SHARP CRITERIA OF SCATTERING FOR THE FRACTIONAL NLS

QING GUO AND SHIHUI ZHU

Abstract. In this paper, the sharp threshold of scattering for the fractional nonlinear Schrödinger equation in the $L^2$-supercritical case is obtained, i.e., if $1 + \frac{4s}{N} < p < 1 + \frac{4s}{N-2s}$, and

\[
M[u_0] \frac{\tilde{E}}{\tilde{P}_0} E[u_0] < M[Q] \frac{\tilde{E}}{\tilde{P}_0} E[Q], \quad M[u_0] \frac{\tilde{E}}{\tilde{P}_0} \|u_0\|_{H^s}^2 < M[Q] \frac{\tilde{E}}{\tilde{P}_0} \|Q\|_{H^s}^2,
\]

then the solution $u(t)$ is globally well-posed and scatters. This condition is sharp in the sense that if $1 + \frac{4s}{N} < p < 1 + \frac{4s}{N-2s}$ and

\[
M[u_0] \frac{\tilde{E}}{\tilde{P}_0} E[u_0] < M[Q] \frac{\tilde{E}}{\tilde{P}_0} E[Q], \quad M[u_0] \frac{\tilde{E}}{\tilde{P}_0} \|u_0\|_{H^s}^2 > M[Q] \frac{\tilde{E}}{\tilde{P}_0} \|Q\|_{H^s}^2,
\]

then the corresponding solution $u(t)$ blows up in finite time, according to Boulenger, Himmelsbach, and Lenzmann’s results in [2].

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1. Introduction

From expanding the Feynman path integral from the Brownian-like to the Lévy-like quantum mechanical paths, Laskin in [28, 29] established the fractional Schrödinger equations from the viewpoint of Physics, which have physical applications in the energy spectrum for a hydrogen-like atom-fractional Bohr atom. The studying of the fractional nonlinear Schrödinger equations (fractional NLS, for short) attacking more and more Mathematical researchers (see [1, 2, 7, 10, 12, 13, 14, 21, 24, 31, 33]). In the present paper, we investigate the following Cauchy problem of the $L^2$-supercritical fractional NLS.

\[
iu_t - (-\Delta)^s u + |u|^{p-1}u = 0,
\]

\[
u(0, x) = u_0 \in H^s,
\]

where $0 < s < 1$ and the fractional operator $(-\Delta)^s$ is defined by

\[
(-\Delta)^s u = \frac{1}{(2\pi)^{\frac{N}{2}}} \int e^{ix \cdot \xi} |\xi|^{2s} \hat{u}(\xi) d\xi = \mathcal{F}^{-1}[|\xi|^{2s} \mathcal{F}[u](\xi)],
\]
where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier transform and the Fourier inverse transform in $\mathbb{R}^N$, respectively. $u = u(t, x): \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$ is the wave function. The power exponent $1 + \frac{4s}{N} < p < 1 + \frac{4s}{N-2s}$ (when $N \leq 2s$, $1 + \frac{4s}{N} < p < \infty$).

When $1 + \frac{4s}{N} < p < 1 + \frac{4s}{N-2s}$ for $N > 2s$, and $1 + \frac{4s}{N} < p < \infty$ for $N \leq 2s$, Eq. (1.1) is the $L^2$-supercritical fractional NLS due to the scaling invariance. Indeed, if $u(t, x)$ is a solution of Eq. (1.1), then $u^\lambda(t, x) = \lambda^{\frac{2s}{p-1}} u(\lambda^2 t, \lambda^4 x)$ is also a solution of Eq. (1.1). Then, we see the following invariant norms.

1) $\|u^\lambda\|_{L^p_c} = \|u\|_{L^p_c}$, where $p_c = \frac{N(p-1)}{2s}$. We remark that $p_c > 2$ when $p - 1 > \frac{4s}{N}$, and then Eq. (1.1) is called the $L^2$-supercritical NLS.

2) $\dot{H}^{s_c}$-norm is invariant for Eq. (1.1), i.e., $\|u^\lambda\|_{\dot{H}^{s_c}} = \|u\|_{\dot{H}^{s_c}}$, where $s_c = \frac{N}{2} - \frac{2s}{p-1}$.

Recently, the Cauchy problem (1.1)-(1.2) has been widely studied in the recent years but is not completely settled yet, see, e.g. [8] and [22]. Let $N \geq 2$, $\frac{1}{2} \leq s < 1$ and $1 + \frac{4s}{N} < p < 1 + \frac{4s}{N-2s}$. If $u_0 \in H^s$, then the Cauchy problem (1.1)-(1.2) has a unique solution $u(t, x)$ on $I = [0, T)$ satisfying $u(t, x) \in C(I; H^s) \cap C^1(I; H^{-s})$. Moreover, either $T = +\infty$ (global existence) or both $0 < T < +\infty$ and $\lim_{t \to T} \|u(t, x)\|_{H^s} = +\infty$ (blow-up). Furthermore, $\forall \ t \in I, u(t, x)$ has two important conservation laws.

(i) Conservation of energy:

$$E[u(t)] = \frac{1}{2} \int_{\mathbb{R}^N} \Pi(-\Delta)^s u dx - \frac{1}{p+1} \int_{\mathbb{R}^N} |u(x, t)|^{p+1} dx = E[u_0]. \quad (1.3)$$

(ii) Conservation of mass:

$$M[u(t)] = \int_{\mathbb{R}^N} |u(t, x)|^2 dx = M[u_0]. \quad (1.4)$$

Guided by a analogy to classical NLS, the sufficient criteria for blowup of the solution can be found in [2] in terms of quantities of the ground states $Q \in H^s(\mathbb{R}^N)$, by solving

$$(-\Delta)^s Q + |Q|^{p-1} Q = 0, \quad Q \in H^s(\mathbb{R}^N) \quad (1.5)$$

and the Gagliardo-Nirenberg inequality (see Theorem 3.2 in [36])

$$\int |v(x)|^{p+1} dx \leq C_{GN} \|v\|_2^{p+1-N(p-1)/2s} \|v\|_{\dot{H}^s}^{N(p-1)/2s} \quad (1.6)$$

with

$$C_{GN} = \frac{2s(p+1)}{N(p-1)} \frac{1}{\|Q\|_2^{p+1-N(p-1)/2s}} \|Q\|_{\dot{H}^s}^{N(p-1)/2s}. \quad (1.7)$$

The blow-up and long-time dynamics of the fractional NLS turn out to be very interesting problems. To the best of the authors’ knowledge, the cases that have been successfully addressed by now are: i) for the fractional NLS with nonlocal Hartree-type nonlinearites
and radial data, see, e.g. [7, 30]. Recently, Guo and Zhu [18] obtained a sharp threshold of the scattering versus blow-up for the focusing $L^2$-supercritical case. ii) for the power-type nonlinearities, Boulenger, Himmelsbach, Lenzmann [2] derived a general blowup result for \((1.1)\) in both $L^2$-supercritical and $L^2$-critical cases respectively, subject to certain threshold. Recently, the authors in [20] performed Kenig-Merle type argument [26] to show the global well-posedness of radial solutions and scattering below sharp threshold of ground state solutions. In [33], the authors adapt the strategy in [9] to prove a similar scattering result for the 3D radial focusing cubic fractional NLS, under the restriction that $s \in \left(\frac{4}{3}, 1\right)$.

In this paper, we give a complement of the blowup result given by Boulenger, Himmelsbach, Lenzmann [2] for general dimensions and nonlinearities for $s \in \left(\frac{N}{2N-1}, 1\right)$, with different method from the 3D cubic case. More precisely, we obtain the scattering for the $L^2$-supercritical NLS Eq. \((1.1)\) in terms of the arguments in [15, 23, 26], as follows.

**Theorem 1.1.** Let $N \geq 2$ and $1 + \frac{4s}{N} < p < 1 + \frac{4s}{N-2s}$ Suppose that $u_0 \in H^s$ is radial and $M[u_0]^{\frac{2s}{N-s}} E[u_0] < M[Q]^{\frac{2s}{N-s}} E[Q]$, where $Q$ is the ground-state solution of \((1.3)\). If $\frac{N}{2N-1} \leq s < 1$ and

$$M[u_0]^{\frac{2s}{N-s}} \|u_0\|_{H^s}^2 < M[Q]^{\frac{2s}{N-s}} \|Q\|_{H^s}^2,$$

then the corresponding solution $u(t)$ of \((1.1)-(1.2)\) exists globally in $H^s$. Moreover, $u(t)$ scatters in $H^s$. Specifically, there exists $\phi_\pm \in H^s$ such that $\lim_{t \to \pm \infty} \|u(t) - e^{-it(-\Delta)^s} \phi_\pm\|_{H^s} = 0$.

We should point out that the sharp criteria of scattering for the nonlinear Schrödinger equation is a quite important and interesting problems, and many researchers have devoted on this topics (see e.g. [1, 9, 15, 18, 23, 26, 33]). The scattering involves in the Strichartz estimates and the choice of admissible pairs, which is quite different and difficult with respect to different nonlinearities. Although in [18], we have proved the scattering for the fractional Hartree equation in the $L^2$ supercritical case, that for the fractional NLS \((1.1)\) with power-type nonlinearity is a nontrivial extension(e.g. Proposition 2.6, Theorem 5.1).

At the end of this section, we introduce some notations. $L^q := L^q(\mathbb{R}^N), \| \cdot \|_q := \| \cdot \|_{L^q(\mathbb{R}^N)}$, the time-space mixed norm

$$\|u\|_{L^{q}X} := \left( \int_\mathbb{R} \|u(t, \cdot)\|_{X}^q dt \right)^{\frac{1}{q}},$$

$H^s := H^s(\mathbb{R}^N), \dot{H}^s := \dot{H}^s(\mathbb{R}^N)$, and $\int dx := \int_{\mathbb{R}^N} dx$. $\mathcal{F} v = \hat{v}$ denotes the Fourier transform of $v$, which for $v \in L^1(\mathbb{R}^N)$ is given by $\mathcal{F} v = \hat{v}(\xi) := \int e^{-ix \cdot \xi} v(x) dx$ for all $\xi \in \mathbb{R}^N$, and $\mathcal{F}^{-1} v$ is the inverse Fourier transform of $v(\xi)$. $\Re z$ and $\Im z$ are the real and imaginary parts of the complex number $z$, respectively. $\overline{z}$ denotes the complex conjugate of the complex number $z$. The various positive constants will be denoted by $C$ or $c$. 
2. LOCAL THEORY AND STRICHARTZ ESTIMATE

In fact, the Cauchy problem (1.1)-(1.2) has the following integral equation:

\[ u(t) = U(t)u_0 + i \int_0^t U(t-t^1)|u|^{p-1}u(t^1)dt^1 \]

where

\[ U(t)\phi(x) = e^{-i(-\Delta)^{s/2}}\phi(x) = \frac{1}{(2\pi)^{\frac{N}{2}}} \int e^{i(x-x' -|\xi|^2t)\hat{\phi}(\xi)d\xi}. \]

First, we recall the local theory for Eq. (1.1) by the radial Strichartz estimate ([19]).

Definition 2.1. For the given \( \theta \in [0, s) \), we state that the pair \((q, r)\) is \( \theta \)-level admissible, denoted by \((q, r) \in \Lambda_\theta\), if

\[ q, r \geq 2, \quad \frac{2s}{q} + \frac{N}{r} = \frac{N}{2} - \theta \]

and

\[ \frac{4N + 2}{2N - 1} \leq q \leq \infty, \quad \frac{1}{q} \leq \frac{2N - 1}{2}(\frac{1}{2} - \frac{1}{r}), \quad \text{or} \quad 2 \leq q < \frac{4N + 2}{2N - 1}, \quad \frac{1}{q} < \frac{2N - 1}{2}(\frac{1}{2} - \frac{1}{r}). \]  

(2.2)

Correspondingly, we denote the dual \( \theta \)-level admissible pair by \((q', r') \in \Lambda_\theta^*\) if \((q, r) \in \Lambda_\theta\) with \((q', r')\) is the Hölder dual to \((q, r)\).

Proposition 2.2. (see [19]) Assume that \( N \geq 2 \) and that \( u_0, f \) are radial; then for \( q_j, r_j \geq 2, j = 1, 2, \)

\[ \|U(t)\phi\|_{L^{q_1}L^{r_1}} \leq C\|D^\theta\phi\|_2, \]

(2.3)

where \( D^\theta = (-\Delta)^{\frac{\theta}{2}}, \)

\[ \| \int_0^t U(t-t^1)f(t^1)dt^1 \|_{L^{q_1}L^{r_1}} \leq C\|f\|_{L^{q_2'}L^{r_2'}}, \]

(2.4)

in which \( \theta \in \mathbb{R} \), the pairs \((q_j, r_j)\) satisfy the range conditions (2.2) and the gap condition

\[ \frac{2s}{q_1} + \frac{N}{r_1} = \frac{N}{2} - \theta, \quad \frac{2s}{q_2} + \frac{N}{r_2} = \frac{N}{2} + \theta. \]

Definition 2.3. We define the following Strichartz norm

\[ \|u\|_{S(\Lambda_{sc})} = \sup_{(q, r) \in \Lambda_{sc}} \|u\|_{L^qL^r} \]

and the dual Strichartz norm

\[ \|u\|_{S'(\Lambda_{sc})} = \inf_{(q', r') \in \Lambda_{sc}} \|u\|_{L^{q'}L'^r} = \inf_{(q, r) \in \Lambda_{-sc}} \|u\|_{L^{q'}L'^r}, \]

where \((q', r')\) is the Hölder dual to \((q, r)\).
Remark 2.4. Notice that if
\[ s \in \left[ \frac{N}{2N-1}, 1 \right] \subset \left( \frac{1}{2}, 1 \right), \]
the gap condition (2.1) with \( \theta = 0 \) right implies the range condition (2.2), which further means that \( \Lambda_0 \) is nonempty. That is we have a full set of 0-level admissible Strichartz estimates without loss of derivatives in radial case. By taking
\[ q_c = r_c = \frac{(p-1)(N+2s)}{2s}, \]
we see that \((q_c, r_c) \in \Lambda_{s_c} \neq \emptyset\) is an \( s_c \)-level admissible pair.

When \( \phi, f \) are radial, from Proposition 2.2, we have the following Strichartz estimates.
\[ \|U(t)\phi\|_{S(\Lambda_0)} \leq C\|\phi\|_2 \]
and
\[ \left\| \int_0^t U(t-t^1)f(\cdot, t^1)dt^1 \right\|_{S(\Lambda_0)} \leq C\|f\|_{S'(\Lambda_0)}. \]

Then, we further obtain
\[ \|U(t)\phi\|_{S(\Lambda_{s_c})} \leq c\|\phi\|_{H^{s_c}}, \quad \left\| \int_0^t U(t-t^1)f(\cdot, t^1)dt^1 \right\|_{S(\Lambda_{s_c})} \leq C\|D^{s_c}f\|_{S'(\Lambda_0)} \]
and
\[ \left\| \int_0^t U(t-t^1)f(\cdot, t^1)dt^1 \right\|_{S(\Lambda_{s_c})} \leq C\|f\|_{S'(\Lambda_{-s_c})}, \]
where we use the Sobolev embedding.

Next, we denote \( S(\Lambda_\theta; I) \) to indicate its restriction to a time subinterval \( I \subset (-\infty, +\infty) \).

**Proposition 2.5.** (Small data) Let \( N \geq 2 \) and \( 1 + \frac{4s}{N} < p < 1 + \frac{4s}{N-2s} \). If \( \|u_0\|_{H^{s_c}} \leq A \) is radial, then, there exists \( \delta_{sd} = \delta_{sd}(A) > 0 \) such that when \( \|U(t)u_0\|_{S(\Lambda_{s_c})} \leq \delta_{sd} \), the corresponding solution \( u = u(t) \) solving (1.1) is global, and
\[ \|u\|_{S(\Lambda_{s_c})} \leq 2\|U(t)u_0\|_{S(\Lambda_{s_c})}, \quad \|D^{s_c}u\|_{S(\Lambda_0)} \leq 2c\|u_0\|_{H^{s_c}}. \]
(Note that by the Strichartz estimates, the hypotheses are satisfied if \( \|u_0\|_{H^{s_c}} \leq c\delta_{sd} \).)

**Proof.** Denote
\[ \Phi_{u_0}(v) = U(t)u_0 + i \int_0^t U(t-t^1)|v|^{p-1}v(t^1)dt^1. \]
It follows from the Strichartz estimates that
\[ \|D^{s_c}\Phi_{u_0}(v)\|_{S(\Lambda_0)} \leq c\|u_0\|_{H^{s_c}} + c\|D^{s_c}[|v|^{p-1}v]\|_{L^p'L'} \]
and define

\[ \| \Phi_{u_0}(v) \|_{S(A_{s_c})} \leq \| U(t)u_0 \|_{S(A_{s_c})} + \| |v|^{p-1}v \|_{L^{q'}_p L^{r'}_p} \leq \| U(t)u_0 \|_{S(A_{s_c})} + \| v \|_{L^{p_2} L^{r_2}}^p, \]

where \((q', r') \in \Lambda'_0, (q_2, r_2) \in \Lambda_{s_c}\) and \((\frac{q_2}{p}, \frac{r_2}{p}) \in \Lambda'_c\). Then, by applying the fractional Leibnitz [7, 25], we deduce that

\[
\| D^{s_c}[|v|^{p-1}] \|_{L^{q'}_p L^{r'}_p} \leq c \| |v|^{p-1} \|_{L^{q_2'}_p L^{r_2'}_p} \| D^{s_c}v \|_{L^{q_1} L^{r_1}} \\
\leq c \| v \|_{L^{p_2} L^{r_2}} \| D^{s_c}v \|_{L^{q_1} L^{r_1}},
\]

where the pairs \((q, r), (q_1, r_1) \in \Lambda_0, (q_2, r_2) \in \Lambda_{s_c}\). Now, we take

\[
\delta_{sd} \leq \left( \min \left( \frac{1}{2^p c}, \frac{1}{2^p} \right) \right)^{\frac{1}{p-1}},
\]

and define

\[
B := \left\{ v \| v \|_{S(A_{s_c})} \leq 2\| U(t)u_0 \|_{S(A_{s_c})}, \| D^{s_c}v \|_{S(A_0)} \leq 2c\| u_0 \|_{H^s} \right\}.
\]

Then, we can prove that \( \Phi_{u_0} \) is a contraction mapping from \( B \) to \( B \), which completes the proof.

\[\square\]

**Proposition 2.6.** *(Scattering criterion)* Let \( N \geq 2 \) and \( 1 + \frac{4s}{N} < p < 1 + \frac{4s}{N-2s} \). If \( u_0 \in H^s \) is radial and \( u(t) \) is global with both bounded \( s_c \)-level Strichartz norm \( \| u \|_{S(A_{s_c})} < \infty \) and uniformly bounded \( H^s \) norm \( \sup_{t \in [0, \infty)} \| u \|_{H^s} \leq B \), then \( u(t) \) scatters in \( H^s \) as \( t \to +\infty \). More precisely, there exists \( \phi^+ \in H^s \) such that

\[
\lim_{t \to +\infty} \| u(t) - U(t)\phi^+ \|_{H^s} = 0.
\]

**Proof.** It follows from the integral equation

\[
u(t) = U(t)u_0 + i \int_0^t U(t-t^1)|u|^{p-1}u(t^1)dt^1
\]

that

\[
u(t) - U(t)\phi^+ = -i \int_t^\infty U(t-t^1)|u|^{p-1}u(t^1)dt^1,
\]

where

\[
\phi^+ = u_0 + i \int_0^\infty U(-t^1)|u|^{p-1}u(t^1)dt^1.
\]
Applying Proposition 2.2, we deduce that for $0 \leq \alpha \leq s$, there exist some $(q, r) \in \Lambda_0$, $(q_1, r_1) \in \Lambda_0'$ such that

$$
\left\| D^\alpha \left( \int_I U(t-s) \left( |u|^{p-1} u(s, x) \right) ds \right) \right\|_{L^q_I L^r} \leq C \left\| D^\alpha \left( |u|^{p-1} u \right) \right\|_{L^q_I L^r} \leq C \left\| D^\alpha u \right\|_{L^q_I L^r} \left\| u \right\|_{L^p_I L^r},
$$

where $I \subset [0, +\infty)$,

$$
\frac{1}{q_1} = \frac{p-1}{q_c} + \frac{1}{q}, \quad \frac{1}{r_1} = \frac{1}{r} + \frac{p-1}{r_c}.
$$

Due to $\|u\|_{L^q_{[0,\infty)} L^r} < \infty$, we divide $[0, +\infty)$ into $N$ subintervals: $I_j = [t_j, t_{j+1}], 1 \leq j \leq N$, such that $\|u\|_{L^p_{I_j} L^r} < \delta$ (for small $\delta$) on each subinterval $I_j$. Thus, from (2.6) and (2.8), we see that for $0 \leq \alpha \leq s, \forall 1 \leq j \leq N$,

$$
\left\| D^\alpha u \right\|_{L^q_{I_j} L^r} \leq \left\| U(t)u(t_j) \right\|_{L^q_{I_j} L^r} + \left\| D^\alpha \left( \int_{I_j} U(t-s) \left( |u|^{p-1} u(s, x) \right) ds \right) \right\|_{L^q_{I_j} L^r} \leq \left\| U(t)u(t_j) \right\|_{L^q_{I_j} L^r} + C \left\| D^\alpha u \right\|_{L^q_{I_j} L^r} \left\| u \right\|_{L^p_{I_j} L^r} \leq CB + C\delta^{p-1} \left\| D^\alpha u \right\|_{L^q_{I_j} L^r}.
$$

Let $\delta$ be small and satisfy $C\delta^{p-1} < \frac{1}{2}$. Then $\|D^\alpha u\|_{L^q_{I_j} L^r} < \infty, 1 \leq j \leq N$, and

$$
\left\| D^\alpha u \right\|_{L^q_I L^r} < \infty.
$$

Moreover, from (2.7), we see that for $0 \leq \alpha \leq s$,

$$
\left\| D^\alpha (u(t) - U(t)\phi^+) \right\|_2 \leq \|u\|_{L^q_{[t,\infty)} L^r} \|D^\alpha u\|_{L^q_{[t,\infty)} L^r}.
$$

Therefore, we can obtain the claim by taking $\alpha = 0$ and $\alpha = s$ in (2.9) and letting $t \to +\infty$. \qed

Proposition 2.7. For any given $A$, there exist $\epsilon_0 = \epsilon_0(A, N, p)$ and $c = c(A)$ such that for any $\epsilon \leq \epsilon_0$, any interval $I = (T_1, T_2) \subset \mathbb{R}$ and any $\tilde{u} = \tilde{u}(x, t) \in H^s$ satisfying

$$
i\tilde{u}_t - (-\Delta)^s \tilde{u} - |\tilde{u}|^{p-1} \tilde{u} = e.
$$

If

$$
\|\tilde{u}\|_{S(\Lambda_{sc})} \leq A, \quad \|e\|_{S'(\Lambda_{-sc})} \leq \epsilon \quad \text{and} \quad \|e^{-i(t-t_0)(-\Delta)^s}(u(t_0) - \tilde{u}(t_0))\|_{S(\Lambda_{sc})} \leq \epsilon,
$$

then

$$
\|u\|_{S(\Lambda_{sc})} \leq c = c(A) < \infty.
$$
Proof. Let $u = \tilde{u} + w$, where $\tilde{u}$ is the solution of (2.10) and $w$ is the solution of

$$i\partial_t w - (-\Delta) w - |w + \tilde{u}|^{p-1}(w + \tilde{u}) + |\tilde{u}|^{p-1}\tilde{u} + e = 0. \tag{2.11}$$

For any $t_0 \in I$, $I = (T_1, t_0) \cup [t_0, T_2)$. We need only consider on $I_+ = [t_0, T_2)$, since the case on $I_- = (T_1, t_0]$ can be considered similarly. Since $\|\tilde{u}\|_{S(\Lambda_{sc})} \leq A$, we can partition $[t_0, T_2)$ into $N = N(A)$ intervals $I_j = [t_j, t_{j+1}]$ such that for each $j$, the quantity $\|\tilde{u}\|_{S(\Lambda_{sc}; I_j)} < \delta$ is suitably small with $\delta$ to be chosen later. The integral equation of $w$ with initial time $t_j$ is

$$w(t) = e^{-i(t-t_j)(-\Delta)^s} w(t_j) - i \int_{t_j}^t e^{-i(t-s)(-\Delta)^s} [||w + \tilde{u}|^{p-1}(w + \tilde{u}) - |\tilde{u}|^{p-1}\tilde{u} - e(s)] ds. \tag{2.12}$$

Using the inhomogeneous Strichartz estimates (2.4) on $I_j$, we obtain that for some $(q_1, r_1) \in \Lambda_{-sc}$,

$$\|w\|_{S(\Lambda_{sc}; I_j)} \leq ||e^{-i(t-t_j)(-\Delta)^s} w(t_j)||_{S(\Lambda_{sc}; I_j)} + c \|w + \tilde{u}\|^{p-1}(w + \tilde{u}) + |\tilde{u}|^{p-1}\tilde{u}||_{L^{q_1'}(I; L^{r_1'})} + \|e\|_{S(\Lambda_{-sc})}$$

$$\leq ||e^{-i(t-t_j)(-\Delta)^s} w(t_j)||_{S(\Lambda_{sc}; I_j)} + c \|\tilde{u}\|^{p-1}_{S(\Lambda_{sc}; I_j)} ||w||_{S(\Lambda_{sc}; I_j)} + c \|w\|^{p}_{S(\Lambda_{sc}; I_j)} + \|e\|_{S(\Lambda_{-sc})}$$

$$\leq ||e^{-i(t-t_j)(-\Delta)^s} w(t_j)||_{S(\Lambda_{sc}; I_j)} + c\delta^{p-1} ||w||_{S(\Lambda_{sc}; I_j)} + c \|w\|^{p}_{S(\Lambda_{sc}; I_j)} + c\epsilon_0. \tag{2.13}$$

If

$$\delta \leq \left(\frac{1}{4c}\right)^{\frac{1}{p-1}}, \quad \|e^{-i(t-t_j)(-\Delta)^s} w(t_j)||_{S(\Lambda_{sc}; I_j)} + c\epsilon_0 \leq \frac{1}{2} \left(\frac{1}{4c}\right)^{\frac{1}{p-1}},$$

then

$$\|w\|_{S(\Lambda_{sc}; I_j)} \leq 2 ||e^{-i(t-t_j)(-\Delta)^s} w(t_j)||_{S(\Lambda_{sc}; I_j)} + c\epsilon_0.$$ 

Now take $t = t_{j+1}$ in (2.12), and apply $e^{-i(t-t_{j+1})(-\Delta)^s}$ to both sides to obtain

$$e^{-i(t-t_{j+1})(-\Delta)^s} w(t_{j+1}) = e^{-i(t-t_j)(-\Delta)^s} w(t_j) - i \int_{t_j}^{t_{j+1}} e^{-i(t-s)(-\Delta)^s} [||w + \tilde{u}|^{p-1}(w + \tilde{u}) - |\tilde{u}|^{p-1}\tilde{u} - e](s) ds.$$

Since the Duhamel integral is confined to $I_j$, using the inhomogeneous Strichartz’z estimates and following a similar argument as above, we obtain that

$$\|e^{-i(t-t_{j+1})(-\Delta)^s} w(t_{j+1})||_{S(\Lambda_{sc}; I_j)}$$

$$\leq \|e^{-i(t-t_j)(-\Delta)^s} w(t_j)||_{S(\Lambda_{sc}; I_j)} + c\delta^{p-1} ||w||_{S(\Lambda_{sc}; I_j)} + c \|w\|^{p}_{S(\Lambda_{sc}; I_j)} + c\epsilon_0$$

$$\leq 2 ||e^{-i(t-t_j)(-\Delta)^s} w(t_j)||_{S(\Lambda_{sc}; I_j)} + c\epsilon_0.$$
Iterating beginning with \( j = 0 \), we obtain
\[
\| e^{-i(t-t_j)(-\Delta)^s} w(t_j) \|_{S(\Lambda_{sc};I_j)} \leq 2^j \| e^{-i(t-t_0)(-\Delta)^s} w(t_0) \|_{S(\Lambda_{sc};I_j)} + (2^j - 1)c\epsilon_0 \leq 2^{j+2}c\epsilon_0.
\]
To accommodate the conditions \([2.13]\) for all intervals \( I_j \) with \( 0 \leq j \leq N - 1 \), we require
\[
2^{N+2}c\epsilon_0 \leq \left( \frac{1}{4c} \right)^{p-1}.
\]
Finally,
\[
\| w \|_{S(\Lambda_{sc};I_+)} \leq \sum_{j=0}^{N-1} 2^{j+2}c\epsilon_0 + cN\epsilon_0 \leq c(N)\epsilon_0,
\]
which implies \( \| w \|_{S(\Lambda_{sc};I_+)} \leq c(A)\epsilon_0 \) since \( N = N(A) \), concluding the proof.

3. Variational Characteristic and Invariant Sets

First, we collect some variational properties of \( Q \), as follows.

**Lemma 3.1.** \([36]\) Let \( N \geq 2, \ 0 < s < 1 \) and \( 1 + \frac{4s}{N} < p < 1 + \frac{4s}{N-2s} \). Suppose that \( Q \) is the ground-state solution of \((1.3)\). Then, we have the following properties.

(i) \[ \| Q \|_{p+1}^{p+1} = \frac{2s(p+1)}{N(p-1)} \| Q \|_{H^s}^2 = \frac{2s(p+1)}{2s(p+1) - N(p-1)} \| Q \|_{p+1}^2. \]

(ii) \[ E[Q] = \frac{1}{2} \int \nabla(-\Delta)^s Q dx - \frac{1}{p+1} \| Q \|_{p+1}^{p+1} = \frac{N(p-1) - 4s}{2N(p-1)} \| Q \|_{H^s}^2. \]

(iii) \[ E[Q] M[Q] \frac{2s}{\pi^s} = \frac{N(p-1) - 4s}{4s - (N-2s)(p-1)} \| Q \|_{H^s}^{2s}. \]

(iv) \[ \| Q \|_{H^s}^2 M[Q] \frac{2s}{\pi^s} = \frac{N(p-1)}{4s - (N-2s)(p-1)} \| Q \|_{H^s}^{2s}. \]

(v) \[ C_{GN} = \frac{\| Q \|_{p+1}^{p+1}}{\| Q \|_{p+1}^{p+1}} \frac{\| Q \|_{p+1}^{p+1}}{\| Q \|_{H^s}^{2s}} = \frac{2s(p+1)}{N(p-1)} \frac{\| Q \|_{2s}}{\| Q \|_{H^s}^{2s}} \cdot \frac{1}{\| Q \|_{2s}} \cdot \frac{N(p-1) - 1}{2s} \cdot \frac{N(p-1) - 1}{2s} \cdot \frac{N(p-1) - 1}{2s} \cdot \frac{N(p-1) - 1}{2s}. \]

**Remark 3.2.** In fact, Caffarelli and Silvestre in \([3]\) first proposed a general fractional Laplacian. And then many researchers have been studying the time dependent and independent of fractional nonlinear Schrödinger equations (see \([5, 11, 16, 17, 32, 34, 35]\)).

Let \( u \in H^s \setminus \{0\} \), and define
\[ K_g = \{ \| u \|^2_{H^s} M[\frac{\| u \|}{\pi^s}] \frac{2s}{\pi^s} < \| Q \|^2_{H^s} M[\frac{\| Q \|}{\pi^s}] \frac{2s}{\pi^s}, \quad E[\frac{\| u \|}{\pi^s}] M[\frac{\| u \|}{\pi^s}] \frac{2s}{\pi^s} < E[Q] M[\frac{\| Q \|}{\pi^s}] \frac{2s}{\pi^s} \}. \]

**Proposition 3.3.** Let \( N \geq 2, \ 0 < s < 1 \) and \( 1 + \frac{4s}{N} < p < 1 + \frac{4s}{N-2s} \). Let \( Q \) be the ground-state solution of \((1.3)\). Then \( K_g \) is invariant manifold of \((1.1)\).
Proof. It follows from the conservation of energy and the sharp Gagliardo-Nirenberg inequality (1.6) that

$$M[u] \frac{\delta}{sc} E[u] = \frac{1}{2} \| u(t) \|_2^{\frac{2(p+1)}{p+1}} \| D^s u(t) \|_2^2 - \frac{1}{p+1} \| u \|_p^{p+1} \| D^s u(t) \|_2^{\frac{2(p+1)}{p+1}}$$

$$
\geq \frac{1}{2} \| u(t) \|_2^{\frac{\delta}{sc}} \| D^s u(t) \|_2^2 - \frac{C_{GN}}{p+1} \frac{\delta}{sc} \| D^s u(t) \|_2^\gamma.
$$

Now, we define $f(y) = \frac{1}{2} y^2 - \frac{1}{p+1} C_{GN} y^{\frac{N(p+1)}{2(p+1)}}$. We find that $f(y)$ has the following properties:

$$f'(y) = y \left(1 - C_{GN} N^{\frac{p-1}{2(p+1)}} y^{\frac{N(p+1)-4s}{2s}}\right),$$

and thus, $y_0 = 0$ and $y_1 = \| Q \|_2^{\frac{\delta}{sc}} \| D^s Q \|_2$ are two roots of $f'(y) = 0$, which implies that $f$ has a local minimum at $y_0$ and a local maximum at $y_1$. From Lemma 3.1, we have $f_{max} = f(y_1) = M[Q]^{\frac{\delta}{sc}} E[Q]$, and for all $t \in I$

$$f(\| u(t) \|_2^{\frac{\delta}{sc}} \| D^s u(t) \|_2) \leq M[u(t)]^{\frac{\delta}{sc}} E[u(t)] = M[u_0]^{\frac{\delta}{sc}} E[u_0] < f(y_1). \quad (3.1)$$

If $u_0 \in K_g$, i.e., $\| u_0 \|_2^{\frac{\delta}{sc}} \| D^s u_0 \|_2 < y_1$, then by (3.1) and the continuity of $\| D^s u(t) \|_2$ in $t$, we claim that $t \in \mathbb{R}$,

$$\| u(t) \|_{H^s}^2, M[u(t)]^{\frac{\delta}{sc}} < \| Q \|_{H^s}^2, M[Q]^{\frac{\delta}{sc}}. \quad (3.2)$$

Indeed, if (3.2) is not true, there must be $t_1 \in I$ such that $\| u(t_1) \|_2^{\frac{\delta}{sc}} \| D^s u(t_1) \|_2 \geq y_1$. Since $u(t, x) \in C(I; H^s)$ is continuous with respect to $t$, we can find a $0 < t_0 \leq t_1$ such that $\| u(t_0) \|_2^{\frac{\delta}{sc}} \| D^s u(t_0) \|_2 = y_1$. Thus, by injecting $E[u(t_0)] = E[u_0]$ and $\| u(t_0) \|_2^{\frac{\delta}{sc}} \| D^s u(t_0) \|_2 = y_1$ into (3.1), we see that

$$f(y_1) > M[u_0]^{\frac{\delta}{sc}} E[u_0] = M[u(t_0)]^{\frac{\delta}{sc}} E[u(t_0)] \geq f(\| u(t_0) \|_2^{\frac{\delta}{sc}} \| D^s u(t_0) \|_2) = f(y_1).$$

This is a contradiction. This completes the proof.

\[ \square \]

Remark 3.4. In fact, using the same argument in Proposition 3.3 we can obtain a precise estimate. Specially, if the initial data is such that

$$\| u_0 \|_{H^s}^2, M[u_0]^{\frac{\delta}{sc}} < \| Q \|_{H^s}^2, M[Q]^{\frac{\delta}{sc}},$$

then, we can chose a $\delta > 0$ such that $M[u]^{\frac{\delta}{sc}} E[u] < (1 - \delta)M[Q]^{\frac{\delta}{sc}} E[Q]$. Moreover, for the solution $u = u(t)$ with the above initial data we can find $\delta_0 = \delta_0(\delta)$ such that $\| u(t) \|_2^{\frac{\delta}{sc}} \| D^s u(t) \|_2 < (1 - \delta_0)\| Q \|_2^{\frac{\delta}{sc}} \| D^s Q \|_2$.

By the invariance of $K_g$, we see that (3.2) is true. In particular, the $H^s$-norm of the solution $u$ is bounded, which proves the global existence of the solution in this case.
Proof. Let \( C \) be a constant and \( I \) denote deduce that Theorem 3.5. Let Lemma 3.7. (Comparability of gradient and energy) Let \( u \in K_g \), then \( I = (-\infty, +\infty) \), i.e., \( u(t) \) exists globally in time.

Lemma 3.6. Let \( u_0 \in K_g \). Furthermore, take \( \delta > 0 \) such that \( M[u_0]^{\frac{p}{s+4s}} E[u_0] < (1 - \delta) M[Q]^{\frac{p}{s+4s}} E[Q] \). If \( u \) is a solution to problem (1.1) with initial data \( u_0 \), then there exists \( C_\delta > 0 \) such that for all \( t \in \mathbb{R} \),

\[
\| D^s u \|_2^2 - \frac{N(p-1)}{2s(p+1)} \| u \|_{p+1}^{p+1} \geq C_\delta \| D^s u \|_2^2.
\]  

(3.3)

Proof. Let \( \delta_0 = \delta_0(\delta) > 0 \) be defined in Remark 3.4. Then, for all \( t \in \mathbb{R} \), we have

\[
\| u(t) \|_2^{\frac{2s-4s}{s}} \| D^s u(t) \|_2 < (1 - \delta_0) \| Q \|_2^{\frac{2s-4s}{s}} \| D^s Q \|_2.
\]  

(3.4)

Denote

\[
H(t) = \frac{1}{\| Q \|_2^{\frac{2s-4s}{s}} \| D^s Q \|_2^{\frac{2s-4s}{s}}} \left( \| u(t) \|_2^{\frac{2s-4s}{s}} \| D^s u(t) \|_2^{\frac{2s-4s}{s}} - \frac{N(p-1)}{2s(p+1)} \| u \|_{p+1}^{p+1} \| u(t) \|_2^{\frac{2s-4s}{s}} \right)
\]

and \( G(y) = y^2 - y^{\frac{N(p-1)}{2s}} \). Applying the sharp Gagliardo-Nirenberg inequality in (1.6), we deduce that

\[
H(t) \geq G \left( \frac{\| u(t) \|_2^{\frac{2s-4s}{s}} \| D^s u(t) \|_2}{\| Q \|_2^{\frac{2s-4s}{s}} \| D^s Q \|_2} \right).
\]

When \( 0 \leq y \leq 1 - \delta_0 \), by the properties of the function \( G(y) \), we deduce that there exists a constant \( C_\delta \) such that \( g(y) \geq C_\delta y^2 \) provided \( 0 \leq y \leq 1 - \delta_0 \). This completes the proof.

\[
\square
\]

Lemma 3.7. (Comparability of gradient and energy) Let \( u_0 \in K_g \). Then,

\[
\frac{N(p-1) - 4s}{2N(p-1)} \| D^s u(t) \|_2^2 \leq E[u(t)] \leq \frac{1}{2} \| D^s u(t) \|_2^2.
\]

Proof. The second inequality can be obtained directly by the expression of \( E[u(t)] \). The first inequality follows from (1.6), (1.7) and (3.2) that

\[
\frac{1}{2} \| D^s u \|_2^2 - \frac{1}{p+1} \| u \|_{p+1}^{p+1} \geq \frac{1}{2} \| D^s u \|_2^2 \left( 1 - \frac{4s}{N(p-1)} \left( \| D^s u \|_2 \| u \|_{\frac{p+1}{s+4s}}^{\frac{2s+4s}{s}} \right) \right)
\]

\[
\geq \frac{N(p-1) - 4s}{2N(p-1)} \| D^s u \|_2^2.
\]

\[
\square
\]
At the end of this section, we prove the existence result of the wave operator \( \Omega^+ : \phi^+ \mapsto v_0 \). This is important to establish the scattering theory.

**Proposition 3.8. (Existence of wave operators)** Suppose that \( \phi^+ \in H^s \) and that

\[
\frac{1}{2} M[\phi^+]^{\frac{s-\Delta}{\alpha}} \|D^s \phi^+\|_2^2 < M[Q]^{\frac{s-\Delta}{\alpha}} E[Q].
\]  

(3.5)

Then, there exists \( v_0 \in H^s \) such that \( v \) globally solves (1.1) with initial data \( v_0 \) satisfying

\[
\|D^s v(t)\|_2 \|v_0\|_2^{\frac{s-\Delta}{\alpha}} \leq \|D^s Q\|_2 \|Q\|_2^{\frac{s-\Delta}{\alpha}}, \quad M[v] = \|\phi^+\|_2^2, \quad E[v] = \frac{1}{2} \|D^s \phi^+\|_2^2,
\]

and

\[
\lim_{t \to +\infty} \|v(t) - U(t)\phi^+\|_{H^s} = 0. \quad \text{Moreover, if } \|U(t)\phi^+\|_{\mathcal{S}(\Lambda_{sc})} \leq \delta_{sd}, \text{ where } \delta_{sd} \text{ is defined in Proposition 2.5,} \text{ then}
\]

\[
\|v\|_{\mathcal{S}(\Lambda_{sc})} \leq 2 \|U(t)\phi^+\|_{\mathcal{S}(\Lambda_{sc})}, \quad \|D^s v\|_{\mathcal{S}(\Lambda_0)} \leq 2c \|\phi^+\|_{H^s}.
\]

Proof. Let \( v(t) = FNLS(t)v_0 \) be the solution \( v(t) \) of the fractional NLS (1.1) with the initial data \( v(0) = v_0 \). According to the scattering theory of small initial data (see Proposition 2.5), we consider the integral equation

\[
v(t) = U(t)\phi^+ - i \int_t^\infty U(t-t^1)|v|^{p-1}v(t^1)dt^1
\]

(3.6)

for \( t \geq T \) with \( T \) large. From Proposition 2.5 for sufficiently large \( T \), we deduce that

\[
\|v\|_{\mathcal{S}(\Lambda_{sc};[T,\infty))} \leq 2\delta_{sd}, \quad \text{and}
\]

\[
\|v\|_{\mathcal{S}(\Lambda_0;[T,\infty))} + \|D^s v\|_{\mathcal{S}(\Lambda_0;[T,\infty))} < 2c \|\phi^+\|_{H^s}.
\]

Thus, by a similar argument when \( t > T \), we obtain

\[
\|v - U(t)\phi^+\|_{\mathcal{S}(\Lambda_0;[T,\infty))} + \|D^s (v - e^{it\Delta} \phi^+)\|_{\mathcal{S}(\Lambda_0;[T,\infty))} \to 0 \quad \text{as } T \to \infty.
\]

Hence, \( v(t) - U(t)\phi^+ \to 0 \) in \( H^s \), and thus, \( M[v] = \|\phi^+\|_2^2 \). By the fact \( U(t)\phi^+ \to 0 \) in \( L^q \) for any \( q \in (2, \frac{2N}{N-2s}] \) as \( t \to \infty \), we have \( \|U(t)\phi^+\|_{p+1} \to 0 \). Moreover, combining this with that \( \|D^s U(t)\phi^+\|_2 \) is conserved, we deduce that

\[
E[v] = \lim_{t \to \infty} \left( \frac{1}{2} \|D^s U(t)\phi^+\|_2^2 - \frac{1}{p+1} \|U(t)\phi^+\|_{p+1}^2 \right) = \frac{1}{2} \|D^s \phi^+\|_2^2.
\]

By the assumption (3.5), then we obtain

\[
M[v]^{\frac{s-\Delta}{\alpha}} E[v] < E[Q] M[Q]^{\frac{s-\Delta}{\alpha}}. \quad \text{According to (3.5) and Remark 3.1,} \text{ we deduce that}
\]

\[
\lim_{t \to \infty} \|v(t)\|_2^{\frac{2(s-\Delta)}{s}} \|D^s v(t)\|_2 = \lim_{t \to \infty} \|U(t)\phi^+\|_2^{\frac{2(s-\Delta)}{s}} \|D^s U(t)\phi^+\|_2^2 = \|\phi^+\|_2^{\frac{2(s-\Delta)}{s}} \|D^s \phi^+\|_2^2 \leq 2E[Q] M[Q]^{\frac{s-\Delta}{s}} = \frac{N(p-1) - 4s}{N(p-1)} \|Q\|_2^{\frac{2(s-\Delta)}{s}} \|D^s Q\|_2^2.
\]

Finally, we can evolve \( v(t) \) from \( T \) back to time 0 as in Theorem 3.5. \qed
4. Critical solution and compactness

In the previous, we have proved the the global existence part of Theorem 1.1 (see Theorem 3.5). From now on, we begin to prove the scattering part of Theorem 1.1. $u(t)$ is globally well-posed. According to Proposition 2.6, we just need to show that

$$\|u\|_{S(A_{sc})} < \infty.$$ (4.1)

Then, the $H^s$ scattering of the solution for Eq. (1.1) follows.

We say that $SC(u_0)$ holds if (4.1) is true for the solution $u$ with the initial data $u_0$.

From Proposition 2.5, we see that there exists $\delta > 0$ such that if $E[u_0]M[u_0]^{\frac{1}{s-\varepsilon}} < \delta$ and $\|u_0\|_2^{\frac{s}{s-\varepsilon}} \|D^s u_0\|_2 < \|Q\|_2^{\frac{s}{s-\varepsilon}} \|D^s Q\|_2$, then (4.1) holds. Now, for each $\delta$, we define the set $S_\delta$ to be the collection of all such initial data in $H^s$:

$$S_\delta = \{u_0 \in H^s : E[u_0]M[u_0]^{\frac{1}{s-\varepsilon}} < \delta \text{ and } M[u_0]^{\frac{1}{s-\varepsilon}} \|D^s u_0\|_2 < M[Q]^{\frac{s}{s-\varepsilon}} \|D^s Q\|_2^2 \}.$$ 

We also define that $(ME)_c = \sup \{\delta : u_0 \in S_\delta \Rightarrow SC(u_0) \text{ holds} \}$. If $(ME)_c = M[Q]^{\frac{s}{s-\varepsilon}} E[Q]$, then we are done. Thus, we assume that

$$(ME)_c < M[Q]^{\frac{s}{s-\varepsilon}} E[Q].$$ (4.2)

Then, there exist solutions $u_n$ of (1.1) with $H^s$ initial data $u_{n,0}$ (after rescaling, we may inquire that $u_n$ satisfies $\|u_n\|_2 = 1$) such that $\|D^s u_{n,0}\|_2 < \|Q\|_2^{\frac{s}{s-\varepsilon}} \|D^s Q\|_2$ and $E[u_{n,0}] \downarrow (ME)_c$ as $n \to \infty$, and $SC(u_0)$ does not hold for any $n$.

In this section, we will prove that there exists a critical $H^s$ solution $u_c$ to (1.1) with initial data $u_{c,0}$ such that $\|u_{c,0}\|_2^{\frac{s}{s-\varepsilon}} \|D^s u_{c,0}\|_2 < \|Q\|_2^{\frac{s}{s-\varepsilon}} \|D^s Q\|_2$ and $M[u_c]^{\frac{1}{s-\varepsilon}} E[u_c] = (ME)_c$ for which $SC(u_{c,0})$ does not hold. Then, we will show that the set $\{u_c(\cdot, t) | 0 \leq t < +\infty \}$ is precompact in $H^s$. Finally, we will use these properties to obtain the rigidity theorem in Section 5, which will use to conduct a contradiction. This can be used to finish the proof of Theorem 1.1.

First, we will introduce a profile decomposition lemma that is highly similar to that in [23], which were firstly proposed by Keraani [27] for the cubic Schrödinger equation, as follows.

**Lemma 4.1.** (Profile expansion) Let $\phi_n(x)$ be a radial and uniformly bounded sequence in $H^s$. Then, for each $M$, there exists a subsequence of $\phi_n$, also denoted by $\phi_n$, and

1. for each $1 \leq j \leq M$, there exists a (fixed in $n$) profile $\tilde{\psi}_j(x)$ in $H^s$,
2. for each $1 \leq j \leq M$, there exists a sequence (in $n$) of time shifts $t^j_n$,
(3) There exists a sequence (in $n$) of remainders $W_n^M(x)$ in $H^s$ such that
\[
\phi_n(x) = \sum_{j=1}^{M} U(-t_n^j) \psi^j(x) + W_n^M(x).
\]
The time and space sequences have a pairwise divergence property, i.e., for $1 \leq j \neq k \leq M$, we have
\[
\lim_{n \to +\infty} |t_n^j - t_n^k| = +\infty.
\]
(4.3)
The remainder sequence has the following asymptotic smallness property:
\[
\lim_{M \to +\infty} \left[ \lim_{n \to +\infty} \|U(t) W_n^M \|_{S(\Lambda_{sc})} \right] = 0.
\]
(4.4)
For fixed $M$ and any $0 \leq \alpha \leq s$, we have the asymptotic Pythagorean expansion:
\[
\|\phi_n\|^2_{H^\alpha} = \sum_{j=1}^{M} \|\psi^j\|^2_{H^\alpha} + \|W_n^M\|^2_{H^\alpha} + o_n(1).
\]
(4.5)

Proof. The proof of the above linear profile decomposition for the fractional NLS is quite similar with that for the fourth-order nonlinear Schrödinger equation in [15]. Here, we omit the main proof. But we should point out that (4.4) could be improved to
\[
\lim_{M \to +\infty} \left[ \lim_{n \to +\infty} \|U(t) W_n^M \|_{L^q L^r} \right] = 0, \quad \forall (q, r) \text{ satisfies (2.1) with } \theta = s_c.
\]
(4.6)
More precisely,
\[
\lim_{M \to +\infty} \left[ \lim_{n \to +\infty} \|U(t) W_n^M \|_{L^\infty L^{\frac{2N}{N-2s_c}}} \right] = 0.
\]
(4.7)

Using the profile expansion, similar argument as in [15] could just be applied to obtain the following results: Energy expansion, Existence of a critical solution and Precompactness of the flow of the critical solution. Note that we have also proved similar counterparts of these results with respect to the fractional Hartree equation [18], and we omit the proof here.

Lemma 4.2. (Energy Pythagorean expansion) In the situation of Lemma 4.1, we have
\[
E[\phi_n] = \sum_{j=1}^{M} E[U(-t_n^j) \psi^j] + E[W_n^M] + o_n(1).
\]
(4.8)

Proposition 4.3. (Existence of a critical solution) There exists a global solution $u_c$ in $H^s$ with initial data $u_{c,0}$ such that $\|u_{c,0}\|_2 = 1$,

\[
E[u_c] = (ME)_c < M[Q]^{\frac{s}{s_c}} E[Q], \quad \|D^s u_c\|^2_2 < M[Q]^{\frac{s}{s_c}} \|D^s Q\|^2_2, \text{ for all } 0 \leq t < \infty,
\]
and
\[ \|u_c\|_{\mathcal{S}(\Lambda_{sc})} = +\infty. \]

**Proposition 4.4.** (Precompactness of the flow of the critical solution) Let \( u_c \) be as in Proposition 4.3, then, if \( \|u_c\|_{\mathcal{S}(0, +\infty); \Lambda_{sc}} = \infty \),
\[ \{ u_c(\cdot, t) | t \in [0, +\infty) \} \subset H^s \]
is precompact in \( H^s \). A corresponding conclusion is reached if \( \|u_c\|_{\mathcal{S}((-\infty, 0]; \Lambda_{sc})} = \infty \).

**Corollary 4.5.** Let \( u = u(t) \) be a solution to \((1.1)\) such that \( K^+ = \{ u(\cdot, t) | t \in [0, +\infty) \} \) is precompact in \( H^s_\tau \). Then, for each \( \epsilon > 0 \), there exists \( R > 0 \) such that
\[ \int_{|x| > R} |D^s u(x, t)|^2 + |u(x, t)|^2 + |u(x, t)|^{p+1} dx \leq \epsilon. \]

**Proof.** If not, for any \( R > 0 \), there exists \( \epsilon_0 > 0 \) and a sequence \( t_n \) such that
\[ \int_{|x| > R} |D^s u(x, t_n)|^2 + |u(x, t_n)|^2 + |u(x, t_n)|^{p+1} dx \geq \epsilon_0. \]
By the precompactness of \( K^+ \), there exists \( \phi \in H^s \) such that, up to a subsequence of \( t_n \), we have \( u(\cdot, t_n) \to \phi \) in \( H^s \). Thus, for any \( R > 0 \), we obtain
\[ \int_{|x| > R} |D^s \phi(x)|^2 + |\phi(x)|^2 + |\phi(x)|^{p+1} dx \geq \epsilon_0, \]
from which we can easily obtain a contradiction because \( \phi \in H^s \) and \( \|\phi\|_{p+1} \leq c\|\phi\|_{H^s} \) by the Sobolev inequality. \( \square \)

### 5. Rigidity theorem

In order to finish the proof of Theorem 1.1, we need the following rigidity theorem.

**Theorem 5.1.** Let \( N \geq 2 \) and \( 1 + \frac{4s}{N} < p < 1 + \frac{4s}{N-2s} \). Assume that the initial data \( u_0 \in H^s \) is radial and \( u_0 \in K_g \), i.e.,
\[ M[u_0] \xrightarrow{\mathcal{S}} E[u_0] < M[Q] \xrightarrow{\mathcal{S}} E[Q], \tag{5.1} \]
and
\[ M[u_0] \xrightarrow{\mathcal{S}} \|u_0\|_{H^s}^2 < M[Q] \xrightarrow{\mathcal{S}} \|Q\|_{H^s}^2, \tag{5.2} \]
Let \( u = u(t) \) be the radially global solution of \((1.1)\) with initial data \( u_0 \). If it holds that \( K^+ = \{ u(\cdot, t) | t \in [0, +\infty) \} \) is precompact in \( H^s \), then \( u_0 = 0 \). The same conclusion holds if \( K^- = \{ u(\cdot, t) : t \in (-\infty, 0] \} \) is precompact in \( H^s \).
Now, we introduce the localized virial estimate for the radial solutions of (1.1) in terms of the idea in [2]. Let $u \in H^s$ with $s \geq \frac{1}{2}$. we define the auxiliary function $u_m = u_m(t, x)$ as

$$u_m := c_s \frac{1}{-\Delta + m} u(t) = c_s \mathcal{F}^{-1} \frac{\hat{u}(t, \xi)}{|\xi|^2 + m} \quad (5.3)$$

where $c_s = \sqrt{\frac{\sin \pi s}{\pi}}$. It follows from [2] that, for any $u \in H^s$,

$$\int_0^\infty m^s \int_{\mathbb{R}^N} |\nabla u_m|^2 dx dm = s \|(-\Delta)^\frac{s}{2} u\|^2_2. \quad (5.4)$$

Then, we obtain the following corollary, which is a counterpart of Corollary 4.5.

**Corollary 5.2.** Let $u = u(t)$ be a solution to (1.1) such that $K = \{u(\cdot, t) | t \in [0, +\infty)\}$ is precompact in $H^s_r$. Then, for each $\epsilon > 0$, there exists $R > 0$ such that

$$\int_0^\infty m^s \int_{|x| > R} |\nabla u_m|^2 dx dm + \int_{|x| > R} |u(x, t)|^2 + |u(x, t)|^{p+1} dx \leq \epsilon.$$

**The Proof of Theorem 5.1.** It suffices to address the $K^+$ case, since the $K^-$ case follows similarly.

Let $\varphi \in C_c^\infty$ be a radially real-valued function, and defined by

$$\varphi(x) = \begin{cases} |x|^2 & \text{for } |x| \leq 1 \\ 0 & \text{for } |x| \geq 2. \end{cases}$$

For any $R > 0$, take $\varphi_R(x) = R^2 \varphi(\frac{x}{R})$ and define the localized virial of $u \in H^s$ by

$$J_R(t) := 2Im \int_{\mathbb{R}^N} \bar{u}(t, x) \nabla \varphi_R(x) \cdot \nabla u(t, x) dx.$$

Similar to the method in [2], we obtain the identity

$$J_R'(t) = \int_0^\infty m^s \int_{\mathbb{R}^N} \left(4\partial_k u_m \partial_k^2 \varphi(\frac{x}{R}) - \Delta^2 \varphi_R(x) |u_m|^2 \right) dx dm - \frac{2(p-1)}{p+1} \int_{\mathbb{R}^N} (\Delta \varphi)(\frac{x}{R}) |u|^{p+1} dx.$$
By the properties of \( \varphi \), we deduce that
\[
\mathcal{J}_R'(t) \geq 8 \int_0^\infty m^s \int_{|x| \leq R} |\nabla u_m|^2 \, dx + 4 \int_0^\infty m^s \int_{R < |x| < 2R} \frac{\partial^2_r \varphi \left( \frac{x}{R} \right)}{R} |\nabla u_m|^2 \, dx dm \tag{5.5}
\]
\[
- \int_0^\infty m^s \int_{|x| > R} \Delta^2 \varphi_R(x) |u_m|^2 \, dx dm
- \frac{2(p-1)}{p+1} \int_{|x| \leq R} |u|^{p+1} \, dx - c \int_{R < |x| < 2R} |u|^{p+1} \, dx
\geq \left( 8 \int_0^\infty m^s \int_\mathbb{R}^N |\nabla u_m|^2 \, dx - \frac{4N(p-1)}{P+1} \int |u|^{p+1} \, dx \right)
- \left( 8 \int_0^\infty m^s \int_{|x| > R} |\nabla u_m|^2 \, dx - \frac{4N(p-1)}{P+1} \int_{|x| > R} |u|^{p+1} \, dx \right)
- c \left( \int_0^\infty m^s \int_{R < |x| < 2R} |\nabla u_m|^2 \, dx + \int_{R < |x| < 2R} |u|^{p+1} \, dx \right) - \frac{c}{R^{2s}} \|u\|_2^2.
\]
Here, we use the following estimate in the last step (see [2]),
\[
\int_0^\infty m^s \int_{|x| > R} \Delta^2 \varphi_R(x) |u_m|^2 \, dx dm \leq c R^{-2s} \|u\|_2^2.
\]
Now, let \( \delta \in (0, 1) \) satisfy \( E[u_0] < (1 - \delta) E[Q] M[Q] \). From Lemma 3.6 and Lemma 3.7, we see that
\[
8 \int_0^\infty m^s \int_\mathbb{R}^N |\nabla u_m|^2 \, dx - \frac{4N(p-1)}{P+1} \int |u|^{p+1} \, dx = 8s \gamma \|D^s u_m\|_2^2
- \frac{4N(p-1)}{P+1} \int |u|^{p+1} \, dx \geq C_s \|D^s u_0\|_2^2.
\]
Then, we can take \( R \) large enough to obtain the following estimate
\[
\mathcal{J}_R'(t) \geq C \|D^s u_0\|_2^2. \tag{5.6}
\]
Integrate (5.6) over \([0, t]\).
\[
|\mathcal{J}_R(t) - \mathcal{J}_R(0)| \geq C \|D^s u_0\|_2^2
\]
However, by [2], we should have
\[
|\mathcal{J}_R(t) - \mathcal{J}_R(0)| \leq C_R (\|u\|_{H^s_2}^2 + \|u_0\|_{H^s_2}^2) \leq C_R (\|u\|_{H^s}^2 + \|u_0\|_{H^s}^2) \leq C_R \|Q\|_{H^s}^2,
\]
which is a contradiction for large \( t \) unless \( u_0 = 0 \).
\[\square\]

Now, we can finish the proof of Theorem 1.1.

**The Proof of Theorem 1.1.**
Note that by Proposition 4.4, the critical solution $u_c$ constructed in Section 4 satisfies the hypotheses in Theorem 5.1. Therefore, to complete the proof of Theorem 1.1, we should apply Theorem 5.1 to $u_c$ and find that $u_{c,0} = 0$, which contradicts the fact that $\|u_c\|_{S(A_{sc})} = +\infty$. This contradiction shows that $\text{SC}(u_0)$ holds. Thus, by Proposition 2.6, we have shown that $H^s$ scattering holds. □

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