Abstract

Well known results in string thermodynamics show that there is always a negative specific heat phase in the microcanonical description of a gas of closed free strings whenever there are no winding modes present. We will carefully compute the number of strings in the gas to show how this negative specific heat is related to the fact that the system does not have thermodynamic extensivity. We will also discuss the consequences for a system of having a microcanonical negative specific heat versus the exact result that such a thing cannot happen in any canonical (fixed temperature) description.

1E-mail addresses: cobas, osorio, maria@string1.ciencias.uniovi.es
1 Introduction and foundations

Unclear progress has been made during the last years in order to unambiguously settle the longstanding problem posed by Hagedorn [1] about the statistical behavior of strings. It is established that, in general, there is not equivalence between the canonical (fixed temperature) and the microcanonical (fixed energy) descriptions. This non-equivalence is related, although not equivalent, to the appearance of a negative specific heat in the microcanonical description. This fact has been used by some authors as a sort of selection principle to discard any model with a microcanonical negative specific heat regime. We will comment on this issue in the conclusions to mainly remind to the reader that it is only in the canonical descriptions that the specific heat cannot be negative. Microcanonical negative specific heats are always related to a thermodynamic non-extensivity of the system. More precisely, in terms of the entropy, the entropy of the system is not the raw sum of the entropies of the independent subsystems when equilibrium among them is supposed to exist. This special behavior may be interpreted as a sign of a first order phase transition, but can also be taken as an indication of the appearance of a non-equilibrium state. Anyhow, what seems to be the key ingredient for closed strings is the fact that the negative specific region covers the whole high energy regime. This happens, at least, as long as no meaning can be found to $\beta$-duality and its corresponding version in terms of energy.

On the other hand, this thermodynamic non-extensivity for the Hagedorn gas is related to the well known image in which a single fat string absorbs energy thorough its vibrational modes preventing the increasing of the temperature that gets closer to the Hagedorn one as long as energy increases. What we give here, among other things, is an explicit calculation of the number of strings in the particular case in which there is no winding mode state in the gas. We use the bosonic string living in 25 big dimensions as the tool instead of, for example, the SSTII, simply because its high dimensionality makes easier the convergence of some plots we present in our work.

To begin with, to properly make Thermodynamics, one should put the system inside a sort of a box as it is done with a classical system of particles. We do this for strings in a fundamental way by taking a particular compact geometry for the space. A limit like the high volume limit has to be understood and physically defined over this finite volume picture as it is done for the analogous point-particle system. Now, we know more clearly that this is a subtle point when extended objects as strings are involved [2, 3]. Other important aspect to remark is that the treatments usually called canonical (see for instance [1]), are really macrocanonical. The number of strings in the gas is a function of the temperature and the volume and not a free variable. Indeed, it is in this ensemble where the implementation of a null chemical potential is more natural, because $\mu$ is a variable of the grand partition function. When classical Maxwell-Boltzmann
(MB) statistical counting is applicable, it is easy to implement this equilibrium condition over the canonical ensemble too. Things are far more difficult when quantum statistics is involved because, as it is well known, it is hard to get an explicit expression for the canonical partition function and it is really troublesome, it seems that analytically impossible, to compute, in a closed form, the derivative of the Helmholtz free energy with respect to \( N \), the number of constituents.

In Statistical Mechanics, when \( \Gamma (E, V, 1) \), the number of one object accessible states of a system, is much bigger than the number \( N \) of objects, the subtleties of Bose-Einstein (BE) or Fermi-Dirac (FD) statistics become superfluous; it is highly improbable for two objects to try to get into the same states since there are lots of rooms to accommodate into. From \( \Gamma (E, V, 1) \), by taking one derivative with respect to the energy, one can obtain the single-object density of states \( \Omega (E, V, 1) \) and from it it is easy to obtain the single object partition function as \( q = Z (\beta, V, 1) = \sum_E \Omega (E, V, 1) e^{-\beta E} \) where \( \beta = 1/T \) being \( T \) the temperature in units in which the Boltzmann constant \( k \) equals one. When use is made of MB counting, the canonical partition function for \( N \) independent objects can be written as \( \frac{Z (\beta, V, N)}{N!} \). This is really an approximate calculation valid for free (independent) objects or, in a more realistic manner, dilute gases at sufficiently high temperatures. The macrocanonical description rests upon the grand partition function \( \Theta (\beta, V, \lambda) = \sum_N Z (\beta, V, N) \lambda^N = \sum_N \lambda^N \left[ q (\beta, V) \right]^N /N! \), where \( \lambda \) is finally identified with \( e^{\beta \mu} \) and the second equality holds as long as MB statistics is a good approximation. When physical conditions are such that \( \mu = 0 (\lambda = 1) \) we get \( \Theta (\beta, V, 1) = \sum_N \left[ q (\beta, V) \right]_N /N! = e^q \), again when MB statistics is applicable. Imposing \(-\beta \mu = \partial \ln Z (\beta, V, N) /\partial N = 0\) gives, for big \( N \) that is what one needs to make thermodynamics, \( N = Z (\beta, V, 1) \) as the number of objects for the system at a given temperature \( T \) when MB statistics is applicable. When BE or FD statistics are necessarily involved, the grand canonical partition function is the best starting point to face the more complicated combinatorics, although actual computations can finally be written in terms of \( q \).

Our main goal here is to face the problem of the black-body radiation looking at the system as one for which energy is conserved. The property of the system to have null chemical potential, when one assumes that, at least at low energies, quantum statistics is relevant, makes useful the use of a system for which the chemical potential is fixed (to zero in our case) and the number of objects depends finally on energy and volume. The ensemble adapted to these conditions has a partition function given by \( \Lambda (E, V, \mu) = \sum_N \Omega (E, V, N) \lambda^N \) that, for \( \mu = 0 \), gives \( \Lambda (E, V, 0) = \sum_N \Omega (E, V, N) \). Here \( \Omega (E, V, N) \) is the microcanonical density of states for the system with \( N \) objects. Thermodynamics can be gotten from \( \beta H \) where \( H \) is the enthalpy \( E + PV \) with \( P \) the pressure. Namely, \( H = E + PV = TS + N\mu \). Since in our case \( \mu = 0 \), we have that \( \beta H = S = \ln \Lambda (E, V, 0) \).
The computation of the average number of objects then gives

\[ \overline{N} (E, V, \lambda = 1) = \left[ \lambda \left( \partial \ln \Lambda / \partial \lambda \right)_{V,E} \right]_{\lambda = 1} = \frac{\sum_{N=0}^{\infty} N \Omega (E, V, N)}{\sum_{N=0}^{\infty} \Omega (E, V, N)} \]  

(1.1)

If it is assumed that, in the summations appearing in the numerator and the denominator, there is a value of the label \( N \) equal to \( N^* \) such that each sum can be well approximated by the single \( N^* \) contribution, we get that \( \overline{N} = N^* \). Assuming that \( N^* \) is the same for the sum in the numerator and the denominator is equivalent to assuming that no fluctuations in the number of objects are present. Quantitatively, \( N^* \) is such that \( \left[ \partial \Omega / \partial N \right]_{N=N^*} = 0 \) which is just the number of particles we get by imposing \( \mu = 0 \) in the (genuine) microcanonical ensemble characterized by the density of states \( \Omega (E, V, N) \). In other words, we assume the equivalence of the microcanonical and the "enthalpic" thermodynamical descriptions based on the fact that \( S = \ln \Lambda (E, V, 0) = \ln \sum_{N=0}^{\infty} \Omega (E, V, N) \approx \ln \Omega (E, V, N^*) \).

It is also interesting to try to express the average number of objects in terms of the single object partition function \( q \equiv Z_1 (\beta) \) (it is a function of temperature and volume although only its dependence thorough \( \beta \) will be kept explicit for convenience as it will happen for other functions. The result is

\[ \overline{N} (E) = \frac{\sum_{n=0}^{\infty} \mathcal{L}^{-1} \left\{ \sum_{r=1}^{\infty} (\pm)^{r+1} Z_1 (r\beta) \right\} \ast \widetilde{\Omega}_n (E)}{\sum_{n=0}^{\infty} \widetilde{\Omega}_n (E)} \]  

(1.2)

where \( \mathcal{L}^{-1} \) means inverse Laplace transformation with respect to the variable \( \beta \) to \( E, \widetilde{\Omega}_n (E) \equiv (1/n!){\mathcal{L}}^{-1} \left\{ \left[ \sum_{r=1}^{\infty} (\pm)^{r+1} Z_1 (r\beta) / r \right]^n \right\} \), the star means convolution and, in the sums over \( r \), the plus sign corresponds to BE statistics and the minus sign to FD statistics\(^2\). Everything comes from rewriting the entropy as

\[ \Lambda (E, V, 0) = \mathcal{L}^{-1} \left\{ \Theta (\beta, V, 1) \right\} = \sum_{n=0}^{\infty} \tilde{\Omega}_n (E) \]  

(1.3)

where \( \tilde{\Omega}_0 (E) = \delta (E) \) which corresponds to the fact that there is only one state with zero objects and it is the one with zero energy. It is very important to remark that, for example, \( \tilde{\Omega}_1 (E) \) is not the single object density of states \( \Omega (E, V, 1) = \mathcal{L}^{-1} \{ Z_1 (\beta) \} \). For instance, using BE statistics, what \( \tilde{\Omega}_1 (E) \) really does is to take into account all the possibilities for allocating several bosons in a way in which all of them are in the same state. The counting is then made in terms of clusters or bunches of bosons (or in terms of the number of different filled quantum states).

\(^2\)Here: \( \tilde{\Omega}_n (E) = (1/n!) \mathcal{L}^{-1} \left\{ \ln^n \Theta (\beta, V, 1) \right\} \), with \( \ln \Theta (\beta, V, \lambda) = \pm \sum_i \ln \left( 1 + \lambda e^{-\beta \epsilon_i} \right) = \sum_{r=1}^{\infty} (\mp)^{r+1} \lambda r Z_1 (r\beta) / r \). FD statistics corresponds to the upper sign.
This can be seen if we look at the way of computing the canonical partition function with quantum statistics. One can write

\[ Z(\beta, N) = \sum_{\{n, \lambda\}} \prod_i \frac{[Z_1(n_i \beta)]^{\lambda_i}}{n_i^{\lambda_i} \lambda_i!} \]  

(1.4)

where the sum is restricted over the set of positive integer values \( \lambda_i, n_i \) such that \( \sum_i \lambda_i n_i = N \). From (1.4) we get that the contribution to the canonical partition function \( Z(\beta, N) \) coming from having the \( N \) objects in the same quantum state is given by \( Z_1(N \beta)/N \). Now, if we sum over the number of objects \( N \) from one to infinity we get \( \sum_{N=1}^{\infty} Z_1(N \beta)/N \) whose inverse Laplace transform just gives \( \tilde{\Omega}_1(E) \) for BE counting. As an example, \( \tilde{\Omega}_2(E) \) gives the density of states for a system of 2-clusters, i.e., two objects in different states, three objects distributed two of them in the same state and a single object in a different one, and, in general, \( N \) objects allocated in a manner such that only two different single object states are occupied. \( \Omega(E, V, N) \) and \( \tilde{\Omega}_{n=N}(E) \) only coincide when classical (MB) statistics is applied because then, as explained at the beginning of this introduction, \( Z_{MB}(\beta, N) = q^N/N! \) and so \( \tilde{\Omega}_n = \mathcal{L}^{-1}\{q^n/n!\} = \Omega_{MB}(E, n) \). If one is interested in knowing the dependence on the volume, one finds that, when \( q \) depends linearly on the volume, \( \Omega(E, V, N) \) goes as \( A_1 V^N + A_2 V^{N-1} + \ldots + A_N V \) as long as \( \tilde{\Omega}_N \sim V^N \). Anyhow, everything finally fits in place because, in our case, what is relevant to physics is the sum over \( N \). One can then reorder the sum regrouping the terms proportional to \( V^N \) and so get the same sum over the label \( n \) of \( \tilde{\Omega}_n \) instead of the sum over the number of particles \( N \) of \( \Omega_N \).

One of the goals of this work is to show that the number of strings, as a function of energy and volume, can be computed even when quantum statistics has to be taken into account (we will see that this would be the case for a low number of open spatial dimensions). In particular, we will exemplify thorough the gas of closed bosonic strings that the number of strings tends toward a constant value \( N_H \) as long as energy increases once the microcanonical negative specific heat phase has been reached. To try to make a clear exposition, section two will be devoted to present a microcanonical description of the traditional blackbody radiation problem. In section three, we will present the problem of computing thermodynamic quantities for a gas of strings that has a big number of open spatial dimensions; this will be exemplified thorough a gas of bosonic strings, although the results clearly hold for other closed strings under the same conditions. After the computation of \( \overline{N}(E, V) \), emphasis will be made on the non extensivity of the entropy \( S(E, V) \) at high energy. Finally, we will present a discussion about what can be said when there is a negative specific heat in a microcanonical approach, the equivalence of microcanonical and canonical descriptions and under which conditions a negative specific heat can be used as a kind of selection rule to say that a system is unphysical.
2 A microcanonical description of the blackbody radiation problem

The blackbody radiation problem is a Statistical Mechanics classic. Although the problem is posed in such a way that a microcanonical (fixed energy) description should be used, this is not the standard procedure in textbooks. Here we adopt such point of view stressing the fact that an “enthalpic” description in terms of an ensemble in which the chemical potential $\mu$ is fixed and null (at a given energy and volume) would be the most natural first attempt to describe that system. Eq. (1.2) gives now

$$\bar{N}(E; d) = \frac{\zeta(d)}{\zeta(d+1)} \frac{\sum_{n=1}^{\infty} n \tilde{\Omega}_n^{BE}(E)}{\sum_{n=0}^{\infty} \tilde{\Omega}_n^{BE}(E)}$$

(2.1)

$\tilde{\Omega}_n^{BE}(E)$ is easily calculated to give

$$\tilde{\Omega}_n^{BE}(E; d) = \frac{a_0^n}{\Gamma(dn) n!} \left[ \frac{V \zeta(d+1) \Gamma(d)}{2^{d-1} \pi^{d/2} \Gamma(d/2)} \right]^n E^{dn-1}$$

(2.2)

both functions are expressed, for later convenience, as depending on the number of space dimensions, $d$. The factor of $a_0^n$ comes from the contribution of the spin degrees of freedom for a massless object and it will equal $2^n$ for a photon gas. The next step to really make (2.1) useful is to approximate the sums in the numerator and the denominator by the contribution of an $n^*$ such that the logarithm of each sum is dominated by the logarithm of that term $\Omega_n^{*}$. The terms of maximum contribution for the numerator and the denominator only differ in $\ln n$ which is negligible as compared to $n$ when $n$ is big. If one uses that $n^*$ is big and Stirling’s approximation, one gets that

$$n_{BE}^*(E; d) = \left[ \frac{a_0 d^{-d} \zeta(d+1) \Gamma(d)}{2^{d-1} \pi^{d/2} \Gamma(d/2)} V E^d \right]^{1/(d+1)}$$

(2.3)

For the photon gas with $a_0 = 2$ in three space dimensions one exactly gets

$$\bar{N}(E; 3) = \frac{\zeta(3)}{\zeta(4)} n_{BE}^*(E; 3)$$

(2.4)

that, of course, coincides with the value given by the macrocanonical description once the expression of $n_{BE}^*$ is substituted.

The ensemble average inverse temperature can be computed for the photon gas along the same lines to give

$$\bar{\beta} = \left[ \frac{\partial S}{\partial E} \right]_V = \frac{3 n_{BE}^*(E; 3)}{E} = \frac{3 \zeta(4) N}{\zeta(3) E}$$

(2.5)
It is interesting to emphasize that \( n_{BE}^* \) is not the average number of particles because \( \zeta(3) \neq \zeta(4) \). In fact, this difference is what gives a quantum correction to the classical ideal gas equation of state. It may also be instructive for treating fermionic strings to get the same results in the enthalpic description for a gas of massless fermions at \( \mu = 0 \). For them, the main basic ingredient is the single-cluster density given by

\[
\widetilde{\Omega}_1^{FD} (E) = \left( 1 - \frac{1}{2^d} \right) \widetilde{\Omega}_1^{BE}
\]  

(2.6)

As for bosons, from here, after repeated convolutions one may easily get \( \widetilde{\Omega}_n^{FD} (E) \). From the results for bosons and fermions it is easy to get the thermodynamics for the supersymmetric system.

Before ending this section, some comments come into place. The connection between Statistical Mechanics and Thermodynamics rests upon the fact that the number which connects atomic Physics and the scale of grams is Avogadro’s number; a big number indeed. Consequently, many approximations as the use of Stirling’s formula over \( \ln N! \) are based on the big number of objects involved. Here, however, what has been assumed to be a big number is \( n^* \) which refers to the number of clusters or different occupied states. Assuming that the main contributions come from big \( n \)'s, we are then supposing that one gets the most relevant contributions to the entropy from the situations in which a big number of objects have different quantum numbers. The fermionic and supersymmetric calculations show that there will be no difference between bosons and fermions when the space dimension grows big. With a big number of dimensions there would be a lot of room to move in and, consequently, a big number of accessible states even when energy is not too high.

### 3 The stringy blackbody

The approach to the problem of a system of strings in equilibrium such that the chemical potential is null has been widely treated in numerous references in the past. In many cases, the question about what ensemble is used to describe the problem has not been clear. This way, descriptions that should be called grand canonical have been named canonical at the same time it was sustained that, at least from the beginning, a quantum description in terms of bosons and fermions were in course. Perhaps this is, in some cases, a remnant of the presentation made by classical text books in Statistical Mechanics when treating the gas of photons (cf. [4]). Even though \( \mu \) cannot be computed in the quantum canonical description, we know that the equivalence of this picture and the one gotten from an ensemble in which the chemical potential is fixed and not an averaged quantity will rests upon the possibility of substituting the sum over the
number of objects in the grand canonical description by the contribution of a
preponderant single term \( N = N^* \). This is equivalent to assuming that there are
no big fluctuations in the number of objects and this, if true, may happen for a
picture in which temperature is fixed or for one in which energy is given and not
an averaged quantity. It has been exposed in the introduction that, it is only after
taking something like the dilute gas approximation to make proper use of MB
statistics, when one can certainly compute in a canonical description the number
of objects that corresponds to a vanishing chemical potential and compare it to
the more natural description in terms of an ensemble with fixed and vanishing
chemical potential and an averaged number of objects. In fact, when making the
approximation of a free theory, what finally plays the most important role for the
MB approximation to hold is the number of big dimensions one finally has.

By assuming that this is the physical picture to make thermodynamics, a
fundamental question arises about whether a decompactified description in which
no winding mode contribution appears can be reached from a particular finite
volume description for the gas by taking an infinite volume limit. That question
can be resolved in affirmative if one realizes that the mass of a winding mode
is proportional to the radius. Then, there must exist a radius big enough so as
to get that, for a given energy, the masses of the oscillators (proportional to a
positive integer number) are much lighter than the mass of the winding modes.
The degeneration of the oscillators increases much faster than that of the winding
modes and this is the explanation for the negative specific heat in the oscillator
ruled high energy phase and this is what a decompactified phase for the string
gas really means. Among other things, we are going to study a standard bosonic
string gas to see the characteristics of the transition from the low to the high
energy regime. For one single bosonic string in \( d = 25 \) we have that \( \tilde{\Omega} (E, V, 1) \)
is given by

\[
\tilde{\Omega}_1 (E) = \frac{V \pi^{-25/2}}{2^{24} \Gamma (25/2)} \sum_{k=0}^{\infty} \sum_{r=1}^{\infty} r^{-26} a_k^2 E \left( E^2 - \frac{4kr^2}{\alpha'} \right)^{23/2} \theta \left( E^2 - \frac{4kr^2}{\alpha'} \right)
\]

(3.1)

Here, the tachyonic contribution with \( k = -1 \) has been suppressed. This result
amounts to summing upon the mass levels of the string for which the masses are
\( m_k = \sqrt{\frac{2k}{\alpha'}} \) with \( k \) a zero or natural number. The coefficients \( a_k^2 \) measure the
degeneration of the corresponding mass level.

Now if \( k \neq 0 \), the sum over \( r \) cannot be analytically performed. For the
massless contribution we have an equation of state given by \( PV = \frac{\zeta(26)}{\zeta(25)} N \). But
now, \( \zeta (26) / \zeta (25) = 1 - 1.49 \cdot 10^{-8} \sim 1 \). With this approximation, that is very
natural, we are making classic the gas of massless modes in 25 dimensions. This
amounts also to taking only into account the \( r = 1 \) term in the second sum because
the second term with \( r = 2 \) would be suppressed by a factor \( 2^{-26} \sim 1.40 \cdot 10^{-8} \).
We can then make the approximation of using MB statistics to get

\[ \tilde{\Omega}_1(E) = \frac{V \pi^{-25/2}}{2^{24} \Gamma(25/2)} \sum_{k=0}^{\infty} a_k^2 E \left( E^2 - \frac{4k}{\alpha'} \right)^{23/2} \theta \left( E^2 - \frac{4k}{\alpha'} \right) \]  

(3.2)

Then, one has \( \tilde{\Omega}_1 = \Omega(E, 1) \) and, in general, \( \tilde{\Omega}_n = \Omega(E, n) \). Now the average number of objects can be computed to be

\[ \bar{N}(E) = \frac{\sum_{n=0}^{\infty} n \Omega_n(E)}{\sum_{n=0}^{\infty} \Omega_n(E)} \]  

(3.3)

When there is a number \( n = n^* \) that gives a maximum contribution to both sums, one obtains \( \bar{N} = n^* \). So, the next step would be to compute \( \Omega_n \) for a very big \( n \) from the single string density of states. This can, in principle, be done by convolution, but the integral to compute \( \Omega_2 \) is already too complicated to perform it analytically. It seems more clever to get the behavior of \( \Omega_1(E) \) at high energy from the asymptotic behavior of \( a_k \sim \frac{1}{\sqrt{k}} a_0 (k-1)^{-27/4} e^{\frac{4\pi}{\sqrt{k-1}}} \). When computing

\[ \Omega_2(E) = \frac{1}{2! \Gamma(50)} \left( \frac{24^2 \Gamma(25)}{2^{24} \pi^{25/2} \Gamma(25/2)} \right)^2 V^2 E^{49} \]  

(3.4)

where \( l \) stands for low energy that actually means that there are only massless modes. After going up thorough \( E = 2 \) increasing the energy the massive levels appear. One can define \( S_2 = \ln \tilde{\Omega}_2 \) and from here one gets \( 1/T_2 = \beta_2 = \partial S_2/\partial E \) which gives a sort of fictitious temperature for the system with two strings shearing an amount of energy \( E \). Of course, the real microcanonical temperature will be \( T_n(E) \) for big \( n \) that, after substituting \( \bar{N} \) as a function of the volume an energy, will give the temperature as a function only of volume and energy. \( T_2(E) \) is linear with the energy for the low energy phase. The massive modes start to curve the straight line in such a way that after passing \( T_2 = T_H \) there appears a maximum (see Fig. [1] below). This behavior is something normal whenever a new channel opens in any multiple channel relativistic theory because energy is spent in mass and not in the kinetic energy of the objects in the system. What is stringy is the fact that no phase of positive specific heat reappears because the increasing degeneration of the fat string prevents it. Equipartition is broken favoring the existence of a heavy string [6]. We can use this to get an approximate form for \( \Omega_n \). Let us suppose that we have a gas that at low temperature is a gas of massless modes. For this system is

\[ T_n(E) = \frac{E}{25n - 1} \approx \frac{E}{25n} \]  

(3.5)

where the approximation is natural because the number of convolutions is supposed to be very big.
The Hagedorn temperature is reached at an energy $E_n$ such that $T_H = \frac{E_n}{25n-1}$. For energies bigger than this one, $n-1$ strings in the gas have an energy $E_{n-1} = T_H(25n - 26)$ and the fat string takes $(E - E_{n-1})$ with $E \gg E_{n-1}$. So our model to approximate the behavior of the system can be expressed in the following way

$$\Omega_n(E) \approx \begin{cases} \Omega_n^l(E) & E < E_n^c \\ \Omega_n^h(E) & E > E_n^c \end{cases}$$

where $C_n$ is a function of the number of strings chosen so as to get a continuous curve in $E_n^c$, the transition point, which belongs to the transition region in which our approximation is not pretty good and then the election is rather arbitrary. The important point is that this intersection point depends on $n$. For $n \gg 1$ we take

$$C_n = \left( \frac{\Gamma(25) 24^2}{2^{24} \pi^{25/2} \Gamma(25/2)} \right)^n \frac{(4\beta_H)^{-27/2} E_{n-1}^{25n-1}}{\Gamma(25n) n! (5400n)^{-27/4}}$$

$$E_n^c = \frac{25n}{\beta_H}$$

From this approximation it is easy to get Fig. 1 in which the numerical computation of the fictitious temperature $T_2(E)$ is compared with $T_2(E)$ obtained from our approximation for $\Omega_n(E)$ in (3.6) for $n = 2$.

![Figure 1](image)

**Figure 1:** $T_2(E)$ numerically computed compared with our approximation

## 4 The average number of strings

To compute $\overline{N}(E)$ we are going to use the approximation of the maximum term for the sums in (3.3). However, now, in the sum, we get two different functions. When energy is big, most of the terms of the sum will correspond to the low energy regime as long as $n > n_0 = \beta_H E/25$, but when energy is high enough
$\Omega_n(E)$ will be described by the high energy approximation for all $n < n_0$. It is then convenient to distinguish the high energy function $\Omega^h_n(E)$ from the low energy one, $\Omega^l_n(E)$ as they are defined in (3.6). For the low energy function we have, for the term with maximum contribution

$$n^* = \overline{N}^l(E) = D_{25}^{1/26} E^{25/26} V^{1/26}$$  \hspace{1cm} (4.1)$$

where, to simplify the equations, it has been taken $D_{25} \equiv \frac{\Gamma(25/2)}{\Gamma(25/2) 25^{25/2} \pi^{25/2}}$ and the temperature is given by (3.5). It is then straightforward to get the temperature as a function only of volume and energy in a way perfectly compatible with the classical blackbody but adapted to the polarizations of the massless states of the closed bosonic string and the number of spatial dimensions. The Hagedorn temperature is then reached from the low energy phase at an energy

$$E_H = D_{25}^{25} V \beta_H^{25}$$  \hspace{1cm} (4.2)$$

and the averaged number of objects at $E_H$ is

$$N_H = 25^{25} D_{25} V \beta_H^{25} = \beta_H E_H / 25$$  \hspace{1cm} (4.3)$$

On the other hand, the computation of $n^*$ for $\Omega^h_n(E)$ gives it as a solution of

$$\ln (V D_{25}^{25}) - \ln n^* - 25 \ln \beta_H + \frac{27}{4n^*} + \frac{27}{2} \frac{25}{\beta_H (E - E_{n^*-1})} = 0$$  \hspace{1cm} (4.4)$$

To solve this equation we use that $n^*$ will be very big and that $E \gg 25n^*/\beta_H$ which means that we are really in the high energy regime. One has to find the roots for a second order polynomial. One of the solutions is not compatible with the assumption that we are in the high energy regime and the other one gives

$$\overline{N}^h(E) = n^* = \frac{\beta_H}{50} \left[ E + E_H - \sqrt{(E - E_H)^2 - \frac{1350}{\beta_H^2} E_H} \right]$$  \hspace{1cm} (4.5)$$

It is immediate to get that $\lim_{E \to \infty} \overline{N}^h(E) = N_H$ (here $E \to \infty$ physically means $E \gg E_H$). Fig. 2 shows the average number of objects for the low and high energy regimes as a function of $E$ for a given volume that it is fixed giving a value to $E_H$ (we also take $\alpha' = 1$ to get the plot). The straight line we see for the low energy part is the graphical result of the fact that $\overline{N}^l(E) \sim E^{25/26} \approx E$. The part of $C_n$ that survives in this computation is precisely what gives the degeneration of the objects that form the gas that remains at a temperature $T_H$. It is easy to check that for energies lower than $E_H$ the maximum $n^*$ really appears in the low energy region and for energies above $E_H$ it does in the high energy part. Finally it is easy to get from our computation the form of the density of states for the high
energy regime as the result of approximating the infinite sum giving $\Lambda(E, V, 0)$ by the single contribution corresponding to $n^* \approx N_h$

$$\Omega^h(E, V) = \beta H^{-25/2} (5400 N_h(V)/16)^{27/4} (E - E_H)^{-27/2} e^{N_h(V)+\beta H E} (4.6)$$

that is compatible with what was computed longtime ago. We have made explicit to the reader that there is a dependence on the volume for $N_H$ as shown in (4.3). The entropy at high energy then contains terms of the form $\frac{27}{4} \ln V$ and $-\frac{27}{2} \ln E$ that break thermodynamic extensivity because, for systems with $\mu = 0$, the extensivity of entropy amounts to the fact that $S(E, V)$, which depends only on energy and volume after substituting $\overline{N}(E, V)$, must be a homogeneous function of degree one of both variables. Those terms are added to others of the form $\beta H E$ and $\beta H V$ that preserve extensivity. This means that the limiting system obtained at very high volume and energy, which has an infinite calorific capacity, preserves thermodynamic extensivity. The term $-\frac{27}{2} \ln E$ is actually responsible for $1/C_V$ to be different from zero and, in fact, negative. This is the way we thermodynamically recover what we have physically implemented, as the result of the breaking of equipartition, in the statistical mechanics construction of $\Omega_N$. In it, from the very beginning, we have distinguished the single fat string from the sea and then the procedure is inherently non extensive.

5 Comments

Following [8], the negative specific heat for the gas of closed strings that appears in the microcanonical description when no winding modes are present has been promoted to a selection principle ruling out those systems. We do not know what the basics to prefer the situation in which the specific heat is positive are. Moreover, when it is used to the extent of automatically ruling out every model for which, in a microcanonical description, a negative specific heat appears. Perhaps what has happened is that the fact that no negative specific heat can appear in any canonical (fixed temperature) description (based in Boltzmann-Gibbs statistics)
has been unjustifiably transferred from canonical thermodynamics to the microcanonical one. Anyhow, what is well established \cite{9} is that the microcanonical entropy $S$ and the microcanonical inverse temperature $\beta$ are single valued, multiple differentiable, and smooth. Furthermore, as $\beta > 0$ and $\partial^2 S/\partial E^2 \neq \pm \infty$, the microcanonical $C_V$ can be positive or negative but never zero. In conclusion, a priori there is nothing wrong with having negative microcanonical $C_V(E)$. Furthermore, from a thermodynamic point of view, the occurrence of a first-order phase transition in the canonical ensemble defined by the presence of a non differentiable point of the free energy may appear as a non concavity of the entropy as a function of energy and, consequently, as a situation of non equivalence of ensembles \cite{10}. Several empirical systems in atomic and condensed matter physics are also known for which negative specific heat has been measured (cf. for example \cite{11}). In any case, as for strings, the negative specific heat for these systems is related to a loss of thermodynamic extensivity. However, for the laboratory examples, non extensivity results from the presence of long-range interactions when the system is of any size or short-range interactions in the case it is a small one.

Non equivalence of ensembles is a topic also related to the appearance of fluctuations \cite{12}. This is the case for the big fluctuations that emerge in the canonical internal energy near Hagedorn and that imply the need to distinguish canonical and microcanonical descriptions. Fluctuations in the number of particles are also related to the possibility of approaching both sums in (1.1) for the single contribution given by the same $N = N^*$.

A possible alternative to try to understand the Hagedorn phase is to think that we really have an almost out-of-equilibrium phenomenon for which a sort of generalized statistics like the one in \cite{13} has to be used. It is immediate to see that such attempt implies the use of a factor different from the Boltzmann factor $e^{-\beta H}$ and that would certainly change (may be suppress) the Hagedorn behavior in the canonical descriptions. The problem is that, as lucidly exposed in \cite{13}, there are very serious objections against this kind of non extensive generalized statistics.

**Acknowledgments**

M.A.C. acknowledges partial support by a Spanish MCYT-FPI fellowship. M. S. is partially supported by a Spanish MEC-FPU fellowship. We all are partially supported in our work by the MCYT project BFM2003-00313/FISI.

**References**

\[1\] R. Hagedorn, *Nuovo Cim. Suppl.* 3, 147 (1965).
[2] M. A. Cobas, M. A. R. Osorio, M. Suárez, [hep-th/0211211]

[3] N. Deo, S. Jain, C. -I. Tan, Phys. Rev. D 45, 3641 (1992).
   M. Axenides, S. D. Ellis, C. Kounnas, Phys. Rev. D 37, 2964 (1988).

[4] S. Frautsschi, Phys. Rev. D 3, 2821 (1971).

[5] K. Huang, Statistical Mechanics, John Wiley, New York, 1987.

[6] M. Laucelli Meana, M. A. R. Osorio, J. Puente Peñalba, Phys. Lett. B 400, 275 (1997), Phys. Lett. B 408, 183 (1997).

[7] M. Laucelli Meana, J. Puente Peñalba, Nucl. Phys. B 560, 154 (1999).

[8] R. H. Brandenberger, C. Vafa, Nucl. Phys. B 316, 391 (1988).

[9] D. H. E. Gross, Microcanonical thermodynamics: Phase transitions in "small" systems, Lecture Notes in Physics, Vol. 66, World Scientific, Singapore, 2001.

[10] H. Touchette, Equivalence and Nonequivalence of the Microcanonical and Canonical Ensembles: A Large Deviations Study, Ph. D Thesis (2003), McGill University.

[11] M. Schmidt, R. Kusche, T. Hippler, J. Donges, W. Kronmuller, B. von Issendorff and H. Haberland, Phys. Rev. Lett. 86, 1191 (2001).

[12] R. D. Carlitz, Phys. Rev. D 5, 3231 (1972).

[13] A. Réyni, in Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability (University of California Press, Berkeley, 1960, Vol.1).
   C. Tsallis, J. Stat. Phys. 52, 479 (1988).

[14] M. Nauenberg, Phys. Rev. E 67, 036114 (2003).