Integration-by-parts reductions via unitarity cuts and algebraic geometry

Kasper J. Larsen
University of Southampton

Taming the complexity of multi-loop integrals

ETH Zurich, 4th of June 2018

Based on PRD 93(2016)041701, 1712.09737 and 1805.01873 with Yang Zhang et al.
Overview

1. Motivation

2. Baikov representation

3. IBP identities on cuts

4. Syzygy equation and its solution

5. Main example:
Integration-by-parts reductions

Integration-by-parts identities arise from the vanishing integration of total derivatives,

\[
\int \prod_{i=1}^{L} \frac{d^D \ell_i}{\pi^{D/2}} \sum_{j=1}^{L} \frac{\partial}{\partial \ell_j^\mu} \frac{v_j^\mu P}{D_1^{a_1} \cdots D_k^{a_k}} = 0
\]

where \( P \) and \( v_j^\mu \) are polynomials in \( \ell_i, p_j, \) and \( a_i \in \mathbb{N} \).

Role in perturbative QFT calculations:

- **Reduction.** IBP identities reduce any set of loop integrals to a typically much smaller set of master integrals.

- **Computing master integrals.** Using IBP reduction, the master integrals \( \mathcal{I}_j \) can be computed via differential equations:

\[
\frac{\partial}{\partial x_m} \mathcal{I}(x, \epsilon) = A_m(x, \epsilon) \mathcal{I}(x, \epsilon)
\]

where \( x_m \) denotes a kinematical invariant.
IBP reductions on unitarity cuts

Standard approach: enumerate all linear relations and apply Gauss-Jordan elimination to large linear systems

[Laporta, Int.J.Mod.Phys. A 15 (2000) 5087-5159]

Idea here: use unitarity cuts to block-diagonalize system

\[
\begin{pmatrix}
\vdots \\
\vdots \\
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\vdots \\
\vdots \\
\end{pmatrix}
\]

We use the Baikov representation \((k = \frac{L(L+1)}{2} + LE)\),

\[
I(N; a) = \prod_{j=1}^{L} \frac{d^D \ell_j}{i \pi^{D/2}} \frac{N}{D_1^{a_1} \cdots D_k^{a_k}} = \int \frac{dz_1 \cdots dz_k}{z_1^{a_1} \cdots z_k^{a_k}} \text{Gram}(z) \frac{D-L-E-1}{2} N
\]

[Baikov, Phys.Lett. B 385 (1996) 404-410]

in which cuts are straightforward to apply,

\[
\int \frac{dz_i}{z_i^{a_i}} \rightarrow \oint_{\Gamma_{\epsilon}(0)} \frac{dz_i}{z_i^{a_i}} \quad i \in S_{\text{cut}}
\]
Consider a generic loop integral,

\[ I(N; \alpha_1, \ldots, \alpha_m; D) = \int \prod_{j=1}^{L} \frac{d^D \ell_j}{i\pi^{D/2}} \frac{D_{\alpha_{k+1}} \cdots D_{\alpha_m}}{D_{\alpha_1} \cdots D_{\alpha_k}} \]

Let \( \{v_1, \ldots, v_{E+L}\} \equiv \{p_1, \ldots, p_E, \ell_1, \ldots, \ell_L\} \) and, with \( x_{i,j} = v_i \cdot v_j \)

\[
U = \begin{vmatrix}
  x_{1,1} & \cdots & x_{1,E} \\
  \vdots & \ddots & \vdots \\
  x_{E,1} & \cdots & x_{E,E}
\end{vmatrix}
\quad \text{and} \quad
F = \begin{vmatrix}
  x_{1,1} & \cdots & x_{1,E+L} \\
  \vdots & \ddots & \vdots \\
  x_{E+L,1} & \cdots & x_{E+L,E+L}
\end{vmatrix}
\]

The Baikov variables are the inverse propagators \( (m = LE + \frac{L(L+1)}{2}) \)

\[
z_\alpha = D_\alpha = \sum_{\beta=1}^{m} A_{\alpha\beta} x_\beta + \sum_{1 \leq i \leq j \leq E} (B_\alpha)_{ij} \lambda_{ij} \quad \text{with} \quad A_{\alpha\beta}, (B_\alpha)_{ij} \in \mathbb{Z}
\]

The Baikov representation is

\[
I(N; \alpha; D) \propto U^{\frac{E-D+1}{2}} \int dz_1 \cdots dz_m \frac{Z_{\alpha_{k+1}}^{\alpha_{k+1}} \cdots Z_{\alpha_m}^{\alpha_m}}{Z_1^{\alpha_1} \cdots Z_k^{\alpha_k}} F^{\frac{D-L-E-1}{2}}
\]
Example: Zurich-flag cut

Let us find the IBP reductions of the double-box integral. We start by allowing only integrals which contain all Zurich-flag propagators:

Define $S_{\text{cut}} = \{1, 2, 4, 5, 7\}$.

After cutting $\frac{1}{\tilde{z}_i} \to \delta(\tilde{z}_i)$, $i \in S_{\text{cut}}$, the double-box integral takes the form

$$I_{\text{cut}}^{\text{DB}} [P] = \int \prod_{i=1}^{9} d\tilde{z}_i \frac{F(\tilde{z})}{\tilde{z}_3 \tilde{z}_6} \prod_{j \in S_{\text{cut}}} \delta(\tilde{z}_j) P(\tilde{z})$$

As the cut sets $\tilde{z}_{\{1,2,4,5,7\}}$ to zero, we set $z_{\{1,2,3,4\}} = \tilde{z}_{\{3,6,8,9\}}$ in the following.
After integrating out the delta functions and relabeling we have

\[ I_{\text{cut}}^{\text{DB}}[P] = \int \frac{dz_1 \, dz_2 \, dz_3 \, dz_4}{z_1 z_2} F(z) \frac{D-6}{2} P(z). \]

An IBP relation corresponds to a total derivative or, equivalently, an exact diff. form. The generic exact diff. form of the form \( I_{\text{cut}}^{\text{DB}} \) is

\[ 0 = \int d \left[ \sum_{i=1}^{4} \frac{(-1)^{i+1} a_i(z) F(z) \frac{D-6}{2}}{z_1 z_2} dz_1 \wedge \cdots \wedge \widehat{dz_i} \wedge \cdots \wedge dz_4 \right] \]

\[ = \int \left[ \sum_{i=1}^{4} \frac{\partial}{\partial z_i} \left( \frac{a_i(z) F(z) \frac{D-6}{2}}{z_1 z_2} \right) \right] dz_1 \wedge \cdots \wedge dz_4 \]

\[ = \int \left[ \sum_{i=1}^{4} \left( \frac{\partial a_i}{\partial z_i} + \frac{D-6}{2F} a_i \frac{\partial F}{\partial z_i} \right) - \sum_{j=1,2} a_j \frac{1}{z_j} \right] F(z) \frac{D-6}{2} dz_1 \wedge \cdots \wedge dz_4. \]

The red term corresponds to an integral in \((D-2)\) dimensions, and the purple term in general produces squared propagators.
To get the generic exact form

\[ 0 = \int \left[ \sum_{i=1}^{4} \left( \frac{\partial a_i}{\partial z_i} + \frac{D - 6}{2F} a_i \frac{\partial F}{\partial z_i} \right) - \sum_{j=1,2} \frac{a_j}{z_j} \right] F(z)^{\frac{D-6}{2}} \frac{dz_1 \wedge \cdots \wedge dz_4}{z_1 z_2} \]

to correspond to an IBP relation in \( D \) dimensions with only single-power propagators, we demand that each term is polynomial,

\[ \sum_{i=1}^{4} \frac{D - 6}{2F} a_i \frac{\partial F}{\partial z_i} = \tilde{b} \quad \implies \quad \sum_{i=1}^{4} a_i \frac{\partial F}{\partial z_i} + bF = 0 \quad \text{(with \( b = \frac{2}{6-D} \tilde{b} \))} \]

\[ a_j = \tilde{b}_j z_j \quad \implies \quad a_j + b_j z_j = 0 \quad \text{(with \( b_j = -\tilde{b}_j \))} , \]

with \( a_i, b_i, b \) polynomials in \( z \). Such equations, with polynomial solutions, are known in algebraic geometry as syzygy equations.

[Gluza, Kajda, Kosower, PRD 83(2011)045012], [Schabinger, JHEP 01(2012)077], [Ita, PRD 94(2016)116015]

Obtain IBPs by plugging \((a_i, b)\) into the top equation.

Note: \((qa_i, qb)\) is also a solution, for polynomial \( q \).
Strategy to solve syzygy equations

Solve syzygy equations with $c$ cuts

\[
\sum_{j=1}^{m-c} a_j \frac{\partial F}{\partial z_k} + bF = 0 \quad (1)
\]

\[
a_j + b_j z_j = 0, \quad j = 1, \ldots, k-c \quad (2)
\]

as follows

1) Find syzygy generators $\mathcal{M}_1 = \langle(a_1, \ldots, a_m, b), \ldots\rangle$ of eq. (1) for the off-shell case $c = 0$.

2) The generators of eq. (2) are trivial:

\[
\mathcal{M}_2 = \langle z_1 e_1, \ldots, z_k e_k, e_{k+1}, \ldots, e_m \rangle
\]

3) Take module intersection $\mathcal{M}_1|_{\text{cut}} \cap \mathcal{M}_2|_{\text{cut}}$. 

Kasper J. Larsen
University of Southampton
A generating set of solutions of
\[ \sum_{\alpha=1}^{m} a_{\alpha} \frac{\partial F}{\partial z_{\alpha}} + bF = 0 \]
can be obtained from Gröbner basis calculations (Schreyer’s thm).

also: [Bern, Enciso, Ita, Zeng, PRD 96(2017)096017]

\[ F \text{ is a determinant} \rightarrow \text{solutions can be explicitly found!} \]

Laplace expansion of generic matrix:
\[
\left[ \sum_{k=1}^{n} r_{jk} \frac{\partial (\det R)}{\partial r_{ik}} \right] - \delta_{ij} \det R = 0
\]
Syzygies from Laplace expansion

A generating set of solutions of
\[ \sum_{\alpha=1}^{m} a_{\alpha} \frac{\partial F}{\partial z_{\alpha}} + bF = 0 \]
can be obtained from Gröbner basis calculations (Schreyer’s thm).

also: [Bern, Enciso, Ita, Zeng, PRD 96(2017)096017]

\( F \) is a determinant \( \implies \) solutions can be explicitly found!

Laplace expansion of a symmetric matrix \( S \):
\[
\left[ \sum_{k=1}^{n} (1+\delta_{ik}) s_{jk} \frac{\partial (\det S)}{\partial s_{ik}} \right] - 2\delta_{ij} \det S = 0
\]
A generating set of solutions of
\[ \sum_{\alpha=1}^{m} a_\alpha \frac{\partial F}{\partial z_\alpha} + bF = 0 \]
can be obtained from Gröbner basis calculations (Schreyer’s thm).

also: [Bern, Enciso, Ita, Zeng, PRD 96(2017)096017]

\( F \) is a determinant \( \rightarrow \) solutions can be explicitly found!

Laplace expansion of \( S = \text{Gram}(v_1, \ldots, v_{E+L}) \):
\[
\left[ \sum_{k=1}^{E+L} (1+\delta_{ik})x_{jk} \frac{\partial F}{\partial x_{ik}} \right] - 2\delta_{ij}F = 0
\]

Using the chain rule this becomes
\[
\sum_{\alpha=1}^{m} \left[ \sum_{k=1}^{E+L} (1+\delta_{ik})x_{jk} \frac{\partial z_\alpha}{\partial x_{ik}} \right] \frac{\partial F}{\partial z_\alpha} - 2\delta_{ij}F = 0 \quad \left\{ \begin{array}{l}
E + 1 \leq i \leq E + L \\
1 \leq j \leq E + L
\end{array} \right.
\]

Proof (based on Józefiak complex) of completeness of syzygies.

[Böhm, Georgoudis, KJL, Schulze, Zhang, 1712.09737]
Example 1: syzygies of planar double box

Set $P_{12} = p_1 + p_2$ and

\[
\begin{align*}
z_1 &= \ell_1^2, & z_2 &= (\ell_1 - p_1)^2, & z_3 &= (\ell_1 - P_{12})^2 \\
z_4 &= (\ell_2 + P_{12})^2, & z_5 &= (\ell_2 - p_4)^2, & z_6 &= \ell_2^2 \\
z_7 &= (\ell_1 + \ell_2)^2, & z_8 &= (\ell_1 + p_4)^2, & z_9 &= (\ell_2 + p_1)^2
\end{align*}
\]

Only need to find explicit relation $z = Ax$. Here

\[
A = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
-2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
-2 & 0 & -2 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 2 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\
-2 & 0 & -2 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

Set $t_{i,j} = (a_\alpha, b)$. The syzygy generators are *linear* in the $z_k$

\[
\begin{align*}
t_{4,1} &= (z_1-z_2, z_1-z_2, -s+z_1-z_2, 0, 0, 0, z_1-z_2-z_6+z_9, t+z_1-z_2, 0, 0) \\
t_{4,2} &= (s+z_2-z_3, z_2-z_3, z_2-z_3, 0, 0, 0, z_2-z_3+z_4-z_9, -t+z_2-z_3, 0, 0) \\
t_{4,3} &= (-s+z_3-z_8, t+z_3-z_8, z_3-z_8, 0, 0, 0, z_3-z_4+z_5-z_8, z_3-z_8, 0, 0) \\
t_{4,4} &= (2z_1, z_1+z_2, -s+z_1+z_3, 0, 0, 0, z_1-z_6+z_7, z_1+z_8, 0, -2) \\
t_{4,5} &= (-z_1-z_6+z_7, -z_1+z_7-z_9, -z_1-z_4+z_7, 0, 0, 0, -z_1+z_6+z_7, -z_1-z_5+z_7, 0, 0) \\
t_{5,1} &= (0, 0, 0, s-z_6+z_9, -t-z_6+z_9, z_9-z_6, z_1-z_2-z_6+z_9, 0, z_9-z_6, 0) \\
t_{5,2} &= (0, 0, 0, z_4-z_9, t+z_4-z_9, -s+z_4-z_9, z_2-z_3+z_4-z_9, 0, z_4-z_9, 0) \\
t_{5,3} &= (0, 0, 0, z_5-z_4, z_5-z_4, s-z_4+z_5, z_3-z_4+z_5-z_8, 0, -t-z_4+z_5, 0) \\
t_{5,4} &= (0, 0, 0, s-z_3-z_6+z_7, -z_6+z_7-z_8, -z_1-z_6+z_7, z_1-z_6+z_7, 0, -z_2-z_6+z_7, 0) \\
t_{5,5} &= (0, 0, 0, -s+z_4+z_6, z_5+z_6, 2z_6, -z_1+z_6+z_7, 0, z_6+z_9, -2)
\end{align*}
\]
$\text{Example 2: syzygies of non-planar double pentagon}$

Set $P_{i,j} \equiv p_i + p_j$ and

\[
\begin{align*}
z_1 &= \ell_1^2, & z_2 &= (\ell_1 - p_1)^2, & z_3 &= (\ell_1 - P_{1,2})^2, \\
z_4 &= (\ell_2 - P_{3,4})^2, & z_5 &= (\ell_2 - p_4)^2, & z_6 &= \ell_2^2, \\
z_7 &= (\ell_1 + \ell_2)^2, & z_8 &= (\ell_1 + \ell_2 + p_5)^2, & z_9 &= (\ell_1 + p_3)^2, \\
z_{10} &= (\ell_1 + p_4)^2, & z_{11} &= (\ell_2 + p_1)^2
\end{align*}
\]

Here $z = Ax$ with

\[
A = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
-2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
-2 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & -2 & 0 & -2 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 \\
-2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & 1 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
\]

and the syzygy generators are again compact:
Computing module intersections

Given $M_1 = \langle v_1, \ldots, v_p \rangle$ and $M_2 = \langle w_1, \ldots, w_q \rangle$ with $v_i, v_w$ $m$-tuples of polynomials. Let $Q$ denote the $m \times (p+q)$ matrix

$$Q = \begin{pmatrix} v_1 & \cdots & v_p & w_1 & \cdots & w_q \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \cdots & \ddots & \vdots & \cdots & \ddots \end{pmatrix}$$

Then compute wrt. POT and variable order $[z_1, \ldots, z_m] \succ [s_{ij}]$

$$\langle h_1, \ldots, h_t \rangle \equiv \text{Gr"obner basis of column space of} \begin{pmatrix} Q \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

Selecting $h_i = (0, \ldots, 0, x_1, \ldots, x_p, y_1, \ldots, y_q)$, we have

$$0 = \sum_{j=1}^{p} x_j v_j + \sum_{k=1}^{q} y_k w_k$$
Computing module intersections

Given $\mathcal{M}_1 = \langle v_1, \ldots, v_p \rangle$ and $\mathcal{M}_2 = \langle w_1, \ldots, w_q \rangle$ with $v_i, v_w$ $m$-tuples of polynomials. Let $Q$ denote the $m \times (p+q)$ matrix

$$Q = \begin{pmatrix}
\vdots & \cdots & \vdots & \cdots & \vdots \\
v_1 & \cdots & v_p & w_1 & \cdots & w_q \\
\vdots & \cdots & \vdots & \cdots & \vdots
\end{pmatrix}$$

Then compute wrt. POT and variable order $[z_1, \ldots, z_m] \succ [s_{ij}]$.

$$\langle h_1, \ldots, h_t \rangle \equiv \text{Gröbner basis of column space of} \begin{pmatrix} Q \\ 1 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

Selecting $h_i = (0, \ldots, 0, x_1, \ldots, x_p, y_1, \ldots, y_q)$, we have

$$0 = \sum_{j=1}^p x_j v_j + \sum_{k=1}^q y_k w_k \implies \sum_{j=1}^p x_j v_j = - \sum_{k=1}^q y_k w_k \in \mathcal{M}_1 \cap \mathcal{M}_2$$

Hence $\sum_{j=1}^p x_j v_j$ generate $\mathcal{M}_1 \cap \mathcal{M}_2$, taking $(x_1, \ldots, x_p)$ from each $h_i$. 
To find the complete IBP reduction, we must consider the cuts associated with “uncollapsible” masters:

A bit more explicitly, the cuts we need to consider are
Main example: non-planar hexagon box

Task: IBP reduce non-planar hexagon box with numerator insertions of degree four in the $z_i$

There are 10 cuts to consider:

where

\[
\begin{align*}
    z_1 &= \ell_1^2, \\
    z_4 &= (\ell_1 - P_{123})^2, \\
    z_7 &= (\ell_2 - p_5)^2, \\
    z_{10} &= (\ell_2 + p_1)^2, \\
    z_2 &= (\ell_1 - p_1)^2, \\
    z_5 &= (\ell_1 + \ell_2 + p_4)^2, \\
    z_8 &= \ell_2^2, \\
    z_{11} &= (\ell_2 + p_2)^2 \\
    z_3 &= (\ell_1 - P_{12})^2, \\
    z_6 &= (\ell_1 + \ell_2)^2, \\
    z_9 &= (\ell_1 + p_5)^2
\end{align*}
\]

[Chicherin, Henn, Mitev JHEP 05(2018)164]
[S. Badger, C. Brønnum-Hansen, H. Hartanto, T. Peraro, PRL 120(2018)092001]
[S. Abreu, F. Cordero, H. Ita, B. Page, M. Zeng, 1712.03946]
[H. Chawdhry, M. Lim, A. Mitov, 1805.09182]
The spanning set of cuts follows from this list of masters, obtained from *Azurite*.

| Cut          | # of master integrals |
|--------------|-----------------------|
| \{1, 5, 7\}  | 26                    |
| \{2, 5, 7\}  | 25                    |
| \{2, 5, 8\}  | 31                    |
| \{2, 6, 7\}  | 31                    |
| \{3, 5, 8\}  | 31                    |
| \{3, 6, 7\}  | 31                    |
| \{3, 6, 8\}  | 25                    |
| \{4, 6, 8\}  | 26                    |
| \{1, 4, 5, 8\}| 13                   |
| \{1, 4, 6, 7\}| 13                   |
Syzygies for ensuring $D$-dimensionality:

$$M_1 = \langle (s_1 - s_2, s_3 - s_4, s_5 - s_6, s_7 - s_8, s_9 - s_{10}, s_1 - s_2 - s_8 + s_{10}, 0, 0, -s_1 + s_2 - s_4 + s_5 + s_9 + s_{10}, 0) \rangle$$

$$M_2 = \langle (s_1 + s_2 + s_3 + s_4 + s_5 + s_6 + s_7, 0, 0, s_9 - s_8 + s_1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0) \rangle$$

Compute intersection of $M_1 \big|_{\text{cut}} \cap M_2 \big|_{\text{cut}}$ on each of the 10 cuts.
Timings and RAM usage $M_1\cap_{cut} \cap M_2\cap_{cut}$ on each of the 10 cuts:

| cut            | time/sec | RAM/GB |
|----------------|----------|--------|
| {1, 5, 7}      | 218      | 4.3    |
| {2, 5, 7}      | 43       | 1.1    |
| {2, 5, 8}      | 303      | 6.7    |
| {2, 6, 7}      | 743      | 9.8    |
| {3, 5, 8}      | 404      | 7.4    |
| {3, 6, 7}      | 699      | 11.0   |
| {3, 6, 8}      | 24       | 1.0    |
| {4, 6, 8}      | 797      | 13.7   |
| {1, 4, 5, 8}   | 53       | 1.7    |
| {1, 4, 6, 7}   | 196      | 3.0    |

The timings are for an Intel Xeon E5-2643 with 24 cores, 3.40 GHz and 384 GB RAM.
Trim the initial overcomplete set of generators, i.e. drop the most complicated ones.

Trimming reduces the string sizes by a factor of $\sim 2^{35}$.

| cut            | original size/MB | trimmed size/MB |
|---------------|------------------|-----------------|
| $\{1, 5, 7\}$ | 68               | 10              |
| $\{2, 5, 7\}$ | 25               | 1.4             |
| $\{2, 5, 8\}$ | 49               | 3.1             |
| $\{2, 6, 7\}$ | 100              | 2.8             |
| $\{3, 5, 8\}$ | 97               | 3.7             |
| $\{3, 6, 7\}$ | 80               | 3.6             |
| $\{3, 6, 8\}$ | 10               | 1.6             |
| $\{4, 6, 8\}$ | 21               | 1.6             |
| $\{1, 4, 5, 8\}$ | 4.4     | 3.6             |
| $\{1, 4, 6, 7\}$ | 9.4     | 4.1             |

Plug resulting generators into ansatz for total derivative:

$$
0 = \int \left[ \sum_{i=1}^{m-c} \left( \frac{\partial a_{ri}}{\partial z_{ri}} + \frac{D-L-E-1}{2F(z)} a_{ri} \frac{\partial F}{\partial z_{ri}} \right) - \sum_{i=1}^{k-c} \frac{a_{ri}}{z_{ri}} \right] F(z) \frac{D-L-E-1}{2} \prod_{i=1}^{k-c} z_{ri} \, dz_{r1} \cdots dz_{rm-c}
$$
Trim the obtained systems of IBP identities.

The resulting systems take up about $\sim 1$ MB each and are sparse.

| cut       | # equations | # integrals | byte size/MB | density |
|-----------|-------------|-------------|--------------|---------|
| \{1, 5, 7\} | 1144        | 1177        | 1.2          | 1.4%    |
| \{2, 5, 7\} | 1170        | 1210        | 0.99         | 1.3%    |
| \{2, 5, 8\} | 1152        | 1190        | 1.1          | 1.5%    |
| \{2, 6, 7\} | 1118        | 1155        | 1.0          | 1.5%    |
| \{3, 5, 8\} | 1160        | 1202        | 1.2          | 1.5%    |
| \{3, 6, 7\} | 1173        | 1217        | 1.3          | 1.7%    |
| \{3, 6, 8\} | 1135        | 1176        | 0.77         | 1.2%    |
| \{4, 6, 8\} | 1140        | 1176        | 0.94         | 1.2%    |
| \{1, 4, 5, 8\} | 700        | 723         | 0.69         | 1.7%    |
| \{1, 4, 6, 7\} | 683        | 706         | 0.66         | 1.6%    |
Gauss-Jordan elimination of IBP systems

To find the IBP reductions, Gauss-Jordan eliminate IBP systems.

Some remarks:

- To preserve sparsity, use a *total pivoting* strategy (i.e., allow column swaps).

- For cut \{1, 4, 6, 7\}, the RREF can be performed fully analytically, requiring 31 minutes on one core and 1.5 GB RAM.

- For \{3, 6, 7\}, assigned numerical values to two $s_{ij}$. Ran 440 points on cluster (2.5 h and 1.8 GB RAM per job). Used interpolation code to get analytical results (23 min and 15 GB RAM on one core).

[von Manteuffel and Schabinger, PLB 744(2015)101]
[Peraro, JHEP12(2016)030]
Merging on-shell IBP reductions

By solving the IBP identities on the following cuts

\[
\begin{align*}
(\cdots) \cdot (\cdots) &= \frac{(D - 4)s^2\chi}{8(D - 3)} - \frac{(3D - 2\chi - 12)s}{4(D - 3)} + \frac{(4 - D)(9\chi + 7)}{4(D - 3)} \\
+ 2 \cdot \frac{(10 - 3D)(2\chi - 13)}{8(D - 4)s} + \frac{2D(\chi + 1) - 8\chi - 7}{2(D - 4)s} \\
+ \frac{9(3D - 10)(3D - 8)}{4(D - 4)^2s^2\chi} + \frac{(3D - 10)(3D - 8)(2\chi + 1)}{2(D - 4)^2(D - 3)s^2}
\end{align*}
\]

we reconstruct the complete IBP reductions by merging the partial results.

An example of an IBP relation produced by our method \((\chi \equiv t/s)\):
Results for IBP reductions

- Fully analytic IBP reductions of the 32 hexagon boxes

\[
\{I(1, 1, 1, 1, 1, 1, 1, 0, 0, -4), I(1, 1, 1, 1, 1, 1, 1, 0, -3, -1),
I(1, 1, 1, 1, 1, 1, 1, -1, -1, -2), I(1, 1, 1, 1, 1, 1, 1, -2, 0, -2),
I(1, 1, 1, 1, 1, 1, 1, -3, 0, -1), I(1, 1, 1, 1, 1, 1, 0, 0, -3),
I(1, 1, 1, 1, 1, 1, 1, 0, -3, 0), I(1, 1, 1, 1, 1, 1, 1, -1, -2, 0),
I(1, 1, 1, 1, 1, 1, 1, -3, 0, 0), I(1, 1, 1, 1, 1, 1, 1, 0, -2, 0),
I(1, 1, 1, 1, 1, 1, 1, 0, 0, -1) \}
\]

\[
I(1, 1, 1, 1, 1, 1, 1, 0, -1, -3), I(1, 1, 1, 1, 1, 1, 1, 0, -4, 0),
I(1, 1, 1, 1, 1, 1, 1, -1, -2, -1), I(1, 1, 1, 1, 1, 1, 1, -2, -1, -1),
I(1, 1, 1, 1, 1, 1, 1, -3, -1, 0), I(1, 1, 1, 1, 1, 1, 1, 0, -1, -2),
I(1, 1, 1, 1, 1, 1, 1, -1, 0, -2), I(1, 1, 1, 1, 1, 1, 1, -2, 0, -1),
I(1, 1, 1, 1, 1, 1, 1, 0, -2, 0), I(1, 1, 1, 1, 1, 1, 1, -1, 0, -1),
I(1, 1, 1, 1, 1, 1, 1, 0, -1, 0)
\]

- can be downloaded from (268 MB compressed / 790 MB uncompressed)

  https://github.com/yzhphy/hexagonbox_reduction/releases/download/1.0.0/hexagon_box_degree_4_Final.zip

- Our results agree with fully numerical results from FIRE5 C++

  (6 hours per point).

  [A. Smirnov, CPC 189(2015)182]
Conclusions

• New formalism for IBP reductions. Main ideas: Baikov rep., cuts, syzygies, module intersection algorithms, total pivoting, rational reconstruction

• Obtained the fully analytic IBP reductions of

\[ \text{with numerator insertions up to degree 4 in the } z_i. \]

• Powerful framework. IBP reductions for further $2 \rightarrow 3$ two-loop processes seem well within reach.