Polyakov conjecture for hyperbolic singularities

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Abstract

We propose the form of the Liouville action satisfying Polyakov conjecture on the accessory parameters for the hyperbolic singularities on the Riemann sphere.

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1 Introduction

Let us consider the Fuchsian equation

$$\partial^2 \psi(z) + \frac{1}{2} T(z) \psi(z) = 0 ,$$

where

$$T(z) = \sum_{j=1}^{n} \left[ \frac{\Delta_j}{(z - z_j)^2} + \frac{c_j}{z - z_j} \right].$$

In the context of the Liouville field theory $T(z)$ plays the role of the $zz$-component of the energy-momentum tensor and the real positive numbers $\Delta_j$ are conformal weights. The complex numbers $c_j$ are called accessory parameters. The requirement that $T(z)$ is regular at the infinity implies the relations

$$\sum_{j=1}^{n} c_j = 0 , \quad \sum_{j=1}^{n} z_j c_j = - \sum_{j=1}^{n} \Delta_j , \quad \sum_{j=1}^{n} z_j^2 c_j = - 2 \sum_{j=1}^{n} z_j \Delta_j .$$

The Polyakov conjecture concerns the following version of the Riemann-Hilbert problem [1–3]. For a given set of positive weights $\{\Delta_j\}_{j=1}^{n}$ one has to adjust the accessory parameters in such a way that the Fuchsian equation (1) admits a fundamental system of solutions with $SU(1,1)$ monodromies around all singularities.

The interest to this problem comes from its close relation to the Liouville equation on the punctured Riemann sphere $X = (\mathbb{C} \cup \{\infty\}) \setminus \{z_1, \ldots, z_n\}$. If $\chi_1(z), \chi_2(z)$ are linearly independent solutions of (1) then the function $\varphi(z, \bar{z})$ determined by the relation

$$e^{\varphi(z, \bar{z})} = \frac{4 |w'|^2}{(1 - |w|^2)^2} , \quad w(z) = \frac{\chi_1(z)}{\chi_2(z)} ,$$

satisfies the Liouville equation

$$\partial \bar{\partial} \varphi = \frac{1}{2} e^\varphi$$

for all that $z$ for which $w(z)$ is well defined. It is convenient to use normalized solutions with Wronskian satisfying:

$$\partial \chi_1(z) \chi_2(z) - \chi_1(z) \partial \chi_2(z) = 1 ,$$

so that the relation (4) can be written in a simple form

$$e^{-\frac{\varphi(z, \bar{z})}{2}} = \pm \frac{1}{2} \left( \overline{\chi_2(z) \chi_2(z)} - \overline{\chi_1(z)} \chi_1(z) \right) .$$

Note that $\varphi(z, \bar{z})$ is real by construction. If in addition $\chi_1(z), \chi_2(z)$ satisfy the $SU(1,1)$ monodromy condition then $\varphi(z, \bar{z})$ is single-valued.

Under some restrictions on conformal weights the relation above can be made more precise. The case of all parabolic singularities was analyzed by Poincaré in the context of the uniformization problem [7]. He showed that the Liouville equation (5) has a unique real-valued regular on $X$ solution with the following behavior at the punctures:

$$\varphi(z, \bar{z}) = \begin{cases} -2 \log |z - z_j| - 2 \log |\log |z - z_j|| + O(1) & \text{as } z \to z_j , \\
-4 \log |z - z_j| + O(1) & \text{as } z \to \infty . \end{cases}$$
This solution defines a metric $ds^2 = e^{\varphi}|dz|^2$ which is complete on $X$. It has constant negative curvature $-1$ and parabolic singularities at each $z_j$. The energy–momentum tensor of the solution $\varphi$,

$$T_\varphi(z) = -\frac{1}{2}(\partial \varphi)^2 + \partial^2 \varphi = -2e^{\frac{\varphi}{2}}\partial^2 e^{-\frac{\varphi}{2}},$$

(8)
is a holomorphic function on $X$ of the form (2) with the conformal weights

$$\Delta_j = \frac{1}{2}, \quad j = 1, \ldots, n.$$

It follows from (8) that there exists a pair of solutions $\chi_1, \chi_2$ to the Fuchsian equation (1) related to $\varphi$ by (4) [1–3, 5, 6]. Since $\varphi$ is real and single-valued this solves the $SU(1,1)$ Riemann-Hilbert problem.

The existence and the uniqueness of the solution to the Liouville equation with the elliptic singularities,

$$\varphi(z, \bar{z}) = \begin{cases} 
-2 \left(1 - \frac{\theta_j}{2\pi}\right) \log |z - z_j| + O(1) & \text{as } z \to z_j, \\
-4 \log |z - z_j| + O(1) & \text{as } z \to \infty,
\end{cases}$$

was proved by Picard [8, 9] (see also [10] for the modern proof). The solution can be interpreted as the conformal factor of the complete, hyperbolic metric on $X$ with the conical singularities of the opening angles $0 < \theta_j < 2\pi$ at the punctures $z_j$. The energy–momentum tensor takes the form (2) with

$$\Delta_j = \frac{1}{2} - \frac{1}{2} \left(\frac{\theta_j}{2\pi}\right)^2, \quad j = 1, \ldots, n.$$

As in the parabolic case one can show that there exists a solution to the corresponding $SU(1,1)$ monodromy problem [1–6].

The Polyakov conjecture states that the (properly defined and normalized) Liouville action functional $S[\varphi]$ evaluated on the classical solution $\varphi(z, \bar{z})$ is a generating function for the accessory parameters of the monodromy problem described above:

$$c_j = -\frac{\partial S[\varphi]}{\partial z_j}.$$

(9)

This formula was derived within path integral approach to the quantum Liouville theory as the quasi-classical limit of the conformal Ward identity [11–13]. In the case of the parabolic singularities on $n$-punctured Riemann sphere a rigorous proof based on the theory of quasi-conformal mappings was given by Zograf and Takhtajan [5]. It was also pointed out that the same technique applies for the elliptic singularities of finite order. Note that only in these two cases the monodromy group of the Fuchsian equation is (up to conjugation in $SL(2, \mathbb{C})$) a discrete subgroup in $SU(1,1)$, and the map $w(z)$ defined by (4) solves the uniformization problem [5].

An alternative method, working both in the case of parabolic and general elliptic singularities, was recently developed by Cantini, Menotti and Seminara [1–3]. Yet another proof,
based on a direct calculation of the regularized Liouville action for parabolic and general elliptic singularities, was proposed by Takhtajan and Zograf in [6].

The aim of this Letter is to find the action which satisfies (9) for the singularities of the hyperbolic type. The $SU(1,1)$ monodromy problem is well posed in this case, but whether it has a solution for arbitrary conformal weights $\Delta_j > \frac{1}{2}$ and arbitrary locations of punctures $z_j$ is up to our knowledge an open problem. In the present Letter we assume that such solution exists.

As one can expect from the case of two punctures [14] the corresponding solution to the Liouville equation determined by the relation (4) develops concentric line singularities around each puncture. We assume that these line singularities do not intersect and that they are the only singularities of the classical solution.

The singular behavior around punctures can be described in terms of local conformal maps. This allows for the construction of an appropriately regularized Liouville action functional. One can then apply the method of Takhtajan and Zograf [6] to prove the Polyakov conjecture. The problem of the existence and the uniqueness of the solution to the Fuchsian equation with the properties stated above goes beyond the scope of the present letter. Let us only mention that in the case of three hyperbolic singularities an explicit solution in terms of the hypergeometric functions exists [4, 15]. There is also a simple geometrical construction yielding a large class of solutions with an arbitrary number of hyperbolic singularities [15].

The choice of the Fuchsian equation (1) as a starting point for the construction of the Liouville field theory is a convenient way to impose the crucial constraint on admissible classical solutions — the holomorphic form (2) of their energy–momentum tensor $T(z)$. We hope that the singular hyperbolic solutions and the corresponding Liouville action will be helpful in understanding the factorization problem in the geometrical approach to the Liouville theory developed by Takhtajan [12, 13, 16, 17]. This was actually our main motivation for the present paper.

Finally let us note that the hyperbolic solutions provide multi black hole solutions of the 3-dim gravity [18–20].

2 Hyperbolic singularities

Let us assume that $\{\chi_1, \chi_2\}$ is a normalized solution to the $SU(1,1)$ monodromy problem for hyperbolic weights

$$\Delta_j = \frac{1 + \lambda_j^2}{2}, \quad \lambda_j \in \mathbb{R}.$$ 

Then the fundamental system defined by

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \sqrt{\frac{i}{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}$$
is also normalized and has $SL(2, \mathbb{R})$ monodromy around all punctures. In terms of $\{\psi_1, \psi_2\}$ the formula (7) reads

$$e^{-\varphi(z, \bar{z})} = \pm \frac{i}{2} \left( \psi_1(z)\overline{\psi_2(z)} - \overline{\psi_1(z)}\psi_2(z) \right).$$  

(10)

The advantage of using solutions with $SL(2, \mathbb{R})$ monodromy is that any hyperbolic element $M \in SL(2, \mathbb{R})$ ($\text{tr}(M) > 2$) is $SL(2, \mathbb{R})$-conjugate to a diagonal matrix. Thus for each singularity $z_j$ there exists an element $B_j \in SL(2, \mathbb{R})$ such that the system

$$\begin{pmatrix} \psi_1 \\
\psi_2 \end{pmatrix} = B_j \begin{pmatrix} \psi_1 \\
\psi_2 \end{pmatrix}$$

has a diagonal monodromy at $z_j$. It follows that $\psi_+^j$ have the canonical form

$$\psi_+^j(z) = \frac{\theta_j}{\sqrt{1 + \lambda_j}} (z - z_j)^{1 + \lambda_j} u_+^j(z),$$  

(11)

where $\theta_j \in \mathbb{R}$, and $u_+^j(z)$ are analytic functions

$$u_+^j(z) = \sum_{l=0}^{\infty} u_{+l}^j (z - z_j)^l, \quad u_{+0}^j = 1,$$

on the disc $D_j = \{ z : |z - z_j| < \min_{i, i \neq j} |z_i - z_j| \}$. Expanding the energy–momentum tensor

$$T(z) = \sum_{j=1}^{n} \left[ \frac{\Delta_j}{(z - z_j)^2} + \frac{c_j}{z - z_j} \right] = \sum_{k=0}^{\infty} t_k^j (z - z_j)^{k-2},$$

one gets

$$t_0^j = \Delta_j, \quad t_1^j = c_j, \quad t_k^j = \sum_{i, i \neq j} \left[ \frac{(k - 1)\Delta_i}{(z_i - z_j)^{k}} - \frac{c_i}{(z_i - z_j)^{k-1}} \right] \quad \text{for } k \geq 2.$$  

(12)

The Fuchsian equation (1) then implies

$$u_{+, l}^j = -\frac{1}{2l(1 + i\lambda_j)} \sum_{k=1}^{l} t_k^j u_{+, l-k}^j \text{ for } l \geq 1.$$  

(13)

It is a well known property of the Schwarz derivative

$$\{f(z), z\} \equiv \frac{f^{(3)}(z)}{f'(z)} - \frac{3}{2} \left( \frac{f''(z)}{f'(z)} \right)^2$$

and the Fuchsian equation (1) that the ratio

$$A_j(z) = \frac{\psi_+^j(z)}{\psi_-^j(z)}$$
satisfies the relation
\[ T(z) = \{ A_j(z), z \} \]  \hspace{1cm} (14)

For each hyperbolic singularity we define
\[ \rho_j(z) = (A_j(z))^{1/\lambda_j} \]  \hspace{1cm} (15)

It follows from (11,12,13) that \( \rho_j(z) \) is an analytic function on \( D_j \) and:
\[ \rho_j(z) = e^{\bar{\psi}_j} \left[ z - z_j + \frac{c_j}{2\Delta_j} (z - z_j)^2 + \mathcal{O} \left( (z - z_j)^3 \right) \right]. \]  \hspace{1cm} (16)

Using (14) and the properties of the Schwarz derivative one gets
\[ T(z) = \left\{ (\rho_j(z))^{i\lambda_j}, z \right\} = \left( \frac{d\rho_j(z)}{dz} \right)^2 \bar{T}_j(\rho_j(z)) + \{ \rho_j(z), z \}, \] where
\[ \bar{T}_j(\rho) = \left\{ \rho^{i\lambda_j}, \rho \right\} = \frac{\Delta_j}{\rho^2}. \]  \hspace{1cm} (17)

Let us consider the Fuchsian equation
\[ \partial^2 \psi(\rho) + \frac{1}{2} \bar{T}_j(\rho)\psi(\rho) = 0 \]
on the complex \( \rho \) plane and a normalized fundamental system of solutions with the diagonal monodromy at \( \rho = 0 \) of the following form
\[ \bar{\psi}_j(\rho_\pm) = (i\lambda_j)^{-\frac{1}{2}} \rho^{\frac{1}{2} \pm i\lambda_j}. \]  \hspace{1cm} (19)

The corresponding solution of the Liouville equation reads [14]
\[ \bar{\varphi}_j(\rho, \bar{\rho}) = \log \left[ \frac{\lambda_j^2}{|\rho|^2 \sin^2 (\lambda_j \log |\rho|)} \right]. \]  \hspace{1cm} (20)

The metric \( e^{\bar{\varphi}_j} d^2 \rho \) has infinitely many closed geodesics:
\[ \bar{\mathcal{G}}_l = \{ \rho \in \mathbb{C} : \lambda_j \log |\rho| = \pi (l - \frac{1}{2}) \}, \quad l \in \mathbb{Z}, \]
and infinitely many singular lines:
\[ \bar{\mathcal{S}}_l = \{ \rho \in \mathbb{C} : \lambda_j \log |\rho| = \pi l \}, \quad l \in \mathbb{Z}. \]

Using the transformation rule (17) and the expansion (16) one can show that on \( D_j \subset X \) the metric \( e^{\bar{\varphi}_j} d^2 \rho \) coincides with the pull-back of the metric \( e^{\bar{\varphi}_j} d^2 \rho \) by the map \( \rho_j(z) \). As \( \rho_j(D_j) \) is an open neighborhood of 0 there are infinitely many geodesics \( \bar{\mathcal{G}}_l \) and singular lines \( \bar{\mathcal{S}}_l \) contained in \( \rho_j(D_j) \). Their inverse images \( \mathcal{G}_l = \rho_j^{-1}(\bar{\mathcal{G}}_l), \mathcal{S}_l = \rho_j^{-1}(\bar{\mathcal{S}}_l) \), are closed singular lines and closed geodesics of the metric \( e^{\bar{\varphi}_j} d^2 \rho \) on \( X \). This provides a detailed description of the singular hyperbolic geometry in a sufficiently small neighborhood of the hyperbolic
singularity: an alternating sequence of the concentric closed geodesics and closed singular lines. Let us stress that all these geodesic have the same length, uniquely determined by the conformal weight:

\[ \ell_j = 2\pi \lambda_j . \]

The question arises what happens to this geometry when one goes away from the singularity. We assume that there exists a set \{\Gamma_j\}_{j=1}^n of closed geodesics with the following properties:

- \( \Gamma_j \) separates \( z_j \) from all other geodesics \( \Gamma_i \) (\( i \neq j \));
- the map \( \rho_j \) extends to a conformal invertable map on the hole \( H_j \) around \( z_j \) defined as the part of \( \hat{\mathbb{C}} \) containing \( z_j \) and bounded by \( \Gamma_j \);
- the metric \( e^\varphi d^2z \) is regular on the surface \( M \equiv \hat{\mathbb{C}} \setminus \bigcup_{j=1}^n H_j \).

The assumption is well justified by the properties of the general 3-puncture solution and by the geometric construction of the \( n \)-puncture solutions [15]. In particular, it implies that each \( \Gamma_j \) is an inverse image by \( \rho_j \) of one of the standard closed geodesics \( G_l \) in the \( \rho \)-plane.

It can be parameterized as

\[ \gamma_j(t) = \rho_j^{-1}(r_j e^{it}) , \quad r_j \equiv e^{\frac{\varphi}{\lambda_j}(l+\frac{1}{2})} , \quad (21) \]

for some \( l \in \mathbb{Z} \). The orientation of the \( j \)-th boundary component \( \partial M_j \equiv \Gamma_j \) corresponds to the parameter \( t \) decreasing from \( 2\pi \) to 0. Using (16) one gets for \( \rho \in \rho_j(H_j) \)

\[ \rho_j^{-1}(\rho) = z_j + e^{-\frac{\varphi}{\lambda_j}} \rho - \frac{c_j}{2\Delta_j} e^{-\frac{\varphi}{\lambda_j}} \rho^2 + O(\rho^3) . \quad (22) \]

### 3 Liouville action

The standard Liouville action on a surface \( M \subset \mathbb{C} \) with regular boundary components reads

\[ S_L[M,\phi] = \frac{1}{2\pi} \int_M d^2z \left( \partial\phi \bar{\partial}\phi + e^\phi \right) + \frac{1}{2\pi} \int_{\partial M} |dz| \kappa_z \phi , \quad (23) \]

where \( d^2z = \frac{i}{2} dz \wedge \bar{dz} \) and \( \kappa_z \) is a geodesic curvature of \( \partial M \) (computed in the flat metric on the complex plane). It yields the boundary conditions

\[ n^a \partial_a \phi + 2\kappa_z = 0 \quad (24) \]

and the equation of motion (5). The classical solution \( \varphi(z,\bar{z}) \) defines on \( M \) a hyperbolic metric \( e^\phi d^2z \) with geodesic boundaries. If \( M \) is unbounded one has to impose an appropriate asymptotic conditions on admissible solutions. It can be done by means of a modified action

\[ S_L^\infty[M,\phi] = \lim_{R \to \infty} S_L^R[M,\phi] , \quad (25) \]

\[ S_L^R[M,\phi] = \frac{1}{2\pi} \int_{M^R} d^2z \left( \partial\phi \bar{\partial}\phi + e^\phi \right) + \frac{1}{2\pi} \int_{\partial M^R} |dz| \kappa_z \phi + \frac{1}{\pi R} \int_{|z|=R} |dz| \phi + 4 \log R , \]
where \( M^R = \{ z \in M : |z| \leq R \} \). The presence of the additional boundary terms forces \( \phi(z, \bar{z}) \) to behave asymptotically as

\[
\phi(z, \bar{z}) \approx -2 \log |z|^2 \quad \text{for} \quad |z| \to \infty .
\]

This implies that \( T(z) \) is regular at infinity and the limit (25) exists.

Let \( \varphi(z, \bar{z}) \) denote a solution of the Liouville equation (5) with the holomorphic component of the energy–momentum tensor of the form (2) and satisfying the regularity conditions formulated at the end of Sect. 2. It defines a surface \( M \) with holes \( H_j \) around each hyperbolic singularity \( z_j \). The shape of \( M \) depends on the conformal weights \( \Delta_j \) and the location of singularities \( z_j \). The starting point of our construction is the Liouville action \( S^\infty_\varphi [M, \varphi] \) on this particular surface. We shall regard \( S^\infty_\varphi [M, \varphi] \) as a functional on the space of all conformal factors \( \varphi(z, \bar{z}) \) on \( M \) with the asymptotic behavior (26) and satisfying the boundary conditions (24). The stationary point coincides by construction with the solution \( \varphi(z, \bar{z}) \) restricted to \( M \). The classical action \( S^\infty_\varphi [M, \varphi] \) (i.e. the action (25) evaluated at the classical solution \( \varphi(z, \bar{z}) \) on \( M \)) does not satisfy the Polyakov conjecture. It turns out that the terms one has to add to \( S^\infty_\varphi [M, \varphi] \) are independent of the “fluctuating field” \( \phi(z, \bar{z}) \) and therefore alter neither the boundary conditions nor the equation of motion.

On each hole \( H_j \) there exists a unique flat metric with the only singularity at \( z_j \) such that the boundary \( \partial H_j = \Gamma_j \) is geodesic and its length is \( 2\pi \lambda_j \). It can be constructed as the pull-back by \( \rho_j(z) \) of the metric \( \frac{\lambda^2}{\rho^2} d^2 \rho \) which yields the following formula for its conformal factor

\[
\varphi_j(z, \bar{z}) = \log \left[ \frac{\lambda^2}{|\rho_j(z)|^2} \right] \left( \frac{d\rho_j(z)}{dz} \right)^2 .
\]

Using the expansion (16) one gets

\[
\varphi_j(z, \bar{z}) = \log \lambda^2 - \log |z - z_j|^2 + \frac{c_j}{2\Delta_j} (z - z_j) + \frac{\bar{c}_j}{2\Delta_j} (\bar{z} - \bar{z}_j) + \mathcal{O}(|z - z_j|^2) .
\]

Let us note that \( \varphi_j(z, \bar{z}) \) satisfies \( C^1 \) sewing relations along the boundary

\[
\varphi(z, \bar{z}) = \varphi_j(z, \bar{z}), \quad n^a \partial_a \varphi(z, \bar{z}) = n^a \partial_a \varphi_j(z, \bar{z}) \quad \text{for} \quad z \in \Gamma_j .
\]

We define on \( H_j \) the regularized classical action

\[
S^\epsilon_\varphi [H_j, \varphi_j] = \frac{1}{2\pi} \int_{H_j^\epsilon} d^2 z \partial\varphi_j \partial \varphi_j + \frac{1}{2\pi} \int_{\Gamma_j} |dz| \kappa z \varphi_j + \log \epsilon ,
\]

where \( H_j^\epsilon \) denotes \( H_j \) with a disc of radii \( \epsilon \) around \( z_j \) cut out. With this notation our proposal for the Liouville action in the case of hyperbolic singularities can be written in the following

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3We shall reserve the symbol \( \varphi(z, \bar{z}) \) for the classical solution of the equation of motion, denoting by \( \phi(z, \bar{z}) \) a general, “fluctuating” field.
form:

\[ S_L[\phi] = \lim_{\epsilon \to 0} S^\epsilon_L[\phi], \quad (30) \]

\[ S^\epsilon_L[\phi] = S^\infty_L[M, \phi] + \sum_{k=1}^{n} S^\epsilon_L[H_k, \varphi_k] - \sum_{k=1}^{n} \lambda_k^2 \log \left| r_k^{-1} \frac{d\rho_k(z_k)}{dz} \right|. \quad (31) \]

It should be stressed that the Polyakov conjecture determines the classical Liouville action \( S_L[\varphi] \) only up to an arbitrary function of conformal weights. This freedom is tacitly assumed in the formula above. Using the sewing relations (29) one can rewrite the classical action \( S^\epsilon_L[\varphi] \) (31) in the form

\[ S^\epsilon_L[\varphi] = \frac{1}{2\pi} \int_M d^2z \left( \partial \bar{\varphi} \partial \varphi + e^\varphi \right) + \frac{1}{2\pi} \sum_{k=1}^{n} \int_{H_k^\epsilon} d^2z \partial \varphi_k \partial \bar{\varphi}_k \]

\[ - \sum_{k=1}^{n} \lambda_k^2 \log \left| r_k^{-1} \frac{d\rho_k(z_k)}{dz} \right| + n \log \epsilon. \]

The modifications of the action related to the asymptotic behavior at infinity are independent of the locations of singularities and are irrelevant for our derivation of the Polyakov conjecture. For the sake of brevity they are suppressed in the formula above.

4 Polyakov conjecture

Using the equations of motion

\[ \partial \bar{\varphi} = \frac{i}{2} e^\varphi, \quad \partial \varphi_j = 0, \quad (33) \]

and the sewing relations (29) one gets

\[ \frac{\partial}{\partial z_j} S^\epsilon_L[\varphi] = \frac{i}{4\pi} \sum_{k=1}^{n} \int_{\partial M_k} d^2z \left( \partial \gamma_k \frac{d\gamma_k}{dz_j} - \partial \bar{\gamma}_k \frac{d\bar{\gamma}_k}{dz_j} \right) + \frac{i}{4\pi} \int_{|z - z_j| = \epsilon} d\bar{z} \partial \varphi_j \partial \bar{\varphi}_j \]

\[ + \frac{i}{4\pi} \sum_{k=1}^{n} \int_{|z - z_k| = \epsilon} \partial \varphi_k d\bar{z} \partial \varphi_k d\bar{z} - \partial \varphi_k d\bar{z} \]

\[ - \sum_{k=1}^{n} \lambda_k^2 \frac{\partial}{\partial z_j} \log \left| \frac{d\rho_k(z_k)}{dz} \right|. \quad (34) \]

The first term on the r.h.s. of (34) results from the change of the shape and the position of the boundary components \( \partial M_k \) induced by the change of \( z_j \). The second one is due to the change of the position of the circle \( |z - z_j| = \epsilon \) (by construction, all the remaining “small holes” preserve their positions; their radii, equal to \( \epsilon \), are fixed). The third term follows (after integration by parts) from the expression

\[ \frac{1}{2\pi} \int_M d^2z \left( \frac{\partial}{\partial z_j} \bar{\varphi} + \partial \bar{\varphi} \frac{\partial}{\partial z_j} \varphi + e^\varphi \frac{\partial}{\partial z_j} \varphi \right) + \frac{1}{2\pi} \sum_{k=1}^{n} \int_{H_k^\epsilon} d^2z \left( \frac{\partial}{\partial z_j} \bar{\varphi}_k + \partial \bar{\varphi}_k \frac{\partial}{\partial z_j} \varphi_k + \partial \varphi_k \frac{\partial}{\partial z_j} \bar{\varphi}_k \right) \]
resulting from the change of the integrands in (30) due to \( z_j \to z_j + \delta z_j \). Using (21), (27), (29) and (22) one gets:

\[
e^\varphi d\bar{\gamma}_k \left|_{\bar{\rho}=r_k e^{-it}} \right. = \frac{\lambda_k^2}{r_k^2} \frac{1}{(\rho_k^{-1})' d\bar{\rho}} \bigg|_{\bar{\rho}=r_k e^{-it}} = -i \frac{\lambda_k^2}{r_k} \left( e^{\frac{\delta_k}{\lambda_k}} e^{-it} + r_k \frac{c_k}{\Delta_k} \right) dt + O(e^{it}) ,
\]

\[
e^\varphi d\gamma_k \left|_{\rho=r_k e^{it}} \right. = \frac{\lambda_k^2}{r_k^2} \frac{1}{(\rho_k^{-1})' d\rho} \bigg|_{\rho=r_k e^{it}} = i \frac{\lambda_k^2}{r_k} \left( e^{\frac{\delta_k}{\lambda_k}} e^{it} + r_k \frac{\bar{c}_k}{\Delta_k} \right) dt + O(e^{-it}) .
\]

From (22) and (21) one also has:

\[
\begin{align*}
\frac{\partial \gamma_k}{\partial z_j} &= \frac{\partial}{\partial z_j} \rho_k^1 \bigg|_{\bar{\rho}=r_k e^{-it}} = \frac{\partial}{\partial z_j} \left( e^{-\frac{\delta_k}{\lambda_k}} r_k e^{-it} + O(e^{-2it}) \right) , \\
\frac{\partial \gamma_k}{\partial z_j} &= \frac{\partial}{\partial z_j} \rho_k^1 \bigg|_{\rho=r_k e^{it}} = \delta_{kj} + \frac{\partial}{\partial z_j} \left( e^{-\frac{\delta_k}{\lambda_k}} r_k e^{it} + O(e^{2it}) \right) ,
\end{align*}
\]

and

\[
\frac{i}{4\pi} \sum_{k=1}^{n} \int \varphi \left( \frac{\partial \gamma_k}{\partial z_j} d\gamma_k - \frac{\partial \gamma_k}{\partial z_j} d\bar{\gamma}_k \right) = - \sum_{k=1}^{n} \frac{\lambda_k^2}{2\Delta_k} \left. \frac{1}{2\pi} \int_{0}^{2\pi} \right[ \frac{c_k}{\Delta_k} \delta_{kj} + e^{\frac{\delta_k}{\lambda_k}} \frac{\partial}{\partial z_j} \left( e^{-\frac{\delta_k}{\lambda_k}} \right) + \ldots \right]
\]

\[
= - \frac{\lambda_k^2}{2\Delta_k} c_j + \sum_{k=1}^{n} \lambda_k \frac{\partial}{\partial z_j} \varphi_k .
\]

The terms in the first line of (35) denoted by dots contain non-zero, integer powers of \( e^{it} \) and vanish upon integration. The expansion (28) implies

\[
\begin{align*}
\partial \varphi_k &= - \frac{1}{z - z_k} + \frac{c_k}{2\Delta_k} + O(z - z_k) , \\
\bar{\partial} \varphi_k &= - \frac{1}{\bar{z} - \bar{z}_k} + \frac{\bar{c}_k}{2\Delta_k} + O(\bar{z} - \bar{z}_k) , \\
\frac{\partial}{\partial z_j} \varphi_k &= \delta_{kj} \left( \frac{1}{z - z_k} - \frac{c_k}{2\Delta_k} \right) + O(|z - z_k|) .
\end{align*}
\]

Hence, up to the terms that vanish in the limit \( \epsilon \to 0 \):

\[
\frac{i}{4\pi} \int_{|z - z_j| = \epsilon} d\bar{z} \partial \varphi_j \bar{\partial} \varphi_j + \frac{i}{4\pi} \sum_{k=1}^{n} \int_{|z - z_k| = \epsilon} \frac{\partial}{\partial z_j} \left( \partial \varphi_k \right) d\bar{z} \partial \varphi_k d\bar{z} = - \frac{c_j}{2\Delta_j} .
\]

Substituting (35) and (36) in (34) and taking into account the relation

\[
\log \left| \frac{d\rho_k}{dz}(z_k) \right| = \frac{\partial}{\partial z_k} \left| \frac{d\rho_k}{dz}(z_k) \right| = \frac{\partial}{\partial z_k} \left| \frac{d\rho_k}{dz}(z_k) \right| = \frac{\partial}{\partial z_k} \left| \frac{d\rho_k}{dz}(z_k) \right|
\]

one finally gets

\[
\frac{\partial}{\partial z_j} S_{\lambda j}^\varphi = \lim_{\epsilon \to 0} \frac{\partial}{\partial z_j} S_{\lambda j}^\varphi = -c_j .
\]
Acknowledgements

The work of L.H. was supported by the EC IHP network HPRN-CT-1999-000161. Laboratoire de Physique Théorique is Unité Mixte du CNRS UMR 8627.

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