On a Variational Definition for the Jensen-Shannon Symmetrization of Distances based on the Information Radius

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Abstract

We generalize the Jensen-Shannon divergence by considering a variational definition with respect to a generic mean extending thereby the notion of Sibson’s information radius. The variational definition applies to any arbitrary distance and yields another way to define a Jensen-Shannon symmetrization of distances. When the variational optimization is further constrained to belong to probability measure families, we get relative Jensen-Shannon divergences and symmetrizations which generalize information projections. Finally, we discuss applications of these divergences and diversity indices to clustering and quantization tasks of probability measures including statistical mixtures.

Keywords: Jensen-Shannon divergence; diversity index; Rényi entropy; information radius; information projection; exponential family; Bregman information; \(q\)-exponential family; centroid; clustering.

1 Introduction: Background and motivations

Let \((\mathcal{X}, \mathcal{F}, \mu)\) denote a measure space \([8]\) with sample \(\mathcal{X}\), \(\sigma\)-algebra \(\mathcal{F}\) on the set \(\mathcal{X}\) and positive measure \(\mu\) on \((\mathcal{F}, \mu)\) (usually the Lebesgue measure or the counting measure). Denote by \(\mathcal{D} = \mathcal{D}(\mathcal{X})\) the set of densities with full support \(\mathcal{X}\) (Radon-Nikodym derivatives of probability measures with respect to \(\mu\)):

\[
\mathcal{D}(\mathcal{X}) := \left\{ p : \mathcal{X} \to \mathbb{R} : p(x) > 0 \ \mu\text{-a.e.}, \int_{\mathcal{X}} p(x) d\mu(x) = 1 \right\}.
\]

The \textit{Jensen-Shannon divergence} \([34]\) (JSD) between two densities \(p\) and \(q\) of \(\mathcal{D}\) is defined by:

\[
D_{\text{JS}}[p, q] := \frac{1}{2} \left( D_{\text{KL}} \left[ p : \frac{p + q}{2} \right] + D_{\text{KL}} \left[ q : \frac{p + q}{2} \right] \right),
\]

where \(D_{\text{KL}}\) denotes the \textit{Kullback-Leibler divergence} \([33, 17]\) (KLD):

\[
D_{\text{KL}}[p : q] := \int_{\mathcal{X}} p(x) \log \left( \frac{p(x)}{q(x)} \right) d\mu(x).
\]

The JSD belongs to the class of \textit{f-divergences} \([37, 18, 1]\), the \textit{invariant decomposable divergences} of information geometry (see \([3]\), pp. 52-57). Although the KLD is asymmetric (i.e., \(D_{\text{KL}}[p : q] \neq D_{\text{KL}}[q : p]\)).
$D_{\text{KL}}(q : p)$, the JSD is symmetric (i.e., $D_{\text{JS}}[p, q] = D_{\text{JS}}[q, p]$). The notation ‘:’ is used as a parameter separator to indicate that the parameters are not permutation invariant.

The 2-point JSD of Eq. 2 can be extended to a weighted set of $n$ densities $\mathcal{P} := \{(w_1, p_1), \ldots, (w_n, p_n)\}$ (with positive $w_i$’s normalized so that $\sum_{i=1}^n w_i = 1$) thus providing a diversity index (i.e., a $n$-point JSD) for $\mathcal{P}$:

$$D_{\text{JS}}(\mathcal{P}) := \sum_{i=1}^n w_i D_{\text{KL}}[p_i : \bar{p}], \quad (3)$$

where $\bar{p} := \sum_{i=1}^n w_i p_i$ denotes the statistical mixture [36] of the densities of $\mathcal{P}$.

The KLD is also called the relative entropy since it can be expressed as the difference between the cross entropy $h[p : q]$ and the entropy $h[p]$:

$$D_{\text{KL}}[p : q] = h[p : q] - h[p], \quad (4)$$

with

$$h[p : q] := -\int_X p(x) \log q(x) d\mu(x), \quad (5)$$

$$h[p] := -\int_X p(x) \log p(x) d\mu(x) = h[p : p]. \quad (6)$$

When $\mu$ is the Lebesgue measure, the Shannon entropy is also called the differential entropy [17]. It follows that the Jensen-Shannon diversity index of Eq. 3 can be rewritten as:

$$D_{\text{JS}}[p, q] := h[\bar{p}] - \sum_{i=1}^n w_i h[p_i]. \quad (7)$$

The JSD representation of Eq. 7 is a Jensen divergence [45] for the strictly convex negentropy $F(p) = -h(p)$ since the entropy function $h[\cdot]$ is strictly concave, hence its name Jensen-Shannon divergence.

Since $\frac{p_i(x)}{p(x)} \leq \frac{p_i(x)}{w_i p_i(x)} = \frac{1}{w_i}$, it can be shown that the Jensen-Shannon diversity index is upper bounded by $H(w) := -\sum_{i=1}^n w_i \log w_i$, the discrete Shannon entropy. In particular, the JSD is bounded by $\log 2$ although the KLD is unbounded and may even be equal to $+\infty$ when the definite integral diverges (e.g., KLD between the standard Cauchy distribution and the standard Gaussian distribution). Another nice property of the JSD is that its square root yields a metric distance [23, 28]. This property further holds for the quantum JSD [58]. Recently, the JSD has gained interest in machine learning. See for example, the Generative Adversarial Networks [30] (GANs) in deep learning [29] where it was proven that minimizing the GAN objective function by adversarial training is equivalent to minimizing a JSD.

The Jensen-Shannon principle of taking the average of the (Kullback-Leibler) divergences between the source parameters to the mid-parameter can be applied to other distances. For example, the Jensen-Bregman divergence is a Jensen-Shannon symmetrization of the Bregman divergence $B_F$ [45]:

$$B_{F}^{\text{JS}}(\theta_1 : \theta_2) := \frac{1}{2} \left( B_F \left( \theta_1 : \frac{\theta_1 + \theta_2}{2} \right) + B_F \left( \theta_2 : \frac{\theta_1 + \theta_2}{2} \right) \right). \quad (8)$$

The Jensen-Bregman divergence $B_{F}^{\text{JS}}$ can also be written as an equivalent Jensen divergence $J_F$:

$$B_{F}^{\text{JS}}(\theta_1 : \theta_2) = J_F(\theta_1 : \theta_2) = \frac{F(\theta_1) + F(\theta_2)}{2} - F \left( \frac{\theta_1 + \theta_2}{2} \right), \quad (9)$$

$$\int X$$
where $F$ is a strictly convex function ensuring $J_F(\theta_1 : \theta_2) \geq 0$ with equality iff $\theta_1 = \theta_2$.

Because of its use in various fields of information sciences [4], various generalizations of the JSD have been proposed: These generalizations are either based on Eq. 1 [43] or are based on Eq. 7 [39, 48, 44]. For example, the (arithmetic) mixture $\bar{p} = \sum_i w_i p_i$ in Eq. 1 was replaced by an abstract statistical mixture with respect to a generic mean $M$ in [43] (e.g., geometric mixture induced by the geometric mean), and the two KLDS defining the JSD in Eq. 1 was further averaged using another abstract mean $N$, thus yielding the following generic $(M, N)$-Jensen-Shannon divergence [43] $(M, N)$-JSD:

$$D_{JS}^{M,N}[p : q] := N \left( D_{KL}[p : (pq)^{\frac{M}{2}}] , D_{KL}[q : (pq)^{\frac{M}{2}}] \right),$$  \hspace{1cm} (10)

where $(pq)^M$ denotes the statistical weighted mixture:

$$(pq)^M := \frac{M_a(p(x), q(x))}{\int_X M_o(p(x), q(x))d\mu(x)}.$$ \hspace{1cm} (11)

Notice that when $M = N = A$ (the arithmetic mean), Eq. 10 of the $(A, A)$-JSD reduces to the ordinary JSD of Eq. 1. When the means $M$ and $N$ are symmetric, the $(M, N)$-JSD is symmetric.

In general, a weighted mean $M_\alpha(a, b)$ for any $\alpha \in [0, 1]$ shall satisfy the in-betweeness property:

$$\min\{a, b\} \leq M_\alpha(a, b) \leq \max\{a, b\}.$$ \hspace{1cm} (12)

A weighted mean (also called barycenter) can be built from a non-weighted mean $M(a, b)$ (i.e., $\alpha = \frac{1}{2}$) using the dyadic expansion of the weight $\alpha$ as detailed in [38].

The three Pythagorean means defined for positive scalars $a > 0$ and $b > 0$ are classic examples of means:

- The arithmetic mean $A(a, b) = \frac{a+b}{2}$,
- the geometric mean $G(a, b) = \sqrt{ab}$, and
- the harmonic mean $H(a, b) = \frac{2ab}{a+b}$.

These Pythagorean means may be interpreted as special instances of another family of means: The power means

$$P_\alpha(a, b) := \left( \frac{a^\alpha + b^\alpha}{2} \right)^{\frac{1}{\alpha}},$$ \hspace{1cm} (13)

defined for $\alpha \in \mathbb{R}\setminus\{0\}$ (also called Hölder means). The power means can be extended to the full range $\alpha \in \mathbb{R}$ by using the property that $\lim_{\alpha \rightarrow 0} P_\alpha(a, b) = G(a, b)$. The power means are homogeneous means: $P_\alpha(\lambda a, \lambda b) = \lambda P_\alpha(a, b)$ for any $\lambda > 0$. We refer to the handbook of means [14] to get definitions and principles of other means beyond these power means.

Choosing the abstract mean $M$ in accordance with the family of the densities allows one to obtain closed-form formula for the $(M, N)$-JSDs which rely on definite integral calculations. For example, the JSD between two Gaussian densities does not admit a closed-form formula because of the log-sum integral, but the $(G, N)$-JSD admits a closed-form formula when using geometric statistical mixtures (i.e., when $M = G$). As an application of these generalized JSDs, Deasy et al. [21] used the skewed geometric JSD (namely, the $(G_{\alpha}, A_{1-\alpha})$-JSD for $\alpha \in (0, 1)$) which admits a closed-form formula between normal densities [43], and showed how regularizing an optimization task
with this G-JSD divergence improved reconstruction and generation of Variational AutoEncoders (VAEs).

More generally, instead of using the KLD, one can also use any arbitrary distance $D$ to define its \textit{JS-symmetrization} as follows:

$$D_{M,N}^{JS}(p : q) := N \left( D \left[ p : \frac{(pq)^{M}}{2} \right], D \left[ q : \frac{(pq)^{M}}{2} \right] \right). \quad (14)$$

These symmetrizations may further be skewed by using $M_{\alpha}$ and/or $N_{\beta}$ for $\alpha \in (0, 1)$ and $\beta \in (0, 1)$ yielding the definition \[43]:

$$D_{M_{\alpha},N_{\beta}}^{JS}(p : q) := N_{\beta} \left( D \left[ p : \frac{(pq)^{M_{\alpha}}}{2} \right], D \left[ q : \frac{(pq)^{M_{\alpha}}}{2} \right] \right). \quad (15)$$

In this work, we consider symmetrizing an arbitrary distance $D$ (including the KLD) generalizing the Jensen-Shannon divergence by using a \textit{variational formula} for the JSD. Namely, we observe that the Jensen-Shannon divergence can also be defined as the following minimization problem:

$$D_{JS}[p, q] := \min_{c \in D} \frac{1}{2} \left( D_{KL}[p : c] + D_{KL}[q : c] \right), \quad (16)$$

since the optimal density $c$ is proven unique using the calculus of variation \[54, 2\] and corresponds to the mid density $p + q / 2$.

The paper is organized as follows: In §2 we recall the rationale and definitions of the Rényi $\alpha$-entropy and the Rényi $\alpha$-divergence \[52\], and explain the information radius of Sibson \[54\] which includes as a special case the ordinary Jensen-Shannon divergence. It is noteworthy to point out that Sibson’s work (1969) includes as a particular case the reference paper of Lin \[34\] (1991). In §3 we present the JS-symmetrization variational formula based on a generalization of the information radius with an abstract mean (Definition 6 and Definition 4). In §4 we constrain the mid density to belong to a class of (parametric) probability densities like an exponential family \[7\], and get relative information radius generalizing information radius and related to information projections. Definition 7 generalizes the (relative) \textit{normal information radius} of Sibson \[54\] who considered the multivariate normal family. As an application of these relative JSDs, we consider clustering and quantization of probability densities in §4.2. Finally, we conclude by summarizing our contributions and discussing related works in §5.

## 2 Rényi entropy and divergence, and Sibson information radius

Rényi \[32\] investigated a generalization of the four axioms of Fadeev \[25\] yielding to the unique Shannon entropy \[19\]. In doing so, Rényi replaced the ordinary weighted arithmetic mean by a more general class of averaging schemes. Namely, Rényi considered the \textit{weighted quasi-arithmetic means} \[32\]. A weighted quasi-arithmetic mean can be induced by a strictly monotonous and continuous function $g$ as follows:

$$M_{g}(x_{1}, \ldots, x_{n}; w_{1}, \ldots, w_{n}) := g^{-1} \left( \sum_{i=1}^{n} w_{i} g(x_{i}) \right), \quad (17)$$

where the $x_i$’s and the $w_i$’s are positive (the weights are normalized so that $\sum_{i=1}^{n} w_{i} = 1$). For example, the power means $P_{\alpha}(a, b) = \left( \frac{a^{\alpha} + b^{\alpha}}{2} \right)^{\frac{1}{\alpha}}$ introduced earlier are quasi-arithmetic means for
the generator \( g^P_\alpha (u) := u^\alpha \):

\[
P_\alpha (a, b) = M_{g^P_\alpha} \left( a, b; \frac{1}{2}, \frac{1}{2} \right).
\]

Rényi proved that among the class of weighted quasi-arithmetic means, only the means induced by

\[
g_\alpha (u) := 2^{(\alpha - 1)u},
\]

\[
g_\alpha^{-1} (v) := \frac{1}{\alpha - 1} \log_2 v,
\]

for \( \alpha > 0 \) and \( \alpha \neq 1 \) yield a proper generalization of Shannon entropy called nowadays the Rényi \( \alpha \)-entropy. The Rényi mean is

\[
M_\alpha^R (x_1, \ldots, x_n; w_1, \ldots, w_n) = M_{g_\alpha} (x_1, \ldots, x_n; w_1, \ldots, w_n),
\]

\[
= \frac{1}{\alpha - 1} \log_2 \sum_{i=1}^n w_i 2^{(\alpha - 1)x_i}.
\]

The Rényi \( \alpha \)-means \( M_\alpha^R \) are not power means: They are not homogeneous means.

The Rényi \( \alpha \)-entropy are defined by:

\[
h_\alpha^R [p] := \frac{1}{1 - \alpha} \log \int_X p^\alpha (x) d\mu (x), \quad \alpha \in (0, 1) \cup (1, \infty).
\]

In the limit case \( \alpha \to 1 \), the Rényi \( \alpha \)-entropy converges to Shannon entropy:

\[
l_{\alpha \to 1} h_\alpha^R [p] = h[p] = -\int_X p(x) \log p(x) d\mu (x).
\]

Rényi \( \alpha \)-entropies are non-increasing with respect to increasing \( \alpha \):

\[
h_\alpha^R [p] \geq h_{\alpha'}^R [p] \text{ for } \alpha < \alpha'.
\]

In the discrete case (i.e., counting measure \( \mu \) on a finite alphabet \( X \)),
we can further define \( h_0[p] = \log |X| \) for \( \alpha = 0 \) (also called max-entropy or Hartley entropy).

The Rényi \( +\infty \)-entropy \( h_{+\infty}[p] = -\log \max_{x \in X} p(x) \) is also called the min-entropy since the sequence \( h_\alpha \) is non-increasing with respect to increasing \( \alpha \).

Similarly, Rényi obtained the \( \alpha \)-divergences for \( \alpha > 0 \) and \( \alpha \neq 1 \) (originally called information gain of order \( \alpha \)):

\[
D_\alpha^R [p : q] := \frac{1}{\alpha - 1} \log_2 \left( \int_X p(x)^\alpha q(x)^{1-\alpha} d\mu (x) \right),
\]

generalizing the Kullback-Leibler divergence since \( \lim_{\alpha \to 1} D_\alpha^R [p : q] = D_{KL}[p : q] \).

Rényi \( \alpha \)-divergences are non-decreasing with respect to increasing \( \alpha \) [57]:

\[
D_\alpha^R [p : q] \leq D_{\alpha'}^R [p : q] \text{ for } \alpha' \geq \alpha.
\]

Sibson [51] considered both the Rényi \( \alpha \)-divergence \( D_\alpha^R \) and the Rényi \( \alpha \)-weighted mean \( M_\alpha^R := M_{g_\alpha} \) to define the information radius \( R_\alpha \) of order \( \alpha \) of a weighted set \( P = \{(w_i, p_i)\}_{i=1}^n \) of densities \( p_i \)’s as the following minimization problem:

\[
R_\alpha (P) := \min_{c \in D} R_\alpha (P, c),
\]

where

\[
R_\alpha (P, c) := M_\alpha^R \left( D_\alpha^R [p_1 : c], \ldots, D_\alpha^R [p_n : c]; w_1, \ldots, w_n \right).
\]

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Robin Sibson (1944-2017) is also renown for inventing the natural neighbor interpolation [55].
Thus we have called Chernoff $\alpha$ to be a density. Provided that the densities respect to $(\alpha - 1)x_i \log 2 + \log w_1, \ldots, (\alpha - 1)x_i \log 2 + \log q(28)$

Function $\text{LSE}(a_1, \ldots, a_n) := \log \left( \sum_{i=1}^{n} e^{a_i} \right)$ denotes the log-sum-exp (convex) function $[10, 49]$. Sibson $[54]$ also considered the limit case $\alpha \to \infty$ when defining the information radius:

$$D^\infty_{\infty}[p : q] := \log_2 \sup_{x \in \mathcal{X}} \frac{p(x)}{q(x)}.$$  

(Sibson reported the following theorem in his information radius study $[54]$:

**Theorem 1** (Theorem 2.2 and Corollary 2.3 of $[54]$). The optimal density $c^*_\alpha = \arg \min_{c \in \mathcal{D}} R_\alpha(\mathcal{P}, c)$ is unique and we have:

$$c^*_\alpha(x) = \frac{\sum_i w_i p_i(x)}{\int_{\mathcal{X}} \sum_i w_i p_i(x)^\alpha d\mu(x)} = \frac{\sum_i w_i p_i(x)}{\int_{\mathcal{X}} \sum_i w_i p_i(x)^\alpha d\mu(x)},$$  

$$c^*_\alpha(x) = \frac{\sum_i w_i p_i(x)^\alpha}{\int_{\mathcal{X}} \sum_i w_i p_i(x)^\alpha d\mu(x)} = \frac{\sum_i w_i p_i(x)^\alpha}{\int_{\mathcal{X}} \sum_i w_i p_i(x)^\alpha d\mu(x)},$$  

$$c^*_\alpha(x) = \frac{\sum_i w_i p_i(x)}{\int_{\mathcal{X}} \sum_i w_i p_i(x)^\alpha d\mu(x)} = \frac{\sum_i w_i p_i(x)}{\int_{\mathcal{X}} \sum_i w_i p_i(x)^\alpha d\mu(x)}.$$  

Observe that $R^\infty(\mathcal{P})$ does not depend on the (positive) weights.

The proof follows from the following decomposition of the information radius:

**Proposition 1.** We have:

$$R_\alpha(\mathcal{P}, c) - R_\alpha(\mathcal{P}, c^*_\alpha) = I_\alpha(c^*_\alpha, c) \geq 0.$$  

Notice that $2^{(\alpha - 1)I_\alpha}[p : q] = \int_{\mathcal{X}} p(x)^\alpha q(x)^{1-\alpha} d\mu(x)$, the Bhattacharyya $\alpha$-coefficient $[45]$ (also called Chernoff $\alpha$-coefficient $[10, 41]$):

$$C_{\text{Bhat}, \alpha}[p : q] := \int_{\mathcal{X}} p(x)^\alpha q(x)^{1-\alpha} d\mu(x).$$  

Thus we have

$$R_\alpha(\mathcal{P}, c) = \frac{1}{\alpha - 1} \log_2 \sum_i w_i C_{\text{Bhat}, \alpha}[p_i : c].$$  

Notice that $c^*_\alpha(x) = \frac{\max \{p_1(x), \ldots, p_n(x)\}}{\int_{\mathcal{X}} \max \{p_1(x), \ldots, p_n(x)\} d\mu(x)}$ is the upper envelope of the densities $p_i(x)$’s normalized to be a density. Provided that the densities $p_i$’s intersect pairwise in at most $s$ locations (i.e., $|\{p_i(x) \cap p_j(x)\}| \leq s$ for $i \neq j$), we can compute efficiently this upper envelope using an output-sensitive algorithm $[50]$ of computational geometry.

In general, the optimal density $c^*_\alpha = \arg \min_{c \in \mathcal{D}} R_\alpha(\mathcal{P}, c)$ yielding the information radius $R_\alpha(\mathcal{P})$ can be interpreted as a generalized centroid (extending the notion of Fréchet means $[27]$) with respect to $(M^R_\alpha, D^R_\alpha)$, where a $(M, D)$-centroid is defined by:
Definition 1 ((M, D)-centroid). Let \( \mathcal{P} = \{(w_1, p_1), \ldots, (w_n, p_n)\} \) be a normalized weighted point set, \( M \) a mean, and \( D \) a distance. Then the \((M, D)\)-centroid is defined as \( c_{M,D}(\mathcal{P}) = \arg \min_c M(D(p_1 : c), \ldots, D(p_n : c); w_1, \ldots, w_n) \).

When all the densities \( p_i \)'s belong to a same exponential family \([7]\) with cumulant function \( F \) (i.e., \( p_i = p_{\theta_i} \)), we have \( D_{KL}[p_{\theta} : p_{\theta_0}] = B_F(\theta : \theta) \) where \( B_F \) denotes the Bregman divergence \([6]\):

\[
B_F(\theta : \theta') := F(\theta) - F(\theta') - (\theta - \theta')^\top \nabla F(\theta').
\]

(33)

Let \( \mathcal{V} = \{(w_1, \theta_1), \ldots, (w_n, \theta_n)\} \) be the parameter set corresponding to \( \mathcal{P} \). Define

\[
R_F(\mathcal{V}, \theta) := \sum_{i=1}^n w_i B_F(\theta_i : \theta).
\]

(34)

Then we have the equivalent decomposition of Proposition 1

\[
R_F(\mathcal{V}, \theta) - R_F(\mathcal{V}, \theta^*) = B_F(\theta^* : \theta),
\]

(35)

with \( \theta^* = \bar{\theta} := \sum_{i=1}^n w_i \theta_i \). (This decomposition is used to prove Proposition 1 of \([6]\).) The quantity \( R_F(\mathcal{V}) = R_F(\mathcal{V}, \theta^*) \) was termed the Bregman information \([6, 17]\). \( R_F(\mathcal{V}) \) could also be called Bregman information radius according to Sibson. Since \( R_F(\mathcal{V}) = \sum_{i=1}^n w_i D_{KL}[p_{\theta} : p_{\theta_0}] \), we can interpret the Bregman information as a Sibson’s information radius for densities of an exponential family with respect to the arithmetic mean \( M_1^R = A \) and the reverse Kullback-Leibler divergence: \( D_{KL}^R[p : q] := D_{KL}[q : p] \). This observation yields us to the JS-symmetrization of distances based on generalized information radii in \([13]\).

Sibson proved that the information radii of any order are all upper bounded (Theorem 2.8 and Theorem 2.9 of \([54]\)) as follows:

\[
R_1(\mathcal{P}) \leq \sum_i w_i \log_2 \sum_j w_j < \infty,
\]

(36)

\[
R_\alpha(\mathcal{P}) \leq \frac{\alpha}{\alpha - 1} \log_2 \left( \sum_i w_i^\alpha \right) \leq \log_2 n < \infty, \quad \alpha \in (0, 1) \cup (1, \infty)
\]

(37)

\[
R_{\infty}(\mathcal{P}) \leq \log_2 n < \infty.
\]

(38)

We interpret Sibson’s upper bound as follows:

Proposition 2 (Information radius upper bound). The information radius of order \( \alpha \) of a weighted set of distributions is upper bounded by the discrete Rényi entropy of order \( \frac{1}{\alpha} \) of the weight distribution: \( R_\alpha(\mathcal{P}) \leq H_\alpha^R[w] \) where \( H_\alpha^R[w] := \frac{1}{1-\alpha} \log \left( \sum_i w_i^\alpha \right) \).

3 JS-symmetrization based on generalized information radius

Let us give the following definitions generalizing the information radius (i.e., Jensen-Shannon symmetrization of the distance when \( |\mathcal{P}| = 2 \)) and the ordinary Jensen-Shannon divergence:

Definition 2 ((M, D)-information radius). Let \( M \) be a weighted mean and \( D \) a distance. Then the generalized information radius for a weighted set of points (e.g., vectors or densities) \( (w_1, p_1), \ldots, (w_n, p_n) \) is:

\[
R_{M,D}(\mathcal{P}) := \min_{c \in D} M(D[p_1 : c], \ldots, D[p_n : c]; w_1, \ldots, w_n).
\]
We also define the \((M, D)\)-centroid as follows:

**Definition 3** \(((M, D)\text{-centroid})\). Let \(M\) be a weighted mean and \(D\) a statistical distance. Then the centroid for a weighted set of densities \((w_1, p_1), \ldots, (w_n, p_n)\) with respect to \((M, D)\) is:

\[
c_{M,D}(P) := \arg \min_{c \in \mathcal{D}} M(D[p_1 : c], \ldots, D[p_n : c]; w_1, \ldots, w_n).
\]

When \(M = A\), we recover the notion of Fréchet mean \([27]\). Notice that although the minimum \(R_{M,D}(P)\) is unique, there may potentially exists several generalized centroids \(c_{M,D}(P)\) depending on \((M, D)\).

The generalized information radius can be interpreted as a diversity index or an \(n\)-point distance. When \(n = 2\), we get the following (2-point) distances which are considered as a generalization of the Jensen-Shannon divergence or Jensen-Shannon symmetrization:

**Definition 4** \((M\text{-vJS symmetrization of } D)\). Let \(M\) be a mean and \(D\) a statistical distance. Then the variational Jensen-Shannon symmetrization of \(D\) is defined by the formula of a generalized information radius:

\[
D_{M}^{\text{vJS}}[p : q] := \min_{c \in \mathcal{D}} M(D[p : c], D[q : c]).
\]

We use the acronym vJS to distinguish it with the JS-symmetrization reported in \([43]\):

\[
D_{M}^{\text{vJS}}[p : q] = D_{M,A}^{\text{vJS}}[p : q] := \frac{1}{2} \left( D \left[ p : (pq)^{M}_{\frac{1}{2}} \right] + D \left[ q : (pq)^{M}_{\frac{1}{2}} \right] \right).
\]

We recover Sibson’s information radius \(R_\alpha[p : q]\) induced by two densities \(p\) and \(q\) from Definition \([3]\) as the \(M_\alpha^{R}\)-vJS symmetrization of the Rényi divergence \(D_\alpha^{R}\). Notice that we may skew these generalized JSDs by taking weighted mean \(M_\beta\) instead of \(M\) for \(\beta \in (0,1)\) yielding to the general definition:

**Definition 5** (Skew \(M_\beta\)-vJS symmetrization of \(D\)). Let \(M_\beta\) be a weighted mean and \(D\) a statistical distance. Then the variational skewed Jensen-Shannon symmetrization of \(D\) is defined by the formula of a generalized information radius:

\[
D_{M_\beta}^{\text{vJS}}[p : q] := \min_{c \in \mathcal{D}} M_\beta(D[p : c], D[q : c]).
\]

Notice that this definition is implicit and can be made explicit when the centroid \(c^*(p, q)\) is unique:

\[
D_{M_\beta}^{\text{vJS}}[p : q] = M_\beta(D[p : c^*(p, q)], D[q : c^*(p, q)]).
\] (39)

In particular, when \(D = D_{\text{KL}}\), the KLD, we obtain generalized skewed Jensen-Shannon divergences:

**Definition 6** (Skewed \(M_\beta\)-vJS divergence). Let \(M_\beta\) be a weighted mean for \(\beta \in (0,1)\). Then the \(M\text{-vJS} \) divergence is defined by the variational formula:

\[
D_{M_\beta}^{\text{vJS}}[p : q] := \min_{c \in \mathcal{D}} M_\beta(D_{\text{KL}}[p : c], D_{\text{KL}}[q : c]).
\]
Amari [2] obtained the \((A, D_A)\)-information radius and its corresponding unique centroid for \(D_A\), the \(\alpha\)-divergence of information geometry [3].

Brekelmans et al. [13] studied the geometric path \((p_1 p_2)_{\beta}^G(x) \propto p_1^{1-\beta} p_2^{\beta}(x)\) between two distributions \(p_1\) and \(p_2\) of \(D\) where \(G_\beta(a, b) = a^{1-\beta} b^\beta\) (with \(a, b > 0\)) is the weighted geometric mean. They proved the variational formula:

\[
(p_1 p_2)_{\beta}^G = \min_{c \in D} (1 - \beta) D_{KL}[c : p_1] + \beta D_{KL}[c : p_2].
\]

That is, \((p_1 p_2)_{\beta}^G\) is a \(G_\beta\)-\(D_{KL}\) centroid, where \(D_{KL}\) is the reverse KLD. The corresponding \((G_\beta, D_{KL})\)-vJSD is studied in [43] and used in deep learning in [21].

It is interesting to study the link between \((M_\beta, D)\)-variational Jensen-Shannon symmetrization of \(D\) and the \((M'_\alpha, N'_\beta)\)-JS symmetrization of of [13]. In particular the link between \(M_\beta\) for averaging in the minimization and \(M'_\alpha\) the mean for generating abstract mixtures.

More generally, Brekelmans et al. [12] considered the \(\alpha\)-divergences extended to positive measures

\[
D^e_\alpha[p : q] = \frac{4}{1 - \alpha^2} \int_X \left( \frac{1 - \alpha}{2} p(x) + \frac{1 + \alpha}{2} q(x) \right) d\mu(x)
\]

and proved that

\[
c^*_\beta = \arg\min_{c \in D} (1 - \beta) D^e_\alpha[p_1 : c] + \beta D^e_\alpha[p_2 : c]
\]

is a density of a likelihood ratio \(q\)-exponential family. \(c^*_\beta = \frac{p(x)}{e^{\beta q(x)}} \exp_q(\beta \log \frac{p_2(x)}{p_1(x)})\) for \(q = \frac{1+\alpha}{2}\). That is, \(c^*_\beta\) is the \((A_\beta, D^e_\alpha)\)-generalized centroid, and the corresponding information radius is the variational JS symmetrization:

\[
D^e_\alpha^{\text{vJS}}[p_1 : p_2] = (1 - \beta) D^e_\alpha[p_1 : c^*_\beta] + \beta D^e_\alpha[p_2 : c^*_\beta]
\]

The \(q\)-divergence \(D_q\) between two densities of a \(q\)-exponential family amounts to a Bregman divergence [3]. Thus \(D^e_q\) for \(M = A\) is a generalized information radius which amounts to a Bregman information.

For the case \(\alpha = \infty\) in Sibson’s information radius, we find that the information radius is related to the total variation:

**Proposition 3** (Lemma 2.4 [54]).

\[
D^\text{vJS, R}_\infty[p : q] = \log_2 (1 + D_{\text{TV}}[p : q]),
\]

where \(D_{\text{TV}}\) denotes the total variation

\[
D_{\text{TV}}[p : q] = \frac{1}{2} \int_X |p(x) - q(x)| d\mu(x).
\]

**Proof.** Since \(\max\{p(x), q(x)\} \leq p(x) + q(x)\), it follows that \(\int_X \max\{p(x), q(x)\} d\mu(x) = 1 + D_{\text{TV}}[p : q]\). From Theorem 1 we have \(R_\infty((\frac{1}{2}, p), (\frac{1}{2}, q)) = \log_2 \int_X \max\{p(x), q(x)\} d\mu(x)\) and therefore \(R_\infty((\frac{1}{2}, p), (\frac{1}{2}, q)) = \log_2 (1 + D_{\text{TV}}[p : q])\).

Notice that when \(M = M_g\) is a quasi-arithmetic mean, we may consider the divergence \(D_g[p : q] = g^{-1}(D[p : q])\) so that the centroid of the \((M_g, D_g)\)-JS symmetrization is:

\[
\arg\min_c g^{-1} \left( \sum_{i=1}^n w_i D[p_i : c] \right) \equiv \arg\min_c \sum_{i=1}^n w_i D[p_i : c].
\]

9
4 Relative information radius and relative Jensen-Shannon divergences

4.1 Relative information radius

In this section, instead of considering the full space of densities $\mathcal{D}$ on $(\mathcal{X}, \mathcal{F}, \mu)$ for performing the optimization of the information radius, we may consider a subfamily of (parametric) densities $\mathcal{R} \subset \mathcal{D}$. Then we define the $\mathcal{R}$-relative Jensen-Shannon divergence ($\mathcal{R}$-JSD for short) as

$$D_{\mathcal{R}}^{\mathcal{R}}[p : q] := \min_{c \in \mathcal{R}} D_{\mathcal{KL}}[p : c] + D_{\mathcal{KL}}[q : c]. \quad (47)$$

In particular, Sibson [54] considered the normal information radius with $\mathcal{R} = \{\mathcal{N}(\mu, \Sigma) : (\mu, \Sigma) \in \mathbb{R}^d \times \mathbb{P}_d^+\}$, where $\mathbb{P}_d^+$ denotes the cone of $d \times d$ positive-definite matrices (positive-definite covariance matrices of Gaussian distributions). More generally, we may consider any exponential family $\mathcal{E}$ [7].

**Definition 7** (Relative ($\mathcal{R}, M$)-JS symmetrization of $D$). Let $M$ be a mean and $D$ a statistical distance. Then

$$D_{M, \mathcal{R}}^{\mathcal{JS}}[p : q] := \min_{c \in \mathcal{R}} M(D[p : c], D[q : c]).$$

We obtain relative Jensen-Shannon divergences when $D = D_{\mathcal{KL}}$.

Grosse et al. [31] considered geometric and moment average paths for annealing. They proved that when $p_1 = p_{\theta_1}$ and $p_2 = p_{\theta_2}$ belong to an exponential family $\mathcal{E}_F$ with cumulant function $F$, we have

$$(p_1 p_2)^\beta = p_1(x)^{1-\beta} p_2(x)^\beta \int p_1(x)^{1-\beta} p_2(x)^\beta d\mu(x) = \arg \min_{c \in \mathcal{E}_F} (1 - \beta)D_{\mathcal{KL}}[c : p_1] + \beta D_{\mathcal{KL}}[c : p_2] \quad (48)$$

and

$$p_{\bar{\beta}} = \arg \min_{c \in \mathcal{E}_F} (1 - \beta)D_{\mathcal{KL}}[p_1 : c] + \beta D_{\mathcal{KL}}[c : p_2], \quad (49)$$

where $\bar{\beta} = (1 - \beta)\eta_1 + \beta \eta_2$, $\eta_i = E_{p_{\theta_i}}[t(x)]$ (this is not an arithmetic mixture but an exponential family density which moment parameter is a mixture of the parameters).

The corresponding minima can be interpreted as relative skewed Jensen-Shannon symmetrization for the reverse KLD $D_{\mathcal{KL}}^*$ (Eq. 48) and the relative skewed Jensen-Shannon divergence (Eq. 49):

$$D_{\mathcal{KL}_{A_{\beta}, \mathcal{E}_F}}^{\mathcal{JS}}[p_1 : p_2] = \min_{c \in \mathcal{E}_F} (1 - \beta)D_{\mathcal{KL}}^*[p_1 : c] + \beta D_{\mathcal{KL}}^*[c : p_2], \quad (50)$$

$$D_{A_{\beta}, \mathcal{E}_F}^{\mathcal{JS}}[p_1 : p_2] = \min_{c \in \mathcal{E}_F} (1 - \beta)D_{\mathcal{KL}}[c : p_1] + \beta D_{\mathcal{KL}}[c : p_2], \quad (51)$$

where $A_{\beta}(a,b) := (1 - \beta)a + \beta b$ is the weighted arithmetic mean for $\beta \in (0,1)$.

Notice that when $p = q$, we have $D_{M, \mathcal{R}}^{\mathcal{JS}}[p : p] = \min_{c \in \mathcal{R}} D[p : c]$ which is the information projection [42] with respect to $D$ of density $q$ to the submanifold $\mathcal{R}$. Thus when $p \notin \mathcal{R}$, we have $D_{M, \mathcal{R}}^{\mathcal{JS}}[p : p] > 0$, i.e., the relative JSDs are not proper divergences since a proper divergence ensures that $D[p : q] \geq 0$ with equality iff $p = q$. 

10
4.2 Relative Jensen-Shannon divergences: Clustering and quantizing densities

Let \( D_{\text{KL}}[p : q_\theta] \) be the KLD between an arbitrary density \( p \) and a density of an exponential family \( q_\theta \). The density of \( q_\theta \) is expressed as \( \exp[\theta^\top t(x) - F(\theta)] \) where \( t(x) \) denotes the sufficient statistics \( t(x) \) and \( F(\theta) \) the cumulant function \cite{17}. Assume we both know (i) \( E_p[t(x)] = m \) and (ii) the Shannon entropy \( h(p) \) of \( p \). Then we can expressed the KLD in a “semi-closed” form:

**Proposition 4.** Let \( q_\theta \) belong to an exponential family and \( p \) a density with \( m = E_p[t(x)] \). Then the Kullback-Leibler divergence is expressed as:

\[
D_{\text{KL}}[p : q_\theta] = F(\theta) - m^\top \theta - h(p),
\]

where \( h^\times(p : q_\theta) = F(\theta) - m^\top \theta \) is the cross-entropy.

**Proof.** The proof is direct since \( \log q_\theta(x) = \theta^\top t(x) - F(\theta) \).

\[
\begin{align*}
D_{\text{KL}}[p : q_\theta] &= h[p : q] - h[p], \\
&= - \int_X p(x) \log q_\theta(x) d\mu(x) - h(p), \\
&= F(\theta) - m^\top \theta - h(p),
\end{align*}
\]

Example 1. For example, when \( q_\theta \) is a Gaussian distribution, we have

\[
D_{\text{KL}}[p : \mathcal{N}(\mu, \Sigma)] = -h(p) + \frac{1}{2} \left( \log 2\pi\Sigma + (\mu - m)^\top \Sigma^{-1} (\mu - m) + \text{tr}(\Sigma^{-1} S) \right),
\]

where \( m = E_p[x] \) and \( S = E_p[x x^\top] \).

The formula of proposition \cite{41} is said semi-closed because although it relies on knowing the entropy of \( h \) and the moment \( E_p[t(x)] \), we can compare \( D_{\text{KL}}[p : q_{\theta_1}] \geq D_{\text{KL}}[p : q_{\theta_2}] \) or not by checking whether \( F(\theta_1) - F(\theta_2) - m^\top (\theta_1 - \theta_2) \geq 0 \) or not. This shall be useful for clustering densities with respect to centroids in \cite{41, 42}.

To find the best density \( q_\theta \) approximating \( p \) by minimizing \( \min_\theta D_{\text{KL}}[p : q_\theta] \), we solve \( \nabla F(\theta) = \eta = m \), and therefore \( \theta = \nabla F^*(m) = (\nabla F)^{-1}(m) \) where \( F^*(\eta) = E_{q_\theta}[\log q_\theta(m)] \) with \( F^* \) denoting the Legendre-Fenchel convex conjugate \cite{17}. In particular, when \( p = \sum w_i p_{\eta_i} \) is a mixture of EFs (with \( m = E_p[t(x)] = \sum w_i \eta_i \) with \( \eta_i = E_{p_{\eta_i}}[t(x)] \) thanks to the linearity of the expectation), then the best density of the EF simplifying \( p \) is

\[
\min_\theta D_{\text{KL}}[p : q_\theta] = \min_\theta F(\theta) - m^\top \theta,
\]

\[
= \min_\theta F(\theta) - \sum w_i \eta_i^\top \theta.
\]

Taking the gradient with respect to \( \theta \), we have \( \nabla F(\theta) = \eta = \sum w_i \eta_i \). This yields another proof without the Pythagoras theorem \cite{51, 53}.

**Proposition 5.** Let \( m(x) = \sum w_i p_{\theta_i}(x) \) be a mixture with components belonging to an exponential family with cumulant function \( F \). Then \( \theta^* = \arg_\theta \min_\theta D_{\text{KL}}[p : q_\theta] \) is \( \nabla F^*(\sum w_i \eta_i) \) where the \( \eta_i = \nabla F(\theta_i) \) are the moment parameters of the mixture components.
Consider the following two problems:

**Problem 1** (Density clustering). Given a set of \( n \) weighted densities \((w_1, p_1), \ldots, (w_n, p_n)\), partition them into \( k \) clusters \( C_1, \ldots, C_k \) in order to minimize the \( k \)-centroid objective function with respect to a statistical divergence \( D \):

\[
\sum_{i=1}^{n} w_i \min_{l \in \{1, \ldots, k\}} D[p_i : c_l],
\]

where \( c_l \) denotes the centroid of cluster \( C_l \) for \( l \in \{1, \ldots, k\} \).

For example, when all densities \( p_i \)'s are isotropic Gaussians, we recover the \( k \)-means objective function [35].

**Problem 2** (Mixture component quantization). Given a statistical mixture \( m(x) = \sum_{i=1}^{n} w_i p_i(x) \), quantize the mixture components into \( k \) densities \( q_1, \ldots, q_k \) in order to minimize

\[
\sum_{i} w_i \min_{l \in \{1, \ldots, k\}} D[p_i : q_l].
\]

Notice that in Problem 1 the input densities \( p_i \)'s may be mixtures, i.e., \( p_i(x) = \sum_{j=1}^{n} w_{i,j} p_{i,j}(x) \).

Using the relative information radius, we can cluster a set of distributions (potentially mixtures) into an exponential family mixture, or quantize an exponential family mixture. Indeed, we can implement an extension of \( k \)-means [35] with \( k \)-centers \( q_{\theta_l} \), to assign density \( p_i \) to cluster \( C_j \) (with center \( q_j \)), we need to perform basic comparison tests \( D_{KL}[p_i : q_{\theta_l}] \geq D_{KL}[p_i : q_{\theta_j}] \). Provided the cumulant \( F \) of the exponential family is in closed-form, we do not need formula for the entropies \( h(p_i) \).

Clustering and quantization of densities/mixtures have been widely studied in the literature, see for example [20, 46, 26, 60, 22, 59, 56].

### 5 Conclusion

To summarize, the ordinary Jensen-Shannon divergence has been defined in three equivalent ways in the literature:

\[
D_{JS}[p, q] := \min_{c \in \mathcal{D}} \frac{1}{2} (D_{KL}[p : c] + D_{KL}[q : c]),
\]

\[
= \frac{1}{2} \left( D_{KL} \left[ p : \frac{p+q}{2} \right] + D_{KL} \left[ q : \frac{p+q}{2} \right] \right),
\]

\[
= h \left[ \frac{p+q}{2} \right] - \frac{h[p] + h[q]}{2}.
\]

The JSD Eq. (58) was studied by Sibson [54] in 1969 with the wider scope of information radius: Sibson relied on the Rényi \( \alpha \)-divergences (relative Rényi \( \alpha \)-entropies [24]) and he recovered the ordinary Jensen-Shannon divergence as a particular case of the information radius when \( \alpha = 1 \). The JSD Eq. (59) was investigated by Lin [34] in 1991 with its connection to the JSD Eq. (59).

Generalizations of the JSD based on Eq. (59) was proposed in [43] using a generic mean instead of the arithmetic mean. Generalization of the bridge between the JSD of Eq. (59) to the JSD of Eq. (60) was studied using a skewing vector in [44]. The JSD is a \( f \)-divergence [18, 44] but the Sibson-M Jensen-Shannon symmetrization of a distance does not belong to the class of \( f \)-divergences in general. The variational JSD definition is implicit while the definitions of Eq. (59) and Eq. (60) are explicit because the unique optimal centroid \( c^* = \frac{p+q}{2} \) has been plugged into the objective function minimized by Eq. (58).
In this paper, we proposed a generalization of the Jensen-Shannon divergence based on the variational definition of the ordinary Jensen-Shannon divergence based on the JSD definition of Eq. 58: \( D_{\text{JS}}[p : q] = \min_c \frac{1}{2}(D_{\text{KL}}[p : c] + D_{\text{KL}}[q : c]) \). We introduced the Jensen-Shannon symmetrization of an arbitrary divergence \( D \) by considering a generalization of the information radius with respect to an abstract weighted mean \( M_\beta: D_{\text{JS}}^\beta[p : q] := \min_c M_\beta(D[p : c], D[q : c]) \). We also consider relative variational JS symmetrization when the centroid has to belong a family. For the case of exponential family, we showed how to compute the relative centroid in closed form extending the work of Sibson who considered the relative normal centroid in the relative normal information radius.

In a similar vein, Chen et al. [16] considered the following minimax symmetrization of the scalar Bregman divergence [11]:

\[
B_{f}^{\text{minmax}}(p, q) := \min_c \max_{\lambda \in [0, 1]} \lambda B_f(p : c) + (1 - \lambda)B_f(q : c),
\]

They proved that \( \sqrt{B_{f}^{\text{minmax}}(p, q)} \) yields a metric when \( 3((\log f''(0))^2 \geq ((\log f''(0))^2) \), and extend the definition to the vector case and conjecture that the square-root metrization still holds in the multivariate case. In a sense, this definition geometrically highlights the notion of radius since the minmax optimization amount to find a smallest enclosing ball enclosing the source distributions. The circumcenter also called the Chebyshev center [15] is then the mid-distribution instead of the centroid for the information radius. Besides providing another viewpoint, variational definitions of divergences are proven useful in practice (e.g., for estimation). For example, a variational definition of the Rényi divergence generalizing the Donsker-Varadhan variational formula of the KLD is given in [9] which is used to estimate the Rényi Divergences.

References

[1] Syed Mumtaz Ali and Samuel D Silvey. A general class of coefficients of divergence of one distribution from another. *Journal of the Royal Statistical Society: Series B (Methodological)*, 28(1):131–142, 1966.

[2] Shun-ichi Amari. Integration of stochastic models by minimizing \( \alpha \)-divergence. *Neural computation*, 19(10):2780–2796, 2007.

[3] Shun-ichi Amari. *Information Geometry and Its Applications*. Applied Mathematical Sciences. Springer Japan, 2016.

[4] J Antolín, JC Angulo, and S López-Rosa. Fisher and Jensen–Shannon divergences: Quantitative comparisons among distributions. application to position and momentum atomic densities. *The Journal of chemical physics*, 130(7):074110, 2009.
[5] Marc Arnaudon and Frank Nielsen. On approximating the Riemannian 1-center. *Computational Geometry*, 46(1):93–104, 2013.

[6] Arindam Banerjee, Srujana Merugu, Inderjit S Dhillon, and Joydeep Ghosh. Clustering with Bregman divergences. *Journal of machine learning research*, 6(Oct):1705–1749, 2005.

[7] Ole Barndorff-Nielsen. *Information and exponential families: in statistical theory*. John Wiley & Sons, 2014.

[8] Patrick Billingsley. *Probability and measure*. John Wiley & Sons, 2008.

[9] Jeremiah Birrell, Paul Dupuis, Markos A Katsoulakis, Luc Rey-Bellet, and Jie Wang. Variational representations and neural network estimation for rényi divergences. *arXiv preprint arXiv:2007.03814*, 2020.

[10] Stephen Boyd, Stephen P Boyd, and Lieven Vandenberghe. *Convex optimization*. Cambridge university press, 2004.

[11] Lev M. Bregman. The relaxation method of finding the common point of convex sets and its application to the solution of problems in convex programming. *USSR computational mathematics and mathematical physics*, 7(3):200–217, 1967.

[12] Rob Brekelmans, Vaden Masrani, Thang Bui, Frank Wood, Aram Galstyan, Greg Ver Steeg, and Frank Nielsen. Annealed importance sampling with q-paths. *arXiv preprint arXiv:2012.07823*, 2020.

[13] Rob Brekelmans, Frank Nielsen, Alireza Makhzani, Aram Galstyan, and Greg Ver Steeg. Likelihood ratio exponential families. *arXiv preprint arXiv:2012.15480*, 2020.

[14] Peter S Bullen. *Handbook of means and their inequalities*, volume 560. Springer Science & Business Media, 2013.

[15] Çagatay Candan. Chebyshev center computation on probability simplex with α-divergence measure. *IEEE Signal Processing Letters*, 27:1515–1519, 2020.

[16] Pengwen Chen, Yunmei Chen, and Murali Rao. Metrics defined by Bregman divergences: Part 2. *Communications in Mathematical Sciences*, 6(4):927–948, 2008.

[17] Thomas M. Cover and Joy A. Thomas. *Elements of information theory*. John Wiley & Sons, 2012.

[18] Imre Csiszár. Eine informationstheoretische ungleichung und ihre anwendung auf beweis der ergodizitaet von markoffschen ketten. *Magyar Tud. Akad. Mat. Kutato Int. Koezl.*, 8:85–108, 1964.

[19] Imre Csiszár. Axiomatic characterizations of information measures. *Entropy*, 10(3):261–273, 2008.

[20] Jason V Davis and Inderjit Dhillon. Differential entropic clustering of multivariate Gaussians. In *Proceedings of the 19th International Conference on Neural Information Processing Systems*, pages 337–344, 2006.
[21] Jacob Deasy, Nikola Simidjievski, and Pietro Liò. Constraining Variational Inference with Geometric Jensen-Shannon Divergence. In Advances in Neural Information Processing Systems, 2020.

[22] Jiuding Duan and Yan Wang. Information-Theoretic Clustering for Gaussian Mixture Model via Divergence Factorization. In Proceedings of 2013 Chinese Intelligent Automation Conference, pages 565–573. Springer, 2013.

[23] Dominik Maria Endres and Johannes E Schindelin. A new metric for probability distributions. IEEE Transactions on Information theory, 49(7):1858–1860, 2003.

[24] Maria Dolores Esteban and Domingo Morales. A summary on entropy statistics. Kybernetika, 31(4):337–346, 1995.

[25] Dmitry Konstantinovich Faddeev. Zum Begriff der Entropie einer endlichen Wahrscheinlichkeitsschemas. Arbeiten zur Informationstheorie I. Deutscher Verlag der Wissenschaften, pages 85–90, 1957.

[26] Aurélie Fischer. Quantization and clustering with Bregman divergences. Journal of Multivariate Analysis, 101(9):2207–2221, 2010.

[27] Maurice Fréchet. Les éléments aléatoires de nature quelconque dans un espace distancié. Annales de l'institut Henri Poincaré, 10(4):215–310, 1948.

[28] Bent Fuglede and Flemming Topsoe. Jensen-Shannon divergence and Hilbert space embedding. In International Symposium on Information Theory, 2004. ISIT 2004. Proceedings., page 31. IEEE, 2004.

[29] Ian Goodfellow, Yoshua Bengio, Aaron Courville, and Yoshua Bengio. Deep learning. MIT press Cambridge, 2016.

[30] Ian J Goodfellow, Jean Pouget-Abadie, Mehdi Mirza, Bing Xu, David Warde-Farley, Sherjil Ozair, Aaron Courville, and Yoshua Bengio. Generative adversarial networks. arXiv preprint arXiv:1406.2661, 2014.

[31] Roger Grosse, Chris J Maddison, and Ruslan Salakhutdinov. Annealing between distributions by averaging moments. In Proceedings of the 26th International Conference on Neural Information Processing Systems, pages 2769–2777, 2013.

[32] Andréï Nikolaevich Kolmogorov and Guido Castelnuovo. Sur la notion de la moyenne. G. Bardi, tip. della R. Accad. dei Lincei, 1930.

[33] Solomon Kullback. Information theory and statistics. Courier Corporation, 1997.

[34] Jianhua Lin. Divergence measures based on the Shannon entropy. IEEE Transactions on Information theory, 37(1):145–151, 1991.

[35] Stuart Lloyd. Least squares quantization in PCM. IEEE transactions on information theory, 28(2):129–137, 1982.

[36] Geoffrey J McLachlan and David Peel. Finite Mixture Models. John Wiley & Sons, 2004.
[37] Tetsuzo Morimoto. Markov processes and the $H$-theorem. *Journal of the Physical Society of Japan*, 18(3):328–331, 1963.

[38] Constantin P Niculescu and Lars-Erik Persson. *Convex Functions and Their Applications: A Contemporary Approach*. Springer, 2018.

[39] Frank Nielsen. A family of statistical symmetric divergences based on Jensen’s inequality. *arXiv preprint arXiv:1009.4004*, 2010.

[40] Frank Nielsen. Chernoff information of exponential families. *arXiv preprint arXiv:1102.2684*, 2011.

[41] Frank Nielsen. An information-geometric characterization of Chernoff information. *IEEE Signal Processing Letters*, 20(3):269–272, 2013.

[42] Frank Nielsen. What is an information projection? *Notices of the AMS*, 65(3):321–324, 2018.

[43] Frank Nielsen. On the Jensen–Shannon symmetrization of distances relying on abstract means. *Entropy*, 21(5):485, 2019.

[44] Frank Nielsen. On a generalization of the Jensen–Shannon divergence and the Jensen–Shannon centroid. *Entropy*, 22(2):221, 2020.

[45] Frank Nielsen and Sylvain Boltz. The Burbea-Rao and Bhattacharyya centroids. *IEEE Transactions on Information Theory*, 57(8):5455–5466, 2011.

[46] Frank Nielsen and Richard Nock. Clustering multivariate normal distributions. In *Emerging Trends in Visual Computing*, pages 164–174. Springer, 2008.

[47] Frank Nielsen and Richard Nock. Sided and symmetrized Bregman centroids. *IEEE Transactions on Information Theory*, 55(6):2882–2904, 2009.

[48] Frank Nielsen and Richard Nock. Generalizing skew Jensen divergences and Bregman divergences with comparative convexity. *IEEE Signal Processing Letters*, 24(8):1123–1127, 2017.

[49] Frank Nielsen and Ke Sun. Guaranteed bounds on information-theoretic measures of univariate mixtures using piecewise log-sum-exp inequalities. *Entropy*, 18(12):442, 2016.

[50] Frank Nielsen and Mariette Yvinec. An output-sensitive convex hull algorithm for planar objects. *International Journal of Computational Geometry & Applications*, 8(01):39–65, 1998.

[51] Bruno Pelletier. Informative barycentres in statistics. *Annals of the Institute of Statistical Mathematics*, 57(4):767–780, 2005.

[52] Alfréd Rényi et al. On measures of entropy and information. In *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, Volume 1: Contributions to the Theory of Statistics*. The Regents of the University of California, 1961.

[53] Olivier Schwander and Frank Nielsen. Learning mixtures by simplifying kernel density estimators. In *Matrix Information Geometry*, pages 403–426. Springer, 2013.
[54] Robin Sibson. Information radius. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 14(2):149–160, 1969.

[55] Robin Sibson. A brief description of natural neighbour interpolation. *Interpreting multivariate data*, 1981.

[56] Przemysław Spurek and Wiesław Palka. Clustering of Gaussian distributions. In *2016 International Joint Conference on Neural Networks (IJCNN)*, pages 3346–3353. IEEE, 2016.

[57] Tim Van Erven and Peter Harremos. Rényi divergence and Kullback-Leibler divergence. *IEEE Transactions on Information Theory*, 60(7):3797–3820, 2014.

[58] Dániel Virosztek. The metric property of the quantum Jensen-Shannon divergence. *Advances in Mathematics*, 380:107595, 2021.

[59] Ju-Chiang Wang, Yi-Hsuan Yang, Hsin-Min Wang, and Shyh-Kang Jeng. Modeling the affective content of music with a Gaussian mixture model. *IEEE Transactions on Affective Computing*, 6(1):56–68, 2015.

[60] Kai Zhang and James T Kwok. Simplifying mixture models through function approximation. *IEEE Transactions on Neural Networks*, 21(4):644–658, 2010.