Analysis of a class of non linear subdivision schemes and associated multi-resolution transforms

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October 7, 2008

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Keywords: Non linear subdivision schemes, Non linear multi-resolutions, convergence, stability

Abstract

This paper is devoted to the convergence and stability analysis of a class of nonlinear subdivision schemes and associated multi-resolution transforms. These schemes are defined as a perturbation of a linear subdivision scheme. Assuming a contractivity property, stability and convergence are derived. These results are then applied to various schemes such as uncentered interpolatory linear scheme, WENO scheme [13], Power-P scheme [16] and a nonlinear scheme using local spherical coordinates [18].

Key Words. Non linear subdivision schemes, convergence, multi-resolution, interpolation, stability, convergence

AMS(MOS) subject classifications. 41A05, 41A10, 65D05, 65D17

*Research partially supported by European network “Breaking complexity” # HPRN-CT-2002-00286
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1 Introduction

Multi-resolution representations of discrete data are useful tools in several areas of application as image compression or adaptive methods for partial differential equations. In these applications, the ability of these representations to approximate the input data with high accuracy using a very small set of coefficients is a central property. Moreover, the stability of these representations in presence of perturbations (generated by compression or due to approximations) is a key point.

In the last decade, several attempts to improve the property of classical linear multi-resolutions have lead to nonlinear multi-resolutions. In many cases, this nonlinear nature hinders the proofs of convergence and stability. In [1], in the context of image compression, a new multi-resolution transform has been presented. This multi-resolution is based on an univariate nonlinear multi-resolution called PPH multi-resolution (see [12] in the context of convexity preserving). It has been analyzed in terms of convergence and stability of an associated subdivision scheme following an approach for data dependent multi-resolutions introduced in [5]. Due to nonlinearity, the stability of the PPH multi-resolution is not a consequence of the convergence of the associated subdivision scheme. It has been established in [2], presenting the PPH subdivision scheme as some perturbation of a linear scheme following [8], [14], [7] and [9].

The aim of the present paper is to generalize the results presented in [2] for a general family of nonlinear multi-resolution schemes associated to an interpolatory subdivision scheme \( S_{NL} : l^\infty(\mathbb{R}) \rightarrow l^\infty(\mathbb{R}) \) of the form:

\[
\forall f \in l^\infty(\mathbb{R}), \quad \forall n \in \mathbb{Z} \quad \begin{cases} 
S_{NL}(f)_{2n+1} = S(f)_{2n+1} + F(\delta f)_{2n+1}, \\
S_{NL}(f)_{2n} = f_n,
\end{cases}
\tag{1}
\]

where \( F \) is a nonlinear operator defined on \( l^\infty(\mathbb{R}) \), \( \delta \) is a linear and continuous operator on \( l^\infty(\mathbb{R}) \) and \( S \) is a linear and convergent subdivision scheme. Considering two subdivision schemes \( S_{NL} \) and \( S \), it is always possible to introduce the difference \( F = S_{NL} - S \). If one assume some properties of polynomial reproduction (see section [3]), as shown in [12], \( F \) is in fact a function of differences, i.e. of \( df \) defined by \( df = f_{n+1} - f_n \).

Theorems [1] and [2] that are the main results of this paper, establish that if \( F, S \) and \( \delta \) satisfy some natural properties, then the subdivision scheme is convergent and the multi-resolution is stable.

The paper is organized as follows. In section [2] we recall briefly the
Harten’s interpolatory multi-resolution framework which is the natural setting for our work. We precise the class of schemes under consideration and we establish the main results in section 3. Various applications are presented in section 4.

2 Harten’s framework and basic definitions

In Harten’s interpolatory multi-resolution, one considers a set of nested bi-infinite regular grids:

$$X^j = \{x^j_n\}_{n \in \mathbb{Z}}, \quad x^j_n = n2^{-j},$$

where $j$ stands for a scale parameter and $n$ controls the position.

The point-value discretization operators (or sampling operators) are defined by

$$D^j : f \in C(\mathbb{R}) \mapsto f^j = (f^j_n)_{n \in \mathbb{Z}} := (f(x^j_n))_{n \in \mathbb{Z}} \in V^j,$$

where $V^j$ is the space of real sequences and $C(\mathbb{R})$ the set of continuous functions on $\mathbb{R}$.

The reconstruction operators $R^j$ associated to this discretization are any right inverses of $D^j$ on $V^j$, that is, any operators $R^j$ satisfying:

$$(R^j f^j)(x^j_n) = f^j_n = f(x^j_n).$$

For any $j$, the operator defined by $D^j R^j+1$ acts between a fine scale $j+1$ and a coarser scale $j$. Here, it is a sub-sampling operator from $V^j+1$ to $V^j$. The operator defined by $D^j+1 R^j$ acts between a coarse scale $j$ and a finer scale $j+1$ and is called a prediction operator. A prediction operator can be considered as a subdivision scheme from $V^j$ to $V^j+1$. We say that the subdivision scheme $S$ defined by $(f^j) \mapsto S(f^j) = D^j+1 R^j(f^j)$ is uniformly convergent if:

$$\forall f \in l^\infty, \exists f^\infty \in C^0(\mathbb{R}) \text{ such that } \lim_{j \to +\infty} \sup_{n \in \mathbb{Z}} |S^j(f)_n - f^\infty(2^{-j}n)| = 0.$$

We note $f^\infty = S^\infty f$.

Since for most function $f$, $D^j+1 R^j f^j \neq f^{j+1}$, details, called $d^j$ and defined by $d^j = f^{j+1} - D^j+1 R^j f^j$, should be added to $D^j+1 R^j f^j$ to recover $f^{j+1}$ from $f^j$. The multi-resolution decomposition (see [3], [10], [11] for...
precisions) of $f^L$ is the sequence \{\(f^0, d^0, \ldots, d^{L-1}\)\}. Moreover, the multi-resolution transform is said to be stable if:

\[
\exists C \text{ such that } \forall f^L, \tilde{f}^L, j \leq L
\]

\[
\|f^j - \tilde{f}^j\|_\infty \leq C \left( \|f^0 - \tilde{f}^0\|_\infty + \sum_{k=0}^{j-1} \|d^k - \tilde{d}^k\|_\infty \right)
\]

(4)

\[
\|f^0 - \tilde{f}^0\|_\infty \leq C \|f^j - \tilde{f}^j\|_\infty,
\]

(5)

\[
\|d^k - \tilde{d}^k\|_\infty \leq C \|f^j - \tilde{f}^j\|_\infty, \quad \forall k, 0 \leq k \leq j - 1,
\]

(6)

where \{\(\tilde{f}^0, \tilde{d}^0, \ldots, \tilde{d}^{L-1}\)\} is the multi-resolution decomposition of \(\tilde{f}^L\).

When the prediction operator \(D_{j+1}R_j f^j\) is linear, the convergence of the associated subdivision scheme implies the stability of the multi-resolution analysis. In the non linear case, it is not the case and there is no general result for the multi-resolution analysis stability.

### 3 A Class of Nonlinear Subdivision Schemes

Introducing \(S\) a linear, reproducing polynomials \(^1\) up to degree \(P\) and convergent interpolatory subdivision scheme we consider nonlinear interpolatory subdivision schemes that write

\[
\left\{ \begin{array}{l}
S_{NL}(f^j)_{2n+1} = S(f^j)_{2n+1} + F(\delta f^j)_{2n+1} \\
S_{NL}(f^j)_{2n} = f^j_n
\end{array} \right.
\]

(7)

where \(F\) is a nonlinear operator defined on \(l^\infty(\mathbb{Z})\) and \(\delta\) is a continuous linear operator on \(l^\infty(\mathbb{Z})\).

#### 3.1 Convergence analysis

We have the following theorem related to the convergence of the nonlinear subdivision scheme \(S_{NL}:\)

**Theorem 1** If \(F, S\) and \(\delta\) verify:

\[
\exists M > 0 \text{ such that } \forall d \in l^\infty \quad \|F(d)\|_\infty \leq M \|d\|_\infty
\]

(8)

\[
\exists c < 1 \text{ such that } \|\delta S(f) + \delta F(\delta f)\|_\infty \leq c \|\delta f\|_\infty,
\]

(9)

\(^1\)The interpolatory subdivision scheme \(S\) reproduces polynomials of degree \(P\) if, for any polynomials \(P\) of degree less or equal to \(P\), if \(f_n = P(x_n^j)\) then \(S(f)_{2n+1} = P(x^j_{2n+1})\).
then the subdivision scheme $S_{NL}$ is uniformly convergent. Moreover, if $S$ is $C^\alpha$ convergent (i.e for all $f \in l^\infty(\mathbb{Z}), S^\infty(f) \in C^{\alpha-2}$) then, for all sequence $f \in l^\infty(\mathbb{Z}), S_{NL}^\infty(f) \in C^\beta$ with $\beta = \min(\alpha, \log_2(c))$.

**Proof** Using hypotheses (8) and (9) and the definition of $S_{NL}$, we get:

$$|S_{NL}(f^j)_{2n+1} - S(f^j)_{2n+1}| \leq M\|\delta f^j\|_\infty,$$

$$\|S_{NL}(f^j) - S(f^j)\|_\infty \leq M\|\delta(S_{NL}f^{j-1})\|_\infty,$$

$$\|S_{NL}(f^j) - S(f^j)\|_\infty \leq Mc\|\delta f^{j-1}\|_\infty,$$

that can be rewritten as:

$$\|S_{NL}(f^j) - S(f^j)\|_\infty \leq Mc^j\|\delta f^0\|_\infty.$$

Writing:

$$\|S_{NL}(f^j) - S(f^j)\|_\infty \leq M\|\delta f^0\|_\infty 2^{j\log_2(c)} \quad (10)$$

the convergence of the subdivision scheme $S_{NL}$ can be obtained applying theorem 3.3 of [7].

In our context, this theorem applies as follows:

If $S$ is a linear $C^\alpha$ convergent subdivision scheme reproducing polynomials up to degree $P$ and if $S_{NL}$ is a perturbation of $S$ in the sense that, calling $f^k := S_{NL}(f^0)$ for all $f^0 \in l^\infty$, 

$$\|S_{NL}(f^k) - S(f^k)\|_\infty = O(2^{-\nu k}),$$

then $S_{NL}$ is $C^\beta$ convergent with $\beta = \geq \min(P, s_L, \nu)$.

It follows that if $S$ is $C^\alpha$ convergent then $S_{NL}$ is at least $C^\beta$ convergent with $\beta = \min(\alpha, \log_2(c))$.

□

**Remark 1** When $F$ is linear, theorem [1] is a consequence of theorem 6.2 in [7].

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2For $0 < \alpha \leq 1$, $C^\alpha = \{f$ continuous, bounded and verifying $\forall \alpha' < \alpha, \exists C > 0, \forall x, y \in \mathbb{R}, |f(x) - f(y)| \leq C|x - y|^{\alpha'} \}$. For $\alpha > 1$ with $\alpha = p + r > 0, p \in \mathbb{N}$ and $0 < r < 1$, $C^\alpha(\mathbb{R}) = \{f$ with $f^{(p}) \in C^r \}$. 

5
Remark 2 In many of our examples, $S$ is the two point centered linear scheme defined by $S(f^j)_{2n+1} = \frac{f_{j} + f_{j+1}^{i}}{2}$ which limit function is in $C^1$. Therefore, as soon as the non linear scheme $S_{NL}$ verifies hypothesis (8) and (9) with $c \geq \frac{1}{2}$, the $S_{NL}$ is $C^{(\log_2(c))^{-1}}$ convergent.

Remark 3 Hypothesis (9) can be weakened as:

$$\exists p \in \mathbb{N} \quad \exists c < 1 \quad \text{such that} \quad \|\delta(S_{NL}^p f)\|_\infty \leq c\|f\|_\infty.$$  

The proof remains the same except that:

$$\|S_{NL}(f^j) - S(f^j)\|_\infty \leq M\|\delta f^j\|_\infty$$

becomes:

$$\|S_{NL}(f^j) - S(f^j)\|_\infty \leq M\|\delta(S_{NL}^p f^j - p)\|_\infty,$$

$$\|S_{NL}(f^j) - S(f^j)\|_\infty \leq M\|\delta f^j - p\|_\infty,$$

that can be rewritten, for $j \equiv i[p]$, as:

$$\|S_{NL}(f^j) - S(f^j)\|_\infty \leq M\frac{j-i}{p} \|\delta f^i\|_\infty.$$  

The conclusion is reached applying theorem 3.3 of [7].

Remark 4 A straightforward generalization of theorem [7] can be obtained introducing two linear operator $\delta_1$, $\delta_2$ and a perturbation of the form $F(\delta_1 f, \delta_2 f)$. Under the following hypotheses:

$$\exists M > 0 \quad \text{such that} \quad |F(d, d')| \leq M \max \{\|d\|_\infty, \|d'\|_\infty\}, \quad (11)$$

$$\exists c > 1 \quad \text{such that} \quad \|\delta_1(S_{NL}(f))\|_\infty \leq c \max \{\|\delta_1 f\|_\infty, \|\delta_2 f\|_\infty\}, \quad (12)$$

$$\|\delta_2(S_{NL}(f))\|_\infty \leq c \max \{\|\delta_1 f\|_\infty, \|\delta_2 f\|_\infty\}, \quad (13)$$

for all $d, d' \in l^\infty, f \in l^\infty$, the scheme $S_{NL}$ is uniformly convergent.

Remark 5 We can also apply theorem [7] to bi-variate schemes written as

$$S_{NL}(x^j, y^j) = \begin{pmatrix} S_{NL_1}(x^j, y^j)_{2n+1} \\ S_{NL_2}(x^j, y^j)_{2n+1} \end{pmatrix} = \begin{pmatrix} x^j_{2n+1} \\ y^j_{2n+1} \end{pmatrix} = \begin{pmatrix} S(x^j)_{2n+1} + F_1(\delta x^j, \delta y^j) \\ S(y^j)_{2n+1} + F_1(\delta x^j, \delta y^j) \end{pmatrix}$$

If the following conditions are satisfied for $i = 1, 2$

$$\exists M > 0 \quad \text{such that} \quad |F_i(d, d')| \leq M \max \{\|d\|_\infty, \|d'\|_\infty\},$$

$$\exists c > 1 \quad \text{such that} \quad \|\delta(S_{NL_i}(x, y))\|_\infty \leq c \max \{\|\delta x\|_\infty, \|\delta y\|_\infty\},$$

for all $d, d', x, y \in l^\infty$, the scheme $S_{NL}$ is uniformly convergent.
3.2 Stability analysis

We now consider the multi-resolution analysis associated to the subdivision scheme (7) recalling that, for any sequence $f^j$, the details $d^j$ are defined by $d^j_n = f^j_{2n+1} - S_{NL}(f^j)_{2n+1}$.

We have the following theorem concerning the stability of the multi-resolution:

**Theorem 2** If $F, S$ and $\delta$ verify: $\exists M > 0, c < 1$ such that $\forall f, g, d_1, d_2$,

\[
\begin{align*}
\|F(d_1) - F(d_2)\|_\infty &\leq M\|d_1 - d_2\|_\infty, \\
\|\delta(S_{NL}f - S_{NL}g)\|_\infty &\leq c\|\delta(f - g)\|_\infty,
\end{align*}
\]

then the multi-resolution transform associated to the non linear subdivision scheme $S_{NL}$ is stable.

**Proof**

We first prove (4):

Due to the interpolatory property, we only consider $|f^j_{2n+1} - \tilde{f}^j_{2n+1}|$.

Since $S$ is a convergent linear scheme, we have, using the stability of the linear scheme $S$: $\exists C' > 0$ such that

\[
|f^j_{2n+1} - \tilde{f}^j_{2n+1}| \leq C' \left( \|f^0 - \tilde{f}^0\|_\infty + \sum_{k=1}^j \|f^k - S(f^{k-1}) - \tilde{f}^k + S(\tilde{f}^{k-1})\|_\infty \right)
\]

\[
\leq C' \left( \|f^0 - \tilde{f}^0\|_\infty + \sum_{k=1}^j \|d^k - \tilde{d}^k\|_\infty + M \sum_{k=1}^j \|\delta(f^{k-1}) - \tilde{\delta}(\tilde{f}^{k-1})\|_\infty \right).
\]

From (14):

\[
|f^j_{2n+1} - \tilde{f}^j_{2n+1}| \leq C' \left( \|f^0 - \tilde{f}^0\|_\infty + \sum_{k=0}^{j-1} \|d^k - \tilde{d}^k\|_\infty + M \sum_{k=1}^j \|\delta(f^{k-1}) - \tilde{\delta}(\tilde{f}^{k-1})\|_\infty \right).
\]

Concentrating on the last right hand side term we get:
\[ \sum_{k=1}^{j} |\delta(f^{k-1}) - \delta(\tilde{f}^{k-1})|_\infty \leq |\delta(f^0) - \delta(\tilde{f}^0)|_\infty \]

\[ + \sum_{k=2}^{j} \left( |\delta(S_{NL}f^{k-2}) - \delta(S_{NL}\tilde{f}^{k-2})|_\infty + |\delta d^{k-2} - \delta \tilde{d}^{k-2}|_\infty \right) \]

From (15) we get:

\[ \sum_{k=1}^{j} |\delta(f^{k-1}) - \delta(\tilde{f}^{k-1})|_\infty \leq |\delta(f^0) - \delta(\tilde{f}^0)|_\infty + \sum_{k=0}^{j-2} \left( c |\delta f^0 - \delta \tilde{f}^0|_\infty + \sum_{l=0}^{k} c^{k-l} |\delta d^l - \delta \tilde{d}^l|_\infty \right) \]

Since \(0 < c < 1\) we get finally:

\[ \|f^j - \tilde{f}^j\|_\infty \leq C'|f^0 - \tilde{f}^0|_\infty + C'' \sum_{k=0}^{j-1} \|d^k - \tilde{d}^k\|_\infty \]

\[ + MC' \frac{1}{1-c} \left( |\delta(f^0) - \delta(\tilde{f}^0)|_\infty + \sum_{k=0}^{j-2} |\delta d^k - \delta \tilde{d}^k|_\infty \right) , \]

and, using the continuity of \(\delta\), we get (11) with a constant

\[ C = C' + \frac{MC'|\delta|_\infty}{1-c} . \]

We now establish (5) et (6).

Equation (5) is a direct consequence of the interpolatory properties.

For (6), we have, for \(0 \leq k \leq j-1\):

\[ |d_n^k - \tilde{d}_n^k| \leq \|f^{k+1} - \tilde{f}^{k+1} - S(f^k) - S(\tilde{f}^k)|_\infty + \|F(\delta f^k) - F(\delta \tilde{f}^k)|_\infty . \]

Using the property (10) for the multi-resolution associated to \(S\), hypothesis (14) and the continuity of \(\delta\), we have:

\[ |d_n^k - \tilde{d}_n^k| \leq C'|f^j - \tilde{f}^j|_\infty + M|\delta|_\infty \|f^k - \tilde{f}^k\|_\infty . \]
From (5) for the multi-resolution associated to $S_{NL}$ we have
\[ |d_n^k - \tilde{d}_n^k| \leq C'||f^j - \tilde{f}^j||_\infty + M||\delta||_\infty||f^{j-1} - \tilde{f}^{j-1}||_\infty, \]
and therefore we get (6) with $C = C' + M||\delta||_\infty$.
\[ \square \]

Remark 6  As previously, we can again consider a weaker formulation for hypothesis (15) such as:
\[ \exists p \in \mathbb{N}, \exists c < 1 \text{ such that } ||\delta(S^{p}_{NL}f - S^{p}_{NL}g)||_\infty \leq c||\delta(f - g)||_\infty. \]
Under this hypothesis, the stability of the subdivision scheme can still be established. However, the multi-resolution stability is not ensured. To get it, a stronger hypothesis like:
\[ \exists p \in \mathbb{N}, \exists c < 1, \text{ such that } \]
\[ ||\delta(f^p f - g^p)||_\infty \leq c||\delta(f - g)||_\infty + M\sum_{k=0}^{p-2} ||d^k (f) - d^k (g)||_\infty, \]
is required.

4  Applications

This section is devoted to applications of the previous results to three specific subdivision schemes (linear and nonlinear) available in the literature. We provide for each of them, the proofs of convergence and stability.

In all this section, given $f^j = (f^j_k)_{k \in \mathbb{Z}}$ we note:
\[ df^j = (df^j_n)_{n \in \mathbb{Z}} \text{ with } df^j_n = f^j_{n+1} - f^j_n, \]
\[ Df^j = (Df^j_n)_{n \in \mathbb{Z}} \text{ with } Df^j_n = f^j_{n+1} - 2f^j_n + f^j_{n-1}, \]
and, more generally \(D^j f^j = (D^j f^j_n)_{n \in \mathbb{Z}}\) with:

\[
D^j f^j_n = D(D^{j-1} f^j)_n = \sum_{i=0}^{2l} (-1)^i C^i_{2l} f_{n-l+i} \quad \text{with} \quad C^i_k = \frac{k!}{i!(k-i)!}.
\]

(19)

### 4.1 Multi-resolution analysis associated to a linear fully non centered Lagrange interpolatory subdivision scheme

As it has been said before, for linear scheme, the stability of the multi-resolution analysis is a consequence of the convergence of the subdivision scheme (see \[11\]). Therefore, we only consider here the convergence of the subdivision scheme.

The convergence of centered linear interpolatory schemes is well known since Delauriers and Debuc \[6\]. For linear but non centered schemes there is no general results of convergence. Moreover the general tools proposed in \[15\] are very fastidious to apply and don’t provide general results.

In this subsection, we focus on completely decentred Lagrange interpolatory linear schemes. In order to apply our theoretical results, we consider \(S\) the two point centered linear scheme and express any right hand side excentred scheme \(S_P\) (where \(P\) stands for the number of point of the considered stencil) as a perturbation of it. Precisely, if we write \(S_P(f^j)_{2n+1} = S(f^j)_{2n+1} + F_P(\delta f^j)_{2n+1}\) we get:

If \(P\) is even,

\[
F_P(\delta f^j)_{2n+1} = + \sum_{k=2}^{P-2} \text{even } k \ D^k f_{n+\frac{k}{2}+1} \frac{(2k-1)!}{2^k (k-1)! (k+1)!} \\
- \sum_{k=1}^{P-3} \text{odd } k \ D^k f_{n+\frac{k+1}{2}+1} \frac{(4k+5)(2k-1)!}{2^{2k+1} (k-1)! (k+2)!}.
\]

and, if \(P\) is odd

\[
F_P(\delta f^j)_{2n+1} = + \sum_{k=2}^{P-3} \text{even } k \ D^k f_{n+\frac{k}{2}+1} \frac{(2k-1)!}{2^{2k} (k-1)! (k+1)!} \\
- \sum_{k=1}^{P-4} \text{odd } k \ D^k f_{n+\frac{k+1}{2}+1} \frac{(4k+5)(2k-1)!}{2^{2k+1} (k-1)! (k+2)!} \\
- D^{\frac{P-1}{2}} f_{n+\frac{N-1}{2}} \frac{(2N-3)!}{2^N (N-3)! (N-1)!}.
\]
Numerical evaluation of the perturbation terms for $4 \leq P \leq 9$ is given in table 1.

| $P$ | $F_P(Df)_{2n+1}$ |
|-----|------------------|
| 4   | $-\frac{3}{16}Df_{n+1} + \frac{1}{16}Df_{n+2}$ |
| 5   | $-\frac{3}{16}Df_{n+1} + \frac{1}{16}Df_{n+2} - \frac{5}{128}D^2f_{n+2}$ |
| 6   | $-\frac{3}{16}Df_{n+1} + \frac{1}{16}Df_{n+2} - \frac{17}{256}D^2f_{n+2} + \frac{7}{256}D^2f_{n+3}$ |
| 7   | $-\frac{3}{16}Df_{n+1} + \frac{1}{16}Df_{n+2} - \frac{17}{256}D^2f_{n+2} + \frac{7}{256}D^2f_{n+3} - \frac{21}{1024}D^3f_{n+3}$ |
| 8   | $-\frac{3}{16}Df_{n+1} + \frac{1}{16}Df_{n+2} - \frac{17}{256}D^2f_{n+2} + \frac{7}{256}D^2f_{n+3} - \frac{75}{2048}D^3f_{n+3} + \frac{33}{2048}D^3f_{n+4}$ |
| 9   | $-\frac{3}{16}Df_{n+1} + \frac{1}{16}Df_{n+2} - \frac{17}{256}D^2f_{n+2} + \frac{7}{256}D^2f_{n+3} - \frac{75}{2048}D^3f_{n+3} + \frac{33}{2048}D^3f_{n+4} - \frac{429}{32768}D^4f_{n+4}$ |

Table 1: Perturbation term $F_P(Df)_{2n+1}$ for different values of $P$

It also appears that $S_P$ can be written naturally as a perturbation of $S_{P-2}$, for even values of $P$ and as a perturbation of $S_{P-1}$ for odd values of $P$. Indeed, we have:

When $P$ is even:

$$S_P(f)_{2n+1} = S_{P-2}(f)_{2n+1} + \frac{(2P-3)!}{2^{2(P-2)}(P-3)!(P-1)!} D^{P-2}_{\frac{P}{2}} f_{n+\frac{P}{2}} - \frac{(2P-3)!}{2^{2(P-1)}(P-1)!(P-1)!} D^{P-2}_{\frac{P}{2}} f_{n+\frac{P}{2}-1},$$
(20)

and when $P$ is odd:

$$S_P(f)_{2n+1} = S_{P-1}(f)_{2n+1} - \frac{(2P-3)!}{2^{2(P-3)}(P-1)!(P-2)!} D^{P-1}_{\frac{P-1}{2}} f_{n+\frac{P-1}{2}}.$$
(21)

In both cases, it is easy to check that the function $F$ defined by $F = F(D^{P-2}_{\frac{P}{2}} f)$ when $P$ is even and by $F = F(D^{P-1}_{\frac{P-1}{2}} f)$ when $P$ is odd, is linear and continuous. Therefore, the convergence can be reached as soon as the contractivity hypothesis (9) for $D^{P-2}_{\frac{P}{2}}$ or $D^{P-1}_{\frac{P-1}{2}}$ is satisfied.

Direct calculations provide the estimates gathered in table 2. It then follows from theorem 1 that all the fully excentred interpolatory subdivision
4.2 The 6 points WENO subdivision scheme

\textit{WENO} subdivision schemes \cite{13}, are constructed using convex combination of different interpolatory polynomials of fixed degree. For degree 3 and therefore a 6 point stencil, the \textit{WENO} – 6 subdivision is given by:

\[
S_{\text{weno}}(f^j)_{2n+1} = \frac{\alpha_2}{16} f^j_{n-2} - \frac{5\alpha_2 + \alpha_1}{16} f^j_{n-1} + (1 + \frac{5\alpha_2 + 2\alpha_1}{8}) f^j_n + (1 + \frac{5\alpha_0 + 2\alpha_1}{8}) f^j_{n+1} - \frac{5\alpha_0 + \alpha_1}{16} f^j_{n+2} + \frac{\alpha_0}{16} f^j_{n+3}
\]
where the coefficients $\alpha_i$ control the convex combination and therefore satisfy $\alpha_i \geq 0$ and $\alpha_0 + \alpha_1 + \alpha_2 = 1$.

In [13], these coefficients are defined as:

$$\alpha_i = \frac{a_i}{a_0 + a_1 + a_2}$$

with

$$a_i = \frac{d_i}{(\epsilon + b_i)^2}$$

where $b_i$, defined as a function of the first difference $df$ is an indicator of smoothness while $d_i$ and $\epsilon$ are fixed positive constants. A set of possible values for these constants is suggested in [17].
The convergence of the associated subdivision scheme has been studied in [5]. We present an alternative proof, using theorem 1.

First, the WENO – 6 subdivision scheme is written as a perturbation of the linear two point interpolation scheme as:

\[
S_{\text{weno}}(f^j)_{2n+1} = \frac{f^j_n + f^j_{n+1}}{2} + \frac{\alpha_0}{16} Df^j_{n+2} - \frac{3\alpha_0 + \alpha_1}{16} Df^j_{n+1} - \frac{\alpha_1 + 3\alpha_2}{16} Df^j_n + \frac{\alpha_2}{16} Df^j_{n-1}
\]  (22)

with \(\alpha_0 = \alpha_0(df^j)\), \(\alpha_1 = \alpha_1(df^j)\) and \(\alpha_2 = \alpha_2(df^j)\).

We then have the following proposition:

**Proposition 1** The WENO – 6 subdivision scheme is convergent and, for any initial sequence \(f^j\), the limit function belongs to \(C^{-\log_2(4)}\).

**Proof**

According to remark 4, the proof can be performed in three steps considering that \(F\) is a function of \(df^j\) and \(Df^j\).

First, according to the definition of \(F\) and to the properties of \(\alpha_i\), we have

\[
|F(d, D)| \leq \frac{1}{2} \max(||d||_\infty, ||D||_\infty).
\]

Second, we prove (10) for the first difference operator \(d\):

We have, for \(f \in l^\infty\):

\[
d(S_{\text{weno}}(f))_k = S_{\text{weno}}(f)_{k+1} - S_{\text{weno}}(f)_k
\]

We have to consider two cases, according to the parity of \(k\). We give the details for \(k = 2n + 1\), the even case being similar.
\[
d(S_{\text{weno}}(f))_{2n+1} = S_{\text{weno}}(f)_{2n+2} - S_{\text{weno}}(f)_{2n+1} \\
= f_{n+1} - \frac{f_n + f_{n+1}}{2} - \frac{\alpha_0}{16} Df_{n+2} + \frac{3\alpha_0 + \alpha_1}{16} Df_{n+1} \\
+ \frac{\alpha_1 + 3\alpha_2}{16} Df_n - \frac{\alpha_2}{16} Df_{n-1} \\
= df_n + \frac{\alpha_0}{16} Df_{n+2} + \frac{3\alpha_0 + \alpha_1}{16} Df_{n+1} \\
+ \frac{\alpha_1 + 3\alpha_2}{16} Df_n - \frac{\alpha_2}{16} Df_{n-1} 
\]

Since \(\alpha_0 + \alpha_1 + \alpha_2 = 1\) and \(0 < \alpha_1 < 1\), we have:

\[
|d(S_{\text{weno}}(f))_{2n+1}| \leq \frac{1}{2} \|df\|_\infty + \left( \frac{\alpha_0}{16} + \frac{3\alpha_0}{16} + \frac{\alpha_1}{16} + \frac{\alpha_1}{16} + \frac{3\alpha_2}{16} + \frac{\alpha_2}{16} \right) \|Df\|_\infty, \\
\leq \frac{1}{2} \|df\|_\infty + \frac{4 - 2\alpha_1}{16} \|Df\|_\infty, \\
\leq \frac{1}{2} \|df\|_\infty + \frac{1}{4} \|Df\|_\infty, \\
\leq \frac{3}{4} \max(\|df\|_\infty, \|Df\|_\infty). \tag{23}
\]

Third, we prove inequality (9) for the second difference operator \(D\).

Again, two cases have to be considered:
• For \(k=2n+1\), then:

\[
D(S_{\text{weno}}(f))_{2n+1} = f_{n+1} - 2S_{\text{weno}}(f)_{2n+1} + f_n, \\
= -\frac{\alpha_0}{8} Df_{n+2} + \frac{3\alpha_0 + \alpha_1}{8} Df_{n+1} + \frac{\alpha_1 + 3\alpha_2}{8} Df_n - \frac{\alpha_2}{8} Df_{n-1}. 
\]

Using \(0 < \alpha_0 < 1\), \(0 < \alpha_1 < 1\), \(0 < \alpha_2 < 1\) and \(\alpha_0 + \alpha_1 + \alpha_2 = 1\), we get:

\[
|D(S_{\text{weno}}(f))_{2n+1}| \leq \frac{4\alpha_0 + 2\alpha_1 + 4\alpha_2}{8} \|Df\|_\infty, \\
\leq \frac{4 - 2\alpha_1}{8} \|Df\|_\infty, \\
\leq \frac{1}{2} \|Df\|_\infty. \tag{24}
\]
• For $k=2n$, then:

\[
D(S_{\text{weno}}(f))_{2n} = S_{\text{weno}}(f)_{2n+1} - 2f_n + S_{\text{weno}}(f)_{2n-1},
\]

\[
= \frac{f_{n+1} - 2f_n + f_{n-1}}{2} + \frac{\alpha_0}{16} Df_{n+2}
\]

\[
- \frac{2\alpha_0 + \alpha_1}{16} Df_{n+1} - \frac{3\alpha_0 + 2\alpha_1}{16} + \frac{3\alpha_2}{16} Df_n
\]

\[
- \frac{\alpha_1 + 2\alpha_2}{16} Df_{n-1} + \frac{\alpha_2}{16} Df_{n-2}.
\]

Using $\alpha_0 + \alpha_1 + \alpha_2 = 1$, we get:

\[
D(S_{\text{weno}}(f))_{2n} = \frac{5}{16} Df_n + \frac{\alpha_0}{16} Df_{n+2} - \frac{2\alpha_0 + \alpha_1}{16} Df_{n+1} + \frac{\alpha_1}{16} Df_n
\]

\[
- \frac{\alpha_1 + 2\alpha_2}{16} Df_{n-1} + \frac{\alpha_2}{16} Df_{n-2}
\]

Then, with $0 < \alpha_0 < 1$, $0 < \alpha_1 < 1$ and $0 < \alpha_2 < 1$:

\[
|D(S_{\text{weno}}(f))_{2n}| \leq \left( \frac{5}{16} + \frac{3\alpha_0 + 3\alpha_1 + 3\alpha_2}{16} \right) \|Df\|_{\infty},
\]

\[
\leq \left( \frac{5}{16} + \frac{3}{16} \right) \|Df\|_{\infty},
\]

\[
\leq \frac{1}{2} \|Df\|_{\infty}.
\]  

(25)

Therefore, from (23), (24) and (25), we obtain the inequality:

\[
\exists c < 1 \quad \forall f \in L^\infty \quad \max (\|d(S_{\text{weno}}(f))\|_{\infty}, \|D(S_{\text{weno}}(f))\|_{\infty}) \leq \frac{3}{4} \max (\|df\|_{\infty}, \|Df\|_{\infty}).
\]

Finally, using remarks 4 and 2 we get the convergence of the \textit{WENO}–6 subdivision scheme to a $C^{-\log_2(\frac{3}{4})}$ function.

\[\square\]
4.3 Power-P subdivision scheme: definition and convergence

In the same vein as the PPH scheme ([1]), the power P scheme is a four point scheme based on a piecewise degree 3 polynomial prediction. Considering $S_L$, the centered four point Lagrange interpolation prediction that reads:

$$(S_L(f))_{2n+1} = \frac{f_n + f_{n+1}}{2} - \frac{1}{8} \frac{Df_{n+1} + Df_n}{2},$$

the definition of the Power-P subdivision scheme is based on the substitution of the arithmetic mean of second order differences, $\frac{Df_{j+1}+Df_j}{2}$, by a general mean $power_p(Df_j, Df_{j+1})$ defined in [16] for any integer $p \geq 1$, and any couple $(x, y)$ as:

$$power_p(x, y) = \frac{\text{sign}(x) + \text{sign}(y)}{2} \frac{x + y}{2} \left(1 - \frac{|x - y|}{x + y}^p\right).$$

(27)

Note that it coincides for $p = 1$, with the arithmetic mean and for $p = 2$ with the geometric mean. The Power-P subdivision scheme then naturally appears as a perturbation of the linear two point interpolation scheme since it is defined by

$$S_{power_p}(f_j)_{2n+1} = \frac{f_n + f_{n+1}}{2} - \frac{1}{8} power_p(Df_j^n, Df_{j+1}^n).$$

(28)

Before establishing the convergence theorem we first prove the following lemma:

**Lemma 1** For any $(x, y), (x', y') \in \mathbb{R}^2$, the function $power_p$ satisfies the following properties:

1. $power_p(x, y) = power_p(y, x)$
2. $power_p(x, y) = 0$ if $xy \leq 0$
3. $power_p(-x, -y) = -power_p(x, y)$
4. $|power_p(x, y)| \leq \max(|x|, |y|)$
5. $|power_p(x, y)| \leq p \min(|x|, |y|)$
Proof

Claims of 1 – 4 are obvious;
Inequality 5 comes from the equality

\[
power_p(x, y) = \frac{\text{sign}(x) + \text{sign}(y)}{2} \min(x, y) \left[1 + \frac{|x - y|}{x + y} + \cdots + \frac{|x - y|^{p-1}}{x + y}\right].
\]

□

We then have the following proposition:

**Proposition 2** The Power P subdivision scheme is uniformly convergent
and, for any initial sequence \(f\), the limit function belongs to \(C^{1-}\) for \(p \leq 4\) and \(C^{-\log_2\left(\frac{3}{4}\right)}\) for \(p \geq 5\).

Proof

Here again, the hypotheses of the general theorem must be checked:
We first check hypothesis (8). Using property 4 of lemma we obtain
for \(d \in l^\infty\):

\[
|F(d)| \leq \frac{1}{8} \max(|d_n|, |d_{n+1}|)
\]

\[
|F(d)| \leq \frac{1}{8} ||d||_\infty
\]

Then we consider hypothesis (9):
We study as before two different cases:

- For \(k = 2n + 1\):

\[
D(S_{power_p}(f))_{2n+1} = f_n - 2S_{power_p}(f)_{2n+1} + f_n + 1
\]

\[
= f_{n+1} + f_n - 2\frac{f_n + f_{n+1}}{2} + 2^\frac{1}{8}power_p(Df_n, Df_{n+1})
\]

\[
= \frac{1}{4}power_p(Df_n, Df_{n+1})
\]
From property 4 of lemma 1 we get:

\[ |D(S_{power}(f))_{2n+1}| \leq \frac{1}{4} \|Df\|_\infty. \]  \hspace{1cm} (29)

• For \( k = 2n \):

\[
D(S_{power}(f))_{2n} = S_{power}(f)_{2n-1} - 2f_n + S_{power}(f)_{2n+1} \\
= \frac{f_n + f_{n+1}}{2} - \frac{1}{8} power_p(Df_n, Df_{n+1}) - 2f_n \\
+ \frac{f_{n-1} + f_n}{2} - \frac{1}{8} power_p(Df_{n-1}, Df_n) \\
= \frac{Df_n}{2} - \frac{1}{8} (power_p(Df_n, Df_{n+1}) + power_p(Df_{n-1}, Df_n))
\]

For \( p \geq 5 \), from property 4 of lemma 1 we get:

\[ |D(S_{power}(f))_{2n}| \leq \frac{3}{4} \|Df\|_\infty. \]  \hspace{1cm} (30)

For \( p \leq 4 \), we note \( D(S_{power}(f))_{2n} = Z(Df_n, Df_{n+1}, Df_{n-1}) \) with

\[ Z(x, y, z) = \frac{x}{2} - \frac{1}{8} (power_p(x, y) + power_p(x, z)) \]

From definition 27 and property 4 and 5 of lemma 1, we have,

if \( x > 0 \),

\[
\frac{x}{2} - \frac{1}{8} (\max(x, y) + \max(x, z)) \leq Z(x, y, z) \leq \frac{x}{2} \\
\frac{x}{4} \leq Z(x, y, z) \leq \frac{x}{2} \\
0 \leq |Z(x, y, z)| \leq \frac{1}{2} |x|
\]
if $x < 0$,

\[
\begin{align*}
\frac{x}{2} & \leq Z(x, y, z) \leq \frac{x}{2} + \frac{p}{8} (\min (|x|, |y|) + \min (|x|, |z|)) \\
\frac{x}{2} & \leq Z(x, y, z) \leq \left(\frac{p}{4} - \frac{1}{2}\right)|x| \\
0 & \leq |Z(x, y, z)| \leq \frac{1}{2}|x|
\end{align*}
\]

Finally,

\[
|D(S_{\text{power}, p}(f))_{2n}| \leq \frac{1}{2} \|Df\|_{\infty}
\]

(31)

From (29), (30) and (31), we obtain:

\[
\begin{align*}
\|D S_{\text{power}, p}(f)\|_{\infty} & \leq \frac{1}{2} \|Df\|_{\infty} \quad \text{for } p \leq 4. \\
\|D S_{\text{power}, p}(f)\|_{\infty} & \leq \frac{3}{4} \|Df\|_{\infty} \quad \text{for } p \geq 5.
\end{align*}
\]

Finally, theorem 1 and remark 2 provides the convergence to a $C^1$ if $p \leq 4$ and $C^{-\log_2(\frac{7}{4})^-} (\mathbb{R})$ if $p \geq 5$.

\[\square\]

4.4 The convergence of a non linear scheme using spherical coordinates

The non linear subdivision scheme studied in this section is defined in [18] where it is considered as a non regular interpolatory subdivision scheme using local spherical coordinates. Here, we consider it as a regular subdivision scheme applied to the $\mathbb{R}^2$ point sequence $P_n^j(x_n^j, f_n^j)_{n \in \mathbb{Z}}$. The resulting scheme reads (see [18]):

\[
\begin{align*}
\left(\begin{array}{c}
x_{2n+1}^{j+1} \\
f_{2n+1}^{j+1}
\end{array}\right) = \left(\begin{array}{c}
\frac{x_n^j + x_{n+1}^j}{2} \\
\frac{f_n^j + f_{n+1}^j}{2}
\end{array}\right) + \frac{r_n^j}{4} \left(\begin{array}{c}
\cos(\theta_n^j + h(\alpha_n^j)) - \cos(\theta_{n+1}^j + h(\beta_{n+1}^j)) \\
\sin(\theta_n^j + h(\alpha_n^j)) - \sin(\theta_{n+1}^j + h(\beta_{n+1}^j))
\end{array}\right)
\end{align*}
\]

(32)
with:

$$r_j^n = \sqrt{(x_j^{n+1} - x_j^n)^2 + (f_j^{n+1} - f_j^n)^2}, \quad (33)$$

$$\theta_j^n = \arctan \left( \frac{f_j^{n+1} - f_j^n}{x_j^{n+1} - x_j^n} \right), \quad (34)$$

$$\gamma_j^n = \arctan \left( \frac{f_j^{n+1} - f_j^n}{x_j^{n+1} - x_j^n} \right), \quad (35)$$

$$\alpha_j^n = \gamma_j^n - \theta_j^n, \quad (36)$$

$$\beta_{n+1}^j = \gamma_{n+1}^j - \theta_{n+1}^j, \quad (37)$$

and, $\theta_j^n, \gamma_j^n \in [-\pi/2, -\pi/2]$.

As explained in [18], the design of $h$, is performed to produce regular limit functions. It is then defined as as a $C^1$ function that is contractive for small values of $\alpha$ and that coincides with identity for large value of $\alpha$. Note that $h = 0$ provides the classical linear two point centered scheme.

In our context, we will note this scheme $S_{spherical}$, and $S_1, S_2$ will stand for the schemes associated to each coordinates: We then get

$$S_1(x, f)_{2n+1} = \frac{x_n + x_{n+1}}{2} + (F_1(dx, df))_{2n+1},$$

$$S_2(x, f)_{2n+1} = \frac{f_n + f_{n+1}}{2} + (F_2(dx, df))_{2n+1},$$

with

$$(F_1(dx, df))_{2n+1} = \frac{r_j^n}{4} \left( \cos (\theta_j^n + h(\alpha_j^n)) - \cos (\theta_{n+1}^j + h(\beta_{n+1}^j)) \right),$$

$$(F_2(dx, df))_{2n+1} = \frac{r_j^n}{4} \left( \sin (\theta_j^n + h(\alpha_j^n)) - \sin (\theta_{n+1}^j + h(\beta_{n+1}^j)) \right).$$

From [33, 34, 35], $r_n, \theta_n$ and $\gamma_n$ can be written using the first divided difference $(df^j, dx^j)$ as:

$$r_n = \sqrt{(dx_j^n)^2 + (df_j^n)^2}$$

$$\theta_n = \arctan \left( \frac{df_j^n + df_{j-1}^n}{dx_j^n + dx_{j-1}^n} \right)$$

$$\gamma_n = \arctan \left( \frac{df_j^n}{dx_j^n} \right)$$
as well as $\alpha_n$ and $\beta_n$ thanks to (36) and (37).

We then have the following proposition:

**Proposition 3**  The scheme $S_{\text{spherical}}$ defined in (32) is convergent.

**Proof**

We again check the hypotheses of theorem (1) generalized to $\mathbb{R}^2$ according to remark 5. We have,

$$r_n \leq \sqrt{2} \max (|dx_n|, |df_n|), \quad (38)$$

and therefore, for $i = 1, 2$:

$$|(F_i(dx, df))_{2n+1}| \leq \frac{\sqrt{2} \max (|dx_n|, |df_n|)}{4},$$

$$\leq \frac{\sqrt{2}}{2} \max (||dx||_{\infty}, ||df||_{\infty}),$$

that shows that the hypothesis (8) of theorem (1) is satisfied.

We now check hypothesis (9). For $f \in l^\infty$ we have

$$d(S_1(x, f))_{2n} = S_1(x, f)_{2n+1} - S_1(x, f)_{2n}$$

$$= \frac{x_n + x_{n+1}}{2} + \frac{r_n}{4} (\cos (\theta_n + h(\alpha_n)) - \cos (\theta_{n+1} + h(\beta_{n+1}))) - x_n$$

$$= \frac{x_{n+1} - x_n}{2} + \frac{r_n}{4} (\cos (\theta_n + h(\alpha_n)) - \cos (\theta_{n+1} + h(\beta_{n+1}))),$$

and therefore

$$|d(S_1(x, f))_{2n}| \leq \frac{||dx||_{\infty}}{2} + \frac{\sqrt{2} \max (||dx||_{\infty}, ||df||_{\infty})}{4} |\theta_n + h(\alpha_n) - \theta_{n+1} - h(\beta_{n+1})|.$$  

Using the definitions of $\alpha_n$ and $\beta_n$ we get

$$|d(S_1(x, f))_{2n}| \leq \frac{||dx||_{\infty}}{2} + \frac{\sqrt{2} \max (||dx||_{\infty}, ||df||_{\infty})}{4} |\theta_n + h(\gamma_n - \theta_n) - (\theta_{n+1} + h(\gamma_n - \theta_{n+1}))|.$$  

22
and
\[
|d(S_1(x, f))_{2n}| \leq \frac{\|dx\|_\infty}{2} + \frac{\sqrt{2}}{4} \max_{x \in [-\pi, \pi]} (\|dx\|_\infty, |df|_\infty) \max_{x \in [-\pi, \pi]} (1 - h'(x)) |\theta_n - \theta_{n+1}|
\]
\[
\leq \left( \frac{1}{2} + \frac{\sqrt{2}\pi}{4} \max_{x \in [-\pi, \pi]} |1 - h'(x)| \right) \max (\|dx\|_\infty, |df|_\infty)
\]

The contractivity hypothesis (9) is therefore satisfied as soon as \( \forall x \in [-\pi, \pi], \quad 1 - \sqrt{2}\pi \leq h'(x) \leq 1 + \sqrt{2}\pi \). For instance, the following function \( h \):

\[
h(x) = \begin{cases} 
  x & \text{if } -\pi < x \leq -\frac{\pi}{2} \\
  x + \frac{63}{125\pi} (x + \frac{\pi}{2})^2 - \frac{3969}{625\pi^2} (x + \frac{\pi}{2})(x + \frac{\pi}{4}) & \text{if } -\frac{\pi}{2} < x \leq -\frac{\pi}{4} \\
  0.55x & \text{if } -\frac{\pi}{4} \leq x \leq \frac{\pi}{4} \\
  x - \frac{63}{125\pi} (x - \frac{\pi}{2})^2 - \frac{3969}{625\pi^2} (x - \frac{\pi}{2})(x - \frac{\pi}{4}) & \text{if } \frac{\pi}{4} < x < \frac{\pi}{2} \\
  x & \text{if } \frac{\pi}{2} \leq x < \pi 
\end{cases}
\]

which is in agreement with the criteria proposed in [18] leads to a scheme satisfying (9).

Since the same sketch of proof also provides the contractivity for \( |d((S_2(x, f))_{2n}| \)
we get the convergence applying theorem [1].

\[
\square
\]

5 Conclusion

We have formulated convergence and stability conditions for non linear subdivision schemes and associated multi-resolutions. These conditions deal with the difference with a suitable linear and convergent subdivision scheme. Many examples show that this formulation lead to simple proofs of convergence and stability.

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