CANONICAL 2-FORMS ON THE MODULI SPACE OF Riemann surfaces

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Abstract. As was shown by Harer [14] [15], the second homology of \( M_g \), the moduli space of compact Riemann surfaces of genus \( g \), is of rank 1, provided \( g \geq 3 \). This means there exists a nontrivial second de Rham cohomology class on \( M_g \) which is unique up to a constant factor. But several canonical 2-forms on the moduli space have been constructed in various geometric contexts, and they differ from each other. In this article we review some constructions of such canonical 2-forms in order to provide material for future research on the “secondary geometry” of the moduli space \( M_g \).

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1. Introduction

Let \( g \geq 2 \) be an integer. The moduli space of compact Riemann surfaces of genus \( g \), \( M_g \), is the quotient space of Teichmüller space \( T_g \) by the natural action of the mapping class group \( \mathcal{M}_g \), \( M_g = T_g / \mathcal{M}_g \). Since Teichmüller space is contractible, the real cohomology of the

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mapping class group is isomorphic to that of the moduli space. As was shown by Harer [14] [15], the second homology of $M_g$ is of rank 1 if $g \geq 3$. This means there exists a nontrivial second de Rham cohomology class on $M_g$ which is unique up to a constant factor. But several canonical 2-forms on the moduli space have been constructed in various geometric contexts, and they differ from each other. In this article we review some constructions of such canonical 2-forms in order to provide material for future research on the “secondary geometry” of the moduli space $M_g$.

The signature of the total space of a fiber bundle is not necessarily equal to the product of the signatures of the base space and the fiber. The first example for this phenomenon was given by Kodaira [27] and Atiyah [6], who constructed a certain branched covering space of the product of two compact Riemann surfaces. The covering space has non-zero signature, while the signature of any compact Riemann surface is zero. We may regard the covering space as a family of compact Riemann surfaces parametrized by a compact Riemann surface, so that it defines a non-trivial 2-cycle on the space $M_g$. As was formulated by Meyer [30] [31], the signature of the total space of a family of compact Riemann surfaces defines a non-trivial 2-cocycle of the mapping class group $M_g$ and this provides a non-trivial cohomology class of degree 2 on the space $M_g$. Nowadays this cocycle is called the Meyer cocycle and it has been playing an essential role in the topological study of fibered complex surfaces. See [4] and [5] for details.

The first and the second Betti numbers of the space $M_g$, or equivalently, those of the group $M_g$, are given by

\begin{align}
(1.1) & \quad b_1(M_g) = 0, \\
(1.2) & \quad b_2(M_g) = 1, \quad \text{if } g \geq 3.
\end{align}

For alternative computations of $b_2(M_g)$, see [2] [28] [44]. The group $H^2(M_g; \mathbb{R})$ is generated by the cohomology class of the Meyer cocycle. In the case $g = 2$ we have $b_2(M_2) = 0$ because of Igusa’s result $M_2 = \mathbb{C}^3/(\mathbb{Z}/5) \simeq * [12]$.

Mumford [42] and Morita [33] independently introduced a series of cohomology classes $e_n = (-1)^{n+1} \kappa_n \in H^{2n}(M_g)$, $n \geq 1$, the Morita-Mumford classes or the tautological classes. They are defined as follows. Let $\pi : \mathbb{C}_g \to M_g$ be the universal family of compact Riemann surfaces of genus $g$. The relative tangent bundle of the map $\pi$, $T_{\mathbb{C}_g/M_g}$, the kernel of the differential $d\pi : T\mathbb{C}_g \to \pi^*TM_g$, is a complex line $V$-bundle over $\mathbb{C}_g$. The $n$-th Morita-Mumford class $e_n = (-1)^{n+1} \kappa_n$, $n \geq 1$, is defined to be the integral of the $(n+1)$-st power of the Chern
class of the bundle $T^X_{\mathcal{C}_g/M_g}$ along the fiber

\[(1.3) \quad e_n = (-1)^{n+1} \kappa_n = \int_{\text{fiber}} c_1(T^X_{\mathcal{C}_g/M_g})^{n+1} \in H^{2n}(M_g).\]

The first one $e_1 = \kappa_1$ is 3 times the cohomology class of the Meyer cocycle. As was proved by Morita [34] and Miller [32], the Morita-Mumford classes are algebraically independent in the stable range $s < \frac{3}{2}g$ [16] of the cohomology algebra $H^*(M_g; \mathbb{R})$. Their proofs generalize the construction of Kodaira and Atiyah. In 2002 Madsen and Weiss [29] proved that the cohomology algebra $H^*(M_g; \mathbb{R})$ in the stable range is generated by the Morita-Mumford classes.

From the results (1.1) and (1.2) the simplest non-trivial cohomology classes on $M_g$ are of degree 2, and they are unique up to a constant factor. But several 2-forms on $M_g$, or equivalently $M_g$-equivariant 2-forms on Teichmüller space $T_g$, have been canonically constructed in various geometric contexts.

From the uniformization theorem any compact Riemann surface $C$ of genus $g \geq 2$ admits a unique hyperbolic metric. The volume form of the hyperbolic metric defines the Weil-Petersson pairing on the cotangent space $T^*_C M_g$ involved with no additional information. As was shown by Wolpert [48] the Weil-Petersson-Kähler form $\omega_{WP}$ represents the first Morita-Mumford class $e_1$. Thus we obtain a canonical 2-form representing $e_1$.

The period map is a canonical map defined on Teichmüller space into the Siegel upper halfspace $\mathcal{H}_g$. We have a canonical 2-form on $\mathcal{H}_g$ whose pullback represents the class $e_1$ on the moduli space $M_g$.

We have another canonical metric on a compact Riemann surface. A natural Hermitian product on the space of holomorphic 1-forms defines the volume form $B$ in §5.3 which induces a Hermitian metric on the Riemann surface. The Arakelov-Green function is derived from the volume form $B$. As will be stated in §7 and §8 a higher analogue of the period map is constructed and yields other canonical 2-forms representing $e_1$. These forms are closely related to the volume form $B$.

All of them differ from each other. As to 2-forms representing non-trivial cohomology classes of degree 2 on the moduli space $M_g$, the term ‘canonical’ does not imply ‘unique’. The difference of such forms should induce some secondary object on the moduli space $M_g$. Assume $g \geq 3$. If we have two real $(1,1)$-forms $\psi_1$ and $\psi_2$ on $M_g$, representing $e_1$, then there exists a real-valued function $f \in C^\infty(M; \mathbb{R})$ such that $\psi_2 - \psi_1 = \sqrt{-1} \partial \bar{\partial} f$. Such a function $f$ is unique up to a constant. See Lemma 8.1. This function captures the difference between these two
forms, so that it should describe a certain relation between the two geometric contexts behind these forms.

In this article we review some constructions of canonical 2-forms. In §2 we give a short review on the cotangent spaces of moduli spaces. They are naturally isomorphic to some spaces of quadratic differentials. In §3 we take a quick glance at the Weil-Petersson Kähler form, which is related to the Virasoro cocycle through the Krichever construction. The most classical 2-form on \( \mathcal{M}_g \) is the pullback of the first Chern form on the Siegel upper halfspace \( \mathcal{H}_g \) by the period map Jac, or equivalently the first Chern form of the Hodge bundle on \( \mathcal{M}_g \). We explain this form in §§4 and 5. The Hodge bundle yields all the odd Morita-Mumford classes but not the even ones. We can obtain other canonical differential forms on the moduli space representing all the Morita-Mumford class \( e_i, i \geq 1 \), through a higher analogue of the period map, and this is described in §§6 and 7. Among them some 2-forms seem to be related to Arakelov geometry, as will be discussed in §8

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2. The cotangent space of the moduli space

Let \( C \) be a compact Riemann surface of genus \( g \geq 2 \), \( P_0 \) a point on \( C \). Then we denote by \( H^q(C; aK + bP_0) \), \( q = 0, 1 \), and \( a, b \in \mathbb{Z} \), the \( q \)-th cohomology group \( H^q(C; \mathcal{O}_C(T^\ast C \otimes [P_0] \otimes b)) \). Moreover we denote by \( \Omega^q(C) \) the complex-valued \( q \)-currents on \( C \) for \( 0 \leq q \leq 2 \). The Hodge \( \ast \)-operator \( \ast : (T^\ast_R C) \otimes \mathbb{C} \rightarrow (T^\ast_R C) \otimes \mathbb{C} \) on the cotangent bundle of \( C \) depends only on the complex structure of \( C \). The \( -\sqrt{-1} \)-eigenspace is the holomorphic cotangent bundle \( T^\ast C \), and the \( \sqrt{-1} \)-eigenspace is the antiholomorphic cotangent bundle \( \overline{T^\ast C} \). The operator \( \ast \) decomposes the space \( \Omega^1(C) \) into the \( \pm\sqrt{-1} \)-eigenspaces

\[
\Omega^1(C) = \Omega^{1,0}(C) \oplus \Omega^{0,1}(C),
\]

where \( \Omega^{1,0}(C) \) is the \( -\sqrt{-1} \)-eigenspace and \( \Omega^{0,1}(C) \) the \( \sqrt{-1} \)-eigenspace. Throughout this article we denote by \( \varphi' \) and \( \varphi'' \) the \( (1,0) \)- and the \( (0,1) \)-parts of \( \varphi \in \Omega^1(C) \), respectively, i.e.,

\[
\varphi = \varphi' + \varphi'', \quad \ast\varphi = -\sqrt{-1}\varphi' + \sqrt{-1}\varphi''.
\]

If \( \varphi \) is harmonic, then \( \varphi' \) is holomorphic and \( \varphi'' \) anti-holomorphic.

The Kodaira-Spencer map gives a natural isomorphism

\[
T_{[C]}\mathcal{M}_g = H^1(C; -K).
\]
To look at the isomorphism (2.1) more explicitly, consider a $C^\infty$ family of compact Riemann surfaces $C_t$, $t \in \mathbb{R}$, $|t| < 1$, with $C_0 = C$. The family $\{C_t\}$ is trivial as a $C^\infty$ fiber bundle over an interval near $t = 0$, so that we have a $C^\infty$ family of $C^\infty$ diffeomorphisms $f^t : C \to C_t$ with $f^0 = 1_C$. In general, if $\phi = \phi_t$ is a “function” in $t \in \mathbb{R}$, $|t| < 1$, then we write simply

$$\phi_t = \frac{d}{dt} \bigg|_{t=0} \phi.$$  

For example, we denote

$$\dot{\mu} = \frac{d}{dt} \bigg|_{t=0} \mu(f^t).$$

Here $\mu(f^t)$ is the complex dilatation of the diffeomorphism $f^t$. Let $z_1$ be a complex coordinate on $C$, and $\zeta_1$ on $C_t$. The complex dilatation $\mu(f^t)$ is defined locally by

$$\mu(f^t) = \mu(f^t)(z_1) \frac{d}{dz_1} \otimes \overline{dz_1} = \frac{(\zeta_1 \circ f^t)_{\mu}}{(\zeta_1 \circ f^t)_{z_1}} \frac{d}{dz_1} \otimes \overline{dz_1},$$

which does not depend on the choice of the coordinates $z_1$ and $\zeta_1$. The Dolbeault cohomology class $[\dot{\mu}] \in H^1(C; -K)$ is exactly the tangent vector $\frac{d}{dt}|_{t=0}[C_t] \in T[C\mathbb{M}_g]$.

We define a linear operator $S = S[\dot{\mu}] : \Omega^1(C) \to \Omega^1(C)$ by

$$S(\varphi) = S(\varphi') + S(\varphi'') := -2\varphi' \dot{\mu} - 2\varphi'' \hat{\mu},$$

for $\varphi = \varphi' + \varphi''$, $\varphi' \in \Omega^{1,0}(C)$, $\varphi'' \in \Omega^{0,1}(C)$. From straightforward computation we have

$$\star = \star S = -S \star : \Omega^1(C) \to \Omega^1(C).$$  

By Serre duality we have a natural isomorphism

$$T^*[C\mathbb{M}_g] = H^0(C; 2K).$$

The space $H^0(C; 2K)$ consists of the holomorphic quadratic differentials on $C$. For any holomorphic quadratic differential $q$ the covariant tensor $q \dot{\mu}$ can be regarded as a $(1, 1)$-form on $C$. The integral $\int_C q \dot{\mu}$ is just the value of the covector $q$ at the tangent vector $[\dot{\mu}] = \frac{d}{dt}|_{t=0}[C_t]$.

Let $\mathbb{C}_g$ denote the moduli space of pointed compact Riemann surfaces $(C, P_0)$ of genus $g$ with $P_0 \in C$. The forgetful map $\pi : \mathbb{C}_g \to \mathbb{M}_g$, $[C, P_0] \mapsto [C]$, can be interpreted as the universal family of compact Riemann surfaces on the moduli space $\mathbb{M}_g$. We identify

$$T_{[C, P_0]} \mathbb{C}_g = H^1(C; -K - P_0), \quad T^*_{[C, P_0]} \mathbb{C}_g = H^0(C; 2K + P_0)$$

with
in a way similar to the space $M_g$.

The relative tangent bundle of the forgetful map $\pi$ with the zero section deleted

$$T^\times_{C_g/M_g} = T_{C_g/M_g} \setminus \text{(zero section)}$$

can be interpreted as the moduli space of triples $(C, P_0, v)$ of genus $g$. Here $C$ is a compact Riemann surface of genus $g$, $P_0 \in C$, and $v \in T_{P_0}C \setminus \{0\}$. Similarly the space of quadratic differentials $H^0(C; 2K + 2P_0)$ is identified with the cotangent space of $T^\times_{C_g/M_g}$

(2.5)  

$$T^\times_{[C,P_0,v]}T^\times_{C_g/M_g} = H^0(C; 2K + 2P_0).$$

Moreover this space is closely related to Ehresmann connections on the bundle $T_{C_g/M_g}$. In general, let $\varpi : L \to M$ be a holomorphic line bundle over a complex manifold $M$, and $L^\times$ the total space with the zero section deleted $L^\times = L \setminus \text{(zero section)}$. We denote by $R_a$ the right action of $a \in \C^\times := \C \setminus \{0\}$ on the space $L^\times$, and by $Z$ the vector field on $L^\times$ generated by the action $R_a$

$$Z := \frac{d}{dt} \bigg|_{t=0} R_{e^{it}}.$$

An Ehresmann connection $A$ (of type $(1,0)$) on the bundle $L$ is a $(1,0)$-form on the space $L^\times$ with the conditions

$$A(Z) = 1, \quad \text{and}$$

$$R_{e^{it}} A = A, \quad \forall t \in \R$$

[20]. In other words, it is a splitting of the extension of holomorphic vector bundles over $M$

$$0 \to T^*M \xrightarrow{\varpi^*} (T^*L^\times)/\C^\times \xrightarrow{Z} \C \to 0.$$

Then there exists a unique $(1,1)$-form $c_1(A)$ on $M$ such that $\frac{\sqrt{-1}}{2\pi} dA = \varpi^* c_1(A)$. The form $c_1(A)$ is, by definition, the Chern form of the connection $A$ and represents the first Chern class of the line bundle $L$

$$[c_1(A)] = c_1(L) \in H^2(M; \R).$$

Now we let $M = C_g$ and $L = T_{C_g/M_g}$. By straightforward computation we have a natural commutative diagram

$$
\begin{array}{cccccccc}
0 & \longrightarrow & T^*_{[C,P_0]}M & \xrightarrow{\varpi^*} & ((T^*L^\times)/\C^\times)_{[C,P_0]} & \overset{Z}{\longrightarrow} & \C & \longrightarrow & 0 \\
& & \| & & \downarrow & & & & \\
0 & \longrightarrow & H^0(C; 2K + P_0) & \longrightarrow & H^0(C; 2K + 2P_0) & \xrightarrow{2\pi\sqrt{-1}\operatorname{Res}_{P_0}} & \C & \longrightarrow & 0
\end{array}
$$
Here $\text{Res}_{P_0} : H^0(C; 2K + 2P_0) \to \mathbb{C}$ is the residue map of quadratic differentials at $P_0$ defined by

$$\text{Res}_{P_0}(q - 2z^{-2} + q_1z^1 + \cdots)dz \otimes 2 = q_{-2},$$

where $z$ is a complex coordinate centered at $P_0$. It is easy to check $q_{-2}$ does not depend on the choice of the coordinate $z$. Consequently any $C^\infty$ family $q = \{q(C, P_0)\}_{[C, P_0] \in \mathbb{M}_g}$ of quadratic differentials parametrized by the space $\mathbb{C}_g$ satisfying the condition $\text{Res}_{P_0} q(C, P_0) = \frac{1}{2\pi\sqrt{-1}}$ for any $[C, P_0] \in \mathbb{C}_g$ corresponds to an Ehresmann connection on the relative tangent bundle $T_{\mathbb{C}_g/\mathbb{M}_g}$. The $(1, 1)$ form $\sqrt{-1}\partial\bar{\partial}q$ on the space $\mathbb{C}_g$ represents the first Chern class of the bundle $T_{\mathbb{C}_g/\mathbb{M}_g}$

$$\frac{-1}{2\pi} [\partial \bar{\partial} q] = c_1(T_{\mathbb{C}_g/\mathbb{M}_g}) \in H^2(\mathbb{C}_g; \mathbb{R}).$$

[20].

3. The Weil-Petersson Kähler form

As was shown in [22] the cotangent space of the moduli space $\mathbb{M}_g$ at $[C]$ is naturally isomorphic to the space of holomorphic quadratic differentials, $H^0(C; 2K)$. Let $d\text{vol}$ denote the hyperbolic volume form on the Riemann surface $C$. It is regarded as a Hermitian metric on the relative tangent bundle $T_{\mathbb{C}_g/\mathbb{M}_g}$. For any two differentials $q_1, q_2 \in H^0(C; 2K)$ the Weil-Petersson pairing $\langle q_1, q_2 \rangle_{WP}$ is defined by the integral

$$\langle q_1, q_2 \rangle_{WP} = \int_C q_1 \overline{q_2} / d\text{vol}.$$ 

Here $q_1 \overline{q_2} / d\text{vol}$ is regarded as a $(1, 1)$-form on $C$. The pairing induces a Hermitian metric on the moduli space $\mathbb{M}_g$, the Weil-Petersson metric. Ahlfors [11] proved it is Kähler. See [10] for an alternative gauge-theoretic proof. Let $\omega_{WP}$ denote the Kähler form of the Weil-Petersson metric.

Now recall the original definition of the $i$-th Morita-Mumford class $e_i = (-1)^{i+1}\kappa_i, i \geq 1$ [42] [33]. It is defined to be the integral along the fiber of the $(i + 1)$-st power of the first Chern class of the relative tangent bundle $T_{\mathbb{C}_g/\mathbb{M}_g}$

$$e_i = (-1)^{i+1}\kappa_i = \int_{\text{fiber}} c_1(T_{\mathbb{C}_g/\mathbb{M}_g})^{i+1} \in H^{2i}(\mathbb{M}_g).$$

It is one of the most orthodox ways to obtain differential forms representing the Morita-Mumford classes to take the integral of powers of the hyperbolic Chern form of the relative tangent bundle $T_{\mathbb{C}_g/\mathbb{M}_g}$ along
the fiber. This was carried out by Wolpert [48]. He computed the Chern form $c_1^{\text{hyperbolic}}(T_{C_g/M_g})$ of the hyperbolic metric explicitly, and he proved

$$\int_{\text{fiber}} c_1^{\text{hyperbolic}}(T_{C_g/M_g})^2 = \frac{1}{2\pi^2} \omega_{WP}$$

as differential forms on the moduli space $\mathbb{M}_g$. As a corollary we have

$$\frac{1}{2\pi^2} [\omega_{WP}] = e_1 \in H^2(\mathbb{M}_g; \mathbb{R}).$$

Furthermore Wolpert [49] gave a description of the Weil-Petersson Kähler form in terms of the Fenchel-Nielsen coordinates $(\tau_j, \ell_j)$, $1 \leq j \leq 3g - 3$, for any pants decomposition of the surface

$$\omega_{WP} = \sum d\ell_i \wedge d\tau_i.$$  

Here $\ell_j$ denotes the geodesic length of each simple closed curve in the decomposition, and $\tau_j \in \mathbb{R}$ the hyperbolic displacement parameter. Penner [43] described explicitly the pullback of $\omega_{WP}$ to the decorated Teichmüller space. Goldman [11] generalized the Weil-Petersson geometry to the space of surface group representations in a reductive Lie group.

Now we consider the Lie algebra $\mathfrak{d}$ of complex analytic vector fields on the punctured disk $\{z \in \mathbb{C}; 0 < |z| < \epsilon\}$, $0 < \epsilon \ll 1$. The 2-cochain $\text{vir}$ on $\mathfrak{d}$ defined by

$$\text{vir} \left( \frac{d}{dz} f_1(z), \frac{d}{dz} f_2(z) \right) := \frac{1}{2\pi \sqrt{-1}} \oint_{|z|=1} \det \begin{pmatrix} f_1'(z) & f_2'(z) \\ f_1''(z) & f_2''(z) \end{pmatrix} \, dz$$

$$= \sqrt{-1} \frac{1}{2\pi} \oint_{|z|=1} \det \begin{pmatrix} f_1'(z) & f_2'(z) \\ f_1''(z) & f_2''(z) \end{pmatrix} \, dz$$

is a cocycle and it is called the Virasoro cocycle. Its cohomology class generates the second Lie algebra cohomology group $H^2(\mathfrak{d}) = \mathbb{C}$.

Arbarello, De Concini, Kac and Procesi [3] established an isomorphism of $H^2(\mathfrak{d})$ onto the second cohomology group of $\mathbb{M}_g$

$$\nu : H^2(\mathfrak{d}) \cong H^2(\mathbb{M}_g; \mathbb{C})$$

induced by the Krichever construction.

For a local coordinate $z$ on a Riemann surface one can define a local differential operator, or a local complex analytic Gel’fand-Fuks 1-cocycle with values in quadratic differentials by

$$\nabla_2^{d/dz} : f(z) \frac{d}{dz} \mapsto \frac{1}{6} f'''(z)(dz)^{\otimes 2}$$
The cocycle $\nabla_{dz}^2$ is equivalent to a projective structure. In fact, if $w$ is another coordinate, then
$$\nabla_{dw}^2 X - \nabla_{dz}^2 X = L_X(\{w, z\}(dz)^2)$$
for any local complex analytic vector field $X$. Here $\{w, z\}$ denotes the Schwarzian derivative. In particular, the hyperbolic structure on a (hyperbolic) Riemann surface defines a global operator $\nabla_{\text{hyperbolic}}^2$.

The Krichever construction relates the 2-cocycle vir with the operator $\nabla_{\text{hyperbolic}}^2$. By straightforward computation using the Bers embedding we have
$$\overline{\partial} \nabla_{\text{hyperbolic}}^2 = 8 \omega_{\text{WP}}$$
as $(1,1)$-forms on the moduli space $\mathcal{M}_g$. This result, the first variation of the hyperbolic structure coincides with $\omega_{\text{WP}}$, was first proved by Zograf and Takhtajan [50, p.310].

4. The first Chern form on the Siegel upper halfspace

The Hodge bundle $\Lambda_{\mathcal{M}_g}$ is defined to be the holomorphic vector bundle on $\mathcal{M}_g$ whose fiber over $[C]$ is the space of holomorphic 1-forms on $C$
$$\Lambda_{\mathcal{M}_g} = \coprod_{[C] \in \mathcal{M}_g} H^0(C; K).$$

We write simply $c_1$ for the first Chern class of $\Lambda_{\mathcal{M}_g}$
$$c_1 = c_1(\Lambda_{\mathcal{M}_g}) \in H^2(\mathcal{M}_g; \mathbb{R}).$$

The bundle $\Lambda_{\mathcal{M}_g}$ comes from a symplectic equivariant vector bundle on the Siegel upper halfspace $\mathfrak{H}_g$. In fact, the space $\mathfrak{H}_g$ can be identified with the space of almost complex structures $J$ on the real $2g$-dimensional symplectic vector space $(\mathbb{R}^{2g}, \cdot)$ with the conditions
$$Jx \cdot Jy = x \cdot y, \quad \forall x, \forall y \in \mathbb{R}^{2g},$$
$$x \cdot Jx > 0, \quad \forall x \in \mathbb{R}^{2g} \setminus \{0\}.$$

We have a holomorphic vector bundle $E'_{\mathfrak{H}_g}$ on $\mathfrak{H}_g$ whose fiber over $J$ is the $-\sqrt{-1}$-eigenspace of $J$. We have a natural isomorphism of vector bundles
$$T^*\mathfrak{H}_g = \text{Sym}^2 E'_{\mathfrak{H}_g}.$$

For each Riemann surface $C$ the Hodge $*$-operator on the 1-forms induces such an almost complex structure on the space of real harmonic 1-forms. This induces a holomorphic map $\text{Jac} : \mathcal{M}_g \to \mathfrak{H}_g/Sp_{2g}(\mathbb{Z})$ known as the period map in the classical context. The pullback of $E'_{\mathfrak{H}_g}$ by the map $\text{Jac}$ is exactly the Hodge bundle $\Lambda_{\mathcal{M}_g}$.
Thus the cohomology class $c_1$ can be regarded as an integral cohomology class of the Siegel modular group $Sp_{2g}(\mathbb{Z})$, $c_1 \in H^2(Sp_{2g}(\mathbb{Z}); \mathbb{Z})$. Meyer [30] proved that the cohomology class of the Meyer cocycle is equal to $4c_1 \in H^2(Sp_{2g}(\mathbb{Z}); \mathbb{Z})$. From the Grothendieck-Riemann-Roch formula, or equivalently the Atiyah-Singer index theorem for families, it follows that

$$\frac{1}{12} e_1 = c_1 \in H^2(\mathcal{M}_g; \mathbb{R}).$$

To describe a canonical 2-form representing $c_1(E'_{\mathcal{H}_g})$ we consider the quotient vector bundle $E''_{\mathcal{H}_g} := (\mathcal{H}_g \times \mathbb{C}^{2g})/E'_{\mathcal{H}_g}$, and the family of projections $\pi = \{\pi_J\}_{J \in \mathcal{H}_g}$ on $\mathbb{C}^{2g}$, $\pi_J := \frac{1}{2}(1 - \sqrt{-1} J)$, parametrized by $\mathcal{H}_g$. Then $\{\pi_J \circ d\}_{J \in \mathcal{H}_g}$ is a covariant derivative $\nabla$ of type $(1,0)$ on the bundle $E''_{\mathcal{H}_g} \simeq \bigsqcup_{J \in \mathcal{H}_g}$ Image $\pi_J$, whose curvature form $R^\nabla$ is given by

$$R^\nabla = \pi(\partial \pi)(\overline{\partial} \pi).$$

The 2-form $c_1(\nabla)$ defined by $c_1(\nabla) = \frac{\sqrt{-1}}{2\pi} \text{trace} R^\nabla$ represents $c_1(E'_{\mathcal{H}_g})$. Let $J_\alpha(t) \in \mathcal{H}_g$, $|t| \ll 1$, $\alpha = 1, 2$, be $C^\infty$ paths on $\mathcal{H}_g$ with $J_1(0) = J_2(0) = J$. Then, one can compute

$$c_1(\nabla)_J = \frac{1}{8\pi} \text{trace}(\dot{J}_1 J \dot{J}_2).$$

In the next section we prove Rauch’s variational formula to obtain the pullback of $c_1(\nabla)_J$ by the period map Jac explicitly.

5. RAUCH’S VARIATIONAL FORMULA

Rauch’s variational formula describes the differential of the period map Jac. Let $C$ be a compact Riemann surface of genus $g$. We denote by $H$ the real first homology group $H_1(C; \mathbb{R})$. Consider the map $H^* = H^1(C; \mathbb{R}) \to \Omega^1(C)$ assigning to each cohomology class the harmonic 1-form representing it. The map can be regarded as an $H$-valued 1-form $\omega(1) \in \Omega^1(C) \otimes H$.

Let $\{X_i, X_{g+i}\}_{i=1}^g$ be a symplectic basis of $H_C = H_1(C; \mathbb{C})$

$$X_i \cdot X_{g+j} = \delta_{ij}, \quad X_i \cdot X_j = X_{g+i} \cdot X_{g+j} = 0, \quad 1 \leq i, j \leq g,$$

and $\{\xi_i, \xi_{g+i}\}_{i=1}^g \subset \Omega^1(C)$ the basis of the harmonic 1-forms dual to $\{X_i, X_{g+i}\}_{i=1}^g$. Then we have

$$\omega(1) = \sum_{i=1}^g \xi_i X_i + \xi_{g+i} X_{g+i} \in \Omega^1(C) \otimes H_C.$$
In particular, if \( \{ \psi_i \}_{i=1}^g \subset H^0(C; K) \) is an orthonormal basis
\[
\frac{\sqrt{-1}}{2} \int_C \psi_i \wedge \overline{\psi}_j = \delta_{ij}, \quad 1 \leq i, j \leq g,
\]
then we obtain
\[
(5.2) \quad \omega^{(1)} = \sum_{i=1}^g \psi_i Y_i + \overline{\psi}_i Y_i,
\]
where \( \{ Y_i, Y_{g+i} \}_{i=1}^g \subset H^*_C \) is the dual basis of the symplectic basis
\( \{ [\psi_i], \frac{\sqrt{-1}}{2} [\overline{\psi}_i] \}_{i=1}^g \) of \( H^*_C = H^1(C; \mathbb{C}) \). Since the complete linear system of the canonical divisor on the complex algebraic curve \( C \) has no basepoint, the 2-form
\[
(5.3) \quad B = \frac{1}{2g} \omega^{(1)} \cdot \omega^{(1)} = \frac{\sqrt{-1}}{2g} \sum_{i=1}^g \psi_i \wedge \overline{\psi}_i \in \Omega^2(C)
\]
is a volume form on \( C \).

Now we recall the Hodge decomposition of the 1-forms on \( C \). We have an exact sequence
\[
0 \to \mathbb{C} \to \Omega^0(C) \xrightarrow{d} \Omega^2(C) \xrightarrow{\int_C} \mathbb{C} \to 0.
\]
The vector space \( \mathbb{C} \) on the left side means the constant functions. A Green operator \( \Psi : \Omega^2(C) \to \Omega^0(C) \) is a linear map satisfying the property
\[
d \ast d \Psi \Omega = \Omega
\]
for any \( \Omega \in \Omega^2(C) \) with \( \int_C \Omega = 0 \). In this article we use two sorts of Green operators \( \widehat{\Phi} = \widehat{\Phi}_C \) and \( \Phi = \Phi^{(C, P_0)} \). The former is characterized by the conditions
\[
(5.4) \quad d \ast d \widehat{\Phi} (\Omega) = \Omega - (\int_C \Omega) B \quad \text{and} \quad \int_C \widehat{\Phi} (\Omega) B = 0
\]
for any \( \Omega \in \Omega^2(C) \). Let \( \delta_{P_0} : C^\infty(C) \to \mathbb{C}, f \mapsto f(P_0) \), be the delta current on \( C \) at the point \( P_0 \). We define the latter \( \Phi \) to be a linear map with values in \( \Omega^0(C)/\mathbb{C} \) instead of \( \Omega^0(C) \). Then the operator \( d \Phi : \Omega^2(C) \to \Omega^1(C) \) makes sense, and the operator \( \Phi \) is defined by the condition
\[
d \ast d \Phi \Omega = \Omega - \left( \int_C \Omega \right) \delta_{P_0}
\]
for any \( \Omega \in \Omega^2(C) \).

Any Green operator \( \Psi \) induces the Hodge decomposition of the 1-currents
\[
(5.5) \quad \varphi = \mathcal{H} \varphi + d \Psi d \ast \varphi + \ast d \Psi d \varphi
\]
for any \( \varphi \in \Omega^1(C) \), where \( \mathcal{H} : \Omega^1(C) \to \Omega^1(C) \) is the harmonic projection on the 1-currents on \( C \).

In the setting of \( \S 2 \) the first variation of \( \omega(1) \) is given by

\[
\omega(1) = -d \Psi d* \omega(1).
\]

In fact, differentiating \( d* \omega(1) = 0 \), we get

\[
d* \omega(1) = -d* \omega(1) = -d* \omega(1).
\]

Since \( f^* \omega(1) \) is cohomologous to \( \omega(1) \), we have some function \( u \) such that \( \omega(1) = du \). Hence from (5.5) we obtain

\[
\omega(1) = d \Psi d* \omega(1) = -d \Psi d* \omega(1),
\]

as was to be shown.

**Theorem 5.1 (Rauch).** The diagram

\[
\begin{array}{ccc}
T^*[C]M_g & \stackrel{(d, \text{Jac})^*}{\longrightarrow} & T^*[\text{Jac}(C)]H_g/Sp_{2g}(\mathbb{Z}) \\
\| & & \|
\end{array}
\]

\[
H^0(C; 2K) \xrightarrow{2\sqrt{-1} \text{(multiplication)}} \text{Sym}^2 H^0(C; K).
\]

commutes. Here the lower horizontal arrow maps \( \psi_1 \otimes \psi_2 \) to the quadratic differential \( 2\sqrt{-1} \psi_1 \psi_2 \) for any 1-forms \( \psi_1 \) and \( \psi_2 \in H^0(C; K) \).

**Proof.** The integral \( \int_C *\omega(1) \wedge \omega(1) \in H \otimes H = H* \otimes H = \text{Hom}(H, H) \) coincides with the almost complex structure on \( H = H_1(C; \mathbb{R}) \) induced by the Hodge \( * \)-operator. Since \( \omega(1) \) is harmonic and \( \omega(1) \) is \( d \)-exact by (5.6), we have

\[
\int_C *\omega(1) \wedge \omega(1) = -\int_C \omega(1) \wedge *\omega(1) = 0.
\]

Hence

\[
(\int_C *\omega(1) \wedge \omega(1))^* = \int_C *\omega(1) \wedge \omega(1) = \int_C (*\omega(1)) \wedge \omega(1)
\]

\[
= 2\sqrt{-1} \int_C \omega(1)' \omega(1)' \mu - 2\sqrt{-1} \left( \int_C \omega(1)' \omega(1)' \mu \right).
\]

This proves the theorem. \( \square \)

Substituting the theorem into the formula (4.4) we have

**Corollary 5.2.**

\[
\text{Jac}^* c_1(\nabla) = \frac{1}{8\pi \sqrt{-1}} \sum_{i,j=1}^g \psi_i \psi_j \otimes \overline{\psi_i} \overline{\psi_j} \in T^*[C]M_g \otimes T^*[C]M_g.
\]
Here \( \{ \psi_i \}_{i=1}^{g} \subset H^0(C; K) \) is any orthonormal basis \( \{ \psi_i \}_{i=1}^{g} \subset H^0(C; K) \).

The elementary polynomials \( \sigma_1, \ldots, \sigma_g \) in indeterminates \( x_1, \ldots, x_g \) are given by \( \prod_{i=1}^{g} (t - x_i) = t^g + \sum_{k=1}^{g} (-1)^k \sigma_k t^{g-k} \). The equation \( \sum_{i=1}^{g} x_i^m = s_m(\sigma_1, \ldots, \sigma_g) \) defines the m-th Newton polynomial \( s_m \). The m-th Newton class of the Hodge bundle \( \Lambda = \Lambda_{M_g} \) is defined by \( s_m(\Lambda) = s_m(c_1(\Lambda), \ldots, c_g(\Lambda)) \in H^{2m}(M_g; \mathbb{R}) \), where \( c_k(\Lambda) \) is the k-th Chern class of the bundle \( \Lambda \).

The complex conjugate \( \Lambda \) satisfies \( s_m(\Lambda) = (-1)^m s_m(\Lambda) \). Since \( \Lambda \oplus \Lambda \) is a flat vector bundle on \( \mathbb{M}_g \) whose fiber over \([C]\) is the homology group \( H_1(C; \mathbb{C}) \), we have

\[
s_{2n}(\Lambda) = \frac{1}{2} s_{2n}(\Lambda \oplus \Lambda) = 0.
\]

From the Grothendieck-Riemann-Roch formula or equivalently the Atiyah-Singer index theorem for families, it follows that

\[
(5.7) \quad e_{2n-1} = (-1)^{n-1} \frac{2n}{B_{2n}} s_{2n-1}(\Lambda) \in H^{4n-2}(\mathbb{M}_g; \mathbb{R}).
\]

Here \( B_{2n} \) is the n-th Bernoulli number. In the case \( n = 1 \) it is exactly the formula \( (4.2) \).

Hence the Hodge bundle yields all the odd Morita-Mumford classes, but not the even ones. To get all the Morita-Mumford classes we introduce a higher analogue of the period map, as will be discussed in the succeeding sections.

6. **The Earle class and the twisted Morita-Mumford classes**

Let \( \Sigma_g \) be a closed oriented \( C^\infty \) surface of genus \( g \), \( p_0 \in \Sigma_g \) a point, and \( v_0 \in T_{p_0} \Sigma_g \setminus \{0\} \) a non-zero tangent vector at the point \( p_0 \). We denote by \( M_g, M_{g,*} \) and \( M_{g,1} \) the mapping class groups for the surface \( \Sigma_g \), the pointed surface \( (\Sigma_g, p_0) \) and the triple \( (\Sigma_g, p_0, v_0) \) respectively. They are the orbifold fundamental groups of the spaces \( \mathbb{M}_g, \mathbb{C}_g \) and \( T_{\mathbb{C}^*_g/M_g} \). The fundamental group \( \pi_1(\Sigma_g, p_0) \) is naturally embedded into the group \( M_{g,*} \).

By abuse of notation let \( H \) denote the real first homology group of \( \Sigma_g, H_1(\Sigma_g; \mathbb{R}) \), on which the mapping class groups act in an obvious way. The module \( H \) can be interpreted as a flat vector bundle on the moduli space \( \mathbb{M}_g \). In 1978 Earle \[9\] constructed an explicit 1-cocycle \( \psi : M_{g,*} \to H \) such that \( (2 - 2g)\psi \) has values in \( H_1(\Sigma_g; \mathbb{Z}) \), and \( \psi|_{\pi_1(\Sigma_g)} \) is equal to the abelianization map of the group \( \pi_1(\Sigma_g) \). Later Morita \[39\]
independently discovered a cohomology class \( k \in H^1(M_{g,*}; H_1(\Sigma_g; \mathbb{Z})) \) which is equal to \([(2 - 2g)\psi]\). Furthermore he proved

\[(6.1) \quad H^1(M_{g,*}; H_1(\Sigma_g; \mathbb{Z})) = \mathbb{Z}k \cong \mathbb{Z}\]

for \( g \geq 2 \). The author would like to propose the class \( k \) should be called the Earle class.

The square of the class \( k \) is related to the first Morita-Mumford class \( \epsilon_1 = \kappa_1 \) through the intersection pairing

\[(6.2) \quad m : H \otimes H = H_1(\Sigma_g; \mathbb{R}) \otimes H_1(\Sigma_g; \mathbb{R}) \to \mathbb{R}.
\]

Morita \([36]\) proved

\[(6.3) \quad m_*(k^{\otimes 2}) = -\epsilon_1 + 2g(2 - 2g)e \in H^2(M_{g,*}).\]

Here \( e \) is the first Chern class of the relative tangent bundle \( c_1(T_{C_g/M_g}) \in H^2(C_g) = H^2(M_{g,*}). \)

These phenomena have a higher analogue. The twisted Morita-Mumford class \( m_{i,j} \in H^{2i+j-2}(M_{g,1}; \Lambda^i H) \), \( i, j \geq 0 \), was introduced in \([21]\). We have \( m_{1,1} = k \) and \( m_{i+1,0} = e_i, \ i \geq 1 \). All the cohomology classes on the mapping class groups with trivial coefficients (even in the unstable range) obtained from any products of the twisted Morita-Mumford classes by contracting the coefficients using the intersection pairing are exactly the polynomials in the Morita-Mumford classes \([25]\).

This fact is closely related to the Johnson homomorphisms on the mapping class group. The fundamental group \( \pi_1(\Sigma_g, p_0, v_0) = \pi_1(\Sigma_g \setminus \{p_0\}, v_0) \) with tangential basepoint \( v_0 \) is a free group of rank \( 2g \). Let \( \Gamma_k, \ k \geq 0 \), denote the lower central series of the free group \( \pi_1(\Sigma_g, p_0, v_0) \). We have \( \Gamma_0 = \pi_1(\Sigma_g, p_0, v_0) \) and \( \Gamma_{k+1} = [\Gamma_k, \Gamma_0] \) for \( k \geq 0 \). The quotient \( \Gamma_1/\Gamma_2 \) is naturally isomorphic to \( \Lambda^2 H_1(\Sigma_g; \mathbb{Z}) \subset \Lambda^2 H \). Let \( I_{g,1} \) be the Torelli group, that is, the kernel of the natural action of \( M_{g,1} \) on the homology group \( H_1(\Sigma_g; \mathbb{Z}) \). For any \( \varphi \in I_{g,1} \) and \( \gamma \in \Gamma_0 \), the difference \( \gamma^{-1}\varphi(\gamma) \) belongs to \( \Gamma_1 \) from the definition of \( I_{g,1} \). Hence we can define a homomorphism

\[\tau_1(\varphi) : H_1(\Sigma_g; \mathbb{Z}) \to \Lambda^2 H_1(\Sigma_g; \mathbb{Z}), \quad [\gamma] \mapsto \gamma^{-1}\varphi(\gamma) \mod \Gamma_2.\]

It is easy to check this induces a homomorphism \( \tau_1 : I_{g,1} \to H^* \otimes \Lambda^2 H \cong H \otimes \Lambda^2 H \). The last isomorphism comes from Poincaré duality. Johnson \([18]\) proved the image \( \tau_1(I_{g,1}) \) is included in \( \Lambda^3 H \). The homomorphism \( \tau_1 \) is called the first Johnson homomorphism. Morita \([38]\) proved there exists a unique cohomology class \( k \in H^1(M_{g,1}; \Lambda^3 H) \) which restricts to \( \tau_1 \) on the Torelli group \( I_{g,1} \). We call it the extended first Johnson homomorphism. See \([41, \S 7]\) for more information on the Johnson homomorphisms.
The class \( \frac{1}{6} m_{0,3} \) is equal to the extended first Johnson homomorphism \( \tilde{k} : \mathcal{M}_{g,1} \to \wedge^3 H \) [25]. Each of the Morita-Mumford classes is obtained from some power of \( \tilde{k} \) by contracting the coefficients using the intersection pairing \( m \) [39]. Conversely for any Sp-module \( V \) and any Sp-homomorphism \( f : (\wedge^3 H)^{\otimes m} \to V \) induced by the intersection pairing, the cohomology class \( f^*(\tilde{k}^{\otimes m}) \) is a polynomial in the twisted Morita-Mumford class [25]. An extension of the second Johnson homomorphism to the whole mapping class group provides a fundamental relation among the twisted Morita-Mumford classes [22].

In the next section we introduce a flat connection on a vector bundle on the space \( \widetilde{T_g,1} \times C_g / \mathcal{M}_g \), whose holonomy is an extension of the Johnson homomorphisms to the whole mapping class group \( \mathcal{M}_{g,1} \).

7. A higher analogue of the period map

A complex-analytic counterpart of the first Johnson homomorphism is the (pointed) harmonic volume introduced by Harris [17] [46]. It is a real analytic section of a fiber bundle on the moduli \( C_g \) whose fiber over \([C, P_0]\) is \( (\wedge^3 H_1(C; \mathbb{Z})) \otimes (\mathbb{R} / \mathbb{Z}) \). The first variation of the (pointed) harmonic volumes is a twisted 1-form representing the cohomology class \([\tilde{k}]\) [23].

To obtain “canonical” differential forms representing all the twisted Morita-Mumford classes and their higher relations, we construct a higher analogue of the classical period map and the harmonic volume, the harmonic Magnus expansion \( \theta : T_{g,1} \to \Theta_{2g} \) [23]. The space \( T_{g,1} = \widetilde{T_g,1} \times C_g / \mathcal{M}_g \) is Teichmüller space of triples \((C, P_0, v)\) of genus \( g \). Here \( C \) is a compact Riemann surface of genus \( g \), \( P_0 \in C \), and \( v \) a non-zero tangent vector of \( C \) at \( P_0 \) as in [22]. For any triple \((C, P_0, v)\) one can define the fundamental group of the complement \( C \setminus \{P_0\} \) with the tangential basepoint \( v \) denoted by \( \pi_1(C, P_0, v) \), which is a free group of rank \( 2g \). The space \( \Theta_n \) is the set of all Magnus expansions of the free group \( F_n \) of rank \( n \geq 2 \) in a wider sense stated as follows.

We denote by \( H \) the first real homology group of the group \( F_n \), \( H_1(F_n; \mathbb{R}) \), \( H^* \) the first real cohomology group of \( F_n \), \( H^1(F_n; \mathbb{R}) \), and \([\gamma] \in H \) the homology class of \( \gamma \in F_n \). The completed tensor algebra generated by \( H \), \( \widehat{T} = \widehat{T}(H) = \prod_{m=0}^{\infty} H^{\otimes m} \), has a decreasing filtration of two-sided ideals \( \{\widehat{T}_p\}_{p \geq 1} \) defined by \( \widehat{T}_p = \prod_{m \geq p} H^{\otimes m} \). The subset \( 1 + \widehat{T}_1 \) is a subgroup of the multiplicative group of the algebra \( \widehat{T} \). We call a map \( \theta : F_n \to 1 + \widehat{T}_1 \) a Magnus expansion of the free group \( F_n \) in a wider sense [22], if \( \theta : F_n \to 1 + \widehat{T}_1 \) is a group homomorphism, and if
\[ \theta(\gamma) \equiv 1 + [\gamma] \pmod{\hat{T}_2} \] for any \( \gamma \in F_n \). One can endow the set of all Magnus expansions \( \Theta_n \) with a natural structure of a (projective limit of) real analytic manifold(s). A certain (projective limit of) Lie group(s) \( \text{IA}(\hat{T}) \) acts on \( \Theta_n \) in a free and transitive way. This induces a series of 1-forms \( \eta_p \in \Omega^1(\Theta_n) \otimes H^* \otimes H^\otimes(p+1), p \geq 1 \), the Maurer-Cartan forms of the action of \( \text{IA}(\hat{T}) \), which are invariant under a natural action of the automorphism group of the group \( F_n, \text{Aut}(F_n) \). The Maurer-Cartan formula \( d\eta = \eta \wedge \eta \) allows us to regard the forms \( \eta_p \) as an equivariant flat connection on the vector bundle \( \Theta_n \times H^* \otimes \hat{T}_2 \). The holonomy of the connection is an extension of all the Johnson homomorphisms to the whole group \( \text{Aut}(F_n) \). The 1-forms \( \eta_p \) represent the twisted Morita-Mumford classes on the group \( \text{Aut}(F_n) \) \[ \text{[22][23].} \]

Let \((C, P_0, v)\) be a triple of genus \( g \). From now on we denote by \( H \) the real first homology group \( H_1(C; \mathbb{R}) \). As in \[ \text{[24]} \] we denote by \( \delta_{P_0} : C^\infty(C) \to \mathbb{R}, f \mapsto f(P_0) \), the delta 2-current on \( C \) at \( P_0 \). Then there exists a \( \hat{T}_1 \)-valued 1-current \( \omega \in \Omega^1(C) \otimes \hat{T}_1 \), satisfying the following 3 conditions

1. \( d\omega = \omega \wedge -I \cdot \delta_{P_0} \), where \( I \in H^\otimes 2 \) is the intersection form.
2. The first term of \( \omega \) is equal to \( \omega(1) \in \Omega^1(C) \otimes H \) introduced in \[ \text{[24]} \] .
3. \( \int_C (\omega - \omega(1)) \wedge * \varphi = 0 \) for any closed 1-form \( \varphi \) and each \( p \geq 2 \).

Using Chen’s iterated integrals \[ \text{[8]}, \] we can define a Magnus expansion
\[ \theta = \theta^{(C, P_0, v)} : \pi_1(C, P_0, v) \to 1 + \hat{T}_1(H_1(C; \mathbb{R})), \quad [\ell] \mapsto 1 + \sum_{m=1}^{\infty} \int_\ell \omega \cdots \omega. \]

Let a point \( p_0 \in \Sigma_g \) and a non-zero tangent vector \( v_0 \in T_{p_0} \Sigma_g \setminus \{0\} \) be fixed as in \[ \text{[26]} \] Moreover we fix an isomorphism \( \pi_1(\Sigma_g, p_0, v_0) \cong F_{2g} \). A marking \( \alpha \) of a triple \((C, P_0, v)\) is an orientation-preserving diffeomorphism of \( \Sigma_g \) onto \( C \) satisfying the conditions \( \alpha(p_0) = P_0 \) and \((d\alpha)_{p_0}(v_0) = v \). For any marked triple \([(C, P_0, v), \alpha]\) we define a Magnus expansion of the free group \( F_{2g} \) by
\[ F_{2g} \cong \pi_1(\Sigma_g, p_0, v_0) \xrightarrow{\alpha} \pi_1(C, P_0, v) \xrightarrow{\theta^{(C, P_0, v)}} 1 + \hat{T}_1(H_1(C; \mathbb{R})) \xrightarrow{\alpha^{-1}} 1 + \hat{T}_1. \]

Consequently, the Magnus expansions \( \theta^{(C, P_0, v)} \) for all the triples \((C, P_0, v)\) define a canonical real analytic map \( \theta : T^\infty_{\Sigma_g/M_g} = T_{g,1} \to \Theta_{2g} \), which we call the harmonic Magnus expansion on the universal family of Riemann surfaces. The pullbacks of the Maurer-Cartan forms \( \eta_p \) define a flat connection on a vector bundle on the space \( T^\infty_{\Sigma_g/M_g} \), and give the canonical differential forms representing the Morita-Mumford classes and their higher relations.
Theorem 7.1 (23). For any $[C, P_0, v, \alpha] \in T_{g,1}$ we have
\[
(\theta^* \eta)[C, P_0, v, \alpha] = 2 \Re (N(\omega' \omega') - 2 \omega(1)' \omega(1)') \in T_{[C, P_0, v, \alpha]}^r T_{g,1} \otimes \hat{T}_3.
\]

Here $N : \hat{T}_1 \to \hat{T}_1$ is defined by $N|_{H^\otimes m} = \sum_{k=0}^{m-1} \begin{pmatrix} 1 & 2 & \cdots & m-1 & m \\ 2 & 3 & \cdots & m & 1 \end{pmatrix}^k$, and the meromorphic quadratic differential $N(\omega' \omega')$ is regarded as a (1, 0)-cotangent vector at $[C, P_0, v, \alpha] \in T_{g,1}$ in a natural way.

The third homogeneous term $N(\omega' \omega')(3) = N(\omega(1)' \omega(2) + \omega(2)' \omega(1))$ is the first variation of the (pointed) harmonic volumes of pointed Riemann surfaces. It represents the extended first Johnson homomorphism $\tilde{k}$. The higher terms provide higher relations among the twisted Morita-Mumford classes. Hence all of the Morita-Mumford classes are represented by some algebraic combinations of $N(\omega' \omega')$.

The second term coincides with $2\omega(1)' \omega(1)'$, which is exactly the first variation of the period matrices given by Rauch’s formula in §5. Hence we may regard the harmonic Magnus expansion as a higher analogue of the classical period map Jac.

8. Secondary objects on the moduli space

The determinant of the Laplacian acting on the space of $k$-differentials on Riemann surfaces is a ‘secondary’ object on the moduli space. Zograf and Takhtajan [51] proved that it yields the difference on the moduli space of compact Riemann surfaces, $\mathbb{M}_g$, between a multiple of the Weil-Petersson form $\omega_{WP}$ and the Chern form of the Hodge line bundle for the $k$-differentials induced by the hyperbolic metric. Moreover, they studied analogous phenomena for punctured Riemann surfaces to introduce their Kähler metric, the Zograf-Takhtajan metric, on the moduli space of punctured Riemann surfaces [52].

In this section we discuss other secondary objects, which come from the higher analogue of the period map introduced in §7. Now we can obtain explicit 2-forms from the connection form $N(\omega' \omega')$ on $T_{\mathbb{C}_g/M_g}^\infty$, $e^J$ on $\mathbb{C}_g$ and $e_1^J$ on $\mathbb{M}_g$. Consider the quadratic differential $\eta_2'$ defined by
\[
\eta_2' = N(\omega' \omega')(4) \in H^0(C; 2K + 2P_0) \otimes H^\otimes 4,
\]
which satisfies
\[
\frac{1}{2g(2g+1)} \text{Res}_{P_0} ((m \otimes m)(\eta_2')) = -\frac{1}{8\pi^2}.
\]
Here $m$ is the intersection pairing $m : H \otimes H \rightarrow \mathbb{R}$ as in (6.2). We define
\[ e^J = \frac{-2}{2g(2g+1)} \overline{\partial}(m \otimes m)(\eta_2) \in \Omega^{1,1}(\mathbb{C}_g). \]
From (2.6) $e^J$ represents the first Chern class of the relative tangent bundle
\[ [e^J] = e = c_1(T_{\mathbb{C}_g/\mathbb{M}_g}) \in H^2(\mathbb{C}_g; \mathbb{R}). \]
We obtain a twisted 1-form $\eta_1^H \in \Omega^1(\mathbb{C}_g; H)$ representing the Earle class $k$ by contracting the coefficients of $\eta_1^H = \overline{N}(\omega' \omega')$. By (6.3) $m(\eta_1^H) \otimes 2 \in \Omega^{1,1}(\mathbb{C}_g)$ represents $-e_1 + 2g(2-2g)e$. So we define
\[ e^J_1 = -m(\eta_1^H) \otimes 2 + 2g(2-2g)e^J \]
which can be regarded as a $(1, 1)$-form on $\mathbb{M}_g$ [23, §8].

Hain and Reed [13] already constructed the same form $e^J_1$ in a Hodge-theoretical context. They applied the following lemma to $\frac{1}{2\pi} e^J_1 - \text{Jac} e_1(\nabla)$ to get a function $\beta_g \in C^\infty(\mathbb{M}_g; \mathbb{R})/\mathbb{R}$, the Hain-Reed function, a secondary object on the moduli space $\mathbb{M}_g$.

**Lemma 8.1.** Let $M$ be a connected complex orbifold with $H^0(M; \mathcal{O}) = \mathbb{C}$ and $H^1(M; \mathbb{C}) = H^1(M; \mathcal{O}) = 0$. If a real $C^\infty(1, 1)$-form $\psi$ is d-exact, then there exists a real-valued function $f \in C^\infty(M; \mathbb{R})$ such that $\psi = \frac{\sqrt{-1}}{2\pi} \overline{\partial} f$. Such a function $f$ is unique up to a constant.

Here we remark all the holomorphic functions on $\mathbb{M}_g$ are constants provided $g \geq 3$. In fact, each of the boundary component of the Satake compactification of $\mathbb{M}_g$ is of complex codimension $\geq 2$. The vanishing of the first cohomology follows from (1.1). See [41]. Hain and Reed also studied the asymptotic behavior of the function $\beta_g$ towards the boundary of the Deligne-Mumford compactification $\mathbb{M}_g^{\text{DM}}$ [13].

We have another ‘secondary’ phenomenon around the 2-forms $e^J$ and $e^J_1$ [24]. Let $B = \frac{1}{2\pi} \omega(1) \cdot \omega(1)$ be the volume form in (5.3). On any pointed Riemann surface $(C, P_0)$ there exists a function $h = h_{P_0} = -\hat{\Phi}(\delta_{P_0})$ with $d^* dh = B - \delta_{P_0}$ and $\int_C hB = 0$. The function $G(P_0, P_1) := \exp(-4\pi h_{P_0}(P_1))$ is just the Arakelov-Green function. We regard $G$ a function on the fiber product $\mathbb{C}_g \times_{\mathbb{M}_g} \mathbb{C}_g$ and define the $(1, 1)$-form $e^A$ on $\mathbb{C}_g$ by
\[ e^A := \frac{1}{2\pi \sqrt{-1}} \overline{\partial} \log G|_{\text{diagonal}} \in \Omega^{1,1}(\mathbb{C}_g) \]
representing the Chern class $e = c_1(T_{\mathbb{C}_g/\mathbb{M}_g})$. In fact, the normal bundle of the diagonal map $\mathbb{C}_g \rightarrow \mathbb{C}_g \times_{\mathbb{M}_g} \mathbb{C}_g$ is exactly the relative tangent bundle $T_{\mathbb{C}_g/\mathbb{M}_g}$. 


Furthermore we introduce an explicit real-valued function $a_g$ on $M_g$ by
\begin{equation}
(8.1) \quad a_g(C) := \int_C \omega(1) \cdot \Phi(\omega(1) \wedge \omega(1)) \cdot \omega(1),
\end{equation}
where $\Phi$ is the Green operator introduced in (5.4). By (3.2) we have
\begin{equation}
(8.2) \quad a_g(C) = -\sum_{i,j=1}^g \int_C \psi_i \wedge \bar{\psi}_j \Phi(\psi_i \wedge \bar{\psi}_j).
\end{equation}
We have $a_g(C) > 0$ if $g \geq 2$. Then comparing $\partial a_g$ with $\eta'_2$ as explicit quadratic differentials, we obtain
\begin{equation}
(8.3) \quad e^A - e^J = \frac{-2\sqrt{-1}}{2g(2g + 1)} \partial \bar{\partial} a_g.
\end{equation}
On the other hand, the integral along the fiber
\begin{equation}
e^F := \int_{\text{fiber}} (e^J)^2 \in \Omega^{1,1}(M_g)
\end{equation}
also represents the first Morita-Mumford class $e_1$. By straightforward computation on $\partial \bar{\partial} a_g$ we deduce

**Theorem 8.2** ([24]).
\begin{equation}
e^A - e^J = \frac{-2\sqrt{-1}}{2g(2g + 1)} \partial \bar{\partial} a_g = \frac{1}{(2 - 2g)^2} (e^F - e_1^J).
\end{equation}

The function $a_g(C)$ is also a secondary object on the moduli space $M_g$, and it defines a conformal invariant of the compact Riemann surface $C$, but the author does not know any of its further properties.

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