ON THE STATIONARY NONLOCAL CAHN-HILLIARD-NAVIER-STOKES SYSTEM: EXISTENCE, UNIQUENESS AND EXPONENTIAL STABILITY

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Abstract. Cahn-Hilliard-Navier-Stokes system describes the evolution of two isothermal, incompressible, immiscible fluids in a bounded domain. In this work, we consider the stationary nonlocal Cahn-Hilliard-Navier-Stokes system in two and three dimensions with singular potential. We prove the existence of a weak solution for the system using pseudo-monotonicity arguments and Browder’s theorem. Further we establish the uniqueness and regularity results for the weak solution of the stationary nonlocal Cahn-Hilliard-Navier-Stokes system for constant mobility parameter and viscosity. Finally, in two dimensions, we establish that the stationary solution is exponentially stable under suitable conditions on mobility parameter and viscosity.

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1. Introduction

We consider a mathematical model of two isothermal, incompressible, immiscible fluids evolving in two or three dimensional bounded domains. This system of equations is well known as Cahn-Hilliard-Navier-Stokes (CHNS) system or is also known as H-Model. Cahn-Hilliard-Navier-Stokes model describes the chemical interactions between the two phases at the interface, which is achieved using a Cahn–Hilliard approach, and also the hydrodynamic properties of the mixture which is obtained using Navier-Stokes equations with surface tension terms acting at the interface (see [23]). If the two fluids have the same constant density, then the temperature differences are negligible.
and the diffusive interface between the two phases has a small but non-zero thickness, and thus we have the well-known “H-Model” (see [27]). The equations for evolution of the Cahn-Hilliard-Navier-Stokes/H-model are given by

\[
\begin{align*}
\varphi_t + \mathbf{u} \cdot \nabla \varphi &= \text{div}(m(\varphi)\nabla \mu), \quad \text{in} \quad \Omega \times (0,T), \\
\mu &= a\varphi - J \ast \varphi + F(\varphi), \\
\rho \mathbf{u}_t - 2\text{div}((\varphi)D\mathbf{u}) + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla \pi &= \mu \nabla \varphi + \mathbf{h}, \quad \text{in} \quad \Omega \times (0,T), \\
\text{div} \mathbf{u} &= 0, \quad \text{in} \quad \Omega \times (0,T), \\
\frac{\partial \mu}{\partial n} &= 0, \quad \mathbf{u} = 0 \quad \text{on} \quad \partial \Omega \times (0,T), \\
\mathbf{u}(0) &= \mathbf{u}_0, \quad \varphi(0) = \varphi_0 \quad \text{in} \quad \Omega,
\end{align*}
\]

in \( \Omega \times (0,T) \), where \( \Omega \subset \mathbb{R}^n \), \( n = 2,3 \) and \( \mathbf{u}(x,t) \) and \( \varphi(x,t) \) denote the average velocity of the fluid and the relative concentration respectively. These equations are of the nonlocal type because of the presence of the term \( J \), which is the spatial-dependent internal kernel and \( J \ast \varphi \) denotes the spatial convolution over \( \Omega \). The mobility parameter is denoted by \( m \), \( \mu \) is the chemical potential, \( \pi \) is the pressure, \( a \) is defined by \( a(x) := \int_{\Omega} J(x - y)dy \), \( F \) is the configuration potential which accounts for the presence of two phases, \( \nu \) is the kinematic viscosity and \( \mathbf{h} \) is the external forcing term acting in the mixture. The strain tensor \( D\mathbf{u} \) is the symmetric part of the gradient of the flow velocity vector, i.e., \( D\mathbf{u} = \frac{1}{2} (\nabla \mathbf{u} + (\nabla \mathbf{u})^\top) \). The chemical potential \( \mu \) is the first variation of the free energy functional:

\[
\mathcal{E}(\varphi) := \frac{1}{4} \int_{\Omega} \int_{\Omega} J(x - y)(\varphi(x) - \varphi(y))^2dxdy + \int_{\Omega} F(\varphi(x))dx.
\]

Various simplified models of this system are studied by several mathematicians and physicists. The local version of the system (see [10, 25]) is obtained by replacing \( \mu \) equation by \( \mu = -\Delta \varphi + F'(\varphi) \) which is first variation of the free energy functional

\[
\mathcal{E}(\varphi) := \int_{\Omega} \left( \frac{1}{2} |\nabla \varphi|^2 + F(\varphi(x)) \right) dx.
\]

Another simplification appeared in the literature is to assume the constant mobility parameter and/or constant viscosity. The solvability, uniqueness and regularity of the system (1.1) and of other simplified Cahn-Hilliard-Navier-Stokes models is well studied in literature though most of the works are recent ones. Typically two types of potentials are considered in literature, regular potential as well as singular potential. In general, singular potentials are difficult to handle and in such cases \( F(\cdot) \) is usually approximated by polynomials in order to make the mathematical analysis easier (see [3]).

The nonlocal Cahn-Hilliard-Navier-Stokes system with regular potential has been analyzed by M. Grasselli et al. in [12, 17, 19]. Using results for regular potential, they have also studied in [18], the existence of weak solution of the system with singular potential. Furthermore, they proved the existence of the Global attractor in 2D and trajectory attractor in 3D. Strong solution for the nonlocal Cahn-Hilliard-Navier-Stokes system was discussed in [24]. Uniqueness results for the same were established in [17]. In [25] authors considered the nonlocal Cahn-Hilliard equation with singular potential and constant mobility and studied well posedness and regularity results. Moreover, they established the strict separation property in dimension 2. Regularity results in case of degenerate mobility were studied in [23]. The local Cahn-Hilliard-Navier-Stokes system with singular free energies has been studied in [1, 10]. Further, along the application side, the optimal control of nonlocal Cahn-Hillard-Navier-Stokes equations and robust control for local Cahn-Hillard-Navier-Stokes equations have been addressed in [20, 22, 30, 31, 7, 8], etc.

Solvability results for the stationary nonlocal Cahn-Hilliard equation with singular potential were discussed in [16] whereas authors in [3] proved the convergence to the equilibrium solution of Cahn
Hilliard system with logarithmic free energy. The existence of the equilibrium solution for steady state Navier-Stokes equation is well known in literature and can be found in the book [38]. In [2], the authors discussed the existence of a weak solution to the stationary local Cahn-Hilliard-Navier-Stokes equations. The author in [29] studied a coupled Cahn-Hilliard-Navier-Stokes model with delays in a two-dimensional bounded domains and discussed the asymptotic behavior of the weak solutions and the stability of the stationary solutions. In this work our main aim is to study the well-posedness of nonlocal steady state system corresponding to the model described in (1.1) in dimensions 2 and 3 and to examine the stability properties of this solution in dimension 2.

Throughout this paper, we consider $F$ to be a singular potential. A typical example is the logarithmic potential:

$$F(\varphi) = \frac{\theta}{2}(\ln(1 + \varphi) + (1 - \varphi) \ln(1 - \varphi)) - \frac{\theta_c}{2} \varphi^2, \quad \varphi \in (-1, 1),$$

where $\theta, \theta_c > 0$. The logarithmic terms are related to the entropy of the system and note that $F$ is convex if and only if $\theta \geq \theta_c$. If $\theta \geq \theta_c$ the mixed phase is stable and if $0 < \theta < \theta_c$, the mixed phase is unstable and phase separation occurs. To the best of our knowledge, the solvability results for stationary nonlocal Cahn-Hilliard-Navier-Stokes equations is not available in the literature. In the current work, using techniques similar to the one developed in [2], we resolve this issue. We prove the existence of a weak solution to the stationary nonlocal Cahn-Hilliard-Navier-Stokes system in dimensions 2 and 3 with variable mobility and viscosity. Further, we answer the questions of well posedness and regularity of the solution for the equations with constant viscosity and mobility parameters. In dimensions 2 and 3, we show that the weak solution possesses higher regularity. The uniqueness of weak solutions is established under certain conditions on the viscosity and mobility parameters. Lastly, for constant viscosity and mobility parameters, we establish that the strong solution of steady state equations in dimension 2, stabilizes exponentially if the boundary data of the chemical potential corresponding to system (1.1) is stationary.

Rest of the paper is organized as follows: In the next section, we explain functional setting for the solvability of stationary nonlocal Cahn-Hilliard-Navier-Stokes equations (2.1) (given below). We define the weak formulation of our system in section 3. The existence of a weak solution to the nonlocal Cahn-Hilliard-Navier-Stokes equations (2.1) is proved using pseudo-monotonicity arguments and Browder’s theorem in this section (see Theorem 3.19). In further study we assume the mobility parameter and viscosity to be constant. The section 4 is devoted to study the uniqueness of a weak solution for the system (2.1). We establish the uniqueness of weak solutions under certain assumptions on mobility parameter and viscosity (see Theorem 4.1). In section 5 we study the regularity properties of the solutions obtained in section 3 (see Theorem 5.1). All the results are established in dimensions 2 as well as 3. Finally, in section 6 we establish that the stationary solution in two dimensions is exponentially stable (see Theorem 6.6) under certain restrictions on mobility parameter and viscosity.

## 2. Stationary Nonlocal Cahn-Hilliard-Navier-Stokes System

In this section, we consider the stationary nonlocal Cahn-Hilliard-Navier-Stokes system in two and three dimensional bounded domains. Here, we consider the case of the coefficient of kinematic viscosity and mobility parameter depending on $\varphi$. Let us consider the following steady state system
associated with the equation (1.1):
\[
\begin{aligned}
\mathbf{u} \cdot \nabla \varphi &= \text{div}(m(\varphi) \nabla \mu), \quad \text{in } \Omega, \\
\mu &= a \varphi - J * \varphi + F'(\varphi), \quad \text{in } \Omega, \\
-2\text{div} (\nu(\varphi) \mathbf{D} \mathbf{u}) + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla \pi = \mu \nabla \varphi + \mathbf{h}, \quad \text{in } \Omega, \\
\text{div } \mathbf{u} &= 0, \quad \text{in } \Omega, \\
\frac{\partial \mu}{\partial n} &= 0, \quad \mathbf{u} = 0 \quad \text{on } \partial \Omega,
\end{aligned}
\tag{2.1}
\]

with average of $\varphi$ given by
\[
\frac{1}{|\Omega|} \int_\Omega \varphi(x) dx = k \in (-1, 1),
\]
where $|\Omega|$ is the Lebesgue measure of $\Omega$. Our main aim in this work is to study the existence, uniqueness, regularity and stability of the system (2.1). For solvability, we formulate the problem in an abstract setup and use the well known Browder’s theorem to establish the existence of a weak solution to the system (2.1). We further study regularity, uniqueness and stability of the system (2.1) with constant viscosity and mobility parameters by establishing a-priori estimates and under certain conditions on viscosity and mobility.

### 2.1. Functional setting.

We first explain the functional spaces needed to obtain our main results. Let us define

\[
\begin{align*}
\mathcal{G}_{\text{div}} &:= \left\{ \mathbf{u} \in L^2(\Omega; \mathbb{R}^n) : \text{div } \mathbf{u} = 0, \text{ } \mathbf{u} \cdot \mathbf{n} |_{\partial \Omega} = 0 \right\}, \\
\mathcal{V}_{\text{div}} &:= \left\{ \mathbf{u} \in H^1_0(\Omega; \mathbb{R}^n) : \text{div } \mathbf{u} = 0 \right\}, \\
H &:= L^2(\Omega; \mathbb{R}), \quad V := H^1(\Omega; \mathbb{R}),
\end{align*}
\]

where $n = 2, 3$. Let us denote $\| \cdot \|$ and $(\cdot, \cdot)$, the norm and the scalar product, respectively, on both $H$ and $\mathcal{G}_{\text{div}}$. The duality between any Hilbert space $X$ and its dual $X'$ is denoted by $X' \langle \cdot, \cdot \rangle_X$. We know that $\mathcal{V}_{\text{div}}$ is endowed with the scalar product

\[
(\mathbf{u}, \mathbf{v})_{\mathcal{V}_{\text{div}}} = (\nabla \mathbf{u}, \nabla \mathbf{v}) = 2(\mathbf{Du}, \mathbf{Dv}), \quad \text{for all } \mathbf{u}, \mathbf{v} \in \mathcal{V}_{\text{div}}.
\]

The norm on $\mathcal{V}_{\text{div}}$ is given by $\|\mathbf{u}\|_{\mathcal{V}_{\text{div}}}^2 := \int_\Omega |\nabla \mathbf{u}(x)|^2 dx = \|\nabla \mathbf{u}\|^2$. In the sequel, we use the notations $H^m(\Omega) := H^m(\Omega; \mathbb{R}^n) = W^{m,2}(\Omega; \mathbb{R}^n)$ and $H^m(\Omega) := H^m(\Omega; \mathbb{R}) = W^{m,2}(\Omega; \mathbb{R})$ for Sobolev spaces of order $m$.

Let us also define
\[
\begin{align*}
L^2_{(k)}(\Omega) &:= \left\{ f \in L^2(\Omega; \mathbb{R}) : \frac{1}{|\Omega|} \int_\Omega f(x) dx = k \right\}, \\
H^1_{(0)}(\Omega) &:= H^1(\Omega; \mathbb{R}) \cap L^2_{(0)}(\Omega) = \left\{ f \in H^1(\Omega; \mathbb{R}) : \frac{1}{|\Omega|} \int_\Omega f(x) dx = 0 \right\}, \\
H^{-1}_{(0)}(\Omega) &:= H^1_{(0)}(\Omega)' .
\end{align*}
\]

Note that $L^2_{(0)}(\Omega)$ is a Hilbert space equipped with the usual inner product in $L^2(\Omega)$. Since $\Omega$ is a bounded smooth domain and the average of $f$ is zero in $H^1_{(0)}(\Omega)$, using the Poincaré-Writtenger inequality (see Lemma 2.2, below), we have $\|f\| \leq C_\Omega \| \nabla f\|$, for all $f \in H^1_{(0)}(\Omega)$. Using this fact, one can also show that $H^1_{(0)}(\Omega)$ is a Hilbert space equipped with the inner product
\[
(\varphi, \psi)_{H^1_{(0)}} = (\nabla \varphi, \nabla \psi), \quad \text{for all } \varphi, \psi \in H^1_{(0)}(\Omega).
\]

We can prove the following dense and continuous embedding:
\[
H^1_{(0)}(\Omega) \hookrightarrow L^2_{(0)}(\Omega) \equiv L^2_{(0)}(\Omega)' \hookrightarrow H^{-1}_{(0)}(\Omega).
\tag{2.2}
\]
Note that the embedding is compact (see for example, Theorem 1, Chapter 5, [13]). The projection $P_0 : L^2(\Omega) \to L^2_{(0)}(\Omega)$ onto $L^2$-space with mean value zero is defined by

$$P_0f := f - \frac{1}{|\Omega|} \int_\Omega f(x)dx$$

(2.3)

for every $f \in L^2(\Omega)$.

For every $f \in \mathcal{V}'$, we denote $\overline{f}$ the average of $f$ over $\Omega$, i.e., $\overline{f} := |\Omega|^{-1} \mathcal{V}(f, 1)_\mathcal{V} = |\Omega|^{-1} \int_\Omega f(x)dx$.

Let us also introduce the spaces (see [17] for more details)

$$\mathcal{V}_0 = H^1_{(0)}(\Omega) = \{v \in \mathcal{V} : \mathbf{r} = 0\},$$

$$\mathcal{V}'_0 = H^{-1}_{(0)}(\Omega) = \{f \in \mathcal{V}' : \overline{f} = 0\},$$

and the operator $A : \mathcal{V} \to \mathcal{V}'$ is defined by

$$\mathcal{V}'(Au, v) := \int_\Omega \nabla u(x) \cdot \nabla v(x)dx, \text{ for all } u, v \in \mathcal{V}.$$  

Clearly $A$ is linear and it maps $\mathcal{V}$ into $\mathcal{V}'_0$ and its restriction $\mathcal{B}$ to $\mathcal{V}_0$ onto $\mathcal{V}'_0$ is an isomorphism. We know that for every $f \in \mathcal{V}'_0$, $\mathcal{B}^{-1} f$ is the unique solution with zero mean value of the *Neumann problem*:

$$\left\{ \begin{array}{lcl}
-\Delta u &=& f, \quad \text{in } \Omega, \\
\frac{\partial u}{\partial n} &=& 0, \quad \text{on } \partial \Omega.
\end{array} \right.$$  

In addition, we have

$$\mathcal{V}'(Au, B^{-1} f)_\mathcal{V} = \mathcal{V}'(f, u)_\mathcal{V}, \quad \text{for all } u \in \mathcal{V}, \ f \in \mathcal{V}'_0,$$

$$\mathcal{V}'(f, B^{-1} g)_\mathcal{V} = \mathcal{V}'(g, B^{-1} f)_\mathcal{V} = \int_\Omega \nabla (B^{-1} f) \cdot \nabla (B^{-1} g)dx, \quad \text{for all } f, g \in \mathcal{V}'_0.$$  

Note that $\mathcal{B}$ can be also viewed as an unbounded linear operator on $\mathcal{H}$ with domain $D(\mathcal{B}) = \{v \in H^2(\Omega) : \frac{\partial v}{\partial n} = 0 \text{ on } \partial \Omega\}$.

### 2.2. Linear and nonlinear operators.

Let us define the Stokes operator $A : D(A) \cap \mathcal{G}_{\mathbf{div}} \to \mathcal{G}_{\mathbf{div}}$.

In the case of no slip boundary condition

$$A = -P\Delta, \quad D(A) = H^2(\Omega) \cap \mathcal{V}_{\mathbf{div}},$$

where $P : L^2(\Omega) \to \mathcal{G}_{\mathbf{div}}$ is the *Helmholtz-Hodge orthogonal projection*. Note also that, we have

$$\mathcal{V}'_{\mathbf{div}}(Au, v)_{\mathbf{div}} = (u, v)_{\mathbf{div}} = (\nabla u, \nabla v), \quad \text{for all } u, v \in \mathcal{V}_{\mathbf{div}}.$$  

It should also be noted that $A^{-1} : \mathcal{G}_{\mathbf{div}} \to \mathcal{G}_{\mathbf{div}}$ is a self-adjoint compact operator on $\mathcal{G}_{\mathbf{div}}$ and by the classical *spectral theorem*, there exists a sequence $\lambda_j$ with $0 < \lambda_1 \leq \lambda_2 \leq \lambda_j \leq \cdots \to +\infty$ and a family of $e_j \in D(A)$ which is orthonormal in $\mathcal{G}_{\mathbf{div}}$ and such that $Ae_j = \lambda_j e_j$. We know that $u \in D(A)$ can be expressed as $u = \sum_{j=1}^{\infty} (u, e_j) e_j$, so that $Au = \sum_{j=1}^{\infty} \lambda_j (u, e_j) e_j$. Thus, it is immediate that

$$\|\nabla u\|^2 = (Au, u) = \sum_{j=1}^{\infty} \lambda_j |(u, e_j)|^2 \geq \lambda_1 \sum_{j=1}^{\infty} |(u, e_j)|^2 = \lambda_1 \|u\|^2.$$  

(2.6)

For $u, v, w \in \mathcal{V}_{\mathbf{div}}$, we define the trilinear operator $b(\cdot, \cdot, \cdot)$ as

$$b(u, v, w) = \int_\Omega (u(x) \cdot \nabla)v(x) \cdot w(x)dx = \sum_{i,j=1}^{n} \int_\Omega u_i(x) \frac{\partial v_j(x)}{\partial x_i} w_j(x)dx,$$

and the bilinear operator $\mathcal{B}$ from $\mathcal{V}_{\mathbf{div}} \times \mathcal{V}_{\mathbf{div}}$ into $\mathcal{V}'_{\mathbf{div}}$ is defined by

$$\mathcal{V}'_{\mathbf{div}}(\mathcal{B}(u, v), w)_{\mathbf{div}} := b(u, v, w), \quad \text{for all } u, v, w \in \mathcal{V}_{\mathbf{div}}.$$
An integration by parts yields,
\[
\left\{
\begin{array}{ll}
b(u, v, v) = 0, & \text{for all } u, v \in \mathbb{V}_{\text{div}}, \\
b(u, v, w) = -b(u, w, v), & \text{for all } u, v, w \in \mathbb{V}_{\text{div}}.
\end{array}
\right.
\]  \tag{2.7}

For more details about the linear and nonlinear operators, we refer the readers to [37].

**Lemma 2.1** (Gagliardo-Nirenberg inequality, Theorem 2.1, [15]). Let \( \Omega \subset \mathbb{R}^n \) and \( u \in W_{0}^{1,p}(\Omega; \mathbb{R}^n) \), \( p \geq 1 \). Then for any fixed number \( p, q, r \geq 1 \), there exists a constant \( C > 0 \) depending only on \( n, p, q \) such that
\[
\|u\|_{L^r} \leq C\|\nabla u\|_{L^p}^{\theta}\|u\|_{L^q}^{1-\theta}, \quad \theta \in [0, 1],
\]  \tag{2.8}

where the numbers \( p, q, r, n \) and \( \theta \) satisfy the relation
\[
\theta = \left( \frac{1}{q} - \frac{1}{r} \right) \left( \frac{1}{n} - \frac{1}{p} + \frac{1}{q} \right)^{-1}.
\]

A particular case of Lemma 2.1 is the well known inequality due to Ladyzhenskaya (see Lemma 1 and 2, Chapter 1, [28]), which is given below:

**Lemma 2.2** (Ladyzhenskaya inequality). For \( u \in C_{0}^{\infty}(\Omega; \mathbb{R}^n) \), \( n = 2, 3 \), there exists a constant \( C \) such that
\[
\|u\|_{L^1} \leq C^{1/4}\|u\|^{\frac{r}{2}}\|\nabla u\|_{L^2}^{\frac{1}{r}}, \quad \text{for } n = 2, 3,
\]  \tag{2.9}

where \( C = 2, 4 \), for \( n = 2, 3 \) respectively.

Note that the above inequality is true even in unbounded domains. For \( n = 3 \), \( r = 6 \), \( p = q = 2 \), from [28], we find \( \theta = 1 \) and
\[
\|u\|_{L^6} \leq C\|\nabla u\| = C\|u\|_{\mathbb{V}_{\text{div}}}.
\]

For \( n = 2 \), the following estimate holds:
\[
|b(u, v, w)| \leq \sqrt{2}\|u\|^{1/2}\|\nabla u\|^{1/2}\|v\|^{1/2}\|\nabla v\|^{1/2}\|\nabla w\|,
\]
for every \( u, v, w \in \mathbb{V}_{\text{div}} \). Hence, for all \( u \in \mathbb{V}_{\text{div}} \), we have
\[
\|B(u, u)\|_{\mathbb{V}_{\text{div}}} \leq \sqrt{2}\|u\|\|\nabla u\| \leq \sqrt{2}\|u\|_{\mathbb{V}_{\text{div}}}^{1/2},
\]  \tag{2.10}

by using the Poincaré inequality. Similarly, for \( n = 3 \), we have
\[
|b(u, v, w)| \leq 2\|u\|^{1/4}\|\nabla u\|^{3/4}\|v\|^{1/4}\|\nabla v\|^{3/4}\|\nabla w\|,
\]
for every \( u, v, w \in \mathbb{V}_{\text{div}} \). Hence, for all \( u \in \mathbb{V}_{\text{div}} \), using the Poincaré inequality, we have
\[
\|B(u, u)\|_{\mathbb{V}_{\text{div}}} \leq 2\|u\|^{1/2}\|\nabla u\|^{3/2} \leq \frac{2}{\lambda_1^{1/4}}\|u\|_{\mathbb{V}_{\text{div}}}^{3/2},
\]  \tag{2.11}

We also need the following general version of the Gagliardo-Nirenberg interpolation inequality and Agmon’s inequality for higher order estimates. For functions \( u : \Omega \to \mathbb{R}^n \) defined on a bounded Lipschitz domain \( \Omega \subset \mathbb{R}^n \), the Gagliardo-Nirenberg interpolation inequality is given by:

**Lemma 2.3** (Gagliardo-Nirenberg interpolation inequality, Theorem 1, [33]). Let \( \Omega \subset \mathbb{R}^n \), \( u \in W^{m,p}(\Omega; \mathbb{R}^n) \), \( p \geq 1 \) and fix \( 1 \leq p, q \leq \infty \) and a natural number \( m \). Suppose also that a real number \( \theta \) and a natural number \( j \) are such that
\[
\theta = \left( \frac{j}{n} + \frac{1}{q} - \frac{1}{r} \right) \left( \frac{m}{n} - \frac{1}{p} + \frac{1}{q} \right)^{-1}
\]  \tag{2.12}
and $\frac{1}{m} \leq \theta \leq 1$. Then for any $u \in W^{m,p}(\Omega; \mathbb{R}^n)$, we have
\[
\|\nabla^j u\|_{L^r} \leq C\left(\|\nabla^m u\|_{L^p}^\theta \|u\|_{L^q}^{1-\theta} + \|u\|_{L^s}\right),
\]  
(2.13)
where $s > 0$ is arbitrary and the constant $C$ depends upon the domain $\Omega$, $m$, $n$.

If $1 < p < \infty$ and $m - j - \frac{p}{q}$ is a non-negative integer, then it is necessary to assume also that $\theta \neq 1$. Note that for $u \in W^{0,p}(\Omega; \mathbb{R}^n)$, Lemma [2.1] is a special case of the above inequality, since for $j = 0$, $m = 1$ and $\frac{1}{s} = \frac{1}{p} + \frac{1-\theta}{q}$ in (2.13), and application of the Poincaré inequality yields (2.8).

It should also be noted that (2.13) can also be written as
\[
\|\nabla^j u\|_{L^r} \leq C\|u\|_{W^{m,p}}^\theta \|u\|_{L^q}^{1-\theta},
\]  
(2.14)
By taking $j = 1$, $r = 4$, $n = m = p = q = s = 2$ in (2.12), we get $\theta = \frac{3}{4}$, and
\[
\|\nabla u\|_{L^4} \leq C\left(\|\Delta u\|^{3/4}\|u\|^{1/4} + \|u\|\right).
\]  
(2.15)
Using Young’s inequality, we obtain
\[
\|\nabla u\|_{L^4}^2 \leq C(\|\Delta u\|^{3/2}\|u\|^{1/2} + \|u\|^2) \leq C(\|\Delta u\|^2 + \|u\|^2) \leq C\|u\|_{L^2}^2.
\]  
(2.16)
Also, taking $j = 1$, $r = 4$, $n = 3$, $m = p = q = s = 2$ in (2.12), we get $\theta = \frac{7}{8}$, and
\[
\|\nabla u\|_{L^4} \leq C\left(\|\Delta u\|^{7/8}\|u\|^{1/8} + \|u\|\right).
\]  
(2.17)
Once again using Young’s inequality, we get
\[
\|\nabla u\|_{L^4}^2 \leq C(\|\Delta u\|^{7/4}\|u\|^{1/4} + \|u\|^2) \leq C(\|\Delta u\|^2 + \|u\|^2) \leq C\|u\|_{L^2}^2.
\]  
(2.18)

Lemma 2.4 (Agmon’s inequality, Lemma 13.2, [4]). For any $u \in H^{s_2}(\Omega; \mathbb{R}^n)$, choose $s_1$ and $s_2$ such that $s_1 < \frac{n}{2} < s_2$. Then, if $0 < \alpha < 1$ and $\frac{1}{q} = \alpha s_1 + (1-\alpha)s_2$, the following inequality holds
\[
\|u\|_{L^q} \leq C\|u\|_{H^{s_1}}^\alpha \|u\|_{H^{s_2}}^{1-\alpha}.
\]

For $u \in H^2(\Omega) \cap H^1_0(\Omega)$, Agmon’s inequality in two and three dimensions states that there exists a constant $C > 0$ such that
\[
\|u\|_{L^\infty} \leq C\|u\|_{H^2}^{1/2} \|u\|_{H^2}^{1/2} \leq C\|u\|_{H^2},
\]  
(2.19)
\[
\|u\|_{L^\infty} \leq C\|u\|_{H^4}^{1/4} \|u\|_{H^2}^{3/4} \leq C\|u\|_{H^2},
\]  
(2.20)
n $n = 2, 3$ respectively. The inequality (2.19) can also be obtained from (2.14), by taking $j = 0$, $r = \infty$, $m = p = q = 2$, so that we have $\theta = \frac{1}{2}$, and (2.20) can also be obtained from (2.13), by taking $j = 0$, $r = \infty$, $m = p = q = 2$, so that we have $\theta = \frac{3}{4}$.

Lemma 2.5 (Poincaré-Wirtinger inequality, Corollary 12.28, [14]). Assume that $1 \leq p < \infty$ and that $\Omega$ is a bounded connected open subset of the $n$-dimensional Euclidean space $\mathbb{R}^n$ whose boundary is of class C. Then there exists a constant $C_{\Omega,p} > 0$, such that for every function $\phi \in W^{1,p}(\Omega)$,
\[
\|\phi - \overline{\phi}\|_{L^p(\Omega)} \leq C_{\Omega,p}\|\nabla \phi\|_{L^p(\Omega)},
\]
where $\overline{\phi} = \frac{1}{|\Omega|}\int_{\Omega} \phi(y)\,dy$ is the average value of $\phi$ over $\Omega$. 

STATIONARY NONLOCAL CAHN-HILLIARD-NAVIER-STOKES SYSTEM
2.3. Basic assumptions. Let us now make the following assumptions on $J$ and $F$ in order to establish the solvability results of the system (2.1). We suppose that the potential $F$ can be written in the following form

$$F = F_1 + F_2$$

where $F_1 \in C^{(2+2q)}(a,b)$ with $q \in \mathbb{N}$ fixed, and $F_2 \in C^2([a,b])$

**Assumption 2.6.** Let $J$ and $F$ satisfy:

1. $J \in W^{1,1}(\mathbb{R}^2, \mathbb{R})$, $J(x) = J(-x)$ and $a(x) = \int_{\Omega} J(x - y)dy \geq 0$, a.e., in $\Omega$.
2. The function $\nu$ is locally Lipschitz on $\mathbb{R}$ and there exists $\nu_1, \nu_2 > 0$ such that
   $$\nu_1 \leq \nu(s) \leq \nu_2, \text{ for all } s \in \mathbb{R}.$$  
3. There exist $C_1 > 0$ and $\epsilon_0 > 0$ such that
   $$F_1^{(2+2q)}(s) \geq C_1, \text{ for all } s \in (a, a + \epsilon_0] \cup [b - \epsilon_0, b).$$
4. There exists $\epsilon_0 > 0$ such that, for each $k = 0, 1, \ldots, 2 + 2q$ and each $j = 0, 1, \ldots, q,$
   $$F_1^{(k)}(s) > 0 \text{ for all } s \in [1 - \epsilon_0, 1)$$
   $$F_1^{(2j+2)}(s) \geq 0, \quad F_1^{(2j+1)}(s) \leq 0, \text{ for all } s \in (a, a + \epsilon_0].$$
5. There exists $\epsilon_0 > 0$ such that $F_1^{(2+2q)}$ is non-decreasing in $[b - \epsilon_0, b)$ and non-increasing in $(a, a + \epsilon_0]$.
6. There exists $\alpha, \beta \in \mathbb{R}$ with $\alpha + \beta > -\min_{[a,b]} F''_1$ such that
   $$F''_1(s) \geq \alpha, \text{ for all } s \in (a,b), \quad a(x) \geq \beta, \text{ a.e. } x \in \Omega.$$

**Assumption 2.7.** We further assume that there exists $C_0 > 0$ such that $\frac{C_0}{2} \geq \|J\|_{L^1}$ and $F''(s) + a(x) \geq C_0$, for all $s \in (a,b)$, a.e. $x \in \Omega$.

**Assumption 2.8.** Let $\Omega \subset \mathbb{R}^d, d = 2, 3$ be a bounded domain with $C^3$-boundary, $\nu : [a,b] \to (0,\infty)$ be continuously differentiable, $m : [a,b] \to (0,\infty)$ be continuous, and $F \in C([a,b]) \cap C^2((a,b))$ such that

$$\lim_{s \to a} F'(s) = -\infty, \quad \lim_{s \to b} F'(s) = \infty.$$  

**Remark 2.9.** We can represent the potential $F$ as a quadratic perturbation of a convex function. That is

$$F(s) = G(s) - \frac{\kappa}{2} s^2, \quad (2.21)$$

where $G$ is strictly convex and $\kappa > 0$.

**Remark 2.10.** Assumption $J \in W^{1,1}(\mathbb{R}^2; \mathbb{R})$ can be weakened. Indeed, it can be replaced by $J \in W^{1,1}(B_\delta; \mathbb{R})$, where $B_\delta := \{z \in \mathbb{R}^2 : |z| < \delta\}$ with $\delta := \text{diam}(\Omega) = \sup_{x,y \in \Omega} d(x,y)$, where $d(\cdot,\cdot)$ is the Euclidean metric on $\mathbb{R}^2$, or also by

$$\sup_{x \in \Omega} \int_{\Omega} (|J(x - y)| + |\nabla J(x - y)|)dy < +\infty. \quad (2.22)$$

**Remark 2.11.** Assumptions 2.6 ((3)-(6)) are satisfied in the case of the physically relevant logarithmic double-well potential for any fixed positive integer $q$. In particular setting

$$F_1(s) = \frac{\theta}{2} ((1 + s) \ln(1 + s) + (1 - s) \ln(1 - s)) \quad F_2(s) = -\frac{\theta c s^2}{2}$$

then Assumption 2.6 (6) satisfied iff $\beta > \theta_c - \theta$. 

3. Existence of Weak Solution

In this section, we establish the existence of a weak solution to the system (2.1) using pseudo-monotonicity arguments and Browder’s theorem. Let us first give the definition of weak solution of the system (2.1).

**Definition 3.1.** Let \( h \in \mathcal{V}'_{\text{div}} \). A triple \((u, \mu, \varphi) \in \mathcal{V}'_{\text{div}} \times H^1(\Omega) \times \left( H^1(\Omega) \cap L^2_k(\Omega) \right)\) is called a weak solution of the problem (2.1) if

\[
\int_{\Omega} (u \cdot \nabla \varphi) \psi \, dx = - \int_{\Omega} m(\varphi) \nabla \mu \cdot \nabla \psi \, dx, \quad (3.1)
\]

\[
\int_{\Omega} \mu \psi \, dx = \int_{\Omega} (a \varphi - J \ast \varphi) \psi \, dx + \int_{\Omega} P_0(F'(^\top \varphi)) \psi \, dx, \quad (3.2)
\]

\[
\int_{\Omega} (u \cdot \nabla)u \cdot v \, dx + \int_{\Omega} 2\nu(\varphi)Du \cdot Dv \, dx = \int_{\Omega} \mu \nabla \varphi \cdot v \, dx + \int_{\Omega} h \cdot v \, dx, \quad (3.3)
\]

for every \( \psi \in \mathcal{V} \) and \( v \in \mathcal{V}'_{\text{div}} \).

Our aim is to establish the existence of a weak solution of (2.1) in the sense of Definition 3.1. But, when working with the above definition, some difficulties arise in the analysis of our problem. The most important one is that \( L^2_k(\Omega) \) is not a vector space for \( k \neq 0 \). But, we can assume that \( k = 0 \) with out loss of generality. Otherwise replace \( \varphi \) by \( \tilde{\varphi} := \varphi - k \) and \( F \) by \( F_k \) with \( F_k(x) := F(x+k) \) for all \( x \in \mathbb{R} \). Note that this implies \( 0 \in (a, b) \). Thus, in order to establish a weak solution of the system (2.1), we first reformulate the problem (3.1)-(3.3). We prove the existence of a solution to the reformulated problem (3.4)-(3.6) (see below) instead of (3.1)-(3.3). We establish the equivalence of these two problems in Lemma 3.2.

We reduce \( \mu \) to \( \mu_0 \) which has mean value 0 which would help in proving coercivity of an operator in the later part of this section. Let us fix \( \mu_0 = \mu - \frac{1}{|H|} \int_{\Omega} F'(^\top \varphi) \, dx \). Then the reformulated problem of (3.1)-(3.3) is given by

\[
\int_{\Omega} (u \cdot \nabla \varphi) \psi \, dx = - \int_{\Omega} m(\varphi) \nabla \mu_0 \cdot \nabla \psi \, dx, \quad (3.4)
\]

\[
\int_{\Omega} \mu_0 \psi \, dx = \int_{\Omega} (a \varphi - J \ast \varphi) \psi \, dx + \int_{\Omega} P_0(F'(^\top \varphi)) \psi \, dx, \quad (3.5)
\]

\[
\int_{\Omega} (u \cdot \nabla)u \cdot v \, dx + \int_{\Omega} 2\nu(\varphi)Du \cdot Dv \, dx = \int_{\Omega} \mu_0 \nabla \varphi \cdot v \, dx + \int_{\Omega} h \cdot v \, dx, \quad (3.6)
\]

where \( u \in \mathcal{V}'_{\text{div}}, \mu_0 \in H^1_0(\Omega), \varphi \in H^1_0(\Omega) \). Now we show that proving the existence of a solution to the equations (3.4)-(3.6) would also give a solution to (3.1)-(3.3).

**Lemma 3.2.** Let \((u, \mu_0, \varphi) \in \mathcal{V}'_{\text{div}} \times H^1_0(\Omega) \times H^1_0(\Omega)\) be a solution to the system (3.4)-(3.6). Then \((u, \mu, \varphi)\) is a solution to the weak formulation (3.1)-(3.3), where \( \mu = \mu_0 - \frac{1}{|H|} \int_{\Omega} F'(^\top \varphi) \, dx \).

**Proof.** Let \((u, \mu_0, \varphi) \in \mathcal{V}'_{\text{div}} \times H^1_0(\Omega) \times H^1_0(\Omega)\) be a solution of the system (3.4)-(3.6). Let \( \varphi = \frac{1}{|H|} \int_{\Omega} F'(^\top \varphi) \, dx \). Since \( \varphi \) is a scalar, from (3.6), using integration by parts, one can easily deduce that \( \int_{\Omega} \mu \nabla \varphi \cdot v \, dx = 0 \). Then, we have

\[
\int_{\Omega} (u \cdot \nabla)u \cdot v \, dx + \int_{\Omega} 2\nu(\varphi)Du \cdot Dv \, dx = \int_{\Omega} \mu_0 \nabla \varphi \cdot v \, dx + \int_{\Omega} h \cdot v \, dx + \int_{\Omega} \varphi \nabla \varphi \cdot v \, dx
\]

\[
= \int_{\Omega} \mu \nabla \varphi \cdot v \, dx + \int_{\Omega} h \cdot v \, dx,
\]
which gives equation (3.3). Once again, since $\mathcal{P}$ is a scalar quantity, we can clearly see (3.1) follows from (3.2). From (3.5), we have
\[
\int_{\Omega} \mu_0 \psi \, dx = \int_{\Omega} (a\varphi - J \ast \varphi) \psi \, dx + \int_{\Omega} P_0(F'(\varphi)) \psi \, dx.
\] (3.7)
Using (2.3) and substituting value of $\mu_0$ in (3.7) we get,
\[
\int_{\Omega} \mu \psi \, dx = \int_{\Omega} (a\varphi - J \ast \varphi) \psi \, dx + \int_{\Omega} (P_0(F'(\varphi)) + \mathcal{P}) \psi \, dx
\]
\[
= \int_{\Omega} (a\varphi - J \ast \varphi) \psi \, dx + \int_{\Omega} F'(\varphi) \psi \, dx,
\]
which completes the proof. \hfill \Box

### 3.1. Preliminaries

In order to formulate the problem (3.4)-(3.6) in the framework of Browder’s Theorem (see Theorem 3.10 below), we need some preliminaries which we state below.

Let $\mathbb{X}$ be a Banach space and $\mathbb{X}'$ be its topological dual. Let $T$ be a function from $\mathbb{X}$ to $\mathbb{X}'$ with domain $D = D(T) \subseteq \mathbb{X}$.

**Definition 3.3** (Definition 2.3, [35]). *The function $T$ is said to be*

(i) *demicontinuous* if for a sequence $u_k \in D, u \in D$ and $u_k \to u$ in $\mathbb{X}$ implies that $T(u_k) \rightharpoonup T(u)$ in $\mathbb{X}'$,

(ii) *hemicontinuous* if $u, v \in \mathbb{X}$ and $u + t_k v \in D$ for a sequence of positive real numbers $t_k$ such that $t_k \to 0$ implies $T(u + t_k v) \rightharpoonup T(u)$ in $\mathbb{X}'$,

(iii) *locally bounded* if for a sequence $u_k \in D, u \in D$ and $u_k \to u$ in $\mathbb{X}$ imply that $T(u_k)$ is bounded in $\mathbb{X}'$.

From the above definition, it is clear that a demicontinuous function is hemicontinuous and locally bounded.

**Definition 3.4** (Definition 2.1(iv), [35]). *We say that $T$ is pseudo-monotone if, for a sequence $u_k$ in $\mathbb{X}$ such that $u_k \to u$ in $\mathbb{X}$ and
\[
\limsup_{k \to \infty} \langle \mathbb{X}'(T(u_k)), u_k - u \rangle_{\mathbb{X}} \leq 0
\]
implies
\[
\liminf_{k \to \infty} \langle \mathbb{X}'(T(u_k)), u_k - u \rangle_{\mathbb{X}} \geq \langle \mathbb{X}'(T(u)), u - v \rangle_{\mathbb{X}}
\]
for every $v \in \mathbb{X}$. Moreover $T$ is said to be monotone if
\[
\langle \mathbb{X}'(T(u)) - T(v), u - v \rangle_{\mathbb{X}} \geq 0, \quad \text{for every } u, v \in D.
\]

**Definition 3.5.** *A mapping $T : \mathbb{X} \to \mathbb{X}'$ is said to be maximal monotone if it is monotone and its graph
\[
G(T) = \{(u, w) : w \in T(u)\} \subset \mathbb{X} \times \mathbb{X}'
\]
is not properly contained in graph of any other monotone operator. In other words, for $u \in \mathbb{X}$ and $w \in \mathbb{X}'$, the inequality $\langle \mathbb{X}(w - T(v)), u - v \rangle_{\mathbb{X}} \geq 0$, for all $v \in \mathbb{X}$ implies $w = T(u)$,***

**Definition 3.6** (Definition 2.3, [35]). *Let $\mathbb{X}$ and $\mathbb{Y}$ be Banach spaces. A bounded linear operator $T : \mathbb{X} \to \mathbb{Y}$ is said to be completely continuous operator if $u_k \to u$ in $\mathbb{X}$ implies $T(u_k) \to T(u)$ in $\mathbb{Y}$.*

We can see that complete continuity implies pseudo-monotonicity (Corollary 2.12, [35]).

**Lemma 3.7** (Lemma 5.1, [2]). *Let $\mathbb{X}$ be a real, reflexive Banach space and $\tilde{T} : \mathbb{X} \times \mathbb{X} \to \mathbb{X}'$ be such that for all $u \in \mathbb{X}$:
(1) $\bar{T}(u, \cdot) : X \to X'$ is monotone and hemicontinuous.
(2) $T(\cdot, u) : X \to X'$ is completely continuous.

Then the operator $T : X \to X'$ defined by $T(u) := \bar{T}(u, u)$ is pseudo-monotone.

**Definition 3.8.** Let $X$ be a real Banach space and $f : X \to (-\infty, \infty]$ be a functional on $X$. A linear functional $g \in X'$ is called subgradient of $f$ at $u$ if $f(u) \neq \infty$ and

$$f(v) \geq f(u) + X'(g, v - u),$$

holds for all $v \in X$.

We know that subgradient of a functional need not be unique. The set of all subgradients of $f$ at $u$ is called subdifferential of $f$ at $u$ and is denoted by $\partial f(u)$. We say that $f$ is Gâteaux differentiable at $u$ in $X$ if $\partial f(u)$ consists of exactly one element (see [3]).

**Lemma 3.9** (Theorem A [33]). If $f$ is a lower semicontinuous, proper convex function on $X$ (i.e., $f$ is a convex function and $f$ takes values in the extended real number line such that $f(u) < +\infty$ for at least one $u \in X$ and $f(u) > -\infty$ for every $u \in X$), then $\partial f$ is a maximal monotone operator from $X$ to $X'$.

Now we state the Browder’s theorem, which we use to prove the existence of solution to the problem (3.4)-(3.6).

**Theorem 3.10** (Theorem 32.A in [39], Browder).

(1) Let $\mathcal{Y}$ be a nonempty, closed and convex subset of a real and reflexive Banach space $X$.
(2) Let $T : \mathcal{Y} \to \mathcal{P}(X')$ be a maximal monotone operator, where $\mathcal{P}(X')$ denotes the power set of $X'$.
(3) Let $S : \mathcal{Y} \to X'$ be a pseudo-monotone, bounded and demicontinuous mapping.
(4) If the set $\mathcal{Y}$ is unbounded, then the operator $S$ is $T$-coercive with respect to the fixed element $b \in X'$ i.e., there exists an element $u_0 \in \mathcal{Y} \cap D(T)$ and $R > 0$ such that

$$X'(S(u), u - u_0)_X > X'(b, u - u_0)_X,$$

for all $u \in \mathcal{Y}$ with $\|u\|_X > R$.

Then the problem

$$b \in T(u) + S(u) \quad (3.8)$$

has a solution $u \in \mathcal{Y} \cap D(T)$.

**3.2. The functional $f$.** We mainly follow the work of [2] (local Cahn-Hilliard-Navier-Stokes equations) to establish the solvability results of the system (3.1). Before we proceed to prove our main result, we first consider the following functional and study its properties. Let us define

$$f(\varphi) := \frac{1}{4} \int_{\Omega} \int_{\Omega} f(x - y)(\varphi(x) - \varphi(y))^2 dx dy + \int_{\Omega} G(\varphi(x)) dx, \quad (3.9)$$

where $\varphi \in L^2_0(\Omega)$ and $G$ is given in [221]. Using Assumption 2.8 we define $G(x) = \infty$ for $x \notin [a, b]$. The domain of $f$ is given by

$$D(f) = \left\{ \varphi \in L^2_0(\Omega) : G(\varphi) \in L^1(\Omega) \right\}.$$

For $\varphi \notin D(f)$, we set $f(\varphi) = +\infty$. Note that $D(f) \neq \emptyset$.

Given a functional $f : L^2_0(\Omega) \to (-\infty, \infty]$, its subgradient maps from $L^2_0(\Omega)$ to $\mathcal{P}(L^2_0(\Omega))'$. We write $\partial_{L^2_0} f$ as the subgradient of the functional $f : L^2_0(\Omega) \to (-\infty, \infty]$. Since $H^1_0(\Omega) \hookrightarrow L^2_0(\Omega)$, we can also consider $f$ as a functional from $H^1_0(\Omega)$ to $(-\infty, \infty]$, and hence we have to distinguish between its different subgradients. If we consider $f : H^1_0(\Omega) \to (-\infty, \infty]$, then the subgradient of $f$ is denoted by $\partial_{H^1_0} f$. 

Furthermore, it holds that
\[ L_\varepsilon \mathcal{L}(\varphi) = a \varphi - J \ast \varphi + P_0 G'(\varphi). \] (3.10)
and
\[ D(L_\varepsilon \mathcal{L}) = \left\{ \varphi \in L_\varepsilon^2(\Omega) : G'(\varphi) \in L^2(\Omega) \right\}. \]

Furthermore, it holds that
\[ \|L_\varepsilon \mathcal{L}(\varphi)\| \leq (\|a\|_{L^\infty} + \|J\|_{L^1})\|\varphi\| + \|G'(\varphi)\| \leq 2a^*\|\varphi\| + \|G'(\varphi)\|. \] (3.11)

**Proof.** Let \( h \in L_\varepsilon^2(\Omega) \). Then, we have
\[
\frac{d}{d\varepsilon} f(\varphi + \varepsilon h) = \frac{1}{4} \int \int \Omega J(x - y)((\varphi + \varepsilon h)(x) - (\varphi + \varepsilon h)(y))^2 \, dx \, dy + \int \Omega G((\varphi + \varepsilon h)(x)) \, dx.
\]

At \( \varepsilon = 0 \), we infer that
\[
\frac{d}{d\varepsilon} f(\varphi + \varepsilon h) \bigg|_{\varepsilon = 0} = \frac{1}{2} \int \int \Omega J(x - y)(\varphi(x) - \varphi(y))(h(x) - h(y)) \, dx \, dy + \int \Omega G'(\varphi(x)) h(x) \, dx.
\]

Using Assumption 2.6(1) and since \( P_0 \) is the orthogonal projection onto \( L_\varepsilon^2(\Omega) \), we get
\[
\frac{d}{d\varepsilon} f(\varphi + \varepsilon h) \bigg|_{\varepsilon = 0} = (a \varphi - J \ast \varphi + P_0 G'(\varphi), h),
\] (3.12)
which proves (3.10). Note that (3.11) is an immediate consequence of (3.10). \( \square \)

**Lemma 3.12.** The functional \( f : L_\varepsilon^2(\Omega) \to (-\infty, \infty] \) defined by (3.9) is proper convex and lower semicontinous with \( D(f) \neq \emptyset \).

**Proof.** **Claim (1).** \( f \) is proper convex with \( D(f) \neq \emptyset \): In order to prove convexity we use the fact that \( f \) is convex if and only if
\[
(L_\varepsilon \mathcal{L}) f(\varphi) - (L_\varepsilon \mathcal{L}) f(\psi), \varphi - \psi \geq 0,
\]
for all \( \varphi, \psi \in L_\varepsilon^2(\Omega) \). Observe that for \( 0 < \theta < 1 \), using Taylor series expansion and the definition of \( P_0 \), we find that
\[
(a \varphi - J \ast \varphi + P_0 G'(\varphi) - (a \psi - J \ast \psi + P_0 G'(\psi)), \varphi - \psi)
\]
\[
= (a \varphi - J \ast \varphi + F'(\varphi) + \kappa \varphi - (a \psi - J \ast \psi + F'(\psi) + \kappa \psi), \varphi - \psi)
\]
\[
= (a(\varphi - \psi) - J \ast (\varphi - \psi) + F'(\varphi) - F'(\psi) + \kappa(\varphi - \psi), \varphi - \psi)
\]
\[
= (a(\varphi - \psi) - J \ast (\varphi - \psi) + F''(\varphi + \theta \psi)(\varphi - \psi) + \kappa(\varphi - \psi), \varphi - \psi)
\]
\[
\geq C_0 \|\varphi - \psi\|^2 - \|J\|_{L^1} \|\varphi - \psi\|^2 + \kappa \|\varphi - \psi\|^2
\]
\[
\geq (C_0 - \|J\|_{L^1})\|\varphi - \psi\|^2
\]
\[
\geq 0,
\]
using Assumption 2.7. Hence \( f \) is a convex functional on \( L^2_\Omega(\Omega) \). Since the domain of \( f \) is 
\[ \text{D}(f) = \left\{ \varphi \in L^2_\Omega(\Omega) : G(\varphi) \in L^1(\Omega) \right\} \neq \emptyset, \]
it is immediate that \( f \) is proper convex.

**Claim (2).** \( f \) is lower semicontinuous: Let \( \varphi_k \in L^2_\Omega(\Omega) \) and \( \varphi_k \to \varphi \) in \( L^2_\Omega(\Omega) \) as \( k \to \infty \). Our aim is to establish that \( f(\varphi) \leq \liminf_{k \to \infty} f(\varphi_k) \). It is enough to consider the case \( \liminf_{k \to \infty} f(\varphi_k) < \infty \).

Thus, for this sequence, we can assume that \( f(\varphi_k) \leq M \), for some \( M \). This implies that \( \varphi_k \in \text{D}(f) \), for all \( k \in \mathbb{N} \). Since \( G : [a,b] \to \mathbb{R} \) is a continuous function, by adding a suitable constant, we can assume without loss of generality that \( G \geq 0 \) and we have 
\[ G(\varphi) \leq \liminf_{k \to \infty} G(\varphi_k), \]
which gives from Fatou’s lemma that 
\[ \int_\Omega G(\varphi(x))dx \leq \liminf_{k \to \infty} \int_\Omega G(\varphi_k(x))dx. \]  
(3.13)

Now consider the functional \( I(\cdot) \) defined by 
\[ I(\varphi) := \int_\Omega \int_\Omega J(x-y)(\varphi(x) - \varphi(y))^2dxdy = (a,\varphi,\varphi) - (J * \varphi, \varphi). \]

We show that the function \( I(\cdot) \) is continuous. We consider 
\[ |I(\varphi_k) - I(\varphi)| = |(a,\varphi_k,\varphi_k) - (J * \varphi_k, \varphi_k) - (a,\varphi,\varphi) - (J * \varphi, \varphi)| \leq |(a(\varphi_k - \varphi), \varphi_k) + (a,\varphi,\varphi_k) - (J * (\varphi_k - \varphi), \varphi_k)| + |(J * (\varphi_k - \varphi), \varphi_k)| + |(J * \varphi, \varphi_k - \varphi)| \leq |a|_1\infty (||\varphi_k|| + ||\varphi||)||\varphi_k - \varphi|| + ||J||_1\infty (||\varphi_k|| + ||\varphi||)||\varphi_k - \varphi|| \]

where we used Young’s inequality, Young’s inequality for convolution and H"{o}lder inequality. Then, we have \( |I(\varphi_k) - I(\varphi)| \to 0 \) as \( k \to \infty \), since \( \varphi_k \to \varphi \) in \( L^2_\Omega(\Omega) \) as \( k \to \infty \). Since continuity implies lower semicontinuity, we have 
\[ \int_\Omega \int_\Omega J(x-y)(\varphi(x) - \varphi(y))^2dxdy = \liminf_{k \to \infty} \int_\Omega \int_\Omega J(x-y)(\varphi_k(x) - \varphi_k(y))^2dxdy. \]  
(3.14)

Combining (3.13) and (3.14), we get 
\[ f(\varphi) \leq \liminf_{k \to \infty} f(\varphi_k). \]

This proves \( f \) is lower semicontinuous.

\[ \square \]

**Remark 3.13.** The proper convexity of \( f : H^1_\Omega(\Omega) \to (-\infty, \infty) \) is immediate, since \( H^1_\Omega(\Omega) \subseteq L^2_\Omega(\Omega) \). Let \( (\varphi_k)_{k \in \mathbb{N}} \in H^1_\Omega(\Omega) \) be such that \( \varphi_k \to \varphi \) in \( H^1_\Omega(\Omega) \). Then from Lemma 2.7 we can easily see that \( \varphi_k \to \varphi \) in \( L^2_\Omega(\Omega) \). Therefore using Lemma 3.12 we get \( f : H^1_\Omega(\Omega) \to (-\infty, \infty) \) is lower semicontinuous.

**Proposition 3.14.** The subgradients \( \partial_{L^2_\Omega} f \) and \( \partial_{H^1_\Omega} f \) are maximal monotone operators.

**Proof.** In lemma 3.12 we have proved that \( f : L^2_\Omega(\Omega) \to \mathbb{R} \) is proper convex and lower semicontinuous. By using Lemma 3.12 we obtain that the operator \( \partial_{L^2_\Omega} f : L^2_\Omega(\Omega) \to \mathcal{P}(L^2_\Omega(\Omega)^\prime) \) is maximal monotone. Now for the operator \( \partial_{H^1_\Omega} f \), Remark 3.13 gives that \( f : H^1_\Omega(\Omega) \to (-\infty, \infty) \) is proper convex and lower semicontinuous. Hence, once again using Lemma 3.12 we get that \( \partial_{H^1_\Omega} f \) is also maximal monotone.  

\[ \square \]

**Lemma 3.15** (Lemma 3.7, [2]). Consider the functional \( f \) as in (3.9). Then for every \( \varphi \in \text{D}(\partial_{L^2_\Omega} f) \) we have that 
\[ \partial_{L^2_\Omega} f(\varphi) \subseteq \partial_{H^1_\Omega} f(\varphi). \]
Lemma 3.16 (Lemma 3.8, [2]). Let \( \varphi \in D(\partial \Omega_H^1 f) \) and \( w \in \partial \Omega_H^1 f(\varphi) \). Suppose \( w \in L^2_0(\Omega) \), then \( \varphi \in D(\partial \Omega_H^1 f) \) and
\[ w = \partial \Omega_H^1 f(\varphi) = a \varphi - J * \varphi + P_0 G'(\varphi). \]

3.3. Abstract Formulation. Now let us define following spaces in order to set up the problem in Browder’s theorem (see Theorem 3.10).
\[ X := \mathbb{V}_{\text{div}} \times \mathbb{H}^1_0(\Omega) \times \mathbb{H}^1_0(\Omega). \]

Let us define,
\[ Z := \left\{ \varphi \in \mathbb{H}^1_0(\Omega) : \varphi(x) \in [a, b] \text{ a.e.} \right\} \quad (3.15) \]
and
\[ Y := \mathbb{V}_{\text{div}} \times \mathbb{H}^1_0(\Omega) \times Z. \]

Clearly \( Y \) is a closed subspace of \( X \). Let \( D(T) := \mathbb{V}_{\text{div}} \times \mathbb{H}^1_0(\Omega) \times D(\partial \Omega_H^1 f) \) and we define a mapping \( T : Y \to \mathcal{P}(X') \) by
\[ T(u, \mu_0, \varphi) := \left\{ \begin{array}{ll}
0 & \text{if } (u, \mu_0, \varphi) \in D(T), \\
\partial \Omega_H^1 f(\varphi) & \text{otherwise}.
\end{array} \right. \quad (3.16) \]

We define the operator \( S : Y \to X' \) as
\[ \langle Sx, y \rangle_X := \int_{\Omega} (u \cdot \nabla)u \cdot v dx + \int_{\Omega} 2\nu(\varphi) Du \cdot Dv dx - \int_{\Omega} \mu_0 \nabla \varphi \cdot v dx + \int_{\Omega} (u \cdot \nabla \varphi) \cdot \eta dx \\
+ \int_{\Omega} m(\varphi) \nabla \mu_0 \cdot \nabla \eta dx - \int_{\Omega} \mu_0 \psi dx - \int_{\Omega} P_0(\kappa \varphi) \psi dx, \quad (3.17) \]
for all \( x = (u, \mu_0, \varphi) \in Y \), \( y = (v, \eta, \psi) \in X \) and \( b \in X' \) is defined by
\[ \langle b, y \rangle_X := \int_{\Omega} h \cdot v dx \]
for all \( y \in X \). From the relations (3.16) and (3.17), the problem \( b \in T(u, \mu_0, \varphi) + S(u, \mu_0, \varphi) \) with \( (u, \mu_0, \varphi) \in Y \cap D(T) \) can be written as
\[ \left( \begin{array}{c}
0 \\
0 \\
\partial \Omega_H^1 f(\varphi)
\end{array} \right) + \left( \begin{array}{c}
(u \cdot \nabla)u - \text{div}(2\nu(\varphi) Du) - \mu_0 \nabla \varphi \\
-m(\varphi) \nabla \mu_0 + u \cdot \nabla \varphi \\
-\mu_0 - P_0(\kappa \varphi)
\end{array} \right) = \left( \begin{array}{c}
h \\
0 \\
0
\end{array} \right), \quad (3.18) \]
in \( X' \).

If we can prove that (3.18) has a solution, then this solution solves the reformulated problem (3.4) - (3.6). This is the content of the next lemma. Later we discuss the existence of solution for (3.18).

Lemma 3.17. Let \( u, \mu_0, \varphi \in Y \cap D(T) \) satisfies \( b \in T(u, \mu_0, \varphi) + S(u, \mu_0, \varphi) \). Then \( u, \mu_0, \varphi \) is a solution of the reformulated problem (3.4) - (3.6).

Proof. Let \( u, \mu_0, \varphi \) be such that \( b \in T(u, \mu_0, \varphi) + S(u, \mu_0, \varphi) \). From first and second equations of (3.18), it clearly follows that (3.4) and (3.6) are satisfied for all \( v \in \mathbb{V}_{\text{div}} \) and \( \psi \in \mathbb{H}^1_0(\Omega) \). Now from the third equation of (3.18), there exists \( w \in \partial \Omega_H^1 f(\varphi) \) such that
\[ w = \mu_0 + P_0(\kappa \varphi) \text{ in } \mathbb{H}^1_0(\Omega). \]
Since $\mu_0 + P_0(\kappa \varphi) \in L^2_{(0)}(\Omega)$ and $\varphi \in D(\partial_{H^1_{(0)}} f)$, we can see from Lemma 3.16 that $w = a \varphi - J \ast \varphi + P_0(G'(\varphi))$ in $H^{-1}_{(0)}(\Omega)$. This gives for every $\psi \in H^1_{(0)}(\Omega)$

$$\int_{\Omega} (\mu_0 + P_0(\kappa \varphi)) \psi dx = \int_{\Omega} (a \varphi - J \ast \varphi) \psi dx + \int_{\Omega} P_0(G'(\varphi)) \psi dx.$$ 

Hence

$$\int_{\Omega} \mu_0 \psi dx = \int_{\Omega} (a \varphi - J \ast \varphi) \psi dx + \int_{\Omega} P_0(G'(\varphi) - \kappa \varphi) \psi dx 
= \int_{\Omega} (a \varphi - J \ast \varphi) \psi dx + \int_{\Omega} P_0(F'(\varphi)) \psi dx,$$

for all $\psi \in H^1_{(0)}(\Omega)$.

In Lemma 3.17 we showed that the existence of $(u, \mu_0, \varphi)$ such that $b \in T(u, \mu_0, \varphi) + S(u, \mu_0, \varphi)$ implies $(u, \mu_0, \varphi)$ satisfies the reformulation (3.4)-(3.6). Now we show that there exists a $(u, \mu_0, \varphi) \in Y \cap D(T)$ which satisfies $b \in T(u, \mu_0, \varphi) + S(u, \mu_0, \varphi)$. To this purpose, we use the Browder’s theorem (see Theorem 3.10).

**Lemma 3.18.** Let $T, S$ be as defined in (3.16) and (3.17). Given $b \in X'$, there exists a triple $x = (u, \mu_0, \varphi) \in Y \cap D(T)$ such that $b \in T(x) + S(x)$.

**Proof.** Let us prove that the operators $T$ and $S$, and the spaces $X$ and $Y$ satisfy the hypothesis of Browder’s theorem (see Theorem 3.10) in the following steps.

1. We can see that the set $Y$ is non-empty, closed, convex subset of $X$. Also $X$ is a reflexive real Banach space, since $V_{\text{div}}$ and $H^1_{(0)}(\Omega)$ are reflexive.

2. Now we show that the operator $T : Y \to \mathcal{P}(X')$ is maximal monotone. Let us first show that $D(\partial_{H^1_{(0)}} f) \subseteq Z$. In order to get this result, let us take $\varphi \in D(\partial_{H^1_{(0)}} f)$. Then we know that $f(\varphi) \neq -\infty$, since $D(\partial_{H^1_{(0)}} f) \subseteq D(f)$. This gives that $\varphi(x) \in [a, b]$, since $G(x) = +\infty$ for $x \notin [a, b]$. Hence $\varphi \in Z$ and $D(\partial_{H^1_{(0)}} f) \subseteq Z$. From Proposition 3.14 the operator $\partial_{H^1_{(0)}} f : H^1_{(0)}(\Omega) \to \mathcal{P}(H^1_{(0)}(\Omega))$ is maximal monotone. This implies that the operator $T : X \to \mathcal{P}(X')$ is maximal monotone. Observe that,

$$D(T) = V_{\text{div}} \times H^1_{(0)}(\Omega) \times D(\partial_{H^1_{(0)}} f) \subseteq V_{\text{div}} \times H^1_{(0)}(\Omega) \times Z = Y \subseteq X.$$

Moreover by the definition of $T$, for $(u, \mu_0, \varphi) \notin D(T)$, $T(u, \mu_0, \varphi) = \emptyset$. Hence it follows that $T : Y \to \mathcal{P}(X')$ is a maximal monotone operator.

3. We write $S = S_1 + \ldots + S_7$ and show that each $S_i$, for $i = 1, \ldots, 7$, is pseudo-monotone. Let us define

$$\langle x', (S_1(u, \mu_0, \varphi), (v, \eta, \psi)) \rangle_X := \int_{\Omega} (u \cdot \nabla) u \cdot v dx,$$

$$\langle x', (S_2(u, \mu_0, \varphi), (v, \eta, \psi)) \rangle_X := \int_{\Omega} 2 \nu(\varphi) Du \cdot Dv dx,$$

$$\langle x', (S_3(u, \mu_0, \varphi), (v, \eta, \psi)) \rangle_X := \int_{\Omega} \mu_0 \nabla \varphi \cdot v dx,$$

$$\langle x', (S_4(u, \mu_0, \varphi), (v, \eta, \psi)) \rangle_X := \int_{\Omega} m(\varphi) \nabla \mu_0 \cdot \nabla \eta dx,$$

$$\langle x', (S_5(u, \mu_0, \varphi), (v, \eta, \psi)) \rangle_X := \int_{\Omega} (u \cdot \nabla \varphi) \eta dx,$$

$$\langle x', (S_6(u, \mu_0, \varphi), (v, \eta, \psi)) \rangle_X := \int_{\Omega} \mu_0 \psi dx,$$

$$\langle x', (S_7(u, \mu_0, \varphi), (v, \eta, \psi)) \rangle_X := \int_{\Omega} \mu_0 \psi dx.$$
Similarly, we have
\[ \chi'(S_7(u, \mu_0, \varphi), (v, \eta, \psi)) := \int_{\Omega} P_0(\kappa \varphi) \psi \, dx. \]

Since completely continuous implies pseudo-monotone, we show that \( S_1, S_3, S_5, S_6 \) and \( S_7 \) are completely continuous operators. Let us denote \( x_n = (u_n, \mu_{n0}, \varphi_n) \), \( x = (u, \mu_0, \varphi) \) and \( y = (v, \eta, \psi) \). Assume that \( x_n \rightharpoonup x \) in \( Y \). This means \( u_n \rightharpoonup u \) in \( V_{\text{div}} \), \( \mu_{n0} \rightharpoonup \mu_0 \) in \( H^1(\Omega) \) and \( \varphi_n \rightharpoonup \varphi \) in \( H^1(\Omega) \), which in turn gives \( u_n \to u \) in \( G_{\text{div}} \), \( \mu_{n0} \to \mu_0 \) in \( L^2(\Omega) \) and \( \varphi_n \to \varphi \) in \( L^2(\Omega) \), using the compact embeddings \( V_{\text{div}} \hookrightarrow G_{\text{div}} \) and \( H^1(\Omega) \hookrightarrow L^2(\Omega) \). We have to show that \( S_1 x_n \) converges strongly to \( S_1 x \) in \( X' \)-norm. Using (2.7), H"older's and Ladyzhenskaya inequalities, for \( n = 2 \), we get
\[ |\chi'(S_1 x_n, y)_X - \chi'(S_1 x, y)_X| = \left| b(u_n, u_n, v) - b(u, u, v) \right| \\
\leq \left\| u_n \right\|_4 \left\| \nabla v \right\| \left\| u_n - u \right\|_4 + \left\| u_n \right\|_4 \left\| \nabla v \right\| \left\| u_n - u \right\|_4 \\
\leq 2^{1/2} \left( \left\| u_n \right\|_4^{1/2} \left\| \nabla u_n \right\|^{1/2}_4 + \left\| u \right\|_4^{1/2} \left\| \nabla u \right\|^{1/2}_4 \right) \left\| u_n - u \right\|_4 \left\| v \right\|_{V_{\text{div}}}. \]

(3.19)

For \( n = 3 \), using H"older's and Ladyzhenskaya inequalities, we get
\[ |\chi'(S_1 x_n, y)_X - \chi'(S_1 x, y)_X| \leq \left\| u_n \right\|_4 \left\| \nabla v \right\| \left\| u_n - u \right\|_4 + \left\| u_n - u \right\|_4 \left\| \nabla v \right\| \left\| u \right\|_4 \\
\leq 2 \left( \left\| u_n \right\|_4^{1/4} \left\| \nabla u_n \right\|^{3/4}_4 + \left\| u \right\|_4^{1/4} \left\| \nabla u \right\|^{3/4}_4 \right) \left\| u_n - u \right\|_4 \left\| v \right\|_{V_{\text{div}}}. \]

(3.20)

Let us now estimate \( |\chi'(S_3 x_n, y) - (S_3 x, y)_X| \) and \( |\chi'(S_5 x_n, y) - (S_5 x, y)_X| \). We perform an integration by parts, use H"older's inequality and the embedding \( H^1(\Omega) \hookrightarrow L^4(\Omega) \) to estimate \( |\chi'(S_3 x_n, y) - (S_3 x, y)_X| \) as
\[ |\chi'(S_3 x_n, y) - (S_3 x, y)_X| = \left| -\int_{\Omega} \mu_{n0} \nabla \varphi_n \cdot v \, dx + \int_{\Omega} \mu_0 \nabla \varphi \cdot v \, dx \right| \\
= \left| \int_{\Omega} \mu_{n0} \nabla (\varphi - \varphi_n) \cdot v \, dx + \int_{\Omega} (\mu_0 - \mu_{n0}) \nabla \varphi \cdot v \, dx \right| \\
\leq \left\| \nabla \mu_{n0} \right\| \left\| v \right\|_{L^4} \left\| \varphi - \varphi_n \right\|_{L^4} + \left\| \mu_0 - \mu_{n0} \right\|_{L^4} \left\| \nabla \varphi \right\| \left\| v \right\|_{L^4} \\
\leq C \left( \left\| \nabla \mu_{n0} \right\| \left\| \varphi - \varphi_n \right\|_{L^4} + \left\| \mu_0 - \mu_{n0} \right\|_{L^4} \left\| \nabla \varphi \right\| \right) \left\| v \right\|_{V_{\text{div}}}. \]

(3.21)

Similarly, we have
\[ |\chi'(S_5 x_n, y) - (S_5 x, y)_X| = \left| \int_{\Omega} (u_n \cdot \nabla \varphi_n) \eta \, dx - \int_{\Omega} (u \cdot \nabla \varphi) \eta \, dx \right| \\
= \left| \int_{\Omega} (u_n - u) \cdot \nabla \varphi_n \eta \, dx - \int_{\Omega} (u \cdot \nabla (\varphi_n - \varphi)) \eta \, dx \right| \\
\leq \left( \left\| u_n - u \right\|_{L^4} \left\| \nabla \varphi_n \right\|_{L^4} + \left\| u \right\|_{L^4} \left\| \varphi_n - \varphi \right\|_{L^4} \right) \left\| \nabla \eta \right\|. \]

(3.22)

Using H"older's inequality, we obtain
\[ |\chi'(S_6 x_n, y) - (S_6 x, y)_X| \leq \int_{\Omega} |\mu_{n0} - \mu_0| |\psi| \, dx \leq \left\| \mu_{n0} - \mu_0 \right\| \left\| \psi \right\|. \]

(3.23)

and
\[ |\chi'(S_7 x_n, y) - (S_7 x, y)_X| = \left| \int_{\Omega} P_0(\kappa \varphi_n) \psi \, dx - \int_{\Omega} P_0(\kappa \varphi) \psi \, dx \right| \leq \kappa \int_{\Omega} |\varphi_n - \varphi| \left| \psi \right| \, dx \\
\leq \kappa \left\| \varphi_n - \varphi \right\| \left\| \psi \right\|. \]
From \([3.19]-[3.20]\), we get
\[
\|S_1\mathbf{x}_n - S_1\mathbf{x}\|_{\mathcal{X}'} \leq 2^{1/2}\left(\|\mathbf{u}_n\|^{1/2}_2 \|\nabla \mathbf{u}_n\|^{1/2}_2 + \|\mathbf{u}\|^{1/2}_2 \|\nabla \mathbf{u}\|^{1/2}_2\right)\|\mathbf{u}_n - \mathbf{u}\|_{L^4}(n = 2),
\]
\[
\|S_1\mathbf{x}_n - S_1\mathbf{x}\|_{\mathcal{X}'} \leq 2\left(\|\mathbf{u}_n\|^{1/4}_4 \|\nabla \mathbf{u}_n\|^{3/4}_4 + \|\mathbf{u}\|^{1/4}_4 \|\nabla \mathbf{u}\|^{3/4}_4\right)\|\mathbf{u}_n - \mathbf{u}\|_{L^4}(n = 3),
\]
and both converges to 0 as \(n \to \infty\) using the compact embedding of \(L^1_0(\Omega) \hookrightarrow L^4(\Omega)\). Using the compact embedding \(H^1(\Omega) \hookrightarrow L^4(\Omega)\), from \([3.21]-[3.24]\), we have
\[
\|S_3\mathbf{x}_n - S_3\mathbf{x}\|_{\mathcal{X}'} \leq \|\nabla \mathbf{u}_n\|_2 \|\mathbf{v} - \varphi_n\|_{L^4} + \|\mathbf{u}_n - \mathbf{u}\|_{L^4} \|\nabla \varphi\| \to 0,
\]
\[
\|S_5\mathbf{x}_n - S_5\mathbf{x}\|_{\mathcal{X}'} \leq \|\mathbf{u}_n - \mathbf{u}\|_{L^4} \|\varphi_n - \mathbf{v}\|_{L^4} \to 0,
\]
\[
\|S_6\mathbf{x}_n - S_6\mathbf{x}\|_{\mathcal{X}'} \leq \|\mathbf{u}_n - \mathbf{u}\|_{L^4} \|\varphi_n - \mathbf{v}\|_{L^4} \to 0,
\]
\[
\|S_7\mathbf{x}_n - S_7\mathbf{x}\|_{\mathcal{X}'} \leq \kappa \|\varphi_n - \mathbf{v}\| \to 0,
\]
as \(n \to \infty\). This proves that \(S_1, S_3, S_5, S_6,\) and \(S_7\) are completely continuous and hence pseudomonotone. In order to prove the pseudo-monotonicity of \(S_4\), we use Lemma \(3.7\). Let us define the operator \(\tilde{S}_4 : \mathcal{X} \times \mathcal{X} \to \mathcal{X}'\) by
\[
\langle \tilde{S}_4(\mathbf{x}_1, \mathbf{x}_2), \mathbf{y} \rangle_{\mathcal{X}} := \int_\Omega m(\varphi_1)\nabla \mu_0 \cdot \nabla \eta d\mathbf{x},
\]
where \(\mathbf{x}_i = (\mathbf{u}_i, \mu_0, \varphi_i) \in \mathcal{X}, i = 1, 2\), respectively and \(\mathbf{y} = (\mathbf{v}, \eta, \psi) \in \mathcal{X}\). If \(\mathbf{x} = (\mathbf{u}, \mu_0, \varphi) \in \mathcal{X}\), then
\[
\langle \tilde{S}_4(\mathbf{x}_1, \mathbf{x}_2 - \mathbf{x}_1), \mathbf{x}_1 - \mathbf{x}_2 \rangle_{\mathcal{X}} = \int_\Omega m(\varphi_1)\nabla(\mu_0 - \mu_2) \cdot \nabla(\mu_1 - \mu_2) d\mathbf{x}
\]
\[
= \int_\Omega m(\varphi_1)\nabla(\mu_0 - \mu_2)^2 d\mathbf{x} \geq 0,
\]
since \(m(\cdot) \geq 0\). This implies \(\tilde{S}_4(\mathbf{x}, \cdot)\) is monotone. Now for each fixed \(\mathbf{x} \in \mathcal{X}\), it can be easily seen that \(\langle \tilde{S}_4(\mathbf{x}, \mathbf{x}_1 + t_k \mathbf{x}_2), \mathbf{y} \rangle_{\mathcal{X}} \to \langle \tilde{S}_4(\mathbf{x}, \mathbf{x}_1), \mathbf{y} \rangle_{\mathcal{X}}, \) as \(t_k \to 0\). Hence, the operator \(\tilde{S}_4(\mathbf{x}, \cdot)\) is hemicontinuous for each fixed \(\mathbf{x} \in \mathcal{X}\). In order to prove complete continuity of \(\tilde{S}_4(\cdot, \mathbf{x})\) for all \(\mathbf{x} \in \mathcal{X}\), we consider a sequence \(\bar{\mathbf{x}}_k \subseteq \mathcal{X}\) such that \(\bar{\mathbf{x}}_k \to \bar{\mathbf{x}}\) in \(\mathcal{X}\) and \(\bar{\mathbf{x}} = (\bar{\mathbf{u}}, \bar{\mu}_0, \bar{\varphi}) \in \mathcal{X}\). Then
\[
\langle \tilde{S}_4(\bar{\mathbf{x}}_k, \mathbf{x} - \bar{\mathbf{x}}_k), \bar{\mathbf{x}} - \bar{\mathbf{x}}_k \rangle_{\mathcal{X}} = \int_\Omega (m(\bar{\varphi}_k)\nabla \mu_0 - m(\bar{\varphi})\nabla \mu_0) \cdot \nabla \mathbf{v} d\mathbf{x}
\]
\[
\leq \|m(\bar{\varphi}_k)\nabla \mu_0 - m(\bar{\varphi})\nabla \mu_0\| \|\mathbf{v}\|_{W^{1, \infty}}.
\]
We know that \(\bar{\varphi}_k \to \bar{\varphi}\) in \(L^2_0(\Omega)\) and \(m\) is a continuous function, we have to show that \(\|m(\bar{\varphi}_k)\nabla \mu_0 - m(\bar{\varphi})\nabla \mu_0\| \to 0\). Let us define
\[
H(\bar{\varphi})(x) := g(x, \bar{\varphi}(x)) = m(\bar{\varphi}(x))\nabla \mu_0(x).
\]
Then Lemma 1.19, \([3.2]\) yields that \(H : L^2_0(\Omega) \to L^2(\Omega)\) is continuous and bounded. Since \(\bar{\varphi}_k \to \bar{\varphi}\) in \(L^2_0(\Omega)\), this implies that \(\|m(\bar{\varphi}_k)\nabla \mu_0 - m(\bar{\varphi})\nabla \mu_0\| \to 0\) as \(k \to \infty\) and hence \(\tilde{S}_4(\cdot, \mathbf{x})\) is completely continuous for all \(\mathbf{x} \in \mathcal{X}\). Using Lemma \(3.7\) we get that the operator \(S_4(\mathbf{x}) = \tilde{S}_4(\mathbf{x}, \mathbf{x})\) is pseudomonotone. A similar proof follows for \(S_2\), since it is of the same form as \(S_4\) and \(\nu(\cdot)\) is a continuously differentiable function.

Since each of the operators \(S_i\)'s are bounded, the operator \(S\) is also bounded. Since pseudomonotonicity and locally boundedess implies demicontinuity (see Lemma 2.4, \([35]\) or Proposition 27.7, \([39]\), \(S\) is demicontinuous.

(4) Observe that \(\mathcal{Y}\) is unbounded. Choose \(\mathbf{x}_0 = (0, 0, 0) \in \mathcal{Y} \cap D(T)\). Then we have to show that there is \(R > 0\) such that
\[
\langle \mathcal{S}(\mathbf{x}) - \mathbf{b}, \mathbf{x} \rangle_{\mathcal{X}} > 0, \text{ for all } \mathbf{x} \in \mathcal{Y} \text{ with } \|\mathbf{x}\|_{\mathcal{X}} > R.
\]
For $x = (u, \mu_0, \varphi) \in \mathbb{Y}$, let us consider
\[
\langle x', \mathbf{S}(x) - \mathbf{b}, x \rangle_x = \int_{\Omega} 2\nu(\varphi)\mathbf{Du} \cdot \mathbf{D}u + \int_{\Omega} m(\varphi)\nabla \mu_0 \cdot \nabla \mu_0 dx - \int_{\Omega} \mu_0 \varphi dx
- \int_{\Omega} P_0(\kappa \varphi) \varphi dx - \int_{\Omega} \mathbf{h} \cdot \mathbf{u} dx
= I_1 + I_2 + I_3 + I_4 + I_5.
\]
(3.25)

Since $|\varphi(x)| \leq \max(|a|, |b|)$, and $\nu(\cdot)$ is a continuously differentiable function, we have
\[
I_1 = \int_{\Omega} 2\nu(\varphi)\mathbf{Du} \cdot \mathbf{D}u \geq 2 \int_{\Omega} \nu(\varphi)|\nabla u|^2 dx \geq \tilde{C}_1 \|u\|_{V_{\text{div}}}^2.
\]
Now, since mean value of $\mu_0$ is 0 and $\mu(\cdot)$ is a continuous function, we also have
\[
I_2 = \int_{\Omega} m(\varphi)\nabla \mu_0 \cdot \nabla \mu_0 dx = \int_{\Omega} m(\varphi)|\nabla \mu_0|^2 dx \geq \tilde{C}_2 \|\mu_0\|_{H^1_0(\Omega)}^2.
\]
Using Hölder’s, Poincaré and Young’s inequalities, we have
\[
I_3 = \int_{\Omega} \mu_0 \varphi dx \leq \max(|a|, |b|) \int_{\Omega} |\mu_0| dx \leq \max(|a|, |b|) |\Omega|^{1/2} \|\mu_0\|_{L^2} \leq \tilde{C}_3 \|\mu_0\|_{H^1_0(\Omega)},
I_4 = \int_{\Omega} P_0(\kappa \varphi) \varphi dx = \kappa \|\varphi\|^2 \leq \tilde{C}_4,
I_5 = \int_{\Omega} \mathbf{h} \cdot \mathbf{u} dx \leq \|\mathbf{h}\|_{V_{\text{div}}'} \|\mathbf{u}\|_{V_{\text{div}}} \leq \frac{1}{2\tilde{C}_1} \|\mathbf{h}\|_{V_{\text{div}}'}^2 + \frac{\tilde{C}_1}{2} \|\mathbf{u}\|_{V_{\text{div}}}^2,
\]
for some constants $\tilde{C}_1, \tilde{C}_2, \tilde{C}_3, \tilde{C}_4 > 0$. Combining the above inequalities and substituting in (3.25) yields
\[
\langle x', \mathbf{S}(x) - \mathbf{b}, x \rangle_x \geq \frac{\tilde{C}_1}{2} \|\mathbf{u}\|_{V_{\text{div}}}^2 + \|\mu_0\|_{H^1_0(\Omega)} \left( \tilde{C}_2 \|\mu_0\|_{H^1_0(\Omega)} - \tilde{C}_3 \right) - \tilde{C}_4 - \frac{1}{2\tilde{C}_1} \|\mathbf{h}\|_{V_{\text{div}}'}^2 
=: g(\|\mu_0\|_{H^1_0(\Omega)})
\]
for $x = (u, \mu_0, \varphi) \in \mathbb{Y}$. Since $\mathbf{h} \in G$ is fixed, we can choose a constant $M > 0$ large enough such that
\[
\tilde{C}_4 + \frac{1}{2\tilde{C}_1} \|\mathbf{h}\|_{V_{\text{div}}'}^2 \leq M.
\]
Furthermore, it can be easily seen that
\[
\lim_{x \to +\infty} g(x) = +\infty.
\]
Since $M$ is chosen and $g(\cdot)$ grows to infinity, we can choose $R > 0$ such that
\[
\langle x', \mathbf{S}(x) - \mathbf{b}, x \rangle_x > 0 \quad \text{for all} \quad x = (u, \mu_0, \varphi) \in \mathbb{Y} \quad \text{such that} \quad \|x\|_x > R.
\]
Hence using Theorem 3.10, there exists $(u, \mu_0, \varphi) \in \mathbb{Y} \cap D(T)$ such that $\mathbf{b} \in T(u, \mu_0, \varphi) + \mathbf{S}(u, \mu_0, \varphi)$. This completes the proof.

**Theorem 3.19.** Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$ be a bounded subset. Let $\mathbf{h} \in V_{\text{div}}'$ and $k \in (-1, 1)$, then under the Assumptions (2.6), (2.8), there exists a weak solution $(u, \varphi)$ to the system (2.1) such that
\[
u \in V_{\text{div}} \quad \text{and} \quad \varphi \in H^1(\Omega) \cap L^2_k(\Omega)
\]
and satisfies the weak formulation (3.1), (3.3).
Proof. Lemma 3.18 gives the existence of \((u, \mu_0, \varphi) \in Y \cap D(T)\) such that \(b \in T(u, \mu_0, \varphi) + S(u, \mu_0, \varphi)\). From Lemma 3.17 we get that \((u, \mu_0, \varphi)\) satisfies the reformulated problem (3.1)-(3.6). Hence from Lemma 3.2 we know that \((u, \mu_0, \varphi)\) satisfies the weak formulation (3.1)-(3.3), which completes the proof. □

4. Uniqueness of the Weak Solution

In this section, we prove that the weak solution to the system (2.1) is unique, assuming that the viscosity coefficient \(\nu\) and the mobility parameter \(m\) are positive constants. Recall that our existence result ensures that \(\varphi(\cdot) \in [a, b]\), a.e. (see (3.15)). Then, we have the following uniqueness theorem:

**Theorem 4.1.** Let \((u_i, \varphi_i) \in V_{\text{div}} \times (H^1(\Omega) \cap L^2_{\text{loc}}(\Omega))\) for \(i = 1, 2\) be two weak solutions of the following system with chemical potentials \(\mu_i, i = 1, 2:\)

\[
\begin{aligned}
\mathbf{u} \cdot \nabla \varphi &= m \Delta \mu, \quad \text{in } \Omega, \\
\mu &= a \varphi - J \ast \varphi + F'(\varphi) \\
-\nu \Delta u + (\mathbf{u} \cdot \nabla)u + \nabla \pi &= \mu \nabla \varphi + h, \quad \text{in } \Omega, \\
\text{div } u &= 0, \quad \text{in } \Omega, \\
\frac{\partial \mu}{\partial n} &= 0, \quad u = 0, \quad \text{on } \partial \Omega,
\end{aligned}
\]

where \(h \in V'_{\text{div}}\). For \(n = 2\), we have \(u_1 = u_2\) and \(\varphi_1 = \varphi_2\), provided

\[
(i) \quad \nu^2 > \frac{2\sqrt{2}}{\sqrt{\lambda_1}} \|h\|_{V'_{\text{div}}},
\]

\[
(ii) \quad (\nu m)^2 \left(\frac{C_0}{2} - ||J||_{L^1}\right) > \nu m \left(\frac{CM^2}{2\lambda_1}\right) + \frac{2CC^2}{C_0} \left(\frac{2}{\lambda_1}\right)^{\frac{1}{2}} \|h\|_{V'_{\text{div}}}^2.
\]

Similarly, for \(n = 3\), we have \(u_1 = u_2\) and \(\varphi_1 = \varphi_2\), provided

\[
(i) \quad \nu^2 > \left(\frac{16}{\sqrt{\lambda_1}}\right)^{\frac{1}{2}} \|h\|_{V'_{\text{div}}},
\]

\[
(ii) \quad (\nu m)^2 \left(\frac{C_0}{2} - ||J||_{L^1}\right) > \nu m \left(\frac{CM^2}{2\lambda_1}\right) + \frac{2CC^2}{C_0} \left(\frac{4}{\sqrt{\lambda_1}}\right)^{\frac{1}{2}} \|h\|_{V'_{\text{div}}}^2.
\]

where \(M = \max\{a, b\}\) and \(\bar{C}\) is a generic constant.

**Proof.** Let us first find a simple bound for the velocity field. We multiply the first equation in (4.1) with \(\mu\) and third equation with \(u\) to obtain

\[
(u \cdot \nabla \varphi, \mu) = -m \|\nabla \mu\|^2
\]

and

\[
\nu \|\nabla u\|^2 = (\mu \nabla \varphi, u) + \langle h, u \rangle,
\]

where \(\langle \cdot, \cdot \rangle\) denotes the duality pairing between \(V'_{\text{div}}\) and \(V_{\text{div}}\). Adding (4.2) and (4.3), we get

\[
m \|\nabla \mu\|^2 + \nu \|\nabla u\|^2 = \langle h, u \rangle,
\]

where we used the fact that \((u \cdot \nabla \varphi, \mu) = (\mu \nabla \varphi, u)\). From (4.4), we infer that

\[
\nu \|\nabla u\|^2 \leq m \|\nabla \mu\|^2 + \nu \|\nabla u\|^2 = \langle h, u \rangle \leq \|h\|_{V'_{\text{div}}} \|u\|_{V_{\text{div}}}.
\]

Finally, we have

\[
\nu \|\nabla u\| \leq \|h\|_{V'_{\text{div}}}.
\]
Let \((u_1, \varphi_1)\) and \((u_2, \varphi_2)\) be two weak solutions of the system \((4.1)\). Note that the averages of \(\varphi_1\) and \(\varphi_2\) are same and equal to \(k\), which gives \(\overline{\varphi_1} - \overline{\varphi_2} = 0\). We can rewrite the third equation in \((4.1)\) as

\[- \nu \Delta u + (u \cdot \nabla) u + \nabla \overline{u} = - \nabla \frac{\varphi^2}{2} - (J * \varphi) \nabla \varphi, \quad (4.6)\]

where \(\overline{u} := \overline{\varphi} = \pi - \left(F(\varphi) + a \frac{\varphi^2}{2}\right)\). Let us define \(u^e := u_1 - u_2\), \(\varphi^e := \varphi_1 - \varphi_2\) and \(\overline{u} := \overline{\varphi_1} - \overline{\varphi_2}\). Then \((u^e, \varphi^e)\) satisfies the following system:

\[
\begin{cases}
\quad u^e \cdot \nabla \varphi^e + u_2 \cdot \nabla \varphi^e = m \Delta \mu^e, \quad \text{in } \Omega, \\
\quad \mu^e = a \varphi^e - J * \varphi^e + F'(\varphi_1) - F'(\varphi_2), \quad \text{in } \Omega, \\
\quad - \nu \Delta u^e + (u_1 \cdot \nabla) u^e + (u^e \cdot \nabla) u_2 + \nabla \overline{u} = - \varphi^e (\varphi_1 + \varphi_2) \frac{\nabla a}{2} - (J * \varphi^e) \nabla \varphi_2 - (J * \varphi_1) \nabla \varphi^e, \\
\quad \text{in } \Omega, \\
\quad \text{div } u^e = 0, \quad \text{in } \Omega, \\
\quad \frac{\partial \mu^e}{\partial n} = 0, \quad u^e = 0, \quad \text{on } \partial \Omega, \\
\quad \frac{1}{|\Omega|} \int_{\Omega} \varphi^e(x) dx = 0.
\end{cases} \quad (4.7)\]

Now we have the following weak formulation:

\[(u^e \cdot \nabla \varphi_1, \psi) + (u_2 \cdot \nabla \varphi^e, \psi) = m \langle \Delta \mu^e, \psi \rangle,\]

\[
\nu (\nabla u^e, \nabla v) + b(u_1, u^e, v) + b(u^e, u_2, v) = - (\varphi^e(\varphi_1 + \varphi_2) \frac{\nabla a}{2}, v) - ((J * \varphi^e) \nabla \varphi_2, v) - ((J * \varphi_1) \nabla \varphi^e, v),
\]

for every \(v \in V_{div}\) and \(\psi \in V\). Let us choose \(v = u^e\) and \(\psi = B^{-1} \varphi^e\), then we get

\[
(u^e \cdot \nabla \varphi_1, B^{-1} \varphi^e) + (u_2 \cdot \nabla \varphi^e, B^{-1} \varphi^e) = m \langle \Delta \mu^e, B^{-1} \varphi^e \rangle, \quad (4.8)
\]

\[
\nu \|\nabla u^e\|^2 = -b(u_1, u^e, u^e) - \frac{1}{2} (\nabla a \varphi^e(\varphi_1 + \varphi_2), u^e) - ((J * \varphi^e) \nabla \varphi_2, u^e) - ((J * \varphi_1) \nabla \varphi^e, u^e),
\]

where we used the fact that \(b(u_1, u^e, u^e) = 0\). Using \((2.4)\), Taylor’s series expansion and Assumption \((2.6) (2)\), we estimate the term \((\nabla \mu^e, \nabla B^{-1} \varphi^e)\) from \((4.8)\) as

\[
- \langle \Delta \mu^e, B^{-1} \varphi^e \rangle = - \langle \mu^e, \varphi^e \rangle = - \langle \mu^e, \varphi^e \rangle
\]

\[
= - (a \varphi^e - J * \varphi^e + F'(\varphi_1) - F'(\varphi_2), \varphi^e)
\]

\[
= - ((a + F''(\varphi_1 + \theta \varphi_2)) \varphi^e, \varphi^e) + (J * \varphi^e, \varphi^e)
\]

\[
\leq - C_0 \|\varphi^e\|^2 + (J * \varphi^e, \varphi^e).
\]

Using \((4.10)\) in \((4.8)\), we obtain

\[
mC_0 \|\varphi^e\|^2 \leq - (u^e \cdot \nabla \varphi_1, B^{-1} \varphi^e) - (u_2 \cdot \nabla \varphi^e, B^{-1} \varphi^e) + m(J * \varphi^e, \varphi^e).
\]

Let us now estimate the terms in the right hand side of \((4.9)\) and \((4.10)\) one by one. We use the Hölder’s, Ladyzhenskaya, Young’s and Poincaré’s inequalities to estimate \(|b(u^e, u_2, u^e)|\). For \(n = 2\), we get

\[
|b(u^e, u_2, u^e)| \leq \|u^e\|_{L^4}^2 \|\nabla u_2\| \leq \sqrt{2} \|u^e\| \|\nabla u^e\| \|\nabla u_2\| \leq \left(\frac{2}{3}\right)^\frac{1}{2} \|\nabla u^e\| \|\nabla u_2\|,
\]

\[
(4.12)
\]
and for \( n = 3 \), we have

\[
|b(u^e, u_2, u^e)| \leq \|u^e\|_{L^2}^2 \|\nabla u^e\| \leq 2\|u^e\|^2 \|\nabla u^e\|^2 \|\nabla u_2\| \leq \left( \frac{4}{\sqrt{\lambda_1}} \right)^2 \|\nabla u^e\|_2 \|\nabla u_2\|. \tag{4.13}
\]

Since \((u_i, \varphi_i)\) are weak solutions of (4.11) for \( i = 1, 2 \), such that \( \varphi_i \in [a, b] \), a.e., there exists constant \( M \) such that

\[
|\varphi_i(x)| \leq \max\{|a|, |b|\} = M. \tag{4.14}
\]

Using the Hölder, Ladyzhenskaya and Young’s inequalities and (4.14), we obtain

\[
\left| \frac{1}{2} (\nabla a \varphi^e(\varphi_1 + \varphi_2), u^e) \right| \leq \frac{1}{2} \|\nabla a\|_{L^\infty} \|\varphi^e\| \|\nabla (\varphi_1 + \varphi_2)\| \|u^e\|
\leq \frac{1}{2} \|\nabla a\|_{L^\infty} \|\varphi^e\| (\|\varphi_1\|_{L^\infty} + \|\varphi_2\|_{L^\infty}) \|u^e\|
\leq \frac{mC_0}{8} \|\varphi^e\|^2 + \frac{2M^2}{mC_0} \|\nabla a\|^2_{L^\infty} \|u^e\|^2. \tag{4.15}
\]

In order to estimate \(((J * \varphi^e)\nabla \varphi_2, u^e)\) and \(((J * \varphi_1)\nabla \varphi^e, u^e)\), we write these terms using an integration by parts and the divergence free condition as

\[
((J * \varphi^e)\nabla \varphi_2, u^e) = -((\nabla J * \varphi^e)\varphi_2, u^e),
((J * \varphi_1)\nabla \varphi^e, u^e) = -((\nabla J * \varphi_1)\varphi^e, u^e).
\]

Using Hölder’s and Ladyzhenskaya inequalities, and Young’s inequality for convolution, we estimate \(((\nabla J * \varphi^e)\varphi_2, u^e)\) as

\[
|((\nabla J * \varphi^e)\varphi_2, u^e)| \leq \|\nabla J * \varphi^e\|_{L^2} \|\varphi_2\|_{L^\infty} \|u^e\|
\leq M\|\nabla J\|_{L^1} \|\varphi^e\| \|u^e\|
\leq \frac{mC_0}{8} \|\varphi^e\|^2 + \frac{2M^2}{mC_0} \|\nabla J\|^2_{L^1} \|u^e\|^2. \tag{4.16}
\]

Similarly, we obtain

\[
|((\nabla J * \varphi_1)\varphi^e, u^e)| \leq \frac{mC_0}{8} \|\varphi^e\|^2 + \frac{2M^2}{mC_0} \|\nabla J\|^2_{L^1} \|u^e\|^2. \tag{4.17}
\]

Substituting (4.12), (4.15)-(4.17) in (4.9), then using (4.5) and the fact that \( \|\nabla a\|_{L^\infty} \leq \|\nabla J\|_{L^1} \), for \( n = 2 \), we obtain

\[
\nu \|\nabla u^e\|^2 \leq \frac{3mC_0}{8} \|\varphi^e\|^2 + \frac{1}{\nu} \left( \frac{2}{\lambda_1} \right)^\frac{1}{2} \|h\|_{\mathcal{V}_{div}} \|\nabla u^e\|^2 + \frac{6M^2}{mC_0} \|\nabla J\|^2_{L^1} \|u^e\|^2. \tag{4.18a}
\]

Combining (4.13), (4.15)-(4.17) and substituting it in (4.9), for \( n = 3 \), we get

\[
\nu \|\nabla u^e\|^2 \leq \frac{3mC_0}{8} \|\varphi^e\|^2 + \frac{1}{\nu} \left( \frac{4}{\sqrt{\lambda_1}} \right)^\frac{1}{2} \|h\|_{\mathcal{V}_{div}} \|\nabla u^e\|^2 + \frac{6M^2}{mC_0} \|\nabla J\|^2_{L^1} \|u^e\|^2. \tag{4.18b}
\]

Now we estimate the terms in the right hand side of (4.11). To estimate \((u^e \cdot \nabla \varphi_1, B^{-1} \varphi^e)\), we use an integration by parts, \( u^e|_{\partial \Omega} = 0 \) and the divergence free condition of \( u^e \) to obtain

\[
(u^e \cdot \nabla \varphi_1, B^{-1} \varphi^e) = -(u^e \cdot \nabla B^{-1} \varphi^e, \varphi_1).
\]

Using Hölder’s, Ladyzhenskaya, Poincaré and Young’s inequalities, we estimate the above term as

\[
|((u^e \cdot \nabla B^{-1} \varphi^e, \varphi_1)| \leq \|u^e\| \|\nabla B^{-1} \varphi^e\| \|\varphi_1\|_{L^\infty}
\leq M \|u^e\| \|\nabla B^{-1} \varphi^e\|.
\]
Combining (4.19), (4.20), (4.21a) and (4.22) and substituting it in (4.11), for \( n = 2 \), we infer
\[
\frac{m C_0}{8} \| \varphi^e \|^2 \leq \frac{M^2}{2 \nu \lambda_1} \| B^{-1/2} \varphi^e \|^2 + \frac{2 \tilde{C} \tilde{C}_2}{m C_0} \| \varphi^e \|^2 \left( \frac{4}{\nu^2 \sqrt{\lambda_1}} \right)^{\frac{1}{2}} \| h \|^2_{\| h \|^2_{H^1}^2

(4.23a)
\]
Combining (4.19), (4.20), (4.21b) and (4.22) and substituting it in (4.11), for \( n = 3 \), we find
\[
\frac{m C_0}{8} \| \varphi^e \|^2 \leq \frac{M^2}{2 \nu \lambda_1} \| B^{-1/2} \varphi^e \|^2 + \frac{2 \tilde{C} \tilde{C}_2}{m C_0} \| \varphi^e \|^2 \left( \frac{4}{\nu^2 \sqrt{\lambda_1}} \right)^{\frac{1}{2}} \| h \|^2_{\| h \|^2_{H^1}^2}
\]

(4.23b)
Adding \((4.18a)\) and \((4.23a)\), for \(n = 2\), we obtain
\[
\left[\frac{\nu}{2} - \frac{1}{\nu} \left(\frac{2}{\lambda_1}\right)^\frac{1}{2} \parallel h\parallel_{\text{div}}\right] \parallel \nabla u^e\parallel^2 + \frac{6M^2}{mC_0} \parallel \nabla J\parallel_{L^1}^2 \parallel u^e\parallel^2 + \left[\frac{mC_0}{2} - m\parallel J\parallel_{L^1}\right] \parallel \varphi^e\parallel^2 \\
\leq \frac{2C\tilde{C}^2}{\nu^2mC_0} \left(\frac{2}{\lambda_1}\right)^\frac{1}{2} \parallel h\parallel_{\text{div}}^2 \parallel \varphi^e\parallel^2 + \frac{M^2}{2\nu\lambda_1} \parallel B^{-1/2} \varphi^e\parallel^2 \tag{4.24a}
\]

Combining \((4.18b)\) and \((4.23b)\), for \(n = 3\), we get
\[
\left(\frac{\nu}{2} - \frac{1}{\nu} \left(\frac{4}{\sqrt{\lambda_1}}\right)^\frac{1}{2} \parallel h\parallel_{\text{div}}\right) \parallel \nabla u^e\parallel^2 + \frac{6M^2}{mC_0} \parallel \nabla J\parallel_{L^1}^2 \parallel u^e\parallel^2 + \left[\frac{mC_0}{2} - m\parallel J\parallel_{L^1}\right] \parallel \varphi^e\parallel^2 \\
\leq \frac{M^2}{2\nu\lambda_1} \parallel B^{-1/2} \varphi^e\parallel^2 + \frac{2C\tilde{C}^2}{\nu^2mC_0} \left(\frac{4}{\sqrt{\lambda_1}}\right)^\frac{1}{2} \parallel h\parallel_{\text{div}}^2 \parallel \varphi^e\parallel^2 \tag{4.24b}
\]

Now using the continuous embedding of \(L^2(\Omega) \hookrightarrow V_0\), i.e. \(\parallel B^{-1/2} \varphi^e\parallel^2 \leq C \parallel \varphi^e\parallel\), we further obtain
\[
\left[\frac{\nu}{2} - \frac{1}{\nu} \left(\frac{2}{\lambda_1}\right)^\frac{1}{2} \parallel h\parallel_{\text{div}}\right] \parallel \nabla u^e\parallel^2 + \left[\frac{mC_0}{2} - m\parallel J\parallel_{L^1}\right] - \frac{C\tilde{C}^2}{2\nu\lambda_1} - \frac{2C\tilde{C}^2}{\nu^2mC_0} \left(\frac{2}{\lambda_1}\right)^\frac{1}{2} \parallel h\parallel_{\text{div}}^2 \parallel \varphi^e\parallel^2 \leq 0. \tag{4.25}
\]

From \((4.25)\), uniqueness for \(n = 2\) follows provided quantities in both brackets in above inequality are strictly positive. Thus we conclude that for
\[
\begin{align*}
(i) & \quad \nu^2 > \frac{2\sqrt{\frac{7}{\sqrt{\lambda_1}}}}{\parallel h\parallel_{\text{div}}} , \\
(ii) & \quad (\nu m)^2 \left(\frac{C_0}{2} - \parallel J\parallel_{L^1}\right) > \nu m \left(\frac{C\tilde{C}^2}{2\nu\lambda_1} + \frac{2C\tilde{C}^2}{\nu^2mC_0} \left(\frac{2}{\lambda_1}\right)^\frac{1}{2} \parallel h\parallel_{\text{div}}^2\right) ,
\end{align*}
\]
uniqueness follows in two dimensions. Similarly for \(n = 3\), we obtain
\[
\left(\frac{\nu}{2} - \frac{1}{\nu} \left(\frac{4}{\sqrt{\lambda_1}}\right)^\frac{1}{2} \parallel h\parallel_{\text{div}}\right) \parallel \nabla u^e\parallel^2 + \left[\frac{mC_0}{2} - m\parallel J\parallel_{L^1}\right] - \frac{C\tilde{C}^2}{2\nu\lambda_1} - \frac{2C\tilde{C}^2}{\nu^2mC_0} \left(\frac{4}{\sqrt{\lambda_1}}\right)^\frac{1}{2} \parallel h\parallel_{\text{div}}^2 \parallel \varphi^e\parallel^2 \leq 0.
\]
Hence, the uniqueness follows provided
\[
\begin{align*}
(i) & \quad \nu^2 > \left(\frac{16}{\sqrt{\lambda_1}}\right)^\frac{1}{2} \parallel h\parallel_{\text{div}} , \\
(ii) & \quad (\nu m)^2 \left(\frac{C_0}{2} - \parallel J\parallel_{L^1}\right) > \nu m \left(\frac{C\tilde{C}^2}{2\nu\lambda_1} + \frac{4C\tilde{C}^2}{\nu^2mC_0} \left(\frac{4}{\sqrt{\lambda_1}}\right)^\frac{1}{2} \parallel h\parallel_{\text{div}}^2\right) ,
\end{align*}
\]
which completes the proof. \(\square\)

5. Regularity of the Weak Solution

In this section, we establish the regularity results for the weak solution to the system \((4.1)\). Let \((u, \varphi, \mu) \in V_{\text{div}} \times (H^1(\Omega) \cap L^2_{(k)}(\Omega)) \times H^1(\Omega)\) and \(\varphi(x) \in (a, b)\) a.e. be the unique weak solution of the system \((4.1)\). Let us now establish the higher order regularity results for the system \((4.1)\).

Theorem 5.1 (2-D regularity). If \(J \in W^{2,1}(\mathbb{R}^2; \mathbb{R})\), \(F'''(\varphi) \in L^q(\Omega)\) for some \(2 \leq q \leq \infty\) and \(h \in \mathcal{G}_{\text{div}}\), then for \(n = 2\), the weak solution \((u, \varphi)\) of the system \((4.1)\) has the following regularity:
\[
\varphi \in H^2(\Omega) \quad \text{and} \quad u \in H^2(\Omega) , \tag{5.1}
\]
that is, \((\mathbf{u}, \varphi)\) is a strong solution.

**Proof.** By multiplying first equation in (4.1) by \(\mu\) and third by \(\mathbf{u}\), calculations similar to (4.5) yields

\[
m\|\nabla\mu\|^2 + \frac{\nu}{2}\|\nabla\mathbf{u}\|^2 \leq \frac{1}{2\nu}\|\mathbf{h}\|_{\text{div}}^2 < +\infty. \tag{5.2}
\]

Using the Poincaré’s inequality, we know that \(\sqrt{\lambda_1}\|\mathbf{u}\| \leq \|\nabla\mathbf{u}\|\), where \(\lambda_1\) is the first eigen value of the stokes operator, we also get \(\|\mathbf{u}\| < +\infty\). Thus by using Ladyzhenskaya inequality \((2.9)\), we obtain \(\|\mathbf{u}\|_{L^1} \leq C^{1/4}\|\mathbf{u}\|^{1-\frac{2}{p}}\|\nabla\mathbf{u}\|^\frac{2}{p} < +\infty\), where \(C = 2, 4\) for \(n = 2, 3\), respectively. Let us now multiply the first equation in (4.1) by \(\varphi\) to obtain

\[
m(\Delta \mu, \varphi) = (\mathbf{u} \cdot \nabla \varphi, \varphi) = 0, \tag{5.3}
\]

since an integration by parts and the divergence free condition yields

\[
(\mathbf{u} \cdot \nabla \varphi, \varphi) = \sum_{i=1}^{n} \int_{\Omega} \mathbf{u}_i \frac{\partial \varphi}{\partial x_i} \varphi dx = \frac{1}{2} \sum_{i=1}^{n} \int_{\Omega} \mathbf{u}_i \frac{\partial (\varphi^2)}{\partial x_i} dx = -\frac{1}{2} \sum_{i=1}^{n} \int_{\Omega} \frac{\partial \mathbf{u}_i}{\partial x_i} (\varphi^2) dx = 0.
\]

Now we consider

\[
-(\Delta \mu, \varphi) = (\nabla \mu, \nabla \varphi) = (\nabla (a\varphi - J * \varphi + F'(\varphi)), \nabla \varphi) = (a\nabla \varphi + \nabla a\varphi - J * \varphi + F''(\varphi)\nabla \varphi, \nabla \varphi) \geq C_0\|\nabla \varphi\|^2 + (\nabla a\varphi - J * \varphi, \nabla \varphi) \geq C_0\|\nabla \varphi\|^2 - \|\nabla a\|_{L^\infty}\|\varphi\|\|\nabla \varphi\| - \|\nabla J\|_{L^1}\|\varphi\|\|\nabla \varphi\| \geq \frac{C_0}{2}\|\nabla \varphi\|^2 - \frac{2}{C_0}\|\nabla J\|_{L^1}\|\varphi\|^2, \tag{5.4}
\]

where we used the Assumption 2.6 (2), Hölder’s inequality, Young’s inequality for convolution and Young’s inequality. Using (5.3) in (5.4) to get

\[
\|\nabla \varphi\| \leq \frac{2}{C_0}\|\nabla J\|_{L^1}\|\varphi\| < +\infty. \tag{5.5}
\]

if \(\|\varphi\| < +\infty\), which is true since \((\mathbf{u}, \varphi)\) is a weak solution of the system (4.1). Moreover, by the Poincaré-Wirtinger inequality, we have

\[
\|\varphi - \varphi^e\| \leq C_\Omega\|\nabla \varphi\|,
\]

where \(C_\Omega\) is the Poincaré-Wirtinger constant.

Using the Gagliardo-Nirenberg interpolation inequality \((2.13)\), for \(2 \leq p < \infty\), we also infer that

\[
\|\varphi\|_{L^p} \leq C\left(\|\nabla \varphi\|^{1-\frac{2}{p}}\|\varphi\|^\frac{2}{p} + \|\varphi\|\right) < +\infty, \tag{5.6}
\]

and since \(\mu \in H^1(\Omega)\), we have

\[
\|\mu\|_{L^p} \leq C\left(\|\nabla \mu\|^{1-\frac{2}{p}}\|\mu\|^\frac{2}{p} + \|\mu\|\right) < \infty, \tag{5.7}
\]

where \(C\) is the constant appearing in the Gagliardo-Nirenberg inequality.

In order to get the higher order estimates, we now consider the first equation of (4.1). Using Hölder’s inequality, we have

\[
m\|\Delta \mu\| \leq \|\mathbf{u} \cdot \nabla \varphi\| \leq \|\mathbf{u}\|_{L^4}\|\nabla \varphi\|_{L^4}. \tag{5.8}
\]

Now, we control \(\|\nabla \varphi\|_{L^p}\) in terms of \(\|\nabla \mu\|_{L^p}\), for every \(2 \leq p < \infty\). Let us take the gradient of \(\mu = a\varphi - J * \varphi + F'(\varphi)\), multiply it by \(\nabla \varphi|\nabla \varphi|^{p-2}\), integrate the resulting identity over \(\Omega\), and use
the Assumption \( \text{(2.6)} \) (2), Hölder’s and Young’s inequalities and Young’s inequality for convolution to obtain

\[
\int_{\Omega} \nabla \varphi |\nabla \varphi|^{p-2} \cdot \nabla \mu dx = \int_{\Omega} \nabla \varphi |\nabla \varphi|^{p-2} \cdot (a \nabla \varphi + \nabla a \varphi - \nabla J * \varphi + F''(\varphi) \nabla \varphi) dx \\
= \int_{\Omega} (a + F''(\varphi)) |\nabla \varphi|^{p} dx + \int_{\Omega} \nabla \varphi |\nabla \varphi|^2 \cdot (\nabla a \varphi - \nabla J * \varphi) dx \\
\geq C_0 \int_{\Omega} |\nabla \varphi|^p dx - (\|\nabla a\|_{L^\infty} + \|\nabla J\|_{L^1}) \|\varphi\|_{L^p} \|\nabla \varphi\|_{L^{p-1}}^p \\
\geq C_0 \frac{2}{\Omega} \|\nabla \varphi\|_{L^p}^{p} - \frac{1}{p} \left( \frac{2(p-1)}{C_0p} \right)^{p-1} (\|\nabla a\|_{L^\infty} + \|\nabla J\|_{L^1})^p \|\varphi\|_{L^p}^p.
\]

(5.9)

Using Young’s inequality, we also have

\[
\left| \int_{\Omega} \nabla \varphi |\nabla \varphi|^{p-2} \cdot \nabla \mu dx \right| \leq \int_{\Omega} |\nabla \varphi|^{p-1} |\nabla \mu| dx \\
\leq C_0 \frac{4}{\Omega} \|\nabla \varphi\|_{L^p}^{p} + \frac{1}{p} \left( \frac{4(p-1)}{C_0p} \right)^{p-1} \|\nabla \mu\|_{L^p}^{p}.
\]

(5.10)

Combining (5.9) and (5.10), we get

\[
\frac{C_0}{4} \|\nabla \varphi\|_{L^p}^{p} \leq \frac{1}{p} \left( \frac{2(p-1)}{C_0p} \right)^{p-1} \left[ 2^{p-1} \|\nabla \mu\|_{L^p}^{p} + (\|\nabla a\|_{L^\infty} + \|\nabla J\|_{L^1}) \|\varphi\|_{L^p}^{p} \right],
\]

(5.11)

for \( 2 \leq p < \infty \). An application of the Gagliardo-Nirenberg interpolation inequality (see \( \text{(2.14)} \)) yields

\[
\|\nabla \mu\|_{L^p} \leq C \|\nabla \mu\|_{H^{1/2}}^{1-\frac{2}{p}} \|\nabla \mu\|_{L^2}^{\frac{2}{p}} \leq C \|\nabla \mu\|_{H^{1/2}}^{1-\frac{2}{p}} \|\nabla \mu\|_{L^2}^{\frac{2}{p}} = C - \Delta \mu + \mu \left| \frac{1}{2} - \frac{2}{p} \right| \|\nabla \mu\|_{L^2}^{\frac{2}{p}} \\
\leq 2^{1-\frac{2}{p}} C \left( \|\Delta \mu\|_{H^{1/2}}^{1-\frac{2}{p}} + \|\mu\|_{H^{1/2}}^{1-\frac{2}{p}} \right) \|\nabla \mu\|_{L^2}^{\frac{2}{p}}.
\]

(5.12)

Combining (5.11) and (5.12), it is immediate that

\[
\|\nabla \varphi\|_{L^p}^{p} \leq \frac{4}{pC_0} \left( \frac{2(p-1)}{C_0p} \right)^{p-1} \left[ 2^{p-1} \left( 2^{1-\frac{2}{p}} C \left( \|\Delta \mu\|_{H^{1/2}}^{1-\frac{2}{p}} + \|\mu\|_{H^{1/2}}^{1-\frac{2}{p}} \right) \|\nabla \mu\|_{L^2}^{\frac{2}{p}} \right)^p \\
+ (\|\nabla a\|_{L^\infty} + \|\nabla J\|_{L^1}) \|\varphi\|_{L^p}^{p} \right] \\
\leq \frac{4}{pC_0} \left( \frac{2(p-1)}{C_0p} \right)^{p-1} \left[ 2^{2p-4} C^p \left( \|\Delta \mu\|_{H^{1/2}}^{p-2} \|\nabla \mu\|_{L^2}^2 + \|\mu\|_{p-2}^2 \|\nabla \mu\|_{L^2}^2 \right) \\
+ (\|\nabla a\|_{L^\infty} + \|\nabla J\|_{L^1}) \|\varphi\|_{L^p}^{p} \right].
\]

(5.13)

For \( p = 4 \), let us use \( \text{(5.13)} \) in (5.8) to obtain

\[
m^4 \|\Delta \mu\|_{L^4}^4 \leq \frac{27}{8C_0^4} \|u\|_{L^4}^4 \left[ 2^8 C^4 \|\Delta \mu\|_{L^2}^2 \|\nabla \mu\|_{L^2}^2 + 2^8 C^4 \|\mu\|_{L^2}^2 \|\nabla \mu\|_{L^2}^2 \\
+ (\|\nabla a\|_{L^\infty} + \|\nabla J\|_{L^1}) \|\varphi\|_{L^4}^4 \right] \\
\leq \frac{m^4}{2} \|\Delta \mu\|_{L^4}^4 + \frac{4}{9m^4} \left( \frac{3C}{C_0} \right)^8 \|u\|_{L^4}^8 \|\nabla \mu\|_{L^2}^4 + \frac{2}{3} \left( \frac{6C}{C_0} \right)^4 \|u\|_{L^4}^4 \|\mu\|_{L^2}^2 \|\nabla \mu\|_{L^2}^2 \\
+ \frac{27}{8C_0^4} \|u\|_{L^4}^4 (\|\nabla a\|_{L^\infty} + \|\nabla J\|_{L^1}) \|\varphi\|_{L^4}^4.
\]

(5.14)
Thus, we have
\[
\frac{m^4}{2} \|\Delta \mu\|^4 \leq \frac{4}{9m^4} \left( \frac{3C}{C_0} \right)^8 \| \mathbf{u} \|_{L^4}^8 \| \nabla \mu \|^4 + \frac{2}{3} \left( \frac{6C}{C_0} \right)^4 \| \mathbf{u} \|_{L^4}^4 \| \mu \|^2 \| \nabla \mu \|^2 \\
+ \frac{27}{8C_0} \| \mathbf{u} \|_{L^4}^4 (\| \nabla \mathbf{a} \|_{L^\infty} + \| \nabla \mathbf{J} \|_{L^4})^4 \| \varphi \|^4_{L^4} < +\infty, \tag{5.15}
\]
using (5.2), (5.6) and (5.7). Hence, for all \(2 \leq p < \infty\), it is immediate from (5.13) that
\[
\| \nabla \varphi \|_{L^p} < +\infty \tag{5.16}
\]
Finally, we need to prove that \(\| \Delta \varphi \| < +\infty\). In order to prove that, we consider
\[
(\Delta \mu, \Delta \varphi) = (\Delta (a \varphi - J * \varphi + F'(\varphi)), \Delta \varphi) \\
= (a \Delta \varphi + 2 \nabla a \nabla \varphi + \Delta a \varphi - \Delta J * \varphi + \Delta \varphi, \Delta \varphi) \\
\geq C_0 \| \Delta \varphi \|^2 + 2 \| \nabla \varphi \|_{L^\infty} \| \nabla \varphi \|_{L^2} \| \Delta \varphi \| \\
+ (F''(\varphi)(\nabla \varphi)^2, \Delta \varphi) \\
= : C_0 \| \Delta \varphi \|^2 + \sum_{i=1}^{4} I_i, \tag{5.17}
\]
where we used the Assumption (2.6) (2). Let us now estimate each \(I_i\) for \(i = 1, \ldots, 4\) using Cauchy-Schwarz, Hölder and Young’s inequalities as
\[
|I_1| \leq 2 \| \nabla \mathbf{a} \|_{L^\infty} \| \Delta \varphi \| \| \Delta \varphi \| \leq 2 \| \nabla \mathbf{a} \|_{L^\infty} \| \nabla \varphi \| \| \Delta \varphi \| \\
\leq \frac{C_0}{8} \| \Delta \varphi \|^2 + \frac{8}{C_0} \| \nabla \mathbf{a} \|_{L^\infty} \| \nabla \varphi \|^2, \tag{5.18}
\]
\[
|I_2| \leq \| \Delta J * \varphi \| \| \Delta \varphi \| \leq \| \Delta J \|_{L^4} \| \varphi \| \| \Delta \varphi \| \\
\leq \frac{C_0}{8} \| \Delta \varphi \|^2 + \frac{2}{C_0} \| \Delta J \|^2_{L^4} \| \varphi \|^2, \tag{5.19}
\]
\[
|I_3| \leq \| \Delta a \varphi \| \| \Delta \varphi \| \leq \| \Delta a \|_{L^\infty} \| \varphi \| \| \Delta \varphi \| \\
\leq \frac{C_0}{8} \| \Delta \varphi \|^2 + \frac{2}{C_0} \| \Delta a \|_{L^\infty} \| \varphi \|^2, \tag{5.20}
\]
\[
|I_4| \leq \| F''(\varphi)(\nabla \varphi)^2 \| \| \Delta \varphi \| \leq \| F''(\varphi) \|_{L^2} \| \nabla \varphi \|^2_{L^4} \| \Delta \varphi \| \\
\leq \frac{C_0}{8} \| \Delta \varphi \|^2 + \frac{2}{C_0} \| F''(\varphi) \|^2_{L^2} \| \nabla \varphi \|^4_{L^4}, \tag{5.21}
\]
where \(p > 4\) and \(\frac{2}{p} + \frac{1}{q} = \frac{1}{2}\). Combining (5.18), (5.21) and substituting it in (5.17) to obtain
\[
(\Delta \mu, \Delta \varphi) \geq \frac{C_0}{2} \| \Delta \varphi \|^2 - \frac{4}{C_0} (2 \| \nabla \mathbf{a} \|_{L^\infty} + \| \Delta J \|^2_{L^4}) \| \nabla \varphi \|^2 - \frac{2}{C_0} \| F''(\varphi) \|^2_{L^2} \| \nabla \varphi \|^4_{L^4}. \tag{5.22}
\]
But we also know that
\[
| (\Delta \mu, \Delta \varphi) | \leq \frac{C_0}{4} \| \Delta \varphi \|^2 + \frac{1}{C_0} \| \Delta \mu \|^2. \tag{5.23}
\]
Combining (5.22) and (5.23), it is immediate that
\[
\| \Delta \varphi \|^2 \leq \frac{4}{C_0} \| \Delta \mu \|^2 - \frac{16}{C_0} (2 \| \nabla \mathbf{a} \|_{L^\infty} + \| \Delta J \|^2_{L^4}) \| \nabla \varphi \|^2 \\
+ \frac{8}{C_0} \| F''(\varphi) \|^2_{L^2} \| \nabla \varphi \|^4_{L^4} < +\infty, \tag{5.24}
\]
since \(J \in W^{2,1}(\mathbb{R}^2; \mathbb{R})\), and using (5.5), (5.6), (5.15) and (5.16).
Let us now take inner product with $\Delta u$ in the third equation in (4.11) to obtain
\[ \nu \|\Delta u\|^2 = \langle (u \cdot \nabla)u, \Delta u \rangle - (\mu \nabla \varphi, \Delta u) - (h, \Delta u), \]  
(5.25)
where we used the divergence free condition to get rid of the pressure term. We estimate the first term from the right hand side of the equality (5.25) using Cauchy-Schwarz, Hölder, Gagliardo-Nirenberg and Young’s inequalities to get
\[ \|((u \cdot \nabla)u, \Delta u)\| \leq \|((u \cdot \nabla)u)\|\|\Delta u\| \leq \|u\|_{L^4} \|\nabla u\|_{L^4} \|\Delta u\| \leq C \|u\|_{L^4} \left( \|\Delta u\|^{3/4}\|u\|^{1/4} + \|u\| \right) \|\Delta u\| \]
\[ \leq \frac{\nu}{8} \|\Delta u\|^2 + \frac{1}{8} \left(\frac{7}{\nu}\right)^7 C^8 \|u\|^8 \|\nabla u\|^2 + \frac{\nu}{8} \|\Delta u\|^2 + \frac{2}{\nu} \|\nabla \varphi\|^2 \|u\|^2 \|
\leq \frac{\nu}{4} \|\Delta u\|^2 + \frac{1}{\nu} \|\varphi\|^2 \|\nabla \varphi\|^2 \|u\|^2, \]  
(5.26)
where $C$ is the constant appearing in Gagliardo-Nirenberg interpolation inequality (2.15). Let us use the Cauchy-Schwarz, Hölder’s and Young’s inequalities to estimate the term $|\langle \mu \nabla \varphi, \Delta u \rangle|$ as
\[ \|\mu \nabla \varphi, \Delta u\| \leq \|\mu\|_{L^4} \|\nabla \varphi\| \|\Delta u\| \]
\[ \leq \frac{\nu}{4} \|\Delta u\|^2 + \frac{1}{\nu} \|\mu\|^2 \|\nabla \varphi\|^2 \|u\|^2. \]  
(5.27)
Finally, we estimate $\|(h, \Delta u)\|$ as
\[ \|(h, \Delta u)\| \leq \|h\|\|\Delta u\| \leq \frac{\nu}{4} \|\Delta u\|^2 + \frac{1}{\nu} \|h\|^2. \]  
(5.28)
Combining (5.26), (5.27) and (5.28) and using it in (5.25) to get
\[ \|\Delta u\|^2 \leq \frac{8C^2}{\nu^2} \left(\frac{7}{16} \frac{7C}{\nu}\right)^6 \|u\|^6 \|\Delta u\|^2 + \frac{4}{\nu^2} \|\mu\|^2 \|\nabla \varphi\|^2 \|u\|^2 + \frac{4}{\nu^2} \|h\|^2 < +\infty, \]  
(5.29)
Hence, from (5.29) and (5.24) we get the required regularity given in (5.1). Note that we have got all above estiamtes without using $\varphi \in [a, b]$, a.e. Thus we get the higher regularity results in the two dimensional case without assuming $L^\infty$ bound on $\varphi$. □

**Remark 5.2 (3D-Regularity).** For $n = 3$ case, note that (5.6) is valid only for $2 \leq p \leq 6$ i.e.,
\[ \|\varphi\|_{L^p} \leq C \left( \|\nabla \varphi\|^{\left(\frac{1-\frac{2}{p}}{2}\right)} \|\varphi\|^{\frac{6}{2p}} + \|\varphi\| \right) < +\infty, \]  
(5.30)
and
\[ \|\mu\|_{L^p} \leq C \left( \|\nabla \mu\|^{\left(\frac{1-\frac{2}{p}}{2}\right)} \|\mu\|^{\frac{6}{2p}} + \|\mu\| \right) < +\infty, \]  
(5.31)
Now in case of $n = 3$, using Gagliardo-Nirenberg inequality (2.14), for $2 \leq p \leq 6$, we get
\[ \|\nabla \mu\|_{L^p} \leq C \|\Delta \mu\|^{\left(\frac{1-\frac{2}{p}}{2}\right)} \|\Delta \mu\|^{\frac{6}{2p}} \leq C \|\mu\|^{\left(\frac{1-\frac{2}{p}}{2}\right)} \|\nabla \mu\|^{\frac{6}{2p}} = C|\Delta \mu + \mu|^{\left(\frac{1-\frac{2}{p}}{2}\right)} \|\nabla \mu\|^{\frac{6}{2p}}, \]
\[ \leq 2^{\left(\frac{1-\frac{2}{p}}{2}\right)} C \left( \|\Delta \mu\|^{\left(\frac{1-\frac{2}{p}}{2}\right)} + \|\mu\|^{\left(\frac{1-\frac{2}{p}}{2}\right)} \right) \|\nabla \mu\|^{\frac{6}{2p}}. \]  
(5.32)
Combining (5.11) and (5.32)
\[ \frac{C_0}{4} \|\nabla \varphi\|_{L^p}^p \leq \frac{1}{p} \left( \frac{2(p-1)}{C_0 p} \right)^{p-1} \left[ 2^{p-1} \left( C^2 \left(\frac{1-\frac{2}{p}}{2}\right) \|\Delta \mu\|^{\left(\frac{1-\frac{2}{p}}{2}\right)} + \|\mu\|^{\left(\frac{1-\frac{2}{p}}{2}\right)} \right) \|\nabla \mu\|^{\frac{6}{2p}} \right]^p. \]
This gives for $2 \leq p \leq 6$,

$$
\|\nabla \varphi\|_{L^p} < +\infty.
$$

(5.34)

Indeed from (3.15), we know that $\varphi \in [a, b]$ a.e., and hence (5.30) and (5.34) holds true for $2 \leq p \leq \infty$. This easily implies that (5.31) is true for all $r \geq 2$. Using (5.33) (for $p = 4$) in (5.8), we get

$$
m^4 \|\Delta \mu\|^4 \leq \frac{27}{8C_0^4} \left\| \frac{\|\nabla a\|_{L^4}}{C_0} \right\|^4 \left[ 2^9 C^4 \|\Delta \mu\|^3 \|\nabla \mu\| + \|\mu\|^2 \|\nabla \mu\| \right] + \left( \|\nabla a\|_{L^\infty} + \|\nabla J\|_{L^4} \right)^4 \|\varphi\|^4_{L^4},
$$

and

$$
\|\|\nabla a\|_{L^\infty} + \|\nabla J\|_{L^4}\|^4 \|\varphi\|^4_{L^4}.
$$

(5.35)

Then we have

$$
m^4 \|\Delta \mu\|^4 \leq \frac{8}{3m^{12}} \left( \frac{3C}{C_0} \right)^{16} \|\nabla a\|^4_{L^4} \|\nabla \mu\|^4 + \frac{4}{3} \left( \frac{3C}{C_0} \right)^{12} \|\nabla a\|^4_{L^4} \|\mu\|^2 \|\nabla \mu\| + \frac{27}{8C_0^4} \left( \|\nabla a\|_{L^\infty} + \|\nabla J\|_{L^4} \right)^4 \|\varphi\|^4_{L^4} < \infty.
$$

As in the two dimensional case, for $n = 3$, (5.21) is valid for $4 < p \leq 6$ and

$$
|I_4| \leq \frac{C_0}{8} \|\Delta \varphi\|^2 + \frac{2}{C_0} \|F''(\varphi)\|^2_{L^4} \|\nabla \varphi\|^4_{L^4}.
$$

(5.36)

Using (5.18), (5.20), (5.36) in (5.17) and making use of

$$
|\langle \Delta \mu, \Delta \varphi \rangle| \leq \frac{C_0}{4} \|\Delta \varphi\|^2 + \frac{1}{C_0} \|\Delta \mu\|^2,
$$

we get

$$
\|\Delta \varphi\|^2 \leq \frac{4}{C_0} \|\Delta \mu\|^2 + \frac{16}{C_0} \frac{1}{2} \|\Delta a\|_{L^\infty}^2 + \frac{8}{C_0} \|F''(\varphi)\|^2_{L^4} \|\nabla \varphi\|^4_{L^4} < +\infty.
$$

(5.37)

Finally, the estimate in (5.26) becomes

$$
\|(u \cdot \nabla)u, \Delta u\| \leq \|(u \cdot \nabla)u\| \|\Delta u\| \leq \|u\|_{L^1} \|\nabla u\|_{L^4} \|\Delta u\|
$$

$$
\leq C\|u\|_{L^1} \left( \|\Delta u\|^2 + \frac{16}{C_0} \frac{1}{2} \|\Delta a\|^2 + \frac{2}{C_0} \|F''(\varphi)\|^2_{L^4} \|\nabla \varphi\|^4_{L^4} \right) \|\Delta u\|.
$$

(5.38)
Using (5.38), (5.27) and (5.28), for \( n = 3 \) we find
\[
\| \Delta u \|^2 \leq \frac{4}{\nu} \left( 16C \left( \frac{15C}{2\nu} \right)^{15} \| u \|_{L^4}^4 + \frac{2}{\nu} C^2 \right) \| u \|_{L^4}^2 \| u \|^2 + \frac{4}{\nu^2} \| \mu \|_{L^4}^2 \| \nabla \varphi \|_{L^4}^2 + \frac{4}{\nu^2} \| h \|^2 < +\infty.
\]

Hence, from above inequality and (5.37) we can conclude the required regularity given in (5.1).

**Remark 5.3.** If \( \varphi = 0 \), then from (5.5), we have
\[
\| \nabla \varphi \| \leq \frac{2}{C_0} \| \nabla J \|_{L^1} \| \varphi \| \leq \frac{2C_0}{C_0} \| \varphi \|_{L^1} \| \nabla \varphi \|,
\]
where we used the Poincaré-Wirtinger inequality. That is, we have
\[
\left( 1 - \frac{2C_0}{C_0} \| \nabla J \|_{L^1} \right) \| \nabla \varphi \| \leq 0.
\]
If \( \| \nabla J \|_{L^1} < \frac{C_0}{2C_0} \), then we get \( \| \nabla \varphi \| = 0 \) and hence \( \varphi = C \), a constant. Since \( \varphi = 0 \), we have \( \varphi \equiv 0 \). In this case (4.1) reduces to the stationary Navier-Stokes equations, if \( F'(0) = 0 \).

### 6. Exponential Stability

The stability analysis of nonlinear dynamical systems has a long history starting from the works of Lyapunov. For the solutions of ordinary or partial differential equations describing dynamical systems, different kinds of stability may be described. One of the most important types of stability is that concerning the stability of solutions near to a point of equilibrium (stationary solutions). In the qualitative theory of ordinary and partial differential equations, and control theory, Lyapunov’s notion of (global) asymptotic stability of an equilibrium is a key concept. It is important to note that the asymptotic stability do not quantify the rate of convergence. In fact, there is a strong form of stability which demands an exponential rate of convergence. The notion of exponential stability is far stronger and it assures a minimum rate of decay, that is, an estimate of how fast the solutions converge to its equilibrium. In particular, exponential stability implies uniform asymptotic stability. Stability analysis of fluid dynamic models has been one of the essential areas of applied mathematics with a good number of applications in engineering and physics (see [37, 6]). In this section, we consider the singular potential \( F \) to be singular in \((-1,1)\). For example,
\[
F(\varphi) = \frac{\theta}{2} ((1 + \varphi) \log(1 + \varphi) + (1 - \varphi) \log(1 - \varphi)) - \frac{\theta}{2} \varphi^2, \quad \varphi \in (-1,1).
\]

Then for such potentials, we prove that the stationary solution \((u^e, \varphi^e)\) of the system (2.1) with constant mobility parameter \( m \) and coefficient of kinematic viscosity \( \nu \) is exponentially stable in 2-D. That is, our aim is to establish that:

- there exists constants \( M > 0 \) and \( \alpha > 0 \) such that
\[
\| u(t) - u^e \|^2 + \| \varphi(t) - \varphi^e \|^2 \leq M e^{-\alpha t},
\]
    for all \( t \geq 0 \).
6.1. Global solvability of two dimensional CHNS system. We consider the following initial and boundary value problem:

\[
\begin{align*}
\phi_t + u \cdot \nabla \phi &= m \Delta \mu, \quad \text{in } \Omega \times (0, T), \\
\mu &= a \phi - J * \phi + F'(\phi), \\
u \Delta u + (u \cdot \nabla)u + \nabla \pi &= \mu \nabla \phi + h, \quad \text{in } \Omega \times (0, T), \\
\nabla \cdot u &= 0, \quad \text{in } \Omega \times (0, T), \\
\frac{\partial \mu}{\partial n} &= 0, \quad u = 0, \quad \text{on } \partial \Omega \times (0, T), \\
u(t) &= u_0, \quad \phi(0) = \phi_0, \quad \text{in } \Omega.
\end{align*}
\]  
\tag{6.1}

Let us now give the global solvability results available in the literature for the system (6.1). We first give the definition of weak solution for the system (6.1).

**Definition 6.1 (weak solution).** Let \( u_0 \in G_{\text{div}}, \phi_0 \in H \) with \( F(\phi_0) \in L^1(\Omega) \) and \( 0 < T < \infty \) be given. Then \( (u, \phi) \) is a weak solution to the system (6.1) on \([0, T]\) corresponding to the initial conditions \( u_0 \) and \( \phi_0 \) if

(i) \( u, \phi \) and \( \mu \) satisfy

\[
\begin{align*}
u(t) &= u_0, \quad \phi(0) = \phi_0, \quad \text{in } \Omega.
\end{align*}
\]

and

\[
\phi \in L^\infty(Q), \quad |\phi(x, t)| < 1 \quad \text{a.e. } (x, t) \in Q := \Omega \times (0, T);
\]

(ii) For every \( \psi \in V \), every \( v \in V_{\text{div}} \) and for almost any \( t \in (0, T) \), we have

\[
\langle \phi_t, \psi \rangle + (\nabla \phi, \nabla \psi) = \int_\Omega (u \cdot \nabla \psi) \phi \, dx,
\]

\[
\langle u_t, v \rangle + \nu(\nabla u, \nabla v) + b(u, u, v) = -\int_\Omega (v \cdot \nabla \mu) \phi \, dx + \langle h, v \rangle.
\]

(iii) Moreover, the following initial conditions hold in the weak sense

\[
u(t) = u_0, \quad \phi(0) = \phi_0, \quad \text{in } \Omega.
\]

\[
\phi \in L^\infty(Q), \quad |\phi(x, t)| < 1 \quad \text{a.e. } (x, t) \in Q := \Omega \times (0, T);
\]

Next, we discuss the existence and uniqueness of weak solution results available in the literature for the system (6.1).

**Theorem 6.2 (Existence, Theorem 1 [18]).** Let the Assumption 2.6 be satisfied for some fixed positive integer \( q \). Let \( u_0 \in G_{\text{div}}, \phi_0 \in L^\infty(\Omega) \) such that \( F(\phi_0) \in L^1(\Omega) \) and \( h \in L^2_{\text{loc}}([0, \infty), V_{\text{div}}') \). In addition, assume that \( |\phi_0| < 1 \). Then, for every given \( T > 0 \), there exists a weak solution \( (u, \phi) \) to the equation (6.1) such that \( \phi(t) = \phi_0 \) for all \( t \in [0, T] \) and

\[
\phi \in L^\infty(0, T; L^{2+2q}(\Omega))
\]

Furthermore, setting

\[
\mathcal{E}(u(t), \phi(t)) = \frac{1}{2} \|u(t)\|^2 + \frac{1}{4} \int_\Omega \int_\Omega J(x - y)(\phi(x, t) - \phi(y, t))^2 \, dx \, dy + \int_\Omega F(\phi(t)) \, dx,
\]

\[
\mathcal{E}(u(t), \phi(t)) = \frac{1}{2} \|u(t)\|^2 + \frac{1}{4} \int_\Omega \int_\Omega J(x - y)(\phi(x, t) - \phi(y, t))^2 \, dx \, dy + \int_\Omega F(\phi(t)) \, dx,
\]

\[
\mathcal{E}(u(t), \phi(t)) = \frac{1}{2} \|u(t)\|^2 + \frac{1}{4} \int_\Omega \int_\Omega J(x - y)(\phi(x, t) - \phi(y, t))^2 \, dx \, dy + \int_\Omega F(\phi(t)) \, dx,
\]

\[
\mathcal{E}(u(t), \phi(t)) = \frac{1}{2} \|u(t)\|^2 + \frac{1}{4} \int_\Omega \int_\Omega J(x - y)(\phi(x, t) - \phi(y, t))^2 \, dx \, dy + \int_\Omega F(\phi(t)) \, dx,
\]
the following energy estimate holds
\[ E(u(t), \varphi(t)) + \int_s^t \left( 2\|\sqrt{\nu(\varphi)} Du(s)\|^2 + \|\nabla \mu(s)\|^2 \right) ds \leq E(u(s), \varphi(s)) + \int_s^t (h(s), u(s)) ds, \]
for all \( t \geq s \) and for a.s. \( s \in [0, \infty) \). If \( d = 2 \), the weak solution \((u, \varphi)\) satisfies the following energy identity,
\[ \frac{d}{dt} E(u(t), \varphi(t)) + 2\|\sqrt{\nu(\varphi)} Du(s)\|^2 + \|\nabla \mu(t)\|^2 = (h(t), u(t)). \]
i.e., equality in (6.7) holds for every \( t \geq 0 \).

**Remark 6.3.** We denote by \( Q \), a continuous monotone increasing function with respect to each of its arguments. As a consequence of energy inequality (6.7), we have the following bound:
\[ \|u\|_{L^\infty(0,T; \mathcal{G}_{div}) \cap L^2(0,T; \mathcal{V}_{div})} + \|\varphi\|_{L^\infty(0,T; H)} + \|F(\varphi)\|_{L^\infty(0,T; H)} \leq Q \left( E(u_0, \varphi_0), \|h\|_{L^2(0,T; \mathcal{V}_{div})} \right), \]
where \( Q \) also depends on \( F, J, \nu \) and \( \Omega \).

**Theorem 6.4** (Uniqueness, Theorem 3, [17]). Let \( d = 2 \). Suppose that the Assumption [2, 7] is satisfied. Let \( u_0 \in \mathcal{G}_{div}, \varphi_0 \in L^\infty(\Omega) \) with \( F(\varphi_0) \in L^1(\Omega), |\varphi_0| < 1 \) and \( h \in L^2_{loc}(0, \infty; \mathcal{V}'_{div}) \). Then, the weak solution \((u, \varphi)\) corresponding to \((u_0, \varphi_0)\) and given by Theorem 6.3 is unique. Furthermore, for \( i = 1, 2 \), let \( z_i := (u_i, \varphi_i) \) be two weak solutions corresponding to two initial data \( z_{0i} := (u_{0i}, \varphi_{0i}) \) and external forces \( h_i \), with \( u_{0i} \in \mathcal{G}_{div}, \varphi_{0i} \in L^\infty(\Omega) \) such that \( F(\varphi_{0i}) \in L^1(\Omega), |\varphi_{0i}| < \eta \) for some constant \( \eta \in (0, 1) \) and \( h_i \in L^2_{loc}(0, \infty; \mathcal{V}'_{div}) \). Then the following continuous dependence estimate holds:
\[ \|u_2(t) - u_1(t)\|^2 + \|\varphi_2(t) - \varphi_1(t)\|^2 \leq \left( \left\| \frac{C_0}{2}\|\varphi_2(\tau) - \varphi_1(\tau)\|^2 + \frac{\nu}{4}\|\nabla (u_2(\tau) - u_1(\tau))\|^2 \right\| \right) d\tau \]
\[ \leq \left( \left\| u_2(0) - u_1(0)\|^2 + \|\varphi_2(0) - \varphi_1(0)\|^2 \right\| \right) \Lambda_0(t) + \|\varphi_2(0) - \varphi_1(0)\| Q \left( E(z_{01}), E(z_{02}), \|h_1\|_{L^2(0,T; \mathcal{V}_{div})}, \|h_2\|_{L^2(0,T; \mathcal{V}_{div})} \right) \Lambda_1(t) \]
\[ + \|h_2 - h_1\|^2 \Lambda_2(t), \]
for all \( t \in [0, T] \), where \( \Lambda_0(t), \Lambda_1(t) \) and \( \Lambda_2(t) \) are continuous functions which depend on the norms of the two solutions. The functions \( \Lambda_i(t) \) also depend on \( F, J, \Omega \), and \( Q \) depending on \( F, J, \Omega \) and \( \eta \).

**Remark 6.5.** The above theorems also imply \( u \in C([0, T]; \mathcal{G}_{div}) \) and \( \varphi \in C([0, T]; H) \), for all \( T > 0 \). Therefore, the initial conditions \( u(0) = u_0 \) and \( \varphi(0) = \varphi_0 \) make sense.

### 6.2. Exponential stability of the stationary solution
Let us now prove that the stationary solution of (6.1) with \( \mu = \mu^2 \) on \( \partial \Omega \) is exponentially stable in two dimensions. Let \((u^e, \varphi^e)\) be the steady-state solution of the system (4.1). From Theorem 3.19 we know that there exists a weak solution for the system (4.1). Remember that if \( J \in W^{2,1}(\mathbb{R}^2; \mathbb{R}), F''(\varphi) \in L^2(\Omega) \) for \( 2 \leq q \leq \infty \) and \( h \in \mathcal{G}_{div} \), then the weak solution \((u^e, \varphi^e)\) of the system (4.1) has the following regularity:
\[ \varphi^e \in H^2(\Omega) \] \[ u^e \in \mathbb{H}^2(\Omega), \]
that is, \((u^e, \varphi^e)\) is a strong solution. To prove exponential stability, furthermore we assume that \( F''(\varphi) \in L^2(\Omega) \).
Theorem 6.6. Let \( u_0 \in \mathcal{V}_{div}, \varphi_0 \in V \cap L^\infty(\Omega) \) and \( h \in \mathbb{G}_{div} \). For
\[
(i) \quad \nu^2 > \frac{4}{\lambda_1} \|
abla u^e\|^2;
\]
\[
(ii) \quad m(C_0 - \| J \|_{L^1})^2 > C_{1\Omega}^2 \left( \frac{1}{m} \| u^e \|^2_{L^\infty} + \frac{4}{\nu \sqrt{\lambda_1}} \| \nabla \mu^e \|^2_{L^2} \right),
\]
the stationary solution \((u^e, \varphi^e)\) of (4.11), with the regularity given in Theorem 5.1, is exponentially stable. That is, there exists constants \( M > 0 \) and \( \rho > 0 \) such that
\[
\| u(t) - u^e \|^2 + \| \varphi(t) - \varphi^e \|^2 \leq M e^{-\rho t},
\]
for all \( t \geq 0 \), where \((u, \varphi)\) is a solution of the system (6.11) with \( \mu = \mu^e \) on \( \partial \Omega \),
\[
M = \frac{\| y_0 \|^2 + 2 \| J \|_{L^1} \| \psi_0 \|^2 + \| F'(\varphi^e + \theta \psi_0) \| \| \psi_0 \|_{L^\infty} \| \psi_0 \|}{\min\{(C_0 - \| J \|_{L^1}), 1\}}
\]
and
\[
\rho = \min\left\{ \left( \lambda_1 \nu - \frac{4}{\nu} \| \nabla u^e \|^2 \right), \left[ \frac{m}{C_{1\Omega}^2} - \frac{1}{(C_0 - \| J \|_{L^1})^2} \left( \frac{1}{m} \| u^e \|^2_{L^\infty} + \frac{4}{\nu \sqrt{\lambda_1}} \| \nabla \mu^e \|^2_{L^2} \right) \right] \right\} > 0. \tag{6.9}
\]

Proof. Let us define \( y := u - u^e, \varphi := \varphi - \varphi^e, \tilde{\mu} = \mu - \mu^e \) and \( \tilde{\pi} := \pi - \pi^e \). Then we know that \((y, \psi)\) satisfies the following system:
\[
\begin{aligned}
\psi_t + y \cdot \nabla \psi + y \cdot \nabla \varphi^e + u^e \cdot \nabla \psi &= m \Delta \tilde{\mu}, \quad \text{in} \quad \Omega \times (0, T), \\
\tilde{\mu} &= a \varphi - J \ast \psi + F'(\psi + \varphi^e) - F'(\varphi^e), \\
y_t - \nu \Delta y + (y \cdot \nabla) y + (y \cdot \nabla) u^e + (u^e \cdot \nabla) y + \nabla \tilde{\pi} = \tilde{\mu} \nabla \psi + \tilde{\mu} \nabla \varphi^e + \mu^e \nabla \psi, \\
\text{in} \quad \Omega \times (0, T),
\end{aligned}
\tag{6.10}
\]
\[
\begin{gathered}
\text{div } y = 0, \quad \text{in} \quad \Omega \times (0, T), \\
\partial \tilde{\mu} / \partial n = 0, \quad y = 0, \quad \text{on} \quad \partial \Omega \times (0, T), \\
y(0) = y_0, \quad \psi(0) = \psi_0, \quad \text{in} \quad \Omega.
\end{gathered}
\]

Now consider third equation of (6.10) and take inner product with \( y(\cdot) \) to obtain
\[
\frac{1}{2} \frac{d}{dt} \| y(t) \|^2 + \nu \| \nabla y(t) \|^2 + b(y(t), u^e, y(t))
= (\tilde{\mu}(t) \nabla \psi(t), y(t)) + (\tilde{\mu}(t) \nabla \varphi^e, y(t)) + (\mu^e \nabla \psi(t), y(t)), \tag{6.11}
\]
where we used the fact that \( b(y, y, y) = b(u^e, y, y) = 0 \) and \( (\nabla \tilde{\pi}, y) = (\tilde{\pi}, \nabla \cdot y) = 0 \). An integration by parts yields
\[
(\tilde{\mu} \nabla \psi, y) = -(\psi \nabla \tilde{\mu}, y) - (\psi \tilde{\mu}, \nabla \cdot y) = -(\psi \nabla \tilde{\mu}, y),
\]
where we use the boundary data and divergence free condition of \( y \). Similarly, we have \((\tilde{\mu} \nabla \varphi^e, y) = -(\varphi^e \nabla \tilde{\mu}, y) \) and \((\mu^e \nabla \psi, y) = -(\psi \nabla \mu^e, y) \). Thus from (6.11), we have
\[
\frac{1}{2} \frac{d}{dt} \| y(t) \|^2 + \nu \| \nabla y(t) \|^2 + b(y(t), u^e, y(t))
= -(\psi(t) \nabla \tilde{\mu}(t), y(t)) - (\varphi^e \nabla \tilde{\mu}(t), y(t)) - (\psi(t) \nabla \mu^e, y(t)). \tag{6.12}
\]
Taking inner product of the third equation in (6.10) with \( \tilde{\mu}(\cdot) \), we obtain
\[
(\psi(t), \tilde{\mu}(t)) + m \| \nabla \tilde{\mu}(t) \|^2 = -(y(t) \cdot \nabla \psi(t), \tilde{\mu}(t)) - (y(t) \cdot \nabla \varphi^e, \tilde{\mu}(t)) - (u^e \cdot \nabla \psi(t), \tilde{\mu}(t)). \tag{6.13}
\]
Using an integration by parts, divergence free condition and boundary value of \( y \), we get
\[
(y \cdot \nabla \psi, \tilde{\mu}) = -(\psi \nabla \cdot y, \tilde{\mu}) - (\psi \nabla \tilde{\mu}, y) = -(\psi \nabla \tilde{\mu}, y).
\]
Similarly, we have \((y \cdot \nabla \varphi^e, \bar{\mu}) = -(\varphi^e \nabla \bar{\mu}, y)\) and \((u^e \cdot \nabla \psi, \bar{\mu}) = -(\psi \nabla \bar{\mu}, u^e)\). Thus, from (6.13), it is immediate that

\[
(\psi_t(t), \bar{\mu}(t)) + m\|\nabla \bar{\mu}(t)\|^2 = (\psi(t) \nabla \bar{\mu}(t), y(t)) + (\varphi^e \nabla \bar{\mu}(t), y(t)) + (\psi(t) \nabla \bar{\mu}(t), u^e).
\]  
(6.14)

Adding (6.12) and (6.14), we infer

\[
\frac{1}{2} \frac{d}{dt} \|y(t)\|^2 + \nu \|\nabla y(t)\|^2 + (\psi_t(t), \bar{\mu}(t)) + m\|\nabla \bar{\mu}(t)\|^2
= -b(y(t), u^e, y(t)) - (\psi(t) \nabla \mu^e, y(t)) + (\psi(t) \nabla \bar{\mu}(t), u^e).
\]  
(6.15)

We estimate the term \((\psi_t, \bar{\mu})\) from (6.15) as

\[
(\psi_t, \bar{\mu}) = (\psi_t, a \psi - J * \psi + F'(\psi + \varphi^e) - F'(\varphi^e))
\]  
(6.16)

\[
= \frac{d}{dt} \left\{ \frac{1}{2} \|\sqrt{a} \psi\|^2 - \frac{1}{2} \int_{\Omega} F(\psi + \varphi^e) dx - (F'(\varphi^e), \psi) \right\}
\]

\[
= \frac{d}{dt} \left\{ \frac{1}{2} \|\sqrt{a} \psi\|^2 - \frac{1}{2} \int_{\Omega} F(\psi + \varphi^e) dx - \int_{\Omega} F(\varphi^e) dx - \int_{\Omega} F'(\varphi^e) \psi dx \right\}.
\]  
(6.17)

Since \(\frac{d}{dt}(\int_{\Omega} F(\varphi^e) dx) = 0\). Using Taylor’s formula, we have

\[
\int_{\Omega} [F(\psi + \varphi^e) - F(\varphi^e)] dx = \frac{1}{2} \int_{\Omega} F''(\varphi^e + \theta \psi) \psi^2 dx,
\]

for some 0 < \theta < 1. Thus, from (6.16), we get

\[
(\psi_t, \bar{\mu}) = \frac{1}{2} \frac{d}{dt} \left\{ \|\sqrt{a} \psi\|^2 - (J * \psi, \psi) + \int_{\Omega} F''(\varphi^e + \theta \psi) \psi^2 dx \right\}
\]

\[
= \frac{1}{2} \frac{d}{dt} \left\{ \int (a + F''(\varphi^e + \theta \psi)) \psi^2 dx - (J * \psi, \psi) \right\}.
\]  
(6.18)

Note also that

\[
(\psi_t, \bar{\mu}) = \frac{d}{dt} \left\{ \frac{1}{4} \int_{\Omega} \int_{\Omega} J(x - y)(\psi(x) - \psi(y))^2 dx dy + \frac{1}{2} \int_{\Omega} F''(\varphi^e + \theta \psi) \psi^2 dx \right\}.
\]

Let us use (6.18) in (6.15) to get

\[
\frac{1}{2} \frac{d}{dt} \left\{ \|y\|^2 + \int_{\Omega} (a + F''(\varphi^e + \theta \psi)) \psi^2 dx - (J * \psi, \psi) \right\} + \nu \|\nabla y\|^2 + m\|\nabla \bar{\mu}\|^2
= -b(y, u^e, y) - (\psi \nabla \mu^e, y) + (\psi \nabla \bar{\mu}, u^e).
\]  
(6.19)

Using Hölder, Ladyzhenskaya and Young’s inequalities, we estimate \(b(y, u^e, y)\) as

\[
|b(y, u^e, y)| \leq \|\nabla u^e\| \|y\|_2 \leq \sqrt{2} \|\nabla u^e\| \|y\| \|\nabla y\| \leq \frac{\nu}{4} \|\nabla y\|^2 + \frac{2}{\nu} \|\nabla u^e\|^2 \|y\|^2.
\]  
(6.20)

We estimate the term \((\psi \nabla \mu^e, y)\) from (6.19) using Hölder, Ladyzhenskaya and Young’s inequalities as

\[
|(\psi \nabla \mu^e, y)| \leq \|\psi\| \|\nabla \mu^e\| \|y\|_2 \leq \sqrt{2} \|\psi\| \|\nabla \mu^e\| \|y\| \|\nabla y\| \leq \sqrt{2(\lambda_1)}^{1/4} \|\psi\| \|\nabla \mu^e\| \|\nabla y\| \leq \frac{\nu}{4} \|\nabla y\|^2 + \frac{2}{\nu} \|\nabla \mu^e\|^2 \|\psi\|^2.
\]  
(6.21)

Similarly, we estimate the term \((\psi \nabla \bar{\mu}, u^e)\) from (6.19) as

\[
|(\psi \nabla \bar{\mu}, u^e)| \leq \|\psi\| \|\nabla \bar{\mu}\| \|u^e\| \|\nabla y\| \leq \frac{m}{2} \|\nabla \bar{\mu}\|^2 + \frac{1}{2m} \|u^e\|^2 \|\psi\|^2.
\]  
(6.22)
Combining (6.20)–(6.22) and substituting it in (6.19), we obtain
\[
\frac{1}{2} \frac{d}{dt} \left\{ \|y\|^2 + \int_\Omega (a + F''(\varphi^e + \theta \psi)) \psi^2 dx - (J * \psi, \psi) \right\} + \frac{\nu}{2} \| \nabla y \|^2 + \frac{m}{2} \| \nabla \bar{\mu} \|^2 \\
\leq \left( \frac{1}{2m} \nu \| u^e \|^2_\infty + \frac{2}{\nu \sqrt{\lambda_1}} \| \nabla \mu^e \|^2 \| \nabla \psi \|^2 \right) \| \psi \|^2 + \frac{2}{\nu} \| \nabla u^e \|^2 \| y \|^2. \tag{6.23}
\]
Since \( \bar{\mu} = 0 \) on \( \partial \Omega \), from the above inequality, it is immediate that
\[
\frac{d}{dt} \left\{ \|y\|^2 + \int_\Omega (a + F''(\varphi^e + \theta \psi)) \psi^2 dx - (J * \psi, \psi) \right\} \\
+ \left( \nu \lambda_1 - \frac{4}{\nu} \| \nabla u^e \|^2 \right) \| y \|^2 + \frac{m}{C_0^2} \| \bar{\mu} \|^2 \\
\leq \left( \frac{1}{m} \nu \| u^e \|^2_\infty + \frac{4}{\nu \sqrt{\lambda_1}} \| \nabla \mu^e \|^2 \| \nabla \psi \|^2 \right) \| \psi \|^2, \tag{6.24}
\]
where we used the Poincaré and Poincaré-Wirtinger inequalities. Using Assumption [2.6] (2) and Young’s inequality for convolutions, we know that
\[
\int_\Omega (a + F''(\varphi^e + \theta \psi)) \psi^2 dx \geq C_0 \| \psi \|^2 \geq \| J \|_{L^1} \| \psi \|^2 \geq (J * \psi, \psi).
\]
Thus, we have
\[
\int_\Omega (a + F''(\varphi^e + \theta \psi)) \psi^2 dx - (J * \psi, \psi) \geq 0.
\]
Using Taylor’s series expansion and Assumption (2.6), we also obtain
\[
(\bar{\mu}, \psi) = (a \psi - (J * \psi) + F'(\psi + \varphi^e) - F'(\varphi^e), \psi) \\
= (a \psi + F''(\varphi^e + \theta \psi) \psi, \psi) - (J * \psi, \psi) \\
\geq (C_0 - \| J \|_{L^1}) \| \psi \|^2, \tag{6.25}
\]
for some \( 0 < \theta < 1 \). From the above relation, it is immediate that
\[
\| \psi \| \leq \frac{1}{(C_0 - \| J \|_{L^1})} \| \bar{\mu} \|, \tag{6.26}
\]
using Cauchy-Schwarz inequality. We use (6.26) in (6.24) to find
\[
\frac{d}{dt} \left\{ \|y\|^2 + \int_\Omega (a + F''(\varphi^e + \theta \psi)) \psi^2 dx - (J * \psi, \psi) \right\} \\
+ \left( \nu \lambda_1 - \frac{4}{\nu} \| \nabla u^e \|^2 \right) \| y \|^2 + \left[ \frac{m}{C_0^2} - \frac{1}{(C_0 - \| J \|_{L^1})^2} \left( \frac{1}{m} \nu \| u^e \|^2_\infty + \frac{4}{\nu \sqrt{\lambda_1}} \| \nabla \mu^e \|^2 \right) \right] \| \bar{\mu} \|^2 \\
\leq 0. \tag{6.27}
\]
But we also know that
\[
\int_\Omega (a + F''(\varphi^e + \theta \psi)) \psi^2 dx - (J * \psi, \psi) = (\bar{\mu}, \psi) \leq \| \bar{\mu} \| \| \psi \| \leq \frac{1}{(C_0 - \| J \|_{L^1})} \| \bar{\mu} \|^2,
\]
using Cauchy-Schwarz inequality and (6.26). Thus, from (6.27), with the assumption
\[
m(C_0 - \| J \|_{L^1}) > C_0^2 \left( \frac{1}{m} \nu \| u^e \|^2_\infty + \frac{4}{\nu \sqrt{\lambda_1}} \| \nabla \mu^e \|^2 \right),
\]
it is immediate that
\[
\frac{d}{dt} \left[ \|y\|^2 + (\bar{\mu}, \psi) \right] + \left( \nu \lambda_1 - \frac{4}{\nu} \| \nabla u^e \|^2 \right) \| y \|^2
\]
\[+ (C_0 - \|J\|_{L^1}) \left[ \frac{m}{C_Ω^2} - \frac{1}{(C_0 - \|J\|_{L^1})^2} \left( \frac{1}{m} \|u^e\|_{L^\infty}^2 + \frac{4}{\nu \sqrt{\lambda_1}} \|\nabla \mu^e\|_{L^1}^2 \right) \right] (\bar{\mu}, \psi) \leq 0. \]

Now for
\[\nu^2 > \frac{4}{\lambda_1} \|\nabla u^e\|^2 \text{ and } m(C_0 - \|J\|_{L^1})^2 > C_Ω^2 \left( \frac{1}{m} \|u^e\|_{L^\infty}^2 + \frac{4}{\nu \sqrt{\lambda_1}} \|\nabla \mu^e\|_{L^1}^2 \right),\]
we use the variation of constants formula (see Lemma A.1) to find
\[\|y(t)\|^2 + (\bar{\mu}(t), \psi(t)) \leq (\|y(0)\|^2 + (\bar{\mu}(0), \psi(0))) e^{-\rho t},\]
for all \(t \geq 0\), where
\[\rho = \min \left\{ \lambda_1 \nu - \frac{4}{\nu} \|\nabla u^e\|^2, \left[ \frac{m}{C_Ω^2} - \frac{1}{(C_0 - \|J\|_{L^1})^2} \left( \frac{1}{m} \|u^e\|_{L^\infty}^2 + \frac{4}{\nu \sqrt{\lambda_1}} \|\nabla \mu^e\|_{L^1}^2 \right) \right] \right\} > 0. \]

From (6.25), we also have \((C_0 - \|J\|_{L^1}) \|\psi\|^2 \leq (\bar{\mu}, \psi)\), so that from (6.30), we infer that
\[\|y(t)\|^2 + (C_0 - \|J\|_{L^1}) \|\psi\|^2 \leq (\|y(0)\|^2 + (\bar{\mu}(0), \psi(0))) e^{-\rho t}.\]

Using H"older’s inequality, Young’s inequality for convolutions and Assumption 2.6, we obtain
\[(\bar{\mu}(0), \psi(0)) = \int_Ω (a + F''(\varphi^e + \theta \psi_0)) \psi_0^2 dx - (J * \psi_0, \psi_0) \leq (\|a\|_{L^\infty} + \|J\|_{L^1}) \|\psi_0\|^2 + \|F''(\varphi^e + \theta \psi_0)\| \|\psi_0\|^2 \leq 2 \|J\|_{L^1} \|\psi_0\|^2 + \|F''(\varphi^e + \theta \psi_0)\| \|\psi_0\|_{L^\infty} \|\psi_0\|^2 \]

where we used boundedness of \(\psi_0\) and \(\|a\|_{L^\infty} \leq \|J\|_{L^1}\). Hence from (6.32), we finally have
\[\|y(t)\|^2 + \|\psi(t)\|^2 \leq \left( \frac{\|y(0)\|^2 + 2 \|J\|_{L^1} \|\psi_0\|^2 + \|F''(\varphi^e + \theta \psi_0)\| \|\psi_0\|_{L^\infty} \|\psi_0\|^2}{\min \{ (C_0 - \|J\|_{L^1}), 1 \}} \right) e^{-\rho t}, \]
which completes the proof. \(\square\)

**Appendix A. Variation of Constants Formula**

In this appendix, we give a variant of variation of constants formula, which is useful when we have two or more differentiable functions with different constant coefficients.

**Lemma A.1** (Lemma C.3, [32]). Assume that the differentiable functions \(y(\cdot), z(\cdot) : [0, T] \to [0, \infty)\) and the constants \(a_1, a_2, k_1, k_2, k_3 > 0\) satisfy:
\[
\frac{d}{dt} (a_1 y(t) + a_2 z(t)) + k_1 y(t) + k_2 z(t) \leq 0, \quad (A.1)
\]
for all \(t \in [0, T]\). Then, we have
\[y(t) + z(t) \leq C (y(0) + z(0)) e^{-\rho t}, \text{ where } C = \frac{\max \{a_1, a_2\}}{\min \{a_1, a_2\}} \text{ and } \rho = \min \left\{ \frac{k_1}{a_1}, \frac{k_2}{a_2} \right\}. \quad (A.2)\]

**Proof.** Since \(a_1 > 0\), from (A.1), we have
\[
\frac{d}{dt} \left( y(t) + \frac{a_2}{a_1} z(t) \right) + \frac{k_1}{a_1} \left( y(t) + \frac{k_2}{k_1} z(t) \right) \leq 0. \]
Now, for \(\frac{a_2}{a_1} \leq \frac{k_2}{k_1}\), from the above inequality, we also have
\[
\frac{d}{dt} \left( y(t) + \frac{a_2}{a_1} z(t) \right) + \frac{k_1}{a_1} \left( y(t) + \frac{a_2}{a_1} z(t) \right) \leq 0. \]
From the above relation, it is immediate that
\[
\frac{d}{dt} \left[ \frac{k_1}{a_1} \left( y(t) + \frac{a_2}{a_1} z(t) \right) \right] \leq 0,
\]
which easily implies
\[
a_1 y(t) + a_2 z(t) \leq (a_1 y(0) + a_2 z(0)) e^{-\frac{k_1}{a_1} t}. \tag{A.3}
\]
We can do a similar calculation by a division with \(a_2 > 0\) and for \(\frac{a_1}{a_2} \leq \frac{k_1}{k_2}\), we arrive at
\[
a_1 y(t) + a_2 z(t) \leq (a_1 y(0) + a_2 z(0)) e^{-\frac{k_2}{a_2} t}. \tag{A.4}
\]
Combining (A.3) and (A.4), we finally obtain (A.2). \(\square\)

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