Prepotentials in $N = 2$ supersymmetric Yang-Mills theories are known to obey non-linear partial differential equations called Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations. In this paper, the prepotentials at one-instanton level in $N = 2$ supersymmetric SU(4) Yang-Mills theory are studied from the standpoint of WDVV equations. Especially, it is shown that the one-instanton prepotentials are obtained from WDVV equations by assuming the perturbative prepotential and by using the scaling relation as a subsidiary condition but are determined without introducing Seiberg-Witten curve. In this way, various one-instanton prepotentials which satisfy both WDVV equations and scaling relation can be derived, but it turns out that among them there exist one-instanton prepotentials which coincide with the instanton calculus.

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I. INTRODUCTION

A class of developments of quantum field theory in the ninety of this century may be represented by two keywords: $N = 2$ supersymmetry and duality. For example, the mirror symmetry established in the beginning of the ninety was based on the (trivial) isomorphism between left and right $U(1)$ currents of $(2, 2)$-superconformal field theory and this isomorphism predicted the existence of a pair of Calabi-Yau manifolds whose axes of Hodge diamond were exchanged. Candelas et al. skillfully used the consequence expected from this duality of Hodge structure and showed that the numbers of rational curves on Calabi-Yau quintic 3-fold could be determined from mirror symmetry. The coincidence of their result with mathematically rigorous results gave a great surprising!

On the other hand, also in the recent studies of low energy effective dynamics of $N = 2$ supersymmetric Yang-Mills theory, $N = 2$ supersymmetry and duality play a crucial role. Before the arrival of Seiberg and Witten’s proposal by using electro-magnetic duality for the description of the low energy effective action of SU(2) gauge theory, though it has been known that the prepotential which is a generating function of the low energy effective action is not renormalized beyond one-loop in perturbative calculation due to $N = 2$ supersymmetry, actually the prepotential was expected to receive instanton corrections. Unfortunately, such corrections were not so extensively discussed, but thanks to their proposal, it made possible to extract informations on instanton effects from a Riemann surface and periods of meromorphic differential on it. Namely, the low energy effective theory was turned out to be parameterized by a Riemann surface. The validity of their proposal was discussed by Klemm et al. with the aid of Picard-Fuchs equation and the instanton corrections to the prepotential was revealed. The instanton corrections obtained in this way showed extremely good agreement with the prediction of instanton calculus.

However, deeper and striking features of prepotentials of $N = 2$ supersymmetric Yang-Mills theories may be nicely interpreted in terms of differential equations satisfied by prepotentials. For instance, it is well-known that the prepotentials satisfy an Euler equation called scaling relation, and in fact this simple equation simplified and accelerated the study of prepotentials. As for another characteristic equations, we can mention that there is a non-linear system of partial differential equations called Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations (rigorously speaking, the WDVV equations in $N = 2$ Yang-Mills theory are not equivalent to those arising in two-dimensional topological field theory). Actually, these equations hold not
only in four-dimensional gauge theories but also in higher dimensions even if hypermultiplets are included. Accordingly, it becomes possible to regard the prepotentials in various gauge theories as a member of solutions to WDVV equations. Then, what is the most general solution (function form of prepotentials) to the WDVV equations? Unfortunately, we can not precisely know the answer to this question, but Braden et al. partially found the answer. They assumed the function form of prepotential which is expected from known examples and found a new prepotential which is considered as that in five-dimensional gauge theory, although their study was restricted to perturbative part. Of course, among the solutions found by them we can see the existence of the prepotential in four-dimensional Yang-Mills theory. This seems to indicate that the prepotentials can be constructed without introducing Riemann surface, provided the WDVV equations are used. Finding whether non-perturbative prepotentials are available from WDVV equations without using Riemann surface is the subject of this paper.

The paper is organized as follows. In Sec. II, the construction of perturbative solution to WDVV equations for SU(4) gauge theory in four dimensions discussed by Braden et al. is summarized. We can see that the perturbative prepotential is in fact obtained from WDVV equations. In Sec. III, we add the non-perturbative part for this perturbative prepotential and try to solve the WDVV equations. Though the non-perturbative part satisfies a non-linear differential equation, restricting it at one-instanton level, we can reduce it to a linear differential equation satisfied by one-instanton prepotential. For this reason, the one-instanton prepotential is investigated in this paper. To solve this equation, the scaling relation is used as a subsidiary condition, but it turns out that there are miscellaneous solutions which do not contradict to both WDVV equations and scaling relation. In Sec. IV, we compare our result with the prediction of one-instanton calculus. It is shown that among our one-instanton prepotentials obtained from WDVV equations there are one-instanton prepotentials which agree the prediction of instanton calculus. In this way, we conclude that it is possible to obtain non-perturbative prepotential from WDVV equations without relying on Riemann surface. Sec. V is a brief summary.

II. PERTURBATIVE PREPOTENTIAL FROM WDVV EQUATIONS

In this section, we briefly outline the construction to get perturbative prepotential from WDVV equations in the SU(4) gauge theory presented by Braden et al. Note that the SU(4) model is the
simplest and non-trivial example for a study of WDVV equations.

In this case, the WDVV equations for the prepotential $F$ take the form

$$(F_i)(F_k)^{-1}(F_j) = (F_j)(F_k)^{-1}(F_i), \ i, j, k = 1, \ldots, 3, \quad (2.1)$$

where

$$(F_i) \equiv (F_i)_{jk} = \frac{\partial^3 F}{\partial a_i \partial a_j \partial a_k} \quad (2.2)$$

are the matrix notations and in this paper the brackets are always added as $(F_i)$ when “$F_i$” mean matrices. The coordinates $a_i$ are the periods of the SU(4) gauge theory (see Appendix).

Braden et al. considered the perturbative prepotential in the form

$$F_{\text{per}}(a_1, a_2, a_3) = \sum_{i<j=1}^4 f(a_{ij}), \ a_{ij} = a_i - a_j, \ \sum_{i=1}^4 a_i = 0. \quad (2.3)$$

Of course $F_{\text{per}}$ may depend on the mass scale $\Lambda \equiv \Lambda_{S_{\text{SU(4)}}}$ of the theory, but we can ignore its dependence for the moment because $\Lambda$-differentiation is not included in the WDVV equations (2.1).

Under the assumption (2.3), we can find that when $F_{\text{per}}$ satisfies (2.1) there is a functional relation

$$g(a_{12})g(a_{34}) - g(a_{13})g(a_{24}) + g(a_{14})g(a_{23}) = 0, \quad (2.4)$$

where

$$g(a) \equiv \left(\frac{\partial^3 f}{\partial a^3}\right)^{-1} \quad (2.5)$$

With the aid of several conditions, we can conclude that $g$ is an odd function with

$$g(0) = g''(0) = 0, \quad (2.6)$$

where the prime means the differentiation over the argument.

There are several functions enjoying the properties (2.4) and (2.6), but a function which is necessary for us among them is the function of the form $g(a) = a$. Namely,

$$f(a) = \frac{a^2}{2} \ln a + O(a^2). \quad (2.7)$$

Note that $O(a^2)$-term can not be fixed from the WDVV equations because they are third-order differential equations. Namely, $\Lambda$-dependence of one-loop contribution is not fixed. It is easy to see
that substituting (2.7) back to (2.3) in fact yields the perturbative part of the SU(4) prepotential (in a suitable normalization).

Marshakov et al. give a general proof that the perturbative prepotentials in various gauge theories satisfy WDVV equations, but this is also confirmed by Ito and Yang in their study of these equations.

III. ONE-INSTANTON PREPOTENTIALS

A. Differential equation for one-instanton prepotential

Next, let us consider whether non-perturbative prepotential \( F \) is available from the WDVV equations by assuming the form

\[
F(a_1, a_2, a_3, \Lambda) = F_{\text{per}}(a_1, a_2, a_3) + F_{\text{ins}}(a_1, a_2, a_3, \Lambda),
\]

where

\[
F_{\text{ins}}(a_1, a_2, a_3, \Lambda) = \sum_{k=1}^{\infty} F_k(a_1, a_2, a_3) \Lambda^k.
\]

In order to derive differential equations for \( F_{\text{ins}} \), we assume that \( F_{\text{per}} \) is already given by (2.3) with (2.7).

Then substituting \( F \) into (2.1) we can obtain a single non-linear differential equation for \( F_{\text{ins}} \), but if we restrict only the case \( k = 1 \) (one-instanton level), the equation reduces to

\[
\frac{\partial_1^3 F_1}{a_{12}a_{13}a_{14}} - \frac{\partial_1^2 F_1}{a_{12}a_{23}a_{24}} + \frac{\partial_1^3 F_1}{a_{13}a_{23}a_{34}} + \frac{6\partial_1 \partial_2 \partial_3 F_1}{a_{14}a_{24}a_{34}} - \frac{A_{012} \partial_2 \partial_3^2 F_1 + A_{021} \partial_2^2 \partial_3 F_1}{a_{12}a_{13}a_{23}a_{24}a_{34}}
\]

\[
- \frac{A_{102} \partial_1 \partial_3^2 F_1 - A_{201} \partial_1 \partial_2^2 F_1 + A_{210} \partial_1 \partial_2 \partial_3 F_1}{a_{12}a_{13}a_{14}a_{23}a_{24}} = 0,
\]

where \( \partial_i \equiv \partial / \partial a_i \) and

\[
A_{012} = a_1 a_2 - 3 a_2^2 - 2 a_1 a_3 + 4 a_2 a_3 + a_1 a_4 + a_2 a_4 - a_3 a_4,
\]

\[
A_{021} = 2 a_1 a_2 - a_1 a_3 - 4 a_2 a_3 + 3 a_3^2 - a_1 a_4 + 2 a_2 a_4 - a_3 a_4,
\]

\[
A_{102} = 3 a_1^2 - a_1 a_2 - 4 a_1 a_3 + 2 a_2 a_3 - a_1 a_4 - a_2 a_4 + 2 a_3 a_4,
\]

\[
A_{201} = 2 a_1 a_2 - 4 a_1 a_3 - a_2 a_3 + 3 a_3^2 + 2 a_1 a_4 - a_2 a_4 - a_3 a_4,
\]

\[
A_{120} = 3 a_1^2 - 4 a_1 a_2 - a_1 a_3 + 2 a_2 a_3 - a_1 a_4 + 2 a_2 a_4 - a_3 a_4,
\]

\[
A_{210} = 4 a_1 a_2 - 3 a_2^2 - 2 a_1 a_3 + 2 a_2 a_3 - 2 a_1 a_4 + a_2 a_4 + a_3 a_4.
\]
B. The solutions

In order to solve (3.3), let us introduce the new variables

\[ x = a_{12}, \quad y = a_{13}, \quad z = a_{14}. \]  

(3.5)

In addition, using Euler derivatives \( \theta_x = x \partial / \partial x \) etc, we can rewrite (3.3) as

\[ L(\theta_x, \theta_y, \theta_z)F_1 = 0, \]  

(3.6)

where

\[
L(\theta_x, \theta_y, \theta_z) = yz(y - z)\theta_x(\theta_x - 1)(\theta_x - 2) + z(4xy - 3y^2 - 2xz + yz)(\theta_x - 1)\theta_y \theta_y \\
+ z(3x^2 - 4xy - xz + 2yz)(\theta_y - 1)\theta_x \theta_y - xz(x - z)\theta_y(\theta_y - 1)(\theta_y - 2) \\
-y(3x^2 - xy - 4xz + 2yz)(\theta_x - 1)\theta_x \theta_y - y(4xz + yz - 3z^2 - 2xy)(\theta_x - 1)\theta_x \theta_z \\
+ 6(x - y)(x - z)(y - z)\theta_x \theta_y \theta_z + x(xz + 4yz - 3z^2 - 2xy)(\theta_y - 1)\theta_y \theta_z \\
+ x(3y^2 + 2xz - 4yz - xy)(\theta_x - 1)\theta_y \theta_z + xy(x - y)\theta_x(\theta_x - 1)(\theta_x - 2). \]  

(3.7)

Here, suppose that \( F_1 \) is given by

\[ F_1 = x^{\nu_1}y^{\nu_2}z^{\nu_3}F(x, y, z), \]  

(3.8)

where

\[
F(x, y, z) = \sum_{i,j,k=0}^{\infty} B_{i,j}x^i y^j z^k. \]  

(3.9)

and the expansion coefficients are assumed to be independent of \( x, y \) and \( z \). In (3.9), \( \epsilon \) etc are signature symbols (the choice of signatures depends on where is the convergence region), e.g., \( \epsilon_x = \pm \).

Then from (3.3) we get the differential equation for \( F \)

\[ L(\theta_x + \nu_1, \theta_y + \nu_2, \theta_z + \nu_3)F = 0. \]  

(3.10)

and the indicial equations for \( \nu_i \)

\[
\nu_1(\nu_1 - 2\nu_2 - 1)(\nu_1 + \nu_2 - 3\nu_3 - 2) = 0, \quad \nu_2(2\nu_1 - \nu_2 + 1)(\nu_1 + \nu_2 - 3\nu_3 - 2) = 0,
\]

\[
\nu_2(\nu_2 - 2\nu_3 - 1)(3\nu_1 - \nu_2 - \nu_3 + 2) = 0, \quad (\nu_2 - \nu_3)(\nu_2^2 - \nu_1 \nu_2 + \nu_2 \nu_3) = 0,
\]

\[
\nu_1(\nu_1^2 - 3\nu_1 + 5\nu_2 - 3\nu_1 \nu_2 + \nu_3 - \nu_1 \nu_3 + 4\nu_2 \nu_3 + 2) = 0,
\]

\[
\nu_2^2(\nu_3 - \nu_2 - 3) + 5\nu_1 \nu_2 \nu_3 + 3\nu_2 \nu_3 - 2\nu_1^2 \nu_3 + 2\nu_1 \nu_3 + 2\nu_3 + \nu_1 \nu_2 = 0,
\]

\[
\nu_3^2(\nu_3 - \nu_2 - 3) - 2\nu_2^2 \nu_3 + 3\nu_1 \nu_2 \nu_3 + 3\nu_2 \nu_3 + 3\nu_3 \nu_2 + 2\nu_3 = 0.
\]  

(3.11)
The sets of possible $\nu_i$ are thus given by

$$\nu \equiv (\nu_1, \nu_2, \nu_3) = (-2, -2, -2), (-1, -1, -1), (0, -1, -1)^4, (0, 0, 0)^2, (0, 0, 1), (0, 0, 2)^2,$$
$$\quad (0, 1, 0), (0, 1, 1), (0, 2, 0)^2, (1, 0, 0)^2, (1, 0, 1), (1, 0, 2), (2, 0, 0)^2,$$

(3.12)

where the superscript means degeneracy, e.g., $(0, 0, 0)^2$ is composed of two $(0, 0, 0)$, but we do not discuss the consequence of degeneracy in this paper. For the indices (3.12), it is straightforward to obtain $F$. Though we have expressed $F$ in (3.9) as an infinite series, actually we can restrict possible terms in (3.3) by considering the degree counting of one-instanton prepotential.

C. Scaling relation for one-instanton prepotential

If the WDVV equations can in fact yield a physically acceptable prepotential, the prepotential obtained from those equations must also satisfy the fundamental homogeneity condition called scaling relation.\(^\text{22,23,24,25}\) Therefore, we may use it as a subsidiary condition for the problem how to solve WDVV equations in gauge theory.

To see this, firstly, let us recall the scaling relation:\(^\text{22,23,24,25}\)

$$\sum_{i=1}^{3} a_i \frac{\partial F}{\partial a_i} + \Lambda_{SU(4)} \frac{\partial F}{\partial \Lambda_{SU(4)}} = 2F. \quad (3.13)$$

We need a scaling relation for $F_1$ not for $F$ itself, but in order to extract it from (3.13), the $\Lambda$-dependence of perturbative prepotential which can not be fixed from WDVV equations must be included appropriately. For this, our choice here is

$$F_{\text{per}} = \sum_{i<j=1}^{4} \frac{a_{ij}^2}{2} \ln \frac{a_{ij}}{\Lambda_{SU(4)}}. \quad (3.14)$$

Then from (3.13), $F_1$ is found to satisfy

$$\sum_{i=1}^{3} a_i \frac{\partial F_1}{\partial a_i} + 6F_1 = 0, \quad (3.15)$$

which indicates that $F_1$ is a homogeneous function of degree $-6$.

In the variables (3.5), (3.15) becomes

$$x\partial_x F_1 + y\partial_y F_1 + z\partial_z F_1 + 6F_1 = 0. \quad (3.16)$$

Accordingly, from (3.8), (3.9) and (3.13) it must be always true that

$$\nu_1 + \nu_2 + \nu_3 + \epsilon_x i + \epsilon_y j + \epsilon_z k = -6. \quad (3.17)$$
D. Examples of one-instanton prepotentials

We have now enough informations to construct explicit one-instanton prepotentials which do not contradict to WDVV equations and scaling relation.

To begin with, let us consider the case $\epsilon \equiv (\epsilon_x, \epsilon_y, \epsilon_z) = (+, +, +)$. In this case, we can easily find that there exists only one solution which satisfies (3.17). It is the solution with $\nu = (-2, -2, -2)$, and thus

$$F_1 = \frac{B_{0,0,0}}{x^2 y^2 z^2}.$$  \hspace{1cm} (3.18)

However, when $\epsilon = (-, -, -)$ or one entry of $\epsilon$ differs to the others, e.g., $\epsilon = (-, -, +)$, the situation changes, in particular, drastically in the latter case. In the former case, it is easy to see that $F$ consists of finite number of terms for all indices in (3.12), but in the latter case $F$ is generally represented by infinite number of terms as long as (3.17) is satisfied. We do not know whether it is possible to find any physical meaning for this type of one-instanton prepotential, but it may be interesting to recall that a similar one was observed in the one-instanton prepotential in the five-dimensional gauge theory.

Since the latter case mentioned above is slightly intractable, let us consider an example of the former case instead. In the case of $\nu = (-1, -1, -1)$ with $\epsilon = (-, -, -)$, for instance, we have

$$F_1 = \frac{1}{xyz} \left[ B_{0,-1,-2} \left( \frac{1}{y^2 z^2} + \frac{1}{y^2 z^2} \right) + B_{-1,0,-2} \left( \frac{1}{x^2 y^2} + \frac{1}{x^2 y^2} \right) + B_{-1,-2,0} \left( \frac{1}{x^2 y} + \frac{1}{x^2 y} \right) + B_{-1,-1,-1} \frac{1}{xyz} \right].$$  \hspace{1cm} (3.19)

Note that (3.19) includes the one-instanton prepotential of the form (3.18).

In a similar manner, we can construct one-instanton prepotentials for all other possible values of $\nu_i$, which do not contradict to both WDVV equations and scaling relation, but it would not be necessary to explicitly show them here. However, we should point out that since (3.3) is a partial differential system we can expect that there exist more and more various solutions. In fact, this observation is right, and we can show that also in the variables

$$(x, y, z) = (a_{12}, a_{23}, a_{24}), \ (a_{13}, a_{23}, a_{34})$$  \hspace{1cm} (3.20)

we can construct miscellaneous one-instanton prepotentials. Among them one-instanton prepotentials of the form (3.18) are included.
Remark: The function form of the one-instanton prepotentials can be determined by solving the WDVV equations, but its numerical factors, i.e., instanton expansion coefficients, are not obtained because they correspond to integration constants. In order to determine them, it is necessary to rewrite the scaling relation as a relation between prepotential and moduli. Then substituting the one-instanton prepotential obtained from WDVV equations into this scaling relation, we will be able to get the expansion coefficients. Of course, in this case the moduli must be represented as a function of periods and its expansion coefficients must be determined. However, since knowing moduli is equivalent to introduce a Seiberg-Witten curve, this method based on scaling relation represented by using moduli is not preferable in the formalism of WDVV equations because prepotentials available from WDVV equations should be determined without introduction of Seiberg-Witten curves. Accordingly, when the determination of instanton expansion coefficients is required, they should be determined from the result of instanton calculus.

IV. ONE-INSTANTON PREPOTENTIAL FROM INSTANTON CALCULUS

We have derived one-instanton prepotentials by solving the WDVV equations in the previous section. Though these one-instanton prepotentials satisfies the WDVV equations and the scaling relation, unfortunately in view of WDVV equations we can not determine which ones are physically acceptable. For this reason, in order to extract physically meaningful one-instanton prepotentials among them, we must compare our result with the one-instanton prediction of instanton calculus.

In the case of SU(4) gauge theory, one-instanton contribution for prepotential is given by

$$F_1 = \Delta_4', \quad \Delta_4' = \sum_{i=1}^{4} \prod_{k,l\neq i}^{4} (a_k - a_l)^2, \quad \Delta_4 = \prod_{k<l=1}^{4} (a_k - a_l)^2. \quad (4.2)$$

The closed form of one-instanton prepotential for SU($N_c$) gauge theory is also obtained by solving Picard-Fuchs equations and direct calculation of period integrals.

Note that (4.1) is a sum of (3.18) and those for (3.20)

$$F_1 = \frac{1}{(a_{12}a_{13}a_{14})^2} + \frac{1}{(a_{12}a_{23}a_{24})^2} + \frac{1}{(a_{13}a_{23}a_{34})^2} \quad (4.3)$$
up to constant factors. Accordingly, we can conclude that the WDVV equations can yield physical prepotential in spite of without introducing Riemann surface.

V. SUMMARY

In this paper, we have discussed the non-perturbative prepotential of $N = 2$ supersymmetric SU(4) Yang-Mills theory in the standpoint of WDVV equations. Especially, we have found a differential equation for one-instanton prepotential and constructed its solutions. The method to get prepotentials based on WDVV equations is fascinating in the point that the prepotentials can be obtained without introducing Seiberg-Witten curves, but it has been shown that unfortunately too many prepotentials exist in contrast with the approach based on Seiberg-Witten curves which uniquely determines a prepotential. Nevertheless, we have succeeded to show that one-instanton prepotentials which coincide with the one-instanton calculus can be obtained from WDVV equations.

As for another aspect of WDVV equations, we should mention a connection to topological field theory in two dimensions. From the appearance of WDVV equations, it may be natural to think that the low energy effective theory is actually a kind of topological field theory, but we must notice that a priori there is no reason that the effective theory must be a topological field theory. Therefore, topological or not topological: that is the question.

An approach to argue this implication more explicitly is to regard the Seiberg-Witten curves often identified with spectral curves of integrable system as if they were superpotentials of topological $\mathbb{CP}^1$ model. Although this observation strongly relying on the existence of Riemann surface (Seiberg-Witten curve), it enables us to find a connection to topological field theory, specifically, topological string theory at genus zero level. Then, even if we do not assume a Riemann surface, can we find a topological nature of the effective theory? Probably WDVV equations give the answer, but the study is a subject in the future.

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In this appendix, we briefly summarize the SU(4) Seiberg-Witten solution. The SU(4) Seiberg-Witten curve is given by the hyperelliptic curve of genus three

$$y^2 = (x^4 - ux^2 - vx - w)^2 - \Lambda_{\text{SU}(4)}^8,$$

(A1)

where \((x, y) \in \mathbb{C}^2\) is the local coordinate, \(u, v\) and \(w\) are moduli of the theory. Then the Seiberg-Witten differential and its periods are given by

$$\lambda_{\text{SW}} = \frac{x \partial_x W}{y} \, dx$$

(A2)

and

$$a_i = \oint_{\alpha_i} \lambda_{\text{SW}}, \quad a_{D_i} = \oint_{\beta_i} \lambda_{\text{SW}}, \quad i = 1, 2, 3,$$

(A3)

respectively, where \(\alpha_i\) and \(\beta_i\) are the canonical bases of the 1-cycles on the curve and the numerical normalization factor of the Seiberg-Witten differential is ignored. It is convenient to use the period vector

$$\Pi = \begin{pmatrix} a_{D_i} \\ a_i \end{pmatrix}.$$  

(A4)

In general, these periods satisfy Fuchsian differential equations and in the case at hand they are given by

$$\mathcal{L}_1 \Pi \equiv \left[ \partial_v^2 - \partial_v \partial_w \right] \Pi = 0,$$

$$\mathcal{L}_2 \Pi \equiv \left[ 4\partial_u^2 - 2u \partial_u \partial_w - v \partial_v \partial_w - \partial_w \right] \Pi = 0,$$

$$\mathcal{L}_3 \Pi \equiv \left[ v \partial_w^2 + 2u \partial_v \partial_w - 4 \partial_u \partial_v \right] \Pi = 0,$$

$$\left[ 4(u^2 + 24w)\partial_u^2 + 9v^2 \partial_v^2 - 16(\Lambda_{\text{SU}(4)}^8 - w^2)\partial_w^2 + 12uv \partial_u \partial_v \right.$$

$$\left. -32uw \partial_u \partial_w + 3v \partial_v - 16w \partial_w + 1 \right] \Pi = 0,$$

(A5)

which can be summarized as

$$\left[ \theta_v (\theta_v - 1) - \frac{v^2}{uw} \theta_u \theta_w \right] \Pi = 0,$$

$$\left[ (2\theta_u + \theta_v + 1)\theta_w - \frac{4w}{u^2} \theta_u (\theta_u - 1) \right] \Pi = 0,$$

$$\left[ (2\theta_u + 3\theta_v + 4\theta_w - 1)^2 - \frac{16\Lambda_{\text{SU}(4)}^8}{w^2} \theta_w (\theta_w - 1) \right] \Pi = 0,$$

(A6)
where $\theta_w = w \partial_w$ etc are Euler derivatives, provided the first, second and last equations in (A5) are chosen as independent equations. Note that the third one in (A5) is not an independent equation because $(v \partial_v \mathcal{L}_1 + \partial_v \mathcal{L}_2 + \partial_v \mathcal{L}_3)\Pi = 0.$

Introducing new variables $x, y$ and $z$ by

$$x = \frac{\Lambda_{SU(4)}^8}{4w^2}, \quad y = \frac{v^2}{4uw}, \quad z = \frac{w}{u^2},$$

we find that (A6) is converted into

$$[(8\theta_x + 1)^2 - 64x(2\theta_x + \theta_y - \theta_z)(2\theta_x + \theta_y - \theta_z + 1)] \Pi = 0,$$

$$\left[\theta_y(2\theta_y - 1) - 2y(\theta_y + 2\theta_z)(2\theta_x + \theta_y - \theta_z)\right] \Pi = 0,$$

$$\left[(2\theta_x + \theta_y - \theta_z)(4\theta_z - 1) - 4z(\theta_y + 2\theta_z)(\theta_y + 2\theta_z + 1)\right] \Pi = 0. \tag{A8}$$

This system (A8) further simplifies to

$$\left[\theta_x - x\left(2\theta_x + \theta_y - \theta_z - \frac{1}{2}\right)\left(2\theta_x + \theta_y - \theta_z + \frac{1}{2}\right)\right] \bar{\Pi} = 0,$$

$$\left[\theta_y\left(\theta_y - \frac{1}{2}\right) - y\left(\theta_y + 2\theta_z + \frac{1}{2}\right)\left(2\theta_x + \theta_y - \theta_z - \frac{1}{2}\right)\right] \bar{\Pi} = 0,$$

$$\left[\left(2\theta_x + \theta_y - \theta_z - \frac{1}{2}\right)\theta_z - z\left(\theta_y + 2\theta_z + \frac{1}{2}\right)\left(\theta_y + 2\theta_z + \frac{3}{2}\right)\right] \bar{\Pi} = 0 \tag{A9}$$

by $\Pi = x^{-1/8} z^{1/4} \bar{\Pi}$. An analytic solution around $(x, y, z) = (0, 0, 0)$ is given by

$$\bar{\Pi} = \sum_{m, n, p=0}^\infty \frac{(1/2)^{n+2p}(-1/2)^{2m+n-p} x^m y^n z^p}{(1/m)(1/2)_n} m! n! p!, \tag{A10}$$

which is known as the type 54b Srivastava and Karlsson’s (Gaussian) hypergeometric function in three variables, $\text{H}_3$ and we denote it by

$$G_{54b}[\alpha, \beta; \gamma, \delta; x, y, z] = \sum_{m, n, p=0}^\infty \frac{(\alpha)_{n+2p}(\beta)_{2m+n-p} x^m y^n z^p}{(\gamma)_{m}(\delta)_{n}} m! n! p!, \tag{A11}$$

which recovers Horn’s $\mathcal{H}_4$ for $p = 0$. Note that $G_{54b}$ satisfies

$$\left[(\theta_x + \gamma - 1)\theta_x - x\left(2\theta_x + \theta_y - \theta_z + \beta\right)(2\theta_x + \theta_y - \theta_z + \beta + 1)\right]G_{54b} = 0,$$

$$\left[(\theta_y + \delta - 1)\theta_y - y\left(\theta_y + 2\theta_z + \alpha\right)(2\theta_x + \theta_y - \theta_z + \beta)\right]G_{54b} = 0,$$

$$\left[(2\theta_x + \theta_y - \theta_z + \beta)\theta_z - z\left(\theta_y + 2\theta_z + \alpha\right)(\theta_y + 2\theta_z + \alpha + 1)\right]G_{54b} = 0. \tag{A12}$$
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