

Research Article

The Product-Type Operators from Hardy Spaces into n
th Weighted-Type Spaces

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The main goal of this paper is to investigate the boundedness and essential norm of a class of product-type operators \( T^m_{\omega,p} \), \( m \in \mathbb{N} \) from Hardy spaces into \( n \)th weighted-type spaces. As a corollary, we obtain some equivalent conditions for compactness of such operators.

1. Introduction

Let \( \mathbb{D} \) denote the open unit disc of the complex plane \( \mathbb{C} \) and \( H(\mathbb{D}) \) denotes the space of all holomorphic functions on \( \mathbb{D} \). The space of bounded holomorphic functions on \( \mathbb{D} \) is denoted by \( H^\infty \); it is a Banach space with the equipped norm

\[
\|g\|_{H^\infty} = \sup_{z \in \mathbb{D}} |g(z)|. \tag{1}
\]

Let \( 0 < p < \infty \). A Hardy space \( H^p \) consists of all \( g \in H(\mathbb{D}) \) such that

\[
\|g\|_{H^p} = \sup_{0 < r < 1} \left( \frac{1}{2\pi} \int_0^{2\pi} \left| g(re^{i\theta}) \right|^p d\theta \right)^{1/p} < \infty. \tag{2}
\]

When \( 1 \leq p < \infty \), \( H^p \) is a Banach space with the norm \( \|\cdot\|_{H^p} \). If \( 0 < p < 1 \), \( H^p \) is a nonlocally convex topological vector space and it is a complete metric space (see [1]).

Let \( n \in \mathbb{N}_0 = \{0, 1, 2, \cdots\} \) and \( \mu(z) \) be a weight, continuous, and positive function on \( \mathbb{D} \). The \( n \)th weighted-type space \( \mathcal{W}_\mu^{(n)} \), consists of all \( g \in H(\mathbb{D}) \) such that

\[
b_{\mathcal{W}_\mu^{(n)}}(g) = \sup_{z \in \mathbb{D}} |\mu(z)| \left| g^{(n)}(z) \right| < \infty. \tag{3}
\]

It is a Banach space with the following norm

\[
\|g\|_{\mathcal{W}_\mu^{(n)}} = \sum_{j=0}^{n-1} a_j^{(n)}(g) + b_{\mathcal{W}_\mu^{(n)}}(g). \tag{4}
\]

The little \( n \)th weighted-type space \( \mathcal{W}_\mu^{(n)} \) is a closed subspace of \( \mathcal{W}_\mu^{(n)} \) such that for any \( g \in \mathcal{W}_\mu^{(n)} \)

\[
\lim_{|z| \to 1} \mu(z) \left| g^{(n)}(z) \right| = 0. \tag{5}
\]

For more information about \( n \)th weighted-type spaces, see [2–4]. Let \( \alpha > 0 \). Then, \( H^\alpha = H^{-\alpha}(\text{growth space}) \), \( \mathcal{W}_\mu^{(1)}(1-|z|^2)^\alpha = \mathcal{B}^\alpha(\text{Bloch-type space}) \), and \( \mathcal{W}_\mu^{(2)}(1-|z|^2)^\alpha = \mathcal{L}^\alpha(\text{Zygmund-type space}) \). Also \( H^\alpha = H^\alpha(\text{weighted-type space}) \), \( \mathcal{W}_\mu^{(1)} = \mathcal{B}(\text{weighted Bloch space}) \), \( \mathcal{W}_\mu^{(2)} = \mathcal{L}(\text{weighted Zygmund space}) \), and \( \mathcal{W}_\mu^{(1)}(1-|z|^2)\log(2/(1-|z|^2)) \) coincide with the logarithmic Bloch space \( \mathcal{B}_{\log} \).

Let \( n, k \in \mathbb{N}_0 \) such that \( k \leq n \); the partial Bell polynomials are triangular

\[
\sum_{j=1}^n \frac{1}{j! \cdots j_{k+1}!} (x_{j_1} \cdots x_{j_{k+1}})_{g^{(n-k+1)}(z)}^{\sum_{j=1}^n \frac{1}{j! \cdots j_{k+1}!} (x_{j_1} \cdots x_{j_{k+1}})_{g^{(n-k+1)}(z)}} \tag{6}
\]
where the sum is taken over all nonnegative integers \( j_1, \ldots, j_{n-k+1} \) such that

\[
j_1 + \cdots + (n - k + 1)j_{n-k+1} = n, \\
j_1 + \cdots + j_{n-k+1} = k.
\]

(7)

More information about Bell polynomials can be found in ([5], p 134).

Let \( m \in \mathbb{N}_0, u, v \in H(D) \) and \( \varphi \in S(D) \) be the set of all holomorphic self-map of \( D \). In [6], Stević', Sharma and Krishan defined a new product-type operator \( T^m_{u,v,\varphi} \) as follows:

\[
T^m_{u,v,\varphi}g(z) = u(z)g^{(m)}(\varphi(z)) + v(z)g^{(m+1)}(\varphi(z)), \quad g \in H(D), z \in D.
\]

(8)

When \( m = 0 \), we obtain the Stević'-Sharma-type operator, and for \( v \equiv 0 \), we get the generalized weighted composition operators \( D^m_{u,\varphi} \). Product-type operators on some spaces of analytic functions on the unit disc have become a subject of increasing interest in the recent years. We refer the reader to [6–10] and the references therein.

Liu and Yu have considered boundedness and compactness of operator \( T^0_{u,v,\varphi} \) from Hardy spaces and \( H^\infty \) into the logarithmic Bloch space in [11, 12]. Also, Zhang and Liu have found some characterizations for boundedness and compactness of operator \( T^0_{u,v,\varphi} \) from Hardy spaces into the weighted Zygmund space in [10]. Recently, the boundedness, compactness, and norm of operator \( T^m_{u,v,\varphi} : H^p \to \mathcal{H}^n_{\mu} \) are considered in [13].

Motivated by previous works, the results found in them will be generalized for operator \( T^m_{u,v,\varphi} \). For this purpose in the second section of this paper, we give some characterizations for boundedness of \( T^m_{u,v,\varphi} \) and \( H^p \to \mathcal{H}^n_{\mu} \) where \( m, n \in \mathbb{N}_0 \) and \( 0 < p \leq \infty \). In the third section, some new estimates are obtained for the essential norm of such operators. As a corollary, some equivalent conditions are acquired for compactness of such operators.

Throughout this paper, if there exists a constant \( c \) such that \( a \geq cb \), we use the notation \( a \geq b \). The symbol \( a \approx b \) means that \( a \geq b \geq a \).

2. Boundedness

In this section, some equivalent conditions are found for the boundedness of operator \( T^m_{u,v,\varphi} \) from \( H^p(0 < p \leq \infty) \) into \( n \)th weighted-type spaces. Firstly, we state some lemmas.

**Lemma 2** (see [15], Lemma 2.1). Let \( \alpha > 0 \) and \( \mathcal{B}^n_{\alpha} = \mathcal{H}^n_{\alpha} \). The sequence \( \{ f^{-1}(z) \}_{i=0}^\infty \) is bounded in \( \mathcal{B}^n_{\alpha} \) and

\[
\lim_{j \to \infty} \frac{f^j(z)}{j!} = 0.
\]

From Lemma 1, Proposition 5.1.2 [16] and [1], the next lemma is obtained.

**Lemma 3**. Let \( 0 < p \leq \infty, n \in \mathbb{N}_0 \) and \( g \in H^p \). Then,

\[
\left\| g^{(n)}(z) \right\|_{H^p} \leq \frac{\|g\|_{H^p}}{(1 - |z|^2)^{(1/p)n}}, \quad z \in D.
\]

(11)

**Lemma 4** (see [4]). Let \( \varphi \in S(D) \) and \( u, v \in H(D) \). Then, for any \( m, n \in \mathbb{N}_0 \)

\[
\left( u(z)g^{(m)}(\varphi(z)) \right)^{(n)} = \sum_{i=0}^{\infty} \binom{n}{i} u^{(i)}(z)B_{i} \left( \varphi^{(i)}(z), \varphi^{(i+1)}(z), \ldots, \varphi^{(i+n)}(z) \right).
\]

(12)

In this paper, we set

\[
f_{i,a}(z) = \frac{1 - |a|^2}{1 - a \bar{z}} z^i, \quad 0 \neq a, z \in D, i \in \mathbb{N},
\]

\[
P_{i,a}(z) = \sum_{l=1}^{n} \binom{n}{l} u^{(l)}(z)B_{i} \left( \varphi^{(l)}(z), \varphi^{(l+1)}(z), \ldots, \varphi^{(l+n)}(z) \right),
\]

\[
P_{m,a}(z) = T_{m,a}(z) = 0, \quad p_{i}(z) = z^i.
\]

(13)

By using the functions \( f_{i,a} \), we obtain the following lemma. Since the proof of it resembles the proof of Lemma 1 [2], therefore, it is omitted.

**Lemma 5**. Let \( \delta_{ak} \) be Kronecker delta. For any \( 0 \neq a \in D, m \in \mathbb{N}_0 \) and \( i \in \{0, \ldots, n+1\} \), there exists a function \( g_{i,a} \) in \( H^p \) such that

\[
g_{i,a}^{(m+k)}(a) = \frac{\delta_{ak}a^{m+k}}{(1 - |a|^2)^{(m+k)(1/p)}}.
\]

(14)

In this case, \( g_{i,a}(z) = \sum_{j=0}^{n+1} c^j f_{j,a}(z) \), where \( c^j \) are independent of choice \( a \).

**Theorem 6**. Let \( m, n \in \mathbb{N}_0, 0 < p \leq \infty, u, v \in H(D), \mu \) be a weight and \( \varphi \in S(D) \). Then, the following statements are equivalent:

(a) The operator \( T^m_{u,v,\varphi} : H^p \to \mathcal{H}^n_{\mu} \) is bounded
(b) The operator $T_{u,v}^m : \mathcal{B}^{1+1/(p)} \to \mathcal{W}^m$ is bounded.

(c) The operator $T_{u,v}^m : \mathcal{B}_0^{1+1/(p)} \to \mathcal{W}^m$ is bounded.

(d) sup$_{j \geq 1} j^{1/p} \|T_{u,v}^m p_j\|_{\mathcal{W}^m} < \infty$

(e) For each $i \in \{0, 1, \ldots, n + 1\}$, sup$_{z \in \Omega} \|T_{u,v}^m f_{i+1,a} \|_{\mathcal{W}^m} < \infty$ and sup$_{z \in \Omega} \|\phi(z) I_{n_u, v}^m(z) + \sum_{k=1}^{j} \frac{(j + m)!}{(j - k)!} \phi(z)^{j-k}(I_{n_u, v}^m + I_{n_v, v}^m) (z)\|_{\mathcal{W}^m} < \infty$.

(f) For each $i \in \{0, 1, \ldots, n + 1\}$,

$$\sup_{\Omega} \|\mu(z) \left(I_{n_u, v}^m + I_{n_v, v}^m(z)\right)\|_{\mathcal{W}^m} < \infty$$

Theorem 2. Since $\mathcal{B}_0^{1+1/(p)} \subset \mathcal{B}^{1+1/(p)}$, we get (c).

Proof. $(b) \implies (c)$ follows from Lemma 2.

$(c) \implies (e)$ For each $i \in \{0, 1, a \in \mathbb{D}\}$, $f_{i+1,a}(z) = (1 - |a|^2)^{i+1} \sum_{j=0}^{\infty} \frac{\Gamma(i + 1 + (1/p) + j)}{j! \Gamma(i + 1 + (1/p))} a^j$. (16)

So,

$$\|T_{u,v}^m f_{i+1,a}\|_{\mathcal{W}^m} \leq \sup_{j \geq 1} j^{1/p} \|T_{u,v}^m p_j\|_{\mathcal{W}^m} (1 - |a|^2)^{i+1} \sum_{j=0}^{\infty} \frac{\Gamma(i + 1 + (1/p) + j)}{j! \Gamma(i + 1 + (1/p))} |a|^j.$$

$$\leq \sup_{j \geq 1} j^{1/p} \|T_{u,v}^m p_j\|_{\mathcal{W}^m} (1 - |a|^2)^{i+1} \sum_{j=0}^{\infty} \frac{\Gamma(i + 1 + j)}{j! \Gamma(i + 1)} |a|^j.$$

$$\leq 2^{i+1} \sup_{j \geq 1} j^{1/p} \|T_{u,v}^m p_j\|_{\mathcal{W}^m}.$$

Hence, sup$_{z \in \Omega} \|T_{u,v}^m f_{i+1,a}\|_{\mathcal{W}^m} < \infty$. It remains to show that for each $i \in \{0, 1, \ldots, n + 1\}$, sup$_{z \in \Omega} \mu(z) I_{n_u, v}^m(z) + \sum_{k=1}^{j} \frac{(j + m)!}{(j - k)!} \phi(z)^{j-k}(I_{n_u, v}^m + I_{n_v, v}^m) (z)\|_{\mathcal{W}^m} < \infty$. Applying the operator $T_{u,v}^m$ to $\mu(z)$, by using Lemma 4, we have

$$\sup_{z \in \Omega} \mu(z) I_{n_u, v}^m(z) + \sum_{k=1}^{j} \frac{(j + m)!}{(j - k)!} \phi(z)^{j-k}(I_{n_u, v}^m + I_{n_v, v}^m) (z)\|_{\mathcal{W}^m} < \infty.$$ (18)

Now, assume that we have the following inequalities for $0 \leq i \leq j - 1$,

$$\sup_{z \in \Omega} \mu(z) \left(I_{n_u, v}^m + I_{n_v, v}^m(z)\right) < \infty,$$ (19)

where $j \leq n + 1$. By applying the operator $T_{u,v}^m$ for $p_{j+m}(z)$ and using Lemma 4, we get

$$\sup_{z \in \Omega} \mu(z) \left(I_{n_u, v}^m + I_{n_v, v}^m(z)\right) < \infty.$$

Also for each $0 \leq k < n$, we have

$$\left|\left(T_{u,v}^m f\right)^{(k)}(0)\right| \leq \|f\|_{\mathcal{B}^{1+1/(p)}} \sum_{i=0}^{n=k} \frac{\Gamma(n+1)}{\Gamma(n+1-k)} |z|^{n+1-k}.$$ (26)

So, the operator $T_{u,v}^m : \mathcal{B}^{1+1/(p)} \to \mathcal{W}^m$ is bounded.
(b) \implies (a) From Lemma 3, \(H^p \subset \mathcal{B}^{1+(1/p)}\), so we obtain (a).

(a) \implies (c) It is clear that \(f_{i,a} \in H^p\) and \(\sup_{a \in D} \|f_{i,a}\|_{H^p} < \infty\). Hence, for each \(i \in \{0, \ldots, n + 1\}\),

\[
\sup_{a \in D} \left\| T^{m}_{u,v,p} f_{i+1,a} \right\|_{\mathcal{Y}^{(n)}_{\mu}} \leq \left\| T^{m}_{u,v,p} \right\|_{H^p \to \mathcal{Y}^{(n)}_{\mu}} \sup_{a \in D} \|f_{i+1,a}\|_{H^p} < \infty.
\]

(27)

The proof of the second part of (e) is similar to the proof (d) \implies (e), so it is omitted. The proof is complete. \(\square\)

### 3. Essential Norm

In this section, we find some approximations for the essential norm of operator \(T^{m}_{u,v,p}\) from Hardy spaces into \(n\)th weighted type-spaces. As a corollary, we give some equivalent conditions for compactness of such operators.

Let \(X\) and \(Y\) be Banach spaces and \(T : X \to Y\) be the continuous linear operator. The essential norm of \(T\) is the distance from \(T\) to the compact operators, that is,

\[
\|T\|_{e,X \to Y} = \inf \{ \|T - K\| : K : X \to Y \text{ is compact} \}.
\]

(28)

It is clear that \(T\) is compact if and only if \(\|T\|_{e,X \to Y} = 0\).

**Theorem 7.** Let \(m, n \in \mathbb{N}, 0 < p \leq \infty, u, v \in H(D), \varphi \in S(D),\) and \(\mu\) be a weight such that \(T^{m}_{u,v,p} : \mathcal{B}^{1+(1/p)} \to \mathcal{Y}^{(n)}_{\mu}\) is bounded. Then

\[
\left\| T^{m}_{u,v,p} \right\|_{e,\mathcal{B}^{1+(1/p)} \to \mathcal{Y}^{(n)}_{\mu}} \approx \max \left\{ A_i \right\}_{i=0}^{n+1} \approx \max \left\{ B_j \right\}_{j=0}^{n+1},
\]

(29)

where

\[
A_i = \limsup_{|a| \to 1} \left\| T^{m}_{u,v,p} f_{i+1,a} \right\|_{\mathcal{Y}^{(n)}_{\mu}},
\]

\[
B_j = \frac{\mu(z)}{|\phi(z)|} \left( \left| f_{i+1,a}^{(m)} + f_{i-1,a}^{(m)} \right| (z) \right).
\]

(30)

**Proof.** For each \(i \in \{0, \ldots, n + 1\}\), \(\sup_{a \in D} \|f_{i+1,a}\|_{\mathcal{B}^{1+(1/p)}} < \infty\) and \(f_{i+1,a} \to 0\) uniformly on compact subsets of \(D\) as \(|a| \to 1\). Applying Lemma 2.10 from [17], for any compact operator \(K\) from \(\mathcal{B}^{1+(1/p)}\) into \(\mathcal{Y}^{(n)}_{\mu}\), we have

\[
\lim_{|a| \to 1} \left\| K f_{i+1,a} \right\|_{\mathcal{Y}^{(n)}_{\mu}} = 0.
\]

(31)

Hence, for any \(i \in \{0, \ldots, n + 1\}\),

\[
A_i = \limsup_{|a| \to 1} \left\| T^{m}_{u,v,p} f_{i+1,a} \right\|_{\mathcal{Y}^{(n)}_{\mu}} - \lim_{|a| \to 1} \left\| K f_{i+1,a} \right\|_{\mathcal{Y}^{(n)}_{\mu}} \leq \limsup_{|a| \to 1} \left\| \left( T^{m}_{u,v,p} - K \right) f_{i+1,a} \left\|_{\mathcal{Y}^{(n)}_{\mu}} \right. \right. \] 

\[
\leq \left\| T^{m}_{u,v,p} - K \right\|_{\mathcal{B}^{1+(1/p)} \to \mathcal{Y}^{(n)}_{\mu}}.
\]

(32)

So,

\[
\max \left\{ A_i \right\}_{i=0}^{n+1} \leq \inf \left\| T^{m}_{u,v,p} - K \right\|_{\mathcal{B}^{1+(1/p)} \to \mathcal{Y}^{(n)}_{\mu}} = \inf \left\| T^{m}_{u,v,p} \right\|_{\mathcal{B}^{1+(1/p)} \to \mathcal{Y}^{(n)}_{\mu}}.
\]

(33)

Now, we prove that

\[
\max \left\{ B_j \right\}_{j=0}^{n+1} \leq \inf \left\| T^{m}_{u,v,p} \right\|_{e,\mathcal{B}^{1+(1/p)} \to \mathcal{Y}^{(n)}_{\mu}}.
\]

(34)

Let \(\{z_j\}_{j=2}^{\infty}\) be a sequence in \(D\) such that \(\lim_{j \to \infty} |\varphi(z_j)| = 1\). Since \(T^{m}_{u,v,p} : \mathcal{B}^{1+(1/p)} \to \mathcal{Y}^{(n)}_{\mu}\) is bounded, by using Lemmas 4 and 5 for any compact operator \(K : \mathcal{B}^{1+(1/p)} \to \mathcal{Y}^{(n)}_{\mu}\) and \(i \in \{0, \ldots, n + 1\}\), we obtain

\[
\left\| \left( T^{m}_{u,v,p} - K \right) \left( \frac{\mu(z)}{|\phi(z)|} \left( |f_{i+1,a}^{(m)} + f_{i-1,a}^{(m)}| (z) \right) \right. \right. \]

\[
\left\| \left( T^{m}_{u,v,p} - K \right) \right\|_{\mathcal{B}^{1+(1/p)} \to \mathcal{Y}^{(n)}_{\mu}} \leq \limsup_{j \to \infty} \left| \left| \varphi(z_j) \right| \frac{1}{1 - |\varphi(z_j)|^2} \left( n_{u,a}^{(1)} + n_{v,a}^{(1)} \right) \right| \left( T^{m}_{u,v,p} - K \right) \left| \varphi(z_j) \right| \frac{1}{1 - |\varphi(z_j)|^2} \left( n_{u,a}^{(1)} + n_{v,a}^{(1)} \right) \left( z_j \right).
\]

(35)

So, from the definition of the essential norm, we get (34). For \(r \in [0,1]\), we define \(K_r f(z) = f(rz)\). It is apparent that \(K_r\) is a compact operator on \(\mathcal{B}^{1+(1/p)}\). Let \(\{r_j\}_{j=0}^{\infty}\) be a sequence such that \(r_j \to 1\) as \(j \to \infty\). Since \(f \to f\) uniformly on compact subsets of \(D\) as \(r \to 1\), then, for any positive integer \(j\), the operator \(T^{m}_{u,v,p} K_{r_j} : \mathcal{B}^{1+(1/p)} \to \mathcal{Y}^{(n)}_{\mu}\) is compact. Based on the definition of the essential norm, we obtain

\[
\left\| T^{m}_{u,v,p} \right\|_{e,\mathcal{B}^{1+(1/p)} \to \mathcal{Y}^{(n)}_{\mu}} \leq \limsup_{j \to \infty} \left\| T^{m}_{u,v,p} - T^{m}_{u,v,p} K_{r_j} \right\|_{\mathcal{B}^{1+(1/p)} \to \mathcal{Y}^{(n)}_{\mu}}.
\]

(36)

So, it is sufficient to show that

\[
\limsup_{j \to \infty} \left\| T^{m}_{u,v,p} - T^{m}_{u,v,p} K_{r_j} \right\| \leq \max \left\{ A_i \right\}_{i=0}^{n+1} \max \left\{ B_j \right\}_{j=0}^{n+1}.
\]

(37)

Let \(f \in \mathcal{B}^{1+(1/p)}\) such that \(\|f\|_{\mathcal{B}^{1+(1/p)}} \leq 1\) and for all \(j \geq N, r_j \geq (3/4)\), therefore,
uniformly, hence, from Theorem 6, we obtain

\[ \limsup_{j \to \infty} M_k \leq \sum_{i=0}^{n+1} \left( T^m_{u,v,\varrho} f_{i+1,a} \right)_{\mathfrak{E}_{\mu}^{(n)}} \leq \max \{ A_i \}_{i=0}^{n+1} , \]

likewise, we have

\[ \limsup_{j \to \infty} N_k \leq \sum_{i=0}^{n+1} \left( T^m_{u,v,\varrho} f_{i+1,a} \right)_{\mathfrak{E}_{\mu}^{(n)}} \leq \max \{ B_i \}_{i=0}^{n+1} . \]

Thus, by using (38), (39), (40), (42) and (43), we obtain

\[ \limsup_{j \to \infty} \left\| T^m_{u,v,\varrho} - T^m_{w,v,\varrho} K_r \right\|_{\mathfrak{E}_{\mu}^{(n+1)} \to \mathfrak{E}_{\mu}^{(n)}} = \limsup_{j \to \infty} \sup \left\{ \left\| T^m_{u,v,\varrho} - T^m_{w,v,\varrho} K_r \right\|_{\mathfrak{E}_{\mu}^{(n+1)} \to \mathfrak{E}_{\mu}^{(n)}} : \max \{ A_i \}_{i=0}^{n+1} \right\} . \]

Hence, from (36),

\[ \left\| T^m_{u,v,\varrho} \right\|_{\mathfrak{E}_{\mu}^{(n+1)} \to \mathfrak{E}_{\mu}^{(n)}} \leq \min \left\{ \max \{ A_i \}_{i=0}^{n+1} , \max \{ B_i \}_{i=0}^{n+1} \right\} . \]

Consequently,

\[ \left\| T^m_{u,v,\varrho} \right\|_{\mathfrak{E}_{\mu}^{(n+1)} \to \mathfrak{E}_{\mu}^{(n)}} \approx \max \{ A_i \}_{i=0}^{n+1} = \max \{ B_i \}_{i=0}^{n+1} . \]

The proof is complete. \( \square \)

**Theorem 8.** Let \( m, n \in \mathbb{N}, 0 < p \leq \infty, u, v \in H(D), \varrho \in S(D), \mu \) be a weight. If \( T^m_{u,v,\varrho} : H^p \to \mathfrak{E}_{\mu}^{(n)} \) be bounded then

\[ \left\| T^m_{u,v,\varrho} \right\|_{c,H^p \to \mathfrak{E}_{\mu}^{(n)}} \approx \left\| T^m_{u,v,\varrho} \right\|_{c,\mathfrak{E}_{\mu}^{(n+1)} \to \mathfrak{E}_{\mu}^{(n)}} . \]

**Proof.** It is evident that

\[ \left\| T^m_{u,v,\varrho} \right\|_{c,H^p \to \mathfrak{E}_{\mu}^{(n)}} \approx \left\| T^m_{u,v,\varrho} \right\|_{c,\mathfrak{E}_{\mu}^{(n+1)} \to \mathfrak{E}_{\mu}^{(n)}} . \]

On the other hand, since \( f_{i,a}(z) = (1 - |a|^2)^i / (1 - \bar{a}z)^{(i+1)/p} \in H^p \), for any compact operator \( K : H^p \to \mathfrak{E}_{\mu}^{(n)} \), from Lemma 2.10 in [17], for any \( i \in \{0, \cdots, n+1\} \), we get
\begin{align*}
&\|T_{u,v,p}^m - K\|_{eH^p \to \mathscr{W}^\mu_p} \geq \limsup_{[a] \to 1} \left( R_{T_{u,v,p}^m - K} \|_{eH^p} \right)_{\mathscr{W}^\mu_p} \\
&\geq \limsup_{[a] \to 1} \left( R_{T_{u,v,p}^m f_{i+1,a}} \right)_{\mathscr{W}^\mu_p} - \lim_{[a] \to 1} \|Kf_{i+1,a}\|_{\mathscr{W}^\mu_p} = A_i.
\end{align*}

(49)

So, from the last inequality and Theorem 7

\begin{align*}
\|T_{u,v,p}^m\|_{eH^p \to \mathscr{W}^\mu_p} \geq \max \{A_i\}_{i=0}^{n+1} = \|T_{u,v,p}^m\|_{eH^{1+1}(p) \to \mathscr{W}^\mu_p}.
\end{align*}

(50)

The proof is complete.

\textbf{Theorem 9.} Let \(m, n \in \mathbb{N}, 0 < p \leq \infty, u, v \in H(D), \mu \) be a weight and \(\varphi \in S(D)\) such that \(T_{u,v,p}^m : \mathscr{B}^{1+1}(p) \to \mathscr{W}^\mu(p)\) be bounded. Then,

\begin{align*}
\limsup_{j \to \infty} \|T_{u,v,p}^m f_j\|_{\mathscr{W}^\mu_p} = \|T_{u,v,p}^m\|_{eH^{1+1}(p) \to \mathscr{W}^\mu_p}.
\end{align*}

(51)

Proof. Let \(j\) be any positive integer and \(h_j(z) = f^{1/p}_j(z)\). It is clear that \(\|h_j\|_{\mathscr{B}^{1+1}(p)} = 1, h_j \in \mathscr{B}^{1+1}(p)\) and \(h_j \to 0\) for all \(j \in \mathbb{N}\), converging uniformly on compact subsets of \(D\). By using Lemma 2.10 in [17], for any compact operator \(K\) from \(\mathscr{B}_0^{1+1}(p)\) to \(\mathscr{W}^\mu\), we get

\begin{align*}
\lim_{j \to \infty} \|Kh_j\|_{\mathscr{W}^\mu_p} = 0.
\end{align*}

(52)

Hence,

\begin{align*}
\|T_{u,v,p}^m - K\|_{\mathscr{B}^{1+1}(p) \to \mathscr{W}^\mu_p} \geq \limsup_{j \to \infty} \left( T_{u,v,p}^m - K \right) h_j = \|T_{u,v,p}^m - K\|_{\mathscr{W}^\mu_p} \\
\geq \limsup_{j \to \infty} \left( T_{u,v,p}^m h_j \right) - \limsup_{j \to \infty} \|Kh_j\|_{\mathscr{W}^\mu_p} = \limsup_{j \to \infty} f^{1/p}_j \|T_{u,v,p}^m f_j\|_{\mathscr{W}^\mu_p}.
\end{align*}

(53)

So,

\begin{align*}
\|T_{u,v,p}^m\|_{\mathscr{B}^{1+1}(p) \to \mathscr{W}^\mu_p} \geq \limsup_{j \to \infty} f^{1/p}_j \|T_{u,v,p}^m f_j\|_{\mathscr{W}^\mu_p}.
\end{align*}

Now, we prove that

\begin{align*}
\limsup_{j \to \infty} f^{1/p}_j \|T_{u,v,p}^m f_j\|_{\mathscr{W}^\mu_p} \geq \|T_{u,v,p}^m\|_{\mathscr{B}^{1+1}(p) \to \mathscr{W}^\mu_p}.
\end{align*}

(54)

From Theorem 6, for any fixed positive integer \(k \geq m\) and \(0 \leq i \leq n + 1\), we have

\begin{align*}
\|T_{u,v,p}^m f_{i+1,a}\|_{\mathscr{W}^\mu_p} \leq C_{i+1} (1 - |a|)^{i+1} \sup_{j \in \mathbb{N}} \|T_{u,v,p}^m f_{j}^1\|_{\mathscr{W}^\mu_p}.
\end{align*}

(55)

where \(Q = \sup_{j \in \mathbb{N}} f^{1/p}_j \|T_{u,v,p}^m f_j\|_{\mathscr{W}^\mu_p}\). Letting \(|a| \to 1\), we obtain

\begin{align*}
A_{i+1} = \limsup_{[a] \to 1} \|T_{u,v,p}^m f_{i+1,a}\|_{\mathscr{W}^\mu_p} \leq \sup_{j \in \mathbb{N}} f^{1/p}_j \|T_{u,v,p}^m f_{j}^1\|_{\mathscr{W}^\mu_p}.
\end{align*}

(56)

Applying Theorem 7, we get

\begin{align*}
\|T_{u,v,p}^m\|_{\mathscr{B}^{1+1}(p) \to \mathscr{W}^\mu_p} \geq \max \{A_i\}_{i=0}^{n+1} = \|T_{u,v,p}^m\|_{\mathscr{B}^{1+1}(p) \to \mathscr{W}^\mu_p}.
\end{align*}

(57)

It is clear that \(\|T_{u,v,p}^m\|_{\mathscr{B}^{1+1}(p) \to \mathscr{W}^\mu_p} \leq \|T_{u,v,p}^m\|_{\mathscr{B}^{1+1}(p) \to \mathscr{W}^\mu_p} \to \mathscr{W}^\mu_p\); so, from the last inequalities, we have

\begin{align*}
\limsup_{j \to \infty} f^{1/p}_j \|T_{u,v,p}^m f_j\|_{\mathscr{W}^\mu_p} \approx \|T_{u,v,p}^m\|_{\mathscr{B}^{1+1}(p) \to \mathscr{W}^\mu_p}.
\end{align*}

(58)

The proof is complete.

\textbf{4. Some Applications}

For \(0 < p < \infty\), by using Lemma 3, we have \(H^p \subset \mathscr{B}_0^{1+1}(p)\). Also, for \(p = \infty\), \(H^\infty \not\subset \mathscr{B}_0^{1+1}(p)\) and \(H^\infty \cap \mathscr{B}_0^{1+1}(p)\) are a Banach space with the norm \(\|\cdot\|_{H^\infty}\). In this case, we get the following corollary.

\textbf{Corollary 10.} Let \(m, n \in \mathbb{N}, u, v \in H(D), \mu \) be a weight and \(\varphi \in S(D)\). The operator \(T_{u,v,p}^m : H^\infty \to \mathscr{W}^\mu(\mathbb{N})\) is bounded if and only if the operator \(T_{u,v,p}^m : H^\infty \cap \mathscr{B}_0^{1+1}(p) \to \mathscr{W}^\mu(\mathbb{N})\) be bounded.

\textbf{Corollary 11.} Let \(m, n \in \mathbb{N}, u, v \in H(D), \varphi \in S(D), \mu \) be a weight. If \(T_{u,v,p}^m : H^\infty \to \mathscr{W}^\mu(\mathbb{N})\) be bounded, then,

\begin{align*}
\|T_{u,v,p}^m\|_{eH^\infty \to \mathscr{W}^\mu(\mathbb{N})} \approx \|T_{u,v,p}^m\|_{eH^\infty \cap \mathscr{B}_0^{1+1}(p) \to \mathscr{W}^\mu(\mathbb{N})}.
\end{align*}

(59)
Proof. It is clear that \( \|T_{u,v,p}^m\|_{c,H^\infty \cap \mathcal{B}_0 \rightarrow \mathcal{W}_\mu^{(n)}}^\infty \)
\( \leq \|T_{u,v,p}^m\|_{c,H^\infty \cap \mathcal{B}_0 \rightarrow \mathcal{W}_\mu^{(n)}}^\infty \) and \( f_{i,u}(z) = ((1 - |a|)^2,1 - |a\tilde{z}|) \in H^\infty \)
\( \cap \mathcal{B}_0 \). So, for any compact operator \( K : H^\infty \cap \mathcal{B}_0 \rightarrow \mathcal{W}_\mu^{(n)} \), from Lemma 2.10 in [17], for any \( i \in \{0, \ldots, n+1\} \), we obtain
\[
\|T_{u,v,p}^m\|_{c,H^\infty \cap \mathcal{B}_0 \rightarrow \mathcal{W}_\mu^{(n)}}^\infty \geq \limsup_{|i| \rightarrow 1} \left( T_{u,v,p}^m - K \right)_{f_i+1,a} \mathcal{W}_\mu^{(n)} \]
\[
\geq \limsup_{|i| \rightarrow 1} \|T_{u,v,p}^m f_{i+1,a}\|_{\mathcal{W}_\mu^{(n)}} - \lim_{|i| \rightarrow 1} \|K f_{i+1,a}\|_{\mathcal{W}_\mu^{(n)}} = A_i.
\]
(60)

Hence, from the last inequality and Theorem 7,
\[
\|T_{u,v,p}^m\|_{c,H^\infty \cap \mathcal{B}_0 \rightarrow \mathcal{W}_\mu^{(n)}}^\infty \geq \max \{A_i\}_{i=0}^{n+1} = \|T_{u,v,p}^m\|_{c,\mathcal{B}_0^{(n)} \rightarrow \mathcal{W}_\mu^{(n)}}^\infty.
\]
(61)

The proof is complete. \( \square \)

From Theorems 7, 8 and 9 and Corollary 11, the next corollaries are obtained.

Corollary 12. Let \( m, n \in \mathbb{N} \), \( 0 < p < \infty \), \( u, v \in H(D) \), \( \phi \in S(D) \), and \( \mu \) be a weight such that \( T_{u,v,p}^m : H^p \rightarrow \mathcal{W}_\mu^{(n)} \) is bounded. Then, the following statements are equivalent:

(a) The operator \( T_{u,v,p}^m : H^p \rightarrow \mathcal{W}_\mu^{(n)} \) is compact
(b) The operator \( T_{u,v,p}^m : \mathcal{B}^{1+i/p} \rightarrow \mathcal{W}_\mu^{(n)} \) is compact
(c) The operator \( T_{u,v,p}^m : \mathcal{B}^{1+i/p} \rightarrow \mathcal{W}_\mu^{(n)} \) is compact
(d) \( \lim_{i \rightarrow \infty} \|T_{u,v,p}^m\|_{c,\mathcal{B}^{1+i/p}} = 0 \)
(e) For each \( i \in \{0, \ldots, n+1\} \), \( \limsup_{|i| \rightarrow 1} \|T_{u,v,p}^m f_{i+1,a}\|_{\mathcal{W}_\mu^{(n)}} = 0 \)
(f) For each \( i \in \{0, \ldots, n+1\} \),
\[
\limsup_{|\phi(z)| \rightarrow 1} \frac{\mu(z) \|I_{\phi}^m + I_{\phi}^n\|_{\mathcal{W}_\mu^{(n)}}}{(1 - |\phi(z)|^2)^{m+i+1/p}} = 0
\]
(62)

Corollary 13. Let \( m, n \in \mathbb{N} \), \( u, v \in H(D) \), \( \phi \in S(D) \), and \( \mu \) be a weight such that \( T_{u,v,p}^m : H^\infty \rightarrow \mathcal{W}_\mu^{(n)} \) is bounded. Then, the following statements are equivalent:

(a) The operator \( T_{u,v,p}^m : H^\infty \rightarrow \mathcal{W}_\mu^{(n)} \) is compact
(b) The operator \( T_{u,v,p}^m : H^\infty \cap \mathcal{B}_0 \rightarrow \mathcal{W}_\mu^{(n)} \) is compact
(c) The operator \( T_{u,v,p}^m : \mathcal{B} \rightarrow \mathcal{W}_\mu^{(n)} \) is compact
(d) The operator \( T_{u,v,p}^m : \mathcal{B} \rightarrow \mathcal{W}_\mu^{(n)} \) is compact
(e) \( \lim_{i \rightarrow \infty} \|T_{u,v,p}^m\|_{c,\mathcal{B}_0^{(n)}} = 0 \)
(f) For each \( i \in \{0, \ldots, n+1\} \), \( \limsup_{|i| \rightarrow 1} \|T_{u,v,p}^m f_{i+1,a}\|_{\mathcal{W}_\mu^{(n)}} = 0 \)
(g) For each \( i \in \{0, \ldots, n+1\} \)
\[
\limsup_{|\phi(z)| \rightarrow 1} \frac{\mu(z) \|I_{\phi}^m + I_{\phi}^n\|_{\mathcal{W}_\mu^{(n)}}}{(1 - |\phi(z)|^2)^{m+i+1/p}} = 0
\]
(63)

Remark 14. By putting \( \nu \equiv 0 \) in Theorems 6, 7, 8, and 9 and Corollaries 12 and 13, some characterizations are acquired for boundedness, essential norm, and compactness of the generalized weighted composition operator from Hardy spaces \( (0 < p \leq \infty) \) into \( n \)th weighted-type spaces.

Since
\[
\left( I_{\phi}^m + I_{\phi}^n \right)(z) = u'(z),
\]
\[
\left( I_{\phi}^m + I_{\phi}^n \right)(z) = u(z)\phi'(z) + v'(z),
\]
\[
\left( I_{\phi}^m + I_{\phi}^n \right)(z) = v(z)\phi'(z),
\]
we obtain the next remark.

Remark 15. Let \( \alpha > 0 \). Setting \( n = 1(\mu(z)) = (1 - |z|^2)\alpha, (1 - |z|^2) \alpha(2/(1 - |z|))) \) in Theorems 6, 7, 8, and 9 and Corollaries 12 and 13 using (64) we get similar results for operator \( T_{u,v,p}^m : H^\infty \rightarrow \mathcal{B}_\mu(\mu, \mathcal{B}_0^{(n)} : H^p \rightarrow \mathcal{B}_\mu^{(n)} : H^p \rightarrow \mathcal{B}_\mu^{(n)} : H^p \rightarrow \mathcal{B}_0^{(n)}(\mu) \) (see [11, 12]).

\[
\left( I_{\phi}^m + I_{\phi}^n \right)(z) = u'(z),
\]
\[
\left( I_{\phi}^m + I_{\phi}^n \right)(z) = u(z)\phi'(z) + v'(z),
\]
\[
\left( I_{\phi}^m + I_{\phi}^n \right)(z) = v(z)\phi'(z),
\]
(65)

Data Availability

No data were used to support this study.
Conflicts of Interest

The authors declare that they have no conflicts of interest.

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