Coherent States of the Creation Operator from Fully Developed Bose-Einstein Condensates

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Abstract. A fully developed Bose-Einstein condensate, containing macroscopically large number of bosons can under certain conditions be considered as a generalized vacuum state. Applying the annihilation operator to the condensate hole states can be defined. Infinite ladders of such hole states can be considered as generalized coherent states of the creation operator. Dedicated to the memory of Professor V. N. Gribov.

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1. Introduction

Coherent states of the annihilation operator are well known in quantum optics. They were first described by E. Schrödinger as classically behaving solutions of the Schrödinger equation with a harmonic potential [1,2]. The importance of coherent states became widely recognized in many branches of physics due to the works of Glauber [3], Klauder [4] and Sudarshan [5].

In section 2, the definition of the coherent states of the annihilation operator is given and the properties of these states are briefly summarized. Section 3, summarizes the pion-laser model, an exactly solvable multiboson wave-packet model that can be solved both in the very rare gas and in the fully condensed limiting case. If the total available energy and thus the number of pions in the condensate is large enough, applying the annihilation operator to the condensate state repeatedly a ladder of holes can be created and by a suitable superposition of these states a...
new kind of state can be defined, as described in Section 4.

2. Coherent States

For the harmonic oscillator, the coherent states $|\alpha\rangle$ can be equivalently defined with the help of the displacement operator method, the ladder (annihilation operator) method and the minimum uncertainty method, see ref. [3] for an elegant summary.

The coherent states of the annihilation operator are solutions of the equation

$$a|\alpha\rangle = \alpha|\alpha\rangle,$$

where the annihilation and creation operators $a$ and $a^\dagger$ satisfy the canonical commutation relations

$$[a, a^\dagger] = 1.$$

For the harmonic oscillator, the above coherent states are given by the unitary displacement operator $D(\alpha)$ acting on the ground (or vacuum) state $|0\rangle$ as

$$D(\alpha)|0\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle = |\alpha\rangle,$$

$$D(\alpha) = \exp[\alpha a^\dagger - \alpha^* a] = \exp[-|\alpha|^2/2] \exp[\alpha a^\dagger] \exp[\alpha^* a]$$

It is straightforward to show that the coherent states of the harmonic oscillator correspond to minimum uncertainty wave-packets with $(\Delta x)^2 (\Delta p)^2 = 1/4$ that retain their Gaussian shape during their time evolution and whose mean position and coordinate values follow the oscillations of classical motion of the harmonic oscillator. The coordinate space representation of these coherent states is

$$\langle x|\alpha\rangle = \left[ \frac{m\omega}{\pi} \right]^{1/4} \exp \left[ -\frac{m\omega(x-x_0)^2}{2} + ip_0 x \right]$$

where $m$ is the mass of the classical particle in the harmonic oscillator potential, $\omega$ is the frequency of the oscillator and $x_0$, $p_0$ correspond to the coordinate and momentum expectation values at the initial time $t_0$. The complex eigenvalue $\alpha$ of these coherent states is given by

$$\alpha = \sqrt{\frac{m\omega}{2}} x_0 + i \frac{1}{\sqrt{2m\omega}} p_0.$$

3. Fully condensed limit of the pion laser model

In high energy heavy ion reactions, hundreds of identical bosons (mostly $\pi$ mesons) can be created. These bosons are described by rather complicated fields and as the density of these bosons is increased multi-particle symmetrization effects are becoming increasingly important. The related possibility of Bose-Einstein condensation
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and the development of partial coherence was studied recently in a large number
of papers. Let us follow refs. [6] in describing an analytically solved multiparticle
wave-packet system, with full symmetrization and the possibility of condensation
of wave-packets to the least energetic wave-packet mode.

This solvable model is described by a multiparticle density matrix
\[ \hat{\rho} = \sum_{n=0}^{\infty} p_n \hat{\rho}_n, \]  
normalized to one. Here \( \hat{\rho}_n \) is the density matrix for events with fixed particle
number \( n \), which is normalized also to one. The probability for such an event is \( p_n \).

The multiplicity distribution is described by the set \( \{ p_n \}_{n=0}^{\infty} \), also normalized to 1.

The density matrix of a system with a fixed number of boson wave packets has
the form
\[ \hat{\rho}_n = \int d\alpha_1...d\alpha_n \rho_n(\alpha_1, ..., \alpha_n) |\alpha_1, ..., \alpha_n\rangle \langle \alpha_1, ..., \alpha_n|, \]  
where \( |\alpha_1, ..., \alpha_n\rangle \) denote properly normalized \( n \)-particle wave-packet boson states.

In Heisenberg picture, the wave packet creation operator is given as
\[ \alpha_i^\dagger = \int \frac{d^3p}{(\pi\sigma_i^2)^{\frac{3}{2}}} e^{-\frac{(p-\pi_i)^2}{2\sigma_i^2}} i\xi_i(p-\pi_i) \hat{a}^\dagger(p). \]  
The commutator
\[ [\alpha_i, \alpha_j^\dagger] = \langle \alpha_i | \alpha_j \rangle \]  
vanishes only in the case, when the wave packets do not overlap.

Here \( \alpha_i = (\xi_i, \pi_i) \) refers to the center of the wave-packets in coordinate space
and in momentum space. It is assumed that the widths \( \sigma_i \) of the wave-packets in
momentum space and the production time for each of the wave-packets coincide.

We call the attention to the fact that although one cannot attribute exactly
defined values for space and momentum at the same time, one can define precisely
the \( \pi_i, \xi_i \) parameters.

The \( n \) boson states, normalized to unity, are given as
\[ |\alpha_1, ..., \alpha_n\rangle = \frac{1}{\sqrt{\sum_{\sigma^{(n)} \sigma_i=1} \prod_{i=1}^{n} \langle \alpha_i | \alpha_{\sigma_i} \rangle}} \alpha_1^\dagger ... \alpha_n^\dagger |0\rangle. \]  
Here \( \sigma^{(n)} \) denotes the set of all the permutations of the indexes \( \{1, 2, ..., n\} \) and the
subscript \( \sigma_i \) denotes the index that replaces the index \( i \) in a given permutation from
\( \sigma^{(n)} \). The normalization factor contains a sum of \( n! \) term. These terms contain \( n \)
different \( \alpha_i \) parameters.

There is one special density matrix, for which one can overcome the difficulty,
related to the non-vanishing overlap of many hundreds of wave-packets, even in an
explicit analytical manner. This density matrix is the product uncorrelated single particle matrices multiplied with a correlation factor, related to stimulated emission of wave-packets

$$
\rho_n(a_1, ..., a_n) = \frac{1}{\mathcal{N}(n)} \left( \prod_{i=1}^{n} \rho_1(a_i) \right) \left( \sum_{\sigma(n)} \prod_{k=1}^{n} \langle a_k | a_{\sigma(k)} \rangle \right).
$$

(12)

Normalization to one yields $\mathcal{N}(n)$.

For the sake of simplicity we assume a factorizable Gaussian form for the distribution of the parameters of the single-particle states:

$$
\rho_1(\alpha) = \rho_x(\xi) \rho_p(\pi) \delta(t - t_0),
$$

(13)

$$
\rho_x(\xi) = \frac{1}{(2\pi R^2)^{\frac{3}{2}}} \exp(-\xi^2/(2R^2)),
$$

(14)

$$
\rho_p(\pi) = \frac{1}{(2\pi mT)^{\frac{3}{2}}} \exp(-\pi^2/(2mT)).
$$

(15)

These expressions are given in the frame where we have a non-expanding static source at rest.

A multiplicity distribution when Bose-Einstein effects are switched off (denoted by $p_n^{(0)}$), is a free choice in the model. We assume independent emission,

$$
p_n^{(0)} = \frac{n!}{n^n} \exp(-n_0),
$$

(16)

so that correlations arise only due to multiparticle Bose-Einstein symmetrization. This completes the specification of the model.

It has been shown in refs. [6,7] that the above model features a critical density. If the boson source is sufficiently small and cold, the overlap between the various wave-packets can become sufficiently large so that Bose-Einstein condensation starts to develop as soon as $n_0$, the mean multiplicity without symmetrization reaches a critical value $n_c$.

In the highly condensed $R^2 T << 1$ and $n_0 >> n_c$ Bose gas limit a kind of lasing behaviour and an optically coherent behaviour is obtained, which is characterized by the disappearance of the bump in the two-particle intensity correlation function:

$$
C_2(k_1, k_2) = 1
$$

(17)

Suppose that $n_f = E_{tot}/m_\pi$ quanta are in the condensed state. It was shown in refs. [6] that the condensation happens to the wave-packet mode with the minimal energy, i.e. $\alpha = \alpha_0 = (0, 0)$. The density matrix of the condensate can be easily given as the fully developed Bose-Einstein condensate corresponds to the $T \to 0$ and the $R \to 0$ limiting case, when the Gaussian factors in $\rho(\alpha)$ tend to Dirac delta functions. In this particular limiting case, the density matrix of eq. (12) simplifies as:

$$
\rho_c = \frac{1}{n_f^{n_f}} (\alpha_0^\dagger)^{n_f} |0\rangle \langle 0| (\alpha_0)^{n_f}
$$

(18)
which means that the fully developed Bose-Einstein condensate corresponds to \( n_f \) bosons in the same minimal wave-packet state that is centered at the origin with zero mean momentum, \( \alpha_0 = (0, 0) \). Note also that

\[ \rho_c^2 = \rho_c \]  

which implies that the fully developed Bose-Einstein condensate is in a pure state.

An important feature of such a Bose-Einstein condensate of massive quanta is that it becomes impossible to add more than \( n_f \) pions to the condensate as all the available energy can be used up by the rest mass of these bosons.

4. Coherent states of creation operators

Observe, that the fully developed Bose-Einstein condensate (BEC) of the previous section corresponds to filling a single (wave-packet) quantum state with macroscopic amount of quanta. As the number of quanta in the BEC is macroscopically large, we can treat this number first to be the infinitely large limit of the quanta in the BEC. Formally, such a quantum state of the condensate can be defined as

\[ |\text{BEC}\rangle = \frac{1}{\sqrt{n_f!}} (\alpha_0^\dagger)^{n_f} |0\rangle, \quad (n_f >> 1). \]  

(20)

In what follows, it will be irrelevant that the Bose-Einstein condensation happened to a wave-packet mode in the pion-laser model. The relevant essential feature of the fully developed Bose-Einstein condensate will be that in contains a macroscopically large number of bosons in the same quantum state created by certain creation operator \( a^\dagger \)

\[ |\text{BEC}\rangle = \frac{1}{\sqrt{n_f!}} (a^\dagger)^{n_f} |0\rangle, \quad (n_f >> 1). \]  

(21)

Due to the macroscopically large number of quanta in the same state, a large number of wave-packets can be taken away from this state. Due to the finite energy constraint, \( n_f m = E_{\text{tot}} \), it is impossible to add one more particle to the condensate at the prescribed \( E_{\text{tot}} \) available energy. We thus have

\[ a^\dagger |\text{BEC}\rangle = 0 \]  

(22)

On the other hand, we have

\[ a^m |\text{BEC}\rangle \neq 0 \quad \text{for all} \quad 0 \leq m \leq n_f. \]  

(23)

Hence a Bose-Einstein condensate with macroscopically large amount of quanta and with a constraint that the condensate is fully developed, can be considered as a generalized vacuum state of the creation operator,

\[ |\text{BEC}\rangle = |0\rangle_{\dagger}, \]  

(24)
and generalized hole-states can be defined as removing particles from the condensate:
\[ |n\rangle_\dagger = \frac{1}{\sqrt{n!}} a^n |0\rangle_\dagger \] (25)

The following calculations and equations are to be done first at \( n_f \) kept finite, and then performing the \( n_f \to \infty \) limiting case. This corresponds to the limit of a macroscopically large Bose-Einstein condensate. One obtains:
\[ |0\rangle_\dagger = |n_f\rangle = |n_f \to \infty\rangle, \]
\[ |1\rangle_\dagger = |n_f - 1\rangle = a|0\rangle_\dagger, \]
\[ ... \]
\[ |j\rangle_\dagger = |n_f - j\rangle = \frac{a^j}{\sqrt{j!}} |0\rangle_\dagger, \]
\[ ... \] (28)

The above states can be considered as the number states related to the creation of \( n \) holes in the fully developed Bose-Einstein condensate. These form a ladder that is built up with the help of the annihilation operator. The creation and annihilation operators act on these states as
\[ a|n\rangle_\dagger = \sqrt{n + 1}|n + 1\rangle_\dagger, \]
\[ a^\dagger |n\rangle_\dagger = \sqrt{n}|n - 1\rangle_\dagger. \] (29), (30)

The number operator \( N_\dagger \) that counts the number of holes is given as
\[ N_\dagger = aa^\dagger. \] (31)

With other words, the creation and annihilation operators change role if the ground state for our considerations is chosen to be the quantum state of a fully developed Bose-Einstein condensate.

If the number of quanta in the Bose-Einstein condensate is macroscopically large, \( (\lim n_f \to \infty) \), an infinite ladder can be formed from these states that is not bounded from below. Hence, the coherent states of the creation operator can be defined as follows:
\[ |\alpha\rangle_\dagger = \exp(-|\alpha|^2/2) \sum_{n=0}^{\infty} \frac{(\alpha^*)^n}{\sqrt{n!}} |n\rangle_\dagger \] (32)

Note that the above defined coherent state is an eigenstate of the creation operator,
\[ a^\dagger |\alpha\rangle_\dagger = \alpha^* |\alpha\rangle_\dagger. \] (33)

It is tempting to note that the coherent states of the creation operator are also expressible as the action of the displacement operator \( D^\dagger(\alpha) \) on the fully developed Bose-Einstein condensate state \( |BEC\rangle = |0\rangle_\dagger \) as
\[ |\alpha\rangle_\dagger = D^\dagger(\alpha)|0\rangle_\dagger. \] (34)
It is straightforward to generalize the results to different modes characterized with a momentum $k$. In that case, the state $|0\rangle_{k^+}$ has to be introduced as a state of fully developed Bose-Einstein condensate where each boson moves with momentum $k$. Such moving Bose-Einstein condensates [8] are frequently referred to as atom lasers [9] in atomic physics.

5. Interpretation and summary

Coherent states of the annihilation operator correspond to semiclassical excitations of the vacuum, that follow a classical equation of motion and keep their shape minimizing the Heisenberg uncertainty relations all the time. They correspond to a displaced ground state of the harmonic oscillators.

What is the physical interpretation of the new kind of coherent states of the creation operator described in the present work? We have shown that these new states correspond to coherently excited holes in a fully developed Bose-Einstein condensate. We have found that displaced Bose-Einstein condensates in a harmonic oscillator potential can be considered as generalized coherent states of the creation operator.

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What we call the beginning is often the end
And to make an end is to make a beginning.
The end is where we start from.

from Little Gidding by T. S. Eliot

Dedicated to the memory of Volodja Gribov.
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Appendix: Constraints and the ladder representation

The introduction of the coherent states of the creation operator essentially relied on the possibility of putting macroscopically large, \( n_f \gg 1 \) amount of neutral bosons to the same quantum state. The equations describing these states as given in the body of the paper are correct only to the precision given by \( \frac{1}{n_f} \). We argue in this section, that such a limitation in fact is not generic to the coherent states of the creation operator, but also appears when energy constraints are taken into account in the description of the well-known coherent states of the annihilation operator.

Let us note, that the eigenmode \( n \) of the harmonic oscillator has an energy of \( E_n - E_0 = n\omega \) (after removing the contribution of the ground or vacuum state). Hence in the superposition given by eq. (3) modes with arbitrarily large energy component are mixed into (but the weight decreases as a Poisson-tail with increasing values of \( n \)). Although the expectation value of the total energy in the coherent states is finite, the admixture of extremely high energy components can never be perfectly realized in any experiment as with increasing the energy of the included modes new physical phenomena like particle and antiparticle creation, deviations from the harmonic shape of the oscillator potential or other non-ideal phenomena have to occur.

Suppose that \( E_{max} \) is the maximal available energy and states with energy larger than \( E_{max} \) are not allowed either due to e.g. constraints from energy conservation or due to the break-down of the harmonic approximation to the Hamiltonian after some energy scale. In this case modes are limited to \( n \leq n_f = E_{max}/\omega \). In case of photons in electromagnetic fields, \( \omega_k = |k| \), hence for sufficiently soft modes \( n_f \) can be always made so large that the contribution of states with \( n \geq n_f \) to the coherent states \( |\alpha\rangle \) can be made arbitrarily small. However, for massive bosons like bosonic atoms or mesons created in high energy physics, \( \omega_k = \sqrt{m^2 + |k|^2} \), hence \( n_f \leq E_{max}/m \) for any mode of the field which yields

\[
|\alpha\rangle \rightarrow |\alpha\rangle_m = \sum_{n=0}^{n_f} \frac{\alpha^n (a^\dagger)^n}{n!} |0\rangle \equiv D_{n_f}(\alpha)|0\rangle \quad (35)
\]

As each mode of a free boson field is approximately a harmonic oscillator mode, this suggests that coherent states of massive bosons with finite energy constraints can only be approximately realized.