INFINITE DIMENSIONAL GEOMETRY AND QUANTUM FIELD
THEORY OF STRINGS
I. INFINITE DIMENSIONAL GEOMETRY OF SECOND QUANTIZED
FREE STRING

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Abstract. There are investigated several objects of an infinite dimensional geometry, appearing
from the second quantization of a free string.

This paper is devoted to structures of an infinite dimensional geometry, appearing in quantum
field theory of (closed) strings; the objects connected with second quantization of a free string
are described in the first part of the paper, when an analogous material for self–interacting string
field is proposed to be discussed in the second part.

In the present paper we follow the general ideology of string theory presented in [1]. It should
be mentioned that the detailed exposition of used formalism of the first quantization of a closed
string is contained in the book [2]. This publication maybe considered as a continuation of a
previous one [3] devoted to geometric aspects of quantum conformal field theory.

The first part contains two chapters: the first one is devoted to the infinite dimensional ge-
ometry of flag, fundamental and Π–spaces for the Virasoro–Bott group and its nonassociative
deformation defined by Gelfand–Fuchs 3–cocycle (which will be called Gelfand–Fuchs loop) as
well as of infinite–dimensional non–Euclidean symplectic grassmannian, to the construction of
Verma modules, their models and skladens over the Virasoro algebra and their properties; in the
second chapter there is described an infinite dimensional geometry of the configuration space for
the second quantized free string in flat and curved backgrounds, as well as an author version of
Bowick–Rajeev formalism of the separation of internal and external degrees of freedom of a closed
string.

The great attention is paid to an interaction of various geometric structures: in the first
chapter they are infinite dimensional Lie algebras, groups and loops, homogeneous, Kähler, Finsler,
contact and symmetric spaces, complex, real and CR–manifolds, determinant sheaves, manifolds
with subsymmetries, polarizations and Fock spaces, bibundles and objects of integral geometry,
nonholonomic spaces, deformations of geometric structures and moduli spaces; in the second one
— gauge fields, Faddeev–Popov ghosts, Gauss–Manin connections, Kostant–Blattner–Sternberg
pairings, BRST–operators. The text does not contain any essential terminological innovations
— our purpose is rather to play an interaction of known classical concepts on a nice infinite–
dimensional example.
1. INFINITE DIMENSIONAL GEOMETRY OF THE FLAG MANIFOLD $M(\text{Vir})$, THE FUNDAMENTAL
AFFINE SPACE $A(\text{Vir})$ AND $\Pi$–SPACE $\Pi(\text{Vir})$ FOR THE VIRAŠO–BOČT GROUP,
THE UNIVERSAL DEFORMATION OF A COMPLEX DISC, THE INFINITE DIMENSIONAL
SYMPLECTIC NON–EUCLIDEAN GRASSMANNIAN $A(\text{Vir})$, THE SPACES OF ANGLES
$\text{Ang}(\text{Vir})$ AND PUNCTURED MIRRORS $\Sigma(\text{Vir})$ ON THE FLAG MANIFOLD $M(\text{Vir})$. THE
VERMA MODULES OVER THE VIRAŠO–BOČT GROUP, THEIR MODELS AND SKŁADENS

1.1. The Lie algebra $\text{Vect}(S^1)$ of vector fields on a circle, the group $\text{Diff}_+(S^1)$ of
diffeomorphisms of a circle, the Virašo–Bott group $\text{Vir}$, the Neretin semigroup $\text{Ner}$ (the mantle $\text{Mantle}(\text{Diff}_+(S^1))$ of the group of diffeomorphisms of a circle). Kirillov construction and class $S$ of univalent functions. The Gelfand–Fuchs 3–cocycle, Finsler geometry of $M(\text{Vir})$ and the Gelfand–Fuchs loop $G_\text{f}$, the non–associative deformation of the Virašo–Bott group $\text{Vir}$.

Let $\text{Diff}(S^1) = \text{Diff}_+(S^1) \cup \text{Diff}_-(S^1)$ be the group of analytic diffeomorphisms of a circle $S^1$: diffeomorphisms from the subgroup $\text{Diff}_+(S^1)$ preserve an orientation on a circle $S^1$, ones from the coset $\text{Diff}_-(S^1)$ change it; Lie algebra of the group $\text{Diff}_+(S^1)$ is identified with the vector space $\text{Vect}(S^1)$ of analytic vector fields $v(t)/d$ on a circle $S^1$; the structural constants of the complexification $\text{C}\text{Vect}(S^1)$ of the Lie algebra $\text{Vect}(S^1)$ have the form $c_{jk}^i = (j - k)\delta^i_{j+k}$ in the basis $e_k = i \exp(ikt)/d$. In 1968 L.M. Gelfand and D.B. Fuchs discovered a non–trivial central extension of Lie algebra $\text{Vect}(S^1)$: the corresponding 2–cocycle maybe written as $c(u, v) = \int u'(t) dv'(t)$ or as $c(u, v) = \det(A(t_0))$, where $A(t) = \begin{bmatrix} u'(t) & v'(t) \\ u''(t) & v''(t) \end{bmatrix}$; independently this central extension was discovered in 1969 by M.Virašo and was called later the Virašo–Bott group $\text{vir}$ (the same name belongs to the complexification $\text{C}\text{vir}$ of this algebra); the Virašo algebra $\text{Vir}$ is generated by the vectors $e_k$ and the central element $c$, the commutation relations in it has the form $[e_j, e_k] = (j - k)e_{j+k} + \delta(j + k)\cdot \frac{j^3 - j}{12} \cdot c$. One may correspond an infinite dimensional group $\text{Vir}$ to the Lie algebra $\text{Vir}$ which is a central extension of the group $\text{Diff}_+(S^1)$, the corresponding 2–cocycle was calculated by R. Bott in 1977: $c(\phi, g) = \int \log(\phi \cdot \phi') d \log(x')$. The group $\text{Vir}$ is called the Virašo–Bott group.

There are no any infinite dimensional groups corresponding to Lie algebras $\text{C}\text{Vect}(S^1)$ and $\text{C}\text{vir}$ but it is useful to consider the following construction, which is attributed to Yu.Neretin and developed by M.Kontsevich [4,5]. Let us denote accordingly to Yu.Neretin by $\text{LDiff}_+(S^1)$ a set of all analytic mappings $g : S^1 \rightarrow \mathbb{C}\setminus\{0\}$ with a Jordan image $g(S^1)$ such that zero belongs to the interior of the contour $g(S^1)$, the orientations of $g(S^1)$ and $S^1$ coincide and the value of $g(z)$ is not equal to zero anywhere; $\text{LDiff}_+(S^1)$ is a local group in the following sense: let $g_1$ and $g_2$ belong to $\text{LDiff}_+(S^1)$ and $g_1$ admits an analytic extension to an area which contains the contour $g_2(S^1)$, then the composition $g_1 \circ g_2$ is correctly defined. Let’s denote by $\text{L Ner}$ the local semigroup in $\text{LDiff}_+(S^1)$ consisting of mappings $g$ such that $|g(\exp(it))| < 1$; as it was shown by Yu.Neretin [4] local semigroup $\text{L Ner}$ maybe supplied by a natural structure of a global semigroup $\text{Ner}$. There exist two different constructions of the Neretin semigroup. The first construction (Yu.Neretin [4,5]), an element of semigroup $\text{Ner}$ — a formal product $p A(t)q(\ast)$, where $p, q \in \text{Diff}_+(S^1)$, $p(1) = 1, t > 0, A(t) : \mathbb{C} \rightarrow \mathbb{C}$ such that $A(t) x = \exp(-t) x$. To define a multiplication in $\text{Ner}$ it is necessary to describe a rule of transformation of formal product $A(s)pA(t)$ to the form $A(t)$). If $t$ is so small that the diffeomorphism $p$ maybe analytically extended to the ring $\exp(-t) \leq |z| \leq 1$, then there is correctly defined a product $g = A(s)pA(t)$; let $K$ be a domain bounded by $S^1$ and $g(S^1)$, and $Q$ is a canonical conformal mapping from $K$ onto the ring $\exp(-t') \leq |z| \leq 1$ such that $Q(1) = 1$, then $g = p^\prime A(t')q^\prime$, where $p^\prime = Q^{-1}g_1$, and $q^\prime$ is defined from the relation $A(s)pA(t) = p^\prime A(t')q^\prime$. If $t$ is an arbitrary real number then there exist $t/n$ so small that a product $A(s)pA(t) = \ast\ast\ast A(t/n)\ast\ast\ast$ calculated accordingly the previous construction. The obtained multiplication is associative [4]. The second construction (M.Kontsevich, in the version of Yu.Neretin). An element $g$ of semigroup $\text{Ner}$ is a triple $(K, p, q)$, where $K$ is a Riemann surface with a boundary, which is biholomorphically equivalent to the ring, $p : S^1 \rightarrow \partial K$ are fixed analytic parametrizations of the boundary $\partial K$ of the surface $K$, so that $K$ is on the right side from $p(\exp(it))$ and from the left side from $q(\exp(it))$. Two elements $g_1 = (K_1, p_1, q_1)$ and $g_2 = (K_2, p_2, q_2)$ are equivalent if there exists a conformal mapping $R : K_1 \rightarrow K_2$ such that $p_2 = Rp_1$, $q_2 = Rq_1$; the product of two elements $g_1$ and $g_2$ of the semigroup $\text{Ner}$ is the element $g_3 = (K_3, p_3, q_3)$ of this semigroup, where $K_3 = K_1 \cup K_2$, and $p_3, q_3$ are $p, q$ on $K_1$ and $K_2$, respectively. The Neretin–Chevyrev $\text{Ner}$ is called the
mantle of the group $\text{Diff}_+(S^1)$ of diffeomorphisms of a circle and is denoted by $\text{Mantle}(\text{Diff}_+(S^1))$. The Neretin semigroup admits a central extension; the corresponding 2–cocycle was calculated by Yu.Neretin in 1989 [4], the explicit formulas for that cocycle is rather cumbersome so that they are omitted.

The flag manifold $M(\text{Vir})$ for the Virasoro–Bott group Vir is the homogeneous space $\text{Diff}_+(S^1)/S^1$ with the group of motions $\text{Diff}_+(S^1)$ and the isotropy group $S^1$ [6-8]. There exist several realizations of this manifold. The realization of $M(\text{Vir})$ as an infinite dimensional homogeneous space $\text{Diff}_+(S^1)/S^1$ is called algebraic; in this realization the space $M(\text{Vir})$ maybe also identified with the quotient of the Neretin semigroup Ner by its subsemigroup $\text{Ner}^0$ consisting of elements $g$ of the semigroup Ner, which admit an analytic extension to $D_+$ ($D_+ = \{ z \in \mathbb{C} : |z| \geq 1 \}$). The probabilistic realization: the group $\text{Diff}_+(S^1)$ acts on the space of all probabilistic measures $u(t) \, dt$ on a circle with an analytic positive density $u(t)$ in a natural way, the stabilizer of the point $(2\pi)^{-1} \, dt$ is isomorphic to $S^1$, therefore, from the transitivity of the action of the group $\text{Diff}_+(S^1)$ on the space of probabilistic measures it follows that this space maybe identified with $M(\text{Vir})$. Orbital realization: the space $M(\text{Vir})$ may be considered as an orbit of the coadjoint representation for the groups $\text{Diff}_+(S^1)$ or Vir [9-11], namely, elements of the space $\text{vir}^*$ dual to the Virasoro algebra vir are identified with pairs $(p(t)\, dt^2, b)$ so that the coadjoint action of the group Vir has the following form $K(g)(p, b) = (gp - bS(g), b)$, where $S(g)$ is the Schwarz derivative of a function $g$; the orbit of the point $(a \cdot dt^2, b)$ coincides with $M(\text{Vir})$ if and only if $a/b = -n^2/2$, $n = 1, 2, 3, 4, \ldots$. So there is defined a family of symplectic structures $\omega_{a,b}$ on the space $M(\text{Vir})$. Analytic realization (Kirillov construction) [6,7]: let us consider the space $S$ of functions $f(z)$ analytic and univalent in the closed unit disc $D_+$ normalized by the conditions $f(0) = 0, f'(0) = 1, f'(\exp(it)) \neq 0$; the Taylor coefficients $c_1, c_2, c_3, c_4, \ldots$, of a function $f(z) = z + c_1 z^2 + c_2 z^3 + \ldots$ determine a coordinate system on $S$; necessary and sufficient conditions for the univalency of a function $f(z)$, which describe the domain $S$ in the linear space $\mathbb{C}[z]$, are contained in [12-14].

In 1986 A.A.Kirillov [6] (see also [7]) showed that the class $S$ maybe identified in a natural way with $\text{Diff}_+(S^1)/S^1$, namely for each element $f$ from $S$ there is defined the unique function $g$ holomorphic in the exterior of the unit disc, mapping it onto the exterior of the contour $f(S^1)$ and normalized by the conditions $g(\infty) = \infty, g'(\infty) > 0$; let us denote by $\gamma_f$ the diffeomorphism of the unit circle defined by the formula $\gamma_f = f^{-1} \circ g$, then the correspondence $f \mapsto \gamma_f$ defines a bijection of $S$ onto $M(\text{Vir})$; let us construct the inverse mapping from $M(\text{Vir})$ onto $S$; for each diffeomorphism let us consider a manifold $\mathcal{T}_\gamma = \mathcal{T}_+ \cup \mathcal{T}_-$, which is diffeomorphic and, therefore, holomorphic to $\mathbb{C}$; let us normalise the biholomorphic mapping $F : \mathbb{C}_+ \to \mathbb{C}$ by the conditions $F(0) = 0, F'(0) = 1, F(\infty) = \infty$ and define $f_\gamma$ from $S$ corresponding to the diffeomorphism $\gamma$ as $F|_{D_+}$. The action of C$\text{Vec}(S^1)$ on $\text{Diff}_+(S^1)/S^1$ in the coordinate system $\{c_k\}$ has the form

$$
\mathcal{L}_\psi f(z) = -i f^2(z) \int \frac{w f'(w)}{f(w)}^2 \frac{v(w)}{f(w) - f(z)} \frac{dw}{w};
$$

$$
L_p = \frac{\partial}{\partial c_p} + \sum_{k \geq 1} (k + 1) c_k \frac{\partial}{\partial c_{k+p}} \quad (p > 0),
$$

$$
L_0 = \sum_{k \geq 1} k c_k \frac{\partial}{\partial c_k},
$$

$$
L_{-1} = \sum_{k \geq 1} ((k + 2) c_{k+1} - 2 c_1 c_k) \frac{\partial}{\partial c_k},
$$

$$
L_{-2} = \sum_{k \geq 1} ((k + 3) c_{k+2} - (4 c_2 - c_1^2) c_k - b_k (c_1, \ldots, c_{k+2})) \frac{\partial}{\partial c_k},
$$

$$
L_{-n} = \frac{1}{(n-2)!} \text{ad}^{n-2} L_{-1} \cdot L_{-2} \quad (n > 2),
$$

where $b_k$ are the Laurent coefficients of the function $1/(w f(w))$; the connection of the described formulas with the classical variational formulas of the theory of univalent functions is described in [8,16-18]. The symplectic structures $\omega_{a,b}$ coupled with the complex structure on $M(\text{Vir})$ form the two–parameter family of Kähler matrices $w_{a,b}$; the more detailed information on the Kähler geometry on the flag manifold for the Virasoro–Bott group maybe received from the original papers [6-8,17,18] or the monographs [18].
In 1968 I.M.Gelfand and D.B.Fuchs discovered a non–trivial 3–cocycle of Lie algebra $\text{Vect}(S^1)$, it maybe written as $c(u,v,w) = \det(B(t_0))$, where $B(t) = 
abla(t)$, or as $c(u,v,w) = \int \det(B(t)) \, dt$ [20]. This cocycle defines a nonassociative deformation of the Virasoro–Bott group, a loop [21-25], which will be called Gelfand–Fuchs loop and will be denoted by Gf.

The set of all fixed points of a subsymmetry (mirror) is completely geodesic Lagrange submanifolds $\Lambda(Vir)$ and any point of $\Lambda(Vir)$ and any point of $\Lambda(M(Vir))$ and any point of $\Lambda(M(Vir))$ are the oriented mirrors $V_\delta$, pairs $(V,a)$, where $V$ is a mirror form $\Lambda(Vir)$, and $a$ is a point of the absolute, which belongs to $V$. More detailed information on the infinite dimensional geometry on the absolute of $\Lambda(M(Vir))$ may be found in the papers [18,28,29].

For the following purposes we shall consider an equivariant mapping of the flag manifold for the Virasoro–Bott group into the infinite dimensional classical domain of the third type [16-18,4]; let $H$ be the completion of the space of the smooth real–valued 1–forms $u(\exp(it))dt$ on a circle such that $f u(\exp(it))dt = 0$ by the norm $||u||^2 = \sum |u_n|^2/n$; let $H^C$ be its complexification, $H^C_\pm$ be the transversal spaces, consisting of 1–forms $u(\exp(it))dt$, which maybe holomorphically extended to the discs $D_\pm$; $H^C \simeq \mathcal{O}(S^1)/\text{Const}$, namely $f(z) \in \mathcal{O}(S^1) \mapsto df(z) \in H^C$, $H^C_\pm \simeq O(D_\pm)/\text{Const}$; there are the symplectic and the pseudohermitean structures defined on $H^C$: $(f(z), g(z)) = \int f(z) dg(z), \quad \langle f(z), g(z) \rangle = \int f(z) \overline{dg(z)} (f, g \in \mathcal{O}(S^1))$; let $Sp(H^C, \mathbb{C})$ and $U(H^C_+, H^C_-)$ be the groups of invariance of these structures, $Sp(H, \mathbb{R}) = Sp(H^C, \mathbb{C})U(H^C_+, H^C_-)$. Let us consider the Grassmannian $Gr(H^C)$ – the set of all complex Lagrange subspaces in $H^C$, $Gr(H^C)$ is an infinite dimensional homogeneous space with the group of transformations $Sp(H^C, \mathbb{C})$. Let us consider the action of $Sp(H, \mathbb{R})$ on $Gr(H^C)$; the orbit of the point $H^C_\pm$ is an open subspace $R$ in $Gr(H^C)$ isomorphic to $Sp(H, \mathbb{R})/U$, where $U = \{ A + \tilde{A}, A \in U(H^C_+), \tilde{A} \in U(H^C_-) \}$, the space $R$ is an infinite dimensional classical homogeneous domain of the third type (i.e. an infinite dimensional analogue of finite dimensional classical domains of this type [30]), $R$ is mapped in the linear space $\text{Hom}(H^C_+, H^C_-)$ so that the elements of $R$ are represented by symmetric matrices $Z$ such that $E - ZZ > 0$. The detailed information on infinite dimensional Grassmannians maybe received from the papers [31-33]: the construction of the mapping of $M(Vir)$ into $R$ was described in the papers [16-18,4], namely, the representation of $Diff_+(S^1)$ in $H$ defines a monomorphism $Diff_+(S^1) \to Sp(H, \mathbb{R})$, hence $Diff_+(S^1)$ acts on $R$ the orbit of the initial point under this action.
coincides with $\text{Diff}_+ (S^1) / \text{PSL}(2, \mathbb{R})$, therefore we have an equivariant mapping of $M(\text{Vir})$ into $R$, its explicit form can be found in [16-18]. The matrix $Z_f$ from $R$ which corresponds to the function $f \in S$ is called the Grunsky matrix [34] (the definitions of the grunsky matrix maybe found also in [14,16]), the mapping $f \mapsto Z_f$ is the partial case of the Krichever mapping [32,35].

The skeleton of the domain $R$ consists of all symmetric unitary matrices $Z$, therefore, the skeleton of $S = \text{Diff}_+ (S^1)/\mathbb{S}^1$ consists of univalent functions which Grunsky matrices are unitary, accordingly to the Milin theorem [36] the skeleton of the space $S$ consists of all univalent functions $f$ such that $\text{mes}(C\setminus D^2_z) = 0$, but the action of the group $\text{Diff}_+ (S^1)$ on the skeleton is not transitive; the structure of the skeleton of class $S$ is described in the paper [29]. Following it let us consider the $\mathbb{R}$–analytic manifold $E$ whose elements are cuts $K$ of the complex plane $\mathbb{C}$ with one end at infinity, such that the conformal radius of $\mathbb{C}\setminus K$ is equal to one. Let us consider the mapping $E \mapsto \nu_+(\text{Vir})$, defined as $f(z) \mapsto (s,a)$, where $f(D_+^a) \cap K = \mathbb{C}$, $f(0) = 0$, $f'(0) = 1$, $f(a) = \infty$, $f(s(z)) = f(z)$; it was shown in the paper [29] that this mapping is an isomorphism. Let us mention that the action of $\text{Diff}_+ (S^1)$ on $S$ maybe analytically extended on $E$, so that $E$ and $\nu_+(\text{Vir})$ are isomorphic as homogeneous spaces. It should be mentioned that the infinite dimensional non–Euclidean oriented symplectic grassmannian $\nu_+(\text{Vir})$ is a symmetric space: $(s_1, a_1) \circ (s_2, a_2) = (s_1 s_2 s_1, a_1 a_2)$ with the group of transvections $\text{Diff}_+ (S^1)$ and the isotropy group $G_0 = \{g \in \text{Diff}_+ (S^1) : g(z) = \overline{g(z)}, g(1) = 1\}$, the tangent space $T$ to $\nu_+(\text{Vir})$ at the point $(s, 1)$ maybe identified with the space of odd vector fields on $S^1$. The lines in $V$ invariant with respect to $G_0$, which are determined by the generalized vectors $\delta_{±}(t)d/dt$, correspond to the nonholonomic generalized invariant fields $\xi_{±}$ on $\nu_+(\text{Vir})$; let $O(E)$ and $O(\nu_+(\text{Vir}))$ be the structural rings of $E$ and $\nu_+(\text{Vir})$; $O(\nu_+(\text{Vir})/\xi_{±}) = \{f \in O(\nu_+(\text{Vir})) : \xi_{±} f = 0\}$; the isomorphism of the rings holds: $(E, O(E)) \simeq (\nu_+(\text{Vir}), O(\nu_+(\text{Vir})/\xi_{±}))$ [29].

Let us consider the mapping $M(\nu_+(\text{Vir})) \mapsto \Gamma_{\nu}(\nu_+(\text{Vir}))$, where $\Gamma_{\nu}(\nu_+(\text{Vir}))$ is the space of all closed geodesics on $\nu_+(\text{Vir})$, namely, we shall assign to a point $x$ the set of all oriented mirrors which pass through it; this mapping is an isomorphism [29]. Under the identification with $\Gamma_{\nu}(\nu_+(\text{Vir}))$ the symplectic structure on $M(\nu_+(\text{Vir}))$ has the form $\omega_{\nu_+(\text{Vir})} = \int_{\gamma_1} (AX, Y) ds$, where $s$ is a natural parameter on $\gamma_1$, $X$ and $Y$ are the Jacobi fields orthogonal to the field $\dot{s}$ for the unique (up to a multiplication by a real number) invariant degenerate pseudoriemannian metric on $\nu_+(\text{Vir})$, $A = a\nabla_s + b\nabla_{\nu} s$, where $\nabla_s$ is the covariant derivative along $\dot{s}$. Let $O^0(\nu_+(\text{Vir}))$ be the class of all holomorphic functionals on $S$ which admit an analytic extension to $E$ then $\text{Re}^0(S) \simeq O(\nu_+(\text{Vir})/\xi_{±})$, namely [29] $F(f) = \int_{\nu_+(\text{Vir})} F(f_s) ds$ (**). Let us denote by $\Lambda^h_{\nu(\text{Vir})}$ the nonholonomic manifold (cf.[37]) the "spectrum" of $O(\nu_+(\text{Vir})/\xi_{±})$, let $O_{\nu_+(\text{Vir})}(\Lambda^h_{\nu(\text{Vir})})$ be the inverse image of $O^0(\nu_+(\text{Vir})/\xi_{±})$ under the mapping (**).

An angle $\hat{\alpha}$ on the flag space $M(\nu_+(\text{Vir}))$ is a triple $\hat{\alpha} = (x, U, V)$, $x \in M(\nu_+(\text{Vir}))$, $U, V \in \nu_+(\text{Vir})$, $x \in U \cap V$, the point $x$ is called the vertex of an angle and the mirrors $U$ and $V$ are called the sides of the angle $\hat{\alpha}$; for two angles with a common vertex and a common side there is defined a sum $\hat{\alpha} + \hat{\beta} = (x, U, W)$, where $\hat{\alpha} = (x, U, V)$, $\hat{\beta} = (x, V, W)$; two angles are called congruent if they may be mapped one into another by one element of the group $\text{Diff}_+ (S^1)$, acts on $M(\nu_+(\text{Vir}))$. This action conserves an additive invariant of angles — their value, the real number defined up to $2\pi k$, $k \in \mathbb{Z}$; the angles with the vertex $x$ form a compact riemannian manifold — two–dimensional torus $T^2$; an angle is called rational if and only if its value is a rational number and irrational otherwise; an angle $\hat{\alpha} = (x, U, V)$ is irrational if and only if $U \cap V = \{x\}$. The space of angles $\text{Ang}(\nu_+(\text{Vir}))$ is a biuniverse: $M(\nu_+(\text{Vir})) \leftrightarrow \text{Ang}(\nu_+(\text{Vir})) \mapsto \nu_+(\text{Vir}) \times \nu_+(\text{Vir})$ ($\pi_1 : \text{Ang}(\nu_+(\text{Vir})) \rightarrow M(\nu_+(\text{Vir}))$, $\pi_2 : \text{Ang}(\nu_+(\text{Vir})) \rightarrow \nu_+(\text{Vir}) \times \nu_+(\text{Vir})$); there is defined a mapping $\pi_1 \circ \pi_2^{-1}$ on $\text{Ang}(\nu_+(\text{Vir}))$, where $\text{Ang}(\nu_+(\text{Vir}))$ is the space of irrational angles; the image of this mapping coincides with $M(\nu_+(\text{Vir}))$. Let us consider a mapping from $O_{\nu_+(\text{Vir})}(\Lambda^h_{\nu(\text{Vir})})$ to $O(M(\nu_+(\text{Vir}))$ defined as $\tilde{F}(x) = \int_{\hat{\alpha}; \pi_1(\hat{\alpha}) = x} F(\pi_2(\hat{\alpha})) d\mu_{\nu_+(\text{Vir})}$, where $d\mu_{\nu_+(\text{Vir})}$ is the canonical measure on torus $T^2$.

**Theorem 1.** The mapping $\tilde{F} \rightarrow F$ is injective and has a dense image in $O(M(\nu_+(\text{Vir}))$ after the restriction on the solutions of the system of equations $\Box_p F = 0$, $(X^1 - X^2) F = 0$ ($\Box_p = \sum_{i+j=p} (i-j)(L_{i}^{(1)} L_{j}^{(2)} - L_{j}^{(1)} L_{i}^{(2)})$, $X$ is the vector field that generate a geodesics passing through points 1 and 2).

The statement of the theorem is an evident consequence of the integral formulas (**).
modules, which were investigated by several authors [38–40]. Namely, let $\text{Cvir}^+ = \text{span}(e_k, k \geq 0)$, $\chi_{h,c}$ be the character of $\text{Cvir}$, defined by the condition $\chi_{h,c}(e_k) = 0$ if $k > 0$, $\chi_{h,c}(e_0) = h$, $\chi_{h,c}(e) = c$; the Verma module $V_{h,c}$ is the $\text{Cvir}$-module induced by the character $\chi_{h,c}$ of the subalgebra $\text{Cvir}^+$; otherwise, $V_{h,c} = \mathcal{U}(\text{Cvir}) \otimes \mathcal{U}(\text{Cvir}^+) V_{h,c}$, where $V_{h,c}$ is the $\text{Cvir}$-module defined by the character $\chi_{h,c}$, $\mathcal{U}(\text{Cvir})$ and $\mathcal{U}(\text{Cvir}^+)$ are the universal enveloping algebras of Lie algebras $\text{Cvir}$ and $\text{Cvir}^+$, the Verma module $V_{h,c}$ is a graded $\text{Cvir}$-module; if $h$ and $c$ are real (that will be supposed below) there is defined the unique up to a multiple invariant hermitean form on $V_{h,c}$, the Verma module $V_{h,c}$ is unitarizable if and only if the hermitean form is positive definite; let us denote by $D_n(h,c)$ the determinant of this form in $n$-th homogeneous component of $V_{h,c}$ in the basis $L_{k_1}^n \cdots L_{k_j}^n V$, $k_j \geq 0$ ($L_{0\nu} = h
u$, $L_{m\nu} = 0$), then it was shown by V.G.Kac, B.L.Feigin and D.B.Fuchs $D_n(h,c) = A \prod_{\alpha < \beta, \alpha < \beta \leq n} \Phi_{\alpha,\beta}^{\nu(n-\nu)}$, where $\Phi_{\alpha,\beta}(h,c) = h + \frac{c-13}{24} (\alpha^2 - 1)$, $\Phi_{\alpha,\beta}(h,c) = (h + \frac{c-13}{24} (\beta^2 - 1) + \frac{\alpha-1}{2} (h + \frac{c-13}{24} (\alpha^2 - 1) + \frac{\alpha-1}{2} + \frac{\alpha^2-2^2}{16}$). If any $\alpha$, $\beta$ $\Phi_{\alpha,\beta}(h,c) \neq 0$ then the module $V_{h,c}$ is irreducible and is not contained in any other Verma module; if there exist exactly one pair $\alpha, \beta$ such that $\Phi_{\alpha,\beta}(h,c) = 0$ then there are three possibilities maybe realized: 1) $\alpha \beta < 0$, then $V_{h,c}$ maybe imbedded into the Verma module $V_{h+\alpha,\beta,c}$, 2) $\alpha \beta > 0$, then $V_{h,c}$ contains a submodule $V_{h+\alpha,\beta,c}$, 3) $\alpha = 0$ or $\beta = 0$, then $V_{h,c}$ is irreducible and is not a submodule of another Verma module; if there exist two pairs $(\alpha_1, \beta_1)$ and $(\alpha_2, \beta_2)$ such that $\Phi_{\alpha_1,\beta_1}(h,c) = 0$ then there exist an infinite number of pairs $(\alpha, \beta)$, which possess such property — this situation is realized if

\begin{align}
(1A) & \quad c_{12} = 1 - \frac{6((\alpha_1 + \alpha_2) - (\beta_1 + \beta_2))^2}{(\alpha_1 + \alpha_2)(\beta_1 + \beta_2)} \\
(1B) & \quad c_{12} = \frac{(\alpha_2 \beta_1 - \alpha - 1 \beta_2)^2 - ((\alpha_1 + \alpha_2) - (\beta_1 + \beta_2))^2}{4(\alpha_1 + \alpha_2 - 2)(\beta_1 + \beta_2)}
\end{align}

In this case the structure of the Verma modules is described by the Feigin–Fuchs theory.

The Verma module $V_{h,c}$ is unitarizable if $h > 0$, $c > 1$; the Verma module $V_{h,c}$ contains an unitarizable quotient if (a) $h > 0$, $c > 1$; (b) $c = 1 - \frac{6}{p(p+1)}$, $h = \frac{(p-2)(p+1)}{4p(p+1)}$, $\alpha, \beta, p \in \mathbb{Z}$; $p \geq 2$; $1 \leq \alpha \leq p$, $1 \leq \beta \leq p + 1$.

Let us describe a geometric way of the construction of the Verma modules over the Virasoro algebra, based on the orbit method; it is described by the following facts [16–18]: (1) To each $\text{Vir}^+$-invariant Kähler metric $w_{h,c}$ on the space $M(\text{Vir})$ one should correspond the linear holomorphic bundle $E_{h,c}$ over $M(\text{Vir})$ with the following properties: (a) $E_{h,c}$ is the hermitean bundle with the metric $\exp(-U_{h,c})d\lambda d\bar{\lambda}$, where $\lambda$ is a coordinate in a fiber, $K_{h,c} = \exp(U_{h,c})$ is the Bergman kernel, the exponential of the Kähler potential $U_{h,c} = h \log |g(\infty)| - c \log \det(E - Z \bar{Z}_f)$ of the metric $w_{h,c}$, (b) algebra $\text{Cvir}$ holomorphically acts in the prescribed bundle by covariant derivatives with respect to the hermitean connection with the curvature form being equal to $2\pi \omega_{h,c}$; (2) let $O(E_{h,c})$ be the space of all polynomial (in the trivialization of the paper [17]) germs of sections of the bundle $E_{h,c}$, $O(E_{h,c})$ is the graded $\text{Cvir}$-module, the action of the Lie algebra $\text{Cvir}$ in which is defined by the following formulas [17,18]

\begin{align}
L_p &= \frac{\partial}{\partial c_k} + \sum_{k \geq 1} (k+1)c_k \frac{\partial}{\partial c_k} + (p > 0), \\
L_0 &= \sum_{k \geq 1} k c_k \frac{\partial}{\partial c_k} + h, \\
L_{-1} &= \sum_{k \geq 1} ((k+2)c_{k+1} - 2c_1 c_k) \frac{\partial}{\partial c_k} + 2hc_1, \\
L_{-2} &= \sum_{k \geq 1} ((k+3)c_{k+2} - (4c_2 - c_1^2)c_k - b_k(c_1, \ldots, c_{k+2})) \frac{\partial}{\partial c_k} + h(4c_2 - c_1^2) + \frac{c}{2}(c_2 - c_1^2), \\
L_{-n} &= \frac{1}{(n-2)!} \text{ad}^{n-2} L_{-1} \cdot L_{-2} \quad (n > 2)
\end{align}

let us fix the basis $e_{\alpha_1,\ldots,\alpha_n} = e_{\alpha_1} \cdots e_{\alpha_n}$ in $O(E_{h,c}^*)$ and let $O^*(E_{h,c}^*)$ be the space of all linear functionals on $O(E_{h,c}^*)$ which obey the next property: if $n(p) > 2$ then $\text{deg}(p) < N$, the
space $O^*(E^*_h,c)$ is called the Fock space of the pair $(M(Vir), E_{h,c})$ [41,p.117] (the Fock spaces play an essential role in the method of geometric quantization, the standard Fock space is a partial case of the spaces introduced in [41]), the Verma module $V_{h,c}$ is realized in the Fock space $F(E_{h,c})$ of the pair $(M(Vir), E_{h,c})$; let us fix the basis $e_{a_1 \ldots a_n} = e_1^{a_1} \ldots e_n^{a_n}$ in $F(E_{h,c})$ such that 

$$< e_{a_1 \ldots a_n}, e^{b_1 \ldots b_m} > = a_1 \ldots a_n b_1^{b_1} \ldots b_m^{b_m};$$

the action of the Virasoro algebra in such basis is defined by the next formulas [17,18]

$$L_{-p} = c_p + \sum_{k \geq 1} (k + 1)c_{k+p} \frac{\partial}{\partial c_k} \quad (p > 0),$$

$$L_0 = \sum_{k \geq 1} k c_k \frac{\partial}{\partial c_k} + h,$$

$$L_1 = \sum_{k \geq 1} c_k ((k + 2) \frac{\partial}{\partial c_{k+1}} - 2 \frac{\partial}{\partial c_1} \frac{\partial}{\partial c_k}) + 2h \frac{\partial}{\partial c_1},$$

$$L_2 = \sum_{k \geq 1} c_k ((k + 3) \frac{\partial}{\partial c_{k+2}} - 4 \frac{\partial}{\partial c_2} - (\frac{\partial}{\partial c_1})^2) \frac{\partial}{\partial c_k} - 2h \frac{\partial}{\partial c_1} +$$

$$+ h(4 \frac{\partial}{\partial c_2} - (\frac{\partial}{\partial c_1})^2) + \frac{c}{2} (\frac{\partial}{\partial c_2} - (\frac{\partial}{\partial c_1})^2),$$

$$L_n = \frac{(-1)^n}{(n-2)!} \text{ad}^{n-2} L_1 \cdot L_2 \quad (n > 2).$$

1.4. The fundamental affine space $A(Vir)$ and the $\Pi$–space $\Pi(Vir)$ for the Virasoro–Bott group, the model and the skladen of the Verma modules over the virasoro algebra. The space $\Sigma(Vir)$ of punctured oriented mirrors and the model of the Verma modules over the Virasoro algebra. the space $\Sigma^*(Vir)$, the skladen of the Verma modules over the Virasoro algebra and the Aleksseev–Shatashvili construction. The universal deformation of a complex disc, the Manin–kontsevich–Beilinson–Schechtman construction and the model of the Verma modules over the Virasoro algebra.

It is well–known from the theory of representations of compact groups that the model of the representations of a compact Lie group $U$ is realised in the space of functions on the fundamental affine space for its complexification $G^C$; the model of the representations of the group $U$ is a representation which maybe expanded in the direct sum of the irreducible ones so that each such representation appear in the sum exactly one time; the fundamental affine space is a homogeneous space $G^C/N$ of the complex reductive group $G^C$ with the stationary subgroup being isomorphic to the maximal nilpotent subgroup $N$. An analogous construction holds for the real noncompact simple Lie groups of Hermitian type (there is an inaccuracy in the paper [41]: the supposition of caspidality should be changed on the condition formulated above), namely, let $G$ be a real noncompact simple Lie group of Hermitian type, $G^C$ be its complexification, Mantle($G$) be the semigroup lying in $G^C$ (so–called partial complexification of the group $G$ [42,43]); the quotient Mantle($G$)/Mantle($G$) $\cap N$ is called the fundamental affine space for the Lie group $G$; the fundamental affine space of the universal covering $\tilde{G}$ of the group $G$ is the universal covering of its fundamental affine space. as it was shown in the paper [41] in the Fock space of the pair (the fundamental affine space of the group $G$, trivial holomorphic bundle over it) there is realised a direct integral of the Verma modules over the Lie algebra $g^C$ (the model of the Verma modules over it). In the case of the Virasoro–Bott group Vir the corresponding complex group CVir does not exist, but the semigroup Mantle(Vir), the Neretin semigroup Ner exists, so the construction of the fundamental affine space maybe extended on the infinite dimensional case; the fundamental affine space of the group Vir consists of the pairs $(f,t)$, where $f$ is an univalent function from $S$, and $t$ is a non–zero complex number, $|t| > 1$ [41]; the Virasoro algebra generators have the form [41]

$$L_p = \frac{\partial}{\partial c_p} + \sum_{k \geq 1} (k + 1)c_k \frac{\partial}{\partial c_{k+p}} \quad (p > 0),$$

$$L_0 = \sum_{k \geq 1} k c_k \frac{\partial}{\partial c_k} + t \frac{\partial}{\partial t}.$$
The model of the Verma modules over the Virasoro algebra is realised in the Fock space of the bundle $E$ over the flag manifold; the Virasoro algebra generators in the Fock space of the bundle $E$ have the form\cite{41}

\[
L_{p} = c_{p} + \sum_{k \geq 1} (k + 1) c_{k+p} \frac{\partial}{\partial c_{k}} \quad (p > 0),
\]

\[
L_{0} = \sum_{k \geq 1} k c_{k} \frac{\partial}{\partial c_{k}} - t \frac{\partial}{\partial t},
\]

\[
L_{1} = \sum_{k \geq 1} c_{k}((k + 2) \frac{\partial}{\partial c_{k+1}} - 2 \frac{\partial}{\partial c_{1}} \frac{\partial}{\partial c_{k}}) - 2 t \frac{\partial}{\partial t} \frac{\partial}{\partial c_{1}},
\]

\[
L_{2} = \sum_{k \geq 1} c_{k}((k + 3) \frac{\partial}{\partial c_{k+2}} - \frac{4}{2} \frac{\partial}{\partial c_{2}} - (\frac{1}{2} \frac{\partial}{\partial c_{1}})^{2} \frac{\partial}{\partial c_{k}} - b_{k} \frac{\partial}{\partial c_{1}} \frac{\partial}{\partial c_{k+1}} - \frac{1}{2} \frac{\partial}{\partial c_{2}} \frac{\partial}{\partial c_{k+2}}) - \frac{c}{2} \frac{\partial}{\partial c_{1}} - \frac{1}{2} \frac{\partial}{\partial c_{2}} - \frac{1}{2} \frac{\partial}{\partial c_{1}}^{2},
\]

\[
L_{n} = \frac{(-1)^{n}}{(n - 2)!} \text{ad}^{-2} L_{-1} \cdot L_{-2} \quad (n > 2).
\]

Let us consider the bundle $E(c)$ over the fundamental affine space, the inverse image of the bundle $E_{0,c}$ under the projection onto the flag manifold; the Virasoro algebra generators in the Fock space of the bundle $E(c)$ have the form\cite{44}

\[
L_{-1} = \sum_{k \geq 1} ((k + 2)c_{k+1} - 2c_{1}c_{k}) \frac{\partial}{\partial c_{k}} + 2c_{1}t \frac{\partial}{\partial t},
\]

\[
L_{-2} = \sum_{k \geq 1} ((k + 3)c_{k+2} - (4c_{2} - c_{1}^{2})c_{k} - b_{k}(c_{1}, \ldots c_{k+2})) \frac{\partial}{\partial c_{k}} + (4c_{2} - c_{1}^{2})t \frac{\partial}{\partial t},
\]

\[
L_{-n} = \frac{1}{(n - 2)!} \text{ad}^{-2} L_{-1} \cdot L_{-2} \quad (n > 2).
\]

The model of the Verma modules over the Virasoro algebra is realised in the Fock space of the bundle $E(c)$ over the fundamental affine space for the group $\tilde{G}$.

As it was mentioned in the paper \cite{44} the skladen of the Verma modules over the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ is realised in the Fock space over the fundamental affine space for the group $\Pi(G)$, where a non–zero complex number, $k > 0$, $\bar{c}$ is a direct integral $\int_{\mathbb{R}} V_{\pi(x)} dx$ of the Verma modules over this algebra of the weight $\pi(x)$, where $W = \bar{c}$, $\pi : W \rightarrow \bar{c}$, $\bar{c}$ is the Cartan subalgebra in $\mathfrak{g}^{C}$, $\bar{c}$ is the weight space; the II–space of the real simple Lie group $G$ of Hermitean type maybe defined by the noncommutative diagram

\[
\begin{array}{ccc}
\mathbb{C}^{r} & \simeq & \mathbb{C}^{r} \\
\downarrow & & \downarrow \\
T^{*}(\mathbb{D}_{+}^{*} \Gamma) & \hookrightarrow & \Pi(G) \hookrightarrow M(G) \\
\downarrow & & \downarrow \\
\mathbb{D}_{+}^{*} \Gamma & \hookrightarrow & A(G) \hookrightarrow M(G)
\end{array}
\]

the II–space of the universal covering $\tilde{G}$ of the Lie group $G$ is the universal covering of the II–space $\Pi(G)$. The II–space of the group Vir consists of triples $(f, t, s)$, where $f$ is an univalent function from $S$, $t$ is a non–zero complex number, $|t| < 1$, $s$ is an arbitrary complex number (or an element of the universal covering $\mathbb{E}^{*}$ of the complex plane without zero; the Virasoro algebra generators have the form

\[
L_{p} = \frac{\partial}{\partial c_{p}} + \sum_{k \geq 1} (k + 1) c_{k} \frac{\partial}{\partial c_{k+p}} \quad (p > 0),
\]

\[
L_{0} = \sum_{k \geq 1} k c_{k} \frac{\partial}{\partial c_{k}} + t \frac{\partial}{\partial t} + s \frac{\partial}{\partial s},
\]

\[
L_{1} = \sum_{k \geq 1} c_{k}((k + 2) \frac{\partial}{\partial c_{k+1}} - 2 \frac{\partial}{\partial c_{1}} \frac{\partial}{\partial c_{k}}) - 2 t \frac{\partial}{\partial t} \frac{\partial}{\partial c_{1}},
\]

\[
L_{n} = \frac{(-1)^{n}}{(n - 2)!} \text{ad}^{-2} L_{-1} \cdot L_{-2} \quad (n > 2).
\]
\[ L_{-2} = \sum_{k \geq 1} ((k + 3)c_{k+2} - (4c_2 - c_1^2) c_k - b_k(c_1, \ldots, c_{k+2})) \frac{\partial}{\partial c_k} + (4c_2 - c_1^2) \frac{\partial}{\partial t} + (4c_2 - c_1^2) s \frac{\partial}{\partial s}, \]
\[ L_{-n} = \frac{1}{(n-2)!} \text{ad}^{n-2} L_{-1} \cdot L_{-2} \quad (n > 2). \]

Let us consider the bundle \( E(c) \) over the \( \Pi \)-space \( \Pi(Vir) \), the inverse image of the bundle \( E(c) \) over the fundamental affine space \( A(Vir) \) under the projection of \( \Pi(Vir) \) on \( A(Vir) \); the Virasoro algebra generators in the Fock space of the bundle \( E(c) \) have the form

\[ L_{-p} = c_p + \sum_{k \geq 1} (k + 1)c_{k+p} \frac{\partial}{\partial c_k} \quad (p > 0), \]
\[ L_0 = \sum_{k \geq 1} k c_k \frac{\partial}{\partial c_k} - t \frac{\partial}{\partial t} - s \frac{\partial}{\partial s}, \]
\[ L_1 = \sum_{k \geq 1} c_k ((k + 2) \frac{\partial}{\partial c_{k+1}} - 2 \frac{\partial}{\partial c_1} \frac{\partial}{\partial c_k}) - 2t \frac{\partial}{\partial t} \frac{\partial}{\partial c_1} - 2s \frac{\partial}{\partial s} \frac{\partial}{\partial c_1}, \]
\[ L_2 = \sum_{k \geq 1} c_k ((k + 3) \frac{\partial}{\partial c_{k+2}} - (4 \frac{\partial}{\partial c_2} - (\frac{\partial}{\partial c_1})^2) \frac{\partial}{\partial c_k} - b_k(\frac{\partial}{\partial c_1}, \ldots, \frac{\partial}{\partial c_{k+2}})) - t \frac{\partial}{\partial t} (4 \frac{\partial}{\partial c_2} - (\frac{\partial}{\partial c_1})^2) - s \frac{\partial}{\partial s} (4 \frac{\partial}{\partial c_2} - (\frac{\partial}{\partial c_1})^2) + \frac{c}{2} (\frac{\partial}{\partial c_2} - (\frac{\partial}{\partial c_1})^2), \]
\[ L_n = \frac{(-1)^n}{(n-2)!} \text{ad}^{n-2} L_{1} \cdot L_{2} \quad (n > 2). \]

**THEOREM 2.** The skladen of the Verma modules over the Virasoro algebra \( \mathcal{C}_{Vir} \) is realised in the Fock space of the pair \((\Pi(Vir), E(c))\).

Namely, the highest weights in this Fock space have the form \( t^h \varphi(t/s) \).

A punctured oriented mirror \( V^+_a \) on the flag manifold \( M(Vir) \) is a pair \((V_a, x)\), where \( V_a \) is an oriented mirror and \( x \) is a point of \( M(Vir) \) which belongs to \( V_a \); the space \( \Sigma(Vir) \) of the punctured oriented mirrors is a bobundle \( M(Vir) \leftarrow \Sigma(Vir) \rightarrow \Lambda_+(Vir) \), where \( \Sigma(Vir) \mapsto M(Vir) \) is the subsymmetry bundle \([28,18]\) and \( \Sigma(Vir) \mapsto \Lambda_+(Vir) \) is the tautological bundle. On the space \( \Sigma(Vir) \) of the punctured oriented mirrors there is defined a structure of a contact CR–manifold \([45]\) by the canonical connection of the prequantisation \([28,18]\) in the subsymmetry bundle, the contact structure is defined by the connection form \( \theta^\text{can} \) and the structure of CR–manifold on the space \( \Sigma(Vir) \) of the punctured oriented mirrors is defined by the complex structure on horizontal subspaces of the subsymmetry bundle lifted from the base of the bundle, the flag manifold \( M(Vir) \); the Levy form of the CR–manifold \( \Sigma(Vir) \) coincides with the Kähler form on the base. Let us denote by \( \Sigma(Vir) \) the universal covering of the space \( \Sigma(Vir) \). Let us denote by \( E(c) \) the CR–analytic bundle over the space \( \Sigma(Vir) \) of the punctured oriented mirrors, lifted from the bundle \( E_{o,c} \) on the base.

**THEOREM 3.** The model of the Verma modules over the Virasoro algebra is realised in the Fock space of the pair \((\Sigma(Vir), E(c))\).

The statement of the theorem is the consequence of the result obtained in the third paragraph of the paper \([41]\).

Let us consider the space \( \Sigma^#(Vir) \) defined by the commutative diagram

\[
\begin{array}{ccc}
\mathbb{R} & \overset{\cong}{\to} & \mathbb{R} \\
\downarrow & & \downarrow \\
T^*(\mathbb{RP}^1) & \mapsto & \Sigma^#(Vir) \mapsto M(Vir) \\
\downarrow & & \| \\
\mathbb{RP}^1 & \mapsto & A(Vir) \mapsto M(Vir)
\end{array}
\]

On \( \Sigma^#(Vir) \) there is defined a structure of the CR–manifold; let us denote by \( \Sigma^#(\tilde{Vir}) \) the universal covering of the space \( \Sigma^#(Vir) \), and by \( E(c) \) the CR–analytic bundle over \( \Sigma^#(\tilde{Vir}) \) lifted from \( \Sigma(Vir) \).
THEOREM 4. The skladen of the Verma modules over the Virasoro algebra is realised in the Fock space of the pair $(\Sigma^\#(\mathfrak{Vir}), E(c))$.

The statement of the theorem follows from the theorem 1.

On $\Sigma^\#(\mathfrak{Vir})$ there is defined a structure of symplectic manifold; the symplectic form on $\Sigma^\#(\mathfrak{Vir})$, the Alekseev–Shatashvili form $\omega_{AS}$ maybe obtained by the hamiltonian reduction of the canonical symplectic structure on $T^*(\text{PSL}(2,\mathbb{R}))$ [46,47]. In the papers [46,47] there is considered a procedure of the quantisation of such symplectic structure, the value of the central charge of the corresponding bundle $E(c)$ maybe determined from the following characteristics of the Alekseev–Shatashvili form: there is defined a fiberwise action of the abelian group $\mathbb{Z}$ in the bundle $\mathbb{R}^2 \leftarrow \Sigma^\#(\mathfrak{Vir}) \rightarrow M(\mathfrak{Vir})$, the volume $\alpha$ of the fundamental domain in the fiber determines the central charge $c = 1 - 6(\alpha - \alpha^{-1})^2$; the condition of rationality of such volume extract the values of central charge from the spectrum (2A) [47]. the attentive reading of the papers of the mentioned authors shows that the starting point for the receiving of the Kac spectrum (2) in them is the skladen more than the model of the verma modules over the Virasoro algebra, this circumstance explains the appearing of two parameters in Kac spectrum.

Let us consider the representation of elements of the homogeneous space $\text{Diff}_+(\mathbb{S}^1)/\mathbb{S}^1$ by univalent functions $f$ from class $S$: the function $f(1-tf)^{-1}$ belongs to class $S$, if $t^{-1}$ does not belong to the image of $f$; the set of such points $t$ forms a domain $\mathbb{C}\setminus(f(D_+))$ in the complex plane $\mathbb{C}$, the family of pairs $(f,t)$ is the space of the universal deformation of a complex disc [41,p.217], $S$ is the universal Teichmüller space [48-51]; the action of the Virasoro algebra $\mathfrak{C}v$ on pairs $(f,t)$ is defined by formulas

\[
L_p = \frac{\partial}{\partial c_p} + \sum_{k \geq 1} (k+1)c_k \frac{\partial}{\partial c_{k+p}} \quad (p > 0),
\]

\[
L_0 = \sum_{k \geq 1} kc_k \frac{\partial}{\partial c_k} + t \frac{\partial}{\partial t},
\]

\[
L_{-1} = \sum_{k \geq 1} ((k+2)c_{k+1} - 2c_1c_k) \frac{\partial}{\partial c_k} + 2c_1t \frac{\partial}{\partial t} + t^2 \frac{\partial}{\partial t},
\]

\[
L_{-2} = \sum_{k \geq 1} ((k+3)c_{k+2} - (4c_2 - c_1^2)c_k - b_k(c_1, \ldots c_{k+2})) \frac{\partial}{\partial c_k} + (4c_2 - c_1^2)t \frac{\partial}{\partial t} + 3c_1t^2 \frac{\partial}{\partial t} + t^3 \frac{\partial}{\partial t},
\]

\[
L_{-n} = \frac{1}{(n-2)!} \text{ad}^{n-2} L_{-1} \cdot L_{-2} \quad (n > 2);
\]

this action maybe obtained by the Manin–Kontsevich–Beilinson–Schechtman construction [52-57]; the Virasoro algebra generators in the Fock space of the bundle $E(c)$ over the universal deformation of a complex disc have the form

\[
L_p = c_p + \sum_{k \geq 1} (k+1)c_{k+p} \frac{\partial}{\partial c_k} \quad (p > 0),
\]

\[
L_0 = \sum_{k \geq 1} kc_k \frac{\partial}{\partial c_k} - t \frac{\partial}{\partial t},
\]

\[
L_1 = \sum_{k \geq 1} c_k((k+2) \frac{\partial}{\partial c_{k+1}} - 2 \frac{\partial}{\partial c_1} \frac{\partial}{\partial c_k}) - 2 \frac{\partial}{\partial t} \frac{\partial}{\partial c_1} + t^2 \frac{\partial}{\partial t},
\]

\[
L_2 = \sum_{k \geq 1} c_k((k+3) \frac{\partial}{\partial c_{k+2}} - (4 \frac{\partial}{\partial c_2} - (\frac{\partial}{\partial c_1})^2) \frac{\partial}{\partial c_k} - b_k(\frac{\partial}{\partial c_1}, \ldots \frac{\partial}{\partial c_{k+2}})) -
\]

\[
t \frac{\partial}{\partial t}(4 \frac{\partial}{\partial c_2} - (\frac{\partial}{\partial c_1})^2) + 3t^2 \frac{\partial}{\partial t} \frac{\partial}{\partial c_1} - t^3 \frac{\partial}{\partial t} + c \frac{\partial}{\partial c_2} - (\frac{\partial}{\partial c_1})^2,
\]

\[
L_n = \frac{(-1)^n}{(n-2)!} \text{ad}^{n-2} L_1 \cdot L_2 \quad (n > 2);
\]

these generators define a structure of the model of the Verma modules over the Virasoro algebra in the Fock space of the bundle $E(c)$ over the universal covering of the universal deformation of a complex disc.
let us consider the determinant sheaf $\text{DET}_\lambda$ over $M(\text{Vir})$: $\text{DET}_\lambda = R^0 p_* (\Omega^\lambda(\tilde{C}/M(\text{Vir})))$, where $p : C \mapsto M(\text{Vir})$ is the projection of the universal deformation of the punctured complex disc $D_+^*$ onto the base $M(\text{Vir})$, $\tilde{C}$ is the universal covering of $C$ fibred over $M(\text{Vir})$ with fiber $\tilde{D}_+^*$, $\Omega^\lambda$ is the sheaf of holomorphic $\lambda$-differentials on $D_+^*$.

THEOREM 5. The model of the Verma modules over the Virasoro algebra $\text{C}_{\text{Vir}}$ with the central charge $c = 2(6\lambda^2 - 6\lambda + 1)$ maybe realised in the space of sections of the sheaf $\text{DET}^\lambda$.  

The statement of the theorem is a consequence of the fact that the Virasoro algebra representations obtained by the determinant construction and non–isomorphic to the Verma modules over the Virasoro algebra have the zero measure in the space of parameters [39,40], the relation between the central charge and the parameter $\lambda$ is also described in these papers.

2. INFINITE DIMENSIONAL GEOMETRY AND QUANTUM FIELD THEORY OF NON–INTERACTING STRINGS: SECOND QUANTIZED FREE STRING ON FLAT AND CURVED BACKGROUND, THE BOWICK–RAJEEV FORMALISM OF THE SEPARATION OF THE INTERNAL AND EXTERNAL DEGREES OF FREEDOM OF A STRING

The purpose of this chapter is to give a mathematical description of the second quantization of a closed string based on infinite dimensional geometry. The configuration space of the quantum field theory is an infinite dimensional space, its elements are classical fields (functions, distributions, differential forms, connections) on the support manifold. If these fields are free of constraints, then the configuration space is flat, and in the presence of constraints it has a rather complicated structure. if the number of fields is infinite then the support manifold has an infinite dimension, the configuration space has a dimension $\infty^2$. This situation is realized in the case of the closed string–field theory.

2.1. Flat background: the Banks–Peskin differential forms, the Feigin–Frenkel–Garland–Zukerman formalism, the Siegel string fields and the Kato–Ogawa BRST–operator.

Let us introduce the basic notions [1,58,59]: a closed string is an arbitrary contour $C$ in $D$–dimensional space $\mathbb{R}^D$ (it is usually supposed that $\mathbb{R}^D$ possesses a Minkowsky metric and the transition to the Euclidean one is done later). a collection of such contours will be denoted by $Q_0$. One might define a classical string field as a magnitude on $Q_0$. But this is not convenient. the fact is that $Q_0$ is not a smooth manifold. It has singularities. To define a string field one should specify its behaviour near singularity. It is opportune to make occasion of the possibility of a representation of $Q_0$ as an orbifold to achieve our object. Indeed, let us parametrise a contour $C$ by a function $x^\mu(s) : [-\pi, \pi] \mapsto \mathbb{R}^D$. We denote the set of such functions by $Q$, which admits an action of the group $\text{Diff}_+(\mathbb{S}^1)$. The initial space $Q_0$ is a quotient $Q/\text{Diff}_+(\mathbb{S}^1)$. So a classical scalar string field maybe defined as a magnitude on $Q$. In a parametrisation $x^\mu(s)$ a string field has the form $\Phi(x^\mu)$. The independence of $\Phi$ on a choice of parametrisation is called reparametrisation invariance. Let us find a law of string–field transformation under reparametrisations, supposing that it is determined by the law of transformations of the first quantised string coordinates $x^\mu_n$: $x^\mu_n = \text{Re} \sum_n x^\mu_n z^{-n} (s = \exp(is))$. This law should be derived from the first quantised closed string action. However, the action can not be received in a unique way, because it should be obtained only from some physical assumptions on the behaviour of magnitudes on $Q_0$ instead of $Q$. Hence, the action will contain a gauge parameter. The choice of the first quantised closed string action is one of A.M.Polyakov [60]:

\begin{equation}
\tag{2A}
S(x^\mu) = \int g(z, \bar{z}) \partial_\bar{z} x^\mu(z) \partial_z \bar{x}^\mu(z) \, dz \, d\bar{z}.
\end{equation}

Here we assume the holomorphy of the coordinates $x^\mu(z)$ on a world surface of a string in view of the Hamiltonian character of its evolution. Let us interpret the metric $g(z, \bar{z})$ as a gauge parameter. In the gauge $g(z, \bar{z}) = 1$ one gets the next expression for the action

\begin{equation}
\tag{2B}
S(x^\mu) = \int_{D_+} \partial_\bar{z} x^\mu(z) \partial_z \bar{x}^\mu(z) \, dz \, d\bar{z} = \sum_{n>0} n x^\mu_n \bar{x}^\mu_n,
\end{equation}

where $D_+ = \{z : |z| \leq 1\}$. The first quantisation of a string with such action is described in [1,58,59]. In the space of states of a first quantised string (functions of the infinite number of
variables $x_n^\mu$ obeying the equation $(\frac{\partial}{\partial x_n^\mu} - e_\mu)\Phi = 0$ the Virasoro algebra generators $L_{-p}$ with $p \geq 0$ naturally act:

$$L_{-p} = e_\mu px_p^\mu + \frac{1}{2} \sum_{k=1}^{p-1} k(p-k)x_k^\mu x_{p-k}^\mu + \sum_{k \geq 1} (k+p)x_{k+p}^\mu \frac{\partial}{\partial x_k^\mu}.$$  

After a transition to the second quantisation the Fock space $F(Q^*)$ dual to the space of the first quantised string states is interpreted as a configuration space for the second quantised string.

However, the action (2B) is not gauge invariant. The mechanism accounting for the non-invariance of the gauge was proposed in the papers [61,62,19,63]. Accordingly to [19] expression (2A) depends on a choice of a complex structure on $Q^*$. The space of such structures is isomorphic to $M(Vir) = \text{Diff}^+(S^1)/S^1$. An element of the space $M(Vir)$, an univalent function $f$, generates global transformations of the fields $\Phi(x^\mu)$ preserving the family of Polyakov actions. In such case one should postulate a locality of transformations. The relation between the gauge parameters $g(z, \bar{z})$ and $f(z)$ has the form $g(z, \bar{z})dx^\mu d\bar{x}^\mu = dx^\mu(f) d\bar{x}^\mu(f)$. Let us now define the configuration space of the closed string field as a space of magnitudes $\Phi(x^\mu, f)$. The decomposition of a field by coefficients $c_k$ independent of $x_n^\mu$ determines a multicomponent string field $\Phi(x^\mu, f) = \sum \xi c_1^{\xi_1} \ldots c_n^{\xi_n} \Phi_\xi(x^\mu)$. The new action has the form [63]

$$(2C) \quad S(x^\mu, f) = \int dx^\mu(f) d\bar{x}^\mu(f) + U_{h,c},$$

where $U_{h,c}(f)$ are the Kähler potentials on $M(Vir)$ and $\int dx^\mu d\bar{x}^\mu$ is the Kähler potential on $Q^*$. The space dual to the space of string fields consists of all holomorphic sections of the Hermitian bundle $E_{h,c}$, the action of the Virasoro algebra in $O(E_{h,c})$ has the form

$$L_p = \frac{\partial}{\partial c_p} + \sum_{k \geq 1} (k+1)c_k \frac{\partial}{\partial c_k + p} \quad (p > 0),$$

$$L_0 = \sum_{k \geq 1} kc_k \frac{\partial}{\partial c_k} + \sum_{k \geq 1} kx_k^\mu \frac{\partial}{\partial x_k^\mu} + h,$$

$$L_{-1} = \sum_{k \geq 1} \frac{((k+2)c_{k+1} - 2c_1c_k)}{2c_1c_k} \frac{\partial}{\partial c_k} + 2c_1 \sum_{k \geq 1} kx_k^\mu \frac{\partial}{\partial x_k^\mu} +$$

$$+ \sum_{k \geq 1} (k+1)x_{k+1}^\mu \frac{\partial}{\partial x_{k+1}^\mu} + 2hc_1 + e_\mu x_1^\mu,$$

$$L_{-2} = \sum_{k \geq 1} \frac{((k+3)c_{k+2} - (4c_2 - c_1^2)c_k - b_k(c_1 \ldots c_{k+2}))}{2c_1 \ldots c_{k+2}} \frac{\partial}{\partial c_k} + (4c_2 - c_1^2) \sum_{k \geq 1} kx_k^\mu \frac{\partial}{\partial x_k^\mu} +$$

$$+ \frac{3c_1}{2} \sum_{k \geq 1} x_k^\mu \frac{\partial}{\partial x_k^\mu} + \sum_{k \geq 1} (k+2)x_{k+2}^\mu \frac{\partial}{\partial x_{k+2}^\mu} +$$

$$+ \frac{x_2^2}{2} + h(4c_2 - c_1^2) + \frac{c}{2}(c_2 - c_1^2) + 3e_\mu c_1 x_1^\mu + 2e_\mu x_2^\mu,$$

$$L_{-n} = \frac{1}{(n-2)!} \text{ad}^{n-2} L_{-1} \cdot L_{-2} \quad (n > 2).$$

The dual space to $O(E_{h,c})$ (the Fock space which is a semidirect product of the Fock space over the flag manifold for the Virasoro–bott group considered in the first chapter and the standard (flat) Fock space) is the configuration space for the closed string–field theory (without ghosts). The action of the Virasoro algebra in it has the form

$$L_{-p} = c_p + \sum_{k \geq 1} (k+1)c_{k+p} \frac{\partial}{\partial c_k} \quad (p > 0),$$

$$L_0 = \sum_{k \geq 1} kc_k \frac{\partial}{\partial c_k} + \sum_{k \geq 1} kx_k^\mu \frac{\partial}{\partial x_k^\mu} + h.$$
\[ L_1 = \sum_{k \geq 1} c_k ((k + 2) \frac{\partial}{\partial c_{k+1}} - 2 \frac{\partial}{\partial c_1} \frac{\partial}{\partial c_k}) + 2 \sum_{k \geq 1} kx_k^\mu \frac{\partial}{\partial c_1} \frac{\partial}{\partial x_k^\mu} + \sum_{k \geq 1} (k + 1)x_k^\mu \frac{\partial}{\partial x_{k+1}^\mu} + 2h \frac{\partial}{\partial c_1} + e^\mu \frac{\partial}{\partial x_1^\mu}, \]

(3A) \[ L_2 = \sum_{k \geq 1} c_k ((k + 3) \frac{\partial}{\partial c_{k+2}} - 4 \frac{\partial}{\partial c_2} - (\frac{\partial}{\partial c_1})^2 \frac{\partial}{\partial c_k} - b_k (\frac{\partial}{\partial c_1}, \ldots, \frac{\partial}{\partial c_{k+2}})) + 2 \sum_{k \geq 1} kx_k^\mu \frac{\partial}{\partial c_2} - (\frac{\partial}{\partial c_1})^2 + 3 \sum_{k \geq 1} (k + 1)x_k^\mu \frac{\partial}{\partial x_{k+1}^\mu} + \sum_{k \geq 1} (k + 2)x_k^\mu \frac{\partial}{\partial x_{k+2}^\mu} + \frac{1}{2} \frac{\partial^2}{\partial x_1^\mu} + h(4 \frac{\partial}{\partial c_2} - (\frac{\partial}{\partial c_1})^2) + \frac{c}{2} (\frac{\partial}{\partial c_2} - (\frac{\partial}{\partial c_1})^2) + 3e^\mu \frac{\partial}{\partial c_1} + 2e^\mu \frac{\partial}{\partial x_2^\mu}, \]

\[ L_n = \frac{(-1)^n}{(n - 2)!} \text{ad}^{n-2} L_1 \cdot L_2 \quad (n > 2). \]

A metric on the configuration space has the form

\[ (\Phi, \Psi) = (\Phi|\exp(S)|\Psi) =: \int \exp(-S(x^\mu, f))\Phi(x^\mu, f)\Psi(x^\mu, f) DXD\bar{X}DfD\bar{f}. \]

The area of integration in the last integral is determined by the univalency conditions for the function \( f \). It should be mentioned that \( \exp(S) = \exp(\int dx^\mu(f)\bar{dx}^\mu(f)) \cdot K_{h,c}(f) \). The second term in the formula is the Bergman kernel function on the space of the univalent functions (after the identification of this space with the universal Teichmüller space the Bergman kernel function will coincide with the Polyakov measure [64,65] as it was shown by A. Morozov and A. Rosly in the paper [66], namely, \( K_{h,c}(f) = \exp(U_{h,c}(f)) = (\text{cap}(f(D_+)))^h \cdot \text{det}^{-c}(1 - Z \bar{Z}_f) \), where \( \text{cap}(f(D_+)) \) is the conformal capacity of the domain \( f(D_+) \).

To quantise a field \( \Phi(x^\mu, f) \) it is necessary to supplement the support manifold by the anticommuting Faddeev–Popov ghosts. They are transformed accordingly to the adjoint representation of the constraint algebra \( \text{Vect}(S^1) \) (usually parallel with ghosts one consider also elements of dual space, which transform accordingly to the coadjoint representation of the constraint algebra). It is convenient to think ghosts as vector fields on \( M(\text{Vir}) \ltimes Q^* \) tangent to the constraint foliation. The dual family of differential forms (antighosts) \( \xi_p \) with the law of transformation \( L_p \xi_q = -(q + 2p)\xi_{p+q} \) generates the space of the Banks–Peškin string differential forms [61]. So the string differential forms depend on the string coordinates \( x_n^\mu \) in the external space, the internal degrees of freedom of the string \( c_k \) and Faddeev–Popov antighosts \( \xi_p \). The subspace of such forms in \( \Omega(M(\text{Vir}) \ltimes Q^*) \) will be denoted as \( \Omega_{BP} \). Let us introduce according to Feigin, Frenkel, Garland and Zuckerman [67,68] the vector space corresponding to the filled ghost Dirac sea relative to the subalgebra \( \{ L_p, p > 0 \} \) as vac = \( \xi_1 \wedge \xi_2 \wedge \xi_3 \wedge \ldots \). Consider now the space \( \Omega_{BP}^\infty \) of semi–infinite string differential forms (the strict definition of the semi–infinite form maybe found in [67,68]), and also \( \Omega^\infty_{BP}(E_{h,c}) = \Omega^\infty_{BP} \otimes \Omega(M(\text{Vir}) \ltimes Q^*) \otimes O(E_{h,c}) \). The Virasoro algebra acts in \( \Omega^\infty_{BP} \) with the central charge \( c = -26 \) and the value of \( L_0 \) on the vacuum vac being equal to \( h - 1 \). The formulas for the action have the form

(3B) \[ L_{-p}' = L_{-p} + L_{-p}^\text{ghost}, \quad L_{-p}^\text{ghost} = \sum_q (p - 2q)\xi_{q-p} \frac{\partial}{\partial \xi_q}. \]

Let us introduce the Kato–Ogawa BRST–operator [69] as a partial differential along the constraints \( Q = \sum_p L_{-p}S_p \). A request for the absence of the conformal anomaly (the nilpotency of the BRST–operator \( Q : Q^2 = 0 \)) picks out the value of the parameter \( c : c = 26 \).

**DEFINITION** [62,63]. The Siegel string field is an element of the space \( \Omega_{BP}^\infty(E_{h,c})^* \) dual to the space of the Banks–Peškin differential forms.

Therefore, the space of Siegel string fields is a product of the Fock space of the bundle \( E_{h,c} \), over \( M(\text{Vir}) \), with \( \Omega^* \) and the space of the semi–infinite short forms, the operator \( \Omega^* \) conjugate to the BRST–operator.
Q (which is also called by the Kato–Ogawa BRST–operator) has the form \( Q^* = \sum_{p,q} (L_p \xi^*_p - \frac{1}{2} (p-q) \xi^*_p \xi^*_q) \frac{\partial}{\partial x^p} \), where \((\xi^*_p, \xi^*_q) = -\delta(p+q)\). The scalar product in \( \Omega_{h,c}^* \) maybe defined as

\[
(Q, \tilde{Q}) = \int \exp(-S) \Phi^{(\alpha)} \bar{\Psi}^{(\beta)} D X D \bar{X} D f D \bar{f} (\xi^*_\alpha, \xi^*_\beta),
\]

where \( \Phi = \Phi^{(4)} \), \( \Psi = \Psi^{(4)} \) and \((\xi^*_\alpha, \xi^*_\beta)\) is the scalar product in the space of the semi–infinite ghost forms [67,68].

At the end of this article we shall give a cohomological interpretation of the Nambu–Goto action [1,58,59], accordingly to [70]. For such purpose let us consider a representation of the Virasoro algebra in the space \( V_D \) of the variables \( c_1, \ldots, c_n, \ldots x_1^\mu, \ldots x_k^\mu, \ldots \) by the differential operators

\[
\begin{align*}
L_p &= \frac{\partial}{\partial c_p} + \sum_{k \geq 1} (k+1)c_k \frac{\partial}{\partial c_{k+p}} \quad (p > 0), \\
L_0 &= \sum_{k \geq 1} kc_k \frac{\partial}{\partial c_k} + \sum_{k \geq 1} kx_k^\mu \frac{\partial}{\partial x_k^\mu}, \\
L_{-1} &= \sum_{k \geq 1} ((k+2)c_{k+1} - 2c_1 c_k) \frac{\partial}{\partial c_k} + 2c_1 \sum_{k \geq 1} kx_k^\mu \frac{\partial}{\partial x_k^\mu}, \\
L_{-2} &= \sum_{k \geq 1} ((k+3)c_{k+2} - (4c_2 - c_1^2)c_k - b_k(c_1, \ldots, c_{k+2})) \frac{\partial}{\partial c_k} + (4c_2 - c_1^2) \sum_{k \geq 1} kx_k^\mu \frac{\partial}{\partial x_k^\mu} + 3c_1 \sum_{k \geq 1} x_k^\mu \frac{\partial}{\partial x_k^\mu} + \sum_{k \geq 1} (k+2)x_k^\mu \frac{\partial}{\partial x_k^\mu}, \\
L_{-n} &= \frac{1}{(n-2)!} \text{ad}^{n-2} L_{-1} \cdot L_{-2} \quad (n > 2).
\end{align*}
\]

THEOREM 6 [70]. \( H^1(\text{Cvri}, V_D) = \text{Sym} (\mathbb{C}, D) + \mathbb{C}^D + \mathbb{C}_c \). The matrix cohomology class is determined by the Nambu–Goto action \( S_{NG}(G_{ab}) = \sum_n nG_{ab}x_n^a x_n^b : \)

\[
L'_{-p} = L_{-p} + F_{-p}^{NG}, \quad F_{-p}^{NG} = \frac{1}{2} \sum_{m+n=p} nmG_{ab}x_m^a x_n^b, \quad \text{if } p > 0 \text{ and } 0 \text{ otherwise.}
\]

The vector class has the form

\[
L'_{-p} = L_{-p} + F_{-p}^\text{vect}, \quad F_{-p}^\text{vect} = b_p (p-1)x_p^a, \quad \text{if } p > 0 \text{ and } 0 \text{ otherwise.}
\]

The class corresponding to parameter \( c \) is defined by the formulas written in the first chapter. Here \( x_n^a = (x^a(f))_n \).

The statement of the theorem maybe obtained by the straightforward calculations.

2.2. Curved background [71].

It should be mentioned that the Banks–Peskin differential forms have no a geometric meaning; on the contrary, the string fields are the correctly defined objects on the background. Let \( v_{k_1 \ldots k_D} = (x_1^1 k_1 \ldots (x_D)^k_D) \) be a vector field on the background then the natural action of the field \( v_{k_1 \ldots k_D} \) in the space of string fields has the form [71]

\[
T(v_{k_1 \ldots k_D}^j) = \sum_{m_{ij} = M} \tilde{x}_{m_{11}} \ldots \tilde{x}_{m_{1k_1}} \ldots \tilde{x}_{m_{D1}} \ldots \tilde{x}_{m_{Dk_D}} \frac{\partial}{\partial \tilde{x}_{M}},
\]

\[
\tilde{x}_{m_{Dk_D}} = (x^a(f))_n.
\]
If $G_{ab} = 0$ then $[T(v_{1}^{a}...v_{D}^{a}), L_{p}] = 0$, where $L_{p}$ is defined by formulas dual to (4).

Let us fix a non-constant metric $G_{\alpha \beta}$ on the background. The Laplace operator for $G_{\alpha \beta}$ will have the form

$$\Delta(G_{\alpha \beta}) = G^{ij}(x^1, \ldots x^D) \frac{\partial^2}{\partial x^i \partial x^j} + H^i(x^1, \ldots x^D) \frac{\partial}{\partial x^i},$$

where

$$H^i = G^{-\frac{1}{2}} \frac{\partial}{\partial x^j}(G^{\frac{1}{2}} \cdot G^{ij}).$$

Let us now consider the operators $[\Delta(G_{\alpha \beta})]_p$ defined as

$$[\Delta(G_{\alpha \beta})]_p = \sum_{b+c-|a|=p} f(a,b,c) \tilde{G}^{ij}(x^1_{a_{11}} \ldots x^1_{a_{1m_1}} \ldots x^D_{a_{D1}} \ldots x^D_{a_{Dm_D}}) \frac{\partial^2}{\partial x^i_b \partial x^j_c} +$$

$$\frac{1}{6} \sum_{b+c-|a|=p} (b^3 - b) \tilde{H}^i(x^1_{a_{11}} \ldots x^1_{a_{1m_1}} \ldots x^D_{a_{D1}} \ldots x^D_{a_{Dm_D}}) \frac{\partial}{\partial x^i_b},$$

where $\tilde{G}^{ij}$ and $\tilde{H}^i$ are the polylinearisations of $G^{ij}$ and $H^i$, the function $f(a,b,c)$ is defined as

$$f(a,b,c) = \sum_{A' \cup A''} (b - \sum_{i \in A'} a_i)(c - \sum_{i \in A''} a_i).$$

**THEOREM 7 [71].** The operators

$$G_{\alpha \beta}(L_p) = L_p$$

of (4) if $p > 0$ and $L_p$ of (4) + $\frac{1}{2} [\Delta(G_{\alpha \beta})]_{-p}(\bar{x}_p^*)$ if $p \leq 0$,

where $\bar{x}_p^n = (x^{(f)})_n$, define an action of the Virasoro algebra in the space of string fields. The correspondence $G_{\alpha \beta} \longrightarrow G_{\alpha \beta}(\cdot)$ is generally covariant.

The corresponding BRST–operator $G_{\alpha \beta}(Q)$ has the form

$$G_{\alpha \beta}(Q) = \sum (L_{-p} \xi_p^* - \frac{1}{2} (p-q) \xi_p^* \xi_q^* \frac{\partial}{\partial \xi_{-p-q}^*}).$$

The BRST–operator is nilpotent if $c = 26$.

**2.3. The Bowick–Rajeev formalism.**

The Bowick–Rajeev formalism [19] describes a separation of variables, characterising external and internal degrees of freedom of a string in quantum field theory. Let us present the Bowick–Rajeev formalism for the flat background following to [72]. Let us consider the space $\Omega_{BB}(E_{b,c})$ of the semi–infinite Banks–Peskin differential forms. The action of the Virasoro algebra has the form

$$L_p = \frac{\partial}{\partial \xi^*_p} + \sum_{k \geq 1} (k+1)c_k \frac{\partial}{\partial \xi^*_{k+p}} - \sum_{k \geq 1} (p+2k) \xi_{k+p} \frac{\partial}{\partial \xi^*_k}$$

$$L_0 = \sum_{k \geq 1} c_k \frac{\partial}{\partial \xi^*_k} + \sum_{k \geq 1} k c_k \frac{\partial}{\partial \xi^*_k} - 2 \sum_{k \geq 1} k \xi_k \frac{\partial}{\partial \xi^*_k} + h,$$

$$L_{-1} = \sum_{k \geq 1} ((k+1) c_{k+1} - 2 c_1 c_k) \frac{\partial}{\partial \xi^*_k} + 2 c_1 \sum_{k \geq 1} k c_k \frac{\partial}{\partial \xi^*_k} +$$

$$\sum_{k \geq 1} (k+1) x_{k+1}^\mu \frac{\partial}{\partial x^\mu_k} + \sum_{k \geq 1} (1-2k) \xi_{k-1} \frac{\partial}{\partial \xi^*_k} + 2hc_1 + e_1 x_1^\mu,$$

$$L_{-2} = \sum_{k \geq 1} ((k+3) c_{k+2} - 4(c_2 - c_1^2) c_k - b_k(c_1, \ldots c_{k+2})) \frac{\partial}{\partial \xi^*_k} + (4c_2 - c_1^2) \sum_{k \geq 1} k c_k \frac{\partial}{\partial \xi^*_k} +$$

$$3c_1 \sum_{k \geq 1} x_{k+1}^\mu \frac{\partial}{\partial x^\mu_k} + \sum_{k \geq 1} (k+2) x_{k+2}^\mu \frac{\partial}{\partial x^\mu_k} +$$
DEFINITION. The covariantly constant section of the bundle $\mathcal{F}G_{h,c}$ projector $P$ Rajeev vacuum.

Let then as

The pairings $B : \Omega^*\mathcal{F}G_{h,c}$ determined by a choice of a gauge parameter value (i.e. if

Define a connection

The curvature tensor of the connection $\nabla$ of which is isomorphic to $\mathcal{F}G_{h,c}(\text{Vir})$, a fiber $H_f(h,c)$ of which is isomorphic to $H(h,c) = F(Q) \otimes \Lambda^1(S^1)$. We shall denote this bundle by $\mathcal{F}h,c(M(\text{Vir}))$, the variables $x^\mu_n, c_k$ define its trivialisation.

A gauge of a Banks–Peskin string differential form is a relation $c_k = c_k(x^\mu)$, and a gauge–fixing projector $P$ is the operator $P : \Omega^*\mathcal{F}B_{h,c} \mapsto H(h,c)$ determined by the formula

Let $f(z) = z + c_1^2 z^2 + c_2^2 z^3 + c_3^2 z^4 + \ldots$ be an arbitrary univalent function, the $f$–gauge is the gauge defined by the relation $c_k = c_k^0$. The gauge–fixing projector has the form

As it was shown in \[72\] there exists an imbedding $I_f : H(h,c) \mapsto \Omega^*\mathcal{F}B_{h,c}$ such that

Define a connection $\nabla$ in $\mathcal{F}h,c(M(\text{Vir}))$ such as

or

The connection $\nabla$ maybe considered as an infinite dimensional analogue of the Gauss–Manin connection. the connection is not always flat. the condition of an absence of a curvature put a restrictions on values of parameters $d, c, h, e_\mu$.

THEOREM 8 \[72\]. The curvature tensor of the connection $\nabla$ in the bundle $\mathcal{F}h,c(M(\text{Vir}))$ is equal to

DEFINITION. The covariantly constant section of the bundle $\mathcal{F}h,c(M(\text{Vir}))$ is called the Bowick–Rajeev vacuum.

The Bowick–Rajeev vacuum exists if and only if $D = c, h = e^2/2$.

Define following \[72\] the family of parings

as

The pairings $B_{f_1,f_2} : ((\Omega^*\mathcal{F}B_{h,c}))^\otimes 2 \mapsto \mathbb{C}$ play a role of the Kostant–Blattner–Sternberg pairings \[72\]. Indeed, as it was shown in \[72\], if $c > 1, h > 0$ the connection $\nabla$ maybe defined in the following way

(\nabla_f \Phi = 0) \Leftrightarrow (\Phi_{f,t}X_f, f(\Phi, \Psi) - B_{f,f}(\Phi, \Psi) = o(t), \forall \Psi).
DEFINITION. A covariantly constant with respect to $\nabla^*$ element of the space $\Omega_{\text{BP}}^\text{SI}(E_{h,c}^*)^*$ of the string fields is called a gauge–invariant string field; the space of such fields will be denoted by $\Omega_{\text{BP}}^\text{SI}(E_{h,c}^*)^*_{\text{GI}}$.

the space of the gauge–invariant string fields is dual to the space of the Bowick–Rajeev vacuums.

**THEOREM 9A** [72]. The space of the gauge–invariant string fields $\Omega_{\text{BP}}^\text{SI}(E_{h,c}^*)^*_{\text{GI}}$ is invariant under the Virasoro algebra action. Let us identify the space $\Omega_{\text{BP}}^\text{SI}(E_{h,c}^*)^*_{\text{GI}}$ with $H_f(h,c)$ by the operator $I_f$. In $H_f(h,c)$ the Virasoro algebra generators act by operators $T_f(L_k)$. If $f = f_0$ then

$$T_{f_0}(L_k) = L_k^V + L_k^{\text{ghost}},$$

where $L_k^V$ are the generators in the Virasoro representation

$$L_p^V = \sum_{k \geq 1} (k + p)x_k^\mu \frac{\partial}{\partial x_{k+p}^\mu} + \frac{1}{2} \sum_{k=1}^{p-1} k(p - k) \frac{\partial^2}{\partial x_k^\mu \partial x_{p-k}^\mu} + pe^\mu \frac{\partial}{\partial x_p^\mu},$$

$$L_0^V = \sum_{k \geq 1} kx_k^\mu \frac{\partial}{\partial x_k^\mu} + \frac{e^2}{2},$$

$$L_{-p}^V = \sum_{k \geq 1} kx_k^\mu \frac{\partial}{\partial x_{k-p}^\mu} + \frac{1}{2} \sum_{k=1}^{p-1} x_k^\mu x_{p-k}^\mu + e^\mu x_p^\mu,$$

and $L_k^{\text{ghost}}$ are the generators of the Virasoro algebra in the ghost space

$$L_k^{\text{ghost}} = \sum_{q} (q - p)\xi_q^* \frac{\partial}{\partial \xi_{p+q}^*}.$$  

**THEOREM 9B** [72]. The action of the Virasoro algebra in $(\mathcal{O}(\mathcal{F}_{h,c}^*(\mathcal{M}(\text{Vir})))^*)^*$ has the form

$$L_K|_f = \nabla(L_k)|_f^* + T_f(L_k),$$

$$D = c, \ e^2 = 2h.$$  

It should be mentioned that the space $\Omega_{\text{BP}}^\text{SI}(E_{h,c}^*)^*$ of the gauge–invariant string fields is $Q^*$–invariant, where $Q^*$ is the conjugate Kato–Ogawa BRST–operator [69]. Let us define a fiber BRST–operator $Q_0^*$ as $Q_0^*|_{\Omega_{\text{BP}}^\text{SI}(E_{h,c}^*)^*_{\text{GI}}}$, where $\Omega_{\text{BP}}^\text{SI}(E_{h,c}^*)^*_{\text{GI}}$ is identified with $H(h,c)$ by $I_f^*.$

**THEOREM 9C** [72]. $Q^* = D_\nabla + Q_0^*$. The fiber BRST–operator $Q_0^*$ under the initial fiber has the form $Q_0^* = \sum_{p} (L_{-p}^V + \frac{1}{2} L_{-p}^{\text{ghost}})\xi_p^*.$

The Bowick–Rajeev formalism maybe expanded on a curved background: the gauges of the Banks–Peskin string differential forms, the projectors, which fix these gauges, the Connection $\nabla$, the Bowick–Rajeev vacua, the gauge–invariant string fields, the fiber BRST–operators can be introduced analogously to the flat case. The main problem in the case problem, which is not solved completely yet, is to characterise the values of parameters of the curved background for which the Bowick–Rajeev vacua and the gauge–invariant string fields exist. The differential equations on such parameters providing the presence of the gauge–invariant string fields are called the string Einshitein equations.

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