Six-Dimensional Tensionless Strings
In The Large $N$ Limit

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Abstract

When $N$ five-branes of M-theory coincide the world-volume theory contains tensionless strings, according to Strominger’s construction. This suggests a large $N$ limit of tensionless string theories. For the small $E_8$ instanton theories, the definition would be a large instanton number. An adiabatic argument suggests that in the large $N$ limit an effective extra uncompactified dimension might be observed. We also propose “surface-equations”, which are an analog of Makeenko-Migdal loop-equations, and might describe correlators in the tensionless string theories. In these equations, the anti-self-dual two forms of 6D and the tensionless strings enter on an equal footing. Addition of strings with CFTs on their world-sheet is analogous to addition of matter in 4D QCD.
1 Introduction

Although all known quantum field theories in 6D are trivial in the IR limit, it has been suggested [1], on the basis of non-perturbative string theory, that non-trivial quantum theories in flat $\mathbb{R}^{5,1}$ do exist. These theories have been called "tensionless non-critical strings". They acquired that name because by a relevant perturbation (Higgs-ing the tensor multiplet) of the tensionless non-critical string theory we can reach the "tensile non-critical string". That theory is described in the IR by a free anti-self-dual tensor field. The massive BPS spectrum contains a 1-brane which is a source for the 2-form field.

There are various constructions of such theories from string theory.

In [1] it was argued that when type-IIB is compactified on a $K_3$ with a very small 2-cycle the resulting 6D theory contains a BPS 1-brane with a tension proportional to the area of the 2-cycle. When the 2-cycle shrinks to zero and the $K_3$ develops an $A_1$ singularity the string becomes of zero tension. In [2] the same theory was constructed as the world volume theory of two parallel 5-branes in M-theory that become very close. The connection between the two examples was made explicit in [3]. This example has $\mathcal{N} = 2$ SUSY in 6D.

An example with $\mathcal{N} = 1$ was constructed in [4] as a small heterotic $E_8 \times E_8$ instanton and an extensive description of $\mathcal{N} = 1$ tensionless string theories that arise at phase transition points of heterotic vacua on $K_3$ was given in [7].

All the constructions as well as new ones can be described in the framework of F-theory [6, 7, 12, 8, 9] as type-IIB vacua on a base manifold with a 2-cycle that shrinks to zero.

What are the variables in terms of which six-dimensional non-critical string theories should be formulated?

It is unknown if a Lagrangian formulation exists.

Could there be a world-sheet formulation as a string theory? In that case there are some features of the theories that seem very hard to incorporate in a world-sheet description:

1. The graviton does not appear in the spectrum [1].

2. The non-critical string is charged under an anti-symmetric anti-self-dual tensor field $B_{\mu\nu}^{(-)}$ [5]. It is thus dual to itself in the "electric-magnetic duality" sense [1].

3. The string coupling constant is fixed and is inevitably not weakly coupled. This follows
from the anti-self-duality of the string and the Dirac quantization condition [1].

4. The tension of the string can be taken to be either non-zero or zero according to the VEVs of the tensor multiplet. When the tension vanishes we obtain a scale invariant theory that is non-trivial [1, 7].

5. Compactifying to 4D on a torus there are no multiply-wound modes of the string (except $(p, q)$ for co-prime $p$ and $q$ which is a winding number 1 on another 1-cycle) [10].

In particular, the fact that in 6D the genus expansion is not an expansion in a small parameter seems to suggest that any world-sheet approximation would be outside its region of applicability.

One of the motivations to this paper was to explore a third possibility which is neither a field-theory nor a world-sheet theory.

We know that upon compactification to 4D on a torus $T^2$ one obtains a gauge theory [1]. As argued in [2], when $N$ 5-branes of the M-theory come close together there emerges a six-dimensional world-volume theory that contains $N$ tensor multiplets and $N(N - 1)$ kinds of strings on the 6D world-volume which correspond to 2-branes connecting the 5-branes. When this theory is compactified to 4D one obtains $U(N)$ $\mathcal{N} = 4$ Yang-Mills. The gauge group is generically broken to $U(1)^N$. The $N$ photon fields come from the dimensional reduction of the $B^{(-)}_{\mu\nu}$ that live on the 5-branes. The $(N^2 - N)$ W-bosons in 4D arise as winding modes of the 6D strings around one of the cycles of $T^2$. Winding states around other cycles will give monopoles and dyons.

When the $N$ 5-branes coincide we obtain the scale invariant theory in 6D that gives rise to the un-Higgsed $\mathcal{N} = 4$ $U(N)$ in 4D.

It is interesting that in 4D the winding modes of the six-dimensional tensionless strings as well as the reduction of the 2-forms are related by a non-abelian gauge symmetry. In an $SU(2)$ gauge theory there is no real distinction between the $W^3$ boson and $W^\pm$. This suggests that maybe in 6D there is a formulation where the 2-forms $B^{(-)}_{\mu\nu}$ and the tensionless strings enter on an equal footing.

Gauge theories can be formulated in terms of geometrical variables namely the Wilson loops. The variables of the theory are labeled by loops and the correlators are related by the Makeenko-Migdal loop equations [11]. Furthermore, in the large $N$ limit the whole formula-
tion simplifies because relations among loops (that say wind $N$ times) that are a consequence of the finite $N$ nature of the matrices might be neglected. A further simplification arises because different closed loops are uncorrelated up to $O\left(\frac{1}{N^2}\right)$.

It is quite possible that some of the features of tensionless string theories simplify in the large $N$ limit. We will argue, on the basis of adiabatic considerations for type-II on an $A_{N-1}$ singularity that the 6D tensionless string theory effectively grows an extra dimension. An adiabatic argument that is similar in spirit can be given for the small $E_8$ instanton string as well, by using the CHS solution $[25]$ at a large instanton number.

We also wish to propose surface equations that maybe describe correlators of degrees of freedom of the tensionless strings in the limit when $N \to \infty$. The surfaces would be non-abelian generalizations of the integrals $e^{i \int_s B^{(-)}_{\mu \nu}}$.

Our guidelines in guessing the equations were the requirements of locality and unitarity.

The anti-self-duality of $B^{(-)}_{\mu \nu}$ causes extra complications because the time derivative is first order and the Bianchi identity is no longer satisfied. We will discuss first an operator algebra that leads to surface equations which are "interacting" generalizations of the equations that a free 2-form field should satisfy. Then we will discuss the more complicated case of anti-self-duality.

The paper is organized as follows:

- **Section (2):** A review of the constructions of tensionless string theories from non-perturbative string theory.
- **Section (3):** A generalization of loop equations to surface equations.
- **Section (4):** Anti-self-duality is incorporated into the equations.
- **Section (5):** We add a few words about the super-symmetric case.
- **Section (6):** An adiabatic argument based on the CHS solution at instanton number $N$ and a similar argument for type-II on an $A_{N-1}$ singularity, both for large $N$, is presented. We argue that an extra effective dimension might arise.
- **Section (7):** Discussion.

Section (6), is independent of the previous sections, but we decided to postpone it till after the description of the surface equations in order to discuss how the implications of an
extra dimension can be visible in the surface correlators.

I recently learned of another work that is under preparation [19], which discusses various aspects of possible world-sheet formulations of tensionless strings and might have some overlap with this introduction.

2 Review of tensionless strings

The existence of a tensionless string theory was first deduced in [1] by considering compactification of type-IIB on a $K_3$ with a 2-cycle whose area is very small compared to the string scale. The resulting spectrum in 6D contains an $\mathcal{N} = 2$ tensor multiplet with an anti-self-dual 2-form $B_{\mu\nu}^{(-)}$. It is obtained from that mode of the 4-form of type-IIB which is proportional to the harmonic anti-self-dual 2-form in the $K_3$ Poincaré dual to the small 2-cycle. The spectrum also contains a BPS 1-brane obtained by wrapping the 3-brane of type-IIB around the 2-cycle. This 1-brane is charged under the $B_{\mu\nu}^{(-)}$. When the area of the 2-cycle is much smaller than the string scale the 2-form $B_{\mu\nu}^{(-)}$ (and its super-partner) and the BPS 1-brane, which has a very small tension, can be considered separately from the other string modes for energies much smaller than the string scale. It was thus predicted that there exists a 6D theory which describes a string interacting with an anti-self-dual 2-form but which does not have the graviton in its spectrum. Such a string was previously constructed in [5] as a solitonic object.

Another construction of the same theory appeared in [2]. There it was shown that a 2-brane of M-theory can end on the 5-brane. When $N$ 5-branes are very close an IR observer sees a 6D theory which contains $N$ tensor multiplets and $(N^2 - N)$ strings. Each string is related to a 2-brane that connects two of the $N$ 5-branes (the strings have an orientation). The tension of the string is proportional to the distance between the 5-branes that it connects. It was further shown in [2, 13] that the endpoint of a 2-brane is a source for $B_{\mu\nu}^{(-)}$ on the corresponding 5-brane. Thus each string in the 6D theory is charged under the difference between the two $B_{\mu\nu}^{(-)}$ corresponding to the two 5-branes that it connects.

The two constructions – as the world-volume theory on the 5-brane and type-IIB on $K_3$ – were shown to be equivalent in [3] as a consequence of the equivalence of M-theory on $T^5/Z_2$. 
with 16 5-branes and type-IIB on $K_3$. When two 5-branes approach each other on the M-theory side, there is a 2-cycle that shrinks in the $K_3$ on the type-IIB side.

In these constructions the string has a small tension. When the $K_3$ is singular with an $A_{N-1}$ singularity or when the $N$ 5-branes coincide, the string in 6D becomes “figuratively” tensionless. A more precise statement would be that the 6D theory has non-trivial IR dynamics – a property that no known quantum field theory has in 6D.

The above construction has $\mathcal{N} = 2$ in 6D and upon compactification on a torus $T^2$ and going to the IR limit, one obtains the non-trivial scale invariant $\mathcal{N} = 4$ Yang-Mills theory in 4D. The modular parameter $\tau$ of the torus becomes the coupling constant and $\theta$-angle of Yang-Mills and it was suggested in [1] that this would be a natural explanation for S-duality of $\mathcal{N} = 4$ Yang-Mills.

When the scalars of $\mathcal{N} = 4$ YM get Higgsed, the gauge group can be broken to a maximal abelian subgroup with massive charged particles (the $W$-bosons) and massive monopoles. This theory would be obtained by compactifying the 6D theory of strings with small tension on a torus $T^2$ of size much smaller compared to the tension. The $W$-bosons and monopoles would correspond to strings in 6D winding around the different cycles of $T^2$.

So far the constructions had $\mathcal{N} = 2$ in 6D. A tensionless string theory with less supersymmetry, namely $\mathcal{N} = 1$ can be constructed from a heterotic $E_8$ instanton of zero size [4]. Upon compactification of this theory to 4D one obtains $\mathcal{N} = 2$ Yang-Mills with matter. The gauge group is $Sp(k)$ and there is some matter in the fundamental $2k$ of $Sp(k)$ (as in [13]). In 6D there are additional tensionless strings with a CFT on their world-sheet and those are the ones that go over to the matter in 4D [4, 13, 12].

A unifying framework for all the above constructions was given by F-theory [6]. Tensionless string theories arise when F-theory is compactified on a base manifold with a shrinking 2-cycle. The $E_8$ tensionless string arises on a 4D manifold with a blown-up point [7, 12, 8], and there are other constructions as well [12, 9].

Tensionless string theories can be phase-transition points in the moduli space of $\mathcal{N} = 1$ theories in 6D as was shown in [4]. In the example of a small $E_8$ instanton, the tensionless string theory connects the “nonzero-instanton” phase with another phase that contains 1 more tensor multiplet and 29 less hyper-multiplets.
Understanding the spectrum of states of tensionless string theory seems to be also relevant for a microscopic understanding of the entropy of certain cases of black holes [17]. It has also been suggested recently that a six-dimensional non-critical string theory (though not tensionless) can describe the world-volume theory of the 5-brane of the M-theory [18].

To summarize, we repeat again some of the properties of the non-critical string theories:

1. The graviton does not appear in the spectrum of the non-critical string [1].
2. The non-critical string is charged under an anti-symmetric anti-self-dual tensor field $B_{\mu\nu}^{(-)}$ [5]. It is thus dual to itself in the “electric-magnetic duality” sense [1].
3. The string coupling constant is fixed and is inevitably not weakly coupled. This follows from the anti-self-duality of the string and the Dirac quantization condition [1].
4. The tension of the string can be taken to be either non-zero or zero. When the tension vanishes we obtain a scale invariant theory that is non-trivial [1, 7].
5. The theory can have either (1, 0) or (2, 0) super-symmetry in six-dimensions. When the super-symmetry is (2, 0) (which reduces to $N = 4$ in 4D) the possibilities seem to be limited [1, 2]. On the other hand, there might be a diversity of (1, 0) theories. In particular, those theories can carry a world-sheet current algebra [4, 7, 12].
6. Non-critical string theories in 6D provide insight into 4D gauge theories and their BPS states [1, 16].
7. Tensionless string theories can be phase transition points between two phases in 6D $\mathcal{N} = 1$ theories [7].

3 Higher dimensional generalizations of Yang-Mills theory

Yang-Mills theories lead to a set of loop equations [20]. Although it is hard to specify the exact boundary conditions of these equations, they provide a geometrical realization of the theory in terms of a complete set of gauge invariant quantities which are the Wilson loop

\footnote{I do not know if no super-symmetry at all is possible or is ruled out.}
expectation values:

\[ W[C] = \langle \text{tr} \{ P e^{i \oint_C A_\mu dx_\mu} \} \rangle. \] (1)

and

\[ W[C_1, C_2, \ldots] = \langle \text{tr} \{ P e^{i \oint_{C_1} A_\mu dx_\mu} \text{tr} \{ P e^{i \oint_{C_2} A_\mu dx_\mu} \} \ldots \rangle \] (2)

The equations for SU(\(N\)) are (see [20] for a comprehensive review):

\[ \partial_\mu(x) \frac{\delta}{\delta \sigma_{\mu\nu}(x)} W[C] = g_B^2 N \int_C dy_\nu \delta^{(D)}(x - y) \{ W[C_{xy}, C_{yx}] - \frac{1}{N} W[C] \} \] (3)

\( \frac{\delta}{\delta \sigma_{\mu\nu}(x)} \) is the “area-derivative” and \( \partial_\mu(x) \) is the “path-derivative”.

The non-abelian nature of Yang-Mills theories is captured by a non-linear term in the loop equations which describes the splitting of a self-intersecting loop as in Fig 1.

\[ \text{Fig.1: The splitting of a self-intersecting loop in the RHS of the loop equations.} \]

Naive scaling dimensions of (3) imply that the coupling constant \( g_B \) is dimensionless in 4D. In higher dimensions Yang-Mills theories are non-renormalizable and are IR trivial.

However, we can contemplate a different kind of generalization of (3) from 4D to higher dimensions. Instead of using correlators that are parameterized by loops we can try to use correlators that are labeled by surfaces in \( D \) dimensions. The non-linear term on the RHS can be straightforwardly generalized to surfaces and we obtain a set of equations with a naive scaling dimension that makes the coupling constant dimensionless in 6D.

The resulting surface equations will be described below, but we wish to precede them with a discussion of their relevance to tensionless strings. We will begin with what we believe is a direct but inappropriate approach to tensionless strings coupled to an anti-symmetric 2-form. In this direct approach the strings and the 2-form do not enter on an equal footing. In
the subsection that will follow we will remedy that asymmetry by introducing the non-linear surface equations.

Throughout this chapter we will simplify our lives by leaving outside both super-symmetry and the anti-self-duality of the 2-form. That is, the “abelian part” of the equations will correspond to a full two-form with both self-dual and anti-self-dual parts, and hence a Lagrangian description.

The next section will be devoted to the anti-self-dual theories.

3.1 Interacting tensionless strings and two-forms

3.1.1 Abelian surface equations

Let us start with a free field theory of 2-forms in 6D. The field $B_{\mu\nu}$ is an anti-symmetric tensor with action

$$S = \frac{1}{2} \int |dB|^2 d^6x$$

This action is gauge invariant under

$$B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu.$$  

Instead of Wilson loops in 4D we have “Wilson-surfaces”:

$$W[S] = e^{i \int_S B}$$

The surface equations are derived in the usual way:

$$\partial_\alpha \frac{\delta}{\delta \sigma_{\alpha\mu\nu}(x)} W[S] = \gamma_{\mu\nu}(S) W[S]$$

$$\epsilon_{\alpha\beta\gamma\delta\mu\nu} \partial_\alpha \frac{\delta}{\delta \sigma_{\beta\gamma\delta}(x)} W[S] = 0$$

where

$$\gamma_{\mu\nu}(S) dx^\mu \wedge dx^\nu$$

is a 2-form that is supported on the surface $S$ and is Poincaré dual to the surface. The first equation is derived from the equation of motion

$$d^* dB = 0$$

while the second is the Bianchi identity $dK = 0$ where $K = dB$. 

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3.1.2 Stringy matter in 6D

Incorporation of dynamical matter fields in 4D QED is accomplished by adding correlators of open loops. The end-points are thought of as creation and annihilation operators for charged particles. For scalar particles, each end is supplemented with the equation

\[ \partial_\mu(x) \partial^\mu(x) \langle \Phi(C_{xy}) \rangle = m^2 \langle \Phi(C_{xy}) \rangle \] (11)

Here \( \partial_\mu \) is the path derivative defined in [20] and \( m \) is the mass of the particle at the corresponding end.

The equation for an “electric” \( B_{\mu \nu} \) source is (formally):

\[ \partial_\alpha \frac{\delta}{\delta \sigma_{\alpha \mu\nu}(x)} \mathcal{W}[S] = \mathcal{W}[S; \Theta^{(e)}_{\mu\nu}(x)] + \gamma_{\mu\nu}(S) \mathcal{W}[S] \] (12)

\( \Theta^{(e)}_{\mu\nu}(x) \) is the “electric” current.

It would seem that the six-dimensional generalization of QED with matter would be to include both closed and open surfaces in the correlators. The boundaries of open surfaces will describe creation and annihilation of “stringy” matter.

In the “particle” case in 4D, there was an equation in terms of derivatives of the boundary position (i.e. the end-points of the loops) which expressed the mass-shell condition \( p^2 = m^2 \) for the end-point particles (in the scalar case). What is the analog of that term in 6D for strings?

The operator that is the 6D analog of \( p^2 \) should be given by a second-order differential operator on the boundary and as such has to be constructed from the loop differential operators

\[ \frac{\delta}{\delta \sigma_{\mu\nu}(x)}, \quad \partial_\mu(x) \] (13)

The RHS should be the tension \( T \) of the string.

We can try to make the following Ansatz. Let us consider a specific boundary \( C \) of the surface \( S \). \( C \) is a simple closed loop. Forgetting for a moment about \( S \), let \( C \) be parameterized by \( x^\mu(s) \). Let us fix the reparameterization invariance by choosing \( s \) to be the length of the string. A piece of the string from \( s \) to \( s + ds \) has mass \( T ds \). On the other hand, its momenta in the directions transverse to the string are

\[ P_\mu = (ds) \times \left( \frac{dx^\nu}{ds} \right) \frac{\delta}{\delta \sigma_{\mu\nu}(x)} \] (14)
So we can implement

\[ T^2 \eta_{\mu\nu} dx^\mu dx^\nu = T^2 ds^2 = \eta_{\alpha\beta} P^\alpha P^\beta = \eta_{\alpha\beta} dx^\nu \frac{\delta^2}{\delta \sigma_{\mu\alpha}(x) \delta \sigma_{\nu\beta}(x)} \]

we find formally:

\[ \eta_{\alpha\beta} \frac{\delta^2}{\delta \sigma_{\mu\alpha}(x) \delta \sigma_{\nu\beta}(x)} \mathcal{W}[S] = \eta_{\mu\nu} T^2 \mathcal{W}[S] \]

where the differentiations act on the boundary. There is however the issue of divergences. We know from [20], for example, that two area derivatives at the same point is an ill-defined object. However, it might be that a renormalization of the tension \( T \) will be sufficient. \footnote{I am grateful to David Gross for suggesting this possibility.}

Another possibility is that with enough super-symmetry there will not be a divergence.

### 3.1.3 Analogy with Yang-Mills theories

Let us re-assess the situation so far by recalling the guideline – the tensionless string theory in 6D.

The “non-critical string” has twice the minimum \( B^{(-)}_{\mu\nu} \) charge. It is analogous to the \( W^\pm \) gauge bosons. Indeed after compactification to 4D the winding states give rise to \( W^\pm \) and \( B^{(-)}_{\mu\nu} \) will give rise to \( W^3 \). It follows that at each string boundary two (overlapping) sheets must end. In the discussion up to now, the surface should have really been interpreted as \textit{double layered}. But now we can just as well allow two different single layer sheets to end at each boundary. Allowing them to be non-overlapping we end up with configurations with surfaces on which there are strings (the “strings” are what we called up to now “boundaries”). On a surface, each string is the boundary of two sheets (inside and outside) which agrees with the fact that the string has twice the minimum \( B^{(-)}_{\mu\nu} \) charge.

However, if we would have applied a “similar” reasoning for 4D Yang-Mills we would have ended up with a \( U(1) \) gauge theory coupled to the \( W^\pm \) “matter”. Instead of closed non-abelian Wilson loops we would have used closed \( U(1) \) Wilson loops and open loops ending on \( W^\pm \) points. We know however, that this is not the correct formulation of \( SU(2) \) Yang-Mills. The correct formulation has just closed loops and the \( W^\pm \) and \( W^3 \) are treated on an \textit{equal footing}. 
We are lead to believe that we have done the same error in 6D. It would thus seem that the correct set of variables in 6D is just the set of \textit{closed} surfaces. The “marked” strings that we drew on them before “mingle” with the $B_{\mu\nu}^{(-)}$ fields and together they form one surface operator, just like the $W^\pm$ bosons and $W^3$ enter on an equal footing in a Wilson loop in 4D.

### 3.2 Surface equations

In the large $N$ limit, the variables will be functionals $W[S]$ of \textit{surfaces} $S$ embedded in 6D. The surfaces are oriented but can be of arbitrary genus. The surfaces are defined by a map from some “world-sheet” Riemann surface of genus $g$ to $\mathbb{R}^{5,1}$ with the equivalence of surfaces that differ just by reparametrization. This “world-sheet” is related to the surface-variable and is \textit{not} the world-sheet of the tensionless string. For the moment we will assume that there are no anomalies with respect to reparameterization. We think this is plausible as long as we don’t deal with a chiral theory. (We will return to the question of gravitational anomalies on the world-sheet later for the anti-self-dual case. Indeed, in that case we expect gravitational anomalies to be canceled only after addition of tensionless strings with a CFT with $c - \bar{c} = 8$ on their world-sheet, though we were not able to prove it in this framework.)

Surfaces can self-intersect. because we used a world-sheet in the definition, two surfaces that look the same as sets in 6D might be inequivalent (and this can happen only if the surfaces self-intersect).

Similarly to large $N$ 4D QCD Wilson loop rules, we might expect that a functional of a surface that is disconnected is, up to $O(\frac{1}{N^2})$ corrections, the product of the functionals of its connected components.

There are only \textit{closed} surfaces.

A direct generalization of the area-derivative $\frac{\delta}{\delta \sigma_{\mu\nu}}$ for loop functionals is the \textit{volume-derivative} for surface functionals:

$$\frac{\delta}{\delta \sigma_{\mu\nu\tau}(x)} W[S]$$

which is defined by adding to the surface a small bubble in the direction $\mu\nu\tau$ at the position $x$ on the surface, and then calculating the change in $W[S]$ and dividing by the volume of the bubble.
We also need to define an operator $K_p(S)$ which acts on surfaces $S$ where $p$ is a point of self-intersection of the surface. $K_p(S)$ will be another surface that as a set in 6D is identical to $S$, but has a different world-sheet map and thus is distinct from $S$ in our sense. It is defined as follows. Let us consider the vicinity of the self-intersection point $p$. The surface will look like two planes that intersect transversely. Let us denote the planes by $x_1 - x_2$ and $x_3 - x_4$ (see Fig 2).

![Fig.2: The transverse intersection of two surfaces](image)

The intersection point will be the origin in each of the planes. Now cut a small disk of radius $\epsilon$ around each of the origins of the two planes and glue the two planes along the circles of radius $\epsilon$ keeping track of orientation. As we take $\epsilon \to 0$ we obtain $K_p(S)$. The resulting surface is described in Fig 3.
This procedure is a direct generalization of the way the RHS of the QCD loop equation is defined.

Heuristically we might interpret the construction of $K_p$ as an infinitesimal string that is exchanged between the two surfaces at the point of intersection.

The $K_p$ operator increases the genus of $S$ by one. For example, it transforms a sphere that is smashed such that its north pole and south pole touch – into a torus (Fig 4).

If $x, y \in S$ are two points that are distinct on the world-sheet but $x = y$ at target space, we will use the notation $K_{xy}(S)$ for the surface that is obtained by passing the infinitesimal string from $x$ to $y$. 
Our suggestion for the surface equation is thus:

\[ L_{\mu\nu}(x) = \partial_\tau \frac{\delta}{\delta \sigma_{\mu\nu}(x)} \]

\[ L_{\mu\nu}(x) W[S] = g^2 N \int_S dy_\mu \wedge dy_\nu \delta^{(6)}(y - x) \{ \mathcal{W}[K_{xy}(S)] - \frac{1}{N} \mathcal{W}[S] \} \] (18)

The surface variables should also satisfy:

\[ \epsilon_{\alpha\beta\gamma\mu\nu\tau} \partial_\mu(x) \frac{\delta}{\delta \sigma_{\alpha\beta\gamma}(x)} \mathcal{W}[S] = 0 \] (19)

Note that by naive scaling arguments the equations are indeed scale invariant precisely in six dimensions, as was expected.

In spite of the Bianchi identity, the surface functionals cannot be of Stokes-type, i.e. expandable as:

\[ \sum_{r=0}^{\infty} \int_{S \times S} F_{\mu_1 \nu_1, \ldots, \mu_r \nu_r}(x^{(1)}, \ldots, x^{(r)}) \prod_{i=1}^{r} (dx_{\mu_i}^{(i)} \wedge dx_{\nu_i}^{(i)}) \] (20)

If this expansion were true, it would essentially mean that \( \mathcal{W}[S] \) is given by a set of correlators of points, i.e. by a field theory.

In fact, if \( \mathcal{W}[S] \) would be of Stokes-type, then (18) could not be satisfied. Unlike loops, where expanding \( \partial_\mu \frac{\delta}{\delta \sigma_{\mu\nu}} \) in terms of the \( F_{i_1, \ldots, i_n} \) of the Stokes-type expansion produces several terms which correspond to the non-abelian terms in QCD (see [20])

\[ \frac{\delta}{\delta \sigma_{\mu\nu}(x)} : F_{\mu_1 \ldots \mu_n}(x_1, \ldots, x_n) \rightarrow \partial_{[\mu} F_{\nu]\mu_1 \ldots \mu_n}(x, x_1, \ldots, x_n) + F_{[\mu\nu]\mu_1 \ldots \mu_n}(x, x, \ldots, x_n) \] (21)

the surface case would produce only one term:

\[ \frac{\delta}{\delta \sigma_{\mu\nu}(x)} : F_{\mu_1 \ldots \mu_n \nu_1 \ldots \nu_n}(x_1, \ldots, x_n) \rightarrow \partial_{[\mu} F_{\nu]\mu_1 \ldots \mu_n}(x, x_1, \ldots, x_n). \] (22)

One can try to solve (18) perturbatively in \( g^2 N \). Let’s put \( N = \infty \) for simplicity. Then, for the first iteration we would set \( \mathcal{W}[S] = 1 \). The next iteration would produce

\[ \mathcal{W}[S] \sim \int_{S \times S} D_{\mu\nu,\tau\sigma}(x - y)(dx_\mu \wedge dx_\nu)(dy_\tau \wedge dy_\sigma) \] (23)

where \( D_{\mu\nu,\tau\sigma}(x - y) \) is the Abelian propagator. Now, plugging this back in the RHS of (18) we seem to be getting an abelian perturbative expansion. The same procedure in QCD

\footnote{I am grateful to A.A. Migdal for pointing this out, as well as the following issue about perturbative solutions.}
would also produce non-abelian “gluon-interactions” because plugging the substitution in the LHS as well produces additional terms. But we saw above that those terms are absent for surfaces. So we seem to get only the abelian solution. This cannot be true since the abelian solution satisfies the abelian surface equation and not (18). What goes wrong is that when we plug (24) in \( W[K_{xy}(S)] \) we get a divergence. Here is precisely where the “stringy” nature enters! In QCD we didn’t get a divergence because the contraction in the analogous \( W[K_{xy}(C)] \) involved only two points which coincide. Here, the contraction in \( W[S] \) indeed involves two points in target space, because the surfaces intersect transversely, but on the world-sheet \( K_{xy}(S) \) contains a whole loop that shrunk. Indeed it seems like a “string” of zero length is propagating from point \( y \) on one sheet to point \( x \) on the other sheet. The definition of the integral of the singular propagator on \( K_{xy}(S) \) cannot be regularized because it is a limit of a shrunk loop and not just two points that coincide on the world-sheet.

### 3.3 Operator formalism

Without any underlying Lagrangian, how can we tell whether the set of equations (18) corresponds to a consistent quantum theory? We can try to find a set of operators whose expectation values reproduce (18). We will construct these equations in a completely “geometrical” way. The operators will be labeled by surfaces and locality will be implemented by commutation relations that are non-zero only when two surfaces intersect.

In this way unitarity is easy to check but 6D Lorentz invariance is not manifest. As in QFTs, 6D Lorentz invariance becomes apparent when Schwinger-Dyson equations for correlators are written. In our case reproduction of (18) will make Lorentz invariance obvious.

In principle, there might be a flaw in this argument because of world-sheet gravitational anomalies. In constructing (18) we have defined the surfaces as maps from some world-sheet to the 6D target-space and so we tacitly assumed that there are no anomalies under reparameterization of the world-sheet. In an operator formalism, such anomalies might cause anomalies in the 6D Poincaré algebra. Nevertheless, we believe that as long as we are not including chiral fermions or anti-self-dual fields, we will be safe.

We proceed to describe how the geometrical operator formalism works in Yang-Mills and in the present case.
3.3.1 Loop operators in QCD

The loop representation of QCD was discussed in [21]. Working in the temporal gauge \( A_0 = 0 \), the operators are Wilson loops that lie on an equal-time plane together with equal-time operators of the form

\[
W_{i_1i_2\ldots i_k}(C; x_1, \ldots x_k) = \text{tr}\{ \mathcal{P} \exp i \int_{C_{x_1}} A \cdot dy \hat{A}_{i_1}(x_1) e^{i \int_{C_{x_2}} A \cdot dy \hat{A}_{i_2}(x_2)} \ldots e^{i \int_{C_{x_k}} A \cdot dy \hat{A}_{i_k}(x_k)} \}
\]

where \( C_{xy} \) is the part of the loop \( C \) from \( x \) to \( y \).

The commutation relations among the \( W \) operators are (see eqs (3.9 – 3.10) of [21]):

\[
[W(C), W(C')] = 0
\]

\[
[W(C), W_i(C'; x)] = g \int_C dy_i \delta^3(y - x) \left\{ W(C_{0y}C_{yx}C_{y0}) - \frac{1}{N} W(C)W(C') \right\}
\]

and similar equations for an arbitrary number of insertions of \( \hat{A}_i \).

The equation of motion for \( W \) that is not self-intersecting is (\( \hat{i}_r \) means that \( i_r \) is excluded):

\[
\frac{d}{dt} W_{i_1i_2\ldots i_k}(C; x_1, \ldots x_k) = \sum_{r=1}^{k} \int_{C_{x_{r-1}x_r}} W_{i_1\ldots i_{r-1}i_{r}i_{r+1}\ldots i_k}(C; x_1, \ldots x_{r-1}, y, x_r \ldots x_k) dy_j
\]

\[
+ g^2 \sum_{r=1}^{k} \partial_j(x_r) \frac{\delta}{\delta \sigma_{i_r,j}(x_r)} W_{i_1\ldots i_{r-1}i_{r}i_{r+1}\ldots i_k}(C; x_1, \ldots \hat{x}_r \ldots x_k)
\]

For a loop \( W \) that does self-intersect, say \( x(s_1) = x(s_2) \), we have to worry about the normal ordering of the operators \( A_i(x(s_1)) \) and \( \hat{A}_i(x(s_2)) \). Thus we get an extra term in the time derivative of \( W \):

\[
\frac{d}{dt} W = \cdots + g \int_{C \times C} \delta^{(3)}(y - x) \left\{ W(C_{xy})W(C_{yx}) - \frac{1}{N} W(C) \right\} dx_i dy_i
\]

Here \( C_{xy} \) is the closed loop from \( x \) to \( y \) as in the loop equations. Now we can check whether the equations of motion are consistent by requiring

\[
\frac{d}{dt} [\mathcal{O}_1, \mathcal{O}_2] = [\hat{\mathcal{O}}_1, \mathcal{O}_2] + [\mathcal{O}_1, \hat{\mathcal{O}}_2]
\]

Unitarity requires (eq. (3.17-3.18) of [21]):

\[
W(\bar{C}) = W^\dagger(C)
\]
and is supplemented by

$$|\langle \Psi | W(C) | \Psi \rangle| \leq \langle \Psi | W(C_0) | \Psi \rangle$$  \hspace{1cm} (27)$$

where $C_0$ is the null loop and $\bar{C}$ is the loop $C$ with a reversed orientation.

The Hilbert space is formally the space of functionals of loops. We can think of it as being regularized on a finite lattice. Then, in order to correspond to $SU(N)$ Yang-Mills we have to add the extra requirements when a link belongs to more than $N$ loops (see [20] for details). All the relations of this kind will make the Hilbert space finite.

### 3.3.2 Surface operators in 6D

Generalizing to $5 + 1$ dimensions we would have operators $\hat{W}\{S\}$ labeled by surfaces $S$ of various genera in 5D. Those will commute among themselves, but we need to add surfaces with an arbitrary number of marked points. Each point will be supplemented with a pair of indices. This will be analogous to loops with field strength $\hat{A}_i$ insertions.

$$\hat{W}_{i_1j_1,\ldots i_kj_k}\{S; x_1, \ldots, x_k\}$$

The operator is antisymmetric with respect to interchange within each pair (independently) $i_\ell j_\ell \rightarrow j_\ell i_\ell$.

The commutation relations among those operators will be nonzero only in case one of the $x_i$-s of one operator lies on the surface of the other operator:

$$[\hat{W}_{i_1j_1,\ldots i_kj_k}\{S; x_1, \ldots, x_k\}, \hat{W}_{i'_1j'_1,\ldots i'_lj'_l}\{S'; x'_1, \ldots, x'_l\}]$$

$$= g \sum_{r=1}^{k} \int_{S'} \left( \hat{W}_{i_1j_1,\ldots \tilde{i}_rj_r,\ldots i_kj_k}\{S; x_1, \ldots, \tilde{x}_r, \ldots, x_k\} \hat{W}_{i'_1j'_1,\ldots i'_lj'_l}\{S'; x'_1, \ldots, x'_l\} \right) \delta^5(x_r - y) dy_{i_r} \wedge dy_{j_r}$$

$$- \frac{1}{N} \hat{W}_{i_1j_1,\ldots i_kj_k}\{S; x_1, \ldots, \tilde{x}_r, \ldots, x_k\} \hat{W}_{i'_1j'_1,\ldots i'_lj'_l,\ldots \tilde{i}_rj_r}\{S'; x'_1, \ldots, x'_l, \ldots, \tilde{x}_r, \ldots, x'_l\}$$

$$\left( K_{y\ell,x_r} (S, S'); x_1, \ldots, x_k, x'_1, \ldots, \tilde{x}_r, \ldots, x'_l \right)$$

$$- \frac{1}{N} \hat{W}_{i_1j_1,\ldots i_kj_k}\{S; x_1, \ldots, x_k\} \hat{W}_{i'_1j'_1,\ldots i'_lj'_l,\ldots \tilde{i}_rj_r}\{S'; x'_1, \ldots, x'_l, \ldots, \tilde{x}_r, \ldots, x'_l\}$$

$$\left( K_{y\ell,x'_r} (S, S'); x_1, \ldots, x_k, x'_1, \ldots, \tilde{x}_r, \ldots, x'_l \right)$$

$$\delta^5(x_r - y) dy_{i'_r} \wedge dy_{j'_r}$$

(29)

where $\tilde{x}_r$ means that $x_r$ is excluded. $K_{y\ell,x'} (S, S')$ is the operation of gluing two surfaces $S$ and $S'$ at an intersection point $S \ni y = y' \in S'$ which was described above (in Figs 2-4).
The time evolution for a single insertion is given by:

$$\frac{d}{dt} \hat{W}_{ij}\{S; x\} = $$

$$\int_S \hat{W}_{i,j'; k'}\{S; x, y\} dy_i \wedge dy_{j'} + \delta_k(x) \frac{\delta}{\delta \sigma_{ijk}(x)} \hat{W}\{S\}$$

$$+ g^2 \int_{S \times S} \delta(5)(y - y') \left( \hat{W}_{ij}\{K_{y,y'}(S); x\} - \frac{1}{N} \hat{W}_{ij}\{S; x\} \right) (dy_k \wedge dy_l)(dy_k' \wedge dy_l')$$

(30)

The last line is similar to the last line in (24). The case of many insertions is similar.

We have to check that these rules are consistent with the chain rule for differentiation:

$$\frac{d}{dt} [O_1, O_2] = [\dot{O}_1, O_2] + [O_1, \dot{O}_2]$$

(31)

The only non-trivial case is when the surface corresponding to $O_1$ intersects itself at $x = y$ and the surface corresponding to $O_2$ intersects the self-intersecting surface at the same point $z = x = y$. In this case it is a subtle question of regularization. We will discuss a similar case in section (4.3).

The unitarity requirement will be that orientation reversal of the world-sheet is the same as complex conjugation.

4 The anti-self-dual tensionless string

The tensionless strings that are derived from the (10D/11D) string theory constructions couple to an anti-self-dual 2-form $B_{\mu\nu}^{(-)}$. Supersymmetry in 6D allows only anti-self-dual 2-forms, if we do not wish to include the graviton.

The previous section dealt with a theory that has a full 2-form (and no supersymmetry, of course). In what follows we will discuss the anti-self-dual case.

4.1 The abelian anti-self-dual two-form

The abelian anti-self-dual surface equations are obtained from the free anti-self-dual two-form theory in 6D. We consider only surfaces that are all in a constant time hyper-plane. The operators of the theory satisfy the relations

$$H_{ijk}^{(-)} = \partial_k B_{ij}^{(-)}$$
\[
\partial_0 B_{lm}^{(-)} = \frac{1}{3!} \epsilon_{ijklm} H_{ij}^{(-)} \\
[H_{ijk}^{(-)}(x), B_{lm}^{(-)}(y)] = i \epsilon_{ijklm} \delta^{(5)}(x - y)
\]

Second quantization is done by writing out the second quantized operator for the full (self-dual plus anti-self-dual) theory with Lagrangian \( \int |dB|^2 d^6x \) and then discarding the self-dual part of the resulting \( B_{\mu\nu} \) operator.

The surface operators are
\[
\hat{\mathcal{W}}\{S\} = : \exp \{ ig \int_S B^{(-)} \} : = : \exp \{ ig \int_{M,S=\partial M} H \} : \quad (32)
\]

Here \( M \) is any manifold whose boundary is \( S \). If \( S \) and \( S' \) do not intersect then they commute. They satisfy (see also \([24]\)):
\[
\hat{\mathcal{W}}\{S\}\hat{\mathcal{W}}\{S'\} = e^{ig^2 \mathcal{L}(S,S')} \hat{\mathcal{W}}\{S'\}\hat{\mathcal{W}}\{S\} \quad (33)
\]

where \( \mathcal{L}(S,S') \) is the integer linking number of \( S \) and \( S' \). In principle, \( g \) could have any value, but if \( \hat{\mathcal{W}}\{S\} \) is to describe the action of some stringy physical objects, they must satisfy Dirac’s quantization condition
\[
g^2 = 2\pi \quad (34)
\]

and \( \hat{\mathcal{W}}\{S\} \) commutes with \( \hat{\mathcal{W}}\{S'\} \) if \( S \) and \( S' \) do not intersect.

Since
\[
\frac{\delta}{\delta \sigma_{ijk}(x)} \mathcal{L}(S,S') = \epsilon_{ijklm} \int_{S'} \delta^{(5)}(x - y) dy_l \wedge dy_m \quad (35)
\]

we find for \( g^2 = 2\pi \):
\[
\left[ \frac{\delta \hat{\mathcal{W}}\{S\}}{\delta \sigma_{ijk}(x)}, \hat{\mathcal{W}}\{S'\} \right] = 2\pi i \epsilon_{ijklm} \int_{S'} \hat{\mathcal{W}}\{S,S'\} \delta^{(5)}(x - y) dy_l \wedge dy_m \quad (36)
\]

where \( \hat{\mathcal{W}}\{S,S'\} \) is the normal-ordered product.

From the equations of motion we find the time derivative for surfaces that do not intersect themselves:
\[
\frac{d}{dt} \hat{\mathcal{W}}\{S\} = \epsilon_{ijklm} \int_S \frac{\delta \hat{\mathcal{W}}\{S\}}{\delta \sigma_{ijk}(x)} dx_l \wedge dx_m \quad (37)
\]

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4.2 The non-abelian equations

What could be a possible non-abelian generalization of the operator algebra of the abelian case?

We should write down equations for
\[ \frac{d}{dt} \hat{\mathcal{W}} \{ S \} = \cdots \] (38)
as well as commutator relations.

Because of anti-self-duality we expect something like
\[ \frac{d}{dt} \hat{\mathcal{W}} \{ S \} = \int_S \epsilon_{ijklm} \frac{\delta \hat{\mathcal{W}} \{ S \}}{\delta \sigma_{ijk}} dx^i \wedge dx^m + \text{(contact terms)} \] (39)
The “contact-terms” should appear only for a self-intersecting loop. In fact, we should be able to define a Hamiltonian. Although the operators are in general non-local it might be the case that there exist a local energy momentum tensor. In this case we can write the Hamiltonian as
\[ \hat{\mathcal{H}} = \int \hat{\mathcal{H}} (x) d^5 x \] (40)
In the abelian case
\[ \hat{\mathcal{H}} (x) = \frac{1}{2} (H_{ijk}^{-}(x))^2 \] (41)
where $H_{ijk}^{-}$ is the (anti-self-dual) field strength. $\hat{\mathcal{H}} (x)$ can be defined as a limit of a second derivative of a small spherical surface containing $x$:
\[ \frac{1}{2} \lim_{S \to \{ x \}} \frac{\delta^2}{\delta \sigma_{ijk}(x) \delta \sigma_{ijk}(y)} \hat{\mathcal{W}} \{ S \} \] (42)
here as $S$ shrinks to the point $x$, $y \to x$ too. Let us assume that the same expression holds in the non-abelian case as well. After all, in Yang-Mills theory a similar expression (with the addition of a $\text{tr} \{ E_i^2 \}$ describes the Hamiltonian too). The problem is to find a consistent set of equations for products of operators.

As in the field theory case, where it is more convenient to write the commutation relation $[H_{ijk}^{-}(x), B_{mn}^{-}(y)]$ than $[B_{jk}^{-}(x), B_{mn}^{-}(y)]$ we will make a guess for the commutation relation:
\[ \left[ \frac{\delta \hat{\mathcal{W}} \{ S \}}{\delta \sigma_{ijk}(x)}, \hat{\mathcal{W}} \{ S' \} \right] = \text{(contact terms)} \] (43)

\(^4\) This was suggested by E. Witten.
we will then have to check the identities
\[
\frac{\delta}{\delta \sigma_{lmn}(y)} \left[ \frac{\delta \hat{W}\{S\}}{\delta \sigma_{ijk}(x)} ; \hat{W}\{S'\} \right] = \frac{\delta}{\delta \sigma_{ijk}(x)} \left[ \frac{\delta \hat{W}\{S\}}{\delta \sigma_{lmn}(y)} ; \hat{W}\{S'\} \right], \quad x, y \in S
\]
\[
\frac{\delta}{\delta \sigma_{lmn}(y)} \left[ \frac{\delta \hat{W}\{S\}}{\delta \sigma_{ijk}(x)}, \hat{W}\{S'\} \right] = \frac{\delta}{\delta \sigma_{ijk}(x)} \left[ \frac{\delta \hat{W}\{S\}}{\delta \sigma_{lmn}(y)}, \hat{W}\{S'\} \right], \quad x \in S, \quad y \in S'
\]
as well as the Jacobi identity
\[
\left[ \frac{\delta \hat{W}\{S\}}{\delta \sigma_{ijk}(x)}, \left[ \frac{\delta \hat{W}\{S'\}}{\delta \sigma_{lmn}(y)}, \hat{W}\{S''\} \right] \right] = \left[ \frac{\delta \hat{W}\{S''\}}{\delta \sigma_{ijk}(x)}, \left[ \frac{\delta \hat{W}\{S\}}{\delta \sigma_{lmn}(y)}, \hat{W}\{S'\} \right] \right]
\]
\[
\quad = \left[ \frac{\delta \hat{W}\{S''\}}{\delta \sigma_{ijk}(x)}, \left[ \frac{\delta \hat{W}\{S\}}{\delta \sigma_{ijk}(x)}, \hat{W}\{S''\} \right] \right] = \left[ \frac{\delta \hat{W}\{S''\}}{\delta \sigma_{ijk}(x)}, \frac{\delta \hat{W}\{S''\}}{\delta \sigma_{ijk}(x)} \right] \hat{W}\{S''\} \quad (46)
\]

Our guess for the commutator is
\[
\left[ \frac{\delta \hat{W}\{S\}}{\delta \sigma_{ijk}(x)}, \hat{W}\{S'\} \right] = \epsilon_{ijklm} \int_{S'} \left( \hat{W}\{K_{xy}(S, S')\} - \frac{1}{N} \hat{W}\{S, S'\} \delta^{(5)}(x - y) dy_t \wedge dy_m \right) \quad (47)
\]
Here we assume that \( S \) and \( S' \) do not intersect at points other than \( x \). If they do there are more contact terms that have to be taken care of by including more \( \frac{\delta}{\delta \sigma_{ijk}} \) derivatives.

They seem to satisfy (44,13,16). Of course, the only non-trivial check is the singular contribution when \( x = y \) in those equations. The check for (44,13) only involves derivatives of the \( \delta \)-functions and not the special nature of \( K_{xy} \). Since (44,13) are certainly satisfied for the abelian free field case, they continue to do so for the non-abelian case as well. On the other hand (46) depends on the specific nature of the new term with \( K_{xy} \). Thus, the precise regularization here is important. We can define \( K_{xy}(S, S') \) with a world-sheet that looks locally like a cylinder at \( x = y \), parameterized by \( -\infty < r < \infty \) and \( 0 \leq \theta \leq 2\pi \) such that the circle at \( r = 0 \) maps to \( x = y \) whereas the part \( r < 0 \) maps to \( S \) and the part \( r > 0 \) maps to \( S' \). We can then use this world sheet for the RHS in \( \left[ \frac{\delta \hat{W}\{S\}}{\delta \sigma_{lmn}(y)}, \hat{W}\{S''\} \right] \) in (46). Then we have to commute this with \( \frac{\delta \hat{W}\{S\}}{\delta \sigma_{ijk}(x)} \). Using (47) again, we would get some terms with a delta function \( \delta^{(5)}(x - \xi) \) where \( \xi \) is a coordinate on \( K_y(S', S'') \). If we regularize it by a Gaussian \( e^{-|x - \xi|^2} \) for example, then we pick contributions from both \( S' \) and \( S'' \). Thus, in (46) the first term picks contributions from \( S, S' \) and from \( S, S'' \), the second term from \( S', S'' \) and \( S, S'' \) and the RHS from \( S, S'' \) and \( S', S'' \). So they indeed seem to cancel. I do not know if there are any subtleties in the regularization that invalidate this argument.
5 Supersymmetry

The $\mathcal{N} = 1$ supersymmetry multiplets in 6D are:

- **Supergravity Multiplet** $G_{MN}, \Psi^{A+}, B^+_{MN}$
- **Tensor Multiplet** $B^-_{MN}, \chi^A, \Phi$
- **Hypermultiplet** $\psi^{a-}, \phi^a$
- **Yang–Mills Multiplet** $A_M, \lambda^A$

All spinors are symplectic Majorana–Weyl. The two-forms $B^+_{MN}$ and $B^-_{MN}$ have three-form field strengths that are self-dual and anti-self-dual, respectively.

For a super-symmetric theory of surface operators, we need to add operators labeled by surfaces with marked points. These would be generalizations of loop operators in 4D QCD like

$$\text{tr}\{P\lambda^a(x)e^{i\int_{G_{xx}} A_k dx_k}\}$$

where $\lambda$ is the gluino in Super-Yang-Mills (see also [23] for a related discussion). In 4D $\mathcal{N} = 2$ QCD there are also bosonic fields in the adjoint representation and so we need to add loops with marked points that correspond to the bosonic fields as well.

In the surface case, we will try to do the same. We add surfaces with marked points. Each point corresponds either to a bosonic $\Phi$ field or to a fermionic $\chi^{(-)}$ field of the tensor multiplet. Each fermionic marked point is supplemented with a spinor index. Each bosonic marked point can be either a $\Phi$ or a $\dot{\Phi}$ insertion. In the 4D QCD case, the commutation relations among the different operators are all derived from the commutation relations among the fields. For example, when a $\Phi(x)^a\tau^a$ insertion on one loop meets a $\dot{\Phi}(y)^b\tau^b$ insertion on another loop (when $x = y$) we get a term

$$[(\Phi(x)^a\tau^a)_s, (\dot{\Phi}(y)^b\tau^b)_q] \sim \delta(x - y)(\delta^r_p \delta^s_q - \frac{1}{N} \delta^r_p \delta^s_q)$$

here $p,q,r,s$ are $SU(N)$ indices and $\tau^a, a = 1 \ldots N^2 - 1$ are generators of $SU(N)$. In the QCD case, these rules imply the same kind of geometrical operation on the loops, so we guess that in 6D whenever a $\Phi$ insertion meets a $\dot{\Phi}$ insertion on another surface, the commutator will include a term like the RHS of (47). Surface operators with an odd number of fermionic

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5 This table is taken from [22].
insertions will be anti-commuting and their anti-commutator will contain the appropriate spinor index matrix.

6 Adiabatic arguments for large $N$

We would like to learn more about the large $N$ limit of tensionless string theories, directly from the 10D/11D string theory construction.

We will discuss two cases. The first case will be the $\mathcal{N} = 2$ theory, obtained as type-IIB on an $A_{N-1}$ singularity which is modeled by $\mathbb{C}^2/\mathbb{Z}_N$ where the $\mathbb{Z}_N$ acts as

$$Z_N : (z_1, z_2) \rightarrow (e^{\frac{2\pi i}{N} z_1}, e^{-\frac{2\pi i}{N} z_2})$$

(it has to be mentioned that we need the conifold and not the orbifold, which defer by a $\theta$-angle [28]).

The second case is the small $E_8$ instanton at instanton number $N$. Here we will use the effective field-theory solution of Callan-Harvey-Strominger (the symmetric five-brane) in the large $N$ limit.

We will find that in both cases an extra dimension decompactifies.

6.1 Type-IIB on $A_{N-1}$ at large $N$

Parameterizing the conifold as:

$$Z_N : (z_1, z_2) \rightarrow (e^{\frac{2\pi i}{N} z_1}, e^{-\frac{2\pi i}{N} z_2})$$

we can think of the space as an angle of $\frac{2\pi}{N}$ in the $z_1$-plane, over each point of which we have a $z_2$ plane. At the two boundaries of the $\frac{2\pi}{N}$ angle the $z_2$ planes are identified with a twist of $\frac{-2\pi}{N}$. When $N$ is large, the two sides of the angle in the $z_1$ plane are almost parallel. Moreover, if we restrict to the vicinity of the origin in the $z_2$ plane, the $\frac{-2\pi}{N}$ twist is also small. Thus, we can think of the close vicinity of the $z_2$ origin as part of a compactification on $S^1$ of radius $\frac{R}{N}$, where $R$ is the distance to the origin in the $z_1$ plane (see Fig 5).
In the adiabatic approximation, we will T-dualize type-IIB on this construction to, approximately, type-IIA on a circle of radius $\frac{R}{N}$ at the vicinity of the origin. The circle grows to infinity as $R \to 0$. T-duality also multiplies the 10D coupling constant by $\frac{N}{R}$ so that we have a strongly coupled type-IIA which grows an 11th dimension \[26\].

Let us check precisely the transformations and the region of applicability of the adiabatic argument, i.e. the region where the curvature is small.

We parameterize the conifold by $r_1, \theta_1, r_2, \theta_2$ with

$$0 \leq r_1, \quad 0 \leq r_2, \quad 0 \leq \theta_2 \leq 2\pi, \quad 0 \leq \theta_1 \leq \frac{2\pi}{N} \quad (53)$$
\( r_1, \theta_1 \) are polar coordinates for the \( z_1 \)-plane and \( r_2, \theta_2 \) are polar coordinates for the \( z_2 \)-plane. The \( \theta_1 \) and \( \theta_2 \) coordinates live on a torus. The original type-IIB metric is

\[
ds^2 = dr_1^2 + dr_2^2 + r_1^2 d\theta_1^2 + r_2^2 d\theta_2^2
\]  

(54)

Now we define

\[
\theta_1 = \frac{y}{N}, \quad \theta_2 = x + \frac{y}{N}
\]  

(55)

so \( x, y \) live on a square torus of unit size, and the metric is

\[
ds^2 = dr_1^2 + dr_2^2 + \frac{r_1^2 + r_2^2}{N^2} dy^2 + r_2^2 dx^2 + \frac{2r_2^2}{N} dxdy
\]  

(56)

T-duality on a torus with metric \( G_{11}, G_{12}, G_{22} \) replaces the metric with

\[
G'_{11} = G_{11}
\]

\[
G'_{12} = 0
\]

\[
G'_{22} = \frac{G_{11}}{G_{22} - G_{12}}
\]

and now there is also a \( B_{12} \) field

\[
B_{12} = \frac{G_{11}G_{12}}{G_{11}G_{22} - G_{12}G_{12}}
\]  

(57)

So after T-duality in the \( y \) direction we get a metric (using the adiabatic approximation):

\[
dr_1^2 + dr_2^2 + \frac{N^2}{r_1^2} dy^2
\]  

(58)

and a \( B \) field

\[
B_{xy} = \frac{Nr_2^2}{r_1^2}
\]  

(59)

The coupling constant becomes:

\[
\lambda_A = \lambda_B \frac{N}{r_1}
\]  

(60)

Transforming to M-theory we find the metric

\[
ds^2 = \frac{9}{16} \lambda_B^{-2/3} N^{-2/3} d(r_1^{4/3})^2 + \lambda_B^{-2/3} N^{-2/3} r_1^{2/3} dr_2^2 + \lambda_B^{-2/3} N^{-2/3} r_1^{2/3} r_2^2 dx^2 + \lambda_B^{-2/3} N^{-2/3} r_1^{-2/3} N^{-4/3} r_1^{4/3} (dx_{11})^2
\]

and

\[
A_{xy11} = \frac{Nr_2^2}{r_1^2}
\]  

(61)
Defining
\[ u = \lambda_B^{-1/3} N^{-1/3} r_1^{4/3}, \quad v = \lambda_B^{-1/4} N^{-1/4} r_2 \]  
we obtain
\[ ds^2 = \frac{9}{16} du^2 + u^{1/2} dv^2 + u^{1/2} v^2 dx^2 + \lambda_B^{-1} N u^{-1} dy^2 + \lambda_B N u^{-1} (dx^{11})^2 \]
\[ A_{xy11} = v^2 u^{-3/2} \]

An estimate of the tensor \( R_{mnpq}R^{mnpq} \) gives, for fixed \( \lambda_B \), the conditions (of small curvature):
\[ u^2 \gg 1, \quad u^{1/2} v^2 \gg 1 \]  
(63)

Similarly, a calculation of \(|dA|^2\) gives the condition (of small gradient):
\[ \frac{1}{N^2 u^2} \ll 1, \quad \frac{v^2}{N^2 u^{5/2}} \ll 1 \]  
(64)

which together give
\[ 1 \ll u, \quad u^{-1/2} \ll v^2 \ll N^2 u^{5/2} \]  
(65)

The conditions for decompactification in the large \( N \) limit were that the sizes of the \( dy \) and \( dx^{11} \) directions be large. This gives
\[ u \ll N \]  
(66)

which is consistent with (63) for a certain range of \( u \).

One might be worried about the fact that this range corresponds to large \( r_1 \) in the original variables. This range stops approximately where \( r_1 \sim N^{1/4} \). However, the phenomenon that we are looking for is generic, in the sense that it should correspond to “something new” even if we take type-IIB on an infinite \( \mathbb{R}^4 \) divided by \( \mathbb{Z}_N \). The extra dimensions that decompactify close to the singularity have sizes much larger than the distance to the origin.

To be more precise, if we start with type-IIB on \( \mathbb{R}^4/\mathbb{Z}_N \) for \( N \) finite, a low-energy observer in 10D would describe the situation as having type-IIB super-gravity in the 10D bulk which interacts with a tensionless string theory on the 6D singularity. From the above formulae, we learn that for \( u \sim N^\epsilon \) (for a small positive \( \epsilon \)) the extra uncompactified dimensions have sizes of order \( N^{3-\frac{5}{2} \epsilon} \) in M-theory units. This will correspond to Kaluza-Klein masses of order \( N^{-1/2+\epsilon/2} \) in these units. Thus, if we probe energies which are much smaller than \( N^{-\epsilon} \) but not smaller than \( N^{-1/2+\epsilon/2} \), we will think that the phenomenon that we are observing
is coming directly from the singularity \( u \) smaller than our sensors) but we will observe the extra dimensions.

Another thing that we notice about the metric is that even if we take the limit of weak \( \lambda_B \) or strong \( \lambda_B \), at least one of the dimensions \( dy \) or \( dx^{11} \) will be large.

On the other hand, it could also be that the states of masses \( N^{-1/2} \) have nothing to do with the tensionless string theory but are only related to type-IIB on \( R^4/Z_N \). However, we note that the applicability of the argument ceased because curvatures became large and not because the size of the extra dimensions became small. The fact that the other tensionless string theory related to small \( E_8 \) instantons also exhibits a similar phenomenon, as we will discuss shortly, might be considered as supporting evidence for relating the phenomenon to tensionless string theory.

### 6.2 A check on type-IIA on \( A_{N-1} \)

Let us see what would happen to a similar argument for type-IIA on an \( A_{N-1} \) singularity. The argument should not work in this case, of course, since type-IIA on \( A_{N-1} \) produces an enhanced gauge symmetry and not an exotic tensionless string theory.

In this case we would get again

\[
\begin{align*}
  ds^2 &= dr_1^2 + dr_2^2 + r_2^2 dx^2 + \frac{N^2}{r_1^2} dy^2 \\
  B_{xy}^{(NS)} &= \frac{Nr_2^2}{r_1^2}
\end{align*}
\]

(67)

and a \( B \) field

\[
B_{xy}^{(NS)} = \frac{Nr_2^2}{r_1^2}
\]

(68)

The coupling constant becomes:

\[
\lambda_B = \lambda_A \frac{N}{r_1}
\]

(69)

But now we are in type-IIB so we have to use the S-duality to transform to a weakly coupled theory.

\[
\lambda_{\text{new}} = \frac{1}{\lambda_{\text{old}}}
\]

(70)

This is accompanied by a rescaling

\[
g_{\text{new}} = \frac{1}{\lambda_{\text{old}}} g_{\text{old}}
\]

(71)
so

\[ ds^2 = \lambda^{-1}N^{-1}r_1^2 + \lambda^{-1}N^{-1}r_2^2 + \lambda A N^{-1}r_1^2 dx^2 + \frac{N}{r_1} dy^2 \]  

(72)

 together with a coupling constant of

\[ \lambda = \lambda^{-1}N^{-1}r_1 \]  

(73)

and an RR field

\[ B_{xy}^{(RR)} = \frac{N r_2^2}{r_1^2}. \]  

(74)

Now we have to T-dualize again. The resulting metric is:

\[ ds^2 = \lambda^{-1}N^{-1}r_1^2 + \lambda^{-1}N^{-1}r_2^2 + \lambda A N r_1^{-1}r_2^{-1} dx^2 + \frac{N}{r_1} dy^2 \]  

(75)

An estimate of the tensor \( R_{mnpq} R^{mnpq} \) gives, for fixed \( \lambda_B \), the conditions (of small curvature):

\[ \frac{\lambda A N}{r_1^3} \ll 1, \frac{\lambda A N}{r_1 r_2^2} \ll 1 \]  

(76)

So, for the applicability of the argument, the size of the \( x \) direction has to be small. On top of that, the size of the \( y \) direction is always smaller than \( \sim N^{1/3} \) and so is less than \( r_1 \). Thus, there is no decompactification in this case as it should.

### 6.3 The symmetric five-brane at large \( N \)

The symmetric five-brane solution at instanton number \( N \) is given by [25]:

\[ ds^2 = e^{2\phi} dx^2 + dy^2 \]

\[ e^{2\phi} = e^{2\phi_0} + \frac{Q}{x^2} \]

\[ H = -Q \epsilon \]

where \( ds^2 \) is the 10D metric, \( x \) are the 4D coordinates (where the instanton lives), \( y \) are the 6D coordinates where the tensionless string theory is observed, \( \phi \) is the dilaton field, \( \phi_0 \) is its value at 4D infinity, \( Q = N\alpha' \) is the charge which is proportional to the instanton number \( N \), \( H \) is the 3-form field strength and \( \epsilon \) is a 3-form constant in the directions tangent to the radius in 4D.
The 4D length scale is governed by $\sqrt{N\alpha'}$ so for large $N$ the fields vary slowly. In fact, the curvature is proportional to $\frac{1}{N\alpha'}$ and is small compared to the string scale. If we are dealing with $E_8$ instantons, we can think of locally replacing the heterotic $E_8$ theory with dilaton $\phi$ by M-theory on $S^1/Z_2$ with size [26, 27]:

$$r = e^{2\phi/3} \sim \frac{N^{1/3}}{x^{2/3}}$$

(77)

and the 4D metric is rescaled to

$$ds^2 \to e^{-2\phi/3}ds^2 = e^{4\phi/3}dx^2 \sim N^{2/3}\frac{dx^2}{x^{4/3}}$$

(78)

We see that a 7th dimension becomes uncompactified as before.

The M-theory metric is given by:

$$(\lambda_0^2 + \frac{N}{x^2})^{4/3}dx^2 + (\lambda_0^2 + \frac{N}{x^2})^{2/3}(dx_{11})^2$$

(79)

Neglecting $\lambda_0$, an estimate for the M-theory curvature yields:

$$|x| \ll N^2$$

(80)

which is certainly satisfied at the region of decompactification

$$1 \ll \frac{N}{x^2}$$

(81)

6.4 Remarks on the surface operators and a continuous spectrum

What are the implications of the decompactification? We saw that a region close to the singularity “grows” an extra dimension. We cannot of course extrapolate to the $A_{N-1}$ singularity itself because the curvature there becomes large. It might be that something additional “lives” at the singularity but whatever it is, it seems that it will have to interact with the extra decompactified dimension.

The six-dimensional tensionless string theory is scale-invariant. Thus, if it has any stable massive 1-particle states, the spectrum has to be continuous. The previous calculations suggest that as $N \to \infty$ there is a region close to the singularity where the curvature is small but there is an extra dimension of size at least $O(N^{1/2})$ in M-theory units.
I would guess that at least Kaluza-Klein states of the gravitons exist and are much lighter than the Planck scale. Since the spectrum cannot be discrete it will follow that there is a continuum of massive particles (The states that are lighter than \(O(\frac{1}{N})\) will have to come from the region much closer to the singularity where the above approximations are not valid).

On the other hand, as was mentioned above, it might be that those states of mass \(O(\frac{1}{N})\) have nothing to do with tensionless string theory but exist in type-IIB on \(\mathbb{R}^4/\mathbb{Z}_N\) as massive states that become massless only in the large \(N\) limit.

Even if the states are related to the large \(N\) limit of the tensionless string theory itself, I do not know if these arguments hold for finite \(N\) as well, since it might be that the states discussed above are unstable for finite \(N\).

Can a surface operator formalism incorporate a continuous spectrum?

Let us consider the correlator of two identical closed surfaces of size \(R\) located at points \(x\) and \(y\) in 6D. If we are allowed to assume a similar behaviour as that of large \(N\) QCD, their correlator would behave as

\[
\langle \hat{\mathcal{W}}\{S_x(R)\}\hat{\mathcal{W}}\{S_y(R)\}\rangle \sim O\left(\frac{1}{N^2}\right).
\]  

(82)

The tensionless string theory is assumed to be scale invariant in addition. We therefore expect

\[
\langle \hat{\mathcal{W}}\{S_x(R)\}\hat{\mathcal{W}}\{S_y(R)\}\rangle \sim \frac{1}{N^2} f\left(\frac{|x-y|}{R}\right)
\]  

(83)

A continuous spectrum could be obtained if the function \(f\) contains a piece like \(f(x) \sim \cdots + e^{-cx}\). Then, an operator \(\hat{\mathcal{W}}\{S_x(R)\}\) would create a state of mass proportional to its size \(R\).

7 Discussion

This paper dealt with two different aspects of tensionless string theories in the large \(N\) limit. The first has to do with the variables of the theory and the second concerns space-time.

In the first approach we suggested that surface-equations in 6D analogous to loop-equations in 4D might describe a consistent quantum theory which could be related to tensionless string theories. Here the motivation is to study the tensionless string theories in
6D independently from the framework of 10D/11D string theory. Since the (10D/11D) string theory constructions are non-perturbative one has to rely heavily on BPS arguments. An independent construction of 6D tensionless strings might encompass a larger class of theories, maybe even without super-symmetry, and might shed more light on the non-perturbative superstring-theory phenomena.

The other approach was to try to deduce certain features of the 6D theories using what we know about the non-perturbative superstring theory constructions. The large $N$ limit allows for an adiabatic application of duality transformations where we end up with more than six “large” dimensions. The extra dimensions have sizes proportional to $N^{1/2}$. This has implications on the observed 6D spectrum of states. There is however a caveat – we couldn’t entirely rule out the possibility that those states might just come from the large $N$ limit of type-IIB on $\mathbb{R}^4/\mathbb{Z}_N$ and not from the tensionless string theory.

Returning to the question of what are the variables of tensionless strings, we suggested that within tensionless string theories it might be possible to define correlators of operators labeled by surfaces and we tried to guess what kind of local relations those correlators might satisfy.

In 4D QCD the Wilson loops represent trajectories of charged matter. What do the surfaces in the 6D theory represent?

We have concentrated on the 6D analog of pure Yang-Mills theory so the question arises what would be the 6D analog of the addition of matter.

It was argued in [4, 7, 12] that a heterotic string vacuum in 6D with a zero size $E_8$ instanton in the compactified dimensions can lead to tensionless strings with an $E_8$ current algebra on their “world-sheets”. (There is no known world-sheet formulation of these tensionless strings, so what we really mean is that once we turn on VEVs that increase the tension we can reach a theory with a stringy solitonic object and an $E_8$ current algebra living on it.) The candidates for “stringy-matter” thus seem to be such strings with CFTs on their world-sheets. Indeed, in the above example, upon compactification to 4D on a torus with appropriate $E_8$ Wilson loops the new types of strings become hyper-multiplets in the fundamental representation of the gauge group.

In QCD, quarks are introduced by adding open Wilson loops which correspond to a quark
on one end and an anti-quark at the other end. Going back to 6D, we will include open surfaces as well, with boundaries that have the quantum numbers of the “matter strings” (e.g. a state in the chiral $E_8$ current algebra for the example discussed above).

However, we expect that not all possible CFTs should be allowed. In [4] cancelation of world-sheet anomalies required a CFT with $c - \bar{c} = 8$. In the surface-equations, we have used a world-sheet to define the surface (this was not the world-sheet of the tensionless string, just of the operator). We have assumed invariance under reparameterization of this world-sheet. As long as we have no chirality in the theory – P invariance will probably prevent such anomalies. However such anomalies should arise for the theories with the anti-self-dual string. I do not know if this can be seen in the surface-equations – but this is a necessary check on the equations. Maybe the anomalies arise as failure of 6D Lorentz invariance, in the operator formalism.

Though the critical dimension for the surface equations is six, they can be defined in lower dimensions as well.

Independently from the surface equations, analogy with 2D QCD might suggest that at a low enough dimension, 3D maybe, there will be no propagating degrees of freedom. Maybe at those dimensions, the large $N$ limit can be described by a topological membrane theory, like 2D QCD can be described by a topological string theory [29].

Finally, if indeed tensionless string theories in the large $N$ limit grow a large 11th dimension, as we calculated in section (6), then a better understanding of tensionless string theories might be relevant for M-theory.

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