D-BRANE DUALITIES AS CANONICAL TRANSFORMATIONS

Y. Lozano
Inst. for Theoretical Physics, Utrecht University,
3508 TA Utrecht, The Netherlands

Abstract

We show that the SL(2,R) duality symmetry of type IIB superstring theory can be formulated as the canonical transformation interchanging momenta and magnetic degrees of freedom associated to the abelian world-volume gauge field of the D-3-brane. D-strings are shown to be connected under the corresponding transformation in the world-sheet to the \((m, n)\) family of string solutions of type IIB supergravity constructed by Schwarz. For the type IIA superstring the D-2-brane is mapped under the three-dimensional world-volume electric-magnetic duality to the dimensional reduction of the membrane of M-theory.
1 Introduction

Recent results in the literature show certain equivalences between effective actions of p-branes of the same or different string (or M-) theories. For D-branes \([1]\) a relation between the D-string and the SL(2,Z) multiplet of string solutions of type IIB supergravity \([2]\) has been proved \([3, 4, 5]\), as well as the self-duality of the D-3-brane under SL(2,Z) transformations of the space-time backgrounds \([3, 4, 5]\). In both cases the equivalence goes through under certain transformations in the world-sheet of the D-string or the four-dimensional world-volume of the D-3-brane. In \([3]\) the general world-volume transformation of which the above are particular cases was identified and referred to as vector duality. This vector duality symmetry was interpreted as the world-volume transformation responsible for the strong-weak coupling duality of type IIB \([8, 9, 2, 10]\), in analogy with the world-sheet mapping \(d \rightarrow *d\) underlying T-duality transformations of the space-time background fields \([11]\).

For the type IIA superstring vector duality is also interesting. The p-branes of type IIA supergravity have an alternative interpretation in eleven dimensions which can be used to deduce its \(D = 10\) Lorentz covariant world-volume action \([12, 13, 14]\). In the last of the previous references this was done for the D-membrane and for the D-4-brane (in this case up to quadratic order in the gauge field strength) using the fact that the former can be interpreted as the dimensional reduction of the eleven dimensional membrane \([15]\) and the latter as the double dimensional reduction of the eleven dimensional 5-brane \([16]\). For the membrane in order to prove the complete equivalence between the dimensionally reduced theory and the DBI action a non-trivial transformation involving the abelian world-volume gauge field of the D-brane was shown to be required \([3, 4]\). Tseytlin pointed out that this transformation was just the vector duality symmetry in the three dimensional world-volume \([5]\). This observation is in agreement with the conjecture that M-theory compactified on a circle is the strong coupling limit of type IIA \([13, 3]\). This is the reason why a strong-weak coupling duality transformation is needed to connect the type IIA D-2-brane with the membrane of M-theory.

In this letter we show that vector duality can be realized in the phase space of D-branes as the canonical transformation interchanging the momenta and the magnetic degrees of freedom associated to the abelian world-volume gauge field, i.e. the same transformation responsible for the SL(2,Z) symmetry of electromagnetism\([5]\). In \(d\) dimensions this transformation maps \(r\)-forms onto \((d - r - 2)\)-forms.

We start in section 2 considering the D-3-brane of type IIB. In the four-dimensional world-volume the dual of the abelian gauge field is also a 1-form, and we in fact show that under the four-dimensional world-volume canonical transformation a D-3-brane is obtained in SL(2,R)-transformed backgrounds. The SL(2,R) self-duality of the D-3-brane has been shown previously in \([3, 4, 7]\). This was an expected result \([2]\) due to the non-existence of multiplets of 3-brane solutions in type IIB \([18]\) nor bound states of D-3-branes \([19]\). For the D-string, considered in section 3, \(n\) D-strings are shown to be equivalent under the two-dimensional canonical transformation to the \((m, n)\) family of string solutions constructed by Schwarz in \([3]\), supporting the SL(2,Z) symmetry of type IIB. In this case the dual variables

---

2 In \([7]\) it was shown at the level of the equations of motion.
3 SL(2,Z) transformations in non-linear electrodynamics and in particular of four-dimensional DBI actions have been studied in \([17]\).
are scalars satisfying a quantization condition, being this the origin of the \( m \) appearing in the dual theory. The SL(2,Z) multiplet of string solutions, corresponding to bound states of elementary and D-strings \[19\], are then shown to be connected by electric-magnetic world-sheet duality to D-strings.

In section 4 we consider the D-membrane of type IIA. The result of the canonical transformation is in this case an eleven dimensional action, where the eleventh coordinate has arisen as the dual of the abelian gauge field in the three-dimensional world-volume. This eleven dimensional action is interpreted as the dimensional reduction of the membrane of M-theory \[3, 4, 5\].

The eleven dimensional 5-brane gives upon double dimensional reduction the D-4-brane of type IIA \[14, 20, 21\]. As indicated in the last of these references the equivalence between the bosonic parts of the actions follows after a vector duality transformation is performed in the dimensionally reduced action. In our description this transformation is formulated as the five-dimensional world-volume canonical transformation interchanging momenta and magnetic variables. Details on this construction will be presented elsewhere.

Before considering DBI actions let us recall briefly how S-duality in Maxwell’s theory is formulated as a canonical transformation \[22\]. We start by considering the electromagnetic Lagrangian with \( \theta \)-term:

\[
L = \frac{1}{8\pi} \left( \frac{4\pi}{g^2} F_{mn} F^{mn} + \frac{\theta}{4\pi} \epsilon^{mnpq} F_{mn} F_{pq} \right)
= \frac{1}{g^2} \left( -\vec{E}^2 + \vec{B}^2 \right) + \frac{\theta}{4\pi^2} \vec{E} \cdot \vec{B}
\]

(1.1)

where \( E_\alpha = F_{0\alpha} \), \( B_\alpha = \frac{1}{2} \epsilon^{\alpha\beta\gamma} F_{\beta\gamma} \) and we are taking a Minkowskian space-time. The canonical momenta are given by:

\[
\Pi_0 = 0 \\
\Pi_\alpha = -\frac{2}{g^2} E_\alpha + \frac{\theta}{4\pi^2} B_\alpha,
\]

(1.2)

and the Hamiltonian:

\[
H = -(\frac{g^2}{4} \vec{\Pi}^2 + \frac{g^2 \theta}{8\pi^2} \vec{\Pi} \cdot \vec{B} + \frac{g^2 \theta^2}{64\pi^4} + \frac{1}{g^2}) \vec{B}^2) + \Pi^\alpha \partial_\alpha A_0.
\]

(1.3)

The primary constraint \( \Pi_0 = 0 \) implies the secondary constraint \( \partial_\alpha \Pi^\alpha = 0 \), therefore the term \( \Pi^\alpha \partial_\alpha A_0 \) can be dropped out of the Hamiltonian but keeping in mind that we are working in the restricted phase space defined by the two constraints.

It is easy to see that the canonical transformation:

\[
\Pi^\alpha = -\frac{1}{2\pi} \vec{B}^\alpha \\
\vec{\Pi}^\alpha = \frac{1}{2\pi} B_\alpha,
\]

(1.4)

where \( \vec{F} = d\vec{A} \), i.e. the interchange between electric and magnetic degrees of freedom\[1\] yields the S-dual Hamiltonian with \( \vec{\tau} = -1/\tau \) and \( \tau = \theta/2\pi + 4\pi i/g^2 \).

\[4\]Note that in the definition of \( \Pi^\alpha \) there is also a contribution from \( B^\alpha \) when \( \theta \neq 0 \) \[23\].

2
The generating functional is given by:

$$ G = -\frac{1}{2\pi} \int_{D_4/\partial D_4 = M_4} dA \wedge d\tilde{A} = -\frac{1}{4\pi} \int_{M_3} d^3x \left( \tilde{A}_\alpha B^\alpha + A_\alpha \tilde{B}^\alpha \right) $$

where $D_4$ is the four-dimensional manifold whose boundary is the three-dimensional spatial submanifold $M_3$. In order to show the complete equivalence between the initial and dual Hamiltonians we have to prove that they are defined in the same restricted phase spaces. This is easily seen by noting that the Bianchi identity component $\partial_\alpha F^{0\alpha} = 0$ of the initial theory implies in the dual $\partial_\alpha \tilde{\Pi}^\alpha = 0$, and $\tilde{\Pi}_0 = 0$ comes directly from the generating functional above since it has no $\tilde{A}_0$ dependence. The secondary constraint $\partial_\alpha \Pi^\alpha = 0$ of the original theory implies in the dual the Bianchi identity component $\partial_\alpha * F^{0\alpha} = 0$.

This same idea can be straightforwardly generalized to abelian gauge theories of $r$-forms in $d$ dimensions\footnote{Our conventions are: $* F^{m_1...m_d - r - 1} = \frac{1}{(r+1)!} e^{m_1...m_d} F_{m_{d-r}...m_d}; F \wedge G = e^{m_1...m_d} F_{m_1...m_r} G_{m_{r+1}...m_d}$. In the particular case $d = 2(r+1)$, $r$ odd, a $\theta$-term can also be introduced in the Lagrangian.}:

$$ S = \int d^d x \frac{1}{2g^2} F \wedge * F. $$

The dual theory\footnote{Now and in the following sections we choose the normalization of the canonical transformation such that in the dual $\tilde{g} = 1/g$.}:

$$ \tilde{S} = \int d^d x \frac{g^2}{2} \tilde{F} \wedge * \tilde{F} $$

with $\tilde{F} = d\tilde{A}$ and $\tilde{A}$ a $(d - r - 2)$-form, is obtained as the result of the transformation:

$$ \Pi^{\alpha_1...\alpha_r} = \frac{\delta G}{\delta \tilde{A}_{\alpha_1...\alpha_r}} = -* \tilde{F}^{0\alpha_1...\alpha_r} $$

$$ \tilde{\Pi}^{\alpha_1...\alpha_{d-r-2}} = - \frac{\delta G}{\delta \tilde{A}_{\alpha_1...\alpha_{d-r-2}}} = (-1)^{d+r-1} * F^{0\alpha_1...\alpha_{d-r-2}}, $$

with generating functional:

$$ G = - \int_{D_{d+1}/\partial D_{d+1} = M_{d+1}} d^{d+1} x dA \wedge d\tilde{A}. $$

Now the restricted phase space is defined by: $\Pi^{\alpha_1...0...\alpha_r} = 0$, $\partial_\alpha \Pi^{\alpha_1...\alpha_i...\alpha_r} = 0$. The Bianchi identity component $\partial_\alpha * F^{0\alpha_1...\alpha_i...\alpha_{d-r-2}} = 0$ of the initial theory implies $\partial_\alpha \tilde{\Pi}^{\alpha_1...\alpha_i...\alpha_{d-r-2}} = 0$ in the dual, and $\tilde{\Pi}^{\alpha_1...0...\alpha_{d-r-2}} = 0$ holds immediately from the generating functional. Therefore the original and dual Hamiltonians are defined in the same restricted phase spaces.

The generating functionals $(1.5)$ and $(1.9)$ are linear in the original and dual variables. Hence we can follow \cite{21} and write a relation between the Hilbert spaces of the original and dual theories\footnote{Although still renormalization effects need to be considered.}:

$$ \psi_k[A] = N(k) \int \mathcal{D}A(x^\alpha) e^{ig[\tilde{A},A(x^\alpha)]} \phi_k[A(x^\alpha)]. $$

Here $\psi_k$ and $\phi_k$ are eigenfunctions of the respective Hamiltonians with the same eigenvalue, labelled by $k$, and $N(k)$ is a normalization factor. From this relation it is possible to derive
global properties in the dual theory. For instance it is easy to see that the original theory satisfies a Dirac quantization condition the same holds true for the dual theory. The equivalence between the original and the dual partition functions also holds trivially in phase space.

In the next sections we will show that starting with D-brane effective actions we can prove the dualities mentioned in the introduction as results of the same type of canonical transformations.

2 The D-3-brane of type IIB

The effective action of a D-p-brane in RR background fields contains the usual DBI part\cite{25}:

\[ S_p = \int d^{p+1}x e^{-\phi} \sqrt{-\det(G_{mn} + F_{mn})}, \tag{2. 1} \]

where the backgrounds are those induced in the D-brane from ten dimensions, \( F = B + F \) with \( F = dA \) and \( B \) the NS-NS two-form; plus a WZ term \cite{26, 3, 4}:

\[ S_{WZ} = \int d^{p+1}\theta_0 d^{p+1}x \sum_{r=0}^{p+1} C_r(\theta_0) e^{\frac{1}{2} F_{mn}\theta_0^m \theta_0^n}, \tag{2. 2} \]

where \( \theta_0 \) denote the fermionic zero modes and \( C_r(\theta_0) = \frac{1}{m_r!} C_{m_1 \ldots m_r} \theta_0^{m_1} \ldots \theta_0^{m_r} \) are the induced rank \( r \) RR background fields. We will focus on the bosonic part of the actions, assuming that the fermionic part will be fixed by supersymmetry and fermionic kappa symmetry\cite{27}.

The effective action of the D-3-brane in presence of RR background fields is then given by:

\[ S_3 = \int d^4x [e^{-\phi} \sqrt{1 + e^{\phi} |\vec{\cal E}|^2 + e^{-2\phi} (\vec{\cal F})^2} + \sqrt{1 + \frac{1}{2} e^{-\phi} \vec{\cal F}^2 + \frac{1}{8} e^{-2\phi} (\vec{\cal F}^2)^2 - \frac{1}{4} e^{-2\phi} \vec{\cal F}^4}], \tag{2. 3} \]

Introducing the dilaton inside the square root and working with the canonical metric \( g_{mn} = e^{-\phi/2} G_{mn} \) we can write:

\[ e^{-\phi} \sqrt{-\det(G_{mn} + F_{mn})} = \sqrt{-}\det g \sqrt{\frac{1}{2} e^{-\phi} \vec{\cal F}^2 + \frac{1}{8} e^{-2\phi} (\vec{\cal F}^2)^2 - \frac{1}{4} e^{-2\phi} \vec{\cal F}^4}, \tag{2. 4} \]

where \( \vec{\cal F}^2 = \vec{\cal F}_{mn} \vec{\cal F}^{mn} \) and \( \vec{\cal F}^4 = \vec{\cal F}_{mn} \vec{\cal F}_{np} \vec{\cal F}_{pq} \vec{\cal F}^{nm} \). Restricting for simplicity to a Minkowskian canonical metric \( g_{mn} = \eta_{mn} \) and introducing electric and magnetic variables: \( \cal E_\alpha = \vec{\cal F}_{0\alpha} \), \( \cal B^\alpha = \frac{1}{2} e^{\theta_0^a} F_{\beta\gamma}^a \), where the letters \( \cal E, \cal B \) are used to recall their dependence on the NS-NS two-form, we can write:

\[ S_3 = \int d^4x [\sqrt{1 - e^{-\phi} \vec{\cal E}^2 + e^{-\phi} \vec{\cal B}^2} - e^{-2\phi} (\vec{\cal E} \cdot \vec{\cal B})^2 + \vec{\cal C} \cdot \vec{\cal B} + \vec{\cal D} \cdot \vec{\cal E} + \vec{\cal F} \cdot \vec{\cal B} + \frac{1}{24} e^{mnpq} C_{mnpq}], \tag{2. 5} \]

\[ \text{We are taking a Minkowskian signature space-time.} \]
where we have defined $C_\alpha \equiv C_{0\alpha}$ and $D^\alpha \equiv \ast C_{0\alpha}$. The $A_m$-conjugate momenta are given by:

$$
\Pi_0 = 0 \\
\Pi_\alpha = \frac{e^{-\phi}}{\sqrt{1 - e^{-\phi} \tilde{E}^2 + e^{-\phi} \tilde{B}^2 - e^{-2\phi} (\tilde{E} \cdot \tilde{B})^2}} (\mathcal{E}_\alpha + e^{-\phi} (\tilde{E} \cdot \tilde{B}) B_\alpha) + D_\alpha + C B_\alpha, \quad (2. 6)
$$

and the Hamiltonian:

$$
H_3 = -\sqrt{1 + e^{-\phi} \tilde{B}^2 + e^\phi (\tilde{\Pi} - (\tilde{D} + C \tilde{B}))^2 + \tilde{B}^2 (\tilde{\Pi} - (\tilde{D} + C \tilde{B}))^2 - (\tilde{B} (\tilde{\Pi} - (\tilde{D} + C \tilde{B})))^2} \\
-\tilde{C} \tilde{B} - \Pi^\alpha B_{0\alpha} - \frac{1}{24} \epsilon^{mnpq} C_{mnpq} + \Pi^\alpha \partial_\alpha A_0.
$$

The same canonical transformation responsible for S-duality in electromagnetism

$$
\Pi_\alpha = -\tilde{B}_\alpha \\
\tilde{\Pi}_\alpha = B_\alpha, \quad (2. 8)
$$

(in the four-dimensional world-volume the dual theory is also defined in terms of 1-forms $\tilde{A}$) can be shown to yield the effective action of a D-3-brane in the SL(2,R) transformed backgrounds:

$$
e^{-\tilde{\phi}} = \frac{1}{e^{-\phi} + e^\phi C^2}, \quad \tilde{C} = -\frac{C e^\phi}{e^{-\phi} + e^\phi C^2}, \quad (2. 9)
$$

or, equivalently:

$$
\tilde{\lambda} = -\frac{1}{\lambda}, \quad \lambda \equiv C + i e^{-\phi}, \quad (2. 10)
$$

and

$$
\tilde{B}_{mn} = C_{mn}, \quad \tilde{C}_{mn} = -B_{mn}, \quad \tilde{C}_{mnpq} = C_{mnpq}. \quad (2. 11)
$$

This transformation together with the invariance of the action under constant shifts of the RR scalar field generates the whole SL(2,R) invariance, which is broken to SL(2,Z) due to quantum effects. The same calculation for an arbitrary metric $g_{mn}$ yields $\tilde{g}_{mn} = g_{mn}$, as $g_{mn}$ is the canonical metric.

Substituting in (2. 8) the expressions for the canonical momenta the non-local change of variables in configuration space responsible for SL(2,R) transformations is obtained. This change of variables is the one induced by the Fourier transformation of (2. 3) with respect to the field strength of the abelian world-volume gauge field $F$. Due to the complicated and non-linear expressions for the momenta this cannot be interpreted as a simple generalization of $d \rightarrow \ast d$ (or the interchange between electric and magnetic degrees of freedom). However in phase space the transformation is very simple being just the definition of S-duality in a four dimensional abelian gauge theory.

We have shown that the D-3-brane is self-dual under SL(2,R) transformations, with self-duality meaning that the dual is also a D-3-brane, although defined in the SL(2,R)

\footnote{Now we have chosen different normalizations.}
\footnote{With the same remarks concerning the constraints.}
transformed backgrounds. In our formalism it is clear that the reason for this is the general 
\[ A_r \leftrightarrow \tilde{A}_{d-r-2} \] 
electric-magnetic duality, that particularized to \( d = 4 \) and \( r = 1 \) yields also a 
dual 1-form, and it is consistent with the non-existence of multiplets of 3-brane solutions in 
type IIB supergravity \([18]\).

As was mentioned by Tseytlin \([5]\) this transformation in the world-volume of the D-brane 
can be interpreted as the world-volume mapping responsible for SL(2,R) transformations of 
the space-time backgrounds, in analogy with the \( d \to *d \) world-sheet mapping underlying 
the T-duality transformation \( R \to 1/R \) in space-time. In our construction this symmetry 
is understood as the same type of canonical transformation responsible for S-duality in 
abelian gauge theories. Therefore SL(2,Z) can be viewed as a subgroup of the whole group of 
simplectic diffeomorphisms on the world-volume of the 3-brane. Also, we can implement this 
transformation in the path integral since both the action and the measure remain invariant 
in phase space. This generalizes previous results in the literature \([3, 11]\), where a saddle point 
approximation had to be made.

Finally let us mention that four-dimensional BI actions coupled to an SL(2,R) invariant 
four-dimensional bulk were considered in \([7]\) in relation to the S-duality of the heterotic 
string in four dimensions \([28]\). In our case the D-3-brane is coupled to a ten dimensional 
SL(2,R) invariant bulk \([29]\) and the S-duality under consideration is the one of the ten 
dimensional type IIB superstring.

### 3 \((m,n)\) strings in type IIB

The same type of analysis above can be made for the D-string. In this case we find a 
correspondence with the \((m,n)\) string solutions of type IIB supergravity constructed by 
Schwarz in \([2]\).

We just need to recall the results stated in the introduction about electric-magnetic 
duality in \( d \) dimensional abelian gauge theories of arbitrary \( r \)-forms. Our starting point is 
the action describing the bound state of \( n \) D-strings\(^{11}\)

\[
S_1 = \int d^2 x n [e^{-\phi} \sqrt{-\det(G_{mn} + F_{mn})} + \frac{1}{2} \epsilon^{mn}(C_{mn} + C F_{mn})].
\]  

(3.1)

In a two-dimensional world-volume there are only electric degrees of freedom. Redefining 
\( g_{mn} = e^{-\phi} G_{mn} \) and taking \( g_{mn} = \eta_{mn} \) for simplicity we can write:

\[
S_1 = \int d^2 x n [\sqrt{1 - e^{-2\phi}} E_1^2 + C_{01} + C E_1].
\]  

(3.2)

The canonical momenta are given by:

\[
\Pi_0 = 0
\]

\[
\Pi_1 = -\frac{n e^{-2\phi}}{\sqrt{1 - e^{-2\phi} E_1^2}} E_1 + n C
\]  

(3.3)

\(^{11}\)Since \( n \) parallel D-p-branes are described by a \( U(1)^n \) gauge theory in the \((p + 1)\)-dimensional world-volume \([14]\) we effectively get a factor of \( n \) in front of the one D-string action.
and the Hamiltonian:

\[ H_1 = -\sqrt{n^2 + e^{2\phi}(\Pi_1 - nC)^2} - \Pi_1 B_{01} - nC_{01} + \Pi_1 \partial_1 A_0. \]  

The canonical transformation responsible for electric-magnetic duality is in this case:

\[ \Pi^1 = -\epsilon^{01} \tilde{F}^{01} = -\epsilon^{01} \tilde{\Lambda} \]

\[ \tilde{\Pi} = *F^0 = \epsilon^{01} A_1, \]  

with generating functional:

\[ G_1 = -\int dx \epsilon^{01} A_1 \tilde{\Lambda}. \]  

Since we only have electric degrees of freedom the two transformations in (3.5) cannot be independent. One checks easily that the first one is enough to obtain the dual theory and the second gives the corresponding equation of motion. The secondary constraint \( \partial_1 \Pi^1 = 0 \), associated to \( \Pi_0 = 0 \), implies that \( \tilde{\Lambda} \) must be a function of time, and moreover from (1.10) (note that \( G_1 \) is linear in the original and dual variables) we obtain that \( \tilde{\Lambda} \) must be an integer due to Dirac quantization condition in the original theory. Setting \( \tilde{\Lambda} = -m \) we find a dual action:

\[ \tilde{S}_1 = \int d^2 x [\sqrt{n^2 + e^{2\phi}(nC - m)^2} + \frac{1}{2} \epsilon^{mn}(nC_{mn} + mB_{mn})]. \]  

For arbitrary metric the result is:

\[ \tilde{S}_1 = \int d^2 x [\sqrt{e^{-2\phi}n^2 + (nC - m)^2} \sqrt{-\det G_{mn} + \frac{1}{2} \epsilon^{mn}(nC_{mn} + mB_{mn})}], \]

i.e. the Nambu-Goto action of a fundamental string with tension \( T = \sqrt{e^{-2\phi}n^2 + (nC - m)^2} \) and charges \((m, n)\) with respect to the NS-NS and RR 2-forms. The existence of this string multiplet is required by the SL(2,\mathbb{Z}) symmetry of type IIB, since it is obtained from the elementary type IIB superstring (the (1,0) string in this notation):

\[ S_{(1,0)} = \int d^2 x [\sqrt{-\det G_{mn} + \frac{1}{2} \epsilon^{mn} B_{mn}}] \]  

after an SL(2,\mathbb{Z}) transformation of parameters

\[ \Lambda = \left( \begin{array}{cc} p & q \\ -n & m \end{array} \right), \quad pm + qn = 1. \]  

This multiplet was first constructed by Schwarz in [2] as a set of solutions of type IIB supergravity. There the argument requiring that \( m, n \) were integers and relatively prime was Dirac quantization condition plus stability since if this was not the case a given \((m, n)\) string would be at the threshold of decaying into \( p \) \((\frac{m}{p}, \frac{n}{p})\) strings, with \( p \) the maximum common divisor of \( m \) and \( n \). In [11] Witten showed that these string solutions correspond to bound states of \( m \) fundamental strings and \( n \) D-strings. From the world-volume point of view its existence can be proved starting from \( n \) D-strings, as shown here and, previously, in [3, 4, 5]. We have seen that the transformation that is required is the corresponding electric-magnetic duality in a two-dimensional world-volume, which we have formulated as a canonical transformation. The equivalence between the partition functions also holds straightforwardly in our formalism.
The D-membrane of type IIA

Since SL(2,R) is not a symmetry of type IIA supergravity we don’t expect any relation between the D-membrane of type IIA and other objects in the same theory, under the three-dimensional world-volume transformation. Instead we expect some connection with M-theory (compactified on a circle) since it is conjectured to be the strong coupling limit of ten dimensional type IIA. As we mentioned in the introduction it is in fact known that the vector dual of the D-membrane is the dimensional reduction of the membrane of M-theory \[3, 4, 5\]. Here we show that this duality can also be formulated as a canonical transformation in the three-dimensional world-volume.

We start by considering the effective action of the D-membrane:

\[ S_2 = \int \! d^3x \left[ e^{-\phi} \sqrt{-\det(G_{mn} + F_{mn})} + \frac{1}{2} \epsilon^{mnp} \left( \frac{1}{3} C_{mnp} + C_m F_{np} \right) \right]. \] (4.1)

Redefining \( g_{mn} = e^{-2\phi/3} G_{mn} \) and choosing \( g_{mn} \) to be Minkowski for simplicity we can write in electric and magnetic variables (our conventions are: \( E_\alpha = F_{0\alpha} \), \( B = \frac{1}{2} \epsilon^{0\alpha\beta} F_{\alpha\beta} \)):

\[ S_2 = \int \! d^3x \left[ \sqrt{1 - e^{-4\phi/3} \vec{E}^2} + e^{-4\phi/3} \vec{B}^2 + \frac{1}{6} \epsilon^{mnp} C_{mnp} + C_0 B + \vec{D}.\vec{E} \right] \] (4.2)

where \( D^\alpha \equiv * C^0 = \epsilon^{0\alpha\beta} C_\beta \). The canonical momenta are given by:

\[ \Pi_0 = 0 \]
\[ \Pi_\alpha = - \frac{e^{-4\phi/3} E_\alpha}{\sqrt{1 - e^{-4\phi/3} \vec{E}^2} + e^{-4\phi/3} \vec{B}^2}} + D_\alpha, \] (4.3)

and the Hamiltonian:

\[ H_2 = -\sqrt{1 + e^{4\phi/3}(\vec{\Pi} - \vec{D})^2 + e^{-4\phi/3} \vec{B}^2} + \vec{B}^2(\vec{\Pi} - \vec{D})^2 - B_0 \vec{B} \Pi_0 - \Pi_\alpha B_\alpha - \frac{1}{6} \epsilon^{mnp} C_{mnp} + \Pi_\alpha \partial_\alpha A_0. \] (4.4)

The canonical transformation:

\[ \Pi^\alpha = - \vec{F}^0 = - \epsilon^{0\alpha\beta} \partial_\beta \vec{A} \]
\[ \vec{\Pi} = B \] (4.5)

with generating functional:

\[ G_2 = - \int_{M_2} \! d^2x \epsilon^{0\alpha\beta} A_\alpha \partial_\beta \vec{A}, \] (4.6)

yields the following dual action (for arbitrary metric):

\[ S_2 = \int \! d^3x \left[ \sqrt{-\det(e^{-2\phi/3} G_{mn} + e^{4\phi/3}(\partial_m \vec{A} + C_m)(\partial_n \vec{A} + C_n))} + \frac{1}{6} \epsilon^{mnp} (C_{mnp} - 3 \partial_m \vec{A} B_{np}) \right]. \] (4.7)

This is the Nambu-Goto action of the dimensionally reduced eleven dimensional supermembrane \[11\], with:

\[ C_{mn}^{(11)} = e^{-2\phi/3} G_{mn}^{(10)} + e^{4\phi/3}(\partial_m \vec{A} + C_m)(\partial_n \vec{A} + C_n) \]
\[ B_{mnp}^{(11)} = C_{mnp}^{(10)} - 3 \partial_m \vec{A} B_{np}. \] (4.8)
In [14] the connection between the D-membrane and the dimensionally reduced eleven dimensional supermembrane was pointed out, and further developed in [3, 4, 5] where it was shown that a given world-volume transformation on the D-membrane was required. Here we have seen that it is just the strong-weak coupling transformation (defined in phase space), needed to connect ten dimensional type IIA with M-theory.

5 Conclusions

To summarize, we have seen that certain S-dualities between D-branes and p-branes of some string (or M-) theories can be formulated as canonical transformations in the phase space defined by the abelian world-volume gauge field of the D-brane and its conjugate momentum[^12]. This generalizes previous results in the literature [3, 4, 5, 6, 7] where a saddle point approximation is required in order to prove the equivalence with the corresponding dual theory. In our description the equivalence holds straightforwardly at a quantum mechanical level[^13].

The canonical transformation description provides a unified picture in which duality symmetries in string theories can be formulated. The same kind of transformation responsible for S-duality in abelian gauge theories is responsible for S-duality in the world-volume of D-branes. This transformation is just the interchange between momenta and magnetic degrees of freedom. For abelian gauge theories it reduces to electric-magnetic exchange whereas for the D-branes it is quite more complicated due to the non-trivial dependence of the DBI action on the electric degrees of freedom.

As mentioned in the introduction the D-4-brane of type IIA is equivalent to the double dimensional reduction of the eleven dimensional 5-brane after a vector duality transformation is performed in the five-dimensional world-volume. The dual theory is then defined in terms of a 2-form. The double dimensional “oxidation” of this theory could shed some light on the determination of the action of the eleven dimensional 5-brane [14, 20, 21, 30, 31].

There are more results in the literature which would be interesting to formulate in the present description. In [30] it is shown that vector duality in the eleven dimensional membrane gives upon dimensional reduction T-duality in the ten dimensional type II superstring. Given that both symmetries have well-defined descriptions in terms of canonical transformations (see the second of [22]) it could be interesting to show their explicit connexion.

Finally it could also be interesting to study the role of electric-magnetic duality transformations in the relation between the p-branes of type IIB and F theories [32] (see for instance [33] for some recent results).

Acknowledgements

I would like to thank E. Alvarez, A. González-Ruiz and E. Verlinde for useful discussions. Work supported by the European Commission TMR programme ERBFMRX-CT96-0045.

[^12]: Although the transformation is involutive it is more instructive to start from the D-brane action because then all the dualities can be formulated in a more unified way.

[^13]: Up to renormalization effects, as mentioned previously.
References

[1] J. Dai, R.G. Leigh and J. Polchinski, Mod. Phys. Lett. A4 (1989) 2073; J. Polchinski, Phys. Rev. Lett. 75 (1995) 4724; “TASI Lectures on D-branes”, hep-th/9611050.

[2] J.H. Schwarz, Phys. Lett. B360 (1995) 13.

[3] C. Schmidhuber, Nucl. Phys. B467 (1996) 146.

[4] S. P. de Alwis and K. Sato, Phys. Rev. D53 (1996) 7187.

[5] A.A. Tseytlin, Nucl. Phys. B469 (1996) 51.

[6] S. Ryang, hep-th/9608118.

[7] M.B. Green and M. Gutperle, Phys. Lett. B377 (1996) 28.

[8] C.M. Hull and P.K. Townsend, Nucl. Phys. B438 (1995) 109; C.M. Hull, Phys. Lett. B357 (1995) 545.

[9] E. Witten, Nucl. Phys. B443 (1995) 85.

[10] J.H. Schwarz, Phys. Lett. B367 (1996) 97.

[11] A. Giveon, M. Porrati and E. Rabinovici, Phys. Rep. 244 (1994) 77; E. Alvarez, L. Alvarez-Gaumé and Y. Lozano, Nucl. Phys. B (Proc. Supp.) 41 (1995) 1.

[12] M.J. Duff, P.S. Howe, T. Inami and K.S. Stelle, Phys. Lett. B191 (1987) 70; M.J. Duff and K.S. Stelle, Phys. Lett. B253 (1991) 113; M.J. Duff and J.X. Lu, Nucl. Phys. B390 (1993) 276; M.J. Duff, J.T. Liu and R. Minasian, Nucl. Phys. B452 (1995) 261.

[13] P.K. Townsend, Phys. Lett. B350 (1995) 184.

[14] P.K. Townsend, Phys. Lett. B373 (1996) 68.

[15] E. Bergshoeff, E. Sezgin and P.K. Townsend, Phys. Lett. B189 (1987) 75.

[16] R. Güven, Phys. Lett. B276 (1992) 49.

[17] G.W. Gibbons and D.A. Rasheed, Nucl. Phys. B454 (1995) 185; Phys. Lett. B365 (1996) 46.

[18] G.T. Horowitz and A. Strominger, Nucl. Phys. B360 (1991) 197; M.J. Duff and J.X. Lu, Phys. Lett. B273 (1991) 409.

[19] E. Witten, Nucl. Phys. B460 (1996) 335.

[20] E. Bergshoeff, M. de Roo and T. Ortín, Phys. Lett. B386 (1996) 85.

[21] P. Pasti, D. Sorokin and M. Tonin, hep-th/9701037; M. Aganagic, J. Park, C. Popescu and J.H. Schwarz, hep-th/9701116.
[22] Y. Lozano, Phys. Lett. B364 (1995) 19; Mod. Phys. Lett. A11 (1996) 2893.

[23] E. Witten, Phys. Lett. B86 (1979) 283.

[24] G.I. Ghandour, Phys. Rev. D35 (1987) 1289; T. Curtright, in “Differential Geometrical Methods in Theoretical Physics: Physics and Geometry”, eds. L.L. Chau and W. Nahm, Plenum, New York, (1990) 279; T. Curtright and G.I. Ghandour, in “Quantum Field Theory, Statistical Mechanics, Quantum Groups and Topology”, eds. T. Curtright, L. Mezincescu and R. Nepomechie, World Scientific, (1992). hep-th/9503080.

[25] R.G. Leigh, Mod. Phys. Lett. A4 (1989) 2767.

[26] M. Li, Nucl. Phys. B460 (1996) 351; C.G. Callan and I.R. Klebanov, Nucl. Phys. B465 (1996) 473; M. Douglas, hep-th/9512077. C.G. Callan, C. Lovelace, C.R. Nappi and S.A. Yost, Nucl. Phys. B308 (1988) 221.

[27] M. Cederwall, A. von Gussich, B.E.W. Nilsson and A. Westerberg, hep-th/9610148. M. Cederwall, A. von Gussich, B.E.W. Nilsson, P. Sundell and A. Westerberg, hep-th/9611159. E. Bergshoeff and P.K. Townsend, hep-th/9611173. M. Aganagic, C. Popescu and J.H. Schwarz, hep-th/9612080.

[28] A. Font, L.E. Ibañez, D. Lüst and F. Quevedo, Phys. Lett. B249 (1990) 35; A. Sen, Int. J. Mod. Phys. A9 (1994) 3707.

[29] J.H. Schwarz, Nucl. Phys. B226 (1983) 269; P.S. Howe and P.C. West, Nucl. Phys. B238 (1984) 181; E. Bergshoeff, C.M. Hull and T. Ortín, Nucl. Phys. B451 (1995) 547.

[30] O. Aharoni, Nucl. Phys. B476 (1996) 470.

[31] E. Witten, hep-th/9610234; M. Perry and J.H. Schwarz, hep-th/9611067. J.H. Schwarz, hep-th/9701008.

[32] C. Vafa, Nucl. Phys. B469 (1996) 403.

[33] D.P. Jatkar and S.K. Rama, Phys. Lett. B388 (1996) 283.