Forward scattering amplitude of the virtual longitudinal photon in QED

A.V. Samsonov

Institute of Theoretical and Experimental Physics
B. Cheremushkinskaya 25, 117259, Moscow, Russia

Abstract

Forward scattering amplitude of the virtual longitudinal photon at zero energy on electron in QED in the limit of small photon virtualities $Q^2$ is calculated. The first radiation corrections are taken into account. Two terms in the expansion over $Q^2$ are obtained.

B.L. Ioffe [1] proved the theorem, determining the forward scattering amplitude of the virtual longitudinal photon at zero energy on any target in the limit of small photon virtualities $Q^2$. The only assumption about interaction was the presence of a gap in the mass spectrum. It was shown that two terms in the expansion of the amplitude over $Q^2$ are universal and expressed in terms of static target properties. It is interesting to study, if such theorem is valid in the case where is no gap in the mass spectrum. An example of the theory with no gap in the mass spectrum is QED.

In this paper the forward scattering amplitude of the virtual longitudinal photon on electron at zero energy is calculated in QED at order $\alpha^2$ in the limit of small $Q^2$ and two terms of the expansion over $Q^2$ are retained. It was found that in the leading $Q^2$ term the universality indeed takes place and the theorem [1] is valid, but in the next $Q^2$ term it is violated; i.e. the results in presence and absence of the mass gap are different.
For real (transverse) photon the low energy theorem for Compton scattering was proved by Thirring [2] and Kroll and Ruderman [3]; it gives Thomson formula for the amplitude: \( f = -\alpha/m \).

Now we consider the forward scattering amplitude of the virtual longitudinal photon at zero energy on electron in QED. We assume that \( Q^2 \) tends to zero and calculate all non-vanishing terms at order \( e^4 \). This amplitude is the function of two invariants \( \nu = pq \) and \( q^2 \), where \( p \) and \( q \) are the electron and photon 4-momenta. Our conditions are:

\[
\nu = pq = 0 \, , \quad q^2 < 0 \, , \quad q^2 \to 0 .
\]

We use metric \((1, -1, -1, -1)\). We can choose coordinate system, where \( z \) axis is the collision one and \( q_0 = 0 \). Then \( q^2 = -q_z^2 \), \( q_\mu = (0, 0, 0, q_z) \) and \( p_\mu = (m, 0, 0, 0) \). If \( e_\mu \) is the photon polarization vector, then \( e_\mu q_\mu = 0 \) and \( e_\mu = (1, 0, 0, 0) \), because we consider longitudinal photon.

At first we find our amplitude in the tree approximation \( M_0 \) (Fig 1).

\[
M_0 = \frac{1}{2im} \overline{u}(p) \gamma_0 e_0(-ie) \left( \frac{i}{p + \hat{q} - m} + \frac{i}{\hat{p} - \hat{q} - m} \right)(-ie) e_0 \gamma_0 u(p) = -\frac{e^2}{mq^2} 4m^2 ,
\]

or, introducing \( Q^2 = -q^2 \),

\[
M_0 = \frac{4me^2}{Q^2} .
\]

It is easy to see singular behaviour of the amplitude when \( Q^2 \) becomes zero. The sign of the amplitude is opposite to one in the Thomson formula.

Now we consider changes from diagrams with radiation corrections of the first order. Let us first in all expressions retain \( 1/q^2 \) terms only.

1. Diagrams with mass operator \( M_1 \) (Fig 2).

Exact electron propogator \( \hat{G}_r(p') \) has the form:

\[
\hat{G}_r(p') = \frac{1}{\hat{p}' - m - \hat{\Sigma}_r(p')} ,
\]

where \( p' = p + q \), and \( \hat{\Sigma}_r(p') \) -exact mass operator

\[
\hat{\Sigma}_r(p') = \frac{1}{1 - \hat{\Sigma}'(m)} \left[ \hat{\Sigma}(p') - \hat{\Sigma}(m) - (\hat{p}' - m) \hat{\Sigma}'(m) \right] ,
\]

where

\[
\hat{\Sigma}'(m) = (\partial \hat{\Sigma}(p')/\partial p')|_{p'=m} .
\]
Therefore:
\[
\hat{G}_r(p') = \frac{1 - \hat{\Sigma}'(m)}{\hat{p}' - m - [\hat{\Sigma}(p') - \hat{\Sigma}(m)]}.
\]

Using expansion
\[
\hat{\Sigma}(p') - \hat{\Sigma}(m) = (\hat{p}' - m) \hat{\Sigma}'(m) + (\hat{p}' - m)^2 \hat{\Sigma}''(m) + \ldots ,
\]
we obtain for \(\hat{G}_r(p')\):
\[
\hat{G}_r(p') = \frac{1}{\hat{p} + \hat{q} - m + b(\hat{p} + \hat{q} - m)^2},
\]
where \(b\) - constant, which comes out of the contributions from radiation corrections when \(q_z \to 0\).

But
\[
(\hat{p} + \hat{q} - m)^2 = 2m^2 + q^2 - 2m (\hat{p} + \hat{q}) ,
\]
because \(\nu = 0\), and therefore (1) has the following form:
\[
\hat{G}_r(p') = \frac{1}{(\hat{p} + \hat{q})(1 - 2mb) - m(1 - 2mb) + bq^2}.
\]

Substituting this expression (and the expression for another diagram, for which we have to replace \(q_\mu\) by \(-q_\mu\) ) and retaining \(1/q^2\) terms only, we obtain (\(u(\hat{q}q = 0)\):
\[
M_1 = -\frac{e^2}{m}u(p) \frac{(\hat{p} + \hat{q} + m)(1 - 2mb)}{q^2(1 - 2mb)^2 + 2mbq^2(1 - 2mb)} u(p) = M_0.
\]

We see that radiation corrections to propogator (i.e. constant \(b\)) don’t give contribution (proportional to \(1/q^2\)) to the amplitude \(M_0\).

2. Vertex diagrams \(M_2\) (Fig 3).

Vertex has the form:
\[
\Gamma_{r\mu} = \gamma_\mu + b(\gamma_\mu \hat{q} - \hat{q} \gamma_\mu) .
\]
Using that \(e_\mu = \frac{1}{m}P_\mu\), we obtain:
\[
M_2 = -\frac{e^2}{m}u(p) \frac{\hat{p}}{m(\hat{p} + \hat{q} - m)} \frac{1}{m} \frac{1}{m}(\hat{p} + 2b\hat{p}\hat{q}) u(p)
\]
\[
-\frac{e^2}{m} \frac{1}{u(p)} \frac{\hat{p}}{m} \frac{1}{m(\hat{p} + 2b\hat{q})} \frac{1}{(\hat{p} + \hat{q} - m)} u(p) = M_0.
\]
Just as in the first case, constant $b$ vanishes from the answer. So vertex corrections don’t give additional contribution to $M_0$.

3. Box diagrams $M_3$ (Fig 4).
If $\hat{G}(p)$ - electron propagator: $\hat{G}(p) = 1/(\hat{p} - m)$, then $\partial \hat{G}(p)/\partial p_\mu$ is the vertex with zero photon momentum. Therefore we obtain two $M_3$ diagrams (in the limit $q^2 \to 0$) after double differentiation of mass operator with respect to incoming momentum. It is easy to see that this expression has no singularity $1/q^2$.

4. Diagrams with vacuum polarization $M_4$ (Fig 5).
Polarization operator has the following form ($-q^2 << m^2$):

$$\Pi_{\mu\nu} = \Pi(q^2) (g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}), \quad \Pi(q^2) = -\frac{e^2}{15\pi} \frac{q^4}{m^2}.$$ 

That’s why $M_4$ diagrams don’t contain $1/q^2$ terms.

As a result we can conclude that the forward scattering amplitude of the photon with $q_\mu = (0, 0, 0, q_z)$ and $e_\mu = (1, 0, 0, 0)$ on electron with $p_\mu = (m, 0, 0, 0)$, being calculated with the first radiation corrections, has the form:

$$M = 4 \frac{me^2}{Q^2} + R,$$ 

where $Q^2 = -q^2$, and $R$ contains non-negative degrees of $Q^2$.

In paper [1] for such amplitude was obtained the following expression (assuming the presence of the gap):

$$f_L(0, Q^2) = \frac{4e^2m}{Q^2} [1 - \frac{Q^2}{2m^2} \mu_a - \frac{1}{3} \langle r_E^2 \rangle Q^2],$$ 

where $\mu_a$ and $\langle r_E^2 \rangle$ are anomalous magnetic moment and target mean square radius. We see that the first terms of expansion over $Q^2$ coincide in equations (2) and (3).

In order to obtain the second term of the expansion of the amplitude over $Q^2$ (i.e. constant $R$ in (2)), we have to calculate directly all above-mentioned diagrams.

Let us introduce two new parameters:

$$\rho = -q^2/m^2, \quad s = \lambda^2/m^2,$$

where $\lambda$ is the photon mass.

We assume that

$$0 < s << \rho << 1, \quad \rho \to 0.$$

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Under such conditions we obtain:

\[ M_1 = \frac{e^4}{\pi m} \left[ \int_0^1 dx \frac{4}{\rho^2} (1 + x) \ln \frac{x^2 + \rho x (1-x) + s(1-x)}{x^2 + s(1-x)} + \right. \]

\[ + \int_0^1 dx \frac{1}{\rho} ((1-3x) \ln \frac{x^2 + \rho x (1-x) + s(1-x)}{x^2 + s(1-x)} - \frac{4x(1-x^2)}{x^2 + s(1-x)}) \right] = \]

\[ = \frac{e^4}{\pi m} \left( \frac{4}{\rho} - \frac{4}{\rho} \ln \rho + \frac{2}{\rho} \ln s - 3 \ln \rho + \frac{1}{2} \right). \]

\[ M_2 = 2 \frac{e^4}{\pi m \rho} \int_0^1 dx \int_0^1 dy \left[ \frac{2(y^2 + 2y - 2) + \rho(2x^2 - 1 + y - 2xy)}{y^2 + \rho x (1-x) + s(1-y)} \right. \]

\[ + \left. 2 \ln \frac{y^2 + s(1-y)}{y^2 + \rho x (1-x) + s(1-y)} - \frac{2(y^2 + 2y - 2)}{y^2 + s(1-y)} \right] = \]

\[ = \frac{e^4}{\pi m} \left( -\frac{8}{\rho} + 8 \ln \rho - \frac{4}{\rho} \ln s + 6 \ln \rho - \frac{277}{15} \right). \]

\[ M_3 = -\frac{e^4}{\pi m} \int_0^1 dx \int_0^1 dy (y - x) \left( \frac{y^3 + 3y^2 - 6y + 4 + \rho(3yx^2 - x^2 - 2x - 2xy)}{(y^2 + \rho x (1-x) + s(1-y))^2} \right. \]

\[ - \left. \frac{3 - 3y}{y^2 + \rho x (1-x) + s(1-y)} \right) = \]

\[ = \frac{e^4}{\pi m} \left( \frac{4}{\rho} - \frac{4}{\rho} \ln \rho + \frac{2}{\rho} \ln s - 3 \ln \rho + \frac{11}{6} \right). \]

\[ M_4 = \frac{8}{15} \frac{e^4}{\pi m}. \]

In this expressions we retain only singular terms and constants.

\[ R = M_1 + M_2 + M_3 + M_4 = -\frac{78}{5} \frac{e^4}{\pi m}. \]
The final result for the forward scattering amplitude \( M \) has the following form:

\[
M = 4 \frac{m e^2}{Q^2} - \frac{78}{5} \frac{e^4}{\pi m} .
\] (5)

According to the first part of this article, radiation corrections diagrams don’t give any contribution to the singular term.

The next to leading term in the expansion over \( Q^2 \) of the amplitude (3) in QED is equal to \( 4/3 (e^4/\pi m) \ln s \). (The expressions for \( \mu_a \) and \( \langle r^2_E \rangle \) at order \( e^4 \) are substituted from [4]). That’s why (3) at order \( e^4 \) has the following form:

\[
f_L(0, Q^2) = 4 \frac{m e^2}{Q^2} + \frac{4}{3} \frac{e^4}{\pi m} \ln s .
\] (6)

The second terms in (5) and (6) don’t coincide. Expression (5) contains neither photon mass explicitly (it has been obvious from the beginning) nor experimental resolution \( \Delta \epsilon \).

It can be shown that formula (6) is reproduced in the case of the finite mass gap, which in our notation corresponds to condition

\[
0 < \rho << s << 1 , \quad s \to 0 .
\]

In all calculations \( \nu \) equals zero, and \( \lambda \) equals very small but nonzero quantity. But if we put nonzero \( \nu \) and change condition (4) to the following one:

\[
0 < s << \frac{\nu}{m^2} << \rho << 1 , \quad \rho \to 0 , \quad \nu \to 0 ,
\]

we obtain the same answer (5) for the amplitude \( M \).

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References.

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Figures.
Figure captions.

Fig. 1 -tree diagrams.
Fig. 2 -diagrams with mass operator.
Fig. 3 -vertex diagrams.
Fig. 4 -box diagrams.
Fig. 5 -diagrams with polarization operator.