Computation of Analytical Zoom Locus Using Padé Approximation

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Abstract: When the number of lens groups is large, the zoom locus becomes complicated and thus cannot be determined by analytical means. By the conventional calculation method, it is possible to calculate the zoom locus only when a specific lens group is fixed or the number of lens groups is small. To solve this problem, we employed the Padé approximation to find the locus of each group of zoom lenses as an analytic form of a rational function consisting of the ratio of polynomials, programmed in MATLAB. The Padé approximation is obtained from the initial data of the locus of each lens group. Subsequently, we verify that the obtained locus of lens groups satisfies the effective focal length (EFL) and the back focal length (BFL). Afterwards, the Padé approximation was applied again to confirm that the error of BFL is within the depth of focus for all zoom positions. In this way, the zoom locus for each lens group of the optical system with many moving lens groups was obtained as an analytical rational function. The practicality of this method was verified by application to a complicated zoom lens system with five or more lens groups using preset patents.

Keywords: lens design; zoom locus; Padé approximation

1. Introduction

A zoom lens is an optical system, where the effective focal length (EFL) or magnification can change continuously, whereas the image distance remains fixed [1]. To commercialize this setup, when moving the lens group, the zoom locus must be calculated such that the EFL or magnification continuously changes to the desired value. The function for the path of the moving lens group must hence be differentiable.

Each lens group must move continuously in the real product; however, the lens group is discontinuously placed at a specific position in the optical design process. Therefore, according to the optical design, it is necessary to find the zoom locus of the continuous lens group by interpolation from these discontinuous nodes. The most accessible method to connect each node is to use linear interpolation [2], as shown in Figure 1a. However, in the cam, which is a component converting rotational motion into linear motion processes with linear interpolation, the lens group does not move smoothly around each node [3,4]. Therefore, in the case of a zoom lens, the locus of the lens group should be calculated to smoothly connect the node, as shown in Figure 1b. By calculating the locus of each lens group in such a way that it is continuous and differentiable, the cam barrel can be made, as shown in Figure 1c. In Figure 1a,b, the arrows and circles represent cam roller components and zoom locus, respectively.
There are several ways to find the zoom locus. Basically, we want to find the unknown variables in the zoom equations. In general, however, there are many more unknown variables than the number of zoom equations, hence the solution cannot be obtained. However, in the past, a specific lens group was often fixed in a zoom lens, hence there was an analytical method considering the conditions of a fixed lens group relative to an image plane [5–7].

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![Figure 1. (a) Linear interpolation, (b) curved interpolation, and (c) barrel for cam [8].](image)

Recently, however, all lens groups have been moved to increase zoom magnification and reduce product size. Particularly in the case of a zoom lens system with a large change in magnification, the aberration change is also large when the lens group is moved from the wide position to the tele position [9]. Therefore, the number of moving groups is increased to correct this aberration, and thus the zoom locus becomes complicated. Further, since there is no fixed lens group, there are more unknown variables than the number of zoom equations. Therefore, to solve this problem, a method estimating the movement of some lens groups to satisfy a desired back focal length (BFL) or effective focal length (EFL) after spline interpolation from nodes of each lens group has been proposed. At this time, when calculating the movement of lens groups, it is possible to calculate the differential matrix and iterative calculation by considering the zoom equations and constraints as nonlinear simultaneous equations [10,11]. However, this iterative calculation method has the disadvantage of a complicated calculation method, which takes a long time. To solve this problem, an analytical method calculating the displacement of one or two lens groups with the desired value of BFL or EFL has been proposed [12,13].

However, this method also results in a numerically calculated total locus of the lens group. Therefore, the gap is made up of very tight discrete values. If the zoom locus is to be made smoother, the locus needs to be recalculated at even tighter intervals. Further, to find the minimum spacing of the lens group, these methods cannot be accurately obtained.

To solve this problem, we aim to calculate the locus of all lens groups of the zoom lens in the form of an analytical polynomial, unlike conventional methods. The interpolation method from these discrete values to the polynomial rational function is referred to as the Padé approximation [14]. The Padé approximation expresses the function values in the form of a rational function, which is the ratio of two polynomials, to solve the shortcoming of the Taylor series’ narrow convergence radius [15]. There is the advantage that the function shape is smooth, and the inflection point is small. Therefore, even with an optical system with high zoom magnification, a smooth zoom locus can be obtained, and the locus can be expressed as a function. Further, this method can be a definite contribution to improve the operational feeling of the product. It is possible to calculate the locus of each group from the existing zoom locus calculation method, however none of the studied methods have obtained the locus of each lens group in the form of an analytic function.

The optical products, used as an example to confirm the zoom locus calculation method, are shown in Figure 2. The patents for Figure 2a,b are in [16] and [17], respectively. These optical systems are
a product currently on the market, and they have a large diameter optical system with a small F/#. The detailed design method of this zoom lens system is mentioned in [18], and this study only focuses on the locus calculation method.

![Camera Lenses](a) Samsung NX 16-50 mm F2.0-2.8 ED OIS [16]; (b) Samsung 50–150 mm F2.8 S ED OIS [17].

2. Materials and Methods

Figure 3 shows the optical layout for a zoom lens comprising a total of N thin lens groups. To calculate the zoom locus, we need to trace paraxial rays. Gaussian brackets are very convenient for programming [19–21]. In this case, the calculation method of the Gaussian bracket is as shown in Equations (1)–(5).

\[
[] = 1 \\
[\alpha_1] = \alpha_1 \\
[\alpha_1, \alpha_2] = \alpha_1 \alpha_2 + 1 \\
[\alpha_1, \alpha_2, \alpha_3] = [\alpha_1, \alpha_2] \alpha_3 + [\alpha_1] \\
[\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_i] = [\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{i-1}] \alpha_i + \alpha_i, \alpha_{i+1}, \ldots, \alpha_{i+2}],
\]

where \( \alpha \) is the refractive power of the lens group or the spacing between each lens group. Thus, the heights of the rays at the image plane and the EFL of the optical system are shown in Equations (6) and (7). This is referred to as the zooming equation.

\[
[k_1, -z_1, k_2, -z_2, \ldots, -z_{i-1}, k_i, -z_i, \ldots, -z_{j-1}, k_j, -z_j, \ldots, k_{N-1}, -z_{N-1}, k_N] = K \tag{6}
\]

\[
[k_1, -z_1, k_2, \ldots, -z_{i-1}, k_i, -z_i, \ldots, -z_{j-1}, k_j, -z_j, \ldots, k_{N-1}, -z_{N-1}, k_N, -z_N] = 0 \tag{7}
\]

where \( k \) is the refractive power of each lens group, and \( z \) is the spacing of each lens group. In addition, the subscript is the number of the lens group, and \( K \) is the inverse of the EFL as the refractive power of the entire optical system.

If variances in \( K \) change the field angle, \( z \) must change accordingly. \( z \) must be interpolated in a suitable method. In this study, we employ the Padé approximation. However, in some sections, Equations (6) and (7) may not be satisfied. In particular, \( z_N \) is the BFL of the optical system in Equation (7), and if this value is within the depth of focus, there is no issue. However, it is assumed that Equations (6) and (7) are not satisfied with the interpolation method. Therefore, a specific lens group must be moved for this case. Namely, if the \( i \)-th and \( j \)-th lens groups are moved by \( \Delta z_i \) and \( \Delta z_j \) as shown in Figure 3, Equations (6) and (7) are changed to Equations (8) and (9).

\[
[k_1, -z_1, k_2, \ldots, -z_{i-1} + \Delta z_i, k_i, -z_i - \Delta z_i, \ldots, -z_{j-1} + \Delta z_j, k_j, -z_j - \Delta z_j, \ldots, k_{N-1}, -z_{N-1}, k_N] = K \tag{8}
\]

\[
[k_1, -z_1, k_2, \ldots, -z_{i-1} + \Delta z_i, k_i, -z_i - \Delta z_i, \ldots, -z_{j-1} + \Delta z_j, k_j, -z_j - \Delta z_j, \ldots, k_{N-1}, -z_{N-1}, k_N, -z_N] = 0 \tag{9}
\]
Solving Equations (8) and (9), $z_i$ and $z_j$ do not satisfy the formula calculated by the Padé approximation. Therefore, $z_i$ and $z_j$ must obtain the rational function again by the Padé approximation. In this method, $z_i$ and $z_j$ must compute the equation by the Padé approximation until they satisfy Equations (8) and (9). If a particular lens group needs to be moved linearly and satisfy only the desired BFL, then moving just one lens group is not a problem. In this case, therefore, Equation (10) may be satisfied. Likewise, until $z_i$ is satisfied, $z_i$ must be determined repeatedly by the Padé approximation.

$$[k_1, -z_1, k_2, \ldots, -z_{i-1} + \Delta z_i, k_i, -z_i - \Delta z_i, \ldots, k_{N-1}, -z_{N-1}, k_N, -z_N] = 0$$ \hspace{1cm} (10)

Naturally, Equations (8)–(10) cannot be fully satisfied with the Padé approximation, and the error in $z_N$, i.e., the distance from the rear surface of the last lens group to the image plane, is due to the depth of focus (DOF). If there is no issue, we repeat this several times to calculate $z$ with a sufficiently small error. This is because the Padé approximation is a relatively accurate calculation method [22]. This process is summarized in Figure 4, where it showed the flow chart of the method for calculating the zoom locus in this study. The programming tool used is MATLAB [23].

![Figure 3. Optical layout with i-th and j-th group movement.](image)

![Figure 4. Flow chart for calculating zoom locus with Padé approximation.](image)
Subsequently, Equations (8)–(10) are calculated in further detail. Using the characteristics of the Gaussian bracket, the unknown $k_i$ is exported out of the Gaussian bracket and written as the first equation for $k_i$ [24–27].

\[
[k_1, -z_1, k_2, \ldots, -z_{i-1} + \Delta z_i] \cdot k_i \cdot [-z_1 - \Delta z_i, \ldots, k_{N-1}, -z_{N-1}, k_N, -z_N] + [k_1, -z_1, k_2, \ldots, -z_{i-1} - z_i, \ldots, k_{N-1}, -z_{N-1}, k_N, -z_N] = 0
\] (11)

If $\Delta z_i$ is taken out of the Gaussian bracket and arranged in Equation (11), it can be summarized as a quadratic equation for $\Delta z_i$, as shown in Equation (12) below.

\[
(( -z_{i-1} + \Delta z_i) \cdot [k_1, -z_1, \ldots, k_{i-1}] + [k_i, -z_i - \Delta z_i]) \cdot k_i \cdot (( -z_{i-1} + \Delta z_i) \cdot [k_{i+1}, \ldots, k_N, -z_N] + [z_{i+1}, \ldots, k_N, -z_N]) + [k_1, -z_1, k_2, \ldots, -z_{i-1} - z_i, \ldots, k_{N-1}, -z_{N-1}, k_N, -z_N] = 0
\] (12)

Equation (12), with respect to $\Delta z_i$, is identical to Equation (13). Solving Equation (13) yields two solutions, hence we have to choose a solution with a small absolute value.

\[
-BDk_i \cdot \Delta z_i^2 + k_i(BC - AD)\Delta z_i + k_iAC + E = 0
\]

\[
A \equiv [-z_0, k_1, -z_1, k_2, \ldots, -z_{i-1}]
\]

\[
B \equiv [-z_0, k_1, -z_1, k_2, \ldots, k_{i-1}]
\]

\[
C \equiv [-z_i, \ldots, k_{N-1}, -z_{N-1}, k_N, -z_N]
\]

\[
D \equiv [-k_{i+1}, \ldots, k_N, -z_N, k_N, -z_N]
\]

\[
E \equiv [-z_0, k_1, -z_1, k_2, \ldots, -z_{i-1} - z_i, \ldots, k_{N-1}, -z_{N-1} - z_N, k_N, -z_N]
\] (13)

Equations (8) and (9) apply the properties of Gaussian brackets in a similar manner. However, the formula is complicated, and the results are mentioned in References [12,13]. Nevertheless, Equations (8) and (9) can be solved using the “solve” command in MATLAB. Thereby, the solution of the unknown $\Delta z_i$ can be found and reflected in the zoom locus. The Padé approximation is a generalized form of Taylor series expansion, and it is defined as Equation (14) below as the ratio of two polynomials [28]:

\[
[L/M] = \frac{P_L(x)}{Q_M(x)}
\] (14)

where $P_L(x)$ is an $L$-order polynomial such as Equation (15), and $Q_M(x)$ is an $M$-order polynomial such as Equation (16).

\[
P_L(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots + a_L(x - x_0)^L
\] (15)

\[
Q_M(x) = f_0 + f_1(x - x_0) + f_2(x - x_0)^2 + f_3(x - x_0)^3 + \cdots + f_M(x - x_0)^M
\] (16)

Generally, we make an approximation by assuming that $L$ and $M$ are the same. Here, the constant term of the denominator is normalized to 1 to express the equation as shown in Equation (17).

\[
f(x) \approx \frac{a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0) + f_0}{\beta_1(x - x_0)^n + \beta_2(x - x_0)^{n-1} + \cdots + \beta_n(x - x_0) + 1}
\] (17)

where $\alpha$ is the coefficient of the numerator term, $\beta$ is the coefficient of the denominator term, $x$ is the position to start approximation, and the $x_0$ value represents the translation of the function to the x-axis. Since each coefficient is paired, an even number of data is required. To calculate the coefficients of the numerator and denominator in Equation (17), we can arrange as Equations (18)–(21).

\[
a_1(x_i - x_0)^n + \cdots + a_n(x_i - x_0) + f_0 = f_i[\beta_1(x_i - x_0)^n + \cdots + \beta_n(x_i - x_0) + 1]
\] (18)

\[
a_1(x_i - x_0)^n + \cdots + a_n(x_i - x_0) = f_i[\beta_1(x_i - x_0)^n + \cdots + \beta_n(x_i - x_0)] + f_i - f_0
\] (19)

\[
a_1(x_i - x_0)^{n-1} + \cdots + a_n = f_i[\beta_1(x_i - x_0)^{n-1} + \cdots + \beta_n] + (f_i - f_0) / (x_i - x_0)
\] (20)
Equations (18)–(21) in matrix form are the same as in Equation (22).

Equations (18)–(21) in matrix form are the same as in Equation (22).

\[
\begin{bmatrix}
    (x_1 - x_0)^{n-1} & \cdots & 1 & -f_1(x_1 - x_0)^{n-1} & \cdots & -f_1 \\
    (x_2 - x_0)^{n-1} & \cdots & 1 & -f_2(x_2 - x_0)^{n-1} & \cdots & -f_2 \\
    \vdots & \cdots & \vdots & \cdots & \vdots & \cdots \\
    (x_n - x_0)^{n-1} & \cdots & 1 & -f_n(x_n - x_0)^{n-1} & \cdots & -f_n \\
    (x_{n+1} - x_0)^{n-1} & \cdots & 1 & -f_{n+1}(x_{n+1} - x_0)^{n-1} & \cdots & -f_{n+1} \\
    \vdots & \cdots & \vdots & \cdots & \vdots & \cdots \\
    (x_{2n} - x_0)^{n-1} & \cdots & 1 & -f_{2n}(x_{2n} - x_0)^{n-1} & \cdots & -f_{2n}
\end{bmatrix}
\begin{bmatrix}
    \alpha_1 \\
    \alpha_2 \\
    \vdots \\
    \alpha_n \\
    \beta_1 \\
    \vdots \\
    \beta_n
\end{bmatrix}
= \begin{bmatrix}
    \frac{f_1 - f_0}{x_1 - x_0} \\
    \frac{f_2 - f_0}{x_2 - x_0} \\
    \vdots \\
    \frac{f_n - f_0}{x_n - x_0} \\
    \frac{f_{n+1} - f_0}{x_{n+1} - x_0} \\
    \vdots \\
    \frac{f_{2n} - f_0}{x_{2n} - x_0}
\end{bmatrix}
\tag{22}
\]

In the above matrix equations, the values except coefficients are determined by the optical design and are unknown variables \( \alpha \) and \( \beta \). To determine these, both sides are multiplied by the inverse matrix of the right term, and the unknown variables can be calculated as shown in Equation (23).

\[
\begin{bmatrix}
    \alpha_1 \\
    \alpha_2 \\
    \vdots \\
    \alpha_n \\
    \beta_1 \\
    \vdots \\
    \beta_n
\end{bmatrix}
= \begin{bmatrix}
    (x_1 - x_0)^{n-1} & \cdots & 1 & -f_1(x_1 - x_0)^{n-1} & \cdots & -f_1 \\
    (x_2 - x_0)^{n-1} & \cdots & 1 & -f_2(x_2 - x_0)^{n-1} & \cdots & -f_2 \\
    \vdots & \cdots & \vdots & \cdots & \vdots & \cdots \\
    (x_n - x_0)^{n-1} & \cdots & 1 & -f_n(x_n - x_0)^{n-1} & \cdots & -f_n \\
    (x_{n+1} - x_0)^{n-1} & \cdots & 1 & -f_{n+1}(x_{n+1} - x_0)^{n-1} & \cdots & -f_{n+1} \\
    \vdots & \cdots & \vdots & \cdots & \vdots & \cdots \\
    (x_{2n} - x_0)^{n-1} & \cdots & 1 & -f_{2n}(x_{2n} - x_0)^{n-1} & \cdots & -f_{2n}
\end{bmatrix}^{-1}\begin{bmatrix}
    \frac{f_1 - f_0}{x_1 - x_0} \\
    \frac{f_2 - f_0}{x_2 - x_0} \\
    \vdots \\
    \frac{f_n - f_0}{x_n - x_0} \\
    \frac{f_{n+1} - f_0}{x_{n+1} - x_0} \\
    \vdots \\
    \frac{f_{2n} - f_0}{x_{2n} - x_0}
\end{bmatrix}
\tag{23}
\]

Algorithm 1 shows the MATLAB code for calculating Equation (23). Thus, the coefficients of each term can be calculated to obtain an analytic function, as shown in Equation (17). The description of the MATLAB code is shown in Algorithm A1 of Appendix A.

---

**Algorithm 1: MATLAB code for Padé approximation**

1: \textbf{function} \texttt{0: PadéApproximation}(ff, x, x0, n)
2: \texttt{X} = \texttt{x-x0}
3: \texttt{df} = (ff-\texttt{f0})/\texttt{X}
4: \texttt{A} = \texttt{zeros}(2^n,2^n)
5: \texttt{for} \texttt{i} = \texttt{1:n-1}
6: \texttt{A(\texttt{i},\texttt{i})} = \texttt{X}^\texttt{i}
7: \texttt{A(\texttt{i},\texttt{n+i}) = \texttt{-f}.*(\texttt{X}^\texttt{i})}
8: \texttt{end}
9: \texttt{A(\texttt{i},\texttt{n})} = \texttt{ones}(2^n,1)
10: \texttt{A(\texttt{:}2^n) = -f}
11: \texttt{B} = \texttt{A}\backslash\texttt{df}
12: \texttt{a} = \texttt{B(1:n,:)}
13: \texttt{b} = \texttt{B(n+1:2^n,:)}

---

To analyze the error, we need to find the BFL which corresponds to \( z_N \) in Equation (9). Therefore, using the property of the Gaussian bracket, Equation (24) is constructed [29]. If the BFL falls within the DOF, the error can be ignored. DOF is given by Equation (25) [30]. The pixel size of the sensor used in
this study is assumed to be 5 µm. If the error of the BFL is smaller than the DOF, there is no need to reapply the Padé approximation.

\[
BFL = z_N = \left[ k_1, -z_1, k_2, -z_2, \cdots, k_{i-1}, -z_{i-1}\right] \\
\left[ k_1, -z_1, k_2, -z_2, \cdots, k_{i-1}, -z_{i-1}, k_i\right]
\]

(24)

\[
DOF = 2 \cdot p \cdot FNO
\]

(25)

where \(FNO\) in Equation (25) is the \(F/#\) (f-number) of the optical system, and \(p\) is the pixel size of the sensor.

### 3. Results and Discussion

We attempt to verify this method by calculating the zoom locus of a complex zoom system with a large number of moving groups. Table 1 summarizes the results of the zoom locus calculated by applying the Padé approximation for two optical systems. The two optical systems were calculated for every two cases. As a result of the calculation of the zoom locus, the error between the method mentioned in this paper and method by the conventional locus calculation is less than about 0.001 mm, and this error is within DOF, so it does not affect the resolution performances of the optical systems. Figures and Tables which are not mentioned in Table 1 are the results of the initial locus calculation before applying the Padé approximation.

Table 1. Summary of examples of optical systems and locus calculation results used in this paper.

| Optical Layout | KR10-1890304 (Figure 2a) | KR-2014-0111861 (Figure 2b) |
|----------------|---------------------------|-----------------------------|
| Linearization for the 1st group | Figure 5 | Figure 10 |
| Linearization for the 2nd group | Table 3 (locus), Figure 6 (locus), Table 4 (coefficients) | Table 11 (locus), Figure 12 (locus), Table 12 (coefficients) |
| Linearization for EFL | Table 7 (locus), Figure 9 (locus), Table 8 (coefficients) | Table 13 (locus), Figure 13 (locus), Table 14 (coefficients) |

Table 2 shows the distance between the lens groups in the optical system for Example 1 of patent KR 10-1890304 [16], which is assumed to be a zoom lens product of 16–50 mm, \(F/2.0\) to \(F/2.8\). The optical layout for this optical system is shown in Figure 5, and it can be seen that it consists of a total of five lens groups. You can also see that the second lens group is fixed relative to the image plane.

Table 2. Zoom data of the first example in KR10-1890304.

| Zoom | EFL | THI S5 | THI S11 | THI S14 | THI S20 | THI S32 | BFL |
|------|-----|-------|--------|--------|--------|--------|-----|
| 1 (wide) | 16.5995 | 1.2000 | 10.4100 | 12.3700 | 10.6500 | 20.7200 | 0.5004 |
| 2 | 18.6691 | 4.5000 | 12.3581 | 10.0464 | 8.3049 | 23.4319 | 0.5104 |
| 3 | 23.6565 | 10.0000 | 13.3228 | 6.7996 | 5.2673 | 28.7509 | 0.5082 |
| 4 | 26.3337 | 12.2000 | 12.9100 | 5.7200 | 4.2801 | 31.2020 | 0.5072 |
| 5 | 39.5063 | 19.6800 | 8.0926 | 2.9859 | 1.9437 | 41.0420 | 0.5103 |
| 6 (tele) | 48.5032 | 23.2000 | 4.0000 | 2.2000 | 1.4700 | 46.3700 | 0.5053 |

THI S5 is the distance between the first and second lens groups, THI S5 is the distance between the second and third lens groups, THI S14 is the distance between the third and fourth lens groups, THI S20 is the distance between the fourth and fifth lens groups, THI S32 is the distance between the fifth lens group and cover glass, and back focal length (BFL) is the distance between the cover glass and image plane.
the cam [31–34]. Typically, the products rotate the ring to change the field of view. The rotation angle of the zoom ring is determined for each product, however the maximum angle of the zoom ring was normalized in this study.

Table 3. Results of Padé approximation assuming the first lens group moves linearly.

| Normalization Cam Angle | EFL   | THI S5 | THI S11 | THI S14 | THI S20 | THI S32 | BFL   |
|-------------------------|-------|--------|---------|---------|---------|---------|-------|
| 1 (tele)                | 48.5032 | 23.2000 | 4.0000  | 2.2000  | 1.4700  | 46.3700 | 0.5053|
| 0.9                    | 42.6389 | 21.0000 | 6.6948  | 2.6536  | 1.7127  | 43.0024 | 0.5088|
| 0.8                    | 37.5687 | 18.8000 | 8.9332  | 3.2324  | 2.1306  | 39.7675 | 0.5111|
| 0.7                    | 33.2143 | 16.6000 | 10.7152 | 3.9364  | 2.7054  | 36.7110 | 0.5106|
| 0.6                    | 29.4954 | 14.4000 | 12.0408 | 4.7656  | 3.4246  | 33.8540 | 0.5082|
| 0.5                    | 26.3337 | 12.2000 | 12.9100 | 5.7200  | 4.2801  | 31.2020 | 0.5072|
| 0.4                    | 23.6565 | 10.0000 | 13.3228 | 6.7996  | 5.2673  | 28.7509 | 0.5083|
| 0.3                    | 21.3981 | 8.8000  | 13.2792 | 8.0444  | 6.3843  | 26.4910 | 0.5103|
| 0.2                    | 19.4996 | 7.6000  | 12.7792 | 9.3344  | 7.6316  | 24.4098 | 0.5116|
| 0.1                    | 17.9099 | 5.4000  | 11.8229 | 10.7897 | 9.0119  | 22.4935 | 0.5072|
| 0 (wide)               | 16.5995 | 1.2000  | 10.4100 | 12.3700 | 10.650  | 20.7200 | 0.5004|

Figure 6 shows the zoom locus, EFL change, and BFL error drawn from the data according to Table 3. Here, Figure 6 is a graph drawn up to the image plane and actually shows the locus of the lens group movement. As shown in Figure 6c, the error of BFL is within the DOF, hence there is no need for additional correction or iteration calculation. Table 4 shows the coefficients of the rational function for the displacement of each lens group calculated by the Padé approximation.
Figure 6. Movement of each lens group assuming linear movement of the first lens group. (a) Zoom locus of each lens group, (b) effective focal length (EFL) of the optical system, and (c) error for image plane.

Table 4. Coefficients of rational function for each lens group calculated by the Padé approximation.

| n | Group 1 | Group 2 | Group 3 |
|---|---------|---------|---------|
|   | $\alpha_n$ | $\beta_n$ | $\alpha_n$ | $\beta_n$ | $\alpha_n$ | $\beta_n$ |
| 1 | $f(x) = 22x + 1.2$ | $-1.332760 \times 10^{15}$ | $3.733451 \times 10^{9}$ | $9.246665 \times 10^{14}$ | $3.991029 \times 10^{10}$ |
| 2 | $9.583397 \times 10^{14}$ | $-6.689493 \times 10^{9}$ | $-2.427209 \times 10^{15}$ | $3.880115 \times 10^{10}$ |
| 3 | $6.080430 \times 10^{14}$ | $5.840851 \times 10^{13}$ | $1.827784 \times 10^{15}$ | $1.477584 \times 10^{14}$ |

Table 5 shows the results of the Padé approximation for the spacing of each lens group, assuming that the EFL varies linearly with the cam angle. Figure 7 likewise shows the locus, EFL changes, and BFL errors of each lens group.
Table 5. Linearization of EFL using the Padé approximation.

| Normalization Cam Angle | EFL   | THI S5 | THI S11 | THI S14 | THI S20 | THI S32 | BFL       |
|-------------------------|-------|--------|---------|---------|---------|---------|-----------|
| 1                       | 48.5032 | 23.2000 | 4.0000  | 2.2000  | 1.4700  | 46.3700 | 0.5053    |
| 0.9                     | 45.3128 | 22.0439 | 5.4732  | 2.7003  | 1.7412  | 42.6806 | 0.5373    |
| 0.8                     | 42.1225 | 20.7916 | 6.9264  | 3.0560  | 2.0012  | 40.6701 | 0.5070    |
| 0.7                     | 38.9321 | 19.4244 | 8.3443  | 3.5032  | 2.4502  | 38.5226 | 0.4935    |
| 0.6                     | 35.7417 | 17.1619 | 9.7042  | 4.0032  | 2.9570  | 36.2236 | 0.4918    |
| 0.5                     | 32.5514 | 14.8301 | 10.9704 | 4.5032  | 3.4502  | 34.0226 | 0.4991    |
| 0.4                     | 29.3610 | 12.0761 | 12.0848 | 5.5032  | 4.0012  | 31.8226 | 0.5072    |
| 0.3                     | 26.1706 | 8.3994  | 12.9453 | 6.5032  | 4.7512  | 29.6226 | 0.5029    |
| 0.2                     | 22.9803 | 5.3994  | 12.8942 | 8.5032  | 5.7012  | 27.4226 | 0.5004    |
| 0.1                     | 19.7899 | 1.2000  | 10.4100 | 12.3700 | 7.1012  | 25.2226 | 0.4991    |
| 0.0                     | 16.5995 | 1.2000  | 0.5000  | 12.3700 | 10.6500 | 20.7200 | 0.5004    |

As shown in Figure 7c, the error of BFL is larger than DOF. Therefore, the solution must be made by combining Equations (8) and (9). At this time, the first and the fourth lens groups were corrected. The results for this case are shown in Table 6 and Figure 8.
Table 6. Result of correcting the error for the image plane and EFL by moving the first and fourth lens groups.

| Normalization Cam Angle | EFL   | THI S5 | THI S11 | THI S14 | THI S20 | THI S32 | BFL  |
|-------------------------|-------|--------|---------|---------|---------|---------|------|
| 1                       | 48.5032| 23.2070| 4.0000  | 2.2000  | 1.4693  | 46.3700 | 0.5004 |
| 0.9                     | 45.3128| 22.0489| 5.4732  | 2.4259  | 1.5604  | 44.5806 | 0.5004 |
| 0.8                     | 42.1225| 20.8026| 6.9264  | 2.7061  | 1.7354  | 42.6845 | 0.5004 |
| 0.7                     | 38.9321| 19.4419| 8.3443  | 3.0572  | 1.9950  | 40.6701 | 0.5004 |
| 0.6                     | 35.7417| 17.9377| 9.7042  | 3.5007  | 2.3501  | 38.5226 | 0.5004 |
| 0.5                     | 32.5514| 16.2513| 10.9704 | 4.0671  | 2.8242  | 36.2236 | 0.5004 |
| 0.4                     | 29.3610| 14.3296| 12.0848 | 4.8023  | 3.4594  | 33.7479 | 0.5004 |
| 0.3                     | 26.1706| 12.0918| 12.9453 | 5.7800  | 4.3295  | 31.0586 | 0.5004 |
| 0.2                     | 22.9803| 9.4019 | 13.3569 | 7.1301  | 5.5667  | 28.0955 | 0.5004 |
| 0.1                     | 19.7899| 5.9946 | 12.8942 | 9.1115  | 7.4207  | 24.7435 | 0.5004 |
| 0                      | 16.5995| 1.2000 | 10.4100 | 12.3700 | 10.6500 | 20.7200 | 0.5004 |

Figure 8. Result of correcting error for image plane and EFL by moving first and fourth lens groups, as shown in Table 6. (a) Zoom locus of each lens group, (b) EFL of this optical system, and (c) error for image plane.

The results obtained by applying the Padé approximation to the data shown in Table 6 and Figure 8 are illustrated in Table 7 and Figure 9. Further, the coefficients for the equations of each lens group are shown in Table 8.
Table 7. Results of reapplying the Padé approximation to results in Table 6.

| Normalization Cam Angle | EFL   | THI S5 | THI S11 | THI S14 | THI S20 | THI S32 | BFL  |
|-------------------------|-------|--------|---------|---------|---------|---------|------|
| 1                       | 48.5032 | 23.2070 | 4.0000  | 2.2000  | 1.4693  | 46.3700 | 0.5004 |
| 0.9                     | 45.3128 | 22.0488 | 5.4732  | 2.4259  | 1.5604  | 44.5806 | 0.5004 |
| 0.8                     | 42.1225 | 20.8025 | 6.9264  | 2.7061  | 1.7354  | 42.6845 | 0.5004 |
| 0.7                     | 38.9321 | 19.4419 | 8.3443  | 3.0572  | 1.9950  | 40.6701 | 0.5004 |
| 0.6                     | 35.7417 | 17.9377 | 9.7042  | 3.5007  | 2.3501  | 38.5226 | 0.5004 |
| 0.5                     | 32.5514 | 16.2512 | 10.9704 | 4.0671  | 2.8242  | 36.2236 | 0.5004 |
| 0.4                     | 29.3610 | 14.3295 | 12.0848 | 4.8023  | 3.4594  | 33.7479 | 0.5004 |
| 0.3                     | 26.1706 | 12.092  | 12.9453 | 5.7800  | 4.3295  | 31.0586 | 0.5004 |
| 0.2                     | 22.9803 | 9.4019  | 13.3569 | 7.1301  | 5.5667  | 28.0955 | 0.5004 |
| 0.1                     | 19.7899 | 5.9946  | 12.8942 | 9.1115  | 7.4207  | 24.7435 | 0.5004 |
| 0                     | 16.5995 | 1.2000  | 10.4100 | 12.3700 | 10.650  | 20.7200 | 0.5004 |

Figure 9. Results of the reapplication of the Padé approximation to results in Table 6. (a) Zoom locus of each lens group, (b) EFL of this optical system, and (c) error for image plane.
Table 8. Coefficients of the rational function for each lens group calculated by the Padé approximation, shown in Table 5.

| n  | \( a_n \) | \( \beta_n \) | \( a_n \) | \( \beta_n \) | \( a_n \) | \( \beta_n \) |
|----|-----------|-----------|-----------|-----------|-----------|-----------|
| 1  | -2.036482 \( \times 10^{13} \) | -4.344269 \( \times 10^{11} \) | 2.513836 \( \times 10^{14} \) | -1.295515 \( \times 10^{12} \) | -1.251574 \( \times 10^{13} \) | -5.637689 \( \times 10^{12} \) |
| 2  | 4.065631 \( \times 10^{13} \) | 3.101308 \( \times 10^{11} \) | -7.79830 \( \times 10^{14} \) | -1.166387 \( \times 10^{13} \) | 3.884953 \( \times 10^{13} \) | -5.173036 \( \times 10^{12} \) |
| 3  | -3.779113 \( \times 10^{12} \) | 1.425480 \( \times 10^{12} \) | 8.215034 \( \times 10^{13} \) | 2.422190 \( \times 10^{13} \) | 1.034836 \( \times 10^{14} \) | 6.488446 \( \times 10^{12} \) |
| 4  | -1.301273 \( \times 10^{13} \) | -6.974780 \( \times 10^{11} \) | -3.303085 \( \times 10^{14} \) | 9.862482 \( \times 10^{12} \) | 9.636515 \( \times 10^{13} \) | -1.495082 \( \times 10^{12} \) |
| 5  | 1.640220 \( \times 10^{12} \) | -1.282749 \( \times 10^{11} \) | 2.398983 \( \times 10^{13} \) | -7.652251 \( \times 10^{11} \) | -3.162410 \( \times 10^{13} \) | -1.122007 \( \times 10^{12} \) |
| 6  | 3.796005 \( \times 10^{10} \) | 3.063338 \( \times 10^{11} \) | 7.709175 \( \times 10^{12} \) | 7.395548 \( \times 10^{11} \) | -3.042863 \( \times 10^{13} \) | -2.460873 \( \times 10^{12} \) |

Table 9 shows the distance between the lens groups in the optical system for Example 1 of patent KR-2014-0111861 [17], which is assumed to be a zoom lens product of 50-150 mm, F/2.8. The optical layout for this optical system is shown in Figure 10, and it can be seen that it consists of a total of five lens groups. You can also see that the first and the fifth lens group are fixed relative to the image plane. Similarly, the results of the Padé approximation assuming that the second lens group moves in a straight line from the data shown in Table 9 are listed in Table 10. Further, from this data, the loci of each lens group, EFL, BFL error of the whole optical system are shown in Figure 11.

Table 9. Zoom data of the first example in KR10-2014-011186.

| Number of Zoom | EFL | THI S7 | THI S13 | THI S15 | THI S20 | THI S39 | BFL |
|----------------|-----|--------|--------|--------|--------|--------|-----|
| 1 (Wide)       | 51.4947 | 2.2569 | 6.3454 | 18.1084 | 23.6443 | 25.3410 | 1.0063 |
| 2              | 56.2267 | 6.1254 | 6.0855 | 17.3773 | 20.7905 | 25.3410 | 1.0063 |
| 3              | 61.8577 | 10.0161 | 6.1163 | 16.4780 | 17.7378 | 25.3410 | 1.0183 |
| 4              | 68.6140 | 13.8965 | 6.4696 | 15.3905 | 14.5932 | 25.3410 | 1.0110 |
| 5              | 77.0188 | 17.7743 | 7.1378 | 14.0380 | 11.4052 | 25.3410 | 1.0063 |
| 6 (Tele)       | 145.3643 | 32.0985 | 12.3055 | 3.6140 | 2.3370 | 25.3410 | 1.0063 |

THI S7 is the distance between the first and second groups, THI S13 is the distance between the second and third lens groups, THI S15 is the distance between the third and fourth lens groups, THI S20 is the distance between the fourth and fifth groups, THI S39 is the distance between the fifth lens group and cover glass, and BFL is the distance between the cover glass and image plane.

Table 10. Padé approximation linearization of the second lens group for the optical system shown in Figure 10.

| Normalization Cam Angle | EFL | THI S7 | THI S13 | THI S15 | THI S20 | THI S39 | BFL |
|------------------------|-----|--------|--------|--------|--------|--------|-----|
| 0                      | 51.4947 | 2.2569 | 6.3454 | 18.1084 | 23.6443 | 25.3410 | 1.0063 |
| 0.1                    | 55.0677 | 5.2410 | 6.1247 | 17.5608 | 21.4672 | 25.3410 | 0.9984 |
| 0.2                    | 59.1477 | 8.2252 | 6.0614 | 16.9121 | 19.1375 | 25.3410 | 1.0166 |
| 0.3                    | 63.7943 | 11.2094 | 6.1910 | 16.1668 | 16.7796 | 25.3410 | 1.0170 |
| 0.4                    | 69.1913 | 14.1935 | 6.5097 | 15.2975 | 14.3497 | 25.3410 | 1.0102 |
| 0.5                    | 75.5902 | 17.1777 | 7.0147 | 14.2674 | 11.8935 | 25.3410 | 1.0061 |
| 0.6                    | 83.3501 | 20.1619 | 7.7041 | 13.0250 | 9.4613 | 25.3410 | 1.0133 |
| 0.7                    | 93.0991 | 23.1460 | 8.5772 | 11.4949 | 7.1209 | 25.3410 | 1.0380 |
| 0.8                    | 105.4162 | 26.1302 | 9.6340 | 9.5620 | 4.9944 | 25.3410 | 1.0772 |
| 0.9                    | 121.9990 | 29.1143 | 10.8763 | 7.0412 | 3.2833 | 25.3410 | 1.1020 |
| 1                      | 145.3643 | 32.0985 | 12.3055 | 3.6140 | 2.3370 | 25.3410 | 1.0064 |
Figure 10. Optical layout of the first example in KR-2014-0111861 [17].

Table 10. Padé approximation linearization of the second lens group for the optical system shown in Figure 10.

| Normalization | Cam Angle | EFL  | THI S7 | THI S13 | THI S15 | THI S20 | THI S39 | BFL    |
|---------------|-----------|------|--------|---------|---------|---------|---------|--------|
| 0             | 51.4947   | 2.25690| 6.34540| 18.1084 | 23.6443 | 25.3410 | 1.0063  |
| 0.1           | 55.0677   | 5.24110| 6.12470| 17.5608 | 21.4672 | 25.3410 | 0.9984  |
| 0.2           | 59.1477   | 8.22520| 6.06140| 16.9121 | 19.1575 | 25.3410 | 1.0166  |
| 0.3           | 63.7943   | 11.2094| 6.19100| 16.1668 | 16.7796 | 25.3410 | 1.0170  |
| 0.4           | 69.1913   | 14.1935| 6.50970| 15.2975 | 14.3497 | 25.3410 | 1.0102  |
| 0.5           | 75.5902   | 17.1777| 7.01470| 14.2674 | 11.8953 | 25.3410 | 1.0061  |
| 0.6           | 83.3501   | 20.1619| 7.70410| 13.0250 | 9.4613  | 25.3410 | 1.0133  |
| 0.7           | 93.0091   | 23.1460| 8.57720| 11.4949 | 7.1209  | 25.3410 | 1.0380  |
| 0.8           | 105.4162  | 26.1302| 9.63430| 9.5620  | 4.9944  | 25.3410 | 1.0772  |
| 0.9           | 121.9990  | 29.1143| 10.8763| 7.0412  | 3.2833  | 25.3410 | 1.1020  |
| 1             | 145.3643  | 32.0985| 12.3055| 3.6140  | 2.3370  | 25.3410 | 1.0064  |

Figure 11. Linearization of the second lens group using the Padé approximation, as shown in Table 10. (a) Zoom locus of each lens group, (b) EFL of this optical system, and (c) error for image plane (BFL).
As shown in Figure 11c, the error is larger than DOF, hence the fourth lens group needs to move to correct it. The Padé approximation of the fourth lens group used in the calibration is shown in Table 11. Similarly, Table 11 lists the locus, EFL, and BFL errors of each lens group, as shown in Figure 12. Moreover, the coefficients of the function according to the movement of each lens group are shown in Table 12. Figure 12c indicates that the error for BFL is small enough to be ignored.

Table 11. Results of reapplying the Padé approximation after correcting BFL by moving the fourth lens group from the results in Table 10.

| Normalization Cam Angle | EFL       | THI S7 | THI S13 | THI S15 | THI S20 | THI S39 | BFL    |
|-------------------------|-----------|--------|---------|---------|---------|---------|--------|
| 1                       | 145.3645  | 32.0985| 12.3055 | 3.6140  | 2.3370  | 25.3410 | 1.00629|
| 0.9                     | 121.9377  | 29.1143| 10.8763 | 7.0885  | 3.2359  | 25.3410 | 1.00629|
| 0.8                     | 105.3703  | 26.1302| 9.6343  | 9.5971  | 4.9593  | 25.3410 | 1.00634|
| 0.7                     | 92.9890   | 23.1460| 8.5772  | 11.5106 | 7.1052  | 25.3410 | 1.00633|
| 0.6                     | 83.3463   | 20.1619| 7.7041  | 13.0285 | 9.4579  | 25.3410 | 1.00628|
| 0.5                     | 75.5900   | 17.1777| 6.1910  | 14.1722 | 11.8952 | 25.3410 | 1.00626|
| 0.4                     | 69.1856   | 14.1935| 6.5097  | 15.2995 | 14.3477 | 25.3410 | 1.00623|
| 0.3                     | 69.1856   | 14.1935| 6.5097  | 15.2995 | 14.3477 | 25.3410 | 1.00623|
| 0.2                     | 63.7811   | 11.2094| 6.1910  | 16.1722 | 16.7742 | 25.3410 | 1.00618|
| 0.1                     | 59.1353   | 8.2252 | 6.0614  | 16.9173 | 19.1523 | 25.3410 | 1.00622|
| 0                       | 55.0774   | 5.2411 | 6.1247  | 17.5569 | 21.4712 | 25.3410 | 1.00628|

Figure 12. Graphs of the reapplication of the Padé approximation after correcting BFL by moving the fourth lens group from the results in Table 11. (a) Zoom locus of each lens group, (b) EFL of this optical system, and (c) error for image plane.
Table 12. Coefficients of the rational function for each lens group calculated by the Padé approximation shown in Table 11.

| n | Group 1 | Group 2 | Group 3 |
|---|---|---|---|
| alpha | beta | alpha | beta | alpha | beta |
| 1 | 9.783476 $\times 10^{12}$ | -4.655744 $\times 10^{10}$ | 7.740465 $\times 10^{12}$ | -2.192537 $\times 10^{12}$ |
| 2 | -1.250194 $\times 10^{13}$ | 1.458901 $\times 10^{11}$ | -1.165847 $\times 10^{14}$ | -2.075092 $\times 10^{12}$ |
| 3 | 1.097399 $\times 10^{13}$ | 9.178999 $\times 10^{11}$ | -9.678796 $\times 10^{14}$ | -3.574061 $\times 10^{13}$ |
| 4 | -6.766977 $\times 10^{12}$ | -9.712168 $\times 10^{14}$ | 1.356786 $\times 10^{15}$ | 7.172526 $\times 10^{13}$ |
| 5 | 6.636467 $\times 10^{12}$ | 2.796794 $\times 10^{11}$ | -2.321274 $\times 10^{14}$ | -4.508796 $\times 10^{12}$ |

At this time, the locus of each lens group was calculated such that the EFL changes linearly. In this case, the third and fourth lens groups were corrected. After correction, the distance between each lens group was calculated using the Padé approximation. Figure 13 shows the locus, EFL change, and BFL error of each lens group from Table 13. In addition, the coefficients of the function for the amount of movement of each lens group are shown in Table 14. Furthermore, in this case, the Padé approximation can represent the distance of each lens group as an analytic function, and the error is sufficiently small, as shown in Figure 13c.

Table 13. After moving the third and fourth lens groups to correct EFL and BFL, results of reapplying the Padé approximation to linearize EFL for the optical system shown in Figure 10.

| Normalization Cam Angle | EFL | THI S7 | THI S13 | THI S15 | THI S20 | THI S39 | BFL |
|---|---|---|---|---|---|---|---|
| 1 | 145.3644 | 32.0985 | 12.3055 | 3.6140 | 2.33700 | 25.3410 | 1.0063 |
| 0.9 | 135.9774 | 31.0361 | 11.7720 | 4.9973 | 2.61080 | 25.3410 | 1.0062 |
| 0.8 | 126.5904 | 29.8029 | 11.1841 | 6.3916 | 3.06450 | 25.3410 | 1.0062 |
| 0.7 | 117.2035 | 28.3552 | 10.5361 | 7.7992 | 3.75100 | 25.3410 | 1.0063 |
| 0.6 | 107.8165 | 26.6339 | 9.82450 | 9.2227 | 4.74020 | 25.3410 | 1.0062 |
| 0.5 | 98.42950 | 24.5369 | 9.04930 | 10.6648 | 6.12260 | 25.3410 | 1.0064 |
| 0.4 | 89.04260 | 22.0077 | 8.22050 | 12.1287 | 8.01330 | 25.3410 | 1.0064 |
| 0.3 | 79.65560 | 19.8174 | 7.37070 | 13.6161 | 10.5531 | 25.3410 | 1.0062 |
| 0.2 | 70.26860 | 17.7666 | 6.58550 | 15.1247 | 13.9042 | 25.3410 | 1.0062 |
| 0.1 | 60.88170 | 15.9360 | 5.80100 | 16.6388 | 18.2339 | 25.3410 | 1.0062 |
| 0 | 51.49470 | 13.2590 | 5.01700 | 18.1084 | 23.6443 | 25.3410 | 1.0063 |

Figure 13. Cont.
Table 14. Coefficients of the rational function for the locus of each lens group calculated by the Padé approximation in Table 13.

| n   | α_n | Group 1 | β_n | α_n | Group 2 | β_n | α_n | Group 3 | β_n   |
|-----|-----|---------|-----|-----|---------|-----|-----|---------|-------|
| 1   | -2.181989 × 10^14 | -4.735851 × 10^12 | 6.212852 × 10^12 | 2.899227 × 10^11 | 8.970257 × 10^14 | 1.268275 × 10^13 |
| 2   | 1.023146 × 10^14 | 2.250370 × 10^11 | 4.658154 × 10^12 | 4.072067 × 10^11 | -1.287229 × 10^15 | -6.935987 × 10^13 |
| 3   | -1.975053 × 10^13 | 3.917116 × 10^11 | -1.525487 × 10^13 | -4.694700 × 10^11 | -2.371261 × 10^14 | 1.789717 × 10^13 |
| 4   | -1.199393 × 10^13 | -3.190916 × 10^11 | 9.108839 × 10^10 | 1.426433 × 10^11 | 9.668934 × 10^13 | 2.990567 × 10^12 |
| 5   | 9.792404 × 10^12 | 3.47322 × 10^10 | 5.102177 × 10^11 | 7.940791 × 10^12 | -5.512953 × 10^13 | -3.045419 × 10^12 |
| 6   | 2.693676 × 10^11 | 1.183512 × 10^11 | 5.965493 × 10^12 | 5.102177 × 10^11 | 7.940791 × 10^12 | -5.512953 × 10^13 | -3.045419 × 10^12 |

4. Conclusions

The analytical zoom locus can be obtained by performing conventional zoom locus calculations. However, these methods require constraints on specific groups. Further, there is a problem of separately solving the zoom equations according to the change of the constraint. In this study, however, the movement of each lens group was determined as an analytic function by interpolating with the Padé approximation. In this process, the specific lens group can be corrected by moving BFL or EFL to the desired value. The magnitude of movement of this particular lens group was then interpolated back to the Padé approximation and expressed as an analytic function. This calculation process was iteratively calculated until the BFL’s error was within DOF. In this manner, we confirm that the movement of all lens groups can be expressed as an analytic function using conventional optical systems.

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Appendix A Description for MATLAB Code for Padé Approximation

Algorithm A1: MATLAB code description

1: function ()
2: X = x-x0 % calculate initial values for cam angle
3: df = (f-f0)./(X % calculate the right side of Equation (22)
4: A = zeros(2*n,2*n) % Initialization for matrix A at Equation (22)
5: % Calculation of matrix A
6: for i = 1:n-1
7: A(:,i) = X.^(n-i)
8: A(:,n+i) = -f.^(X.^(n-i))
9: end
10: A(:,n) = ones(2*n,1)
11: A(:,2*n) = -f
12: B = A
df % Calculation of Equation (22)
13: a = B(1:n,:) % Polynomial coefficients for the denominator
14: b = B((n+1):2*n,:) % Polynomial coefficients for the numerator

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