MORE ON SUPERSYMMETRIC TENSIONLESS ROTATING STRINGS IN $AdS_5 \times S^5$*

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Rotating IIB strings in $AdS_5 \times S^5$ become ultra-relativistic, and hence effectively tensionless, in the limit of large angular momentum on $S^5$. We have shown previously that such tensionless strings may preserve supersymmetry. Here we extend this result to include a class of supersymmetric tensionless strings with arbitrary $SO(6)$ angular momentum. We close with some general comments on tensionless strings in $AdS_5 \times S^5$.

1. Introduction

A recent advance in our understanding of the AdS/CFT correspondence has come about via consideration of classical, closed IIB strings in $AdS_5 \times S^5$ that rotate within $S^5$ and thus carry $SO(6)$ momenta [1, 2]. In the limit...
of large angular momentum \( J \) (precisely, for \( J \gg 1, \sqrt{\lambda} \), where \( \sqrt{\lambda} \) is the string tension) the energy of the string is given by the classical result, and this admits an expansion in positive integer powers of \( \lambda/J^2 \). This yields a prediction for the dimension of the dual operator, with corresponding large \( SO(6) \) R-charge, in perturbative \( N = 4 \) super Yang-Mills (SYM) theory. In the planar limit, this dimension can be computed by spin chain techniques [3] (or by means of an equivalent sigma model [4]) and a precise agreement with the string theory predictions has been found; remarkably, both routes lead to the same integrable model [5, 6, 7, 8].

The cases that were first studied, and which are still best understood, are those for which the \( SO(6) \) angular momentum two-form has only two independent skew-eigenvalues. It was argued in [1, 2] that the agreement between the string and SYM calculations was, in contrast to previous quantitative tests, evidence for the AdS/CFT conjecture in a ‘non-supersymmetric sector’. However, as we pointed out in a previous paper [9], the rotating strings under consideration are effectively tensionless for large angular momentum because almost all of the energy is due to the rotation, and the states of tensionless strings in the same \( SO(6) \) representation preserve either \( 1/4 \) or \( 1/8 \) supersymmetry. Thus, the limit in which the AdS/CFT prediction can be checked by semiclassical, perturbative techniques on both sides of the correspondence is a limit in which supersymmetry is partially restored.

More recently, predictions of the AdS/CFT correspondence have been verified for a larger class of rotating strings for which the \( SO(6) \) angular momentum has three independent skew-eigenvalues [10]. One finds that the energies, like the corresponding anomalous dimensions on the \( N = 4 \) SYM side of the correspondence, are again determined by the solution to an integrable model, but of a more general type. Here we shall show, extending our previous result, that these rotating strings are again supersymmetric in the ultra-relativistic, and hence tensionless, limit. In fact, the tensionless strings are supersymmetric for \( \text{arbitrary} \) \( SO(6) \) angular momentum, generically preserving \( 1/8 \) supersymmetry.

This paper is written with the intention that it be read in conjunction with [9]. We will begin by presenting the tensionless version of the general ‘three-spin’ rotating string solution of [10]. We then present our main new result, which is a demonstration that these tensionless strings generically preserve \( 1/8 \) supersymmetry. We close with some further comments on tensionless strings in \( AdS_5 \times S^5 \).
2. Null rotating strings on $S^5$

Since the strings of interest here lie at the origin of $AdS_5$, the relevant part of the $AdS_5 \times S^5$ metric is

$$ds^2 = -dt^2 + d\theta^2 + \sin^2 \theta d\phi^2 + \sum_{i=1}^{3} a_i^2 (d\alpha_i)^2,$$

where $a_1 = \cos \theta$, $a_2 = \sin \theta \cos \phi$, $a_3 = \sin \theta \sin \phi$, and $\alpha_i$ are polar angles in three orthogonal planes. In order to avoid coordinate singularities, we will assume here that $a_1 a_2 a_3$ is non-zero, which corresponds to the assumption that the angular momentum two-form has maximal rank.

Let $(\tau, \sigma)$ be the worldsheet coordinates. In the gauge $t = \tau$ the phase-space form of the Lagrangian density for a tensionless string in the above background is

$$L = p_{\theta} \dot{\theta} + p_{\phi} \dot{\phi} + \sum_i p_i \dot{\alpha}_i - H - s \left( p_{\theta} \dot{\theta}' + p_{\phi} \dot{\phi}' + \sum_i p_i \alpha_i' \right),$$

where $s$ is the Lagrange multiplier for the string reparametrization constraint, and $H$ is the Hamiltonian density

$$H = \sqrt{p_{\theta}^2 + \frac{p_{\phi}^2}{\sin^2 \theta} + \sum_i \frac{p_i^2}{a_i^2}}.$$ 

Note that the $\alpha_i$ equation of motion is $\dot{\alpha}_i = (sp_i)'$, from which it follows that the angular momentum, $J_i = \oint d\sigma p_i$, is a constant of motion.

We seek solutions for which $\theta$ and $\phi$ are constant and $p_{\theta} = p_{\phi} = 0$. This requires, in order to solve the corresponding equations of motion, that we set

$$a_i^2 = |p_i|/\sum_j |p_j|,$$

from which we see that

$$H = \sum_i |p_i|.$$ 

Given (4), the $p_i$ equations of motion reduce to $\dot{\alpha}_i - sa_i' = 1$, while the constraint imposed by $s$ is $\sum_i p_i \alpha_i' = 0$. We may choose a gauge for which $s = 0$, in which case the equations above have the solution\(^a\)

$$\alpha_i = t + m_i \sigma, \quad p_i = J_i/2\pi,$$

\(^a\)This is not the unique solution, but it is the one of relevance when one considers the tensionless string as a limit of the tensionful string.
for integers $m_i$ satisfying $\sum_i m_i J_i = 0$. Because each $p_i$ has a definite sign, integration of (5) yields the total energy
\[ E = |J_1| + |J_2| + |J_3|. \] (7)
An argument analogous to that of [9] shows that this energy saturates a BPS bound implied by the $PSU(2, 2|4)$ supersymmetry algebra of $AdS_5 \times S^5$. Because of this, the rotating strings are supersymmetric. Since we have assumed that all $J_i$ are non-zero, the fraction of supersymmetry preserved is 1/8, as we will confirm in the following section. If only two $J$’s are non-zero, then 1/4 supersymmetry is preserved. And if only one $J$ is non-zero, then 1/2 supersymmetry is preserved.

3. Supersymmetry
The condition for a IIB superstring to be supersymmetric in the ultra-relativistic limit in which it becomes null is
\[ p_M e^M_A \Gamma^A \epsilon = 0, \] (8)
where $p_M$ is the ten-momentum, $e^M_A$ is the obvious orthonormal frame associated to the metric (1), $\Gamma^A$ are the corresponding tangent-space (constant) ten-dimensional Dirac matrices, and $\epsilon$ is a chiral, complex Killing spinor of the $AdS_5 \times S^5$ background. This spinor can be written in terms of a constant spinor $\epsilon_0$ as
\[ \epsilon = e^{\frac{i}{4} \tilde{\Gamma}^2 \theta} e^{\frac{i}{4} \tilde{\Gamma}^3 \phi} e^{\frac{i}{2} \alpha_1 \Gamma_{\alpha_2 \alpha_3} + \alpha_2 i \Gamma_{\beta \gamma} + \alpha_3 i \Gamma_{\delta \epsilon}} \epsilon_0, \] (9)
where $\tilde{\Gamma}$ commutes with all other matrices in the problem and $\Gamma_i \equiv \Gamma_{\alpha_i}$. It is clear from this expression that the matrices occurring inside the brackets in the last exponential generate, when acting on $\epsilon_0$, rotations in each of the three orthogonal two-planes parametrised by $\alpha_i$. Note that they all square to unity (so they have eigenvalues ±1), they are mutually-commuting, and the product of any two of them yields the third one, up to a sign.

For the rotating string solutions of the previous section, the supersymmetry condition (8) reduces to
\[ a_i \Gamma_i \epsilon = \epsilon, \] (10)
where we have assumed, for definiteness, that all $J_i$ are positive. If the angular momentum two-form has maximum rank then all $a_i$ are non-zero, and equation (10) is equivalent to
\[ i \Gamma_{2 \theta} \epsilon_0 = \epsilon_0, \quad i \Gamma_{3 \phi} \epsilon_0 = \epsilon_0, \quad \Gamma_{t \epsilon} \epsilon_0 = \epsilon_0. \] (11)
Let us first show that (11) implies (10). We see from the first two conditions in (11) that \( \epsilon_0 \) is an eigen-spinor of the last exponential in equation (9), so this exponential cancels on both sides of (10). The first exponential also cancels because \( \tilde{\Gamma} \) commutes with all other matrices in equation (10), so the latter may be rewritten as

\[
a_i \Gamma_{i\ell} \epsilon_0 = e^{-\frac{2}{\lambda} \Gamma_{\phi\phi}} e^{i \Gamma_{\phi 123}} e^{\frac{2}{\lambda} \Gamma_{\phi\phi}} \epsilon_0 = (a_1 + a_2 i \Gamma_{\phi 123} + a_3 i \Gamma_{123\phi}) \epsilon_0 ,
\]

which is identically satisfied by virtue of (11).

With some further algebra, it can also be shown that (10) implies (11), but we will not do this here. Instead, we note that, being mutually-commuting projections, the conditions (11) imply that the fraction of preserved supersymmetry is 1/8, as expected for a string carrying three independent angular momenta. We also recall that, as observed above, they imply that \( \epsilon_0 \) is invariant (up to a phase) under rotations in the \( \alpha_i \)-planes associated to the non-zero components of the angular momentum two-form.

4. Discussion

There are now many string solutions in \( AdS_5 \times S^5 \) whose classical energies have been matched, in the large-energy limit, to anomalous dimensions of composite operators, with a large number of fields, in perturbative \( N = 4 \) SYM theory. All these strings become ultra-relativistic in this limit, and hence effectively tensionless, in the sense that their kinetic energy, \( K \), is much larger than the energy associated to their tension; their classical energy then admits an expansion in positive powers of \( \lambda/K^2 \).b

This property is, of course, necessary for a comparison to perturbative SYM to be possible, and is a typical property of large-energy strings moving on \( S^5 \), as opposed to \( AdS_5 \). The reason is that the size of a generic string moving on \( S^5 \) is bounded above by the radius of the sphere, so in this case the kinetic energy always dominates in the large-energy limit. On the contrary, the size of a string moving in \( AdS_5 \), and hence the contribution of its tension to its energy, can grow without bound. In other words, the large-energy limit for generic strings moving in \( AdS_5 \) is not an ultra-relativistic limit. It is therefore not surprising that the energies of these strings cannot be precisely matched to dimensions of SYM operators computed within perturbation theory.

Almost all known strings that become tensionless in the large-energy limit also become supersymmetric, as we have shown in this article for the

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bSome aspects of these ultra-relativistic strings have been studied in [11].
class of rotating string solutions of [10] with generic $SO(6)$ angular momenta. Of course, in all these cases the energy saturates a BPS bound in the large-energy limit, $E = K = |J_1| + |J_2| + |J_3|$, and so this limit is equivalent to a large-angular-momentum limit. The only apparent exception that we are aware of is a pulsating string solution, for which the classical energy matches an anomalous dimension of a SYM operator computed in perturbation theory despite the fact that this energy is not close to saturating a BPS bound [8]. All string solutions for which it has been checked that the sigma-model quantum corrections to the classical energy vanish in the large-energy limit [1] are also those that happen to be supersymmetric in this limit [9]. The match for the pulsating string is therefore mysterious, since it relies on the neglect of these corrections in the absence of supersymmetry. This might suggest that the key property behind the success of these tests of the AdS/CFT conjecture is the ultra-relativistic nature of the corresponding strings, rather than the nearly-supersymmetric property emphasised in our previous paper.

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