: The sizes of compact subsets of Hilbert space and continuity of Gaussian processes. J. Funct. Anal. 1, 290–330 (1967)

[6] Geman D., Horowitz J.: Occupation densities. Ann. Probab., 8, 1, 1-67 (1980)

[7] Hashorva E., Kobelkov S. G., Piterbarg V. I.: On maximum of Gaussian process with unique maximum point of its variance. Fundamentalnaya i prikladnaya matematika, in press, (2018) See also arXiv:1901.09753v1/math.PR
[8] Fedoruk M.V.: Saddle Point Method, in: Encyclopedia of Mathematics, Springer, New York, (2001)

[9] Feller W.: An Introduction to Probability Theory and its Applications, Volume II, 2nd edition (1971)

[10] Fernique X.: Régularité des trajectoires des fonctions aléatoires gaussiennes, Lecture Notes in Mathematics, 480, 2-187, (1975)

[11] Pickands III,J.: Upcrossing probabilities for Gaussian stationary processes. Trans. Amer. Math. Soc. 145, 51–73 (1969)

[12] Piterbarg V. I. and Prisjažnjuk V. P.: Asymptotic behavior of the probability of a large excursion for a nonstationary Gaussian process, Teor. Verojatnost. i Mat. Statist., Kiev, 18, 121–134 (1978)

[13] Piterbarg V. I.: On the paper by J. Pickands Upcrossing probabilities for stationary Gaussian processes, Vestnik Moskov. Univ. Ser. I, Mat. Meh 27, 5, 25–30 (1972)

[14] Piterbarg V.I.: Asymptotic Methods in Theory of Gaussian Random Processes and Fields, Providence, American Mathematical Society, Ser. Translations of Mathematical Monographies, 148 (2012)

[15] Piterbarg V. I.: Twenty Lectures About Gaussian Processes. Atlantic Financial Press London, NewYork, (2015)

[16] Talagrand, M.: Small tails for the supremum of a gaussian process. Annales de l’I.H.P. Probabilitès et statistiques, Volume 24 (1988) no. 2, pp. 307-315