Analytical and Numerical Study of Biaxial Bending on VHCF Testing Machine

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Abstract. Modern VHCF testing regimes are studied. A new loading scheme applicable to a piezoelectric fatigue testing machine is proposed. The scheme is studied analytically and numerically. The analytical part comprises the calculation of the specimen’s geometry to operate correctly on the VHCF testing machine, as well as the specimen’s stress-strain state calculation during an experiment in order to determine the one’s durability. A model based on a kinetic damage equation is introduced. This model is used in the numerical experiment. The behavior of a finite-element specimen in a fatigue experiment is calculated from the beginning up to the stage of active crack growth.

1. Introduction

1.1. General information
As Japanese researchers [1, 2] showed in the mid-1980s, structural materials may be fractured even at stress levels below the classical fatigue limit while subjected to $10^8$ cycles or more. In the following years the French researcher C. Bathias and his colleagues experimentally approved the concept of very-high cycle fatigue [3]. Traditional fatigue testing methods such as servo hydraulic (maximum loading frequency ~ 35 Hz), electrodynamic or electromagnetic test sets (maximum frequency ~ 100 Hz) are not suitable for very-high cycle fatigue (VHCF), as the test time required to conduct research on $10^9-10^{10}$ cycles is too long (from 1 to 3 years). Thus, the study of fatigue failure of structural materials in the VHCF mode requires quicker testing methods, such as ultrasonic fatigue tests [4].

Subsurface crack initiation is one of the VHCF features. This property allows identifying the cases of fatigue failure of various structural elements subjected to high-frequency loads as failed by the mechanism of VHCF [5]. Disks and blades of gas turbine engines and gearboxes are good examples of such elements.

Various loading conditions for real structural elements require the development of ultrasonic fatigue testing machines capable of operating in such modes as tension-compression, bending, fretting-corrosion and torsion [6]. Some of these testing machines are already on the market (e.g. tensile-compression and torsion testing systems, figure 1). Others are still under development (bending, biaxial loading). This article is devoted to a new VHCF test system.

1.2. On fatigue testing machines
All loading fatigue machines to study the VHCF mode are based on a single principle – the use of standing elastic waves to form deformation fields [7]. For a long period of time, this principle limited the available modes of loading by such machines to a uniaxial tension-compression regime with the coefficient of asymmetry of the cycle equal to -1, lately expanded to a uniaxial pure torsion regime.
The frequency of ultrasonic fatigue tests varies from 15 kHz to 30 kHz with the standard frequency of 20 kHz [7]. A specimen is designed to resonate at this frequency. All mechanical parts of the ultrasonic machine for fatigue tests, including the specimen, resonate at the same loading frequency.

Load parameters are defined and continuously controlled using special software. Oscillations are generated by a piezoelectric element. It converts an ultrasonic sinusoidal electrical signal into mechanical oscillation at the same frequency. These oscillations form a standing elastic wave inside the mechanical components of the machine (i.e. in the horn, amplifier and specimen).

2. Biaxial bending

2.1. Part where some math happens

Nowadays, with the development of numerical methods and 3D modeling, it has become possible to calculate and implement new loading modes. When modeling new modes, it is often necessary to optimize the forms of loading and loading elements (horns and specimen). Resonant lengths of such specimens can be estimated both analytically and numerically. The specimen is supported at the points of standing wave nodes formation, while the loading punch should be located in the central part of the specimen. In the case of a one-dimensional three-point bend, a rectangular bar acts as the specimen. To obtain a two-dimensional bend, a disk or plate is used, figure 2.

In order to determine the geometry of a specimen, the stress-strain state and evaluate durability in the biaxial loading we solved the problem of resonant bending vibrations of the circular plate with intermediate circular support.

The loading scheme is shown in figure 3, where $R$ is the external radius of a plate and $R_n$ is the radius of the intermediate circular support, located at one of the resonance nodes of the specimen.

According to [8, 9], the equation of dynamic axisymmetric bending of the circular plate may be written as (1) where $\zeta(r,t)$ stands for flexural displacement of the centerline of the plate, $\Delta = \partial^2 / \partial r^2 + r^{-1} \partial / \partial r$ is the axisymmetric Laplace operator in polar coordinates, $\rho$ is the density, $E$ is the Young’s modulus, $\nu$ is the Poisson’s ratio, $h$ is the plate thickness, $q$ is the distributed vertical force, $D = Eh\left[12(1-\nu^2)\right]^{-1}$ is the plate stiffness.

$$\Delta^2 \zeta + \frac{12 \rho (1-\nu^2)}{Eh^2} \frac{\partial^2 \zeta}{\partial r^2} = \frac{q}{D}$$

(1)
The flexural displacement to describe harmonic oscillation is 
\[ w(r,t) = w(0)e^{\omega t}, \]
where \( \omega = 2\pi f \), \( f \) is the oscillation frequency, \( w(0) = W_0 \) is the amplitude value. Let’s denote \( k = \omega^2/2(1 - \nu^2)/Eh^2 \). To achieve high displacements, we have to find resonance regime of a plate where \( q = 0 \). With these substitutions equation (1) transforms into much simpler equation (2), which may be divided into two equations with harmonic operator as (3).

\[ \Delta^2 w - k^2 w = 0 \]
\[ \Delta w + k^2 w = 0, \; \Delta w - k^2 w = 0 \]

Using Bessel functions, a solution of equation (3) may be (4).

\[ w(r) = C_1 J_0(kr) + C_2 N_0(kr) + C_3 I_0(kr) \]

As \( r \to 0 \) Neumann and Macdonald functions \( N_0, K_0 \to \infty \) hence \( C_2 = 0 \) and \( C_3 = 0 \). Thus, the solution from (4) simplifies to (5).

\[ w(r) = C_1 J_0(kr) + C_3 I_0(kr) \]

In this solution \( J_0(kr) \) is the Bessel function of the first kind and the solution for \( \Delta w = -k^2 w; I_0(kr) \) is the modified Bessel function of the first kind and the solution for \( \Delta w = k^2 w \). Boundary conditions for displacement \( w \), torque \( M \) and cutting force \( Q \) are expressed by (6). Equation for torque \( M_\theta \) is (7).

\[
\begin{align*}
  w &= W_0 & \quad & \text{at } r = 0 \\
  w &= 0 & \quad & \text{at } r = R_0 \\
  M_\theta &= -D \left( \frac{\partial^2 w}{\partial r^2} + \frac{v}{r} \frac{\partial w}{\partial r} \right) = -D \left( \Delta w - \frac{(1 - \nu)}{r} \frac{\partial w}{\partial r} \right) = 0 & \quad & \text{at } r = R \\
  Q &= -D \frac{\partial}{\partial r} \left( \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} \right) = -D \frac{\partial}{\partial r} (\Delta w) = 0 & \quad & \text{at } r = R \\
  M_\theta &= -D \left( \frac{\nu}{r} \frac{\partial w}{\partial r} + \frac{(1 - \nu)}{r} \frac{\partial w}{\partial r} \right) = -D \left( \nu \Delta w + \frac{(1 - \nu)}{r} \frac{\partial w}{\partial r} \right)
\end{align*}
\]

We can substitute the solution (5) into torque expressions (6) and (7) to obtain more explicit formulae for torques:

**Figure 2.** Schematic overview of a bi-axial bending machine.

**Figure 3.** Estimated loading scheme for bi-axial bending tests.
Let’s study the resonance regime of the plate. In this regime, frequency of one of the oscillation modes is equal to loading frequency. Boundary conditions (6) may be rewritten as follows:

\[ C_1 [I_0(kR) - (1 - \nu) I_1(kR)] / [I_1(kR)] - C_2 [I_0(kR) - (1 - \nu) I_1(kR)] / [I_1(kR)] = 0 \]

\[ C_1 I_1(kR) + C_2 I_1(kR) = 0 \]

The determinant of this system must be zero:

\[ [I_0(kR) - (1 - \nu) I_1(kR)] / [I_1(kR)] \]

Here are two independent values, thereby either \( R \) can be found at given \( k \) or vice versa.

\[ C_1 + C_2 = W_0, \quad C_1 I_1(kR) + C_2 I_1(kR) = 0 \] \( \text{(8)} \)

From the system of equations (8) we can find the constants:

\[ C_1 = W_0 \left( I_1(kR) / (I_1(kR) - J_1(kR)) \right), \quad C_2 = -W_0 \left( I_1(kR) / (I_1(kR) - J_1(kR)) \right) \]

The equation \( R_0 \) is:

\[ C_1 I_0(kR) + C_2 I_0(kR) = 0 \]

Equations for \( R \) and \( R_0 \) may be solved numerically, but while value \( x = kR / 2 \) is within \( 0 < x < 2 \) range it is rather easy to use a series to find an approximate solution:

\[ I_0(2x) \sim 1 - x^2 + x^4 / 4 - x^6 / 36 + x^8 / 576 + \ldots \]

\[ I_1(2x) \sim 1 + x^2 + x^4 / 4 + x^6 / 36 + x^8 / 576 + \ldots \]

\[ J_0(2x) \sim x(1 - x^2 / 2 + x^4 / 12 - x^6 / 144 + x^8 / 2880 + \ldots \]

\[ J_1(2x) \sim x(1 + x^2 / 2 + x^4 / 12 + x^6 / 144 + x^8 / 2880 + \ldots \]

Using the given series, equation for \( R \) transforms into simple equation for \( x = kR / 2 \):

\[ (1 + \nu) / (3 + \nu) x^4 + (5 + \nu) / 12 x^4 = 0 \]

Its approximate solution is:

\[ x^4 = 12(1 + \nu) / (3 + \nu), \quad R = 2R_0 / 2 \]

To estimate an error let’s take \( \nu = 0.2 \) and \( x^4 = 4.5 \) what leads us to an insignificant value \( (5 + \nu) x^4 / 1440 = 0.073 \). Also using the given series, equation for \( R_0 \) gives the result \( y = x / \sqrt{2} \) or \( R_0 = R / \sqrt{2} \), where \( y = kR / 2 \).

Expressions for stress components, the distribution of the maximum tangential stresses and tangential component \( \sigma_{\tau} \) are the following:

\[ \sigma_{\tau} = 12M_{\tau} \frac{z}{h^3} = \frac{12Dk^2W_0}{(I_1(kR) - J_1(kR))} \left( f_{\cdot}(\xi)I_0(kR) + f_{\cdot}(\xi)J_1(kR) \right) \]

\[ \sigma_{\tau} = 12M_{\tau} \frac{z}{h^3} = \frac{12Dk^2W_0}{(I_1(kR) - J_1(kR))} \left( g_{\cdot}(\xi)I_0(kR) + g_{\cdot}(\xi)J_1(kR) \right) \]
\[
\begin{align*}
(s_e - s_a)/2 &= -\frac{3Dk^2W_0(1-v)}{h}\left[\left(1 - \frac{\varepsilon^2}{3} + \frac{\varepsilon^4}{24}\right)I(kR) - \left(1 + \frac{\varepsilon^2}{3} + \frac{\varepsilon^4}{24}\right)J(kR)\right] \\
\sigma_e &= -\frac{6Q}{h^2} - \frac{\varepsilon^2}{4} - \frac{\varepsilon^4}{4} - \frac{\varepsilon^6}{6}
\end{align*}
\]

Thus, derived formulas at a given eigenfrequency determine the plate radius, intermediate support radius and stress components required to evaluate fatigue strength and durability in the adopted scheme of VHCF tests.

2.2. Specimen properties and analytical computations

We chose a titanium alloy called VT3-1 to be the material of our specimen. The alloy’s physical properties are: density \( \rho = 4500 \text{ kg/m}^3 \), Young's modulus \( E = 115 \times 10^9 \text{ Pa} \), Poisson's ratio \( \nu = 0.32 \), classic fatigue limit \( \sigma_o = 360 \times 10^6 \text{ Pa} \), VHCF limit \( \sigma_{B} = 250 \times 10^6 \text{ Pa} \), exponent coefficient \( \beta = -0.3 \). Additional parameters are: loading frequency \( f = 20 \times 10^3 \text{ Hz} \), thickness of the plate \( h = 5 \times 10^{-3} \text{ m} \).

![Figure 4](image)

**Figure 4.** a) \( \sigma_e \) and b) \( \Delta \tau \) distribution within the titanium plate.

Using the formulas above, desired loading frequency and properties of the VT3-1, we calculated geometrical properties of the specimen that are \( R = 23.5 \times 10^{-3} \text{ m} \) and \( R_b = 16 \times 10^{-3} \text{ m} \). Stress distributions on a cross-section that goes along the diameter of the specimen are shown in figure 4.

2.3. Theory for numerical computations

The theory in subsection 2.1 gives us the ability to calculate the stress in the sample at the beginning of a test. Using these results, one can, with a fracture criterion one is comfortable with, determine the specimen’s durability and location of crack initiation.

In numerical computations we decided to use more complex approach to the problem and to use a damage function that was studied in [10-13]. To put it simple, we introduce the damage parameter \( \psi \) on which the properties of the material depends. The kinetic damage equation is (9).

\[
\frac{d\psi}{dN} = B \psi^\gamma / (1 - \psi^\gamma)
\]

The \( B \) parameter depends on the equal stress \( \sigma_{eq} \). There are 2 equations for it depending on the stress level at the studied location within the specimen. The equations are (10).

\[
\begin{align*}
B_{2R} &= 10^4 \left[\left(\sigma_{eq} - \sigma_a\right)/(\sigma_{UTS} - \sigma_a)\right]^{\beta/12}\times(1 + \alpha - \gamma)/(1 - \gamma), \\
B_{2R} &= 10^4 \left[\left(\sigma_{eq} - \sigma_a\right)/(\sigma_{eq} - \sigma_{B})\right]^{\beta/12}\times(1 + \alpha - \gamma)/(1 - \gamma)
\end{align*}
\]

Here \( \sigma_{UTS} \) is ultimate tensile strength, \( \sigma_a \) is classic fatigue limit and \( \sigma_{B} \) is fatigue limit in very-high cycle fatigue so called VHCF limit. The equation with LH subscript is for low- and high-cycle fatigue regimes while the equation with VH subscript is for very-high-cycle fatigue regime.

At each node of the specimen there are not one but two \( B \) values, namely \( B^u \) and \( B^v \), and two \( \sigma_{eq} \) values, \( \sigma_{eq} = \sigma^u, \sigma^v \). For example, in case of low- and high-cycle fatigue regimes equations are:
Values $\sigma^n$ and $\sigma'$ are calculated using different criteria. We chose Smith-Watson-Topper (SWT) criterion and Carpinteri-Spagnoli-Vantadori (CSV) criterion for these values respectively.

**SWT criterion** [14, 15] describes development of normal stress micro-cracks and has the following form:

$$B_* = 10^{-1} \left[ \left( \sigma^n - \sigma_n \right) / (\sigma_{crs} - \sigma_n) \right]^{1/\beta} \alpha / (1 + \alpha - \gamma) / (1 - \gamma).$$

$$B' = 10^{-1} \left[ \left( \sigma' - \sigma_n \right) / (\sigma_{crs} - \sigma_n) \right]^{1/\beta} \alpha / (1 + \alpha - \gamma) / (1 - \gamma).$$

So the equation for the stress level is (11).

$$\sigma^n = \left( \sigma_{crs} \right) / 2 = \sigma_n + \sigma_t N^{-\beta}$$

CSV criterion [16] described development of shear stress micro-cracks and has the following form:

$$\sqrt{\left( \Delta \sigma_{cr} \right) / 2 \Delta \tau_n / 2^2} = \sigma_n + \sigma_t N^{-\beta}$$

So the equation for the stress level is (12).

$$\sigma' = \sqrt{\left( \Delta \sigma_{cr} \right) / 2 \Delta \tau_n / 2^2}$$

It means there are 2 damage values $\psi^n = f(B^*)$ and $\psi' = f(B^{'})$. At every step both $\psi^n$ and $\psi'$ are calculated for every node, then they are compared with each other at every node. When and where one of them becomes greater than 0 the other value is fixed to 0. It means that at every node the process that started first (via either SWT or CSV) prevails over the other one until the end of calculation process. Before the decisive moment both processes have equal rights to be the first one.

In general terms, the program algorithm is as follows. Specimen’s frequency of the desired normal mode of oscillation is found. At this frequency the stress state is calculated. These first two steps are performed using ANSYS software. Next the calculated stress state is exported to the author’s program to perform all other steps that are following. We calculate all $B$ values at each node. After that for each node $k$ a step delta $\Delta N$ is calculated by (13) and the smallest one is chosen. Next for each node a value that we call the total amount of equivalent steps $N_i$ is calculated by (14). Using this value we calculate two damage values $\psi'^n$ at every node with (15). And finally the Young modulus $E_i$ of every element of the specimen is calculated with (16).

$$\min \left\{ \frac{\left( 1 - \psi'^- \right)^2}{2B_i (1 - \gamma)} \right\} \quad \text{when} \quad \psi_i > 0.95$$

$$\min \left\{ \frac{\left( 1 - \psi'^- \right)^2}{4B_i (1 - \gamma)} \right\} \quad \text{when} \quad \psi_i \leq 0.95$$

$$N_i = \Delta N + \frac{2\psi^n - \psi'^n}{k_0 (1 - \gamma)}$$

$$\psi_i = \sqrt{1 - 2N_i B_i (1 - \gamma)}$$

$$E_i = \begin{cases} E_0 (1 - 0.5\psi_i) & \text{when} \quad \psi_i < 0.999 \\ 10^{-3} E_0 & \text{when} \quad \psi_i \geq 0.999 \end{cases}$$

$$\psi^n = \begin{cases} \psi^n (1 - 0.5\psi_i) & \text{when} \quad \psi_i < 0.999 \\ \psi^n & \text{when} \quad \psi_i \geq 0.999 \end{cases}$$
After all these steps the new Young modulus transferred back to ANSYS and the whole procedure is repeated until some result is reached.

2.4. Numerical computation results

We carried out calculations of cyclic loading of the sample until the crack was comparable to the size of the sample. Stress distribution and quasi-crack zone marked with grey area are shown in figure 5. Stress level at the same displacement amplitude was quite similar. Crack-like zone initiated at the center of the specimen as it shown in figure 5a and grew up to half of the specimen’s thickness and as long as 2/3 of the one’s radius. Amount of cycles to crack nucleation was around 4.1e9 cycles. Then additional 0.7e9 cycles were exerted on the specimen for crack to reach the size shown in figure 5c.

It should be noted that even though there were two criteria in the model, only one of them was in charge of the fatigue crack development presented in this paper, namely SWT criterion with development of normal-stress micro-cracks.

3. Conclusions

The perspective scheme of VHCF bi-axial bending tests with cycle asymmetry coefficient equal to -1 was studied. The problem of resonant bending vibrations of a circular plate with intermediate support was solved.

Analytical calculations of resonance parameters of the specimens were carried out, the stress fields were obtained according to the generalized multiaxial criterion applied to the case of VHCF bi-axial bending tests.

Numerical calculations for a sample of analytically calculated geometry were performed. The kinetic damage equation was used in these calculations. This allowed us to get a comprehensive picture of the development of a crack-like defect in the sample up to a long visible crack.

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References

[1] Naito T, Ueda H and Kikuchi M 1984 Fatigue behavior of carburized steel with internal oxides and nonmartensitic microstructure near the surface Metall. Trans. 18A pp 1431-6
[2] Asami K and Sugiyama Y 1985 Fatigue strength of various surface hardened steels Heat Treatment Technol. Assoc. 25 pp 147-50
[3] Bathias C and Ni J 1993 Determination of fatigue limit between 10^5 and 10^9 cycles using an ultrasonic fatigue device ASTM Int. 1211 pp 141-52
[4] Marines I, Bin X and Bathias C 2003 An understanding of very high cycle fatigue of metals Int. J. Fatigue 25 pp 1101-7
[5] Shanyavskiy A A 2007 Modeling of fatigue failure in metals (Ufa: Monography) p 498
[6] Bathias C 2006 Piezoelectric fatigue testing machines and devices Int. J. Fatigue 28 1438-45
[7] Bathias C and Paris P C 2005 Gigacycle Fatigue in Mechanical Practice (NY: Dekker) p 328
[8] Biderman V L 1980 Theory of mechanical oscillation (M: High School) p 408
[9] Nikitin A Stratula B Volkov B 2020 Modern and future schemes of very-high cycle fatigue tests Journal of Physics: Conference Series. Applied Mathematics, Computational Science and Mechanics: Current Problems 012074
[10] Nikitin I S, Burago N G, Nikitin A D and Stratula B A 2020 On kinetic model of damage development Procedia Structural Integrity 28 pp 2032-40
[11] Nikitin I S, Burago N G, Nikitin A D and Stratula B A 2020 Through calculation method of fatigue damage IOP Conference Series: Materials Science and Engineering 927(1) 012019
[12] Nikitin I S, Burago N G, Nikitin A D and Stratula B A 2020 Complex model for fatigue damage development AIP Conference Proceedings 2312 050015
[13] Nikitin I S, Burago N G, Zhuravlev A B and Nikitin A D 2020 Multimode Model for Fatigue Damage Development Mechanics of Solids 55(8) pp 298-306
[14] Smith R N, Watson P and Topper T H 1970 A stress-strain parameter for the fatigue of metals J. of Materials 5(4) pp 767-78
[15] Gates N and Fatemi A 2016 Multiaxial variable amplitude fatigue life analysis including notch effects Int. J. of Fatigue 91 pp 337-51
[16] Carpinteri A Spagnoli A and Vantadori S 2011 Multiaxial assessment using a simplified critical plane based criterion Int. J. of Fatigue 33 pp 969-76