BOUNDARY BUBBLING SOLUTIONS FOR A PLANAR ELLIPTIC PROBLEM WITH EXPONENTIAL NEUMANN DATA

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Abstract. Let Ω be a bounded domain in \( \mathbb{R}^2 \) with smooth boundary, we study the following Neumann boundary value problem
\[
\begin{aligned}
- \Delta \nu + \nu &= 0 \quad \text{in } \Omega, \\
\frac{\partial \nu}{\partial \nu} &= e^\nu - s\phi_1 - h(x) \quad \text{on } \partial \Omega,
\end{aligned}
\]
where \( \nu \) denotes the outer unit normal vector to \( \partial \Omega \), \( h \in C^{0,\alpha}(\partial \Omega) \), \( s > 0 \) is a large parameter and \( \phi_1 \) is a positive first Steklov eigenfunction. We construct solutions of this problem which exhibit multiple boundary concentration behavior around maximum points of \( \phi_1 \) on the boundary as \( s \to +\infty \).

1. Introduction. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^2 \) with smooth boundary. This paper deals with the analysis of solutions of the Neumann boundary value problem
\[
\begin{aligned}
- \Delta \nu + \nu &= 0 \quad \text{in } \Omega, \\
\frac{\partial \nu}{\partial \nu} &= e^\nu - s\phi_1 - h(x) \quad \text{on } \partial \Omega,
\end{aligned}
\]
where \( \nu \) denotes the outer unit normal vector to \( \partial \Omega \), \( h \in C^{0,\alpha}(\partial \Omega) \) is given, \( s > 0 \) is a large parameter and \( \phi_1 \) is a positive first eigenfunction of the Steklov problem (see [2]):
\[
\begin{aligned}
- \Delta \phi + \phi &= 0 \quad \text{in } \Omega, \\
\frac{\partial \phi}{\partial \nu} &= \lambda \phi \quad \text{on } \partial \Omega.
\end{aligned}
\]
We denote eigenvalues of (1.2) as \( 0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \), and set \( \rho(x) \in H^1(\Omega) \) as a unique solution of
\[
\begin{aligned}
- \Delta \rho + \rho &= 0 \quad \text{in } \Omega, \\
\frac{\partial \rho}{\partial \nu} &= h(x) \quad \text{on } \partial \Omega.
\end{aligned}
\]

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Obviously, equation (1.1) is equivalent to solving for \( u = v + \frac{s}{\lambda_1} \phi_1 + \rho \), the problem
\[
\begin{cases}
-\Delta u + u = 0 & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = k(x)e^{-t\phi_1}e^u & \text{on } \partial\Omega,
\end{cases}
\]
where \( k(x) = e^{-\rho(x)} \) and \( t = s/\lambda_1 \). We are interested in solutions of problem (1.4) (or (1.1)) which exhibit the concentration phenomenon as the positive parameter \( t \) tends to infinity.

Let us consider a slightly modified version of (1.4)
\[
\begin{cases}
-\Delta u + u = 0 & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = \varepsilon e^u & \text{on } \partial\Omega,
\end{cases}
\]
where \( \varepsilon > 0 \) is a small parameter. Elliptic equations with this type of nonlinear Neumann boundary condition arise in conformal geometry (prescribing Gaussian curvature of the domain and curvature the boundary), see [21], and also in corrosion modelling which arises from electrochemistry, see [18, 25].

For problem (1.5), Dávila, del Pino and Musso in [12] have analyzed the asymptotic behavior of families of solutions \( u_\varepsilon \) with positive, uniformly bounded mass \( \varepsilon \int_{\partial\Omega} e^{u_\varepsilon} \) as \( \varepsilon \to 0 \). It turns out that, up to subsequences, there is an integer \( m \geq 1 \), such that
\[
\lim_{\varepsilon \to 0} \varepsilon \int_{\partial\Omega} e^{u_\varepsilon} = 2m\pi.
\]
Moreover, \( u_\varepsilon \) makes \( m \) distinct points \( \xi_i, i = 1, \ldots, m \) simple blow up on the boundary such that \( \varepsilon e^{u_\varepsilon} \) approaches the sum of \( m \) Dirac masses centered at these points \( \xi_i \). The location of such points can be characterized as critical points of a functional in terms of the Green’s function for a suitable Neumann problem
\[
\begin{cases}
-\Delta_x G(x, y) + G(x, y) = 0, & x \in \Omega, \\
\frac{\partial G}{\partial \nu_x}(x, y) = 2\pi\delta_y(x), & x \in \partial\Omega,
\end{cases}
\]
and its regular part
\[
H(x, y) = G(x, y) + 2\log |x - y|.
\]
Reciprocally, the authors in [12] also proved the existence of families of boundary bubbling solutions \( u_\varepsilon \) to (1.5) with above properties. More precisely, they showed that, given any integer \( m \geq 1 \), problem (1.5) has at least two distinct families of solutions \( u_\varepsilon \) for which (1.6) holds, and the peaks of these two solutions are located around two distinct critical points of a certain functional of \( m \) points of the boundary.

Here we introduce a natural generalization of equation (1.5), namely the anisotropic elliptic problem
\[
\begin{cases}
-\nabla(a(x)\nabla u) + a(x)u = 0 & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = \varepsilon e^u & \text{on } \partial\Omega,
\end{cases}
\]
where \( a(x) \) is a smooth positive function over \( \Omega \). In [32], the author investigated the asymptotic behavior of solutions \( u_\varepsilon \) to (1.9) for which \( \varepsilon \int_{\partial\Omega} e^{u_\varepsilon} \) is bounded, and
showed that, up to subsequences, it will develop a finite number of bubbles \( \xi_i \in \partial \Omega \) such that
\[
\varepsilon \int_{\partial \Omega} e^{u_*} \to 2\pi \sum_{i=1}^{k} m_i \quad \text{and} \quad \varepsilon e^{u_*} \to 2\pi \sum_{i=1}^{k} m_i \delta_{\xi_i}, \quad \text{as } \lambda \to 0,
\]
for some integers \( m_i, k \geq 1 \), and these bubbling points are nothing but critical points of the function \( a(x) \) on the boundary. Thus in contrast with the above work in \([12]\), a natural problem is posed that whether \((1.9)\) does admit a boundary bubbling solution such that \( m_i > 1 \). Let us point out that \( m_i = 1 \) corresponds to a simple blow up at \( \xi_i \) while \( m_i > 1 \) gives rise to a non-simple (or multiple) blow up. Indeed, the authors in \([31]\) gave a positive answer to this reciprocal question through the construction of solutions to \((1.9)\) with a cluster of multiple bubbles around any isolated local maximum point of \( a(x) \) on the boundary.

This paper is devoted to studying the existence and concentration behavior of boundary bubbling solutions to problem \((1.4)\) (or \((1.1)\)). We prove that there exists a family of solutions of problem \((1.4)\) with the accumulation of arbitrarily many boundary bubbling solutions to problem \((1.4)\) (or \((1.1)\)). We prove that there exists a solution \( u_t \) of problem \((1.4)\) which satisfies
\[
\lim_{t \to +\infty} \int_{\partial \Omega} k(x)e^{-\phi_t} e^{u_t} = 2m\pi.
\]
More precisely, we have
\[
u_k^t(x) = \sum_{i=1}^{m} \left[ \log \frac{1}{|x - \xi_i^t - \varepsilon_i^t \mu_i^t \nu(\xi_i^t)|^2} + H(x, \xi_i^t) \right] + o(1),
\]
where \( o(1) \to 0 \), as \( t \to +\infty \), uniformly on each compact subset of \( \overline{\Omega} \setminus \{\xi_1^t, \ldots, \xi_m^t\} \), \( \nu(\xi) \) denotes the outer unit normal vector to \( \partial \Omega \) at \( \xi_i^t \), the parameters \( \varepsilon_i^t \) and \( \mu_i^t \) satisfy
\[
\varepsilon_i^t = e^{-\phi_t(\xi_i^t)}, \quad \frac{1}{C} \leq \mu_i^t \leq Ct^{m^2+1},
\]
for some \( C > 0 \), and \( (\xi_1^t, \ldots, \xi_m^t) \in \Lambda^m \) satisfies
\[
\text{dist}(\xi_i^t, S) \to 0 \quad \text{for all } i, \quad \text{and} \quad |\xi_i^t - \xi_j^t| > t^{-\frac{m^2+1}{2}} \quad \forall \ i \neq j,
\]
with \( S = \{x \in \Lambda | \phi_t(x) = \sup_{\Lambda} \phi_t\} \).

The corresponding result for problem \((1.1)\) can be read as follows.

**Theorem 1.1.** Let \( \Lambda \) be a subset of \( \partial \Omega \) satisfying \( \sup_{\partial \Omega} \phi_1 < \sup_{\partial \Lambda} \phi_1 \). Then given any positive integer \( m \), there exists \( t_m > 0 \) such that for any \( t > t_m \), there is a solution \( u_t \) of problem \((1.4)\) which satisfies
\[
\lim_{t \to +\infty} \int_{\partial \Omega} k(x)e^{-\phi_t} e^{u_t} = 2m\pi.
\]
More precisely, we have
\[
u_k^t(x) = \sum_{i=1}^{m} \left[ \log \frac{1}{|x - \xi_i^t - \varepsilon_i^t \mu_i^t \nu(\xi_i^t)|^2} + H(x, \xi_i^t) \right] + o(1),
\]
where \( o(1) \to 0 \), as \( t \to +\infty \), uniformly on each compact subset of \( \overline{\Omega} \setminus \{\xi_1^t, \ldots, \xi_m^t\} \), \( \nu(\xi) \) denotes the outer unit normal vector to \( \partial \Omega \) at \( \xi_i^t \), the parameters \( \varepsilon_i^t \) and \( \mu_i^t \) satisfy
\[
\varepsilon_i^t = e^{-\phi_t(\xi_i^t)}, \quad \frac{1}{C} \leq \mu_i^t \leq Ct^{m^2+1},
\]
for some \( C > 0 \), and \( (\xi_1^t, \ldots, \xi_m^t) \in \Lambda^m \) satisfies
\[
\text{dist}(\xi_i^t, S) \to 0 \quad \text{for all } i, \quad \text{and} \quad |\xi_i^t - \xi_j^t| > t^{-\frac{m^2+1}{2}} \quad \forall \ i \neq j,
\]
with \( S = \{x \in \Lambda | \phi_t(x) = \sup_{\Lambda} \phi_t\} \).

The corresponding result for problem \((1.1)\) can be read as follows.

**Theorem 1.2.** Let \( m \) be a positive integer. Then for any \( s \) sufficiently large, there exists a solution \( u_s \) of problem \((1.1)\) such that
\[
\lim_{s \to +\infty} \int_{\partial \Omega} e^{u_s} = 2m\pi.
\]
More precisely, given any subset \( \Lambda \) of \( \partial \Omega \) satisfying \( \sup_{\partial \Omega} \phi_1 < \sup_{\partial \Lambda} \phi_1 \), and a sequence \( s \to +\infty \), there is a subsequence and \( s \) points \( \xi_i \in \Lambda \) with \( \phi_t(\xi_i) = \sup_{\Lambda} \phi_1 \) such that as \( s \to +\infty \),
\[
e^{u_s} \to 2\pi \sum_{i=1}^{m} \delta_{\xi_i}, \quad \text{weakly in the sense of measure in } \partial \Omega.
\]
From Theorems 1.1 and 1.2 we see that problem (1.4) has a solution that exhibits the phenomenon of multiple blow up at boundary maximum points of $\phi_1$, in particular we have the concentration properties

$$k(x)e^{-t\phi_1}e^u \rightharpoonup 2\pi \sum_{i=1}^k m_i \delta_{\xi_i} \quad \text{and} \quad u_t = \sum_{i=1}^k m_i G(x, \xi_i) + o(1),$$

with some integers $m_i > 1$, where $\xi_i$'s are boundary maxima of $\phi_1$.

Let us remark the interesting analogy between these results and those known for the following elliptic problem of Ambrosetti-Prodi type [1]:

$$\begin{cases} -\Delta u = e^u - s\phi_1 - h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.10)$$

where $\Omega$ is a bounded smooth domain in $\mathbb{R}^2$, $h \in C^0(\Omega)$, $s > 0$ is a large parameter and $\phi_1$ is a positive first eigenfunction of the problem $-\Delta \phi = \lambda \phi$ under Dirichlet boundary condition in $\Omega$. In the early 1980s Lazer and McKenna conjectured that equation (1.10) has an unbounded number of solutions as $s \to +\infty$ (see [19]). If we denote its eigenvalues as $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$, and set $\rho(x) = (-\Delta)^{-1}h$ in $H^1_0(\Omega)$, then equation (1.10) is equivalent to solving for $u = v + s\lambda_1 \phi_1 + \rho$, the problem

$$\begin{cases} -\Delta u = k(x)e^{-t\phi_1}e^u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.11)$$

where $k(x) = e^{-\rho(x)}$ and $t = s/\lambda_1$. In [15], del Pino and Muñoz gave a positive answer to the Lazer-McKenna conjecture by constructing solutions of problem (1.11) with the following asymptotic profile

$$u_t = \sum_{i=1}^k m_i G_D(x, \xi_i) + o(1),$$

where $m_i > 1$, $\xi_i$'s are inner maxima of $\phi_1$ and $G_D(x, \xi_i)$ denotes the Green function of the problem

$$\begin{cases} -\Delta x G_D(x, \xi_i) = 8\pi \delta_{\xi_i}(x), & x \in \Omega, \\ G_D(x, \xi_i) = 0, & x \in \partial\Omega. \end{cases}$$

Surprisingly enough, this multiple bubbling phenomenon of solutions of problem (1.11) contrasts with that of the Liouville equation

$$\begin{cases} -\Delta u = \varepsilon^2 k(x)e^u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.12)$$

where $\Omega$ is a bounded smooth domain in $\mathbb{R}^2$, $\varepsilon > 0$ is a small parameter and $k(x)$ is a nonnegative, not identically zero function of class $C^2(\Omega)$. Indeed, the results in [5, 20, 24, 29] shows that the bubbling of solutions of equation (1.12) with positive, uniformly bounded mass $\varepsilon^2 \int_\Omega k(x)e^u$ should be simple, namely all $m_i$'s are equal to one. Reciprocally, solutions of (1.12) with this bubbling behavior have been obtained in [8, 9, 13, 17]. In particular, the construction of solutions with an arbitrarily large number of bubbling points was achieved in [14] in the case that $\Omega$ is not simply connected. Finally, it is necessary to point out that multiple
bubbling has also been built in [35] for solutions of the two-dimensional anisotropic Emden-Fowler equation

\[
\begin{align*}
&-\nabla(a(x)\nabla u) = \varepsilon^2 a(x) e^u \quad \text{in} \quad \Omega, \\
&u = 0 \quad \text{on} \quad \partial\Omega,
\end{align*}
\]

around any isolated local maximum point of the uniformly positive, smooth function \(a(x)\).

We mention that the Lazer-McKenna conjecture holds true for some Ambrosetti-Prodi problems with other type of nonlinearities

\[
\begin{align*}
&-\Delta v = g(v) - s\phi_1 \quad \text{in} \quad \Omega, \\
v = 0 \quad \text{on} \quad \partial\Omega,
\end{align*}
\]

(1.14)

where \(\Omega\) is a bounded smooth domain in \(\mathbb{R}^N\), \(N \geq 3\), \(s > 0\) is a large parameter, \(\phi_1 > 0\) is an eigenfunction of \(-\Delta\) with Dirichlet boundary condition corresponding to the first eigenvalue \(\lambda_1\), and \(g : \mathbb{R} \to \mathbb{R}\) is a continuous function such that \(\lim_{t \to -\infty} g(t) = \alpha < \lim_{t \to +\infty} \frac{g(t)}{t} = \beta\) and \((\alpha, \beta)\) contains some eigenvalues of \(-\Delta\) subject to Dirichlet boundary condition. Here \(\alpha = -\infty\) and \(\beta = +\infty\) are allowed. The reader can refer to [26, 27, 28] for the asymptotically linear case \(g(t) = \beta t + -\alpha t\), for sufficiently large \(\beta\), to [9, 10] for the subcritical case \(g(t) = t^{p_+} + \lambda t, \lambda < \lambda_1, 1 < p < \frac{N+2}{N-2}\) if \(N \geq 3\) and \(1 < p < +\infty\) if \(N = 2\), to [16, 22, 23, 34] for the critical case \(g(t) = t^{\frac{N+2}{N-2}} + \lambda t, 0 < \lambda < \lambda_1\) and \(N \geq 6\), to [7] for the superlinear nonhomogeneous case \(g(t) = t^{p_+} + t^q, 1 < q < p < \frac{N+2}{N-2}\) and \(N \geq 4\), to [3] [11, 13] for the superlinear homogeneous case \(g(t) = |t|^p, 1 < p < \frac{N+2}{N-2}\) if \(N \geq 3\), \(1 < p < +\infty\) if \(N = 2\), and to [8, 33] for other cases, where \(t_+ = \max\{t, 0\}\) and \(t_- = \max\{-t, 0\}\).

The proof of our results relies on a very well-known Lyapunov-Schmidt reduction procedure. In Section 2 we exactly describe the ansatz for the solution of problem (1.4) and rewrite problem (1.4) in terms of a linearized operator for which a solvability theory, subject to suitable orthogonality conditions, is performed through solving a linearized problem in Section 3. In Section 4 we solve an auxiliary nonlinear problem. In Section 5 we reduce the problem of finding boundary bubbling solutions of (1.4) to that of finding a critical point of a finite-dimensional function. In the last section we write an asymptotic expansion for the energy functional appeared in Section 5, and further give the proof of Theorem 1.1.

Throughout this paper, unless otherwise stated, the letters \(c, C\) will always denote various universal positive constants that are independent of \(t\) and \(m\) for \(t\) sufficiently large.

2. Preliminaries and ansatz for the solution. In this section we will provide an ansatz for solutions of problem (1.4). It is well known (see [21, 30, 36]) that any solution of the problem

\[
\begin{align*}
&\Delta u = 0 \quad \text{in} \quad \mathbb{R}^2_+,
\hline
&\frac{\partial u}{\partial \nu} = e^u \quad \text{on} \quad \partial\mathbb{R}^2_+,
\hline
&\int_{\mathbb{R}^2_+} e^u < +\infty,
\end{align*}
\]

(2.1)
where $\mathbb{R}^2_+$ denotes the upper half-plane $\{(x_1, x_2) : x_2 > 0\}$ and $\nu$ the unit exterior normal to $\partial \mathbb{R}^2_+$, must be of the form

$$
\psi_{\mu, \tau}(x) = \psi_{\mu, \tau}(x_1, x_2) = \log \left( \frac{2\mu}{(x_1 - \tau)^2 + (x_2 + \mu)^2} \right),
$$

where $\tau \in \mathbb{R}$ and $\mu > 0$ are parameters. Set

$$
\psi_{\mu}(x) := \psi_{\mu, \tau}(x) \big|_{\tau = 0} = \log \left( \frac{2\mu}{x_1^2 + (x_2 + \mu)^2} \right),
$$

and

$$
z_{\mu 0}(x) := -\frac{1}{\mu} \frac{\partial}{\partial \kappa} \left[ \psi_{\mu}(\kappa x) + \log \kappa \right] |_{\kappa = 1} = \frac{1}{\mu} - 2 \frac{x_2 + \mu}{x_1^2 + (x_2 + \mu)^2},
$$

and

$$
z_{\mu 1}(x) := -\frac{\partial}{\partial \zeta_1} \psi_{\mu}(x_1 - \zeta_1, x_2) \big|_{\zeta_1 = 0} = -2 \frac{x_1}{x_1^2 + (x_2 + \mu)^2},
$$

which exactly correspond to variations of $\psi_{\mu, \tau}$ defined in [2.2] along its parameters of dilation and translation, respectively. Obviously, these objects lie in the kernel of the linearization of equation [2.1] at the solution $\psi_{\mu}$, namely they solve the problem

$$
\begin{cases}
\Delta \phi = 0 & \text{in } \mathbb{R}^2_+,

\frac{\partial \phi}{\partial \nu} - e^{\psi_{\mu}} \phi = 0 & \text{on } \partial \mathbb{R}^2_+.
\end{cases}
$$

Reciprocally, any bounded solution of [2.6] is a linear combination of $z_{\mu 0}$ and $z_{\mu 1}$, see [12] for a proof.

Let us fix a subset $\Lambda$ of $\partial \Omega$ as in the statement of Theorems 1.1-1.2. For the sake of convenience we assume

$$
\sup_{x \in \Lambda} \phi_1(x) = 1.
$$

The configuration space for $m$-tuple $\xi = (\xi_1, \ldots, \xi_m)$ we choose is the following

$$
\mathcal{O}_t := \left\{ \xi = (\xi_1, \ldots, \xi_m) \in \mathbb{K}^m \left| \xi_i - \xi_j \geq \frac{1}{\sqrt{t}}, \ 1 - \phi_1(\xi_i) \leq \frac{1}{\sqrt{t}}, \right. \right. \\
i, j = 1, \ldots, m, \ i \neq j, \left. \right\}
$$

where $\beta$ is given by

$$
\beta = \frac{m^2 + 1}{2}.
$$

Observe that the function

$$
\psi_{\mu}(x) = \psi_{\mu}(x/\varepsilon) - 2 \log \varepsilon = \log \left( \frac{2\mu}{x_1^2 + (x_2 + \varepsilon \mu)^2} \right)
$$

satisfies

$$
\begin{cases}
\Delta u = 0 & \text{in } \mathbb{R}^2_+,

\frac{\partial u}{\partial \nu} = e^{\psi_{\mu}} & \text{on } \partial \mathbb{R}^2_+.
\end{cases}
$$

Thus, for numbers $\mu_i > 0$, $i = 1, \ldots, m$, yet to be chosen, we define

$$
\psi_{\mu_i}(x) = \log \frac{2\mu_i}{k(\xi_i) |x - \xi_i - \varepsilon_i \mu_i(\xi_i)|^2},
$$

where $k(\xi_i) > 0$. The forcing term $k(\xi_i)$ is introduced to ensure the nondegeneracy of the variational problem. By a suitable choice of $\varepsilon_i$ and $\mu_i$, we can ensure that $k(\xi_i)$ is bounded away from zero, thereby guaranteeing the existence of a positive solution to the variational problem.
where
\[ \varepsilon_i = \varepsilon_i(t) \equiv e^{-t\phi_i(x)} . \] (2.11)

We hope to take \( \sum_{i=1}^{m} u_i(x) \) as an initial approximate solution of \( \text{[1.4]} \). So we modify it to be
\[ U(x) := \sum_{i=1}^{m} U_i(x) = \sum_{i=1}^{m} [u_i(x) + H_i(x)] , \] (2.12)
where \( H_i(x) \) is the solution of
\[ \begin{aligned}
-\Delta H_i + H_i &= -u_i \quad \text{in } \Omega, \\
\frac{\partial H_i}{\partial \nu} &= \varepsilon_i k(\xi_i) e^{u_i} - \frac{\partial u_i}{\partial \nu} \quad \text{on } \partial \Omega.
\end{aligned} \] (2.13)
Then \( U_i(x) := u_i + H_i \) satisfies
\[ \begin{aligned}
-\Delta U_i + U_i &= 0 \quad \text{in } \Omega, \\
\frac{\partial U_i}{\partial \nu} &= \varepsilon_i k(\xi_i) e^{u_i} \quad \text{on } \partial \Omega.
\end{aligned} \] (2.14)

**Lemma 2.1.** For any \( 0 < \alpha < 1, \xi = (\xi_1, \ldots, \xi_m) \in \mathcal{O}_t \), then we have
\[ H_i(x) = H(x, \xi_i) - \log 2\mu_i + \log k(\xi_i) + O(\varepsilon_i \mu_i^\alpha) , \] (2.15)
uniformly in \( \overline{\Omega} \), where \( H \) is the regular part of Green’s function defined in \( \text{[1.8]} \).

**Proof.** First, on the boundary, we have
\[ \frac{\partial H_i}{\partial \nu} = \varepsilon_i k(\xi_i) e^{u_i} - \frac{\partial u_i}{\partial \nu} = \frac{2\varepsilon_i \mu_i + 2(x - \xi_i - \varepsilon_i \mu_i \nu(\xi_i)) \cdot \nu(x)}{|x - \xi_i - \varepsilon_i \mu_i \nu(\xi_i)|^2} . \]
Thus
\[ \lim_{\varepsilon_i \mu_i \to 0} \frac{\partial H_i}{\partial \nu}(x) = 2 \frac{(x - \xi_i) \cdot \nu(x)}{|x - \xi_i|^2} \quad \forall x \in \partial \Omega \setminus \{\xi_i\} . \]

On the other hand, the regular part of Green’s function \( H(x, \xi_i) \) satisfies
\[ \begin{aligned}
-\Delta H(x, \xi_i) + H(x, \xi_i) &= -\log \frac{1}{|x - \xi_i|^2} \quad \text{in } \Omega, \\
\frac{\partial H}{\partial \nu}(x, \xi_i) &= 2 \frac{(x - \xi_i) \cdot \nu(x)}{|x - \xi_i|^2} \quad \text{on } \partial \Omega.
\end{aligned} \] (2.16)
Set \( z(x) = H_i(x) - H(x, \xi_i) + \log 2\mu_i - \log k(\xi_i) \), then we get
\[ \begin{aligned}
-\Delta z + z &= \log \frac{|x - \xi_i - \varepsilon_i \mu_i \nu(\xi_i)|^2}{|x - \xi_i|^2} \quad \text{in } \Omega, \\
\frac{\partial z}{\partial \nu} &= \frac{\partial H_i}{\partial \nu} - 2 \frac{(x - \xi_i) \cdot \nu(x)}{|x - \xi_i|^2} \quad \text{on } \partial \Omega.
\end{aligned} \]
Following Lemma 3.1 from \( \text{[12]} \), we can easily prove that for \( 1 < p < 2, \)
\[ \left\| \log \frac{|x - \xi_i - \varepsilon_i \mu_i \nu(\xi_i)|^2}{|x - \xi_i|^2} \right\|_{L^p(\Omega)} \leq C \varepsilon_i \mu_i , \]
and for \( p > 1, \)
\[ \left\| \frac{\partial H_i}{\partial \nu} - 2 \frac{(x - \xi_i) \cdot \nu(x)}{|x - \xi_i|^2} \right\|_{L^p(\partial \Omega)} \leq C (\varepsilon_i \mu_i)^{\frac{1}{p}} . \]
By $L^p$ theory, it follows that
\[ \|z\|_{W^{1,p}((\Omega))} \leq C \left( \|\Delta z + z\|_{L^p(\Omega)} + \left\| \frac{\partial z}{\partial \nu} \right\|_{L^p(\partial \Omega)} \right) \leq C(\varepsilon_i \mu_i)^{\frac{1}{p}}, \]
where $\kappa \in (0, 1/p)$. Then by Morrey embedding,
\[ \|z\|_{C^{\gamma}(\Omega)} \leq C(\varepsilon_i \mu_i)^{\frac{1}{p}}, \]
where $\gamma \in (0, 1/2 + 1/p)$, which derives the asymptotic expansion (2.15) with $\alpha = 1/p$. □

We now hope that for each $i = 1, \ldots, m$, the remainder $U - u_i = H_i + \sum_{j \neq i} (u_j + H_j)$ vanishes at the main order near $\xi_i$, which can be realized by choosing the parameter $\mu_i$ such that
\[
\log 2 \mu_i = \log k(\xi_i) + H(\xi_i, \xi_i) + \sum_{j \neq i} G(\xi_i, \xi_j).
\]
(2.17)
We thus fix $\mu_i$ a priori as a function of $\xi$ in $O_t$ and write $\mu_i = \mu_i(\xi)$ for all $i = 1, \ldots, m$. Since $\xi = (\xi_1, \ldots, \xi_m) \in O_t$, there exists a constant $C > 0$ independent of $t$ large enough,
\[ \frac{1}{C} \leq \mu_i \leq Ct^{2\beta} \quad \text{and} \quad |\partial_{\xi_k} \log \mu_i| \leq Ct^\beta, \quad \forall \ i, k = 1, \ldots, m. \]
(2.18)

Consider the scaling of the solution to equation (1.4)
\[ \omega(y) = u(\varepsilon y) - 2t \quad \forall \ y \in \bar{\Omega}_t, \]
(2.19)
with
\[ \varepsilon = \varepsilon(t) \equiv e^{-t} \quad \text{and} \quad \Omega_t = \varepsilon^{-1} \Omega, \]
(2.20)
then $\omega$ satisfies
\[
\begin{cases}
- \Delta \omega + \varepsilon^2 \omega = -2te^2 & \text{in } \Omega_t, \\
\frac{\partial \omega}{\partial \nu} = q(y, t)e^\omega & \text{on } \partial \Omega_t,
\end{cases}
\]
(2.21)
where
\[ q(y, t) \equiv k(\varepsilon y) \exp \left\{ -t[\phi_1(\varepsilon y) - 1] \right\}. \]
(2.22)
We write $\xi'_i = \xi_i/\varepsilon$ and define the initial approximate solution to (2.21) as
\[ V(y) = U(\varepsilon y) - 2t, \]
(2.23)
where $U$ is defined by (2.12). Moreover, set
\[ W(y) = q(y, t)e^{V(y)}, \]
(2.24)
and the “error term”
\[ R(y) = W(y) - \frac{\partial V(y)}{\partial \nu}. \]
(2.25)
Let us see how well $\frac{\partial V(y)}{\partial \nu}$ matches with $W(y)$ through $V(y)$ so that the “error term” $R(y)$ is sufficiently small for any $y \in \partial \Omega_t$. Assume first $|y - \xi'_i| \leq \delta/(2\varepsilon t^\beta)$.
for some index $i$ and a small constant $\delta > 0$. Then we have
\[
\frac{\partial V(y)}{\partial \nu} = \varepsilon \sum_{j=1}^{m} \left[ \frac{\partial u_j}{\partial \nu} + \frac{\partial H_j}{\partial \nu} \right] (x) = \varepsilon \sum_{j=1}^{m} \varepsilon_j k(\xi_j) e^{u_j(x)} + \frac{2\varepsilon \varepsilon_j \mu_j}{|y - \xi_j - \varepsilon_j \mu_j \nu(\xi_j)|^2} \left[ \frac{2\mu_j}{H_j(\varepsilon_y, \xi_j)} \right] + \sum_{j \neq i} O(\varepsilon \varepsilon_j \mu_j t^{2\beta}), \quad \text{(2.26)}
\]
and by (2.10) and (2.22),
\[
W(y) = \varepsilon^2 q(y, t) \exp \left\{ \sum_{j=1}^{m} \left[ u_j(\varepsilon y) + H_j(\varepsilon y) \right] \right\} = \varepsilon^2 q(y, t) \exp \left\{ \sum_{j=1}^{m} \left[ \log \frac{k(\xi_j) |\varepsilon y - \xi_j - \varepsilon_j \mu_j \nu(\xi_j)|^2}{k(\xi_j)} + H_j(\varepsilon y) \right] \right\} \times \exp \left\{ H_i(\varepsilon y) + \sum_{j \neq i} \left[ \log \frac{2\mu_j}{k(\xi_j) |\varepsilon y - \xi_j - \varepsilon_j \mu_j \nu(\xi_j)|^2} + H_j(\varepsilon y) \right] \right\}, \quad \text{(2.27)}
\]
where
\[
\gamma_i := \frac{1}{\varepsilon} \varepsilon_j \mu_i = \mu_i \exp \left\{ -t [\phi_1(\xi_i) - 1] \right\}. \quad \text{(2.28)}
\]
From (2.15) and the fact that $H$ is $C^1(\partial \Omega)$ we get that for any $y \in \overline{\Omega}$,
\[
H_j(\varepsilon y) = H(\varepsilon y, \xi_j) - \log 2 \mu_j + \log k(\xi_j) + O(\varepsilon_j^\alpha \mu_j^\beta) = H(\xi_i, \xi_j) - \log 2 \mu_j + \log k(\xi_j) + O(|\varepsilon y - \xi_i|) + O(\varepsilon_j^\alpha \mu_j^\beta). \quad \text{(2.29)}
\]
Hence for $|y - \xi_i| \leq \delta/(2\varepsilon t^\beta)$,
\[
H_i(\varepsilon y) + \sum_{j \neq i} \left[ \log \frac{2\mu_j}{k(\xi_j) |\varepsilon y - \xi_j - \varepsilon_j \mu_j \nu(\xi_j)|^2} + H_j(\varepsilon y) \right] = H(\xi_i, \xi_j) - \log 2 \mu_i + \log k(\xi_i) + \sum_{j \neq i} \left[ \log \frac{1}{|\xi_i - \xi_j|^2} + H(\xi_i, \xi_j) \right] + O \left( |\varepsilon y - \xi_i| + \sum_{j=1}^{m} O(\varepsilon_j^\alpha \mu_j^\beta) \right)
\]
\[
= O(t^\beta |\varepsilon y - \xi_i|) + \sum_{j=1}^{m} O(\varepsilon_j^\alpha \mu_j^\beta),
\]
where the last equality is due to the choice of $\mu_i$ in (2.17). Therefore when $|y - \xi_i| \leq \delta/(2\varepsilon \beta)$,

$$W(y) = \frac{2\gamma_i}{|y - \xi_i' - \gamma_i \nu(\xi_i')|^2} \left\{ 1 + O(\varepsilon t^\beta |y - \xi_i'|) + \sum_{j=1}^m O(\varepsilon^a_j \mu_j^a) \right\},$$  \hspace{1cm} (2.30)

which, together with (2.25)-(2.26), concludes that in this region

$$R(y) = \frac{2\gamma_i}{|y - \xi_i' - \gamma_i \nu(\xi_i')|^2} \left\{ O(\varepsilon t^\beta |y - \xi_i'|) + \sum_{j=1}^m O(\varepsilon^a_j \mu_j^a) \right\}$$

$$+ \sum_{j \neq i} O(\varepsilon \mu_j t^{2\beta}).$$  \hspace{1cm} (2.31)

On the other hand, if $|y - \xi_i'| > \delta/(2\varepsilon \beta)$ for all $i$, by (2.10) and (2.15) we obtain

$$\frac{\partial V(y)}{\partial \nu} = \sum_{i=1}^m O(\varepsilon \mu_i t^{2\beta})$$  \hspace{1cm} and

$$W(y) = O \left( \varepsilon^2 q(y,t) e^{V(y)} \right) = O \left( \varepsilon^2 \mu_i t^{2\beta} e^{-t\phi_1(\varepsilon y)} \right).$$  \hspace{1cm} (2.32)

Then

$$R(y) = O \left( \varepsilon^2 t^{2\beta} e^{-t\phi_1(\varepsilon y)} \right) + \sum_{i=1}^m O(\varepsilon \mu_i t^{2\beta}).$$  \hspace{1cm} (2.33)

In the rest of this paper, we try to find a solution of problem (2.21) in the form

$$\omega = V + \phi,$$

where $\phi$ will represent a lower order correction. In terms of $\phi$, problem (2.21) becomes

$$\begin{cases} -\Delta \phi + \varepsilon^2 \phi = 0 & \text{in } \Omega_t, \\ \frac{\partial \phi}{\partial \nu} - W \phi = R + N(\phi) & \text{on } \partial \Omega_t, \end{cases}$$  \hspace{1cm} (3.1)

where the “nonlinear term” $N(\phi)$ is given by

$$N(\phi) = q(y,t) e^V(\varepsilon^\phi - 1 - \phi).$$  \hspace{1cm} (3.2)

3. Solvability of a linear problem. In this section we shall study the solvability of the following linear problem: given $h \in L^\infty(\partial \Omega_t)$ and points $\xi = (\xi_1, \ldots, \xi_m) \in \mathcal{O}_t$, we find a function $\phi$, and scalars $c_1, \ldots, c_m$, such that

$$\begin{cases} -\Delta \phi + \varepsilon^2 \phi = 0 & \text{in } \Omega_t, \\ \frac{\partial \phi}{\partial \nu} - W \phi = h + \sum_{i=1}^m c_i \chi_i Z_{1i} & \text{on } \partial \Omega_t, \\ \int_{\Omega_t} \chi_i Z_{1i} \phi = 0 & \forall \ i = 1, \ldots, m, \end{cases}$$  \hspace{1cm} (3.1)

where $W = q(y,t) e^V$ satisfies (2.30) and (2.33), and $Z_{1i}, \chi_i$ are defined as follows: let $Z_0$, $Z_1$ denote the functions $z_{\mu0}$ and $z_{\mu1}$, respectively given by (2.4) and (2.5).
with the parameter \( \mu = 1 \). More precisely,

\[
Z_0(z) := z_{\mu 0}(z) \bigg|_{\mu = 1} = 1 - 2 \frac{z_2 + 1}{z_1^2 + (z_2 + 1)^2},
\]

\[
Z_1(z) := z_{\mu 1}(z) \bigg|_{\mu = 1} = -2 \frac{z_1}{z_1^2 + (z_2 + 1)^2}.
\]

For each point \( \xi_i \in \partial \Omega \), we define a rotation map \( A_i : \mathbb{R}^2 \to \mathbb{R}^2 \) such that \( A_i \nu_{\Omega} (\xi_i) = \nu_{\Omega} (0) \). Furthermore, let \( d > 0 \) be a small but fixed radius, depending only on the geometry of \( \Omega \), such that

\[
(3.1)
\]

is a C\(^2\) diffeomorphism satisfying \( H_i (B_d(0) \cap A_i (\partial \Omega - \{\xi_i\})) = M \cap \mathbb{R}^2_+ \), where \( M \) is an open neighborhood of the origin. We can select \( C > 0 \) and \( r \) for \( T > T_0 \) such that for any \( i = 1, \ldots, m \), \( j = 0, 1 \), we define

\[
H_i^j(y) = \frac{1}{\varepsilon} A_i (\varepsilon y - \xi_i) \quad \text{and} \quad Z_{ji}(y) = \frac{1}{\gamma_i} Z_j \left( \frac{1}{\gamma_i} H_i^j(y) \right).
\]

Besides, we consider \( R_0 \) a large but fixed positive number and \( \chi(r) \) a radial smooth, non-increasing cut-off function with \( 0 \leq \chi(r) \leq 1 \), \( \chi(r) = 1 \) for \( r \leq R_0 \) and \( \chi(r) = 0 \) for \( r \geq R_0 + 1 \). Let

\[
\chi_i(y) = \chi \left( \frac{1}{\gamma_i} |H_i^j(y)| \right), \quad i = 1, \ldots, m.
\]

Equation (3.1) will be solved for \( h \in L^\infty(\partial \Omega_t) \), but we need to estimate the size of the solution by introducing the following norm:

\[
||h||_{*, \partial \Omega_t} = \sup_{y \in \partial \Omega_t} \left( \frac{1}{\varepsilon} \frac{1}{\gamma_i} \right) \left( \sum_{i=1}^m \frac{\gamma_i^2}{|y - \xi_i| + \gamma_i \varepsilon} \right)^{-1} h(y),
\]

where \( 0 < \sigma < 1 \) is fixed and chosen later on.

**Proposition 3.1.** Let \( m \) be a positive integer. Then there exist constants \( t_m > 1 \) and \( C > 0 \) such that for any \( t > t_m \), any points \( \xi = (\xi_1, \ldots, \xi_m) \in O_t \) and any \( h \in L^\infty(\partial \Omega_t) \), there is a unique solution \( \phi \in L^\infty(\Omega_t) \), \( c_1, \ldots, c_m \in \mathbb{R} \) to problem (3.1), which satisfies

\[
||\phi||_{L^\infty(\Omega_t)} \leq C t ||h||_{*, \partial \Omega_t} \quad \text{and} \quad |c_i| \leq C ||h||_{*, \partial \Omega_t}, \quad i = 1, \ldots, m.
\]

The proof of this result will be divided into four steps which we state and prove next.

**Step 1.** Constructing a suitable barrier.

**Lemma 3.1.** There exist positive constants \( R_1 \) and \( C \), independent of \( t \), such that for any \( t > 1 \) sufficiently large, and \( 0 < \sigma < 1 \), there is

\[
\psi : \Omega_t \setminus \bigcup_{i=1}^m B_{R_1 \gamma_i} (\xi_i^0) \to \mathbb{R}
\]
smooth and positive so that

\[
\begin{cases}
-\Delta \psi + \varepsilon^2 \psi \geq \sum_{i=1}^{m} \frac{\gamma_i^\sigma}{|y - \xi_i^1|^2 + \varepsilon^2} + \varepsilon^2 & \text{in } \Omega_t \setminus \bigcup_{i=1}^{m} B_{R_t \gamma_i}(\xi_i^t), \\
\frac{\partial \psi}{\partial \nu} - W \psi \geq \sum_{i=1}^{m} \frac{\gamma_i^\sigma}{|y - \xi_i^1|^{1+\sigma} + \varepsilon} & \text{on } \partial \Omega_t \setminus \bigcup_{i=1}^{m} B_{R_t \gamma_i}(\xi_i^t), \\
\psi \geq 1 & \text{on } \Omega_t \cap \left( \bigcup_{i=1}^{m} \partial B_{R_t \gamma_i}(\xi_i^t) \right).
\end{cases}
\]  

(3.7)

Moreover, we have a uniform bound

\[0 < \psi \leq C \quad \text{in } \Omega_t \setminus \bigcup_{i=1}^{m} B_{R_t \gamma_i}(\xi_i^t).\]  

(3.8)

**Proof.** Let \( \eta_i \in C_0^\infty(\mathbb{R}^2) \) be such that \( 0 \leq \eta_i \leq 1, \eta_i \equiv 1 \) in \( \Omega_t \cap B_{d/(2\varepsilon)}(\xi_i^t), \eta_i \equiv 0 \) in \( \Omega_t \setminus B_{d/\varepsilon}(\xi_i^t), |\nabla \eta_i| \leq C \varepsilon \) in \( \Omega_t, |\Delta \eta_i| \leq C \varepsilon^2 \) in \( \Omega_t \). Define

\[
\psi = \sum_{i=1}^{m} \eta_i \left[ \gamma_i^\sigma \frac{(y - \xi_i^1) \cdot \nu(\xi_i^t)}{r^{1+\sigma}} + C_1 \left( 1 - \frac{\gamma_i^\sigma}{r^\sigma} \right) \right] + C_1 \psi_0,
\]

where \( r = |y - \xi_i^1 - \gamma_i \nu(\xi_i^t)| \) and \( \psi_0 \) is the solution to

\[
\begin{cases}
-\Delta \psi_0 + \varepsilon^2 \psi_0 = \varepsilon^2 & \text{in } \Omega_t, \\
\frac{\partial \psi_0}{\partial \nu} = \varepsilon & \text{on } \partial \Omega_t.
\end{cases}
\]

Observing that \( \psi_0 \) is uniformly bounded in \( \Omega_t \), it is directly checked that, choosing the positive constant \( C_1 \) larger if necessary, \( \psi \) verifies the required conditions. \( \square \)

**Step 2.** Handling a linear equation. We consider first the linear equation

\[
\begin{cases}
-\Delta \phi + \varepsilon^2 \phi = f & \text{in } \Omega_t, \\
\frac{\partial \phi}{\partial \nu} - W \phi = h & \text{on } \partial \Omega_t,
\end{cases}
\]  

(3.9)

where we use \( \| \cdot \|_{*,\partial \Omega_t} \) to estimate \( h \in L^\infty(\partial \Omega_t) \), and for \( f \in L^\infty(\Omega_t) \) we introduce the following norm:

\[
\|f\|_{*,\Omega_t} = \sup_{y \in \Omega_t} \left| \sum_{\ell=1}^{m} \frac{\gamma_i^\sigma}{(|y - \xi_i^1| + \gamma_i)^{2+\sigma} + \varepsilon^2} f(y) \right|^{-1}.
\]  

(3.10)

For the solution of (3.9) satisfying orthogonality conditions with respect to \( Z_{0i} \) and \( Z_{1i} \), we have the following a priori estimate.

**Lemma 3.2.** There exist \( R_0 > 0 \) and \( t_m > 1 \) such that for any \( t > t_m \) and any solution \( \phi \) of (3.9) with the orthogonality conditions

\[
\int_{\Omega_t} \chi_i Z_{ji} \phi = 0 \quad \forall \ i = 1, \ldots, m, \ j = 0, 1,
\]  

(3.11)

we have

\[
\|\phi\|_{L^\infty(\Omega_t)} \leq C \left( \|h\|_{*,\partial \Omega_t} + \|f\|_{*,\Omega_t} \right),
\]  

(3.12)

where \( C > 0 \) is independent of \( t \).
Proof. Take \( R_0 = 2R_1, R_1 \) being the constant of Lemma 3.1. Thanks to the barrier \( \phi \) of that lemma, we deduce the following maximum principle: if \( \phi \in H^1(\Omega_t \setminus \bigcup_{i=1}^m B_{R_1\gamma_i}(\xi_i')) \) satisfies

\[
\begin{cases}
-\Delta \phi + \varepsilon^2 \phi \geq 0 & \text{in } \Omega_t \setminus \bigcup_{i=1}^m B_{R_1\gamma_i}(\xi_i'), \\
\frac{\partial \phi}{\partial \nu} - W \phi \geq 0 & \text{on } \partial \Omega_t \setminus \bigcup_{i=1}^m B_{R_1\gamma_i}(\xi_i'), \\
\phi \geq 0 & \text{on } \Omega_t \cap \left( \bigcup_{i=1}^m \partial B_{R_1\gamma_i}(\xi_i') \right).
\end{cases}
\]

then \( \phi \geq 0 \) in \( \Omega_t \setminus \bigcup_{i=1}^m B_{R_1\gamma_i}(\xi_i') \).

Let \( f, h \) be bounded and \( \phi \) a solution to (3.9) satisfying (3.11). Consider the “inner norm”

\[
\|\phi\|_i = \sup_{\Omega_t \cap (\bigcup_{i=1}^m B_{R_1\gamma_i}(\xi_i))} |\phi|,
\]

and take

\[
\hat{\phi} = C_1 \psi \left( \|\phi\|_i + \|h\|_{*,\partial \Omega_t} + \|f\|_{*,\Omega_t} \right),
\]

with \( C_1 \) a large constant independent of \( t \). From the above maximum principle it follows that \( -\hat{\phi} \leq \phi \leq \hat{\phi} \) in \( \Omega_t \setminus \bigcup_{i=1}^m B_{R_1\gamma_i}(\xi_i') \). Since \( \psi \) is uniformly bounded, we get

\[
\|\phi\|_{L^\infty(\Omega_t)} \leq C \left( \|\phi\|_i + \|h\|_{*,\partial \Omega_t} + \|f\|_{*,\Omega_t} \right), \tag{3.13}
\]

for some constant \( C \) independent of \( \phi \) and \( t \).

We prove the lemma by contradiction. Assume that there are sequences of parameters \( t_n \to +\infty \), points \( \xi^n = (\xi^n_1, \ldots, \xi^n_m) \in \mathcal{O}_{t_n} \), functions \( f_n \) and \( h_n \), and corresponding solutions \( \phi_n \) of equation (3.9) with the orthogonality conditions (3.11) such that

\[
\|\phi_n\|_{L^\infty(\Omega_{t_n})} = 1, \quad \|h_n\|_{*,\partial \Omega_{t_n}} \to 0, \quad \|f_n\|_{*,\Omega_{t_n}} \to 0 \quad \text{as } n \to +\infty. \tag{3.14}
\]

Using the expansion of \( W^n \) in (2.30) and elliptic regularity, we can deduce that \( \hat{\phi}^n := \phi_n(\mathcal{A}^{\mu^n}_n(\xi^n_1, \ldots, \xi^n_m)) \), where \( \mu^n = (\mu^n_1, \ldots, \mu^n_m) \) and \( \gamma^n_i = \mu^n_i \exp \{ -t_n[\phi_1(\xi^n_1) - 1] \} \) for each \( i \in \{1, \ldots, m\} \), converges uniformly over compact sets to a bounded solution \( \hat{\phi}_i^\infty \) of equation (2.6) with

\[
\int_{\mathbb{R}^2_+} \chi_Z \hat{\phi}_j^\infty = 0 \quad \text{for } j = 0, 1. \tag{3.15}
\]

However, by the result of [12], any bounded solution of equation (2.6) can be expressed as a linear combination of \( Z_0 \) and \( Z_1 \). Therefore, (3.15) implies \( \hat{\phi}_j^\infty = 0 \) or \( \lim_{n \to +\infty} \|\phi_n\|_i = 0 \). But (3.13) and (3.14) tell us \( \liminf_{n \to +\infty} \|\phi_n\|_i > 0 \), which is a contradiction. \( \square \)

Step 3. Establishing an a priori estimate for solutions to (3.9) with the orthogonality condition \( \int_{\Omega_t} \chi_i Z_{1i} \phi = 0 \) only.

Lemma 3.3. For \( t \) sufficiently large, if \( \phi \) is a solution of (3.9) and satisfies

\[
\int_{\Omega_t} \chi_i Z_{1i} \phi = 0 \quad \forall i = 1, \ldots, m, \tag{3.16}
\]
then

\[ \| \phi \|_{L^\infty(\Omega_t)} \leq C t (\| h \|_{\partial \Omega_t} + \| f \|_{\partial \Omega_t}), \tag{3.17} \]

where \( C > 0 \) is independent of \( t \).

\textbf{Proof.} Let \( R > R_0 + 1 \) be large and fixed, \( \delta > 0 \) small but fixed. Denote for \( i = 1, \ldots, m, \)

\[ \tilde{Z}_0(y) = Z_0(y) - \frac{1}{\gamma_i} + a_0 G(\xi_i, \varepsilon y), \tag{3.18} \]

where

\[ a_0 = \frac{1}{\gamma_i [H(\xi_i, \xi_i) - 2 \log(\varepsilon \gamma_i R)]}. \tag{3.19} \]

From estimate (2.18) and definitions (2.11), (2.20) and (2.28) we find

\[ C_1 | \log \varepsilon_i | \leq - \log(\varepsilon \gamma_i R) \leq C_2 | \log \varepsilon_i |, \tag{3.20} \]

and

\[ \tilde{Z}_0(y) = O \left( \frac{G(\varepsilon y, \xi_i)}{\gamma_i | \log \varepsilon_i |} \right). \tag{3.21} \]

Let \( \eta_1 \) and \( \eta_2 \) be radial smooth cut-off functions in \( \mathbb{R}^2 \) such that

\[ 0 \leq \eta_1 \leq 1; \quad |\nabla \eta_1| \leq C \text{ in } \mathbb{R}^2; \quad \eta_1 \equiv 1 \text{ in } B_R(0); \quad \eta_1 \equiv 0 \text{ in } \mathbb{R}^2 \setminus B_{R+1}(0); \]

\[ 0 \leq \eta_2 \leq 1; \quad |\nabla \eta_2| \leq C \text{ in } \mathbb{R}^2; \quad \eta_2 \equiv 1 \text{ in } B_{ \frac{3}{4}\delta}(0); \quad \eta_2 \equiv 0 \text{ in } \mathbb{R}^2 \setminus B_{ \frac{1}{4}\delta}(0). \]

Set

\[ \eta_{1i}(y) = \eta_1 \left( \frac{1}{\gamma_i} |H_i^t(y)| \right), \quad \eta_{2i}(y) = \eta_2 \left( \varepsilon |H_i^t(y)| \right), \tag{3.22} \]

and

\[ \tilde{Z}_0(y) = \eta_{1i} Z_0 + (1 - \eta_{1i}) \eta_{2i} \tilde{Z}_0. \tag{3.23} \]

Given \( \phi \) satisfying (3.9) and (3.16), we define

\[ \tilde{\phi} = \phi + \sum_{i=1}^m d_i \tilde{Z}_0 + \sum_{i=1}^m e_i \chi_i Z_{1i}. \tag{3.24} \]

Let us first prove the existence of \( d_i \) and \( e_i \) such that \( \tilde{\phi} \) satisfies the orthogonality conditions

\[ \int_{\Omega_t} \chi_i Z_{1i} \tilde{\phi} = 0 \quad \forall \ i = 1, \ldots, m, \ j = 0, 1. \tag{3.25} \]

Remark that the map \( H_i^t \) preserves area and \( \tilde{Z}_0 \) coincides with \( Z_0 \) in the region \( \{ y \in \Omega_t : |H_i^t(y)| \leq \gamma_i R \} \), which imply that \( \tilde{Z}_0 \) is orthogonal to \( \chi_i Z_{1i} \) for each \( i = 1, \ldots, m \). Furthermore, using definition (3.24), orthogonality condition (3.25) for \( j = 1 \) and the fact that \( \chi_i \chi_k \equiv 0 \) if \( i \neq k \), we obtain

\[ e_i = - \int_{\Omega_t} \sum_{k \neq i}^m d_k \chi_i Z_{1i} \tilde{Z}_{0k} / \int_{\Omega_t} \chi_i^2 Z_{1i}^2, \quad i = 1, \ldots, m. \tag{3.26} \]

Notice that by (3.21) and (3.23),

\[ \int_{\Omega_t} \chi_i Z_{1i} \tilde{Z}_0 dy = O \left( \frac{\gamma_i | \log t |}{\gamma_k | \log \varepsilon_k |} \right), \quad \forall \ i \neq k, \]

and

\[ \int_{\Omega_t} \chi_i^2 Z_{1i}^2 dy = \int_{\mathbb{R}^2} \chi^2(|z|)|Z_{1i}^2(z)|dz = c > 0, \quad \forall \ i. \]
Then
\[ |e_i| \leq C \sum_{k \neq i} d_k \left| \frac{\gamma_i \log t}{\gamma_k \log \varepsilon_k} \right|. \]  \hspace{1cm} (3.27)

We only need to consider \( d_i \). Testing definition [3.24] against \( \chi_k Z_{0k} \), we obtain a system of \((d_1, \ldots, d_m)\),
\[ \sum_{i=1}^{m} d_i \int_{\Omega_t} \chi_k Z_{0k} \tilde{Z}_{0i} = -\int_{\Omega_t} \chi_k Z_{0k} \phi, \quad \forall \ k = 1, \ldots, m. \]  \hspace{1cm} (3.28)

But
\[ \int_{\Omega_t} \chi_k Z_{0k} \tilde{Z}_{0i} dy = O \left( \frac{\gamma_k \log t}{\gamma_i \log \varepsilon_i} \right), \quad \forall \ i \neq k, \]
and
\[ \int_{\Omega_t} \chi_k Z_{0k} \tilde{Z}_{0k} dy = \int_{\Omega_t} \chi_k Z_{0k}^2 dy = \int_{\mathbb{R}^n} \chi(|z|) Z_0^2(z) dz = C > 0, \quad \forall \ k. \]

We denote \( \mathcal{H} \) the coefficient matrix of system [3.28]. By the above estimates, it is clear that \( P^{-1} \mathcal{H} P \) is diagonally dominant and thus invertible, where \( P = \text{diag}(\gamma_1, \ldots, \gamma_m) \). Hence \( \mathcal{H} \) is also invertible and \((d_1, \ldots, d_m)\) is well defined.

Estimate (3.17) is a direct consequence of the following three claims.

**Claim 1.** Let \( \mathcal{L} = -\Delta_g + \varepsilon^2 \), then for any \( i = 1, \ldots, m, \)
\[ \| \mathcal{L}(\chi_i Z_{1i}) \|_{**; \Omega_t} \leq C, \quad \| \mathcal{L}(\tilde{Z}_{0i}) \|_{**; \Omega_t} \leq C \frac{1}{\gamma_i \log \varepsilon_i}. \]  \hspace{1cm} (3.29)

**Claim 2.** Let \( \mathcal{B} = \frac{\partial}{\partial v} - W \), then for any \( i = 1, \ldots, m, \)
\[ \| \mathcal{B}(\chi_i Z_{1i}) \|_{*, \partial \Omega_t} \leq \frac{C}{\gamma_i}, \quad \| \mathcal{B}(\tilde{Z}_{0i}) \|_{*, \partial \Omega_t} \leq \frac{1}{\gamma_i \log \varepsilon_i}. \]  \hspace{1cm} (3.30)

**Claim 3.** For any \( i = 1, \ldots, m, \)
\[ |d_i| \leq C \gamma_i \log \varepsilon_i (\| h \|_{*, \partial \Omega_t} + \| f \|_{**, \Omega_t}), \]
\[ |e_i| \leq C \gamma_i \log t (\| h \|_{*, \partial \Omega_t} + \| f \|_{**, \Omega_t}). \]  \hspace{1cm} (3.31)

In fact, the definition of \( \tilde{\phi} \) in [3.24] tells us
\[ \begin{cases} \mathcal{L}(\tilde{\phi}) = f + \sum_{i=1}^{m} d_i \mathcal{L}(\tilde{Z}_{0i}) + \sum_{i=1}^{m} e_i \mathcal{L}(\chi_i Z_{1i}) & \text{in } \Omega_t, \\ \mathcal{B}(\tilde{\phi}) = h + \sum_{i=1}^{m} d_i \mathcal{B}(\tilde{Z}_{0i}) + \sum_{i=1}^{m} e_i \mathcal{B}(\chi_i Z_{1i}) & \text{on } \partial \Omega_t. \end{cases} \]  \hspace{1cm} (3.32)

Since [3.25] holds, by Lemma 3.2 we get
\[ \| \tilde{\phi} \|_{L^\infty(\Omega_t)} \leq C \left\{ \| f \|_{**, \Omega_t} + \sum_{i=1}^{m} |d_i| \left( \| \mathcal{L}(\tilde{Z}_{0i}) \|_{**, \Omega_t} + \| \mathcal{B}(\tilde{Z}_{0i}) \|_{*, \partial \Omega_t} \right) \right. \]
\[ + \left. \| h \|_{*, \partial \Omega_t} + \sum_{i=1}^{m} |e_i| \left( \| \mathcal{L}(\chi_i Z_{1i}) \|_{**, \Omega_t} + \| \mathcal{B}(\chi_i Z_{1i}) \|_{*, \partial \Omega_t} \right) \right\}. \]  \hspace{1cm} (3.33)
Furthermore, using the definition of \( \tilde{\phi} \) again and the fact that
\[
\left\| \chi_i Z_{1i} \right\|_{L^\infty(\Omega_i)} \leq \frac{C}{\gamma_i} \quad \text{and} \quad \left\| \tilde{Z}_{0i} \right\|_{L^\infty(\Omega_i)} \leq \frac{C}{\gamma_i}, \quad i = 1, \ldots, m, \tag{3.34}
\]
the Lemma 3.3 then follows from Claims 1-3 and estimate \([3.33]\).

**Proof of Claim 1.** Thanks to \( H^r_i(\xi_i) = (0, 0) \) and \( \nabla H^r_i(\xi_i) = A_{zi} \), we find
\[
-\Delta y = -\Delta z_i + O(\varepsilon|z_i|) \nabla^2 z_i + O(\varepsilon) \nabla z_i, \quad \nabla y = A_i \nabla z_i + O(\varepsilon|z_i|) \nabla z_i, \tag{3.35}
\]
and
\[
\frac{\partial}{\partial \nu_y} = -\frac{\partial}{\partial z_{i,2}} + O(\varepsilon|z_i|) \nabla z_i, \tag{3.36}
\]
where
\[
z_i := H^r_i(y) = \frac{1}{\varepsilon} H_i(A_i(\varepsilon y - \xi_i)) = A_i(y - \xi_i) \{1 + O(\varepsilon A_i(y - \xi_i))\} \tag{3.37}
\]
Note that in the region \(|z_i| \leq \gamma_i(R_0 + 1)\),
\[
\mathcal{L}(\chi_i Z_{1i}) = (-\Delta_y + \varepsilon^2) \left[ \frac{1}{\gamma_i} (\chi_i Z_{1i}) \left( \frac{z_i}{\gamma_i} \right) \right] = O \left( \frac{1}{\gamma_i^2} \right) + O \left( \frac{\varepsilon}{\gamma_i^2} \right) + O \left( \frac{\varepsilon^2}{\gamma_i^2} \right).
\]
By \([3.10]\) we deduce \( \left\| \mathcal{L}(\chi_i Z_{1i}) \right\|_{L^\infty, \Omega_i} = O(1/\gamma_i) \). Let us prove the second inequality in \([3.29]\). Using the definition of \( \tilde{Z}_{0i} \) in \([3.28]\) and the fact that \( \eta_{1i} \xi_i \equiv \eta_{1i} \), we obtain
\[
\mathcal{L}(\tilde{Z}_{0i}) = \eta_{1i} \mathcal{L}(Z_{0i} - \tilde{Z}_{0i}) + \eta_{2i} \mathcal{L}(\tilde{Z}_{0i}) + 2\nabla \eta_{2i} \nabla \tilde{Z}_{0i} + \tilde{Z}_{0i} \Delta \eta_{2i} + 2\nabla \eta_{1i} \nabla (Z_{0i} - \tilde{Z}_{0i}) + (Z_{0i} - \tilde{Z}_{0i}) \Delta \eta_{1i}. \tag{3.38}
\]
Consider four regions
\[
\Omega_1 = (H^r_i)^{-1} \left( \{|z_i| \leq \gamma_i R\} \cap \mathbb{R}^2 \right),
\]
\[
\Omega_2 = (H^r_i)^{-1} \left( \{\gamma_i R < |z_i| \leq \gamma_i(R + 1)\} \cap \mathbb{R}^2 \right),
\]
\[
\Omega_3 = (H^r_i)^{-1} \left( \left\{ \gamma_i(R + 1) < |z_i| \leq \frac{\delta}{4\varepsilon} \right\} \cap \mathbb{R}^2 \right),
\]
\[
\Omega_4 = (H^r_i)^{-1} \left( \left\{ \frac{\delta}{4\varepsilon} < |z_i| \leq \frac{\delta}{3\varepsilon} \right\} \cap \mathbb{R}^2 \right).
\]
Notice that \([1.7]\) and \([3.18]\) imply
\[
\mathcal{L}(Z_{0i} - \tilde{Z}_{0i}) = (-\Delta_y + \varepsilon^2) \left[ \frac{1}{\gamma_i} - a_{0i} G(\xi_i, \varepsilon y) \right] = -\frac{\varepsilon^2}{\gamma_i}. \tag{3.39}
\]
In \( \Omega_1 \cup \Omega_2 \), by \([3.35]\) and since \( \Delta Z_0 = 0 \) we have
\[
\mathcal{L}(Z_{0i}) = (-\Delta_y + \varepsilon^2) \left[ \frac{1}{\gamma_i} Z_{0i} \left( \frac{z_i}{\gamma_i} \right) \right] = O \left( \frac{\varepsilon}{\gamma_i^2} \right) + O \left( \frac{\varepsilon^2}{\gamma_i^2} \right), \tag{3.40}
\]
and then
\[
\eta_{1i} \mathcal{L}(Z_{0i} - \tilde{Z}_{0i}) + \mathcal{L}(\tilde{Z}_{0i}) = (\eta_{1i} - 1) \mathcal{L}(Z_{0i} - \tilde{Z}_{0i}) + \mathcal{L}(Z_{0i})
\]
\[
= O \left( \frac{\varepsilon}{\gamma_i^2} \right) + O \left( \frac{\varepsilon^2}{\gamma_i^2} \right). \tag{3.41}
\]
Hence in $\Omega_1$,
\[
\mathcal{L}(\tilde{Z}_0) = \mathcal{L}(Z_0) = O \left( \frac{\varepsilon}{\gamma_i} \right) + O \left( \frac{\varepsilon^2}{\gamma_i} \right), \tag{3.42}
\]
In $\Omega_2$, by (3.18)-(3.19),
\[
Z_{0i} - \tilde{Z}_{0i} = \frac{1}{\gamma_i} - a_{0i} G(\xi_i, \varepsilon y) - \frac{1}{\gamma_i} \frac{H(\xi_i, \xi_i) - 2 \log(\varepsilon \gamma_i R)}{\gamma_i R} \left( 2 \log \left( 1 + O(\varepsilon |y - \xi_i|) \right) \right), \tag{3.43}
\]
and by (3.20),
\[
|Z_{0i} - \tilde{Z}_{0i}| = O \left( \frac{1}{\gamma_i \log \varepsilon_i} \right) \quad \text{and} \quad |
\nabla (Z_{0i} - \tilde{Z}_{0i})| = O \left( \frac{1}{\gamma_i \log \varepsilon_i} \right). \tag{3.44}
\]
Then, in this region, by (3.38), (3.41) and (3.44) we obtain
\[
\mathcal{L}(\tilde{Z}_0) = O \left( \frac{1}{\gamma_i \log \varepsilon_i} \right). \tag{3.45}
\]
In $\Omega_3$, by (3.2), (3.3), (3.35), (3.38) and (3.39) we deduce
\[
\mathcal{L}(\tilde{Z}_0) = \mathcal{L}(\tilde{Z}_0) = \mathcal{L}(Z_0) - \frac{\varepsilon^2}{\gamma_i} = -\Delta_y + \varepsilon^2 \left[ \frac{1}{\gamma_i} Z_0 \left( \frac{z_i}{\gamma_i} \right) \right] - \frac{\varepsilon^2}{\gamma_i} = O \left( \frac{\varepsilon |z_i|}{\gamma_i} \right) + O \left( \frac{\varepsilon^2 |z_i|}{\gamma_i} \right), \tag{3.46}
\]
Finally in $\Omega_4$, owing to (3.21),
\[
|\tilde{Z}_0| = O \left( \frac{|\log \delta|}{\gamma_i |\log \varepsilon_i|} \right) \quad \text{and} \quad |\nabla \tilde{Z}_0| = O \left( \frac{\varepsilon}{\gamma_i \delta |\log \varepsilon_i|} \right), \tag{3.47}
\]
and by (3.39),
\[
\mathcal{L}(\tilde{Z}_0) = \mathcal{L}(Z_0) - \frac{\varepsilon^2}{\gamma_i} = -\Delta_y \left[ \frac{1}{\gamma_i} Z_0 \left( \frac{z_i}{\gamma_i} \right) \right] - \frac{\varepsilon^2}{\gamma_i} = O \left( \frac{\varepsilon^3}{\gamma_i} \right). \tag{3.48}
\]
Hence in $\Omega_4$,
\[
\mathcal{L}(\tilde{Z}_0) = \eta_{2i} \mathcal{L}(\tilde{Z}_0) + 2 \nabla \eta_{2i} \nabla \tilde{Z}_0 + \tilde{Z}_0 \Delta \eta_{2i} = O \left( \frac{\varepsilon^2 |\log \delta|}{\gamma_i \delta^2 |\log \varepsilon_i|} \right). \tag{3.49}
\]
Putting together (3.10), (3.42), (3.45), (3.46) and (3.49), we conclude
\[
\|\mathcal{L}(\tilde{Z}_0)\|_{** \Omega_1} = O \left( \frac{1}{\gamma_i \log \varepsilon_i} \right).
Proof of Claim 2. By (2.30), (3.36) and (3.37), it follows that for any $|z_i| \leq \frac{\delta}{3(\varepsilon t)^{\beta}}$ and $z_i \in \partial \mathbb{R}^d_+$,

$$
\mathcal{B}(Z_{ji}) = \left( -\frac{\partial}{\partial z_{i,2}} - W + O(\varepsilon |z_i|) \nabla z_i \right) \left[ \frac{1}{\gamma_i} Z_j \left( \frac{z_i}{\gamma_i} \right) \right]
$$

$$
= \frac{2}{|z_i - \gamma_i \eta_0 z_i(0)|^2} \left[ O\left(\varepsilon t^\beta |z_i|\right) + \sum_{l=1}^m O\left(\varepsilon^\alpha \mu_\alpha^l\right) \right], \quad j = 0, 1. \quad (3.50)
$$

Then for any $|z_i| \leq \gamma_i (R_0 + 1)$ and $z_i \in \partial \mathbb{R}^d_+$,

$$
\mathcal{B}(\chi_i Z_{1i}) = \chi_i \mathcal{B}(Z_{1i}) + Z_{1i} \left( -\frac{\partial}{\partial z_{i,2}} + O(\varepsilon |z_i|) \nabla z_i \right) \left[ \chi \left( \frac{z_i}{\gamma_i} \right) \right]
$$

$$
= O\left(\frac{\varepsilon t^\beta}{\gamma_i}\right) + \sum_{l=1}^m O\left(\frac{\varepsilon^\alpha \mu_\alpha^l}{\gamma_i^2}\right),
$$

because of $\frac{\partial \chi_i(\frac{z_i}{\gamma_i})}{\partial z_{i,2}} |_{z_i,2=0} = 0$. Hence by (3.51), we get $\| \mathcal{B}(\chi_i Z_{1i}) \|_{l, \partial \mathcal{N}_i} = O(1/\gamma_i)$.

Let us prove now the second inequality in (3.30). Note that

$$
\mathcal{B}(\tilde{Z}_{0i}) = \eta_1 \mathcal{B}(Z_{0i} - \tilde{Z}_{0i}) + \mathcal{B}(Z_{0i} + \frac{\partial \eta_1^i}{\partial \nu_y} (Z_{0i} - \tilde{Z}_{0i}) + \frac{\partial \eta_2^i}{\partial \nu_y} \tilde{Z}_{0i}. \quad (3.51)
$$

Consider four regions

$$
B_1 = (H_1^i)^{-1} \left( \{|z_i| \leq \gamma_i R \} \cap \partial \mathbb{R}^d_+ \right),
$$

$$
B_2 = (H_1^i)^{-1} \left( \{ \gamma_i R < |z_i| \leq \gamma_i (R + 1) \} \cap \partial \mathbb{R}^d_+ \right),
$$

$$
B_3 = (H_1^i)^{-1} \left( \left\{ \gamma_i (R + 1) < |z_i| \leq \frac{\delta}{4\varepsilon} \right\} \cap \partial \mathbb{R}^d_+ \right),
$$

$$
B_4 = (H_1^i)^{-1} \left( \left\{ \frac{\delta}{4\varepsilon} < |z_i| \leq \frac{\delta}{3\varepsilon} \right\} \cap \partial \mathbb{R}^d_+ \right).
$$

On $B_1$, by (3.50)-(3.51),

$$
\mathcal{B}(\tilde{Z}_{0i}) = \mathcal{B}(Z_{0i}) = \sum_{i=1}^m O\left(\frac{\varepsilon^\alpha \mu_\alpha^i}{\gamma_i^2}\right). \quad (3.52)
$$

On $B_2$, owing to $\frac{\partial \eta_1^i}{\partial z_{i,2}} |_{z_i,2=0} = 0$,

$$
\frac{\partial \eta_1^i}{\partial \nu_y} = \left( -\frac{\partial}{\partial z_{i,2}} + O(\varepsilon |z_i|) \nabla z_i \right) \left[ \eta_1 \left( \frac{z_i}{\gamma_i} \right) \right] = O(\varepsilon),
$$

and then, by (2.30), (3.18), (3.44), (3.50) and (3.51),

$$
\mathcal{B}(\tilde{Z}_{0i}) = \mathcal{B}(Z_{0i}) + (1 - \eta_1^i) W(Z_{0i} - \tilde{Z}_{0i}) + \frac{\partial \eta_1^i}{\partial \nu_y} (Z_{0i} - \tilde{Z}_{0i})
$$

$$
= O\left(\frac{1}{R^2 |z_i| \log \varepsilon_i} \right). \quad (3.53)
$$

On $B_3$, by (3.51),

$$
\mathcal{B}(\tilde{Z}_{0i}) = \mathcal{B}(\tilde{Z}_{0i}) = \mathcal{B}(Z_{0i}) + W \left[ 1 - a_0 G(\xi_i, \varepsilon_y) \right].
$$
For the estimate of these two terms, we decompose $B_3$ to some subregions:

$$B_{3,i} = (H_i^\epsilon)^{-1}\left(\{\gamma_i(R+1) \leq |z_i| \leq \delta/(3\epsilon t^\beta)\} \cap \partial\mathbb{R}^2_+\right),$$

$$B_{3,k} = B_3 \cap (H_k^\epsilon)^{-1}\left(\{|z_k| \leq \delta/(3\epsilon t^\beta)\} \cap \partial\mathbb{R}^2_+\right), \quad k \neq i, \quad \tilde{B}_3 = B_3 \setminus \bigcup_{i=1}^m B_{3,i}.$$

From (2.30), (2.33) and (3.50) we get

$$B(Z_{0i}) = \begin{cases}
\frac{2}{|z_i - \gamma_i\nu_{\mathbb{R}^2_+}(0)|^2} \left[O(\epsilon t^\beta |z_i|) + \sum_{l=1}^m O(\epsilon_{2i}^l \mu_{2i}^l)\right] & \text{on } B_{3,i}, \\
O\left(\gamma_i^{-1} \epsilon t^{2m\beta} e^{-\gamma t_\epsilon}\right) + O\left(\epsilon t^{2\beta}\right) & \text{on } \tilde{B}_3.
\end{cases}$$

Moreover, by (3.19)–(3.20),

$$W\left[1 - a_{0i} G(\xi_i, \epsilon y)\right] = \begin{cases}
\frac{2}{|z_i - \gamma_i\nu_{\mathbb{R}^2_+}(0)|^2} O\left(\frac{\log |z_i| - \log \gamma_i R + \epsilon |z_i|}{\log \epsilon_i}\right) & \text{on } B_{3,i}, \\
O\left(\gamma_i^{-1} \epsilon t^{2m\beta} e^{-\gamma t_\epsilon}\right) & \text{on } \tilde{B}_3.
\end{cases}$$

Thus we have that on $B_{3,i} \cup \tilde{B}_3$,

$$B(\hat{Z}_{0i}) = B(\tilde{Z}_{0i}) = O\left(\frac{\log |z_i| - \log \gamma_i R}{|z_i - \gamma_i\nu_{\mathbb{R}^2_+}(0)|^2}, \frac{1}{\log \epsilon_i}\right). \quad (3.54)$$

On $B_{3,k}$ with $k \neq i$, by (2.30) and (3.21),

$$B(\hat{Z}_{0i}) = B(\tilde{Z}_{0i}) = \left(-\frac{\partial}{\partial z_i} + O(\epsilon |z_i|) \nabla z_i\right) \left[\frac{1}{\gamma_i} Z_0 \left(\frac{z_i}{\gamma_i}\right)\right] - W\hat{Z}_{0i}$$

$$= O\left(\frac{2}{|z_i - \gamma_i\nu_{\mathbb{R}^2_+}(0)|^2}\right) + O\left(\frac{2\gamma_k}{|z_k - \gamma_k\nu_{\mathbb{R}^2_+}(0)|^2}, \frac{G(\epsilon y, \xi_i)}{\gamma_i |\log \epsilon_i|}\right)$$

$$= O\left(\frac{2\gamma_k}{|z_k - \gamma_k\nu_{\mathbb{R}^2_+}(0)|^2}, \frac{\log t}{\gamma_i |\log \epsilon_i|}\right) + O\left(\epsilon t^{2\beta}\right). \quad (3.55)$$

Finally on $B_4$, since $|\hat{Z}_{0i}| = O\left(\frac{\log \delta}{\gamma_i |\log \epsilon_i|}\right)$ and $|\frac{\partial \eta_{2i}}{\partial y}| = O(\frac{\xi}{\gamma_i})$, by (2.33) we deduce

$$B(\tilde{Z}_{0i}) = B(\eta_{2i} \hat{Z}_{0i}) = \eta_{2i} B(\hat{Z}_{0i}) + \frac{\partial \eta_{2i}}{\partial y}\hat{Z}_{0i}$$

$$= \eta_{2i} \left(-\frac{\partial}{\partial z_i} + O(\epsilon |z_i|) \nabla z_i\right) \left[\frac{1}{\gamma_i} Z_0 \left(\frac{z_i}{\gamma_i}\right)\right] - W\eta_{2i} \hat{Z}_{0i} + \frac{\partial \eta_{2i}}{\partial y}\hat{Z}_{0i}$$

$$= O\left(\frac{\epsilon |\log \delta|}{\gamma_i |\log \epsilon_i|}\right). \quad (3.56)$$

Combining (3.5) and estimates (3.52)–(3.56), we arrive at

$$\|B(\tilde{Z}_{0i})\|_{*, \partial\Omega_0} = O\left(\frac{\log t}{\gamma_i |\log \epsilon_i|}\right).$$
Proof of Claim 3. Testing (3.32) against $\tilde{Z}_{0i}$ and using estimates (3.33)-(3.34), we find
\[
\sum_{k=1}^{m} d_k \left[ \int_{\Omega_t} \mathcal{L}(\tilde{Z}_{0k}) \tilde{Z}_{0i} + \int_{\partial \Omega_t} B(\tilde{Z}_{0k}) \tilde{Z}_{0i} \right] - \sum_{k=1}^{m} c_k \left[ \int_{\Omega_t} \chi_k Z_{1k} \mathcal{L}(\tilde{Z}_{0i}) + \int_{\partial \Omega_t} \chi_k Z_{1k} B(\tilde{Z}_{0i}) \right]
+ \int_{\Omega_t} [\phi \mathcal{L}(\tilde{Z}_{0i}) - f \tilde{Z}_{0i}] + \int_{\partial \Omega_t} [\phi B(\tilde{Z}_{0i}) - h \tilde{Z}_{0i}]
\leq C \left[ \| \mathcal{L}(\tilde{Z}_{0i}) \|_{*,\Omega_i} + \| B(\tilde{Z}_{0i}) \|_{*,\partial \Omega_i} \right] \left( \| \phi \|_{L^\infty(\Omega_i)} + \sum_{k=1}^{m} \frac{1}{\gamma_k} |c_k| \right)
+ \frac{C}{\gamma_i} \left( \| f \|_{*,\Omega_i} + \| h \|_{*,\partial \Omega_i} \right)
\leq C \left[ \| \mathcal{L}(\tilde{Z}_{0i}) \|_{*,\Omega_i} + \| B(\tilde{Z}_{0i}) \|_{*,\partial \Omega_i} \right] \left\{ \sum_{k=1}^{m} |d_k| \left[ \| \mathcal{L}(\tilde{Z}_{0k}) \|_{*,\Omega_i} + \| B(\tilde{Z}_{0k}) \|_{*,\partial \Omega_i} \right]
+ \| f \|_{*,\Omega_i} + \| h \|_{*,\partial \Omega_i} + \sum_{k=1}^{m} |c_k| \left[ \frac{1}{\gamma_k} + \| \mathcal{L}(\chi_k Z_{1k}) \|_{*,\Omega_i} + \| B(\chi_k Z_{1k}) \|_{*,\partial \Omega_i} \right] \right\}
+ \frac{C}{\gamma_i} \left( \| f \|_{*,\Omega_i} + \| h \|_{*,\partial \Omega_i} \right),
\]
where we have applied the following two inequalities:
\[
\int_{\Omega_t} \left( \frac{\gamma_i^{2+\sigma}}{|y - \xi_i| + \gamma_i} \right)^2 \leq C \quad \text{and} \quad \int_{\partial \Omega_t} \left( \frac{\gamma_i^{2+\sigma}}{|y - \xi_i| + \gamma_i} \right)^{1+\sigma} \leq C, \quad i = 1, \ldots, m.
\]
From estimates (3.27), (3.29) and (3.30), it follows that for any $i = 1, \ldots, m$,
\[
|d_i| \left[ \int_{\Omega_t} \mathcal{L}(\tilde{Z}_{0i}) \tilde{Z}_{0i} + \int_{\partial \Omega_t} B(\tilde{Z}_{0i}) \tilde{Z}_{0i} \right]
\leq C \left( \frac{\gamma_i^{2+\sigma}}{\gamma_i |y - \xi_i| + \gamma_i} \right) + C \sum_{k=1}^{m} \frac{|d_k| \log^2 t}{\gamma_i \gamma_k |\log \varepsilon_i| \log \varepsilon_k}
+ C \sum_{k \neq i}^{m} |d_k| \left[ \int_{\Omega_t} \mathcal{L}(\tilde{Z}_{0k}) \tilde{Z}_{0k} + \int_{\partial \Omega_t} B(\tilde{Z}_{0k}) \tilde{Z}_{0k} \right].
\]
To achieve the estimates of $d_k$ and $c_k$ in (3.31), we have the following claim.

Claim 4. If $\delta$ is sufficiently small, but $R$ is sufficiently large,
\[
\int_{\Omega_t} \mathcal{L}(\tilde{Z}_{0i}) \tilde{Z}_{0i} + \int_{\partial \Omega_t} B(\tilde{Z}_{0i}) \tilde{Z}_{0i} = -\frac{\pi}{\gamma_i^2} \left[ 1 + o(1) \right].
\]
Besides, if $k \neq i$,
\[
\int_{\Omega_t} \mathcal{L}(\tilde{Z}_{0k}) \tilde{Z}_{0k} + \int_{\partial \Omega_t} B(\tilde{Z}_{0k}) \tilde{Z}_{0k} = O \left( \frac{\log^2 t}{\gamma_i \gamma_k |\log \varepsilon_i| \log \varepsilon_k} \right).
\]
Indeed, once Claim 4 is proven, then substituting (3.58) and (3.59) into (3.57) we conclude
\[
\frac{|d_l|}{\gamma_i} \leq C|\log \varepsilon_i|(||f||_{**\Omega_l} + ||h||_{*,\partial\Omega_l}) + C \sum_{k=1}^m |d_k| \frac{|\log t|}{|\log \varepsilon_k|},
\]
and then, by (2.11),
\[
|d_l| \leq C\gamma_i|\log \varepsilon_i|(||h||_{*,\partial\Omega_l} + ||f||_{**\Omega_l}).
\]
Finally, using estimate (3.27) we derive that
\[
|e_i| \leq C\gamma_i|\log t||h||_{*,\partial\Omega_l} + ||f||_{**\Omega_l}).
\]

**Proof of Claim 4.** Let us first establish the validity of the a priori estimate (3.58). We decompose
\[
\int_{\Omega_l} \mathcal{L}(\tilde{Z}_{0i})\tilde{Z}_{0i} + \int_{\partial\Omega_l} B(\tilde{Z}_{0i})\tilde{Z}_{0i} = \sum_{l=1}^4 \left[ \int_{\Omega_l} \mathcal{L}(\tilde{Z}_{0i})\tilde{Z}_{0i} + \int_{B_l} B(\tilde{Z}_{0i})\tilde{Z}_{0i} \right] = \sum_{l=1}^4 (I_l + J_l).
\]
From (3.42), we get
\[
I_1 = \int_{\Omega_l} \mathcal{L}(Z_{0i})Z_{0i} = \int_{\{\gamma_i \leq \gamma_l \leq \Omega \cap \mathbb{R}^2_+} O \left( \frac{\varepsilon}{\gamma_i^2} \right) \frac{1}{\gamma_i} Z_0(\frac{z_i}{\gamma_i}) = O \left( \frac{\varepsilon}{\gamma_i} \right).
\]
From (3.18)-(3.20) and (3.46), we deduce
\[
I_3 = \int_{\Omega_3} \mathcal{L}(\tilde{Z}_{0i})\tilde{Z}_{0i} = \int_{\Omega_3} \mathcal{L}(\tilde{Z}_{0i}) \left\{ Z_{0i} - a_{0i} \left[ 2 \log \frac{|y - \xi|^2}{\gamma_i R} + O(\varepsilon|y - \xi|) \right] \right\} = \int_{\{\gamma_i \leq \gamma_l \leq \Omega \cap \mathbb{R}^2_+} O \left( \frac{\varepsilon}{\gamma_i} + \frac{\varepsilon^2}{|z_i|^2} \right) \times \left\{ \frac{1}{\gamma_i} + O \left( \frac{1}{|z_i|^2} + \frac{\log |z_i| - \log \gamma_i R + O(\varepsilon|z_i|)}{\gamma_i \log \varepsilon_i} \right) \right\} \ dz_i = O \left( \frac{\varepsilon |\log \varepsilon_i|}{\gamma_i} \right).
\]
From (3.47) and (3.49), we have
\[
I_4 = \int_{\Omega_4} \mathcal{L}(\tilde{Z}_{0i})\eta_i \tilde{Z}_{0i} = \int_{\{\frac{\gamma_i}{2} < |z_i| \leq \frac{\gamma_i}{2} \} \cap \mathbb{R}^2_+} O \left( \frac{\varepsilon^2 |\log \varepsilon_i|^2}{\gamma_i^2 |\log \varepsilon_i|^2} \right) \ dz_i = O \left( \frac{|\log \varepsilon_i|^2}{\gamma_i^2 |\log \varepsilon_i|^2} \right).
\]
As for \(I_2\), we can easily get
\[
I_2 = \int_{\Omega_2} (\tilde{Z}_{0i} - \tilde{Z}_{0i}) \Delta \eta_i + \int_{\Omega_2} 2\tilde{Z}_{0i} \nabla \eta_i \nabla (Z_{0i} - \tilde{Z}_{0i})
\]
Using (3.20), (3.35) and (3.43), we have that
\[ \int_{\Omega_2} Z_{0i} \eta_{i1} \nabla \nabla (Z_{0i} - \tilde{Z}_{0i}) = - \int_{\Omega_2} (Z_{0i} - \tilde{Z}_{0i})^2 |\nabla \eta_{i1}|^2 \]

From (3.53), we deduce
\[ \int_{\Omega_2} (Z_{0i} - \tilde{Z}_{0i}) \eta_{i1} \nabla \tilde{Z}_{0i} + O \left( \frac{\delta}{\gamma_i} \right) \]

On the other hand, from (3.52) we get
\[ \int_{\Omega_2} (Z_{0i} - \tilde{Z}_{0i}) \nabla \nabla (Z_{0i} - \tilde{Z}_{0i}) = I_{21} + I_{22} + I_{23} + O \left( \varepsilon \gamma_i^{-1} \right) . \]

Thus integrating by parts the first term and using estimates (3.41) and (3.44) for the last term, we find
\[ I_{2} = \int_{\Omega_2} Z_{0i} \eta_{i1} \nabla (Z_{0i} - \tilde{Z}_{0i}) - \int_{\Omega_2} (Z_{0i} - \tilde{Z}_{0i})^2 |\nabla \eta_{i1}|^2 \]

Using (3.20), (3.35) and (3.43), we have that |\nabla \eta_{i1}| = O \left( \frac{1}{\gamma_i} \right) and |\nabla \tilde{Z}_{0i}| = O \left( \frac{1}{\gamma_i^2 R^2} \right) in \Omega_2. Furthermore,
\[ I_{22} = O \left( \frac{1}{R^2 \gamma_i^2 |\log \varepsilon_i|^2} \right) and \quad I_{23} = O \left( \frac{1}{R^3 \gamma_i^2 |\log \varepsilon_i|^2} \right) . \]

Since \( \tilde{Z}_{0i} = Z_0 \left[ 1 + O \left( \frac{1}{R |\log \varepsilon_i|} \right) \right] \) in \( \Omega_2 \), by (3.20), (3.35), (3.37) and (3.43) we derive that
\[ I_{21} = \frac{a_{0i}}{\gamma_i^2} \int_{\{ z_i R < |z_i| \leq \gamma_i (R + 1) \cap \mathbb{R}^2 \}} \frac{2}{|z_i|} \eta_i \left( \frac{|z_i|}{\gamma_i} \right) Z_0 \left( \frac{z_i}{\gamma_i} \right) (1 + o(1)) dz_i \]

\[ = \frac{2 \pi a_{0i}}{\gamma_i} \int_{R}^{R + 1} \eta_i(r) \left[ 1 + O \left( \frac{1}{r} \right) \right] dr \]

\[ = - \frac{2 \pi a_{0i}}{\gamma_i^2 |\log \varepsilon_i|} \left[ 1 + O \left( \frac{1}{R} \right) \right] . \]

Hence
\[ \int_{\Omega_1} \mathcal{L}(\tilde{Z}_{0i}) \tilde{Z}_{0i} = - \frac{2 \pi a_{0i}}{\gamma_i^2 |\log \varepsilon_i|} \left[ 1 + O \left( \frac{1}{R} \right) \right] . \]

On the other hand, from (3.52) we get
\[ J_1 = \int_{B_1} B(Z_{0i}) Z_{0i} \]

\[ = \int_{\{|z_i| \leq \gamma_1 R \cap \partial \mathbb{R}^2_i \}} \frac{1}{\gamma_i} Z_0 \left( \frac{z_i}{\gamma_i} \right) \sum_{l=1}^{m} \frac{z_i^l \eta_i^l}{\gamma_i^2} \]

\[ = \sum_{l=1}^{m} \frac{z_i^l \eta_i^l}{\gamma_i^2} . \]

From (3.53), we deduce
\[ J_2 = \int_{B_2} B(\tilde{Z}_{0i}) \left[ Z_{0i} - (1 - \eta_{i1}) (Z_{0i} - \tilde{Z}_{0i}) \right] \]

\[ = \int_{\{ z_i R < |z_i| \leq \gamma_1 (R + 1) \cap \partial \mathbb{R}^2_i \}} O \left( \frac{1}{R^2 \gamma_i |\log \varepsilon_i|} \right) \]

\[ \times \left[ \frac{1}{\gamma_i} Z_0 \left( \frac{z_i}{\gamma_i} \right) + O \left( \frac{1}{R \gamma_i |\log \varepsilon_i|} \right) \right] dz_i \]
Combining estimates (3.69)-(3.72), we conclude that for $\eta_2 > 0$ large enough,

$$J_3 = \int_{B_3 \cup \tilde{B}_3} B(\tilde{Z}_{0i}) \tilde{Z}_{0i} + \sum_{k \neq i} \int_{B_{3,k}} B(\tilde{Z}_{0i}) \tilde{Z}_{0i}$$

$$= O\left(\frac{1}{R^3 \gamma^2 \log \varepsilon_i} \right) + O\left(\frac{\log^2 t}{\gamma^2 \log \varepsilon_i^2} \right).$$

(3.71)

From (3.56), we have

$$J_4 = \int_{B_4} B(\eta_2, \tilde{Z}_{0i}) \eta_2 \tilde{Z}_{0i}$$

$$= \int_{\{\frac{4}{\pi} < |z_i| \leq \frac{4}{\pi}\} \cap \partial B_4^*} O\left(\frac{\varepsilon \log t}{\gamma_i^2 \log \varepsilon_i^2} \right) dz_i$$

$$= O\left(\frac{|\log \delta|}{\gamma_i^2 \log \varepsilon_i^2} \right).$$

(3.72)

Combining estimates (3.69)-(3.72), we conclude that for $R$ and $t$ large enough,

$$\int_{\partial \Omega_t} B(\tilde{Z}_{0i}) \tilde{Z}_{0i} = O\left(\frac{1}{R^3 \gamma^2 \log \varepsilon_i} \right).$$

(3.73)

This, together with (3.68), implies that (3.58) holds.

Now, using the above estimates of $L(\tilde{Z}_{0i}), B(\tilde{Z}_{0i})$ and $\tilde{Z}_{0k}$ we can easily get

$$\int_{\Omega_1} L(\tilde{Z}_{0i}) \tilde{Z}_{0k} = O\left(\frac{\varepsilon \log t}{\gamma_i \gamma_k \log \varepsilon_i} \right),$$

$$\int_{\Omega_2} L(\tilde{Z}_{0i}) \tilde{Z}_{0k} = O\left(\frac{\log t}{\gamma_k \gamma_k \log \varepsilon_k} \log \varepsilon_i \right),$$

$$\int_{\Omega_3} L(\tilde{Z}_{0i}) \tilde{Z}_{0k} = O\left(\frac{\varepsilon \log \varepsilon_i}{\gamma_k} \right),$$

$$\int_{\Omega_4} L(\tilde{Z}_{0i}) \tilde{Z}_{0k} = O\left(\frac{|\log \delta|}{\gamma_k \gamma_k \log \varepsilon_k} \log \varepsilon_i \right),$$

and

$$\int_{B_1} B(\tilde{Z}_{0i}) \tilde{Z}_{0k} = \sum_{l=1}^{m} O\left(\frac{\varepsilon \log t}{\gamma_i \gamma_k \log \varepsilon_i} \right),$$

$$\int_{B_2} B(\tilde{Z}_{0i}) \tilde{Z}_{0k} = O\left(\frac{\log t}{\gamma_k \gamma_k \log \varepsilon_k \log \varepsilon_i} \right),$$

$$\int_{B_4} B(\tilde{Z}_{0i}) \tilde{Z}_{0k} = O\left(\frac{|\log \delta|}{\gamma_k \gamma_k \log \varepsilon_k \log \varepsilon_i} \right),$$

$$\int_{B_{3,1} \cup B_3} B(\tilde{Z}_{0i}) \tilde{Z}_{0k} = O\left(\frac{\log t}{\gamma_i \gamma_k \log \varepsilon_i \log \varepsilon_k} \right),$$

$$\int_{B_{3,1}} B(\tilde{Z}_{0i}) \tilde{Z}_{0k} = O\left(\frac{\log^2 t}{\gamma_i \gamma_k \log \varepsilon_k \log \varepsilon_i} \right) \quad \forall \ l \neq i.$$

Putting together these estimates, we conclude that (3.59) holds. 

\[\square\]
Step 4. Proof of Proposition 3.1. To obtain the solvability of problem (3.1) we first solve the following linear problem: given \( h \in L^\infty(\partial\Omega_t) \) and points \( \xi = (\xi_1, \ldots, \xi_m) \in \mathcal{O}_t \), we find a function \( \phi \in L^\infty(\Omega_t) \), and scalars \( f_1, \ldots, f_m \in \mathbb{R} \), such that

\[
\begin{align*}
- \Delta \phi + \varepsilon^2 \phi &= \sum_{i=1}^m f_i \chi_i \eta_i \quad \text{in} \quad \Omega_t, \\
\frac{\partial \phi}{\partial \nu} - W \phi &= h \quad \text{on} \quad \partial\Omega_t, \\
\int_{\Omega_t} \chi_i \eta_i \phi &= 0 \quad \forall \ i = 1, \ldots, m.
\end{align*}
\]

(3.74)

For any \( \phi \in L^\infty(\Omega_t) \), \( f_1, \ldots, f_m \in \mathbb{R} \) solution of (3.74) we claim that the bound

\[
\|\phi\|_{L^\infty(\Omega_t)} \leq C t \|h\|_{*,\partial\Omega_t}
\]

(3.75)

holds. Indeed, using Lemma 3.3 and the fact that \( \|\chi_i \eta_i\|_{*,\Omega_t} \leq C \gamma_i \), we get

\[
\|\phi\|_{L^\infty(\Omega_t)} \leq C t \left( \|h\|_{*,\partial\Omega_t} + \sum_{i=1}^m \gamma_i |f_i| \right),
\]

(3.76)

hence it suffices to prove that \( |f_i| \leq C \gamma_i^{-1} \|h\|_{*,\partial\Omega_t} \). Let us consider the cut-off function \( \eta_2 \), defined in (3.22). We test equation (3.74) against \( \eta_2 \chi_i \eta_i \eta_1 \) to find

\[
\sum_{k=1}^m f_k \int_{\Omega_t} \eta_2 \chi_i \eta_i \eta_1 \chi_i \eta_i \eta_1 = \int_{\Omega_t} \phi \mathcal{L}(\eta_2 \chi_i \eta_i \eta_1) + \int_{\partial\Omega_t} \left[ \phi \mathcal{B}(\eta_2 \chi_i \eta_i \eta_1) - h \eta_2 \chi_i \eta_i \right].
\]

(3.77)

By (3.35) and (3.37),

\[
\mathcal{L}(\eta_2 \chi_i \eta_i \eta_1) = \left( - \Delta z_i + O(\varepsilon|z_i|) \nabla z_i^2 + O(\varepsilon) \nabla z_i + \varepsilon^2 \right) \left[ \eta_2(\varepsilon z_i) \frac{1}{\gamma_i} Z_1 \left( \frac{z_i}{\gamma_i} \right) \right]
\]

\[= O \left( \frac{\varepsilon}{(\gamma_i + |z_i|)^2} \right) + O \left( \frac{\varepsilon^2}{\gamma_i + |z_i|} \right) + O(\varepsilon^3),
\]

and this implies

\[
\left| \int_{\Omega_t} \phi \mathcal{L}(\eta_2 \chi_i \eta_i \eta_1) \right| \leq C \varepsilon t \|\phi\|_{L^\infty(\Omega_t)}.
\]

(3.78)

On the other hand, by (3.36) we also compute

\[
\mathcal{B}(\eta_2 \chi_i \eta_i \eta_1) = \left( - \frac{\partial}{\partial z_{i,2}} - W + O(\varepsilon|z_i|) \nabla z_i \right) \left[ \eta_2(\varepsilon z_i) \frac{1}{\gamma_i} Z_1 \left( \frac{z_i}{\gamma_i} \right) \right]
\]

\[= \left( - \frac{2\gamma_i}{|z_i - \gamma_i \nu z_2(0)|^2} - W \right) \left[ \eta_2(\varepsilon z_i) \frac{1}{\gamma_i} Z_1 \left( \frac{z_i}{\gamma_i} \right) \right] + O \left( \frac{\varepsilon}{\gamma_i + |z_i|} \right)
\]

\[= \mathcal{B}_i + O \left( \frac{\varepsilon}{\gamma_i + |z_i|} \right).
\]

To estimate \( \mathcal{B}_i \), we decompose \( \text{supp}(\eta_2) \cap \partial\Omega_t \) to some pieces:

\[
\hat{B}_{1k} = \text{supp}(\eta_2) \cap (H^{-1}_k)^{-1} \left( \{|z_k| \leq \delta/(3\varepsilon t^3)\} \cap \partial\mathbb{R}^2 \right), \quad \forall \ k = 1, \ldots, m,
\]

\[
\hat{B}_2 = (\text{supp}(\eta_2) \cap \partial\Omega_t) \setminus \bigcup_{k=1}^m \hat{B}_{1k},
\]
where \( \text{supp}(\eta_{2i}) \cap \partial \Omega_t = (H^1 \cap \partial \Omega_t)^{-1} \{ |z_i| \leq \frac{\delta}{e^{-t}} \} \cap \partial \Omega_t^2 \). Observe that, by (2.33) and (3.77),

\[
|z| \geq C |y - \xi_t^i| \geq C (|\xi_t^i - \xi_t^k| - |y - \xi_t^k|) \geq C (|\xi_t^i - \xi_t^k| - 2|z_k|) \\
\geq C \left( |\xi_t^i - \xi_t^k| - \frac{2\delta}{3} \right) \geq C \left( \frac{2\delta}{3} \right),
\]

uniformly on \( \hat{B}_{1k}, k \neq i \). From expansion (2.30) of \( W \) we get, on \( \hat{B}_{1i} \),

\[
B_i = O \left( \frac{\varepsilon t \gamma_i}{\gamma_i + |z_i|^2} \right),
\]

and on \( \hat{B}_{1k}, k \neq i \), by (3.79),

\[
B_i = \left[ O \left( \frac{\gamma_i}{|z_i|^2} \right) + O \left( \frac{2\gamma_k}{|z_k - \gamma_k v g_k(0)|^2} \right) \right] O \left( \frac{1}{|z_i|} \right).
\]

Then

\[
\left| \int_{\partial \Omega_t} \phi B(\eta_{2i} Z_{1i}) \right| \leq C \varepsilon t \beta \| \phi \|_{L^\infty(\Omega_t)}.
\]

Besides, a direct computation shows that for \( t \) sufficiently large,

\[
\int_{\Omega_t} \eta_{2i} Z_{1i} \chi_k Z_{1k} = \begin{cases} \int_{\mathbb{R}^2_+} \chi(|z|) Z_{1i}^2(z) dz & \text{if } k = i, \\
O \left( \frac{1}{\gamma_i} \gamma_k \varepsilon t \beta \right) & \text{if } k \neq i. \end{cases}
\]

As a consequence, by (3.77)-(3.81) we readily deduce

\[
|f_i| \leq C \left( \varepsilon t \beta \| \phi \|_{L^\infty(\Omega_t)} + \frac{1}{\gamma_i} \| h \|_{L^\infty(\Omega_t)} + \sum_{k \neq i} \frac{1}{\gamma_i} \gamma_k \varepsilon t \beta |f_k| \right),
\]

and then

\[
|f_i| \leq C \left( \varepsilon t \beta \| \phi \|_{L^\infty(\Omega_t)} + \frac{1}{\gamma_i} \| h \|_{L^\infty(\Omega_t)} \right).
\]

This combined with (3.76) implies

\[
|f_i| \leq \frac{C}{\gamma_i} \| h \|_{L^\infty(\Omega_t)}.
\]

Let us consider the Hilbert space \( K_\xi = \{ \phi \in H^1(\Omega_t) : \int_{\Omega_t} \chi_i Z_{1i} \phi = 0, \forall i = 1, \ldots, m \} \) with the norm \( \| \phi \|_{K_\xi}^2 = \int_{\Omega_t} |\nabla \phi|^2 + \varepsilon^2 \phi^2 \). Equation (3.74) is equivalent to find \( \phi \in K_\xi \), such that

\[
\int_{\Omega_t} (\nabla \phi \nabla \psi + \varepsilon^2 \phi \psi) - \int_{\partial \Omega_t} W \phi \psi = \int_{\partial \Omega_t} h \psi \quad \forall \psi \in K_\xi.
\]

By Fredholm’s alternative this is equivalent to the uniqueness of solutions to this problem, which in turn follows from estimate (3.75).

To prove solvability of problem (3.1), let \( Y_t \in L^\infty(\Omega_t), c_1, \ldots, c_m \in \mathbb{R} \) be the solution of equation (3.74) with \( h = \chi_i Z_{1i} \), namely,
\[
- \Delta Y_i + \varepsilon^2 Y_i = \sum_{k=1}^{m} c_{ik} \chi_k Z_{1k} \quad \text{in} \quad \Omega_t,
\]
\[
\frac{\partial Y_i}{\partial \nu} - W Y_i = \chi_i Z_{1i} \quad \text{on} \quad \partial \Omega_t,
\]
\[
\int_{\Omega_t} \chi_k Z_{1k} Y_i = 0 \quad \forall \ k = 1, \ldots, m.
\]
From the above argument there exist a unique solution \( Y_i \in L^\infty(\Omega_t) \), and \( c_{ik} \in \mathbb{R} \), \( k = 1, \ldots, m \), to this equation such that
\[
\| Y_i \|_{L^\infty(\Omega_t)} \leq C t \quad \text{and} \quad |c_{ik}| \leq \frac{C}{\gamma_k}.
\]
Let us first claim that there exist a constant \( A \), independent of \( t \), such that
\[
c_{ik} = \frac{1}{\gamma_k} A \delta_{ik} + O(\varepsilon t^{\beta+1}),
\]
where \( \delta_{ik} \) denotes Kronecker’s symbol. Indeed, testing equation (3.83) against \( \eta_{2k} Z_{1k} \) we obtain
\[
c_{ik} \int_{\Omega_t} \chi_k Z_{1k}^2 + \int_{\partial \Omega_t} \chi_i Z_{1i} \eta_{2k} Z_{1k} + \sum_{l \neq k} c_{il} \int_{\Omega_t} \eta_{2k} Z_{1k} \chi_l Z_{1l}
\]
\[
= \int_{\Omega_t} Y_i \mathcal{L}(\eta_{2k} Z_{1k}) + \int_{\partial \Omega_t} Y_i \mathcal{B}(\eta_{2k} Z_{1k}),
\]
and then, by (3.78), (3.80), (3.81) and (3.84),
\[
c_{ik} \int_{\mathbb{R}^*_+} \chi Z_{1k}^2 + \delta_{ik} \frac{1}{\gamma_k} \int_{\partial \mathbb{R}^*_+} \chi Z_{1k}^2 + (1 - \delta_{ik}) O\left(\frac{1}{\gamma_k} \varepsilon t^\beta\right) + \sum_{l \neq k} O\left(\frac{1}{\gamma_k} \varepsilon t^\beta\right)
\]
\[
= O(\varepsilon t^{\beta+1}),
\]
which concludes the validity of expansion (3.85). Hence the matrix \( D \) with entries \( \gamma_k c_{ik} \) is invertible for sufficiently large \( t \) and \( \|D^{-1}\| \leq C \) uniformly on \( t \).

Now, given \( h \in L^\infty(\partial \Omega_t) \) and points \( \xi = (\xi_1, \ldots, \xi_m) \in \Omega_t \), we have a function \( \phi_1 \in L^\infty(\Omega_t) \) and scalars \( f_1, \ldots, f_m \in \mathbb{R} \) as the solution of equation (3.74), and further define
\[
\phi = \phi_1 + \sum_{i=1}^{m} c_i Y_i,
\]
where \( c_i \) satisfies \( \sum_{i=1}^{m} c_i c_{ik} = -f_k \) for any \( k = 1, \ldots, m \). Then \( \phi \) satisfies equation (3.1) and we deduce
\[
\| \phi \|_{L^\infty(\partial \Omega_t)} \leq \| \phi_1 \|_{L^\infty(\Omega_t)} + C t \sum_{i=1}^{m} |c_i| \leq C t \| h \|_{\partial \Omega_t} + C t \sum_{k=1}^{m} \gamma_k |f_k| \leq C t \| h \|_{\partial \Omega_t},
\]
in view of estimates (3.75) and (3.82).

**Remark 3.1.** A slight modification of the above proof also shows that for any \( h \in L^\infty(\partial \Omega_t) \) and \( f \in L^\infty(\Omega_t) \) the equation
has a unique solution $\phi, c_1, \ldots, c_m$. Moreover, the following estimates hold:

$$
\|\phi\|_{L^\infty(\Omega_t)} \leq C t \left( \|h\|_{*_{\partial\Omega_t}} + \|f\|_{*_{*_{\Omega_t}}} \right),
$$

$$
|c_i| \leq C \left( \|h\|_{*_{\partial\Omega_t}} + \|f\|_{*_{*_{\Omega_t}}} \right), \quad i = 1, \ldots, m.
$$

The result of Proposition 3.1 implies that the unique solution $\phi = T(h)$ of (3.1) defines a continuous linear map from the Banach space $\mathcal{C}_*$ of all functions $h$ in $L^\infty$ for which $\|h\|_{*_{\partial\Omega_t}} < \infty$, into $L^\infty$.

**Lemma 3.4.** The operator $T$ is differentiable with respect to the variables $\xi = (\xi_1, \ldots, \xi_m)$ in $O_t$. Moreover, one has the estimate

$$
\|\partial_{\xi_k} T(h)\|_{L^\infty(\Omega_t)} \leq Ct^2 \|h\|_{*_{\partial\Omega_t}}, \quad \text{for any } k = 1, \ldots, m.
$$

**Proof.** Differentiating equation (3.1) with respect to $\xi_k$, formally $Z = \partial_{\xi_k} \phi$ should satisfy

$$
\begin{cases}
\mathcal{L}(Z) = 0 & \text{in } \Omega_t, \\
\mathcal{B}(Z) = \phi \partial_{\xi_k} W + \sum_{i=1}^{m} \left[ c_i \partial_{\xi_k} (\chi_i Z_{1i}) + \bar{c}_i \chi_i Z_{1i} \right] & \text{on } \partial\Omega_t,
\end{cases}
$$

with (still formally) $\bar{c}_i = \partial_{\xi_k} c_i$, and the orthogonality conditions now become

$$
\int_{\Omega_t} \chi_i Z_{1i} Z = - \int_{\Omega_t} \phi \partial_{\xi_k} (\chi_i Z_{1i}), \quad i = 1, \ldots, m.
$$

We consider the constants $b_i$ defined as

$$
b_i = \int_{\Omega_t} \chi_i^2 |Z_{1i}|^2 = \int_{\Omega_t} \phi \partial_{\xi_k} (\chi_i Z_{1i}),
$$

and the functions

$$
a = \sum_{i=1}^{m} b_i \mathcal{L}(\chi_i Z_{1i}), \quad b = \phi \partial_{\xi_k} W + \sum_{i=1}^{m} \left[ c_i \partial_{\xi_k} (\chi_i Z_{1i}) + b_i \mathcal{B}(\chi_i Z_{1i}) \right].
$$

By (2.30), (3.3) and (3.4), a straightforward but tedious computation shows that $\|\partial_{\xi_k} W\|_{*_{\partial\Omega_t}} = O(1)$,

$$
|b_i| = \begin{cases} O \left( |\partial_{\xi_k} \gamma_i| \right) \|\phi\|_{L^\infty(\Omega_t)} & \text{if } i \neq k, \\
O(1) \|\phi\|_{L^\infty(\Omega_t)} & \text{if } i = k,
\end{cases}
$$

and

$$
\|\partial_{\xi_k} (\chi_i Z_{1i})\|_{*_{\partial\Omega_t}} = \begin{cases} O \left( |\partial_{\xi_k} \gamma_i| / \gamma_i \right) & \text{if } i \neq k, \\
O(1/\gamma_k) & \text{if } i = k.
\end{cases}
$$
Furthermore, using (2.18), (2.28), (3.6), (3.29), (3.30) and the fact that \( \frac{1}{\gamma_i} \leq C \) uniformly on \( t \), we find
\[
\|a\|_{\ast, \Omega_t} \leq Ct\|h\|_{\ast, \partial \Omega_t} \quad \text{and} \quad \|b\|_{\ast, \partial \Omega_t} \leq Ct\|h\|_{\ast, \partial \Omega_t}.
\]
(3.89)
Define
\[
\tilde{Z} = Z + \sum_{i=1}^{m} b_1 \chi_i Z_{i1}.
\]
We then have
\[
\begin{aligned}
\mathcal{L}(\tilde{Z}) &= a \quad \text{in} \quad \Omega_t, \\
\mathcal{B}(\tilde{Z}) &= b + \sum_{i=1}^{m} \tilde{c}_i \chi_i Z_{i1} \quad \text{on} \quad \partial \Omega_t, \\
\int_{\Omega_t} \chi_i Z_{i1} \tilde{Z} &= 0 \quad \forall \; i = 1, \ldots, m.
\end{aligned}
\]
By the remark above it follows that this equation has a unique solution \( \tilde{Z}, \tilde{c}_1, \ldots, \tilde{c}_m \), and thus \( \partial_t T(h) = \tilde{Z} - \sum_{i=1}^{m} b_1 \chi_i Z_{i1} \) is well defined. From (3.87) and (3.89) we get
\[
\|\tilde{Z}\|_{L^\infty(\Omega_t)} \leq Ct\|a\|_{\ast, \Omega_t} + \|b\|_{\ast, \partial \Omega_t} \leq Ct^2\|h\|_{\ast, \partial \Omega_t},
\]
This, together with the fact that \( \|\chi_i Z_{i1}\|_{L^\infty(\Omega_t)} = O(1/\gamma_i) \), implies that (3.88) holds.

4. The nonlinear problem. In what follows we shall consider first the intermediate problem: for any points \( \xi = (\xi_1, \ldots, \xi_m) \in \mathcal{O}_t \), we find a function \( \phi \), and scalars \( c_1, \ldots, c_m \), such that
\[
\begin{aligned}
- \Delta \phi + \varepsilon^2 \phi &= 0 \quad \text{in} \quad \Omega_t, \\
\frac{\partial \phi}{\partial \nu} - W \phi &= R + N(\phi) + \sum_{i=1}^{m} c_i \chi_i Z_{i1} \quad \text{on} \quad \partial \Omega_t, \\
\int_{\Omega_t} \chi_i Z_{i1} \phi &= 0 \quad \forall \; i = 1, \ldots, m,
\end{aligned}
\]
(4.1)
where \( R, N(\phi) \) are given by (2.25) and (2.36), respectively.

**Proposition 4.1.** Let \( m \) be a positive integer. Then there exist constants \( t_m > 1 \) and \( C > 0 \) such that for any \( t > t_m \) and any points \( \xi = (\xi_1, \ldots, \xi_m) \in \mathcal{O}_t \), problem (4.1) admits a unique solution \( \phi \in L^\infty(\Omega_t) \), and scalars \( c_1, \ldots, c_m \in \mathbb{R} \), such that
\[
\|\phi\|_{L^\infty(\Omega_t)} \leq Ct^{\beta + 1} \max_i \{\varepsilon \gamma_i\}.
\]
(4.2)
Furthermore, the map \( \xi' \to \phi(\xi') \in C(\Omega_t) \) is \( C^1 \), precisely for \( k = 1, \ldots, m, \)
\[
\|\partial_{\xi_k} \phi\|_{L^\infty(\Omega_t)} \leq Ct^{\beta + 2} \max_i \{\varepsilon \gamma_i\},
\]
(4.3)
where \( \xi' := (\xi'_1, \ldots, \xi'_m) = (\frac{1}{\varepsilon} \xi_1, \ldots, \frac{1}{\varepsilon} \xi_m) \).

**Proof.** In terms of the operator \( T \) defined in the previous section, problem (4.1) becomes
\[
\phi = A(\phi) := T(N(\phi) + R).
\]
(4.4)
For a given number \( \kappa > 0 \), let us consider the region
\[
\mathcal{F}_\kappa := \left\{ \phi \in C(\Omega_t) : \|\phi\|_{L^\infty(\Omega_t)} \leq \kappa t^{\beta + 1} \max_i \{\varepsilon \gamma_i\} \right\}.
\]
From Proposition 3.1, we get

$$\|A(\phi)\|_{L^\infty(\Omega)} \leq Ct(\|N(\phi)\|_{*,\partial\Omega_t} + \|R\|_{*,\partial\Omega_t}).$$

Estimates (2.31) and (2.34) imply that $$\|R\|_{*,\partial\Omega_t} \leq C t^\beta \max_i \{\varepsilon_{\gamma_i}\}$$. Also, from (2.24), (2.30), (2.33), (2.36) and Lagrange’s theorem we have that for $$\phi, \phi_1, \phi_2 \in \mathcal{F}_\kappa$$,

$$\|N(\phi)\|_{*,\partial\Omega_t} \leq \|W\|_{*,\partial\Omega_t} \|\phi\|_{L^\infty(\Omega_t)}^2 \leq C \|\phi\|_{L^\infty(\Omega_t)}^2.$$

Hence we get

$$\|N(\phi) - N(\phi_2)\|_{*,\partial\Omega_t} \leq C \max_{i=1,2} \|\phi_i\|_{L^\infty(\Omega_t)} \|\phi_1 - \phi_2\|_{L^\infty(\Omega_t)}.$$

Hence we get

$$\|A(\phi)\|_{L^\infty(\Omega)} \leq Ct^{\beta + 1} \max_i \{\varepsilon_{\gamma_i}\} \left(\kappa^2 t^{\beta + 2} \max_i \{\varepsilon_{\gamma_i}\} + 1\right),$$

and

$$\|A(\phi_1) - A(\phi_2)\|_{L^\infty(\Omega)} \leq Ct\|N(\phi_1) - N(\phi_2)\|_{*,\partial\Omega_t} \leq C t^{\beta + 2} \max_i \{\varepsilon_{\gamma_i}\} \|\phi_1 - \phi_2\|_{L^\infty(\Omega_t)},$$

where $$C$$ is independent of $$\kappa$$. Then by (2.23), (2.11), (2.18) and (2.28), it follows that for all $$t$$ sufficiently large $$A$$ is a contraction on $$\mathcal{F}_\kappa$$, and therefore a unique fixed point of $$A$$ exists in this region.

Let us now analyze the differentiability of $$\phi$$. Since $$R$$ depends continuously (in the *-norm) on the $$m$$-tuple $$\xi' = (\xi'_1, \ldots, \xi'_m)$$, the fixed point characterization obviously yields so for the map $$\xi' \mapsto \phi$$. Assume for instance that the partial derivative $$\partial_{\xi'_k} \phi$$ exists. Since $$\phi = T(N(\phi) + R)$$, formally that

$$\partial_{\xi'_k} \phi = \left(\partial_{\xi'_k} T\right) (N(\phi) + R) + T \left(\partial_{\xi'_k} N(\phi) + \partial_{\xi'_k} R\right).$$

From Lemma 3.4 we deduce that

$$\left\|\left(\partial_{\xi'_k} T\right) (N(\phi) + R)\right\|_{L^\infty(\Omega)} \leq Ct^2 \left(\|N(\phi)\|_{*,\partial\Omega_t} + \|R\|_{*,\partial\Omega_t}\right) \leq Ct^{\beta + 2} \max_i \{\varepsilon_{\gamma_i}\}.$$  

Also observe that we have

$$\partial_{\xi'_k} N(\phi) = \partial_{\xi'_k} W(e^\phi - \phi - 1) + W(e^\phi - 1) \partial_{\xi'_k} \phi,$$

so that, using the fact that $$\|\partial_{\xi'_k} W\|_{*,\partial\Omega_t} = O(1)$$,

$$\|\partial_{\xi'_k} N(\phi)\|_{*,\partial\Omega_t} \leq C \|\phi\|_{L^\infty(\Omega_t)} \left(\|\phi\|_{L^\infty(\Omega_t)} + \|\partial_{\xi'_k} \phi\|_{L^\infty(\Omega_t)}\right).$$

Since $$\|\partial_{\xi'_k} R\|_{*,\partial\Omega_t} = O \left(\max_i \{\varepsilon_{\gamma_i}\}\right)$$, and by Proposition 3.1, we conclude from the above computation that

$$\|\partial_{\xi'_k} \phi\|_{L^\infty(\Omega_t)} \leq C t^{\beta + 2} \max_i \{\varepsilon_{\gamma_i}\} \quad \text{for all} \quad k = 1, \ldots, m.$$  

The above computation can be made rigorous by using the implicit function theorem and the fixed point representation (4.4) which guarantees $$C^1$$ regularity of $$\xi'$$. \(\square\)
5. Variational reduction. After problem (4.1) has been solved, we find a solution of problem (2.35) and hence to the original problem (1.4) if \( \xi' \) is such that
\[
c_i(\xi') = 0 \quad \text{for all } i = 1, \ldots, m.
\] (5.1)

This problem is indeed variational: it is equivalent to finding critical points of a function of \( \xi = \varepsilon \xi' \). To realize it we consider the energy function \( J_t \) associated to problem (1.4), namely
\[
J_t(u) = \frac{1}{2} \int_{\Omega} (|\nabla u|^2 + u^2) - \int_{\partial \Omega} k(x)e^{-t\phi_1}e^u \quad \text{for } u \in H^1(\Omega).
\] (5.2)

We define
\[
F_t(\xi) = J_t(U(\xi) + \tilde{\phi}(\xi)) \quad \forall \xi \in \mathcal{O}_t,
\] (5.3)

where \( U \) is the function defined in (2.12) and \( \tilde{\phi}(\xi)(x) = \phi(\tilde{\xi},\tilde{\xi}) \), \( x \in \Omega \), with \( \phi = \phi_t \) the unique solution to problem (4.1) given by Proposition 4.1. Critical points of \( F_t \) correspond to solutions of (5.1) for large \( t \), as the following result states.

**Proposition 5.1.** \( F_t : \mathcal{O}_t \to \mathbb{R} \) is of class \( C^1 \). Moreover, for all \( t \) sufficiently large, if \( D_\xi F_t(\xi) = 0 \), then \( \xi \) satisfies (5.1).

**Proof.** A direct consequence of the results obtained in Proposition 4.1 and the definition of function \( U(\xi) \) is the fact that the map \( \xi \mapsto F_t(\xi) \) is of class \( C^1 \). Define
\[
I_t(\omega) = \frac{1}{2} \int_{\mathcal{O}_t} (|\nabla \omega|^2 + \varepsilon^2 \omega^2) - \int_{\partial \mathcal{O}_t} k(\varepsilon y)e^{-t(\phi_1(\varepsilon y) - 1)}e^\omega.
\] (5.4)

Then, making a change of variables, we have
\[
F_t(\xi) = J_t(U(\xi) + \tilde{\phi}(\xi)) = I_t(V(\xi') + \phi_t).
\] (5.5)

Therefore
\[
\partial_{\xi_k} F_t(\xi) = \frac{1}{\varepsilon} D I_t(V(\xi') + \phi_t) \left[ \partial_{\xi_k} V(\xi') + \partial_{\xi_k} \phi_t \right]
\]
\[
= \frac{1}{\varepsilon} \sum_{i=1}^{m} c_i \int_{\partial \mathcal{O}_t} \chi_i Z_{1i} \left[ \partial_{\xi_k} V(\xi') + \partial_{\xi_k} \phi_t \right].
\]

Let us assume that \( D_\xi F_t(\xi) = 0 \). Then for all \( k = 1, \ldots, m \),
\[
\sum_{i=1}^{m} c_i \int_{\partial \mathcal{O}_t} \chi_i Z_{1i} \left[ \partial_{\xi_k} V(\xi') + \partial_{\xi_k} \phi_t \right] = 0.
\]

Since \( \|\partial_{\xi_1} \phi_t\|_{L^\infty(\Omega_t)} = O\left( e^{-\frac{t}{2}} \right) \) and \( \partial_{\xi_k} V(\xi') = -Z_{1k} + o(1) \) where \( o(1) \) is uniformly small as \( t \to +\infty \), it follows that
\[
\sum_{i=1}^{m} c_i \int_{\partial \mathcal{O}_t} \chi_i Z_{1i} (-Z_{1k} + o(1)) = 0 \quad \text{for all } k = 1, \ldots, m.
\] (5.6)

Note that
\[
\int_{\partial \mathcal{O}_t} \chi_i Z_{1i} Z_{1k} = \begin{cases}
\frac{1}{\gamma_k} \int_{\partial \mathcal{O}_t} \chi(|z|)Z_1^2(z)dz & \text{if } i = k, \\
O(\varepsilon t^\beta) & \text{if } i \neq k.
\end{cases}
\]

This implies that system (5.6) is diagonal dominant, and thus \( c_i = 0 \) for all \( i = 1, \ldots, m \). \(\square\)
We also have the validity of the following lemma.

**Proposition 5.2.** The following expansion holds

\[ F_t(\xi) = J_t(U(\xi)) + \theta_t(\xi), \]  

(5.7)

where \(|\theta_t| + |\nabla \theta_t| = o(1)\), uniformly on points \(\xi = (\xi_1, \ldots, \xi_m) \in \Omega_t\) as \(t \to +\infty\).

**Proof.** First, from (5.4) - (5.5), we have

\[ F_t(\xi) - J_t(U(\xi)) = I_t(V(\xi') + \phi_{\xi'}) - I_t(V(\xi')). \]

Taking into account \(DI_t(V + \phi_{\xi'})[\phi_{\xi'}] = 0\), a Taylor expansion and an integration by parts give

\[ F_t(\xi) - J_t(U(\xi)) = \int_0^1 D^2 I_t(V + \kappa \phi_{\xi'})[\phi_{\xi'}]^2(1 - \kappa) d\kappa \]

(5.8)

Since \(\|\phi_{\xi'}\|_{L^\infty(\Omega_t)} = O\left(t^{\beta+1} \max_i \{\varepsilon_\gamma_i\}\right)\), \(\|N(\phi_{\xi'})\|_{*, \partial \Omega_t} = O\left(t^{2\beta+2} \max_i \{\varepsilon_\gamma_i^2\}\right)\), \(\|R\|_{*, \partial \Omega_t} = O\left(t^{\beta} \max_i \{\varepsilon_\gamma_i\}\right)\) and \(\|W\|_{*, \partial \Omega_t} = O(1)\), by \((2.8), (2.11), (2.18)\) and \((2.28)\), we get

\[ \theta_t(\xi) = F_t(\xi) - J_t(U(\xi)) = O\left(t^{2\beta+1} \max_i \{\varepsilon_\gamma_i^2\}\right) = o(1). \]  

(5.9)

Let us differentiate the representation (5.8) with respect to \(\xi_k^i\), \(k = 1, \ldots, m\),

\[ \partial_{\xi_k^i\xi_k^i} [F_t(\xi) - J_t(U(\xi))] \]

\[ = \int_0^1 \left\{ \int_{\partial \Omega_t} \partial_{\xi_k^i} [(N(\phi_{\xi'}) + R)\phi_{\xi'}] + \partial_{\xi_k^i} W [1 - e^{\kappa \phi_{\xi'}}]\phi_{\xi'}^2 \right\} (1 - \kappa) d\kappa, \]

so we get

\[ \partial_{\xi_k^i\xi_k^i} \theta_t(\xi) = \frac{1}{\varepsilon} \partial_{\xi_k^i\xi_k^i} \theta_t(\xi) = \frac{1}{\varepsilon} \partial_{\xi_k^i\xi_k^i} [F_t(\xi) - J_t(U(\xi))] \]

\[ = O\left(t^{2\beta+2} \max_i \{\varepsilon_\gamma_i^2\}\right) = o(1), \]  

(5.10)

in view of \(\|\partial_{\xi_k^i\xi_k^i} \phi_{\xi'}\|_{L^\infty(\Omega_t)} = O\left(t^{\beta+2} \max_i \{\varepsilon_\gamma_i\}\right)\), \(\|\partial_{\xi_k^i\xi_k^i} R\|_{*, \partial \Omega_t} = O\left(t^{\beta} \max_i \{\varepsilon_\gamma_i\}\right)\), \(\|\partial_{\xi_k^i\xi_k^i} W\|_{*, \partial \Omega_t} = O(1)\) and \(\|\partial_{\xi_k^i\xi_k^i} N(\phi_{\xi'})\|_{*, \partial \Omega_t} = O\left(t^{2\beta+3} \max_i \{\varepsilon_\gamma_i^2\}\right)\). The continuity in \(\xi\) of all these expressions is inherited from that of \(\phi\) and its derivatives in \(\xi\) in the \(L^\infty\) norm. \(\square\)

6. **Expansion of energy and proof of Theorem 1.1.** In this section we write an asymptotic expansion of the energy functional \(J_t\) evaluated at \(U(\xi)\) and further give the proof of Theorem 1.1.

We have

**Proposition 6.1.** Let \(m\) be a positive integer. With the choice (2.17) for the parameters \(\mu_i\), there exists \(t_m > 1\) such that for any \(t > t_m\) and any points
\[ \xi = (\xi_1, \ldots, \xi_m) \in \mathcal{O}, \text{ the following expansion holds uniformly:} \]
\[ J_\Omega(U(\xi)) = 2\pi \sum_{i \neq j}^m \log |\xi_i - \xi_j| + 2\pi t \sum_{i=1}^m \phi_1(\xi_i) + O(1). \quad (6.1) \]

**Proof.** Observe that by (2.12) and (2.14),
\[ \frac{1}{2} \int_{\Omega} (|\nabla U|^2 + U^2) = \frac{1}{2} \int_{\partial \Omega} \frac{\partial U}{\partial \nu} U = \frac{1}{2} \sum_{i,j=1}^m \int_{\partial \Omega} \frac{\partial U_i}{\partial \nu} U_j \]
\[ = \frac{1}{2} \sum_{i,j=1}^m \int_{\partial \Omega} \varepsilon_j k(\xi_i) e^{\mu_i}(u_j + H_j). \]

By (2.10) and (2.15) we obtain
\[ \int_{\partial \Omega} \varepsilon_i k(\xi_i) e^{\mu_i}(u_j + H_j) \]
\[ = \int_{\partial \Omega} \frac{2\varepsilon_i \mu_i}{|x - \xi_i - \varepsilon_i \mu_i \nu(\xi_i)|^2} \left[ \log \frac{1}{|x - \xi_j - \varepsilon_j \mu_j \nu(\xi_j)|^2} + H(x, \xi_j) + O(\varepsilon_j^\alpha \mu_j^\alpha) \right]. \]

Making the change of variables \( \varepsilon_i \mu_i z = A_i(x - \xi_i) \) gives
\[ \int_{\partial \Omega} \varepsilon_i k(\xi_i) e^{\mu_i}(u_j + H_j) \]
\[ = \int_{\partial \Omega \setminus \varepsilon_i \mu_i} \left[ \frac{2}{|z - \nu z^2(0)|^2} \log \frac{1}{|z - \nu z^2(0)|^2} \right] \left[ H(\xi_i + \varepsilon_i \mu_i A_i^{-1} z, \xi_j) + O(\varepsilon_i^\alpha \mu_i^\alpha) \right] dz, \]
where \( \partial \Omega \setminus \varepsilon_i \mu_i = (\varepsilon_i \mu_i)^{-1} A_i(\partial \Omega \setminus \{\xi_i\}) \). Note that
\[ \int_{\partial \Omega \setminus \varepsilon_i \mu_i} \frac{2}{|z - \nu z^2(0)|^2} = 2\pi + O(\varepsilon_i \mu_i), \]
\[ \int_{\partial \Omega \setminus \varepsilon_i \mu_i} \log \frac{1}{|z - \nu z^2(0)|^2} = -4\pi \log 2 + O(\varepsilon_i^\alpha \mu_i^\alpha), \]
and
\[ \int_{\partial \Omega \setminus \varepsilon_i \mu_i} \frac{2}{|z - \nu z^2(0)|^2} \left[ H(\xi_i + \varepsilon_i \mu_i A_i^{-1} z, \xi_j) - H(\xi_i, \xi_j) \right] = O(\varepsilon_i^\alpha \mu_i^\alpha). \]

Then
\[ \int_{\partial \Omega} \varepsilon_i k(\xi_i) e^{\mu_i}(u_j + H_j) = \begin{cases} 2\pi [H(\xi_i, \xi_i) - 2 \log(2\varepsilon_i \mu_i)] + O(\varepsilon_i^\alpha \mu_i^\alpha) & \forall \ i = j, \\ 2\pi G(\xi_i, \xi_j) + O(\varepsilon_i^\alpha \mu_i^\alpha + \varepsilon_j^\alpha \mu_j^\alpha) & \forall \ i \neq j. \end{cases} \]

Hence
\[ \frac{1}{2} \int_{\Omega} (|\nabla U|^2 + U^2) = \pi \sum_{i=1}^m \left[ H(\xi_i, \xi_i) - 2 \log(2\varepsilon_i \mu_i) \right] + \pi \sum_{i \neq j} G(\xi_i, \xi_j) \]
\[ + \sum_{i=1}^m O(\varepsilon_i^\alpha \mu_i^\alpha). \quad (6.2) \]
On the other hand, changing variables \( x = \varepsilon y = e^{-t} y \) and using the definition of \( W \) in (2.24), we have
\[
\int_{\partial \Omega} k(x) e^{-t \phi_1(x)} e^{U(x)} = \int_{\partial \Omega_t} k(\varepsilon y) e^{-t \left[ \phi_1(\varepsilon y) - 1 \right]} e^{U(\varepsilon y) - 2 t} \, dy
= \left( \sum_{i=1}^{m} \int_{\partial \Omega_t \cap B_{\gamma_i/(2\varepsilon^\beta)}(\xi_i')} + \int_{\partial \Omega_t \setminus \bigcup_{i=1}^{m} B_{\gamma_i/(2\varepsilon^\beta)}(\xi_i')} \right) W \, dy.
\]
By (2.30) and (2.33), it follows that
\[
\int_{\partial \Omega_t \setminus \bigcup_{j=1}^{m} B_{\gamma_j/(2\varepsilon^\beta)}(\xi_j')} W \, dy = \int_{\partial \Omega_t \setminus \bigcup_{j=1}^{m} B_{\gamma_j/(2\varepsilon^\beta)}(\xi_j')} O \left( \varepsilon t^{2\alpha_j} e^{-t \phi_1(\varepsilon y)} \right) \, dy
= O \left( \varepsilon t^{2\alpha_j} e^{-t \min_{y \in \partial \Omega_t} \phi_1(\varepsilon y)} \right),
\]
and
\[
\int_{\partial \Omega_t \cap B_{\gamma_i/(2\varepsilon^\beta)}(\xi_i')} W \, dy = \int_{\partial \Omega_t \cap B_{\gamma_i/(2\varepsilon^\beta)}(\xi_i')} \frac{2\gamma_i}{|y - \xi_i'|^2} e^{2 \phi_1(\varepsilon y)} \times \left\{ 1 + O(\varepsilon t^\beta |y - \xi_i'|) + \sum_{j=1}^{m} O(\varepsilon_j^\alpha_j \mu_j^\alpha_j) \right\} \, dy
= 2\pi + O(\varepsilon t^\beta) + \sum_{j=1}^{m} O(\varepsilon_j^\alpha_j \mu_j^\alpha_j).
\]
Then
\[
\int_{\partial \Omega} k(x) e^{-t \phi_1(x)} e^{U(x)} = 2\pi m + O(\varepsilon t^\beta) + \sum_{j=1}^{m} O(\varepsilon_j^\alpha_j \mu_j^\alpha_j)
+ O \left( t^{2\alpha_j} e^{-t \min_{y \in \partial \Omega_t} \phi_1(\varepsilon y)} \right).
\]
As a consequence, from (5.2) and (6.3) we can write an asymptotic expansion of the energy (5.2) evaluated at \( U(\xi) \), namely
\[
J_t(U(\xi)) = \pi \sum_{i=1}^{m} \left[ H(\xi_i, \xi_i) - 2 \log(2\varepsilon_i \mu_i) \right] + \pi \sum_{i\neq j} G(\xi_i, \xi_j) + O(1).
\]
This, together with the definition (2.11) of \( \varepsilon_i \) and the choice (2.17) of \( \mu_i \), implies that (6.1) holds.

We now have all the ingredients needed to give the proof of Theorem 1.1.

**Proof of Theorem 1.1.** According to Proposition 5.1, \( U(\xi') + \hat{\phi}(\xi') \) is a solution to problem (1.4) if \( \xi' \in \Omega_t \) is a critical point of \( F_t \) defined in (5.3). Notice that \( \| \hat{\phi}(\xi') \|_{\infty} \to 0 \) as predicted in Proposition 4.1. It thus suffices to establish that for all sufficiently large \( t \), the following maximization problem
\[
\max_{(\xi_1, \ldots, \xi_m) \in \Omega_t} F_t(\xi_1, \ldots, \xi_m)
\]
has a solution in the interior of \( \Omega_t \).
Let \( \xi^t = (\xi^t_1, \ldots, \xi^t_m) \in \overline{O}_t \) be the maximizer of \( F_t \). We need to prove that \( \xi^t \) belongs to the interior of \( O_t \). First, we obtain a lower bound for \( F_t \) over \( \partial O_t \). Let us fix a point \( \bar{x} \in S \). For a sufficiently small but fixed number \( d > 0 \), we consider a smooth change of variables

\[
H^t_2(y) = e^t H^t_x(e^{-t}y),
\]

where \( H^t_x : B_d(\bar{x}) \to M \) is a diffeomorphism and \( M \) is an open neighborhood of the origin such that \( H^t_x(B_d(x) \cap \Omega) = M \cap \mathbb{R}^2_+ \) and \( H^t_x(B_d(x) \cap \partial \Omega) = M \cap \partial \mathbb{R}^2_+ \). For each \( i = 1, \ldots, m \), let

\[
\xi^0_i = e^{-t}(H^t_2)^{-1} \left( \frac{e^t \xi^0}{\sqrt{t}} \right)
\]

where \( \xi^0_i \in M \cap \partial \mathbb{R}^2_+ \) satisfies \( |\xi^0_i - \xi^0_i+1| = \rho \) for any \( \rho > 0 \) sufficiently small, fixed and independent \( t \). Using the expansion \((H^t_2)^{-1}(y) = e^t \bar{x} + y + O(e^{-t}|y|)\), we find

\[
\xi^0_i = \bar{x} + \frac{1}{\sqrt{t}} \xi^0_i + O \left( \frac{e^{-t}}{\sqrt{t}} \right).
\]

Then it is easy to see \( \xi^0 = (\xi^0_1, \ldots, \xi^0_m) \in O_t \) since \( \beta \geq 1 \) and \( \phi_1(\xi^0_i) = 1 + O(t^{-1}) \). From [6,1] and Proposition 5.2, we obtain

\[
\max_{\xi \in \overline{O}_t} F_t(\xi) \geq J_t(U(\xi^0)) + \theta_t(\xi^0) = 2\pi \sum_{i \neq j}^{m} \log |\xi^0_i - \xi^0_j| + 2\pi t \sum_{i=1}^{m} \phi_1(\xi^0_i) + O(1)
\]

\[
\geq 2\pi mt - \pi m(m - 1) \log t + O(1).
\]

Next, we suppose \( \xi^t = (\xi^t_1, \ldots, \xi^t_m) \in \partial O_t \). There are two possibilities: either there exists an \( i_0 \) such that \( 1 - \phi_1(\xi^t_{i_0}) = \frac{1}{\sqrt{t}} \) or there exist indices \( i_0, j_0 \), \( i_0 \neq j_0 \) such that \( |\xi^t_{i_0} - \xi^t_{j_0}| = t^{-\beta} \). In the first case, we have

\[
\max_{\xi \in \overline{O}_t} F_t(\xi) = F_t(\xi^t) \leq 2\pi \left\{ O\left( \log t \right) + t \left( 1 - \frac{1}{\sqrt{t}} + (m - 1) \right) \right\} + O(1)
\]

\[
= 2\pi mt - 2\pi \sqrt{t} + O(\log t),
\]

which contradicts to [6,4]. In the second case, we have

\[
\max_{\xi \in \overline{O}_t} F_t(\xi) = F_t(\xi^t) \leq 2\pi \left\{ -\beta \log t + t \sum_{i=1}^{m} \phi_1(\xi^t_i) \right\} + O(1)
\]

\[
\leq 2\pi mt - 2\pi \beta \log t + O(1).
\]

This, together with [6,4], implies

\[
2\pi \beta \log t + O(1) \leq \pi m(m - 1) \log t + O(1),
\]

which is impossible by the choice of \( \beta \) in [2,9].

\[ \square \]

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