COUNTING LINES ON SURFACES

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Abstract. This paper deals with surfaces with many lines. It is well-known that a cubic contains 27 of them and that the maximal number for a quartic is 64. In higher degree the question remains open. Here we study classical and new constructions of surfaces with high number of lines. We obtain in particular a symmetric octic with 352 lines.

Cubic surface with 27 lines

1. Introduction

Motivation for this paper is the article in 1943 of Segre [12] which studies the following classical problem: What is the maximum number of lines a surface of degree \( d \) in \( \mathbb{P}^3 \) can have? Segre answers this question for \( d = 4 \) by using some nice geometry, showing that it is exactly 64. For the degree three it is a classical result that each smooth cubic in \( \mathbb{P}^3 \) contains 27 lines, but for \( d \geq 5 \) this number is still not known. In this case, Segre shows in loc. cit. that the maximal number is less or equal to \((d - 2)(11d - 6)\) but this bound is far from being sharp. Indeed, already in degree four it gives 76 lines which is not optimal. So from one hand one can try to improve the upper bound for the number of lines \( \ell(d) \) a surface of degree \( d \) in \( \mathbb{P}^3 \) can have, on the other hand it is interesting to construct surfaces with as many lines as possible to give a lower bound for \( \ell(d) \).

It is notoriously difficult to construct examples of surfaces with many lines. Good examples so far are the surfaces of the kind \( F(x, y, z, t) = \phi(x, y) - \psi(z, t) = 0 \) where \( \phi \) and \( \psi \) are homogeneous polynomials of degree \( d \). Segre in [13] studies the case of \( \deg F = 4 \) showing that in this case the possible numbers of lines are 16, 32, 48, 64. He finds these numbers by studying the automorphisms of \( \mathbb{P}^1 \) between the two sets of four points \( \phi = 0 \) and \( \psi = 0 \). Caporaso-Harris-Mazur in [3], by using similar methods as Segre, then study the maximal number of lines \( N_d \) on such surfaces in

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any degree $d$ showing that $N_d \geq 3d^2$ for each $d$ and $N_4 \geq 64$, $N_6 \geq 180$, $N_8 \geq 256$, $N_{12} \geq 864$, $N_{20} \geq 1600$. In this paper we show the exactness of these results. First we note that it is enough to consider surfaces of the kind $\phi(x,y) - \phi(z,t) = 0$ and by a careful analysis of the automorphisms of the set of points $\phi = 0$ on $\mathbb{P}_1$ we can list all the possible numbers of lines on surfaces of this kind for all $d$ and then we prove:

**Proposition 3.3** The maximal numbers of lines on $F = 0$ are:

- $N_d = 3d^2$ for $d \geq 3$, $d \neq 4, 6, 8, 12, 20$;
- $N_4 = 64$, $N_6 = 180$, $N_8 = 256$, $N_{12} = 864$, $N_{20} = 1600$.

It is well-known that the Fermat surfaces $(x^d - y^d) - (z^d - t^d) = 0$ have $3d^2$ lines. Our proof provides a method to write equations of surfaces $\phi(x,y) - \psi(z,t) = 0$ with each possible number of lines. In particular, our proposition shows that it is not possible, with these surfaces, to obtain better examples and a better lower bound for $\ell(d)$. So, in order to find better examples, one has to use new methods.

In this paper we explore the following kinds of surfaces:

- $d$-covering of the plane $\mathbb{P}_2$ branched over a curve of degree $d$.
- Symmetric surfaces in $\mathbb{P}_3$.

We show that the first method cannot give more than $3d^2$ lines (Proposition 4.2).

The second method is based on the following idea: if a surface has many automorphisms (many symmetries) then possibly it contains many orbits of lines. This idea was used successfully in the study of surfaces with many nodes. In this paper we find a $G_8$-invariant octic with 352 lines, where $G_8 \subset \text{PGL}(3, \mathbb{C})$ has order 576 (Proposition 5.2). This shows $\ell(8) \geq 352$, improving the previous bound of 256.

As stated before, one can also try to improve the upper bound for $\ell(d)$. Following the idea of Segre [12] and imposing some extra conditions on the lines on a surface, we can find the bound $d(7d - 12)$ which surprisingly agrees with the maximal examples in degrees 4, 6, 8, 12 (Section 6).

Finally a related problem to this is to determine the maximal number $m(d)$ of skew-lines a surface of degree $d$ in $\mathbb{P}_3$ can have. It is well-known that $m(3) = 6$ and $m(4) = 16$. For $d \geq 5$, this value is not known. An upper bound $m(d) \leq 2(d(d - 2))$ is given by Miyaoka in [7], which is sharp for $d = 3, 4$. There are results of Rams [9, 10] giving examples of surfaces with $d(d - 2) + 2$ skew-lines ($d \geq 5$) and with 19 skew-lines for $d = 5$. In Proposition 8.2 we improve his examples for $d \geq 7$ and $\gcd(d, d - 2) = 1$ to $d(d - 2) + 4$.

The paper is organized as follows. In Section 2 we give an overview of known results. In Sections 3 and 4 we describe completely the surfaces of the kind $\phi(x,y) - \psi(z,t) = 0$ and the $d$-coverings of the plane $t^d = f(x,y,z)$. Section 5 is devoted to the investigation of symmetric surfaces, and in particular of an octic with 352 lines. In Section 6 we present the uniform bound $d(7d - 12)$ and Section 7 is an application to the problem of the number of rational points on curves. Finally, Section 8 deals with the skew-lines: we give an overview of known results and some new examples.

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2. General results

Our objective is to investigate the number of lines contained in a smooth surface in $\mathbb{P}_3$. We first recall classical results: the generic situation and the bound of Segre.

2.1. Generic situation.

It is a well-known fact that each smooth quadric surface in $\mathbb{P}_3$ contains an infinite number of lines and each smooth cubic surface in $\mathbb{P}_3$ contains exactly 27 lines. What happens for surfaces of higher degree? Generically:

**Proposition 2.1.** A generic smooth surface of degree $d \geq 4$ in $\mathbb{P}_3$ contains no line.

We briefly recall the proof, following [1, 2].

**Proof.** Let $V$ be the vector space of degree $d$ homogeneous polynomials in the coordinates $x, y, z, t$ and $G$ be the Grassmannian of 2-planes in $\mathbb{C}^4$. Consider the incidence variety $F := \{(L, f) \in G \times V \mid f|_L \equiv 0\}$ with its projections $p : F \to G$ and $q : F \to V$.

- Let $L \in G$ and assume that $L$ is generated by the vectors $(1, 0, a, b)$ and $(0, 1, c, d)$ in $\mathbb{C}^4$. Consider the affine neighbourhood of $L$ in $G$: 
  \[ U := \text{Span}\{(1, 0, a, b), (0, 1, c, d)\} \]
  where $a, b, c, d$ are local coordinates. If $f \in p^{-1}(U)$, then 
  \[ f(\lambda(1, 0, a, b) + \mu(0, 1, c, d)) = 0 \quad \forall \lambda, \mu \in \mathbb{C}, \]
  and denoting $f = \sum_{i+j+k+l=d} a_{i,j,k,l}x^iy^jz^k t^l$, the equation 
  \[ \sum_{i+j+k+l=d} a_{i,j,k,l}\lambda^i\mu^j(\lambda a + \mu c)^k(\lambda b + \mu d)^l = 0 \quad \forall \lambda, \mu \in \mathbb{C} \]
gives $d + 1$ linear equations in the coordinates $(a_{i,j,k,l})$ of $f \in V$ whose rank at $a = b = c = d = 0$ is $d + 1$: hence locally in a neighbourhood of $L$, the system has rank $d + 1$ so $p$ is a locally trivial bundle of rank: $\text{dim} V - (d + 1)$.

- Let $X$ be a surface of degree $d$ in $\mathbb{P}_3$, given by a polynomial $f \in V$. Then the Fano scheme parametrizing the lines contained in $X$ is $F(X) := p(q^{-1}(f))$.
- Consider the map $q : F \to V$. Since: 
  \[ \text{dim} F = \text{dim} V - (d + 1) + \text{dim} G = \text{dim} V - (d - 3), \]
for $d \geq 4$ one has $\text{dim} F < \text{dim} V$ hence the map $q$ is not dominant. This means that the generic fibre of $q$ is empty. Otherwise stated, $F(X)$ is empty for $X$ generic. \(\square\)

We shall see in the next section that the number of lines a smooth surface of degree $d \geq 4$ can have is always finite, and bounded. This leads to the problem of finding surfaces with an optimal number of lines.

2.2. Upper bound for lines.

The best upper bound known so far for the number of lines on a smooth surface of degree $d \geq 4$ in $\mathbb{P}_3$ is given by Segre:

**Theorem 2.2** (Segre [12]).

- The number of lines lying on a smooth surface of degree $d \geq 4$ does not exceed $(d - 2)(11d - 6)$.
- The maximum number of lines lying on a quartic surface is exactly 64.
This bound is effective for $d = 4$ (see for instance maximal examples in Section 5.1 but for $d \geq 5$ it is believed that it could be improved. For instance, already for $d = 4$ the uniform bound $(d-2)(11d-6)$ is too big. The next sections are devoted to the study of some families of surfaces with particular properties, containing many lines.

3. Surfaces of the kind $\phi(x, y) = \psi(z, t)$

We consider a surface $S$ given by an equation of the kind:

$$F(x, y, z, t) := \phi(x, y) - \psi(z, t)$$

for two homogeneous polynomials $\phi, \psi$ of degree $d$. Segre gave a complete description of the possible and maximal numbers of lines in the case $d = 4$ (see §VIII). We generalize the method to all degrees: we treat in details the configuration of lines, give a description of all possible numbers, and conclude with the maximal numbers of lines for such surfaces.

3.1. Configuration of the lines.

Let $Z(\phi)$, resp. $Z(\psi)$ denote the set of zeros of $\phi(x, y)$, resp. $\psi(z, t)$ in $\mathbb{P}_1$.

**Theorem 3.1.** Let $F(x, y, z, t) = \phi(x, y) - \psi(z, t)$ be the equation of a smooth surface $S$ of degree $d$ in $\mathbb{P}_3$. The number $N_d$ of lines on $S$ is exactly:

$$N_d = d(d + \alpha_d)$$

where $\alpha_d$ is the order of the group of isomorphisms of $\mathbb{P}_1$ mapping $Z(\phi)$ to $Z(\psi)$.

**Proof.**

- Let $L$ be the line $z = t = 0$ and $L'$ be the line $x = y = 0$. Then $S \cap L = Z(\phi)$ and $S \cap L' = Z(\psi)$. Since the surface $S$ is smooth, the homogeneous polynomials $\phi$ and $\psi$ have simple zeros. Indeed, for example in the case of the polynomial $\phi$, if $[a : b] \in \mathbb{P}_1$ is such that $\phi$ can be factorized by $(bx - ay)^2$, then $\partial_x \phi(a, b) = \partial_y \phi(a, b) = 0$ and the point $[a : b : 0 : 0]$ is a singular point of $S$ (the inverse also holds: if both $\phi$ and $\psi$ have only simple zeros, then $S$ is smooth). Set $Z(\phi) := \{P_1, \ldots, P_d\}$ and $Z(\psi) := \{P'_1, \ldots, P'_d\}$.

- Each line $L_{i,j}$ joining a $P_i$ to a $P'_j$ is contained in $S$: if $P_i = [x_i : y_i : 0 : 0]$ and $P'_j = [0 : 0 : z'_j : t'_j]$ the line joining them consists in points $[\lambda x_i : \lambda y_i : \mu z'_j : \mu t'_j]$, $\lambda, \mu \in \mathbb{C}$, which are all contained in the surface, by homogeneity of the polynomials $\phi$ and $\psi$. This gives $d^2$ lines.

- Each line contained in $S$ and intersecting $L$ and $L'$ is one of the previous lines. Indeed, if $D$ is such a line, set $D \cap L = \{[a : b : 0 : 0]\}$ and $D \cap L' = \{[0 : 0 : c : d]\}$. Then $F(a, b, 0, 0) = \phi(a, b) = 0$ so $[a : b : 0 : 0]$ is one of the points $P_i$ and similarly $[0 : 0 : c : d]$ is one $P'_j$.

- Let $D$ be a line contained in $S$ and not intersecting $L$. Then $D$ does not intersect $L'$ (and vice-versa). Indeed, an equation of such a line $D$ is given by two independent equations:

$$\begin{cases} ax + by + cz + dt = 0 \\ a'x + b'y + c'z + d't = 0 \end{cases}$$

Since $D$ does not intersect $L$, the system

$$\begin{cases} ax + by = 0 \\ a'x + b'y = 0 \end{cases}$$

implies $P_i = P'_j$. Therefore $d = d'(d' + \alpha_{d'})$ and $\alpha_{d'} = \alpha_d$. This is true for every $d'$. Therefore, $d_{d'}$ is the uniform bound of $\alpha_{d'}$ over all $d'$. The case $d = 4$ is a particular case of this.
has rank two, so we can rewrite the equations of $D$ as the following independent equations:

\[
\begin{align*}
x &= \alpha z + \beta t \\
y &= \gamma z + \delta t
\end{align*}
\]

Then $D$ does not intersect $L'$ otherwise the matrix \( \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \) would have rank one.

- Therefore, the equations of the line $D$ define a linear isomorphism between the lines $L'$ and $L$ inducing a bijection between $Z(\psi)$ and $Z(\phi)$. Indeed, setting $P'_j = [0 : 0 : c : d]$, then $a := \alpha x + \beta d$ and $b := \gamma c + \delta d$ have the property that $[a : b : c : d] \in D \subset S$ so $\phi(a, b) = F(a, b, c, d) + \psi(c, d) = 0$ hence $[a : b : 0 : 0]$ is a zero of $\phi$.

- Conversely, let $\sigma : L' \to L$ be an isomorphism mapping the points $P'_j$ to the points $P_i$, and \( \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \) a matrix defining $\sigma$. Consider the smooth quadric $Q_x : x(\gamma z + \delta t) - y(\alpha z + \beta t) = 0$. Its first ruling is the family of lines $(p, \sigma(p))$ for $p \in L'$. For $p = [c : d]$, these lines are given by the equations

\[
I_{[c:d]} : \begin{cases}
(\gamma c + \delta d)x - (\alpha c + \beta d)y = 0 \\
dz - ct = 0
\end{cases}
\]

Its second ruling consists in the family of lines of equations

\[
\mathbb{I}_{[a:b]} : \begin{cases}
ax - b(\alpha z + \beta t) = 0 \\
ay - b(\gamma z + \delta t) = 0
\end{cases}
\]

for $[a : b] \in \mathbb{P}_1$. To this ruling belong the lines $L ([a : b] = [0 : 1]), L' ([a : b] = [1 : 0])$ and $D ([a : b] = [1 : 1])$. It is not true a priori that this $D$ is contained in $S$, since the matrix $\sigma$ is defined up to a scalar factor.

In each ruling, the lines are disjoint to each other, and each line of one ruling intersects each line of the other ruling. Since the intersection $S \cap Q$ contains exactly the $d$ different lines $(P'_j, \sigma(P'_j))$ of the first ruling, it contains also $d$ lines of the second ruling: Consider a line in the first ruling not contained in $S$, then it intersects $S$ in $d$ points, and through each of these points is attached a line of the second ruling, which also intersects the $d$ lines of the first ruling contained in $S$, so these lines of the second ruling intersect $S$ at $d + 1$ points, so are contained in $S$. But it is not clear a priori with our argument that these lines in the second ruling are different. Denote by $U_d$ the group of $d$-th roots of the unit. The group $U_d \times U_d$ acts on $\mathbb{P}_3$ by $(\xi, \eta) \cdot [x : y : z : t] = [\xi x : \xi y : \eta z : \eta t]$, leaving the surface $S$ globally invariant since the polynomials $\phi$ and $\psi$ are homogeneous of degree $d$. Observe that the lines of the first ruling are invariant for the action, but for the second ruling, $(\xi, \eta) \cdot \mathbb{I}_{[a:b]} = \mathbb{I}_{[\xi^{-1}a, \eta^{-1}b]}$ so each line of the second ruling produces a length $d$ orbit through the action. Since the surface $S$ contains at least one line of the second ruling, it contains the whole orbit, this gives us $d$ different lines.

Therefore, each isomorphism $\sigma : L' \to L$ mapping $Z(\psi)$ to $Z(\phi)$ gives $d$ lines, and there are no other lines. Furthermore, for two different isomorphisms, the corresponding lines are different since the matrix defining the isomorphisms are not proportional.

- Denote by $\alpha_d$ the number of isomorphisms $\sigma : L' \to L$ mapping $Z(\psi)$ to $Z(\phi)$. The preceding discussion shows that the exact number of lines contained in the
surface $\mathcal{S}$ is:

$$N_d = d^2 + \alpha_d d.$$  

\[ \square \]

**Remark 3.2.** In the proof of [3, Lemma 5.1], Caporaso-Harris-Mazur proved with a similar argument that the number of lines is at least $d(d + \alpha_d)$ and described some special values. Our argument includes the exactness. In the next subsections we give a full description of the possible values of $\alpha_d$, in particular its maximal values for each $d$.

### 3.2. The possible numbers of lines.

Now we want to find the possible and maximal values of $N_d$, or equivalently $\alpha_d$. If there is at least one isomorphism $\sigma$ (see the proof above), then by composing by $\sigma^{-1}$ we are lead to the problem of determining the possible numbers of automorphisms of $\mathbb{P}_1$ (or projectivities) acting on a given set of $d$ points on $\mathbb{P}_1$. Since a projectivity is defined by its value on three points, we have always $\alpha_3 = 6$, and for $d \geq 4$ there is only a finite number of such isomorphisms, depending on the relative position of the points, encoded in their cross-ratios. The case $d = 4$ was studied by Segre [13] with this point of view. We give a different argument for the general case. The set $\Gamma_d$ of isomorphisms of $\mathbb{P}_1$ acting on $d$ points defines a finite group of automorphisms of $\mathbb{P}_1$. First recall the classical classification:

**Polyhedral groups.** There are five types of finite subgroups of $\text{SO}(3, \mathbb{R})$, or equivalently of $\text{PGL}(2, \mathbb{C})$, called *polyhedral groups*:

- the cyclic groups $C_k \cong \mathbb{Z}/k\mathbb{Z}$ of order $k \geq 2$, isomorphic to the group of isometries of a regular polygon with $k$ vertices in the plane;
- the dihedral groups $D_k \cong \mathbb{Z}/k\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ of order $2k$, $k \geq 2$, isomorphic to the group of isometries of regular polygon with $k$ vertices in the space;
- the group $T$ of positive isometries of a regular tetrahedra, isomorphic to the alternate group $A_4$ of order twelve;
- the group $O$ of positive isometries of a regular octahedra or a cube, isomorphic to the symmetric group $S_4$ of order 24;
- the group $I$ of positive isometries of a regular icosahedra or a regular dodecahedra, isomorphic to the alternate group $A_5$ of order 60.

In the sequel, we shall describe generators of these groups and their orbits on $\mathbb{P}_1$, in order to get explicit constructions of surfaces.

We now proceed to the description of all possible groups of isomorphisms ($d \geq 4$):

1. $\Gamma_d = \{\text{id}\}$. This is not possible for $d = 4$ since there are always at least four automorphisms of a set of four points in $\mathbb{P}_1$ (their cross-ratio takes generically six different values under permutation).
2. $\Gamma_d$ is a cyclic group: $\Gamma_d \cong \mathbb{Z}/k\mathbb{Z}$ $(k \geq 2)$ with generator $\sigma(t) = \xi t$ where $\xi$ is a primitive $k$-th root of the unit. The action of $\sigma$ on $\mathbb{P}_1$ has two fix points $\{0, \infty\}$ and all other points generate a length $k$ orbit. So, depending whether the fix points are in the given set of $d$ points or not we have the decomposition:

$$d = \alpha + \beta k$$

with $\alpha \in \{0, 1, 2\}$ and $\beta \geq 1$:
• \( \alpha = 0 \). The points are:
\[
\{ \mu_1, \mu_1 \xi, \ldots, \mu_1 \xi^{k-1} \}, \ldots, \{ \mu_\beta, \mu_\beta \xi, \ldots, \mu_\beta \xi^{k-1} \}.
\]
This forces \( \beta \geq 3 \) since: if \( \beta = 1 \) or \( \beta = 2 \) then \( t \mapsto 1/t \) or \( t \mapsto \mu_2/\mu_1 t \) generate a dihedral group. For \( \beta \geq 3 \) there are no other isomorphisms.

• \( \alpha = 1 \). The points are:
\[
\{0\}, \{ \mu_1, \mu_1 \xi, \ldots, \mu_1 \xi^{k-1} \}, \ldots, \{ \mu_\beta, \mu_\beta \xi, \ldots, \mu_\beta \xi^{k-1} \}.
\]
There is no other isomorphism whenever \( d = 1 + \beta k \geq 5 \). For \( k = 3 \) and \( \beta = 1 \) there are other isomorphism (a tetrahedral group).

• \( \alpha = 2 \). The points are:
\[
\{0, \infty\}, \{ \mu_1, \mu_1 \xi, \ldots, \mu_1 \xi^{k-1} \}, \ldots, \{ \mu_\beta, \mu_\beta \xi, \ldots, \mu_\beta \xi^{k-1} \}.
\]
As before, this forces \( \beta \geq 3 \).

To summarize, for the group \( \Gamma_d \) be a cyclic group \( \mathbb{Z}/k\mathbb{Z} \) \((d \geq 4, k \geq 2)\):

• \( d = \beta k, \beta \geq 3 \), e.g. \( \phi(x,y) = \prod_{i=1}^{\beta} (x^k - \lambda_i y^k) \);

• \( d = 1 + \beta k \geq 5, \beta \geq 1 \) if \( k = 3 \), e.g. \( \phi(x,y) = x \prod_{i=1}^{\beta} (x^k - \lambda_i y^k) \);

• \( d = 2 + \beta k, \beta \geq 3 \) e.g. \( \phi(x,y) = xy \prod_{i=1}^{\beta} (x^k - \lambda_i y^k) \).

(3) \( \Gamma_d \) is a dihedral group: \( \Gamma_d \cong \mathbb{Z}/k\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) \((k \geq 2)\) with generators \( \sigma(t) = \xi t \) and \( s(t) = 1/t \) where \( \xi \) is a primitive \( k \)-th root of the unit. The action of the dihedral group on \( \mathbb{P}^1 \) has one length 2 orbit \( \{0, \infty\} \) and one length \( k \) orbit generated by 1. So we have the decomposition:
\[
d = 2\alpha + \beta k + \gamma 2k
\]
with \( \alpha, \beta \in \{0, 1\}, \gamma \geq 0 \):

• \( \gamma = 0, \alpha = 0 \) and \( \beta = 1 \). The points are:
\[
\{1, \xi, \ldots, \xi^{k-1} \}
\]
Then \( d = k \) and \( \phi(x,y) = x^k - y^k \). This gives the Fermat surface.

• \( \gamma = 0, \alpha = 1 \) and \( \beta = 1 \). The points are:
\[
\{0, \infty\}, \{1, \xi, \ldots, \xi^{k-1} \}
\]
This forces \( k \neq 2, 4 \): if \( k = 2 \), the configuration is isomorphic to the preceding case (with \( 2k \)) and contains more isomorphisms, and if \( k = 4 \) there are other isomorphisms generating an octahedral group. Then \( d = 2 + k \) and \( \phi(x,y) = xy(x^k - y^k) \).

• \( \gamma \neq 0 \). Then \( d \in \{2k\gamma, 2 + 2k\gamma, k + 2k\gamma, 2 + k + 2k\gamma \} \) and \( \phi \) contains, besides the factors given in the preceding cases, \( \gamma \) factors of the kind \((x^k - \lambda y^k)(x^k - \frac{1}{\lambda} y^k)\).

(4) \( \Gamma_d \) is a tetrahedral group \( \mathcal{T} \). The group \( \mathcal{T} \) is generated by:
\[
\sigma(t) = \omega t, \quad s(t) = \frac{1 - t}{1 + 2t}
\]

\(^2\)Here and in the sequel, the \( \mu_i \)'s are assumed to be generic: they are distinct and in particular they are not the \( \beta \)-th roots of the unit and their \( k \)-powers \( \lambda_i := \mu_i^k \) are distincts.
acting on the set \( \{ 0, 1, \omega, \omega^2 \} \) where \( \omega \) is a primitive third root of the unit. The action of \( \mathcal{T} \) on \( \mathbb{P}_1 \) has two length four orbits:

\[
\{0, 1, \omega, \omega^2\}, \{ \infty, -\frac{1}{2}, -\frac{1}{2}, -\omega^2 \}
\]

and one length six orbit generated by the fix point \( w = -\frac{1+\sqrt{3}}{2} \) of \( s \). These are all the orbits of lengths four or six since the conjugacy classes in \( \mathcal{T} \) are generated by \( s, \sigma, \sigma^2 \). So we have the decomposition:

\[
d = 4\alpha + 6\beta + 12\gamma
\]

with \( \alpha \in \{ 0, 1, 2 \}, \beta \in \{ 0, 1 \}, \gamma \geq 0 \):

- \( \gamma = 0, \beta = 0 \) and \( \alpha = 1 \): the group of isomorphisms is \( \mathcal{T} \).
- \( \gamma = 0, \beta = 0 \) and \( \alpha = 2 \): the group of isomorphisms would be \( \mathcal{O} \) since \( t \mapsto -1/(2t) \) interchanges the two length four orbits.
- \( \gamma = 0, \beta = 1 \) and \( \alpha = 0 \): the group of isomorphisms would be \( \mathcal{O} \) since the length six orbit is stabilized by \( t \mapsto -1/(2t) \).
- \( \gamma = 0, \beta = 1 \) and \( \alpha = 1 \): the group of isomorphisms is \( \mathcal{T} \), because it is not contained in any dihedral group and the groups \( \mathcal{O} \) or \( \mathcal{I} \) have no length four or ten orbit.
- \( \gamma = 0, \beta = 1 \) and \( \alpha = 2 \): as before the group of isomorphisms is \( \mathcal{O} \).
- For \( \gamma \neq 0 \), in general the group of isomorphisms is \( \mathcal{T} \) but for special points this could be \( \mathcal{O} \) or \( \mathcal{I} \).

For example, for the tetrahedral group consider \( \phi(x, y) = x(x^3 - y^3) \).

(5) \( \Gamma_d \) is an octahedral group \( \mathcal{O} \). The group \( \mathcal{O} \) is generated by:

\[
\sigma(t) = it, \quad s(t) = \frac{1}{t}, \quad a(t) = \frac{t+i}{t-1}
\]

acting on the set \( \{ 0, \infty, 1, i, -1, -i \} \). The action of \( \mathcal{O} \) on \( \mathbb{P}_1 \) has one length six orbit, one length eight orbit generated by the fix point \( w = \frac{1+i-\sqrt{3}+i\sqrt{3}}{2} \) of \( a \), and one length twelve orbit generated by the fix point \( z = -1 + \sqrt{2} \) of the isomorphism\(^3 \) \( r(t) = \frac{1+i}{1-t} \). These are all orbits of lengths six, eight or twelve since the conjugacy classes in \( \mathcal{O} \) are generated by \( id, s, \sigma, a, r \). So we have the decomposition:

\[
d = 6\alpha + 8\beta + 12\gamma + 24\delta
\]

with \( \alpha, \beta, \gamma \in \{ 0, 1 \}, \delta \geq 0 \). Since the group \( \mathcal{O} \) is not contained in \( \mathcal{I} \) nor in any dihedral group, all choices of \( \alpha, \beta, \gamma, \delta \) are possible to get \( \Gamma_d \cong \mathcal{O} \).

(6) \( \Gamma_d \) is an icosahedral group \( \mathcal{I} \). The group \( \mathcal{I} \) is generated by:

\[
p_5(t) := \frac{\tau t + \tau - 1 + i}{(-\tau + 1 + i)t + \tau}, \quad q_1(t) := -t, \quad q_2(t) := -\frac{1}{t}
\]

where \( \tau := \frac{1+\sqrt{3}}{2} \). The only length twelve orbit is generated by a fix point of \( p_5 \), the length 20 orbit is generated by a fix point of \( p_5^2 q_2 \) (which has order three) and the length 30 orbit is generated by a fix point of \( q_1 \). Since

\(^3\)The second fix point \( w' = -\frac{1+\sqrt{3}}{2} \) belongs to the same orbit since \( w = \sigma^2 s\sigma(w') \).

\(^4\)The second fix point \( w' = \frac{1+i+\sqrt{3}+i\sqrt{3}}{2} \) belongs to the same orbit since \( w' = sas\sigma(w) \).

\(^5\)The second fix point \( z' = -1 - \sqrt{2} \) belongs to the same orbit since \( z' = \sigma\sigma\sigma(a(z)) \).
the conjugacy classes in $I$ are generated by id, $p_5^2, p_5^5 q_2, q_1$ there are no other orbits. So we have the decomposition:

$$d = 12\alpha + 20\beta + 30\gamma + 60\delta$$

with $\alpha, \beta, \gamma \in \{0, 1\}$, $\delta \geq 0$. All choices give $\Gamma_d \cong I$.

3.3. Maximal number of lines.

As a corollary of Theorem 3.1 and the preceding discussion of cases, we get the following maximality result:

**Proposition 3.3.** The maximal numbers of lines on $S$ are:

- $N_d = 3d^2$ for $d \geq 3$, $d \neq 4, 6, 8, 12, 20$;
- $N_4 = 64$, $N_6 = 180$, $N_8 = 256$, $N_{12} = 864$, $N_{20} = 1600$.

**Proof.** Looking up at the discussion above, it appears that $\alpha_d = 2d$ is maximal when the group of automorphisms can not be a group $T$, $O$ or $I$ and that $\alpha_4 = 12$, $\alpha_6 = \alpha_8 = 24$ and $\alpha_{12} = \alpha_{20} = 60$ are maximal. For other values of $d$, if the automorphism group is $T$, resp. $O$, resp. $I$ then the number of lines is:

$$d^2 + 12d, \ \text{resp.} \ d^2 + 24d, \ \text{resp.} \ d^2 + 60d$$

and these numbers are bigger than $3d^2$ only if

$$d < 6, \ \text{resp.} \ d < 12, \ \text{resp.} \ d < 30.$$ 

So it just remains to check that the degree $d = 10$ is not possible for $O$ and $I$ and that the degrees $d = 14, 16, 18, 22, 24, 26, 28$ are not possible for $I$, that is we cannot decompose such a $d$ as a sum of lengths of orbits for the groups $O$ or $I$. This is clear with the restrictions on the numbers of orbits of each type. $\square$

**Remark 3.4.** Although this result was expected, one has to pass through the study of §3.2 to prove it.

3.4. Examples.

1. For $d$ generic, the Fermat surface $F(x, y, z, t) = (x^d - y^d) - (z^d - t^d)$ gives the best example for surfaces of the kind $\phi(x, y) - \psi(z, t)$.
2. For $d = 4$, $\Gamma_4 \in \{\emptyset, D_2, D_4, T\}$ so the possible numbers of lines for such surfaces are: 16, 32, 48, 64. This agrees with Segre’s result and 64 is the maximal possible number of lines on a quartic surface.
3. For $d = 5$, $\Gamma_5 \in \{\emptyset, \{id\}, C_4, D_3, D_5\}$ so the possible numbers of lines for such surfaces are: 25, 30, 45, 55, 75. The general bound of Segre gives 147.
4. For $d = 6$, $\Gamma_6 \in \{\emptyset, \{id\}, C_2, D_2, D_3, D_6, O\}$ so the possible numbers of lines for such surfaces are: 36, 42, 48, 60, 72, 108, 180. The general bound of Segre gives 240.
5. The discussion of §3.2 gives explicit constructions of surfaces of each group $\Gamma_d$. For the groups $O$ and $I$, see also Section 5.

3.5. Real lines.

It is an interesting problem to find surfaces of any degree $d$ with as many real lines as possible. For surfaces of the kind $\phi(x, y) - \phi(z, t) = 0$, if the zeros of $\phi$ are all real, one gets already $d^2$ real lines (see proof of Theorem 3.1). Then, for each isomorphism in the group $\Gamma_d$ represented by a real matrix, one gets one more real line if $d$ is odd and two more real lines if $d$ is even.
4. Surfaces of the kind $t^d = f(x, y, z)$

We consider smooth surfaces of degree $d \geq 3$ given as covering of $\mathbb{P}_2$ ramified along a plane curve. Let $C : f(x, y, z) = 0$ be a plane curve defined by a homogeneous polynomial $f$ of degree $d$ and consider the surface $S$ in $\mathbb{P}_3$ given by the equation:

$$F(x, y, z, t) := t^d - f(x, y, z).$$

Note that the surface $S$ is smooth if and only if the curve $C$ is.

Set $p = [0 : 0 : 0 : 1] \in \mathbb{P}_3$. The projection:

$$(\mathbb{P}_3 - \{p\}) \to \mathbb{P}_2, [x : y : z : t] \mapsto [x : y : z]$$

induces a $d$-covering $\pi : S \to \mathbb{P}_2$ ramified along the curve $C$.

Recall that a point $x \in C$ is a $d$-point (or total inflection point) if the intersection multiplicity of $C$ and its tangent line at $x$ is equal to $d$.

**Proposition 4.1.**

1. Suppose $L$ is a line contained in $S$. Then $\pi(L)$ is a line.
2. Let $x \in C$ and $L$ the tangent at $C$ in $x$, then the preimage $\pi^{-1}(L)$ consists in $d$ different lines contained in $S$ if and only if $x$ is a $d$-point.
3. Let $L$ be a line in $\mathbb{P}_2$. Then $\pi^{-1}(L)$ contains a line if and only if $L$ is tangent to $C$ at a $d$-point.

**Proof.**

1. It is clear from the definition of the projection $\pi$.
2. Assume $x$ is a $d$-point. Let $\Delta$ be a line of equation $\delta$ intersecting $L$ at $x$. Then $d \cdot (\Delta \cdot L) = (C \cdot L)$ so after restriction to $L$ one has up to a scalar factor $f|_L = \delta|_L^d$ showing that the covering restricted to $L$ is trivial and $\pi^{-1}(L)$ consists in the $d$ lines $t - \xi^i \delta_L = 0, i = 1, \ldots, d$ where $\xi$ is a primitive $d$-th root of the unit. Conversely, if the covering splits, there exists a section $\gamma \in H^0(L, O_L(1))$ such that $\gamma^d = f|_L \in H^0(L, O_L(d))$ so $L$ intersects $C$ at $x$ with multiplicity $d$.
3. If $L$ is the tangent to $C$ at a $d$-point the assertion follows from 2. Assume now that $\pi^{-1}(L)$ contains a line. Let $L$ be given by a linear function $z = l(x, y)$. Then the equation of $\pi^{-1}(L)$ is $t^d - f(x, y, l(x, y)) = 0$. Since it contains a line the equation splits as

$$t^d - f(x, y, l(x, y)) = (t - w(x, y))F_{d-1}(t, x, y)$$

where $w(x, y)$ is a linear form. By comparing the coefficients in $t$ one obtains $f(x, y, l(x, y)) = w(x, y)^d$ hence the preimage consists in the $d$ lines:

$$t^d - f(x, y, l(x, y)) = \prod_{i=0}^{d-1}(t - \xi^i w(x, y))$$

where $\xi$ is a primitive $d$-th root of the unit. This means that the covering is trivial over $L$ so by 2 $x$ is a $d$-point.

We deduce the number of lines contained in such surfaces:
Proposition 4.2. Let $\mathcal{C} : f(x, y, z) = 0$ be a smooth plane curve of degree $d$ with $\beta$ total inflection points. Let $\mathcal{S}$ the surface in $\mathbb{P}_3$ given by the equation:

$$F(x, y, z, t) := t^d - f(x, y, z).$$

Then $\mathcal{S}$ contains exactly $\beta \cdot d$ lines. In particular, it contains no more than $3d^2$ lines.

Proof. The first assertion follows directly from the lemma. For the second one, the inflection points are the intersections of $\mathcal{C}$ with its Hessian curve $\mathcal{H}$ of degree $3(d - 2)$ and at a total inflection point the intersection multiplicity of $\mathcal{C}$ and $\mathcal{H}$ is $d - 2$, so by Bezout one gets $\beta \leq 3d$. \qed

Remark 4.3.

- For $d = 3$, it is well-known that each cubic has nine inflection points, then the induced surface has $3 \cdot 9 = 27$ lines.
- The Fermat curves $x^d + y^d + z^d = 0$ have $3d$ total inflection points hence the Fermat surfaces are examples of surfaces with $3d^2$ lines.

5. Symmetric surfaces

We consider surfaces with many symmetries, since one can expect that such surfaces contain many lines. Indeed, if the surface contains a line then it contains the whole orbit, and if the symmetry group is big, hopefully this orbit has big length. To this purpose, we first take $G \subset \text{PGL}(4, \mathbb{C})$ be a finite group of linear transformations acting on $\mathbb{P}_3$ and construct smooth $G$-invariant surfaces.

5.1. Surfaces with cyclic symmetries.

Denote by $U_d$ the group of $d$-th roots of the unit. The group $U_d \times U_d$ acts on $\mathbb{P}_1$ by $\text{diag}(\xi, \mu)$ for $(\xi, \mu) \in U_d \times U_d$. The graded space of invariant polynomials decomposes as:

$$\mathbb{C}[x, y, z, t]^{U_d \times U_d} \cong \mathbb{C}[x, y]^{U_d} \otimes \mathbb{C}[z, t]^{U_d}.$$

Since $\mathbb{C}[x, y]^{U_d} = 0$ for $d \nmid k$ and $\mathbb{C}[x, y]^{U_d}_k = \mathbb{C}[x, y]_k$ otherwise, all invariant polynomials of degree $d$ for the action of $U_d \times U_d$ are of the kind $\phi(x, y) - \psi(z, t)$ for $\phi$ and $\psi$ homogeneous polynomials of degree $d$. These surfaces were studied in Section 3.

5.2. Surfaces with polyhedral symmetries.

We consider again surfaces of the kind $\phi(x, y) = \phi(z, t)$: we studied such surfaces and their configuration of lines in Section 4. We adopt here a different point of view. Let $\Gamma$ be the group of isomorphisms of $\mathbb{P}_1$ permuting the zeros of $\phi$ in $\mathbb{P}_1$. Then $\phi$ is a projective invariant for the action of $\Gamma$ on $\mathbb{C}^2$, i.e. $\phi(g(x, y)) = \lambda g \phi(x, y)$ for $g \in \Gamma$ and $\lambda_g \in \mathbb{C}^*$. This implies that the surface $F(x, y, z, t) = \phi(x, y) - \phi(z, t)$ is invariant for the diagonal action of $\Gamma$ given by $g(x, y, z, t) = (g(x, y), g(z, t))$. Its number of lines is given by Theorem 3.1.

By using this observation, we can find easily equations for surfaces of this kind with the symmetries of the groups $T, O, I$. The projective invariants are computed for example in Klein [3, I.2, §11-12-13]:

1. A surface of degree six with octahedral symmetries and 180 lines:

$$\phi(x, y) = xy(x^4 - y^4).$$
(2) A surface of degree eight with octahedral symmetries and 256 lines:
\[ \phi(x, y) = x^8 + 14x^4y^4 + y^8. \]

(3) A surface of degree twelve with octahedral symmetries and 432 lines:
\[ \phi(x, y) = x^{12} - 33x^8y^4 - 33x^4y^8 + y^{12}. \]

(4) A surface of degree twelve with icosahedral symmetries and 864 lines:
\[ \phi(x, y) = xy(x^{10} + 11x^5y^5 - y^{10}). \]

(5) A surface of degree 20 with icosahedral symmetries and 1600 lines:
\[ \phi(x, y) = -x^{20} + y^{20} + 228(x^{15}y^5 - x^5y^{15}) - 494x^{10}y^{10}. \]

(6) A surface of degree 30 with icosahedral symmetries and 2700 lines:
\[ \phi(x, y) = (x^{30} + y^{30}) + 522(x^{25}y^5 - x^5y^{25}) - 10005(x^{20}y^{10} + x^{10}y^{20}). \]

5.3. Surfaces with bipolyhedral symmetries.

First recall the construction of the bipolyhedral groups. Start from the exact sequence:
\[ 0 \rightarrow \{\pm 1\} \rightarrow SU(2) \rightarrow SO(3, \mathbb{R}) \rightarrow 0. \]

For any polyhedral group \( G \subset SO(3, \mathbb{R}) \), the inverse image \( \tilde{G} := \phi^{-1}G \) is called a binary polyhedral group. Now consider the exact sequence:
\[ 0 \rightarrow \{\pm 1\} \rightarrow SU(2) \times SU(2) \rightarrow SO(4, \mathbb{R}) \rightarrow 0. \]

For \( \tilde{G} \) a binary polyhedral group, the direct image \( \sigma(\tilde{G} \times \tilde{G}) \subset SO(4, \mathbb{R}) \) is called a bipolyhedral group. We shall make use of the following particular groups:

- \( G_6 = \sigma(\tilde{T} \times \tilde{T}) \) of order 288;
- \( G_8 = \sigma(\tilde{O} \times \tilde{O}) \) of order 1152;
- \( G_{12} = \sigma(\tilde{I} \times \tilde{I}) \) of order 7200.

The polynomial invariants of these groups were studied by Sarti in [11]. First note that the quadratic form: \( Q := x^2 + y^2 + z^2 + t^2 \) is an invariant of the action of these groups.

Theorem 5.1 (Sarti [11, §4]). For \( d = 6, 8, 12 \), there is a one-dimensional family of \( G_d \)-invariant surfaces of degree \( d \). The equation of the family is \( S_d + \lambda Q^{d/2} = 0 \). The base locus of the family consists in \( 2d \) lines, \( d \) in each ruling of \( Q \). The general member of each family is smooth and there are exactly five singular surfaces in each family.

From this theorem immediately follows that each member of the family contains at least \( 2d \) lines.

- **The group** \( G_8 \). Denote by \( S_8 \) the surface \( S_8 = 0 \) where:
  \[ S_8 = x^8 + y^8 + z^8 + t^8 + 168x^2y^2z^2t^2 + 14(x^4y^4 + x^4z^4 + x^4t^4 + y^4z^4 + y^4t^4 + z^4t^4). \]

Proposition 5.2. The surface \( S_8 \) contains exactly 352 lines.
Proof. The proof goes as follows: first we introduce Plücker coordinates for the lines in \( \mathbb{P}_3 \), then we compute explicitly all the lines contained in the surface.

- **Plücker coordinates.** Let \( G(1, 3) \) be the Grassmannian of lines in \( \mathbb{P}_3 \), or equivalently of 2-planes in \( \mathbb{C}^4 \). Such a line \( L \) is given by a rank-two matrix:

\[
\begin{pmatrix}
  a & e \\
  b & f \\
  c & g \\
  d & h
\end{pmatrix}.
\]

The 2-minors (Plücker coordinates):

\[
\begin{align*}
p_{12} &:= af - be & p_{13} &:= ag - ce & p_{14} &:= ah - de \\
p_{23} &:= bg - cf & p_{24} &:= bh - df & p_{34} &:= ch - dg
\end{align*}
\]

are not simultaneously zero, and induce a regular map \( G(1, 3) \to \mathbb{P}_5 \). This map is injective, and its image is the hypersurface \( p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0 \). In order to list once all lines with these coordinates, we inverse the Plücker embedding in the Plücker stratification:

| (1) | (2) | (3) |
|-----|-----|-----|
| \( p_{12} = 1 \) | \( p_{12} = 0, p_{13} = 1 \) | \( p_{12} = 0, p_{13} = 0, p_{14} = 1 \) |
| \[ \begin{pmatrix} 1 & 0 \\
 0 & 1 \\
 -p_{23} & p_{13} \\
 -p_{24} & p_{14} \end{pmatrix} \] | \[ \begin{pmatrix} 1 & 0 & p_{23} & 0 \\
 0 & 1 & -p_{24} & p_{14} \end{pmatrix} \] | \[ \begin{pmatrix} 1 & 0 \\
 p_{24} & 0 \\
 p_{34} & 0 \\
 0 & 1 \end{pmatrix} \] |

- **Counting the lines.** The line \( L \) is contained in the surface \( S_8 \) if and only if the function \( (u, v) \mapsto S_8(ua + ve, ub + vf, uc + vg, ud + vh) \) is identically zero, or equivalently if all coefficients of this polynomial in \( u, v \) are zero. The conditions for the line to be contained in the surface is then given by a set of polynomial equations in \( a, b, c, d, e, f, g, h \). In order to count the lines, we restrict the equations to each Plücker stratum and compute the solutions (this computation is not difficult if left to Singular [3]).

1. **The stratum** \( p_{12} = 1 \). Set \( p_{23} = c, p_{24} = d, p_{13} = g, p_{14} = h \). The equations for such a line to be contained in the surface are:

\[
\begin{align*}
c^7g + d^7h + 7c^3g + 7d^3h + 7c^4d^3h + 7c^3g^4d^4 &= 0 \\
c^6g^2 + d^6h^2 + 3c^4d^2h^2 + 8c^3g^3d^3h + 3c^2g^2d^4 &= 0 \\
+6c^2d^2 + 3c^2g^2 + 3d^2h^2 &= 0 \\
c^5g^3 + d^5h^3 + c^4d^3h + c^3d^4 + 6c^3gd^2h^2 + 6c^2gd^2h + 6cgd^2 &= 0
\end{align*}
\]
Proposition 5.4. The surface \( S \) with many lines. This improves widely the bound \( 256 \).

Although there is no reason for this bound to be maximal, it seems reasonable to expect that an effective construction of a surface with this number of lines is possible in all degrees.

We propose here another lower uniform bound, which interpolates all maximal numbers of lines known so far, including the octic of Section 3.4. Then \( S \) contains exactly 352 lines on the surface.

An easy computation shows that the other strata contain no line, so there are exactly 352 lines on the surface.

Remark 5.3. To our knowledge, this is the best example so far of an octic surface with many lines. This improves widely the bound 256 of Caporaso-Harris-Mazur [3].

- The group \( G_6 \). We take:

\[
S_6 = x^6 + y^6 + z^6 + t^6 + 15(x^2y^2z^2 + x^2y^2t^2 + x^2z^2t^2 + y^2z^2t^2).
\]

Proposition 5.4. The surface \( 8S_6 - 5Q^3 = 0 \) contains exactly 132 lines.

There are surfaces with more lines (see [3,4]), but this shows the existence of a surface with 132 lines. This result can be shown in a similar way as in the \( G_8 \) case.

6. A uniform bound

As we mentioned before, the uniform bound \((d - 2)(11d - 6)\) of Segre is too big already in degree four. We propose here another lower uniform bound, which interpolates all maximal numbers of lines known so far, including the octic of Section 3.4. Although there is no reason for this bound to be maximal, it seems reasonable to expect that an effective construction of a surface with this number of lines is possible in all degrees.

Let \( S \) be a smooth surface of degree \( d \geq 3 \) and \( C \) a line contained in \( S \). Let \(|H|\) be the linear system of planes \( H \) passing through \( C \). Then \( H \cap S = C \cup \Gamma \) where \( \Gamma \) is a succe...
Γ is a curve of degree \( d - 1 \). The system \(|Γ|\) is described by Segre in [12]: it is base-point free and any curve Γ does not contain \( C \) as a component. Then:

**Proposition 6.1** (Segre [12]). Either each curve Γ intersects \( C \) in \( d - 1 \) points which are inflections for Γ, or the points of \( C \) each of which is an inflection for a curve Γ are \( 8d - 14 \) in number. In particular, in this case \( C \) is met by no more than \( 8d - 14 \) lines lying on \( S \).

Following Segre, \( C \) is called a line of the second kind if it intersects each Γ in \( d - 1 \) inflections. A generalization of Segre’s argument in [12, §9] gives the following result:

**Proposition 6.2.** Assume that \( S \) contains \( d \) coplanar lines, none of them of the second kind. Then \( S \) contains at most \( d(7d - 12) \) lines.

**Proof.** Let \( P \) be the plane containing these \( d \) distinct lines. Then they are the complete intersection of \( P \) with \( S \). Hence each other line on \( S \) must intersect \( P \) in some of the lines. By Proposition 6.1, each of the \( d \) lines in the plane meets at most \( 8d - 14 \) lines, so \( 8d - 14 - (d - 1) \) lines not on the plane. The total number of lines is at most:

\[
d + d(7d - 13) = d(7d - 12).
\]

□

This bound takes the following values:

| \( d \) | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 20 |
|-------|----|----|----|----|----|----|----|----|----|----|
| \( 7d^2 - 12d \) | 64 | 115 | 180 | 259 | 352 | 459 | 580 | 715 | 864 | 2560 |

Note that this bound matches perfectly with the maximal known examples in degrees 4, 6, 8, 12.

7. Number of rational points on a plane curve

We give an application of our results to the **universal bound conjecture**, following Caporaso-Harris-Mazur [3]:

**Universal bound conjecture.** Let \( g \geq 2 \) be an integer. There exists a number \( N(g) \) such that for any number field \( K \) there are only finitely many smooth curves of genus \( g \) defined over \( K \) with more than \( N(g) \) \( K \)-rational points.

As mentioned in *loc.cit.* an interesting way to find a lower bound of \( N(g) \), or of the limit:

\[
\overline{N} := \limsup_{g \to \infty} \frac{N(g)}{g}
\]

is to consider plane sections of surfaces with many lines. Indeed, over the common field \( K \) of definition of the surface and its lines, a generic plane section is a curve containing at least as many \( K \)-rational points as the number of lines. In particular, they show that \( N(21) \geq 256 \). Since we obtain an octic surface with 352 lines and a generic plane section of this surface is a smooth curve of genus 21, we get:

**Corollary 7.1.** \( N(21) \geq 352 \).

As we remarked in Section 6, it seems to be possible to construct surfaces with \( d(7d - 12) \) lines. This would improve the lower bound of \( N(g) \) for many \( g \)'s. In particular, this would improve the known estimate \( \overline{N} \geq 8 \) to \( \overline{N} \geq 14 \).
8. Sequences of skew-lines

A natural question related to the number of lines on a surface is the study of maximal sequences of pairwise disjoint lines on a smooth surface in \( \mathbb{P}^3 \). We recall the bound of Miyaoka and give some examples.

8.1. Upper bound for skew-lines.

The best upper bound known so far for the maximal length of a sequence of disjoint lines on a smooth surface of degree \( d \geq 4 \) in \( \mathbb{P}^3 \) is given by Miyaoka:

**Theorem 8.1** (Miyaoka [7, §2.2]). The maximal length of a sequence of skew-lines is \( 2d(d-2) \) for \( d \geq 4 \).

For \( d = 3 \), each cubic surface contains a maximal sequence of 6 skew lines. This comes from the study of the configuration of the 27 lines (see for example [5, Theorem V.4.9] and references therein). For \( d = 4 \), Kummer surfaces contain a maximal sequence of 16 skew lines (see for example [8] and references therein) so the bound is optimal.

But for \( d \geq 5 \), it is not known if it is sharp.

8.2. On Miyaoka’s bound.

We give a quick sketch of the argument of Miyaoka for the bound on the number of skew lines, following [7, §2 Examples 2.1, 2.2].

Let \( X \) be a smooth surface of degree \( d \geq 4 \) in \( \mathbb{P}^3 \). Assume \( X \) contains \( r \) disjoint lines \( D_1, \ldots, D_r \). By adjunction formula, they have self-intersection \( -n = -(d-2) \).

By contracting these lines one gets a surface \( Y \) with \( r \) isolated singular points which locally look like the quotient of \( \mathbb{C}^2 \) by a finite group of order \( n \).

Write \( K_X + \sum_{i=1}^r D_i = P + N' \) with:

\[
P := K_X + \sum_{i=1}^r \frac{n-2}{n} D_i \quad \text{and} \quad N' := \sum_{i=1}^r \frac{n-2}{n} D_i.
\]

This provides a Zariski decomposition in \( \text{Pic}(X) \otimes \mathbb{Q} \) of \( K_X + \sum_{i=1}^r D_i \).

Set \( \nu := 2 - 1/n \), by [7, Theorem 1.1], one has the inequality:

\[
r\nu \leq c_2(X) - \frac{1}{3} P^2.
\]

Using that \( c_2(X) = d(d^2 - 4d + 6) \) and \( K_X^2 = d(d-4)^2 \) one gets \( r \leq 2d(d-2) \).

8.3. Examples.

In [10], Rams considers the surfaces \( x^{d-1}y + y^{d-1}z + z^{d-1}t + t^{d-1}x = 0 \) and proves that they contain a family of \( d(d-2) + 2 \) skew-lines for any \( d \). In [8, Example 2.3], he also gives an example of a surface of degree five containing a sequence of 19 skew-lines. We generalize his result, improving the number of skew-lines to \( d(d-2) + 4 \) in the case \( d \geq 7 \) and \( \gcd(d, d-2) = 1 \).

Consider the surface \( R_d : x^{d-1}y + xy^{d-1} + z^{d-1}t + zt^{d-1} = 0 \). By our study in Section 6.4, this surface contains exactly \( 3d^2 - 4d \) lines if \( d \neq 6 \) and 180 lines for \( d = 6 \). We prove:

**Proposition 8.2.** The surface \( R_d \) with \( \gcd(d, d-2) = 1 \) contains a sequence of \( d(d-2) + 4 \) disjoint lines.
Proof. Denote by $\epsilon$, $\gamma$ the primitive roots of the unit of degrees $d - 2$ and $d$, and let $\eta := \epsilon^l \gamma^s$, with $0 \leq l \leq d - 3$, $0 \leq s \leq d - 1$. Since $\gcd(d, d - 2) = 1$ we have $d(d - 2)$ such $\eta$. Now consider the points

\[(0 : 1 : 0 : -\eta^{d-1}), (-\eta : 0 : 1 : 0)\]

then the line through the two points is

\[C_{l,s} : (-\eta \lambda : \mu : \lambda : -\eta^{d-1} \mu)\]

An easy computation shows that these lines are contained in $R_d$ and are $d(d - 2)$. This form a set of $d(d - 2) + 4$ skew lines together with the lines

\[
\begin{align*}
\{x = 0, z + \epsilon t = 0\}, & \{y = 0, z + t = 0\}, \\
\{z = 0, x + \epsilon y = 0\}, & \{t = 0, x + y = 0\}.
\end{align*}
\]

$\square$

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