A class of solutions for anisotropic stars admitting conformal motion

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Abstract We provide a new class of interior solutions for anisotropic stars admitting conformal motion. The Einstein’s field equations in this construction are solved for specific choices of the density/mass functions. We analyze the behavior of the model parameters like radial and transverse pressures, density and surface tension.

Keywords Conformal motion; Anisotropic star; Exact solution.

1 Introduction

In the modelling of compact objects it is generally assumed that the underlying matter distribution is homogeneous and isotropic i.e., a perfect fluid (1983, Kippenhahn & Weigert 1981). Such an approach is, in general, adopted to model polytropic stars like white dwarfs, compact objects like neutron stars and ultra-compact objects like strange stars (Glendenning 1997). However, theoretical advances show that pressure inside a compact object need not be completely isotropic and various factors may contribute to pressure anisotropy (Ruderman 1972, Canuto 1974, Dev & Gleiser 2002, 2004). Consequently, the pressure inside a compact star may be decomposed into two components: the radial pressure (pr) and the transverse pressure (pt). The later acts in the orthogonal direction to the former one and their difference Δ = (pt − pr), is crucial in the calculations of surface tension of a compact star (Sharma & Maharaj 2007). Ruderman (1972) showed that nuclear matter may become anisotropic in the high density region of order 1015 gm/cc, which is expected at the core of compact terrestrial objects. Though we lack a complete understanding of the microscopic origin of the pressure anisotropy, the role of pressure anisotropy in the modeling of compact stars is a field of active research. In particular, it has been predicted that anisotropy plays a crucial role in the modeling of ultra-compact stars like strange stars (Aktas & Yilmaz 2007). Witten (1984) suggested that if quarks are de-confined from the hadrons then u, d and s quarks may yield a stable ground state of matter. This state of matter is termed as strange matter and a star composed of strange matter is called a strange star. Recent observational and empirical findings related to several compact objects like Her X-1, SAX J 1808.4-3658, RX J185635-3754 and PSR 0943+10 strongly suggest that they are strange stars (Bombaci 1997, Xu et al. 2001). There are several ways a strange star can form: a massive star may go under core collapse after a supernova explosion; alternatively a rapidly spinning star may undergo a phase transition to become a strange star. Since the density inside a strange star is beyond nuclear density anisotropy may develop and, therefore, a relativistic treatment with anisotropic pressure should be a reasonable approach to model such stars.

In the present work, we provide new solutions to model anisotropic stars admitting conformal motion. The plan of the paper is as follows. In section 2 we provide the basic equations which describe an anisotropic stars admitting conformal motion. In sub-sections 2.1 and 2.2, we solve these equations for a specific choices of
the density/mass functions. We conclude by discussing our results in section 3.

2 Anisotropic stars

Inspired by some earlier works (Mak & Harko 2004, Aktas & Yilmaz 2007, Rahman et al. 2010) on anisotropic stars admitting a one parameter group of conformal motions, we look for a new class of anisotropic star solutions admitting conformal motion. It is well known that to find the natural relation between geometry and matter through the Einstein’s equations, it is useful to use the inheritance symmetry. The well known inheritance symmetry is the symmetry under conformal killing vectors (CKV) i.e.,

\[ L_\xi g_{\mu\nu} = \psi g_{\mu\nu}, \quad \mu, \nu = 1, 2, 3, 4. \]  

(1)

The quantity on the left hand side is the Lie derivative of the metric tensor, describing the interior gravitational field of a compact star with respect to the vector field \( \xi \), and \( \psi(r) \) is an arbitrary function of \( r \). If \( \psi \) is a constant then equation (1) generates homeotheties while \( \psi = 0 \) results in killing vectors. Conformal killing vectors provide a deeper insight into the spacetime geometry.

To generate a new class of solutions making use of this symmetry, we choose the static spherically symmetric spacetime in the standard form (chosen units are \( c = 1 = G \))

\[ ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2), \]  

(2)

where \( \nu(r) \) and \( \lambda(r) \) are yet to be determined.

For an anisotropic matter distribution, with the energy-momentum tensor given by \( \pi_{\mu\nu} \), the Einstein’s field equations for the metric (2) are obtained as

\[ e^{-\lambda} \left[ \frac{\lambda'}{r} - \frac{1}{r^2} \right] + \frac{1}{r^2} = \frac{8\pi}{r^2} \rho, \]  

(3)

\[ e^{-\lambda} \left[ \frac{1}{r^2} + \frac{\nu'}{r} \right] - \frac{1}{r^2} = \frac{8\pi}{r^2} p_r, \]  

(4)

\[ \frac{1}{2} e^{-\lambda} \left[ \frac{1}{2} (\nu'') + \nu'' - \frac{1}{2} \lambda' \nu' + \frac{1}{r} (\nu' - \lambda') \right] = \frac{8\pi}{r^2} p_\theta, \]  

(5)

where \( \rho \) and \( p_r, p_\theta, p_\phi \) denote the energy density, radial, tangential and azimuthal pressures. The above equations are yet to be determined.

For the line element given in equation (2) generates

\[ \xi^1 \nu' = \psi, \]  

(7)

\[ \xi^4 = C_1, \]  

(8)

\[ \xi^1 = \frac{\psi r^2}{2}, \]  

(9)

\[ \xi^1 \lambda' + 2\xi^1 = \psi, \]  

(10)

where \( C_1 \) is a constant. These consequently imply

\[ e^\nu = C_2^2 r^2, \]  

(11)

\[ e^\lambda = \left( \frac{C_3^2}{\psi} \right)^2, \]  

(12)

\[ \xi^1 = C_1 \delta^1_1 + \left( \frac{\psi r^2}{2} \right) \delta^1_1, \]  

(13)

where \( C_2 \) and \( C_3 \) are integration constants. Equations (11)-(13), help us to rewrite equations (3)-(5) in the form (Rahman et al. 2010, Ray et al. 2008)

\[ \frac{1}{r^2} \left[ 1 - \frac{\psi^2}{C_3^2} \right] - \frac{2\psi \psi'}{r C_3^2} = 8\pi \rho, \]  

(14)

\[ \frac{1}{r^2} \left[ 1 - \frac{3\psi^2}{C_3^2} \right] = -8\pi p_r, \]  

(15)

\[ \frac{\psi^2}{C_3^2 r^2} + \frac{2\psi \psi'}{r C_3^2} = 8\pi p_\theta. \]  

(16)

We thus have three independent equations and four unknown parameter. We, therefore, are free to choose any physically reasonable ansatz for any one of these four unknown parameters to solve the set of equations.

2.1 Given density profile: \( \rho = \frac{1}{8\pi} \left( \frac{a}{r^2} + 3b \right) \).

In the model of Dev & Gleiser (2002, 2004), an anisotropic star admits two major types of density distributions, \( \rho = \) constant and \( \rho \sim r^{-2} \). These two can be constructed in one simple form as shown above. Here \( a \) and \( b \) are constants which generate various configurations of the star. For example, by choosing \( a = 3/7 \) and \( b = 0 \), one may obtain a relativistic Fermi gas. Making use of the density profile as prescribed by Dev & Gleiser (2002, 2004), we rewrite equation (13) as

\[ \frac{1}{r^2} \left[ 1 - \frac{\psi^2}{C_3^2} \right] - \frac{2\psi \psi'}{r C_3^2} = \left( \frac{a}{r^2} + 3b \right), \]  

(17)

which can be solved easily to yield

\[ \psi = \sqrt{\left[ (1 - a)C_3^2 - C_3^2 b r^2 + \frac{C}{r} \right]}, \]  

(18)

In equation (18), \( C \) is an integration constant. Consequently, we obtain an exact analytical solution in the
The two pressures are obtained as

\[ p_t = \frac{(1 - a)C_0^2 - C_0^2 b r^2 + \frac{C^2}{r^2}}{8\pi C_0^2 r^2}, \quad p_r = \frac{3[(1 - a)C_0^2 - C_0^2 b r^2 + \frac{C^2}{r^2}]}{8\pi C_0^2 r^2} - \frac{1}{8\pi r^2}. \]

The measure of pressure anisotropy is given by

\[ \Delta = \frac{1}{8\pi r^2} \left[ \frac{(1 - a)C_0^2 - C_0^2 b r^2 + \frac{C^2}{r^2}}{4\pi C_0^2 r^2} - \frac{bC_0^2 + \frac{C^2}{r^2}}{4\pi C_0^2 r^2} \right]. \]

In this model, if we set \( C = 0 \), the two metric functions \( \rho(r) \) and \( \lambda(r) \) become well behaved. Though the central singularity in the physical parameters like energy density, pressure and anisotropic parameter can not be avoided in this formalism, the solution may be used to describe the envelope region of a star in a core-envelope type model.

\( \frac{\Delta}{\rho} \) corresponds to a force due to the anisotropic nature of the star. This force will be repulsive if \( \frac{\Delta}{\rho} > 0 \) i.e., \( p_t > p_r \) and attractive if \( \frac{\Delta}{\rho} < 0 \). In the present model, \( \frac{\Delta}{\rho} \) is given by

\[ \frac{\Delta}{\rho} = \frac{1}{8\pi r^2} \left[ \frac{(1 - a)C_0^2 - C_0^2 b r^2 + \frac{C^2}{r^2}}{4\pi C_0^2 r^2} - \frac{bC_0^2 + \frac{C^2}{r^2}}{4\pi C_0^2 r^2} \right]. \]

At the surface of the star \( r = R \), we impose the condition that the radial pressure vanishes, i.e., \( p_r(r = R) = 0 \), which gives

\[ \frac{3[(1 - a)C_0^2 - C_0^2 b R^2 + \frac{C^2}{r^2}]}{8\pi C_0^2 R^2} - \frac{1}{8\pi R^2} = 0. \]

Equation (25) can be solved easily to yield

\[ R = \frac{C}{2bC_0^2} + \sqrt{\left\{ \left( \frac{3a - 2b^2}{729} \right) + \frac{C^2}{4b^2C_0^2} \right\}^\frac{1}{2}}. \]

If a star is composed of quark particles, then the surface tension \( S \) of the star is defined as [Sharma & Maharaj (2007)]

\[ \frac{2S}{R} = r_n \left( \frac{dr}{dr} \right)_{r=R}, \]

where, \( r_n = \left( \frac{1}{n} \right)^\frac{1}{3} \) equals the radius of the quark particles, \( n \) is the baryon number density and \( R \) is the radius of the star. By substituting the value of pressure gradient given by

\[ 8\pi \frac{dr}{dr} = \frac{2}{r^3} \left[ 6 \left( 1 - a \right) C_0^2 - C_0^2 b r^2 + \frac{C^2}{r^2} \right] - 3 \left( 2bC_0^2 r + \frac{C^2}{r^2} \right), \]

one can calculate the surface tension in the present model. Note that \( \frac{dr}{dr} \) is a decreasing function of \( r \) in this model.

- The mass function in this case takes the form

\[ m(r) = 4\pi \int_0^r \rho(x) x^2 dx = \frac{1}{2} (ar + br^3). \]

The characteristics of the model have been shown graphically in Fig. 1-8.

2.2 Given mass function: \( m(r) = \frac{br^3}{2(1 + ar^2)} \).

Let us assume a mass function of the form

\[ m(r) = \frac{br^3}{2(1 + ar^2)}. \]

where, \( a \) and \( b \) are two arbitrary constants. Such a mass function has been found to be relevant in the studies of compact stars like strange stars or dark-energy stars (see e.g., Sharma & Maharaj (2007) and references therein). As the mass \( m(r) \) is defined as

\[ m(r) = 4\pi \int_0^r x^2 \rho(x) dx, \]

this is equivalent to choosing the density profile in the form

\[ 8\pi \rho = \frac{b(3 + ar^2)}{(1 + ar^2)^2}. \]

Equation (13) for the above matter distribution takes the form

\[ \frac{1}{r^2} \left[ 1 - \frac{\psi^2}{C_0^2} \right] - \frac{2\psi \psi'}{r C_0^2} = \frac{b(3 + ar^2)}{(1 + ar^2)^2}, \]

whose solution is given by

\[ \psi = \sqrt{C_0^2 - \frac{bC_0^2 r^2}{(1 + ar^2)^2} + \frac{C}{r}}. \]

Thus the metric functions are obtained as

\[ e^\nu = C_0^2 r^2, \quad e^\lambda = \frac{C_0^2}{(1 + ar^2)^2} + \frac{C}{r}. \]
Note that the metric functions are regular at the centre if we set $C = 0$.

The radial and tangential pressures are obtained as

$$8\pi p_r = -\frac{1}{r^2} + \frac{3(C_3^2 - \frac{bcC_3^2 r^2}{1+ar^2} + \frac{C_1}{r})}{C_3^2 r^2}, \quad (36)$$

$$8\pi p_t = \frac{\left[C_3^2 - \frac{bcC_3^2 r^2}{1+ar^2} + \frac{C_1}{r}\right]}{r^2 C_3^2} - \frac{C}{C_3^2 r^3} - \frac{2b}{(1 + ar^2)^2}, \quad (37)$$

and the measure of anisotropy is given by

$$8\pi \Delta = \frac{1}{r^2} - 2\left[C_3^2 - \frac{bcC_3^2 r^2}{1+ar^2} + \frac{C_1}{r}\right] - \frac{C}{C_3^2 r^3} - \frac{2b}{(1 + ar^2)^2}. \quad (38)$$

At the boundary, radial pressure vanishes $(p_r(r = R) = 0)$, which gives

$$-\frac{1}{R^2} + \frac{3[C_3^2 - \frac{bcC_3^2 R^2}{1+arR^2} + \frac{C_1}{R}]}{C_3^2 R^2} = 0. \quad (39)$$

Equation (39) determines the radius $R$ of the star.

We, thus, obtain all the physical parameters in simple analytic forms. Though energy density is regular throughout the interior of a star, the two pressures still remain singular in this model. The characteristics of the model are shown graphically in Fig. 1-16.

3 Discussions

It is an established fact that the density within a compact star may go beyond nuclear density and anisotropy may develop at the interior of the star. To model a compact star with highly anisotropic matter distribution, we require a relativistic treatment. In this model, the anisotropic star is assumed to be a spherically symmetric fluid distribution where the total pressure of the fluid is decomposed into two pressure terms, the radial and the transverse one. The difference between the two is a measure of the surface tension of the star, vis-a-vis stiffness of the core.

Here we have obtained a new class of solutions in simple analytical forms describing anisotropic stars admitting conformal motion. These are obtained by taking energy density and mass variation profiles in two different cases. The solutions obtained here are in simple closed forms and can be used to study the physical behavior of compact anisotropic stars like neutron stars and strange stars. For a physically meaningful solution, it is imperative to study the behaviour of the physical parameters like energy density, mass, pressure gradient and force inside the star. In this model all these parameters are well-behaved as shown in Fig. 1-16. However, our solutions suffer from a central singularity problem as can be found in some earlier works [Mak & Harko 2004; Aktas & Yilmaz 2007; Rahman et al. 2010] as well and, therefore, is suitable for the description of the envelope region of the star. It will be interesting to examine whether other forms of matter distributions can solve the central singularity problem for anisotropic stars possessing conformal symmetry. This will be taken up elsewhere.

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Fig. 1 The EoS parameter $\omega$ is plotted against the radial parameter. Chosen parameters are $C = .5$, $C_3 = .1$ and $b = .03$. Radius of the star is $R = 13$ km.

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Fig. 2 The density parameter $\rho$ is shown against $r$. Chosen parameters are the same as in Fig.1

Fig. 3 The radial pressure $p_r$ is shown against $r$. Chosen parameters are the same as in Fig.1
Fig. 4  The anisotropy $p_t - p_r$ is shown against $r$. Chosen parameters are the same as in Fig.1

![Graph showing anisotropy](image)

Fig. 5  The force parameter is shown against $r$. Chosen parameters are the same as in Fig.1

![Graph showing force parameter](image)

Fig. 6  The radial pressure gradient is shown against $r$. Chosen parameters are the same as in Fig.1

![Graph showing radial pressure gradient](image)

Fig. 7  The mass parameter $m(r)$ is shown against $r$. Chosen parameters are the same as in Fig.1

![Graph showing mass parameter](image)
Fig. 8 The conformal parameter $\psi(r)$ is shown against $r$. Chosen parameters are the same as in Fig. 1.

Fig. 9 The EoS parameter $\omega$ is plotted against the radial parameter. Chosen parameters are $C = .5$, $C_3 = .1$ and $a = .2$. Radius of the star is $R = 9$ km.

Fig. 10 The density parameter $\rho$ is shown against $r$. Chosen parameters are the same as in Fig. 9.

Fig. 11 The radial pressure $p_r$ is shown against $r$. Chosen parameters are the same as in Fig. 9.
Fig. 12 The anisotropy $p_t - p_r$ is shown against $r$. Chosen parameters are the same as in Fig. 9

Fig. 13 The force parameter is shown against $r$. Chosen parameters are the same as in Fig. 9

Fig. 14 The radial pressure gradient is shown against $r$. Chosen parameters are the same as in Fig. 9

Fig. 15 The mass parameter $m(r)$ is shown against $r$. Chosen parameters are the same as in Fig. 9
Fig. 16  The conformal parameter $\psi(r)$ is shown against $r$. Chosen parameters are the same as in Fig. 9