ON RESOLUTION OF 1-DIMENSIONAL FOLIATIONS ON 3-MANIFOLDS

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Abstract. This paper is devoted to the resolution of singularities of holomorphic vector fields and of one-dimensional holomorphic foliations in dimension 3 and it has two main objectives. First, from the general perspective of one-dimensional foliations, we build upon the work of Cano-Roche-Spivakovsky [10] and essentially complete it. As a consequence, we obtain a general resolution theorem comparable to the resolution theorem of McQuillan-Panazzolo [27] but proved by means of rather different methods.

The second objective of this paper consists of looking at a special class of singularities of foliations containing, in particular, all singularities of complete holomorphic vector fields on complex manifolds of dimension 3. We then prove that for this class of holomorphic foliations, there holds a much sharper resolution theorem. This second result was the initial motivation of this paper and it relies on the combination of the previous resolution theorems for (general) foliations with some classical material on asymptotic expansions for solutions of differential equations.

1. Introduction

The purpose of this introduction is to state the main results obtained in the course of this paper along with the basic notions needed to make their statements intelligible to the non-expert reader. A more detailed discussion about the place of our results in the current state-of-art in the area as well as an outline of our methods and of the structure of the paper will follow in Section 2.

Recall first that a singular, one-dimensional holomorphic foliation $\mathcal{F}$ on $(\mathbb{C}^3, 0)$ is nothing but the (singular) foliation defined by the local orbits of a holomorphic vector field defined on a neighborhood of the origin and having zero-set of codimension at least 2. Unless otherwise stated, throughout this paper the phrase singular holomorphic foliation means a singular, one-dimensional holomorphic foliation. A simple consequence of Hilbert nullstellensatz is that, up to multiplying vector fields by a meromorphic function, every meromorphic vector field $X$ on $(\mathbb{C}^3, 0)$ induces a singular holomorphic foliation on a neighborhood of the origin. This foliation will be called the foliation associated with $X$. Clearly two (meromorphic) vector fields have the same associated foliation if and only if they differ by a multiplicative (meromorphic) function. Conversely, a vector field $X$ inducing a given foliation $\mathcal{F}$ will be called a representative of $\mathcal{F}$ if $X$ is holomorphic and the set of zeros of $X$ has codimension at least 2. In other words, a representative vector field of $\mathcal{F}$ is any holomorphic vector field tangent to $\mathcal{F}$ and having a zero-set of codimension at least 2.

There follows from the preceding that there is no point in considering “singular meromorphic foliations” since all foliations in this category would, in fact, be holomorphic. Similarly, (singular) holomorphic foliations have empty zero-divisor since their singular sets have codimension at least 2. In other words, whenever we are exclusively concerned with foliations, we can freely eliminate any (meromorphic) common factor between the components of a vector field tangent to the foliation to obtain a representative vector field. Naturally this cannot be done if we are focusing on an actual fixed vector field $X$ as it so often happens (more on this below).
In the above mentioned context of singular points, resolution theorems - also known as desingularization theorems - are geared towards foliations in that we are “free” to eliminate non-trivial common factors between components of a vector field whenever these common factors arise from transforming a representative vector field by a birational map. To further clarify these issues, we may recall that the prototype of all “resolution theorems” for foliations is provided by Seidenberg’s theorem [34] which is valid for foliations defined on a two-dimensional ambient. More precisely, if \( \mathcal{F} \) denotes a singular holomorphic foliation defined on a neighborhood of \((0,0) \in \mathbb{C}^2\), then Seidenberg’s theorem asserts the existence of a finite sequence of one-point blow-up maps, along with transformed foliations \( \mathcal{F}_i \) \((i = 1, \ldots, n)\)

\[
\mathcal{F} = \mathcal{F}_0 \xleftarrow{\Pi_1} \mathcal{F}_1 \xleftarrow{\Pi_2} \cdots \xleftarrow{\Pi_n} \mathcal{F}_n
\]

such that the following holds:

- Each blow-up map \( \Pi_i \) \((i = 1, \ldots, n)\) is centered at a singular point of \( \mathcal{F}_{i-1} \).
- All singular points of \( \mathcal{F}_n \) are elementary, i.e. \( \mathcal{F}_n \) is locally given by a representative vector field \( X_n \) whose linear part at the singular point in question has at least one eigenvalue different from zero (cf. below).

Whereas Seidenberg’s theorem is directly concerned with foliations, it is also very effective when applied to vector fields defined on complex surfaces. The general principle to use Seidenberg theorem to study vector fields - as opposed to foliations - consists of applying Seidenberg theorem to the associated foliation while also keeping track of the divisor of zeros/poles of the transformed vector field. In line with this point of view, Seidenberg’s theorem is equally satisfying: the structure of the resolution map (the composition of the blow-ups \( \Pi_i \)) is such that the transform of holomorphic vector fields retains its holomorphic character (here the reader is reminded that the transform of a holomorphic vector field by a birational map is, in general, a meromorphic vector field). More generally, Seidenberg’s procedure allows for an immediate computation of the zero-divisor of the transformed vector field. For example, if we blow-up a vector field \( X \) having an isolated singularity at \((0,0) \in \mathbb{C}^2\) and denote by \( k \) the degree of the first non-zero homogeneous component of the Taylor series of \( X \) at \((0,0)\), then the zero-divisor of the blow-up of \( X \) coincides with the exceptional divisor and has multiplicity \( k - 1 \) (unless \( X \) is actually a multiple of the radial vector field \( x\partial/\partial x + y\partial/\partial y \) in which case the multiplicity is \( k \)).

As it will be seen in the course of the discussion, the generalization of Seidenberg’s theorem to foliations on \((\mathbb{C}^3,0)\) is a very subtle problem. A very satisfactory answer is provided in [27], [26] and it relies heavily on a previous result by Panazzolo in [28]. Slightly later, the topic was revisited from the point of view of valuations in [10]. The “final models” in the resolution theorem proved in [10] are, however, not as accurate as those in [27] (see Section 2 for further details). The present paper grew out of an attempt to use the mentioned results to obtain a sharper resolution result which would hold for the special class of holomorphic foliations which is associated with semicomplete vector fields (cf. Theorem B below). Whereas the class of foliations associated with semicomplete vector fields is rather special, it contains the underlying foliations of all complete vector fields as well as many foliations arising in the context of Mathematical Physics and the importance of these examples justifies the interest in a sharper (or “simpler”) resolution statement valid only for this class of foliations (see Section 2).

However, from the point of view mentioned above, it turned out that the resolution theorem in [10] was not really suited to our needs because the corresponding “final models” were not accurate enough. As to the resolution theorem in [27], we were unsure of the behavior of vector fields - as opposed to foliations - under their procedure. Basically, we did not know if Panazzolo’s algorithm in [28] had one specific property and we raised the issue in a preliminary
version of this paper: we are grateful to the referee for having confirmed that the algorithm in [28] does have the required property. These issues will be detailed in Section 2. In any event, when studying the papers in question, we felt it would be nice to try and complete the work of Cano-Roche-Spivakovsky [10] by deriving “final models” similar to those of [27] (cf. Theorem A below).

To state the mentioned resolution theorems, let us first recall some standard terminology. Let $F$ be a singular holomorphic foliation defined on a neighborhood of the origin in $\mathbb{C}^3$ (or more generally in $\mathbb{C}^n$). The eigenvalues of $F$ at the origin are defined as the eigenvalues of the linear part at the origin of a representative vector field $X$ for $F$. Since two representative vector fields for a same foliation $F$ must differ by an invertible multiplicative holomorphic function, it follows that the eigenvalues of $F$ are well defined only up to a multiplicative constant.

Next, a singular point $p$ for $F$ is said to be elementary if $F$ has at least one eigenvalue different from zero at $p$. Similarly, we will say that $F$ has a nilpotent singularity at $p$ if the linear part at the origin of a (local) representative vector field of $F$ is nilpotent but non-zero. Finally, if the linear part of a representative vector field at the origin is equal to zero, then $p$ is said to be a degenerate singular point of $F$.

We are now able to state Theorem A. Whereas this theorem is in most respects equivalent to the main result in [27], the corresponding proofs are very different.

**Theorem A.** Let $F$ denote a (one-dimensional) singular holomorphic foliation defined on a neighborhood of $(0, 0, 0) \in \mathbb{C}^3$. Then there exists a finite sequence of blow-up maps along with transformed foliations

$$F = F_0 \xleftarrow{\Pi_1} F_1 \xleftarrow{\Pi_2} \cdots \xleftarrow{\Pi_l} F_n$$

satisfying all of the following conditions:

1. The center of the blow-up map $\Pi_i$ is (smooth and) contained in the singular set of $F_{i-1}$, $i = 1, \ldots, n$.
2. The singularities of $F_n$ are either elementary or persistently nilpotent.
3. The number of persistently nilpotent singularities of $F_n$ is finite and each of them can be turned into elementary singular points by performing a single weighted blow-up of weight 2.

As it is implicit in the above statement, persistent nilpotent singular points is a special type of nilpotent singular point which will be detailed described in Section 4 (cf. Theorem 3). As it will be seen, they also play a special role in the resolution theorem of [27]. Namely, they appear as singularities associated with a special type of $\mathbb{Z}/2\mathbb{Z}$-orbifold which, incidentally, require a weight 2 blow-up to be turned into elementary ones. It is also worth pointing out that both statements are sharp in the sense that well known examples by Sancho and Sanz show the use of a weight 2 blow-up cannot be avoided (cf. Sections 2 and 4).

In particular, both Theorem A and the resolution theorem, Theorem 2 in [27] asserts the existence of a birational model for $F$ where all singularities of $F$ are elementary except for finitely many ones that can be turned into elementary singular points by means of a single blow-up of weight 2. In this sense, differences between these two theorems are down to the way in which these rational models are constructed. Alternatively, Theorem A can simply be regarded as a new proof of the resolution theorem in [27].

In terms of the construction of the mentioned rational models, we briefly mention that McQuillan and Panazzolo work in the category of weighted blow up, along with the corresponding orbifolds, while in Theorem A we restrict ourselves as much as possible to the use of standard (i.e. unramified) blow-ups. Once again, additional information on these strategies can be found in Section 2.
At this point, it is convenient to introduce some terminology. Throughout this paper the term blow-up will refer to standard (i.e. homogeneous) blow-ups. This applies, in particular, to the statement of Theorem A. As to blow-ups with weights (i.e. non-homogeneous or ramified blow-ups) which are inevitably also involved in the discussion, these will be explicitly referred to as weighted (or ramified) blow-ups.

Also, we will say that a (germ of) foliation $\mathcal{F}$ can be resolved if there is a sequence of blowing-ups as in (1) leading to a foliation $\mathcal{F}_n$ all of whose singularities are elementary. Similarly, a sequence of blowing-ups as in (1) will be called a resolution of $\mathcal{F}$ if all the singular points of $\mathcal{F}_n$ are elementary. Whenever sequences of weighted blow-ups leading to a foliation having only elementary singular points are considered, they may be referred to as a weighted resolution of $\mathcal{F}$. With this terminology, while every germ of foliation on $(\mathbb{C}^3, 0)$ admits a weighted resolution, as follows from [27] or Theorem A, the mentioned examples of Sancho and Sanz show that not all of them admit a resolution. The reader is referred to Section 2 for a detailed discussion on the mutual interactions involving [10], [27], and our discussion revolving around Theorem A.

We can now go back towards our initial motivation, namely to germs of foliations $\mathcal{F}$ on $(\mathbb{C}^3, 0)$ that are associated with a semicomplete vector field. Since the notion of semicomplete singularity was introduced along with its first applications to the (global) study of complex vector fields ([31]), it has been natural to ask whether all foliations in this class admit a resolution. A special instance of this problem which is of interest in the study of complex Lie group actions consists of asking whether the underlying foliation of a complete holomorphic vector field (on some complex manifold of dimension 3) can be transformed into a foliation all of whose singular points are elementary by means of a sequence of blow-ups as in (1).

To state our results concerning this special class of foliations, let us place ourselves once and for all in the context of semicomplete vector fields. First, it is convenient to recall that a singularity of a holomorphic vector field $X$ is said to be semicomplete if the integral curves of $X$ admit a maximal domain of definition in $\mathbb{C}$, cf. [31]. In particular, whenever $X$ is a complete vector field defined on a complex manifold $M$, every singularity of $X$ is automatically semicomplete. The answer to the above question is then provided by the following theorem:

**Theorem B.** Let $X$ be a semicomplete vector field defined on a neighborhood of the origin in $\mathbb{C}^3$ and denote by $\mathcal{F}$ the holomorphic foliation associated with $X$. Then one of the following holds:

1. The linear part of $X$ at the origin is nilpotent (non-zero).
2. There exists a finite sequence of blowing-up maps along with transformed foliations

$$\mathcal{F} = \mathcal{F}_0 \leftarrow \Pi_1 \mathcal{F}_1 \leftarrow \Pi_2 \mathcal{F}_2 \leftarrow \cdots \leftarrow \Pi_r \mathcal{F}_r$$

such that all of the singular points of $\mathcal{F}_r$ are elementary. Moreover, each blow-up map $\Pi_i$ is centered in the singular set of the corresponding foliation $\mathcal{F}_{i-1}$. In other words, the foliation $\mathcal{F}$ can be resolved.

Let us emphasize that item 1 in Theorem B means that the linear part of $X$ is (nilpotent) non-zero from the outset. In other words, if the foliation $\mathcal{F}$ associated with $X$ cannot be resolved, then $X$ has a non-zero nilpotent linear part and this property is “universal” in the sense that it does not depend on any sequence of blow-ups/blow-downs carried out. In particular, we can choose a “minimal model” for our manifold and the corresponding transform of $X$ will still have non-zero nilpotent linear part at the corresponding point. Moreover, from Theorem 3 about “persistent nilpotent singularities”, it is easy to obtain accurate normal forms for the vector field $X$ (cf. Definition 3).
Also, the statement of Theorem B involves the linear part of the vector field \( X \) rather than the linear part of the associated foliation \( \mathcal{F} \). This makes for a stronger statement which is better emphasized by Corollary C below:

**Corollary C.** Let \( X \) be a semicomplete vector field defined on a neighborhood of \((0, 0, 0) \in \mathbb{C}^3\) and assume that the linear part of \( X \) at the origin is equal to zero. Then item (2) of Theorem B holds.

More precisely, Theorem B asserts that foliations associated with semicomplete vector fields in dimension 3 can be resolved by a sequence of blow-ups centered in the singular set except for a very specific case in which the vector field \( X \) (and hence the foliation \( \mathcal{F} \)) has a “universal” non-zero nilpotent linear part. As mentioned, these statements have the advantage of involving the vector field and not only the underlying foliation. To clarify the meaning of this sentence, consider a holomorphic (seicomplete) vector field \( X \) having the form \( X = fY \), where \( Y \) is another holomorphic vector field and \( f \) is a holomorphic function. Whereas \( X \) and \( Y \) induce the same singular foliation \( \mathcal{F} \), an immediate consequence of Corollary C is that \( \mathcal{F} \) must be as in item (2) of Theorem B provided that \( f \) vanishes at the origin: in fact, if \( f \) and \( Y \) are as indicated, then the linear part of \( X \) vanishes at the origin at that \( \mathcal{F} \) is, indeed, singular (clearly there is nothing to be proved if \( \mathcal{F} \) is regular). In other words, if \( X = fY \) as above with \( f(0, 0, 0) = 0 \) and \( X \) semicomplete, then the foliation associated with \( X \) can certainly be resolved even if \( Y \) has a nilpotent singular point at the origin.

A few additional comments are needed to fully clarify the role of item (1) in Theorem B. First note that more accurate normal forms are available for the vector fields in question: indeed, Theorem 3 provides accurate normal forms for all persistent nilpotent singular points. In addition, not all nilpotent vector fields giving rise to persistent nilpotent singularities are semicomplete and, in this respect, the normal form provided by Theorem 3 will further be refined later on (see Section 6).

Next, taking into account the global setting of complete vector fields, it is natural to wonder if there is, indeed, complete vector fields inducing a foliation with singular points that cannot be resolved. As a consequence of Theorem B, such vector fields would definitely be pretty remarkable since they must have a (non-zero) “universal” nilpotent singular point. To confirm that these global situations do exist, however, it suffices to note that the polynomial vector field
\[
Z = x^2 \partial/\partial x + xz \partial/\partial y + (y - xz) \partial/\partial z
\]
can be extended to a complete vector field defined on a suitable open manifold (see Section 6 for detail). As will be seen, the origin in the above coordinates constitutes a nilpotent singular point of \( Z \) that cannot be resolved by means of blow-ups as in item (2) of Theorem B, albeit this nilpotent singularity can be resolved by using a blow-up centered at the (invariant) \( x \)-axis.

Finally, the question raised above about the existence of singularities as in item (1) of Theorem B in global settings can also be asked in the far more restrictive case of holomorphic vector field defined on compact manifolds of dimension 3. Owing to the compactness of the manifold, every such vector field is automatically complete. In this setting, the methods used in the proof of Theorem B easily yield:

**Corollary D.** Let \( \mathcal{F} \) be the foliation associated with a vector field \( X \) globally defined on some compact manifold \( M \) of dimension 3. Then every singular point of \( \mathcal{F} \) can be resolved.

Let us close this introduction with a couple of remarks inspired by some questions asked to us by A. Glutsyuk. Essentially his questions concern resolution strategies with minimal number of (weighted) blow ups which can also be seen as an analogue of some questions previously considered in the context of Hironaka’s theorem. In this respect, it is clear that being able to work with weighted blow ups, as opposed to standard ones, increases the chances of reducing
the number of blow ups to resolve a given foliation. Indeed, it is easy to produce examples of this phenomenon already in dimension 2 and in the context of Seidenberg’s theorem. Hence, there is no chance that the strategy used in the proof of Theorem A will in general minimize the number of blow ups required to resolve a given foliation. However, we ignore if Panazzolo’s algorithm [28] has minimizing properties in the preceding sense.

A similar question directly motivated by the fact that in dimension 2 standard blow ups suffice to resolve any foliation, consists of trying to minimize the number of weighted blow ups needed to obtain the resolution. In this case, and at least for generic foliations, Theorem A seems to provide a satisfactory answer. Let us try to sketch an argument in this direction. As it follows from Theorem 3, persistent nilpotent singular points are naturally associated with certain formal separatrices (i.e. formal invariant curves) having some special properties. Their “position” in the exceptional divisor obtained after finitely many blow ups is thus determined by the corresponding formal separatrices. In particular, it is possible to talk about these singularities being in “general position” for a given germ in an intrinsic way, i.e. independently of the use of any sequence of (standard) blow ups. At least when these singularities are in “general position” for a foliation $F$, then Theorem A should minimize the number of weighted blow ups needed to turn $F$ into a foliation all of whose singular points are elementary. Indeed, each such singularity requires at least one weighted blow up to be turned into elementary singular points and each such blow up can non-trivially affect only one of these singularities thanks to the “general position” assumption. Thus the number of weighted blow ups needed cannot be smaller than the number of persistent nilpotent singularities and the later is matched by the procedure in Theorem A. We ignore, however, if the “general position assumption” is really needed for this statement. Note that if there is a foliation $F$ that can be resolved by using less weighted blow ups than those prescribed in Theorem A, then $F$ should conceal at least two persistent nilpotent singularities so “close” to each other that they can both be turned into elementary singular points by means of a same weighted blow up.

Finally, we point out that a comprehensive discussion of the interactions between the above results and the results in [10], [27] is provided in the next section along with a brief description of our methods and of the structure of this paper.

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2. Brief review of the literature

As already pointed out, the prototype of every resolution (or desingularization) theorem for (one-dimensional) holomorphic foliations is Seidenberg’s theorem which holds for foliations defined on complex surfaces. It is then natural to begin this section by taking a closer look at Seidenberg theorem for foliations on $(\mathbb{C}^2,0)$, see [34], [2], or [20]. Let $F$ denote a singular holomorphic foliation defined on a neighborhood of $(0,0) \in \mathbb{C}^2$. Since the foliation is defined on
a complex two-dimensional manifold, all blow-ups are necessarily centered at points. Seidenberg theorem is a simple algorithm to transform $\mathcal{F}$ into a foliation all of whose singular points are elementary. The procedure in Seidenberg's algorithm can be summarized as follows. Consider $\mathcal{F}$ as above and its (necessarily isolated) singular point at $(0,0) \in \mathbb{C}^2$. If $(0,0)$ is an elementary singular point for $\mathcal{F}$, then there is nothing to be done and the procedure is trivial. Otherwise, we blow-up $\mathcal{F}$ at $(0,0)$ to obtain a new foliation $\tilde{\mathcal{F}}_1$ whose singular points are denoted by $p_{(1),1}, \ldots, p_{(1),k}$. If all the singular points $p_{(1),1}, \ldots, p_{(1),k}$ are elementary for $\tilde{\mathcal{F}}_1$ then the procedure ends. Otherwise, we carry on by blowing up each non-elementary singular point of $\tilde{\mathcal{F}}_1$. Denote by $\tilde{\mathcal{F}}_2$ the foliation arising from these blow ups and let $p_{(2),1}, \ldots, p_{(2),k}$ be its singular points. The procedure stops at $\tilde{\mathcal{F}}_2$, if all the singular points $p_{(2),1}, \ldots, p_{(2),k}$ are elementary. If not, the procedure is inductively continued. Seidenberg’s theorem claims that this procedure is finite. In other words, we eventually obtain a foliation all of whose singular points are elementary.

The seminal paper [25] by Mattei and Moussu was probably the first work to fully realize how much of an effective tool Seidenberg’s theorem is for the study of foliations on complex surfaces. Indeed, in the course of their paper, a systematic method to study these singular points was introduced. This method then led to several remarkable results (see, for example, [5] and [6]). The success of these and other works, somehow made “popular” the slightly abridged version of Seidenberg’s theorem claiming that for every foliation $\mathcal{F}$ on $(\mathbb{C}^2,0)$, there exists a birational model where the transformed foliation has only elementary singular points. We call this an abridged form of Seidenberg’s theorem because it focuses on the existence of the birational model without keeping track of the way in which this model is constructed.

It turns out, however, that besides foliations, there are many problems where the object of primary interest is a holomorphic vector field. Examples of these problems include:

(a) The classification of semicomplete (germs of) vector fields in dimension 2 (see for example [14] or [17] and its reference list; see also [12] for a nice application of the classification).

(b) The study of parabolic curves and/or Fatou petals for diffeomorphisms of $(\mathbb{C}^2,0)$ tangent to the identity, (see at various levels [15], [19], [1], [4]).

It is an important feature of Seidenberg’s theorem that it is just as effective when applied to vector fields, but in doing so, it is convenient to take into account the full statement of this theorem, rather than its “abridged version”. Indeed, the transform of a holomorphic vector field by an arbitrary birational map is, in general, a meromorphic vector field. This problem, however, can be avoided as explained below.

In fact, it is easy to check that - in full generality - the transform of a holomorphic vector field by a (standard) blow-up remains holomorphic provided that the center of the blow up is invariant by the vector field in question. In view of this remark, the method to apply Seidenberg’s theorem to vector fields can be summarized in two steps as follows. First apply this theorem to the foliation associated with the vector field so as to turn this foliation into another foliation having only elementary singular points. The corresponding vector field $\tilde{X}$ then takes on the local form $X = fY$, where $Y$ is a holomorphic vector field with linear part having at least one eigenvalue different from zero at the singular point and where $f$ is a holomorphic function whose divisor of zeros can easily be computed. In the mentioned (local) nature of $f$, it is encoded the accurate construction provided by Seidenberg’s theorem, namely: every blow-up used in his construction is centered at a singular point of the underlying foliation. In particular, the center of every blow up is invariant by the vector field in question. There follows that the transformed vector field retains its holomorphic character and, whereas in general non-empty
even if we start with a vector field having isolated singular points, the divisor zeros of the final vector field can easily be computed.

Note also that many questions revolving around items (a) and (b) above are actively being studied in dimension 3. In this respect, it reasonable to expect that Theorems B may play a role in facilitating investigations related to item (a). As to item (b), the same can be expected either from McQuillan-Panazzolo resolution theorem in [27], see Theorem 2 or from our Theorem A.

Next, a few additional words about the interest of semicomplete vector fields seem appropriate. The reader is referred to Section 5 for the definition of semicomplete vector fields along with some of their basic properties. For the time being, it is enough to point out the following:

- The germ defined by a complete vector field at any of its singular points automatically belongs to the class of semicomplete vector fields. Thus, Theorem B covers all complete vector fields (on a 3-dimensional ambient) and, in particular, all holomorphic vector fields defined on 3-dimensional compact complex manifolds. Therefore Theorem B is particularly suited to the study of complex Lie groups actions on complex manifolds.
- On a different direction, and in connection with Nevanlinna theory, there is significant activity in Complex Analysis revolving around differential equations possessing meromorphic solutions (see for example [13] for further information and references). It turns out that every equation, or system of equations, possessing meromorphic solutions is automatically semicomplete.
- There is a huge amount of literature in Mathematical Physics devoted to special equations (or systems of equations) possessing the so-called Painlevé property. (see for example [11]). The notion of semicomplete being a sort of “relative” of the notion of Painlevé property, many examples of systems of equations that are naturally semicomplete can be found in the context of Mathematical Physics and/or of Special equations. These include the Painlevé equations P-I, P-II, P-IV as well as the “modified” P-III and P-V. Similarly many Chazy equations are naturally semicomplete [16] and the same remark applies to Garnier’s systems. Also the most interesting examples of Halphen vector fields in the sense of [15], and in particular the Halphen vector field appearing in [3] and the vector field associated with Ramanujan functions $P$, $Q$, and $R$ (the Eisenstein series of weight 2, 4, and 6, cf. [24]) are all semicomplete as well.

Having reviewed Seidenberg theorem in detail and also recalled the interest of semicomplete vector fields, we can now move on to reviewing resolution theorems for singularities of foliations in dimension 3. The discussion will also help us to place our Theorems A and B in proper perspective with respect to previous works.

2.1. Quick chronological review of previous results. At a very basic level, any attempt of generalizing Seidenberg’s theorem to dimension 3 involves deciding whether we are interested in foliations of dimension one or of codimension one (i.e. of dimension two). Whereas for codimension one foliations in 3-dimensional manifolds there is a decisive answer that can hardly be improved on (see [8]), the story involving foliations of dimension one is longer and more elusive. Unless otherwise stated, throughout the sequel the term “foliation” always means a singular holomorphic foliation of dimension one.

Resolution results for foliations on $(\mathbb{C}^3, 0)$ started with [27] where the author proves his formal local resolution theorem. Namely, he shows the existence of a winning strategy for the (formal) Hironaka game. Among other things, in this work it appears for the first time a phenomenon involving singularities possessing a certain formal separatriz (a formal curve invariant by the foliation) which posed some serious difficulty to be resolved by means of standard blow ups.
Here, we remind the reader that in what follows the term blow up always means standard blow ups - as opposed to weighted blow ups.

Building on the work of Cano, Sancho and Sanz were able to work out the first examples of foliations that cannot be resolved by means of blow-ups with invariant centers. Their examples consisted of nilpotent singular points and are briefly recalled in Section 4, see [28] and [9] for detailed accounts.

Naturally, the examples provided by Sancho and Sanz may also appear only after a few blow-ups so that they, indeed, yield numerous examples of foliations that cannot be resolved by standard blow-ups. As an additional simple example of those, we may take the family of foliations $\mathcal{F}_{\alpha,\beta,\lambda}$ associated with the respective vector fields

$$x(y - \lambda x + (1 - \beta)xz)\frac{\partial}{\partial x} + (y^2 - \lambda xy + xz^3 - (\alpha + \beta)xyz)\frac{\partial}{\partial y} + z(y - \lambda x - \beta xz)\frac{\partial}{\partial z}.$$ All foliations in this family have a degenerate (in fact, quadratic) singularity at the origin. Thus none of them belongs to the Sancho-Sanz family since the latter have a non-zero linear part. However, by blowing-up the origin, the blown-up foliation $\tilde{\mathcal{F}}_{\alpha,\beta,\lambda}$ shows a (nilpotent) singularity in belonging to the family of examples by Sancho and Sanz: this singularity lies at the origin of coordinates $(u,v,z)$ with $x = uz$ and $y = vz$. Therefore foliations in the family $\mathcal{F}_{\alpha,\beta,\lambda}$ cannot be resolved by means of standard blow ups. It is a remarkable feature of our Theorem B that none of these constructions gives rise to a singularity associated with a semicomplete vector field. In particular, none of them is realized in the context of complete holomorphic vector fields on manifolds of dimension 3 (not necessarily compact). Theorem B therefore points out a genuinely new phenomenon in the area.

In view of Sancho Sanz examples, the following general question has quickly become popular among experts: is it true that every foliation $\mathcal{F}$ that cannot be resolved by blow ups is such that they can be transformed into a foliation exhibiting a singular point of Sancho Sanz type? Similarly, the question on whether or not these singularities appear in the context of complete vector fields has not gone unnoticed to most experts.

Clearly the resolution theorem of McQuillan-Panazzolo in [27] (cf. Theorem 2) provides an answer to the first question while our Theorem B answers the second one. The answer to the second question provided by Theorem B is essentially sharp as shown by the examples in Section 6. To the best of our knowledge, there is no previous result in the literature pointing out the differences between the resolution problem for general foliations and for the especial class of foliations covered by Theorem B.

On the other hand, as far as the above questions are concerned, all the mentioned results require the notion of Sancho Sanz singular point to be slightly generalized. This generalization appears in [27] under the form of those singularities “intrinsically attached” to orbifolds of type $\mathbb{Z}/2\mathbb{Z}$ while, in the present paper, they are called persistent nilpotent singular points (Section 4) and are characterized by Theorem 3.

Naturally Theorem A also provides an answer to the first question mentioned above. In this regard, the approach used to prove Theorem A is, indeed, such that the answer to the above question is “almost equivalent” to Theorem A itself. To explain this assertion, and also because it naturally fits the structure of our discussion, it is convenient to provide an explicit statement in the form of Theorem 1 below.

Whereas strictly speaking Theorem 1 is a particular case of Theorem A, the two statements are basically equivalent thanks to the work of O. Piltant [29]. In fact, most of our discussion on general foliations will revolve around the proof of Theorem 4. Once this theorem is established, Piltant’s (axiomatic) patching theorem allows us to derive Theorem A by repeating word-by-word a discussion already carried out in [10].
Theorem 1. Assume that $F$ cannot be resolved by a finite sequence of standard blow-ups centered at singular sets. Then there exists a sequence of one-point blow ups (centered at singular points) leading to a foliation $F'$ with a singular point $p$ around which $F$ is given by a vector field of the form

$$(y + zf(x, y, z))\frac{\partial}{\partial x} + zg(x, y, z)\frac{\partial}{\partial y} + z^n\frac{\partial}{\partial z}$$

for some $n \geq 2$ and holomorphic functions $f$ and $g$ of order at least 1 with $\partial g/\partial x(0, 0, 0) \neq 0$.

Furthermore we have:

- The resulting foliation $F'$ admits a formal separatrix at $p$ which is tangent to the $z$-axis;
- The exceptional divisor is locally contained in the plane $\{z = 0\}$.

By now, we can go back to our chronological review of the literature. After the examples found by Sancho and Sanz, the next truly major result in the area is due to D. Panazzolo [28]. In [28], Panazzolo considers singularities of real foliations in (real) dimension 3. He works in the real setting, rather than in the complex one, due to the fact that his original motivation was Hilbert’s problem on the number of limit cycles of a polynomial vector field on $\mathbb{R}^2$. He then shows that the corresponding germs of foliations can always be turned into a foliation all of whose singular points are elementary by means of a finite sequence of weighted blow ups centered at singular sets. The proof of this fact is very elaborate and ultimately relies on a construction associating to a singular foliation an array of six entries along with an order on the resulting family of (possible) arrays. Panazzolo’s theorem follows from showing that this quantity (array) always decreases strictly under a suitable weighted blow up. Panazzolo’s algorithm to choose the weighted blow up to be performed in each situation is, in turn, based on the Newton diagram of the singular point. Here we also mention that his algorithm is well adapted to transform vector fields, and not only foliations, in the sense that for the former we also need to keep track of the divisors of zeros/poles. We will return to this point later in this section.

After Panazzolo’s paper [28], McQuillan and Panazzolo extended the result to the complex case in their original preprint [26] whose published version is [27]. We refrain from providing more information on [27] here since a detailed discussion will be carried out in the next section.

A few years later, the topic of resolution of foliations on $(\mathbb{C}^3, 0)$ was revisited by Cano-Roche-Spivakovsky in [10]. We will close this paragraph with a brief discussion of the material in [10] since our proof of Theorem A builds on their approach.

From the very beginning, the general approach of resolution of singularities due to Zariski is followed in [10] and this makes their paper markedly different from [27]. Since Zariski’s point of view is followed, the notion of valuation becomes central in [10] and the resolution problem is divided in two parts. Namely, there is the local (resolution) problem which consists of “simplifying” - not necessarily all the singularities of a foliation - but only those lying in the center of a given valuation (identified with its transforms, or extensions, through blow ups). Resolution results for singularities lying in the center of a valuation are often referred to as local uniformization theorems. Once a convenient local uniformization result is obtained, the second part of the problem deals with its “globalization”. In other words, once it is proved that for every valuation $\nu$, the singularities lying in the center of $\nu$ can be simplified (in some appropriate sense), we try to conclude that, in fact, all singularities of the foliation can be simplified in the same sense.

In the present case, there is an axiomatic gluing theorem due to O. Piltant [29] providing a very satisfactory general answer to the “globalization problem”. Basically, as noted in [10], any solution to the “local problem” that is obtained in a reasonably natural way can be turned
into a global result by this technique. Owing to Piltant result, the fundamental difficulty of the resolution problem lies in its local version, namely, in obtaining a suitable local uniformization theorem.

The first main result - called Theorem 1 - in [10] asserts that singularities in the center of a valuation can always be simplified until they become log-elementary. We refer the reader to [10] for the accurate definition of log-elementary singularities since, for our purposes, it suffices to know that they are at worst quadratic in the sense that they are locally given by a representative vector field with non-zero second jet.

Theorem 1 is then turned into a global result - Theorem 2 in [10] - by resorting to Piltant’s theorem. Summarizing, Theorem 2 in [10] establishes the existence of a birational model for the initial foliation where singular points are at worst log-elementary. Unlike [27], only standard blow ups are used in the construction of the birational model in question. Nonetheless there is an evident disadvantage in the fact that log-elementary singular points are still significantly harder to be dealt with than elementary singular points.

Apart from Theorems 1 and 2, the paper [10] also contains a few more technical results making additional non-trivial steps towards understanding those singularities that cannot be resolved by means of standard blow ups. Aside some basic observations about the valuations that can pose obstacles to the local uniformization, Theorem 3 of [10] provides a sort of “weak characterization” of foliations that cannot be resolved (by standard blow ups as it was always the case by explicit mention on contrary). If \( \mathcal{F} \) is one of these foliations, Theorem 3 in [10] asserts the existence of a valuation \( \nu \) and of a formal surface \( \hat{W} \) having transverse maximal contact with \( \nu \), cf. [10] or Section 7. Note that the condition about maximal transverse contact can geometrically be interpreted by saying that the for every sequence of blow ups, the transform of \( \hat{W} \) will always pass through the center of \( \nu \).

The material from [10] sketched above will all enter in the proof of our Theorem A. However, bar the results established in [10], the remainder of the proof of Theorem A will require only elementary methods from the theory of foliations/singularities. Indeed, there is another characterization of foliations that cannot be resolved which is more accurate than Theorem 3. This characterization is the content of Proposition 4 which was communicated to us a number of years ago by F. Cano. Roughly speaking, if \( \mathcal{F} \) cannot be resolved, then \( \mathcal{F} \) must admit a formal separatrix giving rise to a sequence of infinitely near singular points (which, in turn, cannot be resolved, see Section 3 for terminology).

We believe that Proposition 4 should be attributed to F. Cano although no proof is available in the literature. For this reason, this paper includes a proof of this proposition relying on Theorem 3 of [10]. The proof given in Section 7 seems to be original in the sense that it may differ from the original argument envisaged by F. Cano. In the present paper, the proof of Proposition 4 is split in two cases according to whether or not the formal surface is invariant by the foliation in question. This, somehow, allows us to keep the discussion essentially elementary while taking some advantage of the 2-dimensional situation. With Proposition 4 in hand, the remainder of the proof of Theorem A is totally elementary with explicit computations.

2.2. On McQuillan Panazzolo [27]. In this section we shall explain in detail the desingularization result proved in [27] as explained to us by D. Panazzolo. We will also compare the construction in [27] with the one carried out in this paper. Finally, we note that in the course of this section the discussion is focused on resolution theorems for general foliations: comments on the additional ideas needed for Theorem B are deferred to Section 2.3.

Let us begin with some basic comments about weighted blow ups on a complex manifold of dimension 3 and the transform of foliations. Unlike standard blow ups that keep the smooth nature of the space, the use of weighted blow ups leads to spaces possessing orbifold-type
singular points. Thus there is a loss of regularity but since orbifold-type singular points are rather easy to handle, this is a minor issue. Up to allowing these singular points to be present, the space resulting from the (weighted) blow-up still is birationally equivalent to the initial one. In particular, foliations can be transformed without any restrictions under weighted blow ups to yield new birational models for them.

The last sentence contrasts a bit with the case of vector fields and this deserves a specific comment. Consider the (standard) blow up $\tilde{X}$ of a holomorphic vector field $X$. It is easy to check that the vector field $\tilde{X}$ retains the holomorphic character provided that the center of the blow up map is invariant by $X$. In particular, this condition is satisfied if blow ups are centered at the singular set of the underlying foliation. This statement, however, does not apply to general weighted blow ups as follows from the example below.

- **Example.** Consider the holomorphic vector field $X = F(x,y,z)\partial/\partial x + G(x,y,z)\partial/\partial y + H(x,y,z)\partial/\partial z$ where $F(x,y,z) = y$ and $G$ and $H$ are such that the $z$-axis $\{x = y = 0\}$ is contained in the singular set of $X$. Let $(x,t,z)$ be coordinates for the weighted blow-up (of weight $2$) centered at the $z$-axis in which the corresponding projection map $\Pi$ is given by $\Pi(x,t,z) = (x^2, tx, z)$. A direct inspection shows that the corresponding transform $\Pi^*X$ of $X$ is given by

$$
\Pi^*X = \frac{1}{2x}F(x^2,tx,z)\frac{\partial}{\partial x} + \left[ -\frac{t}{2x^2}F(x^2,tx,z) + \frac{1}{x}G(x^2,tx,z) \right] \frac{\partial}{\partial t} + H(x^2,tx,z)\frac{\partial}{\partial z}.
$$

Clearly $F(x^2,tx,z)/2x$ and $G(x^2,tx,z)/x$ are both holomorphic but $tF(x^2,tx,z)/2x^2$ is strictly meromorphic. Therefore $\Pi^*X$ is meromorphic with poles over the exceptional divisor.

In fact, for the above blow up, the condition for the blow up of a holomorphic vector field $X$ to retain its holomorphic nature can be explained as follows. For $\lambda \in \mathbb{C}^*$, consider the family of maps $T_\lambda : \mathbb{C}^3 \to \mathbb{C}^3$ given by $T_\lambda(x,y,z) = (\lambda^2 x, \lambda y, z)$. Next, if $T_\lambda^*X$ denotes the pull-back $T_\lambda^*X$, then the blow up of $X$ will be holomorphic if $T_\lambda^*X$ converges to a holomorphic vector field as $\lambda \to 0$. The reader will have no difficulty in working out the general case or to formulate equivalent conditions.

After this short introduction, we are ready to discuss the content of [27]. Basically, this paper consists of two parts, the first one relying heavily on Panazzolo’s previous work [28]. Recall that [28] provides a resolution of singularities (of real analytic foliations on $(\mathbb{R}^3, 0)$) by means of sequences of weighted blow ups. The first part of [27] is devoted to showing that the algorithm of [28] applies equally well in the general case of holomorphic foliations on $(\mathbb{C}^3, 0)$.

As explained above, this provides a birational model for the foliation in question on a space possessing orbifold-type singular points. Furthermore, there is a natural notion of elementary singular point for a (singular) foliation $\mathcal{F}$ defined on this space. Namely, a singular point of $\mathcal{F}$ is said to be elementary if the foliation is represented by a vector field with elementary singular points in an orbifold coordinate for the space. This result summarizes the first part of [27].

In the second part of [27], the authors consider the problem of resolving the singular points of the ambient space while keeping the singular points of the foliation elementary. Then they go on to show that this resolution can always be found except when the singular point correspond to a $\mathbb{Z}/2\mathbb{Z}$-orbifold. Therefore, at least as far as foliations are concerned, they manage to obtain a birational model for the foliation possessing only $\mathbb{Z}/2\mathbb{Z}$-orbifold singular points and where all the singular points of the foliation in question are elementary.
Since resolution theorems are also of interest in the study of singular points of vector fields, rather than foliations, it is natural to ask how the above procedure affects the divisor of zeros/poles of a vector field. This is, indeed, a point that can easily be missed in [27] since it very much hinges in a characteristic of the resolution algorithm in [28], and we thank the referee for having clarified the issue for us. It turns out that the centers of each weighted blow up used in [28], and reproduced in the first part of [27], are what is called strictly invariant with respect to the quasi-homogeneous filtration in question, see [28] for terminology. This means that the transform of holomorphic vector fields remains holomorphic. Taking all these issues together, the resolution theorem in [27] can be formulated as follows.

**Theorem 2. ([25])** Let $\mathcal{F}$ be a singular holomorphic foliation on $(\mathbb{C}^3, 0)$. There is a sequence of weighted blow-ups

$$\mathcal{F}_0 \xleftarrow{\Pi_1} \mathcal{F}_1 \xleftarrow{\Pi_2} \cdots \xleftarrow{\Pi_l} \mathcal{F}_l$$

satisfying the following conditions:

- The center of each weighted blow up is strictly invariant with respect to the quasi-homogeneous filtration in question (hence a holomorphic vector field remains holomorphic).
- The ambient space is an analytic space of dimension 3 whose singular points are $\mathbb{Z}/2\mathbb{Z}$-orbifold type and the total blow-up map $\Pi_1 \circ \cdots \circ \Pi_l$ is birational.
- The singular points of $\mathcal{F}_l$ are elementary in orbifold coordinates.

Hence the basic difference between Theorem 2 and Theorem A lies in the fact that Mcquillan-Panazzolo work in the category of weighted blow ups while we stick with standard blow ups as much as possible. In concrete applications, the choice of one statement over the other will probably go down to a matter of personal taste. For example, it is fair to claim that there is no fundamental reason to privilege standard blow ups over weighted ones so that considering weighted blow ups increases your chances of finding a resolution in a smaller number of steps. It can then be reckoned that in several applications of Seidenberg’s theorem, the number of blow ups used is irrelevant while having essentially a single type of blow up map might make it slightly easier to handle various types of index formulas.

There is an additional couple of differences between these two theorems that we believe are worth drawing the reader’s attention to. First, Panazzolo’s paper [28], and hence [27], does provide an algorithm to obtain a resolution model for a given foliation. In contrast to this, our Theorem A is essentially not algorithmic as it contains arguments based on contradictions arising from assuming the non existence of a sequence of blow ups with certain required properties. The other difference is that, on the other hand, our Theorem A answers the question of deciding how close to a foliation with elementary singular points it is possible to go while sticking with standard blow ups, which might be a natural curiosity for people with background in foliation/differential equations.

It should be clear that these differences between Theorem 2 and Theorem A are, in many senses, minor ones. So the feeling that the choice of which version to use will probably depend on the author’s taste or background seems to be somehow strengthened by them. A point, however, where everyone is likely to agree on is that a new proof of an important result, as it is undoubtedly the case here, is always welcome and helps to increase the general understanding of the problem in question.

2.3. **The structure of the paper and of proofs of Theorems A and B.** Let us close this section by detailing the structure of this article and the inter-dependence of the different sections.
We begin by recalling that a foliation $\mathcal{F}$ can be resolved (or it is resolvable) if there is a finite sequence of blow ups (centered in singular sets) leading to a foliation all of whose singular points are elementary. Recall also that the term blow-up always means a standard blow up. In particular, whenever weights are used we explicitly refer to weighted blow ups.

As already indicated, the initial motivation of this paper was to prove Theorem B. The idea to prove this theorem was based on F. Cano’s comment related to Proposition 4. The sketch of the envisaged proof was as follows. let $\mathcal{F}$ be a foliation tangent to a semicomplete vector field $X$ and assume aiming at a contradiction that $\mathcal{F}$ cannot be resolved. Owing to Proposition 4, there must exist a formal separatrix $S$ for $\mathcal{F}$. If this separatrix were convergent, we might restrict $X$ to $S$ and try to argue from some basic properties of semicomplete vector fields as in [31] provided that the restriction of $X$ to $S$ is not identically zero. Having a merely formal separatrix, however, prevents us from making sense of the restriction of $X$ to $S$ (though, as will be seen, it essentially rules out the inconvenient situation where the restriction of $X$ to $S$ vanishes identically). To remedy for the formal character of $S$, it is natural to consider results about “sectorial normalization” for $\mathcal{F}$ providing asymptotic estimates for its integral curves over conveniently chosen sectors. Naturally, for this approach to be effective, some control about the “angle” of the mentioned sectors is needed. In other words, it should be proved that the sector in question is “large enough”, in some suitable sense.

Modulo taking for granted Proposition 4, the main difficulty to make the above argument accurate clearly lies in obtaining a suitable sectorial normalization for the foliation $\mathcal{F}$. As a matter of fact, Ramis-Sibuya theorem in [30] provides a very general result on the existence of sectorial normalizations for formal separatrices. Nonetheless, at this level of generality, it is virtually impossible to estimate the “angle” where the integral curves satisfy the expected asymptotic conditions. A natural alternative was then to consider classical results due to Malmquist: these come with suitable estimates for the “angle” of the section but they require the foliation $\mathcal{F}$ to have a particularly simple form, cf. [22], [23].

The assumptions in Malmquist theorem [23] immediately led us to consider resolution procedures for $\mathcal{F}$, namely the results in [27] and [10]. The obstacle to apply [10] was evident: their “final models” were still not “simple enough” to satisfy the conditions in Malmquist theorem.

Concerning the possibility of using McQuillan-Panazzolo theorem (Theorem 2), we were with two issues. The smaller issue had to do with the singularities associated with $\mathbb{Z}/2\mathbb{Z}$-orbifolds that cannot be turned into elementary ones unless a blow up with weight 2 is performed. The characterization of these foliations presented in [27] is an invariant one while the use of Malmquist theorem and subsequent derivation of more “quantitative” information requires slightly more explicit normal forms. Of course obtaining normal forms for these singularities from the characterization provided in [27] is rather straightforward so that this was not our main concern. Also, we mention that the corresponding material is “implicitly” included in the present paper (Sections 3 and 4).

On the other hand, we were more seriously concerned about the behavior of the divisor of zeros of a vector field under the sequence of weighted blow ups provided in [27]. Basically, at that point in time, we had no confirmation of the information presented in the first item of the statement of Theorem 2. The issue, once again, stemmed from our strategy to prove Theorem B. More precisely, with the above material about elementary singularities for the underlying foliation in place, Malmquist theorem becomes effective. Namely, this theorem yields suitable asymptotic expansions whose “angle” is directly related to the order of the restriction to $S'$ (the transform of $S$) of a local representative for the foliation $\mathcal{F}'$. The nature of semicomplete vector fields, however, basically requires $X'$ to have a non-empty divisor of
zeros (and an empty divisor of poles) transverse to $S'$ for the desired contradiction to arise. This explains our initial hesitation with respect to using weighted blow ups.

What precedes added to our general feeling that it would be nice to “complete” the work of Cano-Roche-Spivakovsky [10] to obtain a resolution theorem through Zariski classical approach. Besides, Theorem 3 in [10] already provided a characterization of foliations that cannot be resolved which, albeit somewhat “coarse”, looked promising in terms of enabling us to prove Proposition 4.

The remainder of this paper will be devoted to properly implementing the above described strategy.

Section 3 is very elementary and discusses the effect of blow ups on a sequence of singular points determined by a formal separatrix along with its transforms. The basic idea is to consider the multiplicity of the foliation along the separatrix in question and study the way this multiplicity varies under sequences of blow ups. Whereas multiplicity of a foliation along a separatrix is a basic example of valuation, no general result on valuation is required in the course of the discussion which requires only basic knowledge about blow ups.

Section 4 continues the discussion in Section 3 and includes, in particular, the notion of persistent nilpotent singularity. The main result of Section 4 being precisely the characterization of persistent nilpotent singularity, namely Theorem 3. The reader will not fail to note that our “persistent nilpotent singularities” correspond to the singularities associated with $\mathbb{Z}/2\mathbb{Z}$-orbifold type singular points of [27]. Furthermore, the normal form provided by Theorem 3 is equivalent to the invariant characterization of the latter formulated in [27].

In Section 4 we also formulate Proposition 4 in elementary terms, i.e. avoiding any use of valuations. This section ends with the proof of Theorem 1 obtained by combining the general discussion in Section 4 with Theorem 3 and with Proposition 4 (which is taken for granted at this moment). All this material is elementary and requires only some familiarity with (singular) holomorphic foliations at the level of the first chapters of [20]. Yet, we note that readers familiar with valuations and with Piltant’s theorem in [29] will probably be able to easily derive Theorem A from Theorem 1.

Section 5 is devoted to the proof of Theorem B. This proof has basically four ingredients. Besides Theorems 1 and 3, it also requires Malmquist theorem on asymptotic expansion of solutions of certain systems of equations [23] and, of course, some background revolving around the notion of semicomplete singularity. Bar a more specific result on semicomplete vector fields detailed in [17], the background material on semicomplete vector fields is covered in [31] and recalled in the beginning of Section 5 in an effort to make the discussion kind of self-contained. As follows from the preceding, the background material required for this section is significantly larger than for the previous ones, though the discussion is still mostly accessible to readers familiar with the contents of [20] and/or [2].

Section 5 is followed by Section 6 which contains examples illustrating the sharp nature of most of our results as well as some additional constraints on semicomplete persistent nilpotent singularities arising from the holonomy of (formal) separatrices. In particular, in this section we show how to extend the vector field

$$Z = x^2 \partial / \partial x + xz \partial / \partial y + (y - xz) \partial / \partial z$$

in a complete vector field on a suitable open complex manifold of dimension 3. We also provide some explicit examples of persistent nilpotent singular points which cannot lead to any of the Sancho-Sanz vector fields by means of sequences of blow ups (i.e. these examples cannot be “further simplified”).

Finally almost all of Section 7 is devoted to the proof of Proposition 4. This section requires familiarity with resolution techniques based on valuations, including Zariski approach to the
resolution problem. In more concrete and accurate terms, it requires a good knowledge of the material contained in [10]. The proof of Proposition [3] then completes the proof of our Theorem [1]. In turn, at this point, Theorem A becomes little more than a direct blend of Theorem [1] with Piltant’s patching theorem [29].

3. THE MULTIPLICITY OF A FOLIATION ALONG A SEPARATRIZ

For background in the material discussed below, the reader is referred to [2] and to [20]. Consider a singular holomorphic foliation $F$ of dimension 1 defined on a neighborhood of the origin in $\mathbb{C}^3$. By definition, $F$ is given by the local orbits of a holomorphic vector field $X$ whose singular set $\text{Sing}(X)$ has codimension at least 2. The vector field $X$ is said to be a vector field representing $F$. Albeit the representative vector field $X$ is not unique, two of them differ by a multiplicative locally invertible function. The singular set $\text{Sing}(F)$ of a foliation $F$ is defined as the singular set of a representative vector field $X$ so that it has codimension greater than or equal to 2.

Conversely, with every (non-identically zero) germ of holomorphic vector field on $(\mathbb{C}^3, 0)$, it is associated a germ of singular holomorphic foliation $F$. Up to eliminating non-trivial common factors of the components of $X$, we can replace $X$ with another holomorphic vector field $Y$ whose singular set has codimension at least 2. The foliation $F$ is then given by the local orbits of $Y$. A global definition of singular (one-dimensional) holomorphic foliations can be formulated as follows.

**Definition 1.** Let $M$ be a complex manifold. A singular (1-dimensional) holomorphic foliation $F$ on $M$ consists of a covering $\{(U_i, \varphi_i)\}$ of $M$ by coordinate charts together with a collection of holomorphic vector fields $Z_i$ satisfying the following conditions:

- For every $i$, $Z_i$ is a holomorphic vector field having singular set of codimension at least 2 which is defined on $\varphi_i(U_i) \subset \mathbb{C}^n$.
- Whenever $U_i \cap U_j \neq \emptyset$, we have $\varphi_i^*Z_i = g_{ij}\varphi_j^*Z_j$ for some nowhere vanishing holomorphic function $g_{ij}$ defined on $U_i \cap U_j$.

Throughout this section and the next one, all blow ups of foliations (and of vector fields) are standard. Moreover the centers of the blow ups are always contained in the singular set of the foliation in question (associated foliation in the case of vector fields).

Consider now a holomorphic foliation $F$ defined on a complex manifold $M$ of dimension 3 and let $p \in M$ be a singular point of $F$. A separatriz (or analytic separatriz) for $F$ at $p$ is an irreducible analytic curve invariant by $F$, passing through $p$, and not contained in the singular set $\text{Sing}(F)$ of $F$. Along similar lines, a formal separatriz for $F$ at $p$ is a formal irreducible curve $S$ invariant by $F$ and centered at $p$. In other words, in local coordinates $(x, y, z)$ around $p$ where $F$ is represented by the vector field $X = F\partial/\partial x + G\partial/\partial y + H\partial/\partial z$, the formal separatriz $S$ is given by a triplet of formal series $t \mapsto \varphi(t) = (\varphi_1(t), \varphi_2(t), \varphi_3(t))$ satisfying the following (formal) equations

$$\label{3} \varphi'_1(t)(G \circ \varphi)(t) = \varphi'_2(t)(F \circ \varphi)(t) \quad \text{and} \quad \varphi'_2(t)(H \circ \varphi)(t) = \varphi'_3(t)(G \circ \varphi)(t)$$

where:

1. $(F \circ \varphi)(t)$ (resp. $(G \circ \varphi)(t), \ H \circ \varphi(t)$) stands for the formal series obtained by composing the Taylor series of $F$ (resp. $G, H$) at the origin with the formal series of $\varphi$ as indicated.
2. In the preceding it is understood that at least one of the formal series $(F \circ \varphi)(t)$, $(G \circ \varphi)(t)$, and $(H \circ \varphi)(t)$ is not identically zero.

Note that Puiseux theorem allows us to represent an analytic separatriz by a map of the form $t \mapsto \varphi(t) = (\varphi_1(t), \varphi_2(t), \varphi_3(t))$ where the formal series $\varphi_i$ ($i = 1, 2, 3$) are actually convergent.
Thus an analytic separatrix can be viewed as a particular case of a formal separatrix and for this reason our terminology will be such that whenever we refer to a formal separatrix of $\mathcal{F}$ the possibility of having an actual analytic separatrix will not be excluded. If we need to emphasize that a formal separatrix is not analytic, then we will say that the separatrix in question is strictly formal. Finally, we also note that the second condition above is automatically satisfied whenever $\phi(t)$ is a strictly formal curve satisfying Equation (3).

Consider again a singular point $p \in M$ of a holomorphic foliation $\mathcal{F}$. Choose local coordinates $(x, y, z)$ around $p$ and assume that $\mathcal{F}$ has a formal separatrix $S$ at $p$ which is given in the coordinates $(x, y, z)$ by the formal series $\phi(t) = (\phi_1(t), \phi_2(t), \phi_3(t))$. Consider now a local holomorphic vector field $X$ defined around $p$ and tangent to $\mathcal{F}$ but not necessarily representing $\mathcal{F}$. Note that the (formal) pull-back of the restriction of $X$ to $S$ by $\phi$ may be considered since $S$ is a formal separatrix of $\mathcal{F}$. This pull-back is a formal vector field in dimension 1 given by

$$
\phi^*(X|_S) = g(T) \frac{\partial}{\partial T}
$$

where $g$ satisfies

$$(X \circ \phi)(T) = g(T)\phi'(T)
$$

as formal series.

We recall the classical notion of multiplicity of a foliation along a separatrix which is also well known as a basic example of valuation.

**Definition 2.** The multiplicity of $X$ along $S$ is the order of the formal series $g$ at $0 \in \mathbb{C}$

$$\text{mult} (X, S) = \text{ord}(g, 0)
$$

In other words, setting $g(T) = \sum_{k \geq 1} g_k T^k$, mult$(\mathcal{F}, S)$ is the smallest positive integer $k \in \mathbb{N}^*$ for which $g_k \neq 0$. This multiplicity equals zero if and only if the series associated with $g(T)$ vanishes identically.

In turn, the multiplicity of $\mathcal{F}$ along $S$, mult$(\mathcal{F}, S)$, is defined as the multiplicity along $S$ of a vector field $X$ representing $\mathcal{F}$ around $p$. Since $S$ as a separatrix for $\mathcal{F}$ is not contained in the singular set of $\mathcal{F}$, the multiplicity of $\mathcal{F}$ along $S$ is never equal to zero.

It is immediate to check that the notions above are well defined in the sense that they depend neither on the choice of coordinates nor on the choice of the representative vector field $X$.

We begin with a simple albeit important lemma. To fix notation, we will say that $S$ is a (formal) separatrix for a vector field $X$ if $S$ is a (formal) separatrix for the foliation $\mathcal{F}$ associated with $X$. We also recall that the centers of all blow-ups considered in what follows are contained in the singular sets of the corresponding foliations.

**Lemma 1.** The multiplicity of a vector field along a formal separatrix is invariant by blow-ups (with centers contained in the singular set of the foliation in question).

**Proof.** The statement means that the multiplicity of a vector field along a formal separatrix is invariant by blow-ups regardless of whether they are centered at a singular point or at a (locally) smooth analytic curve contained in the singular set of $\mathcal{F}$. We will prove the mentioned invariance in the case of blow-ups centered at a point. The case of blow-ups centered at analytic curves is analogous and thus left to the reader.

Let $X$ be a holomorphic vector field defined on a neighborhood of the origin of $\mathbb{C}^3$ and admitting a formal separatrix $S$. Let $\pi : \tilde{\mathbb{C}}^3 \rightarrow \mathbb{C}^3$ denote the blow-up map centered at the origin and denote by $S$ the transform of $S$ by $\pi$.

Fixed standard $(x, y, z)$-coordinates on $\mathbb{C}^3$, the formal separatrix $S$ is given by a formal map of the form $T \mapsto \phi(T) = (\phi_1(T), \phi_2(T), \phi_3(T))$. Without loss of generality, we assume that $\phi$
The lemma is proved.

Indeed, let \( X \) be a foliation. Indeed, let 

\[
X = F(x, y, z) \frac{\partial}{\partial x} + G(x, y, z) \frac{\partial}{\partial y} + H(x, y, z) \frac{\partial}{\partial z}.
\]

Since \( S \) is a formal solution of the differential equation associated with \( X \), there follows that 

\[
\varphi'(T) \quad \text{and} \quad X \circ \varphi
\]

satisfy Equation (3). Comparing the last component of \( \varphi'(T) \) and of \((X \circ \varphi)(T)\) we conclude that the multiplicative function \( g \) appearing in Equation (1) is of order equal to 

\[
\text{ord}(H \circ \varphi, 0) - p + 1.
\]

Thus 

\[
\text{mult}(X, S) = \text{ord}(H \circ \varphi, 0) - p + 1.
\]

Let us now compute the order of \( \tilde{X} \) along \( \tilde{S} \). For this we consider affine coordinates \( (u, v, z) \) where the blow-up map is given by \( \pi(u, v, z) = (uz, vz, z) = (x, y, z) \). The transform of \( X \) then becomes \( \tilde{X} = (1/z)Z \) where \( Z \) is the vector field given by

\[
Z = (F(uz, vz, z) - uH(uz, vz, z)) \frac{\partial}{\partial x} + G(uz, vz, z) \frac{\partial}{\partial y} + zH(uz, vz, z) \frac{\partial}{\partial z}.
\]

In turn, the transform of \( S \) by \( \pi \) is by definition the formal curve given by \( \psi(T) = \pi^* \varphi(T) = (T^{m-p} + h.o.t., T^{n-p} + h.o.t., T^p) \). Since \( \tilde{S} \) is a formal solution of the differential equation associated to \( \tilde{X} \), there follows again that \( \psi'(T) \) and \((\tilde{X} \circ \psi)(T)\) satisfy Equation (3). By comparing their last components, we conclude that

\[
\text{mult}(\tilde{X}, \tilde{S}) = \text{ord}(H((T^{m-c} + h.o.t)T^c, (T^{n-c} + h.o.t)T^c, T^p), 0) - p + 1
\]

\[
= \text{ord}(H(T^{m} + h.o.t, T^{n} + h.o.t, T^p), 0) - p + 1
\]

\[
= \text{ord}(H \circ \varphi, 0)
\]

\[
= \text{mult}(X, S).
\]

The lemma is proved. \( \square \)

Consider again a foliation \( F \) defined on a neighborhood of the origin of \( \mathbb{C}^3 \) along with a formal separatrix \( S \). Whereas Lemma 1 asserts that the multiplicity of a vector field along a formal separatrix is invariant by blow-ups, the analogous statement does not necessarily hold for a foliation. Indeed, let \( X \) be a vector field representing \( F \) around \((0, 0, 0) \in \mathbb{C}^3 \) so that the zero-set of \( X \) has codimension at least 2. Finally let \( \tilde{X} \) denote the pull-back of \( X \) by the blowing-map centered at the origin. If \( X \) has order at least 2 at the origin, then the singular set of \( \tilde{X} \) has codimension 1 since \( \tilde{X} \) vanishes identically on the corresponding exceptional divisor. More precisely, in the affine coordinates \( (u, v, z) \) where \( x = uz \) and \( y = vz \), we have \( \tilde{X} = z^\alpha Z \) for a certain (holomorphic) vector field \( Z \) having singular set of codimension at least 2 and a certain integer \( \alpha \geq 1 \). In fact, if \( k \) stands for the order of \( X \) at \( 0 \in \mathbb{C}^3 \), then we have \( \alpha = k \) or \( \alpha = k - 1 \) according to whether or not the origin is a dicritical singular point. Here we remind the reader that a singular point is said to be dicritical if the exceptional divisor, given by \( \{z = 0\} \) in the above affine coordinates, is not invariant by \( F \). Next, note that the multiplicity of \( X \) along \( S \) coincides with the multiplicity of \( \tilde{X} \) along \( \tilde{S} \) (Lemma 1). However, the multiplicity of \( \tilde{F} \) along \( \tilde{S} \) is not the multiplicity of \( \tilde{X} \) along \( \tilde{S} \) but rather the multiplicity of \( Z \) along \( \tilde{S} \). More precisely, we have

\[
\mult(\tilde{F}, \tilde{S}) = \mult(Z, \tilde{S})
\]

\[
= \mult(\tilde{X}, \tilde{S}) - \text{ord}(z^\alpha \circ \psi, 0)
\]

\[
= \mult(X, S) - \text{ord}(z^\alpha \circ \psi, 0),
\]
where $\psi$ stands for the triplet of formal series associated with $\tilde{S}$. Since the zero-set of $X$ has codimension at least 2, there also follows that mult $(X,S) = \text{mult} (F,S)$. Summarizing, we have proved the following:

**Proposition 1.** Let $F$ be a holomorphic foliation on $(\mathbb{C}^3,0)$ admitting a formal separatrix $S$. If $F$ has order at least 2 at the origin, then

$$\text{mult} (\tilde{F},\tilde{S}) < \text{mult}(F,S),$$

where $\tilde{F}$ (resp. $\tilde{S}$) stands for the transform of $F$ (resp. $S$) by the one-point blow-up centered at the origin.

In order to state the analogue of Proposition 1 for blow-ups centered at smooth (irreducible) curves contained in $\text{Sing}(F)$, a notion of order for $F$ with respect to the curves in question is needed. A suitable notion can be introduced as follows.

Recall first that the order of the foliation $F$ at the origin is defined as the degree of the first non-zero homogeneous component of a vector field $X$ representing $F$. The mentioned degree, as well as all of the corresponding non-zero homogeneous component, may be recovered through the family of homotheties $\Gamma^\lambda : (x,y,z) \mapsto (\lambda x, \lambda y, \lambda z)$. More precisely, the degree is simply the unique positive integer $d \in \mathbb{N}$ for which

$$\lim_{\lambda \to 0} \frac{1}{\lambda^{d-1}} \Gamma^\lambda X$$

is a non-trivial vector field. Furthermore, the non-trivial vector field obtained as this limit is exactly the first non-zero homogeneous component of $X$. We shall adapt this construction to define the order of $F$ over a curve.

Let then $C$ be a smooth curve contained in $\text{Sing}(F)$. Our purpose is to define the order of $F$ with respect to $C$. Clearly there are local coordinates $(x,y,z)$ in which the curve in question coincides with the $z$-axis, i.e. it is given by $\{x = y = 0\}$ (as usual we only perform blow-ups centered at smooth curves; naturally this is not a very restrictive condition since every curve can be turned into smooth by the standard resolution procedure). The blow-up centered at $\{x = y = 0\}$ is equipped with affine coordinates $(x,t,z)$ and $(u,y,z)$ where the corresponding blow-up map is given by $\pi_z(x,t,z) = (x,tx,z)$ (resp. $\pi_z(u,y,z) = (uy,y,z)$). Consider now the family of automorphisms given by

$$\Lambda^\lambda : (x,y,z) \mapsto (\lambda x, \lambda y, z).$$

The order of $F$ with respect to $C$ (or the order of $F$ over $C$) is defined as the unique integer $d \in \mathbb{N}$ for which

$$\lim_{\lambda \to 0} \frac{1}{\lambda^{d-1}} \Lambda^\lambda X$$

yields a non-trivial vector field. Note that this integer $d$ may be seen as the degree of $X$ with respect to the variables $x$, $y$. In fact, assume that in coordinates $(x,y,z)$ the vector field $X$ is given by $X = X_1(x,y,z) \partial/\partial x + X_2(x,y,z) \partial/\partial y + X_3(x,y,z) \partial/\partial z$. The pull-back of $X$ by $\Lambda^\lambda$ becomes

$$\Lambda^\lambda X = \frac{1}{\lambda} \left[ X_1(\lambda x, \lambda y, z) \frac{\partial}{\partial x} + X_2(\lambda x, \lambda y, z) \frac{\partial}{\partial y} + X_3(\lambda x, \lambda y, z) \frac{\partial}{\partial z} \right].$$

Denote by $k$ (resp. $l$) the maximal power of $\lambda$ that divides $X_1(\lambda x, \lambda y, z) \partial/\partial x$ ($\lambda x, \lambda y, \lambda z) \partial/\partial y$ (resp. $X_3(\lambda x, \lambda y, z) \partial/\partial z$). The order $d$ defined above is simply the minimum between $k$ and $l+1$.

The analogue of Proposition 1 for blow-ups centered at smooth (irreducible) curves can now be stated as follows.
Proposition 2. Let $\mathcal{F}$ be a holomorphic foliation on $(\mathbb{C}^3,0)$ admitting a formal separatriz $S$. Let $\tilde{\mathcal{F}}$ (resp. $\tilde{S}$) stands for the strict transform of $\mathcal{F}$ (resp. $S$) by the blow-up centered at a smooth (irreducible) curve contained in $\text{Sing}(\mathcal{F})$. If $\mathcal{F}$ has order at least 2 with respect to the blow-up center, then
\[ \text{mult} (\tilde{\mathcal{F}}, \tilde{S}) < \text{mult}(\mathcal{F}, S). \]

\[ \square\]

Let us close this section with a first application of Proposition 1 to the reduction of singular points (a slightly more general discussion involving Proposition 2 appears in Section 3). Let $\mathcal{F}$ be a holomorphic foliation defined on a neighborhood of the origin of $\mathbb{C}^3$ and let $X$ be a holomorphic vector field representing $\mathcal{F}$. Recall that a singular point $p$ of $\mathcal{F}$ is said to be elementary if the linear part of $X$ at $p$, $DX(p)$, has at least one eigenvalue different from zero. In the sequel, whenever there is no risk of misunderstanding, we will say that a singular point $p$ is nilpotent if the linear part of $X$ at $p$ is nilpotent and non-zero. Along similar lines, the expression degenerate singularity will be used to refer to singularities where the linear part of $X$ is actually equal to zero.

Consider now a singular foliation $\mathcal{F}_0$ along with a formally smooth separatriz $S_0$ at the origin $((0,0,0) \simeq p_0)$. Consider the blow-up $\mathcal{F}_1$ of $\mathcal{F}_0$ centered at the origin. The transform $S_1$ of $S_0$ selects a singular point $p_1$ of $\mathcal{F}_1$ in the exceptional divisor $\Pi_1^{-1}(0,0,0)$. In fact, if the point $p_1 \in \Pi_1^{-1}(0,0,0)$ selected by $S_1$ were regular for $\mathcal{F}_1$, then $\Pi_1^{-1}(0,0,0)$ would not be invariant by $\mathcal{F}_1$ and the formal separatris $S_1$ (and hence $S$) would actually be analytic and $\mathcal{F}_1$ would be regular on a neighborhood of $p_1$: this situation is excluded in what follows.

Next let $\mathcal{F}_2$ be the blow-up of $\mathcal{F}_1$ at $p_1$. Again the transform $S_2$ of $S_1$ will select a singular point $p_2 \in \Pi_2^{-1}(p_1)$ of $\mathcal{F}_2$. The procedure is then continued by induction so as to produce a (infinite) sequence of foliations $\mathcal{F}_n$

\[(7)\]

\[\mathcal{F}_0 \xleftarrow{\Pi_1} \mathcal{F}_1 \xleftarrow{\Pi_2} \ldots \xleftarrow{\Pi_n} \mathcal{F}_n \xleftarrow{\Pi_{n+1}} \ldots\]

along with singular points $p_n$ and formal separatris $S_n$.\[\]

Lemma 2. Consider a sequence of foliations $\mathcal{F}_n$ as in (7) along with a sequence of formal separatris $S_n$ and singular points $p_n$. Assume that for every $n \in \mathbb{N}$, $p_n$ is not an elementary singular point of $\mathcal{F}_n$. Then there exists $n_0 \in \mathbb{N}$ such that $p_n$ is nilpotent (non-zero) singularity of $\mathcal{F}_n$ for every $n \geq n_0$.

Proof. The statement follows from Proposition 1. Indeed, by assumption, $p_n$ is not an elementary singular point of $\mathcal{F}_n$ (for every $n \in \mathbb{N}$). Assume, in addition, that $\mathcal{F}_1$ is not nilpotent at $p_1$. This means that the order of $\mathcal{F}_1$ at $p_1$ is at least 2 so that the multiplicity of $\mathcal{F}_2$ along $S_2$ is strictly smaller than the multiplicity of $\mathcal{F}_1$ along $S_1$. If $\mathcal{F}_2$ is again non-nilpotent at $p_2$, then the order of $\mathcal{F}_3$ at $p_2$ is again at least 2. There follows that the multiplicity of $\mathcal{F}_3$ along $S_3$ is strictly smaller than the multiplicity of $\mathcal{F}_2$ along $S_2$. When the procedure is continued, the multiplicity of $\mathcal{F}_{n+1}$ along $S_{n+1}$ will be strictly smaller than the multiplicity of $\mathcal{F}_n$ along $S_n$ whenever $p_n$ is not a nilpotent singularity of $\mathcal{F}_n$. Hence we obtain a decreasing - though not necessarily strictly decreasing - sequence of non-negative integers. This sequence must eventually become constant. If $n_0$ is the index for which the sequence is constant for $n \geq n_0$, then Proposition 1 ensures that $\mathcal{F}_n$ has order 1 at $p_n$ for every $n \geq n_0$. Since by assumption $p_n$ is not an elementary singularity of $\mathcal{F}_n$, we conclude that $p_n$ must be a nilpotent singularity of $\mathcal{F}_n$ for $n \geq n_0$. The lemma is proved. \[\square\]
4. On persistent nilpotent singularities

Throughout this section by nilpotent singularity it is always meant a singular foliation whose linear part is nilpotent and different from zero.

Our purpose is to discuss nilpotent singular points that are persistent under blow-up transformations and this will lead to the two main results of the section, namely Theorem 3 and Theorem 4. As mentioned, Theorem 3 generalizes, in a relatively straightforward way, the celebrated examples of vector fields obtained by Sancho and Sanz. Also, they are related to the $\mathbb{Z}/2\mathbb{Z}$-orbifold singular points discussed in the last section of [27] (see Section 2).

First let us make it clear what is meant by persistent nilpotent singularity. In the sequel, the centers of the blow-ups maps are always contained in the singular set of the foliation. Moreover, they are either a single point or a smooth analytic curve. The reader is also reminded that all blow-ups are assumed to be standard.

Let $F_0$ denote a singular foliation along with an irreducible formal separatrix $S_0$ at a chosen singular point $p_0$. Consider a sequence of blow-ups and transformed foliations which is obtained as follows. First, we choose a center $C_0$ with $p_0 \in C_0$ which is contained in the singular set of $F_0$. Then we blow-up $F_0$ with center $C_0$ and let $F_1$ denote the blown-up foliation. The transform $S_1$ of $S_0$ selects a point $p_1$ in the exceptional divisor $\Pi_1^{-1}(C_0)$. In the case where $p_1$ is regular for $F_1$ the sequence of blow-ups stops at this level. Otherwise, $p_1$ is a singular point for $F_1$ and another blow-up will be performed. Let $C_1$ be a center contained in the singular set of $F_1$ and such that $p_1 \in C_1$. The blow-up of $F_1$ with center $C_1$ leads to a foliation $F_2$. Again the transform $S_2$ of $S_1$ will select a point $p_2 \in \Pi_2^{-1}(C_1)$. If $p_2$ is a regular point for $F_2$, then the sequence of blow-ups stops. Otherwise we consider the blow-up of $F_2$ with a center $C_2$ passing through $p_2$. The procedure is then continued by induction so as to produce a sequence of foliations $F_n$

$$F_0 \xleftarrow{\Pi_1} F_1 \xleftarrow{\Pi_2} \cdots \xleftarrow{\Pi_n} F_n \xleftarrow{\Pi_{n+1}} \cdots$$

along with singular points $p_n$ and formal separatrices $S_n$. The mentioned sequence is finite if there exists $n \in \mathbb{N}$ such that $p_n$ is regular for $F_n$. A sequence of points $p_n$ obtained from a formal separatrix $S_0$ as above is often called a sequence of infinitely near singular points.

**Definition 3.** With the preceding notation, assume that $p_0$ is a nilpotent singular point for $F_0$. The point $p_0$ is said to be a persistent nilpotent singularity if there exists a formal separatrix $S_0$ of $F_0$ such that for every sequence of blowing-ups as in (8) the following conditions are satisfied:

(i) The singular points $p_n$ (selected by the transformed separatrices $S_n$) are all nilpotent singular points for the corresponding foliations;

(ii) The multiplicity $\text{mult}(F_n, S_n)$ of $F_n$ along $S_n$ does not depend on $n$.

**Remark 1.** Note that Condition (i) implies Condition (ii) if the blow-up is centered at the nilpotent singular point in question. In the case of blow-ups centered at smooth curves, however, it is possible to have a strictly smaller multiplicity, cf. the proof of Lemma 4.

In view of Seidenberg’s theorem, persistent nilpotent singularities do not exist in dimension 2. In dimension 3, however, the examples of singularities that cannot be resolved by blow-ups with invariant centers found by Sancho and Sanz satisfy the conditions in Definition 3. In fact, Sancho and Sanz have shown that the foliation associated with the vector field

$$X = x \left( x \frac{\partial}{\partial x} - \alpha y \frac{\partial}{\partial y} - \beta z \frac{\partial}{\partial z} \right) + xz \frac{\partial}{\partial y} + (y - \lambda x) \frac{\partial}{\partial z}$$

possesses a strictly formal separatrix $S = S_0$ such that for every sequence of blowing-ups as above, the corresponding sequence of infinitely near singular points consists of nilpotent
singularities. Furthermore the foliations $\mathcal{F}_n$ also satisfy $\text{mult} (\mathcal{F}_n, S_n) = 2$ for every $n \in \mathbb{N}$, where $S_n$ stands for the transform of $S$. The set of persistent nilpotent singular points is thus non-empty. Most of this section will be devoted to the characterization of these persistent singularities and the final result will be summarized by Theorem 3. We begin with the following proposition:

**Proposition 3.** Let $\mathcal{F}$ be a singular holomorphic foliation on $(\mathbb{C}^3, 0)$ and assume that the origin is a persistent nilpotent singularity of $\mathcal{F}$. Then, up to finitely many one-point blow-ups, there exist local coordinates and a holomorphic vector field $X$ representing $\mathcal{F}$ and having the form

$$\begin{align*}
(y + f(x, y, z)) \frac{\partial}{\partial x} + g(x, y, z) \frac{\partial}{\partial y} + z^n \frac{\partial}{\partial z}
\end{align*}$$

for some $n \geq 2 \in \mathbb{N}$ and some holomorphic functions $f$ and $g$ of order at least 2 at the origin. Moreover the orders of the functions $z \mapsto f(0,0,z)$ and of $z \mapsto g(0,0,z)$ can be made arbitrarily large (in particular greater than $2n$).

**Proof.** Let $\mathcal{F}$ be a nilpotent persistent singular point and denote by $S$ a formal separatrix giving rise to a sequence of infinitely near singular points as in Definition 3. Up to finitely many one-point blow-ups the formal separatrix $S$ can be assumed to be smooth in the formal sense. Up to performing an additional one-point blow-up, we may also assume that $\mathcal{F}$ admits an analytic smooth invariant surface which is, in addition, transverse to the formal separatrix $S$. In fact, the resulting exceptional divisor is necessarily invariant under the transformed foliation since the previous singular point is nilpotent and non-zero (recall that the exceptional divisor is not invariant by the blown-up foliation if and only if the first non-zero homogeneous component is a multiple of the radial vector field).

Note also that even if we consider the blow-up of $\mathcal{F}$ centered at a smooth analytic curve contained in the singular set of the foliation (rather than the one-point blow-up) the resulting exceptional divisor is still invariant by the transformed foliation. The argument is similar to the preceding one: if this component were not invariant, then the first non-zero homogeneous component of the foliation with respect to this curve would be a multiple of the vector field $x \partial/\partial x + y \partial/\partial y$ at generic points in this center. Again this cannot happen since the origin is a nilpotent singular point. Finally, denoting by $E_n$ the total exceptional divisor associated with the birational transformation $\Pi_n = \Pi_1 \circ \cdots \circ \Pi_2 \circ \Pi_1$. The argument above also applies to ensure that every irreducible component of the exceptional divisor is invariant by the corresponding foliation $\mathcal{F}_n$.

Summarizing, we can assume without loss of generality that all of the following holds:

- the formal separatrix $S$ is smooth;
- $\mathcal{F}$ possesses a (smooth) analytic invariant surface $E$;
- the formal separatrix $S$ is transverse to $E$ (in the formal sense).

In view of the preceding, consider local coordinates $(x, y, z)$ around $p_0$ where the smooth invariant surface $E$ is given by $\{z = 0\}$. Denote by $H$ a formal change of coordinates preserving $\{z = 0\}$ as invariant surface and taking the formal separatrix $S$ to the $z$-axis given by $\{x = 0, y = 0\}$. Since the vector field obtained by conjugating $X$ through $H$ is merely formal, let $H_m$ denote the polynomial change of coordinates obtained by truncating $H$ at order $m$. We then set

$$Y_m = (DH_m)^{-1}(X \circ H_m).$$

The map $H_m$ is holomorphic and so is the vector field $Y_m$. Denote by $\mathcal{F}_m$ the foliation associated with $Y_m$. The foliation $\mathcal{F}_m$ clearly admits a formal separatrix $S_m$ whose order of tangency with the $z$-axis goes to infinity with $m$ and thus can be assumed arbitrarily large.
Under the above conditions, the vector field \( Y_m \) has the form
\[
Y_m = A(x, y, z) \frac{\partial}{\partial x} + B(x, y, z) \frac{\partial}{\partial y} + C(x, y, z) \frac{\partial}{\partial z}
\]

with

1. \( C(x, y, z) = z^n + g(z) + xP(x, y, z) + yQ(x, y, z) \), for some \( n \in \mathbb{N} \), some holomorphic functions \( P \) and \( Q \) divisible by \( z \) and some holomorphic function \( g \) divisible by \( z^{n+1} \);
2. \( A(0, 0, z) \) and \( B(0, 0, z) \) having order arbitrarily large, say greater than \( 2n \).

Note that the value of \( n = \text{ord} (C(0, 0, z)) \geq 2 \) depends only on the initial foliation \( \mathcal{F} \) and not on the choice of \( m \in \mathbb{N}^* \). In fact, the value of \( n \) is nothing but the multiplicity of \( \mathcal{F} \) along \( S \) and hence it is invariant by (formal) changes of coordinates. The orders of \( A(0, 0, z) \) and of \( B(0, 0, z) \) depend however on \( m \). Note that the orders in question are related to the contact order between \( S_m \) and the \( z \)-axis. In particular these orders can be made arbitrarily large as well.

Naturally the foliation \( \mathcal{F} \) and \( \mathcal{F}_m \) are both nilpotent at the origin. Next we have:

**Claim.** Up to a linear change of coordinates in the variables \( x, y, \) the linear part of \( Y_m \) is given by \( y \partial/\partial x \).

**Proof of the Claim.** The formal Puiseux parametrization \( \varphi \) of \( S_m \) has the form \( \varphi(T) = (T^r + \text{h.o.t.}, T^s + \text{h.o.t.}, T) \) where the integers \( r \) and \( s \) are related to the contact order between \( S_m \) and the \( z \)-axis. In particular both \( r \) and \( s \) can be made arbitrarily large. Now it is clear that both \( \partial A/\partial z \) and \( \partial B/\partial z \) must vanish at the origin provided that \( \varphi \) is invariant by the vector field \( Y_m \). On the other hand, \( \partial C/\partial x \) and \( \partial C/\partial y \) are both zero at the origin since \( P \) and \( Q \) are divisible by \( z \) (cf. condition (1) above). It is also clear that \( \partial C/\partial z \) equals zero at the origin since \( n \geq 2 \). Thus both the third line and the third column in the matrix representing the linear part of \( Y_m \) at the origin are entirely constituted by zeros. Using again the fact that this matrix is nilpotent, the standard Jordan form ensures that a linear change of coordinates involving only the variables \( x, y \) brings the linear part of \( Y_m \) to the form \( y \partial/\partial x \). It is also immediate to check that this linear change of coordinates does not affect the previously established conditions and/or normal forms. The claim is proved. \( \square \)

Consider now the blow-up of \( \mathcal{F} \) centered at the origin. In coordinates \( (u, v, z) \) where \( (x, y, z) = (uz, vz, z) \), the transform \( \tilde{Y}_m \) of \( Y_m \) by the mentioned blow-up is given by
\[
\tilde{Y}_m = \tilde{A}(u, v, z) \frac{\partial}{\partial x} + \tilde{B}(u, v, z) \frac{\partial}{\partial y} + \tilde{C}(u, v, z) \frac{\partial}{\partial z}
\]
where
\[
\tilde{A}(u, v, z) = \frac{A(uz, vz, z) - uC(uz, vz, z)}{z} \quad \text{and} \quad \tilde{B}(u, v, z) = \frac{B(uz, vz, z) - vC(uz, vz, z)}{z}
\]
and where \( \tilde{C}(u, v, z) = C(uz, vz, z) \). In particular \( \tilde{F}_m \) is nilpotent at the origin, with the same linear part as \( \mathcal{F}_m \). Furthermore the above formulas easily imply all of the following:

(a) the order of \( \tilde{C}(0, 0, z) \) coincides with the order of \( C(0, 0, z) \);
(b) the maximal power of \( z \) dividing \( \tilde{C}(u, v, z) - z^n - g(z) \) is strictly greater than the maximal power of \( z \) dividing \( C(x, y, z) - z^n - g(z) \);
(c) \( \text{ord} \tilde{A}(0, 0, z) = \text{ord} A(0, 0, z) - 1 \) and \( \text{ord} \tilde{B}(0, 0, z) = \text{ord} B(0, 0, z) - 1 \).

Also the transform \( \tilde{S}_m \) of \( S_m \) is a formal separatrix tangent to the \( z \)-axis. The tangency order is still large (at least \( 2n \)) since the order in question is related to the orders of \( \tilde{A}(0, 0, z) \) and of \( \tilde{B}(0, 0, z) \) and these orders fall only by one unity (item (c)). In turn, the multiplicity of \( \tilde{F}_m \) along \( \tilde{S}_m \) coincides with the multiplicity of \( \mathcal{F}_m \) along \( S_m \) (from item (a)). Finally the function
$C$ was divisible by $z$. Now, according to item (b), $\tilde{C}$ is divisible by $z^2$. In fact, item (b) ensures that after at most $n$ one-point blow-ups, the corresponding singular point is still a nilpotent singularity for which the component of the representative vector field in the direction transverse to the exceptional divisor (given in local coordinates by $\{z = 0\}$) has the form $z^n I(u, v, z)$ where $I(u, v, z)$ is a holomorphic function satisfying $I(0, 0, 0) \neq 0$. Dividing all the components of the vector field in question by $I$ then yields another representative vector field with the desired normal form. The proposition is proved.

An additional simplification can be made on the normal form \((9)\) of Proposition \(3\). Namely:

\textbf{Lemma 3.} Up to performing an one-point blow-up, the functions $f$ and $g$ in \((9)\) become divisible by $z$.

\textit{Proof.} Again let $\pi$ denote the blow-up map centered at the origin and set $\tilde{X} = \pi^* X$. In the above mentioned affine coordinates $(u, v, z)$, we have

$$\tilde{X} = (y + \tilde{f}(x, y, z)) \frac{\partial}{\partial x} + \tilde{g}(x, y, z) \frac{\partial}{\partial y} + z^n \frac{\partial}{\partial z},$$

where $\tilde{f} = (f uz, vz, z) - uz^n$/z and $\tilde{g} = (g(uz, vz, z) - vz^n)/z$. The functions $\tilde{f}$, $\tilde{g}$ are thus divisible by $z$ since $f$ and $g$ have order at least 2 at the origin. The lemma follows. \(\square\)

Next, we are going to determine conditions on the functions $f$ and $g$ for the singular point $p_0 \simeq (0, 0, 0)$ to be a persistent nilpotent singularity. Thus let $F$ be the foliation associated with a vector field $X$ having the normal form provided by Proposition \(3\) and by Lemma \(3\). In other words, the vector field $X$ is given by

\begin{equation}
X = (y + zf(x, y, z)) \frac{\partial}{\partial x} + zg(x, y, z) \frac{\partial}{\partial y} + z^n \frac{\partial}{\partial z},
\end{equation}

where $f$ and $g$ are holomorphic functions of order at least 1 and $n \in \mathbb{N}$, with $n \geq 2$. Let $S$ be a smooth formal separatrix of $F$ giving rise to the persistent nilpotent singular point (see Definition \(3\)). Without loss of generality, the contact order $k_0(\geq 4)$ between $S$ and the $z$-axis is assumed to be large and, similarly, $f(0, 0, z)$ and $g(0, 0, z)$ are assumed to have order bigger than $2n$ for $n \geq 4$.

Note that the curve locally given by $\{y = 0, z = 0\}$ coincides with the singular set of $F$. We are then allowed to perform either an one-point blow-up centered at $p_0 \simeq (0, 0, 0)$ or a blow-up centered at the mentioned curve. Now we have:

\textbf{Lemma 4.} Assume that $F$ has a persistent nilpotent singularity at the origin and let $S$ denote the corresponding formal separatrix. Then $g(x, 0, 0) = \lambda x + \text{h.o.t.}$ for some constant $\lambda \in \mathbb{C}^*$. 

\textit{Proof.} Denote by $X_1$ (resp. $F_1$, $S_1$) the transform of $X$ (resp. $F$, $S$) by the blow-up map $\pi_1$ centered on $\{y = 0, z = 0\}$. In local coordinates $(x, v, z)$ where $y = vz$ the vector field $X_1$ is given by

\begin{equation}
X_1 = (vz + zf(x, v, z)) \frac{\partial}{\partial x} + (g(x, vz, z) - vz^{n-1}) \frac{\partial}{\partial v} + z^n \frac{\partial}{\partial z}.
\end{equation}

Note that $g(x, 0, 0)$ does not vanish identically, otherwise $g(x, vz, z)$ would be divisible by $z$ and hence the vector field $X_1$ would vanish identically over the exceptional divisor (locally given by $\{z = 0\}$). This is impossible since the multiplicity of $F_1$ along the transform of $S$ would be strictly smaller than the multiplicity of $F$ along $S$, hence contradicting Condition (ii) in Definition \(3\). In particular, the singular set of $F_1$ is locally given by $\{x = 0, z = 0\}$.

On the other hand, the formal separatrix $S_1$ is still tangent to the (transform of the) $z$-axis since the contact between $S$ and the (initial) $z$-axis was greater than 2 (in fact the contact
between $S_1$ and the $z$-axis is at least $k_0 - 1 \geq 3)$. Hence, in the affine coordinates $(x, v, z)$, the foliation $\mathcal{F}_1$ must have a nilpotent singularity at the origin. Combining the conditions that $f(0, 0, 0) = 0$, $n \geq 2$, and the fact that the order of $g(0, 0, z)$ is greater than $2n$, the preceding implies that $\partial g/\partial x$ does not vanish at the origin. In other words, $g(x, 0, 0) = \lambda x + \text{h.o.t.}$ as desired.

Note that the proof above also yields the following sort of converse to Lemma 4.

**Lemma 5.** Keeping the preceding notation, let $\mathcal{F}$ be given by a vector field $X$ as in (10) and assume that $S$ is a formal separatrix of $\mathcal{F}$ with contact at least 3 with the $z$-axis. Assume that $g(x, 0, 0) = \lambda x + \text{h.o.t.}$, with $\lambda \neq 0$. Then the blow-up $\mathcal{F}_1$ of $\mathcal{F}$ centered at the curve \{y = 0, z = 0\} has a nilpotent singularity at the point of the exceptional divisor selected by $S_1$.

Continuing the discussion of Lemma 4, consider again the vector field $X_1$ in (11). We perform the blow-up centered at the curve locally given by \{x = 0, z = 0\} - which is contained in the the singular set of $\mathcal{F}_1$ - and denote by $X_2$ (resp. $\mathcal{F}_2, S_2$) the transform of $X_1$ (resp. $\mathcal{F}_1, S_1$). In affine coordinates $(u, v, z)$ with $x = uz$ we have

$$X_2 = (v + f(uz, vz, z) - uz^{n-1}) \frac{\partial}{\partial x} + (g(uz, vz, z) - vz^{n-1}) \frac{\partial}{\partial v} + z^n \frac{\partial}{\partial z}.$$ 

The contact of $S_2$ with the $z$-axis equals $k_0 - 1 \geq 3$ as follows from a simple computation (cf. also Remark 1). In particular the formal separatrix $S_2$ is still based at the origin of the coordinates $(u, v, z)$ and it is tangent to the $z$-axis. Since the above formula shows that $\mathcal{F}_2$ has a nilpotent singularity at the origin, we conclude:

**Lemma 6.** The foliation $\mathcal{F}_2$ (resp. vector field $X_2$) has a nilpotent singularity at the point of the exceptional divisor selected by $S_2$ (identified with the origin of the coordinates $(u, v, z)$).

The reader will also note that the singular set of $\mathcal{F}_2$ is still locally given by the curve \{v = 0, z = 0\} which clearly contains the origin. As already mentioned, $S_2$ is tangent to the $z$-axis.

**Remark 1.** Let us point out that $X_2$ locally coincides with the transform of $X$ by the one-point blow-up centered at the origin. In this sense, to include blow-ups centered at curves in the current discussion does not lead to additional conditions to have nilpotent singular points.

Consider again a vector field $X$ having the form (10). In the course of the preceding discussion, it was seen that the vector fields obtained through two successive blow-ups centered over the corresponding curves of singular points are respectively given by

$$X_1 = zr(x, v, z) \frac{\partial}{\partial x} + (x + zs(x, v, z)) \frac{\partial}{\partial v} + z^n \frac{\partial}{\partial z},$$

and by

$$X_2 = (v + zf(1)(u, v, z)) \frac{\partial}{\partial u} + zg(1)(u, v, z) \frac{\partial}{\partial v} + z^n \frac{\partial}{\partial z},$$

where $r, s, f(1),$ and $g(1)$ are all holomorphic functions vanishing at the origin of the corresponding coordinates. As usual the coordinates $(x, v, z)$ are determined by $(x, y, z) = (x, vz, z)$ while $(x, y, z) = (uz, vz, z)$. Furthermore the functions $f(1)$ and $g(1)$ satisfy

$$f(1)(u, v, z) = \frac{f(uz, vz, z) - uz^{n-1}}{z} \quad \text{and} \quad g(1)(u, v, z) = \frac{g(uz, vz, z) - vz^{n-1}}{z}.$$ 

The following relations arise immediately:

(1) $\text{ord } r(0, 0, z) = \text{ord } f(0, 0, z)$ and $\text{ord } s(0, 0, z) = \text{ord } g(0, 0, z)$;
vector field \(X\) corresponds to a persistent nilpotent singularity when the separatrix and the analogous comments made in the Introduction. Sanz they want the corresponding separatrix to be strictly formal and further illustrates a foliation is a legitimate invariant center so that, if we are allowed to perform blow-ups with the slightly weaker condition of allowing invariant centers which, in turn, would imply that all the functions \(f\) and \(g\) take on the form (10). In fact, the formal smooth separatrix \(S\) is blown-up at the origin (identified with the singular point selected by the transform of the initial formal separatrix). As pointed out in Remark II here it is convenient to keep in mind that two consecutive blow-ups centered at curves contained in the singular set of the corresponding foliation can be replaced by a single one-point blow-up. To continue the procedure requires us to introduce new affine coordinates for each of these blow-ups and, in doing so, notation is likely to become cumbersome. To avoid this, and since the computations are similar to the previous ones, let us abuse notation and write \((x, y, z)\) for the coordinates \((u, v, z)\): naturally these “new” coordinates \((x, y, z)\) have little to do with the initial ones. Similarly, coordinates for each of these blow-ups and, in doing so, notation is likely to become cumbersome. To explain the last claim, we begin by observing that the \(z\)-axis is not intrinsically determined by Formula (10). In fact, the \(z\)-axis is only subject to having some high contact order with the formal smooth separatrix \(S\) and it is \(S\) - rather than the \(z\)-axis - that has an intrinsic nature in our discussion. In particular, if \(S\) were analytic, we could make \(S\) coincide with the \(z\)-axis which, in turn, would imply that all the functions \(f(0, 0, z)\) and \(g(0, 0, z)\) vanish identically. It would then follow at once that the singularity is necessarily persistent.

Remark 2. It should be emphasized that our definition of persistent singularities requires the centers of all the blow-ups to be contained in the singular set of the corresponding foliations. This accounts for the difference between choosing centers that are contained in the singular set and the slightly weaker condition of allowing invariant centers. An analytic separatrix of a foliation is a legitimate invariant center so that, if we are allowed to perform blow-ups with invariant centers, the preceding singularity would be turned into an elementary one by blowing-up the foliation along the separatrix in question. This explains why in the example of Sancho and Sanz they want the corresponding separatrix to be strictly formal and further illustrates the analogous comments made in the Introduction.

We now go back to the vector field \(X\), which no longer has the form (10). To show that this vector field corresponds to a persistent nilpotent singularity when the separatrix \(S\) is strictly formal we will construct a change of coordinates where the vector field \(X\) still takes on the form (10) but where the orders of the “new” functions \(z \mapsto f(0, 0, z)\) and \(z \mapsto g(0, 0, z)\) take on the form (10) while the second blow-up possesses coordinates \((u, v, z)\). Assuming these identifications are made at every step - i.e. at every pair of blow-ups as indicated above - let \(X_{2i}\) denote the vector field obtained after \(i\)-steps where \(i\) satisfies \(i < k_0\) (recall that \(k_0\) stands for the contact order of the formal separatrix with the “initial \(z\)-axis”). In the (final) coordinates \((u, v, z)\), the vector field \(X_{2i}\) takes on the form (10)

\[
X_{2i} = \frac{x + z f_i(u, v, z)}{\partial x} + z g_i(u, v, z) \frac{x + z \partial x}{\partial x} + z^n \frac{\partial}{\partial z},
\]

with \(\partial g_i/\partial u(0, 0, 0) = \partial g/\partial x(0, 0, 0) = \lambda \neq 0\). In more accurate terms, recall that the orders of \(f_i(0, 0, z)\) and of \(g_i(0, 0, z)\) are directly related to the contact order of the transform of the formal separatrix \(S\) with the corresponding \(z\)-axis. At every step (consisting of a pair of blow-ups), the orders of \(f_i(0, 0, z)\) and of \(g_i(0, 0, z)\) decrease by one unity so that we have

\[
\text{ord}(f_i(0, 0, z)) = \text{ord}(f(0, 0, z)) - i \quad \text{and} \quad \text{ord}(g_i(0, 0, z)) = \text{ord}(g(0, 0, z)) - i.
\]

Thus, for \(i \geq k_0 = \min\{\text{ord}(f(0, 0, z)), \text{ord}(g(0, 0, z))\}\), the vector field \(X_{2i}\) no longer takes on the form (10). At first sight this might suggest that the initial nilpotent singularity may fall short of being persistent, yet it is exactly the opposite that is true: the singularity is necessarily persistent.
increase strictly so as to restore the values of the initial orders. The desired change of coordinates can be made polynomial by truncating a certain formal change of coordinates as in the proof of Proposition \(3\). This is the content of Lemma 7 below.

Consider the vector field \(X_2\) given in \((u, v, z)\) coordinates by Formula (14) along with the initial vector field \(X\) given in \((x, y, z)\) coordinates by Formula (10).

**Lemma 7.** There exists a polynomial change of coordinates \(H\) having the form \((u, v, z) = H(\tilde{x}, \tilde{y}, z) = (h_1(\tilde{x}, z), h_2(\tilde{y}, z), z)\) where the vector field \(X_2\) becomes

\[
X_2 = (\tilde{y} + z\tilde{f}(\tilde{x}, \tilde{y}, z)) \frac{\partial}{\partial \tilde{x}} + z\tilde{g}(\tilde{x}, \tilde{y}, z) \frac{\partial}{\partial \tilde{y}} + z^n \frac{\partial}{\partial z}
\]

with

- (a) \(\text{ord}(\tilde{f}(0, 0, z)) \geq \text{ord}(f(0, 0, z))\) and \(\text{ord}(\tilde{g}(0, 0, z)) \geq \text{ord}(g(0, 0, z))\);
- (b) \(\frac{\partial \tilde{g}}{\partial \tilde{x}}(0, 0, 0) = \frac{\partial g}{\partial x}(0, 0, 0)\).

**Proof.** Denote by \(S_2\) the transform of \(S\) through the one-point blow-up centered at the origin which is therefore a formal separatrix for the foliation associated with \(X_2\). Since \(S_2\) is smooth and tangent to the \(z\)-axis, it can be (formally) parameterized by the variable \(z\). In other words, \(S_2\) is given by \(\varphi(z) = (f(z), g(z), z)\) for suitable formal series \(f\) and \(g\) with zero linear parts.

Consider now the formal map given in local coordinates \((\tilde{x}, \tilde{y}, z)\) by \(H(\tilde{x}, \tilde{y}, z) = (\tilde{x} - f(z), \tilde{y} - g(z), z)\). The linear part of \(H\) at the origin is represented by the identity matrix so that \(H\) is a formal change of coordinates which, in addition, preserves the plane \(\{z = 0\}\). Furthermore, \(H\) takes the formal separatrix \(S_2\) to the \(z\)-axis. As previously mentioned, the formal vector field obtained by conjugating \(X_2\) through \(H\) is strictly formal if \(S_2\) is strictly formal. So, let \(H_m\) stand for the polynomial change of coordinates obtained from \(H\) by truncating it at order \(m\) and let \(Y_m = (DH_m)^{-1}(X \circ H_m)\). Clearly the map \(H_m\) is holomorphic and so is the vector field \(Y_m\). Moreover the foliation \(F_m\) associated to \(Y_m\) possesses a formal separatrix \(T_m\), whose tangency order with the \(z\)-axis goes to infinity with \(m\).

It is straightforward to check that the vector field \(Y_m\) has the form

\[
Y_m = (\tilde{y} + zf_m(\tilde{x}, \tilde{y}, z)) \frac{\partial}{\partial \tilde{x}} + zg_m(\tilde{x}, \tilde{y}, z) \frac{\partial}{\partial \tilde{y}} + z^n \frac{\partial}{\partial z}
\]

with \(\frac{\partial g_m}{\partial \tilde{x}}(0, 0, 0) = \frac{\partial g}{\partial x}(0, 0, 0),\) for every \(m \in \mathbb{Z}\). Furthermore, for \(m\) sufficiently large we have \(\text{ord}(f_m(0, 0, z)) \geq \text{ord}(f(0, 0, z))\) and \(\text{ord}(g_m(0, 0, z)) \geq \text{ord}(f(0, 0, z))\) as well. The lemma is then proved.

The results of this section can now be summarized as follows:

**Theorem 3.** Let \(F\) be a singular holomorphic foliation on \((\mathbb{C}^3, 0)\) and assume that the origin is a persistent nilpotent singularity of \(F\). Let \(S\) denote the corresponding formal separatrix of \(F\). Then, up to finitely many one-point blow-ups, the foliation \(F\) is represented by a vector field \(X\) having the form

\[
(y + zf(x, y, z)) \frac{\partial}{\partial x} + zg(x, y, z) \frac{\partial}{\partial y} + z^n \frac{\partial}{\partial z}
\]

for some \(n \in \mathbb{N}, n \geq 2,\) and some holomorphic functions \(f\) and \(g\) of order at least 1 such that

- (a) The separatrix \(S\) is tangent to the \(z\)-axis. In fact, the contact order of \(S\) and the \(z\)-axis can be made arbitrarily large. Equivalently the orders of \(f(0, 0, z)\) and of \(g(0, 0, z)\) are arbitrarily large.
- (b) \(\frac{\partial g}{\partial x}(0, 0, 0) \neq 0.\)
Conversely, every nilpotent foliation \( F \) represented by a vector field \( X \) as above and possessing a (smooth) formal separatrix \( S \) tangent to the \( z \)-axis gives rise to a persistent nilpotent singularity.

To close the section, let us accurately state Proposition 4 so as to derive Theorem 1.

**Proposition 4.** Let \( F \) be a (germ of) singular 1-dimensional foliation defined on a neighborhood of the origin in \( \mathbb{C}^3 \). Assume that \( F \) cannot be transformed into a foliation all of whose singular points are elementary by means of a finite sequence of blow-ups with centers contained in the singular set of the corresponding foliations. Then, up to performing a finite sequence of blow-ups as above, the foliation \( F \) possesses a formal separatrix \( S \) giving rise to a sequence of infinitely near singular points and such that none of the points in this sequence is elementary.

**Proof of Theorem 1.** Assume that \( F \) is singular foliation on \((\mathbb{C}^3,0)\) whose singularity cannot be resolved by blow-ups centered at the singular set of \( F \). Owing to Proposition 4, let \( S \) denote a formal separatrix of \( F \) giving rise to a sequence of (non-elementary) infinitely near singular points. Next apply a sequence of one-point blow-ups to \( S \) and to its transform. Since the multiplicity of the corresponding foliations along the transforms of \( S \) form a monotone decreasing sequence, this sequence becomes constant after a finite number of steps. Denoting by \( F_k \) (resp. \( S_k \)) the corresponding foliation (resp. transform of \( S \)), there follows that \( F_k \) has a nilpotent singular point at the point in the exceptional divisor selected by \( S_k \). Furthermore, owing to Remark 1, the multiplicity in question does decrease even if blow-ups centered at (smooth) singular curves are allowed. Thus the mentioned nilpotent singular point of \( F_k \) is a persistent one, i.e. it satisfies the conditions in Definition 3. The results of the present section can now be applied to this singular point and the statement follows from Theorem 3.

5. **Proof of Theorem B**

This section is devoted to the proof of Theorem B and of its corollaries while Section 6 will contain some examples complementing our main results as well as a sharper version of the normal form given in Theorem 3 which is valid for foliations tangent to semicomplete vector fields.

Let \( X \) be a holomorphic vector field defined on an open set \( U \) of some complex manifold. According to [31], \( X \) is said to be semicomplete on \( U \) if for every point \( p \in U \) there exists a connected domain \( V_p \subseteq \mathbb{C} \) with \( 0 \in V_p \) and a holomorphic map \( \phi_p : V_p \to U \) satisfying the following conditions:

- \( \phi_p(0) = 0 \) and \( \frac{d\phi_p}{dt}|_{t=t_0} = X(\phi_p(t_0)) \).
- For every sequence \( \{t_i\} \subseteq V_p \) such that \( \lim_{i \to +\infty} t_i \in \partial V_p \), the sequence \( \{\phi_p(t_i)\} \) leaves every compact subset of \( U \).

We also refer to [31] for the basic properties of semicomplete vector fields used in the sequel.

First, a vector field that is semicomplete on \( U \) is semicomplete on every open set \( V \subseteq U \) as well. In particular, the notion of germ of semicomplete vector field makes sense. Furthermore, if \( X \) is a complete vector field on a complex manifold \( M \), then the germ of \( X \) at every singular point is necessarily a germ of semicomplete vector field.

There is a useful criterion (Proposition 5) to detect vector fields that fail to be semicomplete which is as follows. Let \( X \) be a holomorphic vector field defined on an open set \( U \) and denote by \( F \) its associated (singular) holomorphic foliation. Consider a leaf \( L \) of \( F \) which is not contained in the zero set of \( X \). Then leaf \( L \) is then a Riemann surface naturally equipped with a meromorphic abelian 1-form \( dT \) dual to \( X \) in the sense that \( dT.X = 1 \) on \( L \). The 1-form \( dT \) is often referred to as the *time-form* induced by \( X \) on \( L \). The following proposition is taken from [31].
Proposition 5. Let $X$ be a holomorphic semicomplete vector field on an open set $U$. Let $L$ be a leaf of the foliation associated with $X$ on which the time-form $dT$ is defined (i.e. $L$ is not contained in the zero set of $X$). Then we have
\[ \int_c dT \neq 0 \]
for every path $c : [0,1] \rightarrow L$ (one-to-one) embedded in $L$. □

The main result of this section is the following theorem.

Theorem 4. Let $X$ be a holomorphic vector field on $(\mathbb{C}^3,0)$ and denote by $F$ its associated foliation. Assume that the origin is a persistent nilpotent singularity for $F$ and let $S$ denote the corresponding formal separatrix of $F$. Assume also that at least one the following holds:
- The multiplicity $\text{mult}(F,S)$ of $F$ along $S$ is at least 3;
- The linear part $J_0^1 X$ of $X$ at the origin equals zero.
then $X$ is not semicomplete on a neighborhood of the origin.

Our approach to Theorem 4 begins with a couple of remarks. First, it is convenient to remind the reader of the difference between vector fields and foliations in terms of the dimension of their singular sets. In other words, a vector field may have non-trivial common factors among its components so as to give rise to a divisor of zero while singular sets of foliations are always of codimension at least 2. Recall also that a vector field $Y$ is said to be a representative of the foliation $F$ if $Y$ is tangent to $F$ and has singular set of codimension at least 2. With this notation, the vector field $Y$ is a (local) representative of the foliation $F$ associated with $X$.

On a different note, it should also be pointed out that the semicomplete character of a holomorphic vector field is preserved under birational transformations. In particular, it is preserved under blow-ups. It is, however, not necessarily preserved under weighted blow-ups if these are regarded as finite-to-one maps rather than from the birational point of view associated with the orbifold action. Incidentally, blow-ups with weight 2 will be needed in what follows.

Let us now fix a holomorphic vector field $X$ on $(\mathbb{C}^3,0)$ whose associated foliation $F$ is as in Theorem 4. Since blow-ups preserve the semicomplete character of vector fields, up to transforming $X$ through finitely many one-point blow-ups, we can assume that $F$ has a persistent nilpotent singularity with a formal separatrix $S$ giving rise to a sequence of infinitely near (nilpotent) singular points.

Summarizing what precedes, we can assume the existence of local coordinates $(x,y,z)$ where $X$ is given by (cf. Sections 3 and 4)
\[ X = z^k h(x,y,z) \left[ (y + zf(x,y,z)) \frac{\partial}{\partial x} + zg(x,y,z) \frac{\partial}{\partial y} + z^n \frac{\partial}{\partial z} \right] \]
for suitable nonnegative integers $k,n$ and holomorphic functions $f$, $g$, and $h$ satisfying all of the following:
- $n \geq 2$ and $k \geq 0$;
- $f(0,0,0) = g(0,0,0) = 0$. Furthermore the orders at $0 \in \mathbb{C}$ of $f(0,0,z)$ and $g(0,0,z)$ are arbitrarily large, say larger than $2n$ (in particular $S$ is tangent to the $z$-axis);
- $\partial g(0,0,0)/\partial x = \lambda \neq 0$;
- If $h(0,0,0) = 0$, then every irreducible component of the set $\{ h = 0 \}$ is smooth, contains the separatrix $S$, and is not invariant under $F$. 


The above assertion involving the irreducible components of \( \{ h = 0 \} \) requires a couple of comments. Naturally, every analytic surface that does not contain the separatrix \( S \) can be separated from \( S \) by a suitably chosen sequence of blow-ups. Similarly, these components can be made smooth without loss of generality. Finally, for the fact that none of them is invariant under \( \mathcal{F} \), we refer to Corollary 1 which is a by-product of the proof of Proposition 4 in the appendix.

Theorem 4 is thus reduced to proving that \( X \), as given in Formula (10), is not semicomplete on a neighborhood of the origin provided that at least one of the following conditions holds: the linear part of \( X \) at the origin \( J^1_0X \) equals zero (equivalently \( k \geq 1 \)) or \( n = \text{mult} (\mathcal{F}, S) > 2 \), where \( \mathcal{F} \) stands for the foliation associated with \( X \).

Let us begin by showing that \( \mathcal{F} \) can be resolved by using a single blow-up of weight 2. Here these weight 2 blow-ups will be viewed as a two-to-one map. Note also that the lemma below includes some useful explicit formulas for the transformed vector field.

**Lemma 8.** Let \( X \) be as in Formula (10) and denote by \( \Pi \) the blow-up of weight 2 centered at the curve \( \{ y = z = 0 \} \) (the curve of singular points of \( \mathcal{F} \)). Let \( \Pi^*F \) be the transform of \( \mathcal{F} \). Then the singular point of \( \Pi^*F \) selected by \( S \) in the exceptional divisor is elementary and the corresponding eigenvalues of \( \Pi^*F \) are 0, 1, and \(-1\). Furthermore the transform \( \Pi^*X \) of \( X \) is a holomorphic vector field vanishing with order \( 2k + 1 \) on the exceptional divisor.

**Proof.** Let \((x, y, z)\) be the local coordinates where \( X \) is given by Formula (10) and consider the indicated weight 2 blow-up map \( \Pi \). In natural coordinates \((u, v, w)\) the map \( \Pi \) is given by

\[
\Pi(u, v, w) = (u, vw, w^2),
\]

where \( \{w = 0\} \) is contained in the exceptional divisor. Now a straightforward computation shows that \( \Pi^*X \) is given in the \((u, v, w)\) coordinates by

\[
\Pi^*X = w^{2k}h \left[ (vw + w^2f(u, vw, w^2)) \frac{\partial}{\partial u} + \left( wg(u, vw, w^2) - \frac{1}{2}w^2 \right) \frac{\partial}{\partial v} + \frac{1}{2}w^{2n-1} \frac{\partial}{\partial w} \right]
\]

\[
= w^{2k+1}h \left[ (v + w(f(u, vw, w^2)) \frac{\partial}{\partial u} + \left( g(u, vw, w^2) - \frac{1}{2}vw \right) \frac{\partial}{\partial v} + \frac{1}{2}w^{2n-2} \frac{\partial}{\partial w} \right],
\]

where the function \( h \) is evaluated at the point \((u, vw, w^2)\). Since \( h(0, 0, 0) \neq 0 \), there follows that the zero-divisor of \( \Pi^*X \) locally coincides with the exceptional divisor (given by \( \{w = 0\}\)). Moreover the order of vanishing of \( \Pi^*X \) at the exceptional divisor is \( 2k + 1 \). In turn, the foliation \( \Pi^*F \) is represented by the vector field

\[
Y = (v + w(f(u, vw, w^2)) \frac{\partial}{\partial u} + \left( g(u, vw, w^2) - \frac{1}{2}vw \right) \frac{\partial}{\partial v} + \frac{1}{2}w^{2n-2} \frac{\partial}{\partial w}
\]

whose linear part at the origin is given by \( v\partial/\partial u + \lambda u\partial/\partial v \) since \( f(0, 0, 0) = 0 \) and \( n \geq 2 \) (here \( \lambda = \partial g(0, 0, 0)/\partial x \neq 0 \)). Thus the eigenvalues of \( \mathcal{F} \) at the origin are 0 and the two square roots of \( \lambda \) which is clearly equivalent to having eigenvalues 0, 1, and \(-1\). The lemma is proved. \( \square \)

Singularities of foliations on \( \mathbb{C}^3 \) possessing a single eigenvalue equal to 0 are called codimension 1 saddle-nodes. Semicomplete vector fields whose associated foliation is a codimension 1 saddle-node were studied in detail in [32]. However, the case in question where the non-zero eigenvalues belong to the Siegel domain is not covered in the paper in question. Also, in what follows, we will need more specific results, partially due to the fact that the corresponding vector field is not necessarily semicomplete (cf. below). Yet, the reader will note that our argument to prove Theorem 4 overlaps non-trivially with the ideas in [32].

Recall that our purpose is to show that \( X \) is not semicomplete on a neighborhood of \((0, 0, 0)\) provided that \( k \neq 0 \) or \( n > 2 \). Note that, in general, this conclusion does not immediately follow
from proving that the vector field $\Pi^*X$ is not semicomplete on a neighborhood of the origin of the coordinates $(u,v,w)$ since the map $\Pi$ is not one-to-one. It is, in fact, easy to construct examples of semicomplete vector fields whose transforms under ramified coverings are no longer semicomplete. Yet, in our context, the situation can be described in a more accurate form. Consider a regular leaf $L$ of the foliation $\Pi^*\mathcal{F}$ which is equipped with the time-form $dT_{\Pi^*X}$ induced by $\Pi^*X$. Assume that $c : [0,1] \to L$ is open path over which the integral of $dT_{\Pi^*X}$ equals zero so that, in particular, $\Pi^*X$ is not semicomplete (Proposition 5). If $X$ happens to be semicomplete, then we must necessarily have $\Pi(c(0)) = \Pi(c(1))$. Hence, the idea to prove Theorem 4 will be to find open paths $c$ satisfying the following two conditions:

- $c : [0,1] \to L$ is contained in a leaf $L$ of $\Pi^*\mathcal{F}$ and verifies $\Pi(c(0)) \neq \Pi(c(1))$;
- The integral of the time-form $dT_{\Pi^*X}$ over $c$ is equal to zero.

If $c$ is a path as above, then its projection by $\Pi$ is still an open path contained in a leaf of $\mathcal{F}$. Furthermore, the integral of the corresponding time-form induced by $X$ over $\Pi(c)$ is zero so that $X$ cannot be semicomplete.

Before proceeding further, it is convenient to recall the notion of function asymptotic to a formal series. Let then $t \in \mathbb{C}$ be a variable and considers a formal series $\psi(t)$. Consider also a circular sector $V$ of angle $\theta$, vertex at $0 \in \mathbb{C}$, and small radius. A holomorphic function $\psi_V$ defined on $V \setminus \{0\}$ is said to be asymptotic (on $V$) to the formal series $\psi(t)$ if for every $i \in \mathbb{N}$ and for every sector $W \subset V$, of angle strictly smaller than $\theta$ and sufficiently small radius, there exists a constant $\text{Const}_{i,W}$ such that

$$||\psi_V(t) - \psi_i(t)|| \leq \text{Const}_{i,W} \|t\|^{i+1},$$

where $\psi_i$ stands for the $i^{th}$-jet of $\psi$ at $0 \in \mathbb{C}$. The adaptation of the above definition to vector-valued formal series $\psi(t) = (\psi_1(t), \ldots, \psi_n(t))$ and functions $\psi : V \to \mathbb{C}^n$ is straightforward and thus left to the reader.

The following lemma appears in [14] (Lemma 3.12).

**Lemma 9.** Let $V \subset \mathbb{C}$ denote a circular sector with vertex at $0 \in \mathbb{C}$ and angle $2\pi/l$, where $l$ is a strictly positive integer. Assume that $\rho$ is a holomorphic function on $V \setminus \{0\}$ such that

$$||\rho(x) - x^{l+2}|| \leq \text{Const} ||x^{l+3}||$$

for a suitable constant $\text{Const}$. Then for every $r > 0$, there exists an open path $c$ embedded in the intersection of $V$ with the disc of radius $r$ and center at $0 \in \mathbb{C}$ such that the integral of the 1-form $dx/\rho(x)$ equals zero.

**Proof.** It suffices to sketch the argument and refer to [31] for the detail concerning the effect of higher order terms. Consider first the special case where $\rho(x) = x^{l+2}$. In this case the 1-form $dx/\rho(x)$ admits the function $x \mapsto -1/(l+1)x^{l+1}$ as primitive. Thus it is enough to choose a path $c$ of the form $c(t) = x_0e^{2i\pi t/(l+1)}$ where $x_0$ has small absolute value and is such that the resulting path $c$ is still contained in $V$.

In the general case, the leading term of $\rho(x)$ is $x^{l+2}$. In fact, for $||x||$ small, the difference $||\rho(x) - x^{l+2}||$ is bounded by $\text{Const} ||x^{l+3}||$ which is of order larger than $x^{l+2}$ itself. The statement then follows by using the “perturbation” technique in [31].

The proof of Theorem 4 is divided in two cases according to whether or not we have $h(0,0,0) \neq 0$. 
Proof of Theorem 4 when \( h(0,0,0) \neq 0 \). With the notation of Lemma 8 consider the vector field \( \Pi^*X \) and note that \( \Pi^*X = w^{2k+1}h(u, \bar{w}, w^2)Y \) where \( Y \) is given by

\[
(18) \quad Y = (v + wf(u, \bar{w}, w^2)) \frac{\partial}{\partial u} + \left( g(u, \bar{w}, w^2) - \frac{1}{2}v \right) \frac{\partial}{\partial v} + \frac{1}{2}w^{2n-2} \frac{\partial}{\partial w},
\]

for suitable \( k, n, f, g, \) and \( h \) as above. In particular, the vector field \( Y \) is a representative of the foliation \( \Pi^*F \). Fixed a neighborhood \( U \) of the origin, we look for leaves \( L \) of \( \Pi^*F \) along with open paths \( c : [0,1] \to L \) contained in \( U \) such that the two conditions below are satisfied:

- \( \int_c \text{d}T_{\Pi^*X} = 0; \)
- \( \Pi(c(0)) \neq \Pi(c(1)). \)

The existence of the desired paths \( c \) will be obtained with the help of a theorem due to Malmquist in [23] (Théorème 1, page 95) provided that \( n \geq 3 \) or \( k \geq 1 \).

To begin we can assume that \( \lambda = \partial g(0,0,0)/\partial x = 1 \), up to a multiplicative constant, so that the linear part of \( Y \) at the origin has eigenvalues 0, 1, and \(-1\). Consider then the linear change of coordinates \((\bar{u}, \bar{v}, \bar{w}) \mapsto (\bar{u} + \bar{v}, \bar{u} - \bar{v}, \bar{w})\). The pull-back \( \bar{Y} \) of \( Y \) in the coordinates \((\bar{u}, \bar{v}, \bar{w})\) becomes

\[
(19) \quad \bar{Y} = \left[(\bar{u} + \bar{w}A(\bar{u}, \bar{v}, \bar{w})) \frac{\partial}{\partial \bar{u}} + (-\bar{v} + \bar{w}B(\bar{u}, \bar{v}, \bar{w}) + C(\bar{u} + \bar{v})) \frac{\partial}{\partial \bar{v}} + \frac{1}{2}w^{2n-2} \frac{\partial}{\partial \bar{w}} \right]
\]

for suitable holomorphic functions \( A \) and \( B \) of order at least 1 and a holomorphic function \( C \) of order at least 2. Similarly the vector field \( \Pi^*X \) corresponding to the pull-back of \( \Pi^*X \) in the coordinates \((\bar{u}, \bar{v}, \bar{w})\) satisfies \( \Pi^*X = \bar{w}^{2k+1}h(\bar{u} + \bar{v}, \bar{u} - \bar{v}, \bar{w})\bar{Y} \).

Note that the singularity of the foliation associated with \( \bar{Y} \) at the origin is a codimension 1 saddle-node, i.e. it has exactly one eigenvalue equal to zero. In fact, it is a resonant codimension 1 saddle-node in the sense that the non-zero eigenvalues, 1 and \(-1\), are resonant. This type of singularity is closely related to a classical result due to Malmquist involving systems of differential equations with an irregular singular point, cf. [23]. We will state a slightly simplified version of Malmquist results which is adapted to our problem. For \( \delta \in \{0,1\} \), assume that we are given a system of differential equations having the form

\[
(20) \quad \begin{aligned}
\bar{u}^{l+1} \frac{\text{d}u}{\text{d}\bar{w}} &= s_1 \bar{u} + \beta_1(\bar{u}, \bar{v}, \bar{w}) \\
\bar{u}^{l+1} \frac{\text{d}v}{\text{d}\bar{w}} &= s_2 \bar{v} + \delta \bar{u} + \beta_2(\bar{u}, \bar{v}, \bar{w})
\end{aligned}
\]

where \( s_1, s_2 \neq 0 \) and where \( \beta_1, \beta_2 \) are convergent series (in particular conditions (A) and (B) of [23] are necessarily verified). Now let \( \Phi(\bar{w}) = (\psi_1(\bar{w}), \psi_2(\bar{w})) \) be a formal solution for the system in question. Malmquist then shows that for every \( \varepsilon > 0 \), there exist circular sectors of angle \( 2\pi/k - \varepsilon \) in the space of \( \bar{w} \)-variable with respect to which the system \( \Phi \) admits a unique solution which is asymptotic to the formal solution \( \Phi(\bar{w}) \).

The system \( \Phi \) is naturally related to saddle-nodes singularities as those given by the vector field \( \bar{Y} \). In fact, the vector field \( \bar{Y} \) is essentially equivalent to the system of differential equations

\[
\begin{aligned}
\bar{w}^{2n-2} \frac{\text{d}u}{\text{d}\bar{w}} &= \bar{u} + \bar{w}A(\bar{u}, \bar{v}, \bar{w}) \\
\bar{w}^{2n-2} \frac{\text{d}v}{\text{d}\bar{w}} &= -\bar{v} + \bar{w}B(\bar{u}, \bar{v}, \bar{w}) + C(\bar{u} + \bar{v})
\end{aligned}
\]

Thus we have \( s_1 = 1, \ s_2 = -1 \) and \( l = 2n - 3 \) and the formal solution \( \Psi(\bar{w}) = (\psi_1(\bar{w}), \psi_2(\bar{w})) \) is obtained out of the (initial) formal separatrix \( S \) (whose formal parameterization is simply \( \bar{w} \mapsto (\bar{w}, \psi_1(\bar{w}), \psi_2(\bar{w}))) \). Since these statements are clearly invariant by change of coordinates, we can return to the variables \((u, v, \bar{w})\) where the vector field \( \Pi^*X \) is defined. Keeping in mind that \( w = \bar{w} \), the angle of the sector \( \bar{V} \) remains unchanged and the formal parameterization of \( S \) will simply be denoted by \( \Psi(w) = (w, \psi_1(w), \psi_2(w)) \) (where \( \psi_1 = \psi_1 + \psi_2 \) and \( \psi_2 = \psi_1 - \psi_2 \)).
Fix then an arbitrarily small neighborhood of the origin of the coordinates \((u,v,w)\). Owing to Malmquist theorem, we can choose a solution of \(Y\) asymptotic to the formal series \(\Psi(w) = (w, \psi_1(w), \psi_2(w))\) of \(S\) on the above mentioned sector \(V\) (recall that \(w = \tilde{w}\)). In particular there are points \(w_0 \in \mathbb{C}\) with \(\|w_0\|\) arbitrarily small, and there are leaves of \(\Pi^*F\) to which paths of the form \(c(t) = (0,0, w_0 e^{2\pi it/(2n-3)})\) can be lifted (with respect to the fibration given by projection on the \(w\)-axis). Furthermore these lifted paths are contained in arbitrarily small neighborhoods of the origin provided that \(\|w_0\|\) is small enough. In other words, once a convenient circular sector \(V\) of angle \(2\pi/(2n - 3)\) is chosen, we can “parameterize” an open set of a certain leaf \(L\) of \(\Pi^*F\) by a map of the form \(w \mapsto (w, \psi_{1,V}(w), \psi_{2,V}(w))\), \(w \in V\), where the holomorphic functions \(\psi_{i,V}\) are invariant on \(V\) to the formal series \(\psi_i(w), i = 1,2\).

The restriction to \(L\) of \(\Pi^*X = w^{2k+1}h(u,v,w)Y\) can be considered in the \(w\)-coordinate so as to become identified with a certain one-dimensional vector field \(Z(w) = \rho(w)\partial/\partial w\) defined on \(V\). Since \(h(0,0,0) \neq 0\) and the formal series \(\psi_i(w)\) have zero linear terms (\(S\) is tangent to the \(w\)-axis), there follows that \(\rho\) has an asymptotic expansion of the form
\[
 w^{2n+2k-1} + \text{h.o.t.}
\]
up to a multiplicative constant, where h.o.t. stands for terms of order higher than \(2n + 2k - 1\).

Since \(k \geq 0\) and since \(V\) is a sector of angle \(2n - 3\), Lemma 9 implies the existence of an open embedded path \(c \subset V\) over which the integral of the time-form associated with \(Z(w)\) equals zero. Hence the vector field \(\Pi^*X\) is never semicomplete (even if \(n = 2\) and \(k = 0\)).

What precedes shows that \(\Pi^*X\) is not semicomplete but we still need to show that the initial vector field \(X\) is not semicomplete. It is in this part of the argument that the assumption \(n \geq 3\) unless \(k \geq 1\) will play a role. To conclude that \(X\) is not semicomplete we need to consider the possibility of having \(c(0)^2 = c(1)^2\) in the above mentioned path \(c \subset V\). If this happens, it means that the difference of argument between \(c(0)\) and \(c(1)\) is \(\pi\). However, in the preceding discussion (cf. also Lemma 9), it was seen that the constructed path \(c\) is such that the difference of argument between \(c(0)\) and \(c(1)\) can be made arbitrarily close to \(2\pi/(2n+2k-2) = \pi/(n+k-1)\).

Recalling that \(n \geq 2\), there immediately follows that the desired path \(c\) as above satisfying in addition \(c(0)^2 \neq c(1)^2\) can be found provided that \(n \geq 3\) or \(k \geq 1\). Theorem 4 is proved.

Let us now prove Theorem 4 in the remaining case.

**Proof of Theorem 4 when \(h(0,0,0) = 0\).** Consider the foliation \(F\) given in (18). We know that every irreducible component of the set \(\{h = 0\}\) is smooth, contains the separatrix \(S\), and is not invariant under \(\Pi\), cf. Corollary 1. Whereas \(S\) is a formal separatrix for the foliation \(F\) (and hence not contained in the singular set of \(F\)), the vector field \(X\) vanishes identically over \(S\) since \(h = 0\). Hence, the argument employed in the previous case is no longer valid since \(X\) does not induce a time-form on \(S\) (even if \(S\) happens to be convergent). In particular, the existence of an asymptotic leaf over which \(X\) induces a time-form cannot be guaranteed.

To overcome this difficulty, we proceed as follows. To begin, we perform the above indicated blow-up \(\Pi\) of weight 2 so that \(\Pi^*F\) is given by (18). Denote by \(\{h = 0\}\) the transform of \(\{h = 0\}\) by \(\Pi\) and let \(S\) be identified with its own transform by \(\Pi\). The plane \(\{w = 0\}\) is invariant under \(\Pi\) and, in addition, the restriction of \(F\) to this plane yields a foliation having a singularity with eigenvalues 1 and \(-1\) at the origin. In particular, it follows that \(F\) possesses exactly two separatrices, \(S_1\) and \(S_2\), contained in the plane \(\{w = 0\}\). Furthermore, both \(S_1\) and \(S_2\) are smooth and mutually transverse. Indeed, they are tangent to the respective eigenvectors associated with 1 and with \(-1\).

**Claim.** The separatrix \(S_1\) (resp. \(S_2\)) is not contained in strict transform of \(\{h = 0\}\).
Proof of the claim. Without loss of generality we can assume that \( h \) is irreducible (otherwise we apply the argument to each irreducible component of \( h \)). Thus \( \{ h = 0 \} \) is smooth and contains the formal separatrix \( S \) which is tangent to the \( z \)-axis. Thus \( h \) is given by \( h(x, y, z) = ax + by + \text{h.o.t.} \), where at least one between \( a \) and \( b \) is different from zero and where h.o.t. stands for higher order terms. Next, recall that \( \Pi(u, v, w) = (u, vw, w^2) = (x, y, z) \). Thus, if \( a \neq 0 \), then \( \tilde{h}(u, v, w) \) takes on the form \( \tilde{h}(u, v, w) = au + \text{h.o.t.} \) in the previous coordinates \((u, v, w)\). Hence, the surface \( \{ \tilde{h} = 0 \} \) is tangent to the plane \( \{ u = 0 \} \) at the origin. However, as previously seen, the separatrices \( S_1 \) and \( S_2 \) are contained in \( \{ w = 0 \} \) and tangent to \( \{ u = v \} \) and \( \{ u = -v \} \), respectively. Therefore the claim holds provided that \( a \neq 0 \).

Assume now that \( a = 0 \) so that \( b \neq 0 \). If \( h(x, 0, 0) \) vanishes identically, then the strict transform of \( \{ h = 0 \} \) (i.e. ignoring the component associated with the exceptional divisor), is given by a function whose linear part is \( bv \). Thus, now, the corresponding surface is tangent to the plane \( \{ v = 0 \} \) and again cannot contain the separatrices \( S_1 \) or \( S_2 \). Finally, if \( \tau(x) = h(x, 0, 0) \) does not vanish identically, then the intersection of the mentioned surface with the plane \( \{ w = 0 \} \) is given by \( \tau(x) = 0 \). Once again it cannot contain the separatrices \( S_1 \) and \( S_2 \). The claim is proved.

The remainder of the proof consists of generalizing to the present setting a couple of well known properties of 2-dimensional saddle-nodes in the spirit of [17]. Consider the vector field \( \Pi^*X = w^{2k+1} \tilde{h}Y \), where \( Y \) is given by Formula (17). Owing to the above claim, the vector field \( \tilde{h}Y \) is regular at generic points of the separatrix \( S_1 \) (recall that \( S_1 \) is one of the two separatrices of the foliation \( \Pi^*F \) that are contained in the plane \( \{ w = 0 \} \)). If \( T \) is a local coordinate for \( S_1 \) around the origin, then the restriction to \( S_1 \) of the vector field \( \tilde{h}Y \) can naturally be identified with a 1-dimensional vector field of the form \( g(T) \partial/\partial T \), where \( g \) is a holomorphic function. Furthermore, since \( h(0, 0, 0) = 0 \), it follows that \( g(0) = g'(0) = 0 \), i.e. the order at the origin of the restriction of \( \tilde{h}Y \) to \( S_1 \) is at least 2.

Whereas the vector field \( \Pi^*X = w^{2k+1} \tilde{h}Y \) vanishes identically over \( S_1 \subset \{ w = 0 \} \), \( \Pi^*X \) does induce an affine structure on \( S_1 \) (cf. [17]). In the present case, this affine structure has a singular point at the origin whose order coincides with the order of the vector field \( \tilde{h}Y \). In other words, the affine structure in question is given around \( 0 \in S_1 \) by the vector field \( g(T) \partial/\partial T \). This happens because the “index” of the separatrix \( S_1 \) has no effect into the computation of the ramification index of this affine structure which is a phenomenon reminiscent from the fact that the Camacho-Sad index of the strong invariant manifold of a saddle-node in dimension 2 is always zero (see [5], [17]).

To effective use the mentioned affine structure, the holonomy map of \( S_1 \) will also be needed in the sequel. Let \( \Sigma \) denote a local transverse section to \( S_1 \) equipped with coordinates \((\tilde{z}, w)\). The corresponding holonomy map \( \sigma \) then fixes \((0, 0)\) so that it can be viewed as a map from \((\mathbb{C}^2, 0)\) to \((\mathbb{C}^2, 0)\). Now Lemma 10 below asserts that \( \sigma \) never coincides with the identity, though its derivative at \((0, 0)\) is the identity map.

The preceding two paragraphs can be combined to prove Theorem 11 as follows. Consider a loop \( c \subset S_1 \) such that \( c(0) = c(1) = S_1 \cap \Sigma \). Denote by \( \tilde{c}_0 \) the lift of \( c \) in the leaf \( L_0 \) through a point \((\tilde{z}_0, w_0) \in \Sigma \) sufficiently close to \((0, 0) \simeq \Sigma \cap S_1 \). For \( w_0 \neq 0 \), the vector field \( \Pi^*X \) is regular on \( L_0 \) so that the corresponding time-form \( dT_L \) can be considered. Now, the fact that the affine structure induced by \( X \) on \( S_1 \) has order at least 2 at the origin, means that the integral of \( dT_L \) over \( \tilde{c}_0 \) can be assumed to be equal to zero without loss of generality, up to choosing \((\tilde{z}_0, w_0) \) sufficiently close to \((0, 0) \). In more accurate terms, whereas the mentioned integral may not be equal to zero, it is always possible to “slightly perturb” the end-point of \( \tilde{c}_0 \) so as to obtain a new path over which the integral of \( dT_L \) is actually zero. Furthermore,
since we can also assume that $\sigma(\tilde{z}_0, w_0) \neq (\tilde{z}_0, w_0)$, the path $\tilde{c}_0$ is open: the perturbation of the end-point of the lift of $c$ cannot close this path up since the points $\tilde{c}_0(0)$ and $\tilde{c}_0(1)$ are uniformly far apart in the intrinsic distance of $L_0$.

Summarizing the preceding, the integral of the time-form $dT_L$ induced by $\Pi^*X$ on $L_0$ vanishes over the open path $\tilde{c}_0$. In particular, $\Pi^*X$ is not semicomplete. To conclude that the initial vector field $X$ is not semicomplete as well, it is therefore necessary to check that the projection $\Pi(\tilde{c}_0)$ is still open. Denote by $(\tilde{z}_1, w_1)$ the end-point in $\Sigma$ of $\tilde{c}_0$ (viewed as the lift of $c$, i.e. before a possible perturbation of its end-point). Since $\sigma$ is tangent to the identity, it is clear that the argument of $w_1$ converges to the argument of $w_0$ when $(\tilde{z}_0, w_0)$ converges to $(0, 0)$. In particular $w_1 \neq -w_0$ so that $\Pi(\tilde{c}_0)$ is still open. Clearly the same conclusion still holds after a possible “perturbation” of the end-point of $\tilde{c}_0$. This proves that $X$ itself is not semicomplete and establishes Theorem 4. □

To round off the preceding discussion, we state and prove Lemma 10.

**Lemma 10.** The local holonomy map $\sigma$ associated with the separatrix $S_1$ of $\Pi^*F$ does not coincide with the identity. Furthermore it is derivative at the fixed point corresponding to $S_1$ is the identity.

**Proof.** Recall that $\Pi^*(F)$ is given in suitable coordinates $(\bar{u}, \bar{v}, \bar{w})$ by the vector field $\bar{Y}$ of Formula (19). Recall also that $A$ and $B$ are holomorphic functions of order at least 1 while $C$ is holomorphic of order at least 2. In particular, the eigenvalues of $\Pi^*(F)$ at the origin are 1, $-1$, and 0 which implies that the derivative of $\sigma$ at the fixed point corresponding to $S_1$ is the identity. The proof then amounts to checking that $\sigma$ cannot coincide with the identity.

Consider the restriction of $\Pi^*(F)$ to the invariant plane $\{\bar{w} = 0\}$ and the corresponding restriction of $\sigma$. Clearly we can assume that this restriction of $\sigma$ coincides with the identity, otherwise there is nothing to be proved. In this latter case, however, there follows that the restriction of $\Pi^*(F)$ to $\{\bar{w} = 0\}$ is linearizable after an unpublished result due to Mattei (for published generalizations, see [21], [33]). Thus, up to performing a change of coordinates $(u, v, z)$, we can assume that $\Pi^*(F)$ is given by a vector field of the form

$$(u + zF(u, v, z))\frac{\partial}{\partial u} + (-v + zG(u, v, z))\frac{\partial}{\partial v} + z^n\frac{\partial}{\partial z},$$

with $S_1$ given by $\{u = z = 0\}$. Set $v(t) = e^{2\pi it}$ so that $dv/dt = 2\pi i e^{2\pi it}$. Since

$$\frac{du}{dt} = \frac{du}{dv}\frac{dv}{dt} \quad \text{and} \quad \frac{dz}{dt} = \frac{dz}{dv}\frac{dv}{dt},$$

we obtain the following system of equations:

$$\begin{cases}
(-vzG(u, v, z))\frac{du}{dv} = (u + zF(u, v, z))2\pi iv \\
(-v + zG(u, v, z))\frac{dz}{dv} = z^n2\pi iv.
\end{cases} \tag{21}$$

Now let us apply the standard procedure to compute holonomy maps by means of power series. First set $F(u, v, z) = \sum_{k,l \geq 0} F_{kl} u^k z^l$ and $G(u, v, z) = \sum_{k,l \geq 0} G_{kl} u^k z^l$, where the coefficients $F_{kl}$ and $G_{kl}$ are functions of the variable $v$. Similarly, let $u = \sum_{i+j \geq 1} a_{ij}(t) u_0^i z_0^j$ and $z = \sum_{i+j \geq 1} b_{ij}(t) u_0^i z_0^j$. With this notation, the holonomy map $\sigma$ is given by

$$\sigma(u_0, z_0) = (u(1, u_0, z_0), z(1, u_0, z_0)).$$

Since for $t = 0$, the resulting map coincides with the identity, there also follows that $a_{10}(0) = b_{01}(0) = 1$ whereas all the remaining coefficients $a_{ij}$ and $b_{ij}$ are equal to zero.
On the other hand, the following clearly holds:

\[ \frac{du}{dt} = \sum_{i+j \geq 1} a_{ij}(t)u_{01}^{i-j} \quad \text{and} \quad \frac{dz}{dt} = \sum_{i+j \geq 1} b_{ij}(t)u_{01}^{i-j}. \]

Substituting the above formula for \( \frac{dz}{dt} \) in (21) and recalling that \( v = e^{2\pi it} \) yields:

\[
2\pi i e^{2\pi it} \left( \sum_{i+j \geq 1} b_{ij}u_{01}^{i-j} \right) = -e^{2\pi it} + \sum_{k+l \geq 1} G_{kl}(v) \left( \sum_{i+j \geq 1} a_{ij}u_{01}^{i-j} \right)^{k} \left( \sum_{i+j \geq 1} b_{ij}u_{01}^{i-j} \right)^{l+1} \times \left( \sum_{i+j \geq 1} b'_{ij}(t)u_{01}^{i-j} \right).
\]

Comparing monomials in \( u_{01}z_{0} \) in Equation (23), we first note that the left side of this equation does not contain any monomial \( u_{01}^{i-j}z_{0}^{j} \), with \( i + j < n \), and non-zero coefficient. From this, it follows that \( b'_{01}(t) = b_{01}(t) = 0 \) so that they are constant functions of \( t \). In view of the initial conditions for \( t = 0 \), we conclude that \( b_{01}(t) = 0 \) while \( b_{01}(t) = 1 \), for every \( t \in \mathbb{R} \). The evident induction argument then shows that \( b_{i+j}(t) \) is constant equal to zero provided that \( 2 \leq i + j < n \). Finally, in the case of \( b_{n} \) we obtain the equation

\[
-e^{2\pi it}b'_{0n}(t) + \left[ \text{terms involving } b'_{ij} \text{ with } i + j \leq n - 1 \right] = 2\pi i e^{2\pi it}b'_{01}.
\]

Since, for all \( t \in \mathbb{R} \), we have that \( b_{01}(t) = 1 \) and \( b'_{ij}(t) = 0 \) provided that \( i + j \leq n - 1 \), we conclude that \( b'_{0n}(t) = -2\pi i \) and hence \( b_{01}(t) = -2\pi it \) since \( b_{01}(0) = 0 \). In particular,

\[
b_{01}(1) = -2\pi i
\]

so that \( \sigma \) does not coincide with the identity. The proof of the lemma is completed.

Let us close this section with the proof of Theorem B along with its corollaries.

Proof of Theorem B. Let \( X \) be a semicomplete vector field on \((\mathbb{C}^{3}, 0)\) and denote by \( \mathcal{F} \) its associated foliation. Assume that item (2) in the statement of Theorem B does not hold, i.e. that \( \mathcal{F} \) cannot be turned into a foliation all of whose singular points are elementary by means of blow-ups centered at singular sets. Thus owing to Theorem \( \[ \) there is a sequence of one-point blow-ups starting at the origin which leads to a transform \( \tilde{\mathcal{F}} \) of \( \mathcal{F} \) exhibiting a persistent nilpotent singular point \( p \) along with a formal separatrix \( S \) of \( \tilde{\mathcal{F}} \) at \( p \). The corresponding transform of \( X \) will be denoted by \( \tilde{X} \).

Next assume aiming at a contradiction that the linear part of \( X \) at the origin is equal to zero. Then the transform \( \Pi_{1}(X) \) of \( X \) under the first blow-up map vanishes identically over the exceptional divisor. Since the subsequent blow-ups will always be performed at singular points of the foliation which are contained in the zero-divisor of the vector fields in question, there follows that the zero-divisor of \( \tilde{X} \) is not empty on a neighborhood of \( p \). Therefore the linear part of \( X \) at \( p \) is equal to zero so that Theorem \( \[ \) implies that \( X \) is not semicomplete. The resulting contradiction proves Theorem B.

Remark 2. The preceding proof makes it clear why the first condition in Theorem \( \[ \) namely the fact that each weighted blow up is strictly invariant with respect to the quasi-homogeneous filtration in question, would be indispensable if Theorem B were to be proved by means of Theorem \( \[ \). If the singularities of the associated foliation were “reduced” by an arbitrary choice of weighted blow ups, it would not be clear whether or not the transform of \( X \) is holomorphic with a non-empty zero-divisor on a neighborhood of a persistent nilpotent singularity \( p \). This
issue does not appear in the above discussion since only standard blow-ups centered in the singular set of the corresponding foliations were used.

Corollary C is an immediate consequence of Theorem B while Corollary D requires additional explanation.

**Proof of Corollary D.** Strictly speaking, this statement is actually more of a by-product of the proof of Theorem B than a corollary of Theorem B. Consider a compact manifold \( M \) and a holomorphic vector field \( X \) defined on \( M \). Let \( \mathcal{F} \) denote the singular foliation associated with \( X \) and assume for a contradiction that \( \mathcal{F} \) possesses a singular point \( p \) which cannot be resolved by a sequence of blow-ups as in Theorem B. Since finite sequences of (one-point) blow-ups as in Theorem B change neither the compactness of \( M \) nor the holomorphic nature of \( X \), we can assume that \( X \) admits the normal form (16).

Consider then the curve of singular points of \( \mathcal{F} \) locally given by \( \{ y = z = 0 \} \). Since \( M \) is compact this curve of singular points \( \mathcal{C} \) is global and compact on \( M \). Furthermore, up to resolving its singular points (as curve), we can assume \( \mathcal{C} \) to be smooth. Thus \( \mathcal{C} \) can globally be blown-up with weight 2 as in Lemma 8. Again this weighted blow-up is viewed as a two-to-one map as opposed to a birational one. Yet, the resulting manifold \( \tilde{M} \) is still compact. Similarly the computations in Lemma 8 show that the transform \( \Pi^*X \) of \( X \) is still holomorphic. Hence \( \Pi^*X \) is complete on \( \tilde{M} \). Thus, the restriction of \( \Pi^*X \) to an open set \( U \subset \tilde{M} \) is a semicomplete vector field. A contradiction now arises from noting that it was seen in the proof of Theorem 4 that the vector field \( \Pi^*X \) is never semicomplete on a neighborhood of the codimension 2 saddle-node appearing in connection with the transform of \( S \). This ends the proof of Corollary D. \( \square \)

### 6. Examples and complements

The first part of this section is devoted to detailing a couple of examples respectively related to Theorem B and to Theorem 1. The remainder of the section will be devoted to a refinement of Theorem 4 which, of course, can also be used to make Theorem B slightly more accurate.

#### 6.1. A couple of examples

We will first provide an example of complete vector field possessing a persistent nilpotent singular point so as to substantiate the claim made in the Introduction. Then we will see some explicit example of persistent nilpotent singularity that cannot be reduced to the examples of Sancho and Sanz. The argument used here to show that the later example cannot be reduced to those of Sancho and Sanz is elementary and differs from [27].

- **Example:** The vector field

  \[
  Z = x^2 \partial/\partial x + xz \partial/\partial y + (y - xz) \partial/\partial z.
  \]

  Owing to the discussion in Sections 3 and 4, it is clear that the foliation \( \mathcal{F} \) associated with \( Z \) has a persistent nilpotent singular point at the origin which is associated with the convergent separatrix \( \{ y = z = 0 \} \). As a matter of fact, the separatrix giving rise to a sequence of infinitely near (nilpotent) singular points is convergent in this case and, hence, can be blown-up to resolve the singularity in question. Yet, the reader will note that our notion of persistent nilpotent singular point only takes into consideration blow-ups centered in the singular set of foliations so that the preceding observation is of relatively little importance for us.

  We will prove that \( Z \) can be extended as a complete vector field on a suitable open manifold \( M \). To begin with, note that the coordinate \( x(T) \) satisfies

  \[
  x(T) = \frac{x_0}{1 - Tx_0}
  \]
so that \( x(T) \) is defined for every \( T \neq 1/x_0 \). In turn we have \( \frac{d^2y}{dt^2} = zdx/dt + xdz/dt \)
so that the vector field \( Z \) yields
\[
\frac{d^2y}{dt^2} = xy = \frac{x_0}{1 - Tx_0} y
\]
which has a regular singular point at \( T = 1/x_0 \) and is non-singular otherwise. It then follows from the classical theory of Frobenius (see for example [22]) that \( y(T) \) is holomorphic and defined for all \( T \in \mathbb{C} \). Now the vector field \( Z \) also gives us
\[
\frac{dz}{dt} = y - xz = -\frac{x_0}{1 - Tx_0} z + y(T).
\]
Since \( y(T) \) is holomorphic on all of \( \mathbb{C} \), there follows that \( z(T) \) is holomorphic on all of \( \mathbb{C} \) as well.

Summarizing the preceding, the integral curve \( \phi(T) = (x(T), y(T), z(T)) \) of the vector field \( Z \) satisfying \( \phi(0) = (x_0, y_0, z_0) \) is defined for all \( T \in \mathbb{C} \setminus \{1/x_0\} \). Furthermore as \( T \to 1/x_0 \),
the coordinate \( x(T) \) goes off to infinity while \( y(T) \) and \( z(T) \) are holomorphic at \( T = 1/x_0 \). In particular, the vector field \( Z \) is semicomplete on all of \( \mathbb{C}^3 \).

To show that \( Z \) can be extended to a complete vector field on a suitable manifold \( M \) is slightly more involved. Denote by \( \mathcal{F} \) the foliation associated with \( Z \) on \( \mathbb{C}^3 \). Note that the plane \( \{x = 0\} \) is invariant by \( \mathcal{F} \) and that \( \mathcal{F} \) is transverse to the fibers of the projection \( \pi_1(x, y, z) = x \) away from \( \{x = 0\} \). The \( x \)-axis is also invariant by \( \mathcal{F} \) and \( \mathcal{F} \) can be seen as a linear system over the variable \( x \), namely we have \( dy/dx = z/x \) and \( dz/dx = y/x^2 - z/x \), cf. Chapter III of [20].

Let \( L \) be a leaf of \( \mathcal{F} \) which is not contained in \( \{x = 0\} \). The restriction of \( \pi_1 \) to \( L \) is a local diffeomorphism from \( L \) to the \( x \)-axis. In view of the previous discussion, this local diffeomorphism can, in fact, be used to lift paths contained in \( \{y = z = 0\} \setminus \{(0,0,0)\} \) similarly, owing to the description of \( \mathcal{F} \) as a linear system, the parallel transport along leaves of \( \mathcal{F} \) induces linear maps between the fibers of \( \pi_1 \) (isomorphic to \( \mathbb{C}^2 \)). Finally the holonomy (monodromy) arising from the invariant \( x \)-axis coincides with the identity (cf. Lemma [13]). Thus we have proved the following:

**Lemma 11.** Away from \( \{x = 0\} \), the leaves of \( \mathcal{F} \) are graphs over the punctured \( x \)-axis. In particular, the space of these leaves is naturally identified to \( \mathbb{C}^2 \) with coordinates \( (y, z) \). □

The restriction of \( Z \) to the invariant plane \( \{x = 0\} \) being clearly complete, to obtain an extension of \( Z \) as a complete vector field on a suitable open manifold \( M \) we proceed as follows. Fix a leaf \( L \) of \( \mathcal{F} \) with \( L \subset \mathbb{C}^3 \setminus \{x = 0\} \) and denote by \( Z_L \) the restriction of \( Z \) to \( L \). Consider the parameterization of \( L \) having the form \( x \mapsto (x, A(x), B(x)) \) where \( x \in \mathbb{C}^* \) and where \( A \) and \( B \) are holomorphic functions. In the coordinate \( x \), the one-dimensional vector field \( Z_L \) becomes \( x^2 \partial/\partial x \) and thus can be turned in a complete vector field by adding the “point at infinity” to \( L \) (i.e. \( \{u = 0\} \) in the coordinate \( u = 1/x \)). Therefore, to obtain the manifold \( M \), we simply add the “point at infinity” to every leaf \( L \) of \( \mathcal{F} \) (\( L \not\subset \{x = 0\} \)). The description of the leaves of \( \mathcal{F} \) as a linear system and the holomorphic behavior of the functions \( y(T), z(T) \) as \( T \to 1/x_0 \) makes it clear the resulting space can be equipped with the structure of a complex manifold \( M \). Moreover \( Z \) is naturally complete on \( M \) as desired.

**Example:** The (germ of) foliation \( \mathcal{F}_\lambda \) given by
\[
X_\lambda = (y - \lambda z)^2 \frac{\partial}{\partial x} + z x \frac{\partial}{\partial y} + z^3 \frac{\partial}{\partial z},
\]
with \( \lambda \in \mathbb{C} \).

As mentioned the first examples of persistent nilpotent singularities were supplied by Sancho and Sanz. It seems, however, interesting to provide an additional explicit example along with a
and such that the formal separatrix $S_H$ there exists a polynomial change of coordinates origin. If satisfies the conditions in Theorem 3 and thus has a persistent nilpotent singularity at the $y \mapsto \cdots F$ foliation Lemma 12.

(24)

Clearly $\varphi$ a strictly formal separatrix $S$ The leaves of the foliation associated with $\lambda$ The formal separatrix is, in fact, strictly formal if $\sum k \geq 0 a_k z^k$ and $y(z) = \sum k \geq 0 b_k z^k$. By substituting these expressions in the first equation of (24) and comparing both sides, we obtain

$$b_0 = 0, \quad b_1 = \lambda, \quad b_2 = 0, \quad \text{and} \quad b_{k+3} = (k+1) a_{k+1} \quad \text{for} \quad k \geq 0.$$ 

In turn, substitution and comparison in the second equation of (24) yields

$$a_0 = a_1 = 0 \quad \text{and} \quad a_{k+1} = k b_k \quad \text{for} \quad k \geq 1.$$ 

Therefore $b_0 = b_2 = 0, b_1 = \lambda,$ and

$$b_{k+3} = k(k+1) b_k$$

for $k \geq 0$. It then follows that the coefficients of $y(z)$ having the form $b_{3l}$ and $b_{3l+2}$ are zero for all every $l \geq 0$. Furthermore, for $l \geq 1$ we have

$$b_{3l+1} = \lambda \Pi_{j=1}^{l} \frac{(3j-1)!}{(3j-3)!}.$$ 

In the particular case where $\lambda = 0$, the series in question vanishes identically. This means that the curve, given in coordinates $(x, y, z)$ by $\{x = 0, y = 0\}$ is a convergent separatrix for the foliation $\mathcal{F}_0$. Thus we assume from now on that $\lambda \neq 0$. We want to check that the series $y = y(z) = \sum k \geq 0 b_k z^k$ diverges so as to ensure that $z \mapsto (x(z), y(z), z)$ constitutes a strictly formal separatrix for $\mathcal{F}_\lambda$. To do this, just note that the series of $y(z)$ can be reformulated as $z \sum k \geq 0 c_k z^{3k}$, where $c_0 = 0$ and $c_k = \lambda \Pi_{j=1}^{k} \frac{(3j-1)!}{(3j-3)!}$. Up to considering the new variable $w = z^3$, the radius of convergence of this later series is given by

$$\lim_{k \to \infty} \frac{c_k}{c_{k+1}} = \lim_{k \to \infty} \frac{1}{(3k-1)(3k-3)} = 0$$

and the lemma follows. □

Summarizing the preceding, the $z$-axis is invariant by $\mathcal{F}_\lambda$ if $\lambda = 0$. When $\lambda \neq 0$, $\mathcal{F}_\lambda$ admits a strictly formal separatrix $S_\lambda$ parameterized by a triplet of formal series

$$z \mapsto \varphi(z) = \left( \sum_{k \geq 1} a_k z^k, \sum_{k \geq 1} b_k z^k, z \right).$$

Clearly $\varphi'(z) \neq (0, 0, 0)$ so that $S_\lambda$ is formally smooth. In the case $\lambda = 0$, the foliation $\mathcal{F}_0$ satisfies the conditions in Theorem 3 and thus has a persistent nilpotent singularity at the origin. If $\lambda \neq 0$, we note that $S_\lambda$ is not tangent to the $z$-axis. However, arguing as in Lemma 7 there exists a polynomial change of coordinates $H$, of form $H(\tilde{x}, \tilde{y}, z) = (h_1(\tilde{x}, z), h_2(\tilde{y}, z), z)$, and such that the formal separatrix $S_\lambda$ becomes tangent (with arbitrarily large tangency order)
to the $z$-axis. This gives the foliation $\mathcal{F}_\lambda$ the normal form indicated in Theorem 3 and ensures that $\mathcal{F}_\lambda$ gives rise to a persistent nilpotent singularity with a strictly formal separatrix.

Regardless of whether or not $\lambda = 0$, the multiplicity $\text{mult}(\mathcal{F}_\lambda, S_\lambda)$ of $\mathcal{F}_\lambda$ along $S_\lambda$ is equal to 3. This contrasts with the examples of Sancho and Sanz where the corresponding multiplicity is always 2. Since the multiplicity along a formal separatrix is clearly invariant by (formal) change of coordinates, there follows that the singularities $\mathcal{F}_\lambda$ are not conjugate to the singularities of Sancho and Sanz. Furthermore, as shown in Sections 3 and 4, the value of $\text{mult}(\mathcal{F}_\lambda, S_\lambda)$ is invariant by blow-ups centered in the singular set of the corresponding foliations. Therefore the singularities of $\mathcal{F}_\lambda$ cannot give rise to a singularity in the family of Sancho and Sanz by means of any finite sequence of blow-ups as above.

6.2. Local holonomy and semicomplete persistent nilpotent singularities. To close this section, we turn our attention to semicomplete vector fields once again. Assume that $X$ is a vector field whose associated foliation $\mathcal{F}$ possesses a persistent nilpotent singularity at $(0,0,0) \in \mathbb{C}^3$. Assume also that $X$ is semicomplete. Owing to Theorem 4, the vector field $X$ has the normal form in Theorem 3 with $n = 2$. In fact, denoting by $S$ the formal separatrix of $\mathcal{F}$ giving rise to a sequence of infinitely near nilpotent singularities, we have $\text{mult}(X, S) = \text{mult}(\mathcal{F}, S) = 2$. These vector fields are thus very close to the examples of Sancho and Sanz.

This raises the problem of classifying semicomplete vector fields in the Sancho and Sanz family. In what follows we will conduct this classification only in the special case $\lambda = 0$, i.e. when the formal separatrix $S$ is actually convergent. Our purpose in doing so is to point out the role played by the holonomy of this separatrix which shares some ideas with the proof of Theorem 4 in the case $h(0,0,0) = 0$. Furthermore, by dealing only with the case of convergent separatrices, we avoid some technical difficulties that would require a longer discussion: whereas certainly interesting, this discussion is not really indispensable from the point of view of this paper. Finally, we also note that the material developed below includes Lemma 13 already used in the study of the vector field $Z = x^2 \frac{\partial}{\partial x} + xz \frac{\partial}{\partial y} + (y - xz) \frac{\partial}{\partial z}$.

We begin by recalling the context of the proof of Theorem 4. After performing the weight 2 blow-up, we have found an open path $c$ contained in a leaf of the blown-up foliation $\Pi^* \mathcal{F}$ over which the integral of the corresponding time-form is equal to zero. Whereas this implies that $\Pi^* X$ is not semicomplete, we were not able to conclude that $\Pi(c(0)) \neq \Pi(c(1))$ when $n = 2$ and $k = 0$. Thus, if $n = 2$ and $k = 0$ then the possibility of having $X$ semicomplete cannot be ruled out. Let us then consider this problem for the Sancho and Sanz family with $\lambda = 0$, i.e. for the family of vector fields having the form

$$X = x^2 \frac{\partial}{\partial x} + (xz - \alpha xy) \frac{\partial}{\partial y} + (y - \beta xz) \frac{\partial}{\partial z}.$$ 

We note once and for all that the case $\alpha = 0$ and $\beta = 1$ correspond to the previously discussed vector field $Z$.

Let $V$ be a neighborhood of the origin where $X$ is assumed to be semicomplete. Denote by $S$ the separatrix of $\mathcal{F}$ given by the invariant axis $\{y = z = 0\}$. Fix a local transverse section $\Sigma_r$ through a base point $(r,0,0) \in V$. Denote by $L_p$ the leaf of $\mathcal{F}$ passing through the point $(r,p)$ with $p \in \Sigma_r$ (with the evident identifications). If $p$ is close enough to $(0,0)$, then the closed path $c(t) = (re^{2\pi it}, 0, 0)$ can be lifted, with respect to the projection on the $x$-axis, into a path $c_p$ contained in $L_p$. Furthermore we have

$$\int_{c_p} dT_L = \int_c \frac{dx}{x^2} = 0,$$
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where \(dT_L\) stands for the time-form induced on \(L_p\) by \(X\). Thus, the vector field \(X\) cannot be semicomplete unless the holonomy map associated with \(F\) and \(S\) coincides with the identity. Next, we have:

**Lemma 13.** Assume that \(X\) and \(F\) are as above. Then the holonomy map associated with \(F\) and \(S\) coincides with the identity if and only if \(\alpha, \beta \in \mathbb{Z}\) with \(\alpha \neq \beta\).

**Proof.** With the preceding notation, let \(c_p(t) = (x(t), y(t), z(t))\) so that \(x(t) = re^{2\pi it}\). The functions \(y(t)\) and \(z(t)\) satisfy the following differential equations:

\[
\begin{align*}
\frac{dy}{dt} &= dx \frac{dx}{dt} = \frac{zx - \alpha xy}{x^2} 2\pi ix = 2\pi i(z - \alpha y) \\
\frac{dz}{dt} &= dx \frac{dx}{dt} = \frac{y - \beta xz}{x^2} 2\pi ix = 2\pi i(e^{-2\pi it}y - \beta z).
\end{align*}
\]

In terms of matrix representations, this system becomes

\[
\begin{bmatrix}
\dot{y} \\
\dot{z}
\end{bmatrix} =
\begin{bmatrix}
-2\pi i\alpha & 2\pi i \\
2\pi ie^{-2\pi it} & -2\pi i\beta
\end{bmatrix}
\begin{bmatrix}
y \\
z
\end{bmatrix}.
\]

The solution of this (non-autonomous) system can easily be obtained in terms of the coefficient matrix (denoted by \(A(t)\) in the sequel). In particular,

\[
\begin{bmatrix}
y(1) \\
z(1)
\end{bmatrix} = e^{\int_0^1 A(s)\,ds}
\begin{bmatrix}
y(0) \\
z(0)
\end{bmatrix},
\]

where

\[
\int_0^1 A(s)\,ds =
\begin{bmatrix}
-2\pi i\alpha & 2\pi i \\
0 & -2\pi i\beta
\end{bmatrix}.
\]

Hence the matrix \(B = \int_0^1 A(s)\,ds\) has two distinct eigenvalues if and only if \(\alpha \neq \beta\). When \(\alpha = \beta\), the matrix \(e^B\) has the form

\[
\begin{bmatrix}
e^{-2\pi i\alpha} & 2\pi ie^{-2\pi i\alpha} \\
0 & e^{-2\pi i\alpha}
\end{bmatrix}
\]

so that the holonomy map is given by \((y, z) \mapsto e^{-2\pi i\alpha}(y + 2\pi iz, z)\) and hence never coincides with the identity.

Suppose now that \(\alpha \neq \beta\). We then have \(B = PDP^{-1}\) where

\[
D =
\begin{bmatrix}
-2\pi i\alpha & 0 \\
0 & -2\pi i\beta
\end{bmatrix}
\quad \text{and} \quad
P =
\begin{bmatrix}
1 & 1 \\
0 & \alpha - \beta
\end{bmatrix}.
\]

Therefore

\[
e^B =
\begin{bmatrix}
e^{-2\pi i\alpha} & \alpha - \beta(e^{-2\pi i\beta} - e^{-2\pi i\alpha}) \\
0 & e^{-2\pi i\beta}
\end{bmatrix}.
\]

This matrix (and thus the holonomy) coincides with the identity if and only if \(\alpha, \beta \in \mathbb{Z}\). The lemma follows. \(\square\)

Lemma [13] ensures that \(X\) is not semi-complete if \(\alpha = \beta\) or if one of the two parameters \(\alpha\) or \(\beta\) is not an integer. The converse is provided by Lemma [14] below.

**Lemma 14.** The vector field \(X = x^2\partial/\partial x + (xz - \alpha xy)\partial/\partial y + (y - \beta xz)\partial/\partial z\) is semicomplete for every pair \(\alpha, \beta\) in \(\mathbb{Z}\) with \(\alpha \neq \beta\).
Proof. The argument is very much similar to the one employed for the vector field $Z$ ($\alpha = 0$ and $\beta = 1$). Consider an integral curve $(x(T), y(T), z(T))$ of $X$. Clearly $x(T) = x_0/(1-x_0T)$ which is a uniform function on $C \setminus \{1/x_0\}$ (here we use the word uniform as opposed to multi-valued). Thus we need to check that $y = y(T)$ and $z = z(T)$ are also uniform functions of $T$. This being clear for the integral curves contained in the invariant set $\{x = 0\}$, consider the remaining orbits of $X$. These remaining orbits, or rather the leaves of the associated foliation, can locally be parameterized by $x$, i.e. by a map of the form $x \mapsto (x, y(x), z(x))$. Since $x$ is a uniform function of $T$, becomes reduced to showing that $y(x)$ and $z(x)$ are uniform functions of $x$. To do this, note that $dy/dx$ and $dz/dx$ are solutions of the linear system

$$\begin{cases}
\frac{dy}{dx} = \frac{z}{x} - \frac{\alpha y}{x} \\
\frac{dz}{dx} = \frac{y}{x^2} - \frac{\beta z}{x}.
\end{cases}$$

This system has no singularities for $x \neq 0$. Furthermore the parallel transport along leaves gives rise to linear maps. In particular the holonomy map arising from moving around the point $\{x = 0\}$ is linear itself. This last map however is the identity thanks to Lemma 13. The functions $y(x)$ and $z(x)$ are thus uniform functions of $x \in C^*$ (for fuller details see Chapter III of [20]). The lemma is proved. $\blacksquare$

7. Valuations and the proofs of Proposition 4 and of Theorem A

Most of this section is taken by the proof of Proposition 4. As mentioned, this proof relies heavily on the work of Cano-Roche-Spivakovsky [10]. In turn, the proof of Theorem A will follow easily from our previous results combined with Piltant’s theorem [29].

Let $F$ denote a holomorphic foliation defined around $(0, 0, 0) \in \mathbb{C}^3$. The singular set of $F$ will be denoted by $\text{Sing}(F)$. Throughout this section, we only consider sequences of (standard) blow-ups

$$F = F_0 \xleftarrow{\Pi_1} F_1 \xleftarrow{\Pi_2} \cdots \xleftarrow{\Pi_k} F_k,$$

satisfying the following condition: the center of each blow-up map $\Pi_i$ is either a single point in $\text{Sing}(F)$ or a smooth analytic curve contained in $\text{Sing}(F)$.

From now on, we also assume that $F$ is as in Proposition 4. In other words, no sequence of blow-ups as in (25) leads to a foliation all of whose singular points are elementary.

We begin by making accurate a standard piece of terminology so as to avoid misunderstandings. Let $F$ be as above and denote by $\nu$ a valuation over $\mathbb{C}$. In many respects, the authors in [10] follow the Zariski approach to the resolution of singularities. A basic idea in Zariski’s point of view consists of trying to simplify the singularities of $F$ only at the center of $\nu$, as opposed to make all of these singularities simpler. To understand the meaning of the previous assertion, consider a blow up $\pi$ of the ambient manifold where $F$ and $\nu$ are defined. The valuation $\nu$ can be extended (pulled-back by $\pi$) to a valuation on the blown-up manifold. This new valuation - still denoted by $\nu$ - has its center naturally contained in the blown-up manifold where the blown-up foliation $F_1$ is also defined. As a first step towards (global) simplification of the singular points of $F$, we may only consider those singularities of $F_1$ lying in the center of the corresponding extension of $\nu$. As usually happens in the literature, in the sequel we will abuse notation and refer to the center of the extended valuation as the center of $\nu$. In other words, whenever a sequence of blow-ups is considered, the phrase the center of $\nu$ has to be understood as the center of the extended valuation (which will still be denoted by $\nu$) at each stage of the sequence of blow-ups in question.
In this sense, the so-called “local” uniformization (resolution) problem for foliations consists of finding a sequence of blow-ups as in (25) leading to a foliation $\mathcal{F}_k$ all of whose singular points lying in the center of $\nu$ are elementary.

Concerning the paper [10], the first issue that needs to be pointed out is their strategy to turn local results - in the above sense - into global ones. This strategy relies on Piltant’s patching theorem (see [29]) and its structure is summarized as follows.

Assume that for every given valuation $\nu$, the foliation $\mathcal{F}$ can be transformed by a sequence of blow-ups as in (25) into a new foliation $\mathcal{F}'$ whose singularities lying in the center of $\nu$ are all log-elementary (resp. elementary). Then $\mathcal{F}$ can also be turned into a foliation $\mathcal{F}''$ all of whose singular points are log-elementary (resp. elementary). This assertion is proved in Part III of [10]. The proof, in turn, amounts to checking that Piltant’s axioms [29] are satisfied in this setting. Whereas the authors of [10] focus on the case in which the singularities are log-elementary - so that they can deduce their Theorem 2 from the local uniformization statement provided by their Theorem 1 - the argument is insensitive to whether we deal with log-elementary or with elementary singular points.

Applying the preceding to a foliation $\mathcal{F}$ as in the statement of Proposition 4, there follows the existence of a valuation $\nu$ for which no sequence of blow-ups as in (25) yields a foliation having only elementary singularities in the center of $\nu$. Some simple additional assumptions can be made without loss of generality. These are formulated as Lemma 15 below.

**Lemma 15.** Assume that $\mathcal{F}$ is a foliation as in Proposition 4. Then there exists a valuation $\nu$ with residual field coinciding with $\mathbb{C}$ such that for every finite sequence of blow-ups as in (25) the following holds:

1. The center of (the corresponding extension of) $\nu$ is always a single point.
2. The center of $\nu$ is never an elementary singular point for $\mathcal{F}_k$.

Furthermore the rank of $\nu$ is either 1 or 2.

**Proof.** Let $\nu$ be chosen so that it is not possible to turn $\mathcal{F}$ into a foliation having elementary singularities in the center of $\nu$ by means of a sequence of blow-ups as in (25). The existence of $\nu$ is guaranteed by the previous discussions. Now, as explained in [10] (Section 9, Part II of [10]), there is no loss of generality in assuming that the residual field $k_\nu$ of $\nu$ coincides with $\mathbb{C}$. In fact, the remaining cases are essentially cases in which the ambient manifold is of dimension 2 which can be handled with minor modifications of Seidenberg’s theorem. In turn, since the residual field $k_\nu$ of $\nu$ coincides with $\mathbb{C}$, there also follows that the center of $\nu$ consists of a single point and this still holds for all extensions of $\nu$ obtained through blow-ups as above.

Finally, the fact that the rank of $\nu$ must be either 1 or 2 follows directly from Proposition 4 in [10]: the center of a valuation having rank 3 can be turned into an elementary singularity by means of blow-ups as above. The lemma is proved. 

From now on a valuation $\nu$ as in Lemma 15 is assumed to be fixed. Since $k_\nu = \mathbb{C}$, Theorem 3 in [10] ensures the existence of a formal power series $\hat{f}$ having transverse maximal contact with $\nu$. Recall that a formal power series $\hat{f}$ is said to have transverse maximal contact with $\nu$ if it is a Krull-limit of a sequence of (finite) power series $f_i$ at which $\nu$ takes strictly increasing values (see below for a more geometric interpretation of this condition). Naturally, standard desingularization of surfaces implies that the formal surface $\hat{W}$ is smooth at the center of $\nu$. However, a more accurate result in proven in [10]. Namely, up to finitely many blow-ups (some “preparation”), the formal series $\hat{f}$ admits one of the following normal
forms (where \((x, y, z)\) is a regular system of parameters)
\[
\hat{f} = z + \sum_{i,j} c_{i,j} x^i y^j \quad \text{or} \quad \hat{f} = z + \sum_i c_i x^i.
\]

The first case occurs when the rank of the valuation is \(2\) and, in this case, the variables \(x, y\) are such that \(v(x)\) and \(v(y)\) are \(\mathbb{Z}\)-linearly independent. If the rank of \(v\) is \(1\), then we have the second case where \(x\) is such that \(v(x) \neq 0\) and \(\hat{f}\) does not depend on the variable \(y\).

We can now make an important reduction in the statement of Proposition \(\text{[2]}\).

**Lemma 16.** Without loss of generality, we can assume that \(\hat{W}\) is (formally) invariant under \(\mathcal{F} = \mathcal{F}_0\).

To prove Lemma \(\text{[16]}\) let us begin by reminding the reader that the center of \(v\) (in the space where \(\mathcal{F}\) is defined) consists of a single point which can be assumed to coincide with the origin of some local coordinates. We also choose a holomorphic vector field
\[
X = F \partial/\partial x + G \partial/\partial y + H \partial/\partial z
\]
representing \(\mathcal{F}\) on a neighborhood of the center of \(v\) (identified with the origin).

Fixed the power series \(\hat{f}\), the basis of \(\hat{f}\) consists of those points \(q\) at which \(\hat{f}\) naturally defines a formal series. Clearly, the origin lies in the basis of \(\hat{f}\) but the basis of \(\hat{f}\) may or may not contain other points. For example, if \(\hat{f} = z + \sum c_i x^i\), then \(\hat{f}\) can be considered at every point belonging to the \(y\)-axis. This is, however, not necessarily true for \(\hat{f} = z + \sum c_{i,j} x^i y^j\).

Let us now consider the (formal) tangency locus \(\text{Tang}(\mathcal{F}, \hat{W})\) between \(\hat{W}\) and \(\mathcal{F} = \mathcal{F}_0\) based at the origin. The tangency locus \(\text{Tang}(\mathcal{F}, \hat{W})\) (based at the origin) refers to the formal equation
\[
d\hat{f}.X = \frac{\partial \hat{f}}{\partial x} F + \frac{\partial \hat{f}}{\partial y} G + \frac{\partial \hat{f}}{\partial z} H = 0.
\]

A formal curve \(t \mapsto (\gamma_1(t), \gamma_2(t), \gamma_3(t))\) based at the origin is said to be contained in \(\text{Tang}(\mathcal{F}, \hat{W})\) if it satisfies the formal equation \([25]\). By definition, the origin also belongs to \(\text{Tang}(\mathcal{F}, \hat{W})\) (recall that our definition of formal curve requires at least one of the \(\gamma_i\) not to vanish identically).

Similar considerations can be made at any point \(q\) in the basis of \(\hat{f}\) provided that the vector \(X(q)\) belongs to the formal tangent space to \(\hat{W}\) at \(q\). This leads to the notion of tangency locus between \(\hat{W}\) and \(\mathcal{F} = \mathcal{F}_0\) based at \(q\). In the sequel, whenever the basis point is clear from the context, we will simply say the tangency locus \(\text{Tang}(\mathcal{F}, \hat{W})\) between \(\hat{W}\) and \(\mathcal{F}\) without further comments.

Assume now that \(\hat{W}\) is not formally invariant by \(\mathcal{F}\). Since \(\mathcal{F}\) has a singular point at the origin, there follows the existence of a formal curve \(S\) contained in \(\text{Tang}(\mathcal{F}, \hat{W})\). Naturally the formal curve \(S\) must be viewed as given by a formal map \(t \mapsto (\gamma_1(t), \gamma_2(t), \gamma_3(t))\), with \(\gamma_1(0) = \gamma_2(0) = \gamma_3(0) = 0\) where at least one of the power series \(\gamma_1, \gamma_2, \gamma_3\) does not vanish identically. The proof of the existence of \(S\) is a straightforward computation which, in fact, is the same as the standard result for the case of an analytic surface: just note that monomials of degree, say \(d\), in the series of \(\gamma\) depend only on a finite part of the formal series of \(\hat{f}\).

Next recall the geometric interpretation of the fact that \(\hat{W} = \{\hat{f} = 0\}\) has transverse maximal contact with \(v\). For this, consider the formal series \(\hat{f}\) and a blow-up map \(\Pi\) obtained as a composition of finitely many blow-up maps as in \([25]\). The transform of \(\hat{f}\) under \(\Pi\) is nothing but the composition \(\hat{f} \circ \Pi\). Naturally this transform does not make sense as formal series at a generic point of the exceptional divisor associated with \(\Pi\). However, this transform
does make sense at the center of \( \nu \) provided that \( \hat{f} \) and \( \nu \) have transverse maximal contact. Indeed, by definition, \( \hat{f} \) is the Krull-limit of finite series \( f_i \) over which \( \nu \) takes strictly increasing values. The transforms of the series \( f_i \) are well defined (these series are finite) and they are still convergent for the Krull-topology at the center of (the extension of) \( \nu \). Their Krull-limit at the center of (the extension of) \( \nu \) then defines a formal series which naturally provides the transform of \( \hat{f} \) at the center of \( \nu \). This observation is a fundamental issue that allows us to consider the transform under \( \Pi \) of the formal surface \( \hat{W} \) as a formal surface “passing through the center of \( \nu \)”. It also encodes the geometric interpretation of the condition of having transverse maximal contact and is often abridged by saying that the “formal surface \( \hat{W} \) keeps passing through the center of \( \nu \)” for every sequence of blow-ups as in (25). This terminology will also be used in the remainder of our discussion.

Now fix a formal curve \( S \) as above which is contained in \( \text{Tang} (F, \hat{W}) \). Up to performing finitely many blow-ups, \( S \) can be assumed to be formally smooth. Assume that \( S \) is not contained in the singular set of \( F \). Then we claim that for every sequence of blow-ups as in (26), the transform of the curve \( S \) passes through the center of the corresponding extension of \( \nu \). To check our claim, we proceed as follows. First note that the transform of \( \hat{W} \) passes through the center of the extension of \( \nu \) due to the transverse maximal contact assumption. Since the transform of \( F \) is singular at the point in question, the preceding discussion ensures the existence of a (branch of) formal tangency curve \( \hat{S} \) between the transforms of \( F \) and of \( \hat{W} \) stemming from the center of (extension of) \( \nu \). Now note that \( S \) is contained in the center of a blow-up belonging to a reduction procedure as in (25) because these centers are contained in the singular set of the corresponding foliations, unlike the curve \( S \) (by assumption). Since a sequence of blow-ups, starting from \( p = p_0 \), as in (25) cannot produce new tangency points between the transforms of \( F \) and of \( \hat{W} \), we must conclude that \( \hat{S} \) is the transform of \( S \). In particular, the curve \( S \) satisfies the conditions in Proposition 4 unless \( S \) is fully constituted by singular points of \( F \).

Remark 3. It is convenient to point out a fact already implicit in the paragraph above. If \( \Pi \) is a blow-up map as in (26) which is not centered at \( S \), then the center of the corresponding extension of \( \nu \) is determined by the transform of the curve \( S \). Namely, the transform of \( S \) defines a single point in the exceptional divisor associated with \( \pi \) and this point is the center of the extended valuation.

Proof of Lemma 16. Owing to what precedes, we just need to consider the case in which \( S \) is smooth and entirely constituted by singular points of \( F \). Therefore \( S \) is actually an analytic curve contained in the singular set of \( F \) and hence can also be used as center for a blow-up map.

Up to performing finitely many (one-point) blow-ups, we can choose coordinates \( (x, y, z) \) around the center of \( \nu \), identified with \( (0, 0, 0) \), such that the following holds:

1. The exceptional divisor is locally given by \( \{ z = 0 \} \).
2. The curve \( S \) coincides with the \( z \)-axis.
3. In view of the normal forms (20), we can also assume that \( \hat{W} \) is given either by \( x = \sum_{i,j} c_{i,j} y^i z^j \) (with \( c_{i,0} = 0 \) for every \( i \)) or by \( x = \sum_i c_i y^i \).

Note that the curve \( \gamma \subset \{ z = 0 \} \) determined by \( \{ z = 0 \} \cap \hat{W} \) may or may not be contained in the tangency locus of \( \hat{W} \) and \( F \).

Claim 1. We can assume that \( \gamma \) is not contained in the tangency locus of \( \hat{W} \) and \( F \).
Proof of Claim 1. Note that the above described situation is invariant under (one-point) blow-ups at the center of \( \nu \) (in turn determined by the intersection of the transform of \( S \) with the exceptional divisor, cf. Remark \( \text{3} \)).

Now assume that \( \gamma \) is contained in the tangency locus of \( \tilde{W} \) and \( F \). In particular \( \gamma \) is invariant by \( F \) (here it is included the possibility of having \( \gamma \) contained in the singular set of \( F \)). Next, let \( X \) be a local vector field representing \( F \) on a neighborhood of \((0,0,0) \in \mathbb{C}^3 \) (identified with the center of \( \nu \)) and consider its first non-zero homogeneous component \( X_n \) of \( X \) at \((0,0,0) \). The tangent vector to \( \gamma \) at \((0,0,0) \) is clearly invariant by \( X_n \) since \( \gamma \) is invariant under \( F \). The same applies to the vector \((0,0,1) \) since \( F \) is singular all along the \( z \)-axis (identified with \( S \)). Therefore the plane spanned by \((0,0,1) \) and the tangent vector to \( \gamma \) at \((0,0,0) \) is invariant by \( X_n \). Naturally the plane in question is nothing but the tangent space to the surface \( \tilde{W} \), since \( \tilde{W} \) is smooth.

Summarizing what precedes, whenever \( \gamma \) is contained in the tangency locus of \( \tilde{W} \) and \( F \), there follows that the tangent space to \( \tilde{W} \) is invariant under the first non-zero homogeneous component of a vector field representing \( F \) around the center of \( \nu \). However, as previously pointed out, we can apply to this situation any sequence of one-point blow-ups at the center of \( \nu \). Since \( \tilde{W} \) is not invariant under \( F \), we can find a suitable sequence such that the tangent space to the transform of \( \tilde{W} \) is no longer invariant by the first non-zero homogeneous component of a local vector field representing the corresponding transform of \( F \). Therefore the corresponding curve \( \gamma \) will not be contained in the tangency locus of the corresponding transforms of \( F \) and \( \tilde{W} \). The claim is therefore proved.

We now go back to the initial local coordinates \((x,y,z) \). Owing to Claim 1, we can assume that the tangency locus of \( \tilde{W} \) and \( F \) on a neighborhood of \((0,0,0) \) is reduced to the above defined curve \( S \) (locally coinciding with the axis \( z \)). Recalling that \( S \) is smooth and contained in the singular set of \( F \), we will perform blow-ups centered at \( S \). First, we fix a holomorphic vector field \( X \) as in (\ref{27}) which represents \( F \) around the origin. Since \( S \) is contained in the singular set of \( F \), there follows that \( H \) vanishes identically over the \( z \)-axis. Furthermore, the component in the direction \( \partial / \partial z \) of the transform of \( X \) under a blow-up centered at \( S \approx \{x = y = 0\} \) is simply the transform of the function \( H \) under the blow-up in question. In other words, as long as this type of cylindrical blow-up is performed, the singular set \( \text{Sing} (F_1) \) of the resulting blown-up foliation \( F_1 \) is contained in the transform of the surface \( \{H = 0\} \).

Since blow-ups centered at \( S \) will be performed, we need to extend the content of Remark \( \text{3} \) to this type of blow-up. Indeed, if \( \Pi \) denotes the blow-up centered at \( S \), then the center of the extension of \( \nu \) is determined by the fact that it lies in the intersection of \( \Pi^{-1}(0,0,0) \) with the transform of \( \tilde{W} \). More precisely the formal curve \( \gamma \) obtained by intersecting \( \tilde{W} \) and the plane \( \{z = 0\} \) determines a point \( p_1 \) in \( \Pi^{-1}(S) \cap \{z = 0\} \) (where, by abusing notation, \( \{z = 0\} \) stands for both the initial plane \( \{z = 0\} \) and its transform under \( \Pi \)). This point \( p_1 \) is the (new) center of \( \nu \).

At \( p_1 \), let \( S_1 \) denote the tangency locus between the blown-up foliation \( F_1 \) and the (transform of) \( \tilde{W} \). As previously seen, this tangency locus is well defined. Furthermore this tangency locus must contain a formal curve \( S_1 \) since \( p_1 \) has to be a singular point of \( F_1 \). In the present situation, \( S_1 \) is clearly smooth, contained in the exceptional divisor \( \Pi^{-1}(S) \), and transverse to \( \{z = 0\} \). If \( S_1 \) is not contained in \( \text{Sing} (F_1) \), then \( S_1 \) is a formal separatrix satisfying the conditions of Proposition \( \text{4} \) and thus there is nothing else to be proved. Therefore \( S_1 \) can be assumed to be contained \( \text{Sing} (F_1) \). As previously seen, this implies, in particular, that \( S_1 \) is contained in the transform of the surface \( \{H = 0\} \).
The procedure above can now be repeated with center $S_1$ (which is an analytic curve contained in $\text{Sing}(F_1)$). We have:

Claim 2. If $\{H = 0\}$ is not invariant, then we can assume that $S_1$ is not contained in $\text{Sing}(F_1)$.

Proof of Claim 2. For every generic point $z_0 \in S$, the intersection of $\Pi^{-1}_1(z_0)$ and the transform of $\{H = 0\}$ consists of a bounded number of points. Indeed, the context is essentially equivalent to the 2-dimensional one: in particular Seidenberg theorem would lead to elementary singular points sitting over generic points of the z-axis. Thus, by iterating sufficiently many times this type of blow-ups, the generic point of the intersection between the transform of $\{H = 0\}$ and the (cylindrical) exceptional divisor will have to be regular for the corresponding blown-up foliation $F_k$ (since $\{H = 0\}$ is not invariant by $F$). At this point, the tangency locus of $F_k$ and the corresponding transform of $\hat{W}$ yields the desired separatrix proving Proposition 4. This establishes Claim 2.

Thanks to Claim 2, we can assume that the surface $\{H = 0\}$ is invariant by $F$. The rest of the proof consists of proving that $\{H = 0\}$ must have transverse maximal contact with $\nu$, so that it will suffice to deal with invariant surfaces having transverse maximal contact with $\nu$. This is, however, clear by now. In fact, for all blow-ups centered in the curves $S_i$ (as above), $\{H = 0\}$ will have to pass through the center of the valuation. The other two possible types of blow-ups are as follows:

- One-point blow-ups of the center of $\nu$. Again, the new center of $\nu$ will be determined by the transform of $S$ (locally given by the z-axis). As seen in Sections 2 and 3, in the corresponding coordinates $(u, v, z) \mapsto (uz, vz, z)$, the component of $X$ in the direction $\partial/\partial z$ transforms like the function $H$. Thus we obtain the desired separatrix proving Proposition 4 unless $\{H = 0\}$ continues to pass through the center of the valuation.

- Blow-ups centered at smooth curves (contained in $\{z = 0\}$) and passing through the origin (identified with the center of $\nu$). Once again, in suitable coordinates, the component of $X$ in the direction $\partial/\partial z$ transforms like the function $H$.

Summarizing what precedes, we obtain the desired separatrix unless the (analytic) invariant surface given by $\{H = 0\}$ passes through the center of the valuation for every sequence of blow-up maps as in (25). This, however, implies that $\{H = 0\}$ has transverse maximal contact with $\nu$ and completes the proof of the lemma.

The remainder of this appendix is devoted to proving Proposition 4 in the case where the formal surface $\bar{W}$ with transverse maximal contact with $\nu$ is, in addition, invariant under $F$. This case is very close to the 2-dimensional situation considered in Seidenberg’s theorem, as it will be detailed in what follows.

We consider local coordinates around the center of $\nu$ (identified with $(0, 0, 0) \in \mathbb{C}^3$) along with the formal surface $\bar{W} = \{\hat{f} = 0\}$. Since the formal surface $\bar{W}$ is smooth, there is a formal change of coordinates in which $\bar{W}$ becomes identified with a coordinate plane. Naturally, in these formal coordinates, the foliation becomes only formal. In other words, the vector field $X$ of (27) representing $F$ around the origin (i.e. the center of $\nu$) becomes only formal. This, however, is a minor issue since for resolution problems there is essentially no difference between working with a formal vector field or with an actual holomorphic one.

We can then consider local coordinates $(x, y, z)$ where the surface $\bar{W}$ coincides with the plane $\{z = 0\}$ at the expenses of consider $X$ as a formal vector field. In particular, in the coordinates $(x, y, z)$ the (formal) vector field $X$ takes on the form (27), where $F$, $G$, and $H$ are formal series with $H$ being divisible by $z$. 


Remark 4. The reader will note that the choice of formal coordinates \((x, y, z)\) where the surface \(\hat{W}\) becomes identified with the plane \(\{z = 0\}\) is basically a convenient way to abridge notation. Indeed, we can directly work with the initial coordinates and with the formal generator \(\hat{f}\) of the surface \(\hat{W}\) but this would make the notation slightly cumbersome.

The use of formal coordinates as above actually helps to make the argument more transparent since, in most of the discussion, there is no difference between dealing with formal or holomorphic vector fields. Along this direction, we will occasionally allow ourselves to argue as if \(X\) is a holomorphic vector field: this will only be done, however, when the general procedure is straightforward enough to avoid confusion.

Recalling that Seidenberg’s theorem applies equally well to formal vector fields, we can consider the case of the restriction of \(X\) to the invariant plane \(\{z = 0\}\). Applying Seidenberg’s theorem in the present context, however, requires us to distinguish between the restriction of a 3-dimensional foliation to an invariant plane and the foliation on the invariant plane induced by the restriction of the mentioned foliation. In other words, in dimension 2, singularities are always isolated: if the coordinate functions of a vector field have a common factor, then this factor can be eliminated as far as the underlying foliation is concerned. This is no longer true if we are looking at vector fields defined on a 3-dimensional ambient. In more accurate terms, if \(X\) is as in (27), the functions \(F(x, y, 0)\) and \(G(x, y, 0)\) may have a non-trivial common factor that does not divide, for example, \(H\). This gives rise to a curve of singular points of \(X\) contained in the plane \(\{z = 0\}\) which, indeed, constitutes a curve of singularities for the foliation in dimension 3. However, if we look at the foliation induced by restriction of the previous one to the plane \(\{z = 0\}\), then the resulting 2-dimensional foliation can be extended as a regular foliation to all but finitely many points in the curve in question. Lemma 17 below makes these comments accurate.

Lemma 17. Without loss of generality we can assume that the (formal) vector field \(X\) representing the foliation on a neighborhood of \((0,0,0)\) (identified with the center of \(\nu\)) has the following form:

\[
X = x^n y^m (f(x,y)\partial/\partial x + g(x,y)\partial/\partial y) + z(r(x,y,z)\partial/\partial x + s(x,y,z)\partial/\partial y) + zh\partial/\partial z,
\]

where \(h\) is a formal series in \(x, y,\) and \(z\). Furthermore, the following holds:

- The vector field \(Y = f(x,y)\partial/\partial x + g(x,y)\partial/\partial y\) - viewed as a 2-dimensional vector field on the plane \(\{z = 0\}\) - either is regular or has an (isolated) elementary singular point at \((0,0,0)\).
- Both \(m\) and \(n\) are nonnegative integers. In addition, if \(m > 0\) (resp. \(n > 0\)), then the axis \(\{x = 0\}\) (resp. \(\{y = 0\}\)) is invariant by \(Y\).

\[\square\]

Considering the normal form (29), it is clear that at least one between \(m\) and \(n\) must be strictly positive otherwise the origin is an elementary singular point of \(X\). Similarly, \(h(0,0,0)\) must be equal to zero otherwise \(X\) has a non-zero eigenvalue in the direction \(\partial/\partial z\).

Next there is no loss of generality in assuming that the functions \(r\) and \(s\) are divisible by \(x^n y^m\). Indeed, let \(\Pi\) denote the blow-up centered at the \(x\)-axis and consider coordinates \((x, y, v)\) where \(\Pi\) becomes \(\Pi(x, y, v) = (x, y, yv)\). In this case, the transform of \(\{z = 0\}\) coincides with the plane \(\{v = 0\}\) while the transform of \(X\) becomes

\[
\Pi^* X = x^n y^m Y + yv [r(x,y,yv)\partial/\partial x + s(x,y,yv)\partial/\partial y] + v [-x^n y^{m-1}g(x,y) - vs(x,y,yv) + h(x,y,yv)] \partial/\partial v.
\]
Thus the “new” functions $r$ and $s$ have, in particular, acquired a factor of $y$. Hence, by iterating blow-ups as above centered either at the $x$-axis or at the $y$-axis the claim follows.

Formula (33) also yields:

**Lemma 18.** Without loss of generality, the function $h$ admits the decomposition $h = h(0)(x, y) + zh^{(z)}(x, y, z)$ where $h^{(z)}$ is divisible by $x^ny^m$.

**Proof.** Again it follows from formula (33) that every factor of $z$ in $h$ acquires a factor of $y$. Similarly, every new function “$s$” has an additional factor of $y$ (as previously seen). The lemma then follows by repeating the indicated procedure.

**Remark 5.** Clearly the preceding shows that $r$, $s$, and $h^{(z)}$ can be assumed to be divisible by $x^ny^m$, for every a priori given $a, b \in \mathbb{N}$.

Whereas it will not really be necessary in what follows, we may also note that the function $(x, y) \mapsto x^ny^m g(x, y)$ may be assumed to be divisible by $x^ny^m$. Indeed, if $(0, 0)$ is a regular point for $Y$ then we can assume that $g$ vanishes identically. Otherwise $(0, 0)$ is an elementary (irreducible) singularity of $Y$ and, since we are working with formal vector fields, there is no loss of generality in assuming that $g$ is divisible by $y$ in this case.

In the sequel we can also assume that $h$, and hence $h(0)$, is not divisible by either $x$ or $y$, otherwise we can reduce at least one between $m$ and $n$. Now let $P$ denote the (homogeneous) polynomial obtained as the first non-zero homogeneous component of the formal series of $h(0)$.

The degree of $P$ will be denoted by $d \geq 1$. Therefore the polynomial $P$ has the form

$$P = \sum_{i=0}^{d} c_i x^i y^{d-i}.$$ 

Hence the set $\{P = 0\}$ consists of $k$ straight lines $C_1, \ldots, C_k$ through $(0, 0) \in \mathbb{C}^2$, with $k \leq d$. In fact, if we add multiplicity to each one of the lines $C_j$, then we will have $k = d$. The union $C_1 \cup \ldots \cup C_k$ of the mentioned lines naturally forms the tangent cone to the set $\{h(0) = 0\}$ viewed as contained in the plane $\{z = 0\}$ (otherwise the cone in question is simply the “cylinder” over $C_1 \cup \ldots \cup C_k$). Clearly the set $\{h(0) = 0\} \cap \{z = 0\}$ is just a finite number of irreducible (possibly singular) analytic curves.

**Proof of Proposition 4.** We keep the preceding notation. Since $r$, $s$, and $h^{(z)}$ are divisible by $x^ny^m$ and since $d \geq 1$, there follows that the $z$-axis is contained in the singular set of $F$. This axis can thus be used as center for a blow-up.

Here it is convenient to point out a simple issue concerning a blow-up $\Pi$ centered at the $z$-axis and the corresponding center of $\nu$. In contrast with the previous cases, the center of (the extension of) $\nu$ is not immediately detected in the present situation. Clearly, the new center of $\nu$ is contained in the rational curve $\Pi^{-1}(0, 0, 0)$ but this curve is entirely contained in the transform of $\tilde{W} \simeq \{z = 0\}$ so that, a priori, any point in $\Pi^{-1}(0, 0, 0)$ can be the center of $\nu$.

In the sequel, by abuse notation, $\{z = 0\}$ will denote both the initial coordinate plane and its transform under $\Pi$.

Recall that $X$ as in (24) represents $F$ around $(0, 0, 0)$. In turn, the transform of $x^ny^m [(f + zr)\partial/\partial x + (g + zs)\partial/\partial y]$ by $\Pi$ vanishes over the exceptional divisor to the order $m+n$. Similarly the transform under $\Pi$ of $h(0)(x, y)\partial/\partial z$ (resp. $zh^{(z)}(x, y, z)\partial/\partial z$) vanishes over the exceptional divisor to the order $d$ (resp. $m+n$ but actually arbitrarily larger if it were necessary). Hence, the transform of $X$ under $\Pi$ vanishes over the exceptional divisor with order $\min\{d, m+n\}$.

To make the subsequent discussion clearer, it is convenient to first consider two cases:

**Case 1.** Assume that $d > m+n$. 

Moreover the cone tangent to \( h \) with the exceptional divisor, we see that the transform \( \mathcal{F}_1 \) of the foliation \( \mathcal{F} \) is regular at \( \Pi^{-1}(0,0,0) \cap \{z = 0\} \) except at the two points of \( \Pi^{-1}(0,0,0) \cap \{z = 0\} \) determined by the axes \( x \) and \( y \) (both singular for \( \mathcal{F} \)). In particular, the center of \( \nu \) has to be one of these two points. Furthermore, since this center has not become an elementary singular point, the procedure can be continued up to applying again the construction used in Lemma [13] to make sure that the corresponding (new) functions \( r, s \), and \( h^{(z)} \) are as indicated.

When continuing this procedure, note that the degree of the new polynomial “\( P \)” will be strictly smaller than \( d \) provided that \( k \geq 2 \). We will return to this point in the more general discussion below.

\textbf{Case 2.} Assume that \( d \leq m + n \). This is the more interesting case. The foliation \( \mathcal{F}_1 \) is represented by a vector field \( X_1 \) whose component in the direction \( \partial / \partial z \) has the form

\[ z[\tilde{P} + z\tilde{h}^{(z)}] \partial / \partial z \]

where \( \tilde{P} \) (resp. \( \tilde{h}^{(z)} \)) is the transform of \( P \) (resp. \( h^{(z)} \)) after eliminating the above mentioned common factor arising from the exceptional divisor. In the case of \( \tilde{P} \), there follows in particular that \( \tilde{P} \) - viewed as a polynomial of variables \( x, y \) on the plane \( \{z = 0\} \) - vanishes exactly of the transforms of the lines \( C_1, \ldots, C_k \). In particular, the center of \( \nu \) must be coincide with one of the points in \( \Pi^{-1}(0,0,0) \cap \{z = 0\} \) determined by the lines in question.

Up to applying the technique in the proof of Lemma [13] at every stage, a sequence of blow-ups as above can be performed. The outcome of this sequence of blow-ups, which also summarizes the preceding two cases, is the following claim:

\textbf{Claim.} Up to performing the indicated blow-up \( \Pi \) and considering the corresponding transform \( (\mathcal{F}_1) \) of \( \mathcal{F} \) under \( \Pi \), we can assume without loss of generality that the vector field \( X \) representing \( \mathcal{F} \) as in (29) satisfies one of the following conditions:

- if \( n \) and \( m \) are strictly positive. Then \( k = 1 \) (and \( h, h^{(0)} \) are not divisible by either \( x \) or \( y \)).
- if \( n = 0 \), then \( m > 0 \) and \( k \in \{1, 2\} \). However, if \( k = 2 \), then \( h \) is divisible by \( x \) which is therefore naturally associated with one of the lines \( C_1, C_2 \). Finally, again \( h, h^{(0)} \) are not divisible by \( y \).

The above considered blow-up procedure actually leads to a slightly more accurate situation. Recalling that \( h \) does not vanish over the corresponding transforms of the (initial) invariant axes, we will arrive to one of the following two situations:

(1) the tangent cone of \( h^{(0)} \) in the plane \( \{z = 0\} \) does not pass through the points in \( \Pi^{-1}(0,0,0) \cap \{z = 0\} \) determined by the invariant axes of the vector field \( Y \).

(2) Consider the points in \( \Pi^{-1}(0,0,0) \cap \{z = 0\} \) which are determined by the invariant axes of the vector field \( Y \). Then the only component of the tangent cone of \( h^{(0)} \) passing through these points coincides with the curve \( \Pi^{-1}(0,0,0) \cap \{z = 0\} \).

In the first situation, \( \mathcal{F} \) has elementary singular points in \( \Pi^{-1}(0,0,0) \cap \{z = 0\} \) except at the point \( p \) determined by the intersection of \( \Pi^{-1}(0,0,0) \cap \{z = 0\} \). In this case the point \( p \) is regular for the transform of the vector field \( Y \). Thus, there are local coordinates (still denoted by \( (x,y,z) \)) around \( p \), with \( \{y = z = 0\} \subset \Pi^{-1}(0,0,0) \cap \{z = 0\} \) where \( X \) becomes:

\textbf{Case (a)}

\[ X = y^m \partial / \partial x + z(r \partial / \partial x + s \partial / \partial y) + z(h^{(0)} + zh^{(z)}) \partial / \partial z. \]

Moreover the cone tangent to \( h \) (or to \( h^{(0)} \)) coincides with the \( y \)-axis.
In the second situation, we can eliminate the curve $\Pi^{-1}(0,0,0) \cap \{z = 0\}$ from the singular set of $x^ny^mY$. Thus, $X$ becomes:

**Case (b)**

\[
X = x^n[(f + zr)\partial/\partial x + (g + zs)\partial/\partial y] + z(h^{(0)} + zh^{(z)})\partial/\partial z.
\]

Moreover $h^{(0)}$ vanishes only at $\Pi^{-1}(0,0,0) \cap \{z = 0\}$, i.e. $h = y'h_1$ where $h_1(0,0,0) \neq 0$ and $l \geq 1$.

The remainder of the proof amounts to showing how to handle each case.

Assume first that **Case (a)** happens. We consider a blow-up $\Pi$ centered at the $z$-axis which, after the discussion revolving around Lemma 18, is constituted by singular points of $X$. In particular, the function $h$ viewed as the component of $X$ in the direction $\partial/\partial z$ is transformed as function. Hence the blown-up vector field $X_1$ will therefore vanish identically over the cylindrical exceptional divisor. Indeed, the blow-up of $y^m\partial/\partial x + z(r\partial/\partial x + s\partial/\partial y)$ will vanish with order $m - 1$ over the exceptional divisor whereas the zero-set of the transform of $z(h^{(0)} + zh^{(z)})\partial/\partial z$ will consist of the union of the exceptional divisor with the strict transform of $h = 0$. Letting $k$ denote the minimum of the vanishing orders of these two vector fields over the exceptional divisor, the blown-up foliation $\mathcal{F}_1$ is induced by the vector field $X_1$ divided by the $k^{th}$-power of the generator of the ideal associated with the exceptional divisor.

Now, we must have $k < m - 1$ (strictly) since, otherwise, all singularities of $\mathcal{F}_1$ lying in $\Pi^{-1}(0,0,0) \cap \{z = 0\}$ will have a non-zero eigenvalue associated with directions contained in $\{z = 0\}$.

Thus $k < m - 1$, and after dividing $X_1$ by the $k^{th}$-power of the generator of the exceptional divisor, we see that the component of the resulting vector field in the direction $\partial/\partial z$ vanishes only on the strict transform of $\{h = 0\}$. Thus every point in $\Pi^{-1}(0,0,0) \cap \{z = 0\}$ is either regular or elementary for the foliation $\mathcal{F}_1$ except the point determined by the strict transform of $h$. Denoting by $p_1$ the point of $\Pi^{-1}(0,0,0) \cap \{z = 0\}$ in question, we first note that $p_1$ must coincide with the (new) center of $\nu$. Moreover, the structure of the blown-up foliation $\mathcal{F}_1$ around $p_1$ is again as in **Case (a)**. The integer $m$, however, has decreased strictly. Therefore, after finitely many blow-ups as above, the center of $\nu$ will be turned either in a regular point or in an elementary singularity for the corresponding foliation. In any event, this gives a contradiction showing that this situation cannot happen.

Finally, assume now that **Case (b)** happens. Recall that $h = y'h_1$ with $h_1(0,0,0) \neq 0$. Consider again the blow-up $\Pi$ centered at the $z$-axis and note that we will be able to divide the transform of $X$ by the generator of the exceptional divisor to the power $k = \min\{m,l\}$.

Assume that $m \geq l$. Then the blown-up foliation $\mathcal{F}_1$ will have elementary singular points (with a non-zero eigenvalue in the direction $\partial/\partial z$) over the entire curve $\Pi^{-1}(0,0,0) \cap \{z = 0\}$ except at the point determined by the $x$-axis ($\{y = 0\}$). Around this point, we have again the situation described in **Case (b)** except that the value of $m$ is replaced by $(m-l)$ and, therefore, has reduced strictly.

Conversely, if $l > m$, then $h$ will still vanish over the curve $\Pi^{-1}(0,0,0) \cap \{z = 0\}$ (and, in fact, over the cylindrical exceptional divisor). However, the transform of the vector field $(f + zr)\partial/\partial x + (g + zs)\partial/\partial y$ will provide us with regular or elementary singular points over $\Pi^{-1}(0,0,0) \cap \{z = 0\}$, except at the point $q$ determined by the $y$-axis ($\{x = 0\}$). Note that, around $q$, the tangent cone of $\{h = 0\}$ is reduced to $\Pi^{-1}(0,0,0) \cap \{z = 0\}$ since the transform of its initial component given by $\{y = 0\}$ intersects $\Pi^{-1}(0,0,0) \cap \{z = 0\}$ at a point different from $q$. Thus, we have again the situation described in **Case (b)** except that, now, the value of $l$ is replaced by $(l-m)$. 
Hence, at every blow-up as above, at least one between $m$ and $l$ becomes strictly smaller. Once one of them becomes zero, we obtain an elementary singular point at the center of $\nu$ which of course is impossible. The proof Proposition 4 is now completed.

The reader will note that the proof of Proposition 4 also shows that the formal surface $\tilde{W}$ with transverse maximal contact with $\nu$ cannot be invariant by $\mathcal{F}$. Indeed, the preceding discussion shows that singularities in the center of $\nu$ - a valuation with transverse maximal contact with $\tilde{W}$ - can be turned into elementary ones provided that $\tilde{W}$ is invariant. This observation can be reformulated as follows:

**Corollary 1.** Let $\mathcal{F}$ be as in Proposition 4 and consider a formal separatrix $S$ giving rise to a sequence of infinitely near non-elementary singular points. Then $S$ cannot be contained in a formal surface invariant under $\mathcal{F}$.

**Proof.** Clearly a formal surface containing $S$ will always pass through the center of (extended) valuation associated with $S$ so that the statement follows from the previous discussion.

Note that Corollary 4 is hardly surprising since it is, in fact, very much in line with the main result of [9].

Now we close this paper with the proof of Theorem A.

**Proof of Theorem A.** Let $\mathcal{F}$ be a foliation on $(\mathbb{C}^3, 0)$ and fix a valuation $\nu$. Assume that no sequence of blow ups as in (25) transforms $\mathcal{F}$ in a foliation whose singular points contained in the center of (the extension of) $\nu$ are all elementary. Then $\nu$ can be assumed to satisfy the conditions of Lemma 15. In turn, there follows from Proposition 4 and from Theorem 1 that $\mathcal{F}$ can be turned in a foliation possessing a persistent nilpotent singularity at the center of $\nu$. Thus we have improved the Local uniformization theorem (Theorem 1) of [10] to the following statement: the foliation $\mathcal{F}$ can be transformed in a foliation whose singularities at the center of $\nu$ either are elementary or are persistent nilpotent singular points.

Next the blow up procedure we have used all along our construction clearly satisfies the same “naturality” conditions satisfied by the procedure in [10]. Hence this procedure verifies Piltant’s axioms in [20], cf. pages 256-257 of [10]. Thus we obtain the following global result: **Claim:** Every foliation $\mathcal{F}$ on $(\mathbb{C}^3, 0)$ can be transformed by a sequence of blow ups as in (25) into a foliation $\mathcal{F}_k$ whose singular points either are elementary or are persistent nilpotent singularities.

To complete the proof of Theorem A we proceed as follows. Note that the set formed by the persistent nilpotent singular points of $\mathcal{F}_k$ consists of isolated points thanks to Theorem 3. Up to reducing the neighborhood of $(0,0,0) \in \mathbb{C}^3$ under consideration, this set is therefore finite. Finally each of these (finitely many) singular points can be turned into an elementary singular point by means of a blow up with weight 2, cf. Lemma 8. Theorem A is proved.

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