A CHARACTERISATION OF THE \( n\langle 1 \rangle \oplus \langle 3 \rangle \) FORM AND APPLICATIONS TO RATIONAL HOMOLOGY SPHERES

BRENDAN OWENS AND SAŠO STRLE

Abstract. We conjecture two generalisations of Elkies’ theorem on unimodular quadratic forms to non-unimodular forms. We give some evidence for these conjectures including a result for determinant 3. These conjectures, when combined with results of Frøyshov and of Ozsváth and Szabó, would give a simple test of whether a rational homology 3-sphere may bound a negative-definite four-manifold. We verify some predictions using Donaldson’s theorem. Based on this we compute the four-ball genus of some Montesinos knots.

1. Introduction

Let \( Y \) be a rational homology three-sphere and \( X \) a smooth negative-definite four-manifold bounded by \( Y \). For any Spin\(^c\) structure \( t \) on \( Y \) let \( d(Y, \mathfrak{t}) \) denote the correction term invariant of Ozsváth and Szabó [8]. It is shown in [8, Theorem 9.6] that for each Spin\(^c\) structure \( \mathfrak{s} \in \text{Spin}^c(X) \),

\[
(1) \quad c_1(\mathfrak{s})^2 + \text{rk}(H^2(X; \mathbb{Z})) \leq 4d(Y, \mathfrak{s}|_Y).
\]

This is analogous to gauge-theoretic results of Frøyshov. These theorems constrain the possible intersection forms that \( Y \) may bound. The above inequality is used in [7] to constrain intersection forms of a given rank bounded by Seifert fibred spaces, with application to four-ball genus of Montesinos links. In this paper we attempt to get constraints by finding a lower bound on the left-hand side of (1) which applies to forms of any rank. This has been done for unimodular forms by Elkies:

**Theorem 1.1** ([2]). Let \( Q \) be a negative-definite unimodular integral quadratic form of rank \( n \). Then there exists a characteristic vector \( x \) with \( Q(x, x) + n \geq 0 \); moreover the inequality is strict unless \( Q = n\langle -1 \rangle \).

Together with (1) this implies that an integer homology sphere \( Y \) with \( d(Y) < 0 \) cannot bound a negative-definite four-manifold, and if \( d(Y) = 0 \) then the only definite pairing that \( Y \) may bound is the diagonal form. Since \( d(S^3) = 0 \) this generalises Donaldson’s theorem on intersection forms of closed four-manifolds ([1]).

---

*Date: March 29, 2022.*

S.Strle was supported in part by the MSZS of the Republic of Slovenia research program No. P1-0292-0101-04 and research project No. J1-6128-0101-04.
In Section 2 we conjecture two generalisations of Elkies’ theorem to forms of arbitrary determinant. We prove some special cases, including Theorem 3.1 which is a version of Theorem 1.1 for forms of determinant 3. This implies the following

**Theorem 1.2.** Let $Y$ be a rational homology sphere with $H_1(Y;\mathbb{Z}) = \mathbb{Z}/3$ and let $t_0$ be the spin structure on $Y$. If $Y$ bounds a negative-definite four-manifold $X$ then either

$$d(Y, t_0) \geq -\frac{1}{2},$$

or

$$\max_{t \in \text{Spin}^c(Y)} d(Y, t) \geq \frac{1}{6}.$$

If equality holds in both then the intersection form of $X$ is diagonal.

In Section 4 we discuss further topological implications of our conjectures; in particular some predictions for Seifert fibred spaces may be verified using Donaldson’s theorem. We find two families of Seifert fibred rational homology spheres, no multiple of which can bound negative-definite manifolds. We use these results to determine the four-ball genus of two families of Montesinos knots, including one whose members are algebraically slice but not slice.

## 2. Conjectured generalisations of Elkies’ theorem

We begin with some notation. A quadratic form $Q$ of rank $n$ over the integers gives rise to a symmetric matrix with entries $Q(e_i, e_j)$, where $\{e_i\}$ are the standard basis for $\mathbb{Z}^n$; we also denote the matrix by $Q$. Let $Q'$ denote the induced form on the dual $\mathbb{Z}^n$; this is represented by the inverse matrix. Two matrices $Q_1$ and $Q_2$ represent the same form if and only if $Q_1 = P^T Q_2 P$ for some $P \in GL(n, \mathbb{Z})$.

We call $y \in \mathbb{Z}^n$ a characteristic covector for $Q$ if

$$(y, \xi) \equiv Q(\xi, \xi) \pmod{2} \quad \forall \xi \in \mathbb{Z}^n.$$ 

We call $x \in \mathbb{Z}^n$ a characteristic vector for $Q$ if

$$Q(x, \xi) \equiv Q(\xi, \xi) \pmod{2} \quad \forall \xi \in \mathbb{Z}^n.$$ 

Note that the form $Q$ induces an injection $x \mapsto Qx$ from $\mathbb{Z}^n$ to its dual with the quotient group having order $|\det Q|$; with respect to the standard bases this map is multiplication by the matrix $Q$. For unimodular forms this gives a bijection between characteristic vectors and characteristic covectors; in general not every characteristic covector is a characteristic vector. Also for odd determinant, any two characteristic vectors are congruent modulo 2; this is no longer true for even determinant.

Let $Q$ be a negative-definite integral form of rank $n$ and let $\delta$ be the absolute value of its determinant. Denote by $\Delta = \Delta_\delta$ the diagonal form $(n-1)(-1) \oplus (-\delta)$. Both of the following give restatements of Theorem 1.1 when restricted to unimodular forms.
Conjecture 2.1. Every characteristic vector $x_0$ is congruent modulo 2 to a vector $x$ with
$$Q(x, x) + n \geq 1 - \delta;$$
moreover the inequality is strict unless $Q = \Delta$.

Conjecture 2.2. There exists a characteristic covector $y$ with
$$Q'(y, y) + n \geq \begin{cases} 1 - 1/\delta & \text{if } \delta \text{ is odd,} \\ 1 & \text{if } \delta \text{ is even;} \end{cases}$$
moreover the inequality is strict unless $Q = \Delta$.

We will discuss the implications of these conjectures in Section 4.

Proposition 2.3. Conjecture 2.1 is true when restricted to forms of rank $\leq 3$, and Conjecture 2.2 is true when restricted to forms of rank 2 and odd determinant.

Proof. We will first establish Conjecture 2.1 for rank 2 forms. In fact we prove the following stronger statement: if $Q$ is a negative-definite form of rank 2 and determinant $\delta$, then for any $x_0 \in \mathbb{Z}^2$,
$$\max_{x \equiv x_0(2)} Q(x, x) \geq -1 - \delta,$$
and the inequality is strict unless $Q = \Delta$.

Every negative-definite rank 2 form is represented by a reduced matrix
$$Q = \begin{pmatrix} a & b \\ b & c \end{pmatrix},$$
with $0 \geq 2b \geq a \geq c$ and $-1 \geq a$. Any vector $x_0$ is congruent modulo 2 to one of $(0, 0), (1, 0), (0, 1), (1, -1)$; all of these satisfy $x^T Q x \geq a + c - 2b$. Thus it suffices to show
$$a + c - 2b \geq -1 - \delta.$$  
Note that equality holds in (3) if $Q = \Delta$. Suppose now that $Q \neq \Delta$. Let $Q_\tau = \begin{pmatrix} a + 2\tau & b + \tau \\ b + \tau & c \end{pmatrix}$, and let $\delta_\tau = \det Q_\tau$. Then $a_\tau + c_\tau - 2b_\tau$ is constant and $\delta_\tau$ is a strictly decreasing function of $\tau$. Thus (3) will hold for $Q$ if it holds for $Q_\tau$ for some $\tau > 0$. In the same way we may increase both $b$ and $c$ so that $a + c - 2b$ remains constant and the determinant decreases, or we may increase $a$ and decrease $c$. In this way we can find a path $Q_\tau$ in the space of reduced matrices from any given $Q$ to a diagonal matrix $\begin{pmatrix} -1 & 0 \\ 0 & -\delta \end{pmatrix}$, such that $a + b - 2c$ is constant along the path and the determinant decreases. It follows that (3) holds for $Q$, and the inequality is strict unless $Q = \Delta$.

A similar but more involved argument establishes Conjecture 2.1 for rank 3 forms. We briefly sketch the argument. Let $Q$ be represented by a reduced matrix of rank 3.
(see for example [5]) and let \( x_0 \in \mathbb{Z}^3 \). By successively adding \( 2\tau \) to a diagonal entry and \( \pm \tau \) to an off-diagonal entry one may find a path of reduced matrices from \( Q \) to \( \tilde{Q} \) along which \( \max_{x \equiv x_0(2)} x^T \tilde{Q} x \) is constant and the absolute value of the determinant decreases. One cannot always expect that \( \tilde{Q} \) will be diagonal but one can show that the various matrices which arise all satisfy

\[
\max_{x \equiv x_0(2)} x^T \tilde{Q} x \geq -2 - |\det \tilde{Q}|,
\]

(with strict inequality unless \( \tilde{Q} = \Delta \)) from which it follows that this inequality holds for all negative-definite rank 3 forms.

Finally note that for rank 2 forms, the determinant of the adjoint matrix \( \text{ad} Q \) is equal to the determinant of \( Q \). Conjecture [2,2] for rank 2 forms of odd determinant now follows by applying [2] to \( \text{ad} Q \) and dividing by the determinant \( \delta \).

\[\square\]

3. Determinant three

In this section we describe to what extent we can generalise Elkies’ proof of Theorem 1.1 to non-unimodular forms. For convenience we work with positive-definite forms. We obtain the following result.

**Theorem 3.1.** Let \( Q \) be a positive-definite quadratic form over the integers of rank \( n \) and determinant 3. Then either \( Q \) has a characteristic vector \( x \) with \( Q(x, x) \leq n + 2 \) or it has a characteristic covector \( y \) with \( Q'(y, y) \leq n - \frac{2}{3} \). Moreover, at least one of the above inequalities is strict unless \( Q \) is diagonal.

Given a positive-definite integral quadratic form \( Q \) of rank \( n \), we consider lattices \( L \subset L' \) in \( \mathbb{R}^n \) (equipped with the standard inner product), with \( Q \) the intersection pairing of \( L \), and \( L' \) the dual lattice of \( L \). Note that the discriminant of the lattice \( L \) is equal to the determinant of \( Q \).

For any lattice \( L \subset \mathbb{R}^n \) and a vector \( w \in \mathbb{R}^n \) let \( \theta^w_L \) be the generating function for the norms of vectors in \( \frac{w}{2} + L \),

\[
\theta^w_L(z) = \sum_{x \in L} e^{i\pi |x + \frac{w}{2}|^2 z};
\]

this is a holomorphic function on the upper half-plane \( H = \{ z \mid \text{Im}(z) > 0 \} \). The *theta series* of the lattice \( L \) is \( \theta_L = \theta^0_L \).

Recall that the modular group \( \Gamma = \text{PSL}_2(\mathbb{Z}) \) acts on \( H \) and is generated by \( S \) and \( T \), where \( S(z) = -\frac{1}{z} \) and \( T(z) = z + 1 \).
Proposition 3.2. Let $L$ be an integral lattice of odd discriminant $\delta$, and $L'$ its dual lattice. Then

\begin{equation}
\theta_L(S(z)) = \left(\frac{z}{i}\right)^{n/2} \delta^{-1/2} \theta_{L'}(z)
\end{equation}

\begin{equation}
\theta_L(TS(z)) = \left(\frac{z}{i}\right)^{n/2} \delta^{-1/2} \theta_{L'}^w(z)
\end{equation}

\begin{equation}
\theta_{L'}(T^\delta S(z)) = \left(\frac{z}{i}\right)^{n/2} \delta^{1/2} \theta_{L'}^w(z),
\end{equation}

where $w$ is a characteristic vector in $L$.

Remark 3.3. Note that if $w \in L$ is a characteristic vector, then $\theta_{L'}^w$ is a generating function for the squares of characteristic covectors. Under the assumption that the discriminant of $L$ is odd, $\theta_{L'}^w$ is a generating function for the squares of characteristic vectors.

Proof. All the formulas follow from Poisson inversion [10, Ch. VII, Proposition 15]. We only need odd discriminant in (6). Note that in $\theta_{L'}^w(z + \delta)$ we can use

\begin{equation}
\delta |y|^2 \equiv |\delta y|^2 \equiv (\delta y, w) \equiv (y, w) \pmod{2}
\end{equation}

and then apply Poisson inversion. \qed

Corollary 3.4. Let $L_1$ and $L_2$ be integral lattices of the same rank and the same odd discriminant $\delta$. Then

\begin{equation}
R(z) = \frac{\theta_{L_1}(z)}{\theta_{L_2}(z)}
\end{equation}

is invariant under $T^2$ and $ST^{2\delta}S$. Moreover, $R^8$ is invariant under $(T^2S)\delta$ and $ST^{\delta-1}ST^{\delta-1}S$.

Proof. Since $L$ is integral, $\theta_L(z + 2) = \theta_L(z)$, hence $R$ is $T^2$ invariant. The squares of vectors in $L'$ belong to $\frac{1}{\delta} \mathbb{Z}$, so $\theta_{L'}(z + 2\delta) = \theta_{L'}(z)$. From (4) it follows that

\begin{equation}
R(S(z)) = \frac{\theta_{L_1}(z)}{\theta_{L_2}(z)}
\end{equation}

which gives the $ST^{2\delta}S$ invariance of $R$.

To derive the remaining symmetries of $R^8$ we need to use (5) and (6). Let $w$ be a characteristic vector in $L$. Clearly

\begin{equation}
\delta |y + \frac{w}{2}|^2 = \delta |y|^2 + \delta (y, w) + \frac{\delta}{4} |w|^2
\end{equation}

holds for any $y \in L'$, so it follows from (4) that

\begin{equation}
\theta_{L'}^w(z + \delta) = e^{i\pi |w|^2/4} \theta_{L'}^w(z).
\end{equation}

Using (5) we now conclude that $R^8$ is invariant under $TST^\delta ST^{-1} = (ST^{-2})\delta$; the last equality follows from the relation $(ST)^3 = 1$ in the modular group. The remaining invariance of $R^8$ is derived in a similar way from (6). \qed
From now on we restrict our attention to discriminant $\delta = 3$. Consider the subgroup $\Gamma_3$ of $\Gamma$ generated by $T^2$, $ST^3S$ and $ST^2ST^2S$. Clearly $\Gamma_3$ is a subgroup of $\Gamma_+ = \langle S, T^2 \rangle \subset \Gamma$.

**Lemma 3.5.** A full set of coset representatives for $\Gamma_3$ in $\Gamma_+$ is $I, S, ST^2, ST^4$. Hence a fundamental domain $D_3$ for the action of $\Gamma_3$ on the hyperbolic plane $H$ is the hyperbolic polygon with vertices $-1, -\frac{1}{3}, -\frac{1}{5}, 0, 1, i\infty$.

**Proof.** Call $x, y \in \Gamma_+$ equivalent if $y = zx$ for some $z \in \Gamma_3$. Let $x = T^{k_1}ST^{k_2}S \cdots T^{k_n}$ with all $k_i \neq 0$; then the length of $x$, $Sx$, $xS$ and $SxS$ is defined to be $n$. Any element $x \in \Gamma_+$ of length $n \geq 2$ is equivalent to one of the form $ST^kSy$ with $k = 0, \pm 2$ and length at most $n$. If $x = ST^kST^l y$ with $k = \pm 2$ and length $n \geq 2$, then $x$ is equivalent to $ST^{4-k}y$, which has length $\leq n - 1$. It follows by induction on length that any element of $\Gamma_+$ is equivalent to one with length at most 1. Moreover, if the element has length 1, it is equivalent to $ST^k$, $k = 2, 4$.

Finally, recall that a fundamental domain for $\Gamma_+$ is $D_+ = \{z \in H \mid -1 \leq \Re(z) \leq 1, \ |z| \geq 1\}$ so we can take $D_3$ to be the union of $D_+$ and $S(D_+ \cup T^2(D_+) \cup T^4(D_+))$. \hfill $\Box$

**Proof of Theorem 3.4.** Suppose that $L$ is a lattice of discriminant 3 and rank $n$ for which the square of any characteristic vector is at least $n + 2$ and the square of any characteristic covector is at least $n - \frac{2}{3}$. Let $\Delta$ be the lattice with intersection form $(n - 1)(1) \oplus \langle 3 \rangle$; recall from [2] that $\theta_\Delta$ does not vanish on $H$. Then

$$R(z) = \frac{\theta_L(z)}{\theta_\Delta(z)}$$

is holomorphic on $H$ and it follows from Corollary 3.4 that $R^8$ is invariant under $\Gamma_3$. We want to show that $R$ is bounded. We will use the following identities that follow from Proposition 3.2

$$R(S(z)) = \frac{\theta_L(z)}{\theta_\Delta(z)}, \quad R(TS(z)) = \frac{\theta_L^w(z)}{\theta_\Delta^w(z)}, \quad R(ST^2S(z)) = \frac{\theta_L^w(z)}{\theta_\Delta^w(z)}.$$

Since the theta series of any lattice converges to 1 as $z \to i\infty$, $R(z) \to 1$ as $z \to 0, i\infty$. By assumption the square of any characteristic covector for $L$ is at least as large as the square of the shortest characteristic covector for $\Delta$. Since the asymptotic behaviour as $z \to i\infty$ of the generating function for the squares of characteristic covectors is determined by the smallest square, it follows from the middle expression for $R$ above that $R(z)$ is bounded as $z \to 1$. Similarly, using the condition on characteristic vectors and the right-most expression for $R$ as $z \to i\infty$, it follows that $R(z)$ is bounded as $z \to -\frac{1}{3}$. Note that $T^{-2}(1) = -1$ and $ST^0S(1) = -\frac{1}{5}$, so $R(z)$ is also bounded as $z \to -1, -\frac{1}{5}$.

Let $f$ be the function on $\Sigma = H/\Gamma_3$ induced by $R^8$. Then $f$ is holomorphic and bounded, so it extends to a holomorphic function on the compactification of $\Sigma$. It follows that $R(z) = 1$, so the theta series of $L$ is equal to the theta series of $\Delta$. Then
A CHARACTERISATION OF THE $n(1) \oplus (3)$ FORM

$L$ contains $n - 1$ pairwise orthogonal vectors of square 1, so its intersection form is $(n - 1)(1) \oplus (3)$. \hfill \square

4. Applications

In this section we consider applications to rational homology spheres and four-ball genus of knots. We begin with the proof of Theorem 1.2.

Proof of Theorem 1.2. Suppose that $Y = \partial X$ and that $Q$ is the intersection form on $H_2(X; \mathbb{Z})$. Then $Q$ is a quadratic form of determinant $\pm 3$. For any $s \in \text{Spin}^c(X)$, let $c(s)$ denote the image of the first Chern class $c_1(s)$ modulo torsion. Then $c(s)$ is a characteristic covector for $Q$; moreover if $s|_Y$ is spin then $c(s)$ is $Qx$ for some characteristic vector $x$. The result now follows from Theorem 3.1 and (1). \hfill \square

Conjectures 2.1 and 2.2 imply the following more general statement.

Conjecture 4.1. Let $Y$ be a rational homology sphere with $|H_1(Y; \mathbb{Z})| = h$. If $Y$ bounds a negative-definite four-manifold $X$ with no torsion in $H_1(X; \mathbb{Z})$ then

$$\min_{t_0 \in \text{Spin}(Y)} d(Y, t_0) \geq (1 - h)/4,$$

and

$$\max_{t \in \text{Spin}^c(Y)} d(Y, t) \geq \begin{cases} \left(1 - \frac{1}{h}\right)/4 & \text{if } h \text{ is odd,} \\ 1/4 & \text{if } h \text{ is even.} \end{cases}$$

If equality holds in either inequality the intersection form of $X$ is $\Delta_h$.

More generally if $Y$ bounds $X$ with torsion in $H_1(X; \mathbb{Z})$, the absolute value of the determinant of the intersection pairing of $X$ divides $h$ with quotient a square (see for example [7, Lemma 2.1]). One may then deduce inequalities as above corresponding to each choice of determinant; care must be taken since for example not all spin structures on $Y$ extend to spin$^c$ structures on $X$.

Remark 4.2. Given a rational homology sphere $Y$ bounding $X$ with no torsion in $H_1(X; \mathbb{Z})$, the intersection pairing of $X$ gives a presentation matrix for $H^2(Y; \mathbb{Z})$ (and also determines the linking pairing of $Y$). There should be analogues of Conjectures 2.1 and 2.2 which restrict to forms presenting a given group (and inducing a given linking pairing). These should give stronger bounds than those in Conjecture 4.1.

4.1. Seifert fibred examples. In Examples 4.5 and 4.6 we list families of Seifert fibred spaces $Y$ which bound positive-definite but not negative-definite four-manifolds. It follows as in [4, Theorem 10.2] that no multiple of $Y$ bounds a negative-definite four-manifold. In Examples 4.7 through 4.9 we list families of Seifert fibred spaces which can only bound the diagonal negative-definite form $\Delta_\delta$ (or sometimes $\Delta_1$). We found these examples using predictions based on Conjecture 4.1 and verified them using Donaldson’s theorem via Proposition 4.4. Finally, in Example 4.10 we exhibit
a family of Seifert fibred spaces which according to the conjecture can only bound $\Delta_8$. For this family the method of Proposition 4.3 does not apply.

In what follows we extend the definition of $\Delta_1$ to include the trivial form on the trivial lattice. Also note that a lattice uniquely determines a quadratic form, and a form determines an equivalence class of lattices; in the rest of this section we use the terms lattice and form interchangeably.

**Definition 4.3.** Let $L$ be a lattice of rank $m$ and determinant $\delta$. We say $L$ is rigid if any embedding of $L$ in $\mathbb{Z}^n$ is contained in a $\mathbb{Z}^m$ sublattice. We say $L$ is almost-rigid if any embedding of $L$ in $\mathbb{Z}^n$ is either contained in a $\mathbb{Z}^m$ sublattice, or contained in a $\mathbb{Z}^{m+1}$ sublattice with orthogonal complement spanned by a vector $v$ with $|v|^2 = \delta$.

**Proposition 4.4.** Let $Y$ be a rational homology sphere and let $h$ be the order of $H_1(Y; \mathbb{Z})$. Suppose $Y$ bounds a positive-definite four-manifold $X_1$ with $H_1(X_1; \mathbb{Z}) = 0$. Let $Q_1$ be the intersection pairing of $X_1$ and let $m$ denote its rank.

If $Q_1$ does not embed into $\mathbb{Z}^n$ for any $n$ then $Y$ cannot bound a negative-definite four-manifold.

If $Q_1$ is rigid and $Y$ bounds a negative-definite $X_2$ then $h$ is a square and $Q_2 = \Delta_1$; if $h > 1$, then there is torsion in $H_1(X_2; \mathbb{Z})$.

If $Q_1$ is almost-rigid and $Y$ bounds a negative-definite $X_2$ then either

- $Q_2 = \Delta_1$ or
- $Q_1$ embeds in $\mathbb{Z}^m$, $h$ is a square and $Q_2 = \Delta_1$; if $h > 1$, then there is torsion in $H_1(X_2; \mathbb{Z})$.

**Proof.** Suppose $Y$ bounds a negative-definite $X_2$ with intersection pairing $Q_2$. Then $X = X_1 \cup_Y \neg X_2$ is a closed positive-definite manifold. The Mayer-Vietoris sequence for homology and Donaldson’s theorem yield an embedding $\iota : Q_1 \oplus -Q_2 \to \mathbb{Z}^{m+k}$, where $k$ is the rank of $Q_2$.

If the image of $Q_1$ under $\iota$ is contained in a $\mathbb{Z}^m$ sublattice, then the image of $-Q_2$ is contained in the orthogonal $\mathbb{Z}^k$ sublattice. Now consider the Mayer-Vietoris sequence for cohomology:

$$
0 \to H^2(X; \mathbb{Z}) \to H^2(X_1; \mathbb{Z}) \oplus H^2(X_2; \mathbb{Z}) \to H^2(Y; \mathbb{Z}) \cong Q_1' \oplus -Q_2' \oplus T_2
$$

where $T_2$ is the torsion subgroup and $Q'$ denotes the dual lattice to $Q$. This yields an embedding $\iota' : \mathbb{Z}^{m+k} \to Q_1' \oplus -Q_2'$. The mapping $\iota'$ is hom-dual to $\iota$ and hence also decomposes orthogonally, sending $\mathbb{Z}^m$ to $Q_1'$ and $\mathbb{Z}^k$ to $-Q_2'$. The image of $\mathbb{Z}^m$ in $Q_1'$ has index $\sqrt{h}$, since $h$ is the determinant of $Q_1$. (In general if $L_1 \subset L_2$ are lattices of the same rank then the square of the index $[L_2 : L_1]$ is the quotient of their discriminants.) The restriction map from $H^2(X_1; \mathbb{Z})$ to $H^2(Y; \mathbb{Z})$ is onto, so its kernel $K$ is a subgroup of $\mathbb{Z}^m$ of index $\sqrt{h}$. It follows that $\mathbb{Z}^m/K$ injects into $T_2$ and that
the image of $T_2$ in $H^2(Y; \mathbb{Z})$ has order $t \geq \sqrt{h}$. Then by [12, Lemma 2.1], $t = \sqrt{h}$ and $Q_2$ is unimodular. Since $-Q_2$ is a sublattice of $\mathbb{Z}^k$ we have $Q_2 = \Delta_1$.

Suppose now that the image of $Q_1$ under $\iota$ is contained in a $\mathbb{Z}^{m+1}$ sublattice, and its orthogonal complement in $\mathbb{Z}^{m+1}$ is spanned by a vector $v$ with $|v|^2 = h$. Then the image of $-Q_2$ is a sublattice of $(k-1)\langle 1 \rangle \oplus \langle h \rangle$; it therefore has determinant at least $h$. On the other hand its determinant divides $h$ [7, Lemma 2.1]. It follows that $Q_2$ is equal to $\Delta_h$. □

If $Y$ is the Seifert fibred space $Y(e; (\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3))$, let

$$k(Y) = e\alpha_1\alpha_2\alpha_3 + \beta_1\alpha_2\alpha_3 + \alpha_1\beta_2\alpha_3 + \alpha_1\alpha_2\beta_3.$$  

If $k(Y) \neq 0$ then $Y$ is a rational homology sphere and $|k(Y)|$ is the order of $H_1(Y; \mathbb{Z})$. Furthermore, if $k(Y) < 0$ then $Y$ bounds a positive-definite plumbing. For our conventions for lens spaces and Seifert fibred spaces see [7]. Recall in particular that $(\alpha_i, \beta_i)$ are coprime pairs of integers with $\alpha_i \geq 2$. We will also assume here that $1 \leq \beta_i < \alpha_i$.

Example 4.5. Seifert fibred spaces $Y = Y(-2; (\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3))$ with

$$\frac{\alpha_1}{\beta_1} \leq 2, \quad \frac{\alpha_2}{\beta_2} < 2, \quad \frac{\alpha_3}{\beta_3} < 2, \quad k(Y) < 0,$$

cannot bound negative-definite four-manifolds.

![Plumbing graph](image)

Figure 1. Plumbing graph.

Proof. Note that $Y$ is the boundary of the positive-definite plumbing shown in Figure 1 where vertices $u$, $v_1$, $w_1$ and $x_1$ have square 2 and $v_2$ and $w_2$ have square at least 2. This lattice does not admit an embedding in any $\mathbb{Z}^n$. To see this let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{Z}^n$. The vertex $u$ must map to an element of square 2, which we may suppose is $e_1 + e_2$. The 3 adjacent vertices must be mapped to elements of the form $e_1 + e_3$, $e_1 - e_3$ and $e_2 + e_4$. Now we see that it is not possible to map the remaining 2 vertices $v_2$ and $w_2$; we are only able to further extend the map along the leg of the graph emanating from the vertex mapped to $e_2 + e_4$. □
Example 4.6. Seifert fibred spaces \( Y = Y(-2; (\alpha_1, \beta_1), (\alpha_2, \alpha_2 - 1), (\alpha_3, \alpha_3 - 1)) \) with
\[
\alpha_2, \alpha_3 \geq \frac{\alpha_1}{\beta_1}, \quad \alpha_3 \geq 3, \quad k(Y) < 0,
\]
cannot bound negative-definite four-manifolds unless
\[
\beta_1 = 1, \quad \min(\alpha_2, \alpha_3) = \alpha_1.
\]
In the latter case, if \( Y \) bounds a negative-definite \( X \) then the intersection pairing of \( X \) is \( \Delta_1 \) and the torsion subgroup of \( H_1(X; \mathbb{Z}) \) is nontrivial.

Proof. In this case \( Y \) is again the boundary of a positive-definite plumbing as in Figure 4. The vertices \( u, v, \) and \( w \) have square 2, and \( p = \alpha_2 - 1, q = \alpha_3 - 1. \) Vertex \( x_1 \) has square \( a = \left\lfloor \frac{n}{\beta_1} \right\rfloor. \) If \( \frac{n}{\beta_1} = \min(\alpha_2, \alpha_3) = a \) then by inspection this pairing is rigid with determinant \( a^2 > 1; \) otherwise it does not admit any embedding into \( \mathbb{Z}^n. \)
For more details see the proof of Example 4.8. \( \square \)

Example 4.7. The only negative-definite pairing that \( L(p, 1) \) can bound is the diagonal form \( \Delta_p \) unless \( p = 4 \) in which case it may also bound \( \Delta_1. \) (Note that \( L(p, 1) \) is the boundary of the disk bundle over \( S^2 \) with intersection pairing \( \langle -p \rangle). \)

Proof. Observe that \( L(p, 1) \) is the boundary of the positive-definite plumbing \( A_{p-1} \). If \( p \neq 4 \) then up to automorphisms of \( \mathbb{Z}^n \) there is a unique embedding of \( A_{p-1} \) in \( \mathbb{Z}^n; \) the image is contained in a \( \mathbb{Z}^p \) and its orthogonal complement in \( \mathbb{Z}^p \) is generated by the vector \( (1, 1, \ldots, 1). \) Hence \( A_{p-1} \) is almost-rigid and does not embed in \( \mathbb{Z}^{p-1}. \)
However, \( A_3 \) also admits an embedding in \( \mathbb{Z}^3. \) \( \square \)

Example 4.8. If \( Y = Y(-2; (\alpha_2 \beta_1 + 1, \beta_1), (\alpha_2, \alpha_2 - 1), (\alpha_3, \alpha_3 - 1)) \) with \( \alpha_3 > \alpha_2, \)
then the only negative-definite pairing that \( Y \) may bound is the diagonal form \( \Delta_{|k(Y)|} \) unless
\[
\beta_1 = 1, \quad \alpha_3 = \alpha_2 + 1.
\]
In the latter case the only negative-definite pairings that \( Y \) may bound are \( \Delta_{|k(Y)|} \) and \( \Delta_1. \)

Proof. Note this is a borderline case of Example 4.6. In the notation of that example \( \alpha_2 = a - 1. \) The positive-definite plumbing is similar to that in Example 4.6 with \( r = \beta_1; \) also the vertices \( x_l \) with \( l > 1 \) all have square 2. Denote the pairing associated to this plumbing by \( Q. \) We consider an embedding of \( Q \) into \( \mathbb{Z}^n. \) Let \( e_i, f_j \) and \( g_l \) denote unit vectors in \( \mathbb{Z}^n. \) Without loss of generality the vertex \( u \) maps to \( e_1 + f_1. \) Then \( v_i \) maps to \( e_{i-1} + e_i \) and \( w_j \) maps to \( f_{j-1} + f_j. \)

Now consider the image of \( x_1. \) This may map to \( e_1 - e_2 + \cdots + e_{a-1} + g_1; \) then \( x_l \) maps to \( g_{l-1} + g_l \) for \( l > 1. \) Thus the image of \( Q \) is contained in a \( \mathbb{Z}^{p+q+r+2} \) sublattice. The determinant of \( Q \) is \( |k(Y)| = \alpha_2^2 \beta_1 + \alpha_2 + \alpha_3 \) (note \( k(Y) < 0). \) The orthogonal complement of \( Q \) in \( \mathbb{Z}^{p+q+r+2} \) is spanned by the vector \( \sum (-1)^{i-1} e_i + \sum (-1)^j f_j + \alpha_2 \sum (-1)^l g_l, \) whose square is \( |k(Y)|. \) Up to automorphism this is the only embedding
of $Q$ into $\mathbb{Z}^n$ unless $\alpha_3 = a$ and $\beta_1 = 1$. In this case $x_1$ may map to the alternating sum $f_1 - f_2 + \cdots \pm f_a$; the image of the resulting embedding is contained in $\mathbb{Z}^{p+q+r+1}$. 

**Example 4.9.** If $Y = Y(-1; (3, 1), (3a + 1, a), (5b + 3, 2b + 1))$ with $k(Y) < 0$, then the only negative-definite pairing that $Y$ may bound is the diagonal form $\Delta_{|k(Y)|}$ unless $a = b = 1$ in which case it may also bound $\Delta_1$.

**Proof.** Note that the condition $k(Y) < 0$ implies $a = 1$ or $b = 0$ or $a = b + 1 = 2$. Again, $Y$ is the boundary of a positive-definite plumbing as in Figure 11 with $p = a$, $q = b + 1$ and $r = 1$. The vertex $u$ has square 1, $w_1$ and $x_1$ have square 3, $v_1$ has square 4. If $a > 1$ then $v_j$ has square 2 for $j > 1$. If $b > 0$ then $w_2$ has square 3, and any remaining $w_i$ has square 2. Denote the pairing associated to this plumbing by $Q$. We consider an embedding of $Q$ into $\mathbb{Z}^n$. Let $e_i$ denote unit vectors in $\mathbb{Z}^n$. Without loss of generality the vertex $u$ maps to $e_1$, $x_1$ maps to $e_1 + e_2 + e_3$ and $w_1$ maps to $e_1 - e_2 + e_4$. Then $v_1$ has to map to $e_1 - e_3 - e_4 + e_5$. Now $w_2$, if present, has to map to $e_4 + e_5 + e_6$ or $-e_2 + e_3 + e_5$; the second possibility only works if $a = b = 1$. Finally $v_2$, if present, has to map to $e_5 - e_6$. The reader may verify that $Q$ is almost-rigid. 

**Example 4.10.** Let $Y_a = Y(-2; (2, 1), (3, 2), (a, a-1))$ with $a \geq 7$. Then $h = k(Y) = a - 6$,

$$\min_{t_0 \in \text{Spin}(Y)} d(Y, t_0) = (1 - h)/4$$

and

$$\max_{t \in \text{Spin}^+(Y)} d(Y, t) = \begin{cases} (1 - \frac{1}{h})/4 & \text{if } h \text{ is odd,} \\ 1/4 & \text{if } h \text{ is even.} \end{cases}$$

If $a$ is 7 or 9 then the only negative-definite form $Y_a$ bounds is $\Delta_h$. If Conjecture 4.1 holds then the same is true for all $Y_a$.

**Proof.** $Y_a$ is the boundary of the negative-definite plumbing with intersection pairing given by

$$Q = \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & -3 & 0 \\ 1 & 0 & 0 & -a \end{pmatrix},$$

which represents $3(-1) \oplus (-a + 6)$. The computations of $d(Y)$ follow as in [9]. The claim for $Y_7$ follows from the discussion following Theorem 1.1. The claim for $Y_9$ follows from Theorem 1.2.

**4.2. Four-ball genus of Montesinos knots.** Let $K$ be a knot in $S^3$ and let $g$ denote its Seifert genus. The four-ball genus $g^*$ of $K$ is the minimal genus of a smooth surface in $B^4$ with boundary $K$. A classical result of Murasugi states that $g^* \geq |\sigma|/2$, where $\sigma$ is the signature of $K$. If this lower bound is attained then the double
branched cover of $S^3$ along $K$ bounds a definite four-manifold with signature $\sigma$. The double branched cover of the Montesinos knot or link $M(e; (\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3))$ is $Y(-e; (\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3))$. (For more details see [7].)

The following generalises an example of Fintushel and Stern [4].

**Example 4.11.** The pretzel knot $K(p, q, r) = M(2; (p, 1), (q, q-1), (r, r-1))$ for odd $p$, $q$ and $r$ satisfying

$$q, r > p > 0 \quad \text{and} \quad pq + pr - qr \text{ is a square}$$

is algebraically slice but has $g^* = 1$.

**Proof.** The knot has a genus 1 Seifert surface yielding the Seifert matrix

$$M = \left( \begin{array}{cc} \frac{p-r}{2} & \frac{p+1}{2} \\ \frac{p-1}{2} & \frac{p-q}{2} \end{array} \right).$$

The vector $x = (p-l, r-p)$, where $l = \sqrt{pq + pr - qr}$, satisfies $x^T M x = 0$, demonstrating the knot is algebraically slice. The double branched cover $Y$ of the knot has $k(Y) = -l^2$. From Example 4.6 we see that $Y$ does not bound a rational homology ball. It follows that $0 < g^* \leq g = 1$.

It is shown by Livingston [6] that $K(p, q, r)$ has $\tau = 1$, where $\tau$ is the Ozsváth-Szabó knot concordance invariant. This also gives $g^* = 1$. \hfill $\square$

**Example 4.12.** The Montesinos knot $K_{q,r} = M(0; (qr-1, q), (r+1, r), (r+1, r))$ with odd $q \geq 3$ and even $r \geq 2$, has signature $\sigma = 1 - q$ and has

$$g = g^* = \frac{q+1}{2}.$$ Computations suggest that the Taylor invariant of $K_{q,r}$ is $\frac{q-1}{2}$.

**Proof.** The knot $K_{q,r}$ is equal to $M(0; (qr-1, q), (r+1, -1), (r+1, -1))$. It is easily seen that $K_{q,r}$ has a spanning surface with genus $\frac{q+1}{2}$. Using the resulting Seifert matrix one gets the formula for the signature. The double branched cover $Y$ of $K_{q,r}$ has $k(Y) < 0$. From Example 4.6 we see that $Y$ does not bound a negative-definite four-manifold; the genus formula follows.

We have computed the Taylor invariant of $K_{q,r}$ for $q < 10000$ and any $r$. \hfill $\square$

**Remark 4.13.** We have discussed Conjectures 2.1 and 2.2 with Noam Elkies. He has suggested an alternative proof of Theorem 3.1 using gluing of lattices [3]. His proof works for odd determinants $\delta$ up to 11, under the additional assumption that there is an element of $L'$ whose square is congruent to $1/\delta$ modulo 1. He also indicated a way to remove this assumption.

Using his result it follows that $\Delta_{a-6}$ is the only negative-definite form bounded by the manifold $Y_a$ of Example 4.10 for $a = 11, 13, 15$ and 17.
A CHARACTERISATION OF THE $n(1) \oplus (3)$ FORM

References

[1] S. K. Donaldson. *An application of gauge theory to four-dimensional topology*, J. Diff. Geom. *18* 1983, 279–315.

[2] N. D. Elkies. *A characterization of the $\mathbb{Z}^n$ lattice*, Math. Res. Lett. *2* 1995, 321–326.

[3] N. D. Elkies. *Private communication*.

[4] R. Fintushel & R. J. Stern, *Pseudofree orbifolds*, Ann. of Math. *122* 1985, 335–364.

[5] B. W. Jones. *The arithmetic theory of quadratic forms*, Math. Assoc. of America, 1950.

[6] C. Livingston. *Computations of the Ozsváth-Szabó knot concordance invariant*, math.GT/0311036.

[7] B. Owens & S. Strle. *Rational homology spheres and four-ball genus*, math.GT/0308073.

[8] P. Ozsváth & Z. Szabó. *Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary*, Advances in Mathematics *173* 2003, 179–261.

[9] P. Ozsváth & Z. Szabó. *On the Floer homology of plumbed three-manifolds*, Geometry and Topology *7* 2003, 225–254.

[10] J.-P. Serre. *A course in arithmetic*, Springer, 1973.