DERIVATIONS, LOCAL AND 2-LOCAL DERIVATIONS
ON ALGEBRAS OF MEASURABLE OPERATORS

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Abstract. The present paper presents a survey of some recent results devoted to derivations, local derivations and 2-local derivations on various algebras of measurable operators affiliated with von Neumann algebras. We give a complete description of derivation on these algebras, except the case where the von Neumann algebra is of type II_1. In the latter case the result is obtained under an extra condition of measure continuity of derivations. Local and 2-local derivations on the above algebras are also considered. We give sufficient conditions on a von Neumann algebra $M$, under which every local or 2-local derivation on the algebra of measurable operators affiliated with $M$ is automatically becomes a derivation. We also give examples of commutative algebras of measurable operators admitting local and 2-local derivations which are not derivations.

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1991 Mathematics Subject Classification. Primary 46L57, 46L50; Secondary 46L55, 46L60.
Key words and phrases. von Neumann algebras; regular algebra; measurable operator; locally measurable operator; central extensions of von Neumann algebras; inner derivation; spatial derivation; local derivations; 2-local derivation.

The authors would like to acknowledge the hospitality of the California State University, Fullerton, during USA-Uzbekistan Conference on Analysis and Mathematical Physics, May 20-23, 2014.
1. Introduction

The present paper is devoted to some recent results concerning derivations and derivation-type mappings on certain classes of unbounded operator algebras.

The theory of algebras of operators acting on a Hilbert space began in 1930s with a series of papers by Murray and von Neumann (see [46–49]), motivated by the theory of unitary group representations and certain aspects of the quantum mechanical formalism. They analyzed the structure of the family of algebras which are referred nowadays as von Neumann algebras or \( W^* \)-algebras and which have the distinctive property of being closed in the weak operator topology. In 1943 Gelfand and Naimark developed the theory of uniformly closed operator \( * \)-algebras, which are now called \( C^* \)-algebras.

Nowadays the theory of operator algebras plays an important role both in pure mathematical and application aspects. This is motivated by the fact that in terms of operator algebras, their states, representations, groups of automorphisms, and derivations one can describe and investigate properties of model systems in the quantum field theory and statistical physics.

Let \( \mathcal{A} \) be an algebra over the field of complex numbers. A linear (respectively, additive) operator \( D : \mathcal{A} \to \mathcal{A} \) is called a linear (respectively, additive) derivation if it satisfies the identity \( D(xy) = D(x)y +xD(y) \) for all \( x, y \in \mathcal{A} \) (Leibniz rule). Each element \( a \in \mathcal{A} \) defines a linear derivation \( D_a \) on \( \mathcal{A} \) given by \( D_a(x) = ax - xa, x \in \mathcal{A} \). Such derivations \( D_a \) are said to be inner. If the element \( a \) implementing the derivation \( D_a \) on \( \mathcal{A} \), belongs to a larger algebra \( \mathcal{B} \), containing \( \mathcal{A} \) (as a proper ideal as usual) then \( D_a \) is called a spatial derivation.

One of the main problems considered in the theory of derivations is to prove the automatic continuity, innerness or spatialness of derivations, or to show the existence of non inner and discontinuous derivations on various topological algebras.

In particular, it is a general algebraic problem to find algebras which admit only inner derivations.

A more general problem is the following one: given an algebra \( \mathcal{A} \), does there exist an algebra \( \mathcal{B} \) containing \( \mathcal{A} \), such that any derivation of the algebra \( \mathcal{A} \) is spatial and implemented by an element from \( \mathcal{B} \)? (see e.g. [33], [52]).

The theory of derivations in operator algebras is an important and well investigated part of the general theory of operator algebras, with applications in mathematical physics (see, e.g. [29], [52], [53]). It is well known that every derivation of a \( C^* \)-algebra is bounded (i.e. is norm continuous), and that every derivation of a von Neumann algebra is inner. For a detailed exposition of the theory of bounded derivations we refer to the monographs of Sakai [52], [53]. A comprehensive study of derivations in general Banach algebras is given in the monograph of Dales [34] devoted to the study of automatic continuity of derivations on various classes of Banach algebras.

Investigations of general unbounded derivations (and derivations on unbounded operator algebras) began much later and were motivated mainly by needs of mathematical physics, in particular by the problem of constructing the dynamics in quantum statistical mechanics. The kinematical structure of a physical system in the quantum field theory (systems with infinite number of degrees of freedom) is described by an operator algebra \( \mathcal{A} \), where states are positive normalized linear functionals on \( \mathcal{A} \), and observables are elements of this algebra \( \mathcal{A} \). The dynamical evolution of the system is given by a group of \( * \)-automorphism of the operator

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algebra $A$. The infinitesimal motion is described by some form of Hamiltonian formalism, incorporating the interparticle interaction. In quantum field theory the infinitesimal motion is given by a derivation $d$ on the operator algebra $A$ of observables. The basic problem which occurs in this approach is the integration of these infinitesimal motion in order to obtain the dynamical flow. In terms of operator algebras this means: to prove that a given derivation on the algebra of observables is the infinitesimal generator of a one-parameter automorphisms group, moreover it is spatial (i.e. defined by some Hamiltonian operator) or even inner (i.e. the Hamiltonian operator is itself an observable in the considered physical system). For details we refer to [29].

The development of a non commutative integration theory was initiated by Segal [55], who considered new classes of (not necessarily Banach) algebras of unbounded operators, in particular the algebra $S(M)$ of all measurable operators affiliated with a von Neumann algebra $M$. Since the algebraic, order and topological properties of the algebra $S(M)$ are somewhat similar to those of $M$, in [8], [9] the above problems have been considered for derivations on the algebra $S(M)$. If the von Neumann algebra $M$ is abelian then it is $*$-isomorphic to the algebra $L^\infty(\Omega) = L^\infty(\Omega, \Sigma, \mu)$ of all (classes of equivalence of) essentially bounded measurable complex functions on a measure space $(\Omega, \Sigma, \mu)$ and therefore, $S(M) \cong L^0(\Omega)$, where $L^0(\Omega) = L^0(\Omega, \Sigma, \mu)$ is the algebra of all measurable complex functions on $(\Omega, \Sigma, \mu)$, and hence, in this case inner derivations on $S(M)$ are identically zero, i.e. trivial.

In the abelian case Ber, Sukochev, and Chilin in [23] obtained necessary and sufficient conditions for existence of non trivial derivations on commutative regular algebras. In particular they prove that the algebra $L^0(0,1)$ of all measurable complex functions on the interval $(0,1)$ admits non trivial derivations. Independently, Kusraev (see [35], [41], [42]) by means of Boolean-valued analysis establishes necessary and sufficient conditions for existence of non trivial derivations and automorphisms on extended complete complex $f$-algebras. In particular, he also proves that the algebra $L^0(0,1)$ admits non trivial derivations and automorphisms. It is clear that these derivations are discontinuous in the measure topology, and they are neither inner nor spatial. Therefore, the properties of derivations on the algebra $S(M)$ of unbounded operators are very far from being similar to those exhibited by derivations on $C^*$- or von Neumann algebras. But it seems that the existence of such "exotic" examples of derivations is deeply connected with the commutativity of the underlying von Neumann algebra $M$. In view of this conjecture the present authors suggested to investigate the above problems in a non commutative setting (see [1], [2]), by considering derivations on the algebra $LS(M)$ of all locally measurable operators with respect to a semi-finite von Neumann algebra $M$ and on various subalgebras of $LS(M)$. The most complete results concerning derivations on $LS(M)$ have been obtained by the authors and collaborators in the case of type I von Neumann algebras. Some of our results have been confirmed independently in [24] by representation of measurable operators as operator valued functions. Another approach to similar problems in the framework of type I $AW^*$-algebras has been outlined in [35].

The paper is organized as follows. In section 2 we present the preliminaries and basic results on non commutative integration theory and recall definitions of the algebras $S(M)$ of measurable operators, $LS(M)$ of locally measurable operators
affiliated with a von Neumann algebra \( M \). We also consider their subalgebras: \( S(M, \tau) \) of \( \tau \)-measurable operators, and \( S_0(M, \tau) \) of \( \tau \)-compact operators affiliated with the von Neumann algebra \( M \) and a faithful normal semi-finite trace \( \tau \) on \( M \). The latter algebras equipped with the measure topology become metrizable topological algebras. Section 2 contains a complete description of derivations on the algebras \( LS(M) \), \( S(M) \), \( S(M, \tau) \) and \( S_0(M, \tau) \) for a type I von Neumann algebra \( M \).

We give a general construction of derivations which are neither inner nor spatial, and moreover, which are discontinuous in the measure topology on the algebra \( LS(M) = S(M) \) for a finite type I von Neumann algebra \( M \). We show that for properly infinite type I von Neumann algebra \( M \), the algebras \( LS(M) \), \( S(M) \) and \( S(M, \tau) \) admit only inner derivations. Derivations on the algebra \( S_0(M, \tau) \) of \( \tau \)-compact operators are investigated for arbitrary semi-finite (i.e. type II algebras are also included) von Neumann algebras. We show that in the properly infinite case every derivation on this algebra is spatial and implemented by an element of \( S(M, \tau) \). In Section 4 we extend the results of the previous sections to additive derivations on \( S(M) \) for type I\(_\infty \) or type III von Neumann algebras. Here we also present some recent results of [26] which generalize this theorem for arbitrary properly infinite von Neumann algebras. The problem of description of derivations on \( S(M) \) remains open only when \( M \) is of type II\(_1 \). We present a positive solution of this problem in the case of derivations which are continuous in the measure topology.

In Section 5 we study the so-called local derivations on the algebra \( S(M, \tau) \). This notion was introduced by Kadison, who investigated such mappings on von Neumann algebras and some polynomial algebras. Here we extend his results and show that every continuous (in the measure topology) local derivation on \( S(M, \tau) \) is a derivation. In the case of an abelian von Neumann algebra \( M \) we give necessary and sufficient conditions for the existence of local derivations on \( S(M, \tau) \) which are not derivations. Finally, in Section 6 we consider 2-local derivations on algebras of measurable operators. Such mappings were introduced by Semrl, who obtained their description in the case of the algebra \( B(H) \) for infinite dimensional separable Hilbert space \( H \). Here we give the exposition of results which describe 2-local derivations on the algebra \( S(M) \) of measurable operators affiliated with an arbitrary von Neumann algebra \( M \) of type I.

2. Locally measurable operators affiliated with von Neumann algebras

Let \( H \) be a Hilbert space over the field \( \mathbb{C} \) of complex numbers, and let \( B(H) \) be the algebra of all bounded linear operators on \( H \). Denote by \( \mathbf{1} \) the identity operator on \( H \), and let \( P(H) = \{ p \in B(H) : p = p^2 = p^* \} \) be the lattice of projections in \( B(H) \). Consider a von Neumann algebra \( M \) on \( H \), i.e. a \(*\)-subalgebra of \( B(H) \) closed in the weak operator topology and containing the operator \( \mathbf{1} \). Denote by \( \| \cdot \|_M \) the operator norm on \( M \). The set \( P(M) = P(H) \cap M \) is a complete orthomodular lattice with respect to the natural partial order on \( M_h = \{ x \in M : x = x^* \} \), generated by the cone \( M_+ \) of positive operators from \( M \).

Two projections \( e, f \in P(M) \) are said to be equivalent (denoted by \( e \sim f \)) if there exists a partial isometry \( v \in M \) with initial projection \( e \) and final projection \( f \), i.e. \( v^* v = e, vv^* = f \). The relation ” \( \sim \) ” is equivalence relation on the lattice \( P(M) \).

A projection \( e \in P(M) \) is said to be finite, if for \( f \in P(M), f \leq e, f \sim e \) implies that \( e = f \).
A von Neumann algebra $M$ is said to be
- finite if $1$ is a finite projection;
- semi-finite if every non-zero projection in $M$ admits a nonzero finite subprojection;
- infinite if $1$ is not finite;
- properly infinite, if every non-zero central projection in $M$ is infinite (i.e. not finite);
- purely infinite or type III if every non-zero projection in $M$ is infinite.

A projection $e$ in a von Neumann algebra $M$ is said to be abelian if $eMe$ is an abelian von Neumann algebra. Since the lattice of projection $P(M)$ is complete, for every projection $e$ in $M$ there exists the least central projection $z(e)$ containing $e$ as a sub-projection; it is called the central support of $e$. A projection $e$ is said to be faithful if $z(e) = 1$. A von Neumann algebra $M$ is of type I if it contains a faithful abelian projection. A von Neumann algebra $M$ without non-zero abelian projections is called continuous. An arbitrary von Neumann algebra $M$ can be decomposed in a unique way into the direct sum of von Neumann algebras of type $I_{fn}$ (finite type I), type $I_\infty$ (properly infinite type I), type $I_1$ (finite continuous), type $II_{\infty}$ (semi-finite, properly infinite, continuous) and type III.

A linear subspace $D$ in $H$ is said to be affiliated with $M$ (denoted as $D\eta M$), if $u(D) \subset D$ for every unitary $u$ in the commutant $M' = \{ y \in B(H) : xy = yx, \forall x \in M \}$ of the von Neumann algebra $M$ in $B(H)$.

A linear operator $x$ on $H$ with the domain $D(x)$ is said to be affiliated with $M$ (denoted as $x\eta M$) if $D(x)\eta M$ and $u(x(\xi)) = x(u(\xi))$ for all $\xi \in D(x)$ and for every unitary $u$ in $M'$.

A linear subspace $D$ in $H$ is said to be strongly dense in $H$ with respect to the von Neumann algebra $M$, if

1) $D\eta M$;
2) there exists a sequence of projections $\{p_n\}_{n=1}^{\infty}$ in $P(M)$ such that $p_n \uparrow 1$,

$p_n(H) \subset D$ and $p_n^* = 1 - p_n$ is finite in $M$ for all $n \in \mathbb{N}$.

A closed linear operator $x$ acting in the Hilbert space $H$ is said to be measurable with respect to the von Neumann algebra $M$, if $x\eta M$ and $D(x)$ is strongly dense in $H$. Denote by $S(M)$ the set of all measurable operators with respect to $M$.

A closed linear operator $x$ in $H$ is said to be locally measurable with respect to the von Neumann algebra $M$, if $x\eta M$ and there exists a sequence $\{z_n\}_{n=1}^{\infty}$ of central projections in $M$ such that $z_n \uparrow 1$ and $z_n x \in S(M)$ for all $n \in \mathbb{N}$.

It is well-known (see e.g. [44]) that the set $LS(M)$ of all locally measurable operators with respect to $M$ is a unital *-algebra when equipped with the algebraic operations of the strong addition and multiplication and taking the adjoint of an operator.

Let $\tau$ be a faithful normal semi-finite trace on $M$. We recall that a closed linear operator $x$ is said to be $\tau$-measurable with respect to the von Neumann algebra $M$, if $x\eta M$ and $D(x)$ is $\tau$-dense in $H$, i.e. $D(x)\eta M$ and given $\varepsilon > 0$ there exists a projection $p \in M$ such that $p(H) \subset D(x)$ and $\tau(p^\perp) < \varepsilon$. Denote by $S(M, \tau)$ the set of all $\tau$-measurable operators with respect to $M$.

The subalgebra $A \subset LS(M)$ is said to be solid, if $x \in A$, $y \in LS(M)$, $|y| \leq |x|$ implies $y \in A$. 

It is well-known that $S(M)$ and $S(M, \tau)$ are solid *-subalgebras in $LS(M)$ (see [44]).
Consider the topology $t_\tau$ of convergence in measure or measure topology on $S(M, \tau)$, which is defined by the following neighborhoods of zero:

$$V(\varepsilon, \delta) = \{x \in S(M, \tau) : \exists \varepsilon \in P(M), \tau(e^1) \leq \delta, xe \in M, \|xe\|_M \leq \varepsilon\},$$

where $\varepsilon, \delta$ are positive numbers.

It is well-known [50] that $S(M, \tau)$ equipped with the measure topology is a complete metrizable topological *-algebra.

In the algebra $S(M, \tau)$ consider the subset $S_0(M, \tau)$ of all operators $x$ such that given any $\varepsilon > 0$ there is a projection $p \in P(M)$ with $\tau(p^1) < \infty$, $xp \in M$ and $\|xp\|_M < \varepsilon$. Following [57] let us call the elements of $S_0(M, \tau)$ $\tau$-compact operators with respect to $M$. It is known [44, 59] that $S_0(M, \tau)$ is a *-subalgebra in $S(M, \tau)$ and a bimodule over $M$, i.e. $ax, xa \in S_0(M, \tau)$ for all $x \in S_0(M, \tau)$ and $a \in M$.

The following properties of the algebra $S_0(M, \tau)$ are known (see [28, 57]):

Let $M$ be a von Neumann algebra with a faithful normal semi-finite trace $\tau$.

Then

1) $S(M, \tau) = M + S_0(M, \tau)$;

2) $S_0(M, \tau)$ is an ideal in $S(M, \tau)$.

Note that if the trace $\tau$ is finite then

$$S_0(M, \tau) = S(M, \tau) = S(M) = LS(M).$$

The following result describes one of the most important properties of the algebra $LS(M)$ (see [44, 51]).

**Proposition 2.1.** Suppose that the von Neumann algebra $M$ is the C*-product of von Neumann algebras $M_i$, $i \in I$, where $I$ is an arbitrary set of indices, i.e.

$$M = \bigoplus_{i \in I} M_i = \{\{x_i\}_{i \in I} : x_i \in M_i, i \in I, \sup_{i \in I} \|x_i\|_{M_i} < \infty\}$$

with the coordinate-wise algebraic operations and involution and with the C*-norm $\|\{x_i\}_{i \in I}\|_{M} = \sup_{i \in I} \|x_i\|_{M_i}$. Then the algebra $LS(M)$ is *-isomorphic to the algebra $\prod_{i \in I} LS(M_i)$ (with the coordinate-wise operations and involution), i.e.

$$LS(M) \cong \prod_{i \in I} LS(M_i)$$

($\cong$ denotes *-isomorphism of algebras). In particular, if $M$ is finite, then

$$S(M) \cong \prod_{i \in I} S(M_i).$$

It should be noted that such isomorphisms are not valid in general for the algebras $S(M)$, $S(M, \tau)$ (see [44]).

Proposition 2.1 implies that given any family $\{z_i\}_{i \in I}$ of mutually orthogonal central projections in $M$ with $\bigvee_{i \in I} z_i = 1$, and a family of elements $\{x_i\}_{i \in I}$ in $LS(M)$, there exists a unique element $x \in LS(M)$ such that $z_ix = z_ix_i$ for all $i \in I$. This element is denoted by $x = \sum_{i \in I} z_ix_i$.

It is well-known (see e.g. [53]) that every commutative von Neumann algebra $M$ is *-isomorphic to the algebra $L^\infty(\Omega) = L^\infty(\Omega, \Sigma, \mu)$ of all essentially
bounded measurable complex functions on a measure space $(\Omega, \Sigma, \mu)$, and in this case $L^0(S(M)) = S(M) \cong L^0(\Omega)$, where $L^0(\Omega) = L^0(\Omega, \Sigma, \mu)$ is the algebra of all measurable complex functions on $(\Omega, \Sigma, \mu)$.

The following description of the centers of the algebras $S(M)$, $S(M, \tau)$ and $S_0(M, \tau)$ for type I von Neumann algebras is very important in investigation of the structure of these algebras (see [4, 6]).

Proposition 2.2. Let $M$ be a von Neumann algebra of type I with center $Z$ and a faithful normal semi-finite trace $\tau$.

a) If $M$ is finite, then $Z(S(M)) = S(Z)$ and $Z(S(M, \tau)) = S(Z, \tau_Z)$, where $\tau_Z$ is the restriction of the trace $\tau$ on $Z$;

b) If $M$ is of type $I_\infty$, then the centers of the algebras $S(M)$ and $S(M, \tau)$ coincide with $Z$, and the center of the algebra $S_0(M, \tau)$ is trivial, i.e. $Z(S_0(M, \tau)) = \{0\}$.

Let $M$ be a von Neumann algebra of type $I_n$ ($n \in \mathbb{N}$) with center $Z$. Then $M$ is *-isomorphic to the algebra $M_n(Z)$ of $n \times n$ matrices over $Z$ (see [52], Theorem 2.3.3).

In this case the algebras $S(M, \tau)$ and $S(M)$ can be described in the following way (see [4]).

Proposition 2.3. Given a von Neumann algebra $M$ of type $I_n$, $n \in \mathbb{N}$, with a faithful normal semi-finite trace $\tau$, denote by $Z(S(M, \tau))$ and $Z(S(M))$ the centers of the algebras $S(M, \tau)$ and $S(M)$, respectively. Then $S(M, \tau) \cong M_n(Z(S(M, \tau)))$ and $S(M) \cong M_n(Z(S(M)))$.

3. Derivations on algebras of measurable operators for type I von Neumann algebras

In this section we shall give a complete description of derivations on the algebras $LS(M)$, $S(M)$, $S(M, \tau)$ and $S_0(M, \tau)$ for a type I von Neumann algebra $M$.

First we shall present results of Ber, Chilin and Sukochev (see [20, 23]) concerning the existence of nontrivial derivations on the algebras $S(M)$ and $S(M, \tau)$ in the case where $M$ is an abelian von Neumann algebra.

Let $\mathcal{A}$ be a commutative algebra with unit $1$ over the field $\mathbb{C}$ of complex numbers. We denote by $\nabla$ the set $\{e \in \mathcal{A} : e^2 = e\}$ of all idempotents in $\mathcal{A}$. For $e, f \in \nabla$ we set $e \leq f$ if $ef = e$. Equipped with this partial order, lattice operations $e \lor f = e + f - ef$, $e \land f = ef$ and the complement $e^+ = 1 - e$, the set $\nabla$ forms a Boolean algebra. A non zero element $q$ from the Boolean algebra $\nabla$ is called an atom if $0 \neq e \leq q$, $e \in \nabla$, imply that $e = q$. If given any nonzero $e \in \nabla$ there exists an atom $q$ such that $q \leq e$, then the Boolean algebra $\nabla$ is said to be atomic.

An algebra $\mathcal{A}$ is called regular (in the sense of von Neumann) if for any $a \in \mathcal{A}$ there exists $b \in \mathcal{A}$ such that $a = aba$.

Along this section, we shall always assume that $\mathcal{A}$ is a unital commutative regular algebra over $\mathbb{C}$, and that $\nabla$ is the Boolean algebra of all its idempotents. In this case given any element $a \in \mathcal{A}$ there exists an idempotent $e \in \nabla$ such that $ea = a$, and if $ga = a, g \in \nabla$, then $e \leq g$. This idempotent is called the support of $a$ and denoted by $s(a)$ (see [23, P. 111]).

Suppose that $\mu$ is a strictly positive countably additive finite measure on the Boolean algebra $\nabla$ of idempotents in $\mathcal{A}$, and let us consider the metric $\rho(a, b) =$
for all idempotents in $L$ regular algebra $L$. 22 mutative regular algebra to admit nontrivial derivations (see \cite{coincides with tion on an appropriate localizable measure space (Ω, K and L $\tau$ with a faithful normal semi-finite trace, (Ω, ρ) details \cite{L.8 SHA VKAT AYUPOV AND KARIMBERGEN KUDAYBERGENOV 1 85 22. Example 3.3. Corollary 3.3. Let (Ω, $\Sigma$, $\mu$) be a finite measure space and let $L^0(\Omega) = L^0(\Omega, \Sigma, \mu)$ be the algebra of all real or complex valued measurable functions on (Ω, $\Sigma$, $\mu$). The following conditions are equivalent:

(i) the Boolean algebra of all idempotents from $L^0(\Omega)$ is not atomic;
(ii) $L^0(\Omega)$ admits a non-zero derivation.

It is well known \cite{P. 45} that if $M$ is a commutative von Neumann algebra with a faithful normal semi-finite trace $\tau$, then $M$ is *-isomorphic to the algebra $L^\infty(\Omega) = L^\infty(\Omega, \Sigma, \mu)$ of all essentially bounded measurable complex valued function on an appropriate localizable measure space (Ω, $\Sigma$, $\mu$) and $\tau(f) = \int f(t) d\mu(t)$ for $f \in L^\infty(\Omega, \Sigma, \mu)$. In this case the algebra $S(M)$ of all measurable operators
affiliated with \( M \) may be identified with the algebra \( L^0(\Omega) = L^0(\Omega, \Sigma, \mu) \) of all measurable complex valued functions on \( (\Omega, \Sigma, \mu) \), while the algebra \( S(M, \tau) \) of \( \tau \)-measurable operators from \( S(M) \) coincides with the algebra

\[
\{ f \in L^0(\Omega) : \exists F \in \Sigma, \mu(\Omega \setminus F) < +\infty, \chi_F \cdot f \in L^0(\Omega) \}
\]

of all totally \( \tau \)-measurable functions on \( \Omega \), where \( \chi_F \) is the characteristic function of the set \( F \). If the trace \( \tau \) is finite then \( S(M, \tau) = S(M) \cong L^0(\Omega) \) are commutative regular algebras. But if the trace \( \tau \) is not finite, the algebra \( S(M, \tau) \) is not regular. In this case, by considering \( \Omega \) as a union of pairwise disjoint measurable sets with finite measures, we obtain that \( S(M) \) is a direct sum of commutative regular algebras which are metrizable in the above sense, and hence \( S(M, \tau) \) is a solid subalgebra of this direct sum. Therefore Corollary 3.3 implies the following solution of the problem concerning existence of derivations on algebras of measurable operators from \( S(M) \). In the commutative case (see [22], [23]).

**Theorem 3.4.** Let \( M \) be a commutative von Neumann algebra with a faithful normal semi-finite trace \( \tau \). The following conditions are equivalent:

(i) The lattice \( P(M) \) of projections in \( M \) is not atomic;

(ii) The algebra \( S(M) \) (respectively \( S(M, \tau) \)) admits a non-inner derivation.

We are now in position to give a complete description of all derivations on the algebras \( LS(M), S(M), S(M, \tau) \) and \( S_0(M, \tau) \) for a type I von Neumann algebra \( M \). These results were obtained by Albeverio, Ayupov and Kudaybergenov (see [2], [6], and [10]).

It is clear that if a derivation \( D \) on \( LS(M) \) is inner then it is \( Z \)-linear, i.e. \( D(fx) = fD(x) \) for all \( f \in Z, x \in LS(M) \), where \( Z \) is the center of the von Neumann algebra \( M \). The following main result of [2] asserts that the converse is also true.

**Theorem 3.5.** Let \( M \) be a type I von Neumann algebra with center \( Z \). A derivation \( D \) on the algebra \( LS(M) \) is inner if and only if it is \( Z \)-linear, or equivalently it is identically zero on \( Z \).

Let \( A \) be a commutative algebra and let \( M_n(A) \) be the algebra of \( n \times n \) matrices over \( A \). If \( e_{ij}, i, j = 1, ..., n, \) are the matrix units in \( M_n(A) \), then each element \( x \in M_n(A) \) has the form

\[
x = \sum_{i,j=1}^{n} f_{ij} e_{ij}, \quad f_{ij} \in A, \quad i, j = 1, 2, ..., n.
\]

Let \( \delta : A \to A \) be a derivation. Setting

\[
(3.1) \quad D_\delta \left( \sum_{i,j=1}^{n} f_{ij} e_{ij} \right) = \sum_{i,j=1}^{n} \delta(f_{ij}) e_{ij}
\]

we obtain a well-defined linear operator \( D_\delta \) on the algebra \( M_n(A) \). Moreover \( D_\delta \) is a derivation on the algebra \( M_n(A) \), and its restriction onto the center of the algebra \( M_n(A) \) coincides with the given \( \delta \).

Now let us consider arbitrary (non \( Z \)-linear, in general) derivations on \( LS(M) \). The following simple but important remark is crucial in our further considerations.
Let $\mathcal{A}$ be an algebra with center $Z$ and let $D : \mathcal{A} \to \mathcal{A}$ be a derivation. Given any $x \in \mathcal{A}$ and a central element $f \in Z$ we have

$$D(fx) = D(f)x + fD(x)$$

and

$$D(xf) = D(x)f + xD(f).$$

Since $fx = xf$ and $fD(x) = D(x)f$, it follows that $D(f)x = xD(f)$ for any $x \in \mathcal{A}$. This means that $D(f) \in Z$, i.e. $D(Z) \subseteq Z$. Therefore, given any derivation $D$ on the algebra $A$ we can consider its restriction $\delta : Z \to Z$.

Now let $M$ be a homogeneous von Neumann algebra of type $I_n, n \in \mathbb{N}$, with center $Z$. Then the algebra $M$ is *-isomorphic to the algebra $M_n(Z)$ of all $n \times n$-matrices over $Z$, and the algebra $LS(M) = S(M)$ is *-isomorphic to the algebra $M_n(S(Z))$ of all $n \times n$ matrices over $S(Z)$, where $S(Z)$ is the algebra of measurable operators with respect to the commutative von Neumann algebra $Z$.

The algebra $LS(Z) = S(Z)$ is isomorphic to the algebra $L^0(\Omega) = L(\Omega, \Sigma, \mu)$ of all measurable complex functions on a measure space, and therefore it admits (in non atomic cases) non zero derivations (see Theorem 3.4).

The following consideration is the main step in constructing the ”exotic” derivation $D_\delta$ on the algebra $S(M)$ of measurable operators affiliated with a finite type I von Neumann algebra $M$, which admits a non trivial derivation $\delta$ on its center $S(Z)$.

Let $\delta : S(Z) \to S(Z)$ be a derivation and let $D_\delta$ be the derivation on the algebra $M_n(S(Z))$ defined by (3.1).

The following lemma describes the structure of an arbitrary derivation on the algebra of locally measurable operators for homogeneous type $I_n, n \in \mathbb{N}$, von Neumann algebras (see [4]).

**Lemma 3.6.** Let $M$ be a homogenous von Neumann algebra of type $I_n, n \in \mathbb{N}$. Every derivation $D$ on the algebra $LS(M)$ can be uniquely represented as a sum

$$D = D_n + D_\delta,$$

where $D_n$ is an inner derivation implemented by an element $a \in LS(M)$, while $D_\delta$ is the derivation of the form (3.1), generated by a derivation $\delta$ on the center of $LS(M)$ identified with $S(Z)$.

Now let $M$ be an arbitrary finite von Neumann algebra of type I with center $Z$. There exists a family $\{z_n\}_{n \in F}, F \subseteq \mathbb{N}$, of central projections from $M$ with $\sup z_n = 1$, such that the algebra $M$ is *-isomorphic to the C*-product of von Neumann algebras $z_nM$, where each $z_nM$ is of type $I_n$ respectively, $n \in F$, i.e.

$$M \cong \bigoplus_{n \in F} z_nM.$$

By Proposition 2.1 we have that

$$LS(M) \cong \prod_{n \in F} LS(z_nM).$$

Suppose that $D$ is a derivation on $LS(M)$, and $\delta$ is its restriction onto its center $S(Z)$. Since $\delta$ maps each $z_nS(Z) \cong Z(LS(z_nM))$ into itself, for each $n$, it generates a derivation $\delta_n$ on $z_nS(Z)$ for each $n \in F$. 

Let $D_\delta$, be the derivation on the matrix algebra $M_n(z_n Z(\mathcal{L}(M))) \cong \mathcal{L}(z_n M)$
  defined as in (3.1). Put 
  \begin{equation}
  D_\delta(\{ x_n \}_{n \in F}) = \{ D_\delta (x_n) \}, \ \{ x_n \}_{n \in F} \in \mathcal{L}(M).
  \end{equation}
  Then the map $D$ is a derivation on $LS(M)$.

Now Lemma 3.6 implies the following result, which shows, in particular, that $D_\delta$ is the most general form of non-inner derivations on $LS(M)$.

**Lemma 3.7.** Let $M$ be a finite von Neumann algebra of type I. Each derivation $D$ on the algebra $LS(M)$ can be uniquely represented in the form 
  \begin{equation}
  D = D_a + D_\delta,
  \end{equation}
  where $D_a$ is an inner derivation implemented by an element $a \in LS(M)$, and $D_\delta$ is a derivation given as in (3.2).

Now we shall consider derivations on algebras of locally measurable operators affiliated with type I von Neumann algebras.

**Theorem 3.8.** If $M$ is a type $I_\infty$ von Neumann algebra, then any derivation on the algebras $LS(M)$, $S(M)$ and $S(M, \tau)$ is inner.

Finally, let us consider derivations on the algebra $LS(M)$ of locally measurable operators with respect to an arbitrary type I von Neumann algebra $M$.

Let $M$ be a type I von Neumann algebra. There exists a central projection $z_0 \in M$ such that
  
  a) $z_0 M$ is a finite von Neumann algebra;
  b) $z_0^\perp M$ is a von Neumann algebra of type $I_\infty$.

Consider a derivation $D$ on $LS(M)$ and let $\delta$ be its restriction onto its center $Z(S)$. By Theorem 3.8, the restriction $z_0^\perp D$ of the derivation $D$ onto $z_0^\perp LS(M)$ is inner, and thus we have $z_0^\perp \delta = 0$, i.e. $\delta = z_0 \delta$.

Let $D_\delta$ be the derivation on $z_0 LS(M)$ defined as in (3.2) and consider its extension $D_\delta$ on $LS(M) = z_0 LS(M) \oplus z_0^\perp LS(M)$, which is defined as 
  \begin{equation}
  D_\delta(x_1 + x_2) := D_\delta(x_1), \ x_1 \in z_0 LS(M), x_2 \in z_0^\perp LS(M).
  \end{equation}

The following theorem is the main result of this section, and gives the general form of derivations on the algebra $LS(M)$ (see [4]).

**Theorem 3.9.** Let $M$ be a type I von Neumann algebra and let $A$ be one of the algebras $LS(M)$, $S(M)$ or $S(M, \tau)$. Each derivation $D$ on $A$ can be uniquely represented in the form 
  \begin{equation}
  D = D_a + D_\delta
  \end{equation}
  where $D_a$ is an inner derivation implemented by an element $a \in A$, and $D_\delta$ is a derivation of the form (3.3), generated by a derivation $\delta$ on the center of $A$.

If we consider the measure topology $t_\tau$ on the algebra $S(M, \tau)$ then it is clear that every non-zero derivation of the form $D_\delta$ is discontinuous in $t_\tau$. Therefore the above Theorem 3.9 implies:

**Corollary 3.10.** Let $M$ be a type I von Neumann algebra with a faithful normal semi-finite trace $\tau$. A derivation $D$ on the algebra $S(M, \tau)$ is inner if and only if it is continuous in the measure topology.
Now, let $M$ be a type I von Neumann algebra with atomic center $Z$ and let $\{q_i\}_{i \in I}$ be the set of all atoms of $Z$. Consider a derivation $D$ on $LS(M)$. Since $q_i Z \cong q_i \mathbb{C}$ for all $i \in I$, we have $q_i D(f x) = D(q_i f x) = q_i f D(x)$ for all $i \in I, f \in Z, x \in LS(M)$. Thus $D(f x) = f D(x)$ for all $f \in Z$. This means that in the case of $Z$ being atomic, every derivation on $LS(M)$ is automatically $Z$-linear. Combining this fact with Theorem 3.5, we have the following result which is a strengthening of result obtained by Weigt in [55].

**Corollary 3.11.** If $M$ is a von Neumann algebra with atomic lattice of projections, then every derivation on the algebras $LS(M), S(M)$ and $S(M, \tau)$ is inner.

Now let us consider derivations on the algebra $S_0(M, \tau)$ of $\tau$-compact operators affiliated with a semi-finite von Neumann algebra $M$ and a faithful normal semi-finite trace $\tau$ (see [3], [4]).

It should be noted that for an arbitrary von Neumann algebra $M$, the center of the algebra $LS(M)$ coincides with $LS(Z)$, and thus contains $Z$ (see Proposition 2.2). This was an essential point in the proof of theorems describing derivations on the algebra $LS(M)$ of locally measurable operators with respect to a type I von Neumann algebra $M$. Proposition 2.2 shows that this is not the case for the algebra $S_0(M, \tau)$, because the center of this algebra may be trivial. Thus, the methods of the proof of Theorem 4.1 from [4] can not be directly applied for description of derivations on algebras of $\tau$-compact operators with respect to type I von Neumann algebras. Nevertheless, the following result for the algebra $S_0(M, \tau)$ is obtained in [6].

**Theorem 3.12.** Let $M$ be a type I von Neumann algebra with a faithful normal semi-finite trace $\tau$. Each derivation $D$ on $S_0(M, \tau)$ can be uniquely represented in the form

$$D = D_\alpha + D_\delta,$$

where $D_\alpha$ is a spatial derivation implemented by an element $a \in S(M, \tau)$, and $D_\delta$ is a derivation of the form $[\delta, \cdot]$, generated by a derivation $\delta$ on the center of $S_0(M, \tau)$.

Recently, in [14] we have investigated derivations on algebras of $\tau$-compact operators affiliated with an arbitrary semi-finite (i.e. type II algebras are also included) von Neumann algebra $M$ and a faithful normal semi-finite trace $\tau$. Namely, we proved that every $t_\tau$-continuous derivation on the algebra $S_0(M, \tau)$ is spatial and implemented by a $\tau$-measurable operator affiliated with $M$, where $t_\tau$ denotes the measure topology on $S_0(M, \tau)$. We have also shown automatic $t_\tau$-continuity of all derivations on $S_0(M, \tau)$ for properly infinite von Neumann algebras $M$. Thus, in the properly infinite case the condition of $t_\tau$-continuity of the derivation is redundant for its spatiality.

**Theorem 3.13.** Let $M$ be a von Neumann algebra with a faithful normal semi-finite trace $\tau$. Then every $t_\tau$-continuous derivation $D : S_0(M, \tau) \to S_0(M, \tau)$ is spatial and implemented by an element $a \in S(M, \tau)$.

**Theorem 3.14.** Let $M$ be a properly infinite von Neumann algebra with a faithful normal semi-finite trace $\tau$. Then any derivation $D : S_0(M, \tau) \to S_0(M, \tau)$ is $t_\tau$-continuous.

From Theorems 3.13 and 3.14 we obtain the following result.
Theorem 3.15. If $M$ is a properly infinite von Neumann algebra with a faithful normal semi-finite trace $\tau$, then any derivation $D : S_0(M, \tau) \to S_0(M, \tau)$ is spatial and implemented by an element $a \in S(M, \tau)$.

4. Derivations on algebras of measurable operators for arbitrary von Neumann algebras

In the present section we shall consider derivations on the algebras $LS(M)$ and $S(M)$ for an arbitrary von Neumann algebra $M$.

First we consider additive derivations on the algebra $LS(M)$, where $M$ is a properly infinite von Neumann algebra. These results are obtained in the paper of Ayupov and Kudaybergenov (see [11, 13]).

We shall consider the so called central extension $E(M)$ of a von Neumann algebra $M$ and show that $E(M)$ is a *-subalgebra in the algebra $LS(M)$ and this subalgebra coincides with whole $LS(M)$ if and only if $M$ does not contain a direct summand of type II.

As the main result of this section we obtain that if $M$ is a properly infinite von Neumann algebra, then every additive derivation on the algebra $E(M)$ is inner. In particular, every additive derivation on the algebra $LS(M)$, where $M$ is of type $I_\infty$ or $III$, is inner.

Let $E(M)$ denote the set of all elements $x$ from $LS(M)$ for which there exists a sequence of mutually orthogonal central projections $\{z_i\}_{i \in I}$ in $M$ with $\bigvee_{i \in I} z_i = 1$, such that $z_i x \in M$ for all $i \in I$, i.e.

$$E(M) = \{ x \in LS(M) : \exists z_i \in P(Z(M)), z_i z_j = 0, i \neq j, \bigvee_{i \in I} z_i = 1, z_i x \in M, i \in I \},$$

where $Z(M)$ is the center of $M$.

Proposition 4.1. Let $M$ be a von Neumann algebra with the center $Z(M)$. Then

i) $E(M)$ is a *-subalgebra in $LS(M)$ with center $S(Z(M))$, where $S(Z(M))$ is the algebra of measurable operators with respect to $Z(M)$;

ii) $LS(M) = E(M)$ if and only if $M$ does not have direct summands of type II.

A similar notion (i.e. the algebra $E(A)$) for arbitrary *-subalgebras $A \subset LS(M)$ was independently introduced recently by Muratov and Chilin in [45]. They called it the central extension of $A$. Therefore following [45] we shall say that $E(M)$ is the central extension of $M$.

One has the following description of $E(M)$ (see [11, 45]).

Proposition 4.2. Let $M$ be a von Neumann algebra. Then $x \in E(M)$ if only if there exists $f \in S(Z(M))$ such that $|x| \leq f$.

The following theorem is obtained in [11]).

Theorem 4.3. Let $M$ be a properly infinite von Neumann algebra. Then every additive derivation on the algebra $E(M)$ is inner.

The proof of Theorem 4.3 is based on the following lemma which has some interest in its own right (see [11, 25]).
**Lemma 4.4.** Let $M$ be a properly infinite von Neumann algebra, and let $A \subseteq \text{LS}(M)$ be a $*$-subalgebra such that $M \subseteq A$, and suppose that $D : A \to A$ is an additive derivation. Then $D|_{Z(A)} \equiv 0$, in particular, $D$ is $Z(A)$-linear.

From Theorem 4.3 and Proposition 4.1 we obtain the following extension of Theorem 3.8.

**Theorem 4.5.** Let $M$ be a direct sum of von Neumann algebras of type $I_\infty$ and $III$. Then every additive derivation on the algebra $\text{LS}(M)$ is inner.

Since $\text{LS}(M)$ contains $S(M)$ as a solid $*$-subalgebra, and $S(M)$ contains $S(M, \tau)$ as a solid $*$-subalgebra, Theorem 4.5 implies similar results for derivations on the algebras $S(M)$ and $S(M, \tau)$ for type I and type III von Neumann algebras.

Thus, the problem of describing the derivations on the above algebras is reduced to the case, where the underlying von Neumann algebra is of type II.

Recently, Ber, Chilin and Sukochev in [26] have proved that any derivation on the algebra $\text{LS}(M)$ of all locally measurable operators affiliated with a properly infinite von Neumann algebra $M$ is continuous with respect to so-called local measure topology. For type I and type III cases this follows from our Theorem 4.5. But this result is new for the type $II_\infty$ case. Later in [27] they proved the following extension of our Theorem 4.5 for the type $II_\infty$ case.

**Theorem 4.6.** Every derivation on the algebra $\text{LS}(M)$ is inner, provided that $M$ is a properly infinite von Neumann algebra.

Therefore, the problem remains unsolved only in the case when $M$ is a type $II_1$ von Neumann algebra. A partial answer for this case is given by the following theorem from [13].

**Theorem 4.7.** Let $M$ be a finite von Neumann algebra with a faithful normal semi-finite trace $\tau$, equipped with the local measure topology $t$. Then every $t$-continuous derivation $D : S(M) \to S(M)$ is inner.

The above theorem follows also from the above mentioned paper of Ber, Chilin and Sukochev in [27].

Thus the problem of innerness of derivations on algebras of measurable operators is open only for the case of type $II_1$ von Neumann algebras. For finite von Neumann algebras, the above algebras $S(M, \tau)$, $S(M)$, $\text{LS}(M)$ coincide with the algebra of all closed operators affiliated with $M$ (this is so called Murray–von Neumann algebra) (see also [39]).

**Problem 4.8.** Let $M$ be a type $II_1$ von Neumann algebra (in particular – a $II_1$-factor). Prove that every derivation on the algebra $\text{LS}(M) = S(M)$ is inner, or give an example of a $t$-discontinuous derivation on $S(M)$.

5. Local derivations on algebras of measurable operators

In this section we study local derivations on the algebra $S(M, \tau)$ of $\tau$-measurable operators affiliated with a von Neumann algebra $M$ and a faithful normal semi-finite trace $\tau$. The results presented here are due to Albeverio, Ayupov, Kudaybergenov and Nurjanov (see [7, 18]).

There exist various types of linear operators which are close to derivations (see e.g. [31, 32, 38, 43]). In particular Kadison introduced and investigated in [38] so-called local derivations on Banach algebras.
A linear operator $\Delta$ on an algebra $A$ is called a \textit{local derivation} if given any $x \in A$ there exists a derivation $D$ (depending on $x$) such that $\Delta(x) = D(x)$. The main problem concerning these operators is to find conditions under which local derivations become derivations and to present examples of algebras which admit local derivations that are not derivations (see e.g. \cite{38}, \cite{43}). In particular Kadison in \cite{38} proves that every continuous local derivation from a von Neumann algebra $M$ into a dual $M$-bimodule is a derivation. Later this result has been extended in \cite{31} to a larger class of linear operators $\Delta$ from $M$ into a normed $M$-bimodule $E$ satisfying the identity

\[(5.1) \quad \Delta(p) = \Delta(p)p + p\Delta(p)\]

for every idempotent $p \in M$.

It is clear that each local derivation satisfies (5.1) since given any idempotent $p \in M$, we have $\Delta(p) = D(p) = D(p^2) = D(p)p + pD(p) = \Delta(p)p + p\Delta(p)$.

In \cite{32} Brešar and Šemrl proved that every linear operator $\Delta$ on the algebra $M_n(R)$ satisfying (5.1) is automatically a derivation, where $M_n(R)$ is the algebra of $n \times n$ matrices over a unital commutative ring $R$ containing $1/2$.

In \cite{37} Johnson extends Kadison’s result and proves every local derivation from a $C^*$-algebra $A$ into any Banach $A$-bimodule is a derivation. He also shows that every local derivation from a $C^*$-algebra $A$ into any Banach $A$-bimodule is bounded (see \cite{37}, Theorem 5.3).

In \cite{54} it was proved that every local derivation on the maximal $O^*$-algebra $\mathcal{L}^+(D)$ is an inner derivation.

In the present section we study local derivations on the algebra $S(M, \tau)$ of all $\tau$-measurable operators affiliated with a von Neumann algebra $M$ and a faithful normal semi-finite trace $\tau$. One of our main results (Theorem 5.1) presents an unbounded version of Kadison’s Theorem A from \cite{38}, and it asserts that every local derivation on $S(M, \tau)$ which is continuous in the measure topology automatically becomes a derivation. In particular, for type I von Neumann algebras $M$ all such local derivations on $S(M, \tau)$ are inner derivations. We also show that for type I finite von Neumann algebras without abelian direct summands, as well as for von Neumann algebras with the atomic lattice of projections, the continuity condition on local derivations in the above theorem is redundant.

Finally, we consider the problem of existence of local derivations which are not derivations on algebras of measurable operators. The consideration of such examples on various finite- and infinite dimensional algebras was initiated by Kadison, Kaplansky and Jensen (see \cite{38}). We consider this problem on a class of commutative regular algebras, which includes the algebras of measurable functions on a finite measure space, and obtain necessary and sufficient conditions for the algebras of measurable and $\tau$-measurable operators affiliated with a commutative von Neumann algebra to admit local derivations which are not derivations.

Recall that $S(M, \tau)$ is a complete metrizable topological $*$-algebra with respect to the measure topology $t_\tau$. Moreover, the algebra $S(M, \tau)$ is semi-prime, i.e. $aS(M, \tau)a = \{0\}$ for $a \in S(M, \tau)$, implies $a = 0$.

One of the main results of this section is the following (see \cite{7}).

\textbf{Theorem 5.1.} Let $M$ be a von Neumann algebra with a faithful normal semi-finite trace $\tau$. Then every $t_\tau$-continuous linear operator $\Delta$ on the algebra $S(M, \tau)$
satisfying the identity (7.7) is a derivation on $S(M, \tau)$. In particular any $t_\tau$-continuous local derivation on the algebra $S(M, \tau)$ is a derivation.

It should be noted that the proof of the latter theorem essentially relies on a result of Brešar [30, Theorem 1] which asserts that every Jordan derivation on a semi-prime algebra is a (associative) derivation.

For type I von Neumann algebras the above result can be strengthened as follows.

**Corollary 5.2.** Let $M$ be a type I von Neumann algebra with a faithful normal semi-finite trace $\tau$. Then every $t_\tau$-continuous linear operator $\Delta$ satisfying (7.7) (in particular every $t_\tau$-continuous local derivation) on $S(M, \tau)$ is an inner derivation.

Further we have the following technical result, which has some intrinsic interest.

**Lemma 5.3.** Every local derivation $\Delta$ on the algebra $S(M, \tau)$ is necessarily $P(Z)$-homogeneous, i.e.

$$\Delta(zx) = z\Delta(x)$$

for any central projections $z \in P(Z) = P(M) \cap Z$, and for all $x \in S(M, \tau)$.

For finite von Neumann algebras the condition of $t_\tau$-continuity of local derivations can be omitted. Namely, one has the following theorem.

**Theorem 5.4.** Let $M$ be a finite von Neumann algebra of type I without abelian direct summands, and let $\tau$ be a faithful normal semi-finite trace on $M$. Then every local derivation $\Delta$ on the algebra $S(M, \tau)$ is a derivation, and hence can be represented as the sum (3.4) of an inner derivation and a discontinuous derivation.

Recently similar problems in a more general setting were also considered by Hadwin and coauthors in [36]. In particular, Theorem 1 from [36] implies the following extension of the above theorems for general von Neumann algebras.

**Theorem 5.5.** Let $M$ be a von Neumann algebra without abelian direct summands, and let $\mathcal{A}$ be a subalgebra in $LS(M)$ such that $M \subseteq \mathcal{A}$. Then every local derivation $\Delta$ on $\mathcal{A}$ is a derivation.

In the latter theorems the condition on $M$ to have no abelian direct summand is crucial, because in the case of abelian von Neumann algebras the picture is completely different. Therefore, below we shall consider local derivations on the algebras of measurable and $\tau$-measurable operators affiliated with abelian von Neumann algebras.

Now let $D$ be a derivation on a regular commutative algebra $\mathcal{A}$. Since any derivation on $\mathcal{A}$ does not enlarge the supports of elements (see [8, Theorem] and [23, Proposition 2.3]) we have that $s(D(a)) \leq s(a)$ for any $a \in \mathcal{A}$, and also $D|_\mathcal{V} = 0$. Therefore by the definition, each local derivation $\Delta$ on $\mathcal{A}$ satisfies the following two conditions:

\begin{align*}
(5.2) & \quad s(\Delta(a)) \leq s(a), \; \forall a \in \mathcal{A}, \\
(5.3) & \quad \Delta|_\mathcal{V} \equiv 0.
\end{align*}

This means that (5.2) and (5.3) are necessary conditions for a linear operator $\Delta$ to be a local derivation on the algebra $\mathcal{A}$. 
The following lemma which assert that these two condition are also sufficient, is the crucial step for the proofs of the further results in this section.

**Lemma 5.6.** Each linear operator on the algebra $A$ satisfying the conditions (5.2) and (5.3) is a local derivation on $A$.

The following theorem presents conditions for existence of local derivations that are not derivations on commutative regular algebras (cf. Theorem 3.2).

**Theorem 5.7.** Let $A$ be a unital commutative regular algebra over $\mathbb{C}$, and let $\mu$ be a strictly positive countably additive finite measure on the Boolean algebra $\nabla$ of all idempotents in $A$. Suppose that $A$ is complete with respect to the metric $\rho(a, b) = \mu(s(a - b))$, $a, b \in A$. Then the following conditions are equivalent:

1. $K_c(\nabla) \neq A$;
2. The algebra $A$ admits a non-zero derivation;
3. The algebra $A$ admits a non-zero local derivation;
4. The algebra $A$ admits a local derivation which is not a derivation.

The proof of the above theorem is based on the following technical result, which is the main tool for construction of local derivations which are not derivations.

**Lemma 5.8.** If $D$ is a derivation on a commutative regular algebra $A$, then $D^2$ is a derivation if and only if $D \equiv 0$.

An important special case of the latter theorem is the following result concerning the regular algebra $L^0(\Omega, \Sigma, \mu)$.

**Corollary 5.9.** Let $(\Omega, \Sigma, \mu)$ be a finite measure space and let $L^0(\Omega) = L^0(\Omega, \Sigma, \mu)$ be the algebra of all real or complex valued measurable functions on $(\Omega, \Sigma, \mu)$. The following conditions are equivalent:

1. The Boolean algebra of all idempotents from $L^0(\Omega)$ is not atomic;
2. $L^0(\Omega)$ admits a non-zero derivation;
3. $L^0(\Omega)$ admits a non-zero local derivation;
4. $L^0(\Omega)$ admits a local derivation which is not a derivation.

For general commutative von Neumann algebras one has the following result (cf. Theorem 3.4).

**Theorem 5.10.** Let $M$ be a commutative von Neumann algebra with a faithful normal semi-finite trace $\tau$. The following conditions are equivalent:

1. The lattice $P(M)$ of projections in $M$ is not atomic;
2. The algebra $S(M)$ (respectively $S(M, \tau)$) admits a non-inner derivation;
3. The algebra $S(M)$ (respectively $S(M, \tau)$) admits a non-zero local derivation;
4. The algebra $S(M)$ (respectively $S(M, \tau)$) admits a local derivation which is not a derivation.

6. 2-Local derivations on algebras of measurable operators

This section is devoted to 2-local derivations on the algebra $S(M)$ of measurable operators affiliated with a von Neumann algebra $M$ of type I. The results presented here are due to Ayupov, Kudaybergenov and Alauadinov (see [16][17][19]).
In 1997, Semrl [56] introduced the concepts of 2-local derivations and 2-local automorphisms. A map \( \Delta : A \to A \) (not linear in general) is called a 2-local derivation if for every \( x, y \in A \), there exists a derivation \( D_{xy} : A \to A \) such that 
\[
\Delta(x) = D_{xy}(x) \quad \text{and} \quad \Delta(y) = D_{xy}(y).
\]
In this paper he described 2-local derivations and automorphisms of the algebra \( B(H) \) of all bounded linear operators on the infinite-dimensional separable Hilbert space \( H \). A similar description for the finite-dimensional case appeared later in [40]. In our paper [12] we have considered 2-local derivations on the algebra \( B(H) \) of all linear bounded operators on an arbitrary (no separability is assumed) Hilbert space \( H \) and proved that every 2-local derivation on \( B(H) \) is a derivation. Recently, we have extended this result for arbitrary von Neumann algebras [15]. Zhang and Li [60] described 2-local derivations on symmetric digraph algebras and constructed a 2-local derivation which is not a derivation on the algebra of all upper triangular complex \( 2 \times 2 \) matrices.

Throughout this section \( A \) is a unital commutative regular algebra over \( \mathbb{C} \), \( \nabla \) is the Boolean algebra of all its idempotents and \( \mu \) is a strictly positive countably additive finite measure on \( \nabla \). Consider the metric \( \rho(a, b) = \mu(s(a - b)) \), \( a, b \in A \), on the algebra \( A \), and assume that \((A, \rho)\) is a complete metric space (cf. [23]).

The following Theorem (see [17] Theorem 3.5) gives a solution of the problem concerning existence of 2-local derivations which are not derivations on algebras of measurable operator in the abelian case.

**Theorem 6.1.** Let \( M \) be an abelian von Neumann algebra. The following conditions are equivalent:

1. The lattice \( P(M) \) of projections in \( M \) is not atomic;
2. The algebra \( S(M) \) admits a 2-local derivation which is not a derivation.

Further in this section we shall investigate 2-local derivations on matrix algebras over commutative regular algebras.

Let \( M_n(A) \) be the algebra of \( n \times n \) matrices over a commutative regular algebra \( A \). The following result from [17] shows that for \( n \geq 2 \) this algebra has a completely different property compared with the corresponding property of the algebra \( A \) in the previous Theorem.

**Theorem 6.2.** Every 2-local derivation \( \Delta : M_n(A) \to M_n(A) \), \( n \geq 2 \), is a derivation.

The proof of Theorem 6.2 consists of several Lemmata.

For \( x \in M_n(A) \) by \( x_{ij} \) we denote the \((i, j)\)-entry of \( x \), i.e. \( e_{ii}x_{ij}e_{jj} = x_{ij}e_{ij} \), where \( 1 \leq i, j \leq n \).

**Lemma 6.3.** For every 2-local derivation \( \Delta \) on \( M_n(A) \), \( n \geq 2 \), there exists a derivation \( D \) such that \( \Delta(e_{ij}) = D(e_{ij}) \) for all \( i, j \in \{1, 2, ..., n\} \).

**Lemma 6.4.** If \( \Delta(e_{ij}) = 0 \) for all \( i, j \in \{1, 2, ..., n\} \), then the restriction \( \Delta|_A \) is a derivation.

**Lemma 6.5.** If \( \Delta|_A \equiv 0 \) and \( \Delta(e_{ij}) = 0 \) for all \( i, j \in \{1, 2, ..., n\} \), then \( \Delta \equiv 0 \).

Now we outline the sketch of the proof for this Theorem 6.2.

First, according to Lemma 6.3 one can find a derivation \( D \) on \( M_n(A) \) such that \( (\Delta - D)(e_{ij}) = 0 \) for all \( i, j \in \{1, 2, ..., n\} \). Further, by Lemma 6.4 \( \delta = (\Delta - D)|_A \) is a derivation. Finally, passing to the 2-local derivation \( \Delta_0 = \Delta - D - D_\delta \) and
taking into account that \( \Delta_0(e_{ij}) = 0 \) for all \( i, j \in 1, 2, ..., n \), and that \( \Delta_0|_A = 0 \), by Lemma 6.5 we obtain that \( \Delta_0 = 0 \), i.e. \( \Delta = D + D_0 \) is a derivation.

Let \( M \) be a von Neumann algebra and denote by \( S(M) \) the algebra of all measurable operators and by \( LS(M) \) – the algebra of all locally measurable operators affiliated with \( M \). Theorem 6.2 implies the following result.

**Theorem 6.6.** Let \( M \) be a finite von Neumann algebra of type I without abelian direct summands. Then every 2-local derivation on the algebra \( LS(M) = S(M) \) is a derivation.

**Theorem 6.7.** Let \( M \) be an arbitrary von Neumann algebra of type I\( \infty \) and let \( \mathcal{B} \) be a \(*\)-subalgebra of \( LS(M) \) such that \( M \subseteq \mathcal{B} \). Then every 2-local derivation \( \Delta : \mathcal{B} \to \mathcal{B} \) is a derivation.

The proof of Theorem 6.7 (see [16]) is essentially different compared with the proof in the case of finite type I von Neumann algebras. In this case we use the extended center valued trace \( \Phi \) on the set \( M_+ \) of all positive elements \( M \). The following identity is crucial for the proof of the theorem:

\[
\Phi(\Delta(x)y) = -\Phi(x\Delta(y)),
\]

where \( x, y \) are finite range operators from \( LS(M) \).

**Corollary 6.8.** Let \( M \) be an arbitrary von Neumann algebra of type I\( \infty \). Then every 2-local derivation \( \Delta : LS(M) \to LS(M) \) is a derivation.

**Acknowledgements**

The authors are indebted to the referee for valuable suggestions and remarks.

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