VERTEX DEGREE IN EDGE-COLORED DIRECTED GRAPHS

BENTON L. DUNCAN

Abstract. We consider the question of exactness and simplicity of the universal $C^\ast$-algebra for an edge-colored directed graph. For exactness we first consider those edge-colored directed graph where the range map is a constant, saying that these graphs are supported on a single vertex. We introduce the concept of the vertex degree of an edge-colored directed graph and in this context we completely classify exactness of the associated $C^\ast$-algebra. For the general case we consider embeddings of graphs supported on a single vertex and say that the graph is locally exact if there is no such embedding for a non-exact $C^\ast$-algebra. Local exactness is a necessary condition for exactness but it is left open whether it is sufficient.

1. Introduction

The idea of considering an edge-coloring function on a directed graph was considered in [8] and from a different perspective in [3, 2, 1]. It is natural, given the study of directed graph algebras (for a survey of this study see [12]), to consider the algebras associated to edge-colored directed graphs (called separated graphs in [3]). The natural focus then is on the $C^\ast$-algebras associated to an edge-colored directed graph.

In [8] the universal $C^\ast$-algebra for an edge-colored directed graph was studied using a representation of these algebras as universal free products of graph algebras. This perspective allowed a natural extension of many results about graph algebras to the edge-colored directed graph context. In this paper we return to this subject to investigate further some problems left open in [8]. Specifically there are examples of edge-colored directed graphs which give rise to $C^\ast$-algebras which are not exact and one is left with the question of when this is the case. Alternatively the issue of simplicity of edge-colored directed graph $C^\ast$-algebras was considered in [8] but the approach there is flawed in one aspect and further investigation leads to a further analysis of the situation.

2000 Mathematics Subject Classification. 46L05, 46L09.
Key words and phrases. edge-colored directed graph, $C^\ast$-algebra, exact, simple.
Since the $C^*$-algebras of edge-colored directed graph $C^*$-algebras are considered as universal free products, one approach to these questions is to consider exactness of free products. An analysis of exactness for free products of finite dimensional algebras in [7] led us to reconsider the question in the context of edge-colored directed graph $C^*$-algebras. In [7] one quickly realizes the role played by the amalgamating subalgebra in exactness. This proved to be a useful point of view for edge-colored directed graph $C^*$-algebras. This leads to the introduction and consideration of the vertex degree for a vertex in an edge-colored directed graph.

The vertex degree for a vertex in an edge-colored directed graph is a tuple corresponding to the coloring function on those vertices with range equal to the given vertex. This proves to be an important variant in the question of exactness when every edge in the graph shares the same range. This leads one to consider how such graphs embed naturally into edge-colored directed graphs where the range map need not be a constant map. In what follows we say that an edge-colored directed graph $C^*$-algebra is locally exact if there is no such embedding for a graph with a non-exact $C^*$-algebra. Of course if the $C^*$-algebra of an edge-colored directed graph is exact then it must be locally exact.

We now introduce some notation. An edge colored directed graph $G$ is a tuple $(V, E, r, s)$ where $V$ and $E$ are countable and $r, s : E \to V$. An edge-coloring is a function $\chi : E \to \mathbb{N}$. We will denote the edge-colored directed graph by $(G, \chi_G)$ and the associated $C^*$-algebra by $C^*(G, \chi_G)$. It should be pointed out that we are focussed solely on the universal $C^*$-algebra of an edge-colored directed graph. These questions have also been investigated for the reduced algebra studied in [3]. We have little to say about the reduced algebra in the present paper.

2. Subgraphs, paths, and chains

Let $(G, \chi_G)$ be an edge-colored directed graph. If $H$ is a directed subgraph of $G$ we denote by $(H, \chi_G)$ the edge-colored directed graph with coloring $\chi_G|_{E(H)}$. We will call $(H, \chi_G)$ an edge-colored subgraph of $(G, \chi_G)$, symbolically $(H, \chi_G) \subseteq (G, \chi_G)$. Notice that $(H, \chi_G)$ is an edge-colored directed graph in its own right and we will denote this graph with the same symbolism. We have the following results connecting the $C^*$-algebra of an edge-colored subgraph to a subalgebra of the original graph.

Theorem 1. Let $(H, \chi_G) \subseteq (G, \chi_G)$ and let $A$ be the subalgebra of $C^*(G, \chi_G)$ generated by $\{S_e : e \in E(H)\}$ and $\{P_v : v \in V(H)\}$ then there is a surjection $\pi : A \to C^*(H, \chi_G)$.

Proof. We first prove this for 1-colored graphs and then use the free product decomposition of an $n$-colored graph to extend to the general case.
So assume that $H$ is a subgraph of $G$ and $G$ is 1-colorable. Now notice that the subalgebra $A$ will be generated by partial isometries satisfying the following relations:

1. The $P_v$ are mutually orthogonal nonzero projections.
2. $S^*_e S_e = P_{s(e)}$.
3. $\sum_{r(e)=v} S_e S^*_e \leq P_v$.

In addition the gauge action on $C^*(G, f)$ will reduce to a gauge action on the subalgebra $A$, and hence the family $\{P_v, S_e\}$ generating $A$ is a gauge-invariant Toeplitz-Cuntz-Krieger family. Since the graph algebra $C^*(H)$ is co-universal for gauge-invariant Toeplitz-Cuntz-Kreiger families (see [9], [13], and [14] for a general discussion of co-universality of graph algebras) it follows that there is a $*$-representation $\pi : A \to C^*(H)$ which is onto.

Now if $(G, \chi_G)$ is $n$-colored then there exist 1-colored graphs $\{G_i\}_{i=1}^n$ such that $C^*(G, f) = \ast C^*(G_i)$ then consider $A_i = A \cap C^*(G_i)$ which is generated by $\{P_v : v \in A\}$ and $\{S_e : e \in H \cap G_i\}$. Then $C^*(A) = \ast C^*(A_i)$ by [8, Theorem 2]. Now for each $i$ there is $\pi_i : C^*(A_i) \to C^*(H_i)$, where $H_i$ is $H \cap G_i$ which when restricted to the subalgebra $P'_A$ all coincide. It follows that $\ast \pi_i : \ast A \to \ast C^*(H_i)$. The former algebra is of course $A$ and the latter is $C^*(H, \chi_G)$.

**Definition 1.** We say that $(H, \chi_H)$ is a full edge-colored subgraph of $(G, \chi_G)$ if the map in the previous theorem is an injection.

For a 1-colored graph the full subgraphs of $G$ are given by those subgraphs such that $\{e \in H : r(e) = v\} = \{e \in G : r(e) = v\}$ for every vertex in $H$ receiving finite edges (This is just an application of the the gauge-invariant uniqueness theorem for arbitrary graphs, see [4, Theorem 2.1]). The same is true of an $n$-colored graph.

**Proposition 1.** The edge-colored graph $(H, \chi_H) \subseteq (G, \chi_G)$ is full if and only if $\{e \in H : r(e) = v\} = \{e \in G : r(e) = v\}$ for every vertex in $H$ which receives a finite number of edges.

**Proof.** This is just the fact that the third defining relation for a Toeplitz-Cuntz-Krieger family gives a Cuntz-Krieger family if and only if the inequality is in fact equality for any vertex receiving finite edges. In this case the gauge-invariant uniqueness theorem [4, Theorem 2.1] gives an isomorphism. For the reverse direction notice that if the two sets of edges are different then one gets a Toeplitz-Cuntz-Krieger family which is not a Cuntz-Krieger family and hence the associated algebra contains compact operators and so there is a nontrivial quotient.

Recall that a path $\mu$ in an edge-colored directed graph is a finite sequence of edges $e_n e_{n-1} \cdots e_1$ such that $r(e_i) = s(e_{i+1})$. Borrowing from [] we generalize
the notion of path in a directed graph to the edge-colored directed graph setting. If $\mu = e_ne_{n-1} \cdots e_1$ is a path in $G$ we will denote by $\mu^*$ the backward path $e_n^*e_{n-1}^* \cdots e_1^*$ where $e_i^*$ indicates traversing $e_i$ in the backward direction and will write $r(\mu) = r(e_n), s(\mu) = s(e_1)$ and $\chi_G(\mu) = \chi_G(e_n)$, whether we traverse $\mu$ in the forward or backward direction. By a chain $\mu$ we will mean a finite ordered sequence of paths and backward paths such that

- The sequence alternates between paths and backward paths (e.g. $\tilde{\mu} = \mu_n\mu_{n-1}^* \cdots \mu_2^*\mu_1$ where here we assume that $\tilde{\mu}$ begins with and ends with a path, although other possibilities are of course allowed).
- $s(\mu_{i-1}) = s(\mu_i)$ when $\mu_i\mu_{i-1}^*$ is in $\tilde{\mu}$.
- $r(\mu_{k-1}) = r(\mu_k)$ and $\chi_G(\mu_{k-1}) \neq \chi_G(\mu_k)$ when $\mu_k^*\mu_{k-1}$ is in $\tilde{\mu}$.

We will extend the range and source maps to chains as follows:

$$r(\tilde{\mu}) = \begin{cases} r(\mu_n) & \text{if } \tilde{\mu} = \mu_n\mu_{n-1}^* \cdots \\ s(\mu_n) & \text{if } \tilde{\mu} = \mu_n^*\mu_{n-1} \cdots \end{cases}$$

and

$$s(\mu) = \begin{cases} s(\mu_1) & \text{if } \tilde{\mu} = \cdots \mu_2^*\mu_1 \\ r(\mu_1) & \text{if } \tilde{\mu} = \cdots \mu_2\mu_1^* \end{cases}.$$

We will denote by $Ch(G)$ the set of chains in $(G, \chi_G)$ and will denote an arbitrary chain by $\tilde{\mu}$. We will often think of a chain as a path in the undirected graph $\tilde{G} = (V(G), E(G))$ we will denote the chain in this context as $e_ne_{n-1}^* \cdots e_2^*e_1$. We say that a chain $\tilde{\mu}$ is simple if the associated undirected path $\tilde{\mu}$ has no repeated vertices, and we say that $\tilde{\mu}$ is a simple closed chain if $\tilde{\mu}$ is a cycle. Let $\mu$ be a chain then the edges in the chain and the associated vertices form an edge-colored directed subgraph of $(G, \chi_G)$. We will call the associated sub-$C^*$-algebra, as in Theorem 1 the $C^*$-algebra generated by the path, and denote it by $A_{\tilde{\mu}}$.

While chains introduce complications they can often be converted into paths. To see this we introduce a construction. Let $(G, \chi_G)$ be an edge-colored directed graph with $e \in E(G)$. Construct a new graph by reversing the edge $e$, call it $G_e$. Formally we have $V(G) = V(G_e), E(G_e) = (E(G) \setminus \{e\}) \cup \{\overline{\tau}\}$, $r_{G_e}(f) = r(f)$ and $s_{G_e}(f) = s(f)$ for all $f \in E(G) \setminus \{e\}$, and $r(\overline{\tau}) = r(e)$ and $s(\overline{\tau}) = r(e)$. Next define $\chi_{G_e}(f) = \chi_G(f)$ for all $f \in E(G) \setminus \{e\}$ and $\chi_{G_e}((\overline{\tau})) = k+1$ where $k = \max\{\chi_G(f) : r(f) = r(\overline{\tau})\}$. We say that $(G_e, \chi_{G_e})$ is the graph obtained from $(G, \chi_G)$ by reversing the edge $e$.

**Proposition 2.** Let $(G, \chi_G)$ be an edge-colored directed graph and $e$ an edge in $G$. If $\chi_G(e) \neq \chi_G(g)$ for any edge $g$ with $r(g) = r(e)$ then $C^*(G, \chi_G)$ is isomorphic to $C^*(G_e, \chi_{G_e})$.

**Proof.** Notice that if $r(e) = s(e)$ then $(G_e, \chi_{G_e})$ is in fact equal to $(G, \chi_G)$. So we will focus on the case that $r(e) \neq s(e)$. We will assume that $ran\chi_G =
$\{1, 2, \cdots, n\}$ and that $\chi_G(e) = 1$. Now consider the graph $G_1$ and notice that $G_i = (G_e)_i$ if $2 \leq i \leq n$ so if we can show that $C^*((G_e)_1) \cong C^*(((G_e)_n+1)$ then [11] Theorem 2.1 will show us that $C^*((G_e, \chi_{G_e}) \cong C^*(G, \chi_G)$.

Notice that $G_1$ is a graph of the form

\[
\begin{bmatrix}
H & \cdot
\end{bmatrix}
\]

where $H$ is the subgraph of $G$ given by $(V(G) \setminus \{r(e)\}, E(G) \setminus \{e\}, r, s)$. Then $(G_e)_1 = (V(G), E(H), r, s)$ and $(G_e)_n+1 = (V(G), \overline{e}, r', s')$ where $r'(\overline{e}) = s(e)$ and $s'(\overline{e}) = r(e)$. Now if we write $(P, S)$ for the Cuntz-Krieger generating family for $C^*(G_1)$ and $(\tilde{P}, \tilde{S})$ for the Cuntz-Krieger generating family for

\[
C^*(((G_e)_1)_{\tilde{P}} C^*(((G_e)_{n+1}).
\]

Since $V(G_1) = V((G_e))$ we have that there is an equivalence between $P$ and $\tilde{P}$. Similarly since $V(G_1) \setminus \{e\} = V((G_e)_1)$ there is an equivalence between the associated partial isometries $\{S_f : f \neq e\}$ (viewed as either a subset of $S$ or $\tilde{S}$). Finally $S_e \in S$ is a partial isometry satisfying the following $S_e^* S_e = P_{s(e)}$ and $S_e S_e^* = P_{r(e)}$ since there are no other edges with range equal to $r(e)$.

Now the generating partial isometry associated to $\overline{e}$ in $\tilde{S}$ satisfies $T_{\overline{e}} T_{\overline{e}}^* = P_{r(e)}$ and $T_{\overline{e}}^* T_{\overline{e}} = P_{s(e)}$. Then the map $g : S \rightarrow \tilde{S}$ that sends $S_f$ to $S_f$ for $f \neq e$ and $S_e$ to $T_{\overline{e}}^*$ will induce a surjective representations of $C^*(G_1)$ onto $C^*(((G_e)_1)_{\tilde{P}} C^*(((G_e)_{n+1})$ which map is clearly invertible by the universal property for the free product. 

Notice that the preceding proposition in the case that $r(e) = s(e)$ just gives rise to a trivial re-coloring of the edges and hence doesn’t provide any useful change in the graph, however if $r(e) \neq s(e)$ then this operation can be used to simplify some graphs. Consider the following examples.

Examples: Notice that if $(G, \chi_G)$ consists of a single simple chain of length $n$ then $C^*(G, f)$ is isomorphic to $C^*(H)$ where $H$ is the directed graph consisting of a simple path of length $n$. Similarly notice that if $(G, f)$ consists of a simple closed chain then $C^*(G, f)$ is isomorphic to $C^*(C_n)$ where $C_n$ is the length $n$ cycle graph. (These are proved by inductively applying the construction of reversing an edge).
Consider now the edge-colored directed graph

\[
\begin{array}{c}
\bullet \\
\downarrow \\
\bullet \\
\downarrow \\
\bullet \\
\end{array}
\]

where the different arrow types indicate a different coloring. Using this idea of reversing the edge it is straightforward to check that the $C^*$-algebra of this edge-colored directed graph is isomorphic to the $C^*$-algebra of the directed graph

\[
\begin{array}{c}
\bullet \\
\downarrow \\
\bullet \\
\downarrow \\
\bullet \\
\end{array}
\]

Related to the notion of a chain and associated subgraphs is the notion of embedding a edge-colored directed graph $(H, \chi_H)$ into an edge-colored directed graph $(G, \chi_G)$. We will focus on a particular type of embedding which is important in what follows.

Let $H$ be a directed graph with a distinguished vertex $v \in V(H)$ such that every edge $e \in E(H)$ satisfies $r(e) = v$. Further assume that every vertex in $V(H) \setminus \{v\}$ is the source of at least one edge in $E(H)$. We will say that $H$ is based at $v$ and given a coloring on such a graph we say that the edge-colored directed graph $(H, \chi_H)$ is based at a single vertex. Notice that a finite directed graph based at a single vertex is of the form:
where \((k_i)\) denotes the number of edges with the given range and source.

Now given an edge-colored directed graph \((G, \chi_G)\) and a vertex \(w \in V(G)\) we say that there is a \(w\)-embedding of an edge-colored directed graph \((H, \chi_H)\) based at \(v\) if there is a map \(\gamma : E(H) \to Ch(G)\) such that:

- (edges map to chains) \(\gamma(e) = \tilde{\mu}\) is a finite chain in \(G\) such that \(r(\tilde{\mu}) = w\) for all \(e \in E(H)\).
- (no chain is a subchain) if \(\gamma(e) = \hat{e_n} \hat{e_{n-1}} \cdots \hat{e_1}\) and \(\gamma(f) = \hat{f_m} \hat{f_{m-1}} \cdots \hat{f_1}\) with \(e \neq f\) then there is \(1 \leq k \leq \min(m, n)\) such that \(\hat{e_k} \neq \hat{f_k}\). (i.e. if \(e \neq f\) then \(\gamma(e)\) is not a sub-chain of \(\gamma(f)\)).
- (chains preserve coloring) \(\chi_H(e) = \chi_H(f)\) if and only if when \(\gamma(e) = \tilde{e_n} \tilde{e_{n-1}} \cdots \tilde{e_1}\) and \(\gamma(f) = \tilde{f_m} \tilde{f_{m-1}} \cdots \tilde{f_1}\) satisfies \(\chi_G(e_n-k) = \chi_G(f_n-k)\) where \(\tilde{e_{n-k}} \neq \tilde{f_{n-k}}\) but \(e_{(n-k)+j} = f_{(m-k)+j}\) for all \(1 \leq j \leq k\).

In this case we say that \(H\) imbeds into \(G\) and will denote this by \(\gamma_w : (H, \chi_H) \to (G, \chi_G)\).

The reason we call this an embedding is the following result:

**Proposition 3.** Let \((H, \chi_H)\) be an edge-colored directed graph based at \(v\) and \(\gamma_w : (H, \chi_H) \to (G, \chi_G)\) be an embedding of \(H\) into \(G\). Then there is \(A\) a subalgebra of \(C^*(G, \chi_G)\) and a \(\ast\)-homomorphism \(\tilde{\pi} : A \to C^*(H, \chi_H)\) which is onto.

**Proof.** The proof is just some observations about the definition of the map \(\gamma_w\) combined with the argument of Theorem 1. Specifically if \(\gamma_w(e) = \mu\) then the chain \(\mu = \mu_n \mu_{n-1} \cdots \mu_1\) gives rise to a partial isometry \(S_\mu = S_{\mu_n} S_{\mu_{n-1}} \cdots S_1\) which satisfies \(S_\mu^* S_\mu = P_{s(\mu)}\) and \(\sum S_\mu S_\mu^* \leq P_w\). Further since no chain is a subchain of another chain we can define a gauge action on \(A\) for which the family \(\{S_\mu\}\) is gauge-invariant. We now use the co-universality of \(C^*(H, \chi_H)\) as before.

\(\square\)
It is straightforward to see that the map $\tilde{\pi}$ in the preceding proof is injective if $H$ is not row finite or if $\sum S_w S_w^* = P_w$ when the graph $H$ is row finite.

We will return to this notion when we consider the general question of exactness of an edge-colored directed graph algebra. We first wish to introduce another piece of information associated to an edge-colored directed graph which will be useful when we wish to analyze the edge-colored directed graphs which are based at a distinguished vertex $v$.

3. Vertex degree

Notice that if we consider $G_v$ and ignore any isolated vertices we get an edge-colored directed graph based at $v$. This motivates the next definition.

**Definition 2.** The vertex degree of a vertex in an $n$-colored directed graph is an $n$-tuple $(a_1, a_2, a_3, \cdots, a_n)$ where $a_i = |\{e \in H : f(e) = i\}|$.

Notice that given a directed graph $G$ and for each $v \in V(G)$ an $n$-tuple this is not enough to recover the edge-colored directed graph as the following examples illustrate.

\[\begin{array}{c}
\bullet \\
\downarrow \\
\cdots \\
\downarrow \\
\bullet
\end{array}\quad \text{and} \quad \begin{array}{c}
\bullet \\
\downarrow \\
\cdots \\
\downarrow \\
\bullet
\end{array}\]

Notice that the coloring is indicated by different arrows. In this case the vertex degree for the only vertex receiving edges is $(2, 1, 1)$ in both cases. However the graphs are different enough that (as we will see later) the $C^*$-algebra for the first is nuclear, but for the second, the $C^*$-algebra is not exact.

Also notice that we can rewrite Theorem 2 as saying that if we have a directed graph with vertex degree $(a_1, a_2, \cdots, a_i, 1)$ then reversing an edge which is not a loop changes the vertex degree of the vertex to $(m_1, m_2, \cdots, m_i)$. However it introduces a 1 to the vertex degree of the source of the edge, which may involve an overall increase in the number of colors required to color the graph. As in the following 1-colored graph:
which when we reverse the edge on the left we get the 2-colored graph

Proposition 4. Let \((G, \chi_G)\) be an \(n\)-colored directed graph and for a vertex \(v \in V(G)\) let \(d_v = (a_1, a_2, \cdots, a_n)\) be the vertex degree at \(v\). There is a recoloring \(\chi'\) of the edges of \(G\) such that the vertex degree at \(b\) with respect to the new coloring, call it \(d'_v = (a'_1, a'_2, \cdots, a'_n)\) satisfies \(a'_1 \geq a'_2 \geq \cdots \geq a'_n\).

Proof. This is just an application of [8, Corollary 1] in this context since the coloring is a “local” phenomenon.

In effect we can relabel vertex degree so that \(a_1 \geq a_2 \geq \cdots \geq a_n\) without changing the \(C^*\)-algebra of the graph. In what follows we will always assume that the edges have been relabeled to match this convention, and will refer to such an \(n\)-tuple as a decreasing \(n\)-tuple.

We now put an ordering on decreasing \(n\)-tuples as follows. We say that \((a_1, a_2, \cdots, a_n) < (b_1, b_2, \cdots, b_n)\) if there is \(0 \leq j \leq n\) such that \(a_i = b_i\) for all \(i \geq j\) and \(a_j < b_j\). This is just the restriction of the reverse lexicographic ordering on \(\mathbb{N}^n\) on the subset of \(\mathbb{N}^n\) consisting of decreasing sequences. It is thus straightforward to verify that this is a partial ordering on the collection of decreasing \(n\)-tuples.

4. The role of degree in exactness

Since every edge-colored directed graph can be built up from edge-colored directed graphs based at a vertex \(v\) we will first consider only these algebras with respect to the question of uniqueness.

Proposition 5. If \((G, \chi_G)\) is based at \(v\) and \(C^*(G, \chi_G)\) is exact then

\[ d_v < (2, 2, 1, 1, \cdots, 1) \]

and

\[ d_v < (3, 2, 1, 1, \cdots, 1). \]

Proof. We will prove that if \(d_v \geq (2, 2, 1, 1, \cdots, 1)\) or \(d_v \geq (3, 2, 1, 1, \cdots, 1)\) then \(C^*(G, \chi_G)\) is not exact. For the case of \(d_v \geq (2, 2, 2, 1, \cdots, 1)\) notice first that without loss of generality \(d_v = (2, 2, 2, 1, \cdots, 1)\) and the subalgebra generated by the range projections associated to the edges contains a copy of \(C^*(\mathbb{C})C^*(\mathbb{C})C\) which is isomorphic to \(C^*(\mathbb{Z}_2*\mathbb{Z}_2)\) which is an example of a \(C^*\)-algebra which is not exact, see [11] Example 4]. Similarly in the case of \(d_v = (3, 2, 1, 1, \cdots, 1)\) the subalgebra generated by the range projections is \(C^*(\mathbb{C})C^3\) which is isomorphic to \(C^*(\mathbb{Z}_2*\mathbb{Z}_3)\) which again is not exact as in [6]. Since in either case we have a subalgebra which is not exact the corresponding algebras can not be exact.
Notice in the preceding proof that non-exactness was basically a function of considering the $C^*$-subalgebra generated by ranges of the generating partial isometries. To see that the inequalities in the previous proposition are the best possible consider the following example.

Example.

\[
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}
\]

where $\chi_G(e_1) = \chi_G(e_2) = 1$ and $\chi_G(e_3) = \chi_G(e_4) = 2$. Notice that $d_v = (2, 2)$ which is strictly less than both $(2, 2, 2, 1, \cdots, 1)$ and $(3, 2, 1, 1, \cdots, 1)$.

To see that $C^*(G, \chi)$ is nuclear notice that the subalgebra generated by the vertex projections is isomorphic to $\mathbb{C} \oplus \mathbb{C} \oplus A \oplus \mathbb{C} \oplus \mathbb{C}$ where $A$ is isomorphic to

\[
\mathbb{C}^2 \oplus \mathbb{C}^2 = \left\{ \begin{bmatrix} f_{1,1} & f_{1,2} \\ f_{1,2} & f_{2,2} \end{bmatrix} : f_{i,j} \in C([0,1]), f_{1,2}(0) = f_{1,2}(1) = f_{2,1}(0) = f_{2,1}(1) = 0 \right\}.
\]

Now the edges correspond to partial isometries which map the extra copies of $\mathbb{C}$ onto the generating projections in $A$. We can then write the algebra $C^*(G, \chi)$ as a matrix algebra generated by the following partial isometries.
(corresponding to the edge sets)

\[
T_{e_1} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix},
\]

(1) \[
T_{e_2} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix},
\]

(2) \[
T_{e_3} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
sqrt{1-t^2} & 0 & 0 & 0 & 0 & 0 \\
1-t & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}, \text{ and}
\]

(3) \[
T_{e_4} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1-t & 0 & 0 & 0 & 0 & 0 \\
\sqrt{1-t^2} & t & 0 & 0 & 0 & 0 \\
\end{bmatrix}.
\]

If \( w \) denote by \( B \) the ideal of \( A \) given by

\[
\left\{ \begin{bmatrix} f_{1,1} \\ f_{1,2} \\ f_{2,1} \\ f_{2,2} \end{bmatrix} : f_{i,j} \in C([0,1]), f_{1,1}(0) = f_{2,2}(1) = 0 \right\}
\]

then the algebra \( C^*(G, \chi) \) can be written in block matrix form as

\[
\begin{bmatrix}
A & A & B \\
A & A & B \\
B & B & B^1
\end{bmatrix}
\]

where \( B^1 \) indicates the unitization of \( B \). Then notice that \( C^*(G, \chi)/M_3(B) \cong C \oplus C \oplus C \oplus C \oplus C \) and hence \( C^*(G, \chi) \) is nuclear if \( M_3(B) \) is nuclear. Now since \( A \) is nuclear, see [5, IV.1.4.2] and \( B \) is an ideal in \( A \) it follows that \( B \) is nuclear. Hence \( M_3(B) \) is nuclear and finally so is \( C^*(G, \chi) \).

It is important to notice that this vertex degree condition is not enough. For example the directed graph with vertex degree \((3, 1)\) with a single vertex
and four edges is isomorphic to $O_3^* C(\mathbb{T})$ which is not exact. However if $(G, \chi_G)$ is based at $v$ with vertex degree $(3, 1, 1, 1)$ is as follows

![Diagram of a 4-valent graph](image)

with $\chi(e_i) = 1$ and $\chi(f_i) = i$ then this edge-colored directed graph, by reversing edges, gives rise to the 1-colored directed graph

![Diagram of a 1-colored graph](image)

and hence, via Proposition 2 the associated $C^*$-algebra is nuclear.

The next graph obstruction to exactness involves a variant of [8, Proposition 6] as was used above. Again we will focus only on graphs based at a vertex $v$. Here we will consider the notion of embedding a graph supported at a single vertex. We start by defining the graph $B_2(2)$ as follows

![Diagram of B_2(2)](image)

with $\chi_{B_2(2)}(e_i) = i$.

**Proposition 6.** If there is an embedding of $B_2(2)$ into $(G, \chi_G)$ an edge-colored directed graph based at $v$ then $C^*(G, \chi_G)$ is not exact.

**Proof.** This is just the fact that there is a subalgebra of $C^*(G, \chi_G)$ onto $C^*(B_2(2))$ by Proposition 3 and the latter is not exact by [8, Proposition 6]. □

Again this proof reduces to considering the subalgebra generated by the range projections of the powers of the generating partial isometries. We now present two more edge-colored directed graphs based at a vertex $v$ which are not exact.
To see that neither of these are exact notice that the projection corresponding to the middle vertex decomposes as the direct sum of two range projections (coming from the edges of coloring 1) and (coming from the edges of coloring 1) the projection is purely infinite. Hence the \( C^* \)-algebra in either case contains a subalgebra of the form \( C^2 \oplus C^3 \) which is not exact.

Lastly we use Proposition 7 to consider the impact that edges that give rise to a 1 in the vertex degree have on exactness.

**Proposition 7.** Let \((G, \chi_G)\) be an edge-colored directed graph based at \(v\) with vertex degree \(v_d = (a_1, a_2, \ldots, a_{n-1}, 1)\) and let \((H, \chi_H)\) be the subgraph of \((G, \chi_G)\) with vertex degree \((a_1, a_2, \ldots, a_{n-1})\), then \((G, \chi_G)\) is exact (nuclear) if and only if \((H, \chi_H)\) is exact (nuclear).

**Proof.** Certainly if \( C^*(H, \chi_H) \) is not exact then neither is \( C^*(G, \chi_G) \) since the latter contains a subalgebra which has \( C^*(H, \chi_H) \) as a quotient. Alternatively if \( C^*(H, \chi_H) \) is exact then consider the projection \( P_v \in C^*(G, \chi_G) \), then with respect to this projection we have a decomposition of \( C^*(G, \chi_G) \) in matrix form

\[
\begin{pmatrix}
C^*(H, \chi_H) & C \\
C & C
\end{pmatrix}
\]

which can be viewed as a subalgebra of \( M_2(C^*(H, \chi_H)) \) which is exact and hence \( C^*(G, \chi_G) \) is exact. \( \square \)

We are now in a position to completely classify nuclearity/exactness of edge colored directed graph algebras based at a vertex \(v\).
Theorem 2. Let \((G, \chi_G)\) be based at \(v\) and assume that \(d_v < (3, 2, 1, 1, \ldots, 1)\), \(d_w < (2, 2, 1, \ldots, 1)\), and there is no embedding of \(B_2(2)\) into \((G, \chi_G)\) if \(C^*(G, \chi_G)\) is not nuclear then either \(G_{2,1e}\) or \(G_{2,2e}\) embeds into \((G, \chi_G)\) and hence \(C^*(G, \chi_G)\) is not exact.

We now consider embeddings and their impact on exactness. To do this consider an edge-colored directed graph \((G, \chi_G)\) and a vertex \(w \in V(G)\). We say that \(\tilde{\gamma}_w : (H, \chi_H) \to (G, \chi_G)\) if \(\tilde{\gamma}_w(e) = f\) and \(\mu\) the vertex degree at \(w\) by adding a 1 potentially making the graph \(n + 1\) colorable.

Unfortunately this is not necessarily simple to check, since one must consider each vertex and every possible chain extending from the vertex.

In practice there is a related notion which will give some straightforward method to check if \(C^*(G, \chi_G)\) is not exact. We refer to this as vertex degree propagation which is spelled out in the following proposition. We set up some notation. Let \(\tilde{\mu} = e_k e_{k-1} \cdots e_1\) be a simple chain in the \(n\)-colored graph \((G, \chi_G)\) such that \(\tilde{\mu}\) connects the vertex \(v\) to the vertex \(w\) in \(\tilde{G}\). If we assume that \(r(e_k) \neq w\) and \(r(e_1) \neq v\) then we can assume that \(\tilde{\mu}\) is a path. We will assume that this path begins at \(v\) and ends at \(w\), which will increase the vertex degree at \(w\) by adding a 1 potentially making the graph \(n + 1\) colorable. We will call this path \(\mu\).

Proposition 8. Let \(\tilde{\mu} = e_k e_{k-1} \cdots e_1\) be a chain connecting the vertices \(v\) and \(w\) such that \(r(e_k) \neq w\) and \(r(e_1) \neq v\). Assume that the vertex degree of \(v\) is \((a_1, a_2, \ldots, a_n)\) and the vertex degree of \(w\) is \((b_1, b_2, \ldots, b_n)\). There is a subalgebra of \(C^*(G, f)\) which is an edge colored directed graph \(C^*\)-algebra containing a vertex of degree \((k_1, k_2, \ldots, k_{n-1}, k_n, j_1, j_2, \ldots, j_n)\).

Proof. This proof is an application of Proposition 3 given the chain \(\mu e_1, \mu e_2, \mu e_3, \ldots \mu e_r\) and \(f_1, f_2, \ldots, f_s\). Here \(\{e_i\}\) is the set of edge with \(r(e_i) = v\) and \(\{f_i\}\) is the set of edges with \(r(f_i) = w\). \(\square\)

For example,

\begin{figure}
\centering
\begin{tikzpicture}
  \node[shape=circle,fill=black] (v1) at (0,0) {};
  \node[shape=circle,fill=black] (v2) at (-1,-1) {};
  \node[shape=circle,fill=black] (v3) at (1,-1) {};
  \node[shape=circle,fill=black] (v4) at (0,1) {};
  \node (v0) at (0,0) {};

  \draw[->] (v1) -- (v2) node [midway, above] {$e_1$};
  \draw[->] (v1) -- (v3) node [midway, right] {$e_2$};
  \draw[->] (v1) -- (v0) node [midway, below] {$e_3$};
  \draw[->] (v1) -- (v4) node [midway, left] {$e_4$};
  \draw[->] (v0) -- (v2) node [midway, left] {$f$};
\end{tikzpicture}
\end{figure}
where $\chi_G(e_1) = \chi_G(e_2) = \chi_G(f) = 1$ and $\chi_G(e_3) = \chi_G(e_4) = 2$. Then considering the chains $fe_1$, $fe_2$, $e_3$ and $f_4$ we have a subalgebra which maps onto $C^*(H, \chi_H)$ where $(H, \chi_H)$ is the edge colored directed graph

\[
\begin{array}{c}
\bullet \\
\downarrow \\
\bullet \\
& \bullet \\
\uparrow \\
\bullet
\end{array}
\]

where $\chi_G(e_1) = \chi_G(e_2) = 1$ and $\chi_G(e_3) = \chi_G(e_4) = 2$.

Given this situation we will say that the degree at $v$ propagates to $w$ and we combine this with Proposition 5 to get the following corollary.

**Corollary 1.** If $C^*(G, \chi_G)$ is exact then no degree propagates to give a vertex degree greater than or equal to $(2, 2, 2, 1, \ldots, 1)$ or $(3, 2, 1, \ldots, 1)$.

As an example consider the edge-colored directed graph

\[
\begin{array}{c}
\bullet \\
\downarrow \\
\bullet \\
\downarrow \\
\bullet \\
\uparrow \\
\bullet
\end{array}
\]

where $\chi_G(f_1) = 1, \chi_G(g_1) = 2$ and $\chi_G(h_1) = 3$ then the two edges $f_1$ and $f_2$ propagate across $g_2$ to give rise to a subalgebra which has as a quotient (actually in this case is isomorphic to) the $C^*$-algebra of an edge colored directed graph with vertex degree $(4, 2)$ and hence is not exact.

### References

[1] P. Ara, Purely infinite simple reduced $C^*$-algebras of one-relator separated graphs. *J. Math. Anal. Appl.* **393** (2012), 493-508.

[2] P. Ara and K. Goodearl, Leavitt path algebras of separate d graphs. *J. Reine Angew. Math.* **669** (2012), 165–224.

[3] P. Ara and K. Goodearl, $C^*$-algebras of separated graphs. *J. Funct. Anal.* **261** (2011), 2540-2568.

[4] T. Bates, J-H. Hong, I. Raeburn, and W. Szymanski, The ideal structure of the $C^*$-algebras of infinite graphs. *Illinois J. Math.* **46** (2002), 1159-1176.

[5] B. Blackadar, *Operator Algebras: Theory of $C^*$-algebras and von Neumann algebras*. Encyclopaedia of Mathematical Sciences, 122. Operator Algebras and Non-commutative Geometry, III. Springer-Verlag, Berlin, 2006.

[6] M. Choi, A simple $C^*$-algebra generated by two finite-order unitaries. *Canad. J. Math.* **31** (1979), 867-880.
[7] B. Duncan, Exactness of universal free products of finite dimensional $C^*$-algebras with amalgamation. *Oper. Matrices* 6 (2012), 37-48.

[8] B. Duncan, Certain free products of graph operator algebras. *J. Math. Anal. Appl.* 364 (2010), 534-543.

[9] T. Katsura, Ideal structure of $C^*$-algebras associated with $C^*$ correspondences. *Pacific J. Math.* 230 (2007) 107-146.

[10] E. Kirchberg and S. Wassermann, $C^*$-algebras generated by operator systems. *J. Funct. Anal.* 155 (1998), 324-351.

[11] H. Larki, A. Pourabbas, and A. Riazi, A note on the simplicity of $C^*$-algebras of edge-colored graphs. preprint.

[12] I. Raeburn, *Graph algebras.* CBMS Regional Conference Series in Mathematics, 103. American Mathematical Society, Providence, RI, 2005.

[13] A. Sims, The co-universal $C^*$-algebra of a row-finite graph. *New York J. Math.* 16 (2010), 507-524.

[14] A. Sims and S. Webster, A direct approach to co-universal algebras associated to directed graphs. *Bull. Malays. Math. Sci. Soc.* 33 (2010), 211-220.

Department of Mathematics, North Dakota State University, Fargo, North Dakota, USA

E-mail address: benton.duncan@ndsu.edu