Effective action for scalar fields
in two-dimensional gravity

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1 October 2001

Abstract

We consider a general two-dimensional gravity model minimally or non-
minimally coupled to a scalar field. The canonical form of the model is eluci-
dated, and a general solution of the equations of motion in the massless case
is reviewed. In the presence of a scalar field all geometric fields (zweibein
and Lorentz connection) are excluded from the model by solving exactly their
Hamiltonian equations of motion. In this way the effective equations of mo-
tion and the corresponding effective action for a scalar field are obtained. It is
written in a Minkowskian space-time and does not include any geometric vari-
ables. The effective action arises as a boundary term and is nontrivial both
for open and closed universes. The reason is that unphysical degrees of free-
dom cannot be compactly supported because they must satisfy the constraint
equation. As an example we consider spherically reduced gravity minimally
coupled to a massless scalar field. The effective action is used to reproduce
the Fisher and Roberts solutions.

1 Introduction

In recent years great attention has been paid to two-dimensional gravity models
mainly for two reasons: a close relation to string theory and good laboratories to
get deeper insights in classical and quantum properties of gravity models. The first
and the simplest two-dimensional constant curvature gravity was proposed in [1].
It attracted much interest after the papers [2, 3]. Constant curvature surfaces are
described by the integrable Liouville equation, and the main concern was given to
physical interpretation of the solutions and inclusion of matter fields.

In the present paper we consider a wide class of two-dimensional gravity models
with torsion and equivalent generalized dilaton models. A two-dimensional gravity
model with torsion was proposed in [4, 5, 6] to provide dynamics for the metric on

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a string world sheet already at the classical level. Two-dimensional dilaton gravity was proposed as an effective model coming from the string theory. Both models turned out to be integrable. A general solution to the equations of motion of two-dimensional gravity with torsion was first given in the conformal gauge. All solutions are divided into two classes: (i) constant curvature and zero torsion and (ii) nonconstant curvature and nonzero torsion. In this way constant curvature gravity models are included in two-dimensional gravity with torsion with a well-defined purely geometric action and a natural way of introduction of matter fields. The equations of motions of two-dimensional gravity with torsion were integrated in the light-cone gauge and without any gauge fixing. At that time the equivalence of two-dimensional gravity with torsion and dilaton gravity was unknown, and solution of the equations of motion for a generalized dilaton gravity was independently obtained in the light-cone gauge.

Later it was realized that in fact the two-dimensional gravity models with torsion and dilaton gravity models are equivalent. From a geometric point of view the dilaton field is the momentum canonically conjugate to the space component of the Lorentz connection. The equivalence of these models is nontrivial because they contain a different set of fields.

Throughout this paper we consider a general two-dimensional gravity model quadratic in torsion. The integrability of the model is connected to the existence of a Killing vector field in the absence of matter. It is worth noting that without matter fields the two-dimensional gravity models do not describe any propagating degree of freedom. This suggests a generalization of the models by addition of matter fields to attack the problems arising in black hole formation and quantum gravity. Unfortunately, addition of matter destroys integrability. Some exact solutions of two-dimensional gravity with torsion coupled to scalar fields were found. We mention also nontrivial solutions for dilaton gravity coupled to scalar fields found in . In Section 7.2 we consider a Lagrangian with arbitrary dependence on scalar curvature and torsion. There we clarify the statement that this general model yields integrable equations of motion made in the literature.

Let us note that the two-dimensional gravity model nonminimally coupled to a scalar field is important for general relativists working in four dimensions. It is well known that spherically reduced general relativity minimally coupled to a scalar field is equivalent to the dilaton gravity model nonminimally coupled to a scalar field in two-dimensional space-time. This model may be the simplest one to describe the dynamics of a black hole formation by spherical scalar waves. Recently it attracted much interest due to the discovery of critical phenomena in the black hole formation found in numerical simulations. A satisfactory analytical calculation of the critical exponent and scaling period of discrete self-similarity has been missing up to now, and hence our understanding of critical phenomena should be considered incomplete.

In the present paper we consider the two-dimensional gravity model with torsion or the equivalent dilaton gravity coupled to a scalar field. We admit nonminimal coupling to be sufficiently general to include spherically reduced gravity. After describing the Lagrangian we elucidate the Hamiltonian formulation of the model in full detail. The canonical form is essential for our approach and I am not aware how to write down the effective action in the Lagrangian formulation. Afterwards a gen-
eral solution of the equations of motion in the matterless case and the equivalence with the dilaton model are briefly reviewed. In the presence of a scalar field the equations of motion are not integrable. Nevertheless they may be partly integrated. We solve the geometric part of the equations of motion with respect to the zweibein and Lorentz connection assuming the scalar field to be arbitrary. Then the solution is substituted into the equations of motion for a scalar field. In this way we obtain the effective equations of motion only for a scalar field and its conjugate momentum. Next we show that the effective action yielding these equations arises as a boundary term. It appears because unphysical variables cannot be compactly supported functions as the consequence of constraint equations. The effective equations of motion are written in a Minkowskian space-time and provide a general solution to the whole problem because the metric can be easily reconstructed for a given solution to the effective equations of motion. The effective action for spherically reduced gravity in a special case was first derived in [30]. Here we generalize this action by considering the more general two-dimensional gravity part and arbitrary coupling to scalars. For spherically reduced gravity we give a more general effective action depending on one arbitrary function on time to be fixed by boundary conditions on the metric.

The reduction of the whole system of the equations of motion to the equations for a scalar field and its conjugate momentum does not provide a general solution of the model because the effective equations are integro-differential and complicated. As an example I considered spherically reduced gravity. The effective action has clear limiting cases and for a small scalar field reduces to an ordinary action for spherical waves on a Minkowskian or Schwarzschild background. In the static case the equations may be integrated in elementary functions and yield the same solution as found by Fisher [31]. As an example we consider also the Roberts solution [32] which provides a self-similar solution to the effective equations of motion.

2 The action

Let us consider a Lorentzian surface with coordinates \( x^\alpha = \{\tau, \sigma\} \), \( \alpha = 0, 1 \). We assume that it is equipped with a Riemann–Cartan geometry defined by a zweibein \( e^a_\alpha(x) \) and a Lorentz connection \( \omega_\alpha(x) \). A general type action for a scalar field coupled to two-dimensional gravity has the form

\[
S = \int dx^2 (L_G + L_X),
\]

where the Lagrangian for a scalar field \( X(x) \) of mass \( m = \text{const} \) is

\[
L_X = -\frac{1}{\rho} e (\partial X^2 - m^2 X^2),
\]

and we introduce shorthand notations

\[
\partial X^2 = g^{\alpha\beta} \partial_\alpha X \partial_\beta X, \quad e = \det e^a_\alpha.
\]

For \( \rho = \text{const} \) we have a minimally coupled scalar. This Lagrangian is easily generalized to a bosonic string moving in \( D \)-dimensional space-time. To this end one has to replace a scalar by a set of scalars \( X^\mu(x), \mu = 0, 1, \ldots, D - 1 \) and set \( m = 0 \). The
The geometric part of the Lagrangian is given by a two-dimensional gravity with torsion written in the first order form

\[ L_G = -\frac{1}{2}(\pi \hat{R} + p_a \hat{T}^a) - e \left( \frac{1}{2} p_a p^a U + V \right), \]  

(3)

where the densities of two-dimensional scalar curvature and the pseudotrace of torsion are given by

\[ \hat{R} = e R = 2 \hat{\varepsilon}^{\alpha \beta} \partial_\alpha \omega_\beta, \]  

(4)

\[ \hat{T}^a = e T^a = 2 \hat{\varepsilon}^{\alpha \beta} (\partial_\alpha e^a_\beta - \omega_\alpha e^a_\beta e^b_\beta). \]  

(5)

Here \( \hat{\varepsilon}^{\alpha \beta} = e \varepsilon^{\alpha \beta} = e \varepsilon^{ab} e^a_\alpha e^b_\beta \) is the totally antisymmetric tensor density, \( \hat{\varepsilon}^{01} = -\hat{\varepsilon}^{10} = -1, \pi \) and \( p_a \) are considered as independent variables, and \( U = U(\pi) \) and \( V = V(\pi) \) are arbitrary functions of \( \pi \). In the second order form, when it exists, this Lagrangian is quadratic in torsion and arbitrarily depends on the scalar curvature. A hat over a symbol means that it is a tensor density of weight \(-1\). For the nonminimally coupled scalar field we have \( \rho = \rho(\pi) \).

The case \( U = 0 \) and \( V = 0 \) describes surfaces of zero torsion and curvature and is not interesting from the geometric point of view. Therefore we assume that at least one of the functions differs from zero. The Lagrangian (3) is quite general. If \( U = 0 \) then one has the gravity model with zero torsion, and the Lagrangian in the second order form is an arbitrary function of the scalar curvature defined by \( V \). For \( U = 1 \) and \( V = \pi^2 + \text{const} \) one immediately recovers two-dimensional gravity with torsion quadratic in curvature and torsion \([4, 5, 6]\). It is essentially a unique purely geometric invariant model yielding second order equations of motion for the zweibein and Lorentz connection. In this case \( \pi \) and \( p_a \) are proportional to the scalar curvature and the pseudotrace of torsion provided equations of motion are fulfilled.

For arbitrary functions \( U(\pi) \) and \( V(\pi) \) when matter fields do not interact with the Lorentz connection \( \omega_\alpha \) the latter can be excluded from the model by the use of its algebraic equations of motion (Section 6). This leads to a generalized two-dimensional dilaton gravity, the function \( \pi(x) \) being the dilaton field. Then the matter Lagrangian (2) describes scalars minimally, \( \rho = \text{const} \), or nonminimally, \( \rho = \rho(\pi) \), coupled to dilaton gravity. Among these models of particular interest is the spherically reduced gravity (see Section 11) for which

\[ U = \frac{1}{2\pi}, \quad V = -2\kappa K - \frac{\Lambda \pi}{\kappa}, \quad \rho = \frac{\pi}{2\kappa}, \]  

(6)

where \( K = 1, 0, -1 \) for spherical, planar, or hyperbolic reductions, respectively \([33]\). \( \Lambda \) is the four-dimensional cosmological constant, and \( \kappa > 0 \) is the inverse gravitational constant in four dimensions.

Note that in a general case the algebraic equation of motion for \( \pi \) can not be solved in elementary functions and the action in the second order form cannot be written explicitly. An important point is that a gravity with torsion without matter fields as given by (3) can be solved exactly for arbitrary functions \( U \) and \( V \) (see Section 3).
3 Equations of motion

Equations of motion following from the action (1) can be written in the form

\[ \frac{1}{\epsilon} \delta S}{\delta \pi} : \quad \frac{1}{2} \rho R - \left( \frac{1}{2} p^a p_a U' + V' \right) - \frac{1}{2} \rho' \left( \partial X^2 - m^2 X^2 \right) = 0, \quad (7) \]

\[ \frac{1}{\epsilon} \delta S}{\delta p_a} : \quad - \frac{1}{2} T^a - p^a U = 0, \quad (8) \]

\[ \hat{\epsilon}_{\beta \alpha} \delta S}{\delta \omega_\beta} : \quad \partial_\alpha \pi - p_a \epsilon^{a b}_b e_\alpha = 0, \quad (9) \]

\[ \hat{\epsilon}_{\beta \alpha} \delta S}{\delta \epsilon_\beta^a} : \quad \nabla_\alpha p_a + \epsilon_{\alpha a} \left( \frac{1}{2} p^b p_b U + V \right) + \rho T^a_{\beta a} \epsilon^{a \beta} = 0, \quad (10) \]

\[ \frac{1}{\epsilon} \delta S}{\delta X} : \quad \rho \left( g^{\alpha \beta} \nabla_\alpha \nabla_\beta X + m^2 X \right) + \rho' g^{\alpha \beta} \partial_\alpha \pi \partial_\beta X = 0, \quad (11) \]

where

\[ \nabla_\alpha p_a = \partial_\alpha p_a - \omega_\alpha \epsilon^{b}_a p_b \]

is the covariant derivative, and

\[ T_{\alpha \beta} = \partial_\alpha X \partial_\beta X - \frac{1}{2} g_{\alpha \beta} (\partial X^2 - m^2 X^2) \quad (12) \]

is the energy-momentum tensor of the scalar field. Primes on functions \( U, V, \) and \( \rho \)
always mean derivatives with respect to the argument \( \pi \). Transformation of Greek indices into Latin ones and vice versa is everywhere performed using the zweibein field and its inverse, and \( \tilde{\nabla} \) denotes the covariant derivative with Christoffel’s symbols.

The action (1) is invariant under local Lorentz rotations and general coordinate transformations which produce linear relations between the equations. Local Lorentz rotations by the angle \( \omega(x) \)

\[ \delta e_\alpha^a = - e_\alpha^b \epsilon^a_b \omega, \quad \delta \omega_\alpha = \partial_\alpha \omega, \]

\[ \delta p_a = \epsilon^a_b p_b \omega, \quad \delta \pi = 0, \quad (13) \]

lead to the following dependence between equations of motion

\[ \tilde{\nabla}_\alpha \delta S}{\delta \omega_\alpha} + \frac{\delta S}{\delta e_\alpha^a} \epsilon^{b}_a \epsilon^a_b = - \frac{\delta S}{\delta p_a} \epsilon^a_b p_b = 0. \quad (14) \]

Variations of the fields under general coordinate transformations with parameters \( e^a(x) \)

\[ \delta e_\alpha^a = - \partial_\alpha \epsilon^a_\beta \epsilon^a_\beta - \epsilon^a_\beta \partial_\beta e_\alpha^a, \quad \delta \omega_\alpha = - \partial_\alpha \epsilon^a_\beta \omega - \epsilon^a_\beta \partial_\beta \omega_\alpha, \]

\[ \delta p_a = - \epsilon^a_\beta \partial_\beta p_a, \quad \delta \pi = - \epsilon^a_\beta \partial_\beta \pi, \quad (15) \]

\[ \delta X = - \epsilon^a_\beta \partial_\beta X, \]
together with the identity (14) produce two linear relations between equations of motion
\[ e_\alpha^a \nabla_\beta \frac{\delta S}{\delta e_\beta^a} - \frac{\delta S}{\delta e_\beta^a} T_{\alpha \beta}^a + \frac{1}{2} \frac{\delta S}{\delta \omega_\beta} \varepsilon_{\alpha \beta \gamma} R - \frac{\delta S}{\delta \pi} \partial_\alpha \pi - \frac{\delta S}{\delta p_a} \nabla_\alpha p_a - \frac{\delta S}{\delta X} \partial_\alpha X = 0, \tag{16} \]
where the covariant derivative acts with Christoffel’s symbols on Greek indices and the Lorentz connection on Latin ones.

4 Canonical formulation

The analysis of the model (1) turns out to be much simpler in the canonical formulation. Moreover the canonical formulation allows one to write down a general solution to the equations of motion in the matterless case without any gauge fixing, the momenta being arbitrary functions parameterizing a solution. The Hamiltonian structure of the quadratic model was first analysed in the conformal gauge [34].

The canonical momenta for \( \omega_\alpha, e_\alpha^a \), and \( X \) are given by
\[ \pi^0 = \frac{\partial (L_G + L_X)}{\partial (\partial_0 \omega^0)} = 0, \tag{17} \]
\[ \pi^1 = \frac{\partial (L_G + L_X)}{\partial (\partial_0 \omega^1)} = \pi, \tag{18} \]
\[ p^0_a = \frac{\partial (L_G + L_X)}{\partial (\partial_0 e_0^a)} = 0, \tag{19} \]
\[ p^1_a = \frac{\partial (L_G + L_X)}{\partial (\partial_0 e_1^a)} = p_a, \tag{20} \]
\[ P = \frac{\partial (L_G + L_X)}{\partial (\partial_0 X)} = \rho g_{11} \partial_0 X - \rho g_{01} \partial_1 X. \tag{21} \]
The last momentum has dimension \( |P| = 1 \). (For definition of dimensions of the fields see Appendix.) We see that the gravity Lagrangian (3) is already written in the canonical form. The Hamiltonian for the whole system is given by
\[ H = \int d\sigma (\omega_0 G + e_0^a G_a), \tag{22} \]
where
\[ G = -\partial_1 \pi + p_a \varepsilon^a_b e_1^b, \tag{23} \]
\[ G_a = -\partial_1 p_a - \omega_1 p_b e_1^b + e_1^b \varepsilon_{ab} \left( \frac{1}{2} \rho p_c U + V - \frac{\rho}{2} m^2 X^2 \right) \]
\[ + \frac{e_1^b \varepsilon_{ab}}{g_{11}} \left( \frac{1}{2 \rho} P^2 + \frac{\rho}{2} \partial_1 X^2 \right) + \frac{e_{1a}}{g_{11}} P \partial_1 X. \tag{24} \]
The functions \( G \) and \( G_a \) are a Lorentz scalar and vector, respectively. Note that Hamiltonian (22) for polynomial \( U(\pi) \) and \( V(\pi) \) is polynomial in the fields in the absence of scalars. Addition of a scalar field makes these functions nonpolynomial because of the denominator \( g_{11} \) and possible nonminimal coupling \( \rho(\pi) \).
The equal time Poisson brackets are defined as usual
\[
\{e_1^a, p_b'\} = \delta^a_b \delta(\sigma - \sigma'), \\
\{\omega_1, \pi'\} = \delta(\sigma - \sigma'), \\
\{X, P'\} = \delta(\sigma - \sigma'),
\]
(25)
where a prime over a function means that it is taken at a point $\sigma'$. Computing the evolution of primary constraints (17) and (19) one gets the secondary constraints
\[
\partial_0 \pi^0 = \{p^0, H\} = -G = 0, \\
\partial_0 p^0_a = \{p^0_a, H\} = -G_a = 0.
\]
(26) (27)
Thus the Hamiltonian (22) is given by a linear combination of secondary constraints. The secondary constraints form a closed algebra
\[
\{G_a, G'_b\} = \varepsilon_{ab} \left[ U p^\gamma G'_c + \frac{1}{2} p^\gamma p_c U' + V' - \frac{\rho'}{e\rho} L_X \right] G \delta,
\]
(28)
\[
\{G_a, G'\} = \varepsilon_{a}^b G_b \delta,
\]
(29)
\[
\{G, G'\} = 0,
\]
(30)
where $U'$, $V'$, and $\rho'$ denote derivatives with respect to the argument $\pi$, $\delta = \delta(\sigma - \sigma')$, and $L_X$ is the Lagrangian for scalars [2] expressed through canonical variables
\[
L_X = -\frac{1}{2} e \left( -\frac{1}{\rho g_{11}} P^2 + \frac{\rho}{g_{11}} \partial_1 X^2 - m^2 X^2 \right).
\]
The Poisson brackets of the secondary constraints with the primary ones vanish identically. Thus the model with the action [1] possesses six first class constraints, and the Hamiltonian (22) equals the linear combination of three secondary first class constraints.

At this point we are able to count the number of physical continuous degrees of freedom. Suppose we have $D$ scalars. The gravity part in the second order form is described by the zweibein (four components) and the Lorentz connection (two components). Thus the total number of physical degrees of freedom equals $D + 4 + 2 - 6 = D$. If we did not add the gravity part to scalars then the number of physical degrees of freedom would be $D - 2$ due to the symmetry of the Lagrangian (2) under general coordinate transformations.

An interesting point is that the Poisson bracket algebra (28)–(30) closes with $\delta$-functions but not with its derivatives. The ”structure functions” depend not only on the fields but on the functions $U(\pi), V(\pi)$, and $\rho(\pi)$ entering the Lagrangian. In this sense the Poisson bracket algebra depends on the dynamics of the fields. Note that in the usual canonical formulation of a gravity model possessing invariance under general coordinate transformation the constraints algebra closes with the first derivatives of $\delta$-functions.

The Poisson bracket algebra (28)–(30) has a (not invariant) subalgebra generated by the constraint with a vector index
\[
\tilde{G}_a = G_a + k_a \omega_1 G.
\]
(31)
Here for brevity we introduced a light-like vector

\[ k_a = \frac{e_1^a + e_1^b \varepsilon_{ba}}{g_{11}}, \quad k^a k_a = 0, \]  

(32)

with the components

\[ k_0 = k_1 = \frac{e_{10} + e_{11}}{g_{11}}. \]

Straightforward calculations yield

\[ \{ \tilde{G}_a, \tilde{G}'_b \} = \varepsilon_{ab} (U p^c + \omega_1 k^c) \tilde{G}_c \delta, \]  

(33)

\[ \{ \tilde{G}_a, G' b \} = \varepsilon_{a} b \tilde{G}_b \delta - k_a G \delta'. \]  

(34)

The algebra of \( \tilde{G}_a \) is related to the conformal algebra generated by two scalar (with respect to Lorentz rotations) constraints

\[ H_0 = -e_1^a \varepsilon_{a} b \tilde{G}_b = -e_1^a \varepsilon_{a} b G_b + \omega_1 G, \]  

(35)

\[ H_1 = e_1^a \tilde{G}_a = e_1^a G_a + \omega_1 G. \]  

(36)

The constraints \( H_1 \) and \( H_0 \) are Lorentz invariant projections of the vector constraint \( \tilde{G}_a \) on the directions parallel and perpendicular to the vector \( e_1^a \), respectively. Straightforward calculations show that the new set of constraints satisfy the following algebra [35]

\[ \{ H_0, H_0' \} = -(H_1 + H_1') \delta', \]  

(37)

\[ \{ H_0, H_1' \} = -(H_0 + H_0') \delta', \]  

(38)

\[ \{ H_1, H_1' \} = -(H_1 + H_1') \delta', \]  

(39)

\[ \{ H_0, G' \} = \{ H_1, G' \} = -G \delta', \]  

(40)

\[ \{ G, G' \} = 0, \]  

(41)

where

\[ \delta' = \frac{\partial}{\partial \sigma'} \delta (\sigma' - \sigma). \]

The constraints \( H_0 \) and \( H_1 \) form the well known conformal algebra. The total algebra is the ”semidirect product” of the conformal algebra with an invariant abelian subalgebra generated by \( G \) and corresponding to local Lorentz rotations.

The inverse transformations to (35), (36) appear as

\[ G_0 = \frac{1}{g_{11}} \left[ -e_1^1 (H_0 - \omega_1 G) + e_1^0 (H_1 - \omega_1 G) \right], \]  

(42)

\[ G_1 = \frac{1}{g_{11}} \left[ e_1^0 (H_0 - \omega_1 G) - e_1^1 (H_1 - \omega_1 G) \right]. \]  

(43)

For later use we write down explicitly expressions for the constraints \( H_0 \) and \( H_1 \) in terms of the canonical variables

\[ H_0 = -\partial_1 p_a e^a b e_1^b - \omega_1 p_a e_1^a + \omega_1 ( -\partial_1 \pi + p_a e^a b e_1^b ) \]

\[ - g_{11} \left( \frac{1}{2} p^a p_a U + V - \frac{\rho}{2} m^2 X^2 \right) - \frac{1}{2 \rho} P^2 - \frac{\rho}{2} \partial_1 X^2, \]  

(44)

\[ H_1 = -e_1^a \partial_1 p_a - \omega_1 \partial_1 \pi + P \partial_1 X. \]  

(45)

Note that the third term in \( H_0 \) is proportional to the constraint \( G \) [23].
5  The canonical transformation

At the classical level two models related by a canonical transformation are equivalent. At the quantum level this property is not valid in general: There are canonical transformations resulting in different quantum models. Special care must be given to nonlinear canonical transformations. This means that in the canonical quantization the correct choice of canonical variables is of primary importance. Nobody knows the correct choice of variables because gravity is not yet quantized. Therefore one is free to choose any set of canonical variables if it leads to a simpler quantum model. Although we do not consider quantization of the model in the present paper the canonical variables introduced in this section are essential for the solution of the constraints in Section 8 already at the classical level.

In this section we make the canonical transformation $e_1^a, p_a \rightarrow q, q_\perp, p, p_\perp$ which explicitly separates the Lorentz angle and simplifies many formulas [35]. Consider a generating functional depending on old coordinates and new momenta

$$F = \frac{1}{2} \int d\sigma \left( p \ln |g_{11}| + p_\perp \ln \left| \frac{e_1^0 + e_1^1}{e_1^0 - e_1^1} \right| \right).$$  

(46)

Varying it with respect to the old coordinates one obtains the relation between old and new momenta

$$p_a = p \frac{e_1^a}{g_{11}} + p_\perp \frac{e_1^b \varepsilon_{ba}}{g_{11}}.$$  

(47)

It shows that

$$p = e_1^a p_a, \quad p_\perp = p_a \varepsilon^a \varepsilon_{b} \varepsilon_{1}^b,$$

that is, $p$ and $p_\perp$ are projections of the momentum $p_a$ on the vector $e_1^a$ and the perpendicular direction. Variation of the generating functional (46) with respect to the momenta yields the relation between the coordinates

$$e_1^0 = e^q \sinh q_\perp,$$

$$e_1^1 = e^q \cosh q_\perp.$$  

(48)

To drop the modulus signs in Eq. (46) we assume for definiteness that

$$e_1^1 > 0 \quad \text{and} \quad e_1^1 > e_1^0.$$  

(49)

The space component of the metric equals to

$$g_{11} = -e^{2q}$$

and is always negative.

We see that the coordinate $q_\perp$ coincides with the Lorentz angle while $q$ parameterizes the length of the vector $e_1^a$. The square of the momentum is

$$p^a p_a = (p_\perp^2 - p^2) e^{-2q}.$$  

(50)
The constraints in new variables have the form
\[ H_0 = -\partial_1 p_\perp + p \partial_1 q_\perp + p_\perp \partial_1 q - \omega_1 p + \omega_1 (-\partial_1 \pi + p_\perp) \]
\[ + \frac{1}{2}(p_\perp^2 - p^2)U + e^{2q} \left( V - \frac{P}{2} m^2 X^2 \right) - \frac{1}{2} P^2 - \frac{\rho}{2} \partial_1 X^2, \]
\[ H_1 = -\partial_1 p - \omega_1 \partial_1 \pi + p \partial_1 q + p_\perp \partial_1 q_\perp + P \partial_1 X, \]
\[ G = -\partial_1 \pi + p_\perp. \]

(51)

(52)

(53)

For \( U = \text{const} \) the quadratic part in the momenta \( \frac{1}{2}(p_\perp^2 - p^2) \) corresponding to the gravitational degrees of freedom has the same form as for two free particles with indefinite signature. It means that the canonical transformation pushes the nonpolynomiality from the kinetic term to the potential. Quantizing the variables \( q, p \) and \( q_\perp, p_\perp \) one may construct the Fock space representation without referring to a Lorentzian background [35]. We postpone solution of the constraints (51)–(53) to Section 8.

6 Dilatization

If the Lorentz connection does not enter the matter Lagrangian then it can be excluded from the model using its equations of motion. As a consequence one obtains the generalized dilaton model, the dilaton field coinciding with the momentum \( \pi(x) \) conjugate to the space component of the Lorentz connection \( \omega_1 \). The equivalence between gravity with torsion and dilaton gravity was first proved in [17]. There the transformation of variables included the Weyl transformation of the metric and therefore did not preserve global structure of solutions to the equations of motion [17, 36]. Here we follow the improved procedure which does not change global properties of space-time [18, 19].

First, let us note the identity valid in two dimensions
\[ eR = e\tilde{R} + \partial_\alpha (e \varepsilon^\alpha_\beta T^{\beta}), \]
where \( \tilde{R} \) is the scalar curvature constructed from the Christoffel symbols corresponding to zero torsion. After integration by parts the gravity Lagrangian (3) takes the form
\[ L_G = -\frac{1}{2} e\pi \tilde{R} + \frac{1}{2} \partial_\alpha \pi \varepsilon^\alpha_\beta \tilde{T}^{\beta a} - \frac{1}{2} p_\alpha \tilde{T}^{\alpha a} - e \left( \frac{1}{2} p_\alpha p^\alpha U + V \right). \]

(55)

We see that the Lorentz connection enters the Lagrangian only through the torsion components \( \tilde{T}^{\alpha a} \), and they are related by the invertible algebraic equation
\[ \omega_\alpha = e_{\alpha a} \left( \frac{1}{2} T^{\alpha a} - \varepsilon^{\beta \gamma} \partial_\beta e^\gamma_\alpha \right). \]

(56)

Therefore instead of the Lorentz connection one can choose the torsion components \( T^{\alpha a} \) as independent variables. Solving their algebraic equations of motion one arrives at the generalized dilaton model
\[ L_D = -\frac{1}{2} e\pi \tilde{R} + \frac{1}{2} e \partial \pi^2 U - eV. \]

(57)
depending on two arbitrary functions $U(\pi)$ and $V(\pi)$. Here we used the obvious abbreviation $\partial\pi^2 = g^{\alpha\beta}\partial\pi^\alpha\partial\pi^\beta$. Thus, if the matter Lagrangian does not contain the Lorentz connection then the dilaton model is equivalent to the two-dimensional gravity with torsion. In fact, we have proved that if the original variables $e_\alpha^a$, $\omega_\alpha$, $p_a$, $\pi$, and $X$ satisfy equations of motion (7)–(11), then the variables $g_{\alpha\beta}$, $\pi$, and $X$ satisfy equations of motion following from the Lagrangian $L_D + L_X$. The inverse statement is as follows. Let $g_{\alpha\beta}$, $\pi$, and $X$ satisfy equations of motion for $L_D + L_X$. Solve the algebraic equation $e_\alpha^a e_\beta^b \eta_{ab} = g_{\alpha\beta}$ for zweibein (a solution is unique up to a local Lorentz rotations), construct $p_a$ using Equation (9), and solve equations (8) and (56) for $T^a \omega$ and $\omega_\alpha$, respectively. Then the original equations of motion (7)–(11) will be satisfied.

The equivalence is a global one because the transformation of variables is non-degenerate (56) as far as nondegenerate is the zweibein. This equivalence yields the geometric meaning for the dilaton field: It is the momentum conjugate to the space component of the Lorentz connection $\omega_1$.

Variation of the action for the Lagrangian (57) with respect to the metric and the dilaton field yields the equations of motion

$$\frac{1}{e} \delta L_D}{\delta g_{\alpha\beta}} : -\frac{1}{2} (g^{\alpha\beta} \Box_\pi - \tilde{\Box}^{\alpha\beta} \pi) - \frac{1}{2} \tilde{\Box}^{\alpha\pi} \tilde{\Box}^{\beta\pi} U + \frac{1}{2} g^{\alpha\beta} \left( \frac{1}{2} \partial\pi^2 U - V \right) = 0, \quad (58)$$

$$\frac{1}{e} \delta L_D}{\delta \pi} : -\frac{1}{2} R - \tilde{\Box} \pi U - \frac{1}{2} \partial\pi^2 U' - V' = 0, \quad (59)$$

where $\tilde{\Box} = g^{\alpha\beta} \tilde{\Box}_\alpha \tilde{\Box}_\beta$ is the Laplace–Beltrami operator. This system of equations was solved in the conformal gauge in [16], but we are not aware how to solve directly this system of nonlinear equations of motion for arbitrary functions $U$ and $V$ without gauge fixing. In the next section we write down a general solution to the equivalent two-dimensional gravity with torsion we started with in an arbitrary coordinate system. The latter model turns out to be simpler, and a general solution will be written even without gauge fixing.

7 A general solution without matter

The important feature of two-dimensional gravity with torsion described by the Lagrangian (3) alone is its integrability. It has a long history. First the quadratic model was solved in the conformal [9, 10] and light-cone [12] gauge. In [13, 14, 15] a solution for the quadratic model was in fact obtained without gauge fixing. Afterwards this solution was clarified and generalized in the papers [37, 19]. In this section we summarize all approaches and write a general local solution of the equations of motion for arbitrary functions $U$ and $V$ in the absence of matter fields. It is naturally written in the canonical formulation. This solution has one Killing vector field, and using the conformal blocks technique one is able to construct easily all global (maximally extended along extremals) solutions of two-dimensional gravity with torsion or, equivalently, dilaton gravity for arbitrary given functions $U$ and $V$. In this section we set $X = 0$. 

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7.1 Local solution

The integration of the equations of motion is most easily performed for the light cone components of the vectors in the tangent space

\[ p_\pm = \frac{1}{\sqrt{2}}(p_0 \pm p_1), \quad e^\pm_1 = \frac{1}{\sqrt{2}}(e^1_0 \pm e^1_1). \tag{60} \]

The Lorentz metric and antisymmetric tensor for the tangent space indices \( a = \{+, -\} \) become

\[ \eta_{\pm\pm} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \varepsilon_{\pm\pm} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{61} \]

The raising and lowering of the light cone indices is performed according to the rules \( p^+ = p_- \) and \( p^- = p^+ \). Then equations of motion (7)–(10) take the form

\[ -\frac{1}{2} R - p_+ p_- U'' - V' = 0, \tag{62} \]
\[ -\frac{1}{2} T^{*+} - p^+ U = 0, \tag{63} \]
\[ -\frac{1}{2} T^{*-} - p^- U = 0, \tag{64} \]
\[ \partial_\alpha \pi - e^\alpha_+ p_+ + e^\alpha_- p_- = 0, \tag{65} \]
\[ \partial_\alpha p_+ + \omega_\alpha p_+ + e^\alpha_- (p_+ p_- U + V) = 0, \tag{66} \]
\[ \partial_\alpha p_- - \omega_\alpha p_- - e^\alpha_+ (p_+ p_- U + V) = 0, \tag{67} \]

where

\[ T^{*\pm} = 2\varepsilon^{\alpha\beta}(\partial_\alpha \mp \omega_\alpha)e^\beta_{\pm}. \tag{68} \]

Any local solution to these equations for \( U \neq 0 \) and \( V \neq 0 \) belongs to one of the two classes. The first class corresponds to \( p_- = 0 \) or, equivalently, \( p_+ = 0 \). For \( p_- = 0 \) the dilaton field is equal to the constant, \( \pi = \text{const} \), defined by the algebraic equation

\[ V(\pi) = 0 \tag{69} \]

as the consequence of Equation (67). Then \( p_+ = 0 \) due to Equation (63), and Equation (66) is trivially satisfied. The remaining Equations (62)–(64) show that the corresponding space-time is of zero torsion and constant curvature

\[ T^{*\pm} = 0, \quad R = -2V'(\pi) = \text{const.} \tag{70} \]

In this way the constant curvature gravity model [1-3] forms one class of solutions of two-dimensional gravity with torsion.

The second class of solutions, \( p_- \neq 0 \) or \( p_+ \neq 0 \), corresponds to nonzero torsion. For definiteness we assume that \( p_- \neq 0 \) in some domain. The case \( p_+ \neq 0 \) is treated in a similar way. Integrability of the equations of motion relies on two nontrivial observations. The first one is the existence of an integral of motion

\[ A = p_+ p_- e^{-Q} - W = \frac{1}{2}(p_{\perp}^2 - p^2)e^{-2q-Q} - W = \text{const}, \tag{71} \]
where $Q$ and $W$ are primitives

$$Q(\pi) = \int \pi dsU(s), \quad W(\pi) = \int \pi ds V(s) e^{-Q(s)}. \quad (72)$$

They have dimensions

$$[Q] = 1, \quad [W] = [A] = l^{-2},$$

as the consequence of $(283)$ and $(284)$. The constants of integration in Equations $(72)$ do not matter because $A$ by itself is a constant on the equations of motion

$$\partial_\alpha A = 0.$$

This equality is proved by direct calculations using equations of motion $(65)$–$(67)$. Note that this integral of motion depends only on the momenta and its conservation is the consequence of eqs. $(65)$–$(67)$ only. Its existence will be clarified in the next section. This integral of motion was independently found in dilaton gravity $[38, 39]$.

The second important observation is that the form

$$dx^\alpha f_\alpha = \sqrt{\kappa} dx^\alpha e^\alpha e^{Q} \quad (73)$$

is closed on the equations of motion. Here we introduced a dimensionfull gravitational constant $[\kappa] = l^{-2}$ in order for $f_\alpha$ to be dimensionless $[f_\alpha] = 1$. To prove the closedness of the form $(73)$ one has to verify that the expression

$$\varepsilon^{\alpha\beta} \partial_\alpha f_\beta = 0$$

vanishes identically when equations of motion $(65)$, $(66)$, and $(67)$ are satisfied. It means that at least locally the one form $(73)$ may be written as the gradient of some scalar function

$$f_\alpha = \partial_\alpha f. \quad (74)$$

Afterwards a general solution to the equations of motion $(62)$–$(67)$ is written immediately

$$e_\alpha^+ = \frac{1}{\sqrt{\kappa}} p_- e^{-Q} \partial_\alpha f, \quad (75)$$

$$e_\alpha^- = \frac{1}{p_-} \left[ \frac{1}{\sqrt{\kappa}} (A + W) \partial_\alpha f - \partial_\alpha \pi \right], \quad (76)$$

$$\omega_\alpha = \frac{\partial_\alpha p_-}{p_-} - \frac{1}{\sqrt{\kappa}} [(A + W)U + e^{-Q} V] \partial_\alpha f, \quad (77)$$

$$p_+ = \frac{1}{p_-} (A + W)e^Q. \quad (78)$$

Here Equation $(73)$ is the consequence of $(63)$ and $(64)$. Equation $(76)$ follows from $(65)$ and $(71)$. Equation $(77)$ is the consequence of $(71)$ and $(71)$. Equation $(78)$ is a simple rewriting of $(74)$. The next step of the proof is to verify that the remaining equations of motion $(62)$–$(64)$ and $(66)$ are satisfied identically for arbitrary functions $f, \pi$, and $p_-$. This can be checked by direct calculations, and
is the consequence of the linear dependence of the equations of motion given by (14) and (16). Note that the solution (75)–(78) was obtained without any gauge fixing and contains three arbitrary functions $f, \pi$ and $p_-$. The first two functions correspond to the invariance of the model under general coordinate transformations. They must satisfy the restriction

$$d\pi \wedge df \neq 0 \quad \text{or} \quad \epsilon^{\alpha\beta} \partial_\alpha \pi \partial_\beta f \neq 0.$$ (79)

The third function $p_-$ corresponds to a Lorentz rotation and must be strictly positive $p_- > 0$ or negative $p_- < 0$.

The second class of solutions describes surfaces with nonzero torsion and nonconstant scalar curvature which are most easily calculated using Equations (62)–(64). The corresponding metric is

$$ds^2 = 2e_{\alpha}^+ e_{\beta}^- dx^\alpha dx^\beta = 2e^{-Q} \left[ \frac{1}{\kappa} (A + W) df^2 - \frac{1}{\sqrt{\kappa}} df d\pi \right].$$ (80)

We see that the scalar functions $f$ and $\pi$ are the Eddington–Finkelstein coordinates [40, 41], the dilaton field being the lightlike one.

The components of the metric (80) depend only on the dilaton field $\pi$, and hence the solution has one Killing vector field

$$K = \frac{1}{\sqrt{\kappa}} \partial f.$$ (81)

Its square is

$$K^2 = \frac{2}{\kappa^2} e^{-Q} (A + W),$$ (82)

and may be timelike or spacelike depending on the sign of $A + W$. Along horizons defined by the equation $A + W = 0$ the Killing vector is lightlike.

A general solution (75)–(78) in new canonical variables introduced in Section 5 takes the form

$$p = \frac{2}{\sqrt{\kappa}} (A + W) \partial_1 f - \partial_1 \pi,$$ (83)

$$p_\perp = \partial_1 \pi,$$ (84)

$$e^{2q} = \frac{2}{\kappa} \partial_1 f [\sqrt{\kappa} \partial_1 \pi - (A + W) \partial_1 f] e^{-Q},$$ (85)

$$e^{2q_\perp} = \frac{p_-^2 \partial_1 f}{\sqrt{\kappa} \partial_1 \pi - (A + W) \partial_1 f} e^{-Q}.$$ (86)

It is valid only for these functions $\pi$ and $f$ which provide positive definiteness of the right hand sides of (85) and (86) according to our assumption (49).

The above solution is obtained under the restriction $p_- \neq 0$. In a domain with $p_+ \neq 0$ we still have the integral of motion (109) but the form (73) must be replaced because we admit zero of $p_-$. One may check that the form

$$df = \frac{\sqrt{\kappa} dx^\alpha e^-_\alpha}{p_+} e^Q$$
is closed on the equations of motion (64), (65), and (66). A similar procedure results in a general solution to the equations of motion

\[ e^\alpha - \frac{1}{\sqrt{\kappa}} p_+ e^{-Q} \partial_\alpha f, \quad (87) \]

\[ e^\alpha + \frac{1}{p_+} \left[ \frac{1}{\sqrt{\kappa}} (A + W) \partial_\alpha f + \partial_\alpha \pi \right], \quad (88) \]

\[ \omega_\alpha = -\frac{\partial_\alpha p_+}{p_+} - \frac{1}{\sqrt{\kappa}} [(A + W)U + e^{-Q} V] \partial_\alpha f, \quad (89) \]

\[ p_- = \frac{1}{p_+} (A + W) e^Q, \quad (90) \]

where \( p_+ \) is considered as an arbitrary function parameterizing the solution. The corresponding metric differs from (80) by the sign before the second term

\[ ds^2 = 2 e^{-Q} \left[ \frac{1}{\kappa} (A + W) df^2 + \frac{1}{\sqrt{\kappa}} df d\pi \right]. \quad (91) \]

This difference may be eliminated by the redefinition \( f \to -f \).

At the end of this section we make a short comment on the conserved quantity \( A (71) \). For the Schwarzschild solution it equals a mass of the black hole up to a constant factor (see Section [1]). The equation

\[ A = \text{const} \quad (92) \]

is a first class constraint of the model. One may check that its space derivative is expressed in terms of the secondary constraints

\[ \partial_1 A = e^{-Q} \left[ -p^a G_a + \left( \frac{1}{2} p^a p_a U + V \right) G \right]. \quad (93) \]

The Poisson brackets of the constraint (92) with the secondary constraints are zero,

\[ \{ A, G'_a \} = \{ A, G' \} = 0, \]

and thus \( A \) belongs to the center of the algebra (28)–(30). Straightforward calculations yield the following Poisson brackets of this constraint with the other constraints:

\[ \{ A, \tilde{G}_a \} = \left[ k^c p_c \omega_1 + \frac{1}{2} p^c p_c U + V \right] k^a e^{-Q} G\delta, \]

\[ \{ A, \tilde{G}_0 \} = \{ A, \tilde{G}_1 \} = \left[ (p_1 - p) e^{-2q} \omega_1 + \frac{1}{2} (p_1^2 - p^2) e^{-2q} U + V \right] e^{-(q+q_1+Q)} G\delta, \]

\[ \{ A, H'_0 \} = \left[ -\frac{e_1 a p_a}{g_{11}} H_0 - \frac{e_1 a \varepsilon_a b p_b}{g_{11}} H_1 + \left( k^a p_a \omega_1 + \frac{1}{2} p^a p_a U + V \right) G \right] e^{-Q} \delta, \]

\[ \{ A, H'_1 \} = \left[ \frac{e_1 a \varepsilon_a b p_b}{g_{11}} H_0 + \frac{e_1 a p_a}{g_{11}} H_1 + \left( -k^a p_a \omega_1 + \frac{1}{2} p^a p_a U + V \right) G \right] e^{-Q} \delta \]
Here we write the Poisson brackets in the coordinates both before and after the canonical transformation for comparison. We see that the constraint (92) does not form a closed algebra with the conformal constraints \( H_0 \) and \( H_1 \) alone.

### 7.2 Local solution for a general Lagrangian

Local solution to the equations of motion for the gravity Lagrangian found in the previous section relies on the existence of the conserved quantity (71) and the closed form (73). To clarify the appearance of the conservation law (71) we consider a more general gravity Lagrangian

\[
L_G = -\frac{1}{2} \left( \pi \hat{R} + p_a \hat{T}^x_a \right) - U, \tag{94}
\]

where \( U(\varphi, \pi) \) is an arbitrary function of two scalar functions \( \varphi = p^a p_a \) and \( \pi \). For the linear function \( U = \varphi U/2 + V \) in \( \varphi \) we recover the original gravity model (3). In addition we clarify the statement that equations of motion are integrable in this more general case [14, 23, 26].

Equations of motion for the Lagrangian (94) in light-cone coordinates take the form

\[
\frac{\delta S}{\delta \pi} : -\varepsilon^{\alpha\beta}(\partial_\alpha \omega_\beta + U, \pi e_\alpha^+ e_\beta^-) = 0, \tag{95}
\]

\[
\frac{\delta S}{\delta p_+} : -\varepsilon^{\alpha\beta}(\partial_\alpha e_\beta^+ - \omega_\alpha e_\beta^+ + 2p_- U, \varphi e_\alpha^+ e_\beta^-) = 0, \tag{96}
\]

\[
\frac{\delta S}{\delta p_-} : -\varepsilon^{\alpha\beta}(\partial_\alpha e_\beta^- + \omega_\alpha e_\beta^- + 2p_+ U, \varphi e_\alpha^+ e_\beta^-) = 0, \tag{97}
\]

\[
\frac{\delta S}{\delta \omega_\beta} : \partial_\alpha \pi - p_+ e_\alpha^+ + p_- e_\alpha^- = 0, \tag{98}
\]

\[
\frac{\delta S}{\delta e_\beta^+} : \partial_\alpha p_+ + \omega_\alpha p_+ + e_\alpha^- U = 0, \tag{99}
\]

\[
\frac{\delta S}{\delta e_\beta^-} : \partial_\alpha p_- - \omega_\alpha p_- + e_\alpha^+ U = 0, \tag{100}
\]

where

\[
U, \varphi = \frac{\partial U}{\partial \varphi}, \quad U, \pi = \frac{\partial U}{\partial \pi}.
\]

We have nine equations for nine independent variables \( \pi, p_\pm, \omega_\alpha, e_\alpha^\pm \). There are three linear relations between equations of motion (14), (16) due to the symmetry under general coordinate transformations and local Lorentz rotations. It means that to find a general solution to the equations of motion one has to solve only six independent equations.

Equations (95)–(100) may be written in the form

\[
-\frac{1}{2} R - U, \pi = 0, \tag{101}
\]

\[
-\frac{1}{2} T^{\alpha\pm} - 2p^\pm U, \varphi = 0. \tag{102}
\]
where the scalar curvature and torsion are defined by equations (4), (5). If these equations have a unique solution with respect to \( \pi \) and \( p^\pm \) then the Lagrangian (94) is nothing else than the Legendre transform of some function

\[
L_G^{(2)} = eF(R, T^2)
\]

(103)
depending on the scalar curvature \( R \) and the torsion squared term \( T^2 = T_a^a T^{*a} \) with respect to three variables \(-R/2 \) and \(-T^{*a}/2 \). The Lagrangian written in the first order form (94) is more general because we do not assume that a function \( U(\varphi, \pi) \) admits the Legendre transformation.

Let us try to find the six unknowns \( \omega_\alpha, e_\alpha^\pm \) in terms of the conjugate momenta considered as arbitrary functions. For fixed index \( \alpha \) Equations (98)–(100) constitute a set of three linear algebraic inhomogeneous equations for \( \omega_\alpha, e_\alpha^\pm \). Its determinant vanishes identically, and hence for any nontrivial solution there must exist a relation between the momenta. To find it one may take the linear combination of the equations (99) \( p^- + p^+ \) and using equation (98) find

\[
\partial_\alpha \varphi - 2 \partial_\alpha \pi U = 0.
\]

It means that momenta must satisfy the following ordinary differential equation

\[
\frac{d \varphi}{d \pi} = 2U(\varphi, \pi).
\]

(104)

It is an integrability condition for Equations (98)–(100). For reasonable functions \( U \) a general solution of this equations exists and depends on one arbitrary constant which is the integral of motion. For linear \( U \) it is given by Equation (71). Though the integral of motion exists for arbitrary functions \( U \), it is not enough to integrate explicitly equations of motion in a general case.

The existence of this integral of motion means that only two momenta of three are independent. Let it be \( \pi \) and \( p_- \). Afterwards \( p_+ \) is to be found from a solution of Equation (104). Now we use two Equations (98), (100) to find

\[
e_\alpha^- = \frac{1}{p_-}(\partial_\alpha \pi + p_+ e_\alpha^+),
\]

(105)

\[
\omega_\alpha = \frac{1}{p_-}(\partial_\alpha p_- - e_\alpha^+ U).
\]

(106)

To complete integration of the equations of motion one has to find \( e_\alpha^+ \). These two components are to be found as a solution of one of the Equations (95)–(97) or their linear combination because algebraic Equations (98)–(100) are dependent. Taking the linear combination (99) \( p_++p_- \) and using Equations (99), (100) one gets the linear differential equation for \( p_+ e_\alpha^+ \)

\[
\varepsilon^{\alpha\beta} \left[ \partial_\alpha (p_+ e_\beta^+) - 2(U - \psi U, \varphi) \frac{\partial_\alpha \pi}{\varphi} (p_+ e_\beta^+) \right] = 0.
\]

(107)

Without loss of generality we set

\[
p_+ e_\alpha^+ = \varphi^e Q f_\alpha.
\]
where $f_\alpha$ is a one form and $Q(\varphi, \pi)$ is some unknown scalar function of two variables to be specified later. Then Equation (107) takes the form

$$\varphi e^{-Q} \varepsilon^{\alpha\beta} [\partial_\alpha f_\beta - \partial_\alpha \pi f_\beta (2UQ, \varphi + Q, \pi - 2U, \varphi)] = 0.$$ (108)

Solution of this equation always exists for reasonable functions $\pi$, $U$, and $Q$, but a solution cannot be written explicitly in a general case. It can be exactly integrated in a particular case. If we choose the function $Q$ satisfying the partial differential equation

$$2UQ, \varphi + Q, \pi - 2U, \varphi = 0,$$ (109)

then a one form $f_\alpha$ must be closed. It means that locally this one form is exact (14). Hence we get a general solution

$$e_\alpha^+ = 2p_- e^{-Q} \partial_\alpha f,$$ (110)

where $f$ is an arbitrary function, and $Q$ is a solution of the partial differential Equation (109). The rest of the equations of motion are satisfied as the consequence of their linear dependence.

Equations for characteristics $\varphi(\pi), Q(\varphi(\pi), \pi)$ for Equation (104) are

$$\frac{d\varphi}{d\pi} = 2U,$$

$$\frac{dQ}{d\pi} = 2\frac{\partial U}{\partial \varphi}.$$ (111)

Note that the first Equation (111) coincides with the integrability condition (104).

Thus we get a general solution to the equations of motion for arbitrary $U(\varphi, \pi)$ without any gauge fixing. It is given by (105), (106), and (110) where $Q$ is a solution of Equation (109), and $p_-$ is to be found from Equation (104). This solution depends on three arbitrary functions $\pi, f$, and $p_-$, corresponding to general coordinate transformations and local Lorentz rotations with the only restriction (79).

In this way the integration of the equations of motion is reduced to one ordinary differential Equation (104) and one equation with partial derivatives (109). It is hardly justified to say that equations of motion are integrable because solution of the last equation is known to exist for sufficiently smooth $U$ but cannot be written in quadratures in a general case. It is an interesting question to find functions $U$ for which solution of Equation (109) can be written explicitly thus providing integrable two-dimensional gravity models. In this paper we consider only linear functions $U = \varphi U/2 + V$ in $\varphi$ corresponding to gravity models quadratic in torsion. These models are really integrable with a solution for Equation (109) given by (72). Besides, we have the usual kinetic term for the dilaton field after dilatonization which should be modified for more general Lagrangians.

### 7.3 Global solutions

The solution found in the preceding sections is a local one. To give a physical interpretation of the solution and to understand global structure of the space-time
one has to extend the solution along extremals (geodesics). This can be done using the constructive conformal block method developed for the Lorentzian [42, 43, 44] and Euclidean [45] signature metric. An equivalent set of rules in the Eddington–Finkelstein coordinates may be found in [46, 47]. To apply the conformal blocks technique one has to rewrite local solutions obtained in the previous section in the conformal form. This should be done in every domain where the solution is defined because a global solution is obtained by gluing all patches together.

Before doing this we write the metric (80) in the diagonal gauge for comparison with the metric in the matterful case. In a domain with \( A + W > 0 \) we leave the coordinate \( \pi \) as it is and transform the coordinate \( f \) only

\[
f = \frac{1}{2} \tau + g(\pi),
\]

where the function \( g(\pi) \) is defined by the equation

\[
g' = \frac{\sqrt{\kappa^2(A + W)}}{2(A + W)}.
\]

This transformation is nondegenerate at least between horizons defined by the equation \( A + W = 0 \). Then the metric (80) takes a diagonal form

\[
ds^2 = \frac{1}{2} e^{-Q} \left[ \frac{1}{\kappa^2} (A + W) d\tau^2 - \frac{d\pi^2}{A + W} \right].
\]

For \( A + W > 0 \) coordinates \( \tau \) and \( \pi \) are timelike and spacelike, respectively.

This metric can be easily rewritten in the conformal gauge suitable for the global analysis. Introducing the space coordinate \( \sigma \) defined by the equation

\[
\frac{d\pi}{d\sigma} = \pm \frac{A + W}{\sqrt{\kappa}},
\]

and keeping \( \tau \) unchanged, one gets a conformally flat metric

\[
ds^2 = \frac{1}{2\kappa} e^{-Q} (A + W) (d\tau^2 - d\sigma^2), \quad A + W > 0.
\]

Introducing the invariant variable \( \hat{q} \) related to \( \pi \) by ordinary differential equation

\[
\frac{d\hat{q}}{d\pi} = \frac{1}{\sqrt{\kappa}} e^{-Q}
\]

the solution (113) may be rewritten in the form

\[
ds^2 = N(\hat{q})(d\tau^2 - d\sigma^2), \quad N(\hat{q}) > 0,
\]

where

\[
\pm \frac{d\hat{q}}{d\sigma} = N(\hat{q}),
\]
and the conformal factor is

$$N(\hat{q}) = \frac{1}{2\kappa}e^{-Q}(A + W).$$ (119)

In a domain $A + W < 0$ we introduce a space coordinate

$$f = \frac{1}{2}\sigma + g(\pi),$$

where $g(\pi)$ is defined by the same Equation (113). Then solution (80) takes a diagonal form

$$ds^2 = -\frac{1}{2}e^{-Q}\left[\frac{d\pi^2}{A + W} - \frac{1}{\kappa}(A + W)d\sigma^2\right].$$

Introducing the time coordinate

$$\frac{d\pi}{d\tau} = \pm \frac{A + W}{\sqrt{\kappa}},$$

the metric takes a conformally flat form

$$ds^2 = -\frac{1}{2\kappa}e^{-Q}(A + W)(d\tau^2 - d\sigma^2), \quad A + W < 0.$$

In terms of the invariant variable $\hat{q}$ (116) this metric takes the form

$$ds^2 = -N(\hat{q})(d\tau^2 - d\sigma^2), \quad N(\hat{q}) < 0,$$ (120)

where the conformal factor is given by Equation (119) and

$$\pm \frac{d\hat{q}}{d\tau} = -N(\hat{q}).$$

Solutions in domains $A + W > 0$ and $A + W < 0$ may be united in the way

$$ds^2 = |N(\hat{q})|(d\tau^2 - d\sigma^2),$$

$$\left|\frac{d\hat{q}}{d\zeta}\right| = \pm N(\hat{q}),$$

where $\zeta = \sigma$ and $\zeta = \tau$ for positive and negative values of the conformal factor $N(\hat{q})$, respectively. Afterwards one may apply the conformal block technique and construct global (maximally extended) solutions for the matterless models for any functions $U$ and $V$. The method is straightforward and allows one to construct Carter–Penrose diagrams by the analysis of zero and singularities of the conformal factor $N(\hat{q})$. The method is reviewed and several examples are given in [44].

8 Solution of the constraints

From now on we return to the matterfull case $X \neq 0$. To construct the effective action for scalars one has to solve the geometric part of the equations of motion (7)–(10) assuming that a scalar field $X$ and its conjugate momenta $P$ are arbitrary
functions. This will be done in two steps. First, we solve three constraints (51)–(53), which are equivalent to the variational derivatives of the action with respect to the time components of the zweibein and Lorentz connection \( \delta S/\delta e^0_a \) and \( \delta S/\delta \omega_0 \). Afterwards we solve the rest of the geometric part of the equations of motion.

The constraints \( G \) and \( G_a \) in the light cone coordinates are

\[
G = -\partial_1 \pi + e_1^+ p_+ - e_1^- p_- , \\
G_\pm = -\partial_1 p_\pm \mp \omega_1 p_\pm \mp e_1^\mp \left( p_+ p_- U + V - \frac{\rho}{2} m^2 X^2 \right) \mp \frac{\rho}{4 e_1^\pm} P^2_\mp ,
\]

where we introduced a shorthand notation

\[
P_\pm = \frac{1}{\rho} P \pm \partial_1 X .
\]

These constraints were solved in the absence of matter in the previous section. To find a solution to the equations \( G = G_\pm = 0 \) for the arbitrary scalar field we use the canonical transformation introduced in Section 5 and use the equivalent set of constraints \( G = H_0 = H_1 = 0 \). Here we shall see the advantage of the canonical transformation.

To simplify the procedure we fix the gauge; that is, we fix a coordinate system and local Lorentz rotations. This is done in a slightly unusual way. Namely, we consider a wide class of gauges of the form

\[
F_1 = \pi - \pi(\tau, \sigma) = 0 , \\
F_2 = p - p(\tau, \sigma) = 0 , \\
F_3 = q_\perp - q_\perp(\tau, \sigma) = 0.
\]

They state that the two momenta \( \pi, p \) and the Lorentz angle \( q_\perp \) are given functions of space-time coordinates \( \pi(\tau, \sigma), p(\tau, \sigma), \) and \( q_\perp(\tau, \sigma) \). We call them gauge fixing functions. The first two functions specify the coordinate system while the third one fixes a local Lorentz rotation. The form of these functions will be specified later to simplify the effective action for scalars. Let us note that we do not fix the fields by themselves but their conjugate momenta. This shows the strength of the canonical formulation. It is not easy to imagine how one can fix derivatives of the fields keeping the original fields as independent variables within the Lagrangian second order approach. The class of gauges (124) is very wide until we specify the gauge fixing functions. In fact, we only choose the variables which have to be fixed. It is enough to write down a solution for all other geometrical variables in terms of scalars.

The constraints (51)–(53) contain only canonically transformed space components of the geometric variables. Three of them are fixed by gauge conditions (124), and hence we have three equations for three unknown functions \( \omega_1, q, p_\perp \). The last constraint is trivially solved

\[
p_\perp = \partial_1 \pi .
\]

Here we see the importance of the canonical transformation which simplified greatly the Lorentz constraint (23).
Afterwards the two remaining constraints become

\[
H_0 = - \partial_1^2 \pi + p(\partial_1 q_\perp - \omega_1) + \partial_1 \pi \partial_1 q + \frac{1}{2} YU + e^{2q}(V - \frac{\rho}{2} m^2 X^2)
- \frac{1}{2} P^2 - \frac{\rho}{2} \partial_1 X^2 = 0,
\]

\[
H_1 = - \partial_1 p + \partial_1 \pi (\partial_1 q_\perp - \omega_1) + p \partial_1 q + P \partial_1 X = 0,
\]

(126)

(127)

where we introduced a shorthand notation for the quadratic combination of the gauge fixing functions

\[
Y = \partial_1 \pi^2 - p^2, \quad [Y] = l^{-2}.
\]

(128)

There are four different cases depending on the gauge fixing functions

Case A : \( Y \neq 0, \ \partial_1 \pi \neq 0 \)

Case B : \( Y \neq 0, \ \partial_1 \pi = 0 \)

Case C : \( Y = 0, \ \partial_1 \pi \neq 0 \)

Case D : \( Y = 0, \ \partial_1 \pi = 0 \)

(129)

(130)

(131)

(132)

The first case describes a general situation, while the last three are degenerate ones. The fourth possibility, which may be written in an equivalent form \( p = 0 \) and \( \partial_1 \pi = 0 \), restricts the scalars as the consequence of the constraint \( H_1 = 0 \), and does not allow us to find \( \omega_1 \) from the constraints. In the last two cases at least one of two light-cone components of torsion is zero. Indeed, the square of the momenta (50) together with the constraint (53) takes the form

\[
p^a p_a = 2 p_+ p_- = Ye^{-2q}.
\]

(133)

It means that for \( Y = 0 \) at least one of the light cone components of torsion must be zero as the consequence of Equation (8) or (63), (64) which are not affected by the presence of scalars.

The case A. \( Y \neq 0, \ \partial_1 \pi \neq 0 \). For \( \partial_1 \pi \neq 0 \) Equation (127) yields

\[
\partial_1 q_\perp - \omega_1 = \frac{1}{\partial_1 \pi} (\partial_1 p - p \partial_1 q - P \partial_1 X).
\]

(134)

Plugging this solution for \( \omega_1 \) into the constraint (126) one gets the equation for \( q \)

\[
H_0 \partial_1 \pi = \partial_1 qY + e^{2q} \left( V - \frac{\rho}{2} m^2 X^2 \right) \partial_1 \pi - \frac{1}{2} \partial_1 Y + \frac{1}{2} YU \partial_1 \pi - pP \partial_1 X
- \partial_1 \pi \left( \frac{1}{2} P^2 - \frac{\rho}{2} \partial_1 X^2 \right) = 0.
\]

(135)

Provided \( Y \neq 0 \) this first order nonlinear ordinary differential equation for \( q \) has a general solution

\[
2q = \ln \frac{Y}{2(A_m + W_m)} - Q + 2T,
\]

(136)
where $Q$ is defined by the integral (72) and two integrals are introduced,

$$T(\tau, \sigma) = -\int_{\sigma}^{\infty} \frac{d\sigma'}{Y} \left[ pP \partial_1 X + \partial_1 \pi \left( \frac{1}{2\rho} P^2 + \frac{\rho}{2} \partial_1 X^2 \right) \right], \quad (137)$$

$$W_m(\tau, \sigma) = \int_{0}^{\sigma} d\sigma' \partial_1 \pi \left( V - \frac{\rho}{2} m^2 X^2 \right) e^{-Q+2T}. \quad (138)$$

They have dimensions

$$[T] = 1, \quad [W_m] = l^{-2}.$$

This solution depends on one arbitrary function of time $A_m(\tau)$ entering (136). The integrals (137), (138) are assumed to be convergent. If not, then the upper and lower limits of integration in (137) and (138), respectively, have to be redefined. The limits of integration are adopted for spherically reduced gravity considered in Section 11. The solution (136) yields the solution for the space component of the metric

$$-g_{11} = e^{2q} = \frac{Y}{2(A_m + W_m)} e^{-Q+2T}, \quad (139)$$

This solution is valid on those patches where expression (139) is positive. A similar solution may be written for negative values of the right hand side of (139). In the matterless case $T = 0$, $W_m$ coincides with $W$ given by (72), and $A_m = A = \text{const}$ as the consequence of the remaining equations of motion (7)–(10) (this statement will be proved in Section 9).

Substitution of the solution (139) into (134) yields the explicit expression for the space component of the Lorentz connection

$$\omega_1 = \partial_1 q_\perp - \frac{1}{2} \partial_1 \ln \left( \frac{\partial_1 \pi + p}{\partial_1 \pi - p} \right) - \frac{1}{2} pU - \frac{p}{2} \left( V - \frac{\rho}{2} m^2 X^2 \right) e^{-Q+2T}$$

$$+ \frac{\partial_1 \pi}{Y} P \partial_1 X + \frac{p}{Y} \left( \frac{1}{2\rho} P^2 + \frac{\rho}{2} \partial_1 X^2 \right). \quad (140)$$

Together with Equations (125) and (136) it provides a general solution to the constraints in the case $A$ (129).

The case $B$. $Y \neq 0$, $\partial_1 \pi = 0$. Now the constraints are

$$H_0 = p(\partial_1 q_\perp - \omega_1) - \frac{1}{2} \rho^2 U + e^{2q} \left( V - \frac{\rho}{2} m^2 X^2 \right) - \frac{1}{2} \rho P^2 - \frac{\rho}{2} \partial_1 X^2 = 0, \quad (141)$$

$$H_1 = -\partial_1 p + p \partial_1 q + P \partial_1 X = 0. \quad (142)$$

First we solve the last constraint with respect to $q$

$$q = B(\tau) + \ln p - \int_{0}^{\sigma} \frac{d\sigma'}{p} P \partial_1 X, \quad (143)$$

where $B = B(\tau)$ is an arbitrary function of time. Afterwards the constraint $H_0$ is solved with respect to $\omega_1$

$$\omega_1 = \partial_1 q_\perp - \frac{1}{p} \left( \frac{1}{2} \rho^2 U + p^2 \left( V - \frac{\rho}{2} m^2 X^2 \right) e^{2B-2} \int_{p}^{\partial_1 \pi} \frac{dp}{p} \right) + \frac{1}{2} \rho P^2 + \frac{\rho}{2} \partial_1 X^2. \quad (144)$$
Though the solution of the constraints in this case differ from Case A, Equations (143), (144) coincide formally with Equations (136) and (140) for $\partial_1 \pi = 0$ with a suitable redefinition of $B(\tau)$. Therefore Case B may be considered as a subcase of A.

The case C. $Y = 0$, $\partial_1 \pi \neq 0$. In this case equations for $q$ and $\omega_1$ are purely algebraic. The condition $Y = 0$ yields

$$p = \pm \partial_1 \pi.$$  \hfill (145)

For two possible choices of signs the difference $H_0 - H_1$ and the sum $H_0 + H_1$ of the constraints (126), (127) immediately yield

$$-g_{11} = e^{2q} = \frac{\rho P_+^2}{2(V - \frac{\rho}{2} m^2 X^2)},$$  \hfill (146)

where $P_\pm$ is defined by (123). Afterwards the constraint $H_1$ yields a solution for $\omega_1$

$$\omega_1 = \partial_1 q_\perp \pm \partial_1 q + \frac{\partial_1^2 \pi}{\partial_1 \pi} + \frac{P \partial_1 X}{\partial_1 \pi},$$  \hfill (147)

where $q$ is given by Equation (146). The upper and lower signs in Equations (143)–(147) must be chosen simultaneously.

The case D. $Y = 0$, $\partial_1 \pi = 0$. For this case $\pi = \pi (\tau)$ and $p = 0$. For these gauge fixing functions the constraints become

$$H_0 = e^{2q} \left( V - \frac{\rho}{2} m^2 X^2 \right) - \frac{1}{2} \rho P^2 - \frac{\rho}{2} \partial_1 X^2 = 0,$$  \hfill (148)

$$H_1 = P \partial_1 X = 0.$$  \hfill (149)

We see that the space component of the Lorentz connection does not enter the constraint equations and remains arbitrary. For two subcases $P = 0$ and $\partial_1 X = 0$ following from Equation (149) the constraint (148) may be solved with respect to $q$

$$2q = \ln \frac{\rho \partial_1 X^2}{2(V - \frac{\rho}{2} m^2 X^2)}, \quad P = 0,$$  \hfill (150)

$$2q = \ln \frac{\rho^2}{2 \rho (V - \frac{\rho}{2} m^2 X^2)}, \quad \partial_1 X = 0.$$  \hfill (151)

9 Time components of the geometric variables

In the previous section we solved the constraints for $q$, $p_\perp$, and $\omega_1$ with $p$, $q_\perp$, and $\pi$ defined by the gauge fixing function (124). Making the inverse canonical transformation (17) and (18) one immediately gets a general solution to the constraints in terms of the original variables

$$e_1^\pm = \pm \frac{1}{\sqrt{2}} e^{q^\pm q_\perp},$$  \hfill (152)

$$p_\pm = \frac{1}{\sqrt{2}} e^{-q^\pm q_\perp} (\partial_1 \pi \pm p),$$  \hfill (153)

$$\omega_1 = \partial_1 q_\perp - \frac{1}{\partial_1 \pi} (\partial_1 p - p \partial_1 q - P \partial_1 X),$$  \hfill (154)
The function $q$ is given by (136), (143), (144) or (150), (151) in terms of the gauge fixing functions $\pi$, $p$, and $q_{\perp}$ for Cases A, B, C, or D, respectively. We shall keep it as it stands to treat Cases A, B, C simultaneously. In Case D the space component of the Lorentz connection is not defined by the constraints, and the scalar field is restricted by Equation (149). This case will be treated separately in Sections 10.4 and 10.5.

The next problem is to reconstruct time components of the Lorentz connection $\omega_0$ and the zweibein $e_0^\alpha$. To this end one has to solve the remaining six equations of motion: (7), (8) and time components of (9), (10). Let us write them for light-cone components to be as close to the matterless case as possible. The light-cone components of the energy momentum tensor $T_{\pm\pm} = e^\alpha e^\beta T_{\alpha\beta}$ are

$$
T_{++} = \frac{1}{4(e_1^+)^2} p_+^2 = \frac{1}{2} e^{-2q_{\perp} - 2q_{\perp}} P_-, \\
T_{+-} = T_{-+} = \frac{1}{2} m^2 X^2, \\
T_{--} = \frac{1}{4(e_1^-)^2} p_-^2 = \frac{1}{2} e^{-2q_{\perp} + 2q_{\perp}} P_+,
$$

where $P_{\pm}$ is given by Equation (123). Afterwards the remaining equations become

$$
\dot{\omega}_1 - \partial_1 \omega_0 - (e_0^- e_1^- + e_0^+ e_1^+) \left( \frac{1}{2} \rho p_0 U' + V' - \frac{\rho'}{2g_{11}} P_+ P_- \right) = 0,
$$

$$
\dot{\omega}_0 e_0^+ - \omega_1 e_0^- + \omega_1 e_0^+ - (e_0^- e_1^+ - e_0^+ e_1^-) p_- U = 0,
$$

$$
\dot{e}_0^- - \partial_1 e_0^- + \omega_0 e_1^- - \omega_1 e_0^- - (e_0^- e_1^+ - e_0^+ e_1^-) p_+ U = 0,
$$

$$
\dot{\pi} - e_0^+ p_+ + e_0^- p_- = 0,
$$

$$
\dot{p}_+ + \omega_0 p_+ + e_0^- \left( \frac{1}{2} \rho p_0 U + V - \frac{1}{2} \rho m^2 X^2 \right) + e_0^+ \frac{\rho}{4(e_1^+)^2} P_+^2 = 0,
$$

$$
\dot{p}_- - \omega_0 p_- - e_0^+ \left( \frac{1}{2} \rho p_0 U + V - \frac{1}{2} \rho m^2 X^2 \right) - e_0^- \frac{\rho}{4(e_1^-)^2} P_-^2 = 0,
$$

The first three equations are linear inhomogeneous differential equations for $\omega_0$ and $e_0^\pm$. The last three are purely algebraic in these variables. All other geometric variables are known functions and scalars are arbitrary. Equations (153), (160) are dependent between themselves because there are three identities (14), (16) as the consequence of the invariance of the action under local Lorentz rotations and general coordinate transformations. Therefore there is no need to solve all of them.

Let us consider Cases A and B with $Y \neq 0$. In these cases $p_- \neq 0$ and $p_+ \neq 0$ due to Equation (133). First we solve two algebraic Equations (158) and (164) for $e_0^-$ and $\omega_0$ in terms of $e_0^+$

$$
e_0^- = \frac{1}{p_-} ( - \pi + e_0^+ p_+ ),
$$

$$
\omega_0 = \frac{1}{p_-} \left[ \dot{p}_- + \frac{\rho \pi}{4(e_1^-)^2 p_-} P_-^2 - e_0^+ \left( \frac{1}{2} \rho p_0 U + V - \frac{1}{2} \rho m^2 X^2 + \rho \frac{p_+}{4(e_1^+)^2 p_+} P_+^2 \right) \right].
$$
The next step depends on the value of the determinant

\[ D = \frac{\partial_1 \pi - p}{\partial_1 \pi + p} P^2 \left( -1 \right) - \frac{\partial_1 \pi + p}{\partial_1 \pi - p} P^2 \left( +1 \right) \]  

(163)

for the system of linear algebraic Equations (158)–(160) for \( e_0^\pm \) and \( \omega_0 \). Note that in the absence of matter this determinant vanishes identically.

If the determinant is nonzero \( D \neq 0 \) then one may solve the remaining algebraic Equation (159) and get the solution for all time components of the geometric variables

\[ e_0^\pm p_\pm = \frac{2}{\rho D} \left[ -\frac{1}{2} \dot{Y} + Y \dot{q} + \dot{\pi} \left( \frac{1}{2} Y U + V e^{2q} - \frac{1}{2} \rho m^2 X^2 e^{2q} \right) \right] - \frac{\dot{\pi} \partial_1 \pi \pm p}{D \partial_1 \pi \mp p} P^2, \]

(164)

\[ \omega_0 = \dot{q}_- - \frac{1}{Y} (\partial_1 \pi \dot{p} - p \partial_1 \dot{\pi}) + \frac{\rho \dot{\pi} P^2 P_+^2}{DY}, \]

(165)

where the \( p_\pm \) are given by (153). The remaining differential equations are fulfilled as the consequence of gauge invariance given by linear dependencies (14), (16). Thus the whole system of equations of motion is solved for arbitrary scalar field.

If the determinant is zero \( D = 0 \) then the remaining algebraic equation yields the equation for an up to now arbitrary function \( A_m(\tau) \)

\[ -Y \dot{T} + \frac{Y}{2 (A_m + W_m)} \left( \dot{A}_m + \dot{W}_m - \frac{\dot{\pi}}{\partial_1 \pi} \partial_1 W_m \right) + \frac{\rho \dot{\pi}}{2} \frac{\partial_1 \pi + p}{\partial_1 \pi - p} P_+^2 = 0. \]

(166)

In the matterless case \( T = 0 \) and for the gauge fixing function \( \pi = \pi(\sigma) \) depending on the space coordinate only, \( \dot{W}_m = 0 \), and Equation (166) reduces to \( \dot{A}_m = 0 \) and hence \( A_m = \text{const.} \).

To find the zero component \( e_0^+ \) for \( D = 0 \) one is free to solve one of the differential equations. We choose it to be Equation (156). Substituting (161), (162) and the solution for the space components of the geometric variables one finally obtains the first order differential equation

\[ -\partial_1 e_0^+ + e_0^+ F_m + G_m = 0, \]

(167)

where

\[ F_m = \partial_1 \left( q_- + \frac{1}{2} \ln \left| \frac{\partial_1 \pi - p}{\partial_1 \pi + p} (A_m + W_m) \right| \right) - \frac{1}{2} \rho Q + T, \]

(168)

\[ G_m = \frac{1}{\sqrt{2}} e^{q_+ + q_-} \left( 2q - \frac{\partial_1 \pi - \dot{p}}{\partial_1 \pi - p} + \dot{\pi} U - \frac{\rho \dot{\pi}}{(\partial_1 \pi - p)^2} P_+^2 \right). \]

(169)
A general solution to this equation is given by
\[ e_0^+ = e^{\tilde{F}_m} \left( C(\tau) + \int d\sigma G_m e^{-\tilde{F}_m} \right), \] (170)
where \( \tilde{F}_m \) is a primitive
\[ \tilde{F}_m = \int d\sigma F_m. \] (171)

The remaining time components may be found from (161) and (162). This provides a solution for all geometric variables for zero determinant (163).

10 Effective action for a scalar field

In this section we write down effective equations of motion for a scalar field when all geometric variables are excluded by means of Equations (7)–(10). That is, we solve all equations of motion for the geometric variables in terms of a scalar field which is assumed to be arbitrary (this is already done in previous sections) and substitute this solution back into the equation for a scalar field. We are able to do this in the Hamiltonian form. First, we rewrite Equation (11) for a scalar in the Hamiltonian form
\[ \dot{X} = \frac{\rho g_{11}}{g_{01}} P + \frac{\rho \partial_1}{g_{11}} X, \] (172)
\[ \dot{P} = \partial_1 \left( \frac{\rho}{g_{11}} \partial_1 X \right) + \partial_1 \left( \frac{g_{01}}{g_{11}} P \right) + \epsilon m^2 X. \] (173)

From now on for simplicity we shall use the gauge
\[ \pi = \pi(\sigma), \quad p = 0, \quad q_\perp = 0, \] (174)
assuming \( \partial_1 \pi \neq 0 \). This gauge choice corresponds to Case A of a general situation. Then \( Y = \partial_1 \pi^2 \). In this gauge expressions for nonlocal quantities (137)–(139) become
\[ 2T(\tau, \sigma) = - \int_0^\sigma \frac{d\sigma'}{\partial_1 \pi} \left( \frac{1}{\rho} P^2 + \rho \partial_1 X^2 \right), \] (175)
\[ W_m(\tau, \sigma) = \int_0^\sigma d\sigma' \partial_1 \pi \left( V - \frac{\rho}{2} m^2 X^2 \right) e^{-Q+2T}, \] (176)
\[ e^{2q} = \frac{\partial_1 \pi^2}{2(A_m + W_m)} e^{-Q+2T}. \] (177)

The solution (132)–(134) for the space components of geometric variables simplifies
\[ e_1^\pm = \pm \frac{e^q}{\sqrt{2}}, \] (178)
\[ p_\pm = \frac{\partial_1 \pi e^{-q}}{\sqrt{2}}, \] (179)
\[ \omega_1 = \frac{1}{\partial_1 \pi} P \partial_1 X. \] (180)
Remember that we are considering the domain of the space-time where \( A_m + W_m > 0 \) which in the matterless case corresponds to the domain outside a horizon of the Schwarzschild black hole.

Differentiating Equation (176) with respect to \( \sigma \) and using (177) one easily sees that the function \( A_m + W_m \) satisfies the following differential equation

\[
\partial_1 (A_m + W_m) = \frac{2e^{2q}}{\partial_1 \pi} \left( V - \frac{1}{2} \rho m^2 X^2 \right) (A_m + W_m)
\]

which will be used in the following section.

Reconstruction of time components of the geometric variables depends on the value of the determinant (163) which is considerably simplified in the gauge (174)

\[
D = -\frac{4}{\rho} P \partial_1 X.
\]

Roughly speaking in the considered gauge it coincides with the momentum density of a scalar field. We consider the cases \( D \neq 0 \) and \( D = 0 \) separately. Note that the case of zero determinant coincides with the constraint \( H_1 = 0 \) Equation (149) in Case D.

### 10.1 Moving scalar field \( D \neq 0 \)

If the determinant (182) is nonzero \( D \neq 0 \) then the scalar field moves. In this case to reconstruct the time components of geometric variables one has to use Equations (164)–(165). Straightforward calculations yield

\[
e_0^\pm = -\frac{\partial_1 \pi \dot{q} e^q}{\sqrt{2} P \partial_1 X},
\]

\[
\omega_0 = \frac{\dot{q}}{P \partial_1 X} \left( \frac{1}{2} \partial_1 \pi^2 U + V e^{2q} - \frac{\rho}{2} m^2 X^2 e^{2q} + \frac{1}{2} \rho P^2 + \frac{\rho}{2} \partial_1 X^2 \right).
\]

The relation \( e_0^+ = e_0^- \) follows from Equation (161) because \( p_+ = p_- \) and \( \dot{\pi} = 0 \) in the chosen gauge (174). From (178) and (183) one gets expressions for the determinant of the zweibein

\[
e = -\frac{\partial_1 \pi \dot{q} e^q}{P \partial_1 X}
\]

and the metric

\[
ds^2 = e^{2q} \left( \frac{\partial_1 \pi^2 \dot{q}^2}{(P \partial_1 X)^2} d\tau^2 - d\sigma^2 \right).
\]

Substitution of the geometric variables into the equations of motion for a scalar field (172), (173) yields the effective equations

\[
\dot{X} = \frac{\partial_1 \pi \dot{q}}{\rho P \partial_1 X} P,
\]

\[
\dot{P} = \partial_1 \left( \frac{\rho \partial_1 \pi \dot{q}}{P \partial_1 X} \partial_1 X \right) - \rho m^2 \partial_1 \pi \dot{q} e^{2q} \frac{\partial_1 \pi}{P \partial_1 X} X.
\]
They may be rewritten in an equivalent and more suitable form, which does not contain the time derivative $\dot{q}$. Indeed, the function $\partial_1 \pi^2 \dot{q} / P \partial_1 X$ satisfies the following ordinary differential equation

$$\partial_1 \left( \frac{\partial_1 \pi^2 \dot{q}}{P \partial_1 X} \right) = 2 \frac{e^{2q}}{\partial_1 \pi} \left( V - \frac{\rho}{2} m^2 X^2 \right) \frac{\partial_1 \pi^2 \dot{q}}{P \partial_1 X}. \quad (189)$$

This is a crucial observation, and therefore we sketch the proof. Straightforward calculation of the left hand side of (189) and elimination of the time derivatives $\dot{X}$ and $\dot{P}$ using equations of motion (187), (188) yields

$$\partial_1 \left( \frac{\partial_1 \pi^2 \dot{q}}{P \partial_1 X} \right) = 2 \frac{\partial_1 \pi}{\partial_1 \pi} \left( \frac{\partial_1 \pi^2 \dot{q}}{P \partial_1 X} \right) - 2 \partial_1 \pi e^{2q} \left( V - \frac{\rho}{2} m^2 X^2 \right) \frac{\dot{q}}{P \partial_1 X}. \quad (189)$$

This is equivalent to Equation (189). The same equation is satisfied by the function $A_m + W_m$ (181). Therefore they may differ only by an arbitrary function of time which is absorbed by a redefinition of $A_m$. Hence without loss of generality we set

$$\frac{\partial_1 \pi^2 \dot{q}}{P \partial_1 X} = \frac{1}{\sqrt{\kappa}} (A_m + W_m). \quad (190)$$

Here we introduced the coupling constant $\kappa$ for dimensional reasons. This relation is valid only when the effective equations of motion for scalars are satisfied. Afterwards Equations (187), (188) and the metric (186) are rewritten in an equivalent form

$$\sqrt{\kappa} \dot{X} = \frac{1}{\rho \partial_1 \pi} (A_m + W_m) P, \quad (191)$$

$$\sqrt{\kappa} \dot{P} = \partial_1 \left[ \frac{\rho}{\partial_1 \pi} (A_m + W_m) \partial_1 X \right] - \frac{1}{2} \partial_1 \pi e^{-Q+2T} \rho m^2 X, \quad (192)$$

$$dS^2 = \frac{1}{2} e^{-Q+2T} \left[ \frac{1}{\kappa} (A_m + W_m) d\tau^2 - \frac{d\pi^2}{A_m + W_m} \right]. \quad (193)$$

For the matterless case $T = 0$, $A_m = A = \text{const}$, $W_m = W$, and the last expression for the line element coincides with (114). For spherically reduced gravity in the matterless case it yields the Schwarzschild solution (see Section 11). Using relation (190) the solution for the time components (183), (184) becomes

$$e_0^\pm = - \frac{1}{\sqrt{2\kappa}} \frac{A_m + W_m}{\partial_1 \pi} e^\varphi, \quad (194)$$

$$\omega_0 = \frac{A_m + W_m}{\sqrt{\kappa \partial_1 \pi}} \left( \frac{1}{2} \partial_1 \pi^2 U + V e^{2q} - \frac{\rho}{2} m^2 X^2 e^{2q} + \frac{1}{2\rho} P^2 + \frac{\rho}{2} \partial_1 X^2 \right). \quad (195)$$

The important point is that the effective equations of motion for a scalar field (191), (192) do not contain any metric components and are written entirely in terms of scalar fields $X$, $P$, and the gauge fixing function $\pi(\sigma)$. All geometric variables are expressed in terms of the scalar field only by Equations (178)–(180) and (194), (195). In particular, the two-dimensional scalar curvature and torsion squared term are

$$R = e^{-2q} \left( -\frac{\rho'}{\rho} P^2 + \rho' \partial_1 X^2 + U' \partial_1 \pi^2 \right) - 2V' + \rho' m^2 X^2, \quad (196)$$

$$T_a^a = 4e^{-2q} \partial_1 \pi^2 U^2. \quad (197)$$

In other words we have proved the following theorem.
**Theorem 1** Any solution of the matter-gravity equations of motion (7)–(11) in the gauge (174) has the form (178)–(180), (194), (195) where the scalar field satisfies the effective equations of motion (191), (192) under the condition $P \partial_1 X = 0$. Inversely, for arbitrary solution of the effective equations of motion for a scalar field with $P \partial_1 X \neq 0$ the zweibein and Lorentz connection constructed by Equations (178)–(180), (194), (195) satisfies the original equations of motion (7)–(11).

Note that the Lorentz connection does not enter the effective equations for scalars (191), (192) as it should be because a scalar field does not feel Lorentz rotations.

There arises a natural question whether the effective action leading to the effective equations of motion exists or not. The answer is positive

$$S_{\text{eff}} = \int d\tau \left( \int_0^\infty d\sigma P \dot{X} - H_{\text{eff}} \right),$$

where the effective Hamiltonian has the form

$$H_{\text{eff}} = \frac{1}{\sqrt{\kappa}} \int_0^\infty d\sigma \left[ \frac{A_m}{\partial_1 \pi} \left( \frac{1}{2\rho} P^2 + \frac{\rho}{2} \partial_1 X^2 \right) - \frac{1}{2} \partial_1 \pi \left( V - \frac{1}{2} \rho m^2 X^2 \right) e^{-Q+2T} \right].$$

One may check by straightforward calculations that variation of this effective action yields the effective equations of motion for scalars (191), (192). The limits of integration in (199), (175), and (176) are adapted for spherically reduced gravity considered in Section 11. In other models they may be changed. The unusual feature of the effective action is its nonlocality due to the term with $T$.

The intriguing point is the following. The Hamiltonian (22) of the original system is the linear combination of the constraints. If one solved the equations of motion for the geometric variables and substituted the solution back into the Hamiltonian one would get zero because the constraints constitute part of the equations of motion. The existence of the effective action (199) is a very important observation because the value of $H_{\text{eff}}$ is conserved for any trajectory is the phase space. It may be helpful for the analysis of the equations of motion and quantization of the model.

### 10.2 An example

The effective action for a scalar field obtained in previous sections is produced by the boundary term which must be added to the original action (1). To clarify its appearance we consider a simple example in this section. Notations here have nothing to do with the rest of the paper.

Let the model be described by two sets of canonically conjugate variables $q, p$ and $q^*, p^*$ depending on $\tau$ and $\sigma$. Consider the action written in the Hamiltonian form

$$S = \int_{-\infty}^{\infty} d\tau \int_0^\infty d\sigma (p \dot{q} + p^* \dot{q}^* - \lambda G),$$

where the Hamiltonian is given by a single constraint

$$H = \int_0^\infty d\sigma \lambda G, \quad G = -\partial_1 q + H^*(q^*, p^*).$$
Here $H^*(q^*, p^*) \geq 0$ is some nonnegative function which does not contain space derivatives of its arguments, and $\lambda$ is a Lagrange multiplier. For definiteness we specify the limits of integration.

One may pose two variational problems for the action (200). The most common one assumes that variations of all fields are compactly supported functions. In this case one can freely integrate by parts and drop all boundary terms. In the second variational problem variations of the fields are not assumed vanishing at the limits of integration. For this problem the action is sensitive to addition of boundary terms, and besides the equations of motion one obtains boundary conditions on the fields. For example, this problem is considered in the open bosonic string theory where the variational problem produces equations of motion and Neumann boundary conditions.

Let us consider the first variational problem for the action (200). Then one obtains only equations of motion
\begin{align*}
\dot{q} &= 0, \\
\dot{p} &= -\partial_1 \lambda, \\
\dot{q}^* &= \lambda \frac{\partial H^*}{\partial p^*}, \\
\dot{p}^* &= -\lambda \frac{\partial H^*}{\partial q^*}, \\
G &= -\partial_1 q + H^*(q^*, p^*) = 0.
\end{align*}
Equation (205) is a constraint on the canonical variables, and it is obviously of the first class
\[ \{G, G'\} = 0, \]
where $G' = G(\tau, \sigma')$. Therefore $\{G, H\} = 0$, and there are no other constraints. To eliminate the unphysical (nonpropagating) degree of freedom one has to impose one gauge condition. In the canonical gauge fixing approach \[18, 49\] it must form a system of second class constraints together with $G$. We choose it to be
\[ F = p - p(\sigma) = 0, \]
where $p(\sigma)$ is a given function of space coordinate only.

We see that a pair of canonical variables $q, p$ do not describe a physical degree of freedom and may be eliminated from the model explicitly. To this end we solve the equations of motion and the constraint in the chosen gauge. First we solve the constraint (203)
\[ q = \int_0^\sigma d\sigma' H^*(q^*, p^*) + q_0(\tau), \]
where $q_0(\tau)$ is an arbitrary function of time. To be consistent with the equation of motion (201) the following equation must hold $q_0 = 0$. Thus $q_0 = \text{const}$, and is insignificant. The Lagrange multiplier is defined by Equation (202) with $\dot{p} = 0$ which has a general solution
\[ \lambda = \lambda_0(\tau), \]
where $\lambda_0(\tau)$ is an arbitrary function to be fixed, for example, by a boundary condition.

We see that the model describes a single physical degree of freedom $q^*, p^*$ with the effective Hamiltonian $\lambda_0 H^*(q^*, p^*)$. There are two important observations. First, the effective Hamiltonian has the usual form and was obtained by local elimination of the unphysical degree of freedom. It means that the effective Hamiltonian does not depend on whether the space is closed (a circle) or open (line). Second, the effective Hamiltonian cannot be obtained from the action (200) by going to the constraint surface in the fixed gauge because

$$ S \bigg|_{F=0,G=0} = \int d\tau d\sigma p^* \dot{q}^*. $$

To resolve the problem let us consider the second variational problem. We assume that variations of the physical degree of freedom $q^*, p^*$ and the Lagrange multiplier are compactly supported. We are free to do this because the physical degree of freedom is not restricted by the constraint. At the same time the variation of the unphysical degree of freedom $\delta^* q(\tau, \sigma)$ caused by the variations $\delta q^*$ and $\delta p^*$ has the form

$$ \delta^* q = \int_0^\sigma d\sigma' \left( \frac{\partial H^*}{\partial q^*} \delta q^* + \frac{\partial H^*}{\partial p^*} \delta p^* \right). $$

We see that this variation cannot be compactly supported in space as the consequence of the constraint equation. Therefore we are not allowed to drop boundary terms in space containing $q$. At the same time the constraint does not restrict variations of $q$ at time boundaries, and we assume that $\delta^* q(\pm \infty, \sigma) = 0$. Consider the new action

$$ S_{ph} = S - \int d\tau \lambda q \bigg|_{\sigma=0}^\infty $$

(208)

differing from the old one by the boundary term. This boundary term is chosen in such a way as to compensate the boundary contribution to the variation of $S$ with respect to $q$. Therefore the variation of the action (208) yields the same equations of motion (202)–(205), and no boundary conditions arise even for nonvanishing variations $\delta q$. The effective Hamiltonian is now obtained by going to the constraint surface

$$ S_{ph} \bigg|_{F=0,G=0} = \int d\tau \int_0^\infty d\sigma \left( p^* \dot{q}^* - \lambda(\tau, \infty) H^*(q^*, p^*) \right). $$

(209)

The boundary term in the upper limit is given by the integral over the whole space (206) and hence produce the effective Hamiltonian with a well defined Hamiltonian density. The reason for this is that the constraint is a differential equation which does not admit compactly supported solutions.

The above consideration is valid also for a compact space. Let $\sigma \in (0, 2\pi)$. We assume that the physical degree of freedom is described by smooth functions $q^*$ and $p^*$ given on a circle. The unphysical degree of freedom and its variation cannot be smooth functions on a circle because of the constraint (206), and a cut with the appropriate boundary condition must be done. Therefore for a circle one simply has to change the limits of integration over $\sigma$ in (209).
In a more general gauge

\[ F = p - p(\tau, \sigma) = 0, \]

where a function \( p(\tau, \sigma) \) may depend on time, the effective action acquires an additional contribution due to the term \( p\dot{q} \). One can easily check that this contribution does not change the final answer. Indeed,

\[
\Delta S \bigg|_{F=0,G=0} = \int d\tau d\sigma \, p\dot{q} = \int d\tau d\sigma \, (-\dot{p}q).
\]

Using Equation (202) which defines \( \lambda \) and integrating by parts we get

\[
\Delta S \bigg|_{F=0,G=0} = \int d\tau d\sigma (-\lambda \partial_1 q) + \int d\tau \lambda q \bigg|_{\sigma=0}^{\infty}.
\]

The first term now reproduces the effective Hamiltonian \( \lambda H^\ast \) and the second one is cancelled by the boundary term (208). We see that the time dependent gauge also produces correctly the effective Hamiltonian. The considered example shows that this procedure is not the only one, and the origin of the nontrivial Hamiltonian for the physical degree of freedom lies in the boundary term. It arises in the analysis of the constraint equation which does not admit compactly supported solutions.

10.3 The effective action as a boundary term

The example considered in the previous section describes correctly a general situation in gravity models. In our case the constraints \( G = 0 \) and \( H_1 = 0 \) in Section 8 were solved algebraically with respect to \( p_\perp \) and \( \omega_1 \). The constraint \( H_0 = 0 \) is a differential equation with respect to \( q \), and its solution (136) cannot be compactly supported. The Hamiltonian (22) is equal to

\[
H = \int d\sigma \left( \left( \omega_0 + \frac{e-g_{01}}{g_{11}} \omega_1 \right) G - \frac{e}{g_{11}} H_0 + \frac{g_{01}}{g_{11}} H_1 \right).
\]

The only term which contains spatial derivative of \( q \) is the second one. Therefore to compensate the boundary contribution due to the variation of \( q \) we add the boundary term to the action (1)

\[
\Delta S = -\int d\tau \frac{e}{g_{11}} \partial_1 \pi q \bigg|_{\sigma=0}^{\infty}. \tag{210}
\]

Expressing everything in terms of the scalar field and gauge fixing functions we get

\[
\frac{e}{g_{11}} \partial_1 \pi = \frac{1}{\sqrt{\kappa}} (A_m + W_m).
\]

Afterwards the boundary terms are easily calculated

\[
-\frac{1}{\sqrt{\kappa}} (A_m + W_m)q \bigg|_{\sigma=\infty} = \frac{1}{2\sqrt{\kappa}} \int_0^{\infty} d\sigma \partial_1 \pi \left( V - \frac{1}{2} \rho m^2 X^2 \right) e^{-Q+2T},
\]

\[
-\frac{1}{\sqrt{\kappa}} (A_m + W_m)q \bigg|_{\sigma=0} = \frac{1}{2\sqrt{\kappa}} A_m \int_0^{\infty} d\sigma \partial_1 \pi \left( \frac{1}{\rho} P^2 + \rho \partial_1 X^2 \right).
\]

They produce exactly the effective action (198).

The procedure of obtaining the effective action was noted in [30] in the case of spherically reduced gravity.
10.4 Static scalar field $P = 0$

The equation $D = 0$ for zero determinant (182) has two solutions $P = 0$ and $\partial_1 X = 0$ corresponding to static and homogeneous distribution of a scalar matter. In this and the following sections we consider these cases, respectively. They yield another classes of solutions to the whole system of the equations of motion.

For $D = 0$ Equation (166) in the gauge (174) reduces to

$$2\dot{T} = \frac{\dot{A}_m + \dot{W}_m}{A_m + W_m} \iff \dot{q} = 0,$$

(211)

and restricts an arbitrary function $A_m(\tau)$. Equation (167) for the time component of the zweibein $e_0^+$ takes a simple form

$$-\partial_1 e_0^+ + e_0^+ F_m = 0,$$

(212)

where

$$F_m = \partial_1 \left[ \frac{1}{2} \ln (A_m + W_m) - \frac{1}{2} Q + T \right].$$

It has a general solution (194) up to a nonzero factor depending on $\tau$ which may be absorbed by a redefinition of time coordinate. Expressions for the Lorentz connection $\omega_0$ and space components of geometric variables remain the same as before (195), (178)–(179) with $\omega_1 = 0$. The corresponding metric takes the form (193) as in the previous case of moving scalars.

Then the effective equations of motion for a scalar field take precisely the form (191), (192) as in the previous case. Equation (191) together with the restriction $P = 0$ yields $X = 0$. Hence the solution is static $X = X(\sigma)$. We see that the functions $T(\sigma)$ and $W_m(\sigma)$ defined by (175) and (176) depend on the space coordinate only. Therefore Equation (211) yields $A_m = \text{const}$.

Afterwards Equation (192) defines the static distribution of a scalar field

$$\partial_1 \left[ \frac{\rho}{\partial_1 \pi} (A_m + W_m) \partial_1 X \right] - \frac{1}{2} \partial_1 \pi e^{-Q + 2T} \rho m^2 X = 0.$$

(213)

It provides a general solution to the whole system of equations of motion in the static case. In the massless case $m = 0$ this equation simplifies to

$$\frac{\rho}{\partial_1 \pi} (A_m + W_m) \partial_1 X = \text{const}.$$

(214)

It may be rewritten in a local form. Differentiating it two times with respect to $\sigma$ and assuming $\partial_1 X \neq 0$ one obtains the equation

$$\frac{\partial^2 Z}{\partial_1 Z} - \frac{\partial_1 \pi}{\rho Z^2} - \frac{\partial^2 \pi}{\partial_1 \pi} + U \partial_1 \pi - \frac{\partial_1 V}{V} = 0,$$

(215)

where

$$Z(\sigma) = \frac{\partial_1 \pi}{\rho \partial_1 X}.$$

This equation will be solved in Section 11.2 for spherically reduced gravity.
### 10.5 Homogeneous solution $\partial_1 X = 0$

For the homogeneous solution the scalar field depends only on time coordinate $X = X(\tau)$. Expressions for all geometric quantities $e_1^\pm$, $\omega_1$ and $e_0^\pm$, $\omega_0$ are the same as in the previous case (178), (180) and (194), (195). The effective equations of motion reduce to

$$\sqrt{\kappa} \dot{X} = \frac{1}{\rho \partial_1 \pi} (A_m + W_m) P, \quad (216)$$

$$\sqrt{\kappa} \dot{P} = -\frac{1}{2} \partial_1 \pi e^{-Q + 2T} \rho m^2 X. \quad (217)$$

This system of equations defines the homogeneous solution $X = X(\tau)$ and $P = P(\tau, \sigma)$. In general the momenta may depend on both coordinates.

The solution simplifies for zero mass $m = 0$. Then Equation (217) yields $P = P(\sigma)$, and hence nonlocal quantities (173) and (176) depend only on space coordinate $T = T(\sigma)$ and $W_m = W_m(\sigma)$. In this case Equation (211) yields a solution $A_m = \text{const}$ as in the static case. We see that the right hand side of Equation (216) depends on the space coordinate only whereas the left hand side depends only on time. This is possible only for linear dependence on time

$$X = a \tau + b, \quad a, b = \text{const}. \quad (218)$$

Afterwards we obtain the equation for $P = P(\sigma)$

$$\frac{1}{\rho \partial_1 \pi} (A_m + W_m) P = a, \quad (219)$$

which is similar to Equation (214) for the static case. This nonlocal equation may be written in a local form after differentiating it two times. Introducing a new variable for $P \neq 0$

$$Z(\sigma) = \frac{\rho \partial_1 \pi}{P}$$

one gets

$$\frac{\partial^2 Z}{\partial_1 Z} - \frac{\rho \partial_1 \pi}{Z} - \frac{\partial^2 \pi}{\partial_1 \pi} + U \partial_1 \pi - \frac{\partial_1 V}{V} = 0 \quad (220)$$

which differs from (215) in the second term only.

We see that in both degenerate cases $P = 0$ or $\partial_1 X = 0$ one has to solve the same system of the effective equations of motion (191), (192), and the metric has the same form (193) as in a general case. The only difference is that $A_m$ must be constant for static and homogeneous solutions for a massless scalar field.

### 11 Spherically reduced gravity

Let us consider spherically reduced general relativity which is of great importance for our understanding of a black hole formation. We start with the Hilbert–Einstein
action proportional to the four-dimensional scalar curvature \( R^{(4)} \) minimally coupled to a massive scalar field. The four-dimensional Lagrangian is
\[
L = \kappa e R^{(4)} - 2\Lambda e + \frac{1}{2} e (g^{ij} \partial_i X \partial_j X - m^2 X^2), \quad e = \sqrt{|\det g_{ij}|}, \tag{221}
\]
where \( i, j = 0, 1, 2, 3, \kappa > 0 \), and the \( \Lambda \) are gravitational and cosmological constants. Assuming a spherical symmetry for a metric
\[
d s^2 = g_{\alpha \beta} d x^\alpha d x^\beta + \frac{1}{2\kappa} \pi d \Omega \tag{222}
\]
and for a scalar field
\[
X = X(x^\alpha), \quad \alpha = 0, 1, \tag{223}
\]
where
\[
d \Omega = d \theta^2 + \sin^2 \theta d \varphi^2
\]
is the usual metric on a unit sphere, the Lagrangian (221) is reduced to
\[
L = e \left[ -\frac{1}{2 \kappa} \tilde{R} + \frac{\partial \pi^2}{4 \pi} + 2\kappa + \frac{\Lambda \pi}{\kappa} - \frac{\pi}{4 \kappa} (\partial X^2 - m^2 X^2) \right]. \tag{224}
\]
Here \( \tilde{R} \) is the two-dimensional scalar curvature for a two-dimensional metric \( g_{\alpha \beta} \). We introduced a gravitational constant in the spherically symmetric ansatz for the metric (222) in order for the dilaton field to be dimensionless. Thus a minimally coupled scalar field in four dimensions after spherical reduction yields the two-dimensional scalar field nonminimally coupled to two-dimensional dilaton gravity. The Lagrangian (224) coincides with (57) for the functions \( U \) and \( V \) given by (6). We restrict ourselves to negative values of the dilaton field \( \pi < 0 \) to provide the four-dimensional metric with the signature \((+ - - -)\).

The spherical reduction of the four-dimensional action (221) to the two-dimensional one (224) was done on the level of the actions. In general this procedure is not equivalent to a reduction at the level of the equations of motion. In the case of spherically reduced gravity both reductions are equivalent [27]. That is, the substitution of (222) and (223) in the four-dimensional equations of motion yields the system of equations which is equivalent to the equations of motion following directly from the two-dimensional Lagrangian (224).

In this section we consider the case of a massless scalar field \( m = 0 \) and zero cosmological constant \( \Lambda = 0 \) which has attracted much interest in the last years due to the discovery of critical phenomena in gravitational collapse [28] (for a recent review see [29]). Note also that this matter model may be interpreted in terms of "superstiff" perfect liquid with the energy density=pressure equation of state [50]. For comparison we choose the Schwarzschild like coordinates
\[
\pi = -2\kappa \sigma^2, \quad \sigma \in (0, \infty), \tag{225}
\]
with the metric (223) is of the form
\[
d s^2 = g_{\alpha \beta} d x^\alpha d x^\beta - \sigma^2 d \Omega. \tag{226}
\]
For this gauge choice functions defining the two-dimensional gravity model (6) become

\[ U = -\frac{1}{4\kappa}\sigma^2, \quad V = -2\kappa, \quad \rho = -\sigma^2. \quad (227) \]

For spherically reduced gravity we choose the primitive (72) to be \( Q = \ln |\pi|/2 \) which in the gauge (225) yields

\[ e^{-Q} = \frac{1}{\sqrt{2\kappa}\sigma}. \]

To simplify the following analysis we introduce dimensionless coordinates and a scalar field

\[ \sigma = \frac{r}{\sqrt{\kappa}}, \quad \tau = \frac{t}{\sqrt{2\kappa}}, \quad X = 2\sqrt{\kappa}\tilde{X}, \quad P = 2\tilde{P}. \quad (228) \]

Next we redefine the nonlocal quantity

\[ A_m + W_m = 4\sqrt{2}\kappa(\tilde{A}_m + \tilde{W}_m). \quad (229) \]

Dropping tilde signs we arrive at the effective equations of motion

\[ \dot{X} = \frac{1}{r^3}(A_m + W_m)P, \quad (230) \]
\[ \dot{P} = \partial_1 [(A_m + W_m)r\partial_1 X], \quad (231) \]

where dots denote derivatives with respect to time \( t \), \( \partial_1 = \partial_r \), and

\[ 2T(t, r) = -\int_r^\infty \frac{d\sigma}{\sigma} \left( \frac{1}{\sigma^2}P^2 + \sigma^2\partial_1 X^2 \right) \leq 0, \quad \text{(232)} \]
\[ W_m(t, r) = \int_0^r d\sigma e^{2T} \geq 0. \quad (233) \]

Both functions are monotonically increasing functions of \( r \) with the boundary values \( T(t, \infty) = 0 \) and \( W_m(t, 0) = 0 \). The following bound holds \( W_m(t, r) \leq r \) because \( T \) is nonpositive. This is a full set of the effective equations of motion for spherically reduced gravity coupled to a massless scalar field. Here \( A_m = A_m(t) \) is an arbitrary function of time to be defined by a boundary condition. The effective Hamiltonian (199) leading to the effective equations of motion (230), (231) has the form

\[ H_{\text{eff}} = \frac{1}{2} \int_0^\infty dr \left[ \frac{A_m}{r^2} \left( \frac{1}{r^2}P^2 + r^2\partial_1 X^2 \right) + 1 - e^{2T} \right] \]
\[ = -A_m(t)T(t, 0) + \frac{1}{2} \left( \int_0^\infty dr - W_m(t, \infty) \right). \quad (234) \]

Here we inserted the unity in the integrand which does not alter the equations of motion for normalization. The reason for this will become clear later from the comparison with the free spherical waves in flat Minkowskian space-time. The expression
for the effective Hamiltonian may be rewritten in the other form. We integrate by parts
\[ \int_0^\infty dr (1 - e^{2T}) = r(1 - e^{2T}) \bigg|_0^\infty + \int_0^\infty dr e^{2T} \left( \frac{1}{r^2} P^2 + r^2 \partial_1 X^2 \right). \]
The first term in the right hand side may be dropped if the integral (232) converges at the infinity. Then the effective Hamiltonian is
\[ H_{\text{eff}} = \frac{1}{2} \int_0^\infty dr \left( e^{2T} + \frac{A_m}{r} \right) \left( \frac{1}{r^2} P^2 + r^2 \partial_1 X^2 \right). \]
(235)
The Hamiltonian density is clearly positive definite if
\[ e^{2T} + \frac{A_m}{r} > 0. \]
(236)
Let us make a remark concerning the positive energy theorem in asymptotically flat space-time in general relativity (for review, see [51, 48]). The theorem states that the total energy defined via the surface integral is positive under the reasonable assumptions on the matter energy-momentum tensor. The assumption of asymptotic flatness was essential because for closed universes the surface integrals were assumed to vanish. In the present paper we showed that this assumption is not valid because unphysical degrees of freedom cannot be compactly supported functions as a solution of the constraint equations. In our case we have the positive definite Hamiltonian density which is the same for open and closed Universes, and expression (235) proves the positive energy theorem for spherically symmetric solutions under the restriction (236).

The effective Hamiltonian explicitly depends on time through the function $A_m(t)$. Therefore the energy is not conserved
\[ \frac{dE}{dt} = \frac{\dot{A}_m}{2} \int_0^\infty dr \left( \frac{1}{r^2} P^2 + r^2 \partial_1 X^2 \right). \]
The effective Hamiltonian (234) for $A_m = 0$ was obtained in [30]. The arbitrary function $A_m(\tau)$ arose in a solution of the constraints which are differential equations with respect to $\sigma$ and cannot be eliminated by a reparameterization of the time coordinate. It may be fixed by a boundary condition on metric components. The effective equations of motion (230), (231) are similar to the equations of Section 3 in [24] obtained for the spherically reduced gravity in the second order formulation assuming regularity at the center $r = 0$. In the present paper the effective equations for scalars are written in the first order formulation in a general case and for a wider class of models (4).

The solution for a metric (193) in the spherically reduced gravity takes the form
\[ ds^2 = \frac{1}{\kappa} e^{2T} \left( \frac{A_m + W_m}{r} dt^2 - \frac{r}{A_m + W_m} dr^2 \right). \]
(237)
In general relativity the mass function $M(\tau, \sigma)$ in the Schwarzschild like coordinates is usually defined by the equations
\[ g_{00} = 1 - \frac{2M}{r} \quad \text{or} \quad g_{11} = \left( 1 - \frac{2M}{r} \right)^{-1}. \]
For the solution (237) it is
\[ 2M = r - e^{2T}(A_m + W_m) \quad \text{or} \quad 2M = r - e^{-2T}(A_m + W_m). \]

Since \( T(\tau, \infty) = 0 \), both definitions yield the same expression for the total mass \( M_\infty = M(\tau, \infty) \)
\[ 2M_\infty = -A_m + \int_0^\infty dr (1 - e^{2T}). \]

For \( A_m = 0 \) the total mass equals the total energy \( M_\infty = H_{\text{eff}} \) and is conserved in time.

In the case of spherically reduced gravity the effective equations of motion (230), (231) are invariant under the scaling transformation reflecting the absence of mass (length) parameter in the model
\[
\begin{align*}
t' &= kt, \\
r' &= kr, \\
A'_m(t') &= kA_m(t),
\end{align*}
\]
parameterized by a nonzero constant \( k = \text{const} \). It means that if you have some solution to the equations of motion \( X(t, r), P(t, r) \) then the primed functions \( X'(t', r') \), \( P'(t', r') \) satisfy the same set of equations. Under this transformation the integrals (232), (233) transform as
\[
\begin{align*}
T'(t', r') &= T(t, r), \\
W'_m(t', r') &= W_m(t, r).
\end{align*}
\]

11.1 Small scalar field and asymptotics

The zero approximation for a small scalar field is the matterless case \( X = 0 \) and \( P = 0 \) which is obviously a solution to the effective equations of motion (230), (231). Then nonlocal quantities (232), (233) become
\[
T = 0, \quad W_m = r,
\]
Choosing \( A_m = -2M \) which must be constant in the matterless case we immediately get the time-radial part of the Schwarzschild metric
\[
ds^2 = \frac{1}{\kappa} \left[ (1 - \frac{2M}{r}) dt^2 - \frac{dr^2}{1 - \frac{2M}{r}} \right].
\]

Of course for \( M = 0 \) the Minkowskian space-time is also the solution to the whole system of the equations of motion.

Consider a small scalar field
\[
X \sim \epsilon, \quad P \sim \epsilon, \quad \epsilon \ll 1.
\]
Then \( T \sim \epsilon^2 \), and the first correction to \( W_m \) and hence to the metric is of the second order \( \epsilon^2 \) too. Therefore for a small scalar field effective equations of motion are linear
spherical wave equations in the background given by the Schwarzschild or Lorentz metric \((239)\). This is a good approximation because corrections to the metric are of the second order. In fact, this can be clearly seen at the very beginning because the energy-momentum tensor is quadratic in a scalar field and it is the source for corrections to the metric.

Now we show how the effective action \((234)\) reduces to the action of a free scalar field in the Minkowskian space-time or black hole background. For a small scalar field
\[ e^{2T} \approx 1 + 2T. \]
Inserting this approximation into the effective Hamiltonian \((234)\) for \(A_m = 0\) we get
\[ H_{\text{eff}} = -\int_0^\infty dr T = \frac{1}{2} \int_0^\infty dr \int_r^\infty \frac{d\sigma}{\sigma} \left( \frac{1}{\sigma^2} P^2 + \sigma^2 \partial_1 X^2 \right). \]
Exchanging the order of integration one arrives at the usual Hamiltonian for a spherically symmetric waves in Minkowskian space-time
\[ H_{\text{eff}} = \frac{1}{2} \int_0^\infty dr \left( \frac{1}{r^2} P^2 + r^2 \partial_1 X^2 \right). \]
This effective action is clearly positive definite. Here we see the role of the normalization unity in the integrand in \((234)\). Similar calculations for a small scalar field in the Schwarzschild background \(A_m = -2M\) reduce the effective Hamiltonian \((234)\) to
\[ H_{\text{eff}} = \frac{1}{2} \int_0^\infty dr \left( 1 - \frac{2M}{r} \right) \left( \frac{1}{r^2} P^2 + r^2 \partial_1 X^2 \right). \]
The obtained expression is positive definite outside the horizon of a black hole \(r > 2M\).

At large distances \(r \to \infty\) the effective action reduces to spherical waves in a flat background too (wave zone). Assume that all integrals are convergent at \(r \to \infty\). Then \(T(t, r) \to 0\) due to the upper limit of integration, and \(W_m \sim r\). Neglecting \(A_m \ll r\) one gets the effective Hamiltonian \((240)\) for spherical waves in flat Minkowskian space-time. In the wave zone the metric \((237)\) reduces to a Lorentzian one.

At small distances \(r \to 0\) we have
\[ W_m(t, r) \approx r e^{2T_0}, \]
where \(T_0 = T(t, 0) > -\infty\) is assumed to be finite. Then the effective equations of motion and the metric become
\[ \dot{X} = \frac{1}{r^3} (A_m + re^{2T_0}) P, \]
\[ \dot{P} = \partial_1 \left[ (A_m + re^{2T_0}) r \partial_1 X \right], \]
\[ ds^2 = \frac{1}{\kappa} e^{2T_0} \left[ (e^{2T_0} + \frac{A_m}{r}) dt^2 - \frac{1}{e^{2T_0} + \frac{A_m}{r}} dr^2 \right]. \]
11.2 The Fisher solution

For spherically reduced gravity the effective equations of motion can be analytically solved for the massless static scalar field $P = 0$ \[31\]. In this case Equation (215) becomes

$$Z'' = \frac{Z'}{rZ^2}, \quad (245)$$

where

$$Z(r) = \frac{1}{r \partial_1 X}.$$ 

This equation has a general solution depending on two arbitrary constants $k$ and $c$

$$r = k \left( y_1 - y \right)^{1/(y_1 - y_2)} \left( y - y_2 \right)^{1/(y_2 - y_1)}, \quad (246)$$

where

$$y = \frac{1}{Z}.$$ 

Here

$$y_{1,2} = \frac{c}{2} \pm \sqrt{\frac{c^2}{4} + 1},$$

are the roots of the quadratic equation

$$y^2 + cy - 1 = 0 \quad (247)$$

which has real roots $y_1 > 0$ and $y_2 < 0$ for arbitrary values of $c = \text{const}$. The constant $k$ corresponds to a scaling symmetry and is insignificant.

In this way we solved the effective equations of motion for a scalar field in the static case. To write the corresponding metric in elementary functions we choose $y$ as a spacial coordinate instead of $r$. For $r \in (0, \infty)$ the new coordinate varies within the interval $y \in (0, y_1)$ but in reverse direction. Straightforward calculations yield nonlocal quantities (252), (253) and a scalar field

$$2T = \frac{1}{y_1 - y_2} \ln \left( \frac{1 - \frac{y}{y_1}}{1 - \frac{y}{y_2}} \right) y_1^{y_2^{y_2+1} + y_2^{y_2+1}},$$

$$W = ky_1^{-y_1^{y_1+1}} (-y_2)^{-y_2^{y_2+1}} \left( \frac{1}{y} - \frac{1}{y_1} \right),$$

and the expression for a scalar field

$$X = \pm \frac{1}{y_1 - y_2} \ln \left| \frac{y - y_1}{y - y_2} \right|. \quad (248)$$

The scalar field is always singular at the origin of spherical coordinates $r = 0$, or $y = y_1$. Substitution of this solution directly into Equation (214) yields $A_m = k/y_1$ (this is a necessary step because this equation was differentiated).
Dropping the gravitational constant and the insignificant total factor depending on the roots $y_{1,2}$ one finally arrives at the static solution

$$ds^2 = (y_1 - y)^{\frac{y^2}{y^2 + 1}}(y - y_2)^{\frac{y^2}{y^2 + 1}} dt^2 - \frac{k^2}{y^4} (y_1 - y)^{\frac{y^2}{y^2 + 1}} (y - y_2)^{\frac{y^2}{y^2 + 1}} dy^2$$

$$- \frac{k^2}{y^2} (y_1 - y)^{\frac{y^2}{y^2 + 1}} (y - y_2)^{\frac{y^2}{y^2 + 1}} d\Omega.$$

This expression together with (248) solves the problem and describes all static spherically symmetric solutions for general relativity minimally coupled to a massless scalar field. Up to a rescaling a general solution is parameterized by one arbitrary constant $c$ defining the roots (247).

To get physical interpretation of these solutions one has to calculate the four-dimensional curvature components. Nonzero components of Christoffel’s symbols are

$$\Gamma^t_{ty} = \Gamma^t_{yt} = -\frac{y_1 + y_2}{2(y_1 - y)(y - y_2)},$$

$$\Gamma^y_{tt} = -\frac{y^4}{2k^2} (y_1 - y)^{\frac{y^2}{y^2 + 1}} (y - y_2)^{\frac{y^2}{y^2 + 1}} \frac{y_1 + y_2}{(y_1 - y)(y - y_2)},$$

$$\Gamma^y_{yy} = -\frac{2}{y} + \frac{y_1 + y_2}{2(y_1 - y)(y - y_2)},$$

$$\Gamma^\theta_{y\theta} = \Gamma^\theta_{\theta y} = \Gamma^\varphi_{y\varphi} = \Gamma^\varphi_{\varphi y} = -\frac{1}{y(y_1 - y)(y - y_2)},$$

$$\Gamma^t_{\theta t} = y,$$

$$\Gamma^\varphi_{\theta \varphi} = \Gamma^\varphi_{\varphi \theta} = \frac{\cos \theta}{\sin ^2 \theta},$$

$$\Gamma^\varphi_{\varphi y} = y \sin ^2 \theta,$$

$$\Gamma^\varphi_{\varphi \varphi} = -\sin \theta \cos \theta.$$

Afterwards we compute the nonzero curvature components

$$R^t_{tyty} = -\frac{y_1 + y_2}{y} (y_1 - y)^{\frac{y^2}{y^2 + 1}} (y - y_2)^{\frac{y^2}{y^2 + 1}}$$

$$R^t_{t\theta\theta} = \frac{y(y_1 + y_2)}{2} (y_1 - y)^{-\frac{y^2}{y^2 + 1}} (y - y_2)^{-\frac{y^2}{y^2 + 1}},$$

$$R^t_{t\varphi\varphi} = R^t_{t\varphi\varphi} \sin ^2 \theta,$$

$$R^y_{y\theta y\theta} = \frac{k^2(2y - y_1 - y_2)}{2y^3} (y_1 - y)^{-\frac{2y^2}{y^2 + 1}} (y - y_2)^{-\frac{2y^2}{y^2 + 1}},$$

$$R^y_{y\varphi y\varphi} = R^y_{y\varphi y\varphi} \sin ^2 \theta,$$

$$R^y_{\theta y\varphi\varphi} = -\frac{k^2(y - y_1 - y_2)}{y} (y_1 - y)^{-\frac{y^2}{y^2 + 1}} (y - y_2)^{-\frac{y^2}{y^2 + 1}} \sin ^2 \theta.$$

The Ricci tensor has only one nonvanishing component though many components of the full curvature tensor differ from zero

$$R^y_{yy} = -\frac{2}{(y_1 - y)^2 (y - y_2)^2}.$$
Now one can easily verify that the metric (249) together with the scalar field (248) satisfies Einstein equations

\[ R_{ij} = -2 \partial_i X \partial_j X, \quad (250) \]

for all indices \( i, j = 0, 1, 2, 3 \). Note that the Ricci tensor is invariant under rescaling of the metric. Therefore dropping a constant factor in the solution for the metric (249) does not alter this equation.

The four-dimensional scalar curvature for static solutions has the form

\[ R = \frac{2y^4}{k^2} (y_1 - y) \left( \frac{y_1^3}{y_1^2 + 1} (y - y_2) \right)^\frac{3}{y_2 + 1}. \quad (251) \]

At the origin of the spherical coordinate system \( y = y_1 \) the scalar curvature is singular because the exponent is always negative

\[ \frac{y_1^2 + 3}{y_1^2 + 1} < 0. \]

Note that here the scalar field is also singular.

At infinity \( y = 0 \) the scalar curvature tends to zero. The metric (249) is degenerate at this point. Nevertheless one can show that in the spherical coordinates \( t, r, \theta, \varphi \) the metric is asymptotically Lorentzian at infinity. It means that static solutions (249) describe maximally extended space-times with naked timelike singularity at the origin. The \( t, r \) slices are represented by the triangular Carter–Penrose diagram.

Static solution for \( c = 0 \) has a particular simple form and may be written in spherical coordinates explicitly. In this case Equation (246) takes the form

\[ \frac{Z - 1}{r} = \frac{r}{Z + 1}. \]

It yields the solution for the scalar field and nonlocal quantities

\[ X = \pm \frac{1}{2} \ln \frac{\sqrt{1 + r^2} - 1}{\sqrt{1 + r^2} + 1}, \quad (252) \]

\[ 2T = \frac{1}{2} \ln \frac{r^2}{1 + r^2}, \]

\[ W_m = \sqrt{1 + r^2} - 1. \]

Now one easily writes the solution for a metric (237)

\[ ds^2 = dt^2 - \frac{r^2}{1 + r^2} dr^2 - r^2 d\Omega. \quad (253) \]

This metric is degenerate at the origin and asymptotically flat at infinity. A recently discovered family of nonstatic solutions [32] intersects with static solutions by this representative.

The scalar curvature (251) for \( c = 0 \) becomes

\[ R = \frac{2}{r^4}. \quad (254) \]
Let us note that the two-dimensional part of the metric (253) by itself can be easily transformed to the Lorentz metric. Therefore two-dimensional $t, r$ curvature components identically vanish, and there is no singularity in the two-dimensional curvature. The singularity in (254) comes from the mixed radial-angular and angular components of the full four-dimensional curvature tensor.

### 11.3 The Roberts solution

In this section we consider the Roberts solution [32] providing a nontrivial solution to the effective equations of motion (230), (231) depending both on time and space coordinates. It is usually written in the form

$$X = \frac{1}{2} \ln \left( \frac{\sqrt{a^2 w^2 + r^2} - au}{\sqrt{a^2 w^2 + r^2} + au} \right) = \ln \left( \sqrt{1 + a^2 \frac{w^2}{r^2} - a \frac{u}{r}} \right),$$  \hspace{1cm} (255)

$$ds^2 = \left( 1 - \frac{2a^2 u}{\sqrt{a^2 w^2 + r^2}} \right) du^2 - \frac{2r}{\sqrt{a^2 w^2 + r^2}} dr du - r^2 d\Omega, \hspace{1cm} (256)$$

where $a = \text{const} \neq 0$ and $u$ is a light-cone coordinate. Without loss of generality we consider positive $a > 0$ because equations of motion (7)–(11) are invariant under the transformation $X \to -X$. In the wave zone corresponding to the limit

$$r \to \infty, \quad \frac{u}{r} \to 0,$$

the time-radial part of the metric (256) becomes

$$ds^2 = \left( 1 - \frac{2a^2 u}{r} \right) du^2 - 2dr du, \hspace{1cm} (257)$$

where we retained the first correction to the Lorentz metric, and the leading term for the scalar field (255) takes the form

$$X \approx -a \frac{u}{r} = -a \frac{v - t}{r}.$$

In this domain the scalar field indeed represents the free ingoing spherical wave in accord with a general statement in Section 11.1. We see that the Roberts solution describes asymptotically Minkowskian space-time. Note that asymptotic behavior of the Roberts metric (257) differs from the Schwarzschild metric, which in the Eddington–Finkelstein coordinates has the form

$$ds^2 = \left( 1 - \frac{2M}{r} \right) du^2 - 2dr du.$$

The Roberts solution takes a particular simple form in the light-cone coordinates which are useful for the analysis of the global structure of the space-time. Introducing the second light-cone coordinate $v$ related to $r$ and $u$ by the equation

$$r^2 = \frac{1}{4} (u - v)^2 - a^2 u^2, \hspace{1cm} (258)$$
the solution takes the form
\[
X = \frac{1}{2} \ln \frac{(1 - 2a)u - v}{(1 + 2a)u - v}, \quad u > v, \quad (259)
\]
\[
X = \frac{1}{2} \ln \frac{(1 + 2a)u - v}{(1 - 2a)u - v}, \quad u < v, \quad (260)
\]
\[
ds^2 = dudv - r^2d\Omega. \quad (261)
\]
By definition the right hand side of Equation (258) must be nonnegative. This restricts the possible range of lightlike coordinates
\[
[(1 + 2a)u - v][(1 − 2a)u - v] \geq 0. \quad (262)
\]
For definiteness we consider the domain where both multipliers are nonnegative. In this domain \( u \) must be greater than \( v \).

The lines \( r = \text{const} \) on the \( u, v \) plane are deformed ”hyperbolas” given by Equation (258).

The global structure of the space-time depends on the curvature singularities. Using Einstein equations for a massless scalar field (250) one easily obtains expression for the four-dimensional scalar curvature
\[
R^{(4)} = -2\partial X^2,
\]
which for the Roberts solution becomes
\[
R^{(4)} = \frac{2a^2uv}{r^4}. \quad (263)
\]
The scalar curvature is always singular at \( r = 0 \) which on the \( u, v \) plane corresponds to two crossing straight lines
\[
u = (1 \pm 2a)v.
\]
At space infinity \( r \to \infty \) the scalar curvature tends to zero.

To check that the Roberts solution really satisfies the effective equations of motion obtained in the previous sections one has to transform the metric (256) to the diagonal form (237). To this end the function \( u = u(t, r) \) must satisfy the differential equation
\[
\frac{\partial u}{\partial r} = \frac{r}{\sqrt{a^2u^2 + r^2 - 2a^2u}}. \quad (264)
\]
Solution of this equation depends on the value of constant \( a \). There are three cases \( 0 < a < 1/2, \ a = 1/2, \) and \( a > 1/2 \). Omitting all calculations we summarize the result.

11.3.1 The case \( 0 < a < 1/2 \)

The transformation from \( u, r \) to \( t, r \) coordinates is given by a solution of Equation (264)
\[
\Phi_1 = \left| \sqrt{a^2u^2 + r^2 - z_1^2} \right|^{z_1} - C(t)\left| \sqrt{a^2u^2 + r^2 - z_2^2} \right|^{z_2} = 0, \quad (265)
\]
Comparing these expressions with the metric (237) one finds

\[ z_{1,2} = \frac{1 \pm \sqrt{1 - 4a^2}}{2} > 0 \]

are the roots of the quadratic equation

\[ z^2 - z + a^2 = 0, \]

and \( C(t) \) is an arbitrary function of time \( t \) with \( \dot{C} \neq 0 \). The latter appears because the diagonal gauge is defined up to an arbitrary transformation of time coordinate. Straightforward calculations yield the nontrivial metric components

\[ g_{00} = \left(\frac{\dot{C}}{C}\right)^2 \frac{(2a^2u^2 + r^2 - u\sqrt{a^2u^2 + r^2})^2}{(1 - 4a^2)\sqrt{a^2u^2 + r^2}(\sqrt{a^2u^2 + r^2} - 2a^2u)}, \tag{266} \]

\[ g_{11} = -\frac{r^2}{\sqrt{a^2u^2 + r^2}(\sqrt{a^2u^2 + r^2} - 2a^2u)}. \tag{267} \]

Comparing these expressions with the metric (237) one finds

\[
 e^{2T} = \frac{\dot{C}}{C} \frac{r(2a^2u^2 + r^2 - u\sqrt{a^2u^2 + r^2})}{\sqrt{1 - 4a^2}\sqrt{a^2u^2 + r^2}(\sqrt{a^2u^2 + r^2} - 2a^2u)}. \tag{268}
\]

\[
 A_m + W_m = \frac{\dot{C}}{C} \frac{2a^2u^2 + r^2 - u\sqrt{a^2u^2 + r^2}}{\sqrt{1 - 4a^2}}, \tag{269}
\]

Using the equation of motion (230) one easily finds the momentum for a scalar field

\[ P = \frac{ar^3}{\sqrt{a^2u^2 + r^2}(\sqrt{a^2u^2 + r^2} - 2a^2u)}. \tag{270} \]

Straightforward calculations show that expressions (268) and (269) are indeed related to the scalar field (255) and its conjugate momentum (270) through the integrals (232) and (233) if and only if

\[ C(t) = t^{z_1 - z_2}. \tag{271} \]

This happened because in obtaining the effective action we have already fixed the time coordinate. The last equation of motion (231) is checked by direct calculations. This proves that the Roberts solution is the exact solution of the effective equations of motion.

Using Equation (271) the transition function \( \Phi_1 \), the temporal metric component, and nonlocal quantities may be written in the form

\[
 \Phi_1 = \left| \frac{\sqrt{a^2u^2 + r^2} - z_1 u}{t} \right|^{z_1} - \left| \frac{\sqrt{a^2u^2 + r^2} - z_2 u}{t} \right|^{z_2} = 0, \tag{272}
\]

\[
 g_{00} = \frac{(2a^2u^2 + r^2 - u\sqrt{a^2u^2 + r^2})^2}{t^2 \sqrt{a^2u^2 + r^2}(\sqrt{a^2u^2 + r^2} - 2a^2u)}, \tag{273}
\]

\[
 e^{2T} = \frac{r(2a^2u^2 + r^2 - u\sqrt{a^2u^2 + r^2})}{t\sqrt{a^2u^2 + r^2}(\sqrt{a^2u^2 + r^2} - 2a^2u)}. \tag{274}
\]

\[
 A_m + W_m = \frac{1}{t}(2a^2u^2 + r^2 - u\sqrt{a^2u^2 + r^2}). \tag{275}
\]
11.3.2 The case $a = 1/2$

In the second case the transformation to the diagonal gauge is given by the equation

$$
\Phi_2 = \frac{1}{2} \sqrt{u^2 + 4r^2} - \frac{1}{2} u - t \exp \left( \frac{u}{\sqrt{u^2 + 4r^2}} - u \right) = 0, \tag{276}
$$

where we fixed an arbitrary function of time by the same procedure as in the previous section. Afterwards we obtain expressions for the metric components, nonlocal quantities, and momenta

$$
g_{00} = \frac{(\sqrt{u^2 + 4r^2} - u)^3}{4t^2 \sqrt{u^2 + 4r^2}}, \tag{277}
$$

$$
g_{11} = -\frac{\sqrt{u^2 + 4r^2} + u}{\sqrt{u^2 + 4r^2}}, \tag{278}
$$

$$
\epsilon^{2T} = \frac{r(\sqrt{u^2 + 4r^2} - u)}{t \sqrt{u^2 + 4r^2}}, \tag{279}
$$

$$
A_m + W_m = \frac{(\sqrt{u^2 + 4r^2} - u)^2}{4t}, \tag{280}
$$

$$
P = \frac{2r^3}{\sqrt{u^2 + 4r^2}(\sqrt{u^2 + 4r^2} - u)}. \tag{281}
$$

These expressions follow from Equations (273)–(275), (267), and (270) for $a = 1/2$.

11.3.3 The case $a > 1/2$

In the third case the transformation to the diagonal gauge is defined by the equation

$$
\Phi_3 = \sqrt{2a^2 u^2 + r^2} - u \sqrt{a^2 u^2 + r^2} - t \exp \left( \frac{1}{b \arctg\frac{b u}{2\sqrt{a^2 u^2 + r^2} - u}} \right) = 0, \tag{282}
$$

where we introduced a shorthand notation

$$
b = \sqrt{4a^2 - 1} = \text{const.}
$$

Explicit expressions for the metric components, nonlocal quantities, and momentum are the same as before (267), (270), (273)–(275). Note that $\Phi_3 \to \Phi_2$ for $a \to 1/2$.

We see that in all three cases expressions for geometric quantities are the same formally, the only difference being the transition functions (272), (276), and (282). The solution is obviously continuously self-similar.

The Roberts solution is in striking conflict with the intuition gained from the Schwarzschild solution. For example, the line defined by the equation

$$
\frac{1}{g_{11}} = 0 \quad \Leftrightarrow \quad v = (1 - 4a^2)u,
$$

which is often assumed to define a horizon is not lightlike for $a \neq 1/2$. At the singularity $r = 0$ the temporal component of the metric is a nonzero constant for $a \neq 1/2$ and equals zero for $a = 1/2$. The radial component of the metric $g_{11} = 0$ at $r = 0$ for $a \neq 0$. 
The Roberts solution does not describe a formation of a black hole because a true singularity at the origin $r = 0$ exists for all times. On the $u, v$ plane it is timelike in the past $u < 0$ and timelike, lightlike, or spacelike in the future $u > 0$ for $0 < a < 1/2$, $a = 1/2$, or $a > 1/2$, respectively. In the last two cases the type of the singularity changes during the evolution.

12 Conclusion

In the present paper we obtain the effective action for physical degrees of freedom in a general two-dimensional gravity model coupled to a scalar field by a step by step explicit solution of the equations of motion and elimination of all unphysical geometric variables. The resulting effective action contains only a scalar field and its conjugate momentum and is written in an abstract Minkowskian space-time. All geometric variables defining the geometry of the real space-time are given by simple formulas in terms of a scalar field. The whole problem of the solution of the equations of motion is reduced to the following steps. First, one has to find an exact or approximate solution to the effective equations of motion for a scalar field. Second, the metric and other geometrical quantities should be computed. Third, one should analyze the physical properties of the space-time. We checked that the Fisher and Roberts exact solutions are indeed the solutions to the effective equations of motion.

The resulting effective action is nonlocal and complicated, and it is not clear whether it simplifies the solution of the equations of motion or not. In any case its existence guarantees the existence of the conserved quantity which for solutions with the Schwarzschild asymptotic may be identified with the total mass of the space-time.

The effective action was obtained locally by the solution of the equations of motion, and its existence does not depend on the global structure of the space-time. It appears as a boundary term, and the reason for its appearance lies in the constraints. We showed that in general a solution of the constraints does not have a compact support, and to produce the equations of motion without boundary conditions on the fields one must add the boundary term to the action. This boundary term does not depend on the action we started with and is defined entirely by the constraints.

For closed universes the assumption of smoothness for all the fields is in contradiction with a solution of the constraints: There are no such solutions. Therefore this assumption must be weakened. We assume that only physical degrees of freedom are smooth functions in a closed universe. The unphysical degrees of freedom are nonsmooth functions in this case as a solution of the constraints, and the corresponding boundary term must be added to the action. This boundary term is responsible for the effective action which correctly reproduces the effective equations of motion, and its numerical value may be used for the definition of the total energy even for closed universes.

The importance of boundary terms for the definition of the total energy for asymptotically flat space-times was noted long ago [53, 54] (for review see [51, 48]). Roughly speaking, the conclusion was based on two important observations. First, one of the boundary terms which is dropped in obtaining the equations of motion
equals the mass $M$ for the Schwarzschild solution which is assumed to yield a correct asymptotic for any compact distribution of matter. Second, it reproduces the correct equations of motion for physical degrees of freedom in the linear approximation in which the constraints in general relativity may be solved. In the present paper we obtain the same boundary term in full generality without using any asymptotic or approximation. This derivation shows two interesting features: The boundary term is nontrivial even for closed universes, and on the constraints it is given as an integral of some density over the whole space-time as usual. This is because the boundary term contains values of unphysical degrees of freedom on the boundary which are in their turn given by the integral of physical degrees of freedom over the space as a solution to the constraints.

We considered a wide class of two-dimensional gravity models arbitrary coupled to a scalar field which includes the spherically reduced general relativity. This suggests that the situation with the constraints is quite general: The constraints in a gravity model do not admit solutions with compact support, and the boundary term must be added to the action. This boundary term results in the effective Hamiltonian for physical degrees of freedom and should be used for the definition of the total energy of the space-time. In the present paper we have found it by explicit solution of the constraints. The problem remains of how to obtain it without a solution of the constraints. This is important, for example, for the path integral quantization. Indeed, the measure in the path integral [55] in our case contains 6 $\delta$-functions corresponding to the constraints (17), (19), (26), and (27) and 6 $\delta$-functions with gauge conditions (174), (194), and (195) which together form a set of 12 second class constraints. Solution of the constraints and gauge conditions results in a zero effective Hamiltonian if the boundary term is not added from the beginning. We know that it must be added because otherwise the correct equations of motion for the physical degrees of freedom are lost, and the problem is what to add if we are not able to solve the constraints explicitly which is the case for more general gravity models. This problem has a direct influence for the correct covariant perturbation theory.

Many thanks to D. Grumiller, W. Kummer, I. V. Tyutin, and I. V. Volovich for fruitful discussions. The financial support of the Russian Foundation for Basic Research is greatly acknowledged, grants RFBR 96-15-96131 and 99-01-00866. The author thanks the Austrian Academy of Science and Technical University of Vienna where this work began for the hospitality.

**Appendix. Dimensions of the fields**

The analysis of the equations of motion is cumbersome. Therefore we attribute dimensions to every field and coupling constant to be able to check dimensions of expressions at each step of the calculations. By definition, coordinates have dimension of length $[x^\alpha] = l$, the zweibein components and the action are dimensionless $[e_\alpha^\phantom{\alpha}a] = 1, \quad [S] = 1$, and the Lorentz connection has the same dimension as the partial derivative $[\omega_\alpha^\phantom{\alpha}/] = [\partial_\alpha^\phantom{\alpha}/] = l^{-1}$. 


This means that torsion components and the scalar curvature have dimensions

\[ [T^a] = l^{-1}, \quad [R] = l^{-2}. \]

We assume also that a scalar field has the same dimension as in four-dimensional space-time

\[ [X] = l^{-1}, \quad (283) \]

because in section [1] we analyze spherically reduced gravity. Afterwards dimensions of all other variables are uniquely defined. A Lagrangian in two dimensions has dimension \([L] = l^{-2}\). Hence

\[ [\pi] = 1, \quad [p_a] = l^{-1}. \quad (284) \]

Arbitrary functions \(U(\pi)\) and \(V(\pi)\) entering the Lagrangian (3) have dimensions

\[ [U] = 1, \quad [V] = l^{-2}. \quad (285) \]

So one could introduce a dimensionfull coupling constant in front of \(V(\pi)\). The dimension of the mass of a scalar field (2) is the same in any dimensional space-time

\[ [m] = l^{-1}. \]

The dimension of the coupling \(\rho\) is defined by the dimension of a scalar field (283) and dimensionlessness of the action

\[ [\rho] = l^2. \]

The four-dimensional gravitational coupling constant \(\kappa\) has dimension

\[ [\kappa] = l^{-2}. \]

For minimally coupled scalars in two dimensions one would have

\[ [X] = 1, \quad [\rho] = 1. \]

These dimensions are suitable for the analysis of a bosonic string with dynamical geometry when the gravity Lagrangian (3) is added to the string model [4]. We do not consider this model in the present paper.

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