Thermodynamical properties of hairy black holes in $n$ spacetimes dimensions

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Abstract: The issue concerning the existence of exact black hole solutions in presence of non vanishing cosmological constant and scalar fields is reconsidered. With regard to this, in investigating no-hair theorem violations, exact solutions of gravity having as a source an interacting and conformally coupled scalar field are revisited in arbitrary dimensional non asymptotically flat space-times. New and known hairy black hole solutions are discussed. The thermodynamical properties associated with these solutions are investigated and the invariance of the black hole entropy with respect to different conformal frames is proven.

PACS numbers: 04.70.Bw, 04.70.Dy

1 Introduction

It is known that the no-hair conjecture in black hole physics may be violated by the presence of scalar fields. For example, the existence of hairy black holes minimally coupled to a scalar has been considered and studied in [1,2,3,4,5] and even for a phantom field [6]. A four-parameter family of asymptotically dS, flat and AdS solutions, are reported in [10]. Time dependent black holes and their interpretation within the AdS/CFT duality are reported in [7]. Hairy black hole solutions where a phase transition occur near the horizon have been constructed too [8], and a study of black holes in Brans-Dicke-Maxwell theory is in [9]. Then there is the case of conformally coupled non self-interacting or self-interacting scalar field. For vanishing cosmological constant and no interaction, the first solution was found in [11]. This solution has some unphysical features, like the instability (see for example, [12] and the recent survey [13], but also [15]). However, when a non vanishing cosmological constant is taken into account, other hairy black hole solutions have been found. Among them, we would like to mention the black hole solution in $n = 3$ dimensions for a non self-interacting conformally coupled scalar field in AdS [16], and the $n = 4$ solution of the same type presented in [17]. Further solutions have been presented and discussed in [2,18]. At the same time, interesting no hair theorems have recently been proved for $\Lambda > 0$, but without conformal coupling and for convex scalar potentials [19].

The main motivation for looking for new exact BH solutions in presence of scalar fields stems from the renewed interest in BH solutions with non vanishing cosmological constant.

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For instance, the important question of their stability was recently considered in [14]. This motivations was partly triggered by the work by McFadden and Turok [15] concerning the two brane RS-type model, where the low energy effective dynamics contain scalar hairy BH solutions, the scalar here being the radion field which describes dynamically the separation of the two 3-branes.

We also have to recall a recent rather unexpected development: Maeda et al. [20] and Winstanley [13]) proved the existence of BH solutions with scalar hairs in asymptotically non flat space-times AdS and dS. Later Martinez et al. [21] found explicitly some of these hairy BHs.

In this paper, we will revisit conformally coupled self-interacting scalar field in asymptotically $n$ dimensional dS and AdS space-times.

As already mentioned, the interest being related to the fact that recently, McFadden and Turok have solved a low energy brane effective theory recasting the model in the form of Einstein gravity plus a conformally coupled scalar field. Within this new framework, they claim that the old instabilities are no longer present. The conformal, quartic self-interaction arises from branes with cosmological constants.

The presence of conformal coupling for the scalar field is equivalent to work in the so called Jordan frame, traditionally defined as that frame in which there is no direct interaction between the scalar field and the matter fields. In the Jordan frame, if a BH solution exists, it turns out that thermodynamic properties of the BH solution is quite different from the standard ones and typically the entropy is no longer given by the Area Law. Furthermore, its computation is non trivial.

However, there exists a general approach to this problem and the BH entropy may be computed by means of Noether charge method, introduced by Wald [22]. This method is very powerful and has a geometric origin, but sometimes it does not guarantee the positivity of the entropy. This is an issue one just encounters with scalar tensor theories with non minimal coupling, though not only with them. A very general and powerful approach combining the Noether charge method and the quasi-local description of gravity due to Brown and York has been presented in [23, 24]. The former uses an $n$-dimensional Lagrangian density with matter and dilaton satisfying the assumptions of scalar-tensor theories we will discuss in the next Section.

Recall that for theory of gravity of general type, described by a Lagrangian density depending on $f(R)$, $R$ being the Ricci scalar curvature, the Noether charged method, gives a simple expression for the BH entropy (modulo a possible non trivial constant)

$$S_{BH} = \frac{V_F r_H^{n-2}}{4G} f'(R)(r_H) = \frac{A_H}{4G} f'(R)(r_H).$$

(1.1)

Here, $V_F$ is the measure of the fundamental domain of the horizon manifold and $r_H$ the horizon radius (ex. for $S_2$, $V_F = 4\pi$). As a result, since $f'(R)$ might be negative, negative BH entropies could appear.

In principle, in the Jordan frame, the Einstein Eqs. are much more complicated, due to the presence of non minimal coupling with the scalar field. Thus, one may pass to the so called Einstein frame, where the scalar field is minimally coupled to gravity and $f(\hat{R}) = \hat{R}$. In this related conformal frame, one has related BH solutions, and the BH entropy is simply given by the area law

$$S_{BH} = \frac{A_H}{4G}.$$  

(1.2)
However, as soon as the BH thermodynamical quantities are concerned, it is not difficult to prove (see Section III) that there is no difference in dealing with the Jordan or Einstein frame.

The outline of the paper is as follow. In Sec. II we outline some general properties of the scalar tensor theories we are interested in. In Sec. III we show the conformal invariance of Hawking temperature and black hole entropy with respect to conformally related frames. In Sec. IV we revisit the existence of n-dimensional black hole solutions in asymptotically dS and AdS space-times. In Sections V, VI, and VII the solutions are examined in details and new solutions are presented. In Sec. VIII, we discuss the thermodynamics of the BH solutions, with a specific proposal for dealing with possible negative entropies and we conclude with Sec. IX.

2 Scalar-tensor theories

The task of constructing dynamical models involving conventional gravity and a scalar field as well, offers in principle unlimited possibilities (see the textbooks [25] on scalar-tensor theories (ST); see also [26] and T. Clifton’s recent dissertation [27] for a very extensive discussions of ST in cosmology). Their number is considerably reduced if we allow only for second order field equations to be derivable from an invariant action principle. So let us start with the following scalar-tensor theory of gravity, written in the Jordan frame [28, 29, 30], and employing reduced Planck units such that $8\pi G = 1$,

$$I = \int d^n x \sqrt{-g} \left[ \frac{1}{2} F(\phi) R - \Lambda - \frac{1}{2} (\nabla \phi)^2 - V(\phi) \right],$$

(2.1)

where $F(\phi)$ is a positive function, $R$ is the Ricci scalar curvature and $\Lambda$ is the cosmological constant. Below we shall consider the special case $F(\phi) = 1 - \xi \phi^2$, where $\xi$ is a positive coupling parameter. The scalar amplitude will be measured in Planck units, but we recall here that $G = M_p^2 n$ in $n$ space-time dimensions. In scalar-tensor gravity there really is not a “constant Newton constant”, since the effective $G$ is a field; from Eq. (2.1) we see that this effective gravitational coupling is $G = 1/8\pi F$, although this is not the one which is measured in a Cavendish-like experiment, since the scalar field will contribute to the inter-particle force. Coupling constants for scalar-scalar and scalar-gravity interactions are contained in the functions $F(\phi)$ and $V(\phi)$, together with any possible mass term for the scalar field. In principle a function $Z(\phi)$ could have been introduced in front of the kinetic term, but it can be set to $\pm 1$ by a field redefinition without changing the conformal class of the metric. If this is also changed by some conformal transformation, then one can go to a so called Einstein frame where the action takes the usual form of general relativity and the matter action depends on $\phi$, but we prefer to stick to the Jordan frame here from. So if a matter action is added to $I$, say $I_M[\psi_M, g_{ab}]$, it is assumed (with Dicke) that is not directly coupled to the scalar field. This is consistent with the equivalence principle and the fact that the ratio $\alpha$ between the couplings to matter of scalar and tensor field is bounded by about $10^{-3}$ by solar-system experiments. In any case, the only coupling of $\phi$ to matter that can be removed from the action to conform with Dicke’s assumption, is clearly of the form $I_M[\psi_M, \omega^2(\phi) g_{ab}]$, which means that $\phi$ will have matter acting as a source only through the trace of its energy-momentum tensor.

\footnote{In particular, a dilaton coupling to a $U(1)$ gauge filed like $h(\phi) F^2$ is forbidden under Dicke assumption.}
The field equations following from (2.1) take the well known form

\[ F(\phi) \left( R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) + g_{\mu\nu} \nabla^2 F(\phi) - \nabla_\mu \nabla_\nu F(\phi) \]

\[ - \nabla_\mu \phi \nabla_\nu \phi + \frac{1}{2} (\nabla \phi)^2 g_{\mu\nu} + \left( \Lambda + V(\phi) \right) g_{\mu\nu} = 0 \]  

(2.2)

\[ \nabla^2 \phi + \frac{1}{2} F'(\phi) R - V'(\phi) = 0 \]  

(2.3)

If matter is present its energy tensor should be added to the right hand side of (2.2), but Eq. (2.3) will remain unaffected. If the scalar is a constant, say \( \phi = \phi_0 \), the metric solves the ordinary Einstein’s equations with an effective cosmological constant \( \Lambda_{\text{eff}} = (\Lambda + V(\phi_0))/F(\phi_0) \). The constant \( \phi_0 \) can then be related to the Newton constant as measured by a Cavendish experiment as follow. If the field is slowly varying we can expand the action (2.1) around \( \phi_0 \) and keeps first order terms; we obtain the action

\[ I = \frac{1}{2} \int \left( (F_0 + F'_0 \sigma) R - (\nabla \sigma)^2 - V(\phi_0 + \sigma) - \Lambda \right) |g|^{1/2} d^4x \]

We work in 4D for simplicity and define \( F_0 = F(\phi_0) \), \( V_0 = V(\phi_0) \), etc. In a Cavendish experiment we are on space-time scales much less than \( \Lambda_{\text{eff}}^{-1/2} \); the effective vacuum energy of the theory. Redefining \( \varphi = F_0 + F'_0 \sigma \) we obtain the linearized action of Brans-Dicke theory with BD parameter \( \omega = F_0/(F'_0)^2 \) and a scalar potential. The gravitational coupling of this theory in the limit of a massless scalar is (see, e.g. [31])

\[ G = \frac{1}{8\pi \varphi_0} \frac{2\omega + 4}{2\omega + 3} \]

so we obtain \( G \) for the massless (or nearly massless) scalar-tensor theory

\[ G = \frac{1}{8\pi F_0} \frac{2F_0 + 4(F'_0)^2}{2F_0 + 3(F'_0)^2} \]  

(2.4)

In the opposite limit where the kinetic term is unimportant while the potential term dominates, \( G \) reduces to the first factor in (2.4), but it still depend on \( \phi_0 \). An example is the theory with Lagrangian density \( \mathcal{L} = \phi R + V(\phi) \), which is just modified gravity with \( \mathcal{L} = f(R) = V(\phi(R)) - \phi(R)V'(\phi(R)) \), where \( \phi(R) \) is the solution of the equation \( R = -V'(\phi) \). Thus \( \phi_0 \), the constant solution, can be measured by a Cavendish experiment in a slowly varying field. Since \( \phi_0 \) is related to other parameters of the model by the field equations, (2.4) is an important constraint on these parameters. In particular, it implies the obvious requirement \( F_0 > 0 \). For the theory considered in this paper, corresponding to \( F(\phi) = 1 - \xi \phi^2 \), one has a possible pathology when the amplitude \( \phi \sim \xi^{-1/2} \) is of order of the Planck masses, assuming a positive \( \xi \) of order one.

3 Equivalence of Hawking temperature and entropy in conformally related frames

It is known that the surface gravity of a stationary black hole is invariant under conformal transformations of the metric that are the identity at the infinity [32]. An alternative and
simpler proof for static black holes, with an additional result concerning the entropy, goes as follows. Assume that exists a static black hole solution

\[ ds^2 = -A(r)dt^2 + \frac{dr^2}{B(r)} + r^2d\Sigma^2. \]

Here \( \Sigma \) is a maximally symmetric space with volume \( V_F \), representing the event horizon manifold located at \( r_H \), where \( A(r_H) = B(r_H) = 0 \), with \( A'(r_H) \neq 0 \) and \( B'(r_H) \neq 0 \). The Hawking temperature can be computed by means of (see, for example, \([33, 34, 35, 36, 37, 38]\) and references cited therein for a dynamical derivation)

\[ T_H = \frac{\sqrt{A'(r_H)B'(r_H)}}{4\pi}, \quad (3.1) \]

while the entropy, computed by means of the Noether charge method devised by Wald, reads

\[ S_{BH} = 2\pi V_F r_H^{n-2} F(\phi_H). \quad (3.2) \]

where \( \phi_H = \phi(r_H) \). This formula can also be obtained by quasi-local methods \([24]\), but here too the positivity of \( S_{BH} \) cannot be guaranteed without additional assumptions. We note that the condition for the positivity of the Newton constant, which were \( F(\phi) > 0 \) for the relevant range of \( \phi \), is equivalent to that requiring the positivity of the entropy, and also that \( S_{BH} \) is controlled by the effective gravitational coupling \( G_e = 1/8\pi F(\phi_H) \) computed on the horizon, not by the coupling \((2.4)\). This fact is similar to the classical analogue of the non-renormalization theorem of black hole entropy, namely the fact that the effects of high energy degrees of freedom on the entropy of black holes are just the same ultraviolet effects that renormalize the Newton constant.

Now pass from the Jordan frame to the Einstein frame by the conformal transformation

\[ \hat{\Omega} = \Omega^\frac{2}{n-2} g_{\mu\nu}, \quad (3.3) \]

with

\[ \Omega = F(\phi(r)). \quad (3.4) \]

The BH solution in the Einstein frame becomes

\[ d\hat{s}^2 = -dt^2 \hat{A}(r) + \frac{dr^2}{\hat{B}(r)} + \hat{r}^2 d\Sigma^2, \]

\[ \hat{A}(r) = F(\phi)^\frac{2}{n-2} A(r), \]

\[ \hat{B}(r) = F\phi^{-\frac{2}{n-2}} B(r), \]

\[ \hat{r} = F(\phi)^\frac{1}{n-2} r. \quad (3.5) \]

Here (Einstein frame) the scalar field is minimally coupled with a complicated potential. If \( \Omega \) and its first derivatives are well behaved on the horizon, the coordinate location of the horizon will be unaffected by the conformal transformation, and the Einstein frame entropy reads

\[ \hat{S}_{BH} = 2\pi V_F (\hat{r}_H)^{n-2}. \]

\(^5\)recall that \( 4G = 1/2\pi \) with our units.
As a result, from (3.5),
\[ \hat{S}_{BH} = S_{BH}. \]
The Hawking temperature in the Einstein frame is
\[ \hat{T}_H = \sqrt{\frac{A'(r_H)B'(r_H)}{4\pi}}. \]
Since
\[ A'(r_H) = A'(r)F(\phi)^{\frac{2}{n-2}} + O(r - r_H), \]
\[ B'(r_H) = B'(r)F(\phi)^{-\frac{2}{n-2}} + O(r - r_H), \]
one immediately obtains
\[ \hat{T}_H = T_H. \]
We have proved this result: If BH solutions exist, the Hawking temperature and BH entropy are invariant quantities under conformal transformations with factors that have finite values together with finite first order derivatives near and on the event horizon. In particular, they are invariant with respect to different conformally related frames, an important point spelled out from different perspectives also in [9, 39].

4 Static and spherically symmetric solutions

In this Section, we will revisit the existence of static and spherically symmetric solutions which are asymptotically de Sitter (dS) and Anti-de Sitter (AdS) working in the Jordan frame, for the choice \( F(\phi) = 1 - \xi \phi^2 \). Since a generally valid gravitational constant in ST does not exist, we make use of conventional Planck units \( 8\pi G_0 = 1 \), where \( G_0^{-1/2} \) is the Planck mass as measured in a vacuum with a vanishingly small scalar field. Below we relate it to the Planck mass in a vacuum with a constant non zero value of \( \phi \).

4.1 Vacuum solution

To begin with, recall the action for a scalar field \( \phi \) with non-minimal coupling and self-interaction \( V(\phi) \), in \( n \)-dimensions in the Jordan frame:
\[
I = \int d^n x \sqrt{-g} \left[ \frac{1}{2} (R - 2\Lambda) - \frac{1}{2} (\nabla \phi)^2 - \frac{1}{2} \xi R \phi^2 - V(\phi) \right].
\]
(4.1)
The special value, \( \xi_c = (n - 2)/(4(n - 1)) \) and \( V(\phi) \sim \phi^{2n/(n-2)} \), gives a scalar field theory which is conformally invariant on a curved background with a non dynamical metric. The Eqs. of motions read (see for example [15]) or (2.2), (2.3)
\[
(1 - \xi \phi^2) G_{\mu \nu} + g_{\mu \nu} \Lambda = (1 - 2\xi) \nabla_\mu \phi \nabla_\nu \phi + \left( 2\xi - \frac{1}{2} \right) g_{\mu \nu} (\nabla \phi)^2
- 2\xi \phi \nabla_\mu \nabla_\nu \phi + 2\xi g_{\mu \nu} \phi \nabla^2 \phi - g_{\mu \nu} V(\phi),
\]
(4.2)
\[
\nabla^2 \phi - \xi R \phi - \frac{dV}{d\phi} = 0.
\]
(4.3)
where $G_{\mu\nu} = R_{\mu\nu} - \frac{K}{2} g_{\mu\nu}$ is the Einstein tensor. Taking these Eqs. of motion into account and taking the trace, one gets

$$R = -\frac{2(n-1)(\xi - \xi_c)(\nabla \phi)^2 + 2\xi(n-1)\phi \frac{dV}{d\phi} - n(V(\phi) + \Lambda)}{n/2 - 1 + 2(n-1)\xi(\xi - \xi_c)\phi^2}. \quad (4.4)$$

The scalar curvature, and thus the effective potential in Eq. (4.3), has a pole at the critical field $\phi_c$:

$$\phi_c^2 = \frac{\xi_c}{\xi_c - \xi}$$

which is only defined for $0 < \xi < \xi_c$; its 4D cousin is significant for the stability analysis of the constant solutions \[40\]. The other critical points are at $\hat{\phi} = \pm |\xi|^{-1/2}$, where the effective Newton constant vanishes, and are important in the phase space structure of the theory \[41\] \[42\]. Substitution of (4.4) into Eq. (4.3) yields

$$\nabla^2 \phi - U'(\phi) + \frac{\phi}{\phi^2 - \phi_c^2}(\nabla \phi)^2 = 0 \quad (4.5)$$

$$U'(\phi) = -\frac{\phi_c^2}{\phi^2 - \phi_c^2} \left[ (1 - \xi\phi^2)V'(\phi) + \frac{2n\xi}{n-2}\phi(V(\phi) + \Lambda) \right] \quad (4.6)$$

To recover general relativity we require that a constant $\phi$, say $\phi_0$, be a solution of the field equations. With this ansatz, the Eqs. of motion reduce to

$$(1 - \xi\phi_0^2) \left( \frac{n-2}{2} \right) R_0 = \Lambda + V(\phi_0), \quad (4.7)$$

$$\xi R_0 \phi_0 + \frac{dV}{d\phi}(\phi_0) = 0 \quad (4.8)$$

where $R_0$ is the constant scalar curvature. These equations determine $R_0$ and $\phi_0$ of the background manifold, and may or may not have a solution. If there is one, the metric will describe an Einstein manifold, for which one has

$$G^{(0)}_{\mu\nu} = \frac{(2 - n)}{2n} R_0 g^{(0)}_{\mu\nu}, \quad (4.9)$$

but the space will not, in general, be maximally symmetric. Under this stronger requirement it is clear that the theory will admit Minkowski, de Sitter and anti-de Sitter vacuums, if only one chooses appropriately the relevant parameters like $\Lambda$, $V(\phi_0)$, $\xi$ and so on. The stability of these vacuums were analyzed by Hosotani \[40\] in four dimensions for a quartic polynomial potential. A cubic interaction term induces instability unless $\xi = 0$. In higher dimensions, for a potential

$$V(\phi) = \alpha_n \phi^{2n/(n-2)} + \frac{1}{2} m^2 \phi^2$$

Hosotani’s stability criterion against spatially homogeneous perturbations gives: for $\xi \leq 0$ or $\xi \geq \xi_c$ the condition is $2\alpha_4 + m^2 \xi > 0$ for $n = 4$, $\alpha_3 > 0$ for $n = 3$ and $m^2 \xi > 0$ for $n > 4$, while for $0 < \xi < \xi_c$ and $m^2 = 0$ the condition is

$$\alpha_n \left( \frac{\xi_c}{\xi_c - \xi} \right)^{2/(n-2)} + \Lambda \xi^{n/(n-2)} > 0$$
Thus a negative $\alpha_n$ can make sense in curved space. Next we consider just two examples, a potential with mass term and a conformally invariant case. Thus we have, to start with,

$$V(\phi) = \frac{1}{2} \mu^2 \phi^2$$

and the vacuum equations give either $\phi_0 = 0$ and $R_0 = 2\Lambda/(n-2)$, or $R_0 = -\mu^2/\xi$ and $\phi_0^2 = (n-2)\mu^2 + 2\xi \Lambda/(n-3)\mu^2 \xi$. For $n = 3$, $\phi_0$ is either zero or it is undetermined.

Let us consider now a symmetric vacuum solutions with conformal symmetry in the scalar sector, this means that

$$\xi = \xi_c = \frac{n - 2}{4(n-1)}, \quad V(\phi) = \alpha_n \phi^{\frac{2n}{n-2}}, \quad R_0 = \frac{2n}{n-2} \Lambda. \quad (4.10)$$

As a result, we get

$$\xi_c R_0 \phi_0^2 = -\frac{2n}{n-2} \alpha_n \phi_0^{\frac{2n}{n-2}}, \quad (4.11)$$

which satisfies also the equation of motion for the scalar field. Thus, we have found the following relation between the vacuum solutions

$$\xi_c \Lambda = -\alpha_n \phi_0^{\frac{1}{n-2}}. \quad (4.12)$$

For example, when $n = 3$,

$$\phi_0^4 = -\frac{\Lambda}{8\alpha_3}. \quad (4.13)$$

If $n = 4$

$$\phi_0^2 = -\frac{\Lambda}{6\alpha_4}. \quad (4.14)$$

For these values of $n$, in order to have a positive self-interacting coupling constant, one has to deal with negative cosmological constant, in agreement with [33], but a negative $\alpha_n$ can still make sense (this is due to the back reaction of gravity on the scalar field). A positive cosmological constant is consistent only for $\alpha_n < 0$ [2]. For $n > 4$, the relation between $\Lambda$ and $\alpha$ involve non manifestly positive quantities.

### 4.2 Analysis of the exact solutions

It is known that if one allows an Einstein space as horizon manifold, a reasonable ansatz for the BH metric reads (see, for example, [44])

$$ds^2 = -A(r)dt^2 + \frac{dr^2}{A(r)} + r^2 d\Sigma^2,$$

where $d\Sigma$ is the metric of (n-2)-dimensional maximally symmetric Einstein space, whose normalized constant sectional curvature is $k = 0, \pm 1$. Assume also that $\phi(x) = \phi(r)$, and the conformal invariance of the scalar sector, namely $\xi = \xi_c = \frac{n-2}{4(n-1)}$ and

$$V(\phi) = \alpha_n \phi^{\frac{2n}{n-2}}.$$
Thus, Eqs. of motion reduce to (\( \prime \) is the derivative with respect to \( r \))

\[- (1 - 2\xi c)\phi''^2 + 2\xi c\phi'' = 0, \]  

\[\phi' A' + \left( \phi'' + \frac{n - 2}{r} \phi' \right) A - \xi c R\phi - \frac{dV(\phi)}{d\phi} = 0, \]  

\[\frac{n - 2}{2r}(1 - \xi c \phi^2) \left[ A' - \frac{n - 3}{r}(k - A) \right] - \left( 2\xi c - \frac{1}{2} \right) A\phi'^2 + \xi \phi\phi' A' + \]

\[- 2\xi c \phi \nabla^2 \phi + V(\phi) + \Lambda = 0. \] (4.17)

The first equation is a combination of two independent Einstein Eqs., one of which reduces to the third equation, while the second equation is the Eq. of motion for the scalar field.

Within our specific assumptions (the conformal invariance of the scalar sector), the first Eq., is a simple and non linear equation involving only \( \phi(r) \), and the general solution is

\[\phi(r) = c (r + r_0)^{1-n/2}. \] (4.18)

with \( r_0 \) and \( c \) constants of integration.

Once this solution is plugged into the other two equations, the second and third ones become first order differential equations with non constant coefficients for the unknown left function \( A(r) \): a problem of compatibility arises and the solutions for \( A \) may exist or may not exist. Thus a simple procedure is at disposal: solve the second (first order diff.) equation, then verify if the obtained solution satisfies identically the third equation. The solutions, when they exist, lead to relations between the constants of integration \( c \) and \( r_0 \) and the parameters \( \Lambda \) and \( \alpha_n \) of the model.

The general solution of the second diff. equation is standard and reads

\[A(r) = (r + r_0)^{n/2} r^{2-n} \left( k_0 + \int (r + r_0)^{-n/2} r^{n-2} B_n(r) dr \right), \] (4.19)

where \( k_0 \) is a further integration constant and

\[B_n(r) = - \frac{n\Lambda}{(n - 2)(n - 1)} (r + r_0) - \frac{4\alpha_n}{(n - 2)^2} c^{4/(n-2)} (r + r_0)^{-1}. \]

The integral involving the function \( B_n(r) \) may be evaluated for generic \( \Lambda \) and \( n \), but for the sake of simplicity, we prefer to deal with a definite sign of \( \Lambda \) and specific values of \( n \).

5 The \( \Lambda < 0 \) case

Since \( \Lambda < 0 \), we may introduce the length \( l \) by means of \( \Lambda = - \frac{(n-2)(n-1)}{2l^2} \).
5.1 The $n = 3$ solution

For $n = 3$, $\xi_c = 1/8$ and the potential reads $V = \alpha_3 \phi^6$. The solution for the scalar field reduces to

$$\phi(r) = \left(8 - \frac{r_0}{r + r_0}\right)^{1/2}. \quad (5.1)$$

A direct calculation gives

$$A(r) = k_0 \left(\frac{r + r_0}{r}\right)^{3/2} + \frac{r^2}{l^2} - r_0^2 (3 + 2 \frac{r_0}{r}) \Delta, \quad (5.2)$$

where

$$\Delta = \frac{8\alpha_3 c^4}{r_0^2} - \frac{1}{l^2} \quad (5.3)$$

If we plug this solution into the third equation (4.17), one finds that it is satisfied when $k_0 = 0$ and $\Delta = 0$ or under the different set of conditions $k_0 = 0$ and $c^2 = 8r_0$. In the first case,

$$\frac{8\alpha_3 c^4}{r_0^2} = \frac{1}{l^2}, \quad A(r) = \frac{r^2}{l^2}. \quad (5.4)$$

We may anticipate that this form of the lapse function exists for arbitrary $n$.

In the second case, $r_0 = \frac{c^2}{8} > 0$ and

$$A(r) = \frac{r^2}{l^2} + \frac{r_0^2}{l^2} (\omega - 1) \left(3 + \frac{2r_0}{r}\right), \quad (5.5)$$

where

$$\omega = 512l^2\alpha_3 > 0.$$

Recall that one has black hole solution as soon as $A(r)$ has a positive root $r_H$ with $A'(r_H) \neq 0$. The roots are solutions of third order algebraic equation

$$r^3 + 3r_0^2 (\omega - 1) r + 2r_0^3 (\omega - 1) = 0. \quad (5.6)$$

The related discriminant reads

$$D = a^6 (\omega - 1)^2 \omega. \quad (5.7)$$

One has two cases. The first is $\omega = 0$, $D = 0$, corresponding to zero scalar self-interaction, the roots are real and the positive one reads

$$r_H = 2r_0. \quad (5.8)$$

The related solution is

$$A(r) = \frac{r^2}{l^2} - 3 \frac{r_0^2}{l^2} - 2 \frac{r_0^3}{l^2r} \quad (5.9)$$

In the other case, $1 > \omega > 0$, $D > 0$ and there exists only one real positive root

$$r_H = r_0 f(\omega) = r_0 \left(1 - \omega\right)^{1/3} \left[(1 + \sqrt{\omega})^{1/3} + (1 - \sqrt{\omega})^{1/3}\right]$$
5.2 The $n = 4$ solutions

For $n = 4$, the situation changes. First, in order to satisfy the second differential equation, one finds that

$$r_0^2 = 2\alpha_4 c^2 l^2. \quad (5.10)$$

Furthermore, with choice $k_0 = k$, the third equation gives

$$0 = \frac{6k}{r^4}(c^2 - 6r_0^2) \quad (5.11)$$

As a result, one has the set of solutions with $c = \pm \sqrt{6}r_0$, with $r_0 > 0$ and $r_0 < 0$. In general

$$A(r) = k \frac{(r + r_0)^2}{r^2} + \frac{r^2}{l^2}, \quad (5.12)$$

$$\frac{\phi(r)^2}{6} = \frac{r_0^2}{(r + r_0)^2}. \quad (5.13)$$

The self-interacting coupling constant is fixed to be

$$\alpha_4 = \frac{l^2}{12}. \quad (5.14)$$

The solution for $\phi(r)$ is regular everywhere for $r_0 < 0$.

Furthermore, for $k = 1$ (spherical transverse manifold), $A(r)$ is a regular non vanishing lapse function and the solution is asymptotically AdS.

For $k = 0$ (locally flat transverse manifold), the metric is conformally and locally related to Minkowski space-time.

For $k = -1$ (hyperbolic transverse manifold), one has a BH solution, since $A(r)$ has real roots, the biggest one being the radius of the event horizon

$$r_H = \frac{l}{2} \left(1 + \sqrt{1 + 4r_0/l}\right). \quad (5.15)$$

This kind of solutions are conformally related to the solutions found by Martinez et al [2] working in the Einstein frame.

5.3 A charged $n = 4$ black hole solution

Let us consider the $n = 4$ and let us investigate the existence of a charged black hole solution in the presence of a neutral conformally coupled scalar field. For $\Lambda > 0$, a corresponding solution has recently been found in [2]. One has to add the Maxwell term to the scalar gravity action and observes that the only non vanishing component of the EM tensor field is

$$F_{tr} = \frac{Q}{r^2}. \quad (5.16)$$

The Eqs. of motion are supplemented by the EM stress tensor, given by $T^{EM}_{\mu\nu} = \frac{Q^2}{8\pi^2} \varepsilon g_{\mu\nu}$ with $\varepsilon = -1$ for $\mu = t, r$ and $\varepsilon = 1$ for the other spatial coordinates. It is easy to see that the first equation of motion is left unchanged, namely

$$2\phi^2 - \phi\phi'' = 0, \quad (5.17)$$
and the solution is
\[ \phi(r) = \frac{c}{r + r_0}. \tag{5.18} \]

As a consequence, with
\[ \frac{r_0^2}{l^2} = 2\alpha_4 c^2 \tag{5.19} \]
the solution of the second equation is still the same, namely
\[ A(r) = k \frac{(r + r_0)^2}{r^2} + \frac{r^2}{l^2}. \tag{5.20} \]

The third equation, gives the consistency condition regarding the constants of integration. The result is
\[ 6k(c^2 - 6r_0^2) + \frac{Q^2}{8\pi} = 0, \tag{5.21} \]

As a result, within the ansatz we are dealing with, \( k = 0 \) solution is not possible, unless \( Q = 0 \). A solution of this kind, but in the de Sitter space, has been reported in \[ 2 \]. Very recently the case \( k = -1 \), which is a black hole solution, has been reported in \[ 21 \]. The other solution, namely the one with \( k = 1 \) is a regular asymptotically AdS solution.

### 5.4 The solutions for \( n > 4 \)

The previous procedure can be applied to the cases \( n = 5, 6, ... \). The computation becomes more and more involved, and the results are the following:

For arbitrary \( n > 4 \), and with \( k \neq 0 \), it seems unlikely that other solutions exist under our assumptions about the form of the metric (we have not found any up to \( n = 11 \), but were unable to prove this in general). If \( k = 0 \) and if
\[ \alpha_n = \frac{(n-2)^2 r_0^2}{8l^2 c^{n-2}} \tag{5.22} \]
holds, then the unique solution is
\[ A(r) = \frac{r^2}{l^2}. \tag{5.23} \]

In this case, the self-interacting coupling constant \( \alpha_n \) is positive and with the coordinate change \( r = \frac{l^2}{y} \), the lapse function becomes
\[ A(y) = \frac{l^2}{y^2} \left( -dt^2 + dy^2 + l^2 dT_{n-2}^2 \right), \tag{5.24} \]

namely it represents locally the AdS space-time.
6 The $\Lambda > 0$ case

If $\Lambda > 0$, we set $\Lambda = \frac{(n-2)(n-1)}{2l^2}$. The procedure is formally similar to the previous one. There exist non trivial black hole solutions for $n = 3$ and $n = 4$. They may be obtained by the solutions found for $\Lambda < 0$, letting $l^2 \rightarrow -l^2$. Besides event horizons, also cosmological and Cauchy horizons appear and negative BH entropies have been claimed to exist [18].

For example, for $n = 3$, one has $r_0 = \frac{c^2}{8} > 0$ again and

$$A(r) = -\frac{r^2}{l^2} + \frac{r_0^2}{l^2}(\omega + 1) \left( 3 + \frac{2r_0}{r} \right),$$

where

$$\omega = 512l^2\alpha_3,$$

the event, cosmological and Cauchy horizons are defined by the real roots of

$$r^3 - 3r_0^2(\omega + 1)r - 2r_0^3(\omega + 1) = 0. \quad (6.3)$$

First if $\omega = 0$, $D = 0$ corresponding to zero scalar self-interaction, the roots are real and the positive one reads

$$r_H = 2r_0. \quad (6.4)$$

The related solution is

$$A(r) = -\frac{r^2}{l^2} + 3\frac{r_0^2}{l^2} + 2\frac{r_0^3}{l^2 r}. \quad (6.5)$$

If $\omega < 0$, $D > 0$ and there exists only one real root. It is positive and reads

$$r_H = r_0 f(\omega), \quad (6.6)$$

$$f(\omega) = (1 + \omega)^{1/3} \left[ (1 + \sqrt{-\omega})^{1/3} + (1 - \sqrt{-\omega})^{1/3} \right]. \quad (6.7)$$

6.1 The $n = 4$ solution

In order to have a black hole, one must have $k_0 = 1$ and we have

$$A(r) = \frac{(r + r_0)^2}{r^2} - \frac{r^2}{l^2}, \quad (6.8)$$

$$\phi(r) = \frac{\sqrt{6r_0}}{(r + r_0)}.$$ 

This reduces to a vacuum de Sitter space when $r_0 = 0$, so we can think of it as a hairy excitation of de Sitter space. Nevertheless, as we will see, the entropy of the cosmological horizon is less than that of pure de Sitter space. The self-interacting coupling constant is fixed to be negative

$$\alpha_4 = -\frac{l^2}{12}.$$ 

In this case, if $r_0 < 0$, there exist event, cosmological and Cauchy horizons and the transverse manifold is a 2-sphere [2]. It has been proven that this solution is unstable. If $r_0 > 0$, only a cosmological horizon exists.
6.2 A new n-dimensional de Sitter hairy solution

As last case, it is easy to see that for generic \( n \), and for \( k = 0 \), if

\[
\alpha_n = \frac{(n-2)^2 r_0^2}{8l^2 c^{n-2}}
\]  \hspace{1cm} (6.10)

is satisfied, then there exists a solution given by

\[
A(r) = -\frac{r^2}{l^2}.
\]  \hspace{1cm} (6.11)

We may interpret this solution observing that now \( r \) is a time coordinate and \( t \) a space coordinate. Introducing new coordinates \((T, Y)\)

\[
t = \frac{Y}{T_0} \quad r = T_0 e^\frac{T}{l},
\]  \hspace{1cm} (6.12)

one has

\[
ds^2 = -dT^2 + e^{2\frac{T}{l}} \left( dY^2 + dX^2 \right),
\]  \hspace{1cm} (6.13)

where the transverse metric \( dX^2 \) represents the non compact manifold \( R^{n-2} \), then \( dY^2 + dX^2 \) has the metric of \( R^{n-1} \). Introducing spherical coordinates \((R, \theta_1, \theta_2 \ldots)\), the metric can be rewritten as

\[
ds^2 = -dT^2 + e^{2\frac{T}{l}} \left( dR^2 + R^2 dS_{n-2}^2 \right).
\]  \hspace{1cm} (6.14)

This is a de Sitter-like solution in the cosmological synchronous gauge. In three dimensions it emerges as the formal zero temperature limit of the Kerr-de Sitter metric \[45\] and represents a cylindrical universe expanding from a wirelike singularity. The solution is timelike and null geodesically incomplete in the past. It can also be shown that the \( n = 3 \) solution is the asymptotic limit (in time) of the general solution of 3D de Sitter gravity with flat or toroidal spatial topology, a kind of attractor mechanism. The fact that the zero temperature state of three-dimensional de Sitter gravity is not a true ground state makes it hard to give sense to the statistical partition function of the finite temperature states. In fact it seems that without further prescriptions we could get negative entropy states.

As anticipated, the solution for the scalar field also becomes “time-dependent”

\[
\phi(T) = \frac{1}{\left(T_0 e^{T/l} + r_0\right)^{n/2-1}}.
\]  \hspace{1cm} (6.15)

Making use of the standard mapping,

\[
T = \hat{t} + \frac{l}{2} \ln \left(1 - \frac{\rho^2}{l^2}\right),
\]  \hspace{1cm} (6.16)

\[
R = \frac{\rho}{(1 - \rho^2)^{1/2}} e^{-i\hat{t}/l},
\]  \hspace{1cm} (6.17)

Changing \( T \to -T \) one gets a contracting one.
we may pass to the static de Sitter gauge

\[ ds^2 = -dt^2(1 - \frac{\rho^2}{l^2}) + \frac{d\rho^2}{(1 - \frac{\rho^2}{l^2})} + \rho^2 dS_{n-2}^2. \]  

(6.18)

In static gauge, the solution for the scalar field solution is time dependent and assumes the form

\[ \phi(\hat{t}, \rho) = \frac{1}{(r_0 + T_0 \sqrt{1 - \frac{\rho^2}{l^2}})^{n/2 - 1}}. \]  

(6.19)

This solution may interpreted as a n-dimensional hairy scalar de Sitter solution.

7 The non-warped class of black hole solutions

Broadening, in some sense, our initial ansatz in order to find new solutions, we may pass to a setting that will allow us to find metrics that could be regarded as near-horizon approximations of other more standard solutions. In particular, in the case explained below we are also able to find a five dimensional solution, that could serve as a guideline for finding other solutions in \( n \geq 5 \) dimensions.

The ansatz considered here is reminiscent of the solutions describing extremal limits of black holes (cfr. for example [46]):

\[ dS^2 = -A(r)dt^2 + \frac{dr^2}{A(r)} + B^2 d\Sigma^2, \]  

(7.1)

where \( B \) is a suitable real constant, possibly depending on the physical parameters of the solution. We intend for “non-warped solution” the fact that this metric represent a direct product of two manifolds, the one described by coordinates \((r, t)\) and \( \Sigma \), not a warped one. In the examples that follow, the unspecified transverse manifold \( \Sigma \) will always be the \( n-2 \) dimensional sphere.

All the following solution support a constant contravariant electric field \( E \). The value of the electric charge is taken to be \( e \).

7.1 The \( n=3 \) solution

In this setting, the solution for the metric and the field turns out to be

\[ A(r) = \frac{2}{l^2}(r - r_0)^2 + 8\alpha c^4; \]  

(7.2)

\[ \phi(r) = \frac{c}{(r - r_0)^{1/2}}. \]  

(7.3)

The physical constant are related trough \( \Lambda = -1/l^2 = -e^2 \), \( B \) remains a free parameter.

The constants \( c \) can always be regarded as positive thanks to a sign symmetry in the theory, and through a rescaling of the coordinates and of the other constant it can be taken as unit.

So, the event horizon is located at

\[ r_H = r_0 + 2l\sqrt{-\alpha}. \]  

(7.4)
7.2 The n=4 solution

In four space-times dimensions the solution is

\[ A(r) = e^{2(r - r_0)^2 + 2\alpha}; \quad (7.5) \]
\[ \phi(r) = \frac{1}{r - r_0}. \quad (7.6) \]

The horizon manifold is a two-sphere, while there is a more stringent requirement on the physical parameters than before: \( \Lambda = 0, B^2 = e^{-2}, \alpha < 0 \) in order to have an event horizon. Again, the amplitude of the scalar field \( c \) was scaled away.

The horizon is located at radial coordinate

\[ r_H = r_0 + \sqrt{-\frac{2\alpha}{e^2}}. \quad (7.7) \]

7.3 The n=5 solution

The last example of this class of solution, and the one that could actually represent the near-horizon approximation of a yet to be found five dimensional black hole, is given by

\[ A(r) = \frac{2}{3}\Lambda(r - r_0)^2 + \frac{8}{9}\alpha; \quad (7.8) \]
\[ \phi(r) = \frac{1}{(r - r_0)^{3/2}}. \quad (7.9) \]

As before, the charge and cosmological constant are constrained, \( \Lambda = e^2 \), while the direct product factor is given by \( B^2 = 3/2\Lambda \). The horizon is a three-sphere, located at coordinate

\[ r_H = r_0 + 2\sqrt{\frac{\alpha}{3\Lambda}}. \quad (7.10) \]

This expression forces the potential parameter \( \alpha \) to be positive and, as implicit also in the previous cases, constrains \( r_0 \) to be not smaller than a certain value in order to have an event horizon.

8 Thermodynamics of black hole solutions

In this Section, we shall study some thermal properties of the hairy BHs, just as they follow from the form of the solutions. Recall the Hawking temperature can be computed by means of

\[ T_H = \frac{A'(r_H)}{4\pi}, \quad (8.1) \]

and the BH entropy via Wald’s method is

\[ S = \frac{A_H}{4G} \left( 1 - \xi c\phi^2(r_H) \right) = 2\pi A_H \left( 1 - \xi c\phi^2(r_H) \right). \quad (8.2) \]

Note the presence of the non standard and non manifestly positive factor depending on the scalar field. A naive application of the method could lead to problematic results because negative black
hole entropies might appear \cite{47,48,49}.
However, for $\Lambda < 0$, all the BH solutions found so far have non negative entropies. For $\Lambda > 0$, negative entropies may appear and the $n = 4$ case has been considered in \cite{50}.
Here we present a detailed investigation of the thermodynamics associated with all BH hairy solutions previously discussed. We are not going to consider the result for non-warped solutions, as they are not interesting to our purposes and there are no ordinary solution to check if they are really a near horizon limit of some general solution by means of a matching of the thermodynamical parameters.

8.1 The $n = 3$ black hole solution with $\Lambda < 0$
In general, for $0 < \omega < 1$, one has $r_H = f(\omega)r_0$ and

$$T_H = \frac{3r_0}{2\pi l^2}(1 - \omega)\frac{1 + f(\omega)}{f^2(\omega)},$$

where

$$f(\omega) = (1 - \omega)^{1/3}\left[(1 + \sqrt{\omega})^{1/3} + (1 - \sqrt{\omega})^{1/3}\right]. \quad (8.3)$$

The entropy is manifestly non negative

$$S_{BH} = 2\pi A_H \frac{r_H}{r_H + r_0} = 4\pi^2 \frac{f^2(\omega)}{1 + f(\omega)}r_0 \quad (8.4)$$

It is convenient to regard $r_0$ as a physical BH parameter. Then, taking the derivative of the BH entropy with respect to $r_0$

$$dS_{BH} = \frac{1}{T_H}dM, \quad (8.5)$$

with the identification

$$M = \frac{3\pi(1 - \omega)}{l^2}r_0^2. \quad (8.6)$$

One can check the validity of the Smarr relation

$$M = \frac{1}{2}T_H S_{BH}$$

It has been shown \cite{51} that Smarr-like formulas can be proved for many minimally coupled black holes in AdS, due to a scaling symmetry of the reduced action; the remarkable universality of these formulas, expressed by the independence of the scalar potential, has been stressed by these authors. Here we see the validity of the Smarr formula for conformal coupling as well. As a result, we get the first Law of the thermodynamics of the black hole as well as the physical interpretation of the quantity $r_0$, in agreement with \cite{2}, result obtained working in the Einstein frame. Due to our theorem, both the Hawking temperature and the BH entropy are equal. We also have

$$r_0 = \frac{l}{\pi \sqrt{3(1 - \omega)}} \sqrt{M} \quad (8.7)$$
\[ T_H = \frac{\sqrt{3}}{2\pi^2 l} \left( \sqrt{1 - \omega} \right)^{1 + f(\omega)} \frac{f^2(\omega)}{f^2(\omega)} \sqrt{M}. \]  
(8.8)

\[ S_{BH} = \frac{4l}{\pi \sqrt{3(1 - \omega)}} \frac{f^2(\omega)}{1 + f(\omega)} \sqrt{M} = \frac{4l}{\pi \sqrt{3(1 - \omega)}} \frac{f^4(\omega)}{(1 + f(\omega))^2} T_H \]  
(8.9)

Thus, \( M = g(\omega) T_H^2 \) in agreement with the AdS/CFT correspondence.

Remark: Martinez et al. [2], working in the Einstein frame, got an expression for the Hawking temperature formally different from our. The two expression are identical (as they should) if and only if

\[ 2 \text{Im} \left( \frac{(1 - \omega)^{2/3} (1 + i \sqrt{-\omega})^{2/3}}{\sqrt{-\omega}} \right) = \frac{f^2(\omega)}{1 + f(\omega)}. \]  
(8.10)

Only by a numerical computation we were able to confirm it.\(^7\)

8.2 The \( n = 4 \Lambda < 0 \) case

In the \( n = 4 \) case, black hole solutions exist only for \( k = -1 \), namely for hyperbolic horizon. The Hawking temperature and the entropy (which is positive) read

\[ T_H = \frac{1}{2\pi l} \sqrt{1 + 4r_0/l}, \]  
(8.11)

\[ S_{BH} = 2\pi A_H \left( 1 - \frac{r_0^2}{(r_H + r_0)^2} \right) = 2\pi V_F l^2 \left( 1 + \frac{2r_0}{r_H} \right) = 4\pi^2 V_F l^3 T_H, \]  
(8.12)

where \( V_F \) is the fundamental domain of the Riemann surface associate with the hyperbolic horizon and BH entropy goes linearly with the Hawking temperature.

If \( r_0 > 0 \), there is only an event horizon. If \( r_0 < 0 \), a finite static region appears as well as an unbounded static region. This is a quite unusual situation! The divergence of the scalar field lies inside the inner static region, but the metric is otherwise regular there.

As before, we may derive the First Law of the thermodynamics of the black hole and determine the mass as a function of the parameter \( r_0 \), taking the derivative with respect to \( r_0 \),

\[ dS_{BH} = \frac{1}{T_H} d\left( 2V_F r_0 \right), \]  
(8.13)

\[ M = 2r_0 V_F + M_0, \]  
(8.14)

where \( M_0 \) is a constant independent on \( r_0 \). We may determine the constant, requiring \( T_H \) to be real and the mass non negative. Thus,

\[ M_0 = -\frac{l V_F}{2}, \]  
(8.15)

\(^7\)In the range \( \omega \in [-5, 1] \) the numerical agreement was found to be of the order of \( 4 \cdot 10^{-9} \). No analytical method was found to compare the two expressions.
and
\[ M = 2\pi^2 V_F l^3 T_H^2 \]  
(8.16)

Again one has the Smarr formula \( M = T_H S_{BH}/2 \). Remark: The AdS/CFT correspondence would suggest, at least for large mass,
\[ M = B T_H^3. \]  
(8.17)

As a result, it seems that we have an example of violation of AdS/CFT correspondence. Furthermore, the entropy as a function of the mass reads
\[ S_{BH} = \pi (2l)^{3/2} \sqrt{M V_F}, \]  
(8.18)

again in disagreement with the usual AdS/CFT correspondence. However there exists a generalized AdS/CFT correspondence [52] which can be adapted to this case, but it remains to show how the explanation of the apparent disagreement really works. The Hawking temperature and the entropy are equal to the ones computed in the Einstein frame. The specific heat of these solutions is positive, namely there is thermodynamic stability of the BH.

### 8.3 The \( n = 4 \Lambda > 0 \) case

This is a quite interesting case. It has been discussed for a charged BH in [50]. We anticipate that our conclusions will be slightly different. For the sake of simplicity, we consider only the uncharged case, since our results can be easily extended to the charged one.

In order to deal with event horizon \( r = r_E \) and a cosmological horizon \( r_C \), the integration constant \( r_0 \) must be negative, say \( r_0 = -a, a > 0 \). It turns out that event and cosmological horizon have the same Hawking temperature, lukewarm BHs
\[ T_E = T_C = \frac{1}{2\pi l} \sqrt{1 - \frac{4a}{l}}. \]  
(8.19)

Note that it might be possible another interpretation for \( T_C \), due to Klemm and Vanzo [53], in term of negative temperature. We will not pursue this point of view.

In [50] is reported that the cosmological BH entropy is positive and the other one associate with the event horizon is negative, total entropy being vanishing,
\[ S_C = 4\pi^2 V_F l^3 T_C, \]  
(8.20)
\[ S_E = -4\pi^2 V_F l^3 T_E. \]  
(8.21)

One can make use of the First Law of BH thermodynamic and get
\[ a = \frac{M}{8\pi}, \]  
(8.22)

where \( M \) may be interpreted as BH mass. The temperatures may be rewritten as
\[ T_E = T_C = \frac{1}{2\pi l} \sqrt{1 - \frac{M}{2\pi l}}. \]  
(8.23)

As a result, a maximal mass and temperature are present: \( M < 2\pi l \).
A negative entropy, in the statistical sense, does not make sense, so either one removes by hand the negative part or else one interprets the negative entropy as meaning that the lukewarm states have exponentially small probability, because in de Sitter space there are robust arguments to believe that the entropy is the logarithm of the euclidean action (see [54] for a comparison of Noether charge with euclidean methods). However, in de Sitter space it could be that there is not a universal state with zero entropy to which compare all other states, and there is some evidence of this in 3D de Sitter space [45]. In this case only entropy differences make any sense. As emphasized by ’t Hooft [55], it could also be the case that the additive constant diverges if the contribution of the quantum fields around a black hole are not carefully renormalized. So shifting the entropy by a constant not only is a sensible procedure in quantum gravity, it is also a transformation that can be given a precise meaning within the Noether charge method, as discussed extensively in [56].

This motivates our proposal: make use of the ambiguity in the Wald method and fix the constant in the entropy by the following continuity argument. When $M \to 0$, one must get the pure de Sitter solution. Thus, a reasonable expression for the entropy associated with the event horizon is

$$S_E = 2\pi V_F l^2 \left( 1 - \sqrt{1 - \frac{M}{2\pi l}} \right).$$

(8.24)

With this proposal, the total entropy is not vanishing and equal to the de Sitter one

$$S_{dS} = 2\pi V_F l^2 = 8\pi^2 l^2$$

(8.25)

Our proposal can be strengthened by looking at the Smarr formula. Using the “renormalized” entropy (8.24), with a direct verification one has

$$\frac{1}{2} (T_E S_E + T_C S_C) = \frac{1}{4\pi l} \left( \sqrt{1 - \frac{M}{2\pi l}} \right)$$

(8.26)

and it is easily seen that the last term is the vacuum energy in between the two horizons, namely $(4\pi)^{-1} \int \Lambda K_a d\Sigma^a$, where $K$ is the timelike Killing field of the solution, in agreement with [57]. But looking to (8.24), we see that this entropy does not follows anymore the area law.

8.4 The n-dimensional de Sitter case

We conclude this Section with the n dimensional de Sitter solution. For a generic $n$ and $k = 0$, we have shown that there is a de Sitter solution of our gravity conformally coupled to a scalar hair. In the static gauge, there is a cosmological horizon and the scalar field is non static. However, the time dependence disappears on the horizon. Thus, if we compute the entropy with the Noether charge method, we have

$$S_{BH} = 2\pi V_F l^{n-2} \left( 1 - \xi_c \frac{1}{\sqrt{n-2}} \right),$$

(8.27)

or

$$S_{BH} = 2\pi V_F \left[ l^{n-2} - \xi_c \left( \frac{n-2}{2\sqrt{2|\alpha_n|}} \right)^{n-2} \right].$$

(8.28)
The first contribution is the pure de Sitter contribution, the second one is constant and negative and associated with the degrees of freedom of the scalar field.

As a result, as it is well known, pure de Sitter space is the state of maximum entropy.

9 Conclusions

In this paper, we have revisited the Einstein gravity-scalar system in various space-time dimensions, with a conformally coupled scalar field and non vanishing cosmological constant. Motivated by RS models, we have worked in the so called Jordan frame. First, we have provide an elementary proof of the conformal frame independence of Hawking temperature and black hole entropy. Motivated by this result, we have revisited the search for exact n-dimensional black hole solutions in presence of the conformally coupled scalar field. We have recovered the $n = 3$ and $n = 4$ known solutions and found a new de Sitter n-dimensional solution with time dependent scalar field. The thermodynamics properties of these hairy black hole solutions have been discussed in detail and their entropy evaluated with the Noether charge method, which is powerful but does not ensure the positivity of the entropy.

For asymptotically AdS hairy BH solutions, the related BH entropies turn out to be non negative and this fact is certainly related to their stability.

For asymptotically dS hairy BH solutions, a naive evaluation of the BH entropy leads to a negative entropy. Making use of the freedom in the Wald method, a simple continuity argument has been advocated in order to remedy this fact. The resulting expression amusingly satisfy the Smarr relation pertinent to de Sitter space.

As far as the AdS/CFT correspondence, in the negative cosmological constant case, we have seen that the $n = 3$ BH solution is in agreement with it, while the $n = 4$ BH solution, gives rise to a relation between the mass and Hawking temperature in disagreement with the usual AdS/CFT correspondence result. The Mincos/Rivelles generalized AdS/CFT correspondence is probably the way out of this dilemma.

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