Black string instabilities in anti-de Sitter space

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Abstract

We show how to extend the usual black string instability of vacuum or charged black p-branes to the anti-de Sitter background. The string fragments in an analogous fashion to the $\Lambda = 0$ case, the main difference being that instead of a periodic array of black holes forming, an accumulation of “mini” black holes occurs towards the AdS horizon. In the case where the AdS space is of finite extent, such as an orbifold compactification, we show how the instability switches off below a certain compactification scale.

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It has been known for some time that the extended event horizon of a black string or $p$-brane in higher-dimensional gravity is unstable to fragmentation into black holes \[1\], and that this instability extends to a much broader range of charged black holes in string theory – the only exception being extremal solutions \[2\]. There are several ways of understanding this instability intuitively: the event horizon of a black hole can be thought of as analogous to a soap bubble – it has a ‘tension’ ($\kappa$, the surface gravity), and one therefore expects a cylindrical horizon to cease being a stable solution once its length reaches some critical value. Another (related) quantum description is that we can associate an entropy to an event horizon, proportional to its area, and a simple calculation shows that for a cylindrical horizon, there exists a length above which it becomes entropically favourable for the mass to localise in a spherical black hole. These simple arguments can be backed up by a full perturbative analysis of the spacetime of the extended solution, where it is straightforward (if calculationally involved) to show that there exists a transverse-tracefree (TTF) metric perturbation, which is a pure tensor mode in the spacetime orthogonal to the string or brane \[1,2\].

These instability arguments were made purely within the context of asymptotically locally flat spacetimes – in other words, in the absence of a cosmological constant. Recent interest on the other hand has focussed on backgrounds which are not vacuum, but rather negative false vacuum or anti-de Sitter (AdS) spacetimes. Such backgrounds arise in the near-horizon limit of certain D-branes, and the general results of \[1,2\] have been used for example in the context of localisation of AdS$_5$ black holes on the $S^5$ component of the near horizon geometry of the D3-brane \[3\].

A particularly exciting application of AdS spacetime to the ‘real world’ has arisen more recently in the context of superstring cosmological scenarios resulting from the Horava-Witten \[4\] compactification of M-theory. In this case, spacetime is effectively five-dimensional below the Grand Unified scale, \[5\], and our universe arises as a domain wall, or brane, at its boundary. The idea of our universe as a defect in higher dimensions is not new (see e.g. \[6\]), however, Randall and Sundrum \[7\] emphasized out the exciting phenomenological implications of this set-up – not only is gravity essentially four-dimensional at reasonably large length scales, but there is also the possibility of observing fundamental spin-2 particles at LHC (originally pointed out in \[8\]). It is this former feature which is relevant to the current discussion.

If our universe is to be regarded as a domain wall in five-dimensional AdS spacetime, a natural question is what is the gravitational field of a test particle in our universe? In linearized theory, the metric on the wall was shown to be exactly what one would obtain from four-dimensional Einstein gravity, modulo short range corrections due to the massive Kaluza-Klein (KK) graviton modes \[9\]. However, it is obviously of interest to determine the full non-perturbative five-dimensional ‘vacuum’ solution corresponding to a black hole in the brane. This question was explored in \[10\] (see also \[11\]), where the instability of the Einstein black string was used to argue a ‘cigar’ geometry of the event horizon of such a black hole.

Five dimensional AdS spacetime can be written in horospherical coordinates as:

$$ds^2 = e^{-2kz} [\eta_{\mu\nu} dx^\mu dx^\nu] - dz^2$$  \hspace{1cm} (1)

where the cosmological constant is $-6k^2$. The Randall-Sundrum spacetime simply places
a wall of tension $6k$ at $z = 0$, and a second wall of tension $-6k$, if required, at some $z = z_c$. This has the effect of making the metric reflection symmetric around $z = 0$ and $z = z_c$. In either case, it is straightforward to construct the black string, since the flat metric $\eta_{\mu\nu}$ can be replaced by any Ricci-flat metric $g_{\mu\nu}$, for example the four-dimensional Schwarzschild solution. The properties of this Randall Sundrum (RS) black string were explored by Chamblin et. al. in [10]. In particular, it was argued that the Einstein black string instability would come into play near the AdS horizon indicating a cigar type of event horizon. On the other hand, an alternative approach put forward by Emparan et. al. [12], in which the C-metric is used to construct an exact black hole on a 2+1 dimensional universe at the boundary of AdS$_4$, appears to indicate that the horizon of the black hole will appear more as a pancake localised on the wall, indeed these authors used the Einstein black string instability to argue this time the pancake geometry in higher dimensions! Clearly, since the arguments used in these papers involved an application of the instability in a situation other than that in which it was derived, a correct calculation would be useful in such a discussion.

Let us start by recalling the instability for the black string in Einstein gravity. The metric of the string is given by simply adding an extra flat direction to the Schwarzschild metric:

$$ds^2 = \left(1 - \frac{2GM}{r}\right)dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1}dr^2 - r^2d\Omega^2_{II} - dz^2$$ (2)

Writing the perturbation of the metric in the usual fashion as $g_{ab} \rightarrow g_{ab} + h_{ab}$, and choosing the TTF gauge for $h_{ab}$ (i.e. $h_{a}^{a} = 0 = h_{b}^{b}$), then the perturbation $h_{ab}$ satisfies the Lichnerowicz equation:

$$\Delta_L h_{ab} = \left[\delta^c_a \delta^d_b \Box + 2R^c_{a}{}^{d}{}^{b}\right] h_{cd} = 0$$ (3)

It was shown in [1] that this could be interpreted in a Kaluza-Klein spirit as a four-dimensional problem, with the perturbation splitting into scalar, vector and tensor modes. The scalar and vector modes were shown to be zero for any unstable mode, and so the problem reduced to a four-dimensional tensor perturbation

$$\left(\Delta_L^{(4)} + m^2\right) h_{\mu\nu} = 0$$ (4)

where the $z$-dependence of the perturbation $e^{imz}$ simply introduces a mass term for the tensor mode $h_{\mu\nu}$. Since all pure gauge perturbations satisfy the massless equation, any solution to the above massive Lichnerowicz equation must therefore be physical and correspond to an instability of the spacetime.

A single s-wave mode was found in [1] which had the form

$$h_{\mu\nu} = e^{\sigma t} \begin{bmatrix} h_0 & h_1 & 0 & 0 \\ h_1 & h_2 & 0 & 0 \\ 0 & 0 & K & 0 \\ 0 & 0 & 0 & K \sin^2 \theta \end{bmatrix}$$ (5)

Where $h_0, h_1, h_2$ and $K$ are all related via the TTF gauge conditions, and $h_{\mu\nu}$ has the interpretation of a longitudinal graviton mode which perturbs the event horizon, making it ripple with a characteristic wavelength leading to the interpretation of the instability as
being a process which leads ultimately to the fragmentation of the cylindrical event horizon into a line, or array, of black holes.

To sum up, the key (mathematical) features of this instability are that i) it has the form of a tensor perturbation from the four-dimensional point of view; ii) this tensor mode satisfies the TTF gauge conditions\(^1\) and iii) it satisfies a massive four-dimensional Lichnerowicz equation.

Now let us consider the black string in AdS spacetime, which has the metric

\[
 ds^2 = a^2(z) \left[ \left( 1 - \frac{2GM}{r} \right) dt^2 - \left( 1 - \frac{2GM}{r} \right)^{-1} dr^2 - r^2 d\Omega_2^2 \right] - dz^2 \quad (6)
\]

The function \(a(z)\) has been left general in this expression to allow for the simultaneous consideration of instabilities of pure AdS black strings, \(a(z) = e^{\pm kz}\), or the RS black string, \(a(z) = e^{-k|z|}\) (which may or may not have an additional negative tension domain wall at some other value of \(z\)). Note that although this looks from the four-dimensional point of view like a Schwarzschild black hole of mass \(M\), the \(r\)-coordinate is really a ‘comoving’ coordinate, and the ‘ADM’ mass as measured by an observer in a particular \(z_0\) plane is in fact \(M_0 = Me^{-kz_0}\). The spacetime becomes singular as the AdS horizon is approached \([10]\).

Clearly, the only difference between the metrics (6) and (2) is the conformal or warp factor in front of the four-dimensional Schwarzschild part. This is crucial however, in that this warp factor represents the introduction of the negative cosmological constant, and is responsible for the \(de\ facto\) compactification of the spacetime (even though the extra dimension has infinite extent) in the single domain wall universe as explored by Squires and Visser \([13]\). Unlike the standard Kaluza-Klein compactification of spacetime, in which the effective theory is that of coupled massless spin two, one and zero particles, together with a tower of massive spin two states, the ‘exotic’ compactification induced by the AdS spacetime, appears to leave only a massless spin two excitation, the four-dimensional Einstein graviton, and a continuum of massive KK states. We might therefore expect that the instability argument would be severely modified, and calculationally a great deal more involved, nevertheless, for the moment let us proceed innocently\(^2\), and regard this warp factor as simply representing the exotic compactification of spacetime, and look for an tensor instability of a similar form to the vacuum one.

Writing \(R_{\mu\nu\lambda\rho}^{(4)}\) for the Riemann tensor of the four-dimensional Schwarzschild metric, and taking a RS (four-dimensional TTF tensor) perturbation, we find that the perturbation equations for the metric (6) reduce to

\[
 a^{-2} \left( \Box^{(4)} h_{\mu\nu} + 2R_{\mu\lambda\nu\rho}^{(4)} h^{\lambda\rho} \right) - h''_{\mu\nu} + 2 \left( \frac{a''}{a} + \left( \frac{a'}{a} \right)^2 \right) h_{\mu\nu} = 0 \quad (7)
\]

\(^1\)In the context of the Randall-Sundrum braneworld scenario, this gauge, which is simultaneously GN (Gaussian-Normal : \(g_{zz} = 1, g_{z\mu} = 0\)) and TTF, is often called the RS gauge.

\(^2\)If we wish to be less innocent, we can write the perturbation equations in a GN but not TTF gauge, and demonstrate directly that in fact the RS gauge suffices on-shell, since in the absence of additional matter perturbations, the solutions for the trace and divergence of well-behaved perturbations must vanish.
where the greek spacetime indices have been raised by the four-dimensional Schwarzschild metric without the warp factor, and the wave operator similarly represents the four-dimensional Schwarzschild wave operator.

Now, setting $h_{\mu\nu} = \chi_{\mu\nu} u_m(z)$, where

$$u_m(z) = \mathcal{A} J_2 \left( \frac{m}{k} e^{kz} \right) - \mathcal{B} N_2 \left( \frac{m}{k} e^{kz} \right)$$

with the coefficients $\mathcal{A}$ and $\mathcal{B}$ being chosen so that the perturbation satisfies the required boundary conditions for either a pure AdS black string ($\mathcal{B} = 0$), or a RS black string ($\mathcal{A} J_1 \left( \frac{m}{k} \right) = \mathcal{B} N_1 \left( \frac{m}{k} \right)$). Then in either case

$$-h''_{\mu\nu} + 2 \left( \frac{a''}{a} + \left( \frac{a'}{a} \right)^2 \right) h_{\mu\nu} = a^{-2} m^2 h_{\mu\nu}$$

and hence $\chi_{\mu\nu}$ satisfies the equation of motion

$$\left( \Delta^{(4)}_L + m^2 \right) \chi_{\mu\nu} = 0$$

where the $z$-dependence of $u_m(z)$, has simply introduced a mass term for $\chi_{\mu\nu}$, in direct analogy to the flat extra dimension case ($[\text{?}]$). Therefore we can simply read off the solution for $\chi_{\mu\nu}$ from $[\text{?}]$, and see the form of the instability. For $GM = 1$, the instability exists in the range $0 < m < 0.45$, with the most favoured instability (i.e. the one with the shortest half-life) having $m \simeq 0.2$. For general $M$, the perturbation mass varies as $M^{-1}$. 

FIG. 1. The event horizon of the perturbed AdS black string in horospherical coordinates.
An image of the effect of the instability on the horizon of the pure AdS string is shown in figure 1, and for the Randall-Sundrum string in figure 2. Note that in this representation of the instability, we are in ‘comoving’ coordinates, i.e. the unperturbed black string looks cylindrical. Notice how in each case the instability accumulates towards the AdS horizon $z \to \infty$. From the asymptotics of the Bessel functions, we see that successive zeros of $h_{\mu\nu}$ are separated by $\pi e^{-kz}/m$, and hence the ‘mass’ of black string contained in these seedling black holes decreases roughly as $M_s \sim \pi kMe^{-2kz}/m$. Thus we see that the instability, if it proceeds into fragmentation of the black string, will generate an accumulation of mini-black holes towards the AdS horizon. Although such a conclusion seems at first alarming, in that Planck scale black holes will rapidly dominate, it is only fair to point out that since the original black string solution is singular at the AdS horizon [10], perhaps this pathological behaviour of the instability is simply mirroring this fact. In any case, we can always introduce a regulator brane at some finite $z_c$ (see below) to act as a cut-off for such singular behaviour.

Clearly the instability differs for lower values of $z$, where the wall of the Randall-Sundrum string makes its presence felt. For the pure AdS string, the instability is suppressed in the region $z < 0$, or near the boundary of AdS spacetime. This corresponds to the region in which the curvature of the AdS spacetime is large compared to the Schwarzschild curvature. For the RS spacetime, the first node of the instability occurs at a proper distance $z_n$ of approximately 2 from the wall for $GM = k = 1$ and $m = m_{\text{max}} = 0.45$, indicating the typical scale of the central black hole after fragmentation. Since the horizon of the black holes in these units is 2, this seems to indicate neither a pancake geometry nor a cigar one.
for black strings with masses on a similar scale to the AdS curvature (as might be expected). We can however use the asymptotics of the Bessel functions to explore the possible geometry of black holes on the brane in the asymptotic régimes of very large and very small masses. For \( m/k \ll 1 \), this first node of the instability occurs for \( z \simeq -\frac{1}{k} \ln(\frac{m}{k}) \), i.e. \( z_n \propto (\ln M)/k \) for \( M \gg 1 \), which indicates a ‘stottie’ shape for the event horizon. On the other hand, for \( m/k \gg 1 \) we obtain \( z \propto \pi/m \), or \( z \propto M \). This indicates that very small mass black holes are roughly hyperspherical, which is approximately what we might expect since the black hole is at a scale where the AdS curvature is negligible, and we might expect it to look like a five-dimensional black hole. These results are in accord with the exact 2+1-dimensional calculations of [12], as well as with an analysis of the propagator off the brane in [14].

It is interesting to explore what happens if we have a finite fifth dimension, as in the original Randall-Sundrum scenario. In this case, the presence of a second wall at \( z_c \) introduces an additional restriction on the eigenfunctions \( u_m(z) \):

\[
h'_m(z_c) + 2k u_m(z_c) = 0 \quad \Rightarrow \quad J_1 \left( \frac{m}{k} e^{kz_c} \right) N_1 \left( \frac{m}{k} \right) = J_1 \left( \frac{m}{k} \right) N_1 \left( \frac{m}{k} e^{kz_c} \right)
\]

Clearly, if \( kz_c \) is not sufficiently large, there will be no allowed value of \( m \) for which the eigenfunction \( u_m \) exists. For small \( m \), \( kz \propto \ln(\frac{k}{m}) \), and hence for large mass black holes, \( e^{kz_c} \geq 2kGM \) for the existence of the instability.

We can also straightforwardly derive the form of the instability for the quasi-localized gravity model of [15], where the central positive tension wall is flanked by two negative tension walls at \( z = z_c \) in such a fashion that the spacetime is Minkowskian outside the system. The eigenfunctions have the form of (8) inbetween the walls, and the conventional oscillatory form in the exterior flat spacetime. In this case, for \( M \ll e^{kz_c}/k \) the instability proceeds primarily as for the RS string, giving a picture much like figure 4, but for \( M \gg e^{kz_c}/k \), the first node of the instability lies outside the wall system, and we have a central, slightly squashed, black hole forming on the walls, with the usual line of five-dimensional black holes extending off to infinity.

Finally, we note that although we have focussed, mainly for the purposes of clarity, on the case of a black string in AdS$_5$ (or a slice thereof), it is clear how to generalize this to black strings, or indeed branes, in higher dimensional AdS spacetimes. The key feature required is a block diagonal form of the metric, which may or may not depend nontrivially on the extra dimensions. For example, a six-dimensional model in which the universe is a ‘vortex’ [16] (see also [17]), rather than a wall, can be used to analyse the instability of a black membrane. In this case, the analogues of the \( u_m \)'s have been investigated in [18]. Similarly, for higher dimensional branes, the smooth solutions of Olasagasti and Vilenkin [19] can be used to model the behaviour of the massive graviton modes.

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3 A regional bread, of a similar shape to the pancake or tortilla, but somewhat thicker.
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