Universal Fermi gases in mixed dimensions

Yusuke Nishida and Shina Tan
Institute for Nuclear Theory, University of Washington, Seattle, Washington 98195-1550, USA

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We investigate a two-species Fermi gas in which one species is confined in a two-dimensional plane (2D) or one-dimensional line (1D) while the other is free in the three-dimensional space (3D). We discuss the realization of such a system with the interspecies interaction tuned to resonance. When the mass ratio is in the range $0.0351 < m_{2D}/m_{3D} < 6.35$ for the 2D-3D mixture or $0.00646 < m_{1D}/m_{3D} < 2.06$ for the 1D-3D mixture, the resulting system is stable against the Efimov effect and has universal properties. We calculate key quantities in the many-body phase diagram. Other possible scale-invariant systems with short-range few-body interactions are also elucidated.

Introduction.— Experiments using ultracold atomic gases have attracted considerable interest because of their high designability and tunability. Not only can the strength of interaction be varied via the Feshbach resonance, but also the dimensionality of space can be changed by means of strong optical lattices. The Bose-Einstein condensation in trapped atoms has been realized both in one and two dimensions [1]. A strongly interacting one-dimensional Fermi gas has been observed [2], which is an atomic realization of Tomonaga-Luttinger liquid.

What the cold atom community has not paid attention to is the system in mixed dimensions, namely, the system where different types of particles live in different spatial dimensions. Actually the idea of mixed dimensions is commonly used in various subfields of physics. For example, in the brane world model of the Universe, ordinary matter is confined in a three-dimensional space embedded in higher dimensions where gravity can propagate [3]. Also, recently realized graphene can be regarded as a system in mixed dimensions where photons induce the three-dimensional Coulomb interaction between electrons confined in a two-dimensional sheet [4].

In this Letter, we discuss the realization of an analogous system using cold atoms with two fermionic species $A$ and $B$, where $A$ atoms are confined in a two-dimensional plane (2D) or one-dimensional line (1D) while $B$ atoms are free in the three-dimensional space (3D). At low energies, we show that the interspecies interaction in a certain range of mass ratio is characterized by a single parameter, the effective scattering length $a_{\text{eff}}$, whose value is arbitrarily tunable by current experiments. In particular, if the effective interaction is tuned to resonance $|a_{\text{eff}}| = \infty$, such a system is scale invariant and has universal properties. (Here intra-species interactions are neglected and we use the term “universal” in the narrow sense that the interaction does not have any dimensionful parameters.) The corresponding many-body system forms a novel type of unitary Fermi gas in mixed dimensions.

Experimental realization.— To confine only $A$ atoms, we note that the strength of the optical trap experienced by neutral atoms is proportional to the polarizability which depends on the atomic species and the laser frequency $\omega_L$. By tuning $\omega_L$ at a zero of the polarizability of $B$ atoms, which exists between any two nearest resonances, we can confine only $A$ atoms. If the optical trap with such a laser frequency is $d_1$ dimensional with $d_1 = 1$ or 2, $A$ atoms are confined in 2D or 1D, respectively, while $B$ atoms are free from the confinement.

If the confinement potential applied to $A$ atoms is harmonic with the oscillator frequency $\omega_A$, an $A$ atom interacting with a $B$ atom is described by the Hamiltonian:

$$H = \frac{\hbar^2}{2m_A} \nabla_x^2 + \frac{1}{2} m_A \omega_A^2 x_A^2 - \frac{\hbar^2}{2m_B} \nabla_y^2 + V(x_A, x_B).$$

We introduce a notation $x_A = (x_A^A, x_A^B)$, where $x_A^A$ is the $d_1$-dimensional coordinates affected by the confinement potential. $V(x_A, x_B)$ is the bare interspecies interaction. In the limit of zero-range interaction, it is characterized by a single parameter, the $s$-wave scattering length $a$, which is arbitrarily tunable by means of the Feshbach resonance.

Because of the partial confinement of the $A$ atom, the scattering property between $A$ and $B$ atoms is modified from the free-space scattering. In particular, an infinite number of confinement-induced resonances appears. To determine the position of resonances as a function of $l_1/a$ with $l_1 = \sqrt{\frac{\hbar}{m_A \omega_A}}$, we solve the Schrödinger equation for $\Psi(x_A, x_B) = \exp[i \tau L] \Psi(x_A, x_B)$ in the zero-energy limit $E = \frac{\hbar^2}{2m_B} \omega_B \to 0$. After excluding the center-of-mass motion in the parallel direction, the solution is written as

$$\Psi = \int \mathcal{D}y_B \mathcal{G}(x_{A||} - x_{B||}, x_{A\perp}, B\perp; 0, y_B, y_{\perp}) f(y_{\perp}) + C \phi_0(x_{A\perp}),$$

where $\phi_0(x_{A\perp}) = \exp[-x_{A\perp}^2/(2l_{\perp}^2)]$ is the ground state wave function of the $d_1$-dimensional harmonic oscillator. $G$ is the Green’s function given by

$$G = \int_0^\infty \frac{d\tau}{\tau^{3/2}} \left(\frac{e^\tau}{\sinh\tau}\right)^{d_{\perp}/2} \exp\left[\frac{(x_{A||} - x_{B||})^2}{2l_{\perp}^2(u + 1)\tau}\right]$$

$$- \frac{(x_{A\perp}^2 + y_{\perp}^2) \cosh\tau - 2x_{A\perp} \cdot y_{\perp}}{2l_{\perp}^2 \sinh\tau} \frac{(x_{B\perp} - y_{\perp})^2}{2l_{\perp}^2 u \tau}$$

with $u \equiv m_A/m_B$ being the mass ratio. The unknown function $f(y_{\perp})/C$ is uniquely determined by im-
posing the short-range boundary condition $\Psi(x_A, x_B) \propto \frac{1}{|x_A - x_B| - \frac{1}{a}} + O(|x_A - x_B|)$ when $|x_A - x_B| \to 0$. The asymptotic form at a large separation $|x_A - x_B| \to \infty$,

$$\Psi(x_A, x_B) \propto \left[ \frac{1}{D(x_A, x_B)} - \frac{1}{a_{\text{eff}}} \right] \phi_0(x_A, x_B),$$

(3)

gives an effective scattering length $a_{\text{eff}}$, and whenever $a_{\text{eff}}$ diverges, a resonance occurs. Here $D(x_A, x_B) = \sqrt{(|x_A| - |x_B|)^2 + \frac{m_A \cdot |x_B|^2}{m_{AB}^2}}$ with $m_{AB} = \frac{m_A \cdot m_B}{m_A + m_B}$ being the reduced mass is the “interparticle distance” with the anisotropic weights to the parallel and perpendicular directions.

Numerically obtained $a_{\text{eff}}/l_\perp$ as a function of $l_\perp/a$ is plotted for $d_\perp = 1$ (2D-3D mixture) in Fig. 1 and for $d_\perp = 2$ (1D-3D mixture) in Fig. 2. We chose the mass ratio $u = 0.15$ corresponding to the physical case of $A = ^6\text{Li}$ and $B = ^{40}\text{K}$. These figures show that $a_{\text{eff}}$ can be tuned to any desired value by varying $a_0$ or $l_\perp$. In particular, the values of $l_\perp/a$ to achieve the resonances $|a_{\text{eff}}|/l_\perp = \infty$ are plotted as functions of $u$ in Figs. 1 and 2.

If the confinement length $l_\perp$ is much smaller than any other length scales of the system such as $a_{\text{eff}}$ and a mean interparticle distance at finite density, one can neglect the motion of $A$ atoms in the confinement direction. Consequently, the resulting system is mixed-dimensional, namely, $A$ atoms are confined in 2D or 1D while $B$ atoms are in 3D. The interspecies interaction is solely characterized by the effective scattering length $a_{\text{eff}}$ defined in Eq. (3). When $a_{\text{eff}} > 0$, there is a shallow bound state with the binding energy $E_{\text{binding}} = -\hbar^2/(2m_{AB}a_{\text{eff}}^2)$. If the effective interaction is tuned to resonance $|a_{\text{eff}}| = \infty$, the system is scale invariant and universal, i.e., independent of the short-range physics. The corresponding many-body system forms a novel type of unitary Fermi gas in mixed dimensions. In the rest of this Letter, we concentrate on the system at the resonance.

Stability of the system at resonance.— An important question regarding our novel system is its stability. In the usual 3D case, it is known that two heavy and one light fermions with mass ratio greater than 13.6 develop deep bound states in the limit of zero-range interaction (Efimov effect) [6]. Accordingly the corresponding many-body system is not stable toward collapse. To establish the stability of the unitary Fermi gas in mixed dimensions, we study three-body problems and determine the range of $u$ where the Efimov effect is absent.

We first consider the problem of two $A$ atoms interacting with one $B$ atom. The wave function $\Psi(x_A, x_{A2}, x_B)$ for $D(x_A, x_B) > 0$ satisfies the Schrödinger equation:

$$\left[ -\frac{\hbar^2 \nabla_{x_A}^2}{m_A} - \frac{\hbar^2 \nabla_{x_B}^2}{m_B} - \frac{\hbar^2 \nabla^2}{m_{AB}} \right] \Psi = E \Psi.$$ 

Now $x_A = x_{A2}$ is two- or one-dimensional coordinates. The resonant interspecies interaction is taken into account by the short-range boundary condition:

$$\Psi(x_A, x_B) \cong \frac{1}{D(x_A, x_B)^{-1} + O[D(x_A, x_B)]}$$

when $D(x_A, x_B) \to 0$ [see Eq. (3)]. In hyperspherical coordinates, the wave function at a short distance behaves as

$$\Psi(x_A, x_{A2}, x_B) \approx R^\gamma \psi_l(\Omega),$$

where $R$ is the hyperradius and $\Omega$ denotes the hyperangular variables. $\gamma$ is the scaling exponent of the wave function classified by the quantum number $l$ and can be determined by writing $\Psi$ in terms of the Green’s function and imposing the short-range boundary condition [9].

In the 2D-3D mixture, $l = 0, \pm 1, \pm 2, \ldots$ is the orbital angular momentum projected to the 2D plane and $\gamma$ satisfies the equation [9]

$$-\frac{\sqrt{2u + 1}}{u + 1} = \int_0^\pi \frac{d\theta}{\pi} \cos(\theta) \left[ g(\gamma, \arccos \frac{u \cos \theta}{u + 1}) + g(-\gamma - 3, \arccos \frac{u \cos \theta}{u + 1}) \right],$$

where

$$g(\gamma, \beta) \equiv \pi^{-1/2} \Gamma(-\gamma - 1) \Gamma(\gamma + 3/2) e^{-i\theta}\left(\gamma + \frac{3}{2}\right) c^{i\beta(\gamma + 1)} \times 2F1\left(\frac{1}{2}, -\gamma - 1; -\gamma - \frac{1}{2}; e^{2\beta}\right).$$

Here $2F1$ is the hypergeometric function. Equation (5) has a pair of solutions, $\gamma_+$ and $\gamma_-$, related to each other by $\gamma_+ + \gamma_- = -3$. For the $p$-wave channel $|l| = 1$, as the mass ratio $u$ increases from 0 to the critical value $u_{\text{max}} = 6.35$, $\gamma_+$ decreases from 0 to $-\frac{5}{2}$. When $u$ is increased further, $u > u_{\text{max}}$, $\gamma_\pm$ become complex conjugates $\gamma_\pm = -\frac{5}{2} \pm i\delta_0$, indicating that the wave function oscillates as $R^{-3/2}\sin(s_0 \ln R + \delta)$ and the Efimov effect takes place.
The critical mass ratio for other odd \( l \) is greater than 6.35 and the Efimov effect is absent for any even \( l \).

On the other hand, in the 1D-3D mixture, \( l = \pm 1 \) in Eq. (4) corresponds to the parity of the wave function and \( \gamma \) satisfies the equation

\[
\sqrt{2u+1} = \frac{\cos[(\gamma+1)\alpha] + l \cos[(\gamma+1)(\pi - \alpha)]}{(\gamma+1)\sin[\pi (\gamma+1)]}
\]

with \( \alpha \equiv \arccos \frac{\pi}{u+1} \). Similarly, this equation has a pair of solutions, \( \gamma_+ \) and \( \gamma_- \), related to each other by \( \gamma_+ + \gamma_- = -2 \). For the odd-parity channel \( l = -1 \), when \( u \) exceeds the critical value \( u_{\text{max}} = 2.06 \), \( \gamma_\pm \) become complex conjugates \( \gamma_\pm = -1 \pm i\delta_0 \), indicating the Efimov effect. In the even-parity channel, the Efimov effect does not take place.

We then consider the problem of one \( A \) atom interacting with two \( B \) atoms. The wave function \( \Psi(x_A, x_B1, x_B2) \sim R\psi_0(\Omega) \) satisfies the Schrödinger equation and the short-range boundary condition analogous to the previous case. \( \gamma \) in this case satisfies an integral equation [7]. For the \( p \)-wave or odd-parity channel, we obtain

\[
-S_p = \int \frac{d^3q}{2\pi^2} \frac{\hat{p}_p \cdot \hat{q}_p}{p^2 + \frac{q^2}{(u+1)^2} - q_\perp^2} S_q,
\]

where \( p = (p_\parallel, p_\perp) \) and

\[
S_p \equiv S( |p_\perp|/|p_\parallel| )
\]

is an unknown function. There is a pair of solutions, \( \gamma_+ \) and \( \gamma_- \), related to each other by \( \gamma_+ + \gamma_- = -4 \). As the mass ratio \( u \) decreases from \( \infty \) to the critical value \( u_{\text{min}} \), \( \gamma_+ \) decreases from \( 0 \) to \( -2 \). When \( u \) is decreased further, \( u < u_{\text{min}}, \gamma_+ \) become complex conjugates \( \gamma_\pm = -2 \pm i\delta_0 \), indicating the Efimov effect. The critical mass ratios are found to be \( u_{\text{min}} \approx 0.0351 \) for the 2D-3D mixture and \( u_{\text{min}} \approx 0.00646 \) for the 1D-3D mixture. In other channels, the Efimov effect does not take place for the mass ratio greater than \( u_{\text{min}} \).

So far we have supposed that both atomic species \( A \) and \( B \) are fermionic. If either \( A \) or \( B \) atoms are bosonic, one can confirm that the Efimov effect takes place for any mass ratio in the \( s \)-wave or \( \pi \)-even channel. Therefore, for the stability of many-body systems, both atomic species have to be fermionic with the mass ratio between \( u_{\text{min}} \) and \( u_{\text{max}} \). For example, the combination of fermionic atoms, \( \Lambda = ^6\text{Li} \) and \( B = ^{40}\text{K} \) \((u = 0.15)\), can be used to realize both 2D-3D and 1D-3D mixtures, while the opposite combination, \( \Lambda = ^{40}\text{K} \) and \( B = ^6\text{Li} \) \((u = 6.67)\), suffers the Efimov effect in both mixtures.

Many-body physics.— We now proceed to the many-body physics of unitary Fermi gas in mixed dimensions. It is convenient to work in the grand-canonical ensemble by introducing chemical potentials \( \mu_A \) to \( A \) atoms and \( \mu_B \) to \( B \) atoms. The phase diagram of the system in the phase of \( \mu_A \) and \( \mu_B \) is depicted in Fig. 3. There are four distinct regions: vacuum \((\mu_A < 0, \mu_B < 0)\), pure 2D or 1D gas \((\mu_A > 0, \mu_B < \mu_B^{\text{cri}})\), pure 3D gas \((\mu_B > 0, \mu_A < \mu_A^{\text{cri}})\), and mixed gas (rest of the phase diagram). As we discuss below, the mixed-gas phase can be divided into two phases: mixed gas I \((\mu_B < 0)\) and mixed gas II \((\mu_B > 0)\). It is possible that these mixed-gas phases are further divided into several phases by nontrivial many-body physics including inter- and intra-species pairings.

Because of the lack of translational symmetry in the perpendicular direction, the density of \( B \) atoms \( n_B(\{x_\perp\}) \) is a nontrivial function of the distance from the 2D plane or 1D line. If \( \mu_B < 0 \) (mixed gas I), all \( B \) atoms are localized around the 2D plane or 1D line because they are attracted by \( A \) atoms. Accordingly \( B \) atoms have a vanishing density at infinity \( n_B(\{x_\perp\} \to \infty) \to 0 \). When \( \mu_B \) exceeds zero (mixed gas II), a portion of \( B \) atoms are no longer bound due to the Fermi pressure. In this phase, the density of \( B \) atoms at infinity is given by that of noninteracting fermions \( n_B(\{x_\perp\} \to \infty) \to \frac{(2m_B\mu_B)^{3/2}}{6\pi^2\hbar^3} \).

In order to make the discussion more quantitative, we estimate the key parameters in the phase diagram. The chemical potential at the phase boundary \( \mu_B^{\text{cri}} \) normalized by the Fermi energy of majority atoms \( \mu_B^{\text{cri}}/\mu_B \) corresponding to the slope in Fig. 3 is a universal number because of the scale invariance of the interaction. To calculate \( \mu_B^{\text{cri}}/\mu_B \), we employ a many-body variational wave function assuming single particle-hole excitations [11]:

\[
|\chi\rangle = \chi_0 |0_\parallel\rangle_A |\text{FS}_B\rangle_B + \sum_{|p| > \hbar k_{FB}} \chi_{p,q} |p\parallel\rangle_A |p, q\rangle_B |p, q\rangle_B.
\]

Here \( |p\parallel\rangle_A \) is the momentum eigenstate of a single \( A \) atom and \( |\text{FS}_B\rangle_B \) is the unperturbed Fermi sea of \( B \) atoms with the Fermi momentum \( \hbar k_{FB} = \sqrt{2m_B\mu_B} \). The \( |p, q\rangle_B \) describes the perturbed Fermi sea with the particle (momentum \( q \)) and hole (\( p \)) excitations. This simple approximation is known to give reasonable results in the usual 3D case [12]. By minimizing the energy expectation value with respect to the coefficients \( \chi_0 \) and \( \chi_{p,q} \),
we obtain the equation to determine $\mu_A^{\text{cri}}$:

$$
\mu_A^{\text{cri}} = \mu_A^{\text{cri}} \left( \sum_{|q|<\hbar k_{FB}} \sum_q \left[ \frac{\theta(|q|-\hbar k_{FB})}{2m_A} + \frac{|q|^2}{2m_B} - \mu_A^{\text{cri}} \right] \right)^{-1} 
$$

Similarly, it is straightforward to derive the equation to determine $\mu_B^{\text{cri}}$. Numerically obtained critical chemical potentials $\mu_A^{\text{cri}} / \mu_B$ and $\mu_B^{\text{cri}} / \mu_A$ as functions of the mass ratio $u_{\text{min}} < u < u_{\text{max}}$ are plotted in Fig. 4 for the 2D-3D and 1D-3D mixtures.

Other universal systems.— Finally we elucidate other possible scale-invariant systems with short-range few-body interactions. First it is possible for an additional intra-species interaction to produce a resonance in a three-body channel [13]. $A\overline{A}B$ three-body resonance can be introduced if the three-body wave function $\psi$ with $\gamma = \gamma_\pm$ is normalizable at origin $R \to 0$. This corresponds to the range of mass ratio $2.33 < u < 6.35$ for the 2D-3D mixture or $0 < u < 2.06$ for the 1D-3D mixture. Similarly, $A\overline{A}B$ three-body resonance can be introduced if the mass ratio is in the range $0.0351 < u < 0.0661$ for the 2D-3D mixture or $0.00646 < u < 0.0202$ for the 1D-3D mixture. The corresponding many-body systems become universal Fermi gases with both two-body and three-body resonances.

It is also possible to realize scale-invariant systems in different spatial configurations from those considered so far. The key idea is to find a system in which the relative motion is characterized by three coordinates (excluding the center-of-mass translations if exist). In such a system, the interspecies short-range interaction is solely characterized by the effective scattering length and at resonance the scale invariance is achieved. Consider a 1D-2D (2D-2D) mixture where $B$ atoms are confined in a 2D plane having a point-like (linear) interaction with the 1D line (other 2D plane) in which $A$ atoms are confined. Here $A-B$ relative motions are three-dimensional and the resulting systems at resonance provide two more classes of universal two-species Fermi gases in mixed dimensions.

Our idea can be extended to cases with more atomic species. If the number of atomic species is three ($A$, $B$, and $C$), there exist two classes of universal three-species Fermi gases in mixed dimensions. One is a 1D-2D mixture where $A$ and $B$ atoms are confined in the same 1D line embedded in a 2D plane where $C$ atoms are confined. The other is a 1D-1D-1D mixture where $A$, $B$, and $C$ atoms are independently confined in 1D lines and these three lines intersect at a single point. In either case, the interspecies three-body interaction is three-dimensional and can be in principle tuned to resonance.

Furthermore, if we have four atomic species and all of them are confined in the same 1D line, the interspecies four-body interaction becomes three-dimensional. The corresponding many-body system at resonance will be especially interesting because it forms a universal four-species Fermi gas purely in one dimension.

Conclusion.— In addition to the well-known $s$-wave two-body resonant interaction in pure 3D, there exist seven more types of scale-invariant short-range few-body interactions in pure and mixed dimensions: two-body resonant interactions in 2D-3D, 1D-3D, 2D-2D, and 1D-2D mixtures; three-body resonant interactions in 1D$^2$-2D and 1D-1D-1D mixtures; four-body resonant interaction in pure 1D. At finite densities, each of them corresponds to a novel class of universal multispecies Fermi gases that have nontrivial many-body physics. We have given detailed analyses in the cases of two-species Fermi gases in the 2D-3D and 1D-3D mixtures.

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