Analytical Solution of Dirac Equation for Hyperbolic Manning-Rosen-like Potential Using Hypergeometric Method for Exact Spin Symmetry

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Abstract. Analytical solution of Dirac equation for hyperbolic Manning-Rosen-like potential is obtained using hypergeometric method. The analytical solution is aimed to determine energy spectrum and radial wave function for this potential with exact spin symmetry. Behavior of atomic particles can be clearly understood if the energy spectrum and wave function of particle are known. Dirac equation for exact spin symmetry is reduced into a second order differential equation like Schrödinger equation. Hypergeometric method for Dirac equation with hyperbolic Manning-Rosen-like potential is the same as for Dirac equation with general hyperbolic Manning-Rosen potential. Energy spectrum is exactly obtained in the closed form and the radial wave functions are expressed in the form of hypergeometric method.

Keywords: Dirac equation, hyperbolic Manning-Rosen-like potential, Hypergeometric Method.

1. Introduction
In quantum mechanics, a particle that moving around in a strong potential field must pay attention to relativistic effect, one example is the particles in an accelerator[1]. When a particle notice relativistic effects, the behavior of a particle can be described using Dirac or Klein-Gordon equation[2,3]. A charge of particle will be influenced by some potentials energy. Potentials that affect are Coulomb, Manning-Rosen[4,5], Morse, Symmetrical Top, Rosen-Morse[6], Pöschl-Teller, etc. In this paper one kind of potentials will be analyzed, that is hyperbolic Manning-Rosen-like potential which suffered by an electron in an atom. In addition to motion around the nucleus, electrons also do rotate, has a multi-electron structure of the atom, molecules interactions, core deflected, and correlation of quantum states that are affected by such potential like Manning-Rosen potential.

Dirac equation is a difficult equation to be solved in exponential form, need specific functions to simplify the solution of Dirac equation[7]. One method that can be used is the hypergeometric method.

Dirac equation is reduced to Schrödinger equation that for nonrelativistic problems like Dirac equation. With separation of radial variable method, from the reduced Dirac equation then obtained a second-order differential hypergeometric type. If hypergeometric equation has been obtained, spectrum energy levels and wave function can be acquired easily[8].

Dirac equation with vector potential \( V(\vec{r}) \) and scalar potential \( S(\vec{r}) \) shown in equation (1)

\[
[-\hbar^2 \nabla^2 + 2V(r)(E + M)]S(r) = (E + M)(E - M)S(r)
\] (1)
Assuming $\hbar = 1, E = M, V(r) = \frac{1}{2}V$, and $E - M = E_{NR}$ so that equation can be expressed in equation (2)

$$[-\nabla^2 + 2V(r)M]S(r) = 2ME_{NR}S(r)$$

Equation (2) is the Dirac equation that reduced to be similar to the Schrödinger equation for non-relativistic with exact spin symmetry and equation (1) will be used to analysis waves function and spectrum energy levels for particle in hyperbolic Manning-Rosen-like potential with hypergeometric method[9].

2. Dirac Equation for Hyperbolic Manning-Rosen-like Potential

Manning-Rosen potential is a one dimensional potential order of Dirac equation that could be solved by a specific function. Energy and wave function that affected by hyperbolic Manning-Rosen potential. The hyperbolic Manning-Rosen potential shown in equation (3)

$$V(r) = C^2 \left( \frac{Q}{\sinh C} - 2qcoth Cr \right)$$

A constant $C$ is the centrifugal factor that used to obtain potential length, $Q$ and $q$ are non-dimensional parameter in hyperbolic Manning-Rosen potential[9].

In this paper, we solve hyperbolic Manning-Rosen-like potential with exact spin symmetry with hypergeometric method. For hyperbolic Manning-Rosen-like potential is expressed in equation (4)

$$V(r) = V_0 a^2 \sinh^4 ar + V_1 a^2 \sinh^3 ar \cosh ar + V_2 a^2 \sinh^2 ar \cosh^2 ar$$

Equation (4) can be simplify into equation (5)

$$V(r) = a^2 (V_0 + V_1 \coth ar + V_2 \coth^2 ar)$$

Equation (5) is hyperbolic Manning-Rosen-like potential.

In the relativistic case for exact spin symmetry that the Dirac equation shown in equation (6)

$$[-\nabla^2 + V(r)(E + M)]\psi(r) = (E^2 - M^2)\psi(r)$$

The Laplacian operator can be write like equation (7) for spherical coordinate.

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}$$

For this case, we assuming $\hbar = 1, V(r) = \frac{1}{2}V$. Then we solve in radial part only. So that, the radial part shown in equation (8)

$$\left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) - (E + M)V(r) \right] \psi(r) - \frac{l(l+1)}{r^2} \psi(r) = (E^2 - M^2)\psi(r)$$

$$\left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) - \alpha^2 (E + M)(V_0 + V_1 \coth ar + V_2 \coth^2 ar) \right] \psi(r) - \frac{l(l+1)}{r^2} \psi(r) = (E^2 - M^2)\psi(r)$$

If we assuming that wave function in radial part is $\psi(r) = R$ so we have equation (9)

$$\left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) - (E + M)\left(\alpha^2 (V_0 + V_1 \coth ar + V_2 \coth^2 ar)\right) \right] R - \frac{l(l+1)}{r^2} R = (E^2 - M^2)R$$

Where $R = \frac{U}{r}$ and $(E^2 - M^2) = \alpha^2 E'$ so the equation (9) becomes to equation (10)

$$\frac{d^2 U}{dr^2} - \alpha^2 (E + M)(V_0 + V_1 \coth ar + V_2 \coth^2 ar)U - \frac{l(l+1)U}{r^2} = \alpha^2 E'U$$

If $\frac{1}{r^2} = \frac{a^2}{\sinh^2 ar}$, so equation (10) becomes to equation (11)
\[
\frac{d^2U}{dr^2} - a^2 \left[ (E + M)(V_0 + V_z + V_1 \coth ar) + \frac{(E + M)V_2 = l(l+1)}{\sinh^2 ar} \right] U = a^2 E'U
\]

Equation (11) is Dirac equation for Hyperbolic Manning-Rosen like potential. This potential will be solved with hypergeometric method to find energy and wave function.

3. Hypergeometric Method for Dirac Equation with Hyperbolic Manning-Rosen-like Potential

In the principle, the hypergeometric method finds a variable form that will be substituted to the Dirac equation so that it becomes a second-order differential equation hypergeometric function. Equation (12) is the second-order differential equation hypergeometric function

\[
z(1 - z) \frac{d^2\varphi}{dz^2} + (c - (a + b + 1)z) \frac{d\varphi}{dz} - ab\varphi = 0
\]

Equation (12) has two regular singular points

\[
z(1 - z) = 0
\]

Solution for equation (13) are \( z = 0 \) and \( z = 1 \). In this paper we use \( z = 0 \) term because at \( z = 0 \) the equation (12) is more simple to be solved. Solution of equation (12) can be expressed in series form like equation (14)

\[
\varphi = z^s \sum a_n z^n = a_0 z^s + a_1 z^{s+1} + a_2 z^{s+2} + \ldots
\]

For solve this problem, we substituted equation (14) to equation (12) and we have equation (15)

\[
z(1 - z)(a_0 s(s - 1)z^{s-2} + a_1 s(s + 1)z^{s-1} + a_2(s + 2)(s + 1)z^s + \ldots) + (c - (a + b + 1)z)(a_0 sz^{s-1} + a_1(s + 1)z^s + a_2(s + 2)z^{s+1} + \ldots) - ab(a_0 z^s + a_1 z^{s+1} + a_2 z^{s+2} + \ldots) = 0
\]

So we find simple form and equation (15) is the identity equation. Coefficient from each part of \( z \) should be equal to zero, so acquiring equation (16)

\[
a_n = \frac{a(a+1)\ldots(a+n-1)b(b+1)\ldots(b+n-1)}{c(c+1)\ldots(c+n-1)n!}
\]

So the solution of the hypergeometric differential equation is shown equation (17)

\[
F_1^2(a; b; c; z) = \varphi(z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n
\]

In this paper, we will solve hyperbolic Manning-Rosen like potential with hypergeometric method. In the previous part, we find Dirac equation for hyperbolic Manning-Rosen like potential in equation (11). By hyperbolic identity, we can calculated equation (11) in order to equation (12). The hyperbolic identity expressed in equations (18), (19), (20).

\[
\coth ar = 1 - 2z, \quad \csc^2 ar = 4z(1 - z)
\]

\[
\frac{\partial z}{\partial r} = 2ax(z - 1), \quad \frac{d}{dr} = \frac{\partial}{\partial z} = 2ax(z - 1) \frac{\partial}{\partial z}
\]

\[
\frac{d^2}{dr^2} = 4a^2 z^2(z - 1)^2 \frac{\partial^2}{\partial z^2} + 4a^2 z(z - 1)(2z - 1) \frac{\partial}{\partial z}
\]

We substituted equations (18), (19), (20) in to (12) therefore becomes equation (21)

\[
\left[ 4a^2 z^2(z - 1)^2 \frac{\partial^2}{\partial z^2} + 4a^2 z(z - 1)(2z - 1) \frac{\partial}{\partial z} \right] U = \alpha^2 [(E + M)(V_0 + V_z + V_1(1 - 2z)) + (E + M)V_2 = l(l+1)](4z(1 - z)) \right] U = \alpha^2 E'U
\]

In this case, we assuming the parameter

\[
V_3 = (E + M)(V_0 + V_z) \quad V_4 = (E + M)V_1 \quad V_5 = (E + M)V_2 - l(l+1)
\]

So that equation (21) becomes to equation (22)
\[
\left[4\alpha^2 z^2(z-1)^2 \frac{\partial^2}{\partial z^2} + 4\alpha^2 z(z-1)(2z-1) \frac{\partial}{\partial z}\right] U - \alpha^2 [V_3 + V_4(1-2z) + V_5(4z(1-z))]U = E'U
\]

(22)

Equation (24) divided by \(4\alpha^2 z(z-1)\) thus obtained equation (23)

\[
z(z-1) \frac{\partial^2 U}{\partial z^2} + (2z-1) \frac{\partial U}{\partial z} - \left[\frac{[V_3-V_5-E]}{4z(1-z)} - \frac{2V_4}{4(1-z)} - V_5\right] U = 0
\]

(23)

The equation (23) like the equation (12) is the second-order differential equation that has two singular points in \(z = 0\) and \(z = 1\). This equation (23) have the solution was approached by equation (24)

\[
U_z = z^\alpha (1-z)^\beta f_z
\]

(24)

\[
\frac{\partial U_z}{\partial z} = z^\alpha (1-z)^\beta \left[\frac{\alpha}{z} f_z - \frac{\beta}{1-z} f_z + f'_z\right]
\]

(25)

\[
\frac{\partial^2 U_z}{\partial z^2} = z^\alpha (1-z)^\beta \left[\frac{\alpha(\alpha-1)}{z^2} f_z + \frac{\beta(\beta-1)}{(1-z)^2} f_z - \frac{2\alpha\beta}{z(1-z)} f_z + \frac{2\alpha}{z} f'_z - \frac{2\beta}{(1-z)} f'_z + f''_z\right]
\]

(26)

Substituted equations (24), (25), (26) in to equation (23) so we obtained equation (27)

\[
- \frac{\alpha(\alpha-1)(1-z)}{z} f_z + \frac{\alpha(2z-1)}{z} f_z - \frac{\beta z(\beta-1)}{(1-z)^2} f_z - \frac{\beta(2z-1)}{(1-z)^2} f_z - \left[\frac{(V_3-V_5-E)}{4z(1-z)}\right] f_z + \frac{2V_4}{4(1-z)} f_z + V_5 f_z + 2\alpha\beta f_z - 2\alpha(1-z)f'_z + 2\beta z f''_z + (2z-1)f''_z - z(1-z)f''_z = 0
\]

(27)

Taken every coefficients of \(f_z\) to eliminate the parameter \(z\) in the numerator that expressed in equation (28) and (29)

\[
- \frac{\alpha(\alpha-1)(1-z)}{z} = \alpha(\alpha-1) - \frac{\alpha(\alpha-1)}{z}, \quad \frac{\alpha(2z-1)}{z} = 2\alpha - \frac{\alpha}{z}
\]

(28)

\[
- \frac{\beta z(\beta-1)}{(1-z)^2} = \beta(\beta-1) - \frac{\beta(\beta-1)}{(1-z)^2}, \quad - \frac{\beta(2z-1)}{(1-z)^2} = 2\beta - \frac{\beta}{(1-z)}
\]

(29)

Substituted equations (28), (29) to equation (27) so we have a simple equation that expressed in equation (30)

\[
z(1-z)f''_z + [(2\alpha + 1) - (2\alpha + 2\beta + 2)z] f'_z - [V_5 + (\alpha + \beta)(\alpha + \beta + 1)] f_z + \frac{4\alpha^2 - (V_4 + V_5 + E')}{{4z}} f_z + \frac{4\alpha^2 - (V_4 + E')}{4(1-z)} f_z = 0
\]

(30)

The equation (30) is identically same as the general form of the hypergeometric equation

\[
z(1-z) \frac{d^2 \varphi}{dz^2} + (c - (\alpha + b + 1)z) \frac{d\varphi}{dz} - ab\varphi = 0
\]

With

\[
\alpha^2 = \frac{1}{4}(V_3 + E' - V_4), \quad \alpha = \frac{1}{\sqrt{4}}(V_3 + E' - V_4)
\]

(31)

\[
\beta^2 = \frac{1}{4}(V_3 + E' + V_4), \quad \beta = \frac{1}{\sqrt{4}}(V_3 + E' + V_4)
\]

(32)

\[
(\alpha + \beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta = \frac{1}{4}(V_3 + E') + \frac{1}{2} \sqrt{(V_3 + E')^2 - V_4^2}
\]

(33)

\[
\alpha + \beta = \pm \frac{1}{\sqrt{2}}(V_3 + E')^2 - V_4^2 - \frac{1}{2}(V_3 + E')
\]

(34)

Substituted equations (31), (32), (33), (34) back to equation (30) so that we obtained equation (35)

\[
z(1-z)f''_z + [(2\alpha + 1) - (2\alpha + 2\beta + 2)z] f'_z - [V_5 + (\alpha + \beta)(\alpha + \beta + 1)] f_z = 0
\]

(35)
With
\[ a' = (\alpha + \beta + \frac{1}{2}) - \frac{1}{\sqrt{4 - V_5}} = -n \]
\[ b' = -n2 \frac{1}{\sqrt{4 - V_2(E + M) + l(l - 1)}} \]
\[ c' = 2\alpha + 1 \]
So that the spectrum energy as one of the solution for Analytical solution of Dirac equation for Hyperbolic Manning-Rosen-like potential using Hypergeometric method for exact spin symmetry is obtained
\[
E' = \frac{V_1^2}{4\left(\frac{1}{4} - V_5^2 + \frac{1}{2}n\right)} + \left(\frac{1}{4} - V_5^2 + \frac{1}{2}n\right)^2 - V_3
\]  
(36)
\[
E' = \frac{[(E+M)V_2]^2}{4\left(\frac{1}{4} - ((E+M)V_2 - l(l+1))^2 + \frac{1}{2}n\right)} + \left(\frac{1}{4} - ((E+M)V_2 - l(l+1))^2 + \frac{1}{2}n\right)^2 -
\]
\[ [(E + M)(V_0 + V_2)] \]  
(37)
Thus obtained the wave function for radial part is shown in equation (38)
\[
U_r = \left(\frac{1-\coth \alpha}{z}\right)^\alpha \left(\frac{1+\coth \alpha}{z}\right)^\beta f(a', b', c', z)
\]  
(38)
For function \( f(a', b', c', z) \) it can be expressed as
\[
f_n = 1 + \frac{a'b'}{c'} + \frac{\alpha'(a' + 1)b'(b' + 1)z^2}{c'(c' + 1)} + ... \]
Furthermore, wave function for radial part also obtained that expressed in equation (39). The energy bound state \( n_r = 0 \) than \( f_0 = 1 \). Therefore the bound state wave function is shown in equation (39)
\[
U_r = \left(\frac{1-\coth \alpha}{z}\right)^\alpha \left(\frac{1+\coth \alpha}{z}\right)^\beta \left[\frac{V_4}{4\left(\frac{1}{4} - V_5^2 - \frac{1}{2}n\right)} + \frac{1}{4\left(\frac{1}{4} - V_5^2 - \frac{1}{2}n\right)}\right] \]  
(39)
4. Conclusión
The Dirac equation can be reduce to Shrodinger equation. The Schrodinger equation with Manning-Rosen potential can be solve with hypergeometric method.
In this work, we can find energy and wave function to describe condition of particles. Energy spectrum is exactly obtained that shown in equation (37) and wave function is expressed in equation (39). For spherical coordinate, the Dirac equation for hyperbolic Manning-Rosen-like potential in the scheme of centrifugal term approximation is exactly solved using hypergeometric method with spin symmetric cases.

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