ON THE CARLEMAN ULTRADIFFERENTIABLE VECTORS OF A SCALAR TYPE SPECTRAL OPERATOR

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To Academician Yu. M. Berezansky in honor of his 90th jubilee

Abstract. A description of the Carleman classes of vectors, in particular the Gevrey classes, of a scalar type spectral operator in a reflexive complex Banach space is shown to remain true without the reflexivity requirement. A similar nature description of the entire vectors of exponential type, known for a normal operator in a complex Hilbert space, is generalized to the case of a scalar type spectral operator in a complex Banach space.

Never cut what you can untie.
Joseph Joubert

1. Introduction

The description of the Carleman classes of ultradifferentiable vectors, in particular the Gevrey classes, of a normal operator in a complex Hilbert space in terms of its spectral measure established in [10] (see also [12] and [11]) is generalized in [15, Theorem 3.1] to the case of a scalar type spectral operator in a complex reflexive Banach space.

Here, the reflexivity requirement is shown to be superfluous and a similar nature description of the entire vectors of exponential type, known for a normal operator in a complex Hilbert space (see, e.g., [12]), is generalized to the case of a scalar type spectral operator in a complex Banach space.

2. Preliminaries

For the reader’s convenience, we shall outline in this section certain essential preliminaries.

2.1. Scalar type spectral operators. Henceforth, unless specified otherwise, $A$ is supposed to be a scalar type spectral operator in a complex Banach space $(X, \| \cdot \|)$ and $E_A(\cdot)$ to be its spectral measure (the resolution of the identity), the operator’s spectrum $\sigma(A)$ being the support for the latter [1, 4].

In a complex Hilbert space, the scalar type spectral operators are precisely those similar to the normal ones [23].

A scalar type spectral operator in complex Banach space has an operational calculus analogous to that of a normal operator in a complex Hilbert space [1, 8, 4]. To any Borel measurable function $F : \mathbb{C} \to \mathbb{C}$ (or $F : \sigma(A) \to \mathbb{C}$, $\mathbb{C}$ is the complex plane), there corresponds a scalar type spectral operator

$$F(A) := \int_{\mathbb{C}} F(\lambda) \, dE_A(\lambda) = \int_{\sigma(A)} F(\lambda) \, dE_A(\lambda)$$

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defined as follows:

\[ F(A)f := \lim_{n \to \infty} F_n(A)f, \quad f \in D(F(A)), \]

\[ D(F(A)) := \left\{ f \in X \mid \lim_{n \to \infty} F_n(A)f \text{ exists} \right\} \]

\((D(\cdot)\) is the domain of an operator), where

\[ F_n(\cdot) := F(\cdot)\chi_{\{\lambda \in \sigma(A) \mid |F(\lambda)| \leq n\}}(\cdot), \quad n \in \mathbb{N}, \]

\((\chi_{\delta}(\cdot)\) is the characteristic function of a set \(\delta \subseteq \mathbb{C}, \mathbb{N} := \{1, 2, 3, \ldots\}\) is the set of natural numbers) and

\[ F_n(A) := \int_{\sigma(A)} F_n(\lambda) \, dE_A(\lambda), \quad n \in \mathbb{N}, \]

are bounded scalar type spectral operators on \(X\) defined in the same manner as for a normal operator (see, e.g., [3, 21]).

In particular,

\[ A^n = \int_\mathbb{C} \lambda^n \, dE_A(\lambda) = \int_{\sigma(A)} \lambda^n \, dE_A(\lambda), \quad n \in \mathbb{Z}_+, \]

\((\mathbb{Z}_+ := \{0, 1, 2, \ldots\}\) is the set of nonnegative integers).

The properties of the spectral measure \(E_A(\cdot)\) and the operational calculus, exhaustively delineated in [1, 4], underly the entire subsequent discourse. Here, we shall outline a few facts of particular importance.

Due to its strong countable additivity, the spectral measure \(E_A(\cdot)\) is bounded [2, 4], i.e., there is such an \(M > 0\) that, for any Borel set \(\delta \subseteq \mathbb{C},\)

\[ \|E_A(\delta)\| \leq M. \]

The notation \(\| \cdot \|\) has been recycled here to designate the norm in the space \(L(X)\) of all bounded linear operators on \(X\). We shall adhere to this rather common economy of symbols in what follows adopting the same notation for the norm in the dual space \(X^\ast\) as well.

For any \(f \in X\) and \(g^\ast \in X^\ast\), the total variation \(v(f, g^\ast, \cdot)\) of the complex-valued Borel measure \(E_A(\cdot)f, g^\ast\) \((\cdot, \cdot)\) is the pairing between the space \(X\) and its dual \(X^\ast\) is a finite positive Borel measure with

\[ v(f, g^\ast, C) = v(f, g^\ast, \sigma(A)) \leq 4M\|f\|\|g^\ast\| \]

(see, e.g., [15]). Also (Ibid.), \(F : \mathbb{C} \to \mathbb{C}\) (or \(F : \sigma(A) \to \mathbb{C}\) being an arbitrary Borel measurable function, for any \(f \in D(F(A)), g^\ast \in X^\ast\), and an arbitrary Borel set \(\sigma \subseteq \mathbb{C},\)

\[ \int_{\sigma} |F(\lambda)| \, dv(f, g^\ast, \lambda) \leq 4M\|E_A(\sigma)F(A)f\|\|g^\ast\|. \]

In particular,

\[ \int_\mathbb{C} |F(\lambda)| \, dv(f, g^\ast, \lambda) = \int_{\sigma(A)} |F(\lambda)| \, dv(f, g^\ast, \lambda) \leq 4M\|F(A)f\|\|g^\ast\|. \]

The constant \(M > 0\) in [2.3] \(\text{is from}\) [2.4].

The following statement allowing to characterize the domains of the Borel measurable functions of a scalar type spectral operator in terms of positive Borel measures is also fundamental for our discussion.

**Proposition 2.1.** ([14] Proposition 3.1). Let \(A\) be a scalar type spectral operator in a complex Banach space \((X, \| \cdot \|)\) and \(F : \mathbb{C} \to \mathbb{C}\) (or \(F : \sigma(A) \to \mathbb{C}\)) be Borel measurable function. Then \(f \in D(F(A))\) iff

(i) For any \(g^\ast \in X^\ast\),

\[ \int_{\sigma(A)} |F(\lambda)| \, dv(f, g^\ast, \lambda) < \infty. \]
are obvious.

\[ C \left( \frac{2}{7} \right) \quad C \left( \frac{2}{6} \right) \]

\[ \text{The subspaces of } 2.2. \text{ The Carleman classes of vectors.} \]

\[ \text{Let } A \text{ be a densely defined closed linear operator in a complex Banach space } (X, \| \cdot \|) \text{ and } \{ m_n \}_{n=0}^{\infty} \text{ be a sequence of positive numbers and} \]

\[ C^\infty(A) := \bigcap_{n=0}^{\infty} D(A^n). \]

The subspaces of \( C^\infty(A) \)

\[ C_{(m_n)}(A) := \{ f \in C^\infty(A) \mid \exists \alpha > 0 \exists c > 0 : \| A^\alpha f \| \leq c\alpha^m m_n, \ n \in \mathbb{Z}_+ \}, \]

\[ C_{(m_n')} (A) := \{ f \in C^\infty(A) \mid \forall \alpha > 0 \exists c > 0 : \| A^\alpha f \| \leq c\alpha^m m_n, \ n \in \mathbb{Z}_+ \} \]

are called the Carleman classes of ultradifferentiable vectors of the operator \( A \) corresponding to the sequence \( \{ m_n \}_{n=0}^{\infty} \) of Roumieu and Beurling type, respectively.

The inclusions

\[ C_{(m_n)}(A) \subseteq C_{(m_n')} (A) \subseteq C^\infty(A) \subseteq X \]

are obvious.

If two sequences of positive numbers \( \{ m_n \}_{n=0}^{\infty} \) and \( \{ m_n' \}_{n=0}^{\infty} \) are related as follows:

\[ \forall \gamma > 0 \exists c = c(\gamma) > 0 : m_n' \leq c\gamma^n m_n, \quad n \in \mathbb{Z}_+, \]

we also have the inclusion

\[ C_{(m_n')} (A) \subseteq C_{(m_n)}(A), \]

the sequences being subject to the condition

\[ \exists \gamma_1, \gamma_2 > 0, \exists c_1, c_2 > 0 : c_1\gamma_1^n m_n \leq m_n' \leq c_2\gamma_2^n m_n, \quad n \in \mathbb{Z}_+, \]

their corresponding Carleman classes coincide

\[ C_{(m_n)}(A) = C_{(m_n')} (A), \quad C_{(m_n)}(A) = C_{(m_n')} (A). \]

Considering Stirling’s formula and the latter,

\[ E^{(\beta)}(A) := C_{[ln]^{\beta}}(A), \quad C_{[ln]^{\beta}}(A) \]

are the well-known Gevrey classes of strongly ultradifferentiable vectors of \( A \) of order \( \beta \) of Roumieu and Beurling type, respectively (see, e.g., \([10, 11, 12]\)). In particular, \( E^{(1)}(A) \) and \( E^{(1)}(A) \) are the well-known classes of analytic and entire vectors of \( A \), respectively \([6, 19]\). \( E^{(0)}(A) \) and \( E^{(0)}(A) \) (i.e., the classes \( C_{(1)}(A) \) and \( C_{(1)}(A) \) corresponding to the sequence \( m_n = 1 \)) are the classes of entire vectors of exponential and minimal exponential type, respectively (see, e.g., \([22, 12]\)).

If the sequence of positive numbers \( \{ m_n \}_{n=0}^{\infty} \) satisfies the condition

\[ \forall \alpha > 0 \exists c = c(\alpha) > 0 : c\alpha^n \leq m_n, \quad n \in \mathbb{Z}_+, \]

the scalar function

\[ T(\lambda) := m_0 \sum_{n=0}^{\infty} \frac{\lambda^n}{m_n}, \quad \lambda \geq 0, \quad (0^0 := 1) \]

first introduced by S. Mandelbrojt \([13]\), is well-defined (cf. \([12]\)). The function is continuous, strictly increasing, and \( T(0) = 1 \).
As is shown in [10] (see also [12] and [11]), the sequence \( \{m_n\}_{n=0}^{\infty} \) satisfying the condition \((\text{WGR})\), for a normal operator \( A \) in a complex Hilbert space \( X \), the equalities

\[
C_{\{m_n\}}(A) = \bigcup_{t>0} D(T(t|A|)),
\]

(2.11)

\[
C_{\{m_n\}}(A) = \bigcap_{t>0} D(T(t|A|))
\]

are true, the normal operators \( T(t|A|), t>0 \), defined in the sense of the operational calculus for a normal operator (see, e.g., [3, 21]) and the function \( T(\cdot) \) being replaceable with any nonnegative, continuous, and increasing on \([0, \infty)\) function \( F(\cdot) \) satisfying

\[
c_1 F(\gamma_1 \lambda) \leq T(\lambda) \leq c_2 F(\gamma_2 \lambda), \quad \lambda \geq R,
\]

with some \( \gamma_1, \gamma_2, c_1, c_2 > 0 \) and \( R \geq 0 \), in particular, with

\[
S(\lambda) := m_0 \sup_{n \geq 0} \frac{\lambda^n}{m_n}, \quad \lambda \geq 0, \quad \text{or} \quad P(\lambda) := m_0 \sum_{n=0}^{\infty} \frac{\lambda^{2n} \gamma^{1/2}}{m_n^{2n}}, \quad \lambda \geq 0,
\]

(cf. [12]).

In [15, Theorem 3.1], the above is generalized to the case of a scalar type spectral operator \( A \) in a reflexive complex Banach space \( X \). The reflexivity requirement dropped, proved were the inclusions

\[
C_{\{m_n\}}(A) \supseteq \bigcup_{t>0} D(T(t|A|))
\]

(2.13)

\[
C_{\{m_n\}}(A) \supseteq \bigcap_{t>0} D(T(t|A|))
\]

only, which is a deficiency for statements like [16, Theorem 5, WGR] and [18, Theorem 3.2].

3. The Carleman Classes of a Scalar Type Spectral Operator

**Theorem 3.1.** Let \( \{m_n\}_{n=0}^{\infty} \) be a sequence of positive numbers satisfying the condition \((\text{WGR})\) (see (2.9)). Then, for a scalar type spectral operator \( A \) in a complex Banach space \((X, \| \cdot \|)\), equalities (2.11) are true, the scalar type spectral operators \( T(t|A|) \), \( t>0 \), defined in the sense of the operational calculus for a scalar type spectral operator and the function \( T(\cdot) \) being replaceable with any nonnegative, continuous, and increasing on \([0, \infty)\) function \( F(\cdot) \) satisfying (2.12).

Proof. We are only to prove the inclusions inverse to (2.13), the rest, including the latter, having been proved in [15, Theorem 3.1].

Consider an arbitrary vector \( f \in C_{\{m_n\}}(A) \) (\( f \in C_{\{m_n\}}(A) \)). Then necessarily, \( f \in C^\infty(A) \) and for a certain \( \alpha > 0 \) (an arbitrary \( \alpha > 0 \)), there is a \( c > 0 \) such that

\[
\|A^n f\| \leq c \alpha^n m_n, \quad n \in \mathbb{Z}_+,
\]

(see Preliminaries).

For any \( g^* \in X^* \),

\[
\int_{\sigma(A)} T \left( \frac{\lambda}{m^n} \right) dv(f, g^*, \lambda) = \int_{\sigma(A)} \sum_{n=0}^{\infty} \frac{|\lambda|^n}{2^n \alpha^n m_n} dv(f, g^*, \lambda)
\]

by the Monotone Convergence Theorem;

\[
= \sum_{n=0}^{\infty} \int_{\sigma(A)} \frac{|\lambda|^n}{2^n \alpha^n m_n} dv(f, g^*, \lambda) = \sum_{n=0}^{\infty} \frac{1}{2^n \alpha^n m_n} \int_{\sigma(A)} |\lambda|^n dv(f, g^*, \lambda)
\]

by (2.60) and (2.1).
Hence, there is a $K$ whenever
\[
\sup_{n=0}^{\infty} \frac{1}{2^n \alpha^m} 4M \| A^n f \| g^* \leq 4Mc \sum_{n=0}^{\infty} \frac{1}{2^n} \| g^* \| = 8Mc \| g^* \| < \infty.
\]

For an arbitrary $\varepsilon > 0$, one can fix an $N \in \mathbb{N}$ such that,
\[
(3.16) \quad \frac{M^2c}{2^{N-2}} < \varepsilon/2.
\]

Due to the strong continuity of the s.m., for any $n \in \mathbb{N},$
\[
\| E_A \left( \{ \lambda \in \sigma(A) \mid T(\frac{1}{2^n} |\lambda|) > k \} \right) A^n f \| \to 0 \quad \text{as} \quad k \to \infty.
\]
Hence, there is a $K \in \mathbb{N}$ such that
\[
(3.17) \quad \sum_{n=0}^{N} \frac{1}{2^n \alpha^m} 4M \| E_A \left( \{ \lambda \in \sigma(A) \mid T(\frac{1}{2^n} |\lambda|) > k \} \right) A^n f \| < \varepsilon/2
\]
whenever $k \geq K$.

Similarly to (3.15), for $k \geq K$, we have
\[
\sup_{\{g \in X^* \mid \| g^* \|=1\}} \int \left\{ \lambda \in \sigma(A) \mid T(\frac{1}{2^n} |\lambda|) > k \right\} \frac{T(\frac{1}{2^n} |\lambda|)}{\lambda} dv(f, g^*, \lambda)
\]
\[
= \sup_{\{g \in X^* \mid \| g^* \|=1\}} \sum_{n=0}^{\infty} \frac{1}{2^n \alpha^m} \int \left\{ \lambda \in \sigma(A) \mid T(\frac{1}{2^n} |\lambda|) > k \right\} |\lambda|^n dv(f, g^*, \lambda)
\]
by (2.4) and (2.1);
\[
\leq \sup_{\{g \in X^* \mid \| g^* \|=1\}} \sum_{n=0}^{N} \frac{1}{2^n \alpha^m} 4M \| E_A \left( \{ \lambda \in \sigma(A) \mid T(\frac{1}{2^n} |\lambda|) > k \} \right) A^n f \| \| g^* \|
\]
\[
\leq \sup_{\{g \in X^* \mid \| g^* \|=1\}} \left[ \sum_{n=0}^{N} \frac{1}{2^n \alpha^m} 4M \| E_A \left( \{ \lambda \in \sigma(A) \mid T(\frac{1}{2^n} |\lambda|) > k \} \right) A^n f \| \| g^* \|
\right]
+ \sum_{n=N+1}^{\infty} \frac{1}{2^n \alpha^m} \| E_A \left( \{ \lambda \in \sigma(A) \mid T(\frac{1}{2^n} |\lambda|) > k \} \right) \| A^n f \| \| g^* \|
\]
by (2.2) and (3.14);
\[
\leq \sup_{\{g \in X^* \mid \| g^* \|=1\}} \left[ \sum_{n=0}^{N} \frac{1}{2^n \alpha^m} 4M \| E_A \left( \{ \lambda \in \sigma(A) \mid T(\frac{1}{2^n} |\lambda|) > k \} \right) A^n f \| \| g^* \|
\right.
+ \left. 4M^2c \sum_{n=N+1}^{\infty} \frac{1}{2^n} \| g^* \| \right]
\]
\[
\leq \sum_{n=0}^{N} \frac{1}{2^n \alpha^m} 4M \| E_A \left( \{ \lambda \in \sigma(A) \mid T(\frac{1}{2^n} |\lambda|) > k \} \right) A^n f \| + \frac{M^2c}{2^{N-2}}
\]
by (3.16) and (3.17);
\[
< \varepsilon/2 + \varepsilon/2 = \varepsilon
\]
and we conclude that

\[(3.18) \quad \sup_{\{g^* \in X^* : \|g^*\| = 1\}} \int_{\{\lambda \in \sigma(A) : |T(\frac{1}{2\alpha} |\lambda|) > k\}} T\left(\frac{1}{2\alpha} |\lambda|\right) \, dv(f, g^*, \lambda) \to 0 \quad \text{as} \quad k \to \infty.\]

By Proposition 2.1 (3.15) and (3.18) imply

\[f \in D(T(\frac{1}{2\alpha} |A|)).\]

Considering that for \(f \in C_{m_n}(A)\), \(\alpha > 0\) is fixed and for \(f \in C_{m_n}(A)\), \(\alpha > 0\) is arbitrary, we infer that

\[f \in \bigcup_{t > 0} D(T(\frac{1}{2\alpha} |A|))\]

in the former case and

\[f \in \bigcap_{t > 0} D(T(\frac{1}{2\alpha} |A|))\]

in the latter.

Since \(f \in C_{m_n}(A)\) \((f \in C_{m_n}(A))\) is arbitrary, we have proved the inclusions

\[C_{m_n}(A) \subseteq \bigcup_{t > 0} D(T(\frac{1}{2\alpha} |A|)),\]

\[C_{m_n}(A) \subseteq \bigcap_{t > 0} D(T(\frac{1}{2\alpha} |A|)),\]

which along with their inverses (2.13) imply equalities (2.11) to be true. \(\square\)

4. The Gevrey classes

The sequence \(m_n := [n!]^\beta \ (m_n := n^{\beta n})\) with \(\beta > 0\) satisfying the condition \((\text{WGR})\) and the corresponding function \(T(\cdot)\) being replaceable with \(F(\lambda) = e^{\lambda^{1/\beta}}, \lambda \geq 0\), (see [15] for details, cf. also [12]), in [15, Corollary 4.1] describing the Gevrey classes of vectors of a scalar type spectral operator in a reflexive complex Banach space, the reflexivity requirement can be dropped as well and we have the following

Corollary 4.1. Let \(\beta > 0\). Then, for a scalar type spectral operator \(A\) in a complex Banach space \((X, \| \cdot \|)\),

\[E^{(\beta)}(A) = \bigcup_{t > 0} D(e^{t|A|^{1/\beta}}),\]

\[E^{(\beta)}(A) = \bigcap_{t > 0} D(e^{t|A|^{1/\beta}}).\]

Corollary 4.1 generalizes the corresponding result of [10] (see also [12, 11]) for a normal operator \(A\) in a complex Hilbert space and, for \(\beta = 1\), gives a description of the analytic and entire vectors of a scalar type spectral operator \(A\) in a complex Banach space.

5. The entire vectors of exponential type

Observe that the sequence \(m_n \equiv 1\) generating the entire vectors of exponential type does not meet the condition \((\text{WGR})\) (see (2.9)) and thus, this case falls outside the realm of Theorem 3.1.

As is known (cf., e.g., [12, 7]), for a normal operator \(A\) in a complex Hilbert space \(X\),

\[E^{(0)}(A) = \bigcup_{\alpha > 0} E_A(\Delta_\alpha) X\]

and

\[E^{(0)}(A) = \bigcap_{\alpha > 0} E_A(\Delta_\alpha) X = E_A(\{0\}) X = \ker A := \{f \in X : Af = 0\}\]
$(E_A(\cdot))$ is the spectral measure of $A$) with
\[ \Delta_\alpha := \{ \lambda \in \mathbb{C} \mid |\lambda| \leq \alpha \}, \quad \alpha > 0. \]

We are to generalize the above to the case of a scalar type spectral operator $A$ in a complex Banach space $X$.

**Theorem 5.1.** For a scalar type spectral operator $A$ in a complex Banach space $(X, \| \cdot \|)$,

1. $E^{(0)}(A) = \bigcup_{\alpha > 0} E_A(\Delta_\alpha) X$,
2. $E^{(0)}(A) = \bigcap_{\alpha > 0} E_A(\Delta_\alpha) X = E_A(\{0\}) X = \ker A := \{ f \in X \mid Af = 0 \}$,

where $\Delta_\alpha := \{ \lambda \in \mathbb{C} \mid |\lambda| \leq \alpha \}, \alpha > 0$.

**Proof.** Let $f \in \bigcup_{\alpha > 0} E_A(\Delta_\alpha) X$ ($f \in \bigcap_{\alpha > 0} E_A(\Delta_\alpha) X$), i.e.,
\[ f = E_A \left( \{ \lambda \in \mathbb{C} \mid |\lambda| \leq \alpha \} \right) f \]
for some (any) $\alpha > 0$. Then, by the properties of the o.c.,
\[ f \in C^\infty(A). \]

Furthermore, as follows form the Hahn-Banach Theorem and the properties of the o.c. (in particular, (2.1)),
\[ \| A^n f \| = \left\| \int \lambda^n dE_A(\lambda) f \right\| = \left\| \int_{\{ \lambda \in \mathbb{C} \mid |\lambda| \leq \alpha \}} \lambda^n dE_A(\lambda) f \right\| \]
\[ = \sup_{g^* \in X^*, \| g^* \| = 1} \left| \int_{\{ \lambda \in \mathbb{C} \mid |\lambda| \leq \alpha \}} \lambda^n dE_A(\lambda) f, g^* \right| \]
\[ = \sup_{g^* \in X^*, \| g^* \| = 1} \int_{\{ \lambda \in \mathbb{C} \mid |\lambda| \leq \alpha \}} \lambda^n d\langle E_A(\lambda) f, g^* \rangle \]
\[ \leq \sup_{g^* \in X^*, \| g^* \| = 1} \int_{\{ \lambda \in \mathbb{C} \mid |\lambda| \leq \alpha \}} |\lambda|^n dv(f, g^*, \lambda) \]
\[ \leq \sup_{g^* \in X^*, \| g^* \| = 1} \alpha^n v(f, g^*, \{ \lambda \in \mathbb{C} \mid |\lambda| \leq \alpha \}) \]
\[ \leq \sup_{g^* \in X^*, \| g^* \| = 1} 4M \| f \| \| g^* \| \alpha^n \]
which implies that $f \in E^{(0)}(A)$ ($f \in E^{(0)}(A)$).

Conversely, for an arbitrary $f \in E^{(0)}(A)$ ($f \in E^{(0)}(A)$),
\[ \| A^n f \| \leq \alpha^n, \quad n \in \mathbb{Z}_+, \]
with some (any) $\alpha > 0$ and some $c > 0$.

Then, for any $\gamma > \alpha$ and $g^* \in X^*$, we have
\[ \gamma^n v(f, g^*, \{ \lambda \in \mathbb{C} \mid |\lambda| \geq \gamma \}) \leq \int_{\{ \lambda \in \mathbb{C} \mid |\lambda| \geq \gamma \}} |\lambda|^n dv(f, g^*, \lambda) \]
\[ \leq \int_{\mathbb{C}} |\lambda|^n dv(f, g^*, \lambda) \leq 4M \| A^n f \| \| g^* \| \alpha^n, \quad n \in \mathbb{Z}_+. \]

Therefore,
\[ v(f, g^*, \{ \lambda \in \mathbb{C} \mid |\lambda| \geq \gamma \}) \leq 4Mc\| g^* \| \left( \frac{\alpha}{\gamma} \right)^n, \quad n \in \mathbb{Z}_+. \]

Considering that $\alpha/\gamma < 1$ and passing to the limit as $n \to \infty$, we conclude that
\[ v(f, g^*, \{ \lambda \in \mathbb{C} \mid |\lambda| \geq \gamma \}) = 0, \quad g^* \in X^*. \]
and the more so
\[ \langle E_A(\{ \lambda \in \mathbb{C} | ||\lambda|| \geq \gamma \}) f, g^* \rangle = 0, \quad g^* \in X^*. \]

Whence, as follows from the Hahn-Banach Theorem,
\[ E_A(\{ \lambda \in \mathbb{C} | ||\lambda|| \geq \gamma \}) f = 0, \]
which, considering that \( \gamma > \alpha \) is arbitrary, by the strong continuity of the s.m., implies that
\[ E_A(\{ \lambda \in \mathbb{C} | ||\lambda|| > \alpha \}) f = 0. \]

Hence, by the additivity of the s.m.,
\[ f = E_A(\{ \lambda \in \mathbb{C} | ||\lambda|| \leq \alpha \}) f + E_A(\{ \lambda \in \mathbb{C} | ||\lambda|| > \alpha \}) f = E_A(\{ \lambda \in \mathbb{C} | ||\lambda|| \leq \alpha \}) f, \]
which implies that \( f \in \bigcup_{\alpha > 0} E_A(\Delta_\alpha) X \) (\( f \in \bigcap_{\alpha > 0} E_A(\Delta_\alpha) X \)). □

An immediate implication of Theorem 5.1 is the following generalization of the well-known result on the denseness of exponential type vectors of a normal operator in a complex Hilbert space (see, e.g., [12]), which readily follows by the strong continuity of the s.m. and joins a number of similar results of interest for approximation and qualitative theories (see [22, 12, 7, 8, 9]).

Corollary 5.1. For a scalar-type spectral operator \( A \) in a complex Banach space \( (X, ||\cdot||) \),
\[ \overline{E(0)}(A) = X \]
\( (\overline{\cdot} \) is the closure of a set in the strong topology of \( X \)).

Hence, for any positive sequence \( \{m_n\}_{n=0}^\infty \) satisfying the condition (WGR), in particular, for \( m_n = [n!]^\beta \) with \( \beta > 0 \), due to inclusion (2.7) with \( m'_n \equiv 1 \),
\[ E(0)(A) \subseteq C_{(m_n)}(A), \]
which implies
\[ \overline{C_{(m_n)}(A)} = X. \]
ON THE CARLEMAN ULTRADIFFERENTIABLE VECTORS

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