Vortex Dynamics in Selfdual Maxwell-Higgs Systems with Uniform Background Electric Charge Density†

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Abstract

We introduce selfdual Maxwell-Higgs systems with uniform background electric charge density and show that the selfdual equations satisfied by topological vortices can be reduced to the original Bogomol’nyi equations without any background. These vortices are shown to carry no spin but to feel the Magnus force due to the shielding charge carried by the Higgs field. We also study the dynamics of slowly moving vortices and show that the spin-statistics theorem holds to our vortices.
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1. Introduction

We consider the theory of complex and neutral scalar fields coupled to a gauge field with the Maxwell kinetic term in three dimensions. There exist topological vortices of nonzero magnetic flux in the broken phase.\(^1\) We introduce a nondynamical uniform background electric charge density to the system, which would be shielded by the electric charge carried by the Higgs field. In this case, vortices are claimed to carry nonzero spin and feel the fluid dynamical Magnus force, a Lorentz force, due to the shielding charge carried by the Higgs field.\(^2\) In addition, vortices in charged fluid with a nondynamical background magnetic field have been studied to see whether they are anyons.\(^3\) Here we choose a set of special values for coupling constants so that there is a bound on energy, which can be saturated by the vortex configurations satisfying the selfdual equations, generalizing the Bogomol’nyi case.\(^4\) We study in detail the vortex dynamics in this selfdual model.

Recently, there has been a renewed interest in the possibility of the Magnus force in real superconductor.\(^5\) The question is whether there is a nonzero Berry’s phase gained by a vortex wave function due to the Magnus force when it goes around a closed loop on a plane. In real superconductors, the copper pairs condense with net nonzero electric charge, which is neutralized by the background electrons and positive ions. The Maxwell-Higgs system with the background electric charge density is thus more closely related to the real superconductor rather than the Maxwell-Higgs system without as noted by Davis.\(^2\) Our analysis confirms the presence of the Magnus force more clearly. Since our system is selfdual or there is no static interaction between vortices, we can regard our model lying at the boundary between type I and type II superconductors.

The Magnus force is the fluid dynamical force responsible for curve balls. The Magnus force also plays an important role in the vortex dynamics in superfluid.\(^6\) In superfluid, vortices carry infinite angular momentum density per unit length and feel finite amount force per unit length. In large distance, the Magnus force can induce a Berry’s phase. For example, in Chern-Simons-Higgs systems, vortices carry nonzero magnetic flux and charge. These vortices carry nonzero spin and their statistics can be explained only when one put together the naive Aharanov-Bohm phase due to the charge and magnetic flux and that due to the Magnus
In our case, it turns out to be subtle to define the conserved angular momentum in the field theory due to the divergent contribution from the spatial infinity. We provide a satisfactory modification of the Noether angular momentum in the field theory. The selfdual configurations are degenerated in energy but not in angular momentum. The total angular momentum is a complicated function of the vortex positions. We show that there is however no intrinsic spin carried by our vortices, which does not contradict the fact that our vortices carry nonzero magnetic flux but no electric charge. In our systems, the magnus force is in a way decoupled from the spin of vortices.

As there are no massless excitations in our systems as we will see, we expect little or no radiations emitted when vortices are moving very slowly. The field configuration of these slowly moving vortices would be very close to that of vortices at rest. There would be an effective Newtonian action describing these vortices, which would be made of terms linear and quadratic in velocities. The interesting goal would be then to find this effective action. The effective action for slowly moving solitons has been first studied by Manton. There have numerical and analytical studies of this effective action for slowly moving vortices when there is no background charge.

As these approaches are not directly applicable in our case, we take a little bit different approach. We find the first order terms explicitly. The angular momentum calculated from this effective action turns out to be identical to that obtained from the field theory. The linear term has an interesting implication in the vortex dynamics. A single vortex would move a circle due to the Magnus force, which implies a nontrivial Berry’s phase in quantum mechanics. For a system of two vortices, the Magnus force become more complicated when two vortices are close to each other since the charge density is not uniform. When two vortices are separated in large distance, we will show that however there is no additional Berry’s phase which can be attributed to the statistics between vortices, confirming the spin-statistics theorem for our vortices.

This paper is organized as follows. In Sec. 2 we introduce the selfdual Maxwell-Higgs systems with uniform background electric charge density and study their
basic properties. We show that the naive conserved linear and angular momenta
have divergent contributions from the spatial infinity. We show that our selfdual
equations can be reduced to those found by Bogomol’nyi. In Sec. 3, we study the
rotationally symmetric vortices numerically. In Sec. 4 we redefine the linear and
angular momenta and show that our vortices do not carry any spin. In addition,
we provide an explicit expression of the angular momentum as a function of vortex
positions. In Sec. 5, we study the effective action of slowly moving vortices. This
action contains the terms linear and quadratic in vortex velocities. The linear
terms describe the magnetic interaction between vortices themselves and between
vortices and the shielding charge carried by the Higgs field. We use this linear
term to calculate the statistics of our vortices. In Sec. 6, we conclude with some
remarks and questions. In the appendix, we calculate the nonconserved angular
momentum derived from the symmetric energy momentum tensor as a function of
vortex positions.

2. Model

We consider the theory of charged and neutral scalar fields $\phi = f e^{i\theta}/\sqrt{2}, N$
coupled to a photon field $A_\mu$. We assume that there is a uniform background electric
charge density $\rho_e$, which is no dynamical. (A uniform external magnetic field plays
a role of chemical potential for a magnetic flux after a field shift and so no role in
the classical dynamics of vortices.) The lagrangian for this theory is

$$
\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2}(\partial_\mu N)^2 + \frac{1}{2}(\partial_\mu f)^2
+ \frac{1}{2} f^2 (\partial_\mu \theta + e A_\mu)^2 - U(f, N, \rho_e) - \rho_e A_0
$$

The Lorentz symmetry is explicitly broken due to the external electric charge den-
sity. The charge conjugation, $\theta \rightarrow -\theta$, $A_\mu \rightarrow -A_\mu$, is also explicitly broken by the
external charge density. The parity transformation, $(x^1, x^2) \rightarrow (x^1, -x^2), (A_1, A_2) \rightarrow
(A_1, -A_2)$, is not broken. The time reversal, $t \rightarrow -t, A_0 \rightarrow -A_0$, is also explicitly
broken. However, CTP is still a good symmetry. Usually the selfdual systems are
related to the $N = 2$ supersymmetry with a central term, and the neutral scalar
field and the gauge field are a part of the vector multiplet.
The system is invariant under the local gauge transformation, \( \theta \to \theta + \Lambda, A_\mu \to A_\mu - \partial_\mu \Lambda/e \), leading to Gauss’s constraint from the variation of \( A_0 \),

\[
\partial_i F_{0i} + e f^2 (\dot{\theta} + e A_0) - \rho_e = 0 \tag{2.2}
\]

where the dot denotes the time derivative. Since the background charge density is uniform, the action is invariant under the spacetime translation and there is a conserved energy momentum tensor as a Noether current,

\[
T_{\mu \nu} = -F_{\mu \rho} \partial_\nu A^\rho + \partial_\mu N \partial_\nu N + \partial_\mu \dot{f} \partial_\nu f + f^2 (\partial_\mu \theta + e A_\mu) \partial_\nu \theta - \eta_{\mu \nu} \mathcal{L} \tag{2.3}
\]

satisfying, \( \partial^\mu T_{\mu \nu} = 0 \). The total energy \( E = \int d^2 x T_{00} \) can be expressed as

\[
E = \int d^2 x \mathcal{E} \tag{2.4}
\]

after a partial integration and using Gauss’s law (2.2), where the energy density is given by

\[
\mathcal{E} = \frac{1}{2} (F_{0i}^2 + F_{i2}^2) + \frac{1}{2} \left[ \dot{N}^2 + (\partial_i N)^2 \right] + \frac{1}{2} [\dot{f}^2 + (\partial_i f)^2] \\
+ \frac{1}{2} f^2 (\dot{\theta} + e A_0)^2 + \frac{1}{2} f^2 (\partial_i \theta + e A_i)^2 + U \tag{2.5}
\]

The conserved linear momentum \( \int d^2 x T_{0i} \) is

\[
\tilde{P}^i = -\int d^2 x \left\{ F_{0j} \partial_i A_j + \dot{N} \partial_i N + \dot{f} \partial_i f + f^2 (\dot{\theta} + e A_0) \partial_i \theta \right\} \tag{2.6}
\]

Under the local gauge transformation \( \theta \to \theta + e \Lambda \) and \( A_i \to A_i - \partial_i \Lambda \), \( T_{0i} \) is invariant up to a total derivative due to Gauss’s constraint, leaving the linear momentum invariant. As the action is also invariant under rotation, there is a conserved angular momentum current,

\[
\mathcal{J}_\mu = -\epsilon_{ij} F_{\mu i} A_j - \epsilon_{ij} x^i T_{\mu j} \tag{2.7}
\]

with \( T_{\mu \nu} \) given by Eq.(2.3), satisfying \( \partial^\mu \mathcal{J}_\mu = 0 \). The conserved Noether angular
momentum $\bar{J} = \int d^2r J_0$

$$\bar{J} = -\int d^2x \{\epsilon_{ij} F_{0i} A_j + \epsilon_{ij} x^i T_{0j}\}$$

(2.8)

which is also invariant under local gauge transformations. Note that under the translation of the whole system by $x^i \rightarrow x^i + a^i$, $J \rightarrow J + \epsilon_{ij} a^i P^j$.

It turns out that the above definition of the momenta is not entirely correct. When there are $n$ vortices, we will see that $\partial_i \theta \rightarrow -\frac{n}{r} \hat{r}^i$ and $ef^2 A_0 \rightarrow \frac{\rho e}{v}$ for large $r$. Thus, there is a divergent contribution from the spatial infinity to these momenta. The definition of finite, well-defined and conserved momenta will be given in Sec.4.

For the reasons which will be clear in a moment, we consider a specific potential

$$U = \frac{e^2}{2} N^2 f^2 + \frac{e^2}{8} (f^2 - v^2)^2 - \rho v N$$

(2.9)

This potential has a term for the interaction between the external charge and the neutral scalar field. While the physical origin of this interaction is obscure, we can imagine such possibility. This potential is not bounded from below, which we will see is not a problem due to Gauss’s law.

After some algebra and by using Gauss’s law (2.2), we get for any physical configuration

$$\mathcal{E} = \frac{1}{2} (F_{0i} + \partial_i N)^2 + \frac{1}{2} [F_{12} \pm \frac{e}{2} (f^2 - v^2)]^2 + \frac{1}{2} \dot{N}^2$$

$$+ \frac{1}{2} f^2 + \frac{1}{2} f^2 [\theta + eA_0 - eN]^2 + \frac{1}{2} [\partial_i \phi \pm \epsilon_{ij} f (\partial_j \theta + eA_j)]^2$$

(2.10)

$$\pm \frac{e}{2} V_2 F_{12} - \partial_i [F_{0i} N + \frac{1}{2} \epsilon_{ij} f^2 (\partial_j \theta + eA_j)]$$

The ground state of this energy functional will be in phase where $< f^2 > \neq 0$. As there will be no massless mode, the gauge invariant quantities, $F_{0i}, \partial_i \theta + eA_i$ would vanish exponentially at the spatial infinity, making the boundary contribution vanish. After integrating Eq.(2.10) over space, we get a bound on the total
energy,

\[ E \geq \pm \frac{ev^2}{2} \int d^2r F_{12} \]  

(2.11)

because the rest of terms are positive definite.

This bound is saturated by the time independent configurations in a gauge \( \dot{\theta} = 0 \) which satisfy Gauss’s law (2.2) and

\[ A_0 - N = 0 \]
\[ F_{12} \pm \frac{e}{2}(f^2 - v^2) = 0 \]
\[ \partial_i f \pm \epsilon_{ij} f(\partial_j \theta + eA_j) = 0 \]  

(2.12)

For these selfdual configurations Gauss’s law (2.2) becomes

\[ \partial_i^2 A_0 + \rho_e - e^2 f^2 A_0 = 0 \]  

(2.13)

Note that Eq.(2.12) for \( A_i, f, \theta \) is identical to the selfdual vortex equations obtained by Bogomol’nyi.\[ As far as the scalar field and the vector potential are concerned, the selfdual vortex configurations without background charge density is identical to those with. The existence and uniqueness of the selfdual solutions for the scalar field and the vector potential, describing the 2n parametered configurations of n vortices, have been studied and proven.\[ We will see that \( A_0 \) satisfying Eq.(2.13) can be expressed explicitly in terms \( f \). Thus, in our system also there exist unique selfdual configurations of vortices parameterized by the vortex positions.

The ground state is a homogeneous configuration of zero energy. From Eqs. (2.12) and (2.13) we get this ground state described classically by

\[ f = v \]
\[ N = A_0 = \frac{\rho_e}{e^2v^2} \]
\[ \theta = A_i = 0 \]  

(2.14)

in a unitary gauge. (We can imagine inhomogeneous configurations, which have a finite region where \( f = 0 \) and \( N \) being arbitrary large, making the contribution
from the potential energy to be arbitrary negative. The bound (2.11) tells us that these configurations have positive energies.) The small fluctuation analysis around this vacuum implies that in the unitary gauge

\[
(w^2 - \vec{k}^2 - e^2 v^2)\delta f + \frac{2\rho_e}{v}(\delta A_0 - \delta N) = 0
\]

\[
(w^2 - \vec{k}^2 - e^2 v^2)\delta N - \frac{2\rho_e}{v}\delta f = 0
\]

\[
(w^2 - \vec{k}^2 - e^2 v^2)\delta A + \vec{k}\vec{k} \cdot \delta A - w\vec{k}\delta A_0 = 0
\]

\[
(\vec{k}^2 + e^2 v^2)\delta A_0 - w\vec{k} \cdot \delta A + \frac{2\rho_e}{v}\delta f = 0
\]

(2.15)

where \(\delta f, \delta N, \delta A_\mu \sim e^{iwt + i\vec{k} \cdot \vec{r}}\). There are three different massive modes in long distance, or small \(\vec{k}\), with eigenvalues,

\[
w^2 = e^2 v^2
\]

\[
w^2 = e^2 v^2 \left\{ 1 + 2\frac{\rho_e^2}{e^4 v^6}[1 \pm \sqrt{1 + \frac{e^4 v^6}{\rho_e^2}}] \right\}
\]

(2.16)

There is no instability due to these modes. We can see easily that the first spectrum describes a vector boson of spin \(\pm 1\) and the last two spectra describe two scalar bosons.

For the selfdual configurations of the positive magnetic flux, we choose the upper sign of Eq.(2.12),

\[
F_{12} + \frac{e}{2}(f^2 - v^2) = 0
\]

\[
\partial_t \theta + eA_i - \epsilon_{ij} \partial_j \ln f = 0
\]

(2.17)

For the positive magnetic flux configurations, the vorticity in \(\theta\) turns out negative as we will see. We describe \(n\) vortices located at positions \(\vec{q}_a\) by

\[
\theta(\vec{r}) = -\sum_{a=1}^{n} \text{Arg}(\vec{r} - \vec{q}_a)
\]

(2.18)

satisfying

\[
\epsilon_{ij} \partial_i \partial_j \theta = -\partial_i^2 \sum_a \ln |\vec{r} - \vec{q}_a| = -2\pi \sum_a \delta(\vec{r} - \vec{q}_a)
\]

(2.19)

We see that \(f\) should vanish as \(|\vec{r} - \vec{q}_a|\) at the position of each vortex for the
complex scalar field $\phi$ to behave well. Putting together Eqs.(2.17) and (2.19), we get

$$\partial_i^2 \ln f^2 - e^2 (f^2 - v^2) = 4\pi \sum_a \delta(\vec{r} - \vec{q}_a)$$  \hspace{1cm} (2.20)

In addition, we see that the excessive flux $\Psi = \int d^2 x F_{12} = \oint F_{12} \cdot \vec{A} = -\frac{1}{e} \oint F_{12} \cdot \vec{\nabla} \theta = 2\pi n/e$ is positive.

For a given $f$ configuration satisfying Eq.(2.20), Gauss’s law (2.13) determines the $A_0$ configuration. It turns out that $A_0$ can be solved explicitly in terms of $f$,

$$A_0(\vec{r}, \vec{q}) = \frac{\rho_{e}}{e^2 v^2} \left\{ 1 - \sum_{a=1}^{n} (\vec{r} - \vec{q}_a) \cdot \frac{\partial}{\partial \vec{q}_a} \ln f \right\}$$

$$= \frac{\rho_{e}}{e^2 v^2} \left\{ 1 + n - \sum_{a} (\vec{r} - \vec{q}_a) \cdot \frac{\partial}{\partial \vec{q}_a} \ln \left[ \prod_{b} |\vec{r} - \vec{q}_b| \right] \right\}$$  \hspace{1cm} (2.21)

Notice that the quantity in the right side is regular at the vortex positions and approaches the right asymptotic value at spatial infinity and that $\partial / \partial \vec{r} = -\sum_a \partial / \partial \vec{q}_a$ on $f$ due to the space translation invariance.

3. Rotationally symmetric solutions

The rotationally symmetric ansatz for $n$ positive magnetic flux vortices at origin is given in the rotational coordinate $(r, \varphi)$ as $f(r), \theta = -n \varphi, A_{\varphi}(r), A_0(r)$. The total magnetic flux is then $2\pi n/e$. Eq.(2.17) become

$$eA_\varphi = n - r \frac{d \ln f}{dr}$$

$$\frac{1}{r} \frac{dA_\varphi}{dr} = \frac{e}{2} (v^2 - f^2)$$  \hspace{1cm} (3.1)

We introduce dimensionless quantities, $evr = s, f^2 = v^2 F$, Eq.(3.1) can be put as

$$[s(\ln \frac{F}{s^{2n}})]' + s(1 - F) = 0$$  \hspace{1cm} (3.2)

with the prime denotes $d/ds$. The boundary conditions are $F(s) \sim s^{2n}$ for small $s$ and $F(\infty) = 1$.  

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The scalar potential in Eq.(2.21) become

\[ A_0(r) = \frac{\rho_e}{e^2v^2} \left\{ 1 + \frac{s}{2} (\ln F)' \right\} \]

(3.3)

which implies that \( A_0(0) = \rho_e (n + 1)/e^2v^2 \). From Eq.(3.1), we see that \( A_0 \sim (n + 1 - eA_\phi) \).

Since we are interested in regular solutions, there is one parameter near \( s = 0 \) is to be adjusted to get the right asymptotic behavior at \( r = \infty \).

\[ f/v = as^n(1 - \frac{1}{4}s^2 + ...) \]

(3.4)

where the dots indicate the even power series in \( s \) with higher power than what is shown and whose coefficients are fixed in terms of \( a \) and \( b \). For large \( r \), the regular solution would be

\[ f/v = 1 - \frac{c}{\sqrt{s}} e^{-s} \]

(3.5)

in leading orders.

As Eq.(3.2) is identical to the selfdual equations without any background electric charge density, their properties have been well studied numerically. The new aspect here is that the behavior of the electric charge density, \( e^2 f^2 A_0 - \rho_e \), is non-trivial. At the origin the Higgs field vanishes and so the background electric charge is exposed. As this exposed charge is screened, the total electric charge density would go from negative to positive and then falls to zero exponentially.

Fig. 1 shows the field configuration of a unit vorticity after scaled fields in the distance scale \( evr \). It shows the scalar field \( f/v \), the magnetic field \( F_{12}/2ev^2 \), the scalar potential \( A_0/\rho_e/e^2v^2 \), and the electric charge density \( evr[e^2 f^2 A_0 - \rho_e]/\rho_e \). Fig. 2 describes the field configurations for \( n = 2 \).
4. Linear and angular momenta

Our vortices carry nonzero magnetic flux but no net electric charge. Thus, we can consider our system on a large sphere rather than on a plane. On the sphere there would be a conserved angular momentum vector, which remains finite in the infinite volume limit. In this limit, the angular momentum vector would become the linear and angular momenta on the plane. Here we will simply guess the correct answer on the plane. We first notice that when there are vortices, the net magnetic flux on sphere is nonzero and that the vector potential $A_i$ would have a Dirac string. The vector field

$$\tilde{A}_i \equiv \partial_i \theta + eA_i$$

is gauge invariant and well-defined except at the position of vortices where $f = 0$. $\tilde{A}_i$ would also carry the same magnetic flux as $A_i$. Thus there should be Dirac strings for $\tilde{A}_i$, whose only possible positions are at the center of vortices where $\tilde{A}_i$ is ill-defined. We want express the momenta in terms of $\tilde{A}_i$ rather than $A_i$.

First, we use the field equations to rewrite the energy momentum tensor (2.3) as

$$T_{\mu\nu} = T_{\mu\nu}^S + \rho e(\eta_{\mu\nu}A_0 - \eta_{\mu0}A_\nu)$$

where the nonconserved symmetric energy momentum tensor is defined by

$$T_{\mu\nu}^S = -F_{\rho\mu}F^\rho_{\nu} + \partial_\mu N \partial_\nu N + \partial_\mu f \partial_\nu f + f^2(\partial_\mu \theta + eA_\mu)(\partial_\nu \theta + eA_\nu) - \eta_{\mu\nu}(\mathcal{L} + \rho eA_0)$$

Note that $T_{00} = T_{00}^S$ to a total space derivative. The angular momentum current (2.7) can be rewritten as

$$\mathcal{J}_\mu = -\epsilon_{ij}x^iT_{\mu j}^S - \epsilon_{ij}x^iF_{\rho\mu}A_j - \rho e \epsilon_{ij}x^i(\eta_{\mu j}A_0 - \eta_{\mu0}A_j)$$

After throwing away any total derivatives, we still get a divergent contributions to the angular momentum $\tilde{J} = \int d^2x \mathcal{J}_0$ for the vortex configurations.
Now consider a generic $\theta$ configuration given as

$$\theta = -\sum_a (-1)_a \text{Arg}(\vec{r} - \vec{q}_a(t)) + \eta$$  \hspace{1cm} (4.5)$$

where $\eta$ is a single-valued function. We then introduce a current

$$I_\mu = -\frac{\rho_e}{e} \epsilon_{ij} x^i (\eta_{\mu j} \dot{\theta} - \eta_{\mu 0} \partial_j \theta)$$  \hspace{1cm} (4.6)$$

whose divergence is

$$\partial^\mu I_\mu = -\frac{\rho_e}{e} \epsilon_{ij} x^i (\partial_j \partial_0 - \partial_0 \partial_j) \theta$$

$$= \frac{\pi \rho_e}{e} \sum_a (-1)_a \frac{d|\vec{q}_a|^2}{dt} \delta(\vec{r} - \vec{q}_a)$$  \hspace{1cm} (4.7)$$

If we neglect the divergence from the spatial infinity and the mild singularity at the origin, we get a conserved charge,

$$\Delta J = \int d^2 x I_0 - \frac{\pi \rho_e}{e} \sum_a (-1)_a |\vec{q}_a|^2$$  \hspace{1cm} (4.8)$$

We introduce the new angular momentum, as the sum of $\tilde{J} + \Delta J$,

$$J = -\int d^2 x \epsilon_{ij} x^i \left\{ F_{0k} F_{jk} + \tilde{N} \partial_j N + \tilde{f} \partial_j f + f^2 (\dot{\theta} + eA_0) (\partial_j \theta + eA_j) \\ -\frac{\rho_e}{e} (\partial_j \theta + eA_j) \right\} - \frac{\pi \rho_e}{e} \sum_a (-1)_a |\vec{q}_a|^2$$  \hspace{1cm} (4.9)$$

The divergent contributions at the spatial infinity cancel each other in $J$ since $\partial_j \theta + eA_j$ falls off exponentially. There are also no divergent contributions from the vortex positions because $\partial_i \theta \sim 1/|\vec{x} - \vec{q}_a|$. This angular momentum is also conserved. We also note that $J$ would generate the same transformation of the fields as the canonical angular momentum $\tilde{J}$. Hence the angular momentum given by Eq.(4.9) is as good as we hope for. The well-defined linear momentum is obtained
from the simple observation that under the spatial translation, \( \vec{x} \rightarrow \vec{x} + \vec{a}, \) \( J \rightarrow J + \epsilon_{ij}a^jP^i \). This linear momentum is

\[
P^i = - \int d^2x \left\{ F_{0k}F_{ik} + \dot{N} \partial_i N + \dot{f} \partial_i f + f^2(\partial_0 \theta + eA_0)(\partial_i \theta + eA_i) \right. \\
- \frac{\rho_c}{e} (\partial_i + eA_i) \left. \right\} + \frac{2\pi\rho_c}{e} \epsilon_{ij} \sum_a (-1)^a q_a^i
\]

(4.10)

We are now in the position to explore the angular momentum for the self-dual vortex configurations. These configurations are time-independent and satisfy Eqs.(2.17) and (2.13). Since there are no divergent contributions from the vortex position, we can take out these positions from the integration domain without changing the value of the angular momentum. We call this domain of integration as \( R^2 \equiv R^2 - \{ q_a \} \). Eq.(4.9) for these self-dual configuration becomes

\[
J + \frac{\pi\rho_c}{e} \sum_a |q_a|^2 = -\frac{1}{e} \int_{R^2} d^2x \left\{ x^i \partial_i A_0 \bar{F}_{12} + \epsilon_{ij}x^i \partial_k^2 A_0 \bar{A}_j \right\} \\
- \frac{1}{e} \int_{R^2} d^2x \partial_i \left\{ \epsilon_{jk}x^j \left( \partial_i A_0 \bar{A}_k - A_0 \partial_k \bar{A}_i \right) - \epsilon_{ij} A_0 \bar{A}_j \right\} \\
+ \frac{1}{e} \sum_a \oint_{l_a} \left\{ \epsilon_{jk}(q_a^j + l_a^j) \left( \partial_i A_0 \bar{A}_k - A_0 \partial_k \bar{A}_i \right) - \epsilon_{ij} A_0 \bar{A}_j \right\}
\]

(4.11)

where the line integration is over the vortex positions and \( l_a^i = x^i - q_a^i \). The above angular momentum gets separated naturally into two pieces: the extrinsic part proportional to \( q_a^i \) and the intrinsic part proportional to \( l_a^i \).

To evaluate the total angular momentum, let us study the behavior of the fields near \( q_a \). As \( f \sim |\vec{r} - q_a| \) at each vortex, we can expand \( f \) near \( q_a \) to get

\[
\ln f = \ln |\vec{l}_a| + a_a + b_a^i + c_a^{ij}l_a^i l_a^j + \mathcal{O}(l_a^3)
\]

(4.12)

with \( \vec{l} = \vec{r} - q_a \). Eq.(2.20) implies then

\[
\sum_i c_a^{ii} = -\frac{1}{4} e^2 v^2
\]

(4.13)
Eqs. (4.1) and (2.17) then imply

\[ \bar{A}_i = \epsilon_{ij} \left[ \frac{l^j_a}{|l_a|^2} + b^j_a + 2e^j_k l^k_a \right] + \mathcal{O}(l_a^2) \] (4.14)

We also get from Eq. (2.21)

\[ \partial_i A_0 = \frac{\rho e}{e^2 v^2} \left\{ \left[ 1 - \sum_b (\bar{q}_a - \bar{q}_b) \cdot \frac{\partial}{\partial \bar{q}_b} \right] b^j_a + 4 \epsilon^{ij} l^j_a \right\} + \mathcal{O}(|l_a|^2) \] (4.15)

We calculate the line integrals in Eq. (4.11) by using \( \oint dl_a \epsilon_{jk} l^j_a / |l_a|^2 = \pi \delta_{ij} \). By using Eq. (4.14), we get the intrinsic part as

\[ J_{\text{int}} = \frac{1}{e} \sum_a \oint dl_a \left\{ \epsilon_{jk} l^j_a \left( A_0 \partial_k \bar{A}_i - \partial_i A_0 \bar{A}_k \right) + \epsilon_{ij} A_0 \bar{A}_j \right\} \]
\[ = 0 \] (4.16)

The extrinsic part with Eq. (4.14) becomes

\[ J_{\text{ext}} = -\frac{1}{e} \oint dl_a \epsilon_{jk} q^j_a \left( A_0 \partial_k \bar{A}_i - \partial_i A_0 \bar{A}_k \right) \]
\[ = -\frac{1}{e} \sum_a \oint dl_a \epsilon_{jk} q^j_a \left\{ (A_0(\bar{q}_a) + l^i_a \partial_i A_0) \epsilon_{in} \left[ \frac{\delta_{nk}}{|l_a|^2} - \frac{2 l^k_{|a}|}{|l_a|^4} \right] - \partial_i A_0 \epsilon_{kn} \frac{|m^i_{|a}|}{|l_a|^2} \right\} \]
\[ = -\frac{2\pi}{e} \sum_a q^i_a \partial_i A_0 (\bar{r} = \bar{q}_a) \] (4.17)

With Eq. (4.15), the total angular momentum (4.11) can be expressed as

\[ J = -\frac{2\pi \rho e}{e^3 v^2} \sum_a \bar{q}_a \cdot \left( 1 - \sum_b (\bar{q}_a - \bar{q}_b) \cdot \frac{\partial}{\partial \bar{q}_b} \right) \cdot \bar{b}_a - \frac{\pi \rho e}{e} \sum_a |\bar{q}_a|^2 \] (4.18)

The angular momentum is now expressed rather explicitly in terms of the selfdual configurations.
Since the scalar potential $A_0$ is a smooth function of $\vec{r}$ and $\vec{q}_a$, the external part (4.17) is a smooth function of the vortex position. When all vortices come together at the origin, the configuration is symmetric and so the total angular momentum vanishes. This means that these vortices do not carry any spin, contrast to the claims in Ref.[2] The part of the angular momentum which is proportional to $\sum_a |\vec{q}_a|^2$ is due to the Magnus force by the shielding charge. Such a term arises whenever charged particles move in a uniform magnetic field. We will derive Eq.(4.18) in the next section via a rather different path.

If we have considered a general Maxwell-Higgs system, vortices would have a static force between them and would not stay stationary in general. We can still calculate the spin of vortices by considering a single vortex sitting at the origin. Following the similar line of reasoning as before and using Eq.(4.9) which is correct for any potential, one can see easily that the spin of vortices is zero.

For two vortices located at points $\vec{q}_1 = \vec{q}/2$ and $\vec{q}_2 = -\vec{q}/2$, the symmetry of the configuration implies that $f(\vec{r}; \vec{q}) = f(\vec{r}; -\vec{q}) = f(-\vec{r}; -\vec{q})$, which in turn implies $\vec{b}_1(\vec{q}) = -\vec{b}_1(-\vec{q}) = -\vec{b}_2(\vec{q})$. In addition, the $f$ configuration is invariant under the reflection which exchanges two vortices, implying

$$\vec{b}_1 = \vec{q}\mathcal{B}(q)$$  \hspace{1cm} (4.19)

where $q = |\vec{q}|$. The conserved angular momentum (4.18) becomes

$$J = -\frac{\pi \rho_e}{2e} q^2 - \frac{2\pi \rho_e}{e^3 v^2} q (1 + q \frac{d}{dq}) \mathcal{B}(q)$$  \hspace{1cm} (4.20)

We show in the appendix that $\mathcal{B}(q) = 1/q + \mathcal{O}(q)$ near $q = 0$. Thus Eq.(4.20) goes to zero when $q = 0$, which again implies our vortices carry no spin.

A rather similar expression as in Eqs.(4.18) and (4.20) has been obtained for vortices in the selfdual Chern-Simons Higgs systems.\textsuperscript{[7]} In that case, vortices carry nonzero spin $s_v$, and the total angular momentum of two vortices decreases from $4s_v$ to $2s_v$ as the distance between two vortices increases from zero to infinity.
5. Slow motion of vortices

The selfdual configurations of \(n\) vortices are characterized by the vortex positions, \(\vec{q}_a\). Let us now consider the dynamics of slowly moving vortices. When they move slowly enough, we expect that the classical radiation is very small and their dynamics is described by the Newtonian mechanics. Thus, we hope that their dynamics could be summarized by an effective Newtonian lagrangian or action. Our goal would be to find this effective action, which would be expressed in terms of the selfdual configurations.

We expect that the field configuration for a given trajectory \(\{\vec{q}_a(t)\}\) of slowly moving vortices is very close to the selfdual configuration because there is no or little radiation. Even though the Gallileo transformation is no longer a symmetry of the system due to the background electric charge density, we can get a hint for the field configuration of a slowly moving vortices from this transformation. When there is no background charge density, the configuration for uniformly moving vortices is corrected linearly in velocities (especially the vector field) and satisfy the lagrangian field equation to the first order in velocities.\(^7\)

Thus, it seems sensible to assume that the field configuration of slowly moving vortices has first order corrections in general, as \(f(\vec{r}, \vec{q}_a(t)) + \Delta f, N(\vec{r}, \vec{q}(t)) + \Delta N, A_\mu(\vec{r}, \vec{q}_a(t)) + \Delta A_\mu, \) and \(\theta = -\sum_a \text{Arg}(\vec{r} - \vec{q}_a). \) and that they satisfy the field equations to the first order in velocity. We have chosen a gauge where there is no correction to \(\theta.\) The zeroth order terms satisfy the selfdual equations, (2.12) and (2.13) with the upper sign. The first order corrections then satisfy the the field equations to the linear order in velocities,

\[
\partial_i^2 \Delta f + e^2 A_0^2 \Delta f - (\partial_i \theta + eA_i)^2 \Delta f - e^2 N^2 \Delta f - \frac{e^2}{2} (3f^2 - v^2) \Delta f \\
+ 2efA_0(\dot{\theta} + e\Delta A_0) - 2ef(\partial_i \theta + eA_i) \Delta A_i - 2e^2 f N \Delta N = 0
\]

\[
\partial_i \dot{A}_i - \partial_i^2 \Delta A_0 + e f^2 (\dot{\theta} + e\Delta A_0) + 2e^2 A_0 f \Delta f = 0
\]

\[
\partial_i \dot{A}_0 - \epsilon_{ij} \partial_j \Delta F_{12} - e^2 f^2 \Delta A_i - 2e(\partial_i \theta + eA_i) f \Delta f = 0
\]

\[
\partial_i^2 \Delta N - e^2 f^2 \Delta N - 2e^2 N f \Delta f = 0
\]

We evaluate the origin field action for this corrected field configurations for slowly moving vortices. The zeroth order lagrangian will be simply the negative of
the total mass. The first order correction is given by

$$\Delta_1 L = (\dot{A}_i - \partial_i \Delta A_0)(-\partial_i A_0) - F_{12} \Delta F_{12} - \partial_i N \partial_i \Delta N - \partial_i f \partial_i \Delta f$$

$$+ e^2 A_0^2 f \Delta f - (\partial_i \theta + eA_i)^2 f \Delta f + e f^2 A_0(\dot{\theta} + e \Delta A_0) - e f (\partial_i \theta + eA_i) \Delta A_i$$

$$- \frac{e^2}{2} (3f^2 - v^2) \Delta f - e^2 N^2 f \Delta f - e^2 f^2 N \Delta N + \rho e (\Delta N - \Delta A_0)$$

(5.2)

Since the zeroth field configurations satisfy the time independent field equations, the above expression can be simplified as

$$\Delta_1 L = -\partial_i A_0 \dot{A}_i + e f^2 A_0 \dot{\theta}$$

(5.3)

The selfdual equations (2.17) imply that the zeroth order $A_i$ is transverse, making the first term of $\Delta_1 L$ a total derivative. We now use Gauss’s law (2.13) to get

$$\Delta_1 L[q_i^a, \dot{q}_i^a] = \frac{1}{e} \int d^2 x (\rho_e + \partial_k^2 A_0) \dot{\theta}$$

$$- \frac{1}{e} \sum_a \epsilon_{ij} \dot{q}_i^a \int d^2 x (\rho_e + \partial_k^2 A_0) \partial_j \ln |\vec{r} - \vec{q}_a|$$

(5.4)

where $\dot{\theta} = -\sum_a \epsilon_{ij} \dot{q}_i^a \partial_j \ln |\vec{r} - \vec{q}_a|$ is used.

The right hand side of Eq.(5.4) is infrared divergent. To understand the infrared divergent term, let us define

$$V_i^a(\vec{q}_b) = -\frac{\epsilon_{ij}}{e} \int d_R^2 x (\rho_e + \partial_k^2 A_0) \partial_j \ln |\vec{r} - \vec{q}_a|$$

(5.5)

Since the contribution from the point $\vec{r} = \vec{q}_a$ is nonsingular, we take out this point from the integration domain. $V_i^a$ is a vector potential as $\Delta_1 L = \sum_a \dot{q}_i^a V_i^a$. Since
\((\rho_c + \partial_k^2 A_0)(\vec{r} = \vec{q}_a) = 0\), the curl of \(V^i_a\) is

\[
\frac{\partial V^i_a}{\partial q^k_b} = -\frac{\epsilon_{ij}}{e} \int_{R^2 - \vec{q}_a} d^2 x \left\{ \left[ \frac{\partial}{\partial q^k_b} \partial_j A_0 \right] \partial_j \ln |\vec{l}_a| + (\rho_c + \partial_k^2 A_0)(-\delta_{ab}) \partial_k \partial_j \ln |\vec{l}_a| \right\}
\]

\[
= -\frac{\epsilon_{ij}}{e} \int_{R^2 - \vec{q}_a} d^2 x \partial j \left\{ \left[ \partial_l \frac{\partial A_0}{\partial q^k_b} \partial_j \ln |\vec{l}_a| - \frac{\partial A_0}{\partial q^k_b} \partial_l \partial_j \ln |\vec{l}_a| \right] - \delta_{ab} \left[ \partial_l A_0 \partial_k \partial_j \ln |\vec{l}_a| - A_0 \partial_l \partial_k \partial_j \ln |\vec{l}_a| \right] \right\}
\]

\[
= \frac{\epsilon_{ij}}{e} \oint d\vec{l}_a \left\{ \partial_j \frac{\partial A_0}{\partial q^k_b} \partial_j \ln |\vec{l}_a| - \frac{\partial A_0}{\partial q^k_b} \partial_l \partial_j \ln |\vec{l}_a| \right\} - \delta_{ab} \left[ \partial_l A_0 \partial_k \partial_j \ln |\vec{l}_a| - A_0 \partial_l \partial_k \partial_j \ln |\vec{l}_a| \right] \right\}
\]

\[
= \frac{2\pi \epsilon_{ij}}{e} \left\{ \partial_j \frac{\partial A_0}{\partial q^k_b} - \delta_{ab} \left[ \frac{1}{2} \delta_{jk} \partial_l^2 A_0 - \partial_l \partial_j A_0 \right] \right\} \bigg|_{\vec{r} = \vec{q}_a}
\]

(5.6)

where we used \(\oint d\vec{l}^2 / |\vec{l}|^2 = \pi \delta^{ij}\) and its generalizations. We use Eqs. (4.13) and (4.15) to get

\[
\frac{\partial V^i_a}{\partial q^k_b} = \frac{2\pi \rho_c \epsilon_{ij}}{e^3 v^2} \left\{ \frac{e^2 v^2}{2} \delta_{ab} \delta_{jk} + \left[ 2 - \sum_c (\vec{q}_a - \vec{q}_c) \cdot \frac{\partial}{\partial q^k_c} \right] \frac{\partial b^j_a}{\partial q^k_b} \right\}
\]

(5.7)

which indicates that the linearly divergent part is independent of the \(\vec{q}_a\). We have used here the relation \(\sum_c \partial b^i_a / \partial q^k_c = 0\) due to the translation invariance. Integrating Eq.(5.7), we can choose a gauge so that

\[
V^i_a = \frac{2\pi \rho_c \epsilon_{ij}}{e^3 v^2} \left\{ \frac{e^2 v^2}{2} q^j_a + \left[ 1 - \sum_b (\vec{q}_a - \vec{q}_b) \cdot \frac{\partial}{\partial q^k_b} \right] b^j_a \right\}
\]

(5.8)

Even though \(b^i_a\)'s fall to zero exponentially when the mutual distances between vortices increase, it does not mean there is no nontrivial statistical phase when two vortices are exchanges because the above derivation assumes that vortices are not overlapped and so could miss a singular potential with zero magnetic field when vortices are separated. As we will see later, a better guide would be whether a vortex feels any additional magnetic field besides the average uniform magnetic field due to the presence of other vortices.
The second order correction to the effective lagrangian is given by

\[ \Delta_2 \mathcal{L} = \frac{1}{2} (\dot{A}_i - \partial_i \Delta A_0)^2 - \partial_0 \Delta A_i \partial_i A_0 - \frac{1}{2} (\Delta F_{12})^2 + \frac{1}{2} [\dot{\mathcal{N}}^2 - (\partial_i \Delta N)^2] \]

\[ + \frac{1}{2} f^2 [\dot{\mathcal{N}} - e \Delta A_0] \partial_i \dot{A}_0 + e^2 (\Delta A_i)^2 \]

\[ + e^2 (3f^2 - v^2)(\Delta f)^2 - e^2 (\Delta N)^2 - e^2 (\Delta A_i) \Delta A_i \]

\[ - \frac{e^2}{4} (3f^2 - v^2)(\Delta f)^2 - e^2 (\Delta N)^2 - 2e^2 f \Delta f \Delta N \]

We can imagine the possible contribution from the second order corrections of the field configuration to \( \Delta_2 \mathcal{L} \). However, the first order field equation implies this possible contribution vanishing, making our approximation consistent. After using Eq. (5.1) satisfied by the first order corrections, we get

\[ \Delta_2 L[\dot{q}^i_a, \dot{q}^i_a] = \int d^2 x \left\{ \frac{1}{2} [\dot{A}_i^2 + \dot{N}^2 + j^2] + \frac{1}{2} [\Delta A_i \partial_i \dot{A}_0 + \frac{1}{e} \Delta A_0 \partial_i^2 \dot{\theta}] \right\} \]

(5.10)

In principle, we can solve Eq. (5.1) and express the first order corrections of the fields in terms of the selfdual configurations, leading to \( \Delta_2 L \) fully expressed in the selfdual configurations. Samols in Ref. [9] managed to express the second order terms explicitly for the case when there is no background electric charge density. It would be interesting to find such expression in our case too.

We have now an effective lagrangian for slowly moving vortices. Since the energy of vortices at rest does not depend on the vortex positions, there is no static force between them. However, the shielding charge carried by the Higgs field manifest itself as a uniform magnetic field acting on vortices. The Magnus force due to the shielding charge density is now a Lorentz force by this magnetic field. In addition, there is a magnetic interaction between vortices because the shielding charge density around vortices is not uniform.

Let us study the effective action for slowly moving vortices in more detail. The effective lagrangian can be written figuratively as

\[ L_{eff}[\dot{q}^i_a, \dot{q}^i_a] = \frac{1}{2} \sum_{ab,ij} T_{ij}^a (\vec{q}_c) \dot{q}^i_a \dot{q}^i_b + \sum_{a, i} \epsilon_{ij} \dot{q}^i_a [\alpha q^i_a + H^2_a (\vec{q}_c)] \]

(5.11)
where
\[
\alpha = \frac{\pi \rho_e}{e}
\]
and
\[
H_a^i = \frac{2\pi \rho_e}{e^3 v^2} \left\{ 1 - \sum_b (\tilde{q}_a - \tilde{q}_b) \cdot \frac{\partial}{\partial \tilde{q}_b} \right\} b_a^i
\] (5.12)

The dynamical equation for the a-th vortex is
\[
\frac{d(T_{ij}^a \dot{q}_b^j)}{dt} - \frac{1}{2} \frac{\partial T_{jk}^{bc}}{\partial q_a^i} \dot{q}_b^j \dot{q}_c^k = -2\alpha \epsilon_{ij} \dot{q}_a^j + \left\{ \epsilon_{jk} \frac{\partial H_b^k}{\partial q_a^i} - \epsilon_{ik} \frac{\partial H_a^k}{\partial q_b^j} \right\} \dot{q}_b^j
\]
\[
= -2\alpha \epsilon_{ij} \dot{q}_a^j - \epsilon_{ij} \dot{q}_b^j \frac{\partial H_b^k}{\partial q_a^i} + \epsilon_{ij} \dot{q}_b^j \left( \frac{\partial H_b^k}{\partial q_a^i} - \frac{\partial H_a^k}{\partial q_b^j} \right)
\] (5.13)

For a given positive \( \rho_e \), the parameter \( \alpha \) are positive. The Magnus force due to the shielding charge from the above equation of motion makes a moving vortex turn left.

The effective lagrangian (5.11) is invariant under time translation and so the tensors \( T, H \) are independent of time. The conserved energy is then
\[
E_{eff} = \frac{1}{2} \sum_{a,b,i,j} T_{ij}^a \dot{q}_a^i \dot{q}_b^j
\] (5.14)

The space translation invariance implies that the tensors \( T, H \) are invariant under \( \tilde{q}_a \to \tilde{q}_a + \epsilon \), or
\[
\sum_c \frac{\partial T_{jk}^{ab}}{\partial q_c^i} = 0 = \sum_c \frac{\partial H_a^j}{\partial q_c^i}
\] (5.15)

The conserved linear momentum is then
\[
P_{eff}^i = \sum_a \left\{ T_{ij}^a \dot{q}_b^j + \epsilon_{ij} [2\alpha q_a^i + H_a^j] \right\}
\] (5.16)

The rotational invariance implies that
\[
\epsilon_{ij} H_a^j + \epsilon_{kl} q_b^k \frac{\partial H_a^i}{\partial q_b^l} = 0
\]
and
\[
\epsilon_{ik} T_{ab}^{kj} + \epsilon_{jk} T_{ab}^{ik} + \epsilon_{kl} q_c^k \frac{\partial T_{ab}^{ij}}{\partial q_c^l} = 0
\] (5.17)
The conserved angular momentum is given as

\[ J_{\text{eff}} = \epsilon_{ij} q_a^i T^{jk}_{ab} q_b^k - \alpha |q_a|^2 - q_a^i H_a^i \]  

(5.18)

Under the translation \( \vec{x} \rightarrow \vec{x} + \vec{a} \), the angular momentum transforms as \( J \rightarrow J + \epsilon_{ij} a^i P^j \).

We can calculate the energy, linear momentum and angular momentum for our configurations of slowly moving vortices from Eqs.(2.4), (4.10) and (4.9). The interesting question is whether for our slowly moving vortices these field theoretical conserved quantities are identical to those from the above effective action. For vortices at rest, we can compare easily. The total energy in the field theory would be just the total mass. The kinetic energy vanishes. The total angular momentum (5.18) from the effective action with Eq.(5.12) becomes

\[ J_{\text{eff}} = -\frac{\pi \rho_e}{e} \sum_a |q_a|^2 - \frac{2\pi \rho_e}{e^3 v^2} \sum_a \vec{q}_a \cdot \left\{ 1 - \sum_b (q_a - \vec{q}_b) \frac{\partial}{\partial \vec{q}_b} \right\} \vec{b}_a + \mathcal{O}(\dot{q}) \]  

(5.19)

This is exactly what is given in Eq.(4.18). Since these results are agreeing each other, we have some confidence in the linear part of our effective action.

We have the effective lagrangian for slowly moving vortices. Let us first apply the effective action to a single vortex at the position \( \vec{q} \). The effective action would be then just

\[ L = \frac{1}{2} m \dot{q}_a^2 + \frac{\pi \rho_e}{e} \epsilon_{ij} \dot{q}^i q^j \]  

(5.20)

where \( T^{ij}_{aa} = m \delta^{ij} \) with the particle mass \( m \). (We have not shown that this mass is the rest mass of vortices.) This is a lagrangian for a charged particle moving on a uniform magnetic field background. The Magnus force manifests itself as the Lorentz force due to this magnetic field. Since our vortices do not carry any spin and the Magnus force is usually associated with nonzero spin, we are in a somewhat ironic situation. A single vortex would move a circle. Quantum mechanically there will be a nonzero Berry’s phase on the wave function when the vortex moves
around a closed loop. The argument of the phase would be proportional to the total magnetic flux encircled by the loop,
\[
\text{phase} = \exp \left\{ \frac{i \rho_e}{e} \oint_C \epsilon_{ij} dq^i q^j \right\} = \exp \left\{ \frac{i \rho_e}{e} \text{area} \right\} \tag{5.21}
\]

Let us now consider the system of two vortices moving with the positions at \( \tilde{q}_1 = \tilde{q}/2, \tilde{q}_2 = -\tilde{q}/2 \) as before. The total angular is now given in Eq.(4.20). The first order part of the effective action from Eqs.(5.11) and Eq.(5.12) becomes
\[
\Delta_1 L = \epsilon_{ij} \dot{q}^i q^j \left\{ \frac{\pi \rho_e}{2e} + \frac{\pi \rho_e}{e^3 q^2} \frac{d q \mathcal{B}(q)}{dq} \right\} \tag{5.22}
\]
The magnetic field felt by the reduced system is given by
\[
\mathcal{H}_{12}(q) = \frac{d J_{\text{eff}}}{qdq} \tag{5.23}
\]
Since \( q \mathcal{B} = 1 \) at \( q = 0 \) and \( \mathcal{B} \) goes zero exponentially at the spatial infinity, one can deduce from Eqs.(4.20) and (5.22) that the total magnetic flux felt by the reduced particle when it goes around a circle of large \( q \) would be just the total area times the flux \( \rho_e/e \). This implies that there is no nontrivial statistical phase between two vortices in large separation, proving the spin-statistics theorem. In finite distance, the matter is more complicated. Obviously \( \mathcal{B}(q) \) is a complicated function of \( q \) and would lead to an interesting dynamics of two vortices.

By using the spin-statistics theorem, we can argue that our vortices should not carry any spin. The spin-statistics theorem in three dimensions implies that particles and antiparticles carry the same sign spin \( s \). When there is no background magnetic field, the statistics works out because the theorem implies that the orbital angular momentum between two particles is \( 2l + 2s \) and that between a particle and an antiparticle is \( 2l - 2s \). In our case vortices and antivortices have the same charge profile and the opposite electric current. Thus, if they carry any spin, the spin of vortices should have the opposite sign to that of antivortices. Since CTP is a good symmetry of the theory, the spin-statistics theorem should however be correct, implying the same sign. Hence there is no conflict if vortices do not carry any spin, which we have shown.
6. Conclusion

We have studied the vortex dynamics in selfdual Maxwell-Higgs systems with uniform background electric charge density. We have found a well-defined modification of the Noether angular momentum. Our vortices are shown to carry no spin but feels the Magnus force due to the shielding charge carried by the Higgs field. We have studied dynamics of the slowly moving vortices, proving the spin-statistics theorem of vortices. There are many directions we can take from here. The further investigation of the slowly moving vortices, especially quadratic terms would be interesting.

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APPENDIX

Here we calculate the contribution of the symmetric energy tensor to the angular momentum of the selfdual vortex configurations. The angular momentum is

\[ J^S = -\int d^2x \epsilon_{ij} x^i T^S_{0j} \] (A.1)

where the symmetric energy momentum tensor is given in Eq. This is not conserved but well defined without any divergent contribution from the spatial infinity or the vortex positions. It would become a useful quantity to characterize the static configurations. For the selfdual configurations satisfying Eqs.(2.17) and (2.13) \( J^S \) becomes

\[
J^S = -\int d^2x \left\{ x^i \partial_i A_0 F_{12} + e f^2 A_0 \epsilon_{ij} x^i (\partial_j \theta + e A_j) \right\} \\
= \frac{e}{2} \int d^2x x^i \partial_i [A_0 (f^2 - v^2)] \\
= e \int d^2r A_0 (v^2 - f^2) \\
= \frac{\rho_e}{e} \int d^2r \left[ \frac{e^2 v^2 A_0 \rho_e}{\rho_e} - 1 \right]
\] (A.2)
Antivortices would have the same charge profile but the opposite current. Thus $J^S$ of antivortices would have the opposite sign to that of vortices.

We can express $J^S$ more explicitly for the selfdual configurations. To achieve this goal, first we express the angular momentum in terms of $\bar{A}_i \equiv \partial_i \theta + e A_i$. As there is no singular contribution at vortex positions $\{q_a\}$ in Eq.(A.2), we take out these points from the integration domain. From Eqs. (2.21) and (2.17), we get

$$\partial_i A_0 = -\frac{\rho_e}{e^2 v^2} \epsilon_{ij} (1 + D) \bar{A}_j$$ \hspace{1cm} (A.3)

where

$$D \equiv -\sum_a (\vec{r} - \vec{q}_a) \cdot \frac{\partial}{\partial \vec{q}_a}$$ \hspace{1cm} (A.4)

From Eqs.(2.13) and (A.3), we get

$$f^2 A_0 = \frac{\rho_e}{e^2} + \frac{1}{e^2} \partial_i^2 A_0$$

$$= \frac{\rho_e}{e^2} - \frac{\rho_e}{e^4 v^2} [2 + D] F_{12}$$ \hspace{1cm} (A.5)

The nonconserved angular momentum (A.2) becomes

$$J^S = \frac{\rho_e}{e^3 v^2} \int \frac{d^2 x}{R^2} \left\{ -e^2 v^2 \epsilon_{ij} x^i \bar{A}_j + \epsilon_{ij} x^i (3 + D) [\bar{A}_j \bar{F}_{12}] \right\}$$ \hspace{1cm} (A.6)

From Eq.(2.17), we observe that

$$-\epsilon_{ij} x^i \bar{A}_j (e^2 v^2 - 2 \bar{F}_{12}) = -\frac{e^2}{2} x^i \partial_i (1 - f^2)$$

$$= e^2 (v^2 - f^2)$$

$$= 2 \bar{F}_{12}$$ \hspace{1cm} (A.7)

up to a total derivative. With an identity $\epsilon_{ij} \epsilon_{kl} = \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}$, $\partial_i \bar{A}_i = 0$, and
(1 + D) \partial_i = \partial_i D$, we get

\[ J^S = \frac{4\pi n \rho e}{e^3 v^2} + \rho c \int_{R^2} d^2 r \partial_i D \left[ x^j \left\{ \frac{1}{2} \delta_{ij} \vec{A}_k^2 - \vec{A}_i \vec{A}_j \right\} \right] \]

\[ = \frac{4\pi n \rho c}{e^3 v^2} - \sum_a \rho c e^3 v^2 \oint_{\vec{q}_a} dx^i x^j D \left\{ \frac{1}{2} \delta_{ij} \vec{A}_k^2 - \vec{A}_i \vec{A}_j \right\} \]  

(A.8)

where each surface integration is done around the vortex positions \{\vec{q}_a\}.

When all vortices sit together at the origin,

\[ \ln f = n \ln |\vec{x}| + a + O(x^2) \]  

(A.9)

due to the rotational symmetry. Thus,

\[ \vec{A}_i = n \epsilon_{ij} \frac{x^j}{|\vec{x}|^2} + O(x) \]  

(A.10)

Thus, the angular momentum becomes

\[ J^S = \frac{2\pi \rho c}{e^3 v^2} n(n + 2) \]  

(A.11)

for the \( n \) overlapped vortices.

For \( n \) separated vortices, we separate the integration part of Eq.(A.8) to two pieces, by introducing \( \vec{t}_a = \vec{r} - \vec{q}_a \) with \( \oint_{\vec{q}_a} dx^i = \oint_{\vec{t}_a} dx^i \). First, let us calculate the intrinsic part,

\[ - \oint_{\vec{t}_a} dl^i_1 \vec{D} \left[ \frac{1}{2} \delta_{ij} (\vec{A}_k^2) - \vec{A}_i \vec{A}_j \right] = 2\pi \]  

(A.12)

with \( \vec{D} = \vec{t}_a \cdot \partial / \partial \vec{r} - \sum_b (\vec{q}_a - \vec{q}_b) \cdot \partial / \partial \vec{q}_b - \vec{t}_a \cdot (\sum_b \partial / \partial \vec{q}_b + \partial / \partial \vec{r}) \) and \( \oint dl^i j / |\vec{t}|^2 = \pi \delta_{ij} \).
Thus, Eq.(A.8) can be written as

$$J_S = \frac{6\pi \rho_e n}{e^3 v^2} - \sum_a \frac{\rho_e}{e^3 v^2} \oint dl_a q_a^i D \left\{ \frac{1}{2} \delta_{ij} A_k^a - \tilde{A}_i \tilde{A}_j \right\}$$  \hspace{1cm} (A.13)

After some algebra by using Eqs.(4.13) and (A.14), we get

$$J_S = \frac{6\pi \rho_e n}{e^3 v^2} + \frac{2\pi \rho_e}{e^3 v^2} \sum_a \vec{q}_a \cdot \left\{ 1 + \sum_{b \neq a} (\vec{q}_a - \vec{q}_b) \cdot \frac{\partial}{\partial \vec{q}_b} \right\} \vec{b}_a$$  \hspace{1cm} (A.14)

Note that $J_S(\vec{q})$ is a continuous function of the vortex positions.

Consider two vortices at rest with positions $\vec{q}_1 = \vec{q}/2, \vec{q}_2 = -\vec{q}/2$. Following the argument before Eq.(4.19), we get

$$J_S = \frac{12\pi \rho_e}{e^3 v^2} + \frac{2\pi \rho_e}{e^3 v^2} \frac{d}{dq} \left( q(1 - q \frac{d}{dq}) B(q) \right)$$  \hspace{1cm} (A.15)

which should approach $16\pi \rho_e / e^3 v^2$ when $q \to 0$ because of Eq.(A.11). Thus,

$$B = \frac{1}{q} + \mathcal{O}(1)$$  \hspace{1cm} (A.16)

near $q = 0$. 

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FIGURE CAPTIONS

1) The vortex configuration of unit vorticity

2) The vortex configuration of $n = 2$
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