Odd–even staggering in octupole bands of actinides and rare 
earths: Systematics of “beat” patterns

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Abstract

“Beat” patterns are shown to appear in the octupole bands of several actinides and 
rare earths, their appearance being independent from the formula used in order to isolate 
and demonstrate them. It is shown that the recent formalism, making use of discrete 
approximations to derivatives of the transition energies (or of the energy levels) gives results 
consistent with the traditional formulae. In both regions it is seen that the first vanishing 
of the staggering occurs at higher values of the angular momentum I in nuclei exhibiting 
higher staggering at low I. Since these nuclei happen to be good rotators, the observed slow 
decrease of the amplitude of the staggering with increasing I is in good agreement with 
the parameter independent predictions of the su(3) (rotational) limit of several algebraic 
models. In the actinides it has been found that within each series of isotopes the odd–even 
staggering exhibits minima at N = 134 and N = 146, while a local maximum is shown
at $N = 142$, these findings being in agreement with the recent suggestion of a secondary maximum of octupole deformation around $N = 146$.

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Short title: Odd–even staggering in octupole bands

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1 Introduction

Negative parity bands with angular momenta $I^\pi = 1^-, 3^-, 5^-, 7^-, \ldots$, merging with the ground state bands (characterized by $I^\pi = 0^+, 2^+, 4^+, 6^+, \ldots$) into forming a single band with the level sequence $I^\pi = 0^+, 1^-, 2^+, 3^-, 4^+, 5^-, \ldots$ have been known to exist in the light actinides [1, 2], and in particular in the Rn, Ra, and Th isotopes with neutron number $N = 130-138$ [3, 4], as well as in the rare earth region, and in particular in the Ba–Sm isotopes [5, 6, 7, 8] for a long time. These bands have been interpreted as corresponding to reflection asymmetric shapes, i.e. as indicating the existence of permanent octupole deformation [3, 4, 9, 10] of the corresponding nuclei, although alternative interpretations of these bands in terms of alpha clustering have been given [11, 12, 13, 14].

An interesting quantity characterizing the structure of bands with level sequence $I^\pi = 0^+, 1^-, 2^+, 3^-, \ldots$ is the relative displacement of the odd levels with respect to the positions predicted for them by a fitting of the even levels. This quantity, usually called odd–even staggering or $\Delta I = 1$ staggering, should vanish if the even levels and the odd levels form a single band.

Traditionally it has been thought that the odd–even staggering in octupole deformed bands starts from a relatively high value at low $I$ and then gradually decreases down to zero, thus indicating the gradual formation of a band with a reflection asymmetric shape [9, 4, 13, 15]. However, using recent data in the actinide region [16, 17], it has been found that in the light actinides the odd–even staggering exhibits a “beating” behavior [18]. In other words, the quantity measuring the odd–even staggering does not stay near a vanishing value after reaching zero for the first time, but continues to raise and fall (in absolute value) with increasing $I$, producing a figure resembling a sequence of beats.

At this point the following questions are raised:

1) Is there a “beating” behavior of the odd–even staggering in octupole bands of the rare earth region, similar to the one seen in the light actinides?
2) If there is such a behavior, what are the systematics exhibited by these “beats”?

3) Is the “beating” behavior a real effect or is it an artifact of the mathematical method used?

Furthermore, it has been recently argued [19, 20] that in the actinides a second region of octupole deformation exists around the nuclei $^{238}$Pu and $^{240}$Pu, i.e. around $N = 144$, 146. It is therefore of interest to study the systematics of odd–even staggering up to and around this region, since the following questions are raised:

1) Is there any systematic dependence of the odd–even staggering on the neutron number $N$ within each chain of isotopes?

2) Is there any correlation between the amplitude of the odd–even staggering and the angular momentum value at which the vanishing of the staggering occurs?

In section 2 of the present work the formalism will be briefly described and it will be subsequently applied to a few examples in the actinide region in section 3. In section 4 the consistency of several staggering formulae among themselves, as well as with the traditional formalism will be considered. The systematics of odd–even staggering in the actinide region will be examined in section 5, while section 6 will contain examples of “beats” in the rare earth region. The systematics of the odd–even staggering in the rare earth region will be considered in section 7, while section 8 will contain the conclusions drawn from the present results and plans for future work.

2 Formalism

The odd–even staggering (or $\Delta I = 1$ staggering) can be measured by the quantity [15]

$$\Delta E_{1,\gamma}(I) = \frac{1}{16}(6E_{1,\gamma}(I) - 4E_{1,\gamma}(I - 1) - 4E_{1,\gamma}(I + 1) + E_{1,\gamma}(I - 2) + E_{1,\gamma}(I + 2)), \quad (1)$$

where the transition energies

$$E_{1,\gamma}(I) = E(I + 1) - E(I). \quad (2)$$
are determined directly from experiment. This expression is analogous to the one used recently for the study of $\Delta I = 2$ staggering $[21, 22]$ in nuclear superdeformed bands. One of the main properties of eq. (1) is that it vanishes for

$$E(I) = E_0 + AI(I + 1) + B[I(I + 1)]^2,$$

(3)

since it is a (normalized) discrete approximation of the fourth derivative of the function $E_{1,\gamma}(I)$ $[23]$, i.e., essentially the fifth derivative of the function $E(I)$. Therefore eq. (1) is a sensitive probe of deviations from rotational behavior.

3 Examples of “beats” in the actinide region

A couple of examples of staggering plots (for the nuclei $^{220}$Ra and $^{224}$Ra) are shown in fig. 1 for illustrative purposes. More examples of staggering plots for several Rn, Ra, and Th isotopes can be found in ref. [18]. The following comments are in place:

a) The term “staggering” is justified by the fact that the quantity $\Delta E_{1,\gamma}(I)$ exhibits values of alternating sign over extended regions of the angular momentum $I$.

b) Odd–even staggering starts from relatively high values and is then decreasing with increasing angular momentum $I$. After reaching a vanishing value, staggering starts raising and then dropping again, giving an overall picture of “beats”, which has been discussed in ref. [18].

c) When the staggering reaches a vanishing value, a phase change occurs. In other words, while within the first “beat” positive values of staggering occur at even $I$ and negative values of staggering occur at odd $I$, indicating that in this region of $I$ values the odd levels are raised in relation to the corresponding even levels, within the second “beat” the opposite picture holds, i.e. positive values of staggering occur at odd $I$ and negative values of staggering occur at even $I$, indicating that in this region the odd levels are depressed in relation to the corresponding even levels. This phase change has been explained by the simple model of section V of ref. [18]. It should be noticed that this phase
change bears great similarity to the phase change occurring with signature inversion \[23, 26\] in odd–odd nuclei.

d) Because of the phase change mentioned in c), when staggering reaches zero, two consecutive staggering points with the same sign (both positive or both negative) occur. In what follows we are going to use the angular momenta of these two consecutive points in order to indicate the region at which vanishing staggering occurs, labelling them by \( I_{\text{van}} \).

e) In fig. 1 the amplitude of the second “beat” appears to be smaller than the amplitude of the first “beat”. Similarly, the amplitude of the third “beat” appears to be smaller than the amplitude of the second “beat”. These observations are in rough agreement with the expectations \[1, 5\] in the early days of octupole deformed bands, that the staggering should decrease down to zero with increasing \( I \) and then stay at a vanishing value, since a single band with octupole deformation is then formed. The bands shown in fig. 1, as well as several additional bands exhibited in ref. \[18\], indicate that staggering does not stay at a vanishing value after reaching zero for the first time, but the additional “beats” have smaller and smaller amplitudes, in qualitative agreement with the early expectations.

4 Alternative formulae

At this point one might wonder to which extent the characteristics of the odd–even staggering seen in the previous section, as well as in ref. \[18\], depend on the selection of the specific formula given in eq. (1). Indeed, many other choices are possible.

Instead of the normalized discrete approximation of the fourth derivative of the function \( E_{1,\gamma}(I) \), used in eq. (1), one can use the normalized discrete approximation of the second derivative of the same function, calculated using the values of the function at three points \[23\]

\[
\Delta E_{23,\gamma}(I) = \frac{1}{4}(2E_{1,\gamma}(I) - E_{1,\gamma}(I - 1) - E_{1,\gamma}(I + 1)),
\]

(4)
or the normalized discrete approximation of the second derivative of the function \( E_{1,\gamma}(I) \), calculated using the values of the function at five points \[23\]

\[
\Delta E_{25,\gamma}(I) = \frac{1}{64} (30E_{1,\gamma}(I) - 16E_{1,\gamma}(I - 1) - 16E_{1,\gamma}(I + 1) + E_{1,\gamma}(I - 2) + E_{1,\gamma}(I + 2)).
\] (5)

The formulae given above use the transition energies defined in eq. (2). It is possible, however, to use in such formulae directly the energy levels \( E(I) \). Before doing so, it is instructive to write eqs. (1), (4) and (5) in terms of the energy levels \( E(I) \), by substituting eq. (2) into them. The results are

\[
\Delta E_{1,\gamma}(I) = \frac{1}{16} [E(I + 3) - 5E(I + 2) + 10E(I + 1) - 10E(I) + 5E(I - 1) - E(I - 2)],
\] (6)

\[
\Delta E_{23,\gamma}(I) = \frac{1}{4} [-E(I + 2) + 3E(I + 1) - 3E(I) + E(I - 1)],
\] (7)

\[
\Delta E_{25,\gamma}(I) = \frac{1}{64} [E(I + 3) - 17E(I + 2) + 46E(I + 1) - 46E(I) + 17E(I - 1) - E(I - 2)].
\] (8)

We proceed now to use in the derivative formulae the energy levels \( E(I) \) in the place of the transition energies \( E_{1,\gamma}(I) \). In this case the normalized discrete approximation of the fourth derivative of the function \( E(I) \), calculated using the values of the function at five points \[23\] will be

\[
\Delta E_{45,E}(I) = \frac{1}{8} (6E(I) - 4E(I - 1) - 4E(I + 1) + E(I - 2) + E(I + 2)),
\] (9)

which is the analog of eq. (1) in the case in which we use directly the energy levels \( E(I) \) and not the transition energies \( E_{1,\gamma}(I) \). The difference in the normalization factor (by a factor of 2) is justified by comparison to eq. (6), which contains the same quantity as eq. (1), but expressed in terms of the energy levels \( E(I) \). The choice made here is that in both cases the numerator contains a number of terms which is twice the number appearing in the denominator. This choice guarantees that the accumulation of errors of the quantities given in eqs. (6) and (9) will be handled on equal footing.
In a similar manner one can use the normalized discrete approximation of the second derivative of the function \( E(I) \), calculated using the values of the function at three points

\[
\Delta E_{23,E}(I) = \frac{1}{2}(2E(I) - E(I - 1) - E(I + 1)),
\]

which is similar to eq. (4), the normalization being fixed by comparison to eq. (7), or the normalized discrete approximation of the second derivative of the function \( E(I) \), calculated using the values of the function at five points

\[
\Delta E_{25,E}(I) = \frac{1}{32}(30E(I) - 16E(I - 1) - 16E(I + 1) + E(I - 2) + E(I + 2)),
\]

which is an analog of eq. (5), the normalization being fixed by comparison to eq. (8).

The notation used above is of the type \( \Delta E_{ij} \), where by \( i \) we denote the order of the derivative, while by \( j \) we denote the number of values of the function used for the calculation of the derivative each time. According to this notation the quantity of eq. (1) could have been denoted as \( \Delta E_{45,\gamma}(I) \), but we opted for keeping for it the previously used notation \( \Delta E_{1,\gamma}(I) \).

It should be noticed at this point that eq. (10) is equivalent to the expression

\[
\delta E(I) = E(I) - \frac{1}{2}[E(I - 1) + E(I + 1)],
\]

which has been used in staggering studies in the past in octupole bands and gamma bands, as well as in bands of odd–odd nuclei in connection with the effect of signature inversion.

In addition it can be noticed that eq. (7) is equivalent (up to a factor of 2) to the expression

\[
\Delta E_S(I) = [E(I + 1) - E(I)] - \frac{E(I + 2) - E(I + 1) + E(I) - E(I - 1)}{2}
\]

\[
= \frac{1}{2}[-E(I + 2) + 3E(I + 1) - 3E(I - 1) + E(I - 2)],
\]

which has been used in staggering studies in the past in octupole bands and gamma bands, as well as in bands of odd–odd nuclei in connection with the effect of signature inversion.
which has also been used in bands of odd–odd nuclei in relation with the effect of signature inversion \[26\]. The similarity of the phase change occurring in the present case, when a “beat” is completed and the next one is starting, to the phase change occurring at signature inversion in odd–odd nuclei has been already mentioned in comment c) of sec. 3.

Since the early days of octupole deformation, the expression \[15\]
\[
\Delta E_{NO}(I) = E(I) - \frac{(I + 1)E(I - 1) + IE(I + 1)}{2I + 1} \tag{14}
\]
has been widely used. This expression does not correspond to any derivative, but it has the property to vanish for
\[
E(I) = E_0 + AI(I + 1), \tag{15}
\]
giving in this way a measure of deviations from pure rotational behavior.

From the theoretical point of view the main differences of the formulae given above can be described by considering the energy expression
\[
E(I) = E_0 + A'I + AI(I + 1) + B[I(I + 1)]^2. \tag{16}
\]
The following comments can be made:

1) The expression \(\Delta E_{1,\gamma}(I)\) “kills” all terms of eq. (16), i.e. it vanishes if eq. (16) is substituted in it, while the expression \(\Delta E_{45,E}(I)\) gives a constant coming from the last term of eq. (16). This is expected, since \(\Delta E_{45,E}(I)\) corresponds to the fourth derivative of the energies \(E(I)\), while \(\Delta E_{1,\gamma}(I)\) corresponds to the fourth derivative of the transition energies of eq. (2), i.e. essentially to the fifth derivative of \(E(I)\), as seen in eqs. (7) and (8).

2) The expressions \(\Delta E_{23,E}(I)\) and \(\Delta E_{25,E}(I)\) give a constant coming from the third term when the first three terms of eq. (16) are substituted in them, since they correspond to the second derivative of \(E(I)\). The expressions \(\Delta E_{23,\gamma}(I)\) and \(\Delta E_{25,\gamma}(I)\) completely “kill” the first three terms of eq. (16), since they correspond to the second derivative of the transition energies of eq. (2), i.e. essentially to the third derivative of \(E(I)\), as seen in eqs. (7) and (8).
3) The expression $\Delta E_{NO}(I)$ “kills” the first and the third terms of eq. (16), but not the second one (which is of vibrational character), as one can check by simple substitution. The main difference between $\Delta E_{NO}(I)$ and the other quantities is that $E_{NO}(I)$ does not correspond to any derivative.

In order to demonstrate the similarities and differences between these formulae, we give in table 1 detailed results from their application in the set of data of $^{220}$Ra, a nucleus in which the third “beat” is reached. The following observations can be made:

1) $\Delta E_{23,\gamma}(I)$ and $\Delta E_{25,\gamma}(I)$ give results which are almost identical, a fact expected, since both of these quantities represent the second derivative of the transition energies, calculated using the values of the function at a different number of points in each case. The results given by $\Delta E_{1,\gamma}(I)$ are also very similar to the ones just mentioned, indicating that the fourth derivative gives almost the same results as the second derivative.

2) $\Delta E_{23,E}(I)$ and $\Delta E_{25,E}(I)$ give almost identical results, which in addition are very similar to the results given by $\Delta E_{45,E}(I)$, indicating once more that the selection of the order of the derivative (second or fourth), as well as of the number of the values of the function used at each step of the calculation of the derivative (three or five) is not important.

3) In table 1 the quantity $\Delta E_{1,\gamma}(I)$ is accompanied by the relevant errors, calculated from the errors of the experimental data [24]. It is clear that the errors are small in comparison to the magnitude of $\Delta E_{1,\gamma}(I)$. The errors for the rest of the quantities in table 1 are of similar magnitude and therefore are not reported.

4) The overall sign difference between the quantities $\Delta E_{1,\gamma}(I)$, $\Delta E_{23,\gamma}(I)$, $\Delta E_{25,\gamma}(I)$ and the rest is due to the choice made in writing eq. (2), which is used in them. Using $E_{1,\gamma}(I) = E(I) - E(I + 1)$ instead of eq. (2), one would have obtained for these quantities the same results with opposite signs.

In order to check further the similarities and differences between the various results, we have plotted in fig. 2a a representative of case 1) ($\Delta E_{1,\gamma}(I)$), in which the transition energies are used, a representative of case 2) ($\Delta E_{45,E}(I)$), in which the energy levels are
used, and $\Delta E_{NO}(I)$, which does not correspond to any derivative. The following comments can be made:

1) All curves have exactly the same shape. Errors are not shown, since they are smaller than the size of the symbols used for indicating the points, as seen in table 1. The sign difference has been explained in comment 4) above.

2) In all cases the curves cross the zero axis at the same angular momentum values. This can also be seen in table 1, since the crossing of the zero corresponds, as explained above, to a phase change, i.e. to two consecutive values with the same sign.

3) The traditional formula of eq. (14) [15] produces a staggering pattern very similar to the more recent formula of eq. (1) [18, 21, 22]. In early work [1, 2, 15] the plots had been including the values of $\Delta E_{NO}(I)$ at odd $I$ only, thus producing a curve going down to zero with increasing $I$, instead of the first “beat”. In some cases, in which more experimental data was available, the curve was continued with negative values for a few more odd values of $I$, corresponding to the beginning of the second “beat” in the current presentation. In other words, the first three “beats” seen in fig. 2a in the current presentation correspond in the case of the traditional description (using the formula of eq. (14) and plotting only the values at odd $I$) to a curve decreasing from positive values to negative values and then raising to positive values again.

4) The previous point clarifies that staggering is not an artifact constructed through the use of derivatives, but it is corresponding to a physical situation: Within the first and the third “beat” the negative parity band is lying higher than the position corresponding to an interpolation of the levels of the positive parity band, while within the second “beat” the negative parity band is lying lower than the position corresponding to the interpolation of the levels of the positive parity band. What the derivatives do, is to isolate and demonstrate this effect in a clear way. It is remarkable that eq. (14), which is not a derivative, performs this task equally well.
In fig. 2b the same quantities are plotted in the case of $^{224}$Ra, leading to the same conclusions. Similar results have been obtained for the rest of the nuclei used in ref. [18], as well as for the nuclei used in the rest of the present work.

From the above results the following conclusions can be drawn:

a) The staggering pattern does not depend on the selection of the derivative (second or fourth), or on the number of values of the function used each time for the calculation of the derivative (three or five).

b) The staggering pattern does not depend on the use of the energy levels or of the transition energies in the relevant formulae.

c) The staggering pattern is reconciled with the traditional description (using eq. (14) and plotting values only at odd $I$). It is related to the relative position of the levels of the negative parity band with respect to the positions they should have had according to an interpolation of the levels of the positive parity band.

d) The physical question remaining is why in the beginning (within the first “beat”, at low $I$) the negative parity band lies higher than where it should have been, later (within the second “beat”) it lies lower than where it should have been, while at even higher angular momentum $I$ (within the third “beat”) it lies again higher than where it should have been.

5 Systematics of odd–even staggering in the actinide region

In table 2 the first two values of the quantity of eq. (1), i.e. $\Delta E_{1,\gamma}(3)$ and $\Delta E_{1,\gamma}(4)$, are reported for series of Rn, Ra, Th, U, and Pu isotopes, along with the pairs of angular momentum values at which the first vanishing of the odd–even staggering occurs, as explained in comment d) of section 3. The bands used in the table are free from backbendings/upbendings [39]. The following observations can be made:

a) The Rn and Ra series of isotopes clearly show that minimum staggering occurs at $N = 134$, the Th series of isotopes also being consistent with this observation.
b) The U series of isotopes clearly demonstrates that maximum staggering occurs at $N = 142$, the Th and Pu series of isotopes being consistent with this finding.

c) The Pu series of isotopes clearly exhibit minimum staggering at $N = 146$, the U series of isotopes being consistent with this finding.

We conclude that low staggering (i.e. strong octupole correlations) is observed around $N = 134$ (the well known area of octupole deformation in the light actinides [3, 4]) and around $N = 146$ (the recently suggested region of octupole deformation in heavier actinides [19, 20]), while relatively high staggering (weak octupole correlations) is observed around $N = 142$. These findings are consistent with the suggestions of ref. [20], which have been based on systematics of the energies of the lowest $1^-$ states, as well as on systematics of the hindrance factors for alpha decay.

The data in table 2 suggest in addition that the lower the initial staggering (i.e. the staggering at $I = 3, 4$), the lower the angular momentum value at which the staggering vanishes. However, this conclusion is based only on some of the Rn, Ra, and Th isotopes, for which the vanishing of the staggering has been reached experimentally, while in the Pu and U isotopes, as well as in several of the heavier Th isotopes, the vanishing of the staggering has not been reached experimentally yet.

The suggestion that the lower the initial staggering, the lower the angular momentum value at which the staggering vanishes, is also supported by fig. 3, in which the odd–even staggering for the Th and U series of isotopes, calculated from eq. (1), is shown. It is clear that the curves enveloping the staggering pattern of each nucleus (not drawn in the figures for reasons of clarity) do not cross each other, suggesting in this way that the nucleus possessing the lowest initial staggering will reach vanishing staggering first (i.e. at lower angular momentum), while the nucleus possessing the highest initial staggering will reach vanishing staggering last (i.e. at higher angular momentum), provided that no backbending [39] will appear meanwhile. A similar conclusion can be drawn for the Pu series of isotopes from fig. 2 of Ref. [19], where eq. (14) has been used in the traditional way.
In table 3 the $\Delta E_{1,\gamma}(I)$ values for five of the nuclei shown in fig. 3 are given as examples, together with their uncertainties, calculated from the experimental errors accompanying the data on the energy levels. It is clear that in all cases the uncertainties are small, i.e. smaller than the size of the symbols used in fig. 3.

The following comments are here in place:

a) In table 2 it is clear that the nuclei for which the first vanishing of the staggering has been reached experimentally, and therefore the relevant $I_{\text{van}}$ values are reported in the table, are vibrational (with $R_4 = E(4)/E(2)$ ratios in the range $2 \leq R_4 \leq 2.4$), transitional (characterized by $2.4 \leq R_4 \leq 3$), or rotational ($3 \leq R_4 \leq 10/3$) but with $R_4$ up to roughly $3.14$. In contrast, the Ra, Th, U, and Pu isotopes for which long bands are known, but the first vanishing of the staggering has not been reached yet, are all rotational, with $R_4 > 3.2$. It seems therefore that the decrease of the staggering with increasing $I$ is in general slower in good rotors.

b) As a result of a), the good rotors shown in fig. 3 give the impression that their staggering is in rough agreement with the predictions of the su(3) (rotational) limit of several algebraic models, as the spf-Interacting Boson Model [3], the spdf-Interacting Boson Model [9, 10], the Vector Boson Model [40, 41, 42], the Nuclear Vibron Model [11], the details of which have been described in ref. [18] and need not be repeated here. All these models in their su(3) limits predict staggering of constant amplitude, the predictions being parameter independent, thus providing quite strict tests for the validity of these models. Fig. 3 shows that the above mentioned algebraic models pass this parameter independent test quite successfully.

As far as the “beat” patterns are concerned, it should be mentioned that quite satisfactory results can be obtained within the recently suggested [3, 4] extension to negative parity states of the Coherent State Model [13, 14]. Theoretical predictions for the octupole bands of $^{218-222}$Rn and $^{218-226}$Ra have been given in refs [3, 4], by fitting the relevant Hamiltonian, which possesses 7 free parameters, to the experimental data. Plugging the
theoretical predictions of refs [43, 44] into eq. (1) one can easily see that in all of the just mentioned Rn and Ra isotopes the first vanishing of the staggering is reproduced correctly. Furthermore the second vanishing of the staggering is reproduced correctly in $^{224}\text{Ra}$, while in $^{220}\text{Ra}$ and $^{226}\text{Ra}$ it is not reproduced by the specific fits given in refs [43, 44]. However it is interesting to examine if this can be achieved with different parameter sets.

6 Examples of “beats” in the rare earth region

After discussing in some detail the actinide region, we now turn our attention to the rare earth region. Staggering plots for 4 nuclei for which staggering reaches a vanishing value and goes on beyond it are shown in fig. 4. The following comments apply:

a) The staggering patterns seen here resemble the ones seen in sec. 3, but they are shorter. In $^{150}\text{Sm}$ (fig. 4(c)) the second “beat” is almost complete, in $^{144}\text{Ba}$ (fig. 4(a)) about half of the second “beat” is seen, while in $^{146}\text{Ba}$ (fig. 4(b)) and $^{154}\text{Dy}$ (fig. 4(d)) only the beginning of the second “beat” is observed.

b) The best “beat” patterns are seen in the $N = 88$ isotones ($^{144}\text{Ba}$, $^{150}\text{Sm}$, $^{154}\text{Dy}$), which are known to be among the best examples of nuclei showing octupole deformation in the rare earth region [3].

In table 4 the $\Delta E_{1,2}(I)$ values for $^{150}\text{Sm}$ are given as an example, together with their uncertainties, as calculated from the experimental errors accompanying the data on the energy levels. It is clear that the uncertainties are smaller than the size of the symbols used in fig. 4.

7 Systematics of odd–even staggering in the rare earth region

We proceed to a study of the systematics of odd–even staggering in the rare earth region, similar to the one given in sec. 5 for the actinides.
In fig. 5 several Sm, Gd, Dy, and Er isotopes, for which the first vanishing of the staggering is not reached within the angular momentum region observed, are shown. In all cases it is clear that the curves enveloping the staggering patterns of the various isotopes (not shown in the figure for reasons of clarity) do not cross each other. In other words, the higher the initial staggering (at low \( I \)), the higher the angular momentum value at which the first vanishing of the staggering will occur (provided that no backbending \[39\] will occur meanwhile). This is in agreement with what we have seen in sec. 5 for the actinides.

In table 4, which is an analog of table 3, the \( \Delta E_{1,\gamma}(I) \) values for five of the nuclei shown in fig. 5 are given as examples, together with their uncertainties, as calculated from the experimental errors accompanying the data on the energy levels. It is clear that in all cases the uncertainties are smaller than the size of the symbols used in fig. 5.

In table 5, which is an analog of table 2, the first two values of the quantity of eq. (1), i.e. \( \Delta E_{1,\gamma}(3) \) and \( \Delta E_{1,\gamma}(4) \), are reported for several Ba, Ce, Nd, Sm, Gd, Dy, and Er isotopes (including all the isotopes appearing in figs 4 and 5), along with the pairs of angular momentum values at which the first vanishing of the odd–even staggering occurs, as explained in comment d) of sec. 3. The bands used in table 5 (as well as in figs 4 and 5) are free from backbendings \[39\] at least within the angular momentum regions used. The following observations can be made:

a) The Nd, Sm, and Gd series of isotopes show that minimum staggering occurs at \( N = 86 \), the Dy series of isotopes also showing in the same direction.

b) The Ba, Nd, and Sm series of isotopes show that the lower the initial staggering (i.e. the staggering at \( I = 3, 4 \)), the lower the angular momentum value at which the staggering vanishes. This observation is consistent with the suggestions of fig. 5, which are based on Sm, Gd, Dy, and Er isotopes.

c) The nuclei for which the first vanishing of the staggering is reached (and therefore the relevant \( I_{\text{van}} \) values are reported in table 5) are either vibrational (characterized by \( R_4 = E(4)/E(2) \) ratios in the range \( 2 \leq R(4) \leq 2.4 \)) or transitional (having \( 2.4 \leq R(4) \leq 3 \)),

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while most of the nuclei shown in fig. 5, for which the first vanishing of the staggering is not reached up to quite high $I$, are purely rotational, having $3 \leq R(4) \leq 10/3$ (with the notable exception of $^{152}$Gd).

d) Furthermore, the rotational nuclei included in fig. 5 show a slow decrease of the amplitude of the staggering with increasing $I$, being therefore quite in agreement with several algebraic models [9, 10, 11, 40, 41, 42], which in their su(3) (rotational) limits predict staggering of constant amplitude. The predictions of these models (spf-Interacting Boson Model [9], spdf-Interacting Boson Model [3, 10], Vector Boson Model [40, 41, 42], Nuclear Vibron Model [11]) have been described in detail in ref. [18] and need not be repeated here. It is worth mentioning, however, that these predictions of the just mentioned models are parameter independent, thus providing a quite strict test of the validity of the models, which they pass quite successfully.

e) The observations discussed in comments b), c), and d) are similar to the ones made in the actinide region (see sec. 5).

It will be interesting to examine if the recently proposed [43, 44] extension to negative parity states of the Coherent State Model [45, 46], which has been quite successful in describing the “beat” patterns in the actinide region, as already mentioned in sec. 5, can also reproduce the “beat” patterns in the rare earth region.

8 Conclusion

In this paper the odd–even staggering in the octupole bands of several even-even actinides and rare earths and the appearance of “beat” patterns have been considered. The main conclusions are listed here.

a) “Beat” patterns, similar to the ones reported in ref. [18] for the light actinides, are seen in both the actinide and the rare earth regions.

b) The odd–even staggering and the “beat” patterns do not depend on the choice of the formula used for isolating and demonstrating the effect. Several different formulae,
corresponding to discrete approximations of derivatives of the transition energies, or of the energy levels themselves, give results which are perfectly consistent among themselves.

c) The recently used formalism, which makes use of discrete approximations of derivatives, gives results perfectly consistent with the ones provided by the traditional formula of Nazarewicz and Olanders [15].

d) In both the actinide and the rare earth regions it is seen that the higher the initial staggering (i.e. the staggering at low angular momentum $I$), the higher the angular momentum value at which the staggering will reach a vanishing value.

e) In both regions it is also seen that in vibrational and transitional nuclei the first vanishing of the staggering occurs soon, while in rotational nuclei the first vanishing of the staggering is not reached up to much higher angular momentum $I$, giving the impression of staggering with slowly decreasing amplitude, in rough agreement with several algebraic models predicting staggering of constant amplitude in their su(3) (rotational) limits.

f) In both regions it is also seen that the amplitude of the second “beat” appears to be smaller than the amplitude of the first “beat”. In the actinides it is further observed that subsequent “beats” have even smaller amplitudes. These observations are in rough agreement with the early expectations that after the formation of a single band with octupole deformation is reached as $I$ is increasing, the staggering from there on should remain at or near a vanishing value.

g) In the actinide region, in which several Rn, Ra, Th, U, and Pu isotopes have been considered, it has been found that within each series of isotopes the odd–even staggering exhibits minima around $N = 134$ and $N = 146$, while a local maximum appears at $N = 142$, in agreement with the recent suggestion [19, 20] of a secondary maximum of octupole deformation around $N = 142$.

h) In the rare earth region, minimum staggering occurs in the $N = 86, 88$ isotones, which show the best octupole bands and also the best “beat” patterns in this region.
The simple model of ref. [18] explains the main features of staggering, except the decrease of the maximum amplitude of subsequent “beats” with increasing $I$, which remains an open problem, along with a microscopic interpretation of the “beat” patterns in both the rare earth and the actinide regions.

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Figure captions

Fig. 1 $\Delta E_{1,\gamma}(I)$ (in keV), calculated from eq. (1), for octupole bands of a) $^{220}$Ra [24] and b) $^{224}$Ra [16, 17]. The experimental error in all cases is smaller than the symbol used for the points and therefore is not shown. (See table 1 for a list of the errors in the case of $^{220}$Ra.)

Fig. 2 Odd–even staggering quantities $\Delta E_{1,\gamma}(I)$ (from eq. (1), in keV), $\Delta E_{45,\gamma}(I)$ (from eq. (9), in keV), and $\Delta E_{NO}(I)$ (from eq. (14), in keV) for octupole bands of a) $^{220}$Ra [24], b) $^{224}$Ra [16, 17].

Fig. 3 Same as fig. 1, but for a) $^{226}$Th [29], $^{228}$Th [30], $^{230}$Th [17], $^{232}$Th [17], $^{234}$Th [17], b) $^{230}$U [31], $^{232}$U [32], $^{234}$U [33], $^{236}$U [34], $^{238}$U [35].

Fig. 4 Same as fig. 1, but for octupole bands of a) $^{144}$Ba [17], b) $^{146}$Ba [17], c) $^{150}$Sm [31], and d) $^{154}$Dy [33].

Fig. 5 Same as fig. 1, but for a) $^{152}$Sm [32], $^{154}$Sm [33], b) $^{152}$Gd [32], $^{154}$Gd [33], $^{156}$Gd [34], $^{160}$Gd [36], c) $^{156}$Dy [37], $^{162}$Dy [38], d) $^{162}$Er [38], $^{164}$Er [39].
Table 1: Odd–even staggering quantities $\Delta E_{1,\gamma}(I)$ (eq. (1), column 2), $\Delta E_{23,\gamma}(I)$ (eq. (4), column 4), $\Delta E_{25,\gamma}(I)$ (eq. (5), column 5), $\Delta E_{45,E}(I)$ (eq. (9), column 6), $\Delta E_{23,E}(I)$ (eq. (10), column 7), $\Delta E_{25,E}(I)$ (eq. (11), column 8), and $\Delta E_{NO}(I)$ (eq. (14), column 9), all in keV, listed with increasing angular momentum $I$ (column 1) for the nucleus $^{220}$Ra. Experimental data have been taken from ref. [24]. In column 3 the uncertainties of $\Delta E_{1,\gamma}(I)$, occurring from the uncertainties of the experimental data given in ref. [24] are listed, in keV.

| $I$ | $\Delta E_{1,\gamma}$ | error | $\Delta E_{23,\gamma}$ | $\Delta E_{25,\gamma}$ | $-\Delta E_{45,E}$ | $-\Delta E_{23,E}$ | $-\Delta E_{25,E}$ | $-\Delta E_{NO}$ |
|-----|---------------------|-------|------------------------|------------------------|-------------------|-------------------|-------------------|-------------------|
| 2   |                     |       |                        |                        | 222.5             | 265.1             | 259.0             |                   |
| 3   | -165.5              | 0.5   | -162.2                 | -163.0                 | -192.3            | -179.9            | -183.0            | -196.4            |
| 4   | 117.0               | 0.6   | 115.1                  | 115.6                  | 138.6             | 144.4             | 143.0             | 135.5             |
| 5   | -78.4               | 0.7   | -75.8                  | -76.4                  | -95.4             | -85.7             | -88.1             | -98.4             |
| 6   | 48.2                | 0.7   | 47.1                   | 47.3                   | 61.4              | 65.8              | 64.7              | 56.6              |
| 7   | -24.3               | 0.8   | -22.8                  | -23.2                  | -34.9             | -28.4             | -30.0             | -38.8             |
| 8   | 5.3                 | 0.8   | 4.5                    | 4.7                    | 13.7              | 17.2              | 16.3              | 8.6               |
| 9   | 9.5                 | 0.9   | 10.5                   | 10.2                   | 3.0               | 8.2               | 6.9               | -0.8              |
| 10  | -21.3               | 1.0   | -21.8                  | -21.7                  | -16.1             | -12.8             | -13.6             | -20.7             |
| 11  | 30.5                | 1.0   | 31.1                   | 30.9                   | 26.4              | 30.9              | 29.7              | 22.8              |
| 12  | -37.7               | 1.0   | -38.0                  | -37.9                  | -34.6             | -31.3             | -32.1             | -38.7             |
| 13  | 43.2                | 1.0   | 43.7                   | 43.6                   | 40.9              | 44.8              | 43.8              | 37.4              |
| 14  | -47.0               | 1.0   | -47.4                  | -47.3                  | -45.5             | -42.6             | -43.3             | -49.5             |
| 15  | 49.1                | 1.1   | 49.7                   | 49.5                   | 48.5              | 52.1              | 51.2              | 45.4              |
| 16  | -49.5               | 1.2   | -49.9                  | -49.8                  | -49.8             | -47.2             | -47.8             | -53.7             |
| 17  | 48.0                | 1.2   | 48.6                   | 48.5                   | 49.3              | 52.6              | 51.8              | 46.3              |
| 18  | -44.4               | 1.2   | -44.9                  | -44.8                  | -46.8             | -44.6             | -45.1             | -50.7             |
| 19  | 38.7                | 1.2   | 39.3                   | 39.1                   | 42.1              | 45.3              | 44.5              | 39.4              |
| 20  | -31.0               | 1.2   | -31.2                  | -31.2                  | -35.3             | -33.4             | -33.8             | -39.2             |
| 21  | 21.9                | 1.3   | 22.2                   | 22.1                   | 26.7              | 29.1              | 28.5              | 23.6              |
| 22  | -12.1               | 1.4   | -12.1                  | -12.1                  | -17.1             | -15.3             | -15.7             | -20.8             |
| 23  | 1.9                 | 1.4   | 2.1                    | 2.0                    | 7.1               | 9.0               | 8.5               | 3.8               |
| 24  | 8.6                 | 1.4   | 8.5                    | 8.5                    | 3.2               | 4.8               | 4.4               | -0.4              |
| 25  | -19.5               | 1.4   | -19.4                  | -19.4                  | -13.9             | -12.2             | -12.6             | -17.0             |
| 26  | 30.8                | 1.5   | 30.8                   | 30.8                   | 25.1              | 26.6              | 26.2              | 21.6              |
| 27  | -41.7               | 1.6   | -42.1                  | -42.0                  | -36.4             | -35.1             | -35.4             | -39.7             |
| 28  | 51.7                | 1.7   | 51.9                   | 51.9                   | 47.0              | 49.1              | 48.5              | 44.3              |
| 29  | -61.0               |       | -56.5                  | -54.8                  | -55.2             | -59.2             |                   |                   |
| 30  |                     |       |                        |                        |                   |                   | 67.3              |                   |
Table 2: Odd–even staggering quantities $\Delta E_{1,\gamma}(3)$ (column 3) and $\Delta E_{1,\gamma}(4)$ (column 4), in keV, calculated from eq. (1), for octupole bands of several actinides (listed in column 1). The $R_4 = E(4)/E(2)$ ratios of these nuclei are given in column 6. The couple of angular momentum values at which the first vanishing of the staggering occurs (see text for further discussion) is indicated by $I_{\text{van}}$ (column 5), in the cases in which it is known experimentally. Data have been taken from the references indicated in column 2.

| nucleus | Ref. | $\Delta E_{1,\gamma}(3)$ | $\Delta E_{1,\gamma}(4)$ | $I_{\text{van}}$ | $R_4$ |
|---------|------|--------------------------|--------------------------|-----------------|-------|
| $^{218}$Rn$_{132}$ | [16,17] | -322.2 | 241.1 | | 2.014 |
| $^{220}$Rn$_{134}$ | [16,17] | -257.6 | 188.5 | 10, 11 | 2.214 |
| $^{222}$Rn$_{136}$ | [16,17] | -296.9 | 230.3 | 10, 11 | 2.408 |
| $^{220}$Ra$_{132}$ | [24] | -165.5 | 117.0 | 8, 9 | 2.298 |
| $^{222}$Ra$_{134}$ | [16,17] | -103.8 | 71.8 | 7, 8 | 2.715 |
| $^{224}$Ra$_{136}$ | [16,17] | -117.1 | 89.5 | 8, 9 | 2.970 |
| $^{226}$Ra$_{138}$ | [16,17] | -177.0 | 152.3 | 12, 13 | 3.127 |
| $^{228}$Ra$_{140}$ | [17] | -397.1 | 369.4 | | 3.207 |
| $^{224}$Th$_{134}$ | [28] | -110.0 | 78.7 | 8, 9 | 2.896 |
| $^{226}$Th$_{136}$ | [29] | -155.3 | 133.0 | 16, 17 | 3.136 |
| $^{228}$Th$_{138}$ | [30] | -271.8 | 253.2 | | 3.235 |
| $^{230}$Th$_{140}$ | [17] | -456.3 | 437.9 | | 3.271 |
| $^{232}$Th$_{142}$ | [17] | -667.2 | 650.9 | | 3.283 |
| $^{234}$Th$_{144}$ | [17] | | | | 3.308 |
| $^{230}$U$_{138}$ | [31] | -325.6 | 313.2 | | 3.277 |
| $^{232}$U$_{140}$ | [32] | -528.8 | 518.9 | | 3.291 |
| $^{234}$U$_{142}$ | [33] | -759.1 | 752.5 | | 3.296 |
| $^{236}$U$_{144}$ | [34] | -646.2 | 632.2 | | 3.304 |
| $^{238}$U$_{146}$ | [35] | -632.3 | 614.2 | | 3.305 |
| $^{238}$Pu$_{144}$ | [35] | -565.8 | | | 3.311 |
| $^{240}$Pu$_{146}$ | [36,37] | -554.5 | | | 3.311 |
| $^{242}$Pu$_{148}$ | [38] | -733.5 | | | 3.307 |
Table 3: Odd–even staggering quantity $\Delta E_{1,\gamma}(I)$ (eq. (1), in keV) listed with increasing angular momentum $I$ for the actinides $^{226}$Th [28], $^{228}$Th [30], $^{232}$U [32], $^{236}$U [34], and $^{238}$U [35]. The uncertainties, calculated from the experimental errors given in the same references, are given in parentheses, also in keV.

| $I$ | $^{226}$Th | $^{228}$Th | $^{232}$U | $^{236}$U | $^{238}$U |
|-----|------------|------------|------------|------------|------------|
| 3   | -155.3 (0.2) | -271.8 (0.0) | -528.8 (0.2) | -646.2 (0.3) | -632.3 (0.3) |
| 4   | 133.0 (0.3) | 253.2 (0.1) | 518.9 (0.4) | 632.2 (0.6) | 614.2 (0.5) |
| 5   | -110.9 (0.4) | -233.0 (0.2) | -507.7 (0.6) | -616.2 (0.8) | -593.6 (0.6) |
| 6   | 90.4 (0.4) | 212.0 (0.4) | 495.6 (0.8) | 598.7 (0.9) | 570.9 (0.7) |
| 7   | -71.8 (0.4) | -190.7 (0.5) | -482.8 (0.8) | -580.4 (1.0) | -546.7 (0.8) |
| 8   | 55.7 (0.5) | 169.6 (0.6) | 469.7 (0.9) | 561.6 (1.2) | 521.5 (0.9) |
| 9   | -42.1 (0.6) | -148.9 (0.7) | -456.5 (1.0) | -542.8 (1.4) | -496.0 (1.0) |
| 10  | 30.9 (0.7) | 128.9 (0.8) | 524.4 (1.6) | 470.4 (1.0) |          |
| 11  | -22.0 (0.8) | -109.5 (0.8) |          | -506.7 (1.9) | -445.0 (1.1) |
| 12  | 15.1 (0.9) | 90.9 (0.9) | 489.9 (2.1) | 420.2 (1.3) |          |
| 13  | -9.9 (1.0) | -73.3 (1.0) | -473.9 (2.4) | -396.0 (1.4) |          |
| 14  | 6.1 (1.0) | 56.5 (1.0) | 458.9 (2.6) | 372.9 (1.5) |          |
| 15  | -3.3 (1.1) | -40.1 (1.1) | -445.5 (2.9) | -351.0 (1.6) |          |
| 16  | 1.3 (1.2) |          | 433.7 | 330.5 (1.9) |          |
| 17  | 0.3 (1.3) |          | -423.5 | -311.7 (2.3) |          |
| 18  |          |          |          | 294.8 (2.6) |          |
| 19  |          |          |          | -280.7 (2.9) |          |
| 20  |          |          |          | 269.5 (3.1) |          |
| 21  |          |          |          | -260.5 (3.4) |          |
Table 4: Same as table 3, but for the rare earths $^{150}\text{Sm}$ \[51\], $^{152}\text{Sm}$ \[52\], $^{152}\text{Gd}$ \[53\], $^{154}\text{Gd}$ \[53\], $^{156}\text{Gd}$ \[54\], and $^{162}\text{Er}$ \[58\].

| $I$ | $^{150}\text{Sm}$ | $^{152}\text{Sm}$ | $^{152}\text{Gd}$ | $^{154}\text{Gd}$ | $^{156}\text{Gd}$ | $^{162}\text{Er}$ |
|-----|------------------|------------------|------------------|------------------|------------------|------------------|
| 3   | -781.3 (0.0)     | -587.3 (0.1)     | -983.4 (0.0)     | -1070.8 (0.0)    | -1112.0 (0.1)    |
| 4   | 389.8            | 725.3 (0.1)      | 510.4 (0.2)      | 909.5 (0.0)      | 1012.8 (0.0)     | 1026.6 (0.2)     |
| 5   | -311.2           | -672.3 (0.1)     | -464.5 (0.3)     | -842.9 (0.1)     | -955.9 (0.1)     | -940.5 (0.2)     |
| 6   | 245.1 (0.1)      | 623.8 (0.1)      | 420.2 (0.4)      | 783.3 (0.2)      | 901.1 (0.1)      | 854.7 (0.3)      |
| 7   | -185.6 (0.2)     | -579.8 (0.1)     | -376.6 (0.6)     | -729.9 (0.3)     | -849.4 (0.1)     | -769.9 (0.3)     |
| 8   | 129.8 (0.4)      | 540.0 (0.2)      | 333.3 (0.9)      | 681.9 (0.4)      | 801.2 (0.1)      | 687.0 (0.3)      |
| 9   | -77.8 (0.7)      | -504.0 (0.2)     | -290.4 (1.3)     | -638.3 (0.4)     | -757.1 (0.1)     | -606.6 (0.3)     |
| 10  | 30.5 (1.1)       | 471.1 (0.2)      | 248.2 (1.7)      | 598.0 (0.4)      | 717.4 (0.1)      | 529.9 (0.3)      |
| 11  | 11.3 (1.5)       | -440.5 (0.3)     | -208.1 (2.0)     | -560.0 (0.4)     | -681.9 (0.1)     | -459.1 (0.3)     |
| 12  | -45.3 (1.8)      | 172.5 (2.3)      | 523.3 (0.3)      |                  |                  |                  |
| 13  | 65.9 (2.0)       | -146.0 (2.7)     | -486.8 (0.3)     |                  |                  |                  |
| 14  | -70.0 (2.4)      | 131.1 (3.0)      | 451.2 (0.4)      |                  |                  |                  |
| 15  | 56.4 (2.7)       |                  | -416.5 (0.4)     |                  |                  |                  |
| 16  | -25.3 (3.0)      |                  | 380.9 (0.5)      |                  |                  |                  |
| 17  | -18.1 (3.2)      |                  | -346.3 (0.5)     |                  |                  |                  |
| 18  | 64.9 (3.4)       |                  | 318.4 (0.6)      |                  |                  |                  |
| 19  |                  |                  | -297.4 (0.7)     |                  |                  |                  |
| 20  |                  |                  | 279.0 (0.8)      |                  |                  |                  |
| 21  |                  |                  | -260.9 (0.9)     |                  |                  |                  |
| 22  |                  |                  | 242.3 (1.0)      |                  |                  |                  |
| 23  |                  |                  | -225.0 (1.1)     |                  |                  |                  |
Table 5: Same as table 2, but for octupole bands of several rare earths.

| nucleus  | Ref. | $\Delta E_{1,\gamma}(3)$ | $\Delta E_{1,\gamma}(4)$ | $I_{\text{can}}$ | $R_4$ |
|----------|------|---------------------------|---------------------------|-----------------|------|
| $^{144}\text{Ba}_{88}$ | [47] | -442.6 | 351.8 | 9, 10 | 2.662 |
| $^{146}\text{Ba}_{90}$ | [47] | -442.2 | 349.5 | 9, 10 | 2.835 |
| $^{148}\text{Ba}_{92}$ | [48] | -469.1 | 396.9 | 9, 10 | 2.984 |
| $^{146}\text{Ce}_{88}$ | [49] | -456.2 | 336.6 | 8, 9 | 2.586 |
| $^{146}\text{Nd}_{86}$ | [49] | -399.3 | 211.0 | 5, 6 | 2.297 |
| $^{148}\text{Nd}_{88}$ | [50] | -432.2 | 302.5 | 8, 9 | 2.493 |
| $^{150}\text{Nd}_{90}$ | [51] | -666.3 | 615.6 | 8, 9 | 2.929 |
| $^{152}\text{Nd}_{92}$ | [52] | -1084.7 | | | 3.263 |
| $^{148}\text{Sm}_{86}$ | [48] | -287.0 | 124.8 | 5, 6 | 2.145 |
| $^{150}\text{Sm}_{88}$ | [51] | -203.0 | 48.8 | 5, 6 | 2.019 |
| $^{152}\text{Sm}_{90}$ | [52] | -389.8 | 725.3 | 5, 6 | 3.009 |
| $^{154}\text{Sm}_{92}$ | [53] | -832.9 | 801.9 | 5, 6 | 3.254 |
| $^{150}\text{Gd}_{86}$ | [51] | -587.3 | 1012.8 | 5, 6 | 3.015 |
| $^{152}\text{Gd}_{88}$ | [52] | -983.4 | 1009.5 | 5, 6 | 3.239 |
| $^{154}\text{Gd}_{90}$ | [53] | -1070.8 | 1129.1 | 5, 6 | 3.288 |
| $^{156}\text{Gd}_{92}$ | [54] | -1118.0 | 1077.4 | 5, 6 | 3.302 |
| $^{158}\text{Gd}_{94}$ | [55] | -1112.0 | 1026.6 | 5, 6 | 2.233 |
| $^{160}\text{Gd}_{96}$ | [56] | -1112.0 | 1026.6 | 22, 23 | 2.233 |
| $^{154}\text{Dy}_{88}$ | [53] | -1069.4 | 992.7 | 5, 6 | 2.932 |
| $^{156}\text{Dy}_{90}$ | [57] | -1171.9 | 1140.6 | 5, 6 | 3.206 |
| $^{158}\text{Dy}_{92}$ | [55] | -1177.4 | 1140.6 | 5, 6 | 3.294 |
| $^{162}\text{Dy}_{96}$ | [58] | -1112.0 | 1026.6 | 5, 6 | 3.230 |
| $^{164}\text{Er}_{94}$ | [58] | -1214.8 | 1145.5 | 5, 6 | 3.276 |
\[ \Delta E_I \text{ (keV)} \]

Angular Momentum $I$

\[ ^{220}\text{Ra} \]
Angular Momentum $I$ vs. $\Delta E_I$ (keV) for $^{224}\text{Ra}$
$^{220}\text{Ra}$

![Graph showing $\Delta E_1$ vs. Angular Momentum I](image)

- $\Delta E_{1,\gamma}$
- $-\Delta E_{45,\gamma}$
- $-\Delta E_{NO}$

$\Delta E_1$ (keV) vs. Angular Momentum I
Angular Momentum $I$

$\Delta E_i$ (keV)

$^{224}\text{Ra}$

$\Delta E_{1,\gamma}$

$-\Delta E_{45,E}$

$-\Delta E_{NO}$
Angular Momentum $I$

$\Delta E_{I} \text{ (keV)}$

$^\text{230}\text{U}$

$^\text{232}\text{U}$

$^\text{234}\text{U}$

$^\text{236}\text{U}$

$^\text{238}\text{U}$
The graph shows the angular momentum $I$ as a function of the change in energy $\Delta E_1$ (in keV) for the $^{144}\text{Ba}$ isotope. The data points are plotted along the $\Delta E_1$ axis, with a horizontal line at $\Delta E_1 = 0$ indicating the zero energy level.
$\Delta E_1$ (keV) vs. Angular Momentum $I$ for $^{146}$Ba.
\[ \Delta E_1 \text{ (keV)} \]

Angular Momentum $I$

\( ^{152}\text{Sm} \)

\( ^{154}\text{Sm} \)
\[ \Delta E_1 \text{ (keV)} \]

Angular Momentum \( I \)

Graph showing the angular momentum vs. the change in energy for different isotopes of Dy.

- Solid circles: \(^{156}\text{Dy}\)
- Squares: \(^{162}\text{Dy}\)
The graph shows the angular momentum $I$ versus the change in energy $\Delta E_1$ (in keV) for two isotopes of Er: $^{162}\text{Er}$ and $^{164}\text{Er}$. The data points indicate periodic behavior with maxima at specific angular momenta, indicating possible quantum states or transitions in the system.