A FRACTIONAL MODEL FOR THE DYNAMICS OF TUBERCULOSIS (TB) USING ATANGANA-BALEANU DERIVATIVE

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Abstract. In the present paper, we explore the dynamics of fractional tuberculosis model with Atangana-Baleanu (A-B) derivative. The number of confirmed notified cases reported by national tuberculosis control program (NTP) Khyber Pakhtunkhwa, Pakistan, since 2002 to 2017 are used for our analysis and estimation of the model parameters. Initially, the essential properties of the model are presented. We prove the existence of the solution through fixed-point theory. Then, we show the uniqueness of the solution. Modified Adams-Bashforth technique is used to obtain the numerical solution of the fractional model. We obtain numerical results with different values of the fractional order parameters to show the importance of the newly proposed derivative, which provides useful information about the TB dynamics and its control.

1. Introduction. Tuberculosis (TB) is one of a bacterial infectious diseases caused by bacillus *Mycobacterium tuberculosis* (MTB) and ranked one of the top 10 causes of death worldwide. This infection affects mainly the lungs (known as pulmonary TB). Beside this, the other parts of the human body such as kidneys, spine, central nervous, brain, central nervous system or lymphatic system (known as extra pulmonary TB) can also be damaged. The TB disease is also known as air-born infection and transmitted from TB infected people thorough air whenever they sneeze, spit, cough, or speak. Chronic cough, night sweats, weight loss, fever etc are the

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main symptom of TB. According to recent reports of the World Health Organization (WHO) about \((1/4)^{15}\) of the whole world’s population are exposed to TB infection and more than 10.4 million people were infected with TB and about 1.7 million expired due to this infection in 2016 [1]. The person once infected with TB has 5-15 percent life time chance of being infected with TB. It is observed worldwide in middle income countries about the deaths cases from TB. Also, it is observed that more than 60% of the whole TB burden is carrying by the countries, such as Pakistan, China, Nigeria, India, Indonesia, Philippines, and South Africa [1].

Mathematical models play an important role to understand the dynamics and to provide useful techniques to control the communicable diseases. The first transmission model of TB infection was introduced by Waaler et al. [24] in 1962, by subdividing the whole population into three classes. In 1967, a mathematical model of TB infection which depends on the proportion of its prevalence is developed in [18]. Castillo et al. provided a two-strain, an age-structure and time-delay TB infection models in [10, 12]. The authors in [26], presented a mathematical model on TB and discussed its global dynamics with incomplete treatment. The authors in [16], proposed a mathematical model on TB and discussed the results. The TB dynamics with seasonality was explored by Liu et al. [17], by considering the realistic data for the estimation of model parameters. A realistic model to explore the dynamics of TB in different regions of Asia-pacific is studied by Trauer et al. [22]. In [27], Zhang et al. proposed the TB mathematical model incorporating hospitalized and non hospitalized infective population and implemented the TB data of China to simulate his model. The effect of relapse and reinfection of TB infective population is analyzed by Robert [25]. Recently, Kim et al. used the actual data of the Philippines population to simulate the mathematical model with optimal control strategies for TB dynamics [14].

Mathematical models with fractional order are more prominent and helpful to explore the real world phenomena than the integer-order models, due to its memory and description of heredity properties [20, 19]. In real life problems, the integer-order derivative does not explore the dynamics between two different points. Due to these limitations of integer order and local differentiation, various concepts on differentiation having non-local or fractional orders are developed in previous literature. The concept of fractional order derivative was introduced in 1993 [19]. Later on another derivative of fractional order having exponential kernel has been given in [9]. Currently, in literature a new fractional derivative has been introduced by Atangana and Baleanu [2]. In [2], the authors used the concept of the generalized Mittag-Leffler function and the kernel was used as non-singular and non-local. The newly introduced A-B derivative has been applied to many real world complex problems successfully, which can be seen in [6, 7, 8, 3]. The model of Ebola virus with A-B derivative is presented in [15]. An investigation has been made to study the dynamics of HBV which is presented in [23]. Recently, a new model of TB including relapse cases with A-B derivative is developed in [13]. The models with integer order are useful only for local dynamics with no external forces. These models not well explore the model complexities of the TB, because the model can sometimes has a crossover behavior and it is difficult to be analyzed by ordinary classical operators. The use of the concept of A-B derivative in the model made it a novel model with crossover behavior [4, 5].

TB is considered a major public health threat and causes of morbidity and mortality in Pakistan [28]. In the world, there are twenty two countries where the TB
infection cases occurs in greater number and Pakistan is amongst these courtiers, and got the fifth position by producing a large number of cases. Many Pakistani dies each year due to TB infection, and it is documented, that more than 0.5 million new infected cases including 15000 children are notified and approximately more than seventy thousand people died. Pakistan is ranked fourth in the world [1, 28] due to high global prevalence of multi drugs resistant TB. Khyber Pakhtunkhwa, a province of Pakistan, where the TB produced a large number of infected cases and deaths. The reports of national TB program of Khyber Pakhtunkhwa, Pakistan, show that since 2002-2017, approximately 462920 cases were registered and treated [28].

In this work, we formulate a fractional order model with A-B derivative for TB infection. We consider the real cases of TB infections, from 2002-2017 [28], to estimate the parameters for the TB model. The concept of fixed point theory is applied for the TB model and show its existence and uniqueness. The rest of the work on the TB dynamics is organized as follows: The basic definitions and results of A-B derivative are presented in the next section. The model formulation via arbitrary order derivative with the mathematical analysis is shown in section 3. Uniqueness and existence of the fractional model of TB infection is explored in 4. Iterative solution and simulations are given in section 5 and 6 respectively. At the last we summarize the work done on TB fractional model in section 7.

2. Basic concepts of A-B derivative. Here, we give some definitions on A-B fractional derivative which will be used later in our proposed model of TB [2].

**Definition 2.1.** Let $f \in H^1(a, b)$, $b > a$, $\sigma \in [0, 1]$ then the new fractional derivatives in Caputo (ABC) sense is given below:

$$
_{a}^{ABC}D_{t}^{\sigma}(f(t)) = \frac{B(\sigma)}{1-\alpha} \int_{a}^{t} f'(\chi) E_{\sigma} \left[ -\sigma \frac{(t-\chi)^{\sigma}}{1-\sigma} \right] d\chi.
$$

**Definition 2.2.** Let $f \in H^1(a, b)$, $b > a$, $\sigma \in [0, 1]$, and not necessary differentiable then the new A-B fractional derivative in Riemann-Liouville (ABR) sense is given as:

$$
_{a}^{ABR}D_{t}^{\sigma}(f(t)) = \frac{B(\sigma)}{1-\sigma} \int_{a}^{t} f(\chi) E_{\sigma} \left[ -\sigma \frac{(t-\chi)^{\sigma}}{1-\sigma} \right] d\chi.
$$

**Definition 2.3.** The fractional integral associate to the new fractional derivative with non local kernel is defined as:

$$
_{a}^{AB}I_{t}^{\sigma}(f(t)) = \frac{1-\sigma}{B(\sigma)} f(t) + \frac{\sigma}{B(\sigma) \Gamma(\sigma)} \int_{a}^{t} f(y)(t-y)^{\sigma-1} dy.
$$

When the fractional order turns to zero, then the initial function is recovered. Further, by putting the fractional order equal to 1, we obtain the case of classical integral.

**Theorem 2.4.** On $[a, b]$, the following inequality holds for $f$ when the function $f$ is continuous on $[a, b]$.

$$
\|_{a}^{ABR}D_{t}^{\sigma}(f(t))\| < \frac{B(\sigma)}{1-\sigma} \|f(x)\|, \text{ where } \|f(x)\| = \max_{a \leq x \leq b} |f(x)|.
$$

(1)

**Theorem 2.5.** Both of ABC and ABR derivatives satisfy the Lipschitz condition given below:

$$
\|_{a}^{ABC}D_{t}^{\sigma} f_{1}(t) - _{a}^{ABC}D_{t}^{\sigma} f_{2}(t)\| < K \|f_{1}(t) - f_{2}(t)\|,
$$

(2)
Table 1. Estimated parameters for the TB model (6).

| Parameter | Description | Baseline value | Reference |
|-----------|-------------|----------------|-----------|
| $\Lambda$ | Recruitment rate | 450,862.20088626 | Estimated |
| $\beta$ | Transmission rate | 0.5433 | Fitted |
| $\alpha$ | Moving rate from $T$ to $R$ | 0.3968 | Fitted |
| $\gamma$ | Progression rate from $I$ to $T$ | 0.2873 | Fitted |
| $\mu$ | Natural death rate | 1/67.7 [29] | |
| $\sigma_1$ | Disease induced death rate of infected individuals | 0.2202 | Fitted |
| $\sigma_2$ | Disease induced death rate in $T$ | 0.0550 | Fitted |
| $\delta$ | Rate at which treated individuals leave the $T$ | 1.1996 | Fitted |
| $\eta$ | Rate of treatment failure | 0.1500 | Fitted |
| $\epsilon$ | Rate of progression from $L$ to $I$ | 0.2007 | Fitted |

also for ABR derivative we have

$$\|a^{\text{ABR}}D^\sigma f_1(t) - a^{\text{ABR}}D^\sigma f_2(t)\| < K\|f_1(t) - f_2(t)\|. \quad (3)$$

**Theorem 2.6.** The following FDEs

$$a^{\text{ABC}}D^\sigma_t f(t) = s(t), \quad (4)$$

gives a unique solution, by applying the inverse Laplace transform and the convolution result [2]:

$$f(t) = \frac{1 - \sigma}{\text{ABC}(\sigma)s(t)} + \frac{\sigma}{\text{ABC}(\sigma)\Gamma(\sigma)} \int_a^t s(\xi)(t - \xi)^{\sigma-1}d\xi. \quad (5)$$

3. TB model formulation with A-B derivative. Here, we formulate a model on the dynamics of TB on Khyber Pakhtunkhwa, Pakistan with available incidence data since 2002-2017. So, we consider the total population of humans denoted by $N(t)$ and subdividing into five different classes, namely, the susceptible $S(t)$, Exposed $L(t)$, TB active $I(t)$, under treatment $T(t)$, and those recovered after treatment $R(t)$ individuals at any time $t$. The transmission dynamics of fractional TB model with A-B derivative is shown through the following equations:

$$\begin{align*}
0^{\text{ABC}}D^\sigma_t S &= \Lambda - \frac{\beta SI}{N} - \mu S, \\
0^{\text{ABC}}D^\sigma_t L &= \frac{\beta SI}{N} - (\mu + \epsilon)L + (1 - \eta)\delta T, \\
0^{\text{ABC}}D^\sigma_t I &= \epsilon L + \eta\delta T - (\mu + \gamma + \sigma_1)I, \\
0^{\text{ABC}}D^\sigma_t T &= \gamma I - (\mu + \delta + \sigma_2 + \alpha)T, \\
0^{\text{ABC}}D^\sigma_t R &= \alpha T - \mu R,
\end{align*} \quad (6)$$

where $N(t) = S(t) + L(t) + I(t) + T(t) + R(t)$, and

$$S(0) = q_1, L(0) = q_2, I(0) = q_3, T(0) = q_4 \text{ and } R(0) = q_5.$$
3.1. Basic properties of TB model. Here, we explore the basic properties of the TB fractional model. The fractional TB model (6), always has the disease free parameter $\Lambda$ is estimated from the population of Khyber Pakhtunkhwa, while the natural death rate of human is obtained from the available source given in Table 1.
equilibrium (DFE), denoted by $J_0$ and is obtained as follows:

$$J_0 = \left( \frac{\Lambda}{\mu}, 0, 0, 0 \right).$$

The basic reproduction number has the key role in the modeling of infectious diseases. It explores the dynamics of the disease whether the disease is controllable or not or it can be spread or not, or some vaccine to the individuals are required or not. When the value of the basic reproduction number falls less than 1, then the disease will not be spread in the community, whereas, when its values cross the 1, then the disease can be spread in the community, and will produce deaths and sometimes outbreak and epidemics. There are some methods available in literature to find the basic reproduction number, but here for our model (6), we use the most known method, given in [11]. This method involves, the necessary matrices for the computation of the basic reproduction number and then the spectral radius i.e. $\rho(\mathbf{FV}^{-1})$. By using the aforementioned properties of the method, we obtain directly, and present the basic reproduction number for our TB model in the following:

$$R_0 = \frac{\beta \epsilon (\mu + \delta + \sigma_2 + \alpha)}{(\mu + \epsilon)(\mu + \gamma + \sigma_1)(\mu + \delta + \sigma_2 + \alpha)} + \gamma \delta (1 - \eta) + (\mu + \epsilon) \gamma \delta \eta + (\mu + \epsilon) \gamma \delta (1 - \eta) + (\mu + \epsilon) \gamma \delta \eta.$$

We denote the endemic equilibrium of the model by $J_1$, where

$$J_1 = (S^*, L^*, I^*, T^*, R^*)$$

and obtained by setting

$$0 = \Lambda - \frac{\beta SI}{N} - \mu S,$$

$$0 = \frac{\beta SI}{N} - (\mu + \epsilon)L + (1 - \eta)\delta T,$$

$$0 = \epsilon L + \eta \delta T - (\mu + \gamma + \sigma_1)I,$$

$$0 = \gamma I - (\mu + \delta + \sigma_2 + \alpha)T,$$

$$0 = \alpha T - \mu R.$$

After, some calculations, we obtain the following:

$$S^* = \frac{N^*}{R_0},$$

$$L^* = \frac{((\mu + \sigma_1) + \gamma (\mu + \sigma_2 + \alpha) + \gamma \delta (1 - \eta)) N^* (R_0 - 1)}{R_0 ((\mu + \sigma_1) + \gamma (\mu + \sigma_2 + \alpha) + \gamma \delta (1 - \eta) + \epsilon (\gamma + \mu + \delta + \sigma_2 + \alpha))},$$

$$I^* = \frac{\epsilon k_3 N^* (R_0 - 1)}{R_0 ((\mu + \sigma_1) + \gamma (\mu + \sigma_2 + \alpha) + \gamma \delta (1 - \eta) + \epsilon (\gamma + \mu + \delta + \sigma_2 + \alpha))},$$

$$T^* = \frac{\epsilon \gamma N^* (R_0 - 1)}{R_0 ((\mu + \sigma_1) + \gamma (\mu + \sigma_2 + \alpha) + \gamma \delta (1 - \eta) + \epsilon (\gamma + \mu + \delta + \sigma_2 + \alpha))},$$

$$R^* = \frac{\alpha T^*}{\mu}.$$

Hence, we state the following theorem.

**Theorem 3.1.** For the model (6) there exists a unique endemic equilibrium if $R_0 > 1$. 
4. Solution existence. Here, we give in detailed the existence of the model solution by using the application of fixed point theory. Due to the nonlinearity involve in the model (6), it is difficult to obtain its exact solution, and there is no method available in literature to provide its exact solution. However, the model will have the exact solution under some conditions if the existence for the model is proved. The system (6) can be structured as per the following,

\[ \begin{align*}
_0^A D_t^\sigma [S(t)] &= K_1(t, S), \\
_0^A D_t^\sigma [L(t)] &= K_2(t, L), \\
_0^A D_t^\sigma [I(t)] &= K_3(t, I), \\
_0^A D_t^\sigma [T(t)] &= K_4(t, T), \\
_0^A D_t^\sigma [R(t)] &= K_5(t, R).
\end{align*} \]

Now, it follows from Theorem 3, the system (7) can be shown as:

\[ \begin{align*}
S(t) - S(0) &= \frac{(1-\sigma)}{\mathcal{A}(\sigma)} K_1(t, S) + \frac{\sigma}{\mathcal{A}(\sigma) \Gamma(\sigma)} \int_0^t K_1(\xi, S)(t - \xi)^{\sigma-1} d\xi, \\
L(t) - L(0) &= \frac{(1-\sigma)}{\mathcal{A}(\sigma)} K_2(t, L) + \frac{\sigma}{\mathcal{A}(\sigma) \Gamma(\sigma)} \int_0^t K_2(\xi, L)(t - \xi)^{\sigma-1} d\xi, \\
I(t) - I(0) &= \frac{(1-\sigma)}{\mathcal{A}(\sigma)} K_3(t, I) + \frac{\sigma}{\mathcal{A}(\sigma) \Gamma(\sigma)} \int_0^t K_3(\xi, I)(t - \xi)^{\sigma-1} d\xi, \\
T(t) - T(0) &= \frac{(1-\sigma)}{\mathcal{A}(\sigma)} K_4(t, T) + \frac{\sigma}{\mathcal{A}(\sigma) \Gamma(\sigma)} \int_0^t K_4(\xi, T)(t - \xi)^{\sigma-1} d\xi, \\
R(t) - R(0) &= \frac{(1-\sigma)}{\mathcal{A}(\sigma)} K_5(t, R) + \frac{\sigma}{\mathcal{A}(\sigma) \Gamma(\sigma)} \int_0^t K_5(\xi, R)(t - \xi)^{\sigma-1} d\xi.
\end{align*} \]

Now, we prove the Lipschitz condition in the following theorem.

**Theorem 4.1.** The kernel \( K_1 \) satisfies the Lipschitz condition and contraction if the inequality given below holds

\[ 0 \leq (\beta \omega_1 + \mu) < 1. \]

**Proof.** To show the result, we consider the functions, \( S \) and \( S_1 \), then

\[ \| K_1(t, S) - K_1(t, S_1) \| = \| - \frac{\beta I}{N} (S(t) - S(t_1)) - \mu (S(t) - S(t_1)) \| \]

\[ \leq \| I(t) \| \| S(t) - S(t_1) \| + \mu \| S(t) - S(t_1) \| \]

\[ \leq \{ \beta \omega_1 + \mu \} \| S(t) - S(t_1) \| \]

\[ \leq \varrho_1 \| S(t) - S(t_1) \|, \]

where \( \varrho_1 = \{ \beta \omega_1 + \mu \} \) and \( \| I(t) \| \leq \omega_1 \), which implies that

\[ \| K_1(t, S) - K_1(t, S_1) \| \leq \varrho_1 \| S(t) - S(t_1) \|. \]

So, the property of Lipschitz condition for \( K_1 \) is satisfied. Additionally, if \( 0 \leq (\beta \omega_1 + \mu) < 1 \) then a contraction implies. We apply the same analysis to show the
Moreover, the initial conditions are

\[ \|K_2(t, L) - K_2(t, L_1)\| \leq q_2\|L(t) - L(t_1)\|, \]
\[ \|K_3(t, I) - K_3(t, I_1)\| \leq q_3\|I(t) - I(t_1)\|, \]
\[ \|K_4(t, T) - K_4(t, T_1)\| \leq q_4\|T(t) - T(t_1)\|, \]
\[ \|K_5(t, R) - K_5(t, R_1)\| \leq q_5\|R(t) - R(t_1)\|. \]  

(11)

Next, re-writing the model (8) in the following recursive form:

\[ S_n(t) = \frac{1 - \sigma}{\Delta B(\sigma)} K_1(t, S_{n-1}) + \frac{\sigma}{\Delta B(\sigma) L(t, \sigma)} \int_0^t K_1(\xi, S_{n-1})(t - \xi)^\sigma - 1 \, d\xi, \]
\[ L_n(t) = \frac{1 - \sigma}{\Delta B(\sigma)} K_2(t, L_{n-1}) + \frac{\sigma}{\Delta B(\sigma) L(t, \sigma)} \int_0^t K_2(\xi, L_{n-1})(t - \xi)^\sigma - 1 \, d\xi, \]
\[ I_n(t) = \frac{1 - \sigma}{\Delta B(\sigma)} K_3(t, I_{n-1}) + \frac{\sigma}{\Delta B(\sigma) L(t, \sigma)} \int_0^t K_3(\xi, I_{n-1})(t - \xi)^\sigma - 1 \, d\xi, \]
\[ T_n(t) = \frac{1 - \sigma}{\Delta B(\sigma)} K_4(t, T_{n-1}) + \frac{\sigma}{\Delta B(\sigma) L(t, \sigma)} \int_0^t K_4(\xi, T_{n-1})(t - \xi)^\sigma - 1 \, d\xi, \]
\[ R_n(t) = \frac{1 - \sigma}{\Delta B(\sigma)} K_5(t, R_{n-1}) + \frac{\sigma}{\Delta B(\sigma) L(t, \sigma)} \int_0^t K_5(\xi, R_{n-1})(t - \xi)^\sigma - 1 \, d\xi. \]  

(12)

Moreover, the initial conditions are

\[ S_0(t) = S(0), L_0(t) = L(0), I_0(t) = I(0), T_0(t) = T(0), R_0(t) = R(0). \]

For the difference of successive terms, we obtain the below results:

\[ \varphi_{1n}(t) = S_n(t) - S_{n-1}(t) = \frac{1 - \sigma}{\Delta B(\sigma)} \{ K_1(t, S_{n-1}) - K_1(t, S_{n-2}) \} \]
\[ + \frac{\sigma}{\Delta B(\sigma) L(t, \sigma)} \int_0^t (t - \xi)^\sigma - 1 \{ K_1(\xi, S_{n-1}) - K_1(\xi, S_{n-2}) \} d\xi, \]
\[ \varphi_{2n}(t) = L_n(t) - L_{n-1}(t) = \frac{(1 - \sigma)}{\Delta B(\sigma)} \{ K_2(t, L_{n-1}) - K_2(t, L_{n-2}) \} \]
\[ + \frac{\sigma}{\Delta B(\sigma) L(t, \sigma)} \int_0^t (t - \xi)^\sigma - 1 \{ K_2(\xi, L_{n-1}) - K_2(\xi, L_{n-2}) \} d\xi, \]
\[ \varphi_{3n}(t) = I_n(t) - I_{n-1}(t) = \frac{(1 - \sigma)}{\Delta B(\sigma)} \{ K_3(t, I_{n-1}) - K_3(t, I_{n-2}) \} \]
\[ + \frac{\sigma}{\Delta B(\sigma) L(t, \sigma)} \int_0^t (t - \xi)^\sigma - 1 \{ K_3(\xi, I_{n-1}) - K_3(\xi, I_{n-2}) \} d\xi, \]
\[ \varphi_{4n}(t) = T_n(t) - T_{n-1}(t) = \frac{(1 - \sigma)}{\Delta B(\sigma)} \{ K_4(t, T_{n-1}) - K_4(t, T_{n-2}) \} \]
\[ + \frac{\sigma}{\Delta B(\sigma) L(t, \sigma)} \int_0^t (t - \xi)^\sigma - 1 \{ K_4(\xi, T_{n-1}) - K_4(\xi, T_{n-2}) \} d\xi, \]
\[ \varphi_{5n}(t) = R_n(t) - R_{n-1}(t) = \frac{(1 - \sigma)}{\Delta B(\sigma)} \{ K_5(t, R_{n-1}) - K_5(t, R_{n-2}) \} \]
\[ + \frac{\sigma}{\Delta B(\sigma) L(t, \sigma)} \int_0^t (t - \xi)^\sigma - 1 \{ K_5(\xi, R_{n-1}) - K_5(\xi, R_{n-2}) \} d\xi. \]  

(13)
Consider that

\[
\begin{align*}
S_n(t) &= \sum_{i=1}^{n} \varphi_i(t), \\
L_n(t) &= \sum_{i=1}^{n} \varphi_1(t), \\
I_n(t) &= \sum_{i=1}^{n} \varphi_2(t), \\
T_n(t) &= \sum_{i=1}^{n} \varphi_3(t), \\
R_n(t) &= \sum_{i=1}^{n} \varphi_4(t).
\end{align*}
\]

Applying norm on (13), and then the triangular inequality and the condition of Lipschitz shown in (10), we have

\[
\|\varphi_{1n}(t)\| \leq \left(\frac{1-\sigma}{AB(\sigma)}\right)\varrho_1\|\varphi_{1(n-1)}(t)\| + \frac{\sigma}{AB(\sigma)\Gamma(\sigma)}\varrho_1 \int_{0}^{t} \|\varphi_{1(n-1)}(y)\|\,dy. \tag{15}
\]

We obtain the following results after applying the same procedure for the rest of equations,

\[
\begin{align*}
\|\varphi_{2n}(t)\| &\leq \left(\frac{1-\sigma}{AB(\sigma)}\right)\varrho_2\|\varphi_{2(n-1)}(t)\| + \frac{\sigma}{AB(\sigma)\Gamma(\sigma)}\varrho_2 \int_{0}^{t} \|\varphi_{2(n-1)}(\xi)\|\,d\xi, \\
\|\varphi_{3n}(t)\| &\leq \left(\frac{1-\sigma}{AB(\sigma)}\right)\varrho_3\|\varphi_{3(n-1)}(t)\| + \frac{\sigma}{AB(\sigma)\Gamma(\sigma)}\varrho_3 \int_{0}^{t} \|\varphi_{3(n-1)}(\xi)\|\,d\xi, \\
\|\varphi_{4n}(t)\| &\leq \left(\frac{1-\sigma}{AB(\sigma)}\right)\varrho_4\|\varphi_{4(n-1)}(t)\| + \frac{\sigma}{AB(\sigma)\Gamma(\sigma)}\varrho_4 \int_{0}^{t} \|\varphi_{4(n-1)}(\xi)\|\,d\xi, \\
\|\varphi_{5n}(t)\| &\leq \left(\frac{1-\sigma}{AB(\sigma)}\right)\varrho_5\|\varphi_{5(n-1)}(t)\| + \frac{\sigma}{AB(\sigma)\Gamma(\sigma)}\varrho_5 \int_{0}^{t} \|\varphi_{5(n-1)}(\xi)\|\,d\xi.
\end{align*}
\]

Immediately, the following theorem is presented in view of the above result.

**Theorem 4.2.** The solution of the TB fractional model will exist and unique under the conditions that there exist some \(t_0\), such that

\[
\frac{1-\sigma}{AB(\sigma)}\varrho_i + \frac{t_0\sigma}{AB(\sigma)\Gamma(\sigma)}\varrho_i < 1,
\]

for \(i=1,2,\ldots,5\).

**Proof.** As we know that \(S(t)\), \(L(t)\), \(I(t)\), \(T(t)\) and \(R(t)\) are bounded functions and satisfy the Lipschitz conditions. Therefore, we use equations (15) and (16), the following succeeding relation is obtained:

\[
\begin{align*}
\|\varphi_{1n}(t)\| &\leq \|S_n(0)\|\left[\left(\frac{1-\sigma}{AB(\sigma)}\right)\varrho_1 + \frac{t_0\sigma}{AB(\sigma)\Gamma(\sigma)}\varrho_1\right]^n, \\
\|\varphi_{2n}(t)\| &\leq \|L_n(0)\|\left[\left(\frac{1-\sigma}{AB(\sigma)}\right)\varrho_2 + \frac{t_0\sigma}{AB(\sigma)\Gamma(\sigma)}\varrho_2\right]^n, \\
\|\varphi_{3n}(t)\| &\leq \|I_n(0)\|\left[\left(\frac{1-\sigma}{AB(\sigma)}\right)\varrho_3 + \frac{t_0\sigma}{AB(\sigma)\Gamma(\sigma)}\varrho_3\right]^n, \\
\|\varphi_{4n}(t)\| &\leq \|T_n(0)\|\left[\left(\frac{1-\sigma}{AB(\sigma)}\right)\varrho_4 + \frac{t_0\sigma}{AB(\sigma)\Gamma(\sigma)}\varrho_4\right]^n, \\
\|\varphi_{5n}(t)\| &\leq \|R_n(0)\|\left[\left(\frac{1-\sigma}{AB(\sigma)}\right)\varrho_5 + \frac{t_0\sigma}{AB(\sigma)\Gamma(\sigma)}\varrho_5\right]^n.
\end{align*}
\]
The existence and continuity of aforementioned solutions are proved. To prove that the aforementioned functions represent solution for the TB model (6), we define the following:

\[ S(t) - S(0) = S_n(t) - B_{1n}(t), \]
\[ L(t) - L(0) = L_n(t) - B_{2n}(t), \]
\[ I(t) - I(0) = I_n(t) - B_{3n}(t), \]
\[ T(t) - T(0) = T_n(t) - B_{4n}(t), \]
\[ R(t) - R(0) = R_n(t) - B_{5n}(t). \]  

Further, we get

\[
\|B_{1n}(t)\| = \left\| \frac{1 - \sigma}{AB(\sigma)} \{K_1(t, S) - K_1(t, S_{n-1})\} + \frac{\sigma}{AB(\sigma)\Gamma(\sigma)} \int_0^t (t - \xi)^{\sigma - 1}\{K_1(\xi, S) - K_1(\xi, S_{n-1})\}d\xi \right\|
\leq \frac{1 - \sigma}{AB(\sigma)} \|K_1(t, S) - K_1(t, S_{n-1})\|
+ \frac{\sigma}{AB(\sigma)\Gamma(\sigma)} \int_0^t \|K_1(\xi, S) - K_1(\xi, S_{n-1})\|d\xi
\leq \frac{1 - \sigma}{AB(\sigma)} \|S - S_{n-1}\| + \frac{\sigma}{AB(\sigma)\Gamma(\sigma)} \|S - S_{n-1}\| t. \]

On repeating the same procedure at \( t_0 \), we obtained the following:

\[
\|B_{1n}(t)\| \leq \left( \frac{1 - \sigma}{AB(\sigma)} + \frac{\sigma t_0}{AB(\sigma)\Gamma(\sigma)} \right)^{n+1} \phi_1^n 1 + M. \]

Taking limit of equation (20) as \( n \) approaches to \( \infty \), then clearly \( \|B_{1n}(t)\| \) tends to 0. By the same analysis, we have \( \|B_{2n}(t)\|, \|B_{3n}(t)\|, \|B_{4n}(t)\| \) and \( \|B_{5n}(t)\| \) tend to 0 whenever \( n \) tends to \( \infty \).

4.1. **Uniqueness of the solution.** Previously, we proved the existence of solution for the model (6) using fixed point theory. Next, we provide the uniqueness of the solution and we give the following statement.

**Theorem 4.3.** The solution of the non-integer order derivative model given by (6) possess a unique solution, whenever, the assumption holds,

\[
\left(1 - \frac{1 - \sigma}{AB(\sigma)} - \frac{\sigma}{AB(\sigma)\Gamma(\sigma)} t_0\right) \geq 0. \]

**Proof.** We prove the uniqueness of the arbitrary order derivative model (6) by considering that there may be another set of solution such as \( S_1(t), L_1(t), I_1(t), T_1(t) \) and \( R_1(t) \) then,

\[ S(t) - S_1(t) \]
\[ = \frac{1 - \sigma}{AB(\sigma)} \{K_1(t, S) - K_1(t, S_1)\} + \frac{\sigma}{AB(\sigma)\Gamma(\sigma)} \int_0^t (K_1(\xi, S) - K_1(\xi, S_1))d\xi. \]
Applying norm on equation (22), it follows

\[
\|S(t) - S_1(t)\| = \left| 1 - \sigma \left( K_1(t, S) - K_1(t, S_1) \right) + \sigma \int_0^t (K_1(\xi, S) - K_1(\xi, S_1)) d\xi \right| \\
\leq 1 - \frac{\sigma}{AB(\sigma)} \|S(t) - S_1(t)\| + \frac{\sigma}{AB(\sigma) \Gamma(\sigma)} g_1 t \|S(t) - S_1(t)\|. 
\]

(23)

Which gives

\[
\|S(t) - S_1(t)\| \left( 1 - \frac{1 - \sigma}{AB(\sigma)} g_1 - \frac{\sigma}{AB(\sigma) \Gamma(\sigma)} g_1 t \right) \leq 0.
\]

(24)

Clearly, \( S(t) = S_1(t) \), if condition (21) holds. Similarly, \( L(t) = L_1(t) \), \( I(t) = I_1(t) \), \( T(t) = T_1(t) \), and \( R(t) = R_1(t) \). Hence, the solution is unique.

5. Numerical solution. Here, we proceed to obtain the numerical results of the TB fractional model (6). In order to do this, we first express the non-integer order derivative model given by (6) in fractional Volterra type and then applying the results of calculus. We obtain the scheme for the TB model (6) by the same way presented in [21], which is known as modified Adams Bashforth rule for the A-B fractional integral operator. It follows from the results given in [21], that the non-integer order derivative model (6) with A-B derivative after applying the result from calculus on the first equation of TB fractional model (6), we get,

\[
S(t) - S_0 = \frac{1 - \sigma}{AB(\sigma)} K_1(t, S) + \frac{\sigma}{AB(\sigma) \Gamma(\sigma)} \int_0^t K_1(\xi, S)(t - \xi)^{\sigma - 1} d\xi. 
\]

(25)

For \( t = t_n+1 \), \( n = 0, 1, 2, \ldots \), we obtain

\[
S(t_{n+1}) - S_0 = \frac{1 - \sigma}{AB(\sigma)} K_1(t_n, S) + \\
\frac{\sigma}{AB(\sigma) \Gamma(\sigma)} \int_0^{t_{n+1}} K_1(\tau, S)(t_{n+1} - \tau)^{\sigma - 1} d\tau, \\
= 1 - \frac{\sigma}{AB(\sigma)} K_1(t_n, S) + \\
\frac{\sigma}{AB(\sigma) \Gamma(\sigma)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} K_1(\tau, S)(t_{n+1} - \tau)^{\sigma - 1} d\tau. 
\]

(26)

Over \([t_k, t_{k+1}]\), the function \( K_1(\tau, S) \) can be approximated by the interpolation polynomial

\[
P_k(\tau) = \frac{\tau - t_{k-1}}{t_k - t_{k-1}} f(t_k, y(t_k)) - \frac{\tau - t_{k-1}}{t_k - t_{k-1}} f(t_{k-1}, y(t_{k-1})) \\
= \frac{f(t_k, y(t_k))}{h} (\tau - t_{k-1}) - \frac{f(t_{k-1}, y(t_{k-1}))}{h} (\tau - t_k) \\
\approx \frac{f(t_k, y_k)}{h} (\tau - t_{k-1}) - \frac{f(t_{k-1}, y_{k-1})}{h} (\tau - t_k). 
\]

(27)

Which gives

\[
S_{n+1} = S_0 + \frac{1 - \sigma}{AB(\sigma)} K_1(t_n, S) + \\
\frac{\sigma}{AB(\sigma) \Gamma(\sigma)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} f(t_{k+1}, y_{k+1})(\tau - t_{k+1})^{\sigma - 1} d\tau. 
\]
Now,

\[ A_{\sigma,1} = \int_{t_k}^{t_{k+1}} (\tau - t_k)(t_{n+1} - \tau)^{\sigma-1} d\tau, \]

and

\[ A_{\sigma,2} = \int_{t_k}^{t_{k+1}} (\tau - t_k)(t_{n+1} - \tau)^{\sigma-1} d\tau. \]

Calculating these integrals we get

\[ A_{\sigma,1} = h^{\sigma+1} \frac{(n+1-k)^\sigma(n-k+2+\sigma) - (n-k)^\sigma(n-k+2+2\sigma)}{\sigma(\sigma+1)}, \quad (31) \]

\[ A_{\sigma,2} = h^{\sigma+1} \frac{(n+1-k)^\sigma(n-k+1+\sigma)}{\sigma(\sigma+1)}. \quad (32) \]

Finally,

\[
S_{n+1} = S_0 + \frac{1 - \sigma}{AB(\sigma)} K_1(t_n, S) + \frac{\sigma}{AB(\sigma) \Gamma(\sigma)} \sum_{k=0}^{n} \left[ \frac{h^\sigma K_1(t_k, S)}{\Gamma(\sigma+2)} \left( (n+1-k)^\sigma(n-k+2+\sigma) - (n-k)^\sigma(n-k+2+2\sigma) \right) \\
- \frac{h^\sigma K_1(t_k-1, S)}{\Gamma(\sigma+2)} \left( (n+1-k)^{\sigma+1} - (n-k)^\sigma(n-k+1+\sigma) \right) \right]. \quad (33)
\]

In similar way for the rest of equations of system (6), we obtained the recursive formulae as below

\[
L_{n+1} = L_0 + \frac{1 - \sigma}{AB(\sigma)} K_2(t_n, L) + \frac{\sigma}{AB(\sigma) \Gamma(\sigma)} \sum_{k=0}^{n} \left[ \frac{h^\sigma K_2(t_k, L)}{\Gamma(\sigma+2)} \left( (n+1-k)^\sigma(n-k+2+\sigma) - (n-k)^\sigma(n-k+2+2\sigma) \right) \\
- \frac{h^\sigma K_2(t_k-1, L)}{\Gamma(\sigma+2)} \left( (n+1-k)^{\sigma+1} - (n-k)^\sigma(n-k+1+\sigma) \right) \right],
\]

\[
I_{n+1} = I_0 + \frac{1 - \sigma}{AB(\sigma)} K_3(t_n, I) + \frac{\sigma}{AB(\sigma) \Gamma(\sigma)} \sum_{k=0}^{n} \left[ \frac{h^\sigma K_3(t_k, I)}{\Gamma(\sigma+2)} \left( (n+1-k)^\sigma(n-k+2+\sigma) - (n-k)^\sigma(n-k+2+2\sigma) \right) \\
- \frac{h^\sigma K_3(t_k-1, I)}{\Gamma(\sigma+2)} \left( (n+1-k)^{\sigma+1} - (n-k)^\sigma(n-k+1+\sigma) \right) \right],
\]

\[
T_{n+1} = T_0 + \frac{1 - \sigma}{AB(\sigma)} K_4(t_n, T) + \frac{\sigma}{AB(\sigma) \Gamma(\sigma)} \sum_{k=0}^{n} \left[ \frac{h^\sigma K_4(t_k, T)}{\Gamma(\sigma+2)} \left( (n+1-k)^\sigma(n-k+2+\sigma) - (n-k)^\sigma(n-k+2+2\sigma) \right) \\
- \frac{h^\sigma K_4(t_k-1, T)}{\Gamma(\sigma+2)} \left( (n+1-k)^{\sigma+1} - (n-k)^\sigma(n-k+1+\sigma) \right) \right],
\]
\[ -\frac{h^\sigma K_k(t_{k-1},T)}{\Gamma(\sigma + 2)} \left( (n + 1 - k)^{\sigma + 1} - (n - k)^\sigma (n - k + 1 + \sigma) \right). \]

\[ R_{n+1} = R_0 + \frac{1 - \sigma}{AB(\sigma)} K_5(t_n, R) + \frac{\sigma}{AB(\sigma)\Gamma(\sigma)} \sum_{k=0}^{n} \left( \frac{h^\sigma K_5(t_k, R)}{\Gamma(\sigma + 2)} ( (n + 1 - k)^\sigma (n - k + 2 + \sigma) - (n - k)^\sigma (n - k + 2 + 2\sigma) ) - \frac{h^\sigma K_5(t_{k-1}, R)}{\Gamma(\sigma + 2)} ( (n + 1 - k)^{\sigma + 1} - (n - k)^\sigma (n - k + 1 + \sigma) ) \right). \tag{34} \]

6. Simulations results. After the successful implementations of modified Adams-Bashforth numerical scheme on the TB fractional model (6), we find the graphical results of the fractional order TB model (6), by considering and assigning values to the fractional parameter \( \sigma \in [0, 1] \), and biologically model relevant parameters. The time level in the graphical results is taken up to 100 units. The parameters used in the graphical results are estimated based on the available TB data from NTP Pakistan [1, 28] given in Table 1. The graphical behavior of the model (6) and the cumulative TB infective people for \( \sigma = 1 \) is shown in Figure 3. In Figure 4, the simulation of the model variables and total infected individuals are given for \( \sigma = 0.95 \). Figures 5-7 illustrate the behavior of the model (6) and total infected individuals for \( \sigma = 0.90, 0.85, 0.80 \), respectively. In Figures 3-7, the solid line shows the integer case \( \sigma = 1 \), while the dotted line represents the fractional solution. It can be observed from the results provided graphical results, the decrease in the value of \( \sigma \), the population of susceptible people increases, and the rest of the compartments are decreases significantly. Also, we can see that in Figures 3-7, when the fractional parameter \( \sigma \) decreases, then, the proportion of TB infected individuals decrease.

7. Conclusion. We analyzed a TB model with A-B derivative successfully. The model parameters used in numerical simulations are parameterized from the data since 2002-2017 obtained from NTP Khyber Pakhtunkhwa, Pakistan. We have analyzed and briefly discussed the necessary results for the fractional order model. Application of fixed theory was used and proved the model solution, uniqueness and their existence. The solution of the TB fractional model was obtained by an iterative method. Finally, we used the fractional parameter \( \sigma \) with different values and obtained the numerical results and provided a detailed discussion. It can be observed from the graphical results that the newly introduced derivative for the TB model provides flexible results which could be more useful than that of integer order derivative. From the actual data and the result fitted through our differential equation give useful information about the nature of the disease spread and its control. It can be seen from Figures 1 and 2, the behavior of the data and the model fitting, the infection goes onward which shows the increase in the infection cases. The rapid increase in the population of Khyber Pakhtunkhwa, with a less number of health facilities and lack of awareness will badly effect the population of Khyber Pakhtunkhwa, by entering more infective individuals. So, this is a serious problem and the responsible authorities should think over it.

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Figure 3. Simulations of the model (6), (a-e) and total TB infected individuals via A-B derivative (f), for $\sigma = 1$. 
Figure 4. Simulations of TB model (6), (a-e) and total TB infected individuals via A-B derivative (f), for $\sigma = 0.95$. 
Figure 5. Simulations of TB model (6), (a-e) and total TB infected individuals via A-B derivative (f), for $\sigma = 0.90$. 
Figure 6. Simulations of TB model (6), (a-e) and total TB infected individuals via A-B derivative for (f), $\sigma = 0.85$. 
Figure 7. Simulations of TB model (6), (a-e) and total TB infected individuals via A-B derivative (f), for $\sigma = 0.80$. 
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