\[ \mathcal{N} = 2 \text{ SUPERCONFORMAL ALGEBRA AND THE ENTROPY OF CALABI–YAU MANIFOLDS} \]

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Abstract. We use the representation theory of \( \mathcal{N} = 2 \) superconformal algebra to study the elliptic genera of Calabi–Yau (CY) \( D \)-folds. We compute the entropy of CY manifolds from the growth rate of multiplicities of the massive (non-BPS) representations in the decomposition of their elliptic genera. We find that the entropy of CY manifolds of complex dimension \( D \) behaves differently depending on whether \( D \) is even or odd. When \( D \) is odd, CY entropy coincides with the entropy of the corresponding hyperKähler \( (D - 3) \)-folds due to a structural theorem on Jacobi forms. In particular, we find that the Calabi–Yau 3-fold has a vanishing entropy. At \( D > 3 \), using our previous results on hyperKähler manifolds, we find \( S_{CY_D} \sim 2\pi \sqrt{\frac{D-3}{2(D-4)}}^n \). When \( D \) is even, we find the behavior of CY entropy behaving as \( S_{CY_D} \sim 2\pi \sqrt{\frac{D-1}{2}}^n \). These agree with Cardy’s formula at large \( D \).

1. Introduction

The \( \mathcal{N} = 2 \) superconformal algebra (SCA) is a basic tool in the world-sheet analysis of string compactifications on Calabi–Yau (CY) manifold with complex dimension-\( D \). It is well-known that in \( \mathcal{N} = 2 \) SCA there exist two types of representations: BPS (massless) and non-BPS (massive) representations. BPS representations appear when the conformal weight \( h \) of their highest-weight state hits the unitarity bound \( c_2 = \frac{D}{8} \) (in the Ramond sector) where \( c \) denotes the central charge of SCA. On the other hand non-BPS representations appear at \( h > \frac{D}{8} \).

We study the decomposition of the elliptic genus for Calabi–Yau manifold \( CY_D \) in terms of characters of these representations;

\[
\text{elliptic genus of } CY_D = \sum_Q c_{D,Q} [ \text{BPS representations with } U(1) \text{ charge-} Q ] \\
+ \sum_{n=1}^{\infty} \sum_Q p_{D,Q}(n) \left[ \text{non-BPS representations at } h = n + \frac{D}{8} \text{ with } U(1) \text{ charge-} Q \right]. \tag{1.1}
\]

Since there exists only a finite number (of order \( D \)) of BPS representations, the set of their multiplicities \( c_{D,Q} \) is finite. On the other hand, we have an infinite series of multiplicities.
\(p_{D,Q}(n)\) for non-BPS representations and we define the intrinsic CY entropy \(S_{CYD}\) by the rate of its exponential growth

\[
S_{CYD} \sim \log p_{D,Q}(n).
\] (1.2)

Such an analysis has been done for the case of hyperKähler manifolds \([11, 12]\) based on earlier works on the representation theory of the \(\mathcal{N} = 4\) superconformal algebra \([13, 14, 15, 16]\). It was pointed out that the predicted entropy of hyperKähler manifolds coincides with that of the standard \(D1-D5\) black hole \([6, 34]\) when we consider the symmetric product of \(K3\) surfaces. See Ref. 38 for a similar idea. We also note that the expansion of elliptic genera in terms of theta functions has been discussed in Ref. 7.

A key in our analysis is the fact that the characters of the BPS representations are the mock theta functions which were first introduced by Ramanujan (see, e.g., Refs. 1, 9, 18). We rely on recent developments \([3, 40]\) on the understanding of the mock theta function (see Refs. 28, 39 for review): namely, the mock theta function is a holomorphic part of the harmonic Maass form, and that it has a vector-valued modular form with weight-3/2 as its "shadow".

This paper is organized as follows. In Section 2 we review the characters of the \(\mathcal{N} = 2\) superconformal algebra and the elliptic genera of the CY manifolds. We make an extensive use of Jacobi forms, whose properties are collected in Appendix. Decompositions of the elliptic genera for the CY manifold are studied in Sections 3 and 4. We treat the odd-dimensional CY manifolds in Section 3. We notice the close relationship between the characters of \(\mathcal{N} = 2\) and \(\mathcal{N} = 4\) SCA in this case and Gritsenko’s result on the space of Jacobi forms. It turns out that the entropy of CY \(D\)-folds coincides with the entropy of the corresponding hyperKähler manifolds in \((D-3)\)-dimensions. In particular the entropy of CY 3-folds vanishes identically. In Section 4 we discuss the even-dimensional CY manifolds. Adopting the same strategy as in our previous paper \([12]\), we compute the entropy by use of the Poincaré–Maass series and obtain the result

\[
S_{CYD} \sim 2\pi \sqrt{\frac{D-1}{2}} n.
\]

The last section is devoted to concluding remarks.

2. \(\mathcal{N} = 2\) Superconformal Algebras and Elliptic Genera

2.1. \(\mathcal{N} = 2\) Superconformal Algebras. The \(\mathcal{N} = 2\) superconformal algebra with the central charge \(c = 3D\) is a fundamental tool studying the compactification of string theory on Calabi-Yau manifold \(CYD\) with complex dimension \(D\). It is well-known that there exists
an isomorphism \([31]\) of the algebra,

\[
L_n \rightarrow L_n + \alpha J_n + \frac{c}{6} \alpha^2 \delta_{n,0},
\]

\[
J_n \rightarrow J_n + \frac{c}{3} \alpha \delta_{n,0},
\]

\[
G_r^\pm \rightarrow G_r^\pm, \alpha,
\]

where \(\alpha \in \mathbb{Z}/2\). Ramond and NS sectors are exchanged when \(\alpha \in \mathbb{Z} + \frac{1}{2}\). For our purpose of studying string compactification, we need the extended \(\mathcal{N} = 2\) SCA which is invariant under the integral spectral flow \(\alpha \in \mathbb{Z}\). Odake discussed such series of extended SCA’s by adding spectral flow generators to the standard \(\mathcal{N} = 2\) SCA \([24, 25, 26]\).

The highest weight state in the extended \(\mathcal{N} = 2\) SCA are labeled by conformal weight \(h\) and \(U(1)\) charge \(Q\):

\[
L_0 |\Omega\rangle = h |\Omega\rangle, \quad J_0 |\Omega\rangle = Q |\Omega\rangle.
\]

In the Ramond sector, due to a relation between zero-modes of the supercurrents, \(\{G_0^+, G_0^-\} = 2 (L_0 - \frac{c}{24})\), we have the unitarity condition

\[
h \geq \frac{D}{8}.
\]

Characters are defined by

\[
\chi_{D,h,Q}^* (z; \tau) = \text{Tr}_{\mathcal{H}^*} (q^{L_0 - \frac{c}{24}} e^{2\pi i z J_0}),
\]

where \(\ast\) denotes the spin structure, and \(\mathcal{H}^*\) is the Hilbert space of the representation. Under the spectral flow (2.1), the characters in the Ramond/NS sectors are transformed to each other as:

\[
\chi_{\tilde{\text{NS}}}^*(z; \tau) = \chi_{\text{NS}}^* \left( z + \frac{1}{2}; \frac{\tau}{2} \right),
\]

\[
\chi_{\text{R}}^*(z; \tau) = q^{\frac{D}{8}} e^{\frac{D\pi i z}{2}} \chi_{\text{NS}}^* \left( z + \frac{\tau}{2}; \frac{\tau}{2} \right),
\]

\[
\chi_{\tilde{\text{R}}}^*(z; \tau) = e^{\frac{D\pi i z}{2}} \chi_{\text{R}}^* \left( z + \frac{1}{2}; \frac{\tau}{2} \right).
\]

Here the phase factor in the \(\tilde{R}\)-sector is our convention.

By construction, the superconformal character (2.4) is given by the irreducible characters of \(\mathcal{N} = 2\) SCA \([8, 21]\) summed over spectral flow. This fact can be directly checked in the work of Refs. \([25, 26]\).

In the Ramond sector, characters are given explicitly as follows. Here the \(U(1)\) charge takes values

\[
Q \equiv \frac{D}{2} \mod \mathbb{Z}.
\]
massive (non-BPS) representations:
\[ h > \frac{D}{8}; \quad Q = \frac{D}{2}, \frac{D}{2} - 1, \ldots, -(\frac{D}{2} - 1), -\frac{D}{2} \quad \text{and} \quad Q \neq 0 \quad (D = \text{even}) \]

\[
\text{ch}_{D,h,Q > 0}(z; \tau) = (-1)^{Q + \frac{D}{2}} q^{h - \frac{D}{2}} \frac{i \theta_{11}(z; \tau)}{[\eta(\tau)]^3} e^{2\pi i (Q + \frac{1}{2})z} \times \sum_{n \in \mathbb{Z}} q^{\frac{D}{2} - 1} n^2 + (Q + \frac{1}{2})n \left(1 - e^{2\pi i z}\right)^{(D-1)n}, \quad (2.7)
\]

massless (BPS) representations:
\[ h = \frac{D}{8}; \quad Q = \frac{D}{2} - 1, \frac{D}{2} - 2, \ldots, -(\frac{D}{2} - 1), \]

\[
\text{ch}_{D,h=\frac{D}{8},Q \geq 0}(z; \tau) = (-1)^{Q + \frac{D}{2}} \frac{i \theta_{11}(z; \tau)}{[\eta(\tau)]^3} e^{2\pi i (Q + \frac{1}{2})z} \times \sum_{n \in \mathbb{Z}} q^{\frac{D}{2} - 1} n^2 + (Q + \frac{1}{2})n \left(1 - e^{2\pi i z}\right)^{(D-1)n} \left(1 - e^{2\pi i q^n}\right), \quad (2.8)
\]

and for \( h = \frac{D}{8}; \quad Q = \frac{D}{2} \)

\[
\text{ch}_{D,h=\frac{D}{8},Q=\frac{D}{2}}(z; \tau) = (-1)^{\frac{D}{2}} \frac{i \theta_{11}(z; \tau)}{[\eta(\tau)]^3} e^{2\pi i \frac{D+1}{2}z} \times \sum_{n \in \mathbb{Z}} q^{\frac{D}{2} - 1} n^2 + \frac{D+1}{2} n \left(1 - q\right) \left(1 - e^{2\pi i z}\right)^{(D-1)n} \left(1 - e^{2\pi i q^n}\right). \quad (2.9)
\]

See Appendix \( \square \) for the notation of theta functions. The characters for \( Q < 0 \) are given by

\[
\text{ch}_{D,h,\,-Q < 0}(z; \tau) = \text{ch}_{D,h,Q}(\bar{z}; \tau). \quad (2.10)
\]

The Witten index of massless representations are given by

\[
\text{ch}_{D,h=\frac{D}{8},Q \geq 0}(z = 0; \tau) = \begin{cases} (-1)^{Q + \frac{D}{2}}, & \text{for } 0 \leq Q < \frac{D}{2}, \\ 1 + (-1)^{\frac{D}{2}}, & \text{for } Q = \frac{D}{2}, \end{cases} \quad (2.11)
\]

while all massive representations have a vanishing index.

At the unitarity bound \( h = \frac{D}{8} \), a massive character decomposes into a sum of massless characters as

\[
\lim_{h \searrow \frac{D}{8}} \text{ch}_{D,h,Q > 0}(z; \tau) = \text{ch}_{D,h=\frac{D}{8},Q+1}(z; \tau) + \text{ch}_{D,h=\frac{D}{8},Q}(z; \tau), \quad (2.12)
\]

where \( Q \geq 0 \), and

\[
\lim_{h \searrow \frac{D}{8}} \text{ch}_{D,h,\,Q = \frac{D}{2}}(z; \tau)
\]

\[
= \text{ch}_{D,h=\frac{D}{8},Q=\frac{D}{2}}(z; \tau) + \text{ch}_{D,h=\frac{D}{8},Q=\frac{D}{2}-1}(z; \tau) + \text{ch}_{D,h=\frac{D}{8},Q=-(\frac{D}{2}-1)}(z; \tau). \quad (2.13)
\]
2.2. Elliptic Genus. The elliptic genus of the CY manifold with complex dimension $D$ is identified with \[37\]

$$Z_{CY_D}(z;\tau) = \text{Tr}_{H^R \otimes H^R} \left[ (-1)^F e^{2\pi i z J_0} q^{L_0 - \frac{D}{2}} \bar{q}^{-\frac{D}{2}} \right],$$

(2.14)

where $(-1)^F = e^{\pi i (J_0 - \bar{J}_0)}$. Due to the supersymmetry, only the ground state contributes in the right-moving sector, and the elliptic genus is independent of $\bar{q}$. It is known \[20\] that the elliptic genus is the Jacobi form with weight-0 and index-$\frac{D}{2}$. See Appendix \[B\] for the definition of the Jacobi form.

The elliptic genus is related to the topological invariants of manifolds. We have

$$q^{\frac{D}{2}} Z_{CY_D} \left( z = \frac{1 + \tau}{2}; \tau \right) = \hat{\chi}_{CY_D} + \cdots,$$

(2.15)

where $\chi_{CY_D}$ and $\hat{\chi}_{CY_D}$ are respectively the Euler characteristic and the $\hat{\chi}$-genus.

3. Calabi–Yau Manifolds: Odd-Dimension

We study odd-dimensional Calabi-Yau manifolds $CY_D$ ($D = \text{odd}$) throughout this section. The $U(1)$ charge is $Q \in \mathbb{Z} + \frac{1}{2}$.

3.1. Character Decomposition. We see from (2.8) that the combination of $\mathcal{N} = 2$ massless characters which is even in $z$ can be written as

$$\text{ch}_{R,h=\frac{D}{2}, Q=a}^\mathcal{N} = \text{ch}_{R,h=\frac{D}{2}, Q=-\frac{D}{2}}^\mathcal{N} \left( z; \tau \right)$$

$$= (-1)^{\frac{D-1}{2}} \frac{\eta(z)}{\eta(\tau)} \sum_{n \in \mathbb{Z}} q^{\frac{D-1}{2} n^2} e^{2\pi i (\frac{D-1}{2}) n} \frac{1 + e^{2\pi i z} q^n}{1 - e^{2\pi i z} q^n},$$

(3.1)

where $C_D^{\mathcal{N}=4}(z;\tau)$, defined in \[12\], is the isospin-0 massless character in $c = 6k \mathcal{N} = 4$ SCA \[14, 15, 10\]. $\phi_{0,\frac{D}{2}}(z;\tau)$ is a weight-0 index-3/2 Jacobi form listed in Appendix \[B\].

Similarly the even-$z$ combination of massive characters (2.7) can be written as

$$\text{ch}_{R,h,Q}^\mathcal{N} + \text{ch}_{R,h,-Q}^\mathcal{N} = (-1)^{Q+\frac{D}{2}-1} \phi_{0,\frac{D}{2}}(z;\tau) q^{h - \frac{D}{2}} \frac{(q^{1/2})^2}{\eta(z)} B_{D-S^2; Q}^{\mathcal{N}=4, \frac{D}{2}}(z;\tau),$$

(3.2)

where $Q > 0$, and $B_{k,a}^{\mathcal{N}=4}(z;\tau)$, defined in \[11\], is the basis of massive characters in $\mathcal{N} = 4$ SCA \[12\]. So both the (even-$z$ part of) $\mathcal{N} = 2$ massless and massive characters coincide with those of $\mathcal{N} = 4$ SCA up to the Jacobi form $\phi_{0,\frac{D}{2}}(z;\tau)$.

This also happens to the elliptic genus $Z_{CY_D}(z;\tau)$. The elliptic genus for $CY_D$ is a Jacobi form with weight-0 and index-$\frac{D}{2}$ and is an even function of $z$ because of (B.1). Now there exists a structural theorem on the space of Jacobi forms with half-integral index \[19\].
This space is isomorphic to the space \( \mathbb{J}_{0, \frac{D-3}{2}} \in \mathbb{Z} \)
\[
\mathbb{J}_{0, \frac{D-3}{2}} = \phi_{0, \frac{D-3}{2}}(z; \tau) \cdot \mathbb{J}_{0, \frac{D-3}{2}}.
\] (3.3)

In view of the relationship (3.1), (3.2) and the above theorem (3.3), we can conclude that the character decomposition of the elliptic genus of the Calabi–Yau \( D \)-fold is essentially the same as that of the hyperKähler manifolds with complex dimension-(\( D - 3 \)), which was studied in our previous paper [12].

Let us consider the case when the elliptic genus \( Z_{CYD}(z; \tau) \) includes a piece
\[
\phi_{0, \frac{D}{2}}(z; \tau) \left[ \left( \frac{\theta_{10}(z; \tau)}{\theta_{10}(0; \tau)} \right)^{D-3} + \left( \frac{\theta_{00}(z; \tau)}{\theta_{00}(0; \tau)} \right)^{D-3} + \left( \frac{\theta_{01}(z; \tau)}{\theta_{01}(0; \tau)} \right)^{D-3} \right].
\] (3.4)

The above combination contains the identity representation in the NS sector and gives the dominant contribution to the entropy. Using the results of Ref. 12 we can derive the entropy of CY manifolds \( S_{CYD} \) from the increase in the multiplicity of massive representations with \( U(1) \) charge \( Q > 0 \)
\[
S_{CYD} \sim 2\pi \sqrt{\frac{(D - 3)^2}{2(D - 1)} n - \left( \frac{D - 3}{2\pi} - \frac{1}{2} \right)^2}.
\] (3.5)

3.2. Examples.

3.2.1. Calabi–Yau 3-fold \( CY_3 \). The elliptic genus of the Calabi–Yau 3-fold \( CY_3 \) is given by [19, 20, 23]
\[
Z_{CY3}(z; \tau) = \frac{\chi_{CY3}}{2} \phi_{0, \frac{3}{2}}(z; \tau),
\] (3.6)

where \( \chi_{CY3} \) is the Euler number of \( CY_3 \).

With a help of (3.1) we have
\[
\text{ch}_{D=3, h=\frac{3}{2}, Q=\frac{1}{2}}(z; \tau) + \text{ch}_{D=3, h=\frac{3}{2}, Q=-\frac{1}{2}}(z; \tau) = \phi_{0, \frac{3}{2}}(z; \tau),
\]
which proves a simple decomposition formula for the elliptic genus,
\[
Z_{CY3}(z; \tau) = \frac{\chi_{CY3}}{2} \left[ \text{ch}_{D=3, h=\frac{3}{2}, Q=\frac{1}{2}}(z; \tau) + \text{ch}_{D=3, h=\frac{3}{2}, Q=-\frac{1}{2}}(z; \tau) \right].
\] (3.7)

There is no contribution from the massive representations in this decomposition, and we conclude that the entropy of the CY 3-fold vanishes identically,
\[
S_{CY3} = 0.
\] (3.8)

3.2.2. Calabi–Yau 5-fold \( CY_5 \). The elliptic genus of the Calabi–Yau 5-fold \( CY_5 \) is
\[
Z_{CY5}(z; \tau) = \frac{\chi_{CY5}}{24} \phi_{0, 1}(z; \tau) \phi_{0, 1}(z; \tau),
\] (3.9)
where $\chi_{CY_5}$ is the Euler number of $CY_5$ and $\phi_{0,1}$ is one half of the elliptic genus of $K3$ surface (B.5). As was shown in Ref. 11 (see also Ref. 13), character decomposition of the $K3$ elliptic genus is given by

$$2 \phi_{0,1}(z; \tau) = 24 C_1^{N=4}(z; \tau) - q^{-\frac{1}{8}} \left[ 2 - \sum_{n=1}^{\infty} A_n q^n \right] B_{1,1}^{N=4}(z; \tau). \quad (3.10)$$

Here the positive integers $A_n$ are given by the Rademacher expansion as

$$A_n = \frac{-2 \pi i}{(8 n - 1)^2} \sum_{c=1}^{\infty} \frac{1}{\sqrt{c}} I_{\frac{1}{2}}\left(\frac{\pi \sqrt{8 n - 1}}{2 c}\right) \sum_{k \mod 4 \mod 8 c} (-4)^{k/8 c} e^{\frac{2 \pi i}{8 c}}, \quad (3.11)$$

where $(-\frac{1}{4})$ is the Legendre symbol, and $I_{\frac{1}{2}}(x)$ denotes the modified Bessel function,

$$I_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sinh(x). \quad (3.12)$$

Since $N = 2$ characters coincide with those of $N = 4$ up to a factor $\phi_{0,1}$, the character decomposition of (3.9) becomes exactly the same as in (3.10). We conclude that

$$S_{CY_5} = S_{K3} = \log A_n \sim 2 \pi \sqrt{\frac{1}{2} \left( n - \frac{1}{8} \right)}. \quad (3.13)$$

4. Calabi–Yau Manifolds: Even-Dimension

Let us next study the even-dimensional Calabi–Yau manifolds $CY_D$. $D$ is even throughout this section. The $U(1)$ charge is integral $Q \in \mathbb{Z}$. We follow the method of Ref. 12 to construct the character decomposition of the elliptic genus.

4.1. Massless and Massive Characters. For notational simplicity we set $C_D(z; \tau)$ as the massless character with $U(1)$ charge-0,

$$C_D(z; \tau) = (-1)^D \frac{\partial}{\partial z} \chi_{D,h=\frac{D}{2},Q=0}(z; \tau) = \frac{i \theta_{11}(z; \tau)}{[\eta(\tau)]^3} e^{\pi i z} \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{D-1}{2}n^2 + \frac{1}{8} n} e^{2\pi i (D-1)n z} (1 - e^{2\pi i Q} q^n). \quad (4.1)$$

It is identified as

$$C_D(z; \tau) = \frac{i \theta_{11}(z; \tau)}{[\eta(\tau)]^3} e^{\pi i z} f_{D-1}\left( -1 + \frac{\tau}{2 (D - 1)}, z + \frac{-1 + \tau}{2 (D - 1)}; \tau \right), \quad (4.2)$$

where $f_D(u, z; \tau)$ is the Appell function defined in (E.1). Appell function undergoes a modular transformation with a Mordell’s integral (E.2) and is thus a typical mock theta function. One can cure its modular property by the process of completion [40]. By using (E.3), we define the completion

$$\tilde{C}_D(z; \tau) = C_D(z; \tau)$$
where \( \tilde{\vartheta}_{D,a}(z; \tau) \) is a (modified) theta series defined as (A.5), and \( R_{D,a}(z; \tau) \) denotes the non-holomorphic function (E.4). (Note that the combination \( e^{-\frac{a-1}{D-1}\pi}R_{D-1,a-1}(\frac{-1+\tau}{2(D-1)}; \tau) \) is invariant under \( a \to D-a \).

One finds that \( \tilde{C}_D(z; \tau) \) is a real analytic Jacobi form satisfying

\[
\tilde{C}_D \left( \frac{z}{\tau} ; -\frac{1}{\tau} \right) = e^{\pi i D z^2} \tilde{C}_D(z; \tau),
\]

\[
\tilde{C}_D(z + \tau; \tau) = q^{D/2} e^{-2\pi i D z} \tilde{C}_D(z; \tau),
\]

\[
\tilde{C}_D(z; \tau + 1) = \tilde{C}_D(z + 1; \tau) = \tilde{C}_D(z; \tau).
\]

We note that the shadow [39], a weight-3/2 vector modular form, of our system of mock theta functions is given by

\[
\frac{1}{2} \left[ \eta(\tau) \right]^3 \tilde{\vartheta}_{D-1,a} + \tilde{\vartheta}_{D-1,D-a} (0; \tau),
\]

where \( 1 \leq a \leq \frac{D}{2} \).

The theta series \( \tilde{\vartheta}_{D^{-1},a}(\tau) \) (A.5), which is a basis for Jacobi form with index \( \frac{D-1}{2} \in \mathbb{Z} \), may be regarded as the massive characters (2.7) up to a prefactor,

\[
\text{ch}^R_{D,h>\frac{D}{2},Q}(z; \tau) = (-1)^Q \frac{D}{2} q^{h-\frac{D-1}{2} \left( \frac{Q}{2} - \frac{1}{2} \right)^2} \phi_{-1,\frac{1}{2}}(z; \tau) \tilde{\vartheta}_{D^{-1},Q}(z; \tau).
\]

Combining the theta series, we define the basis functions as

\[
B_{D,a}(z; \tau) = \begin{cases} 
\phi_{-1,\frac{1}{2}}(z; \tau) \left[ \tilde{\vartheta}_{D-1,a}(z; \tau) + \tilde{\vartheta}_{D-1,D-a}(z; \tau) \right], & \text{for } 1 \leq a < \frac{D}{2}, \\
\phi_{-1,\frac{1}{2}}(z; \tau) \tilde{\vartheta}_{D^{-1},\frac{D}{2}}(z; \tau), & \text{for } a = \frac{D}{2}.
\end{cases}
\]

Note that \( \tilde{\vartheta}_{D^{-1},D-a}(z; \tau) = -\tilde{\vartheta}_{D^{-1},a}(-z; \tau) \) and \( B_{D,a}(z; \tau) \) is an even-function of \( z \) with a 2nd order zero at \( z = 0 \). One has

\[
\text{ch}^R_{D,h>\frac{D}{2},Q}(z; \tau) + \text{ch}^R_{D,h>\frac{D}{2},-Q}(z; \tau) = (-1)^Q \frac{D}{2} q^{h-\frac{D-1}{2} \left( \frac{Q}{2} - \frac{1}{2} \right)^2} B_{D,Q}(z; \tau).
\]

Thus they describe the even-\( z \) part of the massive characters.
They form a set of vector-valued Jacobi forms satisfying
\[ B_{D,a}(z + 1; \tau) = B_{D,a}(z; \tau), \]
\[ B_{D,a}(z + \tau; \tau) = q^{-\frac{D}{2}} e^{-2\pi Dz} B_{D,a}(z; \tau), \]
\[ B_{D,a}(z; \tau) = \sqrt{\frac{\tau}{\pi}} e^{-\pi Dz^2} \sum_{b=1}^{D} \delta_{a,b} - 2 \sin \left( \frac{(2a - 1)(2b - 1)}{2(D - 1)} \pi \right) B_{D,b} \left( \frac{z}{\tau}; -\frac{1}{\tau} \right), \]
\[ B_{D,a}(z; \tau + 1) = e^{\frac{(a - 1)^2}{D - 1} \pi i} B_{D,a}(z; \tau). \] (4.9)

By using (A.8), the latter two identities are summarized as
\[
B_{D,a} \left( \frac{z}{c \tau + d}; \frac{a \tau + b}{c \tau + d} \right) = \begin{cases} 
\frac{1}{\sqrt{c \tau + d}} e^{D \tau/2 \pi} \sum_{a_1 = 1}^{D} \left( [\rho(\gamma)]_{a_1,a_2} + [\rho(\gamma)]_{D-a_1,a_2} \right) B_{D,a_2}(z; \tau), & \text{for } a_1 < \frac{D}{2}, \\
\frac{1}{\sqrt{c \tau + d}} e^{D \tau/2 \pi} \sum_{a_1 = 1}^{D} [\rho(\gamma)]_{a_1,a_2} B_{D,a_2}(z; \tau), & \text{for } a_1 = \frac{D}{2}. 
\end{cases} \] (4.10)

4.2. Harmonic Maass Forms and Elliptic Genus. By choosing a set of coordinates on the torus, \( w_b \in \mathbb{C}/(\mathbb{Z} \oplus \tau \mathbb{Z}) \) for \( b = 1, \ldots, \frac{D}{2} \), we define functions \( J_D(z; w_1, \ldots, w_{\frac{D}{2}}; \tau) \) as
\[ J_D(z; w_1, \ldots, w_{\frac{D}{2}}; \tau) = \hat{C}_D(z; \tau) - \sum_{a=1}^{\frac{D}{2}} \hat{H}_{D,a}(w_1, \ldots, w_{\frac{D}{2}}; \tau) B_{D,a}(z; \tau) \]
\[ = C_D(z; \tau) - \sum_{a=1}^{\frac{D}{2}} H_{D,a}(w_1, \ldots, w_{\frac{D}{2}}; \tau) B_{D,a}(z; \tau). \] (4.11)

Here
\[ \hat{H}_{D,a}(w_1, \ldots, w_{\frac{D}{2}}; \tau) = \sum_{b=1}^{\frac{D}{2}} [B_D(w; \tau)^{-1}]_{ab} \hat{C}_D(w_b; \tau), \] (4.12)
\[ H_{D,a}(w_1, \ldots, w_{\frac{D}{2}}; \tau) = \sum_{b=1}^{\frac{D}{2}} [B_D(w; \tau)^{-1}]_{ab} C_D(w_b; \tau), \] (4.13)

where the matrix \( B_D \) is defined by
\[ [B_D(w; \tau)]_{ab} = B_{D,b}(w_a; \tau). \]

We note that
\[ \hat{H}_{D,a}(w_1, \ldots, w_{\frac{D}{2}}; \tau) - H_{D,a}(w_1, \ldots, w_{\frac{D}{2}}; \tau) \]
\[
\begin{align*}
= \sum_b [\mathcal{B}_D(w; \tau)^{-1}]_{ab} \left( \widehat{C}_D(w_b; \tau) - C_D(w_b; \tau) \right) \\
= -\frac{1}{2} \sum_b [\mathcal{B}_D(w; \tau)^{-1}]_{ab} \sum_c e^{-\frac{a+\tau}{2}q} q^{-\frac{1}{8(2D-1)}} R_{D-1,a}^{-1} \left( \frac{-1+\tau}{2(D-1)}; \tau \right) B_{D,c}(w_b; \tau) \\
= -\frac{1}{2} e^{-\frac{a+\tau}{2}q} q^{-\frac{1}{8(2D-1)}} R_{D-1,a}^{-1} \left( \frac{-1+\tau}{2(D-1)}; \tau \right).
\end{align*}
\]

Thus \( \hat{H}_{D,a}(w_1, \ldots, w_{\frac{D}{2}}; \tau) \) is a completion of \( H_{D,a}(w_1, \ldots, w_{\frac{D}{2}}; \tau) \). By use of (4.15) and the fact that \( H_{D,a}(w_1, \ldots, w_{\frac{D}{2}}; \tau) \) is \( \bar{\tau} \)-independent, we can check that it is annihilated by the hyperbolic Laplacian (D.3)

\[
\Delta_{\frac{1}{2}} \hat{H}_{D,a}(w_1, \ldots, w_{\frac{D}{2}}; \tau) = 0. \tag{4.15}
\]

We note that

\[
\begin{align*}
J_D(z; w_1, \ldots, w_{\frac{D}{2}}; \tau) &= e^{-\pi i D z^2} J_D \left( \frac{z}{\tau}; w_1, \ldots, w_{\frac{D}{2}}; \frac{1}{\tau} \right), \\
J_D(z + 1; w_1, \ldots, w_{\frac{D}{2}}; \tau) &= J_D(z; w_1, \ldots, w_a + 1, \ldots, w_{\frac{D}{2}}; \tau) \\
&= J_D(z; w_1, \ldots, w_a + \tau, \ldots, w_{\frac{D}{2}}; \tau) \\
&= J_D(z; w_1, \ldots, w_{\frac{D}{2}}; \tau + 1) \\
&= J_D(z; w_1, \ldots, w_{\frac{D}{2}}; \tau), \\
J_D(z + \tau; w_1, \ldots, w_{\frac{D}{2}}; \tau) &= q^{-\frac{D}{2}} e^{-2\pi i D z} J_D(z; w_1, \ldots, w_{\frac{D}{2}}; \tau).
\end{align*}
\]

By construction the function \( J_D(z; w_1, \ldots, w_{\frac{D}{2}}; \tau) \) vanishes at \( z = w_a \) for \( a = 1, \ldots, \frac{D}{2} \)

\[
J_D(z = w_a; w_1, \ldots, w_{\frac{D}{2}}; \tau) = 0, \tag{4.17}
\]

and we also have

\[
J_D(z = 0; w_1, \ldots, w_{\frac{D}{2}}; \tau) = 1. \tag{4.18}
\]

If we choose \( w_1, \ldots, w_{\frac{D}{2}} \) to be half-periods \( \left\{ \frac{1}{2}, \frac{1+k}{2}, \frac{\tau}{2} \right\} \) and use the notation \( w_{(k_2,k_3,k_4)} \),

\[
\begin{align*}
w_{(k_2,k_3,k_4)} &= \left\{ w_1, \ldots, w_{\frac{D}{2}} \right\} k_2 = \# \left( w_a = \frac{1}{2} \right), k_3 = \# \left( w_a = \frac{1+\tau}{2} \right), k_4 = \# \left( w_a = \frac{\tau}{2} \right),
\end{align*}
\]

it is possible from the above conditions (4.16), (4.17), (4.18) to show that

\[
J_D(z; w_{(k_2,k_3,k_4)}; \tau) = \left( \frac{\theta_{10}(z; \tau)}{\theta_{10}(0; \tau)} \right)^2 k_2 \left( \frac{\theta_{00}(z; \tau)}{\theta_{00}(0; \tau)} \right)^2 k_3 \left( \frac{\theta_{01}(z; \tau)}{\theta_{01}(0; \tau)} \right)^2 k_4,
\]

where \( \frac{D}{2} = k_2 + k_3 + k_4 \). In this manner the functions \( J_D(z; w_1, \ldots, w_{\frac{D}{2}}; \tau) \) generate the basis vectors of the space \( J_{0,\frac{D}{2}} \) and their linear combination gives the elliptic genera of \( CY_D \).
4.3. Character Decomposition. We set the Euler number of $CY_D$ to be $\chi_{CY_D}$ and consider a function $Z_{CY_D}(z; \tau) - \chi_{CY_D} C_D(z; \tau)$ which vanishes at $z = 0$. Completion of this function $Z_{CY_D}(z; \tau) - \chi_{CY_D} C_D(z; \tau)$ is a real analytic Jacobi form with weight-0 and index-$\frac{D}{2}$ and can be expanded in terms of $B_{D,a}(z; \tau)$. This is because a function

$$\frac{1}{\phi_{-1, \frac{1}{2}}(z; \tau)} \left( Z_{CY_D}(z; \tau) - \chi_{CY_D} C_D(z; \tau) \right),$$

which is a real analytic Jacobi form with weight-1 and index-$\frac{D}{2} \in \mathbb{Z} + \frac{1}{2}$, can be expanded in terms of $\tilde{\vartheta}_{D,a}(z; \tau)$ as discussed in (C.3). We have

$$\frac{1}{\phi_{-1, \frac{1}{2}}(z; \tau)} \left( Z_{CY_D}(z; \tau) - \chi_{CY_D} C_D(z; \tau) \right) = \sum_{a \text{ mod } (D-1)} \hat{\Sigma}_{D,a}(\tau) \tilde{\vartheta}_{D,a}(z; \tau),$$

which reduces to

$$Z_{CY_D}(z; \tau) - \chi_{CY_D} C_D(z; \tau) = \sum_{a=1}^{\frac{D}{2}} \hat{\Sigma}_{D,a}(\tau) B_{D,a}(z; \tau). \quad (4.20)$$

By taking the holomorphic part, we obtain a character decomposition of the elliptic genus $Z_{CY_D}(z; \tau)$ as

$$Z_{CY_D}(z; \tau) = \chi_{CY_D} C_D(z; \tau) + \sum_{a=1}^{\frac{D}{2}} \Sigma_{D,a}(\tau) B_{D,a}(z; \tau), \quad (4.21)$$

where the function $\Sigma_{D,a}(\tau)$ is computed by the Fourier integral

$$\Sigma_{D,a}(\tau) = q^{-\frac{(a-\frac{1}{2})^2}{2(D-1)}} \int_{z_0}^{z_0+1} \frac{1}{\phi_{-1, \frac{1}{2}}(z; \tau)} \left( Z_{CY_D}(z; \tau) - \chi_{CY_D} C_D(z; \tau) \right) e^{-2\pi i (a-\frac{1}{2}) z} \, dz, \quad (4.22)$$

with an arbitrary $z_0 \in \mathbb{C}$. Since $B_{D,a}(z; \tau)$ is (the even part of) massive characters, the integral Fourier coefficients of $\Sigma_{D,a}(\tau)$ denotes the multiplicities of the massive representations.

We note that the lowest power $q^{-\frac{(a-\frac{1}{2})^2}{2(D-1)}}$ of $\Sigma_{D,a}(\tau)$ corresponds to the unitarity bound, and that it decomposes into a sum of massless characters as (2.12) and (2.13),

$$q^{-\frac{(a-\frac{1}{2})^2}{2(D-1)}} B_{D,a}(z; \tau) = \left\{ \begin{array}{ll}
(-1)^{\frac{a+1}{2}} \sum_{\varepsilon=\pm 1} \left[ \text{ch}^{\tilde{R}}_{D,h=0,Q=\varepsilon a}(z; \tau) + \text{ch}^{\tilde{R}}_{D,h=\frac{D}{2},Q=\varepsilon(a-1)}(z; \tau) \right], & \text{for } 1 \leq a < \frac{D}{2}, \\
- \text{ch}^{\tilde{R}}_{D,h=\frac{D}{8},Q=\frac{D}{2}}(z; \tau) + \sum_{\varepsilon=\pm 1} \text{ch}^{\tilde{R}}_{D,h=\frac{D}{8},Q=\varepsilon\left(\frac{D}{2}-1\right)}(z; \tau), & \text{for } a = \frac{D}{2}.
\end{array} \right. \quad (4.23)$$
4.4. Poincaré–Maass Series. Both sides of (4.20) are real analytic Jacobi forms with weight-0 and index-$\frac{D}{2}$. Because of the transformation formula (4.10) of $B_{D,a}(z;\tau)$, transformation law of the functions $\hat{\Sigma}_{D,a}(\tau)$ is determined as

$$\hat{\Sigma}_{D,a_1}(\gamma(\tau)) = \sqrt{c} \tau + d \sum_{a_2=1}^{D} [\chi(\gamma)]_{a_1,a_2} \hat{\Sigma}_{D,a_2}(\tau), \tag{4.24}$$

where the multiplier system $\chi(\gamma)$ is given by

$$[\chi(\gamma^{-1})]_{a_1,a_2} = \begin{cases} \frac{1}{2\sqrt{i} \sqrt{c(D-1)}} 2^{c-1} \sum_{j=0}^{2c-1} (-1)^j e^{(D-1)\left(j + \frac{1}{2}\right)} e^{\frac{\pi i}{2} \frac{\gamma(a_j^2)}{2}} (j + \frac{1}{2}), & \text{for } a_2 < \frac{D}{2}, \tag{4.25} \\ \frac{1}{2\sqrt{i} \sqrt{c(D-1)}} 2^{c-1} \sum_{j=0}^{2c-1} (-1)^j e^{(D-1)\left(j + \frac{1}{2}\right)} e^{\frac{\pi i}{2} \frac{\gamma(a_j^2)}{2}} (j + \frac{1}{2}), & \text{for } a_2 = \frac{D}{2}. \end{cases}$$

In view that the fact that the basis of Jacobi forms $J_{D}(z;w_1,\cdots,w_{D/2};\tau)$ are decomposed as (4.11) and the elliptic genus is given by the linear combination of these Jacobi forms, we deduce that the functions $\hat{\Sigma}_{D,a}(\tau)$ are also a vector-valued harmonic Maass form as in the case of $\hat{H}_{D,a}(w_1,\ldots,w_{D/2};\tau)$

$$\Delta_{\frac{D}{2}} \hat{\Sigma}_{D,a}(\tau) = 0. \tag{4.26}$$

We construct a solution of above differential equations (4.26) using the Poincaré–Maass series $P_{D,a}(\tau)$ \[3\] \[4\]. We note that the polar part of $P_{D,a}(\tau)$ has the form of

$$P_{D,a}(\tau)|_{\text{polar}} = \sum_{0 \leq n < \frac{(a-\frac{1}{2})^2}{2(D-1)}} p_{D,a}(n) q^n - \frac{(a-\frac{1}{2})^2}{2(D-1)}. \tag{4.27}$$

The Fourier coefficients $p_{D,a}(n)$ of polar parts are fixed from (4.22) once the elliptic genus $Z_{\text{CY}_{D}}(z;\tau)$ is given. For instance, if the elliptic genus is given by a Jacobi form

$$2^{2k_2-1} \left[ \left( \frac{\theta_{10}(z)}{\theta_{10}(0;\tau)} \right)^{2k_2} \left( \frac{\theta_{00}(z;\tau)}{\theta_{00}(0;\tau)} \right)^{2k_3} \left( \frac{\theta_{01}(z;\tau)}{\theta_{01}(0;\tau)} \right)^{2k_4} \right] + \text{other 5 terms obtained by permutations}, \tag{4.28}$$

where $k_2 + k_3 + k_4 = \frac{D}{2}$ and without loss of generality $k_2 \geq k_3 \geq k_4$, we find that

$$p_{D,a}(0) = -2^{k_2-1} \sum_{j=2,3,4} \sum_{m=0}^{k_j-1} \frac{1}{2^{2m}} \binom{2m}{m-a+1} \frac{2a-1}{m+a}, \tag{4.29}$$
where \( \binom{n}{k} \) denotes the binomial coefficient.

By use of the function \( \varphi_{\ell,h,s}^{c}(\tau) \) as an eigenfunction of the hyperbolic Laplacian \( (D.3) \), a solution of \( \Delta_{2} P_{D,a}(\tau) = 0 \) which transforms like \( (4.24) \) with the boundary condition \( (4.27) \) can be constructed in the form of the Poincaré–Maass series as

\[
P_{D,a}(\tau) = \frac{1}{\sqrt{\pi}} \sum_{a_{2}=1}^{D} \sum_{0 \leq m < \frac{(a_{2}-\frac{1}{2})^{2}}{2(D-1)}} p_{D,a_{2}}(m) \times \chi(\gamma^{-1})_{a_{1},a_{2}}^{\frac{1}{\sqrt{c \tau + d}}} \varphi_{\ell,h,s}^{c}(\gamma(\tau)), \quad (4.30)
\]

where \( \Gamma_{\infty} \) is the stabilizer of \( \infty \), \( \Gamma_{\infty} = \{(1 \ 0 \ n) \mid n \in \mathbb{Z} \} \).

We have

\[
\hat{\Sigma}_{D,a}(\tau) = P_{D,a}(\tau) + \Theta_{D,a}(\tau). \quad (4.31)
\]

Here a weight-1/2 theta series \( \Theta_{D,a}(\tau) \) having the same transformation law as \( (4.24) \), and \( \Theta_{D,a}(8(D-1)\tau) \) being a modular forms of \( \Gamma_{0}(64(D-1)^{2}) \) may appear in the right-hand-side of the equation. To have a non-vanishing theta series \( \Theta_{D,a}(\tau) \), \( 64(D-1)^{2} \) is divisible either by \( 64p^{2} \) with odd prime \( p \), or by \( 4(pp')^{2} \) with distinct odd primes \( p \) and \( p' \), due to the Serre–Stark theorem \( (33) \).

The holomorphic part is thus given by

\[
\Sigma_{D,a}(\tau) - \Theta_{D,a}(\tau) = P_{D,a}(\tau)|_{\text{holomorphic}} = q^{-\frac{(a-\frac{1}{2})^{2}}{2(D-1)}} \sum_{n=0}^{\infty} p_{D,a}(n) q^{n}. \quad (4.32)
\]

Following the same analysis with our previous paper \( (12) \) (see Ref. \( 4 \) for a general treatment), we can compute the coefficients \( p_{D,a}(n) \) as

\[
p_{D,a_{1}}(n) = \sum_{a_{2}=1}^{D} \sum_{0 \leq m < \frac{(a_{2}-\frac{1}{2})^{2}}{2(D-1)}} p_{D,a_{2}}(m) A_{D}^{(a_{2},m,a_{1})}(n), \quad (4.33)
\]

where

\[
A_{D}^{(a_{2},m,a_{1})}(n) = \sum_{c=1}^{\infty} \sum_{\substack{d \mod c \mid (c,d)=1}} \chi(\gamma^{-1})_{a_{1},a_{2}}^{\frac{2\pi}{\sqrt{1}}} \left( \frac{(a_{2}-\frac{1}{2})^{2} - 2(D-1)m}{2(D-1)n - (a_{1}-\frac{1}{2})^{2}} \right)^{\frac{1}{2}} \\
\times \frac{1}{c} I_{\frac{1}{2}} \left( \frac{4\pi}{c} \left( n - \frac{(a_{1}-\frac{1}{2})^{2}}{2(D-1)} \right) \left( \frac{(a_{2}-\frac{1}{2})^{2}}{2(D-1)} - m \right) \right) \\
\times e^{-2\pi i \left( \frac{(a_{2}-\frac{1}{2})^{2}}{2(D-1)} - m \right) \frac{1}{2} + 2\pi i \left( n - \frac{(a_{1}-\frac{1}{2})^{2}}{2(D-1)} \right) \frac{1}{2}}. \quad (4.34)
\]
We mean \( a = d^{-1} \mod c \) in the summand.

Since the Fourier coefficients of theta series stay constant, they are negligible as compared with \( p_{D,a}(n) \) which increases exponentially at large \( n \). The dominant terms at large \( n \) read as

\[
p_{D,a_1}(n) \approx \sum_{a_2=1}^{D} p_{D,a_2}(0) \frac{2 \pi \left( \frac{\delta_{a_2, \frac{D}{2}} - 2}{\sqrt{D - 1}} \right)}{\sqrt{D - 1}} \sin \left( \frac{(2 a_1 - 1)(2 a_2 - 1)}{2(D - 1)} \right) \sqrt{D - 1} \sin \left( \frac{2 \pi}{2(D - 1)} \left( n - \frac{(a_1 - \frac{1}{2})^2}{2(D - 1)} \right) \right).
\]

One finds that the exponential growth of \( p_{D,a}(n) \) is determined by the maximum value \( a_2 \) such that \( p_{D,a_2}(0) \neq 0 \). As in the case of hyperKähler manifolds, \( p_{D,a_2}(0) \) is non-zero when the elliptic genus \( Z_{\text{CY}^D}(z; \tau) \) includes a Jacobi form

\[
\left( \frac{\theta_{10}(z; \tau)}{\theta_{10}(0; \tau)} \right)^D + \left( \frac{\theta_{00}(z; \tau)}{\theta_{00}(0; \tau)} \right)^D + \left( \frac{\theta_{01}(z; \tau)}{\theta_{01}(0; \tau)} \right)^D.
\]

as we see from (4.29). In this case we conclude from (4.35) that the entropy \( S_{\text{CY}^D} \) from the \( U(1) \) charge-\( Q \) is given by

\[
S_{\text{CY}^D} = \log |p_{D,Q}(n)| \\
\approx 2 \pi \sqrt{\frac{D - 1}{2} n - \left( \frac{Q - \frac{1}{2}}{2} \right)^2}.
\]

4.5. Examples.

4.5.1. Calabi–Yau 2-folds \( \text{CY}_2 \). It is known that in this case the \( U(1) \) current algebra is enhanced to a level-1 affine \( SU(2) \) current algebra, and that we have \( \mathcal{N} = 4 \) SCA [14, 16]. \( \text{CY}_2 \) is either a complex 2-tori or K3 surface. The elliptic genus for the former vanishes. In the latter case, we have \( Z_{\text{CY}^2}(z; \tau) = 2 \phi_0(z; \tau) \), and its character decomposition (3.10) was studied in detail in Ref. [11]. Our result (4.33) for \( D = 2 \) indeed reproduces multiplicities \( A_n \) (3.11).

4.5.2. Calabi–Yau 4-folds \( \text{CY}_4 \). The dimension of the space of Jacobi forms \( J_{0,2} \) is two, and we set the bases of the Jacobi forms as [10, 12]

\[
Z_{X_4^{(1)}}(z; \tau) = 16 \left[ \left( \frac{\theta_{10}(z; \tau)}{\theta_{10}(0; \tau)} \right)^4 + \left( \frac{\theta_{00}(z; \tau)}{\theta_{00}(0; \tau)} \right)^4 + \left( \frac{\theta_{01}(z; \tau)}{\theta_{01}(0; \tau)} \right)^4 \right],
\]

\[
Z_{X_4^{(2)}}(z; \tau) = 2 \left[ \left( \frac{\theta_{10}(z; \tau)}{\theta_{10}(0; \tau)} \cdot \frac{\theta_{00}(z; \tau)}{\theta_{00}(0; \tau)} \right)^2 + \left( \frac{\theta_{00}(z; \tau)}{\theta_{00}(0; \tau)} \cdot \frac{\theta_{01}(z; \tau)}{\theta_{01}(0; \tau)} \right)^2 \right].
\]
The Eichler–Zagier bases are given by
\[
\left( \begin{array}{c} [\phi_{0,1}]^2 \\ [\phi_{-2,1}]^2 E_4 \end{array} \right) = \left( \begin{array}{cc} 1 & 16 \\ 1 & -8 \end{array} \right) \left( \begin{array}{c} Z_{X_4^{(1)}}(z; \tau) \\ Z_{X_4^{(2)}}(z; \tau) \end{array} \right), \tag{4.39}
\]

In order to derive the character decomposition of these Jacobi forms, we use \( \phi = e^{2\pi iz} \) and find explicit results (we use \( \zeta = e^{2\pi iz} \) for brevity).
\[
\frac{1}{\phi_{-1,\frac{1}{2}}(z; \tau)} \left( Z_{X_4^{(1)}}(z; \tau) - 48 C_4(z; \tau) \right) = \left( -\zeta^\frac{1}{2} + 5 \zeta^\frac{3}{2} + 5 \zeta^\frac{-1}{2} + \zeta^{-\frac{3}{2}} \right) q
\]
\[
+ \left( -5 \zeta^\frac{1}{2} - 207 \zeta^\frac{3}{2} + 790 \zeta^\frac{5}{2} - 790 \zeta^\frac{-1}{2} + 207 \zeta^{-\frac{3}{2}} + 5 \zeta^{-\frac{5}{2}} \right) q^2
\]
\[
+ \left( 5 \zeta^\frac{1}{2} + 790 \zeta^\frac{3}{2} - 5724 \zeta^\frac{5}{2} + 13955 \zeta^\frac{7}{2} - 13955 \zeta^\frac{-1}{2} + 5724 \zeta^{-\frac{3}{2}} - 790 \zeta^{-\frac{5}{2}} - 5 \zeta^{-\frac{7}{2}} \right) q^3
\]
\[
+ \left( \zeta^\frac{1}{2} - 790 \zeta^\frac{3}{2} + 13955 \zeta^\frac{5}{2} - 65385 \zeta^\frac{7}{2} + 132909 \zeta^\frac{9}{2} - 132909 \zeta^{-\frac{1}{2}} + 65385 \zeta^{-\frac{3}{2}} - 13955 \zeta^{-\frac{5}{2}} + 790 \zeta^{-\frac{7}{2}} - \zeta^{-\frac{9}{2}} \right) q^4 + \cdots.
\]

Fourier coefficients in the above expressions give the multiplicities of massive representations, and we obtain the character expansion
\[
Z_{X_4^{(1)}}(z; \tau) = 48 C_4(z; \tau)
\]
\[
+ q^{-\frac{3}{8}} \left( -1 + 5 q + 10 q^2 + 21 q^3 + 31 q^4 + 59 q^5 + \cdots \right) B_{4,1}(z; \tau)
\]
\[
+ q^{-\frac{5}{8}} \left( -1 - 207 q - 5724 q^2 - 65385 q^3 - 494145 q^4 - 2922021 q^5 - \cdots \right) B_{4,2}(z; \tau), \tag{4.40}
\]
\[
Z_{X_4^{(2)}}(z; \tau) = 6 C_4(z; \tau)
\]
\[
+ q^{-\frac{3}{8}} \left( -5 + 790 q + 13955 q^2 + 132909 q^3 + 915248 q^4 + 5070103 q^5 + \cdots \right) B_{4,1}(z; \tau)
\]
\[
+ q^{-\frac{5}{8}} \left( -1 - 207 q - 5724 q^2 - 65385 q^3 - 494145 q^4 - 2922021 q^5 - \cdots \right) B_{4,2}(z; \tau). \tag{4.41}
\]

In Tables 1 and 2 we confirm numerically that the Poincaré–Maass series \( (4.33) \) reproduce the above Fourier coefficients in \( (4.40) \) and \( (4.41) \) accurately, when we set \( (p_{4,1}(0), p_{4,2}(0)) \) to be \( (-5, -1) \) and \( (-1, 0) \), respectively. We plot in Fig. 1 absolute values of the Fourier
We note that these bases are related to the Eichler–Zagier bases \[1 7\] by 4.5.3. **Calabi–Yau 6-folds CY\(_{6}\).** Following the convention of Ref. \[12\] we set the bases of the elliptic genera as

\[
Z_{X_{6}^{(1)}}(z; \tau) = 64 \left[ \left( \frac{\theta_{10}(z; \tau)}{\theta_{10}(0; \tau)} \right)^6 + \left( \frac{\theta_{00}(z; \tau)}{\theta_{00}(0; \tau)} \right)^6 + \left( \frac{\theta_{01}(z; \tau)}{\theta_{01}(0; \tau)} \right)^6 \right], \tag{4.42}
\]

\[
Z_{X_{6}^{(2)}}(z; \tau) = 8 \left[ \left( \frac{\theta_{10}(z; \tau)}{\theta_{10}(0; \tau)} \right)^4 \left( \frac{\theta_{00}(z; \tau)}{\theta_{00}(0; \tau)} \right)^2 + \text{other 5 terms} \right], \tag{4.43}
\]

\[
Z_{X_{6}^{(3)}}(z; \tau) = 4 \left( \frac{\theta_{10}(z; \tau)}{\theta_{10}(0; \tau)} \right)^2 \left( \frac{\theta_{00}(z; \tau)}{\theta_{00}(0; \tau)} \right)^2 \left( \frac{\theta_{01}(z; \tau)}{\theta_{01}(0; \tau)} \right)^2 . \tag{4.44}
\]

We note that these bases are related to the Eichler–Zagier bases \[17\] by

\[
\begin{pmatrix}
[\phi_{0,1}]^3 \\
[\phi_{-1,1}]^2 \phi_{0,1} E_4 \\
[\phi_{-2,1}]^3 E_6
\end{pmatrix} =
\begin{pmatrix}
1 & 24 & 96 \\
1 & 0 & -48 \\
1 & -12 & 96
\end{pmatrix}
\begin{pmatrix}
Z_{X_{6}^{(1)}} \\
Z_{X_{6}^{(2)}} \\
Z_{X_{6}^{(3)}}
\end{pmatrix}. \tag{4.45}
\]
Shown are the absolute values of the Fourier coefficients in the character decomposition of the elliptic genera for the Calabi–Yau 4-folds. Blue (resp. red) dots show the exact multiplicities of massive representations associated with $B_{4,1}(z; \tau)$ (resp. $B_{4,2}(z; \tau)$) in (4.40) (upper part) and (4.41) (lower part). Blue and red curves show the approximation by the dominant term (4.35) of the Poincaré–Maass series with the coefficients of the polar part $(p_{4,1}(0), p_{4,2}(0)) = (-5, -1)$ (upper part) and $(-1, 0)$ (lower part).

For these Jacobi forms, we have character decompositions as follows;

$$Z_{X_6}^{(1)}(z; \tau) = 192 C_6(z; \tau)$$
$$+ q^{-\frac{6}{8}} (-22 + 7133 q + 271635 q^2 + 5130662 q^3 + 63707417 q^4 + \cdots) B_{6,1}(z; \tau)$$
$$+ q^{-\frac{9}{8}} (-7 - 1983 q - 129717 q^2 - 2905560 q^3 - 39223768 q^4 + \cdots) B_{6,2}(z; \tau)$$
$$+ q^{-3} (-1 - 35 q + 26895 q^2 + 887110 q^3 + 14389130 q^4 + \cdots) B_{6,3}(z; \tau),$$

(4.46)

$$Z_{X_6}^{(2)}(z; \tau) = 48 C_6(z; \tau)$$
$$+ q^{-\frac{6}{8}} (-9 + 197 q + 1599 q^2 + 8697 q^3 + 37232 q^4 + \cdots) B_{6,1}(z; \tau)$$
$$+ q^{-\frac{9}{8}} (-1 + 60 q + 474 q^2 + 2457 q^3 + 10932 q^4 + \cdots) B_{6,2}(z; \tau)$$
$$+ q^{-\frac{5}{8}} (-26 q - 528 q^2 - 3954 q^3 - 19432 q^4 + \cdots) B_{6,3}(z; \tau),$$

(4.47)

$$Z_{X_6}^{(3)}(z; \tau) = 4 C_6(z; \tau)$$
$$+ q^{-\frac{6}{8}} (-1 + q + 3 q^2 + 2 q^3 + 7 q^4 + \cdots) B_{6,1}(z; \tau)$$
$$+ q^{-\frac{9}{8}} (3 q + 5 q^2 + 9 q^3 + 12 q^4 + \cdots) B_{6,2}(z; \tau)$$
$$+ q^{-\frac{7}{8}} (2 q + 6 q^2 + 8 q^3 + 14 q^4 + \cdots) B_{6,3}(z; \tau).$$

(4.48)
We have checked numerically that the Poincaré–Maass series \((4.33)\) generates the above multiplicities of the massive representations quite accurately, when we set the polar part \((p_{6,1}(0), p_{6,2}(0), p_{6,3}(0))\) to be \((-22, -7, -1)\), \((-9, -1, 0)\), and \((-1, 0, 0)\), respectively.

5. CONCLUDING REMARKS

We have computed the entropy of the Calabi–Yau \(D\)-folds by use of the Poincaré–Maass series. We have found that, when \(D\) is odd, the entropy coincides with that of the hyperKähler manifolds with a complex \((D - 3)\)-dimension. Especially the entropy of the Calabi–Yau 3-folds vanishes identically. It will be interesting to provide physical interpretation of this result.

In summary, the entropy of Calabi–Yau \(D\)-folds is given by

\[
S_{CYD} \sim \begin{cases} 
2\pi \sqrt{\frac{(D - 3)^2}{2(D - 1)}} n - \left( \frac{D - 3}{D - 1} \frac{Q - \frac{1}{2}}{2} \right)^2, & \text{when } D \text{ is odd}, \\
2\pi \sqrt{\frac{D - 1}{2}} n - \left( \frac{Q - \frac{1}{2}}{2} \right)^2, & \text{when } D \text{ is even}.
\end{cases}
\]  

(A.1)

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APPENDIX A. THETA FUNCTIONS

A.1. Jacobi Theta Functions. The Jacobi theta functions are defined by

\[
\begin{align*}
\theta_{11}(z; \tau) &= \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}(n+\frac{1}{2})^2} e^{2\pi i (n+\frac{1}{2}) (z+\frac{1}{2})}, \\
\theta_{10}(z; \tau) &= \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}(n+\frac{1}{2})^2} e^{2\pi i (n+\frac{1}{2}) z}, \\
\theta_{00}(z; \tau) &= \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}n^2} e^{2\pi i nz}, \\
\theta_{01}(z; \tau) &= \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}n^2} e^{2\pi i (z+\frac{1}{2})}.
\end{align*}
\]

(A.1)

Throughout this paper, we set \(q = e^{2\pi i \tau}\) with \(\tau\) in the upper half plane, \(\tau \in \mathbb{H}\).
A.2. **Theta Function.** We define the theta function for $D \in \mathbb{Z}$ and $a \mod D$ by

$$
\vartheta_{D,a}(z; \tau) = \sum_{n \in \mathbb{Z}} q^{\frac{n^2}{D}} e^{2\pi i n z}.
$$

We have

$$
\vartheta_{D,a}(z; \tau) = \sqrt{\frac{i}{\tau}} \frac{1}{\sqrt{D}} e^{-\pi i D z^2} \sum_{b=0}^{D-1} e^{2\pi i b} \vartheta_{D,b} \left( \frac{z}{\tau} - \frac{1}{\tau} \right).
$$

Note that

$$
(\vartheta_{2,1} - \vartheta_{2,-1})(z; \tau) = -i \theta_{11}(2z; \tau).
$$

For our notational convention, we introduce another set of theta series $\tilde{\vartheta}_{D-1,a}(z; \tau)$ for $D \in \mathbb{Z}$ and $1 \leq a \leq D - 1$

$$
\tilde{\vartheta}_{D-1,a}(z; \tau) = e^{\frac{a-1}{2} \pi i z^2} \vartheta_{D-1,a-1} \left( z + \frac{1 + \tau}{2(D-1)} \right)
$$

$$
= \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{(D-1)n+a-\frac{1}{2}}{2 D-1}} e^{2\pi i ((D-1)n+a-\frac{1}{2})z}.
$$

We see that

$$
\tilde{\vartheta}_{D-1,a}(-z; \tau) = -\tilde{\vartheta}_{D-1,D-a}(z; \tau),
$$

and we have

$$
\tilde{\vartheta}_{D-1,a}(z + 1; \tau) = -\tilde{\vartheta}_{D-1,a}(z; \tau),
$$

$$
\tilde{\vartheta}_{D-1,a}(z + \tau; \tau) = -q^{\frac{D-1}{2}} e^{-2\pi i (D-1)z} \tilde{\vartheta}_{D-1,a}(z; \tau),
$$

$$
\tilde{\vartheta}_{D-1,a}(z; \tau + 1) = \begin{cases} 
    e^{\pi i (\frac{a-1}{D-1})^2} \tilde{\vartheta}_{D-1,a}(z; \tau), & \text{for } 1 \leq a \leq \frac{D}{2}, \\
    e^{\pi i (\frac{D-a-\frac{1}{2}}{D-1})^2} \tilde{\vartheta}_{D-1,a}(z; \tau), & \text{for } a > \frac{D}{2},
\end{cases}
$$

$$
\tilde{\vartheta}_{D-1,a_1}(z; \tau) = \sqrt{\frac{i}{\tau}} e^{-\pi i (D-1) z^2} \frac{1}{\sqrt{D-1}} \sum_{a_2=1}^{D-1} e^{\frac{(2a_1-1)(2a_2-1)}{2(D-1)}} \vartheta_{D-1,a_2} \left( \frac{z}{\tau} - \frac{1}{\tau} \right).
$$

Generally for $\gamma = (a \ b \ c \ d) \in SL(2; \mathbb{Z})$, we have (see, e.g., Refs. [27, 29], also Ref. [30] in which it is used as surgery data in constructing the $SO(3)$ quantum invariants of 3-manifolds)

$$
\tilde{\vartheta}_{D-1,a_1} \left( \frac{z}{c \tau + d}; \frac{a \tau + b}{c \tau + d} \right) = \sqrt{c \tau + d} e^{c \tau + d} \vartheta_{D-1,a_2} \left( \frac{z}{c \tau + d} \right) \prod_{a_2=1}^{D-1} (\rho(\gamma))_{a_1,a_2} \tilde{\vartheta}_{D-1,a_2}(z; \tau),
$$

where $\rho(\gamma)$ is a $(D - 1) \times (D - 1)$ matrix defined by

$$
[\rho(\gamma)]_{a_1,a_2}
$$
\[
\begin{aligned}
&= \begin{cases} 
\frac{e^{\pi i (a_2^2 + 1)} D}{2 \sqrt{-c(D-1)}} \sum_{j=0}^{2c-1} (-1)^j e^{\pi i c(D-1)(j + \frac{a_1^2}{D-1})^2 - 2\pi i a_2 \frac{c}{e} (j + \frac{a_1^2}{D-1})}, & \text{for } a_2 \leq \frac{D}{2}, \\
-\frac{e^{\pi i (a_2-1)^2} D}{2 \sqrt{-c(D-1)}} \sum_{j=0}^{2c-1} (-1)^j e^{\pi i c(D-1)(j + \frac{a_1-1}{D-1})^2 - 2\pi i a_2 - \frac{D}{2} \frac{c}{e} (j + \frac{a_1-1}{D-1})}, & \text{for } a_2 > \frac{D}{2}.
\end{cases}
\end{aligned}
\] (A.9)

(Note: the symbol \( \tilde{\omega} \) should not be confused with the symbol \( \hat{\omega} \) which signifies the completion of mock theta functions).

APPENDIX B. JACOBI FORMS

The Jacobi form with weight-\( k \in \mathbb{Z}/2 \) and index-\( m \in \mathbb{Z}/2 \) fulfills

\[
f\left( \frac{z}{c \tau + d} ; a \tau + b \right) = (c \tau + d)^k e^{2\pi i m (a^2 c \tau + d)} f(z; \tau),
\]

\[
f(z + s \tau + t; \tau) = (-1)^{2m(s+t)} e^{-2\pi i m (s^2 \tau + 2sz)} f(z; \tau),
\]

where \((a, b, c, d) \in SL(2, \mathbb{Z})\) and \(s, t \in \mathbb{Z}\). The weak Jacobi form has a non-negative power of \( q \). See Ref. 17 as a basic reference.

We denote \( J_{k,m} \) as the space of weak Jacobi forms with weight-\( k \) and index-\( m \). The space \( J_{k,m} \) with even weight \( k \) and integral index \( m \in \mathbb{Z} \) is generated by \[17\]

\[
\{ E_4(\tau), E_6(\tau), \phi_{-2,1}(z; \tau), \phi_{0,1}(z; \tau) \}.
\]

Namely bases of \( J_{k,m} \) are

\[
[E_4(\tau)]^a [E_6(\tau)]^b [\phi_{-2,1}(z; \tau)]^c [\phi_{0,1}(z; \tau)]^d,
\]

with non-negative integers \( a, b, c, d \) satisfying

\[
4a + 6b - 2c = k, \quad c + d = m.
\]

Here \( E_4(\tau) \) and \( E_6(\tau) \) are the Eisenstein series defined by

\[
E_{2k}(\tau) = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n,
\]

where \( B_k \) and \( \sigma_k(n) \) are respectively the Bernoulli number and the divisor function

\[
\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!},
\]

\[
\sigma_k(n) = \sum_{1 \leq r | n} r^k.
\]
The remaining two functions with index-1 are

$$\phi_{-2,1}(z; \tau) = -\frac{[\theta_{11}(z; \tau)]^2}{[\eta(\tau)]^6}, \quad (B.4)$$

$$\phi_{0,1}(z; \tau) = 4 \left[ \left( \frac{\theta_{10}(z; \tau)}{\theta_{10}(0; \tau)} \right)^2 + \left( \frac{\theta_{00}(z; \tau)}{\theta_{00}(0; \tau)} \right)^2 + \left( \frac{\theta_{01}(z; \tau)}{\theta_{01}(0; \tau)} \right)^2 \right]. \quad (B.5)$$

In the case of the index being half-odd integral, the space $J_{k,m} + \frac{1}{2}$ is isomorphic to the space with an integral index as follows [2, 19]:

$$J_{2k,m+\frac{1}{2}} = \phi_{0,\frac{1}{2}}(z; \tau) \cdot J_{2k,m}, \quad J_{2k+1,m+\frac{1}{2}} = \phi_{-1,\frac{1}{2}}(z; \tau) \cdot J_{2k+2,m}, \quad (B.6)$$

where $\phi_{0,\frac{1}{2}}(z; \tau)$ and $\phi_{-1,\frac{1}{2}}(z; \tau)$ are defined by

$$\phi_{0,\frac{1}{2}}(z; \tau) = \frac{\theta_{11}(2z; \tau)}{\theta_{11}(z; \tau)} = 2 \frac{\theta_{10}(z; \tau)}{\theta_{10}(0; \tau)} \frac{\theta_{00}(z; \tau)}{\theta_{00}(0; \tau)} \frac{\theta_{01}(z; \tau)}{\theta_{01}(0; \tau)}, \quad (B.7)$$

$$\phi_{-1,\frac{1}{2}}(z; \tau) = \frac{i \theta_{11}(z; \tau)}{[\eta(\tau)]^3}. \quad (B.8)$$

**Appendix C. Jacobi Forms and Theta Functions**

The Jacobi form $f(z; \tau)$ with weight-$k$ and index-$m$ can be expanded in terms of the theta functions $\vartheta_{m,a}(z; \tau)$ (resp. $\tilde{\vartheta}_{m,a}(z; \tau)$) when the index is $m \in \mathbb{Z}$ (resp. $m \in \mathbb{Z} + \frac{1}{2}$).

(a) When $m \in \mathbb{Z}$, we have

$$f(z; \tau) = \sum_{a \mod 2m} \Sigma_a(\tau) \vartheta_{m,a}(z; \tau). \quad (C.1)$$

Here with an arbitrary $z_0 \in \mathbb{C}$, we have

$$\Sigma_a(\tau) = q^{-\frac{a^2}{4m}} \int_{z_0}^{z_0+1} f(z; \tau) e^{-2\pi i az} \, dz. \quad (C.2)$$

(b) When $m \in \mathbb{Z} + \frac{1}{2}$, we have

$$f(z; \tau) = \sum_{a \mod 2m} \Sigma_a(\tau) \tilde{\vartheta}_{m,a}(z; \tau). \quad (C.3)$$

Here with an arbitrary $z_0 \in \mathbb{C}$, we have

$$\Sigma_a(\tau) = q^{-\frac{1}{4m}(a-\frac{1}{2})^2} \int_{z_0}^{z_0+1} f(z; \tau) e^{-2\pi i (a-\frac{1}{2})z} \, dz. \quad (C.4)$$
The former is a standard result (see, e.g., Ref. 17), and the proof is as follows. In the case of an integral index \( m \in \mathbb{Z} \), the Jacobi form is periodic \( f(z + 1; \tau) = f(z; \tau) \), and we have the Fourier expansion

\[
f(z; \tau) = \sum_{n \in \mathbb{Z}} e^{2\pi inz} q^{\frac{n^2}{4m}} \Sigma_n(\tau),
\]

where the prefactor is for our convention. Then the function \( \Sigma_n(\tau) \) is given by (C.2). Using \( f(z + \tau; \tau) = q^{-m} e^{-4\pi imz} f(z; \tau) \) in the integrand, we obtain \( \Sigma_n(\tau) = \Sigma_n + 2m(\tau) \) which gives (C.1).

In the same manner, we can prove the case (b). When the index \( m \) is half-odd integral \( m \in \mathbb{Z} + \frac{1}{2} \), the Jacobi form \( f(z; \tau) \) is anti-periodic, \( f(z + 1; \tau) = -f(z; \tau) \). Then we have the Fourier expansion

\[
f(z; \tau) = \sum_{n \in \mathbb{Z}} e^{2\pi i(n-\frac{1}{2})z} q^{\frac{1}{4m}(n-\frac{1}{2})^2} \Sigma_n(\tau).
\]

Here again the prefactor is chosen for our convention, and the function \( \Sigma_n(\tau) \) is defined by the Fourier integral (C.4). Substituting \( f(z + \tau; \tau) = -q^{-m} e^{-4\pi imz} f(z; \tau) \) for (C.4), we find \( \Sigma_n(\tau) = -\Sigma_n + 2m(\tau) \) which proves (C.3).

### Appendix D. Hyperbolic Laplacian

For \( \tau = u + iv \in \mathbb{H} \), we set \( \Delta_\ell \) to be the hyperbolic Laplacian

\[
\Delta_\ell = -v^2 \left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) + i\ell v \left( \frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right)
\]

\[
= -4 \left( \Im \tau \right)^{2-\ell} \frac{\partial}{\partial \tau} \left( \Im \tau \right)^{\ell} \frac{\partial}{\partial \tau}. \tag{D.1}
\]

Following Ref. 5, we set for \( h > 0 \)

\[
\varphi^\ell_{-h,s}(\tau) = \mathcal{M}^\ell_s(-4\pi h \Im(\tau)) e^{-2\pi i h \Re(\tau)}. \tag{D.2}
\]

Here the function \( \mathcal{M}^\ell_s(v) \) is defined by

\[
\mathcal{M}^\ell_s(v) = |v|^{-\frac{\ell}{2}} M_{\frac{\ell}{2} \text{sgn}(v), s-\frac{1}{2}}(|v|),
\]

where \( M_{\alpha,\beta}(z) \) is the M-Whittaker function [3, 36]. The function \( \varphi^\ell_{-h,s}(\tau) \) is an eigenfunction of \( \Delta_\ell \):

\[
\Delta_\ell \varphi^\ell_{-h,s}(\tau) = \left[ s (1-s) + \frac{\ell}{2} \left( \frac{\ell}{2} - 1 \right) \right] \varphi^\ell_{-h,s}(\tau), \tag{D.3}
\]

which behaves at \( \Im \tau \to +\infty \) as

\[
\varphi^\ell_{-h,s}(\tau) \sim \frac{\Gamma(2s)}{\Gamma\left(\frac{\ell}{2} + s\right)} q^{-h}.
\]
Appendix E. Mock Theta Functions

We set the Appell function \[32, 40\] for \(D \in \mathbb{Z}\),
\[
f_D(u, z; \tau) = \sum_{n \in \mathbb{Z}} q^{Dn^2} \frac{e^{2\pi i Dnz}}{1 - e^{2\pi i (z-u)}} q^n. \tag{E.1}
\]

By use of Watson's method \[35\], we can check that it transforms under the \(S\)-transformation as
\[
f_D(u, z; \tau) = \frac{e^{\pi i D z^2 - \pi i Du^2}}{\tau} f_D \left( \frac{u}{\tau}, \frac{z}{\tau}; -\frac{1}{\tau} \right) = \sum_{a=0}^{D-1} \vartheta_{D,a}(z; \tau) i q^{-\frac{a^2}{D}} e^{-2\pi i au} \int_{-i0}^{R} \frac{e^{\pi i Duw} - 2\pi i (Du+aw)}{1 - e^{2\pi w}} \, dw. \tag{E.2}
\]

Here the integration in the right hand side is the Mordell integral \[22\].

Following Zwegers \[40\] (see Refs. \[28, 39\] for a review; also Refs. \[1, 18\] for a classical review), we define the completion \(\hat{f}_D(u, z; \tau)\) by
\[
\hat{f}_D(u, z; \tau) = f_D(u, z; \tau) - \frac{1}{2} \sum_{a \mod D} R_{D,a}(u; \tau) \vartheta_{D,a}(z; \tau). \tag{E.3}
\]

Here the non-holomorphic partner is defined by
\[
R_{D,a}(u; \tau) = \sum_{m \equiv a \mod D} \left[ \text{sgn} \left( m + \frac{1}{2} \right) - E \left( m + D \frac{3u}{3\tau} \right) \sqrt{\frac{2}{D} \frac{3\tau}{3\tau}} \right] q^{-\frac{m^2}{2}} e^{-2\pi i mu}, \tag{E.4}
\]
where
\[
E(z) = 2 \int_0^z e^{-\pi w^2} \, dw = 1 - \text{erfc} \left( \sqrt{\pi} z \right).
\]

Then the completion \(\hat{f}_D(u, z; \tau)\) has the following modular transformation
\[
\hat{f}_D(u, z; \tau + 1) = \hat{f}_D(u, z; \tau),
\]
\[
\hat{f}_D \left( \frac{u}{\tau}, \frac{z}{\tau}; -\frac{1}{\tau} \right) = \tau e^{\pi i D z^2 - \pi i Du^2} \hat{f}_D(u, z; \tau). \tag{E.5}
\]

Appendix F. \(\mathcal{N} = 4\) Superconformal Algebras

In our previous studies on the \(\mathcal{N} = 4\) superconformal algebras \[10, 11, 12\] with central charge \(c = 6k\), we have used
\[
B_{k,a}^{\mathcal{N}=4}(z; \tau) = \frac{[\theta_{11}(z; \tau)]^2}{[\eta(\tau)]^3} \frac{\vartheta_{k+1,a} - \vartheta_{k+1,-a}}{\vartheta_{2,1} - \vartheta_{2,-1}}(z; \tau), \tag{F.1}
\]
\[
C_k^{\mathcal{N}=4}(z; \tau) = \frac{[\theta_{11}(z; \tau)]^2}{[\eta(\tau)]^3} \frac{i}{\theta_{11}(2z; \tau)} \sum_{n \in \mathbb{Z}} q^{(k+1)n^2} e^{4\pi i (k+1)nz} \frac{1 + e^{2\pi iz} q^n}{1 - e^{2\pi iz} q^n}. \tag{F.2}
\]
Here \( B_{k,a}^{N=4}(z; \tau) \) with \( 1 \leq a \leq k \) is the basis function for the massive characters, and \( C_{k}^{N=4}(z; \tau) \) is the massless character with isospin-0. Base functions \( B_{k,a}^{N=4}(z; \tau) \) are vector-valued Jacobi forms satisfying

\[
B_{k,a}^{N=4}(z; \tau) = -\sqrt{\frac{\tau}{i}} e^{-2\pi i k z^2} \sum_{b=1}^{k} \sqrt{\frac{2}{k+1}} \sin \left( \frac{ab}{k+1} \pi \right) B_{k,b}^{N=4} \left( \frac{z}{\tau}; -\frac{1}{\tau} \right),
\]

while the massless character is a mock theta function satisfying

\[
C_{k}^{N=4}(z; \tau) + e^{-2\pi i k z^2} C_{k}^{N=4} \left( \frac{z}{\tau}; -\frac{1}{\tau} \right) =\sum_{a=0}^{k} B_{k,a}^{N=4}(z; \tau) \frac{1}{2(k+1)} \int_{\mathbb{R}} e^{\pi i \tau w^2/(k+1)} \sin \left( \frac{k+1-a}{k+1} \pi \right) \frac{\sin \left( \frac{k+1-a}{k+1} \pi \right)}{\cosh \left( \frac{w}{k+1} \pi \right) + \cos \left( \frac{k+1-a}{k+1} \pi \right)} \, dw.
\]

Completion of \( C_{k}^{N=4}(z; \tau) \) is given by

\[
\hat{C}_{k}^{N=4}(z; \tau) = C_{k}^{N=4}(z; \tau) - \frac{1}{i \sqrt{2(k+1)}} \sum_{a=1}^{k} B_{k,0}(0; \tau) B_{k,a}^{N=4}(z; \tau),
\]

which is a real analytic Jacobi form satisfying

\[
\hat{C}_{k}^{N=4}(z; \tau) = e^{-2\pi i k z^2} \hat{C}_{k}^{N=4} \left( \frac{z}{\tau}; -\frac{1}{\tau} \right),
\]

\[
\hat{C}_{k}^{N=4}(z; \tau + 1) = \hat{C}_{k}^{N=4}(z+1; \tau) = \hat{C}_{k}^{N=4}(z; \tau),
\]

\[
\hat{C}_{k}^{N=4}(z+\tau; \tau) = q^{-k} e^{-4\pi i k z} \hat{C}_{k}^{N=4}(z; \tau).
\]

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