The additive structure of integers with the lower Wythoff sequence

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Abstract
We have provided a model-theoretic proof for the decidability of the additive structure of integers together with the function \( f \) mapping \( x \) to \( \lfloor \phi x \rfloor \) where \( \phi \) is the golden ratio.

Keywords Decidability · Quantifier elimination · Wythoff sequence

Mathematics Subject Classification 03C10 · 03C57 · 03C98 · 11B39

Introduction
While the theory of the structure \((\mathbb{Z}, +, \cdot)\) is famously undecidable, tame reducts of this structure have been subject of various literature, see for example [4, 9, 12, 13]. A classical result in this direction is the decidability of the theory of the structure \((\mathbb{Z}, +, \{p_n\}_{n \in \mathbb{N}}, 0, 1)\), known as the theory of \(\mathbb{Z}\)-groups. In the mentioned structure, multiplication in \(\mathbb{Z}\) is replaced by infinitely-many unary predicates \(p_n\), where \(p_n(x)\) holds if \(x \equiv 0\). More recent relevant results are, for example, that there are no intermediate structures between the group of integers and Presburger arithmetic (Conant in [4]); and that the theory of integers with a predicate for prime numbers is decidable provided that Dickson’s conjecture holds (Kaplan and Shelah in [9]).

In this paper we prove the decidability of the structure \((\mathbb{Z}, +, f, 0, 1)\) where \(f(x) = \lfloor \phi x \rfloor\), and \(\phi\) is the golden ratio. We are already aware that this follows from the decidability of the theory of \((\mathbb{R}, \mathbb{Z}, \alpha \mathbb{Z}, +, <)\) for a quadratic irrational number \(\alpha\), as proved by Hieronymi in [12]. His proof relies on the continued fractions and Ostrowski representations and interpreting in the structure \((\mathbb{N}, \mathcal{P}(\mathbb{N}), \in, s_{\mathbb{N}})\). Since the latter is
decidable by a classical result of Büchi [10], so is the former. We have also been later informed by the referee that the decidability of our structure can as well be obtained as a consequence of two papers of Shallit et al., [1, 6] where they even propose an automata-based decision algorithms for Fibonacci words.

Nevertheless, although we rely on facts on Fibonacci words and the lower Wythoff sequence to explain the properties of the function \( f \), our approach is pure model-theoretic and based on a quantifier-elimination result in a suitable language. This approach was suggested by Hieronymi, who guessed that a model-theoretic treatment of the properties of Beatty sequences would lead to the decidability of our structure.

Recall that any sequence of the form 
\[
B_r = (\lfloor rn \rfloor)_{n \in \mathbb{N}}
\]
for a positive irrational \( r \), is called a Beatty sequence. For \( r > 1 \) and \( s = r/(r-1) \), \( (B_r, B_s) \) form a so-called pair of complementary Beatty sequences; that is \( B_r \cup B_s = \mathbb{N} \) and \( B_r \cap B_s = \emptyset \) (see \[7, 8\] for more on Beatty sequences). The Beatty sequence \( B_r \), in the special case that \( r = \varphi \), is the golden ratio, is called the lower Wythoff sequence.

We augment the language of \( \mathbb{Z} \)-groups by the unary function symbol \( f \) and denote the obtained language by \( \mathcal{L} \). This choice of the language suggests that in order to have a chance for quantifier-elimination, we need to deal with systems of equations involving congruence relations and the function \( f \). It turns out that the solvability of such systems is closely related to a classical theorem of Kronecker (see Fact 8) that the set of decimal parts of elements of the form \( \varphi n \), for \( n \in \mathbb{N} \), is dense in the unit interval \((0, 1)\). We will deploy this connection as a major means for our axiomatization.

The main idea we rely on is that the order of the decimal parts is definable in \( \mathcal{L} \). That is there is an \( \mathcal{L} \)-formula \( R(x, y) \) such that \( R(m, n) \) holds for two integers \( m, n \) if and only if the decimal part of \( \varphi m \) is smaller than that of \( \varphi n \). Hence we add a binary predicate \( R(x, y) \) to \( \mathcal{L} \) to obtain the language \( \mathcal{L}^* \) (see Notation 2), and our main theorem is the following.

**Theorem** The structure \((\mathbb{Z}, +, f, R, \{p_n\}^n_{n \in \mathbb{N}}, 0, 1)\) admits elimination of quantifiers.

The paper is structured as follows. Basic facts about the properties of \( f \) are gathered in Sect. 1. Note that as \( f(−n) = −f(n) − 1 \) for each natural number \( n \), in the lemmas on the properties of \( f \) in Sects. 1 and 2, (until before Lemma 10) we have restricted the domain to natural numbers. In Sect. 2, some auxiliary lemmas are proved to be used in Sect. 3 as the basis of our axiomatization. The quantifier-elimination result and the decidability that follows immediately from it are established in Sect. 4.

## 1 Preliminaries on the function \( f \)

By properties of the floor function, it is clear that for natural numbers \( m \) and \( n \), we have either \( f(m + n) = f(m) + f(n) \), or \( f(m) + f(n) + 1 \). Hence, for each natural number \( k \), there is \( 0 \leq \ell \leq k - 1 \) such that \( f(kn) = kf(n) + \ell \). Of course \( \ell \) is the unique number such that \( f(kn) \equiv \ell \).

**Lemma 1** For every \( m \in \mathbb{N} \) there is \( n \in \mathbb{N} \) such that either \( m = f(n) \) or \( m = f(n) + n \).
Proof As \( \frac{\varphi}{\varphi - 1} = \varphi + 1 \), \((B_\varphi, B_{\varphi+1})\) is a complementary pair of Beatty sequences, which clearly means that each natural number \( m \) is either equal to \( f(n) \) or \( f(n) + n \) for some natural number \( n \).

Note that Lemma 1 holds also when one replaces \( \mathbb{N} \) with \( \mathbb{Z} \) and forces \( m \neq -1 \); simply because \( f(-n) = -f(n) - 1 \) for all positive \( n \).

Depending on whether \( m \) belongs to the image of \( f \) or \( f + \text{id} \), where \( \text{id} \) denotes the identity function, and by the properties of Beatty sequences one obtains a recursive definition for the function \( f \) in natural numbers as in the following lemma.

Lemma 2 \( f(0) = 0 \), \( f(1) = 1 \), and for each natural number \( n > 1 \), \( f(f(n) + n) = 2f(n) + n \) and \( f(f(n)) = f(n) + n - 1 \).

Proof The fact that \( f(f(n) + n)) = 2f(n) + n \) follows from [7, Theorem 1]. Now \( f(f(n)+n) \) equals either to \( f(f(n)) + f(n) \) or \( f(f(n)) + f(n)+1 \). The former cannot occur since the images of \( f \) and \( f + \text{id} \) are disjoint. So \( f(f(n) + n) = 2f(n) + n = f(f(n)) + f(n)+1 \), and the result follows.

The lemma above implies that for every \( n \in \mathbb{N} \), \( f(n) = \min \{ f(i), f(i) + i : i < n \} \), hence in particular \( f \) is strictly increasing. Again Lemma 2 also holds in \( \mathbb{Z} \) but one needs to add that \( f(-1) = -2 \).

At this point, we aim to establish the connection between the function \( f \) and the Fibonacci sequence \( F_n \) with \( F_0 = F_1 = 1 \). This connection will play a major role in our proofs in this section of the properties of \( f \) in natural numbers.

Consider the sequence \((c_n)_{n \in \mathbb{N}}\), where \( c_n = 1 \) if \( n \) is in the image of \( f \) and 0 otherwise. By the properties of the floor function and the fact that \( \varphi > 1 \) it is easy to check that there are no successive zeros in \((c_n)_{n \in \mathbb{N}}\). A curious way to obtain \((c_n)_{n \in \mathbb{N}}\) is to start with the word 10 (of course of length 2) and then replace 1 with 10 and 0 with 1 (to obtain a word of length 3), and apply the same change to the word obtained (to obtain a word of length 5), and continue the same way. So the length of each such word is a Fibonacci number and the last digit alternates between 0 and 1. So \( c_{F_{2n+1}} = 1 \) and \( c_{F_{2n}} = 0 \) for each \( n \in \mathbb{N} \), and \( c_{F_{n+1}} = c_{1} \) for each \( 1 \leq i \leq F_{n-1} \).

Meanwhile, note that each natural number \( n \) has a unique Fibonacci representation; that is, it can be uniquely written as a sum of non-successive decreasing Fibonacci numbers. To see this one needs to find the largest Fibonacci number \( F_{i_1} < n \), and write \( n = F_{i_1} + G_1 \). Now let \( F_{i_2} \) be the largest Fibonacci number less than \( G_1 \) and write \( n = F_{i_1} + F_{i_2} + G_2 \) and continue with this procedure to end up with a Fibonacci number (see also [3]). The explanation above on the sequence \((c_n)_{n \in \mathbb{N}}\) together with this Fibonacci representation yields the following fact.

Fact 3 For each \( n \), the smallest index appearing in the unique Fibonacci representation of \( n \) determines \( c_n \), where \( c_n = 1 \) if and only if this index is odd.

This in turn provides us with a concrete rule for \( f \) as follows.

Fact 4 1. \( f(F_i) = F_{i+1} \) if \( i \) is even, and \( f(F_i) = F_{i+1} - 1 \), otherwise.
2. If \( m \) has the Fibonacci representation \( m = F_{i_1} + F_{i_2} + \cdots + F_{i_\ell} \) with \( i_1 > i_2 > \cdots > i_\ell \), then \( f(m) = F_{i_1+1} + F_{i_2+1} + \cdots + F_{i_\ell+1} \) if \( i_\ell \) is even, and \( f(m) = F_{i_1+1} + F_{i_2+1} + \cdots + F_{i_\ell+1} - 1 \), otherwise.
Note that \( f(m) \) is equal to the index of the \( m \)th occurrence of 1 in the sequence \((c_n)_{n \in \mathbb{N}}\). Now, by construction, the number of 1’s in the sequence \((c_n)_{n \leq F_{i+1}}\) is \( F_i \). So, \( f(F_i) = F_{i+1} \) if \( i \) is even and \( f(F_i) = F_{i+1} - 1 \) if \( i \) is odd. Similarly the number of 1’s in \((c_n)_{n \leq N}\) with \( N = F_{i_1+1} + F_{i_2+1} + \ldots + F_{i_\ell+1} \), is \( F_{i_1} + F_{i_2} + \ldots + F_{i_\ell} \), and this implies the second item. \( \square \)

It is worth reminding that our sequence \((c_n)_{n \in \mathbb{N}}\) is indeed the complement of the so-called Fibonacci word, given by \( 2 + \lfloor \phi n \rfloor - \lfloor \phi(n+1) \rfloor \).

We end this section by a simple, and yet key fact (for a proof see [2]).

**Fact 5** For any positive integer \( k \) the sequence \((F_i \mod k)_{i \in \mathbb{N}}\) is periodic beginning with 0, 1.

## 2 Auxiliary lemmas for the axiomatization

The main lemmas in this section will correspond to the axioms we present for our structure in the next section. The first lemma below says that the image of \( f \) contains elements in any congruence class.

**Lemma 6** For each \( k, n \in \mathbb{N} \) there is \( m \in \mathbb{N} \) such that \( f(m) \equiv_n^k n \).

**Proof** By Fact 3, if the smallest index in the Fibonacci representation of \( n \) is odd, then \( n \) is in the image of \( f \) and the equation is solved instantly; hence assume for the rest of the proof that this index is even.

Since by Fact 5 the Fibonacci sequence modulo \( k \) is periodic, and the first two elements of this period are 0 and 1, there is a Fibonacci number \( F_i \), with \( i \) larger than the largest index in the Fibonacci representation of \( n \), such that \( F_i \equiv_n^k 1 \). That is there is \( u \in \mathbb{N} \) such that \( F_i - 1 = ku \). Because the index of the smallest Fibonacci number in the representation of \( n \) is even, i.e. \( c_n = 0 \), we have \( c_{F_i-1+n} = c_{n-1} = 1 \). Hence \( F_i - 1 + n \) is in the image of \( f \) and \( F_i - 1 + n \equiv_n^k n \). \( \square \)

The following lemma generalizes the above in the way that \( m \) is also in a desirable congruence class.

**Lemma 7** The system

\[
\begin{align*}
  x & \equiv_n^k n \\
  f(x) & \equiv_n^{k'} n'
\end{align*}
\]

has a solution in \( \mathbb{N} \), for any \( k, n, k', n' \in \mathbb{N} \).

**Proof** It is obvious that \( n \) is a solution of the equation \( x \equiv_n^k n \). Let \( F_{i_1} + F_{i_2} + \ldots + F_{i_\ell} \) be the Fibonacci representation of \( n \) with \( i_1 > i_2 > \ldots > i_\ell \). Suppose that \( n'' \) is such that \( f(n) \equiv_n^{k'} n'' \). If \( n'' \neq n' \), by Fact 5 we can find a Fibonacci number \( F_j \) such that \( F_j \equiv_0^0, F_j \equiv_0^{k'}, f(F_j) = F_{j+1} \equiv_1^1, j > i_1 + 1, \) and \( j \) is an even number. Now \( \square \)
since \( F_j \stackrel{k}{=} 0 \), we have \( n + F_j \stackrel{k}{=} n \). Also because \( j > i_1 + 1 \) is even, by Fact 4, \( f(n + F_j) = f(n) + f(F_j) \stackrel{k'}{=} n'' + 1 \). This procedure gives, after finitely many steps, a natural number \( m \) such that \( m \stackrel{k}{=} n \) and \( f(m) \stackrel{k'}{=} n' \).

Now, as it turns out, the lemma above has a close connection to the following fact about the distribution in the interval \((0, 1)\) of the decimal parts of \( \varphi n \), for all \( n \in \mathbb{N} \) (as explained in Remark 1). Before mentioning the fact, let us fix some notation for the decimal parts.

**Notation 1** We denote the decimal part of \( \varphi n \) by \([\varphi n]\); so \([\varphi n] = \varphi n - f(n)\).

**Fact 8** (*Kronecker* [5, Theorem 439]) If \( r \) is irrational, then \(([rn])_{n \in \mathbb{N}}\) is dense in \((0, 1)\).

To explain the connection in question we need yet another lemma.

**Lemma 9** For each \( k \) and \( 0 \leq i < k \), \( f(n) \stackrel{k}{=} i \) if and only if \([\frac{\varphi}{k} n] \in (\frac{i}{k}, \frac{i+1}{k})\).

**Proof** Write \( \frac{\varphi}{k} n = u + r \), for some integer \( u \) and \( 0 < r < 1 \). So \( f(n) \stackrel{k}{=} i \) if and only if \( \varphi n = ku + kr \), with \( i < kr < i + 1 \), that is \( \frac{i}{k} < r < \frac{i+1}{k} \). Therefore \( f(n) \stackrel{k}{=} i \) if and only if \( \frac{\varphi}{k} n = u + r \) and \( \frac{r}{k} < r < \frac{i+1}{k} \).

**Remark 1** By the lemma above, \([\varphi n] \in (\frac{i}{k}, \frac{i+1}{k})\) if and only if \( f(kn) \stackrel{k}{=} i \). Hence to find a natural number \( n \) such that \([\varphi n]\) is in a desired small subinterval \((\frac{i}{k}, \frac{i+1}{k})\) of \((0, 1)\) one needs to solve the following system of congruence-relation equations:

\[
\begin{cases}
  x \stackrel{k}{=} 0 \\
  f(x) \stackrel{k}{=} i,
\end{cases}
\]

and if \( m \) is the solution of the system above then \( n = \frac{m}{k} \) has the desired property. This means that Lemma 7 indeed implies Kronecker’s theorem that \([\varphi n]_{n \in \mathbb{N}}\) is dense in \((0, 1)\). It is an easy verification that Kronecker’s theorem also implies Lemma 7.

The remark above is interesting, because it suggests that although in our language \( \mathcal{L} \) it is not possible to have any symbol to refer to the decimal part of \( \varphi x \), we are capable of finding an equivalent way of expressing in which interval \([\varphi x]\) lies. Indeed many expressions about the function \( f \) has an equivalent in terms of the decimal parts. For example, \( f(n + m) = f(n) + f(m) \) means that \([\varphi n] + [\varphi m] < 1 \), and similarly \( f(n + m) = f(n) + f(m) + 1 \) means that \([\varphi n] + [\varphi m] > 1 \). As we see in the following lemma, much more can be said about the decimal parts already in the language \( \mathcal{L} \).

**Lemma 10** There is an \( \mathcal{L} \)-formula \( R(x, y) \) such that for all \( m, n \in \mathbb{Z}, (\mathbb{Z}, +, f, 0, 1) \models R(m, n) \) if and only if \([\varphi m] < [\varphi n]\).
Proof Simply let \( R(x, y) \) be the following formula

\[
\forall z \left( f(x + z) = f(x) + f(z) + 1 \rightarrow f(y + z) = f(y) + f(z) + 1 \right). \tag{1}
\]

We first show that if \([\varphi m] < [\varphi n]\), then \((\mathbb{Z}, +, f, 0, 1) \models R(m, n)\). Note that when \([\varphi m] < [\varphi n]\) then for each \( r \in \mathbb{Z}, [\varphi m] + [\varphi r] < [\varphi n] + [\varphi r]\). Hence if \([\varphi m] + [\varphi r] > 1\) then \([\varphi n] + [\varphi r] > 1\). But this is exactly what the formula \( R(m, n) \) says.

For the other direction, note that if \([\varphi m] > [\varphi n]\), then \(1 - [\varphi m] < 1 - [\varphi n]\). Hence by Kronecker’s theorem (Fact 8), there is a natural number \( r \) such that \(1 - [\varphi m] < [\varphi r] < 1 - [\varphi n]\). But this means that \( f(m + r) = f(m) + f(r) + 1 \) and \( f(n + r) = f(n) + f(r)\), which means that the negation of \( R(m, n) \) holds in \((\mathbb{Z}, +, f, 0, 1)\). □

Thus the order of the decimal parts is definable in \((\mathbb{Z}, +, f, 0, 1)\). We expand our language by the binary predicate \( R(x, y) \) which is interpreted by formula (1) above, and defines this order.

**Notation 2** Let \( \mathcal{L}^* \) be the language \( \mathcal{L} \cup \{R\} \).

**Remark 2** We interpret \( R \) in \( \mathbb{Z} \) as in Lemma 10. We will later add to our axioms that \( R \) is indeed a “linear order relation”.

Enriching the language with the predicate \( R \) helps find equivalent expressions in the language \( \mathcal{L}^* \) to the following phrases (where \( m_i, n_i \in \mathbb{N} \) and \( \ell \in \mathbb{Z} \)):

\[
[\varphi x] + [\varphi y] < [\varphi z] \tag{2}
\]

\[
m_1[\varphi x_1] + \cdots + m_k[\varphi x_k] < n_1[\varphi y_1] + \cdots + n_k[\varphi y_k] + \ell. \tag{3}
\]

Phrase 2 can be expressed simply as follows:

\[
(f(x + y) = f(x) + f(y)) \land R(x + y, z) \tag{4}
\]

Finding an equivalent to phrase 3 is also as easy; one needs to consider cases \( f(m_1x_1 + \cdots + m_kx_k) = m_1f(x_1) + \cdots + m_kf(x_k) + \ell_1 \), and \( f(n_1y_1 + \cdots + n_ky_k) = n_1f(y_1) + \cdots + n_kf(y_k) + \ell_2 \) for suitable \( \ell_1, \ell_2 \) and use the relation \( R(m_1x_1 + \cdots + m_kx_k, n_1y_1 + \cdots + n_ky_k) \) accordingly.

Some more power of expression is provided using the following lemma.

**Lemma 11** The decimal part of \( \varphi f(n) \), for an integer \( n \), is determined by the decimal part of \( \varphi n \) as in the following:

\[
[\varphi f(n)] = (1 - \varphi)[\varphi n] + 1 \tag{5}
\]

**Proof** Note that \( \varphi f(n) = \varphi^2(\varphi n - [\varphi n]) = \varphi^2n - \varphi[\varphi n] = \varphi n + n - \varphi[\varphi n] \), where the latter is the case because \( \varphi^2 = \varphi + 1 \). Hence, as \( n \) is an integer, \([\varphi f(n)] = [\varphi n - \varphi[\varphi n]]\).

If \([\varphi n] < \frac{1}{\varphi}\), then obviously \([\varphi[\varphi n]] = \varphi[\varphi n]\). Also it is clear that \([\varphi n] < \varphi[\varphi n]\), hence \([\varphi f(n)] = [\varphi n] - \varphi[\varphi n] + 1\). If \([\varphi n] > \frac{1}{\varphi}\), then \([\varphi[\varphi n]] = \varphi[\varphi n] - 1 < [\varphi n]\), where the latter inequality is always the case. So again \([\varphi f(n)] = [\varphi n] - \varphi[\varphi n] + 1\). □
Thus also the following phrase has an equivalent in the language $\mathcal{L}^*$ (where $m_i, n_i \in \mathbb{N}$ and $\ell, s \in \mathbb{Z}$):

$$m_1 \varphi[x_1] + m_2 [\varphi x_2] + \cdots + m_k [\varphi x_k] < n_1 \varphi[y_1] + n_2 [\varphi y_2] + \cdots + n_k [\varphi y_k] + \ell + \varphi s.$$  

(6)

Note that the difference between the above phrase and phrase 3 is that $\varphi$ itself appears as coefficient in two places, and a constant $\varphi s$ is added to the end. To express the above phrase one needs to simply replace $\varphi[x]$ with $[\varphi x] - [\varphi f(x)] + 1$, $\varphi[y]$ with $[\varphi y] - [\varphi f(y)] + 1$ and $\varphi s$ with $f(s) + [\varphi s]$, to obtain a similar phrase to 3.

In Corollary 13 we will prove that the solvability in $\mathbb{Z}$ of a system of equations involving symbols of $\mathcal{L}^*$ is expressible by a quantifier-free $\mathcal{L}^*$-formula. To explain the required argument more easily, we first deal with a simpler yet essential case in the following lemma.

**Lemma 12** There is a quantifier-free formula $\Phi(y_1, y_2)$ in the language $\mathcal{L}^*$ such that for all integers $n_1, n_2, (\mathbb{Z}, +, f, R, 0, 1) \models \Phi(n_1, n_2)$ if and only if the following system of equations has a solution in $\mathbb{Z}$,

$$\begin{cases}
f(r_1x + s_1 f(x) + n_1) = r_1 f(x) + s_1 f^2(x) + f(n_1) + j_1 \\
f(r_2x + s_2 f(x) + n_2) = r_2 f(x) + s_2 f^2(x) + f(n_2) + j_2
\end{cases}$$

(7)

where $r_1, s_1, j_1, r_2, s_2, j_2$ are fixed natural numbers.

Note that the formula required in lemma above depends on $r_1, s_1, j_1, r_2, s_2, j_2$, but for simplicity we have not reflected this dependence in the notation.

**Proof** The equations in (7) can be rewritten in terms of the decimal parts as follows:

$$\begin{cases}
j_1 - [\varphi n_1] < r_1[\varphi x] + s_1[\varphi f(x)] < j_1 + 1 - [\varphi n_1] \\
j_2 - [\varphi n_2] < r_2[\varphi x] + s_2[\varphi f(x)] < j_2 + 1 - [\varphi n_2].
\end{cases}$$

Replacing $[\varphi f(x)]$ with $(1 - \varphi)[\varphi x] + 1$ as in Lemma 11, we need to consider the following system:

$$(r_1 + s_1 - s_1 \varphi)[\varphi x] \in (j_1 - [\varphi n_1] - s_1, j_1 + 1 - [\varphi n_1] - s_1),$$

$$(r_2 + s_2 - s_2 \varphi)[\varphi x] \in (j_2 - [\varphi n_2] - s_2, j_2 + 1 - [\varphi n_2] - s_2).$$

(8)

Thus the system is essentially of the form

$$A_1[\varphi x] \in (B_1, C_1),$$

$$A_2[\varphi x] \in (B_2, C_2)$$

and is solvable if either $\frac{B_1}{A_1} \in (\frac{B_2}{A_2}, \frac{C_2}{A_2})$ or $\frac{C_1}{A_1} \in (\frac{B_2}{A_2}, \frac{C_2}{A_2})$. But then, each inequality needed to hold (for example that $\frac{B_1}{A_1} > \frac{B_2}{A_2}$, or equivalently $B_1 A_2 > B_2 A_1$) turns into an equality of the form 6 and hence is expressible in the language $\mathcal{L}^*$. \hfill \square
Now the same strategy (that is writing the equations in terms of the decimal parts and analyzing the obtained linear equations in terms of disjoint or intersecting intervals) leads to the following corollary. Note that here other types of equations, for example of the form $R(x, n)$ and $R(f(x), n)$, and two congruence relation equations of the form $x \equiv j$ and $f(x) \equiv j'$ are also added, but this does not essentially change the way we need to treat the system. Indeed all equations describe, in essence, an inequality of the form $[\varphi x] \in (a, b)$ for suitable $a, b$.

**Corollary 13** There is a quantifier free $\mathcal{L}^*$-formula $\theta(y_1, y_2, \ldots, y_k)$ such that for all $n_1, \ldots, n_k \in \mathbb{Z}$ we have $(\mathbb{Z}, +, f, R, 0, 1) \models \theta(n_1, \ldots, n_k)$ if and only if the following system of equations has a solution in $\mathbb{Z}$:

$$
\begin{align*}
& f(r_1 x + s_1 f(x) + n_1) = r_1 f(x) + s_1 f^2(x) + f(n_1) + j_1 \\
& \vdots \\
& f(r_{k-4} x + s_{k-4} f(x) + n_{k-4}) = r_{k-4} f(x) + s_{k-4} f^2(x) + f(n_{k-4}) + j_{k-4} \\
& R(x, n_{k-3}) \\
& R(n_{k-2}, x) \\
& R(f(x), n_{k-1}) \\
& R(n_k, f(x)) \\
& f(x) \equiv j_{k-3} \quad r_{k-4} \\
& x \equiv j_{k-2} \\
\end{align*}
$$

(9)

where $r_i, s_i, j_i \in \mathbb{N}$.

**Proof** Note that the last equation may be discarded, as one can replace $x$ with $r_{k-2} x' + j_{k-2}$ in the rest of the equations and change the system accordingly. The same is true for the last-but-one equation, as it has an equivalent of the form $[\varphi x] \in (a, b)$ for some rational $a, b$ and this information is already in the rest of the equations. So system 9 can be written in the following form:

$$
\begin{align*}
& (r_1 + s_1 \varphi) [\varphi x] \in (j_1 - [\varphi n_1] - s_1, j_1 + 1 - [\varphi n_1] - s_1) \\
& \vdots \\
& (r_{k-4} + s_{k-4} - s_{k-4} \varphi) [\varphi x] \in (j_{k-4} - [\varphi n_{k-4}] - s_{k-4}, j_{k-4} + 1 - [\varphi n_{k-4}] - s_{k-4}) \\
& [\varphi x] \in ([\varphi n_{k-2}], [\varphi n_{k-3}]) \\
& (1 - \varphi)[\varphi x] \in ([\varphi n_k], [\varphi n_{k-1}]).
\end{align*}
$$

(10)

To solve this system one needs to check whether or not the corresponding intervals for $[\varphi x]$ have intersection. Now as in Lemma 12 this can be described by an $\mathcal{L}^*$-formula. 

$\square$
3 Axiomatization

We can now present an axiomatization \( \mathbf{T} \) for our structure, in the language \( \mathcal{L}^* \) as in Notation 2. The axioms are based on what we developed in the previous two sections. More specifically, the axiom-scheme \((T1)\) below expresses the basic properties of \( (\mathbb{Z}, +, \{ p_n \}_{n \in \mathbb{N}}, 0, 1) \) as a \( \mathbb{Z} \)-group. \((T2)\) and \((T3)\) express the main properties of the function \( f \) based on Lemma 1 and Lemma 2. \((T4)\) asserts that \( R(x, y) \) is a linear order relation, which, based on Remark 2, can be naturally thought of as the “order of the decimal parts”. \((T5)\) expresses that this order is dense. In other words, \((T5)\) expresses the Kronecker’s theorem on the distribution of the decimal parts, based on Remark 1; that is it says that if \( R(a, b) \) holds (which can be thought of as \([\varphi a] < [\varphi b]\)) then there is \( c \) such that \( R(a, c) \) and \( R(c, b) \) hold (which again can be thought of as \([\varphi a] < [\varphi c] < [\varphi b]\)). Finally in the light of Corollary 13 the axiom-scheme \((T6)\) expresses when a given system of equations has a solution.

Definition 14 Let \( \mathbf{T} \) be the theory obtained by the axioms expressing the following.

\[(T1)\] The theory of \( \mathbb{Z} \)-groups,
\[(T2)\] \( \forall x \left( x \neq -1 \rightarrow \exists y \left( (x = f(y)) \lor (x = f(y) + y) \right) \right) \land \forall x, y(f(x + y) = f(x) + f(y) \lor f(x + y) = f(x) + f(y) + 1), \)
\[(T3)\] \( (f(0) = 0) \land (f(1) = 1) \land (f(-1) = -2) \land \forall x \left( f(f(x)) = f(x) + x - 1 \land f(f(x) + x) = 2f(x) + x \right), \)
\[(T4)\] \( \forall x \neg R(x, x), \)
\( \forall x, y \left( R(x, y) \rightarrow \neg R(y, x) \right), \)
\( \forall x, y, z \left( R(x, z) \land R(z, y) \rightarrow R(x, y) \right), \)
\( \forall x, y \left( R(x, y) \lor R(y, x) \right), \)
\[(T5)\] \( \forall x, y \left( R(x, y) \rightarrow \exists z \left( R(x, z) \land R(z, y) \right) \right), \)
\[(T6)\] \( \forall y_1, \ldots, y_k \left( \exists x \left( \bigwedge_{i=1}^{k-4} \left( f(r_i x + s_i f(x) + y_i) = r_i f(x) + s_i f^2(x) + f(y_i) + j_i \right) \land R(x, y_{k-3}) \land R(y_{k-2}, x) \land R(f(x), y_{k-1}) \land R(y_k, f(x)) \land p_{rk-3}(f(x) - j_{k-3}) \land p_{rk-2}(x - j_{k-2}) \Leftrightarrow \theta(y_1, \ldots, y_k) \right) \right). \)

Note that \((T1)\) and \((T6)\) are actually axiom schemes.

Theorem 15 \((\mathbb{Z}, +, f, R, \{ p_n \}_{n \in \mathbb{N}}, 0, 1)\) is a model of \( \mathbf{T} \).

The proof of the theorem above is clear by the way we have established the axioms (and the explanation before Definition 14). In the next section we have proved that \( \mathbf{T} \) eliminates quantifiers and this leads to the fact that \( \mathbf{T} \) is complete and decidable.

4 Quantifier-Elimination and decidability

For the rest of the paper, let \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) be models of \( \mathbf{T} \) and \( \mathcal{M} \) be a common substructure. We assume that \( \mathcal{M}_2 \) is \(|\mathcal{M}|\)-saturated. To prove quantifier-elimination,
we will show that any finite system of equations in the language \( L^* \) with parameters in \( M \) which has a solution in \( M_1 \) is also solvable in \( M_2 \).

**Lemma 16** There is \( \mathcal{M}' \models (T1), (T2) \) such that \( \mathcal{M} \subseteq \mathcal{M}' \) and \( \mathcal{M}' \subseteq \mathcal{M}_i \) for \( i = 1, 2 \).

**Proof** Put \( \mathcal{M}' = \{ \frac{s}{n} \mid x \in M, \mathcal{M}_1, \mathcal{M}_2 \models p_n(x) \} \). Indeed \( \mathcal{M}' \) is the algebraically-prime model of \( (T1) \) in the language \( L \) containing \( \mathcal{M} \).

We claim that \( \mathcal{M}' \) is closed under the function \( f \), and hence bears an \( L \)-structure. Suppose that \( t = \frac{a}{n} \in \mathcal{M}' \). Then \( a = nt \) and \( a \in M \). By properties of \( f \), we have \( \mathcal{M}_1 \models f(a) = nf(t) + \ell \), where \( \ell \) is the remainder of the division of \( f(a) \) by \( n \). So \( \mathcal{M}_1 \models p_n(f(a) - \ell) \), and \( f(a) - \ell \in M \). Therefore \( f(t) = \frac{f(a) - \ell}{n} \in M' \).

To prove Axiom \( (T2) \), let \( a \neq -1 \) be an arbitrary element in \( M' \). Since \( \mathcal{M}_1 \) is a model of \( T \), there is \( b \in M_1 \) such that \( \mathcal{M}_1 \models a = f(b) \lor a = f(b) + b \). We will show that \( b \in M' \).

If \( \mathcal{M}_1 \models a = f(b) \), then by \( (T3) \), \( \mathcal{M}_1 \models f(a) = a + b - 1 \) and hence \( \mathcal{M}_1 \models b = f(a) - a + 1 \). It is clear that \( b \in M' \).

If \( \mathcal{M}_1 \models a = f(b) + b \) then by \( (T3) \), \( \mathcal{M}_1 \models f(a) = f(f(b) + b) = 2f(b) + b = a + f(b) \). Therefore \( \mathcal{M}_1 \models f(b) = f(a) - a \). On the other hand, \( a \in M' \) and \( f(a) \in M' \) so \( f(b) = f(a) - a \in M' \). Now since \( f(b) \in M' \), by the above argument we have \( b \in M' \).

The second part of axiom \( (T2) \) is clearly inherited from \( \mathcal{M}_1, \mathcal{M}_2 \). \( \square \)

**Theorem 17** The theory \( T \) admits elimination of quantifiers.

**Proof** According to Lemma 16, we add to the assumptions at the beginning of this section that \( \mathcal{M} \models (T1), (T2), (T3), (T4) \). Note that \( (T3) \) and \( (T4) \) come for free because they are universal and hence inherited from \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \). Now assume that an element \( a \in M_1 \) satisfies finitely-many equations, each of which of one of the forms in Equation 11 below, with parameters \( c, d, e, e', g, \) and \( g' \) in \( M \) and coefficients \( m, n, m', n', r, s, t, u, j \) in \( \mathbb{N} \). Note that the negation of each of the following equations (except for the last one) has the same format as itself.

\[
\begin{align*}
x &\equiv m \\
f(x) &\equiv m' \\
f(rx + sf(x) + c) &= rf(x) + sf^2(x) + f(c) + j \\
R(x, e) & \\
R(e', x) & \\
R(f(x), g) & \\
R(g', f(x)) & \\
rf(x) &= ux + d
\end{align*}
\]  

(11)

Also notice that we do not get more involved equations (say in terms of the powers of \( f \)) simply because the powers of \( f \) reduce to one by Axiom \( (T3) \). Now we aim to find \( b \in M_2 \) satisfying the equations above.

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We first claim that if the system above actually contains an equation of the last form \((tf(x) = ux + d)\), then \(a\) is already in \(M\), hence \(b \in M_2\) can be taken to be \(a\) itself. Indeed such an equation has an “algebraic nature” where the rest of the equations, which only concern with the decimal parts can be thought of being “non-algebraic” (see [11]).

**Claim 1** If for \(d \in M\), \(M_1 \not\models f(a) = \frac{u}{t}a + d\) then \(a \in M\).

**Proof** Suppose that \(M_1 \models f(a) = \frac{u}{t}a + d\), so \(M_1 \models f(f(a)) = f(\frac{u}{t}a + d)\). By axiom \((T3)\), \(M_1 \models f(f(a)) = f(a) + a - 1\), hence similar to the proof of the previous lemma,

\[
M_1 \models f(a) + a - 1 = \frac{uf(a) + \ell}{t} + f(d) + j
\]

for some integer \(\ell\) and natural number \(j\). Replacing \(f(a)\) in both sides of the above formula with \(\frac{u}{t}a + d\) we get a linear equation in terms of \(a\). This forces that \(a \in M\), because the linear equation gives \(a\) by the divisibility relation and \(M\) is a model of \((T1)\).

By the above claim, if there is an equation of the form \(tf(x) = ux + b\) in system \((11)\), then the solution of this system is already in \(M\). Hence, in the rest we drop the last equation from the system.

Meanwhile by the Chinese remainder theorem (which itself is deduced from the theory of \(\mathbb{Z}\)-groups and hence holds in our theory \(T\)) one reduces the congruence equation relations for \(f(x)\) and \(x\) to a single one. Similarly by the properties of a linear order, one can assume that there is only one equation of each form \(R(x, e)\), \(R(e', x)\), \(R(f(x), g)\) and \(R(g', f(x))\) in the system. Now if follows from axiom-scheme \((T6)\) that the system has a solution in \(M_2\). This is because the quantifier-free formula in the mentioned axiom is satisfied in \(M\), and this is because the system clearly possesses a solution, that is \(a\), in \(M_1\).

**Corollary 18** The theory \(T\) is complete and hence equivalent to \(\text{Th}(\mathbb{Z}, +, f, R, \{p_n\}_{n \in \mathbb{N}}, 0, 1)\).

**Proof** Quantifier-elimination implies that \(T\) is model-complete. Also \((\mathbb{Z}, +, f, R, \{p_n\}_{n \in \mathbb{N}}, 0, 1)\) is a prime model of \(T\), and the claim follows.

**Corollary 19** The theory \(T\) is decidable.

**Proof** This follows from the fact that \(T\) is complete and recursively enumerable. Note that to write \((T6)\) recursively, one only needs an algorithm to list all systems of equations and express when they are solvable. Expressing when a system is solvable only involves mentioning (using the power of the language \(L^*\) as in Lemma 12) possible ways certain intervals have intersection. It is important to note that the algorithm is not needed to “solve” the system, but only express when it is solvable using finitely many conditions.

\(\square\)
Remarks

1. In our earlier versions we claimed that the structure \((\mathbb{Z}, +, <, f, \{p_n\}_{n \in \mathbb{N}}, 0, 1)\) eliminates quantifiers, but with the help of the referee’s comments we found out that the proof was flawed. Adjusting our proof in the presence of the order of integers is not as straightforward as we first thought, and we leave this as the following question (whose answer we believe is positive).

   **Question.** Does the structure \((\mathbb{Z}, +, <, f, R, \{p_n\}_{n \in \mathbb{N}}, 0, 1)\) eliminate quantifiers?

2. We do not know if \(R(x, y)\) is equivalent to a quantifier-free formula in \(L\), but adding the binary function subtraction to the language, our quantifier-elimination result can be enhanced as below:

   **Observation** (by A. Valizadeh). The relation \(R(x, y)\) is definable by the formula \(f(y - x) = f(y) - f(x)\). So (the proofs in this paper lead to the fact that) the structure \((\mathbb{Z}, +, -, f, \{p_n\}_{n \in \mathbb{N}}, 0, 1)\) admits elimination of quantifiers.

3. The proofs provided in the last version of this paper, are inspired by [11], where a much more difficult situation is dealt with in a similar manner. We decided to adopt the same technology here, as it made the proofs neater compared to our original proof.

4. The formula \(R(x, y)\) suggests that the structure \((\mathbb{Z}, +, f, 0, 1)\) has the so-called “order property”, which determines its place in terms of the model-theoretic classification of theories. It is reasonable to ask whether this structure is NIP too.

5. We think that \(\varphi\) can be replaced by any algebraic number, and the proofs will be essentially similar. But this needs to be checked. As mentioned, in [11] a much more general case is treated.

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