String vacuum backgrounds
with covariantly constant null Killing vector
and 2d quantum gravity

A.A. Tseytlin †

DAMTP
Cambridge University
Cambridge CB3 9EW
United Kingdom

We consider a 2d sigma model with a 2 + N - dimensional Minkowski signature target space metric having a covariantly constant null Killing vector. We study solutions of the conformal invariance conditions in 2 + N dimensions and find that generic solutions can be represented in terms of the RG flow in N - dimensional “transverse space” theory. The resulting conformal invariant sigma model is interpreted as a quantum action of the 2d scalar (“dilaton”) quantum gravity model coupled to a (non-conformal) ‘transverse’ sigma model. The conformal factor of the 2d metric is identified with a light cone coordinate of the 2 + N - dimensional sigma model. We also discuss the case when the transverse theory is conformal (with or without the antisymmetric tensor background) and reproduce in a systematic way the solutions with flat transverse space known before.

† On leave of absence from the Department of Theoretical Physics, P. N. Lebedev Physics Institute, Moscow 117924, Russia. e-mail: aat11@amtp.cam.ac.uk
1. Introduction

The aim of the present paper to give a systematic discussion of string tree level vacuum backgrounds which have a covariantly constant null Killing vector. Such (plane wave type) solutions of Einstein equations are well known [1] (see also [2]). Some particular examples of such spaces were found to be solutions of the string effective equations to all orders of perturbation theory in $\alpha'$ [3–8]. In a recent paper [9] we have considered the most general metric with a covariantly constant null Killing vector and have shown that if the “transverse” part of the metric satisfies a first order renormalisation group - type equation there exists a dilaton field such that the metric–dilaton background solves the string equations to all orders in $\alpha'$. Below we shall complete the proof given in [9] and explain the relation to particular solutions of refs. [3–8].

Part of our interest in Minkowski signature string backgrounds with a null Killing vector is motivated by the observation that the corresponding sigma model can be interpreted as describing a model of $2d$ quantum gravity with an extra scalar field coupled to $2d$ curvature (see e.g. [10–13,9]). As we shall see below, our solutions provide examples of consistent quantisation of $2d$ scalar quantum gravity coupled to a non-conformal matter being exact ($2 + N$ dimensional) conformal theories satisfying proper “initial conditions”. This is to be contrasted to the case of pure $2d$ gravity (without an additional scalar field) where similar description in terms of an $1 + N$ dimensional conformal theory is not known explicitly.

We shall start in Sec.2 with a discussion of the general form of the “null” metric and remaining freedom of coordinate transformations [1]. Then we shall study the structure of the sigma model Weyl invariance conditions for the backgrounds with null Killing vector. Using the fact that the Weyl invariance coefficients (“$\bar{\beta}$ - functions”) satisfy certain identities [14] reflecting the freedom of coordinate transformations in the target space (and related to the fact that $\bar{\beta}$’s can be derived from a covariant effective action [15,16]) we shall prove that the resulting metric - dilaton equations always have a solution.
In Sec.3 we shall consider the case when the “transverse” theory is Weyl invariant (in particular, when the “transverse” metric is flat [3–8]). We shall discuss a number of explicit solutions, including ones with a non-vanishing antisymmetric tensor.

A relation to 2\textit{d} quantum gravity models will be discussed in Sec.4.

2. Structure of the Weyl invariance conditions and generic solution

1. Let us consider the \( D = N + 2 \) dimensional space with Minkowski signature. The most general metric admitting a covariantly constant null Killing vector\(^1\) can be represented in the form

\[
ds^2 = \hat{g}_{\mu\nu} dx^\mu dx^\nu = -2 du dv + g_{ij}(u, x) dx^i dx^j ,
\]

\[\mu, \nu = 0, 1, \ldots, N, N + 1 \] , \( i, j = 1, \ldots, N \) .

In fact, starting from the null metric [1]

\[
ds^2 = \tilde{g}_{\mu\nu} dx^\mu dx^\nu = -2 du dv + g_{ij}(u, x) dx^i dx^j + 2 A_i(u, x) dx^i du + K(u, x) du^2 ,
\]

one can eliminate \( A_i \) and \( K \) by a change of coordinates which preserves the “null” structure of (2.2) [1]. If

\[
x^i = f^i(u, x') , \quad v = v' + h(u, x') \ ,
\]

we get

\[
A'_m = f'_m, \quad A_i + g_{ij} f'_i, f'_j - h, \quad K' = K + g_{ij} \dot{f}^i \dot{f}^j - 2 \dot{h} ,
\]

\[
g'_{mn} = f'_m, f'_n g_{ij} , \quad f'_{,m} = \frac{\partial f^i}{\partial x^m} , \quad \dot{f} = \frac{\partial f}{\partial u} .
\]

It is clear that redefining \( v \) one can always absorb \( K \) into a longitudinal part of \( A_i \) (or vice versa). The equations \( A'_m = 0 \) , \( K' = 0 \) are first order differential equations in \( \frac{\partial}{\partial u} \) for \( f^i \)

\(^1\) We shall refer to such metrics as “null metrics”.

2
and \( h \) and so they always have a solution (assuming one chooses the initial values in such a way that the matrix \( f^{i,n} \) is non-degenerate) [1].

Thus the most general null metric is parametrized by the functions \( g_{ij}(u,x) \). It is important to keep in mind, however, that (2.1) considered as a generic form of the metric is written using a special choice of coordinates \( v, x^i \). For example, if \( g_{ij}(u,x) \) is a flat metric as a function of \( x^i \) this does not imply that a generic “null” metric with a flat transverse part is just given by \( ds^2 = -2dudv + dx^idx_i \) : transforming the coordinates to make \( g_{ij} \) equal to \( \delta_{ij} \) we will get back the metric (2.2) with non-vanishing \( A_i \) and \( K \).

The metric (2.1) is a natural starting point for a discussion of general properties of solutions while (2.2) may be used if one looks for a solution with a specific ansatz for \( g_{ij}(u,x) \) (e.g. a standard metric of a flat space or a sphere, etc). All solutions can of course be expressed in any of the two equivalent forms (2.1) or (2.2).

In [9] we have studied string vacuum backgrounds represented by the metric (2.1). To reproduce the simplest exact solutions considered in [3–8] (which correspond to the “null” metric with flat transverse part) it will be useful to make a coordinate transformation to put the metric into the “non-diagonal” form (2.2).

2. Let us now discuss the structure of equations which we are going to solve using the ansatz (2.1). We shall follow the notation of refs.[9,16]. The conditions of Weyl invariance of a string sigma model (parametrized by a metric \( G_{\mu\nu} \) and a dilaton \( \phi \)) are equivalent to the tree level string effective equations and have the following general form [15,17,18]

\[
\beta^G_{\mu\nu} = \beta^G_{\mu\nu} + D(\mu W_\nu) + 2\alpha' D_\mu D_\nu \phi = 0 \quad , \\
\beta^G = \beta^G + \frac{1}{2} W^\mu \partial_\mu \phi + \alpha' (\partial_\mu \phi)^2 = 0 \quad , \\
\beta^G_{\mu\nu} = \alpha' R_{\mu\nu} + O(\alpha'^2) \quad , \quad W_\mu = O(\alpha'^3) \quad , \\
\beta^G = c - \gamma \phi + \omega \quad , \\
\gamma = \sum_{n=2}^\infty M^{\mu_1...\mu_n} D_{\mu_1}...D_{\mu_n} = \frac{1}{2} \alpha' D^2 + O(\alpha'^3) \quad ,
\]

3
\[ \omega = O(\alpha'^2) \quad , \quad c = \frac{1}{6}(D - 26) \ . \]

\( \beta^G, \gamma, \omega \) and \( W_\mu \) are covariant functions constructed from the curvature and covariant derivatives. Let us note that there exists a renormalisation scheme (i.e. a definition of \( G_{\mu\nu} \)) in which the leading order term in the ‘anomalous dimension’ differential operator \( \gamma \) is given just by its leading order term \( \frac{1}{2}\alpha'D^2 \) [17]. The Weyl anomaly coefficients \( \beta^G_{\mu\nu} \) and \( \beta^\phi \) satisfy \( D \) differential identities which can be derived from the condition of non-renormalisation of the trace of the energy-momentum tensor of the sigma model [14]. They can be interpreted as being a consequence of the target space reparametrisation invariance given that \( \beta^G_{\mu\nu} \) and \( \beta^\phi \) can be derived from a covariant effective action \( S \) [15,17]

\[
\frac{\delta S}{\delta \phi^A} = k_{AB}\beta^B, \quad \phi^A = (G_{\mu\nu}, \phi) \quad ,
\]

\[
2D\mu\frac{\delta S}{\delta G_{\mu\nu}} - \frac{\delta S}{\delta \phi} D^\nu \phi = 0 \quad .
\]

To the lowest order in \( \alpha' \) one finds [15,17]

\[
\partial_\mu\beta^\phi - \beta^G_{\mu\nu} D^\nu \phi + \frac{1}{2}D^\nu (\beta^G_{\mu\nu} - \frac{1}{2}G_{\mu\nu}G^{\lambda\rho}\beta^G_{\lambda\rho}) + O(\alpha'^2) = 0 \quad .
\]

In general, the identity has the following structure [14,17]

\[
\partial_\mu\beta^\phi - \beta^G_{\mu\nu} D^\nu \phi - V^{\alpha\beta}_{\mu} \beta^G_{\alpha\beta} = 0 \quad ,
\]

where the differential operator \( V^{\alpha\beta}_{\mu} \) depends only on \( G_{\mu\nu} \).

One of the consequences of (2.12) is that \( \beta^\phi = \text{const} \) once (2.5) is satisfied. In general, the identity (2.12) implies that only \( \frac{1}{2}D(D + 1) + 1 - D \) of equations (2.5), (2.6) are independent. It may happen, in particular, that if the “transverse” subset of \( \frac{1}{2}(D - 2)(D - 1) \) equations in (2.5) and the dilaton equation (2.6) are solved, the remaining \( D \) equations (2.5) are satisfied automatically. This observation will be important below.

3. Let us now look for solutions of (2.5), (2.6) which have the form [9]

\[
G_{\mu\nu} = \hat{g}_{\mu\nu}(u, x) \quad , \quad \phi = \phi(v, u, x) \quad , \quad x^\mu = (v, u, x^i) \quad ,
\]

\[
\delta S = k_{AB}\beta^B (u, x) \quad , \quad \phi = \phi(v, u, x) \quad , \quad x^\mu = (v, u, x^i) \quad ,
\]

\[
\frac{\delta S}{\delta \phi^A} = k_{AB}\beta^B (u, x) \quad , \quad \phi^A = (G_{\mu\nu}, \phi) \quad ,
\]

\[
2D\mu\frac{\delta S}{\delta G_{\mu\nu}} - \frac{\delta S}{\delta \phi} D^\nu \phi = 0 \quad .
\]

To the lowest order in \( \alpha' \) one finds [15,17]

\[
\partial_\mu\beta^\phi - \beta^G_{\mu\nu} D^\nu \phi + \frac{1}{2}D^\nu (\beta^G_{\mu\nu} - \frac{1}{2}G_{\mu\nu}G^{\lambda\rho}\beta^G_{\lambda\rho}) + O(\alpha'^2) = 0 \quad .
\]

In general, the identity has the following structure [14,17]

\[
\partial_\mu\beta^\phi - \beta^G_{\mu\nu} D^\nu \phi - V^{\alpha\beta}_{\mu} \beta^G_{\alpha\beta} = 0 \quad ,
\]

where the differential operator \( V^{\alpha\beta}_{\mu} \) depends only on \( G_{\mu\nu} \).
where \( \hat{g}_{\mu\nu} \) is given by (2.1). The non-vanishing components of the Christoffel connection and the curvature of \( \hat{g} \) are

\[
\hat{\Gamma}^i_{jk} = \Gamma^i_{jk}, \quad \hat{\Gamma}^\nu_{ij} = \frac{1}{2} \dot{g}_{ij}, \quad \hat{\Gamma}^i_{ju} = \frac{1}{2} g^{ik} \dot{g}_{kj}, \quad \dot{g}_{ij} \equiv \frac{\partial g_{ij}}{\partial u},
\]

(2.14)

\[
\hat{R}^i_{jk} = R^i_{jk}, \quad \hat{R}^i_{iuju} = T_{ij}, \quad \hat{R}_{uijk} = E_{ijk},
\]

(2.15)

\[
T_{ij} \equiv -\frac{1}{2} \left( \dot{g}_{ij} - \frac{1}{2} g^{mn} \dot{g}_{im} \dot{g}_{nj} \right), \quad E_{ijk} = -D_{[j} \dot{g}_{k]i}.
\]

(2.16)

The ‘covariant’ form of (2.14)–(2.16) is

\[
\hat{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} + g^\lambda_{\rho} \dot{g}_{\rho(l\mu\nu)} - \frac{1}{2} \dot{g}_{\mu\nu} l^\lambda, \quad (2.17)
\]

\[
\hat{R}^\mu_{\nu\rho\sigma} = R^\mu_{\nu\rho\sigma} + 2 l_{[\mu} E_{\nu]\rho\sigma} + 2 l_{[\rho} E_{\sigma]\mu\nu} - 4 l_{[\mu} T_{\nu]\alpha] l^\alpha, \quad (2.18)
\]

\[
\hat{T}^\mu_{\nu\sigma} = R^\mu_{\nu\sigma} - 2 g^{\rho\sigma} E_{\rho\sigma(l\mu\nu)} + l_{\mu} l_{\nu} g^{\alpha\sigma} T_{\alpha\sigma}, \quad (2.19)
\]

where \( l_{\mu} = \partial_{\mu} u = (0, 1, 0, ..., 0) \) is the null Killing vector and \( g_{\mu\nu}, \Gamma^\lambda_{\mu\nu}, R^\mu_{\nu\rho\sigma}, T_{\mu\nu} \) and \( E_{\mu\nu\lambda} \) have only transverse \((i, j, ...)\) components being non-vanishing.

Since \( \beta^G_{\mu\nu}, W_{\mu} \) and hence

\[
\beta^G_{\mu\nu}' = \beta^G_{\mu\nu} + D(\mu W_{\nu})
\]

in (2.5) are covariant functions of the curvature and its derivatives we have

\[
l^\mu \beta^G_{\mu\nu}' \equiv 0, \quad l^\mu W_{\mu} \equiv 0,
\]

i.e. the \((\mu\nu)\) components of (2.19) are identically zero. Then (2.5) gives the following constraint on the dilaton

\[
\partial_\mu \partial_\nu \phi = 0,
\]

i.e.

\[
\phi = pv + \phi(u, x), \quad p = \text{const}.
\]

(2.20)
Here $p$ is an arbitrary integration constant and $\phi(u, x)$ is to be determined. From now on all the functions will depend only on $u$ and $x^i$. Using (2.14), (2.20) we can represent the non-trivial components of (2.5) as follows ($\alpha' = 1$)

$$\bar{\beta}^g_{ij} - p\dot{g}_{ij} = 0 \ ,$$  

$$\bar{\beta}^g_{ij} \equiv \beta^G_{ij} + D_iW_j + 2D_iD_j \phi \ ,$$  

$$\beta^G_{iu} + \frac{1}{2} \partial_i W_u + \frac{1}{2} \dot{W}_i - \dot{g}_{ij} W^j + 2\partial_i \phi - \dot{g}_{ij} D^j \phi = 0 \ ,$$  

$$\beta^G_{uu} + \dot{W}_u + 2\ddot{\phi} = 0 \ .$$

Equation (2.6) takes the form

$$\bar{\beta}^\phi = \frac{1}{3} + \bar{\beta}^{\phi'} + \frac{1}{2} pM^{ij} \dot{g}_{ij} - \frac{1}{2} pW_u - 2p \dot{\phi} = 0 \ ,$$

$$\bar{\beta}^{\phi'} \equiv c' - \gamma' \phi + (\partial_i \phi)^2 + \frac{1}{2} W^i \partial_i \phi + \omega \ , \quad c' = \frac{1}{6} (N - 26) \ .$$

Note that being scalar functions of the curvature (2.15),(2.18) $\gamma$, $\omega$ and hence $\bar{\beta}^{\phi'}$ do not depend on the derivatives of the metric over $u$. The term $\gamma \phi$ in (2.8) reduces to $\gamma' \phi - \frac{1}{2} pM^{ij} \dot{g}_{ij}$ where $\gamma'$ corresponds to the “transverse” theory (i.e. contains only derivatives over $x^i$) and the correction is due to $\partial_v \phi$ ($M^{ij} = \frac{1}{2} g^{ij} + ...$).

The functions $\bar{\beta}^g_{ij}$ (2.22) and $\bar{\beta}^{\phi'}$ (2.26) can be interpreted as the Weyl anomaly coefficients of the “transverse” theory defined by $g_{ij}(u, x)$ and $\phi(u, x)$ at fixed $u$ ($1/3$ in (2.25) corresponds to the central charge contribution of the two light-cone coordinates).

4. Let us first consider the case of non-vanishing $p$. Then (2.21) is a first order differential equation for $g_{ij}(u, x)$ which always has a solution. Eliminating the $u$-derivatives of $g_{ij}$ from (2.25) using (2.21) we find a similar first order equation for $\phi(u, x)$. Eqs. (2.21),(2.25) can be interpreted as renormalisation group equations of the “transverse” theory with $u$ playing the role of the RG “time” [9].
The remaining question is whether the solutions of (2.21) and (2.25) satisfy also (2.23) and (2.24). It can be answered positively using the identity (2.12). Substituting \( \bar{\beta}^G_{ij} = 0 \), \( \bar{\beta}^\phi = 0 \) and the expression (2.20) for the dilaton into (2.12) we get

\[
p\bar{\beta}^G_{iu} - 2V^j_i \bar{\beta}^G_{ju} - V^j_u \bar{\beta}^G_{ju} = 0 \quad ,
\]

\[
p\bar{\beta}^G_{uu} - \bar{\beta}^G_{ui} D^i \phi - 2V^j_i \bar{\beta}^G_{ju} - V^j_u \bar{\beta}^G_{ju} = 0 \quad .
\]

Given that \( V^\mu_{\lambda\nu} \) is a differential operator constructed from the curvature (2.18), its components \( V^j_i \), \( V^j_u \) and \( V^j_u u \) should vanish identically (\( V^j_u \) may be non-vanishing because of possible \( V^\mu \delta^\nu_{\lambda} \) term in \( V^\mu_{\lambda\nu} \)). As a result, equations (2.27),(2.28) take the form

\[
p\bar{\beta}^G_{iu} = 0 \quad ,
\]

\[
p\bar{\beta}^G_{uu} - \bar{\beta}^G_{iu} D^i \phi - 2V^j_i \bar{\beta}^G_{ju} = 0 \quad .
\]

In the leading order approximation (2.11) the identity (2.30) is given by

\[
p\bar{\beta}^G_{uu} - \bar{\beta}^G_{iu} D^i \phi + \frac{1}{2} D^i \bar{\beta}^G_{iu} = 0 \quad .
\]

We conclude that once (2.21) and (2.25) are satisfied for a non-zero \( p \) (2.29) and (2.30) imply that

\[
\bar{\beta}^G_{iu} = 0 \quad , \quad \bar{\beta}^G_{uu} = 0 \quad .
\]

What we have found is that given some initial data \((g_{ij}(x) , \phi(x))\) at \( u = 0 \) there exists a \( u \)-dependent solution \((g_{ij}(u, x) , \phi(u, x))\) of the Weyl invariance conditions (2.5),(2.6). If the initial transverse theory is generic, i.e. \( \bar{\beta}^g_{ij} \) in (2.22) is non-vanishing at \( u = 0 \) then the solution exists only for a non-zero \( p \). If, however, the initial theory is Weyl invariant, i.e.

\[
\bar{\beta}^g_{ij}(u = 0) = 0 \quad , \quad \bar{\beta}^\phi(u = 0) = c'' = \text{const} \quad ,
\]

we have an option. For \( p \neq 0 \) the simplest solution of (2.5),(2.6) is the ‘direct product’ one represented by the fixed point of the RG equations (2.21),(2.25)

\[
g_{ij}(u, x) = g_{ij}(x) \quad , \quad \phi(u, x) = \frac{1}{2p} (\frac{1}{3} + c'' )u + \phi(x) \quad .
\]
There may be also more interesting solutions corresponding to interpolations between different conformal points (one theory at \( u = -\infty \) and another – at \( u = +\infty \)). The case of \( p = 0 \) will be discussed in the next section.

Let us note that if one looks for special solutions with \( g_{ij}(u,x) \) corresponding to a conformal theory for all \( u \) then it is necessary to set \( p = 0 \). In fact, let us try to find, for example, a solution of (2.21)–(2.26) with the metric \( g_{ij}(u,x) \) being flat for arbitrary \( u \) assuming \( p \neq 0 \). Since the transverse components of the curvature (but not necessarily of the connection) are zero, \( \beta_{ij}^G \) and \( W_\mu \) vanish and so the equations for \( g_{ij} \) and \( \phi \) (2.21) and (2.25) take the form

\[
p\dot{g}_{ij} = 2D_i D_j \phi , \tag{2.34}
\]

\[
c - \frac{1}{2} D_i^2 \phi + (\partial_i \phi)^2 + \frac{1}{4} pg^{ij} \dot{g}_{ij} - 2p\dot{\phi} = 0 , \quad c = \frac{1}{6}(N - 24) . \tag{2.35}
\]

Eliminating \( \dot{g}_{ij} \) from (2.35) we get

\[
2p\dot{\phi} = g^{ij} \partial_i \partial_j \phi + c . \tag{2.36}
\]

Since \( p \neq 0 \) the remaining equations (2.23), (2.24) are again satisfied as a consequence of (2.34),(2.36). Equations (2.34),(2.36) must be supplemented by the condition \( (R_{ijkl} = 0) \) that \( g_{ij} \) remains flat for all \( u \). Since the \( u \) - derivatives of \( R_{ijkl} = 0 \) vanish automatically as a consequence of (2.34) (note that \( \dot{\Gamma}_{jk}^i = p^{-1} D_i D_j D_k \phi \), etc) the flatness condition is only a constraint on the initial data \( g_{ij}(0,x) \) for (2.34),(2.35). The resulting system, however, has only the trivial solution: it is straightforward to check that all the components of the curvature (2.18) vanish \( (E_{ijk} = 0, T_{ij} = 0) \). The corresponding metric (2.1) is flat so that the solution for the dilaton must be linear in proper coordinates. In fact, \( \ddot{\phi} = 0 \) and the \( u \) dependence in \( g_{ij} \) can be represented in terms of a coordinate transformation.

\footnote{It is easy to see that \( W_u \) must be zero since as follows from (2.18) the only non-vanishing contributions could come from the terms which are linear in curvature but such terms are absent in \( W_u \) \cite{17}.}
To get a non-trivial solution with a flat $g_{ij}(u, x)$ (or, more generally, conformal transverse theory) one should set $p = 0$. Then (assuming $\phi = \phi(u)$) eqs.(2.21),(2.25) are satisfied automatically but since $p = 0$ the identities (2.29),(2.30) no longer imply that (2.23),(2.24) are also satisfied. To make the analysis of the solutions of the remaining equations (2.23),(2.24) more transparent it is useful to change coordinates, trading the functions $g_{ij}(u, x)$ corresponding to a flat transverse metric for $A_i$ and $K$ in (2.2) (i.e. transforming the metric (2.1) into (2.2) where $g_{ij}(u, x)$ has canonical $\delta_{ij}$ form). This will be discussed in the next section.

3. Solutions with conformal “transverse” part

1. Let us return to the discussion of the case (2.32) when the “transverse” theory $(g_{ij}(u, x), \phi(u, x))$ is Weyl invariant at $u = 0$, assuming now that $p = 0$. If we are looking for a solution of the Weyl invariance conditions for a $N+2$ - dimensional background (2.1) then for $p = 0$ eqs.(2.21),(2.25) imply that the initial Weyl invariance conditions (2.32) are satisfied also for all other values of $u$. Therefore a solution with (2.32),(2.20) and $p = 0$ may exist only if the transverse theory is conformal for all $u$. Since (2.21) holds identically it no longer gives an equation for $g_{ij}(u, x)$. The same is true for (2.25): it does not contain terms with $u$ - derivatives and being a constant (as a consequence of (2.5),(2.12)) it is satisfied for all $u$ if it is for $u = 0$, i.e. if $\frac{1}{3} + c' = 0$. As we already mentioned, for $p = 0$ the identities (2.12) or (2.27),(2.28) do not imply that eqs.(2.23),(2.24) are satisfied automatically. Instead of $N+1$ identities (2.29),(2.30) we are left with just one (2.30). As a result, we get $N$ equations (2.23),(2.24) ((2.30) gives a relation between components of (2.23)) for $\frac{1}{2}N(N + 1) + 1$ functions $g_{ij}(u, x), \phi(u, x)$.

Using (2.7),(2.14)–(2.18) we can represent the leading terms in (2.22),(2.23) in the form [9]

$$g^{jk}E_{jik} + \frac{1}{2}\alpha' E_{mnk}R^m_{i}^{nk} + 2\partial_i\dot{\phi} - \dot{g}_{ij}D^j\phi + O(\alpha')$$

$$= \frac{1}{2}(D^j\dot{g}_{ij} - \partial_i(g^{jk}\dot{g}_{jk})) + 2\partial_i\dot{\phi} - \dot{g}_{ij}D^j\phi + O(\alpha') = 0 , \quad (3.1)$$
\[ g^{ij}T_{ij} + 2\ddot{\phi} + O(\alpha') = -\frac{1}{2}(g^{ij}\ddot{g}_{ij} - \frac{1}{2}g^{ij}g^{mn}\ddot{g}_{im}\ddot{g}_{nj}) + 2\ddot{\phi} + O(\alpha') = 0 \quad . \tag{3.2} \]

The count of powers of \( l_\mu \) in (2.17),(2.18) implies that higher order terms in (3.1) will be proportional to the \( D_i \)-derivatives of the first power of \( \ddot{g}_{ij} \) (originating from the \( O(l_\mu) \) terms in the connection (2.17) or the curvature (2.18)) multiplied by factors constructed from \( D_i \) and \( R_{ijkl} \). In a similar fashion, the higher order terms in (3.2) will be linear in \( D_i \)-derivatives of \( \ddot{g}_{ij} \) or quadratic in \( D_s\dot{g}_{ij} \).

It is easy to check that the identity (2.31) is indeed satisfied by the leading terms in (3.1). Solving formally (3.2) for the dilaton and substituting the result into (3.1) one finds a system of \( N \) equations for \( g_{ij}(u, x) \) with one identity (2.30).

2. Let us now make a specific assumption. Since in any case only \( N - 1 \) of \( \frac{1}{2}N(N+1) \) components of \( g_{ij} \) are constrained by (3.1),(3.2) let us assume that the solution \( g_{ij}(u, x) \) can be represented as a \( u \)-dependent coordinate “rotation” of a \( u \)-independent (e.g. flat) metric

\[ g_{ij}(u, x) = \partial_i y^m \partial_j y^n g_{mn}(y) \quad , \tag{3.3} \]

\[ y^m = y^m(u, x) \quad , \quad y^m(0, x) = x^m \quad , \quad g_{ij}(0, x) = g_{ij}(x) \quad . \]

Substituting (3.3) into (3.1),(3.2) we get a system of equations \( y^i(u, x) \) and \( \phi(u, x) \). In order to simplify the subsequent analysis let us first “undo” the coordinate transformation in (3.3). Represented in terms of the new coordinates \( y^i \) the metric (2.1) takes the non-diagonal form (2.2) with the \( u \)-independent transverse metric (cf.(2.4))

\[ g_{ij}(u, y) = g_{ij}(y) \quad , \quad A_i(u, y) = -g_{ij}\dot{y}^i \quad , \quad K(u, y) = g_{ij}\dot{y}^i\dot{y}^j \quad . \tag{3.4} \]

Instead of solving (3.1),(3.2) for \( y^i(u, x) \) we shall consider eqs.(2.5) for the metric (2.2),(3.4) \( G_{\mu\nu} = \ddot{g}_{\mu\nu} \) and solve them for \( A_i \) , \( K \).

The expressions for the connection and the curvature corresponding to (2.2) are generalisations of (2.14)–(2.19) \[1,6,8\] (in what follows we shall return to the notation \( x^\mu \)

\[ ^3 \text{Note that either } K \text{ or the longitudinal part of } A_i \text{ will not be determined since one of them can be always eliminated by a transformation of } v \text{ (2.3).} \]
for the coordinates in (2.2). If \( \tilde{g}_{\mu\nu} \) denotes the “diagonal” part (2.1) of \( \tilde{g}_{\mu\nu} \) then (see (2.17)–(2.19))
\[
\tilde{g}_{\mu\nu} = \hat{g}_{\mu\nu} + 2 A_{(\mu} l_{\nu)} \quad , \quad A_\mu \equiv (0, \frac{1}{2} K, A_i) \quad ,
\]
\[
l_\mu = \partial_\mu u \quad , \quad \tilde{D}_\mu l_\nu = 0
\]
\[
\tilde{g}^{\mu\nu} = \hat{g}^{\mu\nu} - 2 A^{(\mu} l_{\nu)} + A^\mu l_\nu \quad , \quad A^\mu = \hat{g}^{\mu\nu} A_\nu
\]
\[
\tilde{\Gamma}^\lambda_{\mu\nu} = \hat{\Gamma}^\lambda_{\mu\nu} - (\hat{g}^{\rho\lambda} - A^\rho A^\lambda) F_{\rho(\mu} l_{\nu)} + \hat{D}_{(\mu} A_{\nu)} l^\lambda \quad , \quad F_{\mu\nu} \equiv 2 \partial_\mu A_\nu \quad ,
\]
\[
\tilde{R}^\lambda_{\mu\nu\rho\sigma} = \hat{R}^\lambda_{\mu\nu\rho\sigma} + l_{[\mu} \hat{D}_{\nu]} F_{\rho\sigma} + l_{[\rho} \hat{D}_{\sigma]} F_{\mu\nu} + l_{[\mu} F_{\nu]} l^\lambda F_{\rho\sigma} \quad ,
\]
\[
\tilde{R}_{\mu\nu} = \hat{R}_{\mu\nu} + l_{(\mu} D^\sigma F_{\nu)} + \frac{1}{4} l_\mu l_\nu F_{\rho\sigma} F_{\rho\sigma} \quad .
\]

Explicitly,
\[
\tilde{\Gamma}^i_{jk} = \hat{\Gamma}^i_{jk} \quad , \quad \tilde{\Gamma}^i_{ju} = \frac{1}{2} g^{im} (\hat{g}_{jm} + F_{jm}) \quad , \quad \tilde{\Gamma}^i_{uu} = g^{im} \left( \hat{A}_m - \frac{1}{2} \partial_m K \right) \quad , \quad \tilde{\Gamma}^v_{ij} = \frac{1}{2} \hat{g}_{ij} - D_{(i} A_{j)} \quad ,
\]
\[
\tilde{\Gamma}^v_{iu} = -\frac{1}{2} \partial_i K + \frac{1}{2} A^m (\hat{g}_{im} + F_{im}) \quad , \quad \tilde{\Gamma}^v_{uu} = -\frac{1}{2} \dot{K} + A^m (\hat{A}_m - \frac{1}{2} \partial_m K) \quad , \quad etc.
\]

In general, substituting (2.2) into (2.5) we find again that the dilaton should be linear in \( v \) (2.20). The leading order terms in the \( (iu) \) and \( (uu) \) components of (2.5) take the form (cf.(3.1),(3.2))
\[
\frac{1}{2} (D^j \hat{g}_{ij} - \partial_i (g^{jk} \hat{g}_{jk})) + \frac{1}{2} D^i F_{ij} + 2 \partial_i \phi - (\hat{g}_{ij} - F_{ij}) D^j \phi - p F_{ik} A^k + p \partial_i K + O(\alpha') = 0 \quad ,
\]
\[
-\frac{1}{2} (g^{ij} \hat{g}_{ij} - \frac{1}{2} g^{mn} \hat{g}_{im} \hat{g}_{nj}) - \frac{1}{2} D^2 K + \frac{1}{4} F_{ij} F_{ij} + D^i \hat{A}_i + 2 \dot{\phi} - 2 (\hat{A}_i - \frac{1}{2} \partial_i K) (p A^i + 2 D^i \phi) + p \dot{K} + O(\alpha') = 0 \quad .
\]

Let us note also that the \( (ij) \) component of (2.5) and eq.(2.6) are modified as follows (as compared to (2.21),(2.22) and (2.25))
\[
\beta^G_{ij} + D_{(i} W_{j)} + 2 D_i D_j \phi - p \hat{g}_{ij} + 2 p D_{(i} A_{j)} = 0 \quad ,
\]
\[ \frac{1}{3} + \bar{\beta}^\phi + \frac{1}{2} p M^{ij} (\dot{g}_{ij} - 2D_i A_j) + p^2 (A^i A_i - K) - \frac{1}{2} p W_i - 2p \dot{\phi} = 0 \ . \quad (3.12) \]

3. Let us now consider the special case when
\[ g_{ij}(u, x) = g_{ij}(x) \ , \quad \phi = \phi(u) + \phi'(x) \ , \]
i.e. \( p = 0 \) and \( g_{ij}(x) \), \( \phi'(x) \) represent a Weyl invariant theory with \( \frac{1}{3} + \bar{\beta}^\phi = 0 \). Then we are left with eqs. (3.9),(3.10), i.e.
\[ \frac{1}{2} D^j F_{ij} + F_{ij} D^j \phi + O(\alpha') = 0 \ , \quad (3.13) \]
\[ -\frac{1}{2} D^2 K + \frac{1}{4} F_{ij} F_{ij} + D^i \dot{A}_i + 2\ddot{\phi} - 2(\dot{A}_i - \frac{1}{2} \partial_i K) D^i \phi + O(\alpha') = 0 \ . \quad (3.14) \]
The identity (2.30),(2.31) can now be interpreted as a consequence of the gauge invariance
\[ A'_i = A_i - \partial_i h \ , \quad K' = K - 2\dot{h} \ , \quad (3.15) \]
which originates from the invariance under redefinitions of \( v \) (see (2.3),(2.4)). To avoid (almost all of) higher order corrections to (3.13),(3.14) let us follow refs.[3-8] and further assume that the transverse part of the metric is flat, \( g_{ij} = \delta_{ij} \). If \( p = 0 \) the condition that the transverse theory should be Weyl invariant then implies that \( \phi \) can be at most linear in \( x^i \) and for simplicity we shall set it equal to zero. Then it is easy to see that there are no \( \alpha' \) corrections in (3.13) (the only terms that may contribute to \( \beta_{G_i} \) could originate from the structures \( D^s R \) which are linear in the curvature (3.7) but such higher order terms are actually absent in the \( \beta^G_{\mu\nu} \) - function, cf.[6]). Possible higher order terms in (3.14) could come from the terms \( D^s R D^r R \) in \( \beta^G_{G_i} \) which are quadratic in the curvature and therefore will have the structure \( \partial^s F \partial^r F \). As a result, we are left with the following system
\[ \partial^j F_{ij} = 0 \ , \quad (3.16) \]
\[ -\frac{1}{2} \partial^2 K + \frac{1}{4} F_{ij} F_{ij} + \partial^i \dot{A}_i + 2\ddot{\phi} + \frac{1}{8} \alpha' \partial_i F_{jk} \partial^i F_{jk} + O(\alpha'^s(\partial^s F)^2) = 0 \ . \quad (3.17) \]
The exact solutions will be generated, for example, by solutions of (3.16) for which $F_{ij}$ is a polynomial of finite degree in $x^i$ with $u$-dependent coefficients [6,7]. The simplest solution with $A_i = 0$ was considered in [4,5]. A gauge equivalent (cf.(3.15)) solution with $F_{ij} = 0$ corresponds to

$$A_i = a_{ij}(u)x^j, \quad a_{ij} = a_{ji}, \quad (3.18)$$

where $a_{ij}$ is an arbitrary symmetric matrix (e.g. $a(u)\delta_{ij}$). If one starts with (2.1) with $g_{ij} = \kappa(u)g_{ij}(x)$ then, as it was found in [1], the Einstein equation is satisfied if $\kappa = u^2$. If $g_{ij}(x) = \delta_{ij}, \phi = 0$ this is an exact string vacuum. Making a coordinate transformation to eliminate $\kappa(u)$ from the transverse part of the metric we get (2.2) with $A_i = -u^{-1}x_i$, $K = u^{-2}x^2$ (cf. (3.3),(3.4)), i.e. the equivalent solution of (3.16),(3.17).

A less trivial example of a solution of (3.16),(3.17) is provided by $F_{ij} = F_{ij}(u)$, i.e. by the plane - fronted wave metric [6-8]

$$A_i = -\frac{1}{2}F_{ij}(u)x^j, \quad (3.19)$$

$$\partial^2 K - \frac{1}{2}F^{ij}F_{ij} - 4\ddot{\phi} = 0,$$

$$K = k_{ij}(u)x^ix^j + k_0, \quad k_i = \frac{1}{4}F^{ij}F_{ij} + 2\ddot{\phi}. \quad (3.20)$$

The equivalent metric represented in the form (2.1) (i.e. the equivalent solution of (3.1),(3.2)) corresponds to a special case of a flat transverse metric in (2.1), namely the $x^i$-independent one, $g_{ij}(u,x) = g_{ij}(u)$. In fact, the particular case of the coordinate transformation (2.3),(2.4)

$$x^i = L^i_j(u)x^j', \quad h = s_{ij}x^ix^j, \quad (3.21)$$

where $L^i_j$ is expressed in terms of $F_{ij}$ relates (2.2) with $g_{ij} = \delta_{ij}$ and $A_i , K$ given by (3.19),(3.20) to (2.1) with $g_{ij}(u) = L^m_iL^n_j\delta_{mn}$. Vice versa, if we start with (2.1) with the $x^i$-independent metric $g_{ij}(u,x) = g_{ij}(u)$ we can always make it equal to $\delta_{ij}$ by a
$u$-dependent linear transformation of the transverse coordinates, $x^i = (L^{-1})^i_j(u)y^j$ (the required transformation is a particular case of (3.3),(3.4))

$$ds^2 = -2dudv + g_{ij}(u)dx^i dx^j \quad , \quad g_{ij}(u) = L_i^m(u)L_j^n(u)\delta_{mn} \quad . \quad (3.22)$$

As a result, we get (2.2) with (we rename $y^i \rightarrow x^i$ ; cf.(3.4))

$$g_{ij} = \delta_{ij} \quad , \quad A_i = f_{ij}(u)x^j \quad , \quad K = t_{ij}(u)x^ix^j \quad , \quad (3.23)$$

$$f_{ij} = -\delta_{ik}(\dot{L}L^{-1})^k_j \quad , \quad t_{ij} = \delta_{mn}(\dot{\bar{L}}L^{-1})^m_i(\dot{\bar{L}}L^{-1})^n_j \quad , \quad K = A_iA^i \quad . \quad (3.24)$$

This background is a solution of (3.16),(3.17) if (cf. (3.18)–(3.20))

$$t^i_i = f_{[ij]}f^{[ij]} + \dot{f}_i^i + 2\ddot{\phi} \quad , \quad (3.25)$$

$$F_{ij} = -2f_{[ij]} \quad , \quad a_{ij} = f_{(ij)} \quad . \quad (3.26)$$

Eq.(3.24) is a second order equation for $L_i^j(u)$ equivalent to (3.2) (where there are no higher order corrections if $g_{ij}(u, x) = g_{ij}(u)$). Let us note that as it is clear from (2.3),(2.4) the solutions of (3.9)–(3.12) with $g_{ij}(u, x) = g_{ij}(u)$ (considered in [7,8]) are gauge - equivalent to the solutions with $g_{ij} = \delta_{ij}$ [6] discussed above.

Being equivalent to eqs. (2.21)–(2.25) the system (3.9)–(3.12) does not have a non-trivial solution with flat transverse metric in the case when the coefficient $p$ of the $v$ term in the dilaton (2.20) is non-vanishing. In fact, if $p \neq 0$ one can transform $v$ to absorb $\phi(u, x)$ in (2.20), i.e. to make the dilaton equal to

$$\phi = \phi_0 + pv \quad . \quad (3.25)$$

If $g_{ij} = \delta_{ij}$ we find that (3.9),(3.10),(3.11) and (3.12) reduce to (cf.(3.16),(3.17))

$$\partial^j F_{ij} - 2pF_{ik}A^k + 2p\partial_iK = 0 \quad , \quad (3.26)$$

4 Note that a $v$ - dependent dilaton background breaks the gauge invariance (3.15).
where we have already used (3.28) to simplify (3.27),(3.29). Since \( p \neq 0 \) eqs.(3.26),(3.27) are satisfied as a consequence of (3.28),(3.29). Eqs.(3.26)–(3.29) imply that \( \hat{R}_{\mu\nu} = 0 \) and \( \hat{\Gamma}^\nu_{\mu\nu} = 0 \) and hence all the components of the curvature \( \hat{R}_{\mu\nu\lambda\rho} \) vanish. Note that though (3.19) satisfies (3.28) the solution (3.19),(3.20) is non-trivial since in contrast to \( K \) in (3.29) there in general \( K \neq A_i A^i + k_0 \).

4. It is of interest to generalise the above discussion to the case of non-vanishing antisymmetric tensor background. One of motivations is that WZW models or group spaces “parallelised” by the antisymmetric tensor field strength \([19]\) provide simple explicit examples of conformally invariant backgrounds which can be used to represent the transverse theory. One would like also to find solutions describing interpolation in \( u \) between different conformal points. If the sigma model action contains also the antisymmetric tensor \( B_{\mu\nu} \) coupling

\[
I = \frac{1}{4\pi \alpha'} \int d^2 z \sqrt{\gamma} \left[ (G_{\mu\nu} + B_{\mu\nu})(\gamma^{ab} + i\epsilon^{ab})\partial_a x^\mu \partial_b x^\nu + R^{(2)}(\phi) \right],
\]

then the leading terms in the Weyl anomaly coefficients are given by \([19,15]\)(cf.(2.5),(2.6))

\[
\bar{\beta}^G_{\mu\nu} = \alpha'(R_{\mu\nu} - \frac{1}{4} H_{\mu\lambda\rho} H^\nu_{\rho} + 2 D_\mu D_\nu \phi) + O(\alpha'^2) = 0, \quad H_{\lambda\mu\nu} \equiv 3 \partial_{[\lambda} B_{\mu\nu]} ,
\]

\[
\bar{\beta}^B_{\mu\nu} = -\frac{1}{2} \alpha' D^\lambda H_{\lambda\mu\nu} + \alpha' \partial_\lambda \phi H^\lambda_{\mu\nu} + O(\alpha'^2) = 0 ,
\]

\[
\bar{\beta}^\phi = c - \frac{1}{2} \alpha' D^2 \phi + \alpha' (\partial_\mu \phi)^2 - \frac{1}{24} \alpha'H^2_{\lambda\mu\nu} + O(\alpha'^2) = 0 .
\]

Let us assume that in addition to the metric (2.1) or (2.2) we are given a \( v \)-independent \( \hat{B}_{\mu\nu} \) background

\[
\hat{B}_{ij} = B_{ij}(u,x) , \quad \hat{B}_{iu} = B_i(u,x) ,
\]
\[ \hat{H}_{ijk} = H_{ijk} \equiv 3\partial_i B_{jk} \quad , \quad \hat{H}_{uij} = H_{ij} \quad , \quad H_{ij} \equiv \dot{B}_{ij} + 2\partial_i B_j \], \quad (3.34)

or, in ‘covariant’ notation (cf.(3.5),(3.6))

\[ \hat{B}_{\mu\nu} = B_{\mu\nu} + 2B_{[\mu l\nu]} \quad , \quad \hat{H}_{\lambda\mu\nu} = H_{\lambda\mu\nu} + 3l_{[\lambda} H_{\mu\nu]} \], \quad (3.35)

where \( B_{\mu\nu} \), \( B_\mu \), \( H_{\lambda\mu\nu} \) and \( H_{\mu\nu} \) have only transverse components being non-vanishing.

The remaining gauge symmetry

\[ B'_{ij} = B_{ij} + 2\partial_i \lambda_j \quad , \quad B'_i = B_i + \partial_i \lambda_u - \partial_u \lambda_i \quad , \quad \lambda_u = \lambda_u(u,x) \quad , \quad \lambda_i = \lambda_i(u,x) \], \quad (3.36)

allows one to absorb \( B_i \) into \( B_{ij} \) (\( B_i \) plays the role similar to that of \( A_i \) in (2.2)).

Eqs.(3.30),(3.32) written in components take the form (\( \alpha' = 1 \))

\[ \bar{\beta}^G_{ij} = R_{ij} - \frac{1}{4} H_{imn} H_{jmn}^{\prime} + \ldots = 0 \], \quad (3.37)

\[ \bar{\beta}^G_{ui} = R_{ui} - \frac{1}{4} H_{mn} H_{i}^{mn} + \ldots = 0 \], \quad (3.38)

\[ \bar{\beta}^G_{uu} = R_{uu} - \frac{1}{4} H_{mn} H_{mn}^{\prime} + \ldots = 0 \], \quad (3.39)

\[ \bar{\beta}^B_{ij} = -\frac{1}{2} D^m H_{mij} + \partial m \phi H_{ij}^m - p(\dot{B}_{ij} + 2\partial_i B_j) + \ldots = 0 \], \quad (3.40)

\[ \bar{\beta}^B_{iu} = -\frac{1}{2} D^m H_{mi} + \partial m \phi H_{m}^i + \frac{1}{2} \tilde{\Gamma}_{nu}^m H_{ni}^m + \ldots = 0 \], \quad (3.41)

where \( \tilde{\Gamma}_{nu}^m = \frac{1}{2} g^{mk} \dot{g}_{kn} - \frac{1}{2} F_{mn}^\prime \) (see (3.6),(2.17),(2.14)) and we have assumed that \( \phi \) is given by (2.20) (cf.(2.21)–(2.25)). Like eq.(2.21) \( \bar{\beta}^B_{ij} = 0 \) (eq.(3.40)) can be interpreted (for \( p \neq 0 \)) as the renormalisation group equation for the coupling \( B_{ij}(u,x) \) of the transverse theory.

Let us consider the case when

\[ g_{ij}(u,x) = \kappa(u) g_{ij}(x) \quad , \quad B_{ij}(u,x) = q(u) b_{ij}(x) \], \quad (3.42)

where \( g_{ij} \) and \( b_{ij} \) correspond to a group space and are normalised in such a way that the curvature of the generalised connection with torsion

\[ \tilde{\Gamma}_{jk}^i = \Gamma_{jk}^i - \frac{1}{2} H_{jk}^i = \frac{1}{2} g^{im}(\partial_j \rho_{mk} + \partial_k \rho_{jm} - \partial_m \rho_{jk}) \quad , \quad \rho_{ij} \equiv g_{ij} + b_{ij} \], \quad (3.43)
vanishes. Then equations (3.11),(3.37) and (3.40) take the form

\[ p\dot{\kappa}g_{ij} - 2pD_{(i}A_{j)} = a(1 - \kappa^{-2}q^2)g_{ij} + ... , \quad a \equiv R/N , \quad (3.44) \]

\[ pqb_{ij} + 2p\partial_{[i}B_{j]} = 0 , \]

where we have assumed that the dilaton is homogeneous \((\phi = \phi(u))\) and used that the group space torsion is covariantly constant. If \(p \neq 0\) one should put \(A_i = 0\), \(B_i = 0\) obtaining

\[ p\dot{\kappa} = a(1 - \kappa^{-2}q^2) + ... , \quad p\dot{q} = 0 . \quad (3.45) \]

There exists a renormalisation scheme in which higher order terms in (3.45) are also proportional to \(1 - \kappa^{-2}q^2\). The conformal fixed point corresponds to \(\kappa = \pm q = const [19]\). This point is unstable: if one starts with \(\kappa = \pm q\) at \(u = 0\) one finds \(\kappa(u) \rightarrow u\) at large \(u\). If we take the transverse theory to be at the conformal point for all \(u\) and set \(p = 0\) \((\phi = \phi(u))\) then eqs.(3.37) and (3.40) (and (3.32)) are satisfied automatically so that we are left with eqs.(3.38),(3.39),(3.41) for \(A_i\), \(B_i\) and \(K\). Since in general there are no invariant vector and scalar functions on the group space there seems to be no non-trivial solutions.

One can obtain a simple set of solutions by generalising those with flat transverse space to the presence of the “trivial” antisymmetric tensor background represented by \(B_i\) [3,5,6]. The case of “homogeneous” antisymmetric tensor \(B_{ij} = B_{ij}(u)\) is equivalent to the case of \(B_{ij} = 0\) because of gauge invariance (3.36) (one can absorb \(B_{ij}(u)\) into \(B_i\) by the redefinition \(B_i \rightarrow B_i - \frac{1}{2}\dot{B}_{ij}x^j\)). If

\[ g_{ij} = \delta_{ij} , \quad B_{ij} = 0 , \quad \phi = \phi(u) , \quad p = 0 \]

we get from (3.38),(3.41),(3.39) the following system of equations for \(A_i\), \(B_i\) and \(K\) (cf. (3.16),(3.17))

\[ \partial^jF_{ij} = 0 , \quad (3.46) \]

\[ ^5 \text{It is interesting to note a similarity in the structure of eqs. (3.38) and (3.41) for } A_i \text{ and } B_i \]

\[ \text{which suggests to look for solutions with } A_i = B_i . \]
∂^j H_{ij} = 0 \quad , \quad H_{ij} = 2 \partial_{[i} B_{j]} \quad , \quad (3.47)

\[-\frac{1}{2} \partial^2 K + \frac{1}{4} F^{ij} F_{ij} - \frac{1}{4} H^{ij} H_{ij} + \partial^i \dot{A}_i + 2 \ddot{\phi} + O(\alpha' s (\partial^s F)^2, \alpha' s (\partial^s H)^2) = 0 \quad . \quad (3.48)\]

Note that as in (3.16),(3.46) there are no higher order corrections in eq.(3.47). The simplest solution of (3.47) is provided by \( B_i = -\frac{1}{2} H_{ij} (u) x^j \) [3,5]. This solution is equivalent to the one with \( B_{ij} = B_{ij}(u) \quad , \quad B_i = 0 \).

4. Another representation of generic solution and connection with 2d quantum gravity model

A relation between the model (2.1) and 2d quantum gravity coupled to a ‘transverse’ sigma model was already pointed out in [9]. Below we shall further clarify this relation using a slightly different version of the basic solution discussed in Sec.2.

1. In sec.2 we were solving the Weyl invariance conditions (2.5),(2.6) using the metric (2.1). As we noted, (2.1) is the most general ansatz for a null metric if it is understood that some particular choice of coordinates \( x^i \) and \( v \) have already being made. One may try instead to look for solutions in terms of the metric (2.2) assuming that the freedom to redefine \( v \) was used to fix the form of the dilaton (2.20) as in (3.25). In fact, the metric - dilaton background \( g_{ij}(u,x) \quad , \quad \phi = p v + \phi(u,x) \) is equivalent to \( g_{ij}(u,x) \quad , \quad A_i = p^{-1} \partial_i \phi(u,x) \quad , \quad K = 2 p^{-1} \dot{\phi}(u,x) \quad , \quad \phi = p v \). Let us consider an inequivalent solution of (3.9)–(3.12) for which \( A_i = 0 \) but \( K \neq 0 \), i.e.

\[ ds^2 = -2dudv + g_{ij}(u,x)dx^idx^j + K(u,x)du^2 \quad , \quad \phi = pv \quad . \quad (4.1)\]

When \( A_i = 0 \) eqs.(3.5)–(3.8) simplify as follows [3,5]:

\[ \tilde{g}_{\mu\nu} = \hat{g}_{\mu\nu} + Kl_{\mu}l_{\nu} \quad , \quad \tilde{g}^{\mu\nu} = \hat{g}^{\mu\nu} - Kl^{\mu}l^{\nu} \quad , \quad A_{\mu} = \frac{1}{2} Kl_{\mu} \quad , \quad (4.2)\]

\[ \tilde{\Gamma}_{\mu\nu}^{\lambda} = \hat{\Gamma}_{\mu\nu}^{\lambda} - \frac{1}{2} \hat{g}^{\lambda \rho} \partial_{\rho} Kl_{\mu}l_{\nu} + \partial_{(\mu} Kl_{\nu)}l^{\lambda} \quad , \quad F_{\mu\nu} = \partial_{[\mu} Kl_{\nu]} \quad , \quad (4.3)\]
\[
\tilde{R}_{\mu\nu\rho\sigma} = \hat{R}_{\mu\nu\rho\sigma} + 2\epsilon_{[\mu}\hat{D}_{\nu]}\hat{D}_{[\rho}K_{\sigma]} \quad \text{, } \tilde{R}_{\mu\nu} = \hat{R}_{\mu\nu} - \frac{1}{2}\hat{D}^{2}K\mu\nu \quad .
\]

Then (3.11) and (3.12) take the following form (cf. (2.21), (2.22), (2.25))

\[
\beta_{ij}^{G} + D(iW_{j}) - p\dot{g}_{ij} = 0 \quad ,
\]

\[
\frac{1}{3} + \beta_{ij}^{\phi'} + \frac{1}{2}pM^{ij}\dot{g}_{ij} - \frac{1}{2}pW_{u} - p^{2}K = 0 \quad .
\]

Since \( p \neq 0 \) the remaining equations (3.9),(3.10) (cf.(2.23),(2.24))

\[
\beta_{iu}^{G} + \frac{1}{2}\partial_{i}W_{u} + \frac{1}{2}\dot{W}_{i} - \dot{g}_{ij}W^{j} + p\partial_{i}K = 0 \quad ,
\]

\[
\beta_{uu}^{G} + \dot{W}_{u} - \frac{1}{2}W^{i}\partial_{i}K + p\dot{K} = 0
\]

should again be satisfied as a consequence of (4.5),(4.6) (note that all \( K \)-dependence in (4.7),(4.8) is shown explicitly). Substituting \( \dot{g}_{ij} \) from (4.5) into (4.6) we find the following expression for \( K \) in terms of functions of \( g_{ij} \) only

\[
p^{2}K = \frac{1}{3} + \beta_{ij}^{\phi'} + \frac{1}{2}M^{ij}(\beta_{ij}^{G} + D(iW_{j})) - \frac{1}{2}pW_{u} \quad .
\]

Since (4.5) is a first order equation for \( g_{ij}(u, x) \) and \( K \) is explicitly given by (4.9) we conclude that the system (4.5)–(4.9) always has a solution for generic initial conditions.

It is interesting to note that \( K \) given by (4.9) has a natural interpretation as the basic “central charge” Weyl anomaly coefficient of the transverse theory (a linear combination of \( \beta_{ij}^{\phi'} \) and \( \beta_{ij}^{g} \)) which is changing with ‘time’ \( u \). For example, in the leading order approximation (4.7) and (4.9) take the form

\[
p\dot{g}_{ij} = R_{ij} + O(\alpha') \quad ,
\]

---

6 Eq. (4.8) giving the expression for \( \dot{K} \) may be related to the \( c \)-theorem [20]. The embedding of the RG flow of a non-conformal \( N \)-dimensional “transverse” theory into the Weyl invariance conditions of the \( N+2 \)-dimensional theory [9] may help to clarify the meaning of the \( c \)-theorem in the sigma model context (cf.[16]).
\[ p^2 K = c + \frac{1}{4} R + O(\alpha') \quad . \]  

(4.11)

It is easy to check that eqs. (4.7), (4.8), namely,

\[
\frac{1}{2}(D^j \dot{g}_{ij} - \partial_i (g^{jk} \dot{g}_{jk})) + p \partial_i K + O(\alpha') = 0 \quad ,
\]

(4.12)

\[
-\frac{1}{2}(g^{ij} \ddot{g}_{ij} - \frac{1}{2} g^{ij} g^{mn} \dot{g}_{im} \dot{g}_{nj}) - \frac{1}{2} D^2 K + p \dot{K} + O(\alpha') = 0 \quad ,
\]

(4.13)

are indeed satisfied identically on (4.10), (4.11).

2. The sigma model corresponding to (4.1) \((\alpha' = 1)\)

\[ I = \frac{1}{4\pi} \int d^2 z \sqrt{\gamma} \left[ G_{\mu\nu}(x) \partial_\mu x^i \partial_\nu x^j + R^{(2)}(x) \phi(x) \right] \]

\[ = \frac{1}{4\pi} \int d^2 z \sqrt{\hat{\gamma}} \left[ -2 \partial_a v \partial^a u + g_{ij}(u, x) \partial_a x^i \partial^a x^j + K(u, x) \partial_a u \partial^a u + pv R^{(2)} \right] \quad ,
\]

(4.14)

may be interpreted as a “quantum action” (represented in the conformal gauge) of the scalar-tensor 2\(d\) gravity theory coupled to the transverse sigma model. In fact, consider the following classical action

\[ I_0 = \frac{1}{4\pi} \int d^2 z \sqrt{\gamma} \left[ pv \hat{R}^{(2)} + g_{ij}(x) \partial_a x^i \partial^a x^j \right] \quad ,
\]

(4.15)

where \(u(z)\) is an extra scalar field coupled to 2\(d\) gravity (see e.g. [10–13]). In the conformal gauge

\[ \hat{\gamma}_{ab} = e^{-2u/p} \gamma_{ab} \quad ,
\]

(4.16)

(4.15) takes the form

\[ I_0 = \frac{1}{4\pi} \int d^2 z \sqrt{\hat{\gamma}} \left[ -2 \partial_a v \partial^a u + g_{ij}(x) \partial_a x^i \partial^a x^j + pv R^{(2)} \right] \quad .
\]

(4.17)

Since \(u(z)\) is proportional to the conformal factor of the 2\(d\) metric one expects that at the quantum level \(g_{ij}\) (which in general depends on a cutoff) should start running with \(u\) according to the RG equation (4.10). Also, the conformal anomaly term \((\sim (\partial u)^2)\) should appear. The total theory should be conformal invariant with respect to the background
metric $\gamma_{ab}$ since the 2d metric itself is an integration variable [21–23]. This is precisely the result we have got in (4.14) with $K$ playing indeed the role of the “central charge” coefficient of the transverse ($N$-dimensional) sigma model!

In principle, one could expect the quantum action to contain also another anomaly structure $\phi(u, x)R^{(2)}$. However, as we have seen, the condition of conformal invariance of the theory (4.14) is satisfied without need to introduce such term. If to represent the quantum analog of (4.15) we have used instead of (4.1) the solution of Sec.2 (cf.[9]) then the conformal invariant quantum action would contain the dilaton term $\phi(u, x)R^{(2)}$ instead of the “anomaly” term $K(u, x)\partial_a u \partial^a u$,

$$I = \frac{1}{4\pi} \int d^2 z \sqrt{\gamma} \left[ -2\partial_a v \partial^a u + g_{ij}(u, x) \partial_a x^i \partial^a x^j + pv R^{(2)} + \phi(u, x)R^{(2)} \right] .$$

The important difference between the theory (4.15) and the standard 2d gravity coupled to a sigma model (where one expects both the anomaly term $K(u, x)\partial_a u \partial^a u$ and $\phi(u, x)R^{(2)}$ to appear in the quantum action [23]) is due to the presence of the extra scalar field $v$. Though it could seem that introducing an extra field we would make the theory $N + 2$-dimensional, it, in fact, remains effectively $N + 1$-dimensional as in the absence of $v$ since the couplings do not depend on $v$ (the Killing symmetry is preserved by renormalisation).

It is straightforward to generalise the above discussion to the case when the tachyon coupling $T$ is included into the sigma model action, i.e. when there is a potential term in (4.14). The Weyl invariance condition corresponding to $T$ has the standard form [24,17,25] (cf.(2.5)–(2.8))

$$\beta^T = -\gamma T + (\alpha' \partial^\mu \phi + \frac{1}{2}W^\mu)\partial_\mu T - 2T + b(T)$$

$$= -\frac{1}{2} \alpha' D^2 T + \alpha' \partial^\mu \phi \partial_\mu T - 2T + O(\alpha'^3) + b(T) = 0 ,$$

where $\gamma$ is the same operator as in (2.8) and $b(T)$ denote “non-perturbative” corrections which are of higher order in $T$ (similar terms are present in (2.5),(2.6)). If the metric $G_{\mu\nu}$ is null (2.1) and the dilaton is linear in $v$ (2.20) then for $v$-independent tachyon $T = T(u, x)$
eq.(4.18) takes the form similar to (2.21),(2.25),(3.40), i.e. becomes RG-type equation which is first order in the $u$-derivative

$$p\dot{T} = \bar{\beta}_{x}^{T} \ . \quad (4.19)$$

$\bar{\beta}_{x}^{T}$ denotes the Weyl anomaly coefficient of the “transverse” theory with coupling $T(u, x)$ and $u$ playing the role of the RG “time” ($\bar{\beta}_{x}^{T}$ does not depend on $v$ and contains only derivatives over $x^i$). To provide the simplest example of a solution of (4.18),(4.19) let us drop the dependence on $x^i$ and ignore for a moment “non-perturbative” terms. Then

$$p\dot{T} = 2T \ , \quad T = T_0 e^{2u/p} \ . \quad (4.20)$$

Equivalent solution in the context of 2d gravity model was discussed in the last two papers in [12]. It is interesting to note that this solution is actually the exact one, i.e. it solves (4.18) with all higher order terms included. In fact, it is easy (as compared to the case of the Liouville theory) to see that there are no non-perturbative divergences in the model

$$I = \frac{1}{4\pi} \int d^2 z \sqrt{\gamma} \left[ -2\partial_a v \partial^a u + pvR^{(2)} + T(u) \right] . \quad (4.21)$$

$v$ plays the role of a Lagrange multiplier (for flat $\gamma_{ab}$ background) so that $u$ is effectively non-propagating. As a result, there are essentially no quantum corrections in the theory (a similar observation was made in [26]). Then the condition of conformal invariance is equivalent to the classical conformal invariance relation (4.20).\footnote{One may wonder how to reconcile this conclusion with the expected presence of $O(T^2)$ and $O(\partial T\partial T)$ terms in $\bar{\beta}_T$ and $\bar{\beta}_G$. As discussed in [25], the derivation of such terms (which correspond to analogous terms in the effective action) presumes analytic continuation in momenta and is not, strictly speaking, valid in the case when $T$ depends on just one variable. The question of non-perturbative terms in the $\beta$-functions should be studied separately in each particular theory (e.g. Liouville, sin-Gordon or (4.21)).}

In general, it appears that the 2d gravity model with an extra scalar field is better defined and simpler than pure 2d gravity (which does not have a non-trivial tree level action). What we have demonstrated above is that one can describe the coupling of quantum
2d scalar - gravity system to a non-conformal theory in terms of a conformally invariant sigma model in $N + 2$ dimensions. Similar representations for models describing coupling of pure 2$d$ quantum gravity to a non-conformal matter in terms of $N + 1$ dimensional conformal theories are not explicitly known. Moreover, in an attempt to find such a representation one runs into the problem of ambiguities in choosing proper initial conditions since the corresponding evolution equations are second order in ‘time’ (conformal factor).

It is remarkable that by introducing one extra dimension but at the same time imposing the null Killing symmetry it is possible to interpret the conformal invariance conditions on a higher dimensional theory as the standard (first order) RG equations for the matter theory.

5. Concluding remarks

In this paper we have studied solutions of the string effective equations for the backgrounds with covariantly constant Killing vector. We have generalised the previous discussions [3–8] to the case when the transverse theory is non-conformal and the dilaton contains the term linear in light cone coordinate $v$. The resulting equations can be interpreted as the RG equations for the couplings of the transverse theory [9]. We have proved the existence of the solutions by making use of the general covariance identities for the Weyl anomaly coefficients [14]. We have also clarified the question of gauge equivalence of different backgrounds and reproduced the solutions of refs.[3–8] in a systematic way.

We have suggested the connection between the solutions (conformal invariant $2 + N$ - dimensional sigma models) and the 2$d$ scalar quantum gravity coupled to a non-conformal ‘transverse’ $N$ - dimensional sigma model. The conformal factor of the 2$d$ metric is identified not with time but with the light cone coordinate $u$. The difference as compared to ref.[9] is that we have used the solution for which the analog of the conformal anomaly term appears in the sigma model action. It would be interesting to clarify further the
connection between the corresponding “propagating” conformal theories and 2\(d\) quantum gravity models.

I am grateful to G. Gibbons and G. Horowitz for useful discussions and, in particular, to G. Gibbons for emphasising to me the relevance of ref.\([1]\) and for drawing my attention to refs.\([8]\). I would like to thank the International School for Advanced Studies (SISSA), Trieste and the Aspen Center for Physics for hospitality while parts of this work were done. I would like also to acknowledge the support of Trinity College, Cambridge.
References

[1] H.W. Brinkmann, Math. Ann. 94(1925)119 .

[2] D.Kramer, H. Stephani, E.Herlt and M. MacCallum, Exact solutions of
Einstein's Field Equations (Cambridge U.P., 1980).

[3] R. Guven, Phys. Lett. B191(1987)275 .

[4] D. Amati and C. Klimcik, Phys. Lett. B219(1989)443 .

[5] G. Horowitz and A.R. Steif, Phys. Rev. Lett. 64(1990)260 ;
Phys. Rev. D42(1990)1950 .

[6] G. Horowitz, in: Proceedings of Strings '90,
College Station, Texas, March 1990 (World Scientific,1991).

[7] R.E. Rudd, Nucl. Phys. B352(1991)489 .

[8] C.Duval, G.W. Gibbons and P.A. Horváthy, Phys. Rev. D43(1991)3907 ;
C.Duval, G.W. Gibbons, P.A. Horváthy and M.J. Perry, unpublished (1991) .

[9] A.A. Tseytlin, Phys. Lett. B288(1992)279 .

[10] C. Teitelboim, Phys. Lett. B126(1983)41 ;
R. Jackiw, in: Quantum Theory of Gravity, ed. S.Christensen (Adam
Hilger, Bristol 1984) ;
A.H. Chamseddine, Phys. Lett. B256(1991)2930; Nucl. Phys. B368(1992)98 ;
T. Banks and M. O’Loughlin, Nucl. Phys. B362(1991)649 .

[11] C.G. Callan, S.B. Giddings, J.A. Harvey and A. Strominger,
Phys. Rev. D45(1992)1005 .

[12] J. Russo and A.A. Tseytlin, Nucl. Phys. B382(1992)259 ;
H. Verlinde, preprint PUPT-1303 ;
A. Strominger, preprint UCSBTH-92-18 ;
T.T. Burwick and A.H. Chamseddine, preprint ZU-TH-4/92 ;
S.P. deAlwis, preprint COLO-HEP-280(1992) ;
A. Bilal and C. Callan, preprint PUPT-1320 ;
S.B. Giddings and A. Strominger, preprint UCSBTH-92-28.

[13] B. de Wit, M.T. Grisaru, E. Rabinovici and H. Nicolai,
   Phys. Lett. B286(1992)78.

[14] G. Curci and G. Paffuti, Nucl. Phys. B268(1987)399.

[15] C.G. Callan, D. Friedan, E. Martinec and M.J. Perry, Nucl. Phys.
   B262(1985)593.

[16] A.A. Tseytlin, Int. J. Mod. Phys. A4(1989)1257.

[17] A.A. Tseytlin, Phys. Lett. B178(1986)34; Nucl. Phys. B294(1987)383.

[18] G.M. Shore, Nucl. Phys. B286(1987)349;
   H. Osborn, Nucl. Phys. B294(1987)595.

[19] E. Witten, Commun. Math. Phys. 92(1984)455;
   E. Braaten, T.L. Curtright and C.K. Zachos, Nucl. Phys. B260(1985)630;
   S. Mukhi, Phys. Lett. B162(1985)345.

[20] A.B. Zamolodchikov, JETP Lett. 43(1986)349.

[21] F. David, Mod. Phys. Lett. A3(1988)1651;
   J. Distler and H. Kawai, Nucl.Phys. B321(1989)509;
   S.R. Das, S. Naik and S.R. Wadia, Mod. Phys. Lett. A4(1989)1033;
   J. Polchinski, Nucl. Phys. B324 (1989)123.

[22] T. Banks and J. Lykken, Nucl. Phys. B331(1990)173.

[23] A.A. Tseytlin, Int. J. Mod. Phys. A5(1990)1833.

[24] C. Callan and Z. Gan, Nucl. Phys. B277(1986)647.

[25] A.A. Tseytlin, Phys. Lett. B241(1990)233; Phys. Lett. B264(1991)311.

[26] J. Russo, L. Susskind and L. Thorlacius, preprint SU-ITP-92-24.