Vector duality via conditional extension of dual pairs

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July 9, 2018

ABSTRACT

A Fenchel-Moreau type duality for proper convex and lower semi-continuous functions \( f : X \to \mathbb{L}^0 \) is established where \((X, Y, \langle \cdot, \cdot \rangle)\) is a dual pair of Banach spaces and \(\mathbb{L}^0\) is the set of all extended real-valued measurable functions. We provide a concept of lower semi-continuity which is shown to be equivalent to the existence of a dual representation in terms of elements in the Bochner space \(L^0(Y)\). To derive the duality result, several conditional completions and extensions are constructed.

KEYWORDS: Vector duality, extensions, conditional sets, Bochner spaces, vector optimization

AMSCCLASSIFICATION: 46A20, 03C90, 54D35, 54C20, 46B22

1 Introduction

Duality theory is an important tool in vector optimization and its wide range of applications. This article contributes to vector duality by providing a notion of lower semi-continuity and proving its equivalence to a Fenchel-Moreau type dual representation. For the sake of illustration, let \((\Omega, \mathcal{F}, \mu)\) be a \(\sigma\)-finite measure space, \((X, Y, \langle \cdot, \cdot \rangle)\) a dual pair of Banach spaces and \(\mathbb{L}^0\) the collection of all measurable functions \(x : \Omega \to \mathbb{R} \cup \{\pm \infty\}\) where two of them are identified if they agree almost everywhere. Consider on \(\mathbb{L}^0\) the order of almost everywhere dominance. Let \(f : X \to \mathbb{L}^0\) be a proper convex function. We introduce a concept of lower semi-continuity which is shown to be equivalent to the Fenchel-Moreau type dual representation

\[ f(x) = \operatorname{ess \sup}_{y \in L^0(Y)} \{\langle x, y \rangle - f^*(y)\} \]  \hspace{1cm} (1.1)

for all \(x \in X\), where \(L^0(Y)\) is the Bochner space of all strongly measurable functions \(y : \Omega \to Y\) modulo almost everywhere equality, \(f^*(\cdot) = \operatorname{ess \sup}_{x \in X} \{\langle x, \cdot \rangle - f(x)\}\) is the convex conjugate and \(\langle x, y(\omega) \rangle = \langle x, y(\omega) \rangle\) almost everywhere.

The basic idea is to conditionally extend the whole structure of the problem to conditional set theory where a conditional version of the classical Fenchel-Moreau duality is applied, which translated to the original framework gives us the representation (1.1). The extension procedure consists of three elements. First, we conditionally complete the Banach spaces \(X\) and \(Y\) and prove that they are isometrically isomorphic to the Bochner spaces \(L^0(X)\) and \(L^0(Y)\), respectively. Second, we conditionally extend the duality pairing to a conditional duality pairing on the product \(L^0(X) \times L^0(Y)\). Third, we construct a conditionally lower semi-continuous extension \(f_c\) to \(L^0(X)\) with values in \(L^0(Y)\). By identifying \(\mathbb{L}^0\) with the conditional extended real numbers and applying a conditional version of the Fenchel-Moreau theorem, we obtain a conditional dual representation of \(f_c\) in terms of conditional dual elements in \(L^0(Y)\). By eval-
uating constants in the conditional dual representation, we get the vector duality (1.1). The conditional completions and conditional extensions and the vector duality result are proved for an arbitrary complete Boolean algebra with the corresponding conditional real numbers instead of an associated measure algebra.

We discuss the related literature. Scalarization techniques, see e.g. [3, Chapter 4], are in general not applicable since \( \text{int}(L^0) = \emptyset \) and \( (L^0)^* = \{0\} \). If \( f: X \to L^0 \) is convex and continuous with respect to the topology of convergence in measure, \([22, Theorem 1]\) yields a dual representation \( f(x) = \text{ess sup}_{x \in L(X,L^0)} \{\langle x, x^* \rangle - f^*(x^*)\} \) where \( L(X,L^0) \) is the set of all linear and continuous functions from \( X \) to \( L^0 \) and \( f^* \) is the corresponding convex conjugate. For a convex duality of set-valued functions, we refer to \([10, 11]\) and their references. Vector duality for modules over the ring \( L^0 \) is introduced in \([9]\) which is not applicable since \( X \) is not a module over \( L^0 \). The module-based duality is further studied in \([17, 21]\) where dual representations of conditional convex risk measures on \( L^\infty \)-type modules are established. The possibility of convex duality in modules is investigated in \([14]\). Conditional set theory is briefly introduced and conditional completions of metric spaces are constructed. In Section 3, the conditional extension of dual pairs of normed vector spaces is proved, the concept of lower semi-continuity together with a conditionally lower semi-continuous extension are given and the main vector duality result is shown. Finally, the results are applied to vector duality in Bochner spaces in Section 4.

2 Conditional completion of metric spaces

We start by collecting basic results from conditional set theory as developed in \([7]\). Throughout, we fix a complete non-degenerate Boolean algebra \( \mathcal{A} = (\mathcal{A}, \wedge, \vee, ^c, 0, 1) \). The order on \( \mathcal{A} \) is the relation \( a \leq b \) whenever \( a \wedge b = a \). Denote by \( \vee a_i = \vee_{i \in I} a_i \) and \( \wedge a_i = \wedge_{i \in I} a_i \) the supremum and the infimum of a family \( (a_i) = (a_i)_{i \in I} \) in \( \mathcal{A} \), respectively. A partition in \( \mathcal{A} \) is a family \( (a_i) \) of elements in \( \mathcal{A} \) such that \( a_i \wedge a_j = 0 \) whenever \( i \neq j \) and \( \vee a_i = 1 \). Denote by \( p \) the set of all partitions in \( \mathcal{A} \).

**Definition 2.1.** A conditional set of a non-empty set \( X \) and \( \mathcal{A} \) is a collection \( X \) of objects \( x|a \) where \( x \in X \) and \( a \in \mathcal{A} \) such that

(1) \( x|b = y|a \) implies \( a = b \);

(2) \( x|b = y|b \) and \( a \leq b \) imply \( x|a = y|a \);

(3) for each \( (a_i) \in p \) and every \( (x_i) \) in \( X \) there exists a unique \( x \in X \) such that \( x|a_i = x_i|a_i \) for all \( i \).

The unique element \( x \) is called the concatenation of \( (x_i) \) along \( (a_i) \) and is denoted by \( x = \sum x_i|a_i \).

The property (C2) is called consistency and the property (C3) is named stability. A subset \( Y \subseteq X \) is stable if \( Y \neq \emptyset \) and \( \sum y_i|a_i \in Y \) for every family \( (y_i) \) in \( Y \) and each \( (a_i) \in p \). A stable subset \( Y \) induces a conditional subset \( \mathcal{Y} := \{x|a : x \in Y, a \in \mathcal{A}\} \) of \( \mathcal{X} \). A singleton \( \{x\} \) is stable. The induced conditional subset \( \mathcal{X} := \{x|a : a \in \mathcal{A}\} \) is called a conditional element in \( \mathcal{X} \). It is shown in \([7, Theorem 2.9]\) that the collection of all conditional subsets of \( \mathcal{X} \) with the conditional set operations of conditional intersection \( \cap \), conditional union \( \cup \) and conditional complement \( \subseteq \) form a complete Boolean algebra.

A collection \( \mathcal{V} \) of conditional subsets of \( \mathcal{X} \) is stable if the conditional subset of \( \mathcal{X} \) which is induced by
The conditional open ball \( \{ \sum y_i | a_i : y_i \in Y_i \text{ for all } i \} \) is an element of \( \mathcal{V} \) for all families \( (Y_i) \) in \( \mathcal{V} \) and \( (a_i) \in p \). The conditional Cartesian product of two conditional sets \( X \) and \( Y \) is the conditional set \( X \times Y := \{(x,y) | a : (x,y) \in X \times Y, a \in A \} \). A function \( f : X \to Y \) is said to be stable whenever \( f(\sum x_i | a_i) = \sum f(x_i) | a_i \) for each \( (a_i) \in p \) and every family \( (x_i) \) in \( X \). Its graph \( G_f := \{(x,y) : f(x) = y\} \) is a stable subset of \( X \times Y \). The induced conditional set is the conditional graph of a conditional function \( f : X \to Y \). A conditional function \( f : X \to Y \) is conditionally injective whenever \( f(x) | a = f(y) | a \) implies \( x | a = y | a \).

**Definition 2.2.** Given a non-empty set \( X \), we denote by \( X_s \) the set of all families \((x_i, a_i) \) in \([0, 1] \times A \) such that \((a_i) \in p \) where two families \((x_i, a_i) \) and \((y_j, b_j) \) are identified whenever \( \forall \{a_i : y_i = z\} = \forall \{b_j : y_j = z\} \) for all \( z \in X \). Denote by \([x_i, a_i]\) the equivalence class of \((x_i, a_i)\) in \( X_s \). Then \( X_s \) induces a conditional set \( X_{\mathbb{N}} \) of objects

\[
[x_i, a_i] := \{[y_j, b_j] \in X_{\mathbb{N}} : \forall a_i : x_i = z \land a = \forall \{b_j : y_j = z\} \land a \text{ for all } z \in X \}.
\]

We call \( X_{\mathbb{N}} \) the conditional set of *step functions* with values in \( X \). For two non-empty sets \( X \) and \( Y \), the function \( f : X \to Y \) defined by \( f(\sum x_i | a_i) := \sum f(x_i) | a_i \) is a stable function. We call the induced conditional function \( f_\mathbb{N} : X_{\mathbb{N}} \to Y_{\mathbb{N}} \) a *conditional step function*.

Note that there exists a bijection from \( X \) to \([0, 1] \times X \) as a non-stable subset of \( X_{\mathbb{N}} \) representing the constant step functions in \( X_{\mathbb{N}} \). By stability, each element \([x_i, a_i] \in X_{\mathbb{N}} \) can be written as \( \sum [x_i] | a_i \) and therefore the notation \( \sum x_i | a_i \) is assigned to the elements of \( X_{\mathbb{N}} \). The conditional set \( N_{\mathbb{N}} \) of step functions with values in the natural numbers \( \mathbb{N} \) is called the *conditional natural numbers*.

We summarize the construction of the conditional real numbers. Let \( \mathbb{R}_{a} \) be the conditional set of step functions with values in the real numbers \( \mathbb{R} \). On \( \mathbb{R}_{a} \) define the stable relation \( \sum x_i | a_i \leq \sum y_j | b_j \) whenever \( x_i \leq y_j \) for all \( i, j \) with \( a_i \land b_j > 0 \). We understand \( \sum x_i | a_i \leq \sum y_j | b_j \) if \( x_i < y_j \) for all \( i, j \) with \( a_i \land b_j > 0 \). Moreover, define on \( \mathbb{R}_{a} \) the stable operations \( \sum x_i | a_i + \sum y_j | b_j := \sum x_i + y_j | a_i \land b_j \) and \( \sum x_i | a_i \cdot \sum y_j | b_j := \sum x_i \cdot y_j | a_i \land b_j \). Then \( (\mathbb{R}_{a}, +, \cdot, \leq) \) is a conditionally ordered field. \(^1\) Define the stable absolute value \( | \sum x_i | a_i | := | \sum x_i | a_i |. \) A conditional sequence \(^2\) \( (x_n) \) in \( \mathbb{R}_{a} \) is conditionally Cauchy if for all \( r \in \mathbb{R}_{a} \) with \( r > 0 \) there is \( n_0 \in \mathbb{N}_{a} \) such that \( |x_m - x_n| < r \) for all \( m, n \geq n_0 \). \(^3\) By conditionally identifying \(^4\) two conditional Cauchy sequences \((x_n) \) and \((y_n) \) whenever \( |x_n - y_n| \) conditionally converges \(^5\) to 0, we obtain the *conditional real numbers* which we denote by \( \mathbb{R} \). The conditional elements in \( \mathbb{R} \) are denoted \( x = [x_n] \). The stable set which induces \( \mathbb{R} \) is denoted by \( R \) and consists of equivalence classes of stable Cauchy sequences \( x = [x_n] \). The conditional order on \( \mathbb{R} \) is induced by the stable relation on \( R \) defined by \( [(x_n)] \leq [(y_n)] \) whenever for all \( r \in \mathbb{R}_{a} \) with \( r > 0 \) there exists \( n \in \mathbb{N}_{a} \) such that \( y_m - x_m > -r \) for all \( m \geq n \). We write \( [(x_n)] < [(y_n)] \) if there are \( r \in \mathbb{R}_{a} \) with \( r > 0 \) and \( n \in \mathbb{N}_{a} \) such that \( y_m - x_m > r \) for all \( m \geq n \). We denote \( R_{+} = \{x \in R : x > 0\} \) and \( R_{+ +} = \{x \in R : x > 0\} \). The conditionally ordered set \( (R, \leq) \) is conditionally Dedekind complete which implies that the ordered set \( (R, \leq) \) is Dedekind complete in a classical sense. The conditional arithmetic operations are defined by \( [(x_n)] + [(y_n)] := [(x_n + y_n)] \) and \( [(x_n)] \cdot [(y_n)] := [(x_n \cdot y_n)] \). We have that \( (R_{+}, +, \cdot, \leq) \) is a conditionally ordered field. The conditional absolute value on \( \mathbb{R} \) is induced by the stable mapping \( |x| = x |a + (-x)|a^c \) where \( a = \forall \{b : x |b |b > 0\} \). \(^6\) A conditional open ball with conditional radius \( r \) in \( R_{+ +} \) around \( x \) is the conditional set \( B_r(x) := \{y \in R : |x - y| < r\} \).
collection of all conditional open balls forms a conditional topological base $B$. The conditional topology conditionally generated by $B$ is conditionally complete.

The classical technique of completion extends to general conditional metric spaces.

**Definition 2.3.** A conditional metric is a conditional function $d : X \times X \to \mathbb{R}_+$ such that

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for every $x$ and $y$ in $X$;

(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for every $x$, $y$, and $z$ in $X$.

The pair $(X, d)$ is a conditional metric space. The conditional topology on $X$ is generated by the conditional topological base of conditional sets $B_r(x) := \{ y \in X : d(x, y) < r \}$ for $r \in \mathbb{R}_+$ and $x$ in $X$.

A conditional metric space is called **conditionally complete** if every conditional Cauchy sequence has a conditional limit.

**Theorem 2.4.** Let $(X, d)$ be a conditional metric space. Then there exists a unique, up to conditional isometric isomorphisms, conditionally complete metric space $(X_c, d_c)$ such that $X$ is conditionally isometric to a conditionally dense subset of $X_c$.

We call $X_c$ the conditional completion of $X$.

**Proof.** The relation $(x_n) \sim (y_n)$ whenever $\lim_{n \to \infty} d(x_n, y_n) = 0$ is a conditional equivalence relation on the conditional set $C$ of all conditional Cauchy sequences in $X$. Let $X_c := C/\sim$. For $(x_n)$ in $C$, we denote by $[(x_n)]$ its conditional equivalence class in $X_c$. For $x$ in $X$, we denote by $[x]$ the conditional equivalence class in $X_c$ of the constant conditional sequence $x_n = x$ for all $n$. Let $d_c : X_c \times X_c \to \mathbb{R}_+$ be defined by $d_c([(x_n)], [(y_n)]) = \lim_{n \to \infty} d(x_n, y_n)$. By the conditional triangle inequality, the conditional completeness of $\mathbb{R}$ and since $d$ is a conditional metric, $d_c$ is a well-defined conditional metric on $X_c$. Define $j : X \to X_c$ by $j(x) = [x]$. Inspection shows that $j$ is a conditional isometry. In order to see that $j(X)$ is conditionally dense in $X_c$, let $[(x_n)]$ be in $X_c$. Since $(x_n)$ is a conditional Cauchy sequence in $X$, it follows that the conditional sequence $(x_n) = ([x_n])$ in $j(X)$ conditionally converges to $[(x_n)]$ in $X_c$. To prove the conditional completeness of $(X_c, d_c)$, it suffices to show that every conditional Cauchy sequence in $j(X)$ has a conditional limit in $X_c$. Let $[(x_n)]$ be a conditional Cauchy sequence in $j(X)$. Since $j$ is a conditional isometry, it follows that $[(x_n)]$ conditionally converges to $[(x_n)]$ in $X_c$.

As for the uniqueness, let $Y_1$ and $Y_2$ be two conditional completions of $X$ where $j_1$ and $j_2$ denote the conditional isometric embeddings of $X$ into $Y_1$ and $Y_2$, respectively. Since $j_1$ and $j_2$ are conditionally injective, $h := j_2 \circ j_1^{-1} : j_1(X) \to j_2(X)$ and $k := j_1 \circ j_2^{-1} : j_2(X) \to j_1(X)$ are conditional surjective isometries. Now there exist unique conditional isometries $f : Y_1 \to Y_2$ and $g : Y_2 \to Y_1$ which conditionally extend $h$ and $k$, respectively. One has $f \circ j_1 = j_2$ and $g \circ j_2 = j_1$. Since $j_1(X)$ is conditionally dense in $Y_1$ and $j_2(X)$ is conditionally dense in $Y_2$, $g \circ f$ is the conditional identity of $Y_1$ and $f \circ g$ is the conditional identity of $Y_2$. Hence $f = g^{-1}$, and thus $f : Y_1 \to Y_2$ is the unique conditional surjective isometry such that $f \circ j_1 = j_2$. \qed

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7 See [7, Definition 3.1].

8 See the paragraph succeeding [7, Definition 3.1].

9 See the paragraph succeeding [7, Definition 3.14] for the definition of a conditionally dense subset.

10 By similar arguments as in the proof of Proposition 2.9, it can be shown that there exists a conditionally unique extension.
Remark 2.5. Let \((X, d)\) be a metric space and \(d_a : X_a \times X_a \to \mathbb{R}_+\) induced by the stable function
\[
\left( \sum x_i |a_i|, \sum y_j |b_j| \right) \mapsto d_a \left( \sum x_i |a_i|, \sum y_j |b_j| \right) := \sum d(x_i, y_j) |a_i \wedge b_j|.
\]

Inspection shows that \((X_a, d_a)\) is a conditional metric space. By Theorem 2.4, \((X_a, d_a)\) can be conditionally completed into \((X_c, d_c)\). We denote by \((X_a, d_a)\) and \((X_c, d_c)\) the conditional metric space of step functions and its conditional completion, respectively. ♦

Remark 2.6. Let \(X\) be a metric space. Throughout this article, we identify \(X_a\) with \(j(X_a)\) in \(X_c\). For every \(x \in X\), we denote by \(x\) the conditional element in \(X_c\) of the constant step function \(x/1\). ♦

Definition 2.7. A conditional norm on a conditional vector space\(^{11}\) \((X, +, \cdot)\) is a conditional function \(\|\cdot\| : X \to \mathbb{R}_+\) such that

(i) \(\|x\| = 0\) if and only if \(x = 0\);

(ii) \(\|rx\| = |r| \|x\|\) for all \(x \in X\) and every \(r \in \mathbb{R}\);

(iii) \(\|x + y\| \leq \|x\| + \|y\|\) for all \(x\) and \(y\) in \(X\).

Remark 2.8. By Theorem 2.4, every conditional normed vector space \((X, \|\cdot\|)\) can be conditionally completed into a conditional metric space \(X_c\). Inspection shows that \((X_c, \|\cdot\|_c)\) is a conditional Banach space. Moreover, every normed vector space \((X, \|\cdot\|)\) can be conditionally completed into a conditional Banach space \((X_c, \|\cdot\|_c)\). ♦

Proposition 2.9. Let \((X, d)\) be a metric space and \(f : X \to \mathbb{R}\) a uniformly continuous function. Then there exists a unique conditionally uniformly continuous function \(f_c : X_c \to \mathbb{R}\) such that \(f_c(x) = f(x)\) for all \(x \in X\).

We call \(f_c\) the conditionally uniformly continuous extension of \(f\).

Proof. Define \(f_c : X_c \to \mathbb{R}\) by \(f_c(x) = \lim_{n \to \infty} f_n(x_n)\) where \((x_n)\) is a conditional sequence in \(X_a\) conditionally converging to \(x\). We show that \(\lim_{n \to \infty} f_n(x_n)\) exists. Fix \(r \in \mathbb{R}_{++}\) and choose \(k \in \mathbb{N}_a\) such that \(0 < 1/k < r\). Suppose \(k = \sum k_i |a_i| \in \mathbb{N}_c\). By the uniform continuity of \(f\), there exists \(t_i \in \mathbb{R}_{++}\) such that \(x, y \in X\) with \(d(x, y) < t_i\) implies \(|f(x) - f(y)| < 1/k_i\) for each \(i\). Put \(t = \sum t_i |a_i| \in \mathbb{R}_a\).

Since \((x_n)\) is a conditional Cauchy sequence there exists \(n_0 \in \mathbb{N}_a\) such that \(d_a(x_n, x_m) < t\) for all \(n, m \geq n_0\). By construction, \(|f_n(x_n) - f_n(x_m)| < r\) for all \(n \geq n_0\). The claim follows from the conditional completeness of \(\mathbb{R}\). Inspection shows that \(f_c\) is uniquely determined, independent of the choice of conditional representatives and conditionally uniformly continuous. \(\square\)

Examples 2.10. (i) Let \(X\) be a normed vector space and \(f : X \to \mathbb{R}\) a continuous and linear functional. Its conditionally uniformly continuous extension \(f_c : X_c \to \mathbb{R}\) is conditionally linear.

(ii) Let \(\mathbb{R} := \mathbb{R} \cup \{\pm \infty\}\) be the extended real numbers and let \(\mathbb{R}\) denote the conditional completion of \(\mathbb{R}\).

The conditionally uniformly continuous extension \(\arctan : \mathbb{R} \to \mathbb{R}\) of \(\arctan : \mathbb{R} \to \mathbb{R}\) is conditionally strictly increasing. ◊

\(^{11}\)See [7, Definition 5.1].
3 Conditional extension of lower semi-continuous functions

Throughout this section, let \((X, Y, \langle \cdot, \cdot \rangle)\) be a dual pair of normed vector spaces such that \(|\langle x, y \rangle| \leq \|x\| \|y\|\) for all \(x \in X\) and \(y \in Y\). We assume that \(C^X := \{x \in X : \|x\| \leq 1\}\) is \(\sigma(X, Y)\)-closed and \(C^Y := \{y \in Y : \|y\| \leq 1\}\) is \(\sigma(Y, X)\)-closed.

**Theorem 3.1.** There exists a unique conditional bilinear form \(\langle \cdot, \cdot \rangle_c\) on \(X_c \times Y_c\) with \(\langle x, y \rangle_c = \langle x, y \rangle\) for all \(x \in X\) and \(y \in Y\) such that \((X_c, Y_c, \langle \cdot, \cdot \rangle_c)\) is a conditional dual pair.\(^{12}\)

We call \(\langle \cdot, \cdot \rangle_c\) the *conditional extension* of the duality pairing \(\langle \cdot, \cdot \rangle\).

**Proof.** Let \((x_n)\) be a conditional Cauchy sequence in \(X_s\) and \((y_n)\) a conditional Cauchy sequence in \(Y_s\). By the conditional triangle inequality, for all \(r > 0\) there is \(n_0 \geq 1\) such that

\[
\|x_n - y_n\|_s = \|x_m - y_m\|_s \leq \|x_n - x_m\|_s \|y_n\|_s + \|x_m\|_s \|y_n - y_m\|_s \leq r
\]  

for all \(n, m \geq n_0\). Put \((\langle x_n \rangle, \langle y_n \rangle)_c := \lim_{n \to \infty} \langle x_n, y_n \rangle_s\) for \(\langle x_n \rangle\) in \(X_c\) and \(\langle y_n \rangle\) in \(Y_c\). By (3.1) and the conditional completeness of the conditional real line, \(\langle \cdot, \cdot \rangle_c\) is independent of conditional representatives and conditionally real-valued, and thus well-defined. Since \(\langle \cdot, \cdot \rangle_s\) is a conditional \(\mathbb{R}\)-bilinear form and since the conditional sum and the conditional product of two conditionally convergent sequences in \(\mathbb{R}\) are conditionally convergent, \(\langle \cdot, \cdot \rangle_c\) is a conditional bilinear form on \(X_c \times Y_c\). By (3.1), the conditional pairing \(\langle \cdot, \cdot \rangle_c\) is the unique conditional bilinear form conditionally extending \(\langle \cdot, \cdot \rangle\).

We show that \(\langle \cdot, \cdot \rangle_c\) conditionally separates points. Let \(x\) be a conditionally non-zero element in \(X_c\), that is, \(x|a| \neq 0\) for every \(a > 0\). There is \(r > 0\) with \(r = \sum a_i\) such that \(0\) is conditionally not in the conditional closed ball \(C_r(x) := \{z \in X_c : \|z - x\|_c \leq r\}\), that is, \(0 \notin C_r(x)|a|\) for every \(a > 0\). Since \(X_s\) is conditionally dense in \(X_c\) by Theorem 2.4, we find \(x'\) in \(X_s\) such that \(0\) is conditionally not in \(C_{r/2}(x')\) and \(x' = \sum x'\|a_i\|\).\(^{13}\) Note that

\[
C_{r/2}(x') = \{z \in X_c : z|a_i| \in C_{r/2}(x')|a_i|\text{ for all }i\}.
\]

Since \(C_{r/2}(x') := \{z \in X : \|z\| \leq r/2\}\) is \(\sigma(X, Y)\)-closed, by strong separation, there exists a non-zero \(y_i\) in \(Y\) such that \(\inf_{z \in X_{r/2}(x')} \langle z, y_i \rangle \geq \delta_i > 0\). It follows that \(\inf_{z \in C_{r/2}(x')} \langle z, y \rangle_c \geq \delta > 0\) where \(y = \sum y_i\|a_i\|\) and \(\delta = \sum \delta_i\|a_i\|\). Indeed, let \(z\) be in \(C_{r/2}(x')\) and pick a conditional sequence \((z_n)\) in \(C_{r/2}(x')\) \(\cap X_s\) such that \(\|z_n - z\|_c \to 0\). Since \(z_n\) has the form \(\sum z_{j,n}\|b_{j,n}\|\), it follows that \(\|z_{j,n} - x'\| \leq r_i/2\) whenever \(b_{j,n} \wedge a_i > 0\). By stability of \(\langle \cdot, \cdot \rangle_s\), we obtain

\[
\langle z_n, y \rangle = \sum z_{j,n}\|b_{j,n}\| y_i\|a_i\| = \sum \langle z_{j,n}, y_i \rangle \|b_{j,n} \wedge a_i\| \geq \sum \delta_i\|b_{j,n} \wedge a_i = \sum \delta_i\|a_i = \delta > 0.
\]

By (3.1), it follows that

\[
\langle z, y \rangle_c = \langle z - z_n, y \rangle_c + \langle z_n, y \rangle c \geq \langle z - z_n, y \rangle_c + \delta \quad \text{as } n \to \infty \Rightarrow \delta > 0,
\]

showing that \(\inf_{z \in C_{r/2}(x')} \langle z, y \rangle_c \geq \delta > 0\). Hence \(Y_s\), and therefore \(Y_c\), conditionally separates the points of \(X_c\). By symmetry, \(X_c\) conditionally separates the points of \(Y_c\). Thus \((X_c, Y_c, \langle \cdot, \cdot \rangle_c)\) is a conditional dual pair. \(\Box\)

\(^{12}\)See [7, Definition 5.6].

\(^{13}\)That is, \(x'\) and \(r\) have the same partition. Otherwise consider \(x'\) and \(r\) on the common refinement of the respective partitions.
Examples 3.2. (i) If \( X = L^p \) and \( Y = L^q \) on a finite measure space \( (S, S, \mu) \) with \( 1 \leq p, q \leq \infty \) and \( 1/p + 1/q \leq 1 \), then \( |(x, y)| \leq \|x\|_p \|y\|_q \) for all \( x \in L^p \) and \( y \in L^q \). Moreover, \( C^{L^p} \) is \( L^1 \)-closed and therefore \( \sigma(L^1, L^\infty) \)-closed. Since the identity from \( (L^p, \sigma(L^p, L^q)) \) to \( (L^1, \sigma(L^1, L^\infty)) \) is continuous, it follows that \( C^{L^p} \) is \( \sigma(L^p, L^q) \)-closed. That \( C^{L^p} \) is \( \sigma(L^q, L^p) \)-closed works analogously.

(ii) In the case that \( Y = X^* \) is the norm dual of \( X \), the assumptions of Theorem 3.1 are satisfied for the duality pairing \( \langle x, x^* \rangle := x^*(x) \). Indeed, since \( C^X \) is norm-closed and convex, it is also \( \sigma(X, X^*) \)-closed. On the other hand, since \( C^{X^*} \) is the absolute polar of \( C^X \), it follows from the Banach-Alaoglu theorem that \( C^{X^*} \) is \( \sigma(X^*, X) \)-compact and therefore \( \sigma(X^*, X) \)-closed.

Let \( \overline{\mathbb{R}} \) denote the conditional extended real line which is obtained by a conditional completion of the extended real line \( \overline{\mathbb{R}} := \mathbb{R} \cup \{ \pm \infty \} \). Denote by \( \mathcal{V} \) the stable set which induces \( \overline{\mathbb{R}} \). We endow \( X_c \) with the conditional \( \sigma(X_c, Y_c) \)-topology with conditional neighbourhood basis\(^{14}\)

\[ \mathcal{V}(x) = \left\{ r \in \mathbb{R}^{++}, (y_1)_{1 \leq i \leq n} \text{ conditionally finite family in } Y_c \right\} \]

where

\[ \mathcal{V}^{r,x}_{(y_1)_{1 \leq i \leq n}} := \{ z \in X_c : |z - x, y_i|_c \leq r \} \text{ for all } 1 \leq i \leq n \]

for all \( x \) in \( X_c \). Recall that \( f_s(\sum x_i |a_i|) := \sum f(x_i) |a_i| \) for all \( \sum x_i |a_i| \in X_s \).

**Definition 3.3.** We say that a function \( f : X \to \overline{\mathbb{R}} \) is

- **lower semi-continuous** whenever

\[
\forall x \in X \exists V(x) \text{ such that for all } x \in V \subseteq X \left( f(x) \leq \inf \{ f(z) : z \in V \cap X \} \right) \tag{3.2}
\]

- **proper convex** whenever \( \inf_{x \in X} f(x) = f(0) \) for all \( x \in X \) and \( f(0) \) is defined.

We say that a conditional function \( f : X_c \to \overline{\mathbb{R}} \) is

- **conditionally lower semi-continuous** whenever

\[
\forall x \in X_c \exists V(x) \text{ such that for all } x \in V \subseteq X_c \left( f(x) \leq \inf \{ f(z) : z \in V \} \right)
\]

- **conditionally proper convex** whenever \( \inf_{x \in X_c} f(x) = f(0) \) for all \( x \in X_c \) and \( f(0) \) is defined.

**Remark 3.4.** The definition of conditional lower semi-continuity is a conditional version of the classical definition of lower semi-continuity for extended real-valued functions on a topological space, see e.g. [8, Chapter 6, Section 2], and thus is equivalent to each of the following conditions:

\(^{14}\)For the concept of a conditional neighbourhood basis, see [7, Definition 3.14]. A conditionally finite family is defined in [7, Definitions 2.20 and 2.23].

\(^{15}\)We denote by \( V \subseteq X \) the stable set which induces \( V \subseteq \mathbb{R}^{++} \). Let \( \mathcal{V}(x) \) denote the collection of all stable sets \( V \) such that \( \mathcal{V}(x) \subseteq \mathbb{R}^{++} \). Note that \( \mathcal{V}(x) \cap X \neq \emptyset \) for all \( V \in \mathcal{V}(x) \) and \( x \in X \).

\(^{16}\)On the right-hand side of the following inequality \( \mathbb{R} \) is identified with a non-stable subset of \( \mathbb{R} \), see Remark 2.6.
(i) $f_c(x) \leq \liminf f_c(x_n)$ for every conditional net\textsuperscript{17} $(x_n)$ conditionally converging to $x$;

(ii) the conditional lower level set \{ $x$ in $X_c$: $f_c(x) \leq r$ \} is conditionally closed for all $r$ in $\mathbb{R}$.

If the conditional set $X_c$, seen as a conditional subset of $X_e$, is endowed with the conditional relative $\sigma(X_e, Y_e)$-topology,\textsuperscript{18} then the lower semi-continuity property (3.2) expresses the fact that $f_c$ as a conditional function from $X_c$ to $\overline{R}$ is conditionally lower semi-continuous. In particular, $f: X \rightarrow \overline{R}$ is lower semi-continuous if and only if $f(x) \leq \liminf f_s(x_n)$ for every stable net $(x_n)$ in $X_s$ converging to $x$ with respect to the classical topology on $X_s$ related to the conditional topology on $X_a$\textsuperscript{19}.

Let $\mathcal{V}(x)$ denote a classical weak neighbourhood basis of $x \in X$ induced by the dual pair $(X, Y, \langle \cdot, \cdot \rangle)$ and $f: X \rightarrow \overline{R}$ be a function. By inspection, if

$$f(x) = \sup_{V \in \mathcal{V}(x)} \inf \{ f(z): z \in V \}$$

for all $x \in X$, then $f$ is lower semi-continuous in the above sense.

In the context of Boolean-valued analysis, an extension result for strongly lower semi-continuous functions from a Banach space into the space of continuous functions on an extremally disconnected compact space is stated in [13, Theorem 5.4]. The following theorem establishes a conditional extension result for lower semi-continuous functions in the sense of the previous definition which is necessary for the Fenchel-Moreau type vector duality result.

**Theorem 3.5.** Let $f: X \rightarrow \overline{R}$ be lower semi-continuous. Then there exists a conditionally lower semi-continuous function $f_c: X_c \rightarrow \overline{R}$ such that $f_c(x) = f(x)$ for all $x \in X$. Moreover, if $f$ is proper convex, then $f_c$ is conditionally proper convex.

We call $f_c$ a conditionally lower semi-continuous extension of $f$.

**Proof.** We prove that $f_c: X_c \rightarrow \overline{R}$ given by

$$f_c(x) := \sup_{V \in \mathcal{V}(x)} \inf \{ f_s(z): z \in V \cap X_s \}$$

for all $x$ in $X_c$ is a conditionally lower semi-continuous extension of $f$. If $f_c$ were a conditional function, by definition, $f_c(x) = f(x)$ for all $x \in X$ showing that $f_c$ is a conditional extension.\textsuperscript{20} The remaining proof is divided into three steps.

**Step 1:** We show that for all $x$ in $X_c$ and $V \in \mathcal{V}(x)$ there exists a conditional element $x'$ in $X_a$ such that $x'$ is a conditional element in $V \cap X_a$ which implies that $f_c$ is a well-defined conditional function since $f_c$ is a conditional function on the conditional subset $V \cap X_a$ of $X_a$. Indeed, let $x$ be in $X_c$, $(x_m)$ a conditional sequence in $X_a$ conditionally converging to $x$ and $V^{\tau, x}_{(y_i)_{1 \leq i \leq n}} \in \mathcal{V}(x)$. By (3.1), there is $m_1$ in $N_a$ such that $|\langle x_m - x, y_1 \rangle| \leq r$. Suppose $n$ is induced by the step function $\sum n_i|a_i|$. Fix $i$ and choose $m_i \geq m_1$ in $N_a$ such that $|\langle x_m - x, y_i \rangle| \leq r$ for all $l$ induced by step functions of the form $l|a_i + 1|a_i^l$ with $1 \leq l \leq n_i$. Set $m = \sum m_i|a_i|$. By stability, $|\langle x_m - x, y_1 \rangle| \leq r$ for all $1 \leq l \leq n$. Hence $x' := x_m$ in $X_a$ is in $V^{\tau, x}_{(y_i)_{1 \leq i \leq n}}$.

\textsuperscript{17}See [7, Example 2.21].

\textsuperscript{18}See [7, Example 3.9.1].

\textsuperscript{19}For the relation between a classical topology on the stable set $X_a$ and a conditional topology on $X_a$, see [7, Proposition 3.5].

\textsuperscript{20}If there is a conditional element $x$ in $V \cap X_a$, then $V \cap X_s = V \cap X_a$. For more details on the conditional intersection, we refer to the proof of [7, Theorem 2.9].
Step 2: We show that $f_c$ is conditionally lower semi-continuous. Let $x$ be in $X_c$ and $h$ in $R_{++}$. Since $\arctan_c(f_c(x)) - h < \arctan_c(f_c(x))$, there exists $V = V_{(y_1)_{1 \leq i \leq n}}^{r-x} \in \mathcal{V}(x)$ such that

$$\arctan_c(f_c(x)) - h \leq \inf \{ \arctan_c(f_c(z)) : z \in V \cap X_s \}.$$

Let $w$ be in $V_{(y_1)_{1 \leq i \leq n}}^{r/2-x}$ By the conditional triangle inequality, $V_0 \subseteq V$ for $V_0 = V_{(y_1)_{1 \leq i \leq n}}^{r/2,w} \in \mathcal{V}(w)$. From $V_0 \cap X_s \subseteq V \cap X_s$ it follows that

$$\arctan_c(f_c(x)) - h \leq \inf \{ \arctan_c(f_c(z)) : z \in V_0 \cap X_s \}$$

and thus

$$\arctan_c(f_c(x)) - h \leq \arctan_c(f_c(w)).$$

By the previous argument, for each $h$ in $R_{++}$ there exists $V_h \in \mathcal{V}(x)$ such that $\arctan_c(f_c(x)) - h \leq \arctan_c(f_c(w))$ for all $w$ in $V_h$. This implies that

$$\arctan_c(f_c(x)) - h \leq \sup_{V \in \mathcal{V}(x)} \inf \{ \arctan_c(f_c(z)) : z \in V \}.$$

By letting $h \downarrow 0$ and since $\sup_{V \in \mathcal{V}(x)} \inf \{ \arctan_c(f_c(z)) : z \in V \} \leq \arctan_c(f_c(x))$ is trivially satisfied, it follows from the conditional strict monotonicity of $\arctan_c$ that

$$f_c(x) = \sup_{V \in \mathcal{V}(x)} \inf \{ f_c(z) : z \in V \}.$$

Step 3: First, we show that $f_c$ is conditionally convex. For $x, x'$ in $X_c$ and $\lambda$ in $R_a$ with $0 \leq \lambda \leq 1$, it follows from the hypothesis that

$$f_c(\lambda x + (1 - \lambda)x') \leq \lambda f_c(x) + (1 - \lambda)f_c(x').$$

Let $x, x'$ be in $X_c$ and $\lambda$ in $R_a$ with $0 \leq \lambda \leq 1$. Let $V = V_{(y_1)_{1 \leq i \leq n}}^{r,\lambda x + (1 - \lambda)x'} \in \mathcal{V}(\lambda x + (1 - \lambda)x')$, $W = V_{(y_1)_{1 \leq i \leq n}}^{r, x'} \in \mathcal{V}(x')$ and $W' = V_{(y_1)_{1 \leq i \leq n}}^{r, x'} \in \mathcal{V}(x')$. It can be checked that if $z$ is in $W \cap X_s$ and $z'$ is in $W' \cap X_s$, then $\lambda z + (1 - \lambda)z'$ is in $V \cap X_s$. Therefore, one has

$$\inf \{ f_c(z) : z \in V \cap X_s \} \leq \inf \{ f_c(\lambda z + (1 - \lambda)z') : z \in W \cap X_s, z' \in W' \cap X_s \} \leq \inf \{ \lambda f_c(z) + (1 - \lambda)f_c(z') : z \in W \cap X_s, z' \in W' \cap X_s \} = \lambda \inf \{ f_c(z) : z \in W \cap X_s \} + (1 - \lambda) \inf \{ f_c(z') : z' \in W' \cap X_s \}.$$

We found for every $V \in \mathcal{V}(\lambda x + (1 - \lambda)x')$ conditional neighbourhoods $W \in \mathcal{V}(x)$ and $W' \in \mathcal{V}(x')$ such that

$$\inf \{ f_c(z) : z \in V \cap X_s \} \leq \lambda \inf \{ f_c(z) : z \in W \cap X_s \} + (1 - \lambda) \inf \{ f_c(z) : z \in W' \cap X_s \}.$$

By taking the conditional supremum on both sides of the previous conditional inequality, one obtains

$$f_c(\lambda x + (1 - \lambda)x') \leq \lambda f_c(x) + (1 - \lambda)f_c(x').$$

The last conditional inequality holds for $\lambda$ in $R$ with $0 \leq \lambda \leq 1$ by approximating $\lambda$ with step functions in $R_a$ and the conditional lower semi-continuity of $f_c$.

Second, we show that $f_c$ is conditionally proper. Since $f$ is proper and $f_c$ is a conditional extension there is $x_0$ in $X_c$ with $f_c(x_0)$ in $R$. By way of contradiction, suppose there exist $x$ in $X_c$ and $a > 0$ such that $f_c(x)|a = -\infty|a$. By conditional convexity, one has $f_c(\lambda x_0 + (1 - \lambda)x)|a = -\infty|a$ for all $0 \leq \lambda < 1$. Since $\lambda x_0 + (1 - \lambda)x$ conditionally converges to $x_0$ as $\lambda$ conditionally converges to 1, $f_c(x_0)|a = -\infty|a$ due to the conditional lower semi-continuity of $f_c$ which is the desired contradiction. \[\square\]
As an application of the previous results, we obtain the following Fenchel-Moreau type duality for vector-valued functions.

**Theorem 3.6.** Let \( f : X \to \overline{\mathbb{R}} \) be proper convex. Then \( f \) is lower semi-continuous if and only if
\[
 f(x) = \sup_{y \in Y_c} \{ (x, y)_c - f^*(y) \}
\]
for all \( x \in X \) and for the convex conjugate \( f^*(y) := \sup_{x \in X} \{ (x, y)_c - f(x) \} \) for all \( y \in Y_c \).

**Proof.** Suppose \( f \) is lower semi-continuous. By Theorem 3.5, there exists a conditionally proper convex lower semi-continuous extension \( f_c : X_c \to \overline{\mathbb{R}} \). By a conditional version of the fundamental theorem of duality,\(^{21}\) one has
\[
 (X_c, \sigma(X_c, Y_c))^* = Y_c.
\]
By applying a conditional version of strong separation [7, Theorem 5.5], and following the arguments in the proof of the classical Fenchel-Moreau theorem, one obtains
\[
 f_c(x) = \sup_{y \in Y_c} \{ (x, y)_c - f_c^*(y) \}
\]
for all \( x \) in \( X_c \) and for the conditional convex conjugate \( f_c^*(y) := \sup_{x \in X_c} \{ (x, y)_c - f_c(x) \} \) for all \( y \) in \( Y_c \). Fix \( x \in X \). Since
\[
 f^*(y) = \sup_{x \in X} \{ (x, y)_c - f(x) \} \leq \sup_{x \in X_c} \{ (x, y)_c - f_c(x) \} = f_c^*(y)
\]
and \( (x, y)_c - f^*(y) \leq f(x) \) for all \( y \in Y_c \), one has
\[
 f(x) = f_c(x) = \sup_{y \in Y_c} \{ (x, y)_c - f_c^*(y) \}
\leq \sup_{y \in Y_c} \{ (x, y)_c - f^*(y) \} \leq f(x).
\]
Conversely, suppose (3.3) holds. Fix \( x \in X \). We show that
\[
 f(x) = \sup_{V \subseteq \mathcal{V}(x)} \inf \{ f_s(z) : z \in V \cap X_s \}.
\]
One has \( f(x) \geq \sup_{V \subseteq \mathcal{V}(x)} \inf \{ f_s(z) : z \in V \cap X_s \} \) since \( x \in V \cap X_s \) for all \( V \in \mathcal{V}(x) \). By contradiction, suppose there exist \( a > 0 \) and \( \delta \in \mathbb{R}_{++} \) such that \( f(x)|a - \delta|a \geq \sup_{V \subseteq \mathcal{V}(x)} \inf \{ f_s(z) : z \in V \cap X_s \}|a \). By consistency and stability, we can assume that \( a = 1 \). By stability, there exists a stable net \( (x^V = \sum x_i^V |a_i^V \mid V \in \mathcal{V}(x) \) converging to \( x \) such that
\[
 f(x) - \delta \geq f_s(x^V) = \sum f(x_i^V)|a_i^V
\geq \sum_{y \in Y_c} \sup \{ (x_i^V, y)_c - f^*(y) \} |a_i
\geq \sup_{y \in Y_c} \{ (x_i^V, y)_c - f^*(y) \},
\]
\(^{21}\)In [7, Section 5], a conditional version of basic results in functional analysis is established.
where the last inequality follows from

\[
\langle x^V, y \rangle_c - f^*(y) = \sum \langle x_i^V, y \rangle_c a_i - f^*(y) \\
= \sum (\langle x_i^V, y \rangle_c - f^*(y)) a_i \\
\leq \sum \sup_{y \in Y_c} \{ \langle x_i^V, y \rangle_c - f^*(y) \} a_i
\]

for all \( y \in Y_c \). One concludes

\[
f(x) - \delta \geq \langle x^V, y \rangle_c - f^*(y)
\]

for all \( V \in \mathcal{V}(x) \) and \( y \in Y_c \). By passing to the limit and taking the supremum, one has

\[
f(x) - \delta \geq \sup_{y \in Y_c} \{ \langle x, y \rangle_c - f^*(y) \} = f(x)
\]

which is contradictory. □

**Remark 3.7.** The conditional function \( f_c \) is a conditionally proper convex lower semi-continuous extension of \( f \) in the maximal sense, that is, for every conditionally proper convex lower semi-continuous extension \( g \) of \( f \) one has \( g \leq f_c \). In fact, one has \( f_c^* \leq g^* \) similarly to the argument in (3.4), and thus

\[
f_c(x) = \sup_{y \in Y_c} \{ \langle x, y \rangle_c - f_c^*(y) \} \\
\geq \sup_{y \in Y_c} \{ \langle x, y \rangle_c - g^*(y) \} = g(x)
\]

where the last equality follows by a conditional version of the Fenchel-Moreau theorem as argued in the previous proof. ♦

**Remark 3.8.** Let \( X \) be a normed vector space and \( f: X \to \mathbb{R} \) a proper convex function. With similar arguments as in the proofs of Theorem 3.5 and Theorem 3.6, one can show that \( f \) is lower semi-continuous in the sense that

\[
f(x) = \sup_{n \geq 1} \inf \{ f(z) : z \in C^X_r(x) \}
\]

for all \( x \in X \), where \( C^X_r(x) \) is the closed ball of radius \( r \) around \( x \), if and only if \( f \) admits the dual representation

\[
f(x) = \sup_{x^* \in (X^*)_c} \{ \langle x, x^* \rangle_c - f^*(x^*) \}
\]

where \( f^*(x^*) = \sup_{x \in X} \{ \langle x, x^* \rangle_c - f(x) \} \). ♦

**Example 3.9.** Let \( \mathcal{A} \) be the power set algebra of a finite set of cardinality \( d \) and \( f: X \to \mathbb{R}^d \) proper convex and lower semi-continuous. By Theorem 3.6, one obtains the following dual representation

\[
f(x) = \sup_{y \in Y^d} \{ \langle (x, y_1), \ldots, (x, y_d) \rangle - f^*(y) \}
\]

where \( f^*(y) = \sup_{x \in X} \{ \langle (x, y_1), \ldots, (x, y_d) \rangle - f(x) \} \) for all \( y \in Y^d \). ♦
4 Application to vector duality in Bochner spaces

Throughout, let \((X, Y, \langle \cdot, \cdot \rangle)\) be a dual pair of Banach spaces such that \(|\langle x, y \rangle| \leq \|x\| \|y\|\) for all \(x \in X\) and \(y \in Y\). We assume that \(C^X := \{x \in X : \|x\| \leq 1\}\) is \(\sigma(X, Y)\)-closed and \(C^Y := \{y \in Y : \|y\| \leq 1\}\) is \(\sigma(Y, X)\)-closed.

Let \((\Omega, \mathcal{F}, \mu)\) be a complete \(\sigma\)-finite measure space. Identify in \(\mathcal{F}\) two elements the symmetric difference of which is a \(\mu\)-null set, and thus obtain the associated measure algebra \(\mathcal{A}\). The associated measure algebra is a complete Boolean algebra which satisfies the countable chain condition.\(^{22}\) We denote the equivalence classes in \(\mathcal{A}\) by \(a = [A]\) where \(A \in \mathcal{F}\). We write \(1_A\) for the characteristic function of \(A \in \mathcal{F}\). Let \(\bar{\mathcal{L}}^0\) be the collection of all measurable functions \(x : \Omega \to \bar{\mathbb{R}}\) where two of them are identified if they agree almost everywhere. As usual, denote by \(L^0\) the subset of \(\bar{\mathcal{L}}^0\) consisting of real-valued measurable functions. Henceforth, equalities and inequalities between measurable functions are understood in the almost everywhere sense.

The set \(\bar{\mathcal{L}}^0\) induces a conditional set \(\mathcal{L}^0\) of objects

\[
x | a = \{y \in \bar{\mathcal{L}}^0 : x1_A = y1_A \text{ for some } A \in a\}
\]

where \(x \in \bar{\mathcal{L}}^0\) and \(a \in \mathcal{A}\). We denote by \(L^0\) the conditional subset of \(\bar{\mathcal{L}}^0\) induced by the stable subset \(L^0\). By the countable chain condition, the elements in \(p\) have at most countably many non-zero entries. Let \((a_n)\) be a sequence in \(p\) and \((x_n)\) a sequence in \(\bar{\mathcal{L}}^0\). The concatenation of \((x_n)\) along \((a_n)\) is the equivalence class in \(L^0\) of the measurable function \(\sum x_n1_{A_n}\) where \(a_n = [A_n]\) for all \(n\). In [7, Theorem 4.4], it is shown that the conditional set \(L^0\) is conditionally isometrically isomorphic to the conditional real numbers \(\mathbb{R}\) whenever the latter is constructed with respect to the associated measure algebra. The stable order on \(\bar{\mathcal{L}}^0\) corresponds to the order of almost everywhere dominance on \(\mathcal{L}^0\). The infimum and the supremum in \(\bar{\mathcal{L}}^0\) correspond to the essential infimum and the essential supremum in \(L^0\), respectively.

Recall that a function \(x : \Omega \to X\) is strongly measurable if there exists a sequence of simple functions \((x_n)\) such that \(\|x_n - x\| \to 0\) almost everywhere. We denote by \(L^0(X)\) the Bochner space of equivalence classes of all strongly measurable functions with respect to the equivalence relation of almost everywhere equality.\(^{23}\) The Bochner space \(L^0(X)\) induces a conditional set \(L^0(\mathcal{X})\) similarly to (4.1). The following theorem generalizes [7, Theorem 4.4] and [2, Chapter 7, Theorem 7.1] in the Boolean-valued context, respectively.

**Theorem 4.1.** The conditional set \(L^0(\mathcal{X})\) is a conditional Banach space conditionally isometrically isomorphic to \((\mathcal{X}, \| \cdot \|_\omega)\).

**Proof.** By [7, Theorem 4.4], the conditional real line \(\mathbb{R}\) is conditionally isometrically isomorphic to \(L^0\). On \(L^0(\mathcal{X})\) the conditional addition and conditional scalar multiplication are induced by the stable functions \((x + y)(\omega) = x(\omega) + y(\omega)\) and \((\alpha x)(\omega) = \alpha(\omega)x(\omega)\) almost everywhere, respectively. Let \(\| \cdot \|_0 : L^0(\mathcal{X}) \to L^0\) be the conditional function induced by the stable function \(\| \cdot \|_0 : L^0(X) \to L^0\) defined by \(\|x\|_0(\omega) = \|x(\omega)\|\) for almost all \(\omega \in \Omega\). Direct verification shows that \((L^0(\mathcal{X}), \| \cdot \|_0)\) is a conditional normed vector space. We prove that \(L^0(\mathcal{X})\) is conditionally complete. To this end, let \((x_n)\) be a conditional Cauchy sequence in \(L^0(\mathcal{X})\). By induction, choose for every \(k \in \mathbb{N}\) an \(n_k \in \mathbb{N}\) such that \(n_k \geq n_{k-1}\) and \(\|x_n - x_m\|_0 \leq 1/k\) for all \(n, m \geq n_k\). Then \((x_{nk}(\omega))\) is a Cauchy sequence in \(\mathcal{X}\) almost everywhere. By the completeness of \((\Omega, \mathcal{F}, \mu)\) and the completeness of \(X\), the almost everywhere limit \(x(\omega) := \lim_{k \to \infty} x_{nk}(\omega)\) exists in \(L^0(X)\). By the conditional triangle inequality,

\(^{22}\)For more details about the associated measure algebra, see [16, Chapter 22, Section 2].

\(^{23}\)For more details about Bochner spaces, see [6, Chapter II, Section 2].
By inspection, $4.1 \leq 3.6 < 26$ to $3.6$ one has

In particular, the identity does not admit a Fenchel-Moreau type representation of the form $(\cdot, \cdot)$.

Note that the duality pairing $(\cdot, \cdot)$ on $X \times Y$ extends to $L^0(X) \times L^0(Y)$ by defining

$$
\langle x, y \rangle = \lim_{n \to \infty} \langle x_n, y_n \rangle
$$

where $x \in L^0(X)$ is the limit of a sequence of simples functions $(x_n)$ and $y \in L^0(Y)$ is the limit of a sequence of simples functions $(y_n)$. It can be checked that

$$
\langle x, y \rangle = \langle x, y \rangle_c
$$

for all $x \in L^0(X)$ and $y \in L^0(Y)$ where we identified $X_c$ with $L^0(X)$, $Y_c$ with $L^0(Y)$ and $R$ with $L^0$ by Theorem 4.1 and [7, Theorem 4.4].

The stable set $X_s$ can be identified with

$$
\left\{ \sum x_n 1_{A_n} : (x_n) \text{ is partition of } \Omega \text{ in } F \right\} \subseteq L^0(X),
$$

and $R_{++}$ can be identified with $L^0_{++} := \{ r \in L^0 : r > 0 \}$. Recall that a function $f : X \to L^0$ is lower semi-continuous if $(x_n, y_n)_{c}$ converges to $(x, y)$, implies $f(x) \leq \operatorname{ess} \lim_{n} f(x_n)$ for all stable nets in $X_s$ and $y \in Y_c$. The function $f$ is proper convex if $-\infty < f(x)$ for all $x \in X$ and $f(x_0) \in L^0$ for at least one $x_0 \in X$ and $f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$ for all $\lambda \in \mathbb{R}$ with $0 \leq \lambda \leq 1$ and every $x, y \in X$. A reformulation of Theorem 3.6 in the present context is the following main result in vector duality.

**Theorem 4.2.** Let $f : X \to L^0$ be proper convex. Then $f$ is lower semi-continuous if and only if

$$
f(x) = \operatorname{ess} \sup_{y \in L^0(Y)} \{ \langle x, y \rangle - f^*(y) \}
$$

for all $x \in X$ where $f^*(y) = \operatorname{ess} \sup_{x \in X} \{ \langle x, y \rangle - f(x) \}$ for all $y \in L^0(Y)$.

**Proof.** The result is a consequence of Theorem 3.6 and Theorem 4.1.

**Remark 4.3.** Let $(\Omega, \mathcal{F}, \mu)$ be a complete atomless finite measure space, $L^\infty = L^\infty(\Omega, \mathcal{F}, \mu)$ and $L^1 = L^1(\Omega, \mathcal{F}, \mu)$. Consider the identity map

$$
I : L^\infty \to L^0.
$$

Although the lower level set $\{ x \in L^\infty : I(x) = x \leq r \}$ is $\sigma(L^\infty, L^1)$-closed for all $r \in L^0$ due to the Krein–Smulian theorem, the identity is not lower semi-continuous. In fact, for all $x \in L^\infty$ and $V \in \mathcal{V}(x)$ one has

$$
\operatorname{ess} \inf \{ I_s(z) : z \in V \cap L^\infty_s \} = -\infty.
$$

In particular, the identity does not admit a Fenchel-Moreau type representation of the form (4.2). ♦

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24 By Pettis measurability theorem in the form of [6, Chapter II, Section 1, Corollary 3], every strongly measurable function is the almost uniform limit of a sequence $(x_n)$ of countably valued strongly measurable functions. Put $x_n = \sum x_{n_m} |a_m|$ for every $n \in \mathbb{N}$ with $n = \sum n_m |a_m.$ Then $(x_n)$ is a conditional Cauchy sequence in $X_s$.

25 That is, for all $r \in L^0_{++}$ there exists $\alpha_0$ such that $|\langle x_n, x, y \rangle| \leq r$ for all $\alpha \geq \alpha_0$.

26 That is, $\sum x_{\alpha_n} 1_{A_n} = x_{\sum \alpha_n 1_{A_n}}$ for all partitions $(A_n)$ of $\Omega$ in $\mathcal{F}$ and countable subfamilies $(x_{\alpha_n})$ of $(x_n)$.
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