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To cite this article: Marianna Euler, Norbert Euler, Enrique G Reyes (2017) Multipotentializations and nonlocal symmetries: Kupershmidt, Kaup-Kupershmidt and Sawada-Kotera equations, Journal of Nonlinear Mathematical Physics 24:3, 303–314, DOI: https://doi.org/10.1080/14029251.2017.1341694

To link to this article: https://doi.org/10.1080/14029251.2017.1341694

Published online: 04 January 2021
LETTER TO THE EDITOR

Multipotentializations and nonlocal symmetries: Kupershmidt, Kaup-Kupershmidt and Sawada-Kotera equations

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Received 14 January 2017
Accepted 16 April 2017

In this letter we report a new invariant for the Sawada-Kotera equation that is obtained by a systematic potentialization of the Kupershmidt equation. We show that this result can be derived from nonlocal symmetries and that, conversely, a previously known invariant of the Kaup-Kupershmidt equation can be recovered using potentializations.

Dedicated in memory to Wilhelm I Fushchich (1936-1997)

1. Introduction

We report a new invariant for the Sawada-Kotera equation by a systematic potentialization of the Kupershmidt equation. We deduce an invariant of the Kaup-Kupershmidt equation, obtained in [6] with the help of nonlocal symmetries. We furthermore show that the same invariant of the Sawada-Kotera equation can be derived by considering nonlocal symmetries.

For the benefit of clarity we first review some of the results that have been reported in [2] (see also [1] for more details on recursion operators and multipotentialization of semilinear fifth-order evolution equations).

The Kupershmidt equation

\[ K_t = K_{xxxxx} + \lambda \left( K_x K_{xxx} + K_{xx}^2 \right) - \frac{\lambda^2}{5} \left( K^2 K_{xxx} + 4 K K_x K_{xx} + K_x^3 \right) + \frac{\lambda^4}{125} K^4 K_x \]  

(1.1)

(\( \lambda \) is an arbitrary non-zero constant) potentializes in the so-called first potential Kupershmidt equation,

\[ U_t = U_{xxxx} + \lambda U_x U_{xxx} - \frac{\lambda^2}{5} \left( U_x^2 U_{xxx} + U_{xx}^2 U_{xx} \right) + \left( \frac{\lambda}{5} \right)^4 \frac{U_5^5}{U_x}, \]  

(1.2)

A typing error appeared in this equation in [2] (see eq. (2.7) in [2]), which did however not affect the results reported in [2]
by the potentialization

\[ U_x = K. \]  

Moreover, (1.1) is, by the second potentialization

\[ u_x = -\frac{5}{2\lambda} e^{-2\lambda U/5}, \]  

connected to the equation

\[ u_t = u_{xxxx} - \frac{5}{4u_x^2} u_{xxx} + \frac{15}{4u_x^2} u_{xxxx} - \frac{15}{2u_x^2} u_{xx} - \frac{135}{16u_x^4} u_{xxxx}, \]  

called the second-potential Kupershmidt equation. Combining the potentializations (1.3) and (1.4), we find that (1.5) and (1.1) are related by the differential substitution

\[ K(x,t) = -\frac{5}{2\lambda} u_{xx}. \]  

Diagram 1:

Eq.(1.5) in \( u_1 \)

\[ v_x = u_{1/2}^1 \]

Eq.(1.5) in \( u_2 \)

\[ v_x = u_{2/2}^1 \]

Eq.(1.5) in \( u_3 \)

\[ u_{s,s} = u_{1}^2 \]

Eq.(1.5) in \( u_4 \)

\[ u_{s,s} = v_{s}^4 \]

Equation (1.5) admits the following \( \triangle \)-auto-\( \text{B} \text{"a} \text{c} \text{k} \text{l} \text{e} \text{u} \text{n} \text{d} \) transformations that are obtained by combining potentializations as shown in Diagram 1 (see [2] for details):

\[ u_{j+1,xx} = u_{j+1,x} \left[ \frac{u_{j,xx}}{u_{j,x}} - \frac{2}{u_{j,x}} \frac{u_{j,x}}{u_{j}} \right] + 4u_{j+1,x} \left[ \frac{u_{1/2}^j}{u_{1/4}^j} \right] \]  

(1.7a)

\[ u_{j+1,xx} = u_{j+1,x} \left[ \frac{u_{j,xx}}{u_{j,x}} \right] + 4u_{j+1,x} \left[ \frac{1}{u_{1/4}^j} \right], \]  

(1.7b)

where \( u_j \) and \( u_{j+1} \) satisfy (1.5) for all natural numbers \( j \). On the other hand, (1.5) is also related to

\[ v_t = v_{xxxx} - 5v_x v_{xxxx} + 5v_x^2 v_{xxx}, \]  

(1.8)

as shown in Diagram 2.

\( \text{b} \)A typing error appeared in this equation in [2] (see eq. (2.11) in [2]), which did however not affect the results reported in [2]
Diagram 2:

Eq.(1.8) in $v_1$

$u_x = v_1^2$

Eq.(1.8) in $v_2$

$u_x = v_2^2$

Eq.(1.8) in $v_3$

$v_{3,x} = u_x^{1/2}$

Eq.(1.8) in $v_4$

$v_{4,x} = u_x^{1/2}$

This leads to the $\triangle$-auto-Bäcklund transformations for (1.8) (see Diagram 2 and [2] for details)

$V_{j+1,xx} = V_{j+1,x} \left[ \frac{V_{j,xx}}{V_{j,x}} - \frac{2 V_{j,x}}{V_{j}} \right] + \frac{V_{j}^{2}}{V_{j,x}}$ (1.9a)

$V_{j+1,xx} = V_{j+1,x} \left[ \frac{V_{j,xx}}{V_{j,x}} \right] + \frac{1}{V_{j,x}}$ (1.9b)

$V_{j+1,xx} = V_{j+1,x} \left[ \frac{V_{j,xx}}{V_{j,x}} \right] - \frac{1}{V_{j,x}}$ (1.9c)

where $V_j$ and $V_{j+1}$ satisfy (1.8) for all natural numbers $j$.

We remark that relation (1.9c) follows by applying the discrete symmetry $\nu \mapsto -\nu$ that is admitted by (1.8) to $V_j$ in (1.9b).

2. Connections to the Sawada-Kotera equation

We find that the Sawada-Kotera equation [9]

$S_t = S_{xxxx} + \nu S S_{xxx} + \nu S_x S_{xx} + \frac{1}{5} \nu^2 S_x$ (2.1)

($\nu$ is an arbitrary non-zero constant) is related to the first potential Kupershmidt equation (1.2) by the differential substitution

$S = -\frac{\lambda}{\nu} U_{xx} - \frac{1}{5} \frac{\lambda^2}{\nu} U_x^2$. (2.2)

(Note that a similar differential substitution to (2.2) was given in [4]). On the other hand, (1.2) potentializes in (1.5) by

$u_x = -\frac{5}{2} \frac{1}{\lambda} \exp \left( -\frac{2}{5} \frac{\lambda}{\nu} U \right)$. (2.3)

Combining these transformations, we obtain the following
Proposition 1: The Sawada-Kotera equation (2.1) admits the solutions

\[ S(x,t) = \frac{5}{2\nu} \{u, x\}, \]  

where \( u(x,t) \) is any non-constant solution of the second potential Kupershmidt equation (1.5) and \( \{u, x\} \) is the Schwarzian derivative

\[ \{u, x\} := \left( \frac{u_{xx}}{u_x} \right)_x - \frac{1}{2} \left( \frac{u_{xx}}{u_x} \right)^2. \]  

Moreover, relation (1.6) implies that solutions of the Sawada-Kotera equation (2.1) are

\[ S(x,t) = -\frac{\lambda}{\nu} \left( K_x + \frac{\lambda}{5} K^2 \right), \]  

where \( K(x,t) \) is any solution of the Kupershmidt equation (1.1).

The \( \triangle \)-auto-Bäcklund transformations (1.7a)–(1.7b) can now be applied to generate solutions for (1.5), and hence for (2.1).

We remark that relation (2.6) was previously obtained by Fordy and Gibbons [3] by factorizing a third-order linear operator.

As an example, we apply the \( \triangle \)-auto-Bäcklund transformation (1.7a), viz.

\[ u_{j+1,xx} = u_{j+1,x} \left[ \frac{u_{j,xx}}{u_{j,x}} - 2 \frac{u_{j,x}}{u_j} \right] + 4 u_{j+1,4/3} \left[ \frac{u_j^{1/3}}{u_{j,x}} \right]. \]

Since (1.7a) is in the form of a Bernoulli equation in the variable \( u_{j+1,x} \), we can easily integrate this equation to obtain

\[ u_{j+1,x} = \frac{u_{j,x}}{u_j^{1/3}} \left[ \int \left( \frac{u_j}{u_{j,x}^{1/3}} \right) dx + c_j(t) \right]^4, \]  

where \( c_j(t) \) is an arbitrary function of \( t \) that appears as a constant of integration. Inserting (2.7) with \( u = u_{j+1,x} \) into (2.4), we find that \( c_j(t) \) is an arbitrary constant.

As an explicit example, we use the following seed solution for (1.5):

\[ u_1(x,t) = x^5 - 180t. \]

Applying now relation (2.7), with \( j = 1 \), we obtain

\[ u_2(x,t) = (x^5 - 180t)^{-2} \left[ 5^{3/4} \left( \frac{1}{20} x^5 + 36t \right) + c_1 x \right]^4. \]  

Using \( u = u_2 \) given by (2.8) in relation (2.4), an explicit solution for (2.1) takes the form

\[ S(x,t) = -\frac{30(5x^8 - 14400tx^3 - 40x^4 + 16)}{\nu(x^3 + 720t + 4x)^2}, \]

where we have chosen \( c_1 = 5^{-1/4} \) for simplicity.
Now we obtain an invariant for the Sawada-Kotera equation (2.1). Applying the two potentializations
\begin{align}
    u_x &= v_x^{-2} \\
    u_x &= v^4 v_x^{-2}
\end{align}
(2.9a, 2.9b)
of (1.8), with the connection to the Sawada-Kotera equation given by (2.4), we obtain the following (see Diagram 3)

**Corollary:** The Sawada-Kotera equation (2.1) is invariant under the transformation \( S(x,t) \mapsto \tilde{S}(x,t) \), in which
\[
    \tilde{S}(x,t) = S(x,t) + 30 \left( \ln v \right)_{xx},
\]
(2.10)
where the variables \( S(x,t) \) and \( v(x,t) \) are related by
\[
    S(x,t) = -\frac{5}{v} v_{xxx} v_x
\]
(2.11)
and \( v(x,t) \) is a solution to (1.8), viz.
\[
    v_t = v_{xxxx} - 5 v_{xx} v_{xxx} v_x + 5 \frac{v_x^2 v_{xxx}}{v_x^2}.
\]
This gives a linearization of (1.8) in terms of the Sawada-Kotera equation, namely
\[
    v_t = v_{xxxx} + v v_x \tilde{S}_x.
\]

Using any of the \( \triangle \)-auto-Bäcklund transformations (1.9a), (1.9b) or (1.9c), we can construct solutions of the Sawada-Kotera equation (2.1) with the above Corollary.

### 3. Regarding the Kaup-Kupershmidt equation

In this and the following section we connect the above results with nonlocal symmetries. In the paper [6], Reyes obtained an invariance transformation for the Kaup-Kupershmidt equation
\[
    V_t = V_{xxxx} + 5 V V_{xxx} + \frac{25}{2} V_x V_{xx} + 5 V^2 V_x.
\]
(3.1)
In particular he reported the following
Proposition 2: [6] The Kaup-Kupershmidt equation (3.1) is invariant under the transformation $V \mapsto \bar{V}$, in which

$$\bar{V} = V + 3 (\ln u)_{xx},$$

where the variables $V$ and $u$ are related by

$$V(x,t) = \frac{u_{xxx}}{u_x} + \frac{3}{4} \left( \frac{u_{xx}}{u_x} \right)^2$$

and $u(x,t)$ is a solution to (1.5) viz.

$$u_t = u_{xxxx} - \frac{5u_{xx}u_{xxxx}}{4u_x} - \frac{15u_{xx}^2}{4u_x^2} + \frac{65u_{xx}^2u_{xxx}}{4u_x^2} - \frac{135u_{xx}^4}{16u_x^3}.$$ 

This invariance was obtained with the help of nonlocal symmetries: Equation (3.1) admits a non-local symmetry whose flow can be explicitly computed; consideration of this flow yields (3.2) and (3.3). We present a related computation in the next section. Now, the result given in Proposition 2 can be obtained by our multipotentialization method, namely in a similar way as was done in the
Diagram 4: An invariance transformation for the Kaup-Kupershmidt equation (3.1)

previous section for the Sawada-Kotera equation. Diagram 4 shows the connections between the equations. Besides the second potentialization of the Kupershmidt equation (1.5) (in the variables \(u_1(x, t)\) and \(u_2(x, t)\)) and third potentialization of the Kupershmidt equation (1.8) (in the variable \(v(x, t)\)), Diagram 4 also includes a fourth potentialization of the Kupershmidt equation, namely

\[
\begin{align*}
\frac{\partial w}{\partial t} & = w_{xxxx} - 5w_{xxxx}w_{xxx} - \frac{15}{4}w_{xxx}^2 - \frac{15}{4}w_{xxx}^2 w_{xxx}^2 + 65w_{xxx}^2 w_{xxx}^2 - \frac{15}{16}w_{xxx}^4 w_{xxx}^2 \\
& + \frac{5\beta}{6} \left( w_{xxx}^2 - \frac{7}{4}w_{xxx}^2 w_{xxx}^2 \right) - \frac{5}{36} \beta^2 w_{xxx}^{-1}.
\end{align*}
\]  

(3.4)

Furthermore we have the equation

\[
\begin{align*}
W_t = W_{xxxx} + 5 \left( W_{xx} - W_x^2 + \bar{\lambda} e^{2W} \right) W_{xxx} - 5W_x W_{xx} + 15\bar{\lambda} e^{2W} W_x W_{xx} \\
+ W_x^5 + 5\bar{\lambda}^2 e^{4W} W_x
\end{align*}
\]  

(3.5)

which is related to the fourth potential Kupershmidt equation (3.4) by a potentialization of (3.5), namely

\[
w_x = \frac{\beta}{6\bar{\lambda}} \exp(-2W),
\]

(3.6)

and to the Kaup-Kupershmidt equation (3.1) by the differential substitution (given in [4])

\[
V = 2W_{xx} - W_x^2 + \bar{\lambda} \exp(-2W).
\]

(3.7)
Combining these change of variables (see Diagram 4), we obtain the differential substitution between the third potential Kupershmidt equation (1.8) and the Kaup-Kupershmidt equation (3.1), namely

\[ V(x,t) = -3v_x^{-2}v_{xx}^2 + 2v_x^{-1}v_{xxx} . \]  

(3.8)

The invariance transformation given in Proposition 2 then follows (see Diagram 4).

4. The Sawada-Kotera invariance via nonlocal symmetries

We show that the invariance transformation (2.10) for Sawada-Kotera can be recovered with the help of nonlocal symmetries.

We replace \( S \) for \( (5/v)S \) in (2.1) and we obtain the Sawada-Kotera equation in the standard form

\[ S_t = S_{xxxx} + 5S_{xxx}S_x + 5S_x^2S_x . \]  

(4.1)

This equation is a member of a one-parameter family of equations admitting zero curvature representations. Indeed, we recall from [7, Section 6]:

**Proposition 3:** [7] The family of equations

\[ S_t = S_{xxxx} - \left( 4y + \frac{1}{y} \right) S_{xxx} + 5S_x^2S_x - \left( 2y + \frac{2}{y} \right) S_xS_{xx} , \]  

(4.2)

in which \( y \) is a non-zero real parameter, is the integrability condition of the \( sl(2,\mathbb{R}) \)-valued linear problem \( X\psi = \psi_x, T\psi = \psi_t \) where

\[ X = \begin{bmatrix} 0 & -y/\eta^2 \\ -\eta^2S & 0 \end{bmatrix} \]  

(4.3)

and

\[ T = \begin{bmatrix} yS_{xxx} - S_{xx} & 2y^2S_{xx}/\eta^2 - yS^2/\eta^2 \\ \eta^2(-S_{xxx} + (2y + 1/y)SS_{xx} + S_x^2/y - S^3) & -yS_{xxx} + SS_x \end{bmatrix} . \]  

(4.4)

The real number \( \eta \) appearing in (4.3) and (4.4) is not essential, since this “spectral parameter” can be eliminated via a simple gauge transformation. However, this linear problem does encode non-trivial information on Equation (4.2), as we will see below.

The Kaup-Kupershmidt equation corresponds to (4.2) with \( y = -1/4 \), while the Sawada-Kotera equation (4.1) is (4.2) with \( y = -1 \). This family contains the fifth order Korteweg-de Vries equation as well (it is enough to take \( y = -1/\sqrt{6} \)) but we will not use this observation here. Proposition 3 allows us to find a quadratic pseudo-potential for Equation (4.2):
Lemma: Equation (4.2) admits the quadratic pseudo-potential

\[ \alpha_x = -\eta^2 S + \frac{y}{\eta^2} \alpha^2 \]  
\[ \alpha_t = -\eta^2 S_{xxx} + \left( 2\eta^2 y + \frac{\eta^2}{y} \right) S_{xx} + \frac{\eta^2}{y} S_x^2 - \eta^2 S^3 + (2SS_x - 2yS_{xxx}) \alpha \]

\[ + \left( \frac{y}{\eta^2} S^2 - \frac{2y^2}{\eta^2} S_{xx} \right) \alpha^2 , \]

that is, the system (4.5a) and (4.5b) is completely integrable for \( \alpha(x,t) \) whenever \( S(x,t) \) is a solution to Equation (4.2).

This result generalizes some interesting computations carried out by Nucci in [5]. We write Equation (4.5b) as a conservation law, and define a corresponding potential \( \delta \). We find that \( \delta \) is determined by the following two compatible equations:

\[ \delta_x = \frac{2y}{\eta^2} \alpha \]  
\[ \delta_t = -2yS_{xxx} + 2SS_x + \left( -\frac{4y^2}{\eta^2} S_{xx} + \frac{2y}{\eta^2} S_x^2 \right) \alpha . \]

We would like to find a shadow of a nonlocal symmetry for (4.2), that is, a solution to the formal linearization of (4.2) depending on \( \alpha \) and \( \delta \). For that, the following theorem, given in [7], where the reader can also find further references on nonlocal symmetries, is essential:

**Theorem 1:** [7] Consider the function

\[ G = \alpha \exp(-L(y) \delta) , \]

in which \( \alpha \) and \( \delta \) satisfy Equations (4.5a)-(4.6b). \( G \) is the shadow of a nonlocal symmetry for Equation (4.2) if and only if

\[ L(y) = \frac{4y^2 + 1}{10y^2} , \]

and the parameter \( y \) satisfies the equation

\[ -125y^2(96y^6 - 118y^4 - 1 + 23y^2) = 0 \]

Since \( y \) cannot be equal to zero, Equation (4.9) gives exactly six values for which (4.7) is the shadow of a nonlocal symmetry, namely

\[ y = 1, -1; \frac{1}{4}; -\frac{1}{4}; \frac{1}{\sqrt{6}}; -\frac{1}{\sqrt{6}} . \]

The corresponding values of \( L \) are \( L(1) = L(-1) = 1/2, L(1/4) = L(-1/4) = 2, \) and \( L(1/\sqrt{6}) = L(-1/\sqrt{6}) = 1 \). The equations obtained by replacing these values of \( y \) into (4.2) are, respectively, the Sawada-Kotera, Kaup-Kupershmidt, and fifth-order KdV equations! Thus, the shadow (4.7) recognizes precisely the only 2-homogeneous polynomial evolution equations which possess an infinite number of symmetries, from a whole family of equations which are the integrability condition of overdetermined \( sl(2, \mathbb{R}) \)-valued linear problems, and which admit quadratic pseudo-potentials
We complete the shadow $G$ to a bona-fide nonlocal symmetry of the Sawada-Kotera equation following [7]:

**Theorem 2:** The system of equations formed by the Sawada-Kotera equation (4.1), (4.5a)–(4.6b) with $y = -1$, and the equations

\[
\beta_x = \frac{2\eta^2}{3} \exp(-\frac{1}{2} \delta) \tag{4.10}
\]

and

\[
\beta_t = \frac{2 \exp(-\frac{\delta}{2})}{3\eta^6} \left[ 3\eta^6 S_x \alpha + \eta^8 (S^2 - S_{xx}) \right], \tag{4.11}
\]

admits the classical symmetry

\[
W = \alpha \exp(-\frac{1}{2} \delta) \frac{\partial}{\partial S} - \frac{\eta^4}{3} \exp(-\frac{1}{2} \delta) \frac{\partial}{\partial \alpha} + \beta \frac{\partial}{\partial \delta} - \frac{1}{4} \beta^2 \frac{\partial}{\partial \beta}, \tag{4.12}
\]

and therefore this vector field is a nonlocal symmetry of the Sawada-Kotera equation (4.1).

The flow of the vector field (4.12) is found by solving the system of equations

\[
\frac{\partial S}{\partial \tau} = \alpha \exp(-\frac{1}{2} \delta), \quad \frac{\partial \alpha}{\partial \tau} = -\frac{\eta^4}{3} \exp(-\frac{1}{2} \delta), \tag{4.13a}
\]

\[
\frac{\partial \delta}{\partial \tau} = \beta, \quad \frac{\partial \beta}{\partial \tau} = -\frac{1}{4} \beta^2, \tag{4.13b}
\]

with initial conditions

\[
S(x, t, 0) = S_0; \quad \alpha(x, t, 0) = \alpha_0; \quad \delta(x, t, 0) = \delta_0; \quad \beta(x, t, 0) = \beta_0, \tag{4.14}
\]

in which $S_0$, $\alpha_0$, $\delta_0$ and $\beta_0$ are arbitrary particular solutions to the compatible system of equations given in Theorem 2. The solution to this initial value problem is

\[
\alpha(\tau) = -\frac{4\eta^4}{3\beta_0 \tau + 12} \exp\left(-\frac{1}{2} \delta\right) + \alpha_0 \tag{4.15a}
\]

\[
\delta(\tau) = 4 \ln \left| \frac{\beta_0 \tau + 4}{4} \right| + \delta_0 \tag{4.15b}
\]

\[
\beta(\tau) = \frac{4\beta_0}{\beta_0 \tau + 4}. \tag{4.15c}
\]

The corresponding formula for $S(x, t, \tau)$ is obtained from the first equation in (4.13a) by using (4.15a), (4.15b) and the initial conditions above. We find the family of solutions

\[
S(x, t, \tau) = -\frac{8\eta^4 e^{-\delta_0 \tau^2}}{3(\beta_0 \tau + 4)^2} + \frac{4\tau}{(\beta_0 \tau + 4)} \alpha_0 e^{-(1/2) \delta_0} + S_0. \tag{4.16}
\]
Now we remark that the foregoing analysis allows us to recover transformation (2.10). Indeed, we start from (4.16) and eliminate $\delta_0$ using (4.10). We obtain

$$S = -\frac{6\beta_0^2 \tau^2}{(\beta_0 \tau + 4)^2} + \frac{6\tau \beta_{0,x} \alpha_0}{\eta^2 (\beta_0 \tau + 4)} + S_0 .$$

Now we eliminate $\alpha_0$ using (4.6a) [with $y = -1$]. We find

$$S = -\frac{6\beta_0^2 \tau^2}{(\beta_0 \tau + 4)^2} - \frac{3\tau \beta_{0,x} \alpha_0}{\beta_0 \tau + 4} + S_0 .$$

We re-write the second summand of this expression by using the equation

$$\beta_{0,xx} = (-1/2) \beta_{0,x} \delta_{0,x} , \quad (4.17)$$

which is obtained by differentiating (4.10) with respect to $x$ and simplifying the result using again (4.10). We obtain

$$S = -\frac{6\beta_0^2 \tau^2}{(\beta_0 \tau + 4)^2} + \frac{6\tau \beta_{0,xx}}{\beta_0 \tau + 4} + S_0 ,$$

or, equivalently,

$$S = 6 \frac{\partial^2}{\partial x^2} \ln(B) + S_0 , \quad (4.18)$$

in which $B = \beta_0 \tau + 4$. This is exactly transformation (2.10).

We also recover Equation (2.11). Replacing (4.6a) [with $y = -1$] into Equation (4.17) we obtain

$$\beta_{0,xx} = \frac{1}{\eta^2} \alpha_0 \beta_{0,x} . \quad (4.19)$$

Differentiating (4.19) with respect to $x$ and using (4.5a) [with $y = -1$] we find

$$\beta_{0,xxx} = \frac{\beta_{0,xx}}{\eta^2} \alpha_0 - \beta_{0,x} S_0 - \frac{\beta_{0,x}}{\eta^4} \alpha_0^2 . \quad (4.20)$$

Now we eliminate $\alpha_0$ from (4.20) by means of (4.19) and then we simplify the resulting expression. We obtain $S_0 = -\beta_{0,xxx}/\beta_{0,x}$, which is equivalent to Equation (2.11) for $B$.

Finally, we consider Equation (4.11) for $\beta_0$. Using (4.10) we can write (4.11) as

$$\beta_{0,t} = \frac{3\beta_{0,x} \alpha_0}{\eta^2} + \beta_{0,x} S_0 - \beta_{0,x} S_0 .$$

We eliminate $\alpha_0$ using (4.19), and we eliminate $S_0$ and its derivatives using the relation $S_0 = -\beta_{0,xxx}/\beta_{0,x}$ and its differential consequences. A straightforward calculation then yields

$$\beta_{0,t} = -\frac{5\beta_{0,xxx} \beta_{0,xx}}{\beta_{0,x}} + \frac{5\beta_{0,xxx} \beta_{0,xx}^2}{\beta_{0,x}^2} + \beta_{0,xxxx} .$$

This equation is equivalent to Equation (1.8) for $B$. 

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Co-published by Atlantis Press and Taylor & Francis

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Acknowledgements

We thank Robert Conte for useful discussions and for sharing his personal notes on the Sawada-Kotera and other equations. E.G.R. has been partially supported by the FONDECYT operating grant # 1161691.

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