EIGENVALUES OF THE $p$-LAPLACIAN AND DISCONJUGACY CRITERIA

PABLO L. DE NAPOLI AND JUAN P. PINASCO

Abstract. In this work we derive oscillation and nonoscillation criteria for the one dimensional $p$-laplacian in terms of an eigenvalue inequality for a mixed problem. We generalize the results obtained in the linear case by Nehari and Willet, and the proof is based on a Picone type identity.

1. Introduction

In this work we study the following equation,

$$(1.1) \quad (|u'|^{p-2}u')' + q(t)|u|^{p-2}u = 0.$$ 

Here, $1 < p < \infty$, $t \in [a, +\infty)$, and $q(t)$ is a nonnegative continuous function not vanishing in subintervals of the form $(b, +\infty)$.

The solutions of equation (1.1) are classified as oscillatory or nonoscillatory. In the first case, a solution has an infinite number of isolated zeros; in the second case, a solution has a finite number of zeros. However, from the Sturm-Liouville theory for the $p$-laplacian [1, 12, 16], if one solution is oscillatory (resp., nonoscillatory), then every solution is oscillatory (resp., nonoscillatory). Hence, we may speak about oscillatory or nonoscillatory equations, instead of solutions.

For the $p$-laplacian operator, there are several criteria for oscillation and nonoscillation in the literature, see for example [5, 6, 7, 8]. Among the class of nonoscillatory equations, when any solution has a unique zero in $[a, +\infty)$, the equation is called disconjugate.

The disconjugacy phenomena is considerably more difficult and less understood than nonoscillation, we refer the interested reader to the surveys [3, 4, 17] for the linear case $p = 2$.

In order to fix ideas, we may consider it in a closed interval $[a, b]$ with $b < \infty$. Clearly, the existence of two zeros in $[a, b]$ of a solution $u$ is related with the Dirichlet eigenvalue problem

$$-\big(|u'|^{p-2}u\big)' = \lambda q(t)|u|^{p-2}u, \quad u(a) = 0 = u(b),$$

and the equation (1.1) is disconjugate if the first eigenvalue satisfies $\lambda_1 < 1$.

This observation gives a condition for disconjugacy on finite intervals, namely, the equation is disconjugate if and only if the first Dirichlet eigenvalue is greater than one. Therefore, the problem of find disconjugacy conditions on finite intervals is equivalent to find lower bounds for eigenvalues. This problem has a long history, which can be traced back to [11], similar results for the $p$-laplacian were obtained in [14].

We consider now the disconjugacy problem on $[a, +\infty)$. The relationship between disconjugacy and the eigenvalues of a mixed problem

$$-u'' = \lambda q(t)u, \quad u(a) = 0 = u'(b)$$

1991 Mathematics Subject Classification. 34C10, 34L15.

Key words and phrases. Half-linear differential equation, Picone identity, oscillation theory.
is due to Nehari [13], and was generalized to different equations in [9, 15, 18]. We prove here the following theorems generalizing their results for the \( p \)-laplacian:

**Theorem 1.1.** Let \( q(t) \geq 0 \), not vanishing in subintervals of the form \((b, +\infty)\) and let \( \lambda_1 \) be the first eigenvalue of

\[
(1.2) \quad (|u'|^{p-2}u')' + \lambda q(t)|u|^{p-2}u = 0, \quad u(a) = 0 = u'(b), \quad a < b
\]

then \((1.1)\) is disconjugate in \((a, +\infty)\) if and only if \( \lambda_1 > 1 \) for all \( b > a \).

Also, we have the following result for oscillatory equations:

**Theorem 1.2.** Let \( q(t) \geq 0 \), not vanishing in subintervals of the form \((b, +\infty)\). Then, equation \((1.1)\) is oscillatory if and only if there exists a sequence of intervals \([a_n, b_n]\) with \( a_n \nearrow +\infty \) as \( n \nearrow +\infty \) such that the first eigenvalue \( \lambda_1^{(n)} \) of

\[
(|u'|^{p-2}u')' + \lambda q(t)|u|^{p-2}u = 0, \quad u(a_n) = 0 = u'(b_n)
\]

satisfies \( \lambda_1^{(n)} \leq 1 \) for \( n \geq 1 \).

The proof for the linear case \( p = 2 \) in [17] follows by analyzing a Lagrange identity formed by a positive solution of equation \((1.1)\) and an eigenfunction, and by using Riccati equation techniques. Our main tool for the proof of both theorems is a Picone type identity as in [1], and the variational characterization of the first eigenvalue, which can be obtained from the Rayleigh quotient,

\[
\lambda_1 = \inf_{u \in W} \frac{\int_a^b |u'|^p dt}{\int_a^b q(t)|u|^{p} dt}
\]

where \( W = W_0^{1,p}(a,b) \setminus \{0\} \) for the Dirichlet boundary condition, and \( W = \{W^{1,p}(a,b) : u(a) = 0\} \setminus \{0\} \) for the mixed problem \((1.2)\).

Let us note that the eigenvalue problem for the \( p \)-laplacian has been widely studied in recent years, see for example [2, 16] among several others, and the references therein. Hence, a characterization for disconjugacy in terms of eigenvalues could be an useful tool.

As an application, we consider the so called Roundabout Theorem [7, 8] for bounded intervals, adding another equivalent criterium for disconjugacy. Also, we analyze in this way Hille and Leighton type criteria (see [5, 7, 8]) for oscillation and nonoscillation in the half line.

The paper is organized as follows: in Section 2 we prove the main Theorems 1.1 and 1.2; in section 3 we discuss other oscillation theorems.

## 2. Main Theorems

Our main tool is the following Picone type identity which can be found in [1]:

**Theorem 2.1.** Let \( v > 0, u \geq 0 \) be differentiable a.e. in \((a, b)\). Denote

\[
L(u, v) = |u'|^p + (p - 1) \frac{u^p}{v^p} |v'|^p - p \frac{u^{p-1}}{v^p} |v'|^{p-2} v'u',
\]

\[
R(u, v) = |u'|^p - |v'|^{p-2} v' \left( \frac{u^p}{v^p} \right)' .
\]

Then,

(i) \( L(u, v) = R(u, v) \),

(ii) \( L(u, v) \geq 0 \) a.e., and

(iii) \( L(u, v) = 0 \) a.e. in \((a, b)\) if and only if \( u = kv \) for some \( k \in \mathbb{R} \).

We are ready to proof Theorems 1.1 and 1.2.
Proof of Theorem 1.1. Let us assume that the first eigenvalue \( \lambda_1 \) of

\[
(|u'|^{p-2}u')' + \lambda q(t)|u|^{p-2}u = 0, \quad u(a) = 0 = u'(b), \quad a < b
\]
is greater than 1. If equation (1.1) is not disconjugate in \((a, +\infty)\), there exists a solution \( v \) with two zeros \( t_1, t_2 \in [a, +\infty) \). Now, we choose \( v \) as a test function for the eigenvalue problem in \([a, t_2]\), extending it by zero in \([a, t_1]\). Clearly, the Rayleigh quotient gives

\[
\lambda_1 = \inf_{u \in W^{1,p}(a,b), u(a)=0} \frac{\int_{a}^{b} |u'|^p dt}{\int_{a}^{b} q(t)|u|^p dt} \leq \frac{\int_{t_1}^{t_2} |v'|^p dt}{\int_{t_1}^{t_2} q(t)|v|^p dt} = 1,
\]
a contradiction. Hence, equation (1.1) must be disconjugate.

Let us prove now the converse. Since equation (1.1) is disconjugate, we can consider a positive solution \( v \) in \((a, +\infty)\). Let us suppose now that there exist \( b > a \) and \( \lambda \leq 1 \) such that equation (1.2) has a nontrivial solution \( u \).

Replacing \( u \) and \( v \) in \( R(u,v) \) of Theorem 2.1, we have

\[
R(u, v) = |u'|^p - |v'|^{p-2}v' \left( \frac{u^p}{v^{p-1}} \right)' = (\lambda - 1) \int_{a}^{b} q(t)|u|^p dt.
\]

Here, we use the weak formulation of Problems (1.1) and (1.2), multiplying the first by \((u^p/v^{p-1}) \) and the second by \( u \).

Since \( R(u, v) = L(u, v) \geq 0 \), we have \( \lambda = 1 \), which gives \( R(u, v) = L(u, v) = 0 \). Hence, \( u \equiv kv \) for some constant \( k \in \mathbb{R} \), and we can choose an eigenfunction such that \( k = 1 \) due to the homogeneity of the \( p \)-laplacian.

Let us take \( c > b \), and let us consider the eigenvalue problem in \((a, c)\):

\[
(|u'|^{p-2}u')' + \lambda q(t)|u|^{p-2}u = 0, \quad u(a) = 0 = u'(c).
\]

We extend the eigenfunction \( u \) as \( u(b) \) in \((b, c)\), and let us call it \( \tilde{u} \). Since \( \tilde{u} \) is an admissible function in the variational characterization of the first eigenvalue \( \lambda_1 \) in \((a, c)\), we obtain

\[
\lambda_1^{(c)} = \inf_{\tilde{u} \in W} \frac{\int_{a}^{c} |\tilde{u}'|^p dt}{\int_{a}^{c} q(t)|\tilde{u}|^p dt} \leq \frac{\int_{a}^{b} |\tilde{u}'|^p dt}{\int_{a}^{b} q(t)|\tilde{u}|^p dt} = \frac{\int_{a}^{b} |u'|^p dt}{\int_{a}^{b} q(t)|u|^p dt + \int_{b}^{c} q(t)|u|^p dt}.
\]

Let us observe that \( \lambda_1^{(c)} < 1 \) unless \( q(t) \equiv 0 \) in \((b, c)\). Since this argument is valid for each \( c > b \), and \( q(t) \) cannot vanish identically on intervals of the form \((b, +\infty)\), there exist an interval \((a, c)\) where the first eigenvalue satisfy \( \lambda_1^{(c)} < 1 \).

Applying again the Picone identity now in \((a, c)\), we conclude that \( \lambda_1^{(c)} = 1 \), a contradiction.

Proof of Theorem 1.2. Let us assume first that equation (1.1) is oscillatory. Therefore, there exists a solution \( v \) with infinitely many zeros \( a < t_1 < t_2 < \ldots < \infty \). Let us choose \( a_n = t_n, \quad b_n = t_{n+1} \). The first Dirichlet eigenfunction in \([a_n, b_n]\) coincides with \( u \) up a multiplicative constant, with eigenvalue equal to 1. The eigenvalue \( \lambda_1^{(n)} \) of

\[
(|u'|^{p-2}u')' + \lambda q(t)|u|^{p-2}u = 0, \quad u(a_n) = 0 = u'(b_n)
\]
satisfies \( \lambda_1^{(n)} \leq 1 \), since \( v \) is an admissible function in the variational characterization of it, and

\[
\lambda_1^{(n)} = \inf_{u \in W^{1,p}(a_n, b_n), u(a_n)=0} \frac{\int_{a_n}^{b_n} |u'|^p dt}{\int_{a_n}^{b_n} q(t)|u|^p dt} \leq \frac{\int_{a_n}^{b_n} |v'|^p dt}{\int_{a_n}^{b_n} q(t)|v|^p dt} = 1.
\]
Suppose now that the eigenvalue condition is satisfied for a family of intervals \([a_n, b_n]\). Let us suppose that there exist a nonoscillatory solution \(u\), and let us take one of the intervals with \(a_N\) greater than the last zero of \(u\). Therefore, equation (1.1) is disconjugate in \((a_N, +\infty)\) (if not, there exist a solution with two zeros, and the Sturmian theory implies that \(u\) must have a zero between them). Hence, from Theorem 1.1 we get \(\lambda_1^{(N)} > 1\), a contradiction. \(\square\)

3. Final Remarks

3.1. Disconjugacy on bounded intervals. The arguments of the previous section can be added to the so called Roundabout Theorem (see [7, 8]) as a disconjugacy criteria in an interval \([a, b]\). We state it here with a modification of Theorem 1.1 as condition (v):

**Theorem 3.1** (Roundabout Theorem). The following statements are equivalent:

(i) Equation (1.1) is disconjugate on an interval \(I = [a, b]\), i.e., any nontrivial solution of (1.1) has at most one zero in \(I\).

(ii) There exists a solution of (1.1) having no zero in \([a, b]\).

(iii) There exists a solution \(w\) of the generalized Riccati equation corresponding to equation (1.1) which is defined on the whole interval \([a, b]\).

(iv) The \(p\)-degree functional

\[
F(u; a, b) = \int_a^b |u'|^p - q(t)|u|^p dt
\]

is positive for every \(u \in W_0^{1, p}(a, b)\), \(u\) not identically zero on \(I\).

(v) The first eigenvalue \(\lambda_1\) of

\[
(|u'|^{p-2}u')' + \lambda q(t)|u|^{p-2}u = 0, \quad u(a) = 0 = u(b)
\]

satisfy \(\lambda_1 > 1\).

**Proof.** It is easy to see that (iv) is equivalent to (v). From the variational characterization of the first eigenvalue

\[
\lambda_1 = \inf_{u \in W_0^{1, p}(a, b)} \frac{\int_a^b |u'|^p dt}{\int_a^b q(t)|u|^p dt},
\]

we obtain \(\lambda_1 > 1\) if and only if

\[
\int_a^b |u'|^p dt - \int_a^b q(t)|u|^p dt = (\lambda_1 - 1) \int_a^b q(t)|u|^p dt > 0.
\]

\(\square\)

**Remark 3.2.** The eigenvalue problem in unbounded intervals was studied in [10]. However, it is not know if the eigenvalues can be characterized variationally.

3.2. Oscillation Criteria in the Half Line. Here we consider some oscillation criteria which can be obtained from Theorems 1.1 and 1.2. We begin with a classical oscillation result:

**Theorem 3.3** (Leighton - Wintner Theorem). If \(\int_a^{+\infty} q(t)dt = +\infty\), then equation (1.1) is oscillatory on \([a, +\infty)\).

**Proof.** The proof follows from Theorem 1.2. For any \(a_n \geq a\), we choose \(b_n\) such that

\[
\int_{a_n}^{b_n} q(t)dt \geq 1
\]
and we compute the Rayleigh quotient for the first eigenvalue \( \lambda_1^{(n)} \) of the mixed problem
\[
(|u'|^{p-2}u'| + \lambda q(t)|u|^{p-2}u = 0, \quad u(a_n) = 0 = u'(b_n)
\]
with the test function
\[
v = \begin{cases} 
  t - a_n, & \text{if } t \in [a_n, a_n + 1) \\
  1, & \text{if } t \in [a_n + 1, b_n]
\end{cases}
\]
Hence,
\[
\lambda_1^{(n)} = \inf_{u \in W^{1,p}(a_n, b_n), u(a_n) = 0} \int_{a_n}^{b_n} \frac{|u'|^p dt}{\int_{a_n}^{b_n} q(t)|u|^p dt} \leq \frac{\int_{a_n}^{b_n} |v'|^p dt}{\int_{a_n}^{b_n} q(t)|v|^p dt} < 1.
\]

The following Hille type theorem can be found in [7].

**Theorem 3.4.** (i) Suppose that
\[
\limsup_{t \to +\infty} t^{p-1} \int_{t}^{+\infty} q(t) dt < \frac{1}{p} \left( \frac{p-1}{p} \right)^{p-1}.
\]
Then equation (1.1) is nonoscillatory.

(ii) Suppose that \( \int_{a}^{+\infty} q(t) dt < +\infty \) and
\[
\liminf_{t \to +\infty} t^{p-1} \int_{t}^{+\infty} q(t) dt > \frac{1}{p} \left( \frac{p-1}{p} \right)^{p-1}.
\]
Then equation (1.1) is oscillatory.

It is convenient to introduce the notion of strongly oscillatory and strongly nonoscillatory equations since the results of the inequalities of the previous criteria could be changed by introducing a parameter \( \lambda \) in equation (1.1).

Hence, we will say that equation (1.1) is strongly oscillatory (resp., strongly nonoscillatory) if
\[
(|u'|^{p-2}u'| + \lambda q(t)|u|^{p-2}u = 0
\]
is oscillatory (resp., nonoscillatory) for every \( \lambda > 0 \).

**Remark 3.5.** Strongly nonoscillatory equations were considered in [10] and the references therein.

We give a characterization of strongly oscillatory (resp., nonoscillation) following the ideas of Nehari [13].

**Theorem 3.6.** (i) Equation (1.1) is strongly oscillatory if and only if
\[
\limsup_{t \to +\infty} t^{p-1} \int_{t}^{+\infty} q(t) dt = +\infty.
\]

(ii) Equation (1.1) is strongly nonoscillatory if and only if
\[
\lim_{t \to +\infty} t^{p-1} \int_{t}^{+\infty} q(t) dt = 0.
\]

**Proof.** Let us define
\[
q^* = \limsup_{t \to +\infty} t^{p-1} \int_{t}^{+\infty} q(t) dt
\]
The proof of (i) follows easily from Theorem 3.4, since equation (3.4) is oscillatory if and only if
\[
\lambda q^* > \frac{1}{p} \left( \frac{p-1}{p} \right)^{p-1}
\]
for every $\lambda > 0$, which is valid if and only if $q^* = +\infty$.

The proof of (ii) follows in much the same way, since equation (3.4) is nonoscillatory if and only if

$$\lambda q^* < \frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1},$$

which is valid if and only if $q^* = 0$, and the upper limit coincides with the limit since $q(t) \geq 0$.

\[\square\]

Acknowledgements The first author is supported by Fundacion Antorchas and Conicet. The second author is supported by Fundacion Antorchas and ANPCyT PICT No. 03-05009.

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Pablo L. De Nápoli
FCEyN - Departamento de Matematica
Universidad de Buenos Aires
Ciudad Universitaria, Pabellon I
(1428) Buenos Aires, Argentina.
e-mail: pdenapo@dm.uba.ar
