ERGODICITY OF THE STOCHASTIC COUPLED FRACTIONAL
GINZBURG-LANDAU EQUATIONS DRIVEN BY
α-STABLE NOISE

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ABSTRACT. The current paper is devoted to the ergodicity of stochastic coupled fractional Ginzburg-Landau equations driven by α-stable noise on the Torus T. By the maximal inequality for stochastic α-stable convolution and commutator estimates, the well-posedness of the mild solution for stochastic coupled fractional Ginzburg-Landau equations is established. Due to the discontinuous trajectories and non-Lipschitz nonlinear term, the existence and uniqueness of the invariant measures are obtained by the strong Feller property and the accessibility to zero.

1. Introduction. In recent years, the stochastic partial differential equations driven by Lévy noise have attracted a lot of attention, see [1, 8, 16, 17, 24] and references therein. But most of them required the square integrability for Lévy noise, which clearly rules out the interesting α-stable noise. It is worthy to weak this restriction since α-stable noises have been deeply studied and widely applied to physics, queueing theory, mathematical finances and others. There have been some study of the stochastic equations driven by α-stable noises (see for instance [6, 17, 18, 19, 23, 26, 27]). The authors in [17] investigated the structural properties of solutions for the stochastic nonlinear equations with bounded and Lipschitz non-linearities driven by cylindrical stable processes. While [23] studied the ergodicity of the stochastic equation with unbounded and non-Lipschitz dissipative function driven by α-stable noises with $α \in (1,2)$. The exponential mixing of the SPDEs driven by α-stable noises has been established in [18, 19, 24]. Dong in [6] proved the exponential ergodicity and strong Feller of the stochastic Burgers equations driven by $\frac{2}{5}$-subordinated cylindrical Brownian motions with $α \in (1,2)$. The existence of the invariant measure has been shown for 2D stochastic Navier-Stokes equation forced by α-stable noises with $α \in (1,2)$ in [8]. Specially, Xu in [26] studied the ergodicity of the stochastic real Ginzburg-Landau equation driven by α-stable noises with $α \in (\frac{3}{2},2)$ and established a maximal inequality for the stochastic α-stable convolution which is useful for studying other SPDEs forced by α-stable noise.

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Ginzburg-Landau (GL) equations are usually applied to describe a class of optical fiber materials. There have been extensive study of the GL equations (see [3, 4, 9, 10, 11, 25] and reference therein). The exact homoclinic wave and soliton solution of the GL equations have been studied in [4]. Guo et al in [11] proved the existence of a global attractor for the GL equation. The coupled Ginzburg-Landau (CGL) equations have attracted considerable attention in modeling a class of nonlinear optical fiber materials with active and passive coupled cores. There are also many papers concerning the CGL equations (see [5, 15, 20] and reference therein). The existence of the stable solutions and exponential attractors for the CGL systems has been proved in [5] and [20] respectively. Recently, the well-posedness and dynamics of the stochastic coupled fractional Ginzburg-Landau equation with multiplicative noise have been studied in [21].

Recalling that the fractional Laplace operator is the infinitesimal generator of the Feller semigroup associated with symmetric $\alpha$-stable Lévy noise. Hence, there is a natural question, how about the ergodicity of the stochastic fractional Ginzburg-Landau (CFGL) equations driven by $\alpha$-stable noise? Motivated by the work in [26] and [21], in the present paper, we consider the following coupled fractional Ginzburg-Landau (CFGL) equations driven by $\alpha$-stable noise on the Torus $T$:

$$
\begin{aligned}
&du = \left[ \gamma_1 u - (\gamma_2 + i\gamma_3)(-\Delta)^{\delta} u + (i\sigma_1 - \sigma_2)|u|^2 u + iv \right] dt + dL_t, \\
&dv = \left[ (-\mu_1 + i\mu_2)v - (\mu_3 + i\mu_4)(-\Delta)^{\delta} v + iu \right] dt + dL_t,
\end{aligned}
$$

(1)

with the initial conditions and the periodic boundary conditions:

$$
\begin{aligned}
&u(x, 0) = u_0(x), \\
&v(x, 0) = v_0(x), \\
&u(x + 2\pi, t) = u(x, t), \\
&v(x + 2\pi, t) = v(x, t), \\
&u > 0,
\end{aligned}
$$

(2)

where $\delta \in \left( \frac{1}{2}, 1 \right)$, $u$ and $v$ denote the amplitude of the electromagnetic wave in a dual-core system, $t$ denotes the time, $x$ is the horizontal axis of the plane wave, $\gamma_2 > 0$, $\mu_2 > 0$, $\mu_3 > 0$ and $\mu_4 > 0$ are dissipation coefficients, $\gamma_1 > 0$, $\mu_1 > 3$, $\gamma_3, \sigma_1, \sigma_2$ are real numbers and $L_t$ is cylindrical $\alpha$-stable noise (be specialized later).

The fractional Laplacian $(-\Delta)^{\delta}$ can be regarded as a pseudo differential operator with $|\xi|^{2\delta}$ and can be realized through the Fourier transform [22]:

$$
(-\Delta)^{\delta} u(\xi) = |\xi|^{2\delta} \widehat{u}(\xi),
$$

(3)

where $\widehat{u}$ is the Fourier transform of $u$.

In the current paper, we consider the ergodicity of the stochastic CFGL equations driven by $\alpha$-stable noise with $\alpha \in \left( \frac{1}{3\delta - 2\delta}, 2 \right)$. The main contribution of the current paper is to apply the commutator estimates developed by Kato and Ponce in [13] to overcome the difficulty in the convergence since higher order estimates cannot be obtained caused by the nonlocal fractional diffusion. We apply the maximal inequality for the stochastic $\alpha$-stable convolution developed by Xu in [26] and Banach Fixed point theorem to show the existence and uniqueness of the global mild solution. By a priori estimates for the Galerkin approximation equations and classical Bogoliubov-Krylov theorem, the existence of invariant measure is proved. We truncate the nonlinear term to prove the strong Feller property by a gradient estimate developed by Priola and Zabczyk in [17]. Due to the discontinuous trajectories of $\alpha$-stable noise, we prove that the system (1) is accessible to zero instead of the irreducibility, and present the ergodicity by the criterion in Hairer in [12].
The rest of the paper is organized as follows. In section 2 is devoted to the introduction of the functional setting, the maximal inequality for the stochastic \( \alpha \)-stable convolution and commutator estimates. In section 3, we apply the Banach fixed point theorem and the commutator estimates to show the well-posedness of the mild solution for the stochastic FCGL equation. In section 4, some a priori estimates for the Galerkin approximation equation are presented and the existence of the invariant measures for equation (1) is obtained. Finally, the strong Feller property and the accessibility of the system (1) is shown to obtain the uniqueness of the invariant measure for the stochastic FCGL equations.

In this section, we first introduce some notations for the working function space, and then present the maximal inequality for the stochastic \( \alpha \)-stable convolution and commutator estimates, which are the key tools to show the well-posedness of the solution for stochastic FCGL equations on the torus \( \mathbb{T} \).

Firstly, we introduce some notations as follows

\[
H = L^2_{\text{per}}(\mathbb{T}) = \{u | u \in L^2[0, 2\pi], u(x + 2\pi, t) = u(x, t)\},
\]

with the norm

\[
\|u\|_H^2 = (u, u^*) = \int_\mathbb{T} |u|^2 dx.
\]

Denote

\[
W = H \times H = \{(u, v) | u, v \in H\}
\]

endowed with the norm \( \|\phi\|_{H \times H} = \|u\|_H + \|v\|_H \) for any \( \phi = (u, v) \in H \times H \). We recall some nations to the fractional derivative and fractional Sobolev space.

Since \( u \) is a periodic function, it can be expressed by a Fourier series

\[
u(x) = (\mathcal{F}^{-1}\hat{u})(x) := \frac{1}{2\pi} \sum_{\xi \in \mathbb{Z}} \hat{u}(\xi)e^{i\xi x},
\]

where

\[
\hat{u}(\xi) := \int_\mathbb{T} e^{-i\xi y} u(y) dy.
\]

Then for \( \delta \in \mathbb{R} \), we can write

\[
A^\delta u = \mathcal{F}^{-1}(|\xi|^{2\delta} \hat{u}(\xi)).
\]

Let \( A = -\Delta \), the operator \( A^\delta \) with \( \delta \geq 0 \) can be defined by

\[
A^\delta x = \sum_{k \in \mathbb{Z}_*} \gamma_k^\delta x_k, \quad x \in D(A^\delta), \quad \gamma_k = 4\pi^2|k|^2,
\]

with

\[
D(A^\delta) = \{x \in H : x = \sum_{k \in \mathbb{Z}_*} \gamma_k x_k, \sum_{k \in \mathbb{Z}_*} \gamma_k^2 |x_k|^2 < \infty\},
\]

where \( e_k \) is an orthonormal basis of \( H \) and \( \mathbb{Z}_* = \mathbb{Z} \setminus 0 \).

Denote \( V = D(A^{\frac{3}{2}}) \cap H \) endowed with the norm

\[
\|x\|_V = \|A^{\frac{3}{2}} x\|_H = (\sum_{k \in \mathbb{Z}_*} |\gamma_k|^2 |x_k|^2)^{\frac{1}{2}}, \quad x \in D(A^{\frac{3}{2}}).
\]

Let \( C_\sigma > 0 \) be some constants depending on \( \sigma \), then it follows that

\[
\|A^{\sigma} e^{-A^{\sigma} t}\| \leq C_\sigma t^{-\sigma}, \quad \forall \sigma > 0, \quad t > 0.
\]

Denote

\[ N(\phi) = \left( -\gamma_1 u + (\sigma_2 - i\sigma_1)|u|^2 u - iv \right) \], \quad \phi = (u, v). \tag{5} \]

Equation (1) can be rewritten as

\[
\begin{cases}
  d\phi + N(\phi)dt + A^\delta \phi dt = \left( dL_t \right), \\
  \phi_0 = (u_0, v_0),
\end{cases}
\]

where \( L_t = \sum_{k \in \mathbb{Z}} \beta_k l_k(t)e_k \) is a cylindrical \( \alpha \)-stable processes on \( H \) with \( \{ l_k(t) \}_{k \in \mathbb{Z}} \), being 1 dimensional symmetric \( \alpha \)-stable process sequence with \( \alpha > 1 \). In addition, we assume that there exist \( C_1, C_2 > 0 \) such that

\[ C_1 \gamma_k^{-\beta} \leq |\beta_k| \leq C_2 \gamma_k^{-\beta}, \quad \frac{\delta}{2} + \frac{1}{2\alpha \delta} < \beta < \frac{3}{2} - \frac{1}{\alpha}, \tag{7} \]

where \( \beta > \frac{\delta}{2} + \frac{1}{2\alpha \delta} \) can get that the following convolution \( Z_{1t} \) and \( Z_{2t} \) are in \( V \).

Consider the following Ornstein-Uhlenbeck process

\[
\begin{align*}
  Z_{1t} &= \int_0^t e^{-(\gamma_2 + i\gamma_3)A^\delta(t-s)}dL_s = \sum_{k \in \mathbb{Z}} z_{1k}(t)e_k, \\
  Z_{2t} &= \int_0^t e^{-(\mu_2 + i\mu_3)A^\delta(t-s)}dL_s = \sum_{k \in \mathbb{Z}} z_{2k}(t)e_k,
\end{align*}
\]

where

\[
\begin{align*}
  z_{1k}(t) &= \int_0^t e^{-(\gamma_2 + i\gamma_3)A^\delta(s)}\beta_k dl_k(s), \\
  z_{2k}(t) &= \int_0^t e^{-(\mu_2 + i\mu_3)A^\delta(s)}\beta_k dl_k(s), \quad \gamma_k = 4\pi^2|k|^2. \tag{9}
\end{align*}
\]

Here is an important lemma from [26] that plays a crucial role in proving the well-posedness, strong Feller and accessibility for the solution of equation (1).

**Lemma 1.1.** ([26]) Let \( \theta \in [0, \beta - \frac{1}{2\alpha \delta}) \) be arbitrary. For all \( T > 0 \) and \( \epsilon > 0 \), we have

\[ \mathbb{P}( \sup_{0 \leq t \leq T} \| A^\theta Z_{1t} \|_H \leq \epsilon ) > 0, \quad \mathbb{P}( \sup_{0 \leq t \leq T} \| A^\theta Z_{2t} \|_H \leq \epsilon ) > 0. \tag{10} \]

The following two lemmas which describe commutator estimates developed by Kato and Ponce in [13] and Kening et al in [14] are the key technique tools to show the well-posedness of the mild solution and the estimates on the nonlinearity \( N(\phi) \).

**Lemma 1.2.** ([13]) Assume that \( s > 0 \) and \( p \in (1, +\infty) \). If \( f \) and \( g \) are the Schwartz class, then

\[ \| \Lambda^s(fg) - f\Lambda^s g \|_{L^p} \leq C(\| \nabla f \|_{L^{p_1}} \| g \|_{H^{s-1,p_2}} + \| f \|_{H^{s,p_3}} \| g \|_{L^{p_4}}), \]

and

\[ \| \Lambda^s(fg) \|_{L^p} \leq C(\| f \|_{L^{p_1}} \| g \|_{H^{s,p_2}} + \| f \|_{H^{s,p_3}} \| g \|_{L^{p_4}}), \]

where \( p_2, p_3 \in (1, +\infty) \) such that

\[ \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}, \]

where \( \Lambda \) denotes the square root of the Laplacian \( (-\Delta)^{\frac{1}{2}} \).
Lemma 1.3. (14) Assume that $q > 1$, $p \in [q, +\infty)$, $x \in \mathbb{R}^d$ and
\[
\frac{1}{p} + \frac{s}{d} = \frac{1}{q}.
\]
Suppose that $\Lambda^s f \in L^q$, then $f \in L^p$ and there is a constant $C \geq 0$ such that
\[
\|f\|_{L^p} \leq C\|\Lambda^s f\|_{L^q}.
\]
and if $f = \Lambda^{-s}g$ for $g \in L^q$, then
\[
\|\Lambda^{-s}g\|_{L^p} \leq C\|g\|_{L^q}.
\]

2. Well-posedness of the mild solution. In this section, we will apply the commutator estimates and the Banach fixed point theorem to show the well-posedness of the mild solution for stochastic FCGL equation.

Definition 2.1. Let $I = [a, b]$ be an interval in $\mathbb{R}^+$. A mapping $g : I \to \mathbb{R}^d$ is said to be càdlàg if, for all $t \in [a, b]$, $g$ has a left limit and its right continuous at $t$. Let $D([0, T], H)$ be the space of all càdlàg paths from $[0, T]$ into $H$.

For all $\omega \in \Omega$, denote
\[
    b(\omega) := u(\omega) - Z_{1t}(\omega), \quad c(\omega) := v(\omega) - Z_{2t}(\omega).
\]
Then the equations (1) can be changed into the following equations
\[
\begin{cases}
    db = [\gamma_1(b + Z_{1t}) - (\gamma_2 + i\gamma_3)(-\Delta)^\delta b + (i\sigma_1 - \sigma_2)b + Z_{11}]dt + i(c + Z_{2t})dt, \\
    dc = [(-\mu_1 + i\mu_2)(c + Z_2) - (\mu_3 + i\mu_4)(-\Delta)^\delta c + i(b + Z_1)]dt,
\end{cases}
\]
(11)
\[
b(0, \omega) = u_0, \quad c(0, \omega) = v_0.
\]
For each $T > 0$, define
\[
    K_T(\omega) := \max\{\sup_{0 \leq t \leq T} \|Z_{1t}(\omega)\|, \sup_{0 \leq t \leq T} \|Z_{2t}(\omega)\|\}, \quad \omega \in \Omega.
\]
(12)

Lemma 1.1 yields that for every $k \in \mathbb{N}$, there exists some set $N_k \in \Omega$ such that
\[
P(N_k) = 0 \quad \text{and} \quad K_k(\omega) < \infty, \quad \omega \notin N_k.
\]
(13)
Let $N = \bigcup_{k \geq 1} N_k$, it is easy to see $P(N) = 0$ and that for all $T > 0$
\[
K_T(\omega) < \infty, \quad \omega \notin N.
\]
(14)

We introduce the working space $S$ by
\[
S = \{\psi = (b, c) \in C([0, T], H \times H) : \psi_0 = (u_0, v_0), \psi(t) \in D(A^{\xi_1}) \times D(A^{\xi_2}),
\]
\[
t \in (0, T],
\]
\[
\sup_{0 \leq t \leq T} \|A^{\xi_1}u\|_H + \sup_{0 \leq t \leq T} \|u\|_H + \sup_{0 \leq t \leq T} \|A^{\xi_2}c\|_H + \sup_{0 \leq t \leq T} \|c\|_H \leq B,\]
\]
(15)
where $\xi_1 = \frac{1}{6\sigma}$ and $\xi_2 \in (0, \frac{1}{2})$. For any $\psi = (b, c), \varphi = (d, e) \in S$, we endow $S$ the metric $D(\cdot, \cdot)$ by
\[
D(\psi, \varphi) = \sup_{0 \leq t \leq T} \|A^{\xi_1}(b - d)\|_H + \sup_{0 \leq t \leq T} \|(b - d)\|_H + \sup_{0 \leq t \leq T} \|A^{\xi_2}(c - e)\|_H + \sup_{0 \leq t \leq T} \|(c - e)\|_H,
\]
(16)
then $(S, D)$ is a closed metric space.
Lemma 2.2. If the conditions (1) holds, then we have

(B1) For any \( u_0, v_0 \in H \), and \( \omega \notin N \), there exists some \( 0 < T(\omega) \leq 1 \), depending on \( \|u_0\|_H, \|v_0\|_H \) and \( K_1(\omega) \) such that equation (1) has a unique mild solution \( \psi(\omega) = (b(\omega), c(\omega)) \in C([0, T); H \times H) \) satisfying for all \( \zeta_1 \in (0, 1/2) \)

\[
\|A^{\zeta_1}b\|_H + \|A^{\zeta_2}c\|_H \leq C(t^{-\zeta_1} + t^{-\zeta_2} + 1), \quad t \in (0, T(\omega)),
\]

where \( C \) is some constant depending on \( \|u_0\|_H, \|v_0\|_H \) and \( K_1(\omega) \).

(B2) For \( (u_0, v_0) \in D(A^{\zeta_1}) \times D(A^{\zeta_2}) \), and \( \omega \notin N \), there exists some \( 0 < \tilde{T}(\omega) \leq 1 \), depending on \( \|u_0\|_H, \|v_0\|_H \) and \( K_1(\omega) \) such that equation (1) has a unique mild solution \( \psi(\omega) = (b(\omega), c(\omega)) \in C([0, \tilde{T}(\omega)]; D(A^{\zeta_1}) \times D(A^{\zeta_2})) \) satisfying

\[
\sup_{0 \leq t \leq \tilde{T}(\omega)} (\|A^{\zeta_1}b\|_H + \|A^{\zeta_2}c\|_H) \leq 1 + \|A^{\zeta_1}u_0\|_H + \|A^{\zeta_2}v_0\|_H, \quad t \in [0, \tilde{T}(\omega)].
\]

Specially, when \( \zeta_1 = \zeta_2 = \frac{1}{2} \),

\[
\sup_{0 \leq t \leq \tilde{T}(\omega)} \|\psi(\omega)\|_{V \times V} \leq 1 + \|\psi_0\|_{V \times V}.
\]

Proof. For the sake of simplicity, we will omit the variable \( \omega \). Let \( 0 < T \leq 1 \) and \( B > 0 \) be some constants to be determined later.

For any \( \psi \in S \), define a map \( F : S \to C([0, T]; H \times H) \) by

\[
F \psi = (F_1 b, F_1 c) = (e^{-(\gamma_2+i\gamma_3)A^t}u_0 - \int_0^t e^{-(\gamma_2+i\gamma_3)A^{t-s}}N(b + Z_{1t})ds, e^{-(\mu_3+i\mu_4)A^t}v_0 - \int_0^t e^{-(\mu_3+i\mu_4)A^{t-s}}N(c + Z_{2t})ds).
\]

We claim that there exist \( T_0 > 0 \) and \( B_0 > 0 \) such that the following (a) and (b) hold for \( t \in (0, T_0) \) and \( B \geq B_0 \):

(a) \( F \psi \in S \) for any \( \psi \in S \).

(b) \( D(F \psi, F \varphi) \leq \frac{1}{2} D(\psi, \varphi) \) for any \( \psi, \varphi \in S \).

If fact, it follows from (1), Lemma 1.3 and Young’s inequality that

\[
\|A^{\zeta_1}(F_1 b)\|_H \leq C t^{-\zeta_1} \|u_0\|_H + C \int_0^t \|A^{\zeta_1} e^{-(\gamma_2+i\gamma_3)A^{t-s}}\| \|b + Z_{1t}\|^3 \|H \, ds
\]

\[
+ C \int_0^t \|A^{\zeta_1} e^{-(\gamma_2+i\gamma_3)A^{t-s}}\| \|b + Z_{1t}\|_H \|ds
\]

\[
+ C \int_0^t \|A^{\zeta_1} e^{-(\gamma_2+i\gamma_3)A^{t-s}}\| \|c + Z_{2t}\|_H \|ds
\]

\[
\leq C t^{-\zeta_1} \|u_0\|_H + C \int_0^t (t-s)^{-\zeta_1} \|A^{\zeta_1} (b + Z_{1t})\|^3 \|H \, ds
\]

\[
+ C \int_0^t (t-s)^{-\zeta_1} \|b + Z_{1t}\|_H + \|(c + Z_{2t})\|_H \|ds
\]

\[
\leq C t^{-\zeta_1} \|u_0\|_H + C \int_0^t (t-s)^{-\zeta_1} (\|A^{\zeta_1} b\|^3_H + K_3^3 + B + K_1) \|ds
\]

\[
\leq C t^{-\zeta_1} \|u_0\|_H + C \int_0^t (t-s)^{-\zeta_1} (s^{-3\zeta_1} B^3 + K_3^3 + B + K_1) \|ds,
\]
which yields
\[ t^{\xi_1} \|A^{\xi \delta}(\mathcal{F}_1 b)\|_H \leq C\|u_0\|_H + C t^{\xi_1} \int_0^t (s^{-3\xi_1} B^3 + K_1^3 + B + K_1) ds. \] (22)

Lemma 1.3 and Young’s inequality implies that
\[ \|\mathcal{F}_1 b\|_H \leq \|u_0\|_H + C \int_0^t \|b + Z_{1t}\|^3 \|b + Z_{1t}\|_H ds + C \int_0^t \|b + Z_{1t}\|_H + \|c + Z_{2t}\|_H ds \]
\[ \leq \|u_0\|_H + C \int_0^t (s^{-3\xi_1} B^3 + K_1^3 + B + K_1). \] (23)

Similarly,
\[ \|A^{\xi_2}(\mathcal{F}_2 c)\|_H \leq C t^{-\xi_2} \|v_0\|_H + C \int_0^t (t - s)^{-\xi_2}(\|b + Z_{1t}\|_H + \|c + Z_{2t}\|_H) ds \]
\[ \leq C t^{-\xi_2} \|v_0\|_H + C \int_0^t (t - s)^{-\xi_2}(B + K_1), \] (24)

which gives
\[ t^{\xi_2} \|A^{\xi_2}(\mathcal{F}_2 c)\|_H \leq C\|v_0\|_H + C t^{\xi_2} \int_0^t (B + K_1) ds, \] (25)
and
\[ \|(\mathcal{F}_2 c)\|_H \leq \|v_0\|_H + C \int_0^t (t - s)^{-\xi_2}(B + K_1). \] (26)

Choosing \( T > 0 \) be small enough and \( B \) be large enough, then we derive the claim \( (a) \) holds from (21)–(26). Moreover, (23) and (26) imply the continuity of \( \mathcal{F}_\psi \) in \( H \times H \).

Next, we prove the claim–(b) holds. For any \( \psi, \varphi \in S \), it follows from \( (4) \), Lemma 1.3 and Young’s inequality that
\[ t^{\xi_1} \|A^{\xi_2}(\mathcal{F}_1 b - \mathcal{F}_1 d)\|_H \]
\[ \leq C T^{\xi_1} \int_0^t (t - s)^{-\xi_1} \|b + Z_{1t}\|^3 - |d + Z_{1t}|^3 \|b + Z_{1t}\|_H \]
\[ + C T^{\xi_1} \int_0^t (t - s)^{-\xi_1} (\|b - d\|_H + \|c - e\|_H) ds \]
\[ \leq C T^{\xi_1} \int_0^t (t - s)^{-\xi_1} (\|A^{\xi_2} b\|_H^2 + \|A^{\xi_2} d\|_H^2 + K_1^3) \|A^{\xi_2} (b - d)\|_H ds \]
\[ + C T^{\xi_1} \int_0^t (t - s)^{-\xi_1} (\|b - d\|_H + \|c - e\|_H) ds \]
\[ \leq C T^{\xi_1} \int_0^t (t - s)^{-\xi_1} (s^{-2\xi_1} B^2 + K_1^2) s^{-\xi_1} [s^{\xi_1} \|A^{\xi_2} (b - d)\|_H] ds \]
\[ + C T^{\xi_1} \int_0^t (t - s)^{-\xi_1} (\|b - d\|_H + \|c - e\|_H) ds \]
\[ \leq C [B^2 T^{-3\xi_1} + K_1^2 T^{-1-\xi_1}] \sup_{0 \leq s \leq T} s^{\xi_1} \|A^{\xi_2} (b - d)\|_H \]
\[ + C T \sup_{0 \leq s \leq T} (\|b - d\|_H + \|c - e\|_H). \] (27)
Similarly, 
\[ t^{\zeta_2} \| A^{\zeta_2} (F_2 c - F_2 e) \|_H \leq C T \sup_{0 \leq s \leq T} (\| b - d \|_H + \| c - e \|_H), \] 
(28)

\[ \| F_1 b - F_1 d \|_H \leq C [B^2 T^{1-3\zeta_1} + K_1^2 T^{1-\zeta_1}] \sup_{0 \leq s \leq T} s^{\zeta_1} \| A^{\zeta_1} (b - d) \|_H + 
+ C T \sup_{0 \leq s \leq T} (\| b - d \|_H + \| c - e \|_H), \]
and
\[ \| F_1 c - F_1 e \|_H \leq C T \sup_{0 \leq s \leq T} (\| b - d \|_H + \| c - e \|_H). \] 
(29)

Combining (27) and (29) gives
\[ d(F \psi, F \varphi) \leq C [K_1^2 T^{1-\zeta_1} + B^2 T^{1-3\zeta_1} + C T] d(\psi, \varphi). \] 
(30)

Choosing \( T \) be small enough, then we obtain claim-(b).

By the Banach fixed point theorem and the claims (a) and (b), we obtain that equation (1) admits a unique solution in \( S \).

Let \( \psi \in S \) be the above solution obtained by the Banach fixed point theorem. For every \( \zeta_1 \in \left[ \frac{1}{5}, \frac{3}{2} \right] \) and \( \zeta_2 \in (0, \frac{1}{2}) \), it follows from [1, Lemma 1.3 and Young’s inequality that
\[ \| A^{\zeta_1} b \|_H \leq C t^{-\zeta_1} \| u_0 \|_H + C \int_0^t (t - s)^{-\zeta_1} (\| A^{\frac{1}{2}} b \|_H^3 + K_1^3 + B + K_1) \]
\[ \leq C t^{-\zeta_1} \| u_0 \|_H + C \int_0^t (t - s)^{-\zeta_1} (s^{-\frac{1}{2}} B^3 + K_1^3 + B + K_1), \]
(31)

and
\[ \| A^{\zeta_2} c \|_H \leq C t^{-\zeta_2} \| v_0 \|_H + C \int_0^t (t - s)^{-\zeta_2} (B + K_1), \]
which give the desired inequality.

Now we prove the uniqueness. Let \( \psi, \varphi \in C([0, T], H \times H) \) be two solutions satisfying the above inequality. Similarly to (27), we have
\[ \| b - d \|_H \leq \int_0^t (K_1^2 + \| A^{\frac{1}{2}} b \|_H^2 + \| A^{\frac{1}{2}} d \|_H^2) \| b - d \|_H \]
\[ + \int_0^t (\| b - d \|_H + \| c - e \|_H) ds, \]
(32)

and
\[ \| c - e \|_H \leq \int_0^t (\| b - d \|_H + \| c - e \|_H) ds, \]
(33)
the inequalities (32)-(33) imply \( \| b - d \|_H + \| c - e \|_H = 0 \) for all \( t \in [0, T] \). Thus, we finish the proof of result (B1).

Finally, we prove (B2). Let \( 0 < \tilde{T} < 1 \) and \( \zeta_1 \in \left( \frac{1}{63}, \frac{1}{2} \right), \zeta_2 \in (0, \frac{1}{2}) \), define the space \( \tilde{S} \) by
\[ \tilde{S} = \{ \psi = (b, c) \in C([0, \tilde{T}], D(A^{\zeta_1} \times D(A^{\zeta_2})): \psi_0 = (u_0, v_0), \]
\[ \sup_{0 \leq t \leq T} \| A^{\zeta_1} b \|_H + \sup_{0 \leq t \leq T} \| A^{\zeta_2} c \|_H \leq 1 + \| A^{\zeta_1} u_0 \|_H + \| A^{\zeta_2} v_0 \|_H \}. \]
(34)
endowed with the metric
\[
\bar{D}(\psi, \varphi) = \sup_{0 \leq t \leq T} \|A^{\delta}(b - d)\|_H + \sup_{0 \leq t \leq T} \|A^{\delta}(c - e)\|_H, \quad \forall \psi, \varphi \in \tilde{S}.
\]  
(35)

Then \((\tilde{S}, \bar{D})\) is a closed metric space.

Define a map \(\tilde{F} : \tilde{S} \to C([0, T], D(A^{\delta}) \times D(A^{\delta}))\) by
\[
\tilde{F}\psi = (\tilde{F}_1b, \tilde{F}_2c) = \left( e^{-\sum_{i=1}^{m} \lambda_i t} v_0 - \int_0^t e^{-\sum_{i=1}^{m} \lambda_i (t-s)} f(b + Z_t)ds + \int_0^t e^{-\sum_{i=1}^{m} \lambda_i (t-s)} f(c + Z_t)ds, \right)
\]  
for any \(\psi \in \tilde{S}\).

It follows from \((4)\), Lemma 1.3 and Young’s inequality that
\[
\|A^{\delta}(\tilde{F}_1b)\|_H \leq C \|A^{\delta}v_0\|_H + C \int_0^t (t-s)^{-\gamma_1} (\|A^{\delta}b\|^2 + K_1^2)ds
\]  
and
\[
\|A^{\delta}(\tilde{F}_2c)\|_H \leq C \|A^{\delta}v_0\|_H + C \int_0^t (t-s)^{-\gamma_2} (\|A^{\delta}b\|^2 + \|A^{\delta}c\|^2 + K_1)ds,
\]  
which imply that \(\tilde{F} : \tilde{S} \to \tilde{S}\) provided \(\tilde{T} > 0\) is small enough.

Similarly, we can obtain
\[
\sup_{0 \leq t \leq T} \|A^{\delta}(\tilde{F}_1b - \tilde{F}_1d)\|_H + \sup_{0 \leq t \leq T} \|A^{\delta}(\tilde{F}_2c - \tilde{F}_2e)\|_H
\]  
\[
\leq \frac{1}{2} \left( \sup_{0 \leq t \leq T} \|A^{\delta}(b - d)\|_H + \sup_{0 \leq t \leq T} \|A^{\delta}(c - e)\|_H \right),
\]
which means that \(\tilde{D}(\tilde{F}\psi, \tilde{F}\varphi) \leq \frac{1}{2} \tilde{D}(\psi, \varphi)\). The conclusion (B2) follows from the Banach fixed point theorem. \(\square\)

**Lemma 2.3.** If the conditions \((7)\) holds, then the following statements hold.

(C1) For any \(u_0, v_0 \in H\), and \(\omega \notin N\), equation \((11)\) has a unique global solution \(\psi(\omega) = (b(\omega), c(\omega)) \in C([0, \infty); H \times H)\).

(C2) For \(u_0, v_0 \in V\), and \(\omega \notin N\), \(\psi(\omega) = (b(\omega), c(\omega)) \in C([0, \infty); V \times V)\).

**Proof.** Multiplying the first term and second term of equation \((6)\) with \(u^*\) and \(v^*\), then integrating over \(\mathcal{D}\) leads to
\[
\frac{1}{2} \frac{d}{dt} \|b\|_H^2 + \frac{\sigma_2}{2} \|b\|_L^2 + \gamma_2 \|A^{\frac{1}{2}}b\|_H^2
\]  
\[
\leq (\frac{3}{2} \gamma_1 + 1) \|b\|_H^2 + \frac{1}{2} \|c\|_H^2 + C \|A^{\frac{1}{2}}Z_t\|_H^2 + \frac{1}{2} \|Z_t\|_H^2,
\]  
(39)

\[
\frac{1}{2} \frac{d}{dt} \|c\|_H^2 + \mu_3 \|A^{\frac{1}{2}}c\|_H^2
\]  
\[
\leq (1 - \frac{\mu_1}{2}) \|c\|_H^2 + \frac{1}{2} \|b\|_H^2 + \frac{\mu_1}{2} \|Z_t\|_H^2 + \frac{1}{2} \|Z_t\|_H^2.
\]  
(40)

Combining \((39)\) and \((40)\) gives
\[
\frac{d}{dt} (\|b\|_H^2 + \|c\|_H^2) + \frac{\sigma_2}{2} \|b\|_L^2 + 2R_2 (\|A^{\frac{1}{2}}b\|_H^2 + \|A^{\frac{1}{2}}c\|_H^2)
\]  
\[
\leq 2R_1 (\|b\|_H^2 + \|c\|_H^2) + C (\|A^{\frac{1}{2}}Z_t\|_H^2 + \|A^{\frac{1}{2}}Z_t\|_H^2),
\]  
(41)
where $R_1 = \max\{\frac{3}{2} \gamma_1 + \frac{3}{2}, \frac{3}{2} - \frac{\mu_1}{2}\}$ and $R_2 = \min\{\sigma_2, \mu_3\}$.

It follows from Gronwall’s inequality that
\[
\|b\|^2_H + \|c\|^2_H \leq e^{2R_1t} (\|u_0\|^2_H + \|v_0\|^2_H)
+ C \int_0^t e^{2R_1(t-s)} (\|A^\frac{4}{3} Z_{1t}\|^2_H + \|A^\frac{4}{3} Z_{2t}\|^2_H) ds.
\] (42)

Then integrating (41) over $[0,t]$ yields the following estimate
\[
\begin{align*}
\left(\|b\|^2_H + \|c\|^2_H\right) + \sigma_2 \int_0^t \|b\|^2_{L^4} ds + 2R_2 \int_0^t (\|A^\frac{4}{3} b\|^2_H + \|A^\frac{4}{3} c\|^2_H) ds \\
\leq e^{2R_1t} (\|u_0\|^2_H + \|v_0\|^2_H) + C \int_0^t e^{2R_1(t-s)} (\|A^\frac{4}{3} Z_{1t}\|^2_H + \|A^\frac{4}{3} Z_{2t}\|^2_H) ds.
\end{align*}
\]

Due to the Young’s inequality, it follows
\[
\frac{1}{4} \sigma_2 \|u\|_{L^4}^2 + \frac{9\pi}{2\sigma_2} (\gamma_1 + 1)^2 \geq \frac{3}{2} (\gamma_1 + 1) \|u\|^2_H,
\]
then we have
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} (\|b\|^2_H + \|c\|^2_H) + 2R_3 (\|b\|^2_H + \|c\|^2_H) \\
\leq 9\pi \sigma_2 (\gamma_1 + 1)^2 + C (\|A^\frac{4}{3} Z_{1t}\|^2_H + \|A^\frac{4}{3} Z_{2t}\|^2_H).
\end{align*}
\] (43)

where $R_3 = \min\{\gamma_1 + 1, \mu_1 + 3\} > 0$.

Gronwall’s inequality gives
\[
\|b\|^2_H + \|c\|^2_H \leq e^{-R_3t} (\|u_0\|^2_H + \|v_0\|^2_H) + \frac{9\phi}{\sigma_2} (\gamma_1 + 1)^2 (1 - e^{-R_3t})
+ C \int_0^t e^{-R_3(t-s)} (\|A^\frac{4}{3} Z_{1t}\|^2_H + \|A^\frac{4}{3} Z_{2t}\|^2_H) ds.
\] (44)

Multiplying the first term and second term of equation (6) with $A^\frac{4}{3} u^*$ and $A^\frac{4}{3} v^*$, and denoting $\eta = \frac{2\alpha}{C\sigma_2}$, then integrating over $D$ leads to
\[
\begin{align*}
\frac{d}{dt} \|A^\frac{4}{3} b\|^2_H + 2\gamma_2 \|A^\frac{4}{3} b\|^2_H \\
\leq 3\gamma_1 \|A^\frac{4}{3} b\|^2_H + \gamma_1 \|A^\frac{4}{3} Z_{1t}\|^2_H + (C\sigma_2 C(\eta) + CC\sigma_2 \|u\|_{L^4}^4) \|A^\frac{4}{3} b\|^2_H \\
+ \frac{\gamma_2}{\sigma_2} \|A^\frac{4}{3} b\|^2_H + C\sigma_2 \|Z_{1t}\|^2_L + C\sigma_2 \eta \|A^\frac{4}{3} b\|^2_H + \frac{2}{\gamma_2} \|c\|^2_H + C \|Z_{2t}\|^2_H
\end{align*}
\] (45)

\[
\begin{align*}
\frac{d}{dt} \|A^\frac{4}{3} c\|^2_H + 2\mu_3 \|A^\frac{4}{3} c\|^2_H \\
\leq -\mu_1 \|A^\frac{4}{3} c\|^2_H + \mu_1 \|A^\frac{4}{3} Z_{2t}\|^2_H + \frac{1}{2\mu_3} \|b\|^2_H + 2\mu_3 \|A^\frac{4}{3} c\|^2_H + \frac{1}{2\mu_3} \|Z_{1t}\|^2_H.
\end{align*}
\] (46)

\[
\begin{align*}
\frac{d}{dt} (\|A^\frac{4}{3} b\|^2_H + \|A^\frac{4}{3} c\|^2_H) \leq h_1 (\|A^\frac{4}{3} b\|^2_H + \|A^\frac{4}{3} c\|^2_H) + h_2.
\end{align*}
\]

where $h_1 = \gamma_1 + C\sigma_2 C(\eta) + CC\sigma_2 \|u\|^2_L$, $h_2 = C(\|b\|^2_H + \|c\|^2_H) + C(\|A^\frac{4}{3} Z_{1t}\|^2_H + \|A^\frac{4}{3} Z_{2t}\|^2_H).

It follows from Gronwall’s inequality that
\[
\|A^\frac{4}{3} b\|^2_H + \|A^\frac{4}{3} c\|^2_H
\]
The existence of the invariant measure.

Theorem 2.4. Under the conditions (7), the following statements hold.

(A1) For \( u_0, v_0 \in H \), and \( \omega \in \Omega \), equation (1) possesses a unique mild solution 
\[
\phi(\omega) = (u(\omega), v(\omega)) \in D([0, \infty); H \times H) \cap D([0, \infty); V \times V).
\]
Moreover, \( \phi(\omega) \) has the following form:
\[
\phi(\omega) = e^{-(\gamma_2+i\gamma_3)t}u_0 + \int_0^t e^{-(\gamma_2+i\gamma_3)(t-s)}N(u)ds + \int_0^t e^{-(\gamma_2+i\gamma_3)(t-s)}dL_s(\omega),
\]
\[
e^{-\mu_3+i\mu_4}A^*u_0 + \int_0^t e^{-(\mu_3+i\mu_4)(t-s)}N(v)ds + \int_0^t e^{-(\mu_3+i\mu_4)(t-s)}dL_s(\omega).
\]

(A2) \( \phi \) is a Markov process.

(A3) For every \( u_0, v_0 \in V \) and \( \omega \in \Omega \) a.s., we have \( \phi(\omega) \in D([0, \infty); V \times V) \). For every \( T > 0 \),
\[
\sup_{0 \leq t \leq T} \| \phi(\omega) \|_{V \times V} \leq C,
\]
where \( C \) is some constant depending on \( T, \alpha, \beta \) and \( \omega \).

Proof. It follows from Lemma 3.3 of [26] that \( Z_{t1}, Z_{t2} \in D([0, \infty); V) \). By Lemma 2.3, \( \phi(\omega) = (b+Z(\omega), c+Z(\omega)) \) is the unique solution to equation (1) in \( D([0, \infty); H \times H) \cap D([0, \infty); V \times V) \). The Markov property follows from the uniqueness. (A3) follows from (C2) of Lemma 2.3.

3. The existence of the invariant measure. In this section, we follow the method in [7] to prove the existence of invariant measures.

Let \( e_k, \{ k \in \mathbb{Z} \} \) be an orthogonal basis of \( H \) and define
\[
H_m := \text{span}\{ e_k; |k| \leq m \}.
\]
It is known that \( H_m \) is a finite dimensional Hilbert space equipped with the norm adopted from \( H \). For any \( m > 0 \), let \( \pi_m : H \rightarrow H_m \) be the projection from \( H \) to \( H_m \).

The Galerkin approximation of (1) has the following form
\[
d\phi^m + [N^m(\phi^m)]dt + A^\delta \phi^m = \left( \begin{array}{c} dL^m_t \\ dL^m_t \end{array} \right),
\]
where \( \phi^m = \pi_m \phi, N^m(\phi^m) = \pi_m [N(\phi^m)], L^m_t = \sum_{|k| \leq m} \beta_k l_k(t)e_k. \)

Lemma 3.1. If the conditions (7) holds, then we have the following statements.

(D1) For \( u_0, v_0 \in W \) with \( W = H, V \) and \( \omega \in \Omega \) a.s., there exists a unique mild solution \( \phi^m(\omega) \in D([0, \infty); W_m \times W_m) \) satisfying
\[
\sup_{0 \leq t \leq T} \| \phi^m(\omega) \| \leq C, \quad T > 0,
\]
where \( C \) is some constant depending on \( \| u_0 \|_W, \| v_0 \|_W, T \) and \( K_T(\omega) \).

(D2) For \( u_0, v_0 \in W \) with \( W = H, V \) and \( \omega \in \Omega \) a.s., it follows that
\[
\lim_{n \rightarrow \infty} \| \phi^m(\omega) - \phi(\omega) \|_W = 0, \quad t \geq 0.
\]
Proof. For $t > 0$, Theorem 2.4 implies that
\[
\sup_{0 \leq s \leq T} \|u\|_V \leq \hat{C}_1, \quad \sup_{0 \leq s \leq T} \|u^m\|_V \leq \hat{C}_1, \quad \sup_{0 \leq s \leq T} \|v\|_V \leq \hat{C}_1, \quad \sup_{0 \leq s \leq T} \|v^m\|_V \leq \hat{C}_1.
\]

It follows that
\[
u^m - u = e^{-(\gamma_2+i\gamma_3)A^t}(u^m_0 - u_0) + Z_{1t} - Z_{1t}^m + \int_0^t e^{-(\gamma_2+i\gamma_3)A^t(s-t)}(I - \pi_m)N(u)
\]
\[
+ \int_0^t e^{-(\gamma_2+i\gamma_3)A^t(s-t)}[N^m(u^m) - N^m(u)]ds = I_1 + I_2 + I_3 + I_4,
\]
and
\[
v^m - v = e^{-(\mu_3+i\mu_4)A^t}(v^m_0 - v_0) + Z_{2t} - Z_{2t}^m + \int_0^t e^{-(\mu_3+i\mu_4)A^t(s-t)}(I - \pi_m)N(v)
\]
\[
+ \int_0^t e^{-(\mu_3+i\mu_4)A^t(s-t)}[N^m(v^m) - N^m(v)] = J_1 + J_2 + J_3 + J_4.
\]

Let $m \to \infty$, then
\[
\|I_1\|_V \to 0, \quad \|I_2\|_V \to 0, \quad \|J_1\|_V \to 0, \quad \|J_2\|_V \to 0.
\]
Similarly to (21)-(26), we have
\[
\|I_3\|_V \leq C \int_0^t (t-s)^{-\frac{1}{2}} \|I - \pi_m\|_V \|N(u)\|_H ds
\]
\[
\leq C \|I - \pi_m\|_V (\|u\|_H + \|v\|_H + \|A^{\delta}u\|_H^2) \to 0, \quad m \to \infty,
\]
and
\[
\|J_3\|_V \leq C \int_0^t (t-s)^{-\frac{1}{2}} \|I - \pi_m\|_V \|N(v)\|_H ds
\]
\[
\leq C \|I - \pi_m\|_V (\|u\|_H + \|v\|_H) \to 0, \quad m \to \infty.
\]

Direct calculation shows
\[
\|I_4\|_V \leq C \int_0^t (t-s)^{-\frac{1}{2}} \|I - \pi_m\|_V \|N(u) - N(u^m)\|_H ds
\]
\[
\leq CK \int_0^t (t-s)^{-\frac{1}{2}} \|u - u^m\|_{V_2} ds + C \int_0^t (t-s)^{-\frac{1}{2}} \|v - v^m\|_{V_2} ds
\]
\[
\leq CK \int_0^t (t-s)^{-\frac{1}{2}} ds \left( \int_0^t \|u - u^m\|_{V_2}^2 ds \right)^{\frac{1}{2}}
\]
\[
+ C \left( \int_0^t (t-s)^{-\frac{1}{2}} ds \right)^{\frac{1}{2}} \left( \int_0^t \|v - v^m\|_{V_2}^2 ds \right)^{\frac{1}{2}},
\]
and
\[
\|J_4\|_V \leq C \left( \int_0^t (t-s)^{-\frac{1}{2}} ds \right)^{\frac{1}{2}} \left( \int_0^t \|u - u^m\|_{V_2}^2 ds \right)^{\frac{1}{2}}
\]
\[
+ C \left( \int_0^t (t-s)^{-\frac{1}{2}} ds \right)^{\frac{1}{2}} \left( \int_0^t \|v - v^m\|_{V_2}^2 ds \right)^{\frac{1}{2}}.
\]

where $K = \sup_{0 \leq s \leq t,m}(\|u\|_V + \|u^m\|_V) \leq 2\hat{C}_1$ and $\frac{1}{p} + \frac{1}{q} = 1$ with $1 \leq p \leq 2$. 
Due to $\sup_{0\leq s\leq t} (\|u - u^m\|_V + \|v - v^m\|_V) \leq 4\hat{C}_1$ and Fatou's theorem, we derive
$$
\lim_{m \to \infty} \sup_m (\|u - u^m\|_V + \|v - v^m\|_V) 
\leq C(\hat{K} + 1)t^{\frac{1}{2} - \frac{1}{q}} \lim_{m \to \infty} \sup_m (\int_0^t \|u^m - u\|_V^q \, ds) + (\int_0^t \|v^m - v\|_V^q \, ds) \,\right)^{\frac{1}{q}}
\leq C(\hat{K} + 1)t^{\frac{1}{2} - \frac{1}{q}} (\int_0^t \lim_{m \to \infty} \sup_m \|u^m - u\|_V^q \, ds) \,\right)^{\frac{1}{q}}
+ (\int_0^t \lim_{m \to \infty} \sup_m \|v^m - v\|_V^q \, ds) \,\right)^{\frac{1}{q}},
$$
which implies
$$
\lim_{m \to \infty} \sup_m (\|u - u^m\|_V + \|v - v^m\|_V) \to 0.
$$
Thus, the proof of Lemma [3.1] is completed. 

**Theorem 3.2.** The solution $\phi$ of equation (1) admits at least one invariant measure. The invariant measures are supported on $V \times V$.

**Proof.** We follow the method in [7]. Define
$$
f(\psi) := (\|\psi\|_{H \times H}^2 + 1)^{\frac{1}{2}}, \quad \psi^m \in H_m \times H_m.
$$
Applying Itô formula [2] gives
$$
f(\phi^m) := f(\phi^m_0) - G^m_1(t) + G^m_2(t) + G^m_3(t) + G^m_4(t),
$$
where

$$
\begin{align*}
G^m_1(t) &:= \int_0^t \frac{\|\phi^{m}\|_{H \times H}^2}{1 + \|\phi^{m}\|_{H \times H}^2} \, ds + \int_0^t \frac{\langle \phi^{m}, N(\phi^{m}) \rangle}{1 + \|\phi^{m}\|_{H \times H}^2} \, ds, \\
G^m_2(t) &:= \sum_{|j| < m} \int_0^t \int_{|y| \leq 1} \left[ f(u^m + \gamma^j y) - f(u^m) + f(b^m + \gamma^j y) - f(b^m) \right] \tilde{N}^j(ds, dy), \\
G^m_3(t) &:= \sum_{|j| < m} \int_0^t \int_{|y| \leq 1} \left[ f(u^m + \gamma^j y) - f(u^m) + f(b^m + \gamma^j y) - f(b^m) \right] - \frac{\langle u^m, y \gamma^j e_j \rangle + \langle b^m, y \gamma^j e_j \rangle}{\|\phi^{m}\|_{H \times H}^2 + 1} \nu(dy) \, ds, \\
G^m_4(t) &:= \sum_{|j| < m} \int_0^t \int_{|y| > 1} \left[ f(u^m + \gamma^j y) - f(u^m) + f(b^m + \gamma^j y) - f(b^m) \right] \tilde{N}(ds, dy),
\end{align*}
$$
and $\nu(\cdot)$ is the Lévy measure and satisfies that $\int_{R \setminus \{0\}} 1 \wedge |y|^2 \nu(dy) < \infty$. For $t > 0$ and $\Gamma \in \mathcal{B}(R \setminus \{0\})$, the Poisson random measure associated with $\alpha$ stable Lévy noise is defined by $N^j(t, \Gamma) = \sum_{s \in (0, t]} 1_{\Gamma}(l_j(s) - l_j(s-))$, and the Compensated Poisson random measure is $\tilde{N}^j(t, \Gamma) = N^j(t, \Gamma) - t \nu(\Gamma)$. 

Noticing that \(\langle N(\phi^m), \phi^m \rangle = \langle N(u^m), u^m \rangle + \langle N(v^m), v^m \rangle\). It follows from Young’s inequality that
\[
\langle N(u^m), u^m \rangle \geq -\gamma_1 \|u^m\|_H^2 + \sigma_2 \|u^m\|_H^2 - \mu_1 \|v^m\|_H^2 - 2\pi \|u^m\|_H^2,
\]
\[
\langle N(v^m), v^m \rangle \geq \mu_1 \|v^m\|_H^2 - \mu_1 \|v^m\|_H^2 - \frac{2}{\mu_1} \|u^m\|_H^2,
\]
\[
\sigma_2 \|u^m\|_H^2 - (\gamma_1 + \frac{1}{\mu_1})\|u^m\|_H^2 \geq \frac{\pi(\gamma_1 + \frac{1}{\mu_1})^2}{2\sigma_2}.
\]
Direct calculation derive that
\[
G^m(t) \geq \int_0^t \frac{\|\phi^m\|_V^2}{1 + \|\phi^m\|_H^2} ds - \frac{\pi(\gamma_1 + \frac{1}{\mu_1})^2}{2\sigma_2} t.
\]
(56)

Similar to the argument in [7], we obtain
\[
\mathbb{E}\left[ \sup_{0 < t \leq T} |G^m_2(t)| \right] \leq CT^\frac{1}{2}, \quad \mathbb{E}\left[ \sup_{0 < t \leq T} |G^m_3(t)| \right] \leq CT, \quad \mathbb{E}\left[ \sup_{0 < t \leq T} |G^m_4(t)| \right] \leq CT.
\]

Hence, we have
\[
\mathbb{E}\left[ \sup_{0 < t \leq T} (|\phi_m|^2_{H \times H} + 1)^{\frac{1}{2}} \right] + \mathbb{E}\left[ \int_0^t \frac{\|\phi^m\|_V^2}{1 + \|\phi^m\|_H^2} ds \right]
\]
\[
\leq (|\phi_0|^2_{H \times H} + 1)^{\frac{1}{2}} + CT + CT^\frac{1}{2}.
\]

Lemma 3.1 implies \(\lim_{m \to \infty} \|\phi^m\|_{H \times H} = |\phi|_{H \times H}, \quad \lim_{m \to \infty} \|\phi^m\|_{V \times V} = |\phi|_{V \times V}\).

By Fatou’s Lemma, we obtain
\[
\mathbb{E}\left[ \sup_{0 < t \leq T} (|\phi|^2_{H \times H} + 1)^{\frac{1}{2}} \right] + \mathbb{E}\left[ \int_0^t \frac{\|\phi\|^2_V}{1 + \|\phi\|^2_H} ds \right]
\]
\[
\leq (|\phi_0|^2_{H \times H} + 1)^{\frac{1}{2}} + CT + CT^\frac{1}{2}.
\]

It follows from Young’s inequality that
\[
\mathbb{E}\left( \int_0^T \|\phi\|_{V \times V} ds \right) \leq \mathbb{E}\left( \int_0^T \frac{\|\phi\|^2_V}{1 + \|\phi\|^2_H} ds \right)
\]
\[
\leq C\mathbb{E}\left( \int_0^T \frac{\|\phi\|^2_V + 1}{1 + \|\phi\|^2_H} ds \right)
\]
\[
\leq C(1 + |\phi_0|_{H \times H} + T),
\]
which with the classical Bogoliubov-Krylov’s argument implies the existence of invariant measures and that the support of invariant measure is \(V \times V\).

4. **Uniqueness of the invariant measure.** In this section, we will follow the idea of [20] to show the uniqueness of the invariant measure. Due to the discontinuous trajectories and non-Lipschitz nonlinear term, the existence and uniqueness of the invariant measures are obtained by the strong Feller property and the accessibility to zero instead of the irreducibility.

4.1. **Strong Feller property.** Denote by \(B_0(E)\) the space of bounded measurable function \(f : E \to \mathbb{R}\). For all \(f \in B_0(H \times H)\), define
\[
\mathbb{P}_t(f(u_0, v_0)) = \mathbb{E}[f(u, v)],
\]
where \(t \geq 0\) and \((u_0, v_0) \in H \times H\). It follows from Theorem 2.4 that \((\mathbb{P}_t)_{t \geq 0}\) is a Markov semigroup on \(B_0(H \times H)\).
The noise \((L_t)_{t \geq 0}\) under the norm \(\| \cdot \|_V\) needs to get a gradient estimate for the OU semigroup corresponding to \((Z_{1t})_{t \geq 0}\) and \((Z_{2t})_{t \geq 0}\) to show the strong Feller property of the semigroup \((P_t)_{t \geq 0}\) on \(B_b(V \times V)\).

Since the nonlinear term \(N(\cdot)\) is not bounded, we consider the equation with truncated nonlinearity as follows:

\[
d\phi^\rho + [A^\delta \phi^\rho + N^\rho(\phi^\rho)]dt = \left( \frac{dL^\mu}{dL^\mu_t} \right),
\]

where \(\rho > 0\), \(N^\rho(\phi) = N(\phi)\chi(\frac{\|\phi\|_V}{\rho})\) for all \(\phi \in V \times V\) and \(\chi : \mathbb{R} \to [0, 1]\) is a smooth function such that

\[
\chi(z) = 1, \quad |z| < 1, \quad \chi(z) = 0, \quad |z| \geq 2.
\]

For \(\phi \in V \times V\), we have

\[
\|N^\rho(\phi)\|_{V \times V} = \|N^\rho(u)\|_V + \|N^\rho(v)\|_V,
\]

with the estimates

\[
\|N^\rho(u)\|_V \leq C\|u\|_V + C\|v\|_V + \|A^{\frac{\delta}{2}}u\|_H, \quad \|N^\rho(v)\|_V \leq C\|u\|_V + C\|v\|_V.
\]

It follows from Lemma 1.2 and Lemma 1.3 and \(\delta > \frac{1}{2}\) that

\[
\|A^{\frac{\delta}{2}}u\|_H \leq (\|A^{\frac{\delta}{2}}u\|_H^2 + \|A^{\frac{\delta}{2}}u\|_H^2) \leq C\|A^{\frac{\delta}{2}}u\|_H^2 \leq C\|A^{\frac{\delta}{2}}u\|_H^2.
\]

Thus, we obtain

\[
\|N^\rho(\phi)\|_{V \times V} \leq C(\|u\|_V + \|v\|_V + \|u\|_V^3)\chi(\frac{\|\phi\|_{V \times V}}{\rho}) \leq C(\rho^3 + \rho). \tag{59}
\]

For any \(\phi, \varphi \in V \times V\), it follows that

\[
\|N^\rho(\phi) - N^\rho(\psi)\|_{V \times V} = \|N^\rho(u) - N^\rho(b)\|_V + \|N^\rho(v) - N^\rho(c)\|_V,
\]

\[
\leq C\|u - b\|_V + C\|u - b\|_V + \|A^{\frac{\delta}{2}}(u^3 - b^3)\|_H.
\]

Lemma 1.3 gives

\[
\|A^{\frac{\delta}{2}}(u^3 - b^3)\|_H \leq C(\|u\|_V^2 + \|b\|_V^2)\|u - b\|_V.
\]

Thus, we derive that

\[
\|N^\rho(\phi) - N^\rho(\psi)\|_{V \times V} \leq C(1 + \rho^2)\|\phi - \varphi\|_{V \times V},
\]

which implies that \(N^\rho\) is a Lipschitz function from \(V \times V\) to \(V \times V\). Hence, there exists a unique solution \((u^\rho, v^\rho)\in D([0, \infty); V \times V)\) for equation \((57)\).

For every \(f \in B_b(V \times V)\), define

\[
\mathbb{P}_t^\rho[(f(u_0, v_0))] = \mathbb{E}[f(u^\rho, v^\rho)], \quad t \geq 0, \quad (u_0, v_0) \in V \times V.
\]

Then \((\mathbb{P}_t^\rho)_{t \geq 0}\) is a Markov semigroup.

Define the derivative of \(f \in C_b^k(V \times V)\):

\[
D_hf(x, y) := \lim_{\varepsilon \to 0} \frac{f(x + \varepsilon h_1, y + \varepsilon h_2) - f(x, y)}{\varepsilon}, \quad h = (h_1, h_2).
\]

It follows from Riesz representation theorem that there exists some \(Df(x, y) \in V \times V\) such that

\[
D_hf(x, y) = \langle Df(x, y), h \rangle_{V \times V}, \quad h \in V \times V.
\]
Define
\[ \|D_h f(x,y)\|_\infty = \sup_{(x,y) \in V \times V} \|Df(x,y)\|_{V \times V}. \]

Noticing that \( N^\rho \) is a bounded Lipschitz function. We deduce from Lemma 5.9 of [17] and \( \beta < \frac{3}{2} - \frac{1}{\alpha} \) that following proposition holds.

**Proposition 1.** For \( \alpha \in (\frac{3}{2}, 2) \), there exists some \( \theta \in [\frac{1}{\alpha}, 1) \) such that \[ \|D^\alpha_t f\|_\infty \leq C t^{-\theta} \|f\|_\infty, \quad t > 0, \] where \( f \in B_b(V \times V) \) and \( C > 0 \) depends on \( \rho, \alpha \) and \( \theta \).

**Theorem 4.1.** \((\mathbb{P}_t)_{t \geq 0}\) as a semigroup on \( B_b(V \times V) \) is strong Feller.

**Proof.** Assume \( \|f\|_\infty = 1 \). For \( T_0 > 0 \), it suffices to show that for all \( t \in (0, T_0] \)
\[ \lim_{u_0 \to 0, v_0 \to 0} \mathbb{P}_t(f(u_0', v_0')) = \mathbb{P}_t[f(u_0, v_0)]. \]

Denote
\[ K_{T_0}(\omega) := \max\{ \sup_{0 \leq t \leq T_0} \|Z_{1t}(\omega)\|_V, \sup_{0 \leq t \leq T_0} \|Z_{2t}(\omega)\|_V \}, \quad \forall \omega \in \Omega. \]

Lemma 1.1 and the Markov inequality gives
\[ \mathbb{P}[K_{T_0}(\omega) > \frac{\rho}{4}] \leq \frac{C}{\rho}. \]

Let \( \rho \) be large enough such that \( \|u_0\|_V < \frac{\sqrt{\rho}}{2} \), \( \|v_0\|_V < \frac{\sqrt{\rho}}{2} \) and \( A := \{ K_{T_0} \leq \frac{\rho}{4} \} \). It follows from Lemma 1.1 that there exists some \( 0 < t_0 \leq T_0 \) depending on \( \rho \) such that for all \( \omega \in A \),
\[ \sup_{0 \leq t \leq t_0} (\|u(\omega)\|_V + \|v(\omega)\|_V) \leq 1 + \|u_0\|_V + \|v_0\|_V. \quad (60) \]

Hence, we deduce that
\[ \mathbb{P}( \sup_{0 \leq t \leq t_0} (\|u\|_V + \|v\|_V) \geq \rho) \]
\[ \leq \mathbb{P}( \sup_{0 \leq t \leq t_0} (\|b\|_V + ||c||_V) + \sup_{0 \leq t \leq t_0} \|Z_{1t}\|_V + \sup_{0 \leq t \leq t_0} \|Z_{2t}\|_V \geq \rho) \]
\[ \leq \mathbb{P}(K_{T_0} > \frac{\rho}{4}) + \mathbb{P}( \sup_{0 \leq t \leq t_0} (\|b\|_V + ||c||_V) > \frac{\rho}{2}, A). \]

It follows from \( \omega \in A \) and \( (60) \) that,
\[ \sup_{0 \leq t \leq t_0} (\|b(\omega)\|_V + ||c(\omega)||_V) \leq 1 + \sqrt{\rho} \leq \frac{\rho}{2}. \]

Hence, we have
\[ \mathbb{P}( \sup_{0 \leq t \leq t_0} (\|b\|_V + ||c||_V) \geq \frac{\rho}{2}, A) = 0. \]

and
\[ \mathbb{P}( \sup_{0 \leq t \leq t_0} (\|u\|_V + \|v\|_V) \geq \rho) \leq \mathbb{P}(K_{T_0} > \frac{\rho}{4}) \leq \frac{C}{\rho}. \]

Define the stopping time by
\[ \tau = \inf\{ t > 0, \|u\|_V + \|v\|_V \geq \rho \}. \]
Then for any \( t \in [0, t_0] \),
\[
\mathbb{P} (\tau \leq t) = \mathbb{P} \left( \sup_{0 \leq t \leq t_0} (\|u\|_V + \|v\|_V) \geq \rho \right) \leq \frac{C}{\rho}.
\]
(61)

Since for \( t \in [0, \tau) \), equation \( \text{(1)} \) and equation \( \text{(57)} \) both both have a unique mild solution,
\[
u^0 = u, \quad v^0 = v, \quad \text{a.s.}
\]

Let \( u', v' \in V \) such that \( \|u - u'\|_V \leq 1 \) and \( \|v - v'\|_V \leq 1 \) and \( \rho \) be large enough such that \( \|u\|_V, \|u'\|_V, \|v\|_V, \|v'\|_V \leq \frac{\sqrt{T}}{\varepsilon} \), then for \( t \in (0, t_0) \), we get
\[
|\mathbb{P}_t(f(u_0, v_0)) - \mathbb{P}_t(f(u_0', v_0'))| = |\mathbb{E}[f(u, v)] - \mathbb{E}[f(u', v')]| = I_1 + I_2 + I_3,
\]
where
\[
I_1 = |\mathbb{E}(f(u, v) - f(u', v'))1_{\tau > t}|, \quad I_2 = |\mathbb{E}(f(u, v)1_{\tau \leq t})|, \quad I_3 = |\mathbb{E}(f(u', v'))1_{\tau \leq t})|.
\]

It follows from (61) that
\[
I_2 \leq \frac{C}{\rho}, \quad I_3 \leq \frac{C}{\rho}.
\]

Proposition 1 and 61 give
\[
I_1 = |\mathbb{E}[(f(u, v))1_{\tau > t}] - \mathbb{E}[(f(u', v'))1_{\tau > t}]| \leq Ct^{-\theta}(\|u_0 - u_0'\|_V + \|v_0 - v_0'\|_V) + 2C\frac{\|v_0\|_V}{\rho}.
\]

Thus, we deduce that
\[
|\mathbb{P}_t(f(u_0, v_0)) - \mathbb{P}_t(f(u_0', v_0'))| \leq Ct^{-\theta}(\|u_0 - u_0'\|_V + \|v_0 - v_0'\|_V) + \frac{4C}{\rho}.
\]

For all \( \varepsilon > 0 \), let \( \rho \geq \max \{ \frac{12C}{\varepsilon}, 2\|u_0\|_V^2 + 2, 2\|v_0\|_V^2 + 2 \} \), \( \delta \leq \frac{\varepsilon}{2C} \). Then for \( \|u_0 - u_0'\|_V + \|v_0 - v_0'\|_V < \delta \), we get
\[
|\mathbb{P}_t(f(u_0, v_0)) - \mathbb{P}_t(f(u_0', v_0'))| \leq \varepsilon, \quad t \in (0, t_0].
\]

For \( t_0 < t \leq T_0 \), we derive from Markov property and the strong Feller property that
\[
\mathbb{P}_t(f(u_0, v_0)) - \mathbb{P}_t(f(u_0', v_0')) = \mathbb{P}_{t_0}\left[\mathbb{P}_{t-t_0}(f(u_0, v_0)) - \mathbb{P}_{t_0}\left[\mathbb{P}_{t-t_0}(f(u_0', v_0'))\right]\right] \rightarrow 0,
\]
as \( \|u_0 - u_0'\|_V + \|v_0 - v_0'\|_V \rightarrow 0 \).

\( \square \)

**Theorem 4.2.** \((\mathbb{P}_t)_{t \geq 0}\) as a semigroup on \( B_b(H \times H) \) is strong Feller.

**Proof.** For any \( T_0 > 0 \), it suffices to show that for all \( t \in (0, T_0] \) and \((u_0, b_0) \in H \times H\)
\[
\lim_{\|u_0' - u_0\|_H \rightarrow 0, \|v_0' - v_0\|_H \rightarrow 0} \mathbb{P}_t(f(u_0', v_0')) = \mathbb{P}_t[f(u_0, v_0)].
\]

Let \( \Omega_N := \max \{ \sup_{0 \leq t \leq T_0} \|Z_{1t}\|_V, \sup_{0 \leq t \leq T_0} \|Z_{2t}\|_V \} \leq N \) and it follows from Lemma 1.1 and Chebyshev’s inequality that
\[
\mathbb{P}(\Omega_N^c) \leq \frac{c}{N}.
\]
(62)

Observe that
\[
u - v = I_1 + I_2,
\]
(63)
where

\[ I_1 = e^{-(\gamma_2 + i\gamma_3)A^t}u_0 - e^{-(\gamma_2 + i\gamma_3)A^t}u_0', \quad J_2 = \int_0^t e^{-(\gamma_2 + i\gamma_3)A^t} [N(u) - N(u')] \, ds, \]

and

\[ v - v' = J_1 + J_2, \quad (64) \]

where

\[ J_1 = e^{-(\mu_3 + i\delta)A^t}v_0 - e^{-(\mu_3 + i\delta)A^t}v_0', \quad J_2 = \int_0^t e^{-(\mu_3 + i\delta)A^t} [N(v) - N(v')] \, ds. \]

It follows from (4) that

\[ \|I_1(t)\|_V \leq C t^{-\frac{1}{2}} \|u_0 - u_0'\|_H, \quad \|J_1(t)\|_V \leq C t^{-\frac{1}{2}} \|v_0 - v_0'\|_H. \]

Due to Hölder inequality, we obtain

\[
\begin{align*}
\|I_2(t)\|_V &\leq C \int_0^t (t-s)^{-\frac{1}{2}} (\|u\|_V + \|u'\|_V + 1) \|u - u'\|_V \, ds + C \int_0^t (t-s)^{-\frac{1}{2}} \|v - v'\|_V \, ds, \\
\|J_2(t)\|_V &\leq C \int_0^t (t-s)^{-\frac{1}{2}} (\|u - u'\|_V + \|v - v'\|_V) \, ds.
\end{align*}
\]

Theorem 2.4 implies that

\[ \|u\|_V \leq \|b\|_V + \|Z_{1r}\|_V \leq C, \quad \|v\|_V \leq \|c\|_V + \|Z_{2r}\|_V \leq C. \]

Similarly, \( \|u'\|_V \leq C \) and \( \|v'\|_V \leq C \). Hence,

\[
\begin{align*}
\|I_2(t)\|_V &\leq C \int_0^t (t-s)^{-\frac{1}{2}} (\|v - v'\|_V + \|u - u'\|_V) \, ds, \\
\|J_2(t)\|_V &\leq C \int_0^t (t-s)^{-\frac{1}{2}} (\|v - v'\|_V + \|u - u'\|_V) \, ds.
\end{align*}
\]

For \( r \in (0,t_0] \), define

\[ \Phi_r = \sup_{0 \leq t \leq r} t^{\frac{1}{2}} \|u - u'\|_V, \quad \Psi_r = \sup_{0 \leq t \leq r} t^{\frac{1}{2}} \|v - v'\|_V. \]

Due to (63) and (64), we have

\[ \begin{align*}
\Phi_r &\leq C \|u_0 - u_0'\|_H + C \sup_{0 \leq t \leq r} \left[ t^{\frac{1}{2}} \int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}} \, ds \right] \Phi_r \\
&\quad + C \sup_{0 \leq t \leq r} \left[ t^{\frac{1}{2}} \int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}} \, ds \right] \Psi_r \\
&\leq C \|u_0 - u_0'\|_H + Cr^{\frac{1}{2}} \Phi_r + Cr^{\frac{1}{2}} \Psi_r,
\end{align*} \]

and

\[ \Psi_r \leq C \|v_0 - v_0'\|_H + Cr^{\frac{1}{2}} \Phi_r + Cr^{\frac{1}{2}} \Psi_r. \]

Choosing \( r \) small enough such that \( Cr^{\frac{1}{2}} \leq \frac{1}{4} \), we get

\[ \Phi_r + \Psi_r \leq C (\|u_0 - u_0'\|_H + \|v_0 - v_0'\|_H), \]

which implies that

\[ \|u - u'\|_V + \|v - v'\|_V \leq Ct^{\frac{1}{2}} (\|u_0 - u_0'\|_H + \|v_0 - v_0'\|_H). \quad (65) \]
For all $0 < t \leq T_0$, by the Markov property, we obtain

$$|\mathbb{P}_t(f(u_0, v_0)) - \mathbb{P}_t(f(u'_0, v'_0))| \leq |\mathbb{E}[\mathbb{P}_{t-s} f(u, v) - f(u', v')]| \leq G_1 + G_2,$$

where $s = \frac{t}{2} \land r$ and

$$G_1 = |\mathbb{E}[\mathbb{P}_{t-s} f(u, v) - \mathbb{P}_{t-s} f(u', v')]|_{1\Omega_N}, G_2 = |\mathbb{E}[\mathbb{P}_{t-s} f(u, v) - \mathbb{P}_{t-s} f(u', v')]|_{1\Omega_N}.$$

We derive from (62) that $G_1 \leq 2c \frac{\|u\|}{N}$.

Combining (65), Theorem 4.1 with the dominated convergence theorem, we obtain that

$$G_2 \to 0, \quad \|u_0 - u'_0\|_H + \|v_0 - v'_0\|_H \to 0.$$

Thus, the proof of Theorem 4.2 is completed.

4.2. The accessibility. Because we can’t get the irreducibility, we fail to apply the classical Doob’s Theorem to get the ergodicity. Alternatively, we apply a criterion in [12]. Finally, let’s introduce the conception of accessibility.

Definition 4.3. (Accessibility) Let $(X_t)_{t \geq 0}$ be a stochastic process valued on a metric space $E$ and let $(P_t(x, \cdot))_{x \in E}$ be the transition probability family. $(X_t)_{t \geq 0}$ is said to be accessible to $x_0 \in E$ if the resolvent $\mathcal{R}_\lambda$ satisfies

$$\mathcal{R}_\lambda(x, u) := \int_0^\infty e^{-\lambda t} P_t(x, u) dt > 0,$$

for all $x \in E$ and all neighborhoods $u$ of $x_0$, where $\lambda > 0$ is arbitrary.

The following theorem developed by M.Hairer in [12] is the key tool to show the uniqueness of the invariant measure.

Theorem 4.4. ([12]) If $(X_t)_{t \geq 0}$ is strong Feller at an accessible point $x \in E$, then it can have at most one invariant measure.

Theorem 4.5. Assume that $\alpha \in \left(\frac{1 + 2\beta}{2 - \sigma^2}, 2\right)$ and $\frac{\beta}{2} + \frac{1}{2\lambda_0} \leq \beta \leq \frac{\beta}{2} - \frac{1}{n}$. If $\sigma_2$ is large enough, then the solution $\phi$ of stochastic fractional Coupled Ginzburg-Landau equation (1) has a unique invariant measure.

Proof. Lemma 1.1 implies that for $t > 0$ and $\varepsilon > 0$,

$$\mathbb{P}(\max_{0 \leq s \leq t} (\|Z_{1t}\|_V, \|Z_{2t}\|_V) \leq \varepsilon) \to 0. \quad (66)$$

We can derive from (44) that for $\omega \in \Omega_{\varepsilon, t},$

$$\|b\|^2_H + \|c\|^2_H \leq e^{-Rst}(\|u_0\|^2_H + \|v_0\|^2_H) + \frac{9\pi}{\sigma_2}(\gamma_1 + 1)^2(1 - e^{-Rst}) + c\varepsilon^2. \quad (67)$$

Define

$$B_H(r) := \{u_0, v_0 \in H; \|u_0\| + \|v_0\| \leq r\}, \quad \forall r > 0. \quad (68)$$

Then for all $R > 0$, let $T := T_{R, \delta}$ and $\sigma_2$ be sufficiently large and $\varepsilon := \varepsilon_{R, \delta}$ be sufficiently small. It follows from (69) that for all $\delta > 0$,

$$\|u\|^2_H + \|v\|^2_H \leq e^{-Rst}(\|u_0\|^2_H + \|v_0\|^2_H) + c\varepsilon^2 < \delta, \quad t \geq T, \quad (69)$$

for $x \in B_H(r)$ and $\omega \in \Omega_{\varepsilon, t}$.

We get from (60) that for all $u_0, v_0 \in B_H(r),$

$$\mathbb{P}(t, x, B_H(\delta)) > 0, \quad t \geq T, \quad (70)$$

which implies that

$$\mathcal{R}_\lambda(x, B_H(\delta)) > 0. \quad (71)$$
Since $R > 0$ is arbitrary, inequality (71) is true for all $u_0, v_0 \in H$. Therefore $(u, v)_{t \geq 0}$ is accessible to 0. Theorem 12 guarantees that stochastic equation (1) has a unique invariant measure. Thus, the proof of Theorem 4.5 is completed.

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