Sub-Riemannian cubics in SU(2)

Michael Swaddle and Lyle Noakes
Faculty of Engineering, Computing and Mathematics, The University of Western Australia, Crawley 6009, Australia
(Dated: July 16, 2018)

Sub-Riemannian cubics are a generalisation of Riemannian cubics to a sub-Riemannian manifold. Cubics are curves which minimise the integral of the norm of the covariant acceleration. Sub-Riemannian cubics are cubics which are restricted to move in a horizontal subspace of the tangent space. When the sub-Riemannian manifold is also a Lie group, sub-Riemannian cubics correspond to what we call a sub-Riemannian Lie quadratic in the Lie algebra. The present article studies sub-Riemannian Lie quadratics in the case of \( su(2) \), focusing on the long term dynamics.

I. SUB-RIEMANNIAN CUBICS

Let \( G \) be a matrix Lie group with a positive-definite bi-invariant inner product \( \langle \cdot, \cdot \rangle_B \) on the Lie algebra \( \mathfrak{g} := T_{e}G \). Choose a positive-definite self-adjoint operator \( J \) with respect to \( \langle \cdot, \cdot \rangle_B \). Now define \( \langle \cdot, \cdot \rangle_J \), by

\[
\langle X, Y \rangle_J = \langle X, JY \rangle_B.
\]

Given a basis \( \hat{e}_1, \ldots, \hat{e}_n \) for \( \mathfrak{g} \), define an \( n \times n \) matrix \( J_{ij} \) by \( J_{ij} = \langle \hat{e}_i, \hat{e}_j \rangle_J \). Then given \( V = v_k \hat{e}_k \in \mathfrak{g} \), \( v_k \in \mathbb{R} \), we can compute

\[
J V = v_i J_{ij} \hat{e}_j,
\]

where repeated indices are summed.

A left-invariant metric \( \langle \cdot, \cdot \rangle \) is defined on \( TG \) by the formula \( \langle X, Y \rangle_g := \langle g^{-1}X, g^{-1}Y \rangle_J \). Then given a vector subspace of \( \delta_1 \subset \mathfrak{g} \), we define a left-invariant distribution on \( G \) to be the vector sub-bundle \( \delta \) of \( TG \), whose fibre \( \delta_g \) over \( g \) is \( g\delta_1 \).

Previous work \([1–6]\) has investigated the critical points of the functional

\[
S[\tilde{x}] := \int_0^T \langle \nabla_t \tilde{x}, \nabla_t \tilde{x} \rangle_g dt,
\]

where \( \tilde{x} : [0, T] \to M \), and \( \tilde{x}(0), \tilde{x}(T), \dot{\tilde{x}}(0) \) and \( \dot{\tilde{x}}(T) \) are given. \( \nabla_t \tilde{x} \) denotes the covariant acceleration, and \( M \) is a Riemannian manifold. In this situation critical points of \( S \) are called Riemannian cubics. We now consider the case where, \( M = G \) and \( \tilde{x} \) is constrained to be in the distribution \( \delta \). With this constraint, we will call critical points of \( S \) a sub-Riemannian cubic.

Note that restricting the original Riemannian metric to the distribution makes \( G \) a sub-Riemannian manifold \([7]\). When the metric is not bi-invariant, \( J \neq I \), where \( I \) is the identity matrix, we need an underlying Riemannian metric to define \( \nabla_t \tilde{x} \), which is not necessarily restricted to the distribution.

The equations for normal sub-Riemannian cubics can be derived from the Pontryagin Maximum Principle (PMP). For a reference on the PMP see \([8]\). Usually the PMP applies for control systems on \( \mathbb{R}^n \) but there is a version for control systems on a Lie group \([9]\).

As \( G \) is left-invariant \( \dot{\tilde{x}} \) is constrained by the equation

\[
\dot{\tilde{x}} := \tilde{x} \tilde{V},
\]
where \( \bar{V} : [0, T] \rightarrow \delta_1 \). We also require \( \delta_1 \) to be a bracket generating subset of \( g \).

Let left Lie reduction by \( \bar{x} \) be denoted \( L^{-1} \). Then the left Lie reduction of the covariant acceleration \( \nabla_t \dot{x} \) to \( g \) includes a first order derivative of \( \bar{V} \),

\[
L^{-1}(\nabla_t \dot{x}) = \dot{\bar{V}} - J^{-1} \left[ J\bar{V}, \bar{V} \right].
\]  

(3)

To use the PMP define a new control function \( u : [0, T] \rightarrow g \), and treat \( \bar{V} \) as an additional state variable. So to minimise \( S[\bar{x}] \) subject to the constraints

\[
\dot{\bar{x}} = \bar{x} \bar{V},
\]

(4)

\[
\dot{\bar{V}} = u,
\]

(5)

we form the PMP Hamiltonian, \( H \), given by

\[
H(\bar{x}, \bar{V}, u, \lambda, \vartheta, \nu) := \lambda(\bar{x} \bar{V}) + \vartheta(u) + \frac{\nu}{2} \| u - J^{-1} \left[ J\bar{V}, \bar{V} \right] \|_J^2,
\]

(6)

where \( \lambda \in T^*G_{\bar{x}} \) and \( \vartheta \in g^* \) are the co-states, and \( \nu \leq 0 \). Then the PMP says maximising \( H \) for all \( t \) is a necessary condition for minimising \( S \).

By the PMP, the co-states are required to satisfy

\[
\dot{\lambda}(y) = -dH_{\bar{x}}(y) = -\lambda(y\bar{V}),
\]

\[
\dot{\vartheta}(Z) = -dH_{\bar{V}}(Z) = -\lambda(\bar{x}Z) - \nu \left( -J^{-1} \left[ JZ, \bar{V} \right] + J^{-1} \left[ J\bar{V}, Z \right] - J^{-1} \left[ J\bar{V}, \bar{V} \right] \right)_J.
\]

\( \forall y \in TG_{\bar{x}} \) and \( Z \in g \). \( \lambda \in T^*G_{\bar{x}} \) can be associated with a \( \lambda^* \in g^* \) via left multiplication, \( \Lambda^*(V) = \lambda(L(\bar{x})V) \).

Differentiating \( \Lambda^* \),

\[
\dot{\Lambda}^*(y) = \dot{\lambda}(\bar{x} y) + \lambda(\dot{\bar{x}} y)
\]

\[
= -\lambda(\bar{x} y \bar{V}) + \lambda(\bar{x} \bar{V} y)
\]

\[
= -\Lambda^*(y \bar{V}) + \Lambda^*(\bar{V} y)
\]

\[
= \Lambda^* \left( \left[ \bar{V}, y \right] \right).
\]

Finally \( \Lambda^* \) can be associated with \( \Lambda \in g \) via the bi-invariant inner product, \( \langle \cdot, \cdot \rangle_B \)

\[
\Lambda^*(y) = \langle \Lambda, y \rangle_B.
\]

Likewise for \( \vartheta \) there is an associated \( \Theta \in g \).

\[
\vartheta(w) = \langle \Theta, w \rangle_B.
\]

This gives the following equations for the costates

\[
\dot{\Lambda} = \left[ \Lambda, \bar{V} \right],
\]

\[
\dot{\Theta} = -\Lambda - \nu \left( J [\bar{V}, u] - [J\bar{V}, u] \right)
\]

\[
+ J [\bar{V}, J^{-1} \left[ J\bar{V}, \bar{V} \right]] - [J\bar{V}, J^{-1} \left[ J\bar{V}, \bar{V} \right]].
\]

By the PMP there are two cases to consider.
1. Normal case

In the normal case, $\nu < 0$, the optimal control $u^*$ must maximise $H$. From now on we consider the case when $\langle \cdot, J \cdot \rangle$ is simply bi-invariant and so $J = I$. We now use the notation $\langle \cdot, J \cdot \rangle = \langle \cdot, B \rangle$. Without loss of generality set $\nu = -1$. Hence in the normal case, the PMP Hamiltonian can equivalently written as

$$H = \langle \Lambda, \tilde{V} \rangle + \langle \Theta, u \rangle - \frac{1}{2} \langle u, u \rangle.$$ 

Maxima occur when $dH(u^*)_u = 0$, so

$$\langle \Theta, u^* \rangle - \langle u, u^* \rangle = 0.$$ 

Therefore optimal controls occur when $\text{proj}_{\delta_1}(\Theta) = u = \dot{\tilde{V}}$. The equations for the costates reduce to a single equation, which gives the following theorem.

**Theorem I.1.** $\tilde{x}$ is a normal sub-Riemannian cubic if and only if

$$\dot{\Theta} = \left[ \dot{\Theta}, \tilde{V} \right].$$

**Remark.** Denote the projection of $\Theta$ onto the orthogonal complement, $\perp$, of $\delta_1$ by $\varphi = \text{proj}_\perp(\Theta)$. Let $\delta_1 = g$. Then the resulting equations for normal sub-Riemannian cubics match the bi-invariant Riemannian case, $\tilde{V} = \left[ \tilde{V}, \tilde{V} \right] \mathbb{R}$. In general solutions to (8) are hard to find.

**Remark.** One subclass of solutions are the so called linear Lie quadratics. In this case, $\varphi = \varphi_0$ and $\tilde{V} = (q_0 + q_1 t + q_2 t^2)\tilde{V}_0$, where $q_0, q_1, q_2 \in \mathbb{R}$, $\tilde{V}_0$ is a constant in $\delta_1$, and $\varphi_0$ is a constant in the orthogonal complement of $\delta_1$.

$\dot{\Theta}$ can be found in terms of $\tilde{x}$. Rewrite (8), take the adjoint and integrate,

$$\text{Ad}(\tilde{x}) \left( \dot{\Theta} + [\tilde{V}, \dot{\Theta}] \right) = 0 \quad \implies \quad \dot{\Theta} = \text{Ad}(\tilde{x}^{-1})A$$

where $A \in g$. This simply reflects the fact that $\dot{\Theta}$ satisfies a Lax equation [10] and is therefore isospectral.

2. Abnormal case

The abnormal case is given by $\nu = 0$. As before, the PMP Hamiltonian can be written as

$$H = \langle \Lambda, \tilde{V} \rangle + \langle \Theta, u \rangle.$$ 

Maxima occur when $dH(u^*)_u = 0$. Immediately this requires $\text{proj}_{\delta_1}(\Theta) = 0$, so there is no way to determine $u$ from the PMP.

3. Bounds

Given some function $f$, we say that $f$ is $\mathcal{O}(t^n)$, when for some $c > 0$, $|f| \leq c|t^n|$, for all $t \in \mathbb{R}$. 

Corollary I.1.
\[ \langle \ddot{V}, V \rangle - \frac{1}{2} \langle \dot{\dot{V}}, \dot{V} \rangle = c_1, \]  
\[ \langle \dddot{V}, V \rangle + \langle \ddot{\phi}, \phi \rangle = c_2, \]  
where \( c_1 \in \mathbb{R} \) and \( c_2 \geq 0 \).

**Proof.** First take the inner product of (8) and \( \ddot{V} \) to find \( \langle \ddot{V}, \ddot{V} \rangle = 0 \). Next take the inner product of (8) with \( \dddot{V} + \dot{\phi} \) to find \( \langle \dddot{V}, \dddot{V} \rangle + \langle \dot{\phi}, \dot{\phi} \rangle = 0 \). Integrating these gives the result. \( \Box \)

Corollary I.2.
\[ \frac{1}{2} c_1 t^2 + c_5 t + c_6 \leq \langle \ddot{V}, V \rangle \leq O(t^4). \]

**Proof.** As \( \langle \dot{\phi}, \dot{\phi} \rangle \geq 0 \), we have \( \langle \dddot{V}, V \rangle \leq c_2 \). This argument can be repeated for the components of \( \dddot{V} \), so given \( \dddot{V} = v_k(t) \dot{e}_k \), then \( \dddot{V} \leq c_2 \). As we are working with the bi-invariant metric, we have \( -\sqrt{c_2} \leq \dot{v}_k \leq \sqrt{c_2} \) and so \( |v_k| \leq \frac{1}{2} \sqrt{c_2} t^2 + c_3 t + c_4 \), where \( c_3 \) and \( c_4 \) are some other constants. Therefore \( \langle \dddot{V}, V \rangle \) is bounded above by \( O(t^4) \). The same argument can be used to show \( \langle \dot{\phi}, \dot{\phi} \rangle \) is bounded above a constant, and then \( \| \phi \| \) is bounded above by a linear function. This then shows \( \langle \phi, \phi \rangle \) is bounded above by \( O(t^2) \).

Equation (9) can be written as
\[ \frac{d^2}{dt^2} \langle \ddot{V}, V \rangle = c_1 + \frac{3}{2} \langle \dddot{V}, V \rangle. \]

Immediately this yields the lower bound \( \langle \ddot{V}, V \rangle \geq \frac{1}{2} c_1 t^2 + c_5 t + c_6 \) where \( c_5 \) and \( c_6 \) are other constants. \( \Box \)

II. SUB-RIEMANNIAN LIE QUADRATICS AND SYMMETRIC PAIRS

A. Symmetric pairs

Let \( (\mathfrak{g}, \mathfrak{h}) \) be a symmetric pair, namely \( \mathfrak{g} = \mathfrak{m} + \mathfrak{h} \) where the following properties hold
\[ [\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{h}, \]
\[ [\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}, \]
\[ [\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{m}. \]

An example is \( \mathfrak{su}(2) \) where \( \mathfrak{m} \) is spanned by the Pauli matrices \( i\sigma_1 \) and \( i\sigma_2 \), and \( \mathfrak{h} \) is spanned by \( i\sigma_3 \). Suppose we set \( \delta_1 = \mathfrak{m} \). The equations for normal sub-Riemannian cubics in \( \mathfrak{g} \) separate into two components.
\[ \ddot{\phi} = [\ddot{V}, V], \]
\[ \dddot{V} = [\ddot{\phi}, V]. \]
Integrating the first equation and substituting leaves
\[
\ddot{\bar{V}} = \left[[\dot{\bar{V}}, \bar{V}], \bar{V}\right] + \left[\bar{C}, \bar{V}\right].
\] (12)

We call \(\bar{V}\) which satisfy this equation a *sub-Riemannian Lie quadratic*. One simple solution to this is \(\bar{V} = \bar{V}_0 + \bar{V}_1 t\), where \(\bar{V}_0\) and \(\bar{V}_1\) are chosen so \(\left[\bar{V}_0, \bar{V}_1\right] = \bar{C}\). We call sub-Riemannian Lie quadratics *null* when \(\bar{C} = 0\).

**B. Duality**

We say \(\bar{W}\) is *dual to* \(\bar{V}\), when
\[
\bar{W} = -\text{Ad}(\bar{x})\bar{V},
\] (13)
\[
\dot{\bar{y}} = \bar{y}\bar{W},
\] (14)

where \(\bar{y} = \bar{x}^{-1}\).

In the null case, \(\bar{C} = 0\), \(\bar{V}\) is dual to a non-null *Riemannian Lie quadratic*, \(\dot{V}\), which is defined by the equation
\[
\dot{V} = [\dot{V}, V] + C.
\]

Duality was considered for Riemannian Lie quadratics in [6]. We investigate the sub-Riemannian case.

**Theorem II.1.** \(\bar{V}\) is dual to a rescaled non-null Riemannian Lie quadratic.

**Proof.** Recall that for any other function \(Z\)
\[
\frac{d}{dt} (\text{Ad}(\bar{x}) Z) = \text{Ad}(\bar{x}) \left( \dot{Z} + \left[\bar{V}, Z\right] \right).
\] (15)

Computing derivatives,
\[
\dot{\bar{W}} = -\text{Ad}(\bar{x}) \left(\dot{\bar{V}}\right),
\]
\[
\ddot{\bar{W}} = -\text{Ad}(\bar{x}) \left(\ddot{\bar{V}} + \left[\bar{V}, \dot{\bar{V}}\right]\right),
\]
\[
\dddot{\bar{W}} = -\text{Ad}(\bar{x}) \left(\dddot{\bar{V}} + \left[\dot{\bar{V}}, \dot{\bar{V}}\right] + \left[\bar{V}, \dot{\bar{V}} + \left[\bar{V}, \dot{\bar{V}}\right]\right]\right),
\]

which gives
\[
\dddot{\bar{W}} = -2\text{Ad}(\bar{x}) \left(\left[\bar{V}, \dot{\bar{V}}\right] + \left[\bar{V}, \dot{\bar{V}} + \left[\bar{V}, \dot{\bar{V}}\right]\right]\right).
\]

Then
\[
\left[\dddot{\bar{W}}, \dddot{\bar{W}}\right] = \text{Ad}(\bar{x}) \left(\left[\dddot{\bar{V}}, \dddot{\bar{V}}\right] + \left[\dddot{\bar{V}}, \dddot{\bar{V}}\right]\right),
\]

and so
\[
\dddot{\bar{W}} = 2 \left[\dddot{\bar{W}}, \dddot{\bar{W}}\right].
\]
Integrating this equation leaves a reparameterised non-null Riemannian Lie quadratic
\[ \ddot{\tilde{W}} = 2 \left[ \dot{\tilde{W}}, \tilde{W} \right] + \tilde{D}. \] (16)
Without loss of generality, let \( \tilde{x}(0) = I \). Consider the equations for \( \tilde{W} \) and \( \tilde{V} \) at \( t = 0 \). Clearly \( \tilde{V}(0) = \tilde{W}(0) \), \( \dot{\tilde{V}}(0) = \dot{\tilde{W}}(0) \), and \( \ddot{\tilde{W}}(0) = \ddot{\tilde{V}}(0) + \left[ \tilde{V}(0), \dot{\tilde{V}}(0) \right] \). First this shows \( \tilde{W} \) is no longer constrained to \( \delta_1 \). Additionally we must have \( \tilde{D} = \ddot{\tilde{V}}(0) + 3 \left[ \tilde{V}(0), \dot{\tilde{V}}(0) \right] \).

Now let \( V(t) = a\tilde{W}(bt) \), where \( a, b \in \mathbb{R} \). Setting \( a = 2b \), and \( C = 2b^3\tilde{D} \), \( V \) satisfies
\[ \ddot{V} = \left[ \dot{V}, V \right] + C, \] (17)
which is the equation for a non-null Riemannian Lie quadratic.

Note if we define \( x : [0, T] \to G \), and \( \dot{x} = xV \), then there is no clear relation between \( x \), a non-null Riemannian cubic and \( \tilde{x} \). If \( C = 0 \) then the Lie quadratic is called null [4]. It is possible to integrate the equation for \( V \) in certain groups. This can occur in a non-trivial way if \( \ddot{V}(0) = 0 \) and \( \dot{V}(0) = 0 \) or \( \tilde{V}(0) = 0 \). Let \( W = -\text{Ad}(x)V \).

Computing derivatives, we find
\[ \ddot{W} = 0, \]
which gives \( W = W_0 + W_1 t \), where the \( W_k \) are constant matrices. If \( V \) was known it would be possible to work backwards and compute \( \tilde{x} \) using the work of [10].

C. SU(2)

Let \( G = SU(2) \). Take \( \delta_1 = \text{span} (\{ \hat{e}_1, \hat{e}_2 \}) = \text{span} \left( \frac{i}{\sqrt{2}} \{ \sigma_1, \sigma_2 \} \right) \), where \( \sigma_i \) are the Pauli matrices. Let \( \tilde{C} = C \frac{i}{\sqrt{2}} \sigma_3 \), where \( C \in \mathbb{R} \). Recall \( su(2) \) can be identified with \( so(3) \). \( so(3) \) can then be identified with Euclidean three space, \( \mathbb{E}^3 \), with the cross product. As a consequence of the vector triple product formula, we can write for \( \tilde{V} \) in \( su(2) \)
\[ \ddot{\tilde{V}} = 2 \left( \left< \tilde{V}, \dot{\tilde{V}} \right> \tilde{V} - \left< \tilde{V}, \tilde{V} \right> \dot{\tilde{V}} \right) + [\tilde{C}, \tilde{V}]. \] (18)

We can identify \( \tilde{V} = v_1(t)\hat{e}_1 + v_2(t)\hat{e}_2 \) with a \( v \in \mathbb{C} \) by taking \( v := v_1 + iv_2 \). Then the sub-Riemannian cubic equation in \( \delta_1 \) can be written as
\[ \ddot{v} = \frac{1}{2} v (\dot{v}v - \dot{v}v) - iCv. \]
Assuming \( v(t) \neq 0 \) for all \( t \), define \( \omega : \mathbb{R} \to S^1 \subset \mathbb{E}^2 \cong \mathbb{C} \) by
\[ \omega(t) := \frac{v(t)}{\|v(t)\|}. \]
Define \( \exp(y) = (\cos(y), \sin(y)) \equiv e^{iy} \in \mathbb{C} \). Choose a \( \psi(t_0) \in [0, 2\pi) \) so that \( \exp(\psi(t_0)) = \omega(t_0) \). Then there is a unique continuous function \( \psi : \mathbb{R} \to \mathbb{R} \) such that the diagram
commutes. Then we have \( v(t) = q(t)(\cos(\vartheta(t)), \sin(\vartheta(t))) = q e^{i\vartheta}, \) where \( q(t) = \| v(t) \| = \| \tilde{V} \|. \)

Substituting back, and taking the real and imaginary components gives the two equations

\[
\begin{align*}
\ddot{q} - 3\dot{q} \dot{\vartheta}^2 - 3q \dot{\vartheta} \ddot{\vartheta} &= 0, \\
q \dddot{\vartheta} + 3q \dddot{\vartheta} + 3\dot{q} \dot{\vartheta} - q \ddot{\vartheta}^3 + q^3 \ddot{\vartheta} + C q &= 0.
\end{align*}
\]

Multiplying the first equation by \( r \), and integrating leaves

\[
-\frac{1}{2} \dot{q}^2 + \dddot{q} q - \frac{3}{2} q^2 \dot{\vartheta}^2 + c_1 = 0,
\]

where \( c_1 \in \mathbb{R} \). Note that equation follows directly from equation (11), but we use the complex structure to show several additional properties.

D. \( c_1 > 0 \)

Let \( c_1 > 0 \). Recall \( \langle V, V \rangle \) is at most \( O(t^4) \). Therefore \( q \) increases no faster than \( O(t^2) \). Likewise \( \dot{q}^2 + q^2 \dot{\vartheta}^2 \) must not increase faster than \( O(t^2) \).

Also note \( q^2 \dot{\vartheta}^2 \leq c_2 + c_1 \). Recall \( q^2 \) was bounded below by a quadratic and above by \( O(t^4) \), so at most \( \dot{\vartheta}^2 = O(t^{-2}) \). Additionally \( q^2 \dot{\vartheta}^2 \) is at most \( O(t^{-2}) \). Therefore,

\[
-\frac{1}{2} \dot{q}^2 + \dddot{q} q = c_1 + O(t^{-2}).
\]

First set \( Y = \dot{q}^2 \). Then it follows

\[
-Y + \frac{dY}{dq} q = 2c_1 + O(t^{-2}),
\]

so

\[
\frac{d}{dq} \left( \frac{Y}{q} \right) = \frac{2c_1 + O(t^{-2})}{q^2}.
\]

Integrating with respect to \( q \)

\[
\dot{q}^2 = c_7 q - 2c_1 + O(t^{-2}),
\]

where \( c_7 \geq 0 \). Up to \( O(t^{-2}) \) error, and as \( q \) is at most \( O(t^2) \), we can write

\[
\dot{q} = (c_7 q - 2c_1)^{\frac{1}{2}} + (c_7 q - 2c_1)^{-\frac{1}{2}} O(t^{-2}).
\]

Then

\[
\frac{\dot{q}}{c_7 q - 2c_1} = 1 + O(t^{-2}).
\]
Integrating with respect to $t$
\[
\frac{2(c_7q - 2c_1)^{\frac{1}{2}}}{c_7} = t + \frac{c_8}{c_7} + \mathcal{O}(t^{-1}),
\]
which gives
\[
q = \frac{(c_7t + c_8)^2}{4c_7} + \frac{2c_1}{c_7} + \mathcal{O}(t^{-1}).
\]

III. ASYMPTOTICS

In $G = SU(2)$, and with $c_1 > 0$, it is possible to show that long term asymptotes exist. For Riemannian cubics in $SO(3)$, it was established that a limit exists in [4]. In $SU(2)$, we can show that the limit
\[
\alpha_{\pm}(\tilde{V}) = \lim_{t \to \pm \infty} \frac{\tilde{V}}{\|\tilde{V}\|},
\]
exists. Using the (smooth) identification of $\delta_1$ with $\mathbb{C}$, we can equivalently show $\vartheta$ tends to a constant
\[
\alpha_{\pm} = \lim_{t \to \pm \infty} \vartheta,
\]
recalling the definition of $\vartheta$ from the previous section.

**Theorem III.1.** $\alpha_{\pm}(\tilde{V}) = \lim_{t \to \pm \infty} \frac{\tilde{V}}{\|\tilde{V}\|}$ exists.

**Proof.** Note that we only need to consider $t \to \infty$, as $\tilde{V}$ can be re-parameterised. Using results from section [IID], $\dot{\vartheta}$ behaves at most like $\mathcal{O}(t^{-2})$,
\[
|\dot{\vartheta}| \leq \frac{1}{d_1 t^2},
\]
as $q$ is bounded by a quadratic. First note that for $s \geq r$
\[
|\vartheta(s) - \vartheta(r)| \leq \int_r^s |\dot{\vartheta}| \, dt \\
\leq \frac{1}{d_1} \left( \frac{1}{r} - \frac{1}{s} \right).
\]

We can show that for an unbounded sequence of increasing times, $t_1, t_2, \ldots$, the sequence $\vartheta(t_1), \vartheta(t_2), \ldots$ converges to a limit,
\[
L = \lim_{n \to \infty} \vartheta(t_n).
\]
Given some $\varepsilon \geq 0$, there exists a $N$ such that for all $n, m \geq N$, where $n \geq m$, we have
\[
|\vartheta(t_n) - \vartheta(t_m)| \leq \frac{1}{d_1} \left( \frac{1}{t_m} - \frac{1}{t_n} \right) \\
\leq \frac{1}{d_1} \left( \frac{1}{t_m} \right) \\
\leq \varepsilon \frac{1}{2}.
\]
by choosing $t_N \geq (1 + \sqrt{\frac{2}{d_1}})$ and so the sequence is Cauchy. As $\vartheta$ is a real function, by completeness of $\mathbb{R}$, the sequence converges. So given $\varepsilon > 0$, there exists an $N$ such that for all $n > N$

$$|L - \vartheta(t_n)| \leq \frac{\varepsilon}{2}.$$  

Using a similar argument as before, given an $\varepsilon \geq 0$, there exists a $T$ such that for $s, t \geq T$,

$$|\vartheta(s) - \vartheta(t)| \leq \frac{\varepsilon}{2}.$$  

Now choose $T = t_N$ and for $t \geq T$ we have

$$|L - \vartheta(T)| \leq \frac{\varepsilon}{2},$$

and

$$|\vartheta(T) - \vartheta(t)| \leq \frac{\varepsilon}{2}.$$  

By the triangle inequality this gives

$$|L - \vartheta(t)| \leq \varepsilon.$$  

Therefore

$$L = \lim_{t \to \infty} \vartheta.$$  

Hence take $\alpha_+ = L$. Likewise $\alpha_-$ exists by reparameterising. \hfill \qed

Using the identification this shows $\alpha_+(\tilde{V})$ exists.

For null Riemannian cubics a similar limit was found in Theorem (5) of [11]. A similar approach can be used to establish a more precise statement on the convergence when the sub-Riemannian cubic is null. Define

$$V = \tilde{V} + \frac{1}{2\|\tilde{V}\|} \tilde{V}.$$  

(21)

Recall $q(t) = \|V(t)\|$.

**Theorem III.2.** If $\tilde{c} = 0$, $\left\|\frac{V}{q} - \alpha_{\pm}(\tilde{V})\right\| \leq \frac{\sqrt{c}}{2q^2}$.  

**Proof.** Again, considering $t \to \infty$ as the negative case can be found via re-parameterisation.

$$\frac{d}{dt} \left(\frac{V}{q}\right) = \frac{\dot{V}}{q} - \frac{\dot{q}V}{q^2}$$

$$= \frac{1}{q} \left(\frac{\ddot{V}}{q^3} - \frac{\dot{q}}{q^2} \frac{\dot{V}}{q^3} + \frac{1}{2q^2} \left(2 \left(q\dot{V} - q^2 \ddot{V} + [\tilde{c}, \tilde{V}]\right) - \frac{\dot{q}}{q^2} \left(\tilde{V} + \frac{1}{2q^2} \tilde{V} \right)\right) \right)$$

$$= -\frac{3q\ddot{V}}{2q^3} + \frac{[\tilde{c}, \tilde{V}]}{2q^3}.$$  

9
Then if $\bar{C} = 0$, and noting that $q > 0$, and for large enough $t$, $\dot{q} > 0$, assuming $c_7 > 0$ and $c_1 > 0$,

$$
\| \frac{\mathcal{V}(s)}{q(s)} - \frac{\mathcal{V}(r)}{q(r)} \| \leq \left\| \int_r^s \frac{d}{dt} \left( \frac{\mathcal{V}(t)}{q(t)} \right) \, dt \right\|
$$

$$
\leq \int_r^s \frac{3q\sqrt{c_2}}{2q^4} \, dt
$$

$$
\leq \frac{\sqrt{c_2}}{2q(r)^3} - \frac{\sqrt{c_2}}{2q(s)^3}
$$

$$
\leq \frac{\sqrt{c_2}}{2q(r)^3}.
$$

As $\ddot{\mathcal{V}}$ is bounded,

$$
\lim_{t \to \infty} \frac{\mathcal{V}}{q} = \lim_{t \to \infty} \frac{\mathcal{V}}{q} = \alpha_+(\mathcal{V}).
$$

So finally, taking the limit as $s \to \infty$

$$
\left\| \alpha_+(\mathcal{V}) - \frac{\mathcal{V}(r)}{q(r)} \right\| \leq \frac{\sqrt{c_2}}{2q(r)^3}. \tag{22}
$$

**Corollary III.1.**

$$
\left\| q \alpha_+(\mathcal{V}) - \mathcal{V} \right\| \leq \frac{\sqrt{c_2}}{2q^2}. \tag{23}
$$

**Proof.** Multiply equation (22) through by $q$. \hfill \square

Recall from section (II B), in the long term limit, assuming $c_1, c_7 > 0$, $q$ approaches a quadratic. Hence in the long term $\mathcal{V} = q \alpha_+(\mathcal{V}) + \mathcal{O}(t^{-4})$

In the non-null case, a different estimate can be made for $\mathcal{V}$.

$$
\frac{\mathcal{V}(s)}{q(s)} - \frac{\mathcal{V}(r)}{q(r)} - \int_r^s \frac{[\bar{C}, \mathcal{V}]}{2q^3} \, dt = - \int_r^s \frac{3q\mathcal{V}}{2q^4} \, dt.
$$

As before, taking norms, and letting $s \to \infty$,

$$
\left\| \alpha_+(\mathcal{V}) - \frac{\mathcal{V}(r)}{q(r)} - \int_r^\infty \frac{[\bar{C}, \mathcal{V}]}{2q^3} \, dt \right\| \leq \frac{\sqrt{c_2}}{2q(r)^3}.
$$

Multiplying through by $q$, we can deduce

$$
\mathcal{V}(r) = q(r)\alpha_+(\mathcal{V}) - q(r) \left[ \bar{C}, \int_r^\infty \frac{\mathcal{V}}{2q^3} \, dt \right] + \mathcal{O}(r^{-4}). \tag{24}
$$

With this we can estimate $\mathcal{V}(r)$ recursively up to $\mathcal{O}(r^{-4})$ error.

**Theorem III.3.**

$$
\mathcal{V}(r) = q(r)\alpha_+(\mathcal{V}) - q(r) \int_r^\infty \frac{1}{2q(t)^2} \, dt \left[ \bar{C}, \alpha_+(\mathcal{V}) \right] + \mathcal{O}(r^{-4}).
$$
Proof. By equation (24), substituting \( \vec{V}(r) \) back
\[
\vec{V}(r) = q(r)\alpha_+(\vec{V}) - q(r) \int_r^\infty \frac{1}{2q(t)^2} dt \left[ \vec{C}, \alpha_+(\vec{V}) \right] 
+ q(r) \int_r^\infty \frac{[\vec{C}, O(t^{-4})]}{2q(t)^2} dt + q(r) \int_r^\infty \frac{[\vec{C}, \int_t^\infty \vec{V}(s) \frac{ds}{2q(s)^2}]}{2q(t)^2} dt + O(r^{-4}).
\]
We should ignore terms smaller than \( O(t^{-4}) \). Also recall \( \vec{V} \) behaves like \( O(t^2) \) for large \( t \).
\[
\vec{V}(r) = q(r)\alpha_+(\vec{V}) - q(r) \int_r^\infty \frac{1}{2q(t)^2} dt \left[ \vec{C}, \alpha_+(\vec{V}) \right] 
+ q(r) \int_r^\infty \frac{[\vec{C}, O(t^{-4})]}{2q(t)^3} dt + q(r) \int_r^\infty \frac{[\vec{C}, \alpha_+(\vec{V})]}{2q(t)^2} dt + O(r^{-4})
\]
\[
\vec{V}(r) = q(r)\alpha_+(\vec{V}) - q(r) \int_r^\infty \frac{1}{2q(t)^2} dt \left[ \vec{C}, \alpha_+(\vec{V}) \right] 
+ q(r) O(r^{-9}) + q(r) O(r^{-6}) + O(r^{-4})
\]
\[
\vec{V}(r) = q(r)\alpha_+(\vec{V}) - q(r) \int_r^\infty \frac{1}{2q(t)^2} dt \left[ \vec{C}, \alpha_+(\vec{V}) \right] + O(r^{-4}).
\]

\( \square \)

**EXAMPLE 1**

Equations (20) can be numerically solved with *Mathematica’s NDSolve* function for the components \((q, \vartheta)\) of \( v \). Figure (1) is a parametric plot of \( v_1 = q \cos(\vartheta) \) vs \( v_2 = q \sin(\vartheta) \), where \( C = 1, v_1(0) = 4, v_2(0) = -1.75, \dot{v}_1(0) = -0.1, \dot{v}_2(0) = 2.5, \ddot{v}_1(0) = -5 \) and \( \ddot{v}_2(0) = -5 \).

Initially we see oscillation before stabilising in the long term. Figure (2) shows the radial and angular components of \( v \). Note how \( q \) approaches a quadratic, \( q \) approaches a linear function, and \( \vartheta \) approaches a constant as discussed in the previous sections.

Finally the equation for \( \vec{x} \) can also be numerically integrated, using the previously found \( v \). In \( SU(2) \), \( \vec{x} \) is a matrix with four components which satisfy \( \vec{x}_{11}^2 + \vec{x}_{12}^2 + \vec{x}_{21}^2 + \vec{x}_{22}^2 = 1 \), which is the sphere \( S^3 \). Figure (3) shows a stereographic projection of the components of \( \vec{x} \) onto \( \mathbb{R}^3 \), via
\[
(\vec{x}_{11}, \vec{x}_{12}, \vec{x}_{21}, \vec{x}_{22}) \rightarrow \frac{1}{1 - \vec{x}_{22}} (\vec{x}_{11}, \vec{x}_{12}, \vec{x}_{21}).
\]
FIG. 1: Parametric plot of $v$ per example 1.

FIG. 2: Radial and angular components of $v$ per example 1.

FIG. 3: Components of $\tilde{x}$ projected into $\mathbb{R}^3$ per example 1.
EXAMPLE 2

Setting $C = 0$ can yield just as interesting dynamics as $C \neq 0$. The following figures show $v$ as per equations (20) with $v_1(0) = 2$, $v_2(0) = -1$, $\dot{v}_1(0) = 2$, $\dot{v}_2(0) = -1$, $\ddot{v}_1(0) = 0$ and $\ddot{v}_2(0) = 5$.

FIG. 4: Parametric plot of $v$ per example 2.

FIG. 5: Radial and angular components of $v$ per example 2.
FIG. 6: Components of $\tilde{x}$ projected into $\mathbb{R}^3$ per example 2.

[1] M. Camarinha, F. S. Leite, and P. Crouch, Differential Geometry and its Applications 15, 107 (2001).
[2] M. Pauley and L. Noakes, Differential Geometry and its Applications 30, 694 (2012).
[3] R. GiambA, F. Giannoni, and P. Piccione, IMA Journal of Mathematical Control and Information 19, 445 (2002).
[4] L. Noakes, Journal of Mathematical Physics 44, 1436 (2003).
[5] L. Noakes and T. Ratiu, Communications in Mathematical Sciences 14 (2016).
[6] L. Noakes, Advances in Computational Mathematics 25, 195 (2006).
[7] R. Montgomery, A Tour of Subriemannian Geometries, Their Geodesics and Applications (American Mathematical Society, 2006).
[8] L. S. Pontryagin, V. G. Boltyanski, R. V. Gamkrelidze, E. F. Mishechenko, Journal of Applied Mathematics and Mechanics 43, 514 (1963).
[9] Y. Sachkov, Journal of Mathematical Sciences 156, 381 (2009).
[10] L. Noakes, The Quarterly Journal of Mathematics 57, 527 (2006).
[11] L. Noakes, Siam Journal on Applied Dynamical Systems 7, 437 (2008).