Exact asymptotic for tail of distribution of self-normalized sums of random variables under classical norming.

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Abstract

We derive in this article the asymptotic behavior as well as non-asymptotical estimates of tail of distribution for self-normalized sums of random variables (r.v.) under natural classical norming.

We investigate also the case of non-standard random norming function and the tail asymptotic for the maximum distribution for self-normalized statistics.

We do not suppose the independence or identical distributionness of considered random variables, but we assume the existence and sufficient smoothness of its density.

We show also the exactness of our conditions imposed on the considered random variables by means of building of an appropriate examples (counterexamples).

Key words and phrases: Random variables and vectors (r.v.), exact asymptotics, non-asymptotic upper and lower estimates, H"older’s inequality, density, classical norming, Rademacher’s distribution, anti-Hessian matrix and its entries, self-normalized sums of r.v., Gaussian multivariate distribution, determinant.

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1 Definitions. Notations. Previous results. Statement of problem.

Let \( \{\xi(i)\} \), \( i = 1, 2, 3, \ldots, n; \ n \geq 2 \), be a collection of random variables or equally random vector (r.v.)

\[
\xi = \mathbf{\xi} = \{\xi(1), \xi(2), \ldots, \xi(n)\},
\]

not necessary to be independent, centered or identically distributed, defined on certain probability space, \( \forall i \Rightarrow P(\xi(i) = 0) = 0 \), having a (sufficiently smooth) density of distribution \( f_{\xi}(\mathbf{x}) = f(x) = f(\mathbf{x}), \ x = \mathbf{x} \in \mathbb{R}^n \).
Let us introduce the following self-normalized sequence of sums of r.v. under the classical norming

\[ T = T(n) = \frac{\sum_i \xi(i)}{\sqrt{\sum_i \xi^2(i)}}, \quad (1.1) \]

here and in what follows

\[ \sum = \sum_{i=1}^n, \quad \sum = \sum_{j=2}^n, \quad \prod = \prod_{j=2}^n, \]

and define the correspondent tail probabilities

\[ Q_n = Q_n(B) := P(T(n) > B), \quad B = \text{const} > 0; \]

\[ Q(B) := \sup_n Q_n(B) = \sup_n P(T(n) > B), \quad B = \text{const} > 0. \]

B.Y.Jing, H.Y.Liang and W.Zhou obtained in an article [5] the following uniform estimate of sub-gaussian type for i.i.d. symmetrical non-degenerate random variables \{\xi(i)\}

\[ Q(B) \leq \exp\left(-\frac{B^2}{2}\right). \]

Note first of all that if \( n = 1 \), then the r.v. \( T(1) = \text{sign}(\xi(1)) \) has a Rademacher’s distribution. This case is trivial for us and may be excluded.

Further, it follows from the classical H"older’s inequality that \( T(n) \leq \sqrt{n}, \quad n = 2, 3, \ldots \); therefore

\[ \forall B \geq \sqrt{n} \Rightarrow Q_n(B) = 0. \]

Thus, it is reasonable to suppose \( B = B(\epsilon) = \sqrt{n} - \epsilon, \quad \epsilon \in (0, 1) \); and it is interest by our opinion to investigate the asymptotical behavior as well as the non-asymptotical estimates for the following tail function

\[ q(\epsilon) = q_n(\epsilon) = Q_n(\sqrt{n} - \epsilon) \quad (1.2) \]

as \( \epsilon \to 0+, \quad \epsilon \in (0, 1) \); the value \( n \) will be presumed to be fix and greatest or equal than 2.

The case of the left tail of distribution \( P(T(n) < -\sqrt{n} + \epsilon), \quad \epsilon \in (0, 1) \), as well as the probability \( P(|T(n)| > \sqrt{n} - \epsilon), \quad \epsilon \in (0, 1) \), may be investigated quite analogously.

Our purpose in this short preprint is just obtaining an asymptotical expression of these probabilities, as well as obtaining the non-asymptotical bilateral estimates for ones.

We consider also the case of non-standard random norming function.
The problem of tail investigation for self-normalized random sums with at the same or another self norming sequence was considered in many works, see e.g. [1]-[11]. Note that in these works was considered as a rule only asymptotical approach, or uniform estimates, i.e. when \( n \to \infty \); for instance, was investigated the classical Central Limit Theorem (CLT), Law of Iterated Logarithm (LIL) and Large Deviations (LD) for these variables. Several interest applications of these tail functions, in particular, in the non-parametrical statistics are described in [1], [2], [5], [7]-[8], [11] etc.

2 Main result.

We need to introduce now some needed notions and notations. Introduce for any \( n \) dimensional vector \( x = \tilde{x} = \{x(1), x(2), \ldots, x(n)\} \) its \((n-1)\) - dimensional sub-vector

\[
y = \tilde{y} = \tilde{y}(\tilde{x}) := \{x(2), x(3), \ldots, x(n)\}. \tag{2.1}
\]

Define also for arbitrary \((n-1)\)- dimensional positive vector \( v = \tilde{v} = \{v(2), v(3), \ldots, v(n)\}\) the function

\[
g(v) = g_n(v) = \frac{1 + \sum_j v(j)}{\sqrt{1 + \sum_j v^2(j)}} \tag{2.2}
\]

and introduce the correspondent its anti-Hessian matrix for this function at the extremal point \( \tilde{v}_0 = \vec{1} = (1, 1, \ldots, 1), \dim \tilde{v}_0 = (n-1) \), containing the following entries: \( A = A(n-1) = \{a(j, k)\}, \)

\[
a(j, k) := -\left\{ \frac{\partial^2 g(v)}{\partial v(j) \partial v(k)} \right\} /\vec{v} = \vec{1}, \; j, k = 2, 3, \ldots, n; \tag{2.3}
\]

and we find by the direct computations

\[
a(j, j) = n^{-1/2} - n^{-3/2} - \tag{2.3a}
\]

the diagonal members,

\[
a(j, k) = -n^{-3/2}, \; k \neq j - \tag{2.3b}
\]

off diagonal entries.

Lemma 2.1. Let \( L_m = L_m(x), \; m = 1, 2, 3, \ldots \) be a square matrix of a size \( m \times m \) with entries

\[
l(j, j) = x, \; x \in R; \; l(j, k) = 1; \; j, k = 1, 2, \ldots, m; \; j \neq k.
\]

Then
\[ \det L_m = (x - 1)^{m-1} \cdot (x - m + 1). \]

**Corollary 2.1.** Let \( L_m = L_m(a, b), \ m = 1, 2, 3, \ldots \) be a square matrix of a size \( m \times m \) with entries

\[ l(j, j) = a, \ l(j, k) = b, \ a, b \in \mathbb{R}; \ j, k = 1, 2, \ldots, m; \ j \neq k. \]

Then

\[ \det L_m(a, b) = (a - b)^{m-1} \cdot (a - (m - 1)b). \]

It is no hard to compute by virtue of Corollary 2.1 the determinant of the introduced before matrix \( A \), which will be used further:

\[ \det(A) = n^{-(n-2)/2} \cdot \left(2n^{-1/2} - 3n^{-3/2}\right). \quad (2.3c) \]

Note that this matrix \( A \) is symmetric and positive definite. Further, define a following function as an integral

\[ h(\vec{v}) := \int_{-\infty}^{\infty} f(z, z \vec{v}) \, dz, \quad (2.4) \]

so that

\[ h(\vec{1}) = \int_{-\infty}^{\infty} f_{\vec{z}}(z, \ldots, z) \, dz. \quad (2.4a) \]

**Theorem 2.1.** Suppose that the function \( h(\vec{v}) \) there exists, \( h(\vec{1}) > 0 \), and is continuous at the point \( \vec{v}_0 = \vec{1} \). Then for (positive finite) constant \( K = K(n) : \)

\[ K(n) := 2^{-(n-1)/2} \left(\det A(n-1)\right)^{-1/2} \frac{\pi^{(n-1)/2}}{\Gamma((n+1)/2)} = \]

\[ 2^{-(n-1)/2} \frac{n^{(n-2)/4} \left(2n^{-1/2} - 3n^{-3/2}\right)^{-1/2} \pi^{(n-1)/2}}{\Gamma((n+1)/2)}. \quad (2.5) \]

we have

\[ q_n(\epsilon) \sim K(n) \cdot h(\vec{1}) \cdot \epsilon^{(n-1)/2}. \quad (2.6) \]

**Proof.** Note first of all that the point \( \vec{v}_0 = \vec{1} \) is an unique point of maximum of the smooth function \( v \to g(v) \); and this maximum is equal to \( \sqrt{n} \).

Further, we have as \( \epsilon \to 0^+ \)

\[ q_n(\epsilon) = P \left( \frac{\sum \xi(i)}{\sqrt{\sum \xi^2(i)}} > \sqrt{n} - \epsilon \right) = \]
\[
\int \int \ldots \int_{x(i)/\sqrt{x^2(i)} > \sqrt{n-\epsilon}} f(x(1), x(2), \ldots, x(n)) \, dx(1) \, dx(2), \ldots \, dx(n) = \\
\int \int \ldots \int_{[x(1)+\sum y(j)]/\sqrt{x^2(1)+\sum y^2(j)} > \sqrt{n-\epsilon}} \cdot f(x(1), \bar{y}) \, dx(1) \, dy = \\
\int_{0}^{\infty} \int_{\sum v(j)/\sqrt{1+\sum v^2(j)} > \sqrt{n-\epsilon}} \prod v(j) \cdot f(x(1), x(\bar{v})) \, d\bar{v} = \\
\int_{0}^{\infty} \int_{g(\bar{v}) > \sqrt{n-\epsilon}} \prod v(j) \cdot h(\bar{v}) \, d\bar{v} = \\
\int_{g(\bar{v}) > \max g(\tilde{v}) - \epsilon} \prod v(j) \cdot h(\bar{v}) \, d\bar{v} = \\
\int_{g(\bar{v}) > g(\tilde{v}) - \epsilon} \prod v(j) \cdot h(\bar{v}) \, d\bar{v}. \tag{2.7}
\]

The last integral is localized in some sufficiently small neighborhood of the point of maximum \(\bar{v} = \bar{v}_0 = \tilde{v}\). In detail, notice that as \(\epsilon \to 0^+\) the set \(\{v : g(\bar{v}) > g(\tilde{v}) - \epsilon\}\) is asymptotical equivalent to the ellipsoidal set \(\{v : (A(v-1), (v-1)) < \epsilon\}\), therefore

\[q_n(\epsilon) \sim \int_{\{v : (A(v-1), (v-1)) < \epsilon\}} \prod v(j) \cdot h(\bar{v}) \, d\bar{v},\]

which is in turn asymptotical equivalent to the following integral

\[q_n(\epsilon) \sim h(\tilde{v}) \cdot \int_{\{v : (A(v-1), (v-1)) < \epsilon\}} d\bar{v} = \text{mes}\{v : (A(v-1), (v-1)) < \epsilon\},\]

and we find after simple calculations

\[q_n(\epsilon) \sim h(\tilde{v}) K(n) \cdot \epsilon^{(n-1)/2}, \tag{2.8}\]

Q.E.D.

**Remark 2.1.** If the function \(v \to h(v)\) is not continuous but only integrable in some sufficiently small neighborhood of the point \(\tilde{v}\), or perhaps

\[\lim_{\|\bar{v} - \tilde{v}\| \to 0} \prod v(j) \cdot h(\bar{v}) = 0,\]

then
\[ q_n(\epsilon) \sim \int_{\{v: (A(v-1),(v-1))<\epsilon\}} \prod_j v(j) \cdot h(\vec{v}) \, d\vec{v}, \]

if of course the last integral is finite and non-zero.

Assume for instance that for \( \vec{v} \to \vec{1} \)
\[ \prod_j v(j) \cdot h(\vec{v}) \sim [(A(v-1),(v-1))]^{\gamma/2}, \gamma = \text{const} > 1 - n; \]
then as \( \epsilon \to 0^+ \)
\[ q_n(\epsilon) \sim 2^{-(n-3)/2} \cdot (\det A)^{-1/2} \cdot \frac{\pi^{(n-1)/2}}{\Gamma((n-1)/2)} \cdot \frac{\epsilon^{(n+\gamma-1)/2}}{n+\gamma-1}. \]

Let us return to the promised above case of the left tail of distribution \( P(T(n) < -\sqrt{n+\epsilon}), \epsilon \in (0,1) \), as well as the case of the probability \( P(|T(n)| > \sqrt{n-\epsilon}), \epsilon \in (0,1) \).

**Corollary 2.1.** Suppose that the function \( h(\vec{v}) \) there exists, \( h(-\vec{1}) > 0 \), and is continuous at the point \( \vec{v}^- = -\vec{1} \). Then for at the same positive finite constant \( K = K(n) \) we have
\[ P(T(n) < -\sqrt{n+\epsilon}) \sim K(n) \ h(-\vec{1}) \ \epsilon^{(n-1)/2}. \tag{2.9} \]

**Corollary 2.2.** Suppose that the function \( h(\vec{v}) \) there exists, \( h(\vec{1}) + h(-\vec{1}) > 0 \), and is continuous at both the the points \( \vec{v}_0 = \vec{1} \) and \( \vec{v}^- = -\vec{1} \). Then for at the same positive finite constant \( K = K(n) \) we have as \( \epsilon \to 0^+ \)
\[ P(|T(n)| > \sqrt{n-\epsilon}) \sim K(n) \ [h(\vec{1}) + h(-\vec{1})] \ \epsilon^{(n-1)/2}. \tag{2.10} \]

### 3 Some generalizations: non-standard norming random function.

Let \( \beta = \text{const} > 1 \) and the sequence of r.v. \( \{\xi(i)\}, \ i = 1, 2, \ldots, n; \ n \geq 2 \) is as before. The following statistics was introduced (with applications) at first perhaps by Xiequan Fan [2]:
\[ T_\beta(n) \overset{\text{def}}{=} \frac{\sum_i \xi(i)}{[\sum |\xi(i)|^\beta]^{1/\beta}}. \tag{3.1} \]

Xiequan Fan derived in particular in [2] the following generalization of result belonging to B.Y.Jing, H.Y.Liang and W.Zhou [5] of sub-gaussian type for i, i.d. symmetrical non-degenerate r.v. \( \{\xi(i)\} \).
\[ P(T_\beta(n) > B) \leq \exp\left(-0.5 \cdot B^2 \cdot n^{2/\beta - 1}\right), \quad \beta \in (1, 2]. \]

Since the theoretical attainable maximum of these statistics is following:

\[ \sup_{\xi(i)} T_\beta(n) = n^{1-1/\beta}, \]

it is reasonable to investigate the next tail probability

\[ r_{\beta,n}(\epsilon) = r(\epsilon) := P(T_\beta(n) > n^{1-1/\beta} - \epsilon), \quad \epsilon \to 0+, \epsilon \in (0, 1). \quad (3.2) \]

Define the following modification of the \( g \) function:

\[ g_\beta(\vec{v}) := \left(1 + \sum_j v_j\right) \cdot \left(1 + \sum_j v_j^\beta\right)^{-1/\beta}, \quad \vec{v} \in \mathbb{R}^{n-1} \quad (3.3) \]

which attained its maximal value as before at the point \( v = \vec{v} = \vec{1} \) and herewith

\[ \max_v g_\beta(\vec{v}) = g_\beta(\vec{1}) = n^{1-1/\beta}, \quad (3.4a) \]

\[ -\frac{\partial g_\beta^2}{\partial v_k \partial v_l}(\vec{1}) = -(\beta - 1)n^{-1-1/\beta}, \quad k \neq l; \quad (3.4b) \]

\[ -\frac{\partial g_\beta^2}{\partial v_k^2}(\vec{1}) = (\beta - 1) \left[n^{-1/\beta} - n^{-1-1/\beta}\right]. \quad (3.4c) \]

The correspondent \textit{anti-Hessian} matrix \( A_\beta = A_\beta(n-1) \) at the same extremal point \( \vec{v}_0 = \vec{1} = (1, 1, \ldots, 1) \), \( \dim \vec{v}_0 = (n-1) \) contains the following entries: \( A_\beta = A_\beta(n-1) = \{a_\beta(j,k)\} \), where

\[ a_\beta(j,k) := -\left\{ \frac{\partial^2 g_\beta(v)}{\partial v(j) \partial v(k)} \right\} /\vec{v} = \vec{1}, \quad j,k = 2,3,\ldots,n; \quad (3.5) \]

and we find by direct computations

\[ a_\beta(j,j) = (\beta - 1) \left[n^{-1/\beta} - n^{-1-1/\beta}\right] - \quad (3.6a) \]

diagonal members,

\[ a_\beta(j,k) = -(\beta - 1) \cdot n^{-1-1/\beta}, \quad k \neq j - \quad (3.6b) \]

off diagonal entries.

The correspondent determinant has a form

\[ \det A_\beta(n-1) = (\beta - 1)^{n-1} \cdot n^{-(n-2)/\beta} \cdot \left[2n^{-1/\beta} - 3n^{-1-1/\beta}\right]. \quad (3.7) \]

Of course, the last expressions (3.5)-(3.7) coincides with ones when \( \beta = 2 \) in the second section.

We deduce similar to the second section
Theorem 3.1. Suppose as before that the function $h(\vec{v})$ there exists, $h(\vec{1}) > 0$, and is continuous at the point $\vec{v}_0 = \vec{1}$. Then for (positive finite) constant $K_\beta = K_\beta(n)$:

$$
K_\beta(n) = 2^{-(n-1)/2} \left( \det A_\beta(n-1) \right)^{-1/2} \frac{\pi^{(n-1)/2}}{\Gamma((n+1)/2)} = 
$$

$$
2^{-(n-1)/2} \cdot n^{(n-2)/(2\beta)} \cdot \left[ 2n^{1-1/\beta} - 3n^{-1-1/\beta} \right]^{-1/2} \cdot \frac{\pi^{(n-1)/2}}{\Gamma((n+1)/2)},
$$

we have as $\epsilon \to 0+$

$$
r_{\beta,n}(\epsilon) \sim K_\beta(n) \ h(\vec{1}) \ \epsilon^{(n-1)/2}.
$$

Remark 3.1. It follows immediately from Stirling’s formula that as $n \to \infty$

$$
\log_n K_\beta(n) \sim n \cdot \frac{1-\beta}{2\beta}.
$$

Thus, the sequence $K_\beta(n)$ tends as $n \to \infty$ very rapidly to zero. Recall that we consider the case when $\beta > 1$.

4 Tail of maximum distribution estimates.

Define following Xiequan Fan [2] the tails of maximum distributions

$$
R_n(\epsilon) := P \left( \max_{k=2,3,...,n} \frac{S(k)}{Z(n)} > \sqrt{n} - \epsilon \right),
$$

where

$$
S(k) = \sum_{i=1}^{k} \xi(i), \ Z(n) = \sqrt{\sum_{j} \xi^2(j)},
$$

and

$$
\epsilon \in \left( 0, [2\sqrt{n-1}]^{-1} \right).
$$

We aim to investigate as before the asymptotic behavior as $\epsilon \to 0+$ of this tail probability.

Theorem 4.1. We propose under at the same conditions as in theorem 2.1 that as $\epsilon \to 0+$ and $\epsilon \in \left( 0, [2\sqrt{n-1}]^{-1} \right)$.

$$
R_n(\epsilon) \sim Q_n(\epsilon).
$$
Proof. The lower bound is trivial:

\[ R_n(\varepsilon) := \mathbb{P} \left( \max_{k=2,3,\ldots,n} \frac{S(k)}{Z(n)} > \sqrt{n - \varepsilon} \right) \geq \mathbb{P} \left( \frac{S(n)}{Z(n)} > \sqrt{n - \varepsilon} \right) = Q_n(\varepsilon). \]

It remains to ground the inverse inequality. One can suppose without loss of generality \( \xi(i) > 0 \). We have:

\[ R_n(\varepsilon) \leq \sum_{k=2}^{n} R_{n,k}(\varepsilon), \quad (4.4) \]

where

\[ R_{n,k}(\varepsilon) = \mathbb{P} \left( \frac{S(k)}{Z(n)} > \sqrt{n - \varepsilon} \right). \]

Let now \( 2 \leq k \leq n - 2 \); then

\[ \frac{S(k)}{Z(n)} \leq \frac{S(k)}{Z(k)} \leq \sqrt{k} < \sqrt{n - \varepsilon}, \]

therefore

\[ R_{n,k}(\varepsilon) = 0, \]

if \( \varepsilon \) satisfies the restriction (4.2). Thus,

\[ R_n(\varepsilon) = R_{n,n}(\varepsilon) = Q_n(\varepsilon), \]

Q.E.D.

Note in addition that we have proved in fact that

\[ \overline{R}_n(\varepsilon) := \mathbb{P} \left( \max_{k=2,3,\ldots,n} \frac{S(k)}{Z(k)} > \sqrt{n - \varepsilon} \right) \sim Q_n(\varepsilon), \varepsilon \to 0^+. \quad (4.5) \]

5 Non-asymptotical estimates.

Introduce the following important functional

\[ \lambda = \lambda(g) := \inf_{\vec{v} : ||\vec{v} - \vec{1}|| \leq 1} \left[ \frac{g(\vec{1}) - g(\vec{v})}{||\vec{v} - \vec{1}||^2} \right], \quad (5.1) \]

then \( \lambda(g) \in (0, \infty) \), and as ordinary \( ||\vec{v}||^2 = ||v||^2 = \sum_{j=2}^{n} v^2(j) \), so that

\[ g(\vec{1}) - g(\vec{v}) \geq \lambda(g) \cdot ||\vec{v} - \vec{1}||^2, \quad ||\vec{v} - \vec{1}|| \leq 1. \quad (5.2) \]

Further, let \( \varepsilon \in (0, \lambda(g)) \) and denote also
\[ H_n(\lambda) := \sup_{||\vec{v}-\vec{1}||^2 \leq \lambda} \left[ \prod_j |v(j)| \cdot h(\vec{v}) \right]; \]  

Then

\[ Q_n(\epsilon) \leq H_n(\lambda) \cdot \int_{||\vec{v}-\vec{1}||^2 \leq \lambda} d\vec{v} = \]

\[ H_n(\lambda) \cdot \frac{\pi^{(n-1)/2}}{\Gamma((n+1)/2)} \cdot \left( \frac{\epsilon}{\lambda(g)} \right)^{(n-1)/2}. \]  

The lower bound for this probability may be obtained quite analogously. Denote

\[ \mu = \mu(g) \ := \sup_{\vec{v}, ||\vec{v}-\vec{1}|| \leq 1} \left[ \frac{g(\vec{1}) - g(\vec{v})}{||\vec{v}-\vec{1}||^2} \right], \]

then \( \mu(g) \in (0, \infty) \), and

\[ g(\vec{1}) - g(\vec{v}) \leq \mu(g) \cdot ||\vec{v} - \vec{1}|| \leq 1. \]

Let \( \epsilon \in (0, \mu(g)) \) and set also

\[ G_n(\mu) := \inf_{||\vec{v}-\vec{1}||^2 \leq \epsilon/\mu} \left[ \prod_j |v(j)| \cdot h(\vec{v}) \right]; \]

then

\[ Q_n(\epsilon) \geq G_n(\mu) \cdot \int_{||\vec{v}-\vec{1}||^2 \leq \epsilon/\mu} d\vec{v} = \]

\[ G_n(\mu) \cdot \frac{\pi^{(n-1)/2}}{\Gamma((n+1)/2)} \cdot \left( \frac{\epsilon}{\mu(g)} \right)^{(n-1)/2}. \]

6 Examples. Concluding remarks.

A. An example. It is easily to verify that all the conditions of our theorem 2.1 are satisfied for example for arbitrary non-degenerate Normal (Gaussian) multivariate distribution, as well as in the case when the random variables \( \xi(i) \) are independent and have non-zero continuous density of distribution.

B. Possible generalizations. The offered here method may be easily generalized by our opinion on the asymptotic computation for the distribution of a form

\[ P \left( U(\tilde{\xi}) > \max_{\tilde{x}} U(\tilde{x}) - \epsilon \right), \ \epsilon \to 0+, \]
as well as for computation of integrals of a form

\[ I(\epsilon) = \int_{y: U(y) > \max_x U(x) - \epsilon} Z(y) \mu(dy) \]

etc.

C. **Counterexamples.** Let us prove that the condition about the existence of density function is essential for our conclusions.

**Example 6.1.** Let the r.v. \( \xi(j), j = 2, 3, \ldots, n - 1 \) be arbitrary non-degenerate, say, independent and have the standard Gaussian distribution, and put \( P(\xi(1) = 0) = 1 \). Then both the r.v. \( \sum_j \xi(j), \sum_j \xi^2(j) \) have infinite differentiable bounded densities, but

\[ \sup_{\xi(i)} T(n) = \sqrt{n} - 1, \]

therefore for sufficiently small positive values \( \epsilon \)

\[ Q_n(\epsilon) = P(T(n) > \sqrt{n} - \epsilon) = 0, \]

in contradiction to the propositions of theorem 2.1 and 3.1.

**Example 6.2.** Let now the r.v. \( \xi(i), i = 1, 2, 3, \ldots, n \) be the Rademacher sequence, i.e. the sequence of independent r.v. with distribution

\[ P(\xi(i) = 1) = P(\xi(i) = -1) = 1/2. \]

Then

\[ P(T(n) = \sqrt{n}) = 2^{-n} = P(T(n) > \sqrt{n} - \epsilon), 0 < \epsilon < (2\sqrt{n})^{-1}, \]

in contradiction to the propositions of theorem 2.1 and 3.1.

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