ERGODIC MAXIMIZATION PROBLEM FOR EXPANDING MAPS WITH DIFFERENTIABLE OBSERVABLES

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ABSTRACT. We show that for an expanding map, the maximizing measures of a generic (open and dense) $C^r$ ($r \in \mathbb{N}$) differentiable functions are supported on a single periodic orbit.

1. INTRODUCTION

Ergodic theory relates closely to the iteration of a measure preserving transformation $T$ on a metric space $(X, d)$ with the probability structure $(X, \mathcal{B}, \mu)$. If $\mu(T^{-1}(B)) = \mu(B)$ for any $B \in \mathcal{B}$, then $\mu$ is called an invariant measure. An observable is a continuous function $f : X \to \mathbb{R}$. The time average of the real-valued function $f$ along the orbit is $\lim_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i(x))$ if the limit exists. The space average of the real-valued function $f$ with respect to an invariant measure $\mu$ is $\int_X f \, d\mu$. For an observable $f$, the maximizing orbit for $f$ is the orbit giving the maximum time average of $f$, and the maximizing invariant probability measure for $f$ is the measure giving the maximum space average of $f$. If the invariant measure is ergodic, the classical Birkhoff ergodic theorem tells us that the time average of the observable is equal to the space average of the observable for almost every point in the view point of the ergodic measure. Ergodic optimization is the study on the problems of maximizing orbits and invariant measures, and can be applied in the control of chaos [28, 25], the Aubry-Mather theory in Lagrangian mechanics [10, 23], and the ground state theory in thermodynamics formalism and multifractal analysis [2], and so on.

A core problem in the field of ergodic optimization theory is the Typical Periodic Optimization (TPO) Conjecture for the uniformly expanding maps and the uniformly hyperbolic system, which was proposed by Yuan and Hunt [30]. For the real-valued observables taken from some kind of smooth (Hölder continuous, Lipschitz continuous, differentiable) function spaces, denoted by $V$ the set of observables such that the maximizing measures for each element of $V$ contains a periodic measure, the TPO Conjecture claims that $V$ is an open and dense subset of the function space.

There are many results towards TPO Conjecture. A series of work on the subshift of finite type are obtained: Coelho’s work on the zero temperature limits of equilibrium states and the study on cohomology equation [9], the Walter observable functions by Bousch [5], a fullshift with the space of “super continuous” functions [26], a one-side shift on two symbols with a space of functions with strong modulus of regularity [3], a subshift of finite type with Hölder continuous functions and zero entropy [24].

Key words and phrases. Entropy; ergodic maximization; invariant measure; observable; periodic orbit; shadowing.
On the other hand, there are a lot of important results on TPO Conjecture for maps. Conze and Guivarc’h [14] improved Coelho’s result by showing the existence of a continuous function \( \varphi \) such that \( \tilde{f} := f + \varphi - \varphi \circ T \leq \sup_{\mu \in \mathcal{M}(T)} \int f \, d\mu \), where \( \mathcal{M}(T) \) is the set of invariant measures, the maximizing measures are those invariant probability measures whose support lies in the set of global maxima of the function \( \tilde{f} \). And they also investigated a map \( T(x) = 2x \, (\text{mod} 1) \) with observable \( f_\theta(x) = \cos(2\pi(x - \theta)) \). Later, this model was completely solved based on the work of Hunt and Ott [17], Jenkinson via Sturmian measures [18, 19], and Bousch by studying the function \( \tilde{f} \) [4]. Contreras, Lopes and Thieullen considered a smooth orientation-preserving uniformly expanding map of the circle with the Hölder functions space [12], using techniques inspired by Mañé [22, 23], who had established a similar characterization of \( \tilde{f} \) in the context of Lagrangian systems. In [11], Contreras verified the TPO Conjecture for the expanding maps with Lipschitz observables. Recently, the TPO Conjecture for the hyperbolic maps with \( C^1 \) observables [15], and the Axiom A flows with \( C^1 \) observables [16] were solved. For a survey of recent development in this problem, please refer to [21].

In this paper, we adopt the arguments in [11] to verify the TPO Conjecture for an expanding map with a generic (open and dense) \( C^r \) \((r \in \mathbb{N})\) differentiable functions. The main idea is the combination of the entropy argument in [11] and the smooth perturbation instead of the Lipschitz perturbation used in [11].

**Theorem 1.** Let \( X \) be a compact metric space and \( T : X \to X \) be an expanding map, then there is an open and dense set \( O \subset C^r(X, \mathbb{R}) \) \((r \in \mathbb{N})\) such that for all \( F \in O \), there exists a single \( F \)-maximizing measure and it is supported on a periodic orbit.

The rest is organized as follows. In Section 2, some basic concepts and useful results are introduced. In Section 3, a useful shadowing property is obtained for differentiable functions. In Section 4, the entropy argument is used to verify the main result (Theorem 1). In Section 5, a result of Morris is generalized for expanding map with differentiable functions (Theorem 19), which is used in the entropy argument in Section 4.

## 2. Preliminary

In this section, some basic definitions and useful results are introduced.

**Definition 2.** [11] Let \((X, d)\) be a compact metric space. A map \( T : X \to X \) is called expanding, if \( T \) is Lipschitz continuous and there are constant numbers \( \lambda \in (0, 1) \) and \( N_0 \in \mathbb{N} \) such that for every point \( x \in X \), there are a neighborhood \( U_x \subset X \) of \( x \) and continuous inverse branches \( S_i, i = 1, \ldots, l_x, l_x \leq N_0 \), of \( T \) with disjoint images \( S_i(U_x) \), such that \( T^{-1}(U_x) = \bigcup_{i=1}^{l_x} S_i(U_x) \), \( T \circ S_i = I_{U_x} \) (the identity map restricted to \( U_x \)), and

\[
  d(S_i(y), S_i(z)) \leq \lambda d(y, z) \quad \forall y, z \in U_x.
\]

For \( x \in X \) and \( r \in \mathbb{R}^+ \), set

\[
  B(x, r) := \{ w \in X : d(x, w) < r \}.
\]
Remark 3. By the compactness of $X$, there is a finite subcover of $\{U_x\}_{x \in X}$ in the above definition. So, there exists a constant $e_0 > 0$ such that for every $x \in X$, there is some $U_y$ so that the ball $B(x, e_0) \subset U_y$.

Consider a continuous map $T : X \to X$, the set of invariant measures $\mu \in \mathcal{M}(T)$ with respect to $T$ is given by

$$\mathcal{M}(T) = \{ \mu : \mu(T^{-1}(B)) = \mu(B) \text{ for any Borel subset } B \subset X \}. $$

The following norm is used on the space of Lipschitz functions on $X$, denoted by $\text{Lip}(X, \mathbb{R})$,

$$\|f\|_{\text{Lip}} = \sup_{x \in X} |f(x)| + \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)};$$

the following norm is used on the space of differentiable functions, denoted by $C^r(X, \mathbb{R})$,

$$\|f\|_{C^r} = \sup_{x \in X} |f(x)| + \sum_{1 \leq i \leq r} \sup_{x \in X} |f^{(i)}(x)|,$$

where $f^{(i)}$ is the $i$-th derivative of $f$; if $f : X \to \mathbb{R}$ is continuous, then

$$\|f\|_0 = \sup_{x \in X} |f(x)|.$$

For an observable $f : X \to \mathbb{R}$ with $f \in C^r(X, \mathbb{R})$, the ergodic optimization is the study of the following problem:

$$\sup_{\mu \in \mathcal{M}(T)} \int_M f \, d\mu.$$

Remark 4. It is evident that $C^r(X, \mathbb{R}) \subset \text{Lip}(X, \mathbb{R})$. This fact will be used in the following discussions.

**Definition 5.** [11] Given $F \in C^r(X, \mathbb{R}) \subset \text{Lip}(X, \mathbb{R})$, the Lax operator for $F$ is $

\mathcal{L}_F : \text{Lip}(X, \mathbb{R}) \to \text{Lip}(X, \mathbb{R}) : \mathcal{L}_F(u)(x) := \max_{y \in T^{-1}(x)} \{ \alpha + F(y) + u(y) \},

where

$$\alpha = \alpha(F) = -\max_{\mu \in \mathcal{M}(T)} \int F \, d\mu.$$

The set of maximizing measures for an observable function $F$ is

$$\mathcal{M}_{\text{max}}(F) = \{ \mu \in \mathcal{M}(T) : \int F \, d\mu = -\alpha(F) \}.$$

A calibrated sub-action for $F$ is a fixed point of the Lax operator $\mathcal{L}_F$.

**Lemma 6.** [11, Lemma 2.1]

1. For $u \in \text{Lip}(X, \mathbb{R})$, the Lipschitz constants satisfy

$$\text{Lip}(\mathcal{L}_F(u)) \leq \lambda(\text{Lip}(F) + \text{Lip}(u)).$$

In particular, $\mathcal{L}_F(\text{Lip}(X, \mathbb{R})) \subset \text{Lip}(X, \mathbb{R})$.

2. If $\mathcal{L}_F(u) = u$, set

$$\overline{F} := F + u - u \circ T + \alpha(F),$$

then, we have
Lemma 11. For a calibrated sub-action, the action of every maximizing measure has support on the periodic orbit of $x$. Hence, Proposition 7. [11, Proposition 2.2] There exists a Lipschitz calibrated sub-action.

Remark 8. The Lax operator has an invariant subspace

$$\left\{ u \in \text{Lip}(X, \mathbb{R}) : \text{Lip}(u) \leq \frac{\lambda \text{Lip}(F)}{1 - \lambda} \right\}. \quad (3)$$

Definition 9. [11] For a calibrated sub-action $u$, every point $z \in X$ has a calibrating pre-orbit $\{z_k\}_{k \leq 0}$ such that $T(z_{-k}) = z_{-k+1}, T(z_i) = z_0 = z,$ and

$$u(z_{k+1}) = u(z_k) + \alpha + F(z_k) \forall k \leq -1,$$

or

$$\bar{F}(z_k) = 0 \forall k \leq -1.$$

Hence,

$$u(z_0) = u(z_{-k}) + k\alpha + \sum_{i=k}^{-1} F(z_i) \forall k \leq -1.$$ 

Definition 10. [11] Given $\delta > 0$, a sequence $\{x_n\}_{n \in \mathbb{N}} \subset X$ is said to be a $\delta$-pseudo-orbit if $d(x_{n+1}, T(x_n)) \leq \delta$ for any $n \in \mathbb{N}$.

Given $\epsilon > 0$, we say that the orbit of a point $y \in X$ $\epsilon$-shadows a pseudo-orbit $\{x_n\}_{n \in \mathbb{N}}$ if $d(T^n(y), x_n) < \epsilon$ for any $n \in \mathbb{N}$.

Lemma 11. [11, Lemma 2.3] If there exists a periodic orbit $O(y)$ such that for any calibrated sub-action, the $\alpha$-limit of every calibrating pre-orbit is $O(y)$, then every maximizing measure has support on $O(y)$.

Proposition 12. [11, Proposition 2.4] (Shadowing Lemma) For a $\delta$-pseudo-orbit $\{x_k\}_{k \in \mathbb{N}}$ with $\delta < (1 - \lambda)e_0$, there is a point $y \in X$ $\epsilon$-shadowing $\{x_k\}_{k \in \mathbb{N}}$ with $\epsilon = \frac{\delta}{1 - \lambda}$. Moreover, if $\{x_k\}_{k \in \mathbb{N}}$ is a periodic pseudo-orbit, then $y$ is a periodic orbit with the same period.

Corollary 13. If $T^r(y) = y$ and $\{z_k\}_{k \leq 0}$ is a pre-orbit which $(1 - \lambda)e_0$-shadows the orbit $O(y)$ of $y$, that is, for any $k \leq 0$, $T(z_k) = z_{k+1}$ and $d(z_k, T^k \text{ mod } r(y)) < (1 - \lambda)e_0$, then the $\alpha$-limit of $\{z_k\}_{k \leq 0}$ is $O(y)$.

3. Shadowing property

In this section, a useful shadowing property is derived for differentiable functions.

Let $y \in \text{Per}(T) = \cup_{p \in \mathbb{N}} \text{Fix}(T^p)$ be a periodic point for $T$, $P_y$ be the set of differentiable functions $F \in C^r(X, \mathbb{R})$ so that there is a unique $F$-maximizing measure and it is supported on the periodic orbit of $y$, and $U_y$ be the interior of $P_y$ in $C^r(X, \mathbb{R})$. 
Proposition 14. Let $F, u \in \text{Lip}(X, \mathbb{R})$ with $\mathcal{L}_F(u) = u$ and $\overline{F} = F + \alpha(F) + u - u \circ T$, where $\alpha(F) = -\max_{\mu \in \mathcal{M}(T)} \int F d\mu$.

Suppose that there is $M \in \mathbb{N}^+$ such that for every $Q > 1$ and $\delta_0 > 0$, there exist $0 < \delta < \delta_0$ and a $p(\delta)$-pseudo-orbit $\{x_k^\delta\}_{0 \leq k \leq p(\delta) - 1}$ in $[\overline{F} = 0]$ with at most $M$ jumps such that $\frac{2\delta}{\delta_0} \geq Q$, where $\gamma_\delta := \min_{0 \leq i < j < p(\delta)} d(x_i^\delta, x_j^\delta)$. Then $F$ is in the closure of $\bigcup_{y \in Per(T)} U_y$.

Definition 15. [11] We say that $n_i$, $i = 1, \ldots, l$, $l \leq M$, are the jumps of $\{x_k^\delta\}_{0 \leq k \leq p(\delta) - 1}$, if $d(T(x_k), x_{k+1}) = 0$ for $k \in \{0, 1, \ldots, p(\delta) - 1\} \setminus \{n_1, \ldots, n_l\}$.

Given a positive constant $\epsilon(\approx \sqrt{\delta})$, consider the following constants:

$$
K := \max \left\{ \frac{M \text{Lip}(F)}{(1 - \lambda)^2}, \frac{\text{Lip}(F) + 3}{1 - \lambda} \right\},
$$

$$
\rho := \frac{3K\delta}{\epsilon},
$$

$$
\gamma_2 := \gamma_\delta - \frac{2\delta}{1 - \lambda},
$$

$$
\gamma_3 := \frac{\gamma_2}{\text{Lip}(T)} - \lambda \rho,
$$

$$
\Gamma_1 := \frac{\rho}{12p},
$$

$$
\Gamma_2 := \frac{K\delta}{4p}.
$$

Assume $\delta$ are small enough such that the above constants are all positive, $\rho \ll \epsilon_0$, $\gamma_3 \gg \delta$, $\Gamma_1 < 1$, $\Gamma_2 < 1$, and set

$$
-a := K\delta + K\rho + 2\rho \Gamma_1 + 2\rho \Gamma_2 - \epsilon \gamma_3 < 2K\delta + K\rho - \epsilon \gamma_3 < 0
$$

and

$$
-b := -\epsilon \rho + \frac{K\delta}{\rho} + 2\epsilon \Gamma_1 + 2\Gamma_2 < -3K\delta + K\delta + \frac{K\delta}{2} + \frac{K\delta}{2} = -K\delta < 0.
$$

Suppose $y$ is a periodic point with period $p$, which $\delta \frac{\delta}{1-\lambda}$ shadows the pseudo-orbit $\{x_k\}$. Set

$$
\mathcal{O}(y) := \{T^i(y) : i = 0, 1, \ldots, p - 1\} = \{y_0, y_1, \ldots, y_{p-1}\},
$$

and for any continuous function $G : X \rightarrow \mathbb{R}$, denote by

$$
\langle G \rangle(y) := \frac{1}{p} \sum_{i=0}^{p-1} G(T^i(y))
$$

the average of the function $G$ along the periodic orbit.

Lemma 16. Assume that $d(z, y_k) \leq \rho \ll \epsilon_0$. Choose $w_1 \in T^{-1}(z)$ with $d(w_1, y_{k-1}) < \lambda \rho$. If $w_2 \in T^{-1}(z) \setminus \{w_1\}$, then

$$
d(w_2, \mathcal{O}(y)) \geq \gamma_3 = \frac{\gamma_2}{\text{Lip}(T)} - \lambda \rho \gg \delta.
$$

Proof. This is a claim verified in the proof of [11, Proposition 2.6].
Proof. In the concept of $\overline{F}$ in (2), the calibrated sub-action $u$ is Lipschitz continuous, but it might not be differentiable. By 2. (iii) of Lemma 6, the maximizing measures for $F$ and $\overline{F}$ are the same.

By Proposition 12, we have

$$\left| \sum_{n_i=1+1}^{n_i} \overline{F}(y_k) - \sum_{n_i=1+1}^{n_i} \overline{F}(x_k) \right| \leq \sum_{n_i=1+1}^{n_i} \text{Lip}(\overline{F})d(y_k, x_k)$$

$$\leq \text{Lip}(\overline{F}) \sum_{l=1}^{n_i-n_i-1} \lambda^{k-1} \frac{\delta}{1 - \lambda} \leq \frac{\text{Lip}(\overline{F})}{(1 - \lambda)^2} \delta.$$ 

This, together with the assumptions $\overline{F}(x_k) = 0$, implies that $\sum_{0}^{p-1} \overline{F}(x_k) = 0$, and

$$\sum_{k=0}^{p-1} \overline{F}(y_k) \geq -\frac{M \text{Lip}(\overline{F})}{(1 - \lambda)^2} \delta \geq -K \delta,$$

or

$$\langle \overline{F} \rangle(y) \geq -\frac{K \delta}{p}. \tag{7}$$

We make two perturbations to $\overline{F}$. It follows from [1] that there is a $C^\infty$ function $g$ such that

$$\|g - d(\cdot, O(y))\|_0 < \Gamma_1 \text{ and } \text{Lip}(g) < \text{Lip}(d(\cdot, O(y))) + 1 \leq 2, \tag{8}$$

where the distance function satisfies

$$\|d(\cdot, O(y))\|_0 \leq \text{diam}X \text{ and } \text{Lip}(d(\cdot, O(y))) \leq 1. \tag{9}$$

The first perturbation is $-\epsilon g(x)$. The second one is a perturbation by any function $h \in C^\infty$ with

$$\|h\|_0 < \Gamma_2 \text{ and Lip}(h) < 1. \tag{10}$$

These perturbations depend on $O(y)$ and the period $p$. We will show that the function $G_1 = \overline{F} - \epsilon g + h$ has a unique maximizing measure supported on the periodic orbit $O(y)$, where such function $G_1$ contains an open ball centered at $\overline{F} - \epsilon g$. Note that $\overline{F} = F + \alpha(F) + u - u \circ T$.

Denote by

$$G := \overline{F} - \epsilon g + h + \beta = G_1 + \beta, \tag{11}$$

where

$$\beta = -\sup_{\mu\in\mathcal{M}(T)} \int (\overline{F} - \epsilon g + h)d\mu. \tag{12}$$

It is evident that $G$ and $G_1$ have the same maximizing measures.

By (7), (8), and (10), one has

$$\beta \leq -\langle \overline{F} - \epsilon g + h \rangle(y) = -\langle \overline{F} - \epsilon d(\cdot, O(y)) + \epsilon d(\cdot, O(y)) - \epsilon g + h \rangle(y)$$

$$= -\langle \overline{F} \rangle(y) + \epsilon \Gamma_1 + \|h\|_0$$

$$\leq \frac{K \delta}{p} + \epsilon \Gamma_1 + \Gamma_2. \tag{13}$$

Let $v$ be a calibrated sub-action for $G$, that is, $\mathcal{L}_G(v) = v$. Given any $z \in X$, let $\{z_k\}_{k \leq 0}$ be a pre-orbit of $z$ calibrating $v$. Denote by $0 > t_1 > t_2 > \cdots$ the times on which $d(z_k, O(y)) > \rho$. 

If \( t_{n+1} < t_n - 1 \), there is \( s_n \in \mathbb{Z} \) such that the orbit segment \( \{z_k\}_{k=t_{n+1}+1}^{t_n-1} \) \( \rho \)-shadows \( \{y_{-i+s_n}\}_{i=t_n-t_{n+1}-1}^{1} \), then one has
\[
d(z_{-i+t_n}, y_{-i+s_n}) \leq \lambda^{i-1} \rho, \ \forall n \in \mathbb{N}, \ \forall i = 1, ..., t_n - t_{n+1} - 1.
\]

By Lemma 16, for \( t_{n+1} < t_n - 1 \), we have
\[
d(z_{t_{n+1}}, \mathcal{O}(y)) \geq \gamma_3. \tag{14}
\]

Since both terms in \( \overline{F} \) and \( -d(\cdot, \mathcal{O}(y)) \) are non-positive, it follows from (10) and (13) that
\[
G = \overline{F} - \epsilon d(\cdot, \mathcal{O}(y)) + \rho d(\cdot, \mathcal{O}(y)) - \epsilon g + h + \beta \\
\leq \epsilon \|d(\cdot, \mathcal{O}(y)) - g\|_0 + \|h\|_0 + \beta \leq \epsilon \Gamma_1 + \|h\|_0 + \beta \\
\leq K \frac{\delta}{p} + 2\epsilon \Gamma_1 + 2\Gamma_2. \tag{15}
\]

On a shadowing segment, by (4), (8), (10), (11), we have
\[
\left| \sum_{t_{n+1}+1}^{t_n-1} G(z_k) - \sum_{s_n-t_{n+1}+1}^{s_n-1} G(y_k) \right| \leq \text{Lip}(G) \sum_{i=0}^{\infty} \lambda^i \rho \leq \text{Lip}(G) \frac{\rho}{1 - \lambda} \leq K \rho. \tag{16}
\]

Let
\[
t_n - t_{n+1} - 1 = mp + r \text{ with } 0 \leq r < p,
\]
and separate the shadowing segment in \( m \) loops along the orbit \( \mathcal{O}(y) \).

It follows from the definition of \( \beta \) and \( y \) is a periodic orbit with period \( p \) that \( \langle G \rangle(y) \leq 0 \). Hence, from (15) and (16), it follows that
\[
\sum_{t_{n+1}+1}^{t_n-1} G(z_k) \leq \sum_{s_n-t_{n+1}+1}^{s_n-1} G(y_k) + \text{Lip}(G) \frac{\rho}{1 - \lambda} \\
\leq mp \langle G \rangle(y) + (p - 1) \left( K \frac{\delta}{p} + 2\epsilon \Gamma_1 + 2\Gamma_2 \right) + \text{Lip}(G) \frac{\rho}{1 - \lambda} \\
\leq (p - 1) \left( K \frac{\delta}{p} + 2\epsilon \Gamma_1 + 2\Gamma_2 \right) + K \rho. \tag{17}
\]

Since \( d(z_{t_{n}}, \mathcal{O}(y)) > \rho \), using (13), (8), and (10), we have
\[
G(z_{t_{n}}) \leq \overline{F}(z_{t_{n}}) - \epsilon d(z_{t_{n}}, \mathcal{O}(y)) + \epsilon d(z_{t_{n}}, \mathcal{O}(y)) - \epsilon g(z_{t_{n}}) + h(z_{t_{n}}) + \beta \\
\leq \overline{F}(z_{t_{n}}) - \epsilon \rho + \epsilon d(z_{t_{n}}, \mathcal{O}(y)) - g(z_{t_{n}}) + h(z_{t_{n}}) + \beta \\
\leq 0 - \epsilon \rho + \epsilon \Gamma_1 + \Gamma_2 + \frac{K \delta}{p} + \epsilon \Gamma_1 + \Gamma_2 = -\epsilon \rho + \frac{K \delta}{p} + 2\epsilon \Gamma_1 + 2\Gamma_2 < -b < 0. \tag{18}
\]

Similarly, for \( t_{n+1} < t_n - 1 \), by (14), one has
\[
G(z_{t_{n+1}}) \leq \overline{F}(z_{t_{n+1}}) - \epsilon d(z_{t_{n+1}}, \mathcal{O}(y)) + \epsilon d(z_{t_{n+1}}, \mathcal{O}(y)) - \epsilon g(z_{t_{n+1}}) + h(z_{t_{n+1}}) + \beta \\
\leq \overline{F}(z_{t_{n+1}}) - \epsilon \rho + \epsilon d(z_{t_{n+1}}, \mathcal{O}(y)) - g(z_{t_{n+1}}) + h(z_{t_{n+1}}) + \beta \\
\leq 0 - \epsilon \gamma_3 + \epsilon \Gamma_1 + \Gamma_2 + \frac{K \delta}{p} + \epsilon \Gamma_1 + \Gamma_2 = -\epsilon \gamma_3 + \frac{K \delta}{p} + 2\epsilon \Gamma_1 + 2\Gamma_2. \tag{19}
\]
For \( t_{n+1} < t_n - 1 \), combining (17) and (19), one has

\[
\sum_{t_n+1}^{t_{n-1}} G(z_k) \leq (p-1) \left( \frac{K\delta}{p} + 2\epsilon \Gamma_1 + 2\Gamma_2 \right) + K\rho - \epsilon \gamma_3 + \frac{K\delta}{p} + 2\epsilon \Gamma_1 + 2\Gamma_2
\]

\[= K\delta + K\rho + 2p\epsilon \Gamma_1 + 2p\Gamma_2 - \epsilon \gamma_3 < -a < 0.\] (20)

By (12), \( \alpha(G) = 0 \). Since \( \{z_k\}_{k \leq 0} \) is a calibrating pre-orbit for \( v \), we have

\[v(z) = v(z_k) + \sum_{i=k+1}^{-1} G(z_i) \forall k < 0.\]

Since \( v \) is finite, we have

\[\sum_{-\infty}^{-1} G(z_k) \geq -2\|v\|_0 > -\infty.\]

By (18) and (20), the sequence \( t_n \) is finite. Note that \( \rho < (1 - \lambda)\epsilon_0 \), it follows from Corollary 13 that any calibrating pre-orbit has \( \alpha \)-limit \( O(y) \). This, together with Lemma 11, yields that every maximizing measure for \( G \) has support on \( O(y) \). □

4. Entropy Argument

In this section, the entropy argument is used to verify the main result, Theorem 1, that is, the set \( O = \bigcup_{y \in \text{Per}(T)} U_y \) is open and dense.

The difficult part is the proof of the denseness of this set. We will prove this by contradiction.

Suppose that there is a non-empty open set

\[W \subset C^r(X, \mathbb{R})\]

which is disjoint from \( \bigcup_{y \in \text{Per}(T)} U_y \). It follows from Theorem 19 and Remark 20 that there is \( F \in W \) such that it has an ergodic maximizing measure \( \mu \) with zero measure entropy

\[h_\mu(T) = 0.\] (21)

By 2. (iii) of Lemma 6, \( \text{supp}(\mu) \subset [F = 0] \) for any calibrating subaction \( u \) for \( F \), where \( F \) is specified in (2). Take \( q \in \text{supp}(\mu) \subset [F] \) satisfying

\[\int f d\mu = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} f(T^i(q)),\]

where \( f : X \to \mathbb{R} \) is a continuous function, and \( q \) is called a generic point for \( \mu \).

By the assumption that \( F \) is not in the closure of \( \bigcup_{y \in \text{Per}(T)} U_y \) and Proposition 14 with \( M = 2 \), one has

**Claim** There is \( Q > 1 \) and \( \delta_0 > 0 \) such that if \( 0 < \delta < \delta_0 \) and \( \{x_k\}_{k \geq 0} \subset O(q) \) is a \( p(\delta) \)-periodic \( \delta \)-pseudo-orbit with at most 2 jumps made with elements of the positive orbit of \( q \), then

\[\gamma = \min_{1 \leq i < j < p} d(x_i, x_j) < \frac{1}{2} Q\delta.\]

Take \( N_0 \) satisfying that

\[2Q^{-N_0} < \delta_0.\]
Fix a point \( w \in \text{supp}(\mu) \) satisfying Brin-Katok Theorem [8], that is,

\[
h_\mu(T) = - \lim_{L \to \infty} \frac{1}{L} \log \mu(V(w, L, \epsilon)), \tag{22}
\]

where

\[
V(w, L, \epsilon) = \{ x \in X : d(T^k x, T^k w) < \epsilon, \forall k = 0, \ldots, L \}
\]
is the dynamical ball [6]. Since \( T \) is expanding, we have

\[
V(w, L, \epsilon) = S_1 \circ \cdots \circ S_L(B(T^L(w), \epsilon)),
\]

where \( S_k \) is one branch of the inverse of \( T \) satisfying that

\[
S_k(T^k(w)) = T^k(\epsilon) - 1(w).
\]

So,

\[
V(w, L, \epsilon) \subset B(w, \lambda L \epsilon). \tag{23}
\]

Given \( N > N_0 \), let \( 0 \leq t^N_1 < t^N_2 < \cdots \) be all the \( \frac{1}{2}Q^{-N} \) returns to \( w \), that is,

\[
\{t^N_1, t^N_2, \ldots\} = \{ n \in \mathbb{N} : d(T^n(q), w) \leq \frac{1}{2}Q^{-N} \}.
\]

Proposition 17. [11, Proposition 3.2] For any \( l \geq 0 \), one has

\[
t^N_{i+1} - t^N_i \geq \sqrt{2}^{N-N_0-1}.
\]

Choose \( N \gg N_0 \) and a continuous function \( f_N : X \to \mathbb{R} \) satisfying \( 0 \leq f_N \leq 1 \), \( f_N|_B(w, \frac{1}{2}Q^{-N-1}) \equiv 1 \), and \( \text{supp}(f_N) \subset B(w, \frac{1}{2}Q^{-N}) \). So, by Proposition 17, one has

\[
\mu(B(w, \frac{1}{2}Q^{-N-1})) \leq \mu(B(w, \lambda L \epsilon)) \leq \mu(B(w, \frac{1}{2}Q^{-N-1})) \leq \sqrt{2}^{N-N_0+1}.
\]

Take a sufficiently large \( N \) such that

\[
\frac{1}{2}Q^{-N-2} \leq \lambda L \epsilon \leq \frac{1}{2}Q^{-N-1},
\]

so

\[
-N \leq L \frac{\log \lambda}{\log Q} + \frac{\log(2\epsilon)}{\log Q} + 2.
\]

It follows from (23) and (24) that

\[
\mu(V(w, L, \epsilon)) \leq \mu(B(w, \lambda L \epsilon)) \leq \mu(B(w, \frac{1}{2}Q^{-N-1})) \leq \sqrt{2}^{N-N_0+1},
\]

and

\[
\frac{1}{L} \log \mu(V(w, L, \epsilon)) \leq \frac{1}{L} \log(\sqrt{2})(-N + N_0 + 1) \leq \log(\sqrt{2}) \frac{\log(2\epsilon)}{\log Q} + N_0 + 1.
\]
By (22), one has
\[ h_\mu(T) = \lim_{L \to +\infty} \frac{1}{L} \log \mu(V(w, L, \epsilon)) \geq \frac{\lambda^{-1}}{\log Q} \log \sqrt{2} > 0, \]
this is a contradiction with (21).

This completes the proof of the denseness of \( \mathcal{O} = \bigcup_{\gamma \in \text{Per}(T)} \mathcal{U}_\gamma \). It is obvious that \( \mathcal{O} \) is open.

Therefore, we finish the proof of Theorem 1.

5. Zero entropy

In this section, a result of Morris [24] is generalized for expanding map with differentiable functions, which is used in the entropy argument in Section 4.

Theorem 18. [24] Let \( X \) be a compact metric space and \( T : X \to X \) be an expanding map. There is a residual set \( \mathcal{G} \subset \text{Lip}(X, \mathbb{R}) \) such that if \( F \in \mathcal{G} \), then there is a unique \( F \)-maximizing measure and it has zero metric entropy.

Inspired by this result, we show the following result:

Theorem 19. Let \( X \) be a compact metric space and \( T : X \to X \) be an expanding map. There is a residual set \( \mathcal{G} \subset C^r(X, \mathbb{R}) \) (\( r \in \mathbb{N} \)) such that if \( F \in \mathcal{G} \), then there is a unique \( F \)-maximizing measure and it has zero metric entropy.

Remark 20. By Lemma 6, the ergodic components of a maximizing measure are also maximizing. Hence, the unique maximizing measure in Theorem 19 is ergodic, further, \( T|_{\text{supp}(\mu)} \) is uniquely ergodic.

The arguments below are motivated by the results in [11, 24].

Lemma 21. [11, Lemma 4.1] Let \( a_1, \ldots, a_n \) be non-negative real numbers and \( A = \sum_{i=1}^{n} a_i \geq 0 \), then
\[ \sum_{i=1}^{n} -a_i \log a_i \leq 1 + A \log n, \]
where \( 0 \log 0 = 0 \) is used for convenience.

Lemma 22. [11, Lemma 4.2] Let \( f \in \text{Lip}(X, \mathbb{R}) \) and suppose that \( \mathcal{M}_{\text{max}}(f) = \{ \mu \} \) for some \( \mu \in \mathcal{M}(T) \). Then there is \( C > 0 \) such that for every \( \nu \in \mathcal{M}(T) \),
\[ -\alpha(f) - C \int d(x, K) d\nu \leq \int f d\nu, \]
where \( K = \text{supp} \mu \).

For any \( \gamma \in \mathbb{R}^+ \), denote by
\[ \mathcal{E}_\gamma := \{ f \in C^r(X, \mathbb{R}) : h_\mu(T) < 2\gamma h_{\text{top}}(T) \ \forall \mu \in \mathcal{M}_{\text{max}}(f) \}. \]

Theorem 23. [12, 20] Let \( T : X \to X \) be a continuous map of a compact metric space. Let \( E \) be a topological vector space, which is densely and continuously embedded in \( C^0(X, \mathbb{R}) \). Let
\[ \mathcal{U}(E) = \{ F \in E : \text{there is a unique } F - \text{maximizing measure} \}. \]
Then \( \mathcal{U}(E) \) is a countable intersection of open and dense sets.

Moreover, if \( E \) is a Baire space, then \( \mathcal{U}(E) \) is dense in \( E \).
Definition 24. [13] A topological vector space is a vector space together with a topology such that with this respect to this topology such that addition is continuous, and the scalar multiplication is also continuous.

By Theorem 23, the set
\[ \mathcal{O} = \{ f \in C^r(X, \mathbb{R}) : \#\mathcal{M}_{\text{max}}(f) = 1 \} \]
is residual.

So, it suffices to show that \( \mathcal{E}_\gamma \) is open and dense for any \( \gamma > 0 \), implying that the set
\[ \mathcal{G} = \mathcal{O} \bigcap \bigcap_{n \in \mathbb{N}} \mathcal{E}_{\frac{1}{n}} \]
satisfies the requirements of Theorem 19.

Proof. Step 1. We show that \( \mathcal{E}_\gamma \) is open.

Let \( f \in C^r(X, \mathbb{R}) \), \( f_n \in C^r(X, \mathbb{R}) \setminus \mathcal{E}_\gamma \) with \( \lim_{n \to \infty} f_n = f \) in \( C^r(X, \mathbb{R}) \). So, there are \( \nu_n \in \mathcal{M}_{\text{max}}(f_n) \) with \( h(\nu_n) \geq 2\gamma h_{\text{top}}(T) \). By the compactness of the space \( \mathcal{M}(T) \) in the weak star topology, we can assume that \( \lim_{n \to \infty} \nu_n = \nu \in \mathcal{M}(T) \) in the weak star topology. So,
\[ \int f d\mu - \| f - f_n \|_0 \leq \int f_n d\mu \leq \int f_n d\nu_n \leq \int f d\nu_n + \| f - f_n \|_0, \]
implying that \( \int f d\mu \leq \int f d\nu \) for any \( \mu \in \mathcal{M}(T) \), that is, \( \nu \in \mathcal{M}_{\text{max}}(T) \). It follows from the upper semicontinuity of \( m \to h_m(T) \) that \( h(\nu) \geq 2\gamma h_{\text{top}}(T) \) [29]. Hence, \( f \in C^r(X, \mathbb{R}) \setminus \mathcal{E}_\gamma \), yielding that \( C^r(X, \mathbb{R}) \setminus \mathcal{E}_\gamma \) is closed. Therefore, \( \mathcal{E}_\gamma \) is open.

Step 2. We prove that \( \mathcal{E}_\gamma \) intersects every non-empty open set of \( C^r(X, \mathbb{R}) \).

Let \( \mathcal{U} \subset C^r(X, \mathbb{R}) \) be an open and non-empty subset. It follows from Theorem 23 that there is \( f \in \mathcal{U} \) such that \( \mathcal{M}_{\text{max}}(f) \) contains only one element, denoted by \( \mu \).

By the existence of Markov partitions of arbitrarily small diameter for expanding maps [27], there is a finite collection of sets \( S_i \subset X \), a Markov partition, denoted by \( \mathbb{P} \), satisfying that
\begin{itemize}
  \item \( \bigcup S_i = X \);
  \item \( \text{diam} \mathbb{P} = \text{max} \{ \text{diam} S_i \} < \epsilon_0 \);
  \item \( S_i = \text{int} S_i \);
  \item \( \text{int} S_i \cap \text{int} S_j = \emptyset \) for \( i \neq j \);
  \item \( f(S_i) \) is a union of sets \( S_j \).
\end{itemize}

Set
\[ \mathbb{P}^{(n)} := \bigvee_{i=0}^{n-1} T^{-i}(\mathbb{P}) = \left\{ \bigcap_{i=0}^{n-1} A_i : A_i \in T^{-i}(\mathbb{P}) \right\}. \]
The diameter of the elements of the partition \( \mathbb{P}^{(n)} \) is less than \( \lambda^{n-1} \epsilon_0 \), and this partition generates the Borel \( \sigma \)-algebra \( \mathbb{P}^\infty = \sigma(\bigcup_n \mathbb{P}^{(n)}) = \text{Borel}(X) \).

By [29], for every invariant measure \( \nu \in \mathcal{M}(T) \), one has
\[ h_\nu(T) = \inf_k \frac{1}{k} \sum_{A \in \mathbb{P}^{(k)}} (-\nu(A) \log \nu(A)). \]
If \( \mu \) is a periodic measure, then
\[
h_\mu(T) = \inf_k \frac{1}{k} \sum_{A \in \mathcal{P}(k)} (-\mu(A) \log \mu(A)) = 0,
\]
so, \( f \in \mathcal{E} \cap \mathcal{U} \). Otherwise, if \( \mu \) is not a periodic measure, it follows from (iii) of Lemma 6 that any measure in \( \text{supp}(\mu) \) is also a maximizing measure, implying that
\[
K = \text{supp}(\mu)
\]
does not contain a periodic orbit. By Lemma 22, we have
\[
-\alpha(f) - C \int d(x, K) d\nu \leq \int f d\nu \quad \forall \nu \in \mathcal{M}(T).
\]
(25)

Note that \( f \in U \subset C^r(X, \mathbb{R}) \), for any \( g \in C^\infty(X, \mathbb{R}) \), and \( \beta \in \mathbb{R} \), it is evident that if \( |\beta| \) is sufficiently small, then \( f + \beta g \in U \). By using this basic fact, we will construct a sequence of approximating functions \( f_n \in U \cap \mathcal{E} \) for large enough \( n \).

Step 3. We pick up a sequence of periodic orbits which will be used in the sequel.

Given any \( \theta \in (0, 1) \), there is a sequence of integers \( \{m_n\} \subset \mathbb{N} \) and a sequence of periodic measures \( \mu_n \in \mathcal{M}(T) \) satisfying that
\[
\int d(x, K) d\mu_n = o(\theta^{m_n}) \quad \text{and} \quad \lim_{n \to \infty} \frac{\log n}{m_n} = 0.
\]
By [7, Corollary 3 and Theorem 4], for any given positive integer \( k > 0 \), one has
\[
\lim_{n \to \infty} n^k \left( \inf_{\mu \in \mathcal{M}^n(T)} \int d(x, K) d\mu \right) = 0.
\]
Hence, there exists a sequence of periodic orbits \( \mu_n \in \mathcal{M}^n(T) \) such that
\[
\lim_{n \to \infty} n^k \int d(x, K) d\mu_n = 0.
\]
Set
\[
r_n := \log_\theta \left( \int d(x, K) d\mu_n \right).
\]
(27)

It is evident that, taking \( \log_\theta \) on both sides of
\[
\theta^{r_n} \leq n^k \theta^{r_n} \leq 1,
\]
we have
\[
-\frac{1}{k} \leq \frac{\log_\theta n}{r_n} \leq 0.
\]
So, \( \frac{\log_\theta n}{r_n} \to 0 \). Define
\[
m_n = \left\lfloor \frac{r_n}{2} \right\rfloor,
\]
(28)
where \( \lfloor x \rfloor \) is the largest integer that is less than or equal to \( x \). Hence, \( \frac{\log_\theta n}{m_n} \to 0 \), and
\[
\int d(x, K) d\mu_n = \theta^{r_n} \leq \theta^{m_n + \frac{1}{2}r_n} = o(\theta^{m_n}).
\]
(29)

Step 4. We verify that there is \( N_\gamma > 0 \) such that when \( n \geq N_\gamma \)
\[
\nu \{ x \in X : d(x, L_n) \geq \theta^{m_n} \} > \gamma
\]
(30)
for every invariant measure \( \nu \in \mathcal{M}(T) \) with \( h_\nu(T) \geq 2\gamma h_{\text{top}}(T) \), where
\[
L_n := \text{supp}(\mu_n),
\]
\[
0 < \theta < \min\{\epsilon_0, \lambda, \epsilon_0 \text{Lip}(T)^{-1}\},
\]
\( \lambda \) is introduced in Definition 2, and \( \epsilon_0 \) is specified in Remark 3.

This is the Claim 4.5 in \([11]\).

**Step 5.** Define a sequence of functions \( \{f_n\}_{n \geq 1} \subset C^r(X, \mathbb{R}) \).

For \( L_n \) specified in (31), the function \( d(x, L_n) \) is Lipschitz with Lipschitz constant 1. By \([1]\), there is a \( C^\infty \) function \( \tilde{f}_n(x) \) satisfying that
\[
|d(x, L_n) - \tilde{f}_n(x)| < \gamma \theta^{m_n+\frac{1}{2}r_n} \text{ and Lip}(\tilde{f}_n) \leq 1 + \frac{1}{n},
\]
where \( r_n \) and \( m_n \) are introduced in (27) and (28), respectively.

Define
\[
f_n(x) = f(x) - \beta \tilde{f}_n(x), \quad n \geq 1,
\]
where \( \beta \) is sufficiently small positive constant such that \( f_n \in \mathcal{U} \) and \( \mathcal{U} \) is specified in Step 2, since \( f \in C^r(X, \mathbb{R}), \tilde{f}_n \in C^\infty(X, \mathbb{R}) \), and \( X \) is compact.

It follows from (30) that for sufficiently large \( n \),
\[
\int d(x, L_n)d\nu \geq \theta^{m_n} \nu(\{x \in X : d(x, L_n) \geq \theta^{m_n}\}) \geq \gamma \theta^{m_n}
\]
for all \( \nu \in \mathcal{M}(T) \) with \( h(\nu) \geq 2\gamma h_{\text{top}}(T) \).

By Step 3,
\[
\int d(x, K)d\mu_n = \theta^r n \leq \theta^{m_n+\frac{1}{2}r_n}.
\]

So, we can choose sufficiently large \( n \) such that
\[
\beta \int d(x, L_n)d\nu - 2\beta \gamma \theta^{m_n+\frac{1}{2}r_n} > C \int d(x, K)d\mu_n
\]
for every \( \nu \in \mathcal{M}(T) \) with \( h(\nu) \geq 2\gamma h_{\text{top}}(T) \).

Hence, one has
\[
\int f_n d\nu = \int (f - \beta d(x, L_n))d\nu + \int (\beta d(x, L_n) - \beta \tilde{f}_n(x))d\nu
\]
\[
\leq \int f d\nu - \beta \int d(x, L_n)d\nu + \beta \int |d(x, L_n) - \tilde{f}_n(x)|d\nu
\]
\[
\leq -\alpha(f) - \beta \int d(x, L_n)d\nu + \beta \gamma \theta^{m_n+\frac{1}{2}r_n}
\]
\[
< -\alpha(f) - C \int d(x, K)d\mu_n - \beta \gamma \theta^{m_n+\frac{1}{2}r_n} \leq \int f d\mu_n - \beta \gamma \theta^{m_n+\frac{1}{2}r_n}
\]
\[
= \int (f_n + \beta \tilde{f}_n)d\mu_n - \beta \gamma \theta^{m_n+\frac{1}{2}r_n}
\]
\[
= \int f_n d\mu_n + \int \beta d(x, L_n)d\mu_n + \int (\beta \tilde{f}_n - \beta d(x, L_n))d\mu_n - \beta \gamma \theta^{m_n+\frac{1}{2}r_n}
\]
\[
\leq \int f_n d\mu_n + \beta \int |\tilde{f}_n - d(x, L_n)|d\mu_n - \beta \gamma \theta^{m_n+\frac{1}{2}r_n}
\]
\[
\leq \int f_n d\mu_n \leq -\alpha(f_n).
\]
That is, \( \int f_n d\nu < -\alpha(f_n) \). So, for \( \nu \in \mathcal{M}(T) \) with \( h_\nu(T) \geq 2\gamma h_{\text{top}}(T) \), then \( \nu \not\in \mathcal{M}_{\max}(f_n) \), implying that \( f_n \in \mathcal{E}_\gamma \cap \mathcal{U} \).

Therefore, \( \mathcal{E}_\gamma \) is open and dense in \( C^r(X, \mathbb{R}) \). This completes the whole proof. \( \square \)

**ACKNOWLEDGMENTS**

We would like to thank Prof. Weixiao Shen, who suggested this problem, and thank Prof. Yiwei Zhang for useful discussions.

This work was supported by the National Natural Science Foundation of China (No. 11701328) and Young Scholars Program of Shandong University, Weihai (No. 2017WHWLJH09).

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