DIFFERENTIAL GALOIS THEORY AND INTEGRABILITY

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Abstract

This paper is an overview of our works which are related to investigations of the integrability of natural Hamiltonian systems with homogeneous potentials and Newton’s equations with homogeneous velocity independent forces. The two types of integrability obstructions for these systems are presented. The first, local ones, are related to the analysis of the differential Galois group of variational equations along a non-equilibrium particular solution. The second, global ones, are obtained from the simultaneous analysis of variational equations related to all particular solutions belonging to a certain class. The marriage of these two types of the integrability obstructions enables to realise the classification programme of all integrable homogeneous systems. The main steps of the integrability analysis for systems with two and more degrees of freedom as well as new integrable systems are shown.

keywords: integrability; differential Galois theory; Hamiltonian systems.

1 Introduction

It is hard to imagine yourself a world without integrable models. Teaching in such a world would be rather frustrating. If a theory has no solvable examples, it is difficult to explain that it is useful. Fortunately, in ours we have the harmonic oscillator—the guinea-pig which serves as pedagogical example for all theories.

Integrable models are exceptional, but we do not neglect them. Still, as it was one century ago, finding a new non-trivially integrable model is a discovery. How we can find integrable systems? One way, which seems to be the most natural, is just a search them in the nature. That is, start with more or less general model, and look for some special cases when the model is integrable. The other way is to construct integrable systems. It appears that the first approach is much more difficult than the second one. The reason of this is obvious: we know only few general methods which give strongly enough, and computable necessary conditions for the integrability.

In this paper we consider only classical dynamical systems which are described by ordinary differential equations. There is no a unique definiton of such systems and there is no a unique method for study of their integrability. Nevertheless, in this paper we will concentrate mainly on applications of only one quite new theory. It was developed by Baider, Churchill, Morales, Ramis, Rod, Simó and Singer in the end of the XX century, see [3, 5, 32] and references therein. In the context of Hamiltonian systems it is called the Morales-Ramis
theory. It arose from very long searching for relations between the branching of solutions of differential equations considered as functions of complex time, and the integrability. In the context of Hamiltonian systems these relations were found by S. L. Ziglin \([50, 51]\). His elegant theory expresses necessary conditions for the integrability in terms of properties of the monodromy group of variational equations along a particular solution. The Morales-Ramis theory, in some sense, is an algebraic version of the Ziglin theory. It formulates the necessary conditions for the integrability in terms of the differential Galois group of the variational equations.

During one and half decade the Morales-Ramis theory was applied successfully for study the integrability of numerous systems, see an overview paper \([35]\). Let us mention two biggest successes of this theory. It was applied to prove the non-integrability of the planar three body problem \([12, 47, 48, 49]\), and to prove the non-integrability of the Hill lunar problem \([37]\). Moreover, as it is well known, the first proof of the fact that the problem of the heavy top is integrable only in the classical cases was done by S. L. Ziglin in \([51]\) and it is based on his theory. An alternative proof based on differential Galois approach is given in \([28]\).

The above examples show the real power of the differential Galois methods in a study of the integrability. During last ten years we applied them to study several Hamiltonian and non-Hamiltonian systems which appear in physics and astronomy, see, e.g., \([6, 20, 21, 22, 23, 25, 26, 29, 31, 32, 44]\). Always a hidden dream of those investigations was a strong will to find an unexpectedly integrable system. However, for a long time, neither we, nor other authors succeeded with this respect. All those successful investigations gave negative results: the investigated systems are not integrable except already known integrable cases.

In this paper our aim is to give an overview of our works concerning the integrability of Hamiltonian systems with homogeneous potentials \([24, 27, 41, 42, 43]\), and homogeneous Newton equations \([41]\). Our motivation for those works was an optimistic plan to find the necessary and sufficient conditions for the integrability. In other words, our dream was to find all possible integrable polynomial potentials and forces.

Amazingly enough this plan was not only a dream—we found its quite successful realisation. Our investigations differ in many respects from typical applications of the Morales-Ramis theory. In a ‘typical’ application of this theory there are two difficult problems: the first one is to find a particular solution, and the second is to determine the differential Galois group of the variational equations along this solution. In this paper we investigate ‘global’ multi-parameter problems. We know \(a\) priori a class of particular solutions and we know also the differential Galois group for a given values of the parameters. The goal in our problem is to prove that the system is not integrable for all but finite number of parameters’ values. To achieve the desired result we developed a method which allows to deduce new obstructions for the integrability that come from a global analysis of all possible particular solutions of a given class.

Plan of this paper is the following. In the next section we overview various notions of the integrability of ordinary differential equations. In Section 3 we explain how the differential Galois theory can be used for a study of their integrability. The Morales-Ramis theory, as well as the Ziglin theory are dedicated for Hamiltonian systems. In our presentation we show that, in fact, we can use the differential Galois methods in a general context. One can find necessary conditions for the integrability defined adequately to the geometry of the considered equations and express these conditions in terms of properties of the differential Galois group of variational equations. The next two sections 4 and 5 are devoted the integrability analysis of the class of homogeneous potentials. The homogeneity of the potential guarantees the existence of non-equilibrium particular solutions.
generated by so-called Darboux points. In Section 4 the necessary integrability conditions obtained from an application of the Morales-Ramis theory to a particular solution given by a Darboux point are formulated. But these conditions are too weak for the ambitious classification programme of all integrable potentials. In Section 5 the additional integrability conditions obtained from the simultaneous analysis of all Darboux points are formulated. In Section 6 it was shown that various parts of analysis made for homogeneous potentials can be adapted for systems of Newton equations with homogeneous velocity-independent forces. In final Section 7 open problems and perspectives of the classification program of integrable systems are discussed.

2 Integrability

In this section we overview various notions of the integrability of systems of ordinary differential equations.

Let us consider a system of ordinary differential equations

\[ \frac{d}{dt} x = v(x), \quad x = (x^1, \ldots, x^m) \in \mathbb{R}^m, \quad (2.1) \]

with smooth right-hand sides \( v(x) = (v^1(x), \ldots, v^m(x)) \). As it was observed a long time ago, the knowledge of first integrals or other invariant quantities helps to find explicit solutions of this system. Let us recall that a smooth function \( F \) is a first integral of system (2.1) iff it is constant along its solutions. Thus, a constant value level of \( F \) is a set invariant with respect to the flow of (2.1). Hence, roughly speaking, knowing a first integral we can reduce the dimension of the system by one. Thus, if we know \( m - 1 \) first integrals \( F_1, \ldots, F_{m-1} \), which are functionally independent in a certain domain \( U \subset \mathbb{R}^m \), then we can transform system (2.1) into the following one

\[ \frac{d}{dt} y = w(y), \quad w(y) = (0, \ldots, 0, w^m(y)). \quad (2.2) \]

Solutions of this equation can be find easily

\[ y^i(t) := y^i_0 \quad \text{for} \quad i = 1, \ldots, m - 1, \quad (2.3) \]

and \( y^m(t) \) is given by

\[ \int_{y^m_0}^{y^m(t)} \frac{ds}{w^m(y^1_0, \ldots, y^{m-1}_0, s)} = t. \quad (2.4) \]

In effect, we reduce the problem to calculation just one integral and inversion of a function. In the prescribed situation, we say that the considered system is \textit{integrable by quadratures}. This notion plays a fundamental role. In fact, as we will see later, the other definitions of integrability give necessary conditions for the integrability by quadratures.

A first integral of a given system is just an example of simplest tensor quantity which is invariant with respect to the flow of the system. A smooth tensor field \( T(x) \) is invariant with respect to the flow of the system (2.1) iff

\[ L_v(T) = 0, \quad (2.5) \]

where \( L_v \) denotes the Lie derivative along vector field \( v \). The existence of a certain number of tensor invariants can guarantee that the integration of the system reduces to quadratures. Below we give three examples of results of this type. The first of them, due to S. Lie, tells that system admitting \( m \) linearly independent and commuting symmetries, i.e., invariant vector fields, is integrable by quadratures.
Theorem 2.1 (S. Lie). Assume that system (2.1) admits \( m \) independent and commuting symmetries \( u_1 = v, u_2, \ldots, u_m \). Then the system is integrable by quadratures.

A differential \( m \)-form \( \omega \) in \( \mathbb{R}^m \) is given by

\[
\omega = h(x) \, dx^1 \wedge \cdots \wedge dx^m. \tag{2.6}
\]

It is invariant with respect to system (2.1) iff

\[
L_v(\omega) = \left( \sum_{i=1}^m \frac{\partial (hv_i)}{\partial x^i} \right) dx^1 \wedge \cdots \wedge dx^m = 0.
\]

In the classical literature an invariant \( m \)-form is called the Jacobi last multiplier.

Theorem 2.2 (L. Euler, C. G. J. Jacobi). Assume that system (2.1) admits \( m - 2 \) functionally independent first integrals and invariant differential \( m \)-form. Then the system is integrable by quadratures.

Proofs and detailed discussion of the above two theorems can be find, e.g., in [19; 45].

As we will see later, the following theorem, due to O. I. Bogoyavlensky [10; 11], is very important in the context of applications of differential Galois method to the integrability of non-Hamiltonian systems.

Theorem 2.3 (O. I. Bogoyavlensky). Assume that system (2.1) admits \( 1 \leq k < m \) functionally independent first integrals \( F_1, \ldots, F_k \), and \( m - k \) linearly independent and commuting symmetries \( u_1 = v, u_2, \ldots, u_{m-k} \), such that

\[
L_{u_j}(F_i) = u_j[F_i] = 0, \quad \text{for} \quad 1 \leq i \leq k; \quad 1 \leq j \leq m - k. \tag{2.7}
\]

Then the system is integrable by quadratures.

Assumptions of the above theorems can serve as a source of definitions for specific types of the integrability. Thus, for example, we have the following definition which is frequently used in non-holonomic mechanics.

Definition 2.4. We say that system (2.1) is integrable in the Jacobi sense iff it admits \( m - 2 \) independent first integrals and invariant differential \( m \)-form.

In a similar way, we can take the assumptions of Theorem 2.3 as a basis for the definition of \( B \)-integrability.

Definition 2.5. We say that system (2.1) is \( B \)-integrable iff it admits \( 1 \leq k \leq m \) functionally independent first integrals \( F_1, \ldots, F_k \), and \( m - k \) linearly independent and commuting symmetries \( u_1 = v, \ldots, u_{m-k} \), such that \( u_j[F_i] = 0 \) for \( 1 \leq i \leq k, \ 1 \leq j \leq m - k \).

The above definition arises from a careful analysis of the concept of the integrability of Hamiltonian systems in the Liouville sense (see below).

To describe shortly the most characteristic features of \( B \)-integrable systems let us consider such a system, and let us assume that it admits functionally independent first integrals \( F_1, \ldots, F_k \). With these integrals we associate the momentum map

\[
\mathbb{R}^m \ni x \mapsto \mathbf{F}(x) := (F_1(x), \ldots, F_k(x)) \in \mathbb{R}^k. \tag{2.8}
\]

Let us consider a common level of the first integrals

\[
M_f := \mathbf{F}^{-1}(f) = \{ x \in \mathbb{R}^m \mid \mathbf{F}(x) = f \}, \tag{2.9}
\]
where \( f \in \mathbb{R}^k \). If first integrals are independent on \( M_f \), then \( M_f \) is a smooth manifold. But, even if \( M_f \) is connected and compact, its topology can be very complicated. However, by the assumed \( B \)-integrability, we know that there exist independent and commuting vector fields \( u_1, \ldots, u_{m-k} \) tangent to \( M_f \). Thus, if \( M_f \) is connected and compact it is diffeomorphic to \( m-k \) dimensional torus \( T^{m-k} \), see Chapter 10, Lemma 2 in [2]. In a neighbourhood of \( M_f \) we can introduce local coordinates \((\varphi, I)\), where \( I \in D \subset \mathbb{R}^k \), and \( \varphi = (\varphi_1, \ldots, \varphi_{m-k}) \) are angular coordinates on \( T^{m-k} \). In these coordinates system (2.1) reads

\[
\frac{d}{dt} \varphi = \omega(I), \quad \frac{d}{dt} I = 0. \tag{2.10}
\]

Thus, a solution of this system has the form

\[
I(t) := I_0 \quad \varphi(t) := \omega(I_0)t + \varphi_0. \tag{2.11}
\]

From the above considerations it follows that \( B \)-integrability is similar to the integrability in the Liouville sense of Hamiltonian systems. In fact, the \( B \)-integrability was introduced as a certain generalisation of the integrability in the Liouville sense.

Let us assume that system (2.1) is Hamiltonian. Then \( m = 2n \), and \( v = X_H \) is a Hamiltonian vector field generated by a smooth Hamiltonian function \( H : \mathbb{R}^{2n} \to \mathbb{R} \). Here we consider \( \mathbb{R}^{2n} \) as a linear symplectic space with chosen canonical coordinates \( x = (q, p) \), where \( q = (q_1, \ldots, q_n) \) and \( p = (p_1, \ldots, p_n) \). In these coordinates the symplectic form \( \Omega \) is following

\[
\Omega = \sum_{i=1}^{n} dq_i \wedge dp_i,
\]

and the vector field \( X_H \) reads

\[
X_H = \sum_{i=1}^{n} \left( \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} \right).
\]

Let us recall the definition of the well known notion.

**Definition 2.6.** We say that Hamiltonian vector field \( X_H \) is integrable in the Liouville sense iff it admits \( n \) functionally independent and commuting smooth first integrals \( F_1, \ldots, F_n \).

We notice here that a Hamiltonian vector field \( X_H \) integrable in the Liouville sense is \( B \)-integrable. In fact, \( X_H \) admits \( n \) first integrals, and \( n = 2n - n \) symmetries \( X_{F_1}, \ldots, X_{F_n} \), which satisfy

\[
X_{F_i[F_j]} = \{F_i, F_j\} = 0 \quad \text{for} \quad 1 \leq i, j \leq n, \tag{2.12}
\]

where \( \{\cdot, \cdot\} \) denotes the Poisson bracket.

Obviously, a \( B \)-integrable Hamiltonian system can be non-integrable in the Liouville sense.

### 3 General theory

In this section we show how we can use the differential Galois theory to find necessary conditions for the integrability of ordinary differential equations. To deduce such conditions we have to make several assumptions. The most general one is that the considered system as well as the considered first integrals, or other invariants, have ‘good’ analytical properties. Moreover, the theory requires that the ‘scalars’ form an algebraically closed
field. Thus, we assume that this field is just the field of complex numbers \( \mathbb{C} \). In effect we work with complex functions, complex vector fields, and so on. The above mentioned ‘good’ analytical properties mean that the considered tensors are holomorphic at points where they are defined.

Let us consider a complex holomorphic system of ordinary differential equations

\[
\frac{d}{dt} x = v(x), \quad x \in U \subset \mathbb{C}^m, \quad t \in \mathbb{C},
\]

where \( U \) is an open and connected subset of \( \mathbb{C}^m \). The basic assumption for further considerations is that we know a particular non-equilibrium solution \( \varphi(t) \) of this system. Usually it is not a single-valued function of the complex time \( t \). Thus, we associate with \( \varphi(t) \) a Riemann surface \( \Gamma \) with \( t \) as a local coordinate.

The variational equations along \( \Gamma \) have the form

\[
\frac{d}{dt} \xi = A(t) \xi, \quad A(t) = \frac{\partial v}{\partial x}(\varphi(t)), \quad \xi \in T_\Gamma U.
\]

The entries of matrix \( A(t) \) in the above equation are elements of field \( K := \mathcal{M}(\Gamma) \) of functions meromorphic on \( \Gamma \). This field with the differentiation with respect to \( t \) as a derivation is a differential field. Only constant functions from \( K \) have vanishing derivative, so the sub-field of constants of \( K \) is \( \mathbb{C} \).

It is obvious that solutions of (3.2) are not necessarily elements of \( K^m \). The fundamental theorem of the differential Galois theory guarantees that there exists a differential field \( L \supset K \) such that \( m \) linearly independent (over \( \mathbb{C} \)) solutions of (3.2) are contained in \( L^m \). The smallest differential extension \( L \supset K \) with this property is called the Picard-Vessiot extension of \( K \).

A group \( \mathcal{G} \) of differential automorphisms of \( L \) which do not change \( K \) is called the differential Galois group of equation (3.2). It can be shown that \( \mathcal{G} \) is a linear algebraic group. Thus, in particular, it is a union of a finite number of disjoint connected components. One of them, containing the identity, is called the identity component and is denoted by \( \mathcal{G}^0 \).

Let \( \xi = (\xi^1, \ldots, \xi^m)^T \in L^m \) be a solution of equation (3.2), and \( g \) an element of its differential Galois group \( \mathcal{G} \). Then, \( g(\xi) := (g(\xi^1), \ldots, g(\xi^m))^T \) is also its solution. In fact, by definition \( g \) commutes with the time differentiation, so we have

\[
\frac{d}{dt} g(\xi) = g \left( \frac{d}{dt} x \right) = g(A(t)\xi) = g(A(t))g(\xi) = A(t)g(\xi),
\]

as \( g \) does not change elements of \( K \). Thus, if \( \Xi \in \text{GL}(n, L) \) is a fundamental matrix of (3.2), i.e., its columns are linearly independent solutions of (3.2), then \( g(\Xi) = \Xi M_g \), where \( M_g \in \text{GL}(n, \mathbb{C}) \). In other words, the differential Galois group \( \mathcal{G} \) can be considered as an algebraic subgroup of \( \text{GL}(m, \mathbb{C}) \).

Now, we explain how the existence of first integrals of system (3.1) manifests itself in the properties of the differential Galois group of variational equations. At first, we introduce some definitions. Let us consider a holomorphic function \( F \) defined in a certain connected neighbourhood of solution \( \varphi(t) \). In this neighbourhood we have the expansion

\[
F(\varphi(t) + \xi) = F_l(\xi) + O(\|\xi\|^{l+1}), \quad F_l \neq 0.
\]

Then, the leading term \( f \) of \( F \) is the lowest order term of the above expansion i.e., \( f(\xi) := F_l(\xi) \). Note that \( f(\xi) \) is a homogeneous polynomial of variables \( \xi = (\xi^1, \ldots, \xi^m) \) of degree \( l \). If \( F \) is a meromorphic function, then it can be written as \( F = P/Q \) for certain holomorphic
functions $P$ and $Q$. Then the leading term $f$ of $F$ is defined as $f = p/q$, where $p$ and $q$ are leading terms of $P$ and $Q$, respectively. In this case $f(\xi)$ is a homogeneous rational function of $\xi$.

It is easy to prove that if $F$ is a meromorphic (holomorphic) first integral of equation (3.1), then its leading term $f$ is a rational (polynomial) first integral of variational equation (3.2). If system (3.1) has $k \geq 2$ functionally independent meromorphic first integrals $F_1, \ldots, F_k$, then their leading terms can be functionally dependent. However, by the Ziglin Lemma [3; 5; 50], we can find $k$ polynomials $G_1, \ldots, G_k \in \mathbb{C}[z_1, \ldots, z_k]$ such that leading terms of $G_i(F_1, \ldots, F_k)$, for $1 \leq i \leq k$ are functionally independent.

Additionally, if $\mathcal{S} \subset \text{GL}(m, \mathbb{C})$ is the differential Galois group of (3.2), and $f$ is its rational first integral, then $f(g(\xi)) = f(\xi)$ for every $g \in \mathcal{S}$, see [3; 32], i.e., $f$ is a rational invariant of group $\mathcal{S}$. Thus we have a correspondence between the first integrals of the system (3.1) and invariants of $\mathcal{S}$.

**Lemma 3.1.** If equation (3.1) has $k$ functionally independent first integrals which are meromorphic in a connected neighbourhood of a non-equilibrium solution $\varphi(t)$, then the differential Galois group $\mathcal{S}$ of the variational equations along $\varphi(t)$ has $k$ functionally independent rational invariants.

As it was mentioned above, a differential Galois group is a linear algebraic group, thus, in particular, it is a Lie group, and one can consider its Lie algebra. This Lie algebra reflects only the properties of the identity component of the group. It is easy to show that if a Lie group has an invariant, then also its Lie algebra has an integral. Let us explain what the last sentence means. Let $\mathfrak{g} \subset \text{gl}(m, \mathbb{C})$ denote the Lie algebra of $\mathcal{S}$. Then, an element $Y \in \mathfrak{g} \subset \text{gl}(m, \mathbb{C})$ can be considered as a linear vector field: $x \mapsto Y(x) := Y \cdot x$, for $x \in \mathbb{C}^m$. We say that $f \in \mathbb{C}(x_1, \ldots, x_m)$ is an integral of $\mathfrak{g}$, iff $Y(f)(x) = df(x) \cdot Y(x) = 0$, for all $Y \in \mathfrak{g}$.

**Proposition 3.2.** If $f_1, \ldots, f_k \in \mathbb{C}(x_1, \ldots, x_m)$ are algebraically independent invariants of an algebraic group $\mathcal{S} \subset \text{GL}(m, \mathbb{C})$, then they are algebraically independent first integrals of the Lie algebra $\mathfrak{g}$ of $\mathcal{S}$.

The above facts are the starting points for applications of differential Galois methods to a study of the integrability.

If the considered system is Hamiltonian, then we have additional constrains. First of all, the differential Galois group of variational equations is a subgroup of the symplectic group $\text{Sp}(2n, \mathbb{C})$. Secondly, commutation of first integrals imposed by the Liouville integrability implies commutation of first integrals of variational equations. The following lemma plays the crucial role and this is why it was called The Key Lemma, see Lemma III.3.7 on page 72 in [3].

**Lemma 3.3.** Assume that Lie algebra $\mathfrak{g} \subset \text{sp}(2n, \mathbb{C})$ admits $n$ functionally independent commuting first integrals. Then $\mathfrak{g}$ is Abelian.

Hence, if $\mathfrak{g}$ in the above lemma is the Lie algebra of a Lie group $\mathcal{S}$, then the identity component $\mathcal{S}^0$ of $\mathcal{S}$ is Abelian.

Using all these facts Morales and Ramis proved the following theorem [32; 33].

**Theorem 3.4 (Morales-Ramis).** Assume that a Hamiltonian system is meromorphically integrable in the Liouville sense in a connected neighbourhood of a phase curve $\Gamma$. Then the identity component of the differential Galois group of the variational equations along $\Gamma$ is Abelian.
We have

\[ H = \sum_{i=1}^{m} y_i v^i(x). \]  

(3.4)

where \( (x, y) = (x^1, \ldots, x^m, y_1, \ldots, y_m) \) are canonical coordinates in \( \mathbb{C}^{2m} \). Thus, the Hamiltonian equations have the form

\[ \frac{d}{dt} x^i = \frac{\partial H}{\partial y_i} = v^i(x), \quad \frac{d}{dt} y_i = -\frac{\partial H}{\partial x^i} = -\sum_{j=1}^{m} y_j \frac{\partial v^i}{\partial x^j}(x), \quad 1 \leq i \leq m. \]  

(3.5)

Let us assume that system (3.1) is \( B \)-integrable with \( k \) first integrals \( F_1, \ldots, F_k \), and \( m - k \) symmetries \( u_1, \ldots, u_{m-k} \). Then, we claim that Hamiltonian system (3.5) is integrable in the Liouville sense. Let us define the following functions

\[ F_{k+j}(x, y) := \langle y, u_j(x) \rangle := \sum_{i=1}^{m} y_i u^i_j(x) \quad \text{for} \quad 1 \leq j \leq m - k. \]  

(3.6)

We have

\[ \{ F_{i+k}, H \} = \sum_{i=1}^{m} \left( \frac{\partial F_{i+k}}{\partial x^i} \frac{\partial H}{\partial y_i} - \frac{\partial F_{i+k}}{\partial y_i} \frac{\partial H}{\partial x^i} \right) = \langle y, [u_j, v] \rangle. \]  

(3.7)

But, \( [u_j, v] = 0 \) for \( 1 \leq j \leq m - k \), by assumption, and so \( F_{i+k} \) are first integrals. We show that first integrals \( F_1, \ldots, F_m \) pairwise commute. Obviously, we have \( \{ F_i, F_j \} = 0 \), for \( 1 \leq i, j \leq k \). Moreover, we have

\[ \{ F_{i+k}, F_{i+k} \} = \langle y, [u_j, u_i] \rangle = 0 \quad \text{for} \quad 1 \leq i, j \leq m - k, \]  

(3.8)

as, by assumption, \( [u_j, u_i] = 0 \). Finally, we have also

\[ \{ F_i, F_{j+k} \} = u_j[F_i] = 0, \quad \text{for} \quad 1 \leq i \leq k, \quad 1 \leq j \leq m - k, \]  

(3.9)

because, by assumption, \( F_1, \ldots, F_k \) are common first integrals of \( u_1, \ldots, u_{m-k} \). Thus, we proved our claim.

Now, let \( \varphi(t) = (\varphi^1(t), \ldots, \varphi^m(t)) \) be a particular solution of (3.1). Then,

\[ t \longmapsto \tilde{\varphi}(t) := (\varphi(t), 0) \in \mathbb{C}^{2m}, \]

is a particular solution of the Hamilton equation (3.5). The variational equations for this solution have the following form

\[ \frac{d}{dt} \xi = A(t) \xi, \quad \frac{d}{dt} \eta = -A(t)^T \eta, \quad A(t) = \frac{\partial \varphi}{\partial x}(\varphi(t)). \]  

(3.10)

The first of the above equations is just the variational equation (3.2), and the second one is its adjoint. Thus, if \( \Xi \) is a fundamental matrix of the first equation in (3.10), then \( X := (\Xi^{-1})^T \) is a fundamental matrix of the second equation in (3.10). In effect, the differential Galois group of system (3.10) coincides with the differential Galois group of its first equation, i.e., with the differential Galois group of the original variational equations (3.2). Using the above facts Ayoun and Zung proved in [4] the following theorem.
**Theorem 3.5** (Ayoul-Zhung). Assume that system (3.1) is meromorphically $B$-integrable in a connected neighbourhood of a phase curve $\Gamma$. Then the identity component of the differential Galois group of the variational equations along $\Gamma$ is Abelian.

Let us underline the importance of this theorem. All results which were obtained on the basis of Theorem 3.4 and stating that a given Hamilton system is non-integrable in the Liouville sense are, in fact, much stronger – the considered systems are not $B$-integrable.

Already in the book [32] a very natural extension of described approach was presented. Except the variational equations (3.2) along the phase curve $\phi(t)$, we can consider also the higher order variational equations. To derive them we consider system (3.1) in a neighbourhood of $\Gamma$, where we can write the following expansion

$$x = \phi(t) + \varepsilon \xi^{(1)} + \varepsilon^2 \xi^{(2)} + \cdots + \varepsilon^k \xi^{(k)} + \cdots,$$

where $\varepsilon$ is a formal small parameter. Inserting the above expansion into equation (3.1) and collecting terms of the same order with respect to $\varepsilon$, we obtain the following chain of equations

$$\frac{d}{dt} \xi^{(k)} = A(t) \xi^{(k)} + f_k(\xi^{(1)}, \ldots, \xi^{(k-1)}), \quad k = 1, 2, \ldots,$$

(3.11)

where $f_1 \equiv 0$. For a fixed $k$, this is a system of $k$-th order variational equations, and we denote it by $\text{VE}_k$. It is a linear and, for $k > 1$, non-homogeneous system of equations. Nevertheless, there exists an appropriate framework allowing to define the differential Galois group $G_k$ of $\text{VE}_k$ for any $k$. Obviously, $\text{VE}_1$ coincides with (3.2), so $G_1$ coincides with $G$. A detailed exposition and proofs the reader will find in [36], where among other things the following generalisation of Theorem 3.4 is given.

**Theorem 3.6** (Morales-Ramis-Simó). Assume that a Hamiltonian system is meromorphically integrable in the Liouville sense in a connected neighbourhood of a phase curve $\Gamma$. Then the identity component of the differential Galois group $G_k$ of $k$-th variational equations $\text{VE}_k$ along $\Gamma$ is Abelian, for all $k \in \mathbb{N}$.

Hence, Theorem 3.4 gives only the first order obstructions for the integrability. If $G^\circ_1 = G^\circ_0$ is Abelian, then, having only Theorem 3.4, we cannot be sure whether the system is integrable or not. But knowing the above theorem we can continue our investigations and check if the $G^\circ_2$ is Abelian. If it is not, the system is not integrable, otherwise we have to check if $G^\circ_3$ is Abelian. This process we continue up to such $k$ that $G^\circ_i$ is Abelian for $i < k$, and $G^\circ_k$ is not Abelian. If we are able to find such $k$, then the system is not integrable.

Here it is worth to mention that it is very hard to determine the differential Galois groups $G_k$ with $k > 1$, or even to decide whether $G^\circ_k$ is Abelian or not. This is why we have only a few applications of Theorem 3.6, see [35; 36]. However, all successful applications of this theorem show its real power.

We can derive higher order variational equations for an arbitrary system (3.1). Thus, we can ask if we have a generalisation of Theorem 3.5 similar to that described above for Theorem 3.6. In fact, we have such generalisation.

**Theorem 3.7** (Ayoul-Zhung). Assume that system (3.1) is meromorphically $B$-integrable in a connected neighbourhood of a phase curve $\Gamma$. Then the identity component $G^\circ_k$ of the differential Galois group $G_k$ of the $k$-th variational equations along $\Gamma$ is Abelian, for all $k \in \mathbb{N}$. 

9
4 Integrability of homogeneous potentials – Morales-Ramis theorem

Let us consider Hamiltonian systems with \( n \) degrees of freedom given by a natural Hamiltonian function

\[
H = \frac{1}{2} \sum_{i=1}^{n} p_i^2 + V(q)
\]  

(4.1)

where \( q = (q_1, \ldots, q_n) \) and \( p = (p_1, \ldots, p_n) \) are canonical coordinates and momenta, \( V \) is a homogeneous function of degree \( k \in \mathbb{Z}^* := \mathbb{Z} \setminus \{0\} \). We assume just from the beginning that the considered system is complex, i.e., the phase space is \( \mathbb{C}^{2n} \). The Hamilton equations have the canonical form

\[
\frac{d}{dt}q = p, \quad \frac{d}{dt}p = -V'(q),
\]

(4.2)

where \( V'(q) := \text{grad} \, V(q) \). Moreover, we assume also that the time \( t \) is a complex variable.

We say that a potential \( V \) is integrable iff the Hamiltonian system (4.2) is integrable.

One of the most beautiful applications of the Morales-Ramis Theorem 3.4 concerns Hamiltonian systems of the prescribed above form. The basic assumption in this application is that there exists a non-zero vector \( d \in \mathbb{C}^n \) such that

\[
V'(d) = d,
\]

(4.3)

It is is called a proper Darboux point of potential \( V \). It defines a two dimensional plane in the phase spaces \( \mathbb{C}^{2n} \), given by

\[
\Pi(d) := \{ (q, p) \in \mathbb{C}^{2n} \mid q = qd, \ p = \psi d, \ (q, \psi) \in \mathbb{C}^2 \}.
\]

(4.4)

This plane is invariant with respect to the system (4.2). Equations (4.2) restricted to \( \Pi(d) \) have the form of one degree of freedom Hamilton’s equations

\[
\frac{d}{dt}\varphi = \psi, \quad \frac{d}{dt}\psi = -\varphi^{k-1},
\]

(4.5)

with the following phase curves

\[
\Gamma_{k,\varepsilon} := \{ (\varphi, \psi) \in \mathbb{C}^2 \mid \frac{1}{2}\varphi^2 + \frac{1}{k}\varphi^k = \varepsilon \} \subset \mathbb{C}^2, \quad \varepsilon \in \mathbb{C}.
\]

(4.6)

In this way, a solution \( (\varphi, \psi) = (\varphi(t), \psi(t)) \) of (4.5) gives rise a solution \( (q(t), p(t)) := (\varphi d, \psi d) \) of equations (4.2) with the corresponding phase curve

\[
\Gamma_{k,\varepsilon} := \{ (q, p) \in \mathbb{C}^{2n} \mid (q, p) = (qd, \psi d), \ (q, \psi) \in \Gamma_{k,\varepsilon} \} \subset \Pi(d).
\]

(4.7)

Morales and Ramis obtained necessary conditions for the integrability in the Liouville sense by an analysis of the variational equations along an arbitrary phase curve \( \Gamma_{k,\varepsilon} \) with \( \varepsilon \neq 0 \). These variational equations have the form

\[
\ddot{x} = -\varphi(t)^k x,
\]

(4.8)

where \( \varphi(t)^k \) is the Hessian of \( V \) calculated at \( d \). Let us assume that \( \varphi(t)^k \) is diagonalisable. Then, without loss of the generality, we can assume that \( \varphi(t)^k \) is diagonal, and in such a case system (4.8) splits into a direct product of second order equations

\[
\ddot{x}_i = -\lambda_i \varphi(t)^k x_i, \quad 1 \leq i \leq n,
\]

(4.9)

where \( \lambda_1, \ldots, \lambda_n \) are eigenvalues of \( \varphi(t)^k \). It is easy to show that \( d \) is an eigenvector of \( \varphi(t)^k \) with eigenvalue \( k - 1 \). We always denote this eigenvalue as \( \lambda_n \).

In [34] J. J. Morales-Ruiz and J. P. Ramis proved the following theorem.
Theorem 4.1 (Morales-Ramis). Assume that the Hamiltonian system defined by Hamiltonian (4.1) with a homogeneous potential \( V \in \mathbb{C}(q) \) of degree \( k \in \mathbb{Z}^+ \) satisfies the following conditions:

1. there exists a non-zero \( d \in \mathbb{C}^n \) such that \( V'(d) = d \), and
2. matrix \( V''(d) \) is diagonalisable with eigenvalues \( \lambda_1, \ldots, \lambda_n \);
3. the system is integrable in the Liouville sense with first integrals which are meromorphic in a connected neighbourhood \( U \) of phase curve \( \Gamma_{k,\varepsilon} \) with \( \varepsilon \neq 0 \), and independent on \( U \setminus \Gamma_{k,\varepsilon} \).

Then each pair \((k, \lambda_i)\) for \( i = 1, \ldots, n \) belongs to an item of the following list

| case | \( k \) | \( \lambda \) |
|------|--------|-------------|
| 1.   | \( \pm 2 \) | arbitrary |
| 2.   | \( k \) | \( p + \frac{k}{2}p(p-1) \) |
| 3.   | \( k \) | \( \frac{1}{2} \left( \frac{k-1}{k} + p(p+1)k \right) \) |
| 4.   | 3 | \(-\frac{1}{24} + \frac{1}{6} (1+3p)^2, \quad -\frac{1}{24} + \frac{3}{50} (1+5p)^2 \) |
|      |     | \(-\frac{1}{24} + \frac{3}{50} (1+5p)^2, \quad -\frac{1}{24} + \frac{3}{50} (2+5p)^2 \) |
| 5.   | 4 | \(-\frac{1}{8} + \frac{2}{9} (1+3p)^2 \) |
| 6.   | 5 | \(-\frac{9}{40} + \frac{5}{18} (1+3p)^2, \quad -\frac{9}{40} + \frac{1}{10} (2+5p)^2 \) |
| 7.   | -3 | \(-\frac{25}{24} - \frac{1}{6} (1+3p)^2, \quad -\frac{25}{24} - \frac{3}{32} (1+4p)^2 \) |
|      |     | \(-\frac{25}{24} - \frac{3}{50} (1+5p)^2, \quad -\frac{25}{24} - \frac{3}{50} (2+5p)^2 \) |
| 8.   | -4 | \(-\frac{9}{8} - \frac{2}{9} (1+3p)^2 \) |
| 9.   | -5 | \(-\frac{49}{40} - \frac{5}{18} (1+3p)^2, \quad -\frac{49}{40} - \frac{1}{10} (2+5p)^2 \) |

(4.10)

where \( p \) is an integer.

Let us remark that the above theorem does not give any obstruction for the integrability if \( k = 2 \) or \( k = -2 \).

Remark 4.2 It was explained in [14] that the assumption that \( V''(d) \) is diagonalisable is irrelevant. That is, the necessary conditions for the integrability are the same: if the potential is integrable, then each \( \lambda \in \text{spectr } V''(d) \) must belong to appropriate items of the above list. Additionally, if \( V''(d) \) is not diagonalisable, then new obstacles for the integrability appear. Namely, if the Jordan form of \( V''(d) \) has a block

\[
J_3(\lambda) := \begin{bmatrix}
\lambda & 1 & 0 \\
0 & \lambda & 1 \\
0 & 0 & \lambda \\
\end{bmatrix},
\]

then the system is not integrable. Moreover, if the Jordan form of \( V''(d) \) has a two dimensional block \( J_2(\lambda) \), and \( \lambda \) belongs to the second item of table (4.10), then the system is not integrable. This
fact was proved in [14]. In other words, for \( k \notin \{-2, 0, +2\} \), the presence of a proper Darboux point \( d \) for which \( V''(d) \) has a block of dimension greater than two, or block of dimension two with corresponding \( \lambda_i \) belonging to the second item of table (4.10), implies immediately the non-integrability of the potential.

**Remark 4.3** The case of a homogeneous potential of degree \( k = 0 \) needs a special treatment. Necessary conditions for the integrability in this case were found recently in [13].

We denote by \( M_k \) a subset of rational numbers \( \lambda \) specified by the table in the above theorem for a given \( k \), e.g., for \(|k| > 5\), we have

\[
M_k = \left\{ p + \frac{k}{2} p(p-1) \mid p \in \mathbb{Z} \right\} \cup \left\{ \frac{1}{2} \left[ \frac{k-1}{k} + p(p+1)k \right] \mid p \in \mathbb{Z} \right\}.
\]

(4.11)

The Morales-Ramis Theorem 4.1 is a powerful tool. In fact, the necessary conditions for the integrability of polynomial potentials are reduced to solving algebraic equations: we have to find a Darboux point and then to check if the eigenvalues of the Hessian at this Darboux point are rational numbers which belong to the Morales-Ramis table. However, we note that none of these algebraic problems is trivial. First of all, even if a Darboux point are rational numbers which belong to the Morales-Ramis table. However, we have much more serious problems in a parametric case which is the most important in applications. This is illustrated by the following example.

Let us consider the following potential

\[
V = \frac{1}{3}aq_1^3 + \frac{1}{2}q_1^2q_2 + \frac{1}{3}cq_2^3,
\]

(4.12)

where \( a \) and \( c \) are in general complex parameters. For generic values of these parameters \( V \) has three Darboux points \( d_1, d_2, \) and \( d_3 \). The non-trivial eigenvalues \( \lambda_i = \text{Tr} V''(d_i) - 2 \) of Hessian \( V''(q) \) at these Darboux points are following

\[
\lambda_1 = \frac{1}{c}, \quad \lambda_2 = \frac{2c - 1}{1 + a^2 + \Delta}, \quad \lambda_3 = \frac{2c - 1}{1 + a^2 - \Delta},
\]

(4.13)

where

\[
\Delta = \sqrt{a^2(2 + a^2 - 2c)}.
\]

If potential \( V \) is integrable, then by Theorem 4.1 we have

\[
\lambda_i \in M_3 = \bigcup_{j=1}^{6} M^{(j)}, \quad \text{for} \quad 1 \leq i \leq 3,
\]

(4.14)

where

\[
M^{(1)} := \left\{ p + \frac{3}{2} p(p-1) \mid p \in \mathbb{Z} \right\}, \quad M^{(2)} := \left\{ \frac{1}{2} \left( \frac{2}{3} + 3p(p+1) \right) \mid p \in \mathbb{Z} \right\},
\]

\[
M^{(3)} := \left\{ \frac{1}{6} (1 + 3p)^2 - \frac{1}{24} \mid p \in \mathbb{Z} \right\}, \quad M^{(4)} := \left\{ \frac{3}{32} (1 + 4p)^2 - \frac{1}{24} \mid p \in \mathbb{Z} \right\},
\]

\[
M^{(5)} := \left\{ \frac{3}{50} (1 + 5p)^2 - \frac{1}{24} \mid p \in \mathbb{Z} \right\}, \quad M^{(6)} := \left\{ \frac{3}{50} (2 + 5p)^2 - \frac{1}{24} \mid p \in \mathbb{Z} \right\}.
\]

From Eq. (4.13) we find that

\[
c = \frac{1}{\lambda_1}, \quad a = \frac{\lambda_1 + \lambda_1\lambda_i - 2}{\sqrt{2\lambda_1\lambda_i(2 - \lambda_1 - \lambda_i)}}, \quad \text{for} \quad i \in \{2, 3\}.
\]
Hence, for arbitrary $\lambda_1, \lambda_2, \lambda_3 \in \mathcal{M}_3$, the above defined values of $a$ and $c$ give potential (4.12) which satisfies the necessary conditions for the integrability.

Theorem 4.1 gives only necessary conditions for the integrability, and there are many examples that they are not sufficient. Thus in the above example, we have to check whether infinitely many potentials are integrable or not.

5 Integrability of homogeneous polynomial potentials. Global analysis

In this section we assume that the considered potential $V(q)$ is polynomial and homogeneous of degree $k > 2$. The set of homogeneous polynomials in $n$ variables $q = (q_1, \ldots, q_n)$ of degree $k$, we denote by $\mathbb{C}_k[q]$.

During last few years we worked on the following problem. Is it possible, for a given $k > 2$ and $n > 2$, to distinguish all meromorphically integrable potentials $V \in \mathbb{C}_k[q]$? In other words, is it possible to give a necessary and sufficient conditions for the integrability of homogeneous polynomial potentials? The example given in the end of the previous section shows that, except Theorem 4.1, we need a result which gives stronger necessary conditions. This example shows that even for fixed $k$ and $n$, Theorem 4.1 distinguishes infinitely many parameters’ values for which the potential can be integrable.

5.1 Darboux points

It is clear that the more Darboux points of given potential we know, the more obstructions for its integrability we obtain from the Morales-Ramis Theorem 4.1. Hence, we have to know how many Darboux points a polynomial potential of a given degree can have. An analysis of this and similar problems related to particular solutions of Hamiltonian systems with homogeneous potentials forced us to give a more geometrical definition of Darboux points.

Let $V$ be a homogeneous polynomial potential of degree $k > 2$, i.e., $V \in \mathbb{C}_k[q]$. A direction, i.e., a non-zero $d \in \mathbb{C}^n$, is called a Darboux point of $V$ iff the gradient $V'(d)$ of $V$ at $d$ is parallel to $d$. Hence, $d$ is a Darboux point of $V$ iff

$$d \wedge V'(d) = 0, \quad d \neq 0,$$

or

$$V'(d) = \gamma d, \quad d \neq 0,$$

for a certain $\gamma \in \mathbb{C}$. Obviously, if $d$ satisfies one of the above conditions, then $\tilde{d} = \alpha d$ for any $\alpha \in \mathbb{C}^*$ satisfies them. However, we do not want to distinguish between $d$ and $\tilde{d}$. Hence we consider a Darboux point $d = (d_1, \ldots, d_n) \in \mathbb{C}^n$ as a point $[d] := [d_1 : \cdots : d_n]$ in the projective space $\mathbb{CP}^{n-1}$.

The set $\mathcal{D}(V) \subset \mathbb{CP}^{n-1}$ of all Darboux points of a potential $V$ is a projective algebraic set. In fact, $\mathcal{D}(V)$ is the zero locus in $\mathbb{CP}^{n-1}$ of homogeneous polynomials $R_{i,j} \in \mathbb{C}_k[q]$ which are components of $q \wedge V'(q)$, i.e.

$$R_{i,j} := q_i \frac{\partial V}{\partial q_j} - q_j \frac{\partial V}{\partial q_i}, \quad \text{where} \quad 1 \leq i < j \leq n. \quad (5.3)$$

We say that a Darboux point $[d] \in \mathcal{D}(V)$ is a proper Darboux point of $V$, iff $V'(d) \neq 0$. The set of all proper Darboux points of $V$ is denoted by $\mathcal{D}^*(V)$. If $[d] \in \mathcal{D}(V) \setminus \mathcal{D}^*(V)$, then $[d]$
is called an improper Darboux point of potential $V$. We say that $[d]$ is an isotropic Darboux point, iff

$$d_1^2 + \cdots + d_n^2 = 0. \tag{5.4}$$

We say that potential $V$ is generic iff all its Darboux points are proper and simple. The basic fact concerning Darboux points of generic potentials is given in the following lemma.

**Lemma 5.1.** The set of generic potentials $\mathcal{G}_{n,k} \subset \mathbb{C}_k[q]$ of degree $k$ is a non-empty open set in $\mathbb{C}_k[q]$. A generic $V \in \mathbb{C}_k[q]$ has

$$D(n,k) := \frac{(k-1)^n - 1}{k-2},$$

proper Darboux points.

A non-generic potential can have finite, or infinite number of Darboux points, but for an arbitrary $V \in \mathbb{C}_k[q]$ the set $\mathcal{D}(V)$ is not empty. Moreover, if $V$ does not have improper Darboux points, then it has a finite number of proper Darboux points.

### 5.2 Obstruction for the integrability due to improper Darboux point

Here we must justify the introduced definition of the Darboux point. Let us notice that in Theorem 4.1 only proper Darboux points appear and they give particular solutions. However, we have a more general fact.

**Lemma 5.2.** If $[d]$ is a proper Darboux point of a homogeneous potential $V$ of degree $k > 2$, then

$$q(t) := \varphi(t)d, \quad p(t) := \dot{\varphi}(t)d, \tag{5.5}$$

is a solution of Hamilton’s equation (4.2) provided $\dot{\varphi} = -\varphi^{k-1}$. Moreover, $V''(d) \cdot d = \lambda_n d$ with $\lambda_n = k - 1$, and if additionally $[d]$ is isotropic, then $\lambda_n$ is a multiple eigenvalue of $V''(d)$.

If $[d]$ is an improper Darboux point, then

$$q(t) := td, \quad p(t) := d, \tag{5.6}$$

is a solution of Hamilton’s equations (4.2). Moreover, $V''(d) \cdot d = \lambda_n d$, with $\lambda_n = 0$, and if additionally $[d]$ is isotropic, then $\lambda_n$ is a multiple eigenvalue of $V''(d)$.

Hence, also an improper Darboux point gives the particular solution (5.6) of the considered canonical equations (4.2). However, this solution has an extremely simple form and one can doubt if using it we can obtain any obstruction for the integrability. In fact, it is easy to notice that the monodromy group of the variational equations along solution (5.6) is trivial. Thus, in the frame of the Ziglin theory we do not obtain any obstacle for the integrability. Nevertheless, in [42] the following theorem was proved.

**Theorem 5.3.** Assume that a homogeneous potential $V \in \mathbb{C}_k[q]$ of degree $k > 2$ admits an improper Darboux point $[d] \in \mathbb{CP}^{n-1}$. If $V$ is integrable with rational first integrals, then matrix $V''(d)$ is nilpotent, i.e., all its eigenvalues vanish.

For $n = 2$, the above theorem coincides with Theorem 2.4 in [27].
5.3 Relation among eigenvalues

For a Darboux point \([d] \in V(V)\) we can calculate eigenvalues \(\lambda_1(d), \ldots, \lambda_n(d)\) of the Hessian matrix \(V''(d)\). However, numbers \(\lambda_i(d)\) are not well defined functions of point \([d] \in \mathbb{CP}^{n-1}\), as they depend on its representative \(d\). There are several possibilities to define properly the quantities related to the eigenvalues of \(V''(d)\) which do not depend on a choice of a representative of \([d]\). However, because of some historical reasons and the convention widely accepted in the literature, we choose the one which is a simple normalisation. Namely, if \([d]\) is a proper Darboux point, then the chosen representative is such that it satisfies \(V'(d) = 0\). If \([d]\) is an improper Darboux point, then the representative of \([d]\) can be chosen arbitrarily.

Let \([d]\) be a proper Darboux point of potential \(V\). Then, thanks to our assumption, the eigenvalues \(\lambda_1(d), \ldots, \lambda_n(d)\) of the Hessian matrix \(V''(d)\) can be considered as functions of \(d\). According to our convention \(\lambda_1(d) = k - 1\) is the trivial eigenvalue. Let \(\Lambda(d) = (\lambda_1(d), \ldots, \lambda_{n-1}(d))\). Hence we have the following mapping

\[
\mathcal{D}^*(V) \ni [d] \longmapsto \Lambda(d) \in \mathbb{C}^{n-1}.
\]

Assume that \(\mathcal{D}^*(V)\) is finite. Then the image of \(\mathcal{D}^*(V)\) under the above map is a finite subset of \(\mathbb{C}^{n-1}\). The question is if we can find a potential \(V\) of degree \(k\) such that the elements in the image have values prescribed in advance. We show that the answer to this question is negative. More precisely, we prove that among \(\Lambda(d)\) taken at all proper Darboux points \([d] \in \mathcal{D}^*(V)\) a certain number of universal relations exists. These relations play the central role in our considerations.

To formulate our first theorem we define \(\Lambda(d) = (\Lambda_1(d), \ldots, \Lambda_{n-1}(d))\), where \(\Lambda_i(d) := \lambda_i(d) - 1\) for \(i = 1, \ldots, n - 1\). By \(\tau_r\), for \(0 \leq r \leq n - 1\), we denote the elementary symmetric polynomials in \((n - 1)\) variables of degree \(r\), i.e.,

\[
\tau_r(x) := \tau_r(x_1, \ldots, x_{n-1}) = \sum_{1 \leq i_1 < \cdots < i_r \leq n-1} \prod_{s=1}^{r} x_{i_s}, \quad 1 \leq r \leq n - 1,
\]

and \(\tau_0(x) := 1\).

Our first theorem gives the explicit form of the above mentioned relations among \(\Lambda(d)\), for a generic potential \(V\).

**Theorem 5.4.** Let \(V \in \mathbb{C}_k[q]\) be a homogeneous potential of degree \(k > 2\) and let all its Darboux points be proper and simple. Then

\[
\sum_{[d] \in V(V)} \frac{\tau_1(\Lambda(d))^r}{\tau_{n-1}(\Lambda(d))} = (-1)^{n-1}(n - (k - 2))^r,
\]

and

\[
\sum_{[d] \in V(V)} \frac{\tau_r(\Lambda(d))}{\tau_{n-1}(\Lambda(d))} = (-1)^{r+n-1} \sum_{i=0}^{r} \binom{n-i-1}{r-i}(k-1)^i,
\]

for \(r = 0, \ldots, n - 1\).

Let us explain the importance of Theorem 5.4. To do this we need more definitions.

Let \(\mathcal{C}_m\) denote the set of all unordered tuples \(\Lambda = (\Lambda_1, \ldots, \Lambda_m)\), where \(\Lambda_i \in \mathbb{C}\) for \(i = 1, \ldots, m\). For \(M > 0\), the symbol \(\mathcal{C}_m^M\) denotes the set of all unordered tuples \((\Lambda_1, \ldots, \Lambda_M)\), where \(\Lambda_i \in \mathcal{C}_m\), for \(i = 1, \ldots, M\).
We fix $k > 2$ and $n \geq 2$, and say that a tuple $\mathbf{A} \in \mathbb{C}^{n-1}$ is \textit{admissible} iff $\lambda_i = \Lambda_i + 1 \in \mathbb{M}_k$ for $i = 1, \ldots, n - 1$. In other words, $\Lambda_i$ is admissible iff $\Lambda_i + 1$ belongs to items, appropriate for a given $k$, in the table of the Morales-Ramis Theorem [41] for $i = 1, \ldots, n - 1$. We denote the set of all admissible tuples by $\mathcal{A}_{n,k}$. If the potential $V$ is integrable, then for each $[d] \in \mathcal{D}^* (V)$, the tuple $\Lambda(d)$ is admissible. The set of all admissible elements $\mathcal{A}_{n,k}$ is countable but infinite.

If the set of proper Darboux points of a potential $V$ is non-empty, and $N = \text{card} \mathcal{D}^* (V)$, then the $N$-tuples

$$\mathcal{L}(V) := (\Lambda(d) \mid [d] \in \mathcal{D}^* (V)) \in \mathbb{C}^N_n, \quad (5.10)$$

is called \textit{the spectrum of $V$}. Let $\mathcal{A}^N_{n,k}$ be the subset of $\mathbb{C}^N_n$ consisting of $N$-tuples $(\Lambda_1, \ldots, \Lambda_N)$, such that $\Lambda_i$ is admissible, i.e., $\Lambda_i \in \mathcal{A}_{n,k}$, for $i = 1, \ldots, N$. We say that the spectrum $\mathcal{L}(V)$ of a potential $V$ is admissible iff $\mathcal{L}(V) \in \mathcal{A}^N_{n,k}$. The Morales-Ramis Theorem [41] says that if potential $V$ is integrable, then its spectrum $\mathcal{L}(V)$ is admissible. However, the problem is that the set of admissible spectra $\mathcal{A}^N_{n,k}$ is infinite. We show that from Theorem [5.4] it follows that, in fact, if $V$ is integrable, then its spectrum $\mathcal{L}(V)$ belongs to a certain \textit{finite} subset $\mathcal{Y}^N_{n,k}$ of $\mathcal{A}^N_{n,k}$. We call this set \textit{distinguished one}, and its elements \textit{distinguished spectra}.

**Theorem 5.5.** Let potential $V$ satisfy assumptions of Theorem [5.4]. If $V$ is integrable, then there exists a finite subset $\mathcal{Y}^N_{n,k} \subset \mathcal{A}^N_{n,k}$, where $N = \text{card} \mathcal{D}^* (V)$, such that $\mathcal{L}(V) \in \mathcal{Y}^N_{n,k}$.

Informally speaking, for fixed $k$ and $n$, we restrict the infinite number of possibilities in each line of the Morales-Ramis table to a finite set of choices.

### 5.4 Euler-Jacobi-Kroncker formula and its generalisation

The importance of Theorem [5.5] is clear. Having in mind the our general program of finding all integrable potentials, one would like to have a generalisations of Theorem [5.4] and Theorem [5.5] for non-generic potentials. In [40] we gave a proof of Theorem [5.4] using a certain result of Guillot [17]. Unfortunately, the methods used in [17] do not admit such a generalisation.

Our analysis of case $n = 2$ given in [27] explicitly showed that one can find an alternative proof of Theorem [5.4] which admits a generalisation to non-generic cases. Moreover it also gives a clue that an alternative proof Theorem [5.4] can be done with a help of multidimensional residue technique. We have made many attempts to find such a proof, however all of them failed.

Finally we have found amazingly simple solution of the problem. Here we describe shortly the main construction of our approach. A detailed exposition the reader will find in [42, 43].

Let us introduce local affine coordinates on $\mathbb{C}^{\mathbb{P}^{n-1}}$ where the Darboux points live. We choose chart $(U_1, \theta_1)$, where

$$U_1 := \mathbb{C}^{\mathbb{P}^{n-1}} \setminus \{ [q] \in \mathbb{C}^{\mathbb{P}^{n-1}} \mid q_1 \neq 0 \},$$

and

$$\theta_1 : U_1 \to \mathbb{C}^{n-1}, \quad \bar{x} := (x_1, \ldots, x_{n-1}) = \theta_1 ([q]), \quad (5.11)$$

where

$$x_i = \frac{q_{i+1}}{q_1}, \quad \text{for} \quad i = 1, \ldots, n - 1. \quad (5.12)$$
The image of the set of Darboux points which lie on this chart, i.e., \( \theta_1(\mathcal{D}(V) \cap U_1) \), is an affine algebraic set

\[
\theta_1(\mathcal{D}(V) \cap U_1) = \mathcal{V}(g_1, \ldots, g_{n-1}),
\]

(5.13)

where polynomials \( g_1, \ldots, g_{n-1} \in \mathbb{C}[\tilde{x}] \) are given by

\[
v(\tilde{x}) := V(1, x_1, \ldots, x_{n-1}), \quad g_0 := kv - \sum_{i=1}^{n-1} x_i \frac{\partial v}{\partial x_i},
\]

(5.14)

and

\[
g_i := \frac{\partial v}{\partial x_i} - x_ig_0, \quad \text{for} \quad i = 1, \ldots, n - 1.
\]

(5.15)

Moreover, \([d] \in \mathcal{D}(V) \cap U_1\) is an improper Darboux point iff its affine coordinates \( \tilde{a} := \theta_1([d]) \) satisfy \( g_0(\tilde{a}) = 0 \).

It is instructive to consider first the case \( n = 2 \). In this case a Darboux point \([d]\) on the affine chart is given by one coordinate \( x_\star = \theta_1([d]) \). It is a root of polynomial

\[
g_1(x) := v'(x) - xg_0(x) \quad \text{where} \quad g_0(x) := kv(x) - xv'(x).
\]

Moreover, it is easy to notice that the non-trivial eigenvalue \( \lambda(d) \) of \( V''(d) \) can be calculated from the following formula

\[
\frac{1}{\lambda(d) - 1} = \frac{1}{\Lambda(d)} = \frac{g_0(x_\star)}{g_1'(x_\star)}.
\]

(5.16)

The above formula suggests to introduce the following differential form

\[
\omega = \frac{g_0(x)}{g_1(x)} \, dx,
\]

considered as a differential form on \( \mathbb{C}P^1 \). This form has poles at Darboux points. If a Darboux point \([d]\) is proper and simple, then its affine coordinate \( x_\star = \theta_1([d]) \) is a simple pole of \( \omega \), and the residue of \( \omega \) at this point is

\[
\text{res}(\omega, x_\star) = \frac{g_0(x_\star)}{g_1'(x_\star)} = \frac{1}{\Lambda(d)}.
\]

Without loss of the generality we can assume that all Darboux points are located in the affine part of \( \mathbb{C}P^1 \), and then

\[
\text{res}(\omega, \infty) = 1.
\]

Thus, assuming that all Darboux points are proper and simple and applying the global residue theorem we obtain that

\[
\sum_{[d] \in \mathcal{D}(V)} \frac{1}{\Lambda(d)} = 1.
\]

This is just relation (5.8) for \( n = 2 \) with \( r = 0 \). Note that for \( n = 2 \) it is the only non-trivial relation.

Now, considering cases with \( n > 2 \) one would like to construct differential forms in \( \mathbb{C}P^{n-1} \) which have poles at Darboux points and such that their multidimensional residues at these poles are given by symmetric functions of \( \Lambda(d) \). It is not difficult to define an \((n-1)\)-differential form in affine part of \( \mathbb{C}P^{n-1} \) which has poles at Darboux points with residues given in term of symmetric functions of \( \Lambda(d) \). The problem appears with extension of
this form onto whole \( \mathbb{C}P^{n-1} \). The obtained global differential form has some additional, usually not isolated poles and for this type of forms there is no an appropriate global residue theorem.

Let us recall basic facts about the multi-dimensional residues and the Euler-Jacobi-Kronecker formula. For details the reader is referred to \([1, 15, 16, 18, 46]\). Let \( f_i : \mathbb{C}^n \supset U \to \mathbb{C} \), where \( U \) is an open neighbourhood of the origin, be holomorphic functions for \( i = 1, \ldots, n \), and \( x = 0 \) be an isolated common zero of \( f_i \). We consider differential \( n \)-form

\[
\omega := \frac{p(x)}{f_1(x) \cdots f_n(x)} \, dx_1 \wedge \cdots \wedge dx_n, \tag{5.17}
\]

where \( p : U \to \mathbb{C} \) is a holomorphic function. The residue of the form \( \omega \) at \( x = 0 \) can be defined as

\[
\text{res}(\omega, 0) := \frac{1}{(2\pi i)^n} \int_{\Gamma} \omega, \tag{5.18}
\]

where

\[
\Gamma := \{ x \in U \mid |f_1(x)| = \varepsilon_1, \ldots, |f_n(x)| = \varepsilon_n \}, \tag{5.19}
\]

and \( \varepsilon_1, \ldots, \varepsilon_n \) are sufficiently small positive numbers. The orientation of \( \Gamma \) is fixed by

\[
d(\arg f_1) \wedge \cdots \wedge d(\arg f_n) \geq 0. \tag{5.20}
\]

Let us denote \( f := (f_1, \ldots, f_n) \). It can be shown that if the Jacobian \( \det f'(0) \neq 0 \), then

\[
\text{res}(\omega, 0) = \frac{p(0)}{\det f'(0)}. \tag{5.21}
\]

The following theorem gives the classical Euler-Jacobi-Kronecker formula, see e.g. \([15]\).

**Theorem 5.6 (Euler-Jacobi-Kronecker).** Let \( f_1, \ldots, f_n \in \mathbb{C}[x] \) be non-constant polynomials such that \( V(f) := V(f_1, \ldots, f_n) \) is finite and all points of this set are simple. If \( f_1, \ldots, f_n \) do not intersect at the infinity, then for each \( p \in \mathbb{C}[x] \) such that

\[
\deg p \leq \sum_{i=1}^n \deg f_i - (n + 1), \tag{5.22}
\]

we have

\[
\sum_{d \in V(f)} \text{res}(\omega, d) = \sum_{d \in V(f)} \frac{p(d)}{\det f'(d)} = 0. \tag{5.23}
\]

The above theorem is not sufficient for our investigations. We have to consider cases when \( f_1, \ldots, f_n \) have intersections at the infinity as well as cases when intersections of \( f_1, \ldots, f_n \) are not simple.

The homogenisations of \( f_i \) are given by

\[
F_i(z_0, z_1, \ldots, z_n) := z_0^{-\deg f_i} f_i \left( \frac{z_1}{z_0}, \ldots, \frac{z_n}{z_0} \right), \quad \text{for } i = 1, \ldots, n. \tag{5.24}
\]

They define the projective algebraic set \( V(F) := V(F_1, \ldots, F_n) \subset \mathbb{C}P^n \) whose affine part is homeomorphic to \( V(f) \). Next we extend the form \( \omega \) to a rational form \( \Omega \) defined on \( \mathbb{C}P^n \). To this end we consider \( \omega \) as the expression of \( \Omega \) on the chart \((U_0, \theta_0)\). In order to express \( \Omega \) on other charts we use the standard coordinate transformations of \( n \)-form. Let
\[\text{[9]}\]

where \(U\) as it was shown in \([8]\) we have

\[
\hat{\omega} \quad (5.25)
\]

where \(\hat{\omega}\) denotes form \(\Omega\) expressed in the chart \((U_i, \theta_i)\).

Theorem 5.7. Let \(\nabla^2\) denote form \(\Omega\) expressed in the chart \((U_i, \theta_i)\).

The following theorem is a special version of the global residue theorem.

Let \(\Omega\) be a one dimensional case. We describe it shortly below.

Let us define the following forms

\[
P(z_0, z_1, \ldots, z_n) := z_0^{\deg p} \left( \frac{z_1}{z_0}, \ldots, \frac{z_n}{z_0} \right). \quad (5.26)
\]

To underline the explicit dependence of \(\Omega\) on \(F_i\) and \(P\) we write symbolically \(\Omega = P/F\).

The following theorem is a special version of the global residue theorem.

**Theorem 5.7.** Let \(\nabla F := \nabla(F_1, \ldots, F_n)\) be a finite set. Then for each polynomial \(P\) such that

\[
\deg P \leq \sum_{i=1}^n \deg F_i - (n + 1), \quad (5.27)
\]

we have

\[
\sum_{[s] \in \nabla F} \text{res}(P/F, [s]) = 0. \quad (5.28)
\]

For the proof and the more detailed exposition we refer the reader to \([9, 15]\).

If \(0 \in \nabla f\) is an isolated but not simple point, then we cannot use formula \((5.21)\) to calculate the residue of the form \(\omega\) at this point. In such a case we can apply a very nice method developed by Biernat in \([7, 8]\) that reduces the calculation of multi-dimensional residue to a one dimensional case. We describe it shortly below.

Let us consider the following analytic set

\[
A := \{ x \in U \mid f_2(x) = \cdots = f_n(x) = 0 \}, \quad (5.29)
\]

where \(U \subset \mathbb{C}^n\) is a neighbourhood of the origin. Set \(A\) is a sum of irreducible one dimensional components \(A = A_1 \cup \cdots \cup A_m\). Let \(t \mapsto \varphi_i(t) \in A_i\), \(\varphi_i(0) = 0\), be an injective parametrisation of \(A_i\). Then we define the following forms

\[
\omega_i = \frac{p(\varphi_i(t))}{f_1^{(i)}(\varphi_i(t))} \frac{f_1^{(i)}(\varphi_i(t)) \cdot \varphi_i(t)}{f_1^{(i)}(\varphi_i(t))} dt. \quad (5.30)
\]

As it was shown in \([8]\) we have

\[
\text{res}(\omega, 0) = \sum_{i=1}^m \text{res}(\omega_i, 0). \quad (5.31)
\]

In order to use the above theorems we have to make a kind of blowup. Roughly speaking, the idea is to associate with a Darboux point which is located in \(\mathbb{C}P^{n-1}\), a finite set of points in \(\mathbb{C}P^n\).

Let us define the following \(n\) homogeneous polynomials of \(n+1\) variables \(q := (q_0, q_1, \ldots, q_n)\)

\[
F_i := \frac{\partial V}{\partial q_i} - q_0^{k-2} q_i, \quad i = 1, \ldots, n, \quad (5.32)
\]

and an algebraic set \(\mathcal{D}(V) = \nabla(F_1, \ldots, F_n) \subset \mathbb{C}P^n\).

Assume that \([d] \in \mathcal{D}^*(V)\). Then there exists \(\gamma \in \mathbb{C}^*\), such that \(V'(d) = \gamma d\), so \(k-2\) points \((k-2g+2)\) belong to \(\mathcal{D}(V)\). These points are well defined as they do
not depend on a representative for $[d]$. If $[d]$ is an improper Darboux point, then it defines just one point $[0 : d_1 : \cdots : d_n] \in \mathbb{CP}^n$ which is a point of $\widehat{\mathcal{D}}(V)$.

Set $\widehat{\mathcal{D}}(V)$ is not empty because it contains point $[d_0] := [1 : 0 : \cdots : 0]$. If $[d] = [d_0 : d_1 : \cdots : d_n] \in \widehat{\mathcal{D}}(V) \setminus \{[d_0]\}$, then $[d] = [d_1 : \cdots : d_n]$ is a Darboux point of $V$. Moreover, if $d_0 \neq 0$, then $[d]$ is a proper Darboux point.

The natural projection

$$\pi : \mathbb{CP}^n \setminus \{[d_0]\} \to \mathbb{CP}^{n-1}, \quad \pi([q_0 : q_1 : \cdots : q_n]) = [q_1 : \cdots : q_n], \quad (5.33)$$

maps $\widehat{\mathcal{D}}(V) \setminus \{[d_0]\}$ onto $\mathcal{D}(V)$, and the intersection of the inverse image $\pi^{-1}([d])$ of a Darboux point $[d] \in \mathcal{D}(V)$ with $\widehat{\mathcal{D}}(V)$ is a finite set. We define also

$$\widehat{\pi} : \widehat{\mathcal{D}}(V) \setminus \{[d_0]\} \to \mathcal{D}(V), \quad (5.34)$$

putting $\widehat{\pi}([d]) := \pi([d])$ for $[d] \in \widehat{\mathcal{D}}(V) \setminus \{[d_0]\}$. That is, $\widehat{\pi}$ is the restriction of $\pi$ to $\widehat{\mathcal{D}}(V) \setminus \{[d_0]\}$. This construction is illustrated in the Figure 1. Now, we can consider

Figure 1: Sets $\widehat{\mathcal{D}}(V) \subset \mathbb{CP}^n$ and $\mathcal{D}(V) \subset \mathbb{CP}^{n-1}$

differential form $\Omega := p/F$ in $\mathbb{CP}^n$, which in affine part of $\mathbb{CP}^n$, is given by

$$\omega := \frac{p_r(x)}{f_1(x) \cdots f_n(x)},$$

where $f_i$ is a dehomogenisation of $F_i$, i.e.,

$$f_i(x) := \frac{\partial V}{\partial x_i}(x) - x_i, \quad \text{for} \quad i = 1, \ldots, n, \quad (5.35)$$

and polynomials $p_r$ are of the form

$$p_r(x) = (\text{Tr} \ f'(x) - (k - 2))', \quad \text{with} \quad r \in \{0, \ldots, n - 1\}. \quad (5.36)$$

Now, to obtain relations (5.3) we just make simple calculations according to the following scheme

1. $\mathcal{V}(f) := \mathcal{V}(f_1, \ldots, f_n)$ is the affine part of $\mathcal{V}(F) := \mathcal{V}(F_1, \ldots, F_n) = \widehat{\mathcal{D}}(V)$. 

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2. \( 0 \in \mathcal{V}(f) \), and  
\[
f'(0) = -E_n \quad \text{so:} \quad p_r(0) = (-n - (k - 2))^r, \quad \det f'(0) = (-1)^n.
\]

3. If \( d \in \mathcal{V}(f) \), and \( d \neq 0 \), then \([d] \in \mathcal{D}^*(V) \) and  
\[
\det f'(d) = (k - 2) \prod_{i=1}^{n-1} \Lambda_i(d), \quad \text{and} \quad p_r(d) = \left( \sum_{i=1}^{n-1} \Lambda_i(d) \right)^r.
\]

4. For \( k > 2 \), and \( n \geq 2 \) we have  
\[
\det p_r = r(k - 2) \leq \sum_{i=1}^{n} \deg f_i - (n + 1) = n(k - 1) - n - 1,
\]
for \( r \in \{0, \ldots, n - 1\} \).

5. Moreover, \( d_j := \varepsilon^j d \in \mathcal{V}(f) \), and \( f'(d_j) = f'(d) \) for \( j = 0, \ldots, k - 3 \), where \( \varepsilon \) is a primitive \((k - 2)\)-root of unity.

6. If all Darboux points are proper, then polynomials \( f_1, \ldots, f_n \) do not intersect at the infinity and we can apply the classical Euler-Jacobi-Kronecker formula (5.23) from Theorem 5.6.

In a case of non-generic potential \( V \) having finite number of Darboux points the potential possesses either improper, or multiple Darboux points. In such cases we apply Theorem 5.7. Examples of such calculations are given in [43].

### 5.5 Applications of global analysis

In order to perform a reasonable classification of potentials it is convenient to introduce the following equivalent relations.

Let \( \mathrm{PO}(n, \mathbb{C}) \) be the complex projective orthogonal subgroup of \( \mathrm{GL}(n, \mathbb{C}) \), i.e.,  
\[
\mathrm{PO}(n, \mathbb{C}) = \{ A \in \mathrm{GL}(n, \mathbb{C}), \ | \ \AA^T = \alpha E_n, \ \alpha \in \mathbb{C}^* \}, \quad (5.37)
\]
where \( E_n \) is \( n \)-dimensional identity matrix. We say that \( V \) and \( \tilde{V} \) are equivalent if there exists \( A \in \mathrm{PO}(n, \mathbb{C}) \) such that \( \tilde{V}(q) = V_A(q) := V(Aq) \). Later a potential means a class of equivalent potentials in the above sense.

The general results described in the previous section can be applied to a systematic study the integrability of homogeneous potentials with fixed \( n \) and \( k \). The algorithm is following.

We assume that \( n \geq 2 \) and \( k > 2 \) are fixed. The aim is to distinguish all integrable potentials.

At first we consider generic potentials with \( N = D(n, k) \) proper Darboux points.

By Theorem 5.5 there is only a finite number of distinguished spectra \( j_{n,k}^N \) and we can find all them solving Diophantine equations of the form (5.8) in a subset of rational numbers defined by the Morales-Ramis table. For example, for \( n = 2 \) we have \( D(2, k) = k \), so a generic homogeneous potential of degree \( k \) has \( k \) proper Darboux points. At each proper Darboux point we have one non-trival eigenvalue \( \Lambda \). Thus, in this case elements of a \( j_{n,k}^N = j_{2,k}^k \), are unordered tuples of \( k \) elements. For \( k = 3 \) and \( k = 4 \) they are listed in Table 1.
\begin{table}
\begin{center}
\begin{tabular}{ccc}
\hline
$(\Lambda_1, \Lambda_2, \Lambda_3)$ & $(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4)$ \\
\hline
$(-1, -1, 1)$ & $(-1, -1, 2, 2)$ \\
$(-2/3, 4, 4)$ & $(-5/8, 5, 5, 5)$ \\
$(-7/8, 14, 14)$ & $(-5/8, 2, 20, 20)$ \\
$(-2/3, 7/3, 14)$ & $(-5/8, 27/8, 27/8, 135)$ \\
& $(-5/8, 2, 14, 35)$ \\
\hline
\end{tabular}
\end{center}
\end{table}

Table 1: The distinguished spectra $\mathcal{J}_{2,k}^2$ for $k = 3$ (left), and $k = 4$ (right)

It appears that the determination of the distinguished spectra can be performed only with a help of a computer algebra system. The known algorithms used for these purposes are highly time demanding.

The next step is to find all possible potentials for a given element of the distinguished spectrum. In other words we have to determine all polynomials of a fixed degree such that their Hessians at some points (which are unknown a priori) have specified eigenvalues. At a first glance, it seems that this problem is ill-posed. However, it is not like that. The reason is that the restriction imposed by the fixing of the eigenvalues is very rigid. Moreover, in fact we work only with equivalent classes of potentials and this restricts additionally the number of free parameters in the problem. The algorithm of performing this step is based on determination of the elimination ideal. In all cases, for a fixed distinguished spectrum we obtained, either a finite number of non-equivalent potentials, or a finite number of families of ‘separated’ potentials (a sum of potentials which depend on smaller number of variables).

From the previous step we obtained a finite number of potentials which can be integrable. To check if they are integrable we use two methods. First we try to find an additional polynomial first integral applying a direct method. If it fails we apply the higher order variational equations in order to prove their non-integrability.

In a similar way we investigate non-generic cases.

The first time the prescribed algorithm was used in [24] where it was shown that all integrable homogeneous of degree three polynomial potentials in two variables are already known. Next, in [27] it was shown for $n = 2$ and $k = 4$ all integrable potentials are known except for potential

$$V = \frac{\alpha}{2} q_1^2 (q_1 + i q_2)^2 + \frac{1}{4} (q_1^2 + q_2^2)^2, \quad \alpha \in \mathbb{C}^*, $$

with $\alpha$ such that $\lambda = 1 - \alpha$ belongs to items 2,3 and 5 of table (4.10). All those investigations of homogeneous potentials with two degrees of freedom allowed to find one new non-trivially integrable potential of the form

$$V_{k,l} = (q_2 - i q_1)^l (q_2 + i q_1)^{k-l},$$

with $k = 7$ and $l = 2$, which admits an additional polynomial first integral of degree four in the momenta, see [38].

The case of homogeneous degree three polynomial potentials in three variables, i.e., case $n = k = 3$, was analysed in [42; 43]. In this case a generic potential has seven proper Darboux points $[d_i]$. At each of them we have pair $\Lambda(d_i) := (\Lambda_1^{(i)}, \Lambda_2^{(i)})$ of the shifted
eigenvalues. The relations (5.38) have the following form

\[
\begin{align*}
\sum_{i=1}^{7} \frac{1}{\Lambda_1^{(i)} \Lambda_2^{(i)}} &= 1, \\
\sum_{i=1}^{7} \frac{\Lambda_1^{(i)} + \Lambda_2^{(i)}}{\Lambda_1^{(i)} \Lambda_2^{(i)}} &= -4, \\
\sum_{i=1}^{7} \frac{\left(\Lambda_1^{(i)} + \Lambda_2^{(i)}\right)^2}{\Lambda_1^{(i)} \Lambda_2^{(i)}} &= 16.
\end{align*}
\]

(5.39)

After really long computations we have found ten distinguished spectra. For each of them we reconstructed the general form of the potential. Four among the reconstructed potentials have the form

\[ V(q_1, q_2) = V_1(q_1, q_2) + \frac{1}{3} q_3^3, \]

where \( V_1(q_1, q_2) \) is an integrable potential with two degrees of freedom. Each distinguished spectrum gives several potentials. Some of these potentials are not integrable, and this fact was proved with the help of higher order variational equations. The remaining ones have the forms

\[
\begin{align*}
V_5 &= \frac{3i}{4} q_1^2 + \frac{7i}{3} q_2^3 + \frac{5}{2} q_2 q_3 + \frac{1}{3} q_3^3, \\
V_6 &= 364 \sqrt{17} q_1^3 + 2835i \sqrt{17} q_1^2 q_2 + 1560 \sqrt{17} q_1 q_2^2 + 6552i \sqrt{17} q_2^3 + \\
&\quad + 4355 q_1^2 q_3 + 19074 q_2^2 q_3 + 578 q_3^3, \\
V_7 &= 44 \sqrt{7} q_1^3 + 240i \sqrt{14} q_1^2 q_2 + 330 \sqrt{7} q_1 q_2^2 + 935i \sqrt{14} q_2^3 + 3087 q_2^2 q_3 + 294 q_3^3, \\
V_8 &= \frac{7}{2} q_1^2 q_3 - \frac{5i \sqrt{3}}{2} q_1 q_2 - \frac{9i \sqrt{3}}{2} q_2^3 + \frac{15}{2} q_2 q_3^2 + \frac{1}{3} q_3^3, \\
V_9 &= 27i \sqrt{3990} q_1^3 + 3726 \sqrt{15} q_1^2 q_2 - 456i \sqrt{3990} q_1 q_2^2 - 4092 \sqrt{15} q_2^3 - \\
&\quad - 1125 q_1^2 q_3 - 3000 q_2^2 q_3 - 50 q_3^3, \\
V_{10} &= \frac{4 \sqrt{2} q_1^3}{3} + \frac{5 q_1 q_2^2}{2 \sqrt{2}} + q_2 q_3 + \frac{1}{3} q_3^3.
\end{align*}
\]

All the above potentials are integrable. Each of them admits two commuting additional polynomial first integrals \( I_1 \) and \( I_2 \). They were found with the help of a direct method. All these potentials are integrable in a non-trivial way, i.e., at least one of additional first integral is of degree higher than two with respect the momenta. For example, for the potential \( V_{10} \) the additional first integrals have the forms

\[
\begin{align*}
I_1 &= 12 p_2^4 - 27 q_2^6 - 18 q_2^4 (q_1^2 - 4 \sqrt{2} q_1 q_3 + 2 q_3^2) + 4 (6 p_1^2 - 3 p_3^2 + 16 \sqrt{2} q_1^2 - 2 q_3^2) (3 p_3^2 + 2 q_3^2) + \\
&\quad + 12 q_1^2 (3 p_3^2 (\sqrt{2} q_1 - 4 q_3) + 12 p_1 p_3 (q_1 + \sqrt{2} q_3) - 2 q_3 (12 q_1^2 + \sqrt{2} q_1 q_3 + 2 q_3^2)) \\
&\quad - 12 p_2 q_2 (2 p_3 (16 q_1^2 + 3 q_2^2 + 8 \sqrt{2} q_1 q_3 - 4 q_3^2) + 3 \sqrt{2} p_1 (q_2^2 + 4 q_3^2)) \\
&- 12 p_2^2 (2 p_3 (2 \sqrt{2} p_1 + p_3) - 4 (q_2 - q_3) q_3 (q_2 + q_3) - \sqrt{2} q_1 (5 q_2^2 + 8 q_3^3)),
\end{align*}
\]
These similarities allow to define the notion of Darboux point of a homogeneous force point iff as a non-zero direction such equations, see Definition 2.4.

Integrability of Newton homogeneous equations

\( F \) are not potential, so it can happen that these equations do not admit even a single first integral. Let us notice formal similarities between Newton's (6.2), and Hamilton's (4.2) equations. The analysis of non-generic cases is much more involved but, nevertheless it can be made almost up to the end, see [42] for details.

6 Integrability of Newton homogeneous equations

In this section we consider the following class of Newton's equations

\[ \ddot{q} = -F(q), \quad q = (q_1, \ldots, q_n), \]  

(6.1)

that we rewrite as a system of first order differential equations

\[ \dot{q} = p, \quad \dot{p} = -F(q). \]  

(6.2)

Our aim is to investigate integrability properties of such equations. Generally, the forces \( F \) are not potential, so it can happen that these equations do not admit even a single first integral.

It is easy to observe that equations (6.2) admit

\[ \mu = dq_1 \wedge \cdots \wedge dq_n \wedge dp_1 \wedge \cdots \wedge dp_n, \]

as an invariant 2n-form. Thus, we can talk about the integrability in the Jacobi sense of such equations, see Definition 2.4.

Now, we want to find necessary conditions for the integrability in the Jacobi sense applying the differential Galois framework. Thus, we assume just from the beginning that \( (q, p) \in \mathbb{C}^{2n} \), and we consider only the case when forces are homogeneous of degree \( (k - 1) \).

Let us notice formal similarities between Newton's (6.2), and Hamilton's (1.2) equations. These similarities allow to define the notion of Darboux point of a homogeneous force \( F \), as a non-zero direction \( d \) such that \( F(d) \) is parallel to \( d \). We say that \( d \) is a proper Darboux point iff \( F(d) \neq 0 \). As in the case of homogeneous potentials, a Darboux point \( d \) defines an invariant two dimensional plane

\[ \Pi(d) := \{ (q, p) \in \mathbb{C}^{2n} \mid q = qd, \quad p = pd, \quad (q, p) \in \mathbb{C}^2 \}. \]  

(6.3)
Then pairs (\(F\)) we assume the force we presented in Section 5 for the homogeneous potentials. In order to analyse this question with the following phase curves

\[\Gamma_{k,\varepsilon} := \left\{ (\varphi, \psi) \in \mathbb{C}^2 \mid \frac{1}{2} \psi^2 + \frac{1}{k} \varphi^k = \varepsilon \right\} \subset \mathbb{C}^2, \quad \varepsilon \in \mathbb{C}. \tag{6.5}\]

In this way, a solution \((\varphi, \psi) = (\varphi(t), \psi(t))\) of \((6.4)\) gives rise a solution \((q(t), p(t)) := (\varphi d, \varphi d)\) of equations \((6.2)\) with the corresponding phase curve

\[\Gamma_{k,\varepsilon} := \left\{ (q, p) \in \mathbb{C}^{2n} \mid (q, p) = (\varphi d, \varphi d), \ (\varphi, \psi) \in \Gamma_{k,\varepsilon} \right\} \subset \Pi(d). \tag{6.6}\]

The variational equations along \(\Gamma_{k,\varepsilon}\) have the form

\[\dot{x} = y, \quad \dot{y} = -\varphi(t)^{k-2} F'(d)x, \tag{6.7}\]

or simply

\[\dot{x} = -\varphi(t)^{k-2} F'(d)x, \tag{6.8}\]

where \(F'(d)\) is the Jacobi matrix of \(F\) calculated at a Darboux point \(d\). Let us assume that this matrix is diagonalisable. Then, in an appropriate basis equations \((6.8)\) have the form

\[\dot{\eta}_i = -\lambda_i \varphi(t)^{k-2} \eta_i, \quad i = 1, \ldots, n, \tag{6.9}\]

where \(\lambda_1, \ldots, \lambda_n\) are eigenvalues of \(F'(d)\). It is easy to show using the Euler identity that \(d\) is an eigenvector of \(F'(d)\) with eigenvalue \(k - 1\). We always denote it by \(\lambda_n\). In \([41]\) the following theorem was proved.

**Theorem 6.1.** Assume that the Newton system \((6.1)\) with homogeneous right-hand sides of degree \((k - 1)\) with \(k \in \mathbb{Z}^+\), satisfies the following conditions:

1. force \(F\) admits a proper Darboux point \(d\), and \(\lambda_1, \ldots, \lambda_n\) are eigenvalues of \(F'(d)\),

2. equations \((6.2)\) are integrable in the Jacobi sense with first integrals which are meromorphic in a connected neighbourhood \(U\) of phase curve \(\Gamma_{k,\varepsilon}\) with \(\varepsilon \neq 0\), and independent on \(U \setminus \Gamma_{k,\varepsilon}\).

Then pairs \((k, \lambda_i)\) for \(i = 1, \ldots, n\) belong to the Morales-Ramis Table \((4.10)\).

The above theorem is a generalisation of two theorems (Theorem 1.2 and Theorem 1.3) from \([41]\). Here we remark that the similarity of the theses of Theorem \([4.1]\) and Theorem \([6.1]\) is somewhat misleading. The point is that in the case of Newton equations we do not have any symplectic structure, and thus we have no all geometrical consequences of this fact. On the other hand, the integrability in the Jacobi sense requires ‘big’ number of first integrals and exactly this requirement is the reason why the identity component of the differential Galois group must be Abelian.

Fact that we have Theorem \([6.1]\) allows us to think about a global analysis similar to that we presented in Section \([5]\) for the homogeneous potentials. In order to analyse this question we assume the force \(F\) has polynomial components of the same degree \(l := k - 1 > 1\), i.e., \(F \in (\mathbb{C}[q])^n\). We denote by \(\mathcal{D}(F)\) the set of all Darboux points of \(F\). It appears that a generic force has \(D(n,k) = [(k-1)^n - 1]/(k-2)\) Darboux points (considered as points in \(\mathbb{C}P^{n-1}\)), i.e., as many as a homogeneous potential \(V\) of degree \(k\). It is rather amazing, because
For the Newton equations we can apply all the methods and tools used for the homogeneous potentials that are described in Section 5. In particular, we have the following.

**Theorem 6.2.** Assume that force $F \in (\mathbb{C}_{k-1}[q])^n$ has exactly $D(n,k)$ proper Darboux points $[d] \in \mathcal{D}(F)$. Then $\Lambda(d)$ satisfy the following relations:

$$\sum_{[d] \in \mathcal{D}(F)} \frac{\tau_1(\Lambda(d))^r}{\tau_{n-1}(\Lambda(d))} = (-1)^{n-1+r}(n+k-2)^r,$$

(6.10)

or, alternatively

$$\sum_{[d] \in \mathcal{D}(F)} \frac{\tau_r(\Lambda(d))}{\tau_{n-1}(\Lambda(d))} = (-1)^{r+1-n} \sum_{i=0}^r \left( \binom{n-i-1}{r-i} (k-1)^i \right),$$

(6.11)

for $0 \leq r \leq n-1$.

This theorem has the same consequences for the Jacobi integrability of homogeneous forces as Theorem 5.4 for the integrability of homogeneous potentials. Namely, for a given $k$, there is only a finite set of candidates for $\Lambda(d)$ satisfying the necessary conditions for the Jacobi integrability given by Theorem 6.2. Hence, we can try to perform an analysis similar to that for homogeneous potentials, and find all integrable forces for small $k$. Such an analysis was performed in [41].

It appears that for $n=2$ and $k=3$ almost all forces $F \in \mathbb{C}_2[q] \times \mathbb{C}_2[q]$ are potential. For $k=4$ and $k=5$ several families of integrable forces were found. Among them some intriguing non-trivial examples appear, e.g., force with components

$$F_1 = q_1^2 q_2, \quad F_2 = \frac{11}{6} q_1 q_2^2,$$

is integrable in the Jacobi sense, with two first integrals which are both of degree four with respect to the velocities. They have the following form

$$I_1 = 24 q_1 p_1^3 + 3 q_2^2 (4 q_1^2 p_2^2 + 12 q_1 q_2 p_1 p_2 - 3 q_2^2 p_1^2) + 16 q_1^3 q_2^5,$n

$$I_2 = 162 p_1^3 (q_1 p_2 - q_2 p_1) - 9 q_1^3 (4 q_1^2 p_2^2 - 20 q_1 q_2 p_1 p_2 + 13 q_2^2 p_1^2) + 16 q_1^6 q_2^3.$n

It is also worth to mention a remarkable family of forces given by

$$F_1 = \lambda q_1 q_2^{k-2}, \quad F_2 = q_2^{k-1},$$

(6.12)

where $k > 2$ is and $\lambda \in \mathbb{C}$. The Newton equations with this force admit the following first integral

$$I_1 = \frac{1}{2} p_2^2 + \frac{1}{k} q_2^k.$$

If $(k,\lambda)$ belongs to an item of the Morales-Ramis table (4.10), then the system is integrable in the Jacobi sense with an additional polynomial first integral $I_2$. Moreover, for an arbitrary $M > 0$ we find $\lambda$ such that the degree of $I_2$ with respect to the momenta is greater than $M$, and there is no an additional polynomial first integral independent with $I_1$ and having degree with respect to the momenta smaller or equal to $M$. Additionally, if $(k,\lambda)$ belongs to an item different from 2 in table (4.10), then there exist two additional polynomial first integrals $I_2$ and $I_3$ which are functionally independent together with $I_1$. The above statements were formulated in a form of a well justified conjecture in [41].

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7 Open problems and perspectives

Let us consider a two degrees of freedom Hamiltonian system with homogeneous polynomial potentials of degree \( k > 2 \). In [39] it was shown that if a potential of degree \( k > 4 \) admits a polynomial first integral of degree higher than two with respect to the momenta, then this first integral is a product of polynomial first integrals of lower degrees. One would like to prove the following.

**Conjecture 7.1.** If a homogeneous polynomial potential of degree \( k > 4 \) admits an additional polynomial first integral, then it admits an additional first integral of degree at most two with respect to the momenta.

Let us consider a restricted version of this conjecture. Namely, let us assume that we consider only a generic potential of degree \( k > 5 \). Such potential has exactly \( k \) proper Darboux points, for each \( k \) we know two elements of its distinguished spectrum \( \mathcal{I}_{2,k}^k \), namely

\[
\mathcal{A}_{1,k} = (-1, -1, 1, 2, \ldots, k-2), \quad \mathcal{A}_{2,k} = \left( -\frac{k+1}{2k}, 1, \ldots, k+1 \right).
\]

Let us assume that for an arbitrary \( k > 5 \) set \( \mathcal{I}_{2,k}^k \) has only these two elements. Then we can prove the restricted version of the conjecture showing that the only potential with spectrum \( \mathcal{A}_{1,k} \) is the potential separable in the Cartesian coordinates, and the only potential with spectrum \( \mathcal{A}_{2,k} \) is a potential separable in parabolic coordinates. Unfortunately, the assumption about the number of elements of the set \( \mathcal{I}_{2,k}^k \) is true only for \( k \leq 13 \). For \( k = 14 \), set \( \mathcal{I}_{2,k}^k \) contains two additional elements

\[
\mathcal{A}_{3,k} = \left( -\frac{15}{28}, 12, \ldots, 12, \frac{377}{28}, \frac{377}{28}, 15, 15, 780, 5655 \right),
\]

\[
\mathcal{A}_{4,k} = \left( -\frac{15}{28}, 12, \ldots, 12, \frac{377}{28}, \frac{377}{28}, 39, 39, 5655 \right).
\]

Then, additional elements of \( \mathcal{I}_{2,k}^k \) appear for \( k = 17, 19, 26, 32, \ldots \), and it seems that there is no upper bound for these exceptional values of \( k \).

The presented method can be applied effectively only for small values of \( n \) and \( k \). With the known computational algorithms for determination of distinguished spectra and reconstruction of potential, it seems that limiting values are \( n = 3 \) and \( k < 5 \).

We believe that a substantial progress in this field will be possible after development of the higher order variational equations techniques. Till now, they are used only in cases when the first order variational equations have a very specific form, see [36].

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