A General Framework For Task-Oriented Network Inference

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Abstract
We present a brief introduction to a flexible, general network inference framework which models data as a network space, sampled to optimize network structure to a particular task. We introduce a formal problem statement related to influence maximization in networks, where the network structure is not given as input, but learned jointly with an influence maximization solution.

1 Introduction
Networks are extensively studied in machine learning, mathematics, physics, and other domain sciences [1][4]. Often the entities and relationships are unambiguously known: two users are ‘friends’ in a social network, or two road segments are adjacent in the network if they physically intersect. Often, an underlying social, biological or other process generates data with latent relationships among entities. Rather than studying the process of interest through either coarse population-level statistics or isolated individual-level statistics, networks tend to represent complexity at multiple scales, and are general and reusable representations for different questions of interest on the process generating the original data.

Previous work often focuses on ad-hoc, rule-based network construction, or model-based representational learning. Our flexible, general framework encompasses and formalizes these approaches. Our framework also learns networks subject to many targets: cascade modeling and routing, node and edge classification, or influence in networks.

2 General Framework
Our framework transforms data from individual entities that are unambiguously known (e.g. users, IP addresses, genes) represented as nodes, into a space of networks which are then sampled under a set of cost constraints, and evaluated relative to a problem of interest.

For a set of nodes \( \{v_i \in V\} \), a set of edge weight probability density functions \( \{d_{ij}(\cdot) : (i,j) \in (V \times V)\} \), a node attribute set \( \{a_i \in A\} \), and a node label set \( \{l_i \in L\} \) let \( G = (V, D, A, L) \) be a space of weighted, attributed graphs.\(^1\) A weighted, attributed graph \( G' = (V', E', A, L) \) is drawn from \( G \) by sampling each edge weight distribution: \( E' = \{e'_{ij} \sim d'_{ij}(\cdot) : (i,j) \in (V \times V)\} \).

Our general framework evaluates weighted graphs within \( G \) according to some task \( T(G', \bullet) \) subject to loss \( L_T(G') \). See Figure 1 for a schematic of this formalization.

3 Problem Formulation: Linear Threshold Process
We instantiate a particular task on the above framework, to sample weighted networks which model a set of observed node labels \( L \) as the result of an Linear Threshold spreading process \(^2\).

Given a weighted network, \( G' \), the linear threshold process is initialized with \( k \) labeled nodes. At each time step, unlabeled nodes adopt the label of the neighborhood if the sum of neighbor weights exceeds the node’s threshold. The process continues until label assignments stabilize.\(^3\) For simplicity, we’ll instantiate a global threshold \( (a_i = \alpha, \forall a_i \in A) \) binary label formulation \( l_i \in \{0,1\} \) for all \( l_i \in L \).

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\( ^1 \) Node labels are simply a specific node attribute of interest for a subsequent task, defined separately for notational convenience.\( ^2 \) This assumes the susceptible-infected model. Nodes with \( l_i = 0 \) were necessarily never infected over the process.
network space for 4 nodes. Each node \(v_i\) has a label value \(l_i\) (\(l_i=1\) shaded), and a linear threshold attribute \(a_i\), set globally in this example. Each dashed edge between nodes denotes an edge weight density function. The shown \(d_{ij}()\) has an associated maximum likelihood, \(e_{ij}^*\), an edge weight sample \(e_{ij}'\), and the independent edge loss (red dashed lines).

In the above problem, we are unconstrained by any loss function \(\mathcal{L}(G')\). Therefore we are always ensured at least \(|l_i=1|\) minimal solutions, with \(k=1\).

**Proof sketch:** Selecting any node where \(l_i=1\), we set edges incident to \(v_i\) where \(l_j=1\) such that \(a_j\) is satisfied (simply: \(e_{ij} = 1\)), therefore we infer a star with binary edges of all labeled nodes for each solution.

We allow our method to adjust \(e_{ij}'\) directly rather than sampling randomly from \(d_{ij}()\). Recall that the range of \(d_{ij}()\) is \([0, 1]\). Therefore even if \(P(d_{ij}()) = 0\), we allow setting \(e_{ij}' = 0\). In the constrained case (below) we will be penalized for this unlikely or unobserved edge weight.

### 3.1 Independent Edge Loss

We introduce a loss function measuring edge density function likelihood. This will incur cost when setting edge weights \(E'\), conditioned on the respective edge density function.

The independent edge loss measures the likelihood of a sampled edge, \(e_{ij}'\), against the edge’s maximum likelihood estimate: \(e_{ij}^* = \text{MLE}(d_{ij}())\):

\[
(3.1) \quad \mathcal{L}(e_{ij}', e_{ij}^*) = P(d_{ij}() = e_{ij}') - P(d_{ij}() = e_{ij}^*)
\]

Defined over an entire realized graph, we get:

\[
(3.2) \quad \mathcal{L}(G') = \sum_{(i,j) \in (V \times V)} \mathcal{L}(e_{ij}', e_{ij}^*)
\]

**Problem 2:** Budgeted Network Inference LT \(k\)-Seed Selection

**Given:** Graph-space \(G = (V, D, A, L)\) with \(A\) node linear thresholds, budget \(\lambda\)

**Find:** \(E' \sim D\) and \(k\) nodes: \(S \subseteq V\)

**Where:** \(L\) is realized on \(E'\) through a linear threshold process initialized on seed-set \(S\)

**Minimizing:** \(k\), subject to \(\mathcal{L}(G') \leq \lambda\)

Problem 2 adds the Independent Edge Loss constraint to the initial Network Inference LT \(k\)-Seed Selection problem, also accepting as input a loss budget \(\lambda\).

### 3.2 Existence of solutions

Problem 2 under infinite budget \(\lambda = \infty\) is equivalent to Problem 1, yielding the same \(k=1\) solutions. Depending on finite \(\lambda\), we cannot guarantee the existence of a solution. When \(\lambda = 0\), there exists exactly one potential solution, the maximum likelihood edge weight set \(E^*\), which may not produce \(L\) under any seeding.

**Proof, by example:** Let \(E^*\) be a star with binary edge weights: \(E^* = \{e_{ij}' \in \{0, 1\}\}\). Let the center of the star, \(v_i\) be unlabeled: \(l_i = 0\). The periphery of the star, \(S = V \ v_i\) are all labeled: \(l_j = 1\). \(S\) must therefore be the seeds of the linear process. This is because \(v_i\) is unlabeled therefore will not propagate to any \(v_j \in S\). Because all \(v_j \in S\) are labeled, the binary edge weights incident to \(v_i\) satisfy any threshold \(a_i \in [0, 1]\), so \(v_i\) must be labeled after the linear threshold process: \(l_i = 1\). Therefore \(L\) is not realizable on \(E^*\).

### 3.3 First Approximation

Rather than formulating a solution to the weighted network case, let’s consider only the binary case. In this case, we realize a network which satisfies our loss budget \(\mathcal{L}(G') \leq \lambda\), where \(e_{ij}' \in \{0, 1\}\).

The Influence Maximization \(k\)-seed selection problem generates candidates for the Budgeted Network Inference LT \(k\)-Seed Selection problem, under this added constraint. A selected \(k\) seed labels cannot propagate through unlabeled \(v_i\). Therefore we assume edges incident to unlabeled \(v_i\) will not satisfy \(a_i\). This effectively
disconnects each $v_i$, yielding connected components of nodes labeled in $L$. An accepted seed-set for each sub-problem is the Influence Maximization $k$-seed selection solution which labels all nodes in the component. The total seed set is the union of these sub-problem seed sets.

This approximates the optimal $k$ for one particular $G'$ realization. However, it remains an open problem exploring the graph-space in an efficient way to improve a particular $G'$ realization with respect to $k$.

4 Other Formulations

This general pattern of $\lambda$-constrained graph sampling in $G$, subject to Independent Edge Loss generalizes to diverse tasks. For example, collective classification [5], is instantiated on labels $L$ used to train local classifiers on node attributes $A$. We sample $G'$ on this task to maximizes classifier performance under $\lambda$ loss constraints.

In the area of influence maximization and information networks, this framework can also incorporate different transmission models (e.g. independent cascade), as well as parameterized rates of transmission on edges [2]. In information networks, edge weight density can be empirically measured from delay times between information arrival at nodes. Once again, $\lambda$-constrained graph sampling in $G$ realize graphs to predict known cascades of information, which may perform better for prediction than the MLE graph $G^*$.

5 Conclusion and Open Problems

This is only a brief outline of this general network inference framework for modeling non-network data for a particular task. We sample a space of networks from observed data, subject to loss constraints and a task objective. Future work will focus on efficient search strategies which take advantage of shared problems across tasks, and comparing graph-edit heuristics across different tasks.

References

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