ON SUBELLIPTIC HARMONIC MAPS WITH POTENTIAL

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Abstract. Let $(M, H, g_H; g)$ be a sub-Riemannian manifold and $(N, h)$ be a Riemannian manifold. For a smooth map $u : M \to N$, we consider the energy functional $E_G(u) = \frac{1}{2} \int_M \| du_H \|^2 - 2G(u) dV_M$, where $du_H$ is the horizontal differential of $u$, $G : N \to \mathbb{R}$ is a smooth function on $N$. The critical maps of $E_G(u)$ are referred to as subelliptic harmonic maps with potential $G$. In this paper, we investigate the existence problem for subelliptic harmonic maps with potentials by a subelliptic heat flow. Assuming that the target Riemannian manifold has non-positive sectional curvature and the potential $G$ satisfies various suitable conditions, we prove some Eells-Sampson type existence results when the source manifold is either a step-2 sub-Riemannian manifold or a step-$r$ sub-Riemannian manifold whose sub-Riemannian structure comes from a tense Riemannian foliation.

1. Introduction

Sub-Riemannian geometry can be regarded as a natural generalization of Riemannian geometry. A sub-Riemannian manifold is defined as a triple $(M, H, g_H)$, where $M$ is a connected smooth manifold, $H$ is a subbundle bracket generating for $TM$, and $g_H$ is a smooth fiberwise metric on $H$. Recently, geometric analysis on sub-Riemannian manifolds has been the subject of intense study (cf.[Bau04][Bau18b]).

On the other hand, the equilibrium system of ferromagnetic spin chain and the Neumann Motion which describe two important physical phenomena, have been received much attention. To study them, harmonic maps with potential were introduced in [FR97]. This is a new kind of maps more general than the usual harmonic maps, whose behavior may drastically change in the presence of a potential. Various existence and nonexistence results of harmonic maps with potential have been achieved. Fardoun and Ratto [FR97] obtained some variational properties and existence results of harmonic maps with potential when the target manifolds are spheres. Chen [Che98][Che99] also established uniqueness and existence results of Landau-Lifshitz equations. Later, [FRR00] gave Eells-Sampson type results for harmonic maps with potential.

The main purpose of this paper is to study a natural counterpart of harmonic maps with potential in sub-Riemannian geometry. Let $(M, H, g_H)$ be a sub-Riemannian manifold with a smooth measure $d\mu$ and let $(N, h)$ be a Riemannian manifold.
be a Riemannian manifold. Given a function \( G \in C^\infty(N) \), we consider the following energy functional

\[
E_G(u) = \frac{1}{2} \int_M \left( |du_H|^2 - 2G(u) \right) d\mu
\]

where \( u : M \to N \) is a smooth map and \( du_H \) is the restriction of \( du \) to \( H \). A smooth map \( u : (M, H, g_H) \to (N, h) \) is called a subelliptic harmonic map with potential \( G \) if it is a critical point of (1.1). The subelliptic harmonic maps with potential can be viewed as a generalization of both harmonic maps with potential and subelliptic harmonic maps. In sub-Riemannian geometry, some most important cases are the relatively simple cases that the measures \( d\mu \) for sub-Riemannian manifolds are the volume elements of compatible Riemannian metrics (also called Riemannian extension \( g \) of \( g_H \)), such as the Webster metric in CR geometry, the contact metric in contact geometry, etc. Hence, the present paper concentrates on sub-Riemannian manifolds with such Riemannian extensions. Then the Euler-Lagrange equation of (1.1) is

\[
\tau(u) = \tau_H(u) + (\tilde{\nabla}G)(u) = 0
\]

where \( \tau_H(u) \) denotes the subelliptic tension field associated with the horizontal energy (cf. [Don21]) and \( \tilde{\nabla} \) is the Riemannian connection on \((N, h)\).

As we know, [JX98] first introduced subelliptic harmonic maps whose domain are in the Euclidean space. They investigated the Dirichlet existence problem for such subelliptic harmonic maps. Later, a related uniqueness result was given by [Zhe99]. The pseudo-harmonic maps from pseudoconvex CR manifolds, introduced by Barletta et al. in [BDU01], are actually subelliptic harmonic maps defined with respect to the Webster metrics. Some regularity results for subelliptic harmonic maps from Carnot groups were established in [Wan03], see also [HaS98], [ZF15] for some regularity results of subelliptic p-harmonic maps. In [BD04], the authors investigated the stability problem of pseudo-harmonic maps with potential from strictly pseudoconvex CR manifolds, which are the special cases of the subelliptic harmonic maps with potential.

In Riemannian and Kählerian geometry, there are many fundamental applications based on Eells-Sampson theorem (cf. [JX98], [Tol00]). Chang and Chang [CC13] obtained an Eells-Sampson type result for pseudo-harmonic maps from CR manifolds under some additional analytic condition, which was later generalized by Ren and Yang in [RY18]. Dong [Don21] obtained Eells-Sampson type results for subelliptic harmonic maps from sub-Riemannian manifolds in some general cases. In this paper, we aim to establish Eells-Sampson type theorems for subelliptic harmonic maps with potential from certain kinds of sub-Riemannian manifolds. To this end, we investigate the following subelliptic harmonic map heat flow with potential \( G \) associated
with (1.1) and (1.2):

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \tau(u) \\
u|_{t=0} &= \bar{u}
\end{align*}
\]

where \( \bar{u} : M \to N \) is the initial map, which is assumed to be \( C^\infty \) for simplicity.

Here and afterwards, let \( \text{Hess} \ G \) denote the Hessian matrix of \( G \) with respect to the Riemannian connection \( \tilde{\nabla} \). First we have a long time existence for (1.3) as follows.

**Theorem 1.1.** Let \((M, H, g_H; g)\) be a compact sub-Riemannian manifold and let \((N, h)\) be a complete Riemannian manifold with non-positive sectional curvature. There exists \( T > 0 \) such that the heat flow (1.3) has a unique solution on \( M \times [0, T) \). If, for some constant \( C > 0 \), \( \text{Hess} \ G \leq C \cdot h \), then \( T = +\infty \). In particular, when \((N, h)\) is compact, for any \( G \in \mathcal{C}^\infty(N) \), we have \( T = +\infty \).

For a more precise statement, a generating order was introduced for the sub-Riemannian manifold. We say \( M \) is a step-\( r \) sub-Riemannian manifold if the tangent space \( T_x M \) at each point \( x \) can be spanned by sections of \( H \) together with their Lie brackets up to order \( r \). The simplest nontrivial sub-Riemannian manifolds are step-2 sub-Riemannian manifolds, which include strictly pseudoconvex CR manifolds, contact metric manifolds, and quaternionic contact manifolds, etc. On the other hand, Riemannian foliations provide an important source of sub-Riemannian manifolds as well. For a Riemannian foliation \((M, g; \mathcal{F})\) with a bundle-like metric \( g \), let \( H = (T\mathcal{F})^\perp \) (the horizontal subbundle of the foliation \( \mathcal{F} \) with respect to \( g \) ) and \( g_H = g|_H \).

If \( H \) is bracket generating for \( TM \), then we have a sub-Riemannian manifold \((M, H, g_H; g)\) corresponding to \((M, g; \mathcal{F})\). The Riemannian foliation \((M, g; \mathcal{F})\) will be said to be tense if the mean vector field of \( \mathcal{F} \) is parallel with respect to the Bott connection along the leaves.

In order to establish the Eells-Sampson type results, we need to establish the convergence of \( u(\cdot, t) \) as \( t \to \infty \). Now we consider the following two cases: the source manifold \((M, H, g_H; g)\) is a compact step-2 sub-Riemannian manifold or a compact sub-Riemannian manifold corresponding to a tense Riemannian foliation with the property that \( H \) is bracket generating for \( TM \). Let: \( S(V) \to M \) be the unit sphere bundle of the vertical bundle \( V \), that is, \( S(V) = \{ v \in V : \|v\|_g = 1 \} \). For any \( v \in S(V) \), the \( v \)-component of \( T(\cdot, \cdot) \) is given by \( T^v(\cdot, \cdot) = (T(\cdot, \cdot), v) \), where \( T(\cdot, \cdot) \) is the torsion of the generalized Bott connection \( \nabla^{28} \). In the first case, it turns out that the smooth function \( \eta(v) = \frac{1}{2} ||T^v(\cdot, \cdot)||_g^2 \) can achieve a positive minimal value on \( S(V) \), that is, \( \eta_{\text{min}} = \min_{v \in S(V)} \eta(v) > 0 \).

**Theorem 1.2.** Let \((M, H, g_H; g)\) be a compact step-2 sub-Riemannian manifold and let \((N, h)\) be a compact Riemannian manifold with non-positive sectional curvature. If \( \text{Hess} G < \frac{\eta_{\text{min}}}{2} \cdot h \), then, for any smooth map \( \bar{u} :
$M \to N$, the subelliptic harmonic map heat flow with potential $G$ exists on $M \times [0, +\infty)$ and there exists a sequence $t_i \to \infty$, such that $u(x, t_i)$ converges uniformly to a subelliptic harmonic map $u_\infty(x)$, as $t_i \to \infty$. In particular, any map $\bar{u} \in C^\infty(M, N)$ is homotopic to a $C^\infty$ subelliptic harmonic map with potential $G$.

When $N$ is complete, we need a decay condition on $G$ to obtain Eells-Sampson type results.

**Proposition 1.3.** Suppose $(M, H, g_H; g)$ is a compact step-2 sub-Riemannian manifold. Let $(N, h)$ be a complete Riemannian manifold with non-positive sectional curvature and let $\rho$ denote the distance function on $N$ from a fixed point $P_0 \in N$. If, for some $C > 0$,

$$\text{Hess}G(y) \leq -C(1 + \rho(y))^{-1} \cdot h$$

then $T = +\infty$ and $u$ converges to $u_\infty$, where $u_\infty$ is a constant. Moreover, $\Sigma_G \neq \emptyset$ in this case, where $\Sigma_G = \{y \in N : \nabla G(y) = 0\}$.

For the second case that $M$ is a compact sub-Riemannian manifold corresponding to a tense Riemannian foliation, we have

**Proposition 1.4.** Suppose $(M, H, g_H; g)$ is a compact sub-Riemannian manifold corresponding to a tense Riemannian foliation with the property that $H$ is bracket generating for $TM$. Let $(N, h)$ be a complete Riemannian manifold with non-positive sectional curvature and let $\rho$ denote the distance function on $N$ from a fixed point $P_0 \in N$. If, for some $C > 0$,

$$\text{Hess}G(y) \leq -C(1 + \rho(y))^{-1} \cdot h$$

then $T = +\infty$ and $u$ converges to $u_\infty$, where $u_\infty$ is a constant. Moreover, $\Sigma_G \neq \emptyset$ in this case.

Suppose now that $N$ is embedded isometrically into some Euclidean space $\mathbb{R}^K$. Let $\mathcal{J} : N \to \mathbb{R}^K$ denote this embedding and $A(y) : T_y N \times T_y N \to (T_y N)^\perp$ denote its second fundamental form. We also assume that the potential $G$ is the restriction to $N$ of some smooth function $\mathcal{G} : \mathbb{R}^K \to \mathbb{R}$. For simplicity, we write $y$ instead of $\mathcal{J}(y)$ for all $y \in N$. Then Equation (1.3) becomes

$$\begin{cases}
\Delta_H u - \frac{\partial u}{\partial t} = A(u)(du_H, du_H) - P(D\mathcal{G}(u)) \\
u(x, 0) = \bar{u}(x)
\end{cases}$$

where $P : \mathbb{R}^K \to T_y N$ is the orthogonal projection onto the tangent space of $N$ at $y$, $D$ is the canonical Riemannian connection of $\mathbb{R}^K$.

**Proposition 1.5.** Let $(M, H, g_H; g)$ be either a compact step-2 sub-Riemannian manifold or a compact sub-Riemannian manifold corresponding to a tense Riemannian foliation with the property that $H$ is bracket generating for $TM$. Let $(N, h)$ be a complete Riemannian manifold with non-positive
sectional curvature and let \( \mathcal{J} : N \to \mathbb{R}^K \) be an isometric embedding. Suppose for all \( y \in N, Y \in T_y N \),

\[
\langle A(y)(Y,Y), y \rangle_{\mathbb{R}^K} + |Y|^2_{\mathbb{R}^K} - \langle \nabla G(y), y \rangle_{\mathbb{R}^K} \geq 0.
\]

In the first case, if \( \text{Hess} G < \frac{\text{min}}{2} \cdot h \), then \( T = +\infty \) and \( u \) subconverges to \( u_\infty \), where \( u_\infty \) is a smooth subelliptic harmonic map with potential \( G \). In the second case, if \( \text{Hess} G \leq 0 \), then \( T = +\infty \) and \( u \) subconverges to \( u_\infty \), where \( u_\infty \) is a constant. Moreover, \( \Sigma_G \neq \emptyset \) in this case.

2. Preliminaries

First, we recall some facts on sub-Riemannian manifolds. Suppose that \( M \) is a connected smooth \((m + d)\)-dimensional manifold and \( H \) is a rank \( m \) subbundle of tangent bundle \( TM \). For any \( x \in M \) and any open neighborhood \( U \) of \( x \), we denote the space of smooth sections of \( H \) on \( U \) by \( \Gamma(U, H) \) and define \( \{\Gamma^j(U, H)\}_{j \geq 1} \) inductively by

\[
\Gamma^1(U, H) = \Gamma(U, H)
\]

\[
\Gamma^{j+1}(U, H) = \Gamma^j(U, H) + [\Gamma^1(U, H), \Gamma^j(U, H)]
\]

for each positive integer \( j \), where \([\cdot, \cdot]\) denotes the Lie bracket of vector fields. At each point \( x \), we obtain a subspace \( H^j_x \) of the tangent space \( T_x M \), that is,

\[
H^j_x = \{ X(x) : X \in \Gamma^j(U, H) \}.
\]

The subbundle \( H \) is said to be \( r \)-step bracket generating for \( TM \), if \( H^r_x = T_x M \) for each \( x \in M \) (cf. [Str86], [Mon02]). In this paper, we always assume that \( H \) satisfies the \( r \)-step bracket generating condition for some \( r \geq 2 \).

A sub-Riemannian manifold is a triple \((M, H, g_H)\), where \( g_H \) is a fiber-wise metric on \( H \). According to [Str86], there always exists a Riemannian metric \( g \) on \( M \) such that \( g|_H = g_H \), where \( g \) is referred to as a Riemannian extension of \( g_H \). Henceforth, we always fix a Riemannian extension \( g \) on the sub-Riemannian manifold \((M, H, g_H)\), and consider the quadruple \((M, H, g_H, g)\). In terms of the metric \( g \), the tangent bundle \( TM \) has the following orthogonal decomposition:

\[
TM = H \oplus V
\]

which induces the projections \( \pi_H : TM \to H \) and \( \pi_V : TM \to V \). The distribution \( H \) and \( V = H^\perp \) are called the horizontal distribution and the vertical distribution respectively on \((M, H, g_H, g)\).

We consider the generalized Bott connection on sub-Riemannian manifolds, which is given by (cf. [Ban16], [BF15], [Don21])

\[
\nabla_X^H Y = \begin{cases} 
\pi_H(\nabla_X^H Y), & X, Y \in \Gamma(H) \\
\pi_H([X, Y]), & X \in \Gamma(V), Y \in \Gamma(H) \\
\pi_V([X, Y]), & X \in \Gamma(H), Y \in \Gamma(V) \\
\pi_V(\nabla_X^V Y), & X, Y \in \Gamma(V) 
\end{cases}
\]
where $\nabla^R$ denotes the Riemannian connection of $g$. It is convenient for computations on sub-Riemannian manifolds by using the above connection, since $\nabla^B$ preserves the decomposition (2.1) and it also satisfies
\[
\nabla^B_X g_H = 0 \quad \text{and} \quad \nabla^B_Y g_V = 0
\]
for any $X \in H$ and $Y \in V$, where $g_V = g|_V$. However, in general, $\nabla^B$ is not compatible to $g$.

Let $(M^{m+d}, H^m, g_H; g)$ be a step-$r$ sub-Riemannian manifold with a rank $m$ subbundle $H$. For a smooth function $u$ on $M$, its horizontal gradient is the unique vector field $\nabla^H u$ satisfying
\[
\nabla^H u_q = \pi_H(\nabla^B u_q), \quad \forall q \in M.
\]

Let \{e_i\}_{i=1}^m and \{e_\alpha\}_{\alpha=m+1}^{m+d} be local orthonormal frame fields of the distributions $H$ and $V$ on an open domain $\Omega$ of $(M, g)$. As a result, (2.3)
\[
\nabla^H u = \sum_{i=1}^m (e_i u) e_i.
\]

Since $H$ has the bracket generating property for $TM$, $u$ is constant if and only if $\nabla^H u = 0$. The divergence of a vector field $X$ on $M$ is given by
\[
div_g X = \sum_{A=1}^{m+d} \{e_A \langle X, e_A \rangle - \langle X, \nabla^R_{e_A} e_A \rangle \}.
\]

Then one can define sub-Laplacian of a function $u$ on $(M, H, g_H; g)$ as (2.4)
\[
\Delta^H u = \div_g (\nabla^H u).
\]

By the divergence theorem, it is clear that $\Delta^H$ is symmetric, that is,
\[
\int_M v(\Delta^H u) \, dv_g = \int_M u(\Delta^H v) \, dv_g = -\int_M \langle \nabla^H u, \nabla^H v \rangle \, dv_g
\]
for any $u, v \in C_0^\infty(M)$. In terms of (2.3) and (2.4), we deduce that
\[
\Delta^H u = \sum_{i=1}^m \{e_i \langle \nabla^H u, e_i \rangle - \langle \nabla^H u, \nabla^B_{e_i} e_i \rangle \} - \langle \nabla^H u, \zeta \rangle
\]
\[
= \sum_{i=1}^m e_i^2(u) - \sum_{i=1}^m \nabla^B_{e_i} e_i + \zeta(u)
\]
(2.5)

where $\zeta = \pi_H(\sum_\alpha \nabla^R_{e_\alpha} e_\alpha)$ is referred to as the mean curvature vector field of the vertical distribution $V$.

Suppose $X_1, \cdots, X_m, Y$ are $C^\infty$ vector fields on a manifold $\tilde{M}$, whose commutators up to certain order span the tangent space at each point. The so-called Hörmander operator
\[
\mathcal{D} = \sum_{i=1}^m X_i^2 + Y
\]
were first studied by Hörmander in [Hö67]. He proved a celebrated result that $\mathcal{D}$ is hypoelliptic. In other words, if $u$ is a distribution defined on any open set $\Omega \subset \tilde{M}$, such that $\mathcal{D} u \in C^\infty(\Omega)$, then $u \in C^\infty(\Omega)$. In terms of (2.5), we find that $\Delta_H$ is an operator of Hörmander type, and hence it is hypoelliptic on $M$. Furthermore, the operator $\Delta_H - \frac{\partial}{\partial t}$ is hypoelliptic as well.

Define

$$S_k^p(\Delta_H, \Omega) = \{ u \in L^p(\Omega) | e_{i_1} \cdots e_{i_s}(u) \in L^p(\Omega), 1 \leq i_1, \cdots, i_s \leq m, 0 \leq s \leq k \}$$

and

$$S_k^p(\Delta_H - \frac{\partial}{\partial t}, \Omega \times [0, T)) = \{ u \in L^p(\Omega \times [0, T)) | \partial_t^1 e_{i_1} \cdots e_{i_s}(u) \in L^p(\Omega \times [0, T)), 1 \leq i_1, \cdots, i_s \leq m, 2l + s \leq k \}$$

for any nonnegative integer $k$. Due to regularity theory for hypoelliptic operators of Rothschild and Stein [RS76], we have the following

**Theorem 2.1.** Let $\mathcal{D} = \Delta_H$ (resp. $\Delta_H - \frac{\partial}{\partial t}$) and $\tilde{M} = \Omega$ (resp. $\Omega \times (0, T)$). Suppose $f \in L^p_{loc}(\tilde{M})$, and

$$\mathcal{D} f = g \quad \text{on} \quad \tilde{M}.$$  

If $g \in S_k^p(\mathcal{D}, \tilde{M})$, then $\chi f \in S_k^{p+2}(\mathcal{D}, \tilde{M})$ for any $\chi \in C^\infty_0(\tilde{M})$, $1 < p < \infty, k = 0, 1, 2, \ldots$. In particular, the following inequality holds

$$\| \chi f \|_{S_k^{p+2}(\mathcal{D}, \tilde{M})} \leq C_\chi(\| g \|_{S_k^p(\mathcal{D}, \tilde{M})} + \| f \|_{L^p(\tilde{M})})$$

where $C_\chi$ is a constant independent of $f$ and $g$.

To study the existence problem of subelliptic harmonic maps with potential, we also need some results about the heat kernel and subelliptic heat equation on compact sub-Riemannian manifolds. Let $K(x, y, t)$ be the heat kernel for $\Delta_H$ on a compact sub-Riemannian manifold $(M, H, g_H; g)$, that is

$$\begin{cases}
(\Delta_H - \frac{\partial}{\partial t}) K(x, y, t) = 0 \\
\lim_{t \to 0} K(x, y, t) = \delta_x(y)
\end{cases}$$

According to [Bau01], [Bau18a], [Bis84] and [Str86], we know that $K(x, y, t)$ exists. Some basic properties of $K(x, y, t)$ are listed as follows

1. $K(x, y, t) \in C^\infty(M \times M \times \mathbb{R}^+_0)$;
2. $K(x, y, t) = K(y, x, t)$ for $x, y \in M$ and $t > 0$;
3. $K(x, y, t) > 0$ for $x, y \in M$ and $t > 0$;
4. $\int_M K(x, y, t) dv_g = 1$ for any $x \in M$;
5. $K(x, y, t + s) = \int_M K(x, z, t) K(y, z, s) dv_g(z)$.

**Lemma 2.2.** ([Don21, Lemma 3.3]) For any $\beta \in (0, \frac{1}{2})$, there exists $C_\beta > 0$ such that

$$\int_0^t \int_M | \nabla^H_x K(x, y, s) | dv_g(y) ds \leq C_\beta t^\beta$$
for $0 < t < R_0$ for some positive constant $R_0$.

The following lemma gives both a maximum principle, and a mean value type inequality for subsolutions of the subelliptic heat equation.

**Lemma 2.3.** ([Don21, Lemma 3.4]) Let $M$ be a compact sub-Riemannian manifold. Suppose $\phi$ is a subsolution of the subelliptic heat equation satisfying
\[
(\Delta_H - \frac{\partial}{\partial t})\phi \geq 0
\]
on $M \times [0,T)$ with initial condition $\phi(x,0) = \phi_0(x)$ for any $x \in M$. Then
\[
\sup_M \phi(x,t) \leq \sup_M \phi_0(x).
\]
Furthermore, if $\phi(x,t)$ is nonnegative, then there exist a constant $B$ and an integer $Q$ such that
\[
\sup_{x \in M} \phi(x,t) \leq Bt^{-\frac{Q}{2}} \int_M \phi_0(y) \, dv(y)
\]
for $0 < t < min(R_0^2, T)$, where $R_0$ is as in Lemma 2.2.

Choose an adapted frame field $\{e_A\}_{A=1}^{m+d}$ in $(M,H,g_H;g)$, and denote its dual frame field by $\{\omega^A\}_{A=1}^{m+d}$. Henceforth, we will make use of the following convention on the ranges of indices in $M$:
\[
1 \leq A, B, C, \ldots, \leq m + d; \quad 1 \leq i, j, k, \ldots, \leq m;
\]
\[
m + 1 \leq \alpha, \beta, \gamma, \ldots, \leq m + d,
\]
and the Einstein summation convention.

Let $(N,h)$ be a Riemannian manifold with Riemannian connection $\tilde{\nabla}$. We choose an orthonormal frame field $\{\tilde{e}_I\}_{I=1}^{n}$ in $(N,h)$ and let $\{\tilde{\omega}^I\}$ be its dual frame field. We will make use of the following convention on the ranges of indices in $N$:
\[
I, J, K = 1, \ldots, n.
\]

For a smooth map $f : M \to N$, we have a connection $\nabla^B \otimes \tilde{\nabla}^f$ in $T^*M \otimes f^{-1}TN$, where $\tilde{\nabla}^f$ is the pull-back connection of $\tilde{\nabla}$. Then we can define the second fundamental form with respect to the data $(\nabla^B, \tilde{\nabla}^f)$ as follows
\[
\beta(f; \nabla^B, \tilde{\nabla})(X,Y) = \tilde{\nabla}^f_Y df(X) - df(\nabla^B_X Y).
\]
Using the frame fields in $M$ and $N$, the differential $df$ and the second fundamental form $\beta$ can be written as
\[
df = f^A_A \omega^A \otimes \tilde{e}_I,
\]
and
\[
\beta = f^A_{AB} \omega^A \otimes \omega^B \otimes \tilde{e}_I
\]
respectively. Apart from the differential $df$, we also consider two partial differentials $df_H = df|_H \in \Gamma(H^* \otimes f^{-1}TN)$ and $df_V = df|_V \in \Gamma(V^* \otimes f^{-1}TN)$. By definition, we get

$$|df_H|^2 = (f_H^I)^2, \quad |df_V|^2 = (f_V^I)^2, \quad |df|^2 = (f^A)^2.$$ 

Set

$$e_H(f) = \frac{1}{2}|df_H|^2, \quad e_V(f) = \frac{1}{2}|df_V|^2, \quad e(f) = \frac{1}{2}|df|^2.$$

Then we can define the following two partial energies:

$$E_H(f) = \int_M e_H(f) \, dv_g = \frac{1}{2} \int_M \langle df(e_i), df(e_i) \rangle \, dv_g,$n

$$E_V(f) = \int_M e_V(f) \, dv_g = \frac{1}{2} \int_M \langle df(e_\alpha), df(e_\alpha) \rangle \, dv_g.$$

The partial energies $E_H(f)$ and $E_V(f)$ are referred to as horizontal and vertical energies respectively. Obviously the usual Dirichlet energy $E(f)$ satisfies

$$E(f) = E_H(f) + E_V(f).$$

For any potential function $G \in C^\infty(N)$, we introduce the following energies:

$$E_P(f) = \int_M -G(f) \, dv_g,$n

$$E_G(f) = E_H(f) + E_P(f) = \int_M [e_H(f) - G(f)] \, dv_g.$$

We call energies $E_P(f)$ and $E_G(f)$ potential energy and horizontal energy with potential $G$ respectively.

**Definition 2.4.** A map $f : (M, H, g_H; g) \to (N, h)$ is called a subelliptic harmonic map with potential $G$ if it is a critical point of the energy $E_G(f)$.

For the purpose of deriving the Euler-Lagrange equation for $E_G$, we study a variation of $\{f_t\}_{t < \epsilon}$, which is a family of maps from $(M, H, g_H; g)$ to $(N, h)$ with $f_0 = f$ and $\frac{df_0}{dt} |_{t=0} = \nu \in \Gamma(f^{-1}TN)$. Since $\tilde{\nabla}$ is torsion-free, using divergence theorem on the compact manifold $M$, a computation gives

\begin{equation}
\frac{d}{dt} E_G(f_t) |_{t=0} = \int_M \langle \tilde{\nabla} f_t^i \nu, \, df_t(e_i) \rangle \, dv_g |_{t=0} - \int_M \langle \nu, \tilde{\nabla} G(f) \rangle \, dv_g \\
= \int_M \langle \tilde{\nabla} \nu, \, df(e_i) \rangle \, dv_g - \int_M \langle \nu, \tilde{\nabla} G(f) \rangle \, dv_g \\
= \int_M \langle \nu, \, df(\zeta) \rangle \, dv_g - \int_M \langle \nu, \beta(e_i, e_i) \rangle \, dv_g - \int_M \langle \nu, \tilde{\nabla} G(f) \rangle \, dv_g \\
= \int_M \langle \nu, \tau(f) \rangle \, dv_g,
\end{equation}

where

$$\tau(f) = \tau_H(f) + \tilde{\nabla} G = \beta(e_i, e_i) - df(\zeta) + \tilde{\nabla} G.$$
is called the subelliptic tension field of $f$ with potential $G$. Consequently, we have the following equivalent characterization of subelliptic harmonic maps with potential $G$.

**Proposition 2.5.** A map $f : (M, H, g_H; G) \to (N, h)$ is a subelliptic harmonic map with potential $G$ if and only if it satisfies the Euler-Lagrange equation

$$
\tau(f) = 0.
$$

In order to solve (2.7), we follow Eells-Sampson’s idea to deform a given smooth map $\bar{u} : M \to N$ along the gradient flow of the energy $E_G$. This is equivalent to solving the following subelliptic heat flow with potential $G$

$$
\begin{aligned}
\frac{\partial f}{\partial t} &= \tau(f) \\
\left. f \right|_{t=0} &= \bar{u}
\end{aligned}
$$

where $\tau(f(\cdot, t))$ is the subelliptic tension field with potential $G$ of $f(\cdot, t) : (M, H, g_H; G) \to (N, h)$.

Before proving the existence theory, we intend to give the explicit formu-

lations for both (2.7) and (2.8). According to the Nash embedding theorem, the isometric embedding $\mathfrak{J} : (N, h, \nabla) \to (\mathbb{R}^K, g_{E}, D)$ in some Euclidean space always exists, where $g_{E}$ denotes the standard Euclidean metric, $\nabla$ and $D$ are the Riemannian connections of $(N, h)$ and $(\mathbb{R}^K, g_{E})$ respectively. We also suppose that the potential $G$ is the restriction of some smooth function $G : \mathbb{R}^K \to \mathbb{R}$ to $N$. Applying the composition formula for second fundamental forms [EL83, page 14] to the maps $f : (M, \nabla_B) \to (N, \nabla)$ and $\mathfrak{J} : (N, \nabla) \to (\mathbb{R}^K, D)$, we have

$$
\beta(\mathfrak{J} \circ f ; \nabla_B, D)(\cdot, \cdot) = d\mathfrak{J}(\beta(f ; \nabla_B, \nabla)(\cdot, \cdot)) + \beta(\mathfrak{J} ; \nabla, D)(df(\cdot), df(\cdot)).
$$

To simplify the notation, we identify $N$ with $\mathfrak{J}(N)$, and denote $\mathfrak{J} \circ f$ by $u$. Note that $u$ is a map from $M$ to $\mathbb{R}^K$. Then, we get

$$
\tau_H(u; \nabla_B, D) - \text{tr}_D \beta(\mathfrak{J} ; \nabla, D)(df_H, df_H) = d\mathfrak{J} (\tau_H(f)).
$$

For the submanifold $N$, there exists a tubular neighborhood $B(N)$ of $N$ in $\mathbb{R}^K$ and a natural projection map $\Pi : B(N) \to N$ which is a submersion over $N$. In fact, the map $\Pi$ maps any point in $B(N)$ to its closest point in $N$. Obviously its differential $d\Pi : T_y \mathbb{R}^K \to T_{\Pi(y)} \mathbb{R}^K$ at a point $y \in N$ is given by the identity map when restricted to the tangent space $TN$ of $N$ and maps all the normal vectors to $N$ to the zero vector. Since $\Pi \circ \mathfrak{J} = \mathfrak{J} : N \hookrightarrow \mathbb{R}^K$ and $\beta(\mathfrak{J} ; \nabla, D)$ is normal to $N$, we have

$$
\beta(\mathfrak{J} ; \nabla, D) = \beta(\Pi ; D, D)(d\mathfrak{J}, d\mathfrak{J}).
$$

Choose the natural Euclidean coordinate system $\{y^a\}_{1 \leq a \leq K}$ and set $u^a = y^a \circ u, \Pi^a = y^a \circ \Pi$. According to the above argument, we obtain

$$
\tau_H(u; \nabla_B, D) = \Delta_H u^a \frac{\partial}{\partial y^a}
$$

where $\Delta_H$ is the subelliptic Laplace-Beltrami operator.
and
\[ \text{tr}_g \beta(\mathcal{J}; \nabla, D)(df_H, df_H) = \text{tr}_g \beta(\Pi; D, D)(du_H, du_H) \]
\[ = \Pi^a_{bc} \langle \nabla^H u^b, \nabla^H u^c \rangle \frac{\partial}{\partial y^a} \]
(2.11)

where \( \Pi^a_{bc} = \frac{\partial^2 \Pi^a}{\partial y^b \partial y^c} \). Note that
\[ d\mathcal{J}(\tau(f)) = d\mathcal{J}(\tau_H(f)) + d\Pi D(G). \]

Consequently, (2.9), (2.10), (2.11), (2.12) show that
\[ \Delta_H u^a - \Pi^a_{bc} \langle \nabla^H u^b, \nabla^H u^c \rangle + \Pi^a_b (D \tilde{G}^b) = 0, \]
(2.13)

Thereafter, \( f \) is a subelliptic harmonic map with potential \( G \) if and only if \( u = (u^a) : M \to \mathbb{R}^K \) solves
\[ \Delta_H u^a - \Pi^a_{bc} \langle \nabla^H u^b, \nabla^H u^c \rangle + \Pi^a_b (D \tilde{G}^b) = 0, \]
for second fundamental forms \cite{Eells} and equation (2.14), we have
\[ \frac{\partial}{\partial y^a} \]
(2.14)

whenever \( \text{Imu} \subset B(N) \) for \( t \in [0, T) \), where \( \bar{u}^a = u^a \circ \bar{u} \).

To this end, we need to show \( u(x, t) \in N \) for all \( (x, t) \in M \times [0, T) \), where \( [0, T) \) is the maximal existence domain of \( u \). Define a map \( \rho : B(N) \to \mathbb{R}^K \) by
\[ \rho(y) = y - \Pi(y), \quad y \in B(N). \]

It is easy to see \( \rho(y) \) is normal to \( N \) and \( \rho(y) = 0 \) is equivalent to \( y \in N \). Differentiating both sides of (2.15) twice, we have
\[ \rho^a_b = \delta^a_b - \Pi^a_b \]
and
\[ \rho^a_{bc} = -\Pi^a_{bc} \]
where \( \rho^a_b = \frac{\partial u^a}{\partial y^b}, \Pi^a_b = \frac{\partial \Pi}{\partial y^a \partial y^b} \) and \( \rho^a_{bc} = \frac{\partial^2 \Pi^a}{\partial y^b \partial y^c} \). Using the composition formula for second fundamental forms \cite{Eells} and equation (2.14), we have
\[ (\Delta_H \rho^a(u))^a = \rho^a_b \Delta_H u^b + \rho^a_{bc} \langle \nabla^H u^b, \nabla^H u^c \rangle \]
\[ = \Delta_H u^a - \Pi^a_{bc} \langle \nabla^H u^b, \nabla^H u^c \rangle \]
\[ = \rho^a_b \frac{\partial u^b}{\partial t} + \Pi^a_{bc} \frac{\partial u^b}{\partial t} - \Delta_H u^b + D \tilde{G}^b. \]

Note that \( d\Pi \frac{\partial}{\partial y^a} + \Delta_H u^b + D \tilde{G}^a \) is tangent to \( N \) and \( \rho(u) \) is normal to \( N \), then from (2.16) we obtain
\[ \rho^a(u)(\Delta_H \rho(u))^a = \rho^a(u)\rho^a(u) \frac{\partial u^b}{\partial t}. \]
In terms of divergence theorem and (2.16), we deduce that
\[
\frac{\partial}{\partial t} \int_M (\rho^a(u))^2 \, dv_g = 2 \int_M \rho^a(u) \rho_b^a(u) \frac{\partial u^b}{\partial t} \, dv_g
\]
\[
= 2 \int_M \rho^a(u) (\Delta_H \rho(u))^a \, dv_g
\]
\[
= -2 \int_M |\nabla_H \rho(u)|^2 \, dv_g
\]
\[
\leq 0.
\]
Thus,
\[
\int_M |\rho(u(x,t))|^2 \, dv_g
\]
is non-increasing in \(t\). In particular, if \(\rho(u(x,0)) = 0\), then \(\rho(u(x,t)) = 0\) for all \(t \in [0,T]\), which means \(u(x,t) \in N\) for all \((x,t) \in M \times [0,T]\).

3. Short Time Existence

In this section, we will establish the short-time existence result of (1.3). To achieve this, we need the following Bochner type inequality for \(e(f)\).

Lemma 3.1. Let \((M, H, g_H; g)\) be a compact sub-Riemannian manifold and let \((N, h)\) be a Riemannian manifold with non-positive sectional curvature. Suppose that \(f : M \to N\) is a smooth map. Set \(\tau^I_H = f^I_{kk} - \zeta^k f^I_k\) and \(\tau^I = \tau^I_H + [(\tilde{\nabla} G)(f)]^I\). Then one has

\[
\Delta_H e(f) - f^I_i \tau^I_i - f^I_\alpha \tau^I_\alpha \geq -C_\epsilon e_H(f) - \epsilon e_V(f) + (f^I_k)^2
\]
\[
+ \frac{1}{2} (f^I_{ak})^2 - \text{Hess} G(f_i, f_i) - \text{Hess} G(f_\alpha, f_\alpha)
\]
for any given \(\epsilon > 0\), where \(C_\epsilon\) is a positive number depending only on \(\epsilon\) and

\[
\sup_{M, i, j, k, \alpha} \{ |\zeta^k_i|, |\zeta^k_i|, |\zeta^k_\alpha|, |T^\alpha_{ij}|, |T^\alpha_{ij,k}|, |R^j_{kik}|, |R^j_{kak}| \}.
\]

In particular, we have

\[
\Delta_H e(f) - f^I_i \tau^I_i - f^I_\alpha \tau^I_\alpha \geq -C_\epsilon e(f) - \text{Hess} G(f_i, f_i) - \text{Hess} G(f_\alpha, f_\alpha).
\]

Proof of Lemma 3.1. Let \(T = \frac{1}{2} (T^A_{BC} \omega^B \wedge \omega^C) \otimes e_A\) and \(\Omega^A_B = \frac{1}{2} R^A_{BCD} \omega^C \wedge \omega^D\) be the torsion and curvature of \(\nabla^B\) respectively and let \(\tilde{\Omega}^A_J = \frac{1}{2} R^A_{JKL} \tilde{\omega}^K \wedge \tilde{\omega}^L\) be the curvature of \(\tilde{\nabla}\). Denote covariant derivative of \(\zeta^k\) and \(T^A_{BC}\) by \(\zeta^k_i, A\) and \(T^A_{BC,i}\).
and \( T^A_{BC,D} \) respectively. From \cite{Don21}, we know
\[
\Delta_H e(f) = \Delta_H e_H(f) + \Delta_H e_V(f)
\]
\[
= (f^I_{ik})^2 + f^I_{i} f^I_{ik} - \zeta^I f^I_{i} f^I_{ik} + (f^I_{\alpha k})^2 + f^I_{\alpha} f^I_{\alpha k k}
\]
\[
= (f^I_{ik})^2 + (f^I_{\alpha k})^2 + f^I_{i} \tau^I_{H,i} + f^I_{\alpha} \tau^I_{H,\alpha} + f^I_{i} \zeta^I_{k} f^I_{ik}
\]
\[
+ \zeta^I f^I_{i} f^I_{\alpha} T^\alpha_{ki} + f^I_{i} f^I_{j} R^i_{kik} + 2 f^I_{i} f^I_{\alpha} T^\alpha_{ik} - f^I_{k} f^I_{\alpha} \tilde{R}^I_{KJL} f^J_{i} f^L_{k}
\]
\[
+ f^I_{i} f^I_{\alpha} \tilde{R}^I_{ik,k} + f^I_{i} \zeta^I_{k} f^I_{i} + f^I_{\alpha} f^I_{j} R^i_{kak} - f^I_{k} f^I_{\alpha} \tilde{R}^I_{KJL} f^J_{i} f^L_{k}.
\]
Since \( \tau^I = \tau^I_{H} + [(\nabla G)(f)]^I \), then
\[
\Delta_H e(f) = (f^I_{ik})^2 + (f^I_{\alpha k})^2 + f^I_{i} \tau^I_{i} + f^I_{\alpha} \tau^I_{\alpha} - f^I_{i} G_{IJ} f^J_{i} - f^I_{\alpha} G_{IJ} f^J_{\alpha} + f^I_{i} \zeta^I_{k} f^I_{ik}
\]
\[
+ \zeta^I f^I_{i} f^I_{\alpha} T^\alpha_{ki} + f^I_{i} f^I_{j} R^i_{kik} + 2 f^I_{i} f^I_{\alpha} T^\alpha_{ik} - f^I_{k} f^I_{\alpha} \tilde{R}^I_{KJL} f^J_{i} f^L_{k}
\]
\[
+ f^I_{i} f^I_{\alpha} \tilde{R}^I_{ik,k} + f^I_{i} \zeta^I_{k} f^I_{i} + f^I_{\alpha} f^I_{j} R^i_{kak} - f^I_{k} f^I_{\alpha} \tilde{R}^I_{KJL} f^J_{i} f^L_{k}.
\]
where \( \text{Hess } G = (G_{IJ}) \). Using Schwarz inequality and curvature assumption of \( N \), we have that
\[
f^I_{i} \zeta^I_{k} f^I_{ik} + f^I_{i} f^I_{j} R^i_{kik} \geq -C_{1} e_{H}(f),
\]
\[
\zeta^I f^I_{i} f^I_{\alpha} T^\alpha_{ki} + f^I_{i} f^I_{j} R^i_{kik} + f^I_{i} \zeta^I_{k} f^I_{ik} + f^I_{\alpha} f^I_{j} R^i_{kak} \geq -C_{2}(\varepsilon) e_{H}(f) - \varepsilon e_{V}(f),
\]
\[
2 f^I_{i} f^I_{\alpha} T^\alpha_{ik} \geq -C_{3} e_{H}(f) - \frac{1}{2}(f^I_{\alpha k})^2,
\]
\[
f^I_{k} f^I_{\alpha} \tilde{R}^I_{KJL} f^J_{i} f^L_{k} + f^I_{i} \zeta^I_{k} f^I_{ik} + f^I_{\alpha} f^I_{j} R^i_{kak} \leq 0.
\]
These estimates give \cite{372}. \( \square \)

**Proposition 3.2.** Let \((M^{m+d}, H, g_H; g)\) be a compact sub-Riemannian manifold, and \((N, h)\) be a compact Riemannian manifold. The heat flow \((\ref{Eq:heat_flow})\) admits a unique smooth solution defined on a maximal existence domain \(M \times [0, T)\).

**Proof of Proposition 3.2.** Writing \( u = (u^a(x,t))_{1 \leq a \leq K} \), the subelliptic harmonic map heat flow with potential \( G \) becomes
\[
\begin{cases}
(\Delta_H - \frac{\partial}{\partial t}) u = F(x,t) \\
u(x,0) = \bar{u}(x)
\end{cases}
\]

where \( F(x,t) = \{ \Pi^a_{\beta}(\nabla^b H u^b, \nabla^c H u^c) - \Pi^a_{\beta}(D G^b(u)) \} \).

By Duhamel’s principle, a sequence of approximate solutions can be defined inductively as follows:
\[
u_0(x,t) = \int_M K(x,y,t) \bar{u}(y) \, dv_g(y)
\]
\[
u_k(x,t) = \nu_0(x,t) - \int_0^t \int_M K(x,y,t-s) F_{k-1}(y,s) \, dv_g(y) \, ds
\]

(3.3)
where
\[(3.4)\]
\[F_{k-1}(y, s) = \left( \Pi_{bc}^{a} (\nabla^{H} u_{k-1}^{b}, \nabla^{H} u_{k-1}^{c}) - \Pi_{bc}^{a} (D G^{b}(u_{k-1})) \right), \quad k \geq 1.\]

It is clear that \(u_{0}\) and \(u_{k} : M \to \mathbb{R}^{K}\) solve respectively
\[
\begin{align*}
\begin{cases}
(\Delta_{H} - \frac{\partial}{\partial t}) u_{0} = 0 \\
u_{0}(x, 0) = \bar{u}(x)
\end{cases}
\end{align*}
\]
and
\[
\begin{align*}
\begin{cases}
(\Delta_{H} - \frac{\partial}{\partial t}) u_{k} = F_{k-1}(x, t) \\
u_{k}(x, 0) = \bar{u}(x), \quad k \geq 1.
\end{cases}
\end{align*}
\]
Set
\[
\Lambda = \sup_{B(N), a, b, c, d} \{ |\Pi_{a}^{b}|, |\Pi_{bc}^{a}|, |\frac{\partial \Pi_{bc}^{a}}{\partial y^{d}}| \}
\]
\[
P = \sup_{B(N), I, J} \{ |D \bar{G}|, |\bar{G}_{IJ}| \},
\]
where \((y^{1}, \ldots, y^{K})\) are coordinates of \(\mathbb{R}^{K}\), \(B(N)\) is the tubular neighborhood of \(N\) and \(\Pi\) is the closest point map over \(B(N)\). We denote
\[(3.5)\]
\[p_{k-1}(t) = \sup_{M \times [0, t]} \sqrt{e_{H}(u_{k-1})}, \quad k \geq 1,
\]
which is a non-decreasing function of \(t\). From (3.4) and (3.5), we obtain
\[(3.6)\]
\[
\sup_{M \times [0, t]} |F_{k-1}(x, s)| \leq \Lambda(p_{k-1}^{2} + P).
\]
Since \(\int_{M} K(x, y, t) dy = 1\), we find that
\[(3.7)\]
\[|u_{0}| \leq \|\bar{u}\|_{C^{0}} = \sup_{x \in M} \sqrt{\sum_{a=1}^{K} (\bar{u}^{a}(x))^{2}},
\]
where \(\|\cdot\|_{C^{0}}\) denotes the \(C^{0}\)-norm of functions or tensor fields on \(M\). Using (3.3), (3.4), (3.6) and (3.7), we deduce that
\[(3.8)\]
\[|u_{k} - u_{0}| \leq \Lambda(p_{k-1}^{2} + P)t
\]
\[|u_{k}| \leq \Lambda t(p_{k-1}^{2} + P) + \|\bar{u}\|_{C^{0}}.
\]
Note that \(\tau_{H}(u_{0}) = \Delta_{H} u_{0}\) for the map \(u_{0} : M \to \mathbb{R}^{K}\). By Lemma 3.1 we derive that
\[
(\Delta_{H} - \frac{\partial}{\partial t})(e^{-Ct} e(u_{0})) \geq 0.
\]
As a result, Lemma 2.3 gives that
\[e^{-Ct} e(u_{0}) \leq e(\bar{u})\]
and thus

\[(3.9) \quad p_0(t) \leq \sqrt{e^{C_1e(u)}}.\]

In view of Lemma 2.2, (3.3) and (3.6), we have

\[|\nabla_H^x u_k(x,t)| \leq |\nabla_H^x u_0| + \int_0^t \int_M |\nabla_H^x K(x,y,t-s)| \cdot |F_{k-1}(y,s)| \, dv_g(y) \]
\[\leq |\nabla_H^x u_0| + C_1 \Lambda (p_{k-1}^2(t) + P) t^\beta,\]

which yields

\[(3.10) \quad p_k(t) \leq C_1 \Lambda (p_{k-1}^2(t) + P) t^\beta + p_0(t)\]
\[\leq C_1 \Lambda (p_{k-1}(t) + \sqrt{P})^2 t^\beta + p_0(t).\]

Choosing \(\delta\) sufficiently small, it follows from (3.9) that

\[C_1 \Lambda \delta^\beta (p_0(\delta) + \sqrt{P}) \leq C_1 \Lambda \delta^\beta \left(\sqrt{e^{C_1\delta e(u)}} + \sqrt{P}\right) \leq \frac{\epsilon}{4},\]

for any \(0 < \epsilon < 1\). In terms of (3.10), we get inductively

\[C_1 \Lambda \delta^\beta (p_k(\delta) + \sqrt{P}) \leq \left(C_1 \Lambda \delta^\beta (p_{k-1}(\delta) + \sqrt{P})\right)^2 + C_1 \Lambda \delta^\beta (p_0(\delta) + \sqrt{P})\]
\[\leq \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2},\]

and thus

\[(3.11) \quad C_1 \Lambda \delta^\beta p_k(\delta) \leq C_1 \Lambda \delta^\beta (p_k(\delta) + \sqrt{P}) \leq \frac{\epsilon}{2}.\]

Consequently

\[(3.12) \quad p_k(\delta) \leq C_2 \epsilon \delta^{-\beta}.\]

Let us introduce the following space of functions

\[C^1_H(M, \mathbb{R}^K) = \{f : M \to \mathbb{R}^K | f \in C^0, \nabla^H f \in C^0\},\]

which is equipped with the norm

\[\|f\|_{C^1_H} = \|f\|_{C^0} + \|\nabla^H f\|_{C^0}.\]

It is a fact that \((C^1_H(M, \mathbb{R}^K), \|\cdot\|_{C^1_H})\) is a Banach space. Combining (3.8) with (3.12), one gets

\[\|u_k\|_{C^1_H(M, \mathbb{R}^K)} \leq C_3(C_2, \epsilon, \delta).\]

In terms of (3.8) on \(M \times [0, \delta)\) and using (3.11), one has

\[(3.13) \quad |u_k(x,t) - u_0(x,t)| \leq \Lambda \delta (p_{k-1}^2 + P)(\delta) \leq \Lambda \delta (p_{k-1} + \sqrt{P})^2(\delta) \leq \frac{\epsilon^2 \delta^{1-2\beta}}{4C_1\Lambda}.\]

If we choose a sufficiently small \(\delta\), then the inequality (3.11) is valid. We notice that \(1 - 2\beta > 0\) (see Lemma 2.2). Hence (3.13) suggests that all maps \(u_k (k \geq 1)\) will map \(M\) into \(B(N)\) by choosing \(\delta\) sufficiently small.
since \(|u_0(x, t) - \bar{u}(x)|\) can be chosen to be sufficiently small for small \(t\), owing to continuity of \(u_0\).

The proof of the theorem will be complete, if we show that \(\{u_k(x, t)\}\) is a Cauchy sequence in \(C^1_H(M, \mathbb{R}^k)\), in the case that \(t\) is sufficiently small. To this end, we investigate the following sequence

\[
X_k(t) = \sup_{M \times [0, t]} \{|u_k(x, s) - u_{k-1}(x, s)| + \|\nabla_x u_k(x, s) - \nabla_x u_{k-1}(x, s)\|\}
\]

which is non-decreasing in \(t\). Note that

\[
F_k(x, t) - F_{k-1}(x, t) = \left(\Pi^a_{bc}(u_k) - \Pi^a_{bc}(u_{k-1})\right)\langle \nabla^H u^b_k, \nabla^H u^c_k \rangle
\]

\[
+ \Pi^a_{bc}(u_{k-1})\langle \nabla^H u^b_k - \nabla^H u^b_{k-1}, \nabla^H u^c_k \rangle
\]

\[
+ \Pi^a_{bc}(u_{k-1})\langle \nabla^H u^b_{k-1}, \nabla^H u^c_k - \nabla^H u^c_{k-1} \rangle
\]

\[
+ \Pi^a_{bc}(D \nabla^b(u_k) - D \nabla^b(u_{k-1}))
\]

Using (3.12) and the following estimates

\[
|\Pi^a_{bc}(u_k) - \Pi^a_{bc}(u_{k-1})| \leq \Lambda|u_k - u_{k-1}|;
\]

\[
|D \nabla G(u_k) - D \nabla G(u_{k-1})| \leq P|u_k - u_{k-1}|
\]

we find that

\[
\sup_{M \times [0, t]} |F_k(x, t) - F_{k-1}(x, t)| \leq C_4 X_k(t)(p^2_k + p_k(t) + p_{k-1}(t)) \leq C_5 X_k(t)
\]

for any \(t \leq \delta\). As a result, we get

\[
|u_k - u_{k-1}| \leq \int_0^t \int_M K(x, y, t - s)|F_{k-1}(y, s) - F_{k-2}(y, s)| \, dv_y(s) \, ds \leq C_5 t X_{k-1}(t)
\]

and

\[
|\nabla^H_x u_k - \nabla^H_x u_{k-1}|
\]

\[
\leq \int_0^t \int_M |\nabla^H_x K(x, y, t - s)| \cdot |F_{k-1}(y, s) - F_{k-2}(y, s)| \, dv_y(s) \, ds \leq C_6 t^\beta X_{k-1}(t),
\]

which yield

\[
(3.14) \quad X_k(t) \leq C_7 t^{\beta} X_{k-1}(t)
\]

for \(k \geq 2\). For \(k = 1\), using \(t < 1\), we obtain from (3.3) and (3.9) that

\[
|u_1(x, t) - u_0(x, t)| \leq \int_0^t \int_M K(x, y, t - s)|F_0(y, s)| \, dv_y(s) \, ds \leq t \Lambda p^2_0(t) \leq t \Lambda(e^C e(\bar{u}) + P)
\]
and
\[ |\nabla^H_x u_1(x, t) - \nabla^H_x u_0(x, t)| \leq \int_0^t \int_M |\nabla^H_x K(x, y, t - s)| \cdot |F_0(y, s)| \, dv_y(y) \, ds \]
\[ \leq C_1 t^\beta \Lambda p^0_\beta(t) \leq C_1 t^\beta \Lambda(e^C e(\bar{u}) + P). \]

It follows that
\[ (3.15) \quad X_1(t) \leq C_8(C_7t^\beta)(e(\bar{u}) + P). \]

By iterating (3.14) and using (3.15), we have
\[ (3.16) \quad X_k(t) \leq C_8(C_7t^\beta)^k(e(\bar{u}) + P). \]

Choosing \( \delta_0 \) sufficiently small such that \( 0 < \delta_0 \leq \delta \) and \( C_7\delta^\beta < 1 \), then (3.16) yields that for any \( i \leq j \)
\[ \sup_{[0, \delta_0]} \|u_i(\cdot, t) - u_j(\cdot, t)\|_{C^1_H(M)} \leq \sum_{k=i+1}^{j} X_k(\delta_0) \leq C_9 \sum_{k=i+1}^{j} (C_7\delta^\beta)^k, \]
which vanishes as \( i, j \to \infty \). Therefore there exists \( u \in C^0(M \times [0, \delta_0], B(N)) \) with \( u(\cdot, t) \in C^1_H(M, B(N)) \) for each \( t \in [0, \delta_0] \), such that \( u_k \to u \) and \( \nabla^H u_k \to \nabla^H u \) uniformly on \( M \times [0, \delta_0] \). Thus
\[ F_k(x, t) \to F(x, t) = \Pi_{\overline{bc}}(u)\langle \nabla^H u^b, \nabla^H u^c \rangle - \Pi_{\overline{bc}}(D \tilde{G}(u)) \]
and hence (3.3) implies that \( u \) is given by
\[ u(x, t) = \int_M K(x, y, t) \varphi(y) \, dv_y(y) - \int_0^t \int_M K(x, y, t - s) F(y, s) \, dv_y(y) \, ds. \]

Clearly \( u \) solves the subelliptic harmonic map heat flow with potential weakly. In view of Theorem 2.1 and by a bootstrapping argument, we find that \( u \in C^\infty(M \times (0, \delta_0), N) \) solves (1.3).

Next we will prove the solution of (1.3) is unique. Let \( u \) and \( v \) be solutions on \( M \times [0, \delta) \) to (2.14) with the same initial condition: \( u(x, 0) = v(x, 0) = \bar{u} \). Set \( \Psi = \sum_{a=1}^K (u^a - v^o)^2 \), a similar computation as in the proof of [Don21, Theorem 6.2] shows
\[ (\Delta_H - \frac{\partial}{\partial t})(e^{-\tilde{C}t}\Psi) \geq 0, \]
where \( \tilde{C}(\delta_0, \Lambda, P, e_H(u), e_H(v)) \) is a positive constant. Then the uniqueness follows immediately from Lemma 2.3.

**Remark 3.3.** When \( N \) is complete but not necessarily compact, there exists an open neighborhood \( N' \) of \( \bar{u}(M) \) with compact closure so that \( N' \) can be embedded into \( \mathbb{R}^K \) isometrically, since \( \bar{u} \) is smooth and \( M \) is compact. If necessary, by choosing a smaller neighborhood, we may assume that there exists a bounded tubular neighborhood \( \tilde{N} \) of \( N' \) in \( \mathbb{R}^K \) and the nearest point projection \( \Pi: \tilde{N} \to N \) can be extended smoothly to the whole \( \mathbb{R}^K \) so that each \( \Pi^a \) is compactly supported. Also, \( \tilde{G}|_{\tilde{N}} \), which is the restriction of \( G \) to
\( \tilde{N} \), can be extended smoothly to a smooth function with compact support on \( \mathbb{R}^K \), which we still denote by \( \bar{G} \) for simplicity. Set
\[
\Lambda = \sup_{\mathbb{R}^K, a,b,c,d} \{ |\Pi^a_b|, |\Pi^a_{bc}|, |\frac{\partial \Pi^a_{bc}}{\partial y^d}| \}
\]
\[
P = \sup_{\mathbb{R}^K, I,J} \{ |D \bar{G}|, |\bar{G}_{IJ}| \},
\]
then \( \Lambda \) and \( P \) are bounded. Constructing a sequence of approximate solutions \( u_k \) as we do in the proof of Proposition 3.2, we can get
\[
|u_k(x,t) - u_0(x,t)| \leq \frac{\epsilon^2 \delta^{1-2\beta}}{4C_1^1 \Lambda}
\]
which implies that all maps \( u_k(k \geq 1) \) map \( M \) into \( \tilde{N} \) for \( t \in [0, \delta) \) by choosing \( \delta \) sufficiently small. By showing that \( \{u_k(x,t)\} \) is a Cauchy sequence in \( C^1_H(M, \mathbb{R}^K) \), we can establish the short-time existence result of (1.3) in the case that \( N \) is complete. (cf. [LT91] for a similar discussion for harmonic map heat flows.)

4. LONG TIME EXISTENCE

In this section, we will apply a standard method to prove the long time existence.

Proof of Theorem 1.1. First, we assume that \( N \) is a complete manifold and \( \text{Hess} G \leq C \). The short time existence of solution of (1.3) is a direct consequence of Proposition 3.2 and Remark 3.3. Let \( u(x,t) \) be the solution, and let \( [0,T) \) be its maximal existence domain. Suppose \( T < +\infty \), we want to show:

\[
\begin{align*}
(i) \quad |\frac{\partial u(t)}{\partial t}| & \leq C(T), \\
(ii) \quad |d u(t)| & \leq C(T), \\
(iii) \quad d_N(u(t), \bar{u}) & \leq C(T),
\end{align*}
\]

for some finite number \( C(T) \) on \([0, T)\), where \( d_N \) is the Riemannian distance function on \((N, h)\).

From [Don21, Section 4], we know
\[
u_{At} = u_{tA}
\]
and
\[
u_{AtB} - u_{ABt} = -u_A^K \tilde{R}_{KJL}^I f^I_J f^L_B.
\]
To prove (4.1), a simple computation gives

\begin{equation}
(\frac{\partial}{\partial t} - \Delta_H) \frac{\partial u(t)}{\partial t} = -2|V^H \frac{\partial u(t)}{\partial t}|^2 - 2(\frac{\partial u(t)}{\partial t})_{kk} \frac{\partial u(t)}{\partial t} - 2\zeta^k (\frac{\partial u(t)}{\partial t})_k \frac{\partial u(t)}{\partial t} \\
+ 2 \frac{\partial u(t)}{\partial t} \frac{\partial^2 u(t)}{\partial t^2} \\
= -2|V^H \frac{\partial u(t)}{\partial t}|^2 + 2 \frac{\partial u(t)}{\partial t} \frac{\partial^{\tau_H}(u)}{\partial t} + 2 \frac{\partial u(t)}{\partial t} \frac{\partial^2 u(t)}{\partial t^2} \\
+ 2(R_{\text{Riem}}(du(t)(e_i), \frac{\partial u(t)}{\partial t}du(t)(e_i), \frac{\partial u(t)}{\partial t})).
\end{equation}

Since sectional curvature $R_{\text{Riem}} \leq 0$ and $\text{Hess } G \leq C \cdot h$, then for some constant $C_1 > 0$, we have

\begin{equation}
(\frac{\partial}{\partial t} - \Delta_H) \frac{\partial u(t)}{\partial t} \leq C_1 \frac{\partial u(t)}{\partial t}.
\end{equation}

From Lemma 2.3 we derive that

\begin{equation}
\sup_{x \in M} |\frac{\partial u}{\partial t}|^2(x, t) \leq \sup_{x \in M} e^{C_1t} |\frac{\partial u}{\partial t}|^2(x, 0) \leq \sup_{x \in M} e^{C_1T} |\frac{\partial u}{\partial t}|^2(x, 0),
\end{equation}

which proves (4.1)(i).

The estimate (4.1)(iii) follows from (4.4), since

\begin{equation}
d_N(u(t), \tilde{u}) \leq \int_0^t |\frac{\partial u(s)}{\partial s}| ds \leq \sup_{x \in M} |\frac{\partial u(t)}{\partial t}| \int_0^t e^{C_1} ds \leq \frac{2}{C_1} e^{\frac{C_1}{2}T} \sup_{x \in M} |\frac{\partial u(0)}{\partial t}|.
\end{equation}

Finally, using Lemma 3.1 we may deduce

\begin{equation}
(\frac{\partial}{\partial t} - \Delta_H)e(u(t)) \leq C_0 e(u(t)) + \text{Hess } G(u_i(t), u_i(t)) + \text{Hess } G(u_o(t), u_o(t)).
\end{equation}

Since $\text{Hess } G \leq C \cdot h$, we see that

\begin{equation}
(\frac{\partial}{\partial t} - \Delta_H)e(u(t)) \leq C_2 e(u(t)),
\end{equation}

for some constant $C_2 > 0$. By Lemma 2.3 we get

\begin{equation}
\sup_{x \in M} e(u(x, t)) \leq \sup_{x \in M} e^{C_1t} e(u(x, 0)) \leq \sup_{x \in M} e^{C_1T} e(u(x, 0)).
\end{equation}

The proof of (4.1) is achieved. Since $\tilde{u}(M) \subset N$ is compact and (4.1)(iii), we find that, for any $t \in [0, T)$, $u(t)(M) \subset N'$, where $N' \subset N$ is compact. Consider a sequence $T_i \to T$. Taking $u(\cdot, T_i)$ as a initial map, we may find a solution $u$ of (4.5) on $[0, T + \delta')$ for some positive number $0 < \delta' < \delta$ by Proposition 3.2 if $i$ is sufficiently large.

From above discussion, it is easy to see that if $N$ is compact, then we have the long time existence for any $G \in C^\infty(N)$. \hfill \Box
Remark 4.1. If we take the potential function $G \equiv 0$, we get the following corollary immediately.

Corollary 4.2. Let $(M,H,g_{H};g)$ be a compact sub-Riemannian manifold and let $(N,h)$ be a complete Riemannian manifold with non-positive sectional curvature. Then for any smooth map $\bar{u} : M \to N$, the following heat flow

\[
\begin{cases}
\frac{\partial u}{\partial t} = \tau_H(u) \\
u|_{t=0} = \bar{u}
\end{cases}
\]

admits a unique smooth solution defined on $M \times [0, +\infty)$.

5. EELLS-SAMPSON TYPE THEOREM

In this section, we will establish the Eells-Sampson type theorem for the subelliptic harmonic map heat flow with potential $G$. The only obstacle is to prove the convergence of $u$ at infinity. In order to prove that, we first need to show that $e(u)$ is uniformly bounded with respect to $t \in [0, \infty)$. The next lemma tells us that it is sufficient to estimate the upper bound of $E(u)$.

Lemma 5.1. Let $f : M \to N$ be a solution of the subelliptic harmonic map heat flow with potential $G$ on $[0, \delta)$. Suppose $(N,h)$ has non-positive sectional curvature and $\text{Hess} G \leq C \cdot h$. Set $\alpha = \min\{R_0, \sqrt{\delta}\}$, where $R_0$ is given by Lemma 2.3. Then

\[e(f(\cdot, t)) \leq C(\epsilon_0)E(f(\cdot, t - \epsilon_0))\]

for $t \in [\epsilon_0, \delta)$, where $\epsilon_0$ is a fixed number in $(0, \frac{\alpha^2}{2})$.

Using (4.5), the proof of Lemma 5.1 is similar to [Don21, Lemma 6.4], so we omit it.

From (4.6), it follows that

\[\frac{d}{dt}E_G(u(\cdot, t)) = -\int_M |\tau(u(\cdot, t))|^2 dv_g \leq 0,\]

which implies $E_G(u(\cdot, t)) \leq E_G(\bar{u})$, where $u(\cdot, t)$ is a solution of (1.3). When $\text{Im} u \subset N'$, where $N'$ is a compact subset of $N$, we have $|E_P(u(\cdot, t))| < (\max_{N'}|G|)\text{vol}(M)$. Therefore it is enough to estimate $E_V(u(\cdot, t))$ for our purpose in the case that $\text{Im} u$ has compact closure $N'$.

Let: $S(V) \to M$ be the unit sphere bundle of the vertical bundle $V$. For any $v \in S(V)$, the $v$-component of $T(\cdot, \cdot)$ is given by $T^v(\cdot, \cdot) = \langle T(\cdot, \cdot), v \rangle$. Then we have a smooth function $\eta(v) = \frac{1}{2}\|T^v\|_g^2 : S(V) \to \mathbb{R}$, given by

\[
\eta(v) = \sum_{1 \leq i \leq j \leq m} (T^v_{ij})^2 \langle e_i, v \rangle^2
\]

\[
= \sum_{1 \leq i \leq j \leq m} \langle [e_i, e_j], v \rangle^2.
\]

Lemma 5.2. ([Don21 Lemma 6.6]) $H$ is 2-step bracket generating if and only if $\eta(v) > 0$ for each $v \in S(V)$. 

Lemma 5.3. Let \((M, H, g_H; g)\) be a compact step-2 sub-Riemannian manifold and set \(\eta_{\text{min}} = \min_{v \in S(V)} \eta(v)\). Let \((N, h)\) be a compact Riemannian manifold with non-positive sectional curvature. Suppose \(u: M \times [0, \delta) \to N\) is a solution of the subelliptic harmonic map heat flow with potential \(G\) and \(\text{Hess}G < \frac{\eta_{\text{min}}}{2} \cdot h\). Let \(\epsilon\) be a fixed number with \(0 < \epsilon < \frac{1}{2}(\eta_{\text{min}} - 2\lambda_G)\) in Lemma 5.4, where \(\lambda_G\) is the minimum number such that \(\text{Hess}G \leq \lambda_G \cdot h\). Then, for any given \(t_0 \in (0, \delta)\), we have

\[
E_V(u(\cdot, t)) \leq E_V(u(\cdot, t_0)) + \frac{2}{\eta_{\text{min}} - 2\lambda_G} \left(2 \int_M |\tau(u(\cdot, t))|^2 dv_g \right) + \int_M |\nabla G|^2 dv_g + C(E_G(u(\cdot, t_0)) + (\max_N |G|) \text{vol}(M)).
\]

for any \(t \in (t_0, \delta)\), where \(C(\eta_{\text{min}}, G)\) is a positive constant.

Proof. Since \(M\) is compact, \(S(V)\) is compact. Hence there exists a point \(v \in S(V)\) such that \(\eta_{\text{min}} = \eta(v)\). Note that \((M, H)\) is a step-2 sub-Riemannian manifold, thus from Lemma 5.2, we have \(\eta_{\text{min}} > 0\). Let \(\lambda_G\) be the minimum number such that \(\text{Hess} G \leq \lambda_G \cdot h\), and let \(\epsilon\) be a fixed number with \(0 < \epsilon < \frac{1}{2}(\eta_{\text{min}} - 2\lambda_G)\). From (4.17) in [Don21], we have

\[
(5.3) \quad u_{ij}^l - u_{ji}^l = u_{ij}^l T_{ij}^\alpha.
\]

From (5.2), (5.3) and Lemma 5.1 one has

\[
(5.4) \quad (\Delta_H - \frac{\partial}{\partial t}) e(u) \geq -C\epsilon e_H(u) - \epsilon e_V(u) + (u_{ik}^I)^2 + \frac{1}{2}(u_{ak}^I)^2 - \text{Hess} G(u_i, u_i) - \text{Hess} G(u_{\alpha}, u_{\alpha}) \geq -C\epsilon e_H(u) - \epsilon e_V(u) + \frac{1}{2} \sum_i \sum_{i<j} (u_{ij}^I + u_{ji}^I)^2 + (u_{ij}^I - u_{ji}^I)^2 - \text{Hess} G(u_i, u_i) - \text{Hess} G(u_{\alpha}, u_{\alpha}) \geq -C\epsilon e_H(u) - \epsilon e_V(u) + \frac{1}{2} \sum_i \sum_{i<j} (u_{ij}^I)^2 (T_{ij}^\alpha)^2 - \text{Hess} G(u_i, u_i) - \text{Hess} G(u_{\alpha}, u_{\alpha}) = -C\epsilon e_H(u) - \epsilon e_V(u) + \frac{1}{2} \sum_i \sum_{i<j} (u_{ij}^I)^2 \eta(e_{\alpha}) - \text{Hess} G(u_i, u_i) - \text{Hess} G(u_{\alpha}, u_{\alpha}) \geq - (C\epsilon + 2\lambda_G)e_H(u) + (\eta_{\text{min}} - 2\lambda_G - \epsilon) e_V(u).
\]

Integrating (5.4) over \(M\) shows

\[
\frac{d}{dt} E(u) \leq (C\epsilon + 2\lambda_G) E_H(u) - (\eta_{\text{min}} - 2\lambda_G - \epsilon) E_V(u).
\]
Consequently
\[
\frac{d}{dt} E_V(u) + \frac{(\eta_{\text{min}} - 2\lambda_G)}{2} E_V(u) \leq \frac{d}{dt} E_P(u) - \frac{d}{dt} E_G(u) + (C_s + \eta_{\text{min}})(E_G(u(\cdot, t_0)) + |E_P|).
\]

From (1.3) and (5.1), we get
\[
\frac{d}{dt} E_G(u(\cdot, t)) = -\int_M |\partial(u(\cdot, t))\partial t|^2 dv_g \leq 0.
\]

Clearly, we have (4.3) with \(C_1 = 2\lambda_G\). Integrating (4.3) over \(M \times (t_0, t)\), we
obtain
\[
\int_M |\partial(u(\cdot, t_0))\partial t|^2 dv_g \leq \int_M |\partial(u(\cdot, t_0))\partial t|^2 dv_g ds
\]
since \(2\lambda_G < \eta_{\text{min}}\). On the other hand, by integrating (5.1) on \((t_0, t)\), we obtain
\[
E_G(u(\cdot, t_0)) + |E_P| \geq E_G(u(\cdot, t_0)) - E_G(u(\cdot, t_0)) = \int_{t_0}^{t} \int_M |\partial(u(\cdot, s))\partial t|^2 dv_g ds.
\]

Then we have
\[
\frac{d}{dt} E_G(u(\cdot, t)) \geq \frac{d}{dt} E_G(u(\cdot, t_0)) - \eta_{\text{min}} \left( E_G(u(\cdot, t_0)) + |E_P| \right)
= -\int_M |\partial(u(\cdot, t_0))\partial t|^2 dv_g - \eta_{\text{min}} \left( E_G(u(\cdot, t_0)) + |E_P| \right).
\]

Notice that
\[
|\frac{d}{dt} E_P| = |\int_M \langle \tau, \nabla G \rangle dv_g|
\leq \frac{1}{2} \left( \int_M |\tau(u(\cdot, t))|^2 dv_g + \int_M |\nabla G|^2 dv_g \right)
\leq \frac{1}{2} \left( \int_M |\tau(u(\cdot, t_0))|^2 dv_g + \eta_{\text{min}}(E_G(u(\cdot, t_0)) + |E_P|) + \int_M |\nabla G|^2 dv_g \right)
< \infty
\]
and
\[|E_P| < (\max_N|G|)\text{vol}(M) < \infty,\]
since \(M\) is compact. Set
\[
A = 2 \int_M |\tau(u(\cdot, t_0))|^2 dv_g + \int_M |\nabla G|^2 dv_g
+ (C_s + 3\eta_{\text{min}}) \left( E_G(u(\cdot, t_0)) + (\max_N|G|)\text{vol}(M) \right).
\]

It follows that
\[
\frac{d}{dt} E_V(u(\cdot, t)) + \frac{\eta_{\text{min}} - 2\lambda_G}{2} E_V(u(\cdot, t)) \leq A,
\]
that is,

$$\frac{d}{dt}\left(e^{\frac{\eta_{\min}-2\lambda G}{2}t}E_V(u(\cdot, t))\right) \leq Ae^{\frac{\eta_{\min}-2\lambda G}{2}t}.$$ 

By integrating this over \([t_0, t]\), we get

$$\left(e^{\frac{\eta_{\min}-2\lambda G}{2}s}E_V(u(\cdot, s))\right)\bigg|_{s=t_0}^{s=t} \leq \frac{2A}{\eta_{\min}-2\lambda G}\left(e^{\frac{\eta_{\min}-2\lambda G}{2}s}\right)\bigg|_{s=t_0}^{s=t}.$$ 

Therefore

$$E_V(u(\cdot, t)) \leq e^{\frac{\eta_{\min}-2\lambda G}{2}(t_0-t)}E_V(u(\cdot, t_0)) + \frac{2A}{\eta_{\min}-2\lambda G}(1 - e^{\frac{\eta_{\min}-2\lambda G}{2}(t_0-t)})$$

$$\leq E_V(u(\cdot, t_0)) + \frac{2A}{\eta_{\min}-2\lambda G}.$$ 

\[\Box\]

Proof of Theorem 1.2. From Theorem 1.1, we know that one can solve (1.3) for all time. From (5.6), we get

$$\int_0^\delta \int_M |\frac{\partial u}{\partial t}(s)|^2 \, dv_g \, ds = E_G(\bar{u}) - E_G(u(\delta))$$

$$\leq E_G(\bar{u}) + |E_P(u(\delta))|$$

$$\leq E_G(\bar{u}) + (\max_N|G|)\text{vol}(M)$$

which leads to

$$\int_0^\infty \int_M |\frac{\partial u}{\partial t}(s)|^2 \, dv_g \, ds < \infty.$$ 

Hence there exists a sequence \(s_n \to \infty\) such that

$$\int_M |\frac{\partial u}{\partial t}(s_n)|^2 \, dv_g \to 0.$$ 

Due to (4.3), we know

$$(\Delta_H - \frac{\partial}{\partial t}) \exp(-Ct)|\frac{\partial u}{\partial t}(t)|^2 \geq 0.$$ 

The function

$$\phi(x, t) = \exp\left(-C(s+t)\right)|\frac{\partial u}{\partial t}(s+t)|^2$$

also satisfies

$$(\Delta_H - \frac{\partial}{\partial t})\phi \geq 0.$$ 

By Lemma 2.3, we obtain

$$|\frac{\partial u}{\partial t}(s+t)|^2 \leq B \exp(Ct)t^{-\frac{D}{2}} \int_M |\frac{\partial u}{\partial t}(s)| \, dv_g$$
for $0 < t < R_0^2$. Then, for $t = \frac{R_0^2}{2}$, we have
\[
(5.9) \quad |\frac{\partial u}{\partial t}(s + \frac{R_0^2}{2})|^2 \leq \frac{\gamma^2}{R_0^2} B \exp\left(\frac{CR_0^2}{2}\right) \int_M |\frac{\partial u}{\partial t}(s)|^2 dv_g
\]
for any $s > 0$. From (5.8) and (5.9), it follows that
\[
(5.10) \quad \sup_{x \in M} |\frac{\partial u}{\partial t}(s_n + \frac{R_0^2}{2})|^2 \to 0 \quad \text{as} \quad n \to \infty.
\]
From (5.10) and (5.11), we know that $u_\infty$ solves (2.7) weakly. By Theorem 2.1, $u_\infty$ is smooth. Since $u(t,x)$ is smooth in $t$, then $u_\infty$ is homotopic to $u(0) = \bar{u}$.

When $(N,h)$ is a complete non-compact Riemannian manifold, the solutions $u(\cdot,t)$ of (1.3) may not be uniformly bounded with respect to $t \in [0,\infty)$. However, if we add a decay condition on the potential function $G$ and also the non-positive curvature assumption on the target manifold $N$, the solution $u(\cdot,t)$ will remain uniformly bounded. By a similar argument for Theorem 1.2, we can establish the Eells-Sampson type theorems too.

Proof of Proposition 1.3. The global existence of the solution $u(t)$ is given by Theorem 1.1. It is enough to show that there exists a fixed compact set $N' \subset N$ such that $u(t)(M) \subset N'$ for all $t \in [0,\infty)$.

To this end, we set
\[
f = |\frac{\partial u(t)}{\partial t}|^2 \quad \text{and} \quad \phi(t) = \sup_{x \in M} \sqrt{f(t,x)}.
\]
By the completeness of $N$, for each $x \in M, t \in [0,\infty)$, there exists a minimal geodesic $\gamma_x$ connecting $\bar{u}(x)$ and $u(x,t)$, whose length is $d_N(\bar{u}(x), u(x,t))$. Then we have the following triangle inequality
\[
(5.12) \quad \rho(u(x,t)) \leq \rho(\bar{u}(x)) + d_N(\bar{u}(x), u(x,t))
\]
where $\rho$ denotes the distance function on $N$ from the fixed point $P_0 \in N$. Note that
\[
(5.13) \quad d_N(\bar{u}(x), u(x,t)) \leq \int_0^t \phi(s) \, ds.
\]
From (5.12) and (5.13), we get

\[ \rho(u(x,t)) \leq C_1 + \int_0^t \phi(s) \, ds \]

where \( C_1 = \max_{x \in M} \rho(\bar{u}(x)) \). Since \( f \) satisfies

\[
\frac{\partial}{\partial t} - \Delta_H f = -2|\nabla_H \frac{\partial u(t)}{\partial t}|^2 + 2\text{Hess} G \left( \frac{\partial u(t)}{\partial t}, \frac{\partial u(t)}{\partial t} \right) \\
+ 2 \langle \text{Riem}_N(du(t)(e_i), \frac{\partial u(t)}{\partial t}), \frac{\partial u(t)}{\partial t} \rangle \\
\leq 2\text{Hess} G \left( \frac{\partial u(t)}{\partial t}, \frac{\partial u(t)}{\partial t} \right),
\]

from (1.4) and (5.14), the inequality (5.15) becomes

\[
\frac{\partial}{\partial t} - \Delta_H f \leq -C(1 + \int_0^t \phi(s) \, ds)^{-1} f.
\]

Next, setting \( g = \exp(\psi) f \) with \( \psi(t) = C \int_0^t (1 + \int_0^\tau \phi(s) \, ds)^{-1} d\tau \), we get

\[
\frac{\partial}{\partial t} - \Delta_H g \leq 0
\]

and it follows from Lemma 2.3 that

\[
\sup_{x \in M} g(t, \cdot) \leq \sup_{x \in M} g(0, \cdot),
\]

that is

\[
\phi(t) \leq \phi(0) \exp \left( \frac{-\psi(t)}{2} \right) \\
= \phi(0) \exp \left[ -\frac{C}{2} \int_0^t (1 + \int_0^\tau \phi(s) \, ds)^{-1} d\tau \right].
\]

Since \( 1 + \int_0^\tau \phi(s) \, ds \leq 1 + \phi(0) \tau \), then \( \psi(t) \geq \frac{C \ln(1 + \phi(0) t)}{\phi(0)} \). Substituting this into (5.16), we have

\[
\phi(t) \leq \phi(0) \exp \left( \frac{-C \ln(1 + \phi(0) t)}{2\phi(0)} \right) = \frac{\phi(0)}{(1 + \phi(0) t) \frac{2C}{2\phi(0)}}
\]

which suggests that \( \phi(t) \to 0 \) as \( t \to +\infty \). Then, for any \( C_2 > 0 \), there exists \( t_0 > 0 \) such that

\[
\int_0^t \phi(s) \, ds \leq C_2 t
\]
for all $t \geq t_0$. Hence we have

$$\phi(t) \leq \phi(0) \exp\left[-\frac{C}{2} \int_{t_0}^t (1 + \int_0^\tau \phi(s) \, ds)^{-1} \, d\tau\right]$$

$$\leq \phi(0) \exp\left[-\frac{C}{2} \int_{t_0}^t (1 + C_2 \tau)^{-1} \, d\tau\right]$$

(5.17)

$$= \phi(0) \exp\left[-\frac{C}{2C_2} \left(\ln(1 + C_2 t) - \ln(1 + C_2 t_0)\right)\right]$$

$$= \frac{\phi(0)(1 + C_2 t_0)^{\frac{C}{2C_2}}}{(1 + C_2 t)^{\frac{C}{2C_2}}}$$

for $t \geq t_0$. Choosing a sufficiently small $C_2$ such that $\frac{C}{2C_2} > 1$, integrating (5.17) over $[t_0, t]$ then gives

$$\int_{t_0}^t \phi(s) \, ds \leq C_3 \quad \text{for all} \quad t \geq t_0$$

which leads to

(5.18) \quad \int_0^{+\infty} \phi(s) \, ds \leq C_4$

where $C_3$, $C_4$ are positive constants. Using (5.18) in (5.16), we get for some positive constants $C_5$ and $C_6$

(5.19) \quad \phi(t) \leq C_5 e^{-C_6 t}$

In terms of (5.14) and (5.19), we get $\rho(u(t)) \leq C_0$. So, there exists a compact set $N' \subset N$ such that, for all $t \in [0, +\infty)$, $u(t)(M) \subset N'$. It follows that there exists a sequence $t_k \to +\infty$ such that $u(t_k)$ converges to $u_\infty$ which is a subelliptic harmonic map with potential $G$. To see that $u(t) \to u_\infty$, as $t \to +\infty$, we note that $d_N(u(t), u_\infty) \leq d_N(u(t), u(t_k)) + d_N(u(t_k), u_\infty)$. Thanks to (5.19), we obtain

$$d_N(u(t), u(t_k)) \leq \int_{t_k}^t |\frac{\partial u(s)}{\partial s}| \, ds \leq C_5 \int_{t_k}^t e^{-C_6 s} \, ds \to 0 \quad \text{as} \quad k, t \to \infty.$$  

Finally, from the following formula

$$\Delta_H(-G \circ u) = -dG(\tau_H(u)) - \text{Trace}_g \text{Hess}_G(du_H, du_H)$$

$$= |\nabla G(u)|^2 - \text{Trace}_g \text{Hess}_G(du_H, du_H) \geq 0,$$

we see that $(-G \circ u)$ is a subsolution of subelliptic harmonic equation and so is constant. Since $G$ is strictly concave, we have $\text{Trace}_g \text{Hess}_G(du_H, du_H) = 0$, so $u$ is constant. \[\square\]
The Riemannian foliation \((M, g; \mathcal{F})\) will be said to be tense if the mean vector field of \(\mathcal{F}\) is parallel with respect to the Bott connection along the leaves. For a compact sub-Riemannian manifold \((M^{m+d}, H, g_H; g)\) corresponding to a tense Riemannian foliation \((M, g; \mathcal{F})\), we have the following lemma.

**Lemma 5.4.** Let \((M^{m+d}, H, g_H; g)\) be a compact sub-Riemannian manifold corresponding to a tense Riemannian foliation \((M, g; \mathcal{F})\). Let \(N\) be a complete Riemannian manifold with non-positive sectional curvature. Suppose \(u : M \times [0, \delta) \to N\) is a solution of the subelliptic harmonic map heat flow with potential \(G\) and \(\text{Hess} G \leq 0\), then \(E_V(u(\cdot, t))\) is decreasing. In particular, \(E_V(u(\cdot, t)) \leq E(\bar{u})\).

**Proof.** The assumption that \((M, g; \mathcal{F})\) is tense implying that \(\nabla^B_\xi \zeta = 0\), for any \(\xi \in V\) and the curvature tensor of \(\nabla^B\) satisfies [cf. [Don21]]

\[
R^A_{j\alpha k} = 0.
\]

In particular, we have \(R^I_{k\alpha j} = 0\). From (4.29) in [Don21], we have

\[
\Delta_H e_V(u_t) = (u^I_{\alpha k})^2 + u^I_{\alpha j} R^j_{k\alpha} + u^I_{\alpha j} R^I_{k\alpha} - u^I_{\alpha j} G_{Ij} u^j_{\alpha}
\]

Consequently,

\[
(\Delta_H - \frac{\partial}{\partial t}) e_V(u_t) = (u^I_{\alpha k})^2 + u^I_{\alpha j} R^j_{k\alpha} - u^I_{\alpha j} G_{Ij} u^j_{\alpha}
\]

\[
(5.20)
\]

Integrating (5.20) then gives this lemma. \(\square\)

**Proof of Proposition 1.4.** According to Lemma 5.4, \(E_V(f)\) is uniformly bounded, since \(\text{Hess} G \leq 0\). Using a similar argument for Theorem 1.2 and Proposition 1.3, the proposition follows. \(\square\)

**Proof of Proposition 1.5.** Let \(u(x, t)\) be a solution of (1.5). Assumption (1.6) is equivalent to

\[
(\frac{\partial}{\partial t} - \Delta_H) (u(x, t), u(x, t))_{\mathbb{R}^k} \leq 0.
\]

By Lemma 2.3, we get

\[
\sup_{x \in M} \langle u(x, t), u(x, t) \rangle_{\mathbb{R}^k} \leq \sup_{x \in M} \langle \bar{u}(x), \bar{u}(x) \rangle_{\mathbb{R}^k}.
\]

Since \(\bar{u}(x)\) is included in a compact set, we have \(\sup_{x \in M} \langle \bar{u}(x), \bar{u}(x) \rangle_{\mathbb{R}^k} \leq C\). It follows that \(\sup_{x \in M} \langle u(x, t), u(x, t) \rangle_{\mathbb{R}^k} \leq C\), which implies \(u(t)(M)\) is included in a compact set. By Theorem 1.1 we have \(T = +\infty\). In both cases, since \(E_V(f)\) is uniformly bounded, the convergence follows from Theorem 1.2.
immediately. When \((M, H, g_H; g)\) is a compact sub-Riemannian manifold corresponding to a tense Riemannian foliation, by argument of Proposition 1.3 we know \(u_\infty\) is constant. □

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