Derivative interactions
in de Rham-Gabadadze-Tolley massive gravity

Rampei Kimura and Daisuke Yamauchi

Research Center for the Early Universe,
The University of Tokyo, Tokyo, 113-0033, Japan

Abstract

We investigate the possibility of a new massive gravity theory with derivative interactions as an extension of de Rham-Gabadadze-Tolley massive gravity. We find the most general Lagrangian of derivative interactions using Riemann tensor whose cutoff energy scale is $\Lambda_3$, which is consistent with de Rham-Gabadadze-Tolley massive gravity. Surprisingly, this infinite number of derivative interactions can be resummed with the same method in de Rham-Gabadadze-Tolley massive gravity, and remaining interactions contain only two parameters. We show that the equations of motion for scalar and tensor modes in the decoupling limit contain fourth derivatives with respect to spacetime, which implies the appearance of ghosts at $\Lambda_3$. We claim that consistent derivative interactions in de Rham-Gabadadze-Tolley massive gravity have a mass scale $M$, which is much smaller than the Planck mass $M_{\text{Pl}}$. 
1 Introduction

The construction of consistent theories of a massive spin-2 field has attracted considerable attention from theoretical physicists since Fierz and Pauli proposed the linearized massive gravity in 1939 \cite{1}. This theory consists of the linearized Einstein-Hilbert term and the quadratic mass term described by the fluctuation tensor \( h_{\mu\nu} \) around Minkowski spacetime, \( g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}/M_{\text{Pl}} \). The action of Fierz-Pauli massive gravity is given by

\[
S_{\text{FP}} = \int d^4x \left[ -\frac{1}{4} h^{\mu\nu} \mathcal{E}^{\alpha\beta}_{\mu\nu} h_{\alpha\beta} - m^2 (h_{\mu\nu} h^{\mu\nu} - h^2) + \frac{1}{M_{\text{Pl}}} h_{\mu\nu} T^{\mu\nu} \right],
\]

where \( \mathcal{E}^{\alpha\beta}_{\mu\nu} \) is the Lichnerowicz operator, \( m \) is a graviton mass, \( h \) is the trace of the fluctuation tensor \( h_{\mu\nu} \), and \( T^{\mu\nu} \) is the energy-momentum tensor. This mass term is constructed so that an extra ghost degree of freedom does not appear and the number of propagating degrees of freedom is 5. The massless limit of Fierz-Pauli theory does not reproduce the result of general relativity; i.e., the propagator of Fierz-Pauli theory, in the massless limit does not give the massless one due to the coupling between matter and the scalar mode of the graviton. This problem is called van Dam-Veltman-Zakharov discontinuity \cite{2, 3}. The question of this discontinuity was solved by Vainshtein \cite{4}, who pointed out that nonlinearities become important as the graviton mass goes to zero and they recover general relativity through the so-called Vainshtein mechanism. Nonetheless, in Fierz-Pauli theory the appearance of an extra ghost degree of freedom, called the Boulware-Deser (BD) ghost, is inevitable \cite{5}, and thus the construction of consistent ghost-free theories has been thought to be impossible due to the lack of a Hamiltonian constraint \cite{6}. However, de Rham and Gabadadze constructed a ghost-free theory in the so-called decoupling limit by adding appropriate combinations of nonlinear potential terms \cite{7}. Furthermore, de Rham, Gabadadze, and Tolley (dRGT) showed that the infinite number of potentials can be resummed by using a new tensor, which has a square root structure \cite{8}. The absence of the BD ghost in the full theory has been proven by Hassan and Rosen, and they confirmed the existence of a Hamiltonian constraint, as well as a secondary constraint, which ensures the appropriate number of degrees of freedom of the massive graviton \cite{9}.

Recently, Ref. \cite{10} pointed out a possibility to add a ‘pseudo-linear’ derivative interaction term without introducing a new additional degree of freedom in massive spin-2 theories; it is given by

\[
\mathcal{L}_{2,3} \sim M_{\text{Pl}}^2 \varepsilon^{\mu\nu\rho\sigma} \varepsilon^{\alpha\beta\gamma\delta} \partial_\mu \partial_\alpha h_{\nu\beta} h_{\rho\gamma} h_{\sigma\delta},
\]

in four dimensions.\(^1\) Here \( \varepsilon^{\mu\nu\rho\sigma} \) is the Levi-Civita symbol normalized so that \( \varepsilon^{0123} = 1 \).

\(^1\) These terms, Eq.\((2)\), are not clearly linear in \( h \), but they are the leading terms in the expansion of small fluctuations, which satisfies the gauge symmetry. Therefore, we call them ‘pseudo-linear’ derivative interactions.

\(^2\) In Ref. \cite{11}, the authors already derived the same pseudo-linear derivative interaction term, Eq.\((2)\), in a different way before Ref. \cite{10} appeared.
1. The antisymmetric structure of Eq. (2) prevents \( h_{00} \) from appearing nonlinearly; thus, this term is definitely linear in \( h_{00} \), and it becomes a Lagrange multiplier, which produces a Hamiltonian constraint. However, \( h_{00} \) itself does not give a Hamiltonian constraint dRGT massive gravity; hence, we are not certain that nonlinear generalizations of the derivative interaction Eq. (2) exist.

In this paper, we extend dRGT massive gravity theory by constructing ‘nonlinear’ derivative interactions and we investigate whether these derivative interactions are consistent or not. In Sec 2, we briefly review dRGT massive gravity and \( \Lambda^3 \) theory in the decoupling limit. In Sec 3, we derive the most general Lagrangian of nonlinear derivative interactions using the Riemann tensor. In Sec 4, we investigate the consistency of the nonlinear derivative interactions constructed in Sec 3. Section 5 is devoted to conclusions.

Throughout the paper, we use units in which the speed of light and the Planck constant are unity, \( c = \hbar = 1 \), and \( M_{\text{Pl}} \) is the reduced Planck mass related to Newton’s constant by \( M_{\text{Pl}} = 1/\sqrt{8\pi G} \). We follow the metric signature convention \((- + + +)\). Some contractions of rank-2 tensors are denoted by \( K_{\mu\nu} = [K] \), \( K_{\mu\nu} K_{\mu\nu} = [K^2] \), \( K_{\alpha\beta} K_{\alpha\beta} = [K^3] \), and so on.

2 de Rham-Gabadadze-Tolley massive gravity

The action for ghost-free massive gravity is given by \[^7, 8\]

\[
S_{\text{MG}} = \frac{M_{\text{Pl}}^2}{2} \int d^4 x \sqrt{-g} \left[ R - \frac{m^2}{4} (\mathcal{U}_2 + \alpha_3 \mathcal{U}_3 + \alpha_4 \mathcal{U}_4) \right] + S_m[g_{\mu\nu}, \psi],
\]

where the potentials are given by

\[
\begin{align*}
\mathcal{U}_2 &= 2 \varepsilon_{\mu\alpha\rho\sigma} \varepsilon^{\nu\beta\rho\sigma} K_{\mu\nu} K_{\alpha\beta} = 4 \left([K^2] - [K]^2\right), \\
\mathcal{U}_3 &= \varepsilon_{\mu\alpha\gamma\rho} \varepsilon^{\nu\beta\rho\sigma} K_{\mu\nu} K_{\alpha\beta} K_{\gamma\delta} = -[K]^3 + 3[K][K^2] - 2[K^3], \\
\mathcal{U}_4 &= \varepsilon_{\mu\alpha\gamma\rho} \varepsilon^{\nu\beta\rho\sigma} K_{\mu\nu} K_{\alpha\beta} K_{\gamma\delta} K_{\sigma} \\
&= -[K]^4 + 6[K]^2 [K^2] - 3[K^2]^2 - 8[K][K^3] + 6[K^4],
\end{align*}
\]

and

\[
K_{\mu\nu} = \delta_{\mu\nu} - \sqrt{g} \partial_\mu \partial_\nu - \frac{H_{\mu\nu}}{\sqrt{g}}
= \delta_{\mu\nu} - \sqrt{\eta_{ab}} \partial_\mu \phi^a \partial_\nu \phi^b.
\]

Here \( \alpha_3 \) and \( \alpha_4 \) are constants, \( \varepsilon_{0123} = \sqrt{-g} \), the fluctuation tensor \( H_{\mu\nu} \) is defined by \( H_{\mu\nu} = g_{\mu\nu} - \eta_{ab} \partial_\mu \phi^a \partial_\nu \phi^b \), and \( \phi^a \) is called the St"uckelberg field, which is responsible for restoring general covariance of the theory \[^{12}\]. The choice of the St"uckelberg field is arbitrary, and fixing the unitary gauge, \( \phi^a = \delta^a_\mu x^\mu \), it reduces to Fierz-Pauli massive gravity at linear level.
The decoupling limit is very convenient to capture high energy behavior below the graviton Compton wavelength. Due to the decoupling of vector modes, we can safely ignore the vector modes in the decoupling limit. Usually the St"uckelberg field can be expanded around the unitary gauge,

$$\phi^a = \delta^a_{\mu} x^{\mu} - \eta^{a\mu} \partial_\mu \pi / M_{Pl} m^2,$$

where $\pi$ describes the scalar mode of the massive graviton. We also expand the physical metric as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} / M_{Pl}$. Thus, we can extract the tensor and scalar modes of the massive graviton by taking the following limits,

$$M_{Pl} \to \infty, \quad m \to 0, \quad \Lambda_3 = (M_{Pl} m^2)^{1/3} = \text{fixed}, \quad \frac{T_{\mu\nu}}{M_{Pl}} = \text{fixed}. (7)$$

Then the action in the decoupling limit is given by

$$L_{DL} = -\frac{1}{4} h^{\mu\nu} \varepsilon_{\alpha\beta} h_{\alpha\beta} - h^{\mu\nu} \left[ \frac{1}{4} \varepsilon_{\mu}^\rho \varepsilon_{\nu}^\sigma \Pi_{\alpha\beta}^\rho \varepsilon_{\sigma}^\beta \Pi_{\alpha\beta} \right] + \frac{1}{M_{Pl}} h^{\mu\nu} T_{\mu\nu}. (8)$$

where we defined $\Pi_{\mu\nu} \equiv \partial_\mu \partial_\nu \pi$. The $\Lambda_3$ is the cutoff energy scale of this theory, and the theory above $\Lambda_3$ cannot be trusted. The self-interactions of the scalar mode become the total derivative in the decoupling limit; therefore, the BD ghost does not appear at nonlinear level. In addition, it is obvious that the remaining equations of motion for both $h_{\mu\nu}$ and $\pi$ are the second order differential equations, which prevent the BD ghost from appearing in the theory.

### 3 Construction of Lagrangians

Now we want to construct nonlinear derivative interactions in dRGT massive gravity. To this end, we demand the following restrictions:

1. Linearization of $h_{\mu\nu}$ reproduces Fierz-Pauli massive gravity.
2. The cutoff energy scale is $\Lambda_3$.
3. A derivative interaction term should contribute at the energy scale $\Lambda_3$.
4. The resultant theory does not have a BD ghost.

There are a number of candidates for nonlinear derivative interactions, such as

$$L_{\text{int}} \supset M_{Pl}^2 \sqrt{-g} H R, M_{Pl}^2 \sqrt{-g} H^2 R, M_{Pl}^2 \sqrt{-g} H^3 R, \cdots. (9)$$

\[^3\text{Here we restrict the form of derivative interactions by using only a Riemann tensor. There might be other derivative interactions using the covariant derivative of $H_{\mu\nu}$.}\]
Here we set the mass scale to be $M^2_{Pl}$ for requirement 2, as we will see later. First, we count the energy scales in the decoupling limit. From Eq. (6), $H_{\mu\nu}$ undergoes the following transformation,

$$H_{\mu\nu} \rightarrow h_{\mu\nu} \frac{M_{Pl}}{M_{Pl} m^2} - \frac{\partial_{\mu} \partial_{\nu} \pi}{M_{Pl}^2 m^4},$$

and then the canonically normalized Lagrangian can be schematically written as

$$\mathcal{L}_{int} \sim \Lambda_\lambda^{2-n_h-3n_\pi} h^{n_h-1} \partial^2 h (\partial^2 \pi)^{n_\pi},$$

where we defined the energy scale

$$\Lambda_\lambda = (M_{Pl} m^{\lambda-1})^{1/\lambda}, \quad \lambda = \frac{n_h + 3n_\pi - 2}{n_h + n_\pi - 2}.$$  

Here $n_h \geq 1$ and $n_\pi \geq 1$. For the lowest order of $h_{\mu\nu}$, $n_h = 1$, the energy scales are $\Lambda_5$ for $n_\pi = 2$, $\Lambda_4$ for $n_\pi = 3$, and $\Lambda_{11/3}$ for $n_\pi = 4$, which are lower energy scales than $\Lambda_3$. Therefore, in order to satisfy requirement 2, $\partial^2 h (\partial^2 \pi)^{n_\pi}$ has to be eliminated by the construction of the Lagrangian, and in the next section we show that such eliminations are possible for derivative interactions. For the next order of $h_{\mu\nu}$, $n_h = 2$, the energy scale is always $\Lambda_3$ and does not depend on the value of $n_\pi$, which automatically satisfies requirement 3.

### 3.1 HR order

In this subsection we start with the lowest order terms in a general form,

$$\mathcal{L}_{int,1} = M^2_{Pl} \sqrt{-g} H_{\mu\nu} (R^{\mu\nu} + d R g^{\mu\nu}),$$

where $d$ is a constant. To determine the constant $d$, we first take the unitary gauge, $H_{\mu\nu} = h_{\mu\nu}/M_{Pl}$, and linearize the Lagrangian around Minkowski spacetime, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}/M_{Pl}$. Then the lowest order of $\mathcal{L}_{int,1}$ gives the order of $(\partial h)^2$, which is the same order of the quadratic Lagrangian of the Einstein-Hibert term in (11). In order to satisfy requirement 1, the quadratic action of $\mathcal{L}_{int,1}$ has to be proportional to the Einstein-Hilbert term,

$$\mathcal{L}_{int,1}^{(2)} \propto \left[ \sqrt{-g} R \right]_{h^2}.$$

Therefore, we require $d = -1/2$. Then $\mathcal{L}_{int,1}$ can be written in terms of the Riemann dual tensor,

$$\mathcal{L}_{int,1} = \frac{1}{2} M^2_{Pl} \sqrt{-g} \varepsilon^{\mu\nu\rho\sigma} \varepsilon^{\alpha\beta\gamma}_{\sigma} R_{\mu\nu\alpha\beta} H_{\rho\gamma}.$$  

5
As we stated in the beginning of this section, the energy scale of \( n_h = 1 \) terms in the decoupling limit is potentially dangerous and these terms have to be eliminated. Therefore, we take the decoupling limit of the Lagrangian \( L_{\text{int},1} \). Using the property,

\[
\sqrt{-g} \varepsilon^{\mu\nu\rho\sigma} \varepsilon^{\alpha\beta\gamma} R_{\mu\alpha\nu\beta} = -\frac{1}{M_{\text{Pl}}} \varepsilon^{\mu\nu\rho\sigma} \varepsilon^{\alpha\beta\gamma} \partial_\mu \partial_\alpha h_{\nu\beta},
\]

the lowest order term for \( n_h = 1 \) is given by

\[
L_{\text{int},1} \bigg|_{\partial^2 h \partial^2 \pi} = -\frac{1}{m^2} \varepsilon^{\mu\nu\rho\sigma} \varepsilon^{\alpha\beta\gamma} \partial_\mu \partial_\alpha h_{\nu\beta} \partial_\rho \partial_\gamma \pi
\]

(17)

This is nothing but a total derivative, and the elimination of the \( \partial^2 h \partial^2 \pi \) order term is automatically satisfied by the antisymmetric structure of \( L_{\text{int},1} \). However, the next order \( n_\pi = 2 \) is not a total derivative,

\[
L_{\text{int},1} \bigg|_{\partial^2 h (\partial^2 \pi)^2} = \frac{1}{2 \Lambda_5^2} \varepsilon^{\mu\nu\rho\sigma} \varepsilon^{\alpha\beta\gamma} \partial_\mu \partial_\alpha h_{\nu\beta} \partial_\rho \partial_\kappa \pi \partial_\lambda \partial_\gamma \pi.
\]

(18)

The only way to eliminate this term is to add the next order Lagrangian,

\[
L_{\text{int},1,2} = \frac{1}{8} M_{\text{Pl}}^2 \sqrt{-g} \varepsilon^{\mu\nu\rho\sigma} \varepsilon^{\alpha\beta\gamma} R_{\mu\alpha\nu\beta} H_{\rho\kappa} H_{\kappa}^\gamma.
\]

(19)

This Lagrangian clearly produces the counterterm of Eq.(18), but it contains an \( n_\pi = 3 \) term,

\[
L_{\text{int},1,3} \bigg|_{\partial^2 h (\partial^2 \pi)^3} = \frac{1}{2 \Lambda_4^4} \varepsilon^{\mu\nu\rho\sigma} \varepsilon^{\alpha\beta\gamma} \partial_\mu \partial_\alpha h_{\nu\beta} \partial_\rho \partial_\kappa \pi \partial_\lambda \partial_\gamma \pi \partial_\lambda \partial_\gamma \pi.
\]

(20)

This \( n_\pi = 3 \) term can also be eliminated by adding the Lagrangian,

\[
L_{\text{int},1,3} = \frac{1}{16} M_{\text{Pl}}^2 \sqrt{-g} \varepsilon^{\mu\nu\rho\sigma} \varepsilon^{\alpha\beta\gamma} R_{\mu\alpha\nu\beta} H_{\rho\kappa} H_{\kappa}^\lambda H_{\gamma}^\gamma.
\]

(21)

Then we can use the same procedure to eliminate the \( n_h = 1 \) term in the decoupling limit by introducing appropriate counterterms. The Lagrangian is given by

\[
L_{\text{int},1} = M_{\text{Pl}}^2 \sqrt{-g} \varepsilon^{\mu\nu\rho\sigma} \varepsilon^{\alpha\beta\gamma} R_{\mu\alpha\nu\beta}
\times \left( \frac{1}{2} H_{\rho\gamma} + \frac{1}{8} H_{\rho\kappa} H_{\gamma}^\kappa + \frac{1}{16} H_{\rho\kappa} H_{\lambda}^\kappa H_{\gamma}^\lambda + \frac{5}{128} H_{\rho\kappa} H_{\lambda}^\kappa H_{\gamma}^\lambda H_{\tau}^\tau + \cdots \right)
\]

\[
= -M_{\text{Pl}}^2 \sqrt{-g} \varepsilon^{\mu\nu\rho\sigma} \varepsilon^{\alpha\beta\gamma} R_{\mu\alpha\nu\beta} g_{\rho\lambda} \sum_{n=1}^\infty \tilde{d}_n (H^n)^\lambda_{\gamma},
\]

(22)
where $\bar{d}_n$ is the expansion coefficient, and $(H^n)_\mu^\nu \equiv H^{\mu}_{\alpha_1} H^{\alpha_1}_{\alpha_2} \cdots H^{\alpha_{n-1}}_{\nu}$. One can notice that the coefficients of these counterparts have the following recursive relation,

$$
\bar{d}_n = - \sum_{i=1}^{i \leq n/2} (-1)^i 2^{-2i} n_i C_i \bar{d}_{n-i},
$$

(23)

where the upper bound of the summation $n/2$ is chosen to be the largest integer, $\bar{d}_1 = -1/2$ and $C_r = n!/((n-r)!r!)$. This coefficient is nothing but the expansion coefficients of the $K$ tensor, $\bar{d}_n = (2n)!/((1-2n)(n!)^24^n)$. Using the expanded expression of (5), $K^\mu_\nu = -\sum_{n=1}^{\infty} \bar{d}_n (H^n)_\mu^\nu$, the Lagrangian can be resummed by using the $K$ tensor,

$$
L_{\text{int},1} = M_{Pl}^2 \sqrt{-g} \varepsilon^{\mu\rho\sigma} \varepsilon^{\alpha\beta\gamma\delta} R_{\mu\nu\rho\sigma} K_{\rho\gamma} H_{\sigma\delta}.
$$

(24)

This Lagrangian does not have the terms of the energy scales below $\Lambda_3$, and nonlinear terms contribute at $\Lambda_3$. Note that from the definition of the $K$ tensor, $K^\mu_\nu|_{h_{\mu\nu}=0} \equiv \partial_\mu \partial_\nu \pi$, we have only one $n_h = 1$ term in the decoupling limit, and the $K$ tensor ensures that the $n_h = 1$ term is a total derivative in the decoupling limit.

### 3.2 $H^2 R$ order

Next we want to consider the next order Lagrangian, $O(H^2 R)$. Since we want to eliminate the energy scales below $\Lambda_3$, we perform the same procedure as the $HR$ case. The starting point of the Lagrangian is

$$
L_{\text{int},2} = \frac{1}{4} M_{Pl}^2 \sqrt{-g} \varepsilon^{\mu\rho\sigma} \varepsilon^{\alpha\beta\gamma\delta} R_{\mu\nu\rho\sigma} H_{\mu\nu} H_{\rho\gamma} H_{\sigma\delta}.
$$

(25)

This is the only total derivative combination which eliminates the $\partial^2 h (\partial^2 \pi)^2$ term. If we have different combinations of $H^2 R$, then the $\partial^2 h (\partial^2 \pi)^2$ term remains because higher order Lagrangians $H^3 R$ cannot eliminate the $\partial^2 h (\partial^2 \pi)^2$ term. Now the term (25) produces the $\partial^2 h (\partial^2 \pi)^3$ term, but we can always add counterterms to eliminate order by order. With the same procedure in the previous subsection, the counterterms can be resummed by using the $K$ tensor again,

$$
L_{\text{int},2} = M_{Pl}^2 \sqrt{-g} \varepsilon^{\mu\rho\sigma} \varepsilon^{\alpha\beta\gamma\delta} R_{\mu\nu\rho\sigma} K_{\rho\gamma} K_{\sigma\delta}.
$$

(26)

Apparently, the $\partial^2 h (\partial^2 \pi)^2$ term are a total derivative, and there is no higher order terms of $\pi$ for $n_h = 1$ in the decoupling limit from the definition of the $K$ tensor. Surprisingly, Linearization of (26) in $h_{\mu\nu}$ in unitary gauge gives the ‘pseudo-linear’ derivative interaction term (2).

One might think that we can start with the $O(H^3 R)$ Lagrangian; however, we do not have a total derivative combination of $\partial^2 h (\partial^2 \pi)^3$ due to the number of indices of the antisymmetric tensor in four dimensions, which means there is no higher
order Lagrangian satisfying the restrictions. Therefore, the most general Riemann derivative interaction for dRGT massive gravity is
\[ L_{\text{int}} = \alpha M_{\text{Pl}}^2 \sqrt{-g} \varepsilon^{\mu
u\rho\sigma} \varepsilon_{\alpha\beta\gamma\delta} R_{\mu\alpha\nu\beta} K_{\rho\gamma\delta}, \] (27)
where \( \alpha \) and \( \beta \) are model parameters.

### 3.3 Arbitrary dimensions

The extension to arbitrary dimensions can be easily done with the same method, and the Lagrangian is given by
\[ L_{\text{int}}^{(D,d,n)} = M_{\text{Pl}}^{D-2} m_{\text{ Pl}}^2 \sqrt{-g} \varepsilon^{\mu_1\mu_2\cdots\mu_D} \varepsilon_{\nu_1\nu_2\cdots\nu_D} R_{\mu_1\nu_1\mu_2\nu_2} \cdots R_{\mu_{d-1}\nu_{d-1}\mu_d\nu_d} \]
\[ \times g_{\mu_{d+1}\nu_{d+1}} \cdots g_{\mu_n\nu_n} K_{\mu_{n+1}\nu_{n+1}} \cdots K_{\mu_D\nu_D}, \] (28)
where \( D \) is the number of dimensions, \( d \) is an even number, and \( n \) is an integer. Here, \( d/2 \) is the number of the Riemann tensor, \( n-d \) is the number of the metric tensor, and \( D-n \) is the number of the \( K \) tensor, satisfying the relation, \( 2 \leq d \leq m \leq D-1 \). Similarly, the lowest order of this term in the decoupling limit always becomes a total derivative, so the contributions of the scalar mode in the decoupling limit are always at the energy scale \( \Lambda_{(D+2)/(D-2)} \equiv (m^{4/(D-2)}M_{\text{Pl}})^{(D-2)/(D+2)} \). Note that in the unitary gauge description, these derivative interaction terms in \( D \) dimensions include the derivative interactions discussed in [10].

### 4 Equations of motion of \( \Lambda_3 \) theory

So far we could successfully eliminate all energy scales below \( \Lambda_3 \), and the derivative interactions [29] in the decoupling limit contribute at the energy scale \( \Lambda_3 \). Now we want to check whether these derivative interactions are free of a BD ghost or not in the decoupling limit. For completeness, in the Appendix we show the lack of a Hamiltonian and momentum constraints in the Arnowitt-Deser-Misner formalism.

The Lagrangian \( L_{\text{int},1} \) in the decoupling limit can be written as
\[ L_{\text{int},1} = \frac{1}{\Lambda^3_3} \left[ \sqrt{-g} \varepsilon^{\mu\nu\rho\sigma} \varepsilon_{\alpha\beta\gamma\delta} R_{\mu\alpha\nu\beta} \right] \partial_\mu \partial_\nu h_{\alpha\beta} \]
\[ - \sum_{n=1}^{\infty} \frac{1}{\Lambda^{3n}_3} \frac{1}{\varepsilon^{\mu\nu\rho\sigma} \varepsilon_{\alpha\beta\gamma\delta}} \partial_\mu \partial_\nu h_{\alpha\beta} \left[ K_{\rho\gamma} \right] h_{(\partial^2 \pi)^n}. \] (29)

For \( n = 1 \), the equation of motion for \( h_{\alpha\beta} \) contains third derivative terms,
\[ \frac{\delta L_{\text{int},1}}{\delta h^{\alpha\beta}} \bigg|_{h\partial^2 h \partial^2 \pi} = \frac{1}{\Lambda^3_3} \left[ \frac{1}{4} \left( \partial_\alpha h^{\mu}_\beta + 2 \partial_\mu h_{\alpha\beta} - \partial_\beta h^{\mu}_\alpha \right) \Box \partial_\mu \pi + \frac{1}{2} \left( \partial^\nu h - \partial_\nu h^{\mu} \right) \partial_\nu \partial_\alpha \partial_\beta \pi \right. \]
\[ + \frac{1}{4} \left( \partial_\beta h^{\mu\nu} - \partial^\nu h_{\beta}^{\mu} \right) \partial_\alpha \partial_\nu \partial_\mu \partial_\beta \pi \]
\[ + \frac{1}{4} \eta_{\alpha\beta} \left( \partial_\mu h^{\mu\nu} - \partial^\nu h^{\mu}_\alpha \right) \partial_\nu \partial_\beta \pi \left. \right] \] (30)
Here fourth derivative terms are accidentally canceled due to antisymmetric tensor. One can check that the equation of motion for $\pi$ also contains the third derivative of $h_{\mu\nu}$, and fourth derivative terms are canceled as well. For $n_{\pi} = 2$, the equation of motion for $\pi$ contains the following term,

$$\frac{\delta L_{\text{int,1}}}{\delta h^{\alpha\beta}} \bigg|_{h^{\partial^2 h} (\partial^2 \pi)^2} \supset -\frac{1}{4\Lambda_3^6} \varepsilon^{\mu\rho\sigma} \varepsilon_{\alpha\beta\gamma\delta} h_{\rho\epsilon} \partial_{\mu} \partial_{\nu} \partial_{\sigma} \partial_{\lambda} \pi \partial_{\gamma} \partial_{\delta} \pi. \tag{31}$$

This equation of motion obviously has fourth derivative terms, and one can easily show that the equation of motion of $h_{\mu\nu}$ for $n_{\pi} = 2$ also contains fourth derivative terms. Therefore, a BD ghost appears at $\Lambda_3$ in $L_{\text{int,1}}$.

The Lagrangian $L_{\text{int,2}}$ in the decoupling limit can be written as

$$L_{\text{int,2}} = \frac{1}{4\Lambda_3^6} \left[ \sqrt{-g} \varepsilon^{\mu\nu\rho\sigma} \varepsilon_{\alpha\beta\gamma\delta} R_{\mu\nu\rho\sigma} \right]_{h^{\partial^2 h}} \partial_{\mu} \partial_{\nu} \partial_{\rho} \partial_{\sigma} \partial_{\alpha} \partial_{\beta} \partial_{\gamma} \partial_{\delta} \pi - \sum_{n_{\pi}=1}^{\infty} \frac{1}{\Lambda_3^{3n_{\pi}}} \varepsilon^{\mu\nu\rho\sigma} \varepsilon_{\alpha\beta\gamma\delta} \partial_{\mu} \partial_{\nu} \partial_{\rho} \partial_{\sigma} \partial_{\alpha} \partial_{\beta} \partial_{\gamma} \partial_{\delta} \pi h^{(\partial^2 \pi)^n_{\pi}}. \tag{32}$$

The lowest order term $n_{\pi} = 1$ comes from the second term, and it is given by

$$L_{\text{int,2}} \bigg|_{h^{\partial^2 h} (\partial^2 \pi)^2} = -\frac{1}{\Lambda_3^3} \varepsilon^{\mu\nu\rho\sigma} \varepsilon_{\alpha\beta\gamma\delta} \partial_{\mu} \partial_{\nu} \partial_{\rho} \partial_{\sigma} \partial_{\alpha} \partial_{\beta} \partial_{\gamma} \partial_{\delta} \pi \left[ R^2 - 4R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \right]_{h^2}. \tag{33}$$

This term is the Gauss-Bonnet term with nonminimal coupling of $\pi$ and does not yield higher order derivatives in the equation of motion. Therefore a ghostly extra degree of freedom does not appear from this term. This term is expected from a pseudo-linear derivative interaction in the decoupling limit as found in Ref. [10]. However, the equations of motion for $h_{\mu\nu}$ and $\pi$ of the next order $n_{\pi} = 2$ contain third derivative terms, and the next order $n_{\pi} = 3$ yields fourth derivative terms, similarly to $RK$ term. These higher derivative terms in the equations of motion cannot be eliminated by the choice of the parameters $\alpha$ and $\beta$. Thus BD ghosts appear at $\Lambda_3$ in the derivative interactions in dRGT massive gravity theory.

To avoid BD ghosts from the derivative interactions (29), the mass scale $M$ in the derivative interactions has to be smaller than the Planck mass $M_{Pl}$. Then the derivative interactions do not contribute at $\Lambda_3$ in the decoupling limit, and ghost modes appear at the energy scale higher than the cutoff energy scale $\Lambda_3$. In this case the Lagrangian in the decoupling limit is exactly given by the pure dRGT theory (8). It is interesting to investigate how the derivative interactions affect cosmological solutions in this case, but it is beyond the scope of this paper.
5 Conclusion

In this paper, we discussed the possibility of adding consistent derivative interactions in de Rham-Gabadadze-Tolley massive gravity. On the construction of derivative interactions, we required the following restrictions: (i) Linearization of $h_{\mu\nu}$ reproduces Fierz-Pauli massive gravity, (ii) the cutoff energy scale is $\Lambda_3$, (iii) a derivative interaction term should contribute at the energy scale $\Lambda_3$, and (iv) the resultant theory does not have a Boulware-Deser ghost. We found the derivative interactions, which reproduce the pseudo-linear derivative interaction term discussed in [10]. These Lagrangians yield the problematic terms in the decoupling limit, whose energy scales are below the cutoff energy scale $\Lambda_3$ in dRGT massive gravity. We showed that these terms can be eliminated by adding the appropriate counterterms, and the remaining terms in the decoupling limit have the energy scale $\Lambda_3$. Furthermore, we showed that the infinite number of derivative interactions can be resummed by using the tensor $K_{\mu\nu}$, which is the same tensor introduced in de Rham-Gabadadze-Tolley massive gravity. Thus, the resummed Lagrangian contains only two parameters. However, the equations of motion for the scalar and tensor modes in the decoupling limit are fourth order differential equations, which generically implies the existence of Boulware-Deser ghosts. Thus, the derivative interactions in de Rham-Gabadadze-Tolley massive gravity, constructed under the above restrictions, suffers from ghosts, which appear at $\Lambda_3$ in the decoupling limit. This implies that the mass scale of the derivative interactions has to be smaller than the Planck mass, then Boulware-Deser ghosts of the derivative interactions can be pushed to a higher energy scale than $\Lambda_3$.

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A Hamiltonian analysis

In this appendix, for completeness, we use the Arnowitt-Deser-Misner formalism to show the existence of ghosts.

First we rewrite the Lagrangian in terms of $\Sigma_{\mu\nu} \equiv (\sqrt{g^{-1}\eta})_{\mu\nu}$ instead of $K$. Then the Lagrangian of the derivative interaction is given by

$$L_{\text{int}} = -2(\alpha + \beta)M_{\text{Pl}}^2\sqrt{-g}R$$

$$- M_{\text{Pl}}^2 \sqrt{-g} \varepsilon^{\mu\nu\rho\sigma} \varepsilon_{\alpha\beta\gamma\delta} R_{\mu\alpha\nu\beta} \left[ (\alpha + 2\beta)g_{\rho\gamma}\Sigma_{\delta\sigma} - \beta \Sigma_{\rho\gamma}\Sigma_{\delta\sigma} \right].$$

(34)

Here the first term is nothing but the Einstein-Hilbert term, which comes from derivative interactions. Let us consider that the spacetime can be foliated by a
family of spacelike hypersurfaces $\Sigma_t$, defined by $t = x^0$. The components of the metric can be parametrized by the lapse $N = 1/\sqrt{-g^{00}}$, the shift $N^i = -g^{0i}/g^{00}$, and the metric on $\Sigma_t$ : $\gamma_{ij} = g_{ij}$ with $i, j$ running from 1 to 3. We introduce the future-pointing unit normal vector $u^\mu$ to the surface $\Sigma_t$. In terms of the ADM variables, the components of $u^\mu$ are $u_0 = -N, u_i = 0, u^0 = 1/N, u^i = -N^i/N$. The components of the spacetime metric are described as

$$
g_{00} = -N^2 + N_i N^i, \quad g_{0i} = N_i, \quad g_{ij} = \gamma_{ij}
g^{00} = -\frac{1}{N^2}, \quad g^{0i} = \frac{N^i}{N^2}, \quad g^{ij} = \gamma^{ij} - \frac{N^i N^j}{N^2},$$

and $\sqrt{-g} = N \sqrt{\gamma}$. In the 3+1 decomposition, the projected Riemann tensor can be obtained by using the Gauss, Codazzi, and Ricci relations,

$$
R_{ijkl} = K_{ik} K_{jl} - K_{il} K_{jk} + \gamma_{ij} R_{ijkl},
R_{ijkl} u^\mu = D_i K_{jk} - D_j K_{ik},
R_{ij} u^i u^\nu = -N^{-1} (\partial_i K_{ij} - \mathcal{L}_{N^k} K_{ij}) + K_{ij} K_{jk} + N^{-1} D_i D_j N,
$$

where $D_i$ denotes the covariant derivative associated with $\gamma_{ij}$, $K_{ij}$ is the extrinsic curvature,

$$
K_{ij} = \frac{1}{2N} (\partial_i \gamma_{ij} - D_i N_j - D_j N_i),
$$

and $\mathcal{L}_{N^k} K_{ij} \equiv N^k D_k K_{ij} + K_{ik} D_j N^k + K_{jk} D_i N^k$. The square root tensor $\Sigma^\mu_\nu$ can be decomposed as

$$
\Sigma^\mu_\alpha \Sigma^\alpha_\nu = \frac{1}{N^2} \left( 1 - N^i (N^2 \gamma^{ij} - N^i N^l \delta_{lj}) \right).
$$

In dRGT massive gravity, the Lagrangian is highly nonlinear in the lapse $N$, which implies the lack of a Hamiltonian constraint. However, one can introduce the new variable, which is the combination of the lapse $N$ and shift $N^i$. This new variable is independent of the lapse; hence, the lapse $N$ becomes a Lagrange multiplier, and a Hamiltonian constraint and an associated secondary constraint always exist. In the extension theory of dRGT massive gravity, we also require the existence of a Hamiltonian constraint and a secondary constraint for avoiding a BD ghost. Since the combination of the lapse $N$ and shift $N^i$ should be determined by the mass term in dRGT theory, we investigate the existence of a Hamiltonian constraint in the derivative interaction term using the Hassan and Rosen method.

Following the proof by Hassan and Rosen [9], we introduce a redefined lapse $n^i$,

$$
n^i = (\delta^i_j + N D^i_j) n^j.
$$

Then the square root matrix can be written in the form

$$
N \Sigma^\mu_\nu = A^\mu_\nu + N B^\mu_\nu,
$$
where $A^\mu_\nu$ and $B^\mu_\nu$ are given by

\[
A^\mu_\nu = \frac{1}{\sqrt{x}} \left( \begin{array}{cc} 1 & n^i \delta_{ij} \\ -n_i & -n^i n^j \end{array} \right),
\]

\[
B^\mu_\nu = \left( \begin{array}{cc} 0 & 0 \\ \sqrt{\gamma} \delta_{ij} & 0 \end{array} \right),
\]

\[
\sqrt{x} D^i_j = \sqrt{\gamma} \delta_{ij},
\]

where we defined $x \equiv 1 - \delta_{ab} n^a n^b$. With this lapse $n^i$, the dRGT mass term is linear in the lapse $N$, which gives the Hamiltonian constraint. However, we now have the time derivative of the extrinsic curvature in the Riemann tensor. This might give a time derivative of the lapse $N$ and shift $n^i$. Therefore the first task is to check whether lapse and shift contain a time derivative or not. A problematic term comes from $R_{\mu ij\nu} u^\mu u^\nu$:

\[
\mathcal{L}_{\text{int}} \supset -2M_P^2 N \sqrt{\gamma} \varepsilon^{ijk} \varepsilon^{jkl} R_{\mu ij\nu} u^\mu u^\nu \left[ (\alpha + 2\beta) \gamma_{ac} \Sigma_{bd} - \beta \Sigma_{ac} \Sigma_{bd} \right],
\]

where $\varepsilon^{ijk} \equiv \epsilon^{|ijk|\mu}\epsilon_{\mu}$. Then the $\partial_i K_{ij}$ term produces the time derivative of the lapse and shift. This term is given by

\[
\mathcal{L}_{\text{int}} \supset 2M_P^2 \sqrt{\gamma} \partial_i K_{ij} \gamma^{j\alpha} \varepsilon^{i\beta\gamma} \varepsilon_{abc} \left[ (\alpha + 2\beta) \delta^b_k \Sigma^c_l - \beta \Sigma^b_k \Sigma^c_l \right].
\]

In order to avoid the time derivative of the lapse $N$ and shift $n^i$, we require that

\[
\varepsilon^{i\beta\gamma} \varepsilon_{abc} \left[ (\alpha + 2\beta) \delta^b_k \Sigma^c_l - \beta \Sigma^b_k \Sigma^c_l \right] = \frac{1}{N^2} X^i_a + \frac{1}{N} Y^i_a + Z^i_a
\]

is independent of lapse $N$ and shift $n^i$. First the $X^i_a$ term is automatically zero:

\[
X^i_a = (\alpha + 2\beta) \left( \delta^i_a [A]^2 - \delta^i_a [A]^2 + 2A^j_b A^b_a - 2A^i_a [A] \right) = 0.
\]

In the second equality, we have used the fact that $A^j_i = [A] \hat{n}^i \hat{n}^j$, where $[A] \equiv \text{Tr} A^{j}_i = -(1 - x)/\sqrt{x}$, $\hat{n}^i = n^i/\sqrt{1 - x}$, and $\delta_{ab} \hat{n}^a \hat{n}^b = 1$. Before evaluating $Y^i_a$, we decompose $B^i_j$ as

\[
B^i_j = \hat{B}_0 \hat{n}^k \delta_{kj} + \hat{B}^i \hat{n}^k \delta_{kj} + \hat{B}_j \hat{n}^i + \hat{B}^i_j
\]

where $\hat{B}_0 \hat{n}^i = 0$, $\hat{B}^i \hat{n}^k \delta_{kj} = 0$, $\hat{B}^i \hat{n}^j = 0$, and $\hat{B}^i_j \hat{n}^k \delta_{kj} = 0$. Then the $Y^i_a$ term becomes

\[
Y^i_a = [A] \left\{ (\alpha + 2\beta) - 2\beta \hat{B} \right\} P^i_a + 2\beta \hat{B}^i_a
\]
where $[\hat{B}] \equiv \text{Tr} \hat{B}^i_j$ and $P^i_j$ is the projection tensor, $P^i_j \equiv \delta^i_j - \hat{n}^a \hat{n}^k \delta_{kj}$. To eliminate the $Y^i_a$ term, the only option is $\hat{B}^i_a = b P^i_a$, where $b$ is a constant; then,

$$Y^i_a = [A] P^i_a (\alpha + 2\beta - 2\beta b) \quad (49)$$

Thus the $1/N$ term cancels if $b = 1 + \alpha/2\beta$. Now we have to check whether this solution satisfies Eq. (42). To see this, we project the square of Eq. (42); then, we have three equations,

\begin{align}
\frac{1}{x} \hat{B}^0_0 + \hat{B}^k \hat{B}^k &= \hat{n}^a \delta_{ab} \gamma^{bc} \delta_{cd} \hat{n}^d \\
\frac{1}{x} \hat{B}_0 \hat{B}^i + \hat{B}^k \hat{B}^i_k &= \hat{n}^a \delta_{ab} \gamma^{bc} P^i_c \\
\frac{1}{x} \hat{B}^i \hat{B}_j + \hat{B}_k \hat{B}^k_j &= \gamma^{ab} \delta_{bc} P^i_a P^c_j
\end{align}

(50) (51) (52)

Here we used the fact that $\delta^i_k \hat{B}^k = \hat{B}^i$, which can be obtained from $\delta^i_k D^k_j = \delta^i_k D^k$. It is obvious that the solution of $\hat{B}^i_j$ contains $\gamma_{ij}$, which means $\hat{B}^i_j \neq P^i_j$. This can be checked by substituting the ansatz, $\hat{B}^i_a = b P^i_a$. Thus $Y^i_j$ is not zero, and the lapse $N$ and the shift $n^i$ are dynamical variables.

However, it should be stressed that the new shift variable is introduced so that the dRGT mass term yields a Hamiltonian constraint. Therefore, there might exist a way to avoid dynamical lapse or shift by introducing new variables, but this is still an unsolved problem. Therefore, this proof only shows the lack of Hamiltonian and momentum constraints.

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