We discuss, using the imaginary time method, some aspects of the connection between the Ward identity, the non-analyticity of amplitudes and the causality relation in QED at finite temperature.

I. INTRODUCTION

It is well known that, at finite temperature, field theoretic amplitudes (Greens functions) depend on both external energy and momentum independently. Furthermore, it is also known that new branch cuts develop in a thermal field theory due to the existence of new channels. As a consequence of these new cuts, amplitudes become non-analytic at the origin in the energy-momentum plane and the limits \( k_0 \to 0 \) and \( |\vec{k}| \to 0 \) do not commute in general \([1-4]\). In coordinate space, this simply means that the thermal effects are basically nonlocal \([5,6]\).

In an earlier paper \([7]\), the behavior of thermal perturbative amplitudes in a \(0+1\) dimensional gauge theory was studied in detail and the results (which also generalize to other \(0+1\) dimensional gauge models \([8]\)) are quite interesting. For example, it was found that the self-energy (this is also true for \(n\)-point amplitudes) vanishes when the external energy is non-vanishing, while it is nonzero for a vanishing external energy. Explicitly, one obtains

\[
\Pi^{(0+1)}(k_0) = e^2 \frac{d}{dm} \tanh \frac{m}{2T} \delta_{k_0,0}
\]

Such a behavior is, in fact, required by the Ward identity \([9]\). Nevertheless, it is quite interesting and surprising in that one would not expect a non-analytic behavior in a \(0+1\) dimensional theory.

In this short note, we analyze, in the imaginary time formalism, the origin of such Kronecker delta terms and show that such a behavior naturally arises in all dimensions and is a consequence of the analytic continuation of the external energy to the Minkowski space. We show that the particular analytic continuation, which is conventionally used to ensure good high energy behavior \([10]\), is also the one required by the gauge invariance of the theory and this, in turn, leads to the non-analyticity in the amplitudes. We show in \(3+1\) dimensional QED that these Kronecker delta terms do have a direct physical meaning. The organization of the paper is as follows. In section II, we analyze the origin of the non-analyticity in the \(0+1\) dimensional QED. This is a simple model where many aspects of this problem can be clearly seen. In particular, the analytic continuation that is consistent with the Ward identity is quite clear in this simple model. In section III, we generalize the results of the \(0+1\) dimensional model to QED in \(3+1\) dimensions in the long wave limit. Here, there is a complete parallel of the Kronecker delta terms, which can be shown to be related to the electric and the plasmon masses. In section IV, we discuss QED in \(3+1\) dimensions away from the long wave limit and show that the dispersion relations, to be consistent with the Ward identity, must have a subtraction at finite temperature. We derive a simple subtracted dispersion relation for the Kronecker delta terms, which shows that these are associated with the space-like branch cuts that exist only at finite temperature. Some of these features are known in the literature from earlier studies on the non-analyticity of thermal amplitudes, and we have tried, in this note, to present what is known as well as some new results from a unified point of view.

II. 0 + 1 DIMENSIONAL MODEL

The simplest example where a Kronecker delta term makes its presence manifest is in the \(0+1\) dimensional QED. This model also captures all the essential characteristics of higher dimensional models. Therefore, let us discuss this in some detail.

Let us consider a massive fermion, in \(0+1\) dimension, interacting with an Abelian gauge field described by the Lagrangian

\[
L = \bar{\psi}(i\partial_t - m - eA)\psi
\]
where $A$ is the 0 + 1 dimensional photon field and let us analyze the radiative correction to the photon self-energy due to the fermions at finite temperature. We will use the imaginary time formalism throughout our discussions. We recall that the Ward identities take a very simple form in 0 + 1 dimensions. In particular, for the self-energy, we have

$$k_0 \Pi(k_0) = 0$$  \hspace{1cm} (3)

which implies that

$$\Pi(k_0) = \delta_{k_0,0} \Delta$$  \hspace{1cm} (4)

This, in fact, is the simplest manifestation of Kronecker delta terms and is a direct consequence of the non-analyticity of amplitudes at finite temperature.

It may seem odd that there would be a non-analyticity in a 0 + 1 dimensional model, where the only kinematical variable is energy. So, to understand this better, let us analyze how the Kronecker delta term arises in a calculation. Let us note that, in the imaginary time formalism, the photon self-energy is given by

$$\Pi(k_0) = -e^2 T \sum_n \frac{i((2n+1)\pi T + k_0) + m}{((2n+1)\pi T + k_0)^2 + m^2} - \frac{i((2n+1)\pi T + m)^2}{(2n+1)^2 \pi^2 T^2 + m^2}$$

The sum can be evaluated in a standard manner by noting that

$$T \sum_n f(p = i(2n+1)\pi T) = - \sum_{\text{residues}} f(p) \tanh \frac{\beta p}{2}$$  \hspace{1cm} (6)

where $\beta = \frac{1}{T}$ in units of the Botzmann constant and the sum, on the right hand side, is over the residues of the poles of $f(p)$. However, before evaluating the sum, let us note that the behavior of the result is different depending on whether $k_0 \neq 0$ or $k_0 = 0$. When $k_0 \neq 0$, we note that $f(p)$ will have two distinct poles, while for $k_0 = 0$, the two poles will coincide and we will have a double pole. Basically, this is the reason for the non-analyticity in the amplitude and let us see this in some detail.

Let us note that, for $k_0 \neq 0$, the evaluation of the sum in (5) gives

$$\Pi(k_0) = \frac{e^2}{2i k_0} \left( \tanh \frac{\beta m}{2} - \tanh \frac{\beta (m - i k_0)}{2} \right) + (k_0 \leftrightarrow -k_0)$$  \hspace{1cm} (7)

It is worth noting here that if we assume, at this point, that the amplitude has already been analytically continued in $k_0$, then, this will give a non-vanishing value for $\Pi(k_0)$ for $k_0 \neq 0$, which would be inconsistent with the Ward identity in (3). On the other hand, if we assume that the external energy is not analytically continued yet and use the fact that $k_0 = 2\ell \pi T$, then, it follows from the periodicity of $\tanh \frac{\beta p}{2}$ that

$$\Pi(k_0) = 0, \quad k_0 \neq 0$$  \hspace{1cm} (8)

which is completely consistent with the Ward identity. This also gives an additional reason for the analytic continuation, which is conventionally carried out, in evaluating imaginary time amplitudes.

On the other hand, if $k_0 = 0$, as we have noted, there is a double pole and a direct evaluation of the sum, in (5), gives

$$\Pi(k_0 = 0) = e^2 \frac{d}{dm} \tanh \frac{\beta m}{2} = \Delta$$  \hspace{1cm} (9)

Mathematically, we can write

$$\Pi(k_0) = \tilde{\Pi}(k_0) + \delta_{k_0,0} \Delta$$  \hspace{1cm} (10)

where $\tilde{\Pi}(k_0)$ is the analytic part of the amplitude $\Pi(k_0)$, and this brings out the non-analytic nature of this expression. Alternatively, we can define
which would represent the non-analyticity in the self-energy. Note that, this non-analyticity can be put into the usual framework by noting that the $0 + 1$ dimensional theory can be thought of as the $\vec{k} = 0$ limit of a higher dimensional theory so that the Kronecker delta term simply corresponds, from the point of view of a higher dimensional theory (Note that, in a higher dimensional theory, there will be integration over internal spatial momentum, which does not occur in the $0 + 1$ dimensional theory), to

$$\Delta = \Pi(k_0 = 0, \vec{k} = 0) - \lim_{k_0 \to 0} \Pi(k_0, \vec{k} = 0)$$

which is precisely the difference between the static limit and the long wave limit of the self-energy. We will show later that, in QED, this has a physical meaning, but for the moment, let us note that this non-analyticity is a direct consequence of the analytic continuation used in the imaginary time formalism. In fact, had we treated $k_0$ as analytically continued and calculated $\Pi(k_0)$ in (8), we would have obtained the non-vanishing result

$$\Pi(k_0) = e^2 \frac{\text{sech}^2 \frac{2m}{T}}{1 + \text{tanh}^2 \frac{2m}{T}} \tan \frac{\beta k_0}{k_0}$$

Notice that this is an analytic function, apart from simple poles in the complex $k_0$-plane at $k_0 = \frac{(2n+1)\pi}{\beta} \pm im$. Furthermore, for real values of the energy, this is a well behaved function of $k_0$, which vanishes for $k_0 \to \infty$ and which, in the limit $k_0 \to 0$, coincides with $\Pi(k_0 = 0)$ in (8). This would lead to a vanishing $\Delta$. Namely, if the external energy is not properly analytically continued, the amplitude would not only violate the Ward identity, but also would not lead to any non-analyticity, which has a physical meaning. So, in this model, the main criterion for the correct analytic continuation is provided by the Ward identity as well as the causality condition (to be discussed in section IV). Thus, we see that all these things are quite intricately connected and all these features carry over to higher dimensions as we will see in the subsequent sections (in fact, these features are present in any theory).

### III. Long Wave Limit Kronecker Energy Terms in QED

To see that the Kronecker delta non-analyticity is not simply a feature of the $0 + 1$ dimensional theory and arises in higher dimensions as well, let us analyze $3 + 1$ dimensional QED in the long wave limit. First of all, we note that, at finite temperature, the one loop photon self-energy takes the form (in the imaginary time formalism)

$$\Pi_{\mu\nu}(k_0, \vec{k}) = -\frac{4e^2 T}{(2\pi)^3} \sum_n \int d^3p \frac{\delta_{\mu\nu}(m^2 + p^2 + p \cdot k) - p_\mu k_\nu - p_\nu k_\mu - 2p_\mu p_\nu}{(p^2 + m^2)((p + k)^2 + m^2)}$$

where $p^0 = (2n + 1)\pi T$ and $k_0$ is a multiple of $2\pi T$. We are interested in the thermal contribution which can be obtained through the simple contour representation

$$T \sum_n f(p_0 = i(2n + 1)\pi T) = (T = 0 \text{ part}) - \frac{1}{2\pi i} \int_{-\infty+\epsilon}^{\infty+\epsilon} dp_0 \left( f(p_0 + f(-p_0)) \frac{1}{e^{\beta p_0} + 1} \right)$$

Since the zero temperature part of the amplitude is analytic at the origin, let us concentrate only on the thermal part of the amplitude.

At finite temperature, the self energy can be described in terms of two functions $\Pi_T$ and $\Pi_L$, which are related to $\Pi_{00}, \Pi_{\mu\nu}$ as

$$\Pi_L = \left( -1 + \frac{k_0^2}{|\vec{k}|^2} \right) \Pi_{00}, \quad \Pi_T = -\frac{1}{2} (\Pi_{\mu\nu} + \Pi_L)$$

Therefore, let us look at $\Pi_{00}$ and $\Pi_{\mu\nu}$, which are easier to evaluate. Let us start with the evaluation of the thermal part of $\Pi_{00}$ which has the form (see (14), (15))

$$\Pi_{00}^{\text{thermal}}(k_0, \vec{k}) = \frac{4e^2}{(2\pi)^3} \int d^3p \int_{-\infty+\epsilon}^{\infty+\epsilon} dp_0 n_F(p_0) \left[ \frac{m^2 + p^2 + p \cdot k - 2p_0 k_0 - 2p_0^2}{(p^2 + m^2)((p + k)^2 + m^2)} + (k_0 \leftrightarrow -k_0) \right]$$
where \( n_F(p_0) = \frac{1}{e^{p_0/T} + 1} \) represents the Fermi-Dirac distribution function. It is clear from this that, as in the \( 0 + 1 \) dimensional model, the structure of the integrand for \( \Pi_0^{\text{thermal}}(k_0, 0) \) is different depending on whether \( k_0 = 0 \) or \( k_0 \neq 0 \). When \( k_0 \neq 0 \) the integrand has two distinct poles, while for \( k_0 = 0 \), the two poles coincide, much like in the \( 0 + 1 \) dimensional example.

For \( k_0 \neq 0 \) such that \( |k_0| << m \), a direct evaluation leads to

\[
\Pi_0^{\text{thermal}}(k_0, 0) = \frac{2e^2}{(2\pi)^3} \int d^3 p \frac{1}{k_0} (n_F(E_p + k_0) - n_F(E_p)) + (k_0 \leftrightarrow -k_0)
\]

where \( E_p = (\vec{p}^2 + m^2)^{1/2} \), while for \( k_0 = 0 \), we obtain

\[
\Pi_0^{\text{thermal}}(0, 0) = \frac{4e^2}{(2\pi)^3} \int d^3 p \frac{d n_F(E_p)}{dE_p}
\]

(19)

It is clear that the behavior of these quantities is completely analogous to what we have seen in the \( 0 + 1 \) dimensional model. In particular, let us note that if we evaluate \( \Pi_0^{\text{thermal}}(k_0, 0) \) by analytically continuing \( k_0 \), then, we would obtain \( \lim_{k_0 \to 0} \Pi_0^{\text{thermal}}(k_0, 0) = \Pi_0^{\text{thermal}}(0, 0) \). Namely, in such a case, the amplitude will be analytic, which will be in violation with the Ward identity, as we will show. On the other hand, if we use the fact that \( k_0 = 2\ell \pi T \), then, \( n_F(p_0 + k_0) = n_F(p_0) \) and the result in (18) vanishes, as is the case in the \( 0 + 1 \) dimensional model, and this is completely consistent with the Ward identity. In this case, however, there will be a non-analyticity which can be related to a physical quantity as we will see.

If we define, as in the \( 0 + 1 \) dimensional model (we can add the zero temperature part as well, but it is analytic and, therefore, would drop out of the expression),

\[
\Delta_0 = \Pi_0^{\text{thermal}}(0, 0) - \lim_{k_0 \to 0} \Pi_0^{\text{thermal}}(k_0, 0) = \Pi_0^{\text{thermal}}(0, 0) = \frac{4e^2}{(2\pi)^3} \int d^3 p \frac{d n_F(E_p)}{dE_p}
\]

(20)

then, with the proper analytic continuation that respects Ward identities, we see that this amplitude is non-analytic and the non-analyticity can be determined, at high temperatures, to be

\[
\Delta_0 \approx -\frac{e^2 T^2}{3}
\]

(21)

The thermal part of the trace of the self-energy can, similarly, be obtained from (14) and (15) and has the form

\[
\Pi^{\text{thermal}}(k_0, \vec{k}) = \frac{4e^2}{(2\pi)^4} \int d^4 p \int_{i\epsilon}^{i\epsilon + \epsilon} dp_0 n_F(p_0) \left[ \frac{4m^2 + 2\mu^2 + 2\mu \cdot \vec{k}}{(p^2 + m^2)((p + \vec{k})^2 + m^2)} + (k_0 \leftrightarrow -k_0) \right]
\]

(22)

This integrand, too, has the structure alluded to earlier and we note, without going into details again, that with proper analytic continuation that respects Ward identities, we obtain, for \( k_0 \neq 0 \) with \( |k_0| << m \),

\[
\Pi^{\text{thermal}}(k_0, 0) = -\frac{2e^2}{(2\pi)^3} \int d^3 p \frac{m^2}{E_p k_0} \left[ n_F(E_p + k_0) - n_F(E_p) \right] - n_F(E_p) \left( \frac{m^2}{E_p^2} + 2 \right) \]

\[
= \frac{4e^2}{(2\pi)^3} \int d^3 p \frac{n_F(E_p)}{E_p} \left( \frac{m^2}{E_p^2} + 2 \right)
\]

(23)

while for \( k_0 = 0 \), we obtain

\[
\Pi^{\text{thermal}}(0, 0) = -\frac{4e^2}{(2\pi)^3} \int d^3 p \frac{m^2}{E_p} \frac{d n_F(E_p)}{dE_p} - n_F(E_p) \left( \frac{m^2}{E_p^2} + 2 \right)
\]

(24)

This brings out the non-analytic structure clearly (which is primarily due to the proper analytic continuation that respects Ward identity). We can again define

\[
\Delta_{\mu\nu} = \Pi^{\text{thermal}}_{\mu\nu}(0, 0) - \lim_{k_0 \to 0} \Pi^{\text{thermal}}_{\mu\nu}(k_0, 0) = -\frac{4e^2 m^2}{(2\pi)^3} \int d^3 p \frac{d n_F(E_p)}{E_p}
\]

(25)

which, at high temperature, has a sub-leading behavior compared with (21).
Let us next turn to the Ward identities \([14,15]\) and the physical meaning of these non-analyticities. First of all, we note from the Ward identity that
\[
k_0 \Pi_{0\mu}^{\text{thermal}}(k_0, 0) = 0
\] (26)
which would imply that
\[
\Pi_{0\mu}^{\text{thermal}}(k_0 \neq 0, 0) = 0
\] (27)
As we have seen, this is one of the guiding relations in the proper analytic continuation of the external energy. Second, we have already seen that (see (20,21)), at high temperatures,
\[
\Delta_{00} = \Pi_{00}^{\text{thermal}}(0, 0) \approx -\frac{e^2 T^2}{3}
\] (28)
On the other hand, let us recall that the square of the screening length, in a plasma, is related to \(\lim_{|\vec{k}| \to 0} \Pi_{00}^{\text{thermal}}(0, \vec{k})\).
We will show in the next section that, when \(k_0 = 0\), amplitudes are analytic in \(|\vec{k}|\) so that
\[
\lim_{|\vec{k}| \to 0} \Pi_{00}^{\text{thermal}}(0, \vec{k}) = \Pi_{00}^{\text{thermal}}(0, 0) = \Delta_{00} = -\frac{m^2_{\text{electric}}}{3}
\] (29)
This brings out the physical meaning of this non-analyticity.
Similarly, let us note from (16) that
\[
\Pi_T^{\text{thermal}} = -\frac{1}{2} \left( \Pi_{\mu\mu}^{\text{thermal}} + \left(1 + \frac{k_0^2}{|\vec{k}|^2}\right) \Pi_{00}^{\text{thermal}} \right)
\] (30)
so that we can, correspondingly, introduce
\[
\Delta_T = \Pi_T^{\text{thermal}}(0, 0) - \Pi_T^{\text{thermal}}(k_0 \to 0, 0)
\] (31)
Therefore, we can write
\[
\Delta_T = \frac{1}{2} (\Delta_{00} - \Delta_{\mu\mu}) - \lim_{k_0 \to 0, \vec{k} \to 0} \frac{1}{2} \frac{k_0^2}{|\vec{k}|^2} \Pi_{00}
\] (32)
The last term, in the above expression, appears singular, but because of the Ward identity, \(\Pi_{00} \sim |\vec{k}|^2\) so that it is, in fact, well behaved. At high temperature, this can be evaluated to give
\[
\Delta_T \approx -\frac{e^2 T^2}{9} = -\frac{1}{3} m^2_{\text{electric}} = -m^2_{\text{plasmon}}
\] (33)
This shows that the non-analyticity in \(\Pi_T^{\text{thermal}}\) can be related to the plasmon mass. Let us note that the first term, on the right hand side in (31), is called the magnetic mass, which vanishes to one loop order. In such a case, we recognize that
\[
\Delta_T = -\Pi_T^{\text{thermal}}(k_0 \to 0, 0)
\] (34)
and it can be checked, from the standard results, that it agrees with our result in (33).

**IV. GENERAL STRUCTURE OF KRONECKER ENERGY TERMS IN QED**

Let us next look at the self-energy in QED away from the long wave limit. It would seem that, for small \(\vec{k}\), we can make a Taylor expansion in the external momentum. However, when expanded in this way, every term in the series (except the first term) becomes divergent, when \(k_0 \to 0\), so that such an expansion does not make sense. On the other hand, looking at the expressions in (17,22), we note that, at the poles (the pole can always be chosen to be at \(p^2 + m^2 = 0\) or \(p_0 = E_p\) with appropriate shift), the denominator that needs to be expanded is
\[
\frac{1}{(p + k)^2 + m^2} \bigg|_{p_0 = E_p} = \frac{1}{k^2 + 2k \cdot \vec{p} - k_0^2 - 2E_p k_0}
\]

and can be naturally expanded in powers of \( \frac{|\vec{k}|}{k_0} \). Therefore, let us define

\[
s^2 = \frac{|\vec{k}|^2}{k_0^2}
\]

The parameter \( s \), then, would represent the direction along which one approaches the origin in the \((k_0, |\vec{k}|)\) plane, as \( k_0 \to 0 \), with the ratio \( \frac{|\vec{k}|}{k_0} \) held fixed. For example, \( s = 0 \) will correspond to the long wave limit, while \( s \to \infty \) will denote the static limit. The amplitudes can now be expressed as depending on \((k_0, s)\) and, correspondingly, as before, we can define

\[
\Delta_{00}(s) = \Pi_{00}^{\text{thermal}}(0, 0) - \Pi_{00}^{\text{thermal}}(k_0 \to 0, s), \quad \Delta_{\mu\nu}(s) = \Pi_{\mu\nu}^{\text{thermal}}(0, 0) - \Pi_{\mu\nu}^{\text{thermal}}(k_0 \to 0, s)
\]

As we have seen in the last section, \( \Delta_{\mu\nu} \) leads to sub-leading contributions at high temperature and, therefore, for simplicity, we will confine our discussions to \( \Delta_{00} \) only. This can, in fact, be exactly evaluated, in the high temperature limit, and has the explicit form

\[
\Delta_{00}(s) = -\frac{m_{\text{electric}}^2}{2s} \log \frac{1 + s}{1 - s}
\]

This can be expanded for small values of \( s \) as

\[
\Delta_{00}(s) = -m_{\text{electric}}^2 \left( 1 + \frac{s^2}{3} + \frac{s^4}{5} + \cdots \right)
\]

In particular, it shows that when \( s = 0 \) (namely, in the long wave limit), this discontinuity is related to the square of the electric mass, as was obtained earlier in (29). Furthermore, we note that

\[
\lim_{s \to \infty} \Delta_{00}(s) = 0
\]

This proves, as stated in the last section, that amplitudes are analytic at the origin, in this plane, in the static limit. We have already seen that \( |\Delta_{00}(s = 0)| \) has the physical meaning of being the square of the screening mass. It is, therefore, tempting to speculate that we can think of \( |\Delta_{00}(s)| \), for small \( s \), as the square of the screening mass for quasi-static electric fields at finite temperature.

In the earlier sections, we have shown how the analytic continuation of energy crucially depends on the Ward identity which, in turn, leads to a non-analyticity. We have given a physical meaning to the non-analyticity in a gauge theory. It is, of course, known that the non-analyticity, at finite temperature, arises because of new branch cuts appearing in a thermal field theory. It is because of this, say for example, that \( \Delta_{00}(s) \) depends on the direction along which one approaches the origin. In a sense, therefore, we can think of the non-analyticity as a consequence of the causality in thermal field theory. In this section, we will show how the causality relations, in a thermal gauge theory, are further constrained by the Ward identity to give a physical non-analyticity.

To understand these things a little better, let us simply analyze the behavior of \( \Pi_{00}^{\text{thermal}} \). We note that the amplitudes that we are evaluating, in the imaginary time formalism, correspond to retarded amplitudes. These satisfy a dispersion relation of the form

\[
\text{Re} \Pi_{00}^{\text{dispersion}}(k_0, s) = \int_{-\infty}^{\infty} d\omega \frac{\text{Im} \Pi_{00}(\omega, k)}{\omega - k_0} = \frac{2}{\pi} \int_{0}^{\infty} d\omega \frac{\omega \text{Im} \Pi_{00}(\omega, k)}{\omega^2 - k_0^2}
\]

where we have defined \( k = |\vec{k}| \) and used the fact that \( \text{Im} \Pi_{00}(\omega, k) \) is odd in \( \omega \). At finite temperature, there are two branch cuts in the self-energy. The usual branch cut, which also occurs at zero temperature, satisfies \( \omega^2 - k^2 \geq 4m^2 \) (we will refer to this as the time-like cut), while the new cut that arises in thermal field theory, due to the existence of additional channels of reaction, satisfies \( \omega^2 - k^2 \leq 0 \) (we will refer to this as the space-like cut).

The contribution of the time-like cut is insensitive to how we approach the origin in the \((k_0, k)\) plane and can be obtained from the dispersion relation to be
\[
\text{Re } \Pi_{00}^{\text{time-like}}(0,0) = \frac{e^2}{\pi^2} \int_{2m}^{\infty} d\omega \frac{\omega}{\omega^2} n_F\left(\frac{\omega}{2}\right)
\]
(42)

At high temperature, this has the behavior
\[
\text{Re } \Pi_{00}^{\text{time-like}}(0,0) \approx \frac{e^2 T^2}{3}
\]
(43)

The contribution from the space-like cut, on the other hand, depends on how we approach the origin and has the form
\[
\text{Re } \Pi_{00}^{\text{space-like}}(k_0, k) = \frac{2}{\pi} \int_{0}^{k} d\omega \frac{\omega \text{Im } \Pi_{00}(\omega, k)}{\omega^2 - k_0^2} = \frac{2}{\pi} \int_{0}^{1} dv \frac{v \text{Im } \Pi_{00}(v, k)}{v^2 - \frac{k_0^2}{v^2}}
\]
(44)

where we have defined \(v = \frac{\omega}{k_0}\) and
\[
\text{Im } \Pi_{00}(v, k) = -\frac{2e^2 v}{\pi} \int_{V_{1-\omega}}^{\infty} d\hat{\omega} \hat{\omega} n_P(\hat{\omega})
\]
(45)

It is clear that this depends on how we approach the origin. At high temperature, this can be evaluated to give
\[
\text{Re } \Pi_{00}^{\text{space-like}}(k_0, k) \approx -\frac{e^2 T^2}{3} \left[1 - \frac{1}{2s} \log \left|\frac{1 + s}{1 - s}\right|\right]
\]
(46)

where, as before, we have identified \(s = \frac{k}{k_0}\).

Let us note, however, that the expression in (47) does not vanish when \(k = 0\) (namely, \(s = 0\)), as the Ward identity would require of the amplitude \(\Pi_{00}\). In other words, the naive dispersion relation does not automatically lead to the actual amplitude. On the other hand, at high temperature,
\[
\text{Re } \Pi_{00}^{\text{dispersion}}(k_0, k << m) = \text{Re } \Pi_{00}^{\text{time-like}} + \text{Re } \Pi_{00}^{\text{space-like}} \approx \frac{e^2 T^2}{3} \frac{1}{2s} \log \left|\frac{1 + s}{1 - s}\right|
\]
(47)

Let us note, however, that the expression in (47) does not vanish when \(k = 0\) (namely, \(s = 0\)), as the Ward identity would require of the amplitude \(\Pi_{00}\). In other words, the naive dispersion relation does not automatically lead to the actual amplitude. On the other hand, at high temperature,
\[
\text{Re } \Pi_{00}(k_0, k << m) = \text{Re } \Pi_{00}^{\text{dispersion}}(k_0, k << m) - \frac{e^2 T^2}{3}
\]
(48)

vanishes when \(k = 0\), in accordance with the Ward identity. This implies that the dispersion relation, that is consistent with the Ward identity, can be written as
\[
\text{Re } \Pi_{00}(k_0, k) + \frac{e^2 T^2}{3} = \frac{1}{\pi} \int_{-\infty}^{\infty} d\omega \frac{\text{Im } \Pi_{00}(\omega, k)}{\omega - k_0}
\]
(49)

The result above suggests the following subtracted dispersion relation at finite temperature,
\[
\text{Re } [\Pi_{00}(k_0, k) - \Pi_{00}(0, k)] = \frac{1}{\pi} \int_{-\infty}^{\infty} d\omega \left(\frac{1}{\omega - k_0} - \frac{1}{\omega}\right) \text{Im } \Pi_{00}(\omega, k)
\]
(50)

There are several things to note from here. First, as \(k_0, k \to 0\), the contribution of the time-like cut cancels from this expression so that the entire contribution comes only from the space-like cut. Second, \(\text{Im } \Pi_{00}(0, k) = 0\), since the imaginary part is an odd function of energy. Furthermore, if we define
\[
\Delta_{00}(k_0, k) = \Pi_{00}(0, k) - \Pi_{00}(k_0, k)
\]
(51)

then, it follows from our earlier observations that we can write
\[
\text{Re } \Delta_{00}(k_0, k << m) = \frac{1}{\pi} \int_{-k}^{k} d\omega \left(\frac{1}{\omega - k_0} - \frac{1}{\omega}\right) \text{Im } \Delta_{00}(\omega, k)
\]
(52)
Namely, $\Delta_{00}(k_0, k \ll m)$ satisfies a simple subtracted form of dispersion relation, where only the space-like cut contributes. This shows that the contributions to the Kronecker delta energy terms, which only exist at finite temperature, come from the absorption of virtual, space-like quanta by the real particles present in the plasma.

Relations (50) and (52) can, in fact, be explicitly checked in the simple $0 + 1$ dimensional model, where we can see that they are valid at all temperatures. In this model, there is no branch cut and the only non-analyticity results from the contribution at $k_0 = 0$. From the Ward identity (3), we see that when $k_0 \neq 0$, both the real and the imaginary parts of the self-energy vanish. At $k_0 = 0$, the self-energy is real (see (4)) so that the imaginary part vanishes, but it can be shown to vanish as

$$\text{Im} \Pi(k_0) = \pi \Delta k_0 \delta(k_0)$$

which is manifestly an odd function. Even though this vanishes, it can contribute to the subtracted dispersion relation as follows.

Let us first look at the subtracted relation (50) in the $0 + 1$ dimensional QED model. For $k_0 = 0$, we see that this relation is identically satisfied. When $k_0 \neq 0$, we have $\Pi(k_0) = 0$ so that

$$\frac{1}{\pi} \int_{-\infty}^{\infty} d\omega \left( \frac{1}{\omega - k_0} - \frac{1}{\omega} \right) \pi \Delta \omega \delta(\omega) = -\Delta = -\text{Re} \Pi(0)$$

Therefore, relation (50) is exactly satisfied. Since $\text{Im} \Pi(0) = 0$, an analogous relation to (52), can also be seen to hold in a similar manner.

V. CONCLUSION

In this short note, we have discussed some aspects of the non-analytic terms that arise in thermal gauge theories. We have shown explicitly, in the $0 + 1$ dimensional model, that Ward identities require a particular analytic continuation of the external energy, which is responsible for the Kronecker delta non-analyticity in the amplitudes. This analysis, therefore, provides a further reason for such an analytic continuation (which is conventionally used to ensure good high energy behavior) in other theories as well. The Kronecker delta terms are not particular to $0 + 1$ dimensional gauge theories. They arise in any theory and in higher dimensions as well. In $3 + 1$ QED, such terms, in the long wave limit, have a direct physical meaning, namely, they are related to the electric mass and the plasmon mass. We have shown that the dispersion relations, in QED, need a subtraction at finite temperature to be compatible with the Ward identities. The Kronecker delta energy terms, in particular, satisfy a simple subtracted dispersion relation, which gets contributions only from the space-like cuts that arise at finite temperature.

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