Inductive derivation of formulae by a computer

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Abstract

The aim of the paper is to present visual derivation of classic arithmetic formulae by a computer. The GEOMETER’S SKETCHPAD software is used for the purpose. The approach has been experienced with 6-grade students, who are already familiar with powers. The results are encouraging, since the students show successful application of the formulae that have been taught. Several applications are proposed.

Keywords: positive integer; sum; power; formula; computer; circle; unit square; triangle; geometric figure; area

1. Introduction

The possibility to consider numbers as geometric figures is of interest not only from interdisciplinary point of view. Such a possibility is of considerable importance in Mathematics education, because it helps students to perceive new notions and facts by additional organs of sense, eyesight included. Sometimes, the approach is called “proofs without words” (Nelsen, 1993). Since each formula has two sides, the main well-known combinatorial tool for demonstration is to represent them in two different ways. In our case this means to introduce two geometric objects or notions from Geometry. The GEOMETER’S SKETCHPAD (GSP) software is used for the purpose. Application of other software like GeoNext, GeoGebra, etc., is possible too. The approach has been experienced with 6-grade students. The choice of the age is dependent on the curriculum. Knowledge of powers is necessary. The results are encouraging, since the students show successful application of the formulae that have been taught.

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The authors believe that measurements for the level of perception are not needed in this case since correct uses are enough.

2. Methodology

Usually, mathematical induction is included in the last grades curriculum of secondary school. However, preliminary steps in its main idea learning are possible to be exercised earlier by means of a computer. It is not meant a full induction but small inductive steps that lead students to knowledge. The authors propose an approach, which is similar to the so called Socratic style getting the truth via inquiry. The original Socratic method is not advocated here, because it is not used a questioning to dismantle or discard preexisting ideas. It is emphasized only on the ultimate goal of such a style – to increase understanding through inductive steps. Induction is a hidden reality of science and the computer is a convenient tool to reveal a part of it or at least to give a possibility of investigating to a level, which supports its full study. The induction method could be realized as an experimental approach to a given mathematical problem. This approximates the problem to one from the domain of the experimental science, by which a mathematical relation is discovered. Well-known arithmetic formulae are derived in the sequel and some applications are proposed.

The following formulae for arbitrary positive integers \( n \) are attacked by mathematical induction, usually:

\[
1+2+3+\cdots+n = \frac{n(n+1)}{2}, \quad 1+3+5+\cdots+2n-1 = n^2, \quad 1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6},
\]

\[
l^3 + 2^3 + 3^3 + \cdots + n^3 = (1+2+3+\cdots+n)^2.
\]

However, it is not clear where these formulae come from. A certain clarification will be done using geometric representations by means of GSP.

3. Sum of the first \( n \) positive integers

Usually, the Gauss method is applied to find the sum \( S_n = 1+2+3+\cdots+n \). We have:

\[
2S_n = (1+n) + (2+n-1) + (3+n-2) + \cdots + (n+1).
\]

A geometric variety of this idea has been used in Ancient Greece. Here, the positive integers are identified with corresponding numbers of circles (Fig. 1). Consider the cases: \( S_1 = 1 \), \( S_2 = 1+2 \), \( S_3 = 1+2+3 \) and \( S_4 = 1+2+3+4 \). In each case arrange the circles in a right triangle, copy the triangle and reflect the copy symmetrically, as shown. Finally, “stick” the symmetric image to the initial triangle. Rectangles are obtained of the kind: \( (1 \times 2) \), \( (2 \times 3) \), \( (3 \times 4) \), \( (4 \times 5) \).

![Fig. 1](image-url)
(3×4) and (4×5). Assuming that the area of each circle is equal to 1, then the sums are expressed by the areas of the corresponding rectangles: \( \frac{1}{2} \cdot 1 \cdot 2 , \frac{1}{2} \cdot 2 \cdot 3 , \frac{1}{2} \cdot 3 \cdot 4 \) and \( \frac{1}{2} \cdot 4 \cdot 5 \). The conclusion is that in the general case of \( n \) a rectangle \( (n \times (n+1)) \) is obtained. Thus, \( S_n = \frac{n(n+1)}{2} \). Note, that because of the shapes, by which the representations have been realized, the numbers 1, 3, 6, 10, … are called triangular numbers. The triangular numbers are described by the formula \( \frac{n(n+1)}{2} \).

4. Sum of the first \( n \) odd positive integers

In order to derive a formula for the sum \( P_n = 1 + 3 + 5 + \cdots + 2n - 1 \), consider the cases \( P_1 = 1 \), \( P_2 = 1 + 3 \), \( P_3 = 1 + 3 + 5 \) and \( P_4 = 1 + 3 + 5 + 7 \). Use again a circle for each addend. In each case arrange again the circles in a right triangle, which is with a toothed hypotenuse now (Fig 2.). Rearrange the obtained triangles in squares step by step, as shown. The computer animates the procedure. Inductively we conclude that \( P_n = n^2 \). The shapes are the reasons to call the sums square numbers.

5. Sum of the squares of the first \( n \) positive integers

The formula for the sum \( Q_n = 1^2 + 2^2 + 3^2 + \cdots + n^2 \) could be discovered using the two forms of the square numbers. Firstly, represent each square of the integers from 1 to 8 by the toothed triangles from the previous chapter (Fig. 3). Construct an “Eiffel tower” using full horizontal rows of circles: begin with the longest row, containing \( 2n - 1 \) circles (in our case 15 circles); then put all rows with \( 2n - 3 \) circles (in our case two rows with 13 circles each); continue with the next three rows in a similar way and so, as shown. The “Eiffel tower”, thus obtained, is symmetric and contains as many circles as the value of \( Q_n \). Because of the symmetry it is possible “to stick” two upside down copies of the tower to both sides of the initial one. The obtained figure is a rectangle with sides \((2n-1)+2\) and \(1+2+3+\cdots+n\), containing three times the sum \( Q_n \) (Fig. 4). Consequently, \( 3Q_n = (2n+1)(1+2+3+\cdots+n) \), i.e. \( Q_n = \frac{n(n+1)(2n+1)}{6} \).

Fig. 2
Fig. 3

Fig. 4
6. Sum of the cubes of the first $n$ positive integers

Different geometric ideas exist for the determination of the sum $R_n = 1^3 + 2^3 + 3^3 + \cdots + n^3$. We will consider the one of Warren Lushbaugh (Gardner, 1988). Consider a square $2 \times 2$. Since it contains 4 unit squares, its area is $4.1^2$ (Fig. 5). Pack this square with squares of the same size. Eight squares are needed. The area of the new figure is $4.2^2$. Pack it with $3 \times 3$ squares. We need $12 = 4.3$ such squares and the total new area is $4.3.3^2$. Continue the packing until a pack of $4n$ squares with side length $n$ is reached. The new surface is $4n.n^2$ (Fig. 5). The figure, thus obtained, is a consistent square (without holes and overlapping). Its side length is equal to $2(1 + 2 + 3 + \cdots + n)$, while the area is $\left[2(1 + 2 + 3 + \cdots + n)\right]^2$. On the other hand, from the way of constructing, it follows, that the area is equal to $4\left(1.1^2 + 2.2^2 + 3.3^2 + \cdots + n.n^2\right)$. In such a way we obtain the formula

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2,$$

i.e. $R_n = \left[\frac{n(n+1)}{2}\right]^2$.

7. Exercises and applications

Problem 1. Find the sum $1 + 2 + 3 + \cdots + (n-1) + n + (n-1) + \cdots + 3 + 2 + 1$.

Solution: The students were asked to use the idea for the determination of the sum $S_n$ in the case $n = 10$. All of them constructed isosceles right triangles with pencil like the one in Fig. 6. After that they were asked to take a copy of the triangle and by drawing “to stick” it with the initial triangle in such a way, that their hypotenuses coincide exactly. The obtained figure was a square like the one in Fig. 6. The students counted the circles in the square: $10 \times 10 = 10^2$. The teacher gave the general result $1 + 2 + 3 + \cdots + (n-1) + n + (n-1) + \cdots + 3 + 2 + 1 = n^2$. 
Problem 2. Find the number of all squares, which can be found in a chess-board.

Solution: The classic chess-board is a $(8 \times 8)$ square, but we consider the general situation of a $(n \times n)$ board. Take the cases $n=1$, $n=2$, $n=3$ and $n=4$ (Fig. 7). In each case the first step is to count the initial square. At the second step consider all possible equal squares with side length, which is smaller by one unit square. At the third step consider all possible equal squares with side length, which is smaller by two unit squares, and s. o. Note that at each step the equal squares are grouped in a square and it is enough to consider the dimensions of this square. Thus, the counting of the cases under consideration gives: $1^2$, $1^2 + 2^2$, $1^2 + 2^2 + 3^2$ and $1^2 + 2^2 + 3^2 + 4^2$. The number of the squares in a $(n \times n)$ board is equal to $Q_s = 1^2 + 2^2 + 3^2 + \cdots + n^2$. In the case of a classic chess-board the number is $Q_s = \frac{8 \cdot 9 \cdot 17}{6} = 204$.

The last problem was solved by the teacher, while the next one was given to the students for homework. Instructions were given to follow the idea of the solution of problem 2.

Problem 3. Find the number of all rectangles, which can be found in a chess-board.

8. Conclusion

The formulae and the problems, which are discussed in the present paper, are of arithmetic character. Very often such kind of material is boring. Usually, students feel themselves more comfortably when numbers, terms or notions are visualized. Geometry suggests many possibilities in this direction. Geometric objects and information technologies stimulate cognitive activity. They have the power to transform the profound indifference from the beginning to desirable actions. Then teaching and learning become successful and the proof is in the listed examples.
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