GEOMETRY OF SELF-SIMILAR MEASURES ON INTERVALS WITH OVERLAPS AND APPLICATIONS TO SUB-GAUSSIAN HEAT KERNEL ESTIMATES

QINGSONG GU
Department of Mathematics
The Chinese University of Hong Kong, China

JIAXIN HU
Department of Mathematical Sciences
Tsinghua University, Beijing 100084, China

SZE-MAN NGAI*
Key Laboratory of HPCSIP, Ministry of Education of China
College of Mathematics and Statistics, Hunan Normal University
Changsha, Hunan 410081, China
and Department of Mathematical Sciences
Georgia Southern University, Statesboro, GA 30460-8093, USA
(Communicated by Camil Muscalu)

Abstract. We study the geometric properties of self-similar measures on intervals generated by iterated function systems (IFS’s) that do not satisfy the open set condition (OSC) and have overlaps. The examples studied in this paper are the infinite Bernoulli convolution associated with the golden ratio, and a family of convolutions of Cantor-type measures. We make use of Strichartz second-order identities defined by auxiliary IFS’s to compute measures of cells on different levels. These auxiliary IFS’s do satisfy the OSC and are used to define new metrics. As an application, we obtain sub-Gaussian heat kernel estimates of the time changed Brownian motions with respect to these measures. The walk dimensions obtained under these new metrics are strictly greater than 2 and are closely related to the spectral dimension of fractal Laplacians.

1. Introduction. Heat kernel estimates for local Dirichlet forms on fractals are typically sub-Gaussian. This has been shown by many authors on different classes of fractals, among which are: Barlow and Perkins for the Sierpiński gasket [4], Kumagai [20] for nested fractals, Fitzsimmons, Hambly and Kumagai [8] for affine nested fractals, Hambly and Kumagai [14] (see also [22]) for post-critically finite self-similar sets, Kumagai [21] and Kigami [18] for resistance forms, Barlow and

2000 Mathematics Subject Classification. Primary: 28A80, 35K08; Secondary: 35J05.
Key words and phrases. Heat kernel, sub-Gaussian, self-similar measure, second-order identities.

This research was supported in part by the National Natural Science Foundation of China, grants 11871296, 11771136 and 11271122, by Tsinghua University Initiative Scientific Research Program, and by a Hong Kong RGC grant. SN is also supported in part by Construct Program of the Key Discipline in Hunan Province, the Hunan Province Hundred Talents Program, and the Center of Mathematical Sciences and Applications of Harvard University.

*Corresponding author.
Bass [1, 2] for the Sierpiński carpets, and by Kigami [17, 19] for time changes of self-similar diffusions on self-similar sets. Equivalent conditions for two-sided estimates of heat kernels for local Dirichlet forms on metric measure spaces are given by Grigor’yan, Lau and the second author [10, 11], Grigor’yan and Telcs [13], and others (see also [3] and [15] for certain classes of resistance forms).

The most typical measures on fractals appearing in such studies are $s$-dimensional Hausdorff measures for some number $s > 0$, and they are equivalent to self-similar measures generated by iterated function systems (IFS’s) satisfying the open set condition (OSC). This paper studies self-similar measures generated by IFS’s that do not satisfy the OSC. These measures are not Ahlfors-regular but still possess the doubling property with respect to suitable metrics. Although the self-similar sets themselves are intervals and hence Dirichlet forms can be defined easily, the associated self-similar measures exhibit complicated fractal behavior, and therefore heat kernel estimates become much more awkward.

Let $a, b \in \mathbb{R}$, $a < b$, and set $K := [a, b]$. Let

$$
\mathcal{F} := H^1(a, b) \subset \{ u | u : K \rightarrow \mathbb{R}, u \text{ is continuous} \} =: C(a, b).
$$

(1.1)

Consider the following form $\mathcal{E}$ with domain $\mathcal{F}$ defined as

$$
\mathcal{E}(u, v) = \int_a^b u'(x)v'(x) \, dx, \quad u, v \in \mathcal{F}.
$$

(1.2)

Note that for any $u \in \mathcal{F}$ and any $x, y \in K$,

$$
|u(x) - u(y)|^2 \leq \mathcal{E}(u)|x - y|,
$$

(1.3)

where $\mathcal{E}(u) := \mathcal{E}(u, u)$, since for any $x < y$ and any $u \in \mathcal{F}$,

$$
|u(x) - u(y)|^2 = \left\{ \int_x^y u'(z) \, dz \right\}^2 \leq (y - x) \int_x^y [u'(z)]^2 \, dz \leq (y - x) \mathcal{E}(u).
$$

Let $\mu$ be a Radon measure on $K$ with full support $\text{supp}(\mu) = K = [a, b]$ (that is, $\mu(I) > 0$ for any nonempty open interval $I$ in $K$). Clearly, the form $(\mathcal{E}, \mathcal{F})$ is densely defined, non-negative definite, symmetric, bilinear and Markovian in $L^2(\mu) := L^2(K, \mu)$. By using (1.3), it is not hard to see that $H^1(a, b)$ is complete under the norm $\sqrt{\mathcal{E}(u) + \|u\|_{L^2(\mu)}^2}$. The form $(\mathcal{E}, \mathcal{F})$ given by (1.2) and (1.1) is thus a Dirichlet form in $L^2(\mu)$ (cf. [9]). Moreover, $(\mathcal{E}, \mathcal{F})$ is regular, conservative and strongly local in $L^2(\mu)$ (since $1 \in \mathcal{F}$ and $\mathcal{E}(1) = 0$, the form $(\mathcal{E}, \mathcal{F})$ is conservative).

This paper first investigates the geometric properties of certain self-similar measures with overlaps, and then obtains two-sided estimates of the heat kernel of the form $(\mathcal{E}, \mathcal{F})$ for these measures by using existing machinery. We call a map $S : \mathbb{R}^n \rightarrow \mathbb{R}^n$ a contractive similitude on $\mathbb{R}^n$ if $S(x) = rAx + b$, where $r \in (0, 1)$, $b \in \mathbb{R}^n$ and $A$ is an $n \times n$ orthogonal matrix. Let $\{ S_i \}_{i=0}^N$ be contractive similitudes on $\mathbb{R}$ such that

$$
K = \bigcup_{i=0}^N S_i(K),
$$

(1.4)

and let $\mu$ be a self-similar measure with weight $\{ \rho_i \}_{i=0}^N$:

$$
\mu = \sum_{i=0}^N \rho_i (\mu \circ S_i^{-1}),
$$

(1.5)
where $0 < \rho_i < 1$ for each $i$ and $\sum_{i=0}^{N} \rho_i = 1$. It is easy to see that
\[
\mu(A) = 0
\]
if $A$ is a singleton\(^1\).

Let $\{T_j\}_{j=0}^{m}$ be an auxiliary IFS of contractive similitudes:
\[
|T_j(x) - T_j(y)| = r_j |x - y| \quad \text{for any } x, y \in K,
\]
where $0 < r_j < 1$ for each $j$, such that $\{T_j(K) : j = 0, 1, \ldots, m\}$ forms a partition of $K$, in the sense that
\[
K = \bigcup_{j=0}^{m} T_j(K),
\]
and the intervals $T_i(K)$ and $T_j(K)$ can only intersect at their end-points if $i \neq j$. (Note that the IFS $\{T_j\}_{j=0}^{m}$ does not have overlaps but the IFS $\{S_i\}_{i=0}^{N}$ may have.)

For a word $\omega = \omega_1 \cdots \omega_n$, where $\omega_i \in \{0, \ldots, m\}$ for $1 \leq i \leq n$, we let $|\omega| = n$ denote the length of $\omega$ and use the notation
\[
K_\omega := T_{\omega_1} \circ \cdots \circ T_{\omega_n}(K) \quad \text{and} \quad r_\omega := r_{\omega_1} \cdots r_{\omega_n}.
\]
We call $K_\omega$ an $n$-cell, and write $K_\omega \sim K_\tau$ if two words $\omega, \tau$ satisfy $K_\omega \cap K_\tau \neq \emptyset$. We say that two words $\omega, \tau$ having the same length, or the corresponding cells $K_\omega, K_\tau$ are neighbors or neighboring if $K_\omega \sim K_\tau$ and $\omega \neq \tau$. Note that $\{T_0, T_1, \ldots, T_m\}$, not $\{S_0, S_1, \ldots, S_N\}$, is used to define $K_\omega$. For each $n \in \mathbb{N}$, let $J = \{0, 1, \ldots, m\}$ and let
\[
J^n := \{0, 1, \ldots, m\}^n, \quad J^* := \bigcup_{k=0}^{\infty} J^k
\]
be respectively the sets of words with length $n$ and those with finite length. Here $J^0$ is defined to be the singleton $\{\emptyset\}$ of the empty word $\emptyset$, and we use the convention that
\[
\omega \emptyset = \emptyset \omega = \omega \quad \text{for any word } \omega.
\]
For two finite words $\omega$ and $\tau$, we say $\omega < \tau$ if there exists a non-empty word $\gamma$ such that $\tau = \omega \gamma$. We write $\omega \leq \tau$ (and call $\omega$ a father of $\tau$) if $\omega < \tau$ or $\omega = \tau$.

Let $d_*$ be a metric on $K$ which produces the same topology as the Euclidean metric, and let
\[
V(x, r) := \mu(B_{d_*}(x, r)),
\]
where
\[
B_{d_*}(x, r) = \{y \in K : d_*(y, x) < r\}
\]
denotes the open ball with center $x$ and radius $r$ under the metric $d_*$. Throughout this paper, the sign $f \asymp g$ means that $C^{-1} g \leq f \leq C g$ for some universal constant $C > 0$ independent of the arguments $f$ and $g$.

Fix some number $\beta > 1$. We introduce the following conditions that may or may not be satisfied:

1. comparability of neighboring cells: if $\tau$ and $\sigma$ are neighbors, then
\[
\mu(K_\tau) \asymp \mu(K_\sigma);
\]
and let 

\[ \mu \]

for all \( t \) and the lower estimate

\[ d_\ast(x, z) = d_\ast(x, y) + d_\ast(y, z); \quad (1.11) \]

(3) \text{product} of Euclidean length and \( \mu \)-measure of an interval \([x, y] :\)

\[ |x - y| \mu ([x, y]) \leq d_\ast(x, y)^\beta; \quad (1.12) \]

(4) \text{volume doubling property} (VD): there exists a constant \( C > 0 \) such that

\[ V(x, 2r) \leq CV(x, r) \text{ for all } r > 0 \text{ and all } x \in K; \quad (1.13) \]

(5) \text{ratio} of volumes of two concentric balls \( B_{d_\ast}(x, r) \) and \( B_{d_\ast}(x, \eta r) :\)

\[ \frac{V(x, r)}{V(x, \eta r)} = o (\eta^{-\beta}) \text{ uniformly in } x, r \text{ as } \eta \to 0^+, \]

that is,

\[ \sup_{x \in K, 0 < r < 1} \eta^\beta \frac{V(x, r)}{V(x, \eta r)} \to 0^+ \text{ as } \eta \to 0^+. \quad (1.14) \]

Note that we can take \( \beta = 2 \) in the above conditions if \( \mu \) is the Lebesgue measure and \( d_\ast \) is the Euclidean metric. In this paper we are interested in the situation where \( \beta \) is strictly greater than 2 with respect to suitable \( \mu \) and \( d_\ast \) on \( K \).

Note that condition (1.11) implies the \text{mid-point property}, which in turn implies the \text{chain condition}, see for example [12, Definition 3.4]. (A distance \( d \) on a nonempty set \( X \) is said to have the \text{mid-point property} if for any \( x, y \in X \), there exists some \( z \in X \) such that \( d(x, z) = d(z, y) = d(x, y)/2 \).

Let \( \mu \) be a Radon measure on \( K = [a, b] \) with full support, and let \((E, F)\) be defined by (1.2) and (1.1). Let \( \{T_j\}_{j=0}^m \) be an auxiliary IFS defined by (1.7) such that (1.8) holds. When the metric \( d_\ast \) on \( K \) has been constructed such that conditions (1.10)–(1.14) are all satisfied for \( d_\ast \), then by applying known equivalent conditions for heat kernel estimates given in [18, Theorem 15.10] with \( g(r) = r^{\beta} \), one can conclude that the jointly continuous heat kernel \( p_t(x, y) \) of \((E, F)\) exists and satisfies the upper estimate

\[ p_t(x, y) \leq \frac{C_1}{V(x, t^{1/\beta})} \exp \left( -c_1 \left( \frac{d_\ast(x, y)}{t^{1/\beta}} \right)^{\beta/(\beta - 1)} \right) \quad \text{(UE)} \]

and the lower estimate

\[ p_t(x, y) \geq \frac{C_2}{V(x, t^{1/\beta})} \exp \left( -c_2 \left( \frac{d_\ast(x, y)}{t^{1/\beta}} \right)^{\beta/(\beta - 1)} \right) \quad \text{(LE)} \]

for all \( t \in (0, 1) \) and all \( x, y \in K \).

We consider two classes of specific Radon measures \( \mu \) and introduce a new metric \( d_\ast \) accordingly. The first class consists only of the infinite Bernoulli convolution associated with the golden ratio. Let

\[ S_0(x) = \rho x, \quad S_1(x) = \rho x + (1 - \rho), \quad \rho = \frac{\sqrt{5} - 1}{2}, \quad (1.15) \]

and let \( \mu \) be the self-similar measure with \text{supp}(\mu) = [0, 1] \text{ satisfying:} \]

\[ \mu = \frac{1}{2} \mu \circ S_0^{-1} + \frac{1}{2} \mu \circ S_1^{-1}. \quad (1.16) \]

The metric \( d_\ast \) and the constant \( \alpha \in (0, \frac{1}{2}) \) in the following theorem will be given in Section 2.
Theorem 1.1. Let $\mu$ be defined by (1.16), $\{T_j\}_{j=0}^2$ be defined by (2.1) and $d_*$ by (2.16) below. Let $\alpha \in \left(0, \frac{1}{2}\right)$ be defined by (2.17) and $\beta := 1/\alpha > 2$. Then all the conditions (1.10)–(1.14) are satisfied. Consequently, the heat kernel $p_t(x,y)$ of $(\mathcal{E}, \mathcal{F})$ exists and satisfies the two-sided estimates (UE) and (LE) with this parameter $\beta$.

Theorem 1.1 will be proved in Section 2.

The second class is a family of convolutions of Cantor-type measures. Let $S_0(x) = \frac{1}{m}x$, $S_1(x) = \frac{1}{m}x + \frac{m-1}{m}$, $\mu_m$ be defined by the IFS (1.17) with probability weights $p_0 = p_1 = 1/2$. The $m$-fold convolution $\mu_m = \nu_m \ast m$ of $\nu_m$ is the self-similar measure defined by the following IFS with overlaps (see [25]):

$$S_i(x) = \frac{1}{m}x + \frac{m-1}{m}i, \quad i = 0, 1, \ldots, m,$$

(1.18)

together with probability weights

$$w_i := \frac{1}{2^m} \binom{m}{i}, \quad i = 0, 1, \ldots, m.
$$

That is,

$$\mu_m = \sum_{i=0}^m \frac{1}{2^m} \binom{m}{i} \mu_m \circ S_i^{-1},$$

(1.20)

with $\text{supp}(\mu_m) = [0, m]$.

The metric $d_*$ and the constant $\alpha \in \left(0, \frac{1}{2}\right)$ in the following theorem will be given in Section 3.

Theorem 1.2. Let $m \geq 3$ be an integer and let $\mu_m$ be the $m$-fold convolution of the Cantor measure defined as in (1.20). Let $\{T_j\}_{j=0}^{m-1}$ be defined by (3.1). Let $d_*$ be a metric defined by (3.13) below, let $\alpha \in \left(0, \frac{1}{2}\right)$ be a constant defined by (3.14) and $\beta := 1/\alpha > 2$. Then the same conclusions as those of Theorem 1.1 hold with this value of $\beta$.

We will prove Theorem 1.2 in Section 3. We remark that we cannot deal with the case $m = 2$ for technical reasons. For $m \geq 3$, we can separate the set $\mathcal{J} = \{0, 1, \ldots, m-1\}$ into two subsets $\mathcal{J}_0 = \{0, m-1\}$ and $\mathcal{J}_1 = \{1, \ldots, m-2\}$, so that the equation (3.15) below makes sense as $\mathcal{J}_1$ is not empty. However, this technique does not apply when $m = 2$ since the set $\mathcal{J}_1$ is empty.

Our main efforts in this paper are in constructing the metric $d_*$ on $K$ for the two classes of self-similar measures with overlaps and proving the generalized midpoint property (1.11) of $d_*$, which is crucial in obtaining the full off-diagonal lower estimate of the heat kernel.

The issue of constructing a metric (not necessarily geodesic) that is suitable for describing heat kernel behavior for a given symmetric regular Dirichlet space has been extensively studied by a series of works [17, 18, 19] by Kigami. In particular, for the time change of the reflecting Brownian motion on a compact interval by a measure, he has shown in [17, 19] that, in order to obtain the sub-Gaussian heat kernel estimate in Euclidean spaces, it suffices to verify that the measure is volume doubling with respect to the Euclidean metric only, see [17, Theorem 3.2.3 and Proposition 3.3.1] (or [18, Corollary 15.12]) where the existence of a metric $d_*$ with
respect to which \(\mu\) is volume doubling, off-diagonal upper bound and near-diagonal lower bound of the heat kernel were obtained. What is new in this paper is that we have explicitly constructed a metric that is also geodesic, and have specified the exact value of the walk dimension that is strictly greater than 2. This topic is highly non-trivial for an IFS with overlaps.

We outline the proof of Theorems 1.1 and 1.2 here. First, we use second-order identities and the structure of the associated matrices to show that the measures of two neighboring cells are comparable (Lemmas 2.1 and 3.1) and consequently that \(\mu\) is doubling with respect to the Euclidean metric. Then, we construct the above-mentioned metric \(d_\ast\) and use the spectral dimension formula to prove the generalized mid-point property (Propositions 2.4 and 3.6). Next, we use the spectral dimension formula to show that the resistance metric (or equivalently, the Euclidean metric), the metric \(d_\ast\), and the measure \(\mu\) “match well” (Lemmas 2.7 and 3.8). Finally, we use the above estimates to verify that conditions (1.10)–(1.14) hold.

Our results also have applications in the study of the wave propagation speed problem on fractals (see [27, 6]). In fact, using results in this paper and a generalization of a theorem of Y.-T. Lee [24], Ngai et al. [26] proved that waves defined by the Laplacians in Theorems 1.1 and 1.2 have infinite propagation speed.

2. Infinite Bernoulli convolution associated with the golden ratio. Let \(K = [0, 1]\) and \(\mu\) be given by (1.16) and (1.15). In this section we introduce a new metric \(d_\ast\) on \(K\), and show that conditions (1.10)–(1.14) are all satisfied.

It is shown in [28] that by introducing the auxiliary IFS \(\{T_0, T_1, T_2\}\):

\[
T_0(x) = \rho^2 x, \quad T_1(x) = \rho^3 x + \rho^2, \quad T_2(x) = \rho^2 x + \rho
\]

(2.1)

(see Figure 1), one can obtain the following second-order identities: For all Borel subsets \(A \subset [0, 1]\\),

\[
\begin{bmatrix}
\mu(T_0 T_i A) \\
\mu(T_1 T_i A) \\
\mu(T_2 T_i A)
\end{bmatrix}
= M_i
\begin{bmatrix}
\mu(T_0 A) \\
\mu(T_1 A) \\
\mu(T_2 A)
\end{bmatrix}, \quad i = 0, 1, 2,
\]

(2.2)

where

\[
M_0 = \frac{1}{8}
\begin{bmatrix}
2 & 0 & 0 \\
1 & 2 & 0 \\
0 & 4 & 0
\end{bmatrix}, \quad M_1 = \frac{1}{4}
\begin{bmatrix}
0 & 1 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0
\end{bmatrix}, \quad M_2 = \frac{1}{8}
\begin{bmatrix}
0 & 4 & 0 \\
0 & 2 & 1 \\
0 & 0 & 2
\end{bmatrix}.
\]

(2.3)

\[\text{(a)}\]

\[\text{(b)}\]

**Figure 1.** (a) The IFS \(\{S_0, S_1\}\) has overlaps. (b) The auxiliary IFS \(\{T_0, T_1, T_2\}\) does not have overlaps.
By (2.2) we have

$$\mu(T_0(K)) = \mu(T_1(K)) = \mu(T_2(K)) = \frac{1}{3} \tag{2.4}$$

(see [28, p.109]). Let

$$r_0 = r_2 = \rho^2 \quad \text{and} \quad r_1 = \rho^3 \tag{2.5}$$

be the contraction ratios of the auxiliary IFS \( \{T_0, T_1, T_2\} \).

We will introduce a new metric \( d_\ast \) on \( K \) below. Before this, we show condition (1.10). We use the following notation. For each \( n \in \mathbb{N} \), let

\[
\mathcal{J}^n := \{0, 1, 2\}^n, \quad \mathcal{J}_0^n := \{0, 2\}^n, \quad \mathcal{J}^* := \bigcup_{k=0}^{\infty} \mathcal{J}^k, \quad \mathcal{J}_0^* := \bigcup_{k=0}^{\infty} \mathcal{J}_0^k,
\]

where \( \mathcal{J}^0 \) and \( \mathcal{J}_0^0 \) are defined to be the singleton \( \{\emptyset\} \) of the empty word \( \emptyset \).

For a symbol \( \xi \in \{0, 1, 2\} \), let \( e_\xi \) be the row matrix defined as

\[
e_\xi = \begin{cases} [1 0 0] & \text{if } \xi = 0, \\
[0 1 0] & \text{if } \xi = 1, \\
[0 0 1] & \text{if } \xi = 2. \end{cases}
\]

For \( \omega = (\omega_1, \ldots, \omega_n) \), with \( |\omega| \geq 1 \), by applying (2.2) repeatedly, we obtain

\[
\mu(K_\omega) = \mu(T_\omega(K)) = e_{\omega_1} \begin{bmatrix} \mu(T_0T_{\omega_2} \cdots T_{\omega_n}(K)) \\
\mu(T_1T_{\omega_2} \cdots T_{\omega_n}(K)) \\
\mu(T_2T_{\omega_2} \cdots T_{\omega_n}(K)) \end{bmatrix} = e_{\omega_1} M_{\omega_2} \begin{bmatrix} \mu(T_0(T_{\omega_3} \cdots T_{\omega_n}(K))) \\
\mu(T_1(T_{\omega_3} \cdots T_{\omega_n}(K))) \\
\mu(T_2(T_{\omega_3} \cdots T_{\omega_n}(K))) \end{bmatrix} = e_{\omega_1} M_{\omega_2} \cdots M_{\omega_n} \begin{bmatrix} \mu(T_0(K)) \\
\mu(T_1(K)) \\
\mu(T_2(K)) \end{bmatrix} = \frac{1}{3} e_{\omega_1} M_{\omega_2} \cdots M_{\omega_n} [1] = \frac{1}{3}. \tag{2.6}
\]

Lemma 2.1. For any two neighboring words \( \omega \) and \( \tau \), we have

$$2^{-1} \mu(K_\omega) \leq \mu(K_\omega) \leq 2 \mu(K_\omega). \tag{2.7}$$

Consequently, condition (1.10) is satisfied.

Proof. Without loss of generality, we assume that \( K_\omega \) is on the left of \( K_\tau \). Then either of the following relationships holds for such \( \omega \) and \( \tau \):

\[
\omega = \theta 0^2 \cdots \frac{2}{\ell} \quad \text{and} \quad \tau = \theta 0^1 \cdots 0 \frac{0}{\ell}, \tag{2.8}
\]

or

\[
\omega = \theta 1^2 \cdots \frac{2}{\ell} \quad \text{and} \quad \tau = \theta 2^0 \cdots 0 \frac{0}{\ell}, \tag{2.9}
\]

where \( \theta \) is some finite word (possibly the empty word) and \( \ell \geq 0 \) is some integer.

We deal with the first case; the second one is similar.

When \( |\omega| = |\tau| = 1 \), it follows from (2.4) that \( \mu(K_\omega) = \mu(K_\tau) = 1/3 \) and thus (2.7) holds trivially. We assume that \( |\omega| = |\tau| = n \geq 2 \). Assume that \( |\theta| = s \geq 1 \) and write \( \theta = \theta_1 \theta_2 \cdots \theta_s \), the case \( |\theta| = 0 \) can be treated similarly.
As $\omega = \theta_0 \theta_2 \cdots \theta_{2^\ell}$, by using (2.6), we have

$$\mu(K_\omega) = \frac{1}{3} e_\theta M_{\theta_2} \cdots M_{\theta_{2^\ell}} \cdot M_0 \cdot M_{2^\ell} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \end{bmatrix} \cdot M_0 \cdot M_{2^\ell} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

where $[a \ b \ c]$ is a row vector with $a, b, c \geq 0$. Similarly, we have

$$\mu(K_\tau) = \frac{1}{3} e_\theta M_{\theta_2} \cdots M_{\theta_{2^\ell}} \cdot M_1 \cdot M_{2^\ell} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \end{bmatrix} \cdot M_1 \cdot M_{2^\ell} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$  (2.10)

By comparing $\mu(K_\omega)$ and $\mu(K_\tau)$, it is easy to see that $\mu(K_\omega) \leq 2\mu(K_\tau)$ when $\ell = 0$ (for the second case we have $\mu(K_\tau) \leq 2\mu(K_\omega)$), and hence (2.7) holds. Meanwhile, for any integer $\ell \geq 1$,

$$M_0^\ell = \frac{1}{8^\ell} \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 4 & 0 \end{bmatrix}^\ell = \frac{1}{8^\ell} \begin{bmatrix} 2^{\ell+1} \cdot (\ell + 1) & 2\ell & 0 \\ 2\ell & 2^{\ell+1} \cdot (\ell - 1) & 2^{\ell+1} \\ 0 & 2^{\ell+1} & 2\ell \cdot (\ell + 1) \end{bmatrix},$$

and also

$$M_0^\ell \cdot M_0^* = \frac{1}{8^\ell} \begin{bmatrix} 0 & 2^{\ell+1} \\ 0 & 2^{\ell+1} \cdot (\ell - 1) \\ 1 & 2^{\ell+1} \end{bmatrix}.$$  (2.12)

Substituting (2.12) and (2.13) into (2.11) and (2.10) respectively, we see that

$$\mu(K_\omega) = \frac{1}{8^\ell+1} \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} 2^{\ell+1} \cdot (\ell + 1) \\ 2\ell & 2^{\ell+1} \cdot (\ell + 2) \end{bmatrix},$$

and

$$\mu(K_\tau) = \frac{1}{8^\ell+1} \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} 2\ell & 2^{\ell+1} \\ 2\ell & 2^{\ell+1} \cdot (\ell + 2) \end{bmatrix}.$$  (2.13)

Thus we obtain

$$\frac{\mu(K_\omega)}{\mu(K_\tau)} = \frac{(2\ell + 2)a + (2\ell + 3)b + (2\ell + 4)c}{(\ell + 2)(a + b + c)},$$

which implies that

$$1 \leq \frac{\mu(K_\omega)}{\mu(K_\tau)} \leq 2.$$  (2.7)

Thus (2.7) holds.

For any $0 \leq x < y \leq 1$, we define a set $\mathcal{W}(x, y)$ of finite words as follows:

$$\mathcal{W}(x, y) := \left\{ \omega = \omega_1 \cdots \omega_n \in J^* \setminus \{\emptyset\} : \omega_n = 1, \ K_\omega \subseteq [x, y], \text{ and } \omega \text{ is a father} \right\},$$

where the notion “$\omega = \omega_1 \cdots \omega_n$ is a father” means that no proper ancestor $\omega_1 \cdots \omega_k$ ($k < |\omega|$) of $\omega$ satisfies both of the following conditions:

1. $\omega_k = 1$;
2. $K_{\omega_1 \cdots \omega_k} \subseteq [x, y]$.  (2.14)
In other words, a word $\omega \xi$ with $\xi \neq \emptyset$ cannot belong to $W(x, y)$ if $\omega \in W(x, y)$, and hence $\omega$ has shortest length among all words satisfying the conditions $\omega_n = 1$ and $K_\omega \subseteq [x, y]$ in (2.14).

For example, if $[x, y] = [\rho^2, \rho] = K_1$, then $W(x, y) = \{1\}$, a singleton. Note that the word “11” does not belong to $W(\rho^2, \rho)$ since “1” is its father. As another example, let $[x, y] = [0, \rho^2] = K_0$. In this case,

$$W(x, y) = \{01, 001, 0001, 0021, 0201, 0221, \ldots\}$$

is in $W(0, \rho^2)$, although each word $\omega$ in this set ends with the symbol “1”, and $K_\omega \subseteq [0, \rho^2]$. The reason is that all of them are offspring of the word “01”, which is in $W(0, \rho^2)$.

Note that $W(x, y) \neq \emptyset$ for any $x, y$ with $0 \leq x < y \leq 1$.

Define a function $d_* : K \times K \to [0, \infty]$ as follows:

$$d_*(x, y) = d_*(y, x) = \sum_{\omega \in W(x, y)} (r_\omega \mu(K_\omega))^\alpha$$

if $0 \leq x < y \leq 1$, where $r_0, r_1, r_2$ are given by (2.5) and $\alpha$ is the unique solution of the following equation (see [25, Theorem 1.2 and Lemma 5.7]):

$$\sum_{k=0}^\infty \sum_{J \in J^h_0} (\rho^{2k+3} c_J)^\alpha = 1,$$

with $c_J$ given by

$$c_J := \frac{1}{4} \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} M_J \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \text{ where } M_J := M_{j_1} \cdots M_{j_k}$$

for any index $J = j_1 \cdots j_k \in J^h_0 := \{0, 2\}^k$ and any integer $k \geq 0$. Here we use the convention that $M_{\omega} := I$, the identity matrix, if $\omega$ is the empty word.

**Remark 2.2.** Let $\Delta_\mu$ be the Laplacian defined by $\mu$ (see [5, 16]). Then

$$\dim_s(\mu) = 2\alpha,$$

where $\dim_s(\mu)$ is the spectral dimension of the corresponding Dirichlet and Neumann Laplacians $-\Delta_\mu$ (see [25]). In fact (see [25, p.654]), we have

$$\alpha = \frac{\dim_s(\mu)}{2} \approx \frac{0.998}{2} = 0.499 < 0.5$$

(2.19)

(the value of $\alpha$ is close to but strictly less than 0.5). This is in sharp contrast with the classical case where $\alpha = 0.5$ for the Euclidean metric and the Lebesgue measure.

Note that for any word $\omega$, letting $x = T_\omega(0)$, $y = T_\omega(1)$, we have by definition (2.16),

$$d_*(x, y) = (r_\omega \mu(K_\omega))^\alpha$$

(2.20)
if \( \omega \) ends with the symbol “1” since \( W(x,y) = \{ \omega \} \), while
\[
d_\ast(x,y) = \sum_{k=0}^{\infty} \sum_{J \in J_k^*} (r_{\omega J_1} \mu(K_{\omega J_1}))^\alpha
\] (2.21)

if \( \omega \) ends with the symbols “0” or “2” since \( W(x,y) = \{ \omega J : J \in J_0^* \} \).

**Proposition 2.3.** For any \( 0 \leq x < y \leq 1 \) and for any distinct \( \omega, \tau \in W(x,y) \), we have
\[
K_\omega \cap K_\tau = \emptyset.
\] (2.22)

**Proof.** Assume that there are two distinct words \( \omega, \tau \in W(x,y) \) such that \( K_\omega \cap K_\tau \neq \emptyset \). Since both \( \omega \) and \( \tau \) end with 1, the only possible case is that one cell is contained in the other (Indeed, if \( \omega = \omega_1 \cdots \omega_{n-1} \) and \( \tau = \tau_1 \cdots \tau_{m-1} \) \( n \leq m \), and if \( \omega_k \neq \tau_k \) for some \( k \leq n \), then \( K_\omega \cap K_\tau = \emptyset \)). Without loss of generality, assume that \( K_\omega \subset K_\tau \). Then \( \tau \) is a father of \( \omega \), contradicting the definition of \( W(x,y) \). The proposition follows. \( \square \)

**Proposition 2.4.** The function \( d_\ast \) satisfies
\[
d_\ast(x,z) = d_\ast(x,y) + d_\ast(y,z)
\] (2.23)
for any \( 0 \leq x < y < z \leq 1 \). Moreover, granted that \( d_\ast(K) < \infty \), a fact to be proved in Proposition 2.5, \( d_\ast \) is a metric on \( K \) and consequently, condition (1.11) is satisfied.

**Proof.** If \( d_\ast(x,y) = 0 \) then \( x = y \); otherwise there would exist some nonempty word \( \omega \in W(x,y) \) such that \( \mu(K_\omega) = 0 \), contradicting \( \text{supp}(\mu) = [0,1] \).

It suffices to prove (2.23), since this will imply that \( d_\ast \) satisfies the triangle inequality.

To do this, we first claim that for any \( \omega = \omega_1 \cdots \omega_n \) with \( \omega_n = 1 \), we have
\[
(r_{\omega} \mu(K_\omega))^\alpha = \sum_{J \in J_n^*} (r_{\omega J_1} \mu(K_{\omega J_1}))^\alpha = \sum_{k=0}^{\infty} \sum_{J \in J_k^*} (r_{\omega J_1} \mu(K_{\omega J_1}))^\alpha.
\] (2.24)

The left-hand side of (2.24) is
\[
(r_{\omega} \mu(K_\omega))^\alpha = r_{\omega}^\alpha \cdot \mu(K_\omega)^\alpha,
\] and in view of (2.5), the right-hand side of (2.24) is:
\[
r_{\omega}^{\alpha} \cdot \sum_{k=0}^{\infty} \sum_{J \in J_k^*} (\rho^{2k+3})^\alpha \cdot \mu(K_{\omega J_1})^\alpha.
\]

Thus we only need to show that
\[
\mu(K_\omega)^\alpha = \sum_{k=0}^{\infty} \sum_{J \in J_k^*} (\rho^{2k+3})^\alpha \cdot \mu(K_{\omega J_1})^\alpha.
\] (2.25)

To do this, we first consider the case \( |\omega| \geq 2 \). We use (2.6) to get
\[
\mu(K_\omega) = \frac{1}{3} e_{\omega_1} M_{\omega_2} \cdots M_{\omega_{n-1}} M_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} a & b & c \end{bmatrix} M_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{12} (a + b + c),
\] (2.26)
where $[a \ b \ c] = e_{\omega_1} M_{\omega_2} \cdots M_{\omega_{n-1}}$ is some row vector with nonnegative entries. Similarly, we have

$$
\mu(K_{\omega,1}) = \frac{1}{3} e_{\omega_1} M_{\omega_2} \cdots M_{\omega_{n-1}} \cdot M_1 \cdot M_J \cdot M_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}
$$

$$
= \frac{1}{12} [0 \ a + b + c \ 0] \cdot M_J \cdot M_1 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}
$$

$$
= \frac{1}{12} (a + b + c) \cdot \frac{1}{4} [0 \ 1 \ 0] M_J \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}
$$

$$
= \frac{1}{12} (a + b + c) c_J. \quad (2.27)
$$

We note that the conclusions of equations (2.26) and (2.27) are valid also for $\omega$ with $|\omega| = 1$ as long as $[a \ b \ c]$ is chosen to satisfy $a + b + c = 4$ (even though the computations in these equations do not make sense literally for such $\omega$). Therefore, by (2.17), (2.26) and (2.27), we obtain (2.25), proving (2.24).

Let $x, y, z \in K$ with $x < y < z$. Observe that $W(x, y) \cap W(y, z) = \emptyset$. By the definition of $d_*$, we see that

$$
d_*(x, y) + d_*(y, z) = \sum_{\omega \in W(x,y) \cup W(y,z)} (r_\omega \mu(K_\omega))^\alpha, \quad (2.28)
$$

and that

$$
d_*(x, z) = \sum_{\tau \in W(x,z)} (r_{\tau} \mu(K_\tau))^\alpha. \quad (2.29)
$$

Let $\tau \in W(x,z)$ and $k \in \mathbb{N}$. Then by using (2.24) $k$ times, we have

$$
(r_{\tau} \mu(K_\tau))^\alpha = \sum_{J'_1, \ldots, J'_k \in J_0^*} \left( r_{\tau}$J'_1 \ldots J'_k$ \mu(K_{\omega J'_1 \ldots J'_k}) \right)^\alpha. \quad (2.30)
$$

Since $\{K_{\omega J'_1} \}_{J' \in J_0^*}$ is disjoint for any $\omega \in J^*$, so is $\{K_{\omega J'_1 \ldots J'_k} \}_{J'_1, \ldots, J'_k \in J_0^*}$, and in particular at most one of these cells can contain $y$. Therefore, by classifying all the other terms on the right-hand side of (2.30) according to its unique prefix in $W(x, y) \cup W(y, z)$ and using (2.24), we obtain

$$
(r_{\tau} \mu(K_\tau))^\alpha
$$

$$
= \sum_{j=0}^k \sum_{\omega \in W(x,y) \cup W(y,z)} \sum_{\omega = \omega J'_1 \ldots J'_k \in J_0^*} (r_{\omega J'_1 \ldots J'_k} \mu(K_{\omega J'_1 \ldots J'_k}))^\alpha
$$

$$
+ O(\rho^{2\alpha k})
$$

$$
= \sum_{j=0}^k \sum_{\omega \in W(x,y) \cup W(y,z)} (r_{\omega} \mu(K_{\omega}))^\alpha + O(\rho^{2\alpha k}),
$$
which equals, after letting \( k \to \infty \),
\[
\sum_{j=0}^{\infty} \sum_{\omega \in \mathcal{W}(x,y) \cup \mathcal{W}(y,z)} (r_{\omega} \mu(K_\omega))^\alpha = \sum_{\omega \in \mathcal{W}(x,y) \cup \mathcal{W}(y,z), \tau \omega} (r_{\omega} \mu(K_\omega))^\alpha.
\]

Summing up these equalities over \( \tau \in \mathcal{W}(x,z) \) results in the assertion (2.23) by virtue of (2.29) and (2.28), completing the proof of Proposition 2.4.

For any word \( \omega \), define the diameter \( d_*(K_\omega) \) of the cell \( K_\omega \) under the metric \( d_* \) by
\[
d_*(K_\omega) := \sup_{x,y \in K_\omega} d_*(x,y) = d_*(T_\omega(0), T_\omega(1)).
\]  

**Lemma 2.5.** There exists a constant \( C > 0 \) (depending only on \( \rho \)) such that for any two neighboring words \( \omega \) and \( \tau \),
\[
C^{-1} d_*(K_\tau) \leq d_*(K_\omega) \leq C d_*(K_\tau).
\]  
(\( \text{In fact, one can take } C = \{1 + 2 (8\rho^{-3})^\alpha \} (10\rho^{-1})^\alpha. \text{ In particular, } d_*(K) < \infty \) and thus \( d_* \) is a metric on \( K \).

**Proof.** We first claim that for any finite word \( \omega \), the following holds:
\[
\left( \frac{\rho^2}{10} \right)^\alpha d_*(K_{\omega 1}) \leq d_*(K_{\omega 0}) \leq \left( \frac{8}{\rho^2} \right)^\alpha d_*(K_{\omega 1}),
\]
\[
\left( \frac{\rho^2}{10} \right)^\alpha d_*(K_{\omega 1}) \leq d_*(K_{\omega 2}) \leq \left( \frac{8}{\rho^2} \right)^\alpha d_*(K_{\omega 1}).
\]

Hence,
\[
d_*(K_{\omega 0}) \simeq d_*(K_{\omega 1}) \simeq d_*(K_{\omega 2}).
\]

We first show the ‘\( \leq \)’ part of the first line in (2.33). We first consider the case \( |\omega| \geq 2 \). Indeed, by (2.20),
\[
d_*(K_{\omega 1}) = \mu(K_{\omega 1})^\alpha r_{\omega 1}^\alpha,
\]
and using (2.6), we see that
\[
\mu(K_{\omega 1}) = \frac{1}{3} e_{\omega 1} M_{\omega 2} \cdots M_{\omega_n} M_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{12} e_{\omega 1} M_{\omega 2} \cdots M_{\omega_n} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{12} (a + b + c),
\]
where \( \begin{bmatrix} a & b & c \end{bmatrix} := e_{\omega 1} M_{\omega 2} \cdots M_{\omega_n} \) and \( a, b, c \) are nonnegative numbers. Thus we have
\[
d_*(K_{\omega 1}) = \left( \frac{a + b + c}{12} \right)^\alpha r_{\omega 1}^\alpha = \rho^{3\alpha} \left( \frac{a + b + c}{12} \right)^\alpha \cdot r_{\omega 1}^\alpha.
\]

Again, combining the definition of \( d_* \), (2.6), and (2.21), we get
\[
d_*(K_{\omega 0}) = \sum_{k=0}^{\infty} \sum_{j \in J_k^0} \mu(K_{\omega 0,j 1})^\alpha r_{\omega 0,j 1}^\alpha.
\]
and
\[
\mu(K_{\omega, 0, 1}) = \frac{1}{3} e_{\omega, 1} M_{\omega^2} \cdots M_{\omega^6} M_0 M_J M_1 \left[ \begin{array}{c} 1 \\
1 \\
1 \\
1 \\
1 \\
1 \end{array} \right] = \frac{1}{12} \left[ \begin{array}{c} a \\
\frac{b}{4} \\
c \end{array} \right] M_0 M_J \left[ \begin{array}{c} 1 \\
1 \\
1 \\
1 \\
1 \\
1 \end{array} \right]
\]
\[
= \frac{1}{96} \left[ \begin{array}{c} 2a + b \\
2b + 4c \\
0 \end{array} \right] M_J \left[ \begin{array}{c} 1 \\
1 \\
1 \\
1 \\
1 \\
1 \end{array} \right] \leq \frac{a + b + c}{48} \left[ \begin{array}{c} 1 \\\n2 \\
0 \end{array} \right] M_J \left[ \begin{array}{c} 1 \\
1 \\
1 \\
1 \\
1 \\
1 \end{array} \right]
\]
\[
= \frac{2(a + b + c)}{3} \cdot \left( \frac{1}{4} \left[ \begin{array}{c} 0 \\
1 \\
0 \end{array} \right] M_0 M_J \left[ \begin{array}{c} 1 \\
1 \\
1 \\
1 \\
1 \\
1 \end{array} \right] \right) = \frac{2(a + b + c)}{3} c_{0, J}.
\]
We note that the above computations and estimates of \(\mu(K_{\omega, 1})\) and \(\mu(K_{\omega, 0, 1})\) are valid also for \(\omega\) with \(|\omega| = 1\) as long as \([a \ b \ c]\) is chosen to satisfy \(a + b + c = 4\) (even though the arguments in this proof do not make sense literally for such \(\omega\)). Thus, the right-hand side of (2.35) satisfies
\[
\sum_{k=0}^{\infty} \sum_{J \in \mathcal{J}_0^k} \mu(K_{\omega, 0, 1})^\alpha r_{\omega, 0, 1}^\alpha 
\leq \sum_{k=0}^{\infty} \sum_{J \in \mathcal{J}_0^k} \left( \frac{2(a + b + c)}{3} \right)^\alpha c_{0, J} \cdot r_{\omega, 0}^\alpha \cdot \rho^{(2k+5)^\alpha}
\]
\[
= \left( \frac{2(a + b + c)}{3} \right)^\alpha \cdot r_{\omega, 0}^\alpha \cdot \sum_{k=0}^{\infty} \sum_{J \in \mathcal{J}_0^k} c_{0, J} \cdot \rho^{(2k+5)^\alpha}
\]
\[
\leq \left( \frac{2(a + b + c)}{3} \right)^\alpha \cdot r_{\omega, 0}^\alpha \cdot \sum_{k=0}^{\infty} \sum_{J' \in \mathcal{J}_0^k} c_{J'} \cdot \rho^{(2k+3)^\alpha} \quad \text{(let} \ J' = 0J) \]
\[
= \left( \frac{2(a + b + c)}{3} \right)^\alpha \cdot r_{\omega, 0}^\alpha. \quad \text{(definition of} \ \alpha) \]
Using this and (2.34), we obtain
\[
d_*(K_{\omega, 0}) \leq \left( \frac{8}{\rho^2} \right)^\alpha d_*(K_{\omega, 1}),
\]
thus proving the second ‘\(\leq\)’ in the first line in (2.33).

On the other hand, using Lemma 2.1 and (1.6), for any word \(\theta\), we have
\[
\mu(K_\theta) = \mu(K_{\theta, 0}) + \mu(K_{\theta, 1}) + \mu(K_{\theta, 2}) 
\leq 2\mu(K_{\theta, 1}) + \mu(K_{\theta, 1}) + 2\mu(K_{\theta, 1}) = 5\mu(K_{\theta, 1}). \quad (2.36)
\]
Applying this inequality with \(\theta = \omega 0\) and using Lemma 2.1 again, we obtain
\[
\mu(K_{\omega, 1}) \leq 2\mu(K_{\omega, 0}) \leq 10\mu(K_{\omega, 01}),
\]
which implies, in view of (2.20) again, that
\[
d_*(K_{\omega, 01}) = \mu(K_{\omega, 01})^\alpha r_{\omega, 01}^\alpha \geq 10^{-\alpha} \mu(K_{\omega, 1})^\alpha \rho^{2\alpha} r_{\omega, 1}^\alpha = \left( \frac{\rho^2}{10} \right)^\alpha d_*(K_{\omega, 1}).
\]
Thus, we have
\[
d_*(K_{\omega, 0}) \geq d_*(K_{\omega, 01}) \geq \left( \frac{\rho^2}{10} \right)^\alpha d_*(K_{\omega, 1}), \quad (2.37)
\]
proving the first ‘≤’ in the first line in (2.33). By symmetry, the second line in (2.33) also holds. This proves our claim.

Therefore, for any finite word \( \omega \), by (2.23) and (2.33), we have

\[
d_\bullet(K_{\omega 1}) \leq d_\bullet(K_{\omega}) = d_\bullet(K_{\omega 0}) + d_\bullet(K_{\omega 1}) + d_\bullet(K_{\omega 2})
\]

\[
\leq \left\{ 1 + 2 (8\rho^{-3})^\alpha \right\} d_\bullet(K_{\omega 1}) =: A d_\bullet(K_{\omega 1}),
\]

(2.38)

where \( A := 1 + 2 (8\rho^{-3})^\alpha \).

Finally, for any neighboring words \( \omega \) and \( \tau \), using (2.5) and (2.8) (or (2.9)), we have

\[
\rho r_{\tau} \leq r_{\omega} \leq \rho^{-1} r_{\tau},
\]

(2.39)

which implies, by using Lemma 2.1 and (2.36) with \( \theta = \tau \), that

\[
r_{\omega 1} \mu(K_{\omega 1}) \leq r_{\omega 1} \mu(K_{\omega}) \leq r_1 (\rho^{-1} r_{\tau}) (2\mu(K_\tau))
\]

\[
= 2\rho^{-1} r_{\tau 1} \mu(K_{\tau}) \leq 10\rho^{-1} c r_{\tau 1} \mu(K_{\tau 1}).
\]

Thus, using (2.38), we have

\[
d_\bullet(K_{\omega}) \leq A d_\bullet(K_{\omega 1}) = A [r_{\omega 1} \mu(K_{\omega 1})]^\alpha
\]

\[
\leq A (10\rho^{-1})^\alpha [r_{\tau 1} \mu(K_{\tau 1})]^\alpha = A (10\rho^{-1})^\alpha d_\bullet(K_{\tau 1})
\]

\[
\leq \left\{ 1 + 2 (8\rho^{-3})^\alpha \right\} (10\rho^{-1})^\alpha d_\bullet(K_{\tau}),
\]

(2.40)

completing the proof of (2.32). It now follows by using (2.20) that \( d_\bullet(K) < \infty \).

Combining this with Proposition 2.4 proves that \( d_\bullet \) is a metric on \( K \).

We need the following proposition.

**Proposition 2.6.** Let \( x, y \in K \), \( x < y \) and let \( \omega \) be a shortest word such that \( K_\omega \subseteq [x, y] \). Set \( |\omega| = n \). Then there are at most four \( n \)-cells between the points \( x \) and \( y \).

**Proof.** Assume that \( |\omega| = n \geq 2 \); otherwise nothing needs to be proved. Let \( \omega' \) be the father of \( \omega \); that is, \( \omega \) is one of \( \{\omega', \omega', \omega' \} \). Since \( |\omega| \geq 2 \), we have \( \omega' \neq \emptyset \). Note that \( x, y \) cannot be separated apart by any \((n-1)\)-cell; otherwise, \( \omega \) would not be the shortest. In other words, both \( x \) and \( y \) must lie in the union of two neighboring \((n-1)\)-cells.

We first consider the case \( \omega = \omega' \). The point \( x \) must lie to the left of \( K_{\omega} \), since \( K_{\omega} = K_{\omega' 0} \subseteq [x, y] \). Let \( \tau' \) be the left neighboring \((n-1)\)-word of \( \omega' \) (even though the case \( \omega = 0 \cdots 0 \) does not admit \( \tau' \), this case can still be treated similarly). Then

\[
x \in K_{\omega'} \setminus \{T_{\tau'}(0)\} \quad \text{and} \quad y \in K_{\omega} \setminus \{T_{\omega'}(1)\}
\]

(see Figure 2).

![Figure 2](image-url)

**Figure 2.** Points \( x \), \( y \) and cells \( K_{\omega'}, K_{\tau'} \), where \( \omega = \omega' \).
Clearly, there are at most four $n$-cells between the points $x$ and $y$:

$$ x \in K_{\tau_0} \sim K_{\tau_1} \sim K_{\tau_2} \sim K_{\omega_0} = K_{\omega} \sim K_{\omega'} \sim K_{\omega''} \ni y, $$

thus proving our conclusion.

The proofs for the cases $\omega = \omega'1$ and $\omega = \omega'2$ are similar. \qed

**Lemma 2.7.** Let $x, y \in K$, $x < y$ and let $\omega$ be a shortest word such that $K_{\omega} \subseteq [x, y]$. Then

$$ \mu(K_{\omega}) \leq \mu([x, y]) \leq 30\mu(K_{\omega}), \tag{2.41} $$
$$ r_\omega \leq |x - y| \leq (1 + \rho^{-1}) \rho^{-3} r_\omega, \tag{2.42} $$
$$ C^{-1} |\mu(K_{\omega})r_\omega|^{\alpha} \leq d_+(x, y) \leq C |\mu(K_{\omega})r_\omega|^{\alpha}. \tag{2.43} $$

(In fact, one can take $C = \{1 + 2(8\rho^{-3})^\alpha\} \{20\rho^{-2}\}^\alpha$ in (2.43).) Consequently, condition (1.12) holds with $\beta = 1/\alpha$.

**Proof.** If $\omega = \emptyset$, nothing needs to be proved. Assume that $\omega \neq \emptyset$.

We first consider the case $\omega = \omega'0$ for some word $\omega'$. Without loss of generality assume that $\omega' \neq \emptyset$. Then $y \in K_{\omega'}$ and $x \in K_{\tau}$ for the left neighboring word $\tau'$ of $\omega'$ (see Figure 2 above), and

$$ K_{\omega'0} = K_{\omega} \subseteq [x, y] \subseteq K_{\tau'} \cup K_{\omega}. \tag{2.44} $$

We note that the above estimates are still valid if $\omega = \omega'0 \cdots 0$ as long as we replace $K_{\omega'}$ by the empty set in (2.44), $r_{\tau'}$ by 0 in (2.45) and $d_+(K_{\tau'})$ by 0 in (2.47). It follows that

$$ \mu(K_{\omega}) \leq \mu([x, y]) \leq \mu(K_{\tau'}) + \mu(K_{\omega'}) \leq 3\mu(K_{\omega'}) \quad \text{(using (2.7))}, $$

$$ \leq 15\mu(K_{\omega'}1) \quad \text{(using (2.36))}, $$

$$ \leq 30\mu(K_{\omega'0}) = 30\mu(K_{\omega}) \quad \text{(using (2.7))}, $$

thus proving (2.41).

Observe that by (2.44) and (2.1),

$$ r_\omega \leq |x - y| \leq r_{\omega'} + r_{\tau'}. \tag{2.45} $$

Since $r_\omega = r_{\omega'0} = \rho^3 r_{\omega'}$, using (2.39) we have

$$ r_{\omega'} + r_{\tau'} \leq r_{\omega'} + \rho^{-1} r_{\omega'} = (1 + \rho^{-1}) r_{\omega'} = (1 + \rho^{-1}) \rho^{-2} r_\omega. \tag{2.46} $$

Combining this with (2.45) proves (2.42) for the case $\omega = \omega'0$.

Using (2.32), (2.23) and (2.44), we have

$$ C^{-1} d_+(K_{\omega'1}) \leq d_+(K_{\omega'0}) = d_+(K_{\omega}) \leq d_+(x, y) $$
$$ \leq d_+(K_{\omega'}) + d_+(K_{\tau'}) \leq (1 + C)d_+(K_{\omega'}) $$
$$ \leq (1 + C)Ad_+(K_{\omega'1}) \quad \text{(using (2.38)) with $\omega$ replaced by $\omega'$).} \tag{2.47} $$

By (2.7) and (2.5),

$$ \mu(K_{\omega'1})r_{\omega'1} \leq 2\mu(K_{\omega'0})r_{\omega'1} = 2\rho\mu(K_{\omega})r_\omega, $$
$$ \mu(K_{\omega'1})r_{\omega'1} \geq 2^{-1}\mu(K_{\omega'0})r_{\omega'1} = 2^{-1}\rho\mu(K_{\omega})r_\omega. $$

Combining these inequalities with (2.20) yields

$$ (2^{-1}\rho)^\alpha [\mu(K_{\omega'})r_\omega]^\alpha \leq [\mu(K_{\omega'1})r_{\omega'1}]^\alpha = d_+(K_{\omega'1}) \leq (2\rho)^\alpha [\mu(K_{\omega})r_\omega]^\alpha. $$
From this and using (2.47), we have
\[ C^{-1} (2^{-1} \rho)^\alpha |\mu(K_\omega) r_\omega|^\alpha \leq d_s(x, y) \leq (1 + C) A (2\rho)^\alpha |\mu(K_\omega) r_\omega|^\alpha, \]
thus proving (2.43).

The case \( \omega = \omega' \) can be treated similarly. The case \( \omega = \omega' \) is similar except that (2.46) becomes
\[ r_\omega + r_\tau \leq (1 + \rho^{-1}) \rho^{-3} r_\omega. \]

Finally, the formula (1.12) with \( \beta = 1/\alpha \) follows directly by combining (2.41), (2.42) and (2.43). The proof is complete. \( \square \)

**Lemma 2.8.** Condition (1.13) is satisfied.

**Proof.** It suffices to show that there exists a constant \( C > 1 \) (depending only on \( \rho \)) such that for any \( 0 \leq x < y < z \leq 1 \) with \( d_*(x, y) = d_*(y, z) \), we have
\[ C^{-1} \mu([y, z]) \leq \mu([x, y]) \leq C \mu([y, z]). \tag{2.48} \]

Choose any two shortest words \( \omega \) and \( \tau \) such that \( K_\omega \subseteq [x, y] \) and \( K_\tau \subseteq [y, z] \).

We claim that there exists a universal integer \( k \geq 0 \) (depending only on \( \rho \)) such that
\[ ||\omega|| - ||\tau|| \leq k. \tag{2.49} \]
Indeed, without loss of generality, assume that \( ||\omega|| \geq ||\tau|| \geq 1 \), and let \( \omega' \leq \omega \) be such that \( ||\omega'|| = ||\tau|| \), \( \omega = \omega' \theta \) for some word \( \theta \) (possibly \( \theta = \emptyset \)). Then by applying Proposition 2.6, we see that the number of words with the same length \( ||\tau|| \) and lying between \( y \) and \( z \) is at most 4. See Figure 3 for the worst case when \( \omega' = \omega \). More precisely, the cell \( K_{\omega'} \) can be connected to the cell \( K_\tau \) by at most nine \( ||\tau|| \)-cells.

Thus, by repeatedly using Lemma 2.1, we have
\[ 2^{-5} \mu(K_\omega) \leq \mu(K_{\omega'}) \leq 2^5 \mu(K_\tau), \tag{2.50} \]
and repeatedly using (2.39) yields
\[ r_{\omega'} \leq \rho^{-5} r_\tau. \tag{2.51} \]

By (2.43), we see that
\[ \mu(K_\omega)^\alpha r_\omega^\alpha \approx d_*(x, y) = d_*(y, z) \approx \mu(K_\tau)^\alpha r_\tau^\alpha. \]
Combining this with (2.50) and the inclusion \( K_\omega \subseteq K_{\omega'} \), we see that there exists some \( C_0 > 0 \) such that
\[ \mu(K_\tau) r_\tau \leq C_0 \mu(K_\omega) r_\omega \leq C_0 \mu(K_{\omega'}) r_\omega \leq C_0 2^5 \mu(K_\tau) r_\omega, \tag{2.52} \]
and after dividing by $\mu(K_\tau)$, we get
\[ r_\tau \leq C_0 2^5 r_\omega. \]

Combining this with (2.51), we have
\[ r_\omega' \leq C_0 (2\rho^{-1})^5 r_\omega \leq C_0 (2\rho^{-1})^5 r_\omega \rho^{2(|\omega| - |\tau|)}, \]
where we have used the following fact from (2.5):
\[ r_\omega = r_\omega r_\theta \leq r_\omega \rho^{2|\theta|} = r_\omega \rho^{2(|\omega| - |\omega'|)} = r_\omega \rho^{2(|\omega| - |\tau|)}. \]

This shows that $|\omega| - |\tau|$ is bounded by a universal integer $k$, proving our claim.

Note that by (2.36) and Lemma 2.1,
\[ \mu(K_{\sigma i}) \leq \mu(K_\sigma) \leq 5\mu(K_\sigma) \leq 10\mu(K_\sigma) \tag{2.53} \]
for any word $\sigma$ and any $i \in \{0, 1, 2\}$. From this and using (2.49), we have
\[ \mu(K_\omega') \leq 10\mu(K_{\omega'{\theta}_1}) \leq \cdots \leq 10^k \mu(K_{\omega'{\theta}_1 \cdots \theta_k}) = 10^k \mu(K_\omega), \]
which together with (2.50) gives
\[ \mu(K_\omega) \asymp \mu(K_\tau). \]

Finally, from this and using (2.41), we see that
\[ \mu([x, y]) \asymp \mu(K_\omega) \asymp \mu(K_\tau) \asymp \mu([y, z]), \]
thus proving (2.48). The lemma follows. \qed

**Lemma 2.9.** Condition (1.14) with $\beta = 1/\alpha$ is satisfied.

**Proof.** For any $x \in [0, 1]$, any small $0 < r < d_\ast(K)/2$ and any integer $\ell \geq 2$, choose $z$ and $y_\ell$ in $[0, 1]$ such that either $x < y_\ell < z$ or $z < y_\ell < x$, and such that
\[ d_\ast(x, z) = r = \ell d_\ast(x, y_\ell). \tag{2.54} \]

Then, letting $\eta = 1/\ell$, by (2.48) we have
\[ V(x, r) \asymp \mu([x, z]) \quad \text{and} \quad V(x, \eta r) \asymp \mu([x, y_\ell]). \]

From this and using (1.12), we have
\[ \frac{V(x, r)}{r^3} \asymp \frac{\mu([x, z])}{r^3} \asymp \frac{d_\ast(x, z)^3}{|x - z|r^3} = \frac{1}{|x - z|}, \]
\[ \frac{V(x, \eta r)}{(\eta r)^3} \asymp \frac{\mu([x, y_\ell])}{(\eta r)^3} \asymp \frac{d_\ast(x, y_\ell)^3}{|x - y_\ell|(\eta r)^3} = \frac{1}{|x - y_\ell|}. \]

Thus, in order to prove (1.14), it suffices to show that
\[ \lim_{\ell \to \infty} \frac{|x - y_\ell|}{|x - z|} = 0, \tag{2.55} \]
where the convergence is uniform with respect to $x, r$ and all possible choices of $y_\ell, z$.

We may assume that $x < y_\ell < z$; the other case $z < y_\ell < x$ is similar. Choose a shortest word $\omega := \omega(\ell)$ such that
\[ K_\omega \subseteq [x, y_\ell]. \]

Then by (2.42) we have
\[ |x - y_\ell| \asymp r_\omega. \tag{2.56} \]
Consider a chain of \( k + 1 \) neighboring words starting from \( \omega \) and with length \(|\omega|\). By Proposition 2.4 and Lemma 2.5, there exists a constant \( C_0 > 1 \) such that the sum of \( d_* \) distances of these cells is not more than

\[
d_*(K_\omega)(1 + C_0 + \cdots + C^k_0).
\]

(2.57)

Choose \( k \) to be the largest integer such that

\[
1 + C_0 + \cdots + C^k_0 \leq \ell,
\]

(2.58)

and thus

\[
k \approx \log \ell.
\]

(2.59)

From (2.54), (2.57) and (2.58), we see that the chain of \( k + 1 \) neighboring cells are contained in \([x, z] \) (see Figure 4).

Figure 4. A chain of \( k + 1 \) cells with length \(|\omega|\) contained in \([x, z] \).

Set \( k = 3^{N+1} \) for an integer \( N \geq 0 \). We claim that there exists a word \( \tau \) such that

\[
|\omega| - N = |\tau| > 0 \quad \text{and} \quad K_\tau \subseteq [x, z];
\]

(2.60)

that is, the points \( x \) and \( z \) can be separated apart by at least one \((|\omega| - N)\)-cell. In fact, if \([x, z] \) does not contain any \((|\omega| - N)\)-cell, then the number of \((|\omega| - N)\)-cells outside \([x, z] \) is at least \( 3^{|\omega| - N} - 2 \); this is because the total number of \((|\omega| - N)\)-cells in \( K \) is \( 3^{|\omega| - N} \) whilst the number of \((|\omega| - N)\)-cells intersecting \([x, z] \) is at most 2.

As each \((|\omega| - N)\)-cell contains \( 3^N \) cells of length \(|\omega|\), the total number of cells with length \(|\omega|\) outside \([x, z] \) is thus at least

\[
(3^{|\omega| - N} - 2) \cdot 3^N = 3^{|\omega|} - 2 \cdot 3^N.
\]

However, it follows from earlier discussions that there are \( k + 1 = 3^{N+1} + 1 \) cells with length \(|\omega|\) inside \([x, z] \). Thus, after summing up, the total number of \(|\omega|\)-cells in \( K \) is at least

\[
(3^{|\omega|} - 2 \cdot 3^N) + (3^{N+1} + 1) = 3^{|\omega|} + 3^N + 1 > 3^{|\omega|},
\]

a contradiction, since the number of \(|\omega|\)-cells on \( K \) is \( 3^{|\omega|} \). This proves the claim.

Let \( \omega' < \omega \) satisfy

\[
|\omega'| = |\omega| - N.
\]

Then \( \tau \) satisfying (2.60) can be chosen so that either \( \omega' \) and \( \tau \) are neighboring words or \( \omega' = \tau \). We obtain

\[
|x - z| \geq r_\tau \quad \text{(since \( K_\tau \subseteq [x, z] \) by (2.60))}
\]

\[
\geq \rho r_{\omega'} \quad \text{(by (2.39))}
\]

\[
\geq \rho \cdot \rho^{-2N} r_\omega
\]

\[
\geq C^{-1} \rho^{-2N} |x - y| \quad \text{(by (2.56))}
\]

(2.61)
for some constant $C$ depending only on $\rho$, where (2.61) holds because by setting $\omega = \omega' \theta$ we have
\[
    r_\omega = r_{\omega'} \cdot r_\theta \leq r_{\omega'} (\rho^2)^{|\theta|} = r_{\omega'} (\rho^2)^N
\]
by virtue of (2.5) and (2.60). Therefore, using (2.59), we have
\[
    |x - y| \leq r_{\omega'} (\rho^2)^N \leq C \rho^2 \ell \leq C \rho^{2 \log_3 k} \leq C (\log \ell)^2 \log_3 \rho \to 0
\]
as $\ell \to \infty$, completing the proof. \hfill \qed

Proof of Theorem 1.1. From above, conditions (1.10)–(1.14) are all satisfied with $\beta = 1/\alpha$ and hence Theorem 1.1 follows. \hfill \qed

3. $m$-fold convolution of Cantor-type measures. Let $\{S_i\}_{i=0}^m$ and $\mu := \mu_m$ be defined as in (1.18) and (1.20) respectively, with $m \geq 3$ being an integer, and let $K := [0, m]$. In this section we introduce a new metric $d_*$ on $K$, and again show that conditions (1.10)–(1.14) are all satisfied. Therefore, the conclusion in Theorem 1.2 holds.

Let $\{T_i\}_{i=0}^{m-1}$ be the auxiliary IFS defined by
\[
    T_i(x) = \frac{1}{m} x + i, \quad i = 0, 1, \ldots, m - 1
\]
(see Figure 5 for the case $m = 3$).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.png}
\caption{(a) The IFS $\{S_i\}_{i=0}^3$ has overlaps. (b) The auxiliary IFS $\{T_i\}_{i=0}^2$ does not have overlaps.}
\end{figure}

Let
\[
    \mathcal{J} = \{0, 1, \ldots, m - 1\}, \quad \mathcal{J}_1 := \{1, \ldots, m - 2\}, \quad \mathcal{J}_0 := \{0, m - 1\},
\]
and for each $n \in \mathbb{N}$, let
\[
    \mathcal{J}^n := \{0, 1, \ldots, m - 1\}^n, \quad \mathcal{J}_0^n := \{0, m - 1\}^n, \quad \mathcal{J}^* := \bigcup_{k=0}^\infty \mathcal{J}^k, \quad \mathcal{J}_0^* := \bigcup_{k=0}^\infty \mathcal{J}_0^k,
\]
where $\mathcal{J}_0$ and $\mathcal{J}_0^*$ are defined to be the singleton $\{\emptyset\}$ of the empty word $\emptyset$ as before. It is shown in [23] that $\mu$ satisfies a family of second-order identities with respect to the IFS $\{T_i\}_{i=0}^{m-1}$. More precisely, for $i, j, k \in \mathcal{J}$, define
\[
    a_{j,k}^{(i)} := \begin{cases} w_{\ell}, & \text{if } \exists \ell \ (0 \leq \ell \leq m) \text{ such that } i + mj - (m - 1)\ell = k \\ 0, & \text{otherwise}, \end{cases}
\]
where \( \{w_\ell\}_{\ell=0}^m \) is given by (1.19), and let \( M_i, 0 \leq i \leq m - 1, \) be the matrix
\[
M_i := \left[ a_{p-1,q-1} \right]_{p,q=1}^m. \tag{3.2}
\]

For example, for \( m = 3, \)
\[
M_0 = \begin{bmatrix}
w_0 & 0 & 0 \\
0 & w_1 & 0 \\
w_3 & 0 & w_2
\end{bmatrix}, \quad M_1 = \begin{bmatrix}
0 & w_0 & 0 \\
w_2 & 0 & w_1 \\
0 & w_3 & 0
\end{bmatrix}, \quad M_2 = \begin{bmatrix}
w_1 & 0 & w_0 \\
0 & w_2 & 0 \\
0 & 0 & w_3
\end{bmatrix}.
\]

In general,
\[
M_0 = \begin{bmatrix}
w_0 & 0 & \cdots & 0 & 0 \\
0 & w_1 & 0 & \cdots & 0 \\
\vdots & 0 & w_2 & \cdots & \vdots \\
0 & \cdots & \cdots & \cdots & 0 \\
w_m & \cdots & 0 & 0 & w_{m-1}
\end{bmatrix}, \quad M_1 = \begin{bmatrix}
0 & w_0 & \cdots & 0 \\
\vdots & 0 & w_1 & \cdots \\
0 & \cdots & \cdots & 0 \\
w_{m-1} & 0 & \cdots & 0 \\
0 & w_m & 0 & \cdots & 0
\end{bmatrix},
\]
\[...
\]
\[
M_{m-2} = \begin{bmatrix}
0 & \cdots & 0 & w_0 & 0 \\
w_2 & 0 & \cdots & 0 & w_1 \\
0 & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & w_{m-1} & \cdots & 0 \\
0 & \cdots & 0 & w_m & 0
\end{bmatrix}, \quad M_{m-1} = \begin{bmatrix}
w_1 & 0 & \cdots & 0 \\
0 & w_2 & 0 & \cdots \\
\vdots & \cdots & \cdots & \cdots \\
0 & \cdots & w_{m-1} & 0 \\
0 & 0 & \cdots & 0 & w_m
\end{bmatrix} \tag{3.3}
\]

Then for all Borel subsets \( A \subseteq [0,m], \)
\[
\begin{bmatrix}
\mu(T_0 T_1 A) \\
\vdots \\
\mu(T_{m-1} T_i A)
\end{bmatrix} = M_i \begin{bmatrix}
\mu(T_0 A) \\
\vdots \\
\mu(T_{m-1} A)
\end{bmatrix}, \quad i \in \mathcal{J}. \tag{3.4}
\]

For \( \omega = \omega_1 \cdots \omega_\ell \in \mathcal{J}^\ell, \) we use the notation
\[
K_\omega := T_{\omega_1} \circ \cdots \circ T_{\omega_\ell}(K)
\]
as before. For \( i \in \mathcal{J}, \) let \( e_i \) denote the unit vector in \( \mathbb{R}^m \) whose \((i+1)\)-st coordinate is 1. Applying (3.4) repeatedly yields
\[
\mu(T_\omega A) = e_{\omega_1} M_{\omega_2} \cdots M_{\omega_\ell} \begin{bmatrix}
\mu(T_0 A) \\
\vdots \\
\mu(T_{m-1} A)
\end{bmatrix} \quad \text{for all } A \subseteq [0,m]. \tag{3.5}
\]

Throughout this section we let \( M^t, u^t \) denote the transposes of a matrix \( M \) and a vector \( u \) respectively.

**Lemma 3.1.** Condition (1.10) holds.

**Proof.** Assume, without loss of generality, that \( K_\omega \) is on the left of \( K_\tau. \) Then exactly one of the following relationships holds for \( i = 0,1, \ldots, m-2 \) (\( \theta \) can be empty):
\[
\omega = \theta i (m - 1) \cdots (m - 1) \quad \text{and} \quad \tau = \theta (i+1) \cdots 0.
\]
Assume that $\theta = \theta_1 \cdots \theta_\ell$ for $\ell \geq 1$, $k \geq 1$ and $i = 0$; other cases are similar. Then
\[ \mu(K_\omega) = e^{\theta_1 M_{\theta_2} \cdots M_{\theta_\ell} M_0 M_{m-1}^k} \]
where $[a_0 \cdots a_{m-1}] = e^{\theta_1 M_{\theta_2} \cdots M_{\theta_\ell}}$ and $\mu = [\mu(T_0) \cdots \mu(T_{m-1} K)]$.

Similarly,
\[ \mu(K_\tau) = [a_0 \cdots a_{m-1}] M_1 M_0^k \mu^t. \]  

A direct calculation shows
\[ M_{m-1}^k = \begin{bmatrix} w_1^k & 0 & \cdots & 0 & w_0 \sum_{i=1}^{k} w_1^{k-i} w_m^{i-1} \\ 0 & w_2^k & \cdots & 0 & 0 \\ 0 & 0 & w_3^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 \\ w_{m-1}^k & 0 & \cdots & 0 & w_m^{k-1} \end{bmatrix} \]
\[ = \begin{bmatrix} w_1^k & 0 & \cdots & 0 & w_1^k \\ 0 & w_2^k & \cdots & 0 & 0 \\ 0 & 0 & w_3^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 \\ w_{m-1}^k & 0 & \cdots & 0 & w_m^{k-1} \end{bmatrix} \]

Thus we obtain
\[ M_0 M_{m-1}^k \mu^t \begin{bmatrix} w_1^k & 0 & \cdots & 0 \\ 0 & w_2^k & \cdots & 0 \\ 0 & 0 & w_3^k & \cdots \\ \vdots & \vdots & \ddots & \ddots \\ 0 & \cdots & \cdots & \cdots \\ w_{m-1}^k & 0 & \cdots & 0 \end{bmatrix} \]
\[ = \begin{bmatrix} w_1^k & 0 & \cdots & 0 \\ w_2^k & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots \\ w_{m-1}^k & 0 & \cdots & 0 \end{bmatrix}, \]

and similarly, we have
\[ M_1 M_0^k \mu^t \begin{bmatrix} 0 & w_1^k & 0 & \cdots & 0 \\ 0 & 0 & w_2^k & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ w_{m-1}^k & 0 & \cdots & 0 & w_m^{k-1} \\ 0 & w_1^k & \cdots & 0 & 0 \end{bmatrix} \]
\[ = \begin{bmatrix} 0 & w_1^k \\ 0 & w_2^k \\ \vdots & \vdots \\ w_{m-1}^k & 0 \\ 0 & w_1^k \end{bmatrix}. \]
Substituting (3.8) and (3.9) into (3.6) and (3.7) separately, we obtain

$$\mu(K_\omega) \propto \begin{bmatrix} a_0 & \cdots & a_{m-1} \\ w^1_k & w^2_k & \cdots \\ \vdots \\ w^k_{m-1} & w^k_m \\ w^k_1 \end{bmatrix} \propto \mu(K_r).$$

The assertion follows. \(\square\)

Recall that for \(-(m-1) \leq k \leq m-1\), the \textit{k-diagonal} of an \(m \times m\) matrix \(A = (a_{ij})\) consists of the entries \(a_{ij}\) with \(j = \ell + k\). The main diagonal is the 0-diagonal. We say that \(A\) is of Type 0 (or Type \(m-1\)) if all its \(k\)-diagonals are zero, except possibly for \(k = 0\) or \(k = \pm (m-1)\), and that it is of Type \(i\) \((1 \leq i \leq m-2)\) if all its \(k\)-diagonals are zero, except possibly for \(k = i\) or \(k = -i'\), where

\[i' = m - 1 - i.\]  

An entry \(a_{ij}\) of \(A\) belongs to the \(k\)-diagonal \((-m-1) \leq k \leq m-1\) if and only if \(j - \ell = k\). Note that for each \(i \in \{0, 1, \ldots, m-1\}\), the matrix \(M_i\) defined in (3.2) is of Type \(i\).

**Proposition 3.2.** Let \(i \in \{0, 1, \ldots, m-1\}\) and let \(A\) be an \(m \times m\) matrix of Type \(i\). Then

(a) \(A^t\), the transpose of \(A\), is of Type \(i'\);

(b) the \((m-i)\)-th row of \(A\) is of the form \([*, 0, \ldots, 0, *]\);

(c) the \((i+1)\)-st column of \(A\) is of the form \([*, 0, \ldots, 0, *]^t\).

**Proof.** These are obvious if \(i = 0\) or \(i = m-1\). So we assume that \(1 \leq i \leq m-2\).

(a) The possibly nonzero entries of \(A\) are:

(i) the \(i\)-diagonal: \((1, i+1), (2, i+2), \ldots, (m-i, m)\), and

(ii) the \(-i'\)-diagonal: \((m-i, 1), (m-i+1, 2), \ldots, (m, i+1)\).

Hence, the possibly nonzero entries of \(A^t\) are:

(i') \((1, m-i), (2, m-i+1), \ldots, (i+1, m)\), and

(ii') \((i+1, 1), (i+2, 2), \ldots, (m, m-i)\),

which are respectively the \(i'\) and \(-i = -(i')\) diagonals of \(A^t\). Hence \(A^t\) is of Type \(i'\).

(b) By (i) and (ii), the entries of the two diagonals belong to the same row if and only if the row number is \(m-i\) and the entries are \((m-i, m)\) and \((m-i, 1)\), proving (b).

Finally, (c) follows from (b) by taking transpose and using (a). \(\square\)

We now study products of such matrices.

**Proposition 3.3.** Let \(A, B\) be two \(m \times m\) matrices of Types \(i\) and \(j\), respectively. Then \(AB\) is of Type \(i+j \mod (m-1)\).

**Proof.** Case 1. \(i = 0\) or \(j = 0\). It is obvious that if both \(A\) and \(B\) are of Type 0, then so is \(AB\). Now assume that \(i = 0\) and \(1 \leq j \leq m-2\). Using (i) and (ii) in the proof of Proposition 3.2, we have:

(1) the nonzero entries corresponding to multiplication of \(A\) and the \(j\)-diagonal of \(B\) are \((\ell, j+\ell), 1 \leq \ell \leq m-j\), which belong to the \(j\)-diagonal, and \((m, j+1)\), which belongs to the \((-j')\)-diagonal:
(2) the nonzero entries corresponding to multiplying $A$ with the $(-j')$-diagonal of $B$ are $(m - j + \ell, \ell + 1)$, $0 \leq \ell \leq j$, which belong to the $(-j')$-diagonal, and $(1, j + 1)$, which belongs to the $j$-diagonal.

Hence $AB$ is of Type $j$.

Next, we assume $1 \leq i \leq m - 2$ and $j = 0$. Then $AB = (B^tA^t)^t$. By Proposition 3.2(a), $B^t$ is of Types 0 and $A^t$ is of Type $i'$. By what we have just proved above, $B^tA^t$ is of Type $i'$. By Proposition 3.2(a) again, we see that $AB$ is of Type $i$.

Case 2. $1 \leq i, j \leq m - 2$. The $i$-diagonal of $A$ is: $(1, i + 1), (2, i + 2), \ldots, (m - i, m)$; the $(-i')$-diagonal of $A$ is: $(m - i, 1), (m - i + 1, 2), \ldots, (m, i + 1)$; the $j$-diagonal of $B$ is: $(1, j + 1), (2, j + 2), \ldots, (m - j, m)$; the $(-j')$-diagonal of $B$ is: $(m - j, 1), (m - j + 1, 2), \ldots, (m, j + 1)$. By checking the product of these diagonals, we see that if $i + j \leq m - 1$, then the nonzero entries of $AB$ are along the $(i + j)$-diagonal or the $(-(i + j))'$-diagonal. If $i + j \geq m$, then the nonzero entries of $AB$ are along the $(i + j + m - 1)$-diagonal or the $-(i + j + m - 1)'$-diagonal. In the former case, $AB$ is of Type $i + j$; in the latter case, it is of Type $i + j + m + 1$. This completes the proof. \qed

**Proposition 3.4.** Let $i, \ell \in J$ and let $A$ be an $m \times m$ matrix of Type $\ell$.

(a) If $i + \ell \equiv 0 \pmod{(m - 1)}$, then $e_iA$ is of the form $[*, 0, \ldots, 0, *]$.

(b) If $i + \ell \equiv k \pmod{(m - 1)}$, where $1 \leq k \leq m - 2$, then $e_iA$ is of the form $ae_k$ for some $a \in \mathbb{R}$.

**Proof.** (a) Since $A$ is of Type $\ell$, by Proposition 3.2, the $(m - \ell)$-th row of $A$ is of the form $[*; 0, \ldots, 0, *]$. It follows from $i + \ell \equiv 0 \pmod{(m - 1)}$ that $i = -\ell$ or $i = m - 1 - \ell$ or $i = 2(m - 1) - \ell$. In the second case, $e_i = e_{m-\ell-1}$ and the assertion follows. In the first or the third case $i = \ell = 0$ or $i = \ell = m - 1$, thus $e_i = e_{m-1}$ and $A$ is of Type 0. Again the assertion follows.

(b) Let $i + \ell \equiv k \pmod{(m - 1)}$ and $1 \leq k \leq m - 2$. Since the unique nonzero entry of $A$ in row $(i + 1)$ falls in the column $(k + 1)$,

$$e_iA = [0, \ldots, 0, *, 0 \ldots, 0] = ae_k$$

for some $a \in \mathbb{R}$. \qed

For $i \in \{0, m - 1\}$, let $\widetilde{M}_i$ be the matrix formed from $M_i$ by keeping its first and last rows and assigning 0 to all other entries. For $i \in J_1$, let $\widetilde{M}_i$ denote the matrix formed from $M_i$ by keeping its $(m - i)$-th row and assigning 0 to all other entries. For $J = (j_1, \ldots, j_k) \in J_0^k$, where $k \geq 0$, define (see [23])

$$c_{i,j} := [w_{i+1}, 0, w_i]M_j \begin{bmatrix} w_0 \\ 0 \\ w_m \end{bmatrix} = [w_{i+1}, 0, w_i]\widetilde{M}_j \begin{bmatrix} w_0 \\ 0 \\ w_m \end{bmatrix} = e_iM_i\widetilde{M}_j \begin{bmatrix} w_0 \\ 0 \\ w_m \end{bmatrix}, \quad i \in J_1,$$

where $0$ denotes the zero vector in $\mathbb{R}^{m-2}$, $M_j := M_{j_1} \cdots M_{j_k}$ and $\widetilde{M}_j := \widetilde{M}_{j_1} \cdots \widetilde{M}_{j_k}$

($M_j, \widetilde{M}_j$ are defined to be the identity matrix if $k = 0$).

Define a distance $d_*$ on $K$ as follows. For any $x, y \in [0, m]$ with $x < y$, let

$$W(x, y) := \left\{ \omega = \omega_1 \cdots \omega_n \in J^* \setminus \{\emptyset\} : \begin{array}{l} \omega_n \in J_1, K_\omega \subseteq [x, y], \\
\sum_{i=1}^n \omega_i \equiv 0 \mod{(m - 1)}, \\
and \omega \text{ is a father} \end{array} \right\},$$
where the notion “\( \omega = \omega_1 \cdots \omega_n \) is a father” means that none of the proper ancestors (or prefixes) \( \omega_1 \cdots \omega_k \) \((k < n)\) satisfies all of the following conditions:

- \( \omega_k \in J_1; \)
- \( K_{\omega_1 \cdots \omega_k} \subseteq [x, y]; \)
- \( \sum_{i=1}^{k} \omega_i \equiv 0 \mod (m - 1). \)

For example, let \( m = 3 \). If \([x, y] = [0, 1] = K_0\), then

\[
W(x, y) = \{011, 0011, 0211, 0101, 0121, \ldots \}.
\]

If \([x, y] = [4, 7] = K_{11}\), then \( W(x, y) = \{11\}. \)

Similar to Proposition 2.3, we have

**Proposition 3.5.** For any \( 0 \leq x < y \leq m \) and any distinct \( \omega, \tau \in W(x, y) \), we have

\[
K_\omega \cap K_\tau = \emptyset.
\]

Define a set of words \( S \) by

\[
S = \left\{ \omega_1 \cdots \omega_n \in J^* \setminus \{\emptyset\} : \begin{array}{l}
\omega_1 \in J_1, \omega_n \in J_1, \sum_{i=1}^{n} \omega_i \equiv 0 \mod (m - 1), \\
\sum_{i=1}^{k} \omega_i \not\equiv 0 \mod (m - 1) \text{ for each } k = 1, \ldots, n - 1
\end{array} \right\}.
\]

Now define \( d_*(x, y) := 0 \) if \( x = y \), and if \( x < y \), define

\[
d_*(x, y) := d_*(y, x) := \sum_{\omega \in W(x, y)} \sum_{J \in J_0^*} \sum_{\sigma \in S} (r_{\omega, J_\sigma} \mu(K_{\omega, J_\sigma}))^\alpha,
\]

where \( \alpha \) is the unique solution of the equation

\[
\frac{1}{m^\alpha} \sum_{i=1}^{m-1} w_i^\alpha + \frac{1}{m^{\beta(m+2)\alpha}} \sum_{i=1}^{m-2} c_{i,j}^\alpha = 1,
\]

where \( c_{i,j} \) is given by (3.11). We refer to [25, Theorem 1.3, Lemma 6.12] for the existence of \( \alpha \) and [25, Corollary 1.4] for the estimate \( \alpha < 1/2 \). We remark that \( 2\alpha \) is the spectral dimension of the Laplacian \(-\Delta_\mu\) defined by \( \mu \) [25]. For example if \( m = 3 \), then \( \alpha \approx 0.4985 < 0.5 \) (this value is close to but strictly less than 0.5).

**Proposition 3.6.** Condition (1.11) is satisfied with \( d_* \) defined above.

**Proof.** Following the same spirit as in the proof of Proposition 2.4, we need only show that for any \( \omega = \omega_1 \cdots \omega_n \) with \( \omega_n \in J_1 \) and \( \sum_{i=1}^{n} \omega_i \equiv 0 \mod (m - 1), \)

\[
\sum_{J \in J_0^*} \sum_{\sigma \in S} (r_{\omega, J_\sigma} \mu(K_{\omega, J_\sigma}))^\alpha = \sum_{J \in J_0^*} \sum_{\sigma \in S} (r_{\omega, J_\sigma} \mu(K_{\omega, J_\sigma}))^\alpha.
\]

Indeed, assume that (1.15) is true. Let \( x, y, z \in K \) with \( x < y < z \). Observe that \( W(x, y) \cap W(y, z) = \emptyset \). By the definition of \( d_* \), we see that

\[
d_*(x, y) + d_*(y, z) = \sum_{\omega \in W(x, y) \cup W(y, z)} \sum_{J \in J_0^*} \sum_{\sigma \in S} (r_{\omega, J_\sigma} \mu(K_{\omega, J_\sigma}))^\alpha,
\]

and that

\[
d_*(x, z) = \sum_{\omega \in W(x, z)} \sum_{J \in J_0^*} \sum_{\sigma \in S} (r_{\omega, J_\sigma} \mu(K_{\omega, J_\sigma}))^\alpha.
\]
Let $\tau \in W(x, z)$ and $k \in \mathbb{N}$. Then by using (3.15) $k$ times, we have

$$
\sum_{J \in J_0^*, \sigma \in S} (r_{\tau, J, \sigma} \mu(K_{\tau, J, \sigma}))^\alpha \\
= \sum_{J \in J_0^*, \sigma \in S} \sum_{J_1', \ldots, J_n' \in J_0^*, \sigma_1', \ldots, \sigma_k' \in S} (r_{\tau, J, \sigma_1', \ldots, \sigma_k'} \mu(K_{\tau, J, \sigma_1', \ldots, \sigma_k'}))^{\alpha}. \\
$$

(3.18)

Since $\{K_{\omega J, \sigma}\}_{J \in J_0^*, \sigma \in S}$ is disjoint for any $\omega \in J^*$, so is

$$
\{K_{\tau, J, \sigma_1', \ldots, \sigma_k'}\}_{J \in J_0^*, \sigma_1', \ldots, \sigma_k' \in S},
$$

and in particular at most one of these cells can contain $y$. Therefore, by classifying all the other terms on the right-hand side of (3.18) according to its unique prefix in $W(x, y) \cup W(y, z)$, using (3.15), we obtain

$$
\sum_{J \in J_0^*, \sigma \in S} (r_{\tau, J, \sigma} \mu(K_{\tau, J, \sigma}))^\alpha = \sum_{J \in J_0^*, \sigma \in S} \sum_{j = 0}^k \omega_{\tau, J, \sigma_1', \ldots, \sigma_j'} \mu(K_{\tau, J, \sigma_1', \ldots, \sigma_j'})^{\alpha} + O(m^{-2\alpha k})
$$

which is equal to, upon letting $k \to \infty$,

$$
\sum_{J \in J_0^*, \sigma \in S} \sum_{j = 0}^\infty \omega_{\tau, J, \sigma_1', \ldots, \sigma_j'} \mu(K_{\tau, J, \sigma_1', \ldots, \sigma_j'})^{\alpha} = \sum_{\omega \in W(x, y), \tau \leq \omega} \sum_{J_0^*, \sigma \in S} (r_{\omega, J, \sigma} \mu(K_{\omega, J, \sigma}))^{\alpha}.
$$

Summing up these equalities over $\tau \in W(x, z)$ results in (1.11) by virtue of (3.17) and (3.16), completing the proof of Proposition 3.6.

We now turn to the proof of (3.15). Indeed, let $J \in J_0^*$ and $\sigma \in J^* \setminus \{\emptyset\}$ satisfy $\sigma_1 \in J_1$. Then by (3.5) with $A = K$,

$$
\mu(K_{\omega, J, \sigma}) = e_{\omega_1} M_{\omega_2} \cdots M_{\omega_{n-1}} M_{\omega_n} M_f M_\sigma \begin{bmatrix} \mu(T_0 K) \\ \vdots \\ \mu(T_{m-1} K) \end{bmatrix}.
$$

(3.19)

On the other hand, we can show that

$$
e_{\omega_1} M_{\omega_2} \cdots M_{\omega_{n-1}} M_{\omega_n} = e_{\omega_1} M_{\omega_2} \cdots M_{\omega_{n-1}} e_{\omega_n} \cdot \begin{bmatrix} w_{\omega_{n+1}} & 0 & w_{\omega_n} \end{bmatrix},
$$

(3.20)

where $\omega_n = m-1 - \omega_n$. Indeed, applying Proposition 3.3 to $M_{\omega_2} \cdots M_{\omega_{n-1}}$ and then Proposition 3.4(b) with $i = \omega_1$, $\ell = \omega_2 + \cdots + \omega_{n-1}$, $k = i + \ell = \omega_1 + \cdots + \omega_{n-1} \equiv m-1 - \omega_n = \omega_n$ mod $(m-1)$, we have

$$
e_{\omega_1} M_{\omega_2} \cdots M_{\omega_{n-1}} = a e_{\omega_n}
$$

(3.21)
for some \( a \geq 0 \). Recall that
\[
M_{\omega_n} = \begin{bmatrix}
0 & \cdots & w_0 & \cdots & 0 \\
\vdots & \ddots & \vdots & & \vdots \\
w_{\omega'_n+1} & \cdots & 0 & \cdots & w_{\omega'_n} \\
\vdots & \ddots & \vdots & & \vdots \\
0 & \cdots & w_m & \cdots & 0
\end{bmatrix},
\]
and hence, using (3.21) twice,
\[
\begin{align*}
e_{\omega_1}M_{\omega_2} \cdots M_{\omega_{n-1}}M_{\omega_n} &= a e_{\omega_1}^t M_{\omega_n} = a \begin{bmatrix} w_{\omega'_n+1} & 0 & w_{\omega'_n} \end{bmatrix} \\
&= a e_{\omega_n} \cdot e_{\omega'_n}^t \begin{bmatrix} w_{\omega'_n+1} & 0 & w_{\omega'_n} \end{bmatrix} \\
&= e_{\omega_1}M_{\omega_2} \cdots M_{\omega_{n-1}} \cdot e_{\omega'_n}^t \begin{bmatrix} w_{\omega'_n+1} & 0 & w_{\omega'_n} \end{bmatrix},
\end{align*}
\]
thus showing (3.20).

We show that for any \( J \in J_0^* \) and any \( \sigma_1 \in \mathcal{J}_1 = \{1, 2, \ldots, m-2\} \),
\[
[w_{\omega'_n+1} \ 0 \ w_{\omega'_n}] M_J M_{\sigma_1} = c_{\omega'_n, J} e_{\sigma_1}.
\]
Indeed, the matrix \( M_J \) is of Type-0 by Proposition 3.3, since so is each \( M_{J_k} \) \( 1 \leq k \leq \ell \) if \( J = J_1 \cdots J_\ell \in J_0^* \). Hence
\[
[w_{\omega'_n+1} \ 0 \ w_{\omega'_n}] M_j = w_{\omega'_n+1} \ 0 \ w_{\omega'_n} \begin{bmatrix} * & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & * \end{bmatrix} = \begin{bmatrix} a & 0 & b \end{bmatrix} (3.23)
\]
for some numbers \( a, b \geq 0 \). As \( M_{\sigma_1} \) is of Type-\( \sigma_1 \), its \( (\sigma_1 + 1) \)-st column looks like
\[
\begin{bmatrix} w_0 \\ 0 \\ w_m \end{bmatrix},
\]
and thus,
\[
\begin{align*}
(a & \ 0 & b) \cdot M_{\sigma_1} = 0 & \cdots & aw_0 + bw_m & \cdots & 0 = (aw_0 + bw_m) e_{\sigma_1} \\
&= \begin{bmatrix} a & 0 & b \end{bmatrix} \begin{bmatrix} w_0 \\ 0 \\ w_m \end{bmatrix} e_{\sigma_1} = [w_{\omega'_n+1} \ 0 \ w_{\omega'_n}] M_J \begin{bmatrix} w_0 \\ 0 \\ w_m \end{bmatrix} e_{\sigma_1}.
\end{align*}
\]
Combining this with (3.23) and (3.11), we obtain
\[
[w_{\omega'_n+1} \ 0 \ w_{\omega'_n}] M_J M_{\sigma} = \begin{bmatrix} a & 0 & b \end{bmatrix} M_{\sigma_1}
\]
\[
= [w_{\omega'_n+1} \ 0 \ w_{\omega'_n}] M_J \begin{bmatrix} w_0 \\ 0 \\ w_m \end{bmatrix} e_{\sigma_1} = c_{\omega'_n, J} e_{\sigma_1}.
\]
thus showing (3.22).

For any \( \sigma = \sigma_1 \cdots \sigma_\ell \) with \( \sigma_1 \in \mathcal{J}_1 \), it follows by using (3.20) and (3.22) that
\[
e_{\omega_1}M_{\omega_2} \cdots M_{\omega_{n-1}}M_{\omega_n} \cdot M_J M_{\sigma}
= e_{\omega_1}M_{\omega_2} \cdots M_{\omega_{n-1}}e_{\omega'_n}^t \cdot [w_{\omega'_n+1} \ 0 \ w_{\omega'_n}] M_J M_{\sigma_1} M_{\sigma_2} \cdots M_{\sigma_\ell}
= e_{\omega_1}M_{\omega_2} \cdots M_{\omega_{n-1}}e_{\omega'_n}^t c_{\omega'_n, J} e_{\sigma_1} M_{\sigma_2} \cdots M_{\sigma_\ell}.
\]
we obtain
\[ \omega \] and for any \( J \) we have \( \mu(T_{m-1}K) \), (using (3.5))

we obtain
\[ e_{\omega_1}M_\omega_2 \cdots M_\omega_{n-1}M_\omega_nM_\sigma = e_{\omega_1}M_\omega_2 \cdots M_\omega_{n-1}e_{\omega_n'} \cdot c_{\omega_n',\sigma} \mu(K_\sigma). \] (3.24)

Thus by (3.19), for any \( J \in J_0^* \) and any \( \sigma \in J^* \setminus \{\emptyset\} \) with initial letter \( \sigma_1 \in J_1 \), and for any \( \omega = \omega_1 \cdots \omega_n \) with \( \omega_n \in J_1 \) and \( \sum_{i=1}^{n} \omega_i \equiv 0 \pmod{(m-1)}, \)
\[ \mu(K_\omegaJ_\sigma) = e_{\omega_1}M_\omega_2 \cdots M_\omega_{n-1}e_{\omega_n'} \cdot c_{\omega_n',\sigma} \mu(K_\sigma). \] (3.25)

From this, we only need to prove (3.15) without \( \omega \) and summation of \( J \); that is,
\[ \sum_{\sigma \in S} (r_\sigma \mu(K_\sigma))^\alpha = \sum_{\sigma \in S} \sum_{\sigma' \in S} (r_{\sigma'} \mu(K_{\sigma'}))^\alpha. \] (3.26)

We first claim that for any integer \( k \geq 3 \) and any \( \theta \in J^* \),
\[ \sum_{\sigma \in S, |\sigma| = k} \mu(K_{\sigma\theta})^\alpha = \sum_{j=1}^{m-1} w_j^\alpha \sum_{\sigma \in S, |\sigma| = k-1} \mu(K_{\sigma\theta})^\alpha. \] (3.27)

Indeed, take any \( \sigma = \sigma_1 \sigma_2 \cdots \sigma_{k-1} \in S \) with \( |\sigma| = k-1 \). Let \( S_\sigma \) be a collection of \( m-1 \) words with length \( k \) which are formed by replacing \( \sigma_1 \) in \( \sigma \) by one of the elements
\[ 1(\sigma_1 - 1), 2(\sigma_1 - 2), \ldots, (\sigma_1 - 1)1, \sigma_10, \]
\[ \sigma_1(m-1), (\sigma_1 + 1)(m-2), \ldots, (m-2)(\sigma_1 + 1), \]
whilst keeping the remaining symbols \( \sigma_2 \cdots \sigma_{k-1} \) unchanged. It is not hard to see that
\[ \bigcup_{\sigma \in S, |\sigma| = k-1} S_\sigma = \{ \sigma : \sigma \in S, |\sigma| = k \}, \] (3.28)
where the union is disjoint.

We first look at the element \( 1(\sigma_1 - 1) \sigma_2 \cdots \sigma_{k-1} \) in \( S_\sigma \). By (3.5), we have
\[ \mu(K_{1(\sigma_1 - 1)\sigma_2 \cdots \sigma_{k-1}\theta}) = e_1 M_{\sigma_1 - 1}M_{\sigma_2} \cdots M_{\sigma_{k-1}}M_\theta \left[ \begin{array}{c} \mu(T_{0}K) \\ \vdots \\ \mu(T_{m-1}K) \end{array} \right] \\
= w_1 e_{\sigma_1}M_{\sigma_2} \cdots M_{\sigma_{k-1}}M_\theta \left[ \begin{array}{c} \mu(T_{0}K) \\ \vdots \\ \mu(T_{m-1}K) \end{array} \right] = w_1 \mu(K_{\sigma\theta}). \]

We treat the other elements \( \tau \in S_\sigma \) similarly. Taking the \( \alpha \)-th power and then summing up the equalities, we obtain
\[ \sum_{\tau \in S_\sigma} \mu(K_{\tau\theta})^\alpha = \sum_{j=1}^{m-1} w_j^\alpha \cdot \mu(K_{\sigma\theta})^\alpha. \]
After summing over \( \{ \sigma : \sigma \in S, |\sigma| = k - 1 \} \) and using (3.28), we obtain (3.27), thus proving our claim.

From the claim (3.27), we have

\[
\sum_{\sigma \in S, |\sigma| = k} \frac{1}{m^{|\sigma|}} \mu(K_\sigma) = \left( \frac{1}{m^\alpha} \sum_{j=1}^{m-1} w_j^\alpha \right)^{k-2} \cdot \sum_{\sigma \in S, |\sigma| = 2} \frac{1}{m^{|\sigma|}} \mu(K_\sigma) = (3.29)
\]

From this we have

\[
\sum_{\sigma \in S, |\sigma| = k} \frac{1}{m^{|\sigma|}} \mu(K_\sigma) = \sum_{k=2}^{\infty} \sum_{\sigma \in S, |\sigma| = k} \frac{1}{m^{|\sigma|}} \mu(K_\sigma) \alpha
\]

\[
= \sum_{\sigma \in S, |\sigma| = 2} \frac{1}{m^{|\sigma|}} \mu(K_\sigma) \alpha \cdot \sum_{\ell=0}^{\infty} \left( \frac{1}{m^\alpha} \sum_{j=1}^{m-1} w_j^\alpha \right)^{\ell} \]

\[
= \sum_{\sigma \in S, |\sigma| = 2} \frac{1}{m^{|\sigma|}} \mu(K_\sigma) \alpha \cdot \left( 1 - \frac{1}{m^\alpha} \sum_{j=0}^{m-1} w_j^\alpha \right)^{-1}, \quad (3.30)
\]

where we have used the fact that \( (1/m^\alpha) \sum_{j=1}^{m-1} w_j^\alpha < 1 \) by (3.14). By taking \( \theta = \emptyset \) and using the fact that \( r_\sigma = 1/m^{|\sigma|} \), we see that the left-hand side of (3.26) is

\[
\sum_{\sigma \in S} \frac{1}{m^{|\sigma|}} \mu(K_\sigma) = \sum_{i=1}^{m-2} \mu(K_{ii'}) \cdot \left( 1 - \frac{1}{m^2} \sum_{j=1}^{m-1} w_j^\alpha \right)^{-1}. \quad (3.31)
\]

On the other hand, using (3.30) with \( \theta = J' \sigma' \) and summing up the equalities, we can write the right-hand side of (3.26) as

\[
\sum_{\sigma \in S} \sum_{\sigma' \in S} \left( r_{\sigma,J'\sigma'} \mu(K_{\sigma,J'\sigma'}) \right) = \left( 1 - \frac{1}{m^\alpha} \sum_{j=1}^{m-1} w_j^\alpha \right)^{-1} \sum_{\sigma \in S, |\sigma| = 2} \sum_{J' \in J_0, \sigma' \in S} \left( r_{\sigma,J'\sigma'} \mu(K_{\sigma,J'\sigma'}) \right) = (3.32)
\]

Observing that the set of all the \( \sigma \in S \) with \( |\sigma| = 2 \) is \( \{1(m-2), 2(m-3), \ldots, (m-2)\} \), and that each such \( \sigma \) is of the form \( \sigma = (m-1-i) = i' \) with \( i \in J_1 \), we have by (2.25)

\[
\mu(K_{\sigma,J'\sigma'}) = \mu(K_{ii'J'\sigma'}) = e_i e_j^{\alpha} \cdot c_{i'i'} c_{i'J'} \mu(K_{\sigma'}) = c_{i'J'} \mu(K_{\sigma'}). \quad (3.33)
\]

Substituting (3.33) and (3.31) into (3.32), we have

\[
\sum_{\sigma \in S} \sum_{J' \in J_0, \sigma' \in S} \left( r_{\sigma,J'\sigma'} \mu(K_{\sigma,J'\sigma'}) \right) = \left( 1 - \frac{1}{m^\alpha} \sum_{j=1}^{m-1} w_j^\alpha \right)^{-2} \cdot \sum_{i=1}^{m-2} \mu(K_{ii'}) \cdot \sum_{i=1}^{m-2} \frac{1}{m^{(|J'|+2)\alpha}} c_{i,J'} \cdot (3.34)
\]

Comparing (3.31) and (3.34), we see that (3.26) is equivalent to

\[
1 = \left( 1 - \frac{1}{m^\alpha} \sum_{j=1}^{m-1} w_j^\alpha \right)^{-1} \cdot \sum_{i=1}^{m-2} \frac{1}{m^{(|J'|+2)\alpha}} c_{i,J'}
\]

which is true by using the definition (3.14) of \( \alpha \). \( \square \)
From the identity (3.14) and the fact that 

\[ \omega \equiv b^j \quad \text{for each } j = 1, \ldots, n-1 \]

there exists a constant \( C > 0 \) such that for any finite word \( \omega \),

\[
C^{-1} d_s(K_\omega) \leq r_\omega \mu(K_\omega)^\alpha \leq Cd_s(K_\omega), \tag{3.35}
\]

where \( d_s(K_\omega) := \sup_{x,y \in K_\omega} d_s(x,y) = d_s(T_\omega(0), T_\omega(m)) \). In particular, \( d_s \) is finite and hence a metric on \( K_\omega \).

**Proof.** We first claim that for any finite word \( \omega \), and any \( b \in J = \{0,1, \ldots, m-1\} \),

\[
d_s(K_{\omega b}) \asymp r_\omega \mu(K_\omega)^\alpha. \tag{3.36}
\]

**Case 1.** \( \omega = \omega_1 \cdots \omega_n \) and \( \sum_{k=1}^n \omega_k \equiv m-1-j \quad \text{(mod } m-1) \) for some \( j \in \mathcal{J}_1 \).

If \( b \neq j \), then \( W(T_\omega(0), T_\omega(m)) \) is the set of all words of the form \( \omega b J \tau \) with \( J \in \mathcal{J}_1^* \) and \( \tau \in \mathcal{S}^{J-b} \). Thus by the definition of \( d_s \), we have

\[
d_s(K_{\omega b}) = \sum_{J \in \mathcal{J}_1^*, \tau \in \mathcal{S}^{J-b}} \sum_{j \in \mathcal{S}^J} r_{\omega b J J'}^\alpha \mu(K_{\omega b J J'})^\alpha. \tag{3.37}
\]

Since \( \omega b J \tau \equiv 0 \quad \text{(mod } m-1) \), using (3.25) and writing \( \tau = \tau_1 \cdots \tau_s \), we obtain

\[
\mu(K_{\omega b J J'}) = \mathbf{e}_{\omega_1} \cdots M_{\omega_n} M_b M_j M_{\tau_1} \cdots M_{\tau_{s-1}} t_{\tau_s} \mathbf{e}_{\tau_s} \mathbf{c}_{\tau_s J} \mu(K_{\sigma}). \tag{3.38}
\]

On the other hand, by Proposition 3.4(b), we see that \( \mathbf{e}_{\omega_1} \cdots M_{\omega_n} M_b M_j M_{\tau_1} \cdots M_{\tau_{s-1}} \) can be written as \( a \mathbf{e}_{\tau_s} \). Hence by using (3.5),

\[
\mu(K_{\omega b J J'}) = \mathbf{e}_{\omega_1} \cdots M_{\omega_n} M_b M_j M_{\tau_1} \cdots M_{\tau_{s-1}} t_{\tau_s} \mathbf{e}_{\tau_s} \mathbf{c}_{\tau_s J} \mu(K_{\sigma}). \tag{3.39}
\]

Using (3.31) and (3.39), we have

\[
\sum_{J' \in \mathcal{J}_0^*, \sigma \in \mathcal{S}} r_{\omega b J J'}^\alpha \mu(K_{\omega b J J'})^\alpha \asymp r_{\omega b J}^\alpha \mu(K_{\omega b J})^\alpha \sum_{J' \in \mathcal{J}_0^*, \sigma \in \mathcal{S}} r_{J J'}^\alpha c_{J J'}^\alpha \mu(K_{\sigma})^\alpha
\]

\[
= r_{\omega b J}^\alpha \mu(K_{\omega b J})^\alpha \cdot \frac{\sum_{i=1}^{m-2} \mu(K_i)^\alpha}{m^{\alpha}} \cdot \left( 1 - \frac{1}{m^{\alpha}} \sum_{j=1}^{m-1} w_j \right)^{-1} \cdot \sum_{J' \in \mathcal{J}_0^*} \frac{1}{m^{\alpha \beta}} c_{J J'}^\alpha \mu(K_{\sigma})^\alpha.
\]

From the identity (3.14) and the fact that \( c_{i,J} \asymp c_{j,J} \) for any \( i, j \in \mathcal{J}_1 \), we have

\[
\sum_{J' \in \mathcal{J}_0^*, \sigma \in \mathcal{S}} r_{\omega b J J'}^\alpha \mu(K_{\omega b J J'})^\alpha \asymp r_{\omega b J}^\alpha \mu(K_{\omega b J})^\alpha. \tag{3.40}
\]
For \( J \in J_0' \), denote by \(|J|\) the number of ‘0’ in \( J \) and \(|J(m-1)|\) the number of ‘\( m-1 \)’ in \( J \). Using Proposition 3.4(b), we see that \( e_{\omega_1} \cdot M_{\omega_2} \cdots M_{\omega_n} \cdot M_b \) is of the form \( a e_{b-j} \). Thus

\[
\mu(K_{\omega_{b-j}}) = e_{\omega_1} \cdot M_{\omega_2} \cdots M_{\omega_n} M_b M_0(b - j)^{|J(0)|} \cdot M_{m-1}(b - j)^{|J(m-1)|} \cdot M_\tau \begin{bmatrix}
\mu(T_0 K) \\
\vdots \\
\mu(T_{m-1} K)
\end{bmatrix}
\]

\[
= \mu(K_{\omega_{b-j}}) \cdot M_0(b - j)^{|J(0)|} \cdot M_{m-1}(b - j)^{|J(m-1)|},
\]

where \( M(b - j) \) denotes the \((b - j + 1, b - j + 1)\)-st entry of a matrix \( M \).

Hence we obtain

\[
\sum_{J \in J_0'} r_{\omega_{b-j}}^a \mu(K_{\omega_{b-j}})^\alpha = r_{\omega_{b-j}}^a \mu(K_{\omega_{b-j}})^\alpha \sum_{k \geq 0} \frac{1}{\mu(k)} \cdot M_0(b - j)^{\alpha|J(0)|} \cdot M_{m-1}(b - j)^{\alpha|J(m-1)|}
\]

\[
= r_{\omega_{b-j}}^a \mu(K_{\omega_{b-j}})^\alpha \sum_{k \geq 0} \sum_{|J|=k} \left( \frac{M_0(b - j)}{m} \right)^{\alpha|J(0)|} \cdot \left( \frac{M_{m-1}(b - j)}{m} \right)^{\alpha|J(m-1)|}
\]

\[
= r_{\omega_{b-j}}^a \mu(K_{\omega_{b-j}})^\alpha \sum_{k \geq 0} \left( \frac{M_0(b - j)}{m} \right)^\alpha + \left( \frac{M_{m-1}(b - j)}{m} \right)^\alpha \cdot k^{\alpha-k}. \quad (3.41)
\]

Given this, on one hand, \( M_0(b - j) \) and \( M_{m-1}(b - j) \) both have lower bound \( w_0 \). On the other hand, \( \left( \frac{M_0(b - j)}{m} \right)^\alpha + \left( \frac{M_{m-1}(b - j)}{m} \right)^\alpha \) is maximal when \( b - j = (m + 1)/2 \) for odd \( m \) and when \( b - j = m/2 - 1 \) or \( m/2 \) for even \( m \). Thus the maximum value is \( (\frac{w(m-1/2)}{m})^\alpha + (\frac{w(m+1/2)}{m})^\alpha \) for odd \( m \) and \( (\frac{w(m/2)}{m})^\alpha + (\frac{w(m+2/2)}{m})^\alpha \) for even \( m \). By using (3.14), we see that this value is strictly less than 1 for both the odd and even cases, and hence the series in (3.41) converges and is bounded away from 0. Consequently,

\[
\sum_{J \in J_0'} r_{\omega_{b-j}}^a \mu(K_{\omega_{b-j}})^\alpha \approx r_{\omega_{b-j}}^a \mu(K_{\omega_{b-j}})^\alpha. \quad (3.42)
\]

Using Proposition 3.4(b), we see that \( e_{\omega_1} \cdot M_{\omega_2} \cdots M_{\omega_n} \) can be written as \( a e_{j'} \). Therefore, we have by (3.5),

\[
\mu(K_{\omega_{b-j}}) = a e_{j'} \cdot M_b \cdot M_\tau \begin{bmatrix}
\mu(T_0 K) \\
\vdots \\
\mu(T_{m-1} K)
\end{bmatrix} = a \cdot c \cdot e_{b-j} \cdot M_\tau \begin{bmatrix}
\mu(T_0 K) \\
\vdots \\
\mu(T_{m-1} K)
\end{bmatrix} = \mu(K_\omega) \cdot \mu(K_{b-j})^\tau,
\]

where \( c \) is the only nonzero entry in the \( j' \)-th row of \( M_b \).

Finally, we consider the summation

\[
\sum_{\tau \in S_{J-b}} r_{\omega_{b-j}}^a \mu(K_{\omega_{b-j}})^\alpha \approx \mu(K_\omega)^\alpha \cdot r_{\omega_{b-j}}^a \cdot \sum_{\tau \in S_{J-b}} r_{\tau}^a \mu(K_{b-j})^\tau. \]
By noting that \((b - j)\tau \in S\) and \(j - b \in S^{J - b}\) and using the formula (3.31), we see that
\[
\mu(K_{(b - j)(j - b)}) \leq \sum_{\tau \in S^{J - b}} r_\tau^\alpha \mu(K_{(b - j)\tau})^\alpha \leq m^\alpha \sum_{s \in S} \frac{1}{m|^s|\alpha} \mu(K_s)^\alpha
\]
\[
= \sum_{s = 1}^{m^2 - 2} \mu(K_{s^\alpha})^\alpha \left(1 - \frac{1}{m^\alpha} \sum_{j = 1}^{m^\alpha} w_j^\alpha\right)^{-1}
\]

Thus, \(\sum_{\tau \in S^{J - b}} r_\tau^\alpha \mu(K_{(b - j)\tau})^\alpha\) has an upper bound, and also a lower bound and these bounds are independent of \(\omega\) and \(b\). We conclude that
\[
\sum_{\tau \in S^{J - b}} r_\tau^\alpha \mu(K_{\omega_\tau})^\alpha \leq \mu(K_\omega)^\alpha \cdot r_\omega^\alpha. \quad (3.43)
\]

Combining (3.37), (3.40), (3.42) and (3.43), we obtain (3.36) as desired.

If \(b = j\), we have \(\mathcal{W}(T_{\omega_j}(0), T_{\omega_j}(m)) = \{\omega_j\}\) and hence
\[
d_\ast(K_{\omega_j}) = \sum_{J \in J_\ast, \sigma \in S} r_{\omega_j J \sigma}^\alpha \mu(K_{\omega_j J \sigma})^\alpha.
\]

Using (3.25), (3.31), (3.14) and Proposition 3.4(b), we obtain
\[
d_\ast(K_{\omega_j}) \asymp (e_{\omega_1}, M_{\omega_2}, \ldots, M_{\omega_n}, e_j^J)^\alpha r_{\omega_j}^\alpha \asymp \mu(K_{\omega_j})^\alpha \cdot r_{\omega_j}^\alpha \asymp \mu(K_{\omega_j})^\alpha \cdot r_\omega^\alpha,
\]
where the third asymptotic relation follows from Lemma 3.1. Hence we have shown that (3.36) is true in Case 1.

Case 2. \(\omega = \omega_1 \cdots \omega_n\) and \(\sum_{k=1}^{n} \omega_k \equiv 0 \pmod{m - 1}\).

If \(b \in J_1\), then we can use a similar strategy as in Case 1 to show that (3.36) is true; we omit the details. If \(b \in J_0\), then \(\mathcal{W}(T_{\omega_b}(0), T_{\omega_b}(m)) = \{\omega_b J_\sigma : J \in J_0, \sigma \in S\}\) and hence
\[
d_\ast(K_{\omega_b}) = \sum_{J \in J_0, \sigma \in S} \sum_{J' \in J_\ast, \sigma' \in S} r_{\omega_b J_\sigma J_\sigma'}^\alpha \mu(K_{\omega_b J_\sigma J_\sigma'})^\alpha. \quad (3.44)
\]

Similar to (3.40) in Case 1, we can drop the summations over \(J'\) and \(\sigma'\); that is
\[
d_\ast(K_{\omega_b}) \asymp \sum_{J \in J_0} \sum_{\sigma \in S} r_{\omega_b J \sigma}^\alpha \mu(K_{\omega_b J \sigma})^\alpha.
\]

Using (3.5) and Proposition 3.4(a), we have
\[
\mu(K_{\omega_b J \sigma}) = e_{\omega_1} M_{\omega_2} \cdots M_{\omega_n} M_b M_j M_{\sigma} \mu(T_0 K)
\]
\[
\leq C e_{\omega_1} M_{\omega_2} \cdots M_{\omega_n} \mu(T_0 K) \cdot [w_2, 0, w_1] M_0 M_{\sigma} \mu(T_{m - 1} K)
\]
\[
= C \mu(K_\omega) \cdot c_{1,0,1} \mu(K_s).
\]
Substituting this into (3.44), and using (3.31) and (3.14), we obtain
\[
d_* (K_{\omega 0}) \leq C \sum_{J \in J_0^*} r_{\omega 0, J}^\alpha \mu(K_{\omega})^\alpha \cdot c^\alpha_{0, J} \sum_{\sigma \in \mathcal{S}} r_{\sigma}^\alpha \mu(K_{\sigma})^\alpha \leq C' \sum_{J \in J_0^*} r_{\omega 0, J}^\alpha \mu(K_{\omega})^\alpha \cdot c^\alpha_{0, J}
\]
\[
\leq C \sum_{J' \in \mathcal{J}_0^*} \frac{1}{m |J'|^\alpha} c^\alpha_{1, J'} \leq C \mu(K_{\omega})^\alpha. \tag{3.45}
\]

On the other hand, since \( \omega_1 \) belongs to Case 1, we have
\[
d_* (K_{\omega 0}) \geq d_* (K_{\omega 01}) \geq C^{-1} r_{\omega 01}^\alpha \mu(K_{\omega 01})^\alpha \geq C^{-1} r_{\omega}^\alpha \mu(K_{\omega})^\alpha, \tag{3.46}
\]
where the last inequality is obtained by using Lemma 3.1. We conclude from (3.45) and (3.46) that (3.36) is true in Case 2. Therefore our claim holds.

It follows from Proposition 3.6 and (3.36) that
\[
d_* (K_{\omega}) = \sum_{j = 0}^{m-1} d_* (K_{\omega_j}) \asymp r_{\omega}^\alpha \mu(K_{\omega})^\alpha,
\]
which completes the proof.

\[\square\]

**Lemma 3.8.** Condition (1.12) holds with \( \beta = 1/\alpha \), where \( \alpha \) is given by (3.14).

**Proof.** Choose one of the shortest words, say \( \omega' \), such that \( K_{\omega'} \subseteq \{x, y\} \). Without loss of generality, we assume that \( \omega' \neq \emptyset \). Let \( \omega' = \omega j \), where \( 0 \leq j \leq m - 1 \) and \( \omega \) could be the empty word. If \( \omega \neq \emptyset \), then there exists a neighbor \( \tau \) of \( \omega \) such that
\[
K_{\omega j} \subseteq [x, y] \subseteq K_\omega \cup K_\tau; \tag{3.47}
\]
otherwise we set \( \tau := \emptyset (= \omega) \) and \( K_\emptyset := K \).

Using Lemma 3.1 and (3.47), we have
\[
\mu(K_{\omega j}) \leq \mu([x, y]) \leq \mu(K_\omega) + \mu(K_\tau) \leq (C' + 1) \mu(K_\omega) \leq C \mu(K_{\omega j}). \tag{3.48}
\]
Thus,
\[
\mu([x, y]) \asymp \mu(K_{\omega j}). \tag{3.49}
\]
Also, by (3.47), \( m \cdot r_{\omega j} \leq |x - y| \leq m \cdot (r_\omega + r_\tau) \), which yields
\[
|x - y| \asymp r_{\omega j}. \tag{3.50}
\]

From (3.47) and Proposition 3.6, we also get
\[
d_* (K_{\omega j}) \leq d_* (x, y) \leq d_* (K_\omega) + d_* (K_\tau). \tag{3.51}
\]

Hence, using (3.35) and Lemma 3.1,
\[
C^{-1} \mu(K_{\omega j})^\alpha r_{\omega j}^\alpha \leq d_* (x, y) \leq C (\mu(K_{\omega j})^\alpha r_{\omega}^\alpha + \mu(K_{\tau})^\alpha r_{\tau}^\alpha)
\]
\[
\leq C' (\mu(K_{\omega j})^\alpha r_{\omega}^\alpha \leq C \mu(K_{\omega j})^\alpha r_{\omega j}^\alpha. \tag{3.52}
\]

Finally, combining (3.52), (3.50) and (3.49), we have
\[
d_* (x, y)^{1/\alpha} \asymp \mu(K_{\omega j}) r_{\omega j} \asymp \mu(K_{\omega j}) |x - y| \asymp \mu([x, y]) |x - y|,
\]
which yields (1.12) with \( \beta = 1/\alpha \) and completes the proof.

\[\square\]

**Lemma 3.9.** Condition (1.13) is satisfied.
Proof. By virtue of condition (1.11), it suffices to show that there exists a constant \( c > 1 \) such that for all \( x, y, z \) with \( 0 \leq x < y < z \leq m \) and \( d_\ast(x, y) = d_\ast(y, z) \), we have

\[
 c^{-1} \mu([y, z]) \leq \mu([x, y]) \leq c \mu([y, z]). \tag{3.53}
\]

Choose two shortest words \( \omega \) and \( \tau \) such that

\[
 K_\omega \subseteq [x, y] \quad \text{and} \quad K_\tau \subseteq [y, z],
\]

and such that \( K_\omega \) and \( K_\tau \) are the closest to \( y \) among such words.

**Claim.** There exists a constant \( L \geq 0 \) such that

\[
 ||\omega| - |\tau|| \leq L. \tag{3.54}
\]

To prove the claim we assume, without loss of generality, that \( |\omega| - |\tau| \geq 0 \), and let \( \omega' \leq \omega \) be such that \( |\omega'| = |\tau| \). Then

\[
 r_{\omega'} = r_\tau. \tag{3.55}
\]

The number of words lying between \( K_\omega \) and \( K_\tau \) with length \( |\omega'| \) is less than some constant \( c_1 > 0 \). Thus by Lemma 3.1,

\[
 c_2^{-1} \mu(K_\rho) \leq \mu(K_\omega) \leq c_2 \mu(K_\tau). \tag{3.56}
\]

From (3.52), we have

\[
 d_\ast(x, y) \asymp \mu(K_\omega)^{3/2} \tag{3.57}
\]

and

\[
 d_\ast(y, z) \asymp \mu(K_\tau)^{3/2}. \tag{3.58}
\]

Using this and the equality \( d_\ast(x, y) = d_\ast(y, z) \), we see that there exists some constant \( c_3 > 0 \) such that

\[
 \mu(K_\rho)r_\rho \leq c_3 \mu(K_\omega)r_\omega \leq c_3 \mu(K_\omega)r_\omega. \tag{3.59}
\]

Combining (3.56), (3.55) and (3.57), we have

\[
 r_{\omega'} = r_\tau \leq \frac{c_3 \mu(K_\omega)r_\omega}{\mu(K_\tau)} \leq c_3 c_2 r_\omega = c_3 c_2 r_\omega' \left( \frac{1}{m} \right)^{|\omega| - |\omega'|},
\]

which implies that \( 1 \leq c_3 c_2/m^{(|\omega| - |\omega'|)} \), proving the claim.

Now by (3.54), Lemma 3.1 and (3.56),

\[
 \mu(K_\omega) \asymp \mu(K_\omega) \asymp \mu(K_\tau).
\]

It now follows from (3.49) that \( \mu([x, y]) \asymp \mu([y, z]) \).

\( \square \)

**Lemma 3.10.** Condition (1.14) holds with \( \beta = 1/\alpha \).

**Proof.** For any \( x \in [0, m] \), any small \( r \in (0, 1) \) and any integer \( \ell \geq 2 \), choose \( z \) and \( y_\ell \) in \( [0, m] \) such that either \( x < y_\ell < z \) or \( z < y_\ell < x \), and such that

\[
 d_\ast(x, z) = r = \ell d_\ast(x, y_\ell). \tag{3.58}
\]

We need to show that

\[
 \lim_{\ell \to \infty} \frac{|x - y_\ell|}{|x - z|} = 0, \tag{3.59}
\]

where the convergence is uniform with respect to \( x \), \( r \) and all possible choices of \( y_\ell, z \). We may assume that \( x < y_\ell < z \); the other case is similar. Choose a shortest word \( \omega \) such that \( K_\omega \subseteq [x, y_\ell] \).

Consider a chain of \( k + 1 \) neighboring words starting from \( \omega \) and with length \( |\omega| \), where \( k \) will be determined later. By Lemma 3.7 and Lemma 3.1, \( d_\ast(K_\omega) \asymp d_\ast(K_\tau) \).
if $\tau$ is a neighbor of $\omega$. Hence there exists a constant $c_0 > 1$ such that sum of $d_*$ diameters of the corresponding cells is no more than
\[ d_*(K_\omega)(1 + c_0 + \cdots + c_k^0). \] (3.60)

Let $k$ be the largest integer such that
\[ 1 + c_0 + \cdots + c_k^0 \leq \ell, \] (3.61)
and thus
\[ k \approx \log \ell. \] (3.62)

Using Proposition 3.6, (3.58), (3.60), and the inclusion $K_\omega \subseteq [x,y_\ell]$, we see that there exists a chain of $k+1$ neighboring cells starting from $K_\omega$ that are contained in $[x,z]$. Let $N := \lceil \log_m k \rceil - 1$, where $\lceil \cdot \rceil$ denotes the greatest integer function. Since the Euclidean diameter of $K_\omega$ is $m(\frac{1}{m})^{\lceil \omega \rceil}$ and since $k \geq mN + 1$, we see that
\[ |x - z| \geq k \cdot m \left( \frac{1}{m} \right)^{|\omega|} \geq m^2 \left( \frac{1}{m} \right)^{|\omega| - N} > 2m \left( \frac{1}{m} \right)^{|\omega| - N}. \] (3.60)

This implies that there exists a word $\tau$ such that $|\tau| = |\omega| - N$ and $K_\tau \subseteq [x,z]$.

Let $\omega' \leq \omega$ be such that $|\omega'| = |\tau|$. Then
\[ |x - z| \geq m \cdot r_\tau = m \cdot r_{\omega'} = m^{N+1} r_\omega. \] (3.63)

Note that by (3.50),
\[ |x - y_\ell| \approx r_\omega. \] (3.64)

Combining (3.64), (3.62) and (3.63), we have
\[ \frac{|x - y_\ell|}{|x - z|} \leq \frac{c' r_\omega}{m^{N+1} r_\omega} = \frac{c'}{m^{\lceil \log_m k \rceil}} \leq \frac{mc'}{k} \leq \frac{c}{\log \ell} \to 0 \quad \text{as } \ell \to \infty. \]

This proves (3.59). Finally, it follows from (1.11) and (3.53) that there exists some universal constant $C > 0$ such that for all $x,y \in K$ with $x < y$,
\[ C^{-1} V(x, d_*(x,y)) \leq \mu([x,y]) \leq V(x, d_*(x,y)). \] (3.65)

Consequently, by using (1.12) we have
\[ |x - y| \approx \frac{d_*(x,y)^\beta}{V(x,d_*(x,y))}. \] (3.66)

Now, letting $\eta = \frac{1}{\ell}$ and using (3.66), we have
\[ \sup_{x \in K, 0 < r < 1} \frac{\eta^{1/\alpha} V(x,r)}{V(x,\eta r)} = \sup_{x \in K, 0 < r < 1} \frac{(\eta r)^{\beta}/V(x,\eta r)}{r^{\beta}/V(x,r)} \approx \sup_{x \in K, 0 < r < 1} \frac{|x - y_\ell|}{|x - z|}, \]
which tends to 0 as $\eta \to 0^+$ by (3.59).

Proof of Theorem 1.2. From above, conditions (1.10)–(1.14) are all satisfied with $\beta = 1/\alpha$ and thus Theorem 1.2 follows.
Acknowledgments. QG and SN thank the Department of Mathematical Sciences of Tsinghua University, Hunan Normal University, The Chinese University of Hong Kong, and Harvard University, where part of this research was carried out. SN is grateful to Professor Shing-Tung Yau for the opportunity to visit CMSA and thanks the center for its hospitality and support. The authors are indebted to Ka-Sing Lau for his constant support. The authors thank the anonymous referee for checking an earlier version of the manuscript carefully and providing many valuable suggestions and comments.

REFERENCES

[1] M. T. Barlow and R. F. Bass, Transition densities for Brownian motion on the Sierpiński carpet, *Probab. Theory Related Fields*, 91 (1992), 307–330.

[2] M. T. Barlow and R. F. Bass, Brownian motion and harmonic analysis on Sierpiński carpets, *Canad. J. Math.*, 51 (1999), 673–744.

[3] M. T. Barlow, T. Coulhon and T. Kumagai, Characterization of sub-Gaussian heat kernel estimates on strongly recurrent graphs, *Comm. Pure Appl. Math.*, 58 (2005), 1642–1677.

[4] M. T. Barlow and E. A. Perkins, Brownian motion on the Sierpiński gasket, *Probab. Theory Related Fields*, 79 (1988), 543–624.

[5] E. J. Bird, S.-M. Ngai and A. Teplyaev, Fractal Laplacians on the unit interval, *Ann. Sci. Math. Québec*, 27 (2003), 135–168.

[6] K. Dalrymple, R. S. Strichartz and J. P. Vinson, Fractal differential equations on the Sierpinski gasket, *J. Fourier Anal. Appl.*, 5 (1999), 203–284.

[7] M. Elekes, T. Keleti and A. Máthé, Self-similar and self-affine sets: measure of the intersection of two copies, *Ergodic Theory Dynam. Systems*, 30 (2010), 399–440.

[8] P. J. Fitzsimmons, B. M. Hambly and T. Kumagai, Transition density estimates for Brownian motion on affine nested fractals, *Comm. Math. Phys.*, 165 (1994), 595–620.

[9] M. Fukushima, Y. Oshima and M. Takeda, *Dirichlet forms and symmetric Markov processes*, second revised and extended edition, Walter de Gruyter, Studies in Mathematics, 19, 2011.

[10] A. Grigor’yan and J. Hu, Heat kernels and Green functions on metric measure spaces, *Canad. J. Math.*, 66 (2014), 641–699.

[11] A. Grigor’yan, J. Hu and K.-S. Lau, Generalized capacity, Harnack inequality and heat kernels of Dirichlet forms on metric measure spaces, *J. Math. Soc. Japan*, 67 (2015), 1485–1549.

[12] A. Grigor’yan, J. Hu and K.-S. Lau, Heat kernels on metric-measure spaces and an application to semilinear elliptic equations, *Trans. Amer. Math. Soc.*, 355 (2003), 2065–2095.

[13] A. Grigor’yan and A. Telcs, Two-sided estimates of heat kernels on metric measure spaces, *Ann. Probab.*, 40 (2012), 1212–1284.

[14] B.M. Hambly and T. Kumagai, Transition density estimates for diffusion processes on post critically finite self-similar fractals, *Proc. London Math. Soc. (3)*, 79 (1999), 431–458.

[15] J. Hu, An analytical approach to heat kernel estimates on strongly recurrent metric spaces, *Proc. Edin. Math. Soc.*, 51 (2008), 171–199.

[16] J. Hu, K.-S. Lau and S.-M. Ngai, Laplace operators related to self-similar measures on $\mathbb{R}^d$, *J. Funct. Anal.*, 239 (2006), 542–565.

[17] J. Kigami, Volume doubling measures and heat kernel estimates on self-similar sets, *Mem. Amer. Math. Soc.*, 199 (2009), no. 932.

[18] J. Kigami, Resistance forms, quasisymmetric maps and heat kernel estimates, *Mem. Amer. Math. Soc.*, 216 (2012), no. 1015.

[19] J. Kigami, Time changes of the Brownian motion: Poincaré inequality, heat kernel estimate and protodistance, *Mem. Amer. Math. Soc.*, 209 (2019), no. 1250.

[20] T. Kumagai, Estimation of transition densities for Brownian motion on nested fractals, *Probab. Theory Related Fields*, 96 (1993), 205–224.

[21] T. Kumagai, Heat kernel estimates and parabolic Harnack inequalities on graphs and resistance forms, *Publ. Res. Inst. Math. Sci.*, 40 (2004), 793–818.

[22] T. Kumagai and K. T. Sturm, Construction of diffusion processes on fractals, $d$-sets, and general metric measure spaces, *J. Math. Kyoto Univ.*, 45 (2005), 307–327.

[23] K.-S. Lau and S.-M. Ngai, Second-order self-similar identities and multifractal decompositions, *Indiana Univ. Math. J.*, 49 (2000), 925–972.

[24] Y.-T. Lee, Infinite propagation speed for wave solutions on some p.c.f. fractals, preprint.
[25] S.-M. Ngai, Spectral asymptotics of Laplacians associated with one-dimensional iterated function systems with overlaps, *Canad. J. Math.*, 63 (2011), 648–688.

[26] S.-M. Ngai, W. Tang and Y. Xie, Wave propagation speed on fractals, preprint.

[27] R. S. Strichartz, Analysis on fractals, *Notices Amer. Math. Soc.*, 46 (1999), 1199–1208.

[28] R. S. Strichartz, A. Taylor and T. Zhang, Densities of self-similar measures on the line, *Experiment. Math.*, 4 (1995), 101–128.

Received March 2017; revised May 2019.

E-mail address: 001gqs@163.com
E-mail address: hujiaxin@mail.tsinghua.edu.cn
E-mail address: smngai@georgiasouthern.edu