Automorphisms of elementary adjoint Chevalley groups

of types $A_l, D_l, E_l$ over local rings with $1/2$ \footnote{The work is supported by the Russian President grant MK-2530.2008.1 and by the grant of Russian Fond of Basic Research 08-01-00693.}

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Abstract.

In this paper we prove that every automorphism of an elementary adjoint Chevalley group of types $A_l, D_l, E_l$, over local commutative ring with $1/2$ is a composition of a ring automorphism and conjugation by some matrix from the normalizer of the Chevalley group in $GL(V)$ ($V$ is the adjoint representation space).

INTRODUCTION

Let $G_{ad}$ be a Chevalley–Demazure group scheme associated with an irreducible root system $\Phi$ of type $A_l$ ($l \geq 2$), $D_l$ ($l \geq 4$), $E_l$ ($l = 6, 7, 8$); $G_{ad}(\Phi, R)$ be a set of points $G_{ad}$ with values in a commutative ring $R$; $E_{ad}(\Phi, R)$ be the elementary subgroup of $G_{ad}(\Phi, R)$, where $R$ is a commutative ring with 1. In this paper we describe automorphisms of groups $E_{ad}(\Phi, R)$ over local commutative rings with 1/2.

Similar results for Chevalley groups over fields were proved by R. Steinberg \cite{48} for finite case and by J. Humphreys \cite{34} for infinite case. Many papers were devoted to description of automorphisms of Chevalley groups over different commutative rings, we can mention here the papers of Borel–Tits \cite{10}, Carter–Chen Yu \cite{14}, Chen Yu \cite{15}–\cite{19}. E. Abe \cite{1} proved that automorphisms are standard for Noetherian rings, it could completely close the question about automorphisms of Chevalley groups over arbitrary commutative rings (for the case of systems of rank $\geq 2$ and rings with 1/2), but in consideration of adjoint elementary groups in the paper \cite{1} there is a mistake, that can not be corrected by methods of this paper. Namely, in the proof of Lemma 11 the author uses the fact that $\text{ad}(x_\alpha)^2 = 0$ for all long roots, but it is not true in the adjoint representation. The main problem here is the case of groups of type $E_8$, since in all other cases Chevalley groups have a representation with the property $\text{ad}(x_\alpha)^2 = 0$ for all long roots, but in the case $E_8$ there are no such representations.

In the given paper we consider also this case $E_8$, and it can help to close the question about automorphisms of Chevalley groups over commutative rings with 1/2.

We generalize some methods of V.M. Petechuk \cite{41} to prove the main theorem.

Note that we consider the cases $A_l$, $D_l$, $E_l$, but the case $A_l$ was completely studied by the papers of W.C. Waterhouse \cite{61}, V.M. Petechuk \cite{43}, Fuan Li and Zunxian Li \cite{37}, and also for rings without 1/2. The paper of I.Z. Golubchik and A.V. Mikhailov \cite{29} covers the case $C_l$, that is not considered in the present paper.
1. Definitions and formulation of the main theorem

We fix a root system $\Phi$, that has one of types $A_l$ ($l \geq 2$), $D_l$ ($l \geq 4$), or $E_l$ ($l = 6, 7, 8$), with the system of simple roots $\Delta$, the set of positive (negative) roots $\Phi^+$ ($\Phi^-$), the Weil group $W$. Recall that in our case any two roots are conjugate under the action of the Weil group. Let $|\Phi^+| = m$. More detailed texts about root systems and their properties can be found in the books [35], [12].

Suppose now that we have a semisimple complex Lie algebra $L$ with the Cartan subalgebra $H$ (more details about semisimple Lie algebras can be found in the book [35]).

Lie algebra $L$ has a decomposition $L = H \oplus \sum_{\alpha \neq 0} L_{\alpha}$,

$$L_{\alpha} := \{x \in L \mid [h, x] = \alpha(h)x \text{ for every } h \in H\},$$

and if $L_{\alpha} \neq 0$, then $\dim L_{\alpha} = 1$, all nonzero $\alpha \in H$ such that $L_{\alpha} \neq 0$, form some root system $\Phi$. The root system $\Phi$ and the semisimple Lie algebra $L$ over $\mathbb{C}$ uniquely (up to automorphism) define each other.

On the Lie algebra $L$ we can introduce a bilinear Killing form $\kappa(x, y) = tr(\text{ad} x \text{ ad} y)$, that is non-degenerated on $H$. Therefore we can identify the spaces $H$ and $H^*$. We can choose a basis $\{h_1, \ldots, h_l\}$ in $H$ and for every $\alpha \in \Phi$ elements $x_{\alpha} \in L_{\alpha}$ so that $\{h_i; x_{\alpha}\}$ is a basis in $L$ and for every two elements of this basis their commutator is an integral linear combination of the elements of the same basis. This basis is called a Chevalley basis.

Introduce now elementary Chevalley groups (see [47]).

Let $L$ be a semisimple Lie algebra (over $\mathbb{C}$) with a root system $\Phi$, $\pi : L \to \mathfrak{gl}(V)$ be its finitely dimensional faithful representation (of dimension $n$). If $H$ is a Cartan subalgebra of $L$, then a functional $\lambda \in H^*$ is called a weight of a given representation, if there exists a nonzero vector $v \in V$ (that is called a weight vector) such that for any $h \in H$, $\pi(h)v = \lambda(h)v$.

In the space $V$ in the Chevalley basis all operators $\pi(x_{\alpha})^k/k!$ for $k \in \mathbb{N}$ are written as integral (nilpotent) matrices. An integral matrix also can be considered as a matrix over an arbitrary commutative ring with 1. Let $R$ be such a ring. Consider matrices $n \times n$ over $R$, matrices $\pi(x_{\alpha})^k/k!$ for $\alpha \in \Phi, k \in \mathbb{N}$ are included in $M_n(R)$.

Now consider automorphisms of the free module $R^n$ of the form

$$\exp(tx_{\alpha}) = x_{\alpha}(t) = 1 + t\pi(x_{\alpha}) + t^2\pi(x_{\alpha})^2/2 + \cdots + t^k\pi(x_{\alpha})^k/k! + \ldots$$

Since all matrices $\pi(x_{\alpha})$ are nilpotent, we have that this series is finite. Automorphisms $x_{\alpha}(t)$ are called elementary root elements. The subgroup in $\text{Aut}(R^n)$, generated by all $x_{\alpha}(t), \alpha \in \Phi, t \in R$, is called an elementary Chevalley group (notation: $E_{\pi}(\Phi, R)$).

In elementary Chevalley group we can introduce the following important elements and subgroups:

| $w_{\alpha}(t) = x_{\alpha}(t)x_{-\alpha}(-t^{-1})x_{\alpha}(t), \alpha \in \Phi, t \in R^*$; |
| $h_{\alpha}(t) = w_{\alpha}(t)w_{\alpha}(1)^{-1};$ |
| $N$ is generated by all $w_{\alpha}(t), \alpha \in \Phi, t \in R^*$; |
— $H$ is generated by all $h_\alpha(t)$, $\alpha \in \Phi$, $t \in R^*$.

The action of $x_\alpha(t)$ on the Chevalley basis is described in [13], [57].

It is known that the group $N$ is a normalizer of $H$ in elementary Chevalley group, the quotient group $N/H$ is isomorphic to the Weil group $W(\Phi)$.

All weights of a given representation (by addition) generate a lattice (free Abelian group, where every $\mathbb{Z}$-basis is also a $\mathbb{C}$-basis in $H^*$), that is called the weight lattice $\Lambda_\pi$.

Elementary Chevalley groups are defined not even by a representation of the Chevalley groups, but just by its weight lattice. Namely, up to an abstract isomorphism an elementary Chevalley group is completely defined by a root system $\Phi$, a commutative ring $R$ with 1 and a weight lattice $\Lambda_\pi$.

Among all lattices we can mark two: the lattice corresponding to the adjoint representation, it is generated by all roots (the root lattice $\Lambda_{ad}$) and the lattice generated by all weights of all representations (the lattice of weights $\Lambda_{sc}$). For every faithful representation $\pi$ we have the inclusion $\Lambda_{ad} \subseteq \Lambda_\pi \subseteq \Lambda_{sc}$. Respectively, we have the adjoint and universal elementary Chevalley groups. In this paper we study adjoint elementary Chevalley groups.

Every elementary Chevalley group satisfies the following conditions:

(R1) $\forall \alpha \in \Phi \forall t, u \in R \ x_\alpha(t)x_\alpha(u) = x_\alpha(t + u)$;
(R2) $\forall \alpha, \beta \in \Phi \forall t, u \in R \ \alpha + \beta \neq 0 \Rightarrow$

$$[x_\alpha(t), x_\beta(u)] = x_\alpha(t)x_\beta(u)x_\alpha(-t)x_\beta(-u) = \prod x_{i\alpha + j\beta}(c_{ij}t^iu^j),$$

where $i, j$ are integers, product is taken by all roots $i\alpha + j\beta$, replacing in some fixed order; $c_{ij}$ are integer numbers not depending of $t$ and $u$, but depending of $\alpha$ and $\beta$ and of order of roots in the product. In the cases under consideration always

$$[x_\alpha(t), x_\beta(u)] = x_{\alpha + \beta}(\pm tu).$$

(R3) $\forall \alpha \in \Phi \ w_\alpha = w_\alpha(1)$;
(R4) $\forall \alpha, \beta \in \Phi \forall t \in R^* \ w_\alpha h_\beta(t)w_\alpha^{-1} = h_{w_\alpha(\beta)}(t)$;
(R5) $\forall \alpha, \beta \in \Phi \forall t \in R^* \ w_\alpha x_\beta(t)w_\alpha^{-1} = x_{w_\alpha(\beta)}(ct)$, where $c = c(\alpha, \beta) = \pm 1$;
(R6) $\forall \alpha, \beta \in \Phi \forall t \in R^* \forall u \in R \ h_\alpha(t)x_\beta(u)h_\alpha(t)^{-1} = x_\beta(t^{(\beta, \alpha)}u)$.

By $X_\alpha$ we denote the subgroup consisting of all $x_\alpha(t)$ for $t \in R$.

We will need two types of automorphisms of elementary Chevalley groups $E_{ad}(\Phi, R)$.

**Ring automorphisms.** Let $\rho : R \rightarrow R$ be an automorphism of $R$. The mapping $x \mapsto \rho(x)$ from $E_{ad}(\Phi, R)$ onto itself is an automorphism of the group $E_{ad}(\Phi, R)$, that is denoted by the same letter $\rho$ and is called a ring automorphism of the group $G_{\pi}(\Phi, R)$. Note that for all $\alpha \in \Phi$ and $t \in R$ an element $x_\alpha(t)$ is mapped into $x_\alpha(\rho(t))$.

**Automorphisms-conjugations.** Let $V$ be a representation space of the group $E_{ad}(\Phi, R)$, $C \in \text{GL}(V)$ be some matrix that does not move our Chevalley group:

$$CE_{ad}(\Phi, R)C^{-1} = E_{ad}(\Phi, R).$$

Then the mapping $x \mapsto CxC^{-1}$ from $E_{\pi}(\Phi, R)$ onto itself is an automorphism of the Chevalley group, that is denoted by $i$ and is called an automorphism-conjugation of the group $E(R)$, induced by the element $C$ of $\text{GL}(V)$. 
Theorem 1. Let \( E_{\text{ad}}(\Phi, R) \) be an elementary Chevalley group with an irreducible root system of type \( A_l \) (\( l \geq 2 \)), \( D_l \) (\( l \geq 4 \)), or \( E_l \) (\( l = 6, 7, 8 \)). \( R \) be a commutative local ring with \( 1/2 \). Then every automorphism of \( E_{\text{ad}}(\Phi, R) \) is a composition of a ring automorphism and an automorphism-conjugation.

Next sections are devoted to the proof of Theorem 1.

2. Replacing the initial automorphism to the special one.

From this section we suppose that \( R \) is a local ring with \( 1/2 \), the Chevalley group is adjoint, in this section root system is arbitrary. In this section we use some reasonings from [41].

Let \( J \) be the maximal ideal (radical) of \( R \), \( k \) the residue field \( R/J \). Then \( E_J \) is the greatest normal proper subgroup of \( E_{\text{ad}}(\Phi, R) \) (see [2]). Therefore, \( E_J \) is invariant under the action of \( \varphi \).

By this reason the automorphism \( \varphi : E_{\text{ad}}(\Phi, R) \to E_{\text{ad}}(\Phi, R) \)
induces an automorphism \( \overline{\varphi} : E_{\text{ad}}(\Phi, R)/E_J = E_{\text{ad}}(\Phi, k) \to E_{\text{ad}}(\Phi, k) \).

The group \( E_{\text{ad}}(\Phi, k) \) is a Chevalley group over field, therefore the automorphism \( \overline{\varphi} \) is standard (see [47]), i.e. it has the form
\[
\overline{\varphi} = i_{\overline{g}} \overline{\rho}, \quad \overline{g} \in N(E_{\text{ad}}(\Phi, k)),
\]
where \( \overline{\rho} \) is a ring automorphism, induced by some automorphism of \( k \).

It is clear that there exists a matrix \( g \in GL_n(R) \) such that its image under factorization \( R \) by \( J \) coincides with \( \overline{g} \). We are not sure that \( g \in N(E_{\text{ad}}(\Phi, R)) \).

Consider a mapping \( \varphi' = i_g^{-1} \varphi \). It is an isomorphism of the group \( E_{\text{ad}}(\Phi, R) \subset GL_n(R) \) onto some subgroup in \( GL_n(R) \), with the property that its image under factorization \( R \) by \( J \) coincides with the automorphism \( \overline{\rho} \).

These arguments prove

Proposition 1. Every matrix \( A \in E_{\text{ad}}(\Phi, R) \) with elements from the subring \( R' \) of \( R \), generated by unit, is mapped under the action of \( \varphi' \) to some matrix from the set
\[
A \cdot GL_n(R, J) = \{ B \in GL_n(R) \mid A - B \in M_n(J) \}.
\]

Let \( a \in E_{\text{ad}}(\Phi, R) \), \( a^2 = 1 \). Then the element \( e = \frac{1}{2}(1 + a) \) is an idempotent in the ring \( M_n(R) \). This idempotent \( e \) defines a decomposition of the free \( R \)-module \( V = R^n \):
\[
V = eV \oplus (1 - e)V = V_0 \oplus V_1
\]
(the modules \( V_0, V_1 \) are free, since every projective module over local field is free [39]). Let \( \overline{V} = \overline{V}_0 \oplus \overline{V}_1 \) be decomposition of the \( k \)-module \( \overline{V} \) with respect to \( \overline{\rho} \), and \( \overline{e} = \frac{1}{2}(1 + \overline{a}) \).

Then we have

Proposition 2. The modules (subspaces) \( \overline{V}_0, \overline{V}_1 \) are images of the modules \( V_0, V_1 \) under factorization by \( J \).
Proof. Let us denote the images of $V_0$, $V_1$ under factorization by $J$ by $\widetilde{V}_0$, $\widetilde{V}_1$, respectively. Since $V_0 = \{x \in V | e x = x\}$, $V_1 = \{x \in V | e x = 0\}$, we have $\varnothing(\overline{x}) = \frac{1}{2}(1 + a(x))e(x)$. Then $\widetilde{V}_0 \subseteq V_0$, $\widetilde{V}_1 \subseteq V_1$.

Let $x = x_0 + x_1$, $x_0 \in V_0$, $x_1 \in V_1$. Then $\varnothing(\overline{x}) = \varnothing(\overline{x}_0) + \varnothing(\overline{x}_1) = \overline{x}_0$. If $\overline{x} \in \widetilde{V}_0$, then $\overline{x} = \overline{x}_0$. □

Let $b = \varphi'(a)$. Then $b^2 = 1$ and $b$ is equivalent to a modulo $J$.

**Proposition 3.** Suppose that $a, b \in E_\pi(\Phi, R)$, $a^2 = b^2 = 1$, $a$ is a matrix with elements from the subring of $R$, generated by the unit, $b$ and $a$ are equivalent modulo $J$, $V = V_0 \oplus V_1$ is a decomposition of $V$ with respect to $a$, $V = V'_0 \oplus V'_1$ is a decomposition of $V$ with respect to $b$. Then $\dim V'_0 = \dim V_0$, $\dim V'_1 = \dim V_1$.

Proof. We have an $R$-basis of the module $V \{e_1, \ldots, e_n\}$ such that $\{e_1, \ldots, e_k\} \subset V_0$, $\{e_{k+1}, \ldots, e_n\} \subset V_1$. It is clear that $\overline{ae}_i = \overline{a e}_i = \left(\sum_{j=1}^n a_{ij} e_j\right) = \sum_{j=1}^n a_{ij}e_j$.

Let $\overline{V} = \overline{V}_0 \oplus \overline{V}_1$, $\overline{V}' = \overline{V}'_0 \oplus \overline{V}'_1$ are decompositions of $k$-module (space) $V$ with respect to $\overline{a}$ and $\overline{b}$. It is clear that $\overline{V}_0 = \overline{V}'_0$, $\overline{V}_1 = \overline{V}'_1$. Therefore, by Proposition 2 the images of the modules $V_0$ and $V'_0$, $V_1$ and $V'_1$ under factorization by $J$ coincide. Let us take such $\{f_1, \ldots, f_k\} \subset V'_0$, $\{f_{k+1}, \ldots, f_n\} \subset V'_1$ that $\overline{f}_i = \overline{e}_i$, $i = 1, \ldots, n$. Since the matrix of transformation from $\{e_1, \ldots, e_n\}$ to $\{f_1, \ldots, f_n\}$ is invertible (it is equivalent to the identical matrix modulo $J$) we have that $\{f_1, \ldots, f_n\}$ is a $R$-basis in $V$. It is clear that $\{f_1, \ldots, f_k\}$ is a $R$-basis in $V'_0$, $\{v_{k+1}, \ldots, v_n\}$ is a $R$-basis in $V'_1$. □

3. Images of $w_{\alpha_i}$

We consider some fixed adjoint Chevalley group $E = E_{ad}(\Phi, R)$ with the root system $A_l$ ($l \geq 2$), $D_l$ ($l \geq 4$), $E_6$, $E_7$ or $E_8$, its adjoint representation in the group $GL_n(R)$ ($n = l + 2m$, where $m$ is the number of positive roots of $\Phi$), with the basis of weight vectors $v_1 = x_{\alpha_1}$, $v_{-1} = x_{-\alpha_1}$, $\ldots$, $v_{n} = x_{\alpha_n}$, $v_{-n} = x_{-\alpha_n}$, $V_1 = l_1$, $\ldots$, $V_l = l_l$, corresponding to the Chevalley basis of the system $\Phi$.

We also have the isomorphism $\varphi'$, described in Section 2.

Consider the matrices $h_{\alpha_i}(-1), \ldots, h_{\alpha_i}(-1)$ in our basis. They have the form $h_{\alpha_i}(-1) = \text{diag}[\pm 1, \ldots, \pm 1, 1, \ldots, 1]$, on $(2j - 1)$-th and $(2j)$-th places we have $-1$ if and only if $\langle \alpha_i, \alpha_j \rangle = -1$. As we see, for all $i$ $h_{\alpha_i}(-1)^2 = 1$.

According to Proposition 3 we know that every matrix $h_i = \varphi''(h_{\alpha_i}(-1))$ in some basis is diagonal with $\pm 1$ on the diagonal, and the number of 1 and $-1$ coincides with its number for the matrix $h_{\alpha_i}(-1)$. Since all matrices $h_i$ commutes, there exists a basis, where all $h_i$ have the same form as $h_{\alpha_i}(-1)$. Suppose that we come to this basis with the help of the matrix $g_1$. It is clear that $g_1 \in GL_n(R, J)$. Consider the mapping $\varphi_1 = g^{-1}_1 \varphi'$. It is also an isomorphism of the
group $E$ onto some subgroup of $GL_n(R)$ such that its image under factorization $R$ by $J$ is $\overline{\rho}$, and $\varphi_1(h_{\alpha_i}(-1)) = h_{\alpha_i}(-1)$ for all $i = 1, \ldots, l$.

Let us consider the isomorphism $\varphi_1$.

Matrices $h_{\alpha_k}(a) = \text{diag}[a_1, 1/a_1, a_2, 1/a_2, \ldots, a_m, 1/a_m, 1, \ldots, 1]$ commute with all $h_{\alpha_i}(-1)$, therefore their images under the isomorphism $\varphi_1$ also commute with all $h_{\alpha_i}(-1)$. Thus, these images have the form

$$\begin{pmatrix} C_1 & 0 & \ldots & 0 & 0 \\ 0 & C_2 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & C_m & 0 \\ 0 & 0 & \ldots & 0 & C \end{pmatrix}, \quad C_i \equiv \begin{pmatrix} \overline{\rho}(a_i) & 0 \\ 0 & \overline{\rho}(1/a_i) \end{pmatrix} \mod J, \quad C \in GL_1(R, J).$$

Every element $w_i = w_{\alpha_i}(1)$ maps (by conjugation) diagonal matrices into diagonal ones, so its image has block-monomial form.

From $\varphi_1(w_i) \equiv w_i \mod J$ it follows that blocks of $\varphi_1(w_i)$ are in the same places as blocks of $w_i$.

Consider the first vector of the basis obtained after the last change. Let us denote it by $e$. The Weil group $W$ transitively acts on all roots, therefore for every root $\alpha_i$ there exists such $w^{(\alpha_i)} \in W$ that $w^{(\alpha_i)} a_1 = a_i$. Consider now the basis $e_1, \ldots, e_{2m}, e_{2m+1}, \ldots, e_{2m+l}$, where $e_1 = e$, $e_i = \varphi_1(w^{(\alpha_i)})e$; for $2m < i \leq 2m + 1$ $e_i$ is not changed. It is clear that the matrix of this basis change is equivalent to $1$ modulo $J$. Therefore the obtained set of vectors is a basis.

It is clear that the matrix $\varphi_1(w_i) \ (i = 1, \ldots, l)$ in the part of basis $\{e_1, \ldots, e_{2m}\}$ coincides with the matrix $w_i$ in the initial basis of weight vectors. Since $h_{i}(-1)$ are squares of $w_i$, then their images also are not changed in the new basis.

Besides that, we know that $\varphi_1(w_i)$ is block-diagonal up to the first $2m$ and last $l$ elements. Therefore, the last basis part, consisting of $l$ elements, can be changed independently.

Let us denote matrices $w_i$ and $\varphi_1(w_i)$ on this part of basis by $\tilde{w}_i$ and $\varphi_1(w_i)$, respectively. All these matrices are involutions, they have only one $-1$ in their diagonal forms. Let $\tilde{V} = \tilde{V}_0 \oplus \tilde{V}_1$ be decomposition of the matrix $\varphi_1(w_i)$.

**Lemma 1.** Matrices $\varphi_1(w_i)$ and $\varphi_1(w_j)$, where $i \neq j$, commute if and only if $\tilde{V}_i^i \subseteq \tilde{V}_0$ and $\tilde{V}_i^j \subseteq \tilde{V}_0^j$.

**Proof.** If $\varphi_1(w_i)$ and $\varphi_1(w_j)$ commute, then the (free one-dimensional) submodule $\tilde{V}_i^i$ is proper for $\varphi_1(w_j)$ and the (free one-dimensional) submodule $\tilde{V}_i^j$ is proper for $\varphi_1(w_i)$. Therefore either $\tilde{V}_i^i \subset \tilde{V}_i^j$ or $\tilde{V}_i^i \subset \tilde{V}_0^j$. If $\tilde{V}_i^i \subset \tilde{V}_i^j$ then $\tilde{V}_i^i = \tilde{V}_i^j$. Since the module $V_i^i$ is invariant for $\varphi_1(w_j)$, we have $\tilde{V}_0^i \subset \tilde{V}_0^j$, therefore $\tilde{V}_0^i = \tilde{V}_0^j$, and so $\varphi_1(w_i) = \varphi_1(w_j)$ and we come to contradiction. Consequently, $\tilde{V}_i^i \subset \tilde{V}_0^j$, and similarly $\tilde{V}_i^j \subset \tilde{V}_0^i$. $\square$
Lemma 2. For any root system $\Phi$ there exists such a basis in $\tilde{V}$ that the matrix $\tilde{\varphi}_1(w_1)$ in this basis has the same form as $w_1$, i.e. is equal to
\[
\begin{pmatrix}
-1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & E_{l-2}
\end{pmatrix}.
\]

Proof. Since $\tilde{w}_1$ is an involution and $\tilde{V}_1^1$ has dimension 1, there exists a basis $\{e_1, e_2, \ldots, e_l\}$ where $\tilde{\varphi}_1(w_1)$ has the form $\text{diag}[-1, 1, \ldots, 1]$. In the basis $\{e_1, e_2 - 1/2e_1, e_3, \ldots, e_l\}$ the matrix $\tilde{\varphi}_1(w_1)$ has the obtained form. $\square$

Lemma 3. For the root system $A_2$ there exists such a basis that $\tilde{\varphi}_1(w_1)$ and $\tilde{\varphi}_1(w_2)$ in this basis have the same form as $w_1$ and $w_2$, i.e. are equal to
\[
\begin{pmatrix}
-1 & 1 \\
0 & 1
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
1 & 0 \\
1 & -1
\end{pmatrix},
\]
respectively.

Proof. By Lemma 2 we can find a basis in $\tilde{V}$ such that the matrix $\tilde{\varphi}_1(w_1)$ in this basis has the same form as $w_1$. Let the matrix $\tilde{\varphi}_1(w_2)$ in this basis be
\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}.
\]

Let us make the basis change with the help of the matrix
\[
\begin{pmatrix}
c & (1 - c)/2 \\
0 & 1
\end{pmatrix}.
\]

Under this basis change the matrix $\tilde{\varphi}_1(w_1)$ remains the same form, and the matrix $\tilde{\varphi}_1(w_2)$ becomes
\[
\begin{pmatrix}
a' & b' \\
1 & d'
\end{pmatrix}.
\]

As this matrix is an involution, we have $a' + d' = 0, a'^2 + b' = 1$. So we obtain $d' = -a'$. Now let us use the condition
\[
\left(\begin{pmatrix}
-1 & 1 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
a' & b' \\
1 & -a'
\end{pmatrix}\right)^2 = \begin{pmatrix}
a' & b' \\
1 & -a'
\end{pmatrix} \begin{pmatrix}
-1 & 0 \\
0 & 1
\end{pmatrix}.
\]

This condition gives (its second line and first row) $1 - 2a' = -1$, therefore $a' = 1$. From $a'^2 + b' = 1$ it follows $b' = 0$. $\square$

Lemma 4. For every root system $\Phi \neq A_2$ we can choose a basis in $\tilde{V}$ such that the matrices $\tilde{\varphi}_1(w_1)$ and $\tilde{\varphi}_1(w_2)$ in this basis have the same form as $\tilde{w}_1$ and $\tilde{w}_2$, respectively.

Proof. The intersection of modules $\tilde{V}_0^1$ and $\tilde{V}_0^2$ is a free module of dimension $\geq l - 3$. Therefore we can suppose that $\tilde{\varphi}_1(w_1)$ and $\tilde{\varphi}_1(w_2)$ have the form $\begin{pmatrix} * & 0 \\
0 & E_{l-3} \end{pmatrix}$. Moreover, by Lemma 2 we
can suppose that $\varphi_1(w_1)$ has the same form as $\tilde{w}_1$. We can consider not the whole module $\tilde{V}$, but its limitation to the first three basis vectors. Let

$$\varphi_1(w_1) = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}.$$  

Taking basis change with the matrix

$$\begin{pmatrix} b_1 & (1-b_1)/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we do not change $\varphi_1(w_1)$, but $\varphi_1(w_2)$ becomes

$$\varphi_1(w_2) = \begin{pmatrix} a'_1 & a'_2 & a'_3 \\ b'_1 & b'_2 & b'_3 \\ c'_1 & c'_2 & c'_3 \end{pmatrix}.$$  

Now we will use the same conditions as in the previous lemma. The first is $\varphi_1(w_2)^2 - 1 = 0$ (Cond. 1) and the second is $(\varphi_1(w_1)\varphi_1(w_2))^2 - \varphi_1(w_2)\varphi_1(w_1) = 0$ (Cond. 2). If we subtract Condition 1 from Condition 2 we obtain (line 2, row 1) $a'_1 = 1$, (line 2, row 2) $a'_2 = 0$, then from Cond. 1, line 1, row 3, we obtain $a'_3(1+c'_3) = 0$. As $c'_3 \equiv 1 \mod J$, we have $a'_3 = 0$. The same condition, line 2, row 3, gives $b'_3(b'_2+c'_2) = 0$, as $b'_3 \in R^*$, we have $c'_3 = -b'_2$.

Again taking basis change, but with the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -c'_1 & 1 \end{pmatrix},$$

we do not change $\varphi_1(w_1)$, but $\varphi_1(w_2)$ becomes

$$\varphi_1(w_2) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & b''_2 & b''_3 \\ 0 & c''_2 & -b''_2 \end{pmatrix}.$$  

Then directly from Cond. 1 we obtain $b''_2 = -1$, $c''_2 = 0$, and the last basis change with the matrix $diag[1, 1, b''_2]$ makes the obtained forms of $\varphi_1(w_1)$ and $\varphi_1(w_2)$. \hfill $\Box$

**Lemma 5.** For the root system $D_4$ there exists such a basis that $\varphi_1(w_1), \varphi_1(w_2), \varphi_1(w_3) \text{ and } \varphi_1(w_4)$ in this basis have the same forms as $\tilde{w}_1, \tilde{w}_2, \tilde{w}_3, \text{ and } \tilde{w}_4$, i.e. are equal to

$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix},$$

respectively.

**Proof.** We take such a basis that $\varphi_1(w_1), \varphi_1(w_3), \varphi_1(w_4)$ have the same form as the initial $\tilde{w}_1, \tilde{w}_3, \tilde{w}_4$. We can do it because $w_1, w_3, w_4$ are commuting involutions, there exists a basis
where \( \tilde{\phi}_1(w_1), \tilde{\phi}_1(w_3), \tilde{\phi}_1(w_4) \) have the forms diag\([-1, 1, 1, 1]\), diag\([1, 1, -1, 1]\), diag\([1, 1, 1, -1]\), respectively. Then, conjugating them by the matrix

\[
\begin{pmatrix}
1 & -1/2 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & -1/2 & 1 & 0 \\
0 & -1/2 & 0 & 1
\end{pmatrix},
\]

we come to the obtained basis. Now let us look for \( \tilde{\phi}_1(w_2) \). We have the following conditions:

\( \tilde{w}_2^2 = 1 \) (Cond. 1), \( (\tilde{w}_1\tilde{w}_2)^2 = \tilde{w}_2\tilde{w}_1 \) (Cond. 2), \( (\tilde{w}_3\tilde{w}_2)^2 = \tilde{w}_2\tilde{w}_3 \) (Cond. 3), \( (\tilde{w}_4\tilde{w}_2)^2 = \tilde{w}_2\tilde{w}_4 \) (Cond. 4).

Let

\[
\tilde{\phi}_1(w_2) = \begin{pmatrix}
a_1 & a_2 & a_3 & a_4 \\
b_1 & b_2 & b_3 & b_4 \\
c_1 & c_2 & c_3 & c_4 \\
d_1 & d_2 & d_3 & d_4
\end{pmatrix}.
\]

Taking basis change with the matrix

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & \frac{d_1}{2d_1-d_1} & 0 & \frac{b_1}{b_1-2d_1}
\end{pmatrix},
\]

we do not change \( \tilde{\phi}_1(w_1), \tilde{\phi}_1(w_3), \tilde{\phi}_1(w_4) \), but \( \tilde{\phi}_1(w_2) \) becomes

\[
\begin{pmatrix}
a_1 & a_2 & a_3 & a_4 \\
b_1 & b_2 & b_3 & b_4 \\
c_1 & c_2 & c_3 & c_4 \\
0 & d_2 & d_3 & d_4
\end{pmatrix}
\]

(we do not write primes for simplicity).

Now from the 4-th line of (Cond. 1 - Cond. 2) it follows \( d_2 = d_3 = 0, d_4 = 1 \).

Line 2, row 4 of (Condition 1 - Condition 4) gives \( b_4(b_4-1) = 0 \). Since \( b_4 \in R^* \), we have \( b_4 = 1 \).

Now, taking basis change with the matrix

\[
\begin{pmatrix}
b_3 & b_3 - 2a_3 \\
0 & 2a_3 - b_3 \\
0 & 1 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 1
\end{pmatrix},
\]

we do not change \( \tilde{\phi}_1(w_1), \tilde{\phi}_1(w_3), \tilde{\phi}_1(w_4) \), but \( \tilde{\phi}_1(w_2) \) becomes

\[
\begin{pmatrix}
a_1 & a_2 & 0 & a_4 \\
b_1 & b_2 & b_3 & 1 \\
c_1 & c_2 & c_3 & c_4 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

(again we do not write primes for simplicity).
Then line 1, row 3 of Cond. 1 gives $a_2b_3 = 0 \Rightarrow a_2 = 0$, line 1, row 1 of Cond. 1 gives $a_2^2 = 1 \Rightarrow a_1 = 1$, line 1, row 4 gives $2a_4 = 0 \Rightarrow a_4 = 0$. Line 2, row 4 of Cond. 1—Cond. 2 gives $b_1 = 1$, line 3, row 4 gives $c_1 = c_1$. Line 2, row 3 of Cond. 1 gives $c_3 = -b_2$.

Finally, taking basis change with the matrix

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & \frac{1-b_3}{2} & b_3 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
$$

we do not change $\varphi_1(w_1), \varphi_1(w_3), \varphi_1(w_4)$, but $\varphi_1(w_2)$ becomes

$$
\begin{pmatrix}
1 & 0 & 0 & a_4' \\
1 & b_2' & 1 & 1 \\
c_1' & c_2' & -b_2' & c_1' \\
0 & 0 & 0 & 1
\end{pmatrix}.
$$

After that line 2, row 3 of Cond. 3 gives $b_2' = -1$, and then Cond. 1 gives $c_1' = c_2' = 0$. □

Lemma 6. Suppose that we have some root system $\Phi$ and elements $\varphi_1(w_{i_1}) = \tilde{w}_{i_1}, \ldots, \varphi_1(w_{i_k}) = \tilde{w}_{i_k}$, and also $\tilde{\varphi}_1(w_{i_{k+1}})$, where one of the following cases holds:

a) $\Phi = A_l$, $l \geq 3$, $i_1 = 1$, $i_2 = 2, \ldots, i_k = k$, $i_{k+1} = k + 1$, $k + 1 < l$;

b) $\Phi = D_l$, $l > 3$, $i_1 = 1$, $i_2 = l - 1$, $i_3 = l - 2$, $\ldots$, $i_k = l - k + 1$, $i_{k+1} = l - k$, $4 < k < l$;

c) $\Phi = E_6, E_7$ or $E_8$, $i_1 = 1, i_2 = 2, \ldots, i_k = k$, $i_{k+1} = k + 1$, $4 \leq k < l - 1$.

Then we can choose such a basis of $\tilde{V}$ that $\tilde{\varphi}_1(w_{i_1}) = \tilde{w}_{i_1}, \ldots, \tilde{\varphi}_1(w_{i_{k+1}}) = \tilde{w}_{i_{k+1}}$.

Proof. In all these cases $\tilde{\varphi}_1(w_{i_{k+1}})$ commutes with all $\tilde{\varphi}_1(w_{i_j}) = \tilde{w}_{i_j}, \ldots, \tilde{\varphi}_1(w_{i_{k-1}}) = \tilde{w}_{i_{k-1}}$, therefore (see Lemma 4) for all $j = i_1, \ldots, i_{k-1}$ $V_j^l \subset V_l^{i_{k+1}}$. Since $V_1^{i_1} \oplus \cdots \oplus V_1^{i_{k-1}} = \langle e_{i_1}, \ldots, e_{i_{k-1}} \rangle$, we infer that $\tilde{\varphi}_1(w_{i_{k+1}})$ is identical on the first $k - 1$ basic vectors. As in Lemma 4 we obtain that $\tilde{\varphi}_1(w_{i_{k+1}})$ is identical on the last $l - k - 2$ basic vectors. Therefore we can limit $\tilde{\varphi}_1(w_{i_{k+1}})$ for the part of basis $\{e_k, e_{k+1}, e_{k+2}\}$ (without loss of generality). Now the proof is completely the same as in Lemma 4. □

Proposition 4. For every root system $\Phi = A_l, D_l, E_l$ we can choose a basis in $\tilde{V}$ such that the matrices $\tilde{\varphi}_1(w_1), \ldots, \tilde{\varphi}_1(w_l)$ in this basis have the same form as $\tilde{w}_1, \ldots, \tilde{w}_l$, respectively.

Proof. If we have the system $A_2$, we can use Lemma 3. If $\Phi = A_l$, $l \geq 3$, then we apply Lemma 4 after that Lemma 6 $l - 3$ times, and finally the same arguments as in Lemma 3 for the element $\tilde{\varphi}_1(w_l)$.

If $\Phi = D_l$, then we apply Lemma 5 to the roots $\alpha_{l-3}, \ldots, \alpha_l$, then Lemma 6 $l - 5$ times to the roots $\alpha_{l-4}, \ldots, \alpha_2$, and finally the same arguments as in Lemma 3 for the element $\tilde{\varphi}_1(w_1)$.

If $\Phi = E_l$, then we apply Lemma 5 to the roots $\alpha_2, \ldots, \alpha_5$, then Lemma 6 to the roots $\alpha_6, \ldots, \alpha_{l-1}$, and finally the same arguments as in Lemma 3 for the elements $\tilde{\varphi}_1(w_1)$ and $\tilde{\varphi}_1(w_l)$. □

Therefore, we can now consider the isomorphism $\varphi_2$ with all properties of $\varphi_1$, and such that $\varphi_2(w_i) = w_i$ for all $i = 1, \ldots, l$.

We suppose that we have the isomorphism $\varphi_2$ with these properties.
4. The images of \( x_{\alpha_i}(1) \) and diagonal matrices.

Now we are interested in the images of \( x_{\alpha_i}(t) \). Let \( \varphi_2(x_{\alpha_1}(1)) = x_1 \). Since \( x_1 \) commutes with all \( h_{\alpha_i}(-1), i = 1, 3, \ldots, l \), we have that \( x_1 \) is separated to the blocks of the following form: blocks \( 2 \times 2 \) are corresponded to the part of basis \( \{v_i, v_{-i}\} \), where \( i > 1 \) and \( \langle \alpha_i, \alpha_1 \rangle \neq 0 \); blocks \( 4 \times 4 \) are corresponded to the part of basis \( \{v_i, v_{-i}, v_j, v_{-j}\} \), where \( i > 1, \alpha_i = \alpha_j \pm \alpha_1 \); and we also have the part \( \{v_1, v_{-1}, V_1, \ldots, V_l\} \).

For \( h_{\alpha_2}(-1) \) we know that \( h_{\alpha_2}(-1)x_1h_{\alpha_2}(-1) = x_1^{-1} \). Then, on the blocks \( 2 \times 2 \), described above if \( x_1 \) has the form

\[
\begin{pmatrix}
  a & b \\
  c & d \\
\end{pmatrix},
\]

then this matrix to the second power is 1, therefore \( a^2 + bc = d^2 + bc = 1, b(a + d) = c(a + d) = 0 \). Since \( a + d \equiv 2 \mod J \), we have \( a + d \in R^* \), i.e. \( b = c = 0 \). Since \( a^2 = d^2 = 1 \) and \( a, d \equiv 1 \mod J \), we have \( a = d = 1 \). Thus, on the blocks \( 2 \times 2 \) the matrix \( x_1 \) is always 1 (i.e. it coincides with \( x_{\alpha_1}(t) \) on these blocks), so we can now not to consider these basis elements.

Now let us use the conditions \( w_ix_1w_i^{-1} = x_1 \) for \( i \geq 3 \).

At first, all blocks \( 4 \times 4 \) has the same form, since every two such blocks are conjugate up to the action of \( w_i, i \geq 3 \).

The conditions \( w_ix_1w_i^{-1} = x_1 \) for \( i \geq 3 \) for the rest of the basis together with the condition \( h_2x_1h_2^{-1} = x_1^{-1} \) say that the matrix on the basis subset \( \{v_1, v_{-1}, V_1, V_2, V_3, \ldots, V_l\} \) has the form

\[
\begin{pmatrix}
  * & * & * & * & 0 & \ldots & 0 \\
  * & * & * & * & 0 & \ldots & 0 \\
  * & * & * & * & 0 & \ldots & 0 \\
  * & * & * & * & 0 & \ldots & 0 \\
  * & * & * & 1 & \ldots & 0 \\
  \ldots & & & & & & \\
  * & * & * & * & 0 & \ldots & 1 \\
\end{pmatrix},
\]

and all lines \( 5, \ldots, l \) are expressed via the fourth line. According to the zero corner of this matrix we can restrict the conditions to its left upper submatrix \( 4 \times 4 \).

Suppose that on this part of the basis the matrix \( x_1 \) has the form

\[
\begin{pmatrix}
  a_1 & a_2 & a_3 & a_4 \\
  b_1 & b_2 & b_3 & b_4 \\
  c_1 & c_2 & c_3 & c_4 \\
  d_1 & d_2 & d_3 & d_4 \\
\end{pmatrix},
\]

and on the part of basis \( v_2, v_{-2}, v_{1+2}, v_{-1-2} \) it has the form

\[
\begin{pmatrix}
  e_1 & e_2 & e_3 & e_4 \\
  f_1 & f_2 & f_3 & f_4 \\
  g_1 & g_2 & g_3 & g_4 \\
  h_1 & h_2 & h_3 & h_4 \\
\end{pmatrix}.
\]

We will consider the part of basis \( \{v_1, v_{-1}, v_2, v_{-2}, v_{1+2}, v_{-1-2}, V_1, V_2\} \).
Taking basis change with the block-diagonal matrix, that has the form
\[
\begin{pmatrix}
  1 & -\frac{b_4}{a_4} \\
-\frac{b_4}{a_4} & 1
\end{pmatrix}
\]
on every block \{v_i, v_{i-1}\} (it is possible, because \(b_4 \in J\)), and is identical on the block \{V_1, \ldots, V_l\}, we do not change the elements \(w_i, h_i\), and \(x_1\) now has \(b_4 = 0\). So we can suppose that the isomorphism \(\varphi_2\) is such that \(\varphi_2(x_{a_1}(1))\) has \(b_4 = 0\).

Then we make the basis change with the help of diagonal matrix, having the form \(\frac{1}{a_4} \cdot E\) on the part \{\(v_1, v_{-1}, \ldots, v_m, v_{-m}\}\), and being identical on the part \{\(V_1, \ldots, V_l\)\}. Similarly, all elements \(w_i, h_i\) are not changed, and \(a_4\) is now equal to 1.

So we suppose that \(\varphi_2(x_{a_1}(1))\) has \(b_4 = 0\) and \(a_4 = 1\).

On the part of basis under consideration
\[
w_1 = \begin{pmatrix}
  0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \quad w_2 = \begin{pmatrix}
  0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1
\end{pmatrix}
\]

Since
\[
x_{a_1}(1) = \begin{pmatrix}
  1 & -1 & 0 & 0 & 0 & 0 & -2 & 1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
\]

we have
\[
x_1 = \varphi_2(x_1(1)) = \begin{pmatrix}
a_1 & a_2 & 0 & 0 & 0 & 0 & a_3 & 1 \\
b_1 & b_2 & 0 & 0 & 0 & 0 & b_3 & 0 \\
0 & 0 & e_1 & e_2 & e_3 & e_4 & 0 & 0 \\
0 & 0 & f_1 & f_2 & f_3 & f_4 & 0 & 0 \\
0 & 0 & g_1 & g_2 & g_3 & g_4 & 0 & 0 \\
0 & 0 & h_1 & h_2 & h_3 & h_4 & 0 & 0 \\
c_1 & c_2 & 0 & 0 & 0 & 0 & c_3 & c_4 \\
0 & 0 & 0 & 0 & 0 & 0 & d_3 & d_4
\end{pmatrix},
\]

where \(a_1, b_2, e_1, f_2, f_4, g_3, h_4, c_2, c_3, d_4 \equiv 1 \mod J\), \(a_2, g_1 \equiv -1 \mod J\), \(a_3 \equiv -2 \mod J\), all other entries are in \(J\).
We will use the following conditions:

\[ \text{Con1} := (x_1 x_{1+2} - x_{1+2} x_1 = 0), \quad \text{Con2} := (h_2 x_1 h_2 x_1 - 1 = 0). \]

Position (3,8) of Con1 gives \( f_3 = -e_3 \), position (2,8) of Con1 gives \( h_3 = -b_3 c_4 \), position (2,8) of Con2 gives \( b_1 = b_3 c_4 \), therefore \( h_3 = -b_1 \). From position (1,1) of Con1 we have \( b_1 (a_2 + g_4) = 0 \), therefore \( b_1 = 0 \), since \( a_2 + g_4 \in R^* \).

Now we introduce two more conditions:

\[ \text{Con3} := (x_1 w_1 w_1^{-1} - w_1 h_2 x_1 h_2 = 0), \quad \text{Con4} := (x_2 x_1 - x_{1+2} x_1 x_2 = 0). \]

Denote \( y_1 = a_1 - 1, y_2 = a_2 + 1, y_3 = a_3 + 2, y_4 = b_2 - 1, y_5 = b_3, y_6 = c_1, y_7 = c_2 - 1, y_8 = c_3 - 1, y_9 = c_4, y_{10} = d_1, y_{11} = d_2, y_{12} = d_3, y_{13} = d_4 - 1, y_{14} = e_1 - 1, y_{15} = e_2, y_{16} = e_3, y_{17} = e_4, y_{18} = f_1, y_{19} = f_2 - 1, y_{20} = f_4 - 1, y_{21} = g_1 + 1, y_{22} = g_2, y_{23} = g_3 - 1, y_{24} = g_4, y_{25} = h_1, y_{26} = h_2, y_{27} = h_4 - 1. \) All these \( y_i \) are in \( J \). From the Conditions 1–4 we have the following 27 equations (that are linear with respect to \( y_i \)):
\(y_{23}(-a_2) + y_{24}(a_1 - b_2) + y_{27}a_2 = 0,\)
\(y_{18}(-g_1) + y_{22}(a_1 - c_1) + y_{26}(a_2) = 0,\)
\(y_{1}g_1 + y_{15}(-g_2) + y_{19}(-g_1) + y_{25}a_2 = 0,\)
\(y_{6}(a_3 + 1) + y_{10}(-1) + y_{16}(g_1 + g_2) = 0,\)
\(y_{3}(c_2) + y_{11}(-1) + y_{20} + y_{21}(-f_4) + y_{22}(-e_4) = 0,\)
\(y_{3}(c_3 + c_4 - a_3 + g_1 + g_2) + y_{8}(-1) + y_{9}(-1) + y_{13}(-1) + y_{14}(-1) + y_{23}2^+ + y_{24}(-b_3) = 0,\)
\(y_{9}(-a_3 - 1) + y_{13} + y_{23}(-1) = 0,\)
\(y_{5}c_2 + y_{25}(-f_4) + y_{26}(-e_4) = 0,\)
\(y_{16}a_2 + y_{17}(b_2 - f_2) + y_{18}(-f_4) = 0,\)
\(y_{5}(e_4 - f_4) + y_{16}(1 + 2a_3) = 0,\)
\(y_{15}(-f_1f_4 - 2f_2e_3) + y_{16}(a_1 - a_2 + a_1h_2 + f_1f_2 - f_2^2) + y_{22}(e_3a_2 - f_4b_2) = 0,\)
\(y_{10}(-d_3 - d_4) + y_{11}(-b_1 + 1) + y_{12}e_1 = 0,\)
\(y_{1}(-1) + y_{2}(a_4 + b_2) + y_{3}(-c_2) + y_{4}(-1) + y_{7}2 + y_{11}(-1) = 0,\)
\(y_{5}(-c_2g_3) + y_{16}(b_2 - b_2h_4) + y_{17}a_2 = 0,\)
\(y_{14}g_3 + y_{16}(e_4 - e_3) + y_{21} + y_{23} = 0,\)
\(y_{4}(b_2 + 1) + y_{5}(-c_2) = 0,\)
\(y_{14}(e_1 + 1) + y_{15}f_1 + y_{16}(-g_1) + y_{17}(-h_1) = 0,\)
\(y_{15}(e_1 + f_2) + y_{16}(-g_2) + y_{17}(-h_2) = 0,\)
\(y_{6}(-g_1a_1) + y_{9}(c_3 + c_4 - d_4)(c_1 - d_1) + y_{10}(c_3^2 + c_3c_4 - d_3c_4 - e_1) + y_{11}(-f_1) + y_{25}c_2a_1 = 0,\)
\(y_{16}g_4 + y_{18}e_4 + y_{19} + y_{20}(f_2 - h_4) + y_{27}(-1) = 0,\)
\(y_{14} + y_{21}(g_3 - e_1) + y_{22}f_1 + y_{23}(-1) + y_{24}h_1 = 0,\)
\(y_{17}(-g_1) + y_{22}(-f_4) + y_{24}(g_3 - h_4) = 0,\)
\(y_{4}(-c_2) + y_{6}(-a_2) + y_{8}c_2 + y_{9}d_2 = 0,\)
\(y_{6}(-1) + y_{9}(c_3 + d_4) = 0,\)
\(y_{1}a_2 + y_{2} + y_{4} + y_{6}a_3 + y_{10}(-1 - a_3) = 0,\)
\(y_{19}h_4 + y_{20}(-1) + y_{25}(-g_2) + y_{26}(-h_2) + y_{27} = 0,\)
\(y_{6}(-g_3f_2) + y_{15}(-c_1g_4 - c_2h_4) + y_{16}(d_2 - d_1) + y_{22}(c_4d_2 - c_3c_2 - c_2c_4) + y_{26}(c_4d_1 - c_3c_1 - c_4c_1) = 0,\)
The matrix of this system of linear equations modulo $J$ is

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
$$

Determinant of this matrix is $2^8$, so it is invertible in $R$. Therefore this system has the unique solution $y_1 = \cdots = y_27 = 0$. Consequently, $x_1 = x_{\alpha_1}(1)$ on the part of basis under consideration. Since all roots are conjugate up to the action of $W$, we have that $x_1 = x_{\alpha_1}(1)$ on the whole basis. It is clear that also $x_2 = x_{\alpha_2}(1)$.

Now consider the matrix $d_t = \varphi_1(h_{\alpha_1}(t))$. The matrix $h_{\alpha_1}(t)$ is $\text{diag}[t^2, 1/t^2, 1/t, t, 1/t, 1]$ on the part of basis under consideration.

**Lemma 7.** The matrix $d_t$ is $h_{\alpha_1}(s)$ for some $s \in R^*$. 

*Proof.* For the matrix $d_t$ we have conditions $d_tw_3 = w_3d_t$, \ldots, $d_tw_i = w_id_t$.

Let $l > 2$ and

$$
d_t = \begin{pmatrix}
\gamma_{11} & \gamma_{12} & \cdots & \gamma_{1l} \\
\gamma_{21} & \gamma_{22} & \cdots & \gamma_{2l} \\
\vdots & \vdots & \ddots & \vdots \\
\gamma_{l1} & \gamma_{l2} & \cdots & \gamma_{ll}
\end{pmatrix}
$$

on $\tilde{V}$. 

$$
d_tw_i = w_id_t, \ i > 2, \ \gamma_{ii} = \cdots = \gamma_{i-1,i} = \gamma_{i+1,i} = \cdots = \gamma_{li} = 0. \ \ d_tw_1d_tw_1^{-1} = 1 \ \gamma_{33} = \cdots = \gamma_{li} = 1 \Rightarrow \gamma_{33} = \cdots = \gamma_{li} = 1. \ \ d_tw_1 = w_td_t, \ \gamma_{i-1,j} = \gamma_{l,j}, \ j = 1, 2, \ldots, d_tw_3 = w_3d_t
$$

Every condition $d_tw_i = w_id_t, \ i > 2$, gives $\gamma_{ii} = \cdots = \gamma_{i-1,i} = \gamma_{i+1,i} = \cdots = \gamma_{li} = 0$. From the condition $d_tw_1d_tw_1^{-1} = 1$ now directly follows $\gamma_{33} = \cdots = \gamma_{li} = 1 \Rightarrow \gamma_{33} = \cdots = \gamma_{li} = 1$. Condition $d_tw_1 = w_td_t$ gives that $\gamma_{l-1,j}$ is linearly expressed via $\gamma_{l,j}, \ j = 1, 2, \ldots$, condition $d_tw_3 = w_3d_t$ gives that $\gamma_{2,j}$ is linearly expressed via $\gamma_{l,j}, \ j = 1, 2$. 
According to the zero corner we can consider the conditions for \( d_t \) on the part of basis \( v_1, v_{-1}, v_2, v_{-2}, v_{1+2}, v_{-1-2}, V_1, V_2 \), as we did it for \( x_1 \).

Since \( d_t \) commutes with all \( h_i \), we have that on the part of basis under consideration

\[
d_t = \begin{pmatrix}
k_1 & k_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
k_3 & k_4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & l_1 & l_2 & 0 & 0 & 0 & 0 \\
0 & 0 & l_3 & l_4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & m_1 & m_2 & 0 & 0 \\
0 & 0 & 0 & 0 & m_3 & m_4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & n_1 & n_2 \\
0 & 0 & 0 & 0 & 0 & 0 & n_3 & n_4
\end{pmatrix}.
\]

We know that \( x_2 = x_{\alpha_2}(1) \) and \( x_2^t = \varphi_2(x_{\alpha_2}(t)) = d_t x_2 d_t^{-1} = d_t x_2 w_1 d_t w_1^{-1} \). Using these conditions we obtain the expression of \( x_2^t \) through the entries of \( d_t \). Then we can use the condition Con5 := \( (x_2^t 2 - x_2 2^t - 0) \). Its position (1,6) gives \( k_3(k_2 + k_3) = 0 \Rightarrow k_2 = -k_3 \), pos. (2,1) gives \( k_4(l_3 - m_3) = 0 \Rightarrow m_3 = l_3 \), pos. (2,5) gives \( l_3(l_1 + m_4) = 0 \Rightarrow l_3 = 0 \), pos. (5,6) gives \( k_2(l_1 + m_1) = 0 \Rightarrow k_2 = 0 \). From Con6 := \( (d_t w_1 d_t w_1^{-1} - 1 = 0) \) it follows \( k_1 k_4 = l_1 m_1 = l_4 m_4 = 1 \). Using the condition Con7 := \( (w_2 d_t w_2^{-1} - d_t w_1 d_t w_2^{-1} w_1^{-1} = 0) \), we obtain \( m_2 = l_2 = 0 \) (positions (1,2) and (4,3)) and \( l_4 = l_1 k_1 \) (pos. (1,1)). Position (7,7) of Con6 gives \( n_1^2 - (n_1 + n_2)n_3 = 1 \), position (3,7) of Con5 gives \( n_1^2 - (3n_1 + n_2 - 2n_3 - n_4)n_3 = 1 \). Therefore, \( n_3 = 0 \), \( n_2^2 = 1 \). After that we clearly infer \( n_1 = n_4 = 1 \), \( n_2 = 0 \). Finally, position (3,4) of Con5 gives \( k_1 = 1/l_1^2 \). Denote \( 1/l_1 \) by \( s \).

It is clear that with the help of the elements \( w_i, i = 3, \ldots, l \), we can define all other diagonal elements. Namely, if \( \langle \alpha_1, \alpha_k \rangle = p \), then \( \varphi(h_{\alpha_1}(t))v_k = s^p \cdot v_k \), \( \varphi(h_{\alpha_1}(t))v_{-k} = s^{-p} \cdot v_{-k} \). Whence \( \varphi(h_{\alpha_1}(t)) = h_{\alpha_1}(s) \).

5. Images of the matrices \( x_{\alpha_i}(t) \), proof of the main theorem.

It is clear that \( \varphi_2(h_{\alpha_k}(t)) = h_{\alpha_k}(s) \), \( k = 1, \ldots, n \). Let us denote the mapping \( t \mapsto s \) by \( \rho : R^* \rightarrow R^* \). Note that for \( t \in R^* \), \( \varphi_2(x_1(t)) = \varphi_2(h_{\alpha_2}(t^{-1})x_1(1)h_{\alpha_2}(t)) = h_{\alpha_2}(s^{-1})x_1(1)h_{\alpha_2}(s) = x_1(s) \). If \( t \notin R^* \), then \( t \in J \), i.e. \( t = 1 + t_1 \), where \( t_1 \in R^* \). Then \( \varphi_2(x_1(t)) = \varphi_2(x_1(1)(x_1(t_1))) = x_1(1)x_1(\rho(t_1)) = x_1(1 + \rho(t_1)) \). Therefore, if we extend the mapping \( \rho \) on the whole ring \( R \) (by the formula \( \rho(t) := 1 + \rho(t - 1), t \in R \)), we have \( \varphi_2(x_1(t)) = x_1(\rho(t)) \) for all \( t \in R \). It is clear that \( \rho \) is injective, additive, multiplicative on all invertible elements. Since every element of \( R \) is the sum of two invertible elements, we have that \( \rho \) is an isomorphism from \( R \) onto some its subring \( R' \). Note that in this situation \( CE(\Phi, R)C^{-1} = E(\Phi, R') \) for some matrix \( C \in GL(V) \). Let us show that \( R' = R \).

Let us denote the matrix units by \( E_{ij} \).

**Lemma 8.** Elementary Chevalley group \( E(\Phi, R) \) generates the ring \( M_n(R) \).

**Proof.** The matrix \((x_{\alpha_1}(1) - 1)^2\) has the single nonzero element \(-2 \cdot E_{12} \). Multiplying it to some suitable diagonal matrix we can obtain an arbitrary matrix of the form \( \lambda \cdot E_{12} \) (since \(-2 \in R^* \) and \( R^* \) generates \( R \)). According to the transitive action of the Weil group on the root system \( \Phi \) (for every root \( \alpha_k \) there exists such an element \( w \in W \) that \( w(\alpha_1) = \alpha_k \)) the matrix \( \lambda E_{12} \cdot w \) has the form \( \lambda E_{1,2k} \), and the matrix \( w^{-1} \cdot \lambda E_{12} \) has the form \( \lambda E_{2k-1,2} \). Moreover, according to
the Weil group element that maps the first root to the opposite one, we get an element \( E_{2,1} \).

Taking different combinations of the obtained elements, we can get a non-abelian element \( \lambda E_{ij}, 1 \leq i, j \leq 2m \). Therefore we have always generated the subring \( M_{2m}(R) \). Now let us subtract from \( x_{a_2}(1) - 1 \) suitable matrix units, and we obtain the matrix \( E_{2m+1,2} - 2E_{1,2m+1} + E_{1,2m+2} \). Multiplying its (from the right side) to \( E_{2,i}, 1 \leq i \leq 2m \), we get all \( E_{2m+1,i}, 1 \leq i \leq 2m \).

With the help of Weil group elements we have all \( E_{i,j}, 2m < j \leq 2m + l, 1 \leq j \leq 2n \). Now we have the matrix \( -2E_{1,2m+1} + E_{1,2m+2} \). Multiplying it (from the left side) to \( E_{2m+1,1} \), we get \( E_{2m+1,2m+1} \). With the help of two last matrices we have \( E_{1,2m+1}, \) and, therefore, all \( E_{i,j}, 1 \leq i \leq 2m, 2m < j \leq 2m + l \). It is clear that now we have all matrix units, i.e. the whole matrix ring \( M_n(R) \).

Let us show it for the simplest root system \( A_2 \). In this case \( (x_{a_2}(1) - 1)^2 = -2E_{12}, h_{a_2}(t)(-2E_{12}) = -2tE_{12}, i.e., we can obtain every \( \lambda E_{12} \). Then \( w_{a_1}\lambda E_{12}w_{a_1}^{-1} = \lambda E_{21}, \lambda E_{12}E_{21} = \lambda E_{11}, \lambda E_{21}E_{12} = \lambda E_{22}, w_{a_2}(1)\lambda E_{12} = \lambda E_{52}, w_{a_2}(1)\lambda E_{21} = \lambda E_{61}, \lambda E_{52}E_{21} = \lambda E_{51}, \lambda E_{61}E_{12} = \lambda E_{62}, \lambda E_{12}w_{a_2}(1) = \lambda E_{16}, \lambda E_{21}w_{a_2}(1) = \lambda E_{25}, \lambda E_{21}E_{16} = \lambda E_{26}, \lambda E_{12}E_{25} = \lambda E_{15}, \lambda E_{31}E_{15} = \lambda E_{55}, \lambda E_{61}E_{16} = \lambda E_{66}, \lambda E_{31}E_{16} = \lambda E_{56}, \lambda E_{61}E_{15} = \lambda E_{65}, \lambda E_{56}w_{a_1}(1) = \lambda E_{i3}, i = 1, 2, 5, 6, \lambda E_{i6}w_{a_1}(1) = \lambda E_{i4}, i = 1, 2, 5, 6, \lambda w_{a_1}(1)E_{5i} = \lambda E_{3i}, i = 1, 2, 5, 6, \lambda w_{a_1}(1)E_{6i} = \lambda E_{4i}, i = 1, 2, 5, 6, \lambda E_{i1}E_{13} = \lambda E_{43}, \lambda E_{41}E_{14} = \lambda E_{44}, \lambda E_{31}E_{13} = \lambda E_{33}, \lambda E_{31}E_{14} = \lambda E_{34}, \) so we have all matrix units of the subring \( M_6(R) \).

Then \( y = x_{a_1}(1) - 1 = -E_{12} - 2E_{17} + E_{18} + E_{46} - E_{53} + E_{73}, y' = y + E_{12} - E_{46} + E_{53} = E_{18} - 2E_{17} + E_{72}, (E_{18} - 2E_{17} + E_{72}) \cdot \lambda E_{2i} = \lambda E_{7i}, i = 1, \ldots, 6, (w_{a_2}(1) - 1)\lambda E_{7i} = \lambda E_{8i}, i = 1, \ldots, 6, y'' = y' - E_{72} = E_{18} - 2E_{17}, \lambda E_{81}y'' = \lambda E_{88}, \lambda E_{71}y'' = -2\lambda E_{77}, y'y''E_{88} = \lambda E_{18}, y''\lambda E_{77} = -2\lambda E_{17}, \lambda E_{i1}E_{17} = \lambda E_{i7}, \lambda E_{i1}E_{18} = \lambda E_{i8}, \) so we have generated the whole ring \( M_8(R) \).

**Lemma 9.** If for some \( C \in GL(V) \) we have \( CE(\Phi, R)C^{-1} = E(\Phi, R') \), where \( R' \) is a subring of \( R \), then \( R' = R \).

**Proof.** Suppose that \( R' \) is a proper subring of \( R \).

Then \( CM_n(R)C^{-1} = M_n(R') \), since the group \( E(\Phi, R) \) generates the ring \( M_n(R) \), and the group \( E(\Phi, R') = CE(\Phi, R)C^{-1} \) generates the ring \( M_n(R') \). It is impossible, since \( C \in GL_n(R) \).

Now we have proved that \( \rho \) is an automorphism of the ring \( R \). Therefore, composition of the initial automorphism \( \varphi \) and some basis change with the help of the matrix \( C \in GL_n(R) \) (that maps \( E(\Phi, R') \) onto itself), is a ring automorphism \( \rho \). It proves Theorem 1. □

**References**

[1] Abe E. Automorphisms of Chevalley groups over commutative rings. Algebra and Analysis, 5(2), 1993, 74–90.

[2] Abe E. Chevalley groups over local rings. Tohoku Math. J., 1969, 21(3), 474–494.

[3] Abe E. Chevalley groups over commutative rings. Proc. Conf. Radical Theory, Sendai — 1988, 1–23.

[4] Abe E. Normal subgroups of Chevalley groups over commutative rings. Contemp. Math., 1989, 83, 1–17.

[5] Abe E., Hurley J. Centers of Chevalley groups over commutative rings. Comm. Algebra, 1988, 16(1), 57–74.

[6] Abe E., Suzuki K. On normal subgroups of Chevalley groups over commutative rings. Tohoku Math. J., 1976, 28(1), 185-198.

[7] Bak A. Nonabelian K-theory: The nilpotent class of \( K_1 \) and general stability. K-Theory, 1991, 4, 363–397.
[8] Bak A., Vavilov Normality of the elementary subgroup functors. Math. Proc. Cambridge Philos. Soc., 1995, 118(1), 35–47.

[9] Bloshytsyn V.Ya. Automorphisms of general linear group over a commutative ring, not generated by zero divisors. Algebra and Logic, 1978, 17(6), 639–642.

[10] Borel A., Tits J. Homomorphismes “abstraits” de groupes algébriques simples. Ann. Math., 1973, 73, 499–571.

[11] Borel A. Properties and linear representations of Chevalley groups. Seminar on algebraic groups, 1973, 9–59.

[12] Bourbaki N. Groupes et Algèbres de Lie. Hermann, 1968.

[13] Carter R.W. Simple groups of Lie type, 2nd ed., Wiley, London et al., 1989.

[14] Carter R.W., Chen Yu. Automorphisms of affine Kac–Moody groups and related Chevalley groups over rings. J. Algebra, 1993, 155, 44–94.

[15] Chen Yu. Isomorphic Chevalley groups over integral domains. Rend. Sem. Mat. univ. Padova, 1994, 92, 231–237.

[16] Chen Yu. On representations of elementary subgroups of Chevalley groups over algebras. proc. Amer. Math. Soc., 1995, 123(8), 2357–2361.

[17] Chen Yu. Automorphisms of simple Chevalley groups over Q-algebras. Tohoku Math. J., 1995, 348, 81–97.

[18] Chen Yu. Isomorphisms of adjoint Chevalley groups over integral domains. Trans. Amer. Math. Soc., 1996, 348(2), 1–19.

[19] Chen Yu. Isomorphisms of Chevalley groups over algebras. J. Algebra, 2000, 226, 719–741.

[20] Chevalley C. Certain schemas des groupes semi-simples. Sem. Bourbaki, 1960–1961, 219, 1–16.

[21] Chevalley C. Sur certains groupes simples. Tohoku Math. J., 1955, 2(7), 14–66.

[22] Cohn P., On the structure of the GL_2 of a ring, Publ. Math. Inst. Hautes Et. Sci., 1966, 30, 365–413.

[23] Demazure M., Gabriel P. Groupes algébriques. I. North Holland, Amsterdam et al., 1970, 1–770.

[24] Diedonne J. On the automorphisms of classical groups, Mem. Amer. Math. Soc., 1951, 2.

[25] Diedonne J. Geometry of classical groups, 1974.

[26] Golubchik I.Z., Mikhalev A.V. Isomorphisms of the general linear group over associative ring. Vestnik MSU, ser. math., 1983, 3, 61–72.

[27] Golubchik I.Z. Isomorphisms of the general linear group over an associative ring. Contemp. Math., 1992, 131(1), 123–136.

[28] Golubchik I.Z., Mikhalev A.V. Isomorphisms of unitary groups over associative rings. Zapiski nauchnyh seminarov LOMI, 1983, 132, 97–109 (in Russian).

[29] Grothendieck A. Sur les sous-groupes arithmétiques des groupes semi-simples déployés. Ann. Sci. Ecole Norm. Sup. 4ème sér., 1969, 2, 1–62.

[30] Matsumoto H. Sur les sous-groupes arithmétiques des groupes semi-simples déployés. Ann. Sci. Ecole Norm. Sup. 4ème sér., 1969, 2, 1–62.

[31] McDonald B.R., Automorphisms of GL_n(R), Trans. Amer. Math. Soc., 215, 1976, 145–159.

[32] O’Meara O.T., The automorphisms of linear groups over any integral domain. J. reine angew. Math., 223, 1966, 56–100.

[33] Fuan Li, Zunxian Li. Automorphisms of SL_3(R), GL_3(R). Contemp. Math., 1984, 82, 47–52.

[34] Petechuk V.M. Automorphisms of groups SL_n, GL_n over some local rings. Mathematical Notes, 28(2), 1980, 187–206 (in Russian).
[42] Petechuk V.M. Automorphisms of groups $\text{SL}_3(K)$, $\text{GL}_3(K)$. Mathematical Notes, 31(5), 1982, 657–668 (in Russian).

[43] Petechuk V.M. Automorphisms of matrix groups over commutative rings. Mathematical Sbornik, 1983, 45, 527–542.

[44] Stein M.R. Generators, relations and coverings of Chevalley groups over commutative rings. Amer. J. Math., 1971, 93(4), 965–1004.

[45] Stein M.R. Surjective stability in dimension 0 for $K_2$ and related functors, Trans. Amer. Soc., 1973, 178(1), 165–191.

[46] Stein M.R. Stability theorems for $K_1$, $K_2$ and related functors modeled on Chevalley groups. Japan J. Math., 1978, 4(1), 77–108.

[47] Steinberg R. Lectures on Chevalley groups, Yale University, 1967.

[48] Steinberg R., Automorphisms of finite linear groups, Canad. J. Math., 121, 1960, 606–615.

[49] Suslin A.A., On a theorem of Cohn, J. Sov. Math. 17 (1981), N2, 1801–1803.

[50] Suzuki K., On the automorphisms of Chevalley groups over $p$-adic integer rings, Kumamoto J. Sci. (Math.), 16(1), 1984, 39–47.

[51] Swan R., Generators and relations for certain special linear groups, Adv. Math. 6 (1971), 1–77.

[52] Taddei G. Normalité des groupes élémentaire dans les groupes de Chevalley sur un anneau. Contemp. Math., Part II, 1986, 55, 693–710.

[53] Vaserstein L.N. On normal subgroups of Chevalley groups over commutative rings. Tohoku Math. J., 1986, 36(5), 219–230.

[54] Vavilov N.A. Structure of Chevalley groups over commutative rings. Proc. Conf. Non-associative algebras and related topics (Hiroshima – 1990). World Sci. Publ., London et al., 1991, 219–335.

[55] Vavilov N.A. An $A_3$-proof of structure theorems for Chevalley groups of types $E_6$ and $E_7$. J. Pure Appl. Algebra, 2007, 1-16.

[56] Vavilov N.A. Parabolic subgroups of Chevalley groups over commutative ring. Zapiski nauchnyh seminarov LOMI, 1982, 116, 20–43 (in Russian).

[57] Vavilov N.A., Plotkin E.B. Chevalley groups over commutative rings. I. Elementary calculations. Acta Applicandae Math., 1996, 45, 73–115.

[58] Vavilov N.A., Petrov V.A. On overgroups of $E_p(2l, R)$. Algebra and Analisys, 2003, 15(3), 72–114.

[59] Vavilov N.A., Gavriloich M.R. $A_2$-proof of structure theorems for Chevalley groups of types $E_6$ and $E_7$. Algebra and Analisys, 2004, 116(4), 54–87.

[60] Vavilov N.A., Gavriloich M.R., Nikoleno SI. Structure of Chevalley groups: proof from the book. Zapiski nauchnyh seminarov LOMI, 2006, 330, 36–76 (in Russian).

[61] Waterhouse W.C. Introduction to affine group schemes. Springer-Verlag, N.Y. et al., 1979.

[62] Waterhouse W.C. Automorphisms of $GL_n(R)$. Proc. Amer. Math. Soc., 1980, 79, 347–351.

[63] Waterhouse W.C. Automorphisms of quotients of $\prod GL(n_i)$. Pacif. J. Math., 1982, 79, 221–233.

[64] Waterhouse W.C. Automorphisms of $\det(X_{ij})$: the group scheme approach. Adv. Math., 1987, 65(2), 171–203.

[65] Zalesskiy A.E. Linear groups. Itogi Nauki. M., 1989, 114–228 (in Russian).

[66] Zelmanov E.I. Isomorphisms of general linear groups over associative rings. Siberian Mathematical Journal, 1985, 26(4), 49–67 (in Russian).