Heavy-Light Exotics from QCD Laplace Sum Rules at N2LO in the chiral limit

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Abstract

These talks review and summarize our results in [1, 2] on XYZ-like spectra obtained from QCD Laplace Sum Rules in the chiral limit at next-to-next-leading order (N2LO) of perturbation theory (PT) and including leading order (LO) contributions of dimensions $d \leq 6$ non-perturbative condensates. We conclude that the observed XZ states are good candidates for $1^+$ and $0^+$ molecules or four-quark states while the predictions for $1^−$ and $0^−$ states are about 1.5 GeV above the $Y_{c,b}$ experimental candidates and hadronic thresholds. We (numerically) find that these exotic molecules couple weakly to the corresponding interpolating currents than ordinary $D$, $B$ heavy-light mesons while we observe that these couplings decrease faster [$1/m_{b}^{3/2}$ (resp. $1/m_{b}$)] for the $1^+$, $0^+$ (resp. $1^−$, $0^−$) states than $1/m_{b}^{1/2}$. Our results do not also confirm the existence of the $X(5568)$ state in agreement with LHCb findings.

Keywords: Perturbative and Non-perturbative QCD, QCD spectral sum rules, Exotic hadrons, Masses and Decay constants.

1. Note for the Readers

This paper summarizes the results in our original works [1, 2]. Most of the references fairly quoted there are not repeated here due to space limitations. We sincerely apologize for that.

2. Introduction

A large amount of exotic hadrons which differ from the “standard” $c\bar{c}$ charmonium and $b\bar{b}$ bottomium radial excitation states have been discovered in $D$ and $B$-factories through e.g. $J/\psi\pi^+\pi^−$ and $\Upsilon\pi^+\pi^−$ processes [3, 4]. They are referred as XYZ states [5]. In this talk, we shall present our predictions for the masses and couplings of these states obtained using the Laplace sum rule (LSR) [6-9] version of QCD spectral sum rules (QSSR) [6] known at next-to-next-leading order (N2LO) of PT series and including non-perturbative condensates of dimensions $d \leq 6$. In so doing, we assume a factorization of the four-quark spectral functions into a convolution of two ones built from quark bilinear currents as in [12-14]. We show in [1] that this factorization, though valid to leading order in $1/N_c$, can reproduce with a good accuracy the predictions for the masses and couplings obtained from a complete lowest order expression.

3. Molecules and Four-quark two-point functions

We shall work with the transverse part $\Pi^{(1)}$ of the $B$-factories through e.g. $J/\psi\pi^+\pi^−$ and $\Upsilon\pi^+\pi^−$ processes [3, 4]. They are referred as XYZ states [5]. In this talk, we shall present our predictions for the masses and couplings of these states obtained using the Laplace sum rule (LSR) [6-9] version of QCD spectral sum rules (QSSR) [6] known at next-to-next-leading order (N2LO) of PT series and including non-perturbative condensates of dimensions $d \leq 6$. In so doing, we assume a factorization of the four-quark spectral functions into a convolution of two ones built from quark bilinear currents as in [12-14]. We show in [1] that this factorization, though valid to leading order in $1/N_c$, can reproduce with a good accuracy the predictions for the masses and couplings obtained from a complete lowest order expression.
two-point spectral functions:
\[
\Pi^\mu(q) = i \int d^4x e^{i q \cdot x} \langle 0| T \mathcal{O}^{\mu}(x) \mathcal{O}^\nu(0)|0 \rangle
\]
\[
= -\Pi^{(1)}(q^2) e^{q \cdot p} - \frac{q^2}{q^2} \Pi^{(0)}(q^2) + \frac{q^2}{q^2} \Pi^{(1)}(q^2),
\]
for the spin 1 states while for the spin zero ones, we shall use the two-point functions \(\psi^{(s,p)}(q^2)\) built directly from the (pseudo)scalar currents:
\[
\psi^{(s,p)}(q^2) = i \int d^4x e^{i q \cdot x} \langle 0| T \mathcal{O}^{(s,p)}(x) \mathcal{O}^{(s,p)}(0)|0 \rangle,
\]
which is related to \(\Pi^{(0)}\) appearing in Eq. (1) via Ward identities [10, 11].

- **Interpolating currents**

  The interpolating currents \(\mathcal{O}\) for the molecules (resp. four-quark states) are given in Table 1 (resp. Table 2).

| States | \(J^P\) | Molecule Currents \(\equiv \mathcal{O}_{\mu}(x)\) |
|--------|--------|----------------------------------|
| \(DD, BB\) | 0^+ | \((\bar{q}_\gamma q)(\bar{q}_\gamma q)\) |
| \(D^*D^*, B^*B^*\) | 0^+ | \((\bar{q}_\gamma q)(\bar{q}_\gamma q)\) |
| \(D_0\bar{D}_0\), \(\bar{D}_0D_0\) | 0^- | \((\bar{q}_\gamma q)(\bar{q}_\gamma q)\) |
| \(D^*D_1, \bar{D}_0B_1\) | 1^- | \((\bar{q}_\gamma q)(\bar{q}_\gamma q)\) |
| \(D_0\bar{D}_1\), \(\bar{D}_0B_1\) | 1^- | \((\bar{q}_\gamma q)(\bar{q}_\gamma q)\) |
| \(\bar{D}_0\bar{D}_1\), \(\bar{D}_0B_1\) | 0^- | \((\bar{q}_\gamma q)(\bar{q}_\gamma q)\) |
| \(\bar{D}_0\bar{D}_1\), \(\bar{D}_0B_1\) | 1^- | \((\bar{q}_\gamma q)(\bar{q}_\gamma q)\) |

- **Spectral Function within MDA**

  We shall use the Minimal Duality Ansatz (MDA) given in Eq. 3 for parametrizing the spectral function:
\[
\frac{1}{\pi} \text{Im} \Pi^{(1)}(t) = f_H^2 M_H^2 \delta(t - M_H^2) + "QCD continuum" \theta(t - t_c). \tag{3}
\]
where \(f_H\) is the decay constant defined as:
\[
\langle 0| \mathcal{O}^{(s,p)}(H) = f_H^{(s,p)} M_H \delta(t - M_H^2). \tag{4}
\]
respectively for spin 0 and 1 hadronic states \(H\) with \(\epsilon_u\) the vector polarization. The higher states contributions are smeared by the “QCD continuum” coming from the discontinuity of the QCD diagrams and starting from a constant threshold \(t_c\).

- **NLO and N2LO PT corrections using factorization**

  Assuming a factorization of the four-quark interpolating current as a natural consequence of the molecule definition of the state, we can write the corresponding spectral function as a convolution of the spectral functions associated to quark bilinear current for the \(DD^*\) and \(\bar{D}D\) spin 1 states:
\[
\frac{1}{\pi} \text{Im} \Pi^{(1)}(t) = \theta(t - 4 M_Q^2) \left(\frac{1}{4\pi} \right)^2 \int d^4s \frac{\gamma^5 \gamma^i \gamma^j}{M^2} \int d^4s \frac{\gamma^5 \gamma^i \gamma^j}{M^2} \times 1\frac{1}{\pi} \text{Im} \Pi^{(1)}(t) \left(\frac{1}{4\pi} \right)^2 \int d^4s \frac{\gamma^5 \gamma^i \gamma^j}{M^2} \tag{5}
\]
For the \(DD^*\) spin 0 state, one has:
\[
\frac{1}{\pi} \text{Im} \Pi^{(0)}(t) = \theta(t - 4 M_Q^2) \left(\frac{1}{4\pi} \right)^2 \int d^4s \frac{\gamma^5 \gamma^i \gamma^j}{M^2} \int d^4s \frac{\gamma^5 \gamma^i \gamma^j}{M^2} \times 1\frac{1}{\pi} \text{Im} \Pi^{(0)}(t) \left(\frac{1}{4\pi} \right)^2 \int d^4s \frac{\gamma^5 \gamma^i \gamma^j}{M^2} \tag{6}
\]
and for the \(\bar{D}D\) spin 0 state:
\[
\frac{1}{\pi} \text{Im} \Pi^{(0)}(t) = \theta(t - 4 M_Q^2) \left(\frac{1}{4\pi} \right)^2 \int d^4s \frac{\gamma^5 \gamma^i \gamma^j}{M^2} \int d^4s \frac{\gamma^5 \gamma^i \gamma^j}{M^2} \times 1\frac{1}{\pi} \text{Im} \Pi^{(0)}(t) \left(\frac{1}{4\pi} \right)^2 \int d^4s \frac{\gamma^5 \gamma^i \gamma^j}{M^2} \tag{7}
\]
where:
\[
\lambda = \left( 1 - \frac{\sqrt{t_1 - t_2}}{t} \right) \left( 1 - \frac{\sqrt{t_1 + t_2}}{t} \right). \tag{8}
\]
is the phase space factor and \(M_Q\) is the on-shell heavy quark mass. \(\text{Im} \Pi^{(1)}(t)\) is the spectral function associated to the bilinear \(C_{\gamma\mu}(\gamma_5)q\) vector or axial-vector current, while \(\text{Im} \psi^{(0)}(t)\) is associated to the \(C_{\gamma_{\mu}}(\gamma_5)q\) scalar or pseudoscalar current\(^{3}\). An analogous convolution is assumed for the four-quark states.

\(^{3}\)In the chiral limit \(m_q = 0\), the PT expressions of the vector (resp. scalar) and axial-vector (resp. pseudoscalar) spectral functions are the same.
• The Laplace sum rule (LSR)

The exponential or Laplace sum rule (LSR) and its ratio read\(^4\):

\[
L_{\mu}^{q}(t, \tau_{c}, \mu) = \int_{M_{0}^{2}}^{\infty} dt \, e^{-t} \frac{1}{\pi} \Im \text{Imf}_{\mu}^{(1/0)}(t, \mu),
\]

(9)

\[
R_{\mu}^{q}(t, \tau_{c}, \mu) = \int_{M_{0}^{2}}^{\infty} dt \, e^{-t} \frac{1}{\pi} \Im \text{Imf}_{\mu}^{(1/0)}(t, \mu) = M_{0}^{2},
\]

(10)

where \(\mu\) is the subtraction point which appears in the approximate QCD series when radiative corrections are included and \(\tau\) is the sum rule variable replacing \(q^{2}\). Similar sum rules are obtained for the (pseudo)scalar two-point function \(\psi^{(s,0)}(q^{2})\).

• Stability criteria and some phenomenological tests

The variables \(\tau, \mu\) and \(t_{c}\) are, in principle, free parameters. We shall use stability criteria (if any), with respect to these free 3 parameters, for extracting the optimal results. In the standard Minimal Duality Ansatz (MDA) given in Eq. 3 for parametrizing the spectral function, the “QCD continuum” threshold \(t_{c}\) is constant and is independent on the subtraction point \(\mu\). One should notice that this standard MDA with constant \(t_{c}\) describes quite well the properties of the lowest ground state as explicitly demonstrated in [15] and in various examples [10, 11] after confronting the integrated spectral function within this simple parametrization with all the full data measurements. It has been also successfully tested in the large \(N_{c}\) limit of QCD in [16]. Though it is difficult to estimate with a good precision the systematic error related to this simple model, these features indicate the ability of the model for reproducing accurately the data. We expect that the same feature is reproduced for the case of the XYZ discussed here where complete data are still lacking.

4. QCD input parameters

The QCD parameters which shall appear in the following analysis will be the charm and bottom quark masses \(m_{c,b}\) (we shall neglect the light quark masses \(q \equiv u,d\)), the light quark condensate \(\langle\bar{q}q\rangle\), the gluon condensates \(\langle\alpha_{s}G^{2}\rangle\), \(\langle\alpha_{s}G_{\mu}^{a}G_{\mu}^{a}\rangle\) and \(\langle g^{3}G^{3}\rangle\), the mixed condensate \(\langle\bar{q}Gq\rangle\), \(\langle\bar{q}ge^{\mu\nu}(\lambda_{c}/2)G_{\mu\nu}^{a}q\rangle\), \(\langle\bar{q}G_{\mu}^{a}G_{\mu}^{b}G_{\mu}^{c}\rangle\) and \(\langle\bar{q}G_{\mu}^{a}G_{\mu}^{b}G_{\nu}^{c}\rangle\), the four-quark condensate \(\rho_{a} \equiv \rho_{a}\langle\bar{q}q\rangle\), where \(\rho \equiv 3 - 4\) indicates the deviation from the four-quark vacuum saturation. Their values are given in Table 3. We shall work with the running light quark condensates and masses, which read to leading order in \(\alpha_{s}\):

\[
\langle\bar{q}q\rangle(r) = -\beta_{1}^{2}\langle\bar{q}q\rangle_{0}^{1/3}\beta_{1}(11)
\]

where \(\beta_{1} = -(1/2)(11 - 2n_{f}/3)\) is the first coefficient of the \(\beta\) function for \(n_{f}\) flavours; \(\alpha_{s} \equiv \alpha_{s}(r)/\pi\); \(\hat{\rho}_{q}\) is the spontaneous RGI light quark condensate [17].

Table 3: QCD input parameters: the original errors for \(\langle\alpha_{s}G^{2}\rangle\), \(\langle g^{3}G^{3}\rangle\) and \(\rho\langle\bar{q}q\rangle^{2}\) have been multiplied by about a factor 3 for a conservative estimate of the errors (see also the text).

| Parameters | Values | Ref. |
|------------|--------|------|
| \(\alpha_{s}(M_{Z})\) | 0.125(8) | [18–20] |
| \(\bar{m}_{b}(m_{b})\) | 2.40(12) MeV | average [3, 21, 22] |
| \(\bar{m}_{b}(m_{b})\) | 4177(11) MeV | average [3, 21] |
| \(\hat{\rho}_{q}\) | (253 \pm 6) MeV | [10, 11, 23–25] |
| \(M_{0}^{2}\) | (0.8 \pm 0.2) GeV\(^{2}\) | [26–30] |
| \(\langle\alpha_{s}G^{2}\rangle\) | (7 \pm 5) \times 10\(^{-5}\) GeV\(^{4}\) | [7, 8, 19, 21, 31–34] |
| \(\langle g^{3}G^{3}\rangle\) | (8.2 \pm 2.0) GeV\(^{8}\) \times \langle\alpha_{s}G^{2}\rangle\) | [21] |
| \(\rho_{a} \equiv \rho_{a}\langle\bar{q}q\rangle\) | (5.8 \pm 1.8) \times 10\(^{-5}\) GeV\(^{6}\) | [19, 26, 27, 34, 35] |

5. QCD expressions of the spectral functions

In our works [1, 2], we provide new compact integrated expressions of the spectral functions at LO of PT QCD and including non-perturbative condensates having dimensions \(d \leq 6 - 8\). NLO and N2LO corrections are introduced using the convolution integrals in Eq. 5. The expressions of spectral functions of heavy-light bi-linear currents are known to order \(\alpha_{s}\) (NLO) from [36] and to order \(\alpha_{s}^{2}\) (N2LO) from [37] which are available as a Mathematica Program named Rvs. N3LO corrections are estimated from the geometric growth of the QCD PT series [38] as a source of the PT errors, which we expect to give a good approximation of the uncalculated higher order terms dual to the \(1/q^{2}\) contribution of a tachyonic gluon mass [39, 40] (for reviews see e.g [41, 42]).

In our analysis, we replace the on-shell (pole) mass appearing in the LO spectral functions with the running mass using the relation, to order \(\alpha_{s}^{3}\) [43–52]:

\[
M_{Q} = \bar{m}_{Q}(\mu)
\]

\[
+ \log \left(\frac{\mu}{M_{0}}\right) \left[ a_{0} + (16.2163 - 0.1013a_{0})a_{0}^{2} \right]
\]

\[
+ \log^{2} \left(\frac{\mu}{M_{0}}\right) \left[ (1.9717 - 0.0833a_{0})a_{0}^{2} \right]
\]

(12)

for \(n_{f}\) light flavours where \(\mu\) is the arbitrary subtraction point and \(\alpha_{s} \equiv \alpha_{s}/\pi\).

\(^4\)The last equality in Eq. 10 is obtained when one uses MDA in Eq. 3 for parametrizing the spectral function.
6. Tests of the Factorization Assumption

\* \( D_s^0 D^* (1^-) \) molecule state at LO

In the following, we shall test the factorization assumption if one does it at lowest order (LO) of perturbation theory (PT) by taking the example of the \( D_s^0 D^* (1^-) \) molecule state. To LO of PT, the four-quark correlator can be subdivided into its factorized (Fig. 1a) and its non-factorized (Fig. 1b) parts. The analysis for the decay constant and mass including NP contributions up to dimension \( d = 6 \) is shown in Fig. 2. We conclude from

\[
\left(1 - \frac{\mu^2}{M^2}\right)^{\alpha_s} \approx 0.05 \text{ for the decay constant and } 0.5\% \text{ for the mass which is quite tiny. However, to avoid this (small) effect, we shall work in the following with the full non-factorized \( \text{PT@NP} \) of the LO expressions.}
\]

\* \( B^0 \overline{B}^0 \) four-quark correlator at NLO

For extracting the PT \( \alpha_s \) corrections to the correlator and due to the technical complexity of the calculations, we shall assume that these radiative corrections are dominated by the ones from the factorized diagrams (Fig. 3a,b) while we neglect the ones from non-factorized ones (Fig. 3c to f). This fact has been proven explicitly by \([13, 14]\) in the case of the \( B^0 \overline{B}^0 \) systems (very similar correlator as the ones discussed in the following) where the non-factorized \( \alpha_s \) corrections do not exceed 10% of the total \( \alpha_s \) contributions.

\* Conclusions

We expect from the previous LO example that the masses of the molecules are known with a good accuracy while, for the coupling, we shall have in mind the systematics induced by the radiative corrections estimated by keeping only the factorized diagrams. The contributions of the factorized diagrams will be extracted from the convolution integrals given in Eq. 5. Here, due to a partial cancellation of the corrections, the suppression of the NLO corrections will be more pronounced in the extraction of the meson masses from the ratio of sum rules than to the case of the \( B^0 \overline{B}^0 \) systems.

7. \( \bar{D}D \) molecule decay constant and mass

\* \( \tau \) and \( t_c \) stabilities

We study the behavior of the coupling\(^3\) \( f_{DD} \) and mass \( M_{DD} \) in terms of LSR variable \( \tau \) at different values of \( t_c \) as shown in Fig. 4 at LO, in Fig. 5 at NLO and in Fig. 6 at N2LO. We consider, as a final and conservative result, the one corresponding to the beginning of the \( \tau \)-stability \( (\tau \approx 0.25 \text{ GeV}^{-2}) \) for \( t_c = 22 \text{ GeV}^2 \) until the one where \( t_c \)-stability starts to be reached for \( t_c \approx 32 \text{ GeV}^2 \) and for \( \tau \approx 0.35 \text{ GeV}^{-2} \). In these stability regions, the requirement that the pole contribution is larger than the one of the continuum is automatically satisfied.

\(^3\)Here and in the following : decay constant is the same as : coupling.
- **Running versus the pole quark mass definitions**

  We show in Fig. 7 the effect of the definitions (running and pole) of the heavy quark mass used in the analysis at LO which is relatively important. The difference should be added as errors in the LO analysis. This source of errors is never considered in the current literature.

- **Convergence of the PT series**

  Using \( t_\tau = 32 \text{ GeV}^2 \), we study in Fig. 8 the convergence of the PT series for a given value of \( \mu = 4.5 \text{ GeV} \). We observe (see Table 4) that from NLO to N2LO the mass decreases by about only 1 per mil indicating the good convergence of the PT series.

- **\( \mu \)-stability**

  We improve our previous results by using different values of \( \mu \) (Fig. 9). Using the fact that the final result must be independent of the arbitrary parameter \( \mu \) (plateau / inflexion point for the coupling and minimum for the mass), we consider as an optimal result the one at \( \mu = 4.5 \text{ GeV} \) where we deduce the result in Table 4.

8. **Molecule states masses and couplings**

  The results are given in Table 4 (resp. Table 5) for the charm (resp. bottom) channel where the corresponding hadronic threshold and experimental candidates are shown in the last two columns. The errors come from the QCD parameters and from the range of \( \tau \), \( t_\tau \) and \( \mu \) where the optimal results are extracted.
9. Four-quark states masses and couplings

The results are given in Table 6 (resp. Table 7) for the charm (resp. bottom) channel where the experimental candidates are shown in the last column. The sources of errors are the same as in the molecules case.

10. Confrontation with data and some LO results

- Axial-vector (1++) states

As mentioned in the introduction, there are several observed states in this channel. In addition to the well-established X(3872), we have the X(4147, 4273) and the Z(3900, 4025, 4050, 4430).

For the non-strange states found from their decays into J/ψπ⁺π⁻, one can conclude from the results given in Table 4 that the X(3872) and Z(3900) can be well described with an almost pure D*D molecule or/and four quark [cq̄q̄] states, (q ≡ u, d) while the one of the Z(4200, 4430) might be a D*D1 molecule state. Our results for the X(3872) confirm our LO ones in [53–55].

One can notice that the values of these masses below the corresponding D*D, BB-like thresholds are much lower than the ones predicted ≈ 5.12 (resp 11.32) GeV for the 1++ cc̄c̄ molecule or D*D molecule. While the optimal values of the masses have been extracted, are approximately the mass of the 1st radial excitation, one can deduce that the higher masses experimental states cannot be such radial excitations.

In the bottom sector, experimental checks of our pre-
Predictions are required.

- **Scalar (0++) states**

Our analysis in Tables 4 and 7 predicts that:

The 0++ DD, DD, D*D molecule and four-quark non-strange states are almost degenerate with the 1++ ones and have masses around 3900 MeV. This prediction is comparable with the Zc(3900) quoted by PDG [3] as a 0++ state.

The predicted mass of the D_s^0 D_s^0 molecule is higher [4402(30) MeV] but is still below the D_s^0 D_s^0 threshold.

- **Vector (1−+) states**

Our predictions in Tables 4 to 7 for molecules and four-quark vector states in the range of (5646-5961) MeV are too high compared with the observed Yc(4140) to Yc(4660) states. Our N2LO results confirm previous LO ones in [59, 60] but do not support the result in [61] which are too low.

Our results indicate that the observed states might result from a mixing of the molecule / four-quark with ordinary quarkonia-states (if the description of these states in terms of molecules and/or four-quark states are the correct one). The NP contribution to this kind of mixing has been estimated to leading order in [62]. The same conclusion holds for the Yc(9898, 10260, 10870) where the predicted unmixed molecule / four-quark states are in the range (12326-12829) MeV.

As these pure molecule states are well above the physical threshold, they might not be bound states and could not be separated from backgrounds. Our results go in lines with the ones of [63].

- **Pseudoscalar (0−−) states**

One expects from Tables 4 to 7 that the 0−− molecules will populate the region 5656-6020 (resp 12379-12827) MeV for the charm (resp bottom) channels like in the case of the 1−− vector states. One can notice that these states are much heavier than the predicted 0−−/ 1−− vector states. One can notice that these pseudoscalar states are well above the physical threshold. Therefore, like in the case of vector states, these molecule states should be broad and are difficult to separate from backgrounds.

One can also notice that the D^*_s(0−−) and (0−−) states are almost degenerate despite the opposite signs of the ⟨qq⟩ and ⟨qGq⟩ contributions to the spectral functions in the two channels (see Appendix of [1]).

- **Isospin breakings and almost degenerate states**

In our approach, isospin breakings are controlled by the running light quark mass m_d ~ m_u and condensate ⟨iui−dd⟩ differences which are tiny quantities. Their effects are hardly noticeable within the accuracy of our approach. Therefore, for the neutral combination of currents which we have taken in Table 1, one expects that the molecules built from the corresponding charged currents will be degenerate in masses because their QCD expressions are the same in the chiral limit.

- **Radial excitations**

If one considers the value of the continuum threshold τc, at which the optimal value of the ground state is obtained, as an approximate value of the mass of the 1st radial excitation, one expects that the radial excitations are in the region of about 0.4 to 1.6 GeV.
above the ground state mass. A more accurate prediction can be obtained by combining LSR with Finite Energy Sum Rule (FESR) [53, 54, 59] where the mass-s-splitting is expected to be around 250-300 MeV at LO. Among these different observed states, the $Z_3(4430)$ and $X_0(4506, 4704)$ could eventually be considered as radial excitation candidates.

11. Quark Mass Behaviour of the Decay Constants

The couplings or decay constants given in Tables 4 to 7 are normalized in Eq.4 in the same way as $f_\pi = 130.4(2)$ MeV through its coupling to the pseudoscalar current: $\langle 0 | (\bar{q}q)(i\gamma_5)d|\pi^0\rangle = f_\pi m_\pi^2 \phi_\pi(x)$, where $\phi_\pi(x)$ is the pion field.

One can find from Table 4 that $f_{DD} \approx 170(15)$ keV which is about $10^{-3}$ of $f_\pi$ and of $f_B \approx f_D \approx 206(7)$ MeV [15, 65, 66]. The same observation holds for the other molecule and four-quark states indicating the weak coupling of these states to the associated interpolating currents.

Comparing the size of the couplings in the $c$ and $b$ quark channels (Tables 4 to 7), one can observe that the ratio decreases by a factor about 10 from the $c$ to the $b$ channels for the $0^{++}$ and $1^{++}$ states which is about the value of the ratio $(m_c/m_b)^{3/2}$, while it decreases by about a factor 4 for the $0^{--}$ and $1^{--}$ states which is about the value $(m_c/m_b)$. These behaviours can be compared with the well-known one of $f_B \sim 1/m_b^{1/2}$ from HQET and can motivate further theoretical studies of the molecule and four-quark couplings.

| Nature | $J^P$ | Mass [MeV] | $f_x$ [keV] | $f_x(4.5)$ [keV] |
|--------|-------|------------|-------------|------------------|
| Molecule | $B^*K$ | 1$^+$ | 5186 ± 13 | 4.48 ± 1.45 | 8.02 ± 2.60 |
| | $BK$ | 0$^+$ | 5195 ± 15 | 2.57 ± 0.75 | 8.26 ± 2.40 |
| | $B^*_0\pi$ | 1$^+$ | 5200 ± 18 | 5.61 ± 0.87 | 10.23 ± 1.59 |
| | $B_0\pi$ | 0$^+$ | 5199 ± 24 | 3.15 ± 0.70 | 10.5 ± 2.30 |
| Four-quark ($su(6\bar{d})$) | $A_b$ | 1$^+$ | 5186 ± 16 | 5.05 ± 1.32 | 9.04 ± 2.37 |
| | $S_b$ | 0$^+$ | 5196 ± 17 | 2.98 ± 0.70 | 9.99 ± 2.36 |
| | $A_c$ | 1$^+$ | 2395 ± 48 | 155 ± 36 | 226 ± 52 |
| | $D'K$ | 1$^+$ | 2402 ± 42 | 139 ± 26 | 254 ± 48 |
| | $DK$ | 0$^+$ | 2395 ± 48 | 215 ± 35 | 308 ± 49 |
| | $D_0\pi$ | 0$^+$ | 2404 ± 37 | 160 ± 22 | 331 ± 46 |
| | $A_c$ | 2$^+$ | 2400 ± 47 | 192 ± 41 | 260 ± 55 |
| | $S_c$ | 0$^+$ | 2395 ± 68 | 122 ± 26 | 221 ± 47 |

12. The case of the X(5568)

In [2], we have also studied the $X$ hadron formed by 3 light quarks $uds$ and one heavy quark $Q \equiv c, b$ using the same approach as above by assuming if it is a molecule or four-quark state. We have included NLO and N2LO PT corrections and the contributions of condensates of dimension $d \leq 7$. Our results are summarized in Table 8. Contrary to previous claims in the sum rule literature, our results do not favour a $BK$, $B^*_0\pi$ or $B_0\pi$ molecule or four-quark $(bu)(\bar{d}\bar{s})$ state having a mass around 5568 MeV observed by D0 [67] but not confirmed by LHCb [68]. We also predict the corresponding state in the $c$-quark channel where the $D^*_0(2317)$ seen by BABAR [69] in the $D_\pi$ invariant mass, expected to be an isoscalar-scalar state with a width less than 3.8 MeV [3] could be a good candidate for one of such states.

13. Conclusions

We have presented in these talks a summary of the results obtained in the chiral limit at N2LO of PT [1, 2]. The extension of this work including $SU(3)$ breaking terms which we shall compare with recent experimental states decaying to $J/\psi\phi$ is under investigation.

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