Axisymmetry vs. nonaxisymmetry of hydromagnetic Taylor-Couette flows with axial electric currents

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(Received ?? and in revised form ??)

The stability of a Taylor-Couette flow with resting outer cylinder under the influence of a homogeneous axial electric current is investigated. In the linear theory the critical Reynolds number for axisymmetric perturbations $Re = 68$ is not influenced by the current-induced magnetic field but all the axisymmetric magnetic perturbations decay. The nonaxisymmetric perturbations with $|m| = 1$ are excited even without rotation for large enough Hartmann numbers (‘Taylor Instability’) but the growth rate increases with Reynolds number. In the nonlinear regime shear energy is pumped into the neighboring modes $m = 0$ and $|m| = 2$. The ratio $q$ of the energy of the magnetic $|m| = 1$ modes and the toroidal background field is very small for the pure (non-rotating) Taylor instability and grows strongly if differential rotation is present. For super-Alfvénic rotation the energy in the $|m| = 1$ modes of flow and field are in equipartition, with about 1% of the centrifugal energy of the inner cylinder. If the electric current is strong enough to be Taylor unstable then even for very fast rotation kinetic modes and magnetic modes are of the same order of magnitude so that any transport of angular momentum is accompanied by similarly strong mixing (of chemicals).

1. Introduction

In recent years instabilities in rotating conducting fluids under the influence of magnetic fields became of higher interest. Especially in view of astrophysical applications the consideration of differential rotation is relevant. It is known for a long time that differential rotation with negative shear (‘subrotation’) becomes unstable under the influence of an axial field [Velikhov 1959]. Both ingredients of the MagnetoRotational Instability (MRI) itself, i.e. the axial field and also the not too steep differential rotation, are stable. Nevertheless the full system proves to be unstable. One can show that the fundamental mode of the instability is simply axisymmetric [Gellert et al. 2012] while nonaxisymmetric modes only exist for higher eigenvalues. The opposite is true if the axial field is replaced by an axial current. The toroidal magnetic field due to this (assumed to be homogeneous) current is simply $B_\phi \propto R$ (with $R$ as the distance from the rotation axis). Such a profile is unstable against nonaxisymmetric modes as the necessary and sufficient criterion for stability against nonaxisymmetric perturbations,

$$\frac{d}{dR}(RB_\phi^2) < 0$$

[Tayler 1973], is not fulfilled. Thus the mentioned field profile caused by a homogeneous current can be expected to develop nonaxisymmetric magnetic field patterns. The cri-
terion of stability against axisymmetric perturbations under the presence of differential rotation reads

\[
\frac{1}{R^3} \frac{d}{dR} \left( R^2 \Omega \right)^2 - \frac{R}{\mu_0 \rho} \frac{d}{dR} \left( \frac{B_\phi}{R} \right)^2 > 0,
\]

(Michael 1954), where \( \mu_0 \) is the permeability and \( \rho \) the density. In this relation the influence of the homogeneous current disappears and the existence of axisymmetric kinetic and magnetic perturbations depends solely on the radial profile of the rotation law. Solid-body rotation and all kinds of superrotation (with positive radial shear) act stabilizing while quite steep rotation laws with negative shear (steeper than \( \Omega \propto R^{-2} \)) act destabilizing.

2. The Model

The most simple model to study the interaction of differential rotation and Tayler instability is the classical Taylor-Couette (TC) system. A Reynolds number may represent the rotation of the inner cylinder and forms the only free parameter of rotation if the outer cylinder is at rest.

The equations to describe the problem are the MHD equations

\[
\frac{\partial U}{\partial t} + (U \cdot \nabla) U = -\frac{1}{\rho} \nabla P + \nu \Delta U + \frac{1}{\mu_0 \rho} \text{curl} B \times B
\]

and

\[
\frac{\partial B}{\partial t} = \text{curl}(U \times B) + \eta \Delta B,
\]

\[
\text{div} U = \text{div} B = 0,
\]

for an incompressible fluid, where \( U \) is the velocity, \( B \) the magnetic field, \( P \) the pressure, \( \nu \) the kinematic viscosity, and \( \eta \) the magnetic diffusivity.

The basic state in the cylindrical system \((R, \phi, z)\) is \( U_R = U_z = B_R = B_z = 0 \) and

\[
B_\phi = a_B R + \frac{b_B}{R}, \quad U_\phi = \Omega R = a_\Omega R + \frac{b_\Omega}{R},
\]

where \( a_\Omega, b_\Omega, a_B \) and \( b_B \) are constant values defined by

\[
a_\Omega = \frac{\Omega_{in} \mu_\Omega - \mu_R^2}{1 - \mu_R^2}, \quad b_\Omega = \Omega_{in} R_{in}^2 \frac{1 - \mu_\Omega}{1 - \mu_R^2},
\]

\[
a_B = \frac{B_{in} \mu_R - \mu_\Omega}{R_{in} \left( 1 - \mu_R^2 \right)}, \quad b_B = B_{in} R_{in} \left( 1 - \mu_B \mu_R \right) \frac{1 - \mu_\Omega}{1 - \mu_R^2}.
\]

Here it is

\[
\mu_R = \frac{R_{in}}{R_{out}}, \quad \mu_\Omega = \frac{\Omega_{out}}{\Omega_{in}}, \quad \mu_B = \frac{B_{out}}{B_{in}}.
\]

where \( R_{in} \) and \( R_{out} \) are the radii, \( \Omega_{in} \) and \( \Omega_{out} \) the angular velocities, and \( B_{in} \) and \( B_{out} \) the azimuthal magnetic fields at the inner and the outer cylinders respectively. For \( \mu_B = 1/\mu_R \) the radial magnetic profile simplifies to the relation \( B_\phi/B_{in} = R/R_{in} \), which is the profile that can be considered as the result of a applied homogeneous axial electric current (see Tayler 1957). For the specific configuration taken into account in this paper with \( \mu_R = 0.5 \), the ratio of the field values thus is \( \mu_B = 2 \).

The dimensionless numbers defining the problem are the magnetic Prandtl number
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Pm, the Hartmann number Ha and the Reynolds number Re, i.e.

\[ Pm = \frac{\nu}{\eta}, \quad Ha = \frac{B_{in} D}{\sqrt{\mu_0 \rho \eta}}, \quad Re = \frac{\Omega_{in} D^2}{\nu}, \] (2.6)

where the gap width \( D = R_{out} - R_{in} \) is used as unit of length.

For solving the equations a spectral element code is used, which is based on the hydrodynamic code of Fournier et al. (2005). It works with an expansion of the solution in Fourier modes in the azimuthal direction. The remaining part is collection of meridional problems, each of which is solved using a Legendre spectral element method (see e.g. Deville et al. 2002). Between \( M = 8 \) and \( M = 16 \) Fourier modes are used. The polynomial order is varied between \( N = 10 \) and \( N = 16 \) with two elements in radial direction. The number of elements in axial direction corresponds to the aspect ratio \( \Gamma \), the height of the cylinder in units of the gap width, thus the spatial resolution is the same as for the radial direction. With a semi-implicit approach consisting of second-order backward differentiation formula and third order Adams-Bashforth for the nonlinear forcing terms time stepping is done with second-order accuracy. On the end caps periodic conditions in axial direction are applied to minimize finite size and end wall effects. With \( \Gamma = 8 \) all excitable modes in the analyzed parameter region fit into the system. The boundary conditions at the cylinder walls are assumed to be no-slip and perfect-conducting.

All linear solutions are optimized so that the searched for wave number \( k \) yields the lowest Reynolds number. One can easily show that the system is degenerated under the transformation \( m \rightarrow -m \) so that all eigenvalues (for specific \( Re \) and \( Ha \)) are simultaneously valid for each pair \( m = \pm 1, \pm 2, \ldots \).

3. Taylor instability

Taylor (1957) found magnetohydrodynamical perturbations of the axial current with an azimuthal wave number \( m = 1 \) to be unstable. The critical Hartmann number \( Ha = 35.3 \) of this mode has been given by Rudiger et al. (2007) for \( \mu_R = 0.5 \) and the described boundary conditions. For nonrotating container it does not depend on the magnetic Prandtl number (Rudiger & Schultz 2010). Figure 1 concerns a supercritical realization of the instability with \( Ha = 80 \). One finds that only the mode with \( m = 1 \) is unstable while the neighboring modes with \( m = 0 \) and \( m = 2 \) are nonlinearly coupled to the instability. Without rotation the instability of an axial electric current is of the kink-type, thus the axisymmetric mode does not play an important role.

Next, the same axial electric current may flow through a rotating container. Following the work of Chandrasekhar (1956) and Tataronis & Mond (1987), we expect the decay of the instability for \( Re \geq Ha \) when \( Pm = 1 \), i.e. for \( \Omega > \Omega_A \) with the Alfvén frequency \( \Omega_A = B_{in}/\sqrt{\mu_0 \rho D^2} \). Note that for rigid rotation both the radial profiles of \( U_\phi \) and \( B_\phi \) are identical. The growth maps for the various modes with \( m = 0, 1, 2 \) for two slightly different rotation rates are shown by figure 2. For a Reynolds number of the inner rotation of \( Re = 67 \) one finds the magnetic \( m = 1 \) mode as unstable while the mode with \( m = 2 \) decays and starts to grow only when the \( m = 1 \) mode yet has enough energy. The magnetic mode with \( m = 0 \) at first appears to be uninfluenced by the magnetic background field and grows only as a consequence of the growing \( m = 1 \) mode. Note that the energy of the unstable \( m = 1 \) mode is smaller than the energy of the unstable \( m = 1 \) mode for stationary flows (see figure 1).

The situation changes for faster rotation. The Reynolds number in the right plot of figure 2 with \( Re = 68 \) only slightly exceeds the left-plot value of \( Re = 67 \). Both the
Figure 1. Tayler instability for a resting container. Left: The magnetic energy of the modes with $m = 0 \ldots 2$. Only the mode with $m = 1$ is linearly unstable. Right: The isolines of the radial magnetic field component. $Ha = 80$, $\mu_R = 0.5$, $Pm = 1$.

Figure 2. The suppression of the instability for rigid rotation. The magnetic energy of the modes with $m = 0 \ldots 2$. Left: $Re = 67$ (unstable). Right: $Re = 68$ (stable). Note the passive behavior of the mode $m = 0$. $Ha = 80$, $\mu_\Omega = 1$, $\mu_R = 0.5$, $Pm = 1$.

Figure 3. The stabilizing action of $Pm = 1$ for the TI under the influence of rigid rotation. Weaker fields are unstable if $Pm \neq 1$, $\mu_\Omega = 1$, $\mu_R = 0.5$, $Ha = 80$. (See [Rüdiger et al. 2011].)
nonaxisymmetric modes now decay while again the magnetic mode with \( m = 0 \) behaves uninfluenced. Hence, the excitation condition for the nonaxisymmetric instability for rigid rotation and \( Pm = 1 \) reads

\[
\frac{\Omega_A}{\Omega_m} > 1.19. \tag{3.1}
\]

Surprisingly, the critical value (3.1) becomes smaller for \( Pm \neq 1 \) (figure 3). We find that unequal diffusivities destabilize the rotating fluid conductor. It is thus much more difficult to stabilize the TI by rotation for fluid conductors like liquid metals with their very small magnetic Prandtl numbers (see Rüdiger & Schultz 2010).

The limiting curve in figure 3 clearly represents the rotational suppression of the TI. The instability disappears for too fast rotation. This effect exists for all magnetic Prandtl numbers but it is most effective for \( Pm = 1 \). In order to overcome the rotational stabilization the toroidal field must be rather strong what is hard to imagine without differential rotation. It is also evident that the instability for rigid rotation, if at all, can only excite nonaxisymmetric modes rather than axisymmetric modes.

4. Rotational destabilization

The rotation of the outer cylinder may now vanish hence the rotation law is rather steep. The solid lines in figure 4 (left) describe a completely different instability map. Well known is, that axisymmetric perturbations for this configuration are unstable for \( Re \geq 68 \) even without magnetic field. Again slightly surprising is the behavior of the \( m = 0 \) mode, the solid line marked with \( m = 0 \). It does not show any magnetic influence at all.

In the instability map for the nonaxisymmetric modes the hydrodynamical instability (without magnetic field) with \( Re = 76 \) for \( m = 1 \) and the Tayler instability without rotation with \( Ha = 35 \) are directly connected. If only the lowest Reynolds numbers are considered then a transition exists from axisymmetric perturbations for low \( Ha \) to nonaxisymmetric perturbations for high \( Ha \) which is marked by an asterisk in figure 4 (left). One also finds that even under the influence of an axial electric current only the flow becomes unstable for unstable \( m = 0 \), but not the magnetic field. Figure 5 gives the...
growth rates of the linear theory for the kinetic modes and for $Pm = 1$. The dashed lines denote the axisymmetric flow mode. They are equivalent for all applied field strength. So indeed the magnetic influence vanishes. The reason is that in the linear theory the axisymmetric magnetic mode decouples from the system and decays. Thus this pure hydrodynamical axisymmetric mode leads to the well-known $Re = 68$, and is not suppressed by the toroidal magnetic field (see Rüdiger et al. 2007).

The growth rates for the nonaxisymmetric modes with $m = 1$ in figure 5 behave completely different. For sufficiently strong magnetic fields they become unstable even without global rotation (the green solid line). For slow rotation the growth rate runs with both the Hartmann number and the Reynolds number while for fast rotation the Hartmann number dependence disappears. In the latter case it is simply $\omega_{gr} \propto \Omega$ which is a rotational amplification rather than a rotational suppression. For $\Omega_A/\Omega_{in} \ll 1$ the nonaxisymmetric modes are rapidly growing – in large contrast to the condition (3.1), which holds for rigid rotation. The TI of nonaxisymmetric modes under the presence of differential rotation results in our simulations as rotationally supported. It is not yet clear whether a rotational-suppression of these modes exists nevertheless for much faster rotation.

The growth rates for the magnetic modes are identical with those given in figure 5, but only for the nonaxisymmetric modes. The growth rate for the axisymmetric magnetic mode vanishes for all $Re$ and $Ha$.

5. The nonlinear system

Instability maps and the growth rates both give only a rough overview over the real pattern and appearance of the instability, which is mainly determined by nonlinear interactions. Following the above-formulated achievements, the resulting pattern should be axisymmetric in the flow system and nonaxisymmetric in the field system. Therefore the poloidal components of the flow and the field are considered in more detail, i.e. the vectors without the (strong) toroidal fields. Figure 6 demonstrates for models with differential rotation, how in the linear regime of the numerical simulations all nonaxisymmetric modes are unstable. Among the $m = 0$ modes the kinetic one even dominates the growth of the $m = 1$ mode but the magnetic one proves to be stable. If this would be the final truth...
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Figure 6. The time-dependence of the energy of the modes $m = 0$...2 (without toroidal component). The time is normalized with the microscopic diffusion time. Left: the kinetic modes, right: the magnetic modes. $Re = 700$, $Ha = 60$, $\mu_\Omega = 0$, $\mu_R = 0.5$, $Pm = 1$.

Figure 7. The ratio $\epsilon$ of the energy of the nonaxisymmetric magnetic modes to the energy of the kinetic modes. $m = 1$, $\mu_\Omega = 0$, $\mu_R = 0.5$, $Pm = 1$.

then a magnetic configuration which is mainly axisymmetric could never be the result of the TI under the presence of differential rotation. However, the simulations shown in figure 6 also demonstrate the nonlinear interaction of the modes. After the saturation of the instability the energies in the axisymmetric and nonaxisymmetric modes are almost equal. Never, however, the energy of the axisymmetric modes exceeds the energy of the nonaxisymmetric modes. Hence, also in the nonlinear regime the axisymmetric modes cannot dominate.

The ratio $\epsilon$ of the magnetic and the kinetic energy in figure 7 with $\Omega_A/\Omega_m = 0.08$ is of order unity. For strong differential rotation the instability is not dominated by the magnetic fields. Consequently, also mixing processes of chemicals are induced by the magnetic instability. Nonetheless, also an amplification of the magnetic diffusivity must be expected. Note that only the eddy viscosity is amplified by the magnetic fluctuations but not the diffusion coefficients of magnetic fields and temperature (Vainshtein & Kichatinov 1983).

A detailed analysis of the ratio of both the energies shows that i) the magnetic energy only dominates the kinetic energy for slow rotation but that ii) it is not vice versa for fast rotation. Even for very fast rotation with $\Omega_A/\Omega_m \ll 1$ the magnetic energy does not become dominated by the kinetic energy (figure 7). Even in this approximation the magnetohydrodynamical character of the instability survives. The only exception from
Figure 8. Left: The magnetic energy \( q_{\text{pol}} \) (without toroidal component) of the modes with \( m = 0 \ldots 5 \). The time is normalized with the rotation time. Right: Isolines of the radial magnetic field component. \( Re = 350, Ha = 80, \mu_0 = 0, \mu_R = 0.5, Pm = 1 \).

This rule exists for \( Ha < Ha_{\text{crit}} \), i.e. if the axial current is not strong enough to be Tayler unstable (the green line in figure 7). If the current is strong enough to be Tayler unstable then even very fast rotation leads to \( \epsilon > 1 \), i.e. the rotation does not suppress the magnetic character of the system.

Figure 8 shows for reduced rotation the magnetic structure of the saturated instability pattern in more detail. Again the energy of the mode \( m = 1 \) slightly dominates the energy of the mode \( m = 0 \). The modes with \( m = 2, m = 3, m = 4 \) and \( m = 5 \) are shown to possess about one order of magnitude less energy than the lowest modes with \( m = 1 \) and \( m = 0 \). As a consequence, the right panel of this plot shows the pattern as mainly nonaxisymmetric with \( m = 1 \). Note that the amplitude of the fluctuation \( b_R \) almost approaches the amplitude \( B_{\text{in}} \) of the toroidal background field.

6. Equipartition of magnetic and kinetic energy

The influence of the rotation rate on the magnetic configuration of the instability is shown by the figure 9. For a rather weak field two examples are shown with different Reynolds numbers. In both cases the \( m = 1 \) mode dominates. Its energy is significantly higher for the case of faster rotation. The (enslaved) neighboring modes \( m = 0 \) and \( m = 2 \) possess nearly exactly the same energy. The difference to the \( m = 1 \) mode is reduced by faster rotation. As both models work with the same value of \( Ha \) the reported differences are due to the differential rotation. Obviously, only the rotation pumps the energy into the unstable modes while the magnetic energy in the pure TI modes without rotation is very small. This is insofar unexpected as differential rotation usually also acts smoothing to the nonaxisymmetry of all poloidal magnetic fields. Here it emphasizes nonaxisymmetry.

This phenomenon is demonstrated in detail by figure 10. The magnetic energy shown in this plot stands for the ratio

\[
q_m = \frac{\langle b_m^2 \rangle}{B_{\text{in}}^2},
\]

between the energy of the fluctuations of the magnetic mode \( m \) and the toroidal magnetic background field. It is known for driven turbulence under the influence of a homogeneous magnetic field that the ratio \( q \) runs with the magnetic Reynolds number \( Rm = Re Pm \) in
Figure 9. The energy $q_{\text{pol}}$ of the poloidal magnetic modes (without toroidal component) with $m = 0...2$. Left: $Re = 60$, right: $Re = 140$. $\mu_0 = 0$, $Ha = 30$, $\mu_R = 0.5$, $Pm = 1$. 

Figure 10. The quantity $q_1$ for the magnetic modes with $m = 1$. $\mu_B = 2$, $\mu_0 = 0$, $Pm = 1$. Perfectly conducting cylinders.

the high-conductivity limit and with $Rm^2$ in the low-conductivity limit. The square-root of $q$ yields the ratio of the (averaged) TI component to the magnetic background field.

If unstable, the pure TI without rotation only provides a basal value $q_{\text{bas}}$ of about 0.05. Increasing rotation $\Omega_m$ leads to larger values of $q$ which, however, sinks for growing $B_m$. For fast rotation the curves given in figure 10 suggest the approximative behavior $Ha^2 q_1 \rightarrow \hat{\Omega} Re^2$, hence

$$\frac{\Omega^2_m}{\Omega^2_{in}} \rightarrow \hat{\Omega}.$$  \hspace{1cm} (6.2)

Everything here is demonstrated for $Pm = 1$ only. But this general result also holds for $Pm \neq 1$. The final expression for $m = 1$ is

$$\frac{\langle b^2 \rangle}{\mu_0 \rho} \simeq \hat{\Omega} \Omega^2_{in} D^2$$  \hspace{1cm} (6.3)

with $\hat{\Omega} \simeq 0.01$ (figure 11 left). Note that the molecular diffusivities $\nu$ and $\eta$ do not occur in this relation – in contrast to the nonmagnetic TC flow, where it is the Reynolds number which determines the resulting kinetic energy. Figure 11 (right) shows that a very similar expression also holds for the kinetic energy ($m = 1$). Moreover, both energies are in equipartition. The faster the linear rotation in this regime the more energy is stored in the $m = 1$ mode of the Tayler instability. This result fully complies with the observations.
Figure 11. The quantity \( \hat{q} \) for the magnetic (left) and the kinetic (right) \( m = 1 \) modes. Note that for fast rotation the influence of the Reynolds number disappears so that also the molecular diffusivities do not appear in the expressions (6.3) for both the energies. \( \mu_B = 2, \mu_\Omega = 0, Pm = 1 \).

Perfectly conducting cylinders.

Figure 12. The energy of the poloidal magnetic field (without toroidal component) with \( m = 0 \) (dashed lines) and \( m = 1 \) (solid lines). Left: The ratio \( q \) as a function of Hartmann number and Reynolds number. Right: the same for \( \hat{q} \). \( \mu_\Omega = 0, \mu_R = 0.5, Pm = 1 \).

of the magnetic fields of Ap stars. The background field obviously only acts as a catalyst which does not influence the numerical value of the resulting magnetic energy of the instability.

This result for super-Alfvén rotation is correct for the energy in \( m = 0 \) and \( m = 1 \) of the poloidal magnetic field (figure 12) and for the \( m = 1 \) mode of the toroidal field (figure 13 right). In all these cases the energy contained in the modes – after saturation – is about 1% of the centrifugal energy of the inner cylinder. This holds for weak as well as for strong background fields. One also recognizes in the plots (figures 12 and 13), that the energy of the toroidal \( m = 1 \) mode exceeds the energy of both the modes (\( m = 0 \) and \( m = 1 \)) of the poloidal components.

7. The saturation process

The toroidal \( m = 0 \) mode behaves different compared to the poloidal components and the toroidal \( m = 1 \) mode. The dashed lines in figure 13 show that here the \( \hat{q} \) again does not depend on the Reynolds number but it linearly depends on the Hartmann number. The stronger the toroidal background field the more energy is stored in the \( m = 0 \) mode of the toroidal instability pattern. The relation between \( \hat{q} \) and \( Ha \) seems to be linear. For the particular example with \( Ha = 80 \) and \( Re = 150 \) figure 14 demonstrates that
the sign of the axial electric current, which is formed by the axisymmetric magnetic instability mode, is opposite to the sign of the axial background current. The red dashed line in figure 14 symbolizes the uniform axial background current. Obviously, the instability-induced magnetic diffusivity is of the same order as its molecular value. This current is reduced by the opposite axial current formed by the TI (dashed green line). In the average the resulting total current (blue line) is only 50% of the undisturbed background current. One estimates that the magnetic eddy diffusivity $\eta_T$ in the saturated state is roughly of the order of the molecular magnetic diffusivity (Gellert & Rüdiger 2009). If for the model presented in figure 13 one calculates the instability-induced eddy diffusivity with relation
\begin{equation}
\langle u_R b_\phi - u_\phi b_R \rangle = -\mu_0 \eta_T J_z,
\end{equation}
where $J_z$ is the axial component of the applied electric current, then the ratio $\eta_T/\eta$ results in a value shown in the right panel of figure 13. Indeed, in the radial average we find the relation $\eta_T \simeq \eta$ in perfect correspondence of the results of the middle panel. It becomes clear, therefore, that the system saturates by the effective reduction of the applied electric current, or, what is the same, by the increase of the effective magnetic diffusivity.

8. Summary

The simplest hydromagnetic Taylor-Couette flow system has been considered, which models the basic problem of the interaction of differential rotation and toroidal background field. The background field is thought to be due to a uniform axial electric current, which by itself becomes unstable if its Hartmann number exceeds the value $Ha = 35$. The rotation law in hydrodynamics, without background field, becomes unstable for a Reynolds number of $Re = 68$ of the inner cylinder. This values rely on a TC container with radius ratio $\mu_R = 0.5$. While at the threshold value of the hydrodynamic problem an axisymmetric instability pattern is excited, the threshold value of the current-driven Taylor instability excites a nonaxisymmetric pattern. It is thus the relation between axisymmetry and nonaxisymmetry for the MHD Taylor-Couette flow which is the main focus of the present paper.

The growth rates of the linear theory show basic differences between the modes in the linear regime. One first finds that the growth rates of the kinetic modes increase with increasing Reynolds number. Thus the nonuniform rotation in the container does not
Figure 14. Left: The TI induced axisymmetric magnetic mode (green line) reduces the azimuthal magnetic background field (red line) producing the almost uniform azimuthal field (blue line). Middle: the same for the corresponding electric currents. Right: The ratio $\eta_T/\eta$ after the definition (7.1). All quantities are averaged over the azimuth and vertical axis. $Ha = 80$, $Re = 150$, $\mu_B = 2$, $\mu_\Omega = 0$, $Pm = 1$. Perfectly conducting cylinders.

suppress the instability (as rigid rotation does) but it strongly destabilizes the system. This is also true for both the magnetic modes with $m = \pm 1$, but the magnetic $m = 0$ mode remains stable.

The property of the axisymmetric magnetic mode to be linearly stable does not mean that the resulting pattern in the nonlinear regime is purly of the $m = 1$ type. We have shown with nonlinear numerical simulations that energy is pumped from the unstable $m = 1$ modes into the neighboring modes, i.e. $m = 0$ and $m = 2, 3, ...$. In all cases, however, the azimuthal magnetic spectrum peaks at $m = 1$. For fast rotation this peak energy, given by Eq. (6.3), is only tuned by the rotation speed of the inner cylinder. The amplitude of the magnetic background field, the microscopic viscosity and the magnetic resistivity do not influence the peak energy. The same is true for the kinetic energy of the nonaxisymmetric modes $m = \pm 1$ (see figure 11, right).

The ratio $\epsilon$ of the two peak energies determines the magnetohydrodynamic character of the instability. $\epsilon > 1$ represents magnetic-dominated MHD turbulence. Figure 7 demonstrates that for supercritical currents with $Ha > Ha_{\text{crit}}$. Here it is $\epsilon > 1$ for slow rotation but for fast rotation the magnetic and the kinetic energy are almost in equipartition, thus $\epsilon \simeq 1$. Test calculations have shown that for more flat rotation laws (even for hydromagnetically stable ones with $\mu_\Omega > 0.25$) $\epsilon$ increases to values $\epsilon \lesssim 10$, but does not reaches larger values. Thus for fast rotation the kinetic energy never exceeds the magnetic energy. On the other hand, the kinetic energy in the MHD turbulence cannot be considered as very small compared to the magnetic energy. This result should have severe astrophysical consequences. For fast rotation or weak field the magnetic-induced angular momentum transport does not exist without a turbulent mixing of chemicals with a similar intensity, which should strongly affect the stellar structure and evolution. On the other hand, for slow rotation and/or weak filed the magnetic-induced eddy viscosity strongly dominates the mixing coefficient (which is not influenced by the magnetic fluctuations) so that indeed in this case the angular momentum can drift outwards without any direct influence on the stellar evolution.

The simulations also reveal details about the saturation process of the Tayler instability. The system generates an axisymmetric part of the toroidal magnetic field, which is
equivalent to an electric current in opposite direction of the applied current. The amplitude of the counter-current corresponds to an increase of the effective magnetic diffusivity $\eta + \eta_T$ compared to the microscopic value $\eta$. The eddy diffusivity $\eta_T$ can be computed using relation (7.1), which leads to the relation $\eta_T \simeq \eta$, a rather small value, also known from experiments (see Gellert & Rüdiger 2009).

M.G. acknowledges support by the Helmholtz Alliance LIMTECH.

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