Equations of State in the Brans-Dicke cosmology

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Abstract

We investigate the Brans-Dicke (BD) theory with the potential as cosmological model to explain the present accelerating universe. In this work, we consider the BD field as a perfect fluid with the energy density and pressure in the Jordan frame. Introducing the power-law potential and the interaction with the cold dark matter, we obtain the phantom divide which is confirmed by the native and effective equation of state. Also we can describe the metric $f(R)$ gravity with an appropriate potential, which shows a future crossing of phantom divide in viable $f(R)$ gravity models when employing the native and effective equations of state.

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I. INTRODUCTION

Supernova (SUN Ia) observations has shown that our universe is accelerating \[1\]. Also cosmic microwave background radiation \[2\], large scale structure \[3\], and weak lensing \[4\] have indicated that the universe has been undergoing an accelerating phase since the recent past. Although there exist a number of models explaining an accelerating universe, the two promising candidates are the dark energy of cosmological constant in the general relativity \[5\] and a modified gravitational theory such as \(f(R)\) gravity \[6–8\].

Recently, there was an extensive study of dark energy models based on the Brans-Dicke (BD) theory interacting with the cold dark matter (CDM) \[9\]. However, the total equation of state \(w_{\text{eff}} = p_{\text{tot}}/\rho_{\text{tot}}\) was mainly used to measure the evolution of the universe. In order to see how the BD field describes the accelerating phase, we have to introduce both the native and effective equations of state because the BD field is non-minimally coupled to gravity \[10, 11\]. Especially, we need the effective equation of state to take into account the universe evolution properly because there always exists an interaction between the BD fluid and the CDM \[12\]. For the interacting holographic dark energy models, there is no phantom phase when using the effective equation of state \[13\] instead of the native equation of state \[14\]. For the brane interacting holographic dark energy models, the effective equation of state was used to account the evolution of the universe \[15\]. The effective equation of state could read off from the Bianchi identity which provides a non-standard conservation law.

On the other hand, \(f(R)\) gravity models have been extensively employed to explain the present accelerating universe. The observational data might imply the crossing of the phantom divide \(W_{\text{DE}} = -1\) in the near past \[16\]. In this case, the crossing of the phantom divide could be resolved in the viable \(f(R)\) gravity models \[17–19\]. Especially, we would like to mention that a general approach to phantom divide in \(f(R)\) gravity was investigated in \[18\], where the scalar-tensor version of \(f(R)\) gravity was used to see the phantom divide. However, it was shown that any singular \(f(R)\) gravity may be done non-singular \[20\]. More recently, consistent, viable and non-singular \(f(R)\) gravity was suggested in \[21\].

Interestingly, it was shown that the viable four \(f(R)\) models generally exhibit the crossing of the phantom divide in the future evolution \[22\].

A common feature to all analysis was performed by mapping the Starobinsky model \[23\]
to a scalar-tensor theory of gravity. It seems that the metric $f(R)$ gravity is equivalent to the BD theory with $\omega_{BD} = 0$, while the Palatini $f(R)$ gravity is equivalent to the BD theory with $\omega_{BD} = -3/2$. Despite its mathematical equivalence, two theories may have shown physically non-equivalence: super-accelerating phase in the BD theory describes decelerating phase in $f(R)$ gravity. Also, it was pointed out that the mapping seems to be problematic because the scalar potential defined by $U(\Phi(R)) = R\Phi - f(R)$ with $\Phi = \partial_R f(R)$ induces a singularity in the cosmological evolution.

Before we proceed, we wish to mention the difference between Einstein and Jordan frames. We consider the frame in which non-relativistic matter (CDM, baryons) obey the standard continuity equation with $\rho_m \sim a^{-3}$. This is the Jordan frame as the physical frame in which physical quantities are compared to observations. It is sometimes useful to introduce the Einstein frame where a canonical scalar field is coupled to non-relativistic matter directly. Even though one considers the same physics in both frames, using different time and length scales may offer the apparent difference between the observables in two frames.

In this work we investigate how the present accelerating phase is realized in the scalar-tensor theory (BD cosmology). We consider the BD field as a perfect fluid with the energy density and pressure in the Jordan frame. Introducing the power-law potential and the interaction with the CDM, we confirmed the appearance of phantom divide by using the native and effective equation of state. Especially, inspired by the work of Ref. 22, we study the cosmological implications of the $f(R)$ gravity using the BD theory with an appropriate BD potential, which indicates a future crossing of phantom divide in viable $f(R)$ gravity models when employing the native and effective equations of state. This shows a close connection between BD theory and $f(R)$ gravities for explaining future crossing of phantom divide. In the BD approach, we find a singularity in the past evolution of the universe. Hereafter, we consider the metric $f(R)$ gravity only and thus, we mean $f(R)$ gravity by the “metric $f(R)$” gravity.
II. BD COSMOLOGY WITHOUT A POTENTIAL

For cosmological purpose, we introduce the Brans-Dicke (BD) action with a matter in the Jordan frame

\[
S = \int d^4x \sqrt{-g} \left[ \frac{1}{16\pi G} \left( \Phi R - \omega_{BD} \nabla_{\alpha} \Phi \nabla^{\alpha} \Phi \right) + \mathcal{L}_m \right],
\]

(1)

where \( \Phi \) is the BD scalar, \( \omega_{BD} \) is the parameter of BD theory, and \( \mathcal{L}_m \) represents other matter which takes a perfect fluid form. The field equations for metric \( g_{\mu\nu} \) and BD scalar \( \Phi \) are

\[
G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}^{BD} + \frac{8\pi G}{\Phi} T_{\mu\nu}^m,
\]

(2)

\[
\nabla^2 \Phi = \frac{8\pi G}{2\omega_{BD} + 3} T_{\mu\nu}^{\alpha \alpha},
\]

(3)

where the energy-momentum tensor for the BD scalar is defined by

\[
T_{\mu\nu}^{BD} = \frac{1}{8\pi G} \left[ \frac{\omega_{BD}}{\Phi^2} \left( \nabla_\mu \Phi \nabla_\nu \Phi - \frac{1}{2} g_{\mu\nu} (\nabla \Phi)^2 \right) + \frac{1}{\Phi} \left( \nabla_\mu \nabla_\nu \Phi - g_{\mu\nu} \nabla^2 \Phi \right) \right]
\]

(4)

and the energy-momentum tensor for a perfect fluid takes the form

\[
T_{\mu\nu}^m = p_m g_{\mu\nu} + (\rho_m + p_m) u_\mu u_\nu.
\]

(5)

\( \rho_m \) (\( p_m \)) denote the energy density (pressure) of the matter and \( u_\mu \) is a four velocity vector with \( u_\alpha u^\alpha = 1 \).

Considering that our universe is homogeneous and isotropic, we work with the flat Friedmann-Robertson-Walker (FRW) spacetime

\[
ds^2 = -dt^2 + a^2(t) \left[ dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right].
\]

(6)

In this spacetime, the first Friedmann and BD scalar equations take the forms

\[
H^2 + H \left( \frac{\dot{\Phi}}{\Phi} \right) - \frac{\omega_{BD}}{6} \left( \frac{\dot{\Phi}}{\Phi} \right)^2 = \frac{8\pi G \rho_m}{3 \Phi},
\]

(7)

\[
\ddot{\Phi} + 3H \dot{\Phi} = \frac{8\pi G (\rho_m - 3p_m)}{2\omega_{BD} + 3},
\]

(8)

where \( H = \dot{a}/a \) is the Hubble parameter and the overdot denotes the derivative with respect to time \( t \). Here we note that the case of \( \omega_{BD} = -3/2 \) is not allowed unless a radiation-matter
FIG. 1: The equations of state $W^\pm_{BD}$ for BD scalar versus its parameter $\omega_{BD}$. $W^+_{BD}$ is a monotonically decreasing function of $\omega_{BD}$, while $W^-_{BD}$ is a monotonically increasing function of $\omega_{BD}$. The bound of $W^-_{BD}$ is given by $-1/3 \leq W^-_{BD} \leq 1$ because of $\omega_{BD} \geq -3/2$ and $W^-_{BD} \to 1$ as $\omega_{BD} \to \infty$. At $\omega_{BD} = 0$, one finds that $W^-_{BD} = -1/3$, but $W^+_{BD}$ blows up.

with $p_m = \rho_m / 3$ comes into the BD theory. Regarding the BD field as a perfect fluid, its energy and pressure are defined by kinetic terms as

$$\rho_{BD} = \frac{1}{16\pi G} \left[ \omega_{BD} \left( \frac{\dot{\Phi}}{\Phi} \right)^2 - 6H \frac{\dot{\Phi}}{\Phi} \right],$$

$$p_{BD} = \frac{1}{16\pi G} \left[ \omega_{BD} \left( \frac{\dot{\Phi}}{\Phi} \right)^2 + 4H \frac{\dot{\Phi}}{\Phi} + 2\dddot{\Phi} / \Phi \right].$$

If one does not specify the parameter $\omega_{BD}$, one cannot determine the BD equation of state exactly. However, the Bianchi identity of $\nabla_\mu G^{\mu\nu} = 0$ implies that there exists an energy transfer between BD fluid and matter

$$\dot{\rho}_{BD} + 3H(\rho_{BD} + p_{BD}) = \frac{1}{G} \frac{\dot{\rho}_m}{\dot{\Phi}}.$$  

This continuity equation play a crucial role because it shows manifestly the energy transfer between $\rho_{BD}$ and $\rho_m$ and, thus, it defines the effective equation of state.

On the other hand, we consider action with a minimally coupled scalar $\psi$

$$\tilde{S} = \int d^4\tilde{x} \sqrt{-\tilde{g}} \left[ \frac{1}{16\pi G} \tilde{R} - \frac{1}{2} \tilde{\nabla}_\alpha \psi \tilde{\nabla}^\alpha \psi + L_m \right]$$

in the Einstein frame. The field equations for metric $\tilde{g}_{\mu\nu}$ and a scalar $\psi$ are

$$G_{\mu\nu} = 8\pi GT^\psi_{\mu\nu} + 8\pi GT^m_{\mu\nu},$$

$$\tilde{\nabla}^2 \psi = 0.$$
Its energy density and pressure for $\psi$ are the same given by
\[ \rho_\psi = \frac{\dot{\psi}^2}{2} = p_\psi, \] (15)
which describe a stiff matter with $W_\psi = 1$. Here we obtain canonical forms for $\rho_\psi$ and $p_\psi$, in comparison with non-canonical forms of $\rho_{BD}$ (9) and $p_{BD}$ (10) in the BD-frame. A continuity equation of $\dot{\rho}_\psi + 3H(\rho_\psi + p_\psi) = 0$ [unlike (11)] leads to the $\psi$-scalar equation (14) exactly as
\[ \ddot{\psi} + 3H\dot{\psi} = 0. \] (16)
Importantly, the Bianchi identity leads to the two conservation laws separately
\[ \nabla^\mu T^\psi_{\mu\nu} = 0 \rightarrow \ddot{\psi} + 3H\dot{\psi} = 0, \quad \nabla^\mu T^m_{\mu\nu} = 0. \] (17)
Hence, we need the EOS
\[ W_\psi = \frac{\rho_\psi}{p_\psi}, \] (18)
whereas we do not need to introduce the effective EOS $W^\text{eff}_\psi$ like (44) arisen from (11).

In the absence of matter, the BD scalar plays a role of kinetic matter. This kinetic matter evolves as the conservation law is satisfied by itself
\[ \dot{\rho}_{BD} + 3H(\rho_{BD} + p_{BD}) = 0 \] (19)
whose equation of state (EOS) is defined by
\[ W_{BD} \equiv \frac{p_{BD}}{\rho_{BD}}. \] (20)
The solution to the first Friedmann and BD scalar equations is given by
\[ a(t) = t^{\frac{3(\omega_{BD}+1)\pm\sqrt{3(2\omega_{BD}+3)}}{3(3\omega_{BD}+4)}}, \quad \Phi(t) = t^{\frac{1+\sqrt{3(2\omega_{BD}+3)}}{3\omega_{BD}+4}}. \] (21)
Plugging the above into (20), one finds its EOS as
\[ W^\pm_{BD} = \frac{3(\omega_{BD}+2) \pm 2\sqrt{3(2\omega_{BD}+3)}}{3\omega_{BD}} \text{ with } \omega_{BD} \geq -\frac{3}{2}. \] (22)
In the limit of $\omega_{BD} \to 0$, $W^+_{BD} \to 4/0$, while $W^-_{BD} \to \frac{1}{3}$. Their behavior is shown in Fig. 1. Here we choose $W_{BD} = W^-_{BD}$ as the EOS for the BD kinetic-matter. The EOS bound is
given by \(-1/3 \leq W_{\text{BD}} \leq 1\). If one requires the condition (11) together with \(\rho_m = 0\), the only allowable solution is the case saturating the lower bound

\[
\omega_{\text{BD}} = -\frac{3}{2} \rightarrow W_{\text{BD}} = -\frac{1}{3}
\]  

(23)

which corresponds the solution to the conformal relativity: \(a(t) \sim t, \Phi \sim 1/t^2, \rho_{\text{BD}} \sim 1/a^2, p_{\text{BD}} = -\rho_{\text{BD}}/3\). This case gives a zero acceleration of \(\ddot{a} = 0\). Consequently, the perfect fluid interpretation of the BD scalar is valid only for \(\omega_{\text{BD}} = -3/2\) [10].

When the CDM is present, we have to solve the different equations. In the FRW spacetime, equations take the forms

\[
H^2 = \frac{8\pi G}{3} \left( \rho_{\text{BD}} + \frac{\rho_m}{\Phi} \right),
\]

(24)

\[
\dot{H} = -4\pi G \left( \rho_{\text{BD}} + p_{\text{BD}} + \frac{\rho_m}{\Phi} + \frac{p_m}{\Phi} \right),
\]

(25)

\[
\ddot{\Phi} + 3H\dot{\Phi} = \frac{8\pi G}{2\omega_{\text{BD}} + 3} (\rho_m - 3p_m),
\]

(26)

where \(\rho_m\) is the CDM density given by

\[
\rho_m = \frac{\rho_m^0}{a^3}
\]

(27)

with the present dark matter density \(\rho_m^0\). It is convenient to use new variables as

\[
x = \ln a, \quad \varphi = \frac{\Phi'}{\Phi}, \quad \lambda = -\frac{H'}{H}
\]

(28)

where \(\)'\ denote the derivatie with respect to \(x\). Also we define the density parameters

\[
\Omega_{\text{BD}} \equiv \frac{8\pi G}{3H^2} \rho_{\text{BD}}, \quad \Omega_m \equiv \frac{8\pi G \rho_m}{3H^2 \Phi}.
\]

(29)

Using the relations

\[
\frac{\dot{\Phi}}{\Phi} = \frac{d}{dt} \frac{d\Phi}{dx} \frac{1}{\Phi} = H\varphi,
\]

(30)

\[
\frac{\ddot{\Phi}}{\dot{\Phi}} = H^2 \left( \varphi' + \varphi^2 - \lambda \varphi \right),
\]

(31)

energy density and pressure are given, respectively, by

\[
\rho_{\text{BD}} = \frac{H^2}{16\pi G} \left[ \omega_{\text{BD}} \varphi^2 - 6\varphi \right],
\]

(32)

\[
p_{\text{BD}} = \frac{H^2}{16\pi G} \left[ \omega_{\text{BD}} \varphi^2 + 4\varphi - 2\lambda \varphi + 2 \left( \varphi' + \varphi^2 \right) \right].
\]

(33)
The Bianchi identity (11) takes into account the energy transfer between BD field and CDM, while the CDM evolves according to its own conservation law

\[ \dot{\rho}_m + 3H (\rho_m + p_m) = 0. \] (34)

Eqs. (24), (25) and (26) can be written as

\[ 1 = \Omega_{BD} + \Omega_m, \] (35)
\[ \lambda = \frac{3}{2} + \frac{4\pi G}{H^2} p_{BD}, \] (36)
\[ \varphi' - \lambda \varphi + 3\varphi + \varphi^2 = \frac{3}{2\omega_{BD} + 3} (1 - \Omega_{BD}), \] (37)

where we used the pressureless condition of \( p_m = 0 \) for the CDM. Solving Eq. (37) for \( \varphi' \) and inserting it into \( p_{BD} \) leads to

\[ p_{BD} = \frac{H^2}{16\pi G} \left[ \omega_{BD} \varphi^2 - 2\varphi + \frac{6}{2\omega_{BD} + 3} (1 - \Omega_{BD}) \right]. \] (38)

Substituting this into Eq. (36), we find

\[ \lambda = \frac{3}{2} + \frac{1}{4} \left[ \omega_{BD} \varphi^2 - 2\varphi + \frac{6}{2\omega_{BD} + 3} (1 - \Omega_{BD}) \right]. \] (39)

A further relation is found to be

\[ \Omega_{BD} = \frac{1}{6} (\omega_{BD} \varphi^2 - 6\varphi). \] (40)

Eq. (37) can be rewritten as

\[ \varphi' = -\varphi^2 - 3\varphi + \frac{3(1 - \Omega_{BD})}{2\omega_{BD} + 3} + \lambda \varphi. \] (41)

Let us plug \( \lambda \) and \( \Omega_{BD} \) into Eqs. (41) and solve it numerically with the initial condition.

On the other hand, we obtain the native EOS for the BD fluid

\[ W_{BD} = \frac{p_{BD}}{\rho_{BD}} = \frac{\omega_{BD} \varphi^2 - 2\varphi + \frac{6}{2\omega_{BD} + 3} (1 - \Omega_{BD})}{\omega_{BD} \varphi^2 - 6\varphi}. \] (42)

Considering Eq. (11) as

\[ \dot{\rho}_{BD} + 3H (1 + W_{BD}^{\text{eff}}) \rho_{BD} = 0, \] (43)

we obtain the effective EOS

\[ W_{BD}^{\text{eff}} = W_{BD} - \frac{\rho_m}{3\rho_{BD}^\Phi} \varphi \]
\[ = \frac{\varphi}{3} + \frac{\omega_{BD} \varphi^2 - 4\varphi + \frac{6}{2\omega_{BD} + 3} (1 - \Omega_{BD})}{\omega_{BD} \varphi^2 - 6\varphi}. \] (44)
The initial value for $\phi$ is determined by

$$\phi(0) = 3 \pm \sqrt{3(2\Omega_{BD}^{0}\omega_{BD} + 3)}$$

with $\Omega_{BD}(0) = \Omega_{BD}^{0}$. Here $+$ ($-$) sign correspond to increasing (decreasing) $\Omega_{BD}$ at $x = 0$. For $-$ signature, its evolution induces a singularity. This behavior could be expected from the critical points obtained by solving the equation of $\phi' = 0$. The result is summarized in the Table I. These critical points indicate asymptotic behaviors in the far future and far past. In order to test whether each critical point is or not stable, we need to observe the signature of $d\phi'/d\phi$. If the signature is negative (positive), it may be stable for the far future evolution (far past evolution). The viable parameter range for class (a) is found by requiring the condition of $0 \leq \Omega_{BD} \leq 1$ as

$$\omega_{BD} < -\frac{3}{2}, \quad -\frac{4}{3} \leq \omega_{BD} \leq -\frac{6}{5}.$$  \hspace{1cm} (46)

However, if we demand the positive-definite energy density for the BD fluid $\rho_{BD} > 0$ and the negative-definite pressure $p_{BD} < 0$ \cite{10}, the relevant range is determined solely by

$$\omega_{BD} < -\frac{3}{2}.$$ \hspace{1cm} (47)

In this case, the native and effective equations of state take the bounds

$$W_{BD}, \quad W_{BD}^{\text{eff}} > -\frac{2}{3}$$ \hspace{1cm} (48)

which means that the BD fluid without potential does not explain the future phantom divide. For classes (b) and (c), these are nothing new because we have the condition

$$\omega_{BD} \geq -\frac{3}{2} \rightarrow W_{BD} = W_{BD}^{\text{eff}} \geq -\frac{1}{3}$$ \hspace{1cm} (49)

which corresponds to the absence of the CDM as is shown in Eq.22.
Therefore, we note that the role of BD scalar without potential (equivalently, k-essence with non-canonical kinetic term only) as a source generating the accelerating universe is very restricted because it can at most describe \( W_{\text{BD}}(W_{\text{eff}}) = -2/3 \) acceleration” in the presence of the CDM. In the presence of matters \( \text{[10]} \), the BD scalar \( \Phi \) appears to interpolate smoothly between the matter-dominated and accelerating eras by speeding up the expansion rate of the matter-dominated era like \( a(t) \sim t^{2/3} \rightarrow t^\alpha (\alpha = 2(\omega_{\text{BD}} + 1)/(3\omega_{\text{BD}} + 4) > 2/3) \), while slowing down that of accelerating phase derived by cosmological constant to some degree like \( a(t) \sim e^{\chi t} \rightarrow (1 + \chi t)^{(2\omega_{\text{BD}}+1)/2} \). Hence, we have to include an appropriate potential to obtain the phantom divide of \( W_{\text{BD}}(W_{\text{eff}}) = -1 \).

III. BD COSMOLOGY WITH A POTENTIAL

The action for generalized BD theory is given by \( \text{[11]} \)

\[
S = \int d^4x \sqrt{-g} \left[ \frac{1}{16\pi G} \left( \Phi R - \omega_{\text{BD}} \frac{\nabla_{\alpha} \Phi \nabla^{\alpha} \Phi}{\Phi} - 16\pi G U(\Phi) \right) \right] + S_m, \tag{50}
\]

where \( S_m = \int d^4x \sqrt{-g} \mathcal{L}_m \) is the action for the other matter of the perfect fluid type and \( U(\Phi) \) is a potential for the BD scalar \( \Phi \). The equation of motions can be obtained as

\[
G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{\omega_{\text{BD}}}{\Phi^2} \left\{ \nabla_{\mu} \Phi \nabla_{\nu} \Phi - \frac{1}{2} g_{\mu\nu} (\nabla \Phi)^2 \right\} \\
+ \frac{1}{\Phi} \left\{ \nabla_{\mu} \nabla_{\nu} \Phi - g_{\mu\nu} \nabla^2 \Phi - g_{\mu\nu} 8\pi G U(\Phi) \right\} + \frac{8\pi G}{\Phi} T^m_{\mu\nu} \\
\equiv 8\pi G T^{\text{BD}}_{\mu\nu} + \frac{8\pi G}{\Phi} T^m_{\mu\nu}, \tag{51}
\]

where \( T^{\text{BD}}_{\mu\nu} \) is given by

\[
T^{\text{BD}}_{\mu\nu} = \frac{1}{8\pi G} \left[ \frac{\omega_{\text{BD}}}{\Phi^2} \left\{ \nabla_{\mu} \Phi \nabla_{\nu} \Phi - \frac{1}{2} g_{\mu\nu} (\nabla \Phi)^2 \right\} + \frac{1}{\Phi} \left\{ \nabla_{\mu} \nabla_{\nu} \Phi - g_{\mu\nu} \nabla^2 \Phi - g_{\mu\nu} 8\pi G U(\Phi) \right\} \right] \tag{52}
\]

and \( T^m_{\mu\nu} \) is the energy-momentum tensor \( \text{[5]} \). Equation for \( \Phi \) is changed to be

\[
\nabla^2 \Phi + \frac{16\pi G}{2\omega_{\text{BD}} + 3} \left( 2U(\Phi) - \Phi \frac{dU}{d\Phi} \right) = \frac{8\pi G}{2\omega_{\text{BD}} + 3} T^m_{\alpha\alpha} \tag{53}
\]

In the context of dark energy, it is possible to construct a single scalar model (scalar-tensor theory) based on the BD theory with constant \( \eta = -d\ln U(\phi)/d\phi \). This depends on the choice of the potential \( U(\Phi) \).

10
Setting $\Phi = F(\varphi) = e^{-2Q\varphi}$, the exponential potential $U_0 e^{-\eta \Phi}$ takes the power-law form

$$U(\Phi) = U_0 \Phi^\alpha$$

with the constant $\alpha = \eta/2Q$. In the FRW spacetime, three equations take the forms

$$H^2 = \frac{8\pi G}{3} \left( \rho_{BD} + \rho_m \right),$$

$$\dot{H} = -4\pi G \left( \rho_{BD} - \frac{\rho_m}{\Phi} - p_{BD} - \frac{p_m}{\Phi} \right),$$

$$\ddot{\Phi} + 3H \dot{\Phi} - \frac{16\pi G}{2\omega_{BD} + 3} \left( 2U(\Phi) - \Phi \frac{dU}{d\Phi} \right) = \frac{8\pi G}{2\omega_{BD} + 3} (\rho_m - 3p_m).$$

These equations are consistent with Ref. [29] but are slightly different from Ref. [30]. Regarding the BD field as a perfect fluid, its energy and pressure are changed as [10, 13]

$$\rho_{BD} = \frac{H^2}{16\pi G} \left[ \omega_{BD} \left( \frac{\Phi}{\Phi} \right)^2 - 6H \frac{\dot{\Phi}}{\Phi} + 16\pi G U(\Phi) \right],$$

$$p_{BD} = \frac{H^2}{16\pi G} \left[ \omega_{BD} \left( \frac{\Phi}{\Phi} \right)^2 + 4H \frac{\dot{\Phi}}{\Phi} + 2 \frac{\ddot{\Phi}}{\Phi} - 16\pi G U(\Phi) \right].$$

The Bianchi identity leads to the same relation as in Eq. (11).

In order to solve Eqs. (55), (56) and (57), it is convenient to introduce new variables as

$$\psi = \frac{8\pi G U(\Phi)}{H^2 \Phi}.$$ 

Using (30) and (31), energy density and pressure are expressed in terms of $\varphi$ and $\psi$, respectively, by

$$\rho_{BD} = \frac{H^2}{16\pi G} \left[ \omega_{BD} \varphi^2 - 6\varphi + 2\psi \right],$$

$$p_{BD} = \frac{H^2}{16\pi G} \left[ \omega_{BD} \varphi^2 + 4\varphi - 2\lambda \varphi + 2 (\varphi' + \varphi^2) - 2\psi \right].$$

Then, Eqs. (55), (56) and (57) can be written as

$$\Omega_{BD} + \Omega_m = 1,$$

$$\lambda = \frac{3}{2} + \frac{4\pi G}{H^2} p_{BD},$$

$$\varphi' - \lambda \varphi + 3\varphi + \varphi^2 - \frac{2}{2\omega_{BD} + 3} (2 - \alpha) \psi = \frac{3}{2\omega_{BD} + 3} (1 - \Omega_{BD}).$$

Solving Eq. (65) for $\varphi'$, and inserting it into $p_{BD}$, we obtain the pressure

$$p_{BD} = \frac{H^2}{16\pi G} \left[ \omega_{BD} \varphi^2 - 2\varphi + \frac{4(2 - \alpha)}{2\omega_{BD} + 3} \psi + \frac{6}{2\omega_{BD} + 3} (1 - \Omega_{BD}) - 2\psi \right].$$
Substituting this into Eq. (64), we arrive at
\[
\lambda = \frac{3}{2} + \frac{1}{4} \left[ \omega_{BD} \phi^2 - 2\phi \frac{4(2-\alpha)}{2\omega_{BD}+3} \psi + \frac{6}{2\omega_{BD}+3} (1 - \Omega_{BD}) - 2\psi \right].
\] (67)

Importantly, we note that \( \Omega_{BD} \) can be written as
\[
\Omega_{BD} = \frac{1}{6} (\omega_{BD} \phi^2 - 6\phi + 2\psi).
\] (68)

Plugging this into Eq. (65) leads to
\[
\phi' = -\phi^2 - 3\phi + \frac{2(2-\alpha)}{2\omega_{BD}+3} \psi + \frac{3(1 - \Omega_{BD})}{2\omega_{BD}+3} + \lambda \phi.
\] (69)

On the other hand, from the definition of \( \psi \), we obtain a newly differential equation
\[
\psi' = (\alpha \phi - \phi + 2\lambda) \psi.
\] (70)

Now we have to solve two coupled equations (69) and (70) for \( \phi \) and \( \psi \) numerically with initial conditions.

Considering (61) and (66), we obtain the native EOS for BD field with potential
\[
W_{BD} = \frac{p_{BD}}{\rho_{BD}} = \frac{\omega_{BD} \phi^2 - 2\phi + \frac{4(2-\alpha)}{2\omega_{BD}+3} \psi + \frac{6}{2\omega_{BD}+3} (1 - \Omega_{BD}) - 2\psi}{\omega_{BD} \phi^2 - 6\phi + 2\psi}.
\] (71)

This might not be suitable for representing the true equation of state for the BD scalar because of the non-conservation of this fluid (11). We remind the reader that (11) could be rewritten as
\[
\dot{\rho}_{BD} + 3H (1 + W_{BD}^{\text{eff}}) \rho_{BD} = 0,
\] (72)

which implies an effective EOS
\[
W_{BD}^{\text{eff}} = W_{BD} - \frac{\rho_m}{3\rho_{BD}} \Phi \phi
\]
\[
= \frac{\phi}{3} + \frac{\omega_{BD} \phi^2 - 4\phi + \frac{4(2-\alpha)}{2\omega_{BD}+3} \psi + \frac{6}{2\omega_{BD}+3} (1 - \Omega_{BD}) - 2\psi}{\omega_{BD} \phi^2 - 6\phi + 2\psi}.
\] (73)

A typical solution is given in Fig. 2 for \( \omega_{BD} = 0 \), showing that \( \Omega_{BD} + \Omega_m = 1 \). We observe that there exists a phantom divide (\( W_{BD} = -1, W_{BD}^{\text{eff}} = -1 \)) as confirmed by effective EOS \( W_{BD}^{\text{eff}} \).

In deriving this numerical solution, it was necessary to impose the initial condition at \( a_0 = 1(x=0) \) as an input of the current observation data. It is impossible to construct
FIG. 2: Time evolution of the BD cosmology with potential \( U = U_0 \Phi^\alpha \) with \( \omega_{BD} = 0 \): \( W_{BD} \) (magenta), \( W_{BD}^{\text{eff}} \) (cyan), \( \Omega_m \) (blue), \( \Omega_{BD} \) (green). The initial condition at \( x = 0 \) is imposed by \( \alpha = 1.449 \), \( \psi(0) = 2.253 \), \( \varepsilon(0) = \frac{2.253}{3} - 0.75 = 0.001 \) (corresponding to \( \Omega_{BD}^0 = 0.75 \)), and \( \Phi(0) = 1.0 \).

nearly past, present, and future acceleration phases without fixing the initial condition with the current observation data. Usually, we need one initial condition to solve the first order differential equation. Most of cases are needed to specify \( t = 0 \) as the initial condition. However, since we do not know the origin of the dark energy clearly, we could not use \( t = 0 \) as the initial condition to solve (69) and (70) for the accelerating phase.

Since the effective gravitational constant \( G^{\text{eff}} = G/\Phi \) varies with time, it should satisfy the observed limits defined as

\[
\frac{\dot{G}^{\text{eff}}}{G^{\text{eff}}} = -\frac{\dot{\Phi}}{\Phi} = -H\varepsilon \leq 10^{-13}\text{yr}^{-1}.
\]  

(74)
It means that

\[ H_0 \varphi(0) \leq 10^{-13} \text{yr}^{-1}, \]  

(75)

and, finally

\[ \varphi(0) \leq 0.0013. \]  

(76)

Here we used the present Hubble parameter of \( H_0 = 77 \text{km/s/Mpc} = 2.5 \times 10^{-18} \text{s}^{-1} = 7.88 \times 10^{-11} \text{yr}^{-1}. \)

Considering the density parameter of BD field as

\[ \Omega_{\text{BD}} \equiv \frac{\rho_{\text{BD}}}{\rho_c} = \frac{\omega_{\text{BD}} \varphi^2 - 6 \varphi + 2 \psi}{6}, \quad \rho_c \equiv \frac{8\pi G}{3H^2}, \]  

(77)

\( \Omega_{\text{BD}} \) is regarded as dark energy density parameter and its current value will be determined by observation as \( \Omega_{\text{BD}}^0. \) Hence two initial values of \( \varphi \) and \( \psi \) are not independent, but they are related as

\[ \omega_{\text{BD}} \varphi^2 - 6 \varphi(0) + 2 \psi(0) - 6 \Omega_{\text{BD}}^0 = 0, \]  

(78)

which gives us

\[ \varphi(0) = \frac{3 \pm \sqrt{9 - 2 \omega_{\text{BD}}(\psi(0) - \Omega_{\text{BD}}^0)}}{\omega_{\text{BD}}}. \]  

(79)

For \( \omega_{\text{BD}} \to 0, \) we can choose \(-\) sign so that

\[ \varphi(0) = \frac{\psi(0)}{3} - \Omega_{\text{BD}}^0. \]  

(80)

In order to find the asymptotic values for variable, we need to determine the critical points from Eqs.\((69)\) and \((70)\). We list the critical points and corresponding physical variables in Table II and Table III for \( \omega_{\text{BD}} = 0. \)

Now we wish to analyze which critical point is stable with time evolution. To determine the stability, we consider the perturbation around the critical points. If the coefficients of the perturbation is negative (positive), then it may be stable for future evolution (past evolution). Actually, classes (a), (b) and (c) are nothing new because these are exactly the same classes in the absence of the potential (see Table I). The class (d) is less interesting because both its native and effective equations of state do not provide the phantom divide.
The graph of equation of state $W_{BD}$ as a function of $\alpha$ is shown in Fig. 3 with $\omega_{BD} = 0$. The graph of effective equation of state $W_{eff\ BD}$ as a function of $\alpha$ is given in Fig. 4 with $\omega_{BD} = 0$. The class (b) is not available for $\omega_{BD} = 0$, as was mentioned by the BD kinetic-matter using $W_{BD}^\pm$.

The class (e) is a newly interesting case. Let us study the class (e) more. The solution to the phantom divide of $W_{eff\ BD}(\alpha) = -1$ is given by

$$\alpha_1 = 1, \quad \alpha_2 = 2, \quad (81)$$

while the solution to the dust matter of $W_{eff\ BD}(\alpha) = 0$ takes the forms

$$\alpha_3 = \frac{9 - \sqrt{73}}{4} \simeq 0.114, \quad \alpha_4 = \frac{9 + \sqrt{73}}{4} \simeq 4.386. \quad (82)$$

The solutions to the radiation of $W_{eff\ BD}(\alpha) = \frac{1}{3}$ is given by

$$\alpha_5 = 0, \quad \alpha_6 = 5. \quad (83)$$
FIG. 3: The graphs of $W_{BD}$ as a function of $\alpha$ for the case of $\omega_{BD} = 0$. Class (a): red; (b): N/A; (c): green; (d): dark yellow; (e): blue.

The minimum value of $W_{BD}^{\text{eff}}(\alpha)$

$$W_{BD}^{\text{eff}}(\alpha)|_{\text{min}} = 4\sqrt{\frac{2}{3}} - \frac{13}{3} \simeq -1.067,$$  \hfill (84)

appears at

$$\alpha_{\text{min}} = -1 + \sqrt{6} \simeq 1.449.$$  \hfill (85)

Finally, we mention the $\alpha$-dependent evolutions of two equations of state $W_{BD}$ and $W_{BD}^{\text{eff}}$. As is shown in Fig. 3 for $\alpha \leq 1$, there is no phantom divide, while for $\alpha > 1$, there is phantom divide. For $\alpha > \alpha_{\text{min}}$, there are two crossings of $W_{BD}^{\text{eff}} = -1$. For $1 < \alpha < \alpha_{\text{min}}$, there is one crossing of $W_{BD}^{\text{eff}} = -1$ and $W_{BD}^{\text{eff}}$ approaches de Sitter value of $-1$ for $\alpha = 1, 2$. 

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FIG. 4: The graphs of $W_{\text{BD}}^{\text{eff}}$ as a function of $\alpha$ for the case of $\omega_{\text{BD}} = 0$. Class (a): red (it is overlapped with green); (b): N/A; (c): green; (d): dark yellow; (e): blue.

IV. BD COSMOLOGY AS $f(R)$ GRAVITY

The crossing of the phantom divide could be understood in the viable $f(R)$ gravity models [19]. Recently, it was shown that the viable four $f(R)$ models generally exhibit the crossing of the phantom divide in the future evolution when using the EOS $w_{\text{DE}} = p_{\text{DE}}/\rho_{\text{DE}}$ [22]. Hence, it is very important to see whether the future crossing of phantom divide is available for the BD cosmology with the corresponding potential. Hence, we analyze the BD cosmology with the potential

$$U(\Phi) = U_0 \{1 - C(1 - \Phi)^p\},$$

(86)
FIG. 5: Time evolution of $W_{BD}$ (dotted curve) and $W_{\text{eff}BD}$ (solid curve) for $\alpha = 0.500$ (blue), $\alpha = 1.000$ (green), $\alpha = 1.449$ (red as in Fig. 2), $\alpha = 2.000$ (magenta), $\alpha = 2.200$ (cyan).

which could be obtained by considering the equivalence between the $f(R)$ gravity and the scalar-tensor theory (BD theory with potential and $\omega_{BD} = 0$) \[31\]. Actually, this potential is an approximation to Hu-Sawicki and Starobinsky $f(R)$ models. Here two parameters $C$ and $p$ are chosen

$$0 < C < 1, \quad 0 < p < 1.$$  \hspace{1cm} (87)

Previous analysis could be applied to here when defining $\alpha$ to be

$$\frac{dU(\Phi)}{d\Phi} \equiv \alpha \frac{U(\Phi)}{\Phi},$$ \hspace{1cm} (88)

where $\alpha$ is determined by

$$\alpha(\Phi) = C^p \frac{\Phi(1 - \Phi)^{p-1}}{1 - C(1 - \Phi)^p}.$$ \hspace{1cm} (89)
\[ \frac{-2}{\ln(C[e^{\pi(p-2)}])} \]

TABLE IV: List of critical points with the potential \([86]\). Here \(Z\) is the solution of \(Z(p-1) = \ln(C[e^{\pi(p-2)}])\).

Then, we could use the same equations in Sec. III. Since \(\alpha\) is a function of \(\Phi\), we need to solve the differential equation for \(\Phi\). Hence the full equations to be solved are given by

\[
\begin{align*}
\varphi' &= -\varphi^2 - 3\varphi + \frac{2}{2\omega_{BD} + 3} (2 - \alpha(\Phi)) \psi + \frac{3}{2\omega_{BD} + 3} (1 - \Omega_{BD}) + \lambda \varphi, \\
\psi' &= (\alpha(\Phi)\varphi - \varphi + 2\lambda) \psi, \\
\Phi' &= \varphi \Phi.
\end{align*}
\]

Relevant variables take the forms

\[
\begin{align*}
\lambda &= \frac{3}{2} + \frac{1}{4} \left[ \omega_{BD} \varphi^2 - 2\varphi + \frac{4(2 - \alpha(\Phi))}{2\omega_{BD} + 3} \psi + \frac{6}{2\omega_{BD} + 3} (1 - \Omega_{BD}) - 2\psi \right], \\
\Omega_{BD} &= \frac{1}{6} (\omega_{BD} \varphi^2 - 6\varphi + 2\psi), \\
W_{BD} &= \frac{\omega_{BD} \varphi^2 - 2\varphi + \frac{4(2 - \alpha(\Phi))}{2\omega_{BD} + 3} \psi + \frac{6}{2\omega_{BD} + 3} (1 - \Omega_{BD}) - 2\psi}{\omega_{BD} \varphi^2 - 6\varphi + 2\psi}, \\
W'_{BD} &= \frac{\varphi}{3} + \frac{\omega_{BD} \varphi^2 - 4\varphi + \frac{4(2 - \alpha(\Phi))}{2\omega_{BD} + 3} \psi + \frac{6}{2\omega_{BD} + 3} (1 - \Omega_{BD}) - 2\psi}{\omega_{BD} \varphi^2 - 6\varphi + 2\psi}.
\end{align*}
\]

In order to check that our system is working properly, we first recover Tsujikawa et al’s result \([31]\) by taking \(\omega_{BD} = 9998.5(Q = 0.01)\). We have recovered their result of fig. 1 correctly, which is shown in Fig. \[8\]. This figure shows that for \(0 < C < 1\), the matter-dominated phase with \(w_{eff} \simeq 0\) is followed by the de Sitter phase with \(w_{eff} \simeq -1\). In contrast to this, our equations of states \(W_{BD}\) and \(W'_{BD}\) show that the BD field evolves from a stiff matter with \(W_{BD}(W'_{BD}) = 1\) in the far past to de Sitter phase with \(W_{BD}(W'_{BD}) = -1\) in the far future. We note that the phantom divide appears in the near future. Also, we wish to point out that the initial condition used in \([31]\) do not provide a proper evolution. This means that the density parameter \(\Omega_{BD}\) of BD field becomes negative, showing an unphysical
case. This explains why equations of state diverge at some points. However, our modified initial condition gives a correct evolution for all relevant physical variables. Our initial condition is chosen by requiring the nonnegative density parameter of $\Omega_{BD}(x_{\min}) \geq 0$ on whole evolution. In Figure 6 we have used $x_{\min} = 0$ and $\Omega_{BD}(x_{\min}) = 10^{-3}$. We note that the definition of energy density and pressure for the BD are different from those in [31], but the definition of density parameter $\Omega_{BD}$ is the same. Furthermore, they have analyzed the evolution only for the positive $x \geq 0$ (future direction), as one can see from the Figure 6. However, the general tendency of its whole evolution is almost the same as in the power-law potential $U = U_0 \Phi^\alpha$.

Furthermore, we have to mention that two equations of state $W_{BD}$ and $W_{eff}^{BD}$ defined by Eq. (71) and (73) are slightly different from the notation used in [31]:

$$w_{eff} = -1 - \frac{2}{3} \frac{\dot{H}}{H^2}$$

which was defined from the total conservation law

$$\dot{\rho}_{tot} + 3H(\rho_{tot} + p_{tot}) = 0 \rightarrow \dot{\rho}_{tot} + 3H\rho_{tot}(1 + w_{eff}) = 0, \quad \rho_{tot} = \rho_m + \tilde{\rho}_{BD}$$

with $\tilde{\rho}_{BD} = \Phi \rho_{BD}$. Hence $w_{eff}$ represents the equation of state for the whole matters in the universe.

The asymptotic critical points are listed in Table IV. Actually, two classes (b) and (c) are nothing new because these are nearly the same classes in the absence of the potential (see classes (a) and (c) in Table III). The class (d) is less interesting because it is similar to class (c) in Table III). We note that all of these belong to unstable critical points. On the other hand, the class (a) represents the asymptote of $f(R)$ gravity models and its critical point indicates the par future behavior, showing that $W_{BD}(W_{eff}^{BD}) \rightarrow -1$.

In order to have the correspondence between $f(R)$ gravity and the BD theory, we have to choose the BD parameter to be zero ($\omega_{BD} = 0$). In this case, it is not easy to impose appropriate initial conditions of $\Omega_m = 0.25$ and $\Omega_{BD} = 0.75$ by adjusting two potential parameters $C$ and $p$. A time evolution is depicted in Fig. 7. As this figure is shown, there is a singularity at a past of $x = -3.16$ which reflects that the mapping is problematic. In this case, however, the equation of state $W_{BD}$ approaches $-1$ (de Sitter spacetime) oscillatory as the universe evolves toward the far future. Also, the effective equation of state $W_{eff}^{BD}$ does show the nearly same behavior toward the far future. This confirm the presence of future crossing of phantom divide which appeared in the four viable $f(R)$ models [22] clearly.
FIG. 6: The graphs of $\Omega_m$ (blue), $\Omega_{BD}$ (green), $W_{BD}^{\text{eff}}$ (magenta), $W_{BD}$ (brown), $w_{\text{eff}}$ (red: definition used in Ref.[31]) as a function of $x$ with $\omega_{BD} = 9998.5$. The imposed condition is $C = 0.7$, $Q = 0.01$, $p = 0.2$, $\varphi(0) = 6.47 \times 10^{-4}$, $\psi = 2.85 \times 10^{-3}$ (corresponding to $\Omega_{BD}(0) = 10^{-3}$, $\Phi(0) = 5.0 \times 10^{-13}$).

V. DISCUSSIONS

We have employed the BD cosmology to explain the accelerating universe and future crossing of phantom divide. In this work, we regard the BD field as a perfect fluid model. First, the role of BD scalar without potential (k-essence with non-canonical kinetic term only) as a source generating the accelerating universe is very restricted because it could describe “$W_{BD}(W_{BD}^{\text{eff}}) = -2/3$ acceleration” in the presence of the CDM.

Turning on the power-law potential (54), the BD cosmology could describe the accelerating universe in the interaction with the CDM. In this case, we have used both the equation of
FIG. 7: A time evolution of BD fluid with potential (86) and \( \omega_{BD} = 0 \) (\( f(R) \)-gravity): \( W_{BD} \) (magenta), \( W_{BD}^{eff} \) (red), \( \Omega_m \) (blue), \( \Omega_{BD} \) (green). The initial condition at \( x = 0 \) is imposed as \( C = 0.82, p = 0.09, \varphi(0) = 2.35, \psi(0) = 9.3 \) (corresponding to \( \Omega_{BD}^0 = 0.75 \), \( \Phi(0) = 0.6 \)). A singularity appears at \( x = -3.16 \) when evolving toward the far past, while there is a damped oscillatory evolution toward the far future.

state \( W_{BD} \) and effective equation of state \( W_{BD}^{eff} \) to check whether the phantom phase appears. Explicitly, the BD field acts as a radiation field in the far past, whereas it plays a role of phantom field in the far future. This is compared with the case without the potential where the BD field acts as a radiation field in the far past, while it plays a role of an accelerating matter in the far future. This shows that the presence of the potential is crucial for obtaining a phantom divide in the BD cosmology.

Concerning the BD description of \( f(R) \)-gravity, we have chosen the potential in Eq.(86).
inspired by Hu-Sawicki and Starobinsky $f(R)$ models. For the case of $\omega_{\text{BD}} = 0$, the evolution of $\Omega_m$ and $\Omega_{\text{BD}}$ are similar to the power-law potential [54], but there exists a singularity at $x = -3.16$, which restricts evolving toward the far past of $x = -\infty$ after imposing the initial condition at $x = 0$. However, we have found that the universe evolves toward the far future of $x = \infty$ nicely. Both native and effective equations of state converge to $-1$ (the de Sitter spacetime of class(a) in Table IV) oscillatory, which indicates that the BD description is working for showing a future crossing of phantom divide appeared in the viable $f(R)$ gravity models [22].

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