Self-energy effects in the Polchinski and Wick-ordered renormalization-group approaches

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Abstract

I discuss functional renormalization group (fRG) schemes, which allow for non-perturbative treatment of the self-energy effects and do not rely on the one-particle irreducible functional. In particular, I consider the Polchinski or Wick-ordered scheme with amputation of full (instead of bare) Green functions, as well as more general schemes, and establish their relation to the ‘dynamical adjustment propagator’ scheme by Salmhofer (2007 Ann. Phys., Lpz. 16 171).

While in the Polchinski scheme the amputation of full (instead of bare) Green functions improves treatment of the self-energy effects, the structure of the corresponding equations is not suitable to treat strong-coupling problems; it is also not evident how the mean-field solution of these problems is recovered in this scheme. For the Wick-ordered scheme, fully or partly excluding tadpole diagrams one can obtain forms of fRG hierarchy, which are suitable to treat strong-coupling problems. In particular, I emphasize the usefulness of the schemes, which are local in the cutoff parameter, and compare them to the one-particle irreducible approach.

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1. Introduction

Applications of the functional renormalization group (fRG) approach [1–3] to problems of condensed matter and high energy physics have made substantial progress recently [3–13]. However, correct treatment of the self-energy effects remains one of the most difficult points of many fRG schemes.

Currently, the one-particle irreducible (1PI) scheme [3–9] is mostly used for treatment of the self-energy effects within the fRG. Simple truncations within this scheme, however, do not allow us to fulfill Ward identities and sometimes do not reproduce even the results of the mean-field approach, for which correct treatment of the self-energy effects is crucial. The truncation of 1PI equations of [10] allowed us to reproduce the mean-field results and improve the results of more sophisticated approaches to fulfill Ward identities better. This approach
found its applications in the treatment of weakly and moderately interacting single-impurity problems \[11\], the Hubbard model in the symmetry broken phase \[12\] and, more recently, the two- and three-dimensional Heisenberg model \[13\].

At the same time, the 1PI approach has severe problems in describing the strong coupling regime of many-body systems. The problem of application of this approach in this case is mainly related to the one-loop structure of 1PI hierarchy of fRG equations. Although higher loop contributions can be obtained by reinserting the one-loop contributions from higher to lower order vertices, this procedure is usually spoiled by truncations of the hierarchy and the projection schemes used in approximate solutions, which do not allow exact results to be obtained for non-ladder diagrams \[14\]. In particular, the truncation, which neglects 8-point vertices and accounts for the full frequency and momentum dependence of the 4-point and 6-point vertices (which is already a very complicated problem), yields a correct result for two-loop diagrams only \[14\].

On the other hand, some non-1PI schemes, in particular the Wick-ordered scheme \[2\], already in the original formulation explicitly include many-loop contributions and can, in principle, improve treatment of the strong-coupling regime. The Wick-ordered scheme in its original formulation is however not suited to treat self-energy effects in a non-perturbative way. The change from amputation of the bare to interacting Green functions, proposed in \[2\], is not sufficient for such a treatment, since the function with respect to which the Wick ordering is performed, should also be adjusted appropriately. A way for non-perturbative treatment of the self-energy effects was considered in the ‘dynamical adjustment’ scheme of \[15\]. In this paper, we propose a somewhat different approach for non-perturbative treatment of the self-energy effects and consider in detail both local schemes, which do not remove fully tadpole terms from the RG hierarchy, as well as non-local schemes, removing fully tadpole terms. We compare the approaches of the paper to the dynamical adjusting scheme of \[15\] and the 1PI approach, as well as discuss the results of the application of these methods to the toy model.

2. Self-energy effects in the Polchinski scheme

We consider the model described by an action

\[
S_\Lambda[\overline{\psi}, \psi] = S_0^\Lambda[\overline{\psi}, \psi] + \mathcal{V}[\overline{\psi}, \psi],
\]

where \(\overline{\psi}, \psi\) are the bosonic or fermionic (Grassman) fields, \(\mathcal{V}\) is an interaction and

\[
S_0^\Lambda[\overline{\psi}, \psi] = \int d^d x \int d\tau \overline{\psi}(C_{0,\Lambda})^{-1}\psi
\]

is quadratic in fields. \(C_{0,\Lambda}\) is the cut bare propagator, e.g. having the Fourier transform

\(C_{0,\Lambda} = \chi_{\Lambda,\Lambda}/(i\omega_n - \epsilon_k)\),

where \(i\omega_n\) are bosonic or fermionic Matsubara frequencies, \(\epsilon_k\) is the dispersion and \(\chi_{\Lambda,\Lambda}\) is the cutoff function, which cuts off the low-energy modes, e.g. \(\chi_{\Lambda,\Lambda} = 0\) and \(\chi_{\Lambda,0} = 1\).

The standard procedure of derivation of fRG equations relies on differentiating the generating functional for connected vertices, amputated with the non-interacting Green functions:

\[
\mathcal{V}_\Lambda[\overline{\psi}, \psi] = -\ln \int D[\overline{\chi}, \chi] e^{-S_0^\Lambda[\overline{\chi}, \chi] - \mathcal{V}[\overline{\psi}, \psi, \chi]}.\]

This procedure can be supplemented by a consequent reamputation of external legs, changing their amputation from non-interacting to that with interacting Green functions.

For the purposes of this paper, we consider a somewhat more general procedure, which allows us to obtain the renormalization-group equations accounting for the self-energy effects
with further possible generalizations. To this end, we introduce the counterterm $\Sigma_\Lambda \overline{\psi} \psi$ into the bare action and replace $(C_{0,\Lambda})^{-1} \to C^{-1}_{\Lambda}$:

$$S_\Lambda[\overline{\psi}, \psi] = \overline{\psi} C^{-1}_{\Lambda} \psi + \sum_{\Lambda} \Sigma_\Lambda \psi + \mathcal{V}[\overline{\psi}, \psi].$$

(4)

Although for the choice

$$C^{-1}_{\Lambda} = (C_{0,\Lambda})^{-1} - \Sigma_\Lambda$$

the counterterm is canceled, we will not in general assume the validity of equation (5) during the flow, requiring its fulfillment only at the end of the flow. We also treat $\Sigma_\Lambda$ as a part of the interaction, such that amputation of the vertices by the functions $C_\Lambda$ is achieved naturally in this way, since the non-interacting term now contains $C^{-1}_{\Lambda}$. Introducing the effective interaction in the standard way

$$e^{-\mathcal{V}_\Lambda(\overline{\psi}, \psi)} = \int D[\overline{\psi}, \psi] e^{-\overline{\psi}(C_{\Lambda}^{-1})\psi - \mathcal{V}[\overline{\psi}, \psi]} \Sigma_\Lambda(\overline{\psi}, \psi)$$

(6)

and performing the same steps, as for deriving the Polchinski equation, we obtain

$$\partial_\Lambda \mathcal{V}_\Lambda = \Delta_{C_\Lambda} \mathcal{V}_\Lambda - \Delta_{C_\Lambda}^{(1)} \mathcal{V}_{\Lambda}^{(1)} + \sum_{\Lambda} \frac{\delta \mathcal{V}_\Lambda}{\delta \Sigma_\Lambda},$$

(7)

where $\Delta_{C_\Lambda} \mathcal{V}_\Lambda = \text{Tr}(\hat{C}_\Lambda \delta^2 \mathcal{V}_\Lambda / \delta^2 \Lambda)$, $\Delta_{C_\Lambda}^{(1)} \mathcal{V}_{\Lambda}^{(1)} = \left( \frac{\delta \mathcal{V}_\Lambda}{\delta \overline{\psi}}, \hat{C}_\Lambda \frac{\delta \mathcal{V}_\Lambda}{\delta \overline{\psi}} \right)$, $\psi = (\overline{\psi}, \psi)$, $\partial_\Lambda$ and the dots stand for the derivatives over $\Lambda$. Expressing the variational derivative $(\delta \mathcal{V}_\Lambda / \delta \Sigma_\Lambda)$ through variational derivatives of $\mathcal{V}_\Lambda$ over fields, we obtain

$$\partial_\Lambda (\mathcal{V}_\Lambda - \Sigma_\Lambda \overline{\psi} \psi) = \Delta_{C_{\Lambda - \Sigma_{\Lambda}}} \mathcal{V}_\Lambda - \Delta_{C_{\Lambda - \Sigma_{\Lambda}}} \mathcal{V}_{\Lambda}^{(1)} \mathcal{V}_{\Lambda}^{(2)} - \Sigma_\Lambda \mathcal{C} \left( \frac{\delta \mathcal{V}_\Lambda}{\delta \overline{\psi}} + \frac{\delta \mathcal{V}_\Lambda}{\delta \psi} \right).$$

(8)

Choosing $\Sigma_\Lambda = 0$, we arrive at the hierarchy of rRG equations in the Polchinski scheme [1]

$$\partial_\Lambda \mathcal{V}_\Lambda = \Delta_{C_{0,\Lambda}} \mathcal{V}_\Lambda - \Delta_{C_{0,\Lambda}}^{(1)} \mathcal{V}_{\Lambda}^{(1)} \mathcal{V}_{\Lambda}^{(2)}.$$  

(9)

Considering the expansion of $\mathcal{V}_\Lambda$ in fields

$$\mathcal{V}_\Lambda = \sum_m V_m \Psi(x_1) \ldots \Psi(x_m),$$

this hierarchy reads

$$\dot{V}_2 = -V_2 \dot{C}_{0,\Lambda} V_2 + V_4 \circ \dot{C}_{0,\Lambda},$$

$$\dot{V}_4 = -4V_2 \dot{C}_{0,\Lambda} V_4 + V_6 \circ \dot{C}_{0,\Lambda},$$

$$\dot{V}_6 = -6V_2 \dot{C}_{0,\Lambda} V_6 + 4V_4 \dot{C}_{0,\Lambda} V_4 + V_8 \circ \dot{C}_{0,\Lambda},$$

$$\dot{V}_8 = \ldots,$$

where $\circ$ denotes convolution with respect to internal momenta and frequencies. On the other hand, for the choice $V_2 = 0$ (8) yields equations for the self-energy $\Sigma_\Lambda = \Sigma_\Lambda$ and fully amputated vertex functions, which are denoted in the following as $H_m$:

$$\Sigma_\Lambda = -H_4 \circ (\dot{C}_\Lambda - \Sigma_\Lambda C^2),$$

$$H_4 = H_6 \circ (\dot{C}_\Lambda - \Sigma_\Lambda C^2) - 4H_2 C_\Lambda \Sigma_\Lambda,$$

$$H_6 = H_8 \circ (\dot{C}_\Lambda - \Sigma_\Lambda C^2) + 4H_4 \dot{C}_\Lambda - 6H_6 C_\Lambda \Sigma_\Lambda,$$

(11)

Assuming that relation (5) between the full and bare Green functions is fulfilled during the flow, scheme (11) can be obtained from equations (10) by substituting

$$V_2 = H_2 (C_\Lambda/C_{0,\Lambda})^2 = - (C_\Lambda/C_{0,\Lambda}) \Sigma_\Lambda,$$

$$V_4 = H_4 (C_\Lambda/C_{0,\Lambda})^4, \ldots$$

(12)
For the choice (5) we also have $\hat{C}_\Lambda - \hat{\Sigma}_\Lambda C_\Lambda^2 = \frac{\hat{C}_{0,\Lambda}}{(1 - C_{0,\Lambda} \Sigma_\Lambda)^2}$, which allows us to recover the equations for amputated vertices, derived in [2]. However, as already mentioned, general equations (11) do not assume the validity of relation (5) during the flow; they are valid for any functional $C_\Lambda[\Sigma_\Lambda]$ that fulfills $C_{\Lambda=\Lambda_0} = 0$ at the beginning of the flow and relation (5) at the end of the flow.

The equation for the self-energy (11) is similar to its form in the 1PI scheme (and coincides with this scheme for the choice (5)). However, the equations for higher order vertices are organized differently: the pairs of vertices are connected only by tree-like diagrams, while one- and higher loop contributions are obtained by substituting results from higher to lower order vertices, which may lead to difficulties in the strong-coupling regime. These equations also do not allow us to see easily how the mean-field results are reproduced.

3. Wick-ordered schemes

3.1. General consideration and non-local schemes

To overcome these difficulties, we following [2, 15] consider the Wick-ordered modification of equations (10). This scheme considers an expansion of the effective action in the Wick-ordered monomials $\Omega_D(\Psi(x_1) \ldots \Psi(x_m))$:

$$\mathcal{V}_\Lambda = \sum_m W_m \Omega_D(\Psi(x_1) \ldots \Psi(x_m)) = e^{-\Delta \Lambda} \tilde{V}_\Lambda \quad (13)$$

$$\tilde{V}_\Lambda = \sum_m W_m \Psi(x_1) \ldots \Psi(x_m),$$

where the Wick-ordering propagator $D_\Lambda$ fulfills $D_{\Lambda=0} = 0$. Performing Wick ordering of equation (8) using (13) and assuming $W_2 = 0$, we obtain

$$\hat{\Sigma}_\Lambda = -\left[ \frac{d(D_\Lambda + C_\Lambda)}{d\Lambda} - \hat{\Sigma}_\Lambda C_\Lambda (2D_\Lambda + C_\Lambda) \right] H_4$$

$$+ \sum_{m_1, m_2 \geq 4} H_{m_1} \circ (\hat{C}_\Lambda - \hat{\Sigma}_\Lambda C_\Lambda^2) \circ D_{\Lambda, m_2}^{m_1, m_2 - 2} \circ H_{m_2}$$

$$H_m = \left[ \frac{d(D_\Lambda + C_\Lambda)}{d\Lambda} - \hat{\Sigma}_\Lambda C_\Lambda (2D_\Lambda + C_\Lambda) \right] H_{m+2} - mH_m \hat{\Sigma}_\Lambda C_\Lambda$$

$$- \sum_{m_1, m_2 \geq 4} H_{m_1} \circ (\hat{C}_\Lambda - \hat{\Sigma}_\Lambda C_\Lambda^2) \circ D_{\Lambda, m_2}^{m_1, m_2 - 1} \circ H_{m_2} \quad (m \geq 4). \quad (15)$$

In the square brackets we have grouped together terms, corresponding to generalized tadpole diagrams. The standard choice [2] is $D_\Lambda = -\hat{C}_{\Lambda,0}$. Let us however choose $D$ such that tadpole diagrams, including the contributions proportional to $\Sigma$, cancel out. Then, we have the following differential equation for $D$:

$$\frac{d(D_\Lambda + C_\Lambda)}{d\Lambda} - \hat{\Sigma}_\Lambda C_\Lambda (2D_\Lambda + C_\Lambda) = 0. \quad (16)$$

The resulting hierarchy of equations is simplified and read

$$\hat{\Sigma}_\Lambda = \sum_{m_1, m_2 \geq 4} H_{m_1} \circ (\hat{C}_\Lambda - \hat{\Sigma}_\Lambda C_\Lambda^2) \circ D_{\Lambda, m_2}^{m_1, m_2 - 2} \circ H_{m_2}$$

$$H_m = -mH_m \hat{\Sigma}_\Lambda C_\Lambda - \sum_{m_1, m_2 \geq 4} H_{m_1} \circ (\hat{C}_\Lambda - \hat{\Sigma}_\Lambda C_\Lambda^2) \circ D_{\Lambda, m_2}^{m_1, m_2 - 1} \circ H_{m_2} \quad (m \geq 4). \quad (17)$$
Similar to [15], to guarantee that $D_{\Lambda=0} = 0$ one has to solve equation (16) in terms of the corresponding condition at $\Lambda = 0$, instead of using the initial condition at $\Lambda_0$. The corresponding solution to equation (16) has the form

$$D_\Lambda = -\int_0^\Lambda d\Lambda' (\hat{C}_{\Lambda'} - \hat{\Sigma}_{\Lambda'} C^2_{\Lambda'}) \exp \left( 2 \int_0^\Lambda d\Lambda'' C_{\Lambda''} \hat{\Sigma}_{\Lambda''} \right) .$$

(18)

In particular, for the choice of the propagator (5) we obtain

$$D_\Lambda = \int_0^\Lambda d\Lambda' D'_{\Lambda',0} \exp \left( 2 \int_0^\Lambda d\Lambda'' C_{\Lambda''} \hat{\Sigma}_{\Lambda''} \right)$$

$$= \frac{1}{(1 - C_{0,\Lambda} \Sigma_{\Lambda})^2} \int_0^\Lambda d\Lambda' \hat{C}_{0,\Lambda} \exp \left( 2 \int_0^\Lambda d\Lambda'' C_{\Lambda''} \hat{\Sigma}_{\Lambda''} \right)$$

(19)

and

$$\hat{\Sigma}_{\Lambda} = \frac{1}{2} \sum_{m_1,m_2 \geq 4} D_{m_1} \circ \frac{1}{(1 - C_{0,\Lambda} \Sigma_{\Lambda})^2} \hat{C}_{0,\Lambda} \circ D_{\Lambda} \Delta_{m_1,m_2} \circ [D_{\Lambda} \Delta_{m_1,m_2}]^{-2} \circ \Sigma_{m_2}$$

$$H_m = -m H_m C_{\Lambda} \hat{\Sigma}_{\Lambda}$$

(20)

where $\hat{C}_{0,\Lambda} = \hat{C}_{0,\Lambda} \circ (1 - C_{0,\Lambda} \Sigma_{\Lambda})^2$.

Let us see how the mean-field solution is recovered from equations (15) or (20). Since there are no tadpole diagrams for $d\Sigma / d\Lambda$, we have $d\Sigma_{\Lambda}^{MF} / d\Lambda = 0$, i.e., $\Sigma = \Sigma^{MF}$ is constant in $\Lambda$ and equal to its mean-field value, which is given by a (self-consistently determined) sum of tadpole diagrams, absorbed into the definition of Wick ordering. Equation (18) is easily integrated and yields

$$D_{\Lambda}^{MF} = C_{\Lambda=0}^{MF} - C_{\Lambda}^{MF} .$$

(21)

In the case of fermions and moving Fermi surface ($\text{Re} \Sigma \neq 0$), this function for smooth cutoff and intermediate $\Lambda$ has singularity at both physical and running Fermi surfaces. Since $D_{\Lambda}^{MF} = -C_{\Lambda}^{MF}$, the equation for $H_{\Lambda}$ can then be also solved analytically to reproduce the standard random-phase approximation result

$$H_{\Lambda} = \frac{H_{\Lambda}^{MF}}{1 - H_{\Lambda}^{MF} \text{Tr}(C_{\Lambda}^{MF})^2} .$$

(22)

Returning to the treatment of the interaction beyond mean-field theory, the advantage of equations (19) and (20) is in their simple form in the sharp cutoff limit. Assuming $\chi_{\Lambda} = \theta(|\epsilon_k| - \Lambda)$, $C_{\Lambda} = \chi_{\Lambda} / (i \omega_n - \epsilon_k)$, we obtain

$$D_{\Lambda} = \frac{\theta(\Lambda - |\epsilon_k|)}{i \omega_n - \epsilon_k - \Sigma_{\Lambda=0}|_{\epsilon_k}} ,$$

(23)

which requires knowledge of $\Sigma_{\Lambda}$ at $\Lambda' \ll \Lambda$, which has to be determined self-consistently, and the single-scale propagator

$$F_{\Lambda} = \frac{\delta(\Lambda - |\epsilon_k|)}{i \omega_n - \epsilon_k - \Sigma_{\Lambda}} .$$

(24)

Both propagators have simpler form than those in [15], where additional integration over the cutoff parameter is still present even for the sharp cutoff (see the appendix).
Equations (15) can be somewhat generalized by considering reamputation of vertices $H^{(n)}_\Lambda \equiv \tilde{H}^{(n)}_\Lambda (S_\Lambda)^n$ with respect to the amputation by full Green functions $C_\Lambda$ (corresponding to $S_\Lambda = 1$):

\[
S_\Lambda^{-2} \Sigma_\Lambda = - \left[ \frac{d\tilde{D}_\Lambda}{d\Lambda} - 2\tilde{D}_\Lambda (\Sigma_\Lambda C_\Lambda + \tilde{S}_\Lambda S_\Lambda^{-1}) + \tilde{S}_\Lambda (\tilde{C}_\Lambda - \Sigma_\Lambda C_\Lambda^2) \right] \circ \tilde{H}_4 + \sum_{m_1, m_2 \geq 4} \tilde{H}_{m_1} \circ S_\Lambda^2 (\tilde{C}_\Lambda - \Sigma_\Lambda C_\Lambda^2) \circ \tilde{D}_\Lambda^{m_1-m_2-2} \circ \tilde{H}_{m_2}
\]

\[
\tilde{H}_m = \left[ \frac{d\tilde{D}_\Lambda}{d\Lambda} - 2\tilde{D}_\Lambda (\Sigma_\Lambda C_\Lambda + \tilde{S}_\Lambda S_\Lambda^{-1}) + \tilde{S}_\Lambda (\tilde{C}_\Lambda - \Sigma_\Lambda C_\Lambda^2) \right] \circ \tilde{H}_{m+2} - \sum_{m_1, m_2 \geq 4} \tilde{H}_{m_1} \circ S_\Lambda^2 (\tilde{C}_\Lambda - \Sigma C_\Lambda^2) \circ \tilde{D}_\Lambda^{m_1-m_2-1} \circ \tilde{H}_{m_2} - m\tilde{H}_m (C_\Lambda \Sigma + \tilde{S}_\Lambda S_\Lambda^{-1}),
\]

(25)

where $\tilde{D}_\Lambda = D_\Lambda S_\Lambda$, $S_\Lambda$ being the arbitrary function of the cutoff parameter and the degrees of freedom (e.g., momenta, frequencies), which does not vanish or diverge during the flow, excluding isolated points in momentum–frequency space. Equations (25) allow us to establish connection with the dynamic adjustment propagator scheme of [15]. The explicit relation between equations (25) and those of [15] is discussed in the appendix and involves passing from the function $C_\Lambda$ to a different function $D^{n}_\Lambda$ with the use of equation (A.1).

The amputation factor $S_\Lambda$ can be chosen to simplify further equations (25). In particular, if, similar to [15], we require $C_\Lambda \Sigma_\Lambda + \tilde{S}_\Lambda S_\Lambda^{-1} = 0$, then equations (25) take a simpler form due to cancellation of some terms in this specially chosen amputation of vertices. This way, equations (25) also allow us to establish a simpler view on the result (18). Indeed, let us assume that $\Sigma_\Lambda C_\Lambda + \tilde{S}_\Lambda S_\Lambda^{-1} = 0$ for some $S$. Then, we have

\[
S_\Lambda = \exp \left( \int_\Lambda \Sigma_\Lambda C_\Lambda \, d\Lambda' \right).
\]

(26)

The condition of vanishing generalized tadpole diagrams yields $\frac{d\tilde{D}_\Lambda}{d\Lambda} = -\tilde{S}_\Lambda (\tilde{C}_\Lambda - \Sigma_\Lambda C_\Lambda^2)$, which leads us to equation (18). Although condition (18) generally does not yield Wick propagators, whose derivative coincides with the single-scale propagator $\Sigma_\Lambda C_\Lambda^2 - \tilde{C}_\Lambda$, this is achieved in the amputated scheme with the amputation factor $S_\Lambda$ given by equation (26). In general, analytical or numerical evaluation of the integral in this equation can be, however, rather complicated.

### 3.2. Local schemes

In practical applications, the schemes that are local in $\Lambda$ may have some advantage. The simplest choice of the propagators

\[
D_\Lambda = \frac{\chi_{<,\Lambda}}{C_0^{\Lambda} - \Sigma_\Lambda}; \quad C_\Lambda = \frac{\chi_{>,\Lambda}}{C_0^{\Lambda} - \Sigma_\Lambda}
\]

(27)

suffers from the problem, discussed in [15], that it yields remaining tadpole terms in $\partial_\Lambda H_m$, proportional to $(D_0^{\Lambda} \partial_\Lambda \Sigma_\Lambda) \circ H_{m+2}$, which have potential infrared divergences due to the square of the propagator $D_\Lambda$. To avoid this problem, we choose the propagator $C_\Lambda$ according to equation (5), including the mean-field self-energy in the bare propagator

\[
C_{0,\Lambda} = \chi_{>,\Lambda} / (C_{0,\Lambda}^{\Lambda} - \Sigma^{MF})
\]

and using a similar expression for $D_\Lambda$:

\[
D_\Lambda = \frac{\chi_{<,\Lambda}}{C_{0,\Lambda}^{\Lambda} - \Sigma^{MF} - \chi_{>,\Lambda} \Sigma_\Lambda}; \quad C_\Lambda = \frac{\chi_{>,\Lambda}}{C_{0,\Lambda}^{\Lambda} - \Sigma^{MF} - \chi_{>,\Lambda} \Sigma_\Lambda}
\]

(28)
to obtain from equations (15) the local flow equations
\[
\Sigma_\Lambda = -T_\Lambda \circ H_4 = \sum_{m_1,m_2 \geq 4} H_{m_1} \circ F_\Lambda \circ D_{\Lambda}^{n_1+n_2-2} \circ H_{m_2},
\]
\[
H_m = T_\Lambda \circ H_{m+2} - m H_{m} C_\Lambda \Sigma_\Lambda + \sum_{m_1,m_2 \geq 4} H_{m_1} \circ F_\Lambda \circ D_{\Lambda}^{n_1+n_2-1} \circ H_{m_2},
\]
where
\[
T_\Lambda = \frac{r C_{\Lambda}^{-1} \tilde{\xi}_{\Lambda} m_{\Lambda}}{(C_{\Lambda}^{-1} - \Sigma_{\Lambda}^\text{MF} - \chi_{\Lambda} \Sigma_{\Lambda})^2},
\]
\[
F_\Lambda = -\frac{C_{\Lambda}^{-2} \tilde{\xi}_{\Lambda} m_{\Lambda} + (C_{\Lambda}^{-1} - \Sigma_{\Lambda}^\text{MF}) \tilde{\chi}_{\Lambda}}{(C_{\Lambda}^{-1} - \Sigma_{\Lambda}^\text{MF} - \chi_{\Lambda} \Sigma_{\Lambda})^2},
\]
and we have assumed \( \chi_{\Lambda} + \tilde{\chi}_{\Lambda} = r \). Due to the special choice of propagators, the solution to equations (29) has a natural physical interpretation even at the intermediate stages of the flow, since it corresponds to the flow of the functional with the bare propagator \( C_{\Lambda} \) after the Wick ordering with the propagator \( D_\Lambda \). Equations (29) have to be solved with the initial condition \( \Sigma_\Lambda = 0 \), since the constant (mean-field) initial part of \( \Sigma_\Lambda \) enters propagators explicitly through \( \Sigma_{\Lambda}^\text{MF} \).

Scheme (28)–(30) may be useful for the realization of the interaction [8] and temperature [5, 7, 9] flow in the Wick-ordered scheme. Indeed, the choice \( \chi_{\Lambda} = \Lambda^{1/2}, \tilde{\chi}_{\Lambda} = 1 - \Lambda^{1/2} \) and \( C_{\Lambda} = C_0 \) in equation (27) is analogous to the interaction flow [8], while the choice \( \Lambda = T, \chi_{\Lambda} = T^{-1/2}, \tilde{\chi}_{\Lambda} = T_0^{-1/2} \) and \( C_{\Lambda} = (i \omega_n - \epsilon_k) / T \), where \( T \) is the temperature and \( T_0 \) the final temperature of the flow, is analogous to the temperature flow in the 1PI approach [5, 7, 9]. The disadvantage of the latter scheme in the Wick-ordered approach is that similar to the 1PI flow with counterterms [16] it requires the final temperature of the flow as an input, and therefore does not allow us to obtain the whole temperature evolution of the system for the same input parameters.

Within similar truncations of the hierarchy, the momentum, frequency, or interaction flow in the Wick-ordered scheme with the choice of propagators (28) can already be superior to the 1PI scheme, since the latter contains tadpole contributions to the \( m \)-point vertex of the order \( O(H_{m+2}) \) in the weak-coupling regime, which are needed to generate multiple-loop contributions from fRG hierarchy. In contrast, the Wick-ordered scheme contains multi-loop contributions explicitly, while the ‘dangerous’ tadpole contributions have a smaller value; the initial (mean-field) self-energy is also already included in both propagators \( C_\Lambda \) and \( D_\Lambda \). In particular, in the weak-coupling limit in scheme (29) we obtain \( \Sigma_\Lambda \sim \Sigma_\Lambda = O(H_4^2) \), therefore yielding the correction to \( \Sigma_\Lambda \) from the remaining tadpole diagrams of the order \( O(H_4^2) \) and the corresponding correction to the vertices \( H_{m\geq 4} \) of the order \( O(H_4^2 H_{m+2}) \).

A smaller value of the tadpole terms in the Wick-ordered scheme and explicit presence of multi-loop contributions also implies the possibility of having less fine parametrization of vertices, required for the solution of equations (29) and providing the same accuracy as in the 1PI scheme. Indeed, considering for example contributions to the self-energy, the effort of evaluation of \( r \)-loop contribution in the Wick-ordered scheme scales as \( (n_p n_o)^r \), where \( n_p \) is the number of integration points in momentum–frequency space and \( n_o \) is the number of internal (e.g. spin, orbital, etc) degrees of freedom. At the same time, the parametrization of the \( 2r \)-point vertex in the 1PI scheme, needed to compute \( r \)-loop contribution to the self-energy, requires \( n_{\text{1PI}}^{r+1} n_p n_o^{2r+1} \) operations, where \( n_{\text{1PI}} \) is the number of patches for the 1PI equations. Equal accuracy of the two schemes is achieved for \( n_{\text{1PI}} \sim n_p \), and for not very large number of internal degrees of freedom the required increase of the computational effort in
the 1PI scheme \( \sim \langle n_p n_q \rangle \) can be rather large. While for the lowest-order contributions one can rewrite 1PI equations to include explicitly multi-loop contributions (see e.g. [14]), this is not straightforward to perform in general. Comparison of the contributions to higher order vertices also reveals reduction of necessary operations in the Wick-ordered approach.

Application of the full cancellation of the tadpole diagrams, given by equation (19) or the dynamical adjusting scheme of [15], may further improve applicability of the Wick-ordered equations, but is considerably more complicated because of the necessity to fulfill the self-consistency condition \( D_{\lambda=0} = 0 \) (or \( R_{\lambda=0} = 0 \)), which results in the requirement to know the self-energy \( \Sigma_{\lambda'} \) at stages of the flow later than the current one. Yet, even in this case one can search for easier integrable functions by adjusting \( C_{\lambda} \) in equations (15) and (25).

### 3.3. Toy model

To gain insight into the applicability of the described approaches, we consider the toy \( \phi^4 \)-like model [17]

\[
S_0 = \frac{a}{2} \phi^2; \quad V = \frac{b}{4!} \phi^4.
\]

In the following we put \( a = \frac{1}{\chi_{\lambda}}, \) with \( \chi_{\lambda} = 1 - \Lambda \) and use \( \Lambda \) as a scaling parameter, which changes from 1 (bare model) to zero. In the truncation \( H_6 = 0 \) the flow equations (16) and (20) take the form

\[
\dot{\Sigma}_\lambda = -\frac{1}{2} b^2_\lambda F_\lambda D^2_\lambda
\]

\[
\dot{b}_\lambda = 3 b^2_\lambda F_\lambda D_\lambda - 4 b_\lambda C_\lambda \dot{\Sigma}_\lambda
\]

\[
D_\lambda = F_\lambda + 2 \Sigma_\lambda C_\lambda D_\lambda,
\]

where we choose, according to equation (28),

\[
C_\lambda = \frac{1 - \Lambda}{1 - \Sigma_{\lambda}^{\mathrm{MF}} - (1 - \Lambda) \Sigma_\lambda},
\]

such that

\[
F_\lambda = -\dot{\Sigma}_\lambda + \Sigma_\lambda C_\lambda^2 = \frac{1 - \Sigma_{\lambda}^{\mathrm{MF}}}{(1 - \Sigma_{\lambda}^{\mathrm{MF}} - (1 - \Lambda) \Sigma_\lambda)^2};
\]

the initial conditions \( \Sigma_{\lambda}^{\mathrm{MF}} = \frac{1}{2}(1 - \sqrt{1 + 2 b}) \), \( \Sigma_{\lambda=0} = 0, b_{\lambda=0} = b \) and \( D_{\lambda=0} = 0 \). In numerical calculations, we adjust the initial value \( D_{\lambda=0} \) to achieve vanishing of the propagator at the end of the flow.

On the other hand, the local equations (28) and (29) yield

\[
\Sigma_\lambda = -\frac{1}{2} b_\lambda T_\lambda - \frac{1}{2} b^2_\lambda F_\lambda D^2_\lambda
\]

\[
\dot{b}_\lambda = -4 b_\lambda C_\lambda \dot{\Sigma}_\lambda + 3 b^2_\lambda F_\lambda D_\lambda
\]

with

\[
D_\lambda = \frac{\Lambda}{1 - \Sigma_{\lambda}^{\mathrm{MF}} - (1 - \Lambda) \Sigma_\lambda}
\]

\[
T_\lambda = \frac{\Sigma_\lambda - \Lambda (1 - \Lambda) \dot{\Sigma}_\lambda}{(1 - \Sigma_{\lambda}^{\mathrm{MF}} - (1 - \Lambda) \Sigma_\lambda)^2}.
\]

The results of the solution of equations (32) and (35) for the self-energy and coupling constant at the end of the flow for interaction strength \( b = 1, 2, 3 \) are presented in the table. For comparison, we also present the result of the 1PI approach [17] and the local scheme, suggested in [15], see also equations (A.7); we find numerically, that using the choice (27) of propagators in equations (35) yields the same results as the latter scheme. Note that, as
Table 1. The self-energy and connected fully amputated 4-point vertex $b_R$ of the model (31) obtained by different methods for values of the coupling constant $b = 1, 2, 3$.

| Method                          | $-\Sigma$ (1) | $-\Sigma$ (2) | $-\Sigma$ (3) | $b_R$ (1)   | $b_R$ (2)   | $b_R$ (3)   |
|--------------------------------|----------------|----------------|----------------|-------------|-------------|-------------|
| 1PI, reference [17]            | 0.320 398      | 0.519 824      | 0.677 443      | 0.513 382   | 0.809 275   | 1.056 86    |
| Equations (17), equations (32) | 0.323 881      | 0.527 393      | 0.687 805      | 0.530 763   | 0.855 922   | 1.133 21    |
| Local, reference [15]          | 0.330 197      | 0.546 350      | 0.720 883      | 0.535 013   | 0.874 050   | 1.171 81    |
| Equations (29); equations (35) | 0.329 972      | 0.545 415      | 0.719 051      | 0.542 277   | 0.899 275   | 1.219 13    |
| Exact                          | 0.332 425      | 0.550 557      | 0.726 505      | 0.607 899   | 1.055 430   | 1.464 69    |

discussed in the previous section, the local scheme based on the choice (27), as well as the local scheme of [15], is actually not suitable for treatment of many-body systems because of possible divergences in the flow equations. The non-local (adjusting) scheme of [15] (see also equations (A.5)) is found to yield numerical results very close to equations (32)–(34).

One can see from table 1 that for the model (31) all considered Wick-ordered schemes yield improvement in comparison to the 1PI scheme; equations (35) improve the results for vertices of the local approach of [15]. The self-energies of the two approaches are practically identical. Surprisingly, being computationally more expensive and yielding slight improvement of the 1PI scheme, the approaches, fully excluding tadpole diagrams, yield worse results for the considered toy model than the local schemes. Therefore, the adjustment of the Wick propagator to fully exclude tadpole diagrams does not generally imply smaller truncation errors. This fact can be attributed to the property of the Wick-ordered equations in the general amputation scheme (25) to have derivative of the Wick propagator non-equal to the single-scale propagator. Therefore, some remaining tadpole terms seem necessary to compensate this difference. Note that choosing special amputation of the vertices, which provide equality of the derivative of the Wick propagator to the single-scale propagator, as given by the equation (26) or considered in [15], does not yield further improvement of the truncation, since the amputation itself does not change physically observable quantities. For more complicated problems, one may also expect that schemes that are local in the cutoff parameter but do not fully exclude tadpole diagrams (such as scheme (29)) are more preferable.

4. Summary

In summary, we have considered renormalization-group schemes, which allow us to treat self-energy effects in a non-perturbative way and do not rely on using 1PI functionals. In the Wick-ordered scheme the choice of the propagator, with respect to which the Wick ordering is performed, allowing us to partly or fully remove tadpole diagrams from renormalization-group equations, has some advantage over the 1PI formalism. While fully removing tadpole terms yields the differential equation with 'finite' instead of initial condition and results that are similar to the earlier proposed approach [15], removing tadpole terms partly we have obtained differential equations that are local with respect to the cutoff parameter and avoid potential divergences in the flow equations. We have also argued that in general the suggested local approach may be superior to the 1PI formalism, due to smaller contribution of the tadpole diagrams, which implies that in general it requires less fine parametrization of the vertices than the 1PI approach. Further investigations of the necessity to keep some tadpole contributions in the Wick-ordering hierarchy and comparison of 1PI and Wick-ordered schemes for more complicated models would be of certain interest.
Appendix. Relation to the dynamic adjustment scheme of [15]

In this appendix we consider the relation of the approach of this paper compared with that of \[15\]. To relate equations (25) to those of \[15\], we put

\[
C_\Lambda S^2 = C_\Lambda - D^M_\Lambda
\]  

(A.1)

with some functions \(C_\Lambda\) and \(D^M_\Lambda\), which satisfy \(D^M_\Lambda|_{\Lambda=0} = C_\Lambda|_{\Lambda=0}\). Choosing \(C_\Lambda = S^2_\Lambda (C_0^{-1} - \Sigma_\Lambda)\), we obtain

\[
S^2_\Lambda (C_\Lambda - \hat{\Sigma}_\Lambda C_\Lambda) = S^2_\Lambda \left( \frac{d}{d\Lambda} (C_\Lambda S^2_\Lambda) - \hat{\Sigma}_\Lambda C^2_\Lambda / S^2_\Lambda \right) - \frac{dD^M_\Lambda}{d\Lambda}
\]

\[
+ 2(\hat{\Sigma}_\Lambda / S^2_\Lambda C_\Lambda + S^{-1}_\Lambda \hat{\Sigma}_\Lambda D^M_\Lambda - \hat{\Sigma}_\Lambda / S^2_\Lambda (D^M_\Lambda)^2)
\]

\[
= -\frac{dD^M_\Lambda}{d\Lambda} + 2S^{-1}_\Lambda \hat{\Sigma}_\Lambda D^M_\Lambda - \hat{\Sigma}_\Lambda S^{-2}_\Lambda (D^M_\Lambda)^2 ,
\]  

(A.2)

where in the last line we have assumed \(S_\Lambda = S_1 \Lambda S_2 \Lambda\) and have chosen \(S_1 \Lambda = C_0^{-1} - \Sigma_\Lambda\). This choice of \(S_1 \Lambda\) implies that the remaining factor \(S_2 \Lambda\) represents the reamputation of the vertices, amputated by \(C_\Lambda (C_0^{-1} - \Sigma_\Lambda)\). With this choice we also have

\[
S^{-2}_\Lambda \hat{\Sigma}_\Lambda (C_\Lambda - D^M_\Lambda) + \hat{\Sigma}_\Lambda S^{-1}_\Lambda = -\hat{\Sigma}_\Lambda S^{-2}_\Lambda D^M_\Lambda + \hat{\Sigma}_\Lambda / S_2 \Lambda
\]

and

\[
S^{-2}_\Lambda \hat{\Sigma}_\Lambda = S^{-2}_2 \Lambda (\partial_\Lambda \Sigma_\Lambda) / (C_0^{-1} - \Sigma_\Lambda)^2 = S^{-2}_2 \Lambda (1 / (C_0^{-1} - \Sigma_\Lambda)) = S^{-2}_2 \Lambda \partial_\Lambda G_\Lambda.
\]

Introducing \(Q_\Lambda = S^{-2}_2 \Lambda \partial_\Lambda G_\Lambda\), we obtain

\[
Q_\Lambda = -\left[ \frac{d}{d\Lambda} (\bar{D}_\Lambda - D^M_\Lambda) + 2D_\Lambda (D^M_\Lambda Q_\Lambda - \hat{\Sigma}_2 \Lambda S^{-1}_\Lambda) \right]
\]

\[
- (D^M_\Lambda)^2 Q_\Lambda + 2S^{-1}_2 \Lambda \hat{\Sigma}_2 \Lambda D^M_\Lambda \bar{H}_4
\]

\[
- \sum_{m_1, m_2 \geq 4} \bar{H}_{m_1} \left( \frac{dD^M_\Lambda}{d\Lambda} + Q_\Lambda (D^M_\Lambda)^2 - 2S^{-1}_2 \Lambda \hat{\Sigma}_2 \Lambda D^M_\Lambda \right) \circ D^M_\Lambda \frac{m_1 + m_2}{2} \circ \bar{H}_{m_2}
\]

\[
\hat{H}_m = \left[ \frac{d}{d\Lambda} (\bar{D}_\Lambda - D^M_\Lambda) + 2D_\Lambda (Q_\Lambda D^M_\Lambda - \hat{\Sigma}_2 \Lambda S^{-1}_2 \Lambda) \right]
\]

\[
- (D^M_\Lambda)^2 Q_\Lambda + 2S^{-1}_2 \Lambda \hat{\Sigma}_2 \Lambda D^M_\Lambda \bar{H}_{m+2}
\]

\[
+ \sum_{m_1, m_2 \geq 4} \bar{H}_{m_1} \left( \frac{dD^M_\Lambda}{d\Lambda} + Q_\Lambda (D^M_\Lambda)^2 - 2S^{-1}_2 \Lambda \hat{\Sigma}_2 \Lambda D^M_\Lambda \right) \circ D^M_\Lambda \frac{m_1 + m_2}{2} \circ \bar{H}_{m_2}
\]

\[
+ m\bar{H}_m (D^M_\Lambda Q_\Lambda - \hat{\Sigma}_2 \Lambda / S_2 \Lambda) .
\]  

(A.3)
For $D_M^\Lambda Q_\Lambda - \hat{S}_{2\Lambda}/S_{2\Lambda} = 0$ we obtain equations

$$Q_\Lambda = - \left[ \frac{d}{d\Lambda} (\tilde{D}_\Lambda - D_M^\Lambda) + (D_M^\Lambda)^2 Q_\Lambda \right] \tilde{H}_4$$

$$- \sum_{m_1,m_2 \geq 4} \tilde{H}_{m_1} \circ \left( \frac{dD_M^\Lambda}{d\Lambda} - Q_\Lambda (D_M^\Lambda)^2 \right) \circ D_M^{\Lambda m_1 m_2 - 2} \circ \tilde{H}_{m_2}$$

$$\hat{H}_m = \left[ \frac{d}{d\Lambda} (\tilde{D}_\Lambda - D_M^\Lambda) + (D_M^\Lambda)^2 Q_\Lambda \right] \tilde{H}_{m+2}$$

$$+ \sum_{m_1,m_2 \geq 4} \tilde{H}_{m_1} \circ \left( \frac{dD_M^\Lambda}{d\Lambda} - Q_\Lambda (D_M^\Lambda)^2 \right) \circ D_M^{\Lambda m_1 m_2 - 1} \circ \tilde{H}_{m_2}$$

(A.4)

which coincide with the equations of [15]. Choosing the propagators as in [15], $D_M^\Lambda = \chi_{<,\Lambda}/A_\Lambda$, $S_{2\Lambda} = 1/A_\Lambda$, $\frac{dA_\Lambda}{d\Lambda} = -\chi_{<,\Lambda} Q_\Lambda$, we obtain

$$Q_\Lambda = \sum_{m_1,m_2 \geq 4} \tilde{H}_{m_1} \circ \frac{d\tilde{D}_\Lambda}{d\Lambda} \circ \tilde{D}_\Lambda^{\Lambda m_1 m_2 - 2} \circ \tilde{H}_{m_2}$$

$$\hat{H}_m = \sum_{m_1,m_2 \geq 4} \tilde{H}_{m_1} \circ \frac{d\tilde{D}_\Lambda}{d\Lambda} \circ \tilde{D}_\Lambda^{\Lambda m_1 m_2 - 1} \circ \tilde{H}_{m_2}$$

(A.5)

where the single-scale and Wick propagators are given by

$$\tilde{D}_\Lambda = \int_0^{\Lambda} d\Lambda' F_{\Lambda'}, \quad F_\Lambda = \frac{\dot{\chi}_{<,\Lambda}}{A_\Lambda}$$

$$A_\Lambda = \frac{1}{C_0^{-1} - \Sigma_0 + \int_0^{\Lambda} d\Lambda'' \chi_{<,\Lambda} Q_{\Lambda''}}$$

(A.6)

Relations (A.6) are not easily simplified further for the sharp cutoff because of the involved relation between $Q_\Lambda$ and $\Sigma_\Lambda$, $Q_\Lambda = [A_\Lambda/(C_0^{-1} - \Sigma_\Lambda)]^2 \Sigma_\Lambda$.

In the local scheme $\tilde{D}_\Lambda = D_M^\Lambda$, also considered in [15], we have

$$Q_\Lambda = - \left( \frac{\chi_{<,\Lambda}}{A_\Lambda} \right)^2 Q_\Lambda \circ \tilde{H}_4$$

$$- \sum_{m_1,m_2 \geq 4} \tilde{H}_{m_1} \circ \left( \frac{\ddot{\chi}_{<,\Lambda}}{A_\Lambda} \right) \circ \left( \frac{\chi_{<,\Lambda}}{A_\Lambda} \right)^{\Lambda m_1 m_2 - 2} \circ \tilde{H}_{m_2}$$

$$\hat{H}_m = \left( \frac{\chi_{<,\Lambda}}{A_\Lambda} \right)^2 Q_\Lambda \circ \tilde{H}_{m+2}$$

$$+ \sum_{m_1,m_2 \geq 4} \tilde{H}_{m_1} \circ \left( \frac{\ddot{\chi}_{<,\Lambda}}{A_\Lambda} \right) \circ \left( \frac{\chi_{<,\Lambda}}{A_\Lambda} \right)^{\Lambda m_1 m_2 - 1} \circ \tilde{H}_{m_2}$$

(A.7)

In addition to scheme (27), the latter equations may suffer for general many-body problems from divergences of the convolution of $H_{m \geq 4}$ with the square of the propagator $\chi_{<,\Lambda}/A_\Lambda$ and $Q_\Lambda$, which are not cut in momentum space.

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