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Almost reducibility of quasiperiodic $SL(2, \mathbb{R})$-cocycles in ultradifferentiable classes

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Abstract: Given a quasiperiodic cocycle in $sl(2, \mathbb{R})$ sufficiently close to a constant, we prove that it is almost-reducible in ultradifferentiable class under an adapted arithmetic condition on the frequency vector. We also give a corollary on the Hölder regularity of the Lyapunov exponent.

1 Introduction

1.1 Presentation of the result

Let $d \geq 1$ and $\omega = (\omega_1, \ldots, \omega_d) \in \mathbb{R}^d$ a rationally independent vector (meaning that no non-trivial integer combination of the $(\omega_i)_{i=1}^d$ can vanish). We will assume that $\sup |\omega_i| \leq 1$. We will note $T^d := \mathbb{R}^d/\mathbb{Z}^d$ and $2T^d := \mathbb{R}^d/2\mathbb{Z}^d$. Let $A : T^d \to sl(2, \mathbb{R})$ be in a certain class of continuous matrix-valued functions. We call quasi-periodic cocycle the solution $X : T^d \times \mathbb{R} \to SL(2, \mathbb{R})$ of the differential linear equation

$$\begin{cases} \frac{d}{dt} X^t(\theta) = A(\theta + t\omega)X^t(\theta) \\ X^0(\theta) = Id \end{cases}$$

(1)

One of the main motivations for studying quasi-periodic cocycles is the study of quasi-periodic Schrödinger equations

$$-y''(t) + q(\theta + t\omega)y(t) = Ey(t)$$

where $q : T^d \to \mathbb{R}$ is called the potential, and $E \in \mathbb{R}$ the energy. It gives rise to a cocycle with values in $SL(2, \mathbb{R})$. The cocycle is said to be a constant cocycle if $A$ is a constant matrix. A quasi-periodic cocycle as in (1) is said reducible if it can be conjugated by a quasi-periodic change of variable $Z : T^d \to SL(2, \mathbb{R})$ to a constant cocycle, that is to say, if there exists $B \in sl(2, \mathbb{R})$ such that, for all $\theta \in 2T^d$:

$$\partial_\omega Z(\theta) = A(\theta)Z(\theta) - Z(\theta)B$$

In general, it is important to require the change of variables $Z$ to be regular enough. In this paper, we will be interested in the perturbative setting, that is to say, in quasi-periodic cocycles close to a constant:

$$\begin{cases} \frac{d}{dt} X^t(\theta) = (A + F(\theta + t\omega))X^t(\theta) \\ X^0(\theta) = Id \end{cases}$$

(2)

where $A \in sl(2, \mathbb{R})$ and $F : T^d \to sl(2, \mathbb{R})$ is of ultra-differentiable class and small enough, with a smallness condition depending on $\omega$.

Reducibility is a strong property because it implies that the dynamics will be easily described by the constant equivalent of the system, in particular the Lyapunov exponents, the rotational properties of the solutions, the invariant subbundles etc. On the counterpart, reducibility results generally require many assumptions. Here we are interested in a weaker property which is almost reducibility. A cocycle like (2) is said almost-reducible if it can be conjugated by a sequence of quasi-periodic changes of variables to a cocycle of the form

$$\begin{cases} \frac{d}{dt} X^t(\theta) = (\bar{A}_n(\theta + t\omega) + \bar{F}_n(\theta + t\omega))X^t(\theta) \\ X^0(\theta) = Id \end{cases}$$

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where $A_n$ is reducible and $\bar{F}_n$ is arbitrarily small.

A quantitative version of almost reducibility, that is, almost reducibility together with estimates on the changes of variable, can have interesting corollaries such as approximate solutions, density of reducible cocycles, regularity of Lyapunov exponents.

**Ultra-differentiability**: To quantify the regularity of $F \in C^\infty(\mathbb{T}^d, sl(2,\mathbb{R}))$ and the size of the sequence $(\bar{F}_n)$ above, we introduce the weight function $\Lambda : [0, +\infty[ \to [0, +\infty]$ which we will assume to be increasing and differentiable. Expanding $F$ in Fourier series $F(\theta) = \sum_{k \in \mathbb{Z}^d} \hat{F}(k) e^{2\pi i \langle k, \theta \rangle}$, we will say that $F$ is $\Lambda$-ultra-differentiable if there exists $r > 0$ such that

$$|F|_r = |F|_{\Lambda,r} := \sum_{k \in \mathbb{Z}^d} \|\hat{F}(k)\| e^{2\pi \Lambda(|k|)} r < \infty$$

where $|k|$ is the sum of the absolute values of the components of $k$, and we will denote $F \in U_r(\mathbb{T}^d, sl(2,\mathbb{R})) = U_{\Lambda,r}(T^d, sl(2,\mathbb{R}))$. To make this space a Banach algebra, we will require $\Lambda$ to be subadditive:

$$\Lambda(x + y) \leq \Lambda(x) + \Lambda(y), \quad \forall x, y \geq 0$$

If $\Lambda \equiv id$, it is the analytic case.

**Remark 1.1.** The standard definition of ultra-differentiable functions involves Denjoy-Carleman sequences, that is, real sequences satisfying certain conditions which act as bounds on the successive derivatives of a given function. However, the above definition, introduced by Braun-Meise-Taylor ([8]), can be linked to Denjoy-Carleman classes (see [21], Theorem 11.6). Since Fourier series appear naturally in the problem considered here, we chose to use Braun-Meise-Taylor classes as a starting point.

**Non-resonance condition on the frequency**: An often studied situation is the case where the frequency vector $\omega$ if Diophantine (which we denote by $\omega \in DC(\kappa, \tau)$), for some $0 < \kappa < 1$ and $\tau \geq \max(1, d - 1)$:

$$|\langle k, \omega \rangle| \geq \frac{\kappa}{|k|^\tau}, \quad \forall k \in \mathbb{Z}^d \setminus \{0\}$$

where $\langle \cdot, \cdot \rangle$ is the standard Euclidean inner product. It was proved by Eliasson [13] that in the analytic case, if $\omega \in DC(k, \tau)$, and $F$ is sufficiently small, Equation (2) is almost reducible. This result was improved by Chavaudret [11] who proved that the convergence occurs on analyticity strips of fixed width (whereas Eliasson’s theorem gave the convergence on strips of width going to zero).

One of the aims of the present paper is to weaken this arithmetic condition by introducing the approximating function

$$\Psi : [0, +\infty[ \to [0, +\infty[$$

with $\Psi \geq id$ (which is not restrictive since it is satisfied by the diophantine condition). We will assume $\Psi$ to be increasing, differentiable and satisfying, for all $x, y \in [1, +\infty[$,

$$\Psi(x + y) \geq \Psi(x) + \Psi(y)$$

thus for all $n \in \mathbb{N}$, and for all $x \geq 1$, $\Psi(nx) \geq n\Psi(x)$. In our problem, $\omega$ will satisfy the following arithmetic condition for some $\kappa \in [0, 1]$:

$$|\langle k, \omega \rangle| \geq \frac{\kappa}{\Psi(|k|)}, \quad \forall k \in \mathbb{Z}^d \setminus \{0\}$$

(notice that the case $\Psi(\cdot) = |\cdot|^\tau$, is the Diophantine case).

We will require the following condition:

$$\lim_{t \to +\infty} \frac{\log \Psi(t)}{\Lambda(t)} = 0.$$

and

$$\int_0^\infty \frac{\Lambda'(t) \log \Psi(t)}{\Lambda(t)^2} dt < +\infty$$

This condition, known as the $\Lambda$-Brjuno-Rüssmann condition, will be denoted by $\omega \in BR(\kappa)$. This coincides with the well-known Brjuno condition if $\Lambda$ is the identity.

If $\Lambda$ is elliptic, an almost reducibility theorem was given in [5].

The purpose of this article is to show the following theorem:
Theorem 1.2. Let $r_0 > 0$, $A_0 \in \mathfrak{sl}(2, \mathbb{R})$ and $F_0 \in U_{r_0}(T^d, \mathfrak{sl}(2, \mathbb{R}))$. Then, there exists $\varepsilon_0$ depending only on $A_0, \kappa, \Lambda, \Psi, r_0$ such that, if $|F_0| \leq r_0$, then for all $\varepsilon \leq \varepsilon_0$, there exist

- $r_\varepsilon > 0, \zeta \in [0, \frac{1}{\varepsilon}]$,
- $Z_\varepsilon \in U_{r_\varepsilon}(T^d, SL(2, \mathbb{R}))$,
- $A_\varepsilon \in \mathfrak{sl}(2, \mathbb{R})$,
- $\tilde{A}_\varepsilon, \tilde{F}_\varepsilon \in U_{r_\varepsilon}(T^d, \mathfrak{sl}(2, \mathbb{R}))$,
- $\psi_\varepsilon \in U_{r_\varepsilon}(2T^d, SL(2, \mathbb{R}))$

such that

1. $\tilde{A}_\varepsilon$ is reducible to $A_\varepsilon$ by $\psi_\varepsilon$, with $|\psi_\varepsilon| \leq \varepsilon^{-\frac{1}{2}}$,
2. $|\tilde{F}_\varepsilon| \leq \varepsilon$,
3. $\lim_{\varepsilon \to 0} r_\varepsilon > 0$,
4. For all $\theta \in \mathbb{T}^d$,
   $$\partial_\theta Z_\varepsilon(\theta) = (A_0 + F_0(\theta))Z_\varepsilon(\theta) - Z_\varepsilon(\theta)(\tilde{A}_\varepsilon(\theta) + \tilde{F}_\varepsilon(\theta))$$
5. $|Z_\varepsilon^{\pm1} - I|_{r_\varepsilon} \leq \varepsilon_0^{\frac{1}{2}}$

Moreover, either $\Psi_\varepsilon$ becomes constant as $\varepsilon \to 0$, or there exist arbitrarily small $\varepsilon$ such that $||A_\varepsilon|| \leq \kappa \varepsilon^\zeta$.

This theorem states almost reducibility in an ultradifferentiable class with the same weight function as that of the initial system, but with a smaller parameter $r_\varepsilon$. Notice however that the parameter $r_\varepsilon$ does not shrink to 0. In order to achieve this, the resonance cancellation technique is similar to the one in [11]. Notice that, for topological reasons, a period doubling is necessary in order to preserve the real structure. This phenomenon was already observed in [10].

1.2 Discussion

Almost reducibility in itself is an interesting property of quasi-periodic cocycles, in particular the quantitative version. A perturbative almost reducibility result in arbitrary dimension of space was given in [13], in the analytic framework, under a diophantine condition on the frequency vector. A quantitative version of perturbative almost reducibility, in a similar framework, was then proved in [11]. Using a technique developed in [1], in [16], Hou and You managed to remove the diophantine assumption on the frequency vector, in case it is 2-dimensional ($d = 2$); thus the result became non-perturbative if not global (see also [20] for a non-perturbative reducibility result). It is yet unknown whether arithmetical conditions can be removed in case $d > 2$. However in the analytic case, for any number of frequencies, it is known that the diophantine condition is not optimal for reducibility results and can be replaced by the Brjuno-Rüssmann condition (see [17], [12]). Here we give an almost reducibility result in which the arithmetical condition coincides with the Brjuno-Rüssmann condition in the analytic case.

Concerning the functional framework, a few results were known in the Gevrey class. The reference [11] contains almost reducibility in the Gevrey class as well, under a diophantine condition. The reference [15] gives a result on rigidity of reducibility in the Gevrey class, under a diophantine condition (see also [18] on Gevrey flows). But a simultaneous extension of Eliasson’s reducibility result in [14] to more general ultradifferentiable classes and to a weaker arithmetical condition, which is linked to the considered class of functions, was given in [5] (see also [6] for a result in a hamiltonian setting). Here, we obtain this generalization for almost reducibility, the proof of which is more technical. The link between the arithmetical condition and the functional setting is similar to the one in [5], and also coincides with the Brjuno-Rüssmann condition in the analytic case.
1.3 Comments on the proof

The proof of the main result relies on the well-known KAM algorithm: a step of the algorithm will reduce the size of the perturbation to a power of it, by means of a change of variables which might be far from identity (if resonances have to be cancelled), but is still controlled by a small negative power of the size of the perturbation. The order at which one removes resonances to avoid small divisors has to be suitably chosen in order to decrease the perturbation sufficiently while having a sufficient control on the change of variables. One also has to shrink the parameter of the ultradifferentiable class at every step, and in order to have a strong almost reducibility result (i.e. a sequence of parameters not shrinking to 0), the A-Brjuno-Rüssmann condition comes naturally.

If resonances are cancelled only finitely many times, then the change of variables remains close to identity at every step afterwards, which gives reducibility. Otherwise, the constant part of the system itself becomes small.

The main theorem, which is Theorem 12.1 below, is proved by iterating arbitrarily many times the Lemma 11.1 below; Lemma 11.1 gives a conjugation between two systems $\bar{A} + F$ and $\bar{A}' + \bar{F}'$, where both $\bar{A}$ and $\bar{A}'$ are reducible maps and $\bar{F}'$ is smaller than $\bar{F}$, with a controlled loss of regularity.

The proof of Lemma 11.1 can be sketched by the following diagram:

\[
\begin{align*}
\bar{A} + F & \xrightarrow{\psi} A + F_1 \xrightarrow{\psi^{-1} \Phi^{-1}} F + \Phi \xrightarrow{\epsilon X_1} F + \Phi \xrightarrow{\epsilon X_1} \cdots \xrightarrow{\epsilon X_{l-1}} F + \Phi \xrightarrow{\epsilon X_{l-1}} \bar{A}_l + \bar{F}_l = \bar{A}' + \bar{F}'
\end{align*}
\]

where $A, A_1, \ldots, A_l$ are constant matrices, $\bar{A}, \bar{A}_1, \ldots, \bar{A}_l$ are reducible, and $\bar{F}, \bar{F}_1, \bar{F}_l$ are small.

No non resonance condition is required on $A$, making it necessary to construct the change of variables $\Phi$ which will remove resonances, but may be far from the identity (Lemma 7.1). However, once this is done, the matrices $A_1, \ldots, A_{l-1}$ remain non resonant enough in order to reduce the perturbation a lot without having to remove resonances again.

The superscripts on the arrows refer to the changes of variables. The changes of variables with an exponential expression are close to the identity, therefore the total conjugation, from $A + F$ to $\bar{A}' + \bar{F}'$, is close to identity, which makes it possible to obtain the density of reducible systems in the neighbourhood of a constant.

2 Notations

The notation $E\{x\}$ will refer to the integer part of a number $x$.

If $F \in L^2(2\mathbb{T}^d)$ and $N \in \mathbb{N}$, the truncation of $F$ at order $N$ (denoted $F^N$) is the function we obtain by cutting the Fourier series of $F$:

\[
F^N(\theta) = \sum_{|m| \leq N} \hat{F}(m)e^{2\pi ik \cdot \theta}
\]

In order to simplify the notation throughout this paper, we will write $\Psi(\cdot)$ for $\Psi(|\cdot|)$, and $\Lambda(\cdot)$ for $\Lambda(|\cdot|)$. We will denote by $|| \cdot ||$ the norm of the greatest coefficient for matrices.

3 Decompositions, triviality

We take the following definitions from [11], describing decompositions of $\mathbb{R}^2$ and triviality, which will avoid to double the period more than once.

**Definition 3.1 (Decomposition).**

- If $A \in \mathfrak{sl}(2, \mathbb{R})$ has distinct eigenvalues, we call $A$-decomposition a decomposition of $\mathbb{R}^2$ as the direct sum of two eigenspaces of $A$. If $L$ is an eigenspace of $A$, we write $\sigma(A|_L)$ the spectrum of the restriction of $A$ to the subspace $L$. We shall denote by $\mathcal{L}_A$ the decomposition of $\mathbb{R}^2$ into two distinct eigenspaces of $A$, if the related eigenvalues are distinct.
- If $\mathbb{R}^2 = L_1 \bigoplus L_2$, for all $u \in \mathbb{R}^2$, there exists a unique decomposition $u = u_1 + u_2$, $u_1 \in L_1$, $u_2 \in L_2$. For $i = 1, 2$, we call projection on $L_i$ with respect to $\mathcal{L} = \{L_1, L_2\}$, and we write $P_{L_i}^\mathcal{L}$ the map defined by $P_{L_i}^\mathcal{L} u = u_i$.

Recall the following lemma on estimate of the projection (see [13]):
Lemma 3.2 ([13]). Let \( \kappa' > 0 \) and \( A \in \text{sl}(2, \mathbb{R}) \) with \( \kappa' \)-separated eigenvalues. There exists a constant \( C_0 \geq 1 \) such that, for any subspace \( L \in \mathcal{L} = \mathcal{L}_A \),
\[
\| P_L^C \| \leq C_0 \left( \frac{1}{\kappa'} \right)^6
\]

Remark 3.3. The estimate given in [13] is more general since it also concerns matrices \( A \) with a nilpotent part. Here the setting in \( \text{sl}(2, \mathbb{R}) \) makes the estimate a little better.

Definition 3.4 (Triviality). Let \( \mathcal{L} = \{ L_1, L_2 \} \) such that \( L_1 \bigoplus L_2 = \mathbb{R}^2 \). We say that a function \( \psi \in C^0(2^d, \text{SL}(2\mathbb{R})) \) is trivial with respect to \( \mathcal{L} \) if there exists \( m \in \mathbb{Z}^d \) such that for all \( \theta \in 2^d \),
\[
\psi(\theta) = e^{2i\pi(m,\theta)} P_{L_1}^C + e^{-2i\pi(m,\theta)} P_{L_2}^C
\]

If \( |m| \leq N \), we say that \( \psi \) is trivial of order \( N \).

Remark 3.5. 
- If \( \psi_1, \psi_2 : 2^d \to \text{SL}(2, \mathbb{R}) \) are trivial with respect to \( \mathcal{L} \), then the product \( \psi_1 \psi_2 \) is also trivial with respect to \( \mathcal{L} \) since, for all \( L \neq L' \), \( P_L^C P_{L'}^C = 0 \).
- If \( \psi \) is trivial with respect to a decomposition \( \mathcal{L} \) of \( \mathbb{R}^2 \), then for all \( G \in C^0(2^d, \text{sl}(2, \mathbb{R})) \), we have \( G \psi \psi^{-1} \in C^0(2^d, \text{sl}(2, \mathbb{R})) \). Indeed, notice that if \( \psi = e^{2i\pi(m,\cdot)} P_{L_1}^C + e^{-2i\pi(m,\cdot)} P_{L_2}^C \) for some \( m \in \mathbb{Z}^d \), then
\[
\psi^{-1} = e^{-2i\pi(m,\cdot)} P_{L_1}^C + e^{2i\pi(m,\cdot)} P_{L_2}^C
\]

(it’s a simple calculus to check that with this expression, \( \psi \psi^{-1} = \psi^{-1} \psi = I \).) Then,
\[
\psi G \psi^{-1} = (e^{2i\pi(m,\cdot)} P_{L_1}^C + e^{-2i\pi(m,\cdot)} P_{L_2}^C) G (e^{-2i\pi(m,\cdot)} P_{L_1}^C + e^{2i\pi(m,\cdot)} P_{L_2}^C)
\]
\[
= P_{L_1}^C G P_{L_1}^C + P_{L_2}^C G P_{L_2}^C + e^{2i\pi(2m,\cdot)} P_{L_1}^C G P_{L_2}^C + e^{-2i\pi(2m,\cdot)} P_{L_2}^C G P_{L_1}^C
\]

which is well defined continuously on \( 2^d \). Hence the function \( \psi \) will avoid a period doubling.

4 Choice of parameters

In this section, we define all the constants and parameters used in this paper.

\[
\left\{ \begin{array}{l}
\delta = 100000 \\
\zeta = \frac{1}{2\pi}
\end{array} \right.
\]

Let for all \( r, \varepsilon > 0 \),

\[
N(r, \varepsilon) = \Lambda^{-1} \left( \frac{50|\log \varepsilon|}{\pi r} \right)
\]
\[
R(r, \varepsilon) = \frac{1}{3N(r, \varepsilon)} \Psi^{-1}(\varepsilon^{-\zeta})
\]
\[
\kappa''(\varepsilon) = \kappa \varepsilon^\zeta
\]
\[
r'(r, \varepsilon) = r - \frac{50\delta |\log \varepsilon|}{\pi \Lambda(R(r, \varepsilon) N(r, \varepsilon))}
\]

5 Smallness of the perturbation

Let \( r_0 > 0, A_0 \in \text{sl}(2, \mathbb{R}), F_0 \in U_{r_0}(\mathbb{T}^d, \text{sl}(2, \mathbb{R})) \).

Assumption 1. The functions \( \Lambda, \Psi \) satisfy

\[
\lim_{t \to +\infty} \frac{\ln \Psi(t)}{\Lambda(t)} = 0
\]

and \( \varepsilon_0 \) is small enough as to satisfy conditions of lemma 10.1 below and:

\[
\frac{150\delta |\log \varepsilon_0|}{\pi \Lambda \circ \Psi^{-1}(\varepsilon_0^{-\zeta})} + \frac{150\delta}{\pi \zeta \log(2\delta)} \int_{\varepsilon_0^{-\zeta}}^{+\infty} \frac{\Lambda'(t) \ln \Psi(t)}{\Lambda(t)^2} dt < r_0
\]
These conditions depend on \( r_0, \Lambda, \Psi \).

We shall define the following sequences of parameters used throughout the iteration:

\[
\varepsilon_k := \varepsilon_0^{(2\delta)^k}
\]
\[
\Lambda(N_k) := \Lambda(N(r_k, \varepsilon_k)) = \frac{50|\log \varepsilon_k|}{\pi r_k}
\]
\[
R_k := R(r_k, \varepsilon_k) = \frac{1}{3N(r_k, \varepsilon_k)}\Psi^{-1}(\varepsilon_k^{\delta})
\]
and

\[
 r_k := r_0 - \sum_{i=0}^{k-1} \frac{50\delta|\log \varepsilon_i|}{\pi \Lambda(R(r_i, \varepsilon_i)N(r_i, \varepsilon_i))}
\]

**Lemma 5.1.** Under either the assumption 1, the sequence \( r_k \) converges to a positive limit.

**Proof.** Notice that, for all \( k \), and since \( \Lambda \) is subadditive,

\[
\Lambda(R_k N_k) \geq \frac{1}{3}\Lambda(3R_k N_k) \Rightarrow \frac{1}{\Lambda(R_k N_k)} \leq \frac{3}{\Lambda(3R_k N_k)}
\]

Then

\[
\sum_{k \geq 0} \frac{50\delta|\log \varepsilon_k|}{\pi \Lambda(R_k N_k)} \leq \sum_{k \geq 0} \frac{150\delta|\log \varepsilon_k|}{\pi \Lambda(3R_k N_k)}
\]
\[
\leq \frac{150\delta}{\pi} \sum_{k \geq 0} \frac{(2\delta)^k|\log \varepsilon_0|}{\Lambda(3R_k N_k)}
\]
\[
\leq \frac{150\delta}{\pi} \sum_{k \geq 0} \frac{(2\delta)^k|\log \varepsilon_0|}{\Lambda(\Psi^{-1}(\varepsilon_0^{\delta}))}
\]
\[
\leq \frac{150\delta|\log \varepsilon_0|}{\pi \Lambda \circ \Psi^{-1}(\varepsilon_0^{\delta})} + \frac{150\delta|\log \varepsilon_0|}{\pi} \int_0^{+\infty} \frac{(2\delta)^x}{\Lambda(\Psi^{-1}(\varepsilon_0^{\delta})(2\delta)^x)} dx
\]

With the change of variable \( t := (2\delta)^x \)

\[
\leq \frac{150\delta|\log \varepsilon_0|}{\pi \Lambda \circ \Psi^{-1}(\varepsilon_0^{\delta})} + \frac{150\delta|\log \varepsilon_0|}{\pi} \int_1^{+\infty} \frac{t}{\Lambda(\Psi^{-1}(\varepsilon_0^{\delta})) \log(2\delta)} dt
\]
\[
\leq \frac{150\delta|\log \varepsilon_0|}{\pi \Lambda \circ \Psi^{-1}(\varepsilon_0^{\delta})} + \frac{150\delta|\log \varepsilon_0|}{\pi \log(2\delta)} \int_1^{+\infty} \frac{1}{\Lambda(\Psi^{-1}(\varepsilon_0^{\delta}))} dt
\]

With the change of variable \( v := \Psi^{-1}(\varepsilon_0^{\delta}) \)

\[
\leq \frac{150\delta|\log \varepsilon_0|}{\pi \Lambda \circ \Psi^{-1}(\varepsilon_0^{\delta})} + \frac{150\delta|\log \varepsilon_0|}{\pi \log(2\delta)} \int_{\Psi^{-1}(\varepsilon_0^{\delta})}^{+\infty} \frac{1}{\Lambda(v) \Psi(v)} dv
\]
\[
\leq \frac{150\delta|\log \varepsilon_0|}{\pi \Lambda \circ \Psi^{-1}(\varepsilon_0^{\delta})} + \frac{150\delta}{\pi \zeta \log(2\delta)} \int_{\Psi^{-1}(\varepsilon_0^{\delta})}^{+\infty} \frac{\Psi'(v)}{\Lambda(v) \Psi(v)} dv
\]

After integrating by parts,

\[
\sum_{k \geq 0} \frac{50\delta|\log \varepsilon_k|}{\pi \Lambda(R_k N_k)} \leq \frac{150\delta|\log \varepsilon_0|}{\pi \Lambda \circ \Psi^{-1}(\varepsilon_0^{\delta})} + \frac{150\delta}{\pi \zeta \log(2\delta)} \left[ \log(\varepsilon_0^{\delta}) \right]
\]
\[
+ \int_{\Psi^{-1}(\varepsilon_0^{\delta})}^{+\infty} \frac{\Lambda'(v) \log \Psi(v)}{\Lambda(v)^2} dv
\]
\[
\leq \frac{150\delta|\log \varepsilon_0|}{\pi \Lambda \circ \Psi^{-1}(\varepsilon_0^{\delta})} + \frac{150\delta}{\pi \zeta \log(2\delta)} \int_{\Psi^{-1}(\varepsilon_0^{\delta})}^{+\infty} \frac{\Lambda'(v) \log \Psi(v)}{\Lambda(v)^2} dv
\]

provided \( \lim_{v \to +\infty} \frac{\log \Psi(v)}{\Lambda(v)} = 0 \), thus the assumption 1 implies that \( (r_k) \) converges to a positive limit. \( \square \)
Remark: We naturally find the Bruno-Rüssmann condition with respect to weight function \( \Lambda \), which is the convergence of \( \int \frac{\Lambda'(v) \log \Psi(v)}{\Lambda(v)^2} dv \).

6 Elimination of resonances

Given a matrix \( A \), a useful technique in the KAM iteration will be to remove the resonances in the spectrum of the matrix \( A \). To characterize the non-resonance of \( z \in \mathbb{C} \) (depending on \( \omega \), a constant \( \kappa' > 0 \) and on an order \( N \in \mathbb{N} \)) we will write \( z \in BR^N_\omega(\kappa') \) if and only if:

\[
\forall k \in \mathbb{Z}^d \setminus \{0\}, \quad 0 < |k| \leq N \Rightarrow |z - 2i\pi \langle k, \omega \rangle| \geq \frac{\kappa'}{\Psi(k)}
\]

Definition 6.1. We will say that \( A \) has \( BR^N_\omega(\kappa') \) spectrum if

\[ \sigma(A) = \{ \alpha, \alpha' \} \Rightarrow \alpha - \alpha' \in BR^N_\omega(\kappa') \]

In particular, if the eigenvalues of \( A \) are \( i\alpha \) and \(-i\alpha \) with \( \alpha \in \mathbb{R} \), \( A \) has if \( BR^N_\omega(\kappa') \) spectrum if

\[ 2i\alpha \in BR^N_\omega(\kappa') \]

Remark 6.2. If \( A \) has real eigenvalues \( \alpha, \alpha' \), then for all \( N \in \mathbb{N} \), \( A \) has \( BR^N_\omega(\kappa) \)-spectrum because \( |\alpha - \alpha' - 2i\pi \langle m, \omega \rangle| \geq |2i\pi \langle m, \omega \rangle| \geq \frac{\kappa}{\Psi(m)}. \)

Lemma 6.3. Let \( \alpha \in \mathbb{R} \), \( \tilde{N} \in \mathbb{N}^* \) and \( \kappa' = \frac{\kappa}{\Psi(3N)} \). There exists \( m \in \frac{1}{2}\mathbb{Z}^d \), \( |m| \leq \frac{1}{2}\tilde{N} \) such that, if we denote \( \alpha' = \alpha - 2i\pi \langle m, \omega \rangle \), then \( 2i\alpha' \in BR^\tilde{N}_\omega(\kappa') \) and if \( m \neq 0 \) then \( |\alpha'| \leq \frac{\kappa'}{\Psi}. \)

Proof. We want to remove the resonances between \( i\alpha \) and \(-i\alpha \). If \( 2i\alpha \in BR^\tilde{N}_\omega(\kappa') \), let \( m = 0 \) and we are done. Otherwise, if there exists \( m' \in \frac{1}{2}\mathbb{Z}^d \) with \( |m'| \leq \tilde{N} \) such that

\[
|2\alpha - 2i\pi \langle m', \omega \rangle| < \frac{\kappa}{\Psi(m')}
\]

then let \( m = \frac{m'}{2} \) and \( 2\alpha' = \alpha - 2i\pi \langle m, \omega \rangle \). It’s a simple calculus to check that, in this case, \( |2\alpha'| \leq \kappa' \), hence \( \alpha' \leq \frac{\kappa'}{\Psi}. \) Now, for all \( k \in \frac{1}{2}\mathbb{Z}^d \), \( k \leq \tilde{N} \),

\[
|2i\alpha' - 2i\pi \langle k, \omega \rangle| \geq \frac{\kappa}{\Psi(k)} - \kappa' \geq \frac{\kappa'}{\Psi(k)}
\]

Then \( 2i\alpha' \in BR^{\tilde{N}}_\omega(\kappa'). \)

Lemma 6.4. Let \( \alpha \in \mathbb{R} \). For all \( R \in \mathbb{R}, N \in \mathbb{N}, N \geq 1, R \geq 2 \), there exists \( m \in \frac{1}{2}\mathbb{Z}^d \), \( |m| \leq \frac{1}{2}N \) such that, if we denote \( \kappa'' = \frac{\kappa}{\Psi(3RN)} \) and \( \alpha' = \alpha - 2i\pi \langle m, \omega \rangle \), then \( 2i\alpha' \in BR^{RN}_\omega(\kappa'') \) and if \( m \neq 0 \) then \( |\alpha'| \leq \frac{\kappa''}{\Psi}. \)

Proof. If \( \alpha \in BR^{RN}_\omega(\kappa'') \), then \( m = 0 \). Otherwise, apply the previous Lemma with \( \tilde{N} = N \) and \( \kappa' = \kappa'' \) to obtain \( |m| \leq \frac{1}{2}N \) such that \( |\alpha - \langle m, \omega \rangle| \leq \frac{\kappa''}{\Psi}. \) Therefore for all \( 0 \leq |k| \leq RN \),

\[
|2\alpha' - 2i\pi \langle k, \omega \rangle| \geq |2i\pi \langle k, \omega \rangle| - \kappa'' \geq \frac{\kappa}{\Psi(k)} - \kappa'' \geq \frac{\kappa''}{\Psi(k)}
\]

and then \( 2i\alpha' = 2i\alpha - 2i\pi \langle m, \omega \rangle \in BR^{RN}_\omega(\kappa''). \)
7 Renormalization

We want to define a map $\Phi$ which conjugates $A$ to a matrix with $BR^{RN}_\omega(\kappa'')$ spectrum.

**Lemma 7.1.** Let $A \in sl(2, \mathbb{R}), R \geq 2, N \in \mathbb{N} \setminus \{0\}$. If $\kappa'' = \frac{\kappa}{\Psi(3RN)}$ and $A$ has $\kappa''$-separated eigenvalues, then there exists a map $\Phi \in C^0(2\mathbb{T}^d, SL(2, \mathbb{R}))$ which is trivial with respect to $\mathcal{L}_A$ (the decomposition into eigenspaces of $A$), and a constant $C_0 \geq 1$ such that,

1. For all $r' > 0$,
   \[|\Phi|^{\pm 1}_{r'} \leq 2C_0 e^{2\pi \Lambda(\Phi')r'}\left(\frac{1}{\kappa''}\right)^6\]

2. If $\tilde{A}$ is defined by the following condition: for all $\theta \in 2\mathbb{T}^d$,
   \[\partial_\omega \Phi(\theta) = A\Phi(\theta) - \Phi(\theta)\tilde{A}\]
   (note that $\tilde{A}$ actually does not depend on $\theta$), then $|\tilde{A} - A| \leq \pi N$ and $\tilde{A}$ has $BR^{RN}_\omega(\kappa'')$ spectrum.

3. For any function $G \in C^0(\mathbb{T}^d, sl(2, \mathbb{R}))$, we have $\Phi G \Phi^{-1} \in C^0(\mathbb{T}^d, sl(2, \mathbb{R}))$.

4. If $\tilde{A} \neq A$ then $||\tilde{A}|| \leq \frac{1}{2}\kappa''$.

**Proof.** Let $m$ given by Lemma 6.4 with $\alpha$ the imaginary part of an eigenvalue of matrix $A$. If the eigenvalues of $A$ are in $\mathbb{R}$, then $m = 0$ and $\Phi = I$.

Otherwise, let $L_1$ be the invariant subspace associated to $i\alpha$, $L_2$ associated to $-i\alpha$ and let for all $\theta \in 2\mathbb{T}^d$,

\[\Phi(\theta) = e^{2i\pi (m, \theta)} P^{L_1}_{L_1} + e^{-2i\pi (m, \theta)} P^{L_2}_{L_2}\]

For all $\theta$, since the eigenvalues of $\Phi(\theta)$ are complex conjugate, we have $\Phi(\theta) \in SL(2, \mathbb{R})$, and from Lemma 6.4, $\tilde{A}$ has $BR^{RN}_\omega(\kappa'')$ spectrum (since the eigenvalues of $\tilde{A}$ are $\pm i\alpha$ obtained from lemma 6.4 where $\pm i\alpha$ are the eigenvalues of $A$). Moreover, the spectrum of $\tilde{A} - A$ is $\{\pm 2i\alpha (m, \omega)\}$ and $|2i\alpha (m, \omega)| \leq \pi N$ (remind that $|m| \leq \frac{1}{2}N$), and that we supposed $|\omega| \leq 1$ whence 2. Moreover, because $|m| \leq \frac{1}{2}N$, and from Lemma 3.2,

\[|\Phi|^{r'} \leq (||P_{L_1}|| + ||P_{L_2}||) e^{2\pi \Lambda(\Phi')r'} \leq 2C_0\left(\frac{1}{\kappa''}\right)^6 e^{2\pi \Lambda(\Phi')r'}\]

whence 1. The property 3 follows from the triviality of $\Phi$ (see the remark 3.5).

For the estimate in 4, notice that if $A \neq \tilde{A}$, that is to say if the spectrum of $A$ was resonant, $\Phi \neq I$ conjugates $A$ to $\tilde{A}$. In particular, from lemma 6.4, the two eigenvalues $i\alpha$ and $-i\alpha$ of $\tilde{A}$ (which are the eigenvalues of $A$ translated by $2\pi (m, \omega)$) satisfy $|i\alpha - (-i\alpha)| \leq \kappa''$ and then $||\tilde{A}|| \leq \frac{1}{2}\kappa''$.

**Definition 7.2.** A function $\Phi$ satisfying conclusions of lemma 7.1 will be called renormalization of $A$ of order $R, N$. Here the resonance is removed up to order $RN$ whereas the estimate involves an exponential of $\Lambda(\Phi')$ and $\Psi(3RN)$.

8 Cohomological equation

In order to define a change of variables which will reduce the norm of the perturbation, we will first solve a linearized equation, which has the form:

\[\forall \theta \in \mathbb{T}^d, \partial_\omega \hat{X}(\theta) = [\tilde{A}, \hat{X}(\theta)] + \tilde{F}^N - \hat{F}(0), \quad \hat{X}(0) = 0 \quad (8.1)\]

Here $\tilde{A} \in sl(2, \mathbb{R})$, therefore either it has real non zero eigenvalues, or it is the zero matrix, or it is nilpotent, or it has two eigenvalues $i\alpha, -i\alpha, \alpha \in \mathbb{R}^*$. Only in the latter case can $\tilde{A}$ be resonant.

Assume the eigenvalues are different (so, either they are distinct reals or they are complex conjugates). Let $L_1, L_2$ be the eigenspaces. For all $L, L' \in \{L_1, L_2\}$, define the following operator:

\[A_{L, L'} : gl(2, \mathbb{R}) \rightarrow gl(2, \mathbb{R}), M \mapsto A_{L, L'} M := \tilde{A} P_L M - M P_L \tilde{A}\]

It will be necessary to compute the spectrum of every $A_{L, L'}$ to estimate the solution of the linearized equation. This is done in the following lemma:
Lemma 8.1. Let $L, L' \in \{L_1, L_2\}$, $\beta$ the eigenvalue associated to $L$ and $\gamma$ the eigenvalue associated to $L'$. The spectrum of $A_{L, L'}$ is $\{\beta, -\gamma, \beta - \gamma, 0\}$. Moreover, the operator $A_{L, L'}$ is diagonalizable.

Proof. Let $P \in GL(2, \mathbb{C})$ such that $P^{-1}\tilde{A}P = \begin{pmatrix} \beta & 0 \\ 0 & \gamma \end{pmatrix}$. Notice that $\{\beta, \gamma\} \subset \{\tilde{\beta}, \tilde{\gamma}\}$. Denote by $E_{i,j}$ the elementary matrix which has 1 as the coefficient situated on line $i$ and column $j$, and 0 elsewhere.

Case 1: $L \neq L'$. Here $\{\beta, \gamma\} = \{\tilde{\beta}, \tilde{\gamma}\}$. Without loss of generality, assume $\beta = \tilde{\beta}, \gamma = \tilde{\gamma}$, that is to say, $L = L_1, L' = L_2$. Then $P_L = PE_{11}P^{-1} - 1$ and $P_{L'} = PE_{22}P^{-1}$. Thus $PE_{11}P^{-1}$ is an eigenvector associated to $\beta$ and $PE_{22}P^{-1}$ is an eigenvector associated to $-\gamma$. The matrix $PE_{11}P^{-1}$ is an eigenvector associated to $\beta - \gamma$ and $PE_{22}P^{-1}$ is in the kernel.

Case 2: $L = L' = L_1$. Here $\beta = \gamma = \tilde{\beta}$. Then $PE_{11}P^{-1}$ and $PE_{22}P^{-1}$ are in the kernel and $PE_{11}P^{-1}, PE_{12}P^{-1}$ are eigenvectors associated to $\beta$ and $-\beta$ respectively.

Case 3: $L = L' = L_2$. This case is very similar to the previous one.

Now assume 0 is the only eigenvalue of $\tilde{A}$. If $\tilde{A}$ is the zero matrix, then $ad_{\tilde{A}} = 0$. Otherwise $\tilde{A}$ is nilpotent and in this case one has the following lemma:

Lemma 8.2. Assume $\tilde{A}$ is nilpotent. Then the operator $ad_{\tilde{A}}$ has rank 2 and norm less than 1, and is nilpotent of order 3.

Proof. Let $P$ be such that $P^{-1}\tilde{A}P = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $PE_{12}P^{-1}$ and $P(E_{11} + E_{22})P^{-1}$ are in the kernel. Moreover $ad_{\tilde{A}}PE_{11}P^{-1} = -PE_{12}P^{-1}$ and $ad_{\tilde{A}}PE_{21}P^{-1} = P(E_{11} - E_{22})P^{-1}$, therefore $ad_{\tilde{A}}$ has norm less than 1.

This also implies that $ad^2_{\tilde{A}}(PE_{11}P^{-1}) = 0$ and $ad^3_{\tilde{A}}(PE_{11}P^{-1}) = 0$. Finally $ad_{\tilde{A}}$ is nilpotent of order 3.

Proposition 8.3. Let $N \in \mathbb{N}, \kappa' \in [0, \kappa], r \in [0, r_0]$. Let $\tilde{A} \in sl(2, \mathbb{R})$ with $BR^N_\kappa'(\kappa')$ spectrum. Then there exists a solution $\tilde{X} \in U_r(T^d, sl(2, \mathbb{R}))$ of the equation

$$\forall \theta \in T^d, P_{\tilde{A}}(\tilde{X}(\theta)) + F^N - \hat{F}(0), \quad \tilde{X}(0) = 0 \quad (8.2)$$

The truncation of $\tilde{X}$ at order $N$ is unique.

Moreover,

1. if $\tilde{A}$ is diagonalizable with distinct eigenvalues, let $L_{\tilde{A}} = \{L_1, L_2\}$ (the decomposition into eigenspaces of $\tilde{A}$) and $\Phi = P_{L_1}e^{2\pi i (m_1)} + P_{L_2}e^{-2\pi i (m_1)}$ for some $m \in \mathbb{Z}^d$, $|m| \leq N$, such that for $i \in \{1, 2\}$, $\|P_{L_i}\| \leq \frac{2\kappa}{\kappa'}$, then

$$|\Phi^{-1}\tilde{X}\Phi| < 4C_0^3(\frac{1}{\kappa})^{13}\Psi(N)|\Phi^{-1}\tilde{F}\Phi|_r$$

2. if $\tilde{A}$ is nilpotent,

$$|\tilde{X}| < \frac{3}{\kappa^3}\Psi(N)^3|\tilde{F}|_r$$

3. if $ad_{\tilde{A}} = 0$, then

$$|\tilde{X}| < \frac{1}{\kappa}\Psi(N)|\tilde{F}|_r$$

Proof. About existence, uniqueness and continuity of $\tilde{X} \in sl(2, \mathbb{R})$ on $T^d$, the proof is the same as [11], proposition 3.2. We now have to show the estimate which also follows from [11] and we will adapt the proof to ultra-differentiable setting.

Case 1: $\tilde{A}$ has two $\kappa'$-separated eigenvalues. Let $\Phi = P_{L_1}e^{2\pi i (m_1)} + P_{L_2}e^{-2\pi i (m_1)}$ where $L_1$ and $L_2$ are the eigenspaces of $\tilde{A}$, and $|m_1| \leq N$. For all $L, L' \in L_{\tilde{A}}$, let the linear operator $A_{L, L'} : gl(2, \mathbb{R}) \rightarrow gl(2, \mathbb{R}), M \mapsto A_{L, L'}M := \tilde{A}P_LM - MP_{L'}\tilde{A}$. We decompose (8.2) into blocks, and we get for all $L, L' \in L_{\tilde{A}},$

$$\partial_{\nu}(P_L\tilde{X}(\theta)P_{L'}) = A_{L, L'}P_L\tilde{X}(\theta)P_{L'} + P_L(\tilde{F}^N - \hat{F}(0))P_{L'}.$$
Then for all $m \in \frac{1}{2} \mathbb{Z}^d$, $0 < |m| \leq N$,
\[ 2i\pi \langle m, \omega \rangle (P_L \hat{\Phi}(m)P_{L'}) = A_{L,L'}(P_L \hat{\Phi}(m)P_{L'}) + P_L \hat{F}(m)P_{L'} \]

Let
\[ A_D := (2i\pi \langle m, \omega \rangle I - A_{L,L'}) \]

By Lemma 8.1, $\sigma(A_{L,L'}) = \{\alpha - \alpha', \alpha, -\alpha', 0; \alpha \in \sigma(\hat{A}_L), \alpha' \in \sigma(\hat{A}_{L'})\}$, therefore $\sigma(A_{L,L'} - 2i\pi \langle m, \omega \rangle I) = \{\alpha - \alpha' - 2i\pi \langle m, \omega \rangle, \alpha - 2i\pi \langle m, \omega \rangle, -\alpha' - 2i\pi \langle m, \omega \rangle, -2i\pi \langle m, \omega \rangle; \alpha \in \sigma(\hat{A}_L), \alpha' \in \sigma(\hat{A}_{L'})\}$. Moreover $A_{L,L'}$ is diagonalizable, therefore $A_D$ as well, with non zero eigenvalues, and $\|A_D^{-1}\| = \max\{\|\beta\|, \beta \in \sigma(A_D^{-1})\} = \max\{|\gamma|^{-1}, \gamma \in \sigma(A_D)\}$.

Since $\forall \alpha \in \sigma(\hat{A}_L), \alpha' \in \sigma(\hat{A}_{L'})$, $|\alpha - \alpha' - 2i\pi \langle m, \omega \rangle| \geq \frac{\kappa'}{\Psi(m)}$ (for $m \in \mathbb{Z}^d$ if $L = L'$, $m \in \frac{1}{2} \mathbb{Z}^d$ if $L \neq L'$), then
\[ \|(2i\pi \langle m, \omega \rangle - A_{L,L'})^{-1}\| \leq \left( \frac{\Psi(m)}{\kappa'} \right) \]

Finally, for all $0 < |m| \leq N$,
\[ \|P_L \hat{\Phi}(m)P_{L'}\| = \|(2i\pi \langle m, \omega \rangle - A_{L,L'})^{-1} P_L \hat{\Phi}(m)P_{L'}\| \leq \left( \frac{\Psi(m)}{\kappa'} \right) \|P_L \hat{F}(m)P_{L'}\| \]

Denoting by $m_L$ the vector appearing in $\Phi$ along the projection onto $L$, this estimate implies:
\[ |P_L \hat{\Phi} e^{2i\pi \langle m_L - m_L', \omega \rangle} P_{L'}|_r = \sum_{|m - m_L + m_L'| \leq N} \|P_L \hat{\Phi} (m - m_L + m_L') P_{L'}\| e^{2i\pi \Lambda(m')}$
\[ \leq \sum_{|m - m_L + m_L'| \leq N} \|P_L \hat{\Phi} (m - m_L + m_L') P_{L'}\| e^{2i\pi \Lambda(m')} \frac{\Psi(m - m_L + m_L')}{\kappa'} \]
\[ \leq \frac{\Psi(N)}{\kappa'} |P_L \hat{F} e^{2i\pi \langle m_L - m_L', \omega \rangle} P_{L'}|_r \] (8.3)

We finally estimate $|\Phi^{-1} \hat{\Phi}|_r$.
\[ |\Phi^{-1} \hat{\Phi}|_r = | \sum_{L,L' \in \mathcal{L}} P_L \Phi^{-1} \hat{\Phi} P_{L'}|_r = | \sum_{L,L' \in \mathcal{L}} P_L \hat{\Phi} e^{2i\pi \langle m_L - m_L', \omega \rangle} P_{L'}|_r \]

therefore, from (8.3),
\[ |\Phi^{-1} \hat{\Phi}|_r \leq \frac{\Psi(N)}{\kappa'} \sum_{L,L' \in \mathcal{L}} |P_L \hat{F} e^{2i\pi \langle m_L - m_L', \omega \rangle} P_{L'}|_r = \frac{\Psi(N)}{\kappa'} \sum_{L,L' \in \mathcal{L}} |P_L \Phi^{-1} \hat{F} P_{L'}|_r \]

therefore, since $\|P_L\| \leq \frac{2\kappa}{\kappa'}$, we get the result
\[ |\Phi^{-1} \hat{\Phi}|_r \leq 4C_0^2 \left( \frac{1}{\kappa'} \right)^{13} \Psi(N)|\Phi^{-1} \hat{F} P_{L'}|_r \]

Case 2: $\hat{A}$ is nilpotent. One has to estimate the inverse of the operator $2i\pi \langle m, \omega \rangle I - ad_{\hat{A}}$. By Lemma 8.2,
\[ (2i\pi \langle m, \omega \rangle I - ad_{\hat{A}})^{-1} = (2i\pi \langle m, \omega \rangle)^{-1} \left[ I + (2i\pi \langle m, \omega \rangle)^{-1} ad_{\hat{A}} \right] \]
\[ + (2i\pi \langle m, \omega \rangle)^{-2} ad_{\hat{A}}^2 \] (8.4)

Therefore
\[ \|(2i\pi \langle m, \omega \rangle I - ad_{\hat{A}})^{-1}\| \leq 3|2i\pi \langle m, \omega \rangle|^{-3} \]

Finally, for all $0 < |m| \leq N$,
\[ \| \hat{X}(m) \| = \| (2i\pi \langle m, \omega \rangle - A_{L,L'})^{-1} \hat{F}(m) \| \leq 3 \left( \frac{\Psi(m)}{\kappa} \right)^3 \| \hat{F}(m) \| \]

Thus,
\[ |\hat{X}|_{r'} \leq \frac{3}{\kappa^3} \Psi(N)^3 |\hat{F}|_r \]

**Case 3:** The operator to invert is just \( 2i\pi \langle m, \omega \rangle I \), which makes the estimate much simpler.

9 **Inductive lemma without renormalization**

Before stating the inductive lemma, we will need this next result which will allow us to iterate the inductive lemma without needing a new renormalization map at each step.

**Lemma 9.1.** Let \( \kappa' \in [0,1] \), \( \tilde{F} \in \text{sl}(2, \mathbb{R}) \), \( \xi = \| \tilde{F} \| \), \( \tilde{N} \in \mathbb{N} \), \( \tilde{A} \in \text{sl}(2, \mathbb{R}) \) with \( BR^N(\kappa') \) spectrum.

If
\[ \tilde{\xi} \leq \left( \frac{\kappa'}{32(1 + \| A \|)} \right)^2 \frac{1}{\Psi(N)^2}, \]

then \( \tilde{A} + \tilde{F} \) has \( BR^N(\frac{\kappa'}{4}) \) spectrum.

**Proof.** If \( \tilde{\alpha} \in \sigma(\tilde{A} + \tilde{F}) \), there exists \( \alpha \in \sigma(\tilde{A}) \) such that \( |\alpha - \tilde{\alpha}| \leq 4(\| \tilde{A} \| + 1)\tilde{\xi}^\frac{k}{4} \) (see [11], lemma 4.1). Since \( \tilde{A} \) has \( BR^N(\kappa') \) spectrum, for all \( \alpha, \alpha' \in \sigma(\tilde{A} + \tilde{F}) \), for all \( m \in \frac{1}{2} \mathbb{Z}^d \), \( 0 < |m| \leq \tilde{N} \),
\[ |\alpha - \alpha' - 2i\pi \langle m, \omega \rangle| \geq \frac{\kappa'}{\Psi(m)} - 8(\| \tilde{A} \| + 1)\tilde{\xi}^\frac{k}{4} \]

We have to check that \( 8(\| \tilde{A} \| + 1)\tilde{\xi}^\frac{k}{4} \leq \frac{\kappa}{2\Psi(m)} \), which is satisfied by assumption.

**Lemma 9.2.** Let \( N \geq 1 \). If \( \mathcal{L} = \{L_1, L_2\} \) is a decomposition of \( \mathbb{R}^2 \) into supplementary subspaces, and \( \Phi \) is trivial with respect to \( \mathcal{L} \) of order \( N \), then for all \( 0 < r' < r \) and all \( G \in U_r(\mathbb{T}^d, \text{sl}(2, \mathbb{R})) \),
\[ |\Phi^{-1}(G - G^{3N})|_{r'} \leq e^{-2\pi \Lambda(N)(r-r')} |\Phi^{-1}G|_r \]

**Proof.** Write \( \Phi = P_{L_1}^F e^{2i\pi \langle m_{1,-} \cdot \cdot \cdot \rangle} + P_{L_2}^F e^{2i\pi \langle m_{1,-} \cdot \cdot \cdot \rangle}, \) \( m_1 \in \frac{1}{2} \mathbb{Z}^d \), then
\[ |\Phi^{-1}(G - G^{3N})|_{r'} = | \sum_{L, L' \in \mathcal{L}} P_{L}^F (G - G^{3N}) e^{2i\pi \langle m_L - m_{L'} \cdot \cdot \cdot \rangle} P_{L'}^F |_{r'} \]
\[ = \sum_{k \in \mathbb{Z}^d} | \sum_{L, L'} P_{L}^F (G - G^{3N})(k - m_L + m_{L'}) P_{L'}^F |_{r'} \]
\[ = \sum_{k \in \mathbb{Z}^d} P_{L}^F (G - G^{3N})(k - m_L + m_{L'}) P_{L'}^F |_{r'} \]
\[ = e^{2\pi \Lambda(k)} r \]

Now if \( |k| \leq N \), then for all \( L, L' \in \mathcal{L}, |k - m_L + m_{L'}| \leq 3N \), and \( \sum_{L, L' \in \mathcal{L}} P_{L}^F (G - G^{3N})(k - m_L + m_{L'}) P_{L'}^F = 0 \), therefore
\[ |\Phi^{-1}(G - G^{3N})|_{r'} \leq e^{2\pi \Lambda(N)(r-r')} \sum_{|k|>N} \sum_{L, L' \in \mathcal{L}} P_{L}^F \hat{G}(k - m_L + m_{L'}) P_{L'}^F |_{r'} \]
\[ \leq e^{2\pi \Lambda(N)(r-r')} | \sum_{L, L' \in \mathcal{L}} P_{L}^F G e^{2i\pi \langle m_L - m_{L'} \cdot \cdot \cdot \rangle} P_{L'}^F |_{r'} = e^{2\pi \Lambda(N)(r-r')} |\Phi^{-1}G|_r \]

We can now state the first inductive lemma, which does not require a renormalization map.

**Lemma 9.3.** Let
such that
then there exist
\begin{align*}
\|\hat F(0)\| \leq \xi \leq \left(\frac{\kappa'}{32(1 + \|A\|)}\right)^2 \frac{1}{\Psi(N)^2}
\end{align*}

If
\begin{enumerate}
\item \(\hat A\) has BR_\hat N(\kappa') spectrum,
\item \(\|A' - \hat A\| \leq \varepsilon\),
\end{enumerate}
then we have the following estimates:
\begin{enumerate}
\item \(A' \in U_{r'}(T^d, sl(2, \mathbb{R}))\),
\item \(A' \in sl(2, \mathbb{R})\),
\end{enumerate}
such that
\begin{enumerate}
\item \(A' \in BR_\hat N(\kappa')\) spectrum,
\item \(\|F' - \hat F\| \leq \varepsilon\),
\end{enumerate}
\begin{align*}
\forall \theta \in T^d, \partial_\theta e^{X(\theta)} = (\hat A + \hat F(\theta))e^{X(\theta)} - e^{X(\theta)}(A' + F'(\theta)),
\end{align*}
then we have the following estimates:
\begin{enumerate}
\item If \(\hat A\) has two different eigenvalues, if \(\Phi\) is of the form \(\Phi = P_L e^{2\pi i m,\cdot} + P_L e^{-2\pi i m,\cdot}\) where \(L_1, L_2\) are the two eigenspaces of \(\hat A\), \(|m| \leq \hat N\) and \(\|P_{L_1}\| \leq \frac{2\kappa_0}{\kappa}\).
\item
\begin{align*}
|\Phi^{-1} X \Phi|_{r'} \leq 4C_0^2 \left(\frac{1}{\kappa'}\right)^{13} \Psi(3\hat N)|\Phi^{-1} \hat F \Phi|_{r'},
\end{align*}
\item
\begin{align*}
|\Phi^{-1} F' \Phi|_{r'} \leq 4C_0^2 e^{i|X|_{r'} \cdot \Phi} \left(\frac{1}{\kappa'}\right)^{13} |\Phi^{-1} \hat F \Phi|_{r'} [e^{-2\pi \Lambda(\hat N)(\bar r - r')} + |\Phi^{-1} \hat F \Phi|_{r'} \Psi(3\hat N)(2e + e^{X|_{r'})}].
\end{align*}
\end{enumerate}
If \(\hat A\) is nilpotent:
\begin{enumerate}
\item
\begin{align*}
|X|_{r'} \leq \frac{3}{\kappa^3} \Psi(3\hat N^3)|\hat F|_{r'},
\end{align*}
\item
\begin{align*}
|F'|_{r'} \leq \frac{3}{\kappa^3} e^{X|_{r'} \cdot \Phi} |\hat F|_{r'} [e^{-2\pi \Lambda(\hat N)(\bar r - r')} + |\hat F|_{r'} \Psi(3\hat N^3)(2e + e^{X|_{r'})}].
\end{align*}
\end{enumerate}
If \(\text{ad} \hat A = 0\):
\begin{enumerate}
\item
\begin{align*}
|X|_{r'} \leq \frac{1}{\kappa} \Psi(3\hat N)|\hat F|_{r'},
\end{align*}
\item
\begin{align*}
|F'|_{r'} \leq \frac{1}{\kappa} e^{X|_{r'} \cdot \Phi} |\hat F|_{r'} [e^{-2\pi \Lambda(\hat N)(\bar r - r')} + |\hat F|_{r'} \Psi(3\hat N)(2e + e^{X|_{r'})}].
\end{align*}
\end{enumerate}
In any case, there is the estimate
9. 
\[|\partial_\nu X|_{|p'} \leq 2\|A\| |X|_{|p'} + |F|_{|p'}].\]

Proof. By assumption, \(A\) has \(BR_\nu^N(k')\) spectrum, so we apply Proposition 8.3 with \(N = 3\bar{N}\). Let \(X \in U_\nu(\mathbb{T}^d, sl(2, \mathbb{R}))\) a solution of
\[\forall \theta \in 2\mathbb{T}^d, \partial_\nu X(\theta) = [\hat{A}, X(\theta)] + \hat{F}^3\bar{N}(\theta) - \hat{F}(0)\]
satisfying the conclusion of Proposition 8.3. This obviously implies the property 9.

Let \(A' := \hat{A} + \hat{F}(0)\). We have \(A' \in sl(2, \mathbb{R})\) and \(\|A - A'\| = \|\hat{F}(0)\|\), and then property 2. With assumption (2) we can apply lemma 9.1 to deduce that \(A'\) has \(BR_\nu^N(k')\) spectrum, and then property 1.

If \(F'\) is defined in equation (9.1),
\[F' = e^{-X}(\hat{F} - \hat{F}^3\bar{N}) + e^{-X}\hat{F}(e^X - Id) + (e^{-X} - Id)\hat{F}(0) - e^{-X}\sum_{k \geq 2} \frac{1}{k!} \sum_{l=0}^{k-1} X^l(\hat{F}^3\bar{N} - \hat{F}(0))X^{k-1-l}\]

- Case 1: \(A\) has two different eigenvalues: Let \(\Phi\) be as required, then
\[|\Phi^{-1}F'\Phi|_{|p'} \leq e^{[\Phi^{-1}X\Phi]_{|p'}} |[\Phi^{-1}(\hat{F} - \hat{F}^3\bar{N})\Phi|_{|p'} + |\Phi^{-1}\hat{F}\Phi|_{|p'}|\Phi^{-1}X\Phi|_{|p'}(2e + e|\Phi^{-1}X\Phi|_{|p'})]\]

From proposition 8.3, estimate 1,
\[|\Phi^{-1}X\Phi|_{|p'} \leq |\Phi^{-1}X\Phi|_{|p'} \leq 4C_0^2(\frac{1}{k'})^{13}\Psi(3\bar{N})|\Phi^{-1}\hat{F}\Phi|_{\tilde{r}}\]
whence (3); and from lemma 9.2, since \(\tilde{r}' < \tilde{r}\),
\[|\Phi^{-1}(\hat{F} - \hat{F}^3\bar{N})\Phi|_{|p'} \leq e^{-2\pi\Lambda(\bar{N})(\tilde{r} - \tilde{r}')}|\Phi^{-1}\hat{F}\Phi|_{\tilde{r}}\]
which finally gives
\[|\Phi^{-1}F'\Phi|_{|p'} \leq e^{[\Phi^{-1}X\Phi]_{|p'}} e^{-2\pi\Lambda(\bar{N})(\tilde{r} - \tilde{r}')}|\Phi^{-1}\hat{F}\Phi|_{\tilde{r}}\]
\[+ |\Phi^{-1}\hat{F}\Phi|_{|p'}4C_0^2(\frac{1}{k'})^{13}\Psi(3\bar{N})|\Phi^{-1}\hat{F}\Phi|_{|p'}(2e + e|\Phi^{-1}X\Phi|_{|p'})]\]
\[\leq 4C_0^2e^{[\Phi^{-1}X\Phi]_{|p'}}(\frac{1}{k'})^{13}|\Phi^{-1}\hat{F}\Phi|_{|p'}\left[e^{-2\pi\Lambda(\bar{N})(\tilde{r} - \tilde{r}') + |\Phi^{-1}\hat{F}\Phi|_{|p'}\Psi(3\bar{N})(2e + e|\Phi^{-1}X\Phi|_{|p'})}\right]\]
hence 4 holds.

- Case 2: \(A\) is nilpotent: (9.2) implies
\[|F'|_{|p'} \leq e^{[X]_{|p'}} \|\hat{F} - \hat{F}^3\bar{N}\|_{|p'} + |\hat{F}|_{|p'}|X|_{|p'}(2e + e[X]_{|p'})]\]

From proposition 8.3, estimate 2,
\[|X|_{|p'} \leq |X|_{|p'} \leq \frac{3}{k^3}\Psi(3\bar{N})^3|\hat{F}|_{\tilde{r}}\]
which is estimate 5. Moreover, from Lemma 9.2,
\[|\hat{F} - \hat{F}^3\bar{N}|_{|p'} \leq e^{-2\pi\Lambda(\bar{N})(\tilde{r} - \tilde{r}')}|\hat{F}|_{\tilde{r}}\]

Therefore, similarly to the previous case, we get
\[|F'|_{|p'} \leq \frac{3}{k^3}e^{[X]_{|p'}} |\hat{F}|_{|p'}\left[e^{-2\pi\Lambda(\bar{N})(\tilde{r} - \tilde{r}') + |\hat{F}|_{|p'}\Psi(3\bar{N})^3(2e + e[X]_{|p'})}\right]\]
which is estimate 6.
Case 3: \(ad_A = 0\): From proposition 8.3, estimate 3

\[
|\tilde{X}|_{\tilde{r}} \leq \frac{1}{\kappa} \Psi(3\tilde{N})|\tilde{F}|_{\tilde{r}}
\]

which is estimate (7), and similarly to the two previous cases, we get the estimate (8):

\[
|F'|_{\tilde{r}} \leq \frac{1}{\kappa} e^{X|_{\tilde{r}'}|F'|_{\tilde{r}}[e^{-2\pi\Lambda(\tilde{N})}(\tilde{r} - \tilde{r}')}
\]

\[+ |\tilde{F}|_{\tilde{r}} \Psi(3\tilde{N})(2\epsilon + e^{X|_{\tilde{r}'}})].
\]

\[\square\]

10 Inductive lemma with renormalization

The following Lemma is used to define the smallness assumption on \(\epsilon_0\) mentioned in section 5. This smallness assumption shall be sufficient for Lemmas 10.2 and 11.1.

**Lemma 10.1.** Let \(l = 56\). There exists \(\epsilon_0 > 0\) depending on \(C_0, C', \kappa, b_0\) and \(D_5\), such that, for all \(\epsilon \in [0, \epsilon_0]\), the following inequalities hold for all \(2 \leq j \leq l\):

**In lemma 10.2**

\[
\frac{1}{2} \kappa \epsilon^{\frac{1}{1728}} + \epsilon^{\frac{645}{448}} \leq \frac{3}{4} \kappa \epsilon^{\frac{1}{1728}}
\]

\[
4C_0^2 \kappa^{-13} \epsilon^{-3} \kappa e^{1-2\epsilon} \leq \epsilon^\frac{1}{2}
\]

\[
8C_0^2 \epsilon^{1-2\epsilon} \leq \epsilon^{\frac{1}{2}-4\epsilon} + 3 \epsilon^{1-6\epsilon}
\]

**In lemma 11.1**

\[
\epsilon^{1-576\epsilon} \leq (2C_0)^{-96} \left(\frac{\kappa}{32(\epsilon^{-\frac{1}{2}} + 1)}\right)^{576}
\]

\[
\epsilon^{\frac{3}{4} - \frac{1}{128}} \leq \left(\frac{\frac{3}{4}}{32(1 + (1 + \pi)\epsilon^{-\frac{1}{2}} + \epsilon^{\frac{1}{2}})}\right)^{2\epsilon^2}
\]

\[
\epsilon^{(\frac{3}{2})^{l-1} - \frac{1}{128}} \leq \left(\frac{\frac{3}{4}}{32(1 + \epsilon^{\frac{3}{2}} + (1 + \pi)\epsilon^{-\frac{1}{2}} + \sum_{i=1}^{l-1} \epsilon^{(\frac{3}{2})^{i-1}})\epsilon^{\frac{1}{2}}}ight)^{2\epsilon^2}
\]

\[
256C_0^2 \epsilon^{-14} \left(\frac{1}{(\frac{3}{4})^{j-1} \epsilon_{C_0}}\right)^{13} \epsilon^{(\frac{3}{2})^{j-1}} \left(\epsilon^{\frac{3}{2}} + \epsilon^{(\frac{3}{2})^{j-1}}\right) \leq \epsilon^{(\frac{3}{2})^j}
\]

\[
\epsilon^{\frac{23}{4} + \pi \epsilon^{-\frac{1}{2}} + \sum_{i=1}^{l} \epsilon^{(\frac{3}{2})^{i-1} - \frac{1}{128}}} \leq \epsilon^{-\epsilon}
\]

\[
\frac{1}{2} \kappa \epsilon^{-\epsilon} + 2 \epsilon^{\frac{3}{4} - \frac{1}{128}} \leq \kappa \epsilon^{-\epsilon}
\]

\[
\epsilon^{-\frac{1}{2}} + \epsilon^{\frac{23}{4} + \pi \epsilon^{-\epsilon}} \leq \epsilon^{-2\epsilon}
\]

\[
4 \epsilon^{-2\epsilon + \frac{23}{4}} + 2 \epsilon^{\frac{3}{4} - \frac{1}{128}} \leq \epsilon
\]

\[
2 \epsilon^{\frac{3}{4} + \frac{23}{4}} \leq \epsilon^{\frac{3}{4}}
\]
Proof. Equations in lemma 10.2

Equation (10.1) holds for
\[ \varepsilon \leq \left( \frac{1}{4} \right)^{1728} \left( \frac{\kappa}{2123} \right)^{\frac{1}{24}}. \]

Equation (10.2) holds for
\[ \varepsilon \leq \left( \frac{4C_0^2}{\kappa} \right)^{\frac{432}{864}}. \]

Equation (10.3) holds if
\[ 8C_0^2 \varepsilon^{\frac{432}{864}} (\varepsilon^{1005} + 3\varepsilon^{\frac{287}{288}}) \leq \varepsilon^{\frac{432}{864}} \Leftrightarrow 8C_0^2 (\varepsilon^{1005} + 3\varepsilon^{\frac{287}{288}}) \leq \varepsilon^{\frac{432}{864}} \]
therefore, if we have
\[ \left\{ \begin{array}{l}
8C_0^2 \varepsilon^{1005} \leq \frac{1}{2} \varepsilon^{\frac{432}{864}} \\
24C_0^2 \varepsilon^{\frac{287}{288}} \leq \frac{1}{2} \varepsilon^{\frac{432}{864}}
\end{array} \right. \]
which is satisfied if
\[ \varepsilon \leq (48C_0^2)^{-\frac{432}{864}} \]
then inequality (10.3) holds.

Equations in lemma 11.1

Equation (10.4) holds if
\[ \varepsilon^{\frac{50}{512}} + \varepsilon^{\frac{1}{1024}} \leq (2C_0)^{-\frac{1}{32}} \]
which is satisfied if
\[ \left\{ \begin{array}{l}
\varepsilon^{\frac{50}{512}} \leq \frac{1}{2} (2C_0)^{-\frac{1}{32}} \\
\varepsilon^{\frac{1}{1024}} \leq \frac{1}{2} (2C_0)^{-\frac{1}{32}}
\end{array} \right. \Leftrightarrow \left\{ \begin{array}{l}
\varepsilon \leq \left( \frac{4}{512} \right)^{1152} (2C_0)^{-192} \\
\varepsilon \leq \left( \frac{4}{512} \right)^{864} (2C_0)^{-144}
\end{array} \right. \]
Equation (10.5) is satisfied if
\[ \varepsilon^{\frac{5}{2}-\frac{5}{24}} (1 + \varepsilon^{\frac{5}{24}} + (1 + \pi)\varepsilon^{\frac{5}{24}})^2 \leq \left( \frac{3\kappa}{128C_0} \right)^2. \]
For \( \varepsilon \leq 1 \), we have
\[ 1 + \varepsilon^{\frac{23}{24}} + (1 + \pi)\varepsilon^{\frac{5}{24}} \leq 4\pi \varepsilon^{\frac{5}{24}} \]
then we need
\[ 16\pi^2 \varepsilon^{\frac{5}{24}} \leq \left( \frac{3\kappa}{128C_0} \right)^2 \]
which is satisfied if
\[ \varepsilon \leq \left( \frac{3\kappa}{512\pi C_0} \right)^{\frac{2410}{1728}} \]
Equation (10.6) is satisfied if
\[ \varepsilon^{(\frac{5}{2})^{-1}} (1 + \varepsilon^{\frac{23}{24}} + (1 + \pi)\varepsilon^{\frac{5}{24}} + 2\varepsilon^{\frac{5}{24}}) \leq \left( \frac{3}{4} \right)^{2j} \left( \frac{\kappa}{32C_0} \right)^2. \]
If \( \varepsilon \leq 1 \), then
\[ 1 + \varepsilon^{\frac{23}{24}} + (1 + \pi)\varepsilon^{\frac{5}{24}} + 2\varepsilon^{\frac{5}{24}} \leq 4\pi \varepsilon^{\frac{5}{24}} \]
then it’s enough to have
\[ 16\pi^2 \varepsilon^{\frac{5}{24}} \leq \left( \frac{3}{4} \right)^{2j} \left( \frac{\kappa}{32C_0} \right)^2 \]
which is satisfied if
\[ \varepsilon \leq \left( \frac{3}{4} \right)^{2j} \left( \frac{\kappa}{128\pi C_0} \right)^{\frac{2410}{1728}} \]
Equation (10.7) holds if
\[ 256C_0^2 \left( \frac{1}{3} \right)^{13(j-1)} \left( \frac{C_0}{\kappa} \right)^{13} (\varepsilon^{\frac{504}{512}} + \varepsilon^{(\frac{5}{2})^{-1}}) \leq \varepsilon^{14\zeta + \frac{5}{2}(\frac{5}{2})^{-1}}. \]
We will first show that, for all \( j \in [2, l] \), and for \( \varepsilon \) small enough,
\[
\varepsilon^{5/2} + \varepsilon^{(4)j-1} \leq \varepsilon^{5/4} (\varepsilon^{(4)j-1}.
\]
Since \( l = 56 \), this condition is satisfied if for all \( j \in [2, l] \) if
\[
\begin{align*}
2 & \leq \varepsilon^{5/4} (\varepsilon^{(4)j-1} - \varepsilon^{5/2}) \\
2 & \leq \varepsilon^{5/4} (\varepsilon^{(4)j-1}
\end{align*}
\]
which holds if
\[
\varepsilon \leq 2^{5/4} \varepsilon^{5/2} - \varepsilon^{5/4}
\]
then equation (10.7) is satisfied if
\[
256C_0^2 (\frac{4}{3})^{13(j-1)} \left( \frac{C_0}{\kappa} \right)^{13} \leq \varepsilon^{14}\varepsilon^{(4(j-1))^{-1}} \iff \varepsilon \leq (256C_0^2 (\frac{4}{3})^{13(j-1)} \left( \frac{1}{\kappa} \right)^{13})^{\frac{14}{13}\varepsilon^{(4(j-1))^{-1}}}
\]
Now, as \( C_0 \geq 1 \) and \( 0 < \kappa < 1 \), since \( \varepsilon \leq 1 \),
\[
(\frac{24C_0^2}{\kappa^{11}})^{\frac{14}{13}\varepsilon^{(4(j-1))^{-1}}} \geq (\frac{24C_0^2}{\kappa^{11}})^{\frac{14}{13}\varepsilon^{5/2}} = (\frac{24C_0^2}{\kappa^{11}})^{\frac{14}{13}\varepsilon^{5/2}}
\]
and
\[
(\frac{4}{3})^{\frac{14}{13}\varepsilon^{(4(j-1))^{-1}}} \geq (\frac{4}{3})^{\frac{14}{13}\varepsilon^{5/2}} = (\frac{4}{3})^{\frac{14}{13}\varepsilon^{5/2}}
\]
Finally, equation (10.7) is satisfied with
\[
\varepsilon \leq (\frac{24C_0^2}{\kappa^{11}})^{\frac{14}{13}\varepsilon^{5/2}} (\frac{4}{3})^{\frac{14}{13}\varepsilon^{5/2}} = 1437606^{5/8545}
\]
Equation (10.8) is satisfied if
\[
\varepsilon^{5/2} + \pi \varepsilon^{5/4} - 2\pi \varepsilon^{5/4} \leq \varepsilon^{-\zeta}.
\]
So if we have
\[
\begin{align*}
\varepsilon^{5/2} & \leq \frac{1}{16} \varepsilon^{-\zeta} \\
\pi \varepsilon^{5/4} & \leq \frac{1}{16} \pi \varepsilon^{-\zeta} \\
2 \varepsilon^{5/4} & \leq \frac{1}{16} \varepsilon^{-\zeta}
\end{align*}
\]
then equation (10.8) holds.
Equation (10.9) holds for
\[
\varepsilon \leq (\frac{1}{4})^{1728^{16/11}}
\]
Equation (10.10) holds if
\[
\begin{align*}
\varepsilon^{5/4} & \leq \frac{1}{16} \varepsilon^{-2\zeta} \\
\varepsilon^{5/4} & \leq \frac{1}{16} \varepsilon^{-2\zeta} \\
\varepsilon^{5/4} & \leq \frac{1}{16} \pi \varepsilon^{-2\zeta}
\end{align*}
\]
Equation (10.11) holds if
\[
\begin{align*}
4 \varepsilon^{5/4} & \leq \frac{1}{16} \varepsilon^{-2\zeta} \\
2 \varepsilon^{5/4} & \leq \frac{1}{16} \varepsilon^{-2\zeta}
\end{align*}
\]
and then equation (10.11) holds.
Since, for \( \varepsilon \leq 1 \) we have \( \varepsilon^{5/4} \leq \varepsilon^{5/4} \), equation (10.12) holds if
\[
4 \varepsilon^{5/4} \leq \varepsilon^{5/4}
\]
that's it to say, if
\[
\varepsilon \leq \frac{1}{256}
\]
Now define \( \varepsilon_0 \) in order to satisfy conditions (10.1) to (10.12).
Lemma 10.2 (Inductive lemma with renormalization). Let

• \( A \in \text{sl}(2, \mathbb{R}) \),
• \( r > 0 \),
• \( \bar{A}, \bar{F} \in U_r(\mathbb{T}^d, \text{sl}(2, \mathbb{R})) \), \( \psi \in U_r(2\mathbb{T}^d, \text{SL}(2, \mathbb{R})) \),
• \( |\bar{F}|_r = \varepsilon \),
• \[ N = \Lambda^{-1} \left( \frac{50|\log \varepsilon|}{\pi r} \right) \]
• \[ R = \frac{1}{3N} \Psi^{-1}(\varepsilon^{-\xi}) \]
• \[ r' = r - 50\delta|\log \varepsilon|/\pi \Lambda(RN) \]

Assume \( r' > 0 \). Let \( \kappa'' = \frac{\kappa}{\Psi(3RN)} = \kappa \varepsilon^\xi \). Suppose that \( \varepsilon \leq \varepsilon_0 \) which was defined in Lemma 10.2 and

1. \[ \varepsilon \leq (2C_0)^{-96} \left( \frac{\kappa''}{32(\|A\| + 1)} \right)^{576} \]
2. \( \bar{A} \) is reducible to \( A \) by \( \psi \),
3. \( \|A\| \leq \varepsilon^{-\hat{\xi}} \),
4. for all \( G \in C^0(\mathbb{T}^d, \text{sl}(2, \mathbb{R})) \), \( \psi^{-1}G\psi \in C^0(\mathbb{T}^d, \text{sl}(2, \mathbb{R})) \),
5. \( |\psi^{\pm 1}|_r \leq \varepsilon^{-\zeta} \),

then there exist

• \( Z' \in U_r(\mathbb{T}^d, \text{SL}(2, \mathbb{R})) \),
• \( \bar{A}', \bar{F}' \in U_r(\mathbb{T}^d, \text{sl}(2, \mathbb{R})) \),
• \( \psi' \in U_r(2\mathbb{T}^d, \text{SL}(2, \mathbb{R})) \),
• \( A' \in \text{sl}(2, \mathbb{R}) \)

satisfying the following properties:

1. \( \bar{A}' \) is reducible by \( \psi' \) to \( A' \),
2. for all \( G \in C^0(\mathbb{T}^d, \text{sl}(2, \mathbb{R})) \), \( \psi'^{-1}G\psi' \in C^0(\mathbb{T}^d, \text{sl}(2, \mathbb{R})) \),
3. \( A' \) has \( BR''_{\omega}(\frac{3}{2C_0} \kappa'') \) spectrum, where \( C_0 \) was defined in Lemma 7.1,
4. \[ \partial_\omega Z' = (\bar{A} + \bar{F})Z' - Z'(\bar{A}' + \bar{F}') \]
5. \( \|A'\| \leq \|A\| + \varepsilon^{\hat{\xi}} + \pi N \),
6. \[ |Z'^{\pm 1} - Id|_r' \leq \varepsilon^{\hat{\xi}} \]
7. for all \( s' > 0 \),
   \[ |\psi'^{-1}\psi|_{s'} \leq 2C_0 \left( \frac{1}{\kappa''} \right)^6 e^{2\pi \Lambda(\hat{\xi})s'} \]
   \[ |\psi^{-1}\psi'|_{s'} \leq 2C_0 \left( \frac{1}{\kappa''} \right)^6 e^{2\pi \Lambda(\hat{\xi})s'} \]
8. \[ |\psi'^{\pm 1}|_r \leq \varepsilon^{-\zeta - \hat{\xi}} e^{2\pi \Lambda(\hat{\xi})r} \]
9. \[ |\psi^{-1}\bar{F}'\psi|_{s'} \leq \varepsilon^{\frac{5}{2}} \].
10. If moreover the spectrum of $A$ is not $BR_{\omega}^{RN}(\kappa'')$, then $\|A'\| \leq \frac{3}{4}\kappa''$.

11. If the spectrum of $A$ is $BR_{\omega}^{RN}(\kappa'')$, we have $\Phi \equiv I$, then $\psi' = \psi$ and $\tilde{A} = A$, and then

$$|\psi^{-1}Z'^{\pm 1}\psi|_{s'} \leq e^{\frac{\varepsilon}{2}},$$

$$|\partial_{\omega}(\psi^{-1}Z'^{\pm 1}\psi)|_{s'} \leq e^{\frac{\varepsilon}{2}},$$

**Proof.** Algebraic aspects

If $A$ has a double eigenvalue or $\kappa''$-close eigenvalues, let $\Phi$ be defined on $2T^d$ as constantly equal to $I$ and let $\tilde{A} = A$. Otherwise, let $\Phi$ a renormalization of $A$ of order $R, N$ given by lemma 7.1. Let $\tilde{A} \in sl(2, \mathbb{R})$ such that

$$\forall \theta \in 2T^d, \partial_{\omega}\Phi(\theta) = A\Phi(\theta) - \Phi(\theta)\tilde{A}$$

so $|A - \tilde{A}| \leq \pi N$ and $\tilde{A}$ has $BR_{\omega}^{RN}(\kappa'')$ spectrum. Notice that in this case, $\tilde{A}$ is not nilpotent. Let $\psi' = \psi\Phi$, and

$$\tilde{F} := \psi^{-1}F\psi'$$

Moreover, $\Phi$ is trivial with respect to $L_A$:

$$\Phi = P_{L_1} \varepsilon^{2\pi \langle m, \cdot \rangle} + P_{L_2} \varepsilon^{-2\pi \langle m, \cdot \rangle}$$

with $|m| \leq N$ and $\|P_{L_2}\| \leq \frac{C_0}{\kappa''}$. Since $\Phi$ is trivial with respect to $L_A$, for all $s' \geq 0$, Lemma 7.1 implies

$$|\Phi^\pm|_{s'} \leq 2C_0 \left(\frac{1}{K'}\right)^6 e^{2\pi \Lambda(\frac{\varepsilon}{2})s'},$$

which gives property 7.

Let $\psi' = \psi\Phi$. Let $G \in C^0(T^d, sl(2, \mathbb{R}))$, then by triviality of $\Phi$, $\Phi^{-1}G\Phi \in C^0(T^d, sl(2, \mathbb{R}))$, and by the assumption 4, $\psi'^{-1}G\psi' \in C^0(T^d, sl(2, \mathbb{R}))$. Therefore the property 2 on $\psi'$ holds.

Moreover,

$$\|\tilde{F}(0)\| \leq |\tilde{F}|_0 \leq |\Phi|_0|\Phi^{-1}|_0|\psi|_0|\psi^{-1}|_0|\tilde{F}|_0$$

Therefore by (10.14) and by assumption 5,

$$\|\tilde{F}(0)\| \leq e^{1-2\zeta}(2C_0)^2 \left(\frac{1}{K'}\right)^{12}.\)

Since $\varepsilon \leq \left(\frac{2C_0}{32(\|A\| + 1)}\right)^{576} \leq \left(\frac{2C_0}{96}\right)^{576}$, we get

$$\|\tilde{F}(0)\| \leq e^{1-2\zeta-\frac{\varepsilon}{\kappa''}}.$$

Since $\tilde{A}$ has a $BR_{\omega}^{RN}(\kappa'')$ spectrum, we want to apply lemma 9.3 with

$$\bar{\varepsilon} = e^{1-2\zeta-\frac{\varepsilon}{\kappa''}}, \bar{r} = r, \bar{r}' = r', \kappa'' = \frac{\kappa''}{C_0}, N = RN,$

then we need

$$e^{1-2\zeta-\frac{\varepsilon}{\kappa''}} \leq \left(\frac{1}{C_0} \cdot \frac{\kappa''}{32(1 + \|A\|)}\right)^2 e^{\zeta}\)

or sufficiently

$$e^{1-2\zeta-\frac{\varepsilon}{\kappa''}} \leq \left(\frac{1}{C_0} \cdot \frac{\kappa''}{32(1 + \|A\| + \pi N)}\right)^2 e^{\zeta}\)

which holds true if

$$e^{1-2\zeta-\frac{\varepsilon}{\kappa''}} \leq \left(\frac{1}{C_0} \cdot \frac{\kappa''}{32(2 + \pi)}\right)^2 e^{\zeta}\)

(where we have used the assumption that $\Psi \geq \text{id}$), which holds true by assumption 1. Therefore we can apply lemma 9.3 to get the maps $X \in U_{\omega}(T^d, sl(2, \mathbb{R}))$, $F' \in U_{\omega}(T^d, sl(2, \mathbb{R}))$, and a matrix $A' \in sl(2, \mathbb{R})$ such that
• $A'$ has $BR^RN(\frac{1}{3\delta''})$ spectrum,

• $\|A' - \tilde{A}\| \leq \bar{\varepsilon} \leq \varepsilon^\frac{24}{\pi N}$ (because $1 - 2\xi - \frac{1}{\delta''} \geq \frac{23}{27}$), which implies that

$$\|A' - A\| \leq \|A' - \tilde{A}\| + \|A - \tilde{A}\| \leq \varepsilon^\frac{24}{\pi N}$$

and thus

$$\|A'\| \leq \|A\| + \varepsilon^\frac{24}{\pi N}$$

which is property 5,

• $\partial_\omega e^X = (\tilde{A} + \tilde{F}) e^X - e^X (A' + F').$

Let $\tilde{F}' = \psi' F'(\psi')^{-1} \in C^0(\mathbb{T}^d, sl(2, \mathbb{R}))$ and $\tilde{A}' \in U_r(2\mathbb{T}^d, sl(2, \mathbb{R}))$ such that

$$\partial_\omega \psi' = \tilde{A}' \psi' - \psi' A'$$

(which means that $\tilde{A}'$ is reducible to $A'$, hence Property 1 with $\psi' := \psi\Phi$). Then the function $Z' := \psi' e^X (\psi')^{-1} \in C^0(\mathbb{T}^d, SL(2, \mathbb{R}))$ is solution of

$$\partial_\omega Z' = (\tilde{A} + \tilde{F}) Z' - Z'(\tilde{A}' + \tilde{F}')$$

hence Property 4. This conjugation also implies that $\tilde{A}' \in C^0(\mathbb{T}^d, sl(2, \mathbb{R}))$.

• if $\tilde{A}$ has two different eigenvalues, since $\Phi$ is trivial with respect to $L_A$ which is identical to $L_{\tilde{A}}$, by Lemma 9.3 and the expression (10.13),

$$|\Phi X \Phi^{-1}|_{\rho'} \leq \frac{4C\varepsilon^{15}}{\kappa^{13}} \Psi(3RN)|\tilde{F} \tilde{F} \Phi^{-1}|_{\rho}$$

and

$$|\Phi F' \Phi^{-1}|_{\rho'} \leq \frac{4C\varepsilon^{15}}{\kappa^{13}} e^{|\Phi X \Phi^{-1}|_{\rho'}} (|\Phi \tilde{F} \Phi^{-1}|_{\rho} [e^{-2\pi \Lambda(RN)(r - r')} + |\Phi \tilde{F} \Phi^{-1}|_{\rho} \Psi(3RN)(2e + e|\Phi X \Phi^{-1}|_{\rho'})])$$

otherwise if $\tilde{A}$ is nilpotent,

$$|X|_{\rho'} \leq \frac{3}{\kappa^3} \Psi(3RN)^3 |\tilde{F}|_{\rho}$$

and

$$|F'|_{\rho'} \leq \frac{3}{\kappa^3} e^{|X|_{\rho'}} |\tilde{F}|_{\rho} [e^{-2\pi \Lambda(RN)(r - r')} + |\tilde{F}|_{\rho} \Psi(3RN)^3(2e + e|X|_{\rho'})]$$

and if $ad_{\tilde{A}} = 0$,

$$|X|_{\rho'} \leq \frac{1}{\kappa} \Psi(3RN) |\tilde{F}|_{\rho}$$

and

$$|F'|_{\rho'} \leq \frac{1}{\kappa} e^{|X|_{\rho'}} |\tilde{F}|_{\rho} [e^{-2\pi \Lambda(RN)(r - r')} + |\tilde{F}|_{\rho} \Psi(3RN)(2e + e|X|_{\rho'})].$$

Notice that in any case (since $\Phi \equiv I$ if $\tilde{A}$ is nilpotent or $ad_{\tilde{A}} = 0$), we have

$$|\Phi X \Phi^{-1}|_{\rho'} \leq \frac{4C\varepsilon^{15}}{\kappa^{13}} \Psi(3RN)^3 |\tilde{F} \tilde{F} \Phi^{-1}|_{\rho}$$

and

$$|\Phi F' \Phi^{-1}|_{\rho'} \leq \frac{4C\varepsilon^{15}}{\kappa^{13}} e^{|\Phi X \Phi^{-1}|_{\rho'}} |\Phi \tilde{F} \Phi^{-1}|_{\rho} [e^{-2\pi \Lambda(RN)(r - r')} + |\Phi \tilde{F} \Phi^{-1}|_{\rho} \Psi(3RN)^3(2e + e|\Phi X \Phi^{-1}|_{\rho'})]$$
Estimates

Estimate of $\Psi$, $\Psi^{-1}, A'$

With the assumption $\varepsilon \leq (2C_0)^{-96} \left( \frac{\varepsilon'}{\kappa''} \right)^{576}$, we have

$$|\Phi|_r \leq \varepsilon^{-\frac{1}{2}} e^{2\pi \Lambda(\frac{\varepsilon'}{\kappa''}) r}$$

and similarly for $\Phi^{-1}$. Moreover since $|\psi|_r \leq \varepsilon^{-\frac{1}{2}}$, we get property 8:

$$|\psi'\vert_r = |\psi\Phi|_r \leq |\psi|_r |\Phi|_r \leq \varepsilon^{-\frac{1}{2}} e^{2\pi \Lambda(\frac{\varepsilon'}{\kappa''}) r}$$

and similarly for $\psi'^{-1}$. Notice that this inequality remains true if $\Phi \equiv \text{id}$.

Notice that if $\Phi \not\equiv I$ (that is to say if the spectrum of $A$ is resonant), then from lemma 7.1 we get $\|\tilde{A}\| \leq \frac{1}{2} \kappa''$ and then for $\varepsilon \leq \varepsilon_0$ defined in lemma 10.1 (see equation (10.1)),

$$\|A'\| \leq \|\tilde{A}\| + \|\tilde{F}(0)\|$$

$$\leq \frac{1}{2} \kappa'' \varepsilon + \varepsilon^{1-2\kappa} \frac{1}{\kappa''}$$

$$\leq \frac{1}{2} \kappa'' \frac{\varepsilon^{1+\kappa}}{\kappa'} = \frac{3}{4} \kappa''$$

and property 10 is satisfied.

Estimate of $Z'^{\pm 1} - I, \psi^{-1}(Z'^{\pm 1})\psi$ and its derivative

Since $\tilde{F} = (\psi\Phi)^{-1}\tilde{F}\psi\Phi$, then

$$|\Phi \tilde{F} \Phi^{-1}|_r = |\psi^{-1}\tilde{F}\psi|_{r'} \leq |\tilde{F}|_{r'} \varepsilon^{\frac{1}{2} - 2\kappa} = \varepsilon^{1-2\kappa}$$

(10.17)

Recall the estimate (10.15):

$$|\Phi X \Phi^{-1}|_{r'} \leq \frac{4C_0^{15}}{\kappa''^{13}} \Psi(3RN)^3 |\Phi \tilde{F} \Phi^{-1}|_r$$

(10.18)

therefore by (10.17), and for $\varepsilon \leq \varepsilon_0$ defined in lemma 10.1 (see equation (10.2)),

$$|\Phi X \Phi^{-1}|_{r'} \leq \frac{4C_0^{15}}{\kappa''^{13}} \varepsilon^{-1+\kappa} \varepsilon^{1-2\kappa} \leq \varepsilon^\frac{1}{2}$$

(10.19)

then

$$e^{e|\Phi X \Phi^{-1}|_{r'}} \leq e^{\frac{1}{2}} \leq 2$$

We now estimate $|Z' - I|_{r'} = |\psi\Phi(e^X - I)(\psi\Phi)^{-1}|_{r'}$. From (10.19),

$$|\Phi e^X \Phi^{-1} - Id|_{r'} \leq e^{e|\Phi X \Phi^{-1}|_{r'}} \leq e\varepsilon^\frac{1}{2}$$

Then

$$|Z' - I|_{r'} = |\psi\Phi e^X (\psi\Phi)^{-1} - Id|_{r'} \leq |\psi|_{r'} |\Phi e^X \Phi^{-1} - Id|_{r'} |\psi^{-1}|_{r'} \leq e^\frac{1}{2} \varepsilon^{1-2\kappa}$$

hence property 6 is satisfied. If $\Phi \equiv I$, we have

$$\psi^{-1} Z' \psi = \psi^{-1} \psi e^X (\psi\Phi)^{-1} \psi = e^X,$$

therefore

$$|\psi^{-1} Z' \psi|_{r'} \leq |e^X|_{r'} \leq e^\frac{1}{2}$$
which is the first part of the property 11. Now Lemma 9.3 also states that if $\Phi = I$ (that is, $\tilde{A} = A$),

$$|\partial_w X|_{r'} \leq 2\|A\| |X|_{r'} + |\tilde{F}|_{r'}$$

which implies that

$$|\partial_w X|_{r'} \leq 2\|A\| \varepsilon^2 + \varepsilon^{-2\zeta} \leq \varepsilon^2 (2\|A\| + 1)$$

and by the assumption 1,

$$|\partial_w X|_{r'} \leq \varepsilon^2.$$

Therefore,

$$|\partial_w (\psi^{-1} Z' \psi)|_{r'} = |\partial_w (X)e^X|_{r'} \leq \varepsilon^2 \varepsilon^2 \leq \varepsilon^4$$

hence property 11.

**Estimate of $\psi^{-1} F' \psi = \Phi F' \Phi^{-1}$**

From Equation (10.16),

$$|\Phi F' \Phi^{-1}|_{r'} \leq 4C_0^2 e^{\|\Phi X\Phi^{-1}\|_{r'} \left(\frac{C_0}{3}\right)_r^{13} |\Phi F' \Phi^{-1}|_r [e^{-2\pi \Lambda(RN)(r-r')}$$

$$+ |\Phi F' \Phi^{-1}|_r |\Psi(3RN)^3 (2\epsilon + e^{\|\Phi X\Phi^{-1}\|_{r'}}]$$

Moreover, by definition, we have $\Lambda(RN) = \frac{50\delta}{\pi(r-r')}$, thus

$$e^{-2\pi \Lambda(RN)(r-r')} = \varepsilon^{100\delta}$$

and then, because we assumed $\varepsilon \leq (2C_0)^{-96} \left(\frac{e^\zeta}{\|\Phi X\Phi^{-1}\|_{r'}}\right)^{576}$ and $\Psi(3RN) = \varepsilon^{-\zeta}$,

$$|\Phi F' \Phi^{-1}|_{r'} \leq 8C_0 e^{-1-2\zeta} \varepsilon^{100\delta} + 8\Psi(3RN)^3 \varepsilon^{1-2\zeta}.$$

Thus

$$|\Phi F' \Phi^{-1}|_{r'} \leq 8C_0 e^{-1-2\zeta} \varepsilon^{100\delta} + \varepsilon^{1-6\zeta} \leq \varepsilon^4.$$

Hence property 9 holds for $\varepsilon \leq \varepsilon_0$ as defined in lemma 10.1 (see equation (10.3)).

**11 Inductive step**

Let’s define the following functions which will be used for the complete iterative step:

$$\begin{align*}
\kappa''(\varepsilon) &= \kappa \varepsilon^4 \\
N(r, \varepsilon) &= \Lambda^{-1} \left(\frac{50\delta}{\pi(r-r')} \log \varepsilon \right) \\
R(r, \varepsilon) &= \frac{1}{3N(r, \varepsilon)} \Psi^{-1}(\varepsilon^{-\zeta}) \\
r''(r, \varepsilon) &= r - \frac{50\delta}{\pi \Lambda(R(r, \varepsilon) N(r, \varepsilon))}
\end{align*}$$

Note that these definitions match with lemma 10.2.

**Lemma 11.1.** Let

- $A \in sl(2, \mathbb{R})$,
- $r > 0$,
- $\tilde{A}, \tilde{F} \in U_r(\mathbb{T}^d, sl(2, \mathbb{R})), \psi \in U_r(2\mathbb{T}^d, SL(2, \mathbb{R}))$,
- $|\tilde{F}|_{r'} = \varepsilon$.

Suppose that
1. \( \varepsilon \leq \varepsilon_0 \), where \( \varepsilon_0 \) is defined in Lemma 10.1,
2. \( r'' > 0 \),
3. \( \bar{A} \) is reducible to \( A \) by \( \psi \),
4. for all \( G \in C^0(\mathbb{T}^d, sl(2, \mathbb{R})) \), \( \psi^{-1} G \psi \in C^0(\mathbb{T}^d, sl(2, \mathbb{R})) \),
5. \( |\psi^\pm_1|_r \leq \varepsilon^{-\zeta} \),
6. \( \|A\| \leq \varepsilon^{-\frac{576}{\zeta}} \),

then, there exist

- \( Z' \in U_{r''}(\mathbb{T}^d, SL(2, \mathbb{R})) \),
- \( \bar{A}' \in U_{r''}(\mathbb{T}^d, sl(2, \mathbb{R})) \),
- \( \psi' \in U_r(2\mathbb{T}^d, SL(2, \mathbb{R})) \),
- \( A' \in sl(2, \mathbb{R}) \)

satisfying the following properties:

1. \( \bar{A}' \) is reducible to \( A' \) by \( \psi' \),
2. for all \( G \in C^0(\mathbb{T}^d, sl(2, \mathbb{R})) \), \( \psi^{-1} G \psi \in C^0(\mathbb{T}^d, sl(2, \mathbb{R})) \),
3. \( |\bar{F}'|_{r''} \leq \varepsilon^{25} \),
4. \( |\psi'^\pm_1|_{r''} \leq \varepsilon^{-28\zeta} \),
5. \( \|A'\| \leq \|A\| + \varepsilon^{-\zeta} \leq \varepsilon^{-\frac{(28)^5}{\zeta}} \),
6. \[
\partial_\omega Z' = (\bar{A} + \bar{F})Z' - Z'(\bar{A}' + \bar{F}') \],
7. \[
|Z'^\pm_1 - Id|_{r''} \leq \varepsilon^{\frac{25}{\zeta}}.
\]
8. If moreover the spectrum of \( A \) was not \( BR^{R(r,\varepsilon)}\mathcal{N}(\mathbb{T}^d, \mathbb{R}^d, \kappa''(\varepsilon)) \), we actually have \( ||A'|| \leq \kappa''(\varepsilon) \); 
9. If the spectrum of \( A \) was \( BR^{R(r,\varepsilon)}\mathcal{N}(\mathbb{T}^d, \mathbb{R}^d, \kappa''(\varepsilon)) \), we actually have \( \psi' = \psi \) and then
   \[
   |\psi^{-1} Z'^\pm_1 \psi|_{r''} \leq (1 + 2\varepsilon)\varepsilon^{2\zeta} \]  
   (11.1) and 
   \[
   |\partial_\omega (\psi^{-1} Z'^\pm_1 \psi)|_{r''} \leq \varepsilon^{\frac{25}{\zeta}}.
   \]

**Proof.** Removing the resonances and first step

Let \( R = R(r, \varepsilon), N = N(r, \varepsilon), \kappa'' = \kappa''(r, \varepsilon), r'' = r''(r, \varepsilon) \). Since \( \kappa'' = \kappa \varepsilon^\zeta, \|A\| \leq \varepsilon^{-\frac{576}{\zeta}} \) and \( \varepsilon \leq \varepsilon_0 \) as defined in Lemma 10.1 (see equation (10.4)),

\[
\varepsilon^{1-576\zeta} \leq (2C_0)^{-96 \frac{\kappa}{32(\|A\| + 1)}}
\]

therefore

\[
\varepsilon \leq (2C_0)^{-96 \frac{\kappa''}{32(\|A\| + 1)^{576}}}
\]

and the assumption of lemma 10.2 is satisfied. We can apply lemma 10.2 to get:

- \( Z_1 \in U_{1/2(r'', \varepsilon)}(\mathbb{T}^d, SL(2, \mathbb{R})) \),
- \( \psi' \in U_{1/2(r'', \varepsilon)}(2\mathbb{T}^d, SL(2, \mathbb{R})) \),
- \( A_1 \in sl(2, \mathbb{R}) \),
- \( \bar{A}_1 \in U_{1/2(r', \varepsilon)}(\mathbb{T}^d, sl(2, \mathbb{R})) \),
such that

1. $\tilde{A}_1$ is reducible to $A_1$ by $\psi'$,
2. for all $G \in C^0(\mathbb{T}^d, sl(2, \mathbb{R}))$, $\psi'^{-1}G\psi' \in C^0(\mathbb{T}^d, sl(2, \mathbb{R}))$, which implies that $F_1 \in C^0(\mathbb{T}^d, sl(2, \mathbb{R}))$,
3. $A_1$ has $BR^\omega_\omega(\frac{1}{4}R^N)$ spectrum,
4. $\partial_\omega Z_1 = (\tilde{A} + \tilde{F})Z_1 - Z_1(\tilde{A}_1 + \tilde{F}_1)$,
5. $\|A_1\| \leq \|A\| + \varepsilon^\frac{3}{8} + \pi N$, \quad (11.2)
6. $|Z_1^{\pm 1} - Id|_{r''} \leq \varepsilon^\frac{1}{4}$, \quad (11.3)
7. for all $s' > 0$,
\[ |\psi'^{-1}|_{s'} \leq 2C_0(\frac{1}{\kappa''})^6 e^{2\pi \Lambda(\Psi)} s' \]
\[ |\psi^{-1}|_{s'} \leq 2C_0(\frac{1}{\kappa''})^6 e^{2\pi \Lambda(\Psi)} s' \]
8. $|\psi'^{\pm 1}|_{r''} \leq \varepsilon^{-\frac{1}{2} - \frac{3}{8}} e^{2\pi \Lambda(\Psi)} r$,
9. $|\psi^{-1}F_1|_{r''} \leq \varepsilon^\frac{5}{8}$,
10. If the spectrum of $A$ was not $BR^\omega_\omega(\kappa'')$, $\|A_1\| \leq \frac{1}{2}\kappa''$;
11. If the spectrum of $A$ was $BR^\omega_\omega(\kappa'')$, we actually have $\psi' = \psi$ and then
\[ |\psi^{-1}Z_1^{\pm 1}|_{r''} \leq \varepsilon^\frac{3}{8} \]
\[ |\partial_\omega (\psi^{-1}Z_1^{\pm 1})|_{r''} \leq \varepsilon^\frac{1}{8}. \quad (11.4) \]

**Second step : iteration without resonances**

We will now iterate lemma 9.3 a certain number of times, without renormalization.

Let $l = \frac{E(\log(100\delta))}{\log(\frac{1}{\delta})}$ = 56 which satisfies
\[ \varepsilon(\frac{1}{2})^{l+1} \leq e^{-2\pi \Lambda(\text{RN}) (r - r'')} = \varepsilon^{1008} \leq \varepsilon^l. \]

Define for all $j \geq 0$, the sequences $\varepsilon' = \varepsilon(\frac{1}{2})^{\frac{1}{2}j}$ and $r' = \frac{r''}{2} - j \frac{r''}{2^j}$. Thus $r'_0 = \frac{r''}{2}$ and $r'_1 = r'' < r$.

We want to iterate $l - 1$ times lemma 9.3, from $j = 2$, with

- $\tilde{\varepsilon} = \varepsilon'_{j-1}$,
- $\tilde{\rho} = r'_{j-2}$,
- $\tilde{\rho}' = r'_{j-1}$,
- $\kappa' = (\frac{3}{2})^{j-1} \frac{r''}{r'_0}$,
- $\tilde{N} = \text{RN}$,
- $\tilde{F} = F_{j-1}$,
- $\tilde{A} = A_{j-1}$,
- $\tilde{\Phi} = \psi^{-1}\psi'$,
First iterate of lemma 9.3: From

$$|\psi^{-1}F_1\psi|_0 \leq \varepsilon^\frac{3}{4}$$

and

$$|\psi^{j-1}\psi|_0 \leq 2C_0\left(\frac{1}{\kappa N}\right)^6 \leq \varepsilon^{-\frac{3}{4}}, \quad |\psi^{j-1}\psi|_0 \leq 2C_0\left(\frac{1}{\kappa N}\right)^6 \leq \varepsilon^{-\frac{3}{4}},$$

then

$$\|\hat{F}_1(0)\| \leq |\psi^{j-1}\psi|_0 |\psi^{-1}\psi^{j-1}\psi|_0 \|\psi^{-1}\psi|_0 \leq \varepsilon^\frac{3}{4}.$$

As $A_1$ has $BR^N_{\omega}(\frac{3}{4}, \frac{\varepsilon^{\omega}}{C_0})$ spectrum, to apply lemma 9.3 we need

$$\varepsilon^\frac{3}{4} \leq \left(\frac{\left(\frac{3}{4}\right)^{\omega^{\omega}}}{32(1 + \|A_1\|)}\right)^{2} \frac{1}{\Psi(N)}$$

and since

$$\|A_1\| \leq \|A\| + \varepsilon^\frac{3}{4} + \pi N \leq \varepsilon^{-\frac{3}{4}} + \varepsilon^\frac{3}{4} + \pi \varepsilon^{-\frac{3}{4}}$$

(this last inequality comes from the fact that $\Psi \geq id$, which implies that $N = \frac{1}{2\pi} \Psi^{-1}(\varepsilon^{-\frac{3}{4}}) \leq \varepsilon^{-\frac{3}{4}}$), this remains true if

$$\varepsilon^\frac{3}{4} \leq \left(\frac{\left(\frac{3}{4}\right)^{\omega^{\omega}}}{32(1 + (\pi + 1)\varepsilon^{-\frac{3}{4}} + \varepsilon^\frac{3}{4})}\right)^{2} \frac{1}{\Psi(N)^2}$$

which holds for $\varepsilon \leq \varepsilon_0$ as in lemma 10.1 (see equation (10.5)).

Iteration of lemma 9.3

If for some $j \geq 2$

$$\varepsilon^j \leq \left(\frac{\left(\frac{3}{4}\right)^{j^{\omega^{\omega}}}}{32(1 + \|A_1\| + \sum_{i=1}^{j-1} \varepsilon_i)}\right)^{2} \frac{1}{\Psi(N)^2},$$

which holds true for $\varepsilon \leq \varepsilon_0$ as in lemma 10.1 (see equation (10.6)), then

$$\varepsilon^j \leq \left(\frac{\left(\frac{3}{4}\right)^{j^{\omega^{\omega}}}}{32(1 + \|A_1\| + \sum_{i=1}^{j-1} \varepsilon_i)}\right)^{2} \frac{1}{\Psi(N)^2}.$$

Let $j \geq 2$ and assume that $A_{j-1}$ has $BR^N_{\omega}(\frac{3}{4}, \frac{\varepsilon^{j-1}}{C_0})$ spectrum, $F_{j-1} \in U_{r_{j-2}}(\mathbb{T}^d, sl(2, \mathbb{R})), \text{ and}$

$$\|\hat{F}_{j-1}(0)\| \leq \varepsilon^j_{j-1}; \quad |\psi^{-1}\Psi F_{j-1}\psi^{-1}\Psi|_{r_{j-2}} \leq \varepsilon^{j-1}.$$

We obtain via lemma 9.3 functions $F_j, X_j \in U_{r_{j-1}}(\mathbb{T}^d, sl(2, \mathbb{R}))$ and a matrix $A_j \in sl(2, \mathbb{R})$ such that

1. $A_j$ has $BR^N_{\omega}(\frac{3}{4}, \frac{\varepsilon^{j}}{C_0})$ spectrum,
2. $\|A_j\| \leq \|A_{j-1}\| + \varepsilon^j_{j-1},$
3. $\partial_\omega e^{X_j} = (A_{j-1} + F_{j-1})e^{X_j} - e^{X_j}(A_j + F_j),$
4. the following estimates hold:
   - if $A_{j-1}$ has two different eigenvalues:
     $$|\psi^{-1}\psi X_j\psi^{j-1}\psi|_{r_{j-1}} \leq 4C_0\left(\frac{1}{\left(\frac{3}{4}\right)^{j-1} C_0}\right)^{13} \Psi(3RN)|\psi^{-1}\psi F_j\psi^{j-1}\psi|_{r_{j-1}};$$
     $$|\psi^{-1}\psi F_j\psi^{j-1}\psi|_{r_{j-1}} \leq 4C_0\left(\frac{1}{\left(\frac{3}{4}\right)^{j-1} C_0}\right)^{13} \varepsilon |\psi^{-1}\psi X_j\psi^{j-1}\psi|_{r_{j-1}} + |\psi^{-1}\psi F_j\psi^{j-1}\psi|_{r_{j-1}} + \psi(3RN)(2\varepsilon + e^{\left|\psi^{-1}\psi X_j\psi^{j-1}\psi\right|_{r_{j-1}}}$$
• if $A_{j-1}$ is nilpotent:

$$|X_j|_{r_j-1} \leq \frac{3}{κ^3} \Psi(3RN)^3|F_{j-1}|_{r_j-1},$$

$$|F_j|_{r_j-1} \leq \frac{3}{κ^3} |X_{j-1}|_{r_j-1} |F_{j-1}|_{r_j-2} [e^{-2πλ(3RN)}(r_j' - r_j) + |F_{j-1}|_{r_j-2} |Ψ(3RN)^3(2e + e^{X_{j-1}}_{r_j-1})].$$

• if $ad_{A_{j-1}} = 0$:

$$|X_j|_{r_j-1} \leq \frac{1}{κ} |Ψ(3RN)| |F_{j-1}|_{r_j-1},$$

$$|F_j|_{r_j-1} \leq \frac{1}{κ} |X_{j-1}|_{r_j-1} |F_{j-1}|_{r_j-2} [e^{-2πλ(3RN)}(r_j' - r_j) + |F_{j-1}|_{r_j-2} |Ψ(3RN)(2e + e^{X_{j-1}}_{r_j-1})].$$

Notice that in any case we have

$$|ψ^{-1}ψ'X_jψ'^{-1}ψ|_{r_j-1} \leq 4C_0^2 \left(\frac{1}{(\frac{1}{4})^{\frac{1}{2}} \frac{1}{κ0}}\right)^{13} |Ψ(3RN)| |ψ^{-1}ψ'F_jψ'^{-1}ψ|_{r_j-1},$$

(11.5)

$$|ψ^{-1}ψ'F_jψ'^{-1}ψ|_{r_j-1} \leq 4C_0^2 \left(\frac{1}{(\frac{1}{4})^{\frac{1}{2}} \frac{1}{κ0}}\right)^{13} e^{ψ^{-1}ψ'X_jψ'^{-1}ψ|_{r_j-1} |ψ^{-1}ψ'F_jψ'^{-1}ψ|_{r_j-2}} \left[e^{-2πλ(3RN)}(r_j' - r_j) + 8|ψ^{-1}ψ'F_jψ'^{-1}ψ|_{r_j-2} |Ψ(3RN)| \right],$$

(11.6)

so we will use these estimates to iterate lemma 9.3.

Estimates $ε \leq (2C_0)^{-96} \left(\frac{κ''}{κ[∥A∥]+1}\right)^{576}$ and $|ψ^{-1}ψ'F_jψ'^{-1}ψ|_{r_j-2} \leq e(\frac{κ''}{κ})^{-1}$ give

$$e^{ψ^{-1}ψ'X_jψ'^{-1}ψ|_{r_j-1} \leq 2}$$

Hence from (11.6)

$$|ψ^{-1}ψ'F_jψ'^{-1}ψ|_{r_j-1} \leq 64 \cdot 4C_0^2 |Ψ(3RN)| \left(\frac{Ψ(3RN)}{(\frac{1}{4})^{\frac{1}{2}} \frac{1}{κ0}}\right)^{13} |ψ^{-1}ψ'F_jψ'^{-1}ψ|_{r_j-2} \left[e^{-2πλ(3RN)}(r_j' - r_j) + 8|ψ^{-1}ψ'F_jψ'^{-1}ψ|_{r_j-2} \right]$$

$$\leq 256C_0^2 |Ψ(3RN)|^{14} \left(\frac{1}{(\frac{1}{4})^{\frac{1}{2}} \frac{1}{κ0}}\right)^{13} ε(\frac{κ''}{κ})^{-1} (ε^{504} + ε(\frac{κ''}{κ})^{-1})$$

Since $Ψ(3RN) = ε^{-κ}$ and $ε \leq ε_0$ defined in lemma 10.1 (see equation (10.7)),

$$|ψ^{-1}ψ'F_jψ'^{-1}ψ|_{r_j-1} \leq ε(\frac{κ''}{κ})^{j}$$

(11.7)

We will now estimate $∥\hat{F}_j(0)∥$ to iterate lemma 9.3:

$$∥\hat{F}_j(0)∥ \leq |F_j|_0 = |ψ^{-1}ψ'ψ'F_jψ'^{-1}ψ|_{0} \leq |ψ^{-1}ψ'|_{0} |ψ'^{-1}ψ'|_{0} |ψ^{-1}ψ'F_jψ'^{-1}ψ|_{r_j-1},$$

therefore

$$∥\hat{F}_j(0)∥ \leq |ψ^{-1}ψ'|_{0} |ψ'^{-1}ψ'|_{0} ε(\frac{κ''}{κ})^{j} \leq ε(\frac{κ''}{κ})^{j} e^{-\frac{κ''}{κ}} = ε_j,$$

and we can iterate lemma 9.3, $l - 1$ times.

Equations (11.7) and (11.5) imply that

$$|ψ^{-1}ψ'X_jψ'^{-1}ψ|_{r_j-1} \leq ε_j^{l}$$

(11.8)

and

$$e^{ψ^{-1}ψ'X_jψ'^{-1}ψ|_{r_j-1} \leq 2}$$

Conclusion:
Let $Z = e^{X_2} \cdots e^{X_{l+1}} \in U_{r''}(T^d, SL(2, \mathbb{R}))$.
Let $Z' = Z_1 \psi' Z \psi'^{-1}$, $A' = A_{l+1}$, $F' = F_{l+1}$, $\tilde{F}' = \psi' F' \psi'^{-1}$ (hence Property 2) and $\tilde{A}'$ such that
\[ \partial_{\omega} \psi' = \tilde{A}' \psi' - \psi' A', \]
then
\[ \partial_{\omega} Z' = (\tilde{A} + \tilde{F}') Z' - Z'(\tilde{A}' + \tilde{F}'), \]
hence the properties 1 and 6 hold.

We have
\[ \partial_{\omega} Z = (A_1 + F_1) Z - Z(A_{l+1} + F_{l+1}) \]
and since for all $j \geq 2$, we have
\[ \|A_j\| \leq \|A_{j-1}\| + e^{(\frac{4}{5})^{j-1}} \epsilon^{-\frac{1}{3^5}} \leq \|A_1\| + \sum_{i=1}^{j-1} e^{(\frac{4}{5})^i} \epsilon^{-\frac{1}{3^5}} = \|A_1\| + \sum_{i=1}^{j-1} \epsilon_i, \quad (11.9) \]
then
\[ \|A'\| \leq \|A_1\| + \sum_{i=1}^{l} \epsilon_i \leq \|A\| + \epsilon^{\frac{4}{5}} + \pi N + \sum_{i=1}^{l} e^{(\frac{4}{5})^i} \epsilon^{-\frac{1}{3^5}}. \]
Remind that $\Psi \geq id$ implies
\[ N \leq \epsilon^{-\frac{4}{5}} \]
and then, since $\|A\| \leq \epsilon^{\frac{4}{5}}$,
\[ \|A'\| \leq \|A\| + \epsilon^{-\frac{4}{5}} \leq 2\epsilon^{-\frac{4}{5}} \leq \epsilon^{-\frac{4}{5}} \]
thus the property 5 holds if $\epsilon \leq \epsilon_0$ as defined in lemma 10.1 (see equation (10.8)).

Moreover,
\[ |\psi^{-1} \psi' F_{l+1} \psi'^{-1}|_{r_l} \leq \epsilon^{(\frac{4}{5})^{l+1}} \]
and since $l = 56$, one has
\[ |\psi' F_{l+1} \psi'^{-1}|_{r_l} \leq |\psi|_{r_l} |\psi^{-1}|_{r_l} \epsilon^{(\frac{4}{5})^{l+1}} \leq \epsilon^{(\frac{4}{5})^{57-2\epsilon}} \leq \epsilon^{2\delta} \]
thus the property 3 holds. In the case the spectrum of $A$ was resonant, the function $\Phi$ used in lemma 10.2 is not the identity and we have
\[ \|A'\| \leq \|A_1\| + \sum_{i=1}^{l} \epsilon_i' \leq \frac{1}{2} \kappa''(r, \epsilon) + 2\epsilon'_1 \leq \frac{1}{2} \kappa \epsilon^2 + 2\epsilon^{\frac{4}{5}} \leq \kappa \epsilon^2 = \kappa''(r, \epsilon) \]
since $\epsilon \leq \epsilon_0$ as defined in lemma 10.1 (see equation (10.9)), whence the property 8.

**Estimates**

Now we will show property 4: $|\psi'^{\pm 1}|_{r''} \leq \epsilon^{-2\delta} \epsilon^\delta$.
We know that $|\psi'^{\pm 1}|_{r''} \leq \epsilon^{-\frac{4}{5}} \epsilon^{\frac{4}{5}} e^{2\pi \Lambda(\frac{4}{5}) r}$. But, by definition of $\Lambda(N) = \frac{50 \log \epsilon}{\pi r}$,
\[ e^{2\pi \Lambda(\frac{4}{5}) r} \leq e^{2\pi \Lambda(N) r} \leq e^{\log \epsilon - 100} = \epsilon^{-100} \]
therefore
\[ |\psi'^{\pm 1}|_{r''} \leq \epsilon^{-\frac{4}{5}} \epsilon^{-100} \leq \epsilon^{-2\delta}. \]
which is Property 4. Now we will show the property 3. One has
\[ |\tilde{F}'|_{r''} = |\psi' F' \psi'^{-1}|_{r''} = |\psi |_{r'} |\psi^{-1}|_{r'} |\psi' F' \psi'^{-1}|_{r''} \leq \epsilon^{-\frac{4}{5}} \epsilon^{-100} \epsilon^{2\delta} \leq \epsilon^{-2\delta}. \]
where the last inequality uses equation (11.10), which gives property 3.

According to the estimate
\[|Z_1^{\pm 1} - Id|_{r''} \leq e^{\hat{\omega}}\]
obtained in (11.3), we get
\[
|Z' - Id|_{r''} \leq |Z_1 - Id|_{r'_1} + |Z_1|_{r'_1} |\psi|_r |\psi^{-1}|_r \sum_{j=2}^{l+1} |\psi^{-1}\psi'X_j\psi'^{-1}\psi|_{r'_j}
\]
\[
\leq e^{\hat{\omega}} + (e^{\hat{\omega}} + 1)\varepsilon^{-2\zeta} \sum_{j=2}^{l+1} |\psi^{-1}\psi'X_j\psi'^{-1}\psi|_{r'_j}
\]

Therefore, by the estimate (11.8) and by definition of \(l = E\left(\frac{\log(100\delta)}{\log(\hat{\omega})}\right)\),
\[
|Z' - Id|_{r''} \leq e^{\hat{\omega}} + (e^{\hat{\omega}} + 1)2\varepsilon_1\varepsilon^{-2\zeta} \leq e^{\hat{\omega}}
\]
hence 7.

**Proof of the property 9**

We now have to estimate \(\psi^{-1}Z'\psi\) and its directional derivative in the case \(A\) has a \(B_{R_{\omega}}^{N(r,\varepsilon)}(\kappa''(\varepsilon))\) spectrum. In this case \(\Phi \equiv I\) and \(\psi = \psi'\), therefore \(\psi^{-1}Z'\psi = \psi^{-1}Z_1\psi Z\). Therefore
\[
|\psi^{-1}Z'\psi|_{r''} \leq |\psi^{-1}Z_1\psi|_{r''}|Z|_{r''} \leq (1 + 2\varepsilon)|Z|_{r''}.
\]

(where the last inequality comes from (11.3)). Moreover,
\[
|Z|_{r''} = |\Pi_{k=2}^{l+1}e^{X_k}|_{r''}.
\]

Now for all \(k \in [2, l]\), we have seen in (11.8) that \(|X_k|_{r''} \leq \varepsilon_k\).

Therefore
\[
|Z|_{r''} \leq |\Pi_{k=2}^{l+1}e^{X_k}|_{r''} \leq e^{\sum_{k=2}^{l+1} \varepsilon_k' \leq e^{2\varepsilon}}
\]

and finally,
\[
|\psi^{-1}Z'\psi|_{r''} \leq (1 + 2\varepsilon)e^{2\varepsilon}.
\]

The estimate of \(\psi^{-1}Z'^{-1}\psi\) is obtained in a similar way. This gives the property (11.1).

Moreover,
\[
|\partial_\omega(\psi^{-1}Z'\psi)|_{r''} \leq |\partial_\omega(\psi^{-1}Z_1\psi)Z|_{r''} + |\psi^{-1}Z_1\psi\partial_\omega(Z)|_{r''} \leq e^{\hat{\omega}}|Z|_{r''} + (1 + 2\varepsilon)|\partial_\omega(Z)|_{r''}
\]

where the last inequality comes from (11.4). Now
\[
|\partial_\omega(Z)|_{r''} \leq \sum_{k=2}^{l+1} |\partial_\omega X_k|_{r''} \prod_{j=2}^{l+1} e^{X_j}|_{r''}.
\]

For all \(k \in [2, l + 1]\), by construction of \(X_k\),
\[
|\partial_\omega X_k|_{r''} \leq 2\|A_k\||X_k|_{r''} + |F_k|_{r''} \leq 2\|A_k\||X_k|_{r''} + \varepsilon^{\left(\frac{\hat{\omega}}{2}\right)^k - \frac{1}{\hat{\omega}}}
\]

Now for all \(k \in [2, l + 1]\), by the estimate (11.9), the estimate (11.2) and condition 3 of this lemma, for \(\varepsilon \leq \varepsilon_0\) given by lemma 10.1 (see equation (10.10)),
\[
\|A_k\| \leq e^{-\frac{\hat{\omega}}{2\varepsilon}} + e^{\frac{\hat{\omega}}{2\varepsilon}} + \pi\varepsilon^{-\zeta} \leq e^{-2\zeta}
\]

therefore for \(\varepsilon \leq \varepsilon_0\) given by lemma 10.1 (see equation (10.11)).
![Image of text content]

and finally, for \( \varepsilon \leq \varepsilon_0 \) like in lemma 10.1 (see equation (10.12)),

\[
|\partial_{\omega}(\psi^{-1}Z^i\psi)|_{r''} \leq 2\varepsilon^{\frac{1}{2}} + 2\varepsilon^{\frac{1}{2}} \leq \varepsilon^{\frac{1}{2}}.
\]

The estimate of \( \partial_{\omega}(\psi^{-1}Z^{-1}\psi) \) is similar, which gives property 9. \( \square \)

### 12 Almost reducibility

Here we complete the proof of the main theorem.

**Theorem 12.1.** Let \( r_0 > 0, A \in \text{sl}(2, \mathbb{R}) \) and \( F \in U_{r_0}(T^d, \text{sl}(2, \mathbb{R})) \). Then, if

\[
|F|_r < \varepsilon_0
\]

where \( \varepsilon_0 \) satisfies the assumptions above, and

\[
\|A\| < \varepsilon_0^{\frac{1}{2}},
\]

then for all \( \varepsilon \leq \varepsilon_0 \), there exist

- \( \varepsilon > 0, k \in \mathbb{N} \),
- \( Z_\varepsilon \in U_{r_\varepsilon}(T^d, \text{sl}(2, \mathbb{R})) \),
- \( A_\varepsilon \in \text{sl}(2, \mathbb{R}) \),
- \( \bar{A}_\varepsilon, \bar{F}_\varepsilon \in U_{r_\varepsilon}(T^d, \text{sl}(2, \mathbb{R})) \),
- \( \psi_\varepsilon \in U_{r_\varepsilon}(2T^d, \text{SL}(2, \mathbb{R})) \),

such that

1. \( \bar{A}_\varepsilon \) is reducible to \( A_\varepsilon \) by \( \psi_\varepsilon \), with \( |\psi_\varepsilon|_{r_\varepsilon} \leq \varepsilon^{-\zeta} \),
2. \( |\bar{F}_\varepsilon|_{r_\varepsilon} \leq \varepsilon \),
3. for all \( \theta \in T^d \),

\[
\partial_{\omega}Z_\varepsilon(\theta) = (A + F(\theta))Z_\varepsilon(\theta) - Z_\varepsilon(\theta)(\bar{A}_\varepsilon(\theta) + \bar{F}_\varepsilon(\theta))
\]
4. \( |Z_\varepsilon^{\pm1} - \text{Id}|_{r_\varepsilon} \leq \varepsilon^{\frac{\alpha}{2}} \).

Moreover, either \( |\partial_{\omega}Z_\varepsilon|_{r_\varepsilon} \) is bounded as \( \varepsilon \to 0 \) and \( A + F \) is a reducible cocycle in \( U_{r_\infty}(T^d, \text{sl}(2, \mathbb{R})) \) for some \( r_\infty > 0 \), or for all \( \varepsilon \leq \varepsilon_0 \) there exists \( \varepsilon' \leq \varepsilon \) such that

\[
\|A_{r'}\| \leq \kappa \varepsilon^{\zeta}.
\]

**Proof.** Remind parameters (P) defined in section 11 and define, for all \( k \in \mathbb{N}, k \geq 1 \),

\[
\varepsilon_k := \varepsilon_0^{(2\delta)^k}; \quad r_k := r_0 - \sum_{i=0}^{k-1} \frac{50\delta|\log \varepsilon_i|}{\pi A(R(r_i, \varepsilon_i)} N(r_i, \varepsilon_i))
\]

Notice that, by Lemma 5.1, under assumption 1, for all \( k \in \mathbb{N}, r_k > 0 \).

We can apply a first time lemma 11.1. There exist

- \( Z_1 \in U_{r_1}(T^d, \text{SL}(2, \mathbb{R})) \),
• $\tilde{A}_1, \tilde{F}_1 \in U_{r_1}(T^d, sl(2, \mathbb{R}))$,
• $A_1 \in sl(2, \mathbb{R})$,
• $\psi_0 \in U_{r_1}(2T^d, SL(2, \mathbb{R}))$,

such that

• $\tilde{A}_1$ is reducible to $A_1$ by $\psi_0$,
• for all $G \in C^0(T^d, sl(2, \mathbb{R}))$, $\psi_0^{-1}G\psi_0 \in C^0(T^d, sl(2, \mathbb{R}))$,
• $|\tilde{F}_1|_{r_1} \leq \varepsilon_1$,
• $|\psi_0^{\pm 1}|_{r_1} \leq \varepsilon_1^{-\frac{\delta}{2}}$,
• $\|A_1\| \leq \varepsilon_1^{-\frac{\delta}{2}}$,
• for all $\theta \in T^d$

$$\partial_{\omega}Z_1(\theta) = (A + F(\theta))Z_1(\theta) - Z_1(\theta)(\tilde{A}_1(\theta) + \tilde{F}_1(\theta)),$$

• $|Z_1^{\pm 1} - Id|_{r_1} \leq \varepsilon_0^\frac{\delta}{4}$,

if moreover $A$ had a $BR_\omega^{R(\eta_0, \varepsilon_0)}(N(\eta_0, \varepsilon_0))$ spectrum,

$$|\psi_0^{-1}Z_1\psi_0|_{r_1} \leq (1 + 2\varepsilon_0)e^{2\varepsilon_0}$$

and if not,

$$\|A_1\| \leq \kappa''(\varepsilon_0)$$

and

$$|\partial_{\omega}(\psi_0^{-1}Z_1\psi_0)|_{r_1} \leq \varepsilon_0^\frac{\delta}{4}.$$ (12.2)

Iterative step: let $k \geq 1$ and

• $\tilde{A}_k, \tilde{F}_k \in U_{r_k}(T^d, sl(2, \mathbb{R}))$,
• $A_k \in sl(2, \mathbb{R})$,
• $\psi_{k-1} \in U_{r_k}(2T^d, SL(2, \mathbb{R}))$,

such that

• $\tilde{A}_k$ is reducible to $A_k$ by $\psi_{k-1}$,
• for all $G \in C^0(T^d, sl(2, \mathbb{R}))$, $\psi_{k-1}^{-1}G\psi_{k-1} \in C^0(T^d, sl(2, \mathbb{R}))$,
• $|\tilde{F}_k|_{r_k} \leq \varepsilon_k$,
• $|\psi_{k-1}^{\pm 1}|_{r_k} \leq \varepsilon_k^{-\frac{\delta}{2}}$,
• $\|A_k\| \leq \varepsilon_k^{-\frac{\delta}{2}}$.

We can once again apply lemma 11.1 to get

• $Z_{k+1} \in U_{r_{k+1}}(T^d, SL(2, \mathbb{R}))$,
• $\tilde{A}_{k+1}, \tilde{F}_{k+1} \in U_{r_{k+1}}(T^d, sl(2, \mathbb{R}))$,
• $A_{k+1} \in sl(2, \mathbb{R})$,
• $\psi_k \in U_{r_{k+1}}(2T^d, SL(2, \mathbb{R}))$,

such that

• $\tilde{A}_{k+1}$ is reducible to $A_{k+1}$ by $\psi_k$,
• for all $G \in C^0(T^d, sl(2, \mathbb{R}))$, $\psi_k^{-1}G\psi_k \in C^0(T^d, sl(2, \mathbb{R}))$,
• $|\tilde{F}_{k+1}|_{r_{k+1}} \leq \varepsilon_{k+1}$,
• $|\psi_k^{\pm 1}|_{r_{k+1}} \leq \varepsilon_{k+1}^{-\frac{\delta}{2}}$,
• $\|A_{k+1}\| \leq \varepsilon_{k+1}^{-\frac{\delta}{2}}$,
• for all $\theta \in T^d$

$$\partial_{\omega}Z_{k+1}(\theta) = (\tilde{A}_{k+1}(\theta) + \tilde{F}_{k+1}(\theta))Z_1(\theta) - Z_{k+1}(\theta)(\tilde{A}_{k+1}(\theta) + \tilde{F}_{k+1}(\theta)),$$
\[ |Z_{k+1}^{\pm} - I|_{r_{k+1}} \leq \varepsilon_k^\frac{2}{5} \]

- if moreover \( A_k \) had a \( BR_{\omega}^{N(r_k,\varepsilon_k)}(\kappa''(\varepsilon_k)) \) spectrum,

\[ |\varphi_{k+1}^{-1} Z_k \psi_{k+1}|_{r_{k+1}} \leq (1 + 2\varepsilon_k) e^{2\varepsilon_k} \quad (12.3) \]

\[ \text{and if not,} \]

\[ ||A_{k+1}|| \leq \kappa''(\varepsilon_k) \]

\[ |\partial_\omega (\varphi_{k+1}^{-1} Z_k \psi_{k+1})|_{r_{k+1}} \leq \varepsilon_k^\frac{2}{5} \quad (12.4) \]

**Result:** Let \( \varepsilon \leq \varepsilon_0 \) and \( k_\varepsilon \in \mathbb{N} \) such that \( |F|^{(2\delta)k_\varepsilon} \leq \varepsilon \). Let

\[
\begin{align*}
Z_\varepsilon &= Z_1 \cdots Z_{k_\varepsilon} \\
A_\varepsilon &= A_{k_\varepsilon} \\
F_\varepsilon &= F_{k_\varepsilon} \\
\psi_\varepsilon &= \psi_{k_\varepsilon} \\
r_\varepsilon &= r_{k_\varepsilon}
\end{align*}
\]

then the properties 1 and 2 hold. Moreover for all \( \theta \in \mathbb{T}^d \),

\[ \partial_\omega Z_\varepsilon(\theta) = (A + F(\theta))Z_\varepsilon(\theta) - Z_\varepsilon(\theta)(A_\varepsilon(\theta) + F_\varepsilon(\theta)) \]

and the property 3 holds. Notice that, for all \( k' > k_\varepsilon \), if \( ||A_{k_\varepsilon}|| \leq k_\varepsilon \varepsilon_0^\delta \) (which is satisfied if, for example, the matrix \( A_{k_\varepsilon-1} \) was resonant), then

\[ ||A_{k'}|| \leq ||A_{k_\varepsilon}|| + \sum_{i=k_\varepsilon+1}^{k'} \varepsilon_i \leq \kappa''(\varepsilon) + 2\varepsilon \leq 2\kappa''(\varepsilon). \]

We also have

\[ |Z_{1}^{\pm} - I|_{r_\varepsilon} \leq \varepsilon_0^\frac{2}{5} \Rightarrow |Z_{1}|_{r_\varepsilon} \leq 1 + \varepsilon_0^\frac{2}{5} \]

Let \( k \in \mathbb{N} \) and suppose that for all \( j \leq k - 1 \),

\[ |Z_1 \cdots Z_j|_{r_\varepsilon} \leq 2 \]

then

\[ c_k := |Z_1 \cdots Z_k - I|_{r_\varepsilon} \leq |Z_{k-1} - I|_{r_\varepsilon} |Z_1 \cdots Z_{k-1}|_{r_\varepsilon} + |Z_1 \cdots Z_{k-1} - I|_{r_\varepsilon} \]

\[ \leq 2\varepsilon_0^\frac{2}{5} + c_{k-1} \]

which implies

\[ c_k \leq 2 \sum_{i=0}^{k-2} \varepsilon_i^\frac{2}{5} v \leq 4\varepsilon_0^\frac{2}{5}. \]

Finally

\[ |(Z_1 \cdots Z_k)^{\pm} - I|_{r_\varepsilon} \leq \varepsilon_0^\frac{2}{5} \]

hence the property 4 holds. **Reducible case**

Suppose that there exists \( \bar{k} \) such that for all \( k' \geq \bar{k}, \psi_{k'} \equiv \psi_k \) (which means that for all \( k' \geq \bar{k}, A_{k'} \) has a \( BR_{\omega}^{N(r_{k'},\varepsilon_{k'})} \) spectrum). Then
\begin{align}
\partial_\omega Z_\varepsilon &= \partial_\omega (\prod_{i=1}^{k_\varepsilon} Z_i) \\
&= \partial_\omega (\prod_{i=1}^{k-1} Z_i)(\prod_{j=k}^{k_\varepsilon} Z_j) + (\prod_{i=1}^{k-1} Z_i) \partial_\omega (\prod_{j=k}^{k_\varepsilon} Z_j) \\
&= \partial_\omega (\prod_{i=1}^{k-1} Z_i)(\prod_{j=k}^{k_\varepsilon} Z_j) + \partial_\omega (\prod_{i=1}^{k_\varepsilon} Z_i) \partial_\omega (\prod_{j=k}^{k_\varepsilon} Z_j) \prod_{j=k}^{k_\varepsilon} \psi_j^{-1} Z_j \psi_j^{-1} \\
&= \partial_\omega (\prod_{i=1}^{k-1} Z_i)(\prod_{j=k}^{k_\varepsilon} Z_j) + \partial_\omega [\psi_k^{-1} Z_j \psi_j^{-1} + \partial_\omega (\psi_k^{-1} Z_j \psi_j^{-1})],
\end{align}

\begin{equation}
(12.5)
\end{equation}

Thus

\begin{align}
|\partial_\omega Z_\varepsilon|_{r_\varepsilon} &\leq |\partial_\omega (\prod_{i=1}^{k-1} Z_i)|_{r_\varepsilon} |\prod_{j=k}^{k_\varepsilon} Z_j|_{r_\varepsilon} + \prod_{i=1}^{k-1} Z_i |\psi_k|_{r_\varepsilon} |\partial_\omega (\prod_{j=k}^{k_\varepsilon} \psi_j^{-1} Z_j \psi_j^{-1})|_{r_\varepsilon} \\
&\quad + \prod_{i=1}^{k-1} Z_i |\psi_k^{-1} Z_j \psi_j^{-1}|_{r_\varepsilon} |\partial_\omega |_{r_\varepsilon} \\
&\quad + \prod_{i=1}^{k-1} Z_i |\psi_k^{-1} Z_j \psi_j^{-1}|_{r_\varepsilon} |\partial_\omega |_{r_\varepsilon}.
\end{align}

Since the factors \(|\prod_{i=1}^{k-1} Z_i|_{r_\varepsilon}, |\partial_\omega \prod_{i=1}^{k-1} Z_i|_{r_\varepsilon}, \prod_{i=1}^{k-1} Z_i |\psi_k|_{r_\varepsilon}, |\psi_k^{-1}|_{r_\varepsilon}, \partial_\omega \psi_k|_{r_\varepsilon}, \partial_\omega \psi_k^{-1}|_{r_\varepsilon}|\) are bounded uniformly in \(\varepsilon\) (here we use (12.3)), there exist \(K_1, K_2 \geq 0\) independent of \(\varepsilon\) such that

\begin{equation}
|\partial_\omega Z_\varepsilon|_{r_\varepsilon} \leq K_1 + K_2 |\partial_\omega (\prod_{j=k}^{k_\varepsilon} \psi_j^{-1} Z_j \psi_j^{-1})|_{r_\varepsilon}.
\end{equation}

Moreover, by (12.4) and (12.3),

\begin{align}
|\partial_\omega (\prod_{j=k}^{k_\varepsilon} \psi_j^{-1} Z_j \psi_j^{-1})|_{r_\varepsilon} &\leq \sum_{j=k}^{k_\varepsilon} |\partial_\omega (\psi_j^{-1} Z_j \psi_j^{-1})|_{r_\varepsilon} \prod_{\stackrel{k \leq i \leq k_\varepsilon}{i \neq j}} |\psi_i^{-1} Z_i|_{r_\varepsilon} \\
&\leq \sum_{j=k}^{k_\varepsilon} \varepsilon_j^k \prod_{\stackrel{k \leq i \leq k_\varepsilon}{i \neq j}} (1 + 2e^{\varepsilon_i}) e^{2\varepsilon_i}
\end{align}

therefore

\begin{align}
|\partial_\omega (\prod_{j=k}^{k_\varepsilon} \psi_j^{-1} Z_j \psi_j^{-1})|_{r_\varepsilon} &\leq 2 \sum_{j=k}^{k_\varepsilon} \varepsilon_j^k e^{2\varepsilon_k} \\
&\leq 8 \varepsilon_k^k e^{2\varepsilon_k} \\
&\leq 16 \varepsilon_k^k
\end{align}

and finally, \(|\partial_\omega Z_\varepsilon|_{r_\varepsilon}\) is bounded as \(\varepsilon \to 0\). In this case, \(Z_\varepsilon\) and \(\partial_\omega Z_\varepsilon\) have adherent values; let \(Z_\infty\) be an adherent value of \(Z_\varepsilon\). Since
\[
\partial_\omega (Z_\epsilon \psi_k) = (A + F) Z_\epsilon \psi_k - Z_\epsilon \psi_k (A_\epsilon + \psi_k^{-1} \bar{F}_\epsilon \psi_k)
\]

(where \(A_\epsilon \in \mathfrak{sl}(2, \mathbb{R})\)), and since all factors except \(A_\epsilon\) are known to converge in a subsequence, then there exists a constant \(A_\infty \in \mathfrak{sl}(2, \mathbb{R})\) such that

\[
\partial_\omega (Z_\infty \psi_k) = (A + F) Z_\infty \psi_k - Z_\infty \psi_k A_\infty
\]

and thus \(A + F\) is actually is a reducible cocycle in \(U_{r_\infty}(\mathbb{T}^d, \mathfrak{sl}(2, \mathbb{R}))\) for some \(r_\infty > 0\).

**Non reducible case**

If the system \(A + F\) is not reducible, then for all \(k \geq 1\), there exists \(k' \geq k\) such that \(A_{k'}\) does not have a \(BR^{R_{k'} N_{k'} / (k''(\epsilon_{k'}))}\) spectrum. In this case, \(||A_{k'+1}|| \leq k''(\epsilon_{k'}) = \kappa \epsilon_{k'}^z\).

We will now show a density corollary.

**Corollary 12.1** (Density of reducible cocycles close to a constant cocycle). Let \(r_0 > 0\), \(A \in \mathfrak{sl}(2, \mathbb{R})\), and \(G \in U_{r_0}(2 \mathbb{T}^d, \mathfrak{sl}(2, \mathbb{R}))\) such that \(|G - A|_{r_0} \leq \epsilon_0\) and \(|A| \leq \epsilon_0^{-2}\) with \(\epsilon_0\) as in 10.1, and satisfying the assumption 1. Denote

\[
\rho = r_0 - \frac{150|\log \epsilon_0|}{\pi \Lambda (\epsilon_0)} + \frac{150}{\pi \log (28)} \int_{\epsilon_0}^{+\infty} \frac{\Lambda'(t) \ln \Psi(t)}{\Lambda(t)^2} dt.
\]

Then for all \(\epsilon > 0\) there exists \(H \in U_{\rho}(2 \mathbb{T}^d, \mathfrak{sl}(2, \mathbb{R}))\) such that \(|G - H|_{\rho} \leq \epsilon\) and \(H\) is reducible.

**Proof.** Apply theorem 12.1 with \(F = G - A\). Since \(\rho \leq r_\epsilon\), we in particular get matrices \(Z_\epsilon \in U_{\rho}(\mathbb{T}^d, \mathfrak{sl}(2, \mathbb{R})), \bar{A}_\epsilon, \bar{F}_\epsilon \in U_{\rho}(\mathbb{T}^d, \mathfrak{sl}(2, \mathbb{R}))\) and \(A_\epsilon \in \mathfrak{sl}(2, \mathbb{R})\) such that

- \(\bar{A}_\epsilon\) is reducible to \(A_\epsilon\),
- \(\partial_\omega Z_\epsilon = (A + (G - A)) Z_\epsilon - Z_\epsilon (A_\epsilon + \bar{F}_\epsilon) = G Z_\epsilon - Z_\epsilon (A_\epsilon + \bar{F}_\epsilon),\)
- \(|Z_\epsilon^2|_{\rho} \leq 1 + \epsilon_0^{\frac{1}{10}} \leq 2,\)
- \(|\bar{F}_\epsilon|_{\rho} \leq \frac{1}{10}\)

Let \(H := G - Z_\epsilon \bar{F}_\epsilon Z_\epsilon^{-1}\). We have

\[
\partial_\omega Z_\epsilon = H Z_\epsilon - Z_\epsilon \bar{A}_\epsilon
\]

and then \(H\) is reducible to \(A_\epsilon\) (as \(\bar{A}_\epsilon\) is). Moreover, \(H\) satisfies

\[
|H - G|_{\rho} = |Z_\epsilon^{-1} \bar{F}_\epsilon Z_\epsilon|_{\rho} \leq 4 |\bar{F}_\epsilon|_{\rho} \leq \epsilon.
\]

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