Inverse scattering transformation for the Sasa-Satsuma equation with non-vanishing boundary conditions

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Abstract

We concentrate on inverse scattering transformation for the Sasa-Satsuma equation with $3 \times 3$ matrix spectral and nonzero boundary condition in this article. To circumvent multi valuedness of eigenvalues, we introduce a suitable two-sheet Riemann surface to map the original spectral parameter $k$ into a single-valued parameter $z$. The analyticity of the Jost eigenfunctions and scattering coefficients of Lax pair for the SS equation are analyzed in details. According to the analyticity of eigenfunctions and scattering coefficients, the $z$-complex plane is divided into four analytic regions $D_j$, $j = 1, 2, 3, 4$. Since the second column of Jost eigenfunctions is analytic in $D_j$, $j = 1, 2, 3, 4$, but in upper-half or lower-half plane, we introduce certain auxiliary eigenfunctions which are necessary for deriving the analytic eigenfunctions in $D_j$. We find that for the eigenfunctions, scattering coefficients and the auxiliary eigenfunctions all possess three kinds of symmetries, which characterize the distribution of discrete spectrum. The asymptotic behaviors of eigenfunctions, auxiliary eigenfunctions and scattering coefficients are also systematically derived. Then a matrix Riemann-Hilbert problem with four kind jump conditions associated with the problem of nonzero asymptotic boundary conditions is established, from which $N$-soliton solutions is obtained via the corre-

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sponding reconstruction formulae. The reflectionless soliton solutions are explicitly given. As application of the \( N \)-soliton formula, we present three kinds of single-soliton solutions according to the distribution of discrete spectrum.

**Keywords:** the Sasa-Satsuma equation; nonzero boundary condition; auxiliary eigenfunctions; Riemann-Hilbert problem; soliton solution.

**AMS subject classifications.** 35Q58, 35Q51, 35Q15

1 Introduction

The nonlinear Schrödinger equation

\[
iq_t + \frac{1}{2} q_{xx} + |q|^2 q = 0
\]  

is one of the most important integrable equations appearing in various physical systems such as plasma physics, solid-state physics nonlinear optics and so on. The initial value problem of the NLS equation (1.1) was solved by the inverse scattering transformation (IST) method [1]. Hasegawa and Tappert found the possibility of soliton propagation in optical fibers and showed the stability by numerical computations [2]. In 1980, Mollenauer, Stolen and Gordon observing the soliton propagation experimentally [3]. However, by the advancement of experiment accuracy, several phenomena which can not be explained by the classical equation (1.1) have been observed in experimentally works. In order to explain these phenomena, Kodama and Hasegawa proposed a higher-order nonlinear Schrödinger equation [4, 5]

\[
iq_t + \frac{1}{2} q_{xx} + |q|^2 q + i\varepsilon \{\beta_1 q_{xxx} + \beta_2 |q|^2 q_x + \beta_3 q(|q|^2)_x\} = 0, \quad \beta_j \in \mathbb{R}, \quad (1.2)
\]

which is completely integrable for special parameters \( \beta_1, \beta_2 \) and \( \beta_3 \) [6-9]. For the choice \( \beta_1 : \beta_2 : \beta_3 = 1 : 6 : 3 \), the equation (1.2) reduced to [10]

\[
iq_t + \frac{1}{2} q_{xx} + |q|^2 q + i\varepsilon \{q_{xxx} + 6|q|^2 q_x + 3q(|q|^2)_x\} = 0. \quad (1.3)
\]

If making a transformation [10]

\[
q(x, t) \to q(x, t)e^{\frac{i\varepsilon}{12\varepsilon} (x - \frac{t}{12\varepsilon})}, \quad t \to t, \quad x \to x + \frac{t}{12\varepsilon}
\]
the equation (1.3) reduce to the well-known Sasa-Satsuma (SS) equation

\[ q_t + \varepsilon \{ q_{xxx} + 6 |q|^2 q_x + 3 q (|q|^2)_x \} = 0, \]  

(1.5)

which can be used to describe the nonlinear phenomena in many situations, such as pulse propagation in optical fibers and deep ocean waves [11–15]. It was shown that the SS equation (1.5) is completely integrable and possesses \(3 \times 3\) matrix spectral problem [10]. In recent years, there has been much work on the Sasa-Satsuma equation. For example, bilinearization and multisoliton solutions for the SS equation [15], N-soliton solutions for the SS equation with zero boundary condition by IST method [16], initial-boundary problems of the SS equation on the half-line by Fokas uniformed method [17], N-soliton solutions for the SS equation with zero boundary condition by Riemann-Hilbert approach [18], long-time asymptotic of the SS equation by Deift-Zhou steepest descent method [19], high-order soliton solutions for the SS equation with zero boundary condition by Riemann-Hilbert approach [20], binary Darboux transformation of the SS equation [21], Bright-soliton and optical soliton etc [22, 23].

In this article we are interested in the IST for the SS equation (\(\varepsilon = 1\)) by means of Riemann-Hilbert (RH) method with the following nonzero boundary condition

\[ \lim_{x \to \pm \infty} q(x, t) = q_{\pm} = q_0 e^{i\theta_{\pm}} \]  

(1.6)

where \(q\) and \(q_0\) are the two-component vectors, \(\| \cdot \|\) is the standard Euclidean norm, \(\theta_{\pm}\) are real numbers and \(\| q_0 \|^2 = q_0^2\). Because the SS equation admits a \(3 \times 3\) matrix spectral problem, which causes the analyticity, symmetries and asymptotic to be more complicated than those of \(2 \times 2\) matrix spectral problem. The auxiliary eigenfunctions are also necessary to construct a matrix Riemann-Hilbert problem.

The IST method, first discovered by Gardner, Green, Kruskal and Miura, is one of the most powerful tool to investigate solitons of nonlinear models [24, 25]. The IST for the focusing Schrodinger equation with zero boundary conditions was first developed by Zakharov and Shabat [1], later for the defocusing case.
with nonzero boundary conditions [26]. The next important steps of the development of IST method is the RH method which as the modern version of IST was established by Zakharov and Shabat [27]. It has since become clear that the RH method is applicable to investigate exact solutions and asymptotic analysis of solutions for a wide class of integrable systems [28–34].

The structure of this work is as follows. In section 2, we investigate the spectral problem of SS equation (1.5) with the nonzero boundary condition (1.6) by introducing a Riemann surface. To construct the desired RH problem, we investigate the analyticity of eigenfunctions, auxiliary eigenfunctions and scattering coefficients. In section 3, we show that all eigenfunctions, scattering matrix and reflection coefficient possess three types symmetries. In section 4, we consider the asymptotic of eigenfunctions, auxiliary eigenfunction and scattering coefficient. In section 5, discuss the discrete spectrum and the residue conditions to analyze poles for meromorphic matrices appearing in the RH problem. In section 6, we present a generalized RH problem with four jump matrixes, from which we find a reconstruction formula between solution of the SS equation and the RH problem. In section 7, in reflectionless case, we discuss solvability of the RH problem, from which the $N$-soliton solutions of the SS equation are obtained. In section 8, as an illustrate examples, we explicitly obtain three kinds of one-soliton solutions according to different distribution of the spectrum, and their dynamical features are analyzed. Some proof process are provided in Appendix.

2 Spectral Analysis

2.1 Lax Pair

The Sasa-Satsuma equation (1.5) admits $3 \times 3$ matrix spectral problem [10]

$$\Phi_x = X \Phi, \quad \Phi_t = T \Phi,$$

(2.1)

where

$$X = -ik\Lambda + Q,$$

$$T = -4ik^3\Lambda + 4k^2Q + 2ik\Lambda(Q_x - Q^2) + 2Q^3 - Q_{xx} + [Q_x, Q],$$
\[ Q = \begin{pmatrix} 0 & \mathbf{q} \\ -\mathbf{q}^\dagger & 0 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \mathbf{I} & 0 \\ 0 & -1 \end{pmatrix}, \]

in which some notations in (2.1) are as follows:

\( \mathbf{k} \in \mathbb{C} \) is the spectral parameter.

- vector function \( \mathbf{q} = (q, \bar{q})^T \), the superscript \( T \) denote the matrix transpose and \( \bar{q} \) denotes the complex conjugation of \( q \).

- the superscript "\( \dagger \)" represents the Hermitian of matrix.

- \( \mathbf{I} \) is the identity matrix and \( \mathbf{0} \) is the zero matrix or vector.

- eigenfunction \( \Phi = (\Phi_{ij})_{3 \times 3}, \quad (i, j = 1, 2, 3) \).

Under the nonzero boundary condition (1.6), we obtain the following asymptotic matrix spectral problem

\[ \Phi_x = X_{\pm} \Phi, \quad \Phi_t = T_{\pm} \Phi, \quad (2.2) \]

where

\[ X_{\pm} = \lim_{x \to \pm \infty} X = -i \mathbf{k} \Lambda + Q_{\pm}, \]
\[ T_{\pm} = \lim_{x \to \pm \infty} T = -4ik^3 \Lambda + 4k^2 Q_{\pm} - 2ik \Lambda Q_2^3 + 2Q_3^3, \]
\[ Q_{\pm} = \lim_{x \to \pm \infty} Q = \begin{pmatrix} 0 & \mathbf{q}_{\pm} \\ -\mathbf{q}_{\pm} & 0 \end{pmatrix}. \]

The eigenvalues of matrices \( X_{\pm} \) and \( T_{\pm} \) are respectively

\[ -ik; \quad i\mathbf{\lambda}; \quad -i\mathbf{\lambda}; \]
\[ -4ik^3; \quad 2i\mathbf{\lambda}(2k^2 - q_0^2); \quad -2i\mathbf{\lambda}(2k^2 - q_0^2), \]

where

\[ \mathbf{\lambda}^2 = k^2 + q_0^2 \quad (2.4) \]

which is a double-valued function for complex variable \( k \).

Next, we derive the eigenvector of \( X_{\pm} \) and \( T_{\pm} \). For the convenient of expression, we firstly introduce the definition of orthogonal vector in the following Lemma.
Lemma 1. For any two-component complex-valued vector \( \mathbf{l} = (l_1, l_2)^T \), one can define its orthogonal vector as \( \mathbf{l}^\perp = (l_2, -l_1) \), and holds the property \( \mathbf{l}^\perp \mathbf{l} = \mathbf{l}^T \mathbf{l}^\perp = 0 \).

It is obvious that \( X_{\pm} \) and \( T_{\pm} \) possess the relation \([X_{\pm}, T_{\pm}] = 0\). So \( X_{\pm} \) and \( T_{\pm} \) holds the common eigenvectors. Through some calculations, one can obtain an invertible matrix \( \Gamma_{\pm} \) satisfying the following equation

\[
X_{\pm} \Gamma_{\pm} = i \Gamma_{\pm} J, \quad T_{\pm} \Gamma_{\pm} = i \Gamma_{\pm} \Omega,
\]

where

\[
J = \text{diag} (-\lambda, -k, \lambda),
\]

\[
\Omega = \text{diag} (-2(2k^2 - q_0^2)\lambda, -4k^3, 2(2k^2 - q_0^2)\lambda),
\]

\[
\Gamma_{\pm} = \begin{pmatrix}
-\frac{q_{\pm}}{q_0} & \frac{q_{\pm}}{q_0} & \frac{q_{\mp}}{q_0} \\
\frac{i q_{\mp} q_{\pm}}{k^2 + \lambda} & 0 & \frac{i q_{\pm} q_{\mp}}{k^2 + \lambda} \\
0 & 1 & 1
\end{pmatrix}.
\]

2.2 Riemann Surface

In the scalar case, \( \lambda = \sqrt{k^2 + q_0^2} \) is a branched function. The branch points are the values of \( k = \pm iq_0 \). We take the branch cut on the segment \([-iq_0, iq_0]\). In this case, if we set

\[
k + iq_0 = r_1 e^{i\alpha_1}, \quad k - iq_0 = r_2 e^{i\alpha_2}, \quad -\frac{\pi}{2} < \alpha_j < \frac{3\pi}{2}, j = 1, 2,
\]

we then obtain two single value functions

\[
\lambda = \pm \sqrt{r_1^2 r_2^2 e^{2i(\alpha_1 + \alpha_2)}}.
\]

We introduce an uniformization variable as

\[
z = k + \lambda.
\]

Substituting it into \([2.4]\) gives two single-valued functions

\[
k = \frac{1}{2} \left( z - \frac{q_0^2}{z} \right), \quad \lambda = \frac{1}{2} \left( z + \frac{q_0^2}{z} \right).
\]

Further we can show the following relations between the Riemann surface and the \( k \)-plane.
The region where $\text{Im}\lambda > 0$ come from the upper-half plane of the sheet-I and the lower-half plane of the sheet-II. The region where $\text{Im}\lambda < 0$ come from the upper-half plane of the sheet-II and the lower-half plane of the sheet-I.

On the sheet-I, $z \to \infty$ as $k \to \infty$, and on the sheet-II, $z \to 0$ as $k \to \infty$.

The real $\lambda$ (real $k$) axes is mapped into the real $z$ axes.

The branch cut $i[−q_0, q_0]$ is mapped into the circle $C_0$ of the radius $q_0$ in $z$-plane.

The sheet-I and sheet-II, except for the branch cut, are mapped into the exterior and the interior of $C_0$, respectively.

The jump contour in the complex $z$-plane is denoted by $\Sigma = \mathbb{R} \cup C_0$. The yellow and white regions in Fig.1 denote the analytic region $D_j (j = 1, 2, 3, 4)$, respectively

$D_1 = \{ z \in \mathbb{C} : |z|^2 - q_0^2 > 0 \cap \text{Im}z > 0 \}$, $D_2 = \{ z \in \mathbb{C} : |z|^2 - q_0^2 > 0 \cap \text{Im}z < 0 \}$,

$D_3 = \{ z \in \mathbb{C} : |z|^2 - q_0^2 < 0 \cap \text{Im}z < 0 \}$, $D_4 = \{ z \in \mathbb{C} : |z|^2 - q_0^2 < 0 \cap \text{Im}z > 0 \}$,

and $\bigcup_{j=1}^{4} D_j = \mathbb{C}$.

On the $z$ complex plane, we can rewrite the invertible matrix $\Gamma_\pm$ as

$$\Gamma_\pm = \begin{pmatrix} -\frac{q_\perp}{q_0} & \frac{q_\perp}{q_0} & -iz \frac{q_\perp}{q_0} \\ 0 & 1 & 0 \end{pmatrix},$$

moreover,

$$\text{det} \Gamma_\pm = 1 + \frac{q_0^2}{z^2} \triangleq \gamma(z),$$

$$\Gamma_\pm^{-1} = \frac{1}{\gamma} D(z) \begin{pmatrix} -\frac{q_\perp}{q_0} & -iz \\ 0 & 0 \end{pmatrix},$$

$$D(z) = \text{diag}\{1, \gamma(z), 1\}.$$

We also rewrite the Lax pair (2.1) as the polynomial form

$$\Phi_{\pm,x} = X_{\pm} \Phi_{\pm} + \Delta Q_{\pm} \Phi_{\pm}, \quad \Phi_{\pm,t} = T_{\pm} \Phi_{\pm} + \Delta \hat{Q}_{\pm} \Phi_{\pm}, \quad (2.8)$$
where

$$
\Delta Q_\pm = Q - Q_\pm,
\Delta \hat{Q}_\pm = 4k^2Q + 2ik\Lambda(Q_x - Q^2) + 2Q^3 - Q_{xx} + [Q_x, Q] - (4k^2Q_\pm - 2ik\Lambda Q^2_\pm + 2Q^3_\pm).
$$

### 2.3 Analyticity

In this subsection, we can define the Jost solutions $\Phi_\pm(x, t, z)$ as the simultaneous solutions of the Lax pair (2.1) with the boundary conditions

$$
\Phi_\pm(x, t, z) \sim \Gamma_\pm(z)e^{i\Theta(x, t, z)}, \quad x \to \pm \infty, \quad (2.10)
$$

where

$$
\Theta(x, t, z) = Jx + \Omega t = \text{diag}(\theta_1, \theta_2, -\theta_1),
\theta_1 = -\lambda[x + 2(2k^2 - q_0^2)t], \quad \theta_2 = -k(x + 4k^2t).
$$

To remove the asymptotic exponential oscillations, we introduce the following modified Jost eigenfunction

$$
\mu_\pm(x, t, z) = \Phi_\pm(x, t, z)e^{-i\Theta(x, t, z)}, \quad (2.12)
$$
which implies that
\[
\lim_{x \to \pm \infty} \mu_\pm(x, t, z) = \Gamma_\pm(x, t, z). \tag{2.13}
\]

After some simple calculation, we derive the Lax pair of \( \mu_\pm \)
\[
\begin{align*}
(\Gamma_\pm^{-1} \mu_\pm)_x &= [iJ, \Gamma_\pm^{-1} \mu_\pm] + \Gamma_\pm^{-1} \Delta Q_\pm \mu_\pm, \tag{2.14a} \\
(\Gamma_\pm^{-1} \mu_\pm)_t &= [\Omega, \Gamma_\pm^{-1} \mu_\pm] + \Gamma_\pm^{-1} \Delta \hat{Q}_\pm \mu_\pm, \tag{2.14b}
\end{align*}
\]
which can be written in the full derivative form
\[
d[e^{-i\Theta(x,t,z)} \Gamma_\pm^{-1} \mu_\pm e^{i\Theta(x,t,z)}] = e^{-i\Theta(x,t,z)} [U_1 dx + U_2 dt] e^{i\Theta(x,t,z)}, \tag{2.15}
\]
where
\[
U_1 = \Gamma_\pm^{-1} \Delta Q_\pm (x, t) \mu_\pm (x, t, z), \quad U_2 = \Gamma_\pm^{-1} \Delta \hat{Q}_\pm (x, t) \mu_\pm (x, t, z).
\]

So we can come to the Jost integral equation from (2.15)
\[
\begin{align*}
\mu_-(x, t, z) &= \Gamma_- + \int_{-\infty}^{x} \Gamma_- e^{i(x-y)J(z)} \Gamma_-^{-1} \Delta Q_- \mu_- e^{-i(x-y)J(z)} dy, \tag{2.16a} \\
\mu_+(x, t, z) &= \Gamma_+ - \int_{x}^{+\infty} \Gamma_+ e^{i(x-y)J(z)} \Gamma_+^{-1} \Delta Q_+ \mu_- e^{-i(x-y)J(z)} dy. \tag{2.16b}
\end{align*}
\]

And we also derive the analyticity:

**Theorem 2.1.** If \( q(\cdot, t) - q_+ \in L^1(a, +\infty) \) or \( q(\cdot, t) - q_- \in L^1(-\infty, a) \) for any constant \( a \in \mathbb{R} \), the following columns of \( \mu_+(x, t, z) \) or \( \mu_-(x, t, z) \) can be analytically extended onto the corresponding regions of the complex \( z \)-plane.

| \( \mu_{+,1} \) | \( \mu_{+,2} \) | \( \mu_{+,3} \) | \( \mu_{-,1} \) | \( \mu_{-,2} \) | \( \mu_{-,3} \) |
|---|---|---|---|---|---|
| \( D_1 \) | \( \text{Im} z < 0 \) | \( D_1 \) | \( D_3 \) | \( \text{Im} z > 0 \) | \( D_2 \) |

And (2.12) indicates that the same analyticity and boundedness properties also hold for the columns of \( \Phi_\pm(x, t, z) \).
Proof. For a matrix $M$ with the following product notation

$$
e^{i(x-y)J(z)}Me^{-i(x-y)J(z)} = \begin{pmatrix}
m_{11}e^{i(k-\lambda)(x-y)} & m_{12}e^{-i(k-\lambda)(x-y)} & m_{13}e^{-2i\lambda(x-y)} \\
e^{-i(k+\lambda)(x-y)}m_{21} & m_{22}e^{2i\lambda(x-y)} & m_{23}e^{-i(k+\lambda)(x-y)} \\
e^{i(k+\lambda)(x-y)}m_{31} & m_{32}e^{-2i\lambda(x-y)} & m_{33}e^{-i(k+\lambda)(x-y)}
\end{pmatrix}
$$

(2.17)

We consider the analytic of the first column.

$$e^{-i(k-\lambda)(x-y)} = \frac{\gamma(x)}{e^{i|z|^2/2}}\text{Im}(x-y)e^{-i|z|^2/2}\text{Re}(x-y)$$

(2.18)

Then we derived that

$$\mu_{-,1} \text{ analytic, } \{\text{Im} z < 0 \cup x-y > 0\}, \quad (2.19a)$$

$$\mu_{+,1} \text{ analytic, } \{\text{Im} z > 0 \cup x-y < 0\}. \quad (2.19b)$$

$$e^{2i\lambda(x-y)} = e^{-\frac{|z|^2}{2} - \frac{q_0^2}{2}\text{Im}(x-y)}e^{2i\text{Re}(x-y)}.$$  

(2.20)

So we have the following results

$$\mu_{-,1} \text{ analytic in } D_3, \quad \{\text{Im} z < 0 \cup x-y > 0 \cup |z|^2 - q_0^2 < 0\}, \quad (2.21a)$$

$$\mu_{+,1} \text{ analytic in } D_4, \quad \{\text{Im} z > 0 \cup x-y < 0 \cup |z|^2 - q_0^2 < 0\}. \quad (2.21b)$$

The analytic of the second and third column can be derived in the similar way.

□

Now we introduce the scattering matrix $A(z)$. We note that $\text{tr} X = -ik$ and $\text{tr} T = -4ik^3$. If $\Phi(x, t, z)$ is a solution of (2.1), by using Abel’s theorem, we have

$$(\det \Phi)_x = \text{tr} X \det \Phi, \quad (\det \Phi)_t = \text{tr} T \det \Phi,$$

which can be written into the following differential equations

$$[e^{-i\Theta} \det \Phi]_x = [e^{-i\Theta} \det \Phi]_t = 0.$$

Again by using the boundary condition (2.10), we find that

$$\det \Phi_\pm(x, t, z) = \gamma(z)e^{i\Theta(x, t, z)}.$$  

(2.22)

$\Phi_+(x, t, z)$ and $\Phi_-(x, t, z)$ are the fundamental matrix solutions of the Lax pair.

It is indicates the existence of a $3 \times 3$ matrix $A(z)$, so that

$$\Phi_-(x, t, z) = \Phi_+(x, t, z)A(z), \quad (2.23)$$
where $A(z) = (a_{ij}(z))$ and $\det A(z) = 1$. And we introduce the inverse matrix $A^{-1}(z) = B(z) = (b_{ij}(z))$. In the $2 \times 2$ spectral problem, the analyticity of scattering matrix $A(z)$ can be derived from their representations as Wronskians of [2.23]. In the $3 \times 3$ spectral problem, however, this approach is no longer applicable. The reason is that the second column of Jost eigenfunction is not analytic in one of analytic region $D_j$.

**Theorem 2.2.** Under the same hypotheses as in 2.1, the diagonal scattering coefficient can be analytically extended of $\Sigma$ in the following regions:

| Table 2: the analytically of $a_{jj}$ and $b_{jj}$ |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $a_{11}$ | $a_{22}$ | $a_{33}$ | $b_{11}$ | $b_{22}$ | $b_{33}$ |
| $D_3$ | $\mathrm{Im} z > 0$ | $D_2$ | $D_4$ | $\mathrm{Im} z < 0$ | $D_1$ |

**Proof.** We note that the first part of the (2.8) is equivalent the following problem

$$\Phi_{\pm, x}(x, z) = \tilde{X}_\pm \Phi_{\pm}(x, z) + (Q(x) - Q_f(x)) \Phi_{\pm},$$

where

$$\tilde{X}(x, z) = H(x)X_+(z) + H(-x)X_-(z), \quad Q_f(x) = H(x)Q_+ + H(-x)Q_-,$$

and $H(x)$ is the Heaviside function

$$H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

We introduce the fundamental eigenfunction $\tilde{\Phi}_{\pm}(x, z)$ as the square matrix solution of (2.28)

$$\tilde{\Phi}_{\pm}(x, z) \sim e^{xX_{\pm}(z)}, \quad x \to \pm \infty.$$  (2.26)

By solving (2.28) in the similar way as (2.12), we introduce the following transformation

$$\tilde{\mu}_{\pm} = \tilde{\Phi}_{\pm} e^{-x\tilde{X}},$$

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and we obtain the full derivative
\[ d(e^{-x} \bar{\mu}_\pm e^{x}) = e^{-x}(Q - Q_f)\bar{\mu}_\pm e^{x}. \]

We deriving the integral equation
\[ \Phi_+(x, z) = G_f(x, 0, z) - \int_x^{+\infty} G_f(x, y, z)[Q(y) - Q_f(y)]\Phi_+(y)dy, \quad (2.27a) \]
\[ \Phi_-(x, z) = G_f(x, 0, z) + \int_{-\infty}^x G_f(x, y, z)[Q(y) - Q_f(y)]\Phi_-(y)dy, \quad (2.27b) \]
where
\[ G_f(x, y, z) = \begin{cases} 
  e^{(x-y)X_+}, & x, y \geq 0 \\
  e^{(x-y)X_-}, & x, y \leq 0 \\
  e^{X_+}e^{-yX_-}, & x, -y \geq 0 \\
  e^{X_-}e^{-yX_+}, & x, -y \leq 0 
\end{cases} \]

For equation \((2.27a)\),
\[ \Phi_\pm(x, z) = G_f(x, 0, z)A_\mp, \quad (2.28) \]
where
\[ A_\mp = I \mp \int_{-\infty}^{+\infty} G_f(0, y, z)[Q(y) - Q_f(y)]\Phi_\pm dy. \quad (2.29) \]

Considering the boundary conditions as \(x \to \pm \infty\)
\[ \Phi_\pm e^{xX_\pm} = \Gamma_\pm e^{i\Theta}z^{-1}_\pm, \quad \Phi_\pm \sim \Gamma_\pm e^{i\Theta}. \quad (2.30) \]

Then we obtain
\[ \Phi_\pm(x, z) \sim \Phi_\pm(x, z)\Gamma_\mp(z), \quad (2.31) \]
so \((2.27)\) implied that
\[ \Phi_+(x, z) = G_f(x, 0, z)\Gamma_+(z) - \int_x^{+\infty} G_f(x, y, z)[Q(y) - Q_f(y)]\Phi_+(y)dy, \quad (2.32a) \]
\[ \Phi_-(x, z) = G_f(x, 0, z)\Gamma_-(z) + \int_{-\infty}^x G_f(x, y, z)[Q(y) - Q_f(y)]\Phi_-(y)dy, \quad (2.32b) \]
We compare the asymptotics as \(x \to \infty\) of \(\Phi_-\) from \((2.28)\) with those of \(\Phi_+A(z)\) from \((2.10)\) to obtain
\[ A(z) = \Gamma_+^{-1}(z)A_+(z)\Gamma_-(z). \quad (2.33) \]
The equation (2.33) implied the following expression of scattering matrix

\[ A(z) = \int_{0}^{+\infty} \text{Id}y + \int_{-\infty}^{0} \text{Id}y + \Gamma_{+}^{-1}(z)\Gamma_{-}(z) \]

where the integrands are

\[ I = e^{-iyJ(z)}\Gamma_{+}^{-1}(z)[Q(y) - Q_{+}(y)]\Phi_{-}(y, z), \]

\[ II = \Gamma_{+}^{-1}(z)\Gamma_{-}(z)e^{-iyJ(z)}\Gamma_{-}^{-1}(z)[Q(y) - Q_{-}(z)]\Phi_{-}(y, z). \]

The diagonal elements of I and II are, respectively

\[ I_{11} = \sum_{j=1}^{3} \varrho_{1j}e^{iy\lambda}\Phi_{-j1}, \quad I_{22} = \sum_{j=1}^{3} \varrho_{2j}e^{iyk}\Phi_{-j2}, \quad I_{33} = \sum_{j=1}^{3} \varrho_{3j}e^{-iy\lambda}\Phi_{-j3} \]

\[ II_{11} = \sum_{j=1}^{3} \left( \nu_{11}\vartheta_{1j} + \nu_{12}\vartheta_{2j}e^{iy(k-\lambda)} + \nu_{13}\vartheta_{3j}e^{-2iy\lambda} \right) e^{iy\lambda}\Phi_{-j1} \]

\[ II_{22} = \sum_{j=1}^{3} \left( \nu_{21}\vartheta_{1j} + \nu_{22}\vartheta_{2j}e^{iy(k-\lambda)} + \nu_{23}\vartheta_{3j}e^{-2iy\lambda} \right) e^{iy\lambda}\Phi_{-j2} \]

\[ II_{33} = \sum_{j=1}^{3} \left( \nu_{31}\vartheta_{1j} + \nu_{32}\vartheta_{2j}e^{iy(k-\lambda)} + \nu_{33}\vartheta_{3j}e^{-2iy\lambda} \right) e^{iy\lambda}\Phi_{-j3} \]

where

\[ \Gamma_{+}^{-1}\Gamma_{-} = (\nu_{ij}), \quad \Gamma_{-}^{-1}[Q - Q_{-}] = (\vartheta_{ij}), \quad \Gamma_{+}^{-1}[Q - Q_{+}] = (\vartheta_{ij}). \]

2.4 Auxiliary Eigenfunctions

The second columns of Jost solution are not analytic in a given domain. To circumvent this defect of analyticity, we introduce a modified Lax pair.

\[ \tilde{\Phi}_{x} = X\tilde{\Phi}, \quad \tilde{\Phi}_{t} = T\tilde{\Phi}, \quad (2.36) \]

where \( \tilde{X} = \tilde{X}(x, t, \bar{z}), \tilde{T} = \tilde{T}(x, t, \bar{z}) \). And we note that

\[ \tilde{Q} = -Q^{T}, \quad Q = -Q^{T}, \]

\[ QA = -AQ, \quad Q^{T}\Lambda = -AQ^{T}. \]

Before reconstruct a solution of Lax pair (2.1) from the modified Lax pair (2.36), we need introduce the following lemma.
Lemma 2. For $\forall u, v \in \mathbb{C}^3$ satisfying the following equations:

\[
[(\Lambda u) \times v] + [u \times (\Lambda v)] - [u \times v] - [(\Lambda u) \times (\Lambda v)] = 0,
\]
\[
\Lambda [u \times v] = -[(\Lambda u) \times (\Lambda v)],
\]
\[
Q[u \times v] + [(Q^T u) \times v] + [u \times (Q^T v)] = 0,
\]
\[
\Lambda Q^2[u \times v] + [(\Lambda (Q^T)^2) \times v] + [u \times (\Lambda (Q^T)^2)v] = 0,
\]

where "×" denotes the usual cross product.

Using above identities and Lemma 2 one can straightforward to prove the following theorem.

Theorem 2.3. If $\tilde{v}(x,t,z)$ and $\tilde{w}(x,t,z)$ are the two arbitrary solutions of modified Lax pair (2.36), then

\[
u(x,t,z) = e^{i\theta_2(x,t,z)}[\tilde{v} \times \tilde{w}](x,t,z)
\]

is also a solution of Lax pair (2.1).

Proof. Through some calculations, we have

\[
[\tilde{v} \times \tilde{w}]_x = \tilde{v}_x \times \tilde{w} + \tilde{v} \times \tilde{w}_x
\]

\[
= (ik\Lambda + \tilde{Q})\tilde{v} \times \tilde{w} + \tilde{v} \times (ik\Lambda + \tilde{Q})\tilde{w}
\]

\[
= ik(\Lambda \tilde{v}) \times \tilde{w} + (\tilde{Q} \tilde{v}) \times \tilde{w} + ik[\tilde{v} \times (\Lambda \tilde{w})] + [\tilde{v} \times (\tilde{Q} \tilde{w})]
\]

\[
= ik\{([(\Lambda \tilde{v}) \times \tilde{w}] + [\tilde{v} \times (\Lambda \tilde{w})]) + \{(\tilde{Q} \tilde{v}) \times \tilde{w} + [\tilde{v} \times (\tilde{Q} \tilde{w})]\}
\]

And we note that

\[
[(\tilde{Q} \tilde{v}) \times \tilde{w}] + [\tilde{v} \times (\tilde{Q} \tilde{w})] = Q[\tilde{v} \times \tilde{w}].
\]

So we derived the $x$-part

\[
u_x(x,t,z) = e^{i\theta_2(i\theta_2, x)}[\tilde{v} \times \tilde{w}] + e^{i\theta_2}[\tilde{v} \times \tilde{w}]_x
\]

\[
= e^{i\theta_2}(-ik)[\tilde{v} \times \tilde{w}] + e^{i\theta_2}(ik[\tilde{v} \times \tilde{w}] + ik[(\Lambda \tilde{v}) \times (\Lambda \tilde{w})]) + e^{i\theta_2}Q[\tilde{v} \times \tilde{w}]
\]

\[
= -ik\Lambda e^{i\theta_2}[\tilde{v} \times \tilde{w}] + e^{i\theta_2}Q[\tilde{v} \times \tilde{w}]
\]

\[
= (-ik\Lambda + Q)e^{i\theta_2}[\tilde{v} \times \tilde{w}].
\]
In the similar way, we can derived the $t$-part. □

As $x \to \infty$, we obtain the asymptotic spectral problem of (2.36)

$$\tilde{\Phi}_{\pm,x} = \tilde{X}_{\pm} \tilde{\Phi}_{\pm}, \quad \tilde{\Phi}_{\pm,t} = \tilde{T}_{\pm} \tilde{\Phi}_{\pm}.$$  

The eigenvalues of $\tilde{X}_{\pm}$ and $\tilde{T}_{\pm}$ are

$$ik, \quad -i\lambda, \quad i\lambda, \quad 4ik^3, \quad -2i\lambda(2k^2 - q_0^2), \quad 2i\lambda(2k^2 - q_0^2).$$

We can deriving the eigenvector $\tilde{\Gamma}_{\pm}(z) = \tilde{\Gamma}_{\pm}(\bar{z})$ of matrix $\tilde{X}_{\pm}$ and $\tilde{T}_{\pm}$. And we note that $\text{det}\tilde{\Gamma}_{\pm}(z) = \gamma(z)$. For $z \in \Sigma$, we define the Jost solutions of (2.36) as the simultaneous solutions such that

$$\tilde{\Phi}_{\pm}(x, t, z) \sim \tilde{\Gamma}_{\pm}(z)e^{-i\Theta(x, t, z)}, \quad x \to \infty, \quad z \in \Sigma.$$  

As in section 2.2, we introducing the modified adjoint eigenfunction

$$\bar{\mu}_{\pm}(x, t, z) = \tilde{\Phi}_{\pm}(x, t, z)e^{i\Theta(x, t, z)}. \quad (2.39)$$

One can show the analyticity of $\bar{\mu}_{\pm}$

| Table 3: the analytically of $\bar{\mu}_{\pm}$ |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $\bar{\mu}_{+,1}$ | $\bar{\mu}_{+,2}$ | $\bar{\mu}_{+,3}$ | $\bar{\mu}_{-,1}$ | $\bar{\mu}_{-,2}$ | $\bar{\mu}_{-,3}$ |
| $D_3$ | $\text{Im}z > 0$ | $D_2$ | $D_4$ | $\text{Im}z < 0$ | $D_1$ |

$\tilde{\Phi}_{\pm}(x, t, z)$ are both the fundamental matrix solutions of the same problem, so we introduce the adjoint scattering matrix as

$$\tilde{\Phi}- (x, t, z) = \tilde{\Phi}+(x, t, z)\tilde{A}(z). \quad (2.40)$$

Under the same hypotheses as in 2.1, the diagonal scattering coefficient can be analytically extended of $\Sigma$ in the following regions:

| Table 4: the analytically of $\tilde{a}_{jj}$ and $\tilde{b}_{jj}$ |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $\tilde{a}_{11}$ | $\tilde{a}_{22}$ | $\tilde{a}_{33}$ | $\tilde{b}_{11}$ | $\tilde{b}_{22}$ | $\tilde{b}_{33}$ |
| $D_4$ | $\text{Im}z < 0$ | $D_1$ | $D_3$ | $\text{Im}z > 0$ | $D_2$ |
In light of above results, we can define four auxiliary eigenfunctions \( \chi_j \), \( j = 1, 2, 3, 4 \) as the new solutions of the Lax pair (2.1):

\[
\begin{align*}
\chi_1(x, t, z) &= e^{i\theta_2} [\tilde{\Phi}_{+,2} \times \tilde{\Phi}_{-,3}] (x, t, z), \\
\chi_2(x, t, z) &= e^{i\theta_2} [\tilde{\Phi}_{-,2} \times \tilde{\Phi}_{+,3}] (x, t, z), \\
\chi_3(x, t, z) &= e^{i\theta_2} [\tilde{\Phi}_{+,1} \times \tilde{\Phi}_{-,2}] (x, t, z), \\
\chi_4(x, t, z) &= e^{i\theta_2} [\tilde{\Phi}_{-,1} \times \tilde{\Phi}_{+,2}] (x, t, z).
\end{align*}
\] (2.41)

We note that \( \chi_j \) is analyticity for \( z \in D_j \). And there are some simple relations

exists between the adjoint Jost solutions eigenfunction and the eigenfunctions of the Lax pair (2.1):

**Lemma 3.** For \( z \in \Sigma \) and for all cyclic indices \( j, l, \) and \( m \)

\[
\begin{align*}
\Phi_{\pm,j}(x, t, z) &= e^{i\theta_2} [\Phi_{\pm,l} \times \Phi_{\pm,m}] (x, t, z) / \gamma_j(z), \\
\tilde{\Phi}_{\pm,j}(x, t, z) &= e^{-i\theta_2} [\Phi_{\pm,l} \times \Phi_{\pm,m}] (x, t, z) / \gamma_j(z),
\end{align*}
\] (2.42)

where \( \gamma_1(z) = \gamma_3(z) = 1, \gamma_2(z) = \gamma(z) \).

**Proof.** We prove (2.42a) with \( j = 3 \), the rest of (3) is verified in the similar method. From (2.10) and Theorem 2.3, we deriving

\[
\begin{align*}
u_{\pm}(x, t, z) &= e^{-i\theta_1(x,t,z)} \Gamma_{\pm,3}(z), \quad x \to \infty.
\end{align*}
\]

The vector function \( u_{\pm}(x, t, z) \) must be linear combination of the columns of \( \Phi_{\pm} \), then there exist three scalar functions \( a_{\pm}(z), b_{\pm}(z) \) and \( c_{\pm}(z) \) so that

\[
u_{\pm}(x, t, z) = a_{\pm}(z) \Phi_{\pm,1}(x, t, z) + b_{\pm}(z) \Phi_{\pm,2}(x, t, z) + c_{\pm}(z) \Phi_{\pm,3}(x, t, z).
\]

Comparing with the asymptotics of \( \Phi_{\pm}(x, t, z) \) as \( x \to \pm \infty \) in (2.10) yields \( a_{\pm}(z) = b_{\pm}(z) = 0 \) and \( c_{\pm}(z) = 1 \). □

It is naturally that these relations in Lemma 3 imply the relation of scattering matrix as in Theorem 2.4.

**Theorem 2.4.** The scattering matrix \( A(z) \) and \( \tilde{A}(z) \) admit the relation

\[
\tilde{A}(z) = D(z)(A^{-1}(z))^T D^{-1}(z).
\] (2.43)
Proof. Equation (2.23) and Lemma 3 yield

\[ \tilde{\Phi}_{-1} = (a_{22}a_{33} - a_{32}a_{23})\tilde{\Phi}_{+1} + \gamma(a_{32}a_{13} - a_{12}a_{33})\tilde{\Phi}_{+2} + (a_{12}a_{23} - a_{22}a_{13})\tilde{\Phi}_{+3}. \]

Comparing with (2.40) yields

\[ \tilde{a}_{11} = a_{22}a_{33} - a_{32}a_{23}, \quad \tilde{a}_{21} = \gamma(a_{32}a_{13} - a_{12}a_{33}), \quad \tilde{a}_{31} = a_{12}a_{23} - a_{22}a_{13}. \]

In the similar way, we derived

\[ \tilde{a}_{12} = \frac{1}{\gamma}(a_{23}a_{31} - a_{33}a_{21}), \quad \tilde{a}_{22} = a_{33}a_{11} - a_{13}a_{31}, \quad \tilde{a}_{32} = \frac{1}{\gamma}(a_{13}a_{21} - a_{23}a_{11}), \]
\[ \tilde{a}_{13} = a_{21}a_{32} - a_{31}a_{22}, \quad \tilde{a}_{23} = \gamma(a_{31}a_{12} - a_{11}a_{32}), \quad \tilde{a}_{33} = a_{11}a_{22} - a_{21}a_{12}. \]

We note that

\[ (A^{-1}(z))^T = \begin{pmatrix}
    a_{22}a_{33} - a_{23}a_{32} & a_{23}a_{31} - a_{21}a_{33} & a_{21}a_{32} - a_{22}a_{31} \\
    a_{13}a_{32} - a_{12}a_{33} & a_{11}a_{33} - a_{13}a_{31} & a_{12}a_{31} - a_{11}a_{32} \\
    a_{12}a_{23} - a_{13}a_{22} & a_{13}a_{21} - a_{11}a_{23} & a_{11}a_{22} - a_{12}a_{21}
\end{pmatrix}. \]

We finally obtain

\[ \tilde{A}(z) = D(z)(A^{-1}(z))^T D^{-1}(z). \]

□

Next, combining Lemma 3 and the equation (2.40) in the definition (2.41) deriving Theorem 2.5:

Theorem 2.5. For \( z \in \Sigma \), the Jost eigenfunction have the following relations

\[ \Phi_{-1} = \frac{1}{a_{22}}[\chi_1 + a_{21}\Phi_{-2}] = \frac{1}{a_{33}}[a_{31}\Phi_{-3} + \chi_2], \quad (2.45a) \]
\[ \Phi_{-3} = \frac{1}{a_{22}}[\chi_4 + a_{23}\Phi_{-2}] = \frac{1}{a_{11}}[a_{13}\Phi_{-3} + \chi_3], \quad (2.45b) \]
\[ \Phi_{+1} = \frac{1}{b_{22}}[\chi_2 + b_{21}\Phi_{+2}] = \frac{1}{b_{33}}[b_{31}\Phi_{+3} + \chi_1], \quad (2.45c) \]
\[ \Phi_{+3} = \frac{1}{b_{22}}[\chi_4 + b_{23}\Phi_{+2}] = \frac{1}{b_{11}}[b_{13}\Phi_{+3} + \chi_4]. \quad (2.45d) \]

For convenient, we omit the independent variable.
Proof. Substituting (2.40) into (2.41) obtain the following equations

\[ \chi_1 = \bar{b}_{22}e^{i\theta_2}[\bar{\Phi}_{-1} \times \bar{\Phi}_{-2}] + \bar{b}_{32}e^{i\theta_2}[\bar{\Phi}_{-1} \times \bar{\Phi}_{-3}], \]  
(2.46a)

\[ \chi_4 = \bar{b}_{12}e^{i\theta_2}[\bar{\Phi}_{-1} \times \bar{\Phi}_{-2}] + \bar{b}_{22}e^{i\theta_2}[\bar{\Phi}_{-2} \times \bar{\Phi}_{-3}], \]  
(2.46b)

Substituting (2.42a) into (2.46) yields

\[ \chi_1 = \bar{b}_{22}\bar{\Phi}_{-3} - \bar{b}_{32}a\bar{\Phi}_{-2}, \]  
(2.47a)

\[ \chi_4 = \bar{b}_{12}\bar{\Phi}_{-2} + \bar{b}_{22}a\bar{\Phi}_{-1}. \]  
(2.47b)

Applying (2.43) to (2.47), we obtain

\[ \bar{\Phi}_{-1} = \frac{1}{a_{22}}(\chi_4 + a_{21}\bar{\Phi}_{-2}), \quad \bar{\Phi}_{-3} = \frac{1}{a_{22}}(\chi_1 + a_{23}\bar{\Phi}_{-2}). \]

In the similar way we can obtain the rest of (2.5). □

To remove the exponential oscillations, we define the modified auxiliary eigenfunctions

\[ m_j(x, t, z) = e^{-i\theta_1(x, t, z)}\chi_j(x, t, z), \quad j = 1, 2 \]  
(2.48a)

\[ m_j(x, t, z) = e^{i\theta_1(x, t, z)}\chi_j(x, t, z), \quad j = 3, 4 \]  
(2.48b)

3 Symmetries

For the zero boundary conditions with the only one the symmetry \( k \mapsto \bar{k} \). However, for the nonzero boundary conditions, the symmetries are complicated because the Riemann surface introduced. It not only involved with the symmetries \( k \mapsto \bar{k} \), but also involved with the symmetries \( k \mapsto -q_0^2/z \). It namely that mapping the upper-half plane into the lower-half plane and mapping the exterior of circle \( C_0 \) of radius \( q_0 \) into the interior.

3.1 The First Symmetries

In this subsection, we firstly consider the map that \( z \mapsto \bar{z} \).

Lemma 4. If \( \Phi(x, t, z) \) is a non-singular solution of the Lax pair (2.1), so is \( w(x, t, z) = (\Phi^\dagger(x, t, \bar{z}))^{-1} \).
We also show that

**Proof.** If \( \Phi(x, t, z) \) is a non-singular solution of the Lax pair. Then we have

\[
\Phi_x^\dagger = \Phi^\dagger X^\dagger, \quad \Phi_t^\dagger = \Phi^\dagger T^\dagger. \tag{3.1}
\]

If we substituting \( X \) and \( T \) into (3.1), we have

\[
w_x = - [\Phi^\dagger(\bar{z})]^{-1} \Phi^\dagger(\bar{z})[\Phi^\dagger(\bar{z})]^{-1}
= - [\Phi^\dagger(\bar{z})]^{-1} \Phi^\dagger(\bar{z})(ik\Lambda - Q)[\Phi^\dagger(\bar{z})]^{-1}
= Xw,
\]

\[
w_t = - [\Phi^\dagger(\bar{z})]^{-1} \Phi^\dagger(\bar{z}) \{4ik^3\Lambda - 4k^2Q - 2ik(-Q_x - Q^2)\Lambda - 2Q^3 + 4Q_{xx} - [Q_x, Q]\} [\Phi^\dagger(\bar{z})]^{-1}
= Tw.
\]

Then \( w(x, t, z) \) is a solution of the Lax pair. \( \square \)

The Jost eigenfunctions satisfy the symmetry

**Lemma 5.**

\[
(\Phi^\dagger_\pm(x, t, \bar{z}))^{-1} C(z) = \Phi_\pm(x, t, z), \tag{3.4}
\]

where \( C(z) = \text{diag}(\gamma(z), 1, \gamma(z)). \)

**Proof.** From the Lemma 4, we known that \( \Phi_\pm(x, t, z) \) and \( (\Phi^\dagger_\pm(x, t, \bar{z}))^{-1} \) are the solutions of the Lax pair. And, we note that

\[
(e^{i\Theta(x,t,\bar{z})})^\dagger = e^{-i\Theta(x,t,\bar{z})}.
\]

\[
\Phi_\pm(x, t, z) \sim \Gamma_\pm e^{i\Theta(z)}, \quad x \to \pm\infty \tag{3.5}
\]

\[
(\Phi^\dagger_\pm(x, t, \bar{z}))^{-1} \sim ((\Gamma_\pm(z)e^{i\Theta(z)})^\dagger)^{-1}
= (e^{-i\Theta(z)}\Gamma^\dagger_\pm(z))^{-1} = (\Gamma^\dagger_\pm(z))^{-1} e^{i\Theta(z)}
= (C(z)\Gamma^{-1}_\pm(z)) e^{i\Theta(z)} = \Gamma_\pm(z) C^{-1}(z) e^{i\Theta(z)}
= \Gamma_\pm(z) e^{i\Theta(z)} C^{-1}(z)
\]
Then, we have

\[(\Phi^\dagger_\pm(x, t, \bar{z}))^{-1} C(z) \sim \Phi_\pm(x, t, z), \quad x \to \pm \infty.\]

\[\square\]

We also can show that

**Lemma 6.** For the columns of matrix \(\Phi_\pm(z, t, z)\) with the following property about cross product

\[(\Phi^{-1}_\pm(x, t, z))^T = \frac{1}{\det \Phi_\pm(x, t, z)} (\Phi_\pm \times \Phi_\pm, \Phi_\pm \times \Phi_\pm, \Phi_\pm \times \Phi_\pm, \Phi_\pm \times \Phi_\pm)(x, t, z).\]

(3.7)

Using the Theorem 2.5 and Lemma 5 yields:

**Lemma 7.** The Jost eigenfunctions obey the symmetry relations:

\[\bar{\Phi}_{+,1}(x, t, \bar{z}) = \frac{e^{-i\theta_2}}{b_{22}} [\Phi_{+,2} \times \chi_3](x, t, z), \quad (3.8a)\]

\[\bar{\Phi}_{-,1}(x, t, \bar{z}) = \frac{e^{-i\theta_2}}{a_{22}} [\Phi_{-,2} \times \chi_4](x, t, z), \quad (3.8b)\]

\[\bar{\Phi}_{+,2}(x, t, \bar{z}) = \frac{e^{-i\theta_2}}{\gamma b_{11}} [\chi_4 \times \Phi_{+,1}](x, t, z) = \frac{e^{-i\theta_2}}{\gamma b_{33}} [\Phi_{+,3} \times \chi_1](x, t, z), \quad (3.8c)\]

\[\bar{\Phi}_{-,2}(x, t, \bar{z}) = \frac{e^{-i\theta_2}}{\gamma a_{11}} [\chi_3 \times \Phi_{-,1}](x, t, z) = \frac{e^{-i\theta_2}}{\gamma a_{33}} [\Phi_{-,3} \times \chi_2](x, t, z), \quad (3.8d)\]

\[\bar{\Phi}_{+,3}(x, t, \bar{z}) = \frac{e^{-i\theta_2}}{b_{22}} [\chi_2 \times \Phi_{+,2}](x, t, z), \quad (3.8e)\]

\[\bar{\Phi}_{-,3}(x, t, \bar{z}) = \frac{e^{-i\theta_2}}{a_{22}} [\chi_1 \times \Phi_{-,2}](x, t, z). \quad (3.8f)\]

Using the equation (2.23) and Lemma 5 yields:

**Lemma 8.** The scattering matrix and its inverse satisfy the symmetry relation

\[A^\dagger(z) = C(z) B(z) C^{-1}(z), \quad z \in \Sigma. \quad (3.9)\]

For all \(z \in \Sigma\), the componentwise satisfy:

\[b_{11}(z) = \bar{a}_{11}(\bar{z}), \quad b_{12}(z) = \frac{\bar{a}_{21}(\bar{z})}{\gamma(\bar{z})}, \quad b_{13}(z) = \bar{a}_{31}(\bar{z}), \]

\[b_{21}(z) = \gamma(z) \bar{a}_{12}(\bar{z}), \quad b_{22}(z) = \bar{a}_{22}(\bar{z}), \quad b_{23}(z) = \gamma(z) \bar{a}_{32}(\bar{z}), \]

\[b_{31}(z) = \bar{a}_{13}(\bar{z}), \quad b_{32}(z) = \frac{\bar{a}_{23}(\bar{z})}{\gamma(\bar{z})}, \quad b_{33}(z) = \bar{a}_{33}(\bar{z}).\]
And we also derived the symmetry relations of the auxiliary eigenfunctions:

**Lemma 9.** The auxiliary eigenfunctions holds the symmetry relations:

\begin{align}
\bar{\chi}_1(x, t, \bar{z}) &= e^{-i\theta_2(z)}[\Phi_{+, 2} \times \Phi_{-, 3}](x, t, z), \\
\bar{\chi}_2(x, t, \bar{z}) &= e^{-i\theta_2(z)}[\Phi_{-, 2} \times \Phi_{+, 3}](x, t, z), \\
\bar{\chi}_3(x, t, \bar{z}) &= e^{-i\theta_2(z)}[\Phi_{+, 1} \times \Phi_{-, 2}](x, t, z), \\
\bar{\chi}_4(x, t, \bar{z}) &= e^{-i\theta_2(z)}[\Phi_{-, 1} \times \Phi_{+, 2}](x, t, z). 
\end{align}

(3.11)

The proof of Lemma 9 yields:

\begin{equation}
\bar{\Phi}_{\pm, j}(x, t, \bar{z}) = e^{-i\theta_2(x, t, z)}[\Phi_{\pm, l} \times \Phi_{\pm, m}](x, t, z)/\gamma_j,
\end{equation}

(3.12)

where \( j, l, \) and \( m \) are cyclic indices and \( z \in \Sigma \).

### 3.2 The Second Symmetries

In this subsection, we consider the map \( z \mapsto -\frac{g_0^2}{z} \).

**Lemma 10.** For all \( z \in \Sigma \), the Jost eigenfunctions holds the symmetry relations

\begin{equation}
\Phi_{\pm}(x, t, z) = \Phi_{\pm}(x, t, -\frac{g_0^2}{z})\Pi(z),
\end{equation}

(3.13)

where

\begin{equation}
\Pi(z) = \begin{pmatrix} 0 & 0 & \frac{i g_0}{z} \\
0 & 1 & 0 \\
\frac{i g_0}{z} & 0 & 0 \end{pmatrix}.
\end{equation}

(3.14)

So we derived all of the above relations:

\begin{align}
\Phi_{\pm, 1}(x, t, z) &= \frac{i g_0}{z}\Phi_{\pm, 3}(x, t, -\frac{g_0^2}{z}), \\
\Phi_{\pm, 2}(x, t, z) &= \Phi_{\pm, 2}(x, t, -\frac{g_0^2}{z}), \\
\Phi_{\pm, 3}(x, t, z) &= \frac{i g_0}{z}\Phi_{\pm, 1}(x, t, -\frac{g_0^2}{z}).
\end{align}

(3.15)
Lemma 11. The scattering matrix holds the symmetry
\[ A(-\frac{q_0^2}{z}) = \Pi(z)A(z)\Pi^{-1}(z), \quad B(-\frac{q_0^2}{z}) = \Pi(z)B(z)\Pi^{-1}(z). \tag{3.16} \]

We have the following componentwise relations
\[
\begin{align*}
    a_{11}(z) &= a_{33}(-\frac{q_0^2}{z}), & a_{12}(z) &= \frac{z}{iq_0}a_{32}(-\frac{q_0^2}{z}), & a_{13}(z) &= a_{31}(-\frac{q_0^2}{z}), \\
    a_{21}(z) &= \frac{iq_0}{z}a_{23}(-\frac{q_0^2}{z}), & a_{22}(z) &= a_{22}(-\frac{q_0^2}{z}), & a_{23}(z) &= \frac{iq_0}{z}a_{21}(-\frac{q_0^2}{z}), \\
    a_{31}(z) &= a_{13}(-\frac{q_0^2}{z}), & a_{32}(z) &= \frac{z}{iq_0}a_{12}(-\frac{q_0^2}{z}), & a_{33}(z) &= a_{11}(-\frac{q_0^2}{z}).
\end{align*}
\]

Lemma 12. The auxiliary eigenfunctions holds the symmetries relations
\[
\begin{align*}
    \chi_1(z) &= \frac{iq_0}{z}\chi_4(-\frac{q_0^2}{z}), & z \in D_1 \tag{3.18a} \\
    \chi_2(z) &= \frac{iq_0}{z}\chi_3(-\frac{q_0^2}{z}), & z \in D_2. \tag{3.18b}
\end{align*}
\]

3.3 The Third Symmetry

In this subsection, we introduce the symmetries of reflection coefficients.
\[
\begin{align*}
    \rho_1(z) &= \frac{a_{21}(z)}{a_{11}(z)} = \gamma(z)\overline{b}_{12}(\overline{z}) &, \quad \rho_2(z) &= \frac{a_{31}(z)}{a_{11}(z)} = \overline{b}_{13}(\overline{z}), \tag{3.19a} \\
    \rho_3(z) &= \frac{a_{32}(z)}{a_{22}(z)} = \frac{\overline{b}_{23}(\overline{z})}{\gamma(z)b_{22}(\overline{z})}, & \quad \rho_1(-\frac{q_0^2}{z}) &= \frac{z}{iq_0}a_{23}(z), & \quad \rho_1(-\frac{q_0^2}{z}) &= \frac{z}{iq_0}a_{33}(z), & \tag{3.19b} \\
    \rho_2(-\frac{q_0^2}{z}) &= \frac{a_{13}(z)}{a_{33}(z)} = \frac{\overline{b}_{31}(\overline{z})}{\overline{b}_{33}(\overline{z})}, & \quad \rho_3(-\frac{q_0^2}{z}) &= \frac{iq_0}{z}a_{12}(z), & \quad \rho_3(-\frac{q_0^2}{z}) &= \frac{iq_0}{z}a_{22}(z). \tag{3.19c}
\end{align*}
\]

4 Asymptotic behavior as \( z \to \infty \) and \( z \to 0 \)

In order to normalize the Riemann-Hilbert problem, it is necessary to consider the asymptotic behavior of the eigenfunctions and scattering coefficients as \( k \to \infty \), which corresponds to the behavior both \( z \to \infty \) and \( z \to 0 \). So we consider the following expansion for \( \Gamma_{\pm}^{-1}\mu_{\pm} \)
\[
\begin{align*}
    \Gamma_{\pm}^{-1}\mu_{\pm} &= E_0 + \frac{E_1}{z} + \frac{E_2}{z^2} + \cdots & z \to \infty, \tag{4.1a} \\
    \Gamma_{\pm}^{-1}\mu_{\pm} &= F_0 + F_1z + F_2z^2 + \cdots & z \to 0. \tag{4.1b}
\end{align*}
\]
where $E_j$ and $F_j$ are $3 \times 3$ matrices and independent of $z$. Through some explicitly evaluate, we obtain the asymptotic behavior:

**Theorem 4.1.** The asymptotic of $\mu_\pm$ as $z \to \infty$ and $z \to 0$

$$
\mu_\pm(x, t, z) = \begin{pmatrix} -\frac{q_\pm}{q_0} & \frac{q_\pm}{q_0} & -\frac{iq_\pm}{z} \\ i\frac{q^\dagger q}{zq_0} & i\frac{(q^\dagger q)^t}{zq_0} & 1 \end{pmatrix} + \frac{1}{2} \Gamma_\pm |_{bd}[E_1]_{bd} + o(\frac{1}{z^2}), \quad z \to \infty.
$$

(4.2a)

$$
\mu_\pm(x, t, z) = \begin{pmatrix} -\frac{q_\pm}{q_0} & \frac{q_\pm}{q_0} & -\frac{iq_\pm}{z} \\ i\frac{q^\dagger q}{zq_0} & 0 & 0 \end{pmatrix} + \Gamma_\pm |_{bd}[F_0]_{bo} + o(1), \quad z \to 0.
$$

(4.2b)

**Theorem 4.2.** The asymptotic of $m_j$ as $z \to \infty$

$$
m_1(x, t, z) = \begin{pmatrix} 0 \\ i\frac{q^\dagger q}{zq_0} [-q_-q_+] \end{pmatrix} + o(\frac{1}{z^2}), \quad m_2(x, t, z) = \begin{pmatrix} 0 \\ i\frac{q^\dagger q}{zq_0} [-q_-q_+] \end{pmatrix} + o(\frac{1}{z^2}), \quad (4.3a)
$$

$$
m_3(x, t, z) = \begin{pmatrix} -\frac{iq_\pm}{zq_0} [q_-q_+] \\ \frac{1}{zq_0} [q_-q_+] \end{pmatrix} + o(\frac{1}{z^2}), \quad m_4(x, t, z) = \begin{pmatrix} -\frac{iq_\pm}{zq_0} [q_-q_+] \\ \frac{1}{zq_0} [q_-q_+] \end{pmatrix} + o(\frac{1}{z^2})
$$

(4.3b)

**Theorem 4.3.** The asymptotic of $m_j$ as $z \to 0$

$$
m_1(x, t, z) = \begin{pmatrix} 0 \\ i\frac{q^\dagger q}{zq_0} [-q_-q_+] \end{pmatrix} + o(1), \quad m_2(x, t, z) = \begin{pmatrix} 0 \\ i\frac{q^\dagger q}{zq_0} [-q_-q_+] \end{pmatrix} + o(1), \quad (4.4a)
$$

$$
m_3(x, t, z) = \begin{pmatrix} -\frac{iq_\pm}{z} \\ -\frac{iq_\pm}{z} \\ 0 \end{pmatrix} + o(1), \quad m_4(x, t, z) = \begin{pmatrix} -\frac{iq_\pm}{z} \\ -\frac{iq_\pm}{z} \\ 0 \end{pmatrix} + o(1) \quad (4.4b)
$$

**Proof.** For convenient, the block diagonal and block off-diagonal terms of $3 \times 3$ matrix $S$

$$
S_{bd} = \begin{pmatrix} s_{11} & s_{12} & 0 \\ s_{21} & s_{22} & 0 \\ 0 & 0 & s_{33} \end{pmatrix}, \quad S_{bo} = \begin{pmatrix} 0 & 0 & s_{13} \\ 0 & 0 & s_{23} \\ s_{31} & s_{32} & 0 \end{pmatrix}, \quad [S_{bd}]_o = \begin{pmatrix} 0 & s_{12} & 0 \\ s_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
$$
And we note that for any matrix $S_1$ and $S_2$ holds the following qualities
\[
[UV]_{bd} = U_{bd}V_{bd} + U_{bo}V_{bo}, \quad [UV]_{bo} = U_{bd}V_{bo} + U_{bo}V_{bd},
\]
\[
[U_{bd}V_{bd}]_d = U_dV_d + [U_{bd}]_o[V_{bd}]_o, \quad [U_{bd}V_{bd}]_o = U_d[V_{bd}]_o + [U_{bd}]_oV_d.
\]
Substituting (4.11) into the system (2.14) and compare the coefficients of $z^j$, we deriving the following results
\[
E_0^{(12)} = E_0^{(13)} = E_0^{(21)} = E_0^{(31)} = E_0^{(24)} = E_0^{(32)} = 0,
\]
\[
E_0^{(11)} = E_0^{(22)} = E_0^{(33)} = 1,
\]
\[
E_1^{(13)} = \frac{i}{q_0} [\bar{q}q_\pm + q\bar{q}_\pm - q^2_0], \quad E_1^{(24)} = \frac{i}{q_0} [\bar{q}q_\pm - q\bar{q}_\pm],
\]
\[
E_1^{(31)} = \frac{i}{q_0} [\bar{q}q_\pm + q\bar{q}_\pm - q^2_0], \quad E_1^{(32)} = \frac{i}{q_0} [q\bar{q}_\pm - \bar{q}q_\pm].
\]
\[
F_0^{(12)} = F_0^{(13)} = F_0^{(21)} = F_0^{(31)} = F_0^{(24)} = F_0^{(32)} = 1,
\]
\[
F_0^{(11)} = F_0^{(22)} = F_0^{(33)} = 1,
\]
\[
F_0^{(21)} = \frac{\bar{q}q_\pm + q\bar{q}_\pm}{q_0^2}, \quad F_0^{(23)} = \frac{q^2\bar{q}_\pm^2 - \overline{q^2}q_\pm^2}{q_0^4},
\]
\[
F_1^{(13)} = F_1^{(31)} = -\frac{i}{q_0^3} [\bar{q}q_\pm + q\bar{q}_\pm - q^2_0].
\]

□

**Theorem 4.4.** The asymptotic of $a_{jj}$ as $z \to \infty$
\[
a_{11} = \frac{q_{1j}^1}{q_{0j}^1} q_{-} + o\left(\frac{1}{z}\right), \quad a_{22} = \frac{q_{2j}^1}{q_{0j}^1} q_{-} + o\left(\frac{1}{z}\right), \quad a_{33} = 1 + o\left(\frac{1}{z}\right), \quad (4.7a)
\]
\[
b_{11} = \frac{q_{1j}^1}{q_{0j}^2} q_{+} + o\left(\frac{1}{z}\right), \quad b_{22} = \frac{q_{2j}^1}{q_{0j}^2} q_{+} + o\left(\frac{1}{z}\right), \quad b_{33} = 1 + o\left(\frac{1}{z}\right), \quad (4.7b)
\]

**Theorem 4.5.** The asymptotic of $a_{jj}$ as $z \to 0$
\[
a_{11} = 1 + o(z), \quad a_{22} = \frac{q_{1j}^1}{q_{0j}^1} q_{+} + o(z), \quad a_{33} = \frac{q_{2j}^1}{q_{0j}^1} q_{+} + o(z), \quad (4.8a)
\]
\[
b_{11} = 1 + o(z), \quad b_{22} = \frac{q_{1j}^1}{q_{0j}^2} q_{+} + o(z), \quad b_{33} = \frac{q_{2j}^1}{q_{0j}^2} q_{+} + o(z), \quad (4.8b)
\]

5 Distribution of Discrete Spectral

For the $3 \times 3$ matrix spectral problem, the characters of discrete spectral is more complicate than $2 \times 2$ matrix spectral problem. For convenient, we firstly
introduce the following $3 \times 3$ matrix:

$$
\Xi_1(z) = (\chi_1(z), \Phi_{-2}(z), \Phi_{+3}(z)), \quad z \in D_1
$$

$$
\Xi_2(z) = (\chi_2(z), \Phi_{+2}(z), \Phi_{-3}(z)), \quad z \in D_2
$$

$$
\Xi_3(z) = (\Phi_{-1}(z), \Phi_{+2}(z), \chi_3(z)), \quad z \in D_3
$$

$$
\Xi_4(z) = (\Phi_{+1}(z), \Phi_{-2}(z), \chi_4(z)), \quad z \in D_4
$$

Then the determinant of $\Xi_j(z)$ can be derived via some explicit calculation:

$$
\text{Wr} \Xi_1(z) = a_{22}(z)b_{33}(z)e^{i\theta_2}, \quad \text{Wr} \Xi_2(z) = a_{33}(z)b_{22}(z)e^{i\theta_2},
$$

$$
\text{Wr} \Xi_3(z) = a_{11}(z)b_{22}(z)e^{i\theta_2}, \quad \text{Wr} \Xi_4(z) = a_{22}(z)b_{11}(z)e^{i\theta_2}.
$$

The columns of $\Xi_1(z)$ is linearly dependent at the zeros of $a_{22}(z)$ and $b_{33}(z)$.

In the similar way, the other equations can be obtained analogously.

For the existence of symmetries of scattering coefficient, we can known that the zeros is dependent of each other.

Lemma 13. Let $\text{Im} z_0 > 0$, then

$$
a_{22}(z_0) = 0 \iff b_{22}(\bar{z}_0) = 0 \iff b_{22}(-\frac{q_0^2}{z_0}) = 0 \iff a_{22}(-\frac{q_0^2}{z_0}) = 0.
$$

Lemma 14. Let $\text{Im} z_0 > 0$ and $|z_0|\geq q_0$ then

$$
b_{33}(z_0) = 0 \iff a_{33}(\bar{z}_0) = 0 \iff a_{11}(-\frac{q_0^2}{z_0}) = 0 \iff b_{11}(-\frac{q_0^2}{z_0}) = 0.
$$

Figure 2: The distribution of discrete spectral.
From Lemma 13 and Lemma 14 we can come to the conclusion that the discrete eigenvalues appear in the four kind points: $z_0, \bar{z}_0, -\frac{q_0}{z_0}, -\frac{q_0}{\bar{z}_0}$. The distribution of discrete spectral can be shown in Figure 2. It is similar with the $2 \times 2$ matrix spectral problem. So, it is enough to just study the zeros of $a_{22}(z)$ and $b_{33}(z)$. The zeros are divided into the following three types:

- ▲ the first kind eigenvalue: $a_{22}(z_0) = 0$ and $b_{33}(z_0) \neq 0$.
- ▲ the second kind eigenvalue: $a_{22}(z_0) \neq 0$ and $b_{33}(z_0) = 0$.
- ▲ the third kind eigenvalue: $a_{22}(z_0) = 0$ and $b_{33}(z_0) = 0$.

Then some results can be obtained:

**Lemma 15.** For $\text{Im} z_0$ and $|z_0| > q_0$, the following results are equivalent

1. $\chi_1(z_0) = 0$;
2. $\chi_4(-\frac{q_0^2}{z_0}) = 0$;
3. $\exists b_0, \text{ so that } \Phi_{+2}(\bar{z}_0) = b_0 \Phi_{-3}(\bar{z}_0)$;
4. $\exists \tilde{b}_0, \text{ so that } \Phi_{-2}(-\frac{q_0^2}{\bar{z}_0}) = \tilde{b}_0 \Phi_{+1}(-\frac{q_0^2}{\bar{z}_0})$;

**Proof.**

1 $\iff$ 2: From (3.18a), it can be proofed.

1 $\iff$ 3: From (2.41) and (3.8), it can be derived.

1 $\iff$ 4: From (3.15), it can be obtained.

In the similar way, we have

**Lemma 16.** For $\text{Im} z_0$ and $|z_0| > q_0$, the following results are equivalent

1. $\chi_2(\bar{z}_0) = 0$;
2. $\chi_3(-\frac{q_0^2}{z_0}) = 0$;
3. $\exists \tilde{b}_0, \text{ so that } \Phi_{-2}(z_0) = \tilde{b}_0 \Phi_{+3}(z_0)$;
4. $\exists \tilde{b}_0, \text{ so that } \Phi_{-2}(-\frac{q_0^2}{z_0}) = \tilde{b}_0 \Phi_{+1}(-\frac{q_0^2}{z_0})$;

**Lemma 17.** For $\text{Im} z_0$ and $|z_0| > q_0$, $a_{22}(z_0)b_{33}(z_0) = 0$. 

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If \( z_0 \) is the first kind eigenvalue, then
\[
\Phi_{-2}(z_0) = h_0 \chi_1(z_0), \quad \chi_2(\bar{z}_0) = \hat{h}_0 \Phi_{+2}(\bar{z}_0),
\]
\[
\chi_3(-\frac{q_0^2}{z_0}) = \hat{h}_0 \Phi_{+2}(-\frac{q_0^2}{z_0}), \quad \Phi_{-2}(-\frac{q_0^2}{z_0}) = \hat{h}_0 \chi_4(-\frac{q_0^2}{z_0}).
\]
\[(1)\]

If \( z_0 \) is the second kind eigenvalue, then
\[
\chi_1(z_0) = f_0 \Phi_{+3}(z_0), \quad \Phi_{-3}(\bar{z}_0) = \hat{f}_0 \chi_2(\bar{z}_0),
\]
\[
\Phi_{-1}(-\frac{q_0^2}{z_0}) = \hat{f}_0 \chi_3(-\frac{q_0^2}{z_0}), \quad \chi_4(-\frac{q_0^2}{z_0}) = \hat{f}_0 \Phi_{+1}(-\frac{q_0^2}{z_0}).
\]
\[(2)\]

If \( z_0 \) is the third kind eigenvalue, then \( \chi_1(z_0) = \chi_2(\bar{z}_0) = 0 \) and
\[
\Phi_{-2}(z_0) = g_0 \Phi_{+3}(z_0), \quad \Phi_{-3}(\bar{z}_0) = \hat{g}_0 \Phi_{+2}(\bar{z}_0),
\]
\[
\Phi_{-1}(-\frac{q_0^2}{z_0}) = \hat{g}_0 \Phi_{+2}(-\frac{q_0^2}{z_0}), \quad \Phi_{-2}(-\frac{q_0^2}{z_0}) = \hat{g}_0 \Phi_{+1}(-\frac{q_0^2}{z_0}).
\]
\[(3)\]

Proof. If \( a_{22}(z_0) = 0 \) and \( b_{33}(z_0) \neq 0 \), we can derived \( \Phi_{-2}(z_0) \) and \( \chi_1(z_0) \) are linear dependent. So there exist a constant value \( h_0 \), such that \( \Phi_{-2}(z_0) = h_0 \chi_1(z_0) \). The rest equations of Lemma 17 can be proofed in the similar way. \( \square \)

For convenient when deriving the residue conditions, Lemma 17 can be rewrite in terms of the modified eigenfunctions.

(1) Let \( \{z_n|n=1,2,\cdots N_1\} \) are the first kind eigenvalues
\[
\mu_{-2}(z_n) = h_n e^{i(\theta_1-\theta_2)(z_n)} m_1(z_n), \quad m_2(\bar{z}_n) = \hat{h}_n e^{-i(\theta_1-\theta_2)(\bar{z}_n)} \mu_{+2}(\bar{z}_n),
\]
\[
m_3(-\frac{q_0^2}{\bar{z}_n}) = \hat{h}_n e^{i(\theta_1+\theta_2)(-\frac{q_0^2}{\bar{z}_n})} \mu_{+2}(-\frac{q_0^2}{\bar{z}_n}), \quad \mu_{-2}(-\frac{q_0^2}{\bar{z}_n}) = \hat{h}_n e^{-i(\theta_1+\theta_2)(-\frac{q_0^2}{\bar{z}_n})} m_4(-\frac{q_0^2}{\bar{z}_n}).
\]
\[(1)\]

(2) If \( \{\zeta_n|n=1,2,\cdots N_2\} \) is the second kind eigenvalues, then
\[
m_1(\zeta_n) = f_n e^{-2i\theta_1(\zeta_n)} \mu_{+3}(\zeta_n), \quad \mu_{-3}(\bar{\zeta}_n) = \hat{f}_n e^{2i\theta_1(\bar{\zeta}_n)} m_2(\bar{\zeta}_n),
\]
\[
\mu_{-1}(-\frac{q_0^2}{\bar{\zeta}_n}) = \hat{f}_n e^{-2i\theta_1(-\frac{q_0^2}{\bar{\zeta}_n})} m_3(-\frac{q_0^2}{\bar{\zeta}_n}), \quad m_4(-\frac{q_0^2}{\bar{\zeta}_n}) = \hat{f}_n e^{2i\theta_1(-\frac{q_0^2}{\bar{\zeta}_n})} \mu_{+1}(-\frac{q_0^2}{\bar{\zeta}_n}).
\]
\[(2)\]

(3) If \( \{\omega_n|n=1,2,\cdots N_3\} \) is the third kind eigenvalues
\[
\mu_{-2}(\omega_n) = g_n e^{-i(\theta_1+\theta_2)(\omega_n)} \mu_{+3}(\omega_n), \quad \mu_{-3}(\bar{\omega}_n) = \hat{g}_n e^{i(\theta_1+\theta_2)(\bar{\omega}_n)} \mu_{+2}(\bar{\omega}_n),
\]
\[
\mu_{-1}(-\frac{q_0^2}{\bar{\omega}_n}) = \hat{g}_n e^{-i(\theta_1-\theta_2)(-\frac{q_0^2}{\bar{\omega}_n})} \mu_{+2}(-\frac{q_0^2}{\bar{\omega}_n}), \quad \mu_{-2}(-\frac{q_0^2}{\bar{\omega}_n}) = \hat{g}_n e^{i(\theta_1-\theta_2)(-\frac{q_0^2}{\bar{\omega}_n})} \mu_{+1}(-\frac{q_0^2}{\bar{\omega}_n}).
\]
\[(3)\]
6 Riemann-Hilbert Problem

In this section, we transform solving the asymptotic matrix spectral problem into solving an appropriate RH problem. So we need construct a jump condition that is similar with $2 \times 2$. The difference is that the jump matrixes are constructed four times. For convenient, we make some expressions that $D_+ = D_1 \cup D_3$ and $D_- = D_2 \cup D_4$.

**Theorem 6.1.** The meromorphic function $M(x, t, z) = M_j(x, t, z)$, $z \in D_j (j = 1, 2, 3, 4)$

$$M^+(x, t, z) = \begin{cases} M_1(x, t, z) = \left( \frac{m_1}{a_{33}}, \frac{\mu - \mu_+}{a_{22}}, \mu_+, \mu_+ \right), & z \in D_1 \\ M_3(x, t, z) = \left( \frac{\mu - \mu_+}{a_{11}}, \mu_+, \frac{m_3}{a_{22}}, \mu_+ \right), & z \in D_3 \end{cases}$$

$$M^-(x, t, z) = \begin{cases} M_2(x, t, z) = \left( \frac{m_2}{a_{22}}, \mu_+ + \mu, \frac{\mu - \mu_+}{a_{33}} \right), & z \in D_2 \\ M_4(x, t, z) = \left( \mu_+, \frac{\mu - \mu_+}{a_{22}}, \frac{m_4}{a_{11}} \right), & z \in D_4 \end{cases}$$

and $M_j(x, t, z)$ hold the jump conditions

$$M^+(x, t, z) = M^-(x, t, z)[I - e^{i \Theta(x, t, z)}L(z)e^{-i \Theta(x, t, z)}], \ z \in \Sigma$$

where $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Sigma_4$, here $\Sigma_j = \bar{D}_j \cap \bar{D}_{j+1} (j = 1, 2, 3, 4)$. The matrix
$L(z)$ are given as

$$L(z) = \begin{pmatrix} -\rho_2\left(-\frac{q_0^2}{z^2}\right) & \rho_3(z)\rho_2\left(-\frac{q_0^2}{z^2}\right) & -\frac{\rho}{q_0}\rho_3\left(-\frac{q_0^2}{z^2}\right) & \rho_2\left(-\frac{q_0^2}{z^2}\right) \\ l_1 & l_2 & l_3 \\ -\bar{\rho}_2\left(-\frac{q_0^2}{z^2}\right) & -\rho_3(z) & 0 \end{pmatrix}, \quad z \in \Sigma_1,$$

$$L(z) = \begin{pmatrix} \rho_2(z)\rho_2\left(-\frac{q_0^2}{z^2}\right) & 0 & \rho_2\left(-\frac{q_0^2}{z^2}\right) \\ l_4 & 0 & l_5 \\ \rho_2(z) & 0 & 0 \end{pmatrix}, \quad z \in \Sigma_2,$$

$$L(z) = \begin{pmatrix} l_6 & l_7 & l_8 \\ -\rho_1(z) & 0 & \bar{\rho}_3(z)\gamma(z) \\ -\rho_2(z) + \rho_1(z)\rho_3(z) & \rho_3(z) & -\gamma z\bar{\rho}_3(z)\rho_3(z) \end{pmatrix}, \quad z \in \Sigma_3,$$

$$L(z) = \begin{pmatrix} \bar{\rho}_2\left(-\frac{q_0^2}{z^2}\right)\bar{\rho}_2(\bar{z}) & 0 & -\bar{\rho}_2(\bar{z}) \\ 0 & 0 & 0 \\ \bar{\rho}_2\left(-\frac{q_0^2}{\bar{z}^2}\right) & 0 & 0 \end{pmatrix}, \quad z \in \Sigma_4,$$
where

\[
\begin{align*}
\ell_1 &= -\left(\gamma(z)\frac{z}{i\eta_0} \rho_3(-\frac{q_0^2}{z}) + \rho_2(-\frac{q_0^2}{z}) \tilde{\rho}_3(-\frac{q_0^2}{z}) + \frac{\rho_2(-\frac{q_0^2}{z}) \rho_1(-\frac{q_0^2}{z})}{z}\right), \\
\ell_2 &= \gamma(z)\frac{z}{i\eta_0} \left(\rho_3(z) \rho_2(-\frac{q_0^2}{z}) \tilde{\rho}_3(-\frac{q_0^2}{z}) - \frac{z}{i\eta_0} \rho_3(-\frac{q_0^2}{z}) \tilde{\rho}_3(-\frac{q_0^2}{z}) \right) + \frac{i\eta_0}{z} \rho_3(z) \rho_1(-\frac{q_0^2}{z}), \\
\ell_3 &= \gamma(z)\frac{z}{i\eta_0} \rho_2(-\frac{q_0^2}{z}) \tilde{\rho}_3(-\frac{q_0^2}{z}) + \frac{i\eta_0}{z} \rho_1(-\frac{q_0^2}{z}), \\
\ell_4 &= \gamma(z)\frac{z}{-i\eta_0} \left(\rho_2(z) \rho_2(-\frac{q_0^2}{z}) \tilde{\rho}_3(-\frac{q_0^2}{z}) - \frac{\rho_2(z) \rho_1(-\frac{q_0^2}{z})}{z}\right), \\
\ell_5 &= \gamma(z) \tilde{\rho}_3(-\frac{q_0^2}{z}) + \gamma(z)\frac{z}{-i\eta_0} \rho_2(-\frac{q_0^2}{z}) \tilde{\rho}_3(-\frac{q_0^2}{z}) + \frac{i\eta_0}{z} \rho_1(-\frac{q_0^2}{z}), \\
\ell_6 &= \frac{z}{i\eta_0} \rho_1(z) \rho_3(-\frac{q_0^2}{z}) - \rho_2(z) \tilde{\rho}_2(\bar{z}) + \rho_1(z) \rho_3(z) \tilde{\rho}_2(\bar{z}), \\
\ell_7 &= \frac{z}{i\eta_0} \rho_3(-\frac{q_0^2}{z}) + \rho_3(z) \tilde{\rho}_2(\bar{z}), \\
\ell_8 &= -\gamma(z)\frac{z}{i\eta_0} \tilde{\rho}_3(\bar{z}) (\rho_3(-\frac{q_0^2}{z}) - \gamma(z) \rho_3(z) \tilde{\rho}_3(\bar{z}) \rho_2(\bar{z}) - \bar{\rho}_2(\bar{z}).
\end{align*}
\]

**Proof.** From (6.23) and Theorem 2.5, we can obtain

\[
\frac{\Phi_{-2}}{a_{22}} = \left( \frac{a_{12}}{a_{22}} - \frac{a_{32} a_{13}}{a_{22} a_{33}} \right) \frac{\chi_2}{b_{22}} + \left( 1 + \frac{a_{12} b_{21}}{a_{22} b_{22}} \right) \frac{1}{a_{22}} \frac{a_{32} a_{23}}{a_{33}} \Phi_{+,2} + \frac{a_{32} \Phi_{-,3}}{a_{33}},
\]

\[
\Phi_{+,3} = -\frac{a_{13} \chi_2}{a_{33} b_{22}} + \left( 1 + \frac{a_{13} b_{21}}{a_{33} b_{22}} \right) \frac{\Phi_{+,2}}{a_{33}},
\]

\[
\frac{\chi_1}{b_{33}} = \left( 1 + \frac{b_{31} a_{13}}{b_{33} a_{33}} \right) \frac{\chi_2}{b_{22}} + \left( 1 + \frac{b_{31} a_{13} b_{21}}{b_{33} a_{33} b_{22}} + \frac{b_{31} a_{23}}{b_{33} a_{33}} \right) \Phi_{+,2} - \frac{b_{31} \Phi_{-,3}}{b_{33} a_{33}}.
\]

Then we can obtain the matrix \( L(z) \) as \( z \in \Sigma_1 \). The others can be derived in the similar way. \( \square \)

From the asymptotic behavior of \( \mu_\pm \), \( m_j \) and scattering coefficient in Section 5, the asymptotic behavior of \( M(x, t, z) \) can be obtained:

\[
M(x, t, z) = \begin{pmatrix}
-\frac{q_0^2}{z} & \frac{q_0^2}{z} & 0 \\
0 & 0 & 1 \\
\frac{q_0}{z} & 0 & 0
\end{pmatrix} + o(\frac{1}{z}), \quad z \to \infty, \quad (6.4)
\]

\[
M(x, t, z) = \begin{pmatrix}
0 & 0 & -\frac{i q_0}{z} \\
\frac{i q_0}{z} & 0 & 0
\end{pmatrix} + o(1), \quad z \to 0, \quad (6.5)
\]
For convenient, we define the notation $M^\pm = (m_1^\pm, m_2^\pm, m_3^\pm)$, and $M_{-1,\nu}(x, t)$ represent the residue of $M^\pm$ at $z = \nu$.

**Theorem 6.2.** The meromorphic functions $M^\pm$ satisfy the residue conditions

\begin{align}
M_{-1, \nu}^+ &= \left(0, H_n m_1^+ (z_n) e^{i(\theta_1 - \theta_2) (\nu_n)}, 0\right), \\
M_{-1, \nu}^- &= \left(\hat{H}_n m_2^- (\tilde{z}_n) e^{-i(\theta_1 - \theta_2) (\nu_n)}, 0, 0\right), \\
M_{-1, \nu}^+ &= \left(0, 0, \hat{H}_n m_2^- (\tilde{z}_n) e^{i(\theta_1 + \theta_2) (-\frac{\nu_n}{\nu})}, 0\right), \\
M_{-1, \nu}^- &= \left(0, \frac{\nu_n}{\nu} \hat{H}_n m_1^+ (z_n) e^{-i(\theta_1 + \theta_2) (-\frac{\nu_n}{\nu})}, 0, 0\right),
\end{align}

\begin{align}
M_{-1, \nu}^+ &= \left(0, 0, \hat{F}_n m_1^- (\tilde{z}_n) e^{2i\theta_1 (\nu_n)}, 0\right), \\
M_{-1, \nu}^- &= \left(0, \frac{\nu_n}{\nu} \hat{F}_n m_1^- (\tilde{z}_n) e^{2i\theta_1 (\nu_n)}, 0, 0\right),
\end{align}

where

\begin{align}
H_n &= \frac{h_n b_{33}(z_n)}{a_{22}^\nu(z_n)}, & \hat{H}_n &= \frac{\hat{h}_n}{\tilde{b}_{22}^\nu(\tilde{z}_n)}, & \tilde{H}_n &= \frac{\tilde{h}_n}{\tilde{b}_{22}(\tilde{z}_n)} , & \tilde{\tilde{H}}_n &= \frac{\tilde{\tilde{h}}_n b_{33}(z_n)}{a_{22}^\nu(\tilde{z}_n)}, \\
F_n &= \frac{f_n}{b_{33}(\zeta_n)}, & \hat{F}_n &= \frac{\hat{f}_n b_{22}(\tilde{\zeta}_n)}{a_{33}^\nu(\zeta_n)}, & \hat{\tilde{F}}_n &= \frac{\hat{\tilde{f}}_n}{b_{22}(\tilde{\zeta}_n)} , & \tilde{F}_n &= \frac{\tilde{f}_n b_{22}(\tilde{\zeta}_n)}{a_{33}(\tilde{z}_n)} ,
\end{align}

\begin{align}
G_n &= \frac{g_n}{a_{22}^\nu(\omega_n)}, & \tilde{G}_n &= \frac{\tilde{g}_n}{a_{33}(\tilde{\omega}_n)}, & \tilde{\tilde{G}}_n &= \frac{\tilde{\tilde{g}}_n}{a_{33}^\nu(\tilde{\omega}_n)}.
\end{align}
Lemma 18. The constants in Theorem 17 obey the following relations
\[\tilde{h}_n = \frac{i\gamma_0}{z_n} h_n, \quad \hat{h}_n = \frac{i\gamma_0}{z_n} \hat{h}_n = -\gamma(\bar{z}_n)a_{33}(\bar{z}_n)\tilde{h}_n \quad (6.8a)\]
\[\tilde{f}_n = f_n, \quad \hat{f}_n = \hat{f}_n = -\frac{f_n}{a_{22}(\zeta_n)} \quad (6.8b)\]
\[\tilde{g}_n = \frac{i\gamma_0}{\omega_n} g_n, \quad \hat{g}_n = \frac{i\gamma_0}{\omega_n} \bar{g}_n = -\frac{\bar{a}_3'(\bar{\omega}_n)}{b_{22}(\bar{\omega}_n)} \gamma(\bar{\omega}_n)\bar{g}_n. \quad (6.8c)\]

Lemma 19. The constants in Theorem 6.2 obey the following relations
\[\hat{H}_n = -\gamma(\bar{z}_n)\bar{H}_n, \quad \hat{H}_n = \frac{i\gamma_0}{z_n} \gamma(\bar{z}_n)\bar{H}_n, \quad \hat{H}_n = -\frac{z_n}{i\gamma_0} H_n, \quad (6.9a)\]
\[\hat{F}_n = -\bar{F}_n, \quad \hat{F}_n = -\frac{\bar{a}_2(\zeta_n)}{\zeta_n} F_n, \quad \hat{F}_n = \frac{\bar{a}_2(\zeta_n)}{\zeta_n} F_n, \quad (6.9b)\]
\[\hat{G}_n = -\gamma(\bar{\omega}_n)\bar{G}_n, \quad \hat{G}_n = -\frac{\bar{a}_3(\bar{\omega}_n)}{i\gamma_0} \gamma(\bar{\omega}_n)\bar{G}_n, \quad \hat{G}_n = -\frac{\omega_n}{i\gamma_0} G_n. \quad (6.9c)\]

Proof of Lemma 19
From equation (3.7) and the first symmetry of Jost eigenfunctions
\[\left(\Phi^\dagger_{\pm}(x,t,\bar{z})\right)^{-1}C(z) = \Phi_{\pm}(x,t,z), \quad (6.10)\]
we obtain
\[\Phi_{-,3}(\bar{z}) = \frac{\gamma(\bar{z})}{\det \Phi_{-,2}(\bar{z})} [\Phi_{-,1}(\bar{z}) \times \Phi_{-,2}(\bar{z})], \quad (6.11a)\]
\[\Phi_{+,2}(\bar{z}) = \frac{1}{\det \Phi_{+,2}(\bar{z})} [\Phi_{+,3}(\bar{z}) \times \Phi_{+,1}(\bar{z})]. \quad (6.11b)\]

Substituting them into the first two equations of (3) in Lemma 17 and combine the relation (2.23), we deriving the result
\[\bar{g}_0 \gamma(\bar{z}_0) [a_{11}(z_0)\Phi_{+,3}(z_0)\times \Phi_{+,1}(z_0) + a_{21}(z_0)\Phi_{+,3}(z_0) \times \Phi_{+,2}(z_0)] = \bar{g}_0 \Phi_{+,3}(z_0) \times \Phi_{+,1}(z_0). \quad (6.12)\]

One can dot product the vector $\Phi_{+,2}(z_0)$ on the both sides of equation (6.12).

Then $\bar{g}_0 = -\bar{g}_0 \gamma(\bar{z}_0) a_{11}(z_0)$.

6.1 Trace Formula

By $B = A^{-1}$ and det $A = \det B = 1$, one can obtain
\[a_{22}b_{22} = 1/[1 + \frac{a_{12}b_{21} + a_{32}b_{23}}{a_{22}b_{22}}]. \quad (6.13)\]
Furthermore, taking logarithmic of both sides of (6.13) and combined with the reflection coefficients (3.19), we can get

\[ \log a_{22}(z) - \log \frac{1}{b_{22}(z)} = J_0(z). \]  

(6.14)

where

\[ J_0(z) = \log [1 + \gamma(z) \rho_3(-\frac{q_0^2}{z})\tilde{\rho}_3(-\frac{q_0^2}{\bar{z}}) + \gamma(z) \rho_3(z)\bar{\rho}_3(\bar{z})]. \]

Since \( a_{22} \) and \( b_{22} \) are analytic in the upper-half plane and lower-half plane, equation (6.14) is a jump condition for a scalar, additive Riemann-Hilbert problem. To circumvent the pole singularities coming from the zeros of \( a_{22} \) and \( b_{22} \), we define two analytic function

\[ \beta^+(z) = a_{22}(z)e^{i\Delta \theta} \prod_{n=1}^{N_1} \frac{z - \bar{z}_n}{z - z_n} \frac{z - \bar{\omega}_n}{z - \omega_n} \frac{z - \bar{z}_n}{z - \omega_n} \frac{z - \bar{\omega}_n}{z - \omega_n}, \quad z \in \mathbb{C}^+ \]  

(6.15a)

\[ \beta^-(z) = \frac{1}{b_{22}(z)} e^{i\Delta \theta} \prod_{n=1}^{N_1} \frac{z - \bar{z}_n}{z - z_n} \frac{z - \bar{\omega}_n}{z - \omega_n} \frac{z - \bar{z}_n}{z - \omega_n} \frac{z - \bar{\omega}_n}{z - \omega_n}, \quad z \in \mathbb{C}^- \]  

(6.15b)

where \( \Delta \theta = \theta_+ - \theta_- \), and \( \beta^\pm(z) \) no zeros in \( \mathbb{C}^\pm \) and approaches 1 as \( z \to \infty \) in the analytic region in \( z \)-plane. Using the Plemej’s formula and the projectors \( P^\pm \) we obtain \( \log \beta(z) = P(\log[\beta^+\beta^-]) \) for \( z \in \mathbb{C} \setminus \Sigma \), where \( P \) is the Cauchy projectors

\[ P_\pm(f)(z) = \frac{1}{2\pi i} \int_{\Sigma} \frac{f(\xi)}{\xi - (z \pm i0)}\xi, \]  

(6.16)

Combine (6.14) and taking the exponentials,

\[ a_{22}(z) = \exp \left(-i\Delta \theta - \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(\xi)}{\xi - z} d\xi \right) \prod_{n=1}^{N_1} \frac{z - \bar{z}_n}{z - z_n} \frac{z - \bar{\omega}_n}{z - \omega_n} \frac{z - \bar{z}_n}{z - \omega_n} \frac{z - \bar{\omega}_n}{z - \omega_n}, \]  

(6.17)

In the similar way as above yields

\[ \log b_{33}(z) - \log \frac{1}{a_{33}(z)} = -\log \left[ 1 + \frac{1}{\gamma(z)\rho_1(-\frac{q_0^2}{z})\bar{\rho}_1(-\frac{q_0^2}{\bar{z}}) + \rho_2(-\frac{q_0^2}{z})\bar{\rho}_2(-\frac{q_0^2}{\bar{z}})} \right], \]  

(6.18a)

\[ \log a_{11}(z) - \log \frac{1}{b_{11}(z)} = -\log \left[ 1 + \gamma(z)\rho_1(z)\bar{\rho}_1(z) + \rho_2(z)\bar{\rho}_2(z) \right], \]  

(6.18b)
However, $b_{33}(z)$, $a_{43}(z)$, $a_{11}(z)$ and $b_{11}(z)$ are only analytic in $D_1$, $D_2$, $D_3$, $D_4$. This is the reason that this situation is complicated than above. To formulate an appropriate Riemann-Hilbert problem, we have to introduce a sectionally function which is analytic on the whole $z$-plane. So we obtain

\begin{align}
\log b_{33}(z) - \log \frac{1}{b_{11}(z)} &= \log a_{22}(z) - \log [1 - \beta_2(\bar{z})\bar{\beta}_2(-\frac{q_0^2}{z})], \quad (6.19a) \\
\log a_{11}(z) - \log \frac{1}{a_{33}(z)} &= \log b_{22}(z) - \log [1 - \rho_2(z)\rho_2(-\frac{q_0^2}{z})], \quad (6.19b)
\end{align}

Next, we define the following sectionally analytic function

\begin{align}
\hat{\beta}^+(z) &= \begin{cases} \beta_1(z), & z \in D_1 \\
\beta_2(z), & z \in D_2 \\
\beta_3(z), & z \in D_3 \\
\beta_4(z), & z \in D_4 \end{cases} \\
\hat{\beta}^-(z) &= \begin{cases} \beta_1(z), & z \in D_1 \\
\beta_2(z), & z \in D_2 \\
\beta_3(z), & z \in D_3 \\
\beta_4(z), & z \in D_4 \end{cases}
\end{align}

where

\begin{align}
\beta_1(z) &= \frac{b_{33}(z)}{a_{33}(z)} \prod_{n=1}^{N_1} \frac{z - z_n}{z - \tilde{z}_n} \prod_{n=1}^{N_2} \frac{z - \bar{\zeta}_n}{z - \zeta_n}, \\
\beta_2(z) &= \frac{s_1(z)}{a_{33}(z)} \prod_{n=1}^{N_1} \frac{z - z_n}{z - \tilde{z}_n} \prod_{n=1}^{N_2} \frac{z - \bar{\zeta}_n}{z - \zeta_n}, \\
\beta_3(z) &= \frac{a_{11}(z)}{s_2(z)} \prod_{n=1}^{N_1} \frac{z - z_n}{z - \tilde{z}_n} \prod_{n=1}^{N_2} \frac{z - \bar{\zeta}_n}{z - \zeta_n}, \\
\beta_4(z) &= \frac{s_2(z)}{b_{11}(z)} \prod_{n=1}^{N_1} \frac{z - z_n}{z - \tilde{z}_n} \prod_{n=1}^{N_2} \frac{z - \bar{\zeta}_n}{z - \zeta_n}
\end{align}

where

\begin{align}
s_1(z) &= \prod_{n=1}^{N_1} \frac{z - z_n}{z - \tilde{z}_n} \prod_{n=1}^{N_3} \frac{z - \omega_n}{z - \tilde{\omega}_n}, \\
s_2(z) &= \prod_{n=1}^{N_1} \frac{z - z_n}{z - \tilde{z}_n} \prod_{n=1}^{N_3} \frac{z - \omega_n}{z - \tilde{\omega}_n}
\end{align}

Then (6.18) can be rewritten as

\begin{align}
\log \beta_1(z) - \log \beta_2(z) &= J_1(z), \\
\log \beta_3(z) - \log \beta_4(z) &= J_3(z), 
\end{align}

where

\begin{align}
J_1(z) &= -\log \left[ 1 + \frac{1}{\gamma(z)}\rho_1(-\frac{q_0^2}{z})\bar{\rho}_1(-\frac{q_0^2}{\bar{z}}) + \rho_2(-\frac{q_0^2}{z})\bar{\rho}_2(-\frac{q_0^2}{\bar{z}}) \right], \quad (6.23a) \\
J_3(z) &= -\log \left[ 1 + \gamma(z)\rho_1(z)\bar{\rho}_1(\bar{z}) + \rho_2(z)\bar{\rho}_2(\bar{z}) \right], \quad (6.23b)
\end{align}
Combine (6.17) and $\beta_j(z)$, (6.19) can be rewritten as

$$\log \beta_3(z) - \log \beta_2(z) = J_2(z), \quad \log \beta_1(z) - \log \beta_4(z) = J_4(z),$$  \hspace{1cm} (6.24)

where

$$J_2(z) = \frac{1}{2\pi i} \log \int_{\mathbb{C}} \frac{\tilde{J}_0(\xi)}{\xi - z} d\xi - \log [1 - \tilde{\rho}_2(\xi)\tilde{\rho}_2(-\frac{\eta_0^2}{z})],$$ \hspace{1cm} (6.25a)

$$J_4(z) = -\frac{1}{2\pi i} \log \int_{\mathbb{C}} \frac{J_0(\xi)}{\xi - z} d\xi - \log [1 - \rho_2(z)\rho_2(-\frac{\eta_0^2}{z})].$$ \hspace{1cm} (6.25b)

Now, (6.22) and (6.24) form the jump conditions for the Riemann-Hilbert problem with the analytic functions $\hat{\beta}^\pm(z)$. It can be precisely rewritten as

$$\log \hat{\beta}^+(z) - \log \hat{\beta}^-(z) = J_i(z), \quad J_i(z) = \Sigma_i(i = 1, 2, 3, 4).$$ \hspace{1cm} (6.26)

Applying the projectors $P^\pm$ and Plemelj’s formula yields

$$\log \hat{\beta}(z) = \frac{1}{2\pi i} \int_{\Sigma} \frac{J(\xi)}{\xi - z} d\xi, \quad z \in \mathbb{C} \setminus \Sigma$$ \hspace{1cm} (6.27)

Taking the exponential of both sides with $z \in D_1$ yields

$$\frac{b_{33}(z)}{s_1(z)} = \exp \left( \frac{1}{2\pi i} \int_{\Sigma} \frac{J(\xi)}{\xi - z} d\xi \right) \prod_{n=1}^{N_1} \frac{z - \zeta_n}{z - \zeta_n - \left(-\frac{\eta_0^2}{\zeta_n}\right)} \prod_{n=1}^{N_2} \frac{z - \zeta_n}{z - \zeta_n - \left(-\frac{\eta_0^2}{\zeta_n}\right)} \prod_{n=1}^{N_3} \frac{z - \omega_n}{z - \omega_n},$$ \hspace{1cm} (6.28)

Next, we obtain

$$b_{33}(z) = \exp \left( \frac{1}{2\pi i} \int_{\Sigma} \frac{J(\xi)}{\xi - z} d\xi \right) \prod_{n=1}^{N_1} \frac{z - \zeta_n}{z - \zeta_n - \left(-\frac{\eta_0^2}{\zeta_n}\right)} \prod_{n=1}^{N_2} \frac{z - \zeta_n}{z - \zeta_n - \left(-\frac{\eta_0^2}{\zeta_n}\right)} \prod_{n=1}^{N_3} \frac{z - \omega_n}{z - \omega_n},$$ \hspace{1cm} (6.29)

The asymptotic phase difference $\Delta \theta = \theta_+ - \theta_-$ is given by

$$\Delta \theta = \frac{1}{2\pi} \int_{\Sigma} \frac{J(\xi)}{\xi} d\xi + 2 \sum_{n=1}^{N_1} \arg z_n - 4 \sum_{n=1}^{N_2} \arg \zeta_n - 2 \sum_{n=1}^{N_3} \arg \omega_n$$ \hspace{1cm} (6.30)

7 Solution of Riemann-Hilbert Problem

**Theorem 7.1.** Let $\nu_n$ is the set of all discrete spectral, the solution of the RH problem in Theorem 6.1 is given by

$$M(x, t, z) = \Gamma_+(z) + \sum_{n=1}^{N} \left( \frac{M^+_1}{z - \nu_n} + \frac{M^-_1}{z - \nu_n} + \frac{M^+_2}{z - \left(-\frac{\eta_0^2}{\nu_2}\right)} + \frac{M^-_2}{z - \left(-\frac{\eta_0^2}{\nu_2}\right)} \right) - \frac{1}{2\pi i} \int_{\Sigma} \frac{M^-(\xi)\bar{L}(\xi)}{\xi - z} d\xi,$$ \hspace{1cm} (7.1)
where \( N = N_1 + N_2 + N_3, \hat{L} = e^{i\theta}Le^{-i\theta} \). The modified eigenfunctions in the residue conditions can be given as

\[
m_2^-(z) = \begin{pmatrix} \frac{a_z^+}{q_0} \\ \frac{q_0}{z} \end{pmatrix} + \frac{N_1}{\sum_{n=1}^{N_1}} \left( \frac{H_ne^{i(\theta_1 - \theta_2)(z_n)}}{z - z_n} + \frac{\tilde{H}_ne^{-i(\theta_1 + \theta_2)(-\frac{q_0^2}{z_n})}}{z - (-\frac{q_0^2}{z_n})} \right) m_1^+(z_n)
\]
\[+ \sum_{n=1}^{N_2} \left( \frac{G_ne^{-i(\theta_1 + \theta_2)(\omega_n)}}{z - \omega_n} + \frac{\tilde{G}_ne^{i(\theta_1 - \theta_2)(-\frac{q_0^2}{\omega_n})}}{z - (-\frac{q_0^2}{\omega_n})} \right) m_3^+(\omega_n) - \frac{1}{2\pi i} \int_{\Sigma} \frac{(M - \hat{L}(\xi))z}{\xi - z} d\xi \]

(7.2a)

\[
m_1^+(z) = \begin{pmatrix} \frac{a_z^+}{q_0} \\ \frac{q_0}{z} \end{pmatrix} + \frac{N_1}{\sum_{n=1}^{N_1}} \left( \frac{\tilde{H}_ne^{-i(\theta_1 - \theta_2)(\bar{z}_n)}}{z - \bar{z}_n} m_2^-(\bar{z}_n) + \sum_{n=1}^{N_2} \frac{F_n e^{-2i\theta_1(\bar{z}_n)}}{z - \bar{z}_n} m_3^+(\bar{z}_n) \right)
\]
\[+ \sum_{n=1}^{N_2} \frac{\tilde{F}_n e^{-2i\theta_1(-\frac{q_0^2}{\bar{z}_n})}}{z - (-\frac{q_0^2}{\bar{z}_n})} m_1^-(\bar{z}_n) + \sum_{n=1}^{N_3} \frac{\tilde{G}_n e^{i(\theta_1 - \theta_2)(-\frac{q_0^2}{\bar{z}_n})}}{z - (-\frac{q_0^2}{\bar{z}_n})} m_2^-(\bar{z}_n) - \frac{1}{2\pi i} \int_{\Sigma} \frac{(M - \hat{L}(\xi))}{\xi - z} d\xi \]

(7.2b)

\[
m_1^-(z) = \begin{pmatrix} \frac{a_z^+}{q_0} \\ \frac{q_0}{z} \end{pmatrix} + \frac{N_1}{\sum_{n=1}^{N_1}} \left( \frac{\tilde{H}_ne^{-i(\theta_1 - \theta_2)(\bar{z}_n)}}{z - \bar{z}_n} m_2^-(\bar{z}_n) + \sum_{n=1}^{N_2} \frac{F_n e^{-2i\theta_1(\bar{z}_n)}}{z - \bar{z}_n} m_3^+(\bar{z}_n) \right)
\]
\[+ \sum_{n=1}^{N_2} \frac{\tilde{F}_n e^{-2i\theta_1(-\frac{q_0^2}{\bar{z}_n})}}{z - (-\frac{q_0^2}{\bar{z}_n})} m_1^-(\bar{z}_n) + \sum_{n=1}^{N_3} \frac{\tilde{G}_n e^{i(\theta_1 - \theta_2)(-\frac{q_0^2}{\bar{z}_n})}}{z - (-\frac{q_0^2}{\bar{z}_n})} m_2^-(\bar{z}_n) - \frac{1}{2\pi i} \int_{\Sigma} \frac{(M - \hat{L}(\xi))}{\xi - z} d\xi \]

(7.2c)

\[
m_3^+(z) = \begin{pmatrix} \frac{a_z^+}{q_0} \\ \frac{q_0}{z} \end{pmatrix} + \frac{N_1}{\sum_{n=1}^{N_1}} \left( \frac{H_ne^{i(\theta_1 + \theta_2)(-\frac{q_0^2}{z_n})}}{z - (-\frac{q_0^2}{z_n})} m_2^-(\bar{z}_n) + \sum_{n=1}^{N_2} \frac{\tilde{F}_n e^{2i\theta_1(\bar{z}_n)}}{z - \bar{z}_n} m_1^-(\bar{z}_n) \right)
\]
\[+ \sum_{n=1}^{N_2} \frac{\tilde{F}_n e^{2i\theta_1(-\frac{q_0^2}{\bar{z}_n})}}{z - (-\frac{q_0^2}{\bar{z}_n})} m_3^+(-\omega_n) + \sum_{n=1}^{N_3} \frac{\tilde{G}_n e^{i(\theta_1 + \theta_2)(-\frac{q_0^2}{\bar{z}_n})}}{z - (-\frac{q_0^2}{\bar{z}_n})} m_2^+(\omega_n) - \frac{1}{2\pi i} \int_{\Sigma} \frac{(M - \hat{L}(\xi))^3}{\xi - z} d\xi \]

(7.2d)

Theorem 7.2. The solution of Riemann-Hilbert problem in Theorem 7.1 is
reconstructed as

\[ q(x, t) = q_+ + \sum_{n=1}^{N_1} \hat{H}_n e^{i(\theta_1 + \theta_2)(-\frac{q_n^2}{\bar{z}_n^2})} m_{12}^{-}(\bar{z}_n) + i \sum_{n=1}^{N_2} \zeta_n \hat{F}_n e^{2i\theta_1(-\frac{q_n^2}{\bar{z}_n^2})} m_{13}^+(\zeta_n) \]

\[ + i \sum_{n=1}^{N_2} \hat{F}_n e^{2i\theta_1(\zeta_n)} m_{11}^-(\zeta_n) + i \sum_{n=1}^{N_3} \hat{G}_n e^{i(\theta_1 + \theta_2)(\bar{\omega}_n)} m_{12}^-(\bar{\omega}_n) \]

(7.3)

\[ m_{12}^+(\bar{z}_n) = \frac{q_+}{q_0} + \sum_{n=1}^{N_1} \left( \hat{H}_n e^{i(\theta_1 - \theta_2)(\bar{z}_n)} + z_n \frac{\hat{H}_n e^{-i(\theta_1 + \theta_2)(-\frac{q_n^2}{\bar{z}_n^2})}}{iq_0} \right) m_{11}^+(\bar{z}_n) \]

\[ + \sum_{n=1}^{N_3} \left( \frac{G_n e^{-i(\theta_1 + \theta_2)(\omega_n)}}{\bar{z}_n - \bar{\omega}_n} + \frac{\omega_n \hat{G}_n e^{i(\theta_1 + \theta_2)(-\frac{q_n^2}{\bar{\omega}_n^2})}}{iq_0} \right) m_{13}^+(\omega_n) \]

(7.4a)

\[ m_{12}^-(\bar{\omega}_n) = \frac{q_+}{q_0} + \sum_{n=1}^{N_1} \left( \hat{H}_n e^{i(\theta_1 - \theta_2)(\bar{z}_n)} + \frac{z_n \hat{H}_n e^{-i(\theta_1 + \theta_2)(-\frac{q_n^2}{\bar{z}_n^2})}}{iq_0} \right) m_{11}^+(\bar{z}_n) \]

\[ + \sum_{n=1}^{N_3} \left( \frac{G_n e^{-i(\theta_1 + \theta_2)(\omega_n)}}{\bar{\omega}_n - \omega_n} + \frac{\omega_n \hat{G}_n e^{i(\theta_1 + \theta_2)(-\frac{q_n^2}{\bar{\omega}_n^2})}}{iq_0} \right) m_{13}^+(\omega_n) \]

(7.4b)

\[ m_{11}^+(\bar{z}_n) = -\frac{q_+}{q_0} + \sum_{n=1}^{N_1} \hat{H}_n e^{-i(\theta_1 - \theta_2)(\bar{z}_n)} m_{12}^-(\bar{z}_n) + \sum_{n=1}^{N_2} \hat{F}_n e^{-2i\theta_1(\zeta_n)} m_{13}^-(\zeta_n) \]

\[ + \sum_{n=1}^{N_2} \frac{\zeta_n}{iq_0} \hat{F}_n e^{-2i\theta_1(-\frac{q_n^2}{\zeta_n^2})} m_{11}^-(\zeta_n) + \sum_{n=1}^{N_3} \hat{G}_n e^{-i(\theta_1 + \theta_2)(-\frac{q_n^2}{\zeta_n^2})} m_{12}^-(\zeta_n) \]

(7.4c)

\[ m_{11}^-(\bar{\zeta}_n) = -\frac{q_+}{q_0} + \sum_{n=1}^{N_1} \hat{H}_n e^{-i(\theta_1 - \theta_2)(\bar{\zeta}_n)} m_{12}^-(\bar{\zeta}_n) + \sum_{n=1}^{N_2} \hat{F}_n e^{-2i\theta_1(\bar{\zeta}_n)} m_{13}^+(\bar{\zeta}_n) \]

\[ + \sum_{n=1}^{N_2} \frac{\bar{\zeta}_n}{iq_0} \hat{F}_n e^{-2i\theta_1(-\frac{q_n^2}{\bar{\zeta}_n^2})} m_{11}^-(\bar{\zeta}_n) + \sum_{n=1}^{N_3} \hat{G}_n e^{-i(\theta_1 + \theta_2)(-\frac{q_n^2}{\bar{\zeta}_n^2})} m_{12}^+(\bar{\zeta}_n) \]

(7.4d)
where \( i' = 1, 2, \cdots, N_1, \; j' = 1, 2, \cdots, N_2 \) and \( l' = 1, 2, \cdots, N_3 \).

For convenient, we define the following notations:

\[
m_{13}^+(\zeta') = -\frac{iq_0}{\zeta'} + \sum_{n=1}^{N_1} \frac{\hat{H}_n e^{i(\theta_1 + \theta_2)}(-\frac{\rho_n}{\omega_n})}{\zeta' - (-\frac{\rho_n}{\omega_n})} m_{12}(\zeta_n) + \sum_{n=1}^{N_2} \frac{\hat{F}_n e^{2i\theta_1}(\zeta_n)}{\zeta' - \zeta_n} m_{11}(\zeta_n) + \sum_{n=1}^{N_3} \frac{\hat{G}_n e^{i(\theta_1 + \theta_2)}(\bar{\omega}_n)}{\zeta' - \bar{\omega}_n} m_{12}(\bar{\omega}_n)
\]

(7.4e)

\[
m_{13}^+(\omega') = -\frac{iq_0}{\omega'} + \sum_{n=1}^{N_1} \frac{\hat{H}_n e^{i(\theta_1 + \theta_2)}(-\frac{\rho_n}{\omega_n})}{\omega' - (-\frac{\rho_n}{\omega_n})} m_{12}(\zeta_n) + \sum_{n=1}^{N_2} \frac{\hat{F}_n e^{2i\theta_1}(\zeta_n)}{\omega' - \zeta_n} m_{11}(\zeta_n) + \sum_{n=1}^{N_3} \frac{\hat{G}_n e^{i(\theta_1 + \theta_2)}(\bar{\omega}_n)}{\omega' - \bar{\omega}_n} m_{12}(\bar{\omega}_n)
\]

(7.4f)

\[
\Delta_n^{(1)}(x, t, z) = \frac{H_n e^{i(\theta_1 - \theta_2)(z_n)}}{z - z_n} + z_n \frac{\hat{H}_n e^{i(\theta_1 + \theta_2)}(-\frac{\rho_n}{\omega_n})}{i\omega_0} \frac{z - (-\frac{\rho_n}{\omega_n})}{z_n}
\]

(7.5a)

\[
\Delta_n^{(2)}(x, t, z) = \frac{G_n e^{-i(\theta_1 + \theta_2)}(\omega_n)}{z - \omega_n} + \omega_n \frac{\hat{G}_n e^{i(\theta_1 - \theta_2)}(-\frac{\rho_n}{\omega_n})}{i\omega_0} \frac{z - (-\frac{\rho_n}{\omega_n})}{\omega_n}
\]

(7.5b)

\[
\Delta_n^{(3)}(x, t, z) = \frac{\hat{H}_n e^{i(\theta_1 - \theta_2)}(\bar{\omega}_n)}{z - \bar{\zeta}_n}
\]

(7.5c)

\[
\Delta_n^{(4)}(x, t, z) = \frac{\hat{F}_n e^{2i\theta_1}(\bar{\zeta}_n)}{z - \bar{\zeta}_n}
\]

(7.5d)

\[
\Delta_n^{(5)}(x, t, z) = \frac{\hat{G}_n e^{i(\theta_1 + \theta_2)}(\bar{\omega}_n)}{z - (-\frac{\rho_n}{\omega_n})}
\]

(7.5e)

\[
\Delta_n^{(6)}(x, t, z) = \frac{\hat{F}_n e^{2i\theta_1}(\bar{\zeta}_n)}{z - \bar{\zeta}_n}
\]

(7.5f)
Substituting (7.4c) and (7.4f) into (7.4a) and (7.4b), we obtain

\[
m_{12}(z) = \frac{q_x}{q_0} - \sum_{n=1}^{N_1} \frac{q_x}{q_0} \Delta_n^{(1)}(z) - \sum_{n=1}^{N_3} \frac{iq_x}{\omega_n} \Delta_n^{(2)}(z) + \sum_{n=1}^{N_1} \sum_{n'=1}^{N_2} \Delta_n^{(1)}(z) \Delta_n^{(4)}(z_n) m_{13}^+(\zeta_{n'})
\]

\[
+ \sum_{n=1}^{N_1} \sum_{n'=1}^{N_2} \Delta_n^{(1)}(z) \Delta_n^{(5)}(z_n) m_{11}^+(\zeta_{n'}) + \sum_{n=1}^{N_1} \sum_{n'=1}^{N_3} \Delta_n^{(1)}(z) \Delta_n^{(3)}(z_n) m_{12}^+(\bar{\sigma}_{n'})
\]

\[
+ \sum_{n=1}^{N_1} \sum_{n'=1}^{N_2} \Delta_n^{(2)}(z) \Delta_n^{(6)}(z_n) m_{12}^+(\bar{\sigma}_{n'}) + \sum_{n=1}^{N_1} \sum_{n'=1}^{N_2} \Delta_n^{(2)}(z) \Delta_n^{(9)}(\omega_n) m_{13}^+(\zeta_{n'})
\]

\[
+ \sum_{n=1}^{N_1} \sum_{n'=1}^{N_2} \Delta_n^{(2)}(z) \Delta_n^{(8)}(\omega_n) m_{11}^+(\bar{\sigma}_{n'}) + \sum_{n=1}^{N_1} \sum_{n'=1}^{N_2} \Delta_n^{(2)}(z) \Delta_n^{(7)}(\omega_n) m_{12}^+(\bar{\sigma}_{n'})
\]

\[
+ \sum_{n=1}^{N_1} \sum_{n'=1}^{N_2} \Delta_n^{(2)}(z) \Delta_n^{(10)}(\omega_n) m_{12}^+(\bar{\sigma}_{n'}) .
\]

(7.6)

where \( z = \bar{\sigma}_{n'} \) and \( z = \bar{\sigma}_{n'} \). Together with (7.4d), (7.4f) and (7.6) comprise a linear equations. To solve this equations by Gramer’s rule, we define the vector

\[
x = (x_1, \cdots, x_{N_1+2N_2+N_3}),
\]

\[
x_k = \begin{cases}
  m_{12}^+(z_k), & k = 1, \cdots, N_1, \\
  m_{13}^+(\zeta_{k-N_1}), & k = N_1 + 1, \cdots, N_1 + N_2, \\
  m_{11}^+(\zeta_{k-N_1-N_2}), & k = N_1 + N_2 + 1, \cdots, N_1 + N_2 + N_3, \\
  m_{12}^+(\bar{\sigma}_{k-N_1-N_2}), & k = N_1 + 2N_2 + 1, \cdots, N_1 + 2N_2 + N_3,
\end{cases}
\]

So we obtain a systems equations

\[
(I - F)x_k = Wx_k = b_k,
\]

(7.7)
where

\[
b_k = \begin{cases}
q_+ - \sum_{n=1}^{N_1} \frac{q_+}{q_0} \Delta_n^{(1)}(\bar{z}_k) & k = 1, \cdots, N_1, \\
- \sum_{n=1}^{N_1} \frac{iq_+}{\omega_n} \Delta_n^{(1,2)}(\bar{z}_k), & k = N_1 + 1, \cdots, N_1 + N_2, \\
- \frac{iq_+}{\zeta_k - N_1}, & k = N_1 + N_2 + 1, \cdots, N_1 + 2N_2, \\
q_+ - \sum_{n=1}^{N_1} \frac{q_+}{q_0} \Delta_n^{(3)}(\bar{z}_k - N_1) & k = N_1 + 2N_2 + 1, \cdots, N_1 + 2N_2 + N_3, \\
- \sum_{n=1}^{N_1} \frac{iq_+}{\omega_n} \Delta_n^{(3,2)}(\bar{z}_k - N_1), & k = N_1 + 2N_2 + 1, \cdots, N_1 + 2N_2 + N_3,
\end{cases}
\]

\[
y_k = \begin{cases}
- i\tilde{H}_k e^{i(\theta_1 + \theta_2)(- \frac{q_0}{\omega_n})}, & k = 1, \cdots, N_1, \\
- \frac{\zeta_{k-N_1}}{q_0} \tilde{F}_{k-N_1} e^{2i\theta_1(- \frac{q_0}{\omega_{k-N_1}})}, & k = N_1 + 1, \cdots, N_1 + N_2, \\
- i\tilde{F}_{k-N_1-N_2} e^{2i\theta_1(\bar{z}_k - N_1)}, & k = N_1 + N_2 + 1, \cdots, N_1 + 2N_2, \\
- i\tilde{G}_{k-N_1-2N_2} e^{i(\theta_1 + \theta_2)(\bar{z}_k - N_1 - 2N_2)}, & k = N_1 + 2N_2 + 1, \cdots, N_1 + 2N_2 + N_3.
\end{cases}
\]

The matrix $\mathbf{F}$ are defined as follows:

For $i, j = 1, \cdots, N_1$

\[
F_{ij} = \sum_{n=1}^{N_1} \Delta_n^{(1)}(\bar{z}_i) \Delta_n^{(3)}(z_n) + \sum_{n=1}^{N_3} \Delta_n^{(2)}(\bar{z}_i) \Delta_n^{(7)}(\omega_n)
\]  \hspace{1cm} (7.9)

For $i = 1, \cdots, N_1$ and $j = N_1 + 1, \cdots, N_1 + N_2$

\[
F_{ij} = \sum_{n=1}^{N_1} \Delta_n^{(1)}(\bar{z}_i) \Delta_n^{(4)}(z_n) + \sum_{n=1}^{N_3} \Delta_n^{(2)}(\bar{z}_i) \Delta_n^{(9)}(\omega_n)
\]  \hspace{1cm} (7.10)

For $i = 1, \cdots, N_1$ and $j = N_1 + N_2 + 1, \cdots, N_1 + 2N_2$

\[
F_{ij} = \sum_{n=1}^{N_1} \Delta_n^{(1)}(\bar{z}_i) \Delta_n^{(5)}(z_n) + \sum_{n=1}^{N_3} \Delta_n^{(2)}(\bar{z}_i) \Delta_n^{(8)}(\omega_n)
\]  \hspace{1cm} (7.11)

For $i = 1, \cdots, N_1$ and $j = N_1 + 2N_2 + 1, \cdots, N_1 + 2N_2 + N_3$

\[
F_{ij} = \sum_{n=1}^{N_1} \Delta_n^{(1)}(\bar{z}_i) \Delta_n^{(6)}(z_n) + \sum_{n=1}^{N_3} \Delta_n^{(2)}(\bar{z}_i) \Delta_n^{(10)}(\omega_n)
\]  \hspace{1cm} (7.12)
For $i = N_1 + 1, \cdots, N_2$ and $j = 1, \cdots, N_1$

$$F_{ij} = \Delta_j^{(7)}(\zeta_{i-N_1}) \quad (7.13)$$

For $i, j = N_1 + 1, \cdots, N_2$

$$F_{ij} = \Delta_{j-N_1}^{(9)}(\zeta_{i-N_1}) \quad (7.14)$$

For $i = N_1 + 1, \cdots, N_2$ and $j = N_1 + N_2 + 1, \cdots, N_1 + 2N_2$

$$F_{ij} = \Delta_j^{(8)}(\zeta_{i-N_1}) \quad (7.15)$$

For $i = N_1 + 1, \cdots, N_2$ and $j = N_1 + 2N_2 + 1, \cdots, N_1 + 2N_2 + N_3$

$$F_{ij} = \Delta_j^{(10)}(\zeta_{i-N_1}) \quad (7.16)$$

For $i = N_1 + N_2 + 1, \cdots, N_1 + 2N_2$ and $j = 1, \cdots, N_1$

$$F_{ij} = \Delta_j^{(3)}(\zeta_{i-N_1-N_2}) \quad (7.17)$$

For $i = N_1 + N_2 + 1, \cdots, N_1 + 2N_2$ and $j = N_1 + 1, \cdots, N_1 + N_2$

$$F_{ij} = \Delta_j^{(4)}(\zeta_{i-N_1-N_2}) \quad (7.18)$$

For $i, j = N_1 + N_2 + 1, \cdots, N_1 + 2N_2$

$$F_{ij} = \Delta_j^{(5)}(\zeta_{i-N_1-N_2}) \quad (7.19)$$

For $i = N_1 + N_2 + 1, \cdots, N_1 + 2N_2$ and $j = N_1 + 2N_2 + 1, \cdots, N_1 + 2N_2 + N_3$

$$F_{ij} = \Delta_j^{(6)}(\zeta_{i-N_1-N_2}) \quad (7.20)$$

For $i = N_1 + 2N_2 + 1, \cdots, N_1 + 2N_2 + N_3$ and $j = 1, \cdots, N_1$

$$F_{ij} = \sum_{n=1}^{N_1} \Delta_n^{(1)}(\bar{\omega}_{i-N_1-2N_2}) \Delta_j^{(3)}(z_n) + \sum_{n=1}^{N_3} \Delta_n^{(2)}(\bar{\omega}_{i-N_1-2N_2}) \Delta_j^{(7)}(\omega_n) \quad (7.21)$$

For $i = N_1 + 2N_2 + 1, \cdots, N_1 + 2N_2 + N_3$ and $j = N_1 + 1, \cdots, N_1 + N_2$

$$F_{ij} = \sum_{n=1}^{N_1} \Delta_n^{(1)}(\bar{\omega}_{i-N_1-2N_2}) \Delta_j^{(4)}(z_n) + \sum_{n=1}^{N_3} \Delta_n^{(2)}(\bar{\omega}_{i-N_1-2N_2}) \Delta_j^{(9)}(\omega_n) \quad (7.22)$$
For $i = N_1 + 2N_2 + 1, \cdots, N_1 + 2N_2 + N_3$ and $j = N_1 + N_2 + 1, \cdots, N_1 + 2N_2$

$$F_{ij} = \sum_{n=1}^{N_1} \Delta_n^{(1)}(\bar{\omega}_{i-N_1-2N_2}) \Delta_{j-N_1-N_2}^{(5)}(z_n) + \sum_{n=1}^{N_2} \Delta_n^{(2)}(\bar{\omega}_{i-N_1-2N_2}) \Delta_{j-N_1-N_2}^{(8)}(\omega_n)$$

(7.23)

For $i, j = N_1 + 2N_2 + 1, \cdots, N_1 + 2N_2 + N_3$

$$F_{ij} = \sum_{n=1}^{N_1} \Delta_n^{(1)}(\bar{\omega}_{i-N_1-2N_2}) \Delta_{j-N_1-N_2}^{(6)}(z_n) + \sum_{n=1}^{N_2} \Delta_n^{(2)}(\bar{\omega}_{i-N_1-2N_2}) \Delta_{j-N_1-N_2}^{(10)}(\omega_n)$$

(7.24)

By the Gramer’s rule, one can derive that the components of the solutions of system (7.7)

$$x_k = \frac{\text{det} W_k^{aug}}{\text{det} W}, \quad k = 1, 2, \cdots, N_1 + 2N_2 + N_3, \quad (7.25)$$

where $W_k^{aug} = (W_1, \cdots, W_{k-1}, b, W_{k+1}, \cdots, W_{N_1+2N_2+N_3})$ Substituting $x_k$ into the reconstruction formula (7.3) yields the potential function

$$q(x,t) = \frac{\text{det} W_k^{aug}}{\text{det} W}$$

(7.26)

$$W_k^{aug} = \begin{pmatrix} q+ & y^T \\ b & W \end{pmatrix}$$

(7.27)

where

$$b = (b_1, \cdots, b_{N_1+2N_2+N_3}), \quad y = (y_1, \cdots, y_{N_1+2N_2+N_3})^T$$

(7.28)

### 8 The Single Soliton Solutions

#### 8.1 Case I

In this subsection, we assume the eigenvalues $\nu_n$ is the first kind eigenvalues.

This implies that $N_1 = 1$ and $N_2 = N_3 = 0$. And we also assume that $q_+ = (1,1)^T \quad H_1 = e^{\alpha + i\eta} \quad z = \rho e^{i\delta}, (0 < \delta < \pi)$. So we derived the soliton solution

$$q(x,t) = \frac{\text{det} \begin{pmatrix} q+ & y_1 \\ b_1 & 1-F_{11} \end{pmatrix}}{1-F_{11}},$$

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where

\[ F_{11} = \Delta_1^{(1)}(\bar{z}_1)\Delta_1^{(3)}(z_1), \quad b_1 = \frac{q_+}{q_0} - \frac{q_+}{q_0}\Delta_1^{(1)}(\bar{z}_1), \quad y_1 = -i\tilde{H}_1e^{i(\theta_1 + \theta_2)(-\frac{q_0^2}{2})}, \]

\[ \Delta_1^{(1)}(\bar{z}_1) = \frac{H_1e^{i(\theta_1 - \theta_2)(z_1)}}{\bar{z}_1 - z_1} + \frac{z_1}{iq_0} \frac{\tilde{H}_1e^{-i(\theta_1 + \theta_2)(-\frac{q_0^2}{2})}}{\bar{z}_1 - (-\frac{q_0^2}{z_1})}, \]

\[ \Delta_1^{(3)}(z_1) = \frac{\hat{H}_1e^{-i(\theta_1 - \theta_2)(\bar{z}_1)}}{z_1 - \bar{z}_1}. \]

Figure 3-4 showing the perspective view and propagation view of the one soliton solution as different parameters. Figure 3 showing the situation of \( \rho < q_0 \). Figure 4 showing \( \rho > q_0 \).

Fig. 3. \( q_0 = \sqrt{2}, \rho = 1, \delta = \frac{\pi}{3}, \alpha = 0, \eta = -\frac{\pi}{2} \). (a) the perspective view of one soliton. (b) the propagation view of soliton solution in different time.

Fig. 4. \( q_0 = \sqrt{2}, \rho = 3, \delta = \frac{\pi}{3}, \alpha = 0, \eta = -\frac{\pi}{2} \). (a) the perspective view of one soliton. (b) the propagation view of soliton solution in different time.
8.2 Case II

In this subsection, we assume the eigenvalues $\nu_n$ is the second kind eigenvalues. This implies that $N_2 = 1$ and $N_1 = N_3 = 0$. And we also assume that $\mathbf{q}_+ = (1,1)^T \quad H_1 = e^{\alpha + i\eta} \quad \zeta = i \rho e^{i\delta}$. So we derived the soliton solution

$$q(x,t) = \frac{\det \begin{pmatrix} q_+ & y_1 & y_2 \\ b_1 & 1 - F_{11} & -F_{12} \\ b_2 & -F_{21} & 1 - F_{22} \end{pmatrix}}{\det \begin{pmatrix} 1 - F_{11} & -F_{12} \\ -F_{21} & 1 - F_{22} \end{pmatrix}},$$

where

$$F_{11} = \Delta_{\frac{9}{1}}(\zeta_1), \quad F_{12} = \Delta_{\frac{8}{1}}(\zeta_1), \quad F_{21} = \Delta_{\frac{4}{1}}(\bar{\zeta}_1), \quad F_{22} = \Delta_{\frac{5}{1}}(\bar{\zeta}_1),$$

$$b_1 = -\frac{iq_+}{\zeta_1}, \quad b_2 = -\frac{q_+}{q_0}, \quad y_1 = -\frac{\zeta_1}{q_0} \hat{F}_1 e^{2i\theta_1(-\frac{q_0^2}{q_1^2})}, \quad y_2 = -i \hat{F}_1 e^{2i\theta_1(\bar{\zeta}_1)},$$

$$\Delta_{\frac{9}{1}}(\zeta_1) = \frac{\zeta_1 \hat{F}_1 e^{2i\theta_1(-\frac{q_0^2}{q_1^2})}}{iq_0 \zeta_1 - (-\frac{q_0^2}{q_1^2})}, \quad \Delta_{\frac{8}{1}}(\zeta_1) = \frac{\hat{F}_1 e^{2i\theta_1(\bar{\zeta}_1)}}{\bar{\zeta}_1 - \zeta_1},$$

$$\Delta_{\frac{4}{1}}(\bar{\zeta}_1) = \frac{\hat{F}_1 e^{-2i\theta_1(\zeta_1)}}{\zeta - \bar{\zeta}_1}, \quad \Delta_{\frac{5}{1}}(\bar{\zeta}_1) = \frac{\bar{\zeta}_1 \hat{F}_1 e^{-2i\theta_1(-\frac{q_0^2}{q_1^2})}}{iq_0 \bar{\zeta}_1 - (-\frac{q_0^2}{q_1^2})}.$$

Figure 5 showing the perspective view one soliton solution as $\rho > 0$ and $\rho < 0$. Figure 6 showing the propagation view of one soliton solution as different time.
**Fig. 5.** (a) the perspective view of one soliton solution as $\rho = 2$, $\delta = \frac{\pi}{24}$, (b) the perspective view of one soliton solution $\rho = 4$, $\delta = \frac{\pi}{24}$.

**Fig. 6.** (a) the propagation view of one soliton solution as $\rho = 2$, $\delta = \frac{\pi}{24}$, (b) the propagation view of soliton solution as $\rho = 4$, $\delta = \frac{\pi}{24}$.

### 8.3 Case III

In this subsection, we assume the eigenvalues $\nu_n$ is the third kind eigenvalues. This implies that $N_3 = 1$ and $N_1 = N_2 = 0$. And we also assume that $q_+ = (1, 1)^T$, $H_1 = e^{\alpha + i\eta}$, $\omega_1 = \rho e^{i\delta}$. So we derived the soliton solution

$$q(x, t) = \det \begin{pmatrix} q_+ & y_1 \\ b_1 & 1 - F_{11} \end{pmatrix},$$

where

$$F_{11} = \Delta_1^{(2)}(\bar{\omega}_1) \Delta_1^{(10)}(\omega_1), \quad b_1 = -\frac{i q_+}{\omega_1} \Delta_1^{(2)}(\bar{\omega}_1), \quad y_1 = -i\hat{G}_1 e^{i(\theta_1 + \theta_2)(\bar{\omega}_1)},$$

$$\Delta_1^{(2)}(\bar{\omega}_1) = \frac{G_1 e^{-i(\theta_1 + \theta_2)(\bar{\omega}_1)}}{\bar{\omega}_1 - \omega_1} + \frac{\omega_1}{iq_0} \frac{\hat{G}_1 e^{i(\theta_1 - \theta_2)(-\bar{\omega}_1)}}{\bar{\omega}_1 - (-\bar{\omega}_1)},$$

$$\Delta_1^{(10)}(\omega_1) = \frac{G_1 e^{i(\theta_1 + \theta_2)(\bar{\omega}_1)}}{\omega_1 - \bar{\omega}_1}.$$

Figure 7 and Figure 8 showing the perspective view and propagation view of one soliton solution.
Fig. 7. (a) The perspective view of one soliton solution as $\rho = 2, \delta = \frac{\pi}{4}$.(b) The perspective view of soliton solution as $\rho = 1, \delta = \frac{\pi}{4}$.

Fig. 8. (a) The density view of one soliton solution as $\rho = 2, \delta = \frac{\pi}{4}$.(b) The density view of soliton solution $\rho = 1, \delta = \frac{\pi}{4}$.

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