Constrained heterogeneous facility location games with max-variant cost

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Abstract

In this paper, we propose a constrained heterogeneous facility location model where a set of alternative locations are feasible for building facilities and the number of facilities built at each feasible location is limited. Assuming that a set of agents on the real line can strategically report their locations, and the cost of each agent is her distance to the farthest one that she is interested in, we study deterministic mechanism design without money for constrained heterogeneous $K$-facility location games. Depending on whether agents have optional preference, the problem is considered in two settings: the compulsory setting and the optional setting. For the compulsory setting where all agents are served by $K$ heterogeneous facilities, we provide a 3-approximate deterministic group strategyproof mechanism for the objective of minimizing the sum/maximum cost respectively, which is also the best deterministic strategyproof mechanism under the corresponding social objective. For the optional setting where each agent may be only interested in some of the facilities, we study heterogeneous two-facility location games. We propose a deterministic group strategyproof mechanism with approximation ratio of at most $2n + 1$ for the sum cost objective and a deterministic group strategyproof mechanism with approximation ratio of at most 9 for the maximum cost objective.

Keywords  Mechanism design · Facility location · Strategyproof · Constrained

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1 Introduction

In the mechanism design problem for heterogeneous facility location games, there are a set of strategic agents who are required to report their private information and a social planner who intends to locate several heterogeneous facilities based on the reported information, with the purpose of optimizing some social objective. In this paper, we study the problem of locating heterogeneous facilities with constraint, which means that only a set of alternative locations are feasible for building facilities and the number of facilities built at each feasible location is limited.

In contrast with the classic setting where facilities can be built anywhere in a specific metric space and there is no limit on the number of facilities at each location, our constrained model can be well used to describe many practical scenarios, including urban planning, voter voting, political elections, economic decisions, multi-level adjustment of room temperature, etc. Take the location of communication base stations in urban planning for example. Such facilities can only be built at designated sites and the number of facilities at each site is always limited. To accommodate these constraints, we propose a multiset of feasible locations and assume that at most one facility can be built at each location. Here, the individual cost of each agent is assumed to depend on her distance to the farthest one that she is served by, termed as max-variant (Yuan et al. 2016). The max-variant can be found applications in natural scenarios, such as intelligence gathering, the layout of warehouses and the efficiency of network transmission. Take the layout of warehouses for example. A local authority plans to locate different raw material warehouses for several processing plants. Assuming each plant has multiple transport trucks having the same speed, the time that the plant has to wait depends on its distance to the farthest one if it requires raw materials from different sites.

We discuss the mechanism design problem for constrained heterogeneous $K$-facility location games with max-variant cost in two settings: the first is the compulsory setting, where each agent is served by $K$ heterogeneous facilities; the second is the optional setting, where each agent has an additional preference for facilities and may be only interested in some of the $K$ heterogeneous facilities. Considering that agents may manipulate the facility locations by misreporting their private information, we pursue mechanisms that can perform well under some social objective (e.g., minimizing the sum/maximum cost) while guaranteeing truthful report from agents (i.e., strategyproof or group strategyproof).

1.1 Our results

This paper studies deterministic mechanism design without money for constrained heterogeneous facility location games with max-variant cost under the objective of minimizing the sum/maximum cost.

Our key innovations and results are summarized as follows.

In Sect. 2, we formulate the constrained heterogeneous $K$-facility location game with max-variant cost. We propose a finite multiset of alternative locations which are feasible for building facilities at and require that at most one facility can be built at
each location. Thus, by adjusting the number of same elements in the multiset, the model can accommodate different scenarios where the number of facilities at the same location is limited.

In Sect. 3, we focus on deterministic mechanism design for locating $K$ heterogeneous facilities in the compulsory setting. We prove that no deterministic strategyproof mechanism can have an approximation ratio of less than $3$ under the sum/maximum cost objective. In addition, we present $3$-approximate deterministic group strategyproof mechanisms for both social objectives, which implies that the best deterministic strategyproof mechanisms have been obtained.

In Sect. 4, we discuss mechanism design problem for locating two facilities in the optional setting. For the sum cost objective, we propose a deterministic group strategyproof mechanism with approximation ratio of at most $2n + 1$. For the maximum cost objective, we design a deterministic group strategyproof mechanism with approximation ratio of at most $9$.

1.2 Related work

Mechanism design without money for facility location games has been extensively studied in recent years. Early studies focused on the characterization of strategyproof mechanisms. Moulin (1980) identified all the possible strategyproof mechanisms for one-facility location on the line with single-peaked preferences, whose results were extended by Schummer and Vohra (2002) and Dokow et al. (2012) to tree and cycle networks.

Approximate mechanism design without money was initiated by Procaccia and Tennenholtz (2009), who studied deterministic and randomized strategyproof mechanisms with constant approximation ratio for facility location games under the sum cost and the maximum cost in three settings: one-facility, two-facility and multiple facilities per agent. Following this research agenda, numerous studies have emerged, including improvements on the lower/upper bound of approximation ratio (Lu et al. 2010; Fotakis and Tzamos 2014) and further variants.

Cheng et al. (2013) introduced approximate mechanism design for obnoxious facility location games where the facility is not desirable to each agent. Zou and Li (2015) studied the dual preference setting where the facility can be desirable or undesirable for different agents. Zhang and Li (2014) introduced weights to agents and Filos-Ratsikas et al. (2017) studied one-facility location problem with double-peaked preferences. Serafino and Ventre (2016) introduced heterogeneous two-facility location games where each agent cares about either one facility or both and her cost depends on the sum of distances to her interested facilities (termed as sum-variant). Later, Yuan et al. (2016) considered the min-variant and max-variant instead and Anastasiadis and Deligkas (2018) studied heterogeneous $k$-facility setting with min-variant. Besides, various individual and social objectives were also studied. Mei et al. (2019) introduced a happiness factor to measure each agent’s individual utility. Feigenbaum et al. (2017) considered the $L_p$-form of the vector of agent-costs instead of the classic sum cost. Cai et al. (2016) and Chen et al. (2022) studied facility location problems under the objective of minimizing the maximum envy. Ding et al. (2020) and Liu et al. (2021)
considered the envy ratio objective. Zhou et al. (2021) studied group-fair facility location problems.

Further, motivated by real-world applications, researchers have begun to study the mechanism design problem with constraints on the facilities. Aziz et al. (2020a, b) studied facility location problems with capacity constraints. Chen et al. (2021) studied the two-opposite-facility location problem with maximum distance constraint by imposing a penalty. Xu et al. (2021) studied minimum distance requirement for the heterogeneous two-facility location problem. In addition, considering that in reality the feasible locations that facilities could be built at are usually limited, mechanism design for facility location games with limited locations were also studied. Sui and Boutilier (2015) studied approximately strategyproof mechanisms for facility location games with constraints on the feasible placement of facilities. Feldman et al. (2016) studied the one-facility location setting under the sum cost objective in the context of voting embedded in some underlying metric space. Tang et al. (2020) further considered the maximum cost objective and the two-facility setting. Li et al. (2020) studied the heterogeneous two-facility setting with optional preference, which is also the most related to our work among all reasearch on the constrained heterogeneous facility location problem. However, there are at least three differences between us: (1) our model requires a limit on the number of facilities at each feasible location and Li et al. (2020) does not; (2) each agent’s location is private and her preference on facilities is public in our model while it is the opposite in Li et al. (2020); (3) we consider the max-variant cost while (Li et al. 2020) considers the min-variant where the cost of each agent depends on her distance to the closest facility within her acceptable set.

2 The model

Let $N = \{1, 2, \ldots, n\}$ be a set of agents located on the real line and $\mathcal{F} = \{F_1, \ldots, F_K\}$ be the set of $K$ heterogeneous facilities to be located. Each agent $i \in N$ has a location $x_i \in \mathcal{R}$ as her private information and a public facility preference $p_i \subseteq \mathcal{F}$, where $p_i \neq \emptyset$ is the set of facilities that agent $i$ is interested in or needs to be served by. Denote by $x = (x_1, x_2, \ldots, x_n)$ and $p = (p_1, p_2, \ldots, p_n)$ the $n$ agents’ location profile and facility preference profile, respectively.

Let $A = \{a_1, a_2, \ldots, a_m\} \in \mathcal{R}^m (m \geq K)$ be a multiset of alternative locations which are feasible for building facilities and at most one facility can be built at each location. Assume without loss of generality that $a_1 \leq a_2 \leq \ldots \leq a_m$. Denote by $I(x, p, A)$ an instance of the $n$ agents or simply by $I$ without confusion.

Individual and Social Objectives. Due to the constraint on facility locations, the $K$ heterogeneous facilities $F_1, \ldots, F_K$ should be located at $y_1 \in A, y_k \in A \setminus \{y_j\}_{1 \leq j \leq k - 1}, k = 2, \ldots, K$ respectively. Denote by $y = (y_1, \ldots, y_K)$ the facility location profile. The individual cost of agent $i \in N$ is denoted by $c_i(y, (x_i, p_i)) = \max_{F_j \in p_i} |y_j - x_i|$, since her cost only depends on her distance to the farthest one that she is served by. While each agent seeks to minimize her individual cost, the social planner aims to minimize the sum cost or the maximum cost of all agents. For a location profile and a preference profile $(x, p) \in \mathcal{R}^n \times (2^\mathcal{F})^n$, the sum cost and the maximum
cost under \( y \) are denoted by \( \text{sc}(y, (x, p)) = \sum_{i \in N} c_i(y, (x_i, p_i)) \) and \( \text{mc}(y, (x, p)) = \max_{i \in N} c_i(y, (x_i, p_i)) \), respectively. Let \( \text{OPT}_{\text{sc}}(x, p) \) and \( \text{OPT}_{\text{mc}}(x, p) \) be the optimal solution under the objectives of minimizing the sum cost and the maximum cost, respectively.

**Mechanisms.** A deterministic mechanism \( f \) is a function that maps the \( n \) agents’ location profile and preference profile \((x, p)\) to a facility location profile \( y \), i.e.,

\[
f(x, p) = y = (y_1, \ldots, y_K), \forall (x, p) \in \mathcal{R}^n \times \left(2^F\right)^n,
\]

where \( y = (y_1, \ldots, y_K) \) should satisfy \( y_1 \in A, y_k \in A \setminus \{y_j | 1 \leq j \leq k - 1\}, \forall k = 2, \ldots, K \).

Given a mechanism \( f \) and a reported location profile \( x' \in \mathcal{R}^n \), the cost of agent \( i \in N \) under \( f \) is \( c_i(f(x', p), (x_i, p_i)) \), and the sum cost (or the maximum cost) under \( f \) is \( \text{sc}(f(x', p), (x, p)) = \sum_{i \in N} c_i(f(x', p), (x_i, p_i)) \) (or \( \text{mc}(f(x', p), (x, p)) = \max_{i \in N} c_i(f(x', p), (x_i, p_i)) \)).

**Strategyproofness.** Note that agents can manipulate the output of mechanisms by misreporting their locations. Thus, strategyproofness of mechanisms should be taken into account.

For \( i \in N \), let \( x_{-i} = (x_j)_{j \neq i} \) be the location profile without agent \( i \), then \( x = (x_i, x_{-i}) \). For \( S \subseteq N \), let \( x_{S} = (x_i)_{i \in S}, x_{-S} = (x_i)_{i \notin S} \), then \( x = (x_{S}, x_{-S}) \). The definitions of strategyproofness and group strategyproofness are given below.

**Definition 1** A mechanism \( f \) is strategyproof if each agent cannot benefit from misreporting her location, regardless of the others’ strategies, i.e., for every location profile and preference profile \((x, p) \in \mathcal{R}^n \times \left(2^F\right)^n\), every agent \( i \in N \), and every \( x'_i \in \mathcal{R} \),

\[
c_i(f(x, p), (x_i, p_i)) \leq c_i(f((x'_i, x_{-i}), p), (x_i, p_i)).
\]

**Definition 2** A mechanism \( f \) is group strategyproof if for every nonempty subset of agents misreporting their locations jointly, at least one of them cannot benefit regardless of the others’ strategies, i.e., for every location profile and preference profile \((x, p) \in \mathcal{R}^n \times \left(2^F\right)^n\), every nonempty set \( S \subseteq N \) of agents and every \( x_{S}' \in \mathcal{R}^{|S|} \), there exists \( i \in S \) such that \( c_i(f(x, p), (x_i, p_i)) \leq c_i(f((x'_{S}, x_{-S}), p), (x_i, p_i)) \).

**Approximation Ratio.** We aim at deterministic strategyproof or group strategyproof mechanisms that can perform well under the sum/maximum cost objective. The worst-case approximation ratio is employed to measure the performance of strategyproof mechanisms.

Without confusion, simply denote \( \text{sc}(f(x, p), (x, p)) \), \( \text{sc}(\text{OPT}_{\text{sc}}(x, p), (x, p)) \), \( \text{mc}(f(x, p), (x, p)) \) and \( \text{mc}(\text{OPT}_{\text{mc}}(x, p), (x, p)) \) by \( \text{sc}(f, (x, p)) \), \( \text{sc}(\text{OPT}, (x, p)) \), \( \text{mc}(f, (x, p)) \) and \( \text{mc}(\text{OPT}, (x, p)) \), respectively. Then the approximation ratio under the sum/maximum cost objective is defined as follows.

**Definition 3** A mechanism \( f \) is said to have an approximation ratio of \( \rho (\rho \geq 1) \) under the sum cost objective (or the maximum cost), if

\[
\rho = \sup_{I(x, p, A)} \frac{\text{sc}(f, (x, p))}{\text{sc}(\text{OPT}, (x, p))} \quad \text{or} \quad \sup_{I(x, p, A)} \frac{\text{mc}(f, (x, p))}{\text{mc}(\text{OPT}, (x, p))}.
\]
In this paper, we are interested in deterministic strategyproof or group strategyproof mechanisms with small approximation ratio under the sum/maximum cost objective.

**Notations.** For a location profile $x \in \mathcal{R}^n$, denote the median location in $x$ as $\text{med}(x)$, the leftmost location in $x$ as $\text{lt}(x) = \min_{i \in N} \{x_i\}$, the rightmost location as $\text{rt}(x) = \max_{i \in N} \{x_i\}$, and the center location as $\text{cen}(x) = \frac{\text{lt}(x) + \text{rt}(x)}{2}$.

### 3 Compulsory setting

In this section, we study the compulsory setting where each agent is served by the $K$ heterogeneous facilities, i.e., $p_i = \{F_1, \cdots, F_K\}, \forall i \in N$. For simplicity, we omit $p_i$ or $p$ in this section. For example, replace $(x, p)$ by $x$ and the cost of agent $i \in N$ under the facility location profile $y = (y_1, \cdots, y_K)$ is denoted by $c_i(y, x_i) = \max_{1 \leq j \leq K} |y_j - x_i|$. Note that each agent $i \in N$ always prefers the nearest $K$ facilities to herself, since her cost depends on her distance to the farthest one. For this reason, we introduce the set of all $K$ adjacent locations below.

For the multiset of alternative locations $A = \{a_1, \cdots, a_m\}$ with $a_1 \leq \cdots \leq a_m$, denote $AK = \{(a_1, \cdots, a_K), (a_2, \cdots, a_{K+1}), \cdots, (a_{m-K+1}, \cdots, a_m)\}$. Then the real line can be partitioned into $m - K + 1$ intervals where the $t$-th interval (denoted by $Z_t, t = 1, \cdots, m - K + 1$) represents the set of points whose favorite in $AK$ is $(a_t, \cdots, a_{t+K-1})$. We refer to $Z_t$ as the zone of $K$-tuple $(a_t, \cdots, a_{t+K-1})$. Obviously, it holds that

$$Z_t = \begin{cases} \left(-\infty, \frac{a_t + a_{t+K}}{2}\right], & t = 1 \\ \left[\frac{a_{t-1} + a_{t+K-1}}{2}, \frac{a_t + a_{t+K}}{2}\right], & 2 \leq t \leq m - K \\ \left[\frac{a_{t-1} + a_{t+K-1}}{2}, +\infty\right), & t = m - K + 1 \end{cases} \quad (2)$$

The preferences of all agents over $AK$ are (not strictly) single peaked\(^1\): for each agent $i \in N$ with location $x_i \in Z_t$, her peak (or favorite) in $AK$ is $(a_t, \cdots, a_{t+K-1})$ and her cost under $(a_t, \cdots, a_{t+K-1})$ monotonically increases as $|t - l|$ increases. Based on the single peaked preference, locating at the peak of $x$’s any $t$’th order statistic (denoted by $x_{(i)}$) is group strategyproof.

**Theorem 1** Given a location profile $x \in \mathcal{R}^n$, locating at the peak of $x_{(i)}$ in $AK$ for any $i \in \{1, 2, \cdots, n\}$ is group strategyproof.

**Proof** For any $i$, the set of agents $N$ can be divided into $L(i) = \{j \in N \mid x_j < x_{(i)}\}$, $R(i) = \{j \in N \mid x_j > x_{(i)}\}$, and $M(i) = \{j \in N \mid x_j = x_{(i)}\}$. Let $f$ be the mechanism locating at the peak of $x_{(i)}$ in $AK$.

To show $f$’s group strategyproofness, we only need to prove that for every nonempty $S \subseteq N$ with deviation $x'_S \in \mathcal{R}^{|S|}$, there exists $j \in S$ who cannot benefit from this coalitional deviation. Denote $x' = (x'_S, x_{-S})$.

\(^1\) The theory of single peaked preference was developed by Duncan Black in 1958 (Black 1958). It is defined as the idea that voters have a single most preferred choice among a set of alternatives arranged according to some standards, and the degree of preference or utility of the voter decreases with the degree of the unidirectional deviation.

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If $M(i) \cap S \neq \emptyset$, then any agent $j \in M(i) \cap S$ cannot decrease her cost by the deviation since $f(x)$ is her favorite.

Otherwise, $M(i) \cap S = \emptyset$. Assume w.l.o.g. that $x'(i) < x(i)$. Then there exists some agent $j \in R(i) \cap S$ misreporting her location $x_j(> x(i))$ as $x_j(< x(i))$. Obviously, agent $j$ cannot benefit from the deviation since she prefers the peak of $x(i)$ to that of $x'(i)$. □

Theorem 1 provides a class of group strategyproof mechanisms for the compulsory setting where each agent is served by the $K$ heterogeneous facilities. Next we will choose appropriate mechanisms from this class under the objectives of the sum cost and the maximum cost, respectively.

3.1 Sum cost

For the sum cost objective, we first show that there always exists an optimal solution where the $K$ facilities are located adjacently.

**Lemma 1** Given a location profile $x \in \mathcal{R}^n$, there exists an optimal solution in $AK$ under the sum cost objective.

**Proof** Let $OPT_{sc}(x) = (y_1^*, \ldots, y_K^*)$ be an optimal solution. Assume w.l.o.g. that $y_1^* \leq \cdots \leq y_K^*$. Let $a_i = y_i^*$, then $a_i + K - 1 \leq y_K^*$. We only need to show that $sc((a_1, \ldots, a_i + K - 1), x) \leq sc((y_1^*, \cdots, y_K^*), x)$.

For each agent $i \in N$, note that

$$c_i((a_1, \cdots, a_i + K - 1), x_i) = \max\{|a_i - x_i|, |a_i + K - 1 - x_i|\}, \quad (3)$$

$$c_i((y_1^*, \cdots, y_K^*), x_i) = \max\{|y_1^* - x_i|, |y_K^* - x_i|\}. \quad (4)$$

If $x_i \leq (a_i + K - 1 + y_K^*)/2$, $c_i((a_1, \cdots, a_i + K - 1), x_i) \leq c_i((y_1^*, \cdots, y_K^*), x_i)$; otherwise, $c_i((a_1, \cdots, a_i + K - 1), x_i) = |y_1^* - x_i| = c_i((y_1^*, \cdots, y_K^*), x_i)$.

Thus, we have

$$sc((a_1, \cdots, a_i + K - 1), x) \leq \sum_{i \in N} c_i((a_1, \cdots, a_i + K - 1), x_i) \quad (5)$$

$$= \sum_{i : x_i \leq (a_i + K - 1 + y_K^*)/2} c_i((a_1, \cdots, a_i + K - 1), x_i) \quad (6)$$

$$+ \sum_{i : x_i > (a_i + K - 1 + y_K^*)/2} c_i((a_1, \cdots, a_i + K - 1), x_i) \quad (7)$$

$$\leq \sum_{i : x_i \leq (a_i + K - 1 + y_K^*)/2} c_i((y_1^*, \cdots, y_K^*), x_i) \quad (8)$$

$$+ \sum_{i : x_i > (a_i + K - 1 + y_K^*)/2} c_i((y_1^*, \cdots, y_K^*), x_i) \quad (9)$$

$$\leq \sum_{i : x_i \leq (a_i + K - 1 + y_K^*)/2} c_i((y_1^*, \cdots, y_K^*), x_i) \quad (10)$$
Given a location profile \( \text{Mechanism 1} \).

From Theorem 1, \( \text{Proof} \) under the sum cost objective. Theorem 3 \( \text{Mechanism 1} \) is group strategyproof and has an approximation ratio of 3 deterministic way.

\[
\begin{align*}
\text{sc}(y_1^*, \ldots, y_K^*, x) = \text{sc}(y_1^*, \ldots, y_K^*, x) \quad (11)
\end{align*}
\]

\( \square \)

By Lemma 1, an optimal solution (or mechanism) can be obtained in \( m-K+1 \) steps. However, it cannot guarantee strategyproof. In fact, consider locating two facilities for an instance \( I(x, A) \) with \( x = (0, 2) \) and \( A = [-2, -1, 2.5] \). Note that \( OPT_{sc}(x) = (-1, 2.5) \) and \( c_1(OPT_{sc}(x), x_1) = 2.5 \). Replacing \( x_1 = 0 \) by \( x_1' = -1 \), we have \( OPT_{sc}(x') = (-2, -1) \), \( c_1(OPT_{sc}(x'), x_1) = 2 \). Thus, agent 1 with \( x_1 = 0 \) can benefit herself by misreporting \( x_1' = -1 \).

**Theorem 2** Under the sum cost objective, any deterministic strategyproof mechanism has an approximation ratio of at least 3.

**Proof** Suppose \( f \) is a deterministic strategyproof mechanism with approximation ratio of \( 3-\delta \) for some \( \delta > 0 \).

Consider an instance \( I(x, A) \) with \( x = (-\varepsilon, \varepsilon)(0 < \varepsilon < 1) \) and \( A = \{-1, \ldots, -1, 1, \ldots, 1\} \). \( f(x) \) can be any \( K \)-vector of -1 and 1. Assume w.l.o.g. that \( f(x) \neq (-1, \ldots, -1) \), then the cost of agent 1 is \( c_1(f(x), x_1) = 1 + \varepsilon \).

Consider another instance \( I(x', A) \) with \( x' = (-1, \varepsilon) \). Obviously, \( OPT_{sc}(x') = (-1, \ldots, -1) \) and \( sc(OPT, x') = 1 + \varepsilon \). We claim that \( f(x') = (-1, \ldots, -1) \).

Otherwise, \( sc(f, x') \geq 3 - \varepsilon \). This implies that
\[
\frac{sc(f, x')}{sc(OPT, x')} \geq \frac{3 - \varepsilon}{1 + \varepsilon} > 3 - \delta
\]
for sufficiently small \( \varepsilon > 0 \), which is a contradiction.

Note that \( c_1(f(x'), x_1) = 1 - \varepsilon \). This indicates that agent 1 with \( x_1 = -\varepsilon \) can benefit herself by misreporting \( x_1' = -1 \), which contradicts \( f \)'s strategyproofness. \( \square \)

**Mechanism 1.** Given a location profile \( x \in \mathcal{R}^n \), output the peak of \( \text{med}(x) \) in \( AK \), i.e., \( (y_1, \ldots, y_K) \in \arg\min_{(s_1, \ldots, s_K) \in AK} \max_{1 \leq j \leq K} |s_j - \text{med}(x)| \), breaking ties in any deterministic way.

**Theorem 3** Mechanism 1 is group strategyproof and has an approximation ratio of 3 under the sum cost objective.

**Proof** From Theorem 1, Mechanism 1 is group strategyproof. Let us turn to its approximation ratio. Denote Mechanism 1 by \( f \).

Given a location profile \( x \in \mathcal{R}^n \), let \( OPT_{sc}(x) = (y_1^*, \ldots, y_K^*) \in AK \) be an optimal solution and \( f(x) = (y_1, \ldots, y_K) \). Assume w.l.o.g. that \( y_1 \leq y_1^* \), then \( y_K \leq y_K^* \), since both \( OPT_{sc}(x) \) and \( f(x) \) are \( K \) adjacent locations in \( A \).

Let \( y_K^* \in A \) be the location adjacent to the right of \( y_K \), then \( y' = (y_1 + y_K')/2 \) is the right border of the zone of \( (y_1, \ldots, y_K) \). First, two claims are given below.

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Claim 1. \( |\{i \in N \mid x_i \leq y'\}| \geq |\{i \in N \mid x_i > y'\}| \), since \((y_1, \cdots, y_K)\) is the peak of \(\text{med}(x)\) in \(AK\) implying \(\text{med}(x) \leq y'\).

Claim 2. For agent \(i\) with \(x_i \leq y'\), it holds that \(c_i(f(x), x_i) \leq c_i(OPT_{sc}(x), x_i)\), since the peak of agent \(i\) in \(AK\) is \((y_1, \cdots, y_K)\) or to the left.

Now we compare \(sc(f, x)\) with \(sc(OPT, x)\). The sum cost of Mechanism 1 is

\[
sc(f, x) = \sum_{i \in N} c_i ((y_1, \cdots, y_K), x_i)) = \sum_{i \in N} \max_{1 \leq j \leq K} |x_i - y_j|
\]

where the first term is denoted by \(\alpha\) and the second by \(\beta\). The optimal sum cost is

\[
sc(OPT, x) = \sum_{i \in N} c_i ((y^*_1, \cdots, y^*_K), x_i)) = \sum_{i \in N} \max_{1 \leq j \leq K} |x_i - y^*_j|
\]

where the first term is denoted by \(\gamma\) and the second by \(\delta\).

Note that

\[
\beta = \sum_{i : x_i > y'} \max_{1 \leq j \leq K} |x_i - y_j| \leq \sum_{i : x_i > y'} \max_{1 \leq j \leq K} \{ |x_i - y^*_j| + |y^*_j - y_j| \}
\]

Thus,

\[
\frac{sc(f, x)}{sc(OPT, x)} = \frac{\alpha + \beta}{\gamma + \delta} \leq \frac{\alpha + \gamma + \delta + \alpha}{\gamma + \delta} \leq \frac{3\gamma + \delta}{\gamma + \delta} \leq 3.
\]

Combining with Theorem 2, the approximation ratio of Mechanism 1 is 3. \(\square\)
3.2 Maximum cost

In contrast with the sum cost objective, there is a precise characterization of the optimal solution under the maximum cost objective.

**Lemma 2** Given a location profile \( x \in \mathcal{R}^n \), the peak of \( \text{cen}(x) \) in \( AK \) is exactly an optimal solution under the maximum cost objective.

**Proof** Let \( a = (a_t, \cdots, a_{t+K-1}) \) be the peak of \( \text{cen}(x) \) in \( AK \). If there exists \( s(\in A) < a_t \), then

\[
(s + a_{t+K-1})/2 \leq (a_{t-1} + a_{t+K-1})/2 \leq \text{cen}(x).
\]  

(24)

If there exists \( s(\in A) > a_{t+K-1} \), then

\[
(a_t + s)/2 \geq (a_t + a_{t+K})/2 \geq \text{cen}(x).
\]  

(25)

Let \( y = (y_1, \cdots, y_K) \) be any feasible solution that is different from \( a = (a_t, \cdots, a_{t+K-1}) \). Assume w.l.o.g. that \( y_1 \leq \cdots \leq y_K \), then either \( y_1 < a_t \) or \( a_{t+K-1} < y_K \). By symmetry, we only need to compare \( mc(y, x) \) with \( mc(a, x) \) in the following two cases.

**Case 1:** \( a_t \leq \cdots \leq a_{t+K-1} \leq \text{cen}(x) \). In this case, \( mc(a, x) = rt(x) - a_t \). If \( y_1 < a_t \), then

\[
mc(y, x) \geq rt(x) - y_1 > rt(x) - a_t = mc(a, x).
\]  

(26)

If \( a_{t+K-1} < y_K \), then \( y_K - \text{cen}(x) \geq \text{cen}(x) - a_t \) by Eq. (25). Thus, we have

\[
mc(y, x) \geq y_K - \text{lt}(x) = y_K - \text{cen}(x) + \text{cen}(x) - \text{lt}(x) \geq \text{cen}(x) - a_t + rt(x) - \text{cen}(x) = mc(a, x).
\]  

(27)

(28)

**Case 2:** \( a_t \leq \text{cen}(x) < a_{t+K-1} \). In this case, \( mc(a, x) = \max\{rt(x) - a_t, a_{t+K-1} - \text{lt}(x)\} \). If \( y_1 < a_t \), then \( rt(x) - y_1 > rt(x) - a_t \) and by Eq. (24), it holds that

\[
rt(x) - y_1 = rt(x) - \text{cen}(x) + \text{cen}(x) - y_1 \geq \text{cen}(x) - \text{lt}(x) + a_{t+K-1} - \text{cen}(x) = a_{t+K-1} - \text{lt}(x).
\]  

(29)

(30)

Thus, we have \( mc(y, x) \geq rt(x) - y_1 \geq mc(a, x) \). Similarly if \( a_{t+K-1} < y_K \), then \( y_K - \text{lt}(x) > a_{t+K-1} - \text{lt}(x) \) and \( y_K - \text{lt}(x) = y_K - \text{cen}(x) + \text{cen}(x) - \text{lt}(x) \geq \text{cen}(x) - a_t + rt(x) - \text{cen}(x) = rt(x) - a_t \). Thus, we have \( mc(y, x) \geq y_K - \text{lt}(x) \geq mc(a, x) \).

\( \square \)

Unfortunately, this optimal mechanism is still not strategyproof. Consider locating two facilities for an instance \( I(x, A) \) with \( x = (-0.5, 0.5) \), \( A = \{-1, 1, 1.5\} \). Note that \( OPT_{mc} = (-1, 1) \) and \( c_2(OPT_{mc}(x), x_2) = 1.5 \). Replacing \( x_2 = 0.5 \) by \( x'_2 = 2 \), we have \( OPT_{sc}(x'_2, x_2) = 1.5 \), \( c_2(OPT_{sc}(x'_2), x_2) = 1.5 \). Thus, agent 2 with \( x_2 = 0.5 \) can benefit herself by misreporting \( x'_2 = 2 \).
**Theorem 4** Under the maximum cost objective, any deterministic strategyproof mechanism has an approximation ratio of at least 3.

**Proof** Let $f$ be any deterministic strategyproof mechanism. Consider an instance $I(x, A, K)$ with $x = (-\varepsilon, \varepsilon)(0 < \varepsilon < 1)$ and $A = \{-1, \ldots, -1, 1, \ldots, 1\}$. Note that $f(x)$ can be any $K$-vector of $-1$ and $1$.

If $f(x) \neq (-1, \ldots, -1)$, then the cost of agent 1 is $c_1(f(x), x_1) = 1 + \varepsilon$. Now consider another instance $I(x', A)$ with $x' = (-2 - \varepsilon, \varepsilon)$. Note that $OPT_{mc}(x') = (-1, \ldots, -1)$ and $mc(OPT, x') = 1 + \varepsilon$. We claim that $f(x') \neq (-1, \ldots, -1)$. In fact, if $f(x') = (-1, \ldots, -1)$, then $c_1(f(x'), x_1) = 1 - \varepsilon$, which means that agent 1 can decrease her cost by misreporting her location as $x'_1 = -2 - \varepsilon$, contradicting $f$'s strategyproofness. Thus, $mc(f, x') = 3 + \varepsilon$, and

$$\frac{mc(f, x')}{mc(OPT, x')} = \frac{3 + \varepsilon}{1 + \varepsilon} \to 3. \quad (31)$$

For the case $f(x) = (-1, \ldots, -1)$, there is a similar analysis on another instance corresponding to agent 2, which is not described here. □

**Mechanism 2.** Given a location profile $x \in \mathcal{R}^n$, output the peak of $lt(x)$ in $AK$, i.e., $(y_1, \ldots, y_K) = \arg \min \max_{j \in \{1, 2, \ldots, K\}} |s_j - lt(x)|$, breaking ties in any deterministic way.

**Theorem 5** Mechanism 2 is group strategy-proof and has an approximation ratio of 3 under the maximum cost objective.

**Proof** Denote Mechanism 2 by $f$. We only need to analyze $f$'s approximation ratio.

Given a location profile $x \in \mathcal{R}^n$, let $OPT_{mc}(x) = (y^*_1, \ldots, y^*_K)$ be the peak of $cen(x)$ in $AK$ and $f(x) = (y_1, \ldots, y_K)$.

It is easy to see that $mc(OPT, x) \geq \frac{1}{2}(rt(x) - lt(x))$, and

$$mc(OPT, x) \geq \max_{1 \leq j \leq K} |lt(x) - y^*_j| \geq \max_{1 \leq j \leq K} |lt(x) - y_j|. \quad (32)$$

We compare $mc(f, x)$ with $mc(OPT, x)$ through the following analysis.

**Case 1:** $y_1 \leq y_K \leq lt(x) \leq rt(x)$, or $y_1 \leq lt(x) \leq y_K \leq rt(x)$. We have

$$mc(f, x) = rt(x) - y_1 = (rt(x) - lt(x)) + lt(x) - y_1 \leq 3mc(OPT, x). \quad (33)$$

**Case 2:** $lt(x) \leq y_1 \leq y_K \leq rt(x)$.

$$mc(f, x) \leq rt(x) - lt(x) \leq 2mc(OPT, x). \quad (34)$$
Case 3: \( \text{lt}(x) \leq y_1 \leq \text{rt}(x) \leq y_K \), or \( \text{lt}(x) \leq y_K \leq \text{rt}(x) \leq y_1 \).

In this case, the right border of the zone of \((y_1, \cdots, y_K)\) is no less than \((y_1 + y_K)/2 \geq \text{cen}(x) \geq \text{lt}(x)\). Combining with the fact that \(\text{lt}(x)\) lies in the zone of \((y_1, y_2)\), it holds that \(\text{cen}(x)\) also lies in the zone of \((y_1, \cdots, y_K)\). This implies that \(f(x) = OPT_{mc}(x)\). Thus, we have

\[
mc(f, x) = mc(OPT, x).
\] (35)

Case 4: \( y_1 \leq \text{lt}(x) \leq \text{rt}(x) \leq y_K \). By Eq. (32), we have \(y_K - \text{lt}(x) \leq mc(OPT, x)\) and

\[
\text{rt}(x) - y_1 = \text{rt}(x) - \text{lt}(x) + \text{lt}(x) - y_1 \leq 3mc(OPT, x).
\] (36)

Thus,

\[
mc(f, x) = \max \{y_K - \text{lt}(x), \text{rt}(x) - y_1\} \leq 3mc(OPT, x).
\] (37)

Above all, \(mc(f, x) \leq 3mc(OPT, x)\). Combining with Theorem 4, Mechanism 2 has an approximation ratio of 3.

4 Optional setting

In this section, we focus on the optional setting where agents may be only interested in some of the \(K\) heterogeneous facilities. The cost of agent \(i \in N\) is \(c_i(y, (x_i, p_i)) = \max_{F_k \in p_i} |y_k - x_i|\). Here, we only study heterogeneous two-facility location games with optional preference. For \(K = 2\), \(AK\) consists of \(m - 1\) adjacent location pairs, i.e. \(AK = \{(a_1, a_2), \cdots, (a_{m-1}, a_m)\}\).

Note that even in the optional setting, each agent \(i \in N\) has some kind of single peaked preference: if \(p_i = \{F_1\}\) or \(\{F_2\}\), she has single peaked preference over \(A\); if \(p_i = \{F_1, F_2\}\), she has single peaked preference over \(AK\). Our mechanisms in this section are proposed based on the single peaked preference.

In addition, Mechanism 1 and Mechanism 2 for two-facility location games and two other mechanisms for one-facility location games proposed in Feldman et al. (2016) and Tang et al. (2020) will be implemented as subroutines in our mechanisms for optional preference. Given that a set of \(n\) agents have single peaked preference over the set of alternative locations \(A\), the related results in one-facility location games are listed below.

SC-Mechanism (Feldman et al. 2016). Given \(x \in \mathbb{R}^n\) and \(A\), output \(y \in \arg\min_{a \in A} |a - \text{med}(x)|\), breaking ties in any deterministic way.

Lemma 3 (Feldman et al. 2016) SC-Mechanism is group strategyproof and 3-approximate under the sum cost objective.

MC-Mechanism (Tang et al. 2020). Given \(x \in \mathbb{R}^n\) and \(A\), output \(y \in \arg\min_{a \in A} |a - \text{lt}(x)|\), breaking ties in any deterministic way.

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**Lemma 4** (Tang et al. 2020) **MC-Mechanism** is group strategyproof and 3-approximate under the maximum cost objective.

For a facility preference profile \( p \in (2^F)^n \), denote \( N_k = \{ i \in N \mid p_i = \{ F_k \} \} \) for \( k \in \{1, 2\} \), and \( N_{1,2} = \{ i \in N \mid p_i = \{ F_1, F_2 \} \} \).

### 4.1 Sum cost

**Mechanism 3.** Given a location profile and a preference profile \((x, p) \in \mathcal{R}^n \times (2^F)^n\), output the facility location profile \( y = (y_1, y_2) \) as follows:

- if \( |N_{1,2}| > 0 \), select \((y_1, y_2) \in \arg\min_{(y_1, y_2) \in AK} \max_{j \in \{1, 2\}} |s_j - \text{med}(x_{N_{1,2}})|\), breaking ties in any deterministic way;

- if \( |N_{1,2}| = 0 \) and \( |N_1| \geq |N_2| \), select \( y_1 \in \arg\min_{y \in A} |y - \text{med}(x_{N_1})| \), and \( y_2 \in \) \( \arg\min_{y \in A \setminus \{y_1\}} |y - \text{med}(x_{N_2})| \) (if \( N_2 \neq \emptyset \)), breaking ties in any deterministic way;

- if \( |N_{1,2}| = 0 \) and \( |N_1| < |N_2| \), select \( y_2 \in \arg\min_{y \in A} |y - \text{med}(x_{N_2})| \), and \( y_1 \in \) \arg\min_{y \in A \setminus \{y_2\}} |y - \text{med}(x_{N_1})| \) (if \( N_1 \neq \emptyset \)), breaking ties in any deterministic way.

**Theorem 6** **Mechanism 3** is group strategyproof and has an approximation ratio of at most \( 2n + 1 \) under the sum cost objective.

**Proof** **Group strategyproofness.** Given \((x, p) \in \mathcal{R}^n \times (2^F)^n\), **Mechanism 3** outputs the facility location profile according to the public information \( p \). To show group strategyproofness, we need to prove that for every nonempty \( S \subseteq N \) with deviation \( x'_S \in \mathcal{R}^{|S|}\), there exists \( j \in S \) who cannot benefit from the coalitional deviation. Denote \( x' = (x'_S, x_\_S) \), **Mechanism 3** by \( f \), **Mechanism 1** by \( f_1 \), and **SC-Mechanism** by \( f_2 \).

**Case 1:** \(|N_{1,2}| > 0\), then \( f(x, p) = f_1(x_{N_{1,2}}) \) and \( f(x', p) = f_1(x'_{N_{1,2} \cap S}, x_{N_{1,2} \setminus S}) \). If \( N_{1,2} \cap S \neq \emptyset \), any agent in \( N_{1,2} \cap S \) cannot benefit from the deviation \( x'_{N_{1,2} \cap S} \) by \( f_1 \)'s group strategyproofness. If \( N_{1,2} \cap S = \emptyset \), \( f(x', p) = f_3(x_{N_{1,2}}) \), which implies that any agent in \( S \subseteq N_1 \cup N_2 \) cannot benefit from the deviation.

**Case 2:** \(|N_{1,2}| = 0\) and \(|N_1| \geq |N_2|\). It holds that \( f(x, p) = (f_2(x_{N_1}), f_2(x_{N_2})) \) and \( f(x', p) = (f_2(x'_{N_1 \cap S}, x_{N_1 \setminus S}), f_2(x'_{N_2 \cap S}, x_{N_2 \setminus S})) \), with \( f_2(x_{N_2}) \in A \backslash f_2(x_{N_1}) \) and \( f_2(x'_{N_2 \cap S}, x_{N_2 \setminus S}) \in A \backslash f_2(x'_{N_1 \cap S}, x_{N_1 \setminus S}) \). If \( N_1 \cap S \neq \emptyset \), any agent in \( N_1 \cap S \) cannot benefit from the deviation \( x'_{N_2 \cap S} \) by \( f_2 \)'s group strategyproofness. If \( N_1 \cap S = \emptyset \), \( f(x', p) = (f_2(x_{N_1}), f_2(x'_{N_2 \cap S}, x_{N_2 \setminus S})) \) with \( f_2(x'_{N_2 \cap S}, x_{N_2 \setminus S}) \in A \backslash f_2(x_{N_1}) \). Still by \( f_2 \)'s group strategyproofness, any agent in \( N_2 \cap S \) cannot benefit from the deviation \( x'_{N_2 \cap S} \).

**Case 3:** \(|N_{1,2}| = 0\) and \(|N_1| < |N_2|\). This case is similar to Case 2.

**Approximation ratio.** Given \((x, p) \in \mathcal{R}^n \times (2^F)^n\), let \( OPT_{sc}(x, p) = y^* = (y_1^*, y_2^*) \) be an optimal solution and \( f(x, p) = y = (y_1, y_2) \). We now compare \( sc(f, (x, p)) \) with \( sc(OPT, (x, p)) \).
Case 1: If \( |N_{1,2}| > 0 \), the output of Mechanism 3 on \( I(x, p, A, 2) \) equals to that of Mechanism 1 on \( I(x_{N_{1,2}}, p_{N_{1,2}}, A, 2) \). Denote the optimal solution on \( I(x_{N_{1,2}}, p_{N_{1,2}}, A, 2) \) as \( y_{\text{opt}} \).

By Theorem 3, it holds that

\[
\sum_{i \in N_{1,2}} c_i (y, (x_i, p_i)) \leq 3 \sum_{i \in N_{1,2}} c_i \left( y^\text{opt}, (x_i, p_i) \right) \leq 3 \sum_{i \in N_{1,2}} c_i \left( y^*, (x_i, p_i) \right). \tag{38}
\]

Thus, we have

\[
sc(f, (x, p)) = \sum_{i \in N_1 \cup N_2 \cup N_{1,2}} c_i (y, (x_i, p_i)) \leq \sum_{i \in N_1} |x_i - y_1| + \sum_{i \in N_2} |x_i - y_2| + 3 \sum_{i \in N_{1,2}} c_i \left( y^*, (x_i, p_i) \right) \tag{40}
\]

\[
\leq \sum_{i \in N_1} |x_i - y^*_1| + \sum_{i \in N_2} |x_i - y^*_2| + 3 \sum_{i \in N_{1,2}} c_i \left( y^*, (x_i, p_i) \right) \tag{41}
\]

\[
+ |N_1| \cdot |y_1 - y^*_1| + |N_2| \cdot |y_2 - y^*_2| \tag{42}
\]

\[
\leq 3sc(OPT, (x, p)) + |N_1 \cup N_2| \cdot 2sc(OPT, (x, p)) \tag{43}
\]

\[
\leq (2n + 1)sc(OPT, (x, p)). \tag{44}
\]

Here, the above third inequality holds because for \( j = 1, 2 \),

\[
|y_j - y^*_j| \leq |y_j - \text{med}(x_{N_{1,2}})| + |\text{med}(x_{N_{1,2}}) - y_j^*| \tag{45}
\]

\[
\leq \max_{k \in \{1, 2\}} |y_k - \text{med}(x_{N_{1,2}})| + |\text{med}(x_{N_{1,2}}) - y_j^*| \tag{46}
\]

\[
\leq \max_{k \in \{1, 2\}} |y_k^* - \text{med}(x_{N_{1,2}})| + |\text{med}(x_{N_{1,2}}) - y_j^*| \tag{47}
\]

\[
\leq 2sc(OPT, (x, p)). \tag{48}
\]

Case 2: If \( |N_{1,2}| = 0 \) and \( |N_1| \geq |N_2| \). Without loss of generality, assume that \( N_2 \neq \emptyset \). \( y_1 \) equals to the output of SC-Mechanism on instance \( I_1 = I(x_{N_1}, p_{N_1}, A, 2) \), and \( y_2 \) equals to the output of SC-Mechanism on instance \( I_2 = I(x_{N_2}, p_{N_2}, A \setminus \{y_1\}, 2) \). Denote by \( y_{1,\text{opt}}^* \) the optimal solution on instance \( I_1 \) and \( y_{2,\text{opt}}^* \) the optimal solution on instance \( I_2 \).

For \( k = 1, 2 \), let \( sc(y, I_k) = \sum_{i \in N_k} |x_i - y| \), then

\[
sc(OPT, (x, p)) = \sum_{i \in N_1} |x_i - y^*_1| + \sum_{i \in N_2} |x_i - y^*_2| = sc(y^*_1, I_1) + sc(y^*_2, I_2) \tag{49}
\]

\[
sc(f, (x, p)) = \sum_{i \in N_1} |x_i - y_1| + \sum_{i \in N_2} |x_i - y_2| = sc(y_1, I_1) + sc(y_2, I_2) \tag{50}
\]
For $I_1$, by Proposition 3, it holds that

$$sc (y_1, I_1) \leq 3sc (y_{opt}^1, I_1) \leq 3sc (y_1^*, I_1)$$ (51)

For $I_2$, we consider the following two cases.

Case 2.1: If $y_2^* \in A \setminus \{y_1\}$, by Proposition 3, it holds that

$$sc (y_2, I_2) \leq 3sc (y_{opt}^2, I_2) \leq 3sc (y_2^*, I_2)$$ (52)

Case 2.2: $y_2^* \notin A \setminus \{y_1\}$, then $y_1 = y_2^*$ and $y_1^* \in A \setminus \{y_1\}$. On the one hand, by Proposition 3, we have

$$sc (y_2, I_2) \leq 3sc (y_{opt}^2, I_2) \leq 3sc (y_1^*, I_2) \leq 3sc (y_1, I_2)$$ (53)

On the other hand,

$$sc (y_1^*, I_2) = \sum_{i \in N_2} |x_i - y_1^*| \leq \sum_{i \in N_2} |x_i - y_2^*| + \sum_{i \in N_1} |y_2^* - y_1^*|$$ (54)

$$\leq \sum_{i \in N_2} |x_i - y_2^*| + \sum_{i \in N_1} |y_1^* - x_i| + \sum_{i \in N_1} |x_i - y_1^*|$$ (55)

$$= sc (y_2^*, I_2) + sc (y_1, I_1) + sc (y_1^*, I_1)$$ (56)

$$\leq sc (y_2^*, I_2) + 4sc (y_1^*, I_1)$$ (57)

where the first inequality holds because $|N_1| \geq |N_2|$ and the third holds by Eq. (51). Combining Eqs. (53) and (57), we have

$$sc (y_2, I_2) \leq 3sc (y_2^*, I_2) + 12sc (y_1^*, I_1)$$ (58)

Thus, by Eqs. (51) and (58), it holds that

$$sc (f, (x, p)) = sc (y_1, I_1) + sc (y_2, I_2) \leq 3sc (y_1^*, I_1) + 3sc (y_2^*, I_2) + 12sc (y_1^*, I_1) \leq 15sc (OPT, (x, p))$$ (59) (60) (61)

Case 3: $|N_{1,2}| = 0$ and $|N_1| < |N_2|$. This case is similar to Case 2. Above all, Mechanism 3 has an approximation ratio of at most $2n + 1$.

4.2 Maximum cost

Mechanism 4. Given a location profile and a preference profile $(x, p) \in \mathcal{R}^n \times (2^F)^n$, output the facility location profile $y = (y_1, y_2)$ as follows:
- if \(|N_{1,2}| > 0\), select \((y_1, y_2) \in \arg\min_{(x_1, x_2) \in AK}\max_{j \in \{1, 2\}} |s_j - \text{lt}(x_{N_{1,2}})|\), breaking ties in any deterministic way;
- if \(|N_{1,2}| = 0\), select \(y_1 \in \arg\min_{y \in A} |y - \text{lt}(x_{N_1})|\) (if \(N_1 \neq \emptyset\)), and \(y_2 \in \arg\min_{y \in A \setminus \{y_1\}} |y - \text{lt}(x_{N_2})|\) (if \(N_2 \neq \emptyset\)), breaking ties in any deterministic way.

**Theorem 7** Mechanism 4 is group strategyproof and has an approximation ratio of at most 9 under the maximum cost objective.

**Proof** The proof of Mechanism 4’s group strategyproofness is similar to that of Mechanism 3’s, which is omitted here. Now we focus on the approximation ratio of Mechanism 4.

Denote Mechanism 4 by \(f\). Given \((x, p) \in \mathcal{R}^n \times (2^F)^n\), let \(OPT_{mc}(x, p) = y^* = (y_1^*, y_2^*)\) be an optimal solution and \(f(x, p) = y = (y_1, y_2)\). We now compare \(mc(f, (x, p))\) with \(mc(OPT, (x, p))\).

Case 1: If \(|N_{1,2}| > 0\), the output of Mechanism 4 on \(I(x, p, A, 2)\) equals to that of Mechanism 2 on \(I(x_{N_{1,2}}, p_{N_{1,2}}, A, 2)\). Denote by \(y^{opt} = (y_1^{opt}, y_2^{opt})\) the optimal solution on \(I(x_{N_{1,2}}, p_{N_{1,2}}, A, 2)\).

By Theorem 5, it holds that

\[
\max_{i \in N_{1,2}} c_i(y, (x_i, p_i)) \leq 3 \max_{i \in N_{1,2}} c_i(y^{opt}, (x_i, p_i)) \leq 3 \max_{i \in N_{1,2}} c_i(y^*, (x_i, p_i)).
\]  

Thus, we have

\[
mc(f, (x, p)) = \max_{i \in N_{1} \cup N_{2} \cup N_{1,2}} \{c_i(y, (x_i, p_i))\}
\]

\[
= \max_{i \in N_{1}} \{\max_{j \in \{1, 2\}} |y_1 - x_i|\}, \max_{i \in N_{2}} \{\max_{j \in \{1, 2\}} |y_2 - x_i|\}, \max_{i \in N_{1,2}} \{c_i(y, (x_i, p_i))\}
\]

\[
\leq \max_{i \in N_{1}} \{\max_{j \in \{1, 2\}} |y^* - x_i|\}, \max_{i \in N_{2}} \{\max_{j \in \{1, 2\}} |y^*_j - x_i|\}
\]

\[
+ |y_2 - y^*_2|, \max_{i \in N_{1,2}} \{c_i(y^*, (x_i, p_i))\}
\]

\[
\leq \max_{i \in N_{1}} \{\max_{j \in \{1, 2\}} |y^*_j - x_i| + 2mc(OPT, (x, p))\}, \max_{i \in N_{2}} \{\max_{j \in \{1, 2\}} |y^*_j - x_i| + 2mc(OPT, (x, p))\}, \max_{i \in N_{1,2}} \{c_i(y^*, (x_i, p_i))\}
\]

\[
\leq 3mc(OPT, (x, p)).
\]

Here, the above second inequality holds because for \(j = 1, 2\),

\[
|y_j - y^*_j| \leq |y_j - \text{lt}(x_{N_{1,2}})| + |\text{lt}(x_{N_{1,2}}) - y^*_j|
\]
\[ \leq \max_{k \in \{1, 2\}} \left| y_k - \text{lt}(x_{N_1,2}) \right| + \left| \text{lt}(x_{N_1,2}) - y_j^* \right| \]

(72)

\[ \leq \max_{k \in \{1, 2\}} \left| y_k^* - \text{lt}(x_{N_1,2}) \right| + \left| \text{lt}(x_{N_1,2}) - y_j^* \right| \]

(73)

\[ \leq 2mc(OPT, (x, p)) \]

(74)

Case 2: \(|N_{1,2}| = 0\). Assume w.l.o.g. that \(N_1 \neq \emptyset, N_2 \neq \emptyset\). \(y_1\) equals to the output of \(MC-Mechanism\) on instance \(I_1 = I(x_{N_1, p_{N_1}, A, 2})\), and \(y_2\) equals to the output of \(MC-Mechanism\) on instance \(I_2 = I(x_{N_2, p_{N_2}, A\{y_1\}, 2})\). Denote by \(\hat{y}_1^{opt}\) the optimal solution on instance \(I_1\) and \(\hat{y}_2^{opt}\) the optimal solution on instance \(I_2\).

For \(k = 1, 2\), let \(mc(y, I_k) = \max_{i \in N_k} |x_i - y|\), then

\[ mc(OPT, (x, p)) = \max \left\{ \max_{i \in N_1} |x_i - y_1^*|, \max_{i \in N_2} |x_i - y_2^*| \right\} \]

(75)

\[ = \max \{ mc(y_1^*, I_1), mc(y_2^*, I_2) \} \]

(76)

\[ mc(f, (x, p)) = \max \{ mc(y_1, I_1), mc(y_2, I_2) \} \]

(77)

For \(I_1\), by Proposition 4, it holds that

\[ mc(y_1, I_1) \leq 3mc(y_1^{opt}, I_1) \leq 3mc(y_1^*, I_1) \]

(78)

For \(I_2\), we consider the following two cases.

Case 2.1: If \(y_2^* \in A \{ y_1 \}\), by Proposition 4, it holds that

\[ mc(y_2, I_2) \leq 3mc(y_2^{opt}, I_2) \leq 3mc(y_2^*, I_2) \]

(79)

Case 2.2: \(y_2^* \notin A \{ y_1 \}\), then \(y_1 = y_2^*\) and \(y_1^* \in A \{ y_1 \}\). On the one hand, by Proposition 4, we have

\[ mc(y_2, I_2) \leq 3mc(y_2^{opt}, I_2) \leq 3mc(y_1^*, I_2) \]

(80)

On the other hand,

\[ mc(y_1^*, I_2) = \max_{i \in N_2} |y_1^* - x_i| \leq \max_{i \in N_2} |y_2^* - x_i| + |y_1^* - y_1| \]

(81)

\[ \leq \max_{i \in N_2} |y_2^* - x_i| + |y_1^* - \text{lt}(x_{N_1})| + \left| \text{lt}(x_{N_1}) - y_1 \right| \]

(82)

\[ \leq mc(y_2^*, I_2) + 2mc(y_1^*, I_1) \]

(83)

Combining Eqs. (80) and (83), we have

\[ mc(y_2, I_2) \leq 3mc(y_2^*, I_2) + 6mc(y_1^*, I_1) \]

(84)
Thus, by Eqs. (78) and (84), it holds that

\[
m_{c}(f, (x, p)) = \max \{m_{c}(y_{1}, I_{1}), m_{c}(y_{2}, I_{2})\} \tag{85}
\]

\[
\leq \max \{3m_{c}(y_{1}^{*}, I_{1}), 3m_{c}(y_{2}^{*}, I_{2}) + 6m_{c}(y_{1}^{*}, I_{1})\} \tag{86}
\]

\[
\leq 9m_{c}(OPT, (x, p)) \tag{87}
\]

Above all, Mechanism 4 has an approximation ratio of at most 9. \hfill \Box

Remark 1 By Theorem 2 and Theorem 4, 3 is also a lower bound of the approximation ratio for any deterministic strategyproof mechanism under the sum/maximum cost objective in the optional setting, since the compulsory setting is a special case of the optional setting.

5 Conclusion

In this paper, we proposed a constrained model for heterogeneous \(K\)-facility location games where a set of locations are feasible for building facilities and the number of facilities built at each feasible location is limited. We studied deterministic mechanisms design without money under max-variant cost where the cost of each agent depends on the distance to the farthest one of which she is served by. In the compulsory setting where each agent is served by all the \(K\) facilities, we showed that the optimal solution under the sum/maximum cost objective is not strategyproof and proposed a 3-approximate deterministic group strategyproof mechanism which is also the best deterministic strategyproof mechanism for the corresponding social objective. In the optional setting that agents may be interested in some of the \(K\) heterogeneous facilities, we designed a deterministic group strategyproof mechanism with approximation ratio with at most \(2n + 1\) under the sum cost objective and a deterministic group strategyproof mechanism with approximation ratio with at most 9 under the maximum cost objective.

The constrained model of facility location games has many practical implications and there are several directions for future research. First, there is a severe mismatch between bounds for approximation ratio of deterministic strategyproof mechanisms in the optional setting. Is it possible to obtain more desirable bounds in this setting? Second, what will happen in the optional setting if more than two facilities are considered? Third, it remains an open question to study randomized mechanisms for our constrained model. Further, our model can be extended to more general metric spaces.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.
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