Kazhdan-Lusztig polynomials for $\tilde{B}_2$

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Abstract

In their seminal paper [KL79] Kazhdan and Lusztig define, for an arbitrary Coxeter system $(W, S)$, a family of polynomials indexed by pairs of elements of $W$. Despite their relevance and elementary definition, the explicit computation of these polynomials is still one of the hardest open problems in algebraic combinatorics. In this paper we explicitly compute Kazhdan-Lusztig polynomials for a Coxeter system of type $\tilde{B}_2$.

1 Introduction

Kazhdan-Lusztig polynomials lie at the intersection of representation theory, geometry and algebraic combinatorics. Their relevance derives from the fact that they elegantly express the answers to many difficult problems in the aforementioned areas. Despite their straightforward definition (through a recursive algorithm involving only elementary operations) and enormous efforts made by several authors over the past forty years, in general the explicit computation of Kazhdan-Lusztig polynomials remains elusive.

In this paper we compute Kazhdan-Lusztig polynomials in type $\tilde{B}_2$; for relevant background and related theory we refer the reader to [EMTW20]. In order to present our results we introduce some notation. The main protagonist in this paper is the affine Weyl group of type $\tilde{B}_2$, which we denote by $W$. It is a Coxeter group generated by involutions $S = \{s_1, s_2, s_0\}$ with relations $(s_1 s_2)^4 = (s_2 s_0)^4 = (s_1 s_0)^2 = 1$. As usual, we denote by $\ell(\cdot)$ and $\leq$ the length and Bruhat order on $W$, respectively. It is convenient to recall the realization of $W$ as the group of isometric transformations of the plane generated by reflections in the lines that support the sides of an isosceles right triangle as is illustrated in Figure 1. In this figure the triangle marked with a dot represents the identity of $W$. The colors on the edges of the triangles represent the following simple reflections: red is $s_1$, blue is $s_2$ and green is $s_0$. Given an element $w \in W$ and an expression $s_{i_1} s_{i_2} \ldots s_{i_k}$ (not necessarily reduced) of $w$ we define a sequence of triangles $(\triangle_0, \triangle_1, \ldots, \triangle_k)$ as follows: $\triangle_0$ is the identity triangle. Then $\triangle_{j+1}$ is obtained from $\triangle_j$ by reflecting it through the side colored with $s_{i_j}$. We identify $w$ with $\triangle_k$. Of course, this identification does not depend on the choice of an expression for $w$. Henceforth, we do not distinguish between elements of $W$ and triangles.

We split $W - \{1\}$ into three regions: The big region (the region colored using the lightest gray), the thick region (the region colored by the darkest gray) and the thin region (the region colored with the intermediate gray). The reader familiar with cells in Kazhdan-Lusztig theory might have noticed the small discrepancy between Figure 1 and [Lus85, Figure 2]. The big/thick/thin regions coincide very nearly with two-sided cells, though we will prefer our description in order to keep this paper self-contained.

Let $H$ be the Hecke algebra of $W$ with standard basis $\{H_w\}_{w \in W}$ and Kazhdan-Lusztig basis (canonical basis) $\{\underline{H_w}\}_{w \in W}$. Kazhdan-Lusztig polynomials $\{h_{x,w}(v) \mid x, w \in W\}$ are defined by the equality

$$\underline{H_w} = \sum_{x \in W} h_{x,w}(v)H_x.$$  

There is a group automorphism $\varphi : W \rightarrow W$ which interchanges $s_0$ and $s_1$ and fixes $s_2$. This extends to a Hecke algebra automorphism which we also denote by $\varphi$. We have $\underline{H_{\varphi(w)}} = \varphi(\underline{H_w})$ for all $w \in W$. To condense notation we often write $w' := \varphi(w)$, for $w \in W$.

In this paper we prove an explicit formula for $\underline{H_w}$ for all $w \in W$ located in either the big region or the thick region, and we conjecture explicit formulas for the thin region. The remainder of this introduction is devoted to explaining these formulas.
We define $x \theta$ of the form $x \theta$, for an infinite sequence formed by triangles located to the north (resp. south, east and west) of the identity triangle. Consider the

Furthermore, if $d = 0$ for $m$ even and $t_m = s_1$ for $m$ odd. Then (modulo $\varphi$) all the elements in the big region are of the form $x \theta(m, n)y$, where $x \in \{1, s_0, s_2s_0, s_1s_2s_0\}$ and $y \in \{1, t_m, t_ms_2, t_ms_2 t_m\}$.

For $(m, n) \in \mathbb{N}^2$ we define

$$ \text{Supp}(m, n) = \begin{cases} \{(m - 2i, n - j) \in \mathbb{N}^2 | i, j \in \mathbb{N}\}, & \text{if } m \text{ is odd;} \\ \{(m - 2i, n - j) \in \mathbb{N}^2 | i, j \in \mathbb{N}\} - \{(0, b) | b \neq n \mod 2\}, & \text{if } m \text{ is even.} \end{cases} $$

For $w \in W$ we define

$$ N_w = \sum_{x \leq w} v^{(w) - l(x)} H_x. $$

**Theorem 1.1.** Let $(m, n) \in \mathbb{N}^2$. Then, we have

$$ H_{\theta(m, n)} = \sum_{(a, b) \in \text{Supp}(m, n)} v^{(m-a)+2(n-b)} N_{\theta(a,b)}. $$

Furthermore, if $x \in \{1, s_0, s_2s_0, s_1s_2s_0\}$ and $y \in \{1, t_m, t_ms_2, t_ms_2 t_m\}$ then $H_{x \theta(m, n)y} = XH_{\theta(m, n)}Y$, where

$$ X = \begin{cases} 1, & \text{if } x = 1; \\ H_{s_0}, & \text{if } x = s_0; \\ H_{s_2} H_{s_0} - 1, & \text{if } x = s_2s_0; \\ H_{s_1} H_{s_2} H_{s_0} - H_{s_1} - H_{s_0}, & \text{if } x = s_1s_2s_0; \end{cases} $$

$$ Y = \begin{cases} 1, & \text{if } y = 1; \\ H_{t_m}, & \text{if } y = t_m; \\ H_{t_m} H_{s_2} - 1, & \text{if } y = t_ms_2; \\ H_{t_m} H_{s_2} H_{t_m} - H_{t_m} - H_{t_m}, & \text{if } y = t_mt_m. \end{cases} $$

We now move to the thick region. We denote by $\mathcal{N}$ (resp. $\mathcal{S}$, $\mathcal{E}$ and $\mathcal{W}$) the sub-region of the thick region formed by triangles located to the north (resp. south, east and west) of the identity triangle. Consider the infinite sequence

$$ \{a_n\}_{n=1}^\infty = (s_1, s_2, s_1, s_0, s_2, s_0, s_1, s_2, s_1, s_0, s_0, s_2, s_0, \ldots). $$

We define $x_n = a_1 \cdots a_n$ and $\overline{x}_n = s_1s_2s_0x_{n-3}$. Then, $\mathcal{N} = \{x_n | n \geq 1\} \cup \{\overline{x}_n | n \geq 1\}$. In what follows we refer to $\mathcal{N}$ as the north wall. We notice that $\mathcal{S} = \varphi(\mathcal{N})$. We define $e_n = s_1x_n$. Then,

$$ \mathcal{E} = \{e_n | n \geq 1\} \cup \{e_{3k} | k \geq 1\}. $$

Finally, we define $w_n = s_2e_n$. Then, we have

$$ \mathcal{W} = \{w_n | n \geq 1\} \cup \{w_{3k} | k \geq 1\} \cup \{s_2, s_2s_1, s_2s_0\}. $$
This completes the description of the elements in the thick region.

The following theorem provides formulas for Kazhdan-Lusztig basis elements indexed by elements located in \( \mathcal{N} \). As we already pointed out \( \mathcal{S} = \varphi(\mathcal{N}) \). Therefore, the formulas for elements located in \( \mathcal{S} \) are obtained by applying \( \varphi \) to the formulas in the theorem. For the sake of brevity, in this introduction we omit the formulas for the elements located in \( \mathcal{E} \) and \( \mathcal{W} \). These formulas are presented in §5.2.

**Theorem 1.2.** For all \( k \geq 2 \) we have

\[
\mathbf{H}_{x_f, (k)} = \mathbf{N}_{x_f, (k)} + v \mathbf{N}_{x_f, (k-1)} + \left( \sum_{j=2}^{k-1} v^{j-1}(\mathbf{N}_{u_f, (k-j)} + \mathbf{N}_{u_f, (k-j)}) \right) + v^{k-1} \mathbf{N}_{s_1 s_0}, \tag{1.3}
\]

where \( f(k) := 3k + 1 \) and \( u_n = s_2 x_n \). Furthermore, using the convention of Remark 3.8 we have

\[
\mathbf{H}_{x_{3k+1}, s_0} = \mathbf{H}_{x_{3k+2}, s_0}, \tag{1.4}
\]

\[
\mathbf{H}_{x_{3k+2}} = \begin{cases} \mathbf{H}_{x_{3k+3}} + \mathbf{H}_{x_{3k+1}} + \mathbf{H}_{x_{3k+2}}(k-1, 0) + \mathbf{H}_{s_1}(k-2, 0) + \mathbf{H}_{x_{3k+2}}(k-3, 0), & \text{if } k \text{ is even}; \\ \mathbf{H}_{x_{3k+3}} + \mathbf{H}_{x_{3k+1}} + \mathbf{H}_{x_{3k+2}}(k-1, 0) + \mathbf{H}_{s_1}(k-2, 0) + \mathbf{H}_{x_{3k+2}}(k-3, 0), & \text{if } k \text{ is odd}. \end{cases} \tag{1.5}
\]

Conjectural formulas for \( \mathbf{H}_w \) for \( w \) located in the thin region are presented in §6. It is likely that they could be proved by the methods used to obtain formulas in the other regions, though the accounting becomes more difficult due to the number and type of terms appearing in the explicit formulas. We leave this task for a future investigation.

The alert reader might have noticed that the results presented in this introduction provide formulas for Kazhdan-Lusztig basis elements rather than Kazhdan-Lusztig polynomials. By the nature of our formulas, in order to compute Kazhdan-Lusztig polynomials we need to understand the elements \( \mathbf{N}_w \), or, equivalently, to understand the sets \( \mathcal{S} \). This is exactly the content of §2. We will finish this introduction with an example that illustrates how our formulas can be used in order to efficiently compute Kazhdan-Lusztig polynomials.

Suppose we want to compute \( h_{\tau_{3n}, x_{3m}}(v) \) for integers \( m > n \geq 2, m \) odd and \( n \) even. We notice that taking the coefficient of \( \mathbf{H}_{\tau_{3n}} \) on both sides of (1.5) and using (3.4) we obtain

\[
v^{-1}h_{\tau_{3n}, x_{3m-1}}(v) + h_{x_{3n-1}, x_{3m-1}}(v) = h_{\tau_{3n}, x_{3m}}(v) + h_{\tau_{3n}, x_{3m-2}}(v) + h_{s_1 s_2 s_0}(m-3, 0)(v).
\]

On the other hand, (1.4) and (3.4) imply

\[
h_{\tau_{3n}, x_{3m-1}}(v) = vh_{\tau_{3n}, x_{3m-2}}(v) + h_{s_1 s_2 s_0}(n-2, 0), x_{3n-2}(v),
\]

\[
h_{x_{3n-1}, x_{3m-1}}(v) = v^{-1}h_{x_{3n-1}, x_{3m-2}}(v) + h_{x_{3n-2}, x_{3m-2}}(v).
\]

For an integer \( l \) we define \( F_l(v) = \sum_{i=0}^{l-1} v^{2i} \).

We can use (1.3) and the description of lower intervals in §2 to obtain

\[
h_{x_{3n}, x_{3m-2}}(v) = v^{m-n} (F_{m-n}(v) + F_{m-n-2}(v)),
\]

\[
h_{s_1 s_2 s_0}(n-2, 0), x_{3n-2}(v) = v^{m-n+1} (F_{m-n-1}(v) + F_{m-n-3}(v)),
\]

\[
h_{x_{3n-1}, x_{3m-2}}(v) = v^{m-n+1} (F_{m-n}(v) + F_{m-n-2}(v)),
\]

\[
h_{x_{3n-2}, x_{3m-2}}(v) = v^{m-n} (F_{m-n+1}(v) + F_{m-n-1}(v)).
\]

Similarly, using Theorem 1.1 (or Lemma 4.13) we obtain

\[
h_{x_{3n}, s_1 s_2 s_0}(m-3, 0)(v) = v^{m-n} (F_{m-n}(v) + F_{m-n-2}(v)).
\]

Putting all these together we obtain

\[
h_{\tau_{3n}, x_{3m}}(v) = v^{m-n}(F_{m-n+1}(v) + 2F_{m-n-1}(v) + F_{m-n-3}(v)). \tag{1.6}
\]
Closed formulas for $h_{x,w}(v)$, such as (1.6), can be obtained using the results in this paper for all $x \in W$ and for all $w \in W$ located either in the big region or the thick region. We have chosen this specific example in order to point out a certain discrepancy between an identity appearing in \[Lus97, 1.(a)\] and our results. Concretely, in that paper it is claimed that

$$\mu(\mathfrak{L}_{3n}, x_{3m}) = 1, \quad (1.7)$$

for all $m > n > 0$, $m$ odd, $n$ even, where $\mu(x, w)$ denotes the coefficient of $v$ in the Kazhdan-Lusztig polynomial $h_{x,w}(v)^2$. However, it is clear from (1.6) that $\mu(\mathfrak{L}_{3n}, x_{3m}) = 0$ if $m - n > 1$. Equation (1.7) allows the author to conclude that the $W$-graph of an affine Weyl group of type $\tilde{B}_2$ is not locally finite. This conclusion was later shown to be true by Wang [\textit{Wan11}] through a different set of examples which are consistent with our results.

This paper is organized as follows. In Section 2 we describe geometrically the lower intervals in the Bruhat order of the elements of the big region and the thick region. We specify the region corresponding to each lower interval inductively and obtain closed formulas for their sizes. In Section 3 we introduce the Hecke algebra of $W$ and prove several technical lemmas containing identities which will be necessary for the computations that follow. Our main results lie in Sections 4 and 5, which provide explicit formulas for the Kazhdan-Lusztig basis elements of elements in the big region and the thick region respectively. Finally, in Section 6 we conjecture the explicit formulas of the Kazhdan-Lusztig basis elements corresponding to the elements of the thin region.

\section*{Acknowledgements}

The first author is supported by Fondecyt project 3190144. The third author was partially supported by Fondecyt project 1200341. The authors would like to thank Nicolas Libedinsky for useful discussions and all the contributors and developers of SageMath [\textit{Sag20}] of which we have made indispensable use.

\section{Lower intervals in the Bruhat order}

Throughout this paper $W$ denotes the affine Weyl group of type $\tilde{B}_2$ with generators $S = \{s_0, s_1, s_2\}$. In this section we provide a geometric description for lower Bruhat intervals and obtain formulas for their size. Given $w \in W$ we define $\leq w := \{x \in W \mid x \leq w\}$ and $|w| := |\leq w|$. We also define $D_L(w) := \{s \in S \mid ws < w\}$ and $D_L(w) = D_R(w^{-1})$.

\subsection{Lower intervals for the big region}

We recall from (1.1) the definition of the elements $\theta(m, n)$. We notice that $D_L(\theta(m, n)) = \{s_1, s_2\}$ and $D_R(\theta(m, n)) = \{s_2, t_m\}$. In particular, $D_R(\theta(m, n))$ only depends on the parity of $m$. Left and right descent sets characterize $\theta$-elements in the following sense: If $x \in W$ satisfies $D_L(x) = \{s_1, s_2\}$ and $D_R(x) = \{s_2, t\}$ for some $t \in \{s_0, s_1\}$, then there exists $(m, n) \in \mathbb{N}^2$ such that $x = \theta(m, n)$.

All these $\theta$-elements belong to the same connected component of the big region. We denote this connected component by $\mathcal{C}$. All of the elements in $\mathcal{C}$ are of the form $\theta(m, n)x$ for some $x \in \{1, t_m, t_m s_2, t_m s_2 t_m\}$.

\begin{lemma}
The set $\leq \theta(m, 0)$ is the square $S(m, 0)$ defined as the smallest square satisfying the following properties:
\begin{itemize}
    \item $S(m, 0)$ contains $\theta(m, 0)$.
    \item The center of $S(m, 0)$, say $O$, is the upper vertex of the identity triangle.
    \item The two diagonals of $S(m, 0)$ are the horizontal and vertical lines passing through $O$.
\end{itemize}
\end{lemma}

\textsuperscript{1}We have slightly modified the original statement in order to match the conventions in this paper.
\textsuperscript{2}We remark that throughout this paper we follow Soergel’s normalization in [\textit{Soe97}].
Proof. We proceed by induction on \( m \). For \( m = 0 \) the claim is clear. We now assume the lemma holds for some \( m \geq 0 \). Note that \( \theta(m+1,0) = \theta(m,0)t_m,v_2t_m \). Let \( w \leq x \leq \theta(m,0) \) and \( y \leq t_m,v_2t_m \). By our inductive hypothesis we know that \( x \in S(m,0) \). It follows that \( w \in S(m+1,0) \).

Therefore, \( \theta(m+1,0) \subseteq S(m+1,0) \). Conversely, let \( w \in S(m+1,0) \). If \( w \in S(m,0) \) we are done, so we can assume that \( w \notin S(m,0) \). It is easy to see that \( w \) can be obtained from some \( x \in S(m,0) \) by reflecting it along one of the following sequences: \( t_m, t_mv_2 \) or \( t_m,v_2t_m \). Once again, our inductive hypothesis guarantees that \( w \leq \theta(m+1,0) \). Thus \( S(m+1,0) \subseteq (\leq \theta(m+1,0)) \).

Any square obtained from \( S(0,0) \) by a translation will be called a \( B_2 \)-square. Let \( m, n \in \mathbb{N} \). We recursively define sets \( S(m,n) \) as follows: If \( n = 0 \) then \( S(m,0) \) is defined as in Lemma 2.1. Assume that \( S(m,n) \) has been defined. Then \( S(m,n+1) \) is defined as the set obtained from \( S(m,n) \) by surrounding its boundary with \( B_2 \)-squares. This is illustrated in Figure 2. Starting from \( S(4,0) \) (the black square), we obtain \( S(4,1) \) by adding 20 \( B_2 \)-squares (in dark yellow). Then, we obtain \( S(4,2) \) by adding 24 \( B_2 \)-squares (in light yellow).

Definition 2.2. Let \( x \in W \). We define \( \partial(x) \) to be the set formed by all the elements \( y \leq x \) such that the triangle associated to \( y \) has a side belonging to the boundary of \( \leq x \). We partition \( \partial(x) \) into three (possibly empty) sets according to the color of the relevant side. More precisely, we define \( \partial s_k(x) \) to be the set of all \( y \in \partial(x) \) such that the side of \( y \) belonging to the boundary of \( \leq x \) is colored by \( s_k \).

Definition 2.3. For any \( X, Y \subseteq W \) we define \( XY := \{xy \mid x \in X, y \in Y \} \). In case \( X = \{x\} \), we may also write \( x(Y) \) to mean the same.

Lemma 2.4. For all \( m, n \in \mathbb{N} \) we have \( \leq \theta(m,n) = S(m,n) \).

Proof. We proceed by induction on \( n \). The case \( n = 0 \) is covered by Lemma 2.1. Assume the lemma holds for some fixed \( n \). We have \( \theta(m,n+1) = \theta(m,n)t_m,v_2t_m \) and this implies

\[
\leq \theta(m,n+1) = (\leq \theta(m,n))(\leq t_m,v_2t_m).
\]

(2.1)

From this it is easy to see, using our inductive hypothesis, that \( (\leq \theta(m,n+1)) \subseteq S(m,n+1) \). Conversely, we have

\[
S(m,n+1) = S(m,n) \cup \bigcup_{x \in X} \partial(\theta(m,n))x,
\]

where \( X = \{t_m, t_mv_2, t_m,v_2t_m, t_m,v_2t_m \} \). It follows from our inductive hypothesis and (2.1) that \( S(m,n+1) \subseteq (\leq \theta(m,n+1)) \). We conclude that \( S(m,n+1) = (\leq \theta(m,n+1)) \) as desired.

\[
\end{proof}

Figure 2: Lower intervals: From lightest to darkest gray \( \theta(0,0) \sim \theta(4,0) \). Darkest yellow \( \theta(4,1) \) and lightest yellow \( \theta(4,2) \).
Corollary 2.5. For all \(m, n \in \mathbb{N}\) we have \(|\theta(m, n)| = 8(m^2 + 2m + 4mn + 2n^2 + 2n + 1)\). In particular, we have \(|\theta(m, 0)| = 8(m + 1)^2\).

Proof. The equality follows easily from Lemma 2.1 and Lemma 2.4 by a counting argument. Indeed, \(\leq \theta(m, 0)\) is made of \((m + 1)^2\) (non-intersecting) \(B_2\)-squares, which leads to \(|\leq \theta(m, 0)| = 8(m + 1)^2\). Furthermore, the number of \(B_2\)-squares that one needs to add to \(\leq \theta(m, 0)\) in order to obtain \(\leq \theta(m, n)\) is \(4(m + 1)n + 2n(n - 1)\).

\[
\begin{align*}
&\text{(a) From lightest to darkest grey: Lower intervals of } \\
&\quad \theta(2, 1), \theta(2, 1)s_0, \theta(2, 1)s_0s_2 \text{ and } \theta(2, 1)s_0s_2s_1; \\
&\text{(b) From lightest to darkest grey: Lower intervals of } \\
&\quad s_0\theta(2, 1), s_0\theta(2, 1)s_0, s_0\theta(2, 1)s_0s_2 \text{ and } s_0\theta(2, 1)s_0s_2s_1; \\
&\text{(c) From lightest to darkest grey: Lower intervals of } \\
&\quad s_2s_0\theta(2, 1), s_2s_0\theta(2, 1)s_0, s_2s_0\theta(2, 1)s_0s_2 \text{ and } \\
&\quad s_2s_0\theta(2, 1)s_0s_2s_1; \\
&\text{(d) From lightest to darkest grey: Lower intervals of } \\
&\quad s_1s_2s_0\theta(2, 1), s_1s_2s_0\theta(2, 1)s_0, s_1s_2s_0\theta(2, 1)s_0s_2 \text{ and } \\
&\quad s_1s_2s_0\theta(2, 1)s_0s_2s_1.
\end{align*}
\]

Figure 3: Lower intervals in the big region.

We now continue with the description of the sets \(\leq \theta(m, n)t_m\), \(\leq \theta(m, n)t_ms_2\) and \(\leq \theta(m, n)t_ms_2t'_m\). In order to obtain \(\leq \theta(m, n)t_m\), we take as a starting point \(\leq \theta(m, n) = S(m, n)\) and we add to it the triangles that have an adjoining side matching the color of \(t_m\). Similarly, we can obtain \(\leq \theta(m, n)t_ms_2\) from \(\leq \theta(m, n)t_m\) and \(\leq \theta(m, n)t_ms_2t'_m\) from \(\leq \theta(m, n)t_ms_2\). In formulas we have

\[
\begin{align*}
\leq \theta(m, n)t_m &= \leq \theta(m, n) \cup \partial_m(\theta(m, n))t_m, \quad (2.2) \\
\leq \theta(m, n)t_ms_2 &= \leq \theta(m, n)t_m \cup \partial_s(\theta(m, n)t_m)s_2, \quad (2.3) \\
\leq \theta(m, n)t_ms_2t'_m &= \leq \theta(m, n)t_ms_2 \cup \partial_m(\theta(m, n)t_ms_2)t'_m. \quad (2.4)
\end{align*}
\]

This construction is illustrated in Figure 3a for \(\theta(2, 1)\). With this description in hand we are in position to obtain the sizes of these sets.
Lemma 2.6. Let \( m, n \in \mathbb{N} \). Then we have
\[
\theta(m, n) t_m = 8(m^2 + 3m + 4mn + 2n^2 + 4n + 2), \tag{2.5}
\]
\[
\theta(m, n) t_\theta t_m = 8(m^2 + 4m + 4mn + 2n^2 + 5n + 3), \tag{2.6}
\]
\[
\theta(m, n) t_m s_2 t_m = 8(m^2 + 2m + 5m + 4mn + 2n^2 + 6n + 4). \tag{2.7}
\]

Proof. It is a straightforward counting exercise to show that \( |\partial_{t_m} (\theta(m, n))| = 8(m + 2n + 1) \). Thus (2.5) follows from Corollary 2.5 and (2.2). Similarly, we have that \( |\partial_{s_2} (\theta(m, n) t_m)| = 8(2m + 3n + 2) \). This gives us (2.6) by combining (2.3) and the already proved identity (2.5). Finally, we obtain (2.7) by combining (2.4) and the already proved identity (2.6), together with the equality \( |\partial_{t_m} (\theta(m, n) t_m s_2)| = 8(3m + 4n + 1) \). \( \square \)

We now move on to the description of the remaining components of the big region. We begin with the region \( s_0 \mathcal{C} \). Let \( w = s_0 x \) for some \( x \in \mathcal{C} \). Then \( w = x \cup s_0(x \leq x) \). Geometrically, this implies that \( w \) can be obtained as the union of \( \leq x \) with the image of \( \leq x \) under the reflection through the green line that supports a side of the identity triangle. Since we have already obtained a geometric description of all the sets \( \leq x \) for \( x \in \mathcal{C} \), we now have a geometric description of all the sets \( w \) for \( w \in s_0 \mathcal{C} \). Examples are given in Figure 3b, obtained using those from Figure 3a.

We now describe lower intervals for elements in \( s_2 s_0 \mathcal{C} \). Let \( w \in s_2 s_0 \mathcal{C} \). Then \( w = s_2 x \) for some \( x \in s_0 \mathcal{C} \). Arguing as in the previous paragraph, we get that \( \leq w \) is the union of \( \leq x \) with the image of \( \leq x \) under the reflection through the blue line that supports a side of the identity triangle. This is illustrated in Figure 3c using the examples from Figure 3b.

Finally, we describe the lower intervals for elements in \( s_1 s_2 s_0 \mathcal{C} \). Let \( w = s_1 x \) for some \( x \in s_2 s_0 \mathcal{C} \). This time \( \leq w \) corresponds to the union of \( \leq x \) with the image of \( \leq x \) under the reflection through the red line that supports a side of the identity triangle. We have illustrated this case in Figure 3d starting from the corresponding pictures in Figure 3c.

Having described the lower intervals geometrically it is now an easy (though tedious) task to determine the size of these sets. For space reasons, we omit the proof and leave the reader with the resulting formulas.

Lemma 2.7. Let \( m, n \in \mathbb{N} \). Then,
\[
|s_0 \theta(m, n)| = 8(m^2 + 3m + 4mn + 2n^2 + 4n + 2),
\]
\[
|s_2 s_0 \theta(m, n)| = 8(m^2 + 4m + 4mn + 2n^2 + 6n + 2),
\]
\[
|s_1 s_2 s_0 \theta(m, n)| = 8(m^2 + 5m + 4mn + 2n^2 + 8n + 2),
\]
\[
|s_0 \theta(m, n) t_m| = 8(m^2 + 4m + 4mn + 2n^2 + 6n + 3) + 4,
\]
\[
|s_2 s_0 \theta(m, n) t_m| = 8(m^2 + 5m + 4mn + 2n^2 + 8n + 5),
\]
\[
|s_1 s_2 s_0 \theta(m, n) t_m| = 8(m^2 + 6m + 4mn + 2n^2 + 10n + 6) + 4,
\]
\[
|s_0 \theta(m, n) t_m s_2| = 8(m^2 + 5m + 4mn + 2n^2 + 8n + 4),
\]
\[
|s_2 s_0 \theta(m, n) t_m s_2| = 8(m^2 + 6m + 4mn + 2n^2 + 9n + 5) + 6,
\]
\[
|s_1 s_2 s_0 \theta(m, n) t_m s_2| = 8(m^2 + 7m + 4mn + 2n^2 + 10n + 7) + 4,
\]
\[
|s_0 \theta(m, n) t_m s_2 t_m| = 8(m^2 + 6m + 4mn + 2n^2 + 8n + 6) + 4,
\]
\[
|s_2 s_0 \theta(m, n) t_m s_2 t_m| = 8(m^2 + 7m + 4mn + 2n^2 + 9n + 8) + 4,
\]
\[
|s_1 s_2 s_0 \theta(m, n) t_m s_2 t_m| = 8(m^2 + 8m + 4mn + 2n^2 + 10n + 10) + 4.
\]

Remark 2.8. The big region is made of eight sub-regions, namely, its connected components. In this section we have described just four of them. The remaining four regions are obtained by applying the automorphism \( \varphi \), and therefore, their description follows from the description of the regions already considered. In particular, the size formulas in Corollary 2.5, Lemma 2.6 and Lemma 2.7 are the same for an element and its \( \varphi \)-counterpart. In formulas \( |x| = |x'| \).

2.2 Lower intervals for the thick region

In this section we describe some lower intervals for elements located in the thick region. More precisely, we provide a geometric description of the sets \( \leq x_n \) and \( \leq e_n \). We omit description of the lower intervals of the remaining elements in the thick region as this is not needed in the sequel.
We begin by considering the elements \( x_n \). In this case we can see there is a pattern that appears modulo three: \( x_{3k} \leq x_{3k+1} \leq x_{3k+2} \) all behave differently. Figure 4 shows some examples.

**Lemma 2.9.** Let \( k \geq 1 \). Then \( x_{3k} = \mathcal{S}(k - 1, 0) \setminus \mathcal{Z} \), where \( \mathcal{Z} \) is the subset of \( \mathcal{S}(k - 1, 0) \) formed by the triangles with a side on the line that connects the north and west vertices of \( \mathcal{S}(k - 1, 0) \).

**Proof.** The result follows by induction on \( k \) and the identity \( \leq x_{3k+3} = (\leq x_{3k}) (\leq t'_k s_2) \).

Having described the set \( \leq x_{3k} \), it is now easy to obtain a geometric description of \( \leq x_{3k+1} \) and \( \leq x_{3k+2} \). Indeed, \( \leq x_{3k+1} = \leq x_{3k} \cup (\partial t'_k (x_{3k})) t'_k \). Similarly, we have \( \leq x_{3k+2} = \leq x_{3k+1} \cup (\partial s_2 (x_{3k+1})) s_2 \). Using these descriptions and a straightforward counting argument we obtain the following result.

**Lemma 2.10.** For all \( k \geq 1 \) we have

\[
|x_{3k}| = 8k^2 - 2k, \\
|x_{3k+1}| = 8k^2 + 4k, \\
|x_{3k+2}| = 8k^2 + 12k.
\]

![Figure 4: Lower intervals for elements \( x_n \).](image)

(a) From lightest to darkest grey: Lower intervals of \( x_{3k} \) for \( k = 1, 2, 3 \) and 4. (b) From lightest to darkest grey: Lower intervals of \( x_{3k+1} \) for \( k = 1, 2, 3 \) and 4. (c) From lightest to darkest grey: Lower intervals of \( x_{3k+2} \) for \( k = 1, 2, 3 \) and 4.

We now focus on the elements \( e_n \). Just as the \( x_n \), they obey a pattern that appears modulo 3. Figure 5 shows the different patterns of the lower intervals for the elements \( e_n \). In this case it is easier to first consider the lower intervals of the elements \( e_{3k+2} \).

**Lemma 2.11.** Let \( k \geq 0 \). Then the set \( \leq e_{3k+2} = \mathcal{S}(k) \) where \( \mathcal{S}(k) \) is the smallest square containing \( e_{3k+2} \) and \( x_{3k+2} \).

**Proof.** The result follows by induction on \( k \) and the identity \( \leq e_{3k+5} = (\leq e_{3k+2}) (\leq t_k t'_k s_2) \).

Using Lemma 2.11 as starting point, we can describe the sets \( \leq e_{3k+3} \) and \( \leq e_{3k+4} \). These are

\[
\leq e_{3k+3} = (\leq e_{3k+2}) \cup (\partial t'_k (\leq e_{3k+2})) t_k \\
\leq e_{3k+4} = (\leq e_{3k+3}) \cup (\partial s_2 (\leq e_{3k+3})) t'_k.
\]

An easy counting argument using these descriptions shows the following.

**Lemma 2.12.** For all \( k \geq 0 \), we have

\[
|e_{3k}| = 8k^2 + 4k, \\
|e_{3k+1}| = 8k^2 + 8k + 4, \\
|e_{3k+2}| = 8(k + 1)^2.
\]

### 2.3 Coatoms for lower intervals.

We write \( y \prec w \) to mean that \( y \leq w \) and \( \ell(y) = \ell(w) - 1 \), and \( \prec w \) to denote the set of all such \( y \). If \( w = (s_1, \ldots, s_n) \) is a reduced expression for \( w \), then declare \( \overline{w(i)} := s_1 \cdots s_i \cdots s_n = s_1 \cdots s_{i-1}s_{i+1} \cdots s_n \). Further, let \( x_a \) be the preferred reduced expression \((a_1, \ldots, a_n) \) obtained from the sequence \((1, 2) \) and \( \underline{a}_n = (s_1, a_1, a_2, \ldots, a_n) \). We also consider coatoms for elements \( d_n \) defined as the the product of the first \( n \) symbols of the infinite sequence \( \{b_i\}_{i=1}^\infty = (s_2, s_1, s_2, s_0, s_2, s_1, s_2, s_0, \ldots) \). These elements admit a unique reduced expression, and they comprise the “northwest wall” of the thin region; we will return to them in Section 6.
Lemma 2.13. For \( k \geq 2 \), the sets \( \{ y \mid y < x_n \} \) have the following description.

\[
\ll x_{3k} = \{ x_{3k}(1), x_{3k}(3), x_{3k}(3k-2), x_{3k}(3k) \} \\
= \{ w_{3k-3}, s_1\theta(k-2,0), \theta(k-2,0)\ell_{k-2}, x_{3k-1} \} \\
\ll x_{3k+1} = \{ x_{3k+1}(1), x_{3k+1}(3), x_{3k+1}(3k), x_{3k+1}(3k+1) \} \\
= \{ w_{3k-2}, s_1\theta(k-2,0)\ell_{k-2}, x_{3k}, x_{3k} \} \\
\ll x_{3k+2} = \{ x_{3k+2}(1), x_{3k+2}(3), x_{3k+2}(3k), x_{3k+2}(3k+1), x_{3k+2}(3k+2) \} \\
= \{ w_{3k-1}, s_1\theta(k-2,0)\ell_{k-2,2}, s_1s_2s_0\theta(k-2,0), \theta(k-1,0), x_{3k+1} \}
\]

Furthermore, for \( k \geq 3 \),

\[
\ll y_{3k} = \{ w_{3k-3}, s_1\theta(k-3,0)\ell_{k-3,2}\ell_{k-3,2}, s_1s_2s_0\theta(k-3,0)\ell_{k-3,2}, x_{3k-1} \}.
\]

Proof. Recall that all elements \( y \leq w \) can be obtained as subexpressions of a fixed reduced expression for \( w \), so all elements of \( \ll x_n \) are obtained by removing a single symbol from \( x_n \). We will argue for \( x_{3k} \), the other cases being similar. We examine which individual symbols can be removed from \( x_{3k} \) to obtain a reduced expression. It is clear that \( x_{3k}(1) \) and \( x_{3k}(3k) \) are reduced, and that none of the \( s_2 \)'s in \( x_{3k} \) can be removed to obtain a reduced expression. If we remove any \( a_i = s_1 \) where \( 4 \leq i \leq 3k-3 \), then there is guaranteed to be a substring \( s_0s_2s_0 \) immediately to the right or to the left of \( a_i \). Assume it is to the right; upon removal we obtain a substring of the form \( s_0s_1s_2s_0s_2s_0 = s_0s_1s_2s_0s_2s_0 = s_1s_2s_0s_2 \) which is not reduced. An identical argument applies if the substring is to left, or if \( a_i = s_0 \). Thus, the only other possibilities are \( x_{3k}(3) \) and \( x_{3k}(3k-2) \), and these are indeed reduced.

The latter description of each set \( \ll x_n \) can be easily observed in the geometric realization of \( W \), and the set \( \ll y_{3k} \) is obtained by similar arguments.

We will also need the following in Section 5.2.

Lemma 2.14. For \( k \geq 2 \), the sets \( \ll e_n \) have the following description.

\[
\ll e_{3k} = \{ x_{3k}', y_{3k}, s_1\theta(k-2,0)\ell_{k-2}', e_{3k-1} \} \\
\ll e_{3k+1} = \{ x_{3k+1}', y_{3k+1}, e_{3k+1}', e_{3k} \} \\
\ll e_{3k+2} = \{ x_{3k+2}', y_{3k+2}, s_0\theta(k-1,0), s_1\theta(k-1,0), e_{3k+1} \}
\]

Proof. Multiplying \( x_n \) by \( s_1 \) on the left does not introduce new possibilities for removal of the symbols from \( x_n \); though it can rule them out. We see that removing \( a_3 \) from \( e_n \) gives us a non-reduced substring \( s_1s_0s_2s_1s_2s_1 \), but now we can also remove the first symbol \( s_1 \) from \( e_n \) and get \( x_n \) back, so the size of the coatom set remains the same. The descriptions above are evident from the geometric realization of \( W \).
Lemma 2.15. For $n \geq 7$, the sets $\{y \mid y < d_n\}$ have the following description.

$$< d_n = \begin{cases} \{\varphi_i(\mathfrak{t}), \varphi_i(\mathfrak{p}), \varphi_i(\mathfrak{q}), \varphi_i(\mathfrak{r})\} & \text{if } n \text{ is even}, \\ \{\varphi_i(\mathfrak{t}), \varphi_i(\mathfrak{p}), \varphi_i(\mathfrak{q}), \varphi_i(\mathfrak{r})\} & \text{if } n \text{ is odd}, \end{cases}$$

Proof. No symbol $s_1$ or $s_0$ can be removed from $\varphi_i$, and leave a reduced expression unless it is $b_0$. If $b_1 = s_2$ for $5 \leq i \leq n - 4$, then removal introduces a substring (possibly swapping $s_1$ with $s_0$) of the form

$$s_2s_1s_2s_0s_1s_2s_0s_1 = s_2s_1s_2s_0s_1s_2s_0s_2s_0 = s_1s_2s_1s_0s_0s_0$$

so $\varphi_i(\mathfrak{t})$ is not reduced. The remaining options figure in the lists above.

It is clear from Figure 1 that each element in $\mathcal{C}$ admits a reduced decomposition of the form $x_{6k+3}d_n$ or $x_{6k+3}d_n'$.

Lemma 2.16. For any $w \in \mathcal{C}$, the set $< w$ consists of at most five elements. In particular for $m, n > 0$ we have

$$< \theta(m, n) = \{s_2s_1\theta(m-1, n), \theta(m-1, n)t_{m-1}s_2, s_1s_2\theta(m, n-1), \theta(m-1, n)t_{m}s_2t_{m}'\}$$

$$< \theta(m, n) = \{s_2s_1\theta(m-1, n)t_{m-1}, \theta(m-1, n)t_{m-1}s_2t_{m-1}, s_1s_2\theta(m, n-1)t_{m}, \theta(m, n)t_{m+1}, \theta(m, n)\}$$

$$< \theta(m, n) = \{s_2s_1\theta(m-1, n)t_{m-1}s_2, \theta(m-1, n+1), s_1s_2\theta(m, n-1)t_{m}s_2, \theta(m+1, n-1)t_{m+1}s_2, \theta(m, n)t_{m}\}$$

$$< \theta(m, n) = \{s_2s_1\theta(m-1, n)t_{m-1}s_2t_{m-1}, \theta(m, n+1)t_{m-1}, s_1s_2\theta(m, n-1)t_{m}s_2t_{m}, \theta(m+2, n-1), \theta(m, n)t_{m}s_2\}$$

Proof. We assume $w = x_m d_n$ for $m = 6k$; the other case is identical. From Lemma 2.13, $x_m$ has four removable symbols, but when $d_n$ is appended, the last $a_n$ can no longer be removed to obtain a reduced expression. If $a_3 = s_1$ is removed, then braid relations can be applied to give an expression for $x_m(\mathfrak{t})$ ending in $s_2$ which disqualifies this possibility as well. Thus there are only two possibilities for removal from $x_m$: $a_1$ and $a_{m-2}$.

By Lemma 2.15, the symbols of $d_n$ give four or five more possibilities depending on the parity of $n$. However, removal of $b_3$, leads to a substring of the form $s_0s_2s_0s_1s_0 = s_0s_2s_0s_1$. Further, if $n = 4k$ (respectively, $n = 4k + 2$), removal of $b_n-1$ (respectively, $b_{n-3}$) plus the application of braid relations leads to an expression for $\varphi_i(n-1)$ (respectively, $\varphi_i(n-3)$) that begins with $s_2s_0$. As $x_m$ ends with $s_0s_2s_0$, the result is not a reduced expression. Thus, $d_m$ adds three more possibilities for a maximum total of five elements.

The elements $\theta(m, n)$ in particular can be written as $\theta(m, n) = x_{3(m+1)}d_4$ for $m$ odd and $\theta(m, n) = x_{3(m+1)}d_4'$ for $m$ even. An argument similar to the one of the previous paragraph shows that the removal of $b_4$ (or $b_4'$) as the case may be) from this reduced expression yields something not reduced, so $< \theta(m, n)$ consists of four elements when $m, n > 0$. These are (when $m$ is odd)

$$< \theta(m, n) = \{x_{3(m+1)}(\mathfrak{t})d_4, x_{3(m+1)}(3m+1)d_4, x_{3(m+1)}d_4, x_{3(m+1)}d_4(\mathfrak{t}), x_{3(m+1)}d_4(4m+1)\}. \quad (2.8)$$

The description of this set given in the statement of the lemma can be observed in the geometric realization of $W$. The other cases can be reasoned similarly, with all five possibilities yielding reduced expressions.

Remark 2.17. If $m = 0$, the first two elements of (2.8) are the same, and if $n = 0$ the last two are the same, so the set has three elements in case one of $m$ or $n$ is 0. A similar situation occurs for the elements of $< \theta(m, n) \cap y \in \{t_m, t_m s_2, t_m s_2t_{m}'\}$ in these cases as well. We leave the precise description to the reader.

The following descriptions will be necessary for the arguments of Section 4. They can be deduced by analysis similar to that of Lemma 2.16.

Lemma 2.18. For all $m, n \geq 0$ and $y \in \{t_m, t_m s_2, t_m s_2t_{m}'\}$,

$$< s_0 \theta(m, n) y = \{s_0\} (< \theta(m, n) y) \cup \{\theta(m, n) y\}$$

$$< s_2 s_0 \theta(m, n) y = \{s_2 s_0\} (< \theta(m, n) y) \cup \{s_0 \theta(m, n) y\}$$

$$< s_1 s_2 s_0 \theta(m, n) y = \{s_1 s_2 s_0\} (< \theta(m, n) y) \cup \{s_2 s_0 \theta(m, n) y\}$$
Remark 2.19. Observe that while the description of the set $\prec \theta(m,n)y$ is affected when $m$ or $n$ is zero, the relationship between this set and $\prec x\theta(m,n)y$ is the same in all cases for $x \in \{s_0, s_2s_0, s_1s_2s_0\}$.

3 Multiplicative formulas in the Hecke algebra

In this section we introduce the Hecke algebra of the affine Weyl group of type $\tilde{B}_2$. We also collect several multiplicative identities that will be key in order to obtain formulas for Kazhdan-Lusztig basis elements in the forthcoming sections.

3.1 The Hecke algebra of type $\tilde{B}_2$

Let $\mathcal{H}$ be the Hecke algebra of $W$. It is the $\mathcal{A} := \mathbb{Z}[v,v^{-1}]$-algebra with generators $H_{s_0}, H_{s_1}$ and $H_{s_2}$ and relations $H_{s_i}^2 = (v^{-1} - v)H_{s_i} + 1$.

Given a reduced expression $s_1, s_2, \ldots, s_k$ of an element $w \in W$ we define $H_w := H_{s_1}H_{s_2} \ldots H_{s_k}$. It is well-known that $H_w$ does not depend on the choice of a reduced expression. The set $\{H_w\}_{w \in W}$ forms an $\mathcal{A}$-basis of $\mathcal{H}$, which is called the standard basis. It is easy to see that each generator of $\mathcal{H}$ is invertible and therefore all the elements of the standard basis are invertible. There is a $\mathbb{Z}$-linear involution $d : \mathcal{H} \to \mathcal{H}$ which is determined by $d(v) = v^{-1}$ and $d(H_w) = H_{\bar{w}}^{-1}$. An element invariant under $d$ is called self-dual.

There is another basis $\{H_w\}_{w \in W}$ called the Kazhdan-Lusztig basis whose elements are uniquely determined by two conditions: They are self-dual and

$$H_w = H_w + \sum_{x < w} h_{x,w}(v)H_x,$$

for some polynomials $h_{x,w}(v) \in \mathbb{Z}[v]$. These polynomials are the Kazhdan-Lusztig polynomials.

There is a recursive algorithm to compute this basis. Indeed, if $\mu(x,w)$ denotes the coefficient of $v$ in $h_{x,w}(v)$ then

$$H_x H_w = H_w + \sum_{x < w} \mu(x,w) H_x,$$

for all pairs $(w,s) \in W \times S$ such that $w < ws$. Also, we have

$$H_w H_s = (v + v^{-1})H_w$$

for all pairs $(w,s) \in W \times S$ such that $w > ws$.

We recall the elements defined in the introduction for any $w \in W$,

$$N_w = \sum_{x \leq w} v^{l(w) - l(x)}H_x.$$

Definition 3.1. Given $w \in W$ and $X \in \mathcal{H}$ we denote by $G_w(X)$ the coefficient of $H_w$ in $X$ when it is written in terms of the standard basis. That is,

$$X = \sum_{w \in W} G_w(X)H_w.$$

We also define the content of $X$ by

$$c(X) := \sum_{w \in W} G_w(X)(1) \in \mathbb{Z}.$$

Note that this implies that $c(N_w) = |w|$ and also that $c(XH_s) = 2c(X)$ for any $X \in \mathcal{H}$ and any $s \in S$, since

$$H_w H_s = \begin{cases} H_{ws} + vH_w, & \text{if } w < ws; \\ H_{ws} + v^{-1}H_w, & \text{if } w > ws. \end{cases}$$

(3.3)
Definition 3.2. Let $w \in W$. An element $H \in \mathcal{H}$ is called triangular of height $w$ if $G_w(H) = 1$ and $G_z(H) = 0$ for $x \leq w$. Furthermore, a triangular element $H$ of height $w$ is called monotonic if $G_z(H) \in \mathbb{N}[v, v^{-1}]$ for all $x \in W$ and
\[ G_y(H) - v^{l(x) - l(y)}G_z(H) \in \mathbb{N}[v, v^{-1}], \]
for all $y \leq x \leq w$.

A trivial example of a monotonic element is any $N_w$. We also know that $H_w$ is always monotonic (see [BM01, Pla17]).

Lemma 3.3. Let $H \in \mathcal{H}$ be a monotonic element of height $w$. Suppose that $w > w$. Then, $H_H$ is monotonic of height $w$.

Proof. It is clear that $H_H$ is triangular of height $w$. On the other hand, (3.3) shows that
\[ G_x(H_H) = \begin{cases} vG_x(H) + G_{xs}(H), & \text{if } xS > x; \\ v^{-1}G_x(H) + G_{xs}(H), & \text{if } xS < x. \end{cases} \] (3.4)

Using this, a simple case analysis shows that the monotonicity of $H$ implies the monotonicity of $H_H$. □

Lemma 3.4. Let $(W, S)$ be an arbitrary Coxeter system. Let $w \in W$ and $s \in S$ such that $s \in D_R(w)$. Then, $N_wH_w = (v + v^{-1})N_w$.

Proof. Since $ws < w$ we can use the Lifting Property [BB06, Proposition 2.2.7] to conclude that multiplication on the right by $s$ induces a permutation of the lower interval $[e, w]$. Therefore the result follows by (3.4). □

Definition 3.5. Let $p$ and $q$ in $A$. We write $p \geq q$ if $p - q \in \mathbb{N}[v, v^{-1}]$. Then for $X, Y \in \mathcal{H}$, we write $X \geq Y$ if $G_w(X) \geq G_w(Y)$ for all $w \in W$.

Notice that $X \geq Y$ and $c(X) = c(Y)$ implies $X = Y$. We claim no originality in this observation and refer to [LP20] for its first application in the context of computation of Kazhdan-Lusztig polynomials. In general, it is not an easy task to prove an inequality of the form $X \geq Y$. However, there are certain situations where we can make some simplifications. For instance, let us suppose that
\[ Y = \sum_{w \in Z} p_w(v)N_w \]
where $Z$ is a finite subset of $W$ and $p_z(v) = \sum_{i \in Z} p_z^{i}v^{i} \in \mathbb{N}[v, v^{-1}]$. For each $i \in Z$ we define
\[ Y_i = \sum_{w \in Z} p_w^{-l(w)}N_w. \]

Since the elements $Y_i$ lie in different degrees (in the sense that $G_z(Y_i)$ and $G_z(Y_j)$ are monomials of different degree if $i \neq j$) in order to prove $X \geq Y$ it is enough to show $X \geq Y_i$ for all $i \in Z$. We stress that $Y_i = 0$ for all but finitely many integers. Henceforth, we refer to this simplification as “degree reasons.”

In order to prove inequalities of the form $X \geq Y_i$ we can make some extra reductions if $X$ is monotonic. For example, if $X$ is monotonic and $Y_i$ is made of just one $N$-element, i.e. $Y_i = cv^kN_w$, then in order to prove $X \geq Y_i$ it is enough to check $G_w(X) \geq cv^k$. In contrast, if $Y_i$ is made of two or more $N$-elements then additional analysis of the relationship between the elements involved in $Y_i$ in the Bruhat order is required.

We conclude this section with a useful result that allows us to rule out the occurrence of terms in the sum in (3.1). A proof can be found in [KL79, (2.3.1)] or [BB06, Proposition 5.1.9].

Lemma 3.6. Let $x, w \in W$ such that $\mu(x, w) \neq 0$. Suppose that $l(w) - l(x) > 1$. Then
\[ D_R(w) \subseteq D_R(x) \quad \text{and} \quad D_L(w) \subseteq D_L(x). \]

In particular, if $s \notin D_R(w)$ and $H_w$ appears in the expansion of $H_wH_w$ when written in terms of the Kazhdan-Lusztig basis, then $D_R(w) \cup \{s\} \subseteq D_R(x)$. Similarly, if $s \notin D_L(w)$ and $H_w$ appears in the expansion of $H_wH_w$ when written in terms of the Kazhdan-Lusztig basis, then $D_L(w) \cup \{s\} \subseteq D_L(x)$. 12
3.2 Multiplication formulas for \( N_w \)

In this section we collect several multiplicative formulas involving elements \( N_w \).

**Lemma 3.7.** Let \( m \) and \( n \) be positive integers. Then,

\[
N_{\theta(m,n)H_m} = N_{\theta(m,n)t_m} + \nu N_{\theta(m-1,n)t'_m}. \tag{3.5}
\]

**Proof.** First, let us prove that the contents on both sides of (3.5) coincide. On the left side, Corollary 2.5 implies that

\[
c(N_{\theta(m,n)H_m}) = 2c(N_{\theta(m,n)}) = |\theta(m,n)| = 16(m^2 + 2m + 4mn + 2n^2 + 2n + 1).
\]

On the right side, Lemma 2.6 implies that

\[
c(N_{\theta(m,n)t_m} + \nu N_{\theta(m-1,n)t'_m}) = |\theta(m,n)t_m| + |\theta(m-1,n)t'_m| = 16(m^2 + 2m + 4mn + 2n^2 + 2n + 1).
\]

Thus we only need to show that

\[
N_{\theta(m,n)H_m} \geq N_{\theta(m,n)t_m} + \nu N_{\theta(m-1,n)t'_m}.
\]

By degree reasons it is enough to check

\[
N_{\theta(m,n)H_m} \geq N_{\theta(m,n)t_m}, \tag{3.6}
\]

\[
N_{\theta(m,n)H_m} \geq \nu N_{\theta(m-1,n)t'_m}. \tag{3.7}
\]

Furthermore, by the monotonicity of \( N_{\theta(m,n)H_m} \) ensured by Lemma 3.3, inequalities (3.6) and (3.7) will follow from the inequalities

\[
G_{\theta(m,n)t_m}(N_{\theta(m,n)H_m}) \geq 1, \tag{3.8}
\]

\[
G_{\theta(m-1,n)t'_m}(N_{\theta(m,n)H_m}) \geq \nu, \tag{3.9}
\]

respectively. Inequality (3.8) is clear. On the other hand, a direct computation shows that

\[
G_{\theta(m-1,n)t'_m}(\nu^2 H_{\theta(m-1,n)t'_m} H_m) = \nu,
\]

which proves (3.9). The lemma is proved. \( \square \)

**Remark 3.8.** Henceforth, we adopt the convention that \( N_{\theta(x,y)} \) and \( H_{\theta(x,y)} \) are equal to zero whenever \( m \) or \( n \) is negative, for any \( x, y \in W \). This will allow us to express some results more compactly and consistently.

Using the same arguments as in the proof of Lemma 3.7 we obtain the following four lemmas.

**Lemma 3.9.** Let \( m \) and \( n \) be positive integers. We have

\[
N_{\theta(0,n)H_0} = N_{\theta(0,n)s_0} + \nu^2 N_{\theta(0,n-1)s_0};
\]

\[
N_{\theta(m,0)H_m} = N_{\theta(m,0)t_m} + \nu N_{\theta(m-1,0)t'_m}.
\]

Furthermore, \( N_{\theta(0,0)H_0} = N_{\theta(0,0)s_0} \).

**Lemma 3.10.** For all \( m \in \mathbb{N} \) we have

\[
H_0 N_{\theta(m,0)} = N_{s_0 \theta(m,0)} + \nu N_{s_1 \theta(m-1,0)},
\]

where \( \theta'(m-1,0) \) denotes the image of \( \theta(m-1,0) \) under the automorphism \( \varphi \).

**Lemma 3.11.** Let \( m \) be a positive integer. We have

\[
N_{s_0 \theta(m,0)H_0} = N_{s_0 \theta(m,0)t_m} + \nu N_{s_0 \theta(m-1,0)t'_m}.
\]

Furthermore, \( N_{s_0 \theta(0,0)H_0} = N_{s_0 \theta(0,0)s_0} + \nu^2 N_{s_1 s_0} \).
Lemma 3.12. Let $m$ be a positive integer. We have
\[ N_{s_1s_2s_0}(m, 0)H_{m} = N_{s_1s_2s_0}(m, 0)t_m + vN_{s_1s_2s_0}(m-1, 0)t'_m. \]

The following lemma is proved using essentially the same argument as the one used in the proof of Lemma 3.7. This, however, is the first time so far in which we have a case where we must show $X \geq Y$ with $Y$ consisting of two or more $N$-terms, which leads some extra subtleties in the analysis.

Lemma 3.13. Let $m$ and $n$ be positive integers. Then,
\[ N_{\theta(m,n)}H_{m} = 2N_{\theta(m,n)}H_{m} + N_{\theta(m-1,n)} + N_{\theta(m-1,n+1)} + N_{\theta(m-1,n)} + N_{\theta(m-1,n-1)} + N_{\theta(m-1,n-1)}. \]  

Proof. Let us denote by $L$ and $R$ the left-hand and right-hand side of (3.10), respectively. A repeated application of Corollary 2.5 shows that
\[ c(L) = c(R) = 64(m^2 + 2m + 4mn + 2n^2 + 2n + 1). \]
Therefore, to finish the proof we only need to show that $L \geq R$. Lemma 3.7 implies
\[ R = 2N_{\theta(m,n)t_m} + 2vN_{\theta(m-1,n)t'_m} + N_{\theta(m-1,n)} + N_{\theta(m-1,n+1)} + N_{\theta(m-1,n-1)} + N_{\theta(m-1,n-1)}. \]
Hence, by degree reasons, it is enough to check
\[ L \geq N_{\theta(m+1,n)} \quad (3.11) \]
\[ L \geq N_{\theta(m,n)} \quad (3.12) \]
\[ L \geq 2N_{\theta(m,n)}t_m \quad (3.13) \]
\[ L \geq 2vN_{\theta(m-1,n)}t'_m \quad (3.14) \]
We now notice that $L$ is monotonic of height $\theta(m + 1, n)$ by Lemma 3.2. It is clear that $G_{\theta(m+1,n)}(L) = 1$. Then the monotonicity of $L$ shows (3.11). On the other hand, a direct computation reveals that
\[ G_{\theta(m-1,n)}(L) \geq G_{\theta(m-1,n)}(v^3H_{\theta(m-1,n)}H_{m}H_{m}) = v^2 + 1. \]
Therefore,
\[ G_{\theta(m-1,n)}(L) \geq G_{\theta(m-1,n)}(v^3H_{\theta(m-1,n)}H_{m}H_{m}) = v^2 + 1 \geq 1. \]
The above inequality together with the monotonicity of $L$ proves (3.12).

We now prove (3.13). Since we have two $N$-terms that cannot be separated by degree reasons on the right-hand side, we need to proceed in a different way. Let us explain this more generally. Suppose we want to prove
\[ L \geq c_1v^1N_{z} + c_2v^2N_{w}, \]  
for some positive integers $c_1$ and $c_2$, and where $j - i = l(x) - l(y)$ (otherwise, degree reasons allow us to split the inequality in (3.15) in two inequalities with only one $N$-term on the right-hand side). Further suppose that $\leq x \cap y = \leq z \cup w$ for some elements $z$ and $w$. Then, by the monotonicity of $L$, (3.15) will follow from
\[ L \geq c_1v^1N_{z}, \]
\[ L \geq c_2v^2N_{w}, \]
\[ L \geq (c_1 + c_2)v^{i+l(x)-l(z)}N_{z}, \]  
and
\[ L \geq (c_1 + c_2)v^{i+l(x)-l(w)}N_{w}. \]
Let us apply this general principle to prove (3.13). By the diagrammatic description of lower intervals given in §2.1 we have
\[ (\leq \theta(m - 1, n + 1)) \cap (\leq \theta(m, n)t_m) = (\leq \theta(m - 1, n)l_t, s_2t_m). \]
Therefore, we only need to prove
\[ G_{\theta(m-1,n+1)}(L) \geq 1, \quad G_{\theta(m,n)t_m}(L) \geq 2 \quad \text{and} \quad G_{\theta(m-1,n)l_t, s_2t_m}(L) \geq 3v. \]
These inequalities follow from the following identities which are obtained by a repeated application of (3.3).

\[ G_{\theta(m-1,n+1)}(vH_{\theta(m-1,n)}t_m, H_m, H_m) = 1 \]
\[ G_{\theta(m,n)}(H_m, H_m, H_m) = 2 \]
\[ G_{\theta(m-1,n)}(vH_{\theta(m-1,n)}t_m, H_m, H_m) = v \]
\[ G_{\theta(m-1,n)}(H_m, H_m, H_m) = 2v. \]

The proof of (3.14) is similar, noting that

\[ (\theta(m + 1, n - 1)) \cap (\leq \theta(m - 1, n)t_m) = (\leq \theta(m - 1, n)) \cup (\leq \theta(m, n - 1)t_m). \]

This description of the lower intervals allows one to apply the discussion leading up to (3.16); we leave the details to the reader.

**Proposition 3.14.** Let \( m \) and \( n \) be positive integers. Then,

\[ N_{\theta(m,n)}(H_m, H_m, H_m - 2H_m) = N_{\theta(m+1,n)} + N_{\theta(m-1,n+1)} + N_{\theta(m+1,n+1)} + N_{\theta(m-1,n)}. \]

**Proof.** The result is an immediate consequence of Lemma 3.13.

Similarly, we have the following proposition.

**Proposition 3.15.** Let \( m, n \in \mathbb{N} \). We have

\[
\begin{align*}
N_{\theta(m,0)}(H_m, H_m, H_m - 2H_m) &= N_{\theta(m+1,0)} + N_{\theta(m-1,1)} + (1 + v^2)N_{\theta(m-1,0)}, \\
N_{\theta(0,n)}(H_0, H_0, H_0 - 2H_0) &= N_{\theta(1,n)} + (1 + v^2)N_{\theta(1,n-1)} + v^2N_{\theta(1,n-2)}.
\end{align*}
\]

Propositions 3.14 and 3.15 should be understood as a way to pass from \( N_{\theta(m,n)} \) to \( N_{\theta(m+1,n)} \). We now focus on obtaining a similar statement but this time we want to pass from \( N_{\theta(0,n)} \) to \( N_{\theta(0,n+1)} \).

**Lemma 3.16.** Let \( n \) be a positive integer. We have

\[ N_{\theta(0,n)}(H_0, H_0, H_0, H_0 - 2H_0) = N_{\theta(0,n)} + vN_{\theta(0,n+1)} + v^2N_{\theta(0,n-1)} + v^3N_{\theta(0,n-2)} + v^4N_{\theta(0,n-3)} + \cdots. \]

**Proof.** Once we notice that the element on the left-hand side of (3.17) is monotonic (of height \( \theta(0,n) \)) we can argue as in the proof of Lemma 3.7.

**Lemma 3.17.** Let \( n \) be an integer greater than 2. We have

\[
\begin{align*}
N_{\theta(0,n+1)} &= 2N_{\theta(0,n)} + v^2N_{\theta(0,n)} + N_{\theta(2,n-1)} + N_{\theta(0,n-1)} + vN_{\theta(0,n-2)} + v^2N_{\theta(0,n-3)} + v^3N_{\theta(0,n-4)} + \cdots \\
N_{\theta(0,n)}(H_0, H_0, H_0, H_0, H_0 - 2H_0) &= 3N_{\theta(0,n)} + 2vN_{\theta(0,n-1)} + v^2N_{\theta(0,n-2)} + v^3N_{\theta(0,n-3)} + v^4N_{\theta(0,n-4)} + \cdots
\end{align*}
\]

**Proof.** Let \( L \) and \( R \) be the left and right sides of (3.18), respectively. Corollary 2.5 and Lemma 2.6 imply

\[ c(L) = c(R) = 8(36n^2 + 26n + 18). \]

Therefore, in order to prove \( L = R \) we only need to show that \( L \geq R \). By degree reasons, it is enough to prove the following inequalities

\[
\begin{align*}
L &\geq N_{\theta(0,n+1)}, \\
L &\geq 2N_{\theta(0,n)} + v^2N_{\theta(0,n)} + N_{\theta(2,n-1)}, \\
L &\geq 3N_{\theta(0,n)} + 2vN_{\theta(0,n-1)} + v^2N_{\theta(2,n-2)}, \\
L &\geq vN_{\theta(0,n)} + 3v^2N_{\theta(0,n-1)} + 2v^3N_{\theta(0,n-2)}, \\
L &\geq N_{\theta(0,n-1)}.
\end{align*}
\]
Using Lemma 3.3 we can see that $N_{\theta(0,n)} \mathbf{H}_{\theta_0}, \mathbf{H}_{\theta_1}, \mathbf{H}_{\theta_2}$ is a monotonic element of height $\theta(0,n+1)$. Furthermore, $N_{\theta(0,n-2)s_0s_2}$ is clearly monotonic. Since $\theta(0,n-2)s_0s_2 \leq \theta(0,n+1)$ we conclude that $L$ is monotonic.

An easy computation shows that

$$G_{\theta(0,n+1)}(\mathbf{H}_{\theta(0,n)} \mathbf{H}_{\theta_0}, \mathbf{H}_{\theta_1}, \mathbf{H}_{\theta_2}) = 1,$$

$$G_{\theta(0,n-1)}(v^2 \mathbf{H}_{\theta(0,n-1)s_0}, \mathbf{H}_{\theta_0}, \mathbf{H}_{\theta_1}, \mathbf{H}_{\theta_2}) = 1 + v^2.$$ 

This allows us to conclude that $G_{\theta(0,n+1)}(L) \geq 1$ and $G_{\theta(0,n-1)}(L) \geq 1$. Then, the monotonicity of $L$ proves (3.19) and (3.23).

We now focus on (3.20). We notice that $\theta(0,n)$ is smaller than $\theta(0,n)s_0s_2$ and $\theta(2,n-1)$, but the latter elements are incomparable in Bruhat order. Furthermore, by the description of lower intervals given in §2.1 we have

$$(\leq \theta(0,n)s_0s_2) \cap (\leq \theta(2,n-1)) = (\leq \theta(1,n-1)s_1s_2).$$

Finally, we notice that $\theta(0,n) \leq \theta(1,n-1)s_1s_2$. Thus, by the monotonicity of $L$, in order to prove (3.20) it is enough to show that

$$G_{\theta(0,n)s_0s_2}(L) \geq 2;$$

(3.24)

$$G_{\theta(2,n-1)}(L) \geq 1;$$

(3.25)

$$G_{\theta(1,n-1)s_1s_2}(L) \geq 3v;$$

(3.26)

$$G_{\theta(0,n)}(L) \geq 4v^2.$$ 

(3.27)

A direct computation shows that $G_{\theta(0,n)s_0s_2} \mathbf{H}_{\theta(0,n)} \mathbf{H}_{\theta_0}, \mathbf{H}_{\theta_1}, \mathbf{H}_{\theta_2} = 2$. This proves (3.24). Similarly, we have $G_{\theta(2,n-1)}(v \mathbf{H}_{\theta(0,n-1)s_0s_2}, \mathbf{H}_{\theta_0}, \mathbf{H}_{\theta_1}, \mathbf{H}_{\theta_2}) = 1$, which proves (3.25). On the other hand, we have

$$G_{\theta(1,n-1)s_1s_2}(v \mathbf{H}_{\theta(0,n-1)s_0s_2}, \mathbf{H}_{\theta_0}, \mathbf{H}_{\theta_1}, \mathbf{H}_{\theta_2}) = 2v;$$

(3.26)

$$G_{\theta(1,n-1)s_1s_2}(v^2 \mathbf{H}_{\theta(0,n-1)s_0}, \mathbf{H}_{\theta_0}, \mathbf{H}_{\theta_1}, \mathbf{H}_{\theta_2}) = v.$$ 

This proves (3.26). Finally, we have

$$G_{\theta(0,n)}(v \mathbf{H}_{\theta(0,n-1)s_0s_2}, \mathbf{H}_{\theta_0}, \mathbf{H}_{\theta_1}, \mathbf{H}_{\theta_2}) = v^2;$$

(3.27)

$$G_{\theta(0,n)}(v^2 \mathbf{H}_{\theta(0,n-1)s_0}, \mathbf{H}_{\theta_0}, \mathbf{H}_{\theta_1}, \mathbf{H}_{\theta_2}) = v^2;$$

(3.28)

$$G_{\theta(0,n)}(v^3 \mathbf{H}_{\theta(0,n-1)s_0}, \mathbf{H}_{\theta_0}, \mathbf{H}_{\theta_1}, \mathbf{H}_{\theta_2}) = v^2;$$

$$G_{\theta(0,n)}(v^2 \mathbf{H}_{\theta(0,n-1)s_0s_2}, \mathbf{H}_{\theta_0}, \mathbf{H}_{\theta_1}, \mathbf{H}_{\theta_2}) = v^2.$$ 

These four equalities show (3.27). This completes the proof of inequality (3.20). The remaining two inequalities (3.21) and (3.22) are treated similarly.

We now are in a position to obtain a multiplication formula that relates $N_{\theta(0,n)}$ and $N_{\theta(0,n+1)}$.

**Proposition 3.18.** Let $n$ be an integer greater than 2. Then,

$$N_{\theta(0,n)}(\mathbf{H}_{\theta_0}, \mathbf{H}_{\theta_1}, \mathbf{H}_{\theta_2}) - 2\mathbf{H}_{\theta_0}, \mathbf{H}_{\theta_1} + 1 - (v + v^{-1})^2 = N_{\theta(0,n+1)} + (1 + v^2)N_{\theta(0,n-1)} + v^2N_{\theta(2,n-2)} + N_{\theta(2,n-1)}.$$ 

**Proof.** By combining Lemma 3.9 and Lemma 3.16 we obtain

$$N_{\theta(0,n)} \mathbf{H}_{\theta_0}, \mathbf{H}_{\theta_2} = N_{\theta(0,n)} + vN_{\theta(1,n-1)} + v^2N_{\theta(0,n-1)} + v^3N_{\theta(1,n-2)} - v^4N_{\theta(0,n-2)s_0s_2}.$$ 

(3.28)

Hence, the result follows by combining Lemma 3.17 and (3.28).

So far we have proved several multiplicative identities involving $N$-elements. These will be sufficient to obtain formulas for Kazhdan-Lusztig basis elements corresponding to elements in the big region in §4. We end this section by stating some extra multiplicative identities that will be useful to prove formulas for Kazhdan-Lusztig basis elements corresponding to elements in the thick region in §5.
Lemma 3.19. For $m \geq 1$, the element $(\mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_n - \mathbf{H}_0)$ is equal to
\[
N_{s_1s_2s_0\theta(m,0)} + (v^{-1} + 2v)N_\theta(m,0) + v(N_{\theta(m-2,1)} + N_{s_1\theta'(m-1,0)}) + v^2(N_{s_0\theta(m-2,0)} - N_{s_1s_2s_0\theta(m-2,0)}).
\] (3.29)

Furthermore, we have
\[
(\mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_n - \mathbf{H}_0) \ N_\theta(0,0) = N_{s_1s_2s_0\theta(0,0)} + (v + v^{-1})N_\theta(0,0).
\] (3.30)

Proof. Equation (3.30) can be verified manually. Similarly, we can check (3.29) for $m = 1$. We stress that in this case some terms vanish in accordance with Remark 3.8. From now and on we fix $m \geq 2$. We define
\[
L := (\mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_n - \mathbf{H}_0) \ N_\theta(m,0) + v^2N_{s_1s_2s_0\theta(m-2,0)}
\]
and
\[
R := (\mathbf{H}_0) \ N_\theta(m,0) + N_{s_1s_2s_0\theta(m,0)} + (v^{-1} + 2v)N_\theta(m,0) + v(N_{\theta(m-2,1)} + N_{s_1\theta'(m-1,0)}) + v^2N_{s_0\theta(m-2,0)}.
\]

It is clear that the claim in the lemma is equivalent to $L = R$, which we will now prove. Using Lemma 3.10, we can rewrite $R$ as
\[
R = N_{s_0\theta(m,0)} + N_{s_1s_2s_0\theta(m,0)} + (v^{-1} + 2v)N_{\theta(m,0)} + v(N_{\theta(m-2,1)} + 2N_{s_1\theta'(m-1,0)}) + v^2N_{s_0\theta(m-2,0)}.
\]

A straightforward computation using Corollary 2.5 and Lemma 2.7 shows
\[
c(L) = c(R) = 8(9m^2 + 17m + 4).
\]

To complete the proof we only need to show that $L \geq R$. By degree reasons, this inequality will follow from the next four inequalities
\[
L \geq N_{s_1s_2s_0\theta(m,0)} \quad (3.31)
\]
\[
L \geq N_{s_0\theta(m,0)} + 2vN_{\theta(m,0)} \quad (3.32)
\]
\[
L \geq v^{-1}N_{\theta(m,0)} + 2vN_{s_1\theta'(m-1,0)} + vN_{\theta(m-2,1)} \quad (3.33)
\]
\[
L \geq v^2N_{s_0\theta(m-2,0)} \quad (3.34)
\]

We notice that the left version of Lemma 3.3 shows that $L$ is monotonic. Inequalities (3.31), (3.32) and (3.34) are easily obtained, so we will focus on (3.33). We notice that $\theta(m,0)$ is greater than $s_1\theta'(m-1,0)$ and than $\theta(m-2,1)$ but the last two elements are incomparable in Bruhat order. By the diagrammatic description of the lower intervals given in §2.1 we obtain
\[
(\leq s_1\theta'(m-1,0)) \cap (\leq \theta(m-2,1)) = (\leq s_1s_2s_0\theta(m-2,0)).
\]

Therefore, by the monotonicity of $L$, (3.33) reduces to proving the following polynomial inequalities
\[
G_{\theta(m,0)}(L) \geq v^{-1}, \quad (3.35)
\]
\[
G_{s_1\theta'(m-1,0)}(L) \geq 3v, \quad (3.36)
\]
\[
G_{\theta(m-2,1)}(L) \geq 2v, \quad (3.37)
\]
\[
G_{s_1s_2s_0\theta(m-2,0)}(L) \geq 4v^2. \quad (3.38)
\]

Set $X := \mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_n$. A direct computation (using the left version of (3.3)) shows that $G_{\theta(m,0)}(XH_{\theta(m,0)}) = v^{-1}$, proving (3.35). Similarly, we have
\[
G_{s_1\theta'(m-1,0)}(X(sH_{s_2s_1\theta'(m-1,0)})) = G_{s_1\theta'(m-1,0)}(X(v^2H_{s_1\theta'(m-1,0)})) = G_{s_1\theta'(m-1,0)}(X(v^3H_{\theta'(m-1,0)})) = v.
\]

These three equalities together prove (3.36). We now observe that
\[
G_{\theta(m-2,1)}(X(v^2H_{s_1\theta'(m-1,0)})) = G_{\theta(m-2,1)}(X(v^2H_{\theta(m-2,1)})) = v.
\]

This proves (3.37). Finally, if we set $z := s_1s_2s_0\theta(m-2,0)$ then we have
\[
G_z(X(v^2H_{s_1\theta'(m-1,0)})) = G_z(X(v^2H_{\theta(m-2,1)})) = G_z(X(v^3H_{\theta'(m-1,0)})) = v^2.
\]

These three equalities show that $G_z(XN_{\theta(m,0)}) \geq 3v^2$. Since $G_z(v^2N_z) = v^2$, inequality (3.38) follows from the definition of $L$. The lemma is proved. \(\square\)
Using similar arguments we can prove the following result, recalling the definitions of $x_n$, $e_n$ and $u_n$ from the introduction. The details are left to the reader.

**Lemma 3.20.** Given an integer $k$ we set $f(k) = 3k + 1$. Let $Y = \hat{H}_0 \cdot \hat{H}_3 \cdot \hat{H}_m - \hat{H}_m$. For $k \geq 2$, 
\begin{align*}
N_{x_f(k)}(Y) &= N_{x_f(k+1)} + N_{x_f(k-1)} + N_{\theta(k-1,0)\hat{t}_{k-1}} + N_{s_1 s_2 \theta(k-2,0)\hat{t}_{k-2}} + v N_{\theta(k-2,0)\hat{t}_{k-2}} + v N_{\theta(k-2,0)\hat{t}_{k-2}} \\
N_{u_f(k)}(Y) &= N_{u_f(k+1)} + N_{u_f(k-1)} + N_{\hat{t}_{k-1}} + (v + v^{-1}) N_{\theta(k-1,0)\hat{t}_{k-2}} + v^2 N_{u_f(k-1)} \\
N_{x_f(k)}(Y) &= N_{x_f(k+1)} + N_{x_f(k-1)} + N_{\theta(k-1,0)\hat{t}_{k-1}} + N_{s_1 \theta(k-1,0)\hat{t}_{k-2}} + v N_{\theta(k-2,0)\hat{t}_{k-2}} + v N_{s_1 \theta(k-2,0)\hat{t}_{k-2}}.
\end{align*}
In each case, the identity also holds if $Y$ is replaced by $Y' = \hat{H}_0 \cdot \hat{H}_3 \cdot \hat{H}_m - \hat{H}_m$.

**Remark 3.21.** The reader may have noticed that in the second formula $N_{u_f(k-1)}(Y)$ shows up twice with different coefficients. This may appear to be a slip, but it is not. The reason behind this choice will become clear in Section 5.

4 Kazhdan-Lusztig basis in the big region

In this section we provide an explicit formula for Kazhdan-Lusztig basis elements indexed by elements located in the big region.

4.1 Kazhdan-Lusztig basis in region $C$

Let us recall a definition from the introduction.

**Definition 4.1.** Let $(m, n) \in \mathbb{N}^2$. We define 
\[
\text{Supp}(m, n) = \begin{cases} 
\{(m - 2i, n - j) \in \mathbb{N}^2 \mid i, j \in \mathbb{N}\}, & \text{if } m \text{ is odd;} \\
\{(m - 2i, n - j) \in \mathbb{N}^2 \mid i, j \in \mathbb{N}\} - \{(0, b) \mid b \not\equiv n \mod 2\}, & \text{if } m \text{ is even.}
\end{cases}
\]
We also define 
\[
\hat{H}_{\theta(m,n)} := \sum_{(a,b) \in \text{Supp}(m,n)} v^{(m-a)+2(n-b)} N_{\theta(a,b)}.
\]

The first goal in this section is to prove that $\hat{H}_{\theta(m,n)} = \hat{H}_{\theta(m,n)}$ for all $(m, n) \in \mathbb{N}^2$. To do this we need several preliminary lemmas.

**Lemma 4.2.** Let $m$ and $n$ be positive integers. Then, 
\[
\hat{H}_{\theta(m,n)} = N_{\theta(m,n)} + v^2 \hat{H}_{\theta(m-2,n)} + v^2 \hat{H}_{\theta(m,n-1)} - v^4 \hat{H}_{\theta(m-2,n-1)}.
\]
Furthermore, 
\[
\hat{H}_{\theta(0,n)} = N_{\theta(0,n)} + v^2 \hat{H}_{\theta(0,n-2)} \quad \text{and} \quad \hat{H}_{\theta(m,0)} = N_{\theta(m,0)} + v^2 \hat{H}_{\theta(m-2,0)}.
\]

**Proof.** The result follows directly from the definition of the elements $\hat{H}_{\theta(m,n)}$ once we recall our convention about negative indices in Remark 3.8. \hfill \Box

**Lemma 4.3.** Let $(m, n) \in \mathbb{N}^2$. Then 
\[
\hat{H}_{\theta(m,n)}(\hat{H}_n \cdot \hat{H}_m - 2\hat{H}_m) = \hat{H}_{\theta(m+1,n)} + \hat{H}_{\theta(m-1,n+1)} + \hat{H}_{\theta(m+1,n-1)} + \hat{H}_{\theta(m-1,n)}.
\]

**Proof.** A direct computation shows that (4.2) holds for the pairs $(0,0)$, $(1,0)$, $(0,1)$, $(2,0)$, $(1,1)$ and $(0,2)$. We now fix $u \geq 3$ and assume that (4.2) holds for all pairs $(m', n')$ with $m' + n' < u$. We fix a pair $(m, n)$ with $m + n = u$. We split the proof in four cases. We recall that the set of $D_E(\theta(a,b))$ only depends on the parity of $a$. For the sake of brevity we write 
\[
X^1 := \hat{H}_m H_3 \cdot \hat{H}_m - 2\hat{H}_m.
\]
Case A. Suppose that $m \geq 2$ and $n \geq 1$.
Using Proposition 3.14, Lemma 4.2 and our inductive hypothesis we get
\[
\mathbf{H}_{\theta(m,n)}X_1^1 = N_{\theta(m+1,n)} + v^2H_{\theta(1,n-1)} + v^2H_{\theta(m-1,n)} + v^2H_{\theta(m-1,n-1)} + v^2H_{\theta(m-2,n+1)} + v^2H_{\theta(m-2,n-1)}.
\] (4.3)

Then, by adding by columns in the right-hand side of (4.3) and using Lemma 4.2 we obtain (4.2).

Case B. Suppose that $m = 1$ and $n \geq 2$.
The main difference with respect to the above Case A is that when we decompose $\mathbf{H}_{\theta(1,n)}$ using Lemma 4.2 there are two terms that do not appear. More concretely, we have
\[
\mathbf{H}_{\theta(1,n)} = N_{\theta(1,n)} + v^2H_{\theta(1,n-1)}.
\]

Using Proposition 3.14 and our inductive hypothesis we get
\[
\mathbf{H}_{\theta(1,n)}X_1^1 = N_{\theta(2,n)} + v^2H_{\theta(0,n)} + v^2H_{\theta(0,n-1)} + v^2H_{\theta(0,n-2)} + v^2H_{\theta(0,n-3)}.
\] (4.4)

We now use (4.1) to rewrite $N_{\theta(0,n+1)}$ and $N_{\theta(0,n)}$ as
\[
N_{\theta(0,n+1)} = \mathbf{H}_{\theta(0,n+1)} - v^4H_{\theta(0,n-1)} \quad \text{and} \quad N_{\theta(0,n)} = \mathbf{H}_{\theta(0,n)} - v^4H_{\theta(0,n-2)}
\]
within (4.4). Another application of Lemma 4.2 gives us (4.2).

Case C. Suppose that $m = 0$ and $n \geq 3$.
As in the previous cases we use Lemma 4.2 in order to get
\[
\mathbf{H}_{\theta(0,n)} = N_{\theta(0,n)} + v^4H_{\theta(0,n-2)}.
\]

Using Proposition 3.15 and our inductive hypothesis we get
\[
\mathbf{H}_{\theta(0,n)}X_1^1 = (N_{\theta(1,n)} + v^2N_{\theta(1,n-1)} + v^4H_{\theta(1,n-2)}) + (N_{\theta(1,n-1)} + v^2N_{\theta(1,n-2)} + v^4H_{\theta(1,n-3)}).
\]

Finally, two applications of Lemma 4.2 give (4.2).

Case D. $m \geq 3$ and $n = 0$.

By combining (4.1), Proposition 3.15 and our inductive hypothesis we get
\[
\mathbf{H}_{\theta(m,0)}X_1^1 = (N_{\theta(m+1,0)} + v^2H_{\theta(m,0-1)} + v^2N_{\theta(m-1,0)} + v^2H_{\theta(m-1,0)} + v^2H_{\theta(m,0,3,0)})
\]
\[
= \mathbf{H}_{\theta(m+1,0)} + \mathbf{H}_{\theta(m+1,0)} + \mathbf{H}_{\theta(m+1,0)}.
\]

This completes the proof of the lemma.

\[\square\]

Lemma 4.4. For all $n \in \mathbb{N}$ we have
\[
\mathbf{H}_{\theta(0,n)} = \mathbf{H}_{\theta(0,n)} - 2\mathbf{H}_{\theta(0,n)} + 1 + (v + v^{-1})^2 = \mathbf{H}_{\theta(0,n+1)} + \mathbf{H}_{\theta(0,n-1)}.
\] (4.5)

Proof. We proceed by induction on $n$. The result can be checked directly for $n \leq 3$, so we assume that $n > 3$ and that (4.5) holds for all $n' < n$. We write
\[
X_1^1 := \mathbf{H}_{\theta(0,n)} - 2\mathbf{H}_{\theta(0,n)} + 1 + (v + v^{-1})^2.
\]

By combining Proposition 3.18, Lemma 4.2 and our inductive hypothesis we obtain
\[
\mathbf{H}_{\theta(0,n)}X_1^1 = \begin{cases} \{N_{\theta(0,n+1)} + v^4H_{\theta(0,n-1)}\} + \\ \{N_{\theta(0,n-1)} + v^4H_{\theta(0,n-1)} + v^2N_{\theta(0,n,3,0)} + v^2H_{\theta(0,n,3,0)}\} + \\ \{N_{\theta(0,n-1)} + v^4H_{\theta(0,n-1)}\} \\
\end{cases}
\]
\[
= \mathbf{H}_{\theta(0,n+1)} - \mathbf{H}_{\theta(0,n-1)}.
\]

where for the last equality we have used (4.1) and the identity
\[ \bar{H}_t(2, n−1) = N_\theta(2, n−1) + v^2 N_\theta(2, n−2) + v^2 N_\theta(0, n−1) + v^3 \bar{H}_t(2, n−3), \]
which follows directly from the definition of \( \bar{H}_t(2, n−1) \).

**Theorem 4.5.** For all \((m, n) \in \mathbb{N}^2\) we have \( \bar{H}_t(m, n) = \bar{H}_t(m, n) \).

**Proof.** By definition of \( \bar{H}_t(m, n) \) it is clear that \( G_\theta(m, n) \bar{H}_t(m, n) = 1 \) and that \( G_\theta(\bar{H}_t(m, n)) \in v\mathbb{N}[v] \) for all \( x < \theta(m, n) \). Therefore, in order to prove the theorem it is enough to show that \( \bar{H}_t(m, n) \) is self-dual. To do this we proceed by induction on \( m+n \). The result is clear for the pair \((0, 0)\) since \( \bar{H}_t(0, 0) = N_\theta(0, 0) = \bar{H}_t(0, 0) \).

We fix \( u \in \mathbb{N} \) and assume that \( \bar{H}_t(m, n) \) is self-dual for all pairs \((m, n)\) such that \( m+n \leq u \). By Lemma 4.3 we know that
\[
\bar{H}_t(m, n) \equiv \bar{H}_t(m+1, n) + \bar{H}_t(m−1, n+1) + \bar{H}_t(m−1, n−1).
\]

We notice that \( \bar{H}_t(m, n) \equiv \bar{H}_t(m−1, n+1) + \bar{H}_t(m−1, n−1) \) are self-dual by our inductive hypothesis. We can thus conclude that \( \bar{H}_t(m+1, n) \) is self-dual as well. This shows that \( \bar{H}_t(m, n) \) is self-dual for all pairs \((m, n)\) such that \( m+n \leq u \). By Lemma 4.4 we know that
\[
\bar{H}_t(m, n) \equiv \bar{H}_t(m−1, n+1) + \bar{H}_t(m−1, n−1).
\]

It remains to show that \( \bar{H}_t(0, u+1) \) is self-dual. By Lemma 4.4 we know that
\[
\bar{H}_t(0, u+1) \equiv \bar{H}_t(0, u+1) + \bar{H}_t(0, u−1).
\]

We have already proved its self-duality since \( (2, u−1) \) is a pair \((a, b)\) with \( a+b = u+1 \) and \( a \neq 0 \). As before, we conclude that \( \bar{H}_t(0, u+1) \) is self-dual as well, and the theorem is proved.

**Remark 4.6.** There is an alternative way to prove Theorem 4.5 using the results in [LPP21]. We prefer to present the proof just given in order to keep this paper self-contained and cite only available sources.

**Theorem 4.7.** Let \((m, n) \in \mathbb{N}^2\). Then,
\[
\bar{H}_t(m, n) = \bar{H}_t(m, n) \theta_{m, n} \quad (4.6)
\]
\[
\bar{H}_t(m, n) \theta_{m, n} = \bar{H}_t(m, n) \theta_{m, n}^{m, n} \quad (4.7)
\]
\[
\bar{H}_t(m, n) \theta_{m, n} = \bar{H}_t(m, n) \theta_{m, n} + \bar{H}_t(m, n) \theta_{m, n} \quad (4.8)
\]

**Proof.** By Lemma 2.16, we have \( xt_m > x \) for all \( x < \theta(m, n) \). On the other hand, Theorem 4.5 implies that \( \mu(x, \theta(m, n)) \neq 0 \) if and only if \( x < \theta(m, n) \). Hence, (3.1) implies (4.6).

In order to prove (4.7) we begin by noticing that Lemma 2.16 implies that the only element in \( \angle \theta(m, n) t_m \) satisfying \( xs_2 < x \) is \( \theta(m, n) \). Therefore, a combination of (3.1) and Lemma 3.6 yields
\[
\bar{H}_t(m, n) \theta_{m, n} = \bar{H}_t(m, n) \theta_{m, n}^t + \bar{H}_t(m, n) + \sum_{y \in W} D_y \theta_{m, n} \theta_y,
\]
where \( m_y \in \mathbb{N} \). We stress that \( D_y(\theta(m, n), t_m) = \{t_m, t_m'\} \). Since there is no \( y \in W \) with \( D_y(\theta(m, n), t_m) = \{t_m, t_m', s_2\} \) we conclude that the sum in (4.9) is empty. This gives (4.7).

We now prove (4.8). Once again, Lemma 2.16 shows that the only element in \( \angle \theta(m, n) t_m s_2 \) satisfying \( xt_m' < x \) is \( \theta(m, n) t_m \). Hence (3.1) and Lemma 3.6 give
\[
\bar{H}_t(m, n) \theta_{m, n} = \bar{H}_t(m, n) \theta_{m, n}^{t_m} + \bar{H}_t(m, n) \theta_{m, n} + \sum_{y \in Y} n_y \theta_{m, n} \theta_y,
\]
where \( n_y \in \mathbb{N} \) and \( Y = \{y \in W \mid D_y(\theta(m, n), t_m) = \{t_m', s_2\}, D_y(\theta(m, n), s_2) = \{s_1, s_2\}\} \).

By the discussion at the beginning of §2.1 we conclude that \( y \in Y \) if and only if there exist \((a, b) \in \mathbb{N}^2 \) with \( a \equiv m \) mod 2 such that \( y = \theta(a, b) \). We now multiply (4.10) on the right by \( \theta_{m, n} \) and using (3.2) and (4.6) we obtain
\[
\bar{H}_t(m, n) \theta_{m, n} \theta_{m, n} = \bar{H}_t(m, n) t_m s_2 t_m' \theta_{m, n} + (v^2 + v^{-1}) \bar{H}_t(m, n) \theta_{m, n} + \sum_{y \in Y} n_y \theta_y \theta_{m, n} \theta_y.
\]

20
On the other hand, we combine (3.2), Lemma 4.3, Theorem 4.5, (4.6) and (4.7) to obtain

$$H_{\theta(m,n)t_m} H_{\theta(m,n)} = (v^{-1}H_{\theta(m,n)t_m} + H_{\theta(m+1,n)t_m} + H_{\theta(m-1,n+1)t_m} + H_{\theta(m,n-1)t_m}).$$

(4.12)

We now notice that $H_{\theta(m,n)t_m} H_{\theta(m,n)} = H_{\theta(m,n)t_m}$ and conclude that the right-hand side of (4.11) and (4.12) coincide. Therefore, after cancelling out the term $(v^{-1}H_{\theta(m,n)t_m})$ we get

$$H_{\theta(m,n)t_m \cdot t_m} H_{\theta(m,n)} + \sum_{y \in Y} n_y H_{yt_m} = H_{\theta(m+1,n)t_m} + H_{\theta(m-1,n+1)t_m} + H_{\theta(m,n-1)t_m} + H_{\theta(m,n)t_m}.$$ 

(4.13)

Suppose that $n_y \neq 0$ for some $y = \theta(a,b) \in Y$. Since $a \equiv m \pmod{2}$ we know that $a \neq m \pm 1$. In particular, $\theta(a,b)t_m$ is not equal to any of the elements that index the Kazhdan-Lusztig basis elements on the right-hand side of (4.13). We reach a contradiction and conclude that $n_y = 0$ for all $y \in Y$. If we look back at (4.10) then we see that it reduces to (4.8).

\[\square\]

4.2 Kazhdan-Lusztig basis for the whole big region

Let us remark that Theorem 4.5 and Theorem 4.7 give us all the Kazhdan-Lusztig basis elements indexed by elements in the fundamental region $C$ and also in the region $\varphi(C)$. In this section we extend this result to all the elements in the big region. We stress that we only need specify the description of the Kazhdan-Lusztig basis elements for the regions $s_0C$, $s_2s_0C$ and $s_1s_2s_0C$ since the description for the elements in the regions $s_1\varphi(C)$, $s_2s_1\varphi(C)$ and $s_0s_2s_1\varphi(C)$ will follow from the above by applying the automorphism $\varphi$.

**Theorem 4.8.** Let $(m,n) \in \mathbb{N}^2$. Then,

$$H_{\theta(m,n)} = H_{s_0\theta(m,n)} \quad \text{(4.14)}$$

$$H_{s_0\theta(m,n)t_m} = H_{s_0s_0\theta(m,n)t_m} \quad \text{(4.15)}$$

$$H_{s_0\theta(m,n)t_m t_2} = H_{s_0\theta(m,n)t_m t_2} \quad \text{(4.16)}$$

$$H_{s_0\theta(m,n)t_m s_2t_m} = H_{s_0\theta(m,n)t_m s_2t_m} \quad \text{(4.17)}$$

$$H_{s_0\theta(m,n)} = H_{s_0s_0\theta(m,n) + H_{\theta(m,n)}} \quad \text{(4.18)}$$

$$H_{s_0\theta(m,n)t_m} = H_{s_0s_0\theta(m,n)t_m} + H_{\theta(m,n)t_m} \quad \text{(4.19)}$$

$$H_{s_0\theta(m,n)t_m t_2} = H_{s_0s_0\theta(m,n)t_m t_2} + H_{\theta(m,n)t_m t_2} \quad \text{(4.20)}$$

$$H_{s_0\theta(m,n)t_m s_2t_m} = H_{s_0s_0\theta(m,n)t_m s_2t_m} + H_{\theta(m,n)t_m s_2t_m} \quad \text{(4.21)}$$

$$H_{s_0\theta(m,n)} = H_{s_1s_0s_0\theta(m,n) + H_{s_0\theta(m,n)}} \quad \text{(4.22)}$$

$$H_{s_0\theta(m,n)t_m} = H_{s_1s_0s_0\theta(m,n)t_m} + H_{s_0\theta(m,n)t_m} \quad \text{(4.23)}$$

$$H_{s_0\theta(m,n)t_m s_2t_m} = H_{s_1s_0s_0\theta(m,n)t_m s_2t_m} + H_{s_0\theta(m,n)t_m s_2t_m} \quad \text{(4.24)}$$

$$H_{s_0\theta(m,n)t_m s_2t_m} = H_{s_1s_0s_0\theta(m,n)t_m s_2t_m} + H_{s_0\theta(m,n)t_m s_2t_m} \quad \text{(4.25)}$$

**Proof.** We fix $(m,n) \in \mathbb{N}^2$ and let $R := \{e, t_m, t_m s_2, t_m s_2t_m\}$. By our description of the set of coatoms for lower intervals in Lemma 2.16 we have $s_0x > x$ for all $y \in R$ and $x < \theta(m,n)y$. Therefore, since $D_L(\theta(m,n)y) = \{s_1, s_2\}$ for all $y \in R$ and there is no element $z \in W$ with $D_L(z) = \{s_0, s_1, s_2\}$, we obtain via the left version of (3.1) and Lemma 3.6 all the identities (4.14)–(4.17). We pause our proof for a moment to make an observation that will be useful for the rest of the proof.

**Remark 4.9.** We remark that (4.14)–(4.17) together are equivalent to the following: For any $\varphi \in C$ we have $H_{s_0\theta} = H_{s_0\theta}$. Of course, by applying $\varphi$ to the above equality we get $H_{s_0\theta} = H_{s_0\theta}$ for all $u \in \varphi(C)$.

We continue with the proof of the next four identities. Now, Lemma 2.18 allows us to conclude that for each $y \in R$ the only element in $s_0\theta(m,n)y$ satisfying $s_0x < x$ is $\theta(m,n)y$. Since $D_L(s_0\theta(m,n)y) = \{s_0, s_1\}$ for all $y \in R$, another application of the left version of (3.1) and Lemma 3.6 give us (4.18)–(4.21).

For the last four identities we need to work a little harder. In this case we have $D_L(s_2s_0\theta(m,n)y) = \{s_2\}$ for all $y \in R$. Therefore, the left version of (3.1) and Lemma 3.6 only allow us to conclude that for each $y \in R$ we have

$$H_{s_1s_2s_0\theta(m,n)y} = H_{s_1s_2s_0\theta(m,n)y} + H_{s_0\theta(m,n)y} + \sum_{z \in C} m^y_H_{z}.$$ 

(4.26)


where $m^y \in \mathbb{N}$.

We now multiply (4.26) by $H_m$ on the left, and using (3.2) together with Remark 4.9 we get

$$H_m H_1 H_{s_0 \theta(m,n)y} = H_m H_{s_0 \theta(m,n)y} + (v + v^{-1}) H_{m} \theta(m,n)y + \sum_{z \in C} m^y z H_{m} \theta(z). \quad (4.27)$$

We now compute $H_{s_0 \theta(m,n)y}$ in a different way using the fact that $H_m H_1 = H_1 H_m$. Lemma 2.18 shows that for all $y \in \mathcal{R}$ the only element in $s_2 s_0 \theta(m,n)y$ satisfying $s_0 x < x$ is $s_0 \theta(m,n)y$. On the other hand, we notice that $s_0 s_2 s_0 \theta(m,n)y$ belongs to $\varphi(\mathcal{C})$ since $D_{L}(s_0 s_2 s_0 \theta(m,n)y) = \{s_0, s_2\}$. Using these two facts together with the left version of (3.1) and Lemma 3.6 we obtain

$$H_{s_0 \theta(m,n)y} = \sum_{w \in \varphi(\mathcal{C})} n^w H_m \theta(m,n)y \quad \text{(for some } n^w \in \mathbb{N}).$$

We now fix $y \in \mathcal{R}$ and suppose that $m^y \neq 0$ for some $z \in C$. By comparing the right-hand sides of (4.27) and (4.28), we conclude that $s_0 \theta(m,n)y \in s_1 \varphi(\mathcal{C})$, which is absurd since $s_0 \mathcal{C} \cap s_1 \varphi(\mathcal{C}) = \emptyset$. Therefore, $m^y = 0$ for all $z \in C$ and (4.26) reduces to

$$H_{s_0 \theta(m,n)y} = H_{s_0 s_2 s_0 \theta(m,n)y} + H_{s_0 \theta(m,n)y}.$$  

Since this identity holds for all $y \in \mathcal{R}$ we obtain (4.22)–(4.25).

**Remark 4.10.** The formulas in Theorem 4.7 and Theorem 4.8 are apparently different from the ones presented in Theorem 1.1. It is an easy exercise (which is left to the reader) to show that both are equivalent.

### 4.3 Extra explicit formulas needed in the sequel

Formulas given in §4.1 and §4.2 enable the efficient computation of any Kazhdan-Lusztig polynomial $h_{x,w}(v)$ for all $x \in W$ and all $w$ in the big region. In order to obtain formulas for the thick region in §5, it will be convenient to have more explicit expressions on hand for certain Kazhdan-Lusztig basis elements in the big region. This is the main goal of this section.

**Lemma 4.11.** Let $m \geq 0$. We have $H_{\theta(m,0)t_m} = N_{\theta(m,0)t_m} + v H_{\theta(m-1,0)t_{m-1}}$. 

**Proof.** We proceed by induction on $m$. The cases $m = 0$ and $m = 1$ are easily checked. We assume $m > 1$ and that the lemma holds for $m - 1$. Using Lemma 3.9, the explicit formula for $H_{\theta(m,0)}$, (4.6), our inductive hypothesis and the fact that $t_m = t_{m-2}$ we obtain

$$H_{\theta(m,0)t_m} = H_{\theta(m,0)} H_{\theta(m,0)} = (N_{\theta(m,0)} + v^2 H_{\theta(m-2,0)}) H_{\theta(m,0)} = N_{\theta(m,0)t_m} + v N_{\theta(m-1,0)t_{m-1}} + v^2 H_{\theta(m-2,0)t_m} = N_{\theta(m,0)t_m} + v(N_{\theta(m-1,0)t_{m-1}} + v^2 H_{\theta(m-2,0)t_{m-2}}) = N_{\theta(m,0)t_m} + v H_{\theta(m-1,0)t_{m-1}}.$$

The proof of the following lemma is very similar.

**Lemma 4.12.** For all $m \geq 0$, we have $H_{s_0 \theta(m,0)} = N_{s_0 \theta(m,0)} + v H_{s_0 \theta(m-1,0)}$.

**Lemma 4.13.** For all $m \geq 0$ the element $H_{s_0 s_2 s_0 \theta(m,0)}$ is equal to

$$N_{s_0 s_2 s_0 \theta(m,0)} + v \sum_{i=0}^{\lfloor m/2 \rfloor} v^{2i} (N_{\theta(m-2i,0)} + N_{s_1 \theta'(m-1-2i,0)}) + \sum_{i=1}^{\lfloor m/2 \rfloor} v^{2i} (v^{-1} N_{\theta(m-2i+1,0)} + N_{s_0 \theta(m-2i,0)}). \quad (4.29)$$
Proof. For $m=0$ and $m=1$ the result can be checked by a direct computation. We then assume $m \geq 2$. Definition 4.1 and Theorem 4.5 give us the following identity

$$H_{\theta(m,0)} = \sum_{i=0}^{\frac{m}{2}} v^{2i}N_{\theta(m-2i,0)}.$$  

Then, combining (4.14), (4.18) and (4.22) we get

$$H_{s_1s_2\theta(0,m)} = (H_{s_1}, H_{s_2}, H_{\theta(0,m)} - H_{s_0} - H_{s_1}) H_{\theta(0,m)}$$

$$= (H_{s_1}H_{s_2}, H_{\theta(0,m)} - v + v^{-1}) H_{\theta(0,m)}$$

$$= (H_{s_1}H_{s_2}H_{\theta(0,m)} - H_{s_0}) H_{\theta(0,m)} - (v + v^{-1}) H_{\theta(0,m)}$$

$$= (\sum_{i=0}^{\frac{m}{2}} H_{s_1}H_{s_2}H_{\theta(0,m)} - H_{s_0}) \sum_{i=0}^{\frac{m}{2}} v^{2i}N_{\theta(m-2i,0)} - (v + v^{-1}) \sum_{i=0}^{\frac{m}{2}} v^{2i}N_{\theta(m-2i,0)} + (v + v^{-1}) \sum_{i=0}^{\frac{m}{2}} v^{2i}N_{\theta(m-2i,0)}.$$  

We assume $m$ is odd, the case $m$ even being similar. We then use Lemma 3.19 to conclude that

$$H_{s_1s_2\theta(0,m)} = \sum_{i=0}^{\frac{m}{2}} v^{2i} \left[ N_{s_1s_2\theta(0,m)} - v^{2i}N_{s_1s_2\theta(0,m-2i,0)} \right] + \sum_{i=0}^{\frac{m}{2}} v^{2i} \left[ v(N_{\theta(m-2i,1,1)} + N_{\theta(0,m-2i,1,0)}) \right].$$  

We notice that the first sum in (4.30) telescopes to $N_{s_1s_2\theta(0,m)}$ and therefore the right-hand side of (4.30) reduces to (4.29). We stress that the apparent discrepancy between both expressions is solved by the fact that in (4.30) the term $v^{-1}N_{\theta(m-2i,1,1)} + N_{\theta(0,m-2i,1,0)}$ becomes zero when $i = (m-1)/2$. \hfill \Box

Corollary 4.14. For all $m \geq 1$ we have

$$H_{s_1s_2\theta(0,m)}t_m = N_{s_1s_2\theta(0,m)}t_m + vN_{s_1s_2\theta(0,m-1)}t_{m-1} + v^{k+2}N_{s_1s_0} + \sum_{i=0}^{m} v^{k+1}N_{\theta(m-i,1)}t_{m-i-1} + \sum_{i=2}^{m} v^{i-1}N_{\theta(m-i,1)}t_{m-i-1} + v^{m}N_{\theta(0,m-1)}t_{m-1}.$$  

Consequently,

$$H_{s_1s_2\theta(0,m+1)} = N_{s_1s_2\theta(0,m+1)} + vH_{s_1s_2\theta(0,m)}t_m - v^2N_{s_1s_2\theta(0,m-1)}t_{m-1} + vN_{\theta(m+1,1)}t_{m+1} + vN_{\theta(m+1,1)}t_{m-1} + v^2N_{\theta(0,m)}t_{m-1}.$$  

Proof. We first notice that $(s_1s_2\theta(0,m))^{-1}$ belongs to $\mathcal{C}$ (resp. $\varphi(\mathcal{C})$) if $m$ is even (resp. odd). Thus, using either (4.17) or its image under $\varphi$ we conclude that

$$H_{s_1s_2\theta(0,m)}^{-1} = H_{s_1s_2\theta(0,m)}^{-1}.$$  

It follows that $H_{s_1s_2\theta(0,m)} = H_{s_1s_2\theta(0,m)}H_{s_1s_2\theta(0,m)}^{-1}$. Therefore we can use the expression for $H_{s_1s_2\theta(0,m)}$ given in Lemma 4.13 in order to compute $H_{s_1s_2\theta(0,m)}t_m$. Then (4.31) follows by a combination of Lemma 3.7 with $n=1$, Lemma 3.9, Lemma 3.11 and its $\varphi$-image and Lemma 3.12. Finally, (4.32) is a direct consequence of (4.31). \hfill \Box

The following result can be proved with similar arguments. For the sake of brevity we omit the proof.

Lemma 4.15. Let $m \in \mathbb{N}$. Then, we have

$$H_{\theta(m+1,0)t_{m+1}} = N_{\theta(m+1,0)t_{m+1}} + vN_{s_1\theta^r(m,0)t^r_m} + vH_{\theta(m,0)t_{m+1}}$$

$$H_{s_1s_2\theta(0,m+1)} = N_{s_1s_2\theta(0,m+1)} + vN_{\theta(m,0)t_{m+1}} + vH_{s_1s_2\theta(0,m)}t_m - v^2N_{\theta(0,m)}t_{m-1}.$$  

23
5 Kazhdan-Lusztig polynomials for the thick region

In this section we compute Kazhdan-Lusztig basis elements corresponding to elements in the thick region. We recall from Figure 1 that this region is made of four sub-regions: north (N), south (S), east (E) and west (W). Recall from introduction that \( N = \{ x_n \mid n \geq 1 \} \cup \{ x_n \mid n \geq 1 \} \), where \( x_n = a_1 \cdots a_n \) and \( \tau_n = s_1 s_2 s_0 x_{n-1} \), and the \( a_i \) are defined by the sequence \( \{ a_n \}_{n=1}^{\infty} = (s_1, s_2, s_1, s_0, s_2, s_0, s_1, s_2, s_1, s_0, s_2, s_0, \ldots) \). We also recall that \( S = \varphi(N) \), \( E = \{ e_n = s_1 x'_n \mid n \geq 1 \} \cup \{ e'_n \mid k \geq 1 \} \), and \( W = s_2(E) \cup \{ s_2 \} \).

5.1 North and South

**Definition 5.1.** We recall from the introduction the notation \( f(k) = 3k + 1 \) and \( u_n = s_2 x_n \). For all \( k \geq 2 \) we define

\[
\hat{H}_{f(k)} = N_{x_{f(k)}} + v N_{x_{f(k-1)}} + \left( \sum_{j=2}^{k-1} v^{j-1}(N_{x_{f(k-j)}} + N_{u_{f(k-j)}}) \right) + v^{k-1} N_{s_1 s_0}.
\]  

(5.1)

**Lemma 5.2.** Let \( Y = \hat{H}_t \hat{H}_s \hat{H}_u - \hat{H}_t \). For all \( k \geq 2 \) we have

\[
\hat{H}_{f(k)} Y = \hat{H}_{f(k+1)} + \hat{H}_{f(k-1)} + \hat{H}_{s_1} \theta'(k-2) t_{k-1} + \hat{H}_{s_1} \theta(k-1) t_{k-1} - \hat{H}_{s_1} \theta(k-3) t_{k-3} + \hat{H}_{s_1 s_2} \theta(k-2) t_{k-2}.
\]

**Proof.** We proceed by induction on \( k \). The case \( k = 2 \) follows by a direct computation. We assume the lemma holds for some \( k \geq 2 \). It follows directly from Definition 5.1 that

\[
\hat{H}_{f(k+1)} = N_{x_{f(k+1)}} + v N_{x_{f(k-2)}} + v N_{u_{f(k)}} - v^2 N_{x_{f(k-1)}} + v \hat{H}_{f(k)}.
\]

(5.2)

Multiplying this equality on the right by \( Y \) and using our inductive hypothesis and Lemma 3.20 we obtain that \( \hat{H}_{f(k+1)} Y \) is equal to the sum of all the entries of the following matrix.

| \( N_{x_{f(k+2)}} \) | \( N_{x_{f(k)}} \) | \( N_{\theta(k,0) t_k} \) | \( N_{s_1 s_2} \theta(k-1) t_{k-1} \) | \( N_{s_1 \theta'(k-1) t'_{k-1}} \) | \( v N_{\theta(k-1) t_{k-1}} \) |
|---|---|---|---|---|---|
| \( v N_{u_{f(k)}} \) | \( v N_{u_{f(k-2)}} \) | \( v N_{\theta(k-3) t_{k-2}} \) | \( v^2 N_{\theta(k-2) t_{k-2}} \) | \( N_{s_1 \theta'(k-2) t'_{k-2}} \) | \( v N_{\theta(k-2) t_{k-2}} \) |
| \( v N_{x_{f(k)}} \) | \( v N_{x_{f(k-2)}} \) | \( v N_{\theta(k-3) t_{k-2}} \) | \( v^2 N_{\theta(k-2) t_{k-2}} \) | \( N_{s_1 \theta'(k-2) t'_{k-2}} \) | \( v N_{\theta(k-2) t_{k-2}} \) |
| \( -v^2 N_{x_{f(k)}} \) | \( -v^2 N_{x_{f(k-2)}} \) | \( -v^2 N_{\theta(k-2) t_{k-2}} \) | \( -v^2 N_{s_1 s_2} \theta(k-1) t_{k-1} \) | \( -v^2 N_{s_1 \theta'(k-1) t'_{k-1}} \) | \( -v^2 N_{\theta(k-1) t_{k-1}} \) |
| \( v \hat{H}_{f(k)} \) | \( v \hat{H}_{f(k)} \) | \( v \hat{H}_{s_1} \theta(k-2) t_{k-2} \) | \( v \hat{H}_{s_1 \theta'(k-2) t'_{k-2}} \) | \( v \hat{H}_{s_1 \theta(k-1) t_{k-1}} \) | \( v \hat{H}_{s_1 \theta(k-1) t_{k-1}} \) |

We denote this matrix by \( A = (A_{ij}) \). Using (5.2) we see that

\[
\hat{H}_{f(k+2)} = \sum_{i=1}^{5} A_{i1} \quad \text{and} \quad \hat{H}_{f(k)} = \sum_{i=1}^{5} A_{i2}.
\]

By Lemma 4.11 we have \( \hat{H}_{\theta(k,0) t_k} = A_{13} + A_{54} \) and \( \hat{H}_{s_1 \theta'(k-2) t_{k-2}} = A_{25} + A_{55} \). On the other hand, Lemma 4.15 implies that \( \hat{H}_{s_1} \theta(k-1) t_{k-1} \) is equal to \( A_{15} + A_{33} + A_{33} \). We also observe that \( A_{24} + A_{43} = A_{36} + A_{45} = A_{26} + A_{46} = 0 \), this last equality being a consequence of the fact that \( \theta(k-3,0) u_{k-3} = u_{f(k-2)} \). Finally, we notice that Corollary 4.14 yields \( \hat{H}_{s_1 s_2 s_0} \theta(k-1) t_{k-1} = A_{14} + A_{56} + A_{44} + A_{16} + A_{34} + A_{23} + A_{45} \). Putting all this together we obtain

\[
\hat{H}_{f(k+1)} Y = \hat{H}_{f(k+2)} + \hat{H}_{f(k)} + \hat{H}_{s_1} \theta(k-1) t_{k-1} + \hat{H}_{s_1} \theta(k,0) t_k + \hat{H}_{s_1} \theta(k-1) t_{k-1} + \hat{H}_{s_1 s_2 s_0} \theta(k-1) t_{k-1},
\]

as we wanted to show.

**Theorem 5.3.** For all \( k \geq 2 \) we have \( \hat{H}_{f(k)} = \hat{H}_{f(k)} \).

**Proof.** For \( k = 2 \) the result follows by a direct computation. We now fix some \( k \geq 2 \) and assume the theorem holds for all \( 2 \leq k' < k \). We notice that \( G_{x_{f(k+1)}} \hat{H}_{f(k+1)} = 1 \) and \( G_z(\hat{H}_{f(k+1)}) \in v N[v] \), for all \( z < x_{f(k+1)} \). Thus we only need to show that \( \hat{H}_{f(k+1)} \) is self-dual. This is an immediate consequence of Lemma 5.2 and our inductive hypothesis.

**Theorem 5.4.** Let \( k \geq 2 \). We have

\[
\hat{H}_{s_1 \theta(k-1) t_{k-1}} = \hat{H}_{s_1 \theta(k-1) t_{k-1}}.
\]

(5.3)
\[H_{x_{3k+2}}H_0 = \begin{cases} H_{x_{3k+3}} + H_{x_{3k+4}} + H_{(k-1,0)} + H_{1,0}(k-2,0) + H_{(k-3,0)}, & \text{if } k \text{ is even;} \\
H_{x_{3k+3}} + H_{x_{3k+4}} + H_{1,0}(k-2,0), & \text{if } k \text{ is odd.} \end{cases} \quad (5.4)\]

\[H_{x_{3k+2}}H_1 = \begin{cases} H_{x_{3k+3}} + H_{x_{3k+4}} + H_{1,0}(k-2,0), & \text{if } k \text{ is even;} \\
H_{x_{3k+3}} + H_{x_{3k+4}} + H_{1,0}(k-2,0), & \text{if } k \text{ is odd.} \end{cases} \quad (5.5)\]

**Proof.** The claim can be checked directly for \(k = 2\). From now on we assume \(k \geq 3\). An inspection of the explicit formula for \(H_{x_{3k+1}}\) provided in Definition 5.1 allows us to conclude that \(xs_2 > x\) for all \(x \in W\) such that \(\mu(x, x_{3k+1}) \neq 0\). Thus the sum in (3.1) is empty and (5.3) follows.

We now prove (5.4). We first treat the case \(k\) even. Equation (3.2), Lemma 5.2 and Theorem 5.3 imply

\[H_{x_{3k+1}}H_2H_0H_1 = (v + v^{-1})H_{x_{3k+4}} + H_{x_{3k+2}} + H_{1,0}(k-2,0) + H_{1,0}(k-3,0), \quad (5.6)\]

On the other hand, we can combine (3.4), (5.1) and (5.3) to obtain

\[h_{s_1,0}(k-2,0, x_{3k+2}) = v^{-1}h_{s_1,0}(k-2,0, x_{3k+1}) + h_{x_{3(k-1)}, x_{3k+1}}(v) = v^{-1}(v^2 + v^3) = v + v^3\]

\[h_{\theta(k-3,0), x_{3k+2}}(v) = v^{-1}h_{\theta(k-3,0), x_{3k+1}} + h_{x_{2(k-2), x_{3k+1}}(v) = v^{-1}(v^6 + v^4 + v^2) + (v^7 + v^5 + 2v^3) = v + 3v^3 + 2v^5 + v^7.\]

From this we conclude that

\[\mu(s_1, \theta(k-2,0), x_{3k+2}) = \mu(\theta(k-3,0), x_{3k+2}) = 1. \quad (5.7)\]

We have the following equalities

\[H_{x_{3k+1}}H_2H_0H_1 = H_{x_{3k+2}}H_1H_2 = H_{x_{3k+4}} + H_{x_{3k+1}}H_{1,0}(k-2,0) + H_{1,0}(k-3,0) + \sum_{y \in Y} m_y H_y \]

\[= \left( H_{x_{3k+4}} + \sum_{y \in Y} m_y H_y \right) H_1 H_2 + (v + v^{-1})H_{x_{3k+1}} + H_{1,0}(k-3,0) + \sum_{y \in Y} m_y H_y, \quad (5.8)\]

where \(m_y \in N, Z = \{z \in W \mid D_R(y) = \{s_0, s_1\}\} \)

\[Y = \{y \in W \mid D_R(y) = \{s_0, s_2\}\} \setminus \{s_1, \theta(k-2,0), \theta(k-3,0)\}.\]

Let us explain how to obtain the above equalities. The first equality is a direct consequence of (5.3). For the second one we use (3.1); first consider the elements in \(x_{3k+2}\) satisfying \(w_0 < w\). Lemma 2.13 together with an easy case analysis show that these elements are \(x_{3k+1}\) and \(\theta(k-1,0)\). This justifies the occurrence of the terms \(H_{x_{3k+1}}\) and \(H_{\theta(k-1,0)}\). On the other hand, (5.7) explains the occurrence of the terms \(H_{1,0}(k-2,0)\) and \(H_{\theta(k-3,0)}\). Finally, the equality \(D_R(x_{3k+2}) = \{s_2\}\) allows us to conclude, via Lemma 3.6, that any other term occurring must have right descent set equal to \(\{s_0, s_2\}\). This explains the appearance of the sum. We remark that \(D_R(s_1 \theta(k-2,0)) = D_R(\theta(k-3,0)) = \{s_0, s_2\}\) as well (for \(k\) even). However the elements \(s_1 \theta(k-2,0)\) and \(\theta(k-3,0)\) were already considered, which explains the somewhat strange definition of \(Y\).

For the last equality, the terms inside the parentheses are justified applying the same arguments used for the second equality to the multiplication \(H_{x_{3k+1}}H_2H_1\). We notice that \(D_R(x_{3k+1}) = \{s_0\}\) this explains via Lemma 3.6 the definition of \(Z\). The term \((v + v^{-1})H_{x_{3k+4}}\) follows by (3.2). Finally, the occurrence of all
the other terms follows by applying Remark 4.9, which (in its \( \varphi \)-version) is equivalent to the statement that if \( D_L(w) = \{ s_0, s_2 \} \) then \( \mathcal{H}_w, \mathcal{H}_w = \mathcal{H}_{s_0 w} \). Using inverses to move from left to right, the latter is equivalent to saying that if \( D_R(w) = \{ s_0, s_2 \} \) then \( \mathcal{H}_w, \mathcal{H}_w = \mathcal{H}_{s_1 w} \). This is the version we need to conclude.

A comparison of the expressions for \( \mathcal{H}_{x_{3k-2}} = \mathcal{H}_{x_{3k+1}} + \mathcal{H}_{x_{3k-4}} + \mathcal{H}_{x_{3k-2}} \) given in (5.6) and (5.8) yields

\[
\mathcal{H}_{x_{3k-2}} = \sum_{z \in Z} n_z \mathcal{H}_z + \sum_{y \in Y} m_y \mathcal{H}_{y s_1}.
\]

Suppose that \( x_{3k-2} = y s_1 \) for some \( y \in Y \). Multiplying by \( s_1 \) on the right we obtain that \( \mathcal{H}_{x_{3k-3}} \in Y \) since \( k \)
is even. However, \( D_R(\mathcal{H}_{x_{3k-3}}) = \{ s_0 \} \) and we reach a contradiction. We conclude that \( m_y = 0 \) for all \( y \in Y \).

Therefore,

\[
\mathcal{H}_{x_{3k-2}} = \mathcal{H}_{x_{3k+1}} + \mathcal{H}_{x_{3k+3}} + \mathcal{H}_{x_{3k+4}} + \mathcal{H}_{x_{3k-2}},
\]

as we wanted to show. This completes the proof of (5.4) for \( k \) even.

We now assume \( k \) is odd. As before, Equation (3.2), Lemma 5.2 and Theorem 5.3 imply

\[
\mathcal{H}_{x_{3k+1}} H_{x_{3k+2}} H_{x_{3k+3}} H_{x_{3k+4}} = (v + v^{-1}) \mathcal{H}_{x_{3k+2}} + \mathcal{H}_{x_{3k+4}} + \mathcal{H}_{x_{3k-2}} + \mathcal{H}_{x_{3k+1}} \theta(k-2,0) s_0 + \mathcal{H}_{x_{3k-1}} \theta(k-2,0) s_0 + \mathcal{H}_{x_{3k-1}} s_2 s_0 \theta(k-2,0) s_1.
\]

On the other hand, arguing as in (5.8) we obtain

\[
\mathcal{H}_{x_{3k+1}} H_{x_{3k+2}} H_{x_{3k+3}} H_{x_{3k+4}} = \mathcal{H}_{x_{3k+3}} H_{x_{3k+4}} H_{x_{3k-2}} + \mathcal{H}_{x_{3k+1}} \theta(k-2,0) s_0 + \mathcal{H}_{x_{3k-1}} \theta(k-2,0) s_0 + \mathcal{H}_{x_{3k-1}} s_2 s_0 \theta(k-2,0) s_1 + \sum_{y \in Y} m_y \mathcal{H}_{y s_1}.
\]

where \( m_y, n_z \in \mathbb{N} \), \( Z = \{ z \in W \mid D_R(y) = \{ s_0, s_1 \} \} \) and \( Y = \{ y \in W \mid D_R(y) = \{ s_0, s_2 \} \} \).

A comparison of the expressions for \( \mathcal{H}_{x_{3k+1}} + \mathcal{H}_{x_{3k+2}} + \mathcal{H}_{x_{3k+3}} + \mathcal{H}_{x_{3k+4}} \) given in (5.9) and (5.10) yields

\[
\mathcal{H}_{x_{3k-2}} + \mathcal{H}_{x_{3k-1}} \theta(k-2,0) s_0 + \mathcal{H}_{x_{3k-1}} \theta(k-2,0) s_0 = \sum_{z \in Z} n_z \mathcal{H}_z + \sum_{y \in Y} m_y \mathcal{H}_{y s_1}.
\]

As before we want to show that the sum over \( Y \) is zero. Suppose that \( x_{3k-2} = y s_1 \) for some \( y \in Y \). Multiplying by \( s_1 \) on the right we have that \( x_{3k-3} \in Y \) since \( k \)
is odd. Since \( D_R(x_{3k-3}) = \{ s_0 \} \) we obtain a contradiction. We conclude that term \( \mathcal{H}_{x_{3k-2}} \) does not come from the sum over \( Y \). We now suppose that \( s_1 \theta(k-2,0) s_0 = y s_1 \) for some \( y \in Y \). Multiplying by \( s_1 \) on the right we obtain

\[
y = s_1 \theta(k-2,0) s_0 s_1 = s_1 \theta(k-3,0) s_1 s_2 s_1 s_0 s_1 = s_1 \theta(k-3,0) s_1 s_2 s_0.
\]

It follows that \( s_1 \theta(k-3,0) s_1 s_2 s_0 \in Y \). However, \( D_R(s_1 \theta(k-3,0) s_1 s_2 s_0) = \{ s_0 \} \) and we reach a contradiction. We conclude that term \( \mathcal{H}_{x_{3k-2}} \) does not come from the sum over \( Y \). Similarly, we can prove that \( \mathcal{H}_{x_{3k-1}} \theta(k-2,0) s_0 \) does not come from the sum over \( Y \). It follows that the sum over \( Y \) is zero. Therefore,

\[
\mathcal{H}_{x_{3k+1}} \mathcal{H}_{x_{3k+2}} = \mathcal{H}_{x_{3k+3}} + \mathcal{H}_{x_{3k+1}} \theta(k-2,0),
\]

as we wanted to show.

The proof of (5.5) is dealt with similarity. The details are left to the reader. \( \square \)

Theorem 5.3 and Theorem 5.4 give formulas for all Kazhdan-Lusztig basis elements indexed by elements of \( \mathcal{N} \) of length greater than six; the remainder can be obtained by direct calculation. As we already pointed out at the beginning of this section, \( S = \varphi(\mathcal{N}) \), therefore acting by \( \varphi \) in each one of these formulas we obtain the corresponding formulas for all the Kazhdan-Lusztig basis elements indexed by elements of \( S \).
5.2 East and West.

**Definition 5.5.** For \( k \geq 1 \) we define the following elements

\[
\hat{H}_{3k+1} = \sum_{i=0}^{k} v^i N_{c_3(k-i)+1},
\]

\[
\hat{H}_{3k+2} = N_{c_3k+2} + \sum_{i=1}^{k} 2v^i N_{c_3(k-i)+2},
\]

\[
\hat{H}_{3k+3} = N_{c_3k+3} + v N_{c_3k} + \sum_{i=1}^{k} v^i N_{c_3(k-i)+1} + \sum_{i=1}^{k} v^i \varphi^i (s_0 \theta (k-i-1)),
\]

**Lemma 5.6.** Let \( k \geq 1 \). We have

\[
\hat{H}_{f(k)} H_{3k+1} H_{3k+2} H_{3k+3} = \hat{H}_{f(k+1)} + (v + v^{-1}) \hat{H}_{f(k)} + \hat{H}_{f(k-1)} + H_{0} \theta (k-1) t_k' + H_{1} \theta (k-1) t_k.
\]

**Proof.** We proceed by induction on \( k \). A direct computation gives us the result for \( k = 1 \). We fix \( k \geq 1 \) and assume that (5.13) holds for all \( k' \leq k \). By **Definition 5.5** we have

\[
\hat{H}_{f(k+1)} H_{3k+1} H_{3k+2} H_{3k+3} = \hat{H}_{f(k+1)} + v \hat{H}_{f(k)}.
\]

Then, multiplying this last equation on the right by \( \hat{H}_{3k+1}, \hat{H}_{3k+2}, \hat{H}_{3k+3} \) and using **Lemma 3.20, Lemma 3.4** and our inductive hypothesis we conclude that \( \hat{H}_{f(k+1)}, \hat{H}_{3k+1}, \hat{H}_{3k+2}, \hat{H}_{3k+3} \) is equal to the sum of the entries of the following matrix

| \( N_{c_3(k+2)} \) | \( (v + v^{-1}) N_{c_3(k+1)} \) | \( N_{c_3(k)} \) | \( N_{s_0} \theta (k) t_k \) | \( N_{s_1} \theta (k-1) t_k \) |
|---|---|---|---|---|
| \( v \hat{H}_{f(k+1)} \) | \( v (v + v^{-1}) \hat{H}_{f(k)} \) | \( v \hat{H}_{f(k-1)} \) | \( v N_{s_0} \theta (k-1) t_k' \) | \( v N_{s_1} \theta (k-1) t_k' \) |

| \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) |

Bearing in mind that \( t_k' = t_{k-1} \), **Lemma 4.15** together with (5.14) show that if we add the entries of the above matrix by columns then we obtain

\[
\hat{H}_{f(k+1)} H_{3k+1} H_{3k+2} H_{3k+3} = \hat{H}_{f(k+2)} + (v + v^{-1}) \hat{H}_{f(k-1)} + \hat{H}_{f(k)} + H_{0} \theta (k-1) t_k' + H_{1} \theta (k-1) t_k.
\]

This completes our induction.

**Theorem 5.7.** For all \( k \geq 1 \) and \( j \in \{1, 2, 3\} \) we have

\[
\hat{H}_{3k+j} = H_{3k+j}.
\]

**Proof.** We split the proof of (5.15) in three cases in accordance with the value of \( j \).

**Case A.** \(( j = 1)\) For \( k = 1 \) the result can be checked by hand. We assume \( k > 1 \). It follows directly from the definition of \( H_{3k+1} \) that \( G_{c_3k+1} (H_{3k+1}) = 1 \) and \( G_{z} (H_{3k+1}) \in \mathbb{R}[v] \), for all \( z < c_3k+1 \). Therefore, we only need to check that \( H_{3k+1} \) is self-dual. The latter follows by an inductive argument using **Lemma 5.6**.

**Case B.** \(( j = 2)\) We will prove this identity by showing that \( \phi (H_{3k+2}) = \phi (H_{3k+1}) \) and that \( H_{3k+2} \geq H_{3k+1} \). Inspection of the formula (5.11) and the set \( \phi \) from **Lemma 2.14** allows us to conclude the identity \( H_{3k+2} = H_{3k+1} H_{3k+2} \), applying (3.1) and **Case A**. Therefore, the first step reduces to showing that \( \phi (H_{3k+2}) = 2^k \phi (H_{3k+1}) \) which follows from a straightforward computation using **Lemma 2.12**.

In order to prove that \( H_{3k+2} \geq H_{3k+1} \), by degree reasons and the monotonicity of \( H_{3k+2} \), it is enough to check that \( G_{c_3(k-1)+2} (H_{3k+2}) = G_{c_3(k-1)+2} (H_{3k+1}) \geq 2v^i \) for \( 1 \leq i \leq k \). In view of (5.11), we see that

\[
G_{c_3(k-1)+2} (H_{3k+1} H_{3k+2}) \geq G_{c_3(k-1)+2} ((v^i H_{c_3(k-1)+1} + v^{i+1} H_{3k+1} H_{3k+2}) H_{3k+2}) = 2v^i
\]

as desired.
allow us to conclude
\[
\mathbf{H}_{3k+3} = \mathbf{H}_{3k+2} \mathbf{H}_{3k} - \mathbf{H}_{3k+1} - \mathbf{H}_{3k} \theta(k-1, 0).
\]  
(5.16)

Combining the facts that \( c(\mathbf{H}_{3k+1}^c \mathbf{H}_{3k}) = c(\mathbf{H}_{3k+1} \mathbf{H}_{3k}^c) = 4c(\mathbf{H}_{3k+1}^c) \) and \( c(\mathbf{H}_{3k} \theta(k-1, 0)) = 2c(\mathbf{H}_{k}^c(k-1, 0)) \), a straightforward calculation using Lemma 2.12, Lemma 2.7, Theorem 4.5 and Corollary 2.5 shows that \( c(\mathbf{H}_{3k+3}^c) = c(\mathbf{H}_{3k+3}) \).

Next we show that \( \mathbf{H}_{3k+3} \geq \mathbf{H}_{3k+3}^c \). By degree reasons and monotonicity of \( \mathbf{H}_{3k+3} \), it suffices to prove that \( G_{c_3}(\mathbf{H}_{3k+3}) \geq v \), \( G_{c_1}(\mathbf{H}_{3k+3}) \geq v^k \), and for \( 1 \leq i \leq k-1 \),
\[
\mathbf{H}_{3k+3} \geq v^i \mathbf{N}_{c_3(k-i)+1} + v^i \mathbf{N}_{c_1(k-i)}.
\]  
(5.17)

We will only show (5.17), as the other items follow by similar (and easier) arguments.

Observing that
\[
(\leq e_3(k-i)+1) \cap (\leq \varphi(\theta(k-1-i, 0))) = \leq \varphi'(e_3(k-i)),
\]
the proof of (5.17) reduces to showing that
\[
G_{e_3(k-i)+1}(\mathbf{H}_{3k+3}) \geq v^i, \\
G_{\varphi'(\theta(k-1-i, 0))}(\mathbf{H}_{3k+3}) \geq v^i, \quad \text{and} \quad G_{\varphi'(e_3(k-i))}(\mathbf{H}_{3k+3}) \geq 2v^i+1.
\]

The term \( 2v^i \mathbf{N}_{c_3(k-i)+1} \) in (5.12) has three summands of interest: \( 2v^{i+1} \mathbf{H}_{e_3(k-i)+1} \), \( 2v^{i+1} \mathbf{H}_{\varphi'(\theta(k-1-i, 0))} \), and \( 2v^{i+1} \mathbf{H}_{\varphi'(e_3(k-i))} \). Multiplying these terms by \( \mathbf{H}_{3k} \) contributes \( 2v^i \mathbf{H}_{e_3(k-i)+1} \), \( 2v^i \mathbf{H}_{\varphi'(\theta(k-1-i, 0))} \), and \( 4v^{i+1} \mathbf{H}_{\varphi'(e_3(k-i))} \) to \( \mathbf{H}_{3k+3}^c \). Then using (5.11) and Lemma 4.12 with (5.16), the subtraction of terms \( v^i \mathbf{N}_{c_3(k-i)+1} \) and \( v^i \mathbf{N}_{\varphi'(\theta(k-1-i, 0))} \) has the only effect in the degrees of interest, leaving us with the desired coefficients.

Theorem 5.7 provides formulas for all the Kazhdan-Lusztig basis elements indexed by elements located in \( \mathcal{E} \) of length at least five. Smaller elements can be computed directly. The following theorem provides formulas for all the Kazhdan-Lusztig basis elements indexed by elements located in \( \mathcal{W} \), thus completing our description of Kazhdan-Lusztig basis for the thick region.

**Theorem 5.8.** Let \( n \geq 1 \). Then \( \mathbf{H}_{\varphi_n} = \mathbf{H}_{\varphi_n} \mathbf{H}_{\varphi_n} \).

**Proof.** We notice that \( D_L(e_n) = \{ s_0, s_1 \} \). Then, Lemma 3.6 implies
\[
\mathbf{H}_{s_2} \mathbf{H}_{s_1} = \mathbf{H}_{\varphi_n} + \sum_{2 \leq z \leq 3} \mu(z, e_n) \mathbf{H}_{\varphi_z}.
\]  
(5.18)

However, a case-by-case analysis reveals that if \( z < e_n \) then \( s_2 z > z \). It follows that the sum in (5.18) is empty and the theorem follows.

## 6 Kazhdan-Lusztig basis in the thin region

In this section we present conjectural formulas for \( \mathbf{H}_w \) for \( w \) located in the thin region. We begin by describing the elements in this region. We denote by \( N\mathcal{W} \) (resp. \( N\mathcal{E} \), \( S\mathcal{W} \) and \( S\mathcal{E} \)) the sub-region of the thin region formed by triangles located to the northwest (resp. northeast, southwest and southeast) of the identity triangle. We define \( d_n \) as the product of the first \( n \) symbols of the infinite string \( s_2 s_1 s_2 s_0 s_2 s_1 s_2 s_0 s_2 s_1 s_2 s_0 \ldots \). The elements \( d_n \) comprise the thin wall that goes towards the northwest in our convention, and the elements \( d_n' = \varphi(N\mathcal{W}) \) are the southwestward thin wall (see Figure 1). In other words, \( N\mathcal{W} = \{ d_n \mid n \geq 3 \} \) and \( S\mathcal{W} = \{ d_n' \mid n \geq 3 \} \). Finally, we define \( \overline{d}_n = s_0 d_n \) and with these describe the remaining two thin walls as
\[
S\mathcal{E} = \{ \overline{d}_n \mid n \geq 3 \} \quad \text{and} \quad N\mathcal{E} = \{ \overline{d}_n \mid n \geq 3 \}.
\]

The thin region coincides very nearly with the two-sided cell of \( W \) consisting of the elements with unique reduced expression. Wang has shown [Wan11] that there exist \( u \in W \) for which \( \mu(u, w) \neq 0 \) for infinitely
many w in the thin region, meaning that the W-graph of the group is not locally finite. According to our conjecture, there are in fact infinitely many such u. This indicates an additional level of complexity for the Kazhdan-Lusztig basis in the thin region, reflected in the formulas below.

We first define some notation: For x, z ∈ W, let

\[ D^x_z := \sum_{w \leq x, w \geq z} v^{l(x) - l(w)} H_w \]

Notice that \( D^x_z \) is a truncation of \( N_x \). We further define \( U_x := N_x + D^x_x = U_x' \).

**Conjecture 6.1.** For all \( k ≥ 1 \),

1. \[
    H_{d_{4k+3}} = N_{d_{4k+3}} + v D_{θ(0,k-2)} H_{θ(0,k-2)} + v H_{θ(0,k-2)} + v \sum_{i=3}^{k} \left( H_{θ(0,k-i)} + H_{θ'(0,k-i)} \right)
    \]

2. \[
    H_{d_{4k+1}} H_{d_{4k+1}} = H_{d_{4k+1}} + H_{d_{4k+1}} + \sum_{i=0}^{k-2} H_{θ(0,i)} + \sum_{i=0}^{k-3} H_{θ(0,i)} \cdot
    \]

3. \[
    H_{d_{4k+3}} = H_{d_{4k+3}} + \sum_{i=0}^{k-3} \left( H_{θ'(0,i)} + H_{θ'(0,i)} \right)
    \]

These formulas above have been checked up to \( k = 5 \). Note again that these formulas cover both the northwest and southeast walls of the thin region, while basis elements from the other walls of this region can be obtained by applying the automorphism \( ϕ \).

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