Sufficient and necessary causation are dual

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Abstract

Causation has been the issue of philosophic debate since Hippocrates. Recent work defines actual causation in terms of Pearl/Halpern’s causality framework, formalizing necessary causes (IJCAI’15). This has inspired causality notions in the security domain (CSF’15), which, perhaps surprisingly, formalize sufficient causes instead. We provide an explicit relation between necessary and sufficient causes.

Notation Let \( \mathbb{N} \) be the set of natural numbers and assume that they begin at 0. \( r[v \mapsto \text{val}] := (r \setminus (v, r(v))) \cup (v, \text{val}) \) is short-hand for the function mapping \( v \) to \( \text{val} \) and otherwise behaving like \( r \). We write \( \vec{t} \) for a sequence \( t_1, \ldots, t_n \) if \( n \) is clear from the context and denote the \( i \)th element with \( \vec{t}_i \). We use \( \vec{a} \cdot \vec{b} \) to denote concatenation of vectors \( \vec{a} \) and \( \vec{b} \). We filter a sequence \( l \) by a set \( S \), denoted \( l|_S \), by removing each element that is not in \( S \).

1 Causal model (Review)

We review the causal framework introduced by Pearl and Halpern [Pearl, 2000] [Halpern, 2015], also known as the structural equations model, which provides the notion of causality which we will investigate for the case of security protocols. The causality framework models how random variables influence each other. The set of random variables is partitioned into a set \( \mathcal{U} \) of exogenous variables, variables that are outside the model, e.g., in the case of a security protocol, the attack the adversary decides to mount, and a set \( \mathcal{V} \) of endogenous variables, which are ultimately determined by the value of the exogenous variables. We call the triple consisting of \( \mathcal{U}, \mathcal{V} \) and function \( \mathcal{R} \) associating a range to each variable \( Y \in \mathcal{U} \cup \mathcal{V} \) a signature. A causal model on this signature defines the relation between endogenous variables and exogenous variables or other endogenous variables in terms of a set of functions.

Definition 1 (Causal model). A causal model \( M \) over a signature \( \mathcal{S} = (\mathcal{U}, \mathcal{V}, \mathcal{R}) \) is a pair of said signature \( \mathcal{S} \) and a set of functions \( \mathcal{F} = \{ F_X \}_{X \in \mathcal{V}} \) such that, for each \( X \in \mathcal{V} \),

\[
F_X : (\times_{U \in \mathcal{U}} \mathcal{R}(U)) \times (\times_{Y \in \mathcal{V} \setminus \{X\}} \mathcal{R}(Y)) \rightarrow \mathcal{R}(X)
\]

Each causal model subsumes a causal network, a graph with a node for each variable in \( \mathcal{V} \), and an edge from \( X \) to \( Y \) iff \( F_X \) depends on \( X \). If the causal graph associated to a causal model \( M \) is acyclic, then each setting \( \vec{u} \) of the variables in \( \mathcal{U} \) provides a unique solution to the equations in \( M \). All causal models we will derive in this paper have this property. We call a vector setting the variables in \( \mathcal{U} \) a context, and a pair \( (M, \vec{u}) \) of a causal model and a setting a situation.

As hinted at in the introduction, the definition of causality follows a counterfactual approach, which requires to answer ‘what if’ questions.

Definition 2 (Modified causal model). Given a causal model \( M = (\mathcal{U}, \mathcal{V}, \mathcal{R}, \mathcal{F}) \), we define the modified causal model \( M_{\vec{X} \leftrightarrow \vec{Y}} \) over the signature \( \mathcal{S}_{\vec{X}} = (\mathcal{U}, \mathcal{V} \setminus \vec{X}, \mathcal{R}|_{\mathcal{V} \setminus \vec{X}}) \) by setting the values of each variable in \( \vec{X} \) to the corresponding element \( \vec{x} \) in each equation \( \mathcal{F}_\vec{Y} \in \mathcal{F} \), obtaining \( \mathcal{F}_{\vec{X} \leftrightarrow \vec{Y}} \). Then, \( M_{\vec{X} \leftrightarrow \vec{Y}} = (\mathcal{S}_{\vec{X}}, \mathcal{F}_{\vec{X} \leftrightarrow \vec{Y}}) \).

Definition 3 (Causal formula). A causal formula has the form \( [Y_1 \leftarrow y_1, \ldots, Y_n \leftarrow y_n] \phi \) (abbreviated \( \vec{Y} \leftarrow \vec{y} \phi \)), where

- \( \phi \) is a boolean combination of primitive events, i.e., formulas of form \( X = x \) for \( X \in \mathcal{V}, x \in \mathcal{R}(X) \),
- \( Y_1, \ldots, Y_n \in \mathcal{V} \cup \mathcal{U} \) distinct,
- \( y_i \in \mathcal{R}(Y_i) \).

We write \( (M, \vec{u}) \models [\vec{Y} \leftarrow \vec{y}] \phi \) if the (unique) solution to the equations in \( M_{\vec{X} \leftarrow \vec{Y}} \) in the context \( \vec{u} \) is an element of \( \phi \).

Furthermore, we allow intervention on exogenous variables. This is equivalent to a modification of the context: \( (M, \vec{u}) \models [U_i \leftarrow u'_i] \phi \) equals \( (M, (\vec{u}_{i-1}, u'_i, \vec{u}_i^{i+1})) \models \phi \). We review Halpern’s modification [Halpern, 2015] of Pearl and Halpern’s definition of actual causes [Halpern and Pearl, 2013].
Definition 4 (actual cause + necessary cause). \( \vec{X} = \vec{x} \) is a (minimal) actual cause of \( \varphi \) in \((M, \vec{u})\) if the following three conditions hold.

AC1. \( (M, \vec{u}) \models (\vec{X} = \vec{x}) \land \varphi \).

AC2. There is a set of variables \( \vec{W} \) and a setting \( \vec{x}' \) of the variables in \( \vec{X} \) such that if \( (M, \vec{u}) \models (\vec{W} = \vec{w}) \), then \( (M, \vec{u}) \models (\vec{X} \leftarrow \vec{x}', \vec{W} \leftarrow \vec{w}) \)\( \models \varphi \).

AC3. \( \vec{X} \) is minimal: No strict subset \( \vec{X}' \) of \( \vec{X} \) satisfies AC1 and AC2.

We say \( \vec{X} \) is an actual cause for \( \varphi \) if this is the case for some \( \vec{x} \). For a weaker AC2 as follows, we speak of a (minimal) necessary cause.

NC2. There exists \( \vec{x}' \) such that \( (M, \vec{u}) \models (\vec{X} \leftarrow \vec{x}') \models \varphi \).

2 Sufficient versus necessary causes

The major difference underlying actual causes according to DGKSS [Datta et al., 2015] and Pearl/Halpern [Halpern, 2015] is that the former considers sufficient rather than necessary causes. We transfer this concept to Pearl’s causation framework as follows.

Definition 5 (sufficient cause). Sufficient causes are defined like actual causes (see Definition 4), but with AC2 modified as follows: SF2. For all \( \vec{x} \), \( (M, \vec{u}) \models (\forall \vec{X} \leftarrow \vec{x}) \varphi \).

In this section, we show that sufficient causes and necessary causes (Definition 4) are dual to each other, and that sufficient causes are in fact preferable, as they have a clearer interpretation of what constitutes a part of a cause.

While several formalisations of sufficient causes were proposed [Datta et al., 2015; Gössler and Le Métayer, 2013], so far they were never related to necessary causes. Strongest necessary conditions and weakest sufficient conditions in propositional logic are known to be dual to each other, however, sufficient and necessary causes are first-order predicates, and there is no such result in first-order logic. Even defining these notions is problematic [Lin, 2001].

We fix some finite set \( \mathcal{V}^{\text{res}} \subseteq \mathcal{V} \) and some ordering \( \{V_1, \ldots, V_n\} = \mathcal{V}^{\text{res}} \). We will see in the next section why this restriction is useful. Let \( \mathcal{X} \) denote the bitstring representation of \( X \subseteq \mathcal{V}^{\text{res}} \) relative to \( \mathcal{V}^{\text{res}} \), i.e., \( \mathcal{X} := (1_X(V_1), \ldots, 1_X(V_n)) \). We can now represent the set of necessary causes, or more generally, any set of sets of variables \( \mathcal{X} = X_1, \ldots, X_m \), as a boolean formula in disjunctive normal form (DNF) that is true whenever \( \mathcal{X} \) is the bitstring representation of \( X \subseteq \mathcal{V}^{\text{res}} \) such that \( X \in \mathcal{X} \).

\[
(\mathcal{X} = \mathcal{X}_1 \lor \mathcal{X} = \mathcal{X}_2 \lor \cdots \lor \mathcal{X} = \mathcal{X}_m),
\]

where \( \mathcal{X}_i \) is a conjunction \( \bigwedge_{j \in \mathbb{N}} X_{i,j} = \bar{X}_{i,j} \).

\footnote{If \( \vec{X} = \vec{x} \) is an actual cause under contingency \( \vec{W} = \vec{w} \), then \( \vec{X} \cdot \vec{W} = \vec{x} \cdot \vec{w} \) is a necessary cause. Hence actual causes are parts of necessary causes.}

Theorem 1 (sufficient and necessary causes). For \( \mathcal{X} \) the set of (not necessarily minimal) necessary causes, let \( \overline{\mathcal{X}} \) be the DNF representation of \( \mathcal{X} \). Then the set of (not necessarily minimal) sufficient causes is represented by \( \mathcal{X} \), which is obtained from \( \overline{\mathcal{X}} \) by transforming \( \overline{\mathcal{X}} \) into CNF and switching \( \lor \) and \( \land \). The same holds for the other direction.

Proof. By Definition 4 NC2, we can rephrase the assumption as follows: \( \forall X, \exists x'. \ (M, \vec{u}) \models (X \leftarrow x'). \models \varphi \iff X \in \mathcal{X} \). Now the right-hand side is equivalent to \( (\mathcal{X} = \mathcal{X}_1) \lor \cdots \lor (\mathcal{X} = \mathcal{X}_m) \). This is a boolean function over \( \{0, 1\}^n \). As any boolean function can be transformed into canonical CNF, the right-hand side can be expressed as \( c_1 \land \cdots \land c_k \) with conjunctions \( c_i \) of form \( \lor_{j \in \mathbb{N}} \neg(\mathcal{X}_j) \).

\[
\forall X. c_1 \land \cdots \land c_k \iff \exists x'. \ (M, \vec{u}) \models (X \leftarrow x'). \models \varphi.
\]

Now we can negate both sides of the implication.

\[
\forall X. \neg c_1 \lor \cdots \lor \neg c_k \iff \forall x'. \ (M, \vec{u}) \models (X \leftarrow x'). \models \varphi.
\]

We rename \( X \) to \( Z \) and \( x' \) to \( z' \). Let \( b/a \) denote \( b \) literally replacing \( a \).

\[
\forall Z. \neg c_1 \lor \cdots \lor \neg c_k \iff \forall z'. \ (M, \vec{u}) \models (Z \leftarrow z'). \models \varphi.
\]

We can replace \( Z \) by \( X = \mathcal{V}^{\text{res}} \setminus Z \), as \( Z \models V^{\text{res}} \setminus Z \) is is a bijection between the domain of \( Z \) and the domain of \( X \). Thus

\[
\forall X. \neg c_1 \lor \cdots \lor \neg c_k \iff \forall z'. \ (M, \vec{u}) \models (\mathcal{X} \leftarrow z'). \models \varphi.
\]

As each conjunct \( c_i \) is a disjunction, the negation of \( c_i \) with \( \mathcal{X} \) substituted by \( \neg \mathcal{X} \) can be obtained by switching \( \lor \) and \( \land \). The resulting term is, again, a boolean formula in DNF, so \( \mathcal{X} \) transforms into \( \mathcal{X} \) easily. The reverse direction follows by first applying the above bijection and renaming backwards, and then following the first proof steps.

\( \square \)

To obtain the set of minimally sufficient causes from the set of minimally necessary causes, one saturates the former by adding all non-minimal elements (pick an element, and add its supersets by iteratively switching all zeros to ones until a fixed point is reached) and computes the set of (not-necessarily minimal) elements using the above method. The conversion to CNF can be performed via the Quine–McCluskey algorithm, which is the obvious bottleneck in this computation. Finally, the resulting set representation can be minimised by removing all elements \( X \) such that \( \neg X \land Y \) for some element \( Y \) (where \( \neg \) and \( \land \) are applied bitwise).
| necessary | sufficient |
|-----------|------------|
| Conj(1, 1) | (A), (B) |
| Disj(1, 1) | (A, B) |

Table 1: Comparison: set of all (minimal) necessary/sufficient causes.

References

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