Time parameterizations and spin supplementary conditions of the Mathisson-Papapetrou-Dixon equations

Georgios Lukes-Gerakopoulos

1 Astronomical Institute of the Academy of Sciences of the Czech Republic, Boční II 1401/1a, CZ-141 31 Prague, Czech Republic

The implications of two different time constraints on the Mathisson-Papapetrou-Dixon (MPD) equations are discussed under three spin supplementary conditions (SSC). For this reason the MPD equations are revisited without specifying the affine parameter and several relations are reintroduced in their general form. The latter allows to investigate the consequences of combining the Mathisson-Pirani (MP) SSC, the Tulczyjew-Dixon (TD) SSC and the Ohashi-Kyrian-Semerák (OKS) SSC with two affine parameter types: the proper time on one hand and the parameterizations introduced in [Gen. Rel. Grav. 8, 197 (1977)] on the other. For the MP SSC and the TD SSC it is shown that quantities that are constant of motion for the one affine parameter are not for the other, while for the OKS SSC it is shown that the two affine parameters are the same. To clarify the relation between the two affine parameters in the case of the TD SSC the MPD equations are evolved and discussed.

PACS numbers:

Keywords:

Contents

1. Introduction
2. Some useful relations
3. Choosing a worldline
4. Discussing the constraints
   4.1. The Mathisson-Pirani SSC
   4.2. The Tulczyjew-Dixon SSC
   4.3. The Ohashi-Kyrian-Semerák SSC
5. Numeric comparison for Tulczyjew-Dixon SSC
   5.1. Preliminary considerations
   5.2. Numerical results
6. Conclusions
Acknowledgments
References

1. INTRODUCTION

The motion of a small mass body whose effect on the spacetime background is negligible had been first studied in terms of the multipole moments of the body by Mathisson [1] and Papapetrou [2]. A covariant formalism was achieved by Dixon in [3], who also reformulated the respective equations of motion. These equations of motion are known now as Mathisson-Papapetrou-Dixon (MPD) equations.

In the case of a solely gravitational interaction within the pole-dipole approximation the MPD equations read

\[ \ddot{p}^\mu = -\frac{1}{2} R^\mu_{\nu\kappa\lambda} v^\nu S^\kappa\lambda , \]  

(1)

\[ \dot{S}^{\mu\nu} = p^\mu v^\nu - v^\mu p^\nu \equiv 2p^{[\mu} v^{\nu]} , \]  

(2)

where \( p^\mu \) is the four-momentum, \( v^\mu = \frac{dx^\mu}{d\chi} \) is the tangent vector and \( S^{\mu\nu} \) is the spin tensor of the body. Moreover, \( R^\mu_{\nu\kappa\lambda} \) is the Riemann tensor and the dot denotes a covariant differentiation along the worldline \( x^\mu(\chi) \), where \( \chi \) is an evolution parameter along the worldline not necessarily the proper time \( \tau \). Thus, it is not assumed that the tangent vector is the four-velocity, and the contraction

\[ v^\mu v_\mu \equiv -v^2 \]  

(3)

does not represent necessarily the four-velocity preservation \( v^2 = 1 \).

The notion of mass can be defined either with respect to the momentum \( p^\nu \), i.e.

\[ m^2 \equiv -p^\nu p_\nu , \]  

(4)

or with respect to the tangent vector \( v^\nu \), i.e.

\[ m_v \equiv -v^\nu p_\nu . \]  

(5)

Units and notation: The units employed in this work are geometric (\( G = c = 1 \)), and the signature of the metric \( g_{\mu\nu} \) is (+,+,+,+). Greek letters denote the indices corresponding to spacetime (running from 0 to 3). The Riemann tensor is defined as \( R^\rho_{\beta\gamma\delta} = \Gamma^\rho_{\gamma\lambda} \Gamma^\lambda_{\delta\beta} - \partial_\gamma \Gamma^\rho_{\lambda\delta} - \Gamma^\lambda_{\rho\delta} \Gamma^\rho_{\gamma\beta} + \partial_\delta \Gamma^\rho_{\gamma\beta} \), where \( \Gamma \) are the Christoffel symbols. The Levi-Civita tensor is \( \epsilon_{\kappa\lambda\mu\nu} = \sqrt{-g} \epsilon_{\kappa\lambda\mu\nu} \) with the Levi-Civita symbol defined as \( \epsilon_{0123} = 1 \).
2. SOME USEFUL RELATIONS

By keeping in mind that \( v^2 \) is not necessarily constant, some useful consequences of MPD equations are presented below. These consequences coincide with expressions presented in [4] when \( v^2 = 1 \).

Contracting Eq. (1) with \( v_\mu \) gives
\[
p^{\mu}v_\mu = 0 \quad .
\] (6)

Contraction of Eq. (2) with \( v_\mu \) gives
\[
p^{\mu} = \frac{1}{v^2}(m_v v^{\mu} - \dot{S}^{\mu\nu}v_\nu) \quad ,
\] (7)

while contracting with \( \dot{v}_\mu \) and using relation (7) gives
\[
\dot{v}_\mu \dot{S}^{\mu\nu} = \frac{v^2}{m_v}(v^{\mu} \dot{S}^{\nu\rho} - \dot{S}^{\mu\nu}v^{\rho})v_\rho \quad .
\] (8)

Contracting Eq. (2) with \( p_\mu \) gives
\[
p^{\mu} = \frac{1}{m_v}(m_v v^{\mu} - \dot{S}^{\mu\nu}p_\nu) \quad ,
\] (9)

while contracting with \( \dot{p}_\mu \) and using relation (9) gives
\[
\dot{p}_\mu \dot{S}^{\mu\nu} = \frac{1}{m_v}\dot{p}_\mu \dot{S}^{\nu\rho}p_\rho v^{\mu} \quad .
\] (10)

Contracting Eq. (2) with \( \dot{v}^{\mu} \) leads to
\[
m_v^2 - m_v v^2 = v_\mu \dot{S}^{\mu\nu}p_\nu \quad ,
\] (11)

which combined with Eq. (7) gives
\[
m_v^2 v^4 - m_v v^2 = v^{\mu} \dot{S}^{\mu\nu}v_\nu \quad .
\] (12)

Furthermore, one finds that the evolution equation of the mass \( m \) is
\[
m = -\frac{1}{m_v}p^{\mu}\dot{p}_\mu = \frac{1}{m}\frac{1}{m_v}p_\mu \dot{S}^{\mu\nu}p_\nu \quad ,
\] (13)

for which result Eqs. (11), (10) are used, while the evolution equation of the mass \( m_v \) is
\[
m_v = -\frac{1}{v^2}(v_\mu \dot{S}^{\mu\nu} + m_v v^{\nu})\dot{v}_\nu \quad ,
\] (14)

for which result Eq. (9) is used, and Eq. (6) is taken into account.

The square of the spin’s measure is
\[
S^2 = \frac{1}{2} S_{\mu\nu} S^{\mu\nu} \quad ,
\] (15)

and its evolution equation reads
\[
\dot{S}^2 = 2p_\mu S^{\mu\nu}v_\nu \quad ,
\] (16)
in which calculation Eq (2) is used.

3. CHOOSING A WORLDLINE

The MPD equation system, consisted of (1), (2) and \( dx^{\mu}/d\chi = v^\mu \), is under-defined. Namely, there are only 14 independent equations of motion for the 18 variables \( (x^{\mu}, v^{\mu}, p^{\mu}, S^{\mu\nu}) \). To define a worldline we have to supplement the system with 4 additional constraints.

One of these constraints comes from choosing the evolution parameter \( \chi \). A common choice for the evolution parameter is to identify \( \chi \) with the proper time \( \tau \), see, e.g., [3]. Then, \( v^2 = 1 \) and the tangent vector \( v^\mu \) identifies with the four-velocity. Another interesting choice was introduced in [4], according to which \( \chi \) scales in such way that
\[
v_\mu u^{\mu} = -1 \quad ,
\] (17)

where \( u^{\mu} = p^{\mu}/m \). An apparent consequence of this choice is that \( m_v = m \). This affine parameter is denoted as \( \sigma \).

After having chosen the evolution parameter, the remaining necessary constraints are devoted to choose the center of the mass of the system. The center of mass is called often centroid. By choosing the centroid and the evolution parameter one defines the evolution along the worldline that the body described by the MP equations follows. In particular, the centroid is fixed by choosing an observer through a time-like vector \( V^\mu \) for which \( V_\mu S^{\mu\nu} = 0 \). This constraint is known as spin supplementary condition (SSC). In the bibliography there are five established choices of SSC:

(a) the Mathisson-Pirani (MP) condition \( V^\mu = v^\mu \) [1].

(b) the Tulczyjew-Dixon (TD) condition \( V^\mu = u^\mu \) [6].

(c) the Corinaldesi-Papapetrou condition \( V^\mu = v_{lab}^\mu \), where \( v_{lab} \) is a a congruence of “laboratory” observers.

(d) the Newton-Wigner condition \( V^\mu \propto v_{lab} + u^\mu \) [10].

(e) the Ohashi-Kyrian-Semerák (OKS) condition [11, 12]), for which the \( V^\mu \) is chosen in such way that \( p^{\mu} \parallel \dot{v}^{\mu} \).

4. DISCUSSING THE CONSTRAINTS

In this section are investigated the consequences of combining the time constraints and particularly the condition (17) with the MP SSC, the TD SSC and the OKS SSC. For these three SSCs Eq. (10) shows that the measure of the spin is conserved independently from the time constraint choice.

\[ 1 \quad \text{Because the spin tensor} \ S^{\mu\nu} \ \text{is antisymmetric, it contributes only} \ 6 \ \text{independent equations and variables.} \]
4.1. The Mathisson-Pirani SSC

Contracting the covariant derivative of MP SSC with \( \dot{v}_\nu \) results in \( v_\mu \dot{S}^{\mu\nu} \dot{v}_\nu = 0 \), taking this into account Eq. (14) for the MP SSC gives

\[
\frac{m}{m_v} = \frac{v^2}{2v^2} \Rightarrow \frac{m^2}{v^2} = \text{const. } .
\]

For the condition \( v^2 = 1 \), Eq. (18) gives that \( m_v \) is a constant of motion. For the condition (17), Eq. (18) gives that

\[
\frac{m^2}{v^2} = \text{const. } ,
\]

since \( m = m_v \).

4.2. The Tulczyjew-Dixon SSC

For TD SSC Eq. (13) shows that the mass \( m \) is a constant of motion. Since for the condition (17) \( m = m_v \), then \( m_v \) is constant as well. Thus, Eq. (14) gives

\[
(v_\mu \dot{S}^{\mu\nu} + m_v v^\nu) \dot{v}_\nu = 0 .
\]

According to Eq. (20), if \( v^\nu \dot{v}_\nu = 0 \), then it holds that \( v_\mu \dot{S}^{\mu\nu} \dot{v}_\nu = 0 \) as well. The former implies that \( v^2 \) is a constant, while the latter implies that MP SSC holds along with the assumed TD SSC. The last implication is proven as follows: contracting Eq. (20) with \( v_\nu \) gives \( \dot{S}^{\mu\nu} v_\nu = 0 \), because \( v_\mu \dot{S}^{\mu\nu} \dot{v}_\nu = 0 \) and it is reasonable to assume that \( v^2 = 0 \) is not the case. Since \( v^2 \) and \( m_v \) are constants and it has been shown that \( \dot{S}^{\mu\nu} v_\nu = 0 \), Eq. (17) results in \( p^\mu ||p^\nu \). If \( p^\mu ||p^\nu \), then Eq. (11) gives \( v^2 = 1 \). Therefore, when \( v^\nu \dot{v}_\nu = 0 \) the affine parameter defined by the condition (17) is the proper time, i.e. \( \sigma = \tau \).

The cases of TD SSC for which \( p^\mu ||p^\nu \) holds are very special cases. Thus, it is reasonable to assume that for the condition (17) in general \( v^\nu \dot{v}_\nu \neq 0 \) is true. Under this assumption, Eq. (20) gives

\[
m_v = \frac{v_\mu \dot{S}^{\mu\nu} \dot{v}_\nu}{v^\nu \dot{v}_\nu} = \text{const. } .
\]

In this case Eq. (11) implies that the variation of \( v^2 \) during the evolution is reflected on the \( v_\mu \dot{S}^{\mu\nu} p_\nu \) evolution. Actually, if one uses the \( v^2 = 1 \) condition instead of the condition (17), then the variation of \( m_v^2 \) during the evolution is reflected on the \( v_\mu \dot{S}^{\mu\nu} p_\nu \) evolution.

Another interesting relation comes from eq. (11), when one uses the covariant derivative of the TD SSC and then applies eq. (11), then we get

\[
v^2 = \frac{1}{m^2} \left( m_v^2 - \frac{1}{2} v_\sigma S^{\sigma\mu} R_{\mu\nu\rho\sigma} v^\nu S^{\rho\sigma} \right) .
\]

4.3. The Ohashi-Kyrian-Semerák SSC

Since for OKS SSC by definition \( p^\mu ||u^\mu \), then it holds that \( v_\mu \dot{S}^{\mu\nu} p_\nu = 0 \). Combing the latter with the fact that \( m = m_v \) for the condition (17), Eq. (11) gives that \( v^2 = 1 \). This means that OKS SSC is satisfying both time constraints simultaneously; or in other words the affine parameter \( \sigma \) is identical with the proper time \( \tau \) for OKS SSC. Note that the latter holds when \( (p^\mu ||u^\mu) \) independently of the implemented SSC.

5. NUMERIC COMPARISON FOR TULCZYJEW-DIXON SSC

This section examines the evolution of the MPD equations under the two time choices \( \tau \) and \( \sigma \). In particular, we are going to examine numerically the MPD under the TD SSC, since for OKS SSC the two evolution parameter choices are equivalent and for MP SSC the helical motion introduces an unnecessary complication.

5.1. Preliminary considerations

To do a numerical comparison, the first issue is the initial condition setup, i.e. the initial position, momentum and spin tensor have to be properly chosen. Since we have the same SSC (TD SSC), we have two observers with initially the same position \( x^\mu \). The definitions of the momentum and the spin tensor depend only on the position \( x^\mu \) along the worldline \( \gamma \), hence the initial conditions \( x^\mu \), \( p_\mu \) and \( S^{\mu\nu} \) for two observers are the same.

However, the two observers are equipped with clocks that do not tick the same, i.e. they follow different affine parameters. If the momentum and the spin tensor are not affected by the different choices of the affine parameter, then a time reparametrization of the MPD equations (11)–(22) just means that the MPD equations will reproduce the same worldline under different affine parameter \( \gamma \). The validity of the last statement is what is in this section checked.

To evolve the MPD with TD SSC, one needs the relation

\[
v^\mu = \frac{m_v}{m^2} \left( p^\mu + \frac{2}{4} S^{\mu\nu} R_{\nu\rho\sigma\lambda} p^\rho S^{\sigma\lambda} S^{\nu\delta} \right) ,
\]

which gives \( v^\mu \) as function of \( x^\mu \), \( p_\mu \) and \( S^{\mu\nu} \). An interesting fact about relation (23) is that its derivation does not depend on the time constraint (see, e.g., (15) for the derivation). A related fact is that the relation (23) is invariant under affine parameter changes, since the scalar \( m_v \) contains the tangent vector \( \nu^\mu \) (definition (15)).

The background, on which the MPD are to be evolved, is the Kerr spacetime. The metric tensor of Kerr in...
Boyer-Lindquist (BL) coordinates \( \{ t, r, \theta, \phi \} \) reads

\[
\begin{align*}
g_{tt} &= -1 + \frac{2Mr}{\Sigma}, \quad g_{t\phi} = -\frac{2aMr \sin^2 \theta}{\Sigma}, \\
g_{\phi\phi} &= \frac{\Lambda \sin^2 \theta}{\Sigma}, \quad g_{rr} = \frac{\Delta}{\Sigma}, \quad g_{\theta\theta} = \Sigma,
\end{align*}
\]

where

\[
\Sigma = r^2 + a^2 \cos^2 \theta, \quad \Delta = \omega^2 - 2Mr, \quad \omega = \omega^2 - a^2 \Delta \sin^2 \theta.
\]

and \( M \) defines the mass and \( a \) the Kerr parameter. The motion of a small spinning body in the stationary and axisymmetric Kerr spacetime preserves, respectively, the energy

\[
E = -p_t + \frac{1}{2} g_{\mu\nu} S^{\mu\nu},
\]

and the component of the total angular momentum along the symmetry axis \( z \)

\[
J_z = p_\phi - \frac{1}{2} g_{\phi\nu} S^{\mu\nu}.
\]

The MPD equations are valid when the size of the spinning body is much smaller than the curvature, i.e. when

\[
\lambda = \frac{|R_{\mu\nu\kappa\lambda}|}{\rho^2} \ll 1,
\]

where \( |R_{\mu\nu\kappa\lambda}| \) is the magnitude of the Riemann tensor and \( \rho \) is the radius of the spinning body. If the radius \( \rho \) is approximated by the Møller radius \( |S/m| \), then \( \rho = S/m \). Thus, since \( |R_{\mu\nu\kappa\lambda}| \sim M/r^3 \), we get

\[
\lambda \sim \left( \frac{S}{m M} \right)^2 \left( \frac{M}{r} \right)^3.
\]

For the computations of the MPD equations, the dimensionless counterparts of the involved quantities are employed. For example, in their dimensionless forms the spin of the small body reads \( S/(mM) \), the BL radius reads \( r/M \) and for the momentum holds that \( p^\mu = u^\mu \). Numerically the values of dimensionless quantities are equal to the dimensionful by setting \( M = m = 1 \). From this point on in the article there is no distinction between the dimensionful and the dimensionless quantities. Since our discussion is theoretical, the spin of the small body does not need to be very small\(^2\) as long as \( \lambda \ll 1 \). For a radius \( r \sim 10 \) and \( \lambda \sim 10^{-3} \), approximation \( \chi = 1 \) gives that \( S \sim 1 \). It is advantageous to use large spin values, because the larger the value is, the greater might be the difference between \( v^\mu \) and \( u^\mu \), and consequently the divergence between the two time constraints.

\(^2\) See, e.g., Ref. [14] for thorough discussion the astrophysically relevant spin values

FIG. 1: Two orbits following the MPD equations with TD SSC are depicted in the configuration space. The orbits share the same initial conditions \( r = 10, \theta = \pi/2, u^t = 0.1, S^t = 0.1, S^\theta = 0.01 S \) and constants of motion \( m = 1, S = 0.9, E = 0.97 J_z = 3 \). The black curve shows the orbit for which the affine parameter is the proper time, while the gray curve shows the orbit for which the affine parameter is defined by the constraint \( \chi = 1 \). The left panel shows the evolution of the orbits for \( 0 \leq \chi \leq 10^3 \), while the right for \( 0.99 \times 10^3 \leq \chi \leq 10^5 \).

Following the initial condition setup presented in [14], instead of the spin tensor \( S^{\mu\nu} \) the spin four-vector

\[
S_\mu = - \frac{1}{2} \epsilon_{\mu\rho\sigma} u^\nu S^{\rho\sigma}
\]

is utilized. According to Ref. [14] setup, one can set \( \phi = 0 \) and provide the initial values for \( r, \theta, u^t, S^r, S^\theta \), while the rest of the initial conditions \( u^\phi, u^\sigma, S^t, \) and \( S^\phi, S^\sigma \), are fixed by \( m \) \( (\text{Eq. (1)}), S \) \( (\text{Eq. (14)}),\) \( E \) \( (\text{Eq. (25)}) \), \( J_z \) \( (\text{Eq. (27)}) \), the inverse relation of \( \text{Eq. (20)} \) \( u_{\mu} S^{\mu} = 0 \). The latter constraint is obtained from contracting \( \text{Eq. (26)} \) with \( u^\nu \), while in Eqs. (15), (26), (27) the inverse relation of \( \text{Eq. (25)} \)

\[
S^{\rho\sigma} = - \eta^{\rho\sigma\gamma\delta} S_{\gamma\delta}
\]

is employed.

5.2. Numerical results

To show whether the under discussion time constraints reproduce the same worldline or not, initial conditions leading to a generic non-equatorial orbit has to be chosen. Such initial conditions produce the orbits shown in Fig. 1. These orbits cover a non-zero width spheroidal shell around the central Kerr black hole (left panel). The pseudocartesian coordinates \( (x, y, z) \) used in Fig. 1 relate to the BL ordinates as follows

\[
\begin{align*}
x &= r \cos \phi \sin \theta, \\
y &= r \sin \phi \sin \theta, \\
z &= r \cos \theta.
\end{align*}
\]

The orbits evolve in a non-trivial manner; examples of trivial motion is a circular or a radial orbit. However,
the orbits appear to follow exactly the same paths until the end (right panel of Fig. 1). This implies that the two time constraints reproduce the same worldline.

To ensure that what is shown in Fig. 1 is not just an optical artifact, in the top panel of Fig. 2 is displayed the relative radial difference between the two orbits

$$\Delta r = \left| 1 - \frac{r_\sigma(t)}{r_\tau(t)} \right|$$

as a function of the coordinate time $t$. $r_\tau$ denotes the radial component of the orbit evolved using the proper time, while $r_\sigma$ denotes the radial component of the orbit using the affine parameter $\sigma$ defined by the constraint (17). The coordinate time introduces a third observer at infinity with his own clock. This clock provides a common time by which the orbits can be compared. The top panel of Fig. 2 shows that the discrepancies in the radial component start being at the level of the computational accuracy, which is double precision, and after $t \sim 10^3$ they appear to drift away on average linearly. This drift resembles Fig. 11 in Ref. [15], where the integration scheme of s-stage Gauss Runge-Kutta used in this work was tested, and a similar drift was assigned to the interpolation used in the scheme. Actually, in order to produce the top panel of Fig. 2 interpolation was employed to get from a two component functional $\{r_\chi(\chi), t(\chi)\}$ to the function $r_\chi(t)$, since both orbits were computed using their respective affine parameters ($\chi = \sigma, \tau$). Moreover, the bottom panel of Fig. 2
shows that the relative error of the spin measure
\[ \Delta S^2 = \left| 1 - \frac{S^2(t)}{S^2} \right| \tag{33} \]
increases on average linearly after \( t \sim 10^3 \) as well. \( S^2(t) \) denotes the numerically computed value of \( S^2 \) at time \( t \), while \( S^2 \) denotes the initial value of the spin. In few words, the drift between the two orbits shown in the top panel of Fig. 3 arises for numerical reasons, and the two orbits reproduce the same worldline up to numerical accuracy.

The fact that we do not see this drift in Fig. 3 for the four-velocity conservation \( \mathbf{v} \) in the case of the proper time (black dots, top panel) and for the mass \( m_v \) in the case of the affine parameter \( \sigma \) (gray dots, bottom panel) is that at each step quantities \( v^2 \) and \( m_v \) are normalized in order to compute the velocity through the relation \( v^2 = 1 \). Namely, for the proper time \( v^2 \) is kept equal to 1, while for \( \sigma m_v \) is kept equal to \( m \). Thus, it is no wonder why the relative errors of four-velocity conservation \( v^2 = 1 \) in the first case and of the mass \( m_v = 1 \) in the second case stay at the computational accuracy level for so long. An interesting aspect of Fig. 3 is the evolution of the relative difference between the initial value of \( v^2 \) and the value of \( v^2 \) at time \( t \) for the affine parameter \( \sigma \) (gray dots, top panel), and of the relative difference between the initial value of \( m_v \) and the value of \( m_v \) at time \( t \) for the proper time (black dots, bottom panel). The respective curves of the above two relative differences are practically identical, even the oscillations during the evolution take place at the same time. These curves provide a numerical example of the analysis provided in Sec. 4.2 and show that the orbit does not belong to the special case for which \( p^\mu |v^\mu| \).

It is notable that the phase space of the system does not change its dimensionality for the two time parameterization choices, i.e. the number of the constants of motion the same for \( \tau \) and \( \sigma \). Namely, in the case of the proper time the four-velocity \( \mathbf{v} \) is preserved and the mass \( \mu \) is not, and for the affine parameter the preservation is vice versa. If the number of constants was not the same, then this would imply that the two affine parameter choices alter the nature of the MPD equations and this choice is not just a gauge.

### 6. CONCLUSIONS

This article revisited relations derived from the Mathisson-Papapetrou-Dixon equations without specifying the affine parameter nor the spin supplementary condition. Next, the proper time choice versus the affine parameter choice introduced in \( \mathbf{v} \) were discussed in the case of the Mathisson-Pirani SCC, the Tulczyjew-Dixon SSC and the Ohashi-Kyrian-Semerák SCC, and the implications of this choice were analyzed.

In particular, it was found that under OKSSC the affine parameters are identical, while for the MP SSC the choice of the affine parameter affects the preservation of the mass \( m_v \). Namely, for the proper time choice \( m_v \) is a constant of motion, while for the Ref. \( \mathbf{v} \) choice the quantity \( m_v^2/v^2 \) is preserved instead.

The TD SSC was not only approached analytically, but also numerically. The analytical approach focused on the implications brought by the fact that \( m = m_v \). The numerical approach proved that the affine parameter choices \( \tau \) and \( \sigma \) are just a gauge choice, since both reproduce the same worldline when the MPD equations are evolved from the same initial conditions.

### Acknowledgments

G.L.-G. acknowledges the support from Grant No. GACR-17-06962Y and thanks O. Semerák for useful discussions.

[1] M. Mathisson, “Neue mechanik materieller systemes”, *Acta Phys. Polonica* **6**, 163 (1937)
[2] A. Papapetrou, “Spinning test particles in general relativity. 1.”, *Proc. R. Soc. London Ser. A* **209**, 248 (1951)
[3] W. G. Dixon, “A covariant multipole formalism for extended test bodies in general relativity” *Nuovo Cim.* **34**, 317 (1964)
[4] O. Semerák, “Spinning test particles in a Kerr field - I”, *Mon. Not. R. Astron. S.* **308**, 863 (1999)
[5] J. Ehlers and E. Rudolph, “Dynamics of extended bodies in general relativity: centre-of-mass description and quasirigidity”, *Gen. Rel. Grav.* **8**, 197 (1977)
[6] W. Tulczyjew, “Motion of multipole particles in general relativity theory”, *Acta Phys. Polonica* **18**, 393 (1959)
[7] W. G. Dixon, “Dynamics of extended bodies in general relativity I: Momentum and Angular Momentum”, *Proc. Roy. Soc. Lond. A* **314**, 499 (1970)
[8] F. A. E. Pirani, “On the physical significance of the Riemann tensor”, *Acta Phys. Polonica* **15**, 389 (1956)
[9] E. Corinaldesi and A. Papapetrou, “Spinning test particles in general relativity. 2.”, *Proc. R. Soc. London Ser. A* **209**, 259 (1951)
[10] T. D. Newton and E. P. Wigner, “Localized States for Elementary Systems”, *Rev. Mod. Phys.* **21**, 400 (1949)
[11] A. Ohashi, “Multipole particle in relativity” *Phys. Rev. D* **68**, 044009 (2003)
[12] K. Kyrian and O. Semerák, “Spinning test particles in a Kerr field - II”, *Mon. Not. R. Astron. S.* **382**, 1922 (2007)
[13] C. Møller, “Sur la dynamique des systemes ayant un moment angulaire interne”, *Ann. Inst. Henri Poincaré* **11**, 251 (1949)
[14] M. D. Hartl, “Dynamics of spinning test particles in Kerr spacetime” *Phys. Rev. D* **67**, 024005 (2003), “Survey of
spinning test particle orbits in Kerr spacetime” *Phys. Rev. D* **67**, 104023 (2003)

[15] G. Lukes-Gerakopoulos, J. Seyrich and D. Kunst, “Investigating spinning test particles: Spin supplementary
conditions and the Hamiltonian formalism” *Phys. Rev. D* **90**, 104019 (2014)