ON GROTHENDIECK’S SECTION CONJECTURE FOR ORBICURVES

GIULIO BRESCIANI

ABSTRACT. As already noted by Borne-Emsalem in [BE14, Conjecture 2], there is a natural generalization of the section conjecture for proper orbicurves. Combined with the reformulation by Borne-Vistoli [BV15, §9] of the conjecture in terms of the étale fundamental gerbe, this suggests an even stronger conjecture for orbicurves, asking an equivalence of categories instead of a mere bijection. In the following, we prove that the three versions of the conjecture are in fact equivalent, and that “injectivity” (i.e. full faithfulness) holds in the case of orbicurves. As a byproduct, we obtain a new proof of the fact that the section conjecture for proper curves implies the section conjecture for open curves.

CONTENTS

1. Introduction
2. Acknowledgements
3. Fundamental groups vs. fundamental gerbes
4. Stacky going up and going down theorems
5. Orbicurves
6. Equivalence of the conjectures and full faithfulness
7. Open curves as limits of orbicurves
8. Appendix. Étale fundamental gerbes
References

1. INTRODUCTION

Let $X$ be a quasi-compact, geometrically connected scheme over a field $k$ of characteristic 0. Fix an algebraic closure $\bar{k}/k$ and a geometric point $\bar{x}: \text{Spec } \Omega \to X$. There is short exact sequence of étale fundamental groups

$$1 \to \pi_1(X_{\bar{k}}, \bar{x}) \to \pi_1(X, \bar{x}) \to G_k \to 1$$

where $G_k$ is the absolute Galois group of $\bar{k}/k$. By functoriality, a rational point of $X$ gives a section of this exact sequence, well defined up to conjugacy by an element of $\pi_1(X_{\bar{k}}, \bar{x})$. If we denote by $\text{Hom-Ext}_{G_k}(G_k, \pi_1(X, \bar{x}))$ the set of equivalence classes of sections, we thus obtain a map

$$X(k) \to \text{Hom-Ext}_{G_k}(G_k, \pi_1(X, \bar{x})).$$

In his famous letter to Faltings, Grothendieck stated the so called section conjecture:
Conjecture 1.1 (Grothendieck’s section conjecture). If $k$ is finitely generated over $\mathbb{Q}$ and $X$ is a proper, smooth, geometrically connected curve over $k$ of genus at least 2, then

$$X(k) \to \text{Hom-ext}_{G_k}(G_k, \pi_1(X, \bar{x}))$$

is bijective.

The injectivity part was already known to Grothendieck, and is a consequence of the Mordell-Weil theorem (see for instance \[Sti08, Appendix B\]).

In \[BV15, \S 8,9\], Niels Borne and Angelo Vistoli gave a new interpretation of the conjecture in terms of étale fundamental gerbes. The étale fundamental gerbe of a geometrically connected fibered category $X$ is a pro-étale gerbe $\Pi_{X/k}$ with a morphism $X \to \Pi_{X/k}$ which is universal among morphisms into finite étale stacks (see \[BV15\]). If we were doing topology, $\Pi_{X/K}$ would be the classifying space $B\pi_1$.

Moreover, there is a natural equivalence

$$\Pi_{X/k}(k) \xrightarrow{\sim} \text{Hom-ext}_{G_k}(G_k, \pi_1(X, \bar{x}))$$

whose composition with $X(k) \to \Pi_{X/k}(k)$ is the section map constructed above, see \[BV15, Proposition 9.3\]. Hence, we can reformulate the section conjecture by asking that $X \to \Pi_{X/k}$ induces a bijection between isomorphism classes of rational points.

Consider now a proper orbicurve $\mathcal{X}$ over a field $k$ of characteristic 0 (see \[BE14\] for a precise definition). If $k = \mathbb{C}$, $\mathcal{X}$ is a Riemann surface with an orbifold structure over a finite number of points, in general it is a proper, smooth Deligne-Mumford stack of dimension 1. It is possible to define an Euler characteristic $\chi(\mathcal{X}) \in \mathbb{Q}$ which coincides with the classical one if $\mathcal{X}$ is just a curve. The orbicurve $\mathcal{X}$ is hyperbolic (resp. elliptic, parabolic) if $\chi(\mathcal{X}) < 0$ (resp. $= 0$, $> 0$).

In \[BE14, Conjecture 2\], Borne-Emsalem conjectured the following

Conjecture 1.2 (Orbicurves section conjecture). If $k$ is finitely generated over $\mathbb{Q}$ and $\mathcal{X}$ is a proper, hyperbolic orbicurve over $k$ with a geometric point $\bar{x}$, then

$$\mathcal{X}(k) \to \text{Hom-ext}_{G_k}(G_k, \pi_1(\mathcal{X}, \bar{x}))$$

is bijective on isomorphism classes.

However, if we replace $\text{Hom-ext}$ with $\Pi_{\mathcal{X}/k}(k)$, both $\mathcal{X}(k)$ and $\Pi_{\mathcal{X}/k}(k)$ have a natural structure of small category with only isomorphisms rather than that of a set. Hence, it is natural to conjecture the following:

Conjecture 1.3 (Stacky section conjecture). If $k$ is finitely generated over $\mathbb{Q}$ and $\mathcal{X}$ is a proper, hyperbolic orbicurve over $k$, then

$$\mathcal{X}(k) \to \Pi_{\mathcal{X}/k}(k)$$

is an equivalence of categories.

A priori, the latter is stronger than the one of Borne-Emsalem, which in turn is stronger than Grothendieck’s conjecture. The main result of this paper is that in fact they are all equivalent.

**Theorem 6.3** Conjectures 1.1, 1.2 and 1.3 are equivalent.

Moreover, we generalize the classical result about injectivity of the section map for elliptic and hyperbolic curves.
**Theorem 6.4** Let $k$ be a finitely generated extension of $\mathbb{Q}$, and $X$ a proper orbicurve. Then $\mathcal{X}_k(k') \to \Pi_{\mathcal{X}_k}(k')$ is fully faithful for every finitely generated $k'/k$ if and only if $\chi(X) \leq 0$.

As a byproduct, building on an idea of Borne-Emsalem we give a clear picture of the section conjecture for open (orbi)curves, see §7. This allows us to give a new proof of the fact that the section conjecture for proper curves implies the one for open curves, see Theorem 7.1.

2. **Acknowledgements**

I would like to thank my advisor Angelo Vistoli for his constant support and for the sheer amount of time he spent teaching me algebraic geometry and correcting my mistakes. I would also like to thank Tamás Szamuely for the countless number of answers he gave me about the current state of the section conjecture, and Fabio Tonini for many useful remarks. Finally, I would like to thank an anonymous referee for many useful corrections and observations.

3. **Fundamental Groups vs. Fundamental Gerbes**

We discuss technical details about the étale fundamental gerbe in the appendix §8 at the end of the article. Here we recall briefly what it is and how to compare it with the étale fundamental group.

- If $X$ is a geometrically connected fibered category (for example a geometrically connected scheme, or a geometrically connected algebraic stack), the étale fundamental gerbe exists and it is a pro-étale gerbe with a morphism $\pi_X : X \to \Pi_{X/k}$ (see Theorem 8.13).
- $\pi_X : X \to \Pi_{X/k}$ satisfies a universal property, i.e. it is universal among morphism to finite étale stacks,
- if we fix a rational base point $x \in X(k)$, then $\Pi_{X/k}$ is the classifying stack $B\pi_1(X,x)$ of the étale fundamental group scheme $\pi_1(X,x)$,
- there is a natural isomorphism $\Pi_{X/k}(k) \simeq \text{Hom-ext}_{G_k}(G_k, \pi_1(X,\bar{x}))$, $\Pi_{X/k}$ behaves well with respect to finite, separable extensions $L/k$ of the base field, i.e. $\Pi_{X/L}(k) \simeq \Pi_{X/k} \times_{\text{Spec} k} \text{Spec} L$ (see Proposition 8.14).

The étale fundamental gerbe contains essentially the same information as the étale fundamental group with its natural morphism to $\text{Gal}(k,\bar{k})$. There are some advantages of passing from $\pi_1(X,\bar{x})$ to $\Pi_{X/k}$:

- $\Pi_{X/k}$ has naturally a richer structure of fpqc stack,
- $\pi_X : X \to \Pi_{X/k}$ is an actual morphism, not only a function of sets as $X(k) \to \text{Hom-ext}_{G_k}(G_k, \pi_1(X,\bar{x}))$,
- $\Pi_{X/k}$ does not depend on a base point.

**Remark 3.1.** Since the étale fundamental gerbe is the quotient stack of Deligne’s fundamental groupoid (§BV13. Theorem 8.3), our point of view is similar to the one used in [EH08], but the fact of passing to the quotient gerbe allows us to avoid Tannakian formalism in favour of the one of stacks.
If $\bar{x} \in X(\bar{k})$ is a geometric point, we can reconstruct the étale fundamental group of $X_\bar{k}$ simply as the automorphism group of the image of $\bar{x}$ in $\Pi_{X/\bar{k}}$:

$$\pi_1(X_\bar{k}, \bar{x}) \simeq \text{Aut}_{\Pi_{X/\bar{k}}}(\pi_X(\bar{x})).$$

Moreover, if the base point $\bar{x} \in X(\bar{k})$ comes from a rational point $x \in X(k)$, then we have a natural action by group automorphisms of $\text{Gal}(\bar{k}/k)$ on $\text{Aut}_{\Pi_{X/\bar{k}}}(\bar{x})$ and we can reconstruct the étale fundamental group as

$$\pi_1(X, \bar{x}) \simeq \text{Aut}_{\Pi_{X/k}}(\bar{x}) \rtimes \text{Gal}(\bar{k}/k).$$

We can also reconstruct naturally (without the assumption on the base point) the set $\text{Hom-\text{ext}}_{Gk}(G_k, \pi_1(X, \bar{x}))$. In fact, there is a natural bijective map

$$\Pi_{X/k}(k) \to \text{Hom-\text{ext}}_{Gk}(G_k, \pi_1(X, \bar{x}))$$

whose composition with $\pi_X : X(k) \to \Pi_{X/k}(k)$ is the section map, see [BV15, Proposition 9.3].

Hence, working with $\Pi_{X/k}$ is equivalent to working with $\pi_1(X, \bar{x})$ with its projection to $\text{Gal}(\bar{k}/k)$, but $\Pi_{X/k}$ is defined in greater generality and gives a better point of view for many purposes. In fact, it doesn’t depend on a base point and suggests naturally a lot of geometric constructions which are more complex from the point of view of the étale fundamental group.

For example, if $s : \text{Spec} k \to \Pi_{X/k}$ is a section, the decomposition tower of $s$ (see [Sti13 §4.4]) is simply the fiber product $\tilde{X} = \text{Spec} k \times_{\Pi_{X/k}} X$. Something that was a beautiful idea becomes just a natural and obvious operation: this shows the power of the formalism.

4. STACKY GOING UP AND GOING DOWN THEOREMS

We want now to establish a stacky version of the "going up" and "going down" theorems along étale covers, see [Sti13, Propositions 110, 111]. We also want to stress that in order to get these results in the stacky case it is essential to look at the whole structure of categories of $X(k)$ and $\Pi_{X/k}(k)$, rather than only their sets of isomorphism classes.

Remark 4.1. In [Sti13, Propositions 110, 111], there are hypotheses on the so called centralizers of sections. If $s \in \Pi_{X/k}(k)$ corresponds to a section $s : \text{Gal}(\bar{k}/k) \to \pi_1(X, \bar{x})$, the centralizer of $s$ is the subgroup of elements of $\pi_1(X_\bar{k}, \bar{x})$ centralizing the image of $s$. We don’t need them, since the notion of centralizer of a section (see [Sti13 §3.3]) fits nicely in our point of view without any additional work.

In fact, if we think of $\pi_1(X_\bar{k}, \bar{x})$ as

$$\text{Aut}_{\Pi_{X/\bar{k}}}(\pi_X(\bar{x})) = \text{Aut}_{\Pi_{X/\bar{k}}}(\pi_X(\bar{x}))(\bar{k}) \simeq \text{Aut}_{\Pi_{X/\bar{k}}}(s)(\bar{k}),$$

then the centralizer of $s$ is the group of elements of $\text{Aut}_{\Pi_{X/\bar{k}}}(s)(\bar{k})$ which satisfy Galois descent, i.e. rational points of the group scheme $\text{Aut}_{\Pi_{X/\bar{k}}}(s)$. Hence saying that a section $s \in \Pi_{X/k}(k)$ has trivial centralizer simply means that $\text{Aut}_{\Pi_{X/\bar{k}}}(s)$ has no rational points apart from the identity.

Proposition 4.2 (Going up). Let $X, Y$ be geometrically connected fibered categories and $f : Y \to X$ a representable, finite étale morphism. The following are true:

(i) If $X(k) \to \Pi_{X/k}(k)$ is fully faithful, then $Y(k) \to \Pi_{Y/k}(k)$ is fully faithful, too.
(ii) If \( X(k) \to \Pi_{X/k}(k) \) is an equivalence, then \( Y(k) \to \Pi_{Y/k}(k) \) is an equivalence, too.

**Proof.** By Proposition 8.16 the 2-commutative diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{\pi_Y} & \Pi_{Y/k} \\
\downarrow f & & \downarrow \Pi_f \\
X & \xrightarrow{\pi_X} & \Pi_{X/k}
\end{array}
\]

is actually 2-cartesian.

(i) Since \( X(k) \to \Pi_{X/k}(k) \) is fully faithful, its base change

\[
Y = X(k) \times_{\Pi_{X/k}(k)} \Pi_{Y/k}(k) \to \Pi_{Y/k}(k)
\]

is fully faithful too.

(ii) Since \( X(k) \to \Pi_{X/k}(k) \) is an equivalence, its base change

\[
Y = X(k) \times_{\Pi_{X/k}(k)} \Pi_{Y/k}(k) \to \Pi_{Y/k}(k)
\]

is an equivalence too.

\[\square\]

**Definition 4.3.** Let \( C, D \) be categories and \( f : C \to D \) a functor, \( p \in C \) an object. We say \( f \) is fully faithful at \( p \) if \( \text{Aut}_C(p) \to \text{Aut}_D(f(p)) \) is bijective.

**Remark 4.4.** Suppose that \( C, D \) are small categories in which all morphism are isomorphisms. For example, \( X(S) \) has this form for every stack \( X \) and every scheme \( S \). A functor \( f : C \to D \) is fully faithful if and only if it is fully faithful at every point and is injective on isomorphism classes.

**Lemma 4.5 (Extension of the base field).** Let \( f : A \to B \) be a morphism of fibered categories over \( k \) which are stacks in the étale topology, and \( L/k \) a finite Galois extension. Then the following are true.

(i) Let \( a \in A(k) \) be a rational point, \( a_L \in A(L) \) the pullback of \( a \). If \( A(L) \to B(L) \) is fully faithful at \( a_L \) over \( L \), then \( A(k) \to B(k) \) is fully faithful at \( a \).

(ii) If \( A(L) \to B(L) \) is fully faithful, then \( A(k) \to B(k) \) is fully faithful, too.

(iii) Let \( b \in B(k) \) be a rational point, and suppose that \( A(L) \to B(L) \) is fully faithful. Then \( b \) is in the essential image of \( A(k) \to B(k) \) if and only if \( b_L \) is in the essential image of \( A(L) \to B(L) \).

(iv) If \( A(L) \to B(L) \) is an equivalence, then \( A(k) \to B(k) \) is an equivalence, too.

**Proof.** (i) We have a commutative diagram

\[
\begin{array}{ccc}
\text{Aut}_A(a) & \longrightarrow & \text{Aut}_B(f(a)) \\
\downarrow & & \downarrow \\
\text{Aut}_A(a_L) & \sim & \text{Aut}_B(f(a_L))
\end{array}
\]

where the vertical arrows are injective, and the lower arrow is bijective by hypothesis. Both \( A \) and \( B \) are stacks in the étale topology, hence the Isom functors are sheaves and satisfy Galois descent. This means that the groups in the upper row are just the \( \text{Gal}(L/k) \)-invariant elements of the groups in
the lower row. Since the lower horizontal arrow is clearly equivariant, we get that the upper horizontal row is bijective, too.

(ii) Thanks to point (i), $A(k) \to B(k)$ is fully faithful at every point. If $a, a' \in A(k)$ are such that $f(a) \simeq f(a')$, then $f(a_L) \simeq f(a'_L)$ and hence $a_L \simeq a'_L$. Since we have already identified automorphisms groups, $a_L \simeq a'_L$ descends to $a \simeq a'$, hence $A(k) \to B(k)$ is fully faithful.

(iii) The “only if” part is obvious. Now suppose that $b_L \simeq f(a')$ is in the essential image of $A(L) \to B(L)$. For every $\sigma \in \text{Gal}(L/k)$, we have an isomorphism

$$\psi_\sigma : \sigma^* f(a') \simeq \sigma^* b_L = b_L \simeq f(a')$$

which corresponds to an isomorphism $\psi_{\sigma^*} : \sigma^* (a') \simeq a'$ since $A(L) \to B(L)$ is fully faithful by hypothesis.

Now, we have $\psi_{\sigma^*} = \psi_\sigma \circ \sigma^* \psi_\sigma$ by direct computation. Since $A(L) \to B(L)$ is fully faithful, this means that we also have $\psi_{\sigma^*} = \psi_\sigma \circ \sigma^* \psi_\sigma$ and hence by Galois descent there exists $a \in A(k)$ such that $a_L \simeq a'$. Let us check that $f(a) \simeq b$.

We have a chain of isomorphisms

$$f(a)_L = f(a_L) \simeq f(a') \simeq b_L,$$

we have to check that this is Galois invariant. This amounts to the fact that, by definition, $f(\psi_\sigma) = \psi_\sigma$.

(iv) This is a direct consequence points (ii) and (iii).

\[ \square \]

In the following, we will use without mention the fact that, if $X$ is a geometrically connected fibered category and $L/k$ is a finite, separable extension, the natural morphism $\Pi_{X_L/L} \to \Pi_{X/k} \times_k L$ is an isomorphism (see Proposition 8.14).

**Proposition 4.6 (Going down).** Let $X$ and $Y$ be geometrically connected fibered categories which are stacks in the étale topology, and $f : Y \to X$ a representable, finite étale morphism. The following are true:

(i) If $Y_L(L) \to \Pi_{Y_L(L)}$ is fully faithful for every finite, separable extension $L/k$, then $X(k) \to \Pi_{X/k}(k)$ is fully faithful.

(ii) If $Y_L(L) \to \Pi_{Y_L(L)}$ is an equivalence for every finite, separable extension $L/k$, then $X(k) \to \Pi_{X/k}(k)$ is an equivalence.

**Proof.** As in Proposition 4.2, we are going to use the fact that the 2-commutative diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{\pi_Y} & \Pi_{Y/k} \\
\downarrow f & & \downarrow \Pi_f \\
X & \xrightarrow{\pi_X} & \Pi_{X/k}
\end{array}
$$

is 2-cartesian, see Proposition 8.16. However, the proofs will be much more complex: the main problem is that, while a $k$-rational point of $Y$ defines a $k$-rational point of $X$, the converse is not true, hence we will need to enlarge the base field and then use Galois descent to get back to $k$.

(i) First, let us check that $X(k) \to \Pi_{X/k}(k)$ is fully faithful at every point, next we will show that it is injective on isomorphism classes. Choose $p \in X(k)$, since $Y \to X$ is finite étale there exists a finite Galois extension $L$ and a point
Suppose that we have an isomorphism 
\[ \prod_X \simeq Y, \]
and we also know that 
\[ \text{Aut}_Y(p') \simeq \text{Aut}_{\Pi_Y/k}(\pi_Y(p')). \]
by hypothesis. In particular, \( \text{Aut}_X(p) \to \text{Aut}_{\Pi_X/k}(\pi_X(p)) \) is injective: let us show that it is surjective, too.

Suppose that \( g \in \text{Aut}_{\Pi_X/k}(\pi_X(p)) \) does not come from an element of \( \text{Aut}_X(p) \), thanks to Proposition 8.16 the triple 
\[ (p, g, \pi_Y(p')) \]
gives us a point \( p'' \in Y(k) \) such that \( \pi_Y(p'') \simeq \pi_Y(p') \) and \( f(p'') \simeq p \). Now the fact that \( g \) does not come from \( \text{Aut}_X(p) \) means exactly that the isomorphism \( f(p'') \simeq p \simeq f(p') \) does not lift to an isomorphism \( p'' \simeq p' \), but we have that \( \pi_Y(p'') \simeq \pi_Y(p') \) is a lifting of \( \pi_X(f(p'')) \simeq \pi_X(f(p')) \), which is absurd since \( Y(k) \to \Pi_{Y/k}(k) \) is fully faithful by hypothesis.

Let us check now that \( X(k) \to \Pi_{X/k}(k) \) is injective on isomorphism classes. Suppose that we have an isomorphism \( \alpha : \pi_X(p) \simeq \pi_X(q) \) for some \( p, q \in X(k) \). As before, if we can find a finite Galois extension \( L \) and an isomorphism \( \varphi : p_L \simeq q_L \) such that \( \pi_{X_L}(\varphi) = \alpha_L \), then \( \varphi \) descends to an isomorphism \( p \simeq q \). In fact, if \( \sigma \in \text{Gal}(L/k) \),
\[ \pi_{X_L}(\sigma^* \varphi) = \sigma^* \pi_{X_L}(\varphi), \]
and this implies that \( \pi_{X_L}(\sigma^* \varphi \circ \varphi^{-1}) = \text{id}_{\pi_{X_L}(p_L)} \) and hence \( \sigma^* \varphi \circ \varphi^{-1} = \text{id}_{p_L} \) because we already know that \( \pi_{X_L} \) is fully faithful at every point.

Hence, up to a finite Galois extension we may suppose that there exists \( p' \in Y(k) \) with \( f(p') = p \). Since 
\[ \Pi_f(\pi_Y(p')) = \pi_X(f(p')) = \pi_X(p) \simeq \pi_X(q) \]
and thanks to Proposition 8.16, there exists a point \( q' \in Y(k) \) such that \( \pi_Y(q') \simeq \pi_Y(p') \) and \( f(q') \simeq q \). Now since \( Y(k) \to \Pi_{Y/k}(k) \) is fully faithful by hypothesis and \( \pi_Y(p') \simeq \pi_Y(q') \), we get an isomorphism \( p' \simeq q' \) which induces an isomorphism \( p \simeq q \) as desired.

(ii) This is a direct application of point (i) and Lemma 4.5(iii), plus the observation that every section \( \text{Spec} k \to \Pi_{X/k} \) lifts to a section of \( \Pi_{Y/k} \) up to a finite, separable field extension: in fact, \( \text{Spec} k \times_{\Pi_{X/k}} \Pi_{Y/k} \) is a finite étale scheme. To check that \( \text{Spec} k \times_{\Pi_{X/k}} \Pi_{Y/k} \) is finite étale, observe that up to an extension \( k'/k \) we have 
\[ \text{Spec} k' \times_{\Pi_{X/k}} \Pi_{Y/k} \simeq \text{Spec} k' \times_X Y \]
for some point \( \text{Spec} k' \to X \), since \( \Pi_{X/k} \) is a gerbe and hence all points are fpqc locally isomorphic.
5. ORBICURVES

Consider $X$ a smooth, connected curve over $k$, $(D_i, r_i)_{i=1,\ldots,n}$ a finite family of reduced, effective Cartier divisors $D_i$ together with a positive integer $r_i$. We can define the associated root stack $\mathcal{X}$, and will call such a stack simply an orbicurve. It is a Deligne-Mumford stack of finite type $X$ with a morphism $f : \mathcal{X} \to X$ such that $f^* D_i$ has an $r_i$-th root. Moreover, $\mathcal{X} \to X$ is universal among algebraic stacks $\mathcal{Y}$ with morphisms $\mathcal{Y} \to X$ with this property.

Essentially, we are putting an orbifold structure of ramification $r_i$ on the divisor $D_i$: for example, if $D_i = p$ is a rational point, we are replacing $p$ with a copy of $\mathbb{P}^1_{\mu_{r_i}}$. Outside of the divisors $D_i$, $\mathcal{X} \to X$ is an isomorphism. For a precise definition see [AGV08, Appendix B.2]. In order to be clear, we will use Fraktur letters for orbicurves and normal ones for schemes.

If $\bar{X}$ is the smooth compactification of $X$ and $D_{\infty} = \bar{X} \setminus X$, then the Euler-Poincaré characteristic of $\mathcal{X}$ is

$$\chi(\mathcal{X}) = 2 - 2g - \deg D_{\infty} - \sum_i \frac{r_i-1}{r_i} \deg D_i.$$

If $\mathcal{Y} \to X$ is a finite étale cover of degree $d$, the Riemann-Hurwitz formula implies that

$$\chi(\mathcal{Y}) = d \chi(\mathcal{X}).$$

The orbicurve $\mathcal{X}$ is hyperbolic if $\chi(\mathcal{X}) < 0$, elliptic if $\chi(\mathcal{X}) = 0$ and parabolic if $\chi(\mathcal{X}) > 0$, except one case: if $g = 0$, $\deg D_{\infty} = 2$ and there is no ramification, then we say that $\mathcal{X}$ is parabolic even if it has characteristic $0$. At the end of §7 we explain why we had to make this distinction. Observe that this is coherent with our intuition from complex geometry, since the universal covering of $\mathbb{P}^1_C$ minus two points is the complex plane and not the unit disc: parabolic curves are exactly those covered by the complex plane and $\mathbb{P}^1_C$, while elliptic and hyperbolic ones are covered by the unit disc.

The main fact that allows us to compare curves and orbicurves is that almost every orbicurve has a finite étale covering which is a curve. In fact we can reduce to the complex case, and in turn to a topological problem about surfaces using the Riemann existence theorem.

For surfaces, this problem has been solved by Bundgaard-Nielsen and Fox with a mistake later corrected by Chau (see [Nie48], [BN51], [Fox52] and [Cha83] for the original papers and [Nam87, Theorem 1.2.15] for a more comprehensive treatment). There are some parabolic orbisurfaces supported on the sphere which obviously can’t be covered by ordinary surfaces because they have a finite universal covering which is not a surface, but that’s all, in all other cases it is possible.

**Proposition 5.1.** Let $k$ be a field of characteristic $0$, and $\mathcal{X}/k$ an orbicurve defined over a smooth, connected curve $X$ with smooth compactification $\bar{X}$ by ramification data $(D_i, r_i)_{i=1,\ldots,n}$ with $1 < r_1 < \cdots < r_n$. Set $D_{\infty} = \bar{X} \setminus X$. Suppose that we are not in one of the following cases:

- $D_{\infty} = \emptyset$, $\hat{g}(\bar{X}) = 0$, $n = 1$, $\deg D_1 = 1$;
- $D_{\infty} = \emptyset$, $\hat{g}(\bar{X}) = 0$, $n = 2$, $\deg D_1 = \deg D_2 = 1$.

Then there exists a finite extension $k'/k$ and a smooth connected curve $Y$ defined over $k'$ with a finite étale cover $Y \to \mathcal{X}_{k'}$. 

Proof. Since everything is of finite type, from standard arguments we can obtain the general case once we know the theorem is true for $k$ finitely generated over $Q$. Suppose then that $k$ is finitely generated over $Q$ and fix an immersion of $k \subseteq C$.

Consider the curve $X$ on which $\mathcal{X}$ is supported, the topological set $X^\text{an}_C$ is a compact oriented surface while $X^\text{an}_C$ is a compact orbifold supported on $X^\text{an}_C$. We can regard unramified covers $Y \to X^\text{an}_C$ with $Y$ a compact surface as ramified covers $Y \to X^\text{an}_C$ such that all the points over $D_i \subseteq C$ have ramification $r_i$. By [Nam57, Theorem 1.2.15], such a cover exists for almost all ramification data on oriented surfaces, the only exceptions being the sphere with exactly one critical value and the sphere with two critical values with different ramification.

Hence we have a topological unramified orbifold covering $Y \to X^\text{an}_C$. By applying the Riemann existence theorem to $Y \to X^\text{an}_C$, we can regard $Y$ as a smooth, proper curve over $C$ with a morphism $Y \to X_C$. Consider the closed subset $\mathcal{R}_C \subseteq X_C$ is the ramification locus of $Y \to X_C$. By [Lemma 5.2] there exists a finite extension $k \subseteq k' \subseteq C$ and a morphism of curves $Y' \to X_{k'}$ whose base change to $C$ is isomorphic to $Y \to X_C$. By the universal property of the root stack $\mathcal{X}_{k'}$ this gives a finite étale covering $Y' \to X_{k'}$, as desired.

In the proof of [Proposition 5.1] we have used the following lemma, which is widely known (when $k = Q$ and $X = \mathbb{P}^1$ it is the easy implication of Belyi’s theorem), but for which we could not find a reference.

Lemma 5.2. Let $k \subseteq K$ be fields of characteristic 0, with $K = \bar{K}$. Let $X/k$ and $Y/K$ be smooth, projective curves with a branched covering

$$f : Y \to X_K$$

such that all the ramification values are defined over a finite extension of $k$.

Then there exists a finite extension $k \subseteq k' \subseteq K$ and a branched covering $f' : Z \to X_{k'}$ whose base change to $K$ is isomorphic to $Y \to X_K$.

Proof. Since everything is of finite type, it is enough to find such a covering $Z \to X_{k'}$ for $k' = k \subseteq K$. By hypothesis, there exists an open subset $U \subseteq X$ such that $Y|_U \to U_K$ is unramified.

Since $k$ and $K$ are algebraically closed of characteristic 0, $\pi_1(U_K) = \pi_1(U_k)$ and hence there exists a finite étale morphism $g : V \to U_k$ whose base change to $K$ is $Y|_{U_K} \to U_K$. Let $Z$ be a smooth completion of $V$, $g$ extends to a finite morphism $Z \to X_K$. It is now obvious that the base change of $Z \to X_k$ is isomorphic to $Y \to X_K$. □

6. Equivalence of the Conjectures and Full Faithfulness

For a curve $X$ of genus $g \geq 1$ over a field finitely generated over $Q$ it was already known to Grothendieck that, as a consequence of the Mordell-Weil theorem, $X(k) \to \Pi_{X/k}(k)$ is injective on isomorphism classes. Actually, full faithfulness holds too. The point is to show that the elements of $\Pi_{X/k}(k)$ in the essential image have no nontrivial automorphisms.

By [Remark 4.1], this reduces to proving that sections coming from rational points have trivial centralizers, which is already known (see [Sti13, §9.4]). Anyway, we want to reprove it in our language without passing through centralizers because we think that the proof is rather enlightening (and it doesn’t require much
work). The idea of the proof can already be found in literature, for example in [EH08, Theorem 5.5] where it is used for a different purpose.

**Proposition 6.1.** If $X$ is a smooth, proper curve of genus $g \geq 1$ and $k$ is finitely generated over $\mathbb{Q}$, then $X(k) \to \Pi_{X/k}(k)$ is fully faithful.

**Proof.** Since we already know that $X(k) \to \Pi_{X/k}(k)$ is injective on isomorphism classes, we only need to prove that it is fully faithful at every point, i.e. that the elements of $\Pi_{X/k}(k)$ in the essential image have no nontrivial automorphism.

Let $p \in X(k)$ be a rational point, and set $\tilde{X} = X \times_{\Pi_{X/k}} \text{Spec} k$ where $\text{Spec} k \to \Pi_{X/k}$ is the rational section $\pi_X(p)$ corresponding to $p$. Suppose that $\pi_X(p)$ has a nontrivial automorphism $\phi$, this defines two different rational points $q_1 = (p, \text{id}_{\text{Spec} k}, \text{id}_{\pi_X(p)}) \neq (p, \text{id}_{\text{Spec} k}, \phi) = q_2$ in $\tilde{X}(k)$. Since $\Pi_{X/k} = B\text{Aut}_{\Pi_{X/k}}(\pi_X(p))$ is profinite, there exists a finite étale intermediate covering $\tilde{X} \to Y \to X$ such that the images $q_1', q_2'$ of $q_1$ and $q_2$ in $Y(k)$ are different. By the Riemann-Hurwitz formula, we still have $\chi(Y) \leq 0$. But we have a commutative diagram

\[
\begin{array}{ccc}
\tilde{X} & \longrightarrow & \text{Spec} k \\
\downarrow & & \downarrow \\
Y & \longrightarrow & \Pi_{Y/k}
\end{array}
\]

which implies that the images of $q_1'$ and $q_2'$ in $\Pi_{Y/k}(k)$ must be isomorphic, and this is absurd. \hfill \Box

**Corollary 6.2.** Let $X$ be a smooth, proper curve of genus $g \geq 2$ over a field $k$ finitely generated over $\mathbb{Q}$. If Grothendieck’s section conjecture holds for $X$, then the stacky section conjecture holds for $X$ too, i.e. $X(k) \to \Pi_{X/k}(k)$ is an equivalence of categories. \hfill \Box

We have now all the tools required to prove the equivalence of the three forms of the section conjecture.

**Theorem 6.3.** Conjectures 1.1, 1.2 and 1.3 are equivalent.

**Proof.** Obviously, the stacky section conjecture implies the Borne-Emsalem version of the conjecture, which in turn implies the classical one. On the other hand, suppose that the section conjecture is true. If $X$ is a proper orbicurve with $\chi(X) < 0$, then by Proposition 5.1 we can find a finite extension $k'/k$ and a smooth, proper curve $Y$ with a finite, étale morphism $Y \to X_{k'}$, and $\chi(Y) < 0$. We conclude by applying Corollary 6.2 and Proposition 4.6(ii). \hfill \Box

With the same ideas, we can generalize the fact that the section map is injective for elliptic and hyperbolic proper curves.

**Theorem 6.4.** Let $k$ be a finitely generated extension of $\mathbb{Q}$, and $X$ a proper orbicurve. Then $X_{k'}(k') \to \Pi_{X_{k'}}(k')$ is fully faithful for every finitely generated $k'/k$ if and only if $\chi(X) \leq 0$.

**Proof.** If $\chi(X) > 0$, the fundamental group of $X_{k''}$ is finite since the covers of $X_{k''}$ must have characteristic $\leq 2$ and multiple of $\chi(X) > 0$. With an argument analogous to the one of Proposition 5.1 we can find a finite extension $k'/k$ and a finite
étale covering $\mathcal{Y} \to X_{k'}$ where $\mathcal{Y}$ is a proper orbicurve over $k'$ with trivial fundamental gerbe. Hence clearly $\mathcal{Y}_{k'}(k') \to \Pi_{\mathcal{Y}_{k'}}(k') = \text{Spec} k'(k')$ is not fully faithful as soon as $\mathcal{Y}_{k'}$ has two $k'$-rational points, and applying Proposition 4.2(i) we conclude.

On the other hand, if $\chi(X) \leq 0$, by Proposition 5.1 we find a finite extension $k'/k$ and a smooth, proper curve $Y$ over $k'$ with a finite étale covering $Y \to X_{k'}$. Again, $\chi(Y) \leq 0$ and hence we know that the thesis is true for $Y$ thanks to Proposition 6.1. Applying Proposition 4.6(i), we see that it is true for $X$ too. □

7. OPEN CURVES AS LIMITS OF ORBICURVES

There is a version of the section conjecture for open curves. If $X$ is a smooth geometrically connected curve with smooth completion $\bar{X}$, every “missing” rational point $x \in \bar{X} \backslash X(k)$ defines a so-called packet of cuspidal sections $\mathcal{P}_x \subseteq \Pi_{X/k}(k)$, see for example [EH08]. The section conjecture for open curves says that if $k$ is finitely generated over $\mathbb{Q}$ and $X$ has negative Euler characteristic, every section $s \in \Pi_{X/k}(k)$ comes either from a rational point of $x$ or from a packet of cuspidal sections.

As showed by Niels Borne and Michel Emsalem in [BE14, §2.2.3], the section conjecture for orbicurves (even in the weak form of a bijection) implies easily the section conjecture for open curves. If we put together their observation and Theorem 6.3, we obtain a new proof of the following classical result (see [Sti13, Proposition 103]).

Theorem 7.1. The section conjecture for proper curves implies the section conjecture for open curves. □

Let us show how the ideas of Borne and Emsalem fit nicely in our formalism, giving a clear picture of packets of tangential points and of the section conjecture for open curves. Let $X$ be a smooth connected curve over a field $k$ of characteristic 0 with smooth compactification $\bar{X}$, set $D = \bar{X} \backslash X$. Let $X_n$ be the orbicurve supported over $\bar{X}$ with ramification of degree $n$ along the divisor $D$, and

$$\hat{X} = \varprojlim_n X_n$$

their projective limit: it is an fpqc stack with natural morphisms $X \hookrightarrow \hat{X}$ and $\hat{X} \to \bar{X}$.

Remark 7.2. The proalgebraic stack $\hat{X}$ is the infinite root stack associated to the logarithmic structure given by $D$ on $\bar{X}$, see [TV18].

Moreover, the natural morphism

$$\Pi_{\hat{X}} \to \varprojlim_n \Pi_{X_n}$$

is an isomorphism: in fact, if $\Phi$ is a finite étale stack, thanks to [BV15, Proposition 3.8] we have equivalences

$$\text{Hom}(\varprojlim_n \Pi_{X_n}, \Phi) \simeq \varprojlim_n \text{Hom}(X_n, \Phi) \simeq \varprojlim_n \text{Hom}(X_n, \Phi) \simeq \text{Hom}(\hat{X}, \Phi).$$

Let $\eta_n : X_n \to \hat{X}$, $\eta : \hat{X} \to \bar{X}$ be the natural morphisms. If $p \in \bar{X} \backslash X(k)$ is a rational point, the fiber $\eta_n^{-1}(p)$ over $p$ is non canonically isomorphic to $B_{\mu_n}$,
hence $\eta^{-1}_n(p)(k) \simeq k^*/k^{*n}$ and this implies that $X_n \to \hat{X}$ is surjective on rational points. By taking a coherent sequence of points in $\eta^{-1}_n(p)$ for every $n$ we get an isomorphism $\eta^{-1}(p) \simeq B\hat{Z}(1)$ and hence $\hat{X} \to \hat{X}$ is surjective on rational points too. The packet of tangential points at $p$ is

$$\eta^{-1}(p)(k) \simeq B\hat{Z}(1)(k) = \lim_{n} k^{*}/k^{*n} = \hat{k}^{*}.$$ 

Since we are in characteristic 0, Abhyankar’s lemma implies that the natural map

$$\Pi_{X/k} \to \Pi_{\hat{X}/k}$$

is an isomorphism. In fact, checking that $\Pi_{X/k} \to \Pi_{\hat{X}/k}$ is an isomorphism is equivalent to checking that it induces an isomorphism of $\text{Isom}$ sheaves, and this in turn means asking an isomorphism of étale fundamental groups. For this, see [Bor09, Proposition 3.2.2].

A simple way of seeing this is observing that is enough to prove the isomorphism over an algebraically closed field, over which we can use the standard presentation of the fundamental group. In fact, if we remove a point from a curve, we are adding a generator with infinite order to the presentation, while if we replace it with $B\mu_{n}$ we are adding a generator of order $n$: it is then clear that for $n \to \infty$ we get the desired convergence.

Hence, the section conjecture for an hyperbolic open curve $X$ can be reinterpreted by asking that, if $k/Q$ is finitely generated,

$$\hat{X}(k) \to \Pi_{\hat{X}/k}(k) = \Pi_{X/k}(k)$$

is a bijection (or an equivalence of categories). If $X$ is hyperbolic, $\chi(X) < 0$, and hence $\chi(X_n) < 0$ for $n$ big enough. If we know that the section conjecture for proper curves is true, then it is true for proper orbicurves too thanks to [Theorem 6.3] hence

$$X_n(k) \to \Pi_{X_n/k}(k)$$

is an equivalence of categories for $n$ big enough. Passing to the limit, the same is true for $\hat{X}$, and we get [Theorem 7.1]

At this point, it is natural to see what happens for open orbicurves. If $\hat{X}$ is an open orbicurve we can define $X_n$ and $\hat{X}$ as above. If $\chi(\hat{X}) < 0$ and $k/Q$ is finitely generated, the section conjecture for $\hat{X}$ says that

$$\hat{X}(k) \to \Pi_{\hat{X}/k}(k) = \Pi_{X/k}(k)$$

is an equivalence of categories. The same argument as above shows that this is equivalent to the classical section conjecture for proper curves.

Finally, let us look more closely at the injectivity part of the section conjecture. Recall that an open orbicurve $\hat{X}$ is hyperbolic if $\chi(\hat{X}) < 0$, elliptic if $\chi(\hat{X}) = 0$ and parabolic if $\chi(\hat{X}) > 0$, with one exception: if $\hat{X} = X$ is a curve of genus 0 and $\chi(X) = 0$ (i.e. $\deg(\hat{X} \setminus X) = 2$), $X$ is parabolic even if $\chi(X) = 0$. It is clear now why we had to make this distinction: with this definition, an open orbicurve $\hat{X}$ is hyperbolic (resp. elliptic, parabolic) if and only if $\hat{X}$ can be expressed as a projective limit of hyperbolic (resp. elliptic, parabolic) proper orbicurves. In fact, $\chi(X_n) \geq \chi(\hat{X})$ converges to $\chi(\hat{X})$ from above, and if $\hat{X}$ is not proper we have a strict inequality $\chi(X_n) > \chi(\hat{X})$. Hence, if $\chi(\hat{X}) = 0$ and $\hat{X}$ is not proper,
\( \chi(X_n) > 0 \) for every \( n \). It is immediate to check that this happens only for \( \mathcal{X} \) a curve of genus 0 without a divisor of degree 2.

By a direct application of [Theorem 6.4] we thus get the following.

**Theorem 7.3.** Let \( \mathcal{X} \) be an orbicurve over a field \( k \) which is finitely generated over \( \mathbb{Q} \). Then

\[
\check{\mathcal{X}}(k') \to \Pi_{\check{\mathcal{X}}(k')}(k') = \Pi_{\mathcal{X}(k')}(k')
\]

is fully faithful for every finite extension \( k' / k \) if and only if \( \mathcal{X} \) is elliptic or hyperbolic.

### 8. Appendix. Étale fundamental gerbes

Almost everything in this appendix is already known to the mathematical community, we claim no originality. In particular, most of the ideas and results are already implicit in [BV15] and in the original paper by Deligne [Del89]. Anyway, we could not find a satisfying reference, since [BV15] is mostly concerned with the Nori fundamental gerbe rather than the étale one, and hence the theorems regarding the étale fundamental gerbe are not expressed in the right generality. In particular, they always work with inflexible fibered categories, while geometrically connected is the right hypothesis. See also [TZ17, §2,3,4], where part of what is contained in this appendix is done under minor additional hypotheses. Up to our knowledge, the only thing that is new is the proof of Proposition 8.16 even if the result is widely known (see for example a similar theorem for the Nori fundamental gerbe in [ABETZ17, Theorem III]).

We want to stress out that our effort to state results in maximal generality is not for its own sake: it just happens to work with rather nasty objects that are not even algebraic stacks, like the infinitely ramified orbicurves of §7. Since the theory works for raw fibered categories without any additional hypothesis, we want to give statements in this generality.

#### 8.1. Weil restriction of finite étale stacks.

A basic tool that we are going to use is the Weil restriction of stacks. If \( k' / k \) is a finite extension of fields, the Weil restriction along \( k' \to k \) is the right adjoint to the functor of base change along \( \text{Spec } k' \to \text{Spec } k \). More concretely, if \( X \) is a fibered category over \( k \) and \( Y \) is a fibered category over \( k' \), the Weil restriction \( R_{k'/k} Y \) is a fibered category over \( k \) with an equivalence of categories

\[
\text{Hom}_k(X, R_{k'/k} Y) \simeq \text{Hom}_{k'}(X_{k'}, Y)
\]

functorial in \( X \) and \( Y \). We can construct \( R_{k'/k} Y \) as the fibered product \( Aff / k \times_{Aff / k'} Y \). When \( Y \) is represented by a scheme, \( R_{k'/k} Y \) is represented by its Weil restriction which is a scheme, too. If \( Y \) is represented by a finite stack and \( k' / k \) is separable, then \( R_{k'/k} Y \) is represented by a finite stack too, see [BV15, Lemma 6.2].

**Lemma 8.1.** Let \( k' / k \) be a finite, separable extension, and \( Y \) a finite étale stack over \( k' \). Then \( R_{k'/k} Y \) is a finite étale stack over \( k \), too.

**Proof.** In the proof of [BV15, Lemma 6.2], from a finite groupoid presentation \( R \rightrightarrows U \) of \( Y \) they construct a finite groupoid presentation \( R' \rightrightarrows U' \) of \( R_{k'/k} Y \). Following their construction, it is immediate to check that if \( R \rightrightarrows U \) is étale, \( R' \rightrightarrows U' \) is étale too. \( \square \)
8.2. Geometrically connected fibered categories. Let $X_1, X_2$ be two fibered categories over $k$. It is possible to define the disjoint union $X_1 \sqcup X_2$: is $S$ is a scheme, a morphism $S \to X_1 \sqcup X_2$ is a decomposition of $S = S_1 \sqcup S_2$ with $S_1, S_2$ open and closed plus a pair of morphisms $s_i : S_i \to X_i$.

**Definition 8.2.** A fibered category $X$ is connected if it is nonempty and $X \simeq X_1 \sqcup X_2$ implies that $X_1 = \emptyset$ or $X_2 = \emptyset$.

**Remark 8.3.** If $X$ is an algebraic stack, this is equivalent to asking that the underlying topological space $|X|$ (see [Stacks, Tag 04XE]) is connected. On one hand, if $X = X_1 \sqcup X_2$, then $|X| = |X_1| \sqcup |X_2|$. On the other hand, if $|X| = U_1 \sqcup U_2$ is disconnected, the fact that for every scheme $S$ the natural morphism $|S| \to |X|$ is continuous allows us to define two fibered categories $X_1, X_2$ such that $|X_i| = U_i$ and $X = X_1 \sqcup X_2$.

As shown in the following lemma, our definition of connected fibered category is equivalent to the one given in [TZ17, Definition 2.5].

**Lemma 8.4.** A fibered category $X$ is connected if and only if the scheme $\text{Spec} H^0(X, \mathcal{O}_X)$ is connected.

**Proof.** We have a non trivial disjoint union $X \simeq X_1 \sqcup X_2$ if and only if there exists a surjective morphism

$$X \to \text{Spec } k \sqcup \text{Spec } k,$$

which in turn is equivalent to the existence of a surjective morphism

$$\text{Spec } H^0(X, \mathcal{O}_X) \to \text{Spec } k \sqcup \text{Spec } k.$$

Recall that a fibered category is concentrated [BV16, Definition 4.1] if there exists an affine scheme $U$ and a representable, quasi separated, quasi compact and faithfully flat morphism $U \to X$.

If $X$ is concentrated and $u : U \to X$ is as above, set $R = U \times_X U$, we obtain an fpqc groupoid $(r_1, r_2) : R \rightrightarrows U$ in algebraic spaces. From standard arguments in descent theory we get an exact sequence

$$0 \to H^0(X, \mathcal{O}_X) \xrightarrow{u^*} H^0(U, \mathcal{O}_U) \xrightarrow{r_1^* - r_2^*} H^0(R, \mathcal{O}_R)$$

and hence it follows easily that for any field extension $k'/k$,

$$H^0(X_{k'}, \mathcal{O}_{X_{k'}}) = H^0(X, \mathcal{O}_X) \otimes_k k'.$$

**Lemma 8.5.** Let $X$ be a category fibered over $k$, and $k_s/k$ a separable closure. Consider the following:

(i) $X_{k_s}/k_s$ is connected for every extension $k'/k$,

(ii) $X_k/k$ is connected,

(iii) $X_{k_s}/k_s$ is connected for every finite, separable extension $k'/k$,

(iv) $k$ is the only étale subalgebra of $H^0(X, \mathcal{O}_X),$

(v) $\text{Spec } H^0(X, \mathcal{O}_X)$ is geometrically connected.

We have implications $(i) \iff (ii) \iff (iii) \iff (iv) \iff (v)$.

If $X$ is an algebraic space or it is concentrated, then $(iii) \implies (ii)$ holds, too.

**Proof.** $(i) \implies (ii)$: Obvious.
(ii) ⇒ (i): Suppose that \(X_{k'} = X_1 \sqcup X_2\) is a nontrivial disjoint union. Up to enlarging \(k'\), we may suppose that \(k_s \subseteq k'\). Let \(S\) be a connected scheme over \(k_s\) and \(S \to X_{k_s}\) a morphism. By \[\text{Stacks}, \text{Tag 0363}\], \(S_{k'}\) is connected, hence \(S_{k'} \to X_{k'}\) factors through \(X_i\) for some \(i\). This allows us to write \(X_{k_s}\) as a nontrivial disjoint union, which is absurd.

(ii) ⇒ (iii): Obvious.

(iii) ⇒ (iv): Suppose that \(A \subseteq H^0(X, \mathcal{O}_X)\) is a nontrivial finite étale subalgebra of degree \(d > 1\), there exists a scheme \(S\) with a morphism \(S \to X\) such that the composition \(S \to X \to \text{Spec} A\) is dominant. Now choose \(k'/k\) a finite separable extension which splits \(A\). The base change

\[X_{k'} \to \text{Spec} A_{k'} = \text{Spec} k^{d}\]

is surjective because \(S_{k'} \to X_{k'} \to \text{Spec} k^{d}\) is surjective. But this is absurd, since \(d > 1\) and \(X_{k'}\) is connected.

(iv) ⇒ (iii): Suppose that \(k'/k\) is a finite separable extension and that we can write \(X_{k'} = X_1 \sqcup X_2\) as a nontrivial disjoint union. We hence have a surjective morphism \(X_{k'} \to \text{Spec} k' \sqcup \text{Spec} k'\) which gives a morphism \(X \to R_{k'/k}(\text{Spec} k' \sqcup \text{Spec} k')\). Since \(R_{k'/k}(\text{Spec} k' \sqcup \text{Spec} k')\) is a finite étale scheme, by hypothesis we have a factorization

\[X \to \text{Spec} k \to R_{k'/k}(\text{Spec} k' \sqcup \text{Spec} k').\]

But this gives a factorization

\[X_{k'} \to \text{Spec} k' \to \text{Spec} k' \sqcup \text{Spec} k',\]

which is absurd.

(iv) ⇔ (v): This is well known.

For the implication (iii) ⇒ (ii), if \(X\) is concentrated we have \(H^0(X_{k'}, \mathcal{O}_{X_{k'}}) = H^0(X, \mathcal{O}_X) \otimes_k k'\) for every extension \(k'/k\), hence we can reduce to the case of affine schemes which is well known. If \(X\) is an algebraic space, this is \[\text{Stacks}, \text{Tag 0A17}\].

**Definition 8.6.** Let \(X\) be a fibered category. We say that \(X\) is **geometrically connected** if the equivalent conditions (iii), (iv) and (v) of Lemma 8.5 hold for \(X\).

**8.3. Existence and base change.**

**Definition 8.7.** An fpqc stack \(\Gamma\) over a field \(k\) is **pro-étale** if it is the limit of a projective system of finite, étale stacks over \(k\), in the sense of \[\text{BV15}, \text{Definition 3.5}\].

**Remark 8.8.** In \[\text{BV13}, \text{Definition 3.5}\] they define the limit of a projective system \((\Gamma_i)\) of affine fpqc gerbes as a category fibered in groupoids which turns out to be an fpqc stack. Actually, it is straightforward to check that the definition works without any modification for a projective system \((\Gamma_i)\) of categories fibered in groupoids, and if \(\Gamma_i\) is an fpqc stack for every \(i\) then also the limit is an fpqc stack. Moreover, if \(\Gamma_i\) is an affine fpqc gerbe for every \(i\) and the limit is not empty, then the limit is an fpqc gerbe too, see \[\text{BV15}, \text{Proposition 3.7}\].

**Definition 8.9.** Let \(X\) be a fibered category over \(k\), and \(\Pi\) a pro-étale gerbe with a morphism \(X \to \Pi\). Then \(X \to \Pi\) is an étale fundamental gerbe if, for every finite, étale stack \(\Phi\), the functor

\[\text{Hom}(\Pi, \Phi) \to \text{Hom}(X, \Phi)\]
is an equivalence of categories.

**Lemma 8.10.** Let $X$ be a fibered category with an étale fundamental gerbe $X \to \Xi$, and $\Phi$ a pro-étale stack. Then

$$\text{Hom}(\Xi, \Phi) \to \text{Hom}(X, \Phi)$$

is an equivalence of categories. In particular, the étale fundamental gerbe is unique up to a canonical equivalence.

**Proof.** This is a straightforward application of the definition of the étale fundamental gerbe and of pro-étale stacks. □

The following simple lemma is rather enlightening in the sense that it draws the line between the étale setting and the Nori setting: its failure for finite stacks is what makes Nori’s fundamental gerbe subtler than the étale one.

**Lemma 8.11.** Let $\Phi$ be a finite étale stack. Then the natural morphism

$$\Phi \to \text{Spec} \mathcal{H}^0(\Phi, \mathcal{O}_\Phi)$$

is a gerbe.

**Proof.** We give an elementary proof. See also [TZ17, Proposition 3.2] for a more technical proof for finite, reduced stacks.

If $k'/k$ is an extension, it is easy to check that $\Phi \to \text{Spec} \mathcal{H}^0(\Phi, \mathcal{O}_\Phi)$ is a gerbe if and only if $\Phi_{k'} \to \text{Spec} \mathcal{H}^0(\Phi_{k'}, \mathcal{O}_{\Phi_{k'}})$ is a gerbe. Hence, we may suppose $k = k$.

Choose now a finite étale groupoid $R \to U$ giving a presentation of $\Phi$. Since $k = \overline{k}$ and $R, U$ are finite étale, they are simply finite disjoint unions of points. Hence we can write

$$\Phi = \bigsqcup_i BG_i$$

where $G_i$ are finite discrete groups. Now it is obvious that

$$\Phi = \bigsqcup_i BG_i \to \bigsqcup_i \text{Spec} k$$

is a gerbe. □

**Corollary 8.12.** Let $X$ be a fibered category. Then $X$ is geometrically connected if and only if every morphism $X \to \Gamma$ where $\Gamma$ is a finite étale stack has a factorization

$$X \to \Gamma' \to \Gamma$$

where $\Gamma'$ is a finite étale gerbe.

**Proof.** Suppose that $X$ is geometrically connected. Consider the composition

$$X \to \Gamma \to \text{Spec} \mathcal{H}^0(\Gamma, \mathcal{O}_\Gamma).$$

Since $X$ is geometrically connected and $\mathcal{H}^0(\Gamma, \mathcal{O}_\Gamma)$ is finite étale, we have a factorization

$$X \to \text{Spec} k \to \text{Spec} \mathcal{H}^0(\Gamma, \mathcal{O}_\Gamma).$$

Set $\Gamma' = \text{Spec} k \times_{\text{Spec} \mathcal{H}^0(\Gamma, \mathcal{O}_\Gamma)} \Gamma$, we have a factorization

$$X \to \Gamma' \to \Gamma$$

and $\Gamma'$ is a gerbe over $\text{Spec} k$ thanks to [Lemma 8.11].

On the other hand, if $A \subseteq \mathcal{H}^0(X, \mathcal{O}_X)$ is a nontrivial étale subalgebra, the natural morphism $X \to \text{Spec} A$ cannot factorize through any finite gerbe. □
Theorem 8.13. Let $X$ be a fibered category over $k$. Then $X$ has an étale fundamental gerbe if and only if it is geometrically connected.

Proof. The proof is completely analogous to the proof of [BV15, Theorem 5.7], so we don’t repeat it. The reason why everything works is Corollary 8.12, which shows that geometrically connected fibered categories and finite étale stacks satisfy the same formal property of inflexible fibered categories and finite stacks. See also [TZ17, Proposition 4.3] for a proof under some minor additional hypotheses. □

Proposition 8.14. Let $k'/k$ be an algebraic and separable extension, $X$ a geometrically connected fibered category over $k$. Suppose that either
(a) $k'$ is finite over $k$, or
(b) $X$ is concentrated.
Then $X_{k'}$ is geometrically connected over $k'$ and $\Pi_{X_{k'}/k'} = \text{Spec} k' \times \Pi_{X/k}$.

Proof. Again, the proof is completely analogous to the one of [BV15, Proposition 6.1]. This time, the reason why everything works is Lemma 8.1, which plays the role of [BV15, Lemma 6.2]. □

8.4. Étale coverings of fibered categories.

Lemma 8.15. Let $f : Y \to X$ be a representable, finite étale morphism of fibered categories. If $X$ is connected then $f$ has a constant degree, i.e. there exists an integer $d$ such that for every scheme $S$ and every morphism $s : S \to X$ the étale covering $S \times X Y \to S$ has constant degree $d$.

Proof. If $S$ is a scheme, $s \in X(S)$ an object and $d \geq 0$ an integer, the locus $S_{=d}$ of points $p$ of $S$ such that $Y \times_X S \to S$ has degree $d$ over $p$ is an open and closed subscheme of $S$, set $S_{\neq d} = S \setminus S_{=d}$. This allows to define in an obvious way two fibered categories $X_{=d}$, $X_{\neq d}$ such that $X = X_{=d} \cup X_{\neq d}$. There exists some $d_0$ such that $X_{d_0}$ is nonempty, hence $X_{\neq d_0} = \emptyset$ and $X = X_{=d_0}$. □

Proposition 8.16. Let $Y \to X$ be a representable, finite étale morphism of geometrically connected fibered categories. Then the natural 2-commutative diagram

$$
\begin{array}{ccc}
Y & \longrightarrow & \Pi_{Y/k} \\
\downarrow & & \downarrow \\
X & \longrightarrow & \Pi_{X/k}
\end{array}
$$

is 2-cartesian.

Proof. Thanks to Lemma 8.15 $Y \to X$ is a finite cover of fixed degree $d$. Let $d \times X$ be the disjoint union of $d$ copies of $X$, we have a finite cover $d \times X \to X$ of degree $d$. The group $S_d$ acts on the fibered category $Z = \text{Isom}_X(d \times X, Y)$ by automorphisms of $d \times X$ making it into an $S_d$-torsor over $X$. If $S$ is a scheme with a morphism $S \to Z$, we have a trivialization $d \times S \simeq Y \times_X S$. The first copy of $d \times S$ gives us a morphism $S \to Y$, and thus by Yoneda’s lemma we have a $S_{d-1}$ invariant morphism $Z \to Y$ which is actually a $S_{d-1}$ torsor.
We can repeat the same argument with Π, Φ, which, composed with the projection particular it is an equivalence of categories. Let ρ then we have that morphism, consider the composition is well defined for Γ for every g ∈ Sd, this defines a morphism ρΦ(·, g) : Z → Φ. If h ∈ Sd−1 ⊆ Sd, since Z → Y is Sd−1 invariant we get that ρΦ(·, g) = ρΦ(·, gh) : Z → Φ, hence ρΦ(·, [g]) is well defined for [g] ∈ Sd/Sd−1. This gives us an Sd-equivariant morphism

Z → ΦSd/Sd−1

where Sd acts on ΦSd/Sd−1 via left multiplication on Sd/Sd−1.

On the other hand, if we have an Sd-equivariant morphism Z → ΦSd/Sd−1, in particular it is Sd−1-invariant since Sd−1 acts trivially on Sd/Sd−1. Hence we have an induced morphism

Y → ΦSd/Sd−1,

which, composed with the projection ΦSd/Sd−1 → Φ on the identity component, gives a morphism Y → Φ. It is easy to check that these constructions are inverses and give an equivalence of categories

Hom(Y, Φ) ∼ HomSd(Z, ΦSd/Sd−1).

Since Z → X is an Sd-torsor, we also have an equivalence

HomSd(Z, ΦSd/Sd−1) ∼ HomBSd(X, [ΦSd/Sd−1/Sd])

and their composition

Hom(Y, Φ) ∼ HomBSd(X, [ΦSd/Sd−1/Sd]).

We can repeat the same argument with ΠX/k, Π and Λ instead of X, Y and Z, finding an equivalence

Hom(Π, Φ) ∼ HomBSd(ΠX/k, [ΦSd/Sd−1/Sd]).

But since [ΦSd/Sd−1/Sd] is a finite étale stack there is another equivalence

HomBSd(X, [ΦSd/Sd−1/Sd]) ∼ HomBSd(ΠX/k, [ΦSd/Sd−1/Sd]).
Composing these three, we obtain the desired equivalence

\[ \text{Hom}(Y, \Phi) \xrightarrow{\sim} \text{Hom}(\Pi, \Phi). \]

References

[ABETZ17] M. Antei, I. Biswas, M. Emsalem, F. Tonini, and L. Zhang. “Nori fundamental gerbe of essentially finite covers and Galois closure of towers of torsors”. In: ArXiv e-prints (June 2017). arXiv:1706.00739 [math.AG].

[AGV08] D. Abramovich, T. Graber, and A. Vistoli. “Gromov-Witten Theory of Deligne-Mumford Stacks”. In: American Journal of Mathematics 130.5 (2008), pp. 1337–1398.

[BE14] N. Borne and M. Emsalem. “Un critère d’épointage des sections l-adiques”. In: Bulletin de la société mathématique de France 3.142 (2014), pp. 465–487.

[BN51] S. Bundgaard and J. Nielsen. “On normal subgroups with finite index in F groups”. In: Matematisk Tidsskrift. B (1951), pp. 56–58.

[Bor09] N. Borne. “Sur les Représentations du Groupe Fondamental d’une Variété Privée d’un Diviseur à Croisements Normaux Simples”. In: Indiana University Mathematics Journal 58 (1 2009), pp. 137–180.

[BV15] N. Borne and A. Vistoli. “The Nori fundamental gerbe of a fibered category”. In: Journal of Algebraic Geometry 24 (2 Apr. 2015), pp. 311–353.

[BV16] N. Borne and A. Vistoli. “Fundamental gerbes”. In: ArXiv e-prints (Oct. 2016). arXiv:1610.07341 [math.AG].

[Cha83] T. C. Chau. “A note concerning Fox’s paper on Fenchel’s conjecture”. In: Proceedings of the American Mathematical Society 88 (4 Aug. 1983), pp. 584–586.

[Del89] P. Deligne. “Le groupe fondamental de la droite projective moins trois points”. In: Galois groups over Q (Berkeley, CA, 1987). Vol. 16. Math. Sci. Res. Inst. Publ. Springer, New York, 1989, pp. 79–297.

[EH08] H. Esnault and P. H. Hai. “Packet in Grothendieck’s section conjecture”. In: Advances in Mathematics 218 (2 Feb. 2008), pp. 395–416.

[Fox52] R. H. Fox. “On Fenchel’s Conjecture about F-Groups”. In: Matematisk Tidsskrift. B (1952), pp. 61–65.

[Nam87] M. Namba. Branched coverings and algebraic functions. Pitman research notes in mathematics series. Longman Scientific & Technical, 1987.

[Nie48] J. Nielsen. “Kommutatorgruppen for det frie produkt af cykliske grupper”. In: Matematisk Tidsskrift. B (1948), pp. 49–56.

[Stacks] The Stacks Project Authors. Stacks Project. http://stacks.math.columbia.edu 2017.

[Sti08] J. Stix. “On cuspidal sections of algebraic fundamental groups”. In: ArXiv e-prints (Aug. 2008). arXiv:0809.0017 [math.AG].

[Sti13] J. Stix. Rational points and arithmetic of fundamental groups. Lecture Notes in Mathematics 2054. Springer, 2013.

[TV18] M. Talpo and A. Vistoli. “Infinite root stacks and quasi-coherent sheaves on logarithmic schemes”. In: Proceedings of the London Mathematical Society (2018).
[TZ17] F. Tonini and L. Zhang. “Algebraic and Nori fundamental gerbes”. In: Journal of the Institute of Mathematics of Jussieu (2017), pp. 1–43.