Method of the electron distribution function calculation in a problem of the kinetic rarefied plasma plume modeling

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Abstract. A problem related to the rarefied plasma plume of the stationary plasma thruster (SPT) is considered in the paper. The consideration is conducted fully in terms of kinetics, namely, distribution functions are introduced to describe motion of every plasma component. The system of kinetics equations for the distribution functions should be solved in combination with the Maxwell’s equations. The paper discusses methods for solving the stated problem.

Introduction
The most comprehensive solution of the SPT plume dynamics problem was considered in [1] and [2]. The geometry of such problem statement is shown in the figure 1.

Figure 1. The geometry of the SPT plasma plume problem
A thruster, out of the ring opening of which the ions and neutral particles are flowing into the environment, is modeled with an ABCDA1B1C1D1 parallelepiped. The electrons are emitted by the cathode, which is shown in the figure. The papers mentioned above include the results of previously made estimations that show, that the flowing out medium represents the quazineutral plasma, which requires a kinetic approach for the adequate description of its dynamics.

In the papers [1] and [2], the kinetic approach was used to describe motion of ions and neutrals, i.e. the kinetic equations for the presented distribution functions of the ions and neutrals definition were formulated. To describe the electron component, its motion equations were used, from which the expression for electric field was derived via the A.I. Morozov’s hypothesis of ‘thermalized potential’ [3]. This allowed completing the kinetic equations system for ions and neutrals.

Paper [1] presents also the development of a numerical method for solving the resultant kinetic equations system, and results of the conducted calculations are presented in [2]. The parallelepiped KLMNK1L1M1N1 (figure 1) represents the computational domain, where all calculations were made. It is obvious that they were made both upstream and downstream.

Here we propose to reject the ‘thermalized potential’ hypothesis and to describe the electron component in terms of kinetics.

1. The problem statement

Let’s assume that $f_i = f_i(t, \bar{x}, \bar{\xi})$ is the ion distribution function, where $\bar{\xi} \in D_i$ is the velocity space for ions; and $f_e = f_e(t, \bar{x}, \bar{v})$ is the electron distribution function, where $\bar{v} \in D_e$ is the velocity space for electrons. Therefore, $f_n = f_n(t, \bar{x}, \bar{w})$ is the neutral particle distribution function, where $\bar{w} \in D_n$ is the velocity space for neutrals. It is assumed, that the introduced distribution functions obey the following kinetic equations system:

$$\frac{\partial f_i}{\partial t} + \bar{v} \frac{\partial f_i}{\partial \bar{x}} = e \frac{\partial f_i}{m_i \partial \bar{\xi}} + e \frac{E_i}{m_i c} \frac{\partial f_i}{\partial \bar{\xi}} = \frac{1}{m_i c} \frac{\partial}{\partial \bar{\xi}} (e H_i \frac{\partial f_i}{\partial \bar{\xi}}) = J_{mi}$$

$$\frac{\partial f_e}{\partial t} + \bar{v} \frac{\partial f_e}{\partial \bar{x}} = e \frac{\partial f_e}{m_e \partial \bar{v}} - e \frac{E_e}{m_e c} \frac{\partial f_e}{\partial \bar{v}} = 0$$

$$\frac{\partial f_n}{\partial t} + \bar{w} \frac{\partial f_n}{\partial \bar{x}} = J_{mn}$$

(1)
The equation (1) utilizes the following nomenclature: \( \overline{H} = \{H^i\}, i = 1, 2, 3 \) is the given time-independent magnetic field. Then the electric field \( \overline{E} = \{E^i\} \) is to be a conservative one with the intensity \( E^i = -\frac{\partial \phi}{\partial x^i} \), where \( \phi = \phi(t, \overline{x}) \) is the electrostatic potential, which is defined by the Maxwell’s equation:

\[
\Delta \phi = 4\pi e(n^e - n^i) \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{2}
\]

In both equations (1, 2), \( e \) is an absolute value of the electron charge, \( c \) is the light velocity, \( m_i \) and \( m_e \) are the ion and electron masses, respectively, \( \varepsilon_{ijl} \) is the Levi-Civita symbol. The values \( n^e, n^i \) in equation (2) are the electron and ion densities, respectively, which are defined by the following formulas:

\[
n^e = \int_{\overline{v}} f_e d\overline{v}, \quad n^i = \int_{\overline{v}} f_i d\overline{v}. \tag{3}
\]

where \( \overline{j}_i = \int_{\overline{v}} \overline{\xi} f_i d\overline{\xi}, \quad \overline{j}_e = \int_{\overline{v}} \overline{\nu} f_e d\overline{\nu} \)
\[
\overline{\xi}_i = \frac{\overline{j}_i}{n_i}, \quad \overline{\xi}_e = \frac{\overline{j}_e}{n_e}.
\]

The expressions \( \frac{3}{2} kT = \frac{m}{2} \int_{\overline{v}} (\overline{\xi} - \overline{\nu})^2 f_e d\overline{v}, \quad \overline{\xi} = \frac{3}{2} kT^r = \frac{m}{2} \int_{\overline{v}} (\overline{\xi} - \overline{\nu}) f_e d\overline{v} \) define the ion and electron temperatures.

The right sides of the first and of the third equations of (1) are the integrals of collisions of ions and neutrals with neutrals and ions, correspondingly. They were defined by the following formulas:

\[
J_{in} = v_{in} f_n - v_{ni} f_i, \quad J_{in} = v_{ni} f_i - v_{in} f_n, \quad \text{here } v_{in}, v_{ni} \text{ are the frequencies of ion-neutral and neutral-ion collisions, respectively. The corresponding expressions for these values in a case of resonant charge-exchange reaction were derived in [2].}
\]

It is assumed, that the electrons in the thruster plume are moving in collisionless mode. Note that the formulas considered below summarize by repetitive indices. The indices \( i, e, n \) are introduced only to indicate that the corresponding value relates to ions, electrons or neutrals, regardless of whether the index is superscript or suffix. A.A. Vlasov proposed similar approach to plasma describing in [4].

To obtain the macroscopic equations for ions and electrons, it is needed to integrate the first and the second equations of system (1), each over its own velocity space, and then to multiply the first one by \( \frac{\partial}{\partial t} \), and the second one – by \( \frac{\partial}{\partial t} \); once done, they both are integrated again over their velocity spaces. These steps resulted in:

\[
\frac{\partial n_i}{\partial t} + \frac{\partial j_i^e}{\partial x} = 0, \quad \frac{\partial n^e}{\partial t} + \frac{\partial j^e_i}{\partial x} = 0, \quad \frac{\partial (j^e_i)}{\partial t} + \frac{\partial (j^e_i u^e_i)}{\partial x^i} + \frac{\partial P^i_{ue}}{\partial x^i} = \frac{e}{m^e} n^i E^i - \frac{e}{m^c} \varepsilon_{ai} f_i H^i = \langle \sigma_m g^m \rangle \langle j^i - j^e_i \rangle \tag{3}
\]
\[ \frac{\partial \left( j^s \right)}{\partial t} + \frac{\partial \left( j^s u_k \right)}{\partial x_k} + \frac{\partial P^s_{\perp}}{\partial x_k} + \frac{e}{m^*} n^* E^s + \frac{e}{m^* c^*} \varepsilon_{sli} j^s H^l = 0, s = 1, 2, 3 \]

Let’s make equations (1) and (2) dimensionless. The length scale \( a \) is taken to be equal to the half-width of the ABCD square (figure 1). The scale values of the velocity spaces are taken to be equal to the following values:

\[ \xi_0 = \sqrt{2kT_0^i / m_i}, \nu_0 = \sqrt{2kT_0^e / m_e}, \omega_0 = \sqrt{2kT_0^w / m_w}, \]

where \( T_0^i, T_0^e, T_0^w \) are the representative values of temperatures of respective components. Since \( m_e \ll m_i \), it is commonly fair that

\[ \nu_0 > \xi_0, \]

that is why one can choose \( t_0 = a / \nu_0 \) as the time scale. The self-consistent electric field in the plasma is defined by the motion of its components. And the scale value of the field is taken to be \( kT_e^w / e \).

Generally, the values \( f_i^0 = n_i / \xi_0^3, f_e^0 = n_e / \nu_0^3, f_w^0 = n_w / \omega_0^3 \) are accepted to be the representative values of the corresponding distribution functions. Here, \( n_0 \) is the scale value of ion and electron densities. It is the same for both components. The system (1) in the dimensionless form is as follows:

\[ S_i \frac{\partial f_i}{\partial t} + \xi_0^3 \frac{\partial f_i}{\partial \xi_0^3} - B \frac{\partial \phi}{\partial \xi_0^3} \frac{\partial f_i}{\partial \xi_0^3} + \frac{1}{R_i} \varepsilon_{sli} \xi_0^3 H^l \frac{\partial f_i}{\partial \xi_0^3} = \frac{1}{Kn_{in}} J_{in} \]

\[ \frac{\partial f_e}{\partial t} + \nu_0^3 \frac{\partial f_e}{\partial \nu_0^3} + \frac{\partial \phi}{\partial \nu_0^3} \frac{\partial f_e}{\partial \nu_0^3} - \frac{1}{R_i} \varepsilon_{sle} \nu_0^3 H^l \frac{\partial f_e}{\partial \nu_0^3} = 0 \]  

(4)

\[ Sh_i \frac{\partial f_i}{\partial t} + w_0^3 \frac{\partial f_i}{\partial w_0^3} = \frac{1}{Kn_{ei}} J_{ei} \]

In (4), \( S_i \frac{\nu_0}{\xi_0} = \sqrt{m_i T_0^i / m_e T_0^w} > 1 \) is an analogue of the Strouchal number for ions, \( S_{ei} = \frac{\nu_0}{w_0} \) is the same for neutrals, \( B = \frac{T_e^w}{T_0^w} \geq 1, R = r_i^l / L \), here \( r_i^l = m_i c_i / eH_0 \), \( s = i, e; c_i = \xi_0, c_e = \nu_0 \) are the Larmor radius for ions and electrons, respectively. In the plume problem, the electron Larmor radius is much less than that of the ion, therefore, it is possible to ignore the magnetic field component in the equation for ion. \( Kn_{in}, Kn_{ei} \) are the Knudsen numbers for ion-neutral and neutral-ion interactions, and the components in front of them are the dimensionless values of collision integrals. They are defined in [1] and [2].

Equation for the electric field potential is as follows

\[ \varepsilon^2 \Delta \phi = (n_e - n_i) \]  

(5)
The formula (5) is the Maxwell’s equation in a dimensionless form. Here \( \varepsilon = r_d / L \), with

\[
r_d = \sqrt{\frac{kT_0}{4\pi\varepsilon^2 n_0}}
\]

being the Debye length. In the plume problem, its magnitude is of the order of \( 10^{-3} \), i.e. it is very small.

2. Solution method

The small value of the Debye length stipulates difficulties in finding numerical solutions for the equations (4), (5). Indeed, assuming that the electric field in (4) in the previous time step is known (as it is in [1]), it becomes possible to define \( n', n'' \) by solving the system (4), and typically \( n' \neq n'' \). By substituting the found values in the right side of (5), we obtain very high magnitudes of electric field, which divides the charges in the next step of time in a way that \( n' - n'' \approx 0 \). As the new electric field is weak, this process will be repeated. Hence, the developed numerical scheme reproduces the Langmuir oscillations, which aren’t observed in the plume. That is why it is needed to develop such scheme, which would include the possibility for plasma oscillations damping.

Another, perhaps, fundamental difficulty in obtaining the solution of the previously presented problem concerns the absence of boundary conditions for the equation (5). To overcome the considered difficulties, the following scheme is proposed. Let’s set that \( N = \left( n' - n'' \right) \) and subtract the electron continuity equation from the similar equation for ions. The result is

\[
\frac{\partial N}{\partial t} + \frac{\partial j^e}{\partial x} = 0.
\]

Then, we subtract the momentum conservation equation for electrons from the corresponding equation for ions. After the appropriate manipulation the following equation is derived:

\[
\frac{\partial j^s}{\partial t} + \left( \sigma_n g^{nn} \right) j^s + e \frac{n'}{m'} \frac{\partial \phi}{\partial x} = R^s, s = 1, 2, 3
\]  

(6)

The value \( R^s \) comprises all components needed for the left hand side of (6) to be as it appeared after all the described processing. An immediate deriving of this value is not needed further. The value \( \frac{n'}{m'} \) in (6) exists, if in an accurate deriving \( m_e \ll m_i \) is accounted. By taking the divergence of left and right hand sides of (6) with due regard for (2) and transposing some components to the right-hand side, the following equation may be derived:

\[
\frac{\partial^2 N}{\partial t^2} + \left( \sigma_n g^{nn} \right) \frac{\partial N}{\partial t} + \frac{n'}{m'} e^2 N = Q
\]  

(7)

Here \( Q = \text{div} \vec{R} - \left( \text{grad} \left( \sigma_n g^{nn} \right) j^s - \left( \text{grade} \frac{n'}{m'} \text{grad} \phi \right) \right) \).
It is clear, that the formula (7) was deduced with direct utilizing of the continuity equations. Let us make the equation (7) dimensionless by using the previously introduced formulas. The following is obtained:

\[
\frac{\partial^2 N}{\partial t^2} + \frac{\nu(t, \vec{x})}{2\sqrt{\pi} \varepsilon K_m} \frac{\partial N}{\partial t} + \frac{n^\prime}{\varepsilon} N = Q
\]  

(8)

The presence of the small parameter \( \varepsilon \) in (8) indicates that the motion of the charged plasma components would proceed in a two-scale mode, and to examine it we employ the asymptotic methods, which were developed in [5].

Let us assume that \( N(t_0) \neq 0 \) Then during the plasma evolution in the time interval \( [t_0, t_0 + \Delta t] \), such time interval \( O(\varepsilon) \) exists, when the quasi-neutrality is violated. Let’s introduce ‘fast’ time \( \tau = \frac{t}{\varepsilon} \). The solution of (8) is to be found in the form of asymptotic series expansion \( N(\tau, \vec{x}) = \bar{N}(\tau, \vec{x}) + + \varepsilon N_1(\tau, \vec{x}) + \ldots \). The equation for \( \bar{N}(\tau, \vec{x}) \) will be as follows:

\[
\frac{\partial^2 \bar{N}}{\partial \tau^2} + \frac{\nu(t_0, \vec{x})}{2\sqrt{\pi} K_m} \frac{\partial \bar{N}}{\partial \tau} + n^\prime(\tau, \vec{x}) \bar{N} = \varepsilon^2 \bar{Q}(t_0, \vec{x}),
\]  

(9)

Let us set \( \alpha = \nu(t_0, \vec{x}) / 2\sqrt{\pi} K_m, \omega = \sqrt{n^\prime - \alpha^2 / 4} \).

Under \( n^\prime - \alpha^2 / 4 > 0 \), the solution of (9) is written as:

\[
\bar{N}(\tau, \vec{x}) = A e^{-\varepsilon \tau} \cos(\omega \tau + \varphi) + \frac{\varepsilon^2}{n^\prime(t_0, \vec{x})} Q(t_0, \vec{x})
\]  

(10)

The formula (10) describes the damping of plasma (Langmuir) oscillations in quick time mode. From the equation (10) the following may be derived:

\[
\bar{N}(t_0 + \Delta t, \vec{x}) = A e^{-\varepsilon \frac{\Delta t}{\varepsilon}} \cos\left(\omega \frac{\Delta t}{\varepsilon} + \varphi\right) + \frac{\varepsilon^2}{n^\prime(t_0, \vec{x})} Q(t_0, \vec{x}).
\]

When \( \frac{\Delta t}{\varepsilon} > 1 \), the oscillations will decay. And by the time moment \( t_0 + \Delta t \) the quasi-neutrality sustains. Then, for the definition of the electrostatic potential, the following equation could be utilized: \( \varphi(t_0 + \Delta t, \vec{x}) = -Q(t_0, \vec{x}) \). In this equation the small parameter is missing. The issue would concern the boundary conditions set. Let’s write the solution of the above equation in the following form:
\[
\phi(t_0 + \Delta t, \vec{x}) = \frac{1}{4\pi} \int \frac{Q(t_0, \vec{x})}{\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}} dy_1 + \phi_{ext}
\tag{11}
\]

The integration in (11) is performed over the whole computational domain. The first summand in (11) is a Newtonian potential (specific solution of Poisson equation), and the second one is the known preassigned potential. It is defined as \(\phi_{ext} = 0\). As a result, the following solution scheme for the stated problem is proposed. The electrostatic potential at time \(t_0 + \Delta t\) is defined via formula (11) based on the known values of the macro-variables at time moment \(t_0\). The obtained value is plugged into the kinetic equations system, from the solution of which all macro-variables at time moment \(t_0 + \Delta t\) are defined, and then the process is repeated. The idea of utilizing the conservation equations to solve the kinetic equations has been proposed earlier in [6].

3. Conclusion

A method is proposed, which allows to calculate the electron distribution function in the problem of plasma dynamics calculation as applied to the stationary plasma thruster plume based on describing the motion for every plasma particle with kinetic equations.

The application of a fully kinetic approach to the study of the plasma jet made it possible to obtain a system of conservation equations that govern this motion. The presence of distribution functions made it possible to consider this system as closed, which made it possible, by modernizing the method used in [7], to study qualitatively and quantitatively the process of establishing quasineutrality in a plasma medium, and to use the Maxwell equation directly to determine the electric field.

The principal point in the proposed work of the scheme is the presence of a system of kinetic equations describing the motion of charged and neutral components.

In the case of considering the problem of plasma motion in the thruster channel. The motion of the electrons will no longer be collisionless. We need to simulate the ionization process. There will also be a question about the boundary conditions for the electron distribution function on the walls of the thruster channel.

References

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