Black-holes in asymptotically flat space-times have negative specific heat — they get hotter as they loose energy. A clear statistical mechanical understanding of this has remained a challenge. In this work, we address this issue using fluid-gravity correspondence which aims to associate fluid degrees of freedom to the horizon. Using linear response theory and the teleological nature of event horizon, we show explicitly that the fluctuations of the horizon-fluid lead to negative specific heat for Schwarzschild black Hole. We also point out how the specific heat can be positive for Kerr-Newman or AdS black holes. Our approach constitutes an important advance as it allows us to apply canonical ensemble approach to study thermodynamics of asymptotically flat black-hole space-times.

I. INTRODUCTION

Gravity is universally attractive and a self-gravitating system is unstable to collapse. In Newtonian theory, it is known that a non-relativistic ideal homogeneous fluid is unstable to long wavelength density perturbations [1]. For a gravitating object in a thermal bath, thermal fluctuations will drive the object momentarily hotter or colder than the bath; since, gravity is not screened, gravitational object will evolve in whichever direction it started out with [2]. This is referred to as negative specific heat paradox [2].

Black-holes are thermodynamical objects where quantum effects induce additional instabilities [3, 4]. Broadly, the issues in black-hole thermodynamics can be classified into two categories based on whether or not they can be addressed within the realm of known Physics. It is widely regarded that understanding of black-hole entropy or resolution of information paradox require new Physics [5]. However, the issue of negative specific heat of black-holes and its statistical mechanical description [6], may not require new Physics and, if not resolved, will persist even in quantum gravity. In this work, we provide a physical understanding of the negativity of the specific heat and armed with this go on to construct canonical ensemble description for black-holes.

We use Fluid-Gravity correspondence as described by Damour to understand physical origin of negative specific heat for black-holes [7-9]. Since Damour’s calculation 30 years ago, attempts have been made to use it to gain new physical insight from the Fluid-Gravity correspondence. However, there has not been any progress in understanding how to go beyond the thermodynamic level in this approach. Recently, many interesting features of black-holes have arisen since the formal relation between the equations of gravity (near the black-hole horizon) and equations of fluid dynamics were found [7-9]. More importantly, Fluid-Gravity correspondence allows the possibility to connect macroscopic and microscopic physics through the study of the statistical properties of the fluid on the black-hole horizon. Current authors have shown that horizon-fluid is of physical interest, and an effective theory can be written describing this fluid as a condensate [10-13, 15]. We have shown that Bekenstein-Hawking entropy[13] and Langevin equation for the area expansion (of 2-D fluid) of homogeneous horizon-fluid is given by the Raychaudhury equation [11].

It is well known that fluids can be described by two sets of parameters, namely susceptibilities and transport coefficients. While the first set of parameters correspond to changes in local variables; other set involves fluxes of thermodynamic quantities [16,18]. Obviously, these parameters can not be determined within fluid mechanics, however, these can be derived using the theory of fluctuations that relate susceptibility/transport coefficient to autocorrelation function of a dynamical variable [16,18]. In this paper, we show explicitly that the fluctuations of the horizon-fluid lead to negative specific heat (thermodynamic derivative). Specific heat is a generalized susceptibility or Thermodynamic derivative that quantifies the amount of the change of an extensive quantity (like entropy or internal energy) under the influence of an intensive quantity (like temperature).

We consider fluctuations from the equilibrium value of the total entropy of the horizon-fluid. The change in the total entropy of the horizon-fluid is proportional to the change in the temperature of the fluid and this proportionality constant is the susceptibility. This method of calculating susceptibility has its advantages. First, the theory of Fluctuations is independent of any particular model of quantum gravity. Second, using transport theory, the authors have shown that the teleological nature of the horizon-fluid is responsible for the negative bulk viscosity of the fluid [12].

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The teleological nature refers to the fact that the black-hole area starts increasing long before matter-energy falls in it and stops as the infalling matter-energy crosses the event horizon. This can be viewed as the anti-causal response for globally defined black-holes \[ \text{[1, 7, 8, 12]} \]. This is extended here to the case of susceptibilities to show that the Horizon-Fluid and hence black-holes have negative specific heat. The procedure we shall follow is to first define a suitable dynamic susceptibility for the specific heat of the Horizon-Fluid. The static limit of this quantity is shown to be negative and hence, the specific heat is negative.

The rest of the paper is organised as follows. In the next section, we identify the response function corresponding to the specific heat of the horizon-Fluid. In the third section, we explicitly evaluate the dynamic susceptibility and then show that the specific heat is negative. In section four, we briefly discuss how some black holes (e.g., rotating charged ones) can exhibit positive specific heat. In the final section, the implications of the results obtained here are briefly discussed.

II. IDENTIFICATION OF THE RESPONSE FUNCTION TO DETERMINE SPECIFIC HEAT

Specific heat is given by,

\[ C = T \frac{\partial S}{\partial T}, \]

where, \( S \) is the entropy and \( T \) is the temperature. Since \( T \) is positive definite, the sign of the specific heat is determined by the change of the entropy of the horizon-fluid due to the change in temperature. Defining susceptibility as \( \chi_T \equiv \frac{\partial S}{\partial T} \text{[17]} \); within linear response theory, the change in the entropy under the influence of the external influence \((\delta T)\) is given by,

\[ \delta S = \chi_T \delta T. \]

Typically it is convenient to understand the processes in the frequency \((\omega)\) space and one can define a corresponding susceptibility as \( \chi_c(\omega) \). In general, this susceptibility, \( \chi_c \) is complex (See, for instance, Ref. \[ \text{[16]} \]), i.e.

\[ \chi_c(\omega + i\epsilon) = \chi'(\omega) + i\chi''(\omega). \]

where the imaginary part physically corresponds to absorption. In the static limit, there is no absorption and hence only the real part of \( \chi_c \) contributes to \( \chi_T \). Physically, the static part corresponds to the quasi-static limit (at long timescales) of the process with the system always close to equilibrium. The static part of the susceptibility corresponds to the specific heat of the fluid. The static susceptibility can be found from the dynamic susceptibility using analytic continuation.

Before we proceed with the evaluation of \( \chi_T \), it is useful for the purpose of clarification to compare and contrast \( \chi_T \) with the more familiar magnetic susceptibility \( \chi_M \). \( \chi_M \) satisfies the relation, \( \delta M = \chi_M \delta H \), where, \( \delta H \) is the external influence. In the theory of fluctuations, \( \chi_M \) is the static limit of response function \[ \text{[16, 18]} \]. In the same manner, \( \chi_T \) is the static limit of the response function \( \chi''(\omega) \) given by:

\[ \chi_T = \int \frac{d\omega \chi''(\omega)}{\omega}. \]

However, there are two crucial differences between \( \chi_M \) and \( \chi_T \):

1. It is possible to evaluate \( \chi_M \) from fundamental principle by writing down the Hamiltonian taking into account the quantum effects. In our case, since we do not have complete information about the evolution dynamics, the time evolution of \( \delta S \) is governed by a phenomenological generalized Langevin equation \[ \text{[11]} \]. However, still the excess entropy density and entropy current satisfy continuity equation [See Appendix for details].

2. While the response in the case of magnetic system is causal, here it is anti-causal. This is because, for the event horizon, the response to any external influence is anti-causal. In particular, if matter-energy falls through the event horizon, the area of the event horizon increases till the matter-energy passes through the horizon \[ \text{[8]} \]. This is physical as the event horizon of a black hole is defined globally in the presence of the future light-like infinity \[ \text{[8]} \].

Due to this unusual property of the horizon, the horizon-fluid also exhibits anti-causal response i.e. the response of the horizon takes place before the external influence occurs \[ \text{[8]} \]. This is referred to as the teleological nature of
horizon (See Fig 1). For canonical ensemble, the specific heat is proportional to the energy fluctuations squared i. e.,

$$C = \frac{(\Delta E)^2}{T^2}.$$  

While for a causal process, it is positive; as we will show below, for an anti-causal fluid, the specific heat turns out to be negative.

### III. TELEOLOGICAL BOUNDARY CONDITION AND NEGATIVE SPECIFIC HEAT

Thermal ensemble for small fluctuations away from the equilibrium is characterised by the entropy change due to fluctuations\[11, 12\],

$$\Delta S = \int \delta S = -\frac{1}{4A} \int \delta A d(\delta A) = -\frac{1}{8} \frac{(\delta A)^2}{A}. \quad (5)$$

It is important to note that $\Delta S$ is negative because the system is not in equilibrium and hence the entropy of the system is less than that in the equilibrium state. In the rest of the section we evaluate explicitly the susceptibility ($\chi_T$), and show that $\chi_T$, and hence the specific heat, is negative. The procedure we adopt is the following:

1. First, we express the dynamical susceptibility as an integral involving autocorrelation function of the entropy density fluctuations.

2. We then evaluate the static susceptibility from the dynamic susceptibility using analytic continuation. Using the teleological boundary condition we show that the specific heat for the Horizon-Fluid is negative.

#### A. Dynamic Susceptibility from Autocorrelation of entropy fluctuations

For a linear response, the ensemble average of the fluctuations in entropy ($\delta S$) described by Eq. (2) in the frequency domain is given by:

$$\langle \delta S[\omega] \rangle = \int_0^\infty \chi''(\omega) e^{-i\omega t} \delta T(t) dt, \quad (6)$$

where, $\chi''(\omega)$ is dynamic susceptibility and is given by\[16\]18.

$$\chi''(\omega) = -\frac{1}{8\pi \langle \delta S^2(0) \rangle} \int_{-\infty}^{\infty} \langle \delta S(0) \delta S(t) \rangle e^{i\omega t} dt. \quad (7)$$
The linear response of the system can be written as,

$$\langle \delta S(t) \rangle = - \int_0^\infty d\omega \int_0^\infty dt' dt'' \frac{\delta S(0) \delta S(t'')}{2(\delta S^2(0))} \delta T(t') e^{i\omega(t-t')} e^{i\omega t''}$$

which leads to

$$\langle \delta S(t) \rangle = - \frac{1}{2(\delta S^2(0))} \int_0^\infty \langle \delta S(0) \delta S(t'') \rangle \delta T(t') \delta(t-t' + t'') T(t') dt' dt''$$

Integration of the R.H.S. of the above equation leads to a theta function that decides whether the response is causal or anti-causal. Taking into account, the time translation invariance of the auto-correlation function, we get,

$$\langle \delta S(t) \rangle = - \frac{1}{2(\delta S^2(0))} \int_0^\infty \langle \delta S(t) \delta S(t') \rangle \delta T(t') dt'$$

where, $t' > t$. It is to be noted here that (10) explicitly shows the anti-causal nature of the response function. The linear response can now be expressed as,

$$\langle \delta S(t) \rangle = - \frac{1}{2\pi} Im \int_\omega^\infty \int_0^\infty d\omega' \chi''(\omega) \delta T(t') e^{-i\omega(t-t')} dt'$$

Having obtained the dynamic susceptibility, our next step is to obtain the static susceptibility and use Eq. (2) to evaluate the specific heat.

### B. Evaluation of specific heat

Since we want to look at the static limit of the response, we take $\delta T(t) = \delta T$, i.e. a constant. From Eq. (11), we get

$$\langle \delta S(t) \rangle = \delta T \int_\omega d\omega \chi''(\omega).$$

As mentioned above, we are interested in evaluating the static limit of the susceptibility. Following (3) and using Kramers-Kronig relations, leads to,

$$\chi'(\omega) = -P \int \frac{d\omega'}{\pi} \frac{\chi''(\omega')}{\omega' - \omega}. $$

From (13) and (12), we get,

$$\langle \delta S(0) \rangle = \chi'(0) \delta T.$$  

Since the response at time zero is a response to an adiabatically applied disturbance, $\chi'(0)$ is the static susceptibility, i.e., $\chi'(0) = \chi_T$. Eq. (14) leads to $\langle \delta S(0) \rangle = \chi_T \delta T$. This leads to (4), the expression for the static susceptibility.

Anti-causal response $\langle \delta S(t) \rangle$ (is proportional to the inhomogeneous part of the density that) oscillate slowly in time, has a thin spread $\Delta \omega$ and is given by,

$$\delta S(t) = \theta(-t) \int_{\Delta \omega} e^{i\omega t} d\omega.$$  

Putting these back in (7), one gets,

$$\chi''(\omega) = \frac{-1}{8\pi} Im \int_{\Delta \omega} d\omega \int_{-\infty}^\infty \theta(-t) e^{i\omega t} e^{i\omega t} dt = - \frac{1}{8\pi}. $$

The absorptive part of the dynamic susceptibility comes out to be negative due to the teleological boundary condition for the Horizon-Fluid. It is straightforward to check that $\chi''$ is positive for a causal response, $\delta_C S(t) = \delta S(t) = \theta(t) \int_{\Delta \omega} e^{i\omega t} d\omega$, hence $\chi$ is positive.

Substituting the above value of $\chi''(\omega)$ in Eq. (4), static susceptibility is given by:

$$\chi_T = - \frac{1}{8\pi} \int \frac{d\omega}{\omega}. $$

(17) shows clearly that $\chi_T$ is negative. Hence, the horizon-Fluid specific heat is negative. It is important to note that the integral is well-defined with infrared (corresponding to the horizon size) and ultraviolet cutoffs.
IV. UNDERSTANDING POSITIVE SPECIFIC HEAT OF BLACK-HOLES FROM FLUCTUATION DISSIPATION

In the previous section, we have shown explicitly that within the transport theory it is possible to obtain a fundamental understanding of the negative specific heat of a Schwarzschild black hole. A natural question that arises now is whether the same analysis can provide an explanation for the positive specific heat for certain ranges of parameters, for charged black holes with/without spin [19] and for asymptotic AdS black-holes. In the rest of this section, we provide arguments to show how the analysis changes for these cases and how positive values of specific heat may come about.

A. Asymptotically flat black holes

In order to extend the analysis to general stationary black-hole space-times, let us relook at the calculation for such a black-hole from the point of view of the Raychaudhury equation of a null congruence on the event horizon. Let us first write down the Raychaudhury equation for a quasi-stationary charged, spinning black hole to emphasize the same general form all of them can be expressed in.

\[-\frac{d\theta_H}{dt} + g_H \theta_H = [8\pi (L^H + 4\pi |K|^2) + \sigma_{AB} \sigma^{AB}] - \frac{1}{2} \theta_H^2, \quad (18)\]

where, \(\theta_H\) is the expansion scalar, \(L^H\) denotes the energy flux through the horizon, \(g_H\) is the surface gravity on the horizon, \(K\) is the surface current on the black-hole surface and \(\sigma_{AB}\) is the shear term. Though (18) is exact, in what follows, we shall also assume \(\theta_H\) to be small. Later, we point how a charged and spinning black hole would exhibit different response from one which is neutral and non-rotating.

Since the above equation is non-linear in \(\theta_H\), it is difficult to perform a stability analysis for the black hole horizon starting from it. Rewriting in terms of \(\eta = \sqrt{A}\), i.e.

\[-\ddot{\eta} + 2\pi T \dot{\eta} \approx S, \quad (20)\]

where \(S = \sqrt{A} [8\pi (L^H + 4\pi |K|^2) + \sigma_{AB} \sigma^{AB}]. \quad (21)\)

It is important to note that Eq. (20) holds only up to second order in \(\delta \eta\) and it applicable to any general, stationary black-hole. \(S\) can be viewed as a source term. As we know, the black hole mass or the energy and the entropy of the horizon-fluid increases when matter-energy passes through the horizon [11]. Hence, the matter-energy flux across the horizon, \(L^H\) acts as a source term. But it is not the only term that drives the evolution of \(\eta\). There are two other terms as well, one describing the effects of the shear and is proportional to \(\sigma_{AB} \sigma^{AB}\); the other the effect of an electric current on the black hole horizon and is proportional to \(|K|^2\). As we shall see, whether the specific heat is positive or not depends on which of these three terms is dominant.

1. Schwarzschild black-holes

Let us now focus on the Schwarzschild black hole which is the final stationary state of a nonrotating neutral black hole. Eq. (20) becomes

\[-\ddot{\eta} + 2\pi T \dot{\eta} \approx 4\pi \sqrt{A} L^H, \quad (22)\]

1 Writing \(A = A_0 + \delta A\), where, \(\delta A\) is the change in this area over some constant base value \(A_0\). Similarly, \(\eta\) can also be expressed as \(\eta = \eta_0 + \delta \eta\).
where, \( K = 0 \) and \( \sigma_{AB} = 0 \).

From the LHS of (22), one sees that \( \eta \) would increase exponentially with time. Demanding that the horizon exists in the future necessitates one to impose future boundary condition on \( \eta_H \) \( \frac{\sigma}{\bar{K}} \) instead of the initial boundary condition. (Usually, one imposes the future boundary condition on \( \theta^H \) but it amounts to the same thing.) Otherwise \( \eta \) would increase exponentially and destroy the black-hole. Here we recall that we have assumed that \( \eta \) to be small, which means we have to ensure through the boundary condition that it never grows to a large value. Thus the teleological boundary condition can be viewed as a condition for the stability of the black-hole event horizon. From the fluid perspective, this can be viewed as the condition that small external influences are not going to drive the system far from equilibrium \( \frac{\sigma}{\bar{K}} \). As has already been demonstrated, this can be attributed to the negative specific heat for the Schwarzschild black-hole.

2. Charged black-holes

Now let us look at the charged black-hole. Eq. (20) becomes

\[
- \dot{\eta} + 2\pi T \dot{\eta} \simeq \frac{\sqrt{A}}{2} 8\pi (I^H + 4\pi |K|^2) \tag{23}
\]

In the equilibrium limit, the system settles down to a Reisner-Norstrom black-hole. The electrical potential \( \phi \) is constant on the surface of a black hole in that state. This implies that no current flows at equilibrium. In the fluid picture, although the fluid is charged, there is no current as the velocity of the horizon-fluid for a Reisner-Norstrom black-hole is zero.

Let us consider a process in which the black-hole mass increases, however, the black-hole charge remains a constant. The specific heat \( C_Q \) is given by \( C_Q = T \left( \frac{dS}{dT} \right)_Q \). The electrostatic potential, \( \phi \) on the black-hole surface would not be constant if the black-hole is away from its equilibrium state. Thus, for a charged black-hole undergoing a change in its mass, \( \phi \) vary on the surface. However, the black-hole horizon has a conductivity given by \( \frac{\sigma}{\bar{K}} \). So the inhomogeneous potential would generate an electric field on the black-hole surface. This would produce a current in the fluid that would quickly neutralize the electric field. Such currents would be exponentially damped in time. Now, for a stationary charged black-hole, \( \phi \propto Q \). This implies that \( E = -\nabla \phi \propto Q \). Because Ohm’s law is applicable on the black-hole horizon, we have \( |K| \propto Q \).

Thus, the change from the case of a non-rotating neutral black-hole is the introduction of a second term in (23) as a source along with the term \( I^H \). The two source terms, \( I^H \) and \( |K|^2 \) are different in nature in the sense that individually they would generate anti-causal and causal responses in the horizon-fluid. To see how this comes about, first we note that the matter-energy flux, \( I^H \) that passes through the horizon does not get damped in time(Killing time). However, as we discussed above, the surface current \( K \) for the process we consider would be exponentially damped in time.

To bring out the difference in the response explicitly, we consider what would have happened had \( I^H = 0 \) but \( K \neq 0 \). We emphasize that this example is chosen for demonstrative purpose only and does not correspond to the situation we are considering. In the process we have been considering, if \( I^H = 0 \), then \( K = 0 \) also. Nonetheless, let us proceed with this example and write down the evolution equation for \( \eta \) in this case, which, from (20), we see then would have the form,

\[
- \dot{\eta} + 2\pi T \dot{\eta} \simeq 16\pi^2 \sqrt{A} |K|^2 \tag{24}
\]

As we have already argued that \( K \) has a damping factor of the form \( e^{-\zeta t} \), it is seen that subject to initial boundary conditions, equation (24) has solutions, where \( \eta \) remains finite throughout, if \( \zeta > 2\pi T \). This is in clear contrast to (22), where one needed to impose a future boundary condition to keep \( \theta \) finite. Thus the surface current \( K \) as a source can generate a causal response whereas, for matter-energy flux as the source term, the response is necessarily anti-causal.

It is clear that for the process, we have been considering, both the terms on the right hand side of (23) would compete with each other and depending on the relative magnitude of these, it would be decided whether the response would be causal or anti-causal. Since \( |K|^2 \propto Q^2 \), it is natural that when \( Q \) is small, the response of the horizon would mainly be driven by the matter-energy flux and would be anti-causal, hence giving rise to negative specific heat. However, as \( Q \) increases, there might come a point where the surface current that is generated becomes the main external driving force, thus making the response causal. In this case, the specific heat of a charged black-hole would become positive. To see how this comes about, one has to consider the Transport theory for a charged black-hole in detail, something that is out of the scope of this work. Also, one needs to be careful while considering the fluctuations near the point, where the transition from negative to positive specific heat takes place, since there is some evidence that a phase transition takes place at this point \( \frac{\sigma}{\bar{K}} \).
3. Rotating black-holes

Moving on to a spinning black-hole that is neutral, we see that one can argue in a similar manner. In this case, the specific heat is given by
\[ C_J = T \left( \frac{\partial S}{\partial T} \right)_J. \]

The process that we have to consider here is an increase in the mass of the black-hole while keeping its total angular momentum \( J \) fixed. The equilibrium state for such a spinning black-hole is the Kerr black-hole, which is stationary. The Smarr relation for the Kerr black-hole is given by,
\[ \frac{\kappa A}{4\pi} = M - 2\Omega_H J, \]
where, \( \Omega_H \) is the angular velocity of the black-hole. Comparing this with the Smarr relation for a Reisner-Norstrom black-hole, we see that the structure is similar once \( Q \) is replaced by \( J \) and \( \phi \) by \( \Omega_H \). The argument given in the case of a charged black-hole can be constructed for a spinning black-hole based on this similarity. Away from equilibrium, \( \Omega_H \) should change from its equilibrium value and would vary from point to point on the black-hole surface if we take into account the inhomogeneities.

However, for a spinning black-hole,
\[ v \equiv \Omega_H \frac{\partial}{\partial \phi}, \]
i.e. the angular velocity is the velocity of the horizon-fluid[7]. Hence such variation in the velocity of the horizon-fluid would generate a shear term, \( \sigma_{AB} \), in the fluid. Now we recall that the shear is generated by gravitational waves[8]. Moreover, the gravitational wave modes that generate shear on the black-hole horizon are damped and decay exponentially in time[8]. If we now consider the evolution equation for the shear term on the null congruences on the black-hole event horizon[8], it is seen that the shear term itself would die down exponentially in time. Finally before looking into the transport process, we recall that \( \Omega_H \propto J \) for a Kerr black-hole. For small deviations from the Kerr black-hole, this proportionality is expected to hold.

Now let us look at (20) on the event horizon of a spinning black-hole as in the previous cases,
\[ -\ddot{\eta} + 2\pi T \dot{\eta} \simeq \frac{\sqrt{A}}{2} (8\pi I^H + \sigma_{AB} \sigma^{AB}), \]
where, the surface current, \( K = 0 \). Now one can proceed exactly as before and argue how the positive specific heat of a spinning black-hole might come about. For a spinning, charged black hole, similar considerations will again apply.

B. Black-holes in AdS Background

It is well known that AdS black-holes can have positive specific heat for certain ranges of the value of the Cosmological constant (negative \( \Lambda \)). It is not within the scope of this work to show how this occurs. However we can point out a possible direction along which the explanation may lie. The main difference between a Schwarzschild and an AdS-background black hole is the presence of a negative cosmological constant (\( -\Lambda \)) in the latter case. This, however, does not make any contribution in the equation for the horizon-fluid as such a fluid is defined on a null hypersurface [14]. Hence, black-holes in AdS background should be considered directly as a thermodynamic system and later fluctuation theory needs to be applied. A possible way to proceed is to treat \( \Lambda \) as a thermodynamic variable. Then one can define another thermodynamic variable conjugate to it (See, for instance, [20], [21][22]). Now one has to apply the Fluctuation-Dissipation theorem to this system and try to determine its specific heat. It is clear however that the analysis is going to be quite different from the one done for a Schwarzschild black hole and the argument described here leading to the negative specific heat of a black hole would not be applicable to this case.

C. Dynamical and future outer trapped horizons

In Refs. [25][26], the authors have explicitly shown how to define dynamical horizon or the future outer trapped horizon and also have described the construction of horizon-fluid. Unlike the event horizons, these horizons are defined locally and they are not Killing horizons.

In the context of horizon-fluid, one major difference between the dynamical horizons and event horizons is that the congruence on such a dynamical horizon has a non-zero twist \( \omega \)[26]. This changes the horizon-fluid equation for such a horizon in a qualitative manner as the change in the energy of such a fluid is not given only by the matter-energy
flux through the horizon but there is also an additional heat flux in the form, \( D \left( \frac{\sqrt{g}}{4\pi} \omega \right) \), i.e. proportional to the divergence of the 2-form \( \omega \). Thus the total heat flux across the dynamical horizon is different from that for the event horizon of Black hole in a dynamical spacetime even if the Black hole mass increases by the same amount [29]. From the previous analyses, this suggests that the specific heat can be different for the horizon-fluids for different horizons.

We find this to be the case when we consider the Theory of Fluctuations for these horizons. Since, the dynamic horizon or the Future Outer Trapped Horizon is defined locally, one need not impose a future boundary condition for the evolution of \( \theta \) of a null congruence on the horizon. However, imposing an initial condition means that the response of the horizon-fluid corresponding to a Dynamic Horizon or a Future Outer Trapped Horizon is causal in nature as opposite to the case of Schwarzschild black holes, where the response was anti-causal. This makes the specific heat of the horizon-fluid corresponding to the dynamic horizon of a black hole positive.

V. DISCUSSION

In late 1960’s and early 1970’s, the mathematical analogy between black holes and ordinary thermodynamics was established [3]. However, only after Hawking’s famous discovery of the evaporation of black-holes [4], it was realised that the pairs of analogues between black-holes and thermodynamics are indeed physically similar. Likewise, mathematical similarity between equations of General relativity near the black-hole horizon and fluids was known for a long time [7]. While it is treated as a mathematical curiosity, we have shown that the horizon-fluid itself is of physical interest. Using fluctuation-dissipation theorem [16–18], we have shown explicitly that it is possible to derive the transport coefficients like bulk viscosity or Thermodynamic derivatives.

Till now, there has been no statistical mechanical understanding as to why asymptotically flat space-time black-holes have negative specific heat. In this work — using fluctuation-dissipation theorem and that the black-hole horizon has anti-causal response — we have provided a statistical mechanical understanding of a long-standing problem in black-hole Physics. An important consequence of our approach is that it allows black-holes to be described in canonical ensemble picture. In canonical ensemble, specific heat is related to the mean-squared fluctuations of energy [23].

\[
C = \frac{1}{T^2} \left[ \langle \Delta E^2 \rangle \right].
\] (28)

Since RHS is positive definite, it is not straightforward to describe systems with negative specific heat [23]. The inference above, makes an implicit assumption about the causal response. As we have shown above for systems that are anti-causal, specific heat is negative (See also Evans and Searles [24]). Thus, our analysis explicitly shows that black-holes can be treated within the canonical ensemble in statistical mechanics(provided future boundary condition is used); probability for such a system to be in a state is \( P_{\Delta E} \propto \exp \frac{-\Delta E}{T} \). Summing over all possible states, one can write down a partition function for the system as:

\[
Z = \sum_{\text{all states}} \exp \left[ \frac{-\Delta E}{T} \right].
\] (29)

To keep the physics transparent, we have restricted our analysis to Schwarzschild black-hole. However, fluctuation-dissipation analysis as applied here can be extended to the other black-holes, like Reisner-Norstrom black-hole, Kerr black-hole. Mapping the fluid equations to Raychaudhury equation, we have argued that it is possible to explain positive specific heat for certain ranges of parameters for Reisner-Norstrom and Kerr black-holes. It is interesting to perform the complete analysis for these general cases. This is currently under progress.

It is pertinent here to bring up the issue as to where our analysis stands vis-a-vis the negative specific heat observed for self-gravitating bodies in Newtonian Gravity. Obviously, a future boundary condition is not required for solving the equation for heat transport in such a case and our formalism does not apply to those cases. As mentioned in the Introduction, one can construct a microcanonical ensemble for such a system and calculate the specific heat which turns out to be negative. Unfortunately, things are not so straightforward when we consider black-holes. This is because Black holes differ sharply from any other system composed of matter. First, the entropy of the black-hole is not extensive. Secondly, for a self-gravitating system in Newtonian theory, we know what the underlying degrees of freedom are. This is not the case for Black Holes and there has been lot of debate regarding the relevant degrees of freedom. This greatly restricts our ability to say anything concrete about the negativity of Black hole specific heat by constructing microcanonical ensembles. The only thing on which there seems to be some amount of agreement in the literature is that these underlying degrees of freedom has to be related to the space-time geometry in some way. This is qualitatively different from the case of self-gravitating body in Newtonian gravity, where one looks only at the matter degrees of freedom. Hence the explanation for the negativity of the specif heat for these two entirely different objects may turn out to be quite different. Under this circumstance, it is a step forward in understanding the
specific heat of black-holes if an explanation can be provided for its negativity based only on the general properties of fluctuations and the teleological nature of the event horizon.

Our analysis is applicable in principle to asymptotically flat and AdS black-hole space-times. Until now, such a relation has been obtained only for black-holes in AdS background \cite{9}; attempts to generalize \cite{27} has not been so successful. On the contrary, accumulating evidence from this study and other studies \cite{11-14} suggest that the horizon-fluid fluctuations have deep statistical mechanical significance and relates to Gravity description. Important insight can be obtained if one can map the theory of Fluctuations in AdS black-hole space-times with AdS-CFT, which possibly may provide a field theoretic description for Gravity.

Acknowledgments

The authors thank T. Padmanabhan for comments on an earlier version of the manuscript. The work is supported by Max Planck-India Partner Group on Gravity and Cosmology.

Appendix: Constructing the Continuity Equation

To find out the susceptibility related to the change in the entropy of the fluid, we look at the conserved current, i.e. it satisfies a continuity equation $\cite{16,17}$. This, as will be seen, can be interpreted as the excess entropy current and the excess entropy density of the Horizon Fluid. Here it is important to stress the fact that this however does not constitute the total entropy current but only the part corresponding to the change in the area of the horizon-fluid. The total entropy current must take into account the heat diffusion and shear viscosity processes into account. In what follows, we shall assume that the entropy of the Horizon-Fluid at any point of time is given by, $S = S_0 + \delta S$, where, $S_0$ is the entropy of the system at the equilibrium state and $\delta S$ is the change in the entropy from its equilibrium value. In this case, $N \propto A$ and $A \propto S$, hence $\delta S \propto \delta N$. Keeping this in mind, we define a physical quantity called excess number density, $\delta \rho_N$, where, $\delta \rho_N \propto \delta S$, the entropy current density and

$$\int (\delta \rho_N) dA = \delta N. \quad (A.1)$$

Here we would like to emphasise that the change in entropy $\delta S$ as the system is away from the equilibrium state is negative and so is the change in the total number of excess DOF, $\delta N$. As the horizon-fluid’s equilibrium state is in the future, $\delta N$ and $\delta \rho_N$ always remain negative. The continuity equation for $\delta \rho_N$ is

$$\frac{\partial (\delta \rho_N)}{\partial t} + \nabla. (\delta \rho_N \mathbf{v}) = -S(r, t). \quad (A.2)$$

Here $\mathbf{v}$ is the velocity of the two dimensional horizon-fluid and $S(r, t)$ is a source term. Typically Eq. (A.2) is written by taking the source term to be zero. However, in this case, the source term is non-zero and is responsible for the matter-energy infusion into the horizon-fluid. Since the process is anti-causal, we have taken $S > 0$ physically corresponding to a sink term, also a consequence of having an equilibrium state in the future.

As a consequence of the conservation of the excess number of DOF, we have

$$\frac{d \rho}{dt} - S = \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho. \quad (A.3)$$

From (A.2) and (A.3), one gets,

$$\nabla \cdot \mathbf{v} = -\frac{1}{\rho} \frac{d \rho}{dt}. \quad (A.4)$$

One can express $\delta \rho_N$ as,

$$\delta \rho_N = \frac{\delta N}{\langle A_0 \rangle} + \text{higher order terms}. \quad (A.5)$$

Keeping terms up to first order, we get from (A.4),

$$\nabla \cdot \mathbf{v} = \frac{\partial}{\partial t} (\ln \delta N) = \frac{\partial}{\partial t} (\ln \delta A). \quad (A.6)$$
Because of the homogeneity of the system, \( \text{(A.6)} \) implies
\[
\nabla \cdot \mathbf{v} = \frac{\partial}{\partial t} (\ln A) = \theta,
\]
where \( \theta \) is the volume expansion coefficient of the null geodesic congruence on the horizon. \( \text{(A.7)} \) is a physically reasonable requirement and is known to hold for the horizon fluid \([7],[11]\).

Having written down the continuity equation relevant for this process, we need to show that the change in entropy as the system fluctuates between the equilibrium and the non-equilibrium states is given by \( \text{(5)} \). It is important to note that the Horizon-Fluid is a one parameter system, so the fluctuation away from the equilibrium state is also as the system fluctuates between the equilibrium and the non-equilibrium states is given by \( \text{(5)} \). It is important to note that the change in entropy is positive because \( \delta (\delta S) \) is the difference in entropy of the fluid between \( t_2 \) and \( t_1 \). Since the horizon-fluid is moving towards the equilibrium state, \( \delta S \) is negative and \( \delta (\delta S) > 0 \). It is to be noted that \( \text{(A.9)} \) is valid only up to second order. This establishes that the process we are describing is indeed a fluctuation around the equilibrium state of the system.

\[
\delta S = -\frac{1}{4} \frac{\delta}{\delta A}, \text{ up to second order in } \delta A, \text{ we get,}
\]
\[
\delta t (\delta S) = \delta S (t_2) - \delta S (t_1) = \frac{1}{8} \frac{\delta A^2}{A},
\]
with \( t_2 > t_1 \). It is important to note that the change in entropy is positive because \( \delta \delta (\delta S) \) is the difference in entropy of the fluid between \( t_2 \) and \( t_1 \). Since the horizon-fluid is moving towards the equilibrium state, \( \delta S \) is negative and \( \delta (\delta S) > 0 \). It is to be noted that \( \text{(A.9)} \) is valid only up to second order. This establishes that the process we are describing is indeed a fluctuation around the equilibrium state of the system.
[21] M. M. Caldarelli, G. Cognola, D. Klemm, 17, 399, Class. Quant. Grav. (2000).
[22] D. Kubiznak, R. B. Mann, M. Teo, arxiv: 1608.06147v1 [hep-th] (2016).
[23] K. Huang, Introduction to statistical physics. CRC Press, (2009).
[24] D. J. Evans, D. J. Searles, Phys. Rev. E, 53 , 5808 (1996).
[25] S. A. Hayward, Phys. Rev. D 49, 12, 6467, (1994).
[26] E. Gourgoulhon, J. L. Jaramillo,New Astronomy Reviews 51, 791 (2008).
[27] M. Guica, T. Hartman, W. Song, A. Strominger, Phys. Rev. D, 80, 124008(2009).