Photon and photon-added intelligent states of coupled parametric oscillators

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received 4 June 2021; accepted in final form 20 August 2021
published online 8 November 2021

Abstract – We study a quantum system of coupled oscillators subject to a periodic time dependence of its parameters. Using the Floquet-Lyapunov theory we derive linear non-Hermitian integrals of motion of the system and relate their covariance matrix to that of the canonical observables. The operator integrals allow us to construct states that saturate the Robertson uncertainty relation (intelligent states) for canonical operators and the corresponding photon-added states. We found explicit expressions for the wave function, Wigner function and covariance matrices of the latter.

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Introduction. – The oscillation of a system around its equilibrium position is one of the simplest and most common forms of motion found in nature. This observation makes the harmonic oscillator an indispensable tool in theoretical physics and has inspired the construction of similar models in both classical and quantum mechanical settings. Among these, a class of models of great relevance are oscillators with time-dependent parameters [1–5], also known as parametric oscillators. Such models combine the simplicity of the standard harmonic oscillators with the potential for unveiling interesting non-stationary phenomena in configurations of increasing complexity. The applications are many and range from light propagation through variable media [5] and the dynamical Casimir effect [6], to the motion of trapped ions [7].

A major concern in the study of quantum parametric oscillators has been the construction of coherent states, in close analogy with their famous non-parametric counterpart [8]. While in the latter case, it is well known that the Heisenberg uncertainty relation [9,10] becomes an equality for position and momentum operators [11], when the parameters are allowed to depend on time, the corresponding coherent states do not minimize such inequality, but instead reach the lower bound of Schrödinger’s [12], due to the appearance of non-vanishing covariances between the observables [1]. In the same spirit in which the Heisenberg and Schrödinger relations can be generalized to Robertson’s inequality [13], the coherent states of both stationary and parametric oscillators can be unified into the wider concept of intelligent states. These are states that saturate an uncertainty relation for given observables [14], not necessarily canonical (see [15], for example). This means that coherent states, although not the only ones [16], are also intelligent states for position and momentum operators, both in the stationary and in the parametric case.

Another important property shared by coherent states is Gaussianity. Although Gaussian states occupy a privileged position in quantum information, the departure from this regime might provide a better playground for quantum information processing protocols with no classical analog [17]. A simple way of constructing non-Gaussian quantum states from coherent states is by consecutive application of the creation operator. This defines the photon-added coherent states [18]. Their behavior is intermediate between classical and more general quantum states and provides a scheme to study the transition from particle-like to wave-like behavior of light [19]. They have been previously studied for a single oscillator with a generic time dependence of its parameters [20] and for a generic multimode Gaussian state in the case of a single-photon addition [21].

With the recent increasing interest in non-Gaussian quantum states of the radiation field (see [22,23] and references therein), we believe that the analysis of a non-stationary and multimode generalization of the photon-added coherent states is necessary. With that in mind we will study a system of $N$ quantum parametric oscillators described by the Hamiltonian [2,3,5]

\[ \hat{H}(t) = \frac{1}{2} \hat{x}^T J(t) \hat{x}, \]

(1)
where \( \hat{x} \) is a 2\( N \)-dimensional vector whose first \( N \) components correspond to the position \( \hat{q} \) and the last \( N \) to the momentum operators \( \hat{p} \) of the oscillators. The parametric excitation is described by the explicit time dependence of the frequencies and coupling factors present in \( \Pi(t) \). In particular, we will focus our attention in models in which \( \Pi(t) = \Pi(t+T) \). The canonical commutation relations for the observables, written in matrix form and using natural units \( h \equiv 1 \), read \( [\hat{x}_i, \hat{x}_j] = i \delta_{ij} \). The two \( 2N \times 2N \) matrices appearing on the right-hand side of (1) are, in general, given by \( J = \begin{pmatrix} 0 & 1_N \\ -1_N & 0 \end{pmatrix} \) and \( \Pi(t) = \begin{pmatrix} C_{q+q} & C_{p+q} \\ C_{q+p} & C_{p+q} \end{pmatrix} \), which means that \( J \) is a skew-symmetric (symplectic) matrix \( J^T = J^{-1} = -J \) and \( \Pi(t) \) describes the configuration of the system. That is, its \( C_{q+q} \) block contains the frequencies and coupling factors between the coordinates, \( C_{p+q} \) represents the coupling between momenta and \( C_{q+p} \) the coupling between positions and momenta.

Our goal in this work is twofold, we first show that generalized coherent states of (1) are \( \hat{x} \)-intelligent states for the Robertson uncertainty relation, and then construct their corresponding \( N \)-mode photon-added states, with an arbitrary number of photons (excitations) added to each mode of the system. The remainder of the manuscript is organized as follows. We first construct the linear integrals of motion for (1) exploiting well-known results from the Floquet theory. We will show how these operators relate to creation and annihilation operators in the Schrödinger picture, and how they can be used to introduce the non-stationary analog of the Fock basis. Then, we show that these Fock states do not minimize, in general, the Robertson uncertainty relation. We continue introducing the coherent states of the model by performing a unitary transformation on the ground state. These will be shown to be squeezed, correlated and intelligent states for canonical operators. Then, adding excitations to the previously introduced states, we construct the photon-added intelligent states and obtain their wave function, Wigner function and covariance matrices.

Floquet theory and linear integrals of motion. –

In this section we make use of the periodicity in time of the Hamiltonian operator (1) to employ the Floquet theorem in the construction of linear integrals of motion for the quantum mechanical problem. For completeness we will review the definition of the Floquet-Lyapunov transformation and the Heisenberg picture creation and annihilation operators following ref. [3].

Let us start transforming the Hamiltonian (1) to its Heisenberg picture counterpart,

\[
\hat{H}_h(t) = \frac{1}{2} \hat{x}^T(t)J\Pi(t)\hat{x}(t),
\]

where the \( h \) subscript stands for Heisenberg and the components of \( \hat{x}(t) \) are the Heisenberg operators constructed from the \( \hat{x} \) of the Schrödinger picture. The equations of motion derived from (2) take the form

\[
\dot{\hat{x}}_k(t) = -i[\hat{x}_k(t), \hat{H}_h(t)] = \sum_l \Pi_{kl}(t)\hat{x}_l(t).
\]

This is formally identical to a classical linear dynamical system with periodic coefficients. Hence, according to the Floquet theory [24], it may be described in terms of independent complex modes of oscillation \( \hat{x}(t) \) (Floquet modes), related to \( \hat{x}(t) \) through the Floquet-Lyapunov transformation (FLT), \( F(t) \):

\[
\hat{x}_h(t) = F(t)\hat{x}(t).
\]

The time evolution of the operators \( \hat{x}(t) \) is entirely described, in the Heisenberg picture, by imaginary exponential functions

\[
\dot{\hat{x}}_j(t) = i\hat{x}_j(t)\exp(i\omega_j t),
\]

\[
\dot{\hat{x}}_j(t) = \hat{x}_j(t)\exp(-i\omega_j t),
\]

where the \( j \) index runs from 1 to \( N \) and the \( \omega_j \) are the Floquet exponents of the system. These are the quantized Floquet modes of the classical system. The time-dependent transformation \( F(t) \) shares the period of the Hamiltonian (1), and its elements satisfy

\[
F_{i,j+N}(t) = F_{i,j}(t), \quad j = 1, \ldots, N, \quad i = 1, \ldots, 2N.
\]

Using this last expression we can rewrite the FLT in block form

\[
F(t) = \begin{pmatrix} U & U^* \\ V & V^* \end{pmatrix}.
\]

These blocks are time-dependent \( N \times N \) matrices which share the period of the parameters. They constitute a generalization of the periodic functions appearing in the Floquet theorem [24,25] to the case of several degrees of freedom.

The time-dependent transformation (4) implicitly depends on the configuration \( \Pi(t) \) of the system. In the simple case of uncoupled oscillators, every block of \( \Pi(t) \) is a diagonal matrix, this implies that the blocks of \( F(t) \), that is, \( U \), \( V \) and their complex conjugates, are diagonal, \( F_{ij} = 0 \) if \( i \neq j \). The coupling between the canonical variables would then be reflected in non-vanishing off-diagonal elements of the blocks. We assume non-zero coupling throughout this work.

If we impose the following condition to the FLT,

\[
\sum_{l=1}^N F_{l,i+N}(0)F_{l,m}(0) = \frac{i}{2}\delta_{lm},
\]

or, equivalently, \( V^T(0)U(0) = \frac{i}{2}I \), then the transformation \( F(t) \) will be canonical [3,26]. This will also ensure that the quantized Floquet modes (operators in the Heisenberg picture) satisfy the Weyl-Heisenberg algebra. Written in matrix form, this is

\[
[\hat{x}_i(t), \hat{x}_j(t)] = -\delta_{ij},
\]

and, taking (5) into account, we can identify the Floquet modes, in the Heisenberg picture, with creation \( \{\hat{\chi}_i(t)\} \)

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and annihilation ($\hat{x}_{i+N}(t)$) operators. We will assume the fulfillment of (8) from now on.

In the Schrödinger picture, 
\[ \hat{x} = F(t)\hat{\chi}S(t). \] (10)

The unitarity of the time evolution operator ensures the invariance of the Weyl-Heisenberg algebra under its action. This, in turn, guarantees that also the operators obtained from the Floquet modes in the Schrödinger picture satisfy the commutation relations (9). Then, from these Schrödinger operators we can define the following (explicit) time-dependent operators:
\[ \begin{pmatrix} \hat{A}^\dagger(t) \\ \hat{A}(t) \end{pmatrix} = \begin{pmatrix} e^{-i\Omega t} & 0 \\ 0 & e^{i\Omega t} \end{pmatrix} \hat{x}S(t) \equiv \hat{I}(t), \] (11)

where $\Omega = \text{diag}(\omega_1, \ldots, \omega_N)$. Substituting (11) in (10) and using the inverse of the FLT (see [3]) we can express $\hat{A}$ and $\hat{A}^\dagger$ in terms of the time-independent position and momentum operators
\[ \hat{A}^\dagger(t) = iV^\dagger(t)e^{-i\Omega t} \hat{q} - iU^\dagger(t)e^{-i\Omega t} \hat{p}, \] \[ \hat{A}(t) = -iV(t)e^{i\Omega t} \hat{q} + iU(t)e^{i\Omega t} \hat{p}. \] (12, 13)

It is easy to see that the total derivative of this last expression vanishes, and the following operator equation is satisfied:
\[ \frac{d}{dt}\hat{I}_k(t) = i[\hat{I}_k(t), \hat{H}(t)], \] (14)

where $\hat{H}(t)$ is the Schrödinger Hamiltonian (1). Hence, the time-dependent non-Hermitian operators $\hat{A}(t)$ and $\hat{A}^\dagger(t)$ (grouped together into $\hat{I}(t)$), are quantum integrals of motion of the Dodonov-Malkin-Man’ko type [1,27] and can be used to define an orthonormal basis for the Hilbert space of the oscillators. Moreover, since they satisfy the boson commutation relations (9), we can reinterpret each oscillator as a bosonic system in the second quantization formalism.

The construction of an orthonormal basis is straightforward: We assume that for $t \leq 0$ the system is a stationary one, in such a case the configuration matrix can be redefined as $\Pi(t) \rightarrow (\Pi(0)\Theta(-t) + \Pi(t)\Theta(t))$, where $\Theta(t)$ is the Heaviside step function. Then, for $t \leq 0$, the integrals of motion are just standard creation and annihilation operators and the Fock basis $|\{\bar{n}\}; t\rangle$ can be constructed in the usual way. For $t \geq 0$ the action of the time evolution operator $\hat{U}(t) = \exp\left(\frac{i}{\hbar}\int_0^t H(t')dt'\right)$ upon the stationary Fock basis defines the dynamical Fock states [7] of the parametrically excited system,
\[ |\{\bar{n}\}; t\rangle = \hat{U}(t)|\{\bar{n}\}; t\rangle, \] (15)

where $\{\bar{n}\}$ is the set $\{n_1, \ldots, n_N\}$. We define, in the Schrödinger picture, the number operator for the $i$-th oscillator ($i < N$) as $\hat{n}_i(t) = \hat{A}_i^\dagger(t)\hat{A}_i(t)$. Its eigenvectors are the states defined in (15) and the explicit eigenvalue problem takes the form $\hat{n}_i(t)|\{\bar{n}\}; t\rangle = n_i|\{\bar{n}\}; t\rangle$. This implies that (15) describes the states that evolved from a stationary state $|\{n\}\rangle$ with $n_i$, $i = 1, \ldots, N$ excitations in the $i$-th mode. For definiteness we will be referring to the excitations as photons even if the oscillators system is not necessarily restricted to the modes of the radiation field. The dynamical Fock states, as well as all those discussed in this letter will be multimode states. Non-zero coupling will imply inseparability of the states.

The action of the linear integrals of motion on (15) is analogous to the action of creation and annihilation operators on the Fock basis for the stationary problem. The explicit expressions read
\[ \hat{A}_i(t)|\{\bar{n}\}; t\rangle = \sqrt{n_i}, \ldots, n_i - 1; \ldots, t\rangle, \] \[ \hat{A}^\dagger_i(t)|\{\bar{n}\}; t\rangle = \sqrt{n_i + 1}, \ldots, n_i + 1; \ldots, t\rangle. \] (16, 17)

We would like to emphasize that for the previous treatment to hold it is necessary that the Floquet exponents $\omega_j$ of the system be strictly real numbers. This is equivalent to say that the elements of $\Pi(t)$ have to produce stable oscillations for the classical system.

**Robertson inequality and intelligent states.** – The mathematical description of the uncertainty principle for a set of $2N$ observables of a certain quantum system can be given through the Robertson inequality [13]. If we take the set to be the $2N$ components of the vector $\hat{x}$, this uncertainty relation has the form
\[ \text{Det}(\sigma(\hat{x})) \geq \left(\frac{1}{2}\right)^{2N}. \] (18)

In the left-hand side figures the determinant of the covariance matrix $\sigma$. Its elements, in a state $|\psi; t\rangle$, are given by
\[ \sigma(\hat{x}_i, \hat{x}_j) = \frac{1}{2} \langle \{\hat{x}_i, \hat{x}_j\}\rangle_{\psi, t} - \langle \hat{x}_i \rangle_{\psi, t} \langle \hat{x}_j \rangle_{\psi, t}, \] (19)

where $\{,\}$ denotes the anti-commutator. Identical expressions hold for the integrals of motion $\hat{I}_i(t)$. In fact, the covariance matrix of the canonical observables $\sigma(\hat{x}(t))$ can be related to that of the integrals of motion $\sigma(\hat{I}(t))$ through the Floquet-Lyapunov transformation
\[ \sigma(\hat{x}) = F(t) \exp(i W t) \sigma(\hat{I}) \exp(i W t) F^T(t), \] (20)

where $W = \text{diag}(\omega_1, \ldots, \omega_N, -\omega_1, \ldots, -\omega_N)$. Expression (20) follows directly from (10) and (11). Note that, although $\hat{I}(t)$ are time dependent, $\sigma(\hat{x}(t))$ is not necessarily time dependent too. On the other hand, $\sigma(\hat{x})$ depends explicitly on time, due to the presence of the Floquet-Lyapunov transformation and $\exp(i W t)$ in (20).

The columns of the matrix $F \exp(i W t)$ correspond to the solutions of the classical system of coupled parametric oscillators subject to the initial conditions (8). This means that the covariances of canonical observables, in any quantum mechanical state, are determined by the covariances of $\hat{I}(t)$ and by the classical motion of the system (1). Such a classical motion is, in turn, completely fixed by the the Floquet modes $\exp(i W t)$ and the Floquet-Lyapunov transformation $F(t)$. 

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In what follows we will build states for which (18) becomes the equality, intelligent states for canonical observables of the N-dimensional non-stationary system (1). But, let us first show that in this Fock basis, (15), Robertson’s inequality (18) is not minimized. Using (16) and (17) to calculate the mean values present in (19) for the integrals of motion and making use of the orthonormality of the Fock dynamical basis, we obtain the covariance matrix for $\hat{A}^\dagger(t)$ and $\hat{A}(t)$ in the Fock dynamical states
\[ \sigma_{\alpha,t}(\hat{A}(t)) = \begin{pmatrix} 0 & \text{diag}(\{n\})/2 \\ \text{diag}(\{n\})/2 & 0 \end{pmatrix}, \]
and for the canonical observables $\sigma_{\alpha,t}(\hat{x}) = \frac{1}{2} F(t) F^\dagger(t) + F \text{diag}(\{n\}) F^\dagger$. Notice that the determinant of this matrix will increase with the number of excitations. This means that, as in the non-parametric problem, in general, these Fock states do not minimize the uncertainty relation.

We define then, the coherent states of the system of interacting parametric oscillators, through the action of the displacement operator $D(\{\alpha\})$ on the ground state $|0; t\rangle$,
\[ |\{\alpha\}; t\rangle = D(\{\alpha\})|0; t\rangle = \prod_{i=1}^{N} e^{\alpha_i \hat{A}_i^\dagger(t) + \alpha_i^* \hat{A}_i(t)}|0; t\rangle. \] (21)

These states will turn out to be intelligent states for the Robertson’s uncertainty relation (18). From (21) it follows that $|\{\alpha\}; t\rangle$ are eigenstates of the linear integral of motion $\hat{A}_i(t)|\{\alpha\}; t\rangle = \alpha_i|\{\alpha\}; t\rangle$, where $\alpha_i$ is equal to the initial value of the classical Floquet mode $\chi_{i+N}(0)$. This expression can be used to calculate the coordinate representation wave function [2,3]. It is easy to see that, as in the simple harmonic oscillator, the mean values of the positions $\hat{q}_i$ and momenta $\hat{p}_i$ on these states can reproduce the behavior of any solutions to the classical equation of motion. The mean values of the quadrature operators in coherent states are given by
\[ \langle \hat{x}_i \rangle_{CS} = \sum_{j=1}^{2N} F_{ij}(t) \chi_j(t), \]
or, more explicitly,
\[ \langle \hat{q}_i \rangle_{CS} = \sum_{j=1}^{N} (U_{ij}(t) \chi_j(0)e^{i\omega_j t} + U_{ij}^\dagger(t) \chi_{j+N}(0)e^{-i\omega_j t}), \] (22)
which is exactly the general Floquet solution to the classical equations of motion. In these expressions $\chi(t)$ denotes the classical Floquet modes, they have the same functional form as the operators (5) of the Heisenberg picture.

We now find an explicit expression for the covariance matrices of the integrals of motion and canonical observables in the coherent states (21). To that end we begin by noticing that, since $D^\dagger(\{\alpha\}) = D^{-1}(\{\alpha\})$ and
\[ D^\dagger(\{\alpha\}) \hat{A}(t) D(\{\alpha\}) = \hat{A}(t) + \alpha, \\
D^\dagger(\{\alpha\}) \hat{A}^\dagger(t) D(\{\alpha\}) = \hat{A}^\dagger(t) + \alpha^*, \]
the covariances between the integrals of motion in a given coherent state are invariant under the action of the displacement operator. This is a major simplification in our demonstration, it allows us to consider only the covariance matrix for the ground state of the system, which is shared by all coherent states. Then, for any values of $\{\alpha\}$, the covariance matrix will be given by setting $n_i = 0$, $i = 1, ..., N$ in $\sigma_{\alpha,t}(\hat{I}(t))$. Written explicitly, for the integrals of motion, it reads
\[ \sigma_{\alpha,t}(\hat{I}(t)) = \begin{pmatrix} 0 & 1_{N/2} \\ 1_{N/2} & 0 \end{pmatrix} \] (23)
and for canonical observables,
\[ \sigma_{\alpha,t}(\hat{x}) = \frac{1}{2} F(t) F^\dagger(t) = \sigma_{0,t}(\hat{x}). \] (24)

As in the Fock case, the Floquet exponents cancel out and $F(t)$ is sufficient to know the covariances between the position and momentum operators.

Expression (24) shows that the coherent states (21), of the non-stationary system (1) are squeezed and correlated. Squeezing occurs when the uncertainty in position is different from the uncertainty in momentum. Even in the one-dimensional case, the parametric excitation induces squeezing of the quadratures components [1]. This can be directly seen from (24): the position uncertainty for the $i$-th oscillator ($i < N$) is given by the $i$-th main diagonal element of $FF^\dagger$. On the other hand, the momentum uncertainty corresponds to the $i + N$ element of the diagonal, since these two are in general different, so are the uncertainties. They are correlated in a double sense. Due to the coupling between the position and momentum operators in the Hamiltonian, it is impossible to express the coherent state $|\{\alpha\}; t\rangle$ of the entire system as a product of the single oscillators coherent states $|\alpha_i; t\rangle$. But also, because the parametric excitation guarantees finite correlations between position and momentum operators in each independent oscillator.

To finish this section we want to prove the intelligent character of the (generalized) coherent states (21). In other words, we have to demonstrate that $\det(\sigma_{\alpha,t}(\hat{x})) = 2^{-2N}$. This requires an expression for the determinant of the Floquet-Lyapunov transformation subject to the canonical condition (8). This is a rather simple task if one notices that the matrix $F(t) = \begin{pmatrix} 1_N & 0 \\ 0 & i1_N \end{pmatrix} F^T(t)$, is an element of the complex symplectic group $Sp(\mathbb{C}, 2n)$, which guarantees that its determinant is equal one. Then $\det(F(t)) = \det(\begin{pmatrix} 1_N & 0 \\ 0 & -i1_N \end{pmatrix}) = (-i)^N$. It follows directly from (23) that the determinant of the covariance matrix for the linear integrals of motion, in a coherent state, is equal to $(-4)^{-N}$. We then substitute the last two results in (20) to obtain that the determinant of $\sigma(\hat{x})$ is indeed equal to $2^{-2N}$. This concludes the proof that the coherent states $|\{\alpha\}; t\rangle$ of the parametrically excited oscillators minimize the Robertson uncertainty relation (18).
for any configuration II(t) of the system and hence, are (Robertson) intelligent states for canonical observables. Since this states are also squeezed and correlated, instead of calling them coherent, we will refer to them simply as the \( \hat{x} \)-intelligent states (where \( \hat{x} \) stands for canonical operators).

In summary, so far we have related the (quantized) Fock modes of the classical system with the linear integrals of motion (14) of (1), showing that the work done in [3] for a set of coupled Mathieu equations can be extended to any configuration of oscillators with time-periodic parameters. Moreover, a multimode time-dependent extension of the displacement operator allows the derivation of the intelligent states for position and momentum operators, by its direct application on the ground state of the system. We also showed that the explicit time dependence of covariances between canonical operators, in intelligent and Fock states, is completely determined by the Floquet-Lyapunov transformation. In the next section, we will show that this framework can be used to construct a special case of non-Gaussian states.

**Adding quanta to intelligent states.** – The repeated action of creation operators on canonical coherent states generates a set of non-Gaussian states known as the photon-added coherent states (PACS) (first introduced in [18]). In the non-stationary setting of coupled oscillators (1), the role of the creation operator is played by the integral of motion \( \hat{A} \hat{\imath}(t) \). It is then natural to introduce a dynamical and multimode version of the PACS by the application of \( \hat{A} \hat{\imath}(t) \) on the intelligent states (21). We refer to them as photon-added intelligent states and their concrete definition is given by

\[
|\{\alpha\}, \{m\}; t\rangle = \prod_{k=1}^{N} \frac{(\hat{A}_k(t)^{m_k})|\alpha\rangle}{\sqrt{m_k! L_{m_k}(-|\alpha|^2)}} |\{m\}; t\rangle,
\]

where \( L_{m_k}(-|\alpha|^2) \) is the Laguerre polynomial of order \( m_k \). If \( m_k = 0 \), for all \( k \), we recover the intelligent states \( |\{\alpha\}, \{0\}; t\rangle = |\{\alpha\}; t\rangle \), and Fock states are recovered by setting \( \alpha_k = 0 \), for all \( k \).

The calculation of mean values and covariances in these states is straightforward. By expressing the \( \hat{\imath} \)-operators, as

\[
(\hat{A}_k)^{m_k} = \sqrt{n_k + m_k + 1} \delta_{n_k, n_k + 1} \frac{n_k!}{(2n_k + 1)!} \frac{e^{-|\alpha|^2}}{\sqrt{n_k! L_{m_k}(-|\alpha|^2)}},
\]

we arrive at a formula for the mean value of a generic operator \( \hat{a} \) in a PACS

\[
\langle \hat{a} \rangle_{PACS} = \sum_{i=1}^{N} \prod_{i=1}^{N} \frac{\langle \alpha_i \rangle^{n_i} (\alpha^*_i)^{n_i}}{m_i!} \langle \hat{a} \rangle_{Fock} + \sum_{i=1}^{N} \prod_{i=1}^{N} \frac{\langle \alpha_i \rangle^{n_i} (\alpha^*_i)^{n_i}}{m_i!} \langle \hat{a} \rangle_{Fock},
\]

if the operator \( \hat{a} \) is formed from the integrals of motion, as it is the case we are interested in, its matrix elements in the Fock representation, appearing on the r.h.s. of (26), will be readily obtained from (16) and (17). For example, \( \langle \hat{A}_k \rangle_{Fock} = \sqrt{n_k + m_k + 1} \delta_{n_k, n_k + 1} \). The desired quantities are obtained by direct substitution on (26). As an illustration, let us calculate the mean value of the integral of motion \( \hat{A} \) in photon-added intelligent states. Following the previous recipe, we obtain

\[
\langle \hat{A} \rangle_{PACS} = \alpha_i e^{-|\alpha|^2} \frac{\sum_{n=0}^{\infty} \frac{|\alpha|^n}{n!} (m + 2)_{n}}{L_m(-|\alpha|^2)},
\]

where \( (a)_b = 1 \cdot 2 \cdot 3 \cdot \ldots (a + b - 1) \), \( a, b \in \text{Naturals} \), is the Pochhammer symbol [28]. The sum on the r.h.s. is identified with the confluent hypergeometric function

\[
\langle \hat{A} \rangle_{PACS} = \alpha_i \frac{\sum_{n=0}^{\infty} \frac{|\alpha|^n}{n!} (m + 2)_{n}}{L_m(-|\alpha|^2)}.
\]

The analogous expression for \( \hat{A}^\dagger \) is

\[
\langle \hat{A}^\dagger \rangle_{PACS} = \alpha^*_i \frac{\sum_{n=0}^{\infty} \frac{|\alpha|^n}{n!} (m + 2)_{n}}{L_m(-|\alpha|^2)}.\]

The mean values of canonical operators in PACS are obtained by using (29) and (30), together with (4),

\[
\langle \hat{x}_j \rangle_{PACS} = \sum_{i=1}^{2N} \frac{F_{ij}(t)}{L_m(-|\alpha|^2)} \langle \hat{x}_j \rangle_{Fock}.
\]

The second-order statistical moments of the integrals of motion are

\[

\langle \hat{A}_i \hat{A}_j \rangle_{PACS} = \begin{cases} \langle \hat{A}_i \rangle \langle \hat{A}_j \rangle, & i \neq j, \\ \langle \alpha_i \rangle^2 \frac{L_m(-|\alpha|^2)}{L_m(-|\alpha|^2)}, & i = j, \end{cases}
\]

\[

\langle \hat{A}_i^\dagger \hat{A}_j \rangle_{PACS} = \begin{cases} \langle \hat{A}_i^\dagger \rangle \langle \hat{A}_j \rangle, & i \neq j, \\ \langle \alpha^*_i \rangle^2 \frac{L_m(-|\alpha|^2)}{L_m(-|\alpha|^2)}, & i = j, \end{cases}
\]

\[

\langle \hat{A}_i^\dagger \hat{A}_j \rangle_{PACS} = \begin{cases} \langle \hat{A}_i^\dagger \rangle \langle \hat{A}_j \rangle, & i \neq j, \\ \langle \alpha^*_i \rangle \frac{L_m(-|\alpha|^2)}{L_m(-|\alpha|^2)} + m \delta_{n, m}, & i = j. \end{cases}
\]
From these relations it follows that all non-diagonal elements of each block of the covariance matrix for \(\hat{I}(t)\) are zero, and we only need to calculate the diagonal ones. To avoid a cumbersome notation we define
\[
P_{mn}^i(|\alpha_i|^2) = \frac{L^i_{m_n}(-|\alpha_i|^2)}{L^i_{m_i}(-|\alpha_i|^2)} \left( \frac{L^i_{m_i}(-|\alpha_i|^2)}{L^i_{m_m}(-|\alpha_i|^2)} \right)^2.
\]

The elements of the covariance matrix for the integrals of motion \(\hat{I}(t)\) in \(\{|\alpha\}, \{m\}; t\) are then given by
\[
\sigma_{PACS}(A^\dag_i(t), A_i(t)) = (\alpha_i^*)^2 P_{m_i}^i(-|\alpha_i|^2),
\]
\[
\sigma_{PACS}(A_i^\dag, A_i) = (\alpha_i)^2 P_{m_i}^i(-|\alpha_i|^2),
\]
\[
\sigma_{PACS}(A_i^\dag, A_i) = |\alpha_i|^2 P_{m_i}^i(-|\alpha_i|^2) + m_i + 1/2.
\]

The covariance matrix formed with (34) reduces to that of Fock states if one takes \(\alpha_k = 0\), and to that of coherent states if \(m_k = 0\), for all values of \(k\). The explicit appearance of \(\alpha_i\) and \(m_i\) in (34) implies that the covariances depend on both the \(\hat{x}\)-intelligent state that was excited and the number of excitations added to it. Although all coherent states \(\{|\alpha\}; t\) share the same covariance matrix (24), the states obtained by their excitation \(\{|\alpha\}, \{m\}; t\) do not. Since the excitation of the ground state \((\alpha = 0)\) produces Fock states, which do not minimize the uncertainty relation for \(n \neq 0\), also the excitation of the intelligent states produces states that are not \(\hat{x}\)-intelligent for any combination of \(\{|\alpha\}\) and \(\{m\}\) if \(m \neq 0\).

The corresponding covariance matrix for the position and momentum operators \(\sigma_{PACS}(\hat{x})\) is given by direct substitution of (34) in (20). After some algebraic manipulations and using (5) and (6), we find
\[
\sigma_{PACS}(x_i, x_j) = \frac{2N}{2} \sum_{k=1}^{N} F_{ik} F_{jk} \chi_k^2(t) P_{m_k}^i(-|\alpha_k|^2)
\]
\[
\times \left(\sum_{k=1}^{N} \left(\frac{(F_{ik} F_{jk})^{2} + F_{i,k+n} F_{j,k}}{(F_{ik} F_{jk})^{2}}\right)\right)
\times \left(\alpha_k^2 P_{m_k}^i(-|\alpha_k|^2) + m_i + 1/2\right).
\]

The functions \(\chi_k(t)\) appearing in the first form of the right-hand side, are the classical Floquet modes \(\chi(0)\exp(iWt)\). The initial values \(\chi_k(0)\) are equal to \(\alpha_k\), for \(k < N\) and to \(\alpha_k\) for \(k > N\). The Floquet-Lyapunov transformation is not enough to fully specify the covariances between canonical observables in PACS, we also need the Floquet modes. This is a major difference between the PACS and the intelligent states (24), or Fock states (15).

It is important to note that even if the covariances between position and momentum operators in a PACS are explicit functions of time, the determinant of the covariance matrix is constant, it is not affected by the parametric excitation. This implies that the difference between such a determinant and the lower bound of the Robertson relation is only a function of \(\alpha_i\) and \(m_i\) and does not depend on the Floquet-Lyapunov transformation. This result is a multi-dimensional generalization of that obtained in [20] for the uni-dimensional system.

The wave function describing these states can be directly computed from their definition by proper use of the generating function of multi-dimensional Hermite polynomials [28]. The result is
\[
\psi_{a,m}(q, t) = N_{a,m} \psi_a(q, t) H^M_{2m}(2m) \left(U^{-\ast}q - \alpha\right).
\]

In this expression \(N_{a,m}\) is the normalization constant appearing in (25), \(\psi_a\) denotes the wave function of the coherent \((\hat{x}\)-intelligent) states [2,3] and \(H^M\) is the multi-dimensional Hermite polynomial [28] of order \(m = (m_1, ..., m_N)\). It satisfies the relation
\[
H^M_{2m}(q) = (-1)^{2m} \exp(\sum_{k=1}^{N} \frac{\partial^{2m}}{\partial q_1 \cdots \partial q_N} e^{-q^2} q^q M q).
\]

Expression (36) is a generalization of the single oscillator wave function obtained in [20] for the PACS. Due to the mode coupling, it is not expressible as a product of single oscillators wave functions. This can be directly seen from the definition of the multi-dimensional Hermite polynomials, which cannot be expressed as the product of uni-dimensional polynomials if the matrix \(U^{-\ast}U^{-T}\) is not diagonal.

The Wigner function of the photon-added intelligent states (25) is obtained in a similar manner to that of single-mode non-stationary PACS reported on [20]. The computation will require us to solve an integral of a Gaussian weight times the product of two Hermite polynomials in \(N\) variables. The integral can be reduced to a known form by expressing such a product as a single Hermite polynomial in \(2N\) variables. The result is
\[
W_{PACS}(q, p, t) = 2^N \exp(-2|A(q, p, t) - \alpha|^2)
\]
\[
\times \left(\prod_{k=1}^{N} (-1)^{m_k} \frac{L_{m_k}(-|\alpha_k|^2)}{L_{m_k}(-|\alpha_k|^2)}\right).
\]

The function \(A(q, p, t)\) has the same form (13) as the integral of motion \(A(t)\),
\[
A(q, p, t) = -iV^T(t) e^{i\Omega} q + iU^T(t) e^{i\Omega} p.
\]

If we set \(m_k = 0\) for every value of \(k\), then (38) reduces to a Gaussian distribution. Note that, when coupling is present, (38) cannot be written as the product of single-mode Wigner functions.

This fact has an interesting consequence that does not arise in single-mode cases [20]. To illustrate this we can consider two coupled oscillators \((N = 2)\) in an intelligent state (21), described by the eigenvalues \(\alpha_1\) and \(\alpha_2\). The Wigner function is given by (38), with \(m_1 = m_2 = 0\). It is a Gaussian distribution that takes negative values for any combination of \(\alpha_1\) and \(\alpha_2\). Such state is then a Gaussian state. Let us now add a single excitation \((m_2 = 1)\) to only one of the modes (oscillators), the Wigner function...
will become $4^{-\frac{1}{2}} \sqrt{2} |A_1(q_1, p_1, t) - \alpha_1|^{-1} e^{-\frac{1}{2} |A_1(q_1, p_1, t) - \alpha_1|^2}$, and it can take negative values whenever $|A_1(q_1, p_1, t) - \alpha_1|^2 < 1$. Even if we do not add excitations ($m_1 = 0$) to the mode $\alpha_1$, it cannot be said to be in a Gaussian state. This is a consequence of the non-vanishing correlations between the canonical observables of each mode, caused by the coupling between the oscillators. The lack of separability imposed by the non-zero coupling makes it impossible to describe the system as one mode in a Gaussian state and the other in a non-Gaussian state. The previous analysis works for any number of modes and excitations.

Conclusions. – In this work we have determined the photon-added coherent states of a system of coupled oscillators whose frequencies and coupling factors change periodically with time.

The periodic character of the problem required the use of the Floquet theory. In this context, the quantized Floquet modes in the Schrödinger picture were related to the non-Hermitian integrals of motion, and the Floquet-Lyapunov transformation provided a link between the covariance matrices of such invariants and the canonical observables of the problem. Then, since the linear integrals of motion satisfy the Weyl-Heisenberg algebra, the construction of generalized coherent states followed closely the standard definition of canonical coherent states. The non-stationary states obtained were shown to be squeezed, correlated and intelligent states of Robertson’s inequality for position and momentum operators. Finally, this analog was further extended to construct the photon-added coherent states and to obtain their wave function in coordinate representation and the corresponding Wigner distribution. These are not $\hat{x}$-intelligent states and the determinants of their covariance matrices depend on both the complex eigenvalues of the annihilation integral of motion $A_1(t)$ and the number of excitations added to the coherent state. We obtained that, although the elements of the covariance matrix for canonical observables are explicitly time dependent, the determinant of the covariance matrix is a constant strictly greater than Robertson’s minimum, for non-zero $m$.

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We want to thank E. Aurell, A. Buchleitner and M. Walschaers for the careful reading of the manuscript and useful suggestions.

Data availability statement: No new data were created or analysed in this study.

REFERENCES

[1] Dodonov V., Man’ko O. and Man’ko V., J. Russ. Laser Res., 16 (1995) 1.
[2] Holz A., Lett. Nuovo Cimento, 4 (1970) 26.
[3] Land H., Drewsen M., Reznik B. and Retzker A., J. Phys. A: Math. Theor., 45 (2012) 455305.
[4] Urzúa A., Ramos-Prieto I., Fernández-Guasti M. and Moya-Cessa H. M., Quantum Rep., 1 (2019) 82.
[5] Kryuchkov S. I., Suazo E. and Suslov S. K., Math. Methods Appl. Sci., (2018) 1.
[6] Dodonov V. V. and Dodonov A. V., J. Russ. Laser Res., 26 (2005) 6.
[7] Leibfried D., Blatt R., Monroe C. and Wineland D., Rev. Mod. Phys., 75 (2003) 281.
[8] Glauber R. J., Phys. Rev. Lett., 10 (1963) 84.
[9] Heisenberg W., Z. Phys., 43 (1927) 172.
[10] Kennard E. H., Z. Phys., 44 (1927) 326.
[11] Rosas-Ortiz O., Coherent and squeezed states: Introductory review of basic notions, properties and generalizations, in Integrability, Supersymmetry and Coherent States, CRM Series in Mathematical Physics, Vol. 1 (Springer) 2019, pp. 187-230.
[12] Schrödinger E., Zsitzungber. Preuss. Akad. Wiss., 19 (1932) 105.
[13] Robertson H. P., Phys. Rev., 46 (1934) 794.
[14] Trifonov D. A., J. Phys. A, 30 (1997) 5941.
[15] Aragone C., Guerri G., Salmo S. and Tani J. L., J. Phys. A, 7 (1974) L149.
[16] Zelaya K., Hussin V. and Rosas-Ortiz O., Eur. Phys. J. Plus, 136 (2021) 534.
[17] Adesso G., Ragy S. and Lee A. R., Open Syst. Inf. Dyn., 21 (2014) 1.
[18] Agarwal G. S. and Tara K., Phys. Rev. A, 43 (1991) 492.
[19] Zavatta A., Viciani S. and Bellini M., Science, 306 (2004) 660.
[20] Dodonov V. V., Marchiolli M. A., Korennoy Y. A., Manko V. I. and Moukhin Y. A., Phys. Rev. A, 58 (1998) 4087.
[21] Walschaers M., Fabre C., Parigi V. and Treps N., Phys. Rev. A, 96 (2017) 053835.
[22] Ra Y. S., Dufour A. and Walschaers M., Nat. Phys., 16 (2020) 144.
[23] Wenger J., Tualle-Brouri R. and Grangier P., Phys. Rev. Lett., 92 (2004) 153601.
[24] Floquet G., Ann. Sci. Ecole Norm. Supér., 8 (1879) 3.
[25] McLachlan N. W., Theory and Application of Mathieu Functions (Dover) 1965.
[26] Wolf K. B., J. Math. Phys., 15 (1974) 1295.
[27] Dodonov V. V., Malkin I. A. and Manko V. I., Int. J. Theor. Phys., 14 (1975) 37.
[28] Bateman H., Higher Transcendental Functions, Vol. 2 (McGraw-Hill) 1953.