BOUNDEDNESS FOR GEVREY AND GELFAND-SHILOV KERNELS TO POSITIVE OPERATORS

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ABSTRACT. We study properties of positive operators on Gelfand-Shilov spaces, and distributions which are positive with respect to non-commutative convolutions. We prove that boundedness of kernels $K \in \mathcal{D}'$ to positive operators, are completely determined by the behaviour of $K$ alone the diagonal. We also prove that positive elements $a$ in $\mathcal{S}'$ with respect to twisted convolutions, having Gevrey class property of order $s \geq 1/2$ at the origin, then $a$ belongs to the Gelfand-Shilov space $\mathcal{S}_s$.

0. Introduction

In this paper we deduce boundedness properties for kernels of positive operators and elements with respect to non-commutative convolutions. More precisely, consider a Roumieu distribution (i.e. an ultra-distributions of Roumieu type), which at the same time is a kernel to a positive operator. Roughly speaking, we prove that the kernel is a Gelfand-Shilov distribution of certain degree, if and only if its restriction to the diagonal is also a Gelfand-Shilov distributions of the same degree.

A consequence of this result is that a Roumieu distribution which is positive with respect to a non-commutative convolution algebra, belongs to corresponding space of Gelfand-Shilov distributions.

We remark that the usual convolution as well as the twisted convolution are special cases of these non-commutative convolutions.

For the twisted convolution we perform further investigations when the Roumieu distributions possess stronger regularity. More precisely, if a Roumieu or Gelfand-Shilov distribution is positive with respect to the twisted convolutions, and is of Gevery class of certain degree near the origin, then we prove that the distribution is a Gelfand-Shilov function of the same degree.

These results are analogous to results in [8], where similar properties were deduced after the spaces of Roumieu distributions, Gelfand-Shilov

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distributions and Gelfand-Shilov functions are replaced by corresponding test function and distribution spaces in the standard distribution theory.

In order to describe our results in more details, we recall the definitions of positive operators and elements which are positive with respect to non-commutative convolutions.

Let \( \mathcal{B} \) be a topological vector space with complex dual \( \mathcal{B}' \), and let \( T \) be a linear and continuous operator from \( \mathcal{B} \) to \( \mathcal{B}' \). Then \( T \) is called positive (semi-definite), and is written \( T \geq 0 \), whenever \( (T\varphi, \varphi) \geq 0 \) for every \( \varphi \in \mathcal{B} \). In our situation \( \mathcal{B} \) is usually a Gelfand-Shilov space or a Gevrey class on \( \mathbb{R}^d \), and \( \mathcal{B}' \) corresponding distribution space. However we remark that we may also as in [8] consider the case when \( \mathcal{B} \) is the set of Schwartz functions and \( \mathcal{B}' \) the set of tempered distributions on \( \mathbb{R}^d \).

In any of these situations, the Schwartz kernel theorem is valid in the sense that for any linear and continuous operator from \( \mathcal{B} \) to \( \mathcal{B}' \), there is a distribution \( K = K_T \in \mathcal{B}' \otimes \mathcal{B}' \) such that
\[
(T\varphi, \psi) = (K, \psi \otimes \overline{\varphi}), \quad \text{for every } \varphi, \psi \in \mathcal{B}. \tag{0.1}
\]

We are especially interested in properties possessing by elements \( K \in \mathcal{B}' \otimes \mathcal{B}' \) such that \( T = T_K \), defined by (0.1), are positive as operators. Note here that if \( \mathcal{B} \) is a Gelfand-Shilov space or the set of Schwartz functions on \( \mathbb{R}^d \), then \( \mathcal{B} \otimes \mathcal{B} \) is the corresponding space of functions defined on \( \mathbb{R}^{2d} \), and \( \mathcal{B}' \otimes \mathcal{B}' \) the corresponding distribution space on \( \mathbb{R}^{2d} \).

An important special case of \( T \) concerns non-commutative convolution operators of the form \( \varphi \mapsto a \ast_B \varphi \), defined by
\[
(a \ast_B \varphi)(x) \equiv \int a(x - y) \varphi(y) B(x,y) dy.
\]
In this situation we are for example interested in extensions of Bochner-Schwartz theorem concerning positive elements in non-commutative convolutions.

An important non-commutative convolution is the twisted convolution, which is obtained by choosing \( B \) above as an appropriate complex Gaussian. More precisely, let \( X = (x, \xi) \in \mathbb{R}^{2d} \) and \( Y = (y, \eta) \in \mathbb{R}^{2d} \), and let \( \sigma \) be the standard symplectic form defined by \( \sigma(X,Y) = \langle y, \xi \rangle - \langle x, \eta \rangle \). Then the twisted convolution is defined by
\[
(a \ast_\sigma b)(X) \equiv (2/\pi)^{d/2} \int a(X - Y) b(Y) e^{2i\sigma(X,Y)} dY.
\]

Positivity in the twisted convolution is closely related to positivity in operator theory, especially the Weyl calculus. More precisely, if \( a \in \mathcal{B} \otimes \mathcal{B} \), then \( A_a \) is the operator with Schwartz kernel giving by
\[
(A_a)(x,y) = (2\pi)^{-d/2} \int a((y - x)/2, \xi) e^{-i(x+y, \xi)} d\xi. \tag{0.2}
\]
Here and in what follows we identify operators with their kernels. The operator $A$ in (0.2) is continuous on $B \otimes B$, and extends uniquely to a continuous map on $B' \otimes B'$.

The main relationship on positivity is that $Aa \geq 0$, if and only if $a$ belongs to $(B' \otimes B')_+$, the set of all elements in $B' \otimes B'$ which are positive with respect to the twisted convolution. (Cf. Proposition 1.10 in [8].)

There are also strong links between positivity in operator theory (or equivalently elements in $(B' \otimes B')_+$) and positive Weyl operators. In fact, for $a \in S(R^{2d})$, the Weyl quantization $\text{Op}_w(a)$, is the operator from $S(R^d)$ to $S'(R^d)$, defined by the formula

$$(\text{Op}_w(a)f)(x) = (2\pi)^{-d} \int\int a((x+y)/2,\xi)f(y)e^{i(x-y,\xi)}dyd\xi.$$ 

The integral kernel of $\text{Op}_w(a)$ is equal to

$$(x,y) \mapsto (2\pi)^{-d/2}(Aa)(-x,y). \tag{0.3}$$

For arbitrary $a \in B' \otimes B'$, $\text{Op}_w(a)$ is defined as the operator with kernel given by (0.3).

By straightforward computations we get

$$\text{Op}_w^*(a) = (2\pi)^{-d/2}A(F_\sigma a),$$

where $F_\sigma$ is the symplectic Fourier transform on $B' \otimes B'$, which takes the form

$$(F_\sigma a)(X) = \hat{a}(X) = \frac{1}{\pi} \int a(Y)e^{i\sigma\langle X,Y \rangle}dY,$$

when $a \in \mathcal{S}(R^{2d})$. Consequently, from these identities it follows that $\text{Op}_w^*(a) \geq 0$, if and only if $(F_\sigma a) \in (B' \otimes B')_+$.

In the paper we begin to study kernels in Roumieu distribution spaces $D'_s$, $s > 1$, whose corresponding operators are positive semi-definite. In fact, if $K$ is such kind of kernel whose restriction near the diagonal belongs to corresponding Gelfand-Shilov distribution space $S'_s$, then we prove that $K$ belongs to $S'_s$.

By choosing $K$ such that $T_K$ agree with the convolution operator $\varphi \mapsto a \ast_B \varphi$, for some $a \in D'_s$, it follows that $a \in S'_s$ when the operator is positive. In particular, this holds for elements which are positive with respect to twisted convolution.

We also perform investigations on elements in $(B' \otimes B')_+$ which satisfy certain smoothness condition at origin. More precisely, if $a \in (B' \otimes B')_+$ is Gevrey regular of order $s$, then we prove that $a$ belongs to $S'_s(R^{2d})$. This result is analogous to Theorem 3.13 in [8], which asserts that for any $a \in (B' \otimes B')_+$ which is smooth near origin, belongs to $\mathcal{S}(R^{2d})$. 
1. Preliminaries

In this section we recall some basic results which are needed. In the first part we recall some facts about Gelfand-Shilov spaces.

In the following we let $\alpha, \beta, \gamma$ and $\delta$ denote multi-indices.

Let $s \geq 1/2$ and $h > 0$. Then $S_{s,h}(\mathbb{R}^d)$ is the set of all $\varphi \in C^\infty(\mathbb{R}^d)$ such that the norm

$$
\|\varphi\|_{S_{s,h}} \equiv \sup_{\alpha, \beta \in \mathbb{N}^d} \sup_{x \in \mathbb{R}^d} \left| x^\alpha D^\beta \varphi(x) \right| \left( \alpha! \beta! s h^{\alpha + \beta} \right)
$$

is finite for every multi-indices $\alpha$ and $\beta$. Then the Gelfand-Shilov space $S_s(\mathbb{R}^d)$ is given by

$$
S_s(\mathbb{R}^d) = \lim_{h \to \infty} S_{s,h}(\mathbb{R}^d).
$$

Its dual space is denoted by $S'_s(\mathbb{R}^d)$, which is the Gelfand-Shilov distribution space of order $s$.

Let $s \geq 1/2$, $h > 0$ and let $\Omega$ be an open set in $\mathbb{R}^d$. For a given compact set $K \subset \Omega$, $D_{s,h}(K)$ is the set of all $\varphi \in C^\infty(\mathbb{R}^d)$ such that $\text{supp} \ \varphi \subseteq K$ and the norm

$$
\|\varphi\|_{D_{s,h}} \equiv \sup_{\beta \in \mathbb{N}^d} \sup_{x \in K} \left| D^\beta \varphi(x) \right| \left( \beta! s h^{|\beta|} \right)
$$

is finite for the multi-index $\beta$. Let $(K_n)_n$ be a sequence of compact sets such that $K_n \subset \subset K_{n+1}$, and $\bigcup K_n = \Omega$. Then the space $D_s(\Omega)$ is given by

$$
D_s(\Omega) = \lim_{n \to \infty} (\lim_{h \to \infty} D_{s,h}(K_n)).
$$

Its dual space is $D'_s(\Omega)$, which is the ultra-distribution space of Roumieu type of order $s$. We remark that $D_s$ is equivalent to Gevery class $G^s$ for $s > 1$.

We recall that differentiations of Gevrey or Gelfand-Shilov distributions are defined in the usual way, giving that most of the usual rules hold. Especially it follows from Leibniz rule that

$$
D^\alpha x^\beta f(x) = \sum_{\alpha_0 \leq \alpha, \beta} (-i)^{\alpha_0} \binom{\alpha}{\alpha_0} \binom{\beta}{\alpha_0} \alpha_0! x^{\beta - \alpha_0} D^{\alpha_0} f(x), \quad (1.1)
$$

for admissible distribution $f$. Furthermore, by applying the Fourier transform to this formula we get

$$
x^\alpha D^\beta f(x) = \sum_{\alpha_0 \leq \alpha, \beta} (-1)^{\gamma} \binom{\alpha}{\alpha_0} \binom{\beta}{\alpha_0} \alpha! D^{\beta - \alpha_0} (x^{\alpha - \alpha_0} f(x)). \quad (1.2)
$$

Since we are especially interested in the behaviour of corresponding Schwartz kernels, the following result is important to us. The proof is omitted since it can be found in [5].

**Theorem 1.1.** Let $\Omega_j \subset \mathbb{R}^{d_j}$ be open set, where $j = 1, 2$. Then the following are true.
(1) If \( s > 1 \) and \( T \) is a linear and continuous operator from \( D_s(\Omega_1) \) to \( D_s'(\Omega_2) \), then there is a unique ultradistribution \( K = K_T \in D_s'(\Omega_2 \times \Omega_1) \) such that
\[
(T \varphi_1, \varphi_2) = (K, \varphi_2 \otimes \varphi_1), \quad \varphi_1 \in D_s(\Omega_1), \varphi_2 \in D_s(\Omega_2).
\] (1.3)

Conversely, if \( K = K_T \in D_s'(\Omega_2 \times \Omega_1) \) and \( T \) is defined by (1.3), then \( T \) is a linear and continuous operator from \( D_s(\Omega_1) \) to \( D_s'(\Omega_2) \).

(2) If \( s \geq 1/2 \) and \( T \) is a linear and continuous operator from \( S_s(\mathbb{R}^{d_1}) \) to \( S'_s(\mathbb{R}^{d_2}) \), then there is a unique tempered ultradistribution \( K = K_T \in S'_s(\mathbb{R}^{d_2} \times \mathbb{R}^{d_1}) \) such that
\[
(T \varphi_1, \varphi_2) = (K, \varphi_2 \otimes \varphi_1), \quad \varphi_1 \in S_s(\mathbb{R}^{d_1}), \varphi_2 \in S_s(\mathbb{R}^{d_2}).
\] (1.4)

Conversely, if \( K = K_T \in S'_s(\mathbb{R}^{d_2} \times \mathbb{R}^{d_1}) \) and \( T \) is defined by (1.4), then \( T \) is a linear and continuous operator from \( S_s(\mathbb{R}^{d_1}) \) to \( S'_s(\mathbb{R}^{d_2}) \).

If \( K \) is given, then \( T_K = T \) is defined by (1.3).

Recall that if \( s \geq 1/2 \), and \( T \) is a linear and continuous operator from \( S_s(\mathbb{R}^d) \) to \( S'_s(\mathbb{R}^d) \), then \( T \) is called positive semi-definite if
\[
(T \varphi, \varphi)_{L^2} \geq 0,
\] (1.5)
for every \( \varphi \in S_s(\mathbb{R}^d) \), and then we write \( T \geq 0 \). Furthermore, if more restrictive \( s > 1 \), \( \Omega \subset \mathbb{R}^d \) is an open set, and \( T \) is a linear and continuous operator from \( D_s(\Omega) \) to \( D'_s(\Omega) \), then \( T \) is still called positive semi-definite when (1.5) holds for every \( \varphi \in D_s(\Omega) \).

Since \( D_s(\mathbb{R}^d) \) is dense in \( S_s(\mathbb{R}^d) \) when \( s > 1 \), it follows that if \( T \) from \( D_s(\mathbb{R}^d) \) to \( D'_s(\mathbb{R}^d) \) is positive semi-definite and extendable to a continuous map from \( S_s(\mathbb{R}^d) \) to \( S'_s(\mathbb{R}^d) \), then this extension is unique and \( T \) is positive semi-definite as an operator from \( S_s(\mathbb{R}^d) \) to \( S'_s(\mathbb{R}^d) \).

We have now the following definition.

**Definition 1.2.** Let \( \Omega \subset \mathbb{R}^d \) be open.

1. If \( s > 1 \), then \( D'_{0,s}(\Omega \times \Omega) \) consists of all \( K \in D'_s(\Omega \times \Omega) \) such that \( T_K \) is a positive semi-definite operator from \( D_s(\Omega) \) to \( D'_s(\Omega) \).

2. If \( s \geq 1/2 \), then \( S'_{0,s}(\mathbb{R}^d \times \mathbb{R}^d) \) consists of all \( K \in S'_s(\mathbb{R}^d \times \mathbb{R}^d) \) such that \( T_K \) is a positive semi-definite operator from \( S_s(\mathbb{R}^d) \) to \( S'_s(\mathbb{R}^d) \).

We shall also consider distributions which are positive with respect to a non-commutative convolution.

Let \( s > 1 \), and \( B \in C_c(\mathbb{R}^{2d}) \) such that for every \( \varepsilon > 0 \), it holds
\[
\sup_{x,y} \sup_{\alpha} e^{-\varepsilon (|x|^{1/s} + |y|^{1/s})} \left( \frac{|D^\alpha (B(x,y))^{-1}| \, h |\alpha| \alpha!}{(\alpha!)^{1/s} |\alpha|^{1/s} |\alpha|} + \frac{|D^\alpha B(x,y)| \, h |\alpha| \alpha!}{(\alpha!)^{1/s} |\alpha|^{1/s} |\alpha|} \right) < \infty,
\] (1.6)
for some \( h > 0 \).
Lemma 1.3. Let $B \in C_s(\mathbb{R}^{2d})$ satisfying (1.6). Then the following are true.

1. $\Phi \in S_s(\mathbb{R}^{2d})$ if and only if $B \cdot \Phi \in S_s(\mathbb{R}^{2d})$.
2. $\Phi \in S_s'(\mathbb{R}^{2d})$ if and only if $B \cdot \Phi \in S_s'(\mathbb{R}^{2d})$.

Proof. The result follows from Theorem A in [9]. □

Let $a \in D_s'(\mathbb{R}^d)$ such that

$$(a \ast_B \varphi)(x) \equiv \langle a(x-\cdot), B(x,\cdot)\varphi \rangle,$$

when $\varphi \in D_s(\mathbb{R}^d)$. Then the kernel of the map $\varphi \mapsto u \ast_B \varphi$ is given by $K(x, y) = a(x-y)B(x, y)$. We note that $K(x, y) \in D_s'(\mathbb{R}^{2d})$.

Let $D_{s,+}((\mathbb{R}^d)^{\mathbb{N}})$ be the set of all $a \in D_s'(\mathbb{R}^d)$ such that the map $\varphi \mapsto a \ast_B \varphi$ is positive. Also let $S_{s,+}(\mathbb{R}^d)$ be the set of all $a \in S_s'(\mathbb{R}^d)$ such that the map $\varphi \mapsto a \ast_B \varphi$ is positive.

We consider elements $a$ in Gelfand-Shilov classes of distributions which are positive with respect to the twisted convolution. That is, $a$ should fulfill

$$(a \ast_B \varphi, \varphi) \geq 0,$$

for every $\varphi$. For this reason we make the following definition.

Definition 1.4. Let $s \geq 1/2$.

1. $S_{s,+}(\mathbb{R}^{2d})$ is the set of all $a \in S_s'(\mathbb{R}^{2d})$ such that (1.7) holds for every $\varphi \in S_s(\mathbb{R}^{2d})$.
2. If in addition $s > 1$, then $D_{s,+}((\mathbb{R}^d)^{\mathbb{N}})$ is the set of all $a \in D_s'(\mathbb{R}^{2d})$ such that (1.7) holds for every $\varphi \in D_s(\mathbb{R}^{2d})$.
3. $S_{s,+}(\mathbb{R}^{2d})$ is the set of all $a \in S_s'(\mathbb{R}^{2d})$ such that (1.7) holds for every $\varphi \in S_s(\mathbb{R}^{2d})$.
4. $D_{s,+}(\mathbb{R}^{2d})$ is the set of all $a \in D_s'(\mathbb{R}^{2d})$ such that (1.7) holds for every $\varphi \in C_0^\infty(\mathbb{R}^{2d})$.
5. The set $C_{s,+}(\mathbb{R}^{2d})$ consists of all $a \in C(\mathbb{R}^{2d})$ such that

$$\sum_{j,k} a(X_j - X_k) e^{2i\sigma(X_j,X_k) c_j c_k} \geq 0,$$

for every finite sets

$$\{X_1, X_2, \ldots, X_N\} \subseteq \mathbb{R}^{2d} \quad \text{and} \quad \{c_1, c_2, \ldots, c_N\} \subseteq \mathbb{C}.$$

The following result can be found in the Theorem 2.6, Proposition 3.2 and Theorem 3.13 in [3]. Later on we shall prove an analogue in the frame-work Gelfand-Shilov spaces.

Proposition 1.5. Let $\Omega \subseteq \mathbb{R}^{2d}$ be a neighborhood of the origin. Then the following are true.

1. $D_{s,+}(\mathbb{R}^{2d}) = S_{s,+}(\mathbb{R}^{2d})$.
2. $C_{s,+}(\mathbb{R}^{2d}) \subseteq S_{s,+}(\mathbb{R}^{2d}) \cap L^2(\mathbb{R}^{2d}) \cap L^\infty(\mathbb{R}^{2d}) \cap \mathcal{F} L^\infty(\mathbb{R}^{2d})$.
3. $\mathcal{S}'(\mathbb{R}^{2d}) \cap C(\Omega) = C_{s,+}(\mathbb{R}^{2d})$.
4. $\mathcal{S}'_{s,+}(\mathbb{R}^{2d}) \cap C(\Omega) = C_{s,+}(\mathbb{R}^{2d}) \cap \mathcal{F}(\mathbb{R}^{2d})$. 
By Proposition 1.5 in \([2]\) and the definitions we have the following.

**Proposition 1.6.** Let \(s \geq 1/2, a \in S'_s(\mathbb{R}^{2d})\) be such that \(Aa\) is a trace-class operator on \(L^2(\mathbb{R}^d)\). Then \(a \in L^\infty(\mathbb{R}^{2d})\), and

\[
\|a\|_{L^\infty} \leq (2/\pi)^{d/2}\|Aa\|_{T^d}.
\]

### 2. Gelfand-Shilov properties for kernels to positive operators

In this section, we study the kernel of a positive semi-definite operator. If the kernel belongs to \(S'_s\) along the diagonal, then it belongs to \(S'_s\) everywhere for \(s > 1\). As a application we prove that \(D''_{B,s,+}(\mathbb{R}^d) = S''_{B,s,+}(\mathbb{R}^d)\).

**Theorem 2.1.** Let \(s > 1\). Assume that \(K \in D'_{0,s}(\mathbb{R}^{2d})\), and \(K_\chi(x, y) \in S'_{2d}(\mathbb{R}^{2d})\), where \(K_\chi(x, y) = \chi(x-y)K(x, y)\), for some \(\chi \in D_s(\mathbb{R}^d)\) which satisfies \(\chi(0) \neq 0\). Then \(K \in S'_{0,s}(\mathbb{R}^{2d})\).

We need the following preparations for the proof of Theorem 2.1.

Here let \(s \geq 1/2, h, h_1\) and \(h_2 > 0\), and let \(\psi \in S_s(\mathbb{R}^{2d})\). The semi-norms \(\|\psi\|_{S,s,h}^{(1)}\) and \(\|\psi\|_{S,s,h_1,h_2}^{(2)}\) are given by

\[
\|\psi\|_{S,s,h}^{(1)} \equiv \sup_{\alpha, \beta} \sup_{x,y} \left| x^{\alpha_1}y^{\alpha_2}D^\beta_1 D^\beta_2 \psi(x, y) \right|
\]

\[
\|\psi\|_{S,s,h_1,h_2}^{(2)} \equiv \sup_{\alpha, \beta} \sup_{x,y} \frac{e^{h_2|y|^{1/\alpha}}}{(\alpha\beta_1!\beta_2!)} \left| x^{\alpha}D^\beta_1 D^\beta_2 \psi(x, y) \right|
\]

**Lemma 2.2.** Let \(s \geq 1/2\), and let \(\psi_0 \in C^\infty(\mathbb{R}^{2d})\). Then the following are true:

1. For every \(h > 0\), there are \(C, h_1, h_2 > 0\) such that

\[
\|\psi_0\|_{S_s,h}^{(1)} \leq C\|\psi_0\|_{S_s,h_1,h_2}^{(2)}.
\]

2. For every \(h_1, h_2 > 0\), there are \(C, h > 0\) such that

\[
\|\psi_0\|_{S_s,h_1,h_2}^{(2)} \leq C\|\psi_0\|_{S_s,h}^{(1)}.
\]

3. Let \(\psi_1\) and \(\psi_2\) be given by

\[
\psi_1(x, y) = \psi_0(x, x-y), \quad \text{and} \quad \psi_2(x, y) = \psi_0(x+y, y).
\]

Then

\[
\|\psi_1\|_{S_s,h}^{(1)} \leq \|\psi_0\|_{S_s,h}^{(1)}, \quad \text{and} \quad \|\psi_0\|_{S_s,h}^{(1)} \leq \|\psi_2\|_{S_s,h}^{(1)},
\]

for \(j = 0, 1, 2\).

**Proof.** The assertions (1) and (2) follow from Corollary 2.5 in \([1]\) and its proof. The assertion (3) follows by straightforward elaborations with the semi-norm \(\|\psi_j\|_{S_s,h}^{(1)}\), for \(j = 0, 1, 2\). In order to be self-contained we here give the proof.
We only prove the result in the first inequality in (3) for $j = 2$. We have

$$
\left| \frac{(x-y)^{\alpha_1}y^{\alpha_2}\psi_0(x,y)}{(\alpha_1!\alpha_2!)^* h^{[\alpha_1+\alpha_2]}} \right| \leq \sum_{\gamma \leq \alpha_1} \frac{\alpha_1}{(\alpha_1-\gamma)!(\alpha_1\alpha_2)!^*} \frac{|x^{\alpha_1-\gamma}y^{\alpha_2+\gamma}\psi_0(x,y)|}{h^{[\alpha_1+\alpha_2]}} 
$$

$$
= \sum_{\gamma \leq \alpha_1} \left( \frac{\alpha_1}{(\alpha_1-\gamma)!\gamma!} \right)^{1-s} \left( \frac{\alpha_2+\gamma}{\alpha_2!\gamma!} \right)^s \frac{|x^{\alpha_1-\gamma}y^{\alpha_2+\gamma}\psi_0(x,y)|}{((\alpha_1-\gamma)!(\alpha_2+\gamma)!^* h^{[\alpha_1+\alpha_2]}} 
$$

$$
\leq 2^{\alpha_1(2-s)\alpha_2} \sup_{\gamma} \frac{|x^{\alpha_1-\gamma}y^{\alpha_2+\gamma}\psi_0(x,y)|}{((\alpha_1-\gamma)!(\alpha_2+\gamma)!^* h^{[\alpha_1+\alpha_2]}} 
$$

Similarly,

$$
\frac{|D_x^{\beta_1}D_y^{\beta_2}\psi_2(x,y)|}{(\beta_1!\beta_2!)^* h^{[\beta_1+\beta_2]}} = \frac{|D_x^{\beta_1}D_y^{\beta_2}\psi_0(x+y,y)|}{(\beta_1!\beta_2!)^* h^{[\beta_1+\beta_2]}} 
$$

$$
= \left( \frac{\beta_2}{\delta} \right) \frac{|D_x^{\beta_1+\delta}D_y^{\beta_2-\delta}\psi_0(x+y,y)|}{(\beta_1!\beta_2!)^* h^{[\beta_1+\beta_2]}} 
$$

$$
\leq 4^{[\beta_1+\beta_2]} \sup_{\delta} \frac{|D_x^{\beta_1+\delta}D_y^{\beta_2-\delta}\psi_0(x+y,y)|}{((\beta_1+\delta)!((\beta_2-\delta)!^* h^{[\beta_1+\beta_2]}}. 
$$

A combination of these arguments give

$$
\|\psi_2\|^{(1)}_{S_{s,h}} = \sup_{\alpha_1,\alpha_2,\beta_1,\beta_2} \sup_{x,y} \frac{|x^{\alpha_1}y^{\alpha_2}D_x^{\beta_1}D_y^{\beta_2}\psi_2(x,y)|}{(\alpha_1!\alpha_2!(\beta_1!\beta_2)!^* h^{[\alpha_1+\alpha_2+\beta_1+\beta_2]}} 
$$

$$
\leq 4^{[\alpha_1+\alpha_2+\beta_1+\beta_2]} \sup_{x,y} \frac{|x^{\alpha_1-\gamma}y^{\alpha_2+\gamma}D_x^{\beta_1+\delta}D_y^{\beta_2-\delta}\psi_0(x,y)|}{((\alpha_1-\gamma)!(\alpha_2+\gamma)!(\beta_1+\delta)!((\beta_2-\delta)!^* h^{[\alpha_1+\alpha_2+\beta_1+\beta_2]}}. 
$$

The other cases follow by repeating this argument, and are left for the reader. The proof is complete.

**Lemma 2.3.** Let $s \geq 1/2$, $\psi_0 \in C^\infty(R^{2d}) \cap S'_s(R^{2d})$, and set $\psi_1$ and $\psi_2$ be given by (2.1). If $k = 0, 1, 2$, then the following conditions are equivalent:

1. $\psi_0 \in S_s(R^{2d})$,
2. $\psi_k \in S_s(R^{2d})$,
for some positive constants $C$, $h_1$ and $h_2$, it holds
\[ |x^\alpha D_x^\beta \psi_k(x, y)| \leq C(\alpha!\beta!)^s h_1^{\alpha+\beta} e^{-h_2|y|^1/s}, \]
\[ |\xi^\alpha D_\xi^\beta \psi_k(\xi, \eta)| \leq C(\alpha!\beta!)^s h_1^{\alpha+\beta} e^{-h_2|\eta|^1/s}. \]

Furthermore, the mappings which take $\psi_0$ into $\psi_1$ or $\psi_2$ are homeomorphism on $S_s(\mathbb{R}^d)$.

**Proof.** The result follows from Corollary 2.5 in [1] and its proof together with the fact that $S_s$ is invariant under pullbacks of linear bijections. The details are left for the reader. $\square$

**Proof of Theorem 2.1.** Let $(\cdot, \cdot)_K$ be the semi-scalar product on $D_s(\mathbb{R}^d)$ given by
\[ (\varphi, \psi)_K = (K, \psi \otimes \overline{\varphi}), \] for every $\varphi, \psi \in D_s(\mathbb{R}^d)$. Also let $\| \cdot \|_K$ be the corresponding semi-norm, i.e., $\| \varphi \|_K$ is defined by
\[ \| \varphi \|_K^2 = (\varphi, \varphi)_K = (K, \varphi \otimes \overline{\varphi}), \] when $\varphi \in D_s(\mathbb{R}^d)$. Since $D_s(\mathbb{R}^d)$ is dense in $S_s(\mathbb{R}^d)$, the result follows if we prove that for every positive $h$, there is a positive constant $C = C_h$, such that
\[ |(\varphi, \psi)_K| \leq C \| \varphi \|_{S_{s,h}}^{(1)} \| \psi \|_{S_{s,h}}^{(1)}, \quad (2.2) \] when $\varphi, \psi \in D_s(\mathbb{R}^d) \cap S_{s,h}(\mathbb{R}^d)$.

Since $\chi(0) \neq 0$, and $K_\chi \in S'_s(\mathbb{R}^{2d})$, it follows from Theorem 4.1.23 in [2] that $1/\chi \in C_s$ near the origin, and $\kappa/\chi \in D_s(\mathbb{R}^d)$ for some $\kappa \in D_s(\mathbb{R}^d)$ which is equal to 1 in a neighborhood $\Omega_0$ of the origin. Hence
\[ K_\kappa(x, y) = \frac{\kappa(x - y)}{\chi(x - y)} \cdot K_\chi(x, y) \in S'_s(\mathbb{R}^{2d}), \] since it is obvious that the map $(\varphi, K) \mapsto \varphi(x-y)K(x,y)$ is continuous from $D_s(\mathbb{R}^d) \times S'_s(\mathbb{R}^{2d})$ to $S'_s(\mathbb{R}^{2d})$. Consequently, we may assume that $\chi$ in the assumption is equal to 1 in a neighborhood $\Omega$ of the origin.

Take an even and non-negative function $\phi \in D_s(\mathbb{R}^d)$, such that $\sum_j \phi(\cdot - x_j) = 1$, for some lattice $\{x_j\}_{j \in J} \subset \mathbb{R}^d$, and $\text{supp } \phi + \text{supp } \phi \subset \Omega$. By Cauchy-Schwartz inequality we get
\[ |(\varphi, \psi)_K| \leq \sum_{j,k \in J} |(\varphi_j, \psi_k)_K| \leq \sum_{j,k \in J} \| \varphi_j \|_K \| \psi_k \|_K, \quad \varphi, \psi \in D_s(\mathbb{R}^d), \] where
\[ \varphi_j(x) = \varphi(x)\phi(x-x_j), \quad \psi_j(x) = \psi(x)\phi(x-x_j). \]
Then (2.22) follows if we prove that for every \( h > 0 \), there are \( h_1 > 0, C > 0 \) such that
\[
\| \varphi_j \|_K \leq C \left( \sup_{\alpha, \beta \in \mathbb{N}^d} \sup_{x \in \mathbb{R}^d} \frac{|x^\alpha D^\beta \varphi(x)|}{(\alpha! \beta!) h^{\alpha + \beta}} \right) e^{-h_1|x_j|^{1/s}} = C \| \varphi \|_{S_{s,h}^{(1)}} (1) \cdot e^{-h_1|x_j|^{1/s}}. \tag{2.3}
\]

In order to prove (2.3), we note that the support of \( \varphi_j(\cdot + x_j) \) is contained in \( \text{supp} \phi \). This gives
\[
\| \varphi_j \|_K = (K_{j,\chi}, \varphi_j(\cdot + x_j) \otimes \varphi_j(\cdot + x_j)),
\]
where
\[
K_{j,\chi}(x, y) = K(x + x_j, y + x_j) \chi(x - y).
\]
It follows from the definitions that for every \( \varepsilon > 0 \),
\[
e^{-\varepsilon|x_j|^{1/s}} K_{j,\chi}
\]
is bounded in \( S'_0(\mathbb{R}^{2d}) \) with respect to \( j \in J \). Then for every positive \( \varepsilon \) and \( h \), there is a constant \( C_{\varepsilon,h} \) such that
\[
\| \varphi_j \|_K \leq C_{\varepsilon,h} e^{\varepsilon|x_j|^{1/s}/2} \sup_{\alpha, \beta \in \mathbb{N}^d} \sup_x \frac{|x^\alpha D^\beta_j \varphi_j(x + x_j)|}{(\alpha! \beta!) h^{\alpha + \beta}}. \tag{2.4}
\]

Let \( \psi_0(x, y) = \varphi(x) \phi(y) \), and let \( \psi_1 \) and \( \psi_2 \) be as in Lemma 2.3. Then \( \psi_2(x, x_j) = \varphi_j(x + x_j) \). By Lemmas 2.2 and 2.3, it follows that for every \( h > 0 \), there are constants \( C, h_1, h_2 > 0 \) such that
\[
\| \psi_2 \|_{S_{s,h_1,h_2}^{(2)}} \leq C \| \psi_2 \|_{S_{s,h}^{(1)}} \leq C \| \psi_0 \|_{S_{s,h}^{(1)}}.
\]
Consequently,
\[
\sup_{\alpha, \beta_1, \beta_2, x, y} e^{h_2 |y|^{1/s}} \frac{|x^\alpha D^\beta_2 \varphi_j(x + x_j)|}{(\alpha! \beta_1 \beta_2! h_1^{\alpha + \beta_1 + \beta_2})} \leq C \| \psi_0 \|_{S_{s,h}^{(1)}},
\]
giving that
\[
|x^\alpha D^\beta_x \varphi_j(x + x_j)| = |x^\alpha D^\beta_x \psi_2(x, x_j)| 
\leq C e^{-h_2|x_j|^{1/s}} (\alpha! \beta! h_1^{\alpha + \beta}) ||\psi_0||_{S_{s,h}^{(1)}}, \tag{2.5}
\]
if we choose \( \beta_1 = \beta \) and \( \beta_2 = 0 \).

Then the inequalities (2.4) and (2.5) imply that for every \( h_3 > 0 \), there is a constant \( C_1 \) such that
\[
\| \varphi_j \|_K \leq C_1 e^{\varepsilon(1 + 2)} e^{-h_2|x_j|^{1/s}} \left( \frac{h_1}{h_3} \right)^{\alpha + \beta} \| \psi_0 \|_{S_{s,h}^{(1)}} \leq C_2 e^{-(h_2 - \varepsilon/2)|x_j|^{1/s}} \left( \frac{h_1}{h_3} \right)^{\alpha + \beta} \| \varphi \|_{S_{s,h}^{(1)}},
\]
and (2.2) follows if we choose \( h_3 \) and \( \varepsilon \) such that \( h_1 < h_3 \) and \( \varepsilon < 2h_2 \). \( \square \)
By choosing \( K(x, y) = a(x - y)B(x, y) \) in Theorem 2.1 for suitable \( a \) and \( B \), we get the following result, which shows (1) in Proposition 1.5 has an analogue in the framework of Gelfand-Shilov space or Gevrey class.

**Theorem 2.4.** Let \( s > 1 \) and \( B \in C_s(\mathbb{R}^{2d}) \) be such that (1.6) holds. Then \( \mathcal{D}'_{B,s,+}(\mathbb{R}^d) = \mathcal{S}'_{B,s,+}(\mathbb{R}^d) \). In particular, \( \mathcal{D}'_{s,+}(\mathbb{R}^{2d}) = \mathcal{S}'_{s,+}(\mathbb{R}^{2d}) \).

**Proof.** It is sufficient to prove the first inclusion. By straightforward computation it follows that \( T\varphi = a \ast B \varphi \) and the kernel \( K(x, y) = a(x - y)B(x, y) \in \mathcal{D}'_{0,s}(\mathbb{R}^{2d}) \subseteq \mathcal{S}'_{0,s}(\mathbb{R}^{2d}) \).

Furthermore, Lemma 1.3 giving that \( B(x, y) \cdot K(x, y) = a(x - y) \in \mathcal{S}'_{s}(\mathbb{R}^{2d}) \).

\[ \square \]

### 3. Gelfand-Shilov properties for positive elements for twisted convolutions

In this section, we study elements in \( \mathcal{S}'_+ \), which are smooth and have the Gevrey regularity near the origin, then they are in \( \mathcal{S}_s \) for \( s \geq 1/2 \). Here \( \mathcal{S}'_+ \) is a set such that for all elements \( a \in \mathcal{D}' \) which are positive with respect to the twisted convolution.

The following theorem is the main result. It shows that (4) in Proposition 1.5 has an analogue in the framework of Gelfand-Shilov space or Gevrey class.

**Theorem 3.1.** Let \( s \geq 1/2 \) and \( a \in \mathcal{S}'_{1/2}(\mathbb{R}^{2d}) \cap C^\infty(\Omega) \) be such that

\[ \sup_{\beta \in \mathbb{N}^{2d}} \left( \sup_{X \in \Omega} \frac{|D^\beta a(X)|}{(2!)^{h|\beta|}} \right) < \infty, \quad (3.1) \]

for some neighborhood \( \Omega \) of the origin, and some \( h > 0 \). Then \( a \in \mathcal{S}_s(\mathbb{R}^{2d}) \).

In order to prove the theorem, we need some preparations.

Let

\[ T_j = (2i)^{-1}\partial_{\xi_j} - x_j, \quad \Theta_j = (2i)^{-1}\partial_{x_j} - \xi_j, \]

\[ P_j = (2i)^{-1}\partial_{\xi_j} + x_j, \quad \Pi_j = (2i)^{-1}\partial_{x_j} + \xi_j, \quad (3.2) \]

Then

\[ x_j = (P_j - T_j)/2, \quad \xi_j = (\Pi_j - \Theta_j)/2, \]

\[ \partial_{x_j} = i(\Pi_j + \Theta_j), \quad \partial_{\xi_j} = i(P_j + T_j). \quad (3.3) \]

for \( j = 1, ..., d \).
Lemma 3.2. Let $\Omega \subset \mathbb{R}^{2d}$ be a neighborhood of the origin, and let $a \in C^\infty(\Omega)$ and $s \geq 1/2$. Then the following statements are equivalent.

1. There are positive constants $C$ and $h$ such that
   $$|D^\alpha a(x, \xi)| \leq Ch^{s}(\alpha!)^s, \quad (x, \xi) \in \Omega,$$
   for the multi-index $\alpha$.

2. There are positive constants $C$ and $h$ such that
   $$|(P^{\alpha} \circ T^{\beta} \circ \Theta^{\gamma} \circ \Pi^{\delta} a)(x, \xi)| \leq Ch^{s}(\alpha+\beta+\gamma+\delta)(\alpha!\beta!\gamma!\delta!)^s, \quad (x, \xi) \in \Omega,$$
   for every multi-indices $\alpha$, $\beta$, $\gamma$ and $\delta$.

Lemma 3.3. Let $\alpha$, $\beta$, $P_j$, $T_j$, $\Theta_j$, and $\Pi_j$ be defined as before, then

$$P^{\alpha} \circ T^{\beta} = T^{\beta} \circ P^{\alpha}, \quad \Pi^{\alpha} \circ \Theta^{\beta} = \Theta^{\beta} \circ \Pi^{\alpha}, \quad \Pi^{\alpha} \circ \Pi^{\beta} = \Pi^{\beta} \circ \Pi^{\alpha}, \quad T^{\alpha} \circ \Theta^{\beta} = \Theta^{\beta} \circ T^{\alpha}.$$

and

$$P^{\alpha} \circ \Theta^{\beta} = \sum_{\alpha_0 \leq \alpha, \beta} i^{\alpha_0} \left( \frac{\alpha}{\alpha_0} \right) \left( \frac{\beta}{\alpha_0} \right) \alpha_0! \Theta^{\beta-\alpha_0} \circ P^{\alpha-\alpha_0},$$

$$\Theta^{\alpha} \circ P^{\beta} = \sum_{\alpha_0 \leq \alpha, \beta} (-i)^{\alpha_0} \left( \frac{\alpha}{\alpha_0} \right) \left( \frac{\beta}{\alpha_0} \right) \alpha_0! P^{\beta-\alpha_0} \circ \Theta^{\alpha-\alpha_0},$$

$$T^{\alpha} \circ \Pi^{\beta} = \sum_{\alpha_0 \leq \alpha, \beta} (-i)^{\alpha_0} \left( \frac{\alpha}{\alpha_0} \right) \left( \frac{\beta}{\alpha_0} \right) \alpha_0! \Pi^{\beta-\alpha_0} \circ T^{\alpha-\alpha_0},$$

$$\Pi^{\alpha} \circ T^{\beta} = \sum_{\alpha_0 \leq \alpha, \beta} i^{\alpha_0} \left( \frac{\alpha}{\alpha_0} \right) \left( \frac{\beta}{\alpha_0} \right) \alpha_0! T^{\beta-\alpha_0} \circ \Pi^{\alpha-\alpha_0}.$$

Proof. The identities (3.4) follow by straight-forward computations. The formula (3.5) follows if we prove

$$((P^{\alpha} \circ \Theta^{\beta}) F)(x, \xi) = \sum_{\alpha_0 \leq \alpha, \beta} i^{\alpha_0} \left( \frac{\alpha}{\alpha_0} \right) \left( \frac{\beta}{\alpha_0} \right) \alpha_0! ((\Theta^{\beta-\alpha_0} \circ P^{\alpha-\alpha_0}) F)(x, \xi),$$

for every $F \in \mathcal{S}(\mathbb{R}^{2d})$.

Let $((\mathcal{F}_1 F)(\eta, \xi)$ be the partial Fourier Transform of $F(x, \xi)$ with respect to the $x$-variable. Then $P_j$ and $\Theta_j$ are transformed into the operators

$$\Phi(\eta, \xi) \mapsto \left( \frac{1}{2i} \frac{\partial}{\partial \eta_j} - \frac{1}{2} \frac{\partial}{\partial \xi_j} \right) \Phi(\eta, \xi) \quad \text{and} \quad \Phi(\eta, \xi) \mapsto \left( \frac{1}{2} \eta_j - \xi_j \right) \Phi(\eta, \xi),$$

respectively, and by letting

$$\sigma_j = \frac{1}{2} \eta_j + \xi_j, \quad \tau_j = \frac{1}{2} \eta_j - \xi_j,$$
and letting $G$ be defined by $G(\sigma_j, \tau_j) = (\mathcal{F}_F)(\eta_j, \xi_j)$, it follows that $P_j$ and $\Theta_j$ are transformed into the operators

$$G \mapsto -D_{\tau_j}G \quad \text{and} \quad G \mapsto \tau_jG,$$

respectively. The relative (3.5) is now a consequence of the Leibniz rule (1.1).

The other statements follow by similar arguments, and are left for the reader. □

**Lemma 3.4.** Let $a \in C^\infty(\Omega)$, where $\Omega \subset \mathbb{R}^{2d}$ is open, $s \geq 1/2$, and let $P_j, T_j, \Theta_j$ and $\Pi_j$ be defined as before. Also let $R_{\alpha,\beta,\gamma,\delta}$ be a composition of $P^\alpha, T^\beta, \Theta^\gamma$ and $\Pi^\delta$. Then the following statements are equivalent.

1. There exist positive constants $C$ and $h$ such that

$$|((P^\alpha \circ T^\beta \circ \Theta^\gamma \circ \Pi^\delta)a)(x, \xi)| \leq Ch^{\alpha+\beta+\gamma+\delta}(\alpha!\beta!\gamma!\delta!)^s,$$

when $(x, \xi) \in \Omega$, for every multi-indices $\alpha, \beta, \gamma$ and $\delta$.

2. There exist positive constants $C$ and $h$ such that

$$|R_{\alpha,\beta,\gamma,\delta}a(x, \xi)| \leq Ch^{\alpha+\beta+\gamma+\delta}(\alpha!\beta!\gamma!\delta!)^s,$$

when $(x, \xi) \in \Omega$, for every multi-indices $\alpha, \beta, \gamma$ and $\delta$.

3. $a \in S_s(\mathbb{R}^{2d})$.

**Proof.** We only prove the result for $R_{\alpha,\beta,\gamma,\delta} = \Theta^\gamma \circ P^\alpha \circ T^\beta \circ \Pi^\delta$. The other cases follow by repeating these arguments, and are left for the reader.

First assume that (1) holds, and choose $h \geq 1$ and $C > 0$ such that (3.9) holds. By Lemma 3.3 it follows that

$$R_{\alpha,\beta,\gamma,\delta} = \sum_{\alpha_0 \leq \alpha, \delta} (-i)^{\alpha_0} \binom{\alpha}{\alpha_0} \binom{\gamma}{\alpha_0} \alpha_0! P^{\alpha-\alpha_0} \circ T^\beta \circ \Theta^{\gamma-\alpha_0} \circ \Pi^\delta.$$

Hence (3.9) gives

$$|(R_{\alpha,\beta,\gamma,\delta}a)(x, \xi)| \leq \sum_{\alpha_0 \leq \alpha, \delta} \binom{\alpha}{\alpha_0} \binom{\gamma}{\alpha_0} \alpha_0! |((P^{\alpha-\alpha_0} \circ T^\beta \circ \Theta^{\gamma-\alpha_0} \circ \Pi^\delta)a)(x, \xi)|$$

$$\leq Ch^{\alpha+\beta+\gamma+\delta}(\alpha!\beta!\gamma!\delta!)^s \sum_{\alpha_0 \leq \alpha, \gamma} \left( \binom{\alpha}{\alpha_0} \binom{\gamma}{\alpha_0} \alpha_0! \left( \frac{(\alpha - \alpha_0)!(\gamma - \alpha_0)!}{\alpha!\gamma!} \right) \right)^s.$$

By Cauchy Schwartz’s inequality in combination with the fact that $s \geq 1/2$, it follows that the terms in the sum on the right-hand side
can be estimated as,
\[
\binom{\alpha}{\alpha_0} \binom{\gamma}{\alpha_0} \alpha_0! \left( \frac{\alpha - \alpha_0! \!(\gamma - \alpha_0)!}{\alpha!\gamma!} \right)^s = \binom{\alpha}{\alpha_0} \binom{\gamma}{\alpha_0} \alpha_0! \! (\alpha_0!)^{1-2s}
\]
\[
\leq \binom{\alpha}{\alpha_0} \binom{\gamma}{\alpha_0} \frac{1}{\leq} 1 = \frac{1}{\leq} \frac{1}{\leq} \frac{1}{\leq} 2 \cdot \left( \binom{\alpha}{\alpha_0} \binom{\gamma}{\alpha_0} \right)
\]
A combination of these estimates give
\[
| (R_{\alpha,\beta,\gamma,\delta}) (x, \xi) |
\]
\[
\leq 2^{-1} Ch^{\alpha+\beta+\gamma+\delta} (\alpha!\beta!\gamma!\delta!)^s \left( \sum_{\alpha_0 \leq \alpha} \binom{\alpha}{\alpha_0} + \sum_{\alpha_0 \leq \gamma} \binom{\gamma}{\alpha_0} \right)
\]
\[
= Ch^{\alpha+\beta+\gamma+\delta} (\alpha!\beta!\gamma!\delta!)^s (2^{2|\alpha|} + 2^{2|\gamma|})
\]
which proves statement (2).

Next we prove that (2) gives (3). Therefore assume that (2) holds. We have
\[
| (x^{\alpha_1} \xi^{\alpha_2} D_x^0 D_{\xi}^2 a) (x, \xi) | = | (x^{\alpha_1} \xi^{\alpha_2} D_x^0 D_{\xi}^2 a) (x, \xi) |
\]
By (12) we get
\[
| (x^{\alpha_1} \xi^{\alpha_2} D_x^0 D_{\xi}^2 a) (x, \xi) |
\]
\[
\leq \sum_{\alpha_0 \leq \alpha_2, \beta_2} \binom{\alpha_2}{\alpha_0} \binom{\beta_2}{\alpha_0} \alpha_0! \! | (x^{\alpha_1} D_x^0 \xi^{\alpha_2} \xi^{\alpha_2-\alpha_0} D_x^0 a) (x, \xi) |. \tag{3.10}
\]
For the last factor, using (3.3), we obtain
\[
| (x^{\alpha_1} D_x^0 \xi^{\alpha_2} \xi^{\alpha_2-\alpha_0} D_x^0 a) (x, \xi) |
\]
\[
= 2^{-|\alpha_1+\alpha_2-\alpha_0|} | ((P-T)^{\alpha_1} (P+T)^{\beta_2-a_0} (\Pi-\Theta)^{\alpha_2-\alpha_0} (\Pi+\Theta)^{\beta_1}) a) (x, \xi) |
\]
By the binomial theorem and (2) it follows that the last term can be estimated by
\[
Ch^{\alpha_1+\alpha_2+\beta_1+\beta_2-2\alpha_0} (\alpha_1!(\alpha_2-\alpha_0)!\beta_1!(\beta_2-\alpha_0)!)^s.
\]
Inserting this into (3.10) gives
\[
| (x^{\alpha_1} \xi^{\alpha_2} D_x^0 D_{\xi}^2 a) (x, \xi) |
\]
\[
\leq C \sum_{\alpha_0 \leq \alpha_2, \beta_2} \binom{\alpha_2}{\alpha_0} \binom{\beta_2}{\alpha_0} \alpha_0! h^{\alpha_1+\alpha_2+\beta_1+\beta_2-2\alpha_0} (\alpha_1!(\alpha_2-\alpha_0)!\beta_1!(\beta_2-\alpha_0)!)^s.
\]
Then (3) follows by similar arguments as in the first part of the proof.
Now we prove that (3) gives (1). If (3) holds, we get
\[ |(x^{\alpha_1} \xi^{\alpha_2} D_x^{\beta_1} D_\xi^{\beta_2} a)(x, \xi)| \leq C h^{(\alpha_1 + \alpha_2 + \beta_1 + \beta_2)} (\alpha_1! \alpha_2! \beta_1! \beta_2)! s. \] (3.11)

By (3.2) and the binomial theorem, we obtain that
\[ |(P^\alpha \circ T^\beta \circ \Theta^\gamma \circ \Pi^\delta) a(x, \xi)| \]
\[ = \left| \left( \frac{1}{2i} \partial_\xi + x \right)^\alpha \left( \frac{1}{2i} \partial_x - \xi \right)^\beta \left( \frac{1}{2i} \partial_x + \xi \right)^\gamma \left( \frac{1}{2i} \partial_x \right)^\delta a \right| (x, \xi) \]
\[ \leq \sum_{\alpha_0, \beta_0, \gamma_0, \delta_0} C_{\alpha_0, \beta_0, \gamma_0, \delta_0} |(x^{\alpha_0} \xi^{\beta_0} \partial_\xi^{\beta_0} \partial_x^{\gamma_0} \partial_x^{\delta_0} a)(x, \xi)|, \]
where
\[ C_{\alpha_0, \beta_0, \gamma_0, \delta_0} = \binom{\alpha}{\alpha_0} \binom{\beta}{\beta_0} \binom{\gamma}{\gamma_0} \binom{\delta}{\delta_0}. \]

Here the sum it taken over all \( \alpha_0, \beta_0, \gamma_0, \) and \( \delta_0 \) such that \( \alpha_0 \leq \alpha, \beta_0 \leq \beta, \gamma_0 \leq \gamma, \) and \( \delta_0 \leq \delta. \)

By (1.1), (3.11) and similar arguments as in the first part of the proof we obtain
\[ |(P^\alpha \circ T^\beta \circ \Theta^\gamma \circ \Pi^\delta) a(x, \xi)| \leq C_1 h^{(\alpha + \beta + \gamma + \delta)} \sum_{\alpha_0, \beta_0, \gamma_0, \delta_0} D_{\alpha_0, \beta_0, \gamma_0, \delta_0}, \]
where \( D_{\alpha_0, \beta_0, \gamma_0, \delta_0} \) is given by
\[ C_{\alpha_0, \beta_0, \gamma_0, \delta_0} = ((\alpha + \beta - \alpha_0 - \beta_0)! (\alpha_0 + \beta_0)! (\gamma + \delta - \gamma_0 - \delta_0)! (\gamma_0 + \delta_0)!)^s \]
\[ = ((\alpha + \beta)! (\gamma + \delta)!)^s \binom{\alpha}{\alpha_0} \binom{\beta}{\beta_0} \binom{\gamma}{\gamma_0} \binom{\delta}{\delta_0} \]
\[ \leq ((\alpha + \beta)! (\gamma + \delta)!)^s \binom{\alpha}{\alpha_0} \binom{\beta}{\beta_0} \binom{\gamma}{\gamma_0} \binom{\delta}{\delta_0}. \]

This gives
\[ |(P^\alpha \circ T^\beta \circ \Theta^\gamma \circ \Pi^\delta) a(x, \xi)| \]
\[ \leq C_1 h^{(\alpha + \beta + \gamma + \delta)} \sum_{\alpha_0, \beta_0, \gamma_0, \delta_0} ((\alpha + \beta)! (\gamma + \delta)!)^s C_{\alpha_0, \beta_0, \gamma_0, \delta_0} \]
\[ = C_1 h^{(\alpha + \beta + \gamma + \delta)} \sum_{\alpha_0, \beta_0, \gamma_0, \delta_0} ((\alpha + \beta)! (\gamma + \delta)!)^s. \]

Since \((\alpha + \beta)! \leq 2^{(\alpha + \beta)} \alpha! \beta!,\) we get
\[ |(P^\alpha \circ T^\beta \circ \Theta^\gamma \circ \Pi^\delta) a(x, \xi)| \leq C_1 h^{(\alpha + \beta + \gamma + \delta)! (\alpha + \beta)! (\gamma + \delta)!)} \]
\[ \leq C_1 h^{(\alpha + \beta + \gamma + \delta)! (\alpha + \beta)! (\gamma + \delta)!)} \]
\[ \text{and (1) follows.} \]

The next result is closely related to Lemma 3.4 and can be found implicitly in [2]. In order to be self-contained, we here give a proof.
Lemma 3.5. Let \( \Omega \subset \mathbb{R}^d \) be open, \( f \in C^\infty(\Omega) \), and \( s \geq 1/2 \). Then the following statements are equivalent.

1. There are positive constants \( C \) and \( h \) such that
\[
| x^\alpha (D^\beta f(x)) | \leq C h^{\alpha + \beta} (\alpha! \beta!)^s, \quad x \in \Omega,
\]
for every multi-indices \( \alpha \) and \( \beta \).

2. There are positive constants \( C \) and \( h \) such that
\[
| D^\beta (x^\alpha f(x)) | \leq C h^{\alpha + \beta} (\alpha! \beta!)^s, \quad x \in \Omega,
\]
for every multi-indices \( \alpha \) and \( \beta \).

Proof. Assume that statement (1) holds. By Leibniz rule applied to \( D^\beta (x^\alpha f(x)) \) we get
\[
| D^\beta (x^\alpha f(x)) | \leq C \sum_{\gamma \leq \alpha, \beta} \binom{\alpha}{\gamma} \binom{\beta}{\gamma} \gamma! ((\alpha - \gamma)! (\beta - \gamma)!)^s h^{\alpha + \beta - 2\gamma}
\]
for some constant \( C \) which is independent of \( \alpha \) and \( \beta \). By (1.1), it now follows by the same argument as in the proof of Lemma 3.4 that
\[
| D^\beta (x^\alpha f(x)) | \leq C h^{\alpha + \beta} (\alpha! \beta!)^s (2^{\beta} - 1 + 2^{\alpha - 1}),
\]
and the statement (2) follows for some \( h \geq 1 \).

Assume instead that (2) holds. By (1.2), then statement (1) follows by similar arguments as in the first part of the proof. \( \square \)

The previous lemma can easily be extended to more than one variables. The proof is similar to the proof of Lemmas 3.4 and 3.5, and is left for the reader.

Lemma 3.6. Let \( R_{\alpha, \beta, \gamma, \delta} \) be a composition of the multiplication operators \( x^\alpha, \xi^\beta, \partial_\xi^\gamma, \) and \( \partial_\xi^\delta \). Then the following conditions are equivalent.

1. There are positive constants \( C \) and \( h \) such that
\[
| x^\alpha \xi^\beta \partial_\xi^\gamma a(x, \xi) | \leq C h^{\alpha + \beta + \gamma + \delta} (\alpha! \beta! \gamma! \delta!)^s, \quad (x, \xi) \in \Omega,
\]
for every multi-indices \( \alpha, \beta, \gamma \) and \( \delta \).

2. There are positive constants \( C \) and \( h \) such that
\[
R_{\alpha, \beta, \gamma, \delta} \leq C h^{\alpha + \beta + \gamma + \delta} (\alpha! \beta! \gamma! \delta!)^s, \quad (x, \xi) \in \Omega,
\]
for every multi-indices \( \alpha, \beta, \gamma \) and \( \delta \).

In particular, \( a \in S_s(\mathbb{R}^{2d}) \) if and only if (2) holds.

The lemma follows by similar arguments as in the proofs of Lemmas 3.4 and 3.5. The details are left for the reader.

Proof of Lemma 3.2. Assume that (1) follows, and choose \( R = \max(|x|, |\xi|, 1) \). Then
\[
|((P^\alpha \Theta^\beta \Theta^\gamma \Pi^\delta)a)(x, \xi)| \leq \sum_a (\frac{\alpha}{\alpha_0}) (\frac{\beta}{\beta_0}) (\frac{\gamma}{\gamma_0}) (\frac{\delta}{\delta_0}) Q_{\alpha_0, \beta_0, \gamma_0, \delta_0}(x, \xi),
\]
Since (1) we have \( \alpha \) that particular.

where

\[ Q_{\alpha_0, \beta_0, \gamma_0, \delta_0}(x, \xi) = R^{(a_0 - \alpha_0 | + | \beta_0 | + | \gamma_0 | + | \delta_0 |)} \partial_\xi^a_\alpha \partial_\xi^\beta_\xi \partial_\xi^\gamma_\xi \partial_\xi^\delta_\xi (a)(x, \xi) \],

and the sum is taken over all \( \alpha_0 \leq \alpha, \beta_0 \leq \beta, \gamma_0 \leq \gamma \) and \( \delta_0 \leq \delta \). By (1) we have

\[ Q_{\alpha_0, \beta_0, \gamma_0, \delta_0}(x, \xi) \leq CR^{(a_0 - \alpha_0 | + | \beta_0 | + | \gamma_0 | + | \delta_0 |)} \partial_\xi^a_\alpha \partial_\xi^\beta_\xi \partial_\xi^\gamma_\xi \partial_\xi^\delta_\xi (a)(x, \xi) \]

Since \( s \geq 1/2 \), by inserting this into \( Q \), it follows from binomial theorem that

\[ |((P^\alpha \circ T^\beta \circ \Theta^\gamma \circ \Pi^\delta) a)(x, \xi)| \leq C(R + h)^{a + g + \delta} (\alpha! \beta! \gamma! \delta!)^s. \]

This gives (2).

If instead (2) holds, then for some \( \alpha_1 \) and \( \alpha_2 \), (3.3) gives

\[ |D^a a(x, \xi)| = |D_x^a \partial_x^a a(x, \xi)| \leq |((P + T)^{\alpha_1} (\Pi + \Theta)^{\alpha_2} a)(x, \xi)| \leq \sum \binom{\alpha_1}{\alpha_0} \binom{\alpha_2}{\alpha_0} |((P^a \circ T^{\alpha_1 - \alpha_0} \circ \Theta^{\gamma_0} \circ \Pi^{\alpha_2 - \gamma_0}) a)(x, \xi)|. \]

By similar arguments as in the first part of the proof as well as in earlier proofs, we obtain that the right hand side can be estimated by

\[ CH^{(a_1 + a_2)} (\alpha_1! \alpha_2!)^s. \]

This gives the result.

\[ \square \]

The next lemma is the last step in the proof of Theorem 3.1.

**Lemma 3.7.** Let \( s \geq 1/2 \) and \( a \in C_+(R^{2d}) \cap C^0(R^{2d}) \) be such that

\[ |(\partial_x^a \partial_x^\beta a)(0, 0)| \leq C h^{a + \beta} (\alpha! \beta!)^s, \]

where \( \alpha, \beta \in \mathbb{Z}^d_+ \), then \( a \in S_0(R^{2d}) \).

**Proof.** Since \( S_0 \) is dense in \( S \), it follows from Theorem 3.3 in \( S \) that \( Aa \) is a positive semi-definite trace-class operator on \( L^2(R^{2d}) \). In particular,

\[ (Aa)(x, y) = \sum_j f_j(x) \overline{f_j(y)}, \]

where \((f_j, f_k) = 0 \) when \( j \neq k \), and the trace-norm of \( Aa \) is given by

\[ \|Aa\|_{Tr} = \sum \|f_j\|_{L^2}^2 = (\pi/2)^{d/2} a(0, 0) < \infty. \]

More specific, by Theorem 3.13 in \( S \), it follows that \( a \in S(R^{2d}) \), and that

\[ \sum_j \|x^a D^\gamma f_j\|_{L^2}^2 < \infty, \]

for every multi-indices \( \alpha \) and \( \gamma \). Now let \( a_{\alpha, \gamma} = D^\alpha \circ T^\alpha \circ \Theta^\gamma \circ \Pi^\alpha a \). Then

\[ Aa_{\alpha, \gamma} = \sum (x^a D^\gamma f_j) \otimes (x^a D^\gamma \overline{f_j}). \]
Furthermore, since \( a_{\alpha,\gamma} \in C_+ (\mathbb{R}^d) \), Lemma 3.2 gives
\[
|a_{\alpha,\gamma}(x, \xi)| \leq a_{\alpha,\gamma}(0, 0) \leq C h^{2|\alpha+\gamma|}(\alpha!\gamma!)^{2s},
\]
where \( C \) and \( h \) are independent of \( \alpha \) and \( \gamma \). A combination of these relations and (3.13) give
\[
\|Aa_{\alpha,\gamma}\|_{\mathrm{Tr}} = \sum_j \|x^\alpha D^\gamma f_j\|_{L^2}^2 = (\pi/2)^{d/2} a_{\alpha,\gamma}(0, 0) \leq C h^{2|\alpha+\gamma|}(\alpha!\gamma!)^{2s},
\]
for some constants \( C \) and \( h \).

Next let \( a_{\alpha,\beta,\gamma,\delta} = P^\alpha \circ T^\beta \circ \Theta^\gamma \circ \Pi^\delta a \). Then \( Aa_{\alpha,\beta,\gamma,\delta} \) is a linear combination of terms of the type \( \sum (x^\alpha D^\gamma f_j) \otimes (x^\beta D^\delta f_j) \). By applying Cauchy Schwartz inequality we get
\[
\|Aa_{\alpha,\beta,\gamma,\delta}\|_{\mathrm{Tr}} \leq \sum_j \| (x^\alpha D^\gamma f_j) \otimes (x^\beta D^\delta f_j) \|_{\mathrm{Tr}}
= \sum_j \| (x^\alpha D^\gamma f_j) \otimes (x^\beta D^\delta f_j) \|_{L^2}
= \sum_j \| x^\alpha D^\gamma f_j \|_{L^2} \| x^\beta D^\delta f_j \|_{L^2}
\leq \left( \sum_j \| x^\alpha D^\gamma f_j \|_{L^2}^2 \right)^{1/2} \left( \sum_j \| x^\beta D^\delta f_j \|_{L^2}^2 \right)^{1/2}
\leq C h^{(|\alpha+\beta+\gamma+\delta|)(\alpha!\beta!\gamma!\delta!)^s},
\]
for some constants \( C \) and \( h \). In the first inequality we have used the fact that \( (x^\alpha D^\gamma f_j) \otimes (x^\beta D^\delta f_j) \) is an operator of rank one. By Proposition 1.6 we get
\[
\|a_{\alpha,\beta,\gamma,\delta}\|_{L^\infty} \leq C h^{(|\alpha+\beta+\gamma+\delta|)(\alpha!\beta!\gamma!\delta!)^s},
\]
which implies that \( a \in S_s(\mathbb{R}^d) \). The proof is complete. \( \Box \)

Proof of Theorem 3.13 By theorem 3.13 in \cite{Shilov} it follows that \( a \in S'(\mathbb{R}^d) \), since (3.1) implies (3.12), it follows from Lemma 3.7 that \( a \in S_s(\mathbb{R}^d) \), and the result follows. \( \Box \)

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