Quantum interference of tunneling paths under a double-well barrier

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(Dated: November 10, 2022)

The tunnel effect, a hallmark of the quantum realm, involves motion across a classically forbidden region. In a driven nonlinear system, two or more tunneling paths may coherently interfere, enhancing or cancelling the tunnel effect. Since individual quantum systems are difficult to control, this interference effect has only been studied for the lowest energy states of many-body ensembles. In our experiment, we show a coherent cancellation of the tunneling amplitude in the ground and excited state manifold of an individual squeeze-driven Kerr oscillator, a consequence of the destructive interference of tunneling paths in the classically forbidden region. The tunnel splitting vanishes periodically in the spectrum as a function of the frequency of the squeeze-drive, with the periodicity given by twice the Kerr coefficient. This resonant cancellation, combined with an overall exponential reduction of tunneling as a function of both amplitude and frequency of the squeeze-drive, reduces drastically the well-switching rate under incoherent environment-induced evolution. The control of tunneling via interference effects finds applications in quantum computation, molecular, and nuclear physics.

Introduction – Quantum tunneling was recognized in the heyday of quantum mechanics as an important effect for intramolecular mechanics and in radioactive decay [1]. Later, it was deemed essential for nuclear fusion in stars [2]. Nowadays, the control of tunneling could bring important breakthroughs in applications that include controlled nuclear fusion [3, 4], management of radioactive waste [5], chemical synthesis [6], and transistor miniaturization [7].

In this work, we realize a double-well model system described by a Kerr oscillator submitted to a squeeze-drive [8]. The energy function of this system in phase space is shown in Figure 1A and corresponds to two wells that are connected via two saddle points, under which the system can tunnel. At these saddle points, the momentum is non-zero. By contrast, for a massive particle moving in a quadratic + quartic potential, tunneling through the barrier is associated with only one path under the barrier maximum, corresponding to zero momentum. In the more elaborate situation of Figure 1A, the two tunneling paths can interfere [9]. In this case, oscillations accompany the decay of the wavefunction in the classically forbidden region. This interference can even lead to the coherent cancellation of the tunneling amplitude altogether. This is especially interesting since this may occur for finite barrier height, allowing the tunneling to be restored when the interference is constructive. Whether the interference is destructive or constructive is set by the Hamiltonian parameters, which are adjustable in our experiment.

Model system – We realize this interference experiment in a sinusoidally driven superconducting quantum circuit oscillator described by the time-dependent Hamiltonian

\[
\hat{H}(t) = \omega_o \hat{a}^\dagger \hat{a} + \frac{g_3}{3} (\hat{a} + \hat{a}^\dagger)^3 + \frac{g_4}{4} (\hat{a} + \hat{a}^\dagger)^4 - i \Omega d (\hat{a} - \hat{a}^\dagger) \cos \omega d t, \tag{1}
\]

where \(\omega_o\) is the small oscillation frequency and \(g_3, g_4 \ll \omega_o\) are the third and fourth-rank nonlinearities of the oscillator, \(\hat{a}\) is the bosonic annihilation operator, and where the drive is specified by its amplitude \(\Omega d\) and frequency \(\omega d\). Equation (1) models a SNAIL transmon that is charge-driven at frequency \(\omega d\). It is the electrical circuit analog of an asymmetric mechanical pendulum capable of three- and four-wave mixing [10]. The experimental setup has been described in [11]. The drive is configured so that its second subharmonic \(\omega d/2\) lies in the vicinity of the SNAIL transmon resonance at \(\omega_0 = \omega_d + 3g_4 - 20g_3^2/3\omega_o\).

Under an approximation beyond the rotating-wave that captures the averaged behaviour of this rapidly driven nonlinear superconducting circuit, the dynamics governed by Eq. (1) is well-described by the static effective Hamiltonian [11–13]

\[
\hat{H} = \Delta \hat{a}^\dagger \hat{a} - K \hat{a}^\dagger^2 \hat{a}^2 + \epsilon_2 (\hat{a}^\dagger^2 + \hat{a}^2). \tag{2}
\]

Equation (2) corresponds to an elementary quantum system: a Kerr oscillator dressed by a squeeze-drive. In Eq. (2), the detuning parameter is given by \(\Delta = \Delta^{\text{bare}} + \delta^{\text{ac}},\) where \(\Delta^{\text{bare}} = \omega_o - \omega_d/2,\) with \(|\Delta^{\text{bare}}| \ll \omega_o,\) and where \(\delta^{\text{ac}}\) corresponds to the ac Stark shift: \(\delta^{\text{ac}} = (6g_4 - 9g_3^2/\omega_o)|\Pi|^2,\) where \(|\Pi| = 4\Omega d/3\omega_o.\) For our system, we measure \(\omega_o/2\pi = 6.035 \text{ GHz.}\) The Kerr coefficient arises from the bare \(g_3\) and \(g_4\) nonlinearities of the circuit, which are themselves controlled in situ by a magnetic field, and is given by \(K = 10g_3^2/3\omega_o - 3g_4/2.\) In

\[\text{The detuning parameter } \Delta \text{ is not to be confused with the superconducting gap.}\]
our experiment, we measure it to be $K/2\pi = 316.8$ kHz (see Fig. 1). A crucial component of the experiment, the squeeze-drive amplitude $\epsilon_2 = g_0|\Pi|$, is generated by the near-resonant three-wave mixing process between one drive excitation and two oscillator excitations. Due to the relatively small $K$ compared to a standard transmon [14], our experiment is affected by negligible ac Stark shifts for $\epsilon_2/K \lesssim 1$, so that in this regime $\delta_{ac}/K \lesssim 1\%$. Therefore, in this regime, we can simply take $\Delta = \Delta_{\text{bare}} = \omega_a - \omega_d/2$. By taking $K$ to provide the natural units of our system, the Hamiltonian is completely determined by only two dimensionless parameters: $\Delta/K$ and $\epsilon_2/K$, where the former is controlled by the drive frequency and the latter is directly proportional to the drive amplitude. We thus have independent real-time control of all relevant Hamiltonian parameters. Lastly, in our experiment, the single-photon lifetime of the undriven SNAIL transmon is $T_1 = 20 \mu$s and the Ramsey coherence between its lowest-lying eigenstates is $T_{2R} = 3.8 \mu$s.

To better explain the dynamics governed by Eq. (2), we employ the phase-space formulation of quantum mechanics [15], where the state of the system is represented by the Wigner function, and its evolution is generated by the Hamiltonian as a function of phase space coordinates, which is the Wigner transform of Eq. (2) (not to be confused with the classical limit of the system Hamiltonian, see supplement). Since, in our case, the Hamiltonian surface emerges as the static effective description ruling the averaged dynamics, we distinguish it from the exact time-dependent Hamiltonian Eq. (1) of the driven system, as well as from the equivalent Floquet Hamiltonian, and further refer to it as the metapotential of the squeeze-driven Kerr oscillator.

**Experiment and results** – We first experimentally demonstrate tunneling between the lowest pair of eigenstates, which we call the ground state manifold. In Figure 1A, we show the classical limit of the metapotential surface for $\Delta/K = 3, \epsilon_2/K = 0.11$, as a function of phase-space coordinates. The arrows indicate the two WKB tunneling paths [16]. Furthermore, we show in Figure 1B, the wavefunctions corresponding to the ground state manifold. Note that these are not the energy eigenstates but their even and odd superpositions, which are localized in the left and right wells. Importantly, in the classically forbidden region, marked in grey, oscillations accompany the expected decay of the wavefunctions. To observe coherent cancellation of tunneling in the ground state manifold, we prepare a localized well state and measure its tunneling probability as a function of time for different values of $\Delta$ and $\epsilon_2$. We present the measurement protocol in Figure 1C. The preparation is done by quickly turning on the squeeze-drive until an amplitude of $\epsilon_2/K = 8.7$ is reached. We subsequently wait for $5T_1$ for the system to relax to its steady state in the presence of the squeeze-drive and measure, in a quantum nondemolition (QND) manner, the quadrature containing the which-way information. This measurement projects the system into one of the wells. It is done by the microwave activation of a parametric beam splitter interaction between the squeeze-driven Kerr oscillator and a readout resonator strongly coupled to a quantum-limited amplifier chain. We refer the reader to [11] for experimental details, where the preparation-by-measurement procedure was introduced. This readout protocol yields a stabilized fluorescence signal revealing the quadrature measurement outcome, while the squeeze-drive sustains the circuit oscillation. After the preparation, we adiabatically lower the squeeze-drive amplitude in a duration $1.6 \mu s \gtrsim \pi/K$. The depth of the wells, which increases with $\epsilon_2/K$ (see supplement), is then reduced so that the tunnel effect becomes observable. We then wait for a variable amount of time before adiabatically raising the squeeze-drive amplitude to its initial value. Finally, we measure which well the system has adopted.

The data for this tunneling measurement is shown in Figure 1D where we interpret the oscillating color pattern as tunnel-driven Rabi oscillations. The periodic cancellation of tunneling at $\Delta/K = 2m$, where $m$ is a non-negative integer, is clearly visible as a divergence of the Rabi period. We extract the tunneling amplitude $|\delta E|$ from our data by fitting the oscillation frequency with an exponentially decaying sinusoid and plot this frequency in Figure 1E, where the data-point color corresponds to the value of $\epsilon_2$ (see supplement for calibration of $\epsilon_2$). The black lines, obtained from an exact diagonalization of the static-effective Hamiltonian Eq. (2), correspond to the energy difference between levels in the ground state manifold. The cancellation of tunneling for the ground state manifold in a parametrically modulated oscillator was predicted by [9] where, using a semiclassical WKB method, the authors found that this multi-path interference effect is due to, and accompanied by, oscillations of the wavefunction crossing zero in the classically forbidden region. Here, we find good agreement between our experiment and their WKB prediction (see supplement). Note that, across the zero of the tunneling amplitude, the bonding and anti-bonding superposition of well states alternate as the ground state. Specifically, for $\Delta/K = 4m + 1$, the ground state is the bonding superposition of well states (see supplement). In Fig. 1F, we further plot the extracted decay time of the tunneling oscillations as a function of $\Delta$, and find sharp peaks when $\Delta/K = 2m$, besides an overall continuous increase of the decay time with $\Delta$ and $\epsilon_2$. The peaks at $\Delta/K = 2m$ arise from the degeneracies in the excited state spectrum at this condition and are discussed later in the text.

Importantly, the measurements in Figure 1D can be used as resource for quantum information [17, 18] in what we call a $\Delta$-Kerr-cat qubit. The north and south poles

\[2\] Note that this adiabaticity condition pertains to the gap between the ground and first excited pair of states. We do not need to be adiabatic with respect to the two tunnel split states within the ground state manifold since they have opposite parity and the parity preserving squeeze-drive will not couple them.
FIG. 1: Tunnel-driven Rabi oscillations in the ground state manifold and their periodic cancellation. A Metapotential surface in the classical limit for Δ/K = 3 and ε2/K = 0.11. The orbits shown with black lines are obtained by semiclassical action quantization and represent the ground states (see supplement). Bidirectional arrows represent the two interfering WKB tunneling paths. B Cut of the classical metapotential surface in A at p = 0 (see supplement). The classically forbidden region is marked in grey. The left and right localized wavefunctions are indicated in red and blue. C Pulse sequence for D. The pink line represents the squeeze-drive at frequency ωd and the purple lines represent the preparation and readout drives at frequency ωd/2 − ωr. D Time-domain Rabi oscillation measurement of inter-well tunneling probability (color) as a function of Δbare, taken here as Δ (see text), for ε2/K = 0.11, 0.22, 0.44, and 0.88. The extracted tunneling amplitudes from D are shown as open circles in E. The black lines in E correspond to the transition energy between the lowest eigenstates obtained from an exact diagonalization of Eq. (2). A comparison of the extracted tunneling rate with a semiclassical WKB calculation is presented in the supplement. Green arrows in E denote the condition for constructive interference of tunneling and correspond to the measurements shown in Figure 2. We extract the value of the Kerr coefficient K from this data and note that it is consistent, within experimental inaccuracies, with an independent saturation spectroscopy measurement of the Fock qubit in the absence of the squeeze-drive (see supplement). F Decay time of the tunnel-driven Rabi oscillations for different values of Δ and ε2 in D. Sharp peaks in the decay time are clearly visible for Δ/K = 2m, m being a non-negative integer.

of the corresponding Bloch sphere, a generalization of the Δ = 0 one [11, 13, 17], is defined by the cat states formed by the lowest pair of eigenstates of Eq. (2). In this picture, a tunnel-Rabi cycle in Figure 1D for a fixed Δ/K ≠ 2m corresponds to a travel along the equator. For Δ/K = 2m, this travel is prohibited. Note that when Δ/K = 2m + 1, the tunneling amplitude is maximum and is first-order insensitive to fluctuations in Δ.

From Figure 1E, we also see that, besides the discrete cancellation of tunneling at Δ/K = 2m, tunneling in the ground state manifold is overall continuously reduced with both Δ and ε2. This is because both parameters explicitly control the barrier height and thus exponentially control the tunneling amplitude |δE|. Theory predicts that the larger the detuning Δ, the faster the tunneling reduction with the squeeze-drive amplitude ε2 (see supplement). We have measured this effect by measuring the tunneling amplitude as a function of ε2 for different constructive tunneling conditions corresponding to Δ/K = 2m + 1. The data is presented in Figure 2. The exponential insensitivity to fluctuations in the value of Δ, around Δ = 0, as a function of ε2, was predicted by [17] and thus proposed as a resource for quantum information. This insensitivity was a key motivation for realizing the Kerr-cat qubit experimentally [13]. The insensitivity of the ground state manifold to detuning as a function of ε2/K < 1, the tunneling amplitude |δE| is weakly dependent on ε2, whereas for Δ > 0, it is strongly dependent
FIG. 2: Exponential reduction of tunnel splitting as a function of $\epsilon_2$ in the ground state manifold. Measurement of the tunnel splitting (open circles) for the first five local maxima in Figure 1E as marked by the color coded arrows. Experimental sequence as in Figure 1E. For the raw color data, see Figure 3 in the supplement. Black lines are extracted from Hamiltonian diagonalization with no adjustable parameters. For comparison with a semiclassical WKB calculation, see supplement. Note that for small tunneling amplitude, dissipation plays a relevant role and the Hamiltonian model used here is insufficient.

on $\epsilon_2$. This weak dependence for $\Delta < 0$ is expected since the barrier height vanishes for small values of $\epsilon_2/K$. Our finding shows that new operating points at even, positive values of $\Delta/K$ will increase the resilience of the Kerr-cat qubit to detuning-like noise.

Moving to the pairs of excited states above the ground state manifold, do they also present observable degeneracies as a function of $\Delta/K$? In order to deepen our understanding of this problem, we first examine the classical metapotential surface via the period doubling phase diagram [8], that we show in Figure 3A. In the classical limit (see supplement), the parameter space spanned by $\Delta/K$ and $\epsilon_2/K$ is divided by two phase transitions located at $\Delta = \pm 2\epsilon_2$. The different phases are characterized by the number of stable nodes (attractors) in the classical metapotential and we refer to them as the single-, double-, and triple-node phases. These phases correspond to different metapotential topologies. We show them as contour line insets in Figure 3A, representing classical orbits. The single-node phase occurs for $\Delta < -2\epsilon_2$, and presents only one metapotential well. For $\Delta \geq -2\epsilon_2$, the oscillator has bifurcated and the metapotential acquires two wells. In the presence of dissipation, these wells house stable nodes. The emergent ground state manifold has been exploited, for $\Delta = 0$, in the Kerr-cat qubit [11, 13]. In the interval $-2\epsilon_2 \leq \Delta < 2\epsilon_2$, an unstable extremum (saddle point) appears at the origin. For $\Delta \geq 2\epsilon_2$, the saddle point at the origin splits into two saddle points and an attractor reappears at the origin. The barrier height of the classical metapotential is given by $(\Delta + 2\epsilon_2)^2/4K$ in the double-node phase and by $2\epsilon_2\Delta/K$ in the triple-node phase (see supplement). To count the number of excited states that have sunk under the barrier, we further introduce in Figure 2B a quantum phase diagram of the squeeze-driven Kerr oscillator. Following the Einstein-Brillouin-Keller method, which generalizes the notion of Bohr orbits, we quantize the action enclosed in the well below the height of the barrier and obtain the number of in-well excited states. In Figure 2C, we present the corresponding orbits in the classical metapotential surface for a fixed value of $\epsilon_2/K = 2.17$ and four values of $\Delta/K$. We validate this simple, semiclassical picture with a fully quantum mechanical calculation of the Wigner functions of localized states in the ground and excited state manifold (see supplement). It is clear from this analysis that, by increasing $\epsilon_2$ and $\Delta$, and therefore the barrier height, not only the ground state manifold but even the excited state manifolds become progressively ensconced in the wells, and we thus expect the tunneling between the wells to be drastically reduced.

Besides the overall continuous reduction of tunneling, the excited state manifold of the squeeze-driven Kerr oscillator experiences a discrete cancellation of tunneling when $\Delta/K = 2m$. Since the squeezing interaction preserves photon parity, levels belonging to the even and odd sector of the Kerr Hamiltonian remain decoupled and repeatedly cross at values of $\Delta/K$ corresponding to even integers. This braiding induces $m + 1$ perfect degeneracies at $\Delta/K = 2m$. Moreover, the corresponding eigenstates have a closed-form expression in the Fock basis. Remarkably, these features are independent of the value of $\epsilon_2$, reflecting a particular, underappreciated symmetry of our Hamiltonian Eq. (2) (see supplement).

Both the discrete cancellation and the overall continuous reduction of tunneling now in the excited state manifold of the squeeze-driven Kerr oscillator is accessed by performing spectroscopy measurements as a function of $\Delta$, which we show in Figure 3F for $\epsilon_2/K = 2.17$. The measurement protocol is shown in Figure 3D. We prepare a localized well state in a manner that is similar to the protocols of Figures 1 and 2. To locate the frequency of the excited states, we apply a probe tone at variable frequency in the vicinity of the SNAIL transition resonance $\omega_s$ and measure the well-switching probability. When the probe is resonant with a transition to a state close to the barrier maximum, this probability is increased. The experimental results are shown in Figure 3F. The colored dashed lines (orange and blue) in the lower panel are obtained from an exact diagonalization of the static-effective Hamiltonian Eq. (2) with no adjustable parameters. The crossings of levels are marked with circles. The data also shows that the level crossings are accompanied by a continuous reduction of the braiding amplitude with $\Delta$. The corresponding reduction of the tunnel splitting is the manifestation associated with a generic double-well Hamiltonian while the braiding re-

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3 In the absence of dissipation, the metapotential acquires two wells as soon as $\epsilon_2, \Delta > 0$, i.e. there is no threshold for bifurcation of the driven oscillator. In our quantum experiment, this threshold is finite but is, relatively speaking, extremely small since and is set by $\epsilon_2^2 > (\Delta^2 + T_1^{-2}/4)/4$ (see [10]).
FIG. 3: Spectroscopic measurements of coherent and periodic cancellation of tunnel splitting in the excited state spectrum. A Classical phase diagram for the Kerr oscillator with parametric squeezing, also called the period-doubling bifurcation diagram. B Quantum phase diagram to count in-well excited states. White lines separate single-node, double-node, and triple-node phases. Colors represent contours of constant action. Dashed pink line corresponds to $\epsilon_2/K = 0.88$, the maximum value of squeeze-drive amplitude in Fig. 1. Dashed black line corresponds to $\epsilon_2/K = 2.17$, the value of squeeze-drive amplitude used in Figs. 3F and 3G. Characteristic classical metapotential surfaces for $\epsilon_2/K = 2.17$ and (i) $\Delta/K = 1$, (ii) $\Delta/K = 3$, (iii) $\Delta/K = 7$, and (iv) $\Delta/K = 9$. Bohr-like orbits are indicated as black curves (see supplement for more details). D Pulse sequence for F. The green line represents the weak spectroscopic probe tone at frequency $\omega_{pr}$. The pink line represents the squeeze-drive at frequency $\omega_d$ and the purple lines represent the preparation and readout drives at frequency $\omega_d/2 - \omega_r$. E Pulse sequence for G, F (upper panel) Frequency-domain measurement of well-transition probability (color) via excited states as a function of $\Delta$ for $\epsilon_2/K = 2.17$. The power of the perturbative spectroscopic drive is increased as $\omega_{pr}$ is decreased to compensate for the lower matrix element connecting the ground state with the higher excited levels, yet is kept weak enough to preserve the parity conservation rules of Eq. (2). F (lower panel) Dashed lines plotted on top of experimental data (same as in upper panel) correspond to transition energies obtained by performing an exact diagonalization of Eq. (2) with no adjustable parameters. The Kerr coefficient is calibrated via time-domain measurements in Figure 1E. G Measured well-switching time under incoherent environmental-induced evolution as a function of $\Delta$ for $\epsilon_2/K \approx 2.17$. Background color in G marks the number of excited states per well following semiclassical orbit quantization.

An important consequence of the cancellation of tunneling in the excited state spectrum is the periodic enhancement of the well-switching time under incoherent environment-induced evolution. This time scale corresponds to the transverse relaxation time, $T_X$, of a new bosonic qubit: a $\Delta$-variant of the Kerr-cat qubit [10, 17] as mentioned earlier. To measure $T_X$, we prepare a localized well state by measurement, and wait for a variable amount of time before measuring the which-well information. We show the pulse sequence in Figure 3E. We obtain $T_X$ by fitting a decaying exponential function to the measured well-transition probability for each value of $\Delta$ and plot the result in Figure 3G. Note that we have chosen the squeeze-drive amplitude, identical to that of
FIG. 4: Color plot of $T_X$ as a function of $\Delta/K$ and $\epsilon_2/K$. White line marks the transition from two-node phase to three-node phase. Black solid lines mark contours of constant barrier height. Increasing both $\Delta/K$ and $\epsilon_2/K$ yields fastest enhancement in $T_X$ as predicted by Figure 3B. The additional enhancement by the coherent cancellation of excited-state tunneling at $\Delta/K = 2m$ stands out. The pulse sequence for the measurement is shown in Figure 3E.

Figure 3F, as $\epsilon_2/K = 2.17$. Around values of $\Delta/K$ corresponding to even integers, the variation of $T_X$ presents sharp peaks. The location of the peaks corresponds to the degeneracy condition in the excited state spectrum, associated with coherent cancellation of tunneling and the blocking of noise-induced well-switching pathways via the excited states. The systematic right-offset $\delta/K$ of each peak from an even integer, about 15%. About 5% can be attributed to the ac Stark shift $\delta_{ac}$ for this photon number given the accuracy of our knowledge of the experimental parameters. Note that this explanation is still compatible with the perfect alignment of the cancellation points with even integers in Figure 1F for $\epsilon_2/K < 1$, since for this case the ac Stark shift is negligible. Note also that this offset would provide access, within experimental accuracy, via the ac Stark shift, to the nonlinear parameters of Eq. (1).

The data in Figure 3G also shows that the discrete peaks are accompanied by a monotonic baseline increase, a direct manifestation of the overall continuous tunneling reduction in the spectrum versus $\Delta$. The background colored stripes represent the number of in-well excited states found via the action quantization method discussed above and in the supplement. Continuing with this semiclassical picture, we interpret the slowdown in the growth of $T_X$ for $\Delta/K \gtrsim 5$ as the slowdown in the growth of the barrier height as one crosses over from the double-node, where the barrier height $\propto (\Delta + 2\epsilon_2)^2$, to the triple-node phase, where the barrier height $\propto \Delta \epsilon_2$. Indeed, this is the quantum manifestation of the classical phase transition from the double-node to the triple-node phase.

Thus, whether the theoretical framework is classical, semiclassical, or quantum, the predicted $T_X$ will increase with both $\epsilon_2$ and $\Delta$. While $\epsilon_2$ and $\Delta$ contribute via the overall continuous reduction of tunneling [11], only $\Delta$ controls the discrete cancellation of tunneling. We verify this prediction by measuring $T_X$ while varying simultaneously both Hamiltonian parameters. We present the result of this experiment in Figure 4. We further plot contours of constant barrier height in black and the expected separation between the double-node and triple-node phases as a white line. As expected, following the gradient of the barrier height, and along the boundary separating the double-node and triple-node phases, one observes the fastest gain in $T_X$, with a maximum of $T_X = 1.3$ ms for $\Delta/K = 6$ and $\epsilon_2/K = 4$. Increasing lifetime by increasing $\epsilon_2$ presents limitations, since strong drives are known to cause undesired effects in driven nonlinear systems (see [19, 20] and supplement).

One could argue that $\Delta = 0$ provides an important factorization condition that guarantees that the ground state manifold is spanned by exact coherent states (see [17] and supplement). Indeed, this may be an asset for quantum information, since these states are eigenstates of the single-photon loss operator $\hat{a}$ [21]. However, this desirable property is traded for the advantages discussed earlier when $\Delta/K = 2m, m \geq 1$. Even if the $\Delta$-variant of the Kerr-cat qubit suffers from quantum heating and quantum diffusion [22] at zero temperature resulting from the squeezed nature of its ground states, these effects are small and, as we show in the experiments reported here and in [11], the well-states of the Kerr-cat live longer than its $\Delta = 0$ parent even at finite temperature.

Discussion — Although quantum tunneling was discovered nearly a century ago [1] and observed since in a variety of natural and synthetic systems, the treatment of tunneling is usually limited to the ground states of the system and has rarely been discussed for excited states in the literature, as we elaborate in the following survey. The phenomenology of ground state tunneling has been studied in cold atoms [23] in three-dimensional optical lattices [24], optical tweezers [25], ion traps [26] and in quantum dots [27]. In Josephson tunnel circuits, quantum tunneling of the phase variable was first observed by Devoret, Martinis, and Clarke [28] and since then exploited in several other experiments [29]. Furthermore, the tunnel effect has been involved in quantum simulation [30], in Floquet engineering of topological phases of matter and to generate artificial gauge fields with no static analog [31, 32]. The quantum interference of tunneling for the ground states of a large spin system was measured previously in a cluster of eight iron atoms by Wernsdorfer and Sessoli [33] (see also [34, 35]).

Weilinga and Milburn were the first to identify that the quantum optical model in Eq. (2) can exhibit ground state tunneling [36]. Marthaler and Dykman, using a WKB treatment [9, 16], generalized the analysis of Weilinga and Milburn and predicted that, for this model, the tunnel splitting of the ground state manifold crosses zero periodically and is accompanied by oscillation of the
wavefunction in the classically forbidden region.

Our work is the first experimental realization of the longstanding theoretical proposals of the last paragraph. It is similar, but different, to the phenomenology of the “coherent destruction of tunneling”, discovered theoretically by Grossmann et al. [37] and observed experimentally in cold atoms [38, 39]. Indeed, the dynamical tunneling in our experiment is in sharp contrast with photon-assisted or suppressed tunneling in weakly driven double-well potentials. Firstly, our tunneling is completely dynamical, i.e., the tunneling barrier vanishes in the absence of the drive. Secondly, and most importantly, our work extends the coherent cancellation of tunneling to all the excited states in the well. The periodic resonance condition $\Delta/K = 2m$, shared for the $m+1$ first pairs of excited levels, is independent of the drive amplitude. Remarkably, under this multi-state resonance condition, the first $2(m+1)$ oscillator states have a closed-form expression in the Fock basis (see supplement). We further emphasize that the dynamical tunneling in our work is distinct from chaos-assisted dynamical tunneling [40] observations made in ultracold atoms over three decades ago [40–42]; remarkably our strongly driven nonlinear system remains integrable. To the best of our knowledge, our work is the first demonstration of the cancellation of tunneling amplitude for the ground and excited states. Our data featuring the incoherent dynamics can be qualitatively modeled by a Lindbladian treatment that we implemented for the first time in circuits [13].

As a resource for quantum information, the squeeze-driven Kerr oscillator for $\Delta = 0$, was identified in theory proposals by Cochrane, Milburn, and Munro [43] and Puri, Boutin, and Blais [17] due to its exponential resilience to low frequency noise and was proposed as a bosonic code for amplitude damping. The code was implemented for the first time in circuits [13]. The theory of the bistability for the non-zero $\Delta$ case was studied in [44]. To the best of our knowledge, the work reported in this article is the first experimental demonstration of bistability and the resilience to noise in the non-zero $\Delta$ case.

**Conclusion** – We have observed the interplay between two quantum phenomena, tunneling and interference, via time- and frequency-resolved measurements in the eigenstates of a strongly driven Kerr-parametric oscillator. This was only possible due to a SNAIL oscillator with controllable nonlinearity, low dissipation, high-fidelity QND readout, and the control over all Hamiltonian parameters of a driven system. We find good agreement between our experimental results and theoretical predictions. We believe that the continuous Z-gate [18, 45] demonstrated in this work is a valuable addition to the single qubit gate-set of cat qubits. This provides a new tool for quantum computation [11, 13, 17, 18, 21, 46]. We draw the attention of the reader to the two orders of magnitude enhancement of the transverse relaxation lifetime due to the periodic cancellation and overall continuous reduction of tunneling, a manifestation of the braiding spectrum of our particular system. The coherent control of tunneling in the excited state spectrum suggests a new bosonic code strategy to create highly biased-noise qubits. Our results find immediate technological applications in quantum information and simulation with cat-qubits [18, 21, 44, 45, 47, 48]. Our platform might also be used to simulate driven quantum tunneling in molecular and nuclear physics [49].

After our measurements, we learned that the degeneracies in our squeeze-driven Kerr oscillator were studied theoretically for their value to quantum information processing by our colleagues in the QUANTIC group in INRIA, Paris [50].

**Acknowledgements** – We acknowledge Vladislav Kurilovich for pointing to us the peculiarity of the amplitude independence of the multi-level degeneracies in our model. We thank Charlotte G. L. Bottcher, Steven M. Girvin, Shruti Puri, and Qile Su for useful discussions. R. G. C acknowledges useful discussions with Mazyar Mirrahimi, Diego Ruiz, and Jérémie Guillaud. This research was supported by the U.S. Army Research Office (ARO) under grant W911 NF-18-1-0212 and W911 NF-16-1-0349, and by the U.S. Department of Energy, Office of Science, National Quantum Information Science Research Centers, Co-design Center for Quantum Advantage (C2QA) under contract number DE-SC0012704. The views and conclusions contained in this document are those of the authors and should not be interpreted as representing official policies, either expressed or implied, of the ARO or the U.S. Government. The U.S. Government is authorized to reproduce and distribute reprints for Government purpose notwithstanding any copyright notation herein. Fabrication facilities use was supported by the Yale Institute for Nanoscience and Quantum Engineering (YINQE) and the Yale SEAS Cleanroom.

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Supplemental material:
Quantum interference of tunneling paths under a double-well barrier

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(Dated: November 10, 2022)

The Supplemental text is organized as follows. In Section I, we detail the experimental calibration of Hamiltonian parameters in Eq. (2) of the main text; specifically, in Section I we present a calibration of the squeeze-drive $\epsilon_2$ and in Section II, we present a measurement of the Kerr coefficient $K$. In Sections III and IV, we present further experimental results supplementing Figures 2 and 4 in the main text.

In Section V and the following sections, we switch gears and detail our theoretical models. First, in Section V we formally introduce the notation employed throughout this work. We then present in Section VI two well-known squeeze-driven Kerr oscillator Hamiltonians that were introduced in the literature and comment on the relationship between them. In Section VII, we introduce the operator and phase space formulation of our particular squeeze-driven Kerr oscillator effective Hamiltonian and discuss its classical limit.

In Sections VII A, VIII and IX, we discuss distinct properties of this Hamiltonian and its eigenstates. Specifically, in Section VII A, we discuss the structure of lowest pair of well-localized wavefunctions for different Hamiltonian parameter configurations and distinguish them from those of an ordinary quadratic + quartic double-well potential. We present our semiclassical analyses, namely a WKB analysis of the tunnel splitting in Section VIII A and an overview of action quantization in Section VIII B to discuss the construction of quantized orbits. We discuss in Section IX the robustness of the degeneracies in the squeeze-driven Kerr oscillator. In Section X, we present a simple Lindblad model to capture the qualitative features of the experimentally measured transverse relaxation lifetimes $T_X$ of the $\Delta$ variant of the Kerr-cat qubit.

Finally, in Section XI we present a self-contained tutorial and a concise introduction to the phase space formulation of quantum mechanics.

I. CALIBRATING THE SQUEEZE-DRIVE AMPLITUDE $\epsilon_2$

In this section, we present a measurement that provides an independent calibration of the squeeze-drive amplitude $\epsilon_2$. The pulse sequence is the following: We turn on the squeeze-drive at $\Delta = 0$, for a variable amount of time $t$ during which we also turn on a Rabi drive at amplitude $\epsilon_x$ and frequency $\omega_r/2 = \omega_d$. The squeeze-drive stabilizes the Schrödinger cat states with well-defined parity, and the Rabi drive induces an oscillation in this cat-qubit. We perform this experiment for different values of $\epsilon_2$ and measure $X = |C^+\rangle\langle C^-| + |C^-\rangle\langle C^+|$, where $|C^{\pm}\rangle$ are the Schrödinger cat states. This protocol was introduced in [1, 2] and we refer the reader to these works for further details. The result of our experiment is shown in Figure S1A. From this experimental data, we extract a Rabi oscillation frequency $\Omega_x$ that is related to the amplitude of the Rabi drive as $\epsilon_x = \Omega_x (\epsilon_2 = 0)/2$. The photon-number at $\Delta = 0$ $|\alpha|^2$ is related to $\epsilon_x$ and $\Omega_x$ as $|\alpha|^2 = \Omega_x^2/16\epsilon_x^2$ [1, 2]. In Figure S1B, we plot the experimental data and fit for the extracted photon-number as a function of the digital control amplitude (DAC). With this result, we have a calibration of $\epsilon_2$ as a function of the digital control amplitude (DAC) controlling the squeeze-drive.

II. MEASURING THE KERR COEFFICIENT $K$

In this section, we detail a measurement of the Kerr coefficient $K$ via saturation spectroscopy of the SNAIL transmon. This measurement is performed in the absence of the squeeze-drive. In the following text, the letters $g, e, f$ index the ground, first excited, and second-excited states of the SNAIL transmon oscillator. In Figure S2, we plot the response of the readout as a function of a probe tone, whose frequency is $\omega_{pr}$, and which we vary around the $ge$ transition frequency of the SNAIL transmon oscillator $\omega_{ge}$ corresponding to $\epsilon_2 = 0$. When the probe tone excites the oscillator, the readout signal due to the dispersive coupling [3] changes. The two dips in Figure S2, from left to right, correspond to a two-photon transition that excites the oscillator from $g$ to $f$ and to a resonant excitation of the

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Figure S1: **Calibrating $\epsilon_2$ with cat-Rabi oscillations.** A Color plot of $\langle X \rangle$ as a function of the digital control amplitude (DAC) controlling the squeeze-drive $\epsilon_2$ and duration of the Rabi drive. We find $\epsilon_x/2\pi = 144.93$ kHz using the relation between the Rabi amplitude and Rabi frequency for $\epsilon_2 = 0$, $\epsilon_x = \Omega_x(\epsilon_2 = 0)/2$. A plot of $|\alpha|^2_0 = \epsilon_2/K = \Omega_x^2/16\epsilon_2^2$ [1, 2] as a function of $\epsilon_2$ in DAC units. A line fit gives us a calibration of $|\alpha|^2_0 = \epsilon_2/K$ as a function of the digital control amplitude (DAC) controlling the squeeze-drive.

oscillator from $g$ to $e$ respectively. The $gf/2$ and $ge$ resonances are located at $(\omega_a - K)/2\pi$ and $\omega_a/2\pi$ respectively. Fitting the peaks and subtracting their locations yields a value of $K/2\pi = (329.73 \pm 4.30)$ kHz. This value is consistent with the value of $K/2\pi = 316.83$ kHz, where the latter is extracted from Figure 1E in the main text and is the value for $K$ used throughout the article.

Figure S2: Readout response as a function of the frequency of the saturation (probe) tone. The two readout signal dips in black correspond, from left to right, to the $gf/2$ transition, which is expected to occur at $(\omega_a - K)/2\pi$ and to the $ge$ transition of the SNAIL transmon, which is expected to occur at $(\omega_a)/2\pi$. Here, $gf/2$ refers to a transition induced by two photons from the probe. By fitting the experimental data, we find $K/2\pi = (329.73 \pm 4.30)$ kHz.
Figure S3: **Tunnel-driven Rabi oscillations in the ground state manifold and its exponential reduction as a function of $\epsilon_2$; raw data.** The transition probability as a function of $\epsilon_2/K$ and time $t$ for A $\Delta/K = 1$, B $\Delta/K = 3$, C $\Delta/K = 5$, D $\Delta/K = 7$, and E $\Delta/K = 9$ respectively. This corresponds to the condition of constructive interference of tunneling to occur. By progressively increasing $\epsilon_2$, there is a clear overall continuous reduction of the tunnel-driven Rabi oscillations.

### III. EXPONENTIAL REDUCTION OF TUNNELING WITH $\epsilon_2$

In the main text, we claim that tunneling in the ground state manifold is overall continuously reduced with $\epsilon_2$. The parameter $\epsilon_2$ controls the barrier height, which is given as $(\Delta + 2\epsilon_2)^2/4K$ in the double-node phase and $2\Delta\epsilon_2/K$ in the triple-node phase. Moreover, continuing this reasoning, the larger the detuning, the faster the tunneling reduction as a function of $\epsilon_2$. In Figure S3, we present raw data to further support this claim. We present the measurement protocol in Figure 1C of the main text and recall it for the sake of completeness. First, we prepare, by measurement, a steady-state localized in one of the wells. Following this, we adiabatically lower the squeeze-drive amplitude $\epsilon_2$. Lowering the value of $\epsilon_2$ reduces the barrier depth, and thus the tunnel effect becomes observable. We then wait for a variable amount of time before adiabatically re-raising $\epsilon_2$ to its initial value and finally do which-well readout.

In Figure S3, we present the measured transition probability a function of $\epsilon_2$ for A $\Delta/K = 1$, B $\Delta/K = 3$, C $\Delta/K = 5$, D $\Delta/K = 7$, and E $\Delta/K = 9$ respectively. It is clear from the data that the Rabi-frequency is overall continuously reduced with $\epsilon_2$ and moreover, increasing $\Delta/K$ reduces the Rabi frequency further. We plot in Figure 2 of the main text the extracted tunneling amplitude $|\delta E|$ from our data by fitting the oscillation frequency with an exponentially decaying sinusoid. We find that the extracted tunneling amplitude is in excellent agreement with an exact diagonalization of the static-effective Hamiltonian and in good agreement with a WKB prediction of the tunnel splitting within the expected regime of validity. See Figure S8 for more details.

### IV. TRANSVERSE RELAXATION LIFETIME $T_X$ MEASUREMENTS

In Figure S4, we plot the transverse relaxation lifetime $T_X$ as a function of $\Delta_{\text{bare}} = \omega_a - \omega_d/2$ for different values of $\epsilon_2$. Note that the photon-number at $\Delta = 0$ is given by $|\alpha|^2_0 = \epsilon_2/K$. Importantly, for large photon-numbers $\epsilon_2/K \gtrsim 6.5$, we see that the peaks in lifetime start plateauing and even dropping. This effect is not captured by an ordinary model of the Lindblad master equation as we discuss in Section X. The degradation of the $T_1$ with readout power has been observed for transmon qubits [5]. But other drive-induced effects such as multiphoton nonlinear resonances are present in transmons and disentangling these various sources of lifetime degradation is nontrivial [3, 5–8]. These spurious nonlinear resonances are largely absent in this our SNAIL conducting circuit for values of $\epsilon_2/K \lesssim 5$, thanks to negligible Kerr and stark shifts, but may plague our system for larger mean-photon numbers. Due to this reasoning, $\Delta$ might be a more effective knob to create states with large photon number [4]. Finally, the squeeze-driven Kerr oscillator provides a perfect platform to investigate lifetime degradation under drives.
Figure S4: Measurement of $T_X$ as a function of $\Delta^{\text{bare}} = \omega_0 - \omega_d/2$ for representative values of squeeze-drive amplitude $\epsilon_2$. The measurement protocol is shown in Figure 3E of the main text. We observe a degradation of $T_X$ with increasing $\epsilon_2$, as indicated by red boxes in the legend, and we show representative measurements here. On the other hand, we see no degradation of $T_X$ with increasing $\Delta$. This measurement indicates that $\Delta$ might be a more effective knob to increase $T_X$ than $\epsilon_2$ for cat-states with large photon-number [4].

V. NOTATION

In this work, we note $\hat{X}$ and $\hat{P}$ the position-like and momentum-like coordinates with $[\hat{X}, \hat{P}] = i\hbar$. We build the dimensionless quadratures by introducing the zero point spread of the coordinates as $X_{zps}$ and $P_{zps}$, respecting $X_{zps}P_{zps} = \hbar/2$. We further introduce the complex notation for the dimensionless quadratures as $\hat{a} = (\hat{X}/X_{zps} + i\hat{P}/P_{zps})/2$ and its conjugate operator $\hat{a}^\dagger$, where $[\hat{a}, \hat{a}^\dagger] = 1$ and introduce the rescaled phase space quadratures as $\hat{x} = \sqrt{\lambda/2}\hat{X}/X_{zps} = \sqrt{\lambda/2}(\hat{a} + \hat{a}^\dagger)$ and $\hat{p} = \sqrt{\lambda/2}\hat{P}/P_{zps} = -i\sqrt{\lambda/2}(\hat{a} - \hat{a}^\dagger)$, where $[\hat{x}, \hat{p}] = i\lambda$. These choices induce the definitions $x_{zps} = p_{zps} = \sqrt{\lambda/2}$. Conversely, we have $\hat{a} = (\hat{x} + i\hat{p})/\sqrt{2\lambda}$. At this point, $\lambda$ is a dimensionless rescaling parameter. We will connect it with the Hamiltonian parameters later, while discussing the classical limit ($\lambda \to 0$) of our system, and thereby giving it physical significance. It is also useful to compare our results with those of [9], who have performed a WKB analysis of a driven oscillator. Thus, unless otherwise specified, $\lambda$ should be taken equal to unity $\lambda = 1$.

For a mechanical oscillator with mass $m$ and spring-constant $k$, the small-oscillation frequency is $\omega_0 = \sqrt{k/m}$ and the impedance is $Z_o = 1/\sqrt{km}$. With this, we have $X_{zps} = \sqrt{\hbar Z_o}/2$ and $P_{zps} = \sqrt{\hbar/2Z_o}$. We further remark that there is a direct correspondence between the mechanical harmonic oscillator and a linear LC circuit oscillator [3, 10, 11] under the following relations. The mechanical position coordinate $\hat{X}$ corresponds to the circuit flux $\hat{\Phi}$, the mechanical momentum $\hat{P}$ corresponds to the circuit charge $\hat{Q}$, where $[\hat{\Phi}, \hat{Q}] = i\hbar$, the mechanical oscillator frequency $\omega_0 = \sqrt{k/m}$ corresponds to the circuit oscillator frequency $\omega = 1/\sqrt{LC}$ and the mechanical oscillator impedance $Z_o = 1/\sqrt{km}$ corresponds to the circuit oscillator impedance $Z_o = \sqrt{L/C}$ which amounts to the identification of the mechanical mass $m$ with the circuit capacitance $C$ and the spring constant $k$ with the inverse inductance $1/L$. The expressions for the zero point spreads are given by $\Phi_{zps} = \sqrt{\hbar Z_o}/2$ and $Q_{zps} = \sqrt{\hbar/2Z_o}$. In circuits, it is customary to introduce [3, 12] the reduced flux and charge coordinates: $\hat{\varphi} = \sqrt{2\pi}/\Phi_0$ and $\hat{N} = \sqrt{\lambda Q}/2e$ so that $[\hat{\varphi}, \hat{N}] = i\lambda$, where $e$ is minus the electron charge, and $\Phi_0 = h/2e$ is the magnetic flux quantum. Their respective zero point spreads $\varphi_{zps} = \sqrt{2\pi}\Phi_{zps}/\Phi_0$ and $N_{zps} = \sqrt{\lambda Q_{zps}}/2e$, and are related to the rescaled complex coordinate operators by $\hat{\varphi} = \varphi_{zps}(\hat{a}^\dagger + \hat{a})$ and $\hat{N} = -iN_{zps}(\hat{a} - \hat{a}^\dagger)$ and $\varphi_{zps}N_{zps} = \lambda/2$. We summarize this notation in the following table.

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Note that in this case the adimensionalization of variables is done by fundamental constants and not by linear properties of the oscillator. This comes at the price of a slight notation asymmetry over the reduced operators the electric and mechanical oscillators.
| Mechanical oscillator | Circuit oscillator |
|-----------------------|--------------------|
| $X; \dot{P}$          | $\Phi; \dot{Q}$    |
| $[X, P] = i\hbar$    | $[\Phi, \dot{Q}] = i\hbar$ |
| $\omega_o = \sqrt{k/m}$ | $\omega_o = \sqrt{LC}$ |
| $Z_o = 1/\sqrt{km}$  | $Z_o = \sqrt{LC}$ |
| $X_{zps} = \sqrt{hZ_o}/2$; | $\Phi_{zps} = \sqrt{hZ_o}/2$; |
| $P_{zps} = \sqrt{h/2Z_o}$; | $Q_{zps} = \sqrt{h/2Z_o}$; |
| $\Rightarrow X_{zps}P_{zps} = h/2$; | $\Rightarrow \Phi_{zps}Q_{zps} = h/2$; |
| $\hat{a} = \frac{1}{2} (\hat{X}_{zps} + i \hat{P}_{zps})$ | $\hat{a} = \frac{1}{2} (\hat{\Phi}_{zps} + i \hat{Q}_{zps})$ |
| $\hat{X} = X_{zps} (\hat{a} + \hat{a}^\dagger)$ | $\hat{\Phi} = \Phi_{zps} (\hat{a} + \hat{a}^\dagger)$ |
| $\hat{P} = -iP_{zps} (\hat{a} - \hat{a}^\dagger)$ | $\hat{Q} = -iQ_{zps} (\hat{a} - \hat{a}^\dagger)$ |
| $\hat{a} = \hat{a}, \hat{a}^\dagger = 1$ | $\hat{a}, \hat{a}^\dagger = 1$ |

\[ \hat{x} = \sqrt{\frac{\lambda}{2}} \hat{X}_{zps} = x_{zps} (\hat{a} + \hat{a}^\dagger) \]
\[ \hat{p} = \sqrt{\frac{\lambda}{2}} \hat{P}_{zps} = -i p_{zps} (\hat{a} - \hat{a}^\dagger) \]
\[ \hat{x}_{zps} = p_{zps} = \sqrt{\frac{\lambda}{2}} \]
\[ \hat{p}_{zps} = \sqrt{\frac{\lambda}{2}} \]
\[ \Rightarrow \hat{x}_{zps} \hat{p}_{zps} = \lambda/2 \]
\[ \Rightarrow \hat{x}_{zps} \hat{p}_{zps} = \lambda/2 \]
\[ \hat{a} = \frac{1}{2} \left( \hat{x}_{zps} + i \hat{p}_{zps} \right) \]
\[ \hat{a} = \frac{1}{2} \left( \hat{x}_{zps} + i \hat{p}_{zps} \right) \]
\[ \hat{a} = (\hat{x} + i \hat{p})/\sqrt{2\lambda} \]
\[ \hat{a} = (\hat{x} + i \hat{p})/\sqrt{2\lambda} \]

\[ \hat{\phi} = \sqrt{\lambda x_{zps}} e^{i\phi_{zps}} = \varphi_{zps} (\hat{a} + \hat{a}^\dagger) \]
\[ \hat{N} = \sqrt{\lambda x_{zps}} e^{i\phi_{zps}} = \varphi_{zps} (\hat{a} - \hat{a}^\dagger) \]

VI. THE RELATIONSHIP BETWEEN DIFFERENT SQUEEZE-DRIVEN KERR OSCILLATOR MODELS IN THE LITERATURE AND THEIR CLASSICAL LIMIT

In 1993, Wielinga and Milburn [13] proposed a quantum optical model that they called the dynamical equivalent of the double-well potential. The interest of the problem, to them, was that their model exhibited a double-well structure in classical phase space, and quantum mechanical ground state tunneling between them. The Hamiltonian they addressed is

\[ \hat{H}_{WW} = -K (\hat{a}^\dagger \hat{a})^2 + \epsilon_2 (\hat{a}^2 \hat{a} + \hat{a}^\dagger \hat{a}) ^2. \] (S1)

In 2017, the theoretical discovery of the Kerr-cat qubit by Puri, Boutin, and Blais [14] relied on the fact that the ground states of

\[ \hat{H}_{BB} = -K \hat{a}^2 \hat{a} + \epsilon_2 (\hat{a}^2 \hat{a} + \hat{a}^\dagger \hat{a} ^2) \] (S2)

are fundamentally degenerate and exhibit no tunneling between two wells found in the classical limit (see also [15]). This property can be understood by writing Eq. (S2) into the factorized form [14]

\[ \hat{H}_{BB} = -K (\hat{a}^2 - \epsilon_2/K)(\hat{a}^2 - \epsilon_2/K), \] (S3)

from which it is evident that the two coherent states $|\pm \alpha \rangle$ with $\alpha = \sqrt{\epsilon_2/K}$, which are the eigenstates of the annihilation operator $\hat{a}$, are also degenerate eigenstates of Eq. (S3). Since Eq. (S3) is negative-semidefinite and $\hat{H}_{BB} | \pm \alpha \rangle = 0$, these states are the ground states.

Note that the Hamiltonians $\hat{H}_{WW}$ and $\hat{H}_{BB}$ differ only by an operator-valued commutator since $\hat{a}^2 \hat{a}^\dagger - (\hat{a}^\dagger \hat{a})^2 = \hat{a}^\dagger \hat{a}$. Taking the classical limit loses track of this reordering. Their shared classical limit can be written as

\[ H_{cl} = -K a^2 a^2 + \epsilon_2 (a^* a^2 + a^2) \]
\[ = -K \left( \frac{x^2 + p^2}{2} \right)^2 + \epsilon_2 (x^2 - p^2). \] (S4)
By introducing the more general Hamiltonian
\[ \hat{H} = \Delta \hat{a}^\dagger \hat{a} - K \hat{a}^\dagger \hat{a}^2 + \epsilon_2 (\hat{a}^\dagger \hat{a}^2 + \hat{a}^2), \]  
we identify that \( \hat{H}_{\text{PBB}} \) and \( \hat{H}_{\text{WM}} \) are specific instances of Eq. (S5) with \( \hat{H}_{\text{PBB}} = \hat{H}|_{\Delta=0} \) and \( \hat{H}_{\text{WM}} = \hat{H}|_{\Delta=-K} \). Note that taking \( \Delta \neq 0 \) breaks the simple factorization condition of Eq. (S3). Indeed, the presence of the \( \hat{a}^\dagger \hat{a} \) term is the cause of ground state tunneling in \( \hat{H}_{\text{WM}} \), and its absence is the cause of the complete coherent cancellation of tunneling in \( \hat{H}_{\text{PBB}} \). The lowest eigen-manifold of Eq. (S5) is plotted in Figure S5 while the excited state manifold of Eq. (S5) is plotted in Figure S6.

In 2007, Marthaler and Dykman [9] treated a Hamiltonian similar to Eq. (S5), where \( \Delta \) was kept free for a fixed \( \epsilon_2 \). This led to their prediction of periodic cancellation of tunneling amplitude for the ground state manifold as a function of \( \Delta \). Their work inspired our experiment shown in Figure 1 of the main text. We discuss in detail the mapping of their problem to ours in Section VII 1.

In the following text, we discuss the quantum phase space representation of Eq. (S5).

\section*{VII. Phase Space Formulations of Our Effective Hamiltonian}

Let us reconsider Eq. (S5). For its derivation starting from the circuit Hamiltonian, see appendix A of [2]. We obtain the phase space formulation of Eq. (S5) by taking the invertible Wigner transform \cite{16} \( \mathbb{W} \) as
\[
\hat{x} \to \mathbb{W}\{\hat{x}\} = x; \quad \hat{p} \to \mathbb{W}\{\hat{p}\} = p;
\]
\[
\hat{a} \to \mathbb{W}\{\hat{a}\} = a = (x + ip)/\sqrt{2\lambda}; \quad \hat{a}^\dagger \to \mathbb{W}\{\hat{a}^\dagger\} = a^*;
\]
\[
\hat{a}^\dagger a \to a^* a - a^* a - \frac{1}{2} = \frac{x^2 + p^2}{2\lambda} - \frac{1}{2};
\]
\[
\hat{a}^\dagger a^2 \to a^* a^2 = a^* a^2 - 2a^* a + \frac{1}{2}
\]
\[
= \frac{(x^2 + p^2)^2}{4\lambda^2} - \frac{(x^2 + p^2)}{\lambda} + \frac{1}{2};
\]
\[
\hat{a}^\dagger a^2 \to a^* a^2 + a^2 = \frac{(x^2 - p^2)^2}{\lambda},
\]
where the Groenewold star product \cite{17} is given by \( \mathbb{W}\{AB\} = A \star B = A \exp \left( \frac{1}{2} \{ \partial_a \partial_{a^*} - \partial_{a^*} \partial_a \} \right) B \). The Moyal bracket \cite{16, 18} over \( a \) and \( a^* \) is defined as \( \{A, B\}_{a,a^*} = A \star B - B \star A \) so that we have \( \{a, a^*\} = 1 \). For a pedagogical exposition on the phase space formulation of quantum mechanics, we refer the reader to Section XI and [16, 19, 20]. With Eq. (S6), we write Eq. (S5) in the phase space formulation of quantum mechanics, up-to coordinate-independent terms, as
\[ H = (\Delta + 2K) \left( \frac{x^2 + p^2}{2\lambda} \right) - K \left( \frac{x^2 + p^2}{2\lambda} \right)^2 + \epsilon_2 \left( \frac{x^2 - p^2}{\lambda} \right). \]  

Note that Eq. (S7) is not equal to Eq. (S4) even when \( \Delta = 0 \) and \( \lambda = 1 \). We further rescale Eq. (S7) by \(-K/\lambda^2 \) so as to have a coefficient of order 1 for the nonlinear term and rearrange Eq. (S7) as
\[ \frac{-H\lambda^2}{K} = \left( \frac{x^2 + p^2}{2} \right)^2 - 2\epsilon_2 \frac{x^2}{K} \frac{1}{2} \left( 1 + \frac{(\Delta + 2K)}{2\epsilon_2} \right) + \frac{2\epsilon_2 \lambda}{K} \frac{p^2}{2} \left( 1 - \frac{(\Delta + 2K)}{2\epsilon_2} \right). \]  

By choosing the scale of phase space \( \lambda = K/2\epsilon_2 \) Eq. (S8) becomes
\[ \frac{-H\lambda^2}{K} = \left( \frac{x^2 + p^2}{2} \right)^2 - \frac{x^2}{2} \left( 1 + \frac{\Delta}{2\epsilon_2} + 2\lambda \right) + \frac{p^2}{2} \left( 1 - \frac{\Delta}{2\epsilon_2} - 2\lambda \right). \]  

1. Classical limit

The term proportional to \( \lambda \) in Eq. (S9) involves a commutator, and corresponds to the Lamb shift. The classical limit then consist in dropping this term. This is valid for \( \lambda \ll \min(\Delta/2\epsilon_2, 1) \). This translates to \( \Delta/K, \epsilon_2/K \gg 1 \).
Figure S5: Lowest eigen-manifold of the squeeze-driven Kerr oscillator. Wigner functions of lowest pair of eigenstates of Eq. (S5) (top row) for $\epsilon_2/K = 2$ and A $\Delta/K = -6$, B $\Delta/K = 0$, C $\Delta/K = 2$, and D $\Delta/K = 6$ and $\Delta/K = 10$ respectively. $\Delta/K \ll 0$, the eigenstates are squeezed. For $\Delta/K \ll 0$, increasing $\Delta/K$ yields Schrödinger cat states with increasing photon number. This phenomenon manifests as the monotonically growing baseline in transverse relaxation lifetime $T_X$ in Figure 3 of the main text.

In this limit, the WKB approximation is valid to treat Eq. (1) in the main text. The Equation 5 of Marthaler and Dykman [9] can thus be written as

$$H_{cl} = -\frac{K}{\lambda^2} \left[ \frac{(x^2 + p^2)}{2} - \frac{x^2}{2} \left( 1 + \frac{\Delta}{2\epsilon_2} \right) + \frac{p^2}{2} \left( 1 - \frac{\Delta}{2\epsilon_2} \right) \right]$$

when their parameter $\mu$ is taken to be $\Delta/2\epsilon_2$.

The interpretation of the Hamiltonian classical limit is that the elementary action element $\lambda$ in the phase space defined by $x$ and $p$ must be much smaller than the typical dimensionless action of the system determined by the well-size parameters: $\Delta/K$ and $\epsilon_2/K$. As we discuss in what follows (see section Section VII 2 and [2]), under the condition $\Delta/K, \epsilon_2/K \gg 1$, the wells of the Hamiltonian are large in the sense that they encompass many action quanta $\lambda$. Finally, note that for $\lambda \approx 1$ the classical treatment should not hold.

We call the surface for $H$ in Eq. (S9) the metapotential of the squeeze-driven Kerr oscillator, and the classical limit for $H$ in Eq. (S10) as the classical metapotential surface. Furthermore, as customary, we plot $-H$ rather than $H$ to respect the familiar notion that in the presence of dissipation, stable equilibria correspond to well-bottoms rather than hill-tops.

### 2. Properties of the classical metapotential surface

In the table below, we examine the properties of the classical metapotential surface in the double-well regime. For details on the number of levels inside the well, which we obtain via action quantization following the prescription of Einstein-Brillouin-Keller (EBK) [21], see Section VIII B.

| Phase $\rightarrow$ | Classical metapotential parameter $(x, p \text{ phase space})$ | Double-node $-2\epsilon_2 \leq \Delta \leq 2\epsilon_2$ | Triple-node $\Delta > 2\epsilon_2$ |
|---------------------|-------------------------------------------------|---------------------------------|----------------------------------|
| Area                | $\Delta/K \arccos \left( -\frac{\Delta}{2\epsilon_2} \right) + \frac{2\epsilon_2}{K} \sqrt{1 - \left( \frac{\Delta}{2\epsilon_2} \right)^2}$ | $\frac{4\epsilon_2^2}{K} \sqrt{\frac{\Delta}{2\epsilon_2}} - 1 + \frac{2\Delta}{K} \arcsin \left( \sqrt{\frac{2\epsilon_2}{\Delta}} \right)$ | |
| Number of levels in well (#) | $\text{area}/2\pi - 1/2$ | $\text{area}/2\pi - 1/2$ | |
| Approximation of # | $\frac{\Delta/K}{2} + \frac{\epsilon_2/K - 1}{2}$ | $\frac{\sqrt{8\epsilon_2\Delta}}{K} - \frac{1}{2}$ | |
| Distance between nodes | $2\sqrt{\frac{\Delta + 2\epsilon_2}{K}}$ | $2\sqrt{\frac{\Delta + 2\epsilon_2}{K}}$ | |
| Distance between saddle points | 0 | 0 | |
| Depth of nodes | $\frac{(\Delta + 2\epsilon_2)^2}{4K}$ | $\frac{(\Delta + 2\epsilon_2)^2}{4K}$ | |
| Depth of saddle points | 0 | $\frac{(\Delta - 2\epsilon_2)^2}{4K}$ | |
| Depth of barrier | $\frac{(\Delta + 2\epsilon_2)^2}{4K}$ | $\frac{2\Delta\epsilon_2}{K}$ | |
Figure S6: **Excited well states in the squeeze-driven Kerr oscillator** Localized ground and excited states in the squeeze-driven Kerr oscillator. A Period doubling phase diagram with equi-state contours. B - E. Wigner functions of exact eigenstates’ superpositions, corresponding to localized states, for $\xi_2/K = 4$, and B $\Delta/K = 1$, C $\Delta/K = 4$, D $\Delta/K = 7$, and E $\Delta/K = 9$. The action quantization formulation, detailed in Section VIII and summarized by Eq. (S16), predicts B 0, C 1, D 2, and E 3 excited states respectively. The Wigner functions of states outside this window are seen to have support in the other well too, and larger $\Delta$ helps localize them, thus validating the semiclassical picture discussed in Section VIII quantum mechanically.

### A. Wavefunctions of localized well states

In this section, we examine closely the wave functions of the squeeze-driven Kerr oscillator in the classically forbidden region and contrast them with those of an ordinary quadratic + quartic potential. We define the ordinary double-well Hamiltonian as

$$H = \frac{p^2}{2} + V(x), \quad \text{with} \quad V(x) = -\frac{k_2}{2} x^2 + \frac{k_4}{4} x^4,$$

where $k_2, k_4 > 0$. This potential has a saddle at $x_s = 0$, with $V(x_s) = 0$ and nodes at $x_n = \pm \sqrt{k_2/k_4}$ with the left and right well depth given by $V(x_n) = -k_2^2/(4k_4)$. The barrier height is given by $V(x_n) - V(x_s) = k_2^2/(4k_4)$.

The study of tunneling usually begins by considering a localized wave packet in one well, which is written as the superposition of the wavefunctions of the two lowest laying energy states $\psi_+$ and $\psi_-$.\(^2\) Their energy difference is

\(^2\) Form a perturbation theory point of view this corresponds to the bonding and anti-bonding of the decoupled well states [22]. The zero
noted by $\delta E = E^+ - E^-$ and the left- and right-localized wavefunctions read

$$
\psi_l = \frac{\psi_+ + \psi_-}{\sqrt{2}} \quad \psi_r = \frac{\psi_+ - \psi_-}{\sqrt{2}}. ~ (S11)
$$

On the left column of Figure S7, we plot the left and right-localized wavefunctions in red and blue respectively for $A k_2 = 3, k_4 = 1$, $B k_2 = 2, k_4 = 1$, and $B k_2 = 4, k_4 = 2$ respectively. The wavefunctions are computed by numerical diagonalization of the Hamiltonian. In the classically forbidden region, as one should expect, the wavefunctions display evanescent decay [23].

In the right column of Figure S7, we contrast the localized wavefunctions of the ordinary double-well potential with those of the squeeze-driven Kerr oscillator. The parameters $\Delta, K$, and $\epsilon_2$ were chosen so that a cut of the effective Hamiltonian surface at $p = 0$ yields an identical double-well potential as the left column. The wavefunctions of the full squeeze-driven Kerr oscillator are computed numerically. Importantly, $B, D,$ and $F$ show the localized wavefunctions for $\Delta/K = 2, 1,$ and $0$ respectively, corresponding to the coherent destructive, constructive, and again destructive interference. Interestingly, in the classically forbidden region, in $B$ and $D$, oscillations accompany decay [9, 24]. This is due to the underlying driven nature of our system, providing a quartic term in momentum, which here reflects in the oscillatory nature of the wavefunctions in the classically forbidden region.

### VIII. SEMICLASSICAL ANALYSIS

#### A. WKB calculation of tunnel splitting for the ground state manifold of eq. (1)

The expression for the tunnel splitting following the analysis in [9, 24] is given as

$$
\delta E = f \cos \theta \exp(-A), ~ (S12)
$$

where

$$
\begin{align*}
  f &= 2 \left(\frac{4\epsilon_2}{K}\right)^2 \left(\frac{K}{\pi \Delta}\right)^{1/2} \left(1 + \frac{\Delta}{2\epsilon_2}\right)^{5/4} \\
  \theta &= \frac{\pi}{2} \left(\frac{\Delta}{K} - 1\right) \\
  A &= \frac{2\epsilon_2}{K} \left(\frac{\Delta}{2\epsilon_2} + 1\right)^{1/2} - \frac{\Delta}{K} \log \left(\frac{\left(2\epsilon_2\right)^{1/2}}{\Delta} + \left(1 + \frac{2\epsilon_2}{\Delta}\right)^{1/2}\right),
\end{align*}
$$

(S13)

where, the above expression is only valid for $\Delta/K \gg 1$. There are two failure modes for the WKB approximation. The first condition corresponds to when $\Delta \lesssim K$, and the other is when $\epsilon_2/K \ll 1$. Note that WKB works remarkably well outside its domain of validity ($\epsilon_2/K < 1$). Compare to Figure S7, where the wavelength given by the oscillation period of the wavefunction is of the same magnitude as the potential variation set by the interwell distance. Note that we have applied the formula developed in [9] in a domain that lies beyond the parameter regime where it was produced and we find remarkable agreement with data. The comparison between measured tunneling amplitude and a WKB theory can be found in Figure S8.

#### B. Action quantization via Einstein-Brillouin-Keller (EBK) method

In this section, we present the semiclassical method of obtaining the number of in-well states via action quantization, following the Einstein-Brillouin-Keller method, which generalizes the notion of Bohr orbits.

First, we introduce a polar-coordinate representation of Eq. (S10), which exploits its radial symmetry, as

$$
H_{cl} = \frac{\Delta r^2}{2} - \frac{Kr^4}{4} + \epsilon_2 r^2 \cos 2\theta, ~ (S14)
$$

point energy of the individual wells, in the absence of tunneling, is $E_0 = \sqrt{\epsilon_2}/2$. In the presence of tunneling the system’s energies can be approximated by $E^\pm = E_0 \pm \delta E$. 


Figure S7: Localized wavefunctions of the ground state manifold in the position basis in A, C, E an ordinary double well potential and in B, D, F for a squeeze-driven Kerr oscillator. The Hamiltonian parameters in A, C, and E have been chosen to produce a double-well with the same depth and the well separation as those of B, D, and F respectively. The value of $\Delta/K$ is chosen to be $B \Delta/K = 2$, $D \Delta/K = 1$, and $F \Delta/K = 0$ corresponding to the destructive, constructive, and destructive interference of tunneling respectively. In the right panel, oscillations accompany decay of the wavefunction in the classically forbidden region, marked in grey. In the left panel, the wavefunction exhibits pure decay in the classically forbidden region. In this sense, the cancellation of tunneling amplitude in fig 1 of the main text can be understood as the destructive interference of the wavefunction in the classically forbidden region of the squeeze-driven Kerr oscillator. In [9], Marthaler and Dykman found an analytical expression for the WKB tunnel splitting of the ground state manifold. See Section VIIIA for the WKB expressions for the tunnel splitting and Figure S8B and D for comparisons of the extracted tunnel splitting from experiment with their WKB theory.
Figure S8: Experimentally extracted tunneling amplitude in the ground state manifold as a function of \( \Delta \) (A and C) and \( \epsilon_2 \) (B and D) compared to two different theoretical models. Dots correspond to extracted level splittings from dynamical measurements of the tunneling rate and correspond to the data presented in Figures 1C and 2 respectively. Solid black lines in black in A and B are obtained via exact numerical diagonalization. Solid blue lines in C and D are obtained via a semi-classical WKB treatment developed by Marthaler and Dykman in [9, 24]. As expected, the semi-classical Hamiltonian model, in the domain of its validity \( \Delta/K \gg 1 \) and \( \epsilon_2/K \sim 1 \), agrees well with the measured data.

where \( x = r \cos \theta \) and \( p = r \sin \theta \), for \( r \geq 0 \) and \( \theta \in [0, 2\pi) \).

In a semiclassical treatment, a classical orbit \( C_j \) satisfying the following Einstein-Brillouin-Keller (EBK) quantization condition [21]:

\[
\int_{C_j} dx dp = \hbar \left( N_j + \frac{\beta_j}{4} \right),
\]

(S15)

plays a special role. On the left hand side of Eq. (S15), the action integral corresponds to the area enclosed by the contour \( C_j \). On the right hand side of Eq. (S15), the non-negative integer \( N_j \geq 0 \) represents a quantum number and \( \beta_j \) is called a Maslov index; it counts the number of caustics encountered by the contour \( C_j \). For an orbit in the Kerr-cat metapotential, we have \( \beta_j = 2 \). Thus the condition in Equation (S15) states that only those orbits whose enclosed area satisfy a condition given by non-negative integers \( n_j \) and \( \beta_j = 2 \) correspond to allowed quantum orbits.

With this condition stated, one can ask a simple question: given a set of \( \Delta, \epsilon_2 \), how many in-well or bound states exist in the classical metapotential surface? This will be obtained by computing the number of allowed states at the separatrix, which separates bound and unbound states.

From the calculations detailed in Sections VIII B 1 and VIII B 2, we find the number of bound states as

\[
N \sim \begin{cases} 
\frac{\Delta/K}{2} + \frac{\epsilon_2/K}{\pi} - \frac{1}{2} & \text{if } -2\epsilon_2 \leq \Delta < 2\epsilon_2 \\
\frac{\Delta/K}{2} + \frac{\epsilon_2/K}{\pi} - \frac{1}{2} - \frac{2}{\sqrt{8\epsilon_2 \Delta K \pi}} & \text{if } \Delta \geq 2\epsilon_2.
\end{cases}
\]

(S16)

We demonstrate in Fig. S6 the value of the semi-classical action quantization condition in predicting the locality in phase space of even the excited states of the squeeze-driven Kerr oscillator.

1. **Separatrix area in the double-node phase:** \(-2\epsilon_2 \leq \Delta < 2\epsilon_2\)

In the double-node phase, the separatrix has a special name called the Bernoulli’s lemniscate and its equation is given as

\[
r^2 = \frac{2\Delta}{K} + \frac{4\epsilon_2}{K} \cos 2\theta,
\]

(S17)
and \(-\theta_c \leq \theta \leq \theta_c\), where \(\theta_c = \frac{1}{2} \arccos \frac{\Delta}{2\epsilon_2}\). We compute the area of a half the lemniscate as

\[
\int_{\mathcal{C}_j} dx p = \frac{1}{2} \int_{-\theta_c}^{\theta_c} d\theta r^2
\]

\[
= \int_0^{\theta_c} d\theta \frac{2\Delta}{K} + \frac{4\epsilon_2}{K} \cos 2\theta
\]

\[
= \frac{\Delta}{K} \arccos \left( -\frac{\Delta}{2\epsilon_2} \right) + \frac{2\epsilon_2}{K} \sqrt{1 - \left( \frac{\Delta}{2\epsilon_2} \right)^2}
\]

\[
\sim \frac{\Delta}{K} \left( \frac{\pi}{2} + \frac{\Delta}{2\epsilon_2} \right) + \frac{2\epsilon_2}{K} \left( 1 - \frac{1}{2} \left( \frac{\Delta}{2\epsilon_2} \right)^2 \right)
\]

\[
|\Delta/2\epsilon_2| \ll 1
\]

\[
= \pi \frac{\Delta}{2K} + 2\frac{\epsilon_2}{K}
\]

Note that for \(\Delta = 0\), Eq. (S18) reduces to \(2\epsilon_2/K\).

2. Separatrix area in the triple-node phase: \(\Delta \geq 2\epsilon_2\)

The separatrix in the triple-node phase is given as

\[
r^2_\pm = \frac{\Delta}{K} + \frac{2\epsilon_2}{K} \cos 2\theta \pm \frac{4\epsilon_2 \cos \theta}{K} \sqrt{\frac{\Delta}{2\epsilon_2} - \sin^2 \theta}
\]

(S19)

and \(-\theta_c \leq \theta \leq \theta_c\), where \(\theta_c = \frac{\pi}{2}\). When plotted, this separatrix carves a bean-like shape.

Remarkably, we find an exact analytic expression for the area of this surface as

\[
\int_{\mathcal{C}_j} dx p = \frac{1}{2} \int_{-\theta_c}^{\theta_c} d\theta (r^2_+ - r^2_-)
\]

\[
= \int_{-\pi/2}^{\pi/2} d\theta \frac{4\epsilon_2 \cos \theta}{K} \sqrt{\frac{\Delta}{2\epsilon_2} - \sin^2 \theta} = \frac{4\epsilon_2}{K} \int_0^{\pi/2} dt \sqrt{\frac{\Delta}{2\epsilon_2} - t^2}
\]

\[
= \frac{4\epsilon_2}{K} \left( \frac{\Delta}{2\epsilon_2} - 1 \right) + \frac{2\Delta}{K} \arcsin \left( \sqrt{\frac{2\epsilon_2}{\Delta}} \right)
\]

\[
\sim \frac{2\sqrt{8\epsilon_2 \Delta}}{K}, \quad \Delta/2\epsilon_2 \gg 1.
\]

IX. DEGENERACIES IN THE SQUEEZE-DRIVEN KERR OSCILLATOR

A. Robustness of degeneracies

The squeeze-driven Kerr oscillator we have engineered has the remarkable property: for \(\Delta/K = 2m\), the first \(m + 1\) pairs of levels become decoupled from the rest of the oscillator’s Hilbert space. Their eigenenergies and eigenstates become exactly solvable and present \(m + 1\) robust degeneracies in between states of different photon-number parity. Critically, note that the resonance condition for these degeneracies is independent of the value of the squeeze-drive amplitude \(\epsilon_2\).

First, to show this, we begin by considering the squeeze-drive as a perturbation to the a Kerr oscillator described by the Hamiltonian \(\hat{H}_K/\hbar = \Delta \hat{a}^\dagger \hat{a} - K \hat{a}^2 \hat{a}^\dagger\) which is exactly solvable: its eigenstates are Fock states \(|n\rangle\) and their energies are \(E_n^{(0)} = \Delta n - K(n - 1)\), which, as a function of \(\Delta\), are lines with integer slope that we plot in the top row of Figure S9A. In the second row, we plot the the transition spectrum with respect to the ground state at \(\epsilon_2 = 0\), which, due to the choice of rotating frame, corresponds to the highest energy eigenstate. This is the directly experimentally observable transition spectrum from the ground state. We further note that the ground state changes with \(\Delta\); remarkably, for \(\epsilon_2 = 0\), at \(\Delta/K = 2m\), the ground state \(|m\rangle + |m + 1\rangle)/\sqrt{2}\). This special property of
the squeeze-driven Kerr oscillator has technological applications [25]. In the following rows, we plot the transition spectrum for increasing values of squeeze-drive amplitude \( \epsilon_2 \).

Indeed, it is clear that the squeeze-drive renormalizes the energies of the Kerr oscillator. Level crossings of the Kerr oscillator with different parity remain exact crossings in the presence of the squeeze-drive, since the interaction preserves parity. However, the remarkable feature is that this crossings are locked to where \( \Delta \) equals an even multiple of \( K \). In the following text, we justify this property, first via a perturbative and then provide a to-all-order proof.

1. Perturbative analysis of degeneracies

To first order in perturbation theory, we see that this even and odd Fock states remain decoupled (energy level crossings) under the parity conserving squeeze-drive: \( E^{(1)}_n = \langle n | (\hat{\alpha}^2 + \hat{\alpha}^\dagger)^2 | n + 1 \rangle = 0 \). The condition for crossings of consecutive levels with different parity \( (E_n = E_{n+2}) \) reads instead \( \Delta/K = (2n+1) \). To first order in perturbation theory, the avoided crossing amplitude is \( E^{(1)}_n = \epsilon_2 \sqrt{(n+1)(n+2)} \).

As a next approximation to the problem, we see the robustness of the crossings of consecutive levels with different parity (at \( \Delta = 2nK \)) by computing the second order correction to the \( n \)th energy levels \( E^{(2)}_n \) and comparing it to the correction for the \((n+1)\)th energy level

\[
E^{(2)}_{n+1} = \epsilon_2^2 \left( \frac{(n+3)(n+2)}{-2\Delta + 2\lambda(2n+3)} + \frac{(n+1)n}{2\Delta - 2\lambda(2n+1)} \right),
\]

to find that \( E^{(2)}_n = E^{(2)}_{n+1} \) for \( \Delta/K = 2n \). This robustness can be seen in Figure S9A (all panels), where we see that the crossing shifts in energy but remains locked to \( \Delta/K \) equal to even non-negative integers. The perturbation theory argument is easily generalized to non-consecutive level crossings and anti-crossings to this order.

2. Non-perturbative analysis of degeneracies

To prove that the location of the degeneracies in \( \Delta \) is independent of the squeeze-drive amplitude to all orders we observe that we can write the Hamiltonian in Eq. (1) as

\[
\hat{H} = \lambda_1 (\hat{\alpha}^2 - \alpha^2)(\hat{\alpha}^2 - \alpha^2) + \lambda_2 (\hat{\alpha}^2 - \alpha^2)(\hat{\alpha}^\dagger \hat{\alpha}^\dagger - \alpha^2), \tag{S21}
\]

where, for \( \Delta/K = 2m \) \( (m \) non-negative integer), we have \( \lambda_1 = -K(1 + m/2), \lambda_2 = mK/2 \) and \( \alpha = \pm \sqrt{\epsilon_2/K} \), and which is a generalization of the factorization condition proposed in [14] for \( \Delta = 0 \). We next consider the displaced Hamiltonian \( \hat{H}^+ = D(+\alpha)\hat{H}D(+\alpha) \), which brings one of the wells to the origin of phase space. In this frame, the Hamiltonian operator can be written as\(^3\)

\[
\hat{H}^+ = -K (\hat{\alpha}^2 \hat{\alpha}^\dagger + (4\alpha^2 + 2m)\hat{\alpha}^\dagger \hat{\alpha}) - 2\lambda \hat{\alpha}^\dagger \hat{\alpha} - (m + 1) \hat{\alpha}^\dagger \hat{\alpha} - 2\lambda \hat{\alpha} \hat{\alpha}^\dagger - \epsilon_2/2\hat{\alpha}^\dagger \hat{\alpha}. 
\]

While the first line is number conserving, the two consecutive lines couple only consecutive Fock states. In matrix form, it is tridiagonal in the Fock basis \( |n\rangle \). By examining the square brackets in the above expression, we see that the off-diagonal elements are exactly zero for \( n = m \) and \( n = m + 1 \). Thus, the first \( m + 1 \) states decouple from the rest of the oscillator’s Hilbert space. The finite matrix is Hermitian, negative-semidefinite, and tridiagonal so it is exactly diagonalizable. Finally, we note that in phase space, a displacement of the metapotential surface, which is mirror-symmetric about \( x = 0 \), is identical to an opposite displacement composed with a rotation of 180°.

\(^3\) Note that, without specializing \( \Delta \), one can directly write from Eq. (1) in the main text, or equivalently from Eq. (S21): \( \hat{H}^+ = -K (\hat{\alpha}^2 \hat{\alpha}^\dagger + (4\alpha^2 + \Delta/K)\hat{\alpha}^\dagger \hat{\alpha}) - 2\lambda \hat{\alpha}^\dagger \hat{\alpha} - (\Delta/2K + 1)\hat{\alpha}^\dagger \hat{\alpha} - 2\lambda \hat{\alpha} \hat{\alpha}^\dagger - \Delta/2K\hat{\alpha} \). From this expression one can directly derive the multilevel resonance condition to be \( \Delta/K = 2m \), in an exact manner, without relying in perturbative calculation or any previous knowledge existence of the resonance. The independence of the resonance condition with respect to \( \epsilon_2 \) is explicit.
Figure S9: **Robustness of degeneracies in the squeeze-driven Kerr oscillator.** Spectrum of A Eq. (1) and B $\hat{H}_{\text{eff}} = \Delta \hat{a}^\dagger \hat{a} - K \hat{a}^{12} \hat{a}^2 + \epsilon_2 (\hat{a}^{12} + \hat{a}^2)$ as a function of $\Delta/K$ for different values of $\epsilon_2/K$ and $\epsilon_4/K$ respectively. Dashed lines mark $\Delta/K$ corresponding to even integers. Left panel indicates that even for non-perturbative values of $\epsilon_2/K$, the locations of crossings of even (blue) and odd (orange) parity eigenstates occur at even values of $\Delta/K$. Right panel indicates that even for the parity preserving perturbation controlled by $\epsilon_4/K$, the locations of the crossings of even and odd parity states get renormalized. Red circle tracks one such crossing.
around the origin. Since the photon-number parity operator \( \hat{\Pi} = e^{i\pi \hat{a}^\dagger \hat{a}} \) commutes of the Hamiltonian \((\hat{H}, \hat{\Pi}) = 0\) the rotation is a symmetry of the system. Specifically; \(\hat{H}^- = \hat{D}(-\alpha)\hat{H}\hat{D}^\dagger(-\alpha) \Rightarrow \hat{H}\hat{\Pi} - \hat{\Pi} = \hat{H}^+\). We thus have two sets of equivalent\(^4\) \( m + 1 \) exactly solvable eigenergienes, which implies the existence of \( m + 1 \) degeneracies in the spectrum for \( \Delta/K = 2m \). The \( 2(m+1) \) eigenstates \( |\psi_{k\leq m+1}^\pm\rangle = \hat{D}(\pm\alpha)|\phi_{k\leq m+1}^\pm\rangle \) of \( \hat{H} \), where \( |\phi_{k\leq m+1}^\pm\rangle = \hat{\Pi}|\phi_{k\leq m+1}^\pm\rangle \) and \( \hat{H}^+|\phi_{k\leq m+1}^\pm\rangle = E_{k\leq m+1}|\phi_{k\leq m+1}^\pm\rangle \), found in this way are not orthogonal, but thanks to the two-fold degeneracy condition we can take the superposition of the right \((+)\) and left \((-)\) \( k \)th displaced state to get an orthogonal basis in each of the \( m + 1 \) two-fold degenerate sub-spaces: \( |\mathcal{C}_{k\leq m+1}^\pm\rangle \propto D(+\alpha)|\phi_{k\leq m+1}^\pm\rangle \pm D(-\alpha)|\phi_{k\leq m+1}^\pm\rangle \). These \( 2(m + 1) \) pairwise-degenerate eigenstates of energy are also eigenstates of parity\(^5\). In this work we name these pairs of degenerate states the \( \Delta \)-cats.

Note, that the robustness of the resonance condition is a peculiar symmetry property of the squeeze-driven Kerr oscillator and not a property of generic Kerr parametric oscillators. The existence of these robust symmetry begs the question: what are the symmetries of this bosonic model associated to the degeneracies? We show in Figure S9B, as an example, the spectrum of \( \hat{H} = \Delta \hat{a}^\dagger \hat{a} - K \hat{a}^{\dagger 2} \hat{a}^2 + \epsilon_d (\hat{a}^{4} + \hat{a}^{\dagger 4}) \), where the location in \( \Delta \) of the super-parity resonances depend on the value of the parametric drive amplitude \( \epsilon_d \). Note, also, that even if the multilevel resonances in Figure S9B are displaced with the value of the parametric drive amplitude (red circles), they are locked together to a running resonance condition: the point of exact solvability is changed by the drive. The phenomenon corresponds to deep symmetries\([26, 27]\) of these type of, as of now, engineerable bosonic Hamiltonians and will be discussed in detail in a separate publication.

X. MODELING THE MEASURED TRANSVERSE RELAXATION LIFETIME \( T_X \)

To model the transverse relaxation lifetime measurements \( T_X \) of the Kerr-cat qubit, which we also refer to as the well-switching lifetime of the Kerr-cat system, we use a standard Lindblad master equation as:

\[
\partial_t \hat{\rho} = \frac{1}{i\hbar} [\hat{H}, \hat{\rho}] + \kappa(1 + \bar{n}_{th})\frac{\hat{D}[\hat{a}]\hat{\rho} + \kappa \bar{n}_{th}\frac{\hat{D}[\hat{a}^\dagger]\hat{\rho}}{\hbar}] \quad (S22)
\]

where \( \hat{\rho} \) describes the state of the system, \( \bar{n}_{th} = 1/(\exp(\hbar \omega_a/k_B T) - 1) \) corresponds to the temperature of the environment and \( \kappa \) corresponds to the coupling between system and environment. The Hamiltonian \( \hat{H} \) is given by Eq. (S5) and the dissipator \( \hat{D} \) of the operator \( \hat{O} \) is given by \( \hat{D}[\hat{O}\hat{\rho} \hat{O}^\dagger] = \hat{O}\hat{\rho}\hat{O}^\dagger - (\hat{O}^\dagger \hat{O}\hat{\rho} + \hat{\rho}\hat{O}^\dagger \hat{O})/2 \). In Eq. (S22), these operators correspond to single photon loss \( \hat{D}[\hat{a}] \) and gain \( \hat{D}[\hat{a}^\dagger] \)\([28–30]\). In Figure S10, we compare the data presented in Figure 3 of the main text with the lifetime extracted from Eq. (S22) for different values of \( n_{th} \). The value of \( \kappa \) has been set to \( \kappa = 1/T_1 = 1/20 \) \( \mu s^{-1} \). The current model seems insufficient to accurately predict the observations and more research is needed to understand the decoherence of nonlinear driven systems (see, for example, [31]). Figure S10 emphasizes the need for further measurements and a detailed modeling of possible noise sources affecting particularly driven qubits. See also the note at the end of Section XI. We also present, in Figure S11, the expected \( T_X \) as a function of \( \epsilon_d/K \) for different values of \( \Delta \). This plot indicates that a \( \Delta \)-Kerr-cat, in general, gives larger \( T_X \) lifetimes than a Kerr-cat (\( \Delta = 0 \)).

---

\(^4\) Note that for the off-diagonal elements the parity transformation produces a minus sign \( (\hat{\Pi}|n\rangle \langle n|\hat{\Pi} = -|n\rangle \langle n|) \) that manifest in \( \alpha \to -\alpha; \quad H_{\alpha,n\pm 1}^\pm = -H_{\alpha,n\pm 1} \). This leaves the finite characteristic polynomial invariant.

\(^5\) Specifically \( \hat{\Pi}|\mathcal{C}_{k\leq m+1}^\pm\rangle = \pm|\mathcal{C}_{k\leq m+1}^\pm\rangle \) and are thus orthogonal. We used \( D(+\alpha)\hat{\Pi} = \hat{\Pi}D(-\alpha) \).
Figure S10: **Lindblad simulations of $T_X$ as a function of $\Delta$ for different thermal populations, corresponding to Eq. (S22).** Black dots correspond to experimental data presented in Fig. 3 G in the main text. The value of $\kappa$ has been taken to be $\kappa = 1/T_1 = 1/20 \text{ ms}^{-1}$. The solid curves take the experimentally observed ac Stark shift into account. An ordinary Lindbladian at non-zero temperature is insufficient to predict the experimental data. Beyond-RWA effects may be important to consider [31].

Figure S11: **Ordinary Lindblad simulations of $T_X$ as a function of $\epsilon_2/K$ for different values of $\Delta/K$, corresponding to Eq. (S22).** For both A and B, the value of $\kappa/K = 1/50$ and $\bar{n}_\mathrm{th} = 0.05$. In B for $\epsilon_2/K < 2$ the lifetime is limited by ground state tunneling and is not well captured by our simplified method.

XI. TUTORIAL ON THE PHASE SPACE FORMULATION OF QUANTUM MECHANICS

A full quantum mechanical treatment can be developed in phase space without incurring in any semiclassical approximations [16–18]. For the sake of completeness, here we provide an overview on the mapping from operator-valued Hilbert space to quantum phase space and a few elemental techniques and identities. We focus here on Wigner phase space, and showcase that the Wigner transform is more than a visualization tool for states. We note that our treatment can be equivalently extended to other phase space formulations [16, 19, 32–34].

**From operator Hilbert space to Wigner phase space (and back)**

The Wigner transform [35] of the density matrix $\hat{\rho}$ is the Wigner function $W(X,P)$, where $X$ and $P$ are standard phase space coordinates (not operators) with dimensions of position and momentum (see Section V for notation). We write this as
\[ \mathfrak{W}\{\hat{\rho}\} = W(X, P). \]

Let us remind the reader of some crucial properties of the Wigner function. We have

\[ \int \int dX dP W(X, P) = 1, \]  

(S23)

where each integral runs from \(-\infty\) to \(\infty\) and we suppress the limits in the following text for simplicity. For a pure state, we further have

\[ h \int \int dX dP W(X, P)^2 = 1, \]  

(S24)

where \( h = 2\pi \hbar \).

In general, we have

\[ 0 \leq h^{n-1} \int \int dX dP W(X, P)^n \leq 1, \]  

(S25)

which corresponds to the positivity of the density matrix.

Likewise, for a generic operator \( \hat{F} \), we introduce the phase space function \( F(X, P) = \mathfrak{W}\{\hat{F}\} \).

In this framework, the average value of an Hermitian operator \( \hat{F} \) can be written as

\[ \langle \hat{F} \rangle = \int \int dX dP F(X, P) W(X, P). \]  

(S26)

The transformation \( \mathfrak{W} \) is invertible as appreciated by Groenewold [17]

\[ \mathfrak{W}^{-1}\{W(X, P)\} = \hat{\rho}. \]

The inverse transformation \( \mathfrak{W}^{-1} \) is known as the Weyl transformation [36].

In general, the Weyl transformation is

\[ \hat{\rho} = \mathfrak{W}^{-1}\{W\} = \frac{1}{h} \int \int \int \int dX dP dk dl W(X, P) e^{i(k\hat{X} - \hat{k}X + l\hat{P} - \hat{l}P)}, \]  

(S27)

where the characteristic function \( C(l, k) \) defined as

\[ C(l, k) = \int \int dX dP e^{i(kX + lP)} W(X, P), \]  

(S28)

is the Fourier transform of the Wigner function and \( C \) is dimensionless.

Another useful formula is

\[ W(X, P) = \frac{1}{h} \int dq e^{-i(qP)/\hbar} \langle X + q/2 | \hat{\rho} | X - q/2 \rangle, \]  

(S29)

where \( \hat{\rho} \) is to be understood in the continuous position basis and therefore has the dimension of \( [1/\text{position}] \).

We now review simple operational rules to go from operator space to phase space functions and back without performing cumbersome integrals.

The Wigner and Weyl transformation take a particularly simple form for binomial expansions

\[ \mathfrak{W}\{(\alpha \hat{X} + \beta \hat{P})^n\} = (\alpha X + \beta P)^n. \]

\[ \mathfrak{W}^{-1}\{(\alpha X + \beta P)^n\} = (\alpha \hat{X} + \beta \hat{P})^n. \]

For non-symmetric expressions, the Wigner transform can be evaluated via a non-commutative Wigner phase space product, the celebrated Groenwold’s star product.
A. An introduction to the star product

We introduce the star product as

\[ M( \hat{F} \hat{G} ) = M( \hat{F} ) \star M( \hat{G} ) = F(X, P) \star G(X, P), \]  

defined as (the exponential of the Poisson bracket):

\[ F \star G = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^{k}}{n!} \left( \frac{i\hbar}{2} \right)^{n} \binom{n}{k} \partial_{P}^{n-k} F \times \partial_{P}^{k} G \]

\[ \equiv F \exp \left( \frac{i\hbar}{2} \left( \partial_{X} \partial_{P} - \partial_{P} \partial_{X} \right) \right) G \]

\[ = FG + \frac{i\hbar}{2} \{ F, G \} + \cdots \]

Here \( F \partial_{X} G = (\partial_{X} F)G \) and \( F \partial_{P} G = F(\partial_{X} G) \), and we have introduced the Poisson bracket \( \{ F, G \} = \partial_{X} F \partial_{P} G - \partial_{P} F \partial_{X} G \). The star product can also be conveniently expressed in terms of complex-coordinates \( a \) and \( a^{*} \) as

\[ F \star G \equiv F \exp \left( -\frac{1}{2} \left( \partial_{a} \partial_{a}^{*} - \partial_{a}^{*} \partial_{a} \right) \right) G. \]

It generalizes to a system of many particles (or many modes) as

\[ F \star G = F \exp \left( \frac{i\hbar}{2} \sum_{j} \left( \partial_{X_{j}} \partial_{P_{j}} - \partial_{P_{j}} \partial_{X_{j}} \right) \right) G. \]

In Fourier space the star product becomes a phase factor: \( \star \rightarrow e^{i\frac{1}{2} (k_{X}^{2} - k_{P}^{2})} \) [37]. This phase corresponds to an oriented area in reciprocal phase space. This is the simplest manifestation of the noncommutativity of the algebra of quantum mechanics in phase space.

Remarkably, the scalar product associated with the star product is the usual integral in phase space. For phase space functions in the Wigner representation \( F \) and \( G \), we have

\[ \iint dX dP F(X, P) \star G(X, P) = \iint dX dP F(X, P)G(X, P). \]  

(S32)

Note however that in general for any \( F(X, P), G(X, P), \) and \( H(X, P), \)

\[ \iint dX dP F(X, P) \star G(X, P) \star H(X, P) \neq \iint dX dP F(X, P)G(X, P)H(X, P). \]  

(S33)

For non-symmetric expressions in \( \hat{X} \) and \( \hat{P} \), the above formulae can be employed to evaluate the Wigner transform. For example

\[ M( \hat{X} \hat{P} \hat{P} ) = XP^{2} + i\hbar P \]

\[ M( \hat{P} \hat{P} \hat{X} ) = XP^{2} - i\hbar P \]

\[ M( \hat{P} \hat{X} \hat{P} ) = XP^{2}. \]

We evaluate the Weyl transform of asymmetric expressions by symmetrizing it and replacing phase space functions by their corresponding operators. For example

\[ M^{-1}( XP^{2} ) = \frac{1}{3} ( \hat{X} \hat{P} \hat{P} + \hat{P} \hat{X} \hat{P} + \hat{P} \hat{P} \hat{X} ). \]

To find the Weyl transform of a high-degree polynomial of \( X \) and \( P \), the Weyl-symmetrized form might be too tedious and McCoy [38] provided a shortcut to obtain polynomial expressions in the phase space representation. We review McCoy’s formula in the next section.
The McCoy formula for obtaining ordered operators from phase space functions

While a fully-symmetrized representation is usually inconvenient for polynomials of large degree, McCoy derived a set of formulae [38], each corresponding to a different representation of a Weyl transform. Here, we present two of them that yield operators that privilege the ordering of \( \hat{X} \) (or \( \hat{P} \)).

Consider a phase space function \( F(X, P) \). Its operator-valued correspondent \( \hat{F} \) in normal order with respect to \( X \) is given by the McCoy formula [38] that reads:

\[
\hat{F} = (N_X F) |(\hat{X}, \hat{P}),
\]

The functional (operator over real functions) \( N_X \) is carried out by writing its arguments with \( X \) factors (or \( P \) factors as indicated by the subindex of \( N \)) to the left in each term and replace \( X, P \) with \( \hat{X}, \hat{P} \) respectively. For example, if \( F = XP \), we have \( \hat{F} = \hat{X}\hat{P} - i\hbar/2 \) which gives the correct and now ordered Hermitian expression for the operator \( \hat{F} = \hat{X}\hat{P} + \hat{P}\hat{X} \).

The inverse transform, is simply given \( F(X, P) = e^{i\hbar/2} \partial_X \partial_P F(X, P) \).

In terms of complex coordinates, \( a = \sqrt{2}(x + ip) \) we adapt McCoy’s formula [38]:

\[
\hat{F} = (N_{a*} F) |(\hat{a}, \hat{a}^\dagger),
\]

to get the normal ordered (with respect to \( a^* \)) result. For example, one has classically that \( 1/2(x^2 + p^2) = a^*a \). The correct quantization reads \( F = aa^* \rightarrow \hat{F} = \hat{a}^\dagger\hat{a} + 1/2 \).

Application to our Hamiltonian

If the Wigner phase space Kerr Hamiltonian reads \( H = \Delta a^*a - Ka^2a^2 \) the corresponding operator is

\[
\hat{H} = (\Delta - 2K)\hat{a}^\dagger\hat{a} - K\hat{a}^{12}\hat{a}^2,
\]

where the oscillator frequency is renormalized by \( 2K \). This is the Lamb shift, and its origin is in the non commutativity of \( \hat{a} \) and \( \hat{a}^\dagger \), i.e. the vacuum fluctuations.

Groenewold’s theorem

Note that \( \mathcal{W} \left\{ \frac{1}{i\hbar} [\hat{F}, \hat{G}] \right\} \neq \{ \mathcal{W}(\hat{F}), \mathcal{W}(\hat{G}) \} \). The quantum commutators do not correspond to the Poisson brackets: the theorem [17] states that such a mapping does not exist. We provide a practical consequence of the implications of this theorem to quantum Hamiltonian engineering in Appendix B of [39].
Dynamics of the Wigner function: the Moyal equation

The von-Neumann equation \( \partial_t \hat{\rho} = \frac{1}{i\hbar}[\hat{H}, \hat{\rho}] \) (the density-operator version of the Schrödinger equation) transforms as

\[
\partial_t W = \frac{1}{i\hbar} (H \star W - W \star H),
\]

\[
\partial_t W = \{H, W\}.
\]

Here \( H(X, P) = \mathfrak{M}(\hat{H}) \) is the Hamiltonian function and we have introduced the Moyal bracket notation [18]. We refer the reader to [20] for a derivation of the equation of motion of the Wigner function from Schödinger’s equation for the wavefunction without referring to the star product.

The exponential notation of the star product induces the name “Moyal sine bracket” since it can be written as

\[
\partial_t W = H^2 \frac{2}{\hbar} \sin \left( \frac{\hbar}{2} \langle \partial X \partial P - \partial P \partial X \rangle \right) W.
\]

Note that the Moyal equation is identical to Liouville equation plus quantum corrections coming from the expansion of the sine to higher orders of \( \hbar \).

\[
\partial_t W = \{H, W\} + \mathcal{O}(\hbar^2).
\]

Interestingly, there is no corrections to \( \mathcal{O}(\hbar) \). Importantly, the quantum corrections are proportional to \( \hbar^2 \) and to the nonlinear terms in the Hamiltonian. For quadratic Hamiltonians, all the quantum corrections vanish: the higher-order derivatives exterminate low-order polynomials (see the Appendix of [2]). Specifically, Gaussian transformations, i.e., those generated by quadratic Hamiltonians in the phase space coordinates, are classical in the sense that they are ruled by only the Poisson bracket. Thus, they would not develop negativities in the Wigner distribution if none would be present at the beginning.

Phase space formulation for open quantum systems

So far, we have only discussed the phase space formulation for closed quantum systems. Indeed, one can extend the treatment to open systems as we demonstrate below. The Lindblad equation for single photon loss is given by

\[
\partial_t \hat{\rho} = \frac{1}{i\hbar}[\hat{H}, \hat{\rho}] + \kappa \hat{a} \hat{a}^{\dagger} - \frac{\kappa}{2}(\hat{a}^{\dagger} \hat{\rho} + \hat{\rho} \hat{a}^{\dagger} \hat{a}).
\] (S34)

Using Eq. (S30) and Eq. (S31), we get the phase space formulation of Eq. (S34) as

\[
\mathfrak{M}(\partial_t \hat{\rho}) = \partial_t W,
\]

\[
\mathfrak{M} \left\{ \frac{1}{i\hbar}[\hat{H}, \hat{\rho}] \right\} = \{\mathfrak{M}(\hat{H}), \mathfrak{M}(\hat{\rho})\}
\]

\[
= \{H, W\}
\]

\[
\mathfrak{M}\{\hat{a} \hat{a}^{\dagger}\} = a \star W \star a^*
\]

\[
= a W a^* + \frac{1}{2} \partial_a W + \frac{1}{2} (\partial_{a^*} (W a^*) + \frac{1}{2} \partial_{a a^*}^2 W)
\]

\[
\mathfrak{M}\{\hat{a}^{\dagger} \hat{\rho}\} = \left(a^* a - \frac{1}{2}\right) \star W
\]

\[
= a W a^* - \frac{1}{2} W + \frac{1}{2} (a^* \partial_{a^*} W - a \partial_a W) - \frac{1}{4} \partial_{a a^*}^2 W
\]
\[ \mathfrak{W}\{\hat{\rho}\hat{a}^{\dagger}\hat{a}\} = W \star \left( a^*a - \frac{1}{2} \right) \]
\[ = aW^* - \frac{1}{2}W - \frac{1}{2}(a^*\partial_xW - a\partial_aW) - \frac{1}{4}\partial_{aa^*}^2W \]

Gathering all terms one directly gets
\[ \partial_t W = \{\{H,W\}\} + \kappa \left( \frac{\kappa}{2} + \sqrt{n_{th}} \right) \partial_{aa^*}^2 + \partial_{x}a + \partial_{a^*}a^* \right) W. \]

It is convenient to translate the above to \( x, p \) space
\[ \partial_t W = \{\{H,W\}\} + \frac{\kappa}{2} \left( \partial_x^2 + \partial_p^2 + \partial_xx + \partial_pp \right) W. \]

By expressing the equation in \( x, p \) space in Eq. (S35), the diffusion terms \( \propto (\partial_x^2 + \partial_p^2) \) and the drag terms \( \propto (\partial_xx + \partial_pp) \) associated to the fluctuation and the dissipation become evident. Note, that the Moyal sine bracket has only odd derivatives: the diffusion \( (\partial_x^2 + \partial_p^2) \) cannot be canceled by Hamiltonian dynamics.

For finite temperature \( \bar{n}_{th} \), the Lindblad master equation is
\[ \partial_t \hat{\rho} = \frac{1}{i\hbar} [\hat{H}, \hat{\rho}] + \kappa (1 + \bar{n}_{th}) \hat{D}[\hat{a}] \hat{\rho} + \kappa \bar{n}_{th} \hat{D}[\hat{a}^\dagger] \hat{\rho}, \]

where the dissipator \( \hat{D} \) of the operator \( \hat{\mathcal{O}} \) is given by \( \hat{D}[\hat{\mathcal{O}}] \equiv \hat{\mathcal{O}} \partial_t \hat{\mathcal{O}} - (\hat{\mathcal{O}}^\dagger \hat{\mathcal{O}} + \bullet \hat{\mathcal{O}}^\dagger \hat{\mathcal{O}})/2 \).

It is straightforward to show that in the phase space formulation, Eq. (S36) reads
\[ \partial_t W = \{\{H,W\}\} + \frac{\kappa}{2} \left( \partial_xa + \partial_{a^*}a^* \right) W + \kappa \left( \frac{1}{2} + \bar{n}_{th} \right) \partial_{aa^*}^2W, \]

which reads in \( x, p \) space as
\[ \partial_t W = \{\{H,W\}\} + \frac{\kappa}{2} \left( \partial_xx + \partial_pp \right) W + \frac{\kappa}{2} \left( \frac{1}{2} + \bar{n}_{th} \right) \left( \partial_x^2 + \partial_p^2 \right) W. \]

Equation (S38) is the quantum version of the Fokker-Planck equation, with the Poisson bracket replaced by the Moyal bracket and a quantum diffusion term corresponding to the zero point spread.

Note that for the Hamiltonian corresponding to Eq. (S5), the solution for \( W \) from Eq. (S38) will not yield the Boltzmann distribution in steady state, which perhaps is not surprising for an out-of-equilibrium driven problem [40].

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