The spin-4/3 fractional superstring is characterized by a chiral algebra involving a spin-4/3 current on the world-sheet in addition to the energy-momentum tensor. These currents generate physical state conditions on the fractional superstring Fock space. Scattering amplitudes of these physical states are described which satisfy both spurious state decoupling and cyclic symmetry (duality). Examples of such amplitudes are calculated using an explicit $c = 5$ realization of the spin-4/3 current algebra. This representation has three flat coordinate boson fields and a global SO(2,1) Lorentz symmetry, permitting a particle interpretation of the amplitudes.

String theories are characterized by the local symmetries of two-dimensional field theories on the string world-sheet. The bosonic string is invariant under diffeomorphisms and local Weyl rescalings on the world-sheet; whereas the superstring is characterized by a locally supersymmetric version of these symmetries. It is natural to ask whether other symmetries on the world-sheet can give rise to consistent string theories. Recently, a proposal for a large class of new string theories, called fractional superstrings, was advanced. Since fractional-spin fields exist in two-dimensional theories, one can imagine new local symmetries on the world-sheet involving fractional-spin currents (replacing the spin-3/2 supercurrent of the superstring). Evidence has been presented that fractional superstrings with spin 4/3, 6/5, and 10/9 currents on the world-sheet have potentially interesting phenomenologies in 6, 4 and 3 space-time dimensions, respectively.

This talk will focus exclusively on the spin-4/3 fractional superstring and its tree scattering amplitudes. In the course of the discussion I will simply state many of the properties of the representation theory of the spin-4/3 current algebra on the string world-sheet; for proofs and more details, see Ref. 5. Classically, the spin-4/3 algebra is the constraint algebra arising from gauge-fixing the local world-sheet symmetry. Quantum mechanically, the constraints generate physical state conditions which pick out the propagating degrees of freedom from the larger string
state space. Although the classical world-sheet gauge symmetry giving rise to a spin-4/3 constraint algebra is not understood at present, we can make progress by taking the constraint algebra itself as a starting point, and checking the consistency of the resulting string theory by constructing unitary scattering amplitudes for the physical states.

1. The spin-4/3 fractional superconformal chiral algebra

Before describing the scattering amplitudes, we must first define in more detail the spin-4/3 fractional superstring. The motivating idea behind the construction of this string is to replace the spin-1/2 world-sheet fermions $\psi^\mu$ appearing in the ten-dimensional superstring with spin-1/3 world-sheet fields $\epsilon^\mu$. The world-sheet supersymmetry of the superstring is then replaced with a world-sheet “fractional supersymmetry” which relates coordinate boson fields $X^\mu$ not to fermions but rather to the field $\epsilon^\mu$. The fractional supersymmetry is generated by a generalization of the supercurrent, a new chiral current $G(z)$ of the form $G(z) \sim \epsilon^\mu \partial X^\mu + \cdots$, whose conformal dimension is 4/3.

The fractional current, $G(z)$, and the energy-momentum tensor, $T(z)$, together generate the fractional superconformal (FSC) chiral algebra, encoded in the singular terms of the operator product expansions:

$$
T(z)T(w) = \frac{1}{(z-w)^2} \left\{ \frac{c}{2} + 2(z-w)^2 T(w) + (z-w)^3 \partial T(w) \right\},
$$

$$
T(z)G(w) = \frac{1}{(z-w)^2} \left\{ \frac{4}{3} G(w) + (z-w) \partial G(w) \right\},
$$

$$
G(z)G(w) = \frac{1}{(z-w)^{8/3}} \left\{ \frac{3c}{4} + 2(z-w)^2 T(w) \right\} + \frac{\lambda}{(z-w)^{4/3}} \left\{ G(w) + \frac{1}{2} (z-w) \partial G(w) \right\}.
$$

The first OPE just states that $T(z)$ obeys the Virasoro algebra, while the second implies that $G(z)$ is a dimension-4/3 chiral primary field. We will take this algebra as the constraint algebra generating the physical state conditions for the spin-4/3 fractional string. Although one can imagine other chiral algebras involving dimension-4/3 currents, we have chosen the above FSC algebra to define the spin-4/3 fractional superstring because this algebra is known to have a sensible representation theory.\(^{6,7,8}\) In particular, it is known that associativity fixes $\lambda$ as a function of the central charge $c$:

$$
\lambda^2 = \frac{8-c}{6}.
$$

An important feature of the FSC algebra is the appearance of cuts in the $GG$ OPE. Since there are two different cuts on the right hand side, upon continuation of a correlation function involving $G(z)G(w)$ along a contour interchanging $z$ and $w$ it is not consistent for the correlator to pick up a simple phase. This situation is described by saying that the current $G$ satisfies “nonabelian braid relations.”
For such currents, their OPEs alone do not define the chiral algebra; they must be supplemented by the current braid relations. These braid relations can be described in a simple way. Since only two fractional cuts appear in the $GG$ OPE, one can split the $G$ current into two pieces, $G(z) = G^+(z) + G^-(z)$, which satisfy abelian braid relations:

$$
G_{\pm}(z)G_{\pm}(w) = \frac{\lambda}{(z-w)^{4/3}} \left\{ G^{\mp}(w) + \frac{1}{2} (z-w) \partial G^{\mp}(w) \right\},
$$

$$
G_{\pm}(z)G^{\mp}(w) = \frac{1}{(z-w)^{8/3}} \left\{ \frac{3c}{8} + (z-w)^2 T(w) \right\}. \quad (3)
$$

Under interchange of $z$ and $w$ (along a prescribed path, say a counterclockwise switch) the only consistent phase that $G^+$ or $G^-$ can pick up with itself is $e^{4i\pi/3}$. The phase that develops upon interchange of $G^+$ with $G^-$ can be taken to be $e^{2i\pi/3}$. This serves to define the braid relations satisfied by the full FSC current $G(z)$. It is worth emphasizing that the split algebra currents $G^\pm$ do not generate the physical state conditions for the spin-4/3 superstring; only their combination $G = G^+ + G^-$ does. The split algebra of Eq. (3) is only introduced as a technical tool to describe the braid relations of the fractional supercurrent $G(z)$.

The specification of the FSC algebra, Eq. (1), along with the braid relations of $G$ defines the spin-4/3 fractional superstring. The rest of this talk will outline the first step towards showing that this string in fact exists, by constructing tree scattering amplitudes which are consistent with the physical state conditions following from the FSC algebra. We will see by explicit calculation in an example that these scattering amplitudes have a sensible space-time interpretation.

2. Physical state conditions and FSC highest-weight modules

The first step in constructing tree scattering amplitudes is to derive the algebra of the constraints from the FSC algebra. The physical state conditions are imposed by the requirement that the positive (annihilation) modes of the FSC currents vanish when acting on physical states. Deriving the constraint algebra is thus equivalent to deriving from the OPEs and braid relations the commutation relations satisfied by the modes of the currents. This is a technically complicated step the results of which I will simply state; for a fuller derivation see Ref. 5.

The split algebra, Eq. (3), was studied by Zamolodchikov and Fateev, who noted a $\mathbb{Z}_3$ symmetry which organizes its representation theory. In particular, the currents $G^+$ and $G^-$ can be assigned $\mathbb{Z}_3$ charges $q = 1$ and $-1$, respectively, while the energy-momentum tensor $T$ (as well as the identity) have charge $q = 0$. It is natural to assume that, since the split algebra is supposed to be an organizing symmetry of our theory, all the fields in a representation have definite $\mathbb{Z}_3$ charges.\footnote{One can also introduce fields which are double-valued with respect to the $G^\pm$ currents; this corresponds to enlarging the symmetry of the split algebra from $\mathbb{Z}_3$ to $S_3$, its full automorphism group. The fields in this double-valued sector play a role in the spin-4/3 string analogous to that played by the Ramond sector fields in the superstring.}
Since the split algebra is abelianly braided, the arguments of Ref. 6 can be directly applied to derive the mode expansions and generalized commutation relations following from Eq. (3). The mode expansions of $T, G^+ \text{ and } G^-$ are defined by

$$T(z)\chi_q(0) = \sum_n z^{-n-2} L_n \chi_q(0),$$

$$G^\pm (z)\chi_q(0) = \sum_n z^{n/3} G^\pm_{-1-n-(1/3)q} \chi_q(0),$$  \hspace{1cm} (4)

where $\chi_q$ is an arbitrary state with $\mathbb{Z}_3$ charge $q$. This means, in particular, that the $L_n$’s have integral moding, while the $G_r$’s are moded by one-third integers. The mode expansion for the full fractional superconformal current can be built from the split algebra pieces: $G_r = G^+_r + G^-_r$. The modes of the $G^\pm$ currents and the energy-momentum tensor $T$ satisfy the commutation relations

$$[L_m, L_n] = (m - n) L_{m+n} + \frac{c}{12} (m^3 - m) \delta_{m+n},$$

$$[L_m, G^\pm_r] = \left( \frac{m}{3} - r \right) G^\pm_{m+r},$$  \hspace{1cm} (5)

and the generalized commutation relations

$$\sum_{\ell=0}^{\infty} c^{(-2/3)}_\ell \left[ G^\pm_{\frac{5}{3} + n - \ell} G^\pm_{\frac{2}{3} + m + \ell} - G^\pm_{\frac{5}{3} + m - \ell} G^\pm_{\frac{2}{3} + n + \ell} \right] = \frac{\lambda}{2} (n - m) G^\pm_{\frac{2}{3} + m + n} ,$$

$$\sum_{\ell=0}^{\infty} c^{(-1/3)}_\ell \left[ G^+_\frac{5}{3} + n - \ell G^-_{\frac{1}{3} + m + \ell} + G^-_{\frac{5}{3} + m - \ell} G^+_\frac{2}{3} + n + \ell \right] =$$

$$L_{n+m} + \frac{3 c}{16} \left( n + 1 + \frac{q}{3} \right) \left( n + \frac{q}{3} \right) \delta_{n+m},$$  \hspace{1cm} (6)

when acting on a state with $\mathbb{Z}_3$ charge $q$. The $c^{(\alpha)}_\ell = (-1)^\ell \binom{\alpha}{\ell}$ are fractional binomial coefficients. Because of the infinite sum on the left hand side, the mode algebra in Eq. (6) is not a graded Lie algebra, but a new algebraic structure on the string world-sheet.

The physical state conditions require that a physical state $|\phi\rangle$ be annihilated by the positive modes of $T$ and $G$, and be eigenstates of their zero modes:

$$(L_n - v \delta_{n,0})|\phi\rangle = 0 , \hspace{1cm} 0 \leq n \in \mathbb{Z},$$

$$(G_r - \beta \delta_{r,0})|\phi\rangle = 0 , \hspace{1cm} 0 \leq r \in \mathbb{Z}/3.$$ \hspace{1cm} (7)

Here $v$ and $\beta$ are “intercepts”, normal ordering constants in the definitions of $T$ and $G$. All the positively-moded constraints can be generated from those of the set $\{L_1, L_2, G_{1/3}, G_{2/3}, G_1, G_{4/3}\}$. A state $|s\rangle$ obeying the zero-mode conditions in Eq. (7) is called a spurious state if it is orthogonal to all physical states. Such a state can be written as

$$|s\rangle = \sum_{n>0} L_{-n}|\chi_n\rangle + \sum_{r>0} G_{-r}|\psi_r\rangle ,$$  \hspace{1cm} (8)

in terms of some other states $|\chi_n\rangle$ and $|\psi_r\rangle$. All states not satisfying the physical state conditions must have a spurious component. A physical state can itself be
spurious, in which case it is a null state, and should decouple from all scattering amplitudes. Thus, the decoupling of all spurious states in the scattering amplitudes of physical states is a test of whether Eq. (3) is a sensible constraint algebra.

The physical state conditions imply that the physical state $|\phi\rangle$ is a highest weight state of the FSC algebra with highest weight $v$. The properties of the highest weight modules can be derived from the commutation relations Eqs. (5–6), and it turns out that they fall into two classes. The highest weight state of an S (singlet) module has $Z_3$ charge $q = 0$, the allowed modings of the fractional current acting on it are $G_n$ and its descendants have dimensions $v + n$ and $v + \frac{1}{2} + n$ where $n$ is an integer. The highest weight state of a D (doublet) module is doubly degenerate in the split algebra with $Z_3$ charges $q = \pm 1$, allowed current modings $G_n$ and $G_n - (2/3)$, and descendants of dimensions $v + n$ and $v + \frac{2}{3} + n$. In the D sector, associativity fixes the $G_0$ intercept $\beta$ in terms of $v$ and $c$.

We will see later, by way of an example, that physical states in the S sector are space-time bosons, and include a tachyon; those in the D sector are also space-time bosons, but with the lightest state a massless vector particle (or graviton for a closed string). Henceforth we focus on the properties of the D modules, since it is only for the D-sector states that we can construct scattering amplitudes.

The main properties of D modules are summarized by the OPEs of the current $G$ with the highest weight vertex operator $W_D(z)$ of dimension $v$ and its first descendant operator $V_D(z)$ of dimension $v + \frac{2}{3}$:

$$G(z)W_D = \left(\sqrt{v - \frac{c}{24}}\right)\frac{W_D}{z^{1/3}} + \frac{V_D}{z^{2/3}} + \ldots$$
$$G(z)V_D = \left(v + \frac{c}{12} + \frac{1}{12}\sqrt{(8 - c)(24v - c)}\right)\frac{1}{z^2}\left\{W_D + \frac{z}{3v}\partial W_D\right\}$$
$$+ \left(\sqrt{24v - c} - \sqrt{8 - c}\right)\frac{W_D}{z} + \ldots$$

(9)

For ease of writing, we have inserted the D-module vertex operators $W_D$ etc. at the origin of the complex plane and have dropped their arguments. The first OPE fixes the normalization of the $V_D$ vertex, while the second one introduces the new Virasoro (though not FSC) primary $\tilde{W}_D = (L_{-1} - 3\sqrt{6}vG_{-1})W_D$ of dimension $v + 1$. Note that the fractional current is single-valued with respect to $V_D$, but is nonabelianly braided with respect to $W_D$. These OPEs play an important role in the development of consistent tree scattering amplitudes for D-sector physical states.

3. Tree scattering amplitudes and spurious state decoupling

In what follows, we will construct open fractional superstring tree scattering amplitudes. Closed string scattering amplitudes at tree level are easily formed by

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5As mentioned earlier, there is also a third sector consisting of highest weight modules double-valued with respect to the fractional currents, called the R sector. The appropriate moding of the currents in this sector is half-integral, and the algebra obeyed by the modes is different from Eq. (3). The physical states in this sector correspond to space-time fermions.6
combining two open string amplitudes using a level-matching condition for left- 
and right-movers. The construction we use will be closely analogous to that of 
open Ramond-Neveu-Schwarz superstring tree amplitudes in the “old covariant” 
formalism. (For a detailed explanation of this formalism, see, e.g., Chapter 7 of 
Ref. 9.)

The world-sheet in an open string tree scattering process is conformally 
equivalent to a unit disc with vertex operators \( V(x) \) representing the asymptotic 
scattering states inserted at points on the boundary. Since we must be able to inte-
grate these vertex operators over their insertion positions, they must be dimension-
one operators in the two-dimensional world-sheet theory. We can conformally map 
the disk to the complex upper half-plane, fixing the positions of three of the vertex 
insertions at \( \infty, 1, \) and \( 0 \) on the real axis, with the remaining insertions at points 
\( 1 < x_i < \infty \). This picture is suitable for writing the string amplitude as a correlator 
of operators in the open string Fock space with a radial-ordering prescription:

\[
A_N = \int dx_3 \cdots dx_{N-1} (V_N|V_{N-1}(x_{N-1}) \cdots V_2(1)|V_1), \tag{10}
\]

where the “in” and “out” states are the insertions at \( x_1 = 0 \) and \( x_N = \infty \), and the integration is over all \( x_i \) preserving the order \( 1 < x_3 < \cdots < x_{N-1} < \infty \).

From our previous mapping to the disk, it is clear that \( A_N \) should be invariant 
under cyclic permutations of the vertex ordering. This means that after passing the 
\( V_N \) vertex to the right through all the other vertices in Eq. (10), the value of \( A_N \) 
must be unchanged. This condition on the braiding of the vertex operators can be 
satisfied if two such operators commute when zero space-time momentum is flowing 
into either vertex. The representation theory of the spin-4/3 split algebra implies 
that commuting operators can only have \( \mathbb{Z}_3 \) charge \( q = 0 \). Thus, in particular, the 
\( V_D \) (but not the \( W_D \)) operator in the D sector is a potential candidate for the vertex 
insertions in Eq. (10). For this also to be a dimension-one operator implies that the 
D-sector intercept must be \( v = 1/3 \).

A vertex insertion at \( x \) can be rewritten as \( V(x) = x^{L_0} V(1)x^{-L_0} \), and the posi-
tions of the insertions explicitly integrated over to give the amplitude in the form

\[
A_N = \langle V_N|V_{N-1}(1)\tilde{\Delta} \cdots \tilde{\Delta} V_2(1)|V_1 \rangle \tag{11}
\]

where the propagator is \( \tilde{\Delta} = (L_0 - 1)^{-1} \). For scattering of D-sector states, the vertices 
in Eq. (11) correspond to the \( V_D \) descendant states in a FSC highest-weight mod-
ule. We can convert Eq. (11) to a different “picture” involving the highest-weight 
states \( W_D \) using the general properties of the D modules. In particular, by taking 
appropriate fourier components of the D-sector OPE in Eq. (11), it follows that

\[
[ G_r, V_D(1) ] = \left( L_0 + r - \frac{1}{3} \right) W_D(1) - W_D(1) \left( L_0 - \frac{1}{3} \right) \quad \text{for all } r \in \mathbb{Z}/3. \tag{12}
\]

In deriving this formula we have set \( v = 1/3 \) and used the relation \([L_0, W_D(1)] = (1/3)W_D(1) + \partial W_D(1)\) which is true for any dimension-1/3 Virasoro primary \( W_D \). Note 
that when \( v = 1/3 \) the dimension \( 1 + v \) descendant \( \tilde{W}_D \) (which has no analog in the
superstring) decouples from the OPE in Eq. (3). It is this unexpected decoupling at precisely the physical value of the intercept which allows us to derive Eq. (12), and in turn construct sensible tree scattering amplitudes.

We can now replace the “in” state $|V_D\rangle$ with a physical state using $|V_D\rangle = G^{-2/3}_r |W_D\rangle$ which follows from the first OPE in Eq. (3). The $G^{-2/3}_r$ mode can be commuted to the left using Eq. (12) as well as the relation

$$G_r(L_0 - a - r)^{-1} = (L_0 - a)^{-1} G_r, \quad (13)$$

following from Eq. (3). Acting on the “out” state, $\langle V_D | G^{-2/3}_r = \langle W_D |$, which is a consequence of the second OPE in Eq. (3) with $v = 1/3$. The extra insertions coming from the right-hand side of Eq. (12) vanish by a “cancelled propagator” argument, since setting $r = -2/3$ in Eq. (12) gives factors of $L_0 - 1$ and $L_0 - 1/3$ which cancel the propagators to the left and right, respectively. Thus, the final form we find for the amplitude is

$$A_N = \langle W_N | V_{N-1}(1) \Delta \cdots \Delta V_2(1) | W_1 \rangle, \quad (14)$$

where the propagator in this picture is $\Delta = (L_0 - 1/3)^{-1}$.

Now we can investigate the crucial issue of spurious state decoupling in our amplitudes. If we start with physical states defined as highest-weight vectors of FSC modules, will they scatter only to other physical states? For this to be true, only physical states must contribute to residues of poles in amplitudes when an internal propagator goes on-shell. Suppose we fix the external momenta such that some state $|s\rangle$ in the string Fock space at momentum $\kappa = k_{M+1} + \cdots + k_N$ is on-shell: $(L_0 - 1/3)|s\rangle = 0$. If we factorize the amplitude in Eq. (14) by inserting a sum over a complete set of states of momentum $\kappa$ at the propagator between $V_{M+1}$ and $V_M$, then the $|s\rangle\langle s|$ term in the sum will contribute a pole in momentum space. The requirement of spurious state decoupling is that if $|s\rangle$ is spurious, its contribution to the residue of the pole should vanish:

$$\langle s | V_M(1) \Delta \cdots \Delta V_2(1) | W_1 \rangle = 0. \quad (15)$$

To prove this, consider one term, $\langle \psi | G_r$ with $r > 0$, in the presentation of $|s\rangle$ as a sum of descendant states, Eq. (8). (The $L_n$ descendant pieces can be shown to decouple by a similar argument.) The $G_r$ mode can be commuted to the right in Eq. (13) using Eqs. (12–13). The insertions coming from the right-hand side of Eq. (12) again vanish by a cancelled propagator argument. Finally, the $G_r$ mode acting on the “in” state $|W_1\rangle$ vanishes by the physical state conditions Eq. (7), thus proving spurious state decoupling.

Our prescription for constructing dual N-point tree amplitudes of D-sector states satisfying spurious state decoupling can be extended to include one or two S-sector states by simply replacing the “in” and “out” $W_D$ states in Eq. (14) with S-module physical states $W_S$. The argument for spurious state decoupling then goes

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*Tree amplitudes with cancelled propagators are holomorphic in the Mandelstam invariant of the cancelled propagator channel, and thus vanish if the amplitudes have Regge asymptotic behavior. We will see in the next section that they do have this soft high energy behavior.*
through unchanged. However, it turns out that there is no appropriate dimension-1 commuting vertex in the S sector to play the role of the $V_D$ vertices. Thus, we cannot prove cyclic symmetry of the amplitudes with two S-sector states, nor can we extend the prescription to include scattering of three or more S-sector states. Presumably, as in the Ramond sector of the superstring, this means that there is a nontrivial contribution to S-sector scattering amplitudes coming from the “fractional ghost” fields on the world-sheet.

In section 6 we will argue that a consistent intercept in the S sector is $v = 1/3$, the same as that of the D sector. Note, however, that upon factorizing the D-sector scattering amplitude in Eq. (14) on any propagator, we can never obtain an S-sector intermediate state. The reason is simply that the $W_D$ “in” state has $Z_3$ charge $q = \pm 1$ in the split algebra, and the $V_D$ vertices have charge $q = 0$. Thus, by conservation of $Z_3$ charge, only $q = \mp 1$ intermediate states can contribute, whereas the S-sector physical states $W_S$ have $q = 0$. This selection rule means that it is consistent at tree level to drop the S sector altogether, a desirable feature since it will turn out that the S sector contains tachyons.

4. A $c=5$ free-field representation of the FSC algebra

The scattering amplitude construction we have presented so far has depended only on general properties of the highest-weight modules of the FSC algebra. In this section we will flesh out this construction with an explicit conformal field theory which forms a representation of the FSC algebra at central charge $c = 5$. In the next section we will argue that the critical central charge of the FSC algebra is $c = 10$, so the representation presented below is sub-critical. As is the case with sub-critical representations of the bosonic and superstring, tree amplitudes are perfectly well-behaved. The restriction to the critical central charge is expected only to appear once loop amplitudes are included.

The $c = 5$ representation is constructed from five free (massless) scalar fields on the world-sheet. Three of them, $X^\mu(z)$, $\mu = 0, 1, 2$, are interpreted as the string coordinate fields, and obey the standard OPE

$$X^\mu(z)X^\nu(w) = -\eta^{\mu\nu}\ln(z - w) ,$$

where $\eta^{\mu\nu}$ is the three-dimensional Minkowski metric with signature $(-+++)$. This $X^\mu$ CFT has a global $SO(2,1)$ Lorentz symmetry. The remaining two fields, $\varphi^i(z)$, $i = 1, 2$, are compactified on a triangular lattice—the $su(3)$ root lattice. In a basis in which the $\varphi^i$ boundary conditions are diagonalized, $\varphi^i = \varphi^i + 2\pi$, their OPEs read

$$\varphi^i(z)\varphi^j(w) = -g^{ij}\ln(z - w) , \quad g^{ij} = \frac{1}{3} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} .$$

†The considerations of the last two paragraphs also apply to the R sector: spurious state decoupling works for amplitudes with just two R-sector vertices, though cyclic symmetry is not manifest; and D-sector scattering does not produce R-sector states, though in this case this is interpreted as a space-time spin-statistics selection rule, not as a means of decoupling the R sector.
The vertex operator $V_m = \exp\{im\varphi^i\}$ has dimension $\Delta_m = \frac{1}{2}m_3g^2m_j$ for integer $m_i$. A triplet of dimension-one fields, $U^\mu = \{[V_{(2,1)}+V_{(-1,-1)}], [V_{(-1,1)}+V_{(1,-1)}], [V_{(1,2)}+V_{(-1,-2)}]\}$, generates an SO(2,1) Kac-Moody algebra, organizing all the fields in the $\varphi^i$ CFT in SO(2,1) representations.

Some important fields of low conformal dimension $\Delta$ are:

- $\Delta = 1/3 : \quad \epsilon_\mu = \{V_{(1,0)}, V_{(0,1)}, V_{(-1,1)}\}$, $\epsilon_\mu^\dagger = \{V_{(-1,0)}, V_{(0,-1)}, V_{(1,1)}\}$
- $\Delta = 1 : \quad W_{\mu\nu} = \left\{\frac{1}{2} [V_{(2,1)} - V_{(-2,-1)}], \frac{1}{2} [V_{(-1,1)} - V_{(1,-1)}], \frac{1}{2} [V_{(-1,-2)} - V_{(1,2)}], i\partial \varphi^3\right\}$
- $\Delta = 4/3 : \quad s = \frac{1}{2} [V_{(-2,0)} + V_{(0,-2)} + V_{(2,2)}]$, $s^\dagger = \frac{1}{2} [V_{(2,0)} + V_{(0,2)} + V_{(-2,-2)}]$. (18)

The dimension-1/3 vector fields, $\epsilon_\mu$ and $\epsilon_\mu^\dagger$, are the analogs of the dimension-1/2 $\psi^\mu$ fields in the flat-space superstring representation. The scalars, $s$ and $s^\dagger$, and the spin-2 (symmetric, traceless) $W_{\mu\nu}$ have no superstring analogs.

A current obeying the FSC algebra Eq. (19) is given by

$$G = \frac{1}{\sqrt{2}} \left[ (\epsilon_\mu - \epsilon_\mu^\dagger)\partial X^\mu - \frac{3}{2} (s + s^\dagger) \right].$$

It is manifestly an SO(2,1) scalar Virasoro primary dimension-4/3 field. A splitting of the current satisfying the split algebra in Eq. (18) exists with $\mathbb{Z}_3$ charge assignments $q = 1$ for $\epsilon_\mu$, $s$, $q = -1$ for $\epsilon_\mu^\dagger$ and $s^\dagger$, and $q = 0$ for $X^\mu$, $U^\mu$, and $W_{\mu\nu}$.

The simplest S-sector vertex is $W_S = \exp\{ik\cdot X\}$. The physical state conditions, Eq. (18), applied to $W_S$ imply that $k^2 = 2v$. We will show in the next section that a consistent choice for the S-sector intercept is $v = 1/3$, which implies that $W_S$ is a tachyon.

The simplest D-sector vertex is $W_D = (\zeta \cdot \epsilon + \zeta^\dagger \cdot \epsilon^\dagger)\exp\{ik\cdot X\}$. The physical state conditions acting on $W_D$ constrain $\zeta_\mu$ and $\zeta_\mu^\dagger$ to be expressible in terms of a single polarization vector $\xi_\mu$, such that $k^2 = k \cdot \xi = 0$, and

$$W_D = \left[ \xi_\mu (\epsilon_\mu - \epsilon_\mu^\dagger) - \varepsilon^{\mu\nu\rho} \xi_\mu k_\nu (\epsilon_\rho + \epsilon_\rho^\dagger) \right] e^{ik\cdot X}.$$ (20)

Here $\varepsilon^{\mu\nu\rho}$ is the completely antisymmetric tensor in three dimensions. $W_D$ represents a massless vector particle. In a closed string, we could match a left-moving and a right-moving version of $W_D$ to form the usual graviton, dilaton, and antisymmetric tensor fields. The $V_D$ descendant of $W_D$, defined by the first OPE in Eq. (19), is

$$V_D = -\sqrt{2} \left[ \xi_\mu \partial X_\mu - \varepsilon^{\mu\nu\rho} \xi_\mu k_\nu U_\rho - i k^\mu \varepsilon^{\nu\rho\sigma} \xi_\mu k_\rho W_{\mu\nu} \right] e^{ik\cdot X}.$$ (21)

Note that upon making a gauge transformation $\delta \xi^\mu \propto k^\mu$, one finds $\delta V_D \propto \partial(\exp\{ik\cdot X\})$, a spurious state which decouples by the arguments of the last section. The operators $\partial X_\mu$, $U_\mu$, and $W_{\mu\nu}$ appearing in $V_D$ are all single-valued fields on the world-sheet—they have no cuts in their OPEs with any other field—a property shared by all operators with $\mathbb{Z}_3$ charge $q = 0$. Therefore, only simple commutators, as opposed

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1^Appropriate, though standard, cocycles must be added to the $\varphi^i$ CFT to realize this symmetry.

2^Fields in the R sector appear upon orbifolding the $\varphi^i$ CFT by a 180° rotation of its lattice. The twist fields of this $\mathbb{Z}_3$ orbifold transform in the spinor representation of SO(2,1), and the zero modes of the $\epsilon^\mu$ fields acting on these states satisfy a three-dimensional Clifford algebra.
to the generalized commutators appearing in the constraint algebra Eq. (3), are needed to evaluate an N-point amplitude. Consider, for example, the coupling of three massless vector particles, given by $\mathcal{A}_3 = \langle W_D(k_3, \xi_3) | V_D(k_2, \xi_2; 1) | W_D(k_1, \xi_1) \rangle$. One finds

$$\mathcal{A}_3 = i2\sqrt{2} \left[ (k_1 \cdot \xi_3)(\xi_2 \cdot \xi_1) + (k_2 \cdot \xi_1)(\xi_3 \cdot \xi_2) + (k_3 \cdot \xi_2)(\xi_1 \cdot \xi_3) - 3(\xi_1 \cdot k_2)(\xi_2 \cdot k_3)(\xi_3 \cdot k_1) \right].$$

(22)

The first three terms are precisely the expected Yang-Mills coupling; the last term represents a non-linear correction to the Yang-Mills action which is higher-order in the string tension, and therefore is suppressed at energies far below the Planck scale. The non-linear term also appears in the three-vector coupling in the bosonic string, where it has coefficient $+1$ instead of $-3$; in the superstring no such term appears in the three-point coupling (though string correction terms do appear in higher-point functions).

One can calculate higher-point amplitudes in a similar way. While the details are not illuminating, the main features of these amplitudes are easily understood. Because the contribution from the $e^{ik \cdot X}$ pieces essentially factorizes, it is clear that one will obtain gamma-function factors similar to those that appear in the Veneziano amplitude for the bosonic string. The remaining part of the vertices can only contribute factors which are polynomial in the momenta. Thus, the fractional string amplitudes have the soft high-energy Regge behavior characteristic of bosonic and superstring amplitudes.

5. Determination of the critical central charge

Spurious state decoupling by itself does not imply that tree amplitudes are unitary. One must also prove that there are no negative-norm physical states for the specific representation of the constraint algebra under consideration. In the bosonic and superstrings, for representations with one time-like (space-time) dimension, a non-negative physical state space occurs up to a maximum value of the central charge. As one passes through this critical value of the central charge the norm of some physical states change sign, implying that at the critical central charge there are extra null states. We can check for the existence of a critical central charge in a representation-independent way by searching for the occurrence of extra sets of zero-norm physical states.

Since a null physical state is spurious, consider the simplest (D-sector) spurious state $|\psi_1\rangle = G_{-1/3} |\tilde{\psi}\rangle$, where $\tilde{\psi}$ is an arbitrary state. The physical state conditions Eq. (7) are satisfied by $\psi_1$ only when the intercept $v = 1/3$, irrespective of the value of the central charge. This is the value of the D-sector intercept determined above by the requirement of spurious state decoupling; when $\tilde{\psi} = e^{ik \cdot X}$, $\psi_1$ is just the longitudinal component of the massless vector state in Eq. (20).

To determine the critical central charge we must consider the more complicated general D-sector spurious state

$$|\psi_2\rangle = (G_{-4/3} + \alpha G_{-1/3} L_{-1} + \beta G_{-1} G_{-1/3}) |\tilde{\psi}\rangle.$$

(23)
Applying the physical state conditions, one finds a null state for specific values of $\alpha$ and $\beta$ only when $c = 10^{1,10}$. One can check in the $c = 5$ sub-critical representation of the last section that the lowest-mass state of the form in Eq. (23) (taking $\tilde{\psi} = e^{ik \cdot X}$) is in fact a positive-norm state, as required for tree-level unitarity.

One can also search for sets of S-module null physical states. In particular, for the general spurious states of the form $|\psi_S\rangle = G_{-2/3}|\tilde{\psi}\rangle$, the physical state conditions imply a quadratic relation between the central charge and the S-sector intercept $v_S$. For $c = 10$, the solutions are $v_S = 1/3$ or $-1$. The first possibility gives rise to tachyons, whereas the second describes only massive states.

6. Discussion and outlook

The tree-level considerations discussed so far leave us with a certain amount of arbitrariness in constructing spin-4/3 fractional superstrings. In particular, we are free to include or not S-sector states; we can couple left- and right-moving theories at will on the world-sheet in type II and heterotic constructions; and the choice of CFT representation of the spin-4/3 FSC algebra is constrained by tree unitarity only to have central charge less than or equal to its critical value $c = 10$. The inclusion of string loop amplitudes should remove much of this arbitrariness. As is the case with the bosonic and superstrings, one expects that loop amplitudes will only be consistent at the critical central charge, and modular invariance will determine which left- and right-moving sectors, at which values of their intercepts, can be consistently coupled together.

One difficulty in constructing a critical ($c = 10$) representation of the FSC algebra is its non-linearity. The fact that the FSC structure constant $\lambda$ is a non-linear function of the central charge $c$, Eq. (2), means that the FSC algebra as a whole is non-linear: the tensor product of two representations of this algebra is not itself a representation. In particular, the tensor product of two copies of the $c = 5$ representation described above will not make a $c = 10$ representation of the FSC algebra. One can, however, construct higher-$c$ representations from a given representation by turning on a background charge for one of the $X^\mu(z)$ coordinate boson fields, corresponding to a linear dilaton background field in space-time.

Once given a $c = 10$ representation of the FSC algebra, the construction of loop amplitudes may still not be an easy task. One can imagine “sewing” tree amplitudes in the “old covariant” formalism described above to form one-loop amplitudes by a suitable generalization of the sewing procedure for the bosonic string.\textsuperscript{11} Such an amplitude would not only have to be unitary, but also modular invariant. An indication that this may not be too much to expect may be the existence of the modular-invariant fractional string partition functions proposed in Ref. 2. At higher loops it seems likely that a clearer understanding of the “fractional moduli” describing the sewing of tree amplitudes will be necessary. This is essentially the question of what is the local world-sheet symmetry underlying the FSC constraint algebra. Though the form of the FSC algebra provides a rigid guide to such a symmetry, its identification remains an open question.
Finally, what are the space-time features of critical fractional superstrings? This, also, is an open question, since its answer depends largely on the resolution of the world-sheet issues outlined above. So far, only hints of possibly new space-time structures have been gleaned from the fractional superstring partition function.\textsuperscript{3,4,12,13,14}

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