An analysis of oscillatory hydromagnetic couette flow through a porous medium in a rotating system

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Abstract: Analysis of an exact solution of oscillatory Ekman boundary layer flow through a porous medium bounded by two horizontal flat plates is found. One of the plates is at rest and the other one is oscillating in its own plane. The whole of the system rotates about an axis normal to the plates. The effects of Coriolis force and the permeability of the porous medium on the flow field are studied. It is seen that even in the special case of resonance (w = 2Ω) the solution obtained by Mazumder is incorrect as contended by Ganapathy. It is found that the amplitude of the resultant velocity |A0| for the steady part increases with either an increase in the permeability parameter k0 (or) an increase in the rotation parameter R. But the above profiles have a reverse trend when there is an increase in the Hartmann number.

1. Introduction:
The flow due to rotation is important in many practical and engineering fields such as geophysics, meteorology, cosmical fluid dynamics, gaseous and nuclear reactors, petroleum engineering, in estimating the flight path of rotating wheels and spin-stabilized missiles. In recent years the study of Couette flow through a porous medium in a rotating system attracts the attention among Scholars due to its wide range of applications in these areas.

Das et al [1] discussed about an unsteady MHD Couette flow in a rotating system. Prasad et al [2] studied an unsteady hydro magnetic couette flow through a porous medium in a rotating system. Maitree Jana et al [3] analyzed an unsteady Couette Flow through a Porous Medium in a Rotating System. Singh et al [4] reported exact solution of MHD mixed convection periodic flow in a rotating vertical channel with heat radiation. Gupta et al [5] investigated unsteady MHD couette flow and heat transfer in a rotating horizontal channel with injection/suction. Krishnan Dev Singh [6] discussed about the rotation of oscillatory MHD poiseuille flow. Ramesh Babu et al [7] analyzed about hydro magnetic oscillatory flow of dusty flow in a rotating porous channel through a porous medium. Das et al [8] studied about transient hydro magnetic reactive couette flow and heat transfer in a rotating frame of reference. Vidhya et al [9] discussed about laminar convection through porous medium between two vertical parallel plates with heat source. Govindarajan et al [10] analyzed Chemical reaction effects on unsteady magneto hydrodynamic free convective flow in a rotating porous medium with mass transfer. Mazumder et al [11] discussed about an exact solution of oscillatory couette flow in a rotating system. Ganapathy et al [12] studied about motion of oscillatory couette flow in a rotating system. Singh et al [13] discussed about periodic solution of an oscillatory couette flow through porous medium in a rotating system.
The objective of this present paper is to study the effects of transversely applied uniform magnetic field on the oscillatory flow with permeability between two parallel flat plates when the whole system rotates about an axis perpendicular to the planes of the plates.

2. Mathematical Analysis:
Consider an unsteady flow of a viscous incompressible fluid through a highly porous medium bounded by two horizontal parallel flat plates at a distance \( d \) apart. The lower plate is considered at rest and the upper one is oscillating in its own plane with a velocity \( U(t) \) about a non-zero constant mean velocity \( U_0 \). Choose the origin on the lower plate and the \( x \)-axis parallel to the direction of motion of the upper plate. The \( z \)-axis is taken perpendicular to the plates. It is the axis of rotation about which the entire system is rotating with a constant angular velocity \( \Omega \). A transverse magnetic field of uniform strength \( B_0 \) is applied along the axis of rotation. Here the plates are infinite in length. All the physical quantities except the pressure depends on \( z \) and \( t \) only. Let \( u, v, w \) be the velocity components in the \( x, y, z \) directions. The continuity equation

\[
\frac{\partial u}{\partial t} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial z^2} + 2\Omega v - \frac{\sigma B_0^2}{\rho} u - \frac{\mu v}{k^*} \tag{1}
\]

\[
\frac{\partial v}{\partial t} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \frac{\partial^2 v}{\partial z^2} + 2\Omega u - \frac{\sigma B_0^2}{\rho} v - \frac{\nu v}{k^*} \tag{2}
\]

Where \( \nu \) is Kinematic viscosity, \( t \) is the time, \( \rho \) is the density, \( k^* \) is the permeability of the porous medium \( \sigma \) is the electric conductivity of the fluid, \( p^* \) is the modified pressure.

The boundary conditions of the problem are:

\[
u = v = 0 \text{ at } z = 0
\]

\[
u = U(t) = U_0 (1 + \varepsilon \cos \omega t), \quad v = 0 \text{ at } z = d \tag{3}
\]

Where \( \omega \) is the frequency of oscillations and \( \varepsilon \) is a small positive constant.

Eliminating the modified pressure gradient, under the usual boundary layer approximations, equations (1) and (2) are reduced to

\[
\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial z^2} + \frac{\partial U}{\partial t} + 2\Omega v - (u - U) \left( \frac{\nu}{k^*} + \frac{\sigma B_0^2}{\rho} \right) \tag{4}
\]

\[
\frac{\partial v}{\partial t} = \nu \frac{\partial^2 v}{\partial z^2} - 2\Omega (u - U) - \left( \frac{\nu}{k^*} + \frac{\sigma B_0^2}{\rho} \right) v \tag{5}
\]

Equations (4) and (5) can be combined in complex form as

\[
\frac{\partial q}{\partial t} = \nu \frac{\partial^2 q}{\partial z^2} + \frac{\partial U}{\partial t} - 2i\Omega (q - U) - (q - U) \left( \frac{\sigma B_0^2}{\rho} + \frac{\nu}{k^*} \right) \tag{6}
\]

And the corresponding boundary conditions become \( q = 0 \) at \( z = 0 \)

\[
q = U(t) = U_0 \left[ 1 + \frac{\varepsilon}{2} \left( e^{i\omega t} + e^{-i\omega t} \right) \right] \text{ at } z = d \tag{7}
\]

Where \( q = u + iv \)

The solution sought for equation (6) by Mazumder is not correct. The correct form of the solution, as given by Ganapathy is,

\[
q = (\eta, t) = U_0 \left[ q_0(\eta) + \frac{\varepsilon}{2} q_1(\eta) e^{i\omega t} + q_2 e^{-i\omega t} \right] \tag{8}
\]
Where
\[ \eta = \frac{z}{d}, \quad q_0(\eta) = u_0(\eta) + iv_0(\eta) \quad \text{and} \quad \{q_1(\eta)e^{i\omega t} + q_3(\eta)e^{i\omega t}\} = u_1(\eta, t) + iv_1(\eta, t) \] (9)

Substituting (8) into (6) and (7) and comparing the harmonic and non-harmonic terms, we get
\[ q_0^{11} - \lambda q_0 = -\lambda \] (10)
\[ q_1^{11} - q_1(\lambda + i\lambda) = -\lambda(1 - \lambda) \] (11)
\[ q_2^{11} - q_2(\lambda - i\lambda) = -\lambda(1 - \lambda) \] (12)

with boundary conditions
\[ q_0 = q_1 = q_2 = 0 \text{ at } \eta = 0, \quad q_0 = q_1 = q_2 = 1 \text{ at } \eta = 1 \] (13)

where
\[ R = \frac{\Omega d^2}{\nu} \quad \text{is the rotation parameter,} \]
\[ \lambda = \frac{\omega d^2}{\nu} \quad \text{is the frequency parameter,} \]
\[ k_0 = \frac{k_0^*}{d^2} \quad \text{is the permeability parameter,} \]
\[ M^2 = \frac{\sigma B_0^2 d^2}{\mu} \quad \text{is the Hartmann number,} \]

Solving equations (10) to (12) under the boundary conditions (13), we obtain
\[ q_0(\eta) = 1 - \frac{\sinh \lambda(1 - \eta)}{\sinh \lambda}, \] (14)
\[ q_1(\eta) = 1 - \frac{\sinh m(1 - \eta)}{\sinh m}, \] (15)
\[ q_2(\eta) = 1 - \frac{\sinh n(1 - \eta)}{\sinh n}. \] (16)

where \[ \lambda = \sqrt{2iR + M^2 + k_0^2}, \quad m = \sqrt{2iR + M^2 + k_0^2 + i\lambda}, \quad n = \sqrt{2iR + M^2 + k_0^2 - i\lambda} \]

For \( M= 0 \), the solutions for \( q_0(\eta) \), \( q_1(\eta) \) and \( q_2(\eta) \) are in agreement with those obtained by Ganapathy [12] and Singh KD [13].

As \( K \to \infty \), the solutions for \( q_0(\eta) \), \( q_1(\eta) \) and \( q_2(\eta) \) are in agreement with those obtained by Singh KD [13] and Ganapathy [12].

3. Results and Discussion:
The solution (14) corresponds to the steady part, which gives \( u_0 \) as the primary and \( v_0 \) as the secondary velocity components. The amplitude and phase difference due to these primary and secondary velocities for the steady flow are given by,
\[ |A_0| = \left( u_0^2 + v_0^2 \right) \theta_0 = \tan^{-1} \left( \frac{v_0}{u_0} \right) \] (17)

For large values of \( R \), the expressions for \( u_0 \) and \( v_0 \) can be approximated from equation (14) as
\[ u_0(\eta) \approx 1 - \exp(-\lambda, \eta) \cos \lambda, \eta \] (18)
\[ v_0(\eta) \approx \exp(-\lambda, \eta) \cos \lambda, \eta \] (19)

Where
These approximations for $u_0$ and $v_0$ represent the spiral distribution of velocity and show clearly the existence of a thin boundary layer of order $O(\lambda_\infty^{-1})$ in the neighborhood of the plates and is known as Ekman layer which decreases with the increase of the rotation parameter $R$ (or) the Hartmann number $M$ but increases with the increase of permeability parameter $k_0$.

\[
\lambda_\infty = \left[ \left( \frac{(k_0^{-2} + 4R^2 + M^4)^{1/2} + M^2 + k_0^{-1}}{2} \right)^{1/2} \right]
\]

\[
l_1 = \left[ \left( \frac{(k_0^{-2} + 4R^2 + M^4)^{1/2} - (M^2 + k_0^{-1})}{2} \right)^{1/2} \right]
\]

Figure 1 – The amplitude due to $u_0$ and $v_0$ for steady flow

The amplitude or the resultant velocity $|A_0|$ and the phase angle $\theta_0$ for the steady part are shown graphically in figure 1 and 2 for various values of the rotation parameter $R$, Hartmann number $M$, and the permeability parameter $k_0$. It is observed from figure 1 that the amplitude $|A_0|$ increases with the increase of the permeability parameter $k_0$ for all values of the rotation parameter $R$ large or small. A decrease in $|A_0|$ is noticed with the increasing rotation parameter $R$ and it becomes approximately one for large or small rotation in the upper half of the channel width. It is also noticed that the amplitude $|A_0|$ decreases with the increase of the Hartmann number $M$ for small or large values of rotation parameter ($R$).
Figure 2 – The phase angle due to $u_0$ and $v_0$ for steady flow

Figure 2 shows that the phase angle $\theta_0$ for steady flow increases with increasing permeability $k_0$ of the porous medium for any value of the rotation parameter $R$ large or small. It is also clear from this figure 2 that an increase in small rotation parameter $R$ leads to an increase in $\theta_0$ but an increase in large values of $R$ results in the decrease of phase angle $\theta_0$. The upper half of the channel, the phase angle becomes approximately zero. It is also clear from figure 2 that the phase angle $\theta_0$ decreases with the increasing Hartmann number $M$ for any value of rotation large or small.

The amplitude and the phase differences of shear stresses at the plate $\eta = 0$ for the steady flow can be obtained as

$$\tau_{or} = \left(\tau_{ox}^2 + \tau_{oy}^2\right)^{1/2}, \theta_{or} = \tan^{-1}\left(\frac{\tau_{oy}}{\tau_{ox}}\right)$$

(20)

where $\tau_{ox}$ and $\tau_{oy}$ are respectively, the shear stresses at the plate due to the primary and secondary velocity components.

The numerical values for the resultant shear stress and the phase angle due to the shear stresses are listed in Table 1.

| Sl. No. | R   | M   | $k_0$ | $\tau_{or}$ | $\theta_{or}$ |
|--------|-----|-----|------|-------------|---------------|
| 1      | 1   | 2   | 1    | 2.3519      | 0.1741        |
| 2      | 1   | 4   | 1    | 4.1392      | 0.0583        |
| 3      | 1   | 2   | 5    | 2.1944      | 0.1980        |
| 4      | 1   | 4   | 5    | 4.0424      | 0.0611        |
| 5      | 5   | 2   | 1    | 3.3227      | 0.5560        |
| 6      | 5   | 4   | 1    | 4.4399      | 0.2656        |
| 7      | 5   | 2   | 5    | 3.2695      | 0.5906        |
| 8      | 5   | 4   | 5    | 4.3618      | 0.2762        |
| 9      | 25  | 2   | 1    | 7.0883      | 0.7356        |
| 10     | 25  | 4   | 1    | 7.2671      | 0.6215        |
These values clearly shows that the Shear Stress $\tau_{\alpha\beta}$ increases with increasing Hartmann number $M$ or the rotation parameter $R$. However the increase in the permeability parameter $k_0$ leads to a decrease in the shear stress for small as well as large rotation.

It is also clear from this table that the phase angle $\theta_{a\beta}$ increases with increasing rotation and permeability of the porous medium. However the phase angle $\theta_{a\beta}$ decreases with increase in Hartmann number $M$.

The solutions (15) and (16) together give the unsteady part of the flow, the primary and secondary velocity components $u_1$ and $v_1$ respectively for the fluctuating flow can be obtained from the asymptotic expansion of $q$ for large rotation $R$ as

$$
u_1(\eta, t) \approx e^{-m_0^0} \sin(m_0 \eta - \omega t) + e^{-n_0^0} \sin(n_0 \eta + \omega t)$$

Where

$$m_r = \left\{ \frac{(k_0^{-2} + (2R + \lambda^2) + M^4 \frac{1}{2} + (M^2 + k_0^{-1}))}{2} \right\}^{\frac{1}{2}}$$

$$m_i = \left\{ \frac{(k_0^{-2} + (2R + \lambda^2) + M^4 \frac{1}{2} - (M^2 + k_0^{-1}))}{2} \right\}^{\frac{1}{2}}$$

$$n_r = \left\{ \frac{(k_0^{-2} + (2R - \lambda^2) + M^4 \frac{1}{2} + (k_0^{-1} + M^2))}{2} \right\}^{\frac{1}{2}}$$

$$n_i = \left\{ \frac{(k_0^{-2} + (2R - \lambda^2) + M^4 \frac{1}{2} - (k_0^{-1} + M^2))}{2} \right\}^{\frac{1}{2}}$$

These expressions for $u_1(\eta, t)$ and $v_1(\eta, t)$ show the emergence of a boundary layer of thickness of order $0(mR^{-1})$ superimposed with the boundary layer of thickness of order $(nr^{-1})$. These boundary layers which are due to the cyclonic and anticyclonic components of the impressed harmonic oscillations decrease with the increase of Hartmann number and increase with the increase of the permeability parameter and decrease with the combined effect of rotation and frequency. The combination of Stoke’s layer and Ekman layer is exhibited in these boundary layers which appear in the neighbourhood of both the plates. In the case of resonance, when the natural frequency $2\Omega$ of the
rotating fluid is equal to the forcing frequency \( \omega \), that is for \( 2\Omega \) of the rotating fluid is for \( 2\Omega - \omega = 0 \) (or) \( 2R - \lambda = 0 \). The differential equation (12) for \( q_2 \) reduces to

\[
q_2^{11} - \left(k_0^{-1} + M^2\right) q_2 = -(M^2 + k_0^{-1})
\]

Further, for ordinary medium that is \( K \to \infty \) and in the absence of the magnetic field (that is for \( M = 0 \)). The solution reduces to,

\[
q_2(\eta) = \eta
\]

In the absence of magnetic field (\( M = 0 \)) when the rotation \( R \) is very large the velocity components for the unsteady flow are given by,

\[
u_i(\eta, t) = (1 + \eta) \cos \omega t - e^{-\sigma_1^2} \cos (\sigma_1 \eta - \omega t)\]

\[
u_i(\eta, t) = (1 - \eta) \sin \omega t + e^{-\sigma_1^2} \sin (\sigma_1 \eta - \omega t)
\]

where \( \sigma_1 = \left(R + \left(\frac{\lambda}{2}\right)\right)^{\frac{1}{2}} \). These equations are different from those obtained by Mazumder

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