Detection and Mitigation of Algorithmic Bias via Predictive Rate Parity

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Abstract

Recently, numerous studies have demonstrated the presence of bias in machine learning powered decision-making systems. Although most definitions of algorithmic bias have solid mathematical foundations, the corresponding bias detection techniques often lack statistical rigor, especially for non-iid data. We fill this gap in the literature by presenting a rigorous non-parametric testing procedure for bias according to Predictive Rate Parity, a commonly considered notion of algorithmic bias. We adapt traditional asymptotic results for non-parametric estimators to test for bias in the presence of dependence commonly seen in user-level data generated by technology industry applications and illustrate how these approaches can be leveraged for mitigation. We further propose modifications of this methodology to address bias measured through marginal outcome disparities in classification settings and extend notions of predictive rate parity to multi-objective models. Experimental results on real data show the efficacy of the proposed detection and mitigation methods.

1 INTRODUCTION

Fairness in artificial intelligence (AI) has been receiving much attention in recent years due to its tremendous impact on our everyday lives. Examples of AI-driven systems influencing our lives include social media [Garg and Pahuja 2020], digital assistance [Maedche et al. 2019], web searching [Zhang et al. 2018], online and retail shopping [Pillai et al. 2020], healthcare [Panesar 2019], banking [Caron 2019], and many more. Undoubtedly, as practitioners and researchers, it is our responsibility to make sure that the AI systems are fairly treating all individuals while making critical decisions such as the treatment plan of a patient [Ahmad et al. 2020], approval of the loan to a potential borrower [Mehrabi et al. 2021], etc. An AI system can be unfair to one or more groups of individuals due to the data bias that reflects biases present in society or the algorithmic bias reflecting the imperfection of AI systems in representing the data generating mechanism. There are multiple definitions of group-fairness available in the literature including demographic parity, equality of opportunity, equalized odds and predictive rate parity [Verma and Rubin 2018, Barocas et al. 2019]. This paper focuses on the detection and mitigation of algorithmic bias defined through Predictive Rate Parity for large-scale AI systems.

For binary classifiers, predictive rate parity condition ensures that the observed outcome is conditionally independent of the protected attribute under consideration (e.g., race or gender) given the predicted outcome. In other words, we should observe the same (distribution of) outcome for the same prediction regardless of the protected attribute status. For example, an AI system predicting the chance of developing heart disease should have equal precision irrespective of the gender of a person. In the banking example, customers who got rejected for a loan should have an equal chance of being a defaulter regardless of their race. For general predictive algorithms, predictive rate parity asserts that the mean of the observed outcomes conditional on the model score should be equal across groups. It is also closely tied to the definition of sufficiency in the fairness literature [Barocas et al. 2019]. A closely related notion to Predictive Rate Parity is that of marginal outcome testing [Ayres 2002]. This line of work demonstrates that testing for equal outcomes in binary classification should be performed at the marginal threshold to avoid issues of “infra-marginality” encountered by methods aggregating all outcomes above the threshold.

Although the definition of predictive rate parity is well understood, to the best of our knowledge, there does not exist a formal statistical test of predictive rate parity in the presence of dependent data. Most large-scale AI applications contain multiple records from the same member or user, and as a result, a natural dependency structure exists in the training data. In this paper, we close that gap in literature by
developing statistically rigorous non-parametric regression based tests for predictive rate parity in the presence of dependent observations. We illustrate how Nadaraya-Watson (non-parametric) regression estimators can be used for testing and extend traditional asymptotic results for these estimators to handle the dependence. We also introduce two variants of predictive rate parity which can yield differing results in the dependent observation setting. Next, we propose a mitigation approach based on these non-parametric estimators. We show how they can be used to provide minimally-biased post-processing transformations to achieve predictive rate parity across groups.

Most of the fairness literature focuses on detection and mitigation of algorithmic bias in a single machine learning model. However, most large-scale industry applications use a combination of models that are predicting different objectives and are ultimately combined through weighted summations to give a single score (such as those described in [Agarwal et al., 2014], [Ramanath et al., 2021], [Chen et al., 2020], and [Lada et al., 2021]). In this paper, we also extend our methods to test and mitigate for predictive rate parity in such multi-objective models. We devise methodology to simultaneously achieve predictive rate parity on each of the individual objective models as well as the overall model, which to our knowledge has not been previously addressed.

Finally, we test and mitigate discrepancies in marginal outcomes for binary classification problems. Note that assessing disparity in marginal outcomes has been addressed by [Simoiu et al., 2017], but this work requires the marginal threshold of a classifier be inferred. We suggest applying methodology analogous to the non-parametric regression methodology to provide minimally biased estimates of marginal outcomes for score-based classifiers with known marginal thresholds. An empirical study of our testing and mitigation methodology is presented with real-world datasets to show the efficacy of our approach.

The rest of the paper is organized as follows. In Section 2 we provide a problem statement and precise formulation of predictive rate parity. In Section 3 and 4 we develop the formal testing and mitigation methodology respectively. In Section 5 we extend these methods to multi-objective models. Section 6 discusses the extensions to marginal outcomes. We present the experimental results in Section 7 before concluding with a discussion in Section 8.

1.1 RELATED WORK

While there are many definitions of fairness [Barocas et al., 2019], which are often conflicting, we focus on predictive rate parity, which has also been formulated in the fairness literature as equal model calibration across groups [Chouldechova, 2017]). Calibration in predictive modeling has been well studied, see [Kumar et al., 2019] and references for applications of calibration techniques in and outside of machine learning. In fairness, lack of equal calibration among groups is an issue in many contexts including criminal justice [Kleinberg et al., 2017] and healthcare [Obermeyer et al., 2019].

There have been multiple approaches to yield better calibrated models. [Huang et al., 2020] provides an overview of some such techniques, including as Platt Scaling, Isotonic Regression, and Bayesian Binning. More recent calibration methods include [Hebert-Johnson et al., 2018], which focuses on subgroup calibration. Any post-processing calibration mentioned above can be used to achieve predictive rate parity by individually calibrating for protected groups. However, beyond applying a calibrator onto a model, there is also a question regarding the quality of calibration, which is the focus of this paper. Better methods of calibration testing have been researched in [Kumar et al., 2019], [Widmann et al., 2019]. These tests aim to assess the overall quality of calibration. Recently, [Tygert, 2022] has proposed methods for assessing calibration differences between a subgroup and the full population. Our work is similar in that it also aims to study a similar issue of calibration differences across subgroups. However, the methodology we propose additionally accounts for testing with dependency between samples, which to our knowledge has not been studied before but is highly practical as repeat users appear in many large-scale internet applications.

2 PROBLEM STATEMENT

Assume we observed data as \((Y, G, S)\), where \(Y\) is an outcome being predicted by a machine learning model, \(G\) is a group membership, and \(S\) is a model score. Note that \(Y\) is often binary, but need not be for the remaining discussion. We say that a score based machine learning model satisfies Predictive Rate Parity when the following definition holds.

**Definition 1.** A score based machine learning model satisfies **predictive rate parity** for groups \(g_1\) and \(g_2\) if for each score \(s\),

\[
E(Y|G = g_1, S = s) = E(Y|G = g_2, S = s).
\]

For a binary classifier, this condition asserts that for each possible value of the model score, the proportion of positive outcomes should be identical between the two groups which holds if and only if the outcome is independent of the group label when conditioned on the score.

2.1 TECHNICAL DETAILS

The definition

\[
E(Y|G = g_1, S = s) = E(Y|G = g_2, S = s),
\]

for all \(s\) is a slight abuse of notation, particularly when conditioning on sets of measure zero. For this to be meaningful, it requires that by group, the scores and labels have a bivariate density that is nonzero on the same ranges for each attribute group, which may not always be the case. A more precise formulation is to require that the random variables
Testing for predictive rate parity requires estimating the conditional expected outcomes, \( E(Y|G = g, S) \). A natural estimator of these quantities is the Nadaraya-Watson estimator, which is a class of non-parametric regression estimators. Assume iid pairs \((X_i, Y_i)\) are generated according to a model

\[
Y_i = f(X_i) + \epsilon_i,
\]

where \( f(X_i) = E(Y_i|X_i) \), and \( E(\epsilon_i|X_i) = 0 \). The Nadaraya-Watson estimator is

\[
\hat{f}(x) = \frac{\sum_{i=1}^{n} Y_i K_h(X_i - x)}{\sum_{i=1}^{n} K_h(X_i - x)}
\]

for a Kernel \( K_h(\cdot) \) having bandwidth parameter \( h \). We will further assume that the kernel takes the form \( K_h(x) = K(x/h) \). A common choice of kernel is the Gaussian kernel which specifies

\[
K_h(x) = \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{x^2}{2h^2} \right).
\]

Note that for a fixed “bandwidth” parameter, \( h \), the asymptotics of this estimator follow from a straight-forward application of the delta method (see Theorem 5.5.24 of [Casella and Berger 2002]). However, when the bandwidth is non-vanishing, the estimator is biased, and will not converge to \( f(x) \). In particular, we can decompose the usual pivotal quantity

\[
\frac{\hat{f}(x) - f(x)}{\sqrt{\text{var}(\hat{f}(x))}} = \frac{\hat{f}(x) - E(\hat{f}(x))}{\sqrt{\text{var}(\hat{f}(x))}} + \frac{E(\hat{f}(x)) - f(x)}{\sqrt{\text{var}(\hat{f}(x))}}.
\]

As such, the bandwidth needs to be tending to zero at a suitable rate, otherwise bias will be asymptotically non-zero and the term

\[
\frac{E(\hat{f}(x)) - f(x)}{\sqrt{\text{var}(\hat{f}(x))}}
\]

will be non-vanishing. Consequently, valid inference requires the bandwidth to tend to zero at an appropriate rate. The asymptotics for these estimators with decreasing bandwidth are well studied when the data observed is independent and identically distributed, see for example [McMurty and Politis 2008]. However, we will derive the asymptotic distributions under certain types of dependence that are common to machine learning applications, typically arising from repeated observations from the same member of the platform.

An appropriate model for such dependent data is that we observe tuples \((Y_{m,i}, S_{m,i}, G_m)\) for \( m = 1, \ldots, M \), and \( i = 1, \ldots, n_i \). Here, \( m \) denotes the member, and \( i \) denotes instances at which the member engages with the system. We will assume that the \((Y_{m,i}, S_{m,i}, G_m)\) are identically distributed and that the observations are independent across members, but that \((Y_{m,i}, S_{m,i}, G_m)\) and \((Y_{m,i', S_{m,i'}, G_{m}})\) can be dependent. In these situations, there are several choices for the conditional expectations which may be considered for assessing predictive rate parity which we will now discuss.

**Definition 3.** A predictive algorithm satisfies **user-level predictive rate parity** if Definition 2 is satisfied with respect to the conditional expectations \( E(Y_{m,1}|S_{m,1} = s, G_{m,1} = g) \).

Note that this quantity is not what is typically considered when assessing predictive rate parity. Let \((Y_{1}^*, S_{1}^*, G_1^*)\), \(\ldots\), \((Y_{N}^*, S_{N}^*, G_N^*)\) for \( N = \sum n_i \), be an arbitrary ordering of the observed data, \((Y_{m,i}, S_{m,i}, G_m)\). An alternative formulation of predictive rate parity is as follows.

**Definition 4.** A predictive algorithm satisfies **aggregate predictive rate parity** if Definition 2 is satisfied with respect to the conditional expectations \( E(Y_{i}^*|S_i^* = s, G_i^* = g) \).

In general,

\[
E(Y_{m,1}|S_{m,1} = s, G_{m,1} = g) \neq E(Y_{i}^*|S_i^* = s, G_i^* = g),
\]

and so user-level and aggregate predictive rate parity cannot hold simultaneously. While the aggregate predictive rate
parity condition is what is commonly considered in fairness literature, we recommend considering user-level predictive rate parity, which avoids giving undue influence to users with disproportionately high representation in the data. As a simple example, consider a health care algorithm predicting likelihood of having a disease. A patient may repeatedly take such screening tests as part of a routine check-up until getting a positive outcome. In such cases, affluent users with greater access to health care may tend to test more frequently and have disproportionately many negative outcomes. This can downward-bias the estimates of average outcome for groups tending to have more affluent members. In such cases, the interpretation of calibration is understood as a per-user quantity. For instance, it is typically understood that 60% of randomly chosen people who test with a score of .6 should have the disease, which is in agreement with the per-user formulation of predictive rate parity.

For conciseness, the remainder of this section will focus on testing for user-level predictive rate parity, although obvious modification of the proposed methodology gives valid inference for the aggregate level-predictive rate parity. In order to understand user-level predictive rate parity, we need to estimate the quantities \( f_g(s) = E(Y|S = s, G = g) \), and we will use the Nadaraya-Watson style estimator \( \hat{f}_g(s) = \frac{1}{n_g} \sum_{i=1}^{n_g} Y_{m,i} K((S_{m,i} - s)/h) \). Note that this formulation has the slight modification of averaging over outcomes by user, which is necessary to estimate the per-user conditional average-outcomes. In order to derive a statistical test for predictive rate parity, we will first derive the asymptotic distributions of these estimators following the method of proof used by McMurry and Politis [2008] with appropriate modifications for dependence.

We will prove the following theorem to characterize the appropriate rate of convergence of the bandwidth to zero. For brevity, we push the notation, necessary assumptions, and proof to the Appendix.

**Theorem 1.** Under the Assumptions provided in the Appendix, there exists a function \( \sigma_g^2(x) \) such that
\[
\sqrt{Mh} \left( \hat{f}_g(s) - f_g(s) + o(h^d) \right) \rightarrow N (0, \sigma_g^2(x))
\]
where \( d = \min \{d_1, d_2\} \) (defined in the Appendix). In particular, choosing \( h = O(n^{-1/(2d+1)}) \), we have that
\[
\sqrt{Mh} \left( \hat{f}_g(s) - f_g(s) \right) \rightarrow N (0, \sigma_g^2(x))
\]

Testing for predictive rate parity can be accomplished by testing equality of the conditional expectations at a fixed range of score values and simply applying a Bonferroni (or any other family-wise error-rate controlling) correction to the p-values computed using the result of Theorem 1. With additional effort, methods such as those in Luedtke et al. [2019] or Srihera and Stute [2010] for testing equality of unknown functions can be modified to account for dependence and provide higher-powered tests for predictive rate parity.

**Remark 1.** The choice of bandwidth parameter is an open field of research. Common approaches are to use cross-validation or to use “rule-of-thumb/plug-in” choices such as those proposed by Chu et al. [2013]. Our empirical findings in Section demonstrate that the recommendations for the iid setting carry over well, and that simple heuristics can yield good performance.

### 4 MITIGATION APPROACHES

Writing \( f_g(s) = E(Y|S = s, G = g) \), we could, for members of group \( g_1 \), replace a score \( s \) with the score \( \hat{s} \) satisfying \( f_{g_1}(\hat{s}) = f_{g_2}(s) \), or equivalently, transform the scores of both groups such that \( f_{g_1}(\hat{s}) = f_{g_2}(\hat{s}) \). Any such transformation would satisfy Predictive Rate Parity. Arguably the simplest, and most intuitive approach is to “calibrate” the scores. That is, choose a transformation \( \hat{s} = t(s) \) such that
\[
E(Y|\hat{S} = s, G = g) = s
\]
for all \( g \) and \( s \). This transformation is the ideal transformation in the mean-squared error sense that it solves
\[
E(Y|S, G) = \arg \min_f E(Y - f(G, S))^2.
\]

**Remark 2.** Note that after calibrating model scores for each group under consideration, the scores may no longer have common support across groups, motivating the need for Definition [2].

There are numerous approaches for achieving this calibration including isotonic regression, Platt’s scaling, and binning. Platt’s scaling and isotonic regression have been demonstrated to have good empirical properties, though may fail to yield asymptotically optimal transformation in cases where the model assumptions are not met. Binning can introduce unintended bias, which we will illustrate below. We propose Nadaraya-Watson estimators as an alternative to these methods, and provide an empirical comparison in Section [7].

Binning proceeds as follows. Consider a binning \( B_1, \ldots, B_K \) of the score space with \( B_i = [l_i, u_i] \). For a given score, denote by \( B(s) \) the bin \( B_i \) containing \( s \), i.e. for which \( s \in B_i \). The binning approach defines \( \hat{s} = t(s, g) \) where
\[
t(s, g) = \hat{E}(Y|S \in B(s), G = g) = \frac{\sum_j Y_j \cdot I \{ S_j \in B(s), G_j \in g \}}{\sum_j I \{ S_j \in B(s), G_j \in g \}}.
\]
A drawback of this approach is that it is subject to a bias-variance trade-off with respect to the choice of bins. At one extreme, choosing a single bin amounts to replacing all scores with the average outcome for each group. While this does calibrate the model, it is clearly unsatisfying with respect to model performance. At the other extreme, choosing too many bins will result in many bins with a predicted average outcome of zero, which can fail to yield proper
To overcome these shortcomings, we propose a modification of binning called near-unbiased binning. This approach provides a modification of the typical binning which is consistent in the sense of asymptotically yielding correct calibration, and near-unbiased in the sense that the bias is asymptotically negligible.

4.1 NEAR-UNBIASED BINNING

It was seen in Section 3 that the Nadaraya-Watson estimator provides a consistent estimate of the expected outcome at a given score for suitably chosen kernels and bandwidth parameters. Let \( f_g(\cdot) \) be such an estimator for group \( g \).

Given the score binning, rather than simply transforming according to the average outcome within a bin, a more refined transformation would be to map a score \( s \) to an appropriate weighting of the average outcomes evaluated at the endpoints of the bin containing \( s \). One such transformation is the simple linear interpolation score transformation

\[
    t(s, g) = \frac{\hat{f}_g(u_k) - \hat{f}_g(l_k)}{u_k - l_k} \cdot (s - l_k) + \hat{f}_g(l_k)
\]

where \( k \) is the index such that \( s \in B(k) \). This transformation is asymptotically unbiased at the endpoints of the bins in the sense that the transformation converges to the “ideal” transformation, i.e. \( t(s, g) \rightarrow f_g(s) \) almost surely for scores on the endpoints of the bins. Provided the binning is reasonably fine, the transformation will have small bias over the ideal transformation. Another advantage of this transformation over traditional binning is that it preserves (strict) monotonicity of the average outcome asymptotically, which provides the advantages of isotonic regression while providing robustness against non-monotonicity of the underlying function.

5 MULTI-OBJECTIVE FAIRNESS

Many online recommendation systems including those used by large technology companies such as LinkedIn (Agarwal et al. [2014]), Meta (Lada et al. [2021]) and Pinterest (Chen et al. [2020]), models of various outcomes are combined into a single score through a weighted average.

Suppose that scores \( S^1_i, \ldots, S^K_i \) predict outcomes \( Y^1_i, \ldots, Y^K_i \), and that ranking, recommendation, classification, etc. is then based on a composite score \( S_i = \sum_{k=1}^{K} w_k S^k_i \) where the \( w_k \) are weights chosen to balance the trade-offs between the various objectives. In such cases, it may be interesting to not only have calibration of the individual models, but to have calibration of \( S_i \) as a predictor of the composite outcome \( Y_i = \sum_{k=1}^{K} w_k Y^k_i \).

In general, calibration of the individual model is not enough to guarantee calibration of the composite model. However, a slightly stronger condition is sufficient:

**Lemma 1.** Suppose that \( E(Y^k_i|S_i^1 = s_1, \ldots, S_i^K = s_K) = s_k \) for all \( k \) and all \( s_1, \ldots, s_K \), then \( E(\tilde{Y}_i|S_i = s) = s_i \).

Interestingly, if the individual models are calibrated, then there can exist an individual model for which \( S^k_i \) is not the best linear predictor of \( Y^k_i \) given \( S^1_i, \ldots, S^K_i \). It follows that there can be some other set of weights \( \tilde{w}_1, \ldots, \tilde{w}_K \) such that \( \tilde{S}_i = \sum_{k=1}^{K} \tilde{w}_k Y^k_i \) is a better (in terms of mean-squared error) predictor of \( \tilde{Y}_i \) than \( S_i \).

5.1 MULTI-OBJECTIVE MITIGATION

Similarly to the univariate case, mitigation in the multi-objective setting can be achieved by replacing the original scores with appropriate conditional expectations.

**Theorem 2.** Suppose we observe scores \( S_k = s_k \) Defined transformed scores as

\[
    \tilde{S}_k = E(Y^k|S_1, \ldots, S_K, G)
\]

for \( k = 1, \ldots, K \). That is, replace observe scores \( S_1 = s_1, \ldots, S_K = s_K \), with transformed scores \( \tilde{S}_k = E(Y^k|S_1 = s_1, \ldots, S_K = s_K, G) \). Then, the transformed composite score

\[
    \tilde{S} = \sum_k w_k \cdot \tilde{S}_k
\]

satisfies predictive rate parity for the multi-objective outcome in the sense that

\[
    E\left( \sum_k w_k \cdot Y^k | \sum_k w_k \cdot \tilde{S}_k = s, G = g_1 \right) = E\left( \sum_k w_k \cdot Y^k | \sum_k w_k \cdot \tilde{S}_k = s, G = g_2 \right)
\]

for all \( s \) and weights \( w_1, \ldots, w_K \), while also maintaining predictive rate parity on the individual models in the sense that

\[
    E(Y^k|\tilde{S}_k = s_k, G = g_1) = E(Y^k|\tilde{S}_k = s_k, G = g_2)
\]

for all \( s \) and \( k \).

In particular, if we calibrate the individual models conditional on the scores, then the composite score will satisfy predictive rate parity regardless of the weights chosen. In many internet industry applications, the weights are chosen through online experimentation (for instance, see [Agarwal et al. [2018]]), and this allows for guarantees of predictive rate parity regardless of the weights ultimately chosen.

Even if calibration of the composite outcome is not of interest to the practitioner, the proposed transformations to the individual model scores improve accuracy, i.e.

\[
    E(Y^k_i|S_1, \ldots, S_K, G) = \arg \min_t E \left( Y^k_i - t(S_1, \ldots, S_K, G) \right)^2.
\]
6 MARGINAL OUTCOME FAIRNESS

In classification settings, candidates whose scores are above a common threshold receive the same treatment. Therefore, the matter of fairness comes down to whether a common standard is applied to all groups. While all groups are subjected to a common score threshold, mis-calibration in scores between groups can result in different effective thresholds being applied to different group as measured by the marginal outcome of each group [Corbett-Davies and Goel, 2018]. Marginal outcomes are used to detect discrimination in applications such as police search [Anwar and Fang, 2006] and medical treatment [Anwar and Fang, 2011]. Ayres [2002] notes in such applications that “outcome tests” which compare the average (or infra-marginal) outcomes stemming from classification decisions across groups, can give misleading impressions of algorithmic bias and argues for the need to assess disparities in outcomes associated with marginal (or threshold) decisions. Formally, suppose there exists a function \( \sigma \) analogously to Theorem 1, allowing for rigorous testing of normality of the estimated average marginal outcome holds are missing labels below the marginal threshold. Asymptotic scores above the threshold, since it is now assumed that we can be applied to this context. A consistent and minimally biased estimator of the marginal outcome can be computed as

\[
\hat{f}_g(t^*) = \frac{1}{M} \sum_{m} g_m \hat{f}_{m} \sum_{i:S_i \geq t^*} Y_{m,i} K((S_{m,i} - s)/h)
\]

Note that this has the minor modification of only averaging scores above the threshold, since it is now assumed that we are missing labels below the marginal threshold. Asymptotic normality of the estimated average marginal outcome holds analogously to Theorem 1 allowing for rigorous testing of equality between groups.

Theorem 3. Under the assumptions of Theorem 1 there exists a function \( \sigma^2_g(x) \) such that

\[
\sqrt{Mh} \left( \hat{f}_g(t^*) - E(Y_i|G_i = g, S_i = t^*) \right) \rightarrow N \left( 0, \sigma^2_g(x) \right).
\]

6.1 MARGINAL MITIGATION

When the test rejects, there is evidence that one group is “advantaged” in the sense that the marginal candidates from this group are held to a higher standard than the marginal candidates from the other, “disadvantaged” group. In order to correct for differences in treatment of marginal (or threshold) candidates, we can raise the threshold applied to the advantaged group, lower the threshold applied to the disadvantaged group, or some combination of both. Raising the threshold of the advantaged group may be preferable from a technical standpoint since it avoids the need to infer the outcomes below the marginal threshold, however, it may not be the best choice for the problem at hand (e.g. the practitioner may be satisfied with the threshold applied to the advantaged group and might rather decrease the threshold for the disadvantaged group).

Let \( \hat{f}_g(t) \) be a prediction of the average outcome of group \( g \) at a threshold \( t \). For \( t \) above the marginal threshold, this can simply be the non-parametric regression estimates described in Section 3. Below the marginal threshold, the average outcome can be estimated, for instance, by a linear regression of outcomes near the marginal threshold. Now, the marginal threshold can be replaced with group-specific thresholds \( t^*_g \) which satisfy

\[
\hat{f}_{g_1}(t^*_{g_1}) = \hat{f}_{g_2}(t^*_{g_2}).
\]

Remark 3. While it may seem unfair to apply differing thresholds to each group, this procedure is equivalent to calibrating models and applying the same threshold to each group. Here, the differing thresholds are to account for the differing calibration of the models.

Note that there are typically many thresholds yielding equal marginal outcomes, and additional constraints can be added to choose appropriate thresholds. For instance, thresholds can be modified to mitigate marginal outcome bias while maintaining the same overall rate of positive classification. Let \( F_{g_i}(\cdot) \) be the cumulative distribution function of the scores arising from group \( g_i \). Choosing fair group-level thresholds maintaining the same overall positive classification rate can be accomplished by adding the constraint that

\[
\sum_{i} p_{g_i} \int_{0}^{t^*_{g_i}} \hat{f}_{g_i}(t) \hat{F}_{g_i}(0, \hat{\sigma}^2) dt = \sum_{i} p_{g_i} \int_{0}^{t^*_{g_i}} \hat{f}_{g_i}(t) \hat{F}_{g_i}(0, \hat{\sigma}^2) dt
\]

where \( p_g = P(G = g) \).

Theorem 4. A solution \( (t^*_{g_1}, t^*_{g_2}) \) that mitigates against marginal outcome bias while maintaining the overall positive classification rate, i.e. that solves Equations 7 and 2 exists when the average outcome functions and score distributions are continuous.

Remark 4. In applications without observed outcomes below the marginal threshold, it is important to note that the predicted outcomes below the marginal threshold will be merely suggestive. In practice, one can either either iteratively correct the thresholds and re-test for marginal outcome fairness or temporarily lower the threshold of the disadvantaged group to learn the appropriate threshold.

7 EMPIRICAL FINDINGS

In this section, we demonstrate the efficacy of the non-parametric test for predictive rate parity proposed in Section...
and the linear interpolation reranker for mitigation discussed in Section 4. We have made all the code publicly available[1]. We use three real-world open-source datasets and a proprietary dataset from LinkedIn to show the efficacy of our approach:

• The UCI Adult dataset [Kohavi and Becker, 1996] sources from US Census data where the task is to classify whether a person makes over $50k in annual income or not. We use the race attribute (black/white) to determine group membership.

• The UCI Heart Disease dataset [Jainosi et al., 1996] is a medical dataset where the task is to classify whether a person has heart disease or not. We use the sex attribute (male/female) to determine group membership.

• The Employment US Census data [Ding et al., 2021] also sources from the US Census data but from a much more recent timeframe. We use the 2018, 1-year person survey in the state of Texas for our experiment. The task is to predict if a person between 16-90 is employed or not. We use the sex attribute (male/female) to determine group membership.

• The PYMK data is a proprietary dataset sourced from LinkedIn’s People You May Know product, which suggests connections to its members. Each impression (suggestion) to a viewer has a corresponding pInvite score from a classification model, which estimates the likelihood that a user will send a connection request. We use member visit pattern to determine group membership, namely frequent and infrequent users.

For the publicly available datasets, we split the data into a training and testing set, then tune a random forest model on the training data (without the group attribute) and score the testing set. We then employ bootstrapping on the testing set to gauge the effectiveness of the reranker. Specifically, in each iteration of the bootstrap we sample the scored testing set, then further split it in half into a calibrator training and testing set. Next, one calibration model is trained for each group in the data (namely, ethnicity for the Adult and Census data, sex for the Heart Disease data, and member visit pattern for the PYMK data). Finally, we then rerank the calibrator testing set and evaluate performance of the calibrators with the following metrics:

1. Non-parametric calibration error: Defined as the accuracy of the calibration and computed as the difference between the non-parametric estimated expected outcome at a score $\hat{f}_g(s)$ and the score $s$ itself.

   $$1 \sum_{s \in S} \sum_{g \in G} |\hat{f}_g(s) - s|$$

2. Parity error: Defined as the closeness of the expected outcome for different groups. Even if perfect calibration cannot be achieved, it is still desirable to have the same interpretations of a score for all groups.

   $$\frac{1}{|S|} \sum_{s \in S} \sum_{g, g' \in G, g \neq g'} |\hat{f}_g(s) - \hat{f}_{g'}(s)|$$

3. AUC: A metric for binary classification performance defined as the area under the receiver-operator-characteristic (ROC) curve. Although AUC is invariant under monotonic transformations, separate calibration for different groups could increase the AUC.

4. ECE: The Expected Calibration Error proposed in [Guo et al., 2017], which uses bins to compute a weighted sum of the difference between the average score and the average outcome across each bin. We include this metric to show that the linear interpolation reranker does not exclusively mitigate predictive rate gaps as detected by the non-parametric test.

For the non-parametric test, we evaluate the expected outcome of the model on the testing set at the scores percentiles corresponding to 1%, 5%, 10%, ..., 95%, 99% of scores in the training set (which may be different for each calibration method). We set the bandwidth of the test evaluated at score $s$ to be $\max(1.06 \sqrt{s(1-s)/n}, n^{-\xi/10})$ where the first term corresponds to the rule-of-thumb for Gaussian kernels (Chu et al., 2015) with Bernoulli variance and the second term acts as a lower bound to prevent trivial bandwidths. We use these parameters for all datasets.

Figure 1 shows an illustration of the non-parametric test on the Heart Disease dataset. It can be seen that the large gap in outcomes between the sexes at a given score would be problematic from a fairness perspective as the over-scored females may have easier access to treatment if patients were...
Figure 2: Non-parametric outcome test when assuming user-level (left) vs. aggregate data (right) showing differences in outcome tests of frequent users (blue) and infrequent users (red) prioritized by score alone. After reranking, although the calibration is not perfect, the actual outcomes are much closer between sexes.

We specifically use the PYMK data as it represents a setting where user-level outcome testing is different aggregate as we have many repeat invitees. Figure 2 gives an example of this. At the user-level, our test does not detect a difference in the $s \leq 0.7$ region. But under the aggregation perspective, the outcomes of frequent members are significantly higher than their infrequent counterparts of receiving an invite upon impression. This highlights the importance of distinguishing between user-level and aggregate assumptions when testing as our proposed methodology allows for.

In tables 1-4, we show the mean metric gathered from the bootstrap and indicate whether it improves over the baseline in a statistically significant manner (non-overlapping 95% confidence intervals) by an asterisk.

Table 1: Non-Parametric Calibration Error Comparison

| Dataset   | Baseline | NP      | Platt   | Isotonic |
|-----------|----------|---------|---------|----------|
| Adult     | .10652   | .04967* | .05603* | .05987*  |
| Heart Disease | .04494 | .01587* | .04135  | .02286*  |
| Census    | .03964   | .01954* | .045185 | .027146  |
| PYMK      | 1.06317  | .00777* | .00783* | .00873*  |

Table 2: Parity Error Comparison

| Dataset   | Baseline | NP      | Platt   | Isotonic |
|-----------|----------|---------|---------|----------|
| Adult     | .02683   | .03431  | .02971  | .03678   |
| Heart Disease | .04328 | .00672* | .006611*| .00606*  |
| Census    | .01408   | .01360  | .01540  | .01736   |
| PYMK      | .00237   | .00068* | .00084* | .00073*  |

Table 3: AUC Comparison

| Dataset   | Baseline | NP      | Platt   | Isotonic |
|-----------|----------|---------|---------|----------|
| Adult     | .04807   | .02328* | .03538  | .01661*  |
| Heart Disease | .02532 | .01610* | .03004  | .00657*  |
| Census    | .02370   | .01204* | .03872  | .00825*  |
| PYMK      | .53706   | .00098* | .00141* | .00073*  |

Table 4: Expected Calibration Error Comparison

8 DISCUSSION

This work has proposed using non-parametric regression estimators to test for predictive rate parity and to provide a post-processing mitigation approach that calibrates models in cases where there is evidence of bias. We extended traditional results on the asymptotic normality of these estimators to a non-iid observation setting, allowing for valid statistical testing under the sorts of dependence that are common in industry applications. In dependent data settings, we distinguish between calibration at an instance level versus at a user level, and note that these methods can be applied for either interpretation. We extend this methodology to address multi-objective models which are common in industry as well as to test for disparities in marginal outcomes.

We applied the methodology on several real-world datasets to demonstrate the favorable performance when compared with Platt’s scaling and isotonic regression. We also noted in the PYMK dataset that the calibration can indeed differ between the dependent and independent formulation.

Finally, we would like to remind the practitioners that choosing the proper definition of fairness should be context-specific. Applying a bias mitigation strategy without a good understanding of the context can lead to undesirable consequences due to the conflict between notions of fairness.
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A NOTATION AND ASSUMPTIONS

We will use the following notational conventions and assumptions. Let $f_S(y)$, denote the density of $S_{1,1}$ conditional on $G = g$, and $f_{Y,S}(y,s)$ denote the joint density of $Y_{1,1}$ and $S_{1,1}$ conditional on $G = g$. Further, define

$$
a_g(s) = \int_{-\infty}^{\infty} y f_{Y,S}(y,s) dy,
$$

$$
b_g(s) = \int_{-\infty}^{\infty} y f_{Y,S}(y,s) dy,
$$

$$
Y_{m,i,g} = Y_{m,i} \cdot I \{ G_{m,i} = g \},
$$

$$
\sigma_g^2(S_{n,m};n_m) = \text{var} \left( Y_{m,i,g} \mid S_{n,m}, n_m \right),
$$

$$
f_g(S_{n,m};n_m) = E \left( Y_{m,i,g} \mid S_{n,m}, n_m \right),
$$

$$
\bar{f}_g(s) = E \left( \frac{1}{n_m} f_g(s) ; n_m \right),
$$

$$
\bar{\sigma}_g^2(s) = E \left( \frac{1}{n_m} \sigma_g^2(s) ; n_m \right),
$$

$$
\bar{e}_g(s) = E \left( \frac{1}{n_m} \text{var} \left( Y_{G_{m,i} = g} \mid S_{1,m}, n_m \right) \right),
$$

and

$$
\bar{v}_g(s) = E \left( \frac{1}{n_m} \text{var} \left( Y_{G_{m,i} = g} \mid S_{1,m}, n_m \right) \right).
$$

Assumptions

1. As $M \to \infty$, $h \to 0$ in such a way that $M \cdot h \to \infty$.
2. For convenience, we will assume that conditional on $n_m$, the $(Y_{m,i}, S_{m,i})$ are iid. This assumption is made for notational compactness and ease of exposition. It is readily seen from the proofs that condition is not required, and analogous results hold with appropriate adjustments for dependence. Note that this assumptions still explicitly allows $E \left( Y_{m,i} \mid S_{m,i}, n_{m,i} = n \right)$ to depend on $n$, which encapsulates the case in which calibration depends on user-activity level.
3. $b_g(\cdot)$ has $d_1$ continuous derivatives and $f_g(\cdot)$ has $d_2$ continuous derivatives in a neighborhood of $s$.
4. $K(\cdot)$ integrates to one, the first $d_k \geq \max \{ d_1, d_2 \}$ moments of $K(\cdot)$ are zero, and $K(\cdot)$ has tails decaying faster than $x^{-m}$ for any $m$.
5. $\bar{\sigma}_g^2(s)$ and $\bar{f}_g(s)$ are continuous and bounded below by some constant $b > 0$.
6. There exist constants $C$ and $\delta$ such that for all $s$, $E \left( |Y|^{2+\delta} \mid S = s \right) < C$

B PROOF OF THEOREM 1

We follow the proof of asymptotic normality for the Nadaraya-Watson estimator for the iid case in McMurry and Politis [2008] with needed modifications for dependence. Note that the numerator of the Nadaraya-Watson estimator is an estimator of $a_g(s)$ while the denominator is an estimator of $b_g(s)$.

B.1 SUPPORTING RESULTS

We will begin by establishing the respective rates of convergence. Define

$$
\hat{a}_g(s) = \frac{1}{M} \sum_{m : G_{m,g} = g} \frac{1}{n_m} \sum_{i=1}^{n_m} Y_{m,i} K \left( (S_{m,i} - s) / h \right)
$$

and

$$
\hat{b}_g(s) = \frac{1}{M} \sum_{m : G_{m,g} = g} \frac{1}{n_m} \sum_{i=1}^{n_m} K \left( (S_{m,i} - s) / h \right).
$$

Lemma 2. Under Assumptions 1 – 6,

$$
E \left( \frac{1}{M} \sum_{m : G_{m,g} = g} \frac{1}{n_m} \sum_{i=1}^{n_m} Y_{m,i} K \left( (S_{m,i} - s) / h \right) \mid G_{m,g} = g \right) = P(G_{m,g} = g) \cdot a_g(s) + o(h^d)
$$

and

$$
E \left( \frac{1}{M} \sum_{m : G_{m,g} = g} \frac{1}{n_m} \sum_{i=1}^{n_m} K \left( (S_{m,i} - s) / h \right) \mid G_{m,g} = g \right) = P(G_{m,g} = g) \cdot b_g(s) + o(h^{d_1}).
$$

Proof of Lemma 2 Define $M_g = \{|m : G_m = g|\}$ to be the number of members in group $g$. Note that

$$
E \left( \frac{1}{M} \sum_{m : G_{m,g} = g} \frac{1}{n_m} \sum_{i=1}^{n_m} Y_{m,i} K \left( (S_{m,i} - s) / h \right) \mid n_m, M_g \right)
$$

$$
= E \left( \frac{1}{M} \sum_{m : G_{m,g} = g} \frac{1}{n_m} \sum_{i=1}^{n_m} Y_{m,i} K \left( (S_{m,i} - s) / h \right) \mid n_m, M_g \right)
$$

$$
= E \left( \frac{M_g}{M} Y_{m,1} K \left( (S_{m,1} - s) / h \right) \mid n_m, M_g \right)
$$

$$
= P(G_m = g) E \left( Y_{m,1} K \left( (S_{m,1} - s) / h \right) \mid n_m, M_g \right)
$$

Now,

$$
E \left( Y_{m,1} K \left( (S_{m,1} - s) / h \right) \mid n_m, M_g \right)
$$

$$
= E \left( E \left( Y_{m,1} K \left( (S_{m,1} - s) / h \right) \mid S_{m,i} \right) \right)
$$

$$
= E \left( E \left( Y_{m,1} \mid S_{m,i} \right) K \left( (S_{m,1} - s) / h \right) \right)
$$

$$
= \int_{-\infty}^{\infty} E \left( Y_{m,1} \mid S_{m,i} = s \right) K((s' - s) / h) f_S(s') ds'
$$

$$
= \int_{-\infty}^{\infty} f_g(s') K((s' - s) / h) f_S(s') ds'.
$$

Fix an $\epsilon > 0$, and denote by $N_{\epsilon}(s)$ the neighborhood of radius epsilon centered around $s$. Then, this integral can be
expressed as
\[ \int_{-\infty}^{\infty} f_g(s') K((s' - s)/h) f_S(s') ds' = \int_{N_+(s)} f_g(s') K((s' - s)/h) f_S(s') ds' \]
\[ + \int_{N_+(s)} f_g(s') K((s' - s)/h) f_S(s') ds' \]
The assumption that \( K \) has \( d_k \) finite moments implies that
\[ \int_{N_+(s)} f_g(s') K((s' - s)/h) f_S(s') ds' = o(h^{d_k}) \]
Through a transformation of variables, \( t = (s' - s)/h \)
\[ \int_{N_+(s)} f_g(s') K((s' - s)/h) f_S(s') ds' = \int_{-\epsilon/h}^{\epsilon/h} f_g(s + ht) f_S(s + ht) K(t) dt \]
By assumption, \( f_g(s') f_S(s') \) has \( d \) bounded and continuous derivatives on \( N_+(s) \). It follows that there exists an \( s^* \in [s, s + ht] \) such that
\[ f_g(s + ht) f_S(s + ht) = f_g(s) f_S(s) \sum_{d=1}^{d_k-1} \frac{(ht)^d}{d!} (f_g' f_S)(d)(s) + \frac{(ht)^d}{d!} (f_g' f_S)(d)(s + s^*) \]
and we can write
\[ \int_{-\epsilon/h}^{\epsilon/h} f_g(s + ht) f_S(s + ht) K(t) dt \]
\[ = \int_{-\epsilon/h}^{\epsilon/h} f_g(s) f_S(s) K(t) dt \]
\[ + \sum_{d'=1}^{d_k-1} \int_{-\epsilon/h}^{\epsilon/h} \frac{(ht)^{d'}}{d'!} (f_g' f_S)(d')(s) K(t) dt \]
\[ + \int_{-\epsilon/h}^{\epsilon/h} \frac{(ht)^d}{d!} (f_g' f_S)(d)(s + s^*) K(t) dt \]
By the assumption on the tail decay of \( K(\cdot) \), and the fact that \( K(\cdot) \) integrates to one, we have that
\[ \int_{-\epsilon/h}^{\epsilon/h} f_g(s + ht) f_S(s + ht) K(t) dt \]
\[ = \int_{-\infty}^{\infty} f_g(s) f_S(s) K(t) dt + o(h^{d_k}) \]
\[ = b_g(s) + o(h^{d_k}) \]
Proof of Lemma 3. 

\[
\text{var} \left( \frac{1}{M} \sum_{m=1}^{n_m} \frac{1}{n_m} \sum_{i=1}^{n_m} Y_{m,i}' K((S_{m,i} - s)/h) \right) \\
= \frac{1}{M^2} \sum_{m=1}^{n_m} \text{var} \left( \frac{1}{n_m} \sum_{i=1}^{n_m} Y_{m,i}' K((S_{m,i} - s)/h) \right)
\]

By the law total variances,

\[
\text{var} \left( \frac{1}{n_m} \sum_{i=1}^{n_m} Y_{m,i}' K((S_{m,i} - s)/h) \right) = \left( E \left( \frac{1}{n_m} \sum_{i=1}^{n_m} Y_{m,i}' K((S_{m,i} - s)/h) \right) \right)^2 + E \left( \text{var} \left( \frac{1}{n_m} \sum_{i=1}^{n_m} Y_{m,i}' K((S_{m,i} - s)/h) \right) \right)
\]

The expectation of the first term on the left hand side is constant with respect to \( n_m \), so the term is zero. The inner variance in the second term can be written as

\[
\text{var} \left( \frac{1}{n_m} \sum_{i=1}^{n_m} Y_{m,i}' K((S_{m,i} - s)/h) \right) = \frac{1}{n_m} \sum_{i=1}^{n_m} \text{var} \left( Y_{m,i}' K((S_{m,i} - s)/h) \right)
\]

Note, here, we made use of the assumption that the observations are conditionally independent on the number of repeat measurements, though the following argument hold with non-zero covariances. Again applying the law of total variance, we have

\[
\text{var} \left( Y_{m,i}' K((S_{m,i} - s)/h) \right) = \left( E \left( Y_{m,i}' K((S_{m,i} - s)/h) \right) \right)^2 + E \left( \text{var} \left( Y_{m,i}' K((S_{m,i} - s)/h) \right) \right)
\]

Note that

\[
E \left( Y_{m,i}' K((S_{m,i} - s)/h) \right) = f_g(S; n_m) \cdot K((S_{m,i} - s)/h)
\]

and

\[
\text{var} \left( Y_{m,i}' K((S_{m,i} - s)/h) \right) = K^2((S_{m,i} - s)/h) \text{var} \left( Y_{m,i} \right) = K^2((S_{m,i} - s)/h) \sigma^2_g(S_{m,i}; n_m).
\]

Arguing as in Lemma 2, it is readily seen that

\[
E \left( f_g(S; n_m) \cdot K((S_{m,i} - s)/h) \right) = f_g(s; n_m) + o(h^d)
\]

It follows that

\[
\text{var} \left( \frac{1}{M} \sum_{m=1}^{n_m} \frac{1}{n_m} \sum_{i=1}^{n_m} Y_{m,i}' K((S_{m,i} - s)/h) \right)
= E \left( \frac{1}{M} \sum_{m=1}^{n_m} \frac{1}{n_m} \sum_{i=1}^{n_m} f_g(S; n_m) \cdot \sigma^2_g(S_{m,i}; n_m) \cdot K^2((S_{m,i} - s)/h) \right)
+ \sum_{m=1}^{n_m} \text{var} \left( \frac{1}{n_m} \sum_{i=1}^{n_m} f_g(S; n_m) \cdot \sigma^2_g(S_{m,i}; n_m) \cdot K((S_{m,i} - s)/h) \right)
\]

\[
= \int_{-\infty}^{\infty} \left[ f_g^2(s; n_m) + \sigma^2_g(s; n_m) \right] \cdot f_g(s') \cdot K^2(s'/h) \cdot f_g(s; n_m) + o(h^d)
\]

\[
= \frac{1}{h} \int_{-\infty}^{\infty} \left[ f_g^2(s + th; n_m) + \sigma^2_g(s + th; n_m) \right] \cdot f_g(s + th) \cdot K^2(t) dt + f_g^2(s; n_m) + o(h^d)
\]

\[
= \frac{1}{h} \left( f_g^2(s; n_m) + \sigma^2_g(s; n_m) \right) \cdot f_g(s) \int_{-\infty}^{\infty} K^2(t) dt + f_g^2(s; n_m) + o(h^d)
\]

Therefore,

\[
\text{var} \left( \frac{1}{n_m} \sum_{i=1}^{n_m} Y_{m,i}' K((S_{m,i} - s)/h) \right)
= \frac{1}{h} E \left( \frac{1}{n_m} \sum_{i=1}^{n_m} f_g^2(s; n_m) + \sigma^2_g(S_{m,i}; n_m) \right) f_g(s) \int_{-\infty}^{\infty} K^2(t) dt
+ E \left( f_g^2(s; n_m) \right) + o(h^d)
\]

and so

\[
\text{var} \left( Y_{m,i}' K((S_{m,i} - s)/h) \right) = \frac{1}{n_m} \left( \frac{1}{h} \left( f_g^2(s; n_m) + \sigma^2_g(s) \right) f_g(s) \int_{-\infty}^{\infty} K^2(t) dt + E \left( f_g^2(s; n_m) \right) + o(h^d) \right)
\]

\[
= \frac{1}{n_m} \left( \frac{1}{h} \left( f_g^2(s) + \sigma^2_g(s) \right) f_g(s) \int_{-\infty}^{\infty} K^2(t) dt + E \left( f_g^2(s; n_m) \right) + o(h^d) \right)
\]

\[
= \frac{1}{n_m} \left( \frac{1}{h} \left( f_g^2(s) + \sigma^2_g(s) \right) f_g(s) \int_{-\infty}^{\infty} K^2(t) dt + E \left( f_g^2(s; n_m) \right) + o(h^d) \right)
\]

\[
= \frac{1}{n_m} \left( \frac{1}{h} \left( f_g^2(s) + \sigma^2_g(s) \right) f_g(s) \int_{-\infty}^{\infty} K^2(t) dt + E \left( f_g^2(s; n_m) \right) + o(h^d) \right)
\]

\[
= \frac{1}{n_m} \left( \frac{1}{h} \left( f_g^2(s) + \sigma^2_g(s) \right) f_g(s) \int_{-\infty}^{\infty} K^2(t) dt + E \left( f_g^2(s; n_m) \right) + o(h^d) \right)
\]

\[
= \frac{1}{n_m} \left( \frac{1}{h} \left( f_g^2(s) + \sigma^2_g(s) \right) f_g(s) \int_{-\infty}^{\infty} K^2(t) dt + E \left( f_g^2(s; n_m) \right) + o(h^d) \right)
\]

\[
= \frac{1}{n_m} \left( \frac{1}{h} \left( f_g^2(s) + \sigma^2_g(s) \right) f_g(s) \int_{-\infty}^{\infty} K^2(t) dt + E \left( f_g^2(s; n_m) \right) + o(h^d) \right)
\]

\[
= \frac{1}{n_m} \left( \frac{1}{h} \left( f_g^2(s) + \sigma^2_g(s) \right) f_g(s) \int_{-\infty}^{\infty} K^2(t) dt + E \left( f_g^2(s; n_m) \right) + o(h^d) \right)
\]

which proves the first convergence in the lemma. 

We now show the joint asymptotic normality of \( a_g(s) \) and \( b_g(s) \)

**Lemma 4.** Under Assumptions 1 – 6, \( \sqrt{Mh} (c_1 \cdot a_g(s) + c_2 \cdot b_g(s) - c_1 \cdot a_g(s) + c_2 \cdot b_g(s)) \rightarrow N(0, \gamma(s)) \)

where \( \gamma(s) \) is the appropriate variance according to the variances and covariances given in Lemma 3

**Proof of Lemma 4.** This Lemma follows immediately from the Liapunov Central Limit Theorem where the Liapunov
condition follows from arguing as in \(^2\) and from assumption 6 on the finite moments of the outcome variable.

### B.2 MAIN PROOF

The main asymptotic normality result is a consequence of this Lemma.

**Proof of Theorem** Arguing as in Lemma 6 of McMurry and Politis (2008), it is readily seen that there exists a constant \(c > 0\) such that \(\hat{f}_g(s) > c\) for all \(n\) sufficiently large. Therefore, it follows from the intermediate value theorem and Lemma 2 that the asymptotic normality in Theorem 1 holds with

\[
\hat{f}_g(s) - f_g(s) = \hat{a}_g(s) \left( \frac{1}{b_g(s)} - \frac{1}{\sigma_n} \left( \hat{b}_g(s) - b_g(s) \right) \right) - \frac{a_g(s)}{b_g(s)}
\]

\[
= \frac{1}{b_g(s)} (\hat{a}_g(s) - a_g(s)) + \frac{1}{b_g(s)} \left( \hat{b}_g(s) - b_g(s) \right) + o(h^d)
\]

where

\[
|\delta_n - b_g(s)| \leq |\hat{b}_g(s) - b_g(s)|.
\]

Note that because \(\hat{b}_g(s)\) converges to \(b_g(s)\), we have that \(\delta_n\) also converges to \(b_g(s)\). It follows from Slutsky’s Theorem and Lemma 2 that the asymptotic normality in Theorem 1 holds with

\[
\sigma_g^2(x) = \frac{1}{b_g(s)^2 P(G = g)^2} \cdot \left( \hat{f}_g^2(s) + \hat{a}_g^2(s) \right) f_S(s) \int_{-\infty}^{\infty} K^2(t) dt + \frac{a_g(s)}{b_g(s)^2 P(G = g)^2} \cdot \left( \hat{c}_g^2(s) + \hat{v}_g^2(s) \right) f_S(s) \int_{-\infty}^{\infty} K^2(t) dt
\]

\[
- \frac{2}{b_g(s)^2 P(G = g)^2} \cdot \hat{f}_g(s) f_S(s) \int_{-\infty}^{\infty} K^2(t) dt
\]

### C PROOF OF OTHER RESULTS

**Proof of Lemma**

\[
E(Y_i|S_i = s) = E \left( \sum_{k=1}^{K} w_k Y_i^k \mid S_i^1 = s_1, \ldots, S_i^K = s_k, \sum_{k=1}^{K} w_k S_i^k = s \right)
\]

\[
= E \left( \sum_{k=1}^{K} w_k E(Y_i^k \mid S_i^1 = s_1, \ldots, S_i^K = s_k) \right) \mid \sum_{k=1}^{K} w_k S_i^k = s
\]

\[
= E \left( \sum_{k=1}^{K} w_k \tilde{S}_i^k \right) \sum_{k=1}^{K} w_k S_i^k = s
\]

\[
= s
\]

**Proof of Theorem** The result follows immediately from the fact that the transformed scores satisfy

\[
E(Y_i|\tilde{S}_i = s_1, \ldots, \tilde{S}_K = s_K, G = g) = s_k
\]

for all \(s_1, \ldots, s_K, G\) and \(g_i\).

**Proof of Theorem** The proof readily follows that of Theorem 1.

**Proof of Theorem** Let \(\tilde{S}_i\) denote calibrated scores from group \(g\). Let \(\hat{f}_g(\cdot)\) and \(\hat{F}_g(\cdot)\) denote the conditional average outcome function and distribution function of the calibrated scores, respectively. The system of equations specified in the Theorem now reduces to finding \(t^*\) satisfying the single constraint that

\[
\sum_i p_{g_i} \int_0^{t^*} \tilde{f}_{g_i}(t) \tilde{F}_{g_i}(t) dt = \sum_i p_{g_i} \int_0^{t^*} \tilde{f}_{g_i}(t) \tilde{F}_{g_i}(t) dt = 0
\]

Because

\[
\sum_i p_{g_i} \int_0^{t^*} \tilde{f}_{g_i}(t) \tilde{F}_{g_i}(t) dt = \sum_i p_{g_i} \int_0^{t^*} \tilde{f}_{g_i}(t) \tilde{F}_{g_i}(t) dt = 0
\]

and

\[
\sum_i p_{g_i} \int_0^{t^*} \tilde{f}_{g_i}(t) \tilde{F}_{g_i}(t) dt > \sum_i p_{g_i} \int_0^{t^*} \tilde{f}_{g_i}(t) \tilde{F}_{g_i}(t) dt
\]

the result follows immediately from the intermediate value theorem.