A New Interpretation for Orthofermions

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Abstract

In this article we introduce a simple physical model which realizes the algebra of orthofermions. The model is constructed from a cylinder which can be filled with some balls. The creation and annihilation operators of orthofermions are related to the creation and annihilation operators of balls in certain positions in the cylinder. Relationship between this model and topological symmetries in quantum mechanics[10] is investigated.

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Introduction

The study of general types of statistics dates back to 1940’s. Since then there have been many attempts to generalize the concept of bosons and fermions. Some of the most well known generalized statistics are of anyons \[1, 2\], parafermions and parabosons \[3, 4\], and orthofermions and orthobosons \[5\]. These generalized types of statistics are discussed in different contexts. For examples parafermions and parabosons was originally introduced as general types of fields (parafields) \[3, 4\]. Then they found some applications in generalization of supersymmetric quantum mechanics namely parasupersymmetric quantum mechanics \[6, 7\]. Orthofermi and orthobose statistics originally introduced as new types of statistics \[5\] and subsequently used for another generalization of supersymmetric quantum mechanics called orthosupersymmetric quantum mechanics \[8\]. They also turned out to be the origin of some topological symmetries \[9\] which we have already introduced and explored their algebra \[10, 11\]. A topological symmetry is a generalization of supersymmetry from a topological point of view, namely the Witten index \[12\].

Orthofermi statistics is a generalization of fermi statistics in the sense that an orbital state can not contain more than one particle regardless of its spin.

As is well known, particles with statistics other than bose or fermi statistics have not been seen yet. But the concept of generalized statistics may be still useful as we can construct some models which obey their algebras. The goal of this paper is to introduce a model, called the cylinder model, which realizes the algebra of orthofermions.

The statistics of orthofermions of order \(p\) is given by the following equations \[5\]

\[
\begin{align*}
    c_\alpha c_\beta^\dagger + \delta_{\alpha\beta} \sum_{\gamma=1}^{p} c_\gamma c_\gamma^\dagger &= \delta_{\alpha\beta}, \\
    c_\alpha c_\beta &= 0.
\end{align*}
\]

where \(c_\alpha\) and \(c_\alpha^\dagger\) are annihilation and creation operators respectively. The above algebra is a generalization of fermions in the sense that for \(p = 1\) we get the fermionic algebra

\[
\begin{align*}
    c_1 c_1^\dagger + c_1^\dagger c_1 &= 1, \\
    c_1^2 &= 0.
\end{align*}
\]

The representation of \(c_\alpha\) (up to a unitary equivalence) is given by the \((p+1) \times (p+1)\) matrices

\[
[c_\alpha]_{ij} = \delta_{i,1} \delta_{j,\alpha+1}, \quad i, j = 1, \cdots, p + 1.
\]

The Cylinder Model

Consider a cylinder of radius \(r\) and length \(pr\), where \(p\) is a non negative integer. Suppose that this cylinder is put vertically on a surface so that one can drop some balls of radius \(r\) in it.
Clearly one can drop at most \( p \) balls in the cylinder. To empty the cylinder one should take the balls out of the cylinder in the opposite manner it is filled, i.e. the last ball put in the cylinder should be the first which will be taken out of it. (This cylinder is very similar to some type of memories in computer called LIFO stack. LIFO means that the Last object we put In the stack is the First one we take Out of it).

Let’s label the positions of balls in the cylinder from the bottom of the cylinder with numbers 1, 2, \( \cdots \), \( p \). We denote the state of the cylinder containing \( \alpha \) balls, \( \alpha = 0, 1, \cdots, p \), by \(|\alpha\rangle\).

Therefore \(|0\rangle\) is the state of empty cylinder (the vacuum state).

Furthermore let \( b_\alpha^\dagger \) and \( b_\alpha \) be the creation and annihilation operators of a ball in position \( \alpha (\alpha \in \{1, 2, \cdots, p\}) \) respectively. So we define

\[
|\alpha\rangle = b_\alpha^\dagger b_{\alpha - 1}^\dagger \cdots b_1^\dagger |0\rangle ,
\]

(6)

The vacuum state \(|0\rangle\) should, by its definition, satisfy the following equations

\[
b_\alpha |0\rangle = 0 , \quad \alpha = 1, 2, \cdots, p
\]

(7)

\[
b_\alpha^\dagger |0\rangle = 0 , \quad \alpha = 2, 3, \cdots, p
\]

(8)

\[
b_1^\dagger |0\rangle = |1\rangle
\]

(9)

Using operators \( b_\alpha^\dagger \) and \( b_\alpha \) one can drop a ball in the cylinder or draw a ball out of it respectively considering the following rules:

1. A ball can be created in position \( \alpha \) if and only if the position \( \alpha \) is already empty and all positions 1, 2, \( \cdots \), \( \alpha - 1 \) are filled with balls.

2. A ball can be annihilated in position \( \alpha \) if and only if all positions \( \alpha + 1, \cdots, p \) are empty.

The above two rules is summarized in the following algebraic relations between creation and annihilation operators

\[
b_\beta^\dagger b_\alpha^\dagger = b_\alpha b_\beta = 0 , \quad \beta \neq \alpha + 1
\]

(10)

\[
b_\beta^\dagger b_\alpha = b_\alpha b_\beta^\dagger = 0 , \quad \beta \neq \alpha
\]

(11)

Now let see what is the meaning of \( b_\alpha^\dagger b_\alpha \) and \( b_\alpha b_\alpha^\dagger \). In view of Eq. (6), (10), (11) it is easy to see that

\[
b_\alpha^\dagger b_\alpha |\beta\rangle = 0 , \quad \beta \neq \alpha ,
\]

(12)

\[
b_\alpha b_\alpha^\dagger |\beta\rangle = 0 , \quad \beta \neq \alpha - 1 .
\]

(13)

So \( b_\alpha^\dagger b_\alpha \) and \( b_\alpha b_\alpha^\dagger \) are projection operators on subspaces spanned by \(|\alpha\rangle\) and \(|\alpha - 1\rangle\) respectively. Because the values of \( \alpha \) are constrained to the set \( \{1, 2, \cdots, p\} \), \( b_\alpha^\dagger b_\alpha \) can not project on the
subspace spanned by $|0\rangle$. The same thing is true for $b_\alpha b_\alpha^\dagger$ and $|p\rangle$. Consequently
\[
\sum_{\alpha=1}^p b_\alpha^\dagger b_\alpha + b_1 b_1^\dagger = \sum_{\alpha=1}^p b_\alpha b_\alpha^\dagger + b_p^\dagger b_p = 1 ,
\] (14)

One can also verify that the operator $N := \sum_{\alpha=1}^p b_\alpha^\dagger b_\alpha$ shows that the cylinder is empty or not (if its eigenvalue on a state is zero that state correspond to empty cylinder; otherwise the cylinder is not empty). In the same manner $\sum_{\alpha=1}^p b_\alpha b_\alpha^\dagger$ shows that the cylinder is completely full or not.

Next we derive some more algebraic relations between $b_\alpha$s and $b_\alpha^\dagger$s which will be used in next section to construct the algebra of orthofermions. Eq. (14) together with Eqs. (10) and (11) results in
\[
b_\beta = b_\beta b_\beta^\dagger = b_\beta b_{\beta-1} b_\beta = b_\beta b_{\beta+1} b_\beta^\dagger ,
\]
(15)
\[
b_\beta^\dagger = b_\beta^\dagger b_\beta = b_\beta^\dagger b_{\beta-1} b_\beta = b_{\beta+1} b_\beta^\dagger b_\beta^\dagger .
\] (16)

**Orthofermions and the Cylinder Model**

Let us define a new set of creation and annihilation operators, $c_\alpha^\dagger$ and $c_\alpha$
\[
c_\alpha^\dagger := b_\alpha^\dagger \cdots b_1^\dagger ,
\]
(17)
\[
c_\alpha := b_1 \cdots b_\alpha ,
\] (18)
$c_\alpha^\dagger$ fills the cylinder with $\alpha$ balls starting position 1. In view of Eq. (10) and definitions (17) and (18) one can easily see that
\[
c_\alpha^\dagger c_\beta^\dagger = c_\alpha c_\beta = 0 \, , \alpha, \beta = 1, 2, \cdots, p .
\] (19)

This equation is the first equation of orthofermions algebra, namely Eq. (1). The Physical meaning of this equation is that the cylinder can not contain $\alpha$ and $\beta$ balls in the same time and if we fill it with $\alpha$ balls, we can not put another $\alpha$ balls in the same positions in the cylinder again.

One can also use Eqs. (15)-(18) to get
\[
c_\alpha^\dagger c_\alpha = b_\alpha^\dagger b_\alpha \, , \alpha = 1, \cdots, p
\]
(20)
\[
c_\alpha c_\alpha^\dagger = b_1 b_1^\dagger \, , \alpha = 1, \cdots, p
\] (21)
\[
c_\alpha c_\beta^\dagger = 0 \, , \beta \neq \alpha
\] (22)

Putting the above equations together one arrives at the first equation of orthofermions algebra namely Eq. (1).
Before leaving this section, let us have a look at the inverse of Eqs (17) and (18). Using Eqs. (15) and (16), it can be easily verified that the inverse relations are
\begin{align*}
    b^\dagger_\alpha &= c^\dagger_{\alpha-1}, \\
    b_\alpha &= c^\dagger_{\alpha-1}c_\alpha,
\end{align*}
for \( \alpha > 1 \), and \( b_1 = c_1 \). Thus the relations between \( b_\alpha \)'s and \( c_\alpha \)'s are invertible and therefore having the representation of \( c_\alpha \)'s in hand, the representation of \( b_\alpha \)'s is determined uniquely and vice versa.

**Relation to Topological Symmetries**

As it is mentioned in Introduction topological symmetries are generalizations of supersymmetry. Their physical properties and algebra are investigated in Refs. [10, 11]. Here we are going to have a look at the relation between cylinder model and topological symmetries.

The algebra of a \( \mathbb{Z}_2 \)-graded topological symmetry of type \( (1,p) \) is given by the following equations
\begin{align*}
    [H, Q] &= 0, \\
    \{Q^2, Q^\dagger\} + QQ^\dagger Q &= 2HQ, \\
    Q^3 &= 0, \\
    [H, \tau] &= \{\tau, Q\} = 0, \\
    \tau^2 &= 1, \quad \tau^\dagger = \tau,
\end{align*}
where \( H \) is the Hamiltonian of the system, \( Q \) is the symmetry generator, and \( \tau \) is the grading operator.

Now consider a quantum system with the Hamiltonian
\[
    H = a^\dagger a + \sum_{\gamma=1}^{p} c^\dagger_\gamma c_\gamma,
\]
where \( a \) is the annihilation operator for a bosonic degree of freedom satisfying \([a, a^\dagger] = 1\), and \( c_\gamma \), with \( \gamma = 1, \cdots, p \), are annihilation operators of orthofermions of order \( p \). One can easily verify that this Hamiltonian together with symmetry generator\(^1\)
\[
    Q = \frac{1}{\sqrt{2p}} \left( a \sum_{j=1}^{r} c^\dagger_j + a^\dagger \sum_{j=r+1}^{p} c_j \right),
\]
\(^1\)It should be mentioned that one can equivalently take the more general symmetry generator \( Q = \frac{1}{\sqrt{2p}} \left( a \sum_{j=1}^{r} c^\dagger_j + a^\dagger \sum_{j=r+1}^{p} c_{\gamma_j} \right) \), where \((\gamma_1, \cdots, \gamma_p)\) is an arbitrary permutation of \((1, \cdots, p)\) and \( r \) is an integer between 1 and \( p - 1 \).
satisfy the algebra of a topological $\mathbb{Z}_2$-graded symmetry of type $(1,p)$, provided that the grading operator is given by $\tau = (-1)^N$ where $N := \sum_{\alpha=1}^{p} c_\alpha^\dagger c_\alpha$. Now coming back to the cylinder model one can see that the operator $N$ indicates that the cylinder is empty or not. Comparing $\tau$ with operator $(-1)^F$ in supersymmetry one can see that $N$ is a generalization of fermionic number operator $F$, and this has a clear meaning in the cylinder model of orthofermions.

The algebra of ($\mathbb{Z}_n$-graded) topological symmetry of type $(1,1,\cdots,1)$ is given by

\begin{align}
Q^n &= K , \\
Q_1^n + M_{n-2}Q_1^{n-2} + \cdots &= \left(\frac{1}{\sqrt{2}}\right)^n(K + K^\dagger) , \\
Q_2^n + M_{n-2}Q_2^{n-2} + \cdots &= \left(\frac{1}{\sqrt{2}}\right)^n(i^nK^\dagger + (-i)^nK) ,
\end{align}

\begin{equation}
[\tau, Q]_q = 0 .
\end{equation}

where

$$Q_1 := \frac{1}{\sqrt{2}} (Q + Q^\dagger) \quad \text{and} \quad Q_2 := \frac{-i}{\sqrt{2}} (Q - Q^\dagger).$$

$M_i$s and $K$ are operators commuting with all other operators, $M_i$ and $K$ are Hermitian and $\tau$ is the grading operator satisfying

\begin{align}
\tau^n &= 1 , \\
\tau^\dagger &= \tau^{-1} , \\
[H, \tau] &= 0 .
\end{align}

Here $[.,.]_q$ stands for $q$-commutator defined by $[O_1,O_2]_q := O_1O_2 - qO_2O_1$, and $q := e^{2\pi i/n}$. It is easily verified that Hamiltonian $H$ together with symmetry generator

$$Q = ac_1^\dagger + c_2^\dagger c_1 + \cdots + c_p^\dagger c_{p-1} + a^\dagger c_p .$$

satisfy the above algebra for $n = p + 1$. The grading operator is

$$\tau = q^N , \quad \text{where} \quad N := \sum_{\alpha=1}^{p} \alpha N_\alpha ,$$

and the operator $K$ of Eq. (32) is identified with the Hamiltonian $H$. The operators $M_i$ of Eqs. (33) and (34) are given in terms of $H$ according to

$$M_{n-2k} = (-1)^k \left[ \frac{1}{2^k} \binom{n-k-1}{k} + \frac{1}{2^{k-1}} \binom{n-k-1}{k-1} \right] H ,$$

and

$$\begin{pmatrix} a \\ b \end{pmatrix} := \frac{a^\dagger}{b!(a-b)!}.$$ 

In our cylinder model the (number) operator $N := \sum_{\alpha=1}^{p} \alpha c_\alpha^\dagger c_\alpha$ has a nice meaning. It counts the number of balls in the cylinder. This is another generalization of fermionic number operator.
Conclusion

In this article we introduced a simple physical model which realizes the algebra of orthofermions. We explored how one can get the orthofermionic algebra using the creation and annihilation operators of some balls in certain positions of a cylinder. We also addressed the relationship between two number operators $N = \sum_{\alpha=1}^{p} c^\dagger_\alpha c_\alpha$ and $N := \sum_{\alpha=1}^{p} \alpha c^\dagger_\alpha c_\alpha$ and grading operators of two kinds of topological symmetries namely $\mathbb{Z}_2$-graded topological symmetry of type $(1,p)$ and $\mathbb{Z}_{p+1}$-graded topological symmetry of type $(1,1,\cdots,1)$.

These relationships may shed some light on the meaning of topological symmetries as generalizations of supersymmetry. They also show how orthofermions may be the basic ingredients of topological symmetries.

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