FIDUCIAL MATCHING
FOR
THE APPROXIMATE POSTERIOR: F-ABC

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Summary

Approximate Bayesian Computation (ABC) provides posterior models for a stochastic parameter $\Theta$ when the observed, size $n$ sample $x(\in R^{n \times d})$, has intractable likelihood; $d \geq 1$. For $x$ from c.d.f. $F_\theta$, with unknown $\theta \in \Theta$, the ABC-steps are: a single sample, $x^*$, is drawn from $F_{\theta^*}$, with known $\theta^* \in \Theta$; a nearly sufficient summary, $S(x)$, is determined for matching $x$ and $x^*$ within some $\epsilon$-tolerance for a distance-measure $\rho$; if $x$ and $x^*$ match, $\theta^*$ is included in the approximate posterior with weight, $K(x, x^*; \epsilon)$; $K$ is arbitrary kernel.

We introduce Fiducial (F)-ABC, with $M$ $x^*$ drawn from $F_{\theta^*}$. The goal is a “one-for-all F-models” approach, with $\theta^*$-weights not $K$-artifacts and with universal sufficient $S$ the empirical measure, $\mu_X$, when $d > 1$, the empirical cumulative distribution, $\hat{F}_X$, when $d=1$, and, respectively for $\rho$, the Total-Variation and the Kolmogorov distance, $d_K$. Light is thrown to $\epsilon$’s nature via $d_K$, guidelines are given to determine its value and the “0-1” restrictive influence on $\theta^*$ is reduced. $\theta^*$-weight is the proportion of $x^*$ matching $x$ which, for many models, increases to 1 as $\theta^*$ converges to $\theta$, unlike $K(x, x^*; \epsilon)$. The number of simulations for implementation is moderate. Under few, mild assumptions, F-ABC posterior converges to its target and rates of concentration to $T(F_{\theta})$ are obtained; $T$ functional.

When $M = 1$, F-ABC is reduced to ABC. When $M > 1$, F-ABC posterior includes either selected $\theta^*$ or all $\theta^*$ used, reducing $\epsilon$’s influence. In simulations, nonparametric F-ABC posterior improves the concentration of parametric ABC posterior at $\theta$ and “F-ABC posterior for all $\theta^*$” is satisfactory.
1 Introduction

In Bayesian inference, central theme is the posterior model, $\pi(\theta^*|x)$, of stochastic parameter $\Theta$ given the observed data, sample $x; \theta^* \in \Theta$. Approximate Bayesian Computation (ABC) method provides a posterior model when the data’s likelihood is intractable. Rubin (1984) described the first ABC method for $x$ from the model $f(y|\theta)$ with cumulative distribution function (c.d.f.) $F_\theta(y)$, using simulated $x^*$-samples for several $\theta^*$-values having each $\Theta$-prior $\pi(\theta); \theta, \theta^* \in \Theta, x, x^* \in \mathbb{R}^{n \times d}$, generic sample value $y \in \mathbb{R}^{n \times d}, d \geq 1, n$ is the sample size. The $\theta^*$ for which $x^*$ “matches” (or “looks similar to”) $x$ are $\Theta$’s approximate posterior.

Since then, several research results have been obtained in ABC, creating the new statistical culture of Bayesian-Frequentists. Robert (2017) provides a survey on recent ABC results, including three approximations/concerns:

i) ABC degrades the data precision down to a tolerance level $\epsilon$, replacing the event $X = x$ with the event

$$\rho(X^*, x) \leq \epsilon; \quad (1)$$

$\rho$ is a distance-measure.

ii) ABC substitutes for the likelihood a non-parametric approximation.

iii) ABC summarizes $x$ by an almost always insufficient statistic, $S(x)$, using instead of (1),

$$\rho(S(X^*), S(x)) \leq \epsilon. \quad (2)$$

The basic ABC-rejection algorithm selects $\theta^*$ when either [1] or [2] holds (Tavaré et al. 1997, Pritchard et al., 1999). Recently, $x^*$ are drawn from a Sampler.

There are additional concerns on ABC. a) The dimension of $S$ with Big Data when the statistical nature of $\theta$ is unknown. b) The $\epsilon$-value used and the “0-1” restrictive influence on $\theta^*$, $\epsilon$’s missing sampling interpretation and components, $\epsilon$’s dependence on $n$ and the distance between $F_\theta$ and selected $F_{\theta^*}$. c) The acceptable number of $\theta^*$ in the posterior. d) For continuous $\Theta$, the arbitrary weights “0” or “1” or $K(\frac{x-x^*}{\epsilon})$, given to $\theta^*$ at any distance from $\theta$, using the “one and only” $x^*$ from c.d.f. $F_{\theta^*}$; $K$ is an arbitrary kernel. e) The $\theta^*$-
weights in the approximate posterior create a $K$-dependent artifact; $K$ is usually a normal kernel.  

\textit{f}) It is not clear whether non-selected $\theta^*(\in R)$ is included in the approximate posterior when $\theta_1^*, \theta_2^*$ are selected and $\theta_1^* < \theta^* < \theta_2^*$.  

\textit{g}) For discrete $\Theta$ and with $\theta^*$ drawn twice, it is not clear whether $\theta^*$ is selected if only one of the simulated $x_1^*, x_2^*$ matches $x$.  

\textit{h}) Pure Bayesians and frequentists may question the $\epsilon$-exclusion of non-selected $\theta^*$ from the approximate posterior.

Bernton et al. (2019)\footnote{The details are given for Editors, AEs, referees and readers to avoid confusion, but could be reduced.} propose to solve the choice-problem for $S$ and $\rho$ using, respectively, the “empirical distribution” with “abuse of language” (section 1.1, 1st paragraph) and (basically) Wasserstein distance, $d_W$. The latter is computed for the observed $x$ and the “synthetic” data, $x^*$, and is baptized “distance between empirical distributions” when introducing (5), but no “empirical distributions” appear in $d_W$ even though used in statements. In the abstract, it is stated that the approach avoids “the use of summaries and the ensuing loss of information by instead using the Wasserstein distance between the empirical distributions of the observed and synthetic data.” but in section 1.3, first paragraph, it is instead stated “hoping to avoid the loss of information incurred by the use of summary statistics”. The authors associate “no information loss” with the case $d_W(x, x^*) = 0$ without examining whether there is information loss when $d_W(x, x^*)$ is smaller than positive $\epsilon$. For $d_W$ that “metrizes empirical distributions” consider, for example, the extreme case of only one observation, $n = 1$, the observed data $x = (x_1, \ldots, x_d)$, the synthetic $x^* = (x_1 - \epsilon, \ldots, x_d - \epsilon)$ and $\delta_x, \delta_{x^*}$ the corresponding Dirac functions, with $\epsilon > 0$ much smaller in magnitude than all $x$ coordinates. Using $d_W$ that is sum of absolute differences of the coordinates of $x$ and $x^*$, the distance $d_W(x, x^*) = d \cdot \epsilon$ but the Kolmogorov distance $d_K$ and the Total Variation distance, TV, between $\delta_x$ and $\delta_{x^*}$ take their maximum value 1, and this holds for any $\epsilon$ decreasing to zero. Similar results can be obtained for fixed size $n$ samples $x$ and $x^*$ in $R^d$, with same form. Then, $d_K(\delta_x, \delta_{x^*})/d_W(x, x^*)$ and $TV(\delta_x, \delta_{x^*})/d_W(x, x^*)$ both diverge to infinity as $\epsilon$ converges to zero and $d_W$ is not equivalent to $d_K$ and $TV$, leading to different neighborhoods and convergence. By definition, the $d_K$ and TV values for empirical cumulative distribution functions and empirical measures, respectively, are bounded by 1 but this does not always hold for $d_W(x, x^*)$. Summarizing,
loss of information remains when using $d_W$ and neither empirical distributions, nor empirical measures are used in calculations. The authors avoid the word “sufficiency” in the paper, which is also indicative of potential information loss.

Loss of information due to summary statistics or a method, e.g., Bernton et al. (2019), can be avoided using the empirical cumulative distribution function, $\hat{F}_X$, when $d = 1$ and the empirical measure, $\mu_X(A)$, when $d > 1; A \in B_d$, the Borel sets in $R^d, d > 1$. To match $x$ with $x^*$, $d_K(\hat{F}_x, \hat{F}_{x^*})$ is used when $d = 1$. When $d > 1, \mu_X$ and $\mu_{X^*}$ need to be compared on every Borel set to measure the information difference, and the supremum of the absolute differences over all Borel sets provides the maximum information loss. This is Total Variation (TV) distance to be used as $\rho$-distance herein. TV has the advantage of matching and separating well probabilities $P$ and $Q$ in $R^d$, which are equal when $P(A) = Q(A)$ for every $A \in B_d$, and is useful for $\epsilon$-matching $\mu_X$ with $\mu_{X^*}$.

Concerns a)-h) inspired also the search for an alternative to ABC. The $S$ and $\rho$-choices are: $\mu_X$ with the TV-distance, when $d > 1; \hat{F}_X$ with the Kolmogorov distance, $d_K$, when $d = 1$ (see section 5). Motivated by the Conditional Calibration framework (Rubin, 2019) and a phenomenon observed in several models, the Fiducial (F)-ABC matching is introduced, supported by $M$ $x^*$ drawn from $F_{\theta^*}, 50 \leq M \leq 200$. $p_{\text{match}}(\theta^*)$ is the $x^*$-proportion within the $\epsilon$-tolerance, used as $\theta^*$-weight in the F-ABC posterior. $p_{\text{match}}(\theta^*)$ estimates the $x^*$-matching support probability $\alpha$ of event (2) that provides $\epsilon$’s sampling interpretation and value; $0 \leq \alpha \leq 1$. In practice, $\epsilon$ is determined via $\alpha$ and the Sampler (section 3.1). For several $F_{\theta^*}$-models, $p_{\text{match}}(\theta^*)$ converges to 1 as $\theta^*$ converges to $\theta$, unlike $K(\frac{x-x^*}{\epsilon})$.

In “F-ABC for all” each drawn $\theta^*$ is included in the posterior with weight $p_{\text{match}}(\theta^*)$, reducing $\epsilon$’s influence and without using a kernel. When $M = 1$, F-ABC is ABC. The use of $M$ “pseudo-samples” is non-traditional (see, e.g. Bornn et al., 2017, and references therein), extracting with $p_{\text{match}}(\theta^*)$ useful $\theta$-related information for the posterior. The $\theta^*$-value maximizing $p_{\text{match}}(\theta^*)$ is the Maximum Matching Support Probability Estimate (MMSPE, Yatracos, 2020).

Simulations indicate that nonparametric F-ABC competes well with parametric, flat-kernel ABC and improves very frequently the concentration of the approximate posterior.
The graphs of the F-ABC posterior for all \( \theta^* \) drawn should not pass unnoticed since the Bayesian posterior, \( \pi(\theta^*|x) \), is inclusive of all \( \theta^* \) with different weights.

For the \( X^* \)-matching support probability \( \alpha \) with \( \rho = d_K \), an upper bound \( \epsilon_{n,B} \) on \( \epsilon_n \) is determined; \( 0 < \alpha < 1 \). \( \epsilon_{n,B} \) has two additive components: I) the observed or acceptable discrepancy between \( F_\theta \) and the \( F_{\theta^*} \)-models, and II) a component determined by a confidence related to \( \alpha \) (section 5). Under exchangeability on \( F_\theta(y) \), the ABC and F-ABC posteriors with \( d_K \)-matching converge to \( \pi(\theta|x) \) when \( \epsilon \) converges to zero; \( n \) is fixed. For a continuous linear functional \( T \) on the space of c.d.fs, Bayesian consistency is established and the rate of concentration of \( T(F_{\theta^*}) \) around \( T(F_\theta) \) depends on \( \epsilon_n \), the rate of concentration in probability of \( \hat{F}_X \) around \( F_\theta \), and \( T \)'s modulus of continuity (section 6).

Lintusaari et al (2017) and Fearnhead (2018) provide accessible introductions to ABC presenting, respectively, recent developments and results on asymptotics. Tanaka et al. (2006, p. 1517 and Figure 4) indicate \( \epsilon \)'s choice is crucial for the sampler acceptance rates and the posterior densities. Fearnhead and Prangle (2012) show how to construct appropriate summary \( S \) for ABC to be used in (2) and enable inference about \( \theta \). Frazier et al. (2015) derive conditions under which \( S \) yields consistent Bayesian inference. Biau et al. (2015), analyze ABC as a \( k \)-nearest neighbor method. Frazier et al. (2018) provide for the posterior: its concentration rate on sets containing \( \theta \), its limiting shape and the asymptotic distribution of its mean. Nott et. al. (2018) approximate Bayesian predictive \( p \)-values with Regression ABC. Vihola and Franks (2020) suggest a balanced \( \epsilon \) from a range of tolerances via Bayesian MCMC.

When \( x \) is not obtained from models \( \{f(y, \theta)\pi(\theta), \theta \in \Theta \} \) but \( X^* \) is, Miller and Dunson (2019) propose a robust ABC approach conditioning on \( \rho \)-neighborhoods of empirical c.d.fs \( \hat{F}_X \) and \( \hat{F}_{X^*} \), suggesting among \( \rho \)-distances \( d_K \) (for real valued observations only), but use Kullback-Leibler divergence for their ABC coarsened(c)-posterior.
2 Fiducial ABC

Let \( \pi(\theta) \) be the prior for \( \Theta \) with respect to measure \( \nu \) on \( \Theta \) with \( \sigma \)-field \( C_\Theta \), \( \theta \in \Theta \). \( y \) is generic sample value. \( X \) is a sample of size \( n \) obtained from the unknown \( \theta \)-model with cumulative distribution function \( F_\theta(y) \) and density \( f_\theta(y) \) (or \( f(y|\theta) \)) with respect to measure \( \mu \) on \( Y \) with \( \sigma \)-field \( C_Y \). \( Y \) is usually subset of \( \mathbb{R}^d \) with the Borel \( \sigma \)-field, \( B_d \), \( d \geq 1 \).

\( \pi(\theta|y) \) is the posterior of \( \Theta \). \( X^* \) is a sample of size \( n \) obtained from the sampler with model \( F_{\theta^*} \). \( S(X) \) is a summary for \( X \), \( \rho \) measures the distance between \( S(x) \) and \( S(X^*) \). As statistic \( S(X) \) can be thought of as estimate of \( T(F_\theta) \), \( T \) generic functional of \( F_\theta \).

Definition 2.1 For tolerance \( \epsilon, X \) and \( S \), the \( X^* \)-matching support probability \( \alpha \) for \( \theta^* \) is

\[
P[\rho(S(X^*), S(X)) \leq \epsilon] = \alpha, \quad 0 \leq \alpha \leq 1.
\]

(3)

Given \( \epsilon > 0 \) and \( \Theta^* = \{\theta^*_1, \ldots, \theta^*_N\} \), the matching support probability for \( \Theta^* \) is

\[
\inf\{\alpha_i; i = 1, \ldots, N\};
\]

(4)

\( \alpha_i \) is obtained from (3) for \( \theta^* = \theta^*_i, i = 1, \ldots, N \).

The probability in (3) is not under one probability model as in confidence band calculations since \( X \) and \( X^* \) follow \( F_\theta \) and \( F_{\theta^*} \), respectively. When \( X = x \), \( \epsilon \) is the \( \alpha \)-quantile of \( \rho(S(X^*), S(x)) \) under \( F_{\theta^*} \) and seeing density as “small probability”,

\[
\pi(\theta^*|x) \propto f(x|\theta^*) \propto P_{\theta^*}[\rho(X^*, x) \leq \epsilon],
\]

(5)

for small \( \epsilon \), used in (6) with \( S(x) \) instead of \( x \). The \( \alpha \)-value is omitted from the notation F-ABC since it will be determined in the Algorithm, along with \( \epsilon \).

F-ABC Algorithm

1) Determination of \( \epsilon_n, \alpha_n \) : Sample several \( \theta^* \)-values either from \( \pi(\theta) \) or from a discretization of \( \Theta \) if it is known. Use one of them as base-value, \( \theta^*_b \), and obtain \( x \) generated

\footnote{We consider it part of the algorithm due to repeated samples from \( F_{\theta^*} \). If referees prefer it separated, the change will be made.}
by $\theta_b^*$. Select, e.g., 5-10 $\theta^*$ at increasing standardized distance from $\theta_b^*$ taking into consideration its nature and obtain $M \mathbf{X}^*$-samples from each one of them and $\theta_b^*$. Calculate $\rho(S(\mathbf{X}_i^*), S(\mathbf{x})), i = 1, \ldots, M$, and their empirical quantiles for each one of the selected $\theta^*$ and $\theta_b^*$. Create a table similar to Table 1 in subsection 3.1. After consultation of the quantiles decide on the $\epsilon$ to be used, determined from $\theta_{\epsilon n}^*$ with corresponding quantile $\alpha_n$.

2) Sample i.i.d. $\theta_1^*, \ldots, \theta_{N^*}^*$ from $\Theta$ according to $\pi(\theta)$.

3) Repeat for $i = 1, \ldots, N^*$; F-ABC is potentially used for all $\theta_1^*, \ldots, \theta_{N^*}^*$.

a) Sample $\mathbf{X}_1^*, \ldots, \mathbf{X}_M^*$ from $f(y|\theta_i^*)$.

b) Compute the observed matching support proportion, $p_{\text{match}}(\theta_i^*)$, for the $\mathbf{x}_1^*, \ldots, \mathbf{x}_M^*$:

$$p_{\text{match}}(\theta_i^*) = \frac{\text{Card}(\{\mathbf{x}_i^* : \rho(S(\mathbf{x}_i^*), S(\mathbf{x})) \leq \epsilon_n, i = 1, \ldots, M\})}{M}. \quad (6)$$

c) $\theta^*$-selection criterion: the F-ABC filter.\footnote{Not used in F-ABC for all $\theta^*$. It is intended for users desiring to restrict further the approximate posterior.} Include $\theta_i^*$ in the domain of $\pi(\theta|\mathbf{x})$ when

$$p_{\text{match}}(\theta_i^*) \geq \alpha_n. \quad (7)$$

4) The selected $\theta^*$ in 3) after the end of the algorithm are

$$\Theta_n^* = \{\theta_{\text{sel},i}^* ; i = 1, \ldots, N\}, \; N \leq N^*. \quad (8)$$

Use $\{(\theta_{\text{sel},i}^*, p(\theta_{\text{sel},i}^*)) ; i = 1, \ldots, N\}$ to construct the F-ABC posterior.

Definition 2.2 For $\Theta_n^*$ in (8) the observed matching support probability is $\min\{p(\theta_{\text{sel},i}^*) ; i = 1, \ldots, N\}$.

Remark 2.1 Comparing ABC with F-ABC: When $M = 1$ in 3)a) and $\alpha_n = 1$ in (7), $\rho$-F-ABC is $\rho$-ABC. To compare $\rho_1$-ABC with $\rho_2$-F-ABC, start with $\rho_2$-ABC, use $M$ additional $\mathbf{x}^*$-samples for the selected $\theta^*$ to obtain $p_{\text{match}}(\theta^*)$ for all $(M + 1)$ $\mathbf{x}^*$-drawn, and proceed with 4) to construct the $\rho_2$-F-ABC posterior. When $\alpha_n = 0$ in (7), all $\theta^*$ are selected for the posterior with their corresponding weight, $p_{\text{match}}(\theta^*)$. Let

$$B_{\epsilon n} = \{\mathbf{x}^* : \rho(S(\mathbf{x}^*), S(\mathbf{x})) \leq \epsilon_n\}, \quad (9)$$
without specifying the values of $\alpha$ and $M$ which will be determined by the context. Similarly, the F-ABC posterior of theta is

$$
\pi_f(\theta | B_{c_n}) = \frac{\pi(\theta) \cdot \int_Y I_{B_{c_n}}(y) f(y|\theta) \mu(dy)}{\int_{\Theta} \pi(s) \int_Y I_{B_{c_n}}(y) f(y|s) \mu(dy) \nu(ds)}, = \frac{\pi(\theta) \cdot P_{\theta}^{(n)}(B_{c_n})}{\int_{\Theta} \pi(s) \cdot P_{s}^{(n)}(B_{c_n}) \nu(ds)}. \tag{10}
$$

and for $H \in C_{\Theta}$, its F-ABC probability is

$$
\Pi_f(H | B_{c_n}) = \int_H \pi_f(\theta | B_{c_n}) \nu(d\theta) = \frac{\int_{\Theta} \pi(\theta) \cdot P_{\theta}^{(n)}(H \cap B_{c_n}) \nu(d\theta)}{\int_{\Theta} \pi(s) \cdot P_{s}^{(n)}(B_{c_n}) \nu(ds)}. \tag{11}
$$

For ABC, $\pi_{abc}$ and $\Pi_{abc}$ are used instead.

**Definition 2.3** For any two distribution functions $F, G$ in $\mathbb{R}^d$, $d \geq 1$, their Kolmogorov distance

$$
d_K(F, G) = \sup \{|F(y) - G(y)|; y \in \mathbb{R}^d\}. \tag{12}
$$

**Definition 2.4** For any $n$-size sample $Y = (Y_1, \ldots, Y_n)$ of random vectors in $\mathbb{R}^d$, $\hat{F}_Y(y)$ denotes the number of $Y_i$’s with all their components smaller or equal to the corresponding components of $y$. $\hat{F}_Y$ is the empirical c.d.f. of $Y$.

In section 3 for observations in $R$ use in 1) of the F-ABC Algorithm and in (6): $S(x) = \hat{F}_x, \rho = d_K$. For observations in $\mathbb{R}^d$, $d > 1, \hat{F}$ and $d_K$ will be used over 1-dimensional projections of the samples.

Implementation follows, before the theoretical results for easier reading; could follow the theoretical results, if required.

## 3 Implementation and Comparisons: ABC and F-ABC

The simulation results have no goal to compare for specific data sets F-ABC posteriors with W-ABC or ABC posteriors simply because the comparison does not make sense: F-ABC does not use an arbitrary chosen Kernel, $K(x,x^*;\epsilon)$, and has theoretical advantages with respect to ABC and W-ABC. The simulations compare ABC with $(\hat{F}_x, d_K)$ and F-ABC with parametric ABC to check the concentration of the posteriors and present
posteriors created without the use of Kernel, in particular histograms of the matching support probabilities, before using the by default R-kernel for smoothing. In Figures 1-3, separate graphs are presented, mainly for easier observation and for not mixing domains and ranges of densities having an effect in plots.

3.1 $\epsilon_n$ and matching support probability $\alpha$ in practice

The goal is to implement the selection of $\epsilon_n$ and $\alpha_n$ in 1) of the F-ABC Algorithm. When $\rho = d_K$, upper bound $\epsilon_{n,B}$ for $\epsilon_n$ is provided in section 5, but fine tuning is needed for $\epsilon_{n,B}$ to be used even for real observations. Bayesian-Frequentists and computer scientists use efficiently a powerful tool: the sampler $\mathcal{M}$ for obtaining $X^*$ from $F_{\theta^*}$. As illustration, Table 1 is provided for a sample of $n = 100$ normal random variables with mean $\theta$ and variance 1. With the notation in 1) of F-ABC algorithm, $\theta^*_b = \theta = 0$ and $x$ is obtained. $M = 500$ samples are obtained for each $\theta^* = 0, (.5), 4$ and $d_K$-distances are calculated; .5 corresponds to .5 standard deviation of the model. If $\epsilon = .63$ is used, it is expected that $\theta^*$ in the range $(-1.5, 1.5)$ are selected and the observed matching support probability (Definition 2.2) will be (at least) .95. The dependence of $\epsilon$ and $\epsilon_{n,B}$ in the distance between $F_\theta$ and $F_{\theta^*}$ is confirmed.

3.2 ABC with $d_k$ and a Euclidean distance

The goal is to compare simulated approximate posteriors of parametric ABC and non-parametric ABC with $d_K$. An ABC example in Tavaré (2019, Lectures at Columbia University, # 2, “A Normal example”, p. 35) is revisited. $X_1, \ldots, X_n$ are i.i.d. normal random variables, $\mathcal{N}(\theta, \sigma^2)$. The prior for $\theta$ is uniform $U(a, b)$ with $a \to -\infty$ and $b \to \infty$. Attention is restricted to the sample mean, $\bar{X}_n$, since it is sufficient statistic. For fixed $a, b$ the posterior $\pi(\theta|\bar{X}_n)$ is $\mathcal{N}(\theta, \sigma^2_n)$ truncated in $(a, b)$. For the ABC-simulations and a given $\epsilon^*$ it is assumed the observed $\bar{x}_n = 0$, $\theta^*$ is observed from $U(a, b)$ and is selected when $\rho(\bar{x}_n, \bar{x}_n = 0) = |\bar{x}_n^*| \leq \epsilon^*; |\cdot|$ is absolute value. A flat, “0-1”, kernel is used to select $\theta^*$.

$^4M = 500 > 200$ to increase table’s accuracy, with execution time less than 15 seconds.
Empirical Quantiles of Kolmogorov distances between $\hat{F}_x$ and $\hat{F}_x^\ast$.

| $\theta^\ast$ | MIN | 25th | 50th | 60th | 65th | 70th | 75th | 80th | 85th | 90th | 95th | MAX |
|---------------|-----|------|------|------|------|------|------|------|------|------|------|-----|
| 0             | 0.04| 0.07 | 0.09 | 0.1  | 0.11 | 0.11 | 0.12 | 0.12 | 0.13 | 0.14 | 0.19 |     |
| 0.5           | 0.12| 0.2  | 0.23 | 0.24 | 0.25 | 0.25 | 0.26 | 0.27 | 0.28 | 0.29 | 0.3  | 0.39 |
| 1             | 0.25| 0.38 | 0.41 | 0.42 | 0.42 | 0.43 | 0.44 | 0.45 | 0.46 | 0.48 | 0.55 |     |
| 1.5           | 0.47| 0.55 | 0.57 | 0.58 | 0.59 | 0.59 | 0.6  | 0.61 | 0.61 | 0.62 | 0.63 | 0.69 |
| 2             | 0.6 | 0.68 | 0.71 | 0.71 | 0.72 | 0.72 | 0.73 | 0.73 | 0.74 | 0.75 | 0.76 | 0.79 |
| 2.5           | 0.72| 0.8  | 0.82 | 0.83 | 0.83 | 0.83 | 0.84 | 0.84 | 0.85 | 0.86 | 0.87 | 0.91 |
| 3             | 0.82| 0.89 | 0.9  | 0.91 | 0.91 | 0.91 | 0.92 | 0.92 | 0.92 | 0.93 | 0.93 | 0.95 |
| 3.5           | 0.89| 0.94 | 0.95 | 0.96 | 0.96 | 0.96 | 0.96 | 0.97 | 0.97 | 0.97 | 0.97 | 0.99 |
| 4             | 0.94| 0.97 | 0.98 | 0.98 | 0.98 | 0.99 | 0.99 | 0.99 | 0.99 | 1    | 1    |     |

Table 1: Potential $\epsilon_n$-values the Quantiles, for matching support $\alpha$, $0 < \alpha < 1$.

Approximate posterior densities appear in Figure 1 for nonparametric ABC with $d_K$ and parametric ABC with $|\cdot|$. The Gaussian kernel is used by default in $R$. The observed sample $X = (X_1, \ldots, X_n)$ is from $\mathcal{N}(0, 1)$, $n = 100$. For the parametric ABC, given tolerance $\epsilon^\ast$ the steps in Tavaré (2019) are followed, using $\bar{x}_n = 0$ independently of the observed $\bar{x}_n$.

For nonparametric ABC with $d_K$, $\hat{F}_x$ is used and $\epsilon$ is such that the number of selected $\theta^\ast$ from $U(-1, 1)$ does not differ much from that of the parametric ABC. Randomness remains in the simulations but the number $N^\ast$ of drawn $\theta^\ast$ is large, $N^\ast = 1,000$, such that the number of $\theta^\ast$ selected ($N$ in Figure 1) is also large enough for determining the approximate posterior. $X^\ast_i$ is obtained from $\mathcal{N}(\theta^\ast_i, 1)$ and $\theta^\ast_i$ is selected if $d_K(\hat{F}_x, \hat{F}_{x^\ast_i}) \leq \epsilon$, $i = 1, \ldots, N^\ast$. The process is repeated for four values of $(\epsilon, \epsilon^\ast)$. In Table 2, for the selected $\theta^\ast$ their mean $\bar{\theta}^\ast$, variance and the mean square error of $\bar{\theta}^\ast$ from the mean $\theta = 0$ of the posterior are calculated. When $\epsilon = .45$ and $\epsilon^\ast = 1$, at least 95% of drawn $\theta^\ast$ are selected.

The MSE of parametric ABC posterior improves uniformly in $\epsilon$ the nonparametric ABC.
3.3 Comparison of parametric ABC with F-ABC

The goal is to compare in simulations parametric ABC with the least favorable for concentration F-ABC, i.e., neglecting the additional concentration due to 3) of the F-ABC Algorithm. Remark 2.1 is followed. Start ABC with $d_K$ and $\epsilon$ and for the selected $\theta_i^*$ in ABC, draw $M$ additional $x^*$ to compute $p_{\text{match}}(\theta_i^*)$. The F-ABC posterior for these selected $\theta^*$ is obtained. For the non-selected $\theta^*$ in ABC, $M$ additional $x^*$ are drawn to compute the corresponding $p_{\text{match}}(\theta^*)$. The F-ABC posterior for all $\theta^*$ drawn is then obtained.

In the simulations, very frequently, the concentration (MSE) of the nonparametric F-ABC improves that of parametric ABC. In Tables 3 and 4 and the corresponding Figures 2 and 3, examples are presented where the MSE of each method dominates the other. The set-up in section 3.2 is used: $\epsilon^* = .15, \epsilon = .12, n = 200, \theta = 0, a = -1, b = 1$ and $N^* = 1,000$. For F-ABC, $M = 200 X^*$-samples of size $n$ are drawn for each selected $\theta^*$, but also for non-selected $\theta^*$. A flat, “0-1”, kernel is used for selected $\theta^*$ in parametric ABC.

In Figures 2 and 3, density plots with Gaussian kernel and corresponding histograms are presented for ABC and F-ABC. For the F-ABC approximate posteriors, the bandwidth was set at 0.05. Nonparametric F-ABC for selected $\theta^*$ is satisfactory compared with parametric ABC. F-ABC for all $\theta^*$ seems satisfactory for non-believers of $\theta^*$-exclusion with limited $x^*$-data.

To compare the MSE improvement with F-ABC for selected $\theta^*$, $K = 1,000$ MSE com-
Concentration: Non Parametric ABC, F-ABC selected/drawn-Parametric ABC

| Parameter       | ABC | F-ABC selected $\theta^*$ | F-ABC all drawn $\theta^*$ | ABC |
|-----------------|-----|---------------------------|-----------------------------|-----|
| Mean $\theta^*_\text{select}$ | -0.0916 | -0.0865 | -0.0859 | -0.0117 |
| Variance $\theta^*_\text{select}$ | 0.0182 | 0.0105 | 0.0274 | 0.0107 |
| MSE $\theta^*_\text{select}$ | 0.0266 | 0.018 | 0.0348 | 0.0108 |

Table 3: Mean, Variance and MSE of $\theta^*_\text{select}$

Concentration: Non Parametric ABC, F-ABC selected/drawn-Parametric ABC

| Parameter       | ABC | F-ABC selected $\theta^*$ | F-ABC all drawn $\theta^8$ | ABC |
|-----------------|-----|---------------------------|-----------------------------|-----|
| Mean $\theta^*_\text{select}$ | -0.00198 | -0.00185 | -0.00617 | 0.0112 |
| Variance $\theta^*_\text{select}$ | 0.0187 | 0.0111 | 0.0242 | 0.0138 |
| MSE $\theta^*_\text{select}$ | 0.0187 | 0.0111 | 0.0243 | 0.0139 |

Table 4: Mean, Variance and MSE of $\theta^*_\text{select}$

Comparisons\(^5\) are made and the total number of times, $T$, F-ABC improves ABC is recorded. The parameters are $\epsilon = .12$, $\epsilon^* = .15$, $n = 100$, $\theta = 0$, $a = -1$, $b = 1$, $N^* = 100$, $M = 100$. The process is repeated 50 times out of which 48 times $T > 500$, i.e. F-ABC for selected $\theta^*$ improves the MSE of parametric ABC. A histogram of the results appear in Table 4. To realize 50 comparisons, the process was repeated 55 times because of 5 non-terminations since in F-ABC with $d_K$ there were simulations with no $\hat{F}_{x^*}$ within $\epsilon$ from $\hat{F}_x$. However, in the majority of the remaining cases the number of $x^*$ with F-ABC within $\epsilon$ from $x$ exceeded that of ABC.

\(^5\)Used for higher accuracy. No need to be repeated.
3.4 ABC and F-ABC for all $\theta^*$ in $R^2$ with $d_K$ and half-spaces

ABC and F-ABC for all, are implemented when $X = (X_1, \ldots, X_n) \in R^{nx2}$, with $d_K$ used for $X^*$- matching over all 1-dimensional projections of $X$ and $X^*$, or equivalently in half-spaces, as explained in section 5 for the sufficient, empirical measures $\mu_X, \mu_{X^*}$.

For $a, y \in R^2$, $\langle a, y \rangle$ is the inner product of $y$ and $a$, $|| \cdot ||$ is Euclidean distance in $R^2$. Using the notation in section 2, $S(X) = \mu_X$ and $a \cdot X = (\langle a, X_1 \rangle, \ldots, \langle a, X_n \rangle) \in R^n$,

$$\hat{\rho}_n(\mu_X, \mu_{X^*}) = \max_{a \in \{a_1, \ldots, a_{k_n}\} \subset U_2} d_K(\hat{F}_a X, \hat{F}_a X^*)$$

$a_1, \ldots, a_{k_n}$ are are i.i.d. uniform random vectors in $U_2 = \{u = (u_1, u_2) \in R^2 : ||u|| = 1\}$, independent of $X$ and $X^*$. Direction $a$ used in $\hat{\rho}_n$ has form $(\cos(\phi), \sin(\phi))$, with $\phi$ uniform in $[0, \pi)$. $\hat{\rho}_n$ approximates $\rho$ in (21) when $k_n \uparrow \infty$, but a moderately large $k_n = k$ is adequate. For ABC and F-ABC the number of $X^*$ $\epsilon$-matching $X$ will decrease as $k$ increases.

A sample $x$ of size $n = 50$ is observed from a bivariate normal with means $\theta = (0,2)$, variances 1 and covariance .5. Assume the parameter space is $\Theta = [-1, 2] \times [-2, 3] \subset R^2$. Instead of drawing $\theta^*$ randomly from $\Theta$, a discretization $\Theta^*$ of $\Theta$ is used in order to observe the weights $p_{\text{match}(\theta^*)}$ along $\Theta$. With $NS = 15$ equidistant $\theta_1^*$ and $\theta_2^*$, respectively, in $[-1, 2]$ and $[-2, 3]$, obtain $\theta^* = (\theta_1^*, \theta_2^*)$ in $\Theta^*, N^* = \text{card}(\Theta^*) = 225$. Following Remark 2.1, to obtain $\hat{\rho}_n$-ABC and $\hat{\rho}_n$-F-ABC posteriors, one sample $X^*$ is drawn initially for each $\theta^*$ in $\Theta^*$. 50 $a$-directions are used in $\hat{\rho}_n$, $\epsilon = .33$ and 21 $X^*$ match $X$, thus selecting 21 $\theta^*$ from $\Theta^*$. With F-ABC for all $\theta^* \in \Theta^*$, without using 3c) in the F-ABC Algorithm, $M = 200$ independent copies of $X^*$ are obtained for each $\theta^* \in \Theta^*$. For the same 50 $a$-directions and the $M + 1$ matchings, $p_{\text{match}(\theta^*)}$ in (6) is calculated for $\rho = \hat{\rho}_n$ and $\epsilon = .33$.

In Figure 5, the ABC-posterior density and the F-ABC for all $\theta^*$ posterior histogram and density appear, created with $R$-functions persp, hist3D and persp3D, respectively. Comparison of the ABC and F-ABC densities indicates higher concentration in the latter near the means $(0, 2)$. Outside an area of $(0, 2)$, the $z$-values of the densities and the histogram are 0 in all plots. In ABC (all green), the density’s shape and the 0-values in the $z$-axis are due to the bivariate normal kernel used by default in $R$-function kde2d needed in persp. In F-ABC for all, no kernel is used: the matching propostions, $p_{\text{match}(\theta^*)}$, are
the weights, frequencies and percentages, that provide the 0’s and nearly 0-values in the z-axis. \(\text{hist}^3D\) and \(\text{persp}^3D\) cannot be used in F-ABC for the selected \(\theta^*\).

An additional Example is included for the Editors, AE and referees. New parameters are: \(NS = 10, \text{card}(\Theta^*) = 100, k = 10, M = 50\). There are 11 selected \(\theta^*\) with results in Figure 6. The small number of selected \(\theta^*\) in both Examples indicates the ABC-weakness with the choice of \(\epsilon\)-value, which leads to repeated simulations for various \(\epsilon\) until a “satisfactory” posterior is obtained. F-ABC for all \(\theta^*\) does not face this problem, reducing \(\epsilon\)'s influence.

4 Differences of F-ABC and ABC methods

Main differences, some to appear in section 5, are: the universal sufficient statistics, \(\hat{F}_X\) and \(\mu_X\), and matching via \(d_K\); the F-ABC posterior for all \(\theta^*\) drawn or used; the study and choice of \(\epsilon\); the use of \(M \times^*\) for each \(\theta^*\) to obtain \(p_{\text{match}}(\theta^*)\), which is the \(\theta^*\)-weight and often depends on \(\theta\).

For the last difference, in several models it was observed for \(\tilde{d}, \rho\) generic distances that:

\[
\begin{align*}
\text{when} & \quad d_\Theta(\theta_1^*, \theta) \leq d_\Theta(\theta_2^*, \theta) \Rightarrow \bar{d}(F_{\theta_1^*}, F_\theta) \leq \bar{d}(F_{\theta_2^*}, F_\theta) \\
& \Rightarrow \forall \epsilon > 0, \quad P_{\theta_1^*}[\rho(S(X^*), T(F_\theta)) \leq \epsilon] \geq P_{\theta_2^*}[\rho(S(X^*), T(F_\theta)) \leq \epsilon].
\end{align*}
\]

Implication (13) usually holds. In F-ABC with \(\bar{d} = \rho = d_K, T(F_\theta) = F_\theta, S(X^*) = \hat{F}_X^*\), when (14) holds it will also hold, at least for large \(n\), when \(F_\theta\) is replaced by \(\hat{F}_X\). For families of c.d.fs in \(R\) with densities \(f_\theta\) such that \(f_{\theta_1^*} - f_{\theta_2^*}\) changes sign once, the upper probability of the last implication in (14) increases to 1 with \(n\) if \(\theta^*\) gets closer to \(\theta\) (Yatracos, 2020, Propositions 7.2, 7.4 and Remark 7.2). An inequality similar to (14) holds for the lower bounds of these probabilities (Proposition 5.2). Thus, it is expected the F-ABC approximate posterior concentrates near \(\theta\) more than the ABC-posterior, as observed in the simulations in subsections 3.3 and 3.4.

**Lemma 4.1** The implications leading to (14) hold for i.i.d. normal random variables with mean \(\theta\) and variance 1, \(d_\Theta = \rho = |.|, \bar{d} = d_K, S(X) = \bar{X}_n, T(F_\theta) = \theta\).
For another difference, \( X \) is used without loss of generality instead of \( S(X) \) and measures \( \nu, \mu \) are Lebesgue measures, each in a Euclidean space. For a function \( h(\theta), \theta \in \Theta \), one goal is calculation of

\[
E[h(\Theta)|X = x] = \int_\Theta h(\theta)\pi(\theta|x)d\theta. \tag{15}
\]

In ABC, (15) is approximated using the selected \( \theta^* \) in \( \Theta_{ABC}^* \),

\[
\int_\Theta h(\theta)\pi(\theta|x)d\theta \approx \int_\Theta h(\theta)\pi(\theta)\int f(x^*|\theta)K(\frac{x^*-x}{\epsilon_n})dx^*d\theta \approx \sum_{\Theta_{ABC}^*} h(\theta^*)\Pi_{ABC}(\theta^*|x); \tag{16}
\]

\( \Pi_{ABC}(\theta^*|x) \) depends on \( \pi(\theta), \epsilon_n, K \) and \( f(x^*|\theta) \) which is usually intractable or unknown.

In F-ABC, (15) is approximated using \( \Theta_{n}^* \) in (8),

\[
\int_\Theta h(\theta)\pi(\theta|x)d\theta \approx \sum_{i=1}^{N} h(\theta_{sel,i}^*)p(\theta_{sel,i}^*); \tag{17}
\]

\( p(\theta_{sel,i}^*) \) depends on additional \( x^* \) drawn, Kernel (6) with \( \rho = d_K \) and \( \epsilon_n, i = 1, \ldots, N \).

5 The Matching tools: \( \hat{F}_X, \mu_X, d_K, \tilde{\rho}, \tilde{\rho}_n, \epsilon \) and \( \alpha \)

Sufficiency, \( \hat{F}_X, \mu_X, d_K, \tilde{\rho}, \tilde{\rho}_n \)

In ABC, matching with sufficient \( S \) is preferred since \( \pi(\theta|x) = \pi(\theta|S(x)) \). When \( X \in \mathbb{R}^{nx1} \), \( \hat{F}_X \) is sufficient being equivalent to the order statistic. When \( X = (X_1, \ldots, X_n) \in \mathbb{R}^{nxd}, d > 1, \) and \( X_1, \ldots, X_n \) are either i.i.d. or exchangeable, the empirical measure, \( \mu_X \),

\[
\mu_X(A) = n^{-1}\sum_{i=1}^{n} I_A(X_i), \ A \in \mathcal{B}_d; \tag{18}
\]

is sufficient, respectively by, Dudley (1984, Theorem 10.1.3, p. 95) and de Finetti’s Theorem, e.g., Lauritzen (2007, in Statistical Implications section); \( \mathcal{B}_d \) are the Borel sets in \( \mathbb{R}^d \). When \( d > 1 \), for some models \( \hat{F}_X \) may be nearly sufficient but still better than guessing \( S \).

For \( x \in \mathbb{R}^d, d > 1, \) to guarantee sufficiency, \( \mu_X \) is used for \( \epsilon \)-matching \( X \) with \( X^* \). As explained below, instead of using for matching the usual form of Total Variation distance,

\[
\tilde{\rho}(\mu_X, \mu_X^*) = \sup_{A \in \mathcal{B}_d} |\mu_X(A) - \mu_X^*(A)| = TV(\mu_X, \mu_X^*), \tag{19}
\]
the supremum in (19) is over all half-spaces,
\[ A(a,t) = \{ y \in \mathbb{R}^d : <a,y> \leq t \}, \quad t \in \mathbb{R}, \quad a \in U_d = \{ u = (u_1, \ldots, u_d) \in \mathbb{R}^d : ||u|| = 1 \}; \]
(20)

\(<a,y>\) is the inner product of \(y\) and \(a\), \(||\cdot||\) is Euclidean distance in \(\mathbb{R}^d\). Then,
\[ \tilde{\rho}(\mu_X, \mu_{X^*}) = \sup_{a \in U_d} \sup_{t \in \mathbb{R}} |\mu_X(A(a,t)) - \mu_{X^*}(A(a,t))| = \sup_{a \in U_d} d_K(\hat{F}_{a,X}, \hat{F}_{a,X^*}), \]
(21)

\[ a \cdot X = (<a,X_1>, \ldots, <a,X_n>) \in \mathbb{R}^n. \]

In practice, \(\tilde{\rho}(\mu_X, \mu_{X^*})\) is approximated by
\[ \tilde{\rho}_n(\mu_X, \mu_{X^*}) = \max_{a \in \{a_1, \ldots, a_{k_n}\} \subset U_d} \sup_{t \in \mathbb{R}} |\mu_X(A(a,t)) - \mu_{X^*}(A(a,t))| = \max_{a \in \{a_1, \ldots, a_{k_n}\} \subset U_d} d_K(\hat{F}_{a,X}, \hat{F}_{a,X^*}). \]
(22)

where \(a_1, \ldots, a_{k_n}\) are either a discretization of \(U_d\) or i.i.d. uniform in \(U_d\), independent of \(X\) and \(X^*\), leading to approximate sufficiency. Using \(A = A(a,t)\) in (18),
\[ I_{A(a,t)}(X_i) = 1 \iff <a,X_i> \leq t \Rightarrow \mu_X(A(a,t)) = \frac{\text{Card}(<a,X_i> \leq t, i = 1, \ldots, n)}{n} = \hat{F}_{a,X}(t), \]
(23)

and the last equalities in (21) and (22) follow, relating \(\tilde{\rho}\) over all half-spaces with \(d_K\)-distance over all 1-dimensional projections of \(X, X^*\). Hence, in applications, \(X\) will match \(X^*\) when the last term in (22) is less than or equal to \(\epsilon_n\), with the \(R\)-functions used for \(d = 1\).

\(\tilde{\rho}\) and \(\tilde{\rho}_n\)

If \(P\) and \(Q\) are probabilities in \((\mathbb{R}^d, \mathcal{B}_d)\) which are equal over all half-spaces, \(A(a,t)\), in (20), then \(P\) and \(Q\) are equal for every \(A \in \mathcal{B}_d\) (Cramér and Wold, 1936). When \(X\)'s coordinates follow the unknown probability \(P \in \mathcal{P}\) and \(\tilde{\rho}\) is defined in (21), Beran and Millar (1986, p. 431-433, Theorem 3, p. 436) obtained confidence sets \(\{Q \in \mathcal{P} : \tilde{\rho}(\mu_X, Q) < c\}\) for \(P\) using \(\tilde{\rho}_n\) with \(a_1, \ldots, a_{k_n}\) i.i.d. uniform on \(U_d\), and showed that when \(k_n \uparrow \infty\) as \(n \uparrow \infty\), then \(\lim_{n \to \infty} \tilde{\rho}_n(P,Q) = \tilde{\rho}(P,Q)\) with probability 1 and asymptotically the required coverage is achieved.

\(\text{Pertinent properties of } \hat{F}_X, d_K, \mu_X\)

\(\hat{F}_X\) and \(d_K\) satisfy desired properties for summary statistics (Fearnhead and Prangle, 2012, Frazier \textit{et al.}, 2018) when \(F_\theta\) is the parameter of interest: a) \(E\hat{F}_X = F_\theta, b) F_{\theta_1} = F_{\theta_2}\)
implies $\theta_1 = \theta_2$ due to identifiability, and c) there are various types of $\hat{F}_X$’s convergence to $F_\theta$, including $d_K$-convergence. When $T(F_\theta) = \theta$ and $T$ is continuous with respect to $d_K$ and a metric $d_\Theta$ on $\Theta$, it is expected that $T(\hat{F}_X)$ as estimate of $\theta$ will inherit convergence properties of $\hat{F}_X$ to $F_\theta$. Similar results hold for the empirical measure, $\mu_X$, its corresponding probability $P_\theta$ and the class of half-spaces which is Vapnik-Cervonenkis class of sets with index $(d+1)$, see, e.g. Dudley (1978).

$d_K(\hat{F}_{X^*}, \hat{F}_X)$ is not continuous function in $\mathbb{R}^n$ at $x$ since it cannot be smaller than $\frac{1}{n}$ for all $x^*$ at Euclidean distance $\delta > 0$ from $x$. This makes $d_K$ different from other $\rho$-distances used in ABC, (1), (2); see, e.g. Bernton et al. (2019, p. 39, proof of Proposition 3.1).

**Lemma 5.1** For any observed samples of size $n$, $x^* \neq x_{\sigma(1:n)} \in \mathbb{R}^d$, $d \geq 1$,

$$d_K(\hat{F}_X, \hat{F}_{X^*}) \geq \frac{1}{n}; \quad (24)$$

$x_{\sigma(1:n)}$ denotes a vector, permutation of the $x$ components. Thus,

$$d_K(\hat{F}_X, \hat{F}_{X^*}) = 0 \iff x^* = x_{\sigma(1:n)}. \quad (25)$$

$\epsilon_n, \alpha$ and $d_K$

For matching support probability $\alpha$ in (3), the F-ABC tolerance $\epsilon_n$ satisfies

$$P[d_K(\hat{F}_{X^*}, \hat{F}_X) > \epsilon_n] = 1 - \alpha, \quad 0 \leq \alpha \leq 1. \quad (26)$$

An upper bound $\epsilon_{n,B}$ on $\epsilon_n$ is obtained equating an upper probability bound in (26) with $1 - \alpha$; see Lemma 7.1. Conditionally on $X = x$, $\epsilon_{n,B}(x)$ is similarly obtained under $F_{\theta^*}$.

The $\epsilon_n$ upper bounds follow for $X$ and $X^* \in \mathbb{R}^{n\times d}, d = 1$. When $d > 1$, similar results hold presented after the Proof of Proposition 5.1

**Proposition 5.1** Let $X$ be a sample of $n$ random variables from cumulative distribution $F_\theta$, with $\theta$ unknown, let $X^*$ be a simulated $n$-size sample from a sampler used for $\theta^*$ and let $\alpha$ be the matching support probability for the tolerance $\epsilon_n$ in (26); $0 \leq \alpha < 1$.

a) The upper bound for $\epsilon_n$ is

$$\epsilon_{n,B}(\theta, \theta^*) = d_K(F_\theta, F_{\theta^*}) + \sqrt{\frac{2}{n} \ln \frac{4}{1 - \alpha}} \geq \sqrt{\frac{2}{n} \ln 4}. \quad (27)$$
b) Conditionally on \( X = x \), the upper bound for \( \epsilon_n \) is

\[
\epsilon_{n,B}(x, \theta^*) = d_K(\hat{F}_x, F_{\theta^*}) + \sqrt{\frac{1}{2n} \ln \frac{2}{1 - \alpha}} \geq \delta_n(x, \theta^*) + \sqrt{\frac{1}{2n} \ln 2}. \tag{28}
\]

In practice, \( \min\{\epsilon_n,B(\theta, \theta^*), 1\} \) and \( \min\{\epsilon_n,B(x, \theta^*), 1\} \) are used.

(27) and (28) provide a structure for the tolerance. Since \( F_{\theta} \) is unknown and \( \theta^* \in \Theta^* \), uniform upper bounds are useful. Since \( \hat{F}_X \) is with high probability at \( d_K \)-distance \( \frac{C_n}{\sqrt{m}} \) from \( F_{\theta} \), a plausible choice for the uniform upper bounds of \( d_K(F_{\theta}, F_{\theta^*}) \) and \( d_K(\hat{F}_X, F_{\theta^*}) \) is \( \frac{C_n}{\sqrt{m}} \), with \( C_n^* > C_n > 0 \). Probability bounds are rarely tight and, in practice, \( \epsilon_n \) is determined via simulations; see Table 1 in subsection 3.1.

The next Proposition indicates that for the lower bounds on the Probabilities in (14), the same inequality holds when \( d_K(F_{\theta_1}, F_{\theta}) < d_K(F_{\theta_2}, F_{\theta}) < \epsilon_n \).

**Proposition 5.2** For \( n \) i.i.d. random vectors in \( \mathbb{R}^d \) with c.d.f. \( F_{\theta} \) and \( n \) large:

\[
P_{\theta}[d_K(F_{X^*}, \hat{F}_X) \leq \epsilon_n] \geq 1 - C_1^*(d) \cdot \exp\{-n \cdot C_2^*(d) \cdot (\epsilon_n - d_K(F_{\theta^*}, F_{\theta}))^2\}; \tag{29}
\]

\( C_1^*(d), C_2^*(d) \) are positive constants.

### 6 Asymptotics

Results obtained for Kolmogorov distance, \( d_K \), when \( X \in \mathbb{R}^{nxd} \), hold also for the stronger distance \( (21) \) using \( d_K \) on all half-spaces in \( \mathbb{R}^d \).

In ABC, one question of interest is whether \( \pi_{abc}(\theta|B_\epsilon) \) converges to \( \pi(\theta|x) \) when \( x \) stays fixed and \( \epsilon = \delta_m \downarrow 0 \) as \( m \) increases.

**Proposition 6.1** Use the notation in section 2, for ABC and F-ABC with \( S(X) = \hat{F}_X, \rho = d_K, n \) fixed and \( B_{\epsilon_n} \) in (9). Under the exchangeability assumption, i.e. \( f(y|\theta) = f(y_{\sigma(1:n)}|\theta) \) for any permutation \( y_{\sigma(1:n)} \) of \( y \), and with \( \delta_m \downarrow 0 \) as \( m \) increases,

\[
\lim_{m \to \infty} \pi_u(\theta|B_{\delta_m}) = \pi(\theta|x), \quad u = abc, f-abc. \tag{30}
\]

For continuous \( X \), \( (Y, C_Y) \) is \( \mathbb{R}^{nxd} \) with the Borel sets, \( B \), and \( \Theta \) takes values in \( \mathbb{R}^k, k \leq d \).
Another question of interest for ABC is whether the posterior \( \pi_{\text{abc}}(\theta|B_{\epsilon_n}) \) will place increasing probability mass around \( \theta \) as \( n \) increases to infinity (Fearnhead, 2018), i.e. Bayesian consistency. Posterior concentration is proved for ABC and F-ABC, initially for fixed size \( \zeta \)-neighborhood when \( T(F_\theta) \) is the quantity of interest; \( T \) is a functional, \( \zeta > 0 \).

**Proposition 6.2** Use the notation in section 2 and let \( \mathcal{F}_\Theta = \{F_\theta, \theta \in \Theta\} \) be subset of a metric space \( (\mathcal{F}, d_\mathcal{F}) \) of c.d.fs. Assume

a) \( d_\mathcal{F}(\hat{F}_X, F_\theta) \leq o(k_n), k_n \uparrow \infty \) and \( F_\theta^{(n)} \)-probability \( \uparrow 1 \), as \( n \) increases, and

b) \( T \) is a continuous functional on \( \mathcal{F} \) with values in a metric space \( (T, d_T) \).

Then, for ABC and F-ABC, \( S(X) = \hat{F}_X, \rho = d_\mathcal{F} \) and for any \( \zeta > 0 \)

\[
\lim_{n \to \infty} \Pi_u[\theta^* : d_T(T(F_{\theta^*}), T(F_\theta)) \leq \zeta|B_{\epsilon_n}] = 1, \quad u = \text{abc, f-abc}; \quad (31)
\]

\[
B_{\epsilon_n} = \{x^* : d_\mathcal{F}(\hat{F}(x^*), \hat{F}(x)) \leq \epsilon_n\}, \epsilon_n \downarrow 0 \text{ as } n \uparrow \infty. \quad (32)
\]

**Remark 6.1** In Proposition 6.2, assumption a) holds for \( d_\mathcal{F} = d_K, k_n = \sqrt{n} \); special case of interest in b) when \( T(F_\theta) = \theta \) and \( d_T = d_\Theta \), the metric on \( \Theta \).

To confirm Bayesian consistency for shrinking \( d_T \)-neighborhoods of \( T(F_\theta) \), let \( w \) be the modulus of continuity of \( T \), i.e.

\[
w(\bar{\epsilon}) = \sup\{d_T(T(F_\theta), T(F_\eta)) : d_\mathcal{F}(F_\theta, F_\eta) \leq \bar{\epsilon}; \theta \in \Theta, \eta \in \Theta\}, \quad \bar{\epsilon} > 0. \quad (33)
\]

Consistency was established for \( \zeta \)-\( d_T \)-neighborhood of \( T(F_\theta) \) when (47) holds, i.e. when

\[
\epsilon_n \leq \bar{\epsilon} = \frac{2o(k_n)}{k_n},
\]

thus it holds for the smallest \( \bar{\epsilon} \)-value,

\[
\bar{\epsilon} = \epsilon_n + \frac{2o(k_n)}{k_n} \quad (34)
\]

and since for \( \zeta_n \)-\( d_T \)-neighborhood of \( T(F_\theta) \)

\[
\zeta_n = w(\bar{\epsilon})
\]

it follows that

\[
\zeta_n = w(\epsilon_n + \frac{2o(k_n)}{k_n}) \geq w(\frac{2o(k_n)}{k_n}). \quad (35)
\]


**Lemma 6.1** Under the assumptions of Proposition 6.2, the shortest \( d_T \)-shrinking neighborhood of \( T(F_0) \) for which Bayesian consistency holds has radius \( w(\epsilon_n + \frac{2\alpha(k_n)}{k_n}) \geq w(\frac{2\alpha(k_n)}{k_n}) \).

**Remark 6.2** The rate of posterior concentration around \( T(F_0) \) depends, as expected, on the rate in probability, \( k_n^{-1} \), of the \( d\_\mathcal{F} \)-concentration of \( T(\hat{F}_{x}) \) around \( T(F_0) \) which is not under the user’s control, the tolerance \( \epsilon_n \) and the modulus of continuity, \( w \), of \( T \). Similar conclusions in a different set-up have been obtained by Frazier et al. (2018).

### 7 Annex

**Proof of Lemma 4.1:** The first implication holds from the corresponding models, \( w.l.o.g. \) for \( \theta < \theta^*_1 < \theta^*_2 \), by observing that \( d_K(F_0,F^*_1) = F_0(.5(\theta + \theta^*_1)) - F^*_1(.5(\theta + \theta^*_1)) \) and comparing with \( d_K(F_0,F^*_2) \). The last implication holds from the assumption since

\[
G(\theta^*) = \Phi[\sqrt{n}(\epsilon + \theta - \theta^*)] - \Phi[\sqrt{n}(-\epsilon + \theta - \theta^*)]
\]

is decreasing in \( \theta^* \) when \( \theta^* > \theta \) and increasing in \( \theta^* \) when \( \theta^* < \theta \), and determines the probabilities in (14) for \( \theta^* = \theta^*_1, \theta^*_2 \). Indeed,

\[
P_{\theta^*}(|\bar{X}_n - \theta| \leq \epsilon) = P_{\theta^*}(-\epsilon + \theta \leq \bar{X}_n \leq \epsilon + \theta) = P[\sqrt{n}(\epsilon + \theta - \theta^*) \leq Z \leq \sqrt{n}(\epsilon + \theta - \theta^*)]
\]

\[
= \Phi[\sqrt{n}(\epsilon + \theta - \theta^*)] - \Phi[\sqrt{n}(-\epsilon + \theta - \theta^*)] = G(\theta^*)
\]

\[
G'(\theta^*) = -\sqrt{n}\phi(\sqrt{n}(\epsilon + \theta - \theta^*)) + \sqrt{n}\phi(\sqrt{n}(-\epsilon + \theta - \theta^*)) < 0
\]

\[
\iff \phi(\sqrt{n}(\epsilon + \theta - \theta^*)) < \phi(\sqrt{n}(\epsilon + \theta - \theta^*)) \iff -(\epsilon + \theta - \theta^*)^2 < -(\epsilon + \theta - \theta^*)^2
\]

\[
\iff 2\epsilon(\theta - \theta^*) < -2\epsilon(\theta - \theta^*) \iff 4\epsilon(\theta - \theta^*) < 0 \iff \theta < \theta^*,
\]

hence if \( \theta < \theta^* \), \( G(\theta^*) \) is decreasing in \( \theta^* \), \( \theta < \theta^*_1 < \theta^*_2 \) \( \Rightarrow \) \( G(\theta^*_1) > G(\theta^*_2) \). For \( \theta^* < \theta \), \( G(\theta^*) \) is increasing, \( \theta^*_2 < \theta^*_1 < \theta \) \( \Rightarrow \) \( G(\theta^*_2) < G(\theta^*_1) \).

**Proof of Lemma 5.1:** The smaller \( d_K \)-distance between \( \hat{F}_{x} \) and \( \hat{F}_{x^*} \) occurs when \( x, x^* \) differ by a small \( \delta > 0 \) in one coordinate of one observation and their distance is \( \frac{1}{n} \). \( \square \)
Lemma 7.1 Let $X = x, X^* = x^*$ and let $U(n, \epsilon)$ be positive function defined for positive integers $n$ and $\epsilon > 0, 0 \leq \alpha \leq 1$, such that

$$1 - \alpha = P[d_K(\hat{F}_x, \hat{F}_{x^*}) > \epsilon] \leq U(n, \epsilon).$$  \hspace{1cm} (36)

Let $\epsilon_B : U(n, \epsilon_B) = 1 - \alpha$. Then $\epsilon_B \geq \epsilon$.

Proof of Lemma 7.1 Since $U(n, \epsilon_B) = 1 - \alpha$,

$$P[d_K(\hat{F}_x, \hat{F}_{x^*}) > \epsilon_B] \leq U(n, \epsilon_B) = 1 - \alpha = P[d_K(\hat{F}_x, \hat{F}_{x^*}) > \epsilon]$$

which implies $\epsilon_B \geq \epsilon$. \hfill $\square$

Theorem 7.1 (Dvoretzky, Kiefer and Wolfowitz, 1956, and Massart, 1990, providing the tight constant) Let $\hat{F}_Y$ denote the empirical c.d.f of the size $n$ sample $Y$ of i.i.d. random variables obtained from cumulative distribution $F$. Then, for any $\epsilon > 0$,

$$P[d_K(\hat{F}_Y, F) > \epsilon] \leq U_{DKWM} = 2e^{-2n\epsilon^2}$$ \hspace{1cm} (37)

Proof of Proposition 5.1

a) $$P[d_K(\hat{F}_X^*, \hat{F}_X) > \epsilon_n] \leq P[d_K(\hat{F}_X^*, F_{\theta^*}) + d_K(F_{\theta^*}, F_{\theta}) + d_K(F_{\theta}, \hat{F}_x) > \epsilon_n]$$

$$\leq P[d_K(\hat{F}_X^*, F_{\theta^*}) > \frac{\epsilon_n - d_K(F_{\theta^*}, F_{\theta})}{2}] + P[d_K(\hat{F}_X, F_{\theta}) > \frac{\epsilon_n - d_K(F_{\theta^*}, F_{\theta})}{2}]$$

$$\leq 4 \exp\left\{-\frac{n}{2}(\epsilon_n - d_K(F_{\theta^*}, F_{\theta}))^2\right\}$$

The right side of the last inequality, obtained from (37) is made equal to $1 - \alpha$,

$$4 \exp\left\{-\frac{n}{2}(\epsilon_{n,B} - d_K(F_{\theta^*}, F_{\theta}))^2\right\} = 1 - a \iff \epsilon_{n,B} = d_K(F_{\theta^*}, F_{\theta}) + \sqrt{\frac{2}{n} \ln \frac{4}{1 - \alpha}}.$$  

b) $P[d_K(\hat{F}_X^*, \hat{F}_x) > \epsilon_n] \leq P[d_K(\hat{F}_X^*, F_{\theta^*}) + d_K(F_{\theta^*}, \hat{F}_x) > \epsilon_n] \leq 2 \exp\left\{-2n(\epsilon_n - d_K(F_{\theta^*}, \hat{F}_x))^2\right\}$

obtaining with matching support probability $\alpha$,

$$\epsilon_{n,B}(x) = d_K(F_{\theta^*}, \hat{F}_x) + \sqrt{\frac{1}{2n} \ln \frac{2}{1 - \alpha}}. \hfill \square$$

Generalizations of (37) in $R^d$ have been obtained, at least, by Kiefer and Wolfowitz (1958), Kiefer (1961) and Devroye (1977); $d > 1$. The differences in upper bound $U$ in (37)
are in the multiplicative constant, in the exponent of the exponential and on the sample size for which the exponential bound holds which may also depend on $\epsilon$. The constants used are not determined except for Devroye (1977).

For example, following the Proof in Proposition $5.1 \ b$, conditionally on $X = x$:

i) Using Kiefer and Wolfowitz (1958), with the upper bound in (37) $U_{KW} = C_1(d) e^{-C_2(d) n \epsilon}$,

$$
\epsilon_{n,B}(x, \theta^*) = d_K(\hat{F}_x, F_{\theta^*}) + \sqrt{\frac{1}{nC_2(d)} \ln C_1(d) 1 - \alpha}.
$$

ii) Using Kiefer (1961), with the upper bound in (37) $U_K = C_3(b,d) e^{-(2-b)n \epsilon}$, for every $b \in (0,2)$,

$$
\epsilon_{n,B}(x, \theta^*) = d_K(\hat{F}_x, F_{\theta^*}) + \sqrt{\frac{1}{n(2-b)} \ln C_3(b,d) 1 - \alpha}.
$$

iii) Using Devroye (1977), with the upper bound in (37) $U_{De} = 2e^{2(2n)^d} e^{-2n \epsilon}$ valid for $n \epsilon^2 \geq d^2$,

$$
\epsilon_{n,B}(x, \theta^*) = d_K(\hat{F}_x, F_{\theta^*}) + \sqrt{\frac{1}{2n} [\ln \frac{2}{1 - \alpha} + 2 + d \ln(2n)].}
$$

**Remark 7.1** In (27), (28) and in i)-iii), $\epsilon_{n,B}$ is the sum of the model discrepancy of $F_{\theta^*}$ from either $F_{\theta^*}$ or $\hat{F}_x$ and a confidence term, determined, respectively, under both $F_{\theta^*}$ and $F_{\theta^*}$ or the latter only. $\epsilon_{n,B}$ is independent of $\theta^*$ and $x$ in (27) only. In all cases, since $F_{\theta^*}$ are unknown, a bound will be used for $d_K(F_{\theta^*}, F_{\theta^*}), d_K(\hat{F}_x, F_{\theta^*})$.

**Proof of Proposition 5.2:** Follows along the first three lines in the proof of Proposition $5.1 \ a$, with the exponential upper bound obtained using the $U_{KW}$ above in i) (Kiefer and Wolfowitz, 1958), with $C_1^*(d), C_2^*(d)$ the adjustments of $C_1(d), C_2(d)$.

**Proof of Proposition 6.1:** The arguments used for ABC hold for F-ABC.

a) $Y$ discrete: The ABC posterior with $\rho = d_K$ in (10) is

$$
\pi_{abc}(\theta | B_{\delta_m}) = \frac{\pi(\theta) \cdot \int_{\mathcal{Y}} I_{B_{\delta_m}}(y^*) f(y^* | \theta) \mu(dy^*)}{\int_{\Theta} \pi(s) \int_{\mathcal{Y}} I_{B_{\delta_m}}(y^*) f(y^* | s) \mu(dy^*) \nu(ds)}.
$$

With integral denoting sum, it is enough to prove that the integral in the numerator of $\pi_{abc}(\theta | B_{\delta_m})$ is proportional to $f(x | \theta)$. 

23
For $A \in \mathcal{C}_Y$, let
\[ Q_\theta(A) = \int_A f(y^*|\theta)\mu(dy^*), \quad A \in \mathcal{A}. \]
$Q_\theta$ is a probability measure on $\mathcal{C}_Y$.

Since $n$ and $x$ are fixed, for $\delta_k \geq \frac{1}{n} > \delta_{k+1}$
\[ B_{\delta_1} \supseteq B_{\delta_2} \supseteq \ldots \supseteq B_{\delta_k} \quad (38) \]
and from Lemma 5.1 for $m > k$, $B_{\delta_m} = \{x_{\sigma(1:n)}\}$. Therefore,
\[ \lim_{m \to \infty} B_{\delta_m} = \cap_{m=1}^{\infty} B_{\delta_m} = \{x_{\sigma(1:n)}\} \quad (39) \]
and
\[ \lim_{m \to \infty} \int_Y I_{B_{\delta_m}}(y^*)f(y^*|\theta)\mu(dy^*) = \lim_{m \to \infty} Q_\theta(B_{\delta_m}) = Q_\theta(\cap_{m=1}^{\infty} B_{\delta_m}) = f(x|\theta)\mu(\{x_{\sigma(1:n)}\}), \quad (40) \]
with the last equality due to exchangeability of $f(x|\theta)$.

b) $Y$ continuous: Then, the right side of (40) vanishes, since $\mu(\{x_{\sigma(1:n)}\}) = 0$. A different approach is used, via the notion of regular conditional probability.

When $Y$ is a Euclidean space $\mathbb{R}^{n \times d}$ with Borel $\sigma$-field, $\mathcal{B}_d$, and $\Theta$ takes values in $\mathbb{R}^k, k \leq d$, the integral in the numerator of $\pi_{abc}(\theta|B_{\delta_m})$,
\[ \int_Y I_{B_{\delta_m}}(y^*)f(y^*|\theta)\mu(dy^*) \]
is a regular conditional probability, $P[X^* \in B|\Theta = \theta], \ B = B_{\delta_m}$ (Breiman, 1992, Chapter 4, p. 79, Theorem 4.34), i.e., with $\theta$ fixed, it is a probability for $B \in \mathcal{B}_d$ and with fixed $B$ it is a version of the conditional density, $\theta \in \Theta$. Thus, for fixed $\theta$, from (39),
\[ \lim_{m \to \infty} P[X^* \in B_{\delta_m}|\Theta = \theta] = P[\{x_{\sigma(1:n)}\}|\Theta = \theta] \]
and due to exchangeability is proportional to $f(x|\theta)$ a.s. . \hfill \Box

**Proof of Proposition 6.2**: The arguments used for ABC hold for F-ABC. For the probability in (31), using (11) for ABC with
\[ H = \{\theta^* : d_T(T(F_{\theta^*}), T(F_\theta)) \leq \zeta\}, \quad (41) \]
\[ \Pi_{abc}(H|B_{\epsilon_n}) = \frac{\int_\Theta I_H(\theta^*) \pi(\theta^*) \cdot \int_\mathbb{Y} I_{B_{\epsilon_n}}(y^*) f(y^*|\theta^*) \mu(dy^*) \nu(d\theta^*)}{\int_\Theta \pi(\theta^*) \int_\mathbb{Y} I_{B_{\epsilon_n}}(y^*) f(y^*|s) \mu(dy^*) \nu(ds)} = \frac{\int_\Theta \pi(\theta^*) \cdot P_{\theta^*}^{(n)}(H \cap B_{\epsilon_n}) \nu(d\theta)}{\int_\Theta \pi(s) \cdot P_s^{(n)}(B_{\epsilon_n}) \nu(ds)}. \]  

\( P_{\theta^*}^{(n)}(H \cap B_{\epsilon_n}) \) in the numerators of (42) will be bounded below using continuity of \( T \) and triangular inequality.

Since \( T \) is continuous, for \( \zeta > 0 \) there is \( \tilde{\epsilon} > 0 \) such that if

\[ d_T(F_{\theta^*}, T(F_\theta)) \leq \zeta, \]

and then from (32), (41)

\[ P_{\theta^*}^{(n)}(H \cap B_{\epsilon_n}) \geq P_{\theta^*}^{(n)}[d_T(F_{\theta^*}, F_\theta) \leq \tilde{\epsilon} \cap B_{\epsilon_n}]. \]  

Since

\[ d_T(F_{\theta^*}, F_\theta) \leq d_T(F_{\theta^*}, \hat{F}_{X^*}) + d_T(\hat{F}_{X^*}, \hat{F}_X) + d_T(\hat{F}_X, F_\theta) \]

if

\[ d_T(F_{\theta^*}, \hat{F}_{X^*}) + d_T(\hat{F}_{X^*}, \hat{F}_X) + d_T(\hat{F}_X, F_\theta) \leq \tilde{\epsilon} \]

and therefore, for the right side of (43)

\[ P_{\theta^*}^{(n)}[d_T(F_{\theta^*}, F_\theta) \leq \tilde{\epsilon} \cap B_{\epsilon_n}] \geq P_{\theta^*}^{(n)}[d_T(F_{\theta^*}, \hat{F}_{X^*}) + d_T(\hat{F}_{X^*}, \hat{F}_X) + d_T(\hat{F}_X, F_\theta) \leq \tilde{\epsilon} \cap B_{\epsilon_n}]. \]  

From the assumptions,

\[ d_T(F_{\theta^*}, \hat{F}_{X^*}) \leq \frac{o(k_n)}{k_n} \quad \text{and} \quad d_T(F_\theta, \hat{F}_X) \leq \frac{o(k_n)}{k_n} \]

with \( P_{\theta^*}^{(n)} \) and \( P_{\theta}^{(n)} \) probabilities converging to one, respectively, and assuming \( x^*, x \) are in these subsets the right side of (45)

\[ P_{\theta^*}^{(n)}[d_T(F_{\theta^*}, \hat{F}_{X^*}) + d_T(\hat{F}_{X^*}, \hat{F}_X) + d_T(\hat{F}_X, F_\theta) \leq \tilde{\epsilon} \cap B_{\epsilon_n}] \geq P_{\theta^*}^{(n)}[d_T(\hat{F}_{X^*}, \hat{F}_X) \leq \tilde{\epsilon} - \frac{2o(k_n)}{k_n} \cap B_{\epsilon_n}]. \]  

For \( \epsilon_n \downarrow 0 \) as \( n \) increases, eventually

\[ \epsilon_n \leq \tilde{\epsilon} - \frac{2o(k_n)}{k_n}, \]  

and the right side of (46)

\[ P_{\theta^*}^{(n)}[d_T(\hat{F}_{X^*}, \hat{F}_X) \leq \tilde{\epsilon} - \frac{2o(k_n)}{k_n} \cap B_{\epsilon_n}] = P_{\theta^*}^{(n)}[B_{\epsilon_n}]. \]  

25
follows from (43), (45)-(48) since, when taking the limit in (42) as $n$ increases to infinity, for large $n$ numerator and denominator coincide. \[ \square. \]

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Figure 1: Approximate posterior densities for various tolerance levels
Figure 2: Approximate posterior densities and histograms for ABC and F-ABC # 1
Figure 3: Approximate posterior densities and histograms for ABC and F-ABC # 2
Figure 4: F-ABC improves the MSE of ABC more than 90% of the time.
Figure 5: $d_K$ on discretized half-spaces for ABC and F-ABC for all. ABC: weights from default kernel in $R$. F-ABC for all: weights from repeated samples. K=50 NS=15 M=200
Figure 6: $d_K$ on discretized half-spaces for ABC and F-ABC for all. ABC: weights from default kernel in $R$. F-ABC for all: weights from repeated samples. $K=10$ NS=10 $M=50$