Nonexistence of spontaneous symmetry breakdown of time-translation symmetry:

a review

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The time invariance of equilibrium states has been established in the $C^*$-algebraic framework assuming an infinite-volume Heisenberg time evolution as $C^*$-dynamics. This fundamental property of equilibrium states implies that spontaneous breakdown of time-translation symmetry is impossible in general quantum systems. In particular, any non-trivial order, such as periodic, quasi-periodic, and chaotic order cannot appear in the time direction of equilibrium states for general quantum models, meaning that genuine quantum time crystals are strictly forbidden. We compare our statement on the impossibility of quantum time crystals with the result by Watanabe-Oshikawa which has been considered as a milestone in the study on quantum time crystals. Our no-go statement based on the Kubo-Martin-Schwinger (KMS) condition for $C^*$-dynamics is not only mathematically rigorous but also more general than the main result of Watanabe-Oshikawa and its improvement by Watanabe-Oshikawa-Koma based on temporal correlation functions under cut-off quantum time evolutions.

I. INTRODUCTION

Wilczek [55] conceptualized quantum time crystals as self-organized temporal periodic structures of equilibrium states. These structures are sometimes called genuine quantum time crystals [28] to distinguish them from nonequilibrium time crystalline phenomena, which have been observed experimentally [13] [50] [52].

In this review, we verify that spontaneous breakdown of time-translation symmetry cannot occur in general quantum systems specifying the setup and assumption clearly. As an immediate consequence of this no-go theorem of time-translation symmetry breakdown, we obtain nonexistence of genuine quantum time crystals. To this end, we only recall the time invariance and other basic properties of equilibrium states characterized by the Kubo-Martin-Schwinger (KMS) condition [5]. Thus we do not claim our priority for this no-go theorem. To the best of our knowledge, the KMS condition as a natural, mathematically rigorous, and powerful definition of equilibrium states has been ignored in previous studies on quantum time crystals [24] [46].

In this review, each statement will be given with a precise assumption and sometimes with relevant remarks added to clarify its validity. We present our statements in a self-contained manner so that the readers who are not familiar with $C^*$-algebraic theory can understand them in the context of the issue of quantum time crystals.

The validity of our statements is not guaranteed if the assumptions are not perfectly satisfied. Furthermore, if one employs a different notion of equilibrium states or spontaneous symmetry breakdown from ours, it is not surprising that a different conclusion on the status of genuine quantum time crystals can arise.

II. SETUP AND BASIC FACTS

This section describes our mathematical setup based on the $C^*$-algebraic quantum statistical mechanics. We refer to [5] as our basic reference.

In contrast to the claim of [53] whose motivation was explained in detail by the author in [52], we can prove absence of genuine quantum crystals without novel concepts or tools. To establish our formulation and proof, we need only the well-known definitions of equilibrium states, spontaneous symmetry breakdown, and long-range orders in rigorous statistical mechanics [5] [45] without any alternation of them.

A. Quantum systems

Let $\Gamma$ denote an infinite space such as $\mathbb{R}^\mu$ and $\mathbb{Z}^\mu \ (\mu \in \mathbb{N})$. $\Gamma$ has a natural additive group structure: $\xi_x(y) := y + x \in \Gamma$ for $x, y \in \Gamma$. Let $\mathcal{F}$ be a set of all subsets of $\Gamma$. If $\Lambda \in \mathcal{F}$ has finite volume $|\Lambda| < \infty$, then we denote $\Lambda^* \in \Gamma$.

Let $\mathcal{F}_{\text{loc}}$ denote the set of all finite subsets (or the set of sufficiently many finite subsets of a certain shape) of $\Gamma$. Let $\mathcal{A}$ denote a quasi-local $C^*$-system on $\Gamma$ describing the infinite quantum system under consideration. The total system $\mathcal{A}$ includes subsystems denoted by $\mathcal{A}_{\Lambda} \ (\Lambda \in \mathcal{F})$ indexed by $\mathcal{F}$. The local algebra defined by $\mathcal{A}_{\text{loc}} := \bigcup_{\Lambda \in \mathcal{F}_{\text{loc}}} \mathcal{A}_{\Lambda}$ is a norm dense subalgebra of $\mathcal{A}$. The local commutativity condition is assumed. Namely, for any two disjoint subsets $\Lambda$ and $\Lambda'$ the following commutation relations are satisfied:

$$[A, B] \equiv AB - BA = 0 \quad \forall A \in \mathcal{A}_{\Lambda}, \ B \in \mathcal{A}_{\Lambda'}.$$

Let $\{\tau_x \in \text{Aut}(\mathcal{A}), \ x \in \Gamma\}$ denote the group of space-translation automorphisms on $\mathcal{A}$. Then $\tau_x(\mathcal{A}_{\Lambda}) = \mathcal{A}_{\Lambda + x}$ for any $\Lambda \in \mathcal{F}$ and $x \in \Gamma$.

A state on $\mathcal{A}$ is defined by a normalized positive linear functional on $\mathcal{A}$. We denote the set of all states on $\mathcal{A}$ by...
\(S(A)\). It is an affine space with the affine combination of states. For each \(\omega \in S(A)\) the triple \((H_{\omega}, \pi_{\omega}, \Omega_{\omega})\) denotes the von Neumann algebra \(M\) center of \(\omega\). The GNS representation generates the von Neumann algebra \(\mathcal{M}_{\omega} := \pi_{\omega}(A)'\) on the GNS Hilbert space \(H_{\omega}\). The commutant of \(\mathcal{M}_{\omega}\) is given by \(\mathcal{M}_{\omega}' := \{X \in \mathcal{B}(H_{\omega}); [X, Y] = 0 \forall Y \in \mathcal{M}_{\omega}\}\), and the center of \(\mathcal{M}_{\omega}\) is given by \(\mathcal{Z}_{\omega} := \mathcal{M}_{\omega} \cap \mathcal{M}_{\omega}'\). The center \(\mathcal{Z}_{\omega}\) contains all macroscopic observables with respect to \(\omega\), and thereby \(\mathcal{Z}_{\omega}\) provides macroscopic (or thermodynamical) information about \(\omega\). In general, any order parameter that distinguishes different phases has its corresponding element in the center. For example, the energy density and the magnetization per unit volume for the responding element in the center. For example, the energy density and the magnetization per unit volume for the responding element in the center.

A state \(\omega \in S(A)\) is called a factor state if its center is trivial \(\mathcal{Z}_{\omega} = \mathbb{C}I\), where \(I\) is the identity operator on \(H_{\omega}\).

The set of all factor states on \(A\) is denoted by \(S_{\text{factor}}(A)\). Any \(\omega \in S_{\text{factor}}(A)\) is known to satisfy the uniform cluster property with respect \(\{\tau_{\lambda} \in \text{Aut}(A), x \in \Gamma\}\). Hence factor states are sometimes identified with pure phases. Each \(\omega \in S(A)\) has its factorial (central) decomposition:

\[
\omega = \int d\mu(\omega_{\lambda})\omega_{\lambda}, \quad \omega_{\lambda} \in S_{\text{factor}}(A),
\]

where \(\mu\) denotes the unique probability measure on \(S_{\text{factor}}(A)\) determined by \(\omega\). Note that the above \(\omega \in S(A)\) is not necessarily translation invariant.

**B. Quantum time evolution**

Assume that our stationary Heisenberg-type quantum time evolution is given by a one-parameter group of automorphisms \(\{\alpha_{t} \in \text{Aut}(A), t \in \mathbb{R}\}\) on the quasi-local \(C^*\)-algebra \(A\). We need continuity for \(\{\alpha_{t} \in \text{Aut}(A), t \in \mathbb{R}\}\) with respect to \(t \in \mathbb{R}\). We assume the strong continuity:

\[
\lim_{t \to 0} \alpha_{t}(A) = A \text{ in norm for each fixed } A \in A.
\]

It is known that any short-range quantum spin lattice model generates a strongly continuous time evolution \(\{\alpha_{t} \in \text{Aut}(A), t \in \mathbb{R}\}\). For other quantum systems such as boson systems, in stead of the strong continuity above, we assume \(\sigma\)-weakly continuity for quantum time evolutions realized by the GNS representation of sufficiently many states (including all equilibrium states) on \(A\). In mathematics, such quantum dynamical systems are called \(W^*\)-dynamical systems, see [5, 10].

**C. Equilibrium states**

1. **The KMS condition**

We define equilibrium states for infinite quantum systems by the Kubo-Martin-Schwinger (KMS) condition, named after Kubo [29], Martin and Schwinger [33], which is given in terms of certain temporal correlation (Green’s) functions. The following mathematically sophisticated definition is due to Haag-Hugenholtz-Winnink [22]. Let \(\beta \geq 0\) denote an inverse temperature. Define \(D_{\beta} := \{z \in \mathbb{C}; 0 \leq \text{Im}z \leq \beta\}\) and let \(\tilde{D}_{\beta}\) denote its interior region, i.e. the open strip \(\{z \in \mathbb{C}; 0 < \text{Im}z < \beta\}\). A state \(\varphi\) on \(A\) is called a \(\beta\)-KMS state for the time evolution \(\{\alpha_{t} \in \text{Aut}(A), t \in \mathbb{R}\}\) if for every \(A, B \in A\), there exists a complex function \(F_{A, B}(z)\) of \(z \in D_{\beta}\) such that

\[
F_{A, B}(t) = \varphi(A\alpha_{t}(B)), \quad F_{A, B}(t + i\beta) = \varphi(\alpha_{t}(B)A).
\]

The set of \(\beta\)-KMS states is denoted by \(S_{\text{KMS}}(A, \beta)\). Ground states are straightforwardly given by the KMS condition at \(\beta = \infty\). The set of ground states is denoted by \(S_{\text{ground}}(A)\).

We denote the set of all equilibrium states by \(S_{\text{equil}}(A)\), namely the set of all KMS states \(\bigcup_{\beta \geq 0} S_{\text{KMS}}(A, \beta) \cap S_{\text{ground}}(A)\) for the same time evolution \(\{\alpha_{t} \in \text{Aut}(A), t \in \mathbb{R}\}\) over all inverse temperatures \(\beta\).

2. **Time invariance and factorial decomposition of equilibrium states**

To discuss temporal properties of equilibrium states, the KMS condition has obvious merits over the so called “Gibbs Ansatz” that is based on local Gibbs ensembles on finite boxes, because the KMS condition is directly given by the time evolution as stated above. In fact, the formula (3) of the KMS condition immediately implies that every \(\varphi \in S_{\text{equil}}(A)\) is invariant under \(\{\alpha_{t} \in \text{Aut}(A), t \in \mathbb{R}\}\): For any \(t \in \mathbb{R}\)

\[
\varphi(\alpha_{t}(A)) = \varphi(A) \quad \text{for all } A \in A
\]

regardless of whether \(\varphi \in S_{\text{equil}}(A)\) is a factor state or not.

Next we address the structure of the space of equilibrium states. For any finite \(\beta \geq 0\), the affine space \(S_{\text{KMS}}(A)\) is a Choquet simplex. Precisely, the set of extremal points \(S_{\text{ext}}(A)\) in \(S_{\text{KMS}}(A)\) coincides with \(S_{\text{KMS}}(A) \cap S_{\text{factor}}(A)\). Hence each \(\varphi \in S_{\text{KMS}}(A)\) is uniquely written as an affine sum of \(S_{\text{ext}}(A)\) that coincides with the factorial decomposition as in (1):

\[
\varphi = \int d\mu(\varphi_{\lambda})\varphi_{\lambda}, \quad \varphi_{\lambda} \in S_{\text{ext}}(A) = S_{\text{KMS}}(A) \cap S_{\text{factor}}(A),
\]

where \(\mu\) is a unique probability measure determined by \(\varphi\). Any factor state \(\varphi_{\lambda}\) appearing in the factorial decomposition (5) is a KMS state, and therefore it is obviously time-invariant. The above general structure of KMS states follows from the fact that the center \(\mathcal{Z}_{\omega}\) of
any $\varphi \in S_{\alpha,\beta}(\mathcal{A})$ is pointwise fixed under the time evolution.

For $\beta = \infty$, a similar statement holds by the stronger reason as follows. The affine space $S_{\alpha,\infty}(\mathcal{A})$ is face in $S(\mathcal{A})$. Namely, for any $\varphi \in S_{\alpha,\infty}(\mathcal{A})$, consider its state-decomposition $\varphi = \int d\mu(\varphi_\lambda)\varphi_\lambda$, $\varphi_\lambda \in S(\mathcal{A})$. Then each $\varphi_\lambda$ belongs to $S_{\alpha,\infty}(\mathcal{A})$, and thus it is obviously invariant under the time evolution. The above general structure of ground states follows from the fact that the commutant $\mathcal{M}_\varphi$ (and therefore $\mathcal{F}_\varphi$) of any $\varphi \in S_{\alpha,\infty}(\mathcal{A})$ is pointwise fixed under the time evolution.

Remark 1. Several characterizations of the KMS condition support the idea that the KMS condition is a reasonable definition of equilibrium in general quantum systems. For instance, the KMS condition satisfies passivity by Pusz-Woronowicz [41], which formulates “the second law of thermodynamics” in terms of $\{\alpha_t \in \text{Aut}(\mathcal{A}), \ t \in \mathbb{R}\}$: No work can be obtained from an isolated system in equilibrium by varying adiabatically the external parameters [90]. As passivity necessitates time invariance for states, it excludes non-stationary states and mixtures of such states. If any non-trivial temporal behavior such as quantum-time crystals exists, then passivity is inevitably violated.

3. Symmetry and spontaneous symmetry breakdown

We shall make a detour to restate the time invariance of equilibrium states as “absence of spontaneous breakdown of time-translation symmetry”. For this purpose we introduce some notation related to symmetry and symmetry breakdown in the $C^*$-algebraic formalism.

Let $G$ be a group with its unit element $e$ and let $(G, \theta)$ denote a faithful representation of $G$ into $\text{Aut}(\mathcal{A})$. Namely,

$$
\theta_g \in \text{Aut}(\mathcal{A}), \ g \in G, \quad \theta_e = \text{id} \in \text{Aut}(\mathcal{A}),
$$

$$
\theta_g \neq \text{id} \in \text{Aut}(\mathcal{A}) \text{ for } \forall g \neq e \in G,
$$

$$
\theta_{g_1} \circ \theta_{g_2} = \theta_{g_1g_2}, \ g_1, g_2 \in G.
$$

The action of $G$ upon $S(\mathcal{A})$ is given by $\theta_g^* \omega := \omega \circ \theta_g \in S(\mathcal{A})$ ($g \in G$) for each $\omega \in S(\mathcal{A})$. If $\theta_g^* \omega = \omega \in S(\mathcal{A})$ for all $g \in G$, then $\omega$ is called a $G$-invariant state. The set of all $G$-invariant states is denoted by $S_{\text{inv}}(\mathcal{A})$.

Any $\omega \in S(\mathcal{A})$ that is invariant under the time evolution $\{\alpha_t \in \text{Aut}(\mathcal{A}), \ t \in \mathbb{R}\}$ is called a time-invariant or stationary state. The set of all time-invariant states $S_{\text{inv}}^{\alpha}(\mathcal{A})$ will be denoted simply by $S_{t-\text{inv}}(\mathcal{A})$. As we have seen in [43],

$$
S_{\text{equil}}^{\alpha}(\mathcal{A}) \subset S_{t-\text{inv}}(\mathcal{A}). \tag{6}
$$

Any $\omega \in S(\mathcal{A})$ that is invariant under the space-translation group $\{\tau_x, \ x \in \Gamma\}$ is called a translation-invariant state. Let $\Delta$ be a crystallographic subgroup of $\Gamma$, an infinite sub-lattice of $\Gamma$ such that the quotient group $\Delta/\Gamma$ is finite. If $\omega \in S(\mathcal{A})$ is invariant under $\{\tau_x, \ x \in \Delta\}$, then it is called a spatially periodic state with respect to $\Delta$. We denote the sets of all translation-invariant states and all spatially periodic states by $S_{\text{inv}}^{\tau}(\mathcal{A})$ and $S_{\text{inv}}^{\Delta}(\mathcal{A})$, respectively. Together, translation invariant and spatially periodic states are called homogeneous states. We denote the set of all homogeneous states on $\mathcal{A}$ by $S_{\text{homo}}(\mathcal{A})$.

We now define spontaneous symmetry breakdown (SSB) as follows. If

$$
\alpha_t \circ \theta_g = \theta_g \circ \alpha_t \in \text{Aut}(\mathcal{A}) \quad \text{for all } t \in \mathbb{R}, \ g \in G, \tag{7}
$$

then $(G, \theta)$ is called a dynamical symmetry group. For such $(G, \theta)$, if there exists $\psi \in S_{\alpha,\beta}(\mathcal{A})$ that breaks $G$, namely $\theta_g^* \psi \neq \psi$ for some $g \in G$, then the dynamical symmetry group $(G, \theta)$ is said to be spontaneously broken at $\beta$.

Remark 2. The above definition of spontaneous symmetry breakdown relies on “the set of equilibrium states on $\mathcal{A}$”. On the other hand, the more conventional definition of spontaneous symmetry breakdown is so called Bogoliubov’s method [2]. The relationship between different formulations of SSB has been elucidated in III. 10 of [48] and Theorem 6.2.42 of [8] for spin lattice systems. For boson systems, we refer to [57].

III. NONEXISTENCE OF TIME-TRANSITION SYMMETRY BREAKDOWN

A. No-go statement of temporal symmetry breakdown without homogeneity requirement

In the preceding section, we recalled the following basic properties of equilibrium states and their associated time evolutions.

Proposition 1. All equilibrium states characterized by the KMS condition are invariant under the time evolution. Any macroscopic observable in the center is fixed under the time evolution, irrespective of whether the equilibrium state is factorial or non-factorial.

Remark 3. As the title of [12] we can express the frozen property of the center of any equilibrium state stated in Proposition [1] as “Any macroscopic observable (order parameter) of any equilibrium state moves not.”

The impossibility of spontaneous breakdown for time-translation symmetry in general quantum systems immediately follows from Proposition [1].

Theorem 2. Suppose that a quantum time evolution is given as $C^*$-dynamics, and equilibrium states are given by the KMS condition with respect to the quantum time evolution. Then there is no spontaneous breakdown of time-translation symmetry, and therefore no temporal order exists in equilibrium states. In particular, periodic,
quasi-periodic, and chaotic orders in the time direction are forbidden in equilibrium states.

In the above theorem the one-parameter group of automorphisms \( \{ \alpha_t \in \text{Aut}(A), \ t \in \mathbb{R} \} \) plays two roles. First, it describes a quantum time evolution that determines the model. Second, it describes a specific dynamical symmetry \((G, \theta) = (\mathbb{R}, \alpha)\) of the model, as the requirement \((7)\) is satisfied by the following obvious commutative relation:

\[
\alpha_t \circ \alpha_s = \alpha_s \circ \alpha_t = \alpha_{t+s} \in \text{Aut}(A) \quad \text{for all } t \in \mathbb{R}, \ s \in \mathbb{R}.
\]

Remark 4. We shall address the case \( \beta = 0 \) in order to respond to the additional discussion for this case given in \([54]\), although such special treatment is unnecessary. The KMS condition for \( \beta = 0 \) yields the identity \( \varphi(AB) = \varphi(BA) \) for all \( A, B \in A \), so \( \varphi \) is identical to the tracial state. The tracial state is a factor state (if \( A \) is a simple algebra, which is a standard assumption of quantum statistical mechanics), and so it satisfies the uniform cluster property with respect to space-translations. As the tracial state is invariant under any \( \gamma \in \text{Aut}(A) \), it is automatically invariant under \( \{ \alpha_t \in \text{Aut}(A), \ t \in \mathbb{R} \} \).

Remark 5. The KMS states given above correspond to canonical ensembles determined by the inverse temperature \( \beta \). In a similar manner, one can consider the KMS condition for grand canonical ensembles specified by \( \beta \) and the chemical potential(s) \( \mu \), see \([22]\) and Sec. 5.4.3 of \([3]\). We can derive essentially the same statements as Proposition 4 and Theorem 2 for grand canonical KMS states. If \( U(1) \)-symmetry generated by the number operator breaks spontaneously as in the Bose-Einstein condensation, then the expectation value of the field operator which is responsible for the \( U(1) \)-symmetry breakdown can oscillate periodically in time as noted in \([51, 54]\). However, the time invariance property \((4)\) is valid for all observables, i.e. for all the elements in \( A \) that are fixed by the \( U(1) \)-transformation.

Remark 6. Theorem 2 requires existence of an infinite-volume time evolution as \( C^\ast \)-dynamics. In general, verifying this assumption is not easy. Some known examples are short-range quantum spin models and non-interacting quantum field models which are rather exceptional \([3]\). In particular, the present rigorous knowledge about \( C^\ast \)-dynamics of boson systems is limited; we refer to \([2]\) and also \([8]\) for some recent progress.

Remark 7. The possibility of quantum time crystals in terms of the theory of Einstein relativity was argued in \([33, 34]\). Although we cannot accurately specify the motivation of these authors, we highlight the following seemingly relevant facts from a purely scientific perspective.

1. Thermal equilibrium states in a fixed Lorentz-frame can be characterized by the KMS condition \([5]\).
2. The KMS states always violate the Lorentz-symmetry \([37]\), while those preserve the time-translation symmetry by Proposition 4.

3. The spectrum condition of local quantum physics \([21]\) postulates that the spectrum of Hamiltonian and momentum operators on the Hilbert space of a vacuum state is included in the forward-light cone in Lorentz space-time. (Note that the vacuum state is not necessarily Lorentz-invariant.) The spectrum condition forbids manifestations of crystal structure in space as shown in Theorem 4.6 \([1]\). See also Theorem 3.2.4 \([21]\). Thus relativistic vacuum states allow crystalline structure neither in the space direction nor in the time direction.

**B. Absence of long-range temporal order**

Proposition 4 and Theorem 2 require no particular assumption on the spatial structure. In this subsection we specialize the case where states and time evolution are both assumed to be homogeneous in space.

1. **Densities as macroscopic observables**

Each local observable gives a macroscopic observable on the GNS Hilbert space for any homogeneous state by Cesàro sum under space translations. Let us give this statement a precise mathematical formula following \([20]\).

We introduce the following notations on infinite-volume limits. Suppose that a net \( \{ \Lambda; \Lambda \in \Gamma \} \) of finite subsets of \( \Gamma \) eventually includes any \( I \in \Gamma \). Then, we denote \( \{ \Lambda \uparrow \Gamma; \Lambda \in \Gamma \} \) or simply \( \Lambda \uparrow \Gamma \). Similarly, if a net \( \{ \Lambda; \Lambda \in \Delta \} \) eventually includes any \( I \in \Delta \), then we denote \( \{ \Lambda \uparrow \Delta; \Lambda \in \Delta \} \) or simply \( \Lambda \uparrow \Delta \).

Take any \( A \in \mathcal{A} \). For any finite subset \( \Lambda \in \Gamma \) (or \( \Lambda \in \Delta \)) we take the following averaged sum under space translations:

\[
m_{\Lambda}(A) := \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \tau_x(A) \in A.
\]

For \( \omega \in S(A) \) let \( m^\omega_{\Lambda}(A) := \pi_\omega(m_{\Lambda}(A)) \in \mathcal{M}_{\omega} \). For any \( \omega \in S_{inv}^{\Gamma}(A) \) and any \( \{ \Lambda \uparrow \Gamma; \Lambda \in \Gamma \} \), the net of uniformly bounded operators \( \{ m^\omega_{\Lambda}(A) \in \mathcal{M}_{\omega}; \Lambda \uparrow \Gamma; \Lambda \in \Gamma \} \) has \( n \) accumulation point(s) in the center \( \mathcal{Z}_{\omega} \). Precisely, there exists a subnet of this net that converges to some element of the center in the weak-operator topology. We denote any such accumulation point by \( \mathcal{A}_{\omega}^\omega \). Heuristically, we write

\[
\mathcal{A}_{\omega}^\omega \equiv \lim_{\Lambda \uparrow \Gamma; \Lambda \in \Gamma} m^\omega_{\Lambda}(A) \in \mathcal{Z}_{\omega},
\]

where \( \{ \Lambda \uparrow \Delta; \Lambda \in \Gamma \} \) is a subnet of \( \{ \Lambda \uparrow \Gamma; \Lambda \in \Gamma \} \) having a unique limit. When \( \omega \) is a non-factor state, there can be multiple accumulation points that may depend on chosen subnets. Similarly, we consider \( \omega \in S_{inv}^{\Delta}(A) \). Then every accumulation point of \( \{ m^\omega_{\Lambda}(A) \in \mathcal{M}_{\omega}; \Lambda \uparrow \Delta; \Lambda \in \Delta \} \) in the weak-operator topology belongs to
3ω. If ω ∈ S_{homo.}(A) is a factor state, these macroscopic observables are sharply determined with no dispersion; for each A ∈ A the above accumulation point is uniquely given by the scalar ω(A)|.

2. Formulation of long-range order in C*-systems

We give a general formulation of long-range order (LRO) in the C*-algebraic setup. Take ω ∈ S_{homo.}(A) and ω ∈ B ∈ A. Denote their densities by A_{∞}^ω ∈ 3ω and B_{∞}^ω ∈ 3ω for a net {Λ ↑ Λ; Λ ∈ Γ} or {Λ ↑ ∆; ∆ ∈ Δ} as in (9). Define the following two-point correlation function with respect to ω

\[ f_{A,B}^ω(\tau) = \left( \Omega_ω, \hat{A}_∞^ω(t, τ) \hat{B}_∞^ω(t, τ) \right), \quad \tau \in R. \]

where the GNS representation of A is used. If f_{A,B}^ω is non-zero, then ω is said to exhibit LRO for A, B ∈ A. If f_{A,A}^ω is non-zero for some A ∈ A, then ω is said to exhibit LRO and A ∈ A is called a local order parameter.

Now we consider the group action (G, θ). For A ∈ A and g ∈ G, we define A_{∞}^ω(g) := θ_g(A_{∞}^ω) ∈ 3ω, where θ_g(A) is substituted for A in (9). Consider the following two-point correlation function with respect to ω ∈ S_{homo.}(A):

\[ f_{A,B}^ω(g) = \left( \Omega_ω, \hat{A}_∞^ω(g) \hat{B}_∞^ω(g) \right), \quad g \in G. \]

If f_{A,B}^ω(g) is a non-constant function of g ∈ G, then ω is said to exhibit G-dependent LRO for A, B ∈ A. If f_{A,A}^ω(g) is a non-constant function of g ∈ G for some A ∈ A, then ω is said to exhibit G-dependent LRO, and A ∈ A is called a local order parameter with respect to (G, θ)-symmetry.

Next we consider the quantum time evolution. Let φ ∈ S_{equiv.}(A) ∩ S_{homo.}(A), i.e. an arbitrary homogeneous equilibrium state for \{α_t ∈ Aut(A), t ∈ R\}. Substituting φ for ω ∈ S_{homo.}(A), (R, α) for (G, θ), and t ∈ R for \{Λ ↑ ∆; Λ ∈ Γ\} (defined above), we get A_{∞}^ω(t) ∈ 3ω. Let A, B ∈ A, and define the following two-point temporal correlation function with respect to ϕ:

\[ f_{A,B}^ϕ(t) = \left( \Omega_ϕ, \hat{A}_∞^ϕ(t) \hat{B}_∞^ϕ(t) \right), \quad t \in R. \]

By the pointwise invariance of the center 3ϕ under the time evolution \{α_t ∈ Aut(A), t ∈ R\} as stated in Proposition 1, we have

\[ \hat{A}_∞^ϕ(t) = \hat{A}_∞^ϕ, \quad t \in R. \]

Therefore, by (13) the temporal correlation function is always t-invariant:

\[ f_{A,B}^ϕ(t) = f_{A,B}^ϕ(0) \quad \text{for all } t \in R. \]

We thus obtain the following absence of non-trivial temporal LRO.

Corollary 3. Assume the same assumption of Theorem 2. Assume further that the time evolution is homogeneous in space. Then for any homogeneous equilibrium state there exists no non-trivial temporal LRO.

Remark 8. We have defined A_{∞}^ϕ ∈ 3ϕ for all A ∈ A in (9) which are not necessarily strictly local. This is essential for f_{A,B}^ϕ(t) in (12) to be well defined, since generically the time development α_t(A) of A ∈ A_{loc} does not remain in A_{loc}.

Remark 9. Corollary 3 has the following trivial generalizations. Let ϕ ∈ S_{equiv.}(A) ∩ S_{homo.}(A) and A, B ∈ A as in Corollary 3. Consider

\[ \left( \Omega_ϕ, \hat{A}_∞^ϕ(t) \hat{π}_ϕ(B) \hat{A}_∞^ϕ(t) \right), \quad t \in R, \]

Then from (13) and the time-invariance of ϕ it follows that these temporal two-point correlation functions are constant with respect to t ∈ R. On the other hand, generically, the two-point function \(\langle ϕ(α_t(A)) ϕ(B) ϕ\rangle\) is not constant in t ∈ R as noted in (13).

IV. COMPARISON OF DIFFERENT NO-GO STATEMENTS

In this section, we compare our statements given in Section III with another no-go statement of quantum time crystals given in [53, 54]. First, we note that LRO can be formulated in various ways. The C*-algebraic formalism of LRO is based on states on a quasi-local C*-algebra, while the Griffiths’s formalism of LRO [18] is based on local Gibbs states on finite regions as in Gibbs Ansatz (which is called the box procedure in [21]). We employ the C*-algebraic LRO in Section III B 2 whereas the works [53, 54] utilize a Griffiths-type LRO as presented below.

A. Summary of the statement by Watanabe-Oshikawa-Koma

We recall the formulation and main result of [53, 54] as faithfully as possible, but allowing some modifications for comparison purposes. We embed finite systems in the total system A and explicitly denote the dependence of subsystems.

We consider the cubic lattice Z^d as in [54]. Define the metric on Γ := Z^d by \(\|x - y\| := \max_{i \leq d} \|x_i - y_i\|\) for \(x = (x_i), \ y = (y_i) \in Γ\). Let \(\text{diam}(Λ) := \{\sup \|x - y\|; \ x, y ∈ Λ\}\) for Λ ⊆ Γ.

Assume that the Hamiltonian on the total system is formally given by

\[ Η := \sum_{x ∈ Γ} h_x, \]

\[ (\hat{A}_∞^ϕ(t) \hat{π}_ϕ(B) \hat{A}_∞^ϕ(t)) = \hat{A}_∞^ϕ \hat{π}_ϕ(B) \hat{A}_∞^ϕ, \]

which is a non-constant function of t ∈ R.
where each local Hamiltonian \( \hat{h}_x \in \mathcal{A}_{\text{loc}} \) is a finite-range self-adjoint operator with its support \( \text{supp}(\hat{h}_x) \) centered at site \( x \in \Gamma \). We assume that the range and the norm of \( \{ \hat{h}_x; x \in \Gamma \} \) are uniformly bounded over \( x \in \Gamma \):

\[
\text{diam}(\text{supp}(\hat{h}_x)) \leq R_h, \quad \| \hat{h}_x \| \leq N_h,
\]

where \( R_h \) and \( N_h \) are some positive constants.

For each \( \Lambda \in \Gamma \), let

\[
\hat{H}_\Lambda := \sum_{x \in \Lambda} \hat{h}_x \in \mathcal{A}_{\Lambda; \partial_{\text{ext}} \Lambda},
\]

where \( \partial_{\text{ext}} \Lambda \) denotes some finite region surrounding but not intersecting \( \Lambda \). By (10) the ratio \( \frac{\| \hat{h}_x \|}{N_h} \) tends to 0 as \( \Lambda \uparrow \Gamma \). One may also take the free-boundary local Hamiltonian \( \hat{H}_\Lambda^{\text{free}} := \sum_{x \in \Lambda \setminus \partial_{\text{inside}}} \hat{h}_x \in \mathcal{A}_\Lambda \), where the \( \partial_{\text{inside}} \) is the smallest subset within \( \Lambda \) such that the above sum is in \( \mathcal{A}_\Lambda \). However, we make use of the local Hamiltonian \( \hat{H}_\Lambda \) as in [53].

For \( \Lambda \in \Gamma \), we define the local Heisenberg time evolution by

\[
o_{\Lambda, t} (A) := e^{i t \hat{H}_\Lambda} A e^{-i t \hat{H}_\Lambda} \quad \text{for} \quad A \in \mathcal{A}.
\]

For \( \Lambda \in \Gamma \), we define the local Gibbs state \( \rho^\beta_{\text{Gib}} \in \mathcal{S}(\mathcal{A}) \) at inverse temperature \( \beta \) by the same local Hamiltonian \( \hat{H}_\Lambda \) as

\[
\rho^\beta_{\text{Gib}}(A) := \frac{1}{\text{tr}(e^{-\beta \hat{H}_\Lambda})} \text{tr}(e^{-\beta \hat{H}_\Lambda} A) \quad \text{for} \quad A \in \mathcal{A},
\]

where \( \text{tr} \) denotes the tracial state on \( \mathcal{A} \). Note that the local Gibbs state \( \rho^\beta_{\text{Gib}} \) defined on the total system \( \mathcal{A} \) is the unique \( \beta \)-KMS state for \( \{ o_{\Lambda, t} \in \text{Aut}(\mathcal{A}), t \in \mathbb{R} \} \). Similarly, let \( \rho^\beta_{\Lambda} \in \mathcal{S}(\mathcal{A}_\Lambda) \) denote a ground state for the local Hamiltonian \( \hat{H}_\Lambda \), equivalently, a ground state for \( \{ o_{\Lambda, t} \in \text{Aut}(\mathcal{A}_\Lambda), t \in \mathbb{R} \} \) as defined in Section II C 1.

Take a set of local operators \( \{ A_x; x \in \Gamma \} \), where each \( A_x \) is a local operator with its support \( \text{supp}(A_x) \) centered at \( x \in \Gamma \). Assume that the range and the norm for \( \{ A_x; x \in \Gamma \} \) are uniformly bounded over \( x \in \Gamma \); there are positive constants \( r \) and \( a \) such that for all \( x \in \Gamma \)

\[
\text{diam}(\text{supp}(A_x)) \leq r, \quad \| A_x \| \leq a.
\]

Then for \( \Lambda \in \Gamma \), we define

\[
\hat{\Lambda} := m_{\Lambda}(\{ A_x; x \in \Gamma \}) := \frac{1}{|\Lambda|} \sum_{x \in \Lambda} A_x \in \mathcal{A}_{\text{loc}}.
\]

The notation \( \hat{\Lambda} \) above corresponds to \( \hat{\Lambda} \) in [54]. If \( A_0 \) is a local operator with its support centered at the origin and \( A_x = \tau_x(A_0) \) for all \( x \in \Gamma \), then such \( \{ A_x; x \in \Gamma \} \) is said to be covariant in space-translations, and \( \hat{\Lambda} \) is equal to \( m_{\Lambda}(A) \) of [8] with \( A := A_0 \in \mathcal{A} \).

Consider \( \{ A_x; x \in \Gamma \} \) and \( \{ B_x; x \in \Gamma \} \), both satisfying (20), and obtain their \( \hat{\Lambda} \) and \( \hat{B}_\Lambda \) by (21) for each \( \Lambda \in \Gamma \). For each \( \Lambda \in \Gamma \) we define the following temporal two-point correlation function for the local Gibbs state (19) under the local Heisenberg time evolution (13):

\[
f^\beta_{\Lambda; \hat{\Lambda}, \hat{B}_\Lambda}(t) \equiv \rho^\beta_{\Lambda; t}(\hat{\Lambda}) \hat{B}_\Lambda, \quad t \in \mathbb{R}.
\]

By using these functions, which we call WOK temporal correlation functions, the following results were shown in [53] [54]: For each fixed \( t \in \mathbb{R} \)

\[
\lim_{\Lambda \uparrow \Gamma} \left| f^\beta_{\Lambda; \hat{\Lambda}, \hat{B}_\Lambda}(t) - f^\beta_{\Lambda; \hat{\Lambda}, \hat{B}_\Lambda}(0) \right| = 0 \quad \text{(23)}
\]

and

\[
\lim_{\Lambda \uparrow \Gamma} \left| f^\infty_{\Lambda; \hat{\Lambda}, \hat{B}_\Lambda}(t) - f^\infty_{\Lambda; \hat{\Lambda}, \hat{B}_\Lambda}(0) \right| = 0 \quad \text{(24)}
\]

By triviality of the Griffiths-type LRO with respect to \( t \in \mathbb{R} \) as in [29] [24], Watanabe-Oshikawa-Koma concluded “absence of quantum time crystals for equilibrium states”.

Remark 10. Equation (23) does not imply the identity \( \lim_{\Lambda \uparrow \Gamma} f_{\Lambda; \hat{\Lambda}, \hat{B}_\Lambda}(t) = \lim_{\Lambda \uparrow \Gamma} f_{\Lambda; \hat{\Lambda}, \hat{B}_\Lambda}(0) \). The existence of these limits is unknown. Precisely, the upper limit should be taken in stead of the limit as \( \lim_{\sup_{\Lambda \uparrow \Gamma}} f_{\Lambda; \hat{\Lambda}, \hat{B}_\Lambda}(t) \), see Theorem III.10.2. of [48]. In order to ensure the existence of \( \lim_{\Lambda \uparrow \Gamma} f^\beta_{\Lambda; \hat{\Lambda}, \hat{B}_\Lambda}(t) \) for all \( t \in \mathbb{R} \) (even for \( t = 0 \)) the local Hamiltonians and the local order parameters should not be completely chaotic, but should have a certain spatial regularity. In fact, the derivation of Eq. (9) of [53], which leads to Equation (23), tacitly assumed translation invariance of these quantities.

The recent preprint [38] reported that the lack of translation covariance of local order parameters can yield breakdown of (24) for a translation invariant vacuum state of some simple quantum field model.

B. On Watanabe-Oshikawa-Koma’s no-go statement

In our understanding, the key idea of [53] [54] is based on [27] that developed a method for constructing SSB in quantum spin lattice models. This method may be summarized as follows:

- Consider a local Gibbs state \( \rho^\beta_{\Lambda} \) for each \( \Lambda \in \Gamma \) which is assumed to be symmetric with respect to the pertinent symmetry transformation.
- Then, by taking the infinite-volume limit \( \Lambda \uparrow \Gamma \) one sees whether \( \{ \rho^\beta_{\Lambda}; \Lambda \in \Gamma \} \) gives rise to a statistical mixture of multiple phases which break the pertinent symmetry group to its some subgroup (without adding an external symmetry breaking field).

Following this method Watanabe-Oshikawa-Koma postulated that the existence of quantum time
crystals identifies with the emergent periodicity of $\lim_{\Lambda \uparrow \Gamma} f_{\beta, \text{Gib}}^{\Lambda}(t)$ \[53 \] \[54\]. Before discussing this postulate, we compare the two different formalisms of LRO:

1. The existence of non-trivial $C^*$-algebraic LRO is equivalent to the existence of multiple phases in a certain form. Here the states under consideration are homogeneous states on the quasi-local $C^*$-algebra $\mathcal{A}$, but not necessarily equilibrium states. See §5.2 of [47] for the detail of this equivalence.

2. It appears that only a one-sided implication is known in Griffiths-type LRO: If Griffiths-type LRO manifests in a spin lattice model, then a corresponding spontaneous symmetry breakdown exists. See Theorem 1.3 of [17], [26], [27], and also §5.5 of [47], Sec.III.10 of [48]. However, generally, it is impossible to conclude the absence of symmetry breakdown from the absence of Griffith-type LRO without further information about the model.

To discuss the infinite-volume limit of local Gibbs states and local Heisenberg time evolutions, we introduce some relevant concepts. Let $\rho^{\beta, \text{Gib}}$ denote an arbitrary accumulation point of local Gibbs states $\{\rho^\beta_{\Lambda, t}, \Lambda \in \Gamma\}$ as $\Lambda \uparrow \Gamma$. Heuristically, we write

$$\rho^{\beta, \text{Gib}}(A) = \lim_{\Lambda \uparrow \Gamma} \rho^\beta_{\Lambda, t}(A) = \lim_{\Lambda \uparrow \Gamma} \rho^\beta_{\Lambda, t}(A), \quad A \in \mathcal{A}. \quad (25)$$

Any accumulation point $\rho^{\beta, \text{Gib}}$ is usually called an infinite-volume limit of local Gibbs states, but such $\rho^{\beta, \text{Gib}}$ is not necessarily unique. The quantum time evolution $\{\alpha_t \in \text{Aut}(\mathcal{A}), \ t \in \mathbb{R}\}$ is called approximately inner if

$$\alpha_t(A) = \lim_{\Lambda \uparrow \Gamma} \alpha_{\Lambda, t}(A) \quad \text{for each } A \in \mathcal{A} \text{ and } t \in \mathbb{R}, \quad (26)$$

where the convergence is with respect to the norm (or $\sigma$-weak topology introduced by the GNS representation of a state under consideration).

**Remark 11.** The existence of strongly continuous approximately inner $\{\alpha_t \in \text{Aut}(\mathcal{A}), \ t \in \mathbb{R}\}$ by the formula \[26\] and the existence of infinite-volume limit of local Gibbs states $\rho^{\beta, \text{Gib}}$ as in \[25\] have been verified for general quantum spin models of short-range interactions, see [12] \[40\] and Theorem 6.2.4 of [8]. Also it is known that any such $\rho^{\beta, \text{Gib}}$ is a KMS state for $\{\alpha_{\Lambda, t} \in \text{Aut}(\mathcal{A}), \ t \in \mathbb{R}\}$. However, quantum time evolutions and KMS states do not always exist in the $C^*$-algebra. For example, a class of strong coupling BCS models does not generate an infinite-volume time evolution as $C^*$-dynamics; it exists only in a state-dependent manner [49] [6].

We now compare the assumption on local Hamiltonians of [53] [54] with ours. Finite-range Hamiltonians are considered in [54]. The Lieb-Robinson bound estimate \[32\] for local Hamiltonians is applied to derive \[25\]. It has been known that any finite-range quantum spin lattice model has an approximately inner $\{\alpha_t \in \text{Aut}(\mathcal{A}), \ t \in \mathbb{R}\}$, and that the Lieb-Robinson bound estimate yields an approximately inner time evolution, see [5] [39]. On the other hand, Proposition \[1\] Theorem \[2\] and Corollary \[3\] in this review require no particular restriction on the range of local Hamiltonians, although the existence of $\{\alpha_t \in \text{Aut}(\mathcal{A}), \ t \in \mathbb{R}\}$ excludes some long-range Hamiltonians as noted in Remark \[11\]. We did not directly use the Lieb-Robinson bound estimate anywhere in the proof of our statements. Thus, as the setups of [53] [54] can be formulated in our $C^*$-algebraic formulation, the no-go statement of quantum crystals [53] [54] may be considered as a special case of Corollary \[3\]. However, see Remark \[13\] for more precise argument.

**Remark 12.** This is a historical remark of the Lieb-Robinson bound estimate mentioned above. It is related to the existence problem of quantum time evolutions in infinite quantum systems. In fact, the original paper [32] by Lieb-Robinson emphasized this point addressing the Robinson’s publication [12] that established infinite-volume time evolutions of a general class of quantum spin models. Recently, the Lieb-Robinson bound estimate has become a popular tool for investigating quantum dynamics, but the above important relationship seems to be under-appreciated in the physics literature.

As reviewed above, the infinite-volume limit of WOK temporal correlation functions $\lim_{\Lambda \uparrow \Gamma} f_{\beta, \text{Gib}}^{\Lambda}(t)$ is used to diagnose quantum time crystals in [53] [54]. We agree that this limit describes temporal behavior of $\rho^{\beta, \text{Gib}}$. However, regarding the general status of Griffiths-type LRO, “absence of quantum time crystals for all equilibrium states” cannot be concluded solely from \[25\] \[21\] without more elaborate reasoning on the model under consideration. Non-detection by a particular method is not proof of nonexistence of something, no matter how useful the method is to find its existence.

We now make other relevant remarks on $\rho^{\beta, \text{Gib}}$. It is known that $\rho^{\beta, \text{Gib}}$ satisfies the KMS condition for $\{\alpha_{\Lambda, t} \in \text{Aut}(\mathcal{A}), \ t \in \mathbb{R}\}$ for some class of quantum models as noted in Remark \[11\]. However, the converse implication, whether any symmetric KMS state can be given as an infinite-volume limit of local symmetric Gibbs states, is presently unknown in general quantum spin lattice models. The work [11] suggested that the method for constructing SSB from the infinite-volume limit of local symmetric Gibbs states \[27\] fails in some scenarios. For example, it cannot construct interface phases of the ferromagnetic Ising model [13] associated with the breakdown of a reflection symmetry in space $\mathbb{Z}^d$ (and also the breakdown of the space-translation symmetry). Even when considering only translation-invariant equilibrium states and compact internal symmetry groups for translation-invariant short-range quantum spin models, we lack general knowledge of when infinite-volume limits of local symmetric Gibbs states exhaust all symmetric equilibrium states in the infinite system. The only known cases are trivial models with a unique equilibrium state.
and some special models that have been intensively studied such as the Ising model on $\mathbb{Z}^d$.

Note that $f_{\Lambda; \tilde{A}; \tilde{B}}^{\beta, \text{Gib}}(t)$ defined in (22) has the same $\Lambda$-dependence on the local Gibbs state $\rho_{\Lambda; \tilde{A}}^{\beta, \text{Gib}}$, the cut-off time-translation symmetry $\alpha_{\Lambda, t}$, and the local order parameters $\tilde{A}$ and $\tilde{B}$. In [53, 54] and other subsequent papers in physics, such $\Lambda$-dependence of the WOK temporal correlation functions is not manifestly written. However, the infinite-volume limit of $\{f_{\Lambda; \tilde{A}; \tilde{B}}^{\beta, \text{Gib}}(t); \Lambda \in \Gamma\}$ is far from obvious as noted in Remark [10]. If there are multiple accumulation points, then how to discuss the time dependence of such is not clear, although we have no concrete example to invalidate the heuristic argument of the infinite-volume limit.

To discuss the problem concerning the infinite-volume limit mentioned above, we now suppose that $\{A_x; x \in \Gamma\}$ and $\{B_x; x \in \Gamma\}$ satisfying (20) are both covariant in space-translations. Then by taking the following limits in the order given, we get a $C^*$-algebraic LRO whose existence is verified:

$$\lim_{\Lambda \to \infty} \lim_{\tilde{A}_3 \to \tilde{A} \in \Gamma} \lim_{\tilde{A}_2 \to \tilde{A} \in \Gamma} \lim_{\tilde{A}_1 \to \tilde{A} \in \Gamma} \rho_{\Lambda; \tilde{A}}^{\beta, \text{Gib}}(\alpha_{\tilde{A}_1, t}(\tilde{A}_3)\tilde{B}_{\Lambda}) = f_{\Lambda; \tilde{A}_0, \tilde{B}_0}^{\beta, \text{Gib}}(t), \tag{27}$$

where the above $f_{\Lambda; \tilde{A}_0, \tilde{B}_0}(t)$ is given by $f_{\Lambda, t}(t)$ with the approximately inner time evolution $\{\alpha_t \in \text{Aut}(\Lambda), t \in \mathbb{R}\}$ for its time evolution, and with $A = \Lambda_0$, $B = \Lambda_0$, and $\varphi = \rho_{\Lambda; \tilde{A}}^{\beta, \text{Gib}}$. However, as noted above, the status of $\lim_{\Lambda \to \infty} f_{\Lambda; \tilde{A}; \tilde{B}}^{\beta, \text{Gib}}(t)$ remains unclear. For further discussion, see Lemma 4 and Remark 13 below.

We address another problem with the WOK temporal correlation functions that appears more directly relevant to physics. In [53, 54], the local Gibbs state and local time-translation symmetry are determined by the same local Hamiltonian $\tilde{H}_\Lambda$. This assumption is essential for the proof of their main statement. However, we consider that one should take all possible boundary conditions into account, and so different choices of local Hamiltonians for a local Gibbs state and a local time-translation on the same $\Lambda$ should not be excluded a priori. Furthermore, this artificial coincidence needs justification in terms of physics, because the true quantum time evolution instantly evolves the local observables to nonlocal ones, whereas the cut-off time evolution on $\Lambda$ unnaturally confines the local observables of $A_{\mathbb{Z}^d}$ in its slightly larger subsystem $A_{\Lambda\cup\text{int} \Lambda}$ not allowing them to escape toward outer regions. The following lemma may provide a clue to this problem. It estimates the difference between the cut-off and infinite-volume time evolutions.

**Lemma 4.** Let $\Lambda$ denote a quantum spin system on the lattice $\mathbb{Z}^d$. Suppose that the time evolution $\{\alpha_t \in \text{Aut}(\Lambda), t \in \mathbb{R}\}$ is translation invariant, strongly continuous, and approximately inner. Let $\{\alpha_{\Lambda, t} \in \text{Aut}(\Lambda), t \in \mathbb{R}\}$ denote the local Heisenberg time evolution for $\Lambda \in \Gamma$ as given in (18). Let $\{A_x = \tau_x(A_0); x \in \Gamma\}$, where $A_0$ is a local operator with its support centered at the origin of $\mathbb{Z}^d$. Let $\varepsilon > 0$ and $t_0 > 0$. Then for sufficiently large $\Lambda \in \Gamma$ the following estimate holds for all $t \in [-t_0, t_0]$:

$$\left\|\alpha_t(\tilde{A}_0) - \alpha_{\Lambda, t}(\tilde{A}_0)\right\| \leq \varepsilon. \tag{28}$$

**Proof.** For each $n \in \mathbb{N}$ define

$$\Lambda_0(n) := \left\{(x_1, x_2, \ldots, x_\mu) \in \mathbb{Z}^d; 0 \leq |x_i| \leq \frac{n}{2}\right\} \in \mathbb{Z}^d. \tag{29}$$

It is a box region centered at the origin $0 \in \mathbb{Z}^d$ and $\text{diam}(\Lambda_0(n)) = n$ or $n - 1$. Let $X(n) := \Lambda_0(n) + x$, i.e. the translation of $\Lambda_0(n)$ by $x \in \mathbb{Z}^d$. From the assumption [20] we have

$$\alpha_t(A) = \lim_{n \to \infty} \alpha_{\Lambda_0(n), t}(A) \quad \text{for each } A \in \Lambda \text{ and } t \in \mathbb{R}. \tag{30}$$

By (21) for any $\Lambda \in \Gamma$

$$\alpha_{\Lambda, t}(\tilde{A}_0) = \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \alpha_{\Lambda, t}(A_x). \tag{31}$$

Since the time evolution under consideration is space translation invariant, for any fixed $\varepsilon > 0$ and $t_0 > 0$, there exists a constant $m(> r) \in \mathbb{N}$ (that is independent of $x \in \Gamma$) such that the following estimate holds

$$\left\|\alpha_t(A_x) - \alpha_{\Lambda, t}(A_x)\right\| < \varepsilon/2 \quad \text{for any } t \in [-t_0, t_0] \text{ and } x \in \Gamma, \tag{32}$$

where $I_x$ is any finite subset that includes the box region $\Lambda_x(m)$ centered at $x$:

$$I_x \supset \Lambda_x(m). \tag{33}$$

We now take a sufficiently large $\Lambda \in \Gamma$ such that $\Lambda \ni 0$. We divide $\Lambda$ into the following two complement regions:

$$\tilde{\Lambda}_\varepsilon := \{x \in \Lambda; \Lambda_x(m) \subset \Lambda\}, \quad \text{int} \Lambda \setminus \tilde{\Lambda}_\varepsilon := \Lambda \setminus \tilde{\Lambda}_\varepsilon. \tag{34}$$

where the subscript indicates $\varepsilon$-dependence, but $t_0$ dependence is omitted as there is no fear of confusion. Hence by the obvious inclusion $\Lambda \supset \bigcup_{x \in \Lambda} \Lambda_x(m)$, from (32) it follows that

$$\left\|\alpha_t(A_x) - \alpha_{\Lambda, t}(A_x)\right\| < \varepsilon/2 \quad \text{for any } t \in [-t_0, t_0] \text{ and } x \in \tilde{\Lambda}_\varepsilon. \tag{35}$$
Let \( a := \| A_0 \| \). By using (31) \( \sum \) we obtain

\[
\left\| \alpha_t(\tilde{A}_\Lambda) - \alpha_{\Lambda,t}(\tilde{A}_\Lambda) \right\| = \frac{1}{|\Lambda|} \left\| \sum_{x \in \Lambda} (\alpha_t(x) - \alpha_{\Lambda,t}(x)) \right\|
\]

\[
\leq \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \left\| \alpha_t(x) - \alpha_{\Lambda,t}(x) \right\|
\]

\[
= \frac{1}{|\Lambda|} \sum_{x \in \Lambda_s} \left\| \alpha_t(x) - \alpha_{\Lambda,t}(x) \right\|
\]

\[
+ \frac{1}{|\Lambda|} \sum_{x \in \partial_{\text{ext}} \Lambda_s} \left\| \alpha_t(x) - \alpha_{\Lambda,t}(x) \right\|
\]

\[
\leq \frac{1}{|\Lambda|} \sum_{x \in \Lambda_s} \left\| \alpha_t(x) - \alpha_{\Lambda,t}(x) \right\| + \frac{1}{|\Lambda|} \sum_{x \in \partial_{\text{ext}} \Lambda_s} \| 2A_x \|
\]

\[
\leq \frac{1}{|\Lambda|} \cdot |\tilde{A}_\Lambda| \cdot \varepsilon + \frac{1}{|\Lambda|} |\partial_{\text{ext}} \tilde{A}_\Lambda| \cdot 2a
\]

\[
\leq \frac{\varepsilon}{2} + 2a \frac{|\partial_{\text{ext}} \tilde{A}_\Lambda|}{|\Lambda|}
\]

By (31) for each fixed \( \varepsilon > 0 \) we have

\[
\lim_{\Lambda \to \infty} \frac{|\partial_{\text{ext}} \tilde{A}_\Lambda|}{|\Lambda|} = 0,
\]

where the above infinite-volume limit \( \Lambda \to \infty \) is the so called van Hove limit, see Sec. 6.2.4 of [2]. By (32) \( \sum \) and (36) \( \sum \) for sufficiently large \( \Lambda \in \Gamma \) we obtain the estimate (37).

Remark 13. Lemma 4 shows that the difference between these two different time evolutions of any local order parameter disappears in the infinite-volume limit. However, it does not imply that

\[
\lim_{\Lambda \to \infty} f^{\beta, \Lambda}_\Lambda(t) = f^{\beta, \Lambda}_0(t), \quad t \in \mathbb{R}
\]

This conjectural equation seems to be difficult to prove or disprove. Thus presently, Corollary 3 is logically independent of the no-go statement of [53] \( \sum \).

V. OUTLOOK AND DISCUSSION

A. Summary of this review

We have presented mathematically rigorous statements of nonexistence of spontaneous breakdown of time-translation symmetry. All statements (Proposition 10, Theorem 2, and Corollary 3) are based on the KMS-condition \( \sum \). Although they are not our findings, we refer to them as our statements for convenience.

Theorem 2 excludes any type of quantum time crystals in equilibrium states, such as space-time crystals with periodicity in both the space and time directions as in [31] and also aperiodic time crystals as discussed in [24]. Consequently, interfaces (domain walls) and chaotic orders in the time direction cannot manifest in equilibrium states. As interfaces are a common symmetry breakdown of space-translation symmetry \( \sum \), one may speculate temporal orders for interface equilibrium states. Such weird quantum time crystals seem not to be precluded by the statement of [53] \( \sum \), since the argument used there requires spatial homogeneity. On the other hand, those are surely negated by Theorem 2. In conclusion, genuine quantum time crystals, namely quantum time crystals in equilibrium states, are perfectly forbidden under the assumption of Theorem 2.

On the ground of our no-go statements that seem to cover rather general cases of quantum systems, we consider that the concept of genuine quantum crystals is an imaginary one. Its adjective “genuine” seems somewhat misleading. Furthermore, our view on the history of genuine quantum crystals differs from existing ideas on it shared by physicists as described in [14] \( \sum \) \( \sum \) \( \sum \) \( \sum \). They ignore the basic knowledge about equilibrium states on general quantum systems which immediately negates this concept as shown in this review. Thereby, our standpoint disagrees with repeated advertisements about quantum time crystals made by research papers, non-technical articles, YouTube videos, and so on that allude to reconsideration of fundamental laws of physics with no firm grounds, although maybe we are supposed to enjoy them as SF.

Finally, we emphasize that this review never criticizes the study of quantum time crystals. Those are remarkable nonequilibrium phenomena that do not conflict with fundamental laws of physics.

B. Validity of our assumptions

As noted in Remark 6 and Remark 11, formulating a many-body quantum dynamics as an appropriate \( C^* \)-dynamics is not obvious except a few tractable examples. Thus our no-go statements based on the \( C^* \)-algebraic framework will inevitably have some limitations. In particular, those may not give a completely rigorous answer for interacting quantum field models. Notably, the original quantum crystal model [53] made by many-particles on an Aharonov-Bohm ring is beyond the scope of our \( C^* \)-algebraic formulation. We need to refer to [14] \( \sum \) \( \sum \) \( \sum \) for its detailed analysis. With the above remarks in mind, we will discuss the validity and possible generalization of our no-go statements in the following.

For ground states, the \( C^* \)-algebra formulation is not necessarily required if the given quantum model is implemented by a self-adjoint Hamiltonian on a Hilbert space. However, in general, knowing all possible self-adjoint realizations of the model Hamiltonian of infinitely many degrees of freedom is difficult.

One cannot expect the existence of \( \{ \alpha_t \in \text{Aut}(\Lambda), t \in \mathbb{R} \} \) for slowly decaying models. The work [7] reports recent results of infinite-volume time evolutions and KMS states for some long-range models for which Proposition 44 in the time direction cannot manifest in equilibrium states. As interfaces are a common symmetry breakdown of space-translation symmetry \( \sum \), one may speculate temporal orders for interface equilibrium states. Such weird quantum time crystals seem not to be precluded by the statement of [53] \( \sum \), since the argument used there requires spatial homogeneity. On the other hand, those are surely negated by Theorem 2. In conclusion, genuine quantum time crystals, namely quantum time crystals in equilibrium states, are perfectly forbidden under the assumption of Theorem 2.

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Finally, we emphasize that this review never criticizes the study of quantum time crystals. Those are remarkable nonequilibrium phenomena that do not conflict with fundamental laws of physics.
and Theorem 2 are valid. However, the formulation of quasi-local $C^*$-algebras is invalid for some long-range dynamics that induce the Higgs-phenomenon.

Kozin-Kryienko constructed genuine quantum time crystals by some infinite-range Hamiltonians showing a non-trivial periodic behavior of $\lim_{t \to 1} f_{\Lambda}^{\beta, \text{Gibb}} (t)$ in $t \in \mathbb{R}$. The feasibility of such long-range Hamiltonians and highly entangled ground states has been critically argued in [23]. Additionally, we suspect that some fundamental issues remain in their proposal. Even if $\lim_{t \to 1} f_{\Lambda}^{\beta, \text{Gibb}} (t)$ is available, it is not obvious how to define equilibrium states and quantum time evolutions of such excessively long-range models in huge systems. Mathematically, how can one make sense of the asymptotic formulas in the infinite-volume limit? It looks very difficult to make sense of, because the infinitesimal time translation $\lim_{t \to 1} [H_{\Lambda}, A]$ is not well-defined for generic local operators $A \in A_{\text{loc}}$. Kozin-Kryienko refers to the criterion of Watanabe-Oshikawa-Koma based on Griffiths-type LRO as the conventional definition of quantum time crystals. Lemma 3 will be invalid for general long-range models. The reason for this invalidity is as follows. As the order parameter $m_\Lambda (A)$ ($\Lambda \in \Gamma$) is the mean of $\{ \tau_x (A) \}$ over $x \in \Lambda$ as defined in [38], for any $\Lambda \in \Gamma$, no matter how it is large, most $\tau_x (A)$ with $x \in \Lambda$ do evolve unnaturally within the given region plus some surface under the cut-off time evolution $\alpha_{\Lambda, t}$ ($t \in \mathbb{R}$). Of course, one may argue that these long-range models can be realized in a sufficiently large but finite system under a realistic boundary condition connecting the local system and its outer system. However, engineering a local Hamiltonian that has excessively nonlocal effects in a closed system seems to entail obvious difficulties. We now conclude our speculative argument with the following obvious remark. Our no-go statements assuming the existence of infinite-volume time evolutions cannot be applied to Kozin-Kryienko’s models.

C. Quantum time crystals beyond the case of equilibrium

In the final part of [54], construction of quantum time crystals by a net of local states (densities) invariant under the local Heisenberg time evolutions is considered. In [22], another method of generating quantum time crystals by stationary states is provided with a concrete example of quantum spin lattice models. In $C^*$-algebraic language, a general framework for such quantum time crystals can be formulated as follows. Suppose that $\psi$ is an invariant state under the time-translation symmetry $\{ \alpha_t \in \text{Aut}(A), t \in \mathbb{R} \}$ but it is not an equilibrium state. Suppose that $\psi$ has the following specific factorial decomposition: For some $p > 0$

$$\psi = \int_0^p dt \alpha_t^* \psi, \quad \psi \in S_{\text{factor}} (A), \quad (39)$$

where $\psi$ is a factor state breaking the time-translation symmetry $\{ \alpha_t \in \text{Aut}(A), t \in \mathbb{R} \}$ but invariant under its discrete subgroup $\{ \alpha_t \in \text{Aut}(A), t \in p \mathbb{Z} \}$. If we further assume that $\psi$ is a homogeneous state, then by the equivalence of the existence of non-trivial $C^*$-algebraic LRO and that of multiple phases [47], the function $f_{\Lambda, A}^\psi (t)$ defined by the formula (12) oscillates periodically in time for some local order parameter $A \in A$. The above formulation exploits the identification of factor states and pure phases. It can be considered as a naive generalization of the notion of SSB to stationary states.

Literally, “nonequilibrium” has diverse meanings. Let us restrict our consideration to nonequilibrium states that are not excessively far from equilibrium. As noted in Remark 1, passivity [41] is an expression of the second law of thermodynamics of quantum systems governed by stationary Hamiltonian dynamics. If one requires passivity for any homogeneous stationary state, such as $\psi$ given in the above (39), then it should be a KMS state, see 3.3.27 of [53]. Thus, measuring the deviation of stationary states (or time-periodic states) from equilibrium in terms of passivity is an interesting prospect to look for modest nonequilibrium quantum time crystals. In passing, we mention another framework of investigation of breakdown of time-translation symmetry advocated by [34]. Of course, it does not discuss the usual sense of SSB of time-translation symmetry. It deals with out-of-equilibrium phenomena.

D. Why are quantum time crystals impossible for equilibrium states?

Finally, we return to the fundamental question: why are quantum time crystals impossible for equilibrium states, whereas spatial crystals in equilibrium states are common?

A similar question was argued in [53] from the theory of relativity and the spectrum of Hamiltonian and momentum operators. We have addressed the first point in Remark 1. We now focus on the second point. The spectrum of Hamiltonians (the so-called Liouvillians) of KMS states on infinite quantum systems is generically two-sided unbounded like that of momentum operators. However, as we have justified, quantum time crystals are impossible for KMS states. On the other hand, number operators of ground states are bounded from below, while the $U(1)$ symmetry generated by them can be spontaneously broken, such as Bose-Einstein condensations. The above observations seem inconsistent with the criterion of existence and nonexistence of crystalline structure in equilibrium states based on the spectrum of generators of space and time symmetries suggested by [52]. We consider that it is not possible to attribute the lack of temporal crystals in equilibrium states to merely the information of the spectrum of Hamiltonians.

We give our partial answer to the question. We consider that the absence of spontaneous breakdown of time-
Translation symmetry is due to the rigid stability of equilibrium states. Among others, the passivity [41] and the attainment of the minimum free energy (the minimum energy in the case of $\beta = \infty$) determined by the variational principle as formulated in [13] (in the case of $\beta = \infty$) will be crucial. This is because they are expressions of the second law of thermodynamics of quantum systems [3].

This review has narrowed the chance of genuine quantum time crystals. If you still look for them in their original sense $\delta$, $\delta^2$, the following points should be checked:

- Are quantum time evolution and equilibrium states well-defined for your model without ambiguity?

- How do you formulate the second law of thermodynamics? Does it apply well to your equilibrium model?

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