ACYCLIC JACOBI DIAGRAMS

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Abstract. We propose a simple new combinatorial model to study spaces of acyclic Jacobi diagrams, in which they are identified with algebras of words modulo operations. This provides a starting point for a word-problem type combinatorial investigation of such spaces, and provides fresh insights on known results.

1. Introduction

Jacobi diagrams are a subset of labeled pseudo-graphs whose vertices have valence 1 or 3, with some extra structure. They provide a profound and as yet largely unexplored bridge between the world of Lie algebras and the world of low-dimensional topology, concisely encoding topological invariants which are in some sense “Lie algebra-like”.

We work over a field $k$ of characteristic different from 2. Let $\mathbb{N}_0$ denote the non-negative integers (the natural numbers). Roughly speaking, to a fixed Lie algebra (or more generally to a fixed Lie algebra object $\mathcal{L}$), and a low-dimensional topological object $M$ (a knot, a link, a 3–manifold, a hyper-Kähler manifold, a handlebody...), a Jacobi diagram defines a map called a weight system from the input data to an $\mathbb{N}_0$–graded vector space $V$. Thus, Jacobi diagrams plus weight systems define $V$–valued topological invariants called the finite type invariants of $M$.

If the object of study is the space of finite type invariants (either of a given $M$ or of all $M$), the key space to understand is the space of Jacobi diagrams (see $\mathcal{L}$ for the set-theoretical arguments allowing us to call this a space or a set). In particular, being able to identify this space with a space of Lie algebras (or of Lie algebra objects) with something cut away would be a huge step forward both for low dimensional topology and for the theory of Lie algebras.

For the space of all Jacobi diagrams the question above is wide open (see however $\mathcal{L}$ and $\mathcal{L}$ for recent progress). But when we restrict to acyclic graphs— to Jacobi diagrams without loops— we have a some understanding of the relationship between the Lie algebra world and the world of Jacobi diagrams ($\mathcal{L}$ Section 4.3),

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We should note however that neither side of this relationship is in itself well understood.

**Theorem 1.1.** The space of connected acyclic Jacobi diagrams over $\mathbb{Q}$ is isomorphic to $h_n(p)$ which is given by

\begin{equation}
 h_n(p) \rightarrow L_{n-1}(p) \otimes p \rightarrow L_n(p) \rightarrow 0
\end{equation}

When $L_n(p)$ denotes the length $n$ part of the free Lie algebra over $p$ generators.\(^1\)

One corollary of our main theorem is a simple combinatorial proof of the above result, first proved in the $k = \mathbb{Q}$ case in [10, 14].

An example of Theorem 1.1 for $n = p = 3$ is given below in set-theoretical notation

\[ \begin{array}{c}
1 \\
3 \\
2 \\
\end{array} \rightarrow \begin{array}{c}
[1, 2] \otimes 1 & [1, 2] \otimes 2 & [1, 2] \otimes 3 \\
[1, 3] \otimes 1 & [1, 3] \otimes 2 & [1, 3] \otimes 3 \\
[2, 3] \otimes 1 & [2, 3] \otimes 2 & [2, 3] \otimes 3 \\
\end{array} \rightarrow \begin{array}{c}
[[1, 2], 1] & [[1, 2], 2] & [[1, 2], 3] \\
[[1, 3], 1] & [[1, 3], 2] & [[1, 3], 3] \\
[[2, 3], 1] & [[2, 3], 2] & [[2, 3], 3] \\
\end{array} \rightarrow 0
\]

The topological meaning of the finite type invariants associated to acyclic Jacobi diagrams is well known—these are the Milnor $\bar{\mu}$-invariants which measure linkage ([8, 9, 19]). These invariants play no front-line role in the study of knots and of manifolds (although they play an important role behind the scene as part of what is known as the Associator), but they come to the fore in the study of finite-type invariants of links, braids, tangles.

In this paper we show that the space of acyclic Jacobi diagrams is equivalent to a subspace of itself of graphs of a certain special shape called *swings* modulo a set of moves inherited from the moves on Jacobi diagrams. The space of swings is computationally simpler than the full space of acyclic Jacobi diagrams, and is isomorphic either to an algebra of words we call $\mathcal{W}'(p)$ or to a different algebra of words we call $\mathcal{W}'(p)$. These are defined to be the free associative $k$-algebra on $p$ letters $\text{ASS}(p)$ modulo the families of fold moves $H'$ and $\mathcal{H}$ correspondingly (see Section 2.2). Which of these two algebras of words the space of swings is isomorphic to depends on whether we are dealing with unrooted or with rooted Jacobi diagrams (free Lie algebras).

Formally stated:

**Main Theorem.** The space of (unrooted) acyclic Jacobi diagrams $\mathcal{A}_c(p)_{-1}$ is isomorphic to the algebra of words $\mathcal{W}'(p)$ and the free Lie algebra $L(p)$ is isomorphic to the algebra $\mathcal{W}'(p)$.

An example of the first part of the theorem is given below

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\(^1\)See [14, 15] for the corresponding question over $\mathbb{Z}$.\]
Definitions of the above concepts will be given in Section 2. The space of connected acyclic Jacobi diagram chains \( \mathcal{A}_c(p) \) and its rooted version \( \mathcal{L}(p) \) the free Lie algebra over \( k \) will be defined in Section 2.1. The remaining terminology and notation which pertains to spaces of swings and their moves will be defined in Section 2.2.

In Section 4 we illustrate how our main theorem may be used to calculate by hand the 5373540 basis elements of \( h_9(9) \). This example provides some basis for speculation on the form of a swing-basis for general spaces of acyclic Jacobi diagrams.

We conclude this introduction with a short summary of the current state of knowledge about \( h_n(p) \) and how our result supplements it. Theorem 1.1 gives dimensions of spaces of acyclic Jacobi diagrams in terms of dimensions of free Lie algebras. These are given by Witt’s dimension formula \( [22] \):

\[
\dim(L_n(p)) = \frac{1}{n} \sum_{d \mid n} \mu(d) p^n d!
\]

The sum is over all (positive) divisors \( d \) of \( n \). The Möbius function \( \mu(d) \) is defined by

\[
\mu(d) = \begin{cases} 1, & \text{if } d = 1; \\ (-1)^k, & \text{if } d = p_1 \cdots p_k \text{ (distinct } p_i\text{)}; \\ 0, & \text{if } d \text{ has a square factor.} \end{cases}
\]

Let now \( \mathcal{L}(p)_{(n_1, n_2, \ldots, n_p)} \) denote the subalgebra of the free Lie algebra over \( p \) letters consisting of words in which the \( i \)th element appears \( n_i \) times for \( 1 \leq i \leq p \), where \( \sum_{i=1}^{p} n_i = n \). Then the dimension of this space, the necklace number, is given by the following formula also due to Witt \( [22] \):

\[
\dim(L(p)_{(n_1, n_2, \ldots, n_p)}) = \frac{1}{n} \sum_{d \mid n} \mu(d) \frac{n!}{n_1! \cdots n_p!}
\]

To make further computational progress for acyclic Jacobi diagrams, the best basis one might hope for would be a monomial basis for \( h_n(p) \) in terms of swings. Theorem 1.1 reduces the problem to that of finding a monomial basis for free Lie algebras. Sergei Duzhin suggests that this might be an important step in the calculation of the rational associator in general, not just when restricting to acyclic Jacobi diagrams.

The problem of calculating left-normed bases for free Lie algebras was first discussed by Kukin \( [13] \) who claimed to have solved it. Fifteen years later, deficiencies in his construction were revealed by Blessenohl and Laue \( [4] \) who offered an alternative construction which works over any field \( k \) which contains all roots of unity. Unfortunately this condition rarely holds in a topological setting, where the ground field is usually the rationals, the integers, or some finite field. The main theorem of this paper significantly simplifies the algorithmic calculation of such a basis. Since
it appears unlikely that our set of moves on swings is minimal, this calculation can likely be simplified yet further. We hope to return to this problem in the future.

This paper is a reorganized version of the sections on acyclic Jacobi diagrams in the author’s Master Thesis in the University of Tokyo [17].

2. Basic Definitions and Notation

2.1. Acyclic Jacobi Diagrams. Fix a natural number $p$ and a field $k$ of characteristic different from 2.

**Definition 2.1.** An acyclic Jacobi diagram is a connected vertex-oriented acyclic graph whose vertices have valence 1 or 3 and whose univalent vertices (legs) are labeled by elements of $p$ the ordered set $\{1, 2, \ldots, p\}$ (viewed as a vector space with an ordered basis so that we can tensor it with vector spaces).

All graphs discussed in this paper come equipped with a fixed arbitrary ordering of their vertices and arcs.

Concepts defined for acyclic graphs naturally specialize to the case of acyclic Jacobi diagrams. In the standard terminology for the Jacobi diagram world, a univalent vertex $s$ of a graph $G$ is called a *leg* of $G$ [11].

Two classes of graphs with specified vertex will be used in this paper. A *pointed* graph $\sigma(G)$ is a graph $G$ with a distinguished leg $s$ [11, 6]. A *rooted* Jacobi diagram $rt_s(G)$ is a Jacobi diagram $G$ with a single leg $s$ labeled by a distinguished element $\ast$ called the *root* rather than by an element of $p$ [3]. When $s$ is clear from the context we may omit it from the notation and we may denote such graphs $\sigma(G)$ and $rt(G)$ correspondingly.

Rooted acyclic Jacobi diagrams (acyclic Jacobi diagrams with a single leg labeled $\ast$) are in bijective correspondence with $M(p)$ the free magma over $p$ letters. The binary operation is “connecting at the root” [21].

Returning to Jacobi diagram specific terminology, an *acyclic Jacobi diagram chain* is a linear sum of acyclic Jacobi diagrams over $k$. This space is denoted $D_c(p)_{-1}$. Pointing $D_c(p)_{-1}$ gives us the algebra $k(M(p))$.

**Remark 2.2.** The notation $D_c(p)_{-1}$ is an attempt to be consistent with the standard notation for Jacobi diagrams in general, as outlined in [11]. In the present paper we shall not be interested in what the subscripts, superscripts, and $-1$ in the bracket denote. $D_c(p)_{-1}$ is to be taken as though it were a single compound symbol.

The space of acyclic Jacobi diagrams comes equipped with two operations, which are local moves between acyclic Jacobi diagrams which have embeddings into $\mathbb{R}^2$ which differ inside a dotted circle as indicated below.

\[
\begin{align*}
\text{AS} & \quad = \quad - \\
\text{IHX} & \quad = \quad + 
\end{align*}
\]
Equivalence classes of acyclic Jacobi diagrams modulo $AS$ and $IHX$ are called \textit{acyclic Jacobi diagram classes}. This space is denoted $\mathcal{A}_c(p)_{-1}$.

Remark 2.3. In the literature, Jacobi diagrams, Jacobi diagram chains, and Jacobi diagram classes are all called Jacobi diagrams (except in [18] in which the distinction is made between the first two concepts). In the present paper the distinction between these terms becomes important because of our direct combinatorial approach. From the point of view of notation however we will not distinguish between acyclic Jacobi diagrams and acyclic Jacobi diagram classes.

The rooted version of $\mathcal{A}_c(p)_{-1}$ is $L(p)$, the free Lie algebra on $p$ generators. $AS$ corresponds to the anti-symmetry relation, which $IHX$ is the Jacobi identity (this is the reason for the name ‘Jacobi diagram’ [2]).

The spaces above have a natural degree grading given by the number of legs in the graphs. Denote this degree by $\text{len}(G)$, the length of $G$. When a finer grading is needed, we may refine the length to get $\text{mld}(G)$ the multi-degree of a graph $G$. This is defined to be the vector $(n_1,n_2,\ldots,n_p)$ where $n_i$ is the number of legs of $G$ coloured by the $i$th element of $p$.

2.2. Swings. In this section we introduce the space of words with which we would like to model the space of acyclic Jacobi classes $\mathcal{A}_c(p)_{-1}$. This space is called the space of \textit{swings}. As a set it is a subset of $\mathcal{A}_c(p)_{-1}$, but its set of operations is smaller and more computer-friendly.

A \textit{vertebrate} $G_{s_1,s_2}$ is an acyclic uni-trivalent graph $G$ with distinguished legs $s_1$ and $s_2$ called the head and the tail of $G$ correspondingly. The \textit{vertebral column} of $G$ is the unique elementary path from the tail of $G$ to its head. If $s_1 = s_2$ then $G_{s_1,s_2}$ is said to be \textit{degenerate} [11, 6]. There is a map $\sigma_{s_1,s_2}$ from acyclic uni-trivalent graphs to vertebrates that chooses $s_1$ as the head and $s_2$ as the tail (this map is defined because the vertices are ordered. In Section 3.3 we prove that in a suitable sense this map is independent of this ordering.).

A vertebrate $G_{s_1,s_2}$ is said to be a \textit{swing} if all trivalent vertices of $G_{s_1,s_2}$ are neighbours of legs. The ‘rooted version’ $rt_{s_1}(G_{s_1,s_2})$ is called a \textit{half-swing}. Swings and half-swings are the main objects of this paper, which we shall use to model the whole algebra of acyclic Jacobi diagrams. Define $Sw(-)$ to be the restriction of a $k$–algebra of acyclic Jacobi diagrams to its $k$–subalgebra of $k$–linear sums of swings, or of a $k$–algebra of rooted acyclic Jacobi diagrams to its $k$–subalgebra of $k$–linear sums of half-swings.

In Section 3.2 we construct a well–defined map $\rho$: $\text{ASS}(M(p)) \rightarrow Sw(Dc(p)_{-1})$ from the free associative algebra generated by elements of $M(p)$. Correspondingly, there is a map $\rho^l$ from $\mathcal{A}_c(p)_{-1}$ to $Sw(L(p))$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure1}
\caption{Breaking a tree into branches}
\end{figure}
A vertebrate may be thought of as an element of a free associative algebra. Think of rooted trees (elements of $M(p)$) as generators and the degenerate rooted tree as the identity element, and read off the product along the vertebral column from tail to head. In this way, a swing corresponds to a word on $p$ letters. The free associative $k$–algebra of words on $p$ letters is denoted $ASS(p)$ and the free associative $k$–algebra of words on a set $X$ of rooted trees is denoted $ASS(X)$. The free associative product is given by concatenation of vertebrates, gluing the head of one to the tail of the next.

We now define a number of actions on words in $ASS(p)$. The actions extend to chains by linearity.

**Definition 2.4.**

\[
\eta(\prod_{i=1}^{n} a_i) := \begin{cases} 
a_1 & \text{if } n = 1; 
\, a_n \eta(\prod_{i=1}^{n-1} a_i) - \eta(\prod_{i=1}^{n-1} a_i) a_n & \text{otherwise.}
\end{cases}
\]

$\eta$ is the algebraic operation corresponding to Figure 1 when $T := T_1 T_2$ is a rooted tree corresponding to a left-bracketed word in $M(p)$.

**Definition 2.5.**

- For a word $w \in ASS(p)$:

\[
h_n(w) := \begin{cases} 
(-1)^{n-1} a_n (\eta(\prod_{i=1}^{n-1} a_i)(\prod_{i=n+1}^{\text{len}(w)} a_i)) & \text{for } 2 \leq n \leq \text{len}(w); 
0 & \text{for } \text{len}(w) = 1; 
h_n & \text{for } n = \text{len}(w) > 1;
\end{cases}
\]

$h_n$ and $h'_n$ shall be called fold moves, the image being of folding egg-whites into a mixture. These are the basic operations on swings in this paper.

**Definition 2.6.** Let $^lH$ be the set of all relations of the form $h_n^l(w) - w = 0$ and let $H'$ be the set of all relations of the form $h'_n(w) - w = 0$ for all $n \geq 2$ and for all $w \in ASS(p)$. We refer to $^lH$ and to $H'$ collectively as the $H$–relations or as the collection of fold moves. Define $W'(p)$ to the quotient of $ASS(p)$ by $H'$, and define $^lW(p)$ to the quotient of $ASS(p)$ by $^lH$.

**Remark 2.7.** The naming $^lW$ stems from the fact that $h_n^l$ maps act on the left of the word. The superscript on the upper left hand side serves to make sure that $W'$ and $^lW$ do not look too much alike.

In the next section we shall use the actions we have defined here on $ASS(p)$ to recover well-known facts about free Lie algebras in our language and setting. The basic reference for the material below is Christophe Reutenauer’s book *Free Lie Algebras* [20].

2.3. Free Lie Algebra Identities in Terms of $A$–Spaces.
2.3.1. Relations Involving $\eta$.

Lemma 2.8 ([20], Theorem 1.4 (v)).

$$\eta(\eta(w)) = (-1)^{n-1}n\eta(w)$$

for any $w \in \text{ASS}(p)$.

Proof. It is sufficient to prove the claim for words of multidegree $(1, 1, \ldots, 1)$, and by linearity it is enough to prove it for $w$ a single word with coefficient 1. Our proof shall be by induction. For $\text{len}(w) = 2$, we have $\eta(\eta(w)) = \eta(a_2a_1 - a_1a_2) = 2\eta(w)$.

Let us assume that the claim is true until a certain length $n$. For $w$ of length $n + 1$, then, $n$ elements of $\eta(w)$ with sign $(-1)^{n-1}$ appear in $\eta(\eta(w))$ by induction as $\eta$ of words ending in $a_{n+1}$ in $\eta(w)$. They then appear again as reflections of their reflections which are in $\eta(w)$, when the reflection changes signs or preserves signs both times. These are all the elements in $\eta(\eta(w))$ which start or end in $a_{n+1}$.

Now all we have to show is that there are no words in $\eta(\eta(w))$ in which $a_{n+1}$ does not appear either in the first place or in the last place. Let us assume that there is such a word, of the form $w_1a_{n+1}w_2$. This word may come either from $w_1\tau(w_2)a_{n+1}$ or from $\tau(w_2)w_1a_{n+1}$ (as $a_{n+1}$ must be in the first or last place in $\eta(w)$). These appear the same number of times in $\eta(w)$, with opposite signs— when $n$ is odd, if $w_1\tau(w_2)a_{n+1}$ appears with plus, then $w_2\tau(w_1)a_{n+1}$ would be plus, and then $\tau(w_2)w_1a_{n+1}$ comes out with a minus. When $n$ is even, if $w_1\tau(w_2)a_{n+1}$ appears with plus, then $w_2\tau(w_1)a_{n+1}$ gets a minus sign, and then $\tau(w_2)w_1a_{n+1}$ preserves the sign because $w_1$ and $w_2$ are of the same parity. Then, from both $w_1\tau(w_2)a_{n+1}$ and $\tau(w_2)w_1a_{n+1}$, the sub-word $\tau(w_2)$ is reflected, and every sub-word has the same parity as itself, therefore the sign does not change, and the elements cancel each other out. \hfill $\square$

Corollary 2.9.

$$\eta(\eta(w_1)\eta(w_2) + \eta(w_2)\eta(w_1)) = 0$$
for any $w_1, w_2 \in \text{ASS}(p)$ of lengths $n_1, n_2$ correspondingly, $n_1 + n_2 > 2$. In particular, $\eta(\eta(w_1)\eta(w_1)) = 0$.

Proof. Precisely as in the proof of Lemma 2.8, all elements of $\eta(\eta(w_1)\eta(w_2))$ in which a word in $\eta(w_1)$ is split into two parts are canceled, and we are left with $(-1)^n\eta(\eta(w_2))\eta(\eta(w_1)) + (-1)^{n-1}\eta(\eta(w_1))\eta(\eta(w_2))$ (second term: $(-1)^n$ from the $w_2$) where $n$ is defined to be the combined length of $w_1$ and of $w_2$. By Lemma 2.8 this equals $n(\eta(w_2)\eta(w_1) - \eta(w_1)\eta(w_2))$. For $\eta(\eta(w_1)\eta(w_2))$, we similarly obtain $n(\eta(w_1)\eta(w_2) - \eta(w_2)\eta(w_1))$. The sum of these terms is zero. □

Lemma 2.10 (An Identity of Baker (1905), [20] Section 1.6.6). For any $w_1, w_2$ words of lengths $n_1, n_2$ respectively in $\text{ASS}(p)$,

$$\eta(w_1)\eta(w_2) = (-1)^{n_2}(\eta(w_1)\eta(w_2) - \eta(w_2)\eta(w_1))$$

Proof. Precisely as in the proof of Lemma 2.8, all elements of $\eta(\eta(w_1)\eta(w_2))$ in which a word in $\eta(w_1)$ is split into two parts are canceled, and we are left with $(-1)^{n_2}\eta(\eta(w_2))\eta(\eta(w_1)) + (-1)^{n_2-1}\eta(\eta(w_1))\eta(\eta(w_2))$ (second term: $(-1)^{n_2}$ from the $w_2$) where $n$ is defined to be the combined length of $w_1$ and of $w_2$. By Lemma 2.8 this equals $n(\eta(w_2)\eta(w_1) - \eta(w_1)\eta(w_2))$. For $\eta(\eta(w_1)\eta(w_2))$, we similarly obtain $n(\eta(w_1)\eta(w_2) - \eta(w_2)\eta(w_1))$. The sum of these terms is zero. □

2.3.2. Relations Involving Fold Moves.

Lemma 2.11. For $w := \prod_{i=1}^n a_i$, if $a_i = a_{i+1}$, then $h_i^1(w) = h_{i+1}^1(w)$.

Proof. Direct calculation. □

Lemma 2.12. Let $w := \prod_{i=1}^n a_i$ be a word in $\text{ASS}(p)$. Then for $i > j$, we have $h_i^1h_j^1(w) = h_i^1(w)$.

Proof. Let $w'$ be $\prod_{i=1}^{j-1} a_i$. By Lemma 2.8 we have that

$$h_i^1(w) = \frac{(-1)^j}{j-1} h_i^1(\eta(w') \prod_{i=j}^n a_i)$$

But now we have the equality

$$h_i^1(w) - \frac{(-1)^j}{j-1} h_i^1(w') = \frac{(-1)^j}{j-1} h_i^1(\eta(w') a_j \prod_{i=j+1}^n a_i)$$

But by lemma 2.8 again, this is $\frac{(-1)^j}{j-1} h_i^1(w)$. Subtraction gives equality. □

3. Proof of the Main Theorem

3.1. Outline of Proof. Our basic setup is as follows.

$$\begin{array}{ccccccc}
D_c(p)_{-1} & \xrightarrow{\sigma_1, \sigma_2} & \text{ASS}(M(p)) & \xrightarrow{\rho} & Sw(D_c(p)_{-1}) & \xrightarrow{\psi'} & Sw(D_c(p)_{-1})/H' \\
\downarrow \{\text{ASS, IHX, STU} \} & f \uparrow & \text{f} & \uparrow & g & \\
A_c(p)_{-1} & \xrightarrow{\psi_w} & \text{ASS}(p) & \xrightarrow{\psi_w} & W'(p)
\end{array}$$

Our maps are as follows:
(1) $\sigma_1, \sigma_2$ maps acyclic graphs to vertebrates as in Section 2.2 by selecting $s_1$ as the head and $s_2$ as the tail. This is defined since vertices of elements of $\mathcal{D}_c(p)_{-1}$ are ordered.

(2) $\rho$ is defined by figure 2.2.

(3) $\psi_t$ is the quotient map by the relations $H'$ on trees.

(4) $\psi_w$ is the quotient map by the corresponding relations $H'$ on words.

(5) $f$ and $g$ are natural embeddings.

The corresponding commutative diagram in the rooted world is

$$
\begin{array}{cccc}
\mathbb{K}(M(p)) & \overset{\sigma_1^i}{\longrightarrow} & \mathbb{K}(rt(\text{ASS}(M(p)))) & \overset{\rho^i}{\longrightarrow} & \mathbb{K}(SW(D_c(p)_{-1})) & \overset{\psi^i_t}{\longrightarrow} & \mathbb{K}(SW(D_c(p)_{-1})/H') \\
\downarrow{\{\text{AS, IHX, STU}\}} & & \quad & & \quad & & \quad \\
\mathcal{L}(p) & \quad & \text{ASS}(p) & \quad & \text{ASS}(p) & \quad & \mathcal{W}(p)
\end{array}
$$

The proof of our main theorem proceeds as follows.

(1) We prove in Section 3.2 that the mapping $\rho$ is well-defined.

(2) We prove in Section 3.3 that $\psi_t \rho \sigma_{(i,j)}$ is independent of the mapping $\sigma_{(i,j)}$.

(3) We prove in Section 3.4 that the kernel of the mapping $\psi_w g$ includes the kernels of the $\text{AS}$ and of the $\text{IHX}$ actions (thus $SW(\mathcal{A}_c(p)_{-1})/H' \supseteq \mathcal{A}_c(p)_{-1}$).

As the $H'$ relations come from $\text{AS}$ and $\text{IHX}$ (thus $SW(\mathcal{A}_c(p)_{-1})/H' \subseteq \mathcal{A}_c(p)_{-1}$), this is sufficient to prove isomorphism between $\mathcal{A}_c(p)_{-1}$ and $\mathcal{W'}$. The proof in the rooted world is fully analogous.

Corollaries to the proof, including a new proof to Theorem 1.1 are given in Section 3.5.

3.2. Trees to Swings— That the $\rho$ Map is Well-Defined. The process of breaking down acyclic Jacobi diagrams into sums of swings over $\mathbb{K}$ defines a mapping $\rho$ from the space of twice pointed acyclic Jacobi diagrams $\text{ASS}(\text{M}(p))$ to the space of swings, $SW(\mathcal{D}_c(p)_{-1})$. The aim of this section is to show that the mapping $\rho$ (and its analogous mapping $\rho^i$ from $\mathcal{L}(p)$ to $SW(\mathcal{L}(p))$) is well-defined— that it does not depend on the order in which we break the tree into swings.

Lemma 3.1. The $\rho$ mapping is well defined.

Proof. In order to prove this statement, we have to first show that $\rho$ is independent of the order in which we break down the branches of a tree until we get a swing.

We begin with a twice pointed acyclic Jacobi diagram. For every trivalent vertex not on the vertebral column and which is not the neighbour of two legs, we assign two things— a number to say when it is to be broken down; and a choice of arc adjacent to the vertex connecting it to another trivalent vertex ‘further away’ from the vertebral column. This indicates which subtree is to be broken into which other subtree. An example is given in Figure 3.

The claim that $\rho$ is independent of this labeling is the claim that any two such labelings give the same breakdown of the tree into a sum of swings over $\mathbb{K}$. It is sufficient to prove this for every adjacent pair of trivalent vertices.

Diagrammatically, the claim we have to prove is the claim of Figure 4. Allowing $a_i$’s to signify subtrees as well as individual legs, this is equivalent to Lemma 2.10.

$\blacksquare$
Figure 3. A possible ordering for the breakdown of a tree for the $\rho$ mapping

Figure 4. Independence of labeling on two adjacent vertices.

This immediately implies that $\rho^l$ is also well-defined.

3.3. Heads and Tails—Independence from $\sigma_{i,j}$ Map. We now prove that it does not matter which legs we chose to be the head and the tail of our tree before we break down our tree into a sum of swings—all such choices are equivalent modulo the $^tH$ relations.

Lemma 3.2. For any $w_1, w_2$ words in $ASS(\eta)$

$$w_1 \eta(w_2) = (-1)^{n-1} w_2 \eta(w_1)$$

as elements of $^tW$, where $n := \text{len}(w_1) + \text{len}(w_2)$.

Proof. Let $n_1$ be the length of $w_1$, $n_2$ the length of $w_2$. In order to make the following calculations easier to understand, let us use the notation $w_1 := \prod_{i=1}^{n_1} a_i$ and $w_2 := \prod_{i=1}^{n_2} b_i$.

We start with $w_1 \eta(w_2)$, applying the $h_n^l$ action to all words ending in $b_{n_2}$ (which must be of the form $-w_1 \eta(\prod_{i=1}^{n_2} b_{n_2-i}) b_{n_2}$) and the $h_{n_1+1}^l$ action to the remaining words (which must be of the form $w_1 b_{n_2} \eta(\prod_{i=1}^{n_2} b_{n_2-i})$) and see what happens.

Step 1 The elements of the result of the action we have chosen, for which $\eta(w_2)$ is not split, are $-(-1)^{n-1} b_{n_2} (-1)^{n_2} \eta(\prod_{i=1}^{n_2} b_i) \eta(w_1)$, which by the definition of $h_{n_2}^l$ is simply $(-1)^{n-1} w_2 \eta(w_1)$.

Step 2 It remains to show that what is left of $w_1 \eta(w_2)$ is zero. Let now $w_2'w_2''$ be a word in $\eta(w_2)$, with $\text{len}(w_2') > 1$. Now $w_2' := \prod_{i=1}^{\text{len}(w_2')} b_i$, $w_2'' := \prod_{i=1}^{\text{len}(w_2'')} b_i'$. The word $w_2'\eta(w_1)\tau(w_2'')$ in the image of our action comes from exactly two sources. The first is $h_n^l(w_1 w_2'' \tau(w_2''))$ which contains the linear combination $(-1)^{n-1 \cdot \text{len}(w_2'')} w_2' \eta(w_1) \tau(w_2'')$. The second place it comes from is $h_n^l$ of the linear combination $(-1)^{\text{len}(w_2'')} - w_1 b_{\text{len}(w_2'')} \tau(w_2'') \prod_{i=1}^{\text{len}(w_2'')} b_i'$ in
w_1 \eta(\tau(w_2)). The two terms are of opposite sign, and therefore they cancel each other out.

**Step 3** Let us now see what happens when \text{len}(w'_2) = 1. Let \( w'_2 := b' \cdot w_1 \eta(w''_2) \cdot b' \) all give \((-1)^{n-1} \cdot \text{len}(w''_2) \cdot b' \eta(w_1) \eta(w''_2) \) under the \( h_n^l \) action. Taking now \(-\eta(w_1) b' \eta(w''_2)\) and acting on it with \( h_n^l \cdot h_{n+1}^l \), we get \((-1)^{n-1} \) times the same thing. This is an exact elimination, and it kills all \( b' \eta(w_1) w''_2 \) words.

This exhausts the “remainder”, and our lemma is proved. \( \square \)

For the proof of our main theorem, this lemma is a strong enough result—however we may strengthen it still further.

**Lemma 3.3.** For any \( w_1, w_2, \ldots, w_m \) words in \( \text{ASS}(p) \)

\[
w_1 \prod_{i=2}^{m} \eta(w_i) = (-1)^{m+n-1} w_m \eta(\prod_{i=1}^{m-1} (\eta(w_i)))
\]

as elements of \( \mathcal{W} \), where \( n := \sum_{i=1}^{m} \text{len}(w_i) \). Here the notation \( (\eta(w_i)) \) means that a function applied to the word ‘reads’ \( \eta(w_i) \) as if it were a single letter.

**Proof.** We prove the claim by induction. For \( m = 2 \), this is exactly Lemma 3.2. Let us assume that the claim holds until \( m = M \in \mathbb{N} \). We proceed as in the proof of that lemma, with \( w_{M+1} \) here playing the the role of \( w_2 \), and the proof is exactly analogous. \( \square \)

We now translate Lemma 3.2 to acyclic Jacobi diagrams.

**Corollary 3.4.** When taking a rooted tree to its left-normed bracketed form by breaking down trees, the rooted tree we get is independent of which leg we choose to be the tail of the swing during the construction, as an element of \( \mathcal{W} \).

**Proof.** This follows directly from Lemma 3.2 (see Figure 5). On the left hand side, breaking \( t_2 \) into \( t_1 \) gives \((t_1 \eta(t_2))\), while breaking \( t_1 \) into \( t_2 \) gives \((t_2 \eta(t_1))\). By the lemma, these are then equal. \( \square \)

**Corollary 3.5.** When “breaking down a tree into swings” as in Figure 1, the sum of swings we get is independent of which legs we choose to be the heads of the swings during the construction, as an element of \( \mathcal{W}' \).

**Proof.** Let \((b_1, b_2)\) and \((b'_1, b'_2)\) be two choices of tail and head respectively for a given tree \( T \). Because a tree is 1-connected these exists an arc \( c \) which if removed separates the tree into two subtrees, each of which contains a pair of these four
points. If there exists such an arc $c$ separating the tree into subtrees containing $(b_1, b'_1)$ and $(b_2, b'_2)$ in distinct connected components of $T \setminus \{c\}$, then by Corollary 3.4 the breakdown of these subtrees into sums of half-swings with heads $b_1$ and $b'_1$ (and $b_2$ and $b'_2$ respectively) gives the same pre-image under $f$ as elements of $W'$. ‘Remembering’ $c$ proves the corollary for this case, as the $H$–actions in each $W'$ are also in particular actions of $W'(\text{deg}(T))$. Because of the action $\tau$, the case in which there exists an arc $c$ separating the tree into subtrees containing $(b_1, b'_2)$ and $(b_2, b'_1)$ in distinct connected components of $T \setminus \{c\}$ is analogous.

In the case that there are no such arcs, let $c_1$ and $c_2$ separate $b_1, b'_1, b_2, b'_2$ into separate components respectively. Let us now pick new leaves $b''_1$ in the connected component of $b_1$ and $b'_2$ in the connected component $b''_2$ of $T \setminus \{c_1, c_2\}$.

‘Remembering’ $c_1$ and $c_2$ in turn, the choice of heads $(b''_1, b''_2)$ is equal to the choice $(b_1, b_2)$ (as now $c_1$ separates the tree into subtrees containing $(b_1, b''_2)$ and $(b_2, b''_1)$ respectively) and to the choice $(b'_1, b'_2)$ (same as before except with $c_2$). Therefore these choices give the same sum of swings as an element of $W'$. □

We can now show the following.

**Proposition 3.6.** The kernel of the $IHX$ relation on $A_c(p)_{-1}$, under the $\rho$ mapping, is included in the kernel of the $H$ relations under the $g$ mapping.

**Proof.** Again, it is enough to prove this for a single tree. We must show that $IHX$’s which are not in $H'$ do not ‘impose extra relations on words’. Let us take such a relation, $IHX_a t = t'$.

First, by Lemma 3.2, the relation is not dependent on choices of head and tail. Let us take tails of $t_1$ and of $t_2$ in their left-normed bracketed forms to be our tail and head respectively. Our relation then takes the form of Figure 4 in Section 3.2 where the proof of Lemma 3.1 shows us that this is no new relation. □

We deduce the following.
Corollary 3.7. When “breaking down a tree into swings” the element in $\text{ASS}(p)$ corresponding with the sum of swings we get is independent of how we decide to break the tree down, as an element of $\mathcal{W}'$. The corresponding statement also holds for half-swings.

Proof. This is just a combination of Lemma 3.2, Corollary 3.4, and the Lemma 2.1 as it is used in the proof of Proposition 3.6. □

3.4. The $\text{AS}$ Relations. In order to prove the main theorem, it remains only to show that the kernel of the $\text{AS}$ action on $\mathcal{D}_c(p)_{-1}$ (and on $\mathcal{M}(p)$) is contained in the kernel of the $H'$ relations (the $H$ relations).

First, we move the problem to the level of words, by defining what we shall call the $Y$ relations. We would like the manifestation of these relations as Jacobi diagram relationships to be as pictured in Figure 7.

Definition 3.8. For $v \in \text{ASS}(p)$, $w$ a word in $\text{ASS}(p)$, let

$$y_w(v) := \begin{cases} 0 & \text{if } v = 2w\eta(w); \\ v & \text{otherwise.} \end{cases} \quad (3.3)$$

Remark 3.9. The coefficient 2 in the definition of $y_w(v)$ is there to remind us that the $\text{AS}$ relationships have no kernel over a field of characteristic 2. This coefficient will be ignored from now on, as long as this point is kept in mind.

We would like to find a minimal family of such relations which act on $\mathcal{W}'$. First, we may demand that $w \in \mathcal{W}'$. Secondly we may demand that $\text{len}(w)$ be even, as if it is odd and $y_w(w') = 0$ then $w' = 0$ by Lemma 3.2 anyway.

Definition 3.10. Let $Y$ be the union of all $y_w$’s, $w \in \mathcal{W}'$ a word whose length is an odd number.

In order to show that the $Y$ relations are in the kernel of $H'$, we shall need the following little lemma.

Lemma 3.11 (A version of [20], Theorem 1.4 (v)). For all $w \in \mathcal{W}$ of length $n$,

$$\eta(w) = (-1)^{n-1}nw$$

Proof. For $n = 2$ we have $\eta(12) = 21 - 12 = -2 \cdot 12$ Let $w := \prod_{i=1}^n a_i$. By induction, using Lemma 2.8 we get

$$\eta(wa_{n+1}) = (-1)^{n-1}n \cdot wa_{n+1} + a_{n+1} \eta(w)$$
The second term, by the definition of $h^n_l$, is equal to $(-1)^{n-1}wa_{n+1}$, and induction finishes.

Before we move on, let us point out two pretty little corollaries to this lemma.

**Corollary 3.12.** If $w_1, w_2 \in 1W$, and if $k$ is not of characteristic $\text{len}(w_1)$, then $\eta(w_1) = \eta(w_2)$ if and only if $w_1 = w_2$.

**Corollary 3.13.** For $k$ a field of finite characteristic $n$, there are no words in $1W$ of length greater than $n + 1$.

**Proof.** Use of $h^{n+2}$ twice.

With this proposition, we have at last completed the proof of our main theorem, and as promised we have:

**Main Theorem.** $W'(p)$ is isomorphic to the space $A_c(p)_{-1}$, and $1W(p)$ is isomorphic to the space $\mathcal{L}(p)$.

### 3.5. Corollaries of the Proof

As the following proposition shows, this means that there is no point in picking a head, “playing around” and returning to the same head again in the hope of getting a non-trivial relationship. This looks fairly obvious, but we could find no proof for it in the literature.

**Lemma 3.15.** Let $w := \prod_{i=1}^n a_i$ be a word in $ASS(p)$. Then $h_m^l(w)$ ($m \leq n$) followed by choosing the $i$th letter ($i < n$) to be the head of each of the words in $h_m^l(w)$ by means of the appropriate fold moves is the same as $h_i^l(w)$.

**Proof.** For $i = n - 1$, this is the same as the proof of lemma 3.2.

Now let $c_{a_i}$ be the action on an element of $ASS(p)$ of choosing $a_i$ to be the head.

Then

$$h_i^l(w) = c_{a_i}(h_{i+1}^l(w)) = \cdots = c_{a_i}c_{a_{i+1}} \cdots c_{a_{n-1}}h_n^l(w)$$

Collapsing this again gives $c_{a_i}h_n^l(w)$. □

**Corollary 3.16.** The presentation of an element of $ASS(p)$ with a given letter as the head of each word is unique under the action of $1H$ (and therefore under the action of $H'$), i.e. if we take a word $w \in ASS(p)$ and act on it by arbitrary $h^l$–moves, and then choose $a_1$ to be the head of each summand by $h^l$–moves, we recover $w$.

**Proof.** For a single word, this is just Lemma 3.12 and Lemma 3.15 taken together. □
A further corollary is a proof that the space of acyclic Jacobi diagrams is isomorphic to the kernel of a mapping between free Lie algebras.

**Proof of Theorem 1.1.** By the main theorem we identify:

\[
\begin{align*}
\mathcal{A}_c(p)^n_1 & \simeq \text{ASS}_n(p)/H' \\
\mathcal{L}_{n-1}(p) \otimes p & \simeq \text{ASS}_n(p)/H - \{h^l_n\} \\
\mathcal{L}_n(p) & \simeq \text{ASS}_n(p)/H
\end{align*}
\]

where $\text{ASS}_n(p)$ denotes the span over $k$ of words of length $n$ over $p$ generators.

We begin by defining a map $g: \mathcal{A}_c(p)^n_1 \rightarrow \mathcal{L}_{n-1}(p) \otimes p$ by the equation

\[
g(w) := g'(w) - g'(h^l_n(w))
\]

where the map $g'$ is defined by $g'(b_1 \cdots b_n) \mapsto b_2 \cdots b_n \otimes b_1$ where the leg labeled $b_1$ becomes the root (this is easily seen to be well-defined by assuming for instance that $w$ is lexically minimal in its equivalence class)\(^2\). The mapping $\mathcal{L}_{n-1}(p) \otimes p \rightarrow \mathcal{L}_n(p)$ is given by $a \otimes b \mapsto ba \in \mathcal{L}_n(p)$, a map which we denote $\ell$.

We first note that $\mathcal{A}_c(p)^n_1 \subseteq \mathfrak{h}_n(p)$—in other words that $g(w) = 0$ as an element of $\text{ASS}(p)$ for all $w \in \mathcal{A}_c(p)^n_1$—by choosing $a_1$ as the head of each word and applying Corollary 3.16.

Next, we show the opposite inclusion $\mathfrak{h}_n(p) \subseteq \mathcal{A}_c(p)^n_1$. By the main theorem, $\ker \ell$ is generated by all $h^l_n(w) - w$ for all $n \geq 2$ and for all $w \in \text{ASS}(p)$. To prove that $\text{Im} g$ generates $\ker \ell$, we must show that $\ker \ell$ is generated by all $h^l_n(w) - w$ for all $n \geq 2$ and for $w$ belonging to a maximal set of representatives of equivalence classes in $\mathcal{A}_c(p)^n_1 \simeq \text{ASS}_n(p)/H'$. In other words it is sufficient to show that $h^l_n(h^l'_n(w)) - h^l_n(w) = h^l_n(w) - w$ for all $w \in \mathcal{L}_{n-1}(p) \otimes p$.

For $m < n$ this follows from Lemma \[12\] For $m = n$:

\[
h^l_n(h^l'_n(w)) - h^l_n(w) = (-1)^{2n-1}(a_1a_2\eta(a_n \cdots a_3) - a_1\eta(a_n \cdots a_3)a_2) - h^l_n(w) = \cdots = \sum_{i=2}^{n-1} (-1)^{i-1}a_i \cdots a_i \eta(a_n \cdots a_i + (-1)^{2n-1}w - h^l_n(w)
\]

Finally as in \[13\], note that if we define $\tilde{g}: \mathcal{L}(p) \otimes p \rightarrow \mathcal{A}_c(p)^n_1$ by $a \otimes b \mapsto ba \in \mathcal{A}_c(p)^n_1$, we find that

\[
\tilde{g}g(w) = \tilde{g}(a_2 \cdots a_n \otimes a_1 - h^l_n(a_2 \cdots a_n \otimes a_1)) = w + (-1)^{n-1}a_n \eta(a_1 \cdots a_{n-1}) = w + (-1)^{n-1}a_n \eta(a_1 \cdots a_{n-2})a_{n-1} + (-1)^{n-1}a_n \eta(a_1 \cdots a_{n-2}) = 2w + (-1)^{n-1}a_n \eta(a_1 \cdots a_{n-2})a_{n-1} + (-1)^{n-1}a_n \eta(a_1 \cdots a_{n-2})a_{n-2} = \cdots = nw
\]

where the fourth equality is by $h^l_{n-1}$. \[\square\]

\(^2\)This map coincides with the one used in \[10\] \[13\] \[15\].
4. Sample Calculation

In the present section we illustrate the use of our main theorem to facilitate the calculation by hand of a basis of swings for $h_9(2)$. By Witt's dimension formula, the dimension of this space is:

$$\dim(h_9(2)) = \dim(L_8(2) \otimes 2) - \dim(L_9(2)) = \frac{9}{8}(9^8 - 9^4) - \frac{1}{9}(9^9 - 9^3) = 48420180 - 43046640 = 5373540$$

We list the basis elements by their multi-degree. To simplify notation, in the present section we identify multi-degrees of the form $(n_{\sigma(1)}, n_{\sigma(2)}, \ldots, n_{\sigma(9)})$ for all permutations $\sigma \in \Sigma_9$ of the set with nine elements $\{9\}$. We also suppress zeros in our notation.

We begin by considering the cases where two of the letters appear only once. In this case, we choose the first pair of such letters as the head and the tail of the swing, and count the elements which result.

- $\dim(h_9(1,1,1,1,1,1,1,1)) = 7! = 5040$
- $\dim(h_9(1,1,1,1,1,1,1,2)) = 9 \times 8 \times \binom{3}{2} \times 5! = 181440$
- $\dim(h_9(1,1,1,1,1,1,3)) = \binom{4}{2} \times 7 \times \binom{3}{2} \times 4! = 211680$
- $\dim(h_9(1,1,1,1,1,1,4)) = \binom{2}{2} \times 6 \times \binom{3}{2} \times 3! = 40320$
- $\dim(h_9(1,1,1,1,1,5)) = \binom{3}{3} \times 5 \times \binom{3}{2} \times 2! = 26460$
- $\dim(h_9(1,1,1,1,6)) = \binom{3}{3} \times 4 \times 7 \times 1 = 3528$
- $\dim(h_9(1,1,1,7)) = \binom{3}{3} \times 3 = 3528$
- $\dim(h_9(1,1,1,1,2,2)) = \binom{4}{2} \times \binom{3}{2} \times \binom{3}{2} \times 3! = 952560$
- $\dim(h_9(1,1,1,1,2,3)) = \binom{3}{3} \times 6 \times 5 \times \binom{3}{2} \times 2! = 1058400$
- $\dim(h_9(1,1,1,2,4)) = \binom{3}{3} \times 5 \times 4 \times \binom{3}{2} \times 3 = 264600$
- $\dim(h_9(1,1,2,5)) = \binom{4}{2} \times 4 \times 3 \times \binom{3}{2} = 31752$
- $\dim(h_9(1,1,2,2,2)) = \binom{3}{3} \times \binom{3}{2} \times \binom{3}{2} \times 2! = 1058400$
- $\dim(h_9(1,1,2,2,3)) = \binom{3}{3} \times 5 \times \binom{3}{2} \times \binom{3}{2} \times 2! = 793800$
- $\dim(h_9(1,1,1,1,3,3)) = \binom{3}{3} \times \binom{3}{2} \times \binom{3}{2} \times 4 = 176400$
- $\dim(h_9(1,1,1,3,4)) = \binom{3}{3} \times 12 \times \binom{3}{2} = 52920$

Of the 450468 basis elements remaining we next consider those where only one letter appears only once, which we choose to be the head. We are then left with subspaces of free Lie algebras which are homogenous with respect to multi-degree.

- $\dim(h_9(1,2,2,2,2)) = \binom{4}{2} \times 5 \times 312 = 196560$
- $\dim(h_9(1,2,2,4)) = \binom{3}{3} \times 4 \times 3 \times 51 = 77112$
- $\dim(h_9(1,2,6)) = \binom{3}{3} \times 3! \times 3 = 1512$
- $\dim(h_9(1,2,3,3)) = \binom{3}{3} \times 4 \times 3 \times 70 = 105840$
- $\dim(h_9(1,3,5)) = \binom{3}{3} \times 3! \times 7 = 3528$
- $\dim(h_9(1,4,4)) = \binom{3}{3} \times 3 \times 8 = 2016$

The basis elements for the last two homogenous subspaces of free Lie algebras appearing above are those words generated by two letters 1 and 2 in which there is no consecutive sequence an even number of 2's (this is easily calculated by hand using the \textsuperscript{t}H moves). We ask the following question:
Question. Do words in letters 1 and 2 in which there is no consecutive sequence of an even number of 2’s form a basis for the free Lie algebra over two generators?

This question is implicit in [17], where it is proved for a few special cases. In the same way we attain the basis elements of the other homogenous subspaces of free Lie algebras above (‘no sequence of an even number of larger letters between two smaller letters’) suggesting that there may perhaps be something more general going on.

We next look at those 63900 basis elements in which no letters appear once, and there is a letter which appears twice. We choose this letter as head and tail, and then only reflect.

- $\dim(\mathfrak{h}(9,2,2,3))$ is $\binom{9}{4} \times 4 = 504 \times \frac{1}{2} = 252$ times $1 = 504$ which is 51408.

- $\dim(\mathfrak{h}(9,2,2,5))$ is $\binom{9}{3} \times 3 = 252 \times \frac{1}{2} = 9$ which is 2268.

- $\dim(\mathfrak{h}(9,2,3,4))$ is $\binom{9}{3} \times 6 = 504 \times \frac{1}{2} = 16$ which is 8064.

The final 2160 cases we deal with individually.

- $\dim(\mathfrak{h}(9,3,3,3))$ is $\binom{9}{3} = 84$ times

  (1) Words beginning with 12 and ending with 21. If the third letter is one there are four of these. Otherwise there are three. Altogether ten possibilities.

  (2) Words beginning with 12 and ending with 31. If the seventh letter is three, then if third letter is one we have one possibility, if two there are six, and if three then there are three possibilities, altogether ten. Otherwise if the seventh letter is one we have six possibilities, altogether sixteen.

  (3) Words beginning with 13 and ending with 31 must be of the form 13122331. Altogether 24 possibilities for $84 \times 24 = 2016$.

- $\dim(\mathfrak{h}(9,3,6))$ is $\binom{9}{3} \times 2 = 72$ times one, which is 72.

- $\dim(\mathfrak{h}(9,4,5))$ is $\binom{9}{2} \times 2 = 72$ times a-priori four words of the forms 12122221, 121212221, 121221221, 122112221 which can all be shown to be equivalent modulo the action of $H'$. In total we have 72 possibilities.

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