THE PROFILE OF BOUNDARY GRADIENT BLOWUP
FOR THE DIFFUSIVE HAMILTON-JACOBI EQUATION

Philippe Souplet
(Université Paris 13, LAGA)

joint work with Alessio Porretta (Università Roma 2)
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THE EQUATION

\[ (VHJ) \quad u_t = \Delta u + |\nabla u|^p, \quad x \in \Omega, \; t > 0. \]

- $\Omega \subset \mathbb{R}^n$
- $p > 1$
- Diffusive Hamilton-Jacobi equation (control theory)
- Deterministic Kardar-Parisi-Zhang (KPZ)
  Evolution of the profile of a growing interface in ballistic deposition processes
- Simple model case in the theory of nonlinear parabolic equations
BEHAVIOR

1. Without boundary conditions ($\Omega = \mathbb{R}^n$)
   - Solutions are global and bounded in $C^1$
   - Large-time behavior

[Ben-Artzi, Benachour, Guedda, Gilding, Karch, Kersner, Laurençot, S., Weissler, ...] (1990’s, 2000’s)
TYPES OF BEHAVIORS

1. Without boundary conditions ($\Omega = \mathbb{R}^n$)
   - Solutions are global and bounded in $C^1$
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2. Dirichlet boundary conditions ($\Omega \neq \mathbb{R}^n$)

\[
\begin{aligned}
\left\{ \begin{array}{ll}
  u_t = \Delta u + |\nabla u|^p, & x \in \Omega, \ t > 0 \\
  u = 0, & x \in \partial \Omega, \ t > 0
\end{array} \right.
\end{aligned}
\]

- $p > 2 +$ large initial data $\rightarrow$ singularities in finite time
- Gradient blowup type, $u$ bounded

[Alikakos, Bates, Grant, Conner, Fila, Lieberman, Alaa, Arriera, Rodriguez-Bernal, S., Hesaaraki, Moameni, Barles, Da Lio, Vázquez, Guo, Hu, Li, Q.Zhang, ...] (1990’s, 2000’s)
QUESTION: asymptotic profile of finite time singularities?

1. Classical blowup problem / nonlinear heat equation

\[ u_t - \Delta u = u^p \]

- extensive theory for description of asymptotic profile near a finite time singularity [Giga-Kohn, Herrero-Velázquez, Merle-Zaag, ...] (1985∼2000)
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2. For (VHJ)

   Little is known...

   • Upper estimate in the normal direction [S.-Zhang, 2006]

   \[
   |\nabla u(x, t)| \leq C [\text{dist}(x, \partial \Omega)]^{-1/(p-1)}
   \]

   \[\Rightarrow\] Singular set \( \subset \partial \Omega \)
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- Existence of single-point GBU solutions [Li-Ph.S., Comm. Math. Phys. 2009]

Final blowup profile of $\nabla u$ in the tangential direction: completely unknown

(no information on how the profile is damped away from the point of singularity along the boundary)
NOTATION

\[ \omega = (-\rho, \rho) \times (0, \rho) \subset \mathbb{R}^2, \quad \Gamma := (-\rho, \rho) \times \{0\}, \quad \rho, T > 0 \]

\[ u \in C^{2,1}(\bar{\omega} \times (0, T)) \text{ nonnegative classical solution of (VHJ) in } Q_T = \omega \times (0, T) \]

\[ u = 0 \quad \text{on } \Gamma_T := \Gamma \times (0, T) \]
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**Isolated gradient blowup point at } (0, 0, T):**

\[
\limsup_{(x,y,t) \to (0,0,T)} |\nabla u(x, y, t)| = \infty
\]

and

\[ \nabla u \text{ is bounded on } K \times (0, T) \text{ for any } K \subset \subset \bar{\omega} \setminus \{(0, 0)\}. \]
**MAIN RESULT**

**Theorem.** [Porretta-Ph.S., preprint 2015] Assume

\[ 2 < p \leq 3. \]

Let \( u \in C^{1,2}(\bar{\omega} \times (0, T)) \) be a nonnegative classical solution of (VHJ) in \( Q_T \), with \( u = 0 \) on \( \Gamma_T \). Assume \( u \) has an isolated gradient blowup point at \((0,0,T)\) and satisfies the monotonicity condition

\[ x u_x \leq 0 \quad \text{in} \quad Q_T. \]

Then, in the neighborhood of \((0,0)\), the final profile \( u_y(x,y,T) \) satisfies

\[
d_p \left[ y + C_1 |x|^{2(p-1)/(p-2)} \right]^{-\beta} - C_3 \leq u_y(x,y,T) \leq d_p \left[ y + C_2 |x|^{2(p-1)/(p-2)} \right]^{-\beta} + C_3
\]

with \( \beta = 1/(p - 1) \) and \( d_p = \beta^\beta \). In particular

\[ u_y(x,0,T) \sim |x|^{-2/(p-2)}. \]

Also \( u \leq C, \ |u_x| \leq C \).
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REMARKS

1) Lower estimate true for any $p > 2$. 
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2) GBU profile strongly anisotropic

Exponents of the singularity profile in normal and tangential directions different:

1/$(p - 1) \rightarrow$ self-similar

2/$(p - 2) \rightarrow$ non self-similar
REMARKS

1) Lower estimate true for any $p > 2$.

2) GBU profile **strongly anisotropic**

Exponents of the singularity profile in normal and tangential directions different:

\[ \frac{1}{p - 1} \rightarrow \text{self-similar} \]
\[ \frac{2}{p - 2} \rightarrow \text{non self-similar} \]

3) Comparison with other parabolic blowup problems

- Nonlinear heat equation: stable blowup profile isotropic

\[ u(X, T) \sim c(p)|X|^{-2/(p-1)}|\log|X||^{-1/(p-1)} \quad \text{as } X \to 0. \]

Here $X \in \mathbb{R}^n$ with $n \geq 2$ and $1 < p < (n + 2)/(n - 2)$ (e.g. symmetric, radially decreasing solution).
• Linear heat equation with nonlinear boundary conditions

\[
\begin{cases}
  u_t - \Delta u = 0 \quad \text{in } \Omega \times (0,T), \\
  \frac{\partial u}{\partial \nu} = u^p \quad \text{on } \partial \Omega \times (0,T)
\end{cases}
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[Fila-Quittner, Hu-Yin, Chlebik-Fila, Harada]
Linear heat equation with nonlinear boundary conditions

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Boundary singularities (of \(u\))

Singularity profile [Harada, 2013]: weakly anisotropic

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\begin{aligned}
& u(x, y, T) \sim \begin{cases} 
  y^{-1/(p-1)} & \text{for } y \to 0 \text{ with } |x| = O(y) \\
  x^{-1/(p-1)} |\log x|^{-1/2(p-1)} & \text{for } x \to 0 \text{ and } y = 0
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4) Exponent \( 2/(p-2) \): new

Time rate of GBU = \( 1/(p-2) \) (for monotone in time solutions in 1d)

\[
\| \nabla u(\cdot, t) \|_\infty \sim (T - t)^{-1/(p-2)}
\]

(not self-similar !) [Conner-Grant 96, Guo-Hu 2008]
Heuristic explanation of $1/(p-2)$ and $2/(p-2) \rightarrow$ quasi-stationary approximation

1d steady-states: $V(y) = c_p y^{1-\beta}$,

$V_a(y) = V(y + a) - V(y), \quad y \geq 0, \ a \geq 0.$

$-V''_a = V'_a p, \quad V_a(0) = 0, \quad V'_a(0) = d_p a^{-\beta}.$
Heuristic explanation of $1/(p-2)$ and $2/(p-2) \rightarrow$ quasi-stationary approximation

1d steady-states: \( V(y) = c_p y^{1-\beta}, \)
\[
V_a(y) = V(y + a) - V(y), \quad y \geq 0, \quad a \geq 0.
\]
\[
-V_a'' = V_a'^{p}, \quad V_a(0) = 0, \quad V_a'(0) = d_p a^{-\beta}.
\]

Approximate solution by modulating in \( a \):
\[
U(x, y, t) = V(y + h(t, x)) - V(h(t, x))
\]

$1/(p-2)$ and $2/(p-2)$ = minimal singularity exponents compatible with maximum principle constraints:
\[
|u_t| \leq C, \quad u_{xx} \geq -C.
\]