Abstract: In this paper, we introduce the notion of an elliptical segment as some analogy of the circular segment and we focus on the problem of calculation of its area. Based on the analytical method, we derive the formulas, which can be used for the numerical approximation of the area of the given segment.

Keywords: ellipse; segment; area; approximation

1. Introduction

In this paper, we present the problem of calculation of the area of elliptical segment. The notion elliptical segment is a generalization of the term circular segment. In accordance with this analogy, we consider the elliptical segment as a set of points surrounded by a given elliptical arc and its chord. The figures in Figure 1 are the segments illustrated as the shaded regions.

There exist a few formulas about how to calculate the area $A$ of the given circular segment. Their proving is based on the idea of subtraction of the area of the circular sector and the area of the subsistent triangle portion [1]. In this concept, the term central angle is crucial, but it is undefined for the ellipse [2]. The Gauss–Green formula is used to determine an elliptical sector area. It leads to an algorithm for calculating the area of the elliptical segment [3].

Our approach to the problem of the area of the elliptical segment is different. First, we prefer analytical method. Based on the idea of definite integral, we tessellate the given segment by polygons (trapezoids) and, consecutively, we derive a formula approximately evaluated the reflected area. Finally, we present an interesting corollaries of the results related to the area of the given ellipse.
2. Elliptical Segment

First, we consider an ellipse in its central equation with center in the origin of Cartesian coordinate system \( Oxy \). The straight line, determined by the chord of the conic, is defined in its slope-intercept form \([4]\).

In Cartesian coordinate system \( Oxy \), the ellipse is given by the equation

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{1}
\]

for \( 0 < b < a \), \( a, b \in \mathbb{R} \) and let \( p \) be the line of the form

\[
y = kx + q \tag{2}
\]

for \( k, q \in \mathbb{R} \). The line \( p \) must be a secant line. We show that

\[
q \in \left[ -\sqrt{a^2k^2 + b^2}, \sqrt{a^2k^2 + b^2} \right].
\]

Generally, let us consider an arbitrary parallel line in the equation

\[
y = kx + q_i, \tag{3}
\]

where \( q_i \in \mathbb{R} \).

Suppose that the line in Equation (3) intersects the ellipse in points \( U_i, V_i \). Their coordinates must be derived by solving an equation

\[
\frac{x^2}{a^2} + \frac{(kx + q_i)^2}{b^2} = 1
\]

\[
\vdots
\]

\[
x^2 \left( a^2k^2 + b^2 \right) + 2a^2kq_ix + a^2 \left( q_i^2 - b^2 \right) = 0
\]

We evaluate the discriminant \( D \)

\[
D = 4a^2b^2 \left( a^2k^2 + b^2 - q_i^2 \right) = 4a^2b^2 D_i \tag{4}
\]

where \( D_i = a^2k^2 + b^2 - q_i^2 \).

The points of intersection exist, if and only if \( D_i \geq 0 \). It implies that

\[
-\sqrt{a^2k^2 + b^2} \leq q_i \leq \sqrt{a^2k^2 + b^2}. \tag{5}
\]

The line \( p \) also determines the chord \( UV \) of the given ellipse, if and only if

\[
-\sqrt{a^2k^2 + b^2} < q < \sqrt{a^2k^2 + b^2}, \tag{6}
\]

as it was necessary to prove.

It also holds true that

- if \( D_i > 0 \), then there exists an one-to-one correspondence between the pair of points \( U_i, V_i \) and the intercept \( q_i \).

These points of intersection have the coordinates

\[
x(U_i, V_i) = \frac{-ka^2q_i \pm ab\sqrt{D_i}}{a^2k^2 + b^2},
\]
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3. Arithmetic Sequence of Intercepts

To determine the vertices of the polygons, we introduce the intercepts \( q_i \) as an arithmetic sequence \( \{q_i\}_{i=0}^n \) such that

\[
q_0 = -\sqrt{a^2k^2 + b^2} < q_1 < \cdots < q_i < \cdots < q_n \leq \sqrt{a^2k^2 + b^2},
\]

\[
q_i = -\sqrt{a^2k^2 + b^2} + id, \quad (9)
\]

for \( i = 0, 1, 2, \ldots, n \), where \( n \in \mathbb{N} \) is fixed and \( d > 0 \) is a difference of the sequence. Its value is bounded too. We determine it as follows:

The inequalities (9) result in

\[
q_i = -\sqrt{a^2k^2 + b^2} + id \leq \sqrt{a^2k^2 + b^2}
\]

and we derive

\[
0 < d \leq \frac{2\sqrt{a^2k^2 + b^2}}{i}
\]

for \( i = 1, 2, \ldots, n \). It implies that

\[
0 < d \leq \frac{2\sqrt{a^2k^2 + b^2}}{n}. \quad (11)
\]

This result allows us to put

\[
d = \frac{q + \sqrt{a^2k^2 + b^2}}{n}, \quad (12)
\]

This setting of the difference \( d \) has a geometric meaning. The line segment on the axis \( y \), bounded by the points \( (0, 0), (0, q) \), is fragmented to \( n \) particular segments of the equal length \( d \).

4. Approximation of Elliptical Segment

First, we construct the set of the lines, each in the equation \( y = kx + q_i \) for the arithmetical sequence \( \{q_i\}_{i=0}^n \) with its difference \( d \).

We determine the points of intersection with the ellipse and draw the triangle \( U_0U_1V_1 \) and the trapezoids \( U_iV_iV_{i+1}U_{i+1} \) for \( i = 1, 2, \ldots, n-1 \). They approximate the elliptical segment.
It holds true that
\[ |U_iV_i| = \sqrt{(x_{U_i} - x_{V_i})^2 + (y_{U_i} - y_{V_i})^2} = \cdots = \frac{2ab}{a^2k^2 + b^2} \sqrt{1 + k^2 D_i}. \]

Similarly, we derive
\[ |U_{i+1}V_{i+1}| = \frac{2ab}{a^2k^2 + b^2} \sqrt{1 + k^2 D_{i+1}}. \]

In this setting of the intercept \( q_i \) we derive that for \( D_i \) it holds true
\[ D_i = a^2k^2 + b^2 - q_i^2 = \cdots = 2id \sqrt{a^2k^2 + b^2 - i^2 d^2}, \quad (13) \]
where the difference \( d \) is determined by (12).

The area \( A_e \) of the elliptical segment is approximately equal to the area \( A_{(n,d)} \)
\[ A_{(n,d)} = \sum_{i=0}^{n-1} \frac{|U_iV_i| + |U_{i+1}V_{i+1}|}{2} h_i, \]
where \( h_i \) is a height of the trapezoid.

If we label \( k = \tan \varphi \) (Figure 2), then we derive
\[ h_i = d \cos \varphi = \frac{d}{\sqrt{1 + k^2}}, \quad (14) \]
\[ A_{(n,d)} = \frac{abd}{a^2k^2 + b^2} \sum_{i=0}^{n-1} \left( \sqrt{D_i} + \sqrt{D_{i+1}} \right). \quad (15) \]

**Example 1.** Approximately calculate the area of a semi-ellipse with axes \( a = 3, b = 2 \) for \( n \leq 10 \).

Solution. The semi-ellipse is a special case of the elliptical segment for \( q = 0 \). If we apply it in the Formulas (10), (12), and (13), then we derive
\[ d = \frac{\sqrt{a^2k^2 + b^2}}{n}, \quad D_i = \left( a^2k^2 + b^2 \right) \left( 2 - \frac{i}{n} \right) \frac{i}{n}, \]
and we derive
\[ A_{(n,d)} = \frac{ab}{n} \sum_{i=0}^{n-1} \left( \sqrt{\frac{i}{n} \left( 2 - \frac{i}{n} \right)} + \sqrt{\frac{i + 1}{n} \left( 2 - \frac{i + 1}{n} \right)} \right). \]

It is evident that \( k \) is a free parameter. It has also a geometric explanation-the line \( p \) passing through the center of the ellipse divides the ellipse in half.

Due to the simplicity of the calculation, we put \( k = 0 \) and we organize data in Table 1. Similarly, the reader can also verify some results listed in Table 2.
Figure 2. The secant lines of the ellipse and the points of intersection.

Table 1. Practical calculation for $n = 4$.

| Parameters | $i$ | $i \cdot d$ | $q_i$ | $D_i$ | $D_{i+1}$ | $\sqrt{D_i}$ | $\sqrt{D_{i+1}}$ | $\sqrt{D_i} + \sqrt{D_{i+1}}$ |
|------------|-----|--------------|-------|-------|-----------|--------------|----------------|------------------|
| $a = 3, b = 2$ | 0   | 0.00         | -2.00 | 0.00  | 1.75      | 0.00         | 1.32           | 1.32             |
| $k = 0, q = 0$ | 1   | 0.50         | -1.50 | 1.75  | 3.00      | 1.32         | 1.73           | 3.05             |
| $n = 4$   | 2   | 1.00         | -1.00 | 3.00  | 3.75      | 1.73         | 1.94           | 3.67             |
| $d = 0.50$ | 3   | 1.50         | -0.50 | 3.75  | 4.00      | 1.94         | 2.00           | 3.94             |
|           | 4   | 2.00         | 0.00  | 4.00  | 2.00      |              |                |                  |
| $\sum$    |     |              |       |       |           |              |                | $A_{(n,d)}$ 8.99 |

The visualizations with the different partition number $n$ are shown in Figure 3.

Table 2. The calculations of $A_{(n,d)}$ for $n = 1, 2, ..., 10$.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-----|---|---|---|---|---|---|---|---|---|----|
| $A_{(n,d)}$ | 6.00 | 8.20 | 8.75 | 8.99 | 9.11 | 9.19 | 9.24 | 9.27 | 9.29 | 9.31 |

Figure 3. The approximations for $n = 1, n = 4,$ and $n = 10$. 
5. Estimation of Area of Ellipse

In the case, when the line \( p \) is identical with the tangent line \( t_p \), the elliptical segment is prolonged to the complete ellipse and we have

\[
d = 2 \sqrt{\frac{a^2 k^2 + b^2}{n}}.
\]

It holds that

\[
D_i = a^2 k^2 + b^2 - q_i^2 = \cdots = 4 \left( a^2 k^2 + b^2 \right) \left( \frac{i}{n} - \frac{i^2}{n^2} \right)
\]

and we derive

\[
|U_i V_i| = \frac{2ab}{a^2 k^2 + b^2} \sqrt{1 + k^2} \sqrt{D_i} = \cdots = 4 \frac{ab}{\sqrt{a^2 k^2 + b^2}} \sqrt{1 + k^2} \sqrt{\frac{i}{n} - \frac{i^2}{n^2}}.
\]

By analogy,

\[
|U_{i+1} V_{i+1}| = 4 \frac{ab}{\sqrt{a^2 k^2 + b^2}} \sqrt{1 + k^2} \sqrt{\frac{i+1}{n} - \frac{(i+1)^2}{n^2}}.
\]

The result (15) is simplified as follows:

\[
A_{(n,d)} = \frac{4ab}{n} \sum_{i=0}^{n-1} \left( \sqrt{\frac{i}{n} - \frac{i^2}{n^2}} + \sqrt{\frac{i+1}{n} - \frac{(i+1)^2}{n^2}} \right).
\]  

(17)

We modify the Formula (17) in the form

\[
A_{(n,d)} = \frac{8ab}{n} \sum_{i=1}^{n-1} \sqrt{\frac{i}{n} - \frac{i^2}{n^2}}.
\]

(18)

The independence on the slope \( k \) has a geometric interpretation too. We can approximate the ellipse by many polygonal tessellations related to the value of the slope \( k \). They need to not be congruent, but their areas are equal. Some samples are in (Figure 4).

![Figure 4. The tessellations for \( n = 6 \).](image)

We have demonstrated that the Formula (18) can be used to numerical approximation and its value depends on \( n \)-the number of the polygons which tessellate the given ellipse. We estimate \( A_{(n,d)} \).

Let us consider the function \( f(x) = \sqrt{\frac{x}{n} - \frac{x^2}{n^2}} \) for \( x > 0 \) and fixed \( n \in \mathbb{N} \). We show that this function is bounded.

It is evident that the domain of function is \([0,n]\) and the function is bounded from below. Its minimum is \( m = 0 \) for \( x = 0 \). We find a maximum \( M \) by using its derivatives:

\[
f'(x) = \left( \sqrt{\frac{x}{n} - \frac{x^2}{n^2}} \right)' = \cdots = \frac{n - 2x}{2n^2 \sqrt{\frac{x}{n} - \frac{x^2}{n^2}}}.
\]
To find a stationary point, we set \( f'(x) = 0 \) and we derive \( x_0 = n^2 \).

The second derivative \( f''(x) \) is

\[
\frac{n - 2x}{2n^2 \sqrt{\frac{x}{n^2} - \frac{x^2}{n^2}}} \]

and it holds true that

\[
f''(x_0) = \cdots = -\frac{1}{2n^2} \frac{2\sqrt{\frac{x}{n^2} - \frac{x^2}{n^2}}}{\frac{x}{n^2} - \frac{x^2}{n^2}} = -\frac{1}{n^2} \frac{1}{4} = -\frac{2}{n^2} < 0 \quad (19)
\]

The function \( f(x) \) has in \( x_0 = n^2 \) a global maximum \( M = \frac{1}{2} \) and it implies that

\[
A \approx \frac{4ab}{n} - \frac{2a}{n} \sum_{i=1}^{n-1} \sqrt{\frac{i}{n} - \frac{i^2}{n^2}} < \frac{4ab}{n} \cdot \frac{1}{2} = \frac{4ab}{2} \cdot \frac{n-1}{n} \quad (20)
\]

6. Area of Ellipse-Exactly

The area \( A \) of the given ellipse in Equation (1) can be evaluated by using definite integral [5]. It holds true that

\[
A = \frac{b}{a} \int_0^a \sqrt{a^2 - x^2} \, dx \quad \text{sub. } x = a \sin t, \, dx = a \cos t \, dt = \frac{b}{a} \int_0^\frac{\pi}{2} a \cos t \sqrt{a^2 - a^2 \sin^2 t} \, dt = \cdots = ab \int_0^\frac{\pi}{2} \cos^2 t \, dt = \frac{ab}{2} \left( \frac{t}{2} \right)_0^\frac{\pi}{2} + \frac{ab}{2} \left( \frac{\sin 2t}{2} \right)_0^\frac{\pi}{2} = \cdots = \frac{\pi ab}{4}
\]

The result implies that it holds

\[
A = \pi ab. \quad (21)
\]

Finally, from (18), we can also derive this interesting result

\[
A = \lim_{n \to \infty} \frac{8ab}{n} \sum_{i=1}^{n-1} \sqrt{\frac{i}{n} - \frac{i^2}{n^2}} = \pi ab.
\]

It implies that

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n-1} \sqrt{\frac{i}{n} - \frac{i^2}{n^2}} = \frac{\pi}{8}. \quad (22)
\]

This result can be verified in geometrical way too. If we apply the idea of Darboux integral [6], then we obtain

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n-1} \sqrt{\frac{i}{n} - \frac{i^2}{n^2}} = \int_0^1 \sqrt{x - x^2} \, dx.
\]

The Figure 5 presents the graph of the function \( y = \sqrt{x - x^2} \). It is the semicircle with the center \( O(\frac{1}{2}, 0) \) and the radius \( r = \frac{1}{2} \). It is evident that its area is equal to \( A = \frac{\pi r^2}{2} = \cdots = \frac{\pi}{8} \).
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References

1. Weisstein, E.W. Circular Segment. 2020. Available online: http://mathworld.wolfram.com/CircularSegment.html (accessed on 31 January 2020).
2. Rhoad, R.; Milauskas, G.; Whipple, R. Geometry for Enjoyment and Challenge. New Edition; McDougal Littell: Houghton Mifflin Harcourt: Boston, MA, USA, 1997.
3. Hughes, G.B.; Chraibi, M. Calculating ellipse overlap areas. Comput. Vis. Sci. 2012, 15, 291–301. doi:10.1007/s00791-013-0214-3.
4. Bronshtein, I.N.; Semendyayev, K.A.; Musiol, G.; Mühlig, H. Handbook of Mathematics, 5th ed.; Springer: Berlin/Heidelberg, Germany; New York, NY, USA, 2007.
5. Mendelson, E.; Ayres, F. Schaum’s Outlines: Calculus, 5th ed.; Schaum’s Outline Series; McGraw-Hill: New York, NY, USA, 2008. doi:10.1036/0071508619.
6. Spivak, M. Calculus, 3rd ed.; Publish or Perish, Inc.: Houston, TX, USA, 2008.

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