Magnetohydrodynamics In The Context Of
Nelson’s Stochastic Mechanics

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Abstract

A simple generalization of the MHD model accounting for the fluctuations of the configurations due to kinetic effects in plasmas in short times small scales is considered. The velocity of conductive fluid and the magnetic field are considered as the stochastic fields (or random trial trajectories) for which the classical MHD equations play the role of the mean field equations in the spirit of stochastic mechanics of E. Nelson.

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1 Introduction

Theoretical investigations of cross-field transports in the operating ITER-FEAT (International Thermonuclear Experimental Reactor) calls for an increasing confidence in the modelling efforts that force one to search for the new principles of simulations. The aim of our work is to provide a possible ground for the optimization of existing numerical simulation algorithms for the large scale simulations in hydrodynamics.

The set of equations which describe magnetohydrodynamics (MHD) are a combination of the Navier-Stokes equations (1) of fluid dynamics and Maxwell’s equations (2) of electromagnetism,

\[
\begin{align*}
\dot{v} + \nabla P + (v \cdot \nabla) v - (b \cdot \nabla) b &= \nu \Delta v \\
\dot{b} + (v \cdot \nabla) b - (b \cdot \nabla) v &= \lambda \Delta b,
\end{align*}
\] (1)
in which $\lambda = \frac{c^2}{4\pi\sigma}$ is the resistivity constant, the inverse Prandtl-type constant. The equations (1) describe the dynamics of electrically conducting fluids. The fluid velocity $v(x, t)$, $x \in \mathbb{R}^d$ is supposed to be incompressible, so that the mass continuity equation for that is reduced to the transversally condition $\nabla \cdot v = 0$. The normalized magnetic field $b(x, t)$ is

$$b(x, t) = \frac{B(x, t)}{\sqrt{4\pi\rho}}, \quad \nabla \cdot b = 0,$$

in which $B(x, t)$ is the magnetic induction and $\rho$ is the density of medium. In addition to the usual hydrodynamical interaction presented in the Navier-Stokes equation, in the moving medium, there is the Lorentz force [1] exerted on charged particles in the electromagnetic field,

$$[\text{curl } B \times B] = (B \cdot \nabla)B - \nabla(B^2/2).$$

The first term in (3) is amended to the hydrodynamical interaction while the second term redefines the pressure field in the medium,

$$P(x, t) \rightarrow p(x, t) + \frac{B^2}{2}.$$  

The dynamical equation describing the evolution of magnetic field follows from the simplest form of Ohm’s law [1],

$$j(x, t) = \sigma \left( E(x, t) + \frac{1}{c} [\varphi(x, t) \times B(x, t)] \right),$$

in which $\sigma$ is the conductivity, $c$ is the speed of light, $E$ and $\varphi$ are the electric field and the electrostatic potential respectively, and the Maxwell equations neglecting the displacement current.

The purely longitudinal contributions of the pressure gradient $\nabla P$ and of interactions can be eliminated from (1) by the applying of transverse projection,

$$P_{ij}^{\perp} = \delta_{ij} - \frac{k_i k_j}{k^2},$$

if written in the Fourier space.

The equations (1) are applicable to the plasma if it is strongly collisional, so that the time scale of collisions is shorter than the other characteristic times in the system, and the particle distributions are therefore close to Maxwellian. When this is not the case (for instance in fusion plasmas), we are interested in smaller spatial scales, in which it may be necessary to use a kinetic model which properly accounts for the non-Maxwellian shape of the distribution function. However, because MHD is very simple, and captures many of the important properties of plasma dynamics, it is often qualitatively accurate, and therefore while accounting for the possible kinetic effects in plasmas we are nevertheless interested to stay within the general framework of the MHD approach.
In the present paper, we investigate a simple generalization of the MHD model modelling fluctuations of the configurations \( \{ v, b \} \) due to kinetic effects in plasmas. In the proposed model, we suppose that \( \{ v, b \} \) are the stochastic fields for which (1) plays the role of the mean field equations. Recently, we have implemented the similar approach for the Burgers and Navier-Stokes equations in [2].

2 Stochastic dynamics as the Brownian motion

It is well known that many problems in stochastic dynamics can be treated as a generalized Brownian motion \( \langle \delta (u - u(x, t)) \rangle_\xi \), in which the classical random field indicating the position of a particle \( u(x, t) \) meets a Langevin equation,

\[
\dot{u}(x, t) = K(u) + Q(u) + \xi,
\]

where \( \xi \) is the Gaussian distributed stochastic force characterized by the correlation function

\[
D_\xi = \langle \xi \xi \rangle.
\]

Here the angular brackets \( \langle \ldots \rangle_\xi \) denote an average position of particle with respect to the statistics of \( \xi \). \( K(u) \) is the linear differential operator, and \( Q(u) \) is some \( t \)-local (independent of time derivatives) nonlinear term which depends on the position \( u(x, t) \) and its spatial derivatives. Such a representation was a key idea of the famous Martin-Siggia-Rose (MSR) formalism, [3]-[6].

An elegant way to obtain the field theory representation of stochastic dynamics is given by the functional integral

\[
\langle \delta (u - u(x, t)) \rangle_\xi \equiv \int D\xi \exp \left( -\frac{1}{2} \xi D_\xi \xi \right) \delta (u - u(x, t))
\]

where the \( \text{tr} \)-operation means the integration \( \int dx dt \) and the summation over the discrete indices. The instantaneous positions \( u(x, t) \) meet the dynamical equation that can be taken into account by the change of variables

\[
\delta (u - u(x, t)) \rightarrow \delta (\dot{u}(x, t) - Q(u) - \xi)
\]

should the solution of dynamic equation exists and is unique. The use of integral representation for the \( \delta \)-function in (8) transforms it into

\[
\int D\xi D\xi' \exp \left( -\frac{1}{2} \xi D_\xi \xi - u'\dot{\xi} + u'Q(u) + u'\xi \right) \det M,
\]

in which \( u'(x, t) \) is the auxiliary field that is not inherent to the original model, but appears since we treat its dynamics as a Brownian motion. The Jacobian \( \det M \) relevant to the change of variables [9] is discussed later.

The Gaussian functional integral with respect to the stochastic force \( \xi \) in (10) is calculated

\[
\int DuDu' \exp S(u, u') \det M,
\]
in which
\[ S(u, u') = \text{tr} \left[ -\frac{1}{2} u'D\xi u' - u'u + u'Q(u) \right]. \]  
\[\text{(12)}\]

By means of that all configurations of \( \xi \) compatible with the statistics are taken into account. The integral \( (10) \) identifies the statistical averages \( \langle \ldots \rangle_\xi \) with the functional averages of weight \( \exp S \). The formal convergence requires the field \( u \) to be real and the field \( u' \) to be purely imaginary.

The functional averages in \( (11) \) can be represented by the standard Feynman diagram series exactly matching (diagram by diagram) the usual diagram series found by the direct iterations of the Langevin equation averaged with respect to the random force. This fact justifies the use of functional integrals in stochastic dynamics at least as a convenient language for the proper diagram expansions.

The Jacobian \( \det M \) in \( (10) \) depends upon the nonlinearity \( Q(u) \). If \( Q(u) \) does not depend upon the time derivatives, all diagrams for \( \det M \) are the cycles of retarded lines \( \Delta \propto \theta(t - t') \) and equal to zero excepting for the very first term,
\[ \det M = \text{const} \cdot \exp \text{tr} \Delta, \]  
\[\text{(13)}\]

Then, the convention is used for the Heaviside function of zero argument, \( \theta(0) = 0 \), so that \( \det M = \text{const} \), \( [7] \).

The functional averages computed with respect to the statistical weight \( \exp S \) can be expanded into the series of Feynman diagrams drawn with the interaction vertices determined by the nonlinearity \( Q(u) \) and two propagators (lines) which have the following analytical representations (in the Fourier space)
\[ \Delta_{uu'} = \frac{1}{(i\omega + \hat{K})}, \quad \Delta_{uu} = \frac{D\xi}{(\omega^2 + \hat{K}^2)}, \]  
\[\text{(14)}\]

in which \( \hat{K} \) is the Fourier image of the linear part in the Eq. \( (7) \). The inverse Fourier transform of \( \Delta_{uu'} \) shows that it is retarded, \( \Delta_{uu'} \propto \theta(t - t_0) \).

Many dynamical systems are driven by the non-random external forces. It is worth to mention that if one assumes \( |\xi| \to 0 \) (and consequently \( D\xi \to 0 \)), then the Feynman diagram series is trivial since the propagator vanishes, \( \Delta_{uu} = 0 \). However, the diagram series would be recovered by means of regular external forcing \( [2] \) that gives rise to a branching representation of stochastic dynamics.

3 Probabilistic interpretations for the solutions of elliptic equations

The striking similarity between the Schrödinger equation written for free particles and the diffusion equation motivated the search for a stochastic interpretation of the quantum mechanics. The first attempt had been made by E. Schrödinger himself \( [8] \) and accomplished by J.C. Zambrini who derived the genuine Euclidean version of quantum mechanics \( [9] \).
As a counter motion, E. Nelson [10] had proposed a generalization of the theoretical scheme of classical mechanics known as stochastic mechanics. Classical deterministic trajectories are substituted by random trajectories of well defined stochastic processes. Under appropriate conditions, and for a large class of dynamical systems, the basic equations of stochastic mechanics show a surprising connection with the basic equations of quantum mechanics, [11]-[12]. Therefore, stochastic mechanics gives an approach to quantization of dynamical systems, based on methods of probability theory and stochastic processes.

The original formulation of stochastic mechanics rests on two basic hypothesis, [13]-[14]. The first assumes that the trajectories of the dynamical system are perturbed by an underlying Brownian motion. The second is a particular form of the second principle of dynamics, where the classical acceleration is replaced by a suitable form of stochastic acceleration. Further developments of the theory show that the basic equation of stochastic mechanics can be derived from variational principles, in complete analogy with classical mechanics, based on the same classical action, but exploiting stochastically perturbed trajectories as trial trajectories [14]. The basic equations of stochastic mechanics (the continuity equation and Madelung equation) can be immediately connected with the Schroedinger equation of quantum mechanics. The entire operator structure of the quantum mechanical observables can be easily derived from the general structure of stochastic mechanics. From this point of view, stochastic mechanics can be considered as a kind of probabilistic simulation of quantum mechanics, [13].

Stochastic mechanics can be based on variational principles of Lagrangian type. In order to obtain that it is necessary to generalize the action of classical mechanics to the case where the trial trajectories belong to stochastic processes. This program has been partially realized in [15]-[16] for dynamical systems on curved manifolds. This was the analog of the well known problem of writing the Feynman path integral for a quantum system on a curved manifold, so that all the results could be immediately translated into the language of Feynman path integrals.

Independently of quantum mechanics and quantum field theory, a probabilistic interpretation for the solutions of linear elliptic and parabolic equations with Cauchy and Dirichlet boundary conditions had been proposed [17]. A stochastic process had been defined for which the mean values of some functionals coincide with the solution of the deterministic equations. The problem of existence of such the probabilistic representations for the certain classes of nonlinear equations has been studied extensively by Dynkin [18]. The probabilistic representations of the Fourier transformed Navier-Stokes and Burger’s equations had been discussed in [19] and later extensively developed in the works of Oregon group [20]-[22]. A stochastic representation for the Poisson-Vlasov equation has been derived in [23] recently. In all cases when the appropriate stochastic process has been constructed, its mean values is the solution of the mean field equation, but the process itself always contains more information then that of physical relevance in particular.

A model in which the classical deterministic trajectories $u(x, t), x \in \mathbb{R}^d$, sat-
satisfying the hydrodynamics equations are substituted by the random trajectories of a generalized Brownian motion over the space of fluid velocity configurations \( u \) driven by the stochastic force \( \xi \) is known as stochastic hydrodynamics \([24]\). Here, \( \xi \) is the Gaussian distributed stochastic force characterized by the correlation function \( D\xi = \langle \xi \xi \rangle \), and the angular brackets \( \langle \ldots \rangle_\xi \) denote an average velocity of particle with respect to the statistics of \( \xi \). In such a formulation, the above problem is equivalent to that of Nelson’s \textit{stochastic mechanics}, \([11],[13]\). The relevant variational principle leads to an action functional of the Martin-Siggia-Rose (MSR) type \([3]\). The MSR-theory of stochastic hydrodynamics has been formulated independently by many authors, \([25]-[27]\). Diagram representations for the Green functions of stochastic hydrodynamics exactly reproduce the hydrodynamical diagrams discussed by Wyld, \([28]\). In general, the Green functions of stochastic hydrodynamics diverge for very large moments and therefore require the ultraviolet renormalization that has been discussed in details in \([7]\).

In our previous paper \([2]\), we have pointed that the diagram representations for the Green’s functions in hydrodynamics is still nontrivial if one considers a regular external forcing, instead of random one. In particular, we have studied the \( \delta(x - x_0)\delta(t - t_0) \) external forcing corresponding to the Cauchy problem of the Navier-Stokes equation supplied with an integrable initial condition. Each Feynman graph in the diagram series equals to an average over a forest of multiplicative branching binary trees (of the certain topological structure) implemented in \([20]-[22]\). The branching representations for the Green function of Cauchy problem establishes the direct relation between Nelson’s \textit{stochastic mechanics} \([11]\) and the probabilistic interpretations for the solutions of nonlinear equations with Cauchy and Dirichlet boundary conditions studied in \([18]-[22]\). It is important to note that in contrast to the MSR theory \([7]\), the diagrams of branching representations for the hydrodynamics equations \([2]\) do not diverge, but constitute a regular expansion starting from the standard diffusion kernel. Diagram contributions represent the consequent bifurcations of media resulting in the cascade of consequent partitions of moments, \( k = q + (k - q) \). The magnitude of relevant corrections to the standard diffusion spectrum tends to zero as \( t \to t_0 \), and the saddle-points (instanton) analysis can be then applied to study the ”large order” asymptotic contributions \([2]\). The calculations have shown that the asymptotic coefficients demonstrate the factorial growth like for the most of models in quantum field theory. The asymptotic series for the Green function can be summarized by means of the Borel procedure. In the limit \( t \to t_0 \), the corrections to the diffusion kernel have the closed analytical form.
4 Cauchy problem for the MHD equations and its stochastic mechanics formulation

The Cauchy problem for the MHD equations,
\[
\begin{cases}
\dot{v}_i + \partial_j (v_i v_j - b_i b_j) - \nu \partial^2 v_i = \delta(t - t_0) \delta(x - x_0), \\
\dot{b}_i + \partial_j (v_i b_i - b_i v_i) - \lambda \partial^2 b_i = \delta(t - t_0) \delta(x - x_0),
\end{cases}
\tag{15}
\]
is supplied with the localized integrable initial conditions \( v_0 = v(x, 0) \) and \( b_0 = b(x, 0) \). It is possible to construct a stochastic counterpart of MHD \((15)\) by adding a Gaussian distributed random force into it. This approach leads to the stochastic magnetohydrodynamics \([29],[7],[30]\) developed in order to study the inertial range scaling laws and different regimes of large-scale asymptotic behavior. Alternatively, in the framework of stochastic interpretation, we consider a stochastic model in which the nonlinear dynamical equations \((15)\) play the role of mean field equations, so that their solutions satisfying the given initial conditions play the role of the observables. Instead of the classical deterministic fields \(b(x, t)\) and \(v(x, t)\), we study their stochastic trial analogs, \(\tilde{b}(x, t)\) and \(\tilde{v}(x, t)\). Then the Green functions \(G_\phi, \tilde{\phi} = \{\tilde{v}, \tilde{b}\}\) of the original Cauchy problem \((15)\) can be represented by the functional averages,
\[
G_\phi(x, t; x_0, x'_0; t_0, t'_0) = \frac{\int D\Phi \tilde{\phi}(x, t) \exp S(\Phi)}{\int D\Phi \exp S_0(\Phi)}, \tag{16}
\]
over all possible configurations \(\tilde{v}(x, t)\) and \(\tilde{b}(x, t)\) such that their expectation values satisfy \((15)\) with the given initial conditions. Here, \(\Phi\) are the functional arguments of the action functional \(S(\Phi)\) such that \((15)\) are its saddle-point equations. Its "quadratic" part, \(S_0(\Phi)\), corresponds to the linearized equations of MHD which play the role of an interaction free theory. We discuss the measure of functional integration \(D\Phi\) after we consider \(S(\Phi)\). If we introduce the auxiliary fields \(\tilde{v}'(x, t)\) and \(\tilde{b}'(x, t)\), then, up to an inessential constant factor, the relevant action functional reads as following:
\[
S(\tilde{v}, \tilde{v}', \tilde{b}, \tilde{b}') = \tilde{v}'(x_0, t_0) + \tilde{b}'(x'_0, t'_0) - \text{tr} \left[ \tilde{v}'_i \tilde{v}_i + \nu \tilde{v}'_i \partial_j (g_1 \tilde{v}_i \tilde{v}_j - g_2 \tilde{b}_i \tilde{b}_j) - \nu \tilde{v}'_i \Delta \tilde{v}_i \right] \tag{17}
\]
where as in \((12)\) the \(\text{tr}-\)operator means the integration \(\int dx \, dt\) and summation over the discrete indices. The quadratic part of \((17)\) has the form
\[
S_0(\tilde{v}, \tilde{v}', \tilde{b}, \tilde{b}') = -\text{tr} \left[ \tilde{v}'_i \tilde{v}_i + \tilde{b}'_i \tilde{b}_i - \nu \tilde{v}'_i \Delta \tilde{v}_i - \lambda \tilde{b}'_i \Delta \tilde{b}_i \right], \tag{18}
\]
The action functionals with ultra-local terms like those presented in \((17)\), \(\tilde{v}'(x_0, t_0)\) and \(\tilde{b}'(x_0, t_0)\), had been studied in \([31]\). In order to obtain the formal expansion parameters in the perturbation theory for the above action functional, we have
inserted four coupling constants, \( g_{1,2} \nu \equiv 1 \), and \( g_{3,4} \lambda \equiv 1 \), in front of the relevant interaction terms in (17). Despite their physical dimensions are \( [g_{1,2}] = [\nu] \) and \( [g_{3,4}] = [-\lambda] \), their convectional dimensions would be different. If we put formally \( g_{1,2,3,4} = 0 \) in (17), the action functional \( S(\Phi) \) turns into \( S_0(\Phi) \), free of interactions. The correct normalization of integral in (16) requires that the measure of functional integration \( \mathcal{D}\Phi \) be normalized to the volume of orbits in the free theory (pure diffusion processes),

\[
\mathcal{D}\Phi = \frac{T N \mathcal{D}\tilde{v}' \mathcal{D}\tilde{b}' \mathcal{D}v' \mathcal{D}b'}{Z}, \quad Z \equiv \int \mathcal{D}\Phi \exp S_0(\Phi).
\]

Moreover, it is obvious that the results of functional averages (16) do not change along the set of orbits in the configuration space related by any symmetry transformation of the action (17). The functional integral (16) itself is proportional to the volume of such orbits. The functional (17) possess the Galilean invariance:

\[
\tilde{v}(x, t) = \tilde{v}(x + s(t), t) - u(t), \\
\tilde{v}'(x, t) = \tilde{v}'(x + s(t), t), \\
\tilde{b}(x, t) = \tilde{b}(x + s(t), t), \\
\tilde{b}'(x, t) = \tilde{b}'(x + s(t), t),
\]

where an integrable function \( u(t) \) is a parameter of transformation (the velocity of the frame of reference), and its integral \( s(t) = \int_{-\infty}^{t} u(t') dt' \).

The auxiliary fields introduced in (17) were not inherent to the original physical model, but appear since we treat its dynamics as a Brownian motion. While the first two saddle point equations,

\[
\frac{\delta S}{\delta \tilde{v}'} = 0, \quad \frac{\delta S}{\delta \tilde{b}'} = 0,
\]

recover the original Cauchy problem (16) (in case \( g_{1,2} \nu = 1 \), \( g_{3,4} \lambda = 1 \)), another pair,

\[
\frac{\delta S}{\delta \tilde{v}} = 0, \quad \frac{\delta S}{\delta \tilde{b}} = 0,
\]

describes the dynamics of auxiliary fields:

\[
\begin{align*}
\dot{\tilde{v}}_i + \partial_j (g_1 \nu \tilde{v} \tilde{v}_j - g_2 \lambda \tilde{b} \tilde{b}_j) &= -\nu \Delta \tilde{v}_i, \\
\dot{\tilde{b}}_i + \partial_j (g_3 \lambda \tilde{b} \tilde{v}_j - g_4 \nu \tilde{v} \tilde{b}_j) &= -\lambda \Delta \tilde{b}_i.
\end{align*}
\]

The above equations are characterized by the negative dissipations since \( \nu, \lambda > 0 \). Therefore, physically relevant solutions have to satisfy \( \tilde{v}'(t > 0) = 0 \) and \( \tilde{b}'(t > 0) \). The equations (15,23) give the time evolution of the infinitesimal characteristics of the diffusion processes in MHD and therefore determines them. It is important to mention the striking similarity between the equations (15, 23) and the coupled system of nonlinear equations for the osmotic velocity and the current velocity of stochastically driven Brownian motion. [32, 35].
We conclude the current section with a remark that the action functional (17) can be derived from the standard action functional of the MSR-type for stochastic MHD [29],[7] with the ultra-local interaction terms $v'(x_0,t_0)$ and $b'(x'_0,t'_0)$ added, if one put the stochastic forcing to zero, $|\xi| \rightarrow 0$.

5 Diagram technique for stochastic MHD. The absence of ultraviolet divergences

The formal functional averages (16) computed with respect to the statistical weights $\exp S(\Phi)$ are interpreted as the infinite diagram series,

$$G_\phi(x,t;x_0,t_0;x'_0,t'_0) = \frac{1}{k_1!k_2!k_3!k_4!} \sum_{\{k_1+k_2+k_3+k_4 = 0\}} G_\phi^{(k_1,k_2,k_3,k_4)} g_1^{k_1} g_2^{k_2} g_3^{k_3} g_4^{k_4},$$

(24)

where the sum is taken over all positive integer solutions of the equation

$$k_1 + k_2 + k_3 + k_4 = N, \quad k_i, N \in \mathbb{Z}^+.$$  (25)

The series (24) starts from the standard diffusion kernel in the coupling constants representing the possible physical interactions in MHD: the hydrodynamic dragging ($g_1$), the Lorenz force ($g_2$), the convection ($g_3$), and the stretching ($g_4$). In view of that, the main tasks of the stochastic approach to MHD is the computation of coefficients $G_\phi^{(k_1,k_2,k_3,k_4)}$, and the estimation of the asymptotic properties of the diagram expansions (24).

In general, the computation of $G_\phi^{(k_1,k_2,k_3,k_4)}$ in (24) is a very difficult problem. However, if the functional averages (19) are taken over the Gaussian distributed fields, one can apply Wick’s theorem [36]-[37] to facilitate calculations. In accordance to Wick’s theorem, the functional averages (16) equal to the sums of all complete systems of ”pairings” of the interaction operators. If we denote operators as nodes and the pairings between them as edges, then each system of ”pairings” representing the particular average can be visualized by a graph called a Feynman diagram. The use of Feynman diagrams helps to make the computations visual and clarifies their similarity with the problems of symbolic dynamics [38].

Any Feynman diagram for the stochastic representation of MHD contains one of two ultralocal interaction terms (see Fig. 1), with any number of either $b'$- or $v'$-tails located at the points $(x'_0,t'_0)$ and $(x_0,t_0)$ consequently. They reveal the dependence of the response functions $G_v$ and $G_b$ in MHD upon the initial conditions for the velocity field at $(x_0,t_0)$ and for the magnetic field at $(x'_0,t'_0)$. In quantum field theory, the vertices shown on (1) are called the composite operators, [36].

A diagram consists of the ”interaction vertices” (see Fig. 2), with the factors $V_{imx}^{v'v}$, $V_{imx}^{b'b}$, and $V_{imx}^{b'b}$, which can be written in Fourier space as

$$V_{imx}^{v'v} = \frac{g_1}{2} (i\delta_{im} k_s + i\delta_{im} k_m), \quad V_{imx}^{b'b} = -\frac{g_2}{2} (i\delta_{im} k_s + i\delta_{im} k_m),$$

9
Figure 1: The ultra-local interaction vertices in stochastic representation of magnetohydrodynamics.

Figure 2: The interaction vertices in stochastic representation of magnetohydrodynamics. The vertex factors $V^{\phi^\prime \phi}$ in the Fourier representation are given in (26).

\[ \tilde{V}_{\text{ins}}^{\prime \prime \prime} = \lambda (ig_4 \delta_{is} k_m - ig_3 \delta_{im} k_s). \]  

The $\tilde{b}^\prime$- or $\tilde{v}^\prime$-tails in interaction vertices (see Fig. 2) are to be connected with the $\tilde{b}$- or $\tilde{v}$-tails by means of edges (the response functions of the linearized MHD equations):

\[ R_{ij}^{\prime \prime \prime} = \frac{P_{ij}^{\prime \prime \prime}(k)}{-i\nu k^2}, \quad R_{ij}^{\prime b} = \frac{P_{ij}^{\prime b}(k)}{-i\lambda k^2}. \]  

in which the transverse projector $P_{ij}^{\prime \prime \prime}(k)$ is given in (6). The inverse Fourier transform with respect to the frequency $\omega$ in (27) reveals that the response functions of linearized problem are retarded,

\[ R_{ij}^{\prime \prime \prime} = H(t - t_0)P_{ij}^{\prime \prime \prime}(k)e^{-\nu k^2(t-t_0)}, \quad R_{ij}^{\prime b} = H(t - t_0)P_{ij}^{\prime b}(k)e^{-\lambda k^2(t-t_0)}, \]  

where $H(t)$ is the Heaviside function supplied by the convention $H(0) = 0$. It is required by the the casualty principle that the time arguments correspondent to auxiliary fields $\tilde{v}$ and $\tilde{b}$ always precedes the time arguments of $\tilde{v}$ and $\tilde{b}$. Let us note that the inverse Fourier transforms of response functions (28) correspond to the standard diffusion kernels, (in $d$-dimensional space),

\[ \Delta (x - x_0, t - t_0) = \frac{e^{-(x - x_0)^2}}{4\pi\nu(t - t_0)^d}. \]  

Similarly to quantum electrodynamics where each Feynman diagram represents a certain process of physical interactions between elementary particles, any graph in stochastic mechanics drawn with the interaction vertices Figs. 1, 2 connected by the edges (27) corresponds to a certain process of interactions between
different magnetic and hydrodynamic modes coming along with the cascades of consequent partitions of moments, \( k = q + (k - q) \). The first diagrams relevant to the inverse Green functions (exact response functions) of MHD are shown in Fig. 3 and Fig. 4 consequently. Each diagram in expansions corresponds to a certain analytical integral expression quantifying the contribution of the relevant interactions into the entire behavior (they are given in the Appendix A). In general case, all integrals with an odd number of factors \( k_i \) equals zero as a consequence of space isotropy.

The diagram expansion for the Green function would have a definite physical meaning if it converges. The standard analysis of ultraviolet divergences of graphs is based on the counting of relevant canonical dimensions. Dynamical models have two scales, the time scale \( T \) and the length scale \( L \), consequently the physical dimension of any quantity \( F \) can be defined as \([F] = L^{-d^F_k} T^{-d^F_\omega}\), in which \( d^F_k \) and \( d^F_\omega \) are the momentum and frequency dimensions of \( F \). In diffusion models, these dimensions are always related to each other since \( \partial_t \sim \partial^2_x \) in the diffusion equation that allows us to introduce a combined canonical dimension, \( d^F = d^F_k + 2d^F_\omega \). One can check out that each term in (17) is dimensionless if the following relations hold: \( d_{\phi'} = 0, d_u = d, d_{\nu,\lambda} = -2 + 2 \cdot 1 = 0, \) and \( d_{\phi} = 2 - (d + 1) \). The field theory (17) is logarithmic (the conventional dimension

Figure 3: The first diagrams for the inverse exact response function of the fluid velocity in MHD. The slash marks auxiliary fields. The first diagram corresponds to the response function of the linearized problem. The second diagram expresses the first order correction to the response function due to the hydrodynamic drag. The third diagram stands for the second order correction due to the hydrodynamic drag. The forth diagram stands for the first order correction due to the Lorenz force. The forthcoming diagrams are related to the corrections risen by the combined effect of hydrodynamic and magnetic field fluctuations.
The first diagrams for the inverse exact response function of the magnetic field in MHD. The first diagram corresponds to the response function of the linearized problem, $R_{ij}^{gb}$, in $[25]$. The second diagram gives the second order (in the coupling constants) one-point correction due to fluctuations of the magnetic field. The third diagram stands for the first order two-points contribution. The forth diagram describes the combined effect of the convection and the stretching. The fifth diagram gives the second order two-points correction due to the combined effect of hydrodynamic drag, convection, and stretching.

of the coupling constant $d_{\phi} = 0$) for the Burgers equation ($d = 1$), while in two dimensions $d_{\phi} = -1$, and $d_{\phi} = -2$ for $d = 3$ (the NS equation). Thus, in the infrared region (small moments, large scales) the diagram series in $g$ define just the corrections to the diffusion kernel as $d \geq 2$. However, in the case of Burgers equation, all diagrams look equally essential in large scales.

The diagrams diverge in the ultraviolet region (large moments, small scales) if their canonical dimension

$$d_{\Gamma} = -d_{\phi} N_{\phi} - d_{\phi'} N_{\phi'} \geq 0$$

where $N_{\phi}$ and $N_{\phi'}$ are the numbers of corresponding external legs in the graph $\Gamma$. For the Green function $[14]$, we have $N_{\phi'} = 0$ and $N_{\phi} = 1$. Therefore, there is no ultraviolet divergent graphs in the diagram series $[24]$. At first glance, it seems that any graph having no external $\phi -$legs ($N_{\phi} = 0$) and any number of auxiliary fields $\phi'$ ($N_{\phi'} > 0$) should diverge since $d_{\phi'} = 0$. However, such a graph is also convergent in small scales because of the derivatives in the vertex factors $V(k)$ which are always taken outside the graph onto the external $\phi' -$legs that effectively reduces its canonical dimension to $d_{\Gamma}' = d_{\Gamma} - N_{\phi'} < 0$. Therefore, the field theory with the action functional $[17]$ has no ultraviolet divergences and does not need a renormalization.

Each Feynman diagram in $[24]$ corresponds to a certain magnetohydrodynamic process. For instance, the very first diagrams displayed in Figs. $[34]$ present the solutions of diffusion equations (in $d$-dimensional space). They describe the simple viscous dissipation of a hydrodynamic vortex with no bifurcations and the motion of magnetic field through the fluid, following a diffusion
law with the resistivity of the plasma $\lambda$ serving as a diffusion constant.

The second graph (#2) in Fig. 3 corresponds to a bifurcation characterized by the twofold splitting of the moment, $k = q + (k - q)$. Under the spatial Fourier transformation, it is equivalent to the following analytic expression:

$$\Gamma_1(k, t) = -g \int_0^\infty d t' k \cdot \Delta(k, t - t')$$

$$\times \int \frac{d q}{(2\pi)^d} \Delta(q, t') v_0(q) \Delta(k - q, t') v_0(k - q),$$

(30)

where $\Delta(k, t)$ is the spatial Fourier transform of the diffusion kernel and $v_0(k)$ is the Fourier spectrum of initial condition. It is worth to mention that the diagram expansion for the Green function can be also discussed for the function defined on a finite domain supplied with periodic boundary conditions. In the latter case, it has a discrete set of harmonics, and the integral (30) turns into sums.

The time integration in (30) can be performed easily,

$$\Gamma_1 = -g k \cdot \exp(-\nu k^2 (t - t_0))$$

$$\times \int_{q, k > q^2} \frac{d q}{(2\pi)^d} \left[ v_0(q) v_0(k - q) \exp(-2\nu(q \cdot k - q^2) t_0) \right].$$

(31)

The singularities in $\Gamma_1$ appear at $q = 0$ and $q = k$, when the vortex does not bifurcate. The remaining momentum integral in (31) can be interpreted as an expectation value,

$$\varphi_1(k) = \int_{t_0}^\infty dt \int \frac{d q}{(2\pi)^d} \left[ v_0(q) v_0(k - q) \exp(-2\nu(q \cdot k - q^2)\tau) \right],$$

(32)

over the Poisson process of vortex bifurcation at momentum $k$. Bifurcation of vortexes is the Poisson stochastic process developing with time [2]. Further diagrams shown on Figs. 3,4 represent more complex magnetohydrodynamic processes.

6 The multiplicative factors of Feynman diagrams in MHD

One of fascinating features of the proposed stochastic approach to MHD is that all physically admissible behaviors of the MHD system are encoded by the certain integer solutions of the Eq.(25). It is clear that not all such the solutions are equally contribute to the diagrammatic series (24). While drawing diagrams admissible with respect to the "grammar" prescribed by the standard Feynman rules discussed in the previous section, one can see that only some particular combinations of coupling constants can appear as the factors before diagrams for the Green functions.
In order to find these factors, we use the standard methods of graph theory. Indeed, each diagram constitutes a graph, in which every node representing one of four physical interactions is proportional to the relevant coupling constant. We start drawing a diagram from the point \((x, t)\) by adding vertices and connecting them in a way admissible with respect to the Feynman rules. At any step, the tails representing the fields \(\tilde{V}\) and \(\tilde{b}\) can be fused together at the vertices of the ultra-local interactions (the composite operators) \(\tilde{V}^m(x_0, t_0)\) and \(\tilde{b}^m(x'_0, t'_0)\), \(m \geq 1\). It is convenient to count the powers of coupling constants in the multiplicative factors before diagrams of perturbation theory with the use of the ”grammar matrix” \(G\) given in the Appendix \([13]\).

The grammar matrix is the weighted connectivity matrix, which expresses the fact that the tails \(\phi, \phi'\) belonging to the interaction vertices \(V_{ims}^{\nu \nu'}, V_{ims}^{\nu \nu'}\) and \(V_{ims}^{\nu \nu'}\) (being proportional to \(g_1, g_2, g_3\), and \(g_4\)) can be connected to each other accordingly to the Feynman rules only by means of the propagators \([27]\).

For example, the first row of the grammar matrix \(G\) shows that the \(\tilde{V}\)–tail belonging to the vertex \(V_{ims}^{\nu \nu'}\) may be connected within a Feynman graph either to the \(\tilde{V}\)–tails in the similar interaction vertex \(V_{ims}^{\nu \nu'}\) or to the \(\tilde{V}\)–tails in the interaction vertex \(V_{ims}^{\nu \nu'}\). In both cases, the diagram amplitude acquires the factor \(g_1\). It is then obvious that all possible diagram structures admissible by the Feynman rules can be reproduced by the powers of the grammar matrix, \(G^n\), \(n \geq 1\). Here \(n\) is the number of interaction vertices appearing in the diagram (excepting the ultra-local vertices \(\tilde{V}^m(x_0, t_0)\) and \(\tilde{b}^m(x'_0, t'_0)\) and the starting vertex related to the point \((x, t)\)); it can be interpreted as the number of bifurcations of moments in the certain magnetohydrodynamic process. The entries of \(G^n\) are the monomials, the products of powers of the coupling constants \(g_{1,2,3,4}\).

The use of arguments given in Appendix \([13]\) helps to verify that the multiplicative factors for Feynman diagrams of the Green functions \(G_v\) and \(G_b\) contain powers of \(G_v \equiv g_1^2 g_2 g_3 g_4\), \(G_b \equiv g_2 g_3 g_4^2\) (see Tab. 1). Namely these combinations of coupling constants play the role of the expansion parameters in the diagram series \([24]\) for \(G_v\) and \(G_b\).

**Table 1: The multiplicative factors of Feynman diagrams in MHD**

| # of bifurcations | \(G_v\) | \(G_b\) |
|-------------------|--------|--------|
| \(n = 2k, k \geq 0\) | \(g_1 \cdot G_v^k\), \(g_2 \cdot G_v^k\), \(g_3 \cdot G_v^k\), \(g_4 \cdot G_b^k\) | \(g_1 g_2 g_4 \cdot G_v^k\), \(g_1 g_3 g_4 \cdot G_b^k\), \(g_2 g_3 \cdot G_b^k\), \(g_2 g_3 g_4^2 \cdot G_b^k\) |
| \(n = 2k + 1, k \geq 0\) | \(g_1 g_2 g_4 \cdot G_v^k\), \(g_1 g_2 g_3 \cdot G_v^k\), \(g_1 g_2 g_3 g_4 \cdot G_v^k\), \(g_2 g_3 \cdot G_b^k\), \(g_2 g_3 g_4 \cdot G_b^k\), \(g_2 g_3 g_4^2 \cdot G_b^k\) |

Consequently, if in the equation \(k_1 + k_2 + k_3 + k_4 = N\) we assume that

1. either \(k_1 = 2k + 1, k_2 = k_3 = k_4 = k\), or \(k_1 = 2k, k_2 = k + 1, k_3 = k_4 = k\),

then the diagram series \([24]\) reproduces the Feynman graphs for the Green function of fluid velocity \(G_v\) containing an even number of bifurcation of
moments;

2. either \( k_1 = 2k + 1, k_2 = k_4 = k + 1, k_3 = k, \) or \( k_1 = 2k + 1, k_2 = k_3 = k + 1, k_4 = k, \) or eventually \( k_1 = 2k + 2, k_2 = k + 1, k_3 = k_4 = k, \) then the diagram series \([24]\) reproduces the Feynman graphs for the Green function \( G_v \) containing an odd number of bifurcation of moments.

Diagrams for the Green function of magnetic field \( G_b \) do not contain the vertex responsible for the hydrodynamical interaction \( V_{inv} \), and therefore \( k_1 = 0. \) Then, if we suppose that

1. either \( k_2 = k, k_3 = 2k + 2, k_4 = 2k + 1, \) or \( k_2 = k, k_3 = 2k + 1, k_4 = 2k + 2, \) or eventually \( k_2 = k + 1, k_3 = 2k + 1, k_4 = 2k + 1, \) then the diagram series \([24]\) reproduces the Feynman graphs for the Green function \( G_b \) containing an even number of bifurcation of moments;

2. either \( k_2 = k, k_3 = 2k + 2, k_4 = 2k + 1, \) or \( k_2 = k, k_3 = 2k + 1, k_4 = 2k + 2, \) or eventually \( k_2 = k + 1, k_3 = 2k + 1, k_4 = 2k + 1, \) then the diagram series \([24]\) reproduces the Feynman graphs for the Green function \( G_b \) containing an odd number of bifurcation of moments.

We have suggested everywhere that \( k \in \mathbb{Z}_+ \).

### 7 The large order asymptotic behavior for the MHD Green functions. Instanton approach and Borel summation

It is important to note that we do not know apriori whether the coupling constants \( g_1, g_2, g_3, g_4 \) are small or large. To get the information on the convergence of asymptotic series \([24]\), we should study the asymptotic behavior of large order coefficients \( G_{\phi}^{(k_1, k_2, k_3, k_4)} \), \( k_1 + k_2 + k_3 + k_4 = N \), in the diagram series as \( N \to \infty \), by the asymptotic calculation of the Cauchy integral,

\[
G_{\phi}^{(k_1, k_2, k_3, k_4)} = \frac{1}{(2\pi i)^4} \frac{1}{k_1 k_2 k_3 k_4} \oint \prod_{i=1}^4 \frac{dg_i}{g_i} \prod_{i=1}^4 \frac{1}{k_i} \prod_{i=1}^4 \frac{1}{k_i} \log g_i \times G_{\phi}(x, t; x_0, t_0; x'_0, t'_0) \exp\left(-\sum_{i=1}^4 k_i \log g_i\right),
\]

in which \( G_{\phi}(x, t; x_0, t_0; x'_0, t'_0) \) is the functional integral \([10]\). The contour of integration in the multiple integral \([33]\) embraces the points \( \{g_i = 0\}, i = 1, \ldots, 4, \) in the complex plane.

We estimate the functional integral \([33]\) by the steepest descent method supposing that \( \sum_{i=1}^4 k_i = N \) is large (the instanton method). In so far, the instanton approach has been applied to various problems of stochastic dynamics, see \([39]-[43]\). In contrast to all previous studies, in the MHD system we do not have one coupling constant, but four. In the previous section, we have demonstrated that by applying the additional conditions for the integers \( k_i \) to the equation \( \sum_{i=1}^4 k_i = N \) in the series \([24]\), we can derive the diagram expansions for the
different Green functions being in the diverse statistical regimes (characterized by the even and odd number of momentum bifurcations respectively).

Following the traditional instanton analysis, we perform the uniform rescaling of variables in the action functional (17) in order to extract their dependence upon $N$,

$$\tilde{\phi} \rightarrow \tilde{\phi} \sqrt{N}, \quad \tilde{\phi}' \rightarrow \tilde{\phi}' \sqrt{N},$$

$$g_i \rightarrow g_i / N, \quad \nu \rightarrow \nu \sqrt{N}, \quad \lambda \rightarrow \lambda N, \quad x \rightarrow x \sqrt{N}. \quad (34)$$

This keeps the action functional (17) unchanged, thus each term acquires the multiplier $N$ and then formally gets the same order, as the log $g_i$ in (33) independently of the type of interaction $i = 1, \ldots, 4$. The corresponding Jacobians from the numerator and the denominator of (16) cancel. The saddle point equations are

$$\tilde{\nu}_i + g_1 \nu (\tilde{v} \partial) \tilde{v}_i - g_2 \nu \left( \tilde{b} \partial \right) \tilde{b}_i - \nu \Delta \tilde{v}_i = \delta (x - x_0) \delta (t - t_0),$$

$$\tilde{b}_i + g_3 \lambda (\tilde{v} \partial) \tilde{b}_i = g_4 \lambda \left( \tilde{b} \partial \right) \tilde{b}_i - \lambda \Delta \tilde{b}_i = \delta (x - x'_0) \delta (t - t'_0),$$

$$\tilde{\nu}_i' + g_1 \nu (\tilde{v} \partial) \tilde{v}_i' - g_2 \nu \left( \tilde{b} \partial \right) \tilde{b}_i' + \nu \Delta \tilde{v}_i' = 0,$$

$$\tilde{b}_i' + g_3 \lambda (\tilde{v} \partial) \tilde{b}_i' = g_4 \lambda \left( \tilde{b} \partial \right) \tilde{b}_i' + \lambda \Delta \tilde{b}_i' = 0,$$

$$\nu \tilde{v}_i (\tilde{v} \partial) \tilde{v}_i' = g_1^{-1}, \quad \nu \tilde{v}_i' \left( \tilde{b} \partial \right) \tilde{b}_i = g_2^{-1},$$

$$\lambda \tilde{b}_i' (\tilde{v} \partial) \tilde{b}_i = g_3^{-1}, \quad \lambda \tilde{b}_i \left( \tilde{b} \partial \right) \tilde{v}_i = g_4^{-1}. \quad (35)$$

The first two equations in (35) recover the original Cauchy problem for the MHD equations. The next two equations (#3 and #4) occur within the framework of the stochastic approach since we describe the microscopic dynamics in the MHD system as Brownian motion. The equations for the auxiliary fields are characterized by the negative viscosity and resistivity, and therefore $\tilde{v}' (t > t_0) = \tilde{b}' (t > t_0) = 0$. The last four equations in (35) determine the saddle-point values of the coupling constants that allows to exclude the interaction terms from the previous saddle-point equations and reduce the system (35) to

$$\tilde{v}' K_v \tilde{v} = \tilde{v}' \delta (x - x_0) \delta (t - t_0),$$

$$\tilde{b}' K_b \tilde{b} = \tilde{b}' \delta (x - x'_0) \delta (t - t'_0),$$

$$\tilde{v} K_v^* \tilde{v}' = 1,$$

$$\tilde{b} K_b^* \tilde{b}' = 1. \quad (36)$$

in which we have introduced the diffusion kernels, $K_v = -i \omega + \nu p^2$ and $K_b = -i \omega + \lambda p^2$ (in the $(\omega, p)$ Fourier space) and $K_v^*$, $K_b^*$ are their Hermit conjugated forms.

Bifurcations of vortexes arisen due to the nonlinear interactions in the MHD system do not conclude into a critical regime, and therefore the time spectrum in the nonlinear model is the same as for the free diffusion equations, $T \propto L^2$, that is the reason for the branching processes are Poisson distributed with
the characteristic times $1/(\nu k^2)$ and $1/(\lambda k^2)$. One can see that the saddle-point configurations $\{\bar{v}', \bar{b}', \bar{v}, \bar{b}\}$ which satisfy (36) should be independent of the Poisson branching processes, and therefore, the solutions could exist before bifurcations start that is as $t \to \min(t_0, t'_0)$. With the use of the power model for the Dirac delta function,

$$\delta (x) = \lim_{\varepsilon \to 0} \frac{\varepsilon}{\pi (x^2 + \varepsilon^2)},$$

one can find that in the limit $t \to \min(t_0, t'_0)$, (36) is satisfied by the following radially symmetric solutions $(\phi \equiv \{v, b\}, \phi' \equiv \{v', b'\}, \chi \equiv \{\nu, \lambda\}, \gamma \equiv \{t_0, t'_0\})$,

$$\phi(r, t) = \sqrt{(r - r_0)^2 + (t - t_0)^2}, \quad \phi'(r, t) = \frac{H(t_0 - t)}{t_0 - t} \phi(r, t),$$

$$\phi(r, t) = \frac{H(t - t_0)}{\pi} \arctan \left(\frac{|r - r_0|}{2\chi(t - t_0)}\right), \quad \phi'(r, t) = -\frac{\varepsilon}{2\chi t} \left\{ \frac{4\chi^2(t - t_0)^2 + (r - r_0)^2}{|r - r_0|} \right\},$$

$$\phi(r, t) = \frac{H(t - t_0)}{\pi} \arctan \left(\frac{2\chi(t - t_0)}{|r - r_0|}\right), \quad \phi'(r, t) = \frac{\varepsilon}{2\chi} \left\{ \frac{4\chi^2(t - t_0)^2 + (r - r_0)^2}{|r - r_0|} \right\}.$$  \hspace{1cm} (37)

In the first solution (37), the auxiliary fields $v'$ and $b'$ have poles as either $t \to t_0$ or $t \to t'_0$, and then it follows from (55) that $g^i_1 = g^i_3 = 0$ and $g^i_2 = g'_2 = 0$, the point which definitely lays inside the integration contour in (33). In contrast to it, in the last equation in (37), $v \to 0$ as $t \to t_0$ and $b \to 0$ as $t \to t'_0$ respectively, and therefore $g^i_{1,2,3,4} \to \infty$ that is definitely outside the integration contour. There is a subtle point in Eq. (37) concerning the second solution since $v(r, t = t_0) = b(r, t = t'_0) = H(0)/2$, and the position of the saddle-point configuration charges $g^i_{1,2,3,4}$ depends upon the conventional value for the Heaviside function of zero argument, $H(0)$. While estimating the functional Jacobian, we had assumed following the standard convention [7] that $H(0) = 0$. Then, one can easily verify that in this case we also have $g^i_{1,2,3,4} \to \infty$, so that being interested in the large order asymptotic behavior of the Green functions in MHD we do not need to take the second and the third solutions into account. Even if one takes $H(0) \neq 0$, and then $0 < g^i_{1,2,3,4} \to \infty$, it is always possible to deform the integration contour in (33) in such a way to avoid $g^i_{1,2,3,4}$ to be encircled. Therefore, the first solution in (37) is the only one we need, and substituting it into (55), we can obtain the microscopic power models for the coupling constants,

$$g^i_{1,2} \equiv \lim_{\delta t \to 0} g_{1,2} \approx \frac{\delta t}{\nu \delta r^2}, \quad g^i_{3,4} \equiv \lim_{\delta t \to 0} g_{3,4} \approx \frac{\delta t}{\lambda \delta r^2}. \hspace{1cm} (38)$$

Fields $\phi, \phi'$, and the coupling constants $g_{1,2,3,4}$ fluctuate around their saddle-point values $\phi_0, \phi'_0$, and $g^i_{1,2,3,4}$. By means of the standard shift of variables, $\phi = \phi_0 + \delta \phi, \phi' = \phi'_0 + \delta \phi'$, $g^i_{1,2,3,4} = g^i_{1,2,3,4} + \delta g^i_{1,2,3,4}$, one makes them fluctuate around zero, so that $\delta \phi(\infty) = 0, \delta \phi'(\infty) = 0$. Moreover, if we assume that the MHD system is isotropic (i.e., there is no the global bias of the magnetic fields
and the conductive fluid is isotropic), then $\text{tr} \, \delta \phi = \text{tr} \, \delta \phi' = 0$, and therefore all fluctuations possess the central symmetry, $\delta \phi = \delta \phi(\mathbf{r}, t)$, $\delta \phi' = \delta \phi'(\mathbf{r}, t)$, the same as the saddle point configuration.

The contours of integrations over the variables $\delta g_{1,2,3,4}$ now passes through the origin, and are directed there oppositely to the imaginary axis. The integrals over $\delta g_{1,2,3,4}$ are conducted now on the rectilinear contours in complex planes $(i \infty, -i \infty)$ (in accordance to the standard transformation of contours in the steepest descent method). At the turn of the integration contours $\delta g_{1,2,3,4} \to -i \delta g_{1,2,3,4}$, where the multiplier $(-i)$ appears so that the result $G_{\phi}^{(k_1,k_2,k_3,k_4)}$ is always real. The contributions to the Cauchy integral (33) comes from the poles $\delta g_{1,2,3,4} = -g_{1,2,3,4}$ and tends to zero as $t \to \min(t_0, t_0')$. The values of the functional integrals on the saddle point configurations are proportional to the entire volume of the functional integration, they cancel in the numerator and denominator simultaneously. While calculating the fluctuation integral (16), we take into account that the first order contributions in $\delta \phi$ and $\delta \phi'$ are absent because of the saddle-point condition. We also neglect the high-order interactions between fluctuations, $O(\delta \phi^3)$, $O(\delta \phi^4)$, etc. to arrive at the Gaussian functional integrals for the Green functions $G_{\nu}$ and $G_{b}$,

\[
G_{\nu}^{(N)}(r, \delta t) \simeq \delta t \to 0 \quad \left( G_{\nu}^{*} \right)^{N/N + 1/2} \times \int \int D \delta \phi \, D \delta \phi' \, \exp \left( -\frac{N}{2} \text{tr} \left[ \delta \phi' K_{\phi} \delta \phi + \frac{N}{2} \delta \phi \left( \frac{1}{2} \partial_{r} \right) \delta \phi' \right] \right),
\]

\[
G_{b}^{(N)}(r, \delta t) \simeq \delta t \to 0 \quad \left( G_{b}^{*} \right)^{N/N + 1/2} \times \int \int D \delta \phi \, D \delta \phi' \, \exp \left( -\frac{N}{2} \text{tr} \left[ \delta \phi' K_{\phi} \delta \phi + \frac{N}{2} \delta \phi \left( \frac{1}{2} \partial_{r} \right) \delta \phi' \right] \right),
\]

in which $G_{\nu}$ and $G_{b}$ are the the expansion parameters in the diagram series (24) for $G_{\nu}$ and $G_{b}$ introduced in Sec. 4 in concern with Tab.1. Performing the usual rescaling of fluctuation fields

\[
\delta \phi \to \delta \phi / \sqrt{N}, \quad \delta \phi' \to \delta \phi' / \sqrt{N},
\]

we compute the Gaussian integral, with respect to $\delta \phi$ first, and then the resulting Gaussian integral over the fluctuation of the auxiliary fields $\delta \phi'$,

\[
G_{\nu}^{(N)}(r, \delta t) \simeq \delta t \to 0 \quad N^{N - 1/2} \exp \left( -\frac{N}{2} \log \left( G_{\nu}^{*} \right) \right) \det K_{\phi}^{-1} \left( 1 + O \left( \frac{1}{N} \right) \right),
\]

\[
G_{b}^{(N)}(r, \delta t) \simeq \delta t \to 0 \quad N^{N - 1/2} \exp \left( -\frac{N}{2} \log \left( G_{b}^{*} \right) \right) \det K_{\phi}^{-1} \left( 1 + O \left( \frac{1}{N} \right) \right).
\]

The kernels of the operators $K_{\phi}^{-1}$ are the Green function of the linear diffusion equations. Using the Stirling’s formula, one can check that the coefficients $G_{\phi}^{(N)}$ of the asymptotic series (24) demonstrate the factorial growth (like in the most of quantum field theory models):

\[
G_{\nu}^{(N)}(r, \delta t) \simeq \delta t \to 0 \quad \frac{N!}{2\pi N} \exp N \left( 1 - \frac{1}{2} \log \left( G_{\nu}^{*} \right) \right) \det K_{\phi}^{-1} \left( 1 + O \left( \frac{1}{N} \right) \right),
\]

\[
G_{b}^{(N)}(r, \delta t) \simeq \delta t \to 0 \quad \frac{N!}{2\pi N} \exp N \left( 1 - \frac{1}{2} \log \left( G_{b}^{*} \right) \right) \det K_{\phi}^{-1} \left( 1 + O \left( \frac{1}{N} \right) \right).
\]
Therefore, the asymptotic series can be summed by means of Borel’s procedure. It consists of the following transformation of series

\[
\sum_N G^{(N)}_\phi G^N_\phi = \sum_N \Gamma(N+1) \tilde{G}^{(N)}_\phi G^N_\phi = \sum_N \int_0^\infty d\tau \tilde{G}^{(N)}_\phi (\tau G_\phi)^N e^{-\tau},
\]

where \(\tilde{G}^{(N)}_\phi = G^{(N)}_\phi / \Gamma(N+1)\) are the new expansion coefficients which do not exhibit the factorial growth.

It is traditional, while performing the Borel summation, to change the orders of summation and integration in

\[
\sum_N \int_0^\infty d\tau \tilde{G}^{(N)}_\phi (\tau G_\phi)^N \exp(-\tau) = \int_0^\infty d\tau \exp(-\tau) \sum_N \tilde{G}^{(N)}_\phi (\tau G_\phi)^N.
\]

The we sum over \(N\) in the r.h.s. of (43),

\[
\frac{\det K^{-1}_\phi}{2\pi} \int_0^\infty d\tau \exp(-\tau) \sum_N (\tau G_\phi)^N \exp(N(1-\log G_\phi)),
\]

and obtain

\[
= -\frac{\det K^{-1}_\phi}{2\pi} \int_0^\infty d\tau \exp(-\tau) \log(1 - \frac{G_\phi}{\tilde{G}_\phi}).
\]

The integration of over \(\tau\) gives us

\[
G_\phi \simeq \delta t \to 0 \quad \frac{\det K^{-1}_\phi}{2\pi} \left(1 + \frac{1}{2\pi} \text{Ei}\left(\frac{G_\phi}{\tilde{G}_\phi}\right) \exp\left(-\frac{G_\phi}{\tilde{G}_\phi}\right)\right).
\]

where \(\text{Ei}(x)\) is the exponential integral defined as \(\text{Ei}(x) = -\int_0^\infty y^{-1} e^{-y} dy\) understood in terms of the Cauchy principal value at \(y = 0\). Then we can use the microscopic models and recall that by definition \(g_{1,2} = 1/\nu\) and \(g_{3,4} = 1/\lambda\) in order to estimate the both ratios \(G^*_v/G_v\) and \(G^*_b/G_b\) in (45) as

\[
\frac{G^*_v}{G_v} \simeq \delta t \to 0 \quad \frac{(\delta t)^{5/2}}{(\delta r)^{3}}.
\]

In such a simplified model, the non-Maxwellian corrections to the distribution functions arisen due to the kinetic effects are accounted by the new distribution function,

\[
G_\phi(\delta t, \delta r) \simeq \delta t \to 0 \quad \frac{\exp\left(-\frac{\delta r^2}{4\chi(\delta t)}\right)}{(4\pi\chi(\delta t))^{d/2}} \left[1 + \frac{1}{2\pi} \text{Ei}\left(\frac{(\delta t)^{5/2}}{(\delta r)^{3}}\right) \exp\left(-\frac{(\delta t)^{5/2}}{(\delta r)^{3}}\right)\right].
\]
8 Discussion and Conclusion

Speaking rigorously, the classical MHD equations cannot be applied if the time scale of collisions is comparable or longer than the other characteristic times in plasmas. The particle distributions are far from being of the Maxwellian shape and the certain kinetic models giving an insight into the collision statistics have to be taken into account. The simplest such model which allows accounting the kinetic effects being nevertheless completely in the framework of the classical approach based on the MHD equations is suggested in the present paper.

We investigate a simple generalization of the MHD model (1) modelling fluctuations of the configurations \( \{ v, b \} \) considered as the stochastic fields (or the trial trajectories of the MHD system) for which the classical MHD (1) plays the role of the mean field equations.

The essential point of our approach is that we have used the field theory formulation of the dynamics which allowed us the implementation of various powerful technics borrowed from the quantum field theory. In particular, with the use of the instanton technique and Bord’s summation, we have computed the asymptotic series for the Green functions accounting for the kinetic effects as the corrections to the unperturbed diffusion kernel describing the pure relaxation dynamics. Similarly to the most of quantum field theory models, the high order contributions into the Green functions exhibit a factorial growth.

It is interesting to compare the perturbed diffusion kernel (47) in the 3D space, \( d = 3 \), with the standard diffusion kernel depicted by a Gaussian curve. In Figs. 8, we have sketched the profiles of standard diffusion kernel (the solid lines) calculated at several consequent time steps for \( \nu = 0.2 \) together with the perturbed kernel profiles given by (47).

It is clearly seen that the essential corrections to the standard diffusion kernel are arisen in short times and small scales, while they are negligible in long times large scales.

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Figure 5: The profiles of standard diffusion kernel $G(\delta r)$ (the solid line) calculated at several consequent time steps $t > 0$ for $\nu = 0.2$. The dash-dot lines present the asymptotic kernel $\text{(47)}$ (as $t \to t_0$) accounting for the perturbation due to the kinetic effects modelled by Brownian motion.
Figure 6: The profiles of standard diffusion kernel $G(\delta t)$ (the solid line) calculated at several distant points for $\nu = 0.2$. The dash-dot lines present the asymptotic kernel (47) (as $t \to t_0$) accounting for the perturbation due to the kinetic effects modelled by Brownian motion.
A Analytical expressions for Feynman diagrams

Below, we write down the analytical expressions (in $d$-dimensional Fourier space $(k, t)$) correspondent to diagrams shown in Figs. (34).

\[ G_v(k, t - t_0; k', t' - t'_0) = v_0(k) R_v^{k'}(k, t - t_0) + \int_0^\infty dt' V_v^{k'}(k) R_v^{k'}(k, t - t') \int \frac{dq}{(2\pi)^d} R_v^{q'}(q, t' - t_t) v_0(q) R_v^{k'}(k - q, t' - t_0) \]

\[ v_0(k - q) + \int_0^\infty dt V_v^{k'}(k) R_v^{k'}(k, t - t') \int \frac{dq}{(2\pi)^d} R_v^{q'}(k - q, t' - t_0) v_0(k - q) \]

\[ \int dt'' V_v^{k'}(q) R_v^{k'}(q, t' - t'') \int \frac{dp}{(2\pi)^d} R_v^{p'}(p, t' - t_0) v_0(p) R_v^{q'}(q - p, t' - t_0) \]

\[ v_0(q - p) + \int_0^\infty dt V_v^{k'}(k) R_v^{k'}(k, t - t') \int \frac{dq}{(2\pi)^d} R_v^{k'}(q, t' - t_0) b_0(q) \]

\[ R_v^{k'}(k - q, t - t'_0) b_0(k - q) + \int_0^\infty dt'' V_v^{k'}(k) R_v^{k'}(k, t'' - t_0) \int \frac{dq}{(2\pi)^d} R_v^{k'}(k - q, t'' - t_0) v_0(k - q) \]

\[ \int dt'' V_v^{k'}(p) R_v^{k'}(p, t'' - t'0) \int \frac{dq}{(2\pi)^d} R_v^{k'}(q, t' - t_0) b_0(p) b_0(p - q) \]

\[ V_v^{k'}(p) V_v^{p'}(k - p) \int \frac{dq}{(2\pi)^d} R_v^{k'}(q, t' - t_0) b_0(p) b_0(p - q) \int \frac{ds}{(2\pi)^d} R_v^{k'}(s, t'' - t_0) \]

\[ R_v^{k'}(k - p - s, t'' - t_0) v_0(s) v_0(k - p - s) + \int_0^\infty dt'' V_v^{k'}(k) R_v^{k'}(k, t'' - t_0) \int \frac{dq}{(2\pi)^d} R_v^{k'}(k - q, t'' - t_0) v_0(q) \]

\[ V_v^{k'}(p) R_v^{k'}(p, t'' - t_0) \int \frac{dq}{(2\pi)^d} R_v^{k'}(q, t' - t_0) b_0(p) v_0(q) b_0(q) b_0(q) R_v^{k'}(q, t'' - t'_0) \]

\[ G_b(k, t - t_0; k', t' - t'_0) = b_0(k') R_v^{k'}(k', t - t'_0) + \int_0^\infty dt V_v^{k'}(k') R_v^{k'}(k', t - t') \int \frac{dq}{(2\pi)^d} R_v^{k'}(k' - q, t' - t'_0) b_0(k - q) \]

\[ \int dt'' V_v^{k'}(q) R_v^{k'}(q, t' - t'') \int \frac{dp}{(2\pi)^d} R_v^{k'}(p, t' - t'_0) b_0(p) R_v^{k'}(q - p, t' - t'_0) \]

\[ b_0(q - p) + \int_0^\infty dt V_v^{k'}(k') R_v^{k'}(k', t - t') \int \frac{dq}{(2\pi)^d} b_0(q) R_v^{k'}(k' - q, t'' - t'_0) \]

\[ + \int_0^\infty dt V_v^{k'}(k') R_v^{k'}(k', t - t') \int \frac{dq}{(2\pi)^d} b_0(q) R_v^{k'}(k' - q, t'' - t'_0) \]

\[ V_v^{k'}(k' - q - s) b_0(k' + s) R_v^{k'}(k' - s - q, t' - t_0) R_v^{k'}(k' + s, t'' - t_0) \]

\[ V_v^{k'}(k' - q - s) R_v^{k'}(k' - s - q, t'' - t_0) + \int_0^\infty dt'' V_v^{k'}(k') R_v^{k'}(k', t'' - t_0) \int \frac{dq}{(2\pi)^d} R_v^{k'}(k' - q, t'' - t'_0) \]

\[ V_v^{k'}(k' - q') R_v^{k'}(k' - q', t'' - t_0) R_v^{k'}(q', t'' - t_0) V_v^{k'}(k' - p) \]

(48)

\[ G_b(k, t - t_0; k', t' - t'_0) = b_0(k') R_v^{k'}(k', t - t'_0) + \int_0^\infty dt V_v^{k'}(k') R_v^{k'}(k', t - t') \int \frac{dq}{(2\pi)^d} R_v^{k'}(k' - q, t' - t'_0) b_0(k - q) \]

\[ \int dt'' V_v^{k'}(q) R_v^{k'}(q, t' - t'') \int \frac{dp}{(2\pi)^d} R_v^{k'}(p, t' - t'_0) b_0(p) R_v^{k'}(q - p, t' - t'_0) \]

\[ b_0(q - p) + \int_0^\infty dt V_v^{k'}(k') R_v^{k'}(k', t - t') \int \frac{dq}{(2\pi)^d} b_0(q) R_v^{k'}(k' - q, t'' - t'_0) \]

\[ + \int_0^\infty dt V_v^{k'}(k') R_v^{k'}(k', t - t') \int \frac{dq}{(2\pi)^d} b_0(q) R_v^{k'}(k' - q, t'' - t'_0) \]

\[ V_v^{k'}(k' - q - s) b_0(k' + s) R_v^{k'}(k' - s - q, t' - t_0) R_v^{k'}(k' + s, t'' - t_0) \]

\[ V_v^{k'}(k' - q - s) R_v^{k'}(k' - s - q, t'' - t_0) + \int_0^\infty dt'' V_v^{k'}(k') R_v^{k'}(k', t'' - t_0) \int \frac{dq}{(2\pi)^d} R_v^{k'}(k' - q, t'' - t'_0) \]

\[ V_v^{k'}(k' - q') R_v^{k'}(k' - q', t'' - t_0) R_v^{k'}(q', t'' - t_0) V_v^{k'}(k' - p) \]

(49)
The "grammar" matrix for the Feynman diagram technique

The "grammar" matrix $G$ is the weighted connectivity matrix, which expresses the fact that the tails $\phi, \phi'$ belonging to the interaction vertices $V_{\text{ims}}^{v'vv}$, $V_{\text{ims}}^{v'bb}$, and $V_{\text{ims}}^{b'bv}$ (being proportional to $g_1, g_2, g_3$, and $g_4$) can be connected to each other accordingly to the Feynmann rules only by means of the propagators (27).

|         | $g_1$ | $g_2$ | $g_3$ | $g_4$ |
|---------|-------|-------|-------|-------|
| $v'$    | $v'v$ | $v'v'$| $v'b'$| $v'b'$|
| $v$     | $g_1v$| $g_1v'$| $g_2v$| $g_2v'$|
| $v$     | $g_1v$| $g_1v'$| $g_1v'$| $g_1v'$|
| $v'$    | $g_1v$| $g_1v'$| $g_1v'$| $g_1v'$|
| $b$     | $g_2v$| $g_2v'$| $g_2v'$| $g_2v'$|
| $b$     | $g_2v$| $g_2v'$| $g_2v'$| $g_2v'$|
| $b'$    | $g_2v$| $g_2v'$| $g_2v'$| $g_2v'$|
| $v$     | $g_2v$| $g_2v'$| $g_2v'$| $g_2v'$|
| $b$     | $g_3v$| $g_3v'$| $g_3v'$| $g_3v'$|
| $b$     | $g_3v$| $g_3v'$| $g_3v'$| $g_3v'$|
| $b'$    | $g_3v$| $g_3v'$| $g_3v'$| $g_3v'$|
| $v$     | $g_4v$| $g_4v'$| $g_4v'$| $g_4v'$|

The starting vertex in a diagram contributing into the Green function $G_v$ may be either $V_{\text{ims}}^{v'vv}$ or $V_{\text{ims}}^{v'bb}$. In the first case, the multiplicative factor acquires the additional multiplier $g_1$, and it is $g_2$ if the second vertex is used as the starting one. In order to include the starting nodes into account, we introduce the vector $v^T = [g_1, 0, g_2, 0, 0, 0, 0, 0, 0, 0]$. Then, the multiplicative factors of the diagrams for the Green function $G_v$ are written as following,

$$[G_v]_{\text{factor}} = \{G^n v\}_{n \geq 0}.$$ (50)

Similarly, we can find the multiplicative factors arising in the diagrams for the Green function $G_b$ by

$$[G_b]_{\text{factor}} = \{G^n b\}_{n \geq 0},$$ (51)

in which the vector $b^T = [0, 0, 0, 0, 0, 0, g_3, 0, 0, g_4, 0, 0]$ expresses the fact that $V_{\text{ims}}^{b'bv}$ is the starting vertex for $G_b$.

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