A POSSIBLE SOLUTION TO
THE PROBLEM OF EXTRA DIMENSIONS

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Abstract

We consider a multidimensional universe with the topology $M = \mathbb{R} \times M_1 \times \cdots \times M_n$, where the $M_i$ ($i > 1$) are $d_i$-dimensional Ricci flat spaces. Exploiting a conformal equivalence between minimal coupling models and conformal coupling models, we get exact solutions for such an universe filled by a conformally coupled scalar field. One of the solutions can be used to describe trapped unobservable extra dimensions.

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1 Introduction

Recently models of multidimensional universes $M = \mathbb{R} \times M_1 \times \cdots \times M_n$, where $M_i$, $(i = 1, \ldots, n)$ are Einstein spaces, have received increasing interest \[1\]. The geometry might be minimally coupled to a homogeneous scalar field $\Phi$ with a potential $U(\Phi)$. The class of multidimensional cosmological models (MCM) is rich enough to study the relation and the imprint of internal compactified extra dimensions (like in Kaluza-Klein models \[2, 3\]) on the external space-time. Therein, exactly solvable classical and quantum models were found by \[4\]. Some of these exact solutions describe the compactification of the internal dimensions up to the actual time. Accordingly, all MCM can be divided into two different classes: The first class consists of models where from the very beginning the internal dimensions are assumed to be static with a scale of Planck length $L_{Pl} \sim 10^{-33}$ cm \[5, 6\]. The other class consists of models where the internal dimensions, like external space-time, evolve dynamically. Thereby however, the internal spaces contract for several orders of magnitude relatively to the external one \[5, 7\].

In \[1\]-\[7\] a minimally coupled scalar field as a matter source was considered. More in particular, in \[3\] a generalized Kasner solution was found, with minimally coupled scalar field and all spaces $M_i$ being Ricci flat. Maeda \[9\] (see also \[10\]) has shown the equivalence of a minimally coupled model to a model with arbitrary coupled scalar field. In 4 dimensions, this was used by Page \[11\] to get new solutions with conformally coupled scalar field from some known solutions with minimally coupled scalar field. This idea can be exploited also, more generally, for arbitrary dimensions and coupling constants \[12\]. In the present paper, from the multidimensional solution of \[8\], we obtain a new solution with conformally coupled scalar field. This new solution bears the possibility to solve the problem of extra dimensions. Rubakov and Shaposhnikov \[13\] and others \[14\]-\[22\] consider non-gravitational fields as dynamical variables which are trapped by a potential well (domain wall), which is narrow along corresponding internal dimensions and extended flatly in 4-dimensional space-time. In \[17\] also dynamical variables of gravity are trapped by a potential. We propose that some gravitational degrees of freedom may be trapped in a classically forbidden region and, hence, be invisible. In general the scale factors of all factor spaces $M_i$ are dynamical variables of the MCM. Here we obtain solutions where the internal spaces are still trapped invisibly, while external 3-space is already born. The factor spaces (external or internal) are born in a quantum tunnelling process from “nothing” \[18\]-\[22\], i.e. from a non-real (e.g. imaginary) section in complex geometry. The birth of different factor spaces may happen at different times. Some of them may remain confined forever in a non-classical section. In complex geometry the extra dimensions also correspond to resolutions of simple singularities in a 3 + 1-dimensional space-time, in which their real parts evolve as string tubes \[22\]. In the following, we explore these ideas in the example of a new solution.
2 Classical Multidimensional Universes

We consider a universe described by a (Pseudo-) Riemannian manifold

\[ M = \mathbb{R} \times M_1 \times \ldots \times M_n, \]

with first fundamental form

\[ g \equiv ds^2 = -e^{2\gamma} dt \otimes dt + \sum_{i=1}^{n} a_i^2 ds_i^2, \quad (2.1) \]

where \( a_i = e^{\beta_i} \) is the scale factor of the \( d_i \)-dimensional space \( M_i \). In the following we assume \( M_i \) to be an Einstein space, i.e. its first fundamental form

\[ ds_i^2 = g_{(i)}^{kl} dx_{(i)}^k \otimes dx_{(i)}^l, \quad (2.2) \]

satisfies the equations

\[ R_{(i)}^{kl} = \lambda_i g_{(i)}^{kl}, \quad (2.3) \]

and hence

\[ R^{(i)} = \lambda_i d_i. \quad (2.4) \]

For the metric (2.1) the Ricci scalar curvature of \( M \) is

\[ R = e^{-2\gamma} \left\{ \left[ \sum_{i=1}^{n} (d_i \dot{\beta}_i) \right]^2 + \sum_{i=1}^{n} d_i [(\dot{\beta}_i)^2 - 2\gamma \dot{\beta}_i^2 + 2\beta_i^2] \right\} + \sum_{i=1}^{n} R^{(i)} e^{-2\beta_i}. \quad (2.5) \]

Let us now consider a variation principle with the action

\[ S = S_{EH} + S_{GH} + S_M, \quad (2.6) \]

where

\[ S_{EH} = \frac{1}{2\kappa^2} \int_M \sqrt{|g|} R dx \]

is the Einstein-Hilbert action,

\[ S_{GH} = \frac{1}{\kappa^2} \int_{\partial M} \sqrt{|h|} K dy \]

is the Gibbons-Hawking boundary term \cite{24}, where \( K \) is the trace of the second fundamental form, which just cancels second time derivatives in the equation of motion, and

\[ S_M = \int_M \sqrt{|g|} \left[ -\frac{1}{2} g^{ik} \partial_i \Phi \partial_j \Phi - U(\Phi) \right] dx \]

is a matter term. Here, and in the following, \( \Phi \) is a homogeneous minimally coupled scalar field. In the case of minimal coupling, we denote the lapse function by \( e^{\gamma} \), and the other scale factors
by \( \dot{a}_i \equiv e^{\dot{\beta}^i} \). Then we define the metric on minisuperspace, given in the coordinates \( \dot{\beta}^i \) and \( \Phi \). We set

\[
G_{ij} := d_i \delta_{ij} - d_i d_j
\]

and define the minisuperspace metric as

\[
G = G_{ij} \dot{\beta}^i \otimes \dot{\beta}^j + \kappa^2 d\Phi \otimes d\Phi.
\]

Furthermore we define

\[
N := e^{\dot{\gamma} - \sum_{i=1}^n d_i \dot{\beta}^i}
\]

and a minisuperspace potential \( V = V(\dot{\beta}^i, \Phi) \) via

\[
V := -\frac{\mu}{2} \sum_{i=1}^n R^{(i)} e^{-2\dot{\beta}^i + \dot{\gamma} + \sum_{j=1}^n \dot{d}_j \dot{\beta}^j} + \mu \kappa U(\Phi) e^{\dot{\gamma} + \sum_{j=1}^n d_j \dot{\beta}^j},
\]

where

\[
\mu := \kappa^{-2} \prod_{i=1}^n \sqrt{|\det g^{(i)}|}.
\]

Then the variational principle of (2.6) is equivalent to a Lagrangian variational principle in minisuperspace,

\[
S = \int L dt, \quad \text{where} \quad L = N \left( \frac{\mu}{2} N^{-2} (G_{ij} \dot{\beta}^i \dot{\beta}^j + \kappa^2 \dot{\Phi}^2) - V \right).
\]

Here \( \mu \) is the mass of a classical particle in minisuperspace. Note that \( \mu^2 \) is proportional to the volumes of spaces \( M_i \).

Next let us compare different choices of time \( \tau \) in Eq. (2.1). The time gauge is determined by the function \( \gamma \). There exist few natural time gauges (compare also [12]). In the following we need only:

i) The **synchronous time gauge**

\[
\gamma \equiv 0,
\]

for which \( t \) in Eq. (2.1) is the proper time \( t_s \) of the universe. The clocks of geodesically comoved observers go synchronous to that time.

ii) The **harmonic time gauge**

\[
\gamma \equiv \gamma_h := \sum_{i=1}^n d_i \dot{\beta}^i
\]

yields the time \( t \equiv t_h \), given by

\[
dt_h = \left( \prod_{i=1}^n q_i \right)^{-1} dt_s
\]

In this gauge the time is a harmonic function, i.e., \( \Delta[g] t = 0 \), and \( N \equiv 1 \). The latter is especially convenient when we work in minisuperspace.
In the harmonic time gauge the equations of motion from Eq. (2.12) yield
\[ \mu G_{ij} \ddot{\beta}^j = -\frac{\partial V}{\partial \dot{\beta}^i} + \frac{\partial U}{\partial \Phi} e^{2\gamma} = 0 \] (2.16)
plus the energy constraint
\[ \frac{\mu}{2} (G_{ij} \dot{\beta}^i \dot{\beta}^j + \kappa^2 \Phi^2) + V = 0. \] (2.17)

3 Conformally Related Models

Let us follow [9] and consider an action of the kind
\[ S = \int d^D x \sqrt{|g|} (F(\phi, R) - \frac{\epsilon}{2} (\nabla \phi)^2). \] (3.1)

With
\[ \omega := \frac{1}{D-2} \ln(2\kappa^2 |\partial F / \partial R|) + A, \] (3.2)
where \( D = 1 + \sum_{i=1}^{n} d_i \) and \( A \) is an arbitrary constant, \( g_{\mu\nu} \) is conformally transformed to the minimal metric
\[ \hat{g}_{\mu\nu} = e^{2\omega} g_{\mu\nu}. \] (3.3)

Especially let us consider in the following actions, which are linear in \( R \). With
\[ F(\phi, R) = f(\phi) R - V(\phi). \] (3.4)

the action is
\[ S = \int d^D x \sqrt{|g|} (f(\phi) R - V(\phi) - \frac{\epsilon}{2} (\nabla \phi)^2). \] (3.5)

In this case
\[ \omega = \frac{1}{D-2} \ln(2\kappa^2 |f(\phi)|) + A \] (3.6)

The scalar field in the minimal model is
\[ \Phi = \kappa^{-1} \int d\phi \{ \frac{(D-2)f(\phi) + 2(D-1)(f'(\phi))^2}{2(D-2)f^2(\phi)} \}^{1/2} = \]
\[ = (2\kappa)^{-1} \int d\phi \{ \frac{2\epsilon f(\phi) + \xi_c^{-1} (f'(\phi))^2}{f^2(\phi)} \}^{1/2}, \] (3.7)

where
\[ \xi_c := \frac{D-2}{4(D-1)} \] (3.8)

is the conformal coupling constant.

For the following we define \( \text{sign} x \) to be \( \pm 1 \) for \( x \geq 0 \) resp. \( x < 0 \). Then with the new minimally coupled potential
\[ U(\Phi) = (\text{sign} f(\phi)) [2\kappa^2 |f(\phi)|]^{-D/(D-2)} V(\phi) \] (3.9)
the corresponding minimal action is
\[ S = \text{sign} f \int d^D x \sqrt{|g|} \left( -\frac{1}{2} \left( \hat{\nabla} \Phi \right)^2 - \frac{1}{\kappa^2} \hat{R} - U(\Phi) \right). \]  (3.10)

Example 1:
\[ f(\phi) = \frac{1}{2} \xi \phi^2, \]  (3.11)
\[ V(\phi) = -\lambda \phi^{2D-2}. \]  (3.12)
Substituting this into Eq. (3.9) the corresponding minimal potential \( U \) is constant,
\[ U(\Phi) = (\text{sign} \xi) |\xi \kappa^2|^{-D/D-2} \lambda. \]  (3.13)
It becomes zero precisely for \( \lambda = 0 \), i.e. when \( V \) is zero. With
\[ f'(\phi) = \xi \phi \]  (3.14)
we obtain
\[ \Phi = \kappa^{-1} \int d\phi \left\{ \frac{\left( \frac{\xi}{\xi_c} + \frac{1}{\phi^4} \right) \phi^2}{\phi^4} \right\}^{1/2} = \left( \kappa \sqrt{\xi} \right)^{-1} \sqrt{\frac{1}{\xi_c} + \frac{\epsilon}{\xi}} \int d\phi \frac{1}{|\phi|} \]
\[ = \kappa^{-1} \sqrt{\frac{1}{\xi_c} + \frac{\epsilon}{\xi}} \ln |\phi| + k \]  (3.15)
for \( -\frac{\xi}{\epsilon} \geq \xi_c \), where \( k \) is a constant of integration. Note that for
\[ \frac{\xi}{\epsilon} = -\xi_c, \]  (3.16)
e.g. for \( \epsilon = -1 \) and conformal coupling, we have
\[ \Phi = k. \]  (3.17)
Thus here the conformal coupling theory is equivalent to a theory without scalar field. For \( -\frac{\xi}{\epsilon} < \xi_c \) the field \( \Phi \) would become complex and, for imaginary \( k \), purely imaginary. In any case, the integration constant \( k \) may be a function of the coupling \( \xi \) and the dimension \( D \).

Example 2:
\[ f(\phi) = \frac{1}{2} (1 - \xi \phi^2), \]  (3.18)
\[ V(\phi) = \Lambda. \]  (3.19)
Then the constant potential \( V \) has its minimal correspondence in a non constant \( U \), given by
\[ U(\Phi) = \pm \Lambda |\kappa^2 (1 - \xi \phi^2)|^{-D/D-2} \]  (3.20)
respectively for \( \phi^2 < \xi^{-1} \) or \( \phi^2 > \xi^{-1} \).
Let us set in the following
\[ \epsilon = 1. \]  
(3.21)

Then with
\[ f'(\phi) = -\xi \phi \]  
(3.22)
we obtain
\[ \Phi = \kappa^{-1} \int d\phi \left\{ \frac{1 + c \xi \phi^2}{(1 - \xi \phi^2)^2} \right\}^{1/2}, \]  
(3.23)
where
\[ c := \frac{\xi}{\xi_c} - 1. \]  
(3.24)

For \( \xi = 0 \) it is \( \Phi = \kappa - \frac{1}{1 - \phi^2} + k \), i.e. the coupling remains minimal. To solved this integral for \( \xi \neq 0 \), we substitute \( u := \xi \phi^2 \). To assure a solution of (3.23) to be real, let us assume \( \xi \geq \xi_c \) which yields \( c \geq 0 \). Then we obtain
\[ \Phi = \mathrm{sign}(\phi) \frac{1}{2\kappa \sqrt{\xi}} \int \frac{\sqrt{u^{-1} + c}}{|1 - u|} du + k_< \]
\[ = \frac{\mathrm{sign}((1 - u)\phi)}{2\kappa \sqrt{\xi}} \left[ -\sqrt{c} \ln(2 \sqrt{c} \sqrt{1 + cu} + 2 cu + 1) + \sqrt{1 + c} \ln \left( \frac{2 \sqrt{1 + cu} + 2 cu + 1 + u}{|1 - u|} \right) + k_< \right] + k_>
\]
\[ = \frac{\mathrm{sign}((1 - \xi \phi^2)\phi)}{2\kappa \sqrt{\xi}} \left[ -\sqrt{c} \ln(2 \sqrt{c} \sqrt{1 + c \xi \phi^2 \sqrt{\xi} |\phi| + 2 c \xi \phi^2 + 1}) + \sqrt{1 + c} \ln \left( \frac{2 \sqrt{1 + c \xi \phi^2 \sqrt{\xi} |\phi| + 2 c \xi \phi^2 + 1 + \xi \phi^2}}{|1 - \phi^2|} \right) + k_< \right] + k_>
\]
\[ = \frac{\mathrm{sign}((1 - \xi \phi^2)\phi)}{2\kappa \sqrt{\xi}} \ln \left( \frac{2 \sqrt{1 + c \sqrt{1 + c \xi \phi^2 \sqrt{\xi} |\phi| + 2 c \xi \phi^2 + 1} |\phi| + (2 c + 1) \xi \phi^2 + 1}^{\sqrt{1 + c}} \cdot |1 - \xi \phi^2|^{\sqrt{1 + c}}}{|2 \sqrt{c} \sqrt{1 + c \xi \phi^2 \sqrt{\xi} |\phi| + 2 c \xi \phi^2 + 1}|^{\sqrt{1 + c}}} + k_< \right]. \]  
(3.25)

The integration constants \( k_< \) for \( \phi^2 < \xi^{-1} \) and \( \phi^2 > \xi^{-1} \) respectively may be arbitrary functions of \( \xi \) and the dimension \( D \). The singularities of the transformation \( \phi \rightarrow \Phi \) are located at \( \phi^2 = \xi^{-1} \).

If the coupling is conformal \( \xi = \xi_c \), i.e. \( c = 0 \), the expressions (3.26) simplify to
\[ \kappa \Phi = \frac{1}{\sqrt{\xi_c}} \left[ \arctanh(\sqrt{\xi_c} \phi) + c_< \right] \]  
(3.26)
for \( \phi^2 < \xi^{-1}_c \) and to
\[ \kappa \Phi = \frac{1}{\sqrt{\xi_c}} \left[ \operatorname{arcoth}(\sqrt{\xi_c} \phi) + c_> \right] \]  
(3.27)
for $\phi^2 > \xi_c^{-1}$, with redefined constants of integration $c_\geq$. In the following we restrict to this case of conformal coupling. The inverse formulas expressing the conformal field $\phi$ in terms of the minimal field $\Phi$ are

$$\phi = \frac{1}{\sqrt{\xi_c}} \left[ \tanh(\sqrt{\xi_c} \kappa \Phi - c_\leq) \right]$$

(3.28)

with $\phi^2 < \xi_c^{-1}$ and

$$\phi = \frac{1}{\sqrt{\xi_c}} \left[ \coth(\sqrt{\xi_c} \kappa \Phi - c_\geq) \right]$$

(3.29)

with $\phi^2 > \xi_c^{-1}$ respectively.

The conformal factor is according to Eqs. (3.6) and (3.18) given by

$$\omega = \frac{1}{D-2} \ln(\kappa^2 |1 - \xi_c \phi^2|) + A.$$  

(3.30)

4 Trapped Internal Dimensions

In the following we want to compare the solutions of the minimal model to those of the corresponding conformal model. We specify the geometry for the minimal model to be of MCM type (2.1), with all $M_i$ Ricci flat (when necessary, assumed to be compact), hence $R^{(i)} = 0$ for $i = 1, \ldots, n$. The minimally coupled scalar field is assumed to have zero potential $U \equiv 0$. In the harmonic time gauge (2.14) with harmonic time

$$\tau \equiv t_h^{(m)},$$

(4.1)

we demand this model to be a solution for Eq. (2.16) with vanishing $R^{(i)}$ and $U(\Phi)$. We set $\hat{\beta}^{n+1} := \kappa \Phi$ and obtain as solution a multidimensional (Kasner like) universe, given by

$$\hat{\beta}^i = b^i \tau + c^i$$

and

$$\hat{\gamma} = \sum_{i=1}^{n} d_i \hat{\beta}^i = (\sum_{i=1}^{n} d_i b^i) \tau + (\sum_{i=1}^{n} d_i c^i),$$

(4.2)

with $i = 1, \ldots, n+1$, where with $V \equiv 0$ the constraint Eq. (2.17) simply reads

$$G_{ij} b^i b^j + (b^{n+1})^2 = 0.$$  

(4.3)

With Eq. (3.30) the scaling powers of the universe given by Eqs. (4.2) with $i = 1, \ldots, n$ transform to corresponding scale factors of the conformal universe

$$\beta^i = \hat{\beta}^i - \omega$$

$$= b^i \tau + \frac{1}{2 - D} \ln |1 - \xi_c (\phi)^2| + c^i + \frac{2}{2 - D} \ln \kappa - A$$

(4.4)

and

$$\gamma = \sum_{i=1}^{n} d_i \beta^i - \omega.$$
\[ (\sum_i d_i b^i) \tau + \frac{1}{2 - D} \ln |1 - \xi_e(\phi)^2| + (\sum_i d_i c^i) + \frac{2}{2 - D} \ln \kappa - A. \]  

It should be clear that the variable \( \tau \), when harmonic in the minimal model, in the conformal model cannot be expected to be harmonic either, i.e. in general

\[ \tau \neq t^{(c)}_h. \]  

Actually from

\[ \gamma = \sum_{i=1}^n d_i b^i = \sum_{i=1}^n d_i \hat{b}^i - \omega(D - 1) \]

we see that \( \tau = t^{(c)}_h \) only for \( D = 2 \) (but we have \( D > 2 \) !).

Let us take for simplicity

\[ A = \frac{2}{2 - D} \ln \kappa, \]

which yields the lapse function

\[ e^\gamma = e^{(\sum_i d_i b^i) \tau + (\sum_i d_i c^i) |1 - \xi_e(\phi)^2|^{\frac{1}{2 - D}}} \]  

and for \( i = 1, \ldots, n \) the scale factors

\[ e^{\beta^i} = e^{b^i \tau + c^i |1 - \xi_e(\phi)^2|^{\frac{1}{2 - D}}}. \]

Let us further set for simplicity

\[ c_\prec = c_\succ = \sqrt{\xi_e} c^n + 1. \]

By Eqs. (3.28) or (3.29), the minimally coupled scalar field

\[ \kappa \Phi(\tau) = b^{n+1} \tau + c^{n+1}, \]

substituted into Eqs. (4.8) and (4.9), yields

\[ e^\gamma = e^{(\sum_i d_i b^i) \tau + (\sum_i d_i c^i) |1 - \xi_e(\phi)^2|^{\frac{1}{2 - D}}} \]  

resp.

\[ e^\gamma = e^{(\sum_i d_i b^i) \tau + (\sum_i d_i c^i) |1 - \xi_e(\phi)^2|^{\frac{1}{2 - D}}} \]  

and, with \( i = 1, \ldots, n \), non-singular scale factors

\[ e^{\beta^i} = e^{b^i \tau + c^i \cosh \frac{2}{\sqrt{1 - \xi_e}} (\sqrt{\xi_e} b^{n+1} \tau)} \]

resp. singular scale factors

\[ e^{\beta^i} = e^{b^i \tau + c^i |\sinh \frac{2}{\sqrt{1 - \xi_e}} (\sqrt{\xi_e} b^{n+1} \tau)|} \]

for the conformal model. The scale factor singularity of the minimal coupling model for \( \tau \to -\infty \) vanishes in the conformal model of Eqs. (4.12) and (4.14) for a scalar field \( \phi \) bounded according to (3.28). For \( D = 4 \) this result had already been indicated by [25].
On the other hand in the conformal model of Eqs. (4.13) and (4.15), with \( \phi \) according to (3.29), though the scale factor singularity of the minimal model for \( \tau \to -\infty \) has also disappeared, instead there is another new scale factor singularity at finite (harmonic) time \( \tau = 0 \).

Let us consider a special case of the non-singular solution with \( \phi^2 < \xi_c^{-1} \), where we assume the internal spaces to be static in the minimal model, i.e. \( b^i = 0 \) for \( i = 2, \ldots, n \). Then in the conformal model, the internal spaces are no longer static. Their scale factors (4.14) with \( i > 2 \) have a minimum at \( \tau = 0 \). From Eq. (4.3) with \( G_{11} = d_1(1 - d_1) \) we find

\[
(b^{n+1})^2 = d_1(d_1 - 1)(b^1)^2.
\]

(4.16)

With real \( b_1 \) then also

\[
b^{n+1} = \pm \sqrt{d_1(d_1 - 1)b^1}
\]

(4.17)
is real and by Eq. (4.14) the scale \( a_1 \) of \( M_1 \) has a minimum at

\[
\tau_0 = (\sqrt{\xi_c}b^{n+1})^{-1}\text{artanh}\left(\frac{(2 - D)}{2\sqrt{\xi_c}} b^1\right),
\]

(4.18)

with \( \tau_0 > 0 \) for \( b^1 < 0 \) and \( \tau_0 < 0 \) for \( b^1 > 0 \).

Let \( M_1 \) be the external space with \( b^1 > 0 \) and hence \( \tau_0 < 0 \). Let us start with an Euclidean region of complex geometry given by scale factors

\[
a_k = e^{-ib_k\tau + \tilde{c}_k} |\sin(\sqrt{\xi_c}b^{n+1} \tau)|^{\frac{1}{D-2}}.
\]

Then we can perform an analytic continuation to the Lorentzian region with \( \tau \to i\tau + \pi/(2\sqrt{\xi_c}b^{n+1}) \), and we require \( c_k = \tilde{c}_k - i\pi b_k/(2\sqrt{\xi_c}b^{n+1}) \) to be the real constants of the real geometry.

The quantum creation (via tunnelling) of different factor spaces takes place at different values of \( \tau \) (see Fig. 1). First the factor space \( M_1 \) comes into real existence and after an time interval \( \Delta\tau = |\tau_0| \) the internal factor spaces \( M_2, \ldots, M_n \) appear in the Lorentzian region. Since \( \Delta\tau \) may be arbitrarily large, there is in principle an alternative explanation of the unobservable extra dimensions, independent from concepts of compactification and shrinking to a fundamental length. Similar to the spirit of the idea that internal dimensions might be hidden due to a potential barrier (13-17), they may have been up to now still in the Euclidean region and hence unobservable. This view is also compatible with their interpretation as complex resolutions of simple singularities in external space 23.

Now let us perform a transition from Lorentzian time \( \tau \) to Euclidean time \(-i\tau\). Then with a simultaneous transition from \( b^k \) to \( ib^k \) for \( k = 1, \ldots, n \) the geometry remains real, since \( \tilde{\beta}^k = b^k\tau + c^k \) is unchanged. The analogue of Eq. (4.17) for the Euclidean region then becomes

\[
b^{n+1} = \pm i\sqrt{d_1(d_1 - 1)b^1}.
\]

(4.19)

This solution corresponds to a classical (instanton) wormhole. The sizes of the wormhole throats in the factor spaces \( M_2, \ldots, M_n \) coincide with the sizes of static spaces in the minimal model, i.e. \( \hat{a}_2(0), \ldots, \hat{a}_n(0) \) respectively.
With Eq. (4.17) replaced by (4.19), the Eq. (4.18) remains unchanged in the transition to the Euclidean region, and the minimum of the scale $a_1$ (unchanged geometry!) now corresponds to the throat of the wormhole in the factor space $M_1$.

If one wants to compare the synchronous time pictures of the minimal and the conformal solution, one has to calculate them for both metrics. In the minimal model we have

\[ dt^\text{(m)}_s = e^{\frac{\gamma}{b}} d\tau = e^{(\sum_i d_i b^i)\tau + (\sum_i d_i c^i)\tau} d\tau, \]

which can be integrated to

\[ t^\text{(m)}_s = (\sum_i d_i b^i)^{-1} e^{\frac{\gamma}{b}} + t_0. \]

The latter can be inverted to

\[ \tau = (\sum_i d_i b^i)^{-1} \left\{ \ln(\sum_i d_i b^i)(t^\text{(m)}_s - t_0) - (\sum_i d_i c^i) \right\}. \]

Setting

\[ B := \sum_{i=1}^n d_i b^i \quad \text{and} \quad C := \sum_{i=1}^n d_i c^i, \]

this yields the scale factors

\[ a^i_s = (t^\text{(m)}_s - t_0)^{b_i} e^{b_i(B \ln B + C)/n} + c_i \]

and the scalar field

\[ \kappa \Phi = \frac{b^{n+1}}{B} \left\{ \ln B(t^\text{(m)}_s - t_0) - C \right\} + c^{n+1}. \]

Let us define for $i = 1, \ldots, n + 1$ the numbers

\[ \alpha^i := \frac{b_i}{B}. \]

With (4.23) they satisfy

\[ \sum_{i=1}^n d_i \alpha^i = 1, \]

and by Eq. (4.3) also

\[ \alpha^{n+1} = \sqrt{1 - \sum_{i=1}^n d_i(\alpha^i)^2}. \]

Eqs. (4.24) shows, that the solution (4.2) is really a generalized Kasner universe with exponents $\alpha^i$ satisfying generalized Kasner conditions (4.27) and (4.28).

In the conformal model the synchronous time is given as

\[ t^\text{(c)}_s = \int e^{\gamma} d\tau = \int \cosh \frac{\tau}{\tau} \left( \sqrt{\xi c b^{n+1} \tau} \right) e^{B\tau + C} d\tau \]

resp.

\[ t^\text{(c)}_s = \int e^{\gamma} d\tau = \int \sinh \frac{\tau}{\tau} \left( \sqrt{\xi c b^{n+1} \tau} \right) e^{B\tau + C} d\tau. \]
5 Conclusion

In the first part of this paper, we reexamine the conformal equivalence between a model with minimal coupling and one with non-minimal coupling in the MCM case. The domains of equivalence are separated by certain critical values of the scalar field $\phi$. Furthermore the coupling constant $\xi$ of the coupling between $\phi$ and $R$ is critical at both, the minimal value $\xi = 0$ and the conformal value $\xi_c = \frac{D-2}{4(D-1)}$. In different noncritical regions of $\xi$ a solution of the model behaves qualitatively very different.

In the second part, we applied the conformal equivalence transformation to the multidimensional generalized Kasner universe with minimally coupled scalar field. So we obtained a new exact solution of a universe with Ricci flat factor spaces and conformally coupled scalar field. It has two qualitatively different regions of equivalence: In the first it is singular w.r.t. scale factors, in the other it is regular. For both, static internal spaces in the minimally coupling model become dynamical in the conformal coupling model. For the regular solution, scale factors are highly asymmetric in time. They have minima at different values harmonic time $\tau$. These may naturally considered as the different times of birth of the factor spaces, where they emerge from the classically forbidden region. Hence the extra dimensions of the internal factor spaces may be still trapped, while external space is already born by quantum tunnelling. In particular it is also possible that some internal spaces never leave the classically forbidden region. Analytic continuation of this solution to the Euclidean region (while pertaining geometry and scalar field real), yields a classical wormhole (instanton).

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Fig. 1: Quantum birth with compact Ricci flat spaces and birth time $\tau_0 \leq 0$ of external Lorentzian space $M_1$. The birth of internal factor spaces $M_2, \ldots, M_n$ is delayed by the interval $\Delta \tau = |\tau_0|$. For $\Delta \tau \to \infty$ the internal spaces remain for ever in the (unobservable) classically forbidden region.
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