On some crystalline representations of
$GL_2(\mathbb{Q}_p)$

Vytautas Paškūnas

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Abstract

We show that the universal unitary completion of certain locally algebraic representation of $G := GL_2(\mathbb{Q}_p)$ with $p > 2$ is non-zero, topologically irreducible, admissible and corresponds to a 2-dimensional crystalline representation with non-semisimple Frobenius via the $p$-adic Langlands correspondence for $G$.

1 Introduction

Let $G := GL_2(\mathbb{Q}_p)$ and $B$ be the subgroup of upper-triangular matrices in $G$. Let $L$ be a finite extension of $\mathbb{Q}_p$.

Theorem 1.1. Assume that $p > 2$, let $k \geq 2$ be an integer and let $\chi : \mathbb{Q}_p^\times \rightarrow L^\times$ a smooth character with $\chi(p)^2p^{k-1} \in o_L^\times$. Assume that there exists a $G$-invariant norm $\|\cdot\|$ on $(\text{Ind}_B^G \chi \otimes |\cdot|^{-1}) \otimes \text{Sym}^{k-2} L^2$. Then the completion $E$ is a topologically irreducible, admissible Banach space representation of $G$. Moreover, if we let $E^0$ be the unit ball in $E$ then

$$V_{k,2\chi(p)^{-1}} \otimes (\chi|\chi|) \cong L \otimes_{o_L} \varinjlim V(E^0/\varpi^n E^0),$$

where $V$ is Colmez’s Montreal functor, and $V_{k,2\chi(p)^{-1}}$, is a 2-dimensional irreducible crystalline representation of $G_{\mathbb{Q}_p}$ the absolute Galois group of $\mathbb{Q}_p$, with Hodge-Tate weights $(0,k-1)$ and the trace of crystalline Frobenius equal to $2\chi(p)^{-1}$.
As we explain in [5], the existence of such $G$-invariant norm follows from the recent work of Colmez, [6]. Our result addresses Remarque 5.3.5 in [3]. In other words, the completion $E$ fits into the $p$-adic Langlands correspondence for $GL_2(\mathbb{Q}_p)$.

The idea is to “approximate” $(\text{Ind}_B^G \chi \otimes \chi | \cdot |^{-1}) \otimes \text{Sym}^{k-2} L^2$ with representations $(\text{Ind}_B^G \chi \delta_x \otimes \chi \delta_{x-1} | \cdot |^{-1}) \otimes \text{Sym}^{k-2} L^2$, where $\delta_x : \mathbb{Q}_p^\times \rightarrow L^\times$ is an unramified character with $\delta_x(p) = x \in 1 + pL$. If $x^2 \neq 1$ then $\chi \delta_x \neq \chi \delta_{x-1}$ and the analog of Theorem 1.1 is a result of Berger-Breuil [3]. This allows to deduce admissibility. This “approximation” process relies on the results of Vignéras [14]. Using Colmez’s functor $V$ we may then transfer the question of irreducibility to the Galois side. Here, we use the fact that for $p > 2$ the representation $V_{k,\pm 2p(k-1)/2}$ sits in the $p$-adic family studied by Berger-Li-Zhu in [2].

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2 Notation

We fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. We let $\text{val}$ be the valuation on $\overline{\mathbb{Q}}_p$ such that $\text{val}(p) = 1$, and we set $|x| := p^{-\text{val}(x)}$. Let $L$ be a finite extension of $\mathbb{Q}_p$ contained in $\overline{\mathbb{Q}}_p$, $\mathfrak{o}_L$ the ring of integers of $L$, $\varpi_L$ a uniformizer, and $\mathfrak{p}_L$ the maximal ideal of $\mathfrak{o}_L$. Given a character $\chi : \mathbb{Q}_p^\times \rightarrow L^\times$ we consider $\chi$ as a character of the absolute Galois group $G_{\mathbb{Q}_p}$ of $\mathbb{Q}_p$ via the local class field theory by sending the geometric Frobenius to $p$.

Let $G := GL_2(\mathbb{Q}_p)$, $B$ the subgroup of upper-triangular matrices. Given two characters $\chi_1, \chi_2 : \mathbb{Q}_p^\times \rightarrow L^\times$ we consider $\chi_1 \otimes \chi_2$ as a character of $B$, which sends a matrix $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ to $\chi_1(a) \chi_2(d)$. Let $Z$ be the centre of $G$, $K := GL_2(\mathbb{Z}_p)$, $I := (\begin{smallmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p^\times \end{smallmatrix})$ and for $m \geq 1$ we define

$$K_m := \begin{pmatrix} 1 + p^m \mathbb{Z}_p & p^m \mathbb{Z}_p \\ p^m \mathbb{Z}_p & 1 + p^m \mathbb{Z}_p \end{pmatrix}, \quad I_m := \begin{pmatrix} 1 + p^m \mathbb{Z}_p & p^{m-1} \mathbb{Z}_p \\ p^m \mathbb{Z}_p & 1 + p^m \mathbb{Z}_p \end{pmatrix}.$$ 

Let $\mathfrak{H}_0$ be the $G$-normalizer of $K$, so that $\mathfrak{H}_0 = KZ$, and $\mathfrak{H}_1$ the $G$-normalizer of $I$, so that $\mathfrak{H}_1$ is generated as a group by $I$ and $\Pi := (\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix})$. We note that
if \( m \geq 1 \) then \( K_m \) is normal in \( \mathfrak{K}_0 \) and \( I_m \) is normal in \( \mathfrak{K}_1 \). We denote 
\[ s := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]

3 Diagrams

Let \( R \) be a commutative ring, (typically \( R = L, \mathfrak{o}_L \) or \( \mathfrak{o}_L/\mathfrak{p}^n_L \)). By a diagram \( D \) of \( R \)-modules, we mean the data \((D_0, D_1, r)\), where \( D_0 \) is a \( R[\mathfrak{K}_0] \)-module, \( D_1 \) is \( R[\mathfrak{K}_1] \)-module and \( r : D_1 \to D_0 \) is a \( \mathfrak{K}_0 \cap \mathfrak{K}_1 = IZ \)-equivariant homomorphism of \( R \)-modules. A morphism \( \alpha \) between two diagrams \( D, D' \) is given by \((\alpha_0, \alpha_1)\), where \( \alpha_0 : D_0 \to D'_0 \) is a morphism of \( R[\mathfrak{K}_0] \)-modules, \( \alpha_1 : D_1 \to D'_1 \) is a morphism of \( R[\mathfrak{K}_1] \)-modules, and the diagram

\[
\begin{array}{ccc}
D_0 & \xrightarrow{\alpha_0} & D'_0 \\
\downarrow r & & \downarrow r' \\
D_1 & \xrightarrow{\alpha_1} & D'_1
\end{array}
\]

commutes in the category of \( R[IZ] \)-modules. The condition (1) is important, since one can have two diagrams of \( R \)-modules \( D \) and \( D' \), such that \( D_0 \cong D'_0 \) as \( R[\mathfrak{K}_0] \)-modules, \( D_1 \cong D'_1 \) as \( R[\mathfrak{K}_1] \)-modules, however \( D \not\cong D' \) as diagrams. The diagrams of \( R \)-modules with the above morphisms form an abelian category. To a diagram \( D \) one may associate a complex of \( G \)-representations:

\[
cInd_{\mathfrak{K}_1}^G D_1 \otimes \delta \xrightarrow{\partial} cInd_{\mathfrak{K}_0}^G D_0,
\]

where \( \delta : \mathfrak{K}_1 \to R^\times \) is the character \( \delta(g) := (-1)^{\text{val} \det g} \); \( cInd_{\mathfrak{K}_1}^G D_1 \) denotes the space of functions \( f : G \to D_1 \), such that \( f(kg) = k f(g) \), for \( k \in \mathfrak{K}_1 \) and \( g \in G \), and \( f \) is supported only on finitely many cosets \( \mathfrak{K}_1 g \). To describe \( \partial \), we note that Frobenius reciprocity gives \( \text{Hom}_G(cInd_{\mathfrak{K}_1}^G D_1 \otimes \delta, cInd_{\mathfrak{K}_0}^G D_0) \cong \text{Hom}_{\mathfrak{K}_1}(D_1 \otimes \delta, cInd_{\mathfrak{K}_0}^G D_0) \), now \( \text{Ind}_{IZ}^G D_0 \) is a direct summand of the restriction of \( cInd_{\mathfrak{K}_0}^G D_0 \) to \( \mathfrak{K}_1 \), and \( \text{Hom}_{\mathfrak{K}_1}(D_1 \otimes \delta, \text{Ind}_{IZ}^G D_0) \cong \text{Hom}_{IZ}(D_1, D_0) \), since \( \delta \) is trivial on \( IZ \). Composition of the above maps yields a map \( \text{Hom}_{IZ}(D_1, D_0) \to \text{Hom}_G(cInd_{\mathfrak{K}_1}^G D_1 \otimes \delta, cInd_{\mathfrak{K}_0}^G D_0) \), we let \( \partial \) be the image of \( r \). We define \( H_0(D) \) to be the cokernel of \( \partial \) and \( H_1(D) \) to be the kernel of \( \partial \). So we have an exact sequence of \( G \)-representations:

\[
0 \to H_1(D) \to cInd_{\mathfrak{K}_1}^G D_1 \otimes \delta \xrightarrow{\partial} cInd_{\mathfrak{K}_0}^G D_0 \to H_0(D) \to 0
\]

Further, if \( r \) is injective then one may show that \( H_1(D) = 0 \), see [14] Prop. 0.1. To a diagram \( D \) one may associate a \( G \)-equivariant coefficient system
\(V\) of \(R\)-modules on the Bruhat-Tits tree, see [9, §5], then \(H_0(D)\) and \(H_1(D)\) compute the homology of the coefficient system \(V\) and the map \(\partial\) has a natural interpretation. Assume that \(R = L\) (or any field of characteristic 0), and let \(\pi\) be a smooth irreducible representation of \(G\) on an \(L\)-vector space, so that for all \(v \in \pi\) the subgroup \(\{g \in G : gv = v\}\) is open in \(G\). Since the action of \(G\) is smooth there exists an \(m \geq 0\) such that \(\pi^{l_m} \neq 0\). To \(\pi\) we may associate a diagram \(D := (\pi^{l_m} \hookrightarrow \pi^{K_m})\). As a very special case of a result by Schneider and Stuhler [12, Thm V.1], [11, §3], we obtain that \(H_0(D) \cong \pi\).

We are going to compute such diagrams \(D\), attached to smooth principal series representations of \(G\) on \(L\)-vector spaces. Given smooth characters \(\theta_1, \theta_2 : \mathbb{Z}_p^\times \to L^\times\) and \(\lambda_1, \lambda_2 \in L^\times\) we define a diagram \(D(\lambda_1, \lambda_2, \theta_1, \theta_2)\) as follows. Let \(c \geq 1\) be an integer, such that \(\theta_1\) and \(\theta_2\) are trivial on \(1 + p^n\mathbb{Z}_p\).

We set \(J_c := (K \cap B)K_c = (I \cap B)K_c\), so that \(J_c\) is a subgroup of \(I\). We let \(\theta : J_c \to L^\times\) be the character:

\[
\theta\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) := \theta_1(a)\theta_2(d).
\]

We let \(D_0 := \text{Ind}_K^G \theta\), and we let \(p \in \mathbb{Z}\) act on \(D_0\) by a scalar \(\lambda_1\lambda_2\), so that \(D_0\) is a representation of \(\mathfrak{R}_0\). We set \(D_1 := D_0^l\) so that \(D_1\) is naturally a representation of \(IZ\). We are going to put an action of \(\Pi\) on \(D_1\), so that \(D_1\) is a representation of \(\mathfrak{R}_1\). Let

\[
V_1 := \{f \in D_1 : \text{Supp } f \subseteq I\}, \quad V_s := \{f \in D_1 : \text{Supp } f \subseteq J_c sI\}. \quad (4)
\]

Since \(I\) contains \(K\) we have \(J_c sI = (B \cap K)sI = IsI\), hence \(D_1 = V_1 \oplus V_s\).

For all \(f_1 \in V_1\) and \(f_s \in V_s\), we define \(\Pi \cdot f_1 \in V_s\) and \(\Pi \cdot f_s \in V_1\) such that

\[
[\Pi \cdot f_1](sg) := \lambda_1 f_1(\Pi^{-1} g )\Pi, \quad [\Pi \cdot f_s](g) = \lambda_2 f_s(s \Pi g \Pi^{-1}), \quad \forall g \in I; \quad (5)
\]

Every \(f \in D_1\) can be written uniquely as \(f = f_1 + f_s\), with \(f_1 \in V_1\) and \(f_s \in V_s\), and we define \(\Pi \cdot f := \Pi \cdot f_1 + \Pi \cdot f_s\).

**Lemma 3.1.** The equation (5) defines an action of \(\mathfrak{R}_1\) on \(D_1\). We denote the diagram \(D_1 \hookrightarrow D_0\) by \(D(\lambda_1, \lambda_2, \theta_1, \theta_2)\). Moreover, let \(\pi := \text{Ind}_K^G \chi_1 \otimes \chi_2\) be a smooth principal series representation of \(G\), with \(\chi_1(p) = \lambda_1, \chi_2(p) = \lambda_2, \chi_1|_{\mathbb{Z}_p^\times} = \theta_1\) and \(\chi_2|_{\mathbb{Z}_p^\times} = \theta_2\). There exists an isomorphism of diagrams \(D(\lambda_1, \lambda_2, \theta_1, \theta_2) \cong (\pi^{l_c} \hookrightarrow \pi^{K_c})\). In particular, we have a \(G\)-equivariant isomorphism \(H_0(D(\lambda_1, \lambda_2, \theta_1, \theta_2)) \cong \pi\).

**Proof.** We note that \(p \in \mathbb{Z}\) acts on \(\pi\) by a scalar \(\lambda_1\lambda_2\). Since \(G = BK\), we have \(\pi|_K \cong \text{Ind}_{B \cap K}^G \theta\), and so the map \(f \mapsto [g \mapsto f(g)]\) induces an
isomorphism \( \iota_0 : \pi^{K_c} \cong \text{Ind}_{N}^{G} \theta = D_0 \). Let \( F_1 := \{ f \in \pi : \text{Supp} f \subseteq BI \} \) and \( F_s := \{ f \in \pi : \text{Supp} f \subseteq BsI \} \). Iwasawa decomposition gives \( G = BI \cup BsI \), hence \( \pi = F_1 \oplus F_s \). If \( f_1 \in F_1 \) then Supp(\(\Pi f_1\)) = (Supp \(f_1\))\(\Pi^{-1} = BI\Pi^{-1} = BsI\). Moreover,

\[
[\Pi f_1](sg) = f_1(sg\Pi) = f_1(s\Pi(\Pi^{-1}g\Pi)) = \chi_1(p)f_1(\Pi^{-1}g\Pi), \quad \forall g \in I \tag{6}
\]

Similarly, if \( f_s \in F_s \) then Supp(\(\Pi f_s\)) = (Supp \(f_s\))\(\Pi^{-1} = BsI\Pi^{-1} = BI\), and

\[
[\Pi f_s](g) = f_1(g\Pi) = f_1((\Pi)s(\Pi^{-1}g\Pi)) = \chi_2(p)f_s(s(\Pi^{-1}g\Pi)), \quad \forall g \in I \tag{7}
\]

Now \( \pi^{L_c} = F_1^{L_c} \oplus F_s^{L_c} \subset \pi^{K_c} \). Let \( \iota_1 \) be the restriction of \( \iota_0 \) to \( \pi^{L_c} \) then it is immediate that \( \iota_1(F_1^{L_c}) = V_1 \) and \( \iota_1(F_s^{L_c}) = V_s \), where \( V_1 \) and \( V_s \) are as above. Moreover, if \( f \in D_1 \) and \( \Pi \cdot f \) is given by \( \overline{5} \) then \( \Pi \cdot f = \iota_1(\Pi_{1^{-1}}(f)) \). Since \( \mathfrak{R}_1 \) acts on \( \pi^{L_c} \), we get that \( \overline{5} \) defines an action of \( \mathfrak{R}_1 \) on \( D_1 \), such that \( \iota_1 \) is \( \mathfrak{R}_1 \)-equivariant. Hence, \( (\iota_0, \iota_1) \) is an isomorphism of diagrams \( (\pi^{L_c} \hookrightarrow \pi^{K_c}) \cong (D_1 \hookrightarrow D_0) \).

\textbf{4 Main result}

In this section we prove the main result.

\textbf{Lemma 4.1.} Let \( U \) be a finite dimensional \( L \)-vector space with subspaces \( U_1, U_2 \) such that \( U = U_1 \oplus U_2 \). For \( x \in L \) define a map \( \phi_x : U \rightarrow U \), \( \phi_x(v_1 + v_2) = xv_1 + v_2 \), for all \( v_1 \in U_1 \) and \( v_2 \in U_2 \). Let \( M \) be an \( \mathfrak{a}_L \)-lattice in \( V \), then there exists an integer \( a \geq 1 \) such that for \( x \in 1 + \mathfrak{p}_L^a \) we have \( \phi_x(M) = M \).

\textbf{Proof.} Let \( N \) denote the image of \( M \) in \( U/U_2 \). Then \( N \) contains \( (M \cap U_1) + U_2 \), and both are lattices in \( U/U_2 \). Let \( a \geq 1 \) be the smallest integer, such that \( \mathfrak{p}_L^{-a}(M \cap U_1) + U_2 \) contains \( N \). Suppose that \( x \in 1 + \mathfrak{p}_L^a \) and \( v \in M \). We may write \( v = \lambda v_1 + v_2 \), with \( v_1 \in M \cap U_1 \), \( v_2 \in U_2 \) and \( \lambda \in \mathfrak{p}_L^{-a} \). Now \( \phi_x(v) = v + \lambda(x - 1)v_1 \in M \). Hence we get \( \phi_x(M) \subseteq M \) and \( \phi_x^{-1}(M) \subseteq M \). Applying \( \phi_x^{-1} \) to the first inclusion gives \( M \subseteq \phi_x^{-1}(M) \).

We fix an integer \( k \geq 2 \) and set \( W := \text{Sym}^{k-2} L^2 \), an algebraic representation of \( G \). Let \( \pi := \pi(\chi_1, \chi_2) := \text{Ind}_B^G \chi_1 \otimes \chi_2 \) be a smooth principal series \( L \)-representation of \( G \). We say that \( \pi \otimes W \) admits a \( G \)-invariant norm, if there exists a norm \( \| \cdot \| \) on \( \pi \otimes W \), with respect to which \( \pi \otimes W \) is a normed \( L \)-vector space, such that \( \|gv\| = \|v\| \), for all \( v \in \pi \otimes W \) and \( g \in G \).
Let \( c \geq 1 \) be an integer such that both \( \chi_1 \) and \( \chi_2 \) are trivial on \( 1 + p^c \mathbb{Z}_p \).

Let \( D \) be the diagram \( \pi^c \otimes W \hookrightarrow \pi^{K_c} \otimes W \). Since \( H_0(\pi^c) \hookrightarrow \pi^{K_c} \cong \pi \), by tensoring (2) with \( W \) we obtain \( H_0(D) \cong \pi \otimes W \). Assume that \( \pi \otimes W \) admits a \( G \)-invariant norm \( || \cdot || \), set \( (\pi \otimes W)^0 := \{ v \in \pi \otimes W : ||v|| \leq 1 \} \). Then we may define a diagram \( \mathcal{D} = (D_1 \hookrightarrow D_0) \) of \( \mathfrak{o}_L \)-modules:

\[
\mathcal{D} := ((\pi^c \otimes W) \cap (\pi \otimes W)^0 \hookrightarrow (\pi^{K_c} \otimes W) \cap (\pi \otimes W)^0).
\]

In this case Vignéras [14] has shown that the inclusion \( \mathcal{D} \hookrightarrow D \) induces a \( G \)-equivariant injection \( H_0(\mathcal{D}) \hookrightarrow H_0(D) \), such that \( H_0(\mathcal{D}) \otimes_{\mathfrak{o}_L} L = H_0(D) \); \( H_1(\mathcal{D}) = 0 \). Moreover, \( H_0(\mathcal{D}) \) does not contain an \( \mathfrak{o}_L \)-submodule isomorphic to \( L \), see [14] Prop 0.1. Since \( H_0(D) \) is an \( L \)-vector space of countable dimension, this implies that \( H_0(\mathcal{D}) \) is a free \( \mathfrak{o}_L \)-module. By tensoring (2) with \( \mathfrak{o}_L/\mathfrak{p}_L^n \) we obtain

\[
H_0(\mathcal{D}) \otimes_{\mathfrak{o}_L} \mathfrak{o}_L/\mathfrak{p}_L^n \cong H_0(\mathcal{D} \otimes_{\mathfrak{o}_L} \mathfrak{o}_L/\mathfrak{p}_L^n).
\]

**Proposition 4.2.** Let \( \pi = \pi(\chi_1, \chi_2) \) be a smooth principal series representation, assume that \( \pi \otimes W \) admits a \( G \)-invariant norm and let \( \mathcal{D} \) be as above. Then there exists an integer \( a \geq 1 \) such that for all \( x \in 1 + \mathfrak{p}_F^b \), with \( b \geq a \), there exists a finitely generated \( \mathfrak{o}_L[G] \)-module \( M \) in \( \pi(\chi_1 \delta_x^{-1}, \chi_2 \delta_x) \otimes W \), which is free as an \( \mathfrak{o}_L \)-module and a \( G \)-equivariant isomorphism

\[
M \otimes_{\mathfrak{o}_L} \mathfrak{o}_L/\mathfrak{p}_L^b \cong H_0(\mathcal{D}) \otimes_{\mathfrak{o}_L} \mathfrak{o}_L/\mathfrak{p}_L^b,
\]

where \( \delta_x : \mathbb{Q}_p^\times \to L^\times \) is an unramified character with \( \delta_x(p) = x \).

**Proof.** Apply Lemma [11] to \( U = D_1, U_1 = V_1 \otimes W, U_2 = V_s \otimes W \) and \( M = D_1 \), where \( V_1 \) and \( V_s \) are given by (4). Then we get an integer \( a \geq 1 \), such that for all \( x \in 1 + \mathfrak{p}_F^a \), \( \phi_x(D_1) = D_1 \). It is immediate that \( \phi_x \) is \( \mathfrak{g}_0 \)-equivariant. We define a new action \( \ast \) of \( \Pi \) on \( D_1 \), by setting \( \Pi \ast v := \phi_x(\Pi \phi_x^{-1}(v)) \).

This gives us a new diagram \( D(x) \), so that \( D(x)_0 = D_0 \) as a representation of \( \mathfrak{g}_0 \), \( D(x)_1 = D_1 \) as a representation of \( \mathfrak{g}_0 \), the \( \mathfrak{g}_0 \)-equivariant injection \( D(x)_1 \hookrightarrow D(x)_0 \) is equal to the \( \mathfrak{g}_0 \)-equivariant injection \( D_1 \hookrightarrow D_0 \), but the action of \( \Pi \) on \( D_1 \) is given by \( \ast \), (here by \( \ast \) we really mean an equality, not an isomorphism). If \( f_1 \in V_1 \) and \( f_s \in V_s \) then

\[
\Pi \ast (f_1 \otimes w) = f'_s (\Pi w), \quad \Pi \ast (f_s \otimes w) = f'_1 (\Pi w), \quad \forall w \in W,
\]

where \( f'_s \in V_s, f'_1 \in V_1 \) and for all \( g \in I \) we have:

\[
f'_s(sg) = x^{-1}[\Pi \ast f_1](sg) = x^{-1}l_1 f_1 (\Pi^{-1} g \Pi),
\]

\[
f'_1(g) = x[\Pi \ast f_s](g) = x l_2 f_s(s \Pi g \Pi^{-1}).
\]
Hence, we have an isomorphism of diagrams $D(x) \cong D(x^{-1}\lambda_1, x\lambda_2, \theta_1, \theta_2)$ and so Lemma 3.1 gives $H_0(D(x)) \cong \pi(x_1\delta_{x^{-1}}, x_2\delta_{x}) \otimes W$. Now, let $b \geq a$ be an integer and suppose that $x \in 1 + p_b^L$. Since, $\Pi \cdot D = \phi_x(D) = \phi_x^{-1}(D) = D_1$ we get

$$\Pi \ast (D_0 \cap D_1) = \Pi \ast D_1 = \phi_x((\Pi \phi_x^{-1}(D_1))) = D_1.$$ 

So if we let $D(x)_0 := D_0$ and $D(x)_1 := D(x)_0 \cap D(x)_1$, where $\Pi$ acts on $D(x)_1$ by $\ast$ then the diagram $D(x) := (D(x)_1 \hookrightarrow D(x)_0)$ is an integral structure in $D(x)$ in the sense of [14]. The results of Vigneras cited above imply that $M := H_0(D(x))$ is a finitely generated $\mathfrak{o}_L[G]$-submodule of $\pi(x_1\delta_{x^{-1}}, x_2\delta_{x}) \otimes W$, which is free as an $\mathfrak{o}_L$-module, and $M \otimes_{\mathfrak{o}_L} L \cong \pi(x_1\delta_{x^{-1}}, x_2\delta_{x}) \otimes W$. Moreover, since $\phi_x$ is the identity modulo $p_b^L$, we have $\Pi \ast v \equiv \Pi \cdot v (\mod \omega^b_k(D_1)$, for all $v \in D_1$, and so the identity map $\mathcal{D}(x)_0 \to D_0$ induces an isomorphism of diagrams $\mathcal{D}(x) \otimes_{\mathfrak{o}_L} \mathfrak{o}_L/p_b^L \cong \mathcal{D} \otimes_{\mathfrak{o}_L} \mathfrak{o}_L/p_b^L$. Now (8) gives $H_0(D) \otimes_{\mathfrak{o}_L} \mathfrak{o}_L/p_b^L \cong M \otimes_{\mathfrak{o}_L} \mathfrak{o}_L/p_b^L$. \hfill $\square$

Let $k \geq 2$ be an integer and $a_p \in p_L$, following Breuil [5] we define a filtered $\varphi$-module $D_{k,a_p}$: $D$ is a 2-dimensional $L$-vector space with basis $\{e_1, e_2\}$, an $L$-linear automorphism $\varphi : D \to D$, given by

$$\varphi(e_1) = p^{k-1}e_2, \quad \varphi(e_2) = -e_1 + a_pe_2;$$ 

a decreasing filtration $(\text{Fil}^i D)_{i \in \mathbb{Z}}$ by $L$-subspaces, such that if $i \leq 0$ then $\text{Fil}^i D = D$, if $1 \leq i \leq k-1$ then $\text{Fil}^i D = Le_1$, if $i \geq k$ then $\text{Fil}^i D = 0$. We set $V_{k,a_p} := \text{Hom}_{\varphi,\text{Fil}}(D_{k,a_p}, B_{\text{cris}})$. Then $V_{k,a_p}$ is a 2-dimensional $L$-linear absolutely irreducible crystalline representation of $G_{\mathbb{Q}_p} := \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ with Hodge-Tate weights 0 and $k - 1$. We denote by $\chi_{k,a_p}$ the trace character of $V_{k,a_p}$. Since $G_{\mathbb{Q}_p}$ is compact and the action is continuous, $G_{\mathbb{Q}_p}$ stabilizes some $\mathfrak{o}_L$-lattice in $V_{k,a_p}$ and so $\chi_{k,a_p}$ takes values in $\mathfrak{o}_L$.

**Proposition 4.3.** Let $m$ be the largest integer such that $m \leq (k-2)/(p-1)$. Let $a_p, a_p' \in p_L$, and assume that $\text{val}(a_p) > m$, $\text{val}(a_p') > m$. Let $n \geq em$ be an integer, where $e := e(L/\mathbb{Q}_p)$ is the ramification index. Suppose that $a_p \equiv a_p' (\mod p^n_L)$, then $\chi_{k,a_p}(g) \equiv \chi_{k,a_p'}(g) (\mod p^{n-em}_L)$ for all $g \in G_{\mathbb{Q}_p}$.

**Proof.** This a consequence of a result of Berger-Li-Zhu [2]. In [2] they construct $G_{\mathbb{Q}_p}$-invariant lattices $T_{k,a_p}$ in $V_{k,a_p}$. The assumption $a_p \equiv a_p' (\mod p^n_L)$ implies $T_{k,a_p} \otimes_{\mathfrak{o}_L} \mathfrak{o}_L/p^{n-em}_L \cong T_{k,a_p'} \otimes_{\mathfrak{o}_L} \mathfrak{o}_L/p^{n-em}_L$, see Remark 4.1.2 (2) in [2]. This implies the congruences of characters. \hfill $\square$

Let $k \geq 2$ be an integer and $\lambda_1, \lambda_2 \in L$, such that $\lambda_1 + \lambda_2 = a_p$ and $\lambda_1\lambda_2 = p^{k-1}$ (enlarge $L$ if necessary). Assume that $\text{val}(\lambda_1) \geq \text{val}(\lambda_2) > 0$. Let
\[ \chi_1, \chi_2 : \mathbb{Q}_p^\times \to L^\times \] be unramified characters, with \( \chi_1(p) = \lambda_1^{-1} \) and \( \chi_2(p) = \lambda_2^{-1} \), let \( M \) be a finitely generated \( \mathfrak{o}_L[G] \)-module in \( \pi(\chi_1, \chi_2 \cdot |^{-1}) \otimes W \), where \( W := \text{Sym}^{k-2} L^2 \). If \( \lambda_1 \neq \lambda_2 \) then Berger-Breuil have shown that the unitary \( L \)-Banach space representation of \( G \):

\[
E_{k, \omega} := L \otimes_{\mathfrak{o}_L} \lim_{\rightarrow} M/\omega^n_L M
\]

is non-zero, topologically irreducible, admissible in the sense of [13], and contains \( \pi(\chi_1, \chi_2 \cdot |^{-1}) \otimes W \) as a dense \( G \)-invariant subspace, [3] §5.3]. Moreover, the dual of \( E_{k, \omega} \) is isomorphic to the representation of Borel subgroup \( B \) constructed from \( (\varphi, \Gamma) \)-module of \( V_k, \omega \).

Let \( \text{Rep}_{\mathfrak{o}_L} G \) be the category of finite length \( \mathfrak{o}_L[G] \)-modules with a central character, such that the action of \( G \) is smooth (i.e. the stabilizer of a vector is an open subgroup of \( G \).) Let \( \text{Rep}_{\mathfrak{o}_L} \mathcal{G}_{Q_p} \) be the category of \( \mathfrak{o}_L[G] \)-modules of finite length. Colmez in [6] IV.2.14] has defined an exact covariant functor \( V : \text{Rep}_{\mathfrak{o}_L} G \to \text{Rep}_{\mathfrak{o}_L} \mathcal{G}_{Q_p} \). The constructions in [3] and [6] are mutually inverse to one another. This means if we assume \( \lambda_1 \neq \lambda_2 \) and let \( M \) be as above, then

\[
V_{k, \omega} \cong L \otimes_{\mathfrak{o}_L} \lim_{\rightarrow} V(M/\omega^n_L M). \tag{11}
\]

The fact that \( M/\omega^n_L M \) is an \( \mathfrak{o}_L[G] \)-module of finite length follows from [1] Thm A.

**Theorem 4.4.** Assume that \( p > 2 \), and let \( \lambda = \pm p^{(k-1)/2} \), and \( \chi : \mathbb{Q}_p^\times \to L^\times \) a smooth character, with \( \chi(p) = \lambda^{-1} \). Assume that there exists a \( G \)-invariant norm \( \| \cdot \| \) on \( \pi(\chi, \chi \cdot |^{-1}) \otimes W \), where \( W := \text{Sym}^{k-2} L^2 \). Let \( E \) be the completion of \( \pi(\chi, \chi \cdot |^{-1}) \otimes W \) with respect to \( \| \cdot \| \). Then \( E \) is non-zero, topologically irreducible, admissible Banach space representation of \( G \). Moreover, if we let \( E^0 \) be the unit ball in \( E \) then

\[
V_{k, 2\lambda} \otimes (\chi|\chi|) \cong L \otimes_{\mathfrak{o}_L} \lim_{\rightarrow} V(E^0/\omega^n_L E^0).
\]

**Proof.** Since the character \( \chi|\chi| \) is integral, by twisting we may assume that \( \chi \) is unramified. We denote the diagram

\[
\pi(\chi, \chi \cdot |^{-1})^{\mathcal{D}_1} \otimes W \hookrightarrow \pi(\chi, \chi \cdot |^{-1})^{\mathcal{D}_1} \otimes W
\]

by \( D = (D_1 \hookrightarrow D_0) \). Let \( \mathcal{D} = (\mathcal{D}_1 \hookrightarrow \mathcal{D}_0) \) be the diagram of \( \mathfrak{o}_L \)-modules with \( \mathcal{D}_1 = D_1 \cap E^0 \) and \( \mathcal{D}_0 = D_0 \cap E^0 \). Let \( a \geq 1 \) be the integer given by Proposition 4.2] for each \( j \geq 0 \), we fix \( x_j \in 1 + \mathfrak{p}_L^{x+j}, x_j \neq 1 \) and a finitely
generated $\mathfrak{o}_L[G]$-submodule $M_j$ in $\pi(\chi\delta_x^{-1}, \chi\delta_x | \cdot |^{-1}) \otimes W$, (which is then a free $\mathfrak{o}_L$-module), such that

$$H_0(D) \otimes_{\mathfrak{o}_L} \mathfrak{o}_L/p_L^{a+j} \cong M_j \otimes_{\mathfrak{o}_L} \mathfrak{o}_L/p_L^{a+j}.$$ 

This is possible by Proposition 4.2. To ease the notation we set $M := H_0(D)$. Let $a_p(j) := \lambda x_j^{-1} + \lambda x_j$, $a_p := 2\lambda$ and let $m$ be the largest integer, such that $m \leq (k - 2)/(p - 1)$. Since $p > 2$, $x_j + x_j^{-1}$ is a unit in $\mathfrak{o}_L$, and so $\text{val}(a_p(j)) = \text{val}(a_p) = (k - 1)/2 > m$. (Here we really need $p > 2$.) Moreover, we have $a_p \equiv a_p(j) \pmod{p_L^{j+a+em}}$, where $e := e(L/\mathbb{Q}_p)$ is the ramification index. Now since $x_j \neq 1$ we get that $\lambda x_j \neq x_j^{-1}$, and hence we may apply the results of Berger-Breuil to $\pi(\chi\delta_x^{-1}, \chi\delta_x | \cdot |^{-1}) \otimes W$. Let $T_{k,a_p(j)} := \lim \n V(M_j/\varpi^a L M_j)$. Then (11) gives that $T_{k,a_p(j)}$ is a $G_{\mathbb{Q}_p}$-invariant lattice in $V_{k,a_p(j)}$. Since $M \otimes_{\mathfrak{o}_L} \mathfrak{o}_L/p_L^{a+j} \cong M_j \otimes_{\mathfrak{o}_L} \mathfrak{o}_L/p_L^{a+j}$ we get

$$V(M/\varpi^a L M) \cong V(M_j/\varpi^a L M_j) \cong T_{k,a_p(j)} \otimes_{\mathfrak{o}_L} \mathfrak{o}_L/p_L^{a+j}. \quad (12)$$

Set $V := L \otimes_{\mathfrak{o}_L} \lim \n V(M/\varpi^a L M)$. Then (12) implies that $V$ is a 2-dimensional $L$-vector space. Let $\chi_V$ be the trace character of $V$, then it follows from (12) that $\chi_V \equiv \chi_{k,a_p(j)} \pmod{p_L^{a+j}}$. Since $a_p \equiv a_p(j) \pmod{p_L^{j+a+em}}$, Proposition 4.3 says that $\chi_{k,a_p} \equiv \chi_{k,a_p(j)} \pmod{p_L^{a+j}}$. We obtain $\chi_V \equiv \chi_{k,a_p} \pmod{p_L^{a+j}}$, for all $j \geq 0$. This gives us $\chi_V = \chi_{k,a_p}$. Since $V_{k,a_p}$ is irreducible, the equality of characters implies $V \cong V_{k,a_p}$.

Set $\widehat{M} := \lim \n M/\varpi^a L$, and $E' := \widehat{M} \otimes_{\mathfrak{o}_L} L$. Since $M$ is a free $\mathfrak{o}_L$-module, we get an injection $M \hookrightarrow \widehat{M}$. In particular $E'$ contains $\pi(\chi, \chi | \cdot |^{-1}) \otimes W$ as a dense $G$-invariant subspace. We claim that $E'$ is a topologically irreducible and admissible $G$-representation. Now [2 Thm.4.1.1, Prop.4.1.4] say that the semi-simplification of $T_{k,a_p(j)} \otimes_{\mathfrak{o}_L} k_L$ is irreducible if $p + 1 \nmid k - 1$ and isomorphic to $$\begin{pmatrix} \mu_{\sqrt{-1}}^{(k-1)/(p+1)} & 0 \\ 0 & \mu_{-\sqrt{-1}}^{(k-1)/(p+1)} \end{pmatrix} \otimes \omega(k-1)/(p+1),$$ if $p + 1|k - 1$, where $\mu_{\pm\sqrt{-1}}$ is the unramified character sending arithmetic Frobenius to $\pm\sqrt{-1}$, and $\omega$ is the cyclotomic character. Then [3 Thm A] implies that if $p + 1 \nmid k - 1$ then $M_j \otimes_{\mathfrak{o}_L} k_L$ is an irreducible supersingular representation of $G$, and if $p + 1|k - 1$ then the semi-simplification of $M_j \otimes_{\mathfrak{o}_L} k_L$ is a direct sum of two irreducible principal series. The irreducibility of principal series follows from [11 Thm. 33], since $\sqrt{-1} \neq \pm 1$, as $p > 2$. Since $M \otimes_{\mathfrak{o}_L} k_L \cong M_j \otimes_{\mathfrak{o}_L} k_L$, we get that $M \otimes_{\mathfrak{o}_L} k_L$ is an admissible representation of $G$ (so that for every open subgroup $U$ of $G$, the space of $U$-invariants is finite dimensional). This implies that $E'$ is admissible.
Suppose that $E_1$ is a closed $G$-invariant subspace of $E'$ with $E' \neq E_1$. Let $E_1^0 := E_1 \cap \hat{M}$. We obtain a $G$-equivariant injection $E_1^0 \otimes_{o_L} k_L \hookrightarrow M \otimes_{o_L} k_L$. If $E_1^0 \otimes_{o_L} k_L = 0$ or $M \otimes_{o_L} k_L$ then Nakayama’s lemma gives $E_1^0 = 0$ and $E_1^0 = \hat{M}$, respectively. If $p+1 \dagger k - 1$ then $M \otimes_{o_L} k_L$ is irreducible and we are done. If $p+1|k-1$ then $E_1^0 \otimes_{o_L} k_L$ is an irreducible principal series, and so $V(E_1^0 \otimes_{o_L} k_L)$ is one dimensional, [6 IV.4.17]. But then $V_1 := L \otimes_{o_L} \varprojlim V(E_1^0/\varpi^m E_1^0)$ is a 1-dimensional subspace of $V_{k,ap}$ stable under the action of $G_{Q_p}$. Since $V_{k,ap}$ is irreducible we obtain a contradiction.

Since $E'$ is a completion of $\pi(\chi, \chi| \bullet|^{-1}) \otimes W$ with respect to a finitely generated $o_L[G]$-submodule, it is the universal completion, see eg [7 Prop. 1.17]. In particular, we obtain a non-zero $G$-equivariant map of $L$-Banach space representations $E' \to E$, but since $E'$ is irreducible and $\pi(\chi, \chi| \bullet|^{-1}) \otimes W$ is dense in $E$, this map is an isomorphism.

\[ \square \]

Corollary 4.5. Assume that $p > 2$, and let $\chi : \mathbb{Q}_p^\times \to L^\times$ a smooth character with $\chi(p)^2 p^{k-1} = 1$. Assume that there exists a $G$-invariant norm $\| \bullet \|$ on $\pi(\chi, \chi| \bullet|^{-1}) \otimes W$, where $W := \text{Sym}^{k-2} L^2$. Then every bounded $G$-invariant $o_L$-lattice in $\pi(\chi, \chi| \bullet|^{-1}) \otimes W$ is finitely generated as an $o_L[G]$-module.

**Proof.** The existence of a $G$-invariant norm implies that the universal completion is non-zero. It follows from Theorem 4.4 that the universal completion is topologically irreducible and admissible. The assertion follows from the proof of [3, Cor. 5.3.4].

\[ \square \]

For the purposes of [10] we record the following corollary to the proof of Theorem 4.4.

Corollary 4.6. Assume $p > 2$, and let $\chi : \mathbb{Q}_p^\times \to L^\times$ be a smooth character, such that $\chi^2(p) p^{k-1}$ is a unit in $o_L$. Assume there exists a unitary $L$-Banach space representation $(E, \| \bullet \|)$ of $G$ containing $(\text{Ind}_G^L \chi \otimes \chi| \bullet|^{-1}) \otimes \text{Sym}^{k-2} L^2$ as a dense $G$-invariant subspace, such that $\| E \| \subseteq |L|$. Then there exists $x \in 1+p_L, x^2 \neq 1$ and a unitary completion $E_x$ of $(\text{Ind}_G^L \chi \delta_x \otimes \chi \delta_x^{-1}| \bullet|^{-1}) \otimes \text{Sym}^{k-2} L^2$, such that $E_x \otimes_{o_L} k_L \cong E_x^0 \otimes_{o_L} k_L$, where $E_x^0$ is the unit ball in $E_x$ and $E^0$ is the unit ball in $E$.

**Proof.** Let $\pi := \text{Ind}_G^L \chi \otimes \chi| \bullet|^{-1}$ and $M := (\pi \otimes W) \cap E^0$. Now $M \cap \varpi L E^0 = (\pi \otimes W) \cap \varpi L E^0 = \varpi L M$. So we have a $G$-equivariant injection $\iota : M/\varpi L M \hookrightarrow E^0/\varpi L E^0$. We claim that $\iota$ is a surjection. Let $v \in E^0$, since $\pi \otimes W$ is dense in $E$, there exists a sequence $\{v_n\}_{n \geq 1}$ in $\pi \otimes W$ such that $\lim v_n = v$. We also have $\lim ||v_n|| = ||v||$. Since $\| E \| \subseteq |L| \cong \mathbb{Z}$, there exists $m \geq 0$ such

\[ \square \]
that \( v_n \in M \), for all \( n \geq m \). This implies surjectivity of \( \iota \). So we get 
\[ M \otimes_{\mathfrak{o}_L} k_L \cong E^0 \otimes_{\mathfrak{o}_L} k_L. \]

By Corollary 4.3 we may find \( u_1, \ldots, u_n \in M \) which generate \( M \) as an \( \mathfrak{o}_L[G] \)-module. Further, \( u_i = \sum_{j=1}^{m_i} v_{ij} \otimes w_{ij} \) with \( v_{ij} \in \pi \) and \( w_{ij} \in W \). Since \( \pi \) is a smooth representation of \( G \) there exists an integer \( c \geq 1 \) such that \( v_{ij} \) is fixed by \( K_c \) for all \( 1 \leq i \leq n, 1 \leq j \leq m_i \). Set 
\[ D := (((\pi^{K_c} \otimes W) \cap M) \hookrightarrow (\pi^{K_c} \otimes W) \cap M), \quad D := (\pi^{I_c} \otimes W) \hookrightarrow \pi^{K_c} \otimes W) \]
and let \( M' \) be the image of \( H_0(D) \hookrightarrow H_0(D) \cong \pi \otimes W \). It follows from (3) that 
\( M' \) is generated by \( (\pi^{K_c} \otimes W) \cap M \) as an \( \mathfrak{o}_L[G] \)-module. Hence, \( M' \subseteq M \).

In particular, \( H_0(D) \otimes_{\mathfrak{o}_L} k_L \cong M \otimes_{\mathfrak{o}_L} k_L \). The assertion follows from the proof of Theorem 4.4.

5 Existence

Recent results of Colmez, which appeared after the first version of this note, imply the existence of a \( G \)-invariant norm on \( (\text{Ind}_{\mathfrak{G}}^G \chi \otimes \chi \mid \cdot \mid^{-1}) \otimes \text{Sym}^{k-2} L^2, \chi^2(p) p^{k-1} \in \mathfrak{o}_L^\times \), thus making our results unconditional. We briefly explain this.

We continue to assume \( p > 2, k \geq 2 \) an integer and \( a_p = 2p^{(k-1)/2} \). The representation \( V_{k,a_p} \) of \( \mathfrak{G}_{Q_p} \) sits in the \( p \)-adic family of Berger-Li-Zhu, [2, 3.2.5]. Moreover, all the other points in the family correspond to the crystalline representations with distinct Frobenius eigenvalues, to which the theory of [3] applies. Hence [6] II.3.1, IV.4.11] implies that there exists an irreducible unitary \( L \)-Banach space representation \( \Pi \) of \( \text{GL}_2(Q_p) \), such that \( V(\Pi) \cong V_{k,a_p} \). If \( p \geq 5 \) or \( p = 3 \) and \( k \equiv 3 \) (mod 8) and \( k \equiv 7 \) (mod 8), the existence of such \( \Pi \) also follows from [8]. It follows from [6] VI.6.46] that the set of locally algebraic vectors \( \Pi^{alg} \) of \( \Pi \) is isomorphic to \( (\text{Ind}_{\mathfrak{G}}^G \chi \otimes \chi \mid \cdot \mid^{-1}) \otimes \text{Sym}^{k-2} L^2, \chi \) is an unramified character with \( \chi(p) = p^{-(k-1)/2} \). The restriction of the \( G \)-invariant norm of \( \Pi \) to \( \Pi^{alg} \) solves the problem. Moreover, if \( \delta : \mathbb{Q}_p^\times \rightarrow L^\times \) is a unitary character then we also obtain a \( G \)-invariant norm on \( \Pi^{alg} \otimes \delta \circ \det \).

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