Some Conjectures on the Number of Primes in Certain Intervals

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Abstract. In this paper, we make some conjectures on prime numbers that are sharper than those found in the current literature. First we describe our studies on Legendre’s Conjecture which is still unsolved. Next, we show that Brocard’s Conjecture can be proved assuming our improved version of Legendre’s Conjecture. Finally, we sharpen the Bertrand’s Postulate for prime numbers. Our results are backed by extensive empirical investigation.

Keywords: Bertrand’s Postulate, Brocard’s Conjecture, Legendre’s Conjecture, Prime.

1 Introduction

Investigation on the properties of prime numbers is an interesting area of study for centuries. Starting from Euclid’s result [4] on the infinitude of primes to modern primality testing algorithms [5], research on prime numbers has grown exponentially. There exist many results on the set of prime numbers. Some of them has been proved, such as Bertrand’s Postulate [6]. Some of them has remained conjectures till today, and amongst them, some important ones are: Goldbach’s Conjecture [2], Legendre’s Conjecture [3], Brocard’s Conjecture [9] etc. We concentrate on sharpening Legendre’s Conjecture and Bertrand’s Postulate in this paper. Both of these are related to the number of prime numbers in certain intervals.

In this paper, we propose four novel conjectures on prime numbers.

1. Our analysis of Legendre’s Conjecture brings forth two results.
   (a) In Conjecture 1 we sharpen Legendre’s Conjecture on the lower bound on the number of primes in \([n^2, (n + 1)^2]\).
   (b) In Proposition 8 we prove Brocard’s Conjecture using our Conjecture 1.

2. Our Conjecture 2 improves the upper bound on the number of primes in \([n^2, (n + 1)^2]\) as given by Rosser and Schoenfeld’s Theorem.

3. Our third conjecture and its consequences are two-fold.
   (a) In Conjecture 3 we generalize Bertrand’s Postulate on the number of primes in \([n, kn]\) (Bertrand’s Postulate deals with \(k = 2\) only).
(b) As an implication of the above generalization, in Corollary 2, we propose a stronger upper bound on $n$-th prime than what follows from Bertrand’s Postulate.

4. Our fourth and final conjecture proposes an upper bound on the number of primes in $[n, kn]$. To our knowledge, this problem has not been attempted before this work.

2 Sharpening Legendre’s Conjecture

Legendre’s Conjecture is an important unsolved problem in the domain of prime numbers. It states that:

**Proposition 1. [Legendre’s Conjecture]**

For every positive integer $n$, there exists at least one prime $p$ such that $n^2 < p < (n + 1)^2$.

Although no formal proof has ever been found for this theorem, empirical works suggest that the number of primes in such intervals is not one but more. Let the number of primes between $n^2$ and $(n + 1)^2$ be called $\text{leg}(n)$. The following table contains brief results.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-----|---|---|---|---|---|---|---|---|---|----|
| $\text{leg}(n)$ | 2 | 2 | 2 | 3 | 2 | 4 | 3 | 4 | 3 | 5  |

**Table 1.** No. of primes between $n^2$ and $(n + 1)^2$.

Our empirical observations show that the minimum value of $\text{leg}(n)$ is always 2 or more. It is never 1, as mentioned in Legendre’s Conjecture. We have checked this for many random large values of $n$ and find that $\text{leg}(n)$ oscillates aperiodically, but shows a general upward trend as $n$ increases. Thus, it is extremely unlikely that it would ever be 1. So, we strengthen Legendre’s Conjecture as follows.

**Conjecture 1. [Our Improved version of Legendre’s Conjecture]**

For every positive integer $n$ there exists at least 2 primes $p$ and $q$ such that $n^2 < p < q < (n + 1)^2$.

We can derive a formula for the upper bound of $\text{leg}(n)$ from a known result, namely, Rosser and Schoenfeld’s Theorem.

**Proposition 2. [Rosser and Schoenfeld’s Theorem]**

For any positive integer $n (> 17)$, if $\Pi(n)$ is the number of primes less than or equal to $n$, then $\frac{n}{\ln(n)} \leq \Pi(n) \leq \frac{1.25n}{\ln(n)}$.

**Corollary 1. [Upper Bound on leg(n)]**

$\text{leg}(n) \leq \frac{n^2 + 10n + 5}{8\ln(n)}$.

**Proof.** From Theorem 2, we know that $\Pi((n+1)^2) \leq \frac{5(n+1)^2}{8\ln(n)}$ and $\Pi(n^2) \geq \frac{n^2}{2\ln(n)}$. Subtracting, we have $\text{leg}(n) \leq \frac{n^2 + 10n + 5}{8\ln(n)}$. ☐
Thus, we have an upper bound on $leg(n)$. However, we observe that this theoretical upper limit is extremely loose. This is illustrated in Table 2. We here propose a tighter bound. By plotting the values of $leg(n)$ against consecutive integers $n$, we see a general upward tendency in the curve. Studying the general nature of the curve, we suggest the following conjecture.

**Conjecture 2.** [Our Improved Bound on $leg(n)$]
For any positive integer $n$, $\frac{n^2 + 10n + 5}{3n\ln(n)} \leq leg(n) \leq \frac{n^2 + 10n + 5}{3n}$.

The data in Table 2 supports this conjecture.

| $n$ | $leg(n)$ | Upper Bound from Corollary 1 | Our Bounds from Conjecture 2 |
|-----|---------|----------------------------|-----------------------------|
| 10  | 5       | 11.1                        | 3.0                         |
| 20  | 7       | 25.2                        | 3.4                         |
| 50  | 11      | 96.0                        | 5.1                         |
| 100 | 23      | 298.7                       | 8.0                         |
| 500 | 71      | 5129.1                      | 27.3                        |
| 1000| 152     | 18276.6                     | 48.7                        |
| 2000| 267     | 66110.7                     | 88.2                        |
| 5000| 613     | 367638.8                    | 196.1                       |
| 10000| 2020   | 5051250.9                   | 673.5                       |
| 20000| 4218   | 23629958.8                  | 1400.5                      |

Table 2. Comparing the bounds on $leg(n)$ from Rosser and Schoenfeld’s Theorem and our Conjecture 2

Moreover, we can prove Brocard’s Conjecture [9] using our Conjecture 1.

**Proposition 3.** [Brocard’s Conjecture]
Between $p_n^2$ and $p_{n+1}^2$, there exist at least 4 prime numbers, where $p_n$ is the $n$-th prime number.

**Proof.** From our Conjecture 1, we have at least 2 primes between $p_n^2$ and $(p_n + 1)^2$, and also 2 primes between $(p_{n+1} - 1)^2$ and $p_{n+1}^2$. As the minimum prime gap is 2, we have $p_{n+1} - p_n \geq 2$ Hence, we have $p_{n+1} - 1 \geq p_n + 1$. And hence, $(p_{n+1} - 1)^2 \geq (p_n + 1)^2$. So, the two intervals mentioned above are disjoint. Hence, the number of primes between $p_n^2$ and $p_{n+1}^2$ is at least 4. $\square$

### 3 Generalization of Bertrand’s Postulate

Bertrand’s Postulate [6] is an important theorem about the distribution of prime numbers.

**Proposition 4.** [Bertrand’s Postulate]
For every positive integer $n > 1$ there exists at least one prime $p$ such that $n \leq p < 2n$. 
In [7] it has been shown that for every $n > 25$, there exists at least 1 prime $p$ such that $n \leq p \leq \frac{6n}{5}$.

Inspired by this result, we investigated whether there exists at least $k - 1$ primes between $n$ and $kn$ for any integer $n$. This is a natural generalization of Bertrand’s Postulate. We experimented with $k$ from 2 up to 1000000, and found that for any integer $n \geq a$,

1. there are at least $k - 1$ primes between $n$ and $kn$ and
2. the number of primes in such intervals keep on increasing almost monotonically as $n$ increases.

The number $a$ is a threshold that slowly increases as $n$ increases. The empirical results suggest that the variation in $a$ with $k$ is given by $a = \lceil 1.1 \ln(2.5k) \rceil$.

The variation of the threshold value $a$ is shown in Table 3 below for different values of $k$. The tabulated values of $k$ are those at which the value of $a$ changes.

| $k$ | 2 | 5 | 22 | 65 | 160 | 427 | 1020 | 200000 | 1000000 |
|-----|---|---|----|----|-----|-----|------|--------|---------|
| Actual Threshold | 2 | 3 | 5 | 6 | 7 | 8 | 9 | 14 | 16 |
| $1.1 \ln(2.5k)$ | 1.77 | 2.21 | 4.40 | 5.59 | 6.27 | 7.67 | 8.62 | 14.43 | 16.20 |
| Our Estimate | 2 | 3 | 5 | 6 | 7 | 8 | 9 | 15 | 17 |

**Table 3.** The threshold $a$ for different values of $k$.

Based on the above results, we formally state our conjecture as below:

**Conjecture 3.** [Our Generalization of Bertrand’s Postulate] For any integers $n$, $a$ and $k$, where $a = \lceil 1.1 \ln(2.5k) \rceil$, there are at least $k - 1$ primes between $n$ and $kn$ when $n \geq a$.

Here we have shown the results for certain selected values of $a$ and $k$ just as illustration, however, but we have covered all intermediate values in our experiments.

We also observe that, the actual number of primes in the interval $[n, kn]$ actually increases far beyond $k - 1$ as $n$ increases. When $k = 2$ (Bertrand’s Postulate) it can be derived [1] from Prime Number Theorem that the number of primes in the said interval is roughly $n \ln(n)$, within some error limits.

The known upper bound for the $n$-th prime $p_n$ is $2^n$, i.e. $p_n < 2^n$. This follows from Proposition 4. Using Conjecture 3 we can provide a stronger upper bound.

**Corollary 2.** [Our Upper Bound on $n$-th Prime]

$p_n < 2^a(n - a)$ for any positive integer $a$, and the tightest bound is given by $a = \alpha + 1$, where $\alpha$ is the least positive integral solution of the inequality $2^x > 1.1 \ln(2.5(n - x))$.

**Proof.** We have $p_n < p_a(n - a)$, since the interval $[p_a, p_n]$ contains $(n - a)$ primes. Thus, $p_n < 2^a(n - a)$. We obtain the tightest bound as follows. Using Conjecture 3 we can say that there are at least $n - a - 1$ primes in the interval $[p_a, (n - a)p_a]$, when $p_a > \lceil 1.1 \ln(2.5(n - a)) \rceil$. Hence, the best $a$ will satisfy $2^a > \lceil 1.1 \ln(2.5(n - a)) \rceil$. $\square$
In Table 4, we compare the upper bounds on $n$-th prime as given by Bertrand’s Postulate and our Corollary 2.

| $n$  | $p_n$  | Upper Bound 2nd | Our Upper Bound |
|------|--------|----------------|-----------------|
| 32   | 131    | 4294967296      | 448             |
| 987  | 7703   | 13072 ... (a 298-digit number) | 31424 |
| 2000 | 17389  | 1148 ... (a 603-digit number) | 63840 |

Table 4. Comparing the upper bounds on $n$-th prime.

We studied the number of primes in the intervals $[n, kn]$ for different values of $n$ and $k$. We found from our observations based on plotting and curve-fitting that, for any particular $k$, the number of primes in the said interval increases almost linearly with $n$. Equations of the curves so obtained are roughly of the form $y = \frac{kn}{a} + bk$ where $y$ is the required number of primes, $a$ is close to 10 and $b$ is a number between 1 and 10. $a$ and $b$, however, vary for different values of $n$ and $k$. By manipulating this form, we can suggest an upper bound on the number of primes in the interval $[n, kn]$, as follows:

**Conjecture 4.** [Our Upper Bound on the No. of Primes between $n$ and $kn$] Given a positive integer $k$, the number of primes between $n$ and $kn$, for any positive integer $n$, is bounded by $\frac{kn}{a} + k^2$.

We defend this conjecture by briefly showing our results in Table 5. Each row corresponds to a single value of $n$ and each column corresponds to an individual value of $k$. Each entry in the matrix represents the number of primes between $n$ and $kn$ for the selected values of $n$ and $k$. The value on top denotes the actual number of primes and the value at bottom gives the upper bound on the number of primes as given by Conjecture 4.

| $n / k$ | 2     | 5     | 10    | 50    | 100   |
|---------|-------|-------|-------|-------|-------|
| 10      | 4     | 11    | 21    | 91    | 164   |
|         | 6.2   | 30.6  | 111.1 | 2555.6| 10111.1|
| 50      | 10    | 38    | 80    | 352   | 654   |
|         | 15.1  | 52.8  | 155.6 | 2777.8| 10555.6|
| 100     | 21    | 70    | 143   | 644   | 1204  |
|         | 26.2  | 80.6  | 211.1 | 3055.6| 11111.1|
| 500     | 73    | 272   | 574   | 2667  | 5038  |
|         | 115.1 | 302.8 | 655.6 | 5277.8| 15555.6|
| 1000    | 135   | 501   | 1061  | 4965  | 9424  |
|         | 226.2 | 580.6 | 1211.1| 8055.6| 21111.1|
| 5000    | 560   | 2094  | 4464  | 21375 | 40869 |
|         | 1115.1| 2802.8| 5655.6| 30277.8| 65555.6|

Table 5. Comparing actual no. of primes between $n$ and $kn$ with that predicted by Our Conjecture 4.
We have experimented with the intermediate values of $n$ and $k$ as well, and found the conjecture to hold. But due to lack of space we do not show the results here. However, we find that this bound is quite tight only for small values $k$. As $k$ increases, the upper bound becomes increasingly loose. However, no violation has been detected till $k = 1000000$, and with the upper bound increasing monotonically, it is unlikely that any violation will ever occur. Also, between $n$ and $kn$ there are $kn - n + 1$ numbers. For small values of $n$ and $k$, the upper bound of $\frac{k^n}{9} + k^2$ may be greater than this value, in which our conjecture obviously does not provide any new information. We find that, the suggested upper bound is less than $kn - n + 1$ only for $n \geq \frac{9k^2 - 9}{8k - 9}$. Analyzing this graphically, we find that this approximately requires $n \geq 2k$.

4 Conclusion

This paper attempts at sharpening Legendre’s Conjecture and generalizing Bertrand’s Postulate on prime numbers. We present both formal arguments and empirical supports to defend our new conjectures and their corollaries. Further investigation is required for attempting to prove these results.

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