Invited Paper

Piecewise linear switched dynamical systems: A review

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Abstract: This review paper introduces piecewise linear switched dynamical systems in three topics. In the first topic of autonomous chaotic circuits, we introduce the manifold piecewise linear system and chaotic spiking oscillator. Using piecewise exact solutions and mapping procedure, we obtain rigorous proof of chaos generation. In the second topic of recurrent neural networks, we introduce the hysteresis neural network and its application to associative memories. Performing theoretical analysis based on the piecewise exact solutions, we obtain parameter conditions for guaranteed storage of any desired memories. In the third topic of multiobjective optimization problems, we introduce a two-objective problem in a piecewise linear model of switching power converter with photovoltaic input. Applying a simple multiobjective evolutionary algorithm, we clarify existence of a trade-off between the maximum input power and circuit stability.

Key Words: switched dynamical systems, piecewise linear systems, chaos, chaotic spiking oscillators, recurrent neural networks, hysteresis neural networks, associative memories, multiobjective problems, multiobjective evolutionary algorithms

1. Introduction

A switched dynamical system is a continuous-time nonlinear system defined by multiple subsystems and nonlinear switching rules [1–4]. If the state variable fulfills a condition, a switching event occurs and the future dynamics is determined. Depending on the switching rules and nonlinear characteristics, the system exhibits nonlinear phenomena such as periodic orbits, synchronization, chaos, and bifurcation. Mathematical models of the switched dynamical system are used to describe/analyze various engineering systems including artificial neural networks, analog-to-digital converters, analog signal processing systems, and switching power converters. Analysis and synthesis of the switched dynamical systems are important not only as basic study of nonlinear dynamics but also for engineering applications.

This review paper introduces our previous works on piecewise linear switched dynamical systems. The dynamics is described by multiple linear subsystems connected by nonlinear switching rules. The advantages/disadvantages include the following.

- The systems have piecewise exact solutions. It is suitable for precise/exact analysis of the
dynamics and for understanding physical mechanism of the dynamics.

- The systems are suitable for hardware implementation by analog elements such as operational amplifiers, operational transconductance amplifiers, and analog switches.

- In the case where a piecewise linear approximation is applied to smooth original systems, approximation error is inevitable and only rough reproduction of the original dynamics is possible.

Since general discussion is not easy, we introduce the piecewise linear switched dynamical systems in three topics. The first topic is autonomous chaotic circuits. Since the first autonomous chaotic system, the Lorenz equation, is presented [5], a variety of autonomous chaotic circuits have been studied. It should be noted that the chaotic circuits do not imply just a realization method of existing mathematical models but important real physical systems to investigate interesting nonlinear phenomena. The Chua’s circuit is known as a typical example of autonomous chaotic circuits [6–10]. The circuit consists of three memory elements and one piecewise linear resistor. The dynamics is described by three-dimensional piecewise linear ordinary differential equation. Chaos and related bifurcation phenomena are analyzed theoretically. Presenting a simple hardware, chaos and bifurcation phenomena are confirmed experimentally.

The Chua’s circuit is not a switched dynamical system and we have presented various chaotic circuits as switched dynamical systems. This paper introduces two of them: the manifold piecewise linear system [11–14] and the chaotic spiking oscillator [15, 16]. In these circuits, the dynamics is integrated into one-dimensional return maps. Using the piecewise exact solutions, the return maps are calculated exactly. Referring to theoretical results of one-dimensional maps [17, 18], chaos generation is proven theoretically. These systems are implemented by simple analog circuits and chaotic phenomena are confirmed experimentally. Several related systems are also introduced.

The second topic is recurrent neural networks. As a piecewise linear switched dynamical system in this topic, we introduce the hysteresis neural network and its application to self-associative memories [19, 20]. In the applications, the main problem is storage of desired memories and suppression of spurious memories. Performing theoretical analysis based on the piecewise exact solutions, we give parameters condition that guarantees storage of any desired memories. Applying sparsification of the connection parameters, spurious memories can be suppressed. Application to combinatorial optimization problems solvers is also introduced.

The third topic is multiobjective optimization problems [21]. In this topic, we introduce a two-objective problem in a simple switched dynamical system: a piecewise linear model of switching power converter with photovoltaic input. The first objective represents extraction of the maximum input power. The second objective represents the circuit stability. Applying a simple multiobjective evolutionary algorithm, we clarify existence of trade-off between the maximum input power and strong circuit stability. The trade-off problem is a first step to consider collaboration of multiobjective optimization and nonlinear dynamical systems.

2. Autonomous chaotic circuits
We introduce two autonomous chaotic circuits: the manifold piecewise linear system (MPL [11–13]) and the chaotic spiking oscillator (CSO [15, 16]). Typical chaotic attractors are demonstrated. Main theoretical results for chaos generation are shown.

2.1 Manifold piecewise linear system
In the history of autonomous chaotic systems, the MPL has been recognized as an important example because of the following facts. First, the dynamics is integrated into a one-dimensional piecewise linear return map. The return map is described exactly and chaos generation is proven theoretically [13]. Second, the MPL can be implemented by a simple hardware and chaos generation is confirmed experimentally [13, 14]. Third, the MPL has been applied to various engineering systems including communication systems, signal processor, radar systems, and particle swarm optimizers [22–26]. The MPL is defined by the following second order piecewise linear equation and hysteresis switching rule:
\[ \ddot{x} - 2\delta \dot{x} + x = \begin{cases} p & (+) \\ -p & (-) \end{cases}, \]  

where \( x \) denotes the dimensionless state variable, \( \tau \) denotes the dimensionless time, and \( \dot{x} \equiv \frac{dx}{d\tau} \). Let the right hand side of Eq. (1) be either \((+)\) or \((-)\) at \( \tau = 0 \). In order to define the switching rule, we introduce notations

\[ L \equiv L_+ \cup L_- , \quad L_+ \equiv \{ x \mid x \geq T_h, \ \dot{x} = 0 \} , \quad L_- \equiv \{ x \mid x < T_h, \ \dot{x} = 0 \} , \quad x \equiv (x, \dot{x}). \]

The right hand side of Eq. (1) is switched from \((+)\) to \((-)\) if a trajectory hits \( L_- \) as shown in Fig. 1.

The right hand side of Eq. (1) is switched from \((-)\) to \((+)\) if a trajectory hits \( L_+ \).

The MPL is characterized by three parameters: the damping \( \delta \), the equilibrium point \( p \), and the switching threshold \( T_h \). For simplicity, we assume \( 0 < \delta < 1, \ 0 < p < 1, \ T_h = 0 \).

In this case, Eq. (1) has unstable complex characteristic roots \( \delta \pm j\omega \), where \( \omega \equiv \sqrt{1 - \delta^2} \). As shown in Fig. 1, the trajectories rotate divergently around equilibrium points \( \pm p \). If the trajectory hits negative \( x \)-axis \( (L_-) \) then the equilibrium point is switched from \( p \) to \(-p\). If the trajectory hits positive \( x \)-axis \( (L_+) \) then the equilibrium point is switched from \(-p\) to \( p \). Note that the switching occurs only on \( x \)-axis \( (L) \). Repeating the rotation and switching, the MPL can exhibit chaotic trajectories as shown in Fig. 2(a) and (b). The trajectories can be calculated by the piecewise exact solution

\[ x = p + (x(0) - p)e^{\frac{\delta \tau}{\omega}} \cos \tau + (\dot{x}(0) - \delta(x(0) - p))e^{\frac{\delta \tau}{\omega}} \sin \tau, \]  

where \((x(0), \dot{x}(0))\) is an initial condition at \( \tau = 0 \).

In order to define the return map, we consider a trajectory started from a point \( x_0 \in L \) at \( \tau = 0 \) where a point on \( L \) is represented by its \( x \) coordinate. The trajectory intersects \( L \) at \( \tau = \pi/\omega \) and let \( x_1 \) be the intersection. Since \( x_0 \) determines \( x_1 \), we can define the one-dimensional return map \( F \) from \( L \) to itself. The map is piecewise linear and is described exactly:

\[ x_1 = F(x_0) \equiv \begin{cases} -\beta(x_0 - p) + p & \text{for } x_0 \geq 0 \\ -\beta(x_0 + p) - p & \text{for } x_0 < 0, \end{cases} \]  

where \( \beta \equiv e^{\frac{\delta \pi}{\omega}} \). Now the dynamics is integrated into the iteration \( x_{n+1} = F(x_n) \). The chaos generation is guaranteed if \( 1 < \beta < 2 \). In this case, there exists an invariant interval \( I_1 \) on which the map is expanding:

\[ F(I_1) \subseteq I_1, \ |DF(x)| > 1, \ \text{for } x \in I_1 \equiv (-F(0), F(0)], \]  

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where $DF(x)$ is the slope of $F$ at $x$. In this case, chaos generation is guaranteed theoretically in the sense of ergodic and positive Lyapunov exponent [17]. Figure 3 shows return maps corresponding to chaotic attractors in Fig. 2. Note that trajectories diverge for $\beta > 2$. As stated earlier, the MPL can be implemented by a simple hardware and chaos generation can be confirmed experimentally. Simple hardware examples can be found in [14].

2.2 Chaotic spiking oscillator

In order to realize chaotic behavior in autonomous circuits, an effective nonlinear element is required. The integrate-and-fire switch is used as a key element in the chaotic spiking oscillators (CSO). Note that the integrate-and-fire switching rule is basic to design a variety of spiking neuron models [27–29]. In Fig. 4, $S$ denotes the integrate-and-fire switch and $N$ denotes a linear sub-circuit. If the capacitor voltage $v_1$ reaches a threshold voltage $V_T$, $v_1$ is reset to the constant base voltage $E$, instantaneously. If the sub-circuit $N$ consists of resistors and dependent sources, it can be replaced with the Thevenin equivalent sub-circuit and the circuit can exhibit periodic waveforms as shown in Fig. 4(b). If the sub-circuit $N$ includes one memory element (inductor or capacitor) as shown in Fig. 4(c), the integrate-and-fire switch causes vibrate-and-fire dynamics. If $N$ includes 1 or more memory elements, $v_1$ can vibrate below the threshold and the integrate-and-fire switch can cause chaotic behavior.

The CSO is defined in the case where the sub-circuit consists of one inductor and one negative linear resistor as shown in Fig. 4(c). If the capacitor voltage $v$ is below the threshold $V_T$, the switch $S$ is opened and the circuit dynamics is described by

\[
C \frac{dv_1}{dt} = i, \quad L \frac{di}{dt} = -v_1 + Ri, \quad \text{for } v_1 < V_T.
\]
Fig. 4. Circuits with integrate-and-fire switching.

\( v_1 \) is assumed to vibrate divergently. As \( v_1 \) reaches \( V_T \), the integrate-and-fire switch \( S \) is closed and \( v_1 \) is reset to the base \( E \) instantaneously holding the continuity property of \( i \):

\[
(v_1(t^+), i(t^+)) = (E, i(t)), \quad \text{if } v_1(t) = V_T.
\]  

(6)

Repeating the integrate-and-fire switching, the circuit generates chaos. Using discrete elements, this circuit can be fabricated easily and chaotic behavior can be confirmed experimentally. A simple hardware example can be found in [16].

We assume that Eq. (5) has unstable complex characteristic root

\[
\delta \omega \pm j\omega, \quad \omega^2 = \frac{1}{LC} - \left(\frac{R}{2L}\right)^2 > 0, \quad \delta = \frac{R}{2\omega L} > 0.
\]  

(7)

In this case \( v_1 \) can vibrate divergently below the threshold \( V_T \). The divergent vibration and the integrate-and-fire switching correspond to stretching and folding mechanisms, respectively, which are fundamental for chaos generation. Using the following dimensionless variables and parameters:

\[
\tau = \omega t, \quad q = \frac{E}{V_T}, \quad x = \frac{v_1}{V_T}, \quad y = -\frac{\delta}{V_T} + \frac{1}{\omega C V_T} i,
\]  

(8)

Equations (5) and (6) are transformed into the following.

\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} = \begin{pmatrix}
\delta & 1 \\
-1 & \delta
\end{pmatrix} \begin{pmatrix}
x \\
y
\end{pmatrix} \text{ for } x < 1
\]

(9)

SW: \((x(\tau^+), y(\tau^+)) = (q, y(\tau) - \delta(1 - q)) \text{ if } x_1(\tau) = 1\)

where \( \dot{x} \equiv dx/d\tau \) and \( p = \delta \). This dimensionless equation is characterized by two parameters \( \delta \) and \( q \) which can be controlled by \( -R \) and \( E \), respectively. This equation exhibits chaotic attractor as shown in Fig. 5.

The circuit dynamics can be analyzed using one-dimensional return map. Some objects of the map are shown in Fig. 6: the domain of the map \( L_d = \{(x,y) \mid x = 0, \ y \geq 0\} \), the threshold line \( L_T = \{(x,y) \mid x = 1\} \), and the base line \( L_q = \{(x,y) \mid x = q\} \). Let a point on these objects be represented by their \( y \)-coordinate. Also let \( D \) be a point on \( L_d \) such that a trajectory started from \( D \) touches \( L_T \) within half period.
Let $q < 0$ and let a trajectory start from a point $y_0$ on $L_d$ at $\tau = 0$. If $0 < y_0 < D$, the trajectory return to $L_d$ at $\tau = 2\pi$ without reaching $L_T$. If $D \leq y_0$, the trajectory hits the threshold $L_T$ and is reset to the base $L_q$. Then the trajectory re-starts from $L_q$ and returns to $L_d$. Since any trajectory started from $y_0$ on $L$ must return to $L_d$, a one-dimensional return map can be defined:

$$y_1 = f(y_0), \quad f : L_d \rightarrow L_d$$

where $y_1$ is the return point on $L_d$. That is, the circuit dynamics can be integrated into the iteration $y_{n+1} = f(y_n)$ as shown in Fig. 6. Using piecewise exact solution of Eq. (9), the return map can be described and chaos generation is guaranteed theoretically in the sense of ergodic and positive Lyapunov exponent [16, 17].

### 2.3 Related systems

Chaotic phenomena in the MPL and CSO are analyzed theoretically using one-dimensional return maps. Such theoretical analysis is possible for other piecewise linear switched dynamical systems including hysteresis chaos generators [30–33] and diode chaos generators [34, 35]. These systems are implemented by simple analog hardware and the chaotic phenomena are confirmed experimentally. Especially, we have presented two simple designs of hysteresis chaos generators: an RC-OTA hysteresis chaos generators [32] and a 2-port VCCS hysteresis chaos generators [33]. They are convenient to consider physical mechanism of chaos generation and to consider simple integrated circuits of chaotic circuits. If the number of memory elements increases, we obtain higher-order chaotic circuits that generate hyperchaos and related rich bifurcation phenomena [36–41]. If the pulse-coupling is applied to multiple CSOs, we obtain pulse-coupled network of CSOs [39]. This network exhibits a variety
of synchronization phenomena of periodic/chaotic spike-trains. The real/prospective applications of the network include flexible image processing [42], associative memories [43], and spike-based communication [44].

3. Recurrent neural networks

Recurrent neural networks are continuous-time dynamical systems referring to inhibition/excitation and parallel distributed dynamics of biological neural networks. The network is constructed by connecting simple elements with nonlinear activation function. As a pioneer mathematical model of the network, the Amari-Hopfield network is known [45, 46]. The dynamics is described by

\[
\dot{x}_i = -x_i + \sum_{j=1}^{N} w_{ij} y_j + d_i, \quad y_i = f(x_i), \quad i = 1 \sim N
\]

where \( x \) is the state variable and \( f(x) \) is a smooth activation function such as the sigmoidal function. This network can exhibit a variety of nonlinear phenomena, coexisting equilibrium points, coexisting periodic orbits, synchronization, bifurcation, and chaos. In the classic studies, networks with stable equilibrium points are applied to various systems including associative memories and combinatorial optimization problem solvers [45–48]. These applications are based on Lyapunov-function stability analysis. The study of the recurrent neural networks has continued intensively. In recent years, the dynamic behavior (periodic/chaotic orbits) are used in reservoir computation systems [49]. As a piecewise linear switched dynamical system in recurrent neural networks, we introduce hysteresis neural networks (HNNs [19, 20]).

3.1 Hysteresis neural networks

The HNN is characterized by binary hysteresis activation function as shown in Fig. 7:

\[
h(x) = \begin{cases} +1 & \text{for } x > -Th \\ -1 & \text{for } x < Th \end{cases}
\]

\( h(x) \) is switched from \(-1\) to \(1\) (respectively, \(1\) to \(-1\)) if \( x \) reaches the threshold \( Th > 0 \) (respectively, \(-Th\)). The HNN dynamics is described by

\[
\dot{x}_i = -x_i + \sum_{j=1}^{N} w_{ij} y_j + d_i \equiv -(x_i - p_i), \quad i = 1 \sim N
\]

\[
y_i = h(x_i)
\]

where \( x \equiv (x_1, \cdots, x_N) \) is the state variable vector and \( y \equiv (y_1, \cdots, y_N) \), \( y_i \in \{-1, 1\} \), is the binary output vector. \( w_{ij} \) denotes connection parameters and \( d_i \) denote offset parameters. This piecewise linear switched dynamical system is governed by \( N \) pieces of linear subsystems connected by the hysteresis switching of \( h(x_i) \). If the output vector is fixed, the system is described by linear equation \( \dot{x}_i = -(x_i - p_i) \) for \( i = 1 \sim N \) and the trajectory is calculated precisely by piecewise exact solutions. As parameters vary, the HNN can exhibit various nonlinear phenomena. Even if low dimensional

![Fig. 7. Binary hysteresis activation function.](image)
cases of $N = 2$ and $N = 3$, the HNN exhibits various periodic orbits, synchronization, torus, chaos, and various bifurcation phenomena [50–52]. As $N$ increases, the HNN can exhibit various nonlinear phenomena, however, theoretical analysis of them is not easy.

Here, we introduce application to self-associative memories in the case where the steady state is stable equilibrium points. In the associative memories, key problems are storage of desired memories (e.g., desired images) and suppression of spurious memories (e.g., unnecessary images). The HNN have several advantages: trajectories can be calculated precisely by piecewise exact solutions and test (e.g., desired images) and suppression of spurious memories (e.g., unnecessary images). The HNN storage of desired memories is possible. It should be noted that even if all the desired memories are stored, there usually exist spurious memories. Suppression of the spurious memories is not easy in various associative memories. Here we show basic results for a simple example: 14 desired memories of 8-dimensional binary vectors ($N = 8$, $M = 14$).

$$w_{ij} = w_{ji}, \quad Th = 1, \quad d_i = 0. \quad (14)$$

Theoretical results in more general case can be found in [19]. An equilibrium point of the HNN is given by

$$p(y) \equiv (p_1, \cdots, p_N), \quad p_i = \sum_{j=1}^{N} w_{ij}y_j \quad (15)$$

An equilibrium point $p(y)$ is said to be stable if

$$p_iy_i > -1 \text{ for any } i \quad (16)$$

An equilibrium point $p(y)$ is said to be unstable if

$$p_iy_i \leq -1 \text{ for some } i \quad (17)$$

We can say that any initial state converges to either equilibrium point if

$$0 < w_{ii} + 1 \quad i = 1 \sim N. \quad (18)$$

The desired memories are a set of multiple binary vectors:

$$s^1, \cdots, s^M, \quad s^l \equiv (s^l_1, \cdots, s^l_N), \quad s^l_i \in \{-1, 1\} \quad l = 1 \sim M$$

If we can determine parameters such that the equilibrium point of $s^l$ is stable then the desired memory $s^l$ is said to be stored into the network. If an equilibrium point of some output vector $y$ is stable and the $y$ is not a desired memory, then the $y$ is said to be a spurious memory.

Storage of all the desired memories is guaranteed if

$$0 < w_{ii} + 1 + \min k \sum_{j \neq j} w_{ij}s^k_j s^k_j, \quad k = 1 \sim N. \quad (19)$$

Note that if a desired memory $s^k$ is stored then its inverse vector $-s^k$ is also stored because of the system symmetricity in Eq. (14). Hence half of the desired memories should be inverse vectors of the other half desired memories. It should be noted that this storage condition must be satisfied as the self-feedback parameter $w_{ii}$ is adjusted to satisfy the condition. That is, in this network, storage of any desired memories is possible. It should be noted that even if all the desired memories are stored, there usually exist spurious memories. Suppression of the spurious memories is not easy in various associative memories. Here we show basic results for a simple example: 14 desired memories of 8-dimensional binary vectors ($N = 8$, $M = 14$).

$$s^1 \equiv (-1, 1, -1, 1, 1, 1, -1, 1) = -s^8$$
$$s^2 \equiv (1, -1, 1, -1, -1, 1, -1, -1) = -s^9$$
$$s^3 \equiv (1, 1, -1, 1, -1, -1, 1, 1) = -s^{10}$$
$$s^4 \equiv (1, 1, 1, -1, 1, 1, -1, -1) = -s^{11}$$
$$s^5 \equiv (-1, 1, 1, -1, 1, 1, -1, 1) = -s^{12}$$
$$s^6 \equiv (1, -1, 1, -1, 1, 1, -1, -1) = -s^{13}$$
$$s^7 \equiv (-1, 1, 1, 1, -1, 1, 1, -1) = -s^{14}$$
$s^8$ to $s^{14}$ are inverse patterns of $s^1$ to $s^7$, respectively.
These desired memories can be stored into 8-dimensional HNN \((N = 8)\) if the connection of parameters (connection matrix \(W_b\)) are given by

\[
W_b = \begin{pmatrix}
+3 & -1 & -1 & -1 & +1 & +1 & +1 & -1 \\
-1 & +1 & -1 & -1 & -1 & -1 & -1 & +1 \\
+1 & -1 & +3 & -1 & -1 & +1 & -1 & -1 \\
-1 & -1 & -1 & +1 & -1 & -1 & -1 & +1 \\
+1 & -1 & -1 & -1 & +1 & +1 & -1 & +1 \\
+1 & -1 & +1 & -1 & -1 & +3 & -1 & -1 \\
-1 & +1 & -1 & +1 & -1 & +1 & -1 & +1 \\
-1 & +1 & -1 & +1 & -1 & +1 & +1 & 0
\end{pmatrix}
\] (20)

This full connection matrix is determined by correlation based learning and binalization presented in [53]. However, the example has 88 spurious memories. In order to suppress the spurious memories, we have applied the a simple sparsification algorithm [53] and have obtained a sparse connection matrix \(W_s\) in Eq. (21). In this case, the network has only 4 spurious memories.

\[
W_s = \begin{pmatrix}
+3 & 0 & +1 & 0 & -1 & -1 & 0 & -1 \\
0 & +4 & 0 & -1 & -1 & +1 & -1 & -1 \\
+1 & 0 & +2 & 0 & -1 & -1 & +1 & -1 \\
0 & -1 & 0 & 0 & -1 & -1 & -1 & +1 \\
-1 & -1 & -1 & -1 & +1 & +1 & 0 & -1 \\
-1 & -1 & -1 & -1 & +1 & 0 & -1 & +1 \\
0 & +1 & +1 & -1 & 0 & -1 & +2 & -1 \\
-1 & -1 & -1 & +1 & -1 & +1 & +1 & 0
\end{pmatrix}
\] (21)

### 3.2 Related works

The HNNs are large scale systems and detailed analysis of the nonlinear dynamics is not easy. However, we have analyzed the dynamics almost completely in the case where the connection parameters are uniform [54]. Using a simple potential function, we have clarified theoretically that the HNN exhibits bifurcation from co-existing stable equilibrium points to multi-phase synchronization of stable periodic orbits. The potential function is developed into combinatorial optimization problems solvers such as an N-Queens problem [55], a dynamical N-Queens problems [56], and a Box puzzling problems [57]. In the application to the N-Queen problems (P vs NP problems), computation cost is calculated precisely up to \(N = 150\). The calculation is executed by piecewise exact solutions of the HNN. The results suggest that the parallel processing of the HNN can reduce the computation cost that increases at most polynomial of \(N\).

### 4. Multiobjective optimization problems

Multi-objective optimization problems (MOPs) have been studies in various fields of science [58–63]. Although uniobjective optimization problems require the optimization of only one objective function, the MOPs require the simultaneous optimization of multiple objective functions, e.g., cost and performance in various engineering problems. If the simultaneous optimization is achieved, we can obtain the optimal solution; however, usually the optimal solution is not feasible. In the MOPs, we encounter various difficulties. One typical problem is the presence of conflicting objectives, where an improvement in one objective may cause a deterioration in another objective. The task is to find solutions (Pareto set) which balance a trade-off. The solutions in a design space correspond to the Pareto front in the objective space. In order to find the Pareto front effectively, various evolutionary algorithms have been presented [60–63]. The multiobjective evolutionary algorithm based on decomposition (MOEA/D [60]) is known as one of the most efficient algorithms. The algorithms performance has been investigated in benchmarks such as multiobjective 0-1 knapsack problems.

In this section, we introduce two-objective problem in a piecewise linear switched dynamical system: a piecewise linear circuit model of switching power converter with photovoltaic input [64, 65]. The first
objective corresponds to extraction of the maximum input power. The second objective corresponds to robust/stable circuit operation. These objectives are important to realize efficient low carbon and renewable energy supply [66–71]. The two objectives are described by functions of control parameters of the converter. Using a piecewise linear modeling of the power converter, the two objective functions are described exactly. In order to analyze the two-objective problem, we apply a simple version of the MOEA/D.

4.1 The circuit model and objective problem

Figure 8(a) shows a boost converter with photovoltaic input. In order to analyze the dynamics precisely, we apply two simplifications used in [64]. First, the photovoltaic input of smooth characteristics is simplified into two-segment piecewise linear current-controlled voltage source as shown in Fig. 8(b):

\[ v_{in}(i_{in}) = \begin{cases} -r_a(i_{in} - I_P) + V_P & \text{for } i \leq I_P \\ -r_b(i_{in} - I_P) + V_P & \text{for } i > I_P \end{cases} \]  

(22)

where \((I_P, V_P)\) denotes the breakpoint. In the circuit, the switch \(S\) and diode \(D\) can be either of the 2 states:

- State 1: \(S\) conducting and \(D\) blocking
- State 2: \(S\) blocking and \(D\) conducting

As illustrated in Fig. 8(c), the switching rule is defined by

\[
\begin{align*}
\text{State 1} & \rightarrow \text{State 2} \text{ if } t = nT \\
\text{State 2} & \rightarrow \text{State 1} \text{ if } i_{in} = J_-
\end{align*}
\]

where \(T\) is the clock period and \(J_-\) is the lower threshold of input current. Second, we assume that the time constant of the load is much larger than the clock period \(RC \gg T\). In this case, the load is simplified into a constant voltage source \(V_{out}\). The circuit dynamics is described by

![Fig. 8. Boost converter with photovoltaic input. (a) Circuit model. (b) Piecewise linear model. (c) Switching rule.](image-url)
\[
L \frac{di_{in}}{dt} = \begin{cases} 
   v_{in}(i_{in}) & \text{for State 1} \\
   v_{in}(i_{in}) - V_{out} & \text{for State 2}
\end{cases}
\] (23)

Using the dimensionless variables and parameters

\[
\begin{aligned}
\tau &= \frac{t}{T}, \quad x = i_{in} I_P, \quad y(x) = \frac{v_{in}(I_P x)}{V_P}, \quad \alpha = \frac{r_a I_P}{V_P} \\
\beta &= \frac{r_b I_P}{V_P}, \quad q = \frac{V_{out}}{V_P}, \quad \gamma = \frac{TV_P}{LI_P}, \quad X = \frac{J}{I_P},
\end{aligned}
\] (24)

Equations (23) and (22) are transformed into

\[
\begin{aligned}
dx{\tau} &= \begin{cases} 
   \gamma y(x) & \text{for State 1} \\
   \gamma (y(x) - q) & \text{for State 2}
\end{cases} \\
y(x) &= \begin{cases} 
   -\alpha (x - 1) + 1 & \text{for } x \leq 1 \\
   -\beta (x - 1) + 1 & \text{for } x > 1
\end{cases}
\end{aligned}
\] (25)

Switching rule

\[
\begin{aligned}
\text{State 1} &\rightarrow \text{State 2 if } \tau = n \\
\text{State 2} &\rightarrow \text{State 1 if } x_j = X_-
\end{aligned}
\]

Since Eq. (25) is piecewise linear, we can use piecewise exact solutions in the analysis. Note that the five dimensionless parameters are classified into two categories. The three parameters \((\alpha, \beta, q)\) characterize photovoltaic input and output. The two parameters \((\gamma, X_-)\) characterize switching control. We select \((\gamma, X_-)\) as control parameters and other parameters are fixed:

\[
\alpha = 0.25, \quad \beta = 2.5, \quad q = 1.6
\]

As the control parameters \((\gamma, X_-)\) vary, the circuit exhibits various periodic/chaotic behavior [64]. For simplicity, we consider a periodic orbit with period 1. As shown in Fig. 9, an orbit started from \(x_0\) reaches the lower threshold \(X_-\) and is switched to increase. The orbit increases and reaches a point \(x_1\) at time \(\tau = 1.\) Since \(x_1\) is determined by \(x_0,\) we can define a map

\[
x_1 = F(x_0), \quad X_- < x_0, \quad X_- < x_1.
\] (26)

Using the piecewise exact solutions, the map \(F\) can be calculated exactly. The periodic orbit with period 1 corresponds to a fixed point of the map

\[
p = F(p)
\] (27)

Using \(p\) as the initial point, the periodic orbit can be calculated exactly. This periodic orbit corresponds to input current \(i_{in}\) with period \(T.\) For the periodic orbit, we calculate instantaneous input power \(p_{in}\) with period 1 and the average input power \(P_A:\)

![Fig. 9. Periodic orbit and stability.](image-url)
\[ p_{in}(\tau) = x(\tau)g(x(\tau)), \quad p_{in}(\tau + 1) = p_{in}(\tau), \quad P_A = \int_0^1 p_{in}(\tau) d\tau \] (28)

Figure 10 shows typical examples of the periodic orbits. In order to consider stability of the periodic orbit, we define contraction rate as shown in Fig. 9:

\[ |DF(p)| = \left| \frac{dx_1}{dx_0} \right|_{x_0=p} = \left| \frac{dF(x_0)}{dx_0} \right|_{x_0=p} \] (29)

If \(|DF(p)| < 1\) then the periodic orbit of \(x(\tau)\) and periodic instantaneous input power \(p_{in}(\tau)\) are stable. If \(|DF(p)| > 1\) then they are unstable. Using the piecewise exact solutions, the contraction rate can be calculated exactly [21].

The problem is optimal setting of the control parameters \((\gamma, X_\gamma)\) for (I) extraction of the maximum average input power (characterized by \(P_A\)) and (II) stabilization of the instantaneous input power (characterized by \(|DF(p)|\)). Figure 10 shows three typical examples. (a) The orbit is evaluated by \(P_A \approx 0.96\) and \(|DF(p)| \approx 1.56\): the average power is large but the orbit is unstable. (c) The orbit is evaluated by \(P_A \approx 0.37\) and \(|DF(p)| \approx 0.01\): the average power is small but the orbit is stable. (b) The orbit is evaluated by \(P_A \approx 0.84\), \(|DF(p)| \approx 0.35\): a medium case of (a) and (c). These examples suggest existence of a trade-off between the maximum power and stability.

\[ \begin{array}{c}
(a) \quad x(\tau) \\
(b) \quad x(\tau) \\
(c) \quad x(\tau)
\end{array} \]

\[ \begin{array}{c}
p_{in}(\tau) \\
p_{in}(\tau) \\
p_{in}(\tau)
\end{array} \]

**Fig. 10.** Examples of periodic orbits. (a) \(\gamma = 0.15, X_\gamma = 0.95, P_A \approx 0.96, |DF(p)| \approx 1.56\). (b) \(\gamma = 0.88, X_\gamma = 0.88, P_A \approx 0.84, |DF(p)| \approx 0.35\). (c) \(\gamma = 1.98, X_\gamma = 0.99, P_A \approx 0.37, |DF(p)| \approx 0.01\).

### 4.2 Two-objective problem and MOEA/D

In order to consider the trade-off between the maximum power point and stability, we present two objective functions

\[ f_1(\gamma, X_\gamma) = P_{max} - P_A \quad 0 \leq f_1 \]
\[ f_2(\gamma, X_\gamma) = |DF(p)| / DF_{max} \quad 0 \leq f_2 \] (30)

where the normalized factors\(^1\) are set to be \(P_{max} = 1.1\) and \(DF_{max} = 2\). As \(f_1\) approaches to 0, the input power approaches to the maximum power point. As \(f_2\) approaches to 0, the periodic orbit

---

\(^1\)for visibility. The Pareto front can be shown in the unit range \((0 < f_1 < 1, 0 < f_2 < 1)\) in Fig. 12.
approaches to be super-stable. These two objective functions can be calculated precisely using the piecewise exact solutions. The two-objective problem is defined by

$$\text{Minimize } f(\gamma, X_-) = (f_1(\gamma, X_-), f_2(\gamma, X_-)) \in S_O \text{ subject to } (\gamma, X_-) \in S_D$$

(31)

where $S_O = \{(f_1, f_2)|0 \leq f_1 < 1.1, 0 \leq f_2 < 1.1\}$ and $S_D = \{(\gamma, X_-)|0 < \gamma < 2, 0 < X_- < 1\}$. In the real-valued objective functions $f: S_D \rightarrow S_O$, $S_D$ and $S_O$ are referred to as the design space and the objective space, respectively. Let $d \equiv (\gamma, X_-) \in S_D$ be a decision value. A decision value $d = d_a \in S_D$ is said to dominate another decision value $d_b \in S_D$ if either of the following is satisfied:

$$f_1(d_a) < f_1(d_b) \text{ and } f_2(d_a) < f_2(d_b)$$

(32)

$$f_1(d_a) < f_1(d_b) \text{ and } f_2(d_a) = f_2(d_b)$$

(33)

$$f_2(d_a) < f_2(d_b) \text{ and } f_1(d_a) = f_1(d_b)$$

(34)

A decision value $d_p$ is referred to as a Pareto optimal solution if $d_p$ is not dominated by any other decision value. The set of all the Pareto optimal solutions is named a Pareto set while its image in the objective space is named Pareto front. A point $z^*$ is referred to as a reference point if the $z^*$ gives the minimum of all the objectives. A Pareto front and a reference point are illustrated in Fig. 11.

If the Pareto front can be found, we can clarify existence of a trade-off. In order to solve the two-objective problem efficiently, we use a simplified version of the MOEA/D [21]. In the algorithm, the Tchebycheff decomposition approach is used [60, 62]. This approach uses a reference point $z^* = (z^*_1, z^*_2)$, $z^*_j$ is the minimum value of $f_i(d)$ for $d \equiv (\gamma, X_-)$

(35)

where $i = 1, 2$. In the algorithm, the two-objective problem is decomposed into uniobjective subproblems. The $j$-th subproblem is represented by the $j$-th adaptability function.

$$h^j(d|\lambda^j, z^*) = \max\{h^j_1, h^j_2\}$$

(36)

$$h^j_1(d|\lambda^j, z^*) = \lambda^j_1|f_1(d) - z^*_1|, \ h^j_2(d|\lambda^j, z^*) = \lambda^j_2|f_2(d) - z^*_2|$$

where the weight vectors $(\lambda^j_1, \lambda^j_2)$, $j \in \{1, \cdots, N\}$, are given by $\lambda^j_1 = (j - 1)/(N - 1)$ and $\lambda^j_2 = 1 - \lambda^j_1$. Each weight vector gives one uni-objective problem by the adaptability function. The algorithm tries to minimize the adaptability functions $h^j(d|\lambda^j, z^*)$ for $d$. Let $d^j$ denote the current solution of the $j$-th subproblem: $h^j(d|\lambda^j, z^*)$. A set of $N$ subsolutions, $\{d^1, \cdots, d^N\}$, is referred to as a population. An external population (EP) is used to store nondominated solutions. Let $\lambda^k$ and $\lambda^l$ ($k, l \in B(j)$) denote closest weight vectors to $\lambda^j$ where $B(j) = \{j - 1, j, j + 1\}$ for $j = 2 \sim N - 1$, $B(1) = \{1, 2, 3\}$ for $j = 1$, and $B(N) = \{N - 2, N - 1, N\}$ for $j = N$. The algorithm is defined as the following.

![Objective space and design space.](image)
Step 1 (Initialization):
Set EP = ∅. Generate initial weight vectors, initial population, and initial reference point.

Step 2 (Update): For \( j = 1, \ldots, N \), do
- Randomly select two solutions \( d^k \) and \( d^l \) from \( d^j \), \( j \in B(j) \) and then generate a new solution \( \delta^j \) by using a crossover operator [21].
- Using Eq. (35), the reference point is updated.
- Using \( \delta^j \) and Eq. (36), the closest neighbor set \( B(j) \) is updated.
- Update of EP: Remove from EP all elements dominated by \( f(\delta^j) \). Add \( f(\delta^j) \) to EP if no elements in EP dominate \( f(\delta^j) \).

Step 3 (Termination): \( g \leftarrow g + 1 \), go to Step 2, and repeat until the maximum generation \( g_{\text{max}} \). After the algorithm is terminated, the external population EP gives the Pareto front.

Applying the algorithm to the two-objective problem in Eq. (31), we have obtained the Pareto front as shown in Fig. 12. These results guarantee existence of a trade-off between the maximum power point of the photovoltaic input (in average input power) and stability of periodic orbit (in instantaneous input power). The Pareto set gives important information to set suitable parameter values in design of efficient and stable renewable energy supply systems.

![Fig. 12. Pareto front in objective space and Pareto set in design space. Points a, b, and c are given from periodic orbits in Figs. 10(a), (b), and (c), respectively. \( N = 30 \) and \( g_{\text{max}} = 10 \).](image)

5. Conclusions
Piecewise linear switched dynamical systems in three topics have been reviewed in this paper. In autonomous chaotic circuits, the MPL and CSO have been introduced. The dynamics is described by simple linear subsystems and switching rules with hysteresis. Using the piecewise exact solutions, the embedded one-dimensional return maps can be described exactly and rigorous proof of chaos generation is achieved. The MPL and CSO are implemented by simple analog hardware and chaos generation is confirmed experimentally.

In recurrent neural networks, the HNN has been introduced. The HNN is constructed by connection of artificial neurons with hysteresis activation function. The dynamics of described by multiple piecewise linear subsystems and the hysteresis switching. In application to self-associative memories, storage of any desired memories has been guaranteed theoretically. The HNN is applicable also to combinatorial optimization problems solvers.
In multiobjective optimization problems, a two-objective problem in a piecewise linear model of boost converter with photovoltaic input has been introduced. The first objective represents extraction of the maximum input power. The second objective represents stability of periodic orbit. Using the piecewise exact solutions, the two objective functions are calculated exactly. Applying a simplified version of the MOEA/D, the Pareto front is obtained effectively. The Pareto front guarantees existence of a trade-off between the maximum power point and stability.

Referring to these reviews, one research direction can be considered. That is, analysis of switched dynamical systems in two kinds of perspectives: nonlinear dynamics and multiobjective optimization. In a nonlinear dynamics perspective, a key target is bifurcation sets in the design space. In a multiobjective optimization perspective, a key target is Pareto front in the objective space. An important point is relationship between the two kinds of perspectives. Although bifurcation analysis is important in nonlinear dynamics, the bifurcation sets are not sufficient information to realize desired system operation. Although the Pareto front analysis is important in multiobjective optimization, the Pareto front is not sufficient information to clarify dynamics that causes the trade-off. If we consider both the bifurcation sets and Pareto front, more deep understanding of the objective systems is possible. Such consideration seems to be developed into excellent analysis and synthesis methods of nonlinear circuits and systems.

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