ON ASYMPTOTICALLY ARBITRAGE-FREE APPROXIMATIONS OF THE IMPLIED VOLATILITY

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ABSTRACT. Following-up Fukasawa and Gatheral (Frontiers of Mathematical Finance, 2022), we prove that the BBF formula, the SABR formula, and the rough SABR formula provide asymptotically arbitrage-free approximations of the implied volatility under, respectively, the local volatility model, the SABR model, and the rough SABR model.

1. INTRODUCTION

The implied volatility is one of the basic quantities in financial practice. The option market prices are translated to the implied volatilities to normalize in a sense their dependence on strike price and maturity. The shape of the implied volatility surface characterizes the marginal distributions of the underlying asset price process. Other than a flat surface corresponding to the Black-Scholes dynamics, no exact formula of the surface is available and so, various approximation formulae have been investigated. See [9] for a practical guide for the volatility surface.

One of the most famous and in daily use of financial practice is the SABR formula proposed by Hagan et al. [10] for the SABR model. After its original derivation by [10] based on a formal perturbation expansion, and a verification by [3] based on an asymptotic analysis of PDE, Balland [1] derived the formula (for the so-called lognormal case) by an elegant no-arbitrage argument. The no-arbitrage argument remains valid for non-Markovian models, and Fukasawa and Gatheral [8] derived an extension of the SABR formula to a rough SABR model, where the volatility process is non-Markovian.

It has been known that the implied volatility surface of an equity option market typically exhibits a power-law type term structure. Since classical local-stochastic volatility models including the SABR model are not consistent to such a term structure, the so-called rough volatility model has recently attracted attention, which is the only class of continuous price models that are consistent to the power-law; see Fukasawa [4, 6]. The rough SABR model of [8] (see also [5, 11, 7]) is a rough volatility model and the rough SABR formula derived in [8] explicitly exhibits a power-law term structure.

The aim of this paper is to verify the SABR and rough SABR formulae by the no arbitrage argument beyond the lognormal case.
2. Asymptotically Arbitrage-free Approximation

Here we recall the definition of Asymptotically Arbitrage-free Approximation (AAA for short) given by [8] and derive an alternative expression of the defining equation.

Let $S = \{S_t\}$ be the underlying asset price process of an option market, and $C = \{C_t\}$ be a call option price process with strike price $K > 0$ and maturity $T > 0$. We regard $(S, C)$ as a 2 dimensional stochastic process defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \in [0, T]})$. We assume interest and dividend rates to be zero for brevity. Denote by $\Sigma^{BS} = \{\Sigma^{BS}_t\}$ and $\Sigma^B = \{\Sigma^B_t\}$ respectively the Black-Scholes and the Bachelier implied volatility processes defined by

$$C_t = p^{BS}(S_t, K, T - t, \Sigma^{BS}_t) = p^B(S_t, K, T - t, \Sigma^B_t),$$

where $p^{BS}(S, K, \tau, \sigma)$ and $p^B(S, K, \tau, \sigma)$ are respectively the Black-Scholes and Bachelier call prices with volatility $\sigma$, namely

$$p^{BS}(S, K, \tau, \sigma) = S\Phi(d_+) - K\Phi(d_-), \quad d_\pm = \frac{\log \frac{S}{K} \pm \sigma \sqrt{\tau}}{2},$$

$$p^B(S, K, \tau, \sigma) = \sigma \sqrt{\tau} \phi \left( \frac{S - K}{\sigma \sqrt{\tau}} \right) + \left( S - K \right) \Phi \left( \frac{S - K}{\sigma \sqrt{\tau}} \right).$$

As is well-known, a sufficient and almost necessary condition for $(S, C)$ to be arbitrage-free is that there exists an equivalent probability measure $Q$ to $\mathbb{P}$ such that $(S, C)$ is a local martingale under $Q$. Since $\Sigma^{BS}$ and $\Sigma^B$ are nonlinear transformations of the price $C$, they are not local martingales but should satisfy certain constraints on their finite variation components under $Q$. Assuming that $(S, C)$ is a Brownian local martingale under $Q$ and denoting by $D^{BS} \, dt$ and $D^B \, dt$ respectively the drift components of $d\Sigma^{BS}$ and $d\Sigma^B$, the constraints (the drift conditions) are

$$\frac{d}{dt} \langle \log S \rangle + 2k \frac{d}{dt} \langle \log S, \log \Sigma^{BS} \rangle + k^2 \frac{d}{dt} \langle \log \Sigma^{BS} \rangle - \langle \Sigma^{BS} \rangle^2 + 2\Sigma^{BS} \tau D^{BS}$$

$$= -\langle \Sigma^{BS} \rangle^2 \tau \frac{d}{dt} \langle \log S, \log \Sigma^{BS} \rangle + \left( \frac{\langle \Sigma^{BS} \rangle^4 \tau^2}{4} \frac{d}{dt} \langle \log \Sigma^{BS} \rangleight)$$

and

$$\frac{d}{dt} \langle \log S \rangle + 2x \frac{d}{dt} \langle \log S, \log \Sigma^B \rangle + x^2 \frac{d}{dt} \langle \log \Sigma^B \rangle - \langle \Sigma^B \rangle^2 + 2\Sigma^B \tau D^B = 0,$$

where $k = \log K/S$, $x = K - S$ and $\tau = T - t$. Based on this observation, Fukasawa and Gatheral [8] introduced the following notion:

Definition 2.1. Let $\hat{\Sigma} = \{\hat{\Sigma}_t\}$ be a positive continuous Itô process on $[0, T)$ and denote by $\hat{D} \, dt$ the drift part of $d\hat{\Sigma}$ under $Q$. We say $\hat{\Sigma}$ is an AAA if there exist a continuous function $\varphi$ on $\mathbb{R}$ and a continuous process $\Psi = \{\Psi_t\}$ on $[0, T]$...
such that
\[
\left| \frac{d}{dt} \langle X \rangle + 2k \frac{d}{dt} \langle \log S, \log \hat{\Sigma} \rangle + k^2 \frac{d}{dt} \langle \log \hat{\Sigma} \rangle \right| \leq \varphi(\Psi \hat{\Sigma}) \cdot o_p(1) 
\]
(1)
\[
- \hat{\Sigma}^2 + 2\hat{\Sigma} \partial_t \hat{\Sigma} + \hat{\Sigma}^2 \partial_t \langle \log S, \log \hat{\Sigma} \rangle - \frac{\hat{\Sigma}^4}{4} \frac{d}{dt} \langle \log \hat{\Sigma} \rangle \leq \varphi(\Psi \hat{\Sigma}) \cdot o_p(1) 
\]
as \tau = T - t \to 0. We say \( \hat{\Sigma} \) is an AAA of \( \Sigma^B \) if there exist a continuous function \( \varphi \) on \( \mathbb{R} \) and a continuous process \( \Psi = \{\Psi_t\} \) on \( [0,T] \) such that
\[
\left| \frac{d}{dt} \langle S \rangle + 2x \frac{d}{dt} \langle S, \log \hat{\Sigma} \rangle + x^2 \frac{d}{dt} \langle \log \hat{\Sigma} \rangle - \hat{\Sigma}^2 + 2\hat{\Sigma} \partial_t \hat{\Sigma} \right| \leq \varphi(\Psi \hat{\Sigma}) \cdot o_p(1) 
\]
as \tau = T - t \to 0.

**Remark 2.1.** In [8], as usual, \( o_p(1) \) is interpreted as a term which converges to 0 in probability. This is however not really reflecting the idea behind AAA in that a constant term \( a = a \hat{\Sigma}/\hat{\Sigma} \) also has a form of \( \varphi(\Psi \hat{\Sigma}) \cdot o_p(1) \) when \( \hat{\Sigma} \to \infty \) in probability. We indeed have such a divergence for any fixed \( K > 0 \) when considering the rough SABR approximation. To remedy this, in the present article, we interpret \( o_p(1) \) as a term which converges to 0 uniformly in the strike price \( K > 0 \).

**Remark 2.2.** Any locally bounded function on \( \mathbb{R} \) is dominated by a continuous function. Therefore the continuity property of \( \varphi \) in the definition of AAA can be replaced by the local boundedness of \( \varphi \).

The first observation of this study is the following simplification by Itô’s formula: for a continuous semimartingale \( X \) and a positive continuous semimartingale \( \hat{\Sigma} \),
\[
d\langle X \rangle - 2X d\langle X, \log \hat{\Sigma} \rangle + X^2 d\langle \log \hat{\Sigma} \rangle = \hat{\Sigma}^2 d\left( \frac{X}{\hat{\Sigma}} \right). 
\]
Recalling that \( k = \log K/S \) and \( x = K - S \) and so that \( dk = -\log S \) and \( dx = -dS \), we conclude that (1) and (2) are respectively equivalent to
\[
\left| \frac{d}{dt} \frac{k}{\hat{\Sigma}} - 1 + \frac{2\tau D}{\hat{\Sigma}} - \tau \frac{d}{dt} \langle k, \log \hat{\Sigma} \rangle - \frac{\tau^2}{4} \frac{d}{dt} \langle \hat{\Sigma} \rangle \right| \leq \varphi(\Psi \hat{\Sigma}) \cdot o_p(1) 
\]
and
\[
\left| \frac{d}{dt} \left( \frac{x}{\hat{\Sigma}} \right) - 1 + \frac{2\tau D}{\hat{\Sigma}} \right| \leq \frac{\varphi(\Psi \hat{\Sigma})}{\hat{\Sigma}^2} \cdot o_p(1). 
\]
We conclude this section with some additional definitions.

**Definition 2.2.** We say \( f \) is a \( C^{1a} \) function if \( f \) is differentiable, the derivative \( f' \) is absolutely continuous, and the Radon-Nikodym derivative \( f'' \) of \( f' \) is locally
bounded on the domain of \( f \). We say \( g \) is a \( C^{2+} \) function if \( g \) is \( C^2 \) and

\[
f(x) = \begin{cases} 
\frac{x}{g(x)} & x \neq 0, \\
\frac{1}{g'(0)} & x = 0 
\end{cases}
\]

is a positive \( C^{1a} \) function on \( \mathbb{R} \).

**Remark 2.3.** By the Tanaka formula, Itô’s formula for \( C^2 \) functions remains true for \( C^{1a} \) functions (see e.g., Section 1, Chapter VI of [13]).

**Remark 2.4.** By Lemma [A.1] if \( g \) is \( C^3 \) with \( g(0) = 0 \) and \( g' > 0 \), then \( g \) is a \( C^{2+} \) function.

### 3. The BBF and SABR Formulae

The BBF formula refers to an approximation formula

\[
\hat{\Sigma} = \frac{\log \frac{K}{S}}{\int_S^K \frac{ds}{v(s, T)}}
\]

under a local volatility model

(4) \( dS_t = \nu(S_t, t)dW_t, \)

where \( W \) is a standard Brownian motion. We assume that \( \nu \) is a positive \( C^{2,0} \) function on \((0, \infty) \times [0, T]\) and \( S \) is positive and continuous on \([0, T]\). Here, we interpret

\[
\hat{\Sigma}_t = \lim_{s \to K} \frac{\log \frac{K}{S}}{\int_S^K \frac{ds}{v(s, T)}} = \frac{\nu(K, T)}{K}
\]

when \( S_t = K \). By Berestycki et al. [2], we know that \( \lim_{s \to T} \Sigma_{t}^{\text{BS}} = \hat{\Sigma}_T \). In contrast to a technical argument in [2], here we can easily verify the BBF formula in the sense that it provides an AAA.

**Proposition 3.1.** Let \( f \) be a positive \( C^{1a} \) function on \((0, \infty) \) with \( f(K) \neq 0 \), and let

\[
\hat{\Sigma} = f(S).
\]

Then, \( \hat{\Sigma} \) is an AAA of \( \Sigma_{t}^{\text{BS}} \) under (4) if and only if

\[
f(s) = \frac{\log \frac{K}{s}}{\int_S^K \frac{ds}{v(s, T)}}
\]

for \( s \neq K \) and \( f(K) = \frac{\nu(K, T)}{K} \).

**Proof.** Let

\[
g(s) = \frac{\log \frac{K}{s}}{f(s)}.
\]
Then, \( g \) is a \( C^1 \) function and \( \frac{k}{\Sigma} = g(S) \). By the Itô-Tanaka formula,

\[
d\left( \frac{k}{\Sigma} \right) = g'(S)^2d\langle S \rangle = g'(S)^2\nu(S,t)^2dt
\]

and \( \dot{D}, \dot{\Sigma}^{-1}, \) and \( \frac{d}{dt}\langle \Sigma \rangle \) are \( O_p(1) \) as \( t \to T \). Therefore (3) is satisfied if and only if

\[
|g'(s)v(s,T)| = 1.
\]

Since \( g(0) = 0 \) and \( f \) is positive, this is equivalent to

\[
g(s) = \int_s^K \frac{dx}{v(s,t)}.
\]

\( \square \)

The same argument with \( k \) replaced by \( x = K - S \) shows that

\[
\dot{\Sigma} = \frac{K - S}{\int_s^K ds \beta(s)}
\]

is an AAA of \( \Sigma^B \) under (4).

The SABR formula refers to

(5) \( \dot{\Sigma} = \frac{vk}{g(Y)}, \quad Y = \frac{v}{\alpha} \int_s^K ds \beta(s), \quad g(y) = -\log \frac{\sqrt{1 + 2\rho y + y^2} - y - \rho}{1 - \rho} \)

proposed by Hagan et al. \( [10] \) as an approximation of \( \Sigma^B \) under the SABR model

(6) \( dS_t = \alpha_t\beta(S_t)dZ_t, \quad d\alpha_t = \nu \alpha_t dW_t, \quad d\langle Z, W \rangle_t = \rho dt \)

with \( d\langle Z \rangle_t = d\langle W \rangle_t = dt \), where \( \rho \in (-1, 1) \) and \( \nu > 0 \) are constants, and \( \beta \) is a positive \( C^2 \) function on \((0, \infty)\). Here we assume that \( S \) is positive and continuous on \([0, T]\). The validity of the approximation has been discussed by Berestycki et al. \( [3] \), Osajima \( [12] \) and others. In the lognormal case, that is, \( \beta(s) = s \), Balland \( [1] \) shows that the approximation (5) is, in our terminology, an AAA. Our alternative expression (3) allows us to observe it is the case in general.

**Proposition 3.2.** Let \( g \) be a \( C^{2+} \) function on \( \mathbb{R} \) and

\[
\dot{\Sigma} = \frac{vk}{g(Y)}, \quad Y = \frac{v}{\alpha} \int_s^K ds \beta(s),
\]

Then, \( \dot{\Sigma} \) is an AAA of \( \Sigma^B \) under (6) if and only if

(7) \( g(y) = -\log \frac{\sqrt{1 + 2\rho y + y^2} - y - \rho}{1 - \rho} \).
Proof. Since 
\[ \frac{k}{\hat{\Sigma}} = \frac{g(Y)}{v} \]
and
\[ dY = -\frac{\nu}{\alpha\beta(S)}dS - \frac{Y}{\alpha}da + \text{drift} = -\nu dZ - \nu Y dW + \text{drift}, \]
we have
\[ d\left( \frac{k}{\hat{\Sigma}} \right) = \frac{g'(Y)^2}{\nu^2}d\langle Y \rangle = \frac{g'(Y)^2(1 + 2\rho Y + Y^2)}{1 - \tau}dt. \]
By the Itô-Tanaka formula, \( \hat{D}, \hat{\Sigma}^{-1}, \) and \( \frac{d}{dt}(\hat{\Sigma}) \) are \( O(1) \) as \( t \to T \). Therefore, (3) is satisfied if and only if \( g \) solves
\[ g'(y)^2(1 + 2\rho y + y^2) = 1. \]
The unique solution of this ordinary differential equation with \( g(0) = 0 \) and \( g(y)/y > 0 \) for \( y \neq 0 \) is the one given by (7). □

The same argument with \( k \) replaced by \( x = K - S \) shows that
\[ \hat{\Sigma} = \frac{\nu x}{g(Y)} \]
is an AAA of \( \Sigma^B \) under (6).

4. The rough SABR formula

Here we consider the rough SABR formula proposed by Fukasawa and Gatheral [8]. Suppose a positive continuous local martingale \( S \) follows
\[ dS_t = \alpha_t \xi_t dZ_t, \quad d\xi_t = \zeta(s-t)\xi_t(s) dW_t, \quad d\langle Z, W \rangle_t = \rho dt \]
with \( d\langle Z \rangle_t = d\langle W \rangle_t = dt \), where \( \alpha_t = \sqrt{\xi_t}, \quad \zeta(t) = \eta \sqrt{2H(t)}(1-2H)^{1/2}, \quad \beta \) is a positive \( C^2 \) function on \((0,\infty)\) and \( H \in (0,1/2) \), \( \rho \in (-1,1) \), and \( \eta > 0 \) are constants. The approximation formula to be examined is
\[ \hat{\Sigma} = \frac{\zeta(\tau)k}{g(Y)}, \quad Y = \frac{\zeta(\tau)}{U} \int_S^K \frac{ds}{\beta(s)} \quad U = \sqrt{\frac{1}{\tau} \int_t^{\tau} \xi_t(s) ds}, \]
where \( g \) is a solution of the differential equation
\[ g'(y)^2 \left( 1 + 2\rho \frac{y}{2H + 1} + \frac{y^2}{(2H + 1)^2} \right) = 1 - (1 - 2H) \left( 1 - \frac{yg'(y)}{g(y)} \right) \]
with
\[ g(0) = 0, \quad g'(0) > 0. \]
See Lemma [A.2] below for the unique existence of the solution. Further by Lemmas [A.1] and [A.2] the solution \( g \) is a \( C^2 \) function. In the lognormal case, that is, \( \beta(s) = s \), it is shown in [8] that the formula (9) gives an AAA of \( \Sigma^B^S \). It is however left open in [8] whether (9) is an AAA for general \( \beta \).
Theorem 4.1. Let $g$ be a $C^2$ function and let $\hat{\Sigma}$ be defined as in (9). Then, $\hat{\Sigma}$ is an AAA of $\Sigma^{BS}$ under (8) if and only if $g$ solves (10) with (11).

Proof. Since

$$\frac{k}{\Sigma} = \frac{g(Y)}{\zeta(\tau)}$$

and

$$dY = -\frac{\zeta(\tau)}{U\beta(S)} dS - \frac{Y}{U} dU - \frac{\zeta'(\tau)}{\zeta(\tau)} Y dt$$

(12)

we have

$$d \left( \frac{k}{\Sigma} \right) = \frac{g'(Y)^2}{\zeta(\tau)^2} d\langle Y \rangle = g'(Y)^2 \left( \frac{\alpha^2}{U^2} dt + \frac{2 \alpha Y}{U\zeta(\tau)} d\langle Z \log U \rangle + \frac{Y^2}{\zeta(\tau)^2} d\langle \log U \rangle \right).$$

As is observed in [8],

$$\frac{dU}{U} = \frac{1}{2} \zeta(\tau) R dW + \frac{1}{2} \left( \frac{1}{1 - \alpha^2 U^2} - \frac{1}{4} \zeta(\tau)^2 R^2 \right) dt,$$

(13)

where

$$R_t = \int_t^T \zeta(s-t)\xi_t(s)ds, \quad \frac{1}{\zeta(\tau)} \int_t^T \xi_t(s)ds.$$

Therefore,

$$\frac{d}{dt} \langle Z \log U \rangle = \frac{1}{2} \zeta(\tau) R \rho, \quad \frac{d}{dt} \langle \log U \rangle = \frac{1}{4} \zeta(\tau)^2 R^2$$

and so

$$\frac{d}{dt} \left( \frac{k}{\Sigma} \right) = g'(Y)^2 \left( \frac{\alpha^2}{U^2} + \frac{\alpha}{U} R \rho Y + \frac{R^2}{4} Y^2 \right).$$

(14)

On the other hand, recalling that $f(y) := y/g(y)$ is a positive $C^1$ function,

$$\hat{\Sigma} = Uf(Y)\hat{\Sigma}^0, \quad \hat{\Sigma}^0 = \frac{k}{\int_0^T ds / \beta(s)}$$

and so

$$\frac{d\hat{\Sigma}}{\Sigma} = d \log \hat{\Sigma} + \frac{1}{2} d \langle \log \hat{\Sigma} \rangle$$

$$= d \log U + d \log f(Y) + d \log \hat{\Sigma}^0 + \ldots$$
Therefore, denoting by $\hat{d}dt$ the drift part of $d\hat{\Sigma}$ and using (12) and (13), we have

$$\frac{2\tau\hat{d}dt}{\hat{\Sigma}} = (1 - 2H) \left( 1 - \frac{\alpha^2}{U^2} + \frac{Yf'(Y)}{f(Y)} \frac{\alpha^2}{U^2} \right) + \varphi(Y) \cdot O_p(\tau^{2H})$$

(15)

for some locally bounded function $\varphi$. By the continuity of $f$, $\varphi(Y)$ is dominated by $\hat{\varphi}(\hat{\Sigma})$ with a continuous function $\hat{\varphi}$ and a continuous process $\hat{\varphi}$. By Lemma A.1 of [8], we know $\alpha/U \to 1$ and $R \to 1/(H + 1/2)$. Then (14) and (15) imply that (3) is satisfied if and only if (10) holds. □

The same argument with $k$ replaced by $x = K - S$ shows that

$$\hat{\Sigma} = \frac{\hat{\zeta}(\tau)x}{g(Y)}$$

is an AAA of $\Sigma_B$ under (8).

**Appendix A. Lemmas**

**Lemma A.1.** Let $g$ be a $C^2$ function on $\mathbb{R}$ with

$$g(y) = g'(0)y + g''(0)\frac{y^2}{2} + O(y^3),$$

$$g'(y) = g'(0) + g''(0)y + O(y^2),$$

$$g''(y) = g''(0) + O(y)$$

as $y \to 0$. Then, the function $G$ defined by

$$G(y) = \begin{cases} 
\frac{g(y)}{y} & y \neq 0, \\
g'(0) & y = 0 
\end{cases}$$

is $C^1$ with absolutely continuous derivative $G'$ of which the Radon-Nikodym derivative $G''$ is locally bounded.

**Proof.** By (16),

$$\frac{G(y) - G(0)}{y} = \frac{1}{2}g''(0) + O(y)$$

and

$$G'(y) = \frac{1}{y} \left( g'(y) - \frac{g'(0)}{y} \right) = \frac{1}{2}g''(0) + O(y)$$

as $y \to 0$. In particular, $G$ is $C^1$. Further,

$$G''(y) = \frac{1}{y} \left( g''(y) - \frac{2}{y} \left( g'(y) - \frac{g'(0)}{y} \right) \right) = O(1)$$

as $y \to 0$ by (16), which means that $G'$ is absolutely continuous and $G''$ is locally bounded. □
Lemma A.2. The ordinary differential equation (10) with (11) has a unique $C^1$ solution. The solution $g$ is $C^2$ with (16) and satisfies $g(y)/y > 0$ for all $y \neq 0$.

Proof. Regarding (10) as a quadratic equation in $g'(y)$, we get
\[
g'(y) = \frac{(1 - 2H)f(y) \pm \sqrt{(1 - 2H)^2 f(y)^2 + 8Hq(y)}}{2q(y)},
\]
where
\[
f(y) = \frac{y}{g(y)}, \quad q(y) = 1 + 2\rho \frac{y}{2H + 1} + \frac{y^2}{2(2H + 1)^2}.
\]
Note that $q(y) \geq 1 - \rho^2 > 0$. By (11), we should solve
\[
g'(y) = \varphi\left(y, \frac{y}{g(y)}\right),
\]
where
\[
\varphi(y, z) = \frac{(1 - 2Hz + \sqrt{(1 - 2H)^2 z^2 + 8Hq(y)})}{2q(y)}.
\]
Letting $y \to 0$, we should have
\[
g'(0) = \varphi\left(0, \frac{1}{g'(0)}\right)
\]
of which the unique solution is $g'(0) = 1$. The result then follows from Lemma A.3 below. \qed

Lemma A.3. Let $\varphi : \mathbb{R} \times [0, \infty) \to (0, \infty)$ be a $C^2$ function with
\[
\inf_{(y, z) \in \mathbb{R} \times [0, \infty)} \frac{\partial \varphi}{\partial z}(y, z) \geq 0,
\]
\[
\sup_{(y, z) \in \mathbb{R} \times [0, \infty)} \frac{\partial \varphi}{\partial z}(y, z) < \infty,
\]
(20)
\[
\frac{\partial \varphi}{\partial z}(0, 1) < 1,
\]
(21)
\[
\varphi(0, 1) = 1,
\]
and such that the equation
\[
\alpha = \varphi\left(0, \frac{1}{\varphi(0, \frac{1}{\alpha})}\right), \quad \alpha > 0
\]
has the unique solution $\alpha = 1$. Then, there exists a unique solution of the differential equation
\[
g'(y) = \varphi\left(y, \frac{y}{g(y)}\right), \quad g(0) = 0
\]
The solution $g$ is $C^2$ with (16) and satisfies $g(y)/y > 0$ for $y \neq 0$.\]
Proof. Step 1: Here we show that there exists a $C^1$ function $g$ which satisfies the integral equation $g = \Phi[g]$, where

$$\Phi[g](y) = \int_0^y \varphi\left(u, \frac{u}{g(u)}\right) du.$$  

Since $\varphi(y, z)$ is increasing in $z$, we have $\Phi[g_0](y)/y \geq \Phi[g_1](y)/y$ for all $y \neq 0$ if $0 < g_0(y)/y \leq g_1(y)/y$ for all $y \neq 0$. Let

$$g_0(y) = \int_0^y \varphi(u, 0) du.$$  

and define $g_{n+1} = \Phi[g_n]$ for $n \geq 0$. By (18) we have $0 < g_0(y)/y \leq g_1(y)/y$ for all $y \neq 0$. Then by the above mentioned monotonicity of $\Phi$, we have $g_1(y)/y \geq g_2(y)/y$ for all $y \neq 0$. Again by (18) we have $g_2(y)/y \geq g_0(y)/y$ for all $y \neq 0$. This inequality then implies $g_3(y)/y \leq g_1(y)/y$ for all $y \neq 0$. By induction we obtain

$$0 < \frac{g_0(y)}{y} \leq \frac{g_2n(y)}{y} \leq \frac{g_{2n+2}(y)}{y} \leq \frac{g_{2n+3}(y)}{y} \leq \frac{g_{2n+1}(y)}{y}$$

for all $n \geq 0$ and for all $y \neq 0$. By the monotonicity of the sequences there exist $g_\varepsilon(y)$ and $g_o(y)$ such that

$$\frac{g_{2n}(y)}{y} \leq \lim_{n \to \infty} \frac{g_{2n}(y)}{y} = \frac{g_\varepsilon(y)}{y} \leq \frac{g_o(y)}{y} = \lim_{n \to \infty} \frac{g_{2n+1}(y)}{y} \leq \frac{g_{2n+1}(y)}{y}$$

for each $y \neq 0$. By the dominated convergence theorem, $(g_\varepsilon, g_o)$ is a solution of the coupled integral equation

$$g_\varepsilon = \Phi[g_o], \quad g_o = \Phi[g_\varepsilon].$$

We are going to show $g_\varepsilon = g_o$. Letting $y \to 0$ in (24), we have

$$\alpha_{2n} \leq \liminf_{y \to 0} \frac{g_\varepsilon(y)}{y} \leq \limsup_{y \to 0} \frac{g_o(y)}{y} \leq \alpha_{2n+1},$$

where, by L’Hôpital’s rule,

$$\alpha_n = \lim_{y \to 0} \frac{g_n(y)}{y} = \lim_{y \to 0} \varphi\left(y, \frac{y}{g_{n-1}(y)}\right) = \varphi\left(0, \frac{1}{\alpha_{n-1}}\right)$$

for $n \geq 1$ and $\alpha_0 = \varphi(0, 0) > 0$. By (23), both $\{\alpha_{2n}\}$ and $\{\alpha_{2n+1}\}$ are bounded monotone sequences and so convergent. The limit $\alpha$ of each sequence has to be a solution of (22) and so, $\alpha = 1$ by the assumption. We then conclude that $\alpha_n$ itself converges to 1. Now taking $n \to \infty$ in (25), we have

$$\lim_{y \to 0} \frac{g_\varepsilon(y)}{y} = \lim_{y \to 0} \frac{g_o(y)}{y} = 1.$$  

Therefore for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that if $|y| \leq \delta(\varepsilon)$,

$$1 + \varepsilon \geq \frac{y}{g_o(y)} \geq \frac{y}{g_\varepsilon(y)} \geq 1 - \varepsilon.$$  

(27)
On the other hand, by (20), there exists $\delta_1 > 0$ such that

\begin{equation}
\sup_{|y| \leq \delta_1, |z| \leq \delta_1} \left| \frac{\partial \varphi}{\partial z}(y, z) \right| < 1 - \frac{1}{2} \left( 1 - \frac{\partial \varphi}{\partial z}(0, 1) \right).
\end{equation}

Let $\delta = \min\{\delta_1, \delta(\delta_1), \delta(\epsilon)\}$ and

$$
\|f\|_b = \sup_{|y| \leq \delta} \left| \frac{f(y)}{y} \right|
$$

for a function $f$. Then, using (27) and (28),

\begin{align*}
\|g_0 - g_c\|_b & \leq \sup_{|y| \leq \delta} \frac{1}{y} \int_0^y \left| \varphi\left(u, \frac{u}{g_c(u)}\right) - \varphi\left(u, \frac{u}{g_o(u)}\right) \right| du \\
& \leq \sup_{|y| \leq \delta} \left| \frac{\partial \varphi}{\partial z}(y, z) \right| \sup_{|y| \leq \delta} \frac{1}{y} \int_0^y \frac{u^2}{g_c(u)g_o(u)} \left| \frac{g_c(u)}{u} - \frac{g_o(u)}{u} \right| du \\
& \leq (1 + \epsilon)^2 \left( 1 - \frac{1}{2} \left( 1 - \frac{\partial \varphi}{\partial z}(0, 1) \right) \right) \|g_o - g_c\|_b.
\end{align*}

We can take such $\epsilon > 0$ that

\begin{equation}
(1 + \epsilon)^2 \left( 1 - \frac{1}{2} \left( 1 - \frac{\partial \varphi}{\partial z}(0, 1) \right) \right) < 1
\end{equation}

to conclude that $g_0(y) = g_c(y)$ for $|y| \leq \delta$. For $y > \delta$, by the Lipschitz continuity of $\varphi$ in $z$, (24) and (25), we have

$$
|g_0(y) - g_c(y)| \leq L \int_0^y \frac{y}{g_0(y)^2} |g_0(u) - g_c(u)| du
$$

for a constant $L$. Then, by Gronwall’s lemma we have $g_0(y) = g_c(y)$ for $y \geq \delta$. Similarly we obtain $g_o(y) = g_c(y)$ for $y \leq -\delta$. Thus $g_0 = g_c$ is a solution of $g = \Phi[g]$. By (26), the solution $g$ satisfies

$$
g'(0) = 1 = \varphi(0, 1) = \lim_{y \to 0^-} \varphi\left(y, \frac{y}{g(y)}\right) = \lim_{y \to 0^+} g'(y)
$$

and so, is a $C^1$ function.

**Step 2:** Here we show that a solution of $g = \Phi[g]$ is unique. Let $g$ and $\hat{g}$ be two solutions. By (18), we have

$$
0 < \frac{y}{g(y)} \leq \frac{y}{g_0(y)}, \quad 0 < \frac{y}{\hat{g}(y)} \leq \frac{y}{g_0(y)}.
$$

Therefore by the Lipschitz continuity of $\varphi$ in $z$, if there exists $y_0 > 0$ such that $g(y_0) = \hat{g}(y_0)$, then

\begin{equation}
|g(y) - \hat{g}(y)| \leq L \int_{y_0}^y \frac{y}{g_0(y)^2} |g(u) - \hat{g}(u)| du
\end{equation}

for $y \geq y_0$. We have $g(y) = \hat{g}(y)$ for $y \geq y_0$ by Gronwall’s lemma. Now, suppose that there exists $y > 0$ such that $g(y) > \hat{g}(y)$. Then, from the
above observation, we have \( g(u) > \hat{g}(u) \) for all \( u \in (0, y) \). However, the monotonicity of \( \varphi \) in \( z \) implies that
\[
0 < g(y) - \hat{g}(y) = \int_0^y \varphi \left( u, \frac{u}{g(u)} \right) - \varphi \left( u, \frac{u}{\hat{g}(u)} \right) du \leq 0
\]
that is a contradiction. If there exists \( y < 0 \) such that \( g(y) > \hat{g}(y) \), then, again by a similar argument we conclude that \( g(u) > \hat{g}(u) \) for all \( u \in (y, 0) \). This results in a contradiction as
\[
0 < g(y) - \hat{g}(y) = -\int_y^0 \varphi \left( u, \frac{u}{\hat{g}(u)} \right) - \varphi \left( u, \frac{u}{g(u)} \right) du < 0.
\]
Therefore we have \( g = \hat{g} \).

**Step 3:** It remains to show that the solution \( g \) is \( C^2 \) with (16). Let
\[
\hat{g}(y) = y + \frac{\beta y^2}{2},
\]
where
\[
\beta = \frac{\frac{\partial \varphi}{\partial y}(0, 1)}{1 + \frac{\partial \varphi}{2 \partial z}(0, 1)}.
\]
We have
\[
\frac{g(y)}{y} = \frac{\hat{g}(y)}{y} + O(y^2)
\]
as \( y \to 0 \). Indeed, since
\[
\varphi \left( y, \frac{y}{\hat{g}(y)} \right) = \varphi(0, 1) + \frac{\partial \varphi}{\partial y}(0, 1)y + \frac{\partial \varphi}{\partial z}(0, 1) \left( \frac{1}{1 + \frac{\beta y}{2}} - 1 \right) + O(y^2)
\]
(30)
\[
= 1 + \left( \frac{\partial \varphi}{\partial y}(0, 1) - \frac{\beta}{2} \frac{\partial \varphi}{\partial z}(0, 1) \right) y + O(y^2)
\]
\[
= \hat{g}'(y) + O(y^2),
\]
in light of (28), there exists \( L \in (0, 1) \) such that
\[
\sup_{0 < |y| \leq a} \left| \frac{g(y)}{y} - \frac{\hat{g}(y)}{y} \right| \leq \sup_{0 < |y| \leq a} \left| \frac{1}{y} \int_0^y \varphi \left( u, \frac{u}{g(u)} \right) - \varphi \left( u, \frac{u}{\hat{g}(u)} \right) du \right| + O(a^2)
\]
\[
\leq L \sup_{0 < |y| \leq a} \left| \frac{g(y)}{y} - \frac{\hat{g}(y)}{y} \right| + O(a^2).
\]
This implies
\[
\sup_{0 < |y| \leq a} \left| \frac{g(y)}{y} - \frac{\hat{g}(y)}{y} \right| = O(a^2)
\]
and in particular, (29). Now, from (29), we have

\[
\frac{1}{y} \left( \frac{g(y)}{y} - 1 \right) = \frac{1}{y} \left( \frac{\hat{g}(y)}{y} - 1 \right) + O(y)
\]

(31)

as \( y \to 0 \). Further, by (29) and (30),

\[
\frac{g'(y) - 1}{y} = \frac{\hat{g}'(y) - 1}{y} + O(y) = \beta + O(y)
\]

(32)
as \( y \to 0 \). On the other hand,

\[
g''(y) = \frac{\partial \varphi}{\partial y} \left( y, \frac{y}{g(y)} \right) + \frac{\partial \varphi}{\partial z} \left( y, \frac{y}{g(y)} \right) \left( \frac{y}{g(y)} \right)' \]

(33)

\[
= \frac{\partial \varphi}{\partial y} \left( y, \frac{y}{g(y)} \right) + \frac{\partial \varphi}{\partial z} \left( y, \frac{y}{g(y)} \right) \frac{1}{g(y)} \left( 1 - \frac{y}{g(y)} - \frac{y}{g(y)} (g'(y) - 1) \right) \]

\[
= \frac{\partial \varphi}{\partial y} (0, 1) + \frac{\partial \varphi}{\partial z} (0, 1) \left( \frac{\beta}{2} - \beta \right) + O(y)
\]

\[
= \beta + O(y)
\]
as \( y \to 0 \). Therefore, \( g \) is \( C^2 \) with (16).

\[\square\]

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