A Hybrid Observer for Estimating the State of a Distributed Linear System

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Abstract

A hybrid observer is described for estimating the state of an $m > 0$ channel, $n$-dimensional, continuous-time, linear system of the form $\dot{x} = Ax, y_i = C_i x, i \in \{1, 2, \ldots, m\}$. The system’s state $x$ is simultaneously estimated by $m$ agents assuming each agent $i$ senses $y_i$ and receives appropriately defined data from each of its current neighbors. Neighbor relations are characterized by a time-varying directed graph $\mathcal{N}(t)$ whose vertices correspond to agents and whose arcs depict neighbor relations. Agent $i$ updates its estimate $x_i$ of $x$ at “event times” $t_{(s-1)}, t_s, \ldots$ using a local continuous-time linear observer and a local parameter estimator which iterates $q$ times during each event time interval $[t_{(s-1)}, t_s)$, $s \geq 1$ to obtain an estimate of $x(t_s)$. Subject to the assumptions that none of the $C_i$’s are zero, the neighbor graph $\mathcal{N}(t)$ is strongly connected for all time, and the system whose state is to be estimated is jointly observable, it is shown that for any number $\lambda > 0$, it is possible to choose $q$ and the local observer gains so that each estimate $x_i$ converges to $x$ at least as fast as $e^{-\lambda t}$ does. This result holds whether or not agents communicate synchronously, although in the asynchronous case it is necessary to assume that $\mathcal{N}(t)$ changes in a suitably defined sense. Exponential convergence is also assured if the event time sequences of the $m$ agents are slightly different than each other, although in this case only if the system being observed is exponentially stable; this limitation however, is primarily a robustness issue shared by all state estimators, centralized or not, which are operating in “open loop” in the face of small modeling errors. The result also holds facing abrupt changes in the number of vertices and arcs in the inter-agent communication graph upon which the algorithm depends.

Key words: Hybrid Systems; Distributed Observer; Robustness; Resilience.

1 Introduction

In [2] a distributed observer is described for estimating the state of an $m > 0$ channel, $n$-dimensional, continuous-time, jointly observable linear system of the form $\dot{x} = Ax, y_i = C_i x, i \in \{1, 2, \ldots, m\}$. The state $x \in \mathbb{R}^n$ is simultaneously estimated by $m$ agents assuming that each agent $i$ senses $y_i$ and receives the state of each of its neighbors’ estimates. An attractive feature of the observer described in [2] is that it is able to generate an asymptotically correct estimate of $x$ exponentially fast at a pre-assigned rate, if each agent’s set of neighbors do not change with time and the neighbor graph characterizing neighbor relations is strongly connected. However, a shortcoming of the observer in [2] is that it is unable to function correctly if the network changes with time. Changing neighbor graphs will typically occur if the agents are mobile. A second shortcoming of the observer described in [2] is that it is “fragile” by which we mean that the observer is not able to cope with the situation when there is an arbitrary abrupt change in the topology of the neighbor graph such as the loss or addition of a vertex or an arc. For example, if because of a component failure, a loss of battery power, or some other reasons, an agent drops out of the network, what remains of the overall observer will typically not be able to perform correctly and may become unstable, even if joint observability is not lost and what remains of the neighbor graph is still strongly connected.

This paper breaks new ground by introducing a hy-
The problem of interest is to develop "private estimates" for each agent, which can lead to a distributed observer for continuous-time systems with fixed neighbor graphs assuming only that the neighbor graph is directed and strongly connected. This is done by re-casting the estimation problem as a classical decentralized control problem [21, 22]. Although these observers are limited to discrete-time systems, they have proven possible to make use of the ideas in [15] to obtain a distributed observer for continuous-time systems [2]. In particular, [2] explains how to construct a distributed observer for a continuous-time system with a strongly connected neighbor graph, which is capable of estimating state exponentially fast at a pre-assigned rate. It is straightforward to modify this observer to deal with discrete-time systems.

An interesting idea, suggested in [23], seeks to simplify the structure of a distributed estimator for a continuous-time system at the expense of some design flexibility. This is done, in essence, by exploiting the $A$-invariance of the unobservable spaces of the pairs $(C_i, A)$; this in turn enables one to "split" the estimators into two parts, one based on conventional spectrum assignment techniques and the other based on consensus [23–27]. Reference [23] addresses the problem in continuous time for undirected, connected neighbor graphs. The work of [24, 25] extends the result of [23] to the case when the neighbor graph is directed and strongly connected. Establishing correctness requires one to choose gains to ensure that certain LMIs hold. In [27], motivated by the distributed least squares solver problem, a modified algorithm which can deal with measurement noise is proposed. In [26] a simplified version of the ideas in [24] is presented. Because the "high gain" constructions used in [24] and [26] don't apply in discrete-time, significant modifications are required to exploit these ideas in a
discrete-time context [28].

Despite the preceding advances, until the appearance of [1], which first outlines the idea presented in this paper, there were almost no results for doing state estimation with time varying neighbor graphs for either discrete-time or continuous-time linear systems. For sure, there were a few partial results. For example, [17] suggests a distributed observer using a consensus filter for the state estimation of discrete-time linear distributed system for specially structured, undirected neighbor graphs. Another example, in [18], an $H_\infty$ based observer is described which is intended to function in the face of a time-varying graph with a Markovian randomly varying topology. It is also worth mentioning [29] which tackles the challenging problem of trying to define a distributed observer which can function correctly in the face of intermittent disruptions in available information. Although the problem addressed in [29] is different than the problem to which this paper is addressed, resilience in the face of intermittent disruptions is to some extent similar to the notion of resilience addressed in this paper.

The first paper to provide a definitive solution to the distributed state estimation problem for time varying neighbor graphs under reasonably relaxed assumptions was presented, in abbreviated form at the 2017 IEEE Conference on Decision and Control [1]. The central contribution of [1] and this paper is to describe a distributed observer for a jointly observable, continuous-time linear system with a time-varying neighbor graph $N$ which is capable of estimating the system’s state exponentially fast at any prescribed rate. Assuming “synchronous operation”, the only requirement on the graph is that it be strongly connected for all time.

Since the appearance of [1], several other distributed observers have been suggested which are capable of doing state estimation in the face of changing neighbor graphs. For example, expanding on earlier work in [16], [30] provides a procedure for constructing such an observer which exploits in some detail the structure of $N$ and its relation to the structure of the data matrices defining the system. The resulting algorithm, which is tailored exclusively to discrete-time systems, deals with state estimation under assumptions which are weaker than strong connectivity. Recently we have learned that the split spectrum observer idea first proposed in [23] and later simplified in [24] and [26] can be modified to deal with strongly-connected time-varying neighbor graphs, although only for continuous time systems. See [31] for an unpublished report on the subject.

1.3 Organization

The remainder of the paper is organized as follows. The hybrid observer itself is described in §2 subject to the assumption that all $m$ agents share the same event time sequence. Two cases are considered, one in which the interchanges of information between agents are performed synchronously and the other case being when it is not.

The synchronous case is the one most comparable to the versions of the distributed observer problem treated in [3] - [18]. The main result for this case is Theorem 1 which asserts that so long as the neighbor graph is strongly connected for all time, exponential convergence to zero at a prescribed convergence rate of all $m$ state estimation errors is achieved. This is a new result which has no counterpart in any of the previously cited references. The same result is achieved in the asynchronous case (cf. Theorem 2), but to reach this conclusion it is necessary to assume that the neighbor graph changes in a suitably defined sense. These two theorems are the main contributions of this paper. Their proofs can be found in §3.

The aim of §4 is to explain what happens if the assumption that all $m$ agents share the same event time sequence is not made. For simplicity, this is only done for the case when differing event time sequences are the only cause of asynchronism. As will be seen, the consequence of event-time sequence mismatches turns out to be more of a robustness issue than an issue due to unsynchronized operation. In particular, it will become apparent that if different agents use slightly different event time sequences then asymptotically correct state estimates will not be possible unless $A$ is a stability matrix, i.e., all the eigenvalues of matrix $A$ have strictly negative parts. While at first glance this may appear to be a limitation of the distributed observer under consideration, it is in fact a limitation of virtually all state estimators, distributed or not, which are not used in feedback-loops. Since this easily established observation is apparently not widely appreciated, an explanation is given at the end of the section.

By a (passively) resilient algorithm for a distributed process is meant an algorithm which, by exploiting built-in network and data redundancies, is able to continue to function correctly in the face of abrupt changes in the number of vertices and arcs in the inter-agent communication graph upon which the algorithm depends. In §5, it is briefly explained how to configure things so that that the proposed estimator can cope with the situation when there is an arbitrary abrupt change in the topology of the neighbor graph such as the loss or addition of an arc or a vertex provided connectivity is not lost in an appropriately defined sense. Dealing with a loss or addition of an arc proves to be easy to accomplish because of the ability of the estimator to deal with time-varying graphs. Dealing with the loss or addition of a vertex is much more challenging and for this reason only preliminary results are presented. Finally in §6 simulation results are provided to illustrate the observer’s performance.

1 It is worth noting at this point that many of the subtleties of asynchronous operation are obscured or at least difficult to recognize in a discrete-time setting where there is invariably a single underlying discrete-time clock shared by all $m$ agents.
2 Hybrid Observer

The overall hybrid observer to be considered consists of $m$ private estimators, one for each agent. Agent $i$’s private estimator, whose function is to generate an estimate $x_i$ of $x$, is a hybrid dynamical system consisting of a “local observer” and a “local parameter estimator.” The purpose of local observer $i$ is to generate an asymptotically correct estimate of $L_i x$ where $L_i$ is any pre-specified, full-rank matrix whose kernel equals the kernel of the observability matrix of the pair $(C_i, A)$; roughly speaking, $L_i x$ can be thought of as that “part of $x$” which is observable to agent $i$. Agent $i$’s local observer is then an $n_i$-dimensional continuous-time, linear system of the form

$$\dot{w}_i = (\tilde{A}_i + K_i \tilde{C}_i)w_i - K_i y_i$$

where $n_i = \text{rank } L_i$, $K_i$ is a gain matrix to be specified, and $\tilde{C}_i$ and $\tilde{A}_i$ are unique solutions to the equations $\tilde{C}_i = C_i L_i$ and $L_i A = \tilde{A}_i L_i$, respectively. As is well known, the pair $(\tilde{C}_i, \tilde{A}_i)$ is observable and the local observer estimation error $\hat{e}_i \triangleq w_i - L_i x$ satisfies

$$\hat{e}_i(t) = e^{(\tilde{A}_i + K_i \tilde{C}_i)} \hat{e}_i(0), \quad t \in [0, \infty)$$

Since $(\tilde{C}_i, \tilde{A}_i)$ is an observable pair, $K_i$ can be selected so that $\hat{e}_i(t)$ converges to 0 exponentially fast at any pre-assigned rate. We assume that each $K_i$ is so chosen. Since

$$w_i(t) = L_i x(t) + \hat{e}_i(t), \quad i \in m, \quad t \in [0, \infty)$$

$w_i$ can be viewed as a signal which approximates $L_i x$ in the face of exponentially decaying additive noise, namely $\hat{e}_i$.

The other sub-system comprising agent $i$’s private estimator, is a “local parameter estimator” whose function is to generate estimates of $x$ at each of agent $i$’s pre-specified event times $t_{i1}, t_{i2}, \ldots$. Here $t_{i0}, t_{i1}, t_{i2}, \ldots$ is an ascending sequence of event times with a fixed spacing of $T > 0$ time units between any two successive event times. In this section it is assumed that $t_{i0} = 0$, $i \in m$, and consequently that all event time sequences are the same.

Thus $t_{is} = sT$, $s \geq 0$, $i \in m$. Between event times, each $x_i$ is generated using the equation

$$\dot{x}_i = Ax_i$$

Motivation for the development of the local parameter estimator whose purpose is to enable agent $i$ to estimate $x(t_{is}-1)$ over the event time interval $[t_{is-1}, t_{is})$, stems from the fact that the equations

$$w_j(t_{is-1}) = L_j p + \tilde{e}_j(t_{is-1}), \quad j \in m$$

admit a unique solution, namely $p = x(t_{is-1})$. Existence follows from (4) whereas uniqueness is a consequence of the assumption of joint observability.

The existence and uniqueness of $p$ suggest that an approximate value of $x(t_{is-1})$ can be obtained after a finite number of iterations - say $q$ - using the linear equation solver discussed in [32]. Having obtained such an approximate value of $x(t_{is-1})$, denoted below by $z_{is}(q)$, the desired estimate of $x(t_{is})$ can be taken as

$$x_i(t_{is}) \triangleq e^{AT} z_{is}(q)$$

(6)

This is the architecture which will be considered. The computations needed to update agent $i$’s estimate of $x(t_{is-1})$ are carried out by agent $i$ during the event time interval $[t_{is-1}, t_{is})$. This is done using a local parameter estimator which generates a sequence of $q$ auxiliary states $z_{is}(1), z_{is}(2), \ldots, z_{is}(q)$ where $q > 0$ is a positive integer to be specified below. The sequence is initialized by setting

$$z_{is}(0) = x_i(t_{is-1}),$$

(7)

and is recursively updated by agent $i$ at local iteration times $\tau_{is}(k), k \in q \triangleq \{1, 2, \ldots, q\}$, known only to agent $i$. It is assumed that the $\tau_{is}(k)$ together with the initialization $\tau_{is}(0)$ are of the form

$$\tau_{is}(k) = t_{is-1} + k \Delta + \delta_{is}(k), \quad k \in \{0, 1, \ldots, q\}$$

(8)

where $\delta_{is}(0), \delta_{is}(1), \delta_{is}(2), \ldots, \delta_{is}(q)$ is a sequence of small deviations which satisfy

$$\delta_{is}(k) \in [-\epsilon_i, \epsilon_i], \quad k \in \{0, 1, \ldots, q\}$$

(9)

Here $\epsilon_i$ is a small nonnegative number whose constraints will be described below and $\Delta$ is a positive number satisfying

$$\Delta q + \max_i \{\epsilon_i, i \in m\} \leq T$$

The signal $z_{is}(q)$ is agent i’s updated estimate of $x(t_{is-1})$ and is used to define $x_i(t_{is})$ as in (6).

The transfer of information between agents which is needed to generate the $z_{is}(k)$, is carried out asynchronously as follows. For $k \in q$ and $j \in m$, agent $j$ broadcasts $z_{js}(k-1)$ at time $\tau_{js}(k-1) + \beta$ where $\beta$ is any prescribed nonnegative number chosen smaller than $\Delta$. It is assumed that the bounds $\epsilon_i, i \in m$, appearing in (9) are small enough so that there exist $\beta$ and $\Delta$ satisfying

$$\epsilon_i + \epsilon_j \leq \beta, \quad \text{and} \quad \epsilon_i + \epsilon_j + \beta < \Delta, \quad i, j \in m$$

(10)

These inequalities ensure that for $k \in q$, a broadcast by any agent $j$ at time $\tau_{js}(k-1) + \beta$ will occur within the reception interval $[\tau_{is}(k-1), \tau_{is}(k))$ of agent $i$. Fig. 1
Towards this end, suppose that as a design goal it is desired to pick the $K_i$ and $q$ so that all $m$ state estimation errors
eq x_i - x, \ i \in \mathbf{m} \tag{14}
$ converge to zero as fast as $e^{-\lambda t}$ does where $\lambda > 0$ is some desired convergence rate. The $K_i$ would then have to be chosen using spectrum assignment or some other technique so that the matrix exponentials $e^{(A_i + K_i C_i)t}$ all converge to zero at least as fast as $e^{-\lambda t}$ does. This of course can be accomplished because each matrix pair

$\Delta+\delta_i(1) \Delta-\delta_i(1)$

$\Delta+\delta_j(1) \Delta-\delta_j(1)$

\begin{align*}
t & : \tau_i(1) \tau_i(2) \cdots \tau_i(q) \ t_{is} \\
j & : \tau_j(1) \tau_j(2) \cdots \tau_j(q) \ t_{js}
\end{align*}

\hspace{1cm} Fig. 1. A broadcast by any agent $j$ at time $\tau_{js}(k-1) + \beta$ will occur within the reception interval $[\tau_{is}(k-1), \tau_{is}(k))$ of agent $i$.

provides an example of the update and communication times of two different different agents $i$ and $j$. Accordingly, agent $j$ is a data source or just source for agent $i$ on $[\tau_{is}(k-1), \tau_{is}(k))$ if agent $j$ is in the reception range of agent $i$ at time $\tau_{is}(k-1) + \beta$. Let $S_{is}(k)$ denote the set of labels of such agents; that is

$S_{is}(k) = \{ j : j \in N_i(\tau_{js}(k-1) + \beta) \} \tag{11}$

Note that $i \in S_{is}(k)$, for all $i \in \mathbf{m}$ so $S_{is}(k)$ is never empty. Clearly agent $i$ can use the signals $z_{js}(k-1), j \in S_{is}(k)$, to compute $z_{is}(k)$.

Promoted by [32], the update equation used to recursively generate the $z_{is}(k)$ during agent $i$’s $s$th event time interval $[t_{is}(s-1), t_{is})$ is given by

$z_{is}(k) = z_{is}(k-1) - Q_i(L_i \bar{z}_{is}(k-1) - w_i(t_{is}(s-1))), \ k \in \mathbf{q} \tag{12}$

where $Q_i = L_i^T(L_i L_i^T)^{-1}$, $z_{is}(k-1)$ is an averaged state

$\bar{z}_{is}(k-1) = \frac{1}{m_{is}(k)} \sum_{j \in S_{is}(k)} z_{js}(k-1), \tag{13}$

and $m_{is}(k)$ is the number of labels in $S_{is}(k)$. The overall private estimator for agent $i$ is thus described by the equations (3), (5) - (8) and (11) - (13). In summary, initialize $x_i(t_{i0}), w_i(0)$ randomly. For $t \in [t_{i0}, t_{i1}), \bar{x}_i = Ax_i$. Then for $s = 1, 2, \ldots$, the algorithm of the hybrid estimator for agent $i$ is shown in Algorithm 1.

In order to complete the definition of the hybrid observer, it is necessary to specify values of the $K_i$ and $q$. Towards this end, suppose that as a design goal it is desired to pick the $K_i$ and $q$ so that all $m$ state estimation errors
eq x_i - x, \ i \in \mathbf{m} \tag{14}
$ converge to zero as fast as $e^{-\lambda t}$ does where $\lambda > 0$ is some desired convergence rate. The $K_i$ would then have to be chosen using spectrum assignment or some other technique so that the matrix exponentials $e^{(A_i + K_i C_i)t}$ all converge to zero at least as fast as $e^{-\lambda t}$ does. This of course can be accomplished because each matrix pair

Algorithm 1 The hybrid estimator of agent $i$

1: Initialize $x_i(t_{i0}), w_i(0), K_i, (C_i, A_i), L_i, q$
2: $w_i = (A_i + K_i C_i)w_i - K_i y_i$
3: for $t \in [t_{i0}, t_{i1})$ do
4: $\bar{x}_i = Ax_i$ with $x_i(t_{i0})$
5: end for
6: for $s = 1, 2 \ldots$ do
7: $z_{is}(0) = x_i(t_{is}(s-1))$
8: for $k = 1 : q$ do
9: Agent $i$ gets the sampled value $w_i(t_{is}(s-1))$ from its own estimator, and receives $z_{js}(k-1)$ from its neighbor $j$.

$z_{is}(k) = \bar{z}_{is}(k-1) - Q_i(L_i \bar{z}_{is}(k-1) - w_i(t_{is}(s-1)))$

where $\bar{z}_{is}$ is as defined in Eq. (13)

10: end for
11: for $t \in [t_{is}, t_{is}(s+1))$ do
12: $\bar{x}_i = Ax_i$ with $x_i(t_{is}) = e^{AT}z_{is}(q)$
13: end for
14: end for
15: Output: $x_i$

$(C_i, A_i)$ is observable. In the sequel it will be assumed that for some preselected positive number $\lambda > \lambda$, the $K_i$ have been chosen so that for $i \in \mathbf{m}$ the local observer state estimation errors satisfy

$||\bar{e}_i(t)|| \leq c_i e^{-\lambda(t-\mu)}||\bar{e}_i(\mu)||, \ t \geq \mu \geq 0 \tag{15}$

where the $c_i, i \in \mathbf{m}$ are nonnegative constants and $|| \cdot ||$ denotes the two-norm.

To describe how to define an appropriate value of $q$ to attain the desired convergence rate for the state estimation errors $e_i, i \in \mathbf{m}$, it is necessary to take some preliminary steps. First, for each $i \in \mathbf{m}$, let $P_i$ denote the orthogonal projection on the unobservable space of $(C_i, A_i)$. It is easy to see that $P_i = I - L_i^T(L_i L_i^T)^{-1}L_i, i \in \mathbf{m}$. Moreover, because of the assumption of joint observability,

$\bigcap_{i \in \mathbf{m}} \text{image } P_i = \{0\} \tag{16}$

Next, let $C$ denote the set of all products of the form $P_1 P_2 \cdots P_{\mathbf{m}^{-1}}$, where each projection matrix in $\{P_i : i \in \mathbf{m}\}$ occurs in each of such product at least once. Note that $C$ is a closed subset of $\mathbb{R}^{n \times n}$. Since each projection matrix $P_i, i \in \mathbf{m}$ has a two-norm which is no greater than 1, each matrix $M \in C$ has a two-norm less than or equal to 1. Thus $C$ is also a bounded and thus compact subset. In fact, each product in $C$ actually has two-norm strictly less than 1. This is a consequence of (16) and the requirement that each matrix in $\{P_i : i \in \mathbf{m}\}$ must occur in each product in $C$ at least once [Lemma 2, [33]]. These observations imply that
maximum of the two-norms of the matrices in $C$, namely
\[ \rho \triangleq \max\{||M|| : M \in C\}, \tag{17} \]
exists and is a real non-negative number strictly less than 1. This in turn implies that the attenuation constant
\[ \alpha \triangleq 1 - \frac{(m-1)(1-\rho)}{m(m-1)^2} \tag{18} \]
is also a real non-negative number strictly less than 1. As will become evident below \{cf. (40) and Lemma 5\}, in the idealized case when all $\epsilon_i$ and $\bar{\epsilon}_i$ are zero, for any integer $p > 0$ and any given value of $q$ satisfying
\[ q \geq p((m-1)^2 + 1), \tag{19} \]
the value of the signal
\[ \max\{||z_{is}(k) - x(t_{i(s-1)})|| : i \in m\} \]
attained by at least a factor $\alpha^p$ after $q$ iterations during each event-time interval $[t_{i(s-1)}, t_{i(s)}]$; i.e., for $s \geq 1$,
\[ \max\{||z_{is}(q) - x(t_{i(s-1)})|| : i \in m\} \leq \alpha^p \max\{||z_{is}(0) - x(t_{i(s-1)})|| : i \in m\} \]
It will then be apparent, if it is not already from (6), (7) and (14), that over each event-time interval $[t_{i(s-1)}, t_{i(s)}],$
\[ \max\{||e_i(t_{is})|| : i \in m\} \leq e^{||A||T} \alpha^p \max\{||e_i(t_{i(s-1)})|| : i \in m\} \tag{20} \]
Since each event-time interval is of length $T$, to achieve an exponential convergence rate of $\lambda$ in the idealized case, it is necessary to pick $q$ so that (19) holds where $p$ is any integer satisfying $e^{||A||T} \alpha^p < e^{-qT}$. In other words, the requirement on $q$ is that (19) hold where
\[ p > \left[ \frac{(\lambda + ||A||T)}{\ln(\frac{1}{\alpha})} \right], \tag{21} \]
with $\lceil r \rceil$ here denoting, for any nonnegative number $r$, the smallest integer $k \geq r$. The following theorem, which applies to the synchronous case when all of the $\epsilon_i$ are zero, \{but not necessarily the $\bar{\epsilon}_i\} summarizes these observations.

Theorem 1 Synchronous case: Suppose $\epsilon_i = 0, i \in m,$ and that the neighbor graph $N(t)$ is strongly connected for all $t$. Let $\rho$ and $\alpha$ be defined by (17) and (18) respectively. Then each state estimation $e_i = x_i - x, i \in m,$ of the hybrid observer defined by (3), (5) - (8) and (11) - (13), tends to zero as fast as $e^{-\lambda t}$ does provided $q$ satisfies
\[ q > (m-1)^2 + 1 \left[ \frac{1 - ||A||T}{\ln(\frac{1}{\alpha})} \right] \tag{22} \]
This theorem will be proved in the next section. Several comments are in order. First, the attenuation of $\max\{||e_i|| : i \in m\}$ by $\alpha^p$ over an event time interval is not likely to be tight and a larger attenuation constant can almost certainly be expected. This is important because the larger the attenuation constant the smaller the required value of $q$ needed to achieve a given convergence rate. Second, the hypothesis that $N(t)$ strongly connected is almost certainly stronger than is necessary, the notion of a repeatedly jointly strongly connected sequence of graphs [33] being a likely less stringent alternative.

To deal with the asynchronous case when at least some of the $\epsilon_i$ are nonzero, it is necessary to assume that $N(t)$ is constant on each of the time intervals
\[ I_s(k) = [-\epsilon + sT + (k-1)\Delta + \beta, \epsilon + sT + (k-1)\Delta + \beta], k \in q, s \geq 1 \tag{23} \]
where
\[ \epsilon = \max\{|\epsilon_i| : i \in m\} \tag{24} \]
For this assumption to make sense, these intervals cannot overlap. The following lemma establishes that this is in fact the case.

Lemma 1 Suppose that $q \geq 2$ and that the $\epsilon_i$ are fixed nonnegative numbers satisfying the constraints in (10). Then for each $s \geq 0$, the $q$ time intervals defined by (23) are non-overlapping and each is a subinterval of $[sT, (s+1)T].$

Proof of Lemma 1: Fix $2 \leq k \leq q$. Note that $2\epsilon < \beta$ because of (10). This implies that $\epsilon + sT + (k-2)\Delta + \beta < -\epsilon + sT + (k-1)\Delta + \beta$ and thus that $I_s(k)$ and $I_s(k-1)$ are disjoint. Since this holds for all $k$ satisfying $2 \leq k \leq q$, all $I_s(k), k \in q$ are disjoint. From (10), $\epsilon \leq \beta$ and $\epsilon + \beta < \Delta$. These inequalities imply that $-\epsilon + sT + \beta \geq sT$ and $\epsilon + sT + (q-1)\Delta + \beta < (s+1)\Delta$ respectively. From this it follows that $I_s(1) \subset [sT, (s+1)T], \quad$ that $I_s(q) \subset [sT, (s+1)T]$ and thus that $I_s(k) \subset [sT, (s+1)T], k \in q$. We are led to the asynchronous version of Theorem 1.

Theorem 2 Asynchronous case: Suppose the $\epsilon_i, i \in m,$ satisfy (10) and that the neighbor graph $N(t)$ is constant on each interval $I_s(k), k \in q, s \geq 1$ and strongly connected for all $t$. Let $\rho$ and $\alpha$ be defined by (17) and (18)
respectively and suppose that \( q \) satisfies (22). Then as in Theorem 1, each state estimation \( e_i = x_i - x, \ i \in m \), of the hybrid observer defined by (3), (5) - (8) and (11) - (13), tends to zero as fast as \( e^{-\lambda t} \) does.

The proof of this theorem will be given in the next section. Notice that the asynchronous case here can not be recognized in a discrete-time setting with a discrete-time clock shared by all \( m \) agents considering delays [34].

### 2.1 A special case

It is possible to relax somewhat the lower bound (22) for \( q \) to achieve exponential convergence in the special case when the neighbor graph \( N(t) \) is symmetric and strongly connected for all \( t \). This means that \( N(t) \) is a nonnegative number less than one. In other words, in the special case when \( N(t) \) is symmetric, doubly stochastic matrix with positive diagonals and that \( G \) is its graph. The connection between these matrices and the update rule defined by (25) will become apparent later when assumptions are made which enable us to identify the subsets \( S_k \) appearing in (25) with the neighbor sets of the neighbor graph \( N((s-1)T + (k-1)\delta + \beta) \) [c.f. Lemmas 2 and 3]. Later in this paper it will also be shown that \( P(M_G \otimes I) \) is a contraction in the two norm \{Lemma 6\}. This means that

\[
\sigma = \max_{G \in G} ||P(M_G \otimes I)||
\]

is a nonnegative number less than one.

As will become clear, to achieve a convergence rate of \( \lambda \), it is sufficient to pick \( q \) large enough to that \( e^{||A||T} \sigma^q < e^{-\alpha T} \). In other words, in the special case when \( N(t) \) is symmetric and strongly connected for all time, instead of choosing \( q \) to satisfy (22), to achieve an exponential convergence rate of \( \lambda \) it is enough to choose \( q \) to satisfy the less demanding constraint

\[
q > \left( \frac{\lambda + ||A||T}{\ln(\frac{1}{\sigma})} \right)
\]

Justification for this claim is given in §3. Choosing \( q \) in this way is easier that choosing \( q \) according to (22) because the computation of \( \sigma \) is less demanding than the computation of \( \rho \) and consequently \( \alpha \). On the other hand, this special approach only applies when the neighbor graph is symmetric.

### 3 Analysis

The aim of this section is to analyze the behavior of the hybrid observer defined in the last section. To do this, use will be made of the notion of a “mixed matrix norm” which will now be defined. For any positive integers \( k, m, n, p \), let \( M \) denote the real \( k \times m \times n \times p \) dimensional vector space of block partitioned matrices \( M = [M_{ij}]k \times m \) where each block \( M_{ij} \) is a \( n \times p \) matrix. By the mixed matrix norm of \( M \) in \( M \), written \( |M| \), is meant the infinity norm of the matrix \( ||M_{ij}||_{k \times m} \) where \( ||M_{ij}|| \) is the two-norm of \( M_{ij} \). For example, with \( e \) denoting the “stacked” state estimation error \( \Delta = \text{column} \{e_1, e_2, \ldots, e_m\} \) the quantity \( \max \{||e_i|| : i \in m\} \) mentioned in the last section, is \( |e| \), the mixed matrix norm of \( e \). It is straightforward to verify that \( | \cdot | \) is in fact a norm and that this norm is sub-multiplicative [33].

Recall that the purpose of agent \( i \)'s local parameter estimator defined by (7), (12), and (13) is to estimate \( x(t_{i,s-1}) \) after executing \( q \) iterations during the \( s \)th event time interval of agent \( i \). In view of this, we define the parameter error vectors for \( i \in m \),

\[
\pi_i(k) = z_{i,k} - x(t_{i,s-1}) \quad k = 0, 1, \ldots, q
\]

for all \( s \geq 1 \). This, (7), and the definition of \( e \) in (14) imply that

\[
\pi_i(0) = e_i(t_{i,s-1}), \quad s \geq 1, \quad i \in m
\]

In addition, from (5), (6) and (14) it is clear that that

\[
e_i(t_{i,s}) = e_i^n \pi_i(q), \quad s \geq 1, \quad i \in m
\]

To derive the update equation for \( \pi_i(k) \) as \( k \) ranges from 1 to \( q \), we first note from (13) that

\[
\tilde{z}_{i,k} - x(t_{i,s-1}) = \frac{1}{m_{i,k}} \sum_{j \in S_u(k)} \pi_{j,k} \quad (31)
\]

Next note that because of (4) and (12)

\[
\pi_i(k) = \bar{z}_{i,k} - x(t_{i,s-1})
\]

\[
- Q_i(L_i \bar{z}_{i,k} - x(t_{i,s-1}) - e_i(t_{i,s-1}))
\]

From this and (31) it follows that for \( k \in q, \ i \in m, \ s \geq 1 \),

\[
\pi_i(k) = \frac{1}{m_{i,k}} P_i \sum_{j \in S_u(k)} \pi_{j,k} (k-1) + Q_i \bar{e}_i(t_{i,s-1}) \quad (32)
\]
where as before, $P_t = I - Q_tL_t$. These are the local parameter error equations for the hybrid observer.

The next step in the analysis of the system is studied the evolution of the all-agent parameter error vector

$$\pi_s(k) = \text{column}\{\pi_{is}(k), \pi_{2s}(k), \ldots, \pi_{ms}(k)\}$$

Note first that because of (29) and (30)

$$\pi_s(0) = e(s-1), \ s \geq 1$$

and

$$e(s) = e^{At} \pi_s(q), \ s \geq 1$$

where $e(s)$ is the all-agent state estimation error vector

$$e(s) = \text{column}\{e_1(t_{is}), \ldots, e_m(t_{ms})\}, \ s \geq 0$$

and $A = \text{diagonal}\{A, A, \ldots, A\}$.

In order to develop an update equation for $\pi_s(k)$ as $k$ ranges from 1 to $q$, it is necessary to combine the $m$ update equations in (32) into a single equation and to do this requires a succinct description of the graph determined by the sets $S_{is}, i \in m$ defined in (11). There are two cases to consider: the synchronous case which is when all of the $\epsilon_i = 0$ and the asynchronous case when some or all of the $\epsilon_i$ may be non-zero. The following lemmas cover both cases.

**Lemma 2 Synchronous Case:** Suppose $\epsilon_i = 0, i \in m$. Then for any fixed value of $\beta$ satisfying (10), including $\beta = 0$,

$$S_{is}(k) = \mathcal{N}_i((s-1)T + (k-1)\Delta + \beta), i \in m, k \in q$$

**Proof of Lemma 2:** By hypothesis all $\epsilon_i = 0$. Clearly (10) can be satisfied with $\beta = 0$. Moreover from (8) and (9) and the assumption that $t_i(s-1) = (s-1)T$, it follows that $\mathcal{N}_i((s-1)T + (k-1)\Delta + \beta) = \mathcal{N}_i((s-1)T + (k-1)\Delta + \beta)$. From this and (11) it follows that (36) is true.

The following lemma asserts that (36) still holds in the asynchronous case when some of the $\epsilon_i$ are nonzero, provided $N(t)$ is constant on each interval $I_s(k), k \in q, s \geq 1$.

**Lemma 3 If the $\epsilon_i$ satisfy the constraints in (10) and $N(t)$ is constant on each interval $I_s(k), k \in q, s \geq 1$ then (36) is true.**

**Proof of Lemma 3:** Fix $i \in m, s \geq 1$ and $k \in q$. In light of (8), (9) and the assumption that $t_i(s-1) = (s-1)T$, it is clear that for any $j \in m$,

$$\tau_{js}(k-1) + \beta \in [-\epsilon_j + (s-1)T + (k-1)\Delta + \beta, \epsilon_j + (s-1)T + (k-1)\Delta + \beta] \subset I_s(k)$$

Moreover, $(s-1)T + (k-1)\Delta + \beta \in I_s(k)$. But by assumption, $N(t)$ is constant on $I_s(k)$ which means that $\mathcal{N}_i((s-1)T + (k-1)\Delta + \beta)$ from this and the definition of $S_{is}(k)$ in (11), it follows that (36) is true.

In summary, Lemmas 2 and 3 assert that (36) holds in the synchronous case when all $\epsilon_i = 0$, or alternatively in the asynchronous case when the neighbor graph $N(t)$ is constant on each interval $I_s(k), k \in q, s \geq 1$. Because of this, the following steps to obtain an update equation for $e(s)$ apply to both cases.

Equation (36) implies that the graphs determined by the $S_{is}(k), k \in q, i \in m, s \geq 1$ are the neighbor graphs $N((s-1)T + (k-1)\Delta + \beta), k \in q, s \geq 1$. Since $N(t)$ is assumed to be strongly connected for all $t \geq 0$, each of these neighbor graphs is strongly connected. These graphs are used as follows.

Let $G$ denote the set of all directed graphs on $m$ vertices which have self-arcs at all vertices. Note that $G$ is a finite set and that $N(t) \in G, t \geq 0$. Each graph $G \in G$ uniquely determines a so-called “flocking-matrix” which is an $m \times m$ stochastic matrix of the form $D G^{1} A G$, where $A G$ and $D G$ are respectively the adjacency matrix and diagonal in-degree matrix of of $G$; $D G$ is nonsingular because each graph in $G$ have self-arcs at all vertices.

For $k \in q$ and $s \geq 1$, let $F_s(k)$ denote the flocking matrix determined by $N((s-1)T + (k-1)\Delta + \beta)$. Then (32) implies for $s \geq 1$ that

$$\pi_s(k) = P(F_s(k) \otimes I)\pi_s(k-1) + Q\tilde{e}(s-1), k \in q$$

where

$$\tilde{e}(s) = \text{column}\{\tilde{e}_1(t_{is}), \ldots, \tilde{e}_m(t_{ms})\}, s \geq 0$$

$$P = \text{diagonal}\{P_1, \ldots, P_m\}, Q = \text{diagonal}\{Q_1, \ldots, Q_m\},$$

and $I$ is the $n \times n$ identity. Thus for $s \geq 1$,

$$\pi_s(q) = \Phi_s(0)\pi_s(0) + \left(\sum_{k=1}^{q} \Phi_s(k)\right) Q\tilde{e}(s-1)$$

where $\Phi_s(k)$ is the state transition matrix defined by

$$\Phi_s(k) = P(F_s(q) \otimes I) \cdots P(F_s(k+1) \otimes I)$$

for $0 \leq k < q$ and by $\Phi_s(q) = I$ for $k = q$. From this, (33) and (34) it follows that for $s \geq 1$, the all-agent state estimation error $e(s)$ satisfies

$$e(s) = A(s)e(s-1) + B(s)\tilde{e}(s-1), s \geq 1$$
where
\[ A(s) = e^{AT} \Phi_s(0), \quad s \geq 1 \]  
(42)
\[ B(s) = e^{AT} \sum_{k=1}^{q} \Phi_s(k)Q, \quad s \geq 1 \]  
(43)

To determine the convergence properties of \( e(s) \) as \( s \to \infty \) use will be made of the following lemma which gives bounds on the norms of the coefficient matrices \( A(s) \) and \( B(s) \) appearing in (41).

**Lemma 4** Suppose that \( q \) satisfies the inequality given in Theorem 1. Then
\[ |A(s)| \leq e^{-\lambda sT}, \quad s \geq 1 \]  
(44)
\[ |B(s)| \leq q e^{\mu sT} |Q|, \quad s \geq 1 \]  
(45)

In order to justify the bound on the norm of \( A(s) \) given in (44), use will be made of the following lemma, which is a simple variation on a result in [33].

**Lemma 5** Let \( F \) denote the set of all flocking matrices determined by those graphs in \( G \) which are strongly connected. For any set of \( \mu \geq (m-1)^2 + 1 \) flocking matrices \( S_1, S_2, \ldots, S_n \) in \( F \)
\[ |P(S_n) P(S_{n-1}) \cdots P(S_1)| \leq \alpha \]  
(46)
where \( \alpha \) is the attenuation constant
\[ \alpha = 1 - \frac{(m-1)(1-\rho)}{m(m-1)^2} \]

**Proof of Lemma 5:** Fix \( \mu \geq (m-1)^2 + 1 \), set \( k = (m-1)^2 \) and let and \( S_1, S_2, \ldots, S_n \) be flocking matrices in \( F \).

Then
\[ P(S_n) P(S_{n-1}) \cdots P(S_1) \]
\[ = \{P(S_n) \cdots P(S_{k+1})\} \{S_k \cdots S_1\} \]

But for any flocking matrix \( S \in F, |S| = ||S||_\infty = 1 \) where \( || \cdot ||_\infty \) is the infinity norm. From this, the submultiplicative property of the mixed matrix norm, and the fact that \( |P| \leq 1 \), it follows that
\[ |P(S_n) P(S_{n-1}) \cdots P(S_1)| \leq |P(S_k) \cdots P(S_1)| \]

In view of equation (26) of [33],
\[ |P(S_k) \cdots P(S_1)| \leq 1 - \frac{(m-1)(1-\rho)}{m(m-1)^2} \]

Therefore (46) is true. ■

**Proof of Lemma 4:** Lemma 5 implies that if for a given integer \( p > 0 \), if \( q \geq p(m-1)^2 + 1 \) then for any \( s \geq 1 \),
\[ |(P(F_s(q) \cdots P(F_s(1) \cdots I)| \leq \alpha^p \]  
(47)
Therefore by (40) and (42), if \( q \) is so chosen, then \( |A(s)| \leq e^{||A||T} \alpha^p \). Thus by picking \( p \) so large that
\[ e^{||A||T} \alpha^p < e^{-\lambda sT} \]  
(48)
and then setting \( q = p(m-1)^2 + 1 \) one gets (44). The requirement on \( p \) determined by (48) is equivalent to the requirement on \( p \) determined by (21). It follows that (44) will hold provided \( q \) satisfies the inequality given in Theorem 1.

Recall that \( |P| \leq 1 \) and that \( |S \otimes I| = 1 \) for any \( m \times m \) stochastic matrix \( S \). From this and the sub-multiplicative property of the mixed matrix norm it follows that the matrix \( \Phi_s(k) \) defined by (40) satisfies
\[ |\Phi_s(k)| \leq 1, \quad k \in q, \quad s \geq 1 \]  
(49)
This and the definition of \( B(s) \) in (43) imply that for all \( s \geq 1 \), \( |B(s)| \leq q e^{\mu sT} |Q| \). Thus (45) is true. ■

It is obvious at this point that because (36) holds in both the synchronous and asynchronous cases, the same arguments can be used to prove both Theorem 1 and Theorem 2.

**Proof of Theorems 1 and 2:** In view of (41) and Lemma 4 it is possible to write
\[ |e(s)| < e^{-\lambda sT} |e(s-1)| + b |e(s-1)|, \quad s \geq 1 \]
where \( b = q e^{\mu sT} |Q| \). Therefore
\[ |e(s)| < e^{-\lambda sT} |e(0)| + b \sum_{k=1}^{s} e^{-\lambda(s-k)T} |\tilde{e}(k-1)| \]  
(50)
To deal with the term involving \( \tilde{e} \) in (50), we proceed as follows. Note first from (15) that
\[ ||\tilde{e}_i(t_{is})|| \leq c_i e^{-\lambda_T T} ||\tilde{e}_i(t_{i(s-1)})||, \quad i \in m, \quad s \geq 1 \]
Thus \( ||\tilde{e}_i(t_{is})|| \leq c_i e^{-\lambda T} ||\tilde{e}_i(t_{i0})|| \) for \( i \in m \ s \geq 1 \). It follows from this and the definition of \( e(s) \) in (38) that
\[ |\tilde{e}(s)| \leq c e^{-\lambda T} |\tilde{e}(0)|, \quad s \geq 1 \]
where \( c = \max\{c_i, \ i \in m\} \). Thus for \( s \geq 1 \)
\[
\sum_{k=1}^{s} e^{-\lambda(s-k)T} |\bar{e}(k-1)|
\leq cb \sum_{k=1}^{s} e^{-\lambda(s-k)T} e^{-\bar{\lambda}(k-1)T} |\bar{e}(0)|
= cb e^{-(\lambda s - \bar{\lambda}) T} \sum_{k=1}^{s} e^{-(\bar{\lambda} - \lambda) k T} |\bar{e}(0)|
\leq cb e^{-(\lambda s - \bar{\lambda}) T} \sum_{k=1}^{\infty} e^{-(\bar{\lambda} - \lambda) k T} |\bar{e}(0)|
= cb e^{-(\lambda s - \bar{\lambda}) T} \frac{e^{-(\bar{\lambda} - \lambda) s T}}{1 - e^{-(\bar{\lambda} - \lambda) T}} |\bar{e}(0)|
= cb e^{-\lambda T} \frac{e^{\lambda T}}{1 - e^{-(\bar{\lambda} - \lambda) T}} |\bar{e}(0)|
\]
Using (50) there follows
\[
|e(s)| \leq e^{-\lambda s T} (|e(0)| + d|\bar{e}(0)|), \quad s \geq 1 \tag{51}
\]
where
\[
d = cb \frac{e^{\lambda T}}{1 - e^{-(\bar{\lambda} - \lambda) T}}
\]
Fix \( i \in m \). In view of (51) and the definition of \( e(s) \) in (35),
\[
||e_i(t_{i(s, 1)})|| \leq e^{-\lambda(s-1) T} (|e(0)| + d|\bar{e}(0)|), \quad i \in m, \quad s \geq 1
\]
But for \( t \in (t_{i(s-1)}, t_{is}) \), \( \dot{x}_i = A x_i \); consequently \( \dot{\bar{e}}_i = A \bar{e}_i \) for the same values of \( t \). Therefore
\[
e_i(t) = e^{A(t-(s-1)T)} \bar{e}_i(t_{i(s-1)}), \quad t \in [t_{i(s-1)}, t_{is}), \quad s \geq 1
\]
so
\[
||e_i(t)|| \leq e^{||A|| T} ||e_i(t_{i(s-1)})||, \quad t \in [t_{i(s-1)}, t_{is}), \quad s \geq 1
\]
Therefore for \( t \in [t_{i(s-1)}, t_{is}) \) and \( s \geq 1 \)
\[
||e_i(t)|| \leq e^{(||A|| T - \lambda(s-1) T)} (|e(0)| + d|\bar{e}(0)|)
\]
Now for \( i \in m \),
\[
e^{-\lambda s T} \leq e^{-\lambda t}, \quad t \in [t_{i(s-1)}, t_{is})
\]
so
\[
||e_i(t)|| \leq e^{(||A|| T - \lambda t)} (|e(0)| + d|\bar{e}(0)|), \quad t \in [t_{i(s-1)}, t_{is})
\]
Since this holds for all \( s \geq 1 \)
\[
||e_i(t)|| \leq e^{(||A|| T - \lambda t)} (|e(0)| + d|\bar{e}(0)|), \quad t \geq 0
\]
which proves that the state estimation errors \( e_i, \ i \in m \), all converge to zero as fast as \( e^{-\lambda t} \) does. \( \blacksquare \)

### 3.1 Special case

We now turn to the special case mentioned in §2.1. In this case the definition of the state-transition matrix \( \Phi \) appearing in (39) changes from (40) to
\[
\Phi_s(k) = P(W_s(q) \otimes I) \cdots P(W_s(k+1) \otimes I) \tag{52}
\]
for \( 0 \leq k < q, \) and \( W_s(k) \triangleq M_{\mathbb{N}((s-1)T+(k-1)\Delta+\beta)} \) with graph \( \mathbb{N}((s-1)T+(k-1)\Delta+\beta) \).

Although the formula for \( e(s) \), namely (41), and the definitions of \( A(s) \) and \( B(s) \) in (42) and (43) are as before, the bounds for \( A(s) \) and \( B(s) \) given by (44) and (45) no longer apply. To proceed, use will be made of the following lemma.

**Lemma 6** Let \( F \) be an \( m \times m \) doubly stochastic matrix with positive diagonals and a strongly connected graph. Suppose that \( P_i, \ i \in m, \) is a set of \( n \times n \) orthogonal projection matrices such that
\[
\prod_{i=1}^{m} \text{image } P_i = 0 \tag{53}
\]
then the matrix \( P(F \otimes I) \) is a contraction in the 2-norm where \( P = \text{diag} \{ P_2, P_3, \ldots, P_m \} \).

**Proof:** Write \( S \) for \( F \otimes I \) and note that \( S \) is doubly stochastic with positive diagonals and a strongly connected graph. Since \( ||P|| \leq 1 \), it must be true that that \( ||PS|| \leq ||S|| \). Moreover \( ||S|| \leq 1 \) because \( S \) is stochastic; thus \( ||PS|| \leq 1 \). Hence it is enough to prove that \( ||PS|| \neq 1 \) or equivalently that \( ||S'P|| \neq 1 \)

Suppose that \( ||S'P|| = 1 \) or equivalently that \( PSS'Px = x \) for some nonzero vector \( x \). Clearly \( PSS'P \) is \( P \) which implies that \( Px = x \) and thus that \( x'SS'x = x'x \). Therefore \( ||S'x|| = ||x|| \). From this and Lemma 1 of [35] it follows that \( SS'x = x \). Now \( SS' \) is stochastic. Moreover its graph is strongly connected because \( S \) has a strongly connected graph and positive diagonals, as does \( S' \). Thus by the Perron Frobenius theorem, \( SS' \) has exactly one eigenvalue at 1 and all the rest must be inside the unit circle; in addition the eigenspace for the eigenvalue 1 must be spanned by the one-vector \( 1_{nm} \). Therefore \( x = \mu 1_{nm} \) for some nonzero scalar \( \mu \). Therefore \( P1_{nm} = 1_{nm} \) which implies that \( 1_n = P_i 1_n, \ i \in m \). But this is impossible because of (3). \( \blacksquare \)

The following lemma gives the bounds on \( A(s) \) and \( B(s) \) for the special case under consideration.

**Lemma 7** Suppose that \( q \) satisfies (27). Then
\[
||A(s)|| \leq e^{-\lambda T}, \quad s \geq 1 \tag{54}
\]
\[
||B(s)|| \leq q e^{||A|| T} ||Q||, \quad s \geq 1 \tag{55}
\]
**Proof:** Lemma 6 implies that for each \( 0 \leq k < q, \)
\[
||P(W_s(k) \otimes I)|| < 1. \quad \text{Moreover, } ||P(W_s(k) \otimes I)|| \leq \sigma
\]

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where \( \sigma \) is chosen according to (26). From this and the sub-multiplicative property of the two norm it follows that

\[
\|P(W_s(q) \otimes I) \cdots P(W_s(1) \otimes I)\| \leq \sigma^q
\]

Therefore by (42) and (52), if \( q \) is so chosen to satisfy (27), then

\[
\|A(s)\| \leq e^{\|A\|T} \sigma^q < e^{-\lambda T}
\]

Thus (54) is true. Recall that \( \|P(W_s(k) \otimes I)\| < 1 \) and \( \|W_s(k) \otimes I\| \leq 1 \) for \( 0 \leq k < q \). From this and the sub-multiplicative property of the two norm, the matrix \( \Phi_s(k) \) defined by (52) satisfies

\[
\|\Phi_s(k)\| \leq 1, \quad k \in q, \quad s \geq 1
\]

This and (43) imply (55). ■

Other than the modifications in the bounds on \( A(s) \) and \( B(s) \) given in the above lemma, everything else is the same for both the synchronous and asynchronous versions of the problem. So what one gains in this special case is exponential convergence at a prescribed rate with a smaller value of \( q \).

4 Event-time Mismatch - A Robustness Issue

In the preceding section it was shown that the hybrid observer under discussion will function correctly if local iterations are performed synchronously across the network no matter how fast the associated neighbor graph changes, just so long as it is always strongly connected. Correct performance is also assured in the face of asynchronously executed local iterations across the network during each event time interval, provided the neighbor graph changes in a suitably defined sense. Implicitly assumed in these two cases is that the event time sequences of all \( m \) agents are the same. The aim of this section is to explain what happens if this assumption is not made. For simplicity, this will only be done for the case when differing event time sequences are the only cause of asynchronism. As will be seen, the consequence of event-time sequence mismatches turns out to be more of a robustness issue than an issue due to unsynchronized operation. In particular, it will become apparent that if different agents use slightly different event time sequences then asymptotically correct state estimates will not be possible unless \( A \) is a stability matrix. While at first glance this may appear to be a limitation of the distributed observer under consideration, it is in fact a limitation of virtually all state estimators, distributed or not, which are not used in feedback-loops. Since this easily explained observation is apparently not widely appreciated, an explanation of this simple fact will be given at the end of this section.

There are two differences between the setup to be considered here and the setup considered in the last section. First it will now be assumed that the local deviation times \( \delta_{is}(k) \) appearing in (8) are all zero. Thus in place of (8) the local iteration times for agent \( i \) on \([t_{i(s-1)}, t_{is})\)

\[
\tau_{is}(k) = t_{i(s-1)} + k\Delta, \quad k \in \{0, 1, \ldots, m\}
\]

Second instead of assuming that the initializations \( t_{i0} \). of the \( m \) agents’ event time sequences are all zero, it will be assumed instead that each \( t_{i0} \) is a small number known only to agent \( i \) which lies in the interval \([-\epsilon_i, \epsilon_i]\) where, as before, \( \epsilon_i \) is a small nonnegative number. This means that even though the event time sequences of all \( m \) agents are still periodic with period \( T \), the sequences are not synchronized with each other. As before it is assumed that within event time interval \([t_{j(s-1)}, t_{js})\), agent \( j \) broadcasts iterate \( z_{js}(k-1) \) at time \( \tau_{js}(k-1) + \beta \). To ensure that this time falls within the reception interval \([t_{i(s-1)}, t_{is})\) of each agent \( i \), it will continue to be assumed that (10) holds. Apart from these modifications the setup to be considered here is the same as the one considered previously. As a consequence, many of the steps in the analysis of the hybrid observers performance are the same as they were for the previously considered case.

Our first objective is to develop the relevant equations for the local parameter error vector \( \pi_{is}(k) \) defined by (28). Although (29) and (30) continue to hold without change, (32) requires modification. To understand what needs to be changed, it is necessary to first derive a relationship between \( x(t_{i(s-1)}) \) and \( x(t_{j(s-1)}) \). Towards this end, note that

\[
x(t_{i(s-1)}) = x(t_{j(s-1)}) + x((s-1)T + t_{i0}) - x((s-1)T + t_{j0})
\]

because \( t_{k(s-1)} = t_{k0} + (s-1)T \) for all \( k \in m \). From this and (5) it follows that

\[
x(t_{i(s-1)}) = x(t_{j(s-1)}) + (e^{At_{i0}} - e^{At_{j0}}) x((s-1)T)
\]

Hence (13) can now be used to obtain

\[
\bar{z}_{is}(k-1) - x(t_{i(s-1)}) = \frac{1}{m_{is}(k)} \sum_{j \in S_{is}(k)} \pi_{j(s-1)}(k-1) + \Gamma_{is}(k)x((s-1)T)
\]

where

\[
\Gamma_{is}(k) = \frac{1}{m_{is}(k)} \sum_{j \in S_{is}(k)} (e^{At_{i0}} - e^{At_{j0}})
\]

Next note that because of (4) and (12)

\[
\pi_{is}(k) = \bar{z}_{is}(k-1) - x(t_{i(s-1)}) - Q_i(L_i \bar{z}_{is}(k-1) - x(t_{i(s-1)}) - \bar{e}_i(t_{i(s-1)}))
\]
From this and (57) it follows that
\[
\pi_\sigma(k) = \frac{1}{m_{\sigma}(s)} P_i \sum_{j \in \sigma_i(k)} \pi_{jk}(k-1) + Q_i \bar{e}_i(t_{i(s-1)})
+ P_i \Gamma_{is}(k)x((s-1)T), \ k \in q, \ i \in m, \ s \geq 1 \quad (59)
\]
which is the modified version of (32) needed to proceed. The difference between (32) and (59) is thus the inclusion in (59) of the term \(P_i \Gamma_{is}(k)x((s-1)T)\).

The assumption that the event time sequences of the agents may start at a different time requires us to make the same assumption as before about the neighbor graph \(N(t)\), namely that it is constant on each interval \(I_n(k), \ k \in q, \ s \geq 1\). The assumption makes sense in the present context for the same reason as before, specifically because the interval \(I(k)\) defined by (23) do not overlap. This, in turn, is because the bounds \(\epsilon_i\) have been assumed to satisfy (10) which guarantees that Lemma 1 continues to hold.

The next step in the analysis of the hybrid observer is to study the evolution of the all-agent parameter error vector
\[
\pi_\sigma(k) = \text{column}\{\pi_{1\sigma}(k), \pi_{2\sigma}(k), \ldots, \pi_{m\sigma}(k)\}
\]
As before, (33) and (34) continue to hold where \(e(s)\) is the all-agent state estimation error defined by (35). A simple modification of the proof of Lemma 3 can be used to establish the lemma’s validity in the present context. Consequently a proof will not be given. The lemma enables us to combine the individual update equations in (59), thereby obtaining the update equation
\[
\pi_\sigma(k) = P(F_\sigma(k) \otimes I)\pi_\sigma(k-1) + Q\bar{e}(s-1)
+ P\Gamma_\sigma(k)x((s-1)T), \ k \in q \quad (60)
\]
where
\[
\Gamma_\sigma(k) = \text{column}\{\Gamma_{1\sigma}(k), \ldots, \Gamma_{m\sigma}(k)\}
\]
The steps involved in doing this are essentially the same as the steps involved in deriving (37). Not surprisingly, the only difference between (37) and (60) is the inclusion in the latter of the term \(P\Gamma_\sigma(k)x((s-1)T)\).

From (33), (34), and (60) it follows at once that the all-agent state estimation error vector satisfies
\[
e(s) = A(s)e(s-1) + B(s)\bar{e}(s-1)
+ G(s)x((s-1)T), \ s \geq 1 \quad (61)
\]
where \(A(s)\) and \(B(s)\) are as defined in (42) and (43) respectively, and
\[
G(s) = e^{AT} \sum_{k=1}^{q} \Phi_s(k)P\Gamma_s(k)
\]
The following lemma gives a bound on the mixed matrix norm of \(G(s)\).

**Lemma 8** Suppose that \(q\) satisfies the inequality given in Theorem 1. Then
\[
|G(s)| \leq 2mq\epsilon||A||e||A||(|T+\beta) \quad (62)
\]
Note that this bound is small when \(\epsilon\) is small. This means that small deviations of the agent’s event time sequences from the nominal event time sequence \(0, T, 2T, \ldots\) produce small effects on the error dynamics in (61), provided of course \(x\) is well behaved; i.e., \(A\) is a stability matrix! More will be said about this point below.

**Proof of Lemma 8:** From (58),
\[
||\Gamma_{is}(k)|| \leq \sum_{j \in m} ||e^{At_0} - e^{At_0}||
\]
In general, for any real square matrix \(M\), and real numbers \(t\) and \(\tau\)
\[
||e^{Mt} - e^{Mt_0}|| \leq ||M(t - \tau)||e||M||
\]
so
\[
||\Gamma_{is}(k)|| \leq \sum_{j \in m} ||A(t_0 - t_0)||e||A||T
\]
By assumption \(|t_0| \leq \epsilon_i\) and \(|t_0 - t_0| \leq \epsilon_i + \epsilon_j\). But \(\epsilon_i \leq \beta\) and \(\epsilon = \max\{\epsilon_i, i \in m\}\). Thus \(|t_0| \leq \beta\) and \(|t_0 - t_0| \leq 2\epsilon\). Therefore
\[
||\Gamma_{is}(k)|| \leq 2m\epsilon||A||e||A|||\beta
\]
In view of (49) and the definition of \(G(s)\), \(|G(s)| \leq q\epsilon||A||T||\Gamma_s(k)||\). It follows that (62) holds. \(\blacksquare\)

Taking the construction leading to (50) as a guide, it is not difficult to derive from (61) the inequality
\[
|e(s)| \leq e^{-\lambda s T}(|e(0)| + d|e(0)|)
+ \epsilon g \sum_{k=1}^{s} e^{-\lambda(s-k) T}||x((k-1)T)||, \ s \geq 1 \quad (63)
\]
where \(d\) is as defined just below (51) and \(g = 2mq||A||e||A||(|T+\beta)\). Comparing (51) to (63), we see that the effect of the change in assumptions leads to the inclusion in (63) of the term involving \(x\).

At this point there are two distinct cases to consider - either \(e^{At}\) converges to zero or it does not. Consider first
the case when $e^{At}$ converges to zero. Then there must be positive constants $c_a$ and $\lambda_a$ such that $||e^{At}|| \leq c_a e^{-\lambda_a t}$.

By treating the term involving $x$ in (63) in the same manner as the term involving $\bar{e}$ in (51) was treated, one can easily conclude that for a suitably defined constant $h$

$$|e(s)| \leq e^{-\lambda s T}(|e(0)| + d|\bar{e}(0)| + c h ||x(0)||), \ s \geq 1$$

if $\lambda_a > \lambda$, or

$$|e(s)| \leq e^{-\lambda s T}(|e(0)| + d|\bar{e}(0)| + c e^{-\lambda a s T}||x(0)||), \ s \geq 1$$

if $\lambda_a \leq \lambda$. If the former is true, then the same arguments as were used in the last section can be used to show that the state estimations errors $e_i(t)$ converge to zero as fast as $e^{-Mt}$ does. On the other hand, if the latter is true, by similar reasoning $e_i(t)$ can easily be shown to converge to zero as fast as $e^{-\lambda a T}$ does. Note that in this case, if $\lambda_a$ is small, the effect of the resulting slow convergence of $x$ will to some extent be mitigated by the smallness of $\epsilon$, so even with small $\lambda_a$, the performance of the hybrid observer may be acceptable for sufficiently small perturbations of the start times of the event time sequences from 0.

In the other situation, which is when $A$ is not a stability matrix, the hybrid observer cannot perform acceptably except possibly if finite time state estimation is all that is desired and $\epsilon$ is sufficiently small.

**Key Point:** This limitation applies not only to the hybrid observer discussed in this paper, but to all state estimators, centralized or not, including Kalman filters which are not being used in feedback loops.

Experience has shown that this limitation is not widely recognized, despite its simple justification. Here is the justification.

Suppose one is trying to obtain an estimate $\hat{x}$ of the state $x$ of a single-channel, observable linear system $y = Cx$, $\hat{x} = Ax$ using an observer but approximately correct values of $A$ and $C$ - say $\tilde{A}$ and $\tilde{C}$ - upon which to base the observer design are known. The observer would then be a linear system of the form

$$\hat{x} = \tilde{A}\hat{x} + K(\tilde{C}\hat{x} - y)$$

(64)

with $K$ chosen to exponentially stabilize $\tilde{A} + K\tilde{C}$. Then it is easy to see that the state estimation error $e = \hat{x} - x$ must satisfy

$$\dot{e} = (\tilde{A} + K\tilde{C})e + (\tilde{A} - A + K(\tilde{C} - C))x$$

Therefore if $A$ is not a stability matrix and either $\tilde{A}$ is not exactly equal to $A$ or $\tilde{C}$ is not exactly equal to $C$, then instead of converging to zero, the state estimation error $e$ will grow without bound for almost any initialization. In other words, with robustness in mind, the problem of trying to obtain an estimate of the state of a linear system with an “open-loop” state estimator, does not make sense unless $A$ is a stability matrix. Of course, if one is trying to use a state estimator generate an estimate $\hat{x}$ of the state $x$ of the forced linear system

$$\dot{x} = Ax + BF\hat{x}$$

where $A + BF$ is a stability matrix, this problem does not arise, but to accomplish this one has to change the estimator dynamics defined in (64) to

$$\dot{\hat{x}} = \tilde{A}\hat{x} + K(\tilde{C}\hat{x} - y) + BF\hat{x}$$

While this modification works in the centralized case, it cannot be used in the decentralized case as explained in [36]. In fact, until recently there appeared to be only one of distributed observer which could be used in a feedback configuration thereby avoiding the robustness issue just mentioned [36]. However, recent research suggests other approach may emerge [37].

5 Resilience

By a (passively) resilient algorithm for a distributed process is meant an algorithm which, by exploiting built-in network and data redundancies, is able to continue to function correctly in the face of abrupt changes in the number of vertices and arcs in the inter-agent communication graph upon which the algorithm depends. In this section, it will be shown that the proposed estimator can cope with the situation when there is an arbitrary abrupt change in the topology of the neighbor graph such as the loss or addition of an arc or a vertex provided connectivity is not lost in an appropriately defined sense.

Consider first the situation when there is a potential loss or addition of $a$ arcs in the neighbor graph. Assume the neighbor graph is $\bar{a}$-arc redundantly strongly connected in that the graph is strongly connected and remains strongly connected after any $a \leq \bar{a}$ arcs are removed. With this assumption, strong connectivity of the neighbor graph and joint observability of the system are ensured when any $a \leq \bar{a}$ arcs are lost. Alternatively, if any number of new arcs are added, strong connectivity and joint observability are clearly still ensured. Thus, in the light of Theorem 1, whenever $a \leq \bar{a}$ arcs are lost from or added to the neighbor graph, the hybrid estimator under consideration will still function correctly without the need for any “active” intervention such as redesign of any of the $K_i$ or readjustment of $q$. In fact, Theorem 1 guarantees that correct performance will prevail, even if arcs change over and over, no matter how fast, just so long as strong connectivity is maintained for all time.

Consider next the far more challenging situation when at some time $t^*$ there is a loss of $v < m$ vertices from the
neighbor graph $\mathbb{N}(t)$. For this situation, only preliminary results currently exist. One possible way to deal with this situation is as follows.

As a first step, pick the $K_i$ as before, so that all $m$ local observer state estimator errors converge to zero as fast as $e^{-\lambda t}$ does. Next, assume that the neighbor graph is $\bar{v} < m$-vertex redundantly strongly connected in that it is strongly connected and remains strongly connected after any $v \leq \bar{v}$ vertices are removed. Assume in addition that the system described by (1), (2) is $\bar{v}$ redundantly jointly observable in that the system which results after any $v \leq \bar{v}$ output measurements $y_i$ have been deleted, is still jointly observable. Let $\mathcal{D}$ denote the family of all nonempty subsets $d \subset \mathbb{N}$ such that each subset $d \in \mathcal{D}$ contains at least $m - \bar{v}$ vertices. Thus each loss of at most $\bar{v}$ vertices results in a strongly connected subgraph of $\mathbb{N}(t)$ for some subset $d \in \mathcal{D}$; call this subgraph $\mathbb{N}_d(t)$. Correspondingly, let $\Sigma_d$ denote the multi-channel linear system which results when those outputs $y_i, i \notin d$ are deleted from (1), (2). Thus $\Sigma_d$ is a jointly observable multi-channel linear system whose channel outputs are the $y_i, i \in d$. Fix $\lambda > 0$.

Fix $d \in \mathcal{D}$ and let $m_d$ denote the number of vertices in $\mathbb{N}_d$. Since $\Sigma_d$ is jointly observable it is possible to compute a number $q_d$ which satisfies (17). Using the pair $(\rho_d, m_d)$ in place of the pair $(\rho, m)$ in (18) and (22), it is possible to calculate a value of $q$, for which (22) holds. In other words, for this value of $q$, henceforth labelled $q_d$ Theorem 1 holds for the multichannel system $\Sigma_d$ and neighbor graph $\mathbb{N}_d(t)$. By then picking

$$q^* = \max_{d \in \mathcal{D}} q_d$$

one obtains a value of $q$ for which Theorem 1 holds for all pairs $(\Sigma_d, \mathbb{N}_d(t))$ as $d$ ranges over $\mathcal{D}$. Suppose a hybrid observer using $q = q^*$ is implemented. Suppose in addition that at some time $t^*$, for some specific $d \in \mathcal{D}$, agents with labels in $m - d$ stop functioning. Clearly the remaining agents with labels in $d$ will be able to deliver the desired state estimates with the prescribed convergence rate bounds. In this sense, the observer under consideration is resilient to vertex losses. However, unlike the loss or addition of edges mentioned above, no claim is being made at this point about what might happen if some or all of the lost vertices rejoin the network, especially if this loss-gain process is rapidly reoccurring over and over as time evolves.

A similar approach can be used to deal with the situation when at some time $t^*$, the network gains some additional agents. In this case one would have to specify all possibilities and make sure that for each one, one has a strongly connected graph and a jointly observable system.

A little thought reveals that what makes it possible to deal with a change in the number of vertices in this way, is the fact that there is a single scalar quantity, namely $q$, with the property that for each possible graphical configuration resulting from an anticipated gain or loss of vertices, there is a value of $q$ large enough for the distributed observer to perform correctly and moreover if $q$ is assigned the maximum of these values then the distribute observer will perform correctly no matter which of the anticipated vertex changes is actually encountered. Since the distributed observers described in [23, 24, 26] also require the adjustment of only a single scalar-valued quantity for a given neighbor graphs, the same basic idea just described can be used to make the observers in [23, 24, 26] resilient to a one-time gain or loss of the number of vertices on their associated neighbor graph. On the other hand, some distributed observers such as the ones described in [2, 15, 38] are not really amenable to this kind of generalization because for such observers changes in network topology require completely new designs involving the change of many of the observer’s parameters. There are also papers [39, 40] deal with sensor attacks, where a malicious attacker can manipulate their observations arbitrarily when each sensor only has one dimensional measurement.

6 Simulation

The following simulations are intended to illustrate (i) the performance of the hybrid observer in the face of system noise, (ii) the robustness of the hybrid observer with respect to variations of event time sequences, and (iii) resilience of the hybrid observer to the loss or gain of an agent. Consider the four channel, four-dimensional, continuous-time system described by the equations $\dot{x} = Ax, y_i = C_i x, i \in \{1, 2, 3, 4\}$, where

$$A = \begin{bmatrix} -0.1 & 0.4 & 0 & 0 \\ -0.1 & -0.1 & 0 & 0 \\ 0 & 0 & -0.2 & 0.2 \\ 0 & 0 & -2 & 0.1 \end{bmatrix}$$

and $C_i$ is the $i$th unit row vector in $\mathbb{R}^{1 \times 4}$. Note that $A$ is a stable matrix with two eigenvalues at $-0.1 \pm j0.2$ and a pair of complex eigenvalues at $-0.05 \pm j0.6144$. While the system is jointly observable, no single pair $(C_i, A)$ is observable. However the system is “redundantly jointly observable” in that what remains after the removal of any one output $y_i$, is still jointly observable. For the first two simulations $\mathbb{N}(t)$ is switching back and forth between Figure 2a and Figure 2b, and for the third simulation the neighbor graph is as shown in Figure 2a. Both graphs are strongly connected, and the graph in Figure 2a is redundantly strongly connected in that it is strongly connected and remains strongly connected after any one vertex is removed. Suppose $T = 1$ for this system. To achieve a convergence rate of $\lambda = 2$, $\lambda$ and $q$ are chosen to be $q = 50$ and $\lambda = 3$ respectively. For agent 1: $C_1 = [0 1]$, $\bar{A}_1 = \begin{bmatrix} -0.1 & -0.1 \\ 0.4 & -0.1 \end{bmatrix}, L_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, K_1 = \begin{bmatrix} 13.7 \\ 4.8 \end{bmatrix}$
For agent 2: \( \bar{C}_2 = [0 \ 1] \),
\[
\bar{A}_2 = \begin{bmatrix}
-0.1 & -0.4 \\
0.1 & -0.1
\end{bmatrix}, \quad L_2 = \begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}, \quad K_2 = \begin{bmatrix}
54.7 \\
4.8
\end{bmatrix}
\]

For agent 3: \( \bar{C}_3 = [0 \ 1] \),
\[
\bar{A}_3 = \begin{bmatrix}
0.1 & -2 \\
0.2 & -0.2
\end{bmatrix}, \quad L_3 = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{bmatrix}, \quad K_3 = \begin{bmatrix}
30.6 \\
4.9
\end{bmatrix}
\]

For agent 4: \( \bar{C}_4 = [0 \ 1] \),
\[
\bar{A}_4 = \begin{bmatrix}
-0.2 & -0.2 \\
2 & 0.1
\end{bmatrix}, \quad L_4 = \begin{bmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad K_4 = \begin{bmatrix}
2.32 \\
4.9
\end{bmatrix}
\]

In all four cases the local observer convergence rates are all 2.

This system was simulated with \( x(0) = [3 \ 2 \ 4 \ 1]' \) as the initial state of the process, \( w_3(0) = [5 \ 5]' \), \( w_2(0) = [5 \ 5]' \), \( w_3(0) = [5 \ 5]' \), and \( w_4(0) = [5 \ 5]' \) as the initial states of the four local observers, \( x_1(0) = x_2(0) = [5 \ 5 \ 5 \ 5]' \), and \( x_3(0) = x_4(0) = [4 \ 4 \ 4 \ 4]' \) as the initial estimates of the four local estimators.

Three simulations were performed. The first is intended to demonstrate performance in the face of system noise. For this a modified process dynamic of the form \( \dot{x} = Ax + bw \) is assumed where \( b = [1 \ 1 \ 1 \ 1]' \) and \( \nu = \cos 10t \) is system noise. Traces of this simulation are shown in Figure 3 where \( x^{(3)}_1 \) and \( x^{(3)} \) denote the third components of \( x_1 \) and \( x \) respectively. Only the trajectory of \( x^{(3)}_1 \) is plotted because for agent 1 only the third component is unobservable, and all the other components are observable.

The second simulation, which is without system noise, is intended to demonstrate the hybrid observer’s robustness against a small change in the event time sequence of one of the agents. The change considered presumes that the event times of agent 4 occur \( .2T \) time units before the the event times of the other three agents. Traces of this simulation are shown in Figure 4.

The third simulation, also without system noise, is intended to demonstrate the hybrid observer’s resilience against the disappearance of agent 4 at time \( t = 5 \) and also against agent 4’s re-emergence at time \( t = 7 \). Traces of this simulation are shown in Figure 5.

Disruption appearing at the beginning of the traces for all three simulations are due to initial conditions and

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Fig. 2. \( N(t) \)

Fig. 3. Performance in the face of system noise

Fig. 4. Performance in the face of a perturbed event time sequence

Fig. 5. Performance in the face of abrupt node changes
gain of the agent, namely 2 time units, is large compared to the time constants of the observer. Clearly much more work needs to be done here to better understand rapidly occurring and re-occurring losses and gains of agents.

7 Concluding Remarks

One of the nice properties of the hybrid observer discussed in this paper is that it is resilient. By this we mean that under appropriate conditions it is able to continue to provide asymptotically correct estimates of $x$, even if communications between some agents break down or if one or several of the agents joins or leaves the network. The third simulation provides an example of this capability. As pointed out earlier, further research is needed to more fully understand observer resilience, especially the situation when agents join or leave the network.

Generally one would like to choose $T$ “small” to avoid unnecessarily large error overshooting between event times. Meanwhile it is obvious from (20) that the larger the number $p$ and consequently the number of iterations $q$ on each event-time interval, the faster the convergence. Two considerations limit the value of $q$ - how fast the parameter estimators can compute and how quickly information can be transmitted across the network. We doubt the former consideration will prove very important in most applications, since digital processors can be quite fast and the computations required are not so taxing. On the other hand, transmission delays will almost certainly limit the choice of $q$. A model which explicitly takes such delays into account will be presented in another paper.

A practical issue is that the development in this paper does not take into account measurement noise. On the other hand, the observer provides exponential convergence and this suggests that if noisy measurements are considered, the observer’s performance will degrade gracefully with increasing noise levels. Of course one would like an “optimal” estimator for such situations in the spirit of a Kalman filter. Just how to formulate and solve such a problem is a significant issue for further research.

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