THE NONCOMMUTATIVE CHOQUET BOUNDARY OF PERIODIC WEIGHTED SHIFTS

MARTÍN ARGERAMI AND DOUGLAS FARENICK

Abstract. The noncommutative Choquet boundary and the $C^*$-envelope of operator systems of the form $\text{Span}\{1, T, T^*\}$, where $T$ is a Hilbert space operator with normal-like features, are studied. Such operators include normal operators, $k$-normal operators, subnormal operators, and Toeplitz operators. Our main result is the determination of the noncommutative Choquet boundary for an operator system generated by an irreducible periodic weighted unilateral shift operator.

1. Introduction

If $Y$ is a compact Hausdorff space and $C(Y)$ is the Banach space of all continuous complex-valued functions on $Y$, then the Choquet boundary of a linear subspace $\mathcal{F} \subset C(Y)$ that contains the constants and separates the points of $Y$ is the subset $\partial C\mathcal{F} \subset Y$ of all $y \in Y$ for which the point-mass measure $\delta_y$ on the Borel sets of $Y$ is the only Borel probability measure $\mu$ on $Y$ for which $f(y) = \int_Y f \, d\mu$ for every $f \in \mathcal{F}$. Motivated by the use of the Choquet boundary in the analysis of spaces of continuous complex-valued functions (as in [20], for example), W. Arveson initiated the study of analogous objects in the setting of matricially ordered vector spaces $\mathcal{X}$ of bounded linear operators acting on complex Hilbert spaces $\mathcal{H}$ [1,2].

A notion that is central in Arveson’s work and its subsequent application is that of a boundary representation. A boundary representation for a unital operator space $\mathcal{X} \subset \mathcal{B}(\mathcal{H})$—that is, a subspace $\mathcal{X} \subset \mathcal{B}(\mathcal{H})$ with $1_{\mathcal{B}(\mathcal{H})} \in \mathcal{X}$—is a unital $C^*$-algebra representation $\rho : C^*(\mathcal{X}) \to \mathcal{B}(\mathcal{H}_\rho)$ such that

1. $\rho$ is irreducible and
2. for any unital completely positive (ucp) linear map $\psi : C^*(\mathcal{X}) \to \mathcal{B}(\mathcal{H}_\rho)$ with $\psi|_{\mathcal{X}} = \rho|_{\mathcal{X}}$ we have $\psi = \rho$ (i.e. $\rho|_{\mathcal{X}}$ has a unique completely contractive extension to $C^*(\mathcal{X})$, namely, $\rho$).

If $\partial \mathcal{C}\mathcal{X}$ denotes the subset of the spectrum of $C^*(\mathcal{X})$ consisting of the unitary-equivalence classes $\hat{\rho}$ of boundary representations $\rho$ for $\mathcal{X}$, then the ideal $\mathfrak{S}_\mathcal{X} \subset C^*(\mathcal{X})$ defined by

$$\mathfrak{S}_\mathcal{X} = \bigcap_{\hat{\rho} \in \partial \mathcal{C}\mathcal{X}} \ker \rho,$$

is called the Šilov ideal for $\mathcal{X}$. A unital operator space $\mathcal{X} \subset \mathcal{B}(\mathcal{H})$ is said to have a noncommutative Choquet boundary if the canonical quotient homomorphism $C^*(\mathcal{X}) \to C^*(\mathcal{X})/\mathfrak{S}_\mathcal{X}$ is a complete isometry on $\mathcal{X}$, in which case the subset $\partial \mathcal{C}\mathcal{X}$
of the spectrum of $C^*(\mathcal{A})$ is called the noncommutative Choquet boundary of $\mathcal{A}$. If $\mathcal{A}$ has a noncommutative Choquet boundary, then the $C^*$-algebra $C^*(\mathcal{A})/\mathcal{E}_{\mathcal{A}}$ is called the $C^*$-envelope of $\mathcal{A}$, which we denote by $C^e(\mathcal{A})$. An important theorem of Arveson [4] asserts that every separable unital operator space $\mathcal{A} \subset B(\mathcal{H})$ has a noncommutative Choquet boundary.

The $C^*$-envelope $C^e(\mathcal{A})$ of a unital operator space $\mathcal{A}$ can be viewed as the smallest $C^*$-algebra that is generated by (a copy of) $\mathcal{A}$; concretely, $(C^e(\mathcal{A}), \iota)$ is a $C^*$-envelope for $\mathcal{A}$ if $C^e(\mathcal{A})$ is a $C^*$-algebra and $\iota : \mathcal{A} \rightarrow C^e(\mathcal{A})$ is a complete isometry such that whenever $\phi : \mathcal{A} \rightarrow A$ is a complete isometry into a $C^*$-algebra $A$, there exists an epimorphism of $C^*$-algebras $\pi : C^*(\phi(\mathcal{A})) \rightarrow C^e(\mathcal{A})$ such that $\pi \circ \phi = \iota$. It follows easily from this definition that the $C^*$-envelope of $\mathcal{A}$ is unique up to isomorphism of $C^*$-algebras.

It is not easy, in general, to determine the $C^*$-envelope of a given unital operator space. There is, however, a substantial literature for the case in which $\mathcal{A}$ is an operator algebra (for example, [3, 5, 10, 17]). But our focus in this paper is in the realm of single operator theory, as we consider the smallest possible unital operator algebra (for example, $\mathcal{A} = C(\mathbb{T})$). But our focus in this paper is in the realm of single operator theory, as we consider the smallest possible unital operator algebra (for example, $\mathcal{A} = C(\mathbb{T})$).

2. Preliminaries

Notation 2.1. For an operator system $\mathcal{S} \subset B(\mathcal{H})$, the canonical quotient homomorphism $C^*(\mathcal{S}) \rightarrow C^*(\mathcal{S})/\mathcal{E}_\mathcal{S}$ is denoted by $q_e$, and $\iota_e = q_e|_\mathcal{S}$ denotes the ucp map $\mathcal{S} \rightarrow C^*(\mathcal{S})/\mathcal{E}_\mathcal{S}$.

Note that, by definition, $\mathcal{S}$ has a noncommutative Choquet boundary if and only if $\iota_e$ is a completely isometric embedding.

Lemma 2.2. $C^*(\mathcal{S}_T \otimes M_m(\mathbb{C})) = C^*(\mathcal{S}_T) \otimes M_m(\mathbb{C})$ for every $T \in B(\mathcal{H})$ and $m \in \mathbb{N}$. 
Proof. We can identify $S_T \otimes M_m(\mathbb{C})$ canonically with $M_m(S_T)$, and $C^*(S_T) \otimes M_m(\mathbb{C})$ with $M_m(C^*(S_T))$. And so our assertion reduces to the also canonical identification of $M_m(C^*(S_T))$ with $C^*(M_m(S_T))$.

The ideal $\mathcal{S}_S$ is not an invariant of the operator system $S$, as it depends on the concrete representation of $S$; still it plays a crucial role in Arveson’s theory. An easy but key fact that motivates this significance is as follows:

Lemma 2.3. For any $m \in \mathbb{N}$, there is a canonical isometric embedding (as $C^*$-algebras)

$$f_m : M_m(C^*(S)/\mathcal{S}_S) \rightarrow \prod_{\rho \in \partial C_S} M_m(\rho(C^*(S)))$$

$$X + M_m(\mathcal{S}_S) \rightarrow (\rho^{(m)}(X))$$

Proof. We define $f_m(X + M_m(\mathcal{S}_S)) = \prod_{\rho \in \partial C_S} \rho^{(m)}(X)$, $X \in M_m(C^*(S))$.

Note that $f_m$ is clearly linear, multiplicative, and $*$-preserving. Also,

$$\rho^{(m)}(X) = 0 \ \forall \rho \iff \rho(X_{hk}) = 0, \forall h, k$$

$$\iff X_{hk} \in \mathcal{S}_S, \forall h, k$$

$$\iff X \in M_m(\mathcal{S}_S),$$

which shows that $f_m$ is both well defined and one-to-one.

We shall also compare our analysis of $X_T := \text{Span} \{1, T\}$ with that of the norm-closed algebra generated by $T$.

Definition 2.4. For any operator $T \in B(\mathcal{H})$, the operator algebra generated by $T$ is the subalgebra $P_T \subset B(\mathcal{H})$ given by the norm closure of all operators of the form $p(T)$, for polynomials $p \in \mathbb{C}[t]$.

As mentioned in the Introduction, most of the focus on $C^*$-envelopes in the literature is on operator algebras, while here we focus on operator systems. Although $X_T$ and $P_T$ generate the same $C^*$-subalgebra of $B(\mathcal{H})$, they do not necessarily have the same $C^*$-envelopes. Explicit examples of this occur in Example 3.4 when $|\lambda| \leq 1/2$, and for operators $T = T^*$ with $|\sigma(T)| \geq 3$ (see Proposition [3.4]).

3. Numerical Range and Spectrum

We will see below how the numerical range $W(T)$ and spectrum $\sigma(T)$ of $T$ capture information about the boundary representations of $S_T$. By numerical range we mean the compact convex set

$$W(T) = \{\phi(T) : \phi \text{ is a state on } S_T\},$$

It is well known that the set

$$W_1(T) = \{\langle T\xi, \xi \rangle : \xi \in \mathcal{H}, \|\xi\| = 1\}$$

is convex and dense in $W(T)$.

Proposition 3.1. Let $T \in B(\mathcal{H})$, $\lambda \in \mathbb{C}$.
(1) If \( \lambda = \rho(T) \) for some \( \rho \in \partial_C S_T \), then \( \lambda \in \sigma(T) \cap \partial W(T) \), and \( \lambda \) is an extreme point of \( W(T) \).

(2) Assume that \( \lambda \in \sigma(T) \cap \partial W(T) \). If \( \lambda \) is an extreme point of \( W(T) \) and if the commutator \( [T^*, T] = T^*T - TT^* \) is positive, then \( \lambda = \rho(T) \) for some \( \rho \in \partial_C S_T \).

**Proof.** To prove (1), note first that we have \( \rho(T - \lambda 1) = 0 \). As \( \rho \) is unital and multiplicative, this shows that \( \lambda \in \sigma(T) \). Also, since \( \rho(T) \) is scalar, we have that \( \rho \) is a state on \( S_T \), and thus \( \lambda \in W(T) \). After we prove that \( \lambda \) is an extreme point of \( W(T) \), we will know that \( \lambda \in \partial W(T) \).

Let \( \phi = \rho|_{S_T} \). Suppose that \( \lambda_1, \lambda_2 \in W(T) \) and that \( \lambda = \frac{1}{2} \lambda_1 + \frac{1}{2} \lambda_2 \). As every state on \( S_T \) extends to a state on \( C^*(S_T) \) (by the Hahn–Banach Theorem and some positivity considerations), there are states \( \phi_1 \) and \( \phi_2 \) on \( C^*(S_T) \) such that \( \lambda_j = \phi_j(T) \), \( j = 1, 2 \). Thus, the state \( \psi = \frac{1}{2} \phi_1 + \frac{1}{2} \phi_2 \) is an extension of \( \phi \) to \( C^*(S_T) \). Because \( \rho \) is a boundary representation for \( S_T \), \( \psi = \rho \). That is, \( \rho = \frac{1}{2} \phi_1 + \frac{1}{2} \phi_2 \). But since \( \rho \) is a pure state (because it is multiplicative), we deduce that \( \phi_1 = \phi_2 = \rho \); hence, \( \lambda_1 = \lambda_2 = \lambda \), which implies that \( \lambda \) is an extreme point of \( W(T) \).

For the proof of (2), the hypothesis \( \lambda \in \sigma(T) \cap \partial W(T) \) implies that there is a homomorphism \( \rho : C^*(S_T) \to \mathbb{C} \) such that \( \lambda = \rho(T) \) (Theorem 3.1.2]. Assume that \( \lambda \) is an extreme point of \( W(T) \) and that \( [T^*, T] = T^*T - TT^* \) is positive. Let \( \phi = \rho|_{S_T} \) and suppose that \( \Phi \) is any state on \( C^*(T) \) that extends \( \phi \). Via the GNS construction, there are a Hilbert space \( \mathcal{H}_\pi \), a representation \( \pi : C^*(S_T) \to \mathcal{B}(\mathcal{H}_\pi) \), and a unit vector \( \xi \in \mathcal{H}_\pi \) such that \( \Phi(A) = \langle \pi(A)\xi, \xi \rangle \) for every \( A \in C^*(S_T) \). In particular, \( \lambda = \langle \pi(T)\xi, \xi \rangle \). Now since the numerical range of \( \pi(T) \) is a subset of the numerical range of \( T \), \( \lambda \) is an extreme point of \( W(\pi(T)) \). Moreover, as \( [\pi(T)^*, \pi(T)] = \pi([T^*, T]) \) is positive, \( W(\pi(T)) \) coincides with the convex hull of the spectrum of \( \pi(T) \). Hence, the equation \( \lambda = \langle \pi(T)\xi, \xi \rangle \) together with \( \lambda \in \sigma(\pi(T)) \cap \partial W(\pi(T)) \) imply that \( \pi(T)\xi = \lambda \xi \) and \( \pi(T)^*\xi = \overline{\lambda} \xi \) [16, Satz2]. Thus, \( \Phi \) is a homomorphism and agrees with \( \rho \) on the generating set \( S_T \); hence, \( \Phi = \rho \) and so \( \rho \) is a boundary representation.

It is interesting to contrast (1) of Proposition 3.1 with Theorem 3.1.2 of [1], which states that if \( \lambda \in \sigma(T) \cap \partial W(T) \), then \( \lambda = \rho(T) \) for some boundary representation \( \rho \) for \( P_T \). In this latter assertion, there is no requirement that \( \lambda \) be an extreme point of \( W(T) \), and this is one way in which we see that the operator spaces \( P_T \) and \( S_T \) differ fundamentally.

In general a spectral point \( \lambda \in \sigma(T) \) that also happens to be an extreme point of \( W(T) \) does not give rise to a boundary representation (which explains the extra hypothesis in assertion (2) of Proposition 3.1). For example, with \( T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \frac{1}{2} \), the vectors \( \xi = e_3 \) and \( \eta = \sqrt{\frac{1}{2}}(e_1 + e_2) \) give rise to states \( \rho(X) = \langle X\xi, \xi \rangle \) and \( \psi(X) = \langle X\eta, \eta \rangle \) on \( C^*(T) \) such that \( \rho(T) = \psi(T) = \frac{1}{2} \in \text{ext} W(T) \cap \sigma(T) \); however, \( \rho \) is a representation of \( C^*(T) \) whereas \( \psi \) is not.

In Theorem 5.2 below, numerical range considerations allow us to completely characterise one-dimensional boundary representations for direct sums of operators.
Theorem 3.2. Let $T = \bigoplus_{j=1}^{m} T_j \subset \bigoplus_{j=1}^{m} B(H_j)$, where $m \in \mathbb{N}$ and $k_\ell = 1$ for a fixed $\ell$. Let $\pi_\ell : C^*(T) \to \mathbb{C}$ be the irreducible representation induced by $\bigoplus_{j=1}^{m} T_j \to T_\ell$. Then $\pi_\ell$ is a boundary representation if and only if $T_\ell \notin \text{Conv} \bigcup_{j \neq \ell} W(T_j)$.

Proof. Let us denote $\lambda = T_\ell$, and $\pi_j : C^*(T) \to B(H_j)$ the representation $\bigoplus_{j=1}^{m} T_j \to T_j$.

Assume first that $\lambda \in \text{Conv} \bigcup_{j \neq \ell} W(T_j)$. Therefore, for each $j = 1, \ldots, m$ with $j \neq \ell$ there exists a state $\psi_j$ on $C^*(T_j)$ such that $\lambda = \sum_{j \neq \ell} \alpha_j \psi_j(T_j)$ for some convex coefficients $\alpha_1, \ldots, \alpha_{\ell-1}, \alpha_{\ell+1}, \ldots, \alpha_m$. Define a state $\psi$ on $C^*(T)$ by $\psi = \sum_{j \neq \ell} \alpha_j \psi_j \circ \pi_j$. Because $\psi(T) = \lambda$, we obtain $\psi|_{S_T} = \pi_\ell|_{S_T}$. Now choose some $j$ with $\alpha_j \neq 0$. Then $\psi(1_j) \geq \alpha_j \psi_j(1_j) = \alpha_j > 0$. But, as $j \neq \ell$, $\pi_\ell(1_j) = 0$; thus, $\pi_\ell|_{S_T}$ admits a ucp extension from $S_T$ to $C^*(S_T)$ other than $\pi_\ell$, and so $\pi_\ell$ is not a boundary representation for $S_T$.

Conversely, assume that $\lambda \notin \text{Conv} \bigcup_{j \neq \ell} W(T_j)$. Choose any state $\phi$ on $C^*(S_T)$ for which $\phi|_{S_T} = \pi_\ell|_{S_T}$; that is, $\phi$ is a state such that $\phi(T) = \lambda$. The numerical range $W(T)$ of $T$ is the convex hull of the numerical ranges $W(T_j), \ldots, W(T_m)$. As $W(T_\ell) = \{ \lambda \}$ and $\lambda$ is not in the convex hull of the other numerical ranges we have that $W(T)$ is the convex set generated by the convex set $\text{Conv} \bigcup_{j \neq \ell} W(T_j)$ and the external point $\lambda$; so $\lambda$ is a point of nondifferentiability on the boundary of $W(T)$. By the GNS decomposition, there are a Hilbert space $H_\phi$, a representation $\vartheta : C^*(S_T) \to B(H_\phi)$, and a unit vector $\xi \in H_\phi$ such that $\phi(A) = \langle \vartheta(A)\xi, \xi \rangle$ for every $A \in C^*(S_T)$. In particular, $\lambda = \langle \vartheta(T)\xi, \xi \rangle$. Because the numerical range of $\vartheta(T)$ is a subset of the numerical range of $T$, $\lambda$ is also a point of nondifferentiability on the boundary of $W(\vartheta(T))$; therefore, $\lambda$ is necessarily an eigenvalue of $\vartheta(T)$ (Theorem 1]). Moreover, because this eigenvalue $\lambda$ lies on the boundary of the numerical range of $\vartheta(T)$, the equation $\lambda = \langle \vartheta(T)\xi, \xi \rangle$ implies that $\vartheta(T)\xi = \lambda \xi$ and $\vartheta(T)^*\xi = \overline{\lambda} \xi$ (Satz 1,2). That is, $\phi$ is a homomorphism and it agrees with $\pi_\ell$ on $S_T$; hence, $\phi = \pi_\ell$ on $C^*(S_T)$, which proves that $\pi_\ell$ is a boundary representation.

Remark 3.3. A characterisation of boundary representations of higher order appears in Theorem 5.4. The implications of Theorems 3.2 and 5.3 to direct sums of operators and to Jordan operators in particular will be explored in a further article.

Example 3.4. For each $\lambda \in \mathbb{C}$, let $T_\lambda \in M_3(\mathbb{C})$ be given by

$$T_\lambda = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda \end{bmatrix}.$$ 

Then

$$C^*_c(S_{T_\lambda}) = \begin{cases} M_2(\mathbb{C}) & \text{if } |\lambda| \leq 1/2 \\ M_2(\mathbb{C}) \oplus \mathbb{C} & \text{if } |\lambda| > 1/2 \end{cases}.$$ 

Proof. The $C^*$-algebra generated by $S_{T_\lambda}$ is $M_2(\mathbb{C}) \oplus \mathbb{C}$. Let $\pi : C^*(S_{T_\lambda}) \to \mathbb{C}$ be the map that sends each $X \in C^*(S_{T_\lambda})$ to its (3,3)-entry. Thus, $\pi$ is an irreducible representation of $C^*(S_{T_\lambda})$ on the 1-dimensional Hilbert space $\mathbb{C}$. Another irreducible representation of $C^*(S_{T_\lambda})$ is the map $\rho : C^*(S_{T_\lambda}) \to M_2(\mathbb{C})$ given by

$$\rho(X) = X^* XV,$$

where $V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. Up to unitary equivalence, $\pi$ and $\rho$ are
the only irreducible representations of $\mathcal{S}_T$, and so at least one of these two must be a boundary representation. In fact, regardless of the choice of $\lambda$, $\rho$ is always a boundary representation, for if we were not, then $\pi$ would necessarily be the only boundary representation for $\mathcal{S}_T$, which implies that the Šilov ideal would be given by $\mathcal{S}_{\mathcal{T}_\lambda} = \ker \pi = M_2(\mathbb{C}) \oplus \{0\}$; but if this were true, then the quotient $C^*(\mathcal{S}_T)/\mathcal{S}_{\mathcal{T}_\lambda}$ would be the 1-dimensional algebra $\mathbb{C}$, which would not contain a copy of the 3-dimensional operator system $\mathcal{S}_T$. Hence, $\rho$ is a boundary representation and the only question to resolve is: for which $\lambda$ is a boundary representation? To answer this, it is enough to use Theorem 3.2 and to note that the numerical range of $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is the closed disc of radius $1/2$ centred at the origin. \hfill \Box

4. Normal Operators and Operators with Normal $W$-Dilations

If $T$ is a normal operator, then $C^*(T)$ is abelian; hence, so is $C^*_e(\mathcal{S}_T)$, as it is the image through an epimorphism of $C^*(T)$. We will analyse more carefully which abelian $C^*$-algebras arise in such cases, and we will show that certain non-normal $T$ have abelian $C^*$-envelopes (even though in these cases $C^*(T)$ is non-abelian).

It is well known that positive maps need not be completely positive, but there is a useful “automatic complete positivity” result that we will make use of.

**Proposition 4.1.** ([10] Theorem 3.9) If $\phi : \mathcal{S} \to \mathcal{T}$ is a positive linear map of operator systems, and if $T$ is an operator subsystem $\mathcal{T} \subset \mathcal{A}$ of an abelian $C^*$-algebra $\mathcal{A}$, then $\phi$ is completely positive.

A function system on a compact Hausdorff space $\Omega$ is a subset $\mathcal{F} \subseteq C(\Omega)$ such that: (i) $\mathcal{F}$ is a vector space over $\mathbb{C}$, closed in the topology of $C(\Omega)$; (ii) $f^* \in \mathcal{F}$, for all $f \in \mathcal{F}$; (iii) $1 \in \mathcal{F}$ (the constant function $x \mapsto 1$); and (iv) $\mathcal{F}$ separates the points of $K$. By the Stone–Weierstrass Theorem, the $C^*$-subalgebra of $C(\Omega)$ generated by $\mathcal{F}$ is precisely $C(\Omega)$ itself.

A boundary for $\mathcal{F}$ is a closed subset $\partial_0 \subseteq \Omega$ such that for every $f \in \mathcal{F}$ there is a $t_0 \in \partial_0$ such that $\|f \| = |f(t_0)|$. By a theorem of Šilov, there is a smallest compact subset $\partial_\mathcal{F}$ of $\Omega$ that is contained in every boundary of $\mathcal{F}$ and is itself a boundary of $\mathcal{F}$. The set $\partial_\mathcal{F}$ is known classically as the Šilov boundary of $\mathcal{F}$. In the language of $C^*$-envelopes, Šilov’s theorem takes the following form:

**Theorem 4.2.** (Šilov) If $\mathcal{F}$ is a function system on $\Omega$, then $C^*_e(\mathcal{F}) = C(\partial_\mathcal{F})$.

While for general normal operators there is a great variety of possible operator systems and $C^*$-envelopes, the case of selfadjoint operators is totally rigid:

**Proposition 4.3.** If $T = T^*$, then $C^*_e(\mathcal{S}_T) = \mathbb{C} \oplus \mathbb{C}$.

**Proof.** One can deduce the conclusion from Proposition 3.1 and the fact that the numerical range of $T$ is a line segment (and thus has exactly two extreme points), but we feel the following direct proof is more instructive.

As $C^*(T) \simeq C(\sigma(T))$ as $C^*$-algebras, this isomorphism restricts to a complete isometry on $\mathcal{S}_T$. So $C^*_e(\mathcal{S}_T) = C^*_e(\mathcal{S}_z)$, where $z$ is the function $z : t \mapsto t$ in $C(\sigma(T))$.

So we want to identify the boundary representations of $\mathcal{S}_z$ in $C(\sigma(T))$. Since $C(\sigma(T))$ is an abelian $C^*$-algebra, each of its irreducible representation is one-dimensional, i.e. a character, and it is given by point evaluation.
As $\sigma(T)$ is a compact subset of $\mathbb{R}$, it has a minimum and a maximum, say $t_0$ and $t_1$, and every point in $\sigma(T)$ is a convex combination of $t_0$ and $t_1$. Given any $t \in \sigma(T)$ with $t_0 < t < t_1$, there exists $\alpha \in (0,1)$ with $t = \alpha t_0 + (1 - \alpha)t_1$. The irreducible representation associated with $t$ is the map $\pi_t : f \mapsto f(t)$ in $C(\sigma(T))$. Now consider the state $\psi : f \mapsto af(t_0) + (1 - a)f(t_1)$ on $C(\sigma(T))$. By considering some $f \in C(\sigma(T))$ with $f(t_0) = 1$, $f(t) = 0$, we see that $\pi_t \neq \psi$. But $\pi_t$ and $\psi$ agree on $S_\pi$, indeed: if $f = \beta + \gamma z$,

$$\psi(f) = \alpha(\beta + \gamma t_0) + (1 - \alpha)(\beta + \gamma t_1) = \beta + \gamma(t_0 + (1 - \alpha)t_1) = \beta + \gamma t = \pi_t(f).$$

So $\pi_t|_{S_\pi}$ admits an extension other than $\pi_t$ (provided that $t \neq t_0,t_1$), which shows that $\pi_t$ is not a boundary representation for $S_\pi$.

The only remaining candidates for boundary representations are $\pi_{t_0}$ and $\pi_{t_1}$. Both must be boundary representations because the $C^*$-envelope necessarily contains a copy of $S_T$ and so it has dimension at least 2. By Lemma 4.3 we conclude that $C_e^*(S_T) = \mathbb{C} \oplus \mathbb{C}$. \hfill $\Box$

**Corollary 4.4.** All two-dimensional operator systems are isomorphic.

**Proof.** It is easy to see that a two-dimensional operator system has a selfadjoint generator $T$. By Proposition 4.3 $C_e^*(S_T) = \mathbb{C}^2$. This implies that there exists a unital complete isometry $\psi : S_T \to \mathbb{C}^2$. The image of $\psi$ is two-dimensional, so $\psi$ is onto, and then $S_T \cong \mathbb{C}^2$ as operator systems. \hfill $\Box$

**Definition 4.5.** Assume that $N \in \mathcal{B}(\mathcal{H})$ is a normal operator. The function system associated with $N$ is the operator subsystem $\mathcal{F}_N \subset C(\sigma(N))$ defined by

$$\mathcal{F}_N = \text{Span}\{1, \Gamma(N), \overline{\Gamma(N)}\},$$

where $\Gamma : C^*(N) \to C(\sigma(N))$ is the Gelfand transform.

**Proposition 4.6.** If $N \in \mathcal{B}(\mathcal{H})$ is normal, then $C_e^*(S_N) = C(\partial_\Sigma \mathcal{F}_N)$.

**Proof.** Note that $C^*(S_N) = C^*(N)$. The Gelfand transform $\Gamma : C^*(N) \to C(\sigma(N))$ is an isomorphism of $C^*$-algebras and so the restriction of $\Gamma$ to $S_N$ is a unital completely isometric linear map of $S_N$ onto the operator subsystem $\mathcal{F}_N \subset C(\sigma(N))$. Thus, $\Gamma|_{S_N}$ is a complete order isomorphism and, hence, $C_e^*(S_N) = C_e^*(\mathcal{F}_N) = C(\partial_\Sigma \mathcal{F}_N)$. \hfill $\Box$

**Corollary 4.7.** If $U$ is a unitary operator, then $C_e^*(S_U) = C(\sigma(U))$.

**Proof.** By definition, the Šilov boundary of the function system $\mathcal{F}_U$ is a compact subset of $\sigma(U)$. Therefore, Proposition 4.6 shows that we need only prove the inclusion $\sigma(U) \subset \partial_\Sigma \mathcal{F}_U$. To this end, select $\lambda \in \sigma(U)$ and consider the function $f_\lambda \in \mathcal{F}_U$ defined by

$$f_\lambda(\mu) = \mu + \lambda, \; \mu \in \sigma(U).$$

For any $z \in \mathbb{T}$, $|f_\lambda(z)|$ is the Euclidean distance between $z$ and $-\lambda$, and so the maximum modulus of $f_\lambda$ on $\mathbb{T}$ is attained at $\lambda$ and $|f_\lambda(\lambda)| > |f_\lambda(\mu)|$ for every $\mu \in \sigma(U) \setminus \{\lambda\}$. Hence, $\lambda \in \partial_\Sigma \mathcal{F}_U$. \hfill $\Box$

**Corollary 4.8.** If $U$ is a unitary operator with $\sigma(U) = \mathbb{T}$, then $C_e^*(S_U) = C(\mathbb{T})$. 

There are many operators that behave like normals when one is considering only their numerical range and spectrum. The following definition is meant to capture such a situation.

**Definition 4.9.** An operator $T \in \mathcal{B}(\mathcal{H})$ has a normal $W$-dilation if there is a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ as a subspace and a normal operator $N \in \mathcal{B}(\mathcal{K})$ such that:

1. $N$ is a dilation of $T$ (that is, $T = P_{\mathcal{H}}N|_{\mathcal{H}}$, where $P_{\mathcal{H}} \in \mathcal{B}(\mathcal{K})$ is the projection of $\mathcal{K}$ onto $\mathcal{H}$), and
2. $W(T) = W(N)$.

The class of operators with normal $W$-dilations includes all Toeplitz operators on the Hardy space $H^2(\mathbb{T})$ and all subnormal operators \[15\].

**Proposition 4.10.** If $N$ is a normal $W$-dilation of $T$, then $S_N$ and $S_T$ are completely order isomorphic.

**Proof.** Assume that $\mathcal{K} \supset \mathcal{H}$ and that $N \in \mathcal{B}(\mathcal{K})$ is a normal $W$-dilation of $T \in \mathcal{B}(\mathcal{H})$. Define $\psi : S_N \to S_T$ by $\psi(R) = P_{\mathcal{H}}R|_{\mathcal{H}}$, which is a ucp map that sends $N$ to $T$. Now define a linear map $\phi : S_T \to S_N$ by

$$
\phi(\alpha 1 + \beta T + \gamma T^*) = \alpha 1 + \beta N + \gamma N^*, \quad \text{for all } \alpha, \beta, \gamma \in \mathbb{C}.
$$

As a linear transformation, $\phi = \psi^{-1}$. Thus, it remains to prove that $\phi$ is completely positive. First note that the hypothesis $W(T) = W(N)$ implies that, for $R \in S_T$, $\phi(R)$ is positive if and only if $R$ is positive. Hence, $\phi$ is a positive linear map. The range of $\phi$ is $S_N$, which is an operator subsystem of the C$^*$-algebra $C^*(N)$. Because the C$^*$-algebra $C^*(N)$ is abelian, all positive linear maps into $C^*(N)$ are completely positive (Proposition \[11\]). In particular, $\phi = \psi^{-1}$ must be completely positive, which is to say that $\psi$ is a complete order isomorphism. \qed

**Corollary 4.11.** If $T \in \mathcal{B}(\mathcal{H})$ is a contraction such that $\mathbb{T} \subset \sigma(T)$, then $C^*_e(S_T) = C(\mathbb{T})$.

**Proof.** Every contraction has a unitary dilation \[15\]; explicitly, one such unitary dilation $U$ is given by

$$
U = \begin{bmatrix}
T & (1 - TT^*)^{1/2} \\
-(1 - T^*T)^{1/2} & T^*
\end{bmatrix}.
$$

The condition $\mathbb{T} \subset \sigma(T)$ implies, therefore, that $W(T)$ and $W(U)$ coincide with the closed unit disc and that $\sigma(U) = \mathbb{T}$. Hence, Proposition \[11\] asserts that $S_U$ and $S_T$ are completely order isomorphic, and so $C^*_e(S_T) = C^*_e(S_U)$. Corollary \[7\] yields $C^*_e(S_U) = C(\mathbb{T})$. \qed

Recall that an isometry $V$ is proper if $V$ is not unitary.

**Corollary 4.12.** If $V$ is a proper isometry, then $C^*_e(S_V) = C(\mathbb{T})$.

**Proof.** By the Wold Decomposition, the spectrum of a proper isometry $V$ necessarily contains $\mathbb{T}$. \qed

As was mentioned above, for any operator $T$ one has an epimorphism $\pi : C^*(T) \to C^*_e(S_T)$. Whenever this $\pi$ is not an isomorphism the Silov ideal, being the kernel of $\pi$, is nontrivial; in particular, $C^*(T)$ cannot be simple. Using this straightforward idea, we deduce the following fact from the results of this section:
Corollary 4.13. Let $T$ be an operator that is not a scalar multiple of the identity, and such that any of the following holds:

1. $T$ has a normal $W$-dilation;
2. $T$ is a Toeplitz operator on $H^2(\mathbb{T})$;
3. $T$ is subnormal;
4. $T$ is a contraction with $T \subset \sigma(T)$;
5. $T$ is a proper isometry.

Then $C^*(T)$ is not simple.

5. Finite-Dimensional Boundary Representations

Finite-dimensional irreducible representations of $C^*(\mathcal{S}_T)$ play a role similar to that of an eigenvalue for an operator. We show in this section that such a representation $\rho$ is a boundary representation for $\mathcal{S}_T$ only if $\rho(T)$ is an extremal element in a certain convex set.

Definition 5.1. Let $V$ be a complex vector space and assume that $\mathcal{R}_k \subset M_k(V)$ is a nonempty set, for every $k \in \mathbb{N}$. Let $\mathcal{R} = (\mathcal{R}_k)_{k \in \mathbb{N}}$.

1. The sequence $\mathcal{R}$ is matrix convex in $V$ if, for every $k$, $\sum_{j=1}^{m} A_j^* X_j A_j \in \mathcal{R}_k$,

   whenever $m \in \mathbb{N}$, $X_j \in \mathcal{R}_{n_j}$, $A_j \in M_{n_j,k}(\mathbb{C})$, and $\sum_{j=1}^{m} A_j^* A_j = 1 \in M_k(\mathbb{C})$.

2. An element $X \in \mathcal{R}_k$ is a matrix extreme point of a matrix convex set $\mathcal{R}$ in $V$ if the equation $X = \sum_{j=1}^{m} A_j^* X_j A_j$, where $X_j \in \mathcal{R}_{n_j}$, $A_j \in M_{n_j,k}(\mathbb{C})$ of rank $n_j$, and $\sum_{j=1}^{m} A_j^* A_j = 1 \in M_k(\mathbb{C})$, holds only if each $n_j = k$ and there are unitaries $U_1, \ldots, U_m \in M_k(\mathbb{C})$ such that $X_j = U_j^* X U_j$ for all $j = 1, \ldots, m$.

We shall be interested in the matricial range of an operator, which was introduced by Arveson in \cite{Arveson1976} and which received subsequent study in, for example, \cite{Arveson1976a} \cite{Arveson1976b} \cite{Arveson1976c}.

Definition 5.2. The matricial range of an operator $T \in \mathcal{B}(\mathcal{H})$ is the sequence $\mathcal{W}(T) = (W_k(T))_{k \in \mathbb{N}}$ of subsets $W_k(T) \subset M_k(\mathbb{C})$ defined by

$$W_k(T) = \{ \phi(T) : \phi : \mathcal{S}_T \to M_k(\mathbb{C}) \text{ is a ucp map} \}.$$ 

It is well known that each $W_k(T)$ is compact and that $W(T)$ is matrix convex in $V = \mathbb{C}$. The set $W_1(T)$ coincides with the numerical range of $T$.

Definition 5.3. If $\mathcal{S}$ and $\mathcal{T}$ are operator systems and $\phi, \psi : \mathcal{S} \to \mathcal{T}$ are completely positive linear maps such that $\phi - \psi$ is completely positive, then $\psi$ is said to be subordinate to $\phi$, which is denoted by $\psi \leq_{cp} \phi$. If, for given $\phi$, the only completely positive maps $\psi$ that are subordinate to $\phi$ are those $\psi$ of the form $\psi = t \phi$ for some $t \in [0,1] \subset \mathbb{R}$, then $\phi$ is said to be pure.

A completely positive linear map $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$, where is $\mathcal{A}$ is a unital $C^*$-algebra, is pure if and only if the representation $\pi$ that arises in the minimal Stine-spring decomposition of $\phi$ is irreducible \cite[Corollary 1.4.3]{Arveson1976}. In contrast, very little
can be said in general about pure maps of operator systems that are not $C^*$-algebras, and it is in general very difficult to identify which completely positive linear maps of an operator system are pure. However, for operator systems of the form $\mathcal{S}_T$, a ucp map $\phi : \mathcal{S}_T \to M_k(\mathbb{C})$ is pure if and only if $\phi(T) \in \mathcal{W}_k(T)$ is a matrix extreme point of $\mathcal{W}(T)$ \cite[Theorem 5.1]{argerami2022}.

**Theorem 5.4.** Suppose that $T \in \mathcal{B}(\mathcal{H})$ and that $\rho : C^*(T) \to M_k(\mathbb{C})$ is an irreducible representation.

1. If $\rho$ is a boundary representation for $\mathcal{S}_T$, then $\rho(T)$ is a matrix extreme point of $\mathcal{W}(T)$ and $\rho|_{\mathcal{S}_T}$ is a pure ucp map $\mathcal{S}_T \to M_k(\mathbb{C})$.

2. If $C^*(T)$ is $k$-subhomogeneous and if $\rho|_{\mathcal{S}_T}$ is a pure ucp map of $\mathcal{S}_T \to M_k(\mathbb{C})$, then $\rho$ is a boundary representation for $\mathcal{S}_T$.

**Proof.** Assume that $\rho$ is a boundary representation for $\mathcal{S}_T$. Let $\Lambda = \rho(T)$ and suppose that $\Lambda = \sum_{j=1}^{m} A_j^* \Omega_j A_j$ for $\Omega_j \in \mathcal{W}_{n_j}(T)$ and $n_j \times k$ matrices $A_j$ of rank $n_j$ satisfying $\sum_{j=1}^{m} A_j^* A_j = 1$. As $\Omega_j \in \mathcal{W}_{n_j}(T)$, there are ucp maps $\phi_j : C^*(T) \to M_{n_j}(\mathbb{C})$ such that $\phi_j(T) = \Omega_j$, and so the matricial state $\phi = \sum_{j=1}^{m} A_j^* \phi_j A_j$, whereby $X \mapsto \sum_{j=1}^{m} A_j^* \phi_j(X) A_j$, is a ucp extension of $\rho|_{\mathcal{S}_T}$. By hypothesis, $\phi$ must equal $\rho$; hence, for each $j$,

$$A_j^* \phi_j A_j \leq_{cp} \rho.$$

Since $\rho$ is an irreducible representation, it is pure as a completely positive linear map of $C^*(T)$ into $M_k(\mathbb{C})$ \cite[Corollary 1.4.3]{argerami2022}. Thus, there are $t_j \in [0,1]$ such that $A_j^* \phi_j A_j = t_j \rho$.

Let $U_j = t_j^{-1/2} A_j$. Then evaluation at $1 \in C^*(T)$ gives us $U_j^* U_j = 1 \in M_k(\mathbb{C})$. So $U_j$ is isometric and has rank $k$; we knew that $A_j$ (and so $U_j$) has rank $n_j$, and we conclude that $n_j = k$. Then $U_j \in M_k(\mathbb{C})$ is a unitary. But $U_j^* \phi_j U_j = \rho$ implies that $\Omega_j = U_j^* \Omega_j U_j^*$ for each $j$, which shows that $\Lambda$ is a matrix extreme point of $\mathcal{W}(T)$. Therefore, by \cite[Theorem 5.1]{argerami2022}, $\rho|_{\mathcal{S}_T}$ is a pure ucp map $\mathcal{S}_T \to M_k(\mathbb{C})$.

Conversely, suppose that $C^*(T)$ is $k$-subhomogeneous and that $\rho|_{\mathcal{S}_T}$ is a pure ucp map of $\mathcal{S}_T \to M_k(\mathbb{C})$. Let $C_\rho$ be the BW-compact, convex set of all ucp maps $\psi : C^*(T) \to M_k(\mathbb{C})$ that extend $\rho|_{\mathcal{S}_T}$. By the proof of \cite[Theorem B]{argerami2022}, every extreme point $\phi$ of $C_\rho$ is a pure matrix state of $C^*(T)$, and so we need only show that the only pure extension $\phi$ of $\rho|_{\mathcal{S}_T}$ to $C^*(T)$ is $\phi = \rho$. To this end, let $\phi = v^* \pi v$ be a minimal Stinespring decomposition of $\phi$, where $\pi : C^*(T) \to B(\mathcal{H}_\pi)$ is a representation and $v : \mathbb{C}^k \to \mathcal{H}_\pi$ is an isometry. Because $\phi$ is pure, $\pi$ is necessarily irreducible \cite[Corollary 1.4.3]{argerami2022}. Hence, $\dim \mathcal{H}_\pi \leq k$, as $C^*(T)$ is $k$-subhomogeneous. But because $v$ is an isometry, necessarily $\dim \mathcal{H}_\pi = k$. Thus, $v$ is a unitary; it follows that $\phi = v^* \pi v$ is multiplicative; as it agrees with $\rho$ in the generating set $\mathcal{S}_T$, we get that $\phi = \rho$. \hfill $\square$

An operator $T \in \mathcal{B}(\mathcal{H})$ is $k$-normal if any elements $X_1, \ldots, X_{2k}$ in the von Neumann algebra $N_T$ generated by $T$, satisfies

$$\sum_{\tau \in \mathcal{S}_{2k}} \epsilon(\tau) X_{\tau(1)} \cdots X_{\tau(2k)} = 0,$$
where $S_{2k}$ denotes the group of permutations on $\{1, \ldots, 2k\}$ and $\epsilon(\tau)$ denotes the parity (even or odd) of a permutation $\tau$. Because the C$^*$-algebra generated by a $k$-normal operator is $k$-subhomogeneous [7], we obtain the following result:

**Corollary 5.5.** If $T$ is a $k$-normal operator, then the following statements are equivalent for a representation $\rho : C^*(T) \to M_k(\mathbb{C})$:

1. $\rho$ is a boundary representation for $S_T$;
2. $\rho(T)$ is a matrix extreme point of $W(T)$;
3. $\rho|_{S_T}$ is a pure ucp map.

6. Irreducible Periodic Weighted Shift Operators

In this section we present the main result (Theorem 6.5) of the paper. The operators we consider are irreducible periodic weighted unilateral shifts on $\ell^2(N)$; however, it is instructive to consider first the case of unilateral weighted shifts on finite-dimensional Hilbert spaces.

**Definition 6.1.** If $C^* := \mathbb{C} \setminus \{0\}$ and $\xi = \sum_{i=1}^{d} \xi_i e_i \in (C^*)^d$, then the irreducible weighted unilateral shift with weights $\xi_1, \ldots, \xi_d$ is the operator $W(\xi)$ on $C^{d+1}$ given by the matrix

\[
W(\xi) = \begin{bmatrix}
0 & 0 \\
\xi_1 & 0 \\
& \ddots \\
& & \xi_d & 0
\end{bmatrix}.
\]

**Proposition 6.2.** The C$^*$-envelope of an irreducible weighted unilateral shift acting on $C^{d+1}$ is $M_{d+1}(\mathbb{C})$. Furthermore, if $\xi, \eta \in (C^*)^d$, then the operator systems $S_{W(\xi)}$ and $S_{W(\eta)}$ are unitally completely order isomorphic if and only if $|\xi| = |\eta|$, where, for $\nu \in C^d$, $|\nu|$ denotes the vector of moduli of the coordinates of $\nu$.

**Proof.** If $\xi \in (C^*)^d$, then the operator system $S_{W(\xi)}$ is irreducible and, hence, $C^*(S_{W(\xi)}) = M_{d+1}(\mathbb{C})$, which is simple. Therefore, the Šilov boundary ideal for $S_{W(\xi)}$ is necessarily trivial and so $C^*_e(S_{W(\xi)}) = C^*(S_{W(\xi)}) = M_{d+1}(\mathbb{C})$.

Assume now that there is a unital complete order isomorphism $\phi : S_{W(\xi)} \to S_{W(\eta)}$. As both $W(\xi)$ and $W(\eta)$ are irreducible, $\phi$ is necessarily implemented by an automorphism of $M_{d+1}(\mathbb{C})$ [2 Theorem 0.3]; that is, there is a unitary $U$ such that $W(\xi) = U^*W(\eta)U$. But $W(\xi)$ and $W(\eta)$ are unitarily similar if and only if $|\xi| = |\eta|$ (by direct computation or by applying [12 Theorem 3.2]).

Returning to the case of irreducible $p$-periodic weighted unilateral shifts on $\ell^2(N)$, the image of any such operator in the Calkin algebra generates a $p$-homogeneous C$^*$-algebra, and in this case Theorem 5.4 (or Corollary 5.5) could be invoked. However, Theorem 6.4 is an abstract characterisation which yields limited information in specific cases. Therefore, this section aims to give full information about the noncommutative Choquet boundary and the C$^*$-envelope of $S_W$ for irreducible periodic weighted unilateral shifts $W$. 

Definition 6.3. A weighted unilateral shift operator is an operator \( W \) on \( \ell^2(\mathbb{N}) \) defined on the standard orthonormal basis \( \{ e_n : n \in \mathbb{N} \} \) of \( \ell^2(\mathbb{N}) \) by

\[
W e_n = w_n e_{n+1}, \quad n \in \mathbb{N},
\]

where the weight sequence \( \{ w_n \}_{n \in \mathbb{N}} \) for \( W \) consists of nonnegative real numbers with \( \sup_n w_n < \infty \). If there is a \( p \in \mathbb{N} \) such that \( w_{n+p} = w_n \) for every \( n \in \mathbb{N} \), then \( W \) is called a periodic unilateral weighted shift of period \( p \). If at least one of \( w_1, \ldots, w_p \) is not repeated in the list, we say that \( W \) is distinct.

Proposition 4.12 demonstrates that the \( \mathcal{C}^* \)-envelope of the operator system \( \mathcal{S}_W \) generated by a periodic unilateral weighted shift operator \( W \) of period \( p = 1 \) is the abelian \( \mathcal{C}^* \)-algebra \( C(\mathbb{T}) \). To determine the \( \mathcal{C}^* \)-envelope of an irreducible periodic unilateral weighted shift operator of period \( p > 1 \), a notion related to matrix convexity comes into play.

Definition 6.4. Assume that \( \mathcal{E} \subset M_k(\mathbb{C}) \) is a nonempty set.

1. \( \mathcal{E} \) is \( \mathcal{C}^* \)-convex if \( \sum_{j=1}^m A_j^* X_j A_j \in \mathcal{E} \) for every \( m \in \mathbb{N} \), \( X_1, \ldots, X_m \in \mathcal{E} \), and

\[
A_1, \ldots, A_m \in M_k(\mathbb{C}) \text{ satisfying } \sum_{j=1}^m A_j^* A_j = 1.
\]

2. An element \( X \in \mathcal{E} \) is a \( \mathcal{C}^* \)-extreme point of a \( \mathcal{C}^* \)-convex set \( \mathcal{E} \) if the equation

\[
X = \sum_{j=1}^m A_j^* X_j A_j,
\]

for \( X_1, \ldots, X_m \in \mathcal{E} \) and invertible \( A_1, \ldots, A_m \in M_k(\mathbb{C}) \) with \( \sum_{j=1}^m A_j^* A_j = 1 \), implies that there exist unitaries \( U_1, \ldots, U_m \in M_k(\mathbb{C}) \) such that \( X_j = U_j^* X U_j \) for all \( j = 1, \ldots, m \).

Theorem 6.5. Assume that \( f \) \( W \in \mathcal{B}(\ell^2(\mathbb{N})) \) is an irreducible periodic distinct unilateral weighted shift with smallest period \( p \). Then \( \mathcal{C}^*_c(\mathcal{S}_W) = C(\mathbb{T}) \otimes M_p(\mathbb{C}) \) and \( \mathcal{S}_W = \mathcal{K}(\ell^2(\mathbb{N})) \).

Proof. We will assume that \( w_p \notin \{ w_1, \ldots, w_{p-1} \} \); one such weight exists by \( W \) being distinct; we will assume that it is \( w_p \) because it simplifies the writing a little, but the same idea can be used with any other weight. By periodicity and the fact that \( \ell^2(\mathbb{N}) \cong \bigoplus_1^p \ell^2(\mathbb{N}) \), we may express \( W \) as \( p \times p \) matrix of operators acting on \( \ell^2(\mathbb{N}) \) first paragraph in the proof of Theorem 2.2:

\[
W = \begin{bmatrix}
0 & & w_p S \\
w_1 & 0 & \\
& w_2 & \\
& & \ddots & 0 \\
& & & w_{p-1} & 0
\end{bmatrix},
\]

where unspecified entries of the matrix above are zero and \( S \in \mathcal{B}(\ell^2(\mathbb{N})) \) denotes the unilateral shift operator. The operator system \( \mathcal{S}_W \) is an operator subsystem of \( \mathcal{S}_S \otimes M_p(\mathbb{C}) \).
We aim to show first that $C^*(W) = C^*(S_S \otimes M_p(\mathbb{C}))$. Of course we already have the inclusion $C^*(W) \subset C^*(S_S \otimes M_p(\mathbb{C}))$, and so we consider the converse by a method suggested by the proof of [9, Proposition V.3.1]. Note that $C^*(S_S \otimes M_p(\mathbb{C})) = C^*(S_S) \otimes M_p(\mathbb{C})$. Let $\{E_{ij}\}_{i,j=1}^p \subset M_p(\mathbb{C})$ be the standard matrix units for $M_p(\mathbb{C})$, and let $F_{ij} = 1 \otimes E_{ij} \in C^*(S) \otimes M_p(\mathbb{C})$. Because $W$ is irreducible, $w_k > 0$ for all $k$. Note that $|W| = (W^*W)^{1/2} \in C^*(W)$ is the diagonal operator matrix $|W| = \sum_{k=1}^p w_k F_{kk}$. Now let $f \in \mathbb{C}[t]$ be any polynomial for which $f(w_1) = \cdots = f(w_{p-1}) = 0$ and $f(w_p) = 1$ (here is where we use that $W$ is distinct); then $F_{pp} = f(|W|) \in C^*(W)$.

Now for any $i, j \in \{1, \ldots, p\}$,

$$(W^*)^{p-i}F_{pp}W^{p-j} = \alpha_{ij} F_{ij},$$

where $\alpha_{ij} > 0$ is a product of weights $w_k$. Thus, $C^*(W)$ contains each of the matrix units $F_{ij}$. Moreover, $S \otimes E_{11} = \frac{1}{w_p} F_{1p} W F_{p1} \in C^*(W)$. By multiplying $S \otimes E_{11}$ on the left and right with appropriate matrix units $F_{ij}$ we obtain $S \otimes E_{ij} \in C^*(W)$ for every $i$ and $j$. Hence, $S_S \otimes M_p(\mathbb{C}) \subset C^*(W)$ and so $C^*(S_S \otimes M_p(\mathbb{C})) = C^*(W)$.

Because $S$ is a proper isometry, Proposition 4.12 states that $C^*_e(S_S) = C(T)$. Hence, there is an epimorphism $\pi : C^*(S_S) \to C(T)$ such that $\pi|_{S_S}$ is a completely isometric linear map that maps $S$ to the function $z \in C(T)$ given by $z(e^{i\theta}) = e^{i\theta}$. Due to the fact that $C(T)$ is abelian, it is easy to see that $\pi \equiv 0$ when restricted to the compact operators. Let $\rho = \pi \otimes \text{id}_{M_p}$, which is an epimorphism of $C^*(S_S) \otimes M_p(\mathbb{C})$ onto $C(T) \otimes M_p(\mathbb{C})$ such that $\rho|_{S_S \otimes M_p(\mathbb{C})}$ is a unital completely isometric map. Therefore, $\iota := \rho|_{S_W}$ is a completely isometric embedding of $S_W$ into $C(T) \otimes M_p(\mathbb{C})$:

$$S_W \hookrightarrow C^*(S_W) \hookrightarrow C(T) \otimes M_p(\mathbb{C}).$$

Under this embedding $\iota$, $W$ is mapped to the matrix

$$\iota(W) = \begin{bmatrix} 0 & w_p z \\
 w_1 & 0 \\
 \vdots & \ddots \\
 w_{p-1} & 0 \\
 0 & w_1 \\
 \end{bmatrix}.$$

Because $\rho$ is onto, the $C^*$-algebra $C(T) \otimes M_p(\mathbb{C})$ is generated by $S_{\iota(W)}$, the completely isomorphic copy of $S_W$.

Hence, we need no longer work with $W$ and $C^*(W)$, but may instead study $S_{\iota(W)}$ and $C^*(\iota(W))$. In this regard, we show that the Šilov boundary ideal of $S_{\iota(W)}$ is $\{0\}$, which implies that

$$C^*_e(S_W) = C^*_e(S_{\iota(W)}) = C^*(S_{\iota(W)}) = C(T) \otimes M_p(\mathbb{C}).$$

This is achieved by showing that every irreducible representation of $C(T) \otimes M_p(\mathbb{C})$ is a boundary representation for $S_{\iota(W)}$.

To end this, observe first that the irreducible representations of $C(T) \otimes M_p(\mathbb{C})$ are determined by points $\lambda \in \mathbb{T}$ and are of the form

$$\pi_{\lambda} : C(T) \otimes M_p(\mathbb{C}) \longrightarrow M_p(\mathbb{C}),$$

$$(f_{kj})^p_{k,j=1} \longmapsto (f_{kj}(\lambda))^p_{k,j=1}.$$
where \( f_{kj} \in C(T) \). For each \( \lambda \in T \) let \( \Omega_\lambda \in M_p(C) \) denote the (irreducible) matrix \( \Omega_\lambda = \pi_\lambda(\iota(W)) \). By [6] Theorems 3.9, 3.10, the \( C^* \)-convex hull of the set \( \{ \Omega_\lambda : \lambda \in T \} \) is precisely the set \( M_p \) of all matrices of the form \( \Phi(\iota(W)) \), where \( \Phi : C(T) \otimes M_p(C) \to M_p(C) \) is an arbitrary ucp map, i.e.

\[
M_p = \{ \Phi(\iota(W)) : \Phi : C(T) \otimes M_p(C) \to M_p(C) \text{ ucp } \}.
\]

Because every \( \Omega_\lambda \) is irreducible, every structural element of \( M_p \) is unitarily equivalent to some \( \Omega_\lambda \), by Morenz’s Krein–Milman Theorem [13] Theorem 4.5]. Hence, for at least one \( \lambda_0 \in T \) the matrix \( \Omega_{\lambda_0} \) is a \( C^* \)-extreme point of \( M_p \). We now show that for this particular \( \lambda_0 \) the irreducible representation \( \pi_{\lambda_0} \) is a boundary representation for \( S_\iota(W) \).

The BW-compact set \( C_{\lambda_0} \) of all ucp maps \( \psi : C(T) \otimes M_p(C) \to M_p(C) \) that extend \( \pi_{\lambda_0}|_{S_{\iota(W)}} \) is convex; thus, it is sufficient to show that if \( \phi \) is an extreme point of \( C_{\lambda_0} \), then \( \phi = \pi_{\lambda_0} \). Because \( C^* \)-extreme points of matrix sets are also extreme points, \( \Omega_{\lambda_0} \) is an extreme point of \( M_p \). Hence, by a standard convexity argument, the extreme point \( \phi \) of \( C_{\lambda_0} \) is also an extreme point of the set of all ucp maps \( \theta : C(T) \otimes M_p(C) \to M_p(C) \). Now we write \( \phi = V^* \pi V \) using a minimal Stinespring decomposition, where \( V \) is an isometry \( C^p \to \mathcal{H}_\pi \) and \( \pi : C(T) \otimes M_p(C) \to B(\mathcal{H}_\pi) \) for some Hilbert space \( \mathcal{H}_\pi \). By [11] Theorem 1.4.6, the subspace \( V C^p \subset \mathcal{H}_\pi \) is faithful for the commutant of \( \pi(C(T) \otimes M_p(C)) \). Hence, \( \pi(C(T) \otimes M_p(C))V C^p \)

is dense in \( \mathcal{H}_\pi \), and so \( \mathcal{H}_\pi \) is finite-dimensional. Thus, we can write \( \pi = \bigoplus_{j=1}^m \pi_j \) as a decomposition into a finite direct sum of irreducible (sub)representations \( \pi_j \), where \( \mathcal{H}_{\pi_j} \subset \mathcal{H}_\pi \) is a subspace. Then each \( P_j = \pi_j(1) \) is a central projection in \( \pi(C(T) \otimes M_p(C))' \), and \( \sum_{j=1}^m P_j = 1 \) in \( B(\mathcal{H}_\pi) \).

Because the spectrum of the \( C^* \)-algebra \( C(T) \otimes M_p(C) \) is \( T \), for each \( j = 1, \ldots, m \) there is a \( \lambda_j \in T \) such that \( \pi_j = \pi_{\lambda_j} \). Therefore, we can write, for \( f \in C(T) \otimes M_p(C) \),

\[
\phi(f) = V^* \pi(f)V = V^* \left( \sum_{j=1}^m \pi_{\lambda_j}(f)P_j \right) V = \sum_{j=1}^m (P_j V)^* \pi_{\lambda_j}(f)(P_j V)
\]

Note that

\[
(1) \quad \sum_{j=1}^m (P_j V)^* (P_j V) = \sum_{j=1}^m V^* P_j V = V^* V = 1.
\]

If \( \xi \in C^p \) is a unit vector we define, for nonzero \( P_j V \xi \), \( \hat{\xi}_j = \| P_j V \xi \|^{-1} P_j V \xi \); otherwise we let \( \hat{\xi}_j = 0 \). Then

\[
\langle \Omega_{\lambda_0} \xi, \xi \rangle = \langle \phi(\iota(W)) \xi, \xi \rangle = \sum_{j=1}^m \langle \pi_{\lambda_j}(\iota(W)) P_j V \xi, P_j V \xi \rangle = \sum_{j=1}^m \| P_j V \xi \|^2 \langle \Omega_{\lambda_j} \hat{\xi}_j, \hat{\xi}_j \rangle.
\]
The equality in (11) implies that $\sum_{j=1}^{m} \|P_j V \xi\|^2 = 1$ (i.e. they are convex coefficients), and so we obtain

$$W(\Omega_{\lambda_0}) \subseteq \text{Conv} \left( \bigcup_{j=1}^{m} W(\Omega_{\lambda_j}) \right).$$

If $\zeta, \nu \in \mathbb{T}$ are arbitrary, then the moduli of the weights in the shift matrices $\Omega_{\zeta}$ and $\Omega_{\nu}$ coincide; thus, $\Omega_{\zeta}$ and $\Omega_{\nu}$ have the same numerical radius [22, Lemma 2(2)]. Hence, there is a constant $r > 0$ such that the numerical radius of $\Omega_{\zeta}$ is $r$ for every $\zeta \in \mathbb{T}$. Furthermore, for any $\zeta \in \mathbb{T}$,

$$W(\Omega_{\zeta}) \cap r\mathbb{T} = \{\omega^k \zeta : k = 1, \ldots, p\},$$

where $\omega \in \mathbb{C}$ is a primitive $p$-th root of unity [22, Proposition 3]. Thus, there are exactly $p$ extreme points of the numerical range of any $\Omega_{\zeta}$ on the circle $r\mathbb{T}$. Hence, the only way in which the inclusion (2) can hold is if $\lambda_j = \lambda_0$ for every $j$.

Consequently,

$$\Omega_{\lambda_0} = \phi(\iota(W)) = \sum_{j=1}^{m} (P_j V)^* \pi_{\lambda_0} (\iota(W)) P_j V = \sum_{j=1}^{m} (P_j V)^* \Omega_{\lambda_0} P_j V.$$

Now because $\Omega_{\lambda_0}$ is an irreducible $C^*$-extreme point of $\mathfrak{M}_p$, the expression above holds only if there are unitaries $U_1, \ldots, U_m \in M_p(\mathbb{C})$ and convex coefficients $t_j \in (0, 1)$ such that $P_j V = t_j^{1/2} U_j$ [13, Corollary 1.8]. Thus,

$$\Omega_{\lambda_0} = \sum_{j=1}^{m} t_j U_j^* \Omega_{\lambda_0} U_j.$$

However, every matrix is an extreme point of the convex hull of its unitary orbit and so $U_j^* \Omega_{\lambda_0} U_j = \Omega_{\lambda_0}$ for each $j$. As $\Omega_{\lambda_0}$ is irreducible, each $U_j$ is the identity and so

$$\phi = \sum_{j=1}^{m} (P_j V)^* \pi_{\lambda_0} P_j V = \sum_{j=1}^{m} t_j U_j^* \pi_{\lambda_0} U_j = \sum_{j=1}^{m} t_j \pi_{\lambda_0} = \pi_{\lambda_0}.$$

This completes the proof that $\pi_{\lambda_0}$ is a boundary representation for at least one $\lambda_0 \in \mathbb{T}$.

Note that we have

$$\mathfrak{M}_p = \{ \Phi(W) : \Phi : C^*(W) \to M_p(\mathbb{C}), \text{ucp}, \Phi(K) = 0 \forall K \in C^*(W) \cap \mathcal{K}(\mathcal{H}) \}.$$
\(e^{ip\theta}S = VSV^*\) (one can write this unitary explicitly: it is the diagonal unitary in \(B(\ell^2(\mathbb{N}))\) with diagonal \((1, e^{ip\theta}, e^{2ip\theta}, \ldots)\)). Let \(U\) be the block-diagonal unitary

\[
U = \begin{bmatrix}
V & e^{i\theta}V & \cdots \\
& e^{2i\theta}V & \\
& \& e^{(p-1)i\theta}V
\end{bmatrix}
\]

A straightforward computation then shows that \(UW = e^{i\theta}WU\), and so \(UWU^* = e^{i\theta}W\). We conclude that \(\mathfrak{M}_p\) is closed under multiplication by scalars of modulus 1.

Now select an arbitrary \(\lambda' \in \mathbb{T}\). We aim to show that \(\pi_{\lambda'}\) is a boundary representation. To do so, by the method of proof above applied to \(\pi_{\lambda_0}\) it is sufficient to show that \(\Omega_{\lambda'}\) is an irreducible \(C^*\)-extreme point of \(\mathfrak{M}_p\). Because the weighted shift matrix \(\Omega_{\lambda'}\) differs from \(\Omega_{\lambda_0}\) in the \((1,p)\)-entry only, and because \(|\lambda'| = |\lambda_0|\), there are a unitary \(U' \in M_p(\mathbb{C})\) and a \(\theta \in \mathbb{R}\) such that \(e^{i\theta}\Omega_{\lambda'} = (U')^*\Omega_{\lambda_0}U'\) [22, Lemma 2(2)]. As \(C^*\)-extreme points are closed under unitary similarity and because \(\Omega_{\lambda_0}\) is \(C^*\)-extremal in \(\mathfrak{M}_p\), we deduce that \(e^{i\theta}\Omega_{\lambda'}\) is a \(C^*\)-extreme point of \(\mathfrak{M}_p\). That is, \(\Omega_{\lambda'}\) is a \(C^*\)-extreme point of \(e^{-i\theta}\mathfrak{M}_p = \mathfrak{M}_p\).

Hence, the boundary representations for \(S_W\) are precisely the irreducible representations of \(C^*(W)\) of the form \(\pi_{\lambda} \circ \pi\), for all \(\lambda \in \mathbb{T}\), which is to say that \(C^e_{\pi}(S_W) = C(T) \otimes M_p(\mathbb{C})\) and \(\mathfrak{S}_W = \mathcal{K}\left(\ell^2(\mathbb{N})\right)\). \(\Box\)

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Department of Mathematics and Statistics, University of Regina, Regina, SK S4S 0A2, Canada

E-mail address: argerami@math.uregina.ca

Department of Mathematics and Statistics, University of Regina, Regina, SK S4S 0A2, Canada

E-mail address: douglas.farenick@uregina.ca