Suppose a regularised functional integral depends holomorphically on a parameter that receives only a finite renormalization. Can one expect the correlation functions to retain the analyticity in the parameter after removal of the cutoff(s)? We examine the issue in the Sinh-Gordon theory by computing the intrinsic 4-point coupling as a function of the Lagrangian coupling $\beta$. Drawing on the conjectured triviality of the model in its functional integral formulation for $\beta^2 > 8\pi$, and the weak-strong coupling duality in the bootstrap formulation on the other hand, we conclude that the operations: “Removal of the cutoff(s)” and “analytic continuation in $\beta$” do not commute.
1. **Introduction:** Weak-strong coupling dualities have been a recurrent theme in quantum field theory; see e.g. [1] for a review. Typically either the weak or the strong coupling regime admits a controllable series expansion, while the other regime is ‘elusive’ even in a non-perturbative formulation, e.g. via a lattice-regularized functional integral. Of course this is what makes the duality interesting in the first place, but it also presents a puzzle as to its precise meaning: If the duality is a feature of an already completely defined theory, one should be able to prove or disprove it, – which appears to be utopian in most cases. If on the other hand it should be thought of as a definition, supplementing an incomplete computational scheme, it is hard to see how a non-perturbative functional integral construction should in principle leave room for such a supplementation.

A mathematically controllable variant of this puzzle occurs in the Sinh-Gordon theory. In its bootstrap formulation it is long known to possess a weak-strong coupling duality, mapping the super-renormalizable $\beta^2 < 8\pi$ regime into the ‘elusive’ $\beta^2 > 8\pi$ one, where $\beta$ is the Lagrangian coupling. On the other hand there are good reasons to expect the theory to be “trivial”, i.e. non-interacting in its functional integral formulation, for $\beta^2 > 8\pi$.

Since “triviality” can be expressed as the vanishing of the intrinsic 4-point coupling $g_r$, it is natural to examine the issue in terms of this quantity. In the case at hand $g_r = g_r(\beta)$ is defined by

$$g_r = -\frac{M^2G(0,0,0,0)}{G(0,0)^2},$$

where $M$ is the mass gap and $G$ are the Green functions of the fundamental field $\phi$. They are related to the Fourier transform of the truncated (connected) Euclidean correlation functions by

$$\langle \phi(k_1)\ldots\phi(k_N)\rangle_T = (2\pi)^2\delta^{(2)}(k_1 + \ldots + k_N)G(k_1,\ldots,k_N),$$

where $k_1 + \ldots + k_N = 0$ is understood in the arguments of $G$. In [2] a technique has been developed to compute $g_r$ non-perturbatively and to high accuracy in any integrable QFT, starting from its exact form factors. Later we shall use this technique to compute $g_r$ in the (bootstrap) Sinh-Gordon model and then return to the above issue. Let us first however lay out the problem in more detail.

2. **Triviality versus duality:** Classically the Sinh-Gordon model is described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{M^2}{\beta^2} \cosh \beta \phi(x),$$
where $M_0$ is a (bare) mass parameter and $\beta$ a coupling constant. The QFT is presumed to describe the scattering of a single massive stable particle of (physical) mass $M$. It can be constructed both via a functional integral formulation and in terms of the form factor bootstrap. As this will be relevant for the intrinsic coupling we briefly review the main features known or expected to hold in both approaches.

In a functional integral formulation the model can be regularized by Wick ordering, say with respect to a UV regularized free propagator of mass $\mu$. One can adopt a scheme in which $\beta$ is not renormalized and the model is then thought to have a (asymptotic) super-renormalizable perturbative expansion in $\beta$, for $\beta^2 < 8\pi$. One will require that physical quantities are invariant under the normal-ordering renormalization group, i.e. that they are annihilated by the differential operator $\mu(\partial/\partial\mu) - (\beta^2/8\pi)M_0(\partial/\partial M_0)$. In particular after removal of the cut-off all Green functions will be functions of $\beta, M_0$ and the ratio $\mu/M_0$, that are invariant in the above sense. Moreover one can show (by so-called tadpole dressing [7, 8]) that the dependence on $\mu/M_0$ enters only through the physical mass $M$. The latter, i.e. its relation to the Lagrangian parameters is known exactly [9]

$$M = \frac{4\sqrt{\pi}}{\Gamma(\frac{1}{2} - \frac{B}{4})\Gamma(1 + \frac{B}{4})} \left[ -\frac{\Lambda \Gamma\left(\frac{B}{2-B}\right)}{16\Gamma\left(-\frac{B}{2-B}\right)} \right]^{(2-B)/4},$$

where $B = \frac{2\beta^2}{8\pi + \beta^2}$, $\Lambda = M_0^2 \mu^{\beta^2/4\pi}$. (3)

Note that $\Lambda$ is renormalization group invariant and $\Lambda^{(2-B)/4} = M_0(\mu/M_0)^{B/2}$ produces the correct mass dimension. As it stands the expression (3) is physically meaningful only for $0 < \beta^2/8\pi < 1$, though it happens to be real also for $2n < \beta^2/8\pi < 2n + 1$, with $n$ a positive integer. The eigenvalues of the infinite set of higher conserved charges [8] exhibit the same feature through their dependence on the exact tadpole function (which can be determined from [3]).

For $\beta^2 > 8\pi$ the status of the model defined through the functional integral approach is not rigorously known, however there are good reasons to expect it to be non-interacting. In particular the intrinsic coupling would then vanish for $\beta^2 > 8\pi$ when one attempts to remove the cutoff(s). The rationale behind this expectation are the properties of two closely related models, the Liouville model [3] and the Sine-Gordon model [4], where constructive QFT techniques have been used to derive triviality results for $\beta^2 > 8\pi$. Specifically the methods used for the exponential interaction [3] can be transferred to the Sinh-Gordon case and allow to construct the model at least for $\beta^2 < 4\pi$; for $\beta^2 > \beta_0$ with $\beta_0$ probably $8\pi$ the infrared regularized model is shown to be non-interacting when the
ultraviolet cutoff is removed. Methods of the rigorous renormalization group were used by Dimock and Hurd [4] to study the Sine-Gordon model: their results suggest triviality of that model for \( \beta^2 > 8\pi \); so far they have not been adapted to the exponential interaction or the Sinh-Gordon model.

In the bootstrap approach on the other hand the model is meant to exist and to be interacting for all \( 0 < \beta^2 < \infty \). In particular the intrinsic coupling, which we are going to compute below, is expected to be non-vanishing for all \( \beta \neq 0 \). As usual the bootstrap construction starts from the exact S-matrix. The S-matrix is purely elastic and is postulated to be

\[
S(\theta) = \frac{\text{sh} \theta - i \sin \frac{\pi}{2} B}{\text{sh} \theta + i \sin \frac{\pi}{2} B},
\]

with \( B \) as in (3). The specific dependence on the Lagrangian coupling \( \beta \) has been tested in two-loop perturbation theory [10]. The (relevant) form factors computed from this S-matrix will be supplied below. Both the S-matrix and the form factors are manifestly invariant under the ‘duality’ transformation

\[
\beta \rightarrow \frac{8\pi}{\beta} \quad \text{or} \quad B \rightarrow 2 - B,
\]

mapping the super-renormalizable regime into the ‘elusive’ \( \beta^2 > 8\pi \) one. In principle the Green functions can be constructed from the form factors so that the QFT defined by the form factor bootstrap will exhibit the duality (3), – provided the physical mass is declared to be a finite invariant numerical parameter (unlike (3) which vanishes for \( \beta^2/8\pi \rightarrow 1^- \)).

In quantities like the intrinsic 4-point coupling \( g_r \) where the mass drops out the latter problem is absent. In the following we shall compute \( g_r \) within the form factor bootstrap and find it to be duality invariant as expected

\[
g_r(B) = g_r(2 - B).
\]

A plot of \( g_r(B) \) versus \( B \) in the bootstrap theory can be found in Fig. 1 below.

3. Computation of the intrinsic coupling: In principle it seems straightforward to compute \( g_r \) from the known form factors of a theory. One simply inserts a resolution of the identity in terms of scattering states, once for the two-point function and three times for the four-point function in (2). The technical problem is that a large number of distributional terms will appear that mask the expected analyticity in the momenta and which also
render the expressions rather unwildly. By a careful rearrangement however partial sums of the distributional terms can be decomposed into a regular term and a singular remainder whose coefficient vanishes identically. In particular the analyticity in the momenta then becomes visible and the zero momentum limit relevant for \([L]\) can be taken. Clearly \(g_r\) will decompose in that way into a fourfold infinite sum over the phase space of the inserted multi-particle states. Remarkably the dominant term, coming from the 1-particle form factor contribution in the two-point function and the 1+3-particle contributions in the 4-point function, typically gives already 98% (!) of the full answer \([2]\). Moreover for this dominant contribution a general model-independent expression in terms of the derivative of the S-matrix and the 1-and 3-particle form factors can be obtained. The general formula can be found in \([2]\); here we need only the case where the bootstrap S-matrix \(S(\theta)\) is scalar, diagonal, and without bound state poles. For the \(n\)-particle form factors of a scalar operator one can then write \(F^{(n)} = F(\theta_n, \ldots, \theta_1)\), where the rapidities \(\theta_j\) parameterize the on-shell momenta of the \(n\)-particle scattering state. The formula for the coupling then reads

\[
g_r = -12i \frac{d}{d\theta} S(\theta) \bigg|_{\theta=0} - 24 \int_0^\infty \frac{du}{4\pi} \left[ -\frac{4}{u^2} + \frac{1}{16\cosh^2 u} |F(i\pi, -u, u)|^2 \right] + \ldots . \tag{7}
\]

Here the dots indicate terms coming from \(n \geq 3\)-particle contributions to the two-point function and from the \(n \geq 5\)-particle contributions to the 4-point function. Further since \(g_r\) does not depend on the normalization of \(F^{(1)}\) we normalized \(F^{(3)}\) relative to \(F^{(1)} \equiv 1\).

To evaluate (7) we now prepare the relevant form factors. The form factors of the Sinh-Gordon model factorize into a universal transcendental function (independent of the local operator considered) and a polynomial remainder. The universal part is essentially a product of minimal form factors

\[
\psi(u) = -iC \sinh u \frac{u}{2} \exp \left\{ -2 \int_0^\infty \frac{ds}{s} \frac{\sinh(B-1)s}{\sinh 2s} \sin^2 \frac{s}{2\pi} (i\pi - u) \right\} =: -i\sinh u \frac{u}{2} \psi_0(u) . \tag{8}
\]

\(\psi(u)\) is analytic in the strip \(0 < \text{Im} \, u < \pi\), has a simple zero at \(u = 0\) and no others in the strip of analyticity. The normalization constant \(C = \psi(i\pi)\) is real and is chosen such that \(\psi(u) \to 1\) for \(u \to \pm\infty\). As indicated it is sometimes convenient to split off the sh-term and work with \(\psi_0(u)\). It is roughly ‘bell-shaped’, decays exponentially for \(u \to \pm\infty\) and obeys

\[
\psi_0(u + i\pi)\psi_0(u) = \frac{2i}{\sinh u + i\sin \frac{u}{2}B} . \tag{9}
\]
The form factors of the scalar field \( \phi \) then are \( F^{(1)}(\theta) = 1 \) and

\[
F^{(n)}(\theta) = \left( \frac{8 \sin \frac{n}{2} B}{\psi(i\pi)} \right)^{\frac{n-1}{2}} h^{(n)}(t) \prod_{k>l} \frac{\psi(\theta_{kl})}{t_k + t_l}, \quad n \text{ odd, } n \geq 3.
\]

Here \( t_j = e^{i\theta_j}, \ j = 1, \ldots, n, \) and \( h^{(n)}(t) \) is a symmetric polynomial in \( t_1, \ldots, t_n \) of total degree \( n(n-1)/2 \) and partial degree \( n - 2 \). They are conveniently expressed in terms of elementary symmetric polynomials \( \sigma_k^{(n)}, \ k = 1, \ldots, n. \) For example

\[
h^{(1)}(t) = 1, \quad h^{(3)}(t) = \sigma_3, \quad h^{(5)}(t) = \sigma_5[\sigma_2\sigma_3 - 2\cos \frac{\pi}{2} B \sigma_5],
\]

where we suppressed the superscripts \( (n) \) on the right hand side. Generally \( h^{(n)}(t) \) contains \( \sigma_n \) as a factor. In particular the 3-particle form factor more explicitly reads

\[
F^{(3)}(\theta) = \frac{i \sin \frac{3}{2} B}{\psi(i\pi)} \psi_0(\theta_{31}) \psi_0(\theta_{32}) \psi_0(\theta_{21}) \theta_{31} \frac{\theta_{32}}{2} \theta_{21} \frac{\theta_{21}}{2}.
\]

Inserting (12) and (4) into (7) one arrives at the result

\[
g_r = -\frac{24}{\sin \frac{3}{2} B} - \frac{3}{2\pi} \int_0^\infty du \left\{ \frac{\text{th}_2^2 \text{coth}^4 \frac{u}{2}}{\text{ch}_u^2} \Psi(u) - \frac{16}{u^2} \right\} + \ldots,
\]

where

\[
\Psi(u) = \left| \begin{array}{ccc}
\psi_0(2u) & \psi_0(i\pi - u) & \psi_0(i\pi + u) \\
\psi_0(0) & \psi_0(i\pi) & \psi_0(i\pi) \\
\end{array} \right|^2
= \exp \left\{ -2 \int_0^\infty \frac{dt}{t} \left( \frac{\text{ch}(B-1)t}{\text{ch}_t^2} + \frac{\text{ch}(B-1)t}{\text{ch}_t^2} \right) \frac{1 - \cos \frac{2u}{\pi}}{\text{sh}_t} \right\}.
\]

Fig. 1 shows a plot of \( g_r(B) \) versus \( B \), which exhibits the announced weak-strong coupling duality (6). For small \( B \) the behavior is linear and the slope can be evaluated analytically* 

\[
g_r = (9\pi/2 + \ldots) B + O(B^2).
\]

The dots indicate contributions to the slope coming from the subleading terms in (7).

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*I thank P. Weisz for pointing this out.
On the other hand perturbation theory (PT) predicts $g_r(B)_{\text{PT}} = 4\pi B + O(B^2)$, as $B = \beta^2/4\pi + O(\beta^4)$. The leading term in (7) thus gives only a crude approximation to the slope. Of course the series (7) has been designed to rapidly converge pointwise in $B$, so ‘misusing’ it to extract the coefficients of an asymptotic expansion in $B$ one cannot expect the convergence of the coefficients to be equally rapid. Numerically $g_r(B)$ computed via (13) and $g_r(B)_{\text{PT}}$ give very similar results for $0 < B < 0.1$. In particular a linear fit on the $0 < B < 0.1$ segment of Fig. 1 would give a ‘mock slope’ close to $4\pi$. Generally speaking this illustrates the difficulty to extract the coefficients of an asymptotic expansion from numerical data without having some analytical control over the remainders. Of course next-to-leading contributions could in principle be computed, both in (13) and in PT, but the principle problem would remain. For completeness let us also list the numerical values for $g_r(B)$ in (13) at a few points with $B \geq 0.1$: $g_r(0.1) = 1.270$, $g_r(0.25) = 2.725$, $g_r(0.5) = 4.291$, $g_r(0.75) = 5.107$, $g_r(1) = 5.362$. Guided by the experience with other models [2] we expect them to amount to about 98% of the exact answer.

4. Removal of the cutoff(s) versus analytic continuation: Let us now return to the relation between the functional integral and the bootstrap formulation of the theory. The suspected discrepancy in the $\beta^2 > 8\pi$ behavior is most likely due to the fact that the operations: “Removal of the cutoff(s) in the moments of the functional measure” and “analytic continuation in $\beta$” do not commute. A useful analogy is the behavior of the normalized moments of the ‘Euler integral’ measure, $\langle t^n \rangle_z = \Gamma(z)^{-1} \int_0^\infty dt e^{-t z^{-1+n}}$. Initially defined for Re $z > 0$, they can be analytically extended to Re $z < 0$, and are entire

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*† due to E. Seiler (private communication).*
functions: \( \langle t^N \rangle_z = \prod_{j=0}^{N-1} (z + j) \). On the other hand the integral representation breaks down for \( \text{Re} \, z < 0 \), but can be restored by introducing a cutoff. The moments of the regularized measure

\[
\langle t^N \rangle_{z,l} = \frac{1}{\Gamma(z, 1/l)} \int_1^\infty dt \, e^{-t} t^{z-1+N},
\]

(i.e. a ratio of incomplete Gamma functions) then obey

\[
\lim_{l \to \infty} \langle t^N \rangle_{z,l} = \langle t^N \rangle_z, \quad \text{Re} \, z > 0, \quad \text{but}
\]

\[
\lim_{l \to \infty} \langle t^N \rangle_{z,l} = 0, \quad \text{Re} \, z < 0.
\]

(16)

In fact one can imitate this phenomenon in the bootstrap approach to the Sinh-Gordon theory by describing it as the \( L \to \infty \) limit of auxiliary continuum bootstrap models \( \text{ShG}_L \). This modified bootstrap construction leads to a trivial S-matrix and a vanishing intrinsic coupling at \( B = 1 \). To describe the auxiliary models let \( B_L \) denote the first \( L + 1 \) terms in a Taylor expansion of \( B \) around \( \beta^2/8\pi = 0 \), i.e.

\[
B_L = 2 \left[ \frac{B}{2 - B} - \left( \frac{B}{2 - B} \right)^2 + \ldots + (-1)^L \left( \frac{B}{2 - B} \right)^{L+1} \right], \quad B < 1.
\]

(17)

Let \( S_L(\theta) \) denote the bootstrap S-matrix \( (1) \) with \( B \) replaced by \( B_L \). This coupling dependence of the S-matrix would also pass an \( L \)-loop perturbation theory test, taking for granted that \( (4) \) does. In contrast to \( (4) \) however one has

\[
\lim_{B \to 1^-} S_L(\theta) = 1, \quad \forall L \geq 1, \quad \text{where} \quad S_L(\theta) = \frac{\text{sh} \theta - i \sin \frac{\pi}{2} B_L}{\text{sh} \theta + i \sin \frac{\pi}{2} B_L}.
\]

(18)

Let further \( \text{ShG}_L \) denote the QFT defined by the form factor bootstrap based on the S-matrix \( (18) \). Since the S-matrix is trivial for \( B \to 1^- \) one expects the intrinsic coupling to vanish

\[
\lim_{B \to 1^-} g_{\eta}(B)_{\text{ShG}_L} = 0, \quad \forall L \geq 1.
\]

(19)

This is supported by the explicit computation. The intrinsic couplings of the \( \text{ShG}_L \) models with \( L = 1, 4, 17 \) are shown in Fig. 2.

For \( B > 1 \) the S-matrix \( (18) \) develops a pole in the physical strip and is thus inadequate for the description of a single particle theory. In view of \( (19) \) it is natural to extend

\footnote{Note that for \( B \to 1^- \) the ingredients of the standard bootstrap description degenerate: The S-matrix develops a pole on the boundary of the physical strip and Zamolodchikov’s mass function \( (3) \) (based on the thermodynamic Bethe ansatz) vanishes.}
the bootstrap definition of the ShG\(_L\) theory by taking \(S_L(\theta) = 1\) for \(B > 1\). For large \(L\) the ShG\(_L\) theory then describes a QFT that is practically indistinguishable from the (standard bootstrap) Sinh-Gordon theory for \(B < 1\), but which continuously interpolates to the trivial theory for \(B > 1\). For \(L \to \infty\) the \(B < 1\) match with the conventional bootstrap theory becomes exact and the transition to the trivial \(B > 1\) theory becomes discontinuous. The mechanism is similar to the one in the functional integral approach: The \(L \to \infty\) limit and analytic continuation in \(B\) do not commute. In particular this illustrates, by way of an alternative, that nothing in the bootstrap formalism itself enforces the duality (5), (6). Strictly speaking it has the status of a definition.

In summary we arrive at the following scenario: Both the functional integral and the bootstrap approach most likely provide consistent non-perturbative constructions of the Sinh-Gordon theory, which for \(\beta^2 < 8\pi\) coincide. Each of the construction schemes suggests a natural way to define an extension into the \(\beta^2 > 8\pi\) regime, which however are drastically different: Trivial in the former and ‘dual’ to the \(\beta^2 < 8\pi\) theory in the latter case. The discrepancy can be understood in terms of the non-interchangeability of a (regularizing) limiting procedure and analytic continuation in \(\beta\).

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