The Yangian symmetry and “{V8}” molecule

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Abstract

A new symmetry operator $Q$ is introduced to re-describe the effective Hamiltonian $H_3$ of the Heisenberg spin triangle in the \{V6\} molecule. The operator $Q$ is a special form of Dzyaloshiky-Moriya (DM) interaction for the triangle with three spins itself. By extending $Q$ to a system with four spins we present a new model comprised of four 1/2-spins and show that the ground state of the Hamiltonian $H_4$ for the model may be with $S = 1$ by choosing some suitable exchange constants in $H_4$. In analogy to the molecular \{V6\} where the interaction between two triangles in \{V6\} molecule is assumed also the DM type, we then give the theoretical magnetization of the molecular model comprised of two four-spin systems. Supposed a single parallelogram model with $S = 1$ exists, the local spin moments are shown to be $\frac{1}{10}\mu_B$, $\frac{3}{10}\mu_B$, $\frac{7}{10}\mu_B$, $\frac{9}{10}\mu_B$ respectively for the spin figuration.

Key words: \{V6\} molecule, Yangian, hysteresis, local spin moment

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1. Introduction

The single molecular magnets(SMMs) have attracted much attention both for its scientific importance of studying fundamental issues and for its potential applications. The magnetic molecules \{V6\} and \{V15\} provide us a good platform for exploring these issues, where the total spin is not large. There have been beautiful investigations on these molecules.[1,2,3,4]

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For latter use, we will briefly introduce the structures of the molecular \{V6\} first. As was mentioned in the Ref.\[2\], the molecular \{V6\} is the abbreviation of the magnetic molecule (CN$_3$H$_6$)$_4$Na$_2$[H$_4$V$_6$O$_8$(PO$_4$)$_4$\{(OCH$_2$)$_3$CCH$_2$OH\}]$_2$·14H$_2$O. Its structure is shown in Fig.1. We see that the molecule consists of two pieces, each of which is an isosceles triangle. In each triangle, two of the 2-spin exchange constants are equal (shown in blue $J_a \sim 65K$) and an order of the magnitude of the third (shown in red $J_c \sim 7K$).\[3\] And the experiment has shown that there is some special interaction called Dzyaloshiky-Moriya (DM) interaction\[5\] between the two pieces, whose Hamiltonian can be written as $H_{\text{inter}} = \Delta(S_A \times S_A')$.\[2\] The operators $S_A$ and $S_A'$ are the total spin operators of two triangles, respectively, and the energy gap $\Delta$ is tiny. However, such an interaction can make contributions to the Landau-Zener-Stückelberg (LZS) transition\[6\] when the magnetic field is absent, which can be detected by measuring the magnetization of the molecular \{V6\}.\[2\]

In Ref.\[1, 2, 3, 4\], the wave functions of molecular \{V6\} and the experimental measurements of the system have been shown clearly. However, from the theoretical point of view, each triangle of the molecular \{V6\} is formed by three spins, and the symmetry properties of the triangle desire to be investigated. In this article, we first focus on one piece of the molecular \{V6\}, and introduce a new symmetry operator $Q$ to re-describe the triangular piece. We can see that the commutativity between such a new symmetry operator $Q$ and the Hamiltonian $H_3$ will constrain the parameters in the Hamiltonian $H_3$. The symmetry operator $Q$ looks natural for the triangle model in the molecular \{V6\}, and further we shall extend such a
new symmetry to a four-spin system. Using the extended symmetry operator $Q$ in the four-spin system, we establish its Hamiltonian $H_4$ whose model is usually a parallelogram. By analyzing the Hamiltonian $H_4$, we find that the ground state can be with total spin $S = 1$ in some special cases, which has not been considered before. Based on this assumption, we have made a prediction on the magnetization of such a theoretical molecule comprised of two parallelograms with the DM interaction between them in analogy to the molecular $\{V6\}$.[2] To test the parallelogram model itself, we propose another molecular model and give the prediction of its local spin moments configuration.

This article is organized as follows: In Sec.2 we will investigate the single Heisenberg spin triangle model in $\{V6\}$ and introduce the new quantum number operator $Q$ to describe it. We shall show that the operator $Q$ represents a new symmetry in the three-spin system. In Sec.3 we will extend the symmetry to a system comprised of four particles with each spin 1/2 and establish its Hamiltonian $H_4$. We demonstrate that the ground state of $H_4$ may be with total spin $S = 1$ in some special cases. Besides, we make up a theoretical molecular model called $\{V8\}$ in which there is the DM interaction between the two parallelograms and the prediction of its magnetization in experiment is made. In Sec.4 we propose another molecular model which contains only one parallelogram and give the prediction about its local spin moments configuration. In the Appendix, we shall show details about how the symmetry operator determines the Hamiltonian.

2. the new symmetry in $\{V6\}$

As was mentioned in the Sec.1 the molecular $\{V6\}$ is comprised of two triangles with a tiny DM interaction between them. In this section, we will concentrate on only one triangular piece in the molecular $\{V6\}$. It has been testified that the triangle model is actually isosceles by the experiment.[3] But in this section we would like to introduce a new symmetry operator $Q$ to describe the triangle system.

The Hamiltonian of the usual Heisenberg spin triangle can be written as:

$$H_0 = J_{12}S_1 \cdot S_2 + J_{23}S_2 \cdot S_3 + J_{13}S_1 \cdot S_3,$$

(1)

where $S_1$, $S_2$ and $S_3$ are the spins of each particle, and the relationship among the exchange interaction constants $J_{12}$, $J_{13}$ and $J_{23}$ is unknown. If a magnetic field is
applied along $z$ axis, then the term corresponding to the Zeeman Energy written as $H_{\text{Zeeman}} = \mu B (S_1z + S_2z + S_3z)$ should be included in Eq. (1). The Zeeman term can split the energy with different eigenvalues of $S_z$, where $S = S_1 + S_2 + S_3$ is the total spin operator of the triangle. However, only the quantum numbers $S^2$ and $S_z$ are not adequate to describe a system with three spins. It is easy to see that there are two eigenstates corresponding to the quantum number $S = 1/2, S_z = 1/2$.

The new symmetry operator $Q$ we will introduce is written as $Q = Y^2$, where the operator $Y$ is a special form of the DM interaction in the Heisenberg spin triangle written as:

$$Y = i(S_1 \times S_2 + S_2 \times S_3 + S_3 \times S_1).$$

(2)

It can be verified that such a new operator satisfies the commutation rules as $[Q, S^2] = 0$ and $[Q, S_z] = 0$. So we will use the operator $Q$ to represent a certain symmetry property of the three-spin system just like what we have done in the Hydrogen Atom. (In Mathematics Physics, the operator $Y$ is in fact a special form of the Yangian operator, see Appendix A.) The operator $Q$ can be viewed as a collective quantum number that describe the history besides $S^2$ and $S_z$. If we take the set $\{S^2, S_z, Q\}$ to be the complete operator set of the system, the commutativity $[Q, H] = 0$ must be satisfied. From such an equation, we can easily get the relationship for the exchange interaction constants: $J_{12} = J_{23}$. Fortunately, this relationship $J_{12} = J_{23}$ has been shown to exist in molecular $\{V6\}$. And with this symmetry, the Hamiltonian in Eq. (1) can be simplified as:

$$H_3 = J_{12}(S_1 \cdot S_2 + S_2 \cdot S_3) + J_{13}S_1 \cdot S_3,$$

We emphasize that in the triangle model, the eigenvectors of the operator $Q$ are nondegenerate, and the Yangian symmetry operator $Q$ can uniquely determine the Hamiltonian of the system. So we can use the complete set $\{S^2, S_z, Q\}$ to determine the states in the triangular piece in the $\{V6\}$ model described in Sec 1. By directly diagonalizing the matrix $Q$ in the usual Lie Algebraic representation (see Appendix B), we get the two states with total spin $S = 1/2, S_z = -1/2$ as:

$$|\phi_\alpha\rangle = \frac{-1}{\sqrt{6}} (|\uparrow\downarrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle - 2|\downarrow\uparrow\downarrow\rangle),$$

(3)

$$|\phi_\beta\rangle = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\downarrow\rangle - |\downarrow\downarrow\uparrow\rangle),$$

(4)
which are corresponding to the eigenvalues $\frac{-1}{4}, \frac{-9}{4}$ of $Q$ respectively. It can be verified that these states are the eigenstates of the Hamiltonian $H_3$, and the corresponding levels are $E_{+}^\alpha = \frac{1}{4}J_{13} - J_{12}$ and $E_{-}^\alpha = -\frac{3}{4}J_{13}$.

Hence, the special DM interaction operator $Y$ plays an important role in the Heisenberg spin triangle in $\{V6\}$. Its square form $Q$ is a new quantum number operator called Yangian symmetry operator. Using the symmetry operator $Q$, we can easily determine the form of the Hamiltonian, and directly obtain the proper states of the system. It is reasonable to extend such a Yangian symmetry to a four-spin system, and we will give a discussion about such a system in the next section.

3. an extended $\{V6\}$-type molecular model

3.1. a four-spin system determined by the Yangian symmetry

For a system comprised of four particles with each spin $1/2$, the most general Hamiltonian can be written as:

$$H = \sum_{j>i=1}^{4} a_{ij}S_i \cdot S_j,$$

and the special DM interaction operator for the four-spin system is defined as:

$$Y = i \sum_{j>i=1}^{4} (S_i \times S_j).$$

If there is the Yangian symmetry in the four-spin system, we still take the square of the DM interaction operator $Q = Y^2$ as a quantum number operator to characterize the system. We must emphasize that the DM interaction operator as Eq.(6) is identified with the special form of the Yangian symmetry operator in this four-spin system. (see Appendix.C) Similar to the three-spin system in the Sec.2, we shall get the constraint for the parameters in Eq.(5) from the commutativity $[Q, H] = 0$. To calculate the commutation relationship, we need get the eigenstates of $Q$ first, and then let the Hamiltonian $H$ share the same eigenstates with $Q$. However, it should be noted that in the four-spin system the eigenvectors of the operator $Q$ are partially degenerate in the states with $S = 1$. We need introduce a more viable $\theta$ to form the degenerate states of the Hamiltonian $H_4$ as Eq.(19) shown in the Appendix.C. Through the careful calculation shown in Appendix.C, we obtain the Hamiltonian of the four-spin system as Eq.(21). In a special case($\theta = 0$), we
can get the relationship between the exchange interaction constants in Eq. (5) with the Q-symmetry:

\[
\begin{align*}
    a_{12} &= a_{34}, \quad a_{13} = a_{24}, \\
    a_{14} &= \frac{1}{3} (a_{12} + 2a_{13}), \\
    a_{23} &= \frac{5}{3} a_{12} - \frac{2}{3} a_{13}.
\end{align*}
\]

(7)

So the Hamiltonian can be written as a much simpler form

\[
H_4 = a_{12}(S_1 \cdot S_2 + S_3 \cdot S_4) + a_{13}(S_1 \cdot S_3 + S_2 \cdot S_4)
+ \frac{1}{3}(a_{12} + 2a_{13})S_1 \cdot S_4 + \frac{1}{3}(5a_{12} - 2a_{13})S_2 \cdot S_3.
\]

(8)

The model of such a Hamiltonian is actually a parallelogram model[9] as shown in Fig. 2(a), where there are only two independent exchange constants called \( a_{12}, a_{13} \).

The eigenvalues and eigenstates of the Hamiltonian are as follows:

\[
\begin{align*}
    H_4 \phi_{2,m} &= (a_{12} + \frac{1}{2} a_{13}) \phi_{2,m}, \\
    H_4 \psi_{1,m} &= -\frac{1}{2} a_{13} \psi_{1,m},
\end{align*}
\]

6
\[
H_4 \psi_{1,m}^2 = \left( \frac{1}{3} a_{12} - \frac{5}{6} a_{13} \right) \psi_{1,m}^2,
\]
\[
H_4 \psi_{1,m}^3 = \left( -\frac{4}{3} a_{12} + \frac{5}{6} a_{13} \right) \psi_{1,m}^3,
\]
\[
H_4 \psi_{0,0}^+ = \left( -2 a_{12} + \frac{1}{2} a_{13} \right) \psi_{0,0}^+,
\]
\[
H_4 \psi_{0,0}^- = -\frac{3}{2} a_{13} \psi_{1,m}^0,
\]

where \( m \) is the quantum number of \( S_z \) and the eigenstates are shown as Eq. (18) in the Appendix C. If a magnetic field is applied, the states with different quantum numbers \( m \) will split because of the Zeeman Effect. When the field is weak enough, we emphasize that the state with total spin \( S = 0 \) is not always the ground state. In fact, the ground state depends on the values of the exchange constants. By choosing some suitable constants, we can get the ground states with total spin \( S = 1 \). For example, when the constants satisfy the relationship \( a_{12} < 0, a_{13} > 0 \), and \( a_{12} > -2a_{13} \), the ground state is

\[
| \psi_{1,-1}^2 \rangle = \frac{1}{\sqrt{20}} (3| \uparrow\downarrow\downarrow\downarrow \rangle + | \downarrow\uparrow\downarrow\downarrow \rangle - | \downarrow\downarrow\uparrow\downarrow \rangle - 3| \downarrow\downarrow\downarrow\uparrow \rangle);
\]

When \( a_{12} > 0, a_{13} < 0 \), and \( a_{13} < -2a_{12} \), the ground state is

\[
| \psi_{1,-1}^3 \rangle = \frac{1}{\sqrt{20}} (| \uparrow\downarrow\downarrow\downarrow \rangle - 3| \downarrow\uparrow\downarrow\downarrow \rangle + 3| \downarrow\downarrow\uparrow\uparrow \rangle - | \downarrow\downarrow\downarrow\uparrow \rangle).
\]

In both cases, the total spin of the ground state can be \( S = 1 \). Thus, we have got the Hamiltonian of the four-spin system determined by the Yangian symmetry. In a special case it is a model of parallelogram. It should be emphasized that in such a system, the ground state may be no longer with total spin \( S = 0 \). If we take some suitable constants, the ground state can be with \( S = 1 \). This is an interesting case never considered before, and we will focus on such a case and make some experimental predictions.

3.2. the magnetization of the extended \{V6\}-type molecule

In the molecular \{V6\}, we have shown that there is a Yangian symmetry characterized by the operator \( Q \) in each triangle. Meanwhile it has also been proved by the experiment[2] that there is a DM interaction between the two triangular pieces in \{V6\} that may cause the LZS transition[6]. The effect of the LZS transition can
be reflected on the magnetization of the molecular \{V6\}.

It is natural to extend the LZS transition for \{V6\} to a molecular system comprised of two parallelograms. The molecular model is shown in Fig[2]b), which can be called \{V8\}. We also assume that there is a similar DM interaction written as $H_{\text{inter}} = \Delta(S_A \times S_{A'})$, between the two parallelograms. The operators $S_A$ and $S_{A'}$ are the total spin operators of the two parallelograms, respectively. Obviously, it is complicated to calculate such a system for any spin. To simplify the situation, we only consider the ground state of each parallelogram as in the molecular \{V6\}. [3] If the total spin of ground state is zero, it will make no sense to study the magnetization. However, if the total spin is one, we can make a prediction about its magnetization. The Hamiltonian of such a system will be written as:

$$\mathcal{H} = \hbar \gamma B(t)(S_A z + S_{A'} z) + \Delta(S_A \times S_{A'}),$$

(10)

where the spins $S_A = 1$ and $S_{A'} = 1$, and $\gamma$ is the gyromagnetic ratio. The magnetic field $B(t)$ is along the $z$ axis and varies with the time. The first term in Eq.(10) is about the Zeeman Energy, and the second one is the special DM interaction leading to the LZS transition and is assumed to be tiny. By exactly diagonalizing the Hamiltonian $\mathcal{H}$, we can get the energy levels shown in Fig[3]. The DM interaction plays an important role only near the “crossing point” of the energy levels when the magnetic field is absent. Far away from the “crossing point”, the system behaves as two independent particles with each spin $S = 1$ and we just need to consider one particle to investigate the magnetization of the system. In the experiment, the measurement of the magnetization is related to the relaxation time of the magnetization, and here we assume the relaxation time is long compared with the experimental time so that the hysteresis effect can be observed. The general method to investigate the magnetization has been fully discussed in the Ref.[1], and we shall directly make use of the conclusion in the reference. Using the standard formula

$$\frac{d\rho_{NN}}{dt} = \sum_{N'} W_{N'N}(t)\rho_{N'N'} - \sum_{N'} W_{NN'}(t)\rho_{NN}$$

given in the Ref.[1], where the meaning of $\rho$ and $W$ can be seen clearly later, we can get the magnetization dynamic equation of the spin $S = 1$ written as:

$$\frac{d}{dt}\begin{pmatrix} M \\ \rho_{00} \end{pmatrix} = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}\begin{pmatrix} M \\ \rho_{00} \end{pmatrix} + \begin{pmatrix} E \\ F \end{pmatrix},$$

(11)
Figure 3: The scheme of the energy levels of the system comprised of two parallelograms with a DM interaction between them, where $\Delta$ is the energy gap of the Landau-Zener tunneling.

where $M$ is the magnetization of the system, and $\rho$ is the density operator. The magnetization is written as: $M = -\hbar \gamma \langle S_z \rangle = -\hbar \gamma (\rho_{++} - \rho_{--})$ and the coefficients in Eq. (11) are as follows:

\begin{align*}
C_1 & = -\frac{1}{2}(W_{-0} - W_{+0}) - W_{-+} - W_{+-}, \\
C_2 & = \frac{1}{2}(W_{+0} - W_{-0}) + W_{0+} - W_{0-} + W_{+-} - W_{-+}, \\
C_3 & = \frac{1}{2}(W_{+0} - W_{-0}), \\
C_4 & = -\frac{1}{2}(W_{+0} + W_{-0} + 2W_{0+} + 2W_{0-}), \\
E & = \frac{1}{2}(W_{-0} - W_{+0}) + W_{-+} - W_{+-}, \\
F & = \frac{1}{2}(W_{+0} + W_{-0}).
\end{align*}

The $W_{NN'}$ can be taken as the transition probability from the state $|N\rangle$ to $|N'\rangle$, where $N, N'$ is the magnetic quantum number characterized as $+, 0, -$. If we take the one phonon process approximation, the transition probability can be replaced
by:

\[ W_{NN'} = \frac{A\delta^3}{1 - e^{-\beta\hbar}}, \]

where \( \delta = E_N - E_{N'} = \hbar\gamma B(t)(N - N') \) is the energy difference between two different levels. If a pulsed magnetic field is applied, then a hysteresis loop (see Fig. 4) can be obtained from the Eq. (11), where the magnetization has been normalized with \( M_{\text{max}} = \hbar\gamma \), and all the parameters in the Eq. (11) are taken as the values of the \{V6\} molecule provided in the Ref. [2]. We see that it is the usual hysteresis loop of a molecular magnet with the spin \( S = 1 \).

However, in the \{V8\} model when the magnetic field varies to the vicinity of the “crossing point”, the LZS transition must be considered. The Hamiltonian is written as Eq. (10), and the off-diagonal elements cannot be ignored any more. By diagonalizing the Hamiltonian, the exact energy levels are:

\[
E_{2,0} = E_{1,0} = E_{0,0} = 0; \\
E_{2,\pm 1} = \pm \sqrt{B^2(t) + \Delta^2}; \\
E_{2,\pm 2} = \pm \frac{1}{\sqrt{2}} \left[ 5\mu^2 B^2(t) + 3\Delta^2 + \sqrt{9\mu^4 B^4(t) + 30\mu^2 B^2(t) + \Delta^4} \right]^{\frac{1}{2}}; \\
E_{1,\pm 1} = \pm \frac{1}{\sqrt{2}} \left[ 5\mu^2 B^2(t) + 3\Delta^2 - \sqrt{9\mu^4 B^4(t) + 30\mu^2 B^2(t) + \Delta^4} \right]^{\frac{1}{2}},
\]

where the subscript \((l, m)\) is corresponding to the state \(|l, m\rangle\) in the strong magnetic field in the \(-z\) direction. Here, \( l \) is the quantum number of total spin, and \( m \) is the magnetic quantum number. For simplicity, we just choose the nearest three levels \( E_{2,-1}, E_{2,0}, E_{2,1} \) in a weak field to investigate the effect of the off-diagonal elements. These three levels can be viewed as the eigenvalues of the Hamiltonian with the matrix form

\[
\mathcal{H} = \begin{pmatrix}
B(t) & \frac{\Delta}{\sqrt{2}} & 0 \\
\frac{\Delta}{\sqrt{2}} & 0 & \frac{\Delta}{\sqrt{2}} \\
0 & \frac{\Delta}{\sqrt{2}} & -B(t)
\end{pmatrix},
\]

whose eigenstates are

\[
|E_{+, t}\rangle = \begin{pmatrix}
\frac{1 + \cos \beta}{2} \\
\frac{\sin \beta}{\sqrt{2}} \\
\frac{1 - \cos \beta}{2}
\end{pmatrix}^T,
\]
Figure 4: (color online) The magnetization of the four-spin system in a low temperature. The blue solid line is the magnetization, and the red dashed line is the density matrix $\rho_{00}$. The function of the magnetic field varies with the time as $B(t) = 10 \sin(t)$.

$$|E_0, t\rangle = \left(\frac{-\sin \beta}{\sqrt{2}}, \cos \beta, \frac{\sin \beta}{\sqrt{2}}\right)^T,$$

$$|E_-, t\rangle = \left(\frac{1 - \cos \beta}{2}, -\frac{\sin \beta}{\sqrt{2}}, \frac{1 + \cos \beta}{2}\right)^T,$$  \hfill (13)

where the angle $\beta$ is determined by the formula $\cos \beta = \frac{B(t)}{B_0(t) + \Delta^2}$, and the corresponding eigenvalues are $E_0 = 0$, $E_\pm = \pm \sqrt{B^2(t) + \Delta^2}$. Following the method shown in Ref.[1] and assuming the Landau-Zener tunneling adiabatic, we can get the magnetization for spin $S = 1$:

$$M = -\hbar \gamma \langle S_z \rangle = -\hbar \gamma \left(\tilde{\rho}_{++} S_{zz}^{++} + \tilde{\rho}_{00} S_{zz}^{00} + \tilde{\rho}_{--} S_{zz}^{--}\right).$$

It is easy to calculate the matrix elements of $S_{zz}$, and the Eq.(14) can be simplified as $M = \hbar \gamma \cos \beta \cdot n$. Here, $n$ is defined as $n = \tilde{\rho}_{--} - \tilde{\rho}_{++}$, and obeys the same Bloch equation as Eq.(11):

$$\frac{d}{dt} \begin{pmatrix} n(t) \\ \tilde{\rho}_{00} \end{pmatrix} = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix} \begin{pmatrix} n(t) \\ \tilde{\rho}_{00} \end{pmatrix} + \begin{pmatrix} E \\ F \end{pmatrix}. \hfill (14)$$
Figure 5: (color online) The magnetization of the system comprised of two four-spin system with DM interaction between them. The blue solid line is the magnetization, and the red dashed line is the density matrix $\rho_{00}$. We can see the density matrix $\rho_{00}$ is not affected by the off-diagonal matrix of the Hamiltonian, and its magnetization has a similar behavior to the molecular $\{V6\}$.\[11\]

Combining the Eq. (14) and (14), we can get the magnetization of the system as shown in Fig 5 where the applied magnetic field is taken as $B(t) = 10 \sin(t)$ varying in a period $t \in [0, \pi]$. Similar to the $\{V6\}$ molecule, the Magnetization can be probed because of the LZS effect.

In a summary, we have extended the properties of the molecular $\{V6\}$ to a $\{V8\}$-like molecule comprised of two parallelograms. By investigating its magnetization of the molecular model, we get the conclusion that the $\{V8\}$-like molecule has a similar behavior to the molecular $\{V6\}$.

4. the local spin moment configuration of the four-spin system

In this section, we will make up another molecule whose model can be treated as a parallelogram and give a prediction about its local spin moments configuration.

To make up the theoretical molecular model, we need recall the model of $\{V15\}$ which can be considered as an isosceles triangle in a low temperature first.
The molecular \{V15\} is comprised of 15 \( V^4^+ \) ions with each spin \( s = 1/2 \), and the ions are arranged in a quasispherical layered structure with a triangle sandwiched between two hexagons as is shown in the Fig.6(a). Each hexagon of \{V15\} consists of three pairs of strongly coupled spins with \( J_1 \approx 800K. \) Each spin of the \( V^4^+ \) ions in the central triangle is coupled with the spins in both hexagons with \( J_2 = 150K \) and \( J_3 = 300K. \) resulting in a very weak exchange interaction between the spins within the central triangle with \( J_0 = 2.44K. \) When the temperature is low enough, the molecular \{V15\} has a much simpler approximation: the two hexagons can be omitted because each total spin is zero, and the only part we need to consider is just a simple Heisenberg spin triangle. And it has been revealed that such a Heisenberg spin triangle model is an isosceles triangle with the relationship \( J_{13} > J_{12} = J_{23} \) by measuring the local spin moment configuration in an NMR experiment. So the Yangian symmetry also exists in the \{V15\} molecule in a low temperature, and it can be detected by measuring the local spin moments configuration.

As the next step, we will extend such a molecular model to the four-spin system and give the prediction about the local spin moment configuration. The molecule we make up is shown in the Fig.6(b), whose structure is totally similar to the \{V15\} molecule. The numbers of the ions in the upper and the lower polygons
are both even. And they are coupled to the special parallelogram in the middle, which results in some special exchange constants in the middle parallelogram satisfying Eq. (7). Similar to the \{V15\} molecule, such a molecule can be viewed as a single parallelogram described in Sec 5. We can calculate the local spin moments configuration of this model in the NMR experiment. The local spin moment is usually written as \( \mu_i = -g\mu_B\langle \phi_g | S^z_i | \phi_g \rangle \), where the operator \( S^z_i \) is the \( z \)-component of the spin operator of the \( i \)-th particle, \( g = 2 \), and \( \mu_B \) is the Bohr magneton. The state \( | \phi_g \rangle \) is the ground state of the system. Through direct calculation, we can obtain the local moments of the four particles. It is highly nontrivial that the ground state of a four-spin system is with \( S = 1 \), so we mainly focus on such a case and predict its measurement in the experiment. When the exchange constants satisfy the relationship \( a_{12} < 0, a_{13} > 0, a_{12} > -2a_{13} \), the ground state is \( \psi_{1-1}^2 \), and the corresponding local spin moments are \( \frac{1}{10}\mu_B, \frac{9}{10}\mu_B, \frac{9}{10}\mu_B, \frac{1}{10}\mu_B \), respectively. So the observations are \( \frac{1}{10}\mu_B, \frac{9}{10}\mu_B \) when the NMR experiment is performed. In the other case with \( a_{12} > 0, a_{13} < 0, a_{13} < -2a_{12} \), through similar calculation, we can find that the observations are also \( \frac{1}{10}\mu_B, \frac{9}{10}\mu_B \). Hence, we can see that the measured local spin moments are always \( \frac{1}{10}\mu_B, \frac{9}{10}\mu_B \) when the ground state is with \( S = 1 \). So we can tell whether there is the Yangian symmetry by measuring its local spin moments.

5. summary

By investigating the experimental results on the molecular \{V6\}, we introduced a special form of the DM interaction operator to describe the symmetry of the molecule. Meanwhile we have extended the symmetry operator to the four-spin system with each spin \( 1/2 \). We find that the ground state may be no longer with total spin \( S = 0 \) in a four-spin system determined by the Yangian symmetry. By choosing suitable interaction constants, the state with spin \( S = 1 \) can be the ground state. When the ground state is with spin \( S = 1 \), we extended the DM interaction in the \{V6\} molecule to our new model \{V8\}. By investigating its magnetization, we find that it has a similar behavior to the \{V6\} molecule. At last, we have proposed a theoretical molecular model to make a prediction about its local spin moments configuration which can be measured by the NMR experiment.

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A. the mathematics of Yangian

Yangian algebras were established by Drinfel’d.[16, 17, 18] A Yangian is formed by a set \{I, Y\} obeying the commutation relations:

\[ [I_\lambda, I_\mu] = c_{\lambda\mu\nu} I_\nu, \quad [I_\lambda, Y_\mu] = c_{\lambda\mu\nu} Y_\nu, \]
\[ [[Y_\lambda, [Y_\mu, I_\nu]] - [I_\lambda, [Y_\mu, Y_\nu]] = \hbar^2 a_{\lambda\mu\nu\beta\gamma\delta} \{I_\alpha, I_\beta, Y_\gamma\}, \]
\[ [[Y_\lambda, Y_\mu], [I_\sigma, Y_\tau]] + [[Y_\sigma, Y_\tau], [I_\lambda, Y_\mu]] = \hbar^2 (a_{\lambda\mu\nu\beta\gamma\delta} c_{\sigma\tau\nu} + a_{\sigma\tau\nu\beta\gamma\delta} c_{\lambda\mu\nu}) \{I_\alpha, I_\beta, Y_\gamma\}, \quad (15) \]

where the set of \{I\} forms a simple Lie algebra characterized by \(c_{\lambda\mu\nu}\) and the repeated indices mean summation. The definitions of \(a_{\lambda\mu\nu\beta\gamma\delta}\) and \(\{x_i, x_j, x_k\}\) were given in the Ref.[16],

\[ a_{\lambda\mu\nu\beta\gamma\delta} = \frac{1}{4!} c_{\lambda\alpha\sigma} c_{\mu\beta\tau} c_{\nu\gamma\rho} c_{\sigma\rho\delta}, \]
\[ \{x_1, x_2, x_3\} = \sum_{i\neq j\neq k} x_i x_j x_k. \]

In the system comprised of \(n\) particles, if the operator \(I\) is taken as the total spin operator \(S = \sum_{i=1}^{n} S_i\), the Yangian operator \(Y\) can be realized in terms of

\[ Y = \sum_{i=1}^{n} u_i S_i + i \sum_{i<j} (S_i \times S_j), \quad (16) \]

where \(n\) is the number of the particles, and \(S_i\) is the spin of the \(i\)-th particle. It can be verified that the operators \(S\) and \(Y\) satisfy the Eq.(15) and they form a Yangian. For a system with the Yangian symmetry, we introduce the symmetry operator \(Q = Y^2\) to characterize it. So the commutation relation \([Q, H] = 0\) must be satisfied, which means that both the operator \(Q\) and the Hamiltonian share the same eigenvectors. And the Hamiltonian of the \(n\)-spin system is usually written as:

\[ H = \sum_{i<j} a_{ij} S_i \cdot S_j, \quad (17) \]

where \(a_{ij}\) is the exchange constant between the \(i\)-th particle and \(j\)-th particle. We note that the Hamiltonian is a symmetric matrix in the representation of Lie algebra. To satisfy the commutation relation, the symmetry operator \(Q\) is also demanded to be symmetric in the representation.
B. The three-spin system with the Yangian symmetry

When the system is a triangle system, the Yangian operator \( Y \) is taken as Eq.\((16)\), and the Hamiltonian is as Eq.\((17)\), where \( n \) is taken as \( n = 3 \). We need to calculate the eigenvalues and eigenstates of \( Q \) first. The operator \( Q \) can be calculated as:

\[
Q = \sum_{i=1}^{3} u_i^2 S_i^2 + 2(u_1 u_2 S_1 \cdot S_2 + u_2 u_3 S_2 \cdot S_3 + u_1 u_3 S_1 \cdot S_3) \\
-2i(u_1 - u_2 + u_3)S_1 \cdot (S_2 \times S_3) - \{S_1^2 S_2^2 + S_2^2 S_3^2 + S_1^2 S_3^2 \\
-2(S_1 \cdot S_2 + S_2 \cdot S_3 + S_1 \cdot S_3) + 2S_1^2 (S_2 \cdot S_3) + 2S_3^2 (S_1 \cdot S_2) \\
-2S_2^2 (S_1 \cdot S_3) - (S_1 \cdot S_2 + S_2 \cdot S_3 + S_1 \cdot S_3)^2 \\
+2[(S_1 \cdot S_2)(S_2 \cdot S_3) + (S_2 \cdot S_3)(S_1 \cdot S_2)]\}
\]

And the usual Lie algebraic bases are as follows:

\[
|\phi_{\frac{1}{2}, \frac{1}{2}}^{\pm} \rangle = |↓↓↓\rangle, \\
|\phi_{\frac{1}{2}, \frac{1}{2}}^{\pm} \rangle = \frac{1}{\sqrt{3}}(|↑↓↓\rangle + |↓↑↓\rangle + |↓↓↑\rangle), \\
|\phi_{\frac{1}{2}, \frac{1}{2}}^{\pm} \rangle = \frac{1}{\sqrt{3}}(|↑↑↓\rangle + |↑↓↑\rangle + |↓↑↑\rangle), \\
|\phi_{\frac{1}{2}, \frac{1}{2}}^{\pm} \rangle = |↑↑↑\rangle, \\
|\phi_{\frac{1}{2}, \frac{1}{2}}^{\pm} \rangle = \frac{1}{\sqrt{6}}(|↓↑↑\rangle + |↑↓↑\rangle - 2|↓↓↑\rangle), \\
|\phi_{\frac{1}{2}, \frac{1}{2}}^{\pm} \rangle = \frac{1}{\sqrt{6}}(|↓↑↓\rangle + |↑↓↓\rangle - 2|↓↓↓\rangle), \\
|\phi_{\frac{1}{2}, \frac{1}{2}}^{\pm} \rangle = \frac{1}{\sqrt{2}}(|↓↑↑\rangle + |↑↓↑\rangle), \\
|\phi_{\frac{1}{2}, \frac{1}{2}}^{\pm} \rangle = \frac{1}{\sqrt{2}}(|↓↑↓\rangle + |↑↓↓\rangle).
\]

It can be easily testified that:

\[
Q|\phi_{\frac{1}{2}, m}^{\pm} \rangle = \left[ \frac{3}{4}(u_1^2 + u_2^2 + u_3^2) + \frac{1}{2}(u_1 u_2 + u_2 u_3 + u_1 u_3) - 1 \right]|\phi_{\frac{1}{2}, m}^{\pm} \rangle,
\]

\[
Q|\phi_{\frac{1}{2}, m}^{\prime} \rangle = \left[ \frac{3}{4}(u_1^2 + u_2^2 + u_3^2) + \frac{1}{2}(u_1 u_2 - u_2 u_3 - u_1 u_3 - \frac{7}{4}) \right]|\phi_{\frac{1}{2}, m}^{\prime} \rangle
\]

16
\[-\frac{\sqrt{3}}{2} (u_1 - u_2 + 1)(u_3 + 1) |\phi_{\frac{1}{2}, m}\rangle,\]

\[Q |\phi_{\frac{1}{2}, m}\rangle = -\frac{\sqrt{3}}{2} (u_1 - u_2 - 1)(u_3 - 1) |\phi'_{\frac{1}{2}, m}\rangle + \left[\frac{3}{4} (u_1 - u_2)^2 + \frac{3}{4} u_3^2 - \left(-\frac{3}{2}\right)\right] |\phi_{\frac{1}{2}, m}\rangle.\]

As mentioned in the Appendix A, the Yangian operator must be symmetric, from which we can get the relationship \(u_2 = u_1 + u_3\). If we take \(u_1 = u_2 = u_3 = 0\), we can see that the Yangian operator is reduced to the special form of the DM interaction operator as Eq.(2). In this special case, we can easily obtain the eigenvalues and the eigenvectors of \(Q\) as:

\[Q |\phi_{\frac{1}{2}, m}\rangle = -1 |\phi_{\frac{1}{2}, m}\rangle,\]

\[Q |\psi^a_{\frac{1}{2}, m}\rangle = -\frac{1}{4} |\psi^a_{\frac{1}{2}, m}\rangle,\]

\[Q |\psi^b_{\frac{1}{2}, m}\rangle = -\frac{9}{4} |\psi^b_{\frac{1}{2}, m}\rangle,\]

where \(m\) is the magnetic quantum number, and the eigenvectors are

\[|\phi_{\frac{1}{2}, m}\rangle = |\phi_{\frac{1}{2}, m}\rangle,\]

\[|\psi^a_{\frac{1}{2}, m}\rangle = -\frac{1}{2} |\phi'_{\frac{1}{2}, m}\rangle + \frac{\sqrt{3}}{2} |\phi_{\frac{1}{2}, m}\rangle,\]

\[|\psi^b_{\frac{1}{2}, m}\rangle = -\frac{\sqrt{3}}{2} |\phi'_{\frac{1}{2}, m}\rangle + \frac{1}{2} |\phi_{\frac{1}{2}, m}\rangle.\]

When \(m = -1/2\) the states with total spin \(S = 1/2\) are reduced to the Eq.(3) and Eq.(4). We can see that the operator \(Q\) just mixes the states with the same numbers of \(S^2, S_z\). To calculate the commutativity between the operator \(Q\) and the Hamiltonian, we demand the Hamiltonian share the same eigenstates as \(Q\), i.e.,

\[H |\phi_{\frac{1}{2}, m}\rangle = E_{\frac{1}{2}, m} |\phi_{\frac{1}{2}, m}\rangle,\]

\[H |\psi^a_{\frac{1}{2}, m}\rangle = E^a_{\frac{1}{2}, m} |\psi^a_{\frac{1}{2}, m}\rangle,\]

\[H |\psi^b_{\frac{1}{2}, m}\rangle = E^b_{\frac{1}{2}, m} |\psi^b_{\frac{1}{2}, m}\rangle.\]

By direct calculation, we can get the relationship \(a_{12} = a_{23}\) among the exchange constants in the Hamiltonian(17). And the corresponding energy levels can be obtained as:

\[E^a_{\frac{1}{2}, -\frac{1}{2}} = \frac{1}{4} J_{13} - J_{12}, \quad E^b_{\frac{1}{2}, -\frac{1}{2}} = -\frac{3}{4} J_{13}, \quad E_{\frac{1}{2}, -\frac{1}{2}} = \frac{J_{12}}{2} + \frac{J_{13}}{4}.\]
C. The four-spin system with the Yangian symmetry

For a system comprised of four spins, the similar process will be repeated. The Yangian operator Eq. (16) and the Hamiltonian Eq. (17) are taken as those with \( n = 4 \). The operator \( Q \) in the four-spin system can be calculated as:

\[
Q = \sum_{i=1}^{4} u_i^2 S_i^2 + 2 \sum_{i,j,(i<j)} u_i u_j S_i \cdot S_j \\
+ 2i[(u_1 - u_2 + u_3) S_1 \cdot (S_2 \times S_3) + (u_1 - u_2 + u_4) S_1 \cdot (S_2 \times S_4) + (u_1 - u_3 + u_4) S_1 \cdot (S_3 \times S_4) + (u_2 - u_3 + u_4) S_2 \cdot (S_3 \times S_4)] \\
- \left\{ \sum_{i,j,(i<j)} S_i^2 S_j^2 - \left[ \sum_{i,j,(i<j)} (S_i \cdot S_j) \right]^2 - \sum_{i,j,(i<j)} (S_i \cdot S_j) \right\} \\
+ 2[S_3^2 (S_2 \cdot S_3 + S_2 \cdot S_4 + S_3 \cdot S_4) - S_1 \cdot S_3 + S_3^2 (S_1 \cdot S_2 - S_1 \cdot S_4 - S_2 \cdot S_4)] \\
+ S_3^2 (S_1 \cdot S_2 + S_1 \cdot S_3 + S_2 \cdot S_3)] \\
+ 2[(S_2 \cdot S_3) (S_1 \cdot S_2) + (S_1 \cdot S_2) (S_2 \cdot S_3)] \\
+ 2[(S_2 \cdot S_4) (S_1 \cdot S_2) + (S_1 \cdot S_2) (S_2 \cdot S_4)] \\
+ 2[(S_3 \cdot S_4) (S_1 \cdot S_3) + (S_1 \cdot S_3) (S_3 \cdot S_4)] \\
+ 2[(S_3 \cdot S_4) (S_2 \cdot S_3) + (S_2 \cdot S_3) (S_3 \cdot S_4)] \\
+ 2[(S_1 \cdot S_3) (S_2 \cdot S_4) - (S_1 \cdot S_4) (S_2 \cdot S_3) + 3(S_1 \cdot S_2) (S_3 \cdot S_4)]
\]

The usual Lie algebraic bases are as follows:

\[
|\phi_{2,-2}\rangle = |\downarrow\downarrow\downarrow\rangle, \\
|\phi_{2,-1}\rangle = \frac{1}{2}(|\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle + |\downarrow\downarrow\downarrow\rangle), \\
|\phi_{2,0}\rangle = \frac{1}{\sqrt{6}}(|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle + |\downarrow\downarrow\downarrow\rangle), \\
|\phi_{2,1}\rangle = \frac{1}{2}(|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle + |\uparrow\uparrow\downarrow\rangle), \\
|\phi_{2,2}\rangle = |\uparrow\uparrow\uparrow\rangle, \\
|\phi'_{1,-1}\rangle = \frac{1}{2}(|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\rangle - |\downarrow\uparrow\downarrow\rangle), \\
|\phi'_{1,0}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\downarrow\rangle - |\downarrow\downarrow\uparrow\rangle),
\]
Then the action of $Q$ turns out that

$$Q|\phi_{-1,1}\rangle = \frac{1}{2}(|\downarrow\downarrow\downarrow\rangle + |\downarrow\downarrow\downarrow\rangle - |\downarrow\downarrow\downarrow\rangle - |\downarrow\downarrow\downarrow\rangle - |\downarrow\downarrow\downarrow\rangle),$$

$$Q|\phi_{1,1}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\rangle - |\uparrow\uparrow\uparrow\rangle),$$

$$Q|\phi_{1,0}\rangle = \frac{1}{2}(|\uparrow\uparrow\downarrow\rangle + |\uparrow\uparrow\downarrow\rangle - |\uparrow\uparrow\downarrow\rangle - |\uparrow\uparrow\downarrow\rangle),$$

$$Q|\phi_{2,1}\rangle = \frac{1}{\sqrt{2}}(|\downarrow\downarrow\rangle - |\downarrow\downarrow\rangle).$$

$$Q|\phi_{0,0}\rangle = \frac{1}{2\sqrt{3}}(2(|\uparrow\uparrow\downarrow\rangle + |\downarrow\downarrow\rangle) - (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle + |\uparrow\uparrow\rangle),$$

$$Q|\phi_{2,0}\rangle = \frac{1}{2}(|\uparrow\uparrow\downarrow\rangle + |\downarrow\downarrow\rangle - |\downarrow\downarrow\rangle - |\downarrow\downarrow\rangle).$$

$$Q|\phi_{1,0}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\downarrow\rangle - |\uparrow\uparrow\downarrow\rangle + |\uparrow\uparrow\downarrow\rangle - |\uparrow\downarrow\rangle - |\uparrow\uparrow\rangle).$$

$$Q|\phi_{1,1}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\downarrow\rangle - |\uparrow\downarrow\rangle).$$

$$Q|\phi_{1,2}\rangle = \frac{1}{2\sqrt{2}}(u_1 - u_2 - 1)(u_3 - u_4 + 1)(u_1 - u_2 - 2)|\phi_{1,2}\rangle.$$

$$Q|\phi_{1,3}\rangle = \frac{1}{\sqrt{2}}(u_1 - u_2 - 1)(u_3 - u_4 + 1)|\phi_{1,3}\rangle.$$
In the representation of the Lie algebra, the eigenvalues and the eigenstates of the Yangian operator is identified with special form of the DM interaction operator. We can see that the eigenstates with \( Q \) are:

\[
Q|\phi_{0,0}^1\rangle = \frac{1}{2}(u_1 + u_2 - u_3 - u_4 - 2)(u_1 + u_2 - u_3 - u_4 + 2) + \frac{1}{4}[(u_1 - u_2)^2 + (u_3 - u_4)^2 - 2]|\phi_{0,0}^1\rangle
\]

\[
-\frac{\sqrt{3}}{2}(u_1 - u_2 + 1)(u_3 - u_4 + 1)|\phi_{0,0}^2\rangle;
\]

\[
Q|\phi_{0,0}^2\rangle = -\frac{\sqrt{3}}{2}(u_1 - u_2 + 1)(u_3 - u_4 + 1)|\phi_{0,0}^1\rangle + \frac{3}{4}[(u_1 - u_2)^2 + (u_3 - u_4)^2 - 2]|\phi_{0,0}^1\rangle;
\]

However, when the operator \( Q \) is required to be symmetric, we get the conclusion \( u_1 = u_2 = u_3 = u_4 = 0 \). So the first term of the Eq.(16) must vanish, and the the Yangian operator is identified with special form of the DM interaction operator. In the representation of the Lie algebra, the eigenvalues and the eigenstates of the \( Q \) are:

\[
Q = \begin{pmatrix}
\phi_{2,m} \\
\psi_{1,m}^1 \\
\psi_{1,m}^2 \\
\psi_{1,m}^3 \\
\psi_{0,0}
\end{pmatrix} = \begin{pmatrix}
-\frac{5}{2} & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & -\frac{11}{2} & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & -3
\end{pmatrix}
\begin{pmatrix}
\phi_{2,m} \\
\psi_{1,m}^1 \\
\psi_{1,m}^2 \\
\psi_{1,m}^3 \\
\psi_{0,0}
\end{pmatrix},
\]

where the eigenstates are

\[
|\phi_{2,-2}\rangle = |↓↓↓↓\rangle,
\]

\[
|\psi_{1,-1}\rangle = \frac{1}{2}(|\uparrow\downarrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\downarrow\rangle + |\downarrow\downarrow\uparrow\downarrow\rangle - |\downarrow\downarrow\downarrow\uparrow\rangle),
\]

\[
|\psi_{1,-1}\rangle = \frac{1}{\sqrt{2}}(3|\uparrow\downarrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\downarrow\rangle - |\downarrow\downarrow\uparrow\downarrow\rangle - 3|\downarrow\downarrow\downarrow\uparrow\rangle),
\]

\[
|\psi_{1,-1}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\downarrow\downarrow\rangle - 3|\downarrow\uparrow\downarrow\downarrow\rangle + 3|\downarrow\downarrow\uparrow\downarrow\rangle - |\downarrow\downarrow\downarrow\uparrow\rangle),
\]

\[
|\psi_{0,0}\rangle = \frac{1}{2\sqrt{3}}([|\uparrow\uparrow\downarrow\downarrow\rangle + |\downarrow\downarrow\uparrow\uparrow\rangle] - 2(|\uparrow\uparrow\downarrow\rangle + |\downarrow\uparrow\uparrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle),
\]

\[
|\psi_{0,0}\rangle = \frac{1}{2}(|\uparrow\uparrow\downarrow\downarrow\rangle + |\downarrow\downarrow\uparrow\uparrow\rangle) + (|\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\uparrow\rangle).
\]

We can see that the eigenstates with \( S = 1 \) of \( Q = -1/2 \) are partly degenerate. Here, we need another viable \( \theta \) to form the eigenstates of the Hamiltonian with
$S = 1, S_z = -1, Q = -1/2$ as:

\[
|\psi_{1,-1}^{1}\rangle = \cos \frac{\theta}{2} |\alpha_{1,-1}^{1}\rangle - \sin \frac{\theta}{2} |\alpha_{1,-1}^{3}\rangle \\
|\psi_{1,-1}^{3}\rangle = \sin \frac{\theta}{2} |\alpha_{1,-1}^{1}\rangle + \cos \frac{\theta}{2} |\alpha_{1,-1}^{3}\rangle
\]  

(19)

With the commutativity between the operator $Q$ and the Hamiltonian, we demand these six states $|\phi_{2,-2}\rangle$, $|\psi_{1,-1}^{1}\rangle$, $|\psi_{1,-1}^{2}\rangle$, $|\psi_{1,-1}^{3}\rangle$, $|\psi_{0,0}^{+}\rangle$, $|\psi_{0,0}^{-}\rangle$ to be the eigenstates of the Hamiltonian, from which we can get the relationship among the interaction constants. The relationship is as follows:

\[
a_{24} = \frac{1}{2} (a_{12} + 2a_{13} - a_{34}), \\
a_{14} = \frac{1}{3} (a_{12} + 2a_{13}), \\
a_{23} = \frac{1}{3} (2a_{12} - 2a_{13} + 3a_{34}),
\]  

(20)

and the constants $a_{12}$, $a_{34}$ and $a_{13}$ satisfy the equation:

\[
\cos^2 \frac{\theta}{2} \left( \frac{a_{12}}{2} - \frac{a_{34}}{2} \right) + \sin^2 \frac{\theta}{2} \left( \frac{5}{2} a_{12} - \frac{5}{2} a_{34} \right) + \frac{1}{2} \sin \theta \left( -\frac{2}{3} a_{12} - 2a_{34} + \frac{8}{3} a_{13} \right) = 0.
\]

Hence, the general four-spin Hamiltonian with the Yangian symmetry is written as:

\[
H = a_{12} S_1 \cdot S_2 + a_{34} S_3 \cdot S_4 + a_{13} S_1 \cdot S_3 + \frac{1}{2} (a_{12} + 2a_{13} - a_{34}) S_2 \cdot S_4 + \frac{1}{3} (a_{12} + 2a_{13}) S_1 \cdot S_4 + \frac{1}{3} (2a_{12} - 2a_{13} + 3a_{34}) S_2 \cdot S_3,
\]  

(21)

where $a_{12}$, $a_{13}$ and $a_{34}$ satisfy the Eq.(21). If we take the viable $\theta$ to be zero, then the Eq.(20) and(21) reduced to the Eq.(7) leading to a special parallelogram model, and the Hamiltonian Eq.(21) reduces to Eq.(8).

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