Uniform exponential contraction for viscous Hamilton–Jacobi equations

Konstantin Khanin\(^1\) · Ke Zhang\(^1\) · Lei Zhang\(^2\)

Received: 29 April 2021 / Accepted: 27 December 2022 / Published online: 11 January 2023
© The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2023

Abstract
The well known phenomenon of exponential contraction for solutions to the viscous Hamilton–Jacobi equation in the space-periodic setting is based on the Markov mechanism. However, the corresponding Lyapunov exponent \(\lambda(\nu)\) characterizing the exponential rate of contraction depends on the viscosity \(\nu\). The Markov mechanism provides only a lower bound for \(\lambda(\nu)\) which vanishes in the limit \(\nu \to 0\). At the same time, in the inviscid case \(\nu = 0\) one also has exponential contraction based on a completely different dynamical mechanism. This mechanism is based on hyperbolicity of action-minimizing orbits for the related Lagrangian variational problem. In this paper we consider the discrete time case (kicked forcing), and establish a uniform lower bound for \(\lambda(\nu)\) which is valid for all \(\nu \geq 0\). The proof is based on a nontrivial interplay between the dynamical and Markov mechanisms for exponential contraction. We combine PDE methods with the ideas from the Weak KAM theory.

Mathematics Subject Classification 70H20 · 37J50 · 37D05 · 35K05

1 Introduction

We consider the periodic Hamilton–Jacobi equation with viscosity \(\nu > 0\):

\[
\varphi_t + \frac{1}{2} |\nabla \varphi|^2 = \nu \Delta \varphi + F(x, t), \quad x \in \mathbb{R}^d, \quad t \in \mathbb{R},
\]

\(^1\) University of Toronto, Toronto, Canada
\(^2\) Dalian University of Technology, Dalian, China
where $F(\cdot, t)$ is a $\mathbb{Z}^d$-periodic function, namely $F(x + k, t) = F(x, t)$ for all $k \in \mathbb{Z}^d$. Let $C_{\text{per}}(\mathbb{R}^d)$ and $C^d_{\text{per}}(\mathbb{R}^d)$ denote the space of periodic continuous and $C^d$ functions, respectively. Given an initial condition $\varphi_0 \in C_{\text{per}}(\mathbb{R}^d)$ and $T_0 < T$, consider the solution $\varphi_{T_0}^\nu(x, t)$ to the initial value problem

$$
\varphi_t + \frac{1}{2} |\nabla \varphi|^2 = v \Delta \varphi + F(x, t), \quad x \in \mathbb{R}^d, \ t \in [T_0, T],
$$

$$
\varphi(x, T_0) = \varphi_0(x), \quad x \in \mathbb{R}^d.
$$

Statement of the main result

We are considering the so-called kicked setting in this paper, namely,

$$
F(x, t) = \sum_{j \in \mathbb{Z}} F_j(x) \delta(t - j),
$$

where $\lambda_\nu > 123 \nu > 15$, namely, there exists $\lambda_\nu$ such that $\lambda_\nu \rightarrow 0$ as $\nu \rightarrow 0$. Moreover the convergence is exponential, namely, there is $\lambda_\nu > 0$ such that

$$
\limsup_{T_0 \rightarrow -\infty} \frac{1}{|T_0|} \log \|\varphi_{T_0}^\nu(x, T) - \varphi^\nu(\cdot, T)\|_* < -\lambda_\nu.
$$

The uniqueness of stationary solution also holds in more general settings [7].

In this paper, we are interested in the uniform lower bound for the exponent $\lambda_\nu$. This is related to the property of the viscosity limit $\nu \rightarrow 0$. As $\nu \rightarrow 0$, any limit point of $\varphi^\nu$ in the $\| \cdot \|_*$ norm solves the invicid equation

$$
\varphi_t + \frac{1}{2} |\nabla \varphi|^2 = F(x, t), \quad x \in \mathbb{R}^d, \ t \in (T_0, T),
$$

$$
\varphi(x, T_0) = \varphi_0(x), \quad x \in \mathbb{R}^d.
$$

Under certain non-degeneracy conditions [9, 11, 15], the solution to the invicid problem $\varphi_{T_0}^0(x, t)$ also admits a unique stationary solution $\psi^0$, in the same sense as before:

$$
\lim_{T_0 \rightarrow -\infty} \|\varphi_{T_0}^0(x, T) - \psi^0(\cdot, T)\|_* \rightarrow 0.
$$

The exponential convergence of the invicid solution also hold under similar conditions, (see [10, 15],) namely, there exists $\lambda > 0$ such that

$$
\limsup_{T_0 \rightarrow -\infty} \frac{1}{T} \log \|\varphi_{T_0}^0(x, T) - \psi^0(\cdot, T)\|_* < -\lambda.
$$

The exponential convergence (1.4) comes from the diffusion term $v \Delta \varphi$, and a priori we can think that $\lambda_\nu \rightarrow 0$ as $\nu \rightarrow 0$. However, the exponential convergence in the invicid case makes it plausible to expect a uniform bound $\lambda_\nu > \lambda > 0$ for all $\nu > 0$. The main result of this paper is the proof of this uniform bound. We should point out that the mechanism of the exponential convergence in the invicid case is purely dynamical. Correspondingly, the main difficulty in proving the uniform exponent is to study interaction between the dynamical and Markovian mechanisms asymptotically as $n \rightarrow 0$. 

**Statement of the main result**

where $\lambda_\nu > 123 \nu > 15$, namely, there exists $\lambda_\nu$ such that $\lambda_\nu \rightarrow 0$ as $\nu \rightarrow 0$. Moreover the convergence is exponential, namely, there is $\lambda_\nu > 0$ such that

$$
\limsup_{T_0 \rightarrow -\infty} \frac{1}{|T_0|} \log \|\varphi_{T_0}^\nu(x, T) - \varphi^\nu(\cdot, T)\|_* < -\lambda_\nu.
$$

The uniqueness of stationary solution also holds in more general settings [7].

In this paper, we are interested in the uniform lower bound for the exponent $\lambda_\nu$. This is related to the property of the viscosity limit $\nu \rightarrow 0$. As $\nu \rightarrow 0$, any limit point of $\varphi^\nu$ in the $\| \cdot \|_*$ norm solves the invicid equation

$$
\varphi_t + \frac{1}{2} |\nabla \varphi|^2 = F(x, t), \quad x \in \mathbb{R}^d, \ t \in (T_0, T),
$$

$$
\varphi(x, T_0) = \varphi_0(x), \quad x \in \mathbb{R}^d.
$$

Under certain non-degeneracy conditions [9, 11, 15], the solution to the invicid problem $\varphi_{T_0}^0(x, t)$ also admits a unique stationary solution $\psi^0$, in the same sense as before:

$$
\lim_{T_0 \rightarrow -\infty} \|\varphi_{T_0}^0(x, T) - \psi^0(\cdot, T)\|_* \rightarrow 0.
$$

The exponential convergence of the invicid solution also hold under similar conditions, (see [10, 15],) namely, there exists $\lambda > 0$ such that

$$
\limsup_{T_0 \rightarrow -\infty} \frac{1}{T} \log \|\varphi_{T_0}^0(x, T) - \psi^0(\cdot, T)\|_* < -\lambda.
$$

The exponential convergence (1.4) comes from the diffusion term $v \Delta \varphi$, and a priori we can think that $\lambda_\nu \rightarrow 0$ as $\nu \rightarrow 0$. However, the exponential convergence in the invicid case makes it plausible to expect a uniform bound $\lambda_\nu > \lambda > 0$ for all $\nu > 0$. The main result of this paper is the proof of this uniform bound. We should point out that the mechanism of the exponential convergence in the invicid case is purely dynamical. Correspondingly, the main difficulty in proving the uniform exponent is to study interaction between the dynamical and Markovian mechanisms asymptotically as $n \rightarrow 0$. 

**Statement of the main result**

We are considering the so-called kicked setting in this paper, namely,

$$
F(x, t) = \sum_{j \in \mathbb{Z}} F_j(x) \delta(t - j),
$$

where $\lambda_\nu > 123 \nu > 15$, namely, there exists $\lambda_\nu$ such that $\lambda_\nu \rightarrow 0$ as $\nu \rightarrow 0$. Moreover the convergence is exponential, namely, there is $\lambda_\nu > 0$ such that

$$
\limsup_{T_0 \rightarrow -\infty} \frac{1}{|T_0|} \log \|\varphi_{T_0}^\nu(x, T) - \varphi^\nu(\cdot, T)\|_* < -\lambda_\nu.
$$

The uniqueness of stationary solution also holds in more general settings [7].

In this paper, we are interested in the uniform lower bound for the exponent $\lambda_\nu$. This is related to the property of the viscosity limit $\nu \rightarrow 0$. As $\nu \rightarrow 0$, any limit point of $\varphi^\nu$ in the $\| \cdot \|_*$ norm solves the invicid equation

$$
\varphi_t + \frac{1}{2} |\nabla \varphi|^2 = F(x, t), \quad x \in \mathbb{R}^d, \ t \in (T_0, T),
$$

$$
\varphi(x, T_0) = \varphi_0(x), \quad x \in \mathbb{R}^d.
$$

Under certain non-degeneracy conditions [9, 11, 15], the solution to the invicid problem $\varphi_{T_0}^0(x, t)$ also admits a unique stationary solution $\psi^0$, in the same sense as before:

$$
\lim_{T_0 \rightarrow -\infty} \|\varphi_{T_0}^0(x, T) - \psi^0(\cdot, T)\|_* \rightarrow 0.
$$

The exponential convergence of the invicid solution also hold under similar conditions, (see [10, 15],) namely, there exists $\lambda > 0$ such that

$$
\limsup_{T_0 \rightarrow -\infty} \frac{1}{T} \log \|\varphi_{T_0}^0(x, T) - \psi^0(\cdot, T)\|_* < -\lambda.
$$

The exponential convergence (1.4) comes from the diffusion term $v \Delta \varphi$, and a priori we can think that $\lambda_\nu \rightarrow 0$ as $\nu \rightarrow 0$. However, the exponential convergence in the invicid case makes it plausible to expect a uniform bound $\lambda_\nu > \lambda > 0$ for all $\nu > 0$. The main result of this paper is the proof of this uniform bound. We should point out that the mechanism of the exponential convergence in the invicid case is purely dynamical. Correspondingly, the main difficulty in proving the uniform exponent is to study interaction between the dynamical and Markovian mechanisms asymptotically as $n \rightarrow 0$. 

**Statement of the main result**

We are considering the so-called kicked setting in this paper, namely,
where $F_j \in C^3_{\text{per}}(\mathbb{R}^d)$. The kicked setting retains much of the feature of the system, but is more convenient to study. We fully expect that the results remain true in the continuous time case.

To make our arguments more transparent, we consider the simplest case, where all the kicks are the same ($F_j = F$), and $F$ is a generic potential on $\mathbb{T}^d$. Again, we expect the results to hold in the non-stationary case. In particular, in the case when $\{F_j\}$ form an i.i.d. sequence in the space $C^3(\mathbb{T}^d)$.

**Assumption 1**

$$F(x, t) = \sum_{j \in \mathbb{Z}} F(x) \delta(t - j), \quad x \in \mathbb{R}^d.$$ 

Here $F \in C^3_{\text{per}}(\mathbb{R}^d)$, $\arg\min_{x \in \mathbb{T}^d} F(x) = \{0\}$, $F(0) = 0$, and $D^2 F(0)$ is strictly positive definite.

**Theorem A** Suppose $F$ satisfies Assumption 1, then there exists $\lambda > 0$, $\nu_0 > 0$ and $C > 0$ depending only on $F$, such that for all $\nu \in (0, \nu_0)$ and initial conditions $\phi_0 \in C_{\text{per}}(\mathbb{R}^d)$, the solution $\phi_{\nu}^n(x, t)$ to (1.1) satisfies

$$\|\phi_{\nu}^n(\cdot, 0) - \psi^\nu(\cdot)\|_* \leq B e^{-\lambda n}, \quad n \geq 0,$$

where the constant $B > 0$ depends only on $\|\phi\|_*$ and $C$.

**Discussions of the result and the method**

The standard way to study Eq. (1.2) is to apply the Hopf-Cole transformation $u = e^{-\phi/(2\nu)}$, which transforms our equation to the inhomogeneous heat equation

$$u_t = \nu \Delta u - \frac{1}{2\nu} F(x, t) u, \quad (x, t) \in \mathbb{R}^d \times (-n, 0)$$

$$u(x, -n) = u_0(x) = e^{-\frac{1}{2\nu} \phi_0(x)}, \quad x \in \mathbb{R}^d.$$ 

The solution to this equation (under Assumption 1) is given by $L^n_\nu u_0$, where the operator is $L_\nu = e^{\nu \Delta} e^{F/(2\nu)}$. In this sense, our result can be understood as proving a uniform spectral gap for the operator $L_\nu$. Our proof, however, does not use spectral theory. Instead, we convert the operator $L_\nu$ to a sequence of Markov operators, and prove that the inhomogeneous Markov chain converges exponentially with a uniform rate. As such, our method can be adapted to deal with the case that $F_j$’s are different while sharing some uniform properties, where the spectral approach seems to struggle. One of our main motivations is to apply this approach to the random setting of Ref. [13] and [10], which we will address in a separate paper.

It is natural to expect that the contraction rate $\lambda_\nu$ is a continuous function, and $\lim_{\nu \to 0} \lambda_\nu = \lambda_0$, where $\lambda_0$ is the smallest Lyapunov exponent of the hyperbolic global minimizer. However, the above statement is currently open. It is also interesting to find out whether $\lambda_\nu$ is a monotone increasing function.

We now turn to the method of proof. We note that while the hyperbolicity of the global minimizer in the inviscid case is the guiding heuristical reason for the uniform rate of convergence, the proof itself does not heavily use theories of hyperbolic dynamics. By the Feynman-Kac formula, the solution $L^n_\nu u_0$ can be interpreted as the expectation of an integral over a Brownian path (which becomes a random walk in the kicked case). As $\nu \to 0$, the trajectory of the random walk converges in distribution to the minimizing path of the associated...
Lagrangian. This is where the connection with the invicid Eq. (1.5) lies, as the solution of the invicid equation is given precisely by the integral of the Lagrangian over a minimal path, via the Lax–Oleinik variational principle. We use Weak KAM theory to study the properties of the minimal path, which is used to obtain estimates on the operator $L^n_\nu$, via the classical Laplace’s method.

There is an important technical hurdle to this plan: the Laplace’s method depend strongly on the dimension of the space, while the integral associated to $L^n\nu u_0$ is in $\mathbb{R}^{nd}$. As such, the estimates fail for large $n$. In order to deal with this problem, we devise the following strategy:

Step 1. We first conjugate the operator $L_\nu$ to an operator $\tilde{L}_\nu = e^{\psi/(2\nu)}L_\nu e^{-\psi/(2\nu)}$, where $\psi$ is the stationary solution of the invicid equation.

Step 2. We then obtain uniform estimates for the partition function $\tilde{L}_\nu^n 1$ ($1$ is the constant function $1$), for $0 \leq n \leq N_0(\nu)$, where $N_0(\nu) \sim (\nu \log \frac{1}{\nu})^{-\frac{1}{3}}$.

Step 3. We show that if the estimates of $\tilde{L}_\nu^n 1$ in Step 2 holds for $0 \leq n \leq N(\nu)$, then $\tilde{L}_\nu^n$ contracts exponentially in the same time interval, using a suitable norm.

Step 4. We then bootstrap our estimates: if $\tilde{L}_\nu^n$ contracts exponentially, we can apply it to $\tilde{L}_\nu^{N+1} 1$ to get good estimates for $\tilde{L}_\nu^{n+N+1}$, extending item (2) to longer time intervals.

Heuristically, there are two mechanisms of exponential convergence. The first mechanism is that the invicid problem has a unique global minimizer supported at the minimum of $F$, and all minimizers are attracted to it at an exponential rate. The second comes from the ellipticity of the viscous equation, which provides convergence of the Markov chain, but a priori only at a rate of $e^{-C/\nu}$. The first mechanism says all minimizers are uniformly close, which is why we can estimate $\tilde{L}_\nu^n 1$ for large $n$. This is roughly Step 2. We then apply the second mechanism. Thanks to the good estimates in Step 2, the rate of convergence is uniform in $\nu$. This is roughly Step 3. It turns out that once the second mechanism kicks in, it is self-perpetuating via the bootstrap argument.

**Plan of the paper**

The plan of this paper is as follows.

- In Sects. 2 and 3 we study the invicid equation. Section 2 recalls basic Weak KAM theory and the underlying Lagrangian dynamics, which (in the kicked case) is given by a twist map. Some standard proofs are provided for the benefits of the reader.
- The Lagrangian system has a unique global minimizer which is a hyperbolic fixed point of the twist map. In Sect. 3 we combine hyperbolic theory and variational theory to obtain estimates of the Lagrangian action.
- In Sect. 4, we introduce the Hopf-Cole transformation and state Theorem 4.1, which is the counter part of our main theorem after the transformation. We also state Proposition 4.3 that provides the initial estimates for $\tilde{L}_\nu^n 1$. This is our main technical result, whose proof is postponed to the last two sections.
- In Sect. 5, we prove Proposition 5.2 and Corollary 5.3, implementing Step 3 of our plan. The main tool is the Lyapunov function approach to the convergence of Markov chain (see [8]).
- In Sect. 6, we implement Step 4, namely the bootstrap argument. The main theorem is proved assuming Proposition 4.3 holds.
- Finally, Proposition 4.3 is proven in Sects. 7 and 8. This uses the exponential convergence of the minimizer, and a somewhat delicate application of the Laplace’s method.
2 The variational analysis and the weak KAM solution

The results of this section hold for a general $C^2$ potential $F \in C_{\text{per}}(\mathbb{R}^d)$ such that $\min F = F(0) = 0$ and $F(x) > 0$ for all $x \notin \mathbb{Z}^d$.

Let $\phi \in C_{\text{per}}(\mathbb{R}^d)$, define the Lax-Oleinik operator $T : C_{\text{per}}(\mathbb{R}^d) \to C_{\text{per}}(\mathbb{R}^d)$ by

$$T(\phi)(x) = \min_{y \in \mathbb{R}} \{ \phi(y) + h(y, x) \},$$

where

$$h(y, x) = \frac{1}{2} |x - y|^2 + F(y)$$

is called the generating function. The solution to Eq. (1.5) is given by $\phi(x, n) = T^n(\phi_0)(x)$ for $n \in \mathbb{N}$. The fixed points of $T$ are the stationary solutions of Eq. (1.5).

At this point it is convenient to consider functions and operators defined on $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$. Define, for $x, y \in \mathbb{R}^d$,

$$|x|_{\mathbb{T}} = \min_{k \in \mathbb{Z}^d} |x + k|, \quad A(y, x) = \min_{k \in \mathbb{Z}^d} h(y + k, x).$$

(2.1)

Note in particular, as functions on $\mathbb{R}^d \times \mathbb{R}^d$,

$$A(y, x) = h(y, x), \quad \text{if } |y - x| < \frac{1}{2}.$$

The same functions make sense for $x, y \in \mathbb{T}^d$, which we denote by the same names. The distance on $\mathbb{T}^d$ is given by $|y - x|_{\mathbb{T}}$. If $\phi \in C_{\text{per}}(\mathbb{R}^d)$, let $\phi'$ be the associated function on $\mathbb{T}^d$, define

$$T'(\phi') = \min_{y \in \mathbb{T}^d} \{ \phi'(y) + A(y, x) \}$$

then $T'(\phi')$ lifts to $T(\phi)$ in $\mathbb{R}^d$. From now on we will use the same notation for a function in $C_{\text{per}}(\mathbb{R}^d)$ and its counter part in $C(\mathbb{T}^d)$, and $T$ for the operators in both spaces.

A function $\phi : \mathbb{R}^d \to \mathbb{R}$ is called $C$-semi-concave if for each $x \in \mathbb{R}^d$, there exists $l \in \mathbb{R}^d$ such that

$$\phi(y) - \phi(x) \leq l \cdot (y - x) + C|y - x|^2.$$

The vector $l$ is called a super-gradient, and we use $\partial \phi(x)$ to denote the set of all super-gradients. The function $\phi$ is differentiable at $x$ if and only if $\partial \phi(x)$ is a singleton. A function $\phi \in C(\mathbb{T}^d)$ is semi-concave if the associated function on $C_{\text{per}}(\mathbb{R}^d)$ is semi-concave.

The following properties of semi-concave function are useful.

**Lemma 2.1** (Proposition 4.7.3 of Ref. [6])

1. If $\phi \in C_{\text{per}}(\mathbb{R}^d)$ is $C$-semi-concave, then it is $C'$-Lipschitz with $C'$ depending only on $C$ and the dimension $d$.
2. Suppose $-\phi_1, \phi_2$ are $C$-semi-concave functions on $\mathbb{T}^d$, then for each $x \in \arg \min \{ \phi_2 - \phi_1 \}$, $d\phi_1(x), d\phi_2(x)$ exists and are equal. In particular, any periodic semi-concave function $\phi$ is differentiable on $\arg \min \phi$ and the gradient of $\phi$ vanishes there.

Let $\phi \in C_{\text{per}}(\mathbb{R}^d)$ be semi-concave, then by Radmacher’s theorem, it is differentiable almost everywhere. Denote

$$\mathcal{D}(\phi) = \{ x \in \mathbb{R}^d : \nabla \phi(x) \text{ exists} \}.$$

The Lax-Oleinik operator regularizes continuous functions in the following sense.
Lemma 2.2 (Proposition 6.2.1 of Ref. [6], see also Corollary 3.3 of Ref. [13]) If $\varphi \in C_{\text{per}}(\mathbb{R}^d)$, then $T(\varphi)$ is $C$-semi-concave with $C$ depending only on $\|D^2F\|_{C^0}$.

The weak KAM theorem (see for example [6]) implies there exists a unique $c \in \mathbb{R}$ such that the operator $(T-c)$ admits a fixed point in $C(\mathbb{T}^d)$. Under our assumptions, the value $c = 0$ and the fixed point is unique after normalization.

Proposition 2.3 Suppose $\min_{x \in \mathbb{T}^d} F(x) = F(0) = 0$. Then there is a unique $\psi \in C_{\text{per}}(\mathbb{R}^d)$ satisfying $\psi(0) = 0$, such that

$$T(\psi) = \psi, \quad \text{i.e.} \quad \psi(x) = \min_{y \in \mathbb{R}^d} \{\psi(y) + h(y, x)\}. \quad (2.2)$$

$\psi$ is $C$-semi-concave with $C > 0$ depending only on $\|D^2F\|_{C^0}$, and $\psi(x) > 0$ for all $x \neq 0$.

**Proof** For existence, consider the zero initial function $\varphi_0(x) = 0$, and set $\varphi_n = T(\varphi_{n-1})$, $n \geq 1$. Then noticing that $h(x, y) \geq 0$ for all $x, y \in \mathbb{R}^d$, we have

$$\varphi_n(x) = \min_{x_0=x} \sum_{k=-n}^{-1} h(x_k, x_{k+1}) \leq \min_{x_0=x} \sum_{k=-n+1}^{-1} h(x_k, x_{k+1}) = \varphi_{n-1}(x).$$

Moreover, $\varphi_n(0) = 0$ for all $n$, and all $\varphi_n$ are uniformly semi-concave by Lemma 2.2, hence uniformly Lipschitz. As a result, the monotone sequence $\varphi_n$ converge uniformly to a fixed point $\psi$ of the operator $T$.

For uniqueness, it’s more convenient to consider functions on $\mathbb{T}^d$. Let $(x^{(n)}_k)$ be the sequence that achieves the minimum in

$$\varphi_n(x) = \min_{x_{-n}, \ldots, x_0=\mathbb{T}^d} \sum_{k=-n}^{-1} A(x_k, x_{k+1}),$$

where $A(y, x)$ is defined in (2.1). By compactness, there exists a subsequence $n_j$ such that $(x_j^{(n_j)})$ converge index-wise to a sequence $(x^{(0)}_j)_{j=-\infty}^0$, so that

$$\varphi_n(x) = \sum_{k=-n}^{-1} A(x^{(0)}_k, x^{(0)}_{k+1}).$$

Since $A(y, x) > 0$ except for $(y, x) = (0, 0)$, we must have $(x^{(0)}_n, x^{(0)}_{n+1}) \to (0, 0)$. Suppose $\phi$ is such that $T(\phi) = \phi$, and $\min \phi = 0$. Necessarily we have $\phi(0) = 0$. Then

$$\phi(x) = T^n(\phi)(x) \leq \phi(x^{(0)}_n) + \sum_{k=-n}^{-1} A(x^{(0)}_k, x^{(0)}_{k+1}) = \phi(x^{(0)}_n) + \varphi_n(x).$$

Take $n \to \infty$, we get

$$\phi(x) \leq \phi(0) + \psi(x) = \psi(x).$$

On the other hand, since $\phi(x) \geq 0$, we have $\phi = T^n(\phi) \geq T^n(0) \to \psi$. Hence $\phi = \psi$, as required. \hfill \Box

The Lax-Oleinik operator $T$ is closely related to the twist map $\Phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$, defined by

$$\Phi(x_0, p_0) = (x_1, p_1) \iff v = -\partial_1 h(x_0, x_1), \quad p_1 = \partial_2 h(x_0, x_1). \quad (2.3)$$
Explicitly,\[
\Phi(x_0, p_0) = (x_0 + p_0 + \nabla F(x_0), p_0 + \nabla F(x_0)).
\]
The map \(\Phi\) also projects to a map on \(\mathbb{T}^d \times \mathbb{R}^d\), which we denote by \(\Phi_T\).

Let us denote \(h_\psi(y, x) = \psi(y) + h(y, x)\) for short.

**Lemma 2.4** (See, for example, Lemma 3.2 of Ref. [13]) For all \(x \in \mathbb{R}^d\) and \(y \in \arg \min h_\psi(\cdot, x)\), then \(y \in \mathcal{D}(\psi)\) and
\[
\Phi(y, \nabla \psi(y)) = (x, p),
\]
where \(p\) is a super-gradient of \(\psi\) at \(x\). In particular, if \(x \in \mathcal{D}(\psi)\), then \(y\) is the unique element in \(\arg \min h_\psi(\cdot, x)\) and \(\Phi(y, \nabla \psi(y)) = (x, \nabla \psi(x))\).

Let \(\varphi\) be a semi-concave function on \(\mathbb{R}^d\). Following [4], we define the (overlapping) pseudograph
\[
\mathcal{G}_\varphi = \{(x, \nabla \varphi(x)) : x \in \mathcal{D}(\varphi)\}.
\]
Lemma 2.4 can be rephrased in the pseudograph language as follows.

**Lemma 2.5** (Proposition 2.7 of Ref. [4], see also Lemma 3.2 of Ref. [13] in this setting)
\[
\Phi^{-1} (\mathcal{G}_\psi) \subset \mathcal{G}_\psi.
\]
In particular, we have
\[
\mathcal{G}_\psi = \{(x, p) : (x, p) = \Phi(y, \nabla \psi(y)) \text{ for some } y \in \arg \min h_\psi(\cdot, x)\}.
\]
If \(x \in \mathcal{D}(\psi)\), we denote by \(\tilde{y}(x)\) the unique element of \(\arg \min h_\psi(\cdot, x)\). Then
\[
\tilde{y}(x) = \pi_1 \Phi^{-1}(x, \nabla \psi(x)),
\]
where \(\pi_1 : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d\) is the projection to the first component. Define
\[
\mathcal{D}^-(\psi) = \pi_1 \Phi^{-1}(\mathcal{G}_\psi),
\]
(2.4)
it follows from Lemma 2.5 that
\[
\mathcal{D}^-(\psi) = \bigcup_{x \in \mathbb{R}^d} \arg \min h_\psi(\cdot, x) = \tilde{y}(\mathcal{D}(\psi)) \subset \mathcal{D}(\psi).
\]
Both \(\mathcal{D}(\psi)\) and \(\mathcal{D}^-(\psi)\) are periodic sets, and \(\tilde{y}(x + l) = \tilde{y}(x) + l\) for any \(l \in \mathbb{Z}^d\). We denote the projections of \(\mathcal{D}(\psi)\) and \(\mathcal{D}^-(\psi)\) by \(\mathcal{D}_T(\psi)\) and \(\mathcal{D}_T^-(\psi)\), while keeping the name \(\tilde{y}\) unchanged. In the torus setting,
\[
\{\tilde{y}(x)\} = \arg \min_{y \in \mathbb{T}^d} \{\psi(y) + A(y, x)\}, \quad x \in \mathcal{D}_T(\psi).
\]
While the function \(\nabla \psi\) is only defined at almost every point, it is more regular on the set \(\mathcal{D}^-(\psi)\). This is described in Lemma 2.6 and Corollary 2.8.

**Lemma 2.6** There exists \(C > 0\) such that for every \(y \in \mathcal{D}^-(\psi)\) and \(z \in \mathbb{R}^d\),
\[
|\psi(z) - \psi(y) - \nabla \psi(y) \cdot (z - y)| \leq C|z - y|^2.
\]
Proof Since $\psi$ is $C$-semi-concave, we only need to prove the lower bound.

If $y \in D^-(\psi)$, then there exists $x \in \mathbb{R}^d$, such that $y = \arg \min h_{\psi}(\cdot, x)$. By Lemma 2.6 and (2.3), $\nabla \psi(y) = -\partial_1 h(y, x)$. Since $h(\cdot, x)$ is $C$-semi-concave,

$$0 \leq h_{\psi}(z, x) - h_{\psi}(y, x) = \psi(z) - \psi(y) + h(z, x) - h(y, x)$$

$$\leq \psi(z) - \psi(y) + \partial_1 h(y, x) \cdot (z - y) + C|z - y|^2$$

$$= \psi(z) - \psi(y) - \nabla \psi(y) \cdot (z - y) + C|z - y|^2,$$

implying

$$\psi(z) - \psi(y) - \nabla \psi(y) \cdot (z - y) \geq -C|z - y|^2.$$  

$\Box$

The following holds for general semi-concave functions.

Lemma 2.7 Suppose $f : \mathbb{R}^d \to \mathbb{R}$ is $C$-semi-concave, and suppose for given $x \in \mathcal{D}(f)$, there exists $C' > 0$ such that

$$f(y) - f(x) - \nabla f(x) \cdot (y - x) \geq -C'|y - x|^2, \quad \forall y \in \mathbb{R}^d.$$

Then for any $l_y \in \partial f(y)$, we have

$$|l_y - \nabla f(x)| \leq (4C' + 2C)|y - x|.$$

Proof Consider any $y \in \mathbb{R}^d$ and $l_y \in \partial f(y)$. Set $w = (\nabla f(x) - l_y)/|\nabla f(x) - l_y|$, $\lambda = |y - x|$, we have

$$f(y + \lambda w) - f(y) - l_y \cdot \lambda w \leq C\lambda^2,$$

$$f(y + \lambda w) - f(x) - \nabla f(x) \cdot (y + \lambda w - x) \geq -C|y + \lambda w - x|^2 \geq -4C'|\lambda|^2,$$

the last inequality is due to $|y - x| = |\lambda w| = \lambda$. Subtract the two inequalities, we get

$$-(4C' + C)\lambda^2 \geq f(x) - f(y) + \nabla f(x) \cdot (y - x) + (\nabla f(x) - l_y) \cdot \lambda w$$

$$\geq -C\lambda^2 + \lambda|\nabla f(x) - l_y|,$$

or $|\nabla f(x) - l_y| \leq (4C' + 2C)|y - x|$.  

It is known that $\nabla \psi$ is Lipschitz on $D^-(\psi)$ (see [6]), which is related to Mather’s graph theorem. Our next statement is stronger, and follows directly from Lemmas 2.6 and 2.7.

Corollary 2.8 For all $y \in D(\psi)$ and $x \in D^-(\psi)$, we have

$$|\nabla \psi(y) - \nabla \psi(x)| \leq 6C|y - x|.$$

Proof It follows from Lemma 2.6 that Lemma 2.7 applies to $f = \psi$ and $C' = C$.  

$\Box$

3 Hyperbolicity and weak KAM solution

Let $p$ be a fixed point of a $C^2$ mapping $\Phi : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ (or $\mathbb{T}^{2d} \to \mathbb{T}^{2d}$), we say $p$ is hyperbolic if $D\Phi(p)$ does not admit any eigenvalues on the unit circle. Let $E^s(p)$ and $E^u(p)$ (called the stable and unstable subspaces) be the eigenspaces for the eigenvalues inside and outside the unit circle, respectively. Let us denote

$$W^{s/u}(p) = \{q : \lim_{n \to \infty} \Phi^{\pm n}(q) = p\}$$
called the stable and unstable manifolds of \( p \). The following is a standard result from hyperbolic dynamics.

**Proposition 3.1** (See for example, Theorem 6.2.8 of Ref. [12]) Let \( p \) be a hyperbolic fixed point of \( \Phi \). There exist a neighborhood \( V \) of \( p \) such that the local stable and unstable manifolds \( W^s_{\text{loc}}(p) = W^s(p) \cap V \) and \( W^u_{\text{loc}}(p) = W^u(p) \cap V \) are \( C^2 \) smooth embedded submanifolds tangent to \( E^s(p) \) and \( E^u(p) \) respectively. Moreover, \( W^{s/u}_p(p) = \{ q : \Phi^{\pm n}(q) \in V \} \) for all \( n \geq 0 \).

The map \( \Phi^T \) defined in Eq. (2.3) admits \((0,0)\) as a fixed point. We will show that it is hyperbolic and \( G^\psi \) locally coincides with the unstable manifold. Moreover, we show that \( \psi \) is a Lyapunov function which is strictly contracted by the mapping \( \tilde{\gamma} \).

**Proposition 3.2** There exists \( C > 1 \) depending only on \( F \) such that the following hold.

1. The fixed point \((0,0)\) of \( \Phi^T \) is hyperbolic. Its local unstable manifold \( W^u_{\text{loc}} \) is \( C^2 \) smooth with \( C^2 \) norm bounded by \( C \). The tangent plane to \( W^u_{\text{loc}} \) at \((0,0)\) is given by the graph \( \{(h,S^+h) : h \in \mathbb{R}^d\} \), where \( S^+ \) is a positive definite symmetric matrix.
2. There exists \( r > 0 \) such that for each \( x \in B_r \), we have
   \[
   \{(x, \nabla \psi(x)) : x \in B_r \cap D(T(\psi)) \} = W^u_{\text{loc}} \cap \{(x, v) : x \in B_r \}.
   \]

   We have \( B_r \subseteq \mathcal{D}^{-}(\psi) \subseteq \mathcal{D}(\psi) \), and on \( B_r \), \( \psi \) is \( C^3 \) with uniformly bounded second derivatives. Moreover, \( \psi(x) \geq C^{-1}|x|^2 \) and \( \sqrt{\psi} \) is a \( C \)-Lipschitz function on \( T^d \).
3. We have
   \[
   \{(x, \nabla \psi(x)) : x \in D(T(\psi)) \} \subset W^u := \bigcup_{0 \leq k < C} \Phi^k_T W^u_{\text{loc}}.
   \]
4. There exists \( \kappa \in (0,1) \) such that for each \( x \in D(\psi) \), \( \psi(\tilde{\gamma}(x)) \leq \kappa^2 \psi(x) \).

**Proof** (1): The linearization of \( \Phi^T \) at \((0,0)\) is given in block form by

\[
D\Phi(0,0) = \begin{bmatrix} I_d + D^2 F(0) I_d \\ D^2 F(0) I_d \end{bmatrix}.
\]

To see this matrix is hyperbolic, denote \( M = D^2 F(0) \), and suppose \( S \) is a symmetric matrix such that the hyperplane \( \{(h, Sh) : h \in \mathbb{R}^d\} \) is invariant under \( D\Phi(0,0) \). Since

\[
\begin{bmatrix} I_d + M I_d \\ M I_d \end{bmatrix} \begin{bmatrix} I_d \\ I_d \end{bmatrix} = \begin{bmatrix} I_d + M + S \\ M + S \end{bmatrix}
\]

we get \( M + S = S(I_d + M + S) \) or \( S^2 + SM - M = 0 \). The solutions are given by the quadratic formula \( S^\pm = \frac{1}{2}(-M \pm \sqrt{M^2 + 4M}) \). The mapping \( D\Phi(0,0) \) takes any vector \( (h, S^\pm h) \) to \((h_1, S^\pm h_1)\), where \( h_1 = (I_d + M + S^\pm)h = (I_d + \frac{1}{2}(M \pm \sqrt{M^2 + 4M}))h \). Since \( M + \sqrt{M^2 + 4M} \) is positive definite, there exists \( \kappa_0 \in (0,1) \) such that

\[
|I_d + M + S^\pm|h| > \kappa_0^{-1}|h|.
\]

Moreover \( (I_d + M + S^\pm)(I_d + M + S^{-}) = I_d \), so \( |(I_d + M + S^{-})h| < \kappa_0|h| \). We have proven that \( \{(h, S^\pm h)\} \) is an invariant subspace on which the norm given by \( \|h\| \) is strictly expanded. The complementary subspace \( \{(h, S^- h)\} \) strictly contract \( \|h\| \). This implies hyperbolicity of \((0,0)\), and is hyperbolic with the stable and unstable subspaces given by \( \{(h, S^\pm h)\} \). Also note for future use that \( S^+ \) commutes with \( M \). Item (1) follows from Proposition 3.1.
For $(\kappa, v)$ it suffices to set
\[ A(y, x) \geq C^{-1}(|y|^2 + |x|^2), \quad \forall x, y \in \mathbb{T}^d. \]
Then for each $r > 0$, there exists $\delta > 0$ such that $A(y, x) > \delta$ unless $x, y \in B_r$. Given $n \in \mathbb{N}$, let $m(n, \delta) = \#\{k \in [-n, -1]: |x_k| > r\}$. Then
\[ \psi(x_0) = \psi(x_{n-1}) + \sum_{k=-n}^{-1} A(x_k, x_{k+1}) > \psi(x_{n-1}) + m(n, \delta), \]
since $\psi$ is uniformly Lipschitz, we have $m(n, \delta) < \|\psi\|_{\text{lip}}/\delta$. This argument shows that each backward orbit $\{x_{n-1}\}_{n \in \mathbb{N}}$ can have at most finitely many points outside of $B_r$, i.e. $x_{n-1} \to 0$ in $\mathbb{T}^d$. This convergence is uniform over all $x_0 \in \mathcal{D}_T(\psi)$ since the bound for $m(n, \delta)$ is independent of $x_0$. Moreover $v_{n-1} = \nabla \psi(x_{n-1}) \to 0$ since $\nabla \psi$ is Lipschitz over the set $\mathcal{D}_T(\psi)$. By Proposition 3.1, there exists $r_0 > 0$ such that if the entire orbit $(x_n, v_n)_{n \in \mathbb{N}} \subset B_{r_0}(0, 0)$, then $(x_0, v_0) \in W^{u}_{\text{loc}} \cap B_{r_0}(0, 0)$. This is the case if $x_0$ is sufficiently close to 0. This proves
\[ \{(x, \psi(x)) : x \in B_r \cap \mathcal{D}_T(\psi)\} = W^{u}_{\text{loc}} \cap \{(x, v) : x \in B_r\} \]
and
\[ \{(x, \nabla \psi(x)) : x \in \mathcal{D}_T(\psi)\} \subset \bigcup_{0 \leq k < \infty} \Phi^k W^{u}_{\text{loc}}. \]
The fact that the infinite union can be replaced with a finite one is again due to the uniform convergence of $(x_n, v_n)$ to $(0, 0)$.

Finally, $D^2 \psi(0) = S^+$ is positive definite. Since 0 is the only global minimum of $\psi$, this implies that there exists $C > 0$ such that $\psi(x) \geq C^{-1}|x|^2_{\mathbb{T}^d}$. This also implies $\sqrt{\psi}$ is uniformly Lipschitz with Lipschitz constant depending only on $C$.

(4): First we assume $x \in B_r(0)$. Since the graph of $\nabla \psi$ coincide with the smooth graph $W^{u}_{\text{loc}}$ on $B_r$, the mapping $\tilde{y}$ is smooth on $B_r$, and $D\tilde{y}(0) = (I_d + M + S^+)^{-1}$ as we computed earlier. Since $\tilde{y}(x) = (I_d + M + S^+)^{-1}x + O(x^2), \psi(\tilde{y}) = \psi(\tilde{y}) = (D^2 \psi(0)\tilde{y}, \tilde{y}) + O(|\tilde{y}|^3) = x^T (I_d + M + S^+)^{-T} S^+ (I_d + M + S^+)^{-1} x + O(x^3)$. Since $(I_d + M + S^+)^{-1}$ is strictly contracting and commutes with $S^+$, by diagonalizing the matrices, there exists $\kappa_0 \in (0, 1)$ such that
\[ x^T (I_d + M + S^+)^{-T} S^+ (I_d + M + S^+)^{-1} x < \kappa^2_0 x^T S^+ x^T, \quad |(I_d + M + S^+)^{-1} x| < \kappa_0 |x|. \]

For $\kappa_1 \in (\kappa_0, 1)$, we can choose $r$ sufficiently small such that
\[ \psi(\tilde{y}(x)) \leq \kappa_2 \psi(x), \quad |\tilde{y}(x)| < \kappa |x|, \quad x \in B_r. \]
If $x \not\in B_r(0)$, then there exists $\delta > 0$ such that $A(y, x) \geq \delta$. Denote $\tilde{y} = \tilde{y}(x)$, we have
\[ \frac{\psi(\tilde{y})}{\psi(x)} = \frac{\psi(\tilde{y})}{\psi(\tilde{y}) + A(\tilde{y}, x)} \leq \frac{1}{1 + \delta/\|\psi\|_{C^0}} < 1. \]

It suffices to set $\kappa = \min\{\kappa_1, \sqrt{1/(1 + \delta/\|\psi\|_{C^0})}\}$.
Suppose \( x \in \mathcal{D}(\psi) \), then \( \tilde{y}(x) \) reaches the unique minimum of \( h_\psi(\cdot, x) \). We are interested in the non-degeneracy of this minimum, which turns out to be related to whether \( \psi(x) \) has bounded second derivatives. Lemma 2.6 ensures that this holds for every \( x \in \mathcal{D}^-(\psi) \). Using the fact that \( G_{\psi} \) is contained in a smooth sub-manifold, we can also extend this estimate to a neighborhood.

**Proposition 3.3** There exists an open set \( U \supseteq \mathcal{D}^-(\psi) \) and \( \delta > 0 \) depending only on \( F \), such that the following holds.

1. \( \psi \) is \( C^3 \) on \( U \) with uniformly bounded second derivatives.
2. We have

\[
h_\psi(y, x) - h_\psi(\tilde{y}(x), x) \geq \delta |y - \tilde{y}(x)|^2, \quad \text{for all } x \in U, \ y \in \mathbb{R}^d,
\]

and

\[
h_\psi(y, x) - \min h_\psi(\cdot, x) \geq \delta, \quad \text{for all } y \not\in U.
\]

**Remark** The case \( x \not\in U, \ y \in U \) is not covered by this Proposition, and will be dealt with separately.

We need a number of lemmas.

**Lemma 3.4** Suppose at some \( x \in \mathcal{D}(\psi) \), there exists \( D_x > 0 \) such that

\[
|\psi(z) - \psi(x) - \nabla \psi(x) \cdot (z - x)| \leq D_x |z - x|^2, \quad \forall z \in \mathbb{R}^d,
\]

then for \( \tilde{y} = \tilde{y}(x) \), there exists \( C, r > 0 \) depending only on \( F \), such that

\[
h_\psi(\tilde{y} + v, x) - h_\psi(\tilde{y}, x) \geq \frac{|v|^2}{8(D_x + C)}, \quad \text{for all } |v| < r.
\]

**Proof** Write \( \tilde{y} = \tilde{y}(x) \) and \( D = D_x \). Then for \( v, w \in \mathbb{R}^d \),

\[
h_\psi(\tilde{y} + v, x) - h_\psi(\tilde{y}, x)
= \psi(\tilde{y} + v) + h(\tilde{y} + v, x) - \psi(\tilde{y}) - h(\tilde{y}, x)
= h(\tilde{y} + v, x) - h(\tilde{y} + v, x + w) + [\psi(\tilde{y} + v) + h(\tilde{y} + v, x + w) - \psi(\tilde{y}) - h(\tilde{y}, x)]
\geq h(\tilde{y} + v, x) - h(\tilde{y} + v, x + w) + \psi(x + w) - \psi(x)
= h(\tilde{y} + v, x) - h(\tilde{y} + v, x + w) + \partial_2 h(\tilde{y} + v, x) \cdot w
- (\partial_2 h(\tilde{y} + v, x) - \partial_2 h(\tilde{y}, x)) \cdot w + [\psi(x + w) - \psi(x) - \partial_2 h(\tilde{y}, x) \cdot w]
\geq -\frac{1}{2} \|\partial_2 h\|_{C^0} |w|^2 - \partial_1 h(\tilde{y}, x) v \cdot w - \frac{1}{2} \|\partial_{112} h\| |v|^2 |w| - D|w|^2.
\]

Set \( w = -tv \), and note \( \partial_1 h = I_d, \|\partial_2 h\|, \|\partial_{112} h\| \leq C \), we get

\[
h_\psi(\tilde{y} + v, x) - h_\psi(\tilde{y}, x) \geq t|v|^2 (1 - (D + C)t + C|v|).
\]

Set \( t = \frac{1}{4(D + C)} \) and \( |v| < \frac{1}{4C} \), we have

\[
h_\psi(\tilde{y} + v, x) - h_\psi(\tilde{y}, x) \geq \frac{1}{2} \lambda |v|^2 = \frac{1}{8(D + C)}|v|^2, \quad \forall v < \frac{1}{4C}.
\]

\( \square \)
Lemma 3.5 There exists $C, R > 0$ such that for all $x \in \mathbb{R}^d$ and $\tilde{y} \in \arg \min h_\psi(\cdot, x)$, we have

$$h_\psi(\tilde{y} + v, x) - h_\psi(\tilde{y}, x) \geq \frac{1}{4}|v|^2, \quad \text{for all } |v| > R$$

and

$$h_\psi(\tilde{y} + v, x) - h_\psi(\tilde{y}, x) \leq C|v|^2, \quad \text{for all } v \in \mathbb{R}^d.$$

**Proof** The upper bound follows directly from semi-concavity, we only prove the lower bound.

We first claim that there is a constant $C > 0$ such that $|\tilde{y} - x| < C$ for all $\tilde{y} \in \arg \min h_\psi(\cdot, x)$. Indeed, since $\psi$ is uniformly Lipschitz, $\mathcal{G}_\psi$ has uniformly bounded $p$ component. Therefore $\Phi^{-1}(\mathcal{G}_\psi)$ is of bounded distance away from $\mathcal{G}_\psi$, implying the claim.

By periodicity, it suffices to prove our lemma for $x \in [-\frac{1}{2}, \frac{1}{2})^d$. Since $|\tilde{y} - x| < C$, by resetting $C$

$$h_\psi(\tilde{y} + v, x) - h_\psi(\tilde{y}, x) \geq \frac{1}{2}|\tilde{y} + v - x|^2 - \frac{1}{2}|\tilde{y} - x|^2 - 2\|\psi\|_{C^0} - 2\|F\|_{C^0}$$

$$\geq \frac{1}{2}|v|^2 - |v||\tilde{y} - x| - 2\|\psi\|_{C^0} - 2\|F\|_{C^0} \geq \frac{1}{2}|v|^2 - |v|C - 2\|\psi\|_{C^0} - 2\|F\|_{C^0}.$$ 

Suppose $\frac{1}{8}R > C$ and $\frac{1}{8}R^2 > 2\|\psi\|_{C^0}$, we get $h_\psi(\tilde{y} + v, x) - h(\tilde{y}, x) \geq \frac{1}{4}|v|^2$. \hfill $\square$

**Proof of Proposition 3.3** Let $E \subset W^u \subset \mathbb{T}^d \times \mathbb{R}^d$ be the set of critical points for the projection $\pi_1|W^u : W^u \to \mathbb{T}^d$. By Sard’s lemma, $\pi^E$ is a compact nowhere dense set of zero Lebesgue measure. We claim that $\mathcal{D}^-(\psi) \cap \pi_1 E$ is empty. Otherwise, there must exist $x_0 \in \mathcal{D}^-(\psi) \cap \pi_1 E$ and $\mathcal{D}(\psi) \ni x_k \to x_0$ such that $|\nabla \psi(x_k) - \nabla \psi(x_0)|/|x_k - x_0| \to \infty$ as $k \to \infty$, contradicting Corollary 2.8.

This claim implies that the projection $\pi_1|W^u : W^u \to \mathbb{T}^d$ is regular at every point of $G^- := \{(x, \nabla \psi(x)) : x \in \mathcal{D}^-\}$. Implicit function theorem then implies that $G^-$ is contained in a $C^2$ smooth graph over an open neighborhood $U_1$ of $\mathcal{D}^-(\psi)$. Using Corollary 2.8 again, we conclude that for every $y \in \mathcal{D}(\psi) \cap U_1$, $\nabla \psi$ must be contained in the same graph.

We have now proven $\nabla \psi$ coincides with a $C^2$ function at almost every point in $U_1$, therefore $\psi$ must be $C^3$. The $C^2$ norm of $\psi$ is bounded as long as $U_1 \cap \pi_1 E = \emptyset$. The same hold if we lift $U_1$ to an open set in $\mathbb{R}^d$. This proves item (1) of our Proposition on the set $U_1$.

Set $U = \{z \in U_1 : \dist(z, \mathcal{D}^-(\psi)) \leq \frac{1}{2}\dist(\partial U_1, \mathcal{D}^-(\psi))\}$, we claim that there exists $D > 0$ such that

$$|\psi(z) - \psi(y) - \nabla \psi(y) \cdot (z - y)| \leq D|z - y|^2, \quad \forall z \in \mathbb{R}^d, y \in U. \quad (3.2)$$

We have proven that exists $D_1 > 0$ such that for all $y, x \in U_1$,

$$|\nabla \psi(y) - \nabla \psi(x)| \leq D_1|y - x|.$$

If $z \in U_1$, (3.2) holds since $\psi$ is $C^2$ with an uniform $C^2$ norm. Suppose $z \in \mathbb{R}^d \setminus U_1$. Then for each $y \in U$, there exists $x \in \mathcal{D}^-\langle\psi\rangle$ such that $|y - x| \leq \frac{1}{2}|z - x|$, and hence $|z - y| \geq |z - x| - |y - x| \geq \frac{1}{2}|z - x|$. By Lemma 2.6, there exists $D_2 > 0$ such that

$$|\psi(z) - \psi(x) - \nabla \psi(x) \cdot (z - x)| \leq D_2|z - x|^2.$$

Combine the two, we get

$$|\psi(z) - \psi(y) - \nabla \psi(y) \cdot (z - y)| \leq D_2|z - x|^2 + D_1|y - x|^2 \leq D_3|z - y|^2.$$
for an absolute constant $D_3 > 0$.

We now prove item (2). First, suppose $y \notin U$.

$$\delta_1 = \min_{x \in \mathbb{R}^d, y \notin U} h_\psi(y, x) - h_\psi(\tilde{y}(x), x).$$

Then $\delta_1 > 0$ by compactness argument. If $|y - \tilde{y}(x)| \leq R$ ($R$ is from Lemma 3.5), we have

$$h_\psi(y, x) - h_\psi(\tilde{y}(x), x) \geq \delta \geq \frac{\delta_1}{R^2} |y - \tilde{y}(x)|^2.$$

If $|y - \tilde{y}(x)| > R$, Lemma 3.5 applies.

Suppose $x \in U$. It follows from Eq. (3.2) and Lemma 3.4 that there exists $r > 0$ and $\delta_2 > 0$ such that

$$h_\psi(y, x) - h_\psi(\tilde{y}(x), x) \geq \delta_2 |y - \tilde{y}(x)|^2, \quad \text{for all } x \in U, |y - \tilde{y}(x)| < r.$$

If $|y - \tilde{y}(x)| \geq r$, we set

$$\delta_3 = \min_{x \in U, |y - \tilde{y}(x)| \geq r} h_\psi(y, x) - h_\psi(\tilde{y}(x), x) > 0,$$

and proceed in the same way as the $y \notin U$ case. \hfill \Box

### 4 The viscous equation via Hopf-Cole transformation

Consider the Eq. (1.2) with $T_0 = -n$, $T = 0$. We apply the Hopf-Cole transformation

$$u = \exp\left(\frac{-\varphi}{2v}\right),$$

which transform it to the inhomogeneous heat equation

$$u_t = \nu \Delta u - \frac{1}{2v} F(x, t) u, \quad (x, t) \in \mathbb{R}^d \times (-n, 0)$$

$$u(x, -n) = u_0(x) = e^{-\frac{1}{2v} \varphi_0(x)} \quad x \in \mathbb{R}^d. \quad \text{(4.1)}$$

The solution to Eq. (4.1) is given by the Feynman-Kac formula, namely

$$u(x, t) = \int dy u_0(y) \int \exp\left(-\frac{1}{2v} \int_0^t F(W(\tau)) d\tau\right) d\Pi^{(t, 0)}_x(v; W),$$

where $\Pi^{(t, 0)}_x(v; \cdot)$ is the probability distribution of the Brownian motion $dW = \sqrt{2\nu} dW$ with the condition $W(0) = y$ and $W(t) = x$.

For the kicked case, the formula for the solution is simplified. To solve from time $i$ to $i + 1$, we multiply the function $u^-(\cdot, i)$ by the factor $e^{-\frac{1}{2v} F}$ to obtain $u^+(\cdot, i)$, representing the kicked force. We then solve the heat equation without force on the interval $(i, i + 1)$ to obtain $u^-(\cdot, i + 1)$. Formally, we have $u^-(\cdot, i + 1) = \mathcal{L}_v(u^-(\cdot, i))$, where

$$\mathcal{L}_v(u)(x) = \int K_v(y, x) u(y) dy,$$

and

$$K_v(y, x) = \frac{1}{(4\pi \nu)^{d/2}} \exp\left(-\frac{1}{2v} h(y, x)\right)$$

$$= \frac{1}{(4\pi \nu)^{d/2}} \exp\left(-\frac{1}{2v} \left(\frac{1}{2} (y - x)^2 + F(y)\right)\right).$$
Then for \( n \in \mathbb{N} \), the solution \( u_{-n}(x, 0) \) to Eq. (4.1) satisfy
\[
\begin{aligned}
&u_{-n}(x, 0) = \mathcal{L}^n_v(u_0)(x) \\
&= \int \cdots \int u_0(x_{-n}) K_v(x_{-n}, x_{-n+1}) \cdots K_v(x_{-1}, x) dx_{-n} \cdots dx_{-1}.
\end{aligned}
\]

The counterpart to Theorem 1 in this setting is:

**Theorem 4.1** There exists \( v_0 > 0 \), \( C > 0 \) and \( \lambda > 0 \) depending only on \( F \), such that the following hold for all \( v \in (0, v_0) \). There exists \( u^v \in C_{\text{per}}(\mathbb{R}^d) \) such that \( 0 \leq \log u^v \leq C/v \) and
\[
\| \log \mathcal{L}_v u^v - \log u^v \|_* = 0.
\]
Moreover, for each \( 0 \leq \log u_0, \log v_0 \leq D/v \), we have
\[
\| \log \mathcal{L}_v^n u_0 - \log \mathcal{L}_v^n v_0 \|_* \leq e^{C+D}/v e^{-\lambda n}, \quad \text{for } n \geq C/v.
\]

In particular, \( u^v \) is unique up to a constant.

Theorem 4.1 is proven in Sect. 6. We first show that our main theorem follows from Theorem 4.1.

**Proof of Theorem 1** We normalize \( \nu_0 \) so that \( \min \nu_0 = 0 \), and set \( v_0 = e^{-\frac{1}{2\pi} v_0} \), and \( \nu^v = -2v \log u^v \) where \( u^v \) is from Theorem 4.1. Both \( u^v \) and \( v_0 \) satisfy the assumptions of Theorem 4.1 with \( D = \| \nu_0 \|_* \). It follows that
\[
\| \nu_h^v (\cdot, 0) - \nu^v (\cdot) \|_* = 2v \log \tilde{\mathcal{L}}^n_v v_0 - \log \tilde{\mathcal{L}}^n_v u^v \|_* < e^{\frac{\| \nu_0 \|_* + C}{v}} e^{-\lambda n}
\]

Theorem 1 follows. \( \square \)

### 4.1 The conjugate kernel

Let \( \psi \) be the unique solution to Eq. (2.2). Define
\[
\tilde{h}(y, x) = h(y, x) + \psi(y) - \psi(x).
\]

The conjugate kernel is more convenient to study since \( \arg \min \tilde{h}(\cdot, x) = \arg \min h_{\psi}(\cdot, x) \), but in addition \( \min \tilde{h}(\cdot, x) = 0 \) for all \( x \in \mathbb{R}^d \). Define
\[
\tilde{K}_v(y, x) = (2\pi v)^{-d} \exp \left( -\frac{1}{2v} \tilde{h}(y, x) \right) = e^{\frac{1}{2\pi} \psi(x)} K_v(y, x) e^{-\frac{1}{2\pi} \psi(y)}
\]
\[
\tilde{\mathcal{L}}_v(u)(x) = \int \tilde{K}_v(y, x) u(y) dy = e^{\frac{1}{2\pi} \psi(x)} \mathcal{L}_v \left( e^{-\frac{1}{2\pi} \psi} u \right).
\]

It’s easy to see that \( \tilde{\mathcal{L}}^n_v(u) = e^{\frac{1}{2\pi} \psi} \mathcal{L}^n_v \left( e^{-\frac{1}{2\pi} \psi} u \right) \).

Proposition 3.3 implies:

**Corollary 4.2**

\[
\tilde{h}(y, x) \geq \delta |y - \tilde{y}(x)|^2, \quad \text{for all } x \in U, y \in \mathbb{R}^d,
\]
\[
\tilde{h}(y, x) \geq \delta, \quad \text{for all } y \notin U.
\]
4.2 The Markov kernel

We convert the kernels into Markov ones following [14]. Let 1 denote the constant function 1, define:

\[ \pi_v^{(0)}(y, x) = \frac{\tilde{K}_v(y, x)}{\int \tilde{K}_v(y, x) dy} = \frac{\tilde{K}_v(y, x)}{\mathcal{L}_v(1)(x)}, \]

\[ \pi_v^{(1)}(y, x) = \frac{\tilde{K}_v(y, x)\mathcal{L}_v(1)(y)}{\mathcal{L}_v^2(1)(x)}, \]

\[ \vdots \]

\[ \pi_v^{(n+1)}(y, x) = \frac{\tilde{K}_v(y, x)\mathcal{L}_v^n(1)(y)}{\mathcal{L}_v^{n+1}(1)(x)}, \]

(4.3)

each kernel is Markov in the sense that

\[ \int \pi_v^{(n)}(y, x) dy = \mathcal{P}_v^{(n)}(1)(x) = 1. \]

Define the Markov operators acting on functions

\[ \mathcal{P}_v^n(u)(x) = \int \pi_v^{(n)}(y, x) u(y) dy, \quad n \geq 0, \quad \mathcal{P}_v^0(u) = \mathcal{P}_v^{(n-1)} \ldots \mathcal{P}_v^{(0)} u, \]

then

\[ \frac{\mathcal{L}_v^n u(x)}{\mathcal{L}_v^n 1(x)} = \int \pi_v^{(n-1)}(y, x) \frac{\mathcal{L}_v^{n-1} u(y)}{\mathcal{L}_v^{n-1} 1(y)} dy = \mathcal{P}_v^{(n-1)} \ldots \mathcal{P}_v^{(0)}(u) = \mathcal{P}_v^{0,n}(u). \]

(4.4)

We caution the reader that the similar notations \( \mathcal{P}_v^n \) and \( \mathcal{P}_v^{(n)} \) have very different meanings: the former is a composition of \( n \) operators, while the latter is a normalized version of a single iteration step.

The function \( \mathcal{L}_v^n 1 \) is known as the partition function in statistical mechanics.

Let \( U \) be as in Proposition 3.3, define

\[ \chi_v(x) = \begin{cases} 1 & x \in U, \\ \frac{1}{2} & x \notin U. \end{cases} \]

(4.5)

The following estimate of the partition function is a crucial technical step in our proof.

**Proposition 4.3** There exist \( C > 1, \nu_0 > 0, Q_n > 0 \) satisfying

\[ C^{-1} \leq Q_{n+1}/Q_n \leq C, \]

such that for \( N_1(\nu) = C^{-1}(\nu \log \frac{1}{\nu})^{-\frac{1}{2}}, \)

\[ C^{-1} \leq \frac{\mathcal{L}_v^n(1)(x)}{Q_n} \leq C \chi_v(x), \quad \text{for all} \quad 0 \leq n \leq N_1(\nu), \quad 0 < \nu \leq \nu_0. \]

Proposition 4.3 is proven by applying the classical Laplace method, but trying to obtain uniform estimates in \( n \). The proof is postponed to the last two sections of this paper.
5 Uniform contraction for the Markov operator

5.1 Lyapunov functions

We describe the Lyapunov function approach to the convergence of Markov operators by Hairer and Mattingley [8]. Let \( \pi \) be a positive measurable function on \( \mathbb{R}^d \times \mathbb{R}^d \) such that
\[
P(u)(x) = \int \pi(y, x)u(y)dy
\]
defines a bounded Markov operator from \( L^\infty(\mathbb{R}^d) \) to \( L^\infty(\mathbb{R}^d) \). Assume that:

1. (A1) There exists \( V \in L^\infty(\mathbb{R}^d) \) with \( V \geq 0 \), constants \( M \geq 0 \) and \( \gamma \in (0, 1) \) such that
\[
(PV)(x) \leq \gamma V(x) + M,
\]
for all \( x \in X \).

2. (A2) There exists a constant \( \alpha_0 \in (0, 1) \) and a probability density \( g_0 \) so that
\[
\inf_{x: V(x) \leq R} \pi(\cdot, x) \geq \alpha_0 g_0(\cdot),
\]
where \( R > 2M/(1 - \gamma) \).

Choose the parameters as follows:

\[
\alpha_1 \in (0, \alpha_0), \gamma_0 \in (\gamma + 2M/R, 1), \beta = \frac{\alpha_1}{M}, \alpha = \max \left\{1 - (\alpha_0 - \alpha_1), \frac{2 + R\beta \gamma_0}{2 + R\beta}\right\}.
\]

Theorem 5.1 [8] Suppose \( P \) satisfies (A1) and (A2), and let \( \beta, \alpha \) be chosen as described. Then
\[
\|P(u)\|_{\beta V, *} \leq \alpha \|u\|_{\beta V, *}.\]
The proof for Theorem 5.1 is very short and elementary. We include it in the appendix for the reader’s convenience.

5.2 Choice of Lyapunov functions for the Markov operators

Proposition 5.2 For \( D > 1 \) and \( v_0 > 0 \), let \( \pi_v(y, x), v \in (0, v_0) \) be a family of a periodic Markov kernel on \( \mathbb{R}^d \times \mathbb{R}^d \) such that
\[
\frac{1}{D\chi_v(x)} \leq \frac{\pi_v(y, x)}{K_v(y, x)} \leq D\chi_v(y).
\]
Let \( \mathcal{P}_v \) denote the Markov operator with kernel \( \pi_v(y, x) \).

Then there exists \( C > 1, \alpha_0 \in (0, 1), v_1 > 0, R > 0 \), and a family of probability density \( (g_v)_v \in (0, v_1) \) on \( \mathbb{R}^d \) depending only on \( F \) and the constant \( D \), such that the following hold:
(1) Let \( V(x) = \psi(x)\chi_v^2(x) \), we have

\[
(\mathcal{P}_v V)(x) \leq \gamma V(x) + C_v.
\]

(2) \( R > 2\alpha/(1 - \gamma) \) and

\[
\inf_{V(x) \leq R \psi} \pi_v(y, x) \geq \alpha_0 g_v(y).
\]

We now apply Proposition 5.2 to the kernels \( \pi_v^{(n)}(y, x) \) defined in Eq. (4.3).

**Corollary 5.3** Suppose \( v_0 > 0, N = N(v) \in \mathbb{N} \) where \( v \in (0, v_0) \), \( D > 1 \) and \( Q_n > 0 \) are chosen such that the following hold:

\[
D^{-3} \leq Q_n/Q_{n+1} \leq D^3, \quad \text{for all} \ 0 \leq n \leq N(v),
\]

and

\[
D^{-1} \leq \frac{\tilde{\mathcal{L}}_v^{(n)}(1)(x)}{Q_n} \leq D \chi_v(x), \quad \text{for all} \ 0 \leq n \leq N(v), \ 0 < v < v_0.
\]

Then exists \( C > 1, \alpha, \alpha_1 \in (0, 1), v_1 > 0 \) depending only \( F, D \), such that for each \( v \in (0, v_1) \) and \( 0 \leq n \leq N(v) \), we have

\[
\|\mathcal{P}_v^{(n)} u\|_{\mathcal{B}^*_v} \leq \alpha \|u\|_{\mathcal{B}^*_v},
\]

where \( \beta = \frac{\alpha_1}{\epsilon} \) and \( V(x) = \psi(x)\chi_v^2(x) \).

**Proof of Corollary 5.3** For each \( 0 \leq n \leq N(v) \), we have \( \pi_v^{(n)}(y, x)/\tilde{K}_v(y, x) = \tilde{\mathcal{L}}_v^{(n)}1(y)/\tilde{\mathcal{L}}_v^{(n+1)1}(x) \). It follows from Eqs. (5.3) and (5.4) that

\[
\frac{1}{D^4 \chi_v(x)} \leq \frac{\pi_v^{(n)}(y, x)}{\tilde{K}_v(y, x)} \leq D^4 \chi_v(y),
\]

hence Proposition 5.2 applies with \( D \) replaced with \( D^4 \). We choose parameters according to Eq. (5.2). Note that \( \alpha = \max\{1 - \alpha_0 + \alpha_1, 2 + \frac{\alpha_1}{\epsilon} - R\psi(v_0)\} \) is independent of \( v \). \( \Box \)

**Proof of Proposition 5.2** Item (1), case 1: \( x \in U \). In this case we have

\[
(\mathcal{P}V)(x) = \int \pi_v(y, x)V(y)dy \leq D \int \tilde{K}(y, x)V(y)dy.
\]

By Proposition 3.2, \( \sqrt{\psi} \) is a \( C \)-Lipschitz function. Then for any \( \epsilon > 0 \), and \( x \in U \), we have

\[
\psi(y) \leq \left( \sqrt{\psi}(\tilde{y}(x)) + C|y - \tilde{y}(x)| \right)^2
\]

\[
\leq \psi(\tilde{y}(x)) + C^2|y - \tilde{y}(x)|^2 + 2C\sqrt{\psi}(\tilde{y}(x))|y - \tilde{y}(x)|
\]

\[
\leq (1 + \epsilon^2)\psi(\tilde{y}(x)) + C^2(1 + \epsilon^2)|y - \tilde{y}(x)|^2
\]

\[
\leq (1 + \epsilon^2)k^2\psi(x) + C^2(1 + \epsilon^2)|y - \tilde{y}(x)|^2
\]

\[
\leq \gamma \psi(x) + C_1|y - \tilde{y}(x)|^2,
\]

where we have set \( \gamma = (1 + \epsilon^2)k^2, C_1 = C^2(1 + \epsilon^2) \). We then choose \( \epsilon \) such that \( \gamma \in (0, 1) \).

Write \( \tilde{y} = \tilde{y}(x) \) for short, and let \( r > 0 \) be such that \( B_r(\tilde{y}) \subset U \), then

\[
(\mathcal{P}V)(x) \leq \int_{B_r(\tilde{y})} \pi_v(y, x)V(y)dy + D \int_{\mathbb{R}^d \setminus B_r(\tilde{y})} \tilde{K}(y, x)V(y)dy.
\]
We have
\[
\int_{B_r(y)} \pi_v(y, x) V(y) dy = \int_{B_r(y)} \pi_v(y, x) \psi(y) dy
\]
\[
\leq C_1 D \int_{B_r(y)} \tilde{K}_v(y, x) |y - \bar{y}|^2 dy + \gamma \psi(x) \int_{B_r(y)} \pi_v(x, y) dy
\]
\[
\leq C_1 D (4\pi)^{-\frac{d}{2}} v^{\frac{d}{2}} \int_{B_r(y)} e^{-\frac{1}{\pi \sigma} \tilde{h}(y, x)} |y - \bar{y}|^2 dy + \gamma \psi(x).
\]

By Corollary 4.2,
\[
v^{-\frac{d}{2}} \int_{B_r(y)} e^{-\frac{1}{\pi \sigma} \tilde{h}(y, x)} |y - \bar{y}|^2 dy \leq v^{-\frac{d}{2}} \int_{B_r(y)} e^{-\frac{1}{\pi \sigma} |y - \bar{y}|^2} |y - \bar{y}|^2 dy
\]
\[
= v^{-\frac{d}{2}} \int_{B_r(0)} e^{-\frac{1}{\pi \sigma} |v|^2} |v|^2 dv = v \int_{B_r/\sqrt{\pi}} e^{-\frac{1}{\pi} |v|^2} |v|^2 dv \leq C_2 v,
\]
where \(C_2 = \int_{\mathbb{R}^d} e^{-\frac{1}{\pi} |v|^2} |v|^2 dv\).

On the other hand, since \(V(x) \leq C \chi_v^2(x) \leq C v^{-d}\), we have
\[
\int_{\mathbb{R}^d \setminus B_r(y)} \tilde{K}_v(y, x) V(y) dy
\]
\[
\leq (4\pi)^{\frac{d}{2}} C v^{-\frac{3d}{2}} \int_{\mathbb{R}^d \setminus B_r(y)} e^{-\frac{1}{\pi \sigma} \tilde{h}(y, x)} dy \leq C_3 v^{-\frac{3d}{2}} \int_{\mathbb{R}^d \setminus B_r(y)} e^{-\frac{1}{\pi \sigma} |y - \bar{y}|^2} dy
\]
\[
= C_3 v^{-\frac{3d}{2}} \int_{|v| \geq r} e^{-\frac{1}{\pi \sigma} |v|^2} dv \leq C_3 v^{-\frac{3d}{2}} e^{-\left(\frac{1}{\pi \sigma} - 1\right)r^2} \int_{|v| \geq r} e^{-\frac{1}{\pi \sigma} |v|^2} dv
\]
\[
\leq C_3 e^{r^2} v^{-\frac{3d}{2}} e^{-\frac{1}{\pi \sigma} r^2}.
\]

We now choose \(v_1\) sufficiently small such that \(C_3 e^{r^2} v^{-\frac{3d}{2}} e^{-\frac{1}{\pi \sigma} r^2} < v\).

Combine all the estimate, for \(x \in U\), there is \(C_4 > 1\) depending on \(F\) and \(D\) such that
\[
(PV)(x) \leq C_4 v + \gamma \psi(x) = C_4 v + \gamma V(x).
\]

**Item (1), case 2:** \(x \notin U\). Then
\[
\int \pi_v(y, x) V(y) dy \leq D \int \tilde{K}_v(y, x) \psi(y) \chi_v^3(y) dy.
\]

On one hand,
\[
\int_U \tilde{K}_v(y, x) \psi(y) \chi_v^3(y) dy = \int_U \tilde{K}_v(y, x) \psi(y) dy \leq C \int_U \tilde{K}_v(y, x) dy
\]
\[
\leq C \tilde{L}_v 1(x) \leq C v^{-\frac{d}{2}},
\]
where we used Proposition 4.3. On the other hand,
\[
\int_{U^c} \tilde{K}_v(y, x) \psi(y) \chi_v^2(y) dy \leq C v^{-2d} \int_{U^c} e^{-\frac{1}{\pi \sigma} \tilde{h}(y, x)} dy
\]
\[
\leq C v^{-2d} e^{-\delta(\frac{1}{\pi \sigma} - 1)} \int_{U^c} e^{-\tilde{h}(y, x)} dy \leq C v^{-2d} e^{-\frac{4}{\pi \sigma}}.
\]

\(\odot\) Springer
By choosing \( \nu \) small enough, we can ensure \( C \nu^{-d/2} e^{-\frac{C}{4} \nu} < C \lambda \nu^{-d/2} \). Therefore for \( x \in \mathbb{R}^d \), we have

\[
(PV)(x) \leq 2C \nu^{-d/2} \leq 2C^2 \nu^{d/2} \cdot V(x).
\]

By choosing \( \nu \) again, we can ensure \( (2C^2 \nu^{d/2}) < \gamma \).

Combine the two cases, we have proved item (1).

**Item (2):** Note that

\[
\pi_{\nu}(y, x) \geq \frac{1}{D X_{\nu}(x)} \tilde{K}_{\nu}(y, x) = D^{-1} \tilde{K}_{\nu}(y, x).
\]

Since \( \pi_{\nu} \) is periodic, it suffices to prove (2) for \( x \in [-\frac{1}{2}, \frac{1}{2}]^d \). Note that there exists \( C > 1 \) such that \( V(x) \geq \psi(x) \geq C^{-1}|x|^2 \). It follows that for \( R \nu_0 \) sufficiently small, for all \( \nu \in (0, \nu_0) \),

\[
\{ x : V(x) < \nu \nu_0 \} \subset \{ x : |x| < C \sqrt{R \nu} \} = B_{C \sqrt{R \nu}} \subseteq U.
\]

Proposition 3.2 implies \( \tilde{y}(x) \in B_{C \sqrt{R \nu}} \) for all \( x \in B_{C \sqrt{R \nu}} \). By Corollary 4.2, there exists \( C > 1 \) such that

\[
\tilde{h}(y, x) \leq C|y - \tilde{y}(x)|^2, \quad y \in \mathbb{R}^d.
\]

It follows that

\[
\inf_{V(x) < \nu \nu_0} \pi_{\nu}(y, x) \geq D^{-1} (4\pi \nu)^{-d/2} \inf_{|x| < C \sqrt{R \nu}} e^{-\frac{C}{4\pi} |y - \tilde{y}(x)|^2} \geq D^{-1} (4\pi \nu)^{-d/2} \inf_{|z| < C \sqrt{R \nu}} e^{-\frac{C}{4\pi} |y - z|^2} := G_{\nu}(y).
\]

Set

\[
\alpha_{\nu} = \int G_{\nu}(y)dy, \quad g_{\nu}(y) = \alpha_{\nu}^{-1} G_{\nu},
\]

then \( g_{\nu} \) is a probability density satisfying (2). It suffices to prove that there exists \( \alpha_0 \in (0, 1) \) such that \( \alpha_{\nu} \geq \alpha_0 \).

Indeed,

\[
\int G_{\nu}(y)dy = (4\pi \nu)^{-d/2} \int \inf_{|z| < C \sqrt{R \nu}} e^{-\frac{C}{4\pi} |y - z|^2} = (4\pi)^{-d/2} \int \inf_{|z| < C \sqrt{R}} e^{-\frac{C}{4\pi} |y - z|^2} := \alpha_0
\]

is independent of \( \nu \). Moreover, since \( C > 1 \),

\[
\alpha_0 \leq (4\pi)^{-d/2} \int e^{-\frac{1}{2} |y - x|^2}dy < 1.
\]

To choose parameters, we only need to choose \( R \) sufficiently large such that \( R > 2C_4/(1 - \gamma) \) (\( C_4 \) is from (5.5)) , then choose \( \alpha \) and \( \nu_0 \) depending on \( R \).
6 Bootstrap argument

By Proposition 4.3, there exists \( C > 1, v_0 > 0 \) and \( Q_n > 0 \) satisfying
\[
C^{-1} \leq \frac{Q_{n+1}}{Q_n} \leq C, \tag{6.1}
\]
such that for \( n \leq N_1(v) = C^{-1}(v \log \frac{1}{v})^{-\frac{1}{3}} \) and \( v \in (0, v_0) \), we have
\[
C^{-1} \leq \frac{\tilde{L}_n(x)}{Q_n} \leq C \chi_v(x). \tag{6.2}
\]

This estimate allows us to apply Corollary 5.3 up to \( n = N_1(v) \). Using this corollary, we would like to bootstrap the estimate (6.2) to arbitrary \( n \). More precisely, we will prove the following:

**Theorem 6.1** There exist constants \( M > 0 \) and \( v_0 > 0 \), and \( Q_n > 0 \) depending only on \( F \), such that
\[
M^{-3} \leq \frac{Q_{n+1}}{Q_n} \leq M^3,
\]
and
\[
M^{-1} \leq \frac{\tilde{L}_n(x)}{Q_n} \leq M \chi_v(x)
\]
hold for any \( n \in \mathbb{N} \) and \( v \in (0, v_0) \).

We first show that Theorem 6.1 implies Theorem 4.1, hence our main theorem.

By Theorem 6.1, Corollary 5.3 applies to all \( n \in \mathbb{N} \) with \( D = M \). We conclude that there exists \( C > 1, \alpha, \alpha_1 \in (0, 1) \) depending on \( F \) and \( M \), such that for \( \beta = \alpha_1 (Cv)^{-1} \) and all \( n \in \mathbb{N} \),
\[
\|P_n\|_{\beta V, *} \leq \alpha^n \|u\|_{\beta V, *}. \tag{6.3}
\]

We state two lemmas on norm estimates.

**Lemma 6.2** Let \( \beta = \alpha_1/(D_1 v) \) for \( \alpha_1 \in (0, 1) \) and \( V = \psi \chi_v^2 \). Then there exists \( C > 1 \) depending only on \( F \) such that:
\[
\|u\|_{\beta V, *} \leq \|u\|_*. \tag{6.4}
\]
and
\[
\|u\|_* \leq C D_1 v^{-d-1} \|u\|_{\beta V, *}. \tag{6.5}
\]

**Proof** Let \( \tilde{u} \) be the constant such that \( \|u\|_* = \|u - \tilde{u}\| \) and let \( \tilde{u}_\beta \) be the constant such that \( \|u\|_{\beta V, *} = \|u - \tilde{u}_\beta\|_{\beta V} \). Then,
\[
\|u\|_{\beta V, *} = \left\| \frac{u - \tilde{u}_\beta}{1 + \beta V} \right\| \leq \left\| \frac{u - \tilde{u}}{1 + \beta V} \right\| \leq \|u - \tilde{u}\| = \|u\|_*. \tag{6.6}
\]

Similarly,
\[
\|u\|_* \leq \|u - \tilde{u}_\beta\| \leq (1 + \beta \|V\|_{C^0}) \|u - \tilde{u}_\beta\|_{\beta V} \leq \|\psi\|_{C^0} D_1 v^{-d-1} \|u\|_{\beta V, *}, \tag{6.7}
\]
since \( \|V\|_{C^0} \leq \|\psi\|_{C^0} v^{-d} \), and \( \beta \leq D_1^{-1} v^{-1} \). \qed
Lemma 6.3 Suppose \( u, v \in C_{\text{per}}(\mathbb{R}^d) \) with \( \min u = a, \min v = b, a, b \geq 1 \). Suppose \( \omega := \max\{\|u\|_*, \|v\|_*\} < \frac{1}{4} \).

\[ \| \log \frac{u}{v} \|_* \leq 4\omega. \]

**Proof** Note that \( 0 \leq u - a \leq 2\|u\|_*, \) and \( 0 \leq v - b \leq 2\|v\|_* \). We have

\[ \frac{u}{v} - \frac{a}{b} = \frac{u(b - v) + (u - a)v}{vb} \leq \frac{u - a}{b} \leq \frac{2\|u\|_*}{b} \leq \frac{a}{b} \cdot 2\|u\|_*, \]

by the same calculation,

\[ \frac{u}{v} - \frac{a}{b} \geq -\frac{a}{b} \cdot 2\|v\|_. \]

We get

\[ \left| \frac{u}{v} \frac{a}{b} - 1 \right| < 2\omega < \frac{1}{2}. \]

Note that \( \log(1 + x) \leq 2|x| \) for all \( |x| < \frac{1}{2}, \) we get

\[ \| \log \frac{u}{v} \|_* \leq \| \log \left( \frac{\frac{u}{v}}{\frac{a}{b}} \right) \| \leq 2 \| \frac{u}{v} \frac{a}{b} - 1 \| \leq 4\omega. \]

\[ \square \]

**Proof of Theorem 4.1** Theorem 6.1 implies for all \( n \in \mathbb{N}, \) and \( v \in (0, v_0), \)

\[ \left\| -\frac{\psi}{2v} + \log \widetilde{\mathcal{L}}^n_v \left( e^{\frac{\psi}{2v}} \right) \right\|_* = \| \log \mathcal{L}_v^n 1 \|_* \leq \log(M^2 Q_n) + \frac{d}{2} \log \frac{1}{v}, \]

where we applied (4.2) in the first equality. It follows that

\[ \| \log \widetilde{\mathcal{L}}^n_v \left( e^{\frac{\psi}{2v}} \right) \|_* \leq \log(M^2 Q_n) + \frac{d}{2} \log \frac{1}{v} + \| \psi \|_*/(2v) \leq C/v, \]

if \( C > \| \psi \|_*/2 + 1 \) and \( v_0 \) is small enough. For a fixed \( v > 0, \) the functions \( \log \widetilde{\mathcal{L}}^n_v \left( e^{\frac{\psi}{2v}} \right) \) are uniformly (in \( n \)) Lipschitz (see for example [7]), and therefore \( \log \widetilde{\mathcal{L}}^n_v \left( e^{\frac{\psi}{2v}} \right) \) has a limit point in \( \| \cdot \|_*, \) which we call \( u^v \) and normalize to min log \( u^v = 0. \)

Suppose \( 0 \leq \log u \leq D/v \) for some \( D > 0. \) By Eq. (6.3) and Lemma 6.2, we have

\[ \| \widetilde{\mathcal{L}}^n u/(\widetilde{\mathcal{L}}^n u) 1 \|_* = \| \mathcal{P}^n u \|_* \leq C v^{-d-1} \| \mathcal{P}^n u \|_{\beta^1 V, *}, \]

\[ \leq C \alpha^n v^{-d-1} \| u \|_{\beta^1 V, *}, \quad \text{(6.8)} \]

Note also \( \widetilde{\mathcal{L}}^n u/(\widetilde{\mathcal{L}}^n u) 1 = \mathcal{P}^n u \geq \mathcal{P}^n 1 = 1. \)

We have

\[ \mathcal{L}^n u_0 = e^{-\frac{\psi}{2v}} \widetilde{\mathcal{L}}^n \left( e^{\frac{\psi}{2v}} u_0 \right). \]

Suppose \( 0 \leq \log u_0, \log v_0 \leq D_1/v \) for some \( D_1 > 0, \) we set \( u = e^{\psi/(2v)} u_0, \ v = e^{\psi/2v} v_0, \) then \( 0 \leq \log u, \log v \leq (D_1 + \| \psi \|_{e^0})/v. \) Denote

\[ \omega_n = C \alpha^n v^{-d-1} e^{(D_1 + \| \psi \|_{e^0})/v}. \]
Lemma 6.2. Set
\[\| \log \frac{\mathcal{L}_v^n u_0}{\mathcal{L}_v^n v_0} \|_* = \| \log \frac{\mathcal{L}_v^n u}{\mathcal{L}_v^n v} \|_* = \| \log \frac{\mathcal{L}_v^n u/\mathcal{L}_v^n v}{\mathcal{L}_v^n v/\mathcal{L}_v^n v} \|_* \leq 4\omega_n\]
where we used Eq. (6.8) and Lemma 6.3. By choosing a larger \(C_1 > \| \psi \|_{C^0}\) if needed, we can ensure \(4\omega_n \leq e^{(D_1+D_2)/\nu} \alpha^n\) for all \(n \geq C_1/v\) and \(v \in (0, v_0)\).

Lemma 6.4. There exists a constant \(C > 1\) such that
\[\mathcal{L}_v^{\chi_v} \leq C \chi_v.\]

Proof. The case \(x \in U\) follows directly from Proposition 8.5 for \(n = 1\). The case \(x \notin U\) is identical to the proof of Item (1), case 2 of Proposition 5.2.

Proof of Theorem 6.1. Let \(C\) be the largest of the constants in Eqs. (6.1), (6.2), Lemma 6.4 and Lemma 6.2. Set \(M = 2C\), and let \(C_1 > 1\), \(\alpha, \alpha_1 \in (0, 1)\) and \(v_1\) be the constants obtained by applying Corollary 5.3 with parameter \(D = M\). Choose \(0 < v_2 \leq v_1\) such that
\[2CC_1M^2v^{-3d/2-1}N_{1(v)} < 1\quad \text{for all } v \in (0, v_2).\]
(This is possible because \(N_{1(v)} = C^{-1}(v \log \frac{1}{v})^{-1}\).
First we show that there exists \(Q_n > 0\) such that for all \(n \in \mathbb{N}\) and \(v \in (0, v_2)\),
\[M^{-1} < \frac{\mathcal{L}_v^n 1}{Q_n} < M \chi_v.\] (6.10)

Fix a \(v \in (0, v_2)\), denote \(N = N_{1(v)}\), we proceed by induction in step size \(N\). Suppose that Eq. (6.10) hold for \(0 \leq n \leq kN\) for a given \(k \geq 1\). The inductive hypothesis holds for \(k = 1\) by Proposition 4.3. Corollary 5.3 implies for all \(0 \leq n \leq kN\),
\[\| P^n u \|_{\beta V, *} \leq \alpha^n \| u \|_{\beta V, *},\]
where \(\beta = \alpha_1/(C_1v)\). Set \(R_n = \min \mathcal{L}_v^n 1\), then if Eq. (6.10) is satisfied, we have
\[\| \mathcal{L}_v^n 1/R_n \|_* \leq \frac{\sup \mathcal{L}_v^n 1}{\min \mathcal{L}_v^n 1} \leq M^2 v^{-d/2}.\] (6.11)

Suppose \(kN < n \leq (k + 1)N\), we have
\[\mathcal{L}_v^n 1 = (\mathcal{L}_v^N 1)P^N (\mathcal{L}_v^{n-N} 1) \leq C \chi_v Q_n R_n^{-N} \| P^n (\mathcal{L}_v^{n-N} 1/R_n-N) \|_*\]
\[\leq C \chi_v Q_n R_n^{-N} (1 + 2 \| P^n (\mathcal{L}_v^{n-N} 1/R_n-N) \|_* )\]
\[\leq C \chi_v Q_n R_n^{-N} \left( 1 + 2CC_1v^{-d-1} \| P^n (\mathcal{L}_v^{n-N} 1/R_n-N) \|_{\beta V, *} \right)\]
\[\leq C \chi_v Q_n R_n^{-N} \left( 1 + 2CC_1v^{-d-1} \alpha^n \| \mathcal{L}_v^{n-N} 1/R_n-N \|_* \right)\]
\[\leq C \chi_v Q_n R_n^{-N} \left( 1 + 2CC_1M^2v^{-3d/2-1} \alpha^n \right) \leq 2MC \chi_v Q_n R_n^{-N}.\]
where in the last line we used Eq. (6.9). The converse is easier since
\[\mathcal{L}_v^n 1 = (\mathcal{L}_v^N 1)P^N (\mathcal{L}_v^{n-N} 1) \geq C^{-1} Q_n R_n^{-N}.\]
Therefore Eq. (6.10) holds with \(Q_n = Q_n R_n^{-N} \).
We now show that the constants chosen satisfies

\[ M^{-3} \leq \frac{Q_{n+1}}{Q_n} \leq M^3. \]

By Lemma 6.4

\[ \tilde{L}_v^{n+1} \mathbf{1} = \tilde{L}_v (\tilde{L}_v^n \mathbf{1}) \leq \tilde{L}_v (MQ_n X_v) \leq CM Q_n X_v. \]

For the lower bound,

\[ \tilde{L}_v^{n+1} \mathbf{1} = \tilde{L}_v (\tilde{L}_v^n \mathbf{1}) \geq M^{-1} Q_n \tilde{L}_v \mathbf{1} \geq C^{-1} M^{-1} Q_n. \]

We now use Eq. (6.10) to get

\[ M^{-1} \leq \frac{\tilde{L}_v^{n+1}}{Q_{n+1}} \leq CM X_v \frac{Q_n}{Q_{n+1}}, \quad M X_v \geq \frac{\tilde{L}_v^{n+1} \mathbf{1}}{Q_{n+1}} \geq C^{-1} M^{-1} \frac{Q_n}{Q_{n+1}}, \]

so

\[ C^{-1} M^{-2} \leq \frac{Q_n}{Q_{n+1}} \leq CM^2. \]

We have concluded the proof of the main theorem with the exception of Proposition 4.3, which we prove in the next two sections.

### 7 Estimate of Hessian matrix

Fix \( x \in D(\psi) \) and \( n \in \mathbb{N} \). Let \( X = (x_{-n}, \ldots, x_{-1}) \in \mathbb{R}^{nd} \), \( x_0 = x \in \mathbb{R}^d \), and denote \( H_{n,x}(X) = \sum_{i=-n}^{-1} h(x_i, x_{i+1}) + \psi(x_{-n}) - \psi(x) \), then

\[
\tilde{L}_v^n \mathbf{1}(x) = (4\pi v)^{-\frac{nd}{2}} \int \cdots \int \exp \left( -\frac{1}{2v} H_{n,x}(X) \right) \ dx_{-n} \cdots dx_{-1}. \tag{7.1}
\]

Due to the definition of \( \psi \), the function \( H_{n,x} \geq 0 \), and the minimum is achieved at \( X^* = (x_{-n}^*, \ldots, x_{-1}^*) \), where \( x_{-k}^* = \pi_1 \Phi^{-k}(x, \nabla \psi(x)) = \tilde{y}(x), k = -n, \ldots, -1 \).

The classical Laplace method (see for example [5]) suggests that if the function \( H_{n,x}(X) \) has a unique global minimum at \( X^* \), then

\[ \mathcal{L}_v^n \mathbf{1}(x) \sim e^{-\frac{1}{2v} H_{n,x}(X^*)} \left( \det D^2 H_{n,x}(X^*) \right)^{-\frac{1}{2}} \left( 1 + o_{v \to 0}(1) \right). \]

In this section we carry out preliminary estimates on the Hessian matrix \( D^2 H_{n,x}(X^*) \).

Denote

\[ A_{n,x}(X) = A_n(x_{-n}, \ldots, x_{-1}) = D^2_{x_{-n}, \ldots, x_{-1}} H_{n,x}(X), \quad A^*_{n,x} = A_n(X^*(x)), \]

we have

\[
\begin{align*}
\mathcal{A}_{n,x}(x_{-n}, \ldots, x_{-1}) &= \\
&= \begin{bmatrix}
I_d + D^2(F + \psi)(x_{-n}) & -I_d \\
-I_d & 2I_d + D^2(F(x_{-n-1})) \\
& \ddots & \ddots & \ddots \\
& & 2I_d + D^2(F(x_{-2})) & -I_d \\
& & -I_d & 2I_d + D^2(F(x_{-1}))
\end{bmatrix}. \tag{7.2}
\end{align*}
\]
The goal of this section is to prove the following estimates:

**Proposition 7.1** There exists \( C, \mu > 1 \), independent of \( n \), such that for all \( x \in U \) and \( n \in \mathbb{N} \),

\[
C^{-1} \mu^n \leq \det A_{n,x}^* \leq C \mu^n \tag{7.3}
\]

**Remark** The exponent \( \mu > 1 \) is related to the hyperbolic fixed point \((0, 0)\) of the dynamics. In fact

\[
\mu = \det (I_d + D^2(F + \psi)(0)).
\]

**Proposition 7.2** There exists \( C > 1 \) such that for \( x \in U \)

\[
A_{n,x}^* \geq C^{-1} I_{nd}. \tag{7.4}
\]

The main idea behind Proposition 7.1 is that for each \( x \), the sequence \( x_{-k}(x) \) converges to 0 exponentially, since they are the \( x \) component of a backward orbit on the unstable manifold of \((0, 0)\). We can represent \( \log \det A_{n,x}^* \) as a sum over the orbit (this connection has already appeared in [2], see also [1]), which becomes a uniformly convergent sum. Proposition 7.2 also exploits this connection.

We need a few lemmas for Proposition 7.1.

**Lemma 7.3** Consider

\[
D_n = \begin{bmatrix}
A_1 & -I_d & & \\
-I_d & A_2 & -I_d & \\
& \ddots & \ddots & \ddots \\
& & -I_d & A_{n-1} & -I_d \\
& & & -I_d & A_n
\end{bmatrix}
\tag{7.5}
\]

where \( I_d, A_i \) are \( d \times d \) matrices. Then

\[
det D_n = det \left[ \begin{bmatrix} A_n -I_d \\ I_d & O_d \end{bmatrix} \begin{bmatrix} A_{n-1} -I_d \\ I_d & O_d \end{bmatrix} \cdots \begin{bmatrix} A_1 -I_d \\ I_d & O_d \end{bmatrix} \right]_{11} \tag{7.6}
\]

where \([\cdot]_{11}\) denote the top left element of the \( 2 \times 2 \) block matrix. Conjugate \( \begin{bmatrix} A_i & -I \\ I & O \end{bmatrix} \) with \( \begin{bmatrix} I & -I \\ O & -I \end{bmatrix} \), (7.6) becomes:

\[
det D_n = det \left[ \begin{bmatrix} A_n -I & I \\ A_n -2I & I \end{bmatrix} \cdots \begin{bmatrix} A_1 -I & I \\ A_1 -2I & I \end{bmatrix} \begin{bmatrix} I \\ I \end{bmatrix} \right]_1 \tag{7.7}
\]

**Proof** We consider the equation

\[
(-\lambda I_{nd} + D_n) \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = 0
\tag{7.8}
\]

Expand in components, we have

\[
\begin{bmatrix} x_i \\ x_{i-1} \end{bmatrix} = \begin{bmatrix} -\lambda I_d + A_i -I_d \\ I_d & O_d \end{bmatrix} \cdots \begin{bmatrix} -\lambda I_d + A_1 -I_d \\ I_d & O_d \end{bmatrix} \begin{bmatrix} x_1 \\ 0 \end{bmatrix}
\tag{7.9}
\]
for \( 2 \leq x \leq n - 1 \), and
\[
\begin{pmatrix}
0 \\
X_n
\end{pmatrix} = \begin{pmatrix}
-I_d + A_n & -I_d \\
I_d & O_d
\end{pmatrix} \cdots \begin{pmatrix}
-I_d + A_1 & -I_d \\
I_d & O_d
\end{pmatrix} \begin{pmatrix} x_1 \\ 0 \end{pmatrix}
\tag{7.10}
\]
Denote
\[
M(\lambda) = \begin{pmatrix}
-I_d + A_n & -I_d \\
I_d & O_d
\end{pmatrix} \cdots \begin{pmatrix}
-I_d + A_1 & -I_d \\
I_d & O_d
\end{pmatrix},
\tag{7.11}
\]
then \((x_1, \ldots, x_n) \in \ker(-\lambda I_{nd} + D_n)\) if and only if \(x_1 \in \ker[M(\lambda)]_{11}\). Let
\[
p^{A_1, \ldots, A_n}(\lambda) = \det(-\lambda I_{nd} + D_n), \quad q^{A_1, \ldots, A_n}(\lambda) = \det[M(\lambda)]_{11},
\]
then these two polynomials have the same degree, roots, and leading coefficients. They must be equal if the roots are simple. We claim that the roots of \(p^{A_1, \ldots, A_n}(\lambda)\) are simple on an open set of \(A_1, \ldots, A_n\). Indeed, consider \(A_1, \ldots, A_n \gg 1\) and such that \(\operatorname{diag}\{A_1, \ldots, A_n\}\) has distinct eigenvalues whose mutual distance are also much larger than 1, then \(D_n\) has distinct eigenvalues robustly. Since \(p^{A_1, \ldots, A_n}(0)\) and \(q^{A_1, \ldots, A_n}(0)\) are polynomials of the coefficients of \(A_1, \ldots, A_n\) and they agree on an open set, they must be equal to each other. \(\Box\)

To study the Hessian along the minimizers, notice
\[
D \Phi(x, v) = \begin{pmatrix}
I_d + D^2 F(x) & I_d \\
D^2 F(x) & I_d
\end{pmatrix}
\tag{7.12}
\]
and
\[
D \Phi(x_{-n}, v_{-n}) \begin{pmatrix} I_d \\ D^2 \psi(x_{-n}) \end{pmatrix} = \begin{pmatrix}
D^2 F(x_{-n}) + D^2 \psi(x_{-n}) & I_d \\
D^2 F(x_{-n}) + D^2 \psi(x_{-n}) - I_d & I_d
\end{pmatrix} \begin{pmatrix} I_d \\ I_d \end{pmatrix}
\tag{7.13}
\]
Apply \((7.7)\) to \((7.2)\), if \((x_i, v_i)_{i=-n}^0\) is an orbit of \(\Phi\) such that \(x_0 = x\), we have
\[
\det A_n(x_{-n}, \ldots, x_{-1}) = \det(D^2_{x_{-n}, \ldots, x_{-1}} H(x_{-n}, \ldots, x_{-1})) = \det \pi_1 D \Phi^a(x_{-n}, v_{-n}) \begin{pmatrix} I_d \\ D^2 \psi(x_{-n}) \end{pmatrix}
\tag{7.14}
\]
Since \(W^a\) is an invariant manifold under \(\Phi\), the tangent plane to \(W^a\) is invariant under \(D \Phi\). Apply this to the orbit \((x_i^a, D \psi(x_i^a))\), we get the plane bundle \(\{(h, D^2 \psi(x_i^a)h) : h \in \mathbb{R}^d\}\) is invariant under \(D \Phi\), i.e.
\[
D \Phi(x_i^a) \begin{pmatrix} h \\ D \psi(x_i^a)h \end{pmatrix} = \begin{pmatrix} h_1 \\ D \psi(x_{i+1}^a)h_1 \end{pmatrix},
\]
where
\[
\begin{pmatrix} h \\ D \psi(x_i^a)h \end{pmatrix} = \begin{pmatrix} I_d + D^2 (F + \psi)(x_i^a) \end{pmatrix} h,
\]
hence for any \(h \in \mathbb{R}^d\),
\[
\pi_1 D \Phi^a(x_{-n}^a) h = \prod_{i=-n}^{-1} (I_d + D^2 (F + \psi)(x_i^a)) h.
\]
It follows that
\[
\det A_n^a(x) = \det \prod_{i=-n}^{-1} (I_d + D^2 (F + \psi)(x_i^a))
\tag{7.15}
\]
Proof (Proof of Proposition 7.1) Denote \( \mu = \det \left( I_d + D^2(F + \psi)(0) \right) \), we have \( \mu > 1 \).
Since \( x_{-n}^* \) converge to 0 exponentially fast as \( n \to \infty \), there exists \( N > 0 \) and constants \( C_1 > 0 \) such that for all \( n \in \mathbb{N} \), we have

\[
\left| \det \left( I_d + D^2(F + \psi)(x_{-n}^*) \right) - \mu \right| < \frac{C_1}{n},
\]

which implies

\[
(\mu - \frac{C_1}{n})^n < \det \mathcal{A}_n^*(x) < (\mu + \frac{C_1}{n})^n.
\]

Therefore,

\[
e^{-C_1 \mu \frac{n}{n}} < \det \mathcal{A}_n^*(x) < e^{C_1 \mu \frac{n}{n}}.
\]

\( \Box \)

To prove Proposition 7.2, we need the following classical result, see [3], for example.

**Lemma 7.4 (Poincaré Separation Theorem)** Let \( A \) be an \( n \times n \) symmetric matrix, \((u_1, \cdots, u_r)\) be an orthonormal set in \( \mathbb{R}^n \), \( r < n \). Define \( B = (u_i^T A u_j) \). Let

\[
\lambda_1(A) \leq \lambda_2(A) \leq \cdots \leq \lambda_n(A)
\]

be the eigenvalues of \( A \) and let

\[
\lambda_1(B) \leq \cdots \leq \lambda_r(B)
\]

be the eigenvalues of \( B \). Then

\[
\lambda_k(A) \leq \lambda_k(B) \leq \lambda_{k+n-r}(A), \quad 1 \leq k \leq r.
\]

**Proof of Proposition 7.2** There exists \( \delta > 0 \) such that \( D^2F(0) \geq \delta \). Since \( F \in C^3 \) and \( \lim_{n \to \infty} x_{-n}^* \to 0 \), there exists some \( N \) such that \( D^2F(x_{-n}^*) \geq \frac{\delta}{2} \) for \( n > N \).

Write

\[
\mathcal{A}_n^* = \begin{bmatrix}
A_n & -I_d \\
-I_d & A_{n-1} & -I_d \\
& \ddots & \ddots & \ddots \\
& & -I_d & A_2 & -I_d \\
& & & -I_d & A_1
\end{bmatrix}
\]

and

\[
B_{N,n} = \begin{bmatrix}
A_n & -I_d \\
-I_d & A_{n-1} & -I_d \\
& \ddots & \ddots & \ddots \\
& & -I_d & A_{N+2} & -I_d \\
& & & -I_d & A_{N+1}
\end{bmatrix}
\]

Then we have

\[
B_{N,n} \geq \begin{bmatrix}
(2 + \frac{\delta}{2})I_d & -I_d \\
-I_d & (2 + \frac{\delta}{2})I_d & -I_d \\
& \ddots & \ddots & \ddots \\
& & -I_d & (2 + \frac{\delta}{2})I_d & -I_d \\
& & & -I_d & (2 + \frac{\delta}{2})I_d
\end{bmatrix}
\]
As a result, the minimum eigenvalue of $B_{N,n}$ is bounded below by a constant independent of $n$.

In the following, we use $C(N, d)$ to denote any constant that depends on $N$ and $d$ but does not depend on $n$.

By Eq. (7.15), we have

$$\frac{\det A_n}{\det B_{N,n}} = \det \prod_{i=-N}^{1} \left( I_d + D^2 F(x_i^{(j)}) + D^2 \psi(x_i^{(j)}) \right),$$

which is bounded below and above by a constant independent of $n$.

Let $\lambda_1 \leq \cdots \leq \lambda_{nd}$ be the eigenvalues of $A_n$ and let $\mu_1 \leq \cdots \leq \mu_{(n-N)d}$ be the eigenvalues of $B_{N,n}$. By (7.22), we have

$$\prod_{i=1}^{nd} \lambda_i \prod_{i=1}^{(n-N)d} \mu_i = \frac{\det A_n}{\det B_{N,n}} \geq C(N, d).$$

By Lemma 7.4, we have

$$\lambda_k \leq \mu_k \leq \lambda_k + Nd.$$  (7.24)

Therefore

$$\lambda_1 \geq C(N, d) \prod_{i=q}^{(n-N)d} \mu_i \prod_{i=1}^{nd} \lambda_j \geq C(N, d) \prod_{i=(n-N)d+1}^{nd} \lambda_j$$

Since $\mu_1$ is bounded from below and $\lambda_i$’s are bounded from above by constants independent of $n$, we have

$$\lambda_1 \geq C(N, d).$$  (7.26)

$\square$

8 Laplace’s method for the partition function

In this section, we prove Proposition 4.3, which establishes the estimate

$$C^{-1} \leq \frac{\tilde{L}_n^\nu 1(x)}{Q_n} \leq C \chi_v(x)$$

for $0 \leq n \leq N_1(\nu)$, where $\chi_v(x) = 1$ for $x \in U$ and $\nu^{-d/2}$ for $x \notin U$. The plan of this section is as follows:

- We first prove some technical lemmas on the function $H_{n,x}$ (Lemma 8.1 to 8.4);
- We then prove Proposition 8.5 which establishes the estimate for $x \in U$;
- After that we give the proof of Proposition 4.3.

**Lemma 8.1** There is a constant $C > 0$ depending only on $F$ such that

$$\int e^{-H_{n,x}(X)} dx_{-n} \cdots dx_{-1} \leq C^{nd}.$$
Proof Let $C_0 = \| F \|_{c^0} + \| \psi \|_{c^0}$, then
\[
\int e^{-H_{n,x}(X)}dX = \int \exp \left( -\sum_{k=-n}^{-1} \frac{1}{2} |x_{k+1} - x_k|^2 + F(x_k) \right) dx_{-n} \cdots dx_{-1}
\]
\[
\leq e^{nC_0} \int \exp \left( -\sum_{k=-n}^{-1} \frac{1}{2} |x_{k+1} - x_k|^2 \right) dx_{-n} \cdots dx_{-1} = e^{nC_0} (2\pi)^{nd/2}.
\]
The lemma follows by setting $C = e^{C_0} (2\pi)^{d/2}$. \hfill \Box

Lemma 8.2 There exists $\delta > 0$ depending only on $F$ such that for all nonzero $l \in \mathbb{Z}^d$, $n \in \mathbb{N}$ and $x \in U,$
\[
H_{n,x}(X^*(x) + l) - H_{n,x}(X^*(x)) = H_{n,x}(X^*(x) + l) > \delta.
\]
Proof By the definition of $U$ (see Proposition 3.3), for every $x \in U$, the minimum $h(\cdot, x)$ is achieved at a unique point $\tilde{y}(x)$. As a result, there exists $\delta > 0$ such that
\[
h(\tilde{y}(x) + q, x) > \delta, \quad \text{for all nonzero } q \in \mathbb{Z}^d.
\]
Let $j = \max\{k : l_k \neq 0, -n \leq k \leq 1\}$, then
\[
H_{n,x}(X^*(x) + l) = \sum_{k=-n}^{-1} \tilde{h}(x^*_k + l_k, x^*_{k+1} + l_{k+1}) \geq \sum_{k=j}^{-1} \tilde{h}(x^*_k + l_k, x^*_{k+1} + l_{k+1})
\]
\[
= \tilde{h}(x^*_j + l_j, x^*_{j+1}) > \delta
\]
where we used $\tilde{h} \geq 0$ and $x^*_j = \tilde{y}(x^*_{j+1})$. \hfill \Box

Lemma 8.3 For any $r > 0$, there exists $\delta > 0$ depending only on $r$ and $F$ such that
\[
\inf \{H_{n,x}(X) : x \in U, \| X - X^*(x) \|_{\infty} > r, n \in \mathbb{N}\} > \delta > 0.
\]
Proof Write $\|X\|_{T,\infty} = \sup_{k=1}^d |x_k|_T$. By Lemma 8.2, it suffices to prove
\[
\inf \{H_{n,x}(X) : x \in U, \| X - X^*(x) \|_{T,\infty} > r, n \in \mathbb{N}\} > \delta.
\]
First of all, there exists $\delta_1 > 0$ such that $\inf_{|x| > r} \psi(x) \geq 2\delta_1$ for all $|x| > r$. Secondly, by Proposition 2.3, $\| T^n1 - \psi \|_* \rightarrow 0$ as $n \rightarrow \infty$, therefore there exists $N \in \mathbb{N}$ such that for all $n \geq N$,
\[
\| T^n1 - \psi \|_* < \delta_1.
\]
Thirdly, since $H_{N,x}(X)$ has a unique minimum at $X^*$, by a compactness argument, for a fixed $N$, there exists $\delta_2 > 0$ depending on $N$ such that
\[
\inf \{H_{N,x}(X) : x \in U, \| X - X^* \|_{T,\infty} > r, X \in (\mathbb{R}^d)^N\} > \delta_2.
\]
Suppose $\| X - X^* \|_{T,\infty} > r$, then there exists $1 \leq m \leq n$ such that $|x_{-m} - x^*_{-m}|_T > r$.
If $m < N$, we write $X_{-N}^{-1} = (x_k)_{k=-N}^{-1}$, then $\| X_{-N}^{-1} - X^* \|_{T,\infty} > r$, hence
\[
H_{n,x}(X) = \sum_{k=-N}^{-1} \tilde{h}(x_k, x_{k+1}) + \sum_{k=-n}^{-N-1} \tilde{h}(x_k, x_{k+1}) \geq H_{N,x}(X_{-N}^{-1}) > \delta_2.
\]
since $\tilde{h} \geq 0$. If $m \geq N$, then
\[
H_{n,x}(X) = -\psi(x) + \sum_{k=-m}^{-1} h(x_k, x_{k+1}) + \sum_{k=-n}^{-m-1} h_{x_k, x_{k+1}} + \psi(x_{-n}) \geq -\|T^n 1 - \psi\|_\ast + \psi(x_{-m}) > 2\delta_1 - \delta_1 > \delta_1.
\]
Take $\delta = \min\{\delta_1, \delta_2\}$ and the lemma follows. \hfill $\square$

**Lemma 8.4** Assume $A_n$ is a symmetric and positive definite $nd \times nd$ matrix, and assume there is a uniform lower bound $\mu_{\text{min}} > 0$ of the smallest eigenvalue. Suppose $C \mu_{\text{min}} > 1$ and $0 < \epsilon < 1/(Cn)$, then
\[
\frac{\det(A_n - \epsilon I_{nd})}{\det A_n} \geq (1 - \frac{1}{C \mu_{\text{min}}})^d \text{ and } \frac{\det(A_n + \epsilon I_{nd})}{\det A_n} \leq e^{\epsilon \mu_{\text{min}}}.
\]  

**Proof** For each $A_n$, there exists invertible matrix $E_n$ such that
\[
B_n = E_n^{-1} A_n E_n
\]
is diagonal. Denote $B_n = \text{diag}(b_{n,j})_{1 \leq j \leq nd}$. Then for $\epsilon < \frac{1}{Cn}$, we have
\[
\frac{\det(A_n - \epsilon I_{nd})}{\det A_n} = \frac{\det(B_n - \epsilon I_{nd})}{\det B_n} = \frac{\prod_{j=1}^{nd} (b_{n,j} - \epsilon)}{\prod_{j=1}^{nd} b_{n,j}} \geq (1 - \frac{1}{C \mu_{\text{min}}}n)^d \geq (1 - \frac{1}{C \mu_{\text{min}}})^d.
\]
The other inequality is similar. \hfill $\square$

**Proposition 8.5** Let $\mu$ be as in Proposition 7.1. There exist constants $\nu_0 > 0$ and $C > 1$ such that for all $n \leq N_1(\nu) = C^{-1}(\nu \log \frac{1}{\epsilon})^{-\frac{1}{2}},$
\[
C^{-1} < \nu \cdot \mu \tilde{\eta}_n \cdot 1(x) \leq C, \quad \text{for all } x \in U.
\]

**Proof** Throughout the proof, the notation $C_k$ denote a constant that is greater than 1 and depends only on $F$.

First we note that $H_{n,x}(x_{-n}, \cdots, x_{-1})$ is $C^3$ if all $x_k \in U$. Using Taylor expansion, we have
\[
H_{n,x}(X) = H_{n,x}(X^\ast) + \frac{1}{2}(X - X^\ast)^T A_{n,x}(X^\ast)(X - X^\ast)
\]  
\[
+ \sum_{|\beta|=3}^{\sum_{k=-n}^{-1}} D^3 F(\xi_k)(x_k - x_k^\ast) - D^3 \psi(\xi_k)(x_k - x_k^\ast)
\]
where $\xi_k$ are intermediate points, and $D^3 F$, $D^3 \psi$ are trilinear forms. When $\|X - X^\ast\|_\infty < r$ we have
\[
\sum_{i=-n}^{-1} |x_i - x_i^\ast|^3 \leq r \sum_{i=-n}^{-1} |x_i - x_i^\ast|^2
\]  
\[
|H_{n,x}(X) - H_{n,x}(X^\ast) - \frac{1}{2}(X - X^\ast)^T A_{n,x}^\ast(X - X^\ast)| \leq C_0 r \|X - X^\ast\|^2
\]  
\[
\text{ and } \quad |H_{n,x}(X) - H_{n,x}(X^\ast) - \frac{1}{2}(X - X^\ast)^T A_{n,x}^\ast(X - X^\ast)| \leq C_0 r \|X - X^\ast\|^2
\]  
\[
\text{ Springer}
where \( C_0 = \frac{1}{6} \max_{y \in T^d} (\|DF^3(y)\| + \|D^3\psi(y)\|) \).

By Proposition 7.2, there exists \( r_0 > 0 \) and \( C_1 > 1 \) depending only on \( F \), such that

\[
H_{n,x}(X) > C_1^{-1} \|X - X^*\|^2_2, \quad \text{if } \|X - X^*\|_\infty < r_0.
\]

(8.6)

By Lemma 8.3, there exists \( \delta > 0 \) such that

\[
H_{n,x}(X) \geq \delta, \quad \text{if } \|X - X^*\|_\infty \geq r_0.
\]

(8.7)

Let \( r_1(n) = \frac{C_2}{C_0} \), where \( C_2 \) is large enough so that Lemma 8.4 applies with \( A = A^*_{n,x} \), \( C = C_1 \) and \( \epsilon = C_1 \rho \) (note that the minimum eigenvalue of \( A^*_{n,x} \) is at least \( C_1^{-1} \) due to (8.6)). Set

\[
S_0 = \left\{ X \in (\mathbb{R}^d)^n : \|X - X^*\|_\infty \leq r_0 \right\},
\]

\[
S_1 = S_1(n) = \left\{ X \in (\mathbb{R}^d)^n : \|X - X^*\|_\infty \leq r_1(n) \right\}.
\]

Denote \( \tilde{X} = X - X^* \), we have the following estimates,

\[
\int_{S_1} \exp \left[ -\frac{1}{2} v \right] H_{n,x}(X) \, dX - n \cdots dX_1 \\
\leq \int_{\|\tilde{X}\|_\infty < r_1} \exp \left[ -\frac{1}{2} v \right] \left( H_{n,x}(X^*) + \frac{1}{2} (\tilde{X})^T (A^*_{n,x} - C_0 r_1 I_n) \tilde{X} \right) \, d\tilde{X} \\
\geq (4\pi)^{n/2} \det(A^*_{n,x} - C_0 r_1 I_n)^{-1/2} \left( \frac{\det(A^*_{n,x} - C_0 r_1 I_n)}{\det(A^*_{n,x})} \right)^{-1/2} \\
\leq C_3 (4\pi)^{n/2} \det(A^*_{n,x})^{-1/2} \leq C_4 (4\pi)^{n/2} \mu^n
\]

(8.8)

for some \( C_3, C_4 > 1 \). In the last line we applied Lemma 8.4 and Proposition 7.1.

We now estimate the same integral from below. Indeed, we have

\[
\int_{S_1} \exp \left[ -\frac{1}{2} v \right] H_{n,x}(X) \, dX \\
\geq \int_{\|\tilde{X}\|_\infty \leq r_1(n)} \exp \left[ -\frac{1}{4} \tilde{X}^T (A^*_{n,x} + C_0 r_1 I_n) \tilde{X} \right] \, d\tilde{X} \\
= (2\nu)^{n/2} \int_{\|V\|_\infty \leq r_1(n)/\sqrt{2\nu}} \exp \left[ -\frac{1}{2} V^T (A^*_{n,x} + C_0 r_1 I_n) V \right] \, dV \\
\geq (4\pi)^{n/2} \det(A^*_{n,x} + C_0 r_1 I_n)^{-1/2} \left( 2\pi \right)^{-n/2} \int_{\|V\|_\infty \leq r_1(n)/C_1 \sqrt{2\nu}} \exp \left[ -\frac{1}{2} V^T V \right] \, dV \\
\geq (4\pi)^{n/2} \det(A^*_{n,x} + C_0 r_1 I_n)^{-1/2} \left( 2\pi \right)^{n/2} \int_{\|V\|_\infty \leq r_1(n)/C_1 \sqrt{2\nu}} \exp \left[ -\frac{1}{2} V^T V \right] \, dV \\
\geq C_5^{-1} (4\pi)^{n/2} \mu^n \left( 1 - C_1 n \sqrt{2\nu} e^{-\frac{1}{2} (C_1^2 n \sqrt{2\nu})^2} \right).
\]

(8.9)

In the last formula, we applied the Gaussian tail bound \( \sqrt{2\pi} \int_{|x| > r} e^{-\frac{1}{2} x^2} < \frac{1}{r \sqrt{2\pi}} e^{-\frac{1}{2} r^2} \).
Suppose $n < \nu^{-\frac{1}{3}}$, we will choose $\nu_0$ small enough depending only on $F$ such that

$$C_1^2 n \sqrt{2} e^{-\frac{1}{2}(C_1^2 n \sqrt{2} \nu)^2} < \frac{1}{2n}.$$ 

Indeed,

$$2C_1^2 n \sqrt{2} e^{-\frac{1}{2}(C_1^2 n \sqrt{2} \nu)^2} < 2 \sqrt{2} C_1^2 \nu e^{-\frac{1}{2}(C_1^2 \sqrt{2} \nu - \delta)^2} < 1$$

if $\nu_0$ is small enough. It follows that

$$\int_{S_1} \exp \left[ -\frac{1}{2} \nu H_{n, x}(X) \right] dx_n \cdots dx_{-1} \geq C_5^{-1} (4\pi \nu)^{nd/2} \nu^n (1 - \frac{1}{2n})^{nd}$$

(8.10)

Summarizing, there exist $C_6 > 1$ such that

$$C_6^{-1} \mu^n \leq \frac{1}{(4\pi \nu)^{nd/2}} \int_{S_1} \exp \left[ -\frac{1}{2} \nu H_{n, x}(X) \right] dx_n \cdots dx_{-1} \leq C_6 \mu^n.$$ 

(8.11)

Moreover, the integral over $S_0^c$ can be estimated as follows:

$$\int_{S_0^c} \exp \left[ -\frac{1}{2} \nu H_{n, x}(X) \right] dx_n \cdots dx_{-1} \leq \exp \left[ (1 - \frac{1}{2}) \delta \right] \int_{\mathbb{R}^d} e^{-H_{n, x}(X)} dx_n \cdots dx_{-1} \leq \exp \left[ (1 - \frac{1}{2}) \delta \right] C^n$$

(8.12)

where the last estimate is due to Lemma 8.1.

We prove our proposition by splitting into two cases.

**Case 1:** $r_1(n) \geq r_0$. In this case $S_0^c \supset S_1^c$, we bound the integral on $\mathbb{R}^{nd}$ by the integral on $S_0^c$ and $S_1$ to get

$$C_6^{-1} \mu^n \leq \frac{1}{(4\pi \nu)^{nd/2}} \int_{S_1} \exp \left[ -\frac{1}{2} \nu H_{n, x}(X) \right] dx_n \cdots dx_{-1}$$

$$\leq C_6 \mu^n + (4\pi \nu)^{-nd/2} \exp \left[ (1 - \frac{1}{2}) \delta \right] C^{nd}.$$

Our conclusion holds if we can show

$$\frac{1}{(4\pi \nu)^{nd/2}} e^{(1 - \frac{1}{2}) \delta} C^{nd} \leq C_6 \mu^n. $$

We assume that $\nu_0$ is small enough such that $\nu_0^{nd/2} (4\pi)^{-nd/2} e^{\delta} C^{nd} \mu^n C_6^{-1} \leq 1$, then it suffice to prove

$$\nu^{-nd} e^{-\delta/(2\nu)} \leq 1, \quad \text{or} \quad nd \log \nu^{-1} \leq \delta.$$ 

which holds if $n \leq \frac{\delta}{2\nu} (\nu \log \frac{1}{\nu})^{-1}$.

**Case 2:** $r_1(n) < r_0$. In this case, we split the integral into three domains:

$$\int_{S_1} + \int_{S_0 \setminus S_1} + \int_{S_0^c}. $$

First we assume the same conditions on $\nu_0$ and $n$ so that

$$(4\pi \nu)^{-nd/2} \int_{S_0^c} \exp \left[ -\frac{1}{2} \nu H_{n, x}(X) \right] dx_n \cdots dx_{-1} < C_7 \kappa^n$$
for some $C_7 > 1$. For the integral on $S_0 \setminus S_1$, we use a similar computation to (8.12) to get

$$\int_{S_0 \setminus S_1} \exp \left[ -\frac{1}{2\nu} H_{n,x}(X) \right] dx_{-n} \cdots dx_{-1} \leq \exp \left[ (1 - \frac{1}{2\nu}) C_0^{-1} r_1^2 \right] C^{nd}.$$ 

We will show that under our conditions

$$(4\pi \nu)^{-nd/2} \exp \left[ (1 - \frac{1}{2\nu}) C_0^{-1} r_1^2 \right] C^{nd} \leq C_6 \lambda^n.$$ 

Indeed, plug in $r_1(n) = 1/(C_2 n)$, we set $v_0$ small enough so that all the $\nu$-independent exponential terms are dominated by $\nu^{-nd/2}$, then it suffices to prove

$$\nu^{-nd} e^{C_0^{-1} n^{-2} (2\nu)^{-1}} \leq 1, \quad \text{or} \quad (nd) \log \frac{1}{\nu} \leq C_0^{-1} n^{-2} \nu^{-1},$$

which holds if $n \leq (C_0^3 d)^{-\frac{1}{2}} (v \log \frac{1}{\nu})^{-\frac{1}{2}}$.

Finally, we note that all our estimates hold if $n \leq C^{-1} (v \log \frac{1}{\nu})^{-\frac{1}{2}}$ with $C = \max\{ (C_0^3 d)^{\frac{1}{2}}, \frac{\delta}{2\nu} \}$, and $v_0$ sufficiently small depending only on $F$. \hfill \qed

**Proof** (Proof of Proposition 4.3) In this proof, all constants $C_k$ depend only on $F$.

It suffices to prove the estimate for $x \notin U$. Let $C$ be the constant in Proposition 8.5. For $n \leq N_1(\nu) = C^{-1} (v \log \frac{1}{\nu})^{-\frac{1}{3}}$, set

$$M^n_v = \mu^{-n} v^\frac{d}{2} \sup_{x \notin U} \tilde{L}^n_v(x).$$

First of all, if $x \notin U$ and $\nu < 1$,

$$\tilde{L}^n_v 1(x) = (4\pi \nu)^{-\frac{d}{2}} \int e^{-\frac{1}{2\nu} \tilde{h}(y,x)} dy \leq (4\pi \nu)^{-\frac{d}{2}} \int e^{-\frac{1}{2} \tilde{h}(y,x)} dy \leq C_1 \nu^{-\frac{d}{2}}$$

for some $C_1 > 1$ by Lemma 8.1 (for $n = 1$). We conclude that

$$M^1_v \leq C_1 \mu^{-1}.$$ 

On the other hand, (recall $\tilde{K}_v(y, x) = (4\pi \nu)^{-\frac{d}{2}} e^{-\tilde{h}(y,x)/(2\nu)}$),

$$\tilde{L}^n_v 1(x) = \int \tilde{K}_v(y, x) \tilde{L}^{n-1}_v 1(y) dy = \int_U + \int_{U^c}.$$ 

By Proposition 8.5,

$$\mu^{-n} \int_U \tilde{K}_v(y, x) \tilde{L}^{n-1}_v 1(y) dy \leq C \mu^{-n} \int_U \tilde{K}_v(y, x) dy \leq C \mu^{-n} \tilde{L}_v 1(x) \leq C C_1 \mu^{-1} \nu^{-\frac{d}{2}}.$$ 

By Proposition 3.3,

$$\mu^{-n} \int_{U^c} \tilde{K}_v(y, x) \tilde{L}^{n-1}_v 1(y) dy \leq \mu^{-n} (4\pi \nu)^{-\frac{d}{2}} \int_{U^c} e^{-\frac{\tilde{h}(y,x)}{2\nu}} \tilde{L}^{n-1}_v 1(y) dy$$

$$\leq \mu^{-1} (4\pi \nu)^{-\frac{d}{2}} \nu^{-d} \int_{U^c} e^{-\delta \frac{1}{2\nu} - \frac{\tilde{h}(y,x)}{2\nu}} M^{n-1}_v dy \leq C_2 \mu^{-1} \nu^{-d} e^{-\frac{\delta}{2\nu} M^{n-1}_v}$$

for some $C_2 > 1$. Suppose $v_0$ is small enough such that $C_2 \mu^{-1} \nu^{-d} e^{-\frac{\delta}{2\nu} M^{n-1}_v} < \frac{1}{2}$, then

$$M^n_v \leq C C_1 \mu^{-1} + \frac{1}{2} M^{n-1}_v.$$
which implies $M_n^u \leq 2CC_1\mu^{-1}$ for all $n \leq N(v)$.

For the lower bound, let $x \in \mathbb{R}^d$ and assume $\bar{y} \in \text{arg} \min h(\cdot, x)$. Using the semi-concavity of $\psi$, there is $C_3 > 0$ such that $h(y, x) \leq C_3|y - \bar{y}|^2$ for all $y \in \mathbb{R}^d$. Then

$$\mu^{-n}L^n_\nu(x) \geq \mu^{-n} \int_U \tilde{K}^n_\nu(y, x)\tilde{L}^{n-1}_\nu(y)dy \geq C^{-1} \mu^{-1} \int_U \tilde{K}^n_\nu(y, x)dy$$

$$\geq C^{-1} \mu^{-1} (4\pi \nu)^{-\frac{d}{2}} \int_{y: |y - \bar{y}| \leq \nu^2} e^{-C_3|y - \bar{y}|^2/(2\nu)}dy$$

$$= C^{-1} \mu^{-1} (2\pi)^{-\frac{d}{2}} \int_{|y| \leq 1} e^{-C_3|y|^2}dy = C^{-1} \mu^{-1} C_4$$

for some $C_4 > 1$. The Proposition follows. \qed

Acknowledgements K.K. is supported by NSERC fund RGPIN-2018-04510. K.Z. is supported by NSERC fund RGPIN-2019-07057. L.Z. is supported by Fundamental Research Funds for the Central Universities (No. DUT20RC(3)098) and NSFC grant (No. 12201099).

Appendix A Exponential convergence of the Markov chain

For the reader’s convenience, we record here the proof of Theorem 5.1 as given in Ref. [8], which is short and elementary.

Lemma A.1

$$\|u\|_{\beta V, *} = \sup_{x,y} \frac{|u(x) - u(y)|}{2 + \beta V(x) + \beta V(y)}. \quad (A.1)$$

Proof Let’s denote the right hand side by $B(u)$, we have

$$|u(x) - u(y)| = \inf_C \left( |u(x) + C| - |u(y) + C| \right) \leq \inf_C \|u + C\|_{\beta V} \left( 2 + \beta V(x) + \beta V(y) \right),$$

hence $B(u) \leq \|u\|_{\beta V, *}$. Conversely, let $c = \inf_y (B(u) + \beta B(u)V(y) - u(y))$. On one hand, $c \leq B(u)(1 + \beta V(x)) - u(x)$, hence $u(x) + c \leq B(u)(1 + \beta V(x))$. On the other hand,

$$u(x) + c = \inf_y (u(x) - u(y) + B(u) + \beta B(u)V(y))$$

$$\geq \inf_y \left( B(u)(1 + \beta V(y)) - B(u)(2 + \beta V(x) + \beta V(y)) \right)$$

$$\geq -B(u)(1 + \beta V(x)).$$

We get $\|u\|_{\beta V, *} \leq \|u(x) + c\|_{\beta V} \leq B(u)$. \qed

Proof of Theorem 5.1 By Lemma A.1, it suffices to prove

$$|\mathcal{P}(u)(x) - \mathcal{P}(u)(y)| \leq \alpha \left( 2 + V(x) + V(y) \right)$$

for all $\|u\|_{\beta V, *} \leq 1$. Moreover, by adding a constant, it suffices to consider $\|u\|_{\beta V} \leq 1$, hence $u(x) \leq 1 + \beta V(x)$ for all $x$.

If $V(x) + V(y) \geq R$, we have

$$\mathcal{P}(u)(x) - \mathcal{P}(u)(y) \leq \mathcal{P}(1 + \beta V(x)) + \mathcal{P}(1 + \beta V(y))$$

$$= 2 + \beta \gamma \mathcal{P}(V)(x) + \beta \gamma \mathcal{P}(V)(y) + 2\beta M$$

$$\leq 2 + \beta(\gamma + 2M/R)(V(x) + V(y)) < 2 + \beta \gamma_0(V(x) + V(y)).$$
Set $\gamma_1 = \frac{2+R\beta\gamma_0}{2+R\beta}$, we check that $1-\gamma_0 = R\beta(\gamma_1 - \gamma_0)$. Then

$$\mathcal{P}(u)(x) - \mathcal{P}(u)(y) < 2\gamma_1 + 2(1-\gamma_1) + \beta\gamma_0(V(x) + V(y))$$

$$\leq 2\gamma_1 + \beta R(\gamma_1 - \gamma_0) + \beta\gamma_0(V(x) + V(y))$$

$$\leq \gamma_1(2 + \beta V(x) + \beta V(y)).$$

where we used $V(x) + V(y) \geq R$ in the last line.

If $V(x) + V(y) \leq R$, we define a new Markov operator

$$\tilde{\mathcal{P}}(u) = \frac{1}{1-\alpha_0} \int (\pi(y,x) - \alpha_0 g(y))u(y)dy.$$ 

Note that $\mathcal{P}(u) = (1 - \alpha_0)\tilde{\mathcal{P}}(u) + \alpha_0 \int g(y)u(y)dy$. Then

$$\mathcal{P}(u)(x) - \mathcal{P}(u)(y) = (1 - \alpha_0)\left(\tilde{\mathcal{P}}(u)(x) - \tilde{\mathcal{P}}(u)(y)\right)$$

$$\leq (1 - \alpha_0)\left(\tilde{\mathcal{P}}(1 + \beta V)(x) - \tilde{\mathcal{P}}(1 + \beta V)(y)\right)$$

$$= (1 - \alpha_0)\left(2 + \beta \tilde{\mathcal{P}}(V)(x) + \beta \tilde{\mathcal{P}}V(y)\right)$$

$$\leq 2(1 - \alpha_0)\beta \gamma(V(x) + V(y)) + 2\beta M$$

$$\leq 2(1 - \alpha_0 + 2\alpha_1) + \beta \gamma(V(x) + V(y)).$$

where in the fourth line, we used $\mathcal{P}(V) \geq (1 - \alpha_0)\tilde{\mathcal{P}}(V)$ (since $V \geq 0$). Finally, recall that $\alpha = \max\{1 - \alpha_0 + \alpha_1, \gamma_1\}$ and $\gamma_1 > \gamma$, we get our conclusion.

\[\square\]

References

1. Anantharaman, N.: On the zero-temperature or vanishing viscosity limit for certain Markov processes arising from Lagrangian dynamics. J. Eur. Math. Soc. 6(2), 207–276 (2004)
2. Aubry, S., MacKay, R., Baesens, C.: Equivalence of uniform hyperbolicity for symplectic twist maps and phonon gap for Frenkel–Kontorova models. Phys. D: Nonlinear Phenom. 56(2–3), 123–134 (1992)
3. Bellman, R.: Introduction to Matrix Analysis: Second Edition. Classics in Applied Mathematics. Soc. Ind. Appl. Math. (1997)
4. Bernard, P.: The dynamics of pseudographs in convex Hamiltonian systems. J. Am. Math. Soc. 21(3), 615–669 (2008)
5. de Bruijn, N.G.: Asymptotic methods in analysis, 3rd edn. Dover Publications Inc., New York (1981)
6. Fathi, A.: Weak KAM theorem in Lagrangian dynamics, 10th preliminary version. book preprint (2008)
7. Gomes, D., Iturriaga, R., Khanin, K., Padilla, P.: Viscosity limit of stationary distributions for the random forced Burgers equation. Mosc. Math. J 5(3), 613–631 (2005)
8. Haidier, M., Mattingly, J.C.: Yet another look at Harris’ ergodic theorem for Markov chains. In: Seminar on stochastic analysis, random fields and applications VI, pp. 109–117. Springer (2011)
9. Iturriaga, R., Khanin, K.: Burgers turbulence and random Lagrangian systems. Commun. Math. Phys. 232(3), 377–428 (2003)
10. Iturriaga, R., Khanin, K., Zhang, K.: Exponential convergence of solutions for random Hamilton–Jacobi equations. Stoch. Partial Diff. Equ. Anal. Comput. 8(3), 544–579 (2019)
11. Iturriaga, R., Sánchez-Morgado, H.: Hyperbolicity and exponential convergence of the Lax–Oleinik semigroup. J. Diff. Equ. 246(5), 1744–1753 (2009)
12. Katok, A., Hasselblatt, B.: Introduction to the modern theory of dynamical systems. Cambridge University Press, Cambridge (1995)
13. Khanin, K., Zhang, K.: Hyperbolicity of minimizers and regularity of viscosity solutions for a random Hamilton–Jacobi equation. Commun. Math. Phys. 355(2), 803–837 (2017)
14. Sinai, Y.G.: Two results concerning asymptotic behavior of solutions of the Burgers equation with force. J. Stat. Phys. 64(1–2), 1–12 (1991)
15. Weinan, E., Khanin, K., Mazel, A., Sinai, Y.: Invariant measures for Burgers equation with stochastic forcing. Ann. Math. 151(3), 877–960 (2000)

Publisher’s Note  Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.