Bijective counting of humps and peaks in 
$(k, a)$-paths

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Abstract. Recently, Mansour and Shattuck related the total number of humps in all of the $(k, a)$-paths of order $n$ to the number of super $(k, a)$-paths, which generalized previous results concerning the cases when $k = 1$ and $a = 1$ or $a = \infty$. They also derived a relation on the total number of peaks in all of the $(k, a)$-paths of order $n$ and the number of super $(k, a)$-paths, and asked for bijective proofs. In this paper, we will give bijective proofs of these two relations.

Key words: $(k, a)$-path; hump; peak.

AMS Mathematical Subject Classifications: 05A05, 05C30.

1 Introduction

A $(k, a)$-path of order $n$ is a lattice path in $\mathbb{Z} \times \mathbb{Z}$ from $(0, 0)$ to $(n, 0)$ using up steps $(1, k)$, down steps $(1, -1)$ and horizontal steps $(a, 0)$ and never lying below the $x$-axis. Denote by $\mathcal{P}_n(k, a)$ the set of all $(k, a)$-paths of order $n$. Note that $\mathcal{P}_n(k, \infty)$, $\mathcal{P}_n(1, \infty)$, $\mathcal{P}_n(1, 1)$, and $\mathcal{P}_n(1, 2)$ are the set of k-ary paths \cite{1}, Dyck paths, Motzkin paths and Schröder paths, respectively. If a $(k, a)$-path of order $n$ is allowed to go below the $x$-axis, then it is called a super $(k, a)$-path of order $n$. Denote by $\mathcal{SP}_n(k, a)$ the set of all super $(k, a)$-paths of order $n$. A peak in a $(k, a)$-path is an up step followed by a down step. A hump in a $(k, a)$-path is an up step followed by zero or more horizontal steps followed by a down step. We denote by $#Peaks(P)$ and $#Humps(P)$ the number of peaks and humps in a path $P$.

Using a recurrence relation and the WZ method, Regev \cite{6} proved that

\begin{equation}
2 \sum_{P \in \mathcal{P}_n(1, 1)} #Humps(P) = |\mathcal{SP}_n(1, 1)| - 1.
\end{equation}

\begin{equation}
2 \sum_{P \in \mathcal{P}_n(1, \infty)} #Peaks(P) = |\mathcal{SP}_n(1, \infty)|
\end{equation}
The bijective proofs of Formulae (1.1) and (1.2) were given by Ding and Du [3]. They also derived the following analogous result for Schröder paths
\[ \sum_{P \in \mathcal{P}_n(1,2)} \# \text{Humps}(P) = |\mathcal{SP}_n(1,2)| - 1 \]
(1.3)

Recently, Mansour and Shattuck [4] proved that
\[ (k + 1) \sum_{P \in \mathcal{P}_n(k,a)} \# \text{Humps}(P) = |\mathcal{SP}_n(k,a)| - \delta_{a|n}, \]
(1.4)
\[ (k + 1) \sum_{P \in \mathcal{P}_n(k,a)} \# \text{Peaks}(P) = |\mathcal{SP}_n(k,a)| - |\mathcal{SP}_{n-a}(k,a)|, \]
(1.5)
where \(\delta_{a|n} = 1\) if \(a\) divides \(n\) or 0 otherwise, and asked for bijective proofs. Specializing \(k = 1\) and \(a = 1\), \(a = \infty\) or \(a = 2\) in Formulae (1.4) and (1.5) gives Formulae (1.1), (1.2) and (1.3). The main objective of this paper is to give bijective proofs of Formulae (1.4) and (1.5) in answer to the problem posed by Mansour and Shattuck. As a consequence, our bijection also allows us to get the enumeration of \(k\)-ary paths with respect to the number of peaks.

2 The bijective proofs

In this section, we will give bijective proofs of (1.4) and (1.5). We begin with some necessary definitions and notations.

Throughout this paper we identify a path with a word by encoding each up step by the letter \(U\), each down step by \(D\) and each horizontal step by \(H\). Let \(p\) be a step running from the point \((x_1, y_1)\) to the point \((x_2, y_2)\). Then we say that the point \((x_1, y_1)\) is its starting point, and the point \((x_2, y_2)\) is its ending point. The starting point and the ending point of a path are defined analogously. A point \((x, y)\) of a lattice path \(P\) is said to be a return point if \(y = 0\) and \((x, y)\) is not the starting point of \(P\). An up step is said to intersect the \(x\)-axis if its starting point lies weakly below the \(x\)-axis and its ending point lies weakly above the \(x\)-axis. If \(P = p_1p_2 \ldots p_n\) is a path, then the reverse of the path, denoted by \(\hat{P}\), is defined by \(p_{n}p_{n-1} \ldots p_1\). For example, the reverse of the path \(P = HUDDUDH\) is given by \(HDUDDUH\).

A \((k,a)\)-path is said to be \(m\)-peak (resp. \(m\)-hump) colored if exactly one peak (resp. hump) is assigned by any of the \(m\) colors. Denote by \(\mathcal{PP}_n^m(k,a)\) (resp. \(\mathcal{PM}_n^m(k,a)\)) the set of all \(m\)-peak (resp. \(m\)-hump) colored \((k,a)\)-paths.
Observe that the left-hand sides of (1.4) and (1.5) are equal to $|\mathcal{PM}_{n+1}^k(k,a)|$ and $|\mathcal{PP}_{n+1}^k(k,a)|$, respectively. Moreover, the right-hand side of (1.4) counts the number of super $(k,a)$-paths of order $n$ and with at least one up step. The right-hand side of (1.5) counts the number of super $(k,a)$-paths of order $n$ which do not start with horizontal steps. Denote by $S'_n(k,a)$ and $S''_n(k,a)$ the set of all super $(k,a)$-paths of order $n$ and with at least one up step, and the set of all super $(k,a)$-paths of order $n$ which do not start with horizontal steps, respectively. Thus Formulae (1.4) and (1.5) can be rewritten as

$$|\mathcal{PM}_{n+1}^k(k,a)| = |S'_n(k,a)|,$$

(2.1)

$$|\mathcal{PP}_{n+1}^k(k,a)| = |S''_n(k,a)|.$$  

(2.2)

In order to prove Formulae (2.1) and (2.2), we will establish a bijection $\phi$ between the set $\mathcal{PM}_{n+1}^k(k,a)$ and the set $S'_n(k,a)$. Moreover, we show that the map $\phi$ restricted to the set $\mathcal{PP}_{n+1}^k(k,a)$ gives a bijection between the set $\mathcal{PP}_{n+1}^k(k,a)$ and the set $S''_n(k,a)$.

Now we proceed to describe the map $\phi$ from the set $\mathcal{PM}_{n+1}^k(k,a)$ to the set $S'_n(k,a)$. Let $P = p_1p_2\ldots p_n$ be a $(k+1)$-hump colored $(k,a)$-path of order $n$. Suppose that the hump $p_ip_{i+1}\ldots p_m = UH^{m-i-1}D$ is colored by $c$, where $H^i$ denotes $i$ consecutive horizontal steps. Assume that $p_i$ goes from the point $(x_1,h)$ to the point $(x_1+1,h+k)$ for some nonnegative integers $x_1$ and $h$.

Then the path $P$ can be uniquely decomposed as

$$R_1P'p_1H^{m-l-1}d_1R_2d_2\ldots R_kd_kP'' ,$$

where

- each $d_i$ is the first step after $p_i$ that goes from the line $y = h + k + 1 - i$ to the line $y = h + k - i$;

- $P'$ is the (possibly empty) section of $P$ which is to the left of $p_i$, starts with an up step, and lies strictly above the $x$-axis except for the starting point;

- each $R_i$ is a (possibly empty) $(k,a)$-path;

- $P''$ is the remaining section of $P$ after $d_k$.

Obviously, each $d_i$ is a down step and the subpath $P''$ goes from the line $y = h$ to the $x$-axis in $P$. Now we proceed to construct $\phi(P)$ as follows:
(i) if \( c = 1 \), then set
\[
\phi(P) = H^{m-l-1}p_1R_1d_1R_2d_2 \ldots R_kd_kP''P';
\]

(ii) if \( c = k + 1 \), then set
\[
\phi(P) = H^{m-l-1}\widehat{d_1}\widehat{R_1}\widehat{d_2}\widehat{R_2} \ldots \widehat{d_k}\widehat{R_k}p_lP''P';
\]

(iii) if \( 1 < c < k + 1 \), then set
\[
\phi(P) = H^{m-l-1}\widehat{d_1}\widehat{R_1}\widehat{d_2}\widehat{R_2} \ldots \widehat{d_{c-1}}\widehat{R_{c-1}}p_lR_c \ldots R_kd_kP''P'.
\]

According to the construction of the map \( \phi \), we preserve the number of up steps, the number of down steps and the number of horizontal steps. Moreover, there is at least one up step in the resulting path. Hence, the map \( \phi \) is well defined, that is, \( \phi(P) \in S'_n(k,a) \).

**Remark 2.1** Our map \( \phi \) restricted to case when \( k = 1 \) is different from the bijection given by Ding and Du [3].

**Example 2.2** An example of the decomposition of a \((3,2)\)-path is shown in Figure 1, where the colored hump is marked by a star and \( R_3 \) is an empty path. Suppose that the hump is colored by 2. By applying the map \( \phi \), we get its corresponding super \((3,2)\)-path shown in Figure 2.

In order to show that \( \phi \) is a bijection, we describe a map \( \psi \) from the set \( S'_n(k,a) \) to the set \( \mathcal{P}\mathcal{M}^{k+1}_n(k,a) \). Given a super \((k,a)\)-path \( Q = q_1q_2 \ldots q_n \in S'_n(k,a) \), let \( q_l \) be the leftmost up step that intersects the \( x\)-axis. Suppose that \( q_l \) goes from the point \((x_1,p)\) to the point \((x_1 + 1,q)\), where \( q - p = k \). Let \( A \) be the first return point to the right of the point \((x_1,p)\) and \( B \) be the the lowest point to the right of \((x_1,p)\) in \( Q \). If there are more than one such lowest point, we choose \( B \) to be the rightmost one. Then we generate \( \psi(Q) \) as follows.

(i') If \( q > 0 \) and \( p = 0 \), then \( Q \) can be uniquely decomposed as
\[
Q = H^{l-1}q_lR_1d_1R_2d_2 \ldots R_kd_kQ'Q'',
\]

where
- each \( d_i \) is the first step that goes from the line \( y = k + 1 - i \) to the line \( y = k - i \);
– each $R_i$ is a (possibly empty) $(k,a)$-path;
– $Q'$ is the section of $Q$ which goes from the point $A$ to the point $B$;
– $Q''$ is the remaining section of $Q$.

Then set

$$\psi(Q) = R_1Q'q_1H^{l-1}d_1R_2d_2\ldots R_kd_kQ',$$

where the hump $q_1H^{l-1}d_1$ is colored by $1$.

$(ii')$ If $q = 0$, then $Q$ can be uniquely decomposed as

$$Q = H^md_1\hat{R}_1d_2\hat{R}_2\ldots d_k\hat{R}_kq_1Q'Q'',$$

where

– $m$ is a nonnegative integer;
– each $d_i$ is the last step to the left of the point $(x_1,p)$ that goes from the line $y = -i + 1$ to the line $y = -i$;
– each $R_i$ is a (possibly empty) $(k,a)$-path;
– $Q'$ is the section of $Q$ which goes from the point $A$ to the point $B$;
– $Q''$ is the remaining section of $Q$.

Then set

$$\psi(Q) = R_1Q''q_1H^md_1R_2d_2\ldots R_kd_kQ',$$

where the hump $q_1H^md_1$ is colored by $k + 1$.

$(iii')$ If $q > 0$ and $p < 0$, then $Q$ can be uniquely decomposed as

$$Q = H^md_1\hat{R}_1d_2\hat{R}_2\ldots d_{|p|}\hat{R}_{|p|}q_1R_{|p|+1}d_{|p|+1}\ldots R_kd_kQ'Q'',$$

where

– $m$ is a nonnegative integer;
– for $1 \leq i \leq |p|$, each $d_i$ is the last step to the left of $q_i$ that goes from the line $y = -i + 1$ to the line $y = -i$;
– for $|p| + 1 \leq i \leq k$, each $d_i$ is the first step that goes from the line $y = k + 1 - i$ to the line $y = k - i$;
– each $R_i$ is a (possibly empty) $(k,a)$-path;
– $Q'$ is the section of $Q$ which goes from the point $A$ to the point $B$;
– $Q''$ is the remaining section of $Q$. 

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Then set
\[ \psi(Q) = R_1 Q'' q_H m d_1 R_2 d_2 \ldots R_k d_k Q', \]
where the hump \( q_H m d_1 \) is colored by \(|p| + 1\).

**Example 2.3** The decomposition of a super \((3,2)\)-path \(Q\) is illustrated in Figure 3, where \( A = (14, 0) \) and \( B = (30, -4) \), the path \( R_3 \) is empty, and the leftmost up step that intersects the \( x \)-axis goes from the point \((7, -1)\) to the point \((8, 2)\). By applying the map \( \psi \), we get a \((3,2)\)-path \( \psi(Q) \) shown in Figure 1.

Obviously, each \( d_i \) is a down step. It is easy to check that the map \( \psi \) is well defined, that is, \( \psi(Q) \in PM_{k+1}^{(k,a)} \). From the construction of the map \( \phi \), we see that \( p_l \) is the leftmost up step that intersects the \( x \)-axis and the ending point of \( P'' \) is the last lowest point to the right of the starting point of \( p_l \) in \( \phi(P) \). Moreover, the starting point of \( P'' \) is the first return point to the right of the starting point of \( p_l \) in \( \phi(P) \). Thus, one can easily verify that \((i')\), \((ii')\) and \((iii')\) respectively reverse the procedures of \((i)\), \((ii)\) and \((iii)\). This implies that the maps \( \phi \) and \( \psi \) are inverses of each other. Hence, the map \( \phi \) is a bijection.

**Theorem 2.4** The map \( \phi \) is a bijection between the set \( PM_{k+1}^{(k,a)} \) and the set \( S'(n)_{k,a} \). Moreover, for any \( P \in PM_{k+1}^{(k,a)} \) whose colored hump consists of \( UHlD \), the corresponding super \((k,a)\)-path \( \phi(P) \) starts with exactly \( l \) consecutive horizontal steps.

From Theorem 2.4 it follows that the bijection \( \phi \) restricted to the set \( PP_{n+1}^{(k,a)} \) reduces to a bijection between the set \( PP_{n+1}^{(k,a)} \) and the set \( S''_{n}(k,a) \). Thus we obtain bijective proofs of Formulae (1.4) and (1.5).

Our bijection \( \phi \) also allows us to enumerate \( k \)-ary paths with respect to the number of peaks. Let \( Q_n(k, m) \) be the set of \( k \)-ary paths with \( n \) up steps and \( m \) peaks in which exactly one peak is colored by 1. Denote by \( S_{n}^{UU}(k, m) \) the set of super \( k \)-ary paths with \( n \) up steps and \( m \) peaks which start with at least two consecutive up steps. Denote by \( S_{n}^{UD}(k, m) \) the set of super \( k \)-ary paths with \( n \) up steps and \( m \) peaks which start with an up step followed immediately by a down step.

From the construction of the bijection \( \phi \), it is easily seen that the bijection \( \phi \) restricted to the set \( Q_n(k, m) \) reduces to a bijection between the set \( Q_n(k, m) \) and the set \( S_{n}^{UU}(k, m - 1) \cup S_{n}^{UD}(k, m) \). In order to get the enumeration of \( k \)-ary paths with respect to the number of peaks, we need the following lemma.
Lemma 2.5 For $n, m \geq 1$, we have

$$|S_n^{UU}(k, m)| = \binom{n-1}{m} \binom{kn-1}{m-1}, \quad (2.3)$$

$$|S_n^{UD}(k, m)| = \binom{n-1}{m-1} \binom{kn-1}{m-1}. \quad (2.4)$$

Proof. Each $P \in S_n^{UU}(k, m)$ can be uniquely written as

$$UU^{x_1}D^{y_1}U^{x_2}D^{y_2} \ldots U^{x_m}D^{y_m}U^{x_{m+1}}$$

such that

$$\begin{align*}
1 + x_1 + x_2 + \ldots + x_{m+1} &= n \\
y_1 + y_2 + \ldots + y_m &= kn,
\end{align*}$$

where $x_i, y_i \geq 1$ for $1 \leq i \leq m$ and $x_{m+1} \geq 0$. The solutions of $x_i$’s is equal to $\binom{n-1}{m}$ and the solutions of $y_i$’s is equal to $\binom{kn-1}{m-1}$. Thus, Formula (2.3) is proved.

Each $P \in S_n^{UU}(k, m)$ can be uniquely written as

$$UD^{y_1}U^{x_1}D^{y_2} \ldots U^{x_{m-1}}D^{y_m}U^{x_m}$$

such that

$$\begin{align*}
1 + x_1 + x_2 + \ldots x_m &= n \\
y_1 + y_2 + \ldots + y_m &= kn,
\end{align*}$$

where $x_i, y_i \geq 1$ for $1 \leq i \leq m - 1$, $y_m \geq 1$ and $x_m \geq 0$. The solutions of $x_i$’s is equal to $\binom{n-1}{m-1}$ and the solutions of $y_i$’s is equal to $\binom{kn-1}{m-1}$. This leads to Formula (2.4). This completes the proof.

From Formulae (2.3) and (2.4), we deduce that the number of $k$-ary paths with $n$ up steps and $m$ peaks is equal to

$$\frac{1}{m} \left( \binom{n-1}{m-1} \binom{kn-1}{m-2} + \binom{n-1}{m-1} \binom{kn-1}{m-1} \right) = \frac{1}{n} \binom{n}{m} \binom{kn}{m-1}.$$ 

Note that when $k = 1$, we are led to the Narayana numbers [4, 5].

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Figure 1: The decomposition of a \((3,2)\)-path.

Figure 2: The application of \(\phi\) to the path shown in Figure 1.

Figure 3: The decomposition of a super \((3,2)\)-path \(Q\).