We introduce the Conic Blackwell Algorithm+ (CBA+) regret minimizer, a new parameter- and scale-free regret minimizer for general convex sets. CBA+ is based on Blackwell approachability and attains $O(\sqrt{T})$ regret. We show how to efficiently instantiate CBA+ for many decision sets of interest, including the simplex, $\ell_p$ norm balls, and ellipsoidal confidence regions in the simplex. Based on CBA+ we introduce SP-CBA+ a new parameter-free algorithm for solving convex-concave saddle-point problems, which achieves a $O(1/\sqrt{T})$ ergodic rate of convergence. In our simulations, we demonstrate the wide applicability of SP-CBA+ on several standard saddle-point problems, including matrix games, extensive-form games, distributionally robust logistic regression, and Markov decision processes. In each setting, SP-CBA+ achieves state-of-the-art numerical performance, and outperforms classical methods, without the need for any choice of step sizes or other algorithmic parameters.

1 Introduction

In this paper\footnote{A preliminary version of this paper has appeared as a conference paper by the same authors \cite{Grand-Clément and Kroer, 2021b}.} we develop new algorithms for solving the following convex-concave saddle-point problems (SPPs):

$$\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} F(x, y),$$

where $\mathcal{X} \subset \mathbb{R}^n$, $\mathcal{Y} \subset \mathbb{R}^m$ are convex, compact sets, and $F : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ is a subdifferentiable convex-concave function. The optimization problem (1) arises in a number of practical problems. For example, the problem of computing a Nash equilibrium of a zero-sum game can be formulated as a convex-concave SPP, and this is the foundation of most methods for solving sequential zero-sum games \cite{von Stengel, 1996, Zinkevich et al., 2007, Tammelin et al., 2015, Kroer et al., 2020}. Other instances include imaging \cite{Chambolle and Pock, 2011}, $\ell_\infty$-regression \cite{Sidford and Tian, 2018}, Markov Decision Processes (MDPs) and robust MDPs \cite{Tang, 2005, Wiesemann et al., 2013, Sidford and Tian, 2018}, market equilibrium \cite{Kroer et al., 2021} and distributionally robust logistic regression, where the max term represents the distributional uncertainty \cite{Namkoong and Duchi, 2016, Ben-Tal et al., 2015}. We introduce efficient algorithms for solving (1), focusing on parameter-free algorithms that do not require choosing, learning or tuning any step sizes.

Repeated game framework One way to solve convex-concave SPPs is by viewing the SPP as a repeated game between two players: at each iteration $t$, one player chooses $x_t \in \mathcal{X}$, the other player chooses $y_t \in \mathcal{Y}$, and then the players observe the payoff $F(x_t, y_t)$. If each player employs a regret-minimization algorithm, then a well-known theorem says that the uniform average of the decisions generated by the players converge to a solution to the SPP (see Theorem 2.1 in Section 2). We will call this the “repeated game framework”. There are already well-known algorithms for instantiating the above repeated game framework for (1). For example, one can employ the online mirror descent (OMD) algorithm \cite{Nemirovski and Yudin, 1983}, which generates iterates as follows for the first player (and similarly for the second player):

$$x_{t+1} = \arg \min_{x \in \mathcal{X}} \langle \eta f_t, x \rangle + D(x, x_t),$$

where $\eta$ is the step size.

\cite{arXiv:2202.12277v1 [math.OC] 24 Feb 2022}
where \( f_t \in \partial_x F(x_t, y_t) \) (\( \partial_x \) denotes the set of subgradients as regards the variable \( x \)), \( \eta > 0 \) is an appropriate step size, and \( D \) is a Bregman divergence which measures distance between points. Another example of a regret minimizer is Follow-The-Regularized-Leader (FTRL) [Abernethy et al. 2009], which generates updates as follows:

\[
x_{t+1} = \arg\min_{x \in \chi} \left( \eta \sum_{\tau=1}^{t} f_{\tau}, x \right) + D(x, x_{t}).
\]

The updates (2) and (3) can be computed efficiently for many decision sets \( \chi \) and one can achieve an average regret on the order of \( O(1/\sqrt{T}) \) after \( T \) iterations. This regret can be achieved by choosing a fixed step size \( \eta = \sqrt{2\Omega/LT} \), where \( L \) is an upper bound on the \( l_2 \)-norms of the subgradients \( \{f_t\}_{t \geq 0} \) and \( \Omega = \max\{||x - x'||_2 | x, x' \in \chi\} \). Choosing the step size \( \eta \) is problematic, as it requires choosing in advance the number of iterations \( T \) and to know the upper bound \( L \), which may be hard to obtain in many applications or too conservative in practice. Alternatively, it is possible to choose changing step sizes \( \eta_t = \alpha/\sqrt{t} \), for \( \alpha > 0 \). Still, adequately tuning the parameter \( \alpha \) can be time- and resource-consuming. This is not just a theoretical issue, as we highlight in our numerical experiments (Section 5) and in the appendices (Appendices F).

These issues can be addressed by employing adaptive step sizes, which estimate the parameters through the observed subgradients, e.g., AdaHedge for the simplex setting [De Rooij et al. 2014] or AdaFTRL for general compact convex decisions sets [Orabona and Pál 2015]. These adaptive variants have not seen practical adoption in large-scale game-solving, where variants based on Blackwell approachability are preferred (see the next paragraph). As we show in our experiments, adaptive variants of OMD and FTRL perform much worse than our proposed algorithms. While these adaptive algorithms are referred to as parameter-free, this is only true in the sense that they are able to learn the necessary parameters. Our algorithm is parameter-free in the stronger sense that there are no parameters that even require learning.

**Blackwell approachability** In this paper, we use the framework of Blackwell approachability [Blackwell 1956] to develop novel parameter-free algorithms for solving the convex-concave saddle-point problem (1). In principle, Blackwell approachability arises in the framework of repeated two-player games with vector-valued payoff: the goal of the first-player is to choose a sequence of decisions \( x_1, x_2, \ldots \) such that the average of the visited payoff converges to a known target set \( S \), while the second-player is typically playing adversarially. Blackwell’s celebrated theorem [Blackwell 1956] provides an algorithm for constructing such a sequence of decisions \( x_1, x_2, \ldots \), in the case where the target set \( S \) is half-space forceable (see details in Section 2).

Blackwell approachability is a very general framework and the applications are numerous, ranging from stochastic games [Milman 2006], revenue management, market design, and submodular maximization [Niazadeh et al. 2020], calibration [Perchet 2010], learning in games [Aumann et al. 1995], and fair online learning [Chzhen et al. 2021]. In particular, Blackwell approachability can be used as a regret minimizer [Abernethy et al. 2011], and provides a no-regret algorithm, with an average regret of \( O\left(\frac{1}{\sqrt{T}}\right)\) after \( T \) iterations. Crucially, when applied to online regret minimization, Blackwell approachability can be instantiated without evaluating any of the smoothness or convexity parameters of the objective function \( F \), and the resulting no-regret algorithm does not use any step sizes: this is in contrast to classical regret minimizers such as OMD (2) and FTRL (3), which require choosing step sizes.

Despite its appealing properties from a theoretical standpoint, in practice Blackwell approachability is not widely used to solve classical optimization problems. In fact, to the best of our knowledge, the only practical implementation of Blackwell approachability for solving (1) is for the case of bilinear games on the simplex, where \( F(x, y) = \langle x, Ay \rangle \) for \( A \in \mathbb{R}^{n \times m} \), and \( \chi, \gamma \) are simplices. This simplex instantiation is also used for Extensive-Form Games (EFGs), via the aforementioned CFR decomposition [Zinkevich et al. 2007, Farina et al. 2019a]. In the simplex setting, a particular application of Blackwell approachability yields a no-regret algorithm called regret matching (RM) [Hart and Mas-Colell 2000]. Combining RM with specific weighting, thresholding, and alternating schemes yields an algorithm called regret matching+ (RM+) [Tammelin et al. 2015]. RM+ has been used in every case of solving extremely large-scale EFGs in practice, and in particular it was used in recent poker AI milestones, where poker AIs beat human poker players [Bowling et al. 2015, Moravčík et al. 2017, Brown and Sandholm 2018, 2019b]. In fact, RM+ routinely outperforms theoretically-superior methods, such as optimistic variants of OMD and FTRL [Rakhlin and Sridharan 2013, Chiang et al. 2012], which achieve \( O\left(\frac{1}{T}\right)\) convergence rates in the repeated game framework. Despite its very strong empirical performances, RM+ is only defined when the decision set is the simplex. However, many problems of the form (1) have convex sets \( \chi, \gamma \) that are not simplexes, e.g., box constraints or norm-balls for distributionally robust optimization [Ben-Tal et al. 2015]. Encouraged by the very strong empirical performance of RM+ and CFR+, we will construct parameter-free algorithms based on Blackwell approachability for solving more general instances of the saddle-point problem (1).
1.1 Our Contributions

Our main contributions are as follows.

- **Conic Blackwell Algorithm**\(^+\) (CBA\(^+\)). We start from the general reduction between regret minimization over general convex compact sets and Blackwell approachability \cite{Abernethy2011}. This yields a regret minimizer which we will refer to as the *conic Blackwell algorithm* (CBA). Motivated by the practical performance of RM\(^*\) on simplexes, we construct a variant of CBA which uses a thresholding operation analogous to the one employed by RM\(^*\). We call this regret minimizer CBA\(^+\) (Algorithm \ref{alg:csb}). We show that CBA\(^+\) achieves \(O(1/\sqrt{T})\) average regret in the worst-case. A major selling point of CBA\(^+\) is that it does not require any step size choices. Instead, CBA\(^+\) implicitly adjusts to the structure of the domains and losses by being instantiations of a Blackwell approachability algorithm, which is itself parameter-free.

- **Impacts of weights and alternation.** As regret minimizers, we show that both CBA and CBA\(^+\) are compatible with increasing weighting schemes, that put more weights on more recent decisions and payoffs (Theorem \ref{thm:weights} and Theorem \ref{thm:weights+}), where CBA\(^+\) is compatible with different weighting schemes for the decisions and the payoffs. We then introduce a new algorithm for solving convex-concave saddle-point problems by using CBA\(^+\) in a repeated game framework with linear weights on the sequence of decisions and uniform weights on the payoffs (this is known as *linear averaging* in other algorithms \cite{Tammelin2015,Gao2021}), as well as an alternating payoff scheme. We call this algorithm SP-CBA\(^+\). We quantify the benefits of alternation for solving \(1\) (Theorem \ref{thm:alternation}), and show the first strict improvement guarantee for using alternation; note that prior results only showed that it does not slow the convergence \cite{Burch2019}.

- **Efficient implementation of CBA\(^+\).** We show how to implement CBA and CBA\(^+\) when \(\mathcal{X}\) and \(\mathcal{Y}\) are simplexes, \(\ell_p\) balls, and intersections of the \(\ell_2\) ball with a simplex, which arises naturally as a confidence region. More generally, CBA and CBA\(^+\) can be implemented when we can efficiently compute orthogonal projections onto the set \(\mathcal{X}\) and \(\mathcal{Y}\). Note that the general reduction of regret minimization and Blackwell approachability from \cite{Abernethy2011} yields CBA, but does not yield a practically-implementable algorithm, as the authors do not consider which decision sets allow for efficient projections.

- **Practical performance of SP-CBA\(^+\).** We highlight the practical efficacy of our algorithmic framework on several domains. First, we apply SP-CBA\(^+\) to two-player zero-sum matrix games, where the objective function is bilinear, and we compare with RM\(^*\), as well as with AdaHedge and AdaFTRL, two adaptive first-order algorithms. We then apply SP-CBA\(^+\) to extensive-form games (EFGs), where the RM\(^*\) regret minimizer combined with linear averaging, alternation, and a counterfactual regret (CFR\(^+\)) minimization scheme, leads to state-of-the-art practical algorithms \cite{Tammelin2015,Kroer2020,Gao2021}. For EFGs, we find that SP-CBA\(^+\) leads to comparable performance in terms of the iteration complexity, and for some games it slightly outperforms CFR\(^+\). In the simplex setting we also find that SP-CBA\(^+\) outperforms both AdaHedge and AdaFTRL. These results show that SP-CBA\(^+\) recovers the strong practical performance of RM\(^*\) and CFR\(^*\) in the only setting where these two methods apply. Second, and more importantly, we show that SP-CBA\(^+\) leads to strong practical performance in settings where RM\(^*\) and CFR\(^*\) do not apply. We consider instances of distributionally robust logistic regression and Markov decision processes (MDPs). For these two instances of saddle-point problems, we find that SP-CBA\(^+\) performs orders of magnitude better than online mirror descent and follow-the-regularized leader, as well as their optimistic variants, when using their theoretically-correct fixed step sizes. Even when considering tuned step sizes for the other algorithms, SP-CBA\(^+\) performs better, with only a few cases of comparable performance (at step sizes that lead to divergence for some of the other non-parameter-free methods). The fast practical performance of our algorithm, combined with its simplicity and the total lack of step sizes or parameters tuning, suggests that it should be seriously considered as a practical approach for solving convex-concave optimization instances in various settings.

We conclude our introduction with a brief discussion on the average regret achieved by other methods, and resulting convergence to a saddle point. Our algorithm SP-CBA\(^+\) has a rate of convergence towards a saddle point of \(O(1/\sqrt{T})\), similar to OMD and FTRL. In theory, it is possible to obtain a faster \(O(1/T)\) rate of convergence when \(F\) is differentiable with Lipschitz gradients, for example via mirror prox \cite{Nemirovski2004} or other primal-dual algorithms \cite{Chambolle2016}. However, our experimental results show that SP-CBA\(^+\) is faster than optimistic variants of FTRL and OMD \cite{Syrgkanis2015}, the latter being almost identical to the mirror prox algorithm, and both achieving \(O(1/T)\) rate of convergence. A similar conclusion has been drawn in the context of sequential game solving, where the RM\(^*\)-based algorithms have better practical performance than the theoretically-superior \(O(1/T)\)-rate methods \cite{Kroer2020,Kroer2018}. In a similar vein, using error-bound conditions, it is possible to achieve a linear rate, e.g., when solving bilinear saddle-point problems over polyhedral decision sets, by using the
extragradient method [Tseng 1995] or optimistim gradient descent-ascent [Wei et al. 2020]. However, these linear rates rely on unknown constants, and may not be indicative of practical performance.

2 Repeated game framework and Blackwell approachability

We will solve (1) using a repeated game framework. There are $T$ iterations with indices $t = 1, \ldots, T$. In this framework, each iteration $t$ consists of the following steps:

1. Each player chooses strategies $x_t \in \mathcal{X}$, $y_t \in \mathcal{Y}$.
2. The first player observes $f_t \in \partial_x F(x_t, y_t)$ and uses $f_t$ when computing the next strategy.
3. The second player observes $g_t \in \partial_y F(x_t, y_t)$ and uses $g_t$ when computing the next strategy.

In the repeated game framework described above, the first player chooses strategies from $\mathcal{X}$ to minimize the sequence of payoffs in the repeated game, while the second player chooses strategies from $\mathcal{Y}$ in order to maximize payoffs. The goal of each player is to minimize their regret.

As mentioned in Section 1, for matrix games and EFGs, variants of Blackwell approachability are used in practice (via Regret Matching). Details on Regret Matching

Let $\Delta(n)$ be the $n$-dimensional probability simplex. Regret Matching (RM) arises by instantiating Blackwell approachability with the decision space $\mathcal{X}$ equal to $\Delta(n)$, the target set $\mathcal{S}$ equal to the nonpositive orthant $\mathbb{R}^n_-$, and the vector-valued payoff function $u_t(x_t) = f_t - (f_t, x_t)e$ equal to the regret associated to each of the $n$ actions (which correspond to the corners of $\Delta(n)$). Here $e \in \mathbb{R}^n$ is the all one vector.

Hart and Mas-Colell (2000) showed that with this setup, playing each action with probability proportional to its positive regret
We will simply write where \( \omega \) where we always add the newest payoff to the aggregate, and then project the aggregate onto \( C \) meaning that we do not remember "negative regrets." If \( \tilde{\text{UPDATEPAYOFF}} \) which controls how we aggregate payoffs, is implemented by adding the most recent payoff to the aggregate payoffs, and then projecting onto \( C \). In the next section, we describe a framework by [Abernethy et al. 2011] for using Blackwell's algorithm to construct regret minimizers for more general convex sets \( X \); this will lead to the CBA algorithm, from which we will construct \( \text{CBA}^* \). While we use the framework of [Abernethy et al. 2011], we note that the Lagrangian Hedging framework of [Gordon 2007] could also be used as the basis for developing a general class of Blackwell-approachability-style algorithms. It would be interesting to construct a \( \text{CBA}^* \)-like algorithm and efficient projection approaches for such a framework as well.

### 3 Conic Blackwell Algorithm

#### 3.1 Our algorithm

In this section we introduce our main regret minimizer, Conic Blackwell Algorithm Plus (\( \text{CBA}^* \)), which uses a variation of Blackwell’s approachability procedure [Blackwell 1956] to perform regret minimization on a general convex compact decision set \( X \). We will assume that losses are coming from a bounded set; this occurs, for example, if there exists \( L_x, L_y \) (that we do not need to know), such that

\[
\|f\| \leq L_x, \|g\| \leq L_y, \forall x \in X, y \in Y, \forall f \in \partial_x F(x, y), \forall g \in \partial_y F(x, y).
\]

We will simply write \( L \) for \( L_x \) or \( L_y \) when we focus on the regret of a single player. We will also use the notation \( \kappa = \max_{x \in X} \|x\|_2 \) (recall that \( X \) is compact). \( \text{CBA}^* \) is best understood as a combination of two steps. The first is the basic \( \text{CBA} \) algorithm, derived from Blackwell’s algorithm, which we describe next. To convert Blackwell’s algorithm to a regret minimizer on \( X \), we use the reduction from [Abernethy et al. 2011], which considers the conic hull \( C = \text{cone}(\{ \kappa \} \times X) \subset \mathbb{R}^{n+1} \). The Blackwell approachability problem is then instantiated with \( X \) as the decision set, the target set equal to the polar \( C^0 = \{ z : \langle z, \xi \rangle \leq 0, \forall \xi \in C \} \) of \( C \), and payoff vectors \( \langle (f, x), -f \rangle \in \mathbb{R}^{n+1} \). The conic Blackwell algorithm (\( \text{CBA} \)) is implemented by computing the projection \( \pi_C(u) \) of the average payoff vector \( u \) onto \( C \), noting that the projection can be written as \( \alpha(u, x) \) where \( \alpha \geq 0 \) is scalar, and playing the action \( x \). The second step in \( \text{CBA}^* \) is to replace the average payoff vector \( u \) with a running projected aggregation of the payoffs, where we always add the newest payoff to the aggregate, and then project the aggregate onto \( C \).

More concretely, pseudocode for \( \text{CBA}^* \) is given in Algorithm 1. This pseudocode relies on two functions: \( \text{CHOOSEDECISION}_{\text{CBA}^*} : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n \), which maps the aggregate payoff vector \( u \) to a decision in \( X \), and \( \text{UPDATEPAYOFF}_{\text{CBA}^*} \) which controls how we aggregate payoffs. Given an aggregate payoff vector \( u = (\hat{u}, \tilde{u}) \in \mathbb{R} \times \mathbb{R}^n \), we have

\[
\text{CHOOSEDECISION}_{\text{CBA}^*}(u) = (\kappa/\hat{u}) \tilde{u}.
\]

If \( \hat{u} = 0 \), we just let \( \text{CHOOSEDECISION}_{\text{CBA}^*}(u) = x_0 \) for some arbitrary \( x_0 \in X \). The function \( \text{UPDATEPAYOFF}_{\text{CBA}^*} \) is implemented by adding the most recent payoff to the aggregate payoffs, and then projecting onto \( C \). More formally, it is defined as

\[
\text{UPDATEPAYOFF}_{\text{CBA}^*}(u, x, f, \omega) = \pi_C(u + \omega((f, x)/\kappa, -f)),
\]

where \( \omega \) is the weight assigned to the most recent payoff. Because of the projection step in \( \text{UPDATEPAYOFF}_{\text{CBA}^*} \), we always have \( u \in C \), which in turn guarantees that \( \text{CHOOSEDECISION}_{\text{CBA}^*}(u) \in X \), since \( C = \text{cone}(\{ \kappa \} \times X) \).

Let us give some intuition on the effect of projection onto \( C \). For a geometric intuition, it is easier to visualize the dynamics in \( \mathbb{R}^2 \). Figure 1 illustrates the projection step \( \pi_C(\cdot) \) of \( \text{CBA}^* \). At a high level, from \( u_t \) to \( u_{t+1} \), an instantaneous payoff vector

\[
v_{t+1} = \omega_{t+1} ((f_{t+1}; x_{t+1})/\kappa, -f_{t+1})
\]

is first added to \( u_t \), and then the resulting vector \( u_t^+ = u_t + v_{t+1} \) is projected onto \( C \). The projection \( \pi_C(\cdot) \) moves the vector \( u_t^+ \) along the edges of the cone \( C^0 \), preserving the (orthogonal) distance \( d \) to \( C^0 \). Intuitively, from a game-theoretic perspective in the usual case where \( C = \mathbb{R}^2_+ \), the projection eliminates the negative components of the payoffs, meaning that we do not remember "negative regrets."
Algorithm 1 Conic Blackwell Algorithm Plus (CBA*)

1: **Input** A convex, compact set $X \subset \mathbb{R}^n$, $\kappa = \max \{ \|x\|_2 \mid x \in X \}$.
2: **Algorithm parameters** Weights $(\omega_t)_{t \geq 1} \in \mathbb{R}^n$.
3: **Initialization** $t = 1$, $x_1 \in X$.
4: Observe $f_1$ then set $u_1 = \omega_1 ((f_1, x_1)/\kappa, -f_1) \in \mathbb{R} \times \mathbb{R}^n$.
5: **for** $t \geq 1$ **do**
6: Choose $x_{t+1} = \text{CHOOSEDECISION}_{CBA^+}(u_t)$.
7: Observe the loss $f_{t+1} \in \mathbb{R}^n$.
8: Update $u_{t+1} = \text{UPDATEPAYOFF}_{CBA^+}(u_t, x_{t+1}, f_{t+1}, \omega_{t+1})$.
9: Increment $t \leftarrow t + 1$.

---

Figure 1: Illustration of $\pi_C(\cdot)$ for $C = \mathbb{R}^2_+$ (left-hand side) and $C$ any cone in $\mathbb{R}^2$ (right-hand side).

Let us also note the difference between CBA* and the algorithm introduced in [Abernethy et al., 2011], which we have called CBA. CBA uses different UPDATEPAYOFF and CHOOSEDECISION functions. In CBA the payoff update is defined as

$$\text{UPDATEPAYOFF}_{CBA}(u, x, f, \omega) = u + \omega ((f, x)/\kappa, -f).$$

Note in particular the lack of projection as compared to CBA*, this is analogous to the difference between RM and RM*. The CHOOSEDECISION$_{CBA}$ function then requires a projection onto $\hat{C}$:

$$\text{CHOOSEDECISION}_{CBA}(u) = \text{CHOOSEDECISION}_{CBA^+}(\pi_C(u)).$$

Based upon the analysis in [Blackwell, 1956], [Abernethy et al., 2011] show that CBA with uniform weights (both on payoffs and decisions) guarantees $O(1/\sqrt{T})$ average regret.

### 3.2 Regret bounds for CBA and CBA*

In this section we investigate the theoretical performance guarantees of CBA and CBA* when we vary the weights on decisions and payoffs. This is motivated by practical performance, where it has been observed in several other settings that increasing weights usually perform better [Gao et al., 2021, Tammelin et al., 2015, Brown and Sandholm, 2019a], and that alternating update schemes are helpful [Tammelin et al., 2015, Kroer, 2020]. First, we show that CBA and CBA* are both compatible with varying weights $(\omega_t)_{t \geq 1}$, when those weights are used on both decisions and payoffs. Second, we show that CBA* is compatible with different weights $(\omega_t)_{t \geq 1}$ on payoffs and weights $(\theta_t)_{t \geq 1}$ on decisions.

We start with the following theorem, which shows that CBA with weights on both decisions and payoffs is a no-regret algorithm. This generalizes the result of [Abernethy et al., 2011], which shows that CBA works for uniform weights.
Theorem 3.1. Let \((x_t)_{t \geq 1}\) be the sequence of decisions generated by CBA with payoff weights \((\omega_t)_{t \geq 1}\) and let \(S_t = \sum_{\tau=1}^t \omega_\tau\) for any \(t \geq 1\). Then

\[
\sum_{t=1}^T \omega_t \langle f_t, x_t \rangle - \min_{x \in X} \sum_{t=1}^T \omega_t \langle f_t, x \rangle = O (\kappa \cdot d(u_T, C^o)) .
\]

Additionally,

\[
d(u_T, C^o) = O \left( L \cdot \sqrt{\sum_{t=1}^T \omega_t^2} \right) .
\]

Overall, the average regret is such that

\[
\sum_{t=1}^T \omega_t \langle f_t, x_t \rangle - \min_{x \in X} \sum_{t=1}^T \omega_t \langle f_t, x \rangle \leq \sqrt{2KL} \sqrt{\sum_{t=1}^T \omega_t^2} .
\]

The proof of Theorem 3.1 uses the following facts from conic optimization. Several of these are direct consequences of Moreau’s decomposition theorem. We provide proofs of all statements in Lemma 3.2 in Appendix A.

Lemma 3.2. Let \(C \subset \mathbb{R}^{n+1}\) be a closed convex cone and \(C^o\) its polar.

1. If \(u \in \mathbb{R}^{n+1}\), then \(u - \pi_{C^o}(u) = \pi_C(u)\), \(\langle u - \pi_{C^o}(u), \pi_{C^o}(u) \rangle = 0\), and \(\|u - \pi_{C^o}(u)\|_2 \leq \|u\|_2\).

2. If \(u \in \mathbb{R}^{n+1}\) then

\[
d(u, C) = \max_{w \in C^o \cap B_2(1)} \langle u, w \rangle ,
\]

where \(B_2(1) = \{w \in \mathbb{R}^{n+1} | \|w\|_2 \leq 1\}\).

3. If \(u \in C\), then \(d(u, C^o) = \|u\|_2\).

4. Assume that \(C = \text{cone}(\{\kappa\} \times X)\) with \(X \subset \mathbb{R}^n\) convex compact and \(\kappa = \max_{x \in X} \|x\|_2\). Then \(C^o\) is a closed convex cone. Additionally, if \(u \in C\) we have \(-u \in C^o\).

5. Let us write \(\leq_{C^o}\) for the ordering induced by \(C^o\): \(x \leq_{C^o} y \iff y - x \in C^o\). Then

\[
x \leq_{C^o} y, x' \leq_{C^o} y' \Rightarrow x + x' \leq_{C^o} y + y', \quad \forall \, x, x', y, y' \in \mathbb{R}^{n+1} , \tag{5}
\]

\[
x + x' \leq_{C^o} y \Rightarrow x \leq_{C^o} y, \quad \forall \, x, y \in \mathbb{R}^{n+1} , \forall \, x' \in C^o , \tag{6}
\]

6. Assume that \(x \leq_{C^o} y\) for \(x, y \in \mathbb{R}^{n+1}\). Then \(d(y, C^o) \leq \|x\|_2\).

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. The proof proceeds in two steps. We start by proving

\[
\sum_{t=1}^T \omega_t \langle f_t, x_t \rangle - \min_{x \in X} \sum_{t=1}^T \omega_t \langle f_t, x \rangle = O (\kappa \cdot d(u_T, C^o)) .
\]

We have

\[
d(u_T, C^o) = \max_{w \in \text{cone}(\{\kappa\} \times X) \cap B_2(1)} \left( \sum_{t=1}^T \omega_t v_t, w \right) \tag{7}
\]

\[
\geq \max_{x \in X} \sum_{t=1}^T \omega_t v_t, \frac{\langle \kappa, x \rangle}{\|\kappa, x\|_2} \tag{8}
\]

where (7) follows from Statement 2 in Lemma 3.2 and (8) follows from CBA maintaining

\[
u_t = \left( \sum_{\tau=1}^t \omega_\tau \langle f_\tau, x_\tau \rangle - \sum_{\tau=1}^t \omega_\tau f_\tau \right), \forall \, t \geq 1 .
\]
We now prove that we have weighting schemes for the decisions in the regret definition and the aggregate payoffs. This result is analogous to the fact that for the simplex case, $\text{CBA}^+$ is compatible with polynomial averaging schemes on the decision while using constant weights on the aggregate payoffs [Tammelin et al., 2015, Brown and Sandholm, 2019a].

Consider Theorem 3.3. This is one of the crucial components of Blackwell’s approachability framework: the current decision is chosen to force the next payoff to lie in the hyperplane generated by projecting the aggregate payoffs onto the target set. To see this, first note that $u_t - \pi_C^c(u) = \pi_C(u_t)$. Let us write $\pi = (\tilde{\pi}, \tilde{\pi}) = \pi_C(u_t)$. Note that by definition, $x_{t+1} = (\kappa/\pi)\tilde{\pi}$, and $v_{t+1} = (\langle f_{t+1}, x_{t+1} \rangle / \kappa, -f_{t+1})$. Therefore,

$$\langle u_t - \pi_C^c(u_t), v_{t+1} \rangle = \langle \pi, v_{t+1} \rangle = \langle \langle \tilde{\pi}, \tilde{\pi} \rangle, \langle f_{t+1}, x_{t+1} \rangle / \kappa, -f_{t+1} \rangle = \langle \langle \tilde{\pi}, \tilde{\pi} \rangle, \langle f_{t+1}, (\kappa/\pi)\tilde{\pi} \rangle / \kappa, -f_{t+1} \rangle = 0.$$

Next, recall that $d(u_t, C^o)^2 = \|u_t - \pi_C^c(u_t)\|^2_2$. Applying (9) inductively we obtain

$$d(u_t, C^o)^2 \leq \sum_{\tau=1}^{t} \omega_\tau^2 \|v_\tau\|^2_2 \leq L^2 \cdot \sum_{\tau=1}^{t} \omega_\tau^2,$$

where the last inequality follows from the definition of $v_\tau$ and $L$.

In the next theorem, we show a result that may seem surprising: $\text{CBA}^+$ allows us to use two separate and different weighting schemes for the decisions in the regret definition and the aggregate payoffs. This result is analogous to the fact that for the simplex case, $\text{RM}^+$ is compatible with polynomial averaging schemes on the decision while using constant weights on the aggregate payoffs [Tammelin et al., 2015, Brown and Sandholm, 2019a].

**Theorem 3.3.** Consider $(x_t)_{t \geq 1}$ generated by $\text{CBA}^+$ with aggregate payoff weights $(\omega_t)_{t \geq 1}$, when regret is measured using decision weights $(\theta_t)_{t \geq 1}$, and $S_T = \sum_{t=1}^{T} \theta_t$. Assume that $\frac{\theta_{t+1}}{\theta_t} \geq \frac{\omega_{t+1}}{\omega_t}, \forall t \geq 1$. Then

$$\sum_{t=1}^{T} \theta_t (f_t, x_t) - \min_{\pi \in \mathcal{P}} \sum_{t=1}^{T} \theta_t (f_t, \pi) \leq \sqrt{2} \kappa L \frac{\theta_T}{\omega_T} \sqrt{\sum_{t=1}^{T} \omega_t^2}.$$

Our proof heavily relies on the sequence of payoffs belonging to the cone $C$ at every iteration ($u_t \in C, \forall t \geq 1$), and for this reason it does not extend to $\text{CBA}$. We also note that the use of conic optimization somewhat simplifies the argument compared to the proof that $\text{RM}^+$ is compatible with polynomial averaging on decisions and uniform weights on payoffs.
Proof of Theorem 3.3. Recall that \( v_t = \langle f_t, x_t \rangle / \kappa, -f_t \rangle \). By construction and following the same argument as for the proof of Theorem 3.1 we have
\[
T \sum_{t=1}^T \theta_t \langle f_t, x_t \rangle - \min_{x \in X} \sum_{t=1}^T \theta_t \langle f_t, x \rangle \leq \sqrt{2 \kappa} \cdot d \left( \sum_{t=1}^T \theta_t v_t, C^o \right).
\] (11)
Additionally, we always have
\[
\omega_{t+1} v_{t+1} \geq C^o u_{t+1} - u_t.
\] (12)
This is because
\[
\omega_{t+1} v_{t+1} - u_t + u_t = u_t + \omega_{t+1} v_{t+1} - u_t + u_t
\]
\[
= u_t + \omega_{t+1} v_{t+1} - \pi_C (u_t + \omega_{t+1} v_{t+1})
\]
\[
= \pi_C^o (u_t + \omega_{t+1} v_{t+1}) \in C^o.
\]
Therefore, multiplying (12) by \( \theta_{t+1} \) and dividing by \( \omega_{t+1} \), we obtain
\[
\theta_{t+1} v_{t+1} \geq C^o \frac{\theta_{t+1}}{\omega_{t+1}} (u_{t+1} - u_t).
\]
Reformulating the right-hand side we obtain
\[
\theta_{t+1} v_{t+1} \geq C^o \frac{\theta_{t+1}}{\omega_{t+1}} - \frac{\theta_t}{\omega_t} u_t - \left( \frac{\theta_{t+1}}{\omega_{t+1}} - \frac{\theta_t}{\omega_t} \right) u_t.
\]
Summing up the previous inequalities from \( t = 1 \) to \( t = T - 1 \) and using \( u_1 = v_1 \) we obtain
\[
\sum_{t=1}^T \theta_t v_t \geq C^o \frac{\theta_T}{\omega_T} u_T - \sum_{t=1}^{T-1} \left( \frac{\theta_{t+1}}{\omega_{t+1}} - \frac{\theta_t}{\omega_t} \right) u_t.
\]
Note that \( \sum_{t=1}^{T-1} \left( \frac{\theta_{t+1}}{\omega_{t+1}} - \frac{\theta_t}{\omega_t} \right) u_t \in C \), because \( \frac{\theta_{t+1}}{\omega_{t+1}} - \frac{\theta_t}{\omega_t} \geq 0 \). Therefore, Statement 4 in Lemma 3.2 shows that
\[- \sum_{t=1}^{T-1} \left( \frac{\theta_{t+1}}{\omega_{t+1}} - \frac{\theta_t}{\omega_t} \right) u_t \in C^o.\]
Now, by applying (6) in Statement 5 of Lemma 3.2 we have
\[
\sum_{t=1}^T \theta_t v_t \geq C^o \frac{\theta_T}{\omega_T} u_T - \sum_{t=1}^{T-1} \left( \frac{\theta_{t+1}}{\omega_{t+1}} - \frac{\theta_t}{\omega_t} \right) u_t \Rightarrow \sum_{t=1}^T \theta_t v_t \geq C^o \frac{\theta_T}{\omega_T} u_T.
\]
Since \( \frac{\theta_T}{\omega_T} u_T \in C \), Statement 6 shows that
\[
d \left( \sum_{t=1}^T \theta_t v_t, C^o \right) \leq \left\| \frac{\theta_T}{\omega_T} u_T \right\|_2.
\] (13)
By construction \( u_T \) is the sequence of aggregated payoffs generated by CBA with weights \( (\omega_t)_{t \geq 1} \). We now show that \( d(u_T, C^o) = \| u_T \|_2 = O \left( \sum_{t=1}^T \omega_t^2 \right) \). We have
\[
\| u_{t+1} \|_2^2 = \| \pi_C (u_t + \omega_{t+1} v_{t+1}) \|_2^2
\]
\[
\leq \| u_t + \omega_{t+1} v_{t+1} \|_2^2
\] (14)
where (14) follows from Statement 1 in Lemma 3.2. Therefore,
\[
\| u_{t+1} \|_2^2 \leq \| u_t \|_2^2 + \omega_{t+1}^2 \| v_{t+1} \|_2^2 + 2 \omega_{t+1} \langle u_t, v_{t+1} \rangle.
\]
By construction and for the same reason as for (10), \( \langle u_t, v_{t+1} \rangle = 0 \). Therefore, we have the recursion
\[
\| u_{t+1} \|_2^2 \leq \| u_t \|_2^2 + \omega_{t+1}^2 \| v_{t+1} \|_2^2.
\]
By telescoping the inequality above we obtain
\[
\| u_t \|_2^2 \leq \sum_{t=1}^T \omega_t^2 \| v_t \|_2^2.
\]
By definition of $L$, we conclude that
\[ \|u_T\|_2 \leq L \sqrt{\sum_{t=1}^{T} \omega_t^2}. \]
Therefore, by (13), $d(\sum_{t=1}^{T} \theta_t u_t, c^\circ) \leq L \theta_T \omega_T \sqrt{\sum_{t=1}^{T} \omega_t^2}.$ This shows that
\[ \sum_{t=1}^{T} \theta_t \langle f_t, x_t \rangle - \min_{x \in X} \sum_{t=1}^{T} \theta_t \langle f_t, x \rangle \leq \sqrt{2\kappa L \theta_T \omega_T \sum_{t=1}^{T} \omega_t^2}. \]

3.3 Convergence bounds for saddle-point problems

In this section, we show how the regret bounds from the previous section translate into convergence rates for solving convex-concave saddle-point problems in the repeated game framework. In particular, the following theorem gives the convergence rate of CBA\(^*\) and CBA for solving saddle-point problems of the form (1), based on our bounds on the regret of each player under various weighting schemes. The proof is in Appendix B.

**Theorem 3.4.** Let $L = \max\{L_x, L_y\}$ defined in (4) and $\kappa = \max\{\max\{\|x\|_2, \|y\|_2\} | x \in X, y \in Y\}$.

1. Let $(\bar{x}_T, \bar{y}_T) = \sum_{t=1}^{T} \omega_t (x_t, y_t) / S_T$, where $(x_t)_{t \geq 1}, (y_t)_{t \geq 1}$ are generated by the repeated game framework with CBA with weights $(\omega_t)_{t \geq 1}$ on both decisions and payoffs and $S_T = \sum_{t=1}^{T} \omega_t$. Assume that $\omega_t = t^p, \forall t \geq 1$. Then
   \[ \max_{y \in Y} F(\bar{x}_T, y) - \min_{x \in X} F(x, \bar{y}_T) = O\left(\frac{\kappa L \sqrt{p+1}}{\sqrt{T}}\right). \]

2. Let $p, q \in \mathbb{N}$ with $q \geq p$. Let $(\bar{x}_T, \bar{y}_T) = \sum_{t=1}^{T} \theta_t (x_t, y_t) / S_T$, where $(x_t)_{t \geq 1}, (y_t)_{t \geq 1}$ are generated by the repeated game framework with CBA\(^*\) with aggregate payoff weights $(\omega_t)_{t \geq 1}$, and decision weights $(\theta_t)_{t \geq 1}$ and $S_T = \sum_{t=1}^{T} \theta_t$. Assume that $\theta_t = t^q, \omega_t = t^p, \forall t \geq 1$. Then
   \[ \max_{y \in Y} F(\bar{x}_T, y) - \min_{x \in X} F(x, \bar{y}_T) = O\left(\frac{\kappa L (q + 1)}{\sqrt{p+1} \sqrt{T}}\right). \]

We note that larger weights lead to slightly worse worst-case convergence guarantees. In contrast to this, we will see in our numerical simulations that the strongest empirical performances for CBA\(^*\) are obtained for $q = 1, p = 0$, i.e., linear weights on the decisions and uniform weights on the payoffs.

Let us compare our bounds with the regret bounds of classical first-order methods. We consider $p, q = 0$. CBA and CBA\(^*\) achieve $O\left(\kappa L / \sqrt{T}\right)$ average regret, whereas online mirror descent (OMD) [Nemirovski and Yudin, 1983, Ben-Tal and Nemirovski, 2001] and follow-the-regularized-leader (FTRL) [Abernethy et al., 2009, McMahan, 2011] achieve $O\left(\Omega L / \sqrt{T}\right)$ average regret, where $\Omega = \max\{\|x - x'\|_2 | x, x' \in X\}$. We can always recenter $X$ to contain 0, in which case the bounds for OMD/FTRL and CBA\(^*\) are equivalent since $\kappa \leq \Omega \leq 2\kappa$. The bound on the average regret for optimistic OMD (OOMD, Chiang et al., 2012) and optimistic FTRL (OFTRL, Rakhlin and Sridharan, 2013) is $O\left(\Omega^2 L / T\right)$ in the repeated game framework, a priori better than the bound for CBA\(^*\) as regards the number of iterations $T$. Nonetheless, we will see in Section 5 that the empirical performance of CBA\(^*\) is better than that of $O(1/T)$ methods. A similar situation occurs for RM\(^*\) compared to OOMD and OFTRL for solving extensive-form games such as poker [Farina et al., 2019b, Kroer et al., 2020].

3.4 Improved convergence bounds using alternation

Alternation is a simple variation of the repeated game framework from Section 2. Alternation is known to lead to significant speedup for RM\(^*\) [Tammelin et al., 2015], and we will observe in our simulations (Section 5) that this holds
for CBA* as well. In the repeated game framework with alternation, at iteration $t$, the second player is provided with the decision $x_t$ of the first player for iteration $t$. Because alternation is defined the same way for both CBA and CBA*, we omit the subscripts in CHOOSEDECISION and UPDATEPAYOFF. In particular, at iteration $t$ of the repeated game framework with alternation, the players choose $x_t$ and $y_t$ as follows:

1. Both players start with aggregate payoffs $u_{t-1}^x, u_{t-1}^y$.
2. The first player chooses a decision $x_t$ based on $u_{t-1}^x$:
   $$x_t = \text{CHOOSEDECISION}(u_{t-1}^x).$$
3. For $g_{t-1} = \partial_y F(x_t, y_{t-1})$, the second player updates its aggregate payoff:
   $$u_t^y = \text{UPDATEPAYOFF} \left( u_{t-1}^y, y_{t-1}, g_{t-1}, \omega_t \right).$$
4. The second player chooses a decision $y_t$ based on $u_t^y$:
   $$y_t = \text{CHOOSEDECISION}(u_t^y).$$
5. For $f_t = \partial_x F(x_t, y_t)$, the first player updates its aggregate payoff:
   $$u_t^x = \text{UPDATEPAYOFF} \left( u_{t-1}^x, x_t, f_t, \omega_t \right).$$

Recall that we use the repeated game framework to solve (1) because we can bound the duality gap by the sum of the average regrets of each player using Theorem 2.1. It is known that in the repeated game framework with alternation, it is possible to construct decisions such that Theorem 2.1 fails to hold, because of the mismatch in the sequences of decisions of the players [Farina et al., 2019a]. That said, it was later shown that a modified version of Theorem 2.1 holds [Burch et al., 2019]. Here we state a more general version of that result, which was first shown in a set of lecture notes [Kroer, 2020]. In particular, the following bound holds on the duality gap. For the sake of completeness, we provide the proof in Appendix C.

**Theorem 3.5.** Consider some weights $(\theta_t)_{t \geq 1}$ and $S_T = \sum_{t=1}^T \theta_{t+1}$. Let $(\bar{x}_T, \bar{y}_T) = \sum_{t=1}^T \theta_{t+1} (x_{t+1}, y_t) / S_T$, where $(x_t)_{t \geq 1}, (y_t)_{t \geq 1}$ are generated by the repeated game framework with alternation. Then

$$\max_{y \in \mathcal{Y}} F(\bar{x}_T, y) - \min_{x \in \mathcal{X}} F(x, \bar{y}_T) \leq \frac{1}{S_T} \sum_{t=1}^T \theta_{t+1} \left( \max_{y \in \mathcal{Y}} \sum_{t=1}^T \theta_{t+1} (x_t, y_t) - \sum_{t=1}^T \theta_{t+1} (g_t, y_t) \right)$$

$$+ \frac{1}{S_T} \left( \sum_{t=1}^T \theta_{t+1} (f_t, x_t) - \min_{x \in \mathcal{X}} \sum_{t=1}^T \theta_{t+1} (f_t, x) \right)$$

$$+ \frac{1}{S_T} \left( \sum_{t=1}^T \theta_{t+1} (F(x_{t+1}, y_t) - F(x_t, y_t)) \right).$$

From Theorem 3.5 we see that alternation guarantees convergence to a solution of (1), if

$$\sum_{t=1}^{t+1} \theta_{t+1} (F(x_{t+1}, y_t) - F(x_t, y_t)) \leq 0.$$ (15)

In the framework of RM and RM*, we have $\mathcal{X} = \Delta(n), \mathcal{Y} = \Delta(m)$ and the objective function is bilinear. In this case, it is shown in [Burch et al., 2019] that (15) holds. In particular, for any $t \in [T]$, it holds that $F(x_{t+1}, y_t) - F(x_t, y_t) \leq 0$. We provide the following stronger result for CBA* in the case of an objective function $F$ that is linear in one of the two variables, with any convex compact decision sets $\mathcal{X}$ and $\mathcal{Y}$. The proof is presented in Appendix D.

**Theorem 3.6.** Assume that $(x, y) \mapsto F(x, y)$ is linear in $x$.

1. In the framework of Theorem 3.5, suppose that $(x_t, y_t)_{t \geq 1}$ are generated by CBA* with weights $(\omega_t)_{t \geq 1}$ on the payoffs. We have, for $t \geq 1$,
   $$F(x_{t+1}, y_t) - F(x_t, y_t) \leq -\frac{\kappa}{\omega_t \cdot \|u_t^x\|_\infty} \|u_t^x - u_{t-1}^x\|_2^2.$$

2. In the framework of Theorem 3.5, suppose that $(x_t, y_t)_{t \geq 1}$ are generated by CBA with weights $(\omega_t)_{t \geq 1}$ on the payoffs. We have, for $t \geq 1$,
   $$F(x_{t+1}, y_t) - F(x_t, y_t) \leq -\frac{\kappa}{\omega_t \cdot \|\pi_C (u_t^x)\|_\infty} \|\pi_C (u_t^x) - \pi_C (u_{t-1}^x)\|_2^2.$$
4 Efficient implementations of CBA

We now turn to efficiently implementing CBA and CBA\(^*\). The main bottleneck of both CBA\(^*\) and CBA is to efficiently compute \(\pi_C(u)\), the orthogonal projection of a vector \(u\) on the cone \(C = \text{cone}\{\kappa\} \times \mathcal{X}\):

\[
\pi_C(u) = \arg\min_{y \in C} \|y - u\|_2^2.
\] (16)

Note that this issue is not discussed in Abernethy et al. [2011], which do not provide an efficient implementation of CBA. In this section, we show how to efficiently solve (16) for many important decision sets \(\mathcal{X}\). One of the critical components of our proofs is Moreau’s Decomposition Theorem [Combettes and Reyes, 2013] (Statement [2] in Lemma 3.2), which states that \(\pi_C(u)\) can be recovered from \(\pi_C^\circ(u)\) and vice versa, because for any convex cone \(C\), we have \(\pi_C(u) + \pi_C^\circ(u) = u\). All the proofs for this section are presented in Appendices E.

4.1 Simplex

Assume that \(\mathcal{X} = \Delta(n)\). This setting is standard for matrix games. It is also used for extensive-form games, because CFR decomposes regret minimization over the tree-like decision space into a set of local regret minimizations over simplexes [Zinkevich et al., 2007]. In the game setting, \(n\) is the number of actions of a player and \(x \in \Delta(n)\) represents a randomized strategy. When \(\mathcal{X} = \Delta(n)\), we show that \(\pi_C^\circ(u)\) can be computed in \(O(n \log(n))\) using a sorting trick similar to that for the standard simplex projection, and therefore \(\pi_C(u)\) can be computed in \(O(n \log(n))\) using Moreau’s decomposition. In particular, we provide the following closed-form expression for the polar cone \(C^\circ\).

**Lemma 4.1.** Let \(C = \text{cone}\{1\} \times \Delta(n)\). Then \(C^\circ = \{(\tilde{y}, \hat{y}) \in \mathbb{R}^{n+1} | \max_{i \in [n]} \hat{y}_i \leq -\tilde{y}\}\).

Therefore, computing \(\pi_C^\circ(u)\) is equivalent to solving

\[
\min\{(\tilde{y} - \hat{u})^2 + \|\tilde{y} - \hat{u}\|_2^2 | (\tilde{y}, \hat{y}) \in \mathbb{R}^{n+1}, \max_{i \in [n]} \hat{y}_i \leq -\tilde{y}\}.
\] (17)

We prove the following proposition in Appendix E.

**Proposition 4.2.** Let \(\mathcal{X} = \Delta(n)\). An optimal solution \(\pi_C^\circ(u)\) to (17) can be computed in \(O(n \log(n))\) arithmetic operations. Therefore, \(\pi_C(u)\) can be computed in \(O(n \log(n))\) arithmetic operations.

4.2 \(\ell_p\) balls

For \(p \geq 1\) and \(p = \infty\), we consider the \(\ell_p\) balls \(\mathcal{X} = \{x \in \mathbb{R}^n | \|x\|_p \leq 1\}\). This type of decision set appears in many problems in optimization, including robust optimization [Ben-Tal et al., 2015], distributionally robust logistic regression [Namkoong and Duchi, 2016], \(\ell_\infty\) regression [Sidford and Tian, 2018] and saddle-point reformulation of Markov Decision Processes [Jin and Sidford, 2020]. We first reformulate the cones \(C\) and \(C^\circ\). Recall that \(\kappa = \max\{\|x\|_2 | x \in \mathcal{X}\}\).

**Lemma 4.3.** Let \(\mathcal{X} = \{x \in \mathbb{R}^n | \|x\|_p \leq 1\}\), with \(p \geq 1\) or \(p = \infty\). Let \(q \in \mathbb{R} \cup \{+\infty\}\) be such that \(1/p + 1/q = 1\). Then

\[
C = \{(\tilde{y}, y) \in \mathbb{R} \times \mathbb{R}^n | \|y\|_p \leq \tilde{y} / \kappa\},
\]

\[
C^\circ = \{(\tilde{y}, y) \in \mathbb{R} \times \mathbb{R}^n | \|y\|_q \leq -\kappa \tilde{y}\}.
\]

Based on Lemma 4.3, we can prove the following propositions.

**Proposition 4.4.** Let \(\mathcal{X} = \{x \in \mathbb{R}^n | \|x\|_p \leq 1\}\) for \(p \in \{1, \infty\}\). Then \(\pi_C(u)\) can be computed in \(O(n \log(n))\) operations.

**Proposition 4.5.** Let \(\mathcal{X} = \{x \in \mathbb{R}^n | \|x\|_2 \leq 1\}\). Then \(\pi_C(u)\) can be computed in \(O(n)\) operations.
Experimental setup

Numerical experiments

linear averaging on the decisions and uniform averaging on the payoffs. In Figures 2-3, we compare the performance of these algorithms.

2020, Kroer et al., 2018, Farina et al., 2019b. We also compare with two other scale-free and parameter-free no-regret algorithms, AdaHedge [De Rooij et al., 2014] and AdaFTRL [Orabona and Pál, 2015], with the $\ell_2$ norm as the Bregman divergence. Similarly as for SP-CBA$, for RM$* we use the repeated game framework with alternation, along with linear averaging on the decisions and uniform averaging on the payoffs. In Figures 2-5, we compare the performance

A

normal distribution of mean $A$ and variance $A$. We compare

Matrix games are saddle-point problems with a bilinear objective function and simplexes as decision sets:

\begin{align}
\min_{x \in \Delta(n)} \max_{y \in \Delta(m)} \langle x, Ay \rangle
\end{align}

where $A \in \mathbb{R}^{n \times m}$ is the matrix of payoffs of the game. We can view (20) as a zero-sum game between the first player and the second player, where the coefficient $A_{ij} \in \mathbb{R}$ represents payoffs obtained by the second player when the first player chooses the action $i$ and the second player chooses action $j$.

Experimental setup

We generate 100 synthetic matrices $A$ of size $\mathbb{R}^{n \times m}$ with $(n, m) = (100, 50)$. Similarly as in Chambolle and Pock [2016], Nesterov [2005], for the coefficients of $A$ we consider a uniform distribution in $[0, 1]$ or a normal distribution of mean 0 and variance 1. We compare SP-CBA$^*$ with RM$^*$, which is known to achieve the best empirical performance compared to a wide range of algorithms, including Hedge and other first-order methods [Kroer 2020, Kroer et al., 2018, Farina et al., 2019b]. We also compare with two other scale-free and parameter-free no-regret algorithms, AdaHedge [De Rooij et al., 2014] and AdaFTRL [Orabona and Pál, 2015], with the $\ell_2$ norm as the Bregman divergence. Similarly as for SP-CBA$^*$, for RM$^*$ we use the repeated game framework with alternation, along with linear averaging on the decisions and uniform averaging on the payoffs.
of the four algorithms (SP-CBA\textsuperscript{+}, RM\textsuperscript{+}, AdaHedge and AdaFTRL) for solving (20). In Figure 2a and Figure 2b we let the four algorithms run for $T = 1000$ iterations, and we show the duality gap of the current running average as a function of the number of iterations. This shows the progress made by the algorithms toward solving (20) at each iteration. In Figure 3a and Figure 3b we run the four algorithms for time-max = 10 seconds, and we show the duality gap as a function of the time of computation. We average all the results over 50 randomly generated instances. Note that both axis are in logarithmic scale.

**Results and discussion** When we compare the duality gap as a function of the number of iterations (Figure 2a and Figure 2b), we note that SP-CBA\textsuperscript{+} performs on par with RM\textsuperscript{+}, and both algorithms vastly outperform AdaHedge and AdaFTRL. However, each iteration of SP-CBA\textsuperscript{+} on the simplex requires solving $O(n \log(n))$ arithmetic operations (see Section 4.1), whereas each iteration of RM\textsuperscript{+} can be performed in $O(n)$ operations. Therefore, when we compare the duality gap as a function of the computation time (Figure 3a and Figure 3b), we note that RM\textsuperscript{+} outperforms SP-CBA\textsuperscript{+}, even though after roughly ten seconds of computation, the performances of SP-CBA\textsuperscript{+} and RM\textsuperscript{+} are equivalent.

![Figure 2: Comparison of SP-CBA\textsuperscript{+}, RM\textsuperscript{+}, AdaHedge and AdaFTRL on instances of matrix games, with respect to the number of iterations. The payoffs are chosen randomly, with uniform distribution in Figure 2a and normal distribution in Figure 2b](image)

![Figure 3: Comparison of SP-CBA\textsuperscript{+}, RM\textsuperscript{+}, AdaHedge and AdaFTRL on instances of matrix games, with respect to computation time. The payoffs are chosen randomly, with uniform distribution in Figure 3a and normal distribution in Figure 3b](image)

### 5.2 Extensive-form games

Extensive-form games (EFGs, [von Stengel, 1996; Zinkevich et al., 2007]) are used to model sequential games with imperfect information. For example, they were used for superhuman poker AIs in games such as Texas hold’em [Tammelin et al., 2015; Brown and Sandholm, 2018; 2019b; Moravčík et al., 2017]. EFGs can be written as saddle-point problems, with a bilinear objective functions and polytopes $X, Y$ encoding the players’ decision spaces [von Stengel, 1996]. Based on the counterfactual regret minimization (CFR) framework [Zinkevich et al., 2007], EFGs can be solved
via decomposition into a set of simplex-based regret minimization problems. We point the reader to Farina et al. (2019) for more details.

**Experimental setup** For solving EFGs, we combine the CFR decomposition with CBA\(^+\) as a regret minimizer on the simplex. For the sake of simplicity, we will still call the resulting algorithm SP-CBA\(^+\) (since we use alternation and linear averaging on the decisions), even though the algorithm relies on the CFR decomposition for EFGs (which is not necessary for solving the other saddle-point instances from Section 5.3 and Section 5.4). We compare SP-CBA\(^+\) with CFR\(^+\) (Bowling et al., 2015), the algorithm with the strongest empirical performance for solving EFGs. Note that both SP-CBA\(^+\) and CFR\(^+\) guarantee a \(O(1/\sqrt{T})\) rate of convergence to a Nash equilibrium. We compare SP-CBA\(^+\) and CFR\(^+\) on several Leduc poker benchmark instances, a search game, and sheriff; we refer to Farina et al. (2021) for details about the instances. Similarly as in Section 5.1 we compare the progress of SP-CBA\(^+\) and CFR\(^+\) both as a function of computation time and number of iterations in the repeated game framework. We run the algorithms for \(t = 100 \text{ seconds} \neq 1500 \text{ iterations}\); note that we choose \(t = 100 \text{ seconds} \neq 1500 \text{ iterations}\) for matrix games because the EFG instances are way larger than the matrix games from Section 5.1.

**Results and discussion** If we only consider the duality gap as a function of the number of iterations (Figure 7, SP-CBA\(^+\) performs on par with CFR\(^+\), and significantly outperforms CFR\(^+\) on some EFGs instances (Figure 4a and Figure 4b). However, when we consider the progress made by each algorithm during \(t = 100 \text{ seconds} \neq 1500 \text{ iterations}\) (Figure 5), CFR\(^+\) enjoys better numerical performances than SP-CBA\(^+\). This is because the updates are closed-form in CFR\(^+\), whereas each update of SP-CBA\(^+\) requires to solve an equation, a situation similar as for matrix games over the simplex (Section 5.2). It is interesting to note that for EFGs, the difference in per-iteration computation time has a bigger impact than for matrix games; it is possible that this is due to our python-based implementation of SP-CBA\(^+\). Better implementations of SP-CBA\(^+\) for EFGs could potentially lead to better results. To conclude this section, we note that CFR\(^+\) enjoys the best empirical performances for solving EFGs, and it is not concerning that SP-CBA\(^+\) can not outperform CFR\(^+\) on EFGs (in terms of computation time). Instead, we will see in the next section how SP-CBA\(^+\) carries over these very strong empirical results to saddle-point instances where CFR\(^+\) does not apply and where SP-CBA\(^+\) can be implemented more efficiently.

![Figure 4: Comparison of SP-CBA\(^+\) and CFR\(^+\) for solving extensive-form games, as regards the number of iterations.](image)

![Figure 5: Comparison of SP-CBA\(^+\) and CFR\(^+\) for solving extensive-form games, based on the computation time.](image)

### 5.3 Distributionally robust logistic regression

Distributionally robust optimization exploits knowledge of the statistical properties of the model parameters to obtain risk-averse optimal solutions (Rahimian and Mehrtra 2019). We focus on the following instance of distributionally...
robust logistic regression \cite{Namkoong2016,Ben-Tal2015}. There are $m$ observed feature-label pairs $(a_i, b_i) \in \mathbb{R}^n \times \{-1, 1\}$, and we want to solve

$$\begin{array}{ll}
\min_{x \in \mathbb{R}^n, \|x-x_0\|_2 \leq \epsilon_x} & \max_{y \in \Delta(m), \|y-y_0\|_2 \leq \epsilon_y} \sum_{i=1}^{m} y_i \ell_i(x) + \frac{\mu}{2} \|x\|_2^2 \\
\end{array} \tag{21}$$

where $\ell_i(x) = \log(1 + \exp(-b_i a_i^\top x))$ and $\mu \geq 0$. The formulation (21) takes a worst-case approach to put more weight on misclassified observations and provides some statistical guarantees, e.g., it can be seen as a convex regularization of standard empirical risk minimization instances \cite{Duchi2021}.

**Experimental setup** We compare SP-CBA* with four classical first-order methods (FOMs): Online Mirror Descent (OMD), Optimistic OMD (O-OMD), Follow-The-Regularized-Leader (FTRL) and Optimistic FTRL (O-FTRL). We provide a detailed presentation of our implementations of these algorithms and our experimental setting in Appendix F; we use the $\ell_2$ norm as the Bregman divergence. We compare the performances of these algorithms with SP-CBA* on two synthetic datasets and two real data sets. We use parameters $x_0 = (1, \ldots, 1)/n, \epsilon_x = 10, y_0 = (1, \ldots, 1)/m, \epsilon_y = 1/2m, \mu = 0.1$ in (21), and we initialize all algorithms at $x_0, y_0$. For the synthetic classification instances, we generate a vector $x^* \in \mathbb{R}^n$, sample some vectors $a_i \in \mathbb{R}^n$ at random for $i \in \{1, \ldots, m\}$, set labels $b_i = \text{sign}(a_i^\top x^*)$, and then we flip 10% of the labels. We consider two types of synthetic instances: one where $a_{ij}$ is sampled from a uniform distribution in $[0, 1]$, and one where $a_{ij}$ is sampled from a normal distribution with mean 0 and variance 1. For the real classification instances, we use the following datasets from the libsvm website\footnote{https://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/} adult and splice.

One of the main motivations for SP-CBA* is to obtain a parameter-free algorithm. In contrast, the other FOMs considered in this section require choosing step sizes $\eta_t$ at every iteration $t$. This is a major limitation in practice: if the step sizes are too small, the iterates may be very conservative, while the algorithms may diverge with very large step sizes. We will compare the performances of the FOMs for both the fixed, theoretically-correct step sizes, and for tuned step sizes. The computation of the theoretically-correct step sizes is presented in Appendix F.3. To tune the FOMs, we run them for the first 10 iterations, with step sizes $\eta_1 = \alpha/\sqrt{T+1}$ for OMD and FTRL, and step size $\eta_t = \alpha$ for O-OMD and O-FTRL, and we search for the best $\alpha \in \{0.01, 0.1, 1, 10, 100\}$. We then choose the value of $\alpha$ that lead to the smallest duality gap after 10 iterations, and use this value for the remaining $T = 1000$ iterations. Note that the tuning time and iterations (where the first 10 iterations are repeated with various values of $\alpha$) are counted in the total computation time and number of iterations of the FOMs. We acknowledge that this tuning method is only one possibility and that the multiplicative factor $\alpha$ could be chosen in many different ways. However, any other tuning framework would still be resource-demanding and uncertain. In contrast, SP-CBA* does not require any tuning, and, as we will see, outperforms even the tuned FOMs. Finally, on the $y$-axis we only report the worst-case loss of the current average $\bar{x}_T$; in particular, we do not compute the duality gap at every iteration, because for a fixed value of $y$, computing the optimal $x$ requires solving a (regularized) nominal logistic regression, which would be computationally intensive to do at every iteration.

**Proximal updates for the first-order methods** Note that in (21), SP-CBA* is instantiated on an $\ell_2$ ball (for the first player) and the intersection of an $\ell_2$ ball and the simplex (for the second player). As shown in Section 4.2 and Section 4.3, this leads to closed-form updates for SP-CBA* at every iteration. In contrast, OMD, FTRL, O-OMD and O-FTRL require binary searches for the decision of the second player at each iteration, see Appendix F. The functions used in the binary searches themselves require solving an optimization program (an orthogonal projection onto the simplex) at each evaluation. Even though computing the orthogonal projection of a vector onto the simplex of size $m$ can be done in $O(m \log(m))$, this results in slower overall running time, compared to SP-CBA* with closed-form updates at each iteration. The situation is even worse for O-OMD which requires two proximal updates at each iteration.

**Results and discussion** In Figure 6, we show the progress of all algorithms toward solving (21) as a function of the number of iterations, when the theoretical step sizes are used for the FOMs. We notice that all FOMs are progressing very slowly toward an optimal solution. This is because the theoretical step sizes are very small, relying on upper bounds on the Lipschitz constants of the objective function of (21). In contrast, SP-CBA* quickly converges to an optimal solution, even though we see in Figure 6a that during the first few iterations, SP-CBA* may increase the objective function. In Figure 7, we tune the FOMs for the first 10 iterations, before running them (with the tuned step sizes). We note that depending on the datasets, the tuned FOMs may perform very well (e.g., OMD in Figure 7a) all FOMs in Figure 7b (O-OMD in Figure 7c), but may also fail to converge to an optimal solution, even after very good performances during the first iterations (e.g., O-FTRL in Figure 7c). This is because the convergence guarantees of the FOMs may fail to hold, for large choices of the multiplicative factor $\alpha$. In Figure 8 and Figure 9, we present the same experiments but where we record the computation time on the $x$-axis. Recall that the per-iteration computation time of SP-CBA* is shorter than for the FOMs, because SP-CBA* has closed-form updates in this setting. Therefore, we still observe in Figures 8 and 9 that SP-CBA* outperforms the classical FOMs.
5.4 Markov decision processes

Markov Decision Processes (MDPs) are used as a modeling tool for sequential decision-making problems [Puterman, 1994], and have found applications in game learning [Min et al., 2013] and healthcare [Grand-Clément et al., 2020, Alagoz et al., 2010, Steimle and Denton, 2017]. In a finite MDP, the set of states is \([n]\) and there are \(A\) actions. For each state-action pair \((s, a)\), there is an associated instantaneous reward \(r_{sa}\) as well as a distribution \(P_{sa} \in \Delta(n)\) over the possible next states in \([n]\). We write \(r_{sa} = \max_{s', a} r_{s'a}\) and we assume, without loss of generality, that \(r_{sa} \geq 0, \forall (s, a) \in [n] \times [A]\). Given a discount factor \(\lambda \in (0, 1)\) and an initial probability distribution \(p_0 \in \Delta(n)\), the goal of the decision-making is to maximize the infinite-horizon discounted cumulative reward. This leads to the following linear programming formulation [Puterman, 1994]:

\[
\min \{(1 - \lambda)p_0^\top v \mid v_s \geq r_{sa} + \lambda P_{sa}^\top v, \forall (s, a) \in [n] \times [A]\}
\]

which can be rewritten as a saddle-point problem [Jin and Sidford, 2020]:

\[
\min_{v \in \mathbb{R}^n, \|v\|_2 \leq \sqrt{\max_{s \in [n]} r_{sa} / (1 - \lambda)}} \max_{\mu \in \Delta(n \times A)} (1 - \lambda)p_0^\top v + \sum_{a=1}^A \sum_{s=1}^n \mu_{sa} \left(r_{sa} + \lambda P_{sa}^\top v - v_s\right),\tag{22}
\]

Figure 6: Comparisons of CBA*, OMD, FTRL, OOMD and OFTRL to compute a solution to the distributionally robust logistic regression problem [21], based on the number of iterations. The theoretical choices of step sizes are used in the first-order methods.

Figure 7: Comparisons of CBA*, OMD, FTRL, OOMD and OFTRL to compute a solution to the distributionally robust logistic regression problem [21], based on the number of iterations. The tuned step sizes are used in the first-order methods.

Figure 8: Comparisons of CBA*, OMD, FTRL, OOMD and OFTRL to compute a solution to the distributionally robust logistic regression problem [21], based on the computation time. The theoretical choices of step sizes are used in the first-order methods.
Figure 9: Comparisons of CBA\(^*\), OMD, FTRL, OOMD and OFTRL to compute a solution to the distributionally robust logistic regression problem \(^{(21)}\), based on the computation time. The tuned step sizes are used in the first-order methods.

where we add the constraint \(\|v\|_2 \leq \sqrt{n r_{\infty}} / (1 - \lambda)\) because CBA\(^*\) requires a bounded decision set \(\mathcal{X}\); this is a valid constraint for the optimal solution \(v^* \in \mathbb{R}^n\) to the MDP problem, because \(v^*\) satisfies \(0 \leq v^*_s \leq r_{\infty} / (1 - \lambda), \forall s \in [n]\).

**Experimental setup** We test the performances of SP-CBA\(^*\) for solving \((22)\) on random Garnet MDPs (Generalized Average Reward Non-stationary Environment Test-bench, \(\text{Archibald et al.} [1995] \text{Bhatnagar et al.} [2007]\)), a class of random MDP instances widely used for benchmarking sequential decision-making algorithms. Garnet MDPs are parametrized by a branching factor \(n_b\), which represents the proportion of reachable next states from each state-action pair \((s, a)\). We choose \(S = 100, A = 50, n_b = 50\%, \lambda = 0.95\). We average the performances of our algorithm over 10 random instances of Garnet MDPs, where the reward parameters are drawn at random uniformly in \([0, 10]\). We compare SP-CBA\(^*\) with the same first-order methods as in the previous section: OMD, FTRL, and their optimistic variants, with the same tuning method. The computation of the upper bounds \(L_v\) and \(L_{\infty}\) are detailed in Appendix G. We acknowledge that at the scale of the instances considered in this paper, MDPs can be solved efficiently using policy iteration. This algorithm is specialized to solving MDPs and differs greatly from SP-CBA\(^*\) which is based on the repeated game framework; for this reason, we compare SP-CBA\(^*\) with first-order methods that are widely applicable and that have been developed for larger MDP instances, e.g. online mirror descent for MDPs \(\text{Jin and Sidford} [2020]\).

**Results and discussion** Similarly as in the two previous section, we note that SP-CBA\(^*\) outperforms OMD, FTRL as well as the optimistic variants, even after they are tuned. We note that in our tuning method, choosing the best step sizes after observing the first 10 iterations may lead to algorithms that choose step sizes that are too large and algorithms that fail to converge, such as O-OMD in Figures 11a, 11b. Also, we note in Figure 11b that tuning the FOMs may require a lot of computation time. In contrast, SP-CBA\(^*\) does not need to be tuned and all the computation time in SP-CBA\(^*\) is used to make progress toward solving \((22)\).

Figure 10: Comparisons of CBA\(^*\), OMD, FTRL, OOMD and OFTRL to compute a solution to the MDP problem \((22)\). The theoretical choices of step sizes are used in the first-order methods.
6 Conclusion

We have proposed SP-CBA+, an algorithm based on Blackwell approachability for solving classical instances of saddle-point optimization. Our algorithm is 1) simple to implement for many practical decision sets, 2) completely parameter-free and does not attempt to learn any step sizes, and 3) competitive with, or even better than, state-of-the-art approaches with both theoretical and tuned parameters. Interesting future directions of research include designing efficient implementations for other widespread decision sets (e.g., based on Kullback-Leibler divergence or ϕ-divergence), extending SP-CBA+ to unbounded decision sets, and developing novel accelerated versions based on strong convex-concavity or optimism.

References

Jacob Abernethy, Peter L Bartlett, and Elad Hazan. Blackwell approachability and no-regret learning are equivalent. In Proceedings of the 24th Annual Conference on Learning Theory, pages 27–46. JMLR Workshop and Conference Proceedings, 2011.

Jacob D Abernethy, Elad Hazan, and Alexander Rakhlin. Competing in the dark: An efficient algorithm for bandit linear optimization. 2009.

Oguzhan Alagoz, Heather Hsu, Andrew J Schaefer, and Mark S Roberts. Markov decision processes: a tool for sequential decision making under uncertainty. Medical Decision Making, 30(4):474–483, 2010.

TW Archibald, KIM McKinnon, and LC Thomas. On the generation of Markov decision processes. Journal of the Operational Research Society, 46(3):354–361, 1995.

Robert J Aumann, Michael Maschler, and Richard E Stearns. Repeated games with incomplete information. MIT press, 1995.

Amir Beck and Marc Teboulle. Mirror descent and nonlinear projected subgradient methods for convex optimization. Operations Research Letters, 31(3):167–175, 2003.

Aharon Ben-Tal and Arkadi Nemirovski. Lectures on modern convex optimization: analysis, algorithms, and engineering applications, volume 2. Siam, 2001.

Aharon Ben-Tal, Elad Hazan, Tomer Koren, and Shie Mannor. Oracle-based robust optimization via online learning. Operations Research, 63(3):628–638, 2015.

Dimitris Bertsimas, Dick den Hertog, and Jean Pauphilet. Probabilistic guarantees in robust optimization. 2019.

Shalabh Bhatnagar, Richard S Sutton, Mohammad Ghavamzadeh, and Mark Lee. Naturalgradient actor-critic algorithms. Automatica, 2007.

David Blackwell. An analog of the minimax theorem for vector payoffs. Pacific Journal of Mathematics, 6(1):1–8, 1956.

Michael Bowling, Neil Burch, Michael Johanson, and Oskari Tammelin. Heads-up limit hold’em poker is solved. Science, 347 (6218):145–149, 2015.
Noam Brown and Tuomas Sandholm. Superhuman AI for heads-up no-limit poker: Libratus beats top professionals. *Science*, 359 (6374):418–424, 2018.

Noam Brown and Tuomas Sandholm. Solving imperfect-information games via discounted regret minimization. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 33, pages 1829–1836, 2019a.

Noam Brown and Tuomas Sandholm. Superhuman AI for multiplayer poker. *Science*, 365(6456):885–890, 2019b.

Neil Burch, Matej Moravčík, and Martin Schmid. Revisiting CFR+ and alternating updates. *Journal of Artificial Intelligence Research*, 64:429–443, 2019.

Antonin Chambolle and Thomas Pock. A first-order primal-dual algorithm for convex problems with applications to imaging. *Journal of mathematical imaging and vision*, 40(1):120–145, 2011.

Antonin Chambolle and Thomas Pock. On the ergodic convergence rates of a first-order primal–dual algorithm. *Mathematical Programming*, 159(1–2):253–287, 2016.

Chao-Kai Chiang, Tianbao Yang, Chia-Jung Lee, Mehrdad Mahdavi, Chi-Jen Lu, Rong Jin, and Shenghuo Zhu. Online optimization with gradual variations. In *Conference on Learning Theory*, pages 6–1. JMLR Workshop and Conference Proceedings, 2012.

Evgenii Chzhen, Christophe Giraud, and Gilles Stoltz. A unified approach to fair online learning via Blackwell approachability. *Advances in Neural Information Processing Systems*, 34, 2021.

Patrick L Combettes and Noli N Reyes. Moreau’s decomposition in Banach spaces. *Mathematical Programming*, 139(1):103–114, 2013.

Steven De Rooij, Tim Van Erven, Peter D Grünwald, and Wouter M Koolen. Follow the leader if you can, hedge if you must. *The Journal of Machine Learning Research*, 15(1):1281–1316, 2014.

John Duchi, Shai Shalev-Shwartz, Yoram Singer, and Tushar Chandra. Efficient projections onto the $\ell_1$ ball for learning in high dimensions. In *Proceedings of the 25th international conference on Machine learning*, pages 272–279, 2008.

John C Duchi, Peter W Glynn, and Hongseok Namkoong. Statistics of robust optimization: A generalized empirical likelihood approach. *Mathematics of Operations Research*, 2021.

Juan José Egozcue, Vera Pawlowsky-Glahn, Glòria Mateu-Figueras, and Carles Barcelo-Vidal. Isometric logratio transformations for compositional data analysis. *Mathematical Geology*, 35(3):279–300, 2003.

Gabriele Farina, Christian Kroer, and Tuomas Sandholm. Online convex optimization for sequential decision processes and extensive-form games. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 33, pages 1917–1925, 2019a.

Gabriele Farina, Christian Kroer, and Tuomas Sandholm. Optimistic regret minimization for extensive-form games via dilated distance-generating functions. In *Advances in Neural Information Processing Systems*, pages 5222–5232, 2019b.

Gabriele Farina, Christian Kroer, and Tuomas Sandholm. Regret circuits: Composability of regret minimizers. In *International Conference on Machine Learning*, pages 1863–1872, 2019c.

Gabriele Farina, Christian Kroer, and Tuomas Sandholm. Faster game solving via predictive Blackwell approachability: Connecting regret matching and mirror descent. In *Proceedings of the AAAI Conference on Artificial Intelligence*. AAAI, 2021.

Yuan Gao, Christian Kroer, and Donald Goldfarb. Increasing iterate averaging for solving saddle-point problems. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 35, pages 7537–7544, 2021.

Geoffrey J Gordon. No-regret algorithms for online convex programs. In *Advances in Neural Information Processing Systems*, pages 489–496. Citeseer, 2007.

Vineet Goyal and Julien Grand-Clément. Robust Markov decision process: Beyond rectangularity. *arXiv preprint arXiv:1811.00215*, 2018.

Julien Grand-Clément and Christian Kroer. First-order methods for Wasserstein distributionally robust MDP. *arXiv preprint arXiv:2009.06790*, 2020.

Julien Grand-Clément and Christian Kroer. Scalable first-order methods for robust MDPs. *Proceedings of the AAAI Conference on Artificial Intelligence*, 35(13):12086–12094, May 2021a.

Julien Grand-Clément and Christian Kroer. Conic Blackwell Algorithm: Parameter-free convex-concave saddle-point solving. *Advances in Neural Information Processing Systems*, 34, 2021b.

Julien Grand-Clément, Carri W Chan, Vineet Goyal, and Gabriel Escobar. Robust policies for proactive ICU transfers. *arXiv preprint arXiv:2002.06247*, 2020.
Sergiu Hart and Andreu Mas-Colell. A simple adaptive procedure leading to correlated equilibrium. *Econometrica*, 68(5):1127–1150, 2000.

Garud Iyengar. Robust dynamic programming. *Mathematics of Operations Research*, 30(2):257–280, 2005.

Yuji Jin and Aaron Sidford. Efficiently solving MDPs with stochastic mirror descent. In *International Conference on Machine Learning*, pages 4890–4900. PMLR, 2020.

Christian Kroer. IEOR8100: Economics, AI, and optimization lecture note 5: Computing Nash equilibrium via regret minimization. 2020.

Christian Kroer, Gabriele Farina, and Tuomas Sandholm. Solving large sequential games with the excessive gap technique. In *Advances in Neural Information Processing Systems*, pages 864–874, 2018.

Christian Kroer, Kevin Waugh, Fatma Kılınç-Karzan, and Tuomas Sandholm. Faster algorithms for extensive-form game solving via improved smoothing functions. *Mathematical Programming*, pages 1–33, 2020.

Christian Kroer, Alexander Peysakhovich, Eric Sodomka, and Nicolas E Stier-Moses. Computing large market equilibria using abstractions. *Operations Research*, 2021.

Brendan McMahan. Follow-the-regularized-leader and mirror descent: Equivalence theorems and $l_1$ regularization. In *Proceedings of the Fourteenth International Conference on Artificial Intelligence and Statistics*, pages 525–533. JMLR Workshop and Conference Proceedings, 2011.

Emanuel Milman. Approachable sets of vector payoffs in stochastic games. *Games and Economic Behavior*, 56(1):135–147, 2006.

Volodymyr Mnih, Koray Kavukcuoglu, David Silver, Alex Graves, Ioannis Antonoglou, Daan Wierstra, and Martin Riedmiller. Playing Atari with deep reinforcement learning. *arXiv preprint arXiv:1312.5602*, 2013.

Matej Moravčík, Martin Schmid, Neil Burch, Viliam Lisý, Dustin Morrill, Nolan Bard, Trevor Davis, Kevin Waugh, Michael Johanson, and Michael Bowling. Deepstack: Expert-level artificial intelligence in heads-up no-limit poker. *Science*, 356(6337):508–513, 2017.

Hongseok Namkoong and John C Duchi. Stochastic gradient methods for distributionally robust optimization with f-divergences. In *NIPS*, volume 29, pages 2208–2216, 2016.

Arkadi Nemirovski. Prox-method with rate of convergence $O(1/t)$ for variational inequalities with Lipschitz continuous monotone operators and smooth convex-concave saddle point problems. *SIAM Journal on Optimization*, 15(1):229–251, 2004.

Arkadi Nemirovski and David Yudin. *Problem complexity and method efficiency in optimization*. 1983.

Yu Nesterov. Smooth minimization of non-smooth functions. *Mathematical programming*, 103(1):127–152, 2005.

Rad Niazadeh, Negin Golrezaei, Joshua Wang, Fransisca Susan, and Ashwinkumar Badanidiyuru. Online learning via offline greedy: Applications in market design and optimization. 2020.

Francesco Orabona and Dávid Pál. Scale-free algorithms for online linear optimization. In *International Conference on Algorithmic Learning Theory*, pages 287–301. Springer, 2015.

Vianney Perchet. *Approachability, Calibration and Regret in Games with Partial Observations*. PhD thesis, Université Pierre et Marie Curie, 2010.

M.L. Puterman. *Markov Decision Processes : Discrete Stochastic Dynamic Programming*. John Wiley and Sons, 1994.

Hamed Rahimian and Sanjay Mehrotra. Distributionally robust optimization: A review. *arXiv preprint arXiv:1908.05659*, 2019.

Alexander Rakhlin and Karthik Sridharan. Online learning with predictable sequences. In *Conference on Learning Theory*, pages 993–1019. PMLR, 2013.

Aaron Sidford and Kevin Tian. Coordinate methods for accelerating $l_\infty$ regression and faster approximate maximum flow. In *2018 IEEE 59th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 922–933. IEEE, 2018.

Lauren N Steimle and Brian T Denton. Markov decision processes for screening and treatment of chronic diseases. In *Markov Decision Processes in Practice*, pages 189–222. Springer, 2017.

Vasilis Syrgkanis, Alekh Agarwal, Haipeng Luo, and Robert E Schapire. Fast convergence of regularized learning in games. 28, 2015.

Oskari Tammelin, Neil Burch, Michael Johanson, and Michael Bowling. Solving heads-up limit Texas hold’em. In *Twenty-Fourth International Joint Conference on Artificial Intelligence*, 2015.
Proof of Lemma 3.2

1. The fact that \( u - \pi_{C^*}(u) = \pi_C(u) \in C, (u - \pi_{C^*}(u), \pi_{C^*}(u)) = 0 \) follows from Moreau’s Decomposition Theorem [Combettes and Reyssat 2013]. The fact that \( \| u - \pi_{C^*}(u) \|_2 \leq \| u \|_2 \) is a straightforward consequence of \( \langle u - \pi_C(u), \pi_{C^*}(u) \rangle = 0 \).

2. For any \( w \in C^0 \cap B_2(1) \) we have
\[
\langle u, w \rangle \leq \langle u - \pi_C(u), w \rangle \leq \| w \|_2 \| u - \pi_C(u) \|_2 \leq \| u - \pi_C(u) \|_2.
\]

Conversely, since \( (u - \pi_C(u)) / \| u - \pi_C(u) \|_2 \in C^0 \), we have
\[
\max_{w \in C^0 \cap B_2(1)} \langle u, w \rangle \geq \| u - \pi_C(u) \|_2.
\]

This shows that
\[
\max_{w \in C^0 \cap B_2(1)} \langle u, w \rangle = \| u - \pi_C(u) \|_2 = d(u, C).
\]

3. For any \( u \in \mathbb{R}^{n+1} \), by definition we have \( d(u, C^0) = \| u - \pi_{C^*}(u) \|_2 \). Now if \( u \in C \) we have \( \pi_{C^*}(u) = 0 \) so \( d(u, C^0) = \| u \|_2 \).

4. Let \( u \in C \). Then \( u = \alpha(k, x) \) for \( \alpha \geq 0, x \in \mathcal{X} \). We will show that \( -u \in C^0 \). We have
\[
-u \in C^0 \iff \langle -u, u' \rangle \leq 0, \forall u' \in C
\]
\[
\iff \langle -\alpha(k, x), \alpha'(k, x') \rangle \leq 0, \forall \alpha' \geq 0, \forall x' \in \mathcal{X}
\]
\[
\iff \kappa^2 + \langle x, x' \rangle \geq 0
\]
\[
\iff -\langle x, x' \rangle \leq \kappa^2,
\]
and \( -\langle x, x' \rangle \leq \kappa^2 \) is true by Cauchy-Schwartz and the definition of \( \kappa = \max_{x \in \mathcal{X}} \| x \|_2 \).

5. We start by proving (5). Let \( x, x', y, y' \in \mathbb{R}^{n+1} \), and assume that \( x \leq_{C^0} y, x' \leq_{C^0} y' \). Then \( y - x \in C^0, y' - x' \in C^0 \). Because \( C^0 \) is a convex set, and a cone, we have \( 2 \cdot \left( \frac{y - x}{2} + \frac{y' - x'}{2} \right) \in C^0 \). Therefore, \( y + y' - x - x' \in C^0 \), i.e., \( x + x' \leq_{C^0} y + y' \).

We now prove (6). Let \( x, y \in \mathbb{R}^{n+1}, x' \in C^0 \) and assume that \( x + x' \leq_{C^0} y \). Then by definition \( y - x - x' \in C^0 \). Additionally, \( x' \in C^0 \) by assumption. Since \( C^0 \) is convex, and is a cone, \( 2 \cdot \left( \frac{y - x - x'}{2} + \frac{x'}{2} \right) \in C^0 \), i.e., \( y - x \in C^0 \). Therefore, \( x \leq_{C^0} y \).

6. Let \( x, y \in \mathbb{R}^{n+1} \) such that \( x \leq_{C^0} y \). Then \( y - x \in C^0 \). We have \( d(y, C^0) = \min_{z \in C^0} \| y - z \|_2 \leq \| y - (y - x) \|_2 = \| x \|_2 \).

\( \square \)
**B Proof of Theorem 3.4**

*Proof of Theorem 3.4* We prove the theorem for each part separately.

1. Let 
   \[ \bar{x}_T = \frac{1}{S_T} \sum_{t=1}^{T} \omega_t x_t, \quad \bar{y}_T = \frac{1}{S_T} \sum_{t=1}^{T} \omega_t y_t. \]

Since \( F \) is convex-concave, we first have

\[
\max_{y \in Y} F(\bar{x}_T, y) - \min_{x \in X} F(x, \bar{y}_T) \leq \frac{1}{S_T} \left( \max_{y \in Y} \sum_{t=1}^{T} \omega_t F(x_t, y_t) - \min_{x \in X} \sum_{t=1}^{T} \omega_t F(x_t, y_t) \right).
\]

Now, \[
\max_{y \in Y} \sum_{t=1}^{T} \omega_t F(x_t, y) - \min_{x \in X} \sum_{t=1}^{T} \omega_t F(x_t, y_t) = \max_{y \in Y} \sum_{t=1}^{T} \omega_t F(x_t, y) - \sum_{t=1}^{T} \omega_t F(x_t, y_t)
\]

\[
+ \sum_{t=1}^{T} \omega_t F(x_t, y_t) - \min_{x \in X} \sum_{t=1}^{T} \omega_t F(x, y_t).
\]

Now since \( F \) is convex-concave, we can upper bound each pair of terms using the subgradient inequality:

\[
\sum_{t=1}^{T} \omega_t F(x_t, y_t) - \min_{y \in Y} \sum_{t=1}^{T} \omega_t F(x_t, y_t) \leq \max_{x \in X} \sum_{t=1}^{T} \omega_t (f_t, x_t) - \min_{x \in X} \sum_{t=1}^{T} \omega_t (f_t, x),
\]

where \( f_t \in \partial_x F(x_t, y_t), g_t \in \partial_y F(x_t, y_t) \) (recall the repeated game framework presented at the beginning of Section 2). We recognize the right-hand side as the regrets in the repeated game framework. For CBA with weights on both payoffs and decisions (Theorem 3.1), we have shown that

\[
\frac{1}{S_T} \max_{y \in Y} \sum_{t=1}^{T} \omega_t (g_t, y) - \sum_{t=1}^{T} \omega_t (g_t, y_t) = O \left( \kappa L \sqrt{\frac{\sum_{t=1}^{T} \omega_t^2}{\sum_{t=1}^{T} \omega_t}} \right),
\]

\[
\frac{1}{S_T} \sum_{t=1}^{T} \omega_t (f_t, x_t) - \min_{x \in X} \sum_{t=1}^{T} \omega_t (f_t, x) = O \left( \kappa L \sqrt{\frac{\sum_{t=1}^{T} \omega_t^2}{\sum_{t=1}^{T} \omega_t}} \right).
\]

Recall that \( \omega_t = t^p \). Since \( t \mapsto t^p \) is an increasing function, we have

\[
\int_0^k t^p dt \leq \sum_{t=1}^{k} t^p \leq \int_0^{k+1} t^p dt.
\]

Therefore, we can conclude that

\[
\sum_{t=1}^{T} \omega_t^2 = O \left( \frac{1}{p+1} T^{2p+1} \right),
\]

\[
\frac{1}{p+1} T^{p+1} \leq \sum_{t=1}^{T} \omega_t.
\]

Overall, we obtain that

\[
O \left( \kappa L \sqrt{\frac{\sum_{t=1}^{T} \omega_t^2}{\sum_{t=1}^{T} \omega_t}} \right) = O \left( \kappa L \sqrt{\frac{p+1}{T}} \right).
\]
2. This proof is mostly similar to the first part. We have
\[\theta_T = T^q,\]
\[\frac{T^{q+1}}{q + 1} \leq \sum_{t=1}^{T} \theta_t,\]
\[\sqrt{T} \sum_{t=1}^{T} \omega_t^2 = O\left(\frac{1}{\sqrt{p + 1}} T^{p+1/2}\right),\]
\[\omega_T = T^p.\]
Combining all this we obtain that an upper bound of
\[O\left(\kappa L (q + 1) T^q \frac{T^{p+1/2}}{\sqrt{p + 1} T^p}\right)\]
which is equal to \(O\left(\frac{\kappa L (q + 1)}{\sqrt{p + 1} T}\right).\)

\[\Box\]

C Proof for Theorem 3.5

Proof of Theorem 3.5 The proof of Theorem 3.5 is similar to the proof of Theorem 3.4 presented in Appendix B. Let
\[\bar{x}_T = \frac{1}{S_T} \sum_{t=1}^{T} \theta_{t+1} x_{t+1}, \bar{y}_T = \frac{1}{S_T} \sum_{t=1}^{T} \theta_{t+1} y_t.\]
Since \(F\) is convex-concave, we first have
\[\max_{y \in Y} F(\bar{x}_T, y) - \min_{x \in X} F(x, \bar{y}_T) \leq \frac{1}{S_T} \left(\max_{y \in Y} \sum_{t=1}^{T} \theta_{t+1} F(x_{t+1}, y) - \min_{x \in X} \sum_{t=1}^{T} \theta_{t+1} F(x, y_t)\right).\]
Now we can rewrite
\[\max_{y \in Y} \sum_{t=1}^{T} \theta_{t+1} F(x_{t+1}, y) - \min_{x \in X} \sum_{t=1}^{T} \theta_{t+1} F(x, y_t)\]
as
\[\max_{y \in Y} \sum_{t=1}^{T} \theta_{t+1} F(x_{t+1}, y) - \min_{x \in X} \sum_{t=1}^{T} \theta_{t+1} F(x_{t+1}, y_t)\]
\[+ \sum_{t=1}^{T} \theta_{t+1} F(x_{t+1}, y_t) - \min_{x \in X} \sum_{t=1}^{T} \theta_{t+1} F(x, y_t)\]
Now since \(F\) is convex-concave, we can use the following upper bound:
\[\max_{y \in Y} \sum_{t=1}^{T} \theta_{t+1} F(x_{t+1}, y) - \sum_{t=1}^{T} \theta_{t+1} F(x_{t+1}, y_t) \leq \max_{y \in Y} \sum_{t=1}^{T} (g_t, y) - \sum_{t=1}^{T} \langle g_t, y_t \rangle,\]
\[\sum_{t=1}^{T} \theta_{t+1} F(x_t, y_t) - \min_{x \in X} \sum_{t=1}^{T} \theta_{t+1} F(x, y_t) \leq \sum_{t=1}^{T} \theta_{t+1} \langle f_t, x_t \rangle - \min_{x \in X} \sum_{t=1}^{T} \theta_{t+1} \langle f_t, x \rangle,\]
where \(f_t \in \partial_x F(x_t, y_t), g_t \in \partial_y F(x_{t+1}, y_t).\) This concludes the proof of Theorem 3.5. \(\Box\)
D Proof of Theorem 3.6

We start with the following lemma. It shows that once a non-degenerate update has been chosen ($u_t \neq 0$ for CBA$^+$ and $\pi_C(u_t) \neq 0$ for CBA), all the future updates are also non-degenerate.

**Lemma D.1.**
1. Let $(u_t)_{t \geq 1} \in \left(\mathbb{R}^{n+1}\right)^N$ the sequence of payoffs generated by CBA with weights $(\omega_t)_{t \geq 1}$ on the payoffs. Let $t \geq 1$. If $\pi_C(u_t) \neq 0$, then for all $t' \geq t$ we also have $\pi_C(u_{t'}) \neq 0$.
2. Let $(u_t)_{t \geq 1} \in \left(\mathbb{R}^{n+1}\right)^N$ the sequence of payoffs generated by CBA$^+$ with weights $(\omega_t)_{t \geq 1}$ on the payoffs. Let $t \geq 1$. If $u_t \neq 0$, then for all $t' \geq t$ we also have $u_{t'} \neq 0$.

**Proof of Lemma D.1**
1. Assume that $\pi_C(u_t) \neq 0$. Let $\pi_t = (\pi_t, \bar{\pi}_t)$ such that $\pi_t = \pi_C(u_t)$. In this case, we can define $x_{t+1} = (\kappa/\bar{u}_t) \bar{\pi}_t$. By definition of the updates in CBA, we have
   $$u_{t+1} = u_t + \omega_{t+1} v_{t+1},$$
   for $v_{t+1} = \left(\frac{f_{t+1} - f_t}{\kappa}, -f_{t+1}\right)$. We will show that $u_{t+1} \notin C^\circ$. By definition,
   $$u_{t+1} \notin C^\circ \iff \exists \ z \in C, \langle z, u_{t+1} \rangle > 0.$$
   If we take $z = \pi_t$, we have
   $$\langle \pi_t, u_{t+1} \rangle = \langle \pi_t, u_t + \omega_{t+1} v_{t+1} \rangle = \langle \pi_t, u_t \rangle$$
   since that by definition of $x_{t+1}$, we have $\langle \pi_t, v_{t+1} \rangle = 0$. Now
   $$\langle \pi_t, u_t \rangle = \langle \pi_C(u_t), \pi_C(u_t) + \pi_C(u_t) \rangle = \langle \pi_C(u_t), \pi_C(u_t) \rangle = \|\pi_C(u_t)\|^2 > 0.$$
   This shows that $u_{t+1} \notin C^\circ$. Since $u_{t+1} = \pi_C(u_{t+1}) + \pi_C(u_{t+1})$, this also shows that $\pi_C(u_{t+1}) \neq 0$. By induction, we have shown that $\pi_C(u_t) \neq 0 \Rightarrow \pi_C(u_{t'}) \neq 0, \forall t' \geq t$.
2. The proof is very similar to the proof of the first statement. Suppose that $u_t \neq 0$. In this case, we can define $x_{t+1} = (\kappa/\bar{u}_t) \bar{\pi}_t$. Note that by definition of the updates in CBA$^+$, we have
   $$u_{t+1} = \pi_C(u_t + \omega_{t+1} v_{t+1}).$$
   We will show that
   $$u_t + \omega_{t+1} v_{t+1} \notin C^\circ.$$
   By definition of $C^\circ$,
   $$u_t + \omega_{t+1} v_{t+1} \notin C \iff \exists \ z \in C, \langle z, u_t + v_{t+1} \rangle > 0.$$
   For $z = u_t$, we obtain
   $$\langle u_t, u_t + \omega_{t+1} v_{t+1} \rangle = \langle u_t, u_t \rangle = \|u_t\|^2 > 0,$$
   where
   $$\langle u_t, v_{t+1} \rangle = 0$$
   follows from the choice of $x_{t+1}$ as in Blackwell approachability framework (see [10] in the proof of Theorem 3.1 for more details.) Therefore, for any $t \geq 1$, we have $u_t \neq 0 \Rightarrow u_{t+1} \neq 0$. This concludes the proof of Lemma D.1 by induction.

We are now ready to prove Theorem 3.6.

**Proof of Theorem 3.6** Assume that $(x, y) \mapsto F(x, y)$ is linear in $x$.

1. We want to prove that
   $$F(x_t, y_t) \geq F(x_{t+1}, y_t) + \frac{\kappa}{\omega_t} \|u_t\|_\infty \|u_t - u_{t-1}\|_2^2.$$  \hspace{1cm} (23)
   Let $t \geq 1$. Recall that
   $$x_t = \text{CHOOSEDECISION}_{CBA^+}(u_{t-1}),$$
   $$x_{t+1} = \text{CHOOSEDECISION}_{CBA^+}(u_t).$$
   We consider the following two cases.
2. The proof is identical to the first claim of this theorem. For the sake of conciseness, we omit it in this paper.

(b) Case 2: $u_t \neq 0$. We start from

$$u_t = \pi_C (u_{t-1} + \omega_t v_t)$$

with $v_t = \left( \frac{\langle f_t, x_t \rangle}{\kappa}, -f_t \right)$. The optimality condition for the projection on $C$ shows that

$$\langle u_t - u_{t-1} - \omega_t v_t, u_t - z \rangle \leq 0, \forall z \in C.$$ 

We can apply this with $z = u_{t-1}$ to obtain

$$\langle u_t - u_{t-1} - \omega_t v_t, u_t - u_{t-1} \rangle \leq 0.$$ 

This shows that

$$\|u_t - u_{t-1}\|^2 \leq \langle \omega_t v_t, u_t - u_{t-1} \rangle.$$ 

Recall that by definition of $x_t$ and $v_t$, we have

$$\langle v_t, u_{t-1} \rangle = 0.$$ 

Recall that $u_t = \alpha_{t+1}(\kappa, x_{t+1})$, with $\alpha_{t+1} > 0$ because $u_t \neq 0$. This implies that

$$\langle \omega_t v_t, u_t - u_{t-1} \rangle = \langle \omega_t v_t, u_t \rangle = \omega_t \left( \frac{\langle f_t, x_t \rangle}{\kappa}, -f_t \right), \alpha_{t+1}(\kappa, x_{t+1}) \rangle = \omega_t \alpha_{t+1} \left( \langle f_t, x_t \rangle - \langle f_t, x_{t+1} \rangle \right).$$ 

Overall, we have obtained

$$\langle f_t, x_t \rangle \geq \langle f_t, x_{t+1} \rangle + \frac{1}{\omega_t \alpha_{t+1}} \|u_t - u_{t-1}\|^2.$$ 

Recall that by definition, $u_t = \alpha_{t+1}(\kappa, x_{t+1})$, with $\kappa = \max \{\|x\|_2 | x \in \mathcal{X}\}$. Therefore,

$$\|u_t\|_\infty = \alpha_{t+1} \max \{\kappa, \|x_{t+1}\|_\infty\} = \alpha_{t+1} \kappa,$$

where the last inequality follows from $\|x_{t+1}\|_\infty \leq \|x_{t+1}\|_2 \leq \kappa$. Overall, we have shown that

$$\langle f_t, x_t \rangle \geq \langle f_t, x_{t+1} \rangle + \frac{\kappa}{\omega_t \|u_t\|_\infty} \|u_t - u_{t-1}\|^2.$$ 

Recall that in the repeated game framework with alternation, we have $f_t = \partial_x F(x_t, y_t)$. For an objective function that is linear in $x$, we obtain

$$\langle f_t, x_{t+1} \rangle = F(x_{t+1}, y_t), \quad \langle f_t, x_t \rangle = F(x_t, y_t).$$ 

In this case, we have shown that

$$F(x_t, y_t) \geq F(x_{t+1}, y_t) + \frac{\kappa}{\omega_t \|u_t\|_\infty} \|u_t - u_{t-1}\|^2.$$ 

This concludes the proof of the first statement of Theorem 3.6.
E  Proofs for the efficient projections of Section 4

E.1  Proofs for the simplex

Proof of Lemma 4.1. For $\mathcal{X} = \Delta(n)$, we can choose $\kappa = \max\{\|x\|_2 \mid x \in \mathcal{X}\} = 1$. Therefore, $\mathcal{C} = \{\alpha(1, x) \mid x \in \Delta(n), \alpha \geq 0\}$. For $y = (\tilde{y}, \hat{y}) \in \mathbb{R}^{n+1}$ we have

$$y \in \mathcal{C}^o \iff \langle y, z \rangle \leq 0, \forall z \in \mathcal{C}$$

$$\iff \langle (\tilde{y}, \hat{y}), \alpha(1, x) \rangle \leq 0, \forall x \in \Delta(n), \forall \alpha \geq 0$$

$$\iff \tilde{y} + \langle \hat{y}, x \rangle \leq 0, \forall x \in \Delta(n)$$

$$\iff \max_{x \in \Delta(n)} \langle \hat{y}, x \rangle \leq -\tilde{y}$$

$$\iff \max_{i=1,...,n} \tilde{y}_i \leq -\tilde{y}.$$ 

Proof of Proposition 4.2. Let us fix $\tilde{y} \in \mathbb{R}$ and let us first solve

$$\min_{\hat{y}} \|\hat{y} - \hat{u}\|_2^2 \quad \hat{y} \in \mathbb{R}^n,$$

$$\max_{i \in [n]} \tilde{y}_i \leq -\tilde{y}. \quad (24)$$

This is essentially the projection of $\hat{u}$ on $(-\infty, -\tilde{y})^n$. So a solution to $\hat{u} - \hat{y}(\tilde{y}) = (\hat{u} + \tilde{y}\epsilon)^+$. So overall the orthogonal projection on $\mathcal{C}^o$ boils down to the optimization of the function $\phi : \mathbb{R} \mapsto \mathbb{R}_+$ such that

$$\phi : \hat{y} \mapsto (\hat{y} - \hat{u})^2 + \| (\hat{u} + \tilde{y}\epsilon)^+ \|_2^2. \quad (25)$$

In principle, we could use binary search with a doubling trick to compute a $\epsilon$-minimizer of the convex function $\phi$ in $O(\log(\epsilon^{-1}))$ calls to $\phi$. However, it is possible to find a minimizer $\hat{y}^*$ of $\phi$ using the following remark.

By construction, we know that $u - \pi_{\mathcal{C}^o}(u) \in \mathcal{C}$. Here, $\mathcal{C} = \text{cone}(\{1\} \times \Delta(n))$, and $u - \pi_{\mathcal{C}^o}(u) = \left((\hat{u} - \hat{y}^* (\hat{u} + \tilde{y}\epsilon)^+)\right)$. We first check if $\hat{u} = \hat{y}^*$. This is the case if and only if $u - \pi_{\mathcal{C}^o}(u) = 0$, i.e., if and only if $u \in \mathcal{C}^o$, which is straightforward to check using Lemma 4.1. Now if $\hat{u} \neq \hat{y}^*$, we must have $\hat{u} > \hat{y}^*$, by definition of $\mathcal{C}$. This also implies that

$$\left(\hat{u} + \tilde{y}\epsilon\right)^+ \in \Delta(n),$$

which in turns imply that

$$\hat{y}^* + \sum_{i=1}^n \max\{\hat{u}_i + \hat{y}^*, 0\} = \hat{u}. \quad (26)$$

We can use $26$ to efficiently compute $\hat{y}^*$ without using any binary search. In particular, we can sort the coefficients of $\hat{u}$ in $O(n \log(n))$ arithmetic operations, and use $26$ to find $\hat{y}^*$.

E.2  Proofs for $\ell_p$-balls

Proof of Lemma 4.3. Let us write $B_p(1) = \{z \in \mathbb{R}^n \mid \|z\|_p \leq 1\}$. Here we consider $\mathcal{X} = B_p(1)$. Recall that $\kappa = \max\{\|x\|_2 \mid x \in \mathcal{X}\}$. Therefore, by definition, $\mathcal{C} = \{\alpha(\kappa, x) \mid x \in B_p(1), \alpha \geq 0\}$.

We first provide the reformulation for $\mathcal{C}$. Let $y = (\tilde{y}, \hat{y}) \in \mathcal{C}$. Then $\tilde{y} = \alpha(\kappa, \hat{y}) = \alpha x$ with $\alpha \geq 0$ and with $x$ such that $\|x\|_p \leq 1$. For $\alpha > 0$ we have $\|x\|_p \leq 1 \iff \|\alpha x\|_p \leq \alpha \iff \|\hat{y}\|_p \leq \tilde{y}/\kappa.$
We now provide the reformulation for \( C^\circ \). Note that for \( y = (\hat{y}, \check{y}) \in \mathbb{R}^{n+1} \) we have

\[
y \in C^\circ \iff \langle y, z \rangle \leq 0, \forall z \in C
\]

\[
\iff \langle (\hat{y}, \check{y}), \alpha(\kappa, x) \rangle \leq 0, \forall x \in B_p(1), \forall \alpha \geq 0
\]

\[
\iff \kappa \hat{y} + \langle \check{y}, x \rangle \leq 0, \forall x \in B_p(1)
\]

\[
\iff \max_{x \in B_p(1)} \langle \check{y}, x \rangle \leq -\kappa \hat{y}
\]

\[
\iff \|\check{y}\|_q \leq -\kappa \hat{y},
\]

since \( \| \cdot \|_q \) is the dual norm of \( \| \cdot \|_p \).

Proof of Proposition 4.4. For \( p = 1 \), we have \( \| \cdot \|_1 = \| \cdot \|_\infty \) and we can choose \( \kappa = 1 \). Let us compute the projection of \((\hat{u}, \check{u})\) on \( C^\circ \) using the reformulation of Lemma 4.3:

\[
\min (\hat{y} - \hat{u})^2 + \|\check{y} - \check{u}\|_2^2
\]

\[
\check{y} \in \mathbb{R}, \hat{y} \in \mathbb{R}^n,
\]

\[
\|\check{y}\|_\infty \leq -\hat{y}.
\]

For a fixed \( \check{y} \in \mathbb{R} \), we want to compute \( \min\{ (\|\check{y} - \check{u}\|_2^2) \mid \check{y} \in \mathbb{R}^n, \|\check{y}\|_\infty \leq -\check{y} \} \). This projection can be computed in closed-form as \( \check{y}^*(\check{y}) = \min\{ -\check{y}, \max\{ \check{y}, \hat{u} \} \} \), since this is simply the orthogonal projection of \( \hat{u} \) onto the \( \ell_\infty \) ball of radius \( -\check{y} \). Let us call \( \phi : \mathbb{R} \rightarrow \mathbb{R} \) such that

\[
\phi(\check{y}) = (\hat{y} - \hat{u})^2 + \|\check{y}^*(\check{y}) - \check{u}\|_2^2.
\]

Note that \( \check{y}^*(\check{y}) - \check{u} = (\check{u} + \check{y} e)^\top \), so we have

\[
\phi : \check{y} \mapsto (\hat{y} - \hat{u})^2 + \|\check{u} + \check{y} e\|^2_2.
\]

Assume that we have ordered the coefficients of \( \check{u} \in \mathbb{R}^n \) in decreasing order. This can be done in \( O(n \log(n)) \) arithmetic operations. Then on each of the \( n + 1 \) intervals \( I_1 = (-\infty, -\check{u}_1), I_2 = (-\check{u}_1, -\check{u}_2), \ldots, I_{n+1} = (-\check{u}_n, \infty) \), the map \( \phi \) is a second order polynomial in \( \check{y} \), with a non-negative coefficient in front of \( \check{y}^2 \). Therefore, for each \( i \in [n+1] \), we can find a closed-form expression for the minimum \( \check{y}^*_i \) of \( \phi \) on \( I_i \), and the scalar \( \check{y}^*_i \) attaining this minimum. We can then simply search for a global minimum of \( \phi \) among the scalars

\[
\{ \check{y}^*_i \mid i \in [n+1] \} \bigcup \{ -\check{u}_i \mid i \in [n] \}.
\]

Once we have found \( \check{y}^*_i \) the minimizer of \( \phi \), we obtain the solution of \( \pi_{C^\circ}(\check{u}) \) as \( \pi_{C^\circ}(\check{u}) = (\check{y}^*_i, \check{y}^*_i(\check{y})) \), and we can recover \( \pi_{C^\circ}(u) \) from \( \pi_{C^\circ}(u) = u - \pi_{C^\circ}(\check{u}) \).

Let us now focus on the case \( p = \infty \). We know that \( \| \cdot \|_1 \) and \( \| \cdot \|_\infty \) are dual norms to each other. Therefore, from Lemma 4.3, it is as computationally demanding to compute orthogonal projections onto \( C^\circ \) (when \( p = 1 \)) and onto \( C \) (when \( p = \infty \)). Therefore, the method described in the first part of this proof for computing \( \pi_{C^\circ}(u) \) for \( p = 1 \) can be applied for computing \( \pi_{C}(u) \) in the case \( p = \infty \).

Proof of Proposition 4.3. First, we check if \( \check{u} \in C \), i.e., we check if \( \|\check{u}\|_2 \leq \hat{u} \). If this is the case, then \( \pi_{C}(\check{u}) = \check{u} \). Second, we check if \( \check{u} \in C^\circ \), i.e., we check if \( \|\check{u}\|_2 \leq -\hat{u} \). If this is the case, then \( \pi_{C}(\check{u}) = 0 \). Else, we have \( \|\check{u}\|_2 > |\hat{u}| \), and we can provide a closed-form solution to \( \pi_{C}(\check{u}) \). Let us fix \( \check{y} \in \mathbb{R} \) and define \( \check{y}^*(\check{y}) \) the vector attaining the minimum in \( \min\{ (\|\check{y} - \check{u}\|_2^2) \mid \check{y} \in \mathbb{R}^n, \|\check{y}\|_2 \leq \check{y} \} \). With this notation, we want to find the minimum of \( \phi : \check{y} \mapsto (\check{y}^*(\check{y}) - \check{u})^2 + (\check{y} - \check{u})^2 \). If \( \check{y} \geq \|\check{u}\|_2 \), then \( \check{y}^*(\check{y}) = \check{u} \). This shows that the minimum of \( \phi \) on \( [\|\check{u}\|_2, +\infty) \) is attained at \( \check{y}_1 = \|\check{u}\|_2 \), at a value of \( \phi(\check{y}_1) = (\|\check{u}\|_2 - \check{u})^2 \). When \( \check{y} \in [0, \|\check{u}\|_2] \), we have \( \check{y}^*(\check{y}) = (\check{y}/\|\check{u}\|_2) \check{u} \). Note that here, \( \check{y} \mapsto \check{y}^*(\check{y}) \) is differentiable. Therefore, \( \phi : \check{y} \mapsto (\check{y} - \check{u})^2 + (\check{y}^*(\check{y}) - \check{u})^2 \) is also differentiable. The first-order optimality conditions yield a closed-form solution for the minimum of \( \phi \) on \( [0, \|\check{u}\|_2] \), with \( \check{y}_2 = \frac{\check{u} + \|\check{u}\|_2}{2} \). For this value of \( \check{y}_2 \), we obtain \( \phi(\check{y}_2) = (1/2)(\|\check{u}\|_2 - \check{u})^2 \). Therefore, the global minimum of \( \phi \) on \( [0, +\infty) \) is attained at \( \check{y}_2 \), yielding

\[
\pi_{C}(\check{u}) = \left( \frac{\check{u} + \|\check{u}\|_2}{2}, \frac{\check{u} + \|\check{u}\|_2}{2} \right)
\]
E.3 Proofs for confidence regions in the simplex

Proof of Proposition 4.6. We can write $X = x_0 + \epsilon \hat{B}$, where $\hat{B} = \{ z \in \mathbb{R}^n \mid z^\top e = 0, \|z\|_2 \leq 1 \}$.

Suppose we made a sequence of decisions $x_1, \ldots, x_T$, which can be written as $x_t = x_0 + \epsilon z_t$ for $z_t \in \hat{B}$. Then it is clear that for any sequence of payoffs $f_1, \ldots, f_T$, we have

$$
\sum_{i=1}^T \omega_t(f_i, x_t) - \min_{x \in X} \sum_{i=1}^T \omega_t(f_i, x) = \epsilon \left( \sum_{i=1}^T \omega_t(f_i, z_t) - \min_{z \in \hat{B}} \sum_{i=1}^T \omega_t(f_i, z) \right). \tag{28}
$$

Therefore, if we run CBA* on the set $\hat{B}$ to obtain $O(\sqrt{T})$ growth of the right-hand side of (28), we obtain a no-regret algorithm for $X$. We now show how to run CBA* for the set $\hat{B}$. Let $V = \{ v \in \mathbb{R}^n \mid v^\top e = 0 \}$. We use the following orthonormal basis of $V$: let $v_1, \ldots, v_{n-1} \in \mathbb{R}^n$ be the vectors $v_i = \sqrt{i/(i+1)}(1/i, ..., 1/i, -1, ..., 0, ..., 0)$, $\forall i = 1, ..., n-1$, where the component $1/i$ is repeated $i$ times. The vectors $v_1, \ldots, v_{n-1}$ are orthonormal and constitute a basis of $V$ [Egozcue et al., 2003]. Writing $V = (v_1, ..., v_{n-1}) \in \mathbb{R}^{n \times (n-1)}$, and noting that $V^\top V = I$, we can write $\hat{B} = \{ Vs \mid s \in \mathbb{R}^{n-1}, \|s\|_2 \leq 1 \}$. Now, if $x = x_0 + \epsilon z, z \in V$, we have $x_t = V s_t$, for $s_t \in \mathbb{R}^{n-1}$ and $\|s\|_2 \leq 1$. Finally, $\sum_{t=1}^T \omega_t(f_i, x_t) - \min_{x \in X} \sum_{t=1}^T \omega_t(V f_i, x)$ is equal to

$$
\epsilon \left( \sum_{t=1}^T \omega_t(V^\top f_i, s_t) - \min_{s \in \mathbb{R}^{n-1}, \|s\|_2 \leq 1} \sum_{t=1}^T \omega_t(V^\top f_i, s) \right). \tag{29}
$$

Therefore, to obtain a regret minimizer for (29) with observed payoffs $(f_i)_{i \geq 1}$, we can run CBA* on the right-hand side, where the decision set is an $\ell_2$ ball and the sequence of observed payoffs is $(V^\top f_i)_{i \geq 1}$. In the previous section we showed how to efficiently instantiate CBA* in this setting (see Proposition 4.5).

Remark E.1. In this section we have highlighted a sequence of reformulations of the regret, from (28) to (29). We essentially showed how to instantiate CBA* for settings where the decision set $X$ is the intersection of an $\ell_2$ ball with a hyperplane for which we have an orthonormal basis.

F Details on OMD, FTRL and optimistic variants

F.1 Algorithms

For solving our instances of distributionally robust optimization, we compare SP-CBA* with the following four state-of-the-art algorithms: at iteration $t \geq 1$, for a step size $\eta_t > 0$, the updates are:

1. Follow-The-Regularized-Leader (FTRL) [Abernethy et al., 2009, McMahan 2011]:

$$
x_{t+1} = \arg \min_{x \in X} \left\{ \sum_{\tau=1}^t f_\tau, x \right\} + \frac{1}{\eta_t} \|x\|_2^2. \tag{FTRL}
$$

Optimistic FTRL [Rakhlin and Sridharan, 2013]: given estimation $m^{t+1}$ of loss at iteration $t+1$, choose

$$
x_{t+1} = \arg \min_{x \in X} \left\{ \sum_{\tau=1}^t f_\tau + m^{t+1}, x \right\} + \frac{1}{\eta_t} \|x\|_2^2. \tag{O-FTRL}
$$

2. Online Mirror Descent (OMD) [Nemirovski and Yudin, 1983, Beck and Teboulle, 2003]:

$$
x_{t+1} = \arg \min_{x \in X} \left\{ \langle f, x \rangle + \frac{1}{\eta_t} \|x - x_t\|_2^2 \right\}. \tag{OMD}
$$

Optimistic OMD [Chiang et al., 2012]: given estimation $m^{t+1}$ of loss at iteration $t+1$,

$$
z_{t+1} = \arg \min_{z \in X} \left\{ m^{t+1}, z \right\} + \frac{1}{\eta_t} \|z - x_t\|_2^2,
$$

Observe the loss $f_{t+1}$ related to $z_{t+1}$,

$$
x_{t+1} = \arg \min_{x \in X} \left\{ f_{t+1}, x \right\} + \frac{1}{\eta_t} \|x - x_t\|_2^2. \tag{O-OMD}
$$

29
Note that these algorithms can be written more generally using Bregman divergence (e.g., [Ben-Tal and Nemirovski (2001)]). We choose to work with $\| \cdot \|_2$ instead of Kullback-Leibler divergence as this $\ell_2$-setup is usually associated with faster empirical convergence rates [Chambolle and Pock 2016, Gao et al. 2021]. Additionally, following Chiang et al. [2012], Rakhlin and Sridharan [2013], we use the last observed loss as the predictor for the next loss, i.e., we set $m^{t+1} = f_t$.

### F.2 Implementations

The proximal updates defined in the previous section need to be resolved for the decision sets of both players of the distributionally robust optimization problem (27). We present the details of our implementation here. The results in the rest of this section are reminiscent to the novel tractable proximal setups presented in Grand-Clément and Kroer [2020, 2021a].

**Computing the projection steps for the first player** For $\mathcal{X} = \{x \in \mathbb{R}^n \mid \|x - x_0\|_2 \leq \epsilon_x\}$, $c, x' \in \mathbb{R}^n$ and a step size $\eta > 0$, the prox-update becomes

$$
\min_{\|x - x_0\|_2 \leq \epsilon_x} \langle c, x \rangle + \frac{1}{2\eta} \|x - x'\|_2^2.
$$

Using a change of variable, we find that the optimal solution $x^*$ to the problem above is

$$
x^* = x_0 + \epsilon_x \frac{x' - \eta c - x_0}{\max\{\epsilon_x, \|x' - \eta c - x_0\|_2\}}.
$$

**Computing the projection steps for the second player** For $\mathcal{Y} = \{y \in \Delta(m) \mid \|y - y_0\|_2 \leq \epsilon_y\}$, the proximal update of the second player from a previous point $y'$ and a step size of $\eta > 0$ becomes

$$
\min_{\|y - y_0\|_2 \leq \epsilon_y, y \in \Delta(m)} \langle c, y \rangle + \frac{1}{2\eta} \|y - y'\|_2^2.
$$

(30)

If we dualize the $\ell_2$ constraint with a Lagrangian multiplier $\mu \geq 0$ we obtain the relaxed problem $q(\mu)$ where

$$
q(\mu) = -\frac{1}{2}c^2 y \mu + \min_{y \in \Delta(m)} \langle c, y \rangle + \frac{1}{2\eta} \|y - y'\|_2^2 + \frac{\mu}{2} \|y - y_0\|_2^2.
$$

(31)

Note that the arg min in

$$
\min_{y \in \Delta(m)} \langle c, y \rangle + \frac{1}{2\eta} \|y - y'\|_2^2 + \frac{\mu}{2} \|y - y_0\|_2^2
$$

is the same arg min as in

$$
\min_{y \in \Delta(m)} \|y - \frac{\eta}{\eta \mu + 1} \left(\frac{1}{\eta} y' + \mu y_0 - c\right)\|_2^2.
$$

(32)

Note that (32) is an orthogonal projection onto the simplex. Therefore, it can be solved efficiently [Duchi et al. 2008]. We call $y(\mu)$ an optimal solution of (32). Then $q(\mu)$ can be rewritten

$$
q(\mu) = -(1/2)c^2 \mu + \langle c, y(\mu) \rangle + \frac{1}{2\eta} \|y(\mu) - y'\|_2^2 + \frac{\mu}{2} \|y(\mu) - y_0\|_2^2.
$$

We can therefore binary search $q(\mu)$ as in the previous expression. An upper bound $\bar{\mu}$ for $\mu^*$ can be computed as follows. Note that

$$
q(\mu) \leq -(1/2)c^2 \mu + \langle c, y_0 \rangle + \frac{1}{2\eta} \|y_0 - y'\|_2^2.
$$

Since $\mu \mapsto q(\mu)$ is concave we can choose $\bar{\mu}$ such that $q(\bar{\mu}) \leq q(0)$. Using the previous inequality this yields

$$
\bar{\mu} = \frac{2}{c^2} \left(\langle c, y_0 \rangle + \frac{1}{2\eta} \|y_0 - y'\|_2^2 - q(0)\right).
$$

In our simulations, we search for an optimal $\mu$ using the `minimize_scalar` function from the sklearn Python package, with an accuracy of $\epsilon = 0.001$. 

30
F.3 Computing the theoretical fixed step sizes for Section 5.3

For OMD and FTRL in theory (e.g., [Ben-Tal and Nemirovski 2001]), for a player with decision set $\mathcal{X}$, we can choose $\eta_{th} = \sqrt{2\Omega/LT}$ with $\Omega = \max_{x, x' \in \mathcal{X}} \|x - x'\|_2$, and $L$ an upper bound on the norm of any observed loss $f_t$; $\|f_t\|_2 \leq L$, $\forall t \geq 1$. Note that this requires to know 1) the number of iterations $T$; and 2) the upper bound $L$ on the norm of any observed loss $f_t$, before the losses are generated. For O-OMD we can choose $\eta_{th} = 1/\sqrt{8L}$ (Corollary 6 in Syrgkanis et al. [2015]), and for O-FTRL we can choose $\eta_{th} = 1/2L$ (Corollary 8 in Syrgkanis et al. [2015]). We now show how to compute $L_x$ and $L_y$ (for the first player and the second player) for an instance of the distributionally robust logistic regression problem (2).

1. For the first player we have $f_t = A^t y_t$, where $A^t$ is the matrix of subgradients of $x \mapsto F(x, y_t)$ at $x_t$:

$$A^t_{ij} = -b_i a_{i,j} \exp(-b_i a_{i,j}^T x_t) / (1 + \exp(-b_i a_{i,j}^T x_t)) + \mu x_j, \forall (i,j) \in \{1, \ldots, m\} \times \{1, \ldots, n\}.$$

Therefore, $\|f_t\|_2 \leq \|A^t\|_2 \|y_t\|_2 \leq \|A^t\|_2$, because $y \in \Delta(m)$.

Now we have $\|A^t\|_2 \leq \|A^t\|_F = \sqrt{\sum_{i,j} |A^t_{ij}|^2}$. Note that

$$\sqrt{\sum_{i,j} |A^t_{ij}|^2} \leq \sum_{i,j} |A^t_{ij}|.$$

We also have $|A^t_{ij}| \leq b_i a_{i,j} + \mu |x_j|$. Recall that we have $x \in \mathbb{R}^n$ such that $\|x - x_0\|_2 \leq \epsilon_x$. We obtain the following upper bound:

$$L_x = \sum_{i,j} |b_i a_{i,j}| + \mu \cdot m \cdot (\|x_0\|_1 + \sqrt{\epsilon_x}).$$

2. For the second player, the loss $f_t$ is $f_t = (\ell_i(x_t))_{i \in [1,m]}$, with $\ell_i(x) = \log(1 + \exp(-b_i a_i^T x))$. For each $i \in [1,m]$ we have

$$|\ell_i(x)| \leq \log(1 + \exp(b_i \epsilon_x \|a_{i}\|_2)),$$

and we can conclude that

$$L_y = \sqrt{\sum_{i=1}^m \log(1 + \exp(b_i \epsilon_x \|a_{i}\|_2))}.$$

G Computing the theoretical step sizes for Section 5.4

In the saddle-point formulation of MDP, the objective function is

$$F(v, \mu) = (1 - \lambda)p_0^T v + \sum_{s=1}^n \sum_{a=1}^A \mu_{sa} (r_{sa} + \lambda P_{sa}^T v - v_s),$$

for $v \in \mathbb{R}^n$, $\|v\|_2 \leq \sqrt{n}\epsilon_{\infty}/(1 - \lambda)$ and $\mu \in \Delta(n \times A)$. The function $F$ is differentiable and we have $\nabla_v F(v, \mu) \in \mathbb{R}^n$, $\nabla_\mu F(v, \mu) \in \mathbb{R}^{n \times A}$ with

$$(\nabla_v F(v, \mu))_{sa'} = (1 - \lambda)p_{0s'} + \lambda \sum_{s', a} \mu_{sa} P_{sas'} - \sum_{s'} \mu_{s'a}, \forall s', a \in \{1, \ldots, n\},$$

$$(\nabla_\mu F(v, \mu))_{sa} = r_{sa} + \lambda P_{sa}^T v - v_s, \forall (s, a) \in \{1, \ldots, n\} \times \{1, \ldots, A\}.$$

We now provide upper bounds $L_v$ and $L_\mu$ on $\|\nabla_v F(v, \mu)\|_2$ and $\|\nabla_\mu F(v, \mu)\|_2$. Using the equivalence between $\|\cdot\|_2$ and $\|\cdot\|_1$, we have, for $\mu \in \Delta(n \times A), \|\nabla_v F(v, \mu)\|_2 \leq \|\nabla_v F(v, \mu)\|_1$

$$\leq (1 - \lambda) + \lambda \sum_{s', a, s} \mu_{sa} P_{sas'} + \sum_{s', a} \mu_{s'a}$$

$$\leq (1 - \lambda) + \lambda + 1$$

$$\leq 2.$$
For bounding $\|\nabla_\mu F(v, \mu)\|_2$, we can rely on Cauchy-Schwarz’s inequality and $\|v\|_2 \leq \sqrt{nr_\infty}/(1 - \lambda)$ to obtain

$$\|\nabla_\mu F(v, \mu)\|_2 \leq \|r\|_2 + \frac{\sqrt{nr_\infty}}{1 - \lambda} (A(\lambda n + 1)) .$$

Overall, we can choose

$$L_v = 2, L_\mu = \|r\|_2 + \frac{\sqrt{nr_\infty}}{1 - \lambda} (A(\lambda n + 1)) .$$