Rates of convergence of means of Euclidean functionals

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Abstract

Let $L$ be the Euclidean functional with $p$-th power-weighted edges. Examples include the sum of the $p$-th power-weighted lengths of the edges in minimal spanning trees, traveling salesman tours, and minimal matchings. Motivated by the works of Steele, Redmond and Yukich (1994, 1996) have shown that for $n$ i.i.d. sample points $\{X_1, \ldots, X_n\}$ from $[0,1]^d$, $L(\{X_1, \ldots, X_n\})/n^{(d-p)/d}$ converges a.s. to a finite constant. Here we bound the rate of convergence of $E L(\{X_1, \ldots, X_n\})/n^{(d-p)/d}$.

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1 Introduction.

Let $\{X_1, \ldots, X_n\}$ be $n$ i.i.d. sample points from $\mathbb{R}^d$, $d \geq 2$, and let $0 < p < \infty$. A traveling salesman problem (TSP) is to find a permutation $\pi$ on $\{1, \ldots, n\}$ such that

$$
\sum_{j=1}^{n} |X_{\pi(j+1)} - X_{\pi(j)}|^p
= \min \left\{ \sum_{j=1}^{n} |X_{\pi'(j+1)} - X_{\pi'(j)}|^p : \pi' \text{ a permutation on } \{1, \ldots, n\} \right\},
$$

where $|X_i - X_j|$ is the Euclidean distance between $X_i$ and $X_j$ and where $\pi(n+1) := \pi(1)$ and $\pi'(n + 1) := \pi'(1)$. Let $L_{TSP}(\{X_1, \ldots, X_n\}, p)$ be the sum of the $p$-th power-weighted lengths of the edges in a minimal tour $\pi$. In the case $\{X_1, \ldots, X_n\} = \emptyset$ define

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$L_{TSP}(\emptyset, p) = 0$. Beardwood, Halton, and Hammersley (1959) showed that there exists a strictly positive but finite constant $\alpha(L_{TSP}, d, 1)$ such that for i.i.d. sample points \( \{X_i : i \geq 1\} \) with common distribution $\mu$, which has a compact support in $\mathbb{R}^d$, $d \geq 2$, as $n \to \infty$

\[
\frac{L_{TSP}(\{X_1, \ldots, X_n\}, 1)}{n^{(d-1)/d}} \to \alpha(L_{TSP}, d, 1) \int f^{(d-1)/d}(x)dx \text{ a.s.} \tag{1.1}
\]

where $f$ is the density function of the absolutely continuous part of $\mu$.

The asymptotic behavior (1.1) of the TSP functional is not an isolated one. A minimal matching (MM) on \( \{X_1, \ldots, X_n\} \) is a permutation $\pi$ on \( \{1, \ldots, n\} \) such that

\[
\sum_{j=1}^{[n/2]} |X_{\pi(2j)} - X_{\pi(2j-1)}|^p
= \min \left\{ \sum_{j=1}^{[n/2]} |X_{\pi' (2j)} - X_{\pi' (2j-1)}|^p : \pi' \text{ a permutation on } \{1, \ldots, n\} \right\},
\]

where $[n/2]$ is the largest integer $l$ with $l \leq n/2$. Let $L_{MM}(\{X_1, \ldots, X_n\}, p)$ be the sum of the $p$-th power-weighted lengths of the edges in a minimal matching $\pi$. In the case $\{X_1, \ldots, X_n\} = \emptyset$ define $L_{MM}(\emptyset, p) = 0$.

A minimal spanning tree (MST) on \( \{X_1, \ldots, X_n\} \) is a spanning tree $T$ on the given point set \( \{X_1, \ldots, X_n\} \) such that

\[
\sum_{(X_i, X_j) \in T} |X_i - X_j|^p
= \min \left\{ \sum_{(X_i, X_j) \in T'} |X_i - X_j|^p : T' \text{ a spanning tree on } \{X_1, \ldots, X_n\} \right\}.
\]

Let $L_{MST}(\{X_1, \ldots, X_n\}, p)$ be the sum of the $p$-th power-weighted lengths of the edges in a minimal spanning tree $T$. In the case $\{X_1, \ldots, X_n\} = \emptyset$ define $L_{MST}(\emptyset, p) = 0$.

A Steiner minimal spanning tree (SMST) on \( \{X_1, \ldots, X_n\} \) is a spanning tree $T$ on a point set containing \( \{X_1, \ldots, X_n\} \) (we call such $T$ a Steiner spanning tree on \( \{X_1, \ldots, X_n\} \)) such that

\[
\sum_{(X_i, X_j) \in T} |X_i - X_j|^p
= \min \left\{ \sum_{(X_i, X_j) \in T'} |X_i - X_j|^p : T' \text{ a Steiner spanning tree on } \{X_1, \ldots, X_n\} \right\}.
\]
Let $L_{SMST}(\{X_1, \ldots, X_n\}, p)$ be the sum of the $p$-th power-weighted lengths of the edges in a minimal Steiner spanning tree $T$. In the case $\{X_1, \ldots, X_n\} = \emptyset$ define $L_{SMST}(\emptyset, p) = 0$. Note that for $p > 1$, $L_{SMST} = 0$. So, whenever we talk about $L_{SMST}$, we always consider the case $0 < p \leq 1$.

A rectilinear Steiner minimal spanning tree (RSMST) on $\{X_1, \ldots, X_n\}$ is a Steiner spanning tree $T$ on $\{X_1, \ldots, X_n\}$ in which all the edges are rectilinear (we call such $T$ a rectilinear Steiner spanning tree on $\{X_1, \ldots, X_n\}$) such that

$$\sum_{(X_i, X_j) \in T} |X_i - X_j|^p = \min \left\{ \sum_{(X_i, X_j) \in T'} |X_i - X_j|^p : T' \text{ a rectilinear spanning tree on } \{X_1, \ldots, X_n\} \right\}.$$  

Let $L_{RSMST}(\{X_1, \ldots, X_n\}, p)$ be the sum of the $p$-th power-weighted lengths of the edges in a minimal rectilinear Steiner spanning tree $T$. In the case $\{X_1, \ldots, X_n\} = \emptyset$ define $L_{RSMST}(\emptyset, p) = 0$. Note again that for $p > 1$, $L_{RSMST} = 0$. So, whenever we talk about $L_{RSMST}$, we always consider the case $0 < p \leq 1$.

In a series of papers Steele (1981A, 1981B, 1988, 1990) showed that the asymptotic behavior (1.1) appears for various functionals including some of the above five. Redmond and Yukich (1994, 1996) and Lee (1999) further developed the general conditions providing the asymptotic behavior (1.1). One may consult Section 1.2 of Yukich (1998) for a brief history of this field.

Our results are stated below. But, first we would like to spell out the restrictions on the Euclidean functional $L$. We call $L(\mathcal{A}, B, p)$, $\mathcal{A}$ a finite subset of a box $B = \prod_{i=1}^{d}[x_i, x_i + s]$, $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, $0 < s < \infty$, $d \geq 2$, $0 < p < \infty$, a subadditive Euclidean functional (or a weak Euclidean functional) of power $p$ if the following four conditions are met:

1. $L(\emptyset, B, p) = 0$, 
2. $L(y + t\mathcal{A}, y + tB, p) = t^p L(\mathcal{A}, B, p)$, 
3. $|L(\mathcal{A}, B, p) - L(\mathcal{B}, B, p)| \leq C|\mathcal{A} \Delta \mathcal{B}|^{(d-p)/ds^p}$, 
4. and for a partition $\{Q_i, 1 \leq i \leq m^d\}$ of $[0, 1]^d$ into $m^d$ subboxes of edge length $m^{-1}

\sum_{j=1}^{m^d} L(\mathcal{A} \cap Q_i, Q_i, p) + Cm^{d-p}$.
By (1.2) and (1.4) with \( s = 1 \) and \( B = \emptyset \), we have for a finite subset \( \mathcal{A} \) of the unit box \([0,1]^d\) and for \( d \geq 2, 0 < p < d \),
\[
|L(\mathcal{A}, [0,1]^d, p)| \leq C|\mathcal{A}|^{(d-p)/d}. \tag{1.6}
\]
More strongly, for all the Euclidean functionals \( L \) of our interest in this paper (look at Theorem 2 below for the full list of such \( L \)) based on the space filling curve heuristic, as shown by Steele (1997), there is an extension of the above bound (1.6) which covers both the \( d = 1 \) case and the \( p \geq d \) case; for a finite subset \( \mathcal{A} \) of the unit box \([0,1]^d\) and for \( d \geq 1, p > 0 \),
\[
|L(\mathcal{A}, [0,1]^d, p)| \leq C(|\mathcal{A}|^{(d-p)/d} \lor 1). \tag{1.7}
\]
Note that (1.6) follows from the assumptions of the Euclidean functional whereas (1.7) follows from the specific feature of the Euclidean functional of our interest in this paper.

Let \( U_n \) be \( n \) i.i.d. uniform points in \([0,1]^d\).

**Theorem A.** [Redmond and Yukich (1994, 1996), Lee(1999)] Let \( L \) be a weak Euclidean functional of power \( 0 < p < d \). Then there exists a finite constant \( \alpha := \alpha(L, d, p) \) such that as \( n \to \infty \)
\[
\frac{L(U_n, [0,1]^d, p)}{n^{(d-p)/d}} \to \alpha \text{ c.c. and in } L^1, \tag{1.8}
\]
where \( Y_n \to \alpha \text{ c.c. (complete convergence) means that for any } \varepsilon > 0, \sum_{n=1}^{\infty} P(|Y_n - \alpha| > \varepsilon) < \infty. \)

For a typical weak Euclidean functional \( L \), the limit \( \alpha \) in (1.8) is strictly positive: In most situations of interest, the limit \( \alpha \) is just the subadditive constant and therefore must be strictly positive.

We call \( L^*(A, B, p) \) a superadditive Euclidean functional of power \( p \) if \(-L^*(A, B, p)\) is a subadditive Euclidean functional of power \( p \). We call \( L \) a Euclidean functional of power \( p \) if \( L \) is a subadditive Euclidean functional of power \( p \), if \( L^* \) is a superadditive Euclidean functional of power \( p \), and if
\[
L^*(A, [0,1]^d, p) \leq L(A, [0,1]^d, p), \tag{1.9}
\]
and for the \( n \) uniform points \( U_n \) in \([0,1]^d\)
\[
|EL(U_n, [0,1]^d, p) - EL^*(U_n, [0,1]^d, p)| = o(n^{(d-p)/d}). \tag{1.10}
\]
Theorem B. [Redmond and Yukich (1994, 1996), Lee(1999)] Let $L$ be a Euclidean functional of power $0 < p < d$. Then for i.i.d. sample points $\{X_i : i \geq 1\}$ with common distribution $\mu$, which has a compact support in $[0, 1]^d$, as $n \to \infty$

$$\frac{L(\{X_1, \ldots, X_n\}, [0, 1]^d, p)}{n^{(d-p)/d}} \to \alpha \int_{[0,1]^d} f^{(d-p)/d}(x)dx \text{ c.c. and in } L^1,$$

(1.11)

where $\alpha := \alpha(L, d, p)$ is a finite constant given by (1.8) and where $f$ is the density function of the absolutely continuous part of $\mu$.

If there is no confusion, to save the heavy notations from now on in the case $B = [0, 1]^d$ we use the notation $L(A)$ and $L^*(A)$ instead of $L(A, [0, 1]^d, p)$ and $L^*(A, [0, 1]^d, p)$, respectively.

We call $L$ a strong Euclidean functional of power $0 < p < d$ if $L$ is a Euclidean functional of power $p$ and if $L$ satisfies the add-one bound:

$$|EL(U_n+1) - EL(U_n)| \leq Cn^{-p/d}.$$

(1.15)

Our first result is as follows.
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Theorem 1. (i) For a strong Euclidean functional $L$ of power $0 < p < d$,

$$-Cn^{(d-1-p)/2(d-1)} \leq EL(U_n) - \alpha n^{(d-p)/d} \leq \begin{cases} Cn^{(d-1-p)/d} & \text{for } 0 < p < d - 1 \\ C\log n \lor 1 & \text{for } p = d - 1 \neq 1 \\ C & \text{for } d - 1 < p < d \\ C & \text{for } p = d - 1 = 1, \end{cases}$$

where $\alpha := \alpha(L, d, p)$ is a finite constant given by (1.8).

(ii) For a very strong Euclidean functional $L$ of power $0 < p < d$,

$$-Cn^{\frac{1}{2} - \frac{p}{d}} \leq EL(U_n) - \alpha n^{(d-p)/d} \leq \begin{cases} Cn^{(d-1-p)/d} & \text{for } 0 < p < d - 1 \\ C\log n \lor 1 & \text{for } p = d - 1 \neq 1 \\ C & \text{for } d - 1 < p < d \\ C & \text{for } p = d - 1 = 1, \end{cases}$$

where $\alpha := \alpha(L, d, p)$ is a finite constant given by (1.8).

To see the point of Theorem 1 we compare it with Theorem 5.2 of Yukich (1998). In Theorem 5.2 of Yukich (1998), he showed that, if $L$ satisfies the close in mean approximation, that is, $|EL(U_n) - EL^*(U_n)| = o(n^{(d-p)/d})$, and the add-one bound, then

$$|EL(U_n) - \alpha n^{(d-p)/d}| \leq C(n^{(d-1-p)/d} \lor 1).$$

The provided rate is quite satisfactory. However, as we see in Theorem 2 below, many typical $L$ do not satisfy the add-one bound and hence we cannot apply Theorem 5.2 of Yukich (1998) to those $L$. We wish to provide the same rate without the add-one bound so that typical $L$ has the provided rate of convergence. What we find in Theorem 1 is that this task can be done by strengthening the close in mean approximation to (1.12) and by adding two extra conditions (1.13) and (1.14). We also find that for the case $p = d - 1 \neq 1$ typical $L$ do not satisfy the other condition used in Theorem 5.2 of Yukich (1998) and we properly fix it in (1.12) and (1.13). We also take care the case $0 < p < 1$ which was excluded in Theorem 5.2 of Yukich (1998).

Note that the bounds in Theorem 1 follow from the assumptions of the Euclidean functionals. If we use the specific property of the Euclidean functional, in some cases we can get much better bounds and surprisingly we can even get a strictly positive lower bound for large $n$; see Jaillet (1993), Rhee (1994), Yukich (1998), Lee (2000) for these specific results.

For a given subadditive Euclidean functional $L$, to find the rate of convergence of $EL(U_n)$ using Theorem 1 we have to construct a superadditive Euclidean functional $L^*$ which puts us into the situations considered in Theorem 1. There may be several
ways to construct such an $L^*$. The successful $L^*$ is the one which uses the boundary freely. For example let’s consider the traveling salesman problem. Suppose that there are $n$ cities $A$ in $[0, 1]^d$. Then $L_{TSP}(A, [0, 1]^d, 1)$ is the total mileage to travel all the $n$ cities. Suppose however that there are “free” ways along the boundary $\partial[0, 1]^d$ of $[0, 1]^d$ in which the government pays the gas. In this case we may save some gas by traveling along the boundary and $L^*_{TSP}(A, [0, 1]^d, 1)$ is the total mileage.

In the case $1 \leq p < d$ following the idea of Redmond and Yukich (1994, 1996) we construct superadditive Euclidean functionals $L^*$ for the MM, MST, TSP, SMST, RSMST (of course for the SMST, RSMST we consider the $p = 1$ case only). They are

$$L^*_{MM}(A, B, p) := \min \{L_{MM}(A \cup B, B, p) : B \text{ a finite subset of } \partial B\},$$

$$L^*_{MST}(A, B, p) := L_{MST}(A, B, p) \wedge \min \left\{ \sum_j L_{MST}(A_j \cup \{b_j\}, B, p) \right\},$$

where the minimum is taken over the partition $\{A_j\}$ of $A$ and $b_j \in \partial B$,

$$L^*_{TSP}(A, B, p) := L_{TSP}(A, B, p) \wedge \min \left\{ \sum_j L_{TSP}(A_j \cup \{b_j, b'_j\}, B, p) \right\},$$

where the minimum is taken over the partition $\{A_j\}$ of $A$ and $b_j, b'_j \in \partial B$ and where for a finite subset $\{X_1, \ldots, X_n\}$ of $B$ with $|\{X_1, \ldots, X_n\} \cap \partial B| \geq 2$

$$L^*_{TSP}(\{X_1, \ldots, X_n\}, B, p) := \min \left\{ \frac{1}{n-1} \sum_{j=1}^{n-1} |X_{\pi(j+1)} - X_{\pi(j)}|^p \right\},$$

where the minimum is taken over the permutation $\pi$ on $\{1, \ldots, n\}$ such that $X_{\pi(1)}$, $X_{\pi(n)} \in \partial B$. Note that in the definition of $L^*_{TSP}$ the sum is up to $n - 1$ so that we travel free from $X_{\pi(n)}$ to $X_{\pi(1)}$ along the boundary $\partial B$. Similarly,

$$L^*_{SMST}(A, B, p) := L_{SMST}(A, B, p) \wedge \min \left\{ \sum_j L_{SMST}(A_j \cup \{b_j\}, B, p) \right\},$$

where the minimum is taken over the partition $\{A_j\}$ of $A$ and $b_j \in \partial B$,

$$L^*_{RSMST}(A, B, p) := L_{RSMST}(A, B, p) \wedge \min \left\{ \sum_j L_{RSMST}(A_j \cup \{b_j\}, B, p) \right\},$$

where the minimum is taken over the partition $\{A_j\}$ of $A$ and $b_j \in \partial B$. In the case $A = \emptyset$ define $L^*_{MM}(\emptyset, B, p) = L^*_{MST}(\emptyset, B, p) = L^*_{TSP}(\emptyset, B, p) = L^*_{SMST}(\emptyset, B, p) = L^*_{RSMST}(\emptyset, B, p) = 0.$
In the case $1 \leq p < d$ following the idea of Lee (1999) we construct superadditive Euclidean functionals $L^*$ for the MM, MST, TSP, SMST, and RSMST. They are given in the following way. In the above $L^*$ for $1 \leq p < d$, there are some edges $(X, Y)$ from a boundary point $X \in \partial B$. In the case $0 < p < 1$, for any matching, tree, or tour, we don’t pay the full price for the edge $(X, Y)$ from the boundary point $X \in \partial B$; for this edge we pay half of the full price $|X - Y|^p$, i.e., $|X - Y|^p / 2$.

**Theorem 2.** (i) For the Euclidean functional $L$ of MM, MST, TSP with $0 < p < d$ and SMST, RSMST with $0 < p \leq 1$, $L$ is a strong Euclidean functional (and hence Theorems 1 (i), 3, 4 can be applicable to this $L$).

(ii) For the Euclidean functional $L$ of MST with $0 < p < d$ and TSP, SMST, RSMST with $0 < p \leq 1$, $L$ is a very strong Euclidean functional (and hence Theorems 1 (ii), 3, 4 can be applicable to this $L$).

In Section 2, we develop a theory on the rate of convergence of $EL$ for the uniform sample points, i.e., we prove Theorems 1 and 2. In Section 3, we continue to build a theory on the rate of convergence of $EL$ for the non-uniform sample points which was started by Hero, Costa, and Ma (2003).

## 2 Rates of convergence; the uniform case

In this section we prove Theorems 1 and 2. The main idea comes from the symmetry argument of the patching in Alexander (1994). Using this symmetry argument we get the nice moment estimate.

**Lemma 1.** Let $L$ be a strong Euclidean functional of power $0 < p < d$. Then,

$$-C \leq EL(\mathcal{P}_n) - \alpha n^{(d-p)/d} \leq \begin{cases} 
  Cn^{(d-1-p)/d} & \text{for } 0 < p < d - 1 \\
  C(\log n \lor 1) & \text{for } p = d - 1 \neq 1 \\
  C & \text{for } d - 1 < p < d \\
  C & \text{for } p = d - 1 = 1.
\end{cases}$$

where $\alpha := \alpha(L, d, p)$ is a finite constant given by (1.8).

**Proof.** By the usual subadditive argument (see page 54 of [16]) for $L$,

$$-C \leq EL(\mathcal{P}_n) - \alpha n^{(d-p)/d}.$$  

By the same argument for $L^*$,

$$EL^*(\mathcal{P}_n) - \alpha n^{(d-p)/d} \leq C.$$
Now, use (1.13) to get the Lemma. 

Lemma 2. Let \( N_n \) is a Poisson random variable with mean \( n \) which is independent of i.i.d. uniform points \( \{X_i; 1 \leq i < \infty \} \) on \([0, 1]^d\).

(i) If \( L \) is a strong Euclidean functional of power \( 0 < p < d \), then

\[
|EL(U_n) - EL(U_{N_n})| \leq C(n^{(d-1-p)/2(d-1)} \vee 1). \tag{2.1}
\]

(ii) If \( L \) is a very strong Euclidean functional of power \( 0 < p < d \), then

\[
|EL(U_n) - EL(U_{N_n})| \leq Cn^{1/p-d/d}. \tag{2.2}
\]

Proof. (i) If \( L \) is a strong Euclidean functional of power \( 0 < p < d - 1 \), by (1.14), (1.6) and by Jensen’s inequality we have

\[
|EL(U_n) - EL(U_{N_n})| \leq \left| E(L(U_n) - L(U_{N_n}))1_{\{|N_n - n| \leq n^{(d-1)/d}\}} \right|
+ \left| E(L(U_n) - L(U_{N_n}))1_{\{|n^{(d-1)/d} \leq |N_n - n| \leq n/2\}} \right|
+ \left| E(L(U_n) - L(U_{N_n}))1_{\{N_n < n/2\}} \right|
+ \left| E(L(U_n) - L(U_{N_n}))1_{\{N_n > 3n/2\}} \right|
\leq CE(|N_n - n|^{(d-1-p)/(d-1)} \vee 1) + Cn^{-p/d}E|N_n - n|
+ Cn^{(d-p)/d}P(N_n < n/2) + CEN_n^{(d-p)/d}1_{\{N_n > 3n/2\}}
\leq C(n^{(d-1-p)/(d-1)} \vee 1) + Cn^{1/2-p/d}
+ Cn^{(d-p)/d}P(N_n < n/2) + CEN_n^{(d-p)/d}1_{\{N_n > 3n/2\}}.
\]

Since \( N_n \) has a very light tail, the last two terms of the left hand side in the above inequality are negligible and (2.1) follows. The argument for \( d - 1 \leq p < d \) is same.

(ii) If \( L \) is a very strong Euclidean functional of power \( 0 < p < d \), there is a standard argument from Theorem 5.2 of Yukich (1998): By (1.6) we have

\[
|EL(U_n) - EL(U_{N_n})| \leq Cn^{-p/d}E|N_n - n|1_{\{1 \leq |N_n - n| \leq n/2\}}
+ Cn^{(d-p)/d}P(N_n < n/2) + CEN_n^{(d-p)/d}1_{\{N_n > 3n/2\}}.
\]

Thus, (2.2) follows.

Proof of Theorem 1. Theorem 1 follows from Lemmas 1 and 2.

Lemma 3. For the superadditive Euclidean functional \( L^* \) of MM, MST, TSP with \( 0 < p < d \) and SMST, RSMST with \( 0 < p \leq 1 \), let \( N_B(P_n) \) be the number of points in
\(P_n\) which are connected to the boundary \(\partial[0,1]^d\) in the minimal graph which is used to calculate \(L^*\). Using the same minimal graph let \(L_B(P_n)\) be the sum of the \(p\)-th power-weighted lengths of the edges connecting points in \(P_n\) to the boundary \(\partial[0,1]^d\). Then,

\[
EN_B(P_n) \leq Cn^{(d-1)/d},
\]

\[
EL_B(P_n) \leq \begin{cases} 
  Cn^{(d-1-p)/d} & \text{for } 0 < p < d - 1 \\
  C(\log n \vee 1) & \text{for } p = d - 1 \neq 1 \\
  C & \text{for } d - 1 < p < d \\
  C & \text{for } p = d - 1 = 1.
\end{cases}
\]

The same estimates for \(U_n\) instead of \(P_n\) also holds.

**Proof.** We just follow the argument for Lemma 3.8 of Yukich (1998) or Lemma 4 of Lee (1999) and dig out the quantities of interest. We skip its proof.

**Lemma 4.** For the subadditive Euclidean functional \(L\) of MM, MST, TSP with \(0 < p < d\) and SMST, RSMST with \(0 < p \leq 1\),

\[
0 \leq EL(P_n) - EL^*(P_n) \leq \begin{cases} 
  Cn^{(d-1-p)/d} & \text{for } 0 < p < d - 1 \\
  C(\log n \vee 1) & \text{for } p = d - 1 \neq 1 \\
  C & \text{for } d - 1 < p < d \\
  C & \text{for } p = d - 1 = 1.
\end{cases}
\]  \hspace{1cm} (2.3)

The same estimates for \(U_n\) instead of \(P_n\) also holds.

**Proof.** Since the case \(1 \leq p < d\) has been known quite sometime (see Lemma 3.10 of Yukich (1998)), we consider the case \(0 < p < 1\) only. Even in this case, the argument for the case \(1 \leq p < d - 1\) of Lemma 3.10 of Yukich (1998) works; we just need to use the corresponding estimates for \(0 < p < 1\) in Lemma 3.

**Lemma 5.** For the subadditive Euclidean functional \(L\) of MM, MST, TSP with \(0 < p < d\) and SMST, RSMST with \(0 < p \leq 1\), \(L\) satisfies (1.14).

**Proof.** Since all the arguments are similar, here we prove the Lemma for the MST case only. We leave all the other cases to the reader as an exercise. First we claim that for \(1 \leq k \leq n^{(d-1)/d}\)

\[
EL(U_{n+k}) \leq EL(U_n) + C(k^{(d-1-p)/(d-1)} \vee 1).
\]

(2.4)

Let \(\{X_1, \ldots, X_{n+k}\}\) be the \(n + k\) i.i.d. uniform points on \([0,1]^d\). By renaming these points \(X_i := (X_i^1, \ldots, X_i^d)\) we may assume that the first coordinates of these \(n+k\) points are increasing, i.e., \(X_1^1 < X_2^1 < \cdots < X_{n+k}^1\). With this renaming, let \(\tau := 1 - X_{n+1}^1\).
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Figure 1: A way to construct a spanning tree on \( \{X_{n+1}, \ldots, X_{n+k}\} \); add an edge \((X_i, X_j)\) if an edge \((Y_i, Y_j)\) is in the MST on \( \{Y_{n+1}, \ldots, Y_{n+k}\}\).

After constructing an MST on \( \{X_1, \ldots, X_n\} \) and another MST on \( \{X_{n+1}, \ldots, X_{n+k}\} \), by adding an edge \((X_n, X_{n+1})\) we have a spanning tree on \( U_{n+k} \). So, we have

\[
EL(U_{n+k}) \leq EL(\{X_1, \ldots, X_n\}) + EL(\{X_{n+1}, \ldots, X_{n+k}\}) + C.
\]

Let \( X_j^* := ((1 - \tau)^{-1}X_j^1, X_j^2, X_j^3, \ldots, X_j^d) \), \( 1 \leq j \leq n \). Then, since \( \{X_1^*, \ldots, X_n^*\} \) are \( n \) i.i.d. uniform samples from \([0, 1]^d\),

\[
EL(\{X_1, \ldots, X_n\}) \leq EL(\{X_1^*, \ldots, X_n^*\}) = EL(U_n),
\]

and hence

\[
EL(U_{n+k}) \leq EL(U_n) + EL(\{X_{n+1}, \ldots, X_{n+k}\}) + C. \tag{2.5}
\]

Now, we estimate \( EL(\{X_{n+1}, \ldots, X_{n+k}\}) \). Let \( Y_i, n + 1 \leq i \leq n + k, \) be the projection of \( X_i \) to \([1] \times [0, 1]^{d-1}\). We construct an MST on \( \{Y_{n+1}, \ldots, Y_{n+k}\} \) and then using this MST we construct a spanning tree on \( \{X_{n+1}, \ldots, X_{n+k}\} \) by adding an edge \((X_i, X_j)\) if an edge \((Y_i, Y_j)\) is in this MST (see Figure 1). For this edge \((X_i, X_j)\)

\[
|X_i - X_j|^p \leq (|Y_i - Y_j| + 2\tau)^p \leq C|Y_i - Y_j|^p + C\tau^p.
\]
Figure 2: A way to construct a connected graph on $B_i$; add an edge $(X_i, X'_i)$, $X_i \in B_i$, $X'_i \in B_i$; if $(X_i, X_j)$, $X_j \in W_i$, and $(X'_i, X'_j)$, $X'_j \in W_i$, were edges in the original MST $T(U_n)$ and if $(X_j, X'_j)$ is an edge in the MST $T(W_i)$.

Since $E\tau^p \leq C(k/n)^p$, we have then by (1.7)

$$EL(\{X_{n+1}, \ldots, X_{n+k}\}) \leq CEL(\{Y_{n+1}, \ldots, Y_{n+k}\}, [0,1]^{d-1} \times \{1\}, p) + CkE\tau^p$$

$$\leq C(k^{(d-1-p)/(d-1)} \lor 1) + CkE\tau^p$$

$$\leq C(k^{(d-1-p)/(d-1)} \lor 1). \quad (2.6)$$

Therefore, (2.4) follows from (2.5)-(2.6).

Second, for $n^{(d-1)/d} < k \leq n/2$ we iterate the above argument $\lceil k/n^{(d-1)/d} \rceil + 1$ times and we have

$$EL(U_{n+k}) \leq EL(U_n) + C(\lceil k/n^{(d-1)/d} \rceil + 1)(n^{(d-1-p)/d} \lor 1)$$

$$\leq EL(U_n) + \begin{cases} Ckn^{-p/d} & \text{for } 0 < p < d - 1 \\
Ckn^{-(d-1)/d} & \text{for } d - 1 \leq p < d. \end{cases} \quad (2.7)$$

Third, we claim that for $1 \leq k \leq n^{(d-1)/d}$

$$EL(U_{n-k}) \leq EL(U_n) + C(k^{(d-1-p)/(d-1)} \lor 1). \quad (2.8)$$

As we did before, by renaming the $n$ random points $U_n := \{X_1, \ldots, X_n\}$ we may assume that the first coordinates of these $n$ points are increasing, i.e., $X_1^1 < X_2^1 < \cdots < X_n^1$. 


For $1 \leq i \leq n - k + 1$, let $B_i = U_n \setminus \{X_i, \ldots, X_{i+k-1}\}$ and let $\tau_i := X_{i+k}^1 - X_i^1$. We construct an MST $T(U_n)$ on $U_n$. From the MST $T(U_n)$, we remove all the points $X_j$, $i \leq j \leq i + k - 1$, and all the edges in the MST $T(U_n)$ which use those $X_j$ as one end. The resulting graph $D_i$ on $B_i$ is disconnected. By adding some extra edges to $D_i$ we will construct a connected graph on $B_i$. To do this we first collect those $X_j$, $i \leq j \leq i + k - 1$, for which $X_j$ was connected to a remainder point $X_k \in B_i$ by an edge in the original MST $T(U_n)$. We call the set of all those collected $X_j$ as $W_i$. We construct an MST $T(W_i)$ on $W_i$. Using this MST $T(W_i)$ we have the following addition rule of edges to the disconnected graph $D_i$ on $B_i$; we add an edge $(X_i, X_{j'})$, $X_i \in B_i$, $X_{j'} \in B_i$, if $(X_i, X_j)$, $X_j \in W_i$, and $(X_{j'}, X_{j''})$, $X_{j''} \in W_i$, were edges in the original MST $T(U_n)$ and if $(X_j, X_{j''})$ is an edge in the MST $T(W_i)$. By adding these extra edges to $D_i$ we get a connected graph on $B_i$ (see Figure 2).

In this case

$$|X_i - X_{j'}|^p \leq C|X_i - X_j|^p + C|X_j - X_{j''}|^p + C|X_{j''} - X_{j'}|^p.$$ 

Since the degree of a vertex in an MST on the set of points in $\mathbb{R}^d$ is bounded by a universal constant which depends only on the dimension $d$, by the argument of (2.6)

$$EL(B_i) \leq EL(U_n) + C \sum_{j=i}^{i+k-1} \sum_{(X_j, X_k) \in T(U_n)} E|X_j - X_k|^p + CEL(W_i)$$

$$\leq EL(U_n) + C \sum_{j=i}^{i+k-1} \sum_{(X_j, X_k) \in T(U_n)} E|X_j - X_k|^p + C(k^{(d-1)p}/(d-1) \lor 1). \quad (2.9)$$

Let $X_j^* := X_j - \tau_i e_1$ for $i + k \leq j \leq n$, where $e_1 := (1, 0, \ldots, 0)$ is the first unit vector, and let $B_i^* := \{X_1, \ldots, X_{i-1}, X_i^*, \ldots, X_n^*\}$. $B_i^*$ then consists of the $n - k$ i.i.d. uniform samples from $[0, 1]^{d-1} \times [0, 1 - \tau_i]$. By its construction,

$$L(B_i^*) \leq L(B_i).$$

Multiplying the first coordinates of the points of $B_i^*$ by $(1 - \tau_i)^{-1}$ we have $n$ i.i.d. uniform samples from $[0, 1]^d$. Hence, since $(1 - \tau_i)^{-p} \leq 1 + C\tau_i$ for the case $\tau_i \leq 1/2$,
and since $1 + C\tau_i$ is increasing in $\tau_i$ and $E(L(B_1^i)|\tau_i)$ is decreasing in $\tau_i$, 

$$EL(\mathcal{U}_{n-k}) \leq E((1 - \tau_i)^{-p}E(L(B_1^i)|\tau_i)|\tau_i \leq 1/2) P(\tau_i \leq 1/2) + E(L(\mathcal{U}_{n-k}); \tau_i > 1/2)$$

$$\leq E((1 + C\tau_i)E(L(B_1^i)|\tau_i)|\tau_i \leq 1/2) P(\tau_i \leq 1/2) + e^{-Cn}$$

$$\leq (E(1 + C\tau_i)|\tau_i \leq 1/2) E(L(B_1^i)|\tau_i)|\tau_i \leq 1/2)) P(\tau_i \leq 1/2) + e^{-Cn}$$

$$\leq \left(1 + C\frac{k}{n + 1}\right) E(L(B_1^i)\mid \tau_i \leq 1/2) + e^{-Cn}$$

Since $k \leq n^{(d-1)/d}$, we have

$$\frac{k}{n}EL(\mathcal{U}_n) \leq C\frac{k}{n}n^{-p}n^{-d} \leq Ck^{(d-1-p)/(d-1)}.$$ 

So, by combining the terms $C\frac{k}{n}EL(\mathcal{U}_n)$ and $C(k^{(d-1-p)/(d-1)} \vee 1)$ and by increasing the constant $C$ in the term $C(k^{(d-1-p)/(d-1)} \vee 1)$ we have

$$EL(\mathcal{U}_{n-k}) \leq EL(\mathcal{U}_n) + C\sum_{j=i}^{i+k-1} \sum_{(X_j, X_k) \in T(\mathcal{U}_n)} E|X_j - X_k|^p + C(k^{(d-1-p)/(d-1)} \vee 1). \quad (2.10)$$

Here comes the highlight of the argument; we use the symmetry argument to estimate the term $\sum_{j=i}^{i+k-1} \sum_{(X_j, X_k) \in T(\mathcal{U}_n)} E|X_j - X_k|^p$. Since

$$\sum_{i=1}^{n-k+1} \sum_{j=i}^{i+k-1} \sum_{(X_j, X_k) \in T(\mathcal{U}_n)} E|X_j - X_k|^p \leq CkEL(\mathcal{U}_n),$$

averaging (2.10) over $1 \leq i \leq n - k + 1$ (by (1.6)) we have (2.8).

Last, for $n^{(d-1)/d} < k \leq n/2$ we iterate the above argument $\lceil k/n^{(d-1)/d} \rceil + 1$ times and we have

$$EL(\mathcal{U}_{n-k}) \leq EL(\mathcal{U}_n) + C(\lceil k/n^{(d-1)/d} \rceil + 1)(n^{(d-1-p)/d} \vee 1)$$

$$\leq EL(\mathcal{U}_n) + \begin{cases} Ckn^{-p/d} & \text{for } 0 < p < d - 1 \\ Ckn^{-(d-1)/d} & \text{for } d - 1 \leq p < d. \end{cases} \quad (2.11)$$
For the Euclidean functional $L$ of MST with $0 < p < d$, $L$ satisfies the add-one bound (1.15) as shown by Redmond and Yukich (1994). In Lemma 6 below we show that for the case $0 < p \leq 1$ many typical $L$ also satisfy the add-one bound (1.15) and hence we provide some affirmative answers to the issue raised in p. 55 of Yukich (1998). However, we cannot handle the case $1 < p < d$ for those $L$ and more seriously we cannot prove the add-one bound for the minimal matching Euclidean functional $L_{MM}$. So, we think the add-one bound (1.15) condition is very restrictive.

**Lemma 6.** If $L$ is either the MST Euclidean functional $L$ with $0 < p < d$ or the TSP, SMST, RSMST Euclidean functional with $0 < p \leq 1$, then $L$ satisfies (1.15).

**Proof.** Since all the arguments are similar, here we prove the Lemma for the TSP case only. We leave all the other cases to the reader as an exercise.

Fix $0 < p \leq 1$. First we claim that

$$EL(U_n) \leq EL(U_{n+1}) + Cn^{-p/d}. \quad (2.12)$$

There are two edges $(X_j, X_i)$ and $(X_i, X_k)$ adjacent to $X_i$ in the minimal tour $T(U_{n+1})$. Remove these two and add the edge $(X_j, X_k)$. Then, we have a tour on $U_{n+1} \setminus \{X_i\}$. Since

$$|X_j - X_k|^p \leq (|X_j - X_i| + |X_i - X_k|)^p \leq C|X_j - X_i|^p + C|X_i - X_k|^p,$$

we have

$$EL(U_{n+1} \setminus \{X_i\}) \leq EL(U_{n+1}) + C \sum_{(X_i, X_j) \in T(U_{n+1})} E|X_i - X_j|^p.$$  

Since $EL(U_{n+1} \setminus \{X_i\}) = EL(U_n)$, and since $\sum_{i=1}^{n+1} \sum_{(X_i, X_j) \in T(U_{n+1})} E|X_i - X_j|^p = 2EL(U_{n+1}) \leq Cn^{(d-p)/d}$ (by (1.6)), averaging the above inequality over $1 \leq i \leq n + 1$ we have (2.12).

Now, we claim that

$$EL(U_{n+1}) \leq EL(U_n) + Cn^{-p/d}. \quad (2.13)$$

Let $T(U_n)$ be the minimal tour on $U_n$. For a given $X_{n+1}$, among $n$ i.i.d. uniform points $U_n$ find a nearest point $X_i$ to $X_{n+1}$ and let $(X_i, X_j)$ be an edge in the minimal tour $T(U_n)$. By removing this edge and by adding $(X_i, X_{n+1})$ and $(X_{n+1}, X_j)$ we construct
a tour on $\mathcal{U}_{n+1}$. Since $0 < p \leq 1$, we have $A^p \leq B^p + C^p$ for a triangle with side lengths $A$, $B$, $C$. So, using this triangular inequality

$$L(U_{n+1}) \leq L(U_n) - |X_i - X_j|^p + |X_i - X_{n+1}|^p + |X_j - X_{n+1}|^p$$

$$\leq L(U_n) + 2 |X_i - X_{n+1}|^p$$

$$= L(U_n) + 2 \min_{1 \leq k \leq n} |X_k - X_{n+1}|^p.$$

By taking expectations in the above inequality we have (2.13). □

**Proof of Theorem 2.** (i) follows from Lemmas 4 and 5 and (ii) follows from Lemmas 4, 5, and 6. □

## 3 Rates of convergence; the non-uniform case

In this section, we continue to build a theory on the rate of convergence of $EL$ for the non-uniform sample points which was started by Hero, Costa, and Ma (2003). In Hero, Costa, and Ma (2003) they studied the rate of convergence of $EL$ for the non-uniform sample points based on the very restrictive add-one bound (1.15) condition. Our starting point of this study is to try to remove this condition in their argument and our work in Section 2 is the result of this trial. In this section we build a theory on the rate of convergence of $EL$ for the non-uniform sample points which does not depend on the add-one bound (1.15) condition.

First, we work with a block density function; a probability density function $\phi$ of the form

$$\phi(x) = \sum_{i=1}^{m^d} \phi_i \mathbf{1}_{Q_i}(x),$$

where $\phi_i \geq 0$ is a constant and where $\{Q_i, 1 \leq i \leq m^d\}$ is a partition of $[0, 1]^d$ into $m^d$ subboxes of edge length $m^{-1}$, is a block density function of level $m$.

**Theorem 3.** Let $L$ be a strong Euclidean functional of power $0 < p < d$. Then for i.i.d. sample points $\{X_i : i \geq 1\}$ with common block density function $\phi$ of level $m$ there
exists a constant $C > 0$, independent of $\phi$, $n$, $m$, such that

$$\frac{EL(\{X_1, \ldots, X_n\})}{n^{(d-p)/d}} - \alpha \int_{[0,1]^d} \phi^{(d-p)/d}(x) \, dx \leq \begin{cases} C(nm^{-d})^{-1/d} & \text{for } 0 < p < d - 1 \\ C(nm^{-d})^{-(d-p)/d} (\log(nm^{-d}) \lor 1) & \text{for } p = d - 1 \neq 1 \\ C(nm^{-d})^{-(d-p)/d} & \text{for } d - 1 < p < d \\ C(m^{-d})^{-(d-p)/d} & \text{for } p = d - 1 = 1, \end{cases}$$

(3.1)

where $\alpha$ is a constant given by Theorem A.

**Proof.** Hero, Costa, and Ma (2003) consider the case $1 \leq p < d - 1$ only. However, their argument also works for all the other three cases considered in this theorem without any major changes. For reader’s convenience here we reproduce the argument of Hero, Costa, and Ma (2003) for the case $d - 1 < p < d$ and the case $p = d - 1 \neq 1$ with appropriate (minor) changes.

First, we handle the case $d - 1 < p < d$. Let $X_1, \ldots, X_n$ be $n$ i.i.d. samples with the common block density function $\phi$ of level $m$ and let $L$ be a strong Euclidean functional. Let $n_i$ be the number of samples $\{X_1, \ldots, X_n\}$ falling into $Q_i$. Then, by (1.5), (1.3), Theorem 1 (i)

$$EL(\{X_1, \ldots, X_n\}, [0,1]^d, p) \leq \sum_{i=1}^{m^d} EL(\{X_1, \ldots, X_n\} \cap Q_i, Q_i, p) + Cm^{d-p}$$

$$= m^{-p} \sum_{i=1}^{m^d} EE(L(\{U_1, \ldots, U_{n_i}\}, [0,1]^d, p|n_i) + Cm^{d-p}$$

$$\leq m^{-p} \sum_{i=1}^{m^d} E(\alpha n_i^{(d-p)/d} + C) + Cm^{d-p}$$

$$\leq \alpha m^{-p} n_i^{(d-p)/d} \sum_{i=1}^{m^d} E\left(\frac{n_i}{n}\right)^{(d-p)/d} + Cm^{d-p}.\$$

By Jensen’s inequality, for $0 < \nu < 1$ and $p_i := \phi_i m^{-d}$

$$E\left(\frac{n_i}{n}\right)^\nu \leq p_i^\nu.$$

So,

$$\frac{EL(\{X_1, \ldots, X_n\}, [0,1]^d, p)}{n^{(d-p)/d}} \leq \alpha \sum_{i=1}^{m^d} \phi_i^{(d-p)/d} m^{-d} + \frac{C}{(nm^{-d})^{(d-p)/d}}$$

$$= \alpha \int_{[0,1]^d} \phi^{(d-p)/d}(x) \, dx + \frac{C}{(nm^{-d})^{(d-p)/d}}.$$
Similarly, by (1.12)

\[
EL^*(\{X_1, \ldots, X_n\}, [0, 1]^d, p) \\
\geq m^{-p} \sum_{i=1}^{m^d} EE \left( L^*\left(\{U_1, \ldots, U_{n_i}\}, [0, 1]^d, p\right) \left| n_i \right) \right) - C m^{d-p}
\]

\[
\geq m^{-p} \sum_{i=1}^{m^d} EE \left( L\left(\{U_1, \ldots, U_{n_i}\}, [0, 1]^d, p\right) - C \left| n_i \right) \right) - C m^{d-p}
\]

\[
\geq m^{-p} \sum_{i=1}^{m^d} E \left( \alpha n_i^{(d-p)/d} - C - C \right) - C m^{d-p}
\]

\[
= \alpha m^{-p} n_i^{(d-p)/d} \sum_{i=1}^{m^d} E \left( \frac{n_i}{n} \right)^{(d-p)/d} - C m^{d-p}.
\]

Now, we claim that for \(0 < \nu < 1\) and \(p_i := \phi_i m^{-d}\)

\[
E \left( \frac{n_i}{n} \right)^\nu \geq p_i^\nu - p_i^{\nu-\frac{1}{2}} n^{-1/2}. \tag{3.2}
\]

If \(g \in C^1(0, \infty)\) and if \(g\) is concave over \(x \geq 0\), monotone increasing over \(x \geq 0\), and \(g(0) = 0\), then for any \(x_0 > 0\),

\[
g(x) \geq g(x_0) - \frac{g(x_0)}{x_0} |x - x_0|.
\]

Thus, with \(g(x) := x^\nu, 0 < \nu < 1, x := n_i/n,\) and \(x_0 := p_i,\) we have

\[
\left( \frac{n_i}{n} \right)^\nu \geq p_i^\nu - p_i^{\nu-1} |n_i/n - p_i|.
\]

Take the expectation on both sides. Since by Chebyshev’s inequality

\[
E \left| \frac{n_i}{n} - p_i \right| \leq \left( E \left( \frac{n_i}{n} - p_i \right)^2 \right)^{1/2} \leq \frac{\sqrt{p_i}}{\sqrt{n}},
\]

indeed we have (3.2).

So, by (3.2)

\[
\frac{EL^*(\{X_1, \ldots, X_n\}, [0, 1]^d, p)}{n_i^{(d-p)/d}} \geq \alpha \int_{[0,1]^d} \phi^{(d-p)/d}(x) \, dx - \frac{\alpha}{(nm^{-d})^{\frac{1}{2}}} \int_{[0,1]^d} \phi^{\frac{1}{2} - \frac{\nu}{2}}(x) \, dx - \frac{C}{(nm^{-d})^{(d-p)/d}}.
\]
Therefore, by (1.9) and (1.12) we have

\[
\left| \frac{EL(X_1, \ldots, X_n)}{n^{(d-p)/d}} - \alpha \int_{[0,1]^d} \phi^{(d-p)/d}(x) \, dx \right| \\
\leq \frac{\alpha}{(nm^{-d})^{1/2}} \int_{[0,1]^d} \phi^{1/2 - \frac{p}{d}}(x) \, dx + \frac{C}{(nm^{-d})^{(d-p)/d}}.
\]

Second, we handle the case \( p = d - 1 \neq 1 \) in a similar manner.

\[
EL(\{X_1, \ldots, X_n\}, [0, 1]^d, p) \\
\leq m^{-p} \sum_{i=1}^{m^d} E \left( \alpha n_i^{(d-p)/d} + C (\log n_i \lor 1) \right) + C m^{d-p}
\]

\[
\leq \alpha m^{-\frac{d}{p}} n^{(d-p)/d} \sum_{i=1}^{m^d} E \left( \frac{n_i}{n} \right)^{(d-p)/d} + C m^{-p} \sum_{i=1}^{m^d} E (\log n_i) + C m^{d-p}.
\]

Since \( n_i \) is highly concentrated around its mean \( np_i \), for a large but fixed \( C \) we have \( E \log n_i \leq C \log np_i \). So,

\[
\frac{EL(\{X_1, \ldots, X_n\}, [0, 1]^d, p)}{n^{(d-p)/d}} \\
\leq \alpha \sum_{i=1}^{m^d} \phi_i^{(d-p)/d} m^{-d} + \frac{C \sum_{i=1}^{m^d} (\log \phi_i)m^{-d}}{(nm^{-d})^{(d-p)/d}} + \frac{C \log(nm^{-d})}{(nm^{-d})^{(d-p)/d}} + \frac{C}{(nm^{-d})^{(d-p)/d}}
\]

\[
\leq \alpha \int_{[0,1]^d} \phi^{(d-p)/d}(x) \, dx + \frac{C \int_{[0,1]^d} \log \phi(x) \, dx}{(nm^{-d})^{(d-p)/d}} + \frac{C \log(nm^{-d})}{(nm^{-d})^{(d-p)/d}} + \frac{C}{(nm^{-d})^{(d-p)/d}}.
\]

By the same reasoning,

\[
\frac{EL^*\left(\{X_1, \ldots, X_n\}, [0, 1]^d, p\right)}{n^{(d-p)/d}} \\
\geq \alpha \int_{[0,1]^d} \phi^{(d-p)/d}(x) \, dx - \frac{\alpha}{(nm^{-d})^{1/2}} \int_{[0,1]^d} \phi^{1/2 - \frac{p}{d}}(x) \, dx - \frac{C \int_{[0,1]^d} \log \phi(x) \, dx}{(nm^{-d})^{(d-p)/d}} - \frac{C}{(nm^{-d})^{(d-p)/d}}.
\]

Therefore, by (1.9) and (1.12) we have

\[
\left| \frac{EL(X_1, \ldots, X_n)}{n^{(d-p)/d}} - \alpha \int_{[0,1]^d} \phi^{(d-p)/d}(x) \, dx \right| \\
\leq \frac{C}{(nm^{-d})^{(d-p)/d}} \int_{[0,1]^d} \log \phi(x) \, dx + \frac{\alpha}{(nm^{-d})^{1/2}} \int_{[0,1]^d} \phi^{1/2 - \frac{p}{d}}(x) \, dx
\]

\[
\quad + \frac{C \log \frac{n}{m^d}}{(nm^{-d})^{(d-p)/d}} + \frac{C}{(nm^{-d})^{(d-p)/d}}.
\]
Second, we work with a probability density function \( f \) in the Hölder class \( \sum(\beta, K, [0, 1]^d) \), \( \beta > 0 \); a probability density function \( f : [0, 1]^d \to \mathbb{R} \) is in \( \sum(\beta, K, [0, 1]^d) \) if for \( x, y \in [0, 1]^d \)
\[ |f(y) - p_x^{[\beta]}(y)| \leq K|x - y|^{\beta}, \]
where \( \lfloor \beta \rfloor \) is the largest integer \( l \) with \( l < \beta \) and where \( p_x^{[\beta]} \) is the Taylor expansion of \( f \) at \( x \) up to degree \( \lfloor \beta \rfloor \).

**Lemma 7.** If a probability density function \( f \) is in \( \sum(\beta, K, [0, 1]^d) \), \( 0 < \beta \leq 1 \), then for any partition \( \{Q_i, 1 \leq i \leq m^d\} \) of \( [0, 1]^d \) we define an approximating block density function \( \phi \) with level \( m \) of the form \( \phi(x) = \sum_{i=1}^{m^d} \phi_i 1_{Q_i}(x) \) by \( \phi_i = m^d \int_{Q_i} f(x)dx \). For this approximating block density function \( \phi \),
\[ \int_{[0,1]^d} |f(x) - \phi(x)| \, dx \leq d^{3/2}Km^{-\beta}. \tag{3.3} \]

**Proof.** By the mean value theorem there exists a point \( \eta_i \in Q_i \) such that
\[ \phi_i = m^d \int_{Q_i} f(x) \, dx = f(\eta_i). \]
With this \( \eta_i \),
\[ \int_{[0,1]^d} |\phi(x) - f(x)| \, dx = \sum_{i=1}^{m^d} \int_{Q_i} |f(\eta_i) - f(x)| \, dx \]
\[ \leq \sum_{i=1}^{m^d} \int_{Q_i} K|\eta_i - x|^{\beta} \, dx \]
\[ \leq d^{3/2}Km^{-\beta}. \]

**Theorem 4.** Let \( L \) be a strong Euclidean functional of power \( 0 < p < d \). Then for i.i.d. sample points \( \{X_i : i \geq 1\} \) with common probability density function \( f \in \sum(\beta, K, [0, 1]^d) \), \( 0 < \beta \leq 1 \), there exists a constant \( C := C(d, \beta, p) > 0 \) such that
\[ |EL(X_1, \ldots, X_n) - \alpha \int f(d-p)/d(x) \, dx| \leq \begin{cases} Cn^{-\beta(d-p)/d+1}/d & \text{for } 0 < p < d - 1 \\ Cn^{-\beta(d-p)/d+\beta\log n + 1} & \text{for } p = d - 1 \neq 1 \\ Cn^{-\beta(d-p)/d+\beta+1} & \text{for } d - 1 < p < d \\ Cn^{-\beta(d-p)/d+1} & \text{for } p = d - 1 = 1, \end{cases} \]
where \( \alpha \) is a constant given by Theorem A.

**Proof.** As we did in Lemma 7, for any partition \( \{Q_i, 1 \leq i \leq m^d\} \) of \([0,1]^d\) we define an approximating block density \( \phi \) with level \( m \) of the form \( \phi(x) = \sum_{i=1}^{m^d} \phi_i 1_{Q_i}(x) \) by \( \phi_i = m^d \int_{Q_i} f(x) \, dx \) for which we have (3.3).

Here we consider the case \( 0 < p < d - 1 \) only. We leave all the other cases to the reader as an exercise. Let \( \{\tilde{X}_i\}_{i=1}^n \) be i.i.d. with the common probability density function \( \phi \). Then,

\[
\begin{align*}
\left| &\frac{EL(\{X_1, \ldots, X_n\})}{n^{(d-p)/d}} - \alpha \int_{[0,1]^d} f^{(d-p)/d}(x) \, dx \right| \\
\leq & \left| \frac{EL(\{\tilde{X}_1, \ldots, \tilde{X}_n\})}{n^{(d-p)/d}} - \alpha \int_{[0,1]^d} \phi^{(d-p)/d}(x) \, dx \right| \\
+ & \alpha \left| \int_{[0,1]^d} f^{(d-p)/d}(x) \, dx - \int_{[0,1]^d} \phi^{(d-p)/d}(x) \, dx \right| \\
+ & \left| \frac{EL(\{X_1, \ldots, X_n\}) - EL(\{\tilde{X}_1, \ldots, \tilde{X}_n\})}{n^{(d-p)/d}} \right| := I + II + III.
\end{align*}
\]

The term I can be handled by Theorem 3 and it is bounded by \( C(nm^{-d})^{-1/d} \).

By (3.3), the term II is bounded by \( C m^{-\beta(d-p)/d} \),

\[
II \leq \alpha \int_{[0,1]^d} \left| f^{(d-p)/d}(x) - \phi^{(d-p)/d}(x) \right| \, dx
\]

\[
\leq \alpha \int_{[0,1]^d} \left| f(x) - \phi(x) \right|^{(d-p)/d} \, dx
\]

\[
\leq \alpha \left( \int_{[0,1]^d} \left| f(x) - \phi(x) \right| \, dx \right)^{(d-p)/d}
\]

\[
\leq \alpha \left( d^{\beta/2} K \right)^{(d-p)/d} m^{-\beta(d-p)/d}.
\]

By standard coupling and by (1.6), the term III is bounded by \( C m^{-\beta(d-p)/d} \),

\[
III = \left| EL(\{X_1, \ldots, X_n\}) - EL(\{\tilde{X}_1, \ldots, \tilde{X}_n\}) \right| n^{-(d-p)/d}
\]

\[
\leq CE \left| \{X_1, \ldots, X_n\} \triangle \{\tilde{X}_1, \ldots, \tilde{X}_n\} \right|^{(d-p)/d} n^{-(d-p)/d}
\]

\[
\leq C \left( E \left| \{X_1, \ldots, X_n\} \triangle \{\tilde{X}_1, \ldots, \tilde{X}_n\} \right| \right)^{(d-p)/d} n^{-(d-p)/d}
\]

\[
\leq C \left( \sum_{i=1}^{n} 1_{\{X_i \neq \tilde{X}_i\}} \right)^{(d-p)/d} \left( \sum_{i=1}^{n} 1_{\{X_i \neq \tilde{X}_i\}} \right)^{-(d-p)/d}
\]

\[
\leq C \left( d^{\beta/2} K \right)^{(d-p)/d} m^{-\beta(d-p)/d}.
\]
Therefore,
\[
\left| \frac{EL(X_1, \ldots, X_n)}{n^{(d-p)/d}} - \alpha \int_{[0,1]^d} f^{(d-p)/d}(x) \, dx \right| \leq \frac{C}{(nm^{-d})^{1/d}} + Cm^{-\beta(d-p)/d}.
\]

Now, we choose \( m \) so that 
\[(nm^{-d})^{-1/d} = m^{-\beta(d-p)/d}, \]
i.e., we choose 
\[m = n^{\frac{1}{\beta(d-p)+d}}.\]
Then, Theorem 4 follows from the above bounds for the terms I, II, and III.

It is an interesting problem to generalize Theorem 4 to cover more general continuous probability density functions \( f \). For example, in the case \( f \in \sum(\beta, K, [0, 1]^d) \), \( 1 < \beta \leq 2 \), we may work with a degree-one block density function; a probability density function \( \phi \) of the form
\[
\phi(x) = \sum_{i=1}^{m^d} \phi_i(x)1_{Q_i}(x),
\]
where \( \phi_i(x) \geq 0 \) is a degree-one function and where \( \{Q_i, 1 \leq i \leq m^d\} \) is a partition of \([0, 1]^d\) into \( m^d \) subboxes of edge length \( m^{-1} \), is a degree-one block density function of level \( m \). For this degree-one block density function using Theorem 4 we can get a rate of convergence similar to Theorem 3. If we can get an approximation result using a degree-one block density function \( \phi \) similar to Lemma 7, we can extend Theorem 4 to cover the case \( 1 < \beta \leq 2 \) (and hopefully by iterating this argument) and the case \( 2 < \beta < \infty \). However, we face some technical problems in this approximation procedure. We leave this problem to the interested reader.

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