The periodic temperature oscillations in the near-surface layer of cylindrical structures

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Abstract. A hollow cylinder with thick walls and linear continuously acting variable heat sources are among the most difficult objects to calculate the unsteady temperature field, so this field is the least studied. However, such objects are found in many modern designs of heat generation and distribution systems. The proposed work considers the study of the temperature wave propagation in the wall of a hollow cylinder and in a cylindrical symmetric array around a variable linear heat source with a harmonious change in the temperature of the external or internal environment that occurs during its daily fluctuations or when regulating the heat distribution system. The result of approximate analytical solution of the problem by the method of separation of variables in the complex domain using cylindrical functions is presented. The data of numerical calculation of attenuation of temperature oscillations in the near-surface layer of the cylinder and around the variable linear heat source by means of an explicit finite-difference scheme of increased accuracy under boundary conditions of the first kind and their comparison with the analytical solution for its refinement are presented. The refined analytical dependences for the attenuation coefficient of the temperature wave suitable for use at low thermal inertia of the cylindrical layer and giving physically correct results at the boundary values of the parameters are proposed. The difference in the nature of the damping of oscillations in the direction of the temperature wave outside and inside the cylinder is noted. The presented dependences are proposed to be used for a refined analytical assessment of the amplitude of temperature fluctuations in the near-surface layer of cylindrical heated structures and around the heating and heat supply pipelines in variable modes, which will allow the use of engineering methods to verify compliance with the requirements of industrial safety.

1. Introduction

In the proposed publication, the object of study is the unsteady temperature field of a cylindrical layer of great thickness, in particular, a hollow thick-walled cylinder and a cylindrical area around a linear heat source under periodic thermal effects.

The problem of the temperature waves propagation in bodies of various geometries has been studied for quite a long time, and in recent years, in connection with the development of computer technology, numerical methods for its solution come to the fore. Most often, however, they relate to the one-dimensional case, or the problem is formulated in rectangular coordinates. However, this corresponds to the prevailing part of the problems that are actually encountered, and not only in the non-stationary, but also in the stationary mode [1]. The issues of heating and cooling of solid or thin-walled cylinders [1] are the most extensively studied for cylindrical symmetry.

Recently, a number of publications have appeared, where such issues are considered, both analytically and numerically. However, the results obtained by the authors are, as a rule, extremely
difficult to use in engineering practice [2], [3], [5], [7], or, on the contrary, are too rough [8]. Other solutions options are related to specific types of structures used in limited areas and operating under specific conditions, for example, in supercritical regimes [6], in nuclear power engineering [10] or in the production of composite materials [5], and also in the presence of phase transitions [9], [11], [12] or for underground pipelines [13]. Some studies focus on the applied methods and the computational process, which are considered in relation to very complex objects – a lattice of cylinders or a multi-layer structure [14], [15]. In addition, the authors of some works in this field, on the contrary, solve the inverse problem – on determining the thermophysical characteristics of a material based on the study of temperature fluctuations [16 – 18].

At the same time, for a number of engineering applications, the calculation of a non-stationary temperature field in a hollow cylinder with a wall of significant thickness is of great importance. Such a calculation, in particular, may be required when checking for condensation on the outer surface of the thermal insulation of heating conduits or on the internal surface of chimneys with a daily change in the temperature of the outside air and the heat carrier. This is especially interesting when open laying of heating systems pipelines in case of significant moisture content of the outside air, which happens when its temperature is close to zero and above, when freezing and thawing conditions of atmospheric moisture and snow shall be assessed. In addition, such a calculation may be required to confirm the absence of condensate on the inner surface of the air ducts of exhaust ventilation systems during the day when they are laid externally, which in some cases is permitted when used for constructive purposes, subject to their thermal insulation. The same applies to the outer surface of the air intake ducts, when they, as an exception, are laid heat-insulated inside the building. Finally, this calculation is needed to assess the temperature fluctuations on the outer surface of cylindrical furnaces for changes in their mode of operation to decide whether this temperature is acceptable according to sanitary and hygienic requirements. Consideration of temperature fluctuations around a linear variable heat source may be necessary when assessing the temperature on the surface of heating panels or floors with monolithic pipes or heating cables to check its compliance with sanitary and hygienic standards in variable modes, as well as considering the conditions of heating systems' heating conduits operation in duct-free laying.

In publication [19] the author obtained a fairly simple solution for the propagation of temperature waves in a thick-walled cylinder, but it is directly applicable for the direction of fluctuations from the inside only and has some limitations as to the applicability range, and in work [20] the result is provided for a variable linear source of small diameter, but only on the basis of numerical modeling, without due theoretical justification. Thus, the relevance of the proposed research is the need to search for sufficiently accurate and physically sound, but at the same time acceptable for engineering use dependencies of temperature changes of periodically heated and cooled cylindrical structures for any direction of the temperature wave and for a wide range of geometry. The results obtained may be applicable for a fairly wide range of heat engineering and energy facilities of similar design.

The aim of the work is to refine the calculation of the temperature field for harmonic fluctuations of the ambient temperature at the outer or inner surface of the cylinder, or around a linear variable heat source. The objectives of the study are: - the development of an algorithm that would implement the finite-difference approximation of the thermal conductivity equation in the cylinder wall for a regular mode; - construction of an analytical solution of this equation for a variable linear source; - obtaining analytical dependencies for the amplitude of the temperature wave based on comparison of theoretical results with the program generation data.

2. Methods
The differential equation for transient heat conduction for cylindrical symmetry can be written in the following form [1]:
\[
\frac{\partial t}{\partial \tau} = a \left[ \frac{\partial t}{r \partial r} + \frac{\partial^2 t}{\partial r^2} \right],
\]

where \( \tau \) is the moment of time, s, for which the temperature \( t \), K, is determined; \( r \) – the radial coordinate, or the distance from the axis of the cylinder, m; \( a = \frac{\lambda}{c \rho} \) – the thermal diffusivity of the material of the cylinder array, \( m^2/s \); \( \lambda \) – its thermal conductivity, W/(m·K); \( c \) and \( \rho \) – respectively, the specific heat capacity, J/(kg·K), and the density, kg/m\(^3\).

The boundary condition on the outer surface of the cylinder for a periodic mode in a complex form in the simplest case of harmonic oscillations can be written as follows:

\[
t = A te^{-i \omega \tau},
\]

where \( A te \) is the amplitude of temperature fluctuations of the outside air \( t e \), K; \( \omega = \frac{2 \pi}{T} \) – the wave circular frequency of \( t e \), s\(^{-1}\). Here \( T \) is the oscillation period, c. If the wave propagates from the inside, instead of \( A te \), \( A ti \) should be used – the amplitude of temperature fluctuations of the internal air in the cylindrical structure or on the surface of the linear source \( ti \). The solution in this case will indicate the current temperature field in the temperature wave propagation zone.

Since in practical problems the temperature fluctuations amplitude is of interest, its attenuation rate \( \nu r = A t e / A ti \), can be introduced where \( A ti \) – the amplitude of temperature fluctuations of the cylinder array, K, at the considered radial distance \( r' \) from the surface. If the temperature wave propagates outside, then \( r' = r_0 - r \), where \( r_0 \) and \( r \) are respectively the outer radius of the cylinder and the current radial coordinate, m. The value of \( \nu r \) can be associated with the thermal inertia of the cylindrical layer of \( D \) by analogy with the one-dimensional problem. According to the results of studies conducted by the author in the work [20], the following expression was found for \( \nu r \):

\[
\nu r = \sqrt{\pi D r} \exp \left( \frac{D r}{\sqrt{2}} \right), \text{ where } D r = r \sqrt{\frac{\omega}{a}}.
\]

It is easy to note that the dependence found differs from the one-dimensional case [2] only in an additional factor \( \sqrt{\frac{\pi D r}{2}} \).

The structure of formula (3) is obtained by solving the differential thermal conductivity equation in cylindrical coordinates for the case of a harmonic boundary condition on the outer surface, with due regard for the asymptotic approximation of the Hankel function for large values of the argument and subsequent comparison with the numerical calculations data. The latter were carried out using the computer program developed by the author in the algorithmic language Fortran, which uses a finite-difference approximation of the original differential equation using an explicit scheme of enhanced accuracy. Then the temperature value in the \( i \)-th grid node at \( j + 1 \)-th moment of time can be calculated using the expression:

\[
t_{i,j+1} = F O_\Delta \left( \frac{2i - 1}{2i - 2} t_{i+1,j} + \left( \frac{1}{F O_\Delta} - 2 \right) t_{i,j} + \frac{2i - 3}{2i - 2} t_{i-1,j} \right).
\]

Here \( t_{i,j} \), \( t_{i-1,j} \), and \( t_{i+1,j} \) – temperature values at the \( j \)-th moment of time in the \( i \)-th node and adjacent left and right (numbering of the nodes is from the axis of the cylinder towards the outer surface); \( F O_\Delta = \frac{a \Delta \tau}{(\Delta r)^2} \) – dimensionless local Fourier criterion, where \( \Delta \tau \), \( c \), and \( \Delta r \), m are the time and coordinate steps, respectively, representing the parameters of the finite-difference scheme.
However, it was already noted in work [20] that for small values of $D_r$, the asymptotics used in the solution is in significant error. In fact, for $D_r \to 0$, i.e. directly on the surface, relation (3) instead of the obvious $v_0 = 1$ produces 0, which has no physical meaning. Therefore, it is of interest to clarify the detected dependence for this case and to obtain expressions producing correct results in the entire domain of their definition. This can be achieved with the help of some modification of the previously used algorithm, so that in the course of the calculations, the maximum and minimum temperature values during the fluctuation period for any distance from the cylinder axis was memorized. Thus, we obtain the desired dependence of the fluctuations amplitude on the radial coordinate.

3. Results

The corresponding calculation results for $A_0 = 1$ K, $r_0 = 1$ m, $a = 5.1 \cdot 10^{-7}$ m$^2$/s and $\omega = 2\pi/86400 = 7.27 \cdot 10^{-5}$ s$^{-1}$, i.e. during the daily period of variations, are presented in figure 1. At an inner radius of $r_1 = 0.5$ m, the temperature was maintained constant. Obviously, in this case, the value of $A_t$ is numerically equal to the reciprocal value of the damping coefficient $1/v_r$.

![Figure 1](image_url)

**Figure 1.** The dependence of the amplitude of the temperature wave on the thermal inertia of the cylindrical layer in the surface zone (solid line - numerical calculation, dotted line - approximation as per (5)).

The curve presented is very well approximated using the following expression:

$$\frac{1}{v_r} = \exp(-0.6D_r).$$

(5)

The graph corresponding to this formula, in figure 1, virtually coincides with the original line within its thickness. For other values of the $a$ parameter, characteristic of the commonly used structural and heat-insulating materials, the results change insignificantly. In any case, the detected discrepancies can be explained by the error of approximation of the original differential equation, therefore, the revealed
dependence can be considered universal, at least for the ratio $r_1/r_0 > 0.5$. Thus, for small $D_r$, the decrease in the temperature fluctuations amplitude is of exponential nature, likewise for a flat wall, but with a smaller numerical coefficient for $D_r$. Qualitatively, this confirms the conclusion that was made in the previous work and was reduced just to the fact that in the subsurface layer, in particular, for $D_r < 4/\pi \approx 1.26$, the attenuation of the temperature wave in the cylindrical wall occurs slower than in the one-dimensional case.

At higher values of $D_r$, the nature of the dependence obtained in work [20] for the attenuation coefficient is generally confirmed, which is substantiated by the behavior of the curves in figure 2. Here, the solid line shows the decrease of $A_t$ calculated by the above program, and the dash-dotted curve was calculated using expression (3), in which, for purposes of the results best matching, the numerical coefficient for $D_r$ in the argument of the exponential factor $c 1/\sqrt{2}$ is reduced to 0.5.

![Figure 2](image_url)

**Figure 2.** Attenuation of the temperature wave in the cylindrical layer (solid line – numerical calculation, dash-dotted line – according to the refined formula (3)).

At the same time, it can be assumed that when temperature fluctuations propagate from inside of the cylinder towards its outer surface, the drop in amplitude will have a different character than in the opposite case, due to the gradual increase in the volume of each subsequent cylindrical layer. In figure 3, the solid line shows the $1/\nu$ dependence on $D_r$ obtained from the numerical calculation results using the previously mentioned algorithm for the same values of the parameters as before, but with the condition that the temperature will be maintained constant at $r_0 = 1$, and the amplitude $A_t = 1$ K at the inner radius. The corresponding approximating formula differs from (5) only in even more reduced numerical factor, which in this case equals to 0.5.
Figure 3. Attenuation of the temperature wave during its propagation from inside of the cylinder (solid line – numerical calculation, dotted line – using expression (6)).

A graph calculated using the slightly modified formula (3) is shown for comparison with the dotted curve:

$$\frac{1}{\nu_r} = \frac{\sqrt{\pi D_r} \exp\left(-\frac{D_r}{\sqrt{2}}\right)}{5/3}.$$  \hspace{1cm} (6)

4. Discussion

It is easy to notice that the modification is in that the multiplier $\sqrt{\pi D_r}$ is not proportional now to the attenuation coefficient itself, as it was found for the propagation of the temperature wave from outside to inside of the cylinder, but its inverse value $1/\nu_r$, i.e. indeed, the allowance for cylindrical symmetry in this case leads to slowing down of attenuation. It should only be noted that to ensure the best coincidence of the curves with $D_r$, the numerical factor in the denominator must be assumed to be not 2, but $5/3$.

It is easy to see that here the theoretical solution, the form of which is obtained using the asymptotic approximation of the Hankel function, produces good results for $D_r > 4/\pi \approx 1.26$. At the same time, for the near-surface layer, only the dependence based on the approximation of numerical calculations provides more accurate and physically correct data. In addition, the general nature of temperature changes over the cross section of a cylindrical wall corresponds to the results of some other authors, for example, [11], and their analytical description found, also reveals analogies in theoretical solutions from a number of sources, in particular, [5].

To illustrate the practical use of ratio (6), we will calculate the temperature fluctuations in the thermal insulation layer of the air intake duct with parameters $r_1 = 0.25$ m, $r_0 = 0.4$ m, i.e. with 150 mm thick heat insulation, for the amplitude of temperature variations of outside air $A_{\text{ext}} = 2.7^\circ$C for
average conditions of the heating period. For thermal insulation made of mineral wool $\lambda = 0.044 \text{ W/(m·K)}$, $c = 840 \text{ J/(kg·K)}$, $\rho = 50 \text{ kg/m}^3$, therefore $a = 1.05\times10^{-6} \text{ m}^2$/s. Since we are primarily concerned in the temperature on the outer surface of the thermal insulation $t_s$ in terms of assessing the possibility of water vapor condensation, then $r' = r_0 - r_1 = 0.15 \text{ m}$; and with a daily period of fluctuations using the formula (3), $D_r = 1.25$, i.e. approximately at the limit of applicability (6), whence from expression $1/\nu_r = 0.49$ and, consequently, the amplitude of $t_s$ value variations is equal to $A_{t_\text{ex}}/\nu_r = 2.7/0.49 = 1.33^\circ \text{C}$. This means that when checking the sufficiency of the adopted insulation thickness, it will be necessary to take into account that the minimum daily $t_s$ value will be lower than the average by 1.33 degrees. As a comparison, if we use the original ratio (3) without adjustment, under the same conditions we obtain $\nu_r = 2.48$, whence $A_{t_\text{ex}}/\nu_r = 2.7/2.48 = 1.08^\circ \text{C}$, i.e. significantly less.

If the periodic heat source has a very small diameter compared to the area of space under consideration, for example, when it comes to heating or heating supply pipelines laid in concrete or soil, for periodic changes in the temperature of the heating water, as well as heating electrical cables embedded in concrete panels, the solution takes a slightly different shape. In work [20], the author, on the basis of numerical methods used for solving the initial differential thermal conductivity equation proved that in this case the following expression can be used:

$$\nu_r = \exp\left(\frac{D_r^{4/3}}{4}\right).$$

(7)

The parameter $D_r$ enters here already in the power of 4/3, i.e. in this case, the dependency is more complicated. In work [12], an assumption was made that solution (7) is actually accurate, because under cylindrical symmetry conditions, the propagation and, therefore, attenuation of the temperature wave occurs in two dimensions instead of one for a flat wall, and therefore $D_r$ value must be present to the extent of $4/3 = 1 + 1/3$, where 1/3 is a component associated with the appearance of an additional dimension.

To verify this hypothesis, a comparison was made with the results of an analytical solution obtained using the well-known relationship for the temperature field at an instantaneous linear heat source with intensity $q_l$, J/m, at the time moment $\tau$, c [1]:

$$t(r, \tau) = \frac{q_l}{4\pi\lambda\tau} \exp\left(-\frac{r^2}{4\alpha\tau}\right).$$

(8)

Since in our case the source is, however, continuously acting with the variable $q_l$ value, which, obviously, can be represented as $A_0\cos(\omega \tau)$, it is necessary to integrate (8) in the range from 0 to $\tau$ taking into account the fact that actually, setting one of the boundary conditions of the expression for $q_l$ is not quite equivalent to the change in the temperature of the source that we are primarily interested in. Since the corresponding integral in elementary functions is not used, this can be done by numerical methods. The results are shown in figure 4. It is easy to see that the hypothesis for the accuracy of the solution (7) is confirmed with a sufficient degree of veracity.
Figure 4. Attenuation of the temperature wave from a linear source on the cylinder axis (the solid line – numerical calculation [20], the dotted line – the integration result (8)).

To show the possibility of formula (7) practical application, we will calculate the temperature fluctuations in the layer of 50 mm thick floor heating slab, with the amplitude of temperature fluctuations on the surface of the heating cable $A_t = 1^\circ C$. For the structure of a slab made of cement-sand mortar $\lambda = 0.93$ W/(m·K), $c = 840$ J/(kg·K), $\rho = 1700$ kg/m$^3$, therefore $a = 6.5 \cdot 10^{-7}$ m$^2$/s. Since we are primarily interested in the temperature on the outer surface of the floor $t_s$ in terms of assessing the room comfort level and the safety of human life, then if we assume that the cable is in the lower part of the slab, $r' = 0.05$ m; and for a daily period of fluctuations according to the formula (7) $D_r = 0.52$, whence from the expression (7) $1/\nu_r = 0.9$ and, therefore, the amplitude of $t_s$ fluctuations is equal to $A_{t_s}/\nu_r = 1\cdot 0.9 = 0.9^\circ C$, i.e. almost no different from $A_t$.

5. Conclusion

- It is proved that for any direction of the temperature wave in the cylindrical layer, the theoretical solution, the shape of which is obtained using the asymptotic approximation of the Hankel function, produces good results for $D_r > 4/\pi \approx 1.26$ provided that the numerical coefficients in the solution are adjusted.
- It is noted that the propagation of the temperature wave in the cylindrical wall from the inside to the outside, generally follows the same regularities as with its opposite direction, but with a somewhat different attenuation coefficient.
- It is proved that, in contrast to external temperature waves, a slower decrease in amplitude with increasing distance from the axis is typical for internal waves – the decrease which is due to an exponential factor is partially compensated by a factor proportional to the square root of the radial coordinate.
It was found that for the near-surface layer, more accurate and physically correct data on the attenuation of the temperature wave is provided only by the dependence based on the approximation of numerical calculations.

It is proved that the attenuation of the temperature wave around a variable linear source based on the results of an analytical and numerical solution coincides within the accuracy limits of engineering calculation, and therefore, the obtained dependences are correct.

It was proposed to apply the ratios obtained in the work for analytical assessment of the amplitude of temperature fluctuations on the surfaces of cylindrical heated and cooled structures, primarily to decide on the start of condensation on the inner surface of chimneys when the load of the boiler varies or on the outer surface of heat insulation of heating conduits, on the surfaces open laid insulated air ducts with a daily change in the outside temperature and on the outer surface of cylindrical furnaces operating in alternating modes, and to assess the temperature fluctuations on the surface of the structure heated by internal heat sources, such as piping or electric cables operating in alternating mode, which will allow to use not only software but also the engineering methods of industrial safety compliance verification.

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