THE TIME DERIVATIVE IN A SINGULAR PARABOLIC EQUATION

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Dedicated to Olli Martio on his seventy-fifth birthday

Abstract: We study the Evolutionary $p$-Laplace Equation in the singular case $1 < p < 2$. We prove that a weak solution has a time derivative (in Sobolev’s sense) which is a function belonging (locally) to a $L^q$-space.

1 Introduction

The regularity theory for parabolic partial differential equations of the type

$$\frac{\partial u}{\partial t} = \text{div} A(x, t, u, \nabla u)$$

aims at establishing boundedness and continuity of the solution $u = u(x, t)$ and its gradient

$$\nabla u = \left( \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n} \right).$$

The celebrated methods of DeGiorgi, Nash, and Moser do not directly treat the time derivative $u_t$, which is regarded as merely a distribution. Yet, for many specific equations the time derivative is more than that, it is a function.

\footnotesize
\begin{enumerate}
\item AMS classification 35K67, 35K92, 35B45
\end{enumerate}
in Lebesgue’s theory. We shall prove that the solutions of the Evolutionary $p$-Laplace Equation
\[
\frac{\partial u}{\partial t} = \text{div} \left( |\nabla u|^{p-2} \nabla u \right)
\] (1)
have a first order time derivative $u_t$ in Sobolev’s sense. Thus the time derivative exists as a measurable function satisfying the definition
\[
\int_0^T \int_\Omega u_t \phi \, dx \, dt = - \int_0^T \int_\Omega u \phi \, dx \, dt
\]
for all test functions $\phi \in C^\infty_0(\Omega_T)$. Here $\Omega_T = \Omega \times (0, T)$ and $\Omega$ is a domain in the $n$-dimensional space $\mathbb{R}^n$. The so-called degenerate case $p \geq 2$ (or slow diffusion case) was treated in [L1] and [L2] and now we shall focus our attention on the so-called singular case (or fast diffusion case) $1 < p < 2$, which is much more demanding, because the operator
\[
\text{div}(|\nabla u|^{p-2} \nabla u) = |\nabla u|^{p-2} \Delta u + (p - 2)|\nabla u|^{p-4} \sum_{i,j} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j}
\]
is undefined at the critical points $\nabla u = 0$ when $p < 2$. (It is known that the second derivatives $u_{x_ix_j}$ exist in the singular case, but the negative power $p - 2$ spoils the formula.) — We refer to the books [DB], [WYZ] about the Evolutionary $p$-Laplace Equation. Our method is to differentiate the regularized equation
\[
\frac{\partial u}{\partial t} = \text{div} \left( \{ |\nabla u|^2 + \epsilon^2 \}^{\frac{p-2}{2}} \nabla u \right)
\] (2)
with respect to the $x$-variables and then to derive careful estimates which are passed over to the limit as $\epsilon \to 0$. The appearing identities are, of course, not new. The main formula to start from has been used for other purposes in [Y] and [WZY]. The case $p \geq \frac{3}{2}$ can be extracted from [Y]. See also [AMS] for systems. Unfortunately, there is an extra complication when $p$ is small; in our proofs it appears in the range $1 < p < \frac{3}{2}$. To wit, the natural definition that weak solutions belong only to
\[
\mathcal{H} \subset C(0, T; L^2(\Omega)) \cap L^p(0, T; W^{1,p}(\Omega))
\]
is problematic, because it allows for unbounded solutions, when $1 < p < \frac{2n}{n+2}$. See [DB] and [DH] for this striking phenomenon. Various attempts to deal
with this situation are suggested in [BIV], where it is even proposed to assume that $u_t$ and $\text{div}(\nabla|u|^{p-2}\nabla u)$ both belong to the space $L^1$. (Lemma III.3.6 in [DH] states that $u_t \in L^1_{\text{loc}}$ under the condition that the Bénilan-Crandall estimate

$$u_t \leq \frac{1}{2 - p} \frac{u}{t},$$

is valid, which requires some restrictions.) The common definition is to add the condition

$$u \in L^r(\Omega_T), \quad \text{where } p(n + r) > 2n$$

for some exponent $r$ in this range. This extra assumption has the effect that the weak solutions become locally bounded. See [DH], in particular III.6 and III.7 for further information about this sharp condition.

We shall directly assume that $\|u\|_{\infty} < \infty$, if $1 < p < \frac{3}{2}$. However, in the one-dimensional case ($n = 1$) we have, without any extra hypothesis, a short proof that the time derivative is square summable.

**Theorem 1** Suppose that $u$ is a weak solution of the equation

$$u_t = \text{div}(\nabla|u|^{p-2}\nabla u)$$

in the domain $\Omega_T$. In the case $1 < p < \frac{3}{2}$, $n \geq 2$, we make the extra assumption that $u$ is bounded. Then the time derivative $u_t$ exists in Sobolev’s sense and $u_t \in L^\theta(\Omega_T)$ for some $\theta > 1$.

If $p \geq \frac{3}{2}$ or $n = 1$, we can take $\theta = 2$.

If $1 < p < \frac{3}{2}$ and $n \geq 2$, we have the restriction $1 < \theta < \frac{1}{2 - p}$.

A quantitative proof is the object of this work. It is noteworthy that the proper regularity theory is not invoked.

## 2 Preliminaries

We use standard notation. See [DB] about time dependent Sobolev Spaces. Suppose that $\Omega$ is a bounded domain in $\mathbb{R}^n$ and consider the space $\times$ time cylinder $\Omega_T = \Omega \times (0, T)$. We shall always keep $1 < p \leq 2$, although many

\[\text{Neither is any extra condition needed for } n = 2, \text{ since } p(2 + r) > 2 \cdot 2, \text{ if } r = 2.\]
formulas are valid also for $p > 2$. Denote $\|D^2 u\|^2 = \sum u^2_{x_i x_j}$. Once and for all, we fix a test function $\zeta \in C^\infty_0(\Omega)$, $0 \leq \zeta \leq 1$. In the sequel, the constants in the estimates can depend on $\|\zeta\|_\infty$ and $\|\nabla \zeta\|_\infty$.

**Definition 2** Assume that $u \in C(0, T; L^2(\Omega)) \cap L^p(0, T; W^{1,p}(\Omega))$. We say that $u$ is a weak solution of the equation $u_t = \text{div}(|\nabla u|^{p-2} \nabla u)$ in $\Omega_T$ if

$$\int_0^T \int \nabla u |^{p-2} \langle \nabla u, \nabla \phi \rangle \, dx \, dt \quad \text{when} \quad \phi \in C_0(\Omega_T).$$

Especially, $u \in L^2(\Omega_T)$ by the assumption. The weak solutions for the regularized equation (2) are defined in a similar way, see (4). According to Theorem 4.2 on page 599 in [LSU] they have continuous second derivatives in all variables. We shall use the notation $u_\varepsilon$ for the solution of the regularized equation with boundary values $u$ on the parabolic boundary of $\Omega_T$. The boundary values are taken in the following sense:

- $u_\varepsilon - u \in L^p(0, T; W^{1,p}_0(\Omega))$ and
- $\lim_{\delta \to 0} \frac{1}{\delta} \int_0^\delta \int \|u_\varepsilon - u\|^2 \, dx \, dt = 0$.

### 3 The Time Derivative

Our proof depends on the applicability of the rule

$$\int_0^T \int \nabla u |^{p-2} \langle \nabla u, \phi \rangle \, dx \, dt = \int_0^T \phi \nabla \cdot (|\nabla u|^{p-2} \nabla u) \, dx \, dt,$$

when $\phi \in C^\infty_0(\Omega_T)$. Thus the theorem follows, if we can prove that the derivatives $\partial/\partial x_j (|\nabla u|^{p-2} \nabla u)$ in the formula exist and belong to $L^2_{\text{loc}}(\Omega_T)$. Indeed, that we can do for $p > \frac{3}{2}$. Yet, for smaller values of $p$, the negative exponent $p - 2$ forces us to circumvent this expression, which is problematic when $\nabla u = 0$. We use the regularized equation

$$\int_0^T \int \nabla u_\varepsilon \phi \, dx \, dt = \int_0^T \phi \nabla \cdot \left(\frac{|\nabla u_\varepsilon|^2 + \varepsilon^2}{2} \nabla u_\varepsilon\right) \, dx \, dt$$

(4)
and prove that, as $\varepsilon \to 0$, the derivatives

$$\frac{\partial}{\partial x_j}\left(\{|\nabla u_\varepsilon|^2 + \varepsilon^2\}^{\frac{p-2}{2}} \frac{\partial u_\varepsilon}{\partial x_k}\right)$$

converge weakly in $L^p_{\text{loc}}(\Omega_T)$ with some $\theta > 1$. Since $u_\varepsilon$ converges to $u$ locally in $L^2(\Omega_T)$ by Proposition 4, the Theorem follows from the compactness result below, when we take into account that

$$\left|\frac{\partial}{\partial x_j}\left(\{|\nabla u_\varepsilon|^2 + \varepsilon^2\}^{\frac{p-2}{2}} \frac{\partial u_\varepsilon}{\partial x_k}\right)\right| \leq 2\left\{|\nabla u_\varepsilon|^2 + \varepsilon^2\}^{\frac{p-2}{2}} |\nabla u_\varepsilon|^2\right|$$

Assume that $u \in C(0, T; L^2(\Omega)) \cap L^p(0, T; W^{1,p}(\Omega))$ is a weak solution to $u_t = \text{div}(|\nabla u|^{p-2}\nabla u)$ in $\Omega_T$. Let $u_\varepsilon$ denote the solution of the regularized equation

$$\frac{\partial u_\varepsilon}{\partial t} = \text{div}\left(\{|\nabla u_\varepsilon|^2 + \varepsilon^2\}^{\frac{p-2}{2}} \nabla u_\varepsilon\right)$$

with the same boundary values as $u$ on the parabolic boundary of $\Omega_T$.

**Lemma 3** We have uniformly with respect to $\varepsilon$:

- **$p \geq \frac{3}{2}$**:
  $$\int_0^T \int_\Omega \zeta^2\left(|\nabla u_\varepsilon|^2 + \varepsilon^2\right)^{2(p-2)} |\nabla u_\varepsilon|^2 dx dt \leq L < \infty, \quad \varepsilon \leq 1. \quad (\star)$$

- **$1 < p < \frac{3}{2}$**. Under the extra assumption that $\|u\|_\infty < \infty$, the quantity
  $$\int_0^T \int_\Omega \zeta^2\left(|\nabla u_\varepsilon|^2 + \varepsilon^2\right)^{\theta(p-2)} |\nabla u_\varepsilon|^2 dx dt \leq L(\theta) < \infty, \quad \varepsilon \leq 1.$$
  
  is uniformly bounded in $\varepsilon$ when

  $$1 < \theta < \frac{1}{2 - p}.$$

- **$n = 1$**. In the one-dimensional case $(\star)$ holds for all $p > 1$.

**Proof**: The second case is Proposition 7 and the two other cases are in Section 5. Formally, $(\star)$ is equation (2.16) in [Y]. □
4 Convergence of the Approximation

In this section we shun the extra assumption about the boundedness of the weak solution $u$. This effort complicates the convergence proof for the $u_\varepsilon$'s. Recall the equations
\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \text{div} \left( |\nabla u|^{p-2} \nabla u \right) \\
\frac{\partial u_\varepsilon}{\partial t} &= \text{div} \left( \{ |\nabla u_\varepsilon|^2 + \varepsilon^2 \}^{\frac{p-2}{2}} \nabla u_\varepsilon \right)
\end{aligned}
\]
where $u = u_\varepsilon$ on the parabolic boundary of $\Omega_T$.

**Proposition 4** Under the assumption
\[
u \in C(0, T; L^2(\Omega)) \cap L^p(0, T; W^{1,p}(\Omega))
\]
the convergence
\[
u_\varepsilon \to u \quad \text{in} \quad L^2(\Omega_T), \quad \nabla u_\varepsilon \to \nabla u \quad \text{in} \quad L^p(\Omega_T)
\]
is valid.

*Proof:* Using the test function $\phi = u_\varepsilon - u$ in both equations we get
\[
W_\varepsilon \equiv \int_0^T \int_\Omega \langle |\nabla u_\varepsilon|^2 + \varepsilon^2 \rangle^{\frac{p-2}{2}} \nabla u_\varepsilon - |\nabla u|^{p-2} \nabla u, \nabla u_\varepsilon - \nabla u \rangle \, dxdt
= -\frac{1}{2} \int_\Omega (u_\varepsilon(x, T) - u(x, T))^2 \, dx \leq 0. \tag{5}
\]
Strictly speaking, in the equation for $u$ we must go via a time regularization; the Steklov average works well and the final inequality $W_\varepsilon \leq 0$ follows. Thus
\[
J_\varepsilon \equiv \int_0^T \int_\Omega \{ |\nabla u_\varepsilon|^2 + \varepsilon^2 \}^{\frac{p-2}{2}} |\nabla u_\varepsilon|^2 \, dxdt
\leq - \int_0^T |\nabla u|^p \, dxdt + \int_0^T \int_\Omega \nabla u|^{p-2} \langle \nabla u, \nabla u_\varepsilon \rangle \, dxdt
+ \int_0^T \int_\Omega \{ |\nabla u_\varepsilon|^2 + \varepsilon^2 \}^{\frac{p-2}{2}} \langle \nabla u_\varepsilon, \nabla u \rangle \, dxdt.
\]
By Young’s inequality the last integrand satisfies
\[
\left| \frac{\mu}{q} \langle \nabla u, \nabla u \rangle \right| \leq \frac{1}{p} |\nabla u|^p + \frac{1}{q} \left\{ |\nabla u|^2 + \varepsilon^2 \right\}^{\frac{p}{2}} |\nabla u|^2,
\]
where \( q = p/(p-1) \). In the same way
\[
|\nabla u|^{p-2} \langle \nabla u, \nabla u \rangle \leq \frac{\sigma^p}{p} |\nabla u|^p + \frac{1}{\sigma q} |\nabla u|^p, \quad \sigma > 0.
\]
Upon integration and absorption of a term, we arrive at
\[
J_\varepsilon \leq (p-1)(\sigma^q-1) \int_0^T \int_0^\Omega |\nabla u|^p \, dx \, dt + \sigma \int_0^T \int_0^\Omega |\nabla u_\varepsilon|^p \, dx \, dt. \tag{6}
\]
In order to handle the last integral, we divide the domain of integration into two parts: the set \(|\nabla u_\varepsilon| \leq \varepsilon\) and the set \(|\nabla u_\varepsilon| \geq \varepsilon\). We have
\[
\int_0^T \int_0^\Omega |\nabla u_\varepsilon|^p \, dx \, dt \leq \varepsilon^p \text{mes}(\Omega_T) + \int_0^T \int_{|\nabla u_\varepsilon| \geq \varepsilon} |\nabla u_\varepsilon|^2 |\nabla u_\varepsilon|^{p-2} \, dx \, dt
\]
\[
\leq \varepsilon^p \text{mes}(\Omega_T) + 2^{\frac{2-p}{2}} \int_0^T \int_{|\nabla u_\varepsilon| \geq \varepsilon} \left\{ |\nabla u_\varepsilon|^2 + \varepsilon^2 \right\}^{\frac{p}{2}} |\nabla u_\varepsilon|^2 \, dx \, dt
\]
\[
\leq \varepsilon^p \text{mes}(\Omega_T) + 2^{\frac{2-p}{2}} J_\varepsilon.
\]
We insert this in equation (6) and obtain
\[
J_\varepsilon \leq (p-1)(\sigma^q-1) \int_0^T \int_0^\Omega |\nabla u|^p \, dx \, dt + \sigma \varepsilon^p \text{mes}(\Omega_T) + \sigma^p 2^{\frac{2-p}{2}} J_\varepsilon.
\]
We fix \( \sigma > 0 \) equal to a number, depending only on \( p \), so small that the last \( J_\varepsilon \)-term can be absorbed into the left-hand side. It follows that
\[
J_\varepsilon \leq C_p \left\{ \int_0^T \int_0^\Omega |\nabla u|^p \, dx \, dt + \varepsilon^p \text{mes}(\Omega_T) \right\},
\]
\[
\int_0^T \int_0^\Omega |\nabla u_\varepsilon|^p \, dx \, dt \leq 3C_p \left\{ \int_0^T \int_0^\Omega |\nabla u|^p \, dx \, dt + \varepsilon^p \text{mes}(\Omega_T) \right\}.
\]
In particular, we have a uniform bound:

$$\int_0^T \int_\Omega |\nabla u_\varepsilon|^p \, dx \, dt \leq K, \quad 0 \leq \varepsilon \leq 1.$$  \hspace{1cm} (7)

Now we split $W_\varepsilon$ as

$$W_\varepsilon = \int_0^T \int_\Omega \langle |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon - |\nabla u|^{p-2} \nabla u, \nabla u_\varepsilon - \nabla u \rangle \, dx \, dt$$

$$+ \int_0^T \int_\Omega \{ |\nabla u_\varepsilon|^2 + \varepsilon^2 \}^{\frac{p-2}{2}} \nabla u_\varepsilon - |\nabla u|^{p-2} \nabla u, \nabla u_\varepsilon - \nabla u \rangle \, dx \, dt,$$

and since $W_\varepsilon \leq 0$ by (5),

$$M_\varepsilon \equiv \int_0^T \int_\Omega \langle |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon - |\nabla u|^{p-2} \nabla u, \nabla u_\varepsilon - \nabla u \rangle \, dx \, dt$$

$$\leq \int_0^T \int_\Omega \langle |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon - \{ |\nabla u_\varepsilon|^2 + \varepsilon^2 \}^{\frac{p-2}{2}} \nabla u_\varepsilon, \nabla u_\varepsilon - \nabla u \rangle \, dx \, dt \equiv O_\varepsilon.$$

We claim that $O_\varepsilon \to 0$ as $\varepsilon \to 0$. Recall that $1 < p \leq 2$. Thus the inequality \hspace{1cm} 3

$$0 \leq |a|^{p-2} - (|a|^2 + \varepsilon^2)^{\frac{p-2}{2}} < \frac{2-p}{2} \varepsilon^2 |a|^{p-2} \delta^{-2} \quad |a| \geq \delta,$$

$$0 \leq |a|^{p-2} - (|a|^2 + \varepsilon^2)^{\frac{p-2}{2}} = -\int_0^1 \frac{d}{dt}(|a|^2 + t\varepsilon^2)^{\frac{p-2}{2}} \, dt$$

$$= \frac{2-p}{2} \varepsilon^2 \int_0^1 (|a|^2 + t\varepsilon^2)^{\frac{p-4}{2}} \, dt \leq \frac{2-p}{2} \varepsilon^2 |a|^{p-4}, \quad a \neq 0.$$
is available. Now we split the domain of integration for $O_\varepsilon$ into two parts and achieve

$$
|O_\varepsilon| \leq \frac{2-p}{2} \varepsilon^2 \delta^{-2} \iint_{|\nabla u_\varepsilon| \geq \delta} |\nabla u_\varepsilon|^{p-1} |\nabla u_\varepsilon - \nabla u| \, dx \, dt
$$

$$
+ 2 \iint_{|\nabla u_\varepsilon| \leq \delta} |\nabla u_\varepsilon|^{p-1} |\nabla u_\varepsilon - \nabla u| \, dx \, dt.
$$

By Hölder’s inequality

$$
\iint |\nabla u_\varepsilon|^{p-1} |\nabla u_\varepsilon - \nabla u| \, dx \, dt \leq \left( \iint |\nabla u_\varepsilon|^{p} \, dx \, dt \right)^{\frac{p-1}{p}} \left\{ \|\nabla u_\varepsilon\|_{p} + \|\nabla u\|_{p} \right\}.
$$

Recalling the uniform bound (7), we see that

$$
|O_\varepsilon| \leq \frac{2-p}{2} \varepsilon^2 \delta^{-2} K^{1+\frac{1}{p}} \left( K^{\frac{1}{p}} + \|\nabla u\|_{p} \right) + 2 \varepsilon^{2} \left( K^{\frac{1}{p}} + \|\nabla u\|_{p} \right).
$$

It follows that

$$
\lim_{\varepsilon \to 0} O_\varepsilon = 0.
$$

The inequality\(^4\)

$$
\iint_{0}^{T} \int_{0}^{\Omega} \frac{|\nabla u_\varepsilon - \nabla u|^2 \, dx \, dt}{\left( 1 + |\nabla u|^2 + |\nabla u_\varepsilon|^2 \right)^{\frac{2-p}{2}}} \leq M_\varepsilon \leq O_\varepsilon \quad (8)
$$

shows in combination with

$$
\iint_{0}^{T} \int_{0}^{\Omega} |\nabla u_\varepsilon - \nabla u|^{p} \, dx \, dt = \iint_{0}^{T} \int_{0}^{\Omega} |\nabla u_\varepsilon - \nabla u|^{p} \left\{ \frac{1 + |\nabla u|^2 + |\nabla u_\varepsilon|^2}{1 + |\nabla u|^2 + |\nabla u_\varepsilon|^2} \right\}^{\frac{p(2-p)}{4}} \, dx \, dt
$$

$$
\leq \left\{ \iint_{0}^{T} \int_{0}^{\Omega} \frac{|\nabla u_\varepsilon - \nabla u|^2 \, dx \, dt}{\left( 1 + |\nabla u|^2 + |\nabla u_\varepsilon|^2 \right)^{\frac{2-p}{2}}} \right\}^{\frac{p}{2}} \iint_{0}^{T} \int_{0}^{\Omega} \left( 1 + |\nabla u|^2 + |\nabla u_\varepsilon|^2 \right)^{\frac{p}{2}} \, dx \, dt
$$

\(^4\)For vectors

$$
\langle |b|^{p-2}b - |a|^{p-2}a, b - a \rangle \geq (p-1)|b - a|^2 \left( 1 + |a|^2 + |b|^2 \right)^{\frac{p-2}{2}}, \quad 1 < p \leq 2.
$$
and the uniform bound (7) that
\[
\lim_{\varepsilon \to 0} \int_0^T \int_\Omega |\nabla u_\varepsilon - \nabla u|^p \, dx \, dt = 0. \tag{9}
\]

The convergence \(u_\varepsilon \to u\) in \(L^2(\Omega_T)\) can be extracted from the above proof, according to which
\[
\frac{1}{2} \int_\Omega (u_\varepsilon(x,T) - u(x,T))^2 \, dx = -W_\varepsilon = O_\varepsilon - M_\varepsilon \leq O_\varepsilon \to 0.
\]

When we replace \(T\) by \(t\), \(0 < t < T\), the same bound as before will majorize \(O_\varepsilon\) simultaneously for all \(t\). Integrating with respect to \(t\), we obtain
\[
\int_0^T \int_\Omega (u_\varepsilon(x,t) - u(x,t))^2 \, dx \, dt \leq 2T O_\varepsilon.
\]

This concludes the convergence proof. \(\square\)

5 The Main Identity

In order to derive estimates for the derivatives
\[
\frac{\partial}{\partial x_j} \left( |\nabla u|^{p-2} \nabla u \right)
\]
we differentiate the *regularized* equation
\[
\frac{\partial u_\varepsilon}{\partial t} = \text{div} \left( (|\nabla u_\varepsilon|^2 + \varepsilon^2)^{\frac{p-2}{2}} \nabla u_\varepsilon \right).
\]

Using the abbreviations
\[
\begin{align*}
  u_{\varepsilon,j} &= \frac{\partial}{\partial x_j} u_\varepsilon, \quad v_\varepsilon = |\nabla u_\varepsilon|^2, \quad V_\varepsilon = |\nabla u_\varepsilon|^2 + \varepsilon^2 = v_\varepsilon^2 + \varepsilon^2
\end{align*}
\]
we have
\[
\frac{\partial}{\partial t} u_{\varepsilon,j} = \text{div} \left( V_\varepsilon^{\frac{p-2}{2}} \nabla u_{\varepsilon,j} + \nabla u_{\varepsilon,j} \frac{\partial}{\partial x_j} V_\varepsilon^{\frac{p-2}{2}} \right). \tag{10}
\]
We note
\[\frac{\partial}{\partial x_j} v_\varepsilon = 2 \langle \nabla u_\varepsilon, \nabla u_{\varepsilon,j} \rangle, \quad |\nabla v_\varepsilon|^2 \leq 4|\nabla u_\varepsilon|^2 |\nabla^2 u_\varepsilon|^2 \]
\[\frac{\partial}{\partial x_j} V_\varepsilon^{\varepsilon - 2} = (p - 2) V_\varepsilon^{\varepsilon - 2} \langle \nabla u_\varepsilon, \nabla u_{\varepsilon,j} \rangle.\]

In weak form the equation becomes
\[-\int_0^T \int_\Omega \phi_j \frac{\partial u_{\varepsilon,j}}{\partial t} \, dx \, dt \]
\[= \int_0^T \int_\Omega \left( V_\varepsilon^{\varepsilon - 2} \langle \nabla u_{\varepsilon,j}, \nabla \phi_j \rangle + (p - 2) V_\varepsilon^{\varepsilon - 2} \langle \nabla u_\varepsilon, \nabla u_{\varepsilon,j} \rangle \langle \nabla u_\varepsilon, \nabla \phi_j \rangle \right) \, dx \, dt,\]
valid at least for all test functions \(\phi_j \in C^\infty_0(\Omega_T), \ j = 1, 2, \ldots, n.\) (In fact, it is not needed that \(\phi_j = 0\) when \(t = 0\) or \(t = T.\)) We use the test functions
\[\phi_j = \zeta^2 V_\varepsilon^{\alpha} u_{\varepsilon,j}, \quad \zeta \in C^\infty_0(\Omega_T)\]
and sum the formulas to reach the identity below. (Such identities often serve to derive Caccioppoli inequalities.) We shall keep \(1 - p < 2\alpha < 0.\) Always, \(0 \leq \zeta \leq 1.\)
Fundamental formula

\[
\begin{align*}
\int_0^T \int_\Omega \zeta^2 V_\epsilon^{\frac{p-2+2\alpha}{2}} |\nabla^2 u_\epsilon|^2 \, dx \, dt & \quad \text{Main Term} \tag{I} \\
+ \frac{p - 2 + 2\alpha}{4} \int_0^T \int_\Omega \zeta^2 V_\epsilon^{\frac{p-2+2\alpha}{2}-1} |\nabla v_\epsilon|^2 \, dx \, dt & \tag{II} \\
+ \frac{\alpha(p - 2)}{2} \int_0^T \int_\Omega \zeta^2 V_\epsilon^{\frac{p-2+2\alpha}{2}-2} (\nabla u_\epsilon, \nabla v_\epsilon)^2 \, dx \, dt & \tag{III} \\
+ \frac{1}{2(\alpha + 1)} \left[ \int_\Omega \zeta^2 V_\epsilon^{\alpha+1} \, dx \right]^T_0 & \tag{IV} \\
= (2 - p) \int_0^T \int_\Omega \zeta^2 V_\epsilon^{\frac{p-2+2\alpha}{2}-1} (\nabla u_\epsilon, \nabla v_\epsilon) (\nabla \zeta, \nabla u_\epsilon) \, dx \, dt \tag{V} \\
- \int_0^T \int_\Omega \zeta^2 V_\epsilon^{\frac{p-2+2\alpha}{2}} (\nabla \zeta, \nabla u_\epsilon) \, dx \, dt & \tag{VI} \\
+ \frac{1}{\alpha + 1} \int_0^T \int_\Omega V_\epsilon^{\alpha+1} \zeta \zeta_t \, dx \, dt & \tag{VII}
\end{align*}
\]

The proof is a straightforward calculation. (Compare with formula (2.5) in [Y] and formula (2.20) on page 166 in [WZY].) We only mention how to treat the part with the time derivative:

\[
\phi_j \frac{\partial}{\partial t} u_{\epsilon,j} = \zeta^2 V_\epsilon^\alpha \frac{\partial}{\partial t} \left( \frac{u_{\epsilon,j}^2}{2} \right) \\
\zeta^2 V_\epsilon^\alpha \frac{\partial}{\partial t} \left( \frac{v_\epsilon}{2} \right) = \frac{1}{2} \zeta^2 \frac{\partial}{\partial t} \left( \frac{V_\epsilon^{\alpha+1}}{\alpha + 1} \right).
\]

Thus, upon summation, the left-hand side of (\text{\textsc{\textbf{III}}}\text{)} becomes

\[
- \sum_{j=1}^n \int_0^T \int_\Omega \phi_j \frac{\partial u_{\epsilon,j}}{\partial t} \, dx \, dt = \frac{1}{2} \left[ \int_\Omega \zeta^2 V_\epsilon^{\alpha+1} \, dx \right]^T_0 - \int_0^T \zeta \zeta_t \frac{V_\epsilon^{\alpha+1}}{\alpha + 1} \, dx \, dt.
\]
The right-hand side yields six terms, since the right-hand side of (10) is multiplied by
\[
\nabla \phi_j = \zeta^2 V_\epsilon \nabla u_{\epsilon,j} + \alpha \zeta^2 v_\epsilon^{a-1} u_{\epsilon,j} \nabla V_\epsilon + 2 V_\epsilon^a u_{\epsilon,j} \zeta \nabla \zeta;
\]
two similar terms are joined in term \(I I\).

Always, \(0 > 2\alpha > 1 - p\) and \(1 < p \leq 2\), which means that the factor in front of term \(I I\) is negative. The integral itself is of the same magnitude as term \(I\), and
\[
|\nabla v_\epsilon|^2 \leq 4 v_\epsilon |\nabla^2 u_\epsilon|^2 \leq 4 V_\epsilon |\nabla^2 u_\epsilon|^2.
\]
(12)
This causes the constraint: \(p - 1 + 2\alpha > 0\). Term \(I I\) is positive, but since the expression
\[
\langle \nabla u_\epsilon, \nabla v_\epsilon \rangle = 4 \sum_{i,j=1}^n \frac{\partial u_\epsilon}{\partial x_i} \frac{\partial u_\epsilon}{\partial x_j} \frac{\partial^2 u_\epsilon}{\partial x_i \partial x_j}
\]
may vanish, it is of little use, except in the one dimensional case when term \(I I\) matches term \(I\).

**Estimation of some terms**

**Vanishing of term \(I V\)** It is zero, as \(\zeta\) has compact support also in the time direction.

**Absorption of term \(V\)** We can use Young’s inequality to absorb term \(V\) into the main term \(I\). Now by (12)
\[
|\langle \nabla u_\epsilon, \nabla v_\epsilon \rangle \langle \nabla \zeta, \nabla u_\epsilon \rangle| \leq |\nabla u_\epsilon|^2 |\nabla \zeta| |2 V_\epsilon^{1/2} |\nabla^2 u_\epsilon|
\]
and with a small parameter \(\sigma > 0\)
\[
|V| \leq (2 - p) \sigma I + (2 - p) \sigma^{-1} \int_0^T V_\epsilon^{\frac{p+2\alpha}{2}} |\nabla \zeta|^2 dxdt.
\]
(13)
**Term \(VI\)** Since \(|\langle \nabla \zeta, \nabla v_\epsilon \rangle| \leq 2 |\nabla \zeta| |\nabla u_\epsilon| |\nabla^2 u_\epsilon|\), we get the same as above:
\[
|VI| \leq \sigma I + \sigma^{-1} \int_0^T V_\epsilon^{\frac{p+2\alpha}{2}} |\nabla \zeta|^2 dxdt.
\]
(14)
With these arrangements the main formula yields the estimate

\[(1 - (3 - p)\sigma)I + I + \ II + \ III \leq (3 - p)\sigma^{-1} \int_0^T V_{\varepsilon}^{p+2\alpha} |\nabla \zeta|^2 \, dx \, dt + \ VII\]

The one-dimensional case

In one space dimension we have

\[u'_{\varepsilon} = \frac{\partial u_{\varepsilon}}{\partial x}, \quad u''_{\varepsilon} = \frac{\partial^2 u_{\varepsilon}}{\partial x^2}\]

We fix \(2\alpha = p - 2\), which is negative. Then the sum \(I + I + \ III\) can be written as

\[\int_0^T \int_\Omega \zeta^2 V_{\varepsilon}^{p-2} \left|\frac{\partial^2 u_{\varepsilon}}{\partial x^2}\right|^2 \left\{1 + 2(p - 1)u_{\varepsilon}^2 v_{\varepsilon}^{-1} + (p - 2)^2 u_{\varepsilon}^4 v_{\varepsilon}^{-2}\right\} \, dx \, dt.\]

The expression in braces is a perfect square and can be estimated as

\[\{1 + \ldots v_{\varepsilon}^{-2}\} = \left(\frac{(p - 1)u_{\varepsilon}^2 + \varepsilon^2}{u_{\varepsilon}^2 + \varepsilon^2}\right)^2 \geq (p - 1)^2.\]

Thus the total estimate in one dimension reads

\[((p - 1)^2 - (3 - p)\sigma) \int_0^T \zeta^2 V_{\varepsilon}^{p-2} \left|\frac{\partial^2 u_{\varepsilon}}{\partial x^2}\right|^2 \, dx \, dt\]

\[\leq (3 - p)\sigma^{-1} \int_0^T V_{\varepsilon}^{p-1} |\nabla \zeta|^2 \, dx \, dt + \frac{2}{p} \int_0^T \int_\Omega V_{\varepsilon}^{p} |\zeta\zeta_t| \, dx \, dt.\]

Now we only have to fix \(\sigma\) small enough, noticing that

\[V_{\varepsilon}^{p-1} \leq u_{\varepsilon}^p + 1, \quad V_{\varepsilon}^{p} \leq 2(u_{\varepsilon}^p + \varepsilon^p),\]

to obtain the majorant

\[\int_0^T \int_\Omega \zeta^2 V_{\varepsilon}^{p-2} \left|\frac{\partial^2 u_{\varepsilon}}{\partial x^2}\right|^2 \, dx \, dt \leq C(p) \left\{\int_0^T \int_\Omega |\nabla u_{\varepsilon}|^p \, dx \, dt + 1\right\}. \quad (16)\]
The majorant is finite and, by (7) independent of \( \varepsilon \), but the constant factor \( C (p) \) depends also on \( \| \zeta_t \|_\infty \).

To proceed, use

\[
\frac{\partial}{\partial x} \left\{ u'_\varepsilon^2 + \varepsilon^2 \right\}^{\frac{p-2}{2}} u'_\varepsilon \leq (p - 1) V_{\varepsilon}^{\frac{p-2}{2}} \frac{\partial^2 u'_\varepsilon}{\partial x^2}^2
\]

to conclude that

\[
\frac{\partial}{\partial x} \left\{ u'_\varepsilon^2 + \varepsilon^2 \right\}^{\frac{p-2}{2}} u'_\varepsilon \text{ converges weakly in } L^2_{\text{loc}}(\Omega_T)
\]
at least through a subsequence. Thus we may pass to the limit under the integral signs in

\[
- \iint_0^T \int_\Omega u_\varepsilon \frac{\partial \phi}{\partial t} \, dx \, dt = \iint_0^T \int_\Omega \phi \frac{\partial}{\partial x} \left( \{ u'_\varepsilon + \varepsilon^2 \}^{\frac{p-2}{2}} u'_\varepsilon \right) \, dx \, dt
\]

and conclude that the time derivative \( u_t \) exists and belongs locally to \( L^2 \). The limit is some function.

**General Estimate, 1 < p < 2**

In several space dimensions term III is no longer so useful, so one may as well skip it since it is positive when \( \alpha < 0 \). However, it is convenient to use it to counterbalance a portion of term V:

\[
|V| \leq \text{III} + \frac{2 - p}{|\alpha|} \iint_0^T V_{\varepsilon}^{\frac{p+2\alpha}{2}} |\nabla \zeta|^2 \, dx \, dt,
\]

where Young’s inequality was used. Now we have the general estimate

**Lemma 5 (1 < p < 2.)** Let \( \sigma > 0 \). We have

\[
(p - 1 + 2\alpha - \sigma) \iint_0^T \zeta^2 V_{\varepsilon}^{\frac{p+2\alpha}{2}} |\nabla^2 u'_\varepsilon|^2 \, dx \, dt
\]

\[
\leq \left( \sigma^{-1} + \frac{2 - p}{|\alpha|} \right) \iint_0^T V_{\varepsilon}^{\frac{p+2\alpha}{2}} |\nabla \zeta|^2 \, dx \, dt + \frac{1}{\alpha + 1} \iint_0^T V_{\varepsilon}^{\alpha+1} \zeta_t \, dx \, dt.
\]

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This is worthless if one does not obey the

\[ p - 1 + 2\alpha > 0 \]

6 The case \( \frac{3}{2} < p < 2 \)

Again, we take \( 2\alpha = p - 2 \). Then by (17)

\[
(2p - 3 - \sigma) \int_0^T \int_\Omega \zeta^2 V_{\varepsilon}^{p-2} |\nabla^2 u_{\varepsilon}|^2 \, dx \, dt \\
\leq (\sigma^{-1} + 2) \int_0^T \int_\Omega V_{\varepsilon}^{p-1} |\nabla \zeta|^2 \, dx \, dt + \frac{1}{p - 1} \int_0^T \int_\Omega V_{\varepsilon}^{p} \zeta \zeta_t \, dx \, dt,
\]

Provided that \( 2p > 3 \), this yields the desired local bound with a majorant free of \( \varepsilon \) according to (17).

7 An ”Energy Term” with \( p < \frac{3}{2} \)

In the demanding case \( p < \frac{3}{2} \) we need to estimate the last integral in (17). In this case, we assume that the solution is bounded: \( \|u\|_{\infty} < \infty \). Obviously

\[
V_{\varepsilon}^{\alpha+1} = V_{\varepsilon}^{\alpha} (\varepsilon^2 + |\nabla u_{\varepsilon}|^2) \leq \varepsilon^{2(\alpha+1)} + V_{\varepsilon}^{\alpha} \langle \nabla u_{\varepsilon}, \nabla u_{\varepsilon} \rangle,
\]

since \( \alpha < 0 \). If \( \zeta \in C_0^\infty(\Omega) \), then\(^5\)

\[
\int_0^T \int_\Omega \zeta \zeta_t V_{\varepsilon}^{\alpha+1} \, dx \, dt \leq \varepsilon^{2(\alpha+1)} \int_0^T \int_\Omega |\zeta \zeta_t| \, dx \, dt + \int_0^T \int_\Omega \zeta \zeta_t V_{\varepsilon}^{\alpha} \langle \nabla u_{\varepsilon}, \nabla u_{\varepsilon} \rangle \, dx \, dt.
\]

\(^5\)It is essential that this holds also when \( \zeta \zeta_t < 0 \).
Integration by parts yields
\[
\int_0^T \int_\Omega \zeta_t V^\alpha \langle \nabla u_\varepsilon, \nabla u_\varepsilon \rangle \, dx \, dt = - \int_0^T u_\varepsilon \nabla \cdot (\zeta_t V^\alpha \nabla u_\varepsilon) \, dx \, dt
\]
\[
= - \int_0^T (u_\varepsilon V^\alpha \langle \nabla (\zeta_t), \nabla u_\varepsilon \rangle - 2\alpha \zeta_t u_\varepsilon V^\alpha \langle \nabla u_\varepsilon, \nabla u_\varepsilon \rangle) - u_\varepsilon \zeta_t V^\alpha \Delta u_\varepsilon) \, dx \, dt
\]
\[
\leq \int_0^T \left| u_\varepsilon V^\alpha \nabla (\zeta_t) \right| \| \nabla u_\varepsilon \| \, dx \, dt + (1 + 2|\alpha|) \int_0^T |\zeta_t u_\varepsilon| V^\alpha |\nabla^2 u_\varepsilon| \, dx \, dt.
\]

The last integral can now be absorbed into the main term in (17). To see this, factorize
\[
\zeta_t V^\alpha = \zeta V^\frac{p+2\alpha-2}{4} \cdot \zeta_t V^\frac{2-p+2\alpha}{4}
\]
and select a small \( \kappa > 0 \) for Young’s inequality \( 2ab \leq \kappa a^2 + \kappa^{-1}b^2 \). We arrive at the final bound below, after some arrangements.

**Lemma 6 (Energy Estimate)** We have

\[
\underbrace{\int_0^T \zeta_t V^\alpha \, dx \, dt}_{\text{(Term VII)}} \leq \varepsilon^{2(\alpha+1)} \int_0^T |\zeta_t| \, dx \, dt + 2\| u_\varepsilon \|_\infty \int_0^T |\nabla (\zeta_t)| V^\frac{2\alpha+1}{2} \, dx \, dt
\]
\[
\frac{1 + 2|\alpha|}{2} \left\{ \kappa \int_0^T \zeta^2 V^\frac{p+2\alpha-2}{2} |\nabla^2 u_\varepsilon|^2 \, dx \, dt + \kappa^{-1} \int_0^T \zeta_t^2 V^\frac{2-p+2\alpha}{2} \, dx \, dt \right\} \| u_\varepsilon \|_\infty
\]

8 **The Case \( p < \frac{3}{2} \)**

We combine the estimate in Lemma 6 with the general inequality (17) writing
\[
p - 2 + 2\alpha = \theta(p - 2), \quad 2\alpha = (\theta - 1)(p - 2) > 1 - p
\]
We must obey the restriction \( p - 1 + 2\alpha > 0 \), which means that

\[
1 < \theta < \frac{1}{2 - p}
\]

We obtain

\[
(1 + \theta(p - 2) - \sigma - \frac{2\kappa\|u_\varepsilon\|_\infty}{2 - p}) \int_0^T \int_\Omega \zeta^2 V_\varepsilon^{\theta(p-2)}|\nabla^2 u_\varepsilon|^2 \, dx \, dt
\]

\[
\leq \frac{\varepsilon^{2(\alpha+1)}}{\alpha + 1} \int_0^T \int_\Omega |\nabla \zeta_\varepsilon| \, dx \, dt + \frac{2\|u_\varepsilon\|_\infty}{\alpha + 1} \int_0^T \int_\Omega |\nabla (\zeta_\varepsilon)| V_\varepsilon^{2(\alpha+1)} \, dx \, dt
\]

\[
+ \frac{2\kappa^{-1}\|u_\varepsilon\|_\infty}{2 - p} \int_0^T \int_\Omega \zeta_\varepsilon^2 V_\varepsilon^{-p+2\alpha} \, dx \, dt
\]

\[
+ \left(\sigma^{-1} + \frac{2 - p}{\|\alpha\|}\right) \int_0^T \int_\Omega V_\varepsilon^{\frac{p+2\alpha}{2}} |\nabla \zeta|^2 \, dx \, dt,
\]

where simplifying estimations like \( \frac{1+2|\alpha|}{2(\alpha+1)} \leq \frac{2}{2-p} \) have been made. Notice that \( \alpha + 1 > 0 \). Now the powers of \( V_\varepsilon \) are decisive; they must be positive and no greater than \( p/2 \). Our permanent restriction \( 0 > 2\alpha > 1 - p \) leads to

\[
0 < \frac{2\alpha+1}{2} < \frac{p}{2}, \quad \frac{1}{2} < \frac{p+2\alpha}{2} < \frac{p}{2}, \quad \frac{3-2p}{2} < \frac{2-p+2\alpha}{2} < \frac{p}{2},
\]

but now we need \( p < \frac{3}{2} \) in order to assure

\[
\frac{2 - p + 2\alpha}{2} > 0.
\]

We see that the exponents are in the right range, and for the last three integrals we can use

\[
V_\varepsilon^\beta \leq |\nabla u_\varepsilon|^p + 1 \quad \text{for} \quad 0 < \beta < \frac{p}{2}, \quad 0 < \varepsilon \leq 1
\]

and then the uniform bound \((7)\). By the Maximum Principle \( \|u_\varepsilon\|_\infty \leq \|u\|_\infty \) (equality holds). Hence we have the following result:
Proposition 7  Let $1 < p < \frac{3}{2}$. Fix $\theta$ in the range $1 < \theta < \frac{1}{2-p}$. Then the integral
\[
\int_0^T \int_0^\Omega \zeta^2 \nu^{\theta(p-2)} \|\nabla^2 u_\epsilon\|^2 \, dx \, dt \leq L(\theta), \quad 0 < \epsilon \leq 1,
\]
is uniformly bounded in $\epsilon$. The bound $L(\theta)$ depends also on $p$, $\|u\|_\infty$, $\|\nabla \zeta\|_\infty$, $\|\zeta_t\|_\infty$, and the constant $K$ in [7].

References

[AMS] E. Acerbi, G. Mingione, G. Seregin, Regularity results for parabolic systems related to a class of non-Newtonian fluids, Annales de l’Institut Henri Poincaré - Analyse Non Linéaire 21, 2004, pp. 25–60.

[BIV] M. Bonforte, R. Iagar, J-L. Vázquez, Local smoothing effects, positivity, and Harnack inequalities for the fast $p$-Laplacian equation, Advances in Mathematics 224, 2010, pp.2151–2215.

[C] H. Choe, A regularity theory for a more general class of quasilinear parabolic partial differential equations and variational inequalities, Differential Integral Equations 5, 1992, pp. 915–944.

[DB] E. DiBenedetto, Degenerate Parabolic Equations, Springer-Verlag, New York 1993.

[DH] E. DiBenedetto, M. Herrero, Non-negative solutions of the evolution $p$-Laplacian equation. Initial traces and Cauchy problem when $1 < p < 2$, Archive for Rational Mechanics and Analysis 111, 1990, pp. 225–290.

[DGV] E. DiBenedetto, U. Gianazza, V. Vespri, Harnack’s Inequality for Degenerate and Singular Parabolic Equations, Springer, New York 2012.

[LSU] O. Ladyzhenskaya, V. Solonnikov, N. Uraltseva, Linear and Quasilinear Equations of Parabolic Type, Mathematical Monographs 23, American Mathematical Society, Providence RI 1967.

[L1] P. Lindqvist, On the time derivative in a quasilinear equation, Transactions of The Royal Norwegian Society of Sciences and Letters, 2008 no. 2, pp. 1–7.
[L2] P. Lindqvist, *On the time derivative in an obstacle problem*, Revista Mathematica Iberoamericana 28, 2012, pp. 577–590.

[WZY] Z. Wu, J. Zhao, J. Yin, H. Li, *Non-linear Diffusion Equations*, World Scientific, Singapore 2001.

[Y] C. Yazhe, *Hölder continuity of the gradient of certain degenerate parabolic equations*, Chinese Annals of Mathematics, Series B, 8 no. 3, 1987, pp. 343–356.