GRADED $K$-THEORY, FILTERED $K$-THEORY AND
THE CLASSIFICATION OF GRAPH ALGEBRAS

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Abstract. We prove that an isomorphism of graded Grothendieck groups $K^*_0$ of two Leavitt path algebras induces an isomorphism of their algebraic filtered $K$-theory and consequently an isomorphism of $K$-theory of their associated graph $C^*$-algebras. As an application, we show that, since for a finite graph $E$ with no sinks, $K^*_0(L(E))$ of the Leavitt path algebra $L(E)$ coincides with Krieger’s dimension group of its adjacency matrix $A_E$, our result relates the shift equivalence of graphs to the filtered $K$-theory and consequently gives that two arbitrary shift equivalent matrices give stably isomorphic graph $C^*$-algebras. This result was only known for irreducible graphs.

1. Introduction

One of the beauties of the theory of Leavitt path algebras is that one can obtain a substantial amount of information about the structure of the algebra from the geometry of its associated graph. The first theorem proved in this theory was that the simplicity of a Leavitt path algebra is equivalent to that in the associated graph every cycle has an exit and every vertex connects to every infinite path and every finite path ending in a sink ([1], [2, §2.9]).

The theory of Leavitt path algebras is intrinsically related, via graphs, to the theory of symbolic dynamics and $C^*$-algebras where the major classification programs have been a domain of intense research in the last 50 years. However, it is not yet clear what is the right invariant for the classification of Leavitt path algebras, and for that matter, graph $C^*$-algebras [40]. In the case of simple graph $C^*$-algebras (i.e., algebras with no nontrivial ideals), it is now established that $K$-theory functors $K_0$ and $K_1$ can classify these algebras completely [33, 34]. Following the early work of Rørdam [32] and Restorff [31], it became clear that one way to preserve enough information in the presence of ideals in a $C^*$-algebra, is to further consider the $K$-groups of the ideals, their subquotients and how they are related to each other via the six-term sequence. Over the next ten years since [32, 31] this approach, which is now called filtered $K$-theory, was subsequently investigated and further developed by Eilers, Restorff, Ruiz and Sørensen [13, 14], where it was shown that the sublattice of gauge invariant prime ideals and their subquotient $K$-groups can be used as an invariant. In a major work [15] it was shown that filtered $K$-theory is a complete invariant for unital graph $C^*$-algebras. In [16] the four authors introduced the filtered $K$-theory in the purely algebraic setting and showed that if two Leavitt path algebras with coefficients in complex numbers $\mathbb{C}$ have isomorphic filtered algebraic $K$-theory then the associated graph $C^*$-algebras have isomorphic filtered $K$-theory.

This paper is devoted to graded $K$-theory as a capable invariant for the classification of graph algebras. This approach was initiated in [20] and further studied in [6, 21, 22]. The main aim of this paper is to show that in the setting of graph algebras, graded $K$-theory implies filtered $K$-theory. To be precise, we show that for two Leavitt path algebras over a field, if their graded Grothendieck groups $K^*_0$ are isomorphic, then their filtered $K$-theories are also isomorphic. This shows the richness of a graded Grothendieck group as an invariant. Namely, the single group $K^*_0(L_k(E))$ of a Leavitt path algebra $L_k(E)$, with coefficients in the field $k$, contains all the information about the $K_0$ and $K_1$ groups of the subquotients of graded ideals of $L_k(E)$ and how they are related via the long exact sequence of $K$-theory.

There is a tight connection between the algebraic structure of $L_k(E)$ and the monoid structure of $\nu^\text{st}(L_k(E))$. This is the crucial first step in relating graded $K$-theory to the filtered $K$-theory of these algebras. Namely, we observe that the lattice of graded (prime) ideals of $L_k(E)$ is isomorphic to the lattice of order (prime) ideals of $\nu^\text{st}(L_k(E))$. This allows us to lift an order-preserving $\mathbb{Z}[x, x^{-1}]$-module isomorphism between the graded Grothendieck groups of two Leavitt path algebras

$$\varphi : K^*_0(L_k(E)) \longrightarrow K^*_0(L_k(F)),$$

(1.1)

to a natural homeomorphism between the space of spectrums of their graded prime ideals,

$$\varphi : \text{Spec}^\text{st}(L_k(E)) \longrightarrow \text{Spec}^\text{st}(L_k(F)).$$

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We then proceed to piece together $K_0$ and $K_1$ groups of ideals together from this correspondence. This is possible as we can establish a van den Bergh type exact sequence relating $K^{gr}_0$ to $K_0$ and $K_1$ (Proposition 4.5),

$$K_1(L_k(E)) \rightarrow K^{gr}_0(L_k(E)) \rightarrow K^{gr}_0(L_k(E)) \rightarrow K_0(L_k(E)) \rightarrow 0. \tag{1.2}$$

We then show that the isomorphisms $\varphi$ of (1.1) induce a commutative diagram involving the natural transformation of filtered $K$-theory (Theorem 7.3). Namely for graded prime ideals $I \subseteq J$ of $L_k(E)$, one can obtain induced homomorphism $\alpha_{i,n} : K_n(I) \rightarrow K_n(\varphi(I))$, $\alpha_{j,n} : K_n(J) \rightarrow K_n(\varphi(J))$, and $\alpha_{i,j,n} : K_n(J/I) \rightarrow K_n(\varphi(J)/\varphi(I))$, where $n = 0, 1$, such that the following diagram commutes.

$$
\begin{array}{cccc}
K_1(I) & \rightarrow & K_1(J) & \rightarrow & K_1(J/I) & \rightarrow & K_0(I) & \rightarrow & K_0(J) & \rightarrow & K_0(J/I) \\
\downarrow_{\alpha_{i,1}} & & \downarrow_{\alpha_{j,1}} & & \downarrow_{\alpha_{i,j,1}} & & \downarrow_{\alpha_{i,0}} & & \downarrow_{\alpha_{j,0}} & & \downarrow_{\alpha_{i,j,0}} \\
K_1(\varphi(I)) & \rightarrow & K_1(\varphi(J)) & \rightarrow & K_1(\varphi(J)/\varphi(I)) & \rightarrow & K_0(\varphi(I)) & \rightarrow & K_0(\varphi(J)) & \rightarrow & K_0(\varphi(J)/\varphi(I)).
\end{array}
$$

Here the rows come from the long exact sequence of algebraic $K$-theory (see [11, Theorem 2.4.1]).

Since the isomorphisms of filtered $K$-theory of Leavitt path algebras imply the isomorphisms of filtered $K$-theory of their corresponding graph $C^*$-algebras ([16]), our result allows us to relate the shift equivalent matrices via Krieger’s dimension groups to graded $K$-theory and in return to algebraic and thus analytic filtered $K$-theory ([8]). Then invoking the Eilers, Restorff, Ruiz and Sørensen recent result [15] on the classification of finite graph algebras via filtered $K$-theory, we can conclude that shift equivalent matrices have stably isomorphic Cuntz-Krieger $C^*$-algebras (Proposition 8.1). This was only known in the case of irreducible matrices by a combination of the Franks Theorem on the classification of flow equivalent via Bowen-Franks group [17], Parry and Sullivan’s description of flow matrices in terms of moves [28] and Bates and Pask’s paper [8] translating these moves into the setting of graph $C^*$-algebras.

The following diagram summarizes the connections between the graphs $E$ and $F$, their adjacency matrices $A_E$ and $A_F$, their associated graph algebras and their graded and filtered $K$-theories.

$$
\begin{array}{ccc}
A_E & \overset{\text{shift equivalent of matrices}}{\longrightarrow} & A_F \\
\Delta_E & \overset{\text{iso. of Krieger’s dimension groups}}{\longrightarrow} & \Delta_F \\
K^{gr}_0(L_k(E)) & \overset{\text{iso. of}}{\longrightarrow} & K^{gr}_0(L_k(F)) \\
FK_0,1(L_k(E)) & \overset{\text{iso. of alg. fil. K-groups}}{\longrightarrow} & FK_0,1(L_k(F)) \\
FK_0,1(C^*(E)) & \overset{\text{iso. of alg. fil. K-groups}}{\longrightarrow} & FK_0,1(C^*(F)) \\
C^*(E) & \overset{\text{Morita equivalent}}{\longrightarrow} & C^*(F)
\end{array}
$$

This paper is devoted to row-finite graphs (graphs that each vertex emits finite number of edges). On the presence of infinite emitters, the monoid of a Leavitt path algebra is more involved (see Remark 4.2). Although the majority of the techniques we employ are valid for arbitrary graphs and the statements we establish are consequently hold for arbitrary graphs (albeit more complex and lengthier proofs) there are several instances that the techniques for such graphs need to be yet established, such as working with quotient monoids. This paper is thus devoted to the row-finite case. Our major application is related to symbolic dynamics and it only requires working with finite graphs and thus we are equipped to employ our results in this setting.

We close the introduction by recalling a conjecture posed in [20, Conjecture 1], namely the graded Grothendieck group $K^{gr}_0$ along with its ordering and its module structure is a complete invariants for the class of (finite) Leavitt path algebras (see also [6, 21], [2, § 7.3.4]). The language of homology of groupoids allows us to propose a single invariant which would classify both Leavitt and graph $C^*$-algebras. For an arbitrary graph $E$ and its associated graph groupoid $G_E$, in [24] it was proved that there is an order-preserving $\mathbb{Z}[x, x^{-1}]$-module isomorphism

$$
K^{gr}_0(L_k(E)) \rightarrow H^{gr}_0(G_E),
$$

$$
[L_k(E)] \mapsto [1_{G_E}].
$$
where $H^0_0(G_E)$ is the zeroth homology of the étale groupoid $G_E$. We can then formulate the following conjecture.

**Conjecture 1.** Let $E$ and $F$ be finite graphs and $k$ a field. Then the following are equivalent.

1. There is a gauge preserving isomorphism $\varphi : C^*(E) \to C^*(F)$;
2. There is a graded ring isomorphism $\varphi : L_k(E) \to L_k(F)$;
3. There is an order-preserving $\mathbb{Z}[x, x^{-1}]$-module isomorphism $\phi : H^0_0(G_E) \to H^0_0(G_F)$, such that $\varphi([1_{\omega^0_E}]) = [1_{\omega^0_F}]$.

Here the ordered $\mathbb{Z}[x, x^{-1}]$-module isomorphism $H^0_0(G_E) \to H^0_0(G_F)$ (without preserving the ordered units) should give that these algebras are graded Morita equivalent ([22]).

2. **The lattice of ideals of graph algebras**

2.1. **Lattices.** Throughout the paper we work with the lattice of graded ideals of a Leavitt path algebra and the corresponding lattice of graded order ideals of the associated graded Grothendieck group. They are linked via the lattice of admissible pairs of the graph which defines the algebra (see §2.3). Recall that a lattice is a partially ordered set $M$ which any two elements $a, b \in M$ have meet $a \wedge b$ (the greatest lower bound) and join $a \vee b$ (the least upper bound). A morphism between two lattices is a map which preserves meets and joins and thus preserves the order. An element $x \in M$ is called a prime element if for any $a, b \in M$ with $a \wedge b \leq x$, we have $a \leq x$ or $b \leq x$. We will work with the prime elements in the lattice of graded ideals of a Leavitt path algebra $L(E)$ (§2.3) and the lattice of graded order ideals of $\mathcal{V}^0(L(E))$ (Definition 3.5). Clearly an isomorphism between lattices preserves prime elements.

2.2. **Graphs.** Below we briefly recall the notions that we use throughout the paper.

A (directed) graph $E$ is a tuple $(E^0, E^1, r, s)$, where $E^0$ and $E^1$ are sets and $r, s$ are maps from $E^1$ to $E^0$. A graph $E$ is finite if $E^0$ and $E^1$ are both finite. We think of each $e \in E^1$ as an edge pointing from $s(e)$ to $r(e)$. We use the convention that a (finite) path $p$ in $E$ of length $n \geq 1$ is a sequence $p = \alpha_1\alpha_2 \cdots \alpha_n$ of edges $\alpha_i$ in $E$ such that $r(\alpha_i) = s(\alpha_{i+1})$ for $1 \leq i \leq n - 1$. We define $p(s) = s(\alpha_1)$, and $r(p) = r(\alpha_n)$. A vertex is viewed as a path in $E$ of length 0. If there is a path from a vertex $u$ to a vertex $v$, we write $u \geq v$. A subset $M$ of $E^0$ is downward directed if for any two $u, v \in M$ there exists $w \in M$ such that $u \geq w$ and $v \geq w$ ([2, §4.2], [30, §2]).

A graph $E$ is said to be row-finite if for each vertex $u \in E^0$, there are at most finitely many edges in $s^{-1}(u)$. A vertex $u$ for which $s^{-1}(u)$ is empty is called a sink.

The covering graph $\overline{E} = E \times_1 \mathbb{Z}$ of $E$ is given by

$$
\overline{E}^0 = \{v_n \mid v \in E^0 \text{ and } n \in \mathbb{Z}\},
$$

$$
\overline{E}^1 = \{e_n \mid e \in E^1 \text{ and } n \in \mathbb{Z}\},
$$

where $s(e_n) = s(e)_n$, and $r(e_n) = r(e)_{n-1}$.

As examples, consider the following graphs

$$
E : \\
\begin{array}{ccc}
& e & \\
\odot & f & v \\
& g & \end{array}
$$

$$
F : \\
\begin{array}{ccc}
& f & \\
\odot & e & \\
& & \\
\end{array}
$$

Then

$$
E \times_1 \mathbb{Z} : \\
\begin{array}{ccc}
\cdots & u_1 & e_1 \\
\odot & f_1 & u_0 \\
& e_0 & f_0 \\
\odot & e_0 & u_{-1} \\
\cdots & v_{-1} & g_{-1} \\
\end{array}
$$

and

$$
F \times_1 \mathbb{Z} : \\
\begin{array}{ccc}
\cdots & f_1 & e_1 \\
\odot & u_1 & f_0 \\
& e_0 & f_0 \\
\odot & e_0 & u_{-1} \\
\cdots & v_{-1} & f_{-1} \\
\end{array}
$$

Throughout the note $E$ is a row-finite graph. Recall that a subset $H \subseteq E^0$ is said to be hereditary if for any $e \in E^1$ we have that $s(e) \in H$ implies $r(e) \in H$. A hereditary subset $H \subseteq E^0$ is called saturated if whenever $v$ is not a sink, $\{r(e) : e \in E^1 \text{ and } s(e) = v\} \subseteq H$ implies $v \in H$. We let $\mathcal{S}_E$ denote the set of hereditary saturated subsets of $E^0$, and order two hereditary saturated subsets $H$ and $H'$ by $H \leq H'$ if $H \subseteq H'$. It has been established that the ordered set $\mathcal{S}_E$ is actually a lattice (see [2, Proposition 2.5.6]).

We denote by $E_H$ the restriction graph with $H$ a hereditary saturated subset of $E^0$ such that

$$
E^0_H = H,
$$

$$
E^1_H = \{e \in E^1 \mid s(e) \in H\}
$$
and we restrict \( r \) and \( s \) to \( E \) \( _1 \). On the other hand, for a hereditary saturated subset of \( E \), we denote by \( E / H \) the quotient graph such that

\[
(E / H)^0 = E^0 \setminus H, \\
(E / H)^1 = \{ e \in E^1 \mid r(e) \notin H \}
\]

and we restrict \( r \) and \( s \) to \( (E / H)^1 \).

For hereditary saturated subsets \( H_1 \) and \( H_2 \) of \( E \) with \( H_1 \subseteq H_2 \), define the quotient graph \( H_2 / H_1 \) as a graph such that \((H_2 / H_1)^0 = H_2 \setminus H_1 \) and \((H_2 / H_1)^1 = \{ e \in E^1 \mid s(e) \in H_2, r(e) \notin H_1 \}\). The source and range maps of \( H_2 / H_1 \) are restricted from the graph \( E \). If \( H_2 = E^0 \), then \( H_2 / H_1 \) is the quotient graph \( E / H_1 \) ([2, Definition 2.4.11]).

### 2.3. Cohn algebras and Leavitt path algebras

We refer the reader to [2] for concepts of Cohn path algebras and Leavitt path algebras.

Let \( E \) be a row-finite graph and \( k \) a field. The Cohn path algebra \( C_k(E) \) of \( E \) is the quotient of the free associative \( k \)-algebra generated by the set \( E^0 \cup E^1 \cup \{ e^* \mid e \in E^1 \} \), subject to the relations:

1. \( v \cdot w = \delta_{v,w} \) for \( v, w \in E^0 \);
2. \( s(e) \cdot e = e = e \cdot r(e) \) for \( e \in E^1 \);
3. \( r(e) \cdot e^* = e^* \cdot s(e) \) for \( e \in E^1 \);
4. \( e^* \cdot f = \delta_{e,f} r(e) \) for \( e, f \in E^1 \).

The algebra \( C_k(E) \) is in fact a \(*\)-algebra; it is equipped with an involution \( * : C_k(E) \rightarrow C_k(E)^{op} \) which fixes vertices and maps \( e \) to \( e^* \) for \( e \in E^1 \). Here \( C_k(E)^{op} \) is the opposite algebra of \( C_k(E) \).

Denote by \( K(E) \) the ideal of the Cohn path algebra \( C_k(E) \) generated by the set

\[
\{ v - \sum_{v \in \varepsilon^{-1}(v)} ee^* \mid v \in E^0 \text{ is not a sink} \}.
\]

The Leavitt path algebra \( L_k(E) \) of \( E \) over the field \( k \) (see [2, Definition 1.2.3]) is the quotient algebra \( C_k(E) / K(E) \). Then there is a short exact sequence of rings

\[
0 \rightarrow K(E) \xrightarrow{\iota} C_k(E) \xrightarrow{p} L_k(E) \rightarrow 0.
\]

Throughout this paper we simply write \( C(E) \) instead of \( C_k(E) \) and \( L(E) \) instead of \( L_k(E) \). These algebras are naturally \( \mathbb{Z} \)-graded and this graded structure plays an important role in this paper (see [2, §2.1]).

Denote by \( \mathcal{L}^{gr}(L(E)) \) the lattice of graded (two-sided) ideals of \( L(E) \). There is a lattice isomorphism between the set \( \mathcal{I}_E \) of hereditary saturated subsets of \( E \) and the set \( \mathcal{L}^{gr}(L(E)) \) ([2, Theorem 2.5.8]). The correspondence is

\[
\Phi : \mathcal{I}_E \rightarrow \mathcal{L}^{gr}(L(E)),
\]

\[
H \rightarrow (H),
\]

where \( H \) is a hereditary saturated subsets of \( E \), and \( (H) \) is the graded ideal generated by the set \( H \).

For graded ideals \( I, J \) of \( L(E) \), since \(IJ = I \cap J \), the prime elements of the lattice \( \mathcal{L}^{gr}(L(E)) \) coincide with the graded prime ideals of \( L(E) \) (see 2.1). We denote by \( \text{Spec}^{gr}(L(E)) \) the set of graded prime ideals of \( L(E) \). The prime ideals of the Leavitt path algebra \( L(E) \) are completely characterised in terms of their generators ([30, Theorem 3.12]). Denote by \( \mathcal{I}^1_E \) the set consisting of the hereditary saturated subsets \( H \) of \( E^0 \) such that \( E^0 \setminus H \) is downward directed. Then the correspondence \( \Phi \) of (2.3) restricts to the one-to-one correspondence

\[
\Phi : \mathcal{I}^1_E \rightarrow \text{Spec}^{gr}(L(E)),
\]

\[
H \rightarrow (H),
\]

### 3. The monoid \( \mathcal{V}^{gr} \) and the graded Grothendieck group \( K_0^{gr} \) of graph algebras

Let \( M \) be a commutative monoid with a group \( \Gamma \) acting on it. Throughout we assume that the group \( \Gamma \) is abelian. Indeed in our setting of graph algebras, this group is \( \mathbb{Z} \). We define an ordering on the monoid \( M \) by \( a \leq b \) if \( b = a + c \), for some \( c \in M \). A \( \Gamma \)-order ideal of a monoid \( M \) is a subset \( I \) of \( M \) such that for any \( \alpha, \beta \in \Gamma \), \( \alpha a + \beta b \in I \) if and only if \( a, b \in I \). Equivalently, a \( \Gamma \)-order ideal is a submonoid \( I \) of \( M \) which is closed under the action of \( \Gamma \) and it is hereditary in the sense that \( a \leq b \) and \( b \in I \) implies \( a \in I \). The set \( \mathcal{I}(M) \) of \( \Gamma \)-order ideals of \( M \) forms a (complete) lattice (see [5, §5]).

Let \( G \) be the group completion \( M^+ \) of a commutative monoid \( M \). The action of \( \Gamma \) on \( M \) lifts to an action on \( G \). There is a natural monoid homomorphism \( \Phi : M \rightarrow G \) and by \( M_+ \) we denote the image of \( M \) under this homomorphism. The monoid \( M_+ \) is called the positive cone of \( G \), and induces a pre-ordering on \( G \). We say that \( I \subseteq G \) is a \( \Gamma \)-order ideal of \( G \) if \( I = I_+ - I_- \), where \( I_+ = I \cap M_+ \) and \( I_- \) is a \( \Gamma \)-order ideal of \( M_+ \). It is not difficult to see that there is a lattice isomorphism between the \( \Gamma \)-order ideals of \( M_+ \) and \( \Gamma \)-order ideals of \( G \). In our setting (i.e., the monoid of
graded finitely generated projective modules) the monoid homomorphism $\phi : M \to G$ is injective and thus we can work with the lattice of $G$-order ideals of $M$.

Let $I$ be submonoid of the monoid $M$. Define an equivalence relation $\sim I$ on $M$ as follows: For $a, b \in M$, $a \sim I b$ if there exist $i, j \in I$ such that $a + i = b + j$ in $M$. The quotient monoid $M/I$ is defined as $M/\sim$. Observe that $a \sim I 0$ in $M$ for any $a \in I$. If $I$ is an order-ideal then $a \sim I 0$ if and only if $a \in I$.

3.1. The monoid $\mathcal{V}^g$. For a $\Gamma$-graded ring $A$ with identity, the isomorphism classes of graded finitely generated projective modules with the direct sum $[P] + [Q] = [P \oplus Q]$ as the addition operation constitute a monoid denoted by $\mathcal{V}^g(A)$. There is an action of the group $\Gamma$ on $\mathcal{V}^g(A)$ via the shifting of the modules

$$\alpha[P] \mapsto [P(\alpha)].$$

Here for a $\Gamma$-graded module $P = \bigoplus_{\gamma \in \Gamma} P_\gamma$, the $\alpha$-shifted graded module $P(\alpha)$ is defined as $P(\alpha) := \bigoplus_{\gamma \in \Gamma} P(\alpha)\gamma$, where $P(\alpha)\gamma = P_{\gamma + \alpha}$. Denote by $A$-Gr the category of graded left $A$-modules. For $\alpha, \beta \in \Gamma$, the shift functor

$$T_\alpha : A$-Gr $\to A$-Gr, \quad M \mapsto M(\alpha)$$

is an isomorphism with the property $T_\alpha T_\beta = T_{\alpha + \beta}$ for $\alpha, \beta \in \Gamma$.

The group completion of $\mathcal{V}^g(A)$ is called the graded Grothendieck group $K_0^g(A)$. It naturally inherits the action of $\Gamma$. This is a pre-ordered abelian group and as above the monoid $K_0^g(A)_+$ consisting of isomorphism classes of graded finitely generated projective $A$-modules is the cone of the ordering (see [23, §3.6]).

Denote by $A$-$\text{Mod}$ the category of left $A$-modules. The forgetful functor $U : A$-Gr $\to A$-$\text{Mod}$ (forgetting the graded structure) induces a homomorphism, $U : \mathcal{V}^g(A) \to \mathcal{V}(A)$. Since the (graded) Grothendieck groups are the group completion of these monoids, the homomorphism $U$ extends to the homomorphism of groups $U : K_0^g(A) \to K_0(A)$.

Note that if $\Gamma$ is trivial, the theory reduces to the classical (non-graded) theory, and $K_0^g(A)$ becomes the Grothendieck group $K_0(A)$ ([19, 26]).

When the ring $A$ is not unital, one can define $\mathcal{V}^g(A)$ via idempotent matrices over $A$. We refer the reader to [23] for a comprehensive introduction to graded ring theory and the graded Grothendieck groups.

For the case of a Leavitt path algebra $L(E)$ which is a $\mathbb{Z}$-graded algebra, we can describe the $\mathbb{Z}$-monoid $\mathcal{V}^g(L(E))$ directly from the graph $E$. We do this first for $\mathcal{V}(L(E))$ and then proceed to give the graded version.

A commutative monoid $M_E$ associated to a directed row-finite graph $E$ was constructed in [5]. The monoid $M_E$ is generated by vertices $v \in E^0$ subject to relations

$$v = \sum_{e \in s^{-1}(v)} r(e),$$

for each $v \in E^0$ which is not a sink. It was proved that (see [5, Theorem 3.5]) the natural map

$$M_E \to \mathcal{V}(L(E)), \quad v \mapsto [L(E)v],$$

induces a monoid isomorphism.

There is an explicit description [5, §4] of the congruence on the free commutative monoid given by the defining relations of $M_E$ with $E$ a row-finite graph. Let $F$ be the free commutative monoid on the set $E^0$. The nonzero elements of $F$ can be written in a unique form up to permutation as $\sum_{i=1}^n v_i$, where $v_i \in E^0$. Define a binary relation $\to_1$ on $F \setminus \{0\}$ by $\sum_{i=1}^n v_i \to_1 \sum_{i \neq j} v_i + \sum_{e \in s^{-1}(v_j)} r(e)$ whenever $j \in \{1, \cdots, n\}$ is such that $v_j$ is not a sink. Let $\to$ be the transitive and reflexive closure of $\to_1$ on $F \setminus \{0\}$ and $\sim$ the congruence on $F$ generated by the relation $\to$. Then $M_E = F/\sim$.

Remark 3.1. Observe that for $\alpha, \beta \in F \setminus \{0\}$, we have $\alpha \sim \beta$ if and only if there is a finite string $\alpha = \alpha_0, \alpha_1, \cdots, \alpha_n = \beta$, such that, for each $i = 0, \cdots, n - 1$, either $\alpha_i \to_1 \alpha_{i+1}$ or $\alpha_{i+1} \to_1 \alpha_i$. The number $n$ above will be called the length of the string.

We state the following lemma, given in [5, Lemma 4.3], for later use.

Lemma 3.2. Let $E$ be a row-finite graph, $F$ the free commutative monoid generated by $E^0$ and $M_E$ the monoid of the graph $E$. For $\alpha, \beta \in F \setminus \{0\}$, $\alpha \sim \beta$ in $F$ if and only if there is $\gamma \in F \setminus \{0\}$ such that $\alpha \to \gamma$ and $\beta \to \gamma$.

Next we recall the graded version of $M_E$ with $E$ a row-finite graph, which is a $\mathbb{Z}$-monoid [7, §5.3]. Let $M_E^g$ be an commutative monoid generated by $\{v(i) | v \in E^0, i \in \mathbb{Z}\}$ subject to relations

$$v(i) = \sum_{e \in s^{-1}(v)} r(e)(i - 1)$$

for $v \in E^0$ which is not a sink.
Throughout the paper, we simultaneously use \( v \in E^0 \) as a vertex of \( E \), as an element of \( L(E) \) and the element \( v = v(0) \) in \( M_E^{gr} \), as the meaning will be clear from the context.

The monoid \( \mathcal{V}^g(L(E)) \) for a graph \( E \) was studied in detail in [7]. In the case that \( E \) is a row-finite graph, \( \mathcal{V}^g(L(E)) \) is generated by graded finitely generated projective \( L(E) \)-modules \([L(E)v(i)]\). In [7, Proposition 5.7], \( \mathcal{V}^g(L(E)) \) and \( M_E^{gr} \) were related via the \( Z \)-monoid isomorphism
\[
M_E^{gr} \cong M_{E \times \mathbb{Z}} \cong \mathcal{V}(L(E \times 1 \mathbb{Z})) \cong \mathcal{V}^g(L(E)),
\]
(3.4)
\( \quad (v(i) \mapsto v_i \mapsto L(E \times 1 \mathbb{Z})v_i \mapsto (L(E)v)(-i)) \), see [7, Proposition 5.7]. We correct here that the isomorphism \( M_E^{gr} \cong M_{E \times \mathbb{Z}} \) should be given by \( v(i) \mapsto v_i \) and that the \( Z \)-action on \( M_E^{gr} \) given by Equation (5-10) in [7] should be \( n v(i) = v(i - n) \) for \( n, i \in \mathbb{Z} \) and \( v \in E^0 \).

It was proved in [7, Corollary 5.8] that \( \mathcal{V}^g(L(E)) \) is a cancellative monoid and thus \( \mathcal{V}^g(L(E)) \rightarrow K_0^g(L(E)) \) is injective and therefore the positive cone \( K_0^g(L(E))_+ \cong \mathcal{V}^g(L(E)) \). This shows that, in contrast with the non-graded setting, no information is lost going from the graded monoid to the graded Grothendieck group and as the morphisms involved are order-preserving, one can formulate the statements either on the level of \( K_0^g \) or the monoid \( \mathcal{V}^g \).

Suppose that \( H_1 \) and \( H_2 \) are hereditary saturated subsets of \( E^0 \) with \( H_1 \subseteq H_2 \). We denote by \( M_{H_i}^{gr} \) the monoid \( M_{H_i}^{gr} \), for \( i = 1, 2 \). We claim that \( M_{H_i}^{gr} \) is a submonoid of \( M_{H_2}^{gr} \). In fact, if \( a, b \in M_{H_2}^{gr} \cong M_{E \times H_2 \times \mathbb{Z}} \), with \( a, b \in M_{H_1}^{gr} \), by Lemma 3.2 there exists \( \gamma \) in the free commutative monoid on the set \( \{E_{H_1 \times \mathbb{Z}} \} \) such that \( a \rightarrow \gamma \) and \( b \rightarrow \gamma \). Since \( a, b \in M_{H_1}^{gr} \), we have that \( a \) and \( b \) are sums of finitely many \( v(i) \) with \( v \in H_1 \) and \( i \in \mathbb{Z} \). Since \( H_1 \) is hereditary, all the binary relations \( \rightarrow 1 \) appearing in \( a \rightarrow \gamma \) and \( b \rightarrow \gamma \) are in the free monoid generated by \( (E_{H_1 \times \mathbb{Z}}) \). Thus \( a = \gamma = b \) in \( M_{H_1}^{gr} \cong M_{E \times H_1 \times \mathbb{Z}} \).

**Lemma 3.3.** Suppose that \( H_1 \subseteq H_2 \) with \( H_1 \) and \( H_2 \) two hereditary saturated subsets of \( E^0 \). There is a \( Z \)-monoid isomorphism
\[
M_{H_2 \cap H_1}^{gr} \cong M_{H_2}^{gr}/M_{H_1}^{gr},
\]
(3.5)
In particular, for a hereditary saturated subset \( H \) of \( E^0 \), and the order-ideal \( I = \langle H \rangle \subseteq M_E^{gr} \), we have a \( Z \)-monoid isomorphism
\[
M_{E\slash I}^{gr} \cong M_{E^{gr}}^{gr}/I.
\]
**Proof.** We define a homomorphism of monoids \( f: M_{H_2}^{gr} \rightarrow M_{H_2 \cap H_1}^{gr} \) such that
\[
f(v(i)) = \begin{cases} 
v(i), & \text{if } v \in H_2 \setminus H_1; \\
0, & \text{otherwise.}
\end{cases}
\]
(3.6)
for \( v \in H_2 \) and \( i \in \mathbb{Z} \). Take a vertex \( v \in E_0^0 \), which is not a sink. We claim that the relation (3.3) for the restriction graph \( E_{H_2}^0 \) is preserved by \( f \). If \( v \in H_1 \) and \( i \in \mathbb{Z} \), then \( r(e) \in H_1 \) for all \( e \in s^{-1}(v) \), since \( H_1 \) is hereditary. Thus \( f(\sum_{e \in s^{-1}(v)} r(e)(i - 1)) = 0 = f(v(i)) \). If \( v \in H_2 \setminus H_1 \) and \( i \in \mathbb{Z} \), then
\[
f(\sum_{e \in s^{-1}(v)} r(e)(i - 1)) = \sum_{e \in s^{-1}(v)} f(r(e))(i - 1) + \sum_{e \in s^{-1}(v)} f(r(e))(i - 1) = \sum_{e \in s^{-1}(v)} r(e)(i - 1) = v(i) = f(v(i)).
\]
Here, the second last equality follows from the relation for the monoid \( M_{H_2 \cap H_1}^{gr} \). Thus \( f \) is well-defined. Since \( f(v(i)) = 0 \) for \( v \in H_1 \) and \( i \in \mathbb{Z} \), we have the induced homomorphism \( M_{H_2}^{gr}/M_{H_1}^{gr} \rightarrow M_{H_2 \cap H_1}^{gr} \), still denoted by \( f \).

Now we define a homomorphism of monoids \( g: M_{H_2 \cap H_1}^{gr} \rightarrow M_{H_2}^{gr}/M_{H_1}^{gr} \) such that \( g(v(i)) = [v(i)] \) for \( v \in H_2 \setminus H_1 \) and \( i \in \mathbb{Z} \). Observe that
\[
v(i) = \sum_{e \in s^{-1}(v), r(e) \in H_2 \setminus H_1} r(e)(i - 1),
\]
is in \( M_{H_2 \cap H_1}^{gr} \) for \( i \in \mathbb{Z} \) and any vertex \( v \in H_2 \setminus H_1 \), which is not a sink. Thus \( g \) is well-defined. We can directly check that \( g \circ f = \text{id}_{M_{H_2}^{gr}/M_{H_1}^{gr}} \) and \( f \circ g = \text{id}_{M_{H_2 \cap H_1}^{gr}} \). This finishes the proof. \( \square \)

The following consequence holds immediately.
Corollary 3.4. Suppose that $H_1$ and $H_2$ are hereditary saturated subsets of $E^0$ with $H_1 \subseteq H_2$. There is a short exact sequence of commutative monoids

$$0 \longrightarrow M^\text{gr}_{H_1} \overset{i}{\longrightarrow} M^\text{gr}_{H_2} \overset{\pi}{\longrightarrow} M^\text{gr}_{H_2/H_1} \longrightarrow 0,$$

(3.7)

where $i: M^\text{gr}_{H_1} \to M^\text{gr}_{H_2}$ is the inclusion and $\pi: M^\text{gr}_{H_2} \to M^\text{gr}_{H_2/H_1}$ is the map given by (3.6).

For a hereditary saturated subset $H$ of $E^0$, set $e = \sum_{v \in H} v \in \mathcal{M}(I)$, where $I = \langle H \rangle$ is the ideal of $L(E)$ generated by $H$ and $\mathcal{M}(I)$ denotes the multiplier algebra of $I$. Observe that $e$ is a full idempotent in $\mathcal{M}(I)$, in the sense that $IeI = I$. Then $I$ is graded Morita equivalent to the Leavitt path algebra $L(E_H)$ (see [22, Example 2] and compare [39, Theorem 5.7 (3)]), where the Morita equivalence is indeed a graded Morita equivalence. It follows that

$$\mathcal{V}^\text{gr}(I) \cong \mathcal{V}^\text{gr}(L(E_H)) = M^\text{gr}_{H_1}. \tag{3.8}$$

Suppose that $I_1, I_2$ are two graded ideals of $L(E)$ with $I_1 \subseteq I_2$. Let $H_1$ and $H_2$ be the two hereditary saturated subsets of $E^0$ with $I_1 = \langle H_1 \rangle$ and $I_2 = \langle H_2 \rangle$. The quotient ideal $I_2/I_1$ is a graded ideal of $L(E)/I_1 = L(E/H_1)$. As before, $I_2/I_1$ is graded Morita equivalent to $L((E/H_1)_{H_2/H_1})$ and combining this with (3.5) we have

$$\mathcal{V}^\text{gr}(I_2/I_1) \cong \mathcal{V}^\text{gr}(L((E/H_1)_{H_2/H_1})) \cong M^\text{gr}_{H_2/H_1} \cong M^\text{gr}_{H_2}/M^\text{gr}_{H_1} \cong \mathcal{V}^\text{gr}(I_2)/\mathcal{V}^\text{gr}(I_1). \tag{3.9}$$

It is a fact that for a cancellative commutative monoid $M$, if there is a short exact sequence of commutative monoids

$$0 \longrightarrow N \overset{t}{\longrightarrow} M \overset{t}{\longrightarrow} P \longrightarrow 0$$

with $N$ an order-ideal of $M$, then there is a short exact sequence of their group completions

$$0 \longrightarrow N^+ \overset{\tau}{\longrightarrow} M^+ \overset{\tau}{\longrightarrow} P^+ \longrightarrow 0.$$

By [7, Corollary 5.8] $M^\text{gr}_E$ is cancellative for any graph $E$. Combining this with the fact that $M^\text{gr}_{H_1}$ is an order-ideal of $M^\text{gr}_{H_2}$ if $H_1$ and $H_2$ are hereditary saturated subsets of $E^0$ with $H_1 \subseteq H_2$, we have a short exact sequence

$$0 \longrightarrow K^\text{gr}_0(I_1) \overset{\tau}{\longrightarrow} K^\text{gr}_0(I_2) \overset{\pi}{\longrightarrow} K^\text{gr}_0(I_2/I_1) \longrightarrow 0,$$

(3.10)

where $\tau$ and $\pi$ are induced maps of (3.7).

Denote by $\mathcal{L}(K^\text{gr}_0(L(E)))$ and $\mathcal{L}(\mathcal{V}^\text{gr}(L(E)))$ the lattice of $\mathbb{Z}$-order ideals of $K^\text{gr}_0(L(E))$ and $\mathcal{V}^\text{gr}(L(E))$, respectively. There is a lattice isomorphism between the set $\mathcal{F}_E$ of hereditary saturated subsets of $E^0$ and the set $\mathcal{L}(\mathcal{V}^\text{gr}(L(E)))$ ([7, Theorem 5.11]). The correspondence is

$$\Phi: \mathcal{F}_E \longrightarrow \mathcal{L}(\mathcal{V}^\text{gr}(L(E))),$$

(3.11)

where $H$ is a hereditary saturated subset of $E^0$, and $\Phi(H)$ is the $\mathbb{Z}$-order ideal generated by the set

$$\{v(i) \mid v \in H, i \in \mathbb{Z}\}.$$

This correspondence for finite graphs with no sinks was first established in [20, Theorem 12].

Recall that the lattice of graded ideals of $L(E)$ denoted by $\mathcal{L}(\mathcal{V}^\text{gr}(L(E)))$ from §2.3. Combining the lattice isomorphisms (2.3) and (3.11), there are lattice isomorphisms between the following lattices:

$$\mathcal{L}(\mathcal{V}^\text{gr}(L(E))) \cong \mathcal{F}_E \cong \mathcal{L}(M_E) \cong \mathcal{L}(M^\text{gr}_E) \cong \mathcal{L}(\mathcal{V}^\text{gr}(L(E))). \tag{3.12}$$

The lattice isomorphisms (3.12) allow us to relate the prime elements of these lattices. As mentioned in §2.3, the prime elements of $\mathcal{L}(\mathcal{V}^\text{gr}(L(E)))$ coincide with the graded prime ideals of $L(E)$. We give an explicit definition of prime elements (§2.1) in the setting of a lattice $\mathcal{L}(M)$ for a monoid $M$ with a $\Gamma$-action.

Definition 3.5. Let $M$ be a monoid with the group $\Gamma$ acting on it. A $\Gamma$-order ideal $N$ is called $\Gamma$-prime if for any $\Gamma$-order ideals $N_1, N_2 \subseteq M$, $N_1 \cap N_2 \subseteq N$ implies that $N_1 \subseteq N$ or $N_2 \subseteq N$.

We are in a position to relate the spectrum of $L(E)$ with the set of $\mathbb{Z}$-prime order ideals of $\mathcal{V}^\text{gr}(L(E))$.

Theorem 3.6. Let $E$ be a row-finite graph and $\mathcal{V}^\text{gr}(L(E))$ its associated monoid. Then there is a one-to-one order-preserving correspondence between the set of graded prime ideals of $L(E)$ and the set of $\mathbb{Z}$-prime order ideals of $\mathcal{V}^\text{gr}(L(E))$.

Proof. Since there is a lattice isomorphism between the lattice of graded ideals of $L(E)$ and the lattice of $\mathbb{Z}$-order ideals of $\mathcal{V}^\text{gr}(L(E))$ via (3.12) and the lattice isomorphisms preserve, in particular, prime elements, the statement follows. \qed
4. A short exact sequence of Grothendieck groups for a Leavitt path algebra

In order to relate the graded Grothendieck group $K^g_0$ to the non-graded $K_0$ and $K_1$ in the setting of Leavitt path algebras and ultimately to the filtered $K$-theory of these algebras, we establish a van den Bergh like exact sequence. For a right regular Noetherian $\mathbb{Z}$-graded ring $A$, van den Bergh established the long exact sequence

$$\cdots \longrightarrow K_{n+1}(A) \longrightarrow K^g_n(A) \overset{\tau}{\longrightarrow} K^g_{n}(A) \overset{U}{\longrightarrow} K_n(A) \longrightarrow \cdots.$$  \hspace{1cm} (4.1)

Here $\tau = \tau_1 - \tau_0 = \tau_1 - 1$, where $\tau_1$ is the shift functor (3.1) and $U$ is the forgetful functor [23, §6.4]. Leavitt path algebras are in general neither Noetherian nor regular. However for an arbitrary graph $E$, in Section 4.1 we establish an exact sequence

$$K_1(L(E)) \longrightarrow K^g_0(L(E)) \longrightarrow K^g_0(L(E)) \longrightarrow K_0(L(E)) \longrightarrow 0.$$  

This exact sequence involving only $K_0$-groups and for finite graphs with no sinks was first established in [21, Theorem 3].

4.1. A short exact sequence of Grothendieck groups. Let $E$ be a row-finite graph and $L(E)$ the Leavitt path algebra of $E$ over a field $k$. Recall from [7, Corollary 5.8] that $V(L(E)) \rightarrow K^g_0(L(E))$ is injective and there is a $\mathbb{Z}$-monoid isomorphism $V(L(E)) \cong M^g_E$ (see (3.4)).

We define a group homomorphism $\phi : K^g_0(L(E)) \rightarrow K^g_0(L(E))$ which takes the role of $\tau$ in the sequence (4.1). Let $F^g_E$ be the free commutative monoid generated by $\{v(i) \mid v \in E^0, i \in \mathbb{Z}\}$. We first define a monoid homomorphism $\phi : F^g_E \rightarrow K^g_0(L(E))$ such that $\phi(v(i)) = v(i + 1) - v(i)$. Observe that $\phi$ preserves the relations of $M^g_E$. Thus there is an induced monoid homomorphism $M^g_E \rightarrow K^g_0(L(E))$, still denoted by $\phi$. Since $K^g_0(L(E))$ is the group completion of $M^g_E$, by universality of group completion, $\phi$ extends to a group homomorphism

$$\phi : K^g_0(L(E)) \longrightarrow K^g_0(L(E))$$  

$$v(i) \mapsto v(i + 1) - v(i),$$  \hspace{1cm} (4.2)

where $i, j \in \mathbb{Z}$.

We are in a position to give a short exact sequence relating the graded Grothendieck group to the nongraded one of a Leavitt path algebra over a row-finite graph.

**Proposition 4.1.** Let $E$ be a row-finite graph. We have the following short exact sequence

$$K^g_0(L(E)) \overset{\phi}{\longrightarrow} K^g_0(L(E)) \overset{U}{\longrightarrow} K_0(L(E)) \longrightarrow 0,$$  \hspace{1cm} (4.3)

where $\phi$ is the homomorphism (4.2) and $U$ is the homomorphism induced by the forgetful functor.

**Proof.** Since each element $a \in K^g_0(L(E))$ may be written as $a = x - y$ with $x = \sum_i u_i(s_i), y = \sum_j v_j(t_j)$ in $M^g_E$, we have $(U \circ \phi)(a) = U\left(\sum_i u_i(s_i + 1) - \sum_i u_i(s_i) + \sum_j v_j(t_j + 1) - \sum_j v_j(t_j)\right) = 0$ by the definition of $U$ and $\phi$. Since the canonical homomorphism $M^g_E \rightarrow M_E$ is surjective, the map $U : K^g_0(L(E)) \rightarrow K_0(L(E))$ is surjective.

Now we show that $\text{Ker } U \subseteq \text{Im } \phi$. Observe that

$$u(i) - u(j) = \sum_{k=i}^{j-1} \phi(u(k))$$  \hspace{1cm} (4.4)

with $u \in E^0, i, j \in \mathbb{Z}$ and $i > j$. Suppose that $b \in \text{Ker } U$. We may write $b = x - y$ such that

$$x = \sum_s u_s(k_s)$$

and

$$y = \sum_n v_n(a_n)$$

in $M^g_E$. Since

$$U(b) = \sum_s u_s - \sum_n v_n = 0$$

in $K_0(L(E))$, we have

$$\sum_s u_s = \sum_n v_n$$

in $K_0(L(E))$ and thus

$$\sum_s u_s + z = \sum_n v_n + z$$

in $K_0(L(E))$. Therefore $b = x - y = \sum_{k=0}^{j-1} \phi(u(k))$, and hence $b \in \text{Im } \phi$. This completes the proof.


in $M_E$ for some $z \in M_E$ (see [26, Corollary 1.5]). By Lemma 3.2, there exists $\gamma \in F$ such that
\[ \sum_s u_s + z \rightarrow \gamma \]
and
\[ \sum_n v_n + z \rightarrow \gamma. \]
We will show that $\sum_s u_s + z - \gamma \in \text{Im} \phi$ and $\sum_n v_n + z - \gamma \in \text{Im} \phi$. Then it follows that
\[ \sum_s u_s + z - \gamma = (\sum_n v_n + z - \gamma) = \sum s u_s - \sum_n v_n \in \text{Im} \phi \]
and thus
\begin{align*}

b &= x - y \\
&= \sum_s u_s(k_s) - \sum_n v_n(a_n) \\
&= (\sum_s u_s(k_s) - \sum_n u_s) - (\sum_n v_n(a_n) - \sum_n v_n) + (\sum_s u_s - \sum_n v_n) \in \text{Im} \phi,
\end{align*}
(4.5)
since by (4.4) $\sum_s u_s(k_s) - \sum_s u_s \in \text{Im} \phi$ and $\sum_n v_n(a_n) - \sum_n v_n \in \text{Im} \phi$. We proceed by induction to show that
\[ \sum_s u_s + z - \gamma \in \text{Im} \phi. \]
Suppose that $\sum_s u_s + z \rightarrow \gamma$. We may write $\sum_s u_s + z = \sum_k w_k$ with $w_k \in E^0$. If $w_1 \rightarrow_1 \sum e \in s^{-1}(w_1) r(e)$ for $w_1 \in E^0$ not a sink, then
\[ \sum_s u_s + z - \gamma = \sum_k w_k - (\sum_{k \neq 1} w_k + \sum_{e \in s^{-1}(w_1)} r(e)) = w_1 - w_1(1) \in \text{Im} \phi. \]
By induction we get $\sum_s u_s + z - \gamma \in \text{Im} \phi$. Similarly, we have
\[ \sum_n v_n + z - \gamma \in \text{Im} \phi. \]

Therefore, we proved that the sequence (4.3) is exact. \qed

Remark 4.2. For an arbitrary graph $E$, $M_E^F$ is defined as a commutative monoid [7, §5C] such that the generators \{v(i) \mid v \in E^0, i \in Z\} are supplemented by generators $q_Z(i)$ as $i \in Z$ and $Z$ runs through all nonempty finite subsets of $s^{-1}(u)$ for infinite emitters $u \in E^0$. The relations are
\begin{enumerate}

\item $v(i) = \sum_{e \in s^{-1}(u)} r(e)(i - 1)$ for all regular vertices $v \in E^0$ and $i \in Z$;

\item $u(i) = \sum_{e \in Z} r(e)(i - 1) + q_Z(i)$ for all $i \in Z$, infinite emitters $u \in E^0$ and nonempty finite subsets $Z \subseteq s^{-1}(u)$;

\item $q_Z(i) = \sum_{e \in Z \setminus Z_i} r(e)(i - 1) + q_Z(i)$ for all $i \in Z$, infinite emitters $u \in E^0$ and nonempty finite subsets $Z_i \subseteq Z \subseteq s^{-1}(u)$.

\end{enumerate}

Ara and Goodearl defined analogous monoids $M(E, C, S)$ and constructed natural isomorphisms $M(E, C, S) \cong \mathcal{V}(CL_k(E, C, S))$ for arbitrary separated graphs ([4, Theorem 4.3]). The non-separated case reduces to that of ordinary Leavitt path algebras, and extends the result of [5] to non-row-finite graphs.

For an arbitrary graph $E$, we do have the short exact sequence
\[ K^F_0(L(E)) \xrightarrow{\phi} K^F_0(L(E)) \xrightarrow{\psi} K_0(L(E)) \rightarrow 0 \]
such that $\phi(q_Z(j)) = q_Z(j + 1) - q_Z(j)$ for generators $q_Z(j)$. The proof is similar to that for the row-finite case. This paper is devoted to row-finite graphs. In order to extend the results of this paper to arbitrary graphs, one needs to establish an exact sequence between $K^F_0$ of an arbitrary Leavitt path algebra $L(E)$ with the quotient $L(E)/I$, for any graded ideal $I$ as in Lemma 5.6 (see Remark 5.7).

Decompose the vertices of a row-finite graph $E$ as $E^0 = R \cup S$, where $S$ is the set of sinks and $R = E^0 \setminus S$. With respect to this decomposition we write the adjacency matrix of $E$ as
\[ A_E = \begin{pmatrix} & B_E \\ 0 & C_E \end{pmatrix}. \]
Here, we list the vertices in $R$ first and then the vertices in $S$.

Recall the following computation of $K$-theory for Leavitt path algebra from [3, Corollary 7.7] and [18, Proposition 3.4(ii)-(iii)].

Lemma 4.3. Let $E$ be a row-finite graph and $L(E)$ the Leavitt path algebra over a field $k$. We have
\begin{enumerate}

\item $K_0(L(E)) \cong \text{Coker} \left( \begin{pmatrix} B_E^{k-1} & C_E \\ 0 & C_E \\
\end{pmatrix} : \mathbb{Z}^R \rightarrow \mathbb{Z}^{E_0} \right)$;

\end{enumerate}
(ii) $K_1(L(E))$ is isomorphic to a direct sum:

$$\text{Coker}\left(\frac{(\psi_{k})}{(B_{k}^{E})} : (k^{\times}) \to (k^{\times})^{E^n}\right) \bigoplus \text{Ker}\left(\frac{(\psi_{k})}{(B_{k}^{E})} : \mathbb{Z}^R \to \mathbb{Z}^{E^n}\right).$$

Throughout the paper, for an abelian group $G$ and a set $S$ (possibly infinite), we denote the direct sum $\oplus S G$ by $G^S$.

In order to extend the exact sequence (4.3) to the $K_1$-group, we need the following proposition whose proof follows the ideas in the proof of [29, Theorem 3.2].

**Proposition 4.4.** Let $E$ be a row-finite graph. Define $K : \mathbb{Z}^R \to \mathbb{Z}^R \oplus \mathbb{Z}^S$ by $K(x) = ((B_{k}^{E} - I)(x), C_{k}^{E}x)$ and $\psi : \mathbb{Z}^R \oplus \mathbb{Z}^S \to K^{gr}_0(L(E))$ by $\psi(y) = y(0)$ where an element $y = \sum_{i} n_i v_i$ in $\mathbb{Z}^R \oplus \mathbb{Z}^S$ is identified as $y = \sum_{i} n_i v_i$ with $v_i \in E^0$ and $y(0) = \sum_{i} n_i v_i(0)$. Then $\psi$ restricts to an isomorphism $\psi$ from $\text{Coker} K$ onto $\text{Ker} \phi$ and an isomorphism $\psi$ from $\text{Coker} K$ onto $\text{Coker} \phi$ such that the following diagram commutes:

$$\begin{array}{ccc}
\text{Ker} K & \xrightarrow{\psi} & \mathbb{Z}^R \\
\downarrow & & \downarrow \\
\text{Ker} \phi & \xrightarrow{\psi} & \mathbb{Z}^R \oplus \mathbb{Z}^S \\
\downarrow & & \downarrow \\
\text{Ker} K & \xrightarrow{\psi} & \mathbb{Z}^R \oplus \mathbb{Z}^S \\
\downarrow & & \downarrow \\
\mathbb{Z}^{E^n} & \xrightarrow{\psi} & \mathbb{Z}^{E^n} \\
\end{array}$$

where $\phi$ is given in (4.2).

**Proof.** By typographical reasons, we denote $(v,n) := v_n$ and $(e,n) := e_n$ in the course of the proof.

For integers $m \leq n$, we denote by $E \times [m,n]$ the subgraph of $E \times \mathbb{Z}$ with vertices $\{(v,k) | m \leq k \leq n, v \in E^0\}$ and edges $\{(e,k) | m < k < n, e \in E^1\}$. We allow $m = -\infty$ with the obvious modification of the definition. Recall that the Leavitt path algebra of an acyclic finite graph is a direct sum of copies of matrix algebras, indexed by the sinks in $E$ (see [20, Theorem 2]). Similarly, since $E \times [m,n]$ is an acyclic row-finite graph with paths of length at most $n - m$, $L(E \times [m,n])$ is a direct sum of matrix algebras indexed by the set of sinks in $E \times [m,n]$. The set of sinks in $E \times [m,n]$ is

$$\{(v,m) | v \in E^0\} \cup \{(v,k) | v \in S, m < k \leq n\}.$$ 

We have that $K_0(L(E \times [m,n]))$ is the free abelian group with generators

$$\{[p_{(v,m)}] | v \in R\} \cup \{[p_{(v,k)}] | v \in S, m < k \leq n\}.$$ 

By continuity of $K$-theory we can let $n \to \infty$ and deduce that

$$K_0(L(E \times [m,\infty))) = \bigoplus_{v \in R} \mathbb{Z}[p_{(v,m)}] \bigoplus_{k=0}^{\infty} \mathbb{Z}[p_{(v,m+k)}]$$

$$\cong \mathbb{Z}^R \oplus \mathbb{Z}^{S_1} \oplus \mathbb{Z}^{S_2} \oplus \cdots,$$

where each $S_j$ is a copy of $S$ labelled to indicate its place in the direct sum. Denote by $\iota_m$ the inclusion map from $L(E \times [m+1,\infty))$ to $L(E \times [m,\infty))$ which induces a map from $K_0(L(E \times [m+1,\infty)))$ to $K_0(L(E \times [m,\infty)))$. If $v \in R$, then in $K_0(L(E \times [m,\infty)))$ we have

$$[p_{(v,m+1)}] = \sum_{e \in \iota_{m+1}^{-1}(v)} [(e,m+1)(e,m+1)^*]$$

$$= \sum_{e \in \iota_{m+1}^{-1}(v)} [(e,m+1)^*(e,m+1)]$$

$$= \sum_{w \in E^0} A(v,w)[p_{(v,m)}].$$

If $v \in S$ and $k \geq m+1$, $[p_{(v,k)}]$ is a generator in $K_0(L(E \times [m,\infty)))$. Thus the induced map from $\mathbb{Z}^R \oplus \mathbb{Z}^{S_1} \oplus \mathbb{Z}^{S_2} \oplus \cdots$ to $\mathbb{Z}^R \oplus \mathbb{Z}^{S_1} \oplus \mathbb{Z}^{S_2} \oplus \cdots$ is given by the matrix

$$D = \begin{pmatrix}
B_{0}^{E} & 0 & 0 & & \\
0 & B_{1}^{E} & 0 & & \\
& & & \ddots & \\
& & & & B_{k}^{E}
\end{pmatrix}$$

and $K_0^{gr}(L(E)) \cong K_0(L(E \times [m,\infty)))$ is the limit of the system

$$\mathbb{Z}^R \oplus \mathbb{Z}^{S_1} \oplus \mathbb{Z}^{S_2} \oplus \cdots \xrightarrow{D} \mathbb{Z}^R \oplus \mathbb{Z}^{S_1} \oplus \mathbb{Z}^{S_2} \oplus \cdots \xrightarrow{D} \cdots.$$
Recall that there is a canonical action of \( \mathbb{Z} \) on \( E \times_1 \mathbb{Z} \) which induces an action \( \gamma : \mathbb{Z} \to \text{Aut}(L(E \times_1 \mathbb{Z})) \) characterised by \( \gamma_1((v, n)) = (v, n - 1) \) and \( \gamma_1((e, n)) = (e, n - 1) \). Set \( \beta = \gamma_1 \). Then
\[
\beta^{-1} : L(E \times_1 [m, \infty)) \longrightarrow L(E \times_1 [m, \infty)),
\]
and that
\[
\beta^{-1}_m : K_0(L(E \times_1 [m, \infty))) \longrightarrow K_0(L(E \times_1 [m, \infty))),
\]
viewed as a map on \( \mathbb{Z}^R \oplus \mathbb{Z}^S \oplus \mathbb{Z}^S \oplus \cdots \) is multiplication by \( D \). We have the following commutative diagram
\[
\begin{array}{c}
L(E \times_1 [m, \infty)) \xrightarrow{i_m} L(E \times_1 [m, \infty)) \xrightarrow{\beta^{-1}_m} L(E \times_1 [m, \infty)) \\
\downarrow \beta^{-1}_m \downarrow \downarrow \beta^{-1}_m \\
L(E \times_1 [m+1, \infty)) \xrightarrow{i_m} L(E \times_1 [m, \infty)) \xrightarrow{\beta^{-1}_m} L(E \times_1 [m, \infty))
\end{array}
\]
and thus the following diagram
\[
\begin{array}{c}
\mathbb{Z}^R \oplus \mathbb{Z}^{S_{m+1}} \oplus \mathbb{Z}^{S_{m+2}} \oplus \cdots \xrightarrow{D-1} \mathbb{Z}^R \oplus \mathbb{Z}^{S_{m}} \oplus \mathbb{Z}^{S_{m+1}} \oplus \cdots \xrightarrow{\beta^{-1}_m} K_0(L(E \times_1 \mathbb{Z})) \\
\mathbb{Z}^R \oplus \mathbb{Z}^{S_{m+1}} \oplus \mathbb{Z}^{S_{m+2}} \oplus \cdots \xrightarrow{D-1} \mathbb{Z}^R \oplus \mathbb{Z}^{S_{m}} \oplus \mathbb{Z}^{S_{m+1}} \oplus \cdots \xrightarrow{\beta^{-1}_m} K_0(L(E \times_1 \mathbb{Z}))
\end{array}
\]
commutes.

We can realise \( K_0(L(E \times_1 \mathbb{Z})) \) as the group of equivalence classes \([([x_i])]\) of sequences in \( \prod_{i=0}^{\infty} (\mathbb{Z}^R \oplus \mathbb{Z}^S \oplus \mathbb{Z}^S \oplus \cdots) \) which eventually satisfy \( D x_i = x_{i-1} \) and two sequences are equivalent if they eventually coincide. The natural map \( i^m \) sends \( x \in \mathbb{Z}^R \oplus \mathbb{Z}^{S_m} \oplus \mathbb{Z}^{S_{m+1}} \oplus \cdots \) to the class of the sequence \( (x_i) \) with
\[
x_i = \begin{cases} 
D^{m-i}(x), & \text{if } i \leq m; \\
0, & \text{if } i > m.
\end{cases}
\]

Observe that the homomorphism \( i^0 \) restricts to an isomorphism of \( \text{Ker}(D-1) \) onto \( \text{Ker}(\beta^{-1}_1 - 1) \) and induces an isomorphism \( \tau^0_1 \) of \( \text{Coker}(D-1) \) onto \( \text{Coker}(\beta^{-1}_1 - 1) \). To show that \( i^0 \) is injective on \( \text{Ker}(D-1) \), we check directly that \( i^0(x) \) is the constant sequence \( (x) \) as \( x = D^{m-1}(x) \). If \( z = i^m(y) \in \text{Ker}(\beta^{-1}_1 - 1) \), then we have
\[
(\beta^{-1}_1 - 1)(i^m(y)) = 0 = i^m((D-1)(y)) = i^m(D(y) - y).
\]
Thus there exists a large \( N \) such that \( D^N(y) - y = 0 \). Take \( y' = D^N(y) \). It follows that \( z = i^0(y') \) with \( y' = D(y') \). Therefore \( i^0 \) restricts to an isomorphism of \( \text{Ker}(D-1) \) onto \( \text{Ker}(\beta^{-1}_1 - 1) \). To show that \( \tau^0_1 \) from \( \text{Coker}(D-1) \) to \( \text{Coker}(\beta^{-1}_1 - 1) \) is injective, suppose \( i^0(x) \) is in \( \text{Coker}(\beta^{-1}_1 - 1) \). Then \( ([z_i]) \in \text{Im}(\beta^{-1}_1 - 1) \). For a large enough \( k \), we have \( D^k(x) = z - k = Dy - y \). On the other hand, we have
\[
x = x - D^k(x) = (1 - D)(1 + D + D^2 + \cdots + D^{k-1})x - (D - 1)y_k \in \text{Im}(D-1),
\]
which implies that \( \tau^0_1 \) is injective. To show that \( \tau^0_1 \) is surjective, let \( i^m(y) \in K_0(L(E \times_1 \mathbb{Z})) \) for some \( m \leq 0 \). By (4.7), we have
\[
i^m(Dy) - i^m(y) = i^m((D-1)y) = (\beta^{-1}_1 - 1)(i^m(y)).
\]
Thus \( i^m(y) = i^m(D^{-m}y) \) define the same equivalence class in \( \text{Coker}(\beta^{-1}_1 - 1) \). So \( i^0(y) = i^m(D^{-m}y) \) define the same equivalence class as \( i^m(y) \). Hence \( i^0(y) = i^m(y) \) in \( \text{Coker}(\beta^{-1}_1 - 1) \) and \( \tau^0_1 \) is surjective.

Let \( i \) and \( j \) be the inclusion of \( \mathbb{Z}^R \) and \( \mathbb{Z}^R \oplus \mathbb{Z}^S \) as the first coordinates of \( \mathbb{Z}^R \oplus \mathbb{Z}^S \oplus \mathbb{Z}^S \oplus \cdots \). Then the following diagram commutes
\[
\begin{array}{c}
\mathbb{Z}^R \xrightarrow{k} \mathbb{Z}^R \oplus \mathbb{Z}^S \\
\downarrow i \downarrow \downarrow j \\
\mathbb{Z}^R \oplus \mathbb{Z}^S \oplus \mathbb{Z}^S \oplus \cdots \xrightarrow{D^{-1}} \mathbb{Z}^R \oplus \mathbb{Z}^S \oplus \mathbb{Z}^S \oplus \cdots
\end{array}
\]
\[ \text{commutative diagram} \]
\[
\begin{array}{cccccc}
\text{Ker } K & \to & \mathbb{Z}^R & \to & K & \to & \mathbb{Z}^R \oplus \mathbb{Z}^S & \to & \text{Coker } \psi \\
\downarrow & & \downarrow i & & \downarrow j & & \downarrow \phi & & \downarrow \psi \\
\text{Ker}(D - 1) & \to & \mathbb{Z}^R \oplus \mathbb{Z}^S \oplus \mathbb{Z}^S \oplus \cdots & \to & \mathbb{Z}^R \oplus \mathbb{Z}^S \oplus \mathbb{Z}^S \oplus \cdots & \to & \text{Coker}(D - 1) \\
\downarrow & & \downarrow j & & \downarrow \phi & & \downarrow \psi \\
\text{Ker}(\beta^{-1} - 1) & \to & K_0(L(E \times_1 \mathbb{Z})) & \to & K_0(L(E \times_1 \mathbb{Z})) & \to & \text{Coker}(\beta^{-1} - 1) \\
\downarrow & & \downarrow \phi & & \downarrow \phi & & \downarrow \psi \\
\text{Ker } \phi & \to & K_0^\text{gr}(L(E)) & \to & K_0^\text{gr}(L(E)) & \to & \text{Coker } \phi, \\
\end{array}
\]

implying the commutative diagram in (4.6) follows. The map \( \psi : \mathbb{Z}^R \oplus \mathbb{Z}^S \to K_0^\text{gr}(L(E)) \) is the composition of the maps in the third column and thus is given by \( \psi(y) = \gamma(0) \).

\[ \square \]

**Proposition 4.5.** Let \( E \) be a row-finite graph and \( k \) a field. We have the following short exact sequence

\[ K_1(L(E)) \xrightarrow{T} K_0^\text{gr}(L(E)) \xrightarrow{\phi} K_0^\text{gr}(L(E)) \xrightarrow{U} K_0(L(E)) \to 0, \]  
(4.10)

where \( T \) is the homomorphism given by

\[ K_1(L(E)) \cong \text{Coker} \left( \left( a_{i,j}^{E} - 1 \right) : (k^x)^R \to (k^x)^E \right) \oplus \text{Ker } \phi (\theta) \]

\[ \text{Proposition 4.4.} \]

5. The connecting map for \( K \)-theory of a Leavitt path algebra

In this section, we first briefly recall the notion of lower \( K \)-theory for rings (i.e., \( K_0 \) and \( K_1 \)) in §5.1 and then the long exact sequence for the computation of \( K \)-theory of Leavitt path algebras in §5.2. The group homomorphism \( K_1(L(E)) \to \text{Ker } (A_E^1 - I) \) appears in the long exact sequence where \( A_E \) is the adjacency graph of the graph \( E \). We describe a group homomorphism from \( \text{Ker } (A_E^1 - I) \) to \( K_1(L(E)) \) which is a section for the map \( K_1(L(E)) \to \text{Ker } (A_E^1 - I) \). This will be used in the proof of the main theorem of this paper, Theorem 7.3.

5.1. \( K_0 \) and \( K_1 \) of rings. Let \( R \) be a ring with unit. Write \( M_n(R) \) for the matrix ring. Regard \( M_n(R) \subseteq M_{n+1}(R) \) via \( a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \). Put \( M_\infty(R) = \bigcup_{n=1}^\infty M_n(R) \). Note that \( M_\infty(R) \) is a ring (without unit). We write \( \text{Idem}_n(R) \) and \( \text{Idem}_\infty(R) \) for the set of idempotent elements of \( M_n(R) \) and \( M_\infty(R) \). Thus \( \text{Idem}_\infty(R) = \bigcup_{n=1}^\infty \text{Idem}_n(R) \subseteq M_\infty(R) \).

We write \( GL_n(R) = \left( M_n(R) \right)^* \) for the group of invertible matrices. Regard \( GL_n(R) \subseteq GL_{n+1}(R) \) via \( g \mapsto \begin{pmatrix} g & 0 \\ 0 & 0 \end{pmatrix} \). Put \( GL(R) := \bigcup_{n=1}^\infty GL_n(R) \). Note \( GL(R) \) acts by conjugation on \( M_\infty(R) \), \( \text{Idem}_\infty(R) \) and, of course, \( GL(R) \).

For \( a, b \in M_\infty(R) \) there is a direct sum operation

\[ a \oplus b = \begin{pmatrix} a_{11} & 0 \\ a_{21} & b_{11} \\ a_{12} & 0 \\ a_{22} & b_{22} \\ a_{13} & 0 \\ a_{23} & b_{23} \\ \vdots & \vdots \\ a_{n,1} & 0 \\ b_{1,n} \end{pmatrix}. \]

We remark that if \( a \in M_m(R) \) and \( b \in M_n(R) \) then \( a \oplus b \) belongs to \( M_{m+n}(R) \) and is conjugate, by a permutation matrix, to the usual direct sum \( \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \). One checks that \( \oplus \) is associative and commutative up to conjugation. Thus the coinvariants under the conjugation action \( I(R) := \left( \text{Idem}_\infty(R) \right)_{GL(R)} \) with respect to \( \oplus \) form a commutative monoid.

**Definition 5.1.** Let \( R \) be a ring with unit. We define \( K_0(R) := I(R)^+ \)

and

\[ K_1(R) := \text{GL}(R)/[\text{GL}(R), \text{GL}(R)] = \left( \text{GL}(R) \right)_{ab}. \]

Here \([ , , ]\) denotes the multiplicative commutator subgroup, and the subscript \( ab \) indicates abelianisation.

If \( R \) is any ring (not necessarily unital), then the abelian group \( \tilde{R} = R \oplus \mathbb{Z} \) equipped with the multiplication

\[ (a, n)(b, m) = (ab + ma + nb, nm) \]

for \( a, b \in R, n, m \in \mathbb{Z} \) is a unital ring, with unit element \((0, 1)\), and \( \tilde{R} \to \mathbb{Z}, (a, n) \mapsto n \), is a unital homomorphism. Put \( K_j(R) := \text{Ker } (K_j(\tilde{R}) \to K_j(\mathbb{Z})) \), for \( j = 0, 1 \).
If $R$ happens to have a unit, we have two definitions for $K_j(R)$. To check that they are the same, one observes that the map $\tilde{R} \to R \times \mathbb{Z}, (a,n) \mapsto (a+n, n)$ is a unital ring isomorphism. One verifies that, under this isomorphism, $\tilde{R} \to \mathbb{Z}$ identifies with the projection $R \times \mathbb{Z} \to \mathbb{Z}$, and $\text{Ker} (K_j(\tilde{R}) \to K_j(\mathbb{Z}))$ with $\text{Ker} (K_j(A) \oplus K_j(\mathbb{Z}) \to K_j(\mathbb{Z})) = K_j(A)$.

Define $GL(R) = \text{Ker} (GL(\tilde{R}) \to GL(\mathbb{Z}))$ for all ring $R$. With this definition, $GL$ becomes a left exact functor in the category of rings and ring homomorphisms; thus if $A \subset B$ is an ideal embedding, then

$$GL(A) = \text{Ker} (GL(B) \to GL(B/A)).$$

It is straightforward from this that the group $K_1(R)$ defined above can be described as

$$K_1(R) = GL(R)/(E(\tilde{R}) \cap GL(R)).$$

A little more work shows that $E(\tilde{R}) \cap GL(R)$ is the smallest normal subgroup of $E(\tilde{R})$ which contains the elementary matrices $1 + ac_{ij}$ with $a \in R$ (see [35, 2.5]).

### 5.2. $K$-theory of Leavitt path algebras.

The computation of $K$-theory for Leavitt path algebras was considered in [3, 18]. In [3] the authors consider the Leavitt path algebra of a row-finite graph. The authors of [18] extend the $K$-theory computation of [3, Theorem 7.6] to Leavitt path algebras of graphs that may contain infinite emitters.

Recall that we decompose the vertices of a row-finite graph $E$ as $E^0 = R \sqcup S$, where $S$ is the set of sinks and $R = E^0 \setminus S$. With respect to this decomposition we write the adjacency matrix of $E$ as

$$A_E = \begin{pmatrix} n_{E} & c_{E} \\ 0 & 0 \end{pmatrix}.$$  

Here, we list the vertices in $R$ first and then the vertices in $S$.

The following result is taken from [18, Theorem 3.1] (see also [3, Theorem 7.6]).

**Lemma 5.2.** Let $E$ be a row-finite graph and $k$ a field. There is a long exact sequence

$$\cdots \longrightarrow K_n(k)^R \longrightarrow K_n(k)^{E^0} \longrightarrow K_n(L_k(E)) \longrightarrow K_{n-1}(k)^R \longrightarrow \cdots$$

for any $n \in \mathbb{Z}$.

By Lemma 5.2, setting $n = 1$, we have the following exact sequence

$$k \times R \begin{pmatrix} n_{E} - I \\ c_{E} \end{pmatrix} \longrightarrow k \times E^0 \longrightarrow K_1(L(E)) \longrightarrow \mathbb{Z} \begin{pmatrix} n_{E} - I \\ c_{E} \end{pmatrix} \longrightarrow \mathbb{Z}^{E^0}.$$  

(5.1)

We denote the map $K_1(L(E)) \longrightarrow \mathbb{Z}^R$ appearing in (5.1) by $\xi$. The definition of $\xi$ is given by the composition map $K_1(L(E)) \xrightarrow{\partial} K_0(K(E)) \cong \mathbb{Z}^R$, where $\partial$ is the connecting map of $K$-theory with respect to the short exact sequence given in (2.2). Observe that by (5.1) $\xi$ satisfies

$$\text{Im} \xi = \text{Ker} \left( \begin{pmatrix} n_{E} - I \\ c_{E} \end{pmatrix} : \mathbb{Z}^R \longrightarrow \mathbb{Z}^{E^0} \right).$$

So we have the map

$$\xi : K_1(L(E)) \longrightarrow \text{Ker} \left( \begin{pmatrix} n_{E} - I \\ c_{E} \end{pmatrix} : \mathbb{Z}^R \longrightarrow \mathbb{Z}^{E^0} \right).$$  

(5.2)

The map $k \times E^0 \to K_1(L(E))$ in (5.1) is induced by the homomorphism of algebras

$$\bigoplus_{v \in E^0} k v \longrightarrow L(E),$$

$v \mapsto v$,  

for each $v \in E^0$. By (5.1) we have the map

$$\lambda : \text{Coker} \left( \begin{pmatrix} n_{E} - I \\ c_{E} \end{pmatrix} : k \times R \longrightarrow k \times E^0 \right) \longrightarrow K_1(L(E)),$$

such that for $(\mu_v)_{v \in E^0} \in k \times E^0$,

$$\lambda((\mu_v)_{v \in E^0}) = \sum_{v \in E^0} \mu_v v \in K_1(L(E))$$

(see [9, Proposition 3.4.3]).
Recall that a group isomorphism from $\text{Ker} \left( \frac{C^*_E}{C^*_E} \right)$ to $K_1(C^*(E))$ is described in [10, Proposition 3.8]. Now we consider the Leavitt path algebra and define the map

$$\chi_1 : \text{Ker} \left( \frac{C^*_E}{C^*_E} \right) \longrightarrow K_1(L(E)),$$

and show that $\xi \circ \chi_1 = \text{id}$.

Take $x$ in $\text{Ker} \left( \frac{C^*_E}{C^*_E} \right)$. Note that by definition $x$ has only finitely many nonzero entries $x_v, \cdots, x_w$. We define

$$L^+_x = \left\{ (e,i) \mid e \in E^1, 1 \leq i \leq -x_{s(e)} \right\} \cup \left\{ (v,i) \mid v \in E^0, 1 \leq i \leq -x_v \right\},$$

and

$$L^-_x = \left\{ (e,i) \mid e \in E^1, 1 \leq i \leq -x_{s(e)} \right\} \cup \left\{ (v,i) \mid v \in E^0, 1 \leq i \leq -x_v \right\}.$$

By [10, Lemma 3.1] and [10, Lemma 3.2], $L^+_x$ and $L^-_x$ have the same number of elements and there are bijections

$$\langle \cdot \rangle : L^+_x \longrightarrow \{1, \cdots, h\} \quad \text{and} \quad \langle \cdot \rangle : L^-_x \longrightarrow \{1, \cdots, h\}$$

with the property that $[x,i] = [y,j]$ implies $r(x) = r(y)$. Here $h$ is the common number of elements in $L^+_x$ and $L^-_x$.

With notation as above, define two matrices $V$ and $P$ by

$$V = \sum_{1 \leq i \leq -x_v, s(e) = w} eE_{|w,i|,|e,i|} + \sum_{1 \leq i \leq -x_w, s(e) = w} e^*E_{|e,i|,|w,i|},$$

and

$$P = \sum_{1 \leq i \leq -x_w} wE_{|w,i|,|w,i|} + \sum_{1 \leq i \leq -x_v, s(e) = w} vE_{|e,i|,|e,i|},$$

where $E_{\bullet, \bullet}$ denote the standard matrix units in $h \times h$-matrix algebra $M_h(M(L(E)))$ (see [38, §2] and [10, Definition 3.3]). Here $M(L(E))$ is the multiplier algebra of $L(E)$. Recall that the multiplier of an algebra without unit has a unit.

We write $V_x$ and $P_x$ for the corresponding elements $V$ and $P$, using the added subscript to emphasise the dependence of each of $V$ and $P$ on $x \in \text{Ker} \left( \frac{C^*_E}{C^*_E} \right)$. In addition, we define

$$U_x = V_x + (1 - P_x).$$

The matrix $U_x$ is invertible and $U_x^{-1} = U_x^* = V_x^* + (1 - P_x)$ (compare [10, Lemma 3.4] and [10, Fact 3.6]).

**Proposition 5.3.** Let $E$ be a row-finite graph. We have the following statements.

(i) There exists a group isomorphism $\chi_0 : \text{Coker} \left( \frac{C^*_E}{C^*_E} \right) \longrightarrow K_0(L(E))$ given by

$$\chi_0 \left( \epsilon_v + \text{Im} \left( \frac{C^*_E}{C^*_E} \right) \right) = v$$

for any $v \in E^0$.

(ii) The map $\chi_1 : \text{Ker} \left( \frac{C^*_E}{C^*_E} \right) \longrightarrow K_1(L(E))$ given by

$$\chi_1(x) = U_x$$

satisfies $\xi \circ \chi_1 = \text{id}$, where $\xi : K_1(L(E)) \longrightarrow \text{Ker} \left( \frac{C^*_E}{C^*_E} \right)$ is given in (5.2).

**Proof.** (i) There is a short exact sequence (2.2)

$$0 \longrightarrow K(E) \xrightarrow{i} C(E) \xrightarrow{p} L(E) \longrightarrow 0,$$

where $K(E)$ is the ideal (2.1) of the Cohn path algebra $C(E)$. We then have the induced maps of $K$-groups

$$K_0(K(E)) \xrightarrow{i^*} K_0(C(E)) \xrightarrow{p^*} K_0(L(E)) \xrightarrow{\lambda} \mathbb{Z}^R$$

(5.5)

By [12, Theorem 4.2] we have $\lambda(v) = \epsilon_v$ for any $v \in E^0$. By [12, (4.13)] there is an isomorphism of algebras

$$\bigoplus_{v \in R} M_{P_v} \xrightarrow{\sim} K(E),$$

$$\epsilon_{\alpha, \beta} \mapsto \alpha(v - \sum_{e \in E^{-1}(v)} ee^*) \beta.$$
Observe that the induced isomorphism map $\kappa$ in (5.5) is the composition of
\[ K_0(K(E)) \xrightarrow{\kappa} K_0(\bigoplus_{v \in R} \mathbb{M}_{p_v}) \xrightarrow{\kappa} \mathbb{Z}^R \] (5.7)
which sends $v - \sum_{e \in \epsilon(v)} ee^*$ to $e_v$. Therefore $\chi_0 : \text{Coker} \left( \frac{b_{k, e} - t}{c_{k, e}} I \right) \to K_0(L(E))$ is given by
\[ \chi_0(e_v + \text{Im} \left( \frac{b_{k, e} - t}{c_{k, e}} I \right)) = v, \]
for any $v \in E^0$. Obviously $p_*$ is surjective.

(ii) To show that $\xi \circ \chi_1 = \text{id}$, take $x \in \text{Ker} \left( \frac{b_{k, e} - t}{c_{k, e}} I \right)$. We lift $U_x$ to $\tilde{U}_x = V_x + (1 - \tilde{P}_x) \in M_k(\mathcal{M}(C(E)))$, where $\tilde{V}_x$ and $\tilde{P}_x$ are the elements $V$ and $P$ in $M_k(\mathcal{M}(L(E)))$ which we obtain by using universal property of the Cohn path algebra $C(E)$ (§2.3). Note that $\tilde{U}_x$ is a partial isometry.

By [11, §2.4], we have $\xi(U_z) = [p_h] - [ap_h a^{-1}]$ with
\[ a = \left( \begin{array}{cc} \tilde{\sigma} \sigma^* \tilde{\sigma} & \tilde{\sigma} \sigma^* \tilde{\sigma} - 1 \end{array} \right). \]
We further have
\[ ap_h a^{-1} = \left( \begin{array}{cc} \tilde{\sigma} \sigma^* \tilde{\sigma} & 0 \\ 0 & 1 - \tilde{\sigma} \sigma^* \tilde{\sigma} \end{array} \right), \]
and
\[ 1 - \tilde{U}_x \tilde{U}_x = \tilde{P}_x - \tilde{V}_x \tilde{V}_x = \sum_{1 \leq i \leq \mathbb{Z}^R} (w - \sum_{s \in \mathbb{E}_0} ee^*) E_{w, i, w, i}, \]
and
\[ 1 - \tilde{U}_x \tilde{U}_x = \tilde{P}_x - \tilde{V}_x \tilde{V}_x = \sum_{1 \leq i \leq \mathbb{Z}^R} (w - \sum_{s \in \mathbb{E}_0} ee^*) E_{w, i, w, i}. \]
Since $\kappa$ sends $w - \sum_{s \in \mathbb{E}_0} ee^*$ to $e_w$ by (5.7), we have $(\xi \circ \chi_1)(x) = \sum_{x \neq 0} x_w e_w = x$, implying that $\xi \circ \chi_1 = \text{id}$. This completes the proof.

Let $E$ be a row-finite graph, $H$ a hereditary saturated subset of $E^0$, $R$ the set of regular vertices and $S = E^0 \setminus R$ the set of all sinks in $E^0$. We write the adjacency matrix $A_E$ of $E$ with respect to the decomposition
\[ R \cap H, \ S \cap H, \ R \setminus H, \text{ and } S \setminus H. \]
Therefore
\[ A_E = \left( \begin{array}{ccc} A & 0 & 0 \\ 0 & \xi & 0 \\ 0 & 0 & \beta \end{array} \right). \]

**Lemma 5.4.** Let $E$ be a row-finite graph and $H$ a hereditary saturated subset of $E^0$. There are commutative diagrams

\[
\begin{array}{cccc}
\text{Ker} \left( \frac{A' - \ell}{\alpha'} \right) & \xrightarrow{\sigma} & k \times R \cap H & \xrightarrow{\sigma} & k \times H \\
\text{Ker} \left( \frac{A' - \ell}{\alpha'} \right) & \xrightarrow{\sigma} & k \times R & \xrightarrow{\sigma} & k \times E^0 \\
\text{Ker} \left( \frac{A' - \ell}{\alpha'} \right) & \xrightarrow{\sigma} & k \times R \setminus H & \xrightarrow{\sigma'} & k \times E^0 \setminus H \\
\end{array}
\]

where $\sigma$ is the natural inclusion and $\sigma'$ is the natural projection.

**Proof.** We can check directly that the middle two squares are commutative. \hfill \Box

**Lemma 5.5.** Let $E$ be a row-finite graph, $S$ the set of sink vertices of $E^0$, $H$ a hereditary saturated subset of $E^0$ and $A_E = \left( \begin{array}{cc} A_S \xi \alpha' \\ 0 \beta \end{array} \right)$ the adjacency matrix of $E$ with respect to the decomposition $E^0 = H \cup (E^0 \setminus H)$. Let $S'$ be the set of sink vertices in $E/H$. Then there is a commutative diagram

\[
\begin{array}{cccc}
\text{Ker} \left( \frac{A' - \ell}{\alpha'} \right) & \xrightarrow{\sigma} & k \times R \cap H & \xrightarrow{\sigma} & k \times H \\
\text{Ker} \left( \frac{A' - \ell}{\alpha'} \right) & \xrightarrow{\sigma} & k \times R & \xrightarrow{\sigma} & k \times E^0 \\
\text{Ker} \left( \frac{A' - \ell}{\alpha'} \right) & \xrightarrow{\sigma} & k \times R \setminus H & \xrightarrow{\sigma'} & k \times E^0 \setminus H \\
\end{array}
\]

where $\sigma$ is the natural inclusion and $\sigma'$ is the natural projection.
Lemma 5.6. Let $E$ be a row-finite graph, $S$ the set of sink vertices of $E^0$, $H$ a hereditary saturated subset of $E^0$, $I = \langle H \rangle$ the graded ideal of $L(E)$ generated by $H$, and $A_E$ the adjacency matrix of $E$ with respect to the decomposition $E^0 = H \cup (E^0 \setminus H)$, where the vertices in $H$ are listed first. Let $S'$ be the set of sink vertices in $E/H$. Then there is a commutative diagram

\[
\begin{array}{cccc}
\text{Ker} (A_{E_H}^t - I) & \xrightarrow{\tau} & \text{Ker} (A_E^t - I) & \xrightarrow{\tau'} & \text{Ker} (A_{E/H}^t - I) \\
0 & \xrightarrow{\tau} & \mathbb{Z}^{H \setminus S} & \xrightarrow{\tau'} & 0 \\
\text{Coker} (A_{E_H}^t - I) & \xrightarrow{\tau} & \text{Coker} (A_E^t - I) & \xrightarrow{\tau'} & \text{Coker} (A_{E/H}^t - I) \\
0 & \xrightarrow{\tau} & \mathbb{Z}^{E_0} & \xrightarrow{\tau'} & 0
\end{array}
\]

(5.8)

where $\tau$ is the natural inclusion and $\tau'$ is the natural projection. The connecting map $\delta$ is given by $\delta_{E,H,E}(x) = [X^t x]$ for $x \in \ker(A_{E/H}^t - I)$.

Proof. We observe that $(E^0 \setminus S) \setminus (H \setminus S) = (E^0 \setminus H) \setminus S'$. Then the second row of (5.8) is a short exact sequence. We check directly that the two squares in the middle of the above diagram are commutative. By the Snake Lemma we have the short exact sequence

\[
\begin{align*}
\ker(A_{E_H}^t - I) & \xrightarrow{\tau} \ker(A_E^t - I) \xrightarrow{\tau'} \ker(A_{E/H}^t - I) \\
\xrightarrow{\delta} & \delta \text{Coker}(A_{E_H}^t - I) \xrightarrow{\tau} \text{Coker}(A_E^t - I) \xrightarrow{\tau'} \text{Coker}(A_{E/H}^t - I),
\end{align*}
\]

(5.9)

with the connecting map $\delta_{E,H,E}$ defined by $\delta_{E,H,E}(x) = X^t x$. \hfill \square

such that $\rho \circ \psi = \psi \circ \delta$. (5.10)
\section*{Proof.} By (4.6), we have a commutative diagram

\[ \begin{array}{ccc}
\mathbb{Z}^R & \xrightarrow{A_\mathcal{E}^L - I} & \mathbb{Z}^R \oplus \mathbb{Z}^S \\
\psi & & \phi \\
K_0^\mathcal{G}(L(E)) & \xrightarrow{\phi} & K_0^\mathcal{G}(L(E)).
\end{array} \]

We can check directly that the other squares of (5.10) are commutative. Combining with Lemma 5.5 and (3.10), by the snake Lemma, there exist \( \delta \) and \( \rho \) such that \( \rho \circ \psi = \psi \circ \delta \).

\section*{Remark 5.7.} In order to obtain a similar result as Lemma 5.6 for an arbitrary graph \( E \) (possibly with infinite emitters \( v \in E^0 \) with \( |x^{-1}(v)| = \infty \)), one needs to develop techniques for infinite emitters in order to have similar conditions as Lemma 5.5 to use Snake Lemma and to have a short exact sequence

\[ 0 \longrightarrow K_0^\mathcal{G}(I) \longrightarrow K_0^\mathcal{G}(L(E)) \longrightarrow K_0^\mathcal{G}(L(E)/I) \longrightarrow 0, \]

where \( I \) is a graded ideal of \( L(E) \) so that one can establish Lemma 5.6.

\section*{Lemma 5.8.} Let \( E \) be a row-finite graph, \( S \) the set of sink vertices of \( E \), \( H \) a hereditary saturated subset of \( E^0 \) and \( S' \) the set of sink vertices in \( E/H \). We write the adjacency matrix \( A_E \) of \( E \) by arranging vertices in \( H \) first such that \( A_E = \left( \begin{array}{c} A_{E,H} \\ \lambda_{E,H} \end{array} \right) \). Then there is a commutative diagram

\[ \begin{array}{ccc}
\text{Coker} \left( A_{E,H}^t - I : k^H \to k^H \right) & \overset{\lambda_H}{\longrightarrow} & K_1(L(E_H)) \\
\tau & \longrightarrow & \alpha \\
\text{Coker} \left( A_{E,H}^t - I : k^E \to k^E \right) & \overset{\lambda_E}{\longrightarrow} & K_1(L(E)) \\
\tau' & \longrightarrow & \alpha' \\
\text{Coker} \left( A_{E/H}^t - I : k^{(E^0 \setminus H) \setminus S'} \to k^{(E^0 \setminus H)} \right) & \overset{\lambda_{E/H}}{\longrightarrow} & K_1(L(E/H)) \\
\tau' & \longrightarrow & \alpha' \\
\end{array} \]

with \( \alpha \circ \lambda_{E,H} = \chi_{E,H} \circ \tau \) and \( \alpha' \circ \lambda_{E,H} = \chi_{E/H} \circ \tau' \), where \( \xi_E \) is given by (5.2) and \( \alpha \) and \( \alpha' \) are induced by the canonical homomorphisms of algebras.

\section*{Proof.} We can check directly that \( \alpha \circ \lambda_H = \lambda_E \circ \sigma \) and \( \alpha' \circ \lambda_E = \lambda_{E/H} \circ \sigma' \). In order to show that \( \tau \circ \xi_E = \xi_E \circ \alpha \) and \( \tau' \circ \xi_E = \xi_{E/H} \circ \alpha' \), we use the following commutative diagram

\[ \begin{array}{ccc}
0 & \longrightarrow & K(E_H) \\
\Delta & \longrightarrow & C(E_H) \\
0 & \longrightarrow & L(E_H) \\
\end{array} \]

\[ \begin{array}{ccc}
0 & \longrightarrow & K(E) \\
\Delta' & \longrightarrow & C(E) \\
0 & \longrightarrow & L(E) \\
\end{array} \]

\[ \begin{array}{ccc}
0 & \longrightarrow & K(E/H) \\
\Delta' & \longrightarrow & C(E/H) \\
0 & \longrightarrow & L(E/H) \\
\end{array} \]
where \( l : C(E_H) \to C(E) \) and \( p : C(E) \to C(E/H) \) are canonical homomorphisms of algebras. It follows that the following diagram is commutative

\[
\begin{array}{ccc}
K_1(L(E_H)) & \xrightarrow{\alpha} & K_0(K(E)) & \xrightarrow{\tau} & \mathbb{Z}^{H \setminus S} \\
| & \downarrow & | & | & \\
K_1(L(E/H)) & \xrightarrow{\alpha'} & K_0(K(E/H)) & \xrightarrow{\tau'} & \mathbb{Z}^{(E \setminus H) \setminus S'},
\end{array}
\]

which implies that \( \tau \circ \xi_H = \xi_E \circ \alpha \) and \( \tau' \circ \xi_E = \xi_{E/H} \circ \alpha' \).

Now we show that \( \alpha \circ \chi_{1H} = \chi_{1H} \circ \tau \). Take \( x \in \ker(A_{E_H} - I : \mathbb{Z}^{H \setminus S} \to \mathbb{Z}^{H}) \). Note that \( x \) has finitely many nonzero entries \( x_v, \ldots, x_v \) with \( v_1, \ldots, v_l \in H \setminus S \). By (5.3),

\[
L_x^+(E_H) = \left\{ (e, i) \mid e \in E^1_H, 1 \leq i \leq -x_{s(e)} \right\} \cup \left\{ (v, i) \mid v \in E^0_H, 1 \leq i \leq x_v \right\},
\]

and

\[
L_x^-(E_H) = \left\{ (e, i) \mid e \in E^1_H, 1 \leq i \leq -x_{s(e)} \right\} \cup \left\{ (v, i) \mid v \in E^0_H, 1 \leq i \leq -x_v \right\}.
\]

Let \( h_1 \) be the cardinality of the sets \( L_x^+(E_H) \) and \( L_x^-(E_H) \). By definition of \( \chi_{1H} \), \( \chi_{1H}(x) = U_x \) with \( U_x \) a matrix in \( M_{h_1}(\mathcal{M}(L(E_H))) \). Let

\[
x' = \tau(x) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix}.
\]

The element \( x' \) has the same nonzero entries \( x_v, \ldots, x_v \) as \( x \). Since \( v_1, \ldots, v_l \) are all belong to \( H \), we have

\[
L_x^+(E_H) = \left\{ (e, i) \mid e \in E^1, 1 \leq i \leq -x_{s(e)} \right\} \cup \left\{ (v, i) \mid v \in E^0, 1 \leq i \leq x_v \right\} = L_x^+(E_H),
\]

and similarly \( L_x^- = L_x^-(E_H) \). For the bijections \( [\cdot] : L_x^+ \to \{1, \ldots, h_1 \} \) and \( (\cdot) : L_x^- \to \{1, \ldots, h_1 \} \), we use the same bijections \( [\cdot] : L_x^+(E_H) \to \{1, \ldots, h_1 \} \) and \( (\cdot) : L_x^-(E_H) \to \{1, \ldots, h_1 \} \). Then we have

\[
\chi_{1E}(\tau(x)) = \chi_{1E}(x') = U_x = \chi_{1H}(x) = (\alpha \circ \chi_{1H})(x).
\]

Now we show that \( \alpha' \circ \chi_{1H} = \chi_{1H} \circ \tau' \). Suppose that \( x \in \ker(A_{E_H} - I) \). The element \( x \) finitely many nonzero entries \( x_v, \ldots, x_v, x_{v+1}, \ldots, x_v \) with \( v_1, \ldots, v_l \in H \) and \( v_{l+1}, \ldots, v_n \notin H \). Then

\[
L_x^+ = \left\{ (z, i) \in L_x^+, r(z) \in H \right\} \cup \left\{ (z, i) \in L_x^+, r(z) \notin H \right\},
\]

and

\[
L_x^- = \left\{ (z, i) \in L_x^-, r(z) \in H \right\} \cup \left\{ (z, i) \in L_x^-, r(z) \notin H \right\}.
\]

And \( \{ (z, i) \in L_x^+, r(z) \in H \} \) and \( \{ (z, i) \in L_x^-, r(z) \notin H \} \) have the same cardinality \( h' \); \( \{ (z, i) \in L_x^+, r(z) \notin H \} \) and \( \{ (z, i) \in L_x^-, r(z) \notin H \} \) have the same cardinality \( h'' \). We choose two bijections \( [\cdot] : L_x^+ \to \{1, \ldots, h' + h''\} \) and \( (\cdot) : L_x^- \to \{1, \ldots, h' + h''\} \) such that \( \{x, i\} \to \{y, j\} \) implies \( r(x) = r(y) \) and that \( [\cdot] \) restricts to an isomorphism from \( \{ (x, i) \in L_x^+, r(z) \in H \} \) onto \( \{1, \ldots, h' \} \). In this case \( [\cdot] \) restricts to an isomorphism from \( \{ (x, i) \in L_x^-, r(z) \notin H \} \) onto \( \{h' + 1, \ldots, h' + h''\} \). Then we have

\[
\chi_{1E}(x) = V_x + 1 - P_x \in M_{h' + h''}(\mathcal{M}(L(E))),
\]

where \( V_x = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \) with \( V_{11} \in M_{h'}(\mathcal{M}(L(E_H))), V_{12} \) and \( V_{21} \) matrices with entries in the two sided ideal \( (H) \) of \( L(E) \). On the other hand, \( P_x = \begin{pmatrix} P_{11} & 0 \\ 0 & P_{22} \end{pmatrix} \) is a \( (h' + h'') \times (h' + h'') \)-matrix with \( P_{11} \) a \( h' \times h' \)-matrix with entries in the two sided ideal \( (H) \) of \( L(E) \). The element \( \tau'(x) \) has finitely many nonzero entries \( x_{v+1}, \ldots, x_v \) with \( v_{l+1}, \ldots, v_n \notin H \). Then we have

\[
L_{\tau'(x)}^+(E/H) = \{ (x, i) \in L_x^+, r(z) \notin H \},
\]

and

\[
L_{\tau'(x)}^-(E/H) = \{ (x, i) \in L_x^-, r(z) \notin H \}.
\]
We use the restrictions of $\ell: L^+_2 \to \{1, \cdots, h' + h''\}$ and $\langle \cdot \rangle: L^-_x \to \{1, \cdots, h' + h''\}$. Then we have $\chi_{1E/H}(\tau'(x)) = V_{22} + 1 - P_{22}$. Therefore we have
\[
(\alpha' \circ \chi_{1E})(x) = \alpha'(\chi_{1E}(x)) = \left[\begin{array}{cc} 0 & 0 \\ 0 & v_{22} + 1 - P_{22} \end{array}\right]_1
\]
This shows $\alpha' \circ \chi_{1E} = \chi_{1E/H} \circ \tau'$ and thus the proof is complete. \hfill \square

**Lemma 5.9.** Consider the following commutative diagram consisting of two split short exact sequences of modules over a ring $R$
\[
\begin{array}{cccccccc}
0 & \rightarrow & A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{h} & 0 \\
0 & \rightarrow & A' & \xrightarrow{i'} & B' & \xrightarrow{j'} & C' & \xrightarrow{h'} & 0,
\end{array}
\]
\[
(5.12)
\]
such that $j \circ s = \text{id}_C$, $j' \circ s' = \text{id}_{C'}$ and $g \circ s = s' \circ h$. Then we have the following commutative diagram
\[
\begin{array}{c}
B \xrightarrow{g} A \xleftarrow{f} C \\
B' \xrightarrow{g'} A' \xleftarrow{f'} C'
\end{array}
\]
Combining Lemma 5.8 and Lemma 5.9, we have the following immediate consequence.

**Corollary 5.10.** Let $E$ be a row-finite graph and $H$ a hereditary saturated subset of $E^0$. We write the adjacency matrix $A_E$ of $E$ by arranging vertices in $H$ first such that $A_E = \left(\begin{array}{cc} A_{EH} & 0 \\ 0 & A_{EH/H} \end{array}\right)$. There is a commutative diagram
\[
\begin{array}{cccc}
K_1(L(E_H)) & \xrightarrow{\sim} & \text{Coker}(A_{EH} - I) \oplus \text{Ker}(A_{EH} - I) \\
\downarrow{a} & & \downarrow{\begin{pmatrix} \tau' & 0 \\ 0 & \tau \end{pmatrix}} \\
K_1(L(E)) & \xrightarrow{\sim} & \text{Coker}(A_E - I) \oplus \text{Ker}(A_E - I) \\
\downarrow{a'} & & \downarrow{\begin{pmatrix} \tau' & 0 \\ 0 & \tau' \end{pmatrix}} \\
K_1(L(E/H)) & \xrightarrow{\sim} & \text{Coker}(A_{EH/H} - I) \oplus \text{Ker}(A_{EH/H} - I).
\end{array}
\]

### 5.3. Graded ideals of Leavitt path algebras

Let $E$ be a row-finite graph and $H \subseteq E^0$ a hereditary saturated subset of $E^0$. We define
\[
F(H) = \{\alpha \mid \alpha = e_1 \cdots e_n \text{ is a finite path in } E \text{ with } r(e_n) \in H \text{ and } s(e_n) \notin H\}.
\]
Note that the paths in $F(H)$ must be of length 1 or greater. Let $F^i(H)$ denote another copy of $F(H)$ and write $\overline{\alpha}$ for the copy of $\alpha$ in $F^i(H)$.

Define $\overline{E}_H$ to be the graph with
\[
\overline{E}^0_H = H \cup F(H) \\
\overline{E}^1_H = \{e \in E^1 : s(e) \in H\} \cup F(H)
\]
and we extend $s$ and $r$ to $\overline{E}^i_H$ by defining $s(\overline{e}) = \overline{\alpha}$ and $r(\overline{e}) = r(\overline{\alpha})$ for $\overline{e} \in F^i(H)$.

Recall from [37, Theorem 6.1] the graded ideal $I_H$ generated by $H$ in $L(E)$ is isomorphic to $L(\overline{E}_H)$ with $H \subseteq E^0$ a hereditary saturated subset of $E^0$. More precisely, the isomorphism
\[
L(\overline{E}_H) \rightarrow I_H
\]
of algebras is defined by \( v \mapsto Q_v \) for each \( v \in E_H^0 \) and \( e \mapsto T_e, e^* \mapsto T_e^* \) for each \( e \in E_H^1 \), where

\[
Q_v = \begin{cases} 
  v, & v \in H; \\
  \alpha e^*, & v = \alpha \in F(H);
\end{cases}
\]

\[
T_e = \begin{cases} 
  e, & e \in E^1; \\
  \alpha, & e = \overline{\alpha} \in F(H);
\end{cases}
\]

and

\[
T_e^* = \begin{cases} 
  e^*, & e \in E^1; \\
  \alpha^*, & e = \overline{\alpha} \in F(H).
\end{cases}
\]

Lemma 5.11. Let \( E \) be a row-finite graph and \( H \) a hereditary saturated subset of \( E^0 \). Then the natural homomorphism of algebras

\[
\L(E_H) \to \L(E_H^0)
\]

\[
\begin{array}{c}
  v \mapsto v \\
  e \mapsto e \\
  e^* \mapsto e^*
\end{array}
\]

for \( v \in H \) and \( e \in E_H^1 \) induces an isomorphism \( K_1(\L(E_H)) \to K_1(\L(E_H^0)) \) of \( K \)-groups.

Proof. Denote by \( \overline{A} \) the adjacency matrix for the graph \( \overline{E}_H \). We arrange vertices in \( H \) first such that \( \overline{A} = \begin{pmatrix} A_{E_H} & 0 \\ Z & 0 \end{pmatrix} \) where \( Z \) is a matrix such that each row has exactly one entry with 1 and all the other entries zero.

Clearly the map

\[
\vartheta : \Ker \left( \begin{array}{c} A_{E_H} - I : Z^H \times S \to Z^H \end{array} \right) \to \Ker \left( \begin{array}{c} A_{E_H} - I : Z^H \times S \to Z^H \end{array} \right),
\]

\[
(n_v)^t \mapsto ((n_v), 0)^t
\]

is an isomorphism. It suffices to show that the map

\[
\varsigma : \Coker \left( \begin{array}{c} A_{E_H} - I : k^\times H \times S \to k^\times H \end{array} \right) \to \Coker \left( \begin{array}{c} A_{E_H} - I : k^\times H \times S \to k^\times H \end{array} \right),
\]

\[
(m_1, \ldots, m_k)^t \mapsto (m_1, \ldots, m_k, 1, \ldots, 1)^t
\]

is an isomorphism. Note that \( \varsigma \) is well-defined as \((m_1, \ldots, m_k, 1, \ldots, 1)^t \in \Im \left( \begin{array}{c} A_{E_H} - I \end{array} \right) \) if \((m_1, \ldots, m_k)^t \in \Im \left( \begin{array}{c} A_{E_H} - I \end{array} \right) \). To show that \( \varsigma \) is injective, suppose that

\[
\begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_k \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}
\]

Then \( \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_k \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \) and thus \( \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_k \end{pmatrix} \in \Im \left( \begin{array}{c} A_{E_H} - I \end{array} \right) \). Hence \( \varsigma \) is injective. To show that \( \varsigma \) is surjective, we observe that

\[
\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \\ y_k \end{pmatrix} \Im \left( \begin{array}{c} A_{E_H} - I \end{array} \right) = \begin{pmatrix} y_k \\ y_k \end{pmatrix} \Im \left( \begin{array}{c} A_{E_H} - I \end{array} \right) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}
\]

\[
= \begin{pmatrix} y_k \\ y_k \end{pmatrix} \Im \left( \begin{array}{c} A_{E_H} - I \end{array} \right).
\]

This completes the proof. \( \square \)
6. Filtered $K$-theory

Filtered $K$-theory of a ring is an invariant consisting of $K$-theory of certain ideals and their quotients and the long exact sequence of $K$-theory relating them (Definition 6.1). In its current form, it was first formulated in the setting of $C^*$-algebras by Eilers, Restorff and Ruiz, as a working conjecture that the filtered $K$-theory is a complete invariant for all graph $C^*$-algebras. In major works \[14, 15\], the authors managed to prove the conjecture for unital graph $C^*$-algebras by first proving that when a pair of $C^*$-algebras associated to graphs $E$ and $F$, respectively, have the same filtered $K$-theory, then $E$ may be changed into $F$ by a sequence of geometric “moves” applied to the graphs in a way resembling the Reidemeister moves for knots.

In \[16\] the algebraic version of filtered $K$-theory was formulated and it was shown that, for two graphs, if the algebraic filtered $K$-theories of their Leavitt path algebras are isomorphic then the filtered $K$-theories of their graph $C^*$-algebras are also isomorphic (Theorem 6.2).

Here we briefly recall the notion of algebraic filtered $K$-theory following \[16\] and in §7 we relate the graded $K$-theory of a Leavitt path algebras to its algebraic filtered $K$-theory.

6.1. Prime spectrum for a ring. Let $R$ be a ring. We denote by $\mathcal{I}(R)$ the lattice of ideals in $R$. Let $\mathcal{S}$ be a sublattice of $\mathcal{I}(R)$ closed under arbitrary intersections and containing the trivial ideals $\{0\}$ and $R$. An ideal $p \in \mathcal{S}$ is called $\mathcal{S}$-prime if $p \neq R$ and for any ideals $I, J \in \mathcal{S}$,
$$IJ \subseteq p \implies I \subseteq p \text{ or } J \subseteq p.$$ Define
$$\text{Spec}_\mathcal{S}(R) := \{p \in \mathcal{S} \mid p \text{ is } \mathcal{S}\text{-prime}\}.$$ (6.1)
For each subset $T \subseteq \text{Spec}_\mathcal{S}(R)$, define the kernel of $T$ as
$$\text{Ker}(T) = \bigcap_{p \in T} p.$$ Throughout this section we assume that for any $I \in \mathcal{S}$, we have
$$I = \bigcap_{p \in \mathcal{S}} p, \text{ where } p \in \text{Spec}_\mathcal{S}(R).$$ (6.2)
One can then equip $\text{Spec}_\mathcal{S}(R)$ with the Jacobson topology, where the open sets are
$$W(I) = \{p \in \text{Spec}^\mathcal{S}(R) \mid I \not\subseteq p\},$$ (6.3)
where $I \in \mathcal{S}$ (\[16, \text{Lemma 3}\] and \[16, \text{Theorem 2}\]).

Let $X$ be a topological space and $\mathcal{O}(X)$ denote the set of all open subsets of $X$. Let $Y$ be a subset of $X$. We call $Y$ locally closed if $Y = U \setminus V$, where $U \subseteq V$ are open subsets of $X$. We let $\mathcal{L}(X)$ be the set of locally closed subsets of $X$.

By \[16, \text{Theorem 2}\] there is a lattice isomorphism
$$\phi : \mathcal{O}(\text{Spec}_{\mathcal{S}}(R)) \rightarrow \mathcal{S},$$
$$U \mapsto \text{Ker}(U^c).$$ (6.4)
Note that since $U = W(I)$, for some ideal $I \in \mathcal{S}$, then $\phi(U) = \text{Ker}(U^c) = \bigcap_{p \in \mathcal{S}} p = I$. We will use this correspondence to define our filtered $K$-groups in §6.2.

6.2. Filtered $K$-theory. Suppose $R$ is a ring and $\mathcal{S}$ is a sublattice of ideals as in §6.1. Moreover, assume that every ideal $I \in \mathcal{S}$ has a countable approximate unit consisting of idempotents, i.e., there exists a sequence $\{e_n\}_{n=1}^\infty$ in $I$ such that
$$\bullet \ e_n \text{ is an idempotent for all } n \in \mathbb{N},$$
$$\bullet \ e_n e_{n+1} = e_n \text{ for all } n \in \mathbb{N}, \text{ and}$$
$$\bullet \text{ for all } r \in I, \text{ there exists } n \in \mathbb{N} \text{ such that } re_n = e_n r = r.$$ In keeping with the notation from $C^*$-algebras, for every $U \in \mathcal{O}(\text{Spec}_{\mathcal{S}}(R))$, using the lattice isomorphism $\phi$ from (6.4) we define
$$R[U] := \phi(U).$$ Whenever we have open sets $V \subseteq U$ we can form the quotient $R[U]/R[V]$. Then \[16, \text{Lemma 4}\] shows that the quotient $R[U]/R[V]$ only depends on the set difference $U \setminus V$ up to canonical isomorphism. For $Y = U \setminus V \in \mathcal{L}(\text{Spec}_{\mathcal{S}}(R))$, define
$$R[Y] := R[U]/R[V].$$
Then for any locally closed subset \( Y = U \setminus V \) of \( \text{Spec}_{\mathcal{J}}(R) \), we have a collection of abelian groups \( \{ K_n(R[Y]) \}_{n \in \mathbb{Z}} \), where \( K_n \) are the Quillen algebraic \( K \)-groups. Moreover, for all \( U_1, U_2, U_3 \in \mathcal{O}(\text{Spec}_{\mathcal{J}}(R)) \) with \( U_1 \subseteq U_2 \subseteq U_3 \), by \([36, \text{Lemma 3.10}]\), we have a long exact sequence in algebraic \( K \)-theory

\[
K_n(R[U_2 \setminus U_1]) \xrightarrow{\iota_*} K_n(R[U_3 \setminus U_1]) \xrightarrow{\pi_*} K_n(R[U_3 \setminus U_2]) \xrightarrow{\partial_*} K_{n-1}(R[U_2 \setminus U_1]). \tag{6.5}
\]

We are ready to define the filtered \( K \)-theory of a ring.

**Definition 6.1.** The filtered \( K \)-theory of the ring \( R \) with respect to the sublattice of ideals \( \mathcal{J} \) as above, is the collection

\[
\{ K_n(R[Y]) \}_{k \leq n \leq m, Y \in \mathcal{L}(\text{Spec}_{\mathcal{J}}(R))},
\]
equipped with the natural transformations \( \iota_*, \pi_*, \partial_* \) given in (6.5). We denote this with \( \text{FK}_{k,m}(\text{Spec}_{\mathcal{J}}(R); R) \).

Let \( R, R' \) be rings with sublattices of ideals \( \mathcal{J} \) and \( \mathcal{J}' \), respectively. For \( k, m \in \mathbb{Z} \) with \( k \leq l \leq m \), an isomorphism

\[
\text{FK}_{k,m}(\text{Spec}_{\mathcal{J}}(R); R) \xrightarrow{\phi} \text{FK}_{k,m}(\text{Spec}_{\mathcal{J}'}(R'); R')
\]

consists of a homeomorphism

\[
\phi : \text{Spec}_{\mathcal{J}}(R) \to \text{Spec}_{\mathcal{J}'}(R')
\]

and an isomorphism

\[
a_{Y,n} : K_n(R[Y]) \to K_n(R'[\phi(Y)]),
\]

for each \( n \) with \( k \leq l \leq m \) and for each \( Y \in \mathcal{L}(\text{Spec}_{\mathcal{J}}(R)) \) such that the diagrams involving the natural transformations commute. If the isomorphism (6.6) restricts to an order isomorphism on \( K_0(R[Y]) \) for all \( Y \in \mathcal{L}(\text{Spec}_{\mathcal{J}}(R)) \), we write

\[
\text{FK}_{k,m}^+(\text{Spec}_{\mathcal{J}}(R); R) \cong \text{FK}_{k,m}^+(\text{Spec}_{\mathcal{J}'}(R'); R').
\]

Next we specialise to the case of Leavitt path algebras and graph \( C^* \)-algebras. For a graph \( E \), we consider the set \( \mathcal{J} \) to be the sublattice of all graded ideals of \( L(E) \). Since \( \text{Spec}_{\mathcal{J}}(L(E)) \) by definition (6.1) becomes the set of graded prime ideals of \( L(E) \), we denote it by \( \text{Spec}^{\text{gr}}(L(E)) \). By \([36, \text{Lemma 6}]\), every graded-ideal of \( L(E) \) has a countable approximate unit consisting of idempotents. Thus we can carry out the construction of the filtered \( K \)-theory in this setting.

On the other hand, in the analytic case, filtered \( K \)-theory of graph \( C^* \)-algebras is built from the space of all prime gauge invariant ideals of \( C^*(E) \), denoted by \( \text{Prime}(C^*(E)) \), and the \( K \)-groups involved are topological \( K \)-theory as developed in \([15]\).

The following theorem \([16, \text{Theorem 5}]\) will be used in \( \S 8 \) to relate the graded \( K \)-theory of Leavitt path algebras to the filtered \( K \)-theory of their corresponding graph \( C^* \)-algebras.

**Theorem 6.2.** Let \( E \) and \( F \) be graphs. If there is an isomorphism

\[
\text{FK}_{0,1}(\text{Spec}^{\text{gr}}(L_C(E)); L_C(E)) \cong \text{FK}_{0,1}(\text{Spec}^{\text{gr}}(L_C(F)); L_C(F)),
\]

then there is an isomorphism

\[
\text{FK}_{0,1}(\text{Prime}(C^*(E)); C^*(E)) \cong \text{FK}_{0,1}(\text{Prime}(C^*(F)); C^*(F)).
\]

7. Graded \( K \)-theory gives filtered \( K \)-theory

In this section we prove that the graded Grothendieck group of the Leavitt path algebra of a row-finite graph determines its filtered \( K \)-theory.

**Lemma 7.1.** Let \( E, F \) be row-finite graphs. Suppose that there exists an order-preserving \( \mathbb{Z}[x, x^{-1}] \)-module isomorphism

\[
\varphi : K_0^{\text{gr}}(L(E)) \xrightarrow{\sim} K_0^{\text{gr}}(L(F)).
\]

Then there is a homeomorphism \( \varphi : \text{Spec}^{\text{gr}}(L(E)) \to \text{Spec}^{\text{gr}}(L(F)) \) such that \( \mathcal{V}^{E}(\varphi(I)) = \varphi(\mathcal{V}^{E}(I)) \) for all \( I \in \text{Spec}^{\text{gr}}(L(E)) \).

**Proof.** The order-preserving isomorphism \( \varphi : K_0^{\text{gr}}(L(E)) \to K_0^{\text{gr}}(L(F)) \) sends \( \mathcal{V}^{E}(L(E)) \) to \( \mathcal{V}^{E}(L(F)) \). In particular \( \varphi \) induces a lattice isomorphism, also denoted by \( \varphi \), from \( \mathcal{L}(\mathcal{V}^{E}(L(E))) \) onto \( \mathcal{L}(\mathcal{V}^{E}(L(F))) \). By (3.11) and (3.8), we have that the lattice isomorphism given in (3.12) sends \( I \) to \( \mathcal{V}^{E}(I) \) for any graded ideal \( I \) of \( L(E) \). Therefore it follows from (3.12) that there exists a lattice isomorphism

\[
\varphi : \mathcal{L}^{E}(L(E)) \xrightarrow{\sim} \mathcal{L}^{E}(L(F))
\]
such that \( \mathcal{V}^{E}(\varphi(I)) = \varphi(\mathcal{V}^{E}(I)) \) for any \( I \in \mathcal{L}^{E}(L(E)) \). The restriction of this isomorphism to the set of prime elements gives the desired bijection \( \varphi : \text{Spec}^{\text{gr}}(L(E)) \to \text{Spec}^{\text{gr}}(L(F)) \).
To conclude, it suffices to show that \( \varphi : \text{Spec}^{SF}(L(E)) \to \text{Spec}^{SF}(L(F)) \) is open. By (6.3), open sets of \( \text{Spec}^{SF}(L(E)) \) are precisely the set of the form \( W(I) = \{ p \in \text{Spec}^{SF}(L(E)) \mid p \nsubseteq I \} \) for some proper graded ideal \( I \) of \( L(E) \). Observe that for a proper graded ideal \( I \) of \( L(E) \)

\[
\varphi(W(I)) = \varphi(\{ p \in \text{Spec}^{SF}(L(E)) \mid p \nsubseteq I \}) = \{ \varphi(p) \mid p \in \text{Spec}^{SF}(L(E)), p \nsubseteq I \} = \{ \varphi(p) \in \text{Spec}^{SF}(L(F)) \mid \varphi(p) \nsubseteq \varphi(I) \} = W(\varphi(I))
\]

and \( W(\varphi(I)) \) is open. Thus we prove that \( \varphi : \text{Spec}^{SF}(L(E)) \to \text{Spec}^{SF}(L(F)) \) is an open map. Similar argument shows that the inverse of \( \varphi \) is open.

From now on, let \( E, F \) be row-finite graphs. Suppose that there is an order-preserving \( \mathbb{Z}[x, x^{-1}] \)-isomorphism

\[
\varphi : K_0^{SF}(L(E)) \to K_0^{SF}(L(F)). \quad (7.1)
\]

Let \( I \) be a graded prime ideal of \( L(E) \). By Lemma 7.1 there is a homeomorphism \( \varphi : \text{Spec}^{SF}(L(E)) \to \text{Spec}^{SF}(L(F)) \) such that

\[
\mathcal{V}^{SF}(I) \cong \varphi(\mathcal{V}^{SF}(I)) = \mathcal{V}^{SF}(\varphi(I)). \quad (7.2)
\]

Suppose that \( I, J \in \text{Spec}^{SF}(L(E)) \) with \( I \subseteq J \). Combining (3.9) and (7.2), we have

\[
\mathcal{V}^{SF}(J/I) \cong \mathcal{V}^{SF}(J)/\mathcal{V}^{SF}(I) \cong \mathcal{V}^{SF}(\varphi(J))/\mathcal{V}^{SF}(\varphi(I)) \cong \mathcal{V}^{SF}(\varphi(J)/\varphi(I)). \quad (7.3)
\]

**Lemma 7.2.** Let \( E, F \) be row-finite graphs. Suppose that there exists a homeomorphism \( \varphi : \text{Spec}^{SF}(L(E)) \to \text{Spec}^{SF}(L(F)) \). Then there is a homeomorphism \( \varphi : \text{Spec}^{SF}(L(E)) \to \text{Spec}^{SF}(L(F)) \) and isomorphisms \( K_0^{SF}(J/I) \cong K_0^{SF}(\varphi(J)/\varphi(I)) \) for any \( I \subseteq J \) with \( I, J \in \text{Spec}^{SF}(L(E)) \).

**Proof.** The statement follows immediately from Lemma 7.1 and (7.3).

We are in a position to state the main theorem of this paper.

**Theorem 7.3.** Let \( E, F \) be row-finite graphs and \( k \) a field. Suppose that there exists an order-preserving \( \mathbb{Z}[x, x^{-1}] \)-isomorphism

\[
\varphi : K_0^{SF}(L_k(E)) \to K_0^{SF}(L_k(F)).
\]

Then there is a homeomorphism

\[
\varphi : \text{Spec}^{SF}(L_k(E)) \to \text{Spec}^{SF}(L_k(F))
\]

and isomorphisms \( \varphi : K_0^{SF}(I) \to K_0^{SF}(\varphi(I)) \) and \( K_0^{SF}(J/I) \cong K_0^{SF}(\varphi(J)/\varphi(I)) \), for any \( I, J \in \text{Spec}^{SF}(L_k(E)) \) with \( I \subseteq J \). Moreover, we have

\[
FK_0,1(\text{Spec}^{SF}(L_k(E));L_k(E)) \cong FK_0,1(\text{Spec}^{SF}(L_k(F));L_k(F)).
\]

Namely for any \( I, J \in \text{Spec}^{SF}(L_k(E)) \) with \( I \subseteq J \), there are isomorphisms

\[
\alpha_{I,n} : K_n(I) \to K_n(\varphi(I)), \quad \alpha_{J,n} : K_n(J) \to K_n(\varphi(J)), \quad \alpha_{J/I,n} : K_n(J/I) \to K_n(\varphi(J)/\varphi(I)),
\]

where \( n = 0, 1 \), such that the following diagram commutes

\[
\begin{array}{cccccccc}
K_1(I) & \to & K_1(J) & \to & K_1(J/I) & \to & K_0(I) & \to & K_0(J) & \to & K_0(J/I) \\
\parallel & \parallel & \parallel & \parallel & \parallel & \parallel & \parallel & \parallel & \parallel & \parallel \\
A_1 & \to & A_2 & \to & C & \to & B_1 & \to & B_2 & \to & \alpha_{J/I,0}
\end{array}
\]

Here the exact rows come from the long exact sequence in algebraic \( K \)-theory (see [11, Theorem 2.4.1]).

**Proof.** We denote \( \varphi(I) \) by \( T \) for any \( I \in \text{Spec}^{SF}(L(E)) \) and \( H_I \) denote the hereditary saturated subset of \( E^0 \) such that \( I = \langle H_I \rangle \). Similarly \( H_T \) is the hereditary saturated subset of \( F^0 \) such that \( T = \langle H_T \rangle \).

**Step I:** We show the existence of \( \alpha_{I,0}, \alpha_{J,0} \) and \( \alpha_{J/I,0} \) and check that the squares \( B_1 \) and \( B_2 \) in (7.4) are commutative. Recall that we have the exact sequence given by (4.3). Then we have the following commutative diagram
where the isomorphism \( K_0^G(I) \to K_0^G(L(E_{H_1})) \) is induced by the isomorphism of algebras \( I \cong L(E_{H_1}) \) and the isomorphism \( K_0^G(L(E_{H_1})) \to K_0^G(L(F_{H_1})) \) is given by \( v(i) \mapsto \varphi(v(i)) \) for \( v \in H_1 \) and \( i \in \mathbb{Z} \). Therefore we have the following commutative diagram

\[
\begin{array}{ccccccccc}
K_0^G(I) & \xrightarrow{\phi_I} & K_0^G(J) & \xrightarrow{\phi_J} & K_0^G(J/I) & \xrightarrow{\phi_{J/I}} & K_0^G(J/I) & \xrightarrow{\phi_{J/I}} & 0 \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
K_0^G(L(E_{H_1})) & \xrightarrow{\phi} & K_0^G(L(F_{H_1})) & \xrightarrow{\phi} & K_0^G(L(F_{H_1})) & \xrightarrow{\phi} & K_0^G(L(F_{H_1})) & \xrightarrow{\phi} & 0 \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
K_0^G(T) & \xrightarrow{\phi_T} & K_0^G(T) & \xrightarrow{\phi_T} & K_0^G(T) & \xrightarrow{\phi_T} & K_0^G(T) & \xrightarrow{\phi_T} & 0 \\
\end{array}
\]

Note that the short exact sequence

\[ 0 \to K_0^G(I) \to K_0^G(J) \to K_0^G(J/I) \to 0 \]

in the above diagram is given in (3.10). Therefore \( \alpha_{I,0}, \alpha_{J,0} \) and \( \alpha_{J/I,0} \) exist and \( B_1 \) and \( B_2 \) are commutative.

**Step II:** We define the maps \( \alpha_{I,1} : K_1(I) \to K_1(T), \alpha_{J,1} : K_1(J) \to K_1(T) \) and \( \alpha_{J/I,1} : K_1(J/I) \to K_1(T) \) and check that the squares \( A_1 \) and \( A_2 \) in (7.4) are commutative.

Denote by \( A_I, A_J \) and \( A_{J/I} \) the transposes of the adjacency matrices for \( E_{H_1}, E_{H_1} \), and \( (E/H_1)_{H_{J/I}} \) respectively. Denote by \( A_T, A_{T_1} \) and \( A_{T_I} \) the transposes of the adjacency matrices for \( F_{H_1}, F_{H_1} \) and \( (F/H_1)_{H_{J/I}} \) respectively.
We have the following diagram and in order to show that the squares \(A_1\) and \(A_2\) in (7.4) are commutative, it suffices to show that the squares \(X_1, X_2, Y_1, Y_2, Z_1\) and \(Z_2\) in (7.6) are commutative.

\[
\begin{array}{cccccc}
K_1(I) & 
\xrightarrow{\alpha} & K_1(J) & 
\xrightarrow{\alpha'} & K_1(J/I) & 
\xrightarrow{\pi} & K_0(I) \\
K_1(L(E_{H_1})) & 
\xrightarrow{} & K_1(L(E_{H_2})) & 
\xrightarrow{} & K_1(L(E/H_{H_1})) & 
\xrightarrow{} & K_0(I) \\
Y_1 & 
\xrightarrow{\Theta} & Z_1 & 
\xrightarrow{\Theta} & Z_2 & 
\xrightarrow{\Theta} & Z_3 \\
K_1(L(F_{H_1})) & 
\xrightarrow{} & K_1(L(F_{H_2})) & 
\xrightarrow{} & K_1(L(F/H_{H_1})) & 
\xrightarrow{} & K_0(I) \\
X_1 & 
\xrightarrow{} & X_2 & 
\xrightarrow{} & X_3 & 
\xrightarrow{} & \\
K_1(T) & 
\xrightarrow{} & K_1(J) & 
\xrightarrow{} & K_1(J/I) & 
\xrightarrow{} & K_0(T)
\end{array}
\]

We observe that the squares \(X_1\) and \(X_2\) in (7.6) are commutative as the following diagram whose third row is given by canonical homomorphisms of algebras is commutative

\[
\begin{array}{cccc}
I & 
\xrightarrow{=} & J & 
\xrightarrow{=} & J/I \\
L(E_{H_1}) & 
\xrightarrow{=} & L(E_{H_2}) & 
\xrightarrow{=} & L(E/H_{H_1}) \\
L(E_{H_1}) & 
\xrightarrow{=} & L(E_{H_2}) & 
\xrightarrow{=} & L((E/H_1)_{H_1}).
\end{array}
\]

By Corollary 5.10 the squares \(Y_1\) and \(Y_2\) in (7.6) are commutative.
The map $\Theta_I$ is defined by the composition of the following maps

$$
\begin{align*}
\text{Coker}(A_I - I) \oplus \text{Ker}(A_I - I) & \longrightarrow \text{Coker}(A_{\bar{E}_{H_1}} - I) \oplus \text{Ker}(A_{\bar{E}_{H_1}} - I) \\
& \longrightarrow (K_0(I) \otimes \mathbb{Z}^\times) \oplus \text{Ker}(\phi_I)
\end{align*}
$$

$$
\begin{align*}
\text{Coker}(A_T - I) \oplus \text{Ker}(A_T - I) & \longrightarrow \text{Coker}(A_{\bar{F}_{\mathcal{T}}} - I) \oplus \text{Ker}(A_{\bar{F}_{\mathcal{T}}} - I) \\
& \longrightarrow (K_0(T) \otimes \mathbb{Z}^\times) \oplus \text{Ker}(\phi_T),
\end{align*}
$$

where $\varsigma_{E_{H_1}}, \vartheta_{E_{H_1}}, \varsigma_{F_{\mathcal{T}}} \text{ and } \vartheta_{F_{\mathcal{T}}}$ are given by (5.16) and (5.15). Here the map from $\text{Ker}(A_{\bar{E}_{H_1}} - I)$ to $\text{Ker}(\phi_I)$ is given by the composition of

$$
\text{Ker}(A_{\bar{E}_{H_1}} - I) \cong \text{Ker}(K_0^\mathcal{gr}(L(E_{H_1})) \longrightarrow K_0^\mathcal{gr}(L(E_{H_1}))) \cong \text{Ker}(\phi_I)
$$

with the first isomorphism following from Proposition 4.4. The homomorphisms $\Theta_I$ and $\Theta_{J/I}$ can be defined similarly and we omit the details here.

To show that $Z_1$ and $Z_2$ are commutative, by (7.5) the squares in the following diagram (7.8) with blue colour are commutative. Thus the following diagram (7.8) is commutative.

Similarly by (7.5) the squares in the following diagram (7.9) with blue colour are commutative. Thus the following diagram (7.9) is commutative.

By (7.8) and (7.9) the squares $Z_1$ and $Z_2$ in (7.6) are commutative.
Step III: We check that the square $C$ in (7.4) is commutative. It suffices to show that the squares $X_3, Y_3$ and $Z_3$ in (7.6) are commutative.

We observe that the square $X_3$ in (7.6) is commutative as the algebra homomorphisms $L(E_{H_1}) \to L(E_{H_1})$ induce the maps of $K$-theory $K_1(L(E_{H_1})) \to K_1(L(E_{H_1})) \cong K_1(I)$ which send each element $g \in GL(L(E_{H_1}))$ to itself.

In order to define the map $\delta_{E_{H_1}, E_{H_j}} : \ker(A_{I_{J,I}} - I) \to \text{Coker}(A_I - I)$, note that $H_j \setminus H_I$ is a hereditary saturated subset of $(E/H_I)^0$ and $J/I = H_j \setminus H_I$ is a graded ideal of the Leavitt path algebra $L(E)/I \cong L(E/H_I)$. Then we have $H_{I,J} = H_j \setminus H_I$ and the restriction graph

$$(E/H_I)_{H_{I,J}} := (E/H_I)_{H_j \setminus H_I}.$$ 

Observe that we have $(E/H_I)_{H_{I,J}} = E_{H_{I,J}}/H_I$. By Lemma 5.5 the connecting map $\delta_{E_{H_1}, E_{H_j}}$ is defined. The isomorphism $\pi : K_0(I) \to \text{Coker}(A_I - I)$ is given by the composition

$$K_0(I) \cong K_0(L(E_{H_1})) \cong K_0(L(E_{H_1})) \cong \text{Coker}(A_I - I), \quad (7.10)$$

sending $[ff^*]_0 \in K_0(I)$ to $[e_r(v)] \in \text{Coker}(A_I - I)$ for any $f \in E^1$ with $r(f) \in H_I$.

To show that the square $Y_3$ in (7.6) is commutative, by Lemma 5.9 we need to show that the following diagram is commutative:

$$0 \longrightarrow \text{Coker}(A_{I_{J,I}} - I) \overset{\lambda}{\longrightarrow} K_1(L(E/H_I)_{H_{I,J}}) \overset{\pi}{\longrightarrow} \text{Ker}(A_{I_{J,I}} - I) \overset{\delta_{E_{H_1}, E_{H_j}}}{\longrightarrow} 0 \quad (7.11)$$

Here $\partial : K_1(L(E/H_I)_{H_{I,J}}) \to K_0(I)$ is the map appearing in diagram (7.6), that is, $\partial$ is the composition

$$K_1(L(E/H_I)_{H_{I,J}}) \longrightarrow K_1(J/I) \longrightarrow K_0(I).$$

We can check directly that $\partial \circ \lambda = 0$. We now show that $\delta_{E_{H_1}, E_{H_j}} \circ \xi = \pi \circ \partial$. Observe that we have the following commutative diagram

$$0 \longrightarrow K((E/H_I)_{H_{I,J}}) \overset{\varphi}{\longrightarrow} C((E/H_I)_{H_{I,J}}) \overset{\pi}{\longrightarrow} L(E/H_I)_{H_{I,J}} \longrightarrow 0 \quad (7.12)$$

where the map $\varphi$ is given by $\varphi(v) = v, \varphi(e) = e$ and $\varphi(e^*) = e^*$ for $v \in (E/H_I)^0_{H_{I,J}}$ and $e \in (E/H_I)^1_{H_{I,J}}$. We denote the composition of maps $K((E/H_I)_{H_{I,J}}) \to L(E_{H_I}) \to I$ by $\varpi$. Note that we have the following equality in $J$, for each $v \in (H_j \setminus H_I) \cap R$,

$$v - \sum_{e \in s^{-1}(v), r(e) \notin H_I} ee^* = \sum_{f \in s^{-1}(v), r(f) \in H_I} ff^*.$$

The map $\varpi$ satisfies

$$\varpi(v - \sum_{e \in s^{-1}(v), r(e) \notin H_I} ee^*) = \sum_{f \in s^{-1}(v), r(f) \in H_I} ff^* \in I.$$

We denote by $\mu : \ker(A_{I_{J,I}} - I) \to K_0(K((E/H_I)_{H_{I,J}}))$ the natural inclusion map.

By (7.12) and the naturality of the connecting map in $K$-theory, we have the following commutative diagram:

$$K_1(L((E/H_I)_{H_{I,J}})) \overset{\xi}{\longrightarrow} \ker(A_{I_{J,I}} - I) \overset{\mu}{\longrightarrow} K_0(K((E/H_I)_{H_{I,J}})) \overset{K_0(\pi)}{\longrightarrow} K_0(I) \overset{\pi}{\longrightarrow} \text{Coker}(A_I - I) \quad (7.13)$$

To show that $\delta_{E_{H_1}, E_{H_j}} \circ \xi = \pi \circ \partial$, by (7.13) it suffices to prove that $\delta_{E_{H_1}, E_{H_j}} = \pi \circ K_0(\varpi) \circ \mu$. Recall from (5.7) that $[v - \sum_{e \in s^{-1}(v), r(e) \notin H_I} ee^*]_0$, for $v \in (H_j \setminus H_I) \cap R$, are the canonical generators of the free abelian group.
$K_0(K((E/H_I)_{H_J/i}))$. Now for $v \in (H_J \setminus H_I) \cap R$ we have

$$(\pi \circ K_0(\pi))(v - \sum_{e \in s^{-1}(v)} ee^*|_0) = \pi(\sum_{f \in e^{-1}((v))} ff^*|_0)$$

$$= \sum_{f \in e^{-1}(v)} [e_f].$$

(7.14)

On the other hand, it follows from Lemma 5.5 that

$$\delta_{E_{H_I}, E_{H_J}}(x) = [X^tx]$$

for every $x \in \text{Ker}(A_{J/I} - I)$, where $X$ is the matrix such that $X(v, w) = A_E(v, w)$ for each $v \in (H_J \setminus H_I) \cap R$ and $w \in H_I$. It follows that $\delta_{E_{H_I}, E_{H_J}} = \pi \circ K_0(\pi) \circ \mu$, as desired.

To show that $Z_3$ in (7.6) is commutative, by (7.5) and the snake Lemma, we have connecting maps $\rho : \text{Ker}(\phi_{J/I}) \to K_0(I)$ and $\pi : \text{Ker}(\phi_{\overline{J}/\overline{I}}) \to K_0(\overline{T})$ such that the blue square in the following diagram is commutative (similarly as in Lemma 5.6).

\[
\begin{array}{ccc}
\Psi & \xrightarrow{\rho} & K_0(I) \\
\downarrow \delta_{E_{H_I}, E_{H_J}} & & \downarrow \pi^{-1} \\
\text{Ker}(A_{J/I} - I) & \xrightarrow{\phi_{J/I}} & \text{Coker}(A_I - I) \\
\downarrow \phi_{\overline{J}/\overline{I}} & & \downarrow \pi \\
\text{Ker}(A_{\overline{J}/\overline{I}} - I) & \xrightarrow{\delta_{H_{\overline{J}}, H_{\overline{I}}}} & \text{Coker}(A_{\overline{T}} - I) \\
\end{array}
\]

(7.15)

We need to check that the top and bottom faces in the above diagram (7.15) are commutative. Hence we obtain that $Z_3$ in (7.6) which is the front face of the above diagram is commutative. By Lemma 5.6 we have $\rho \circ \Psi = \pi^{-1} \circ \delta_{E_{H_I}, E_{H_J}}$, implying that the top face in the above diagram (7.15) is commutative as well. This completes the proof.

\section{8. Shift equivalent and the stability of corresponding graph $C^*$-algebras}

Let $A$ be an integral $n \times n$ matrix with nonnegative entries. Consider the following directed system of abelian groups with $A$ acting as an order-preserving group homomorphism

$$\mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^n \xrightarrow{A} \cdots,$$

where the ordering in $\mathbb{Z}^n$ is defined point-wise. The direct limit of this system, $\Delta_A := \lim_{\longrightarrow} \mathbb{Z}^n$, along with its positive cone, $\Delta^+ := \lim_{\longrightarrow} \mathbb{Z}^n_+$, and the automorphism which is induced by multiplication by $A$ on the direct limit, $\delta_A : \Delta_A \to \Delta_A$, is called Krieger’s dimension group. Following [27], we denote this triple by $(\Delta_A, \Delta^+_A, \delta_A)$. It can be shown that two matrices $A$ and $B$ are shift equivalent if and only if their associated Krieger’s dimension groups are isomorphic ([25, Theorem 4.2], and [27, Theorem 7.5.8], see also [27, §7.5] for a detailed algebraic treatment). We say two finite graphs $E$ and $F$ are shift equivalent if their adjacency matrices $A_E$ and $A_F$ are shift equivalent. If the matrices $A_E$ and $A_F$ are shift equivalent then by [27, Theorem 7.4.17], their Bowen-Franks groups are isomorphic, namely, $\text{BF}(A_E) \cong \text{BF}(A_F)$. On the other hand, by [27, Exercise 7.4.4, for $p(t) = 1 - t$, $\text{det}(1 - A_F) = \text{det}(1 - A_F)$. Recall that a graph $E$ is irreducible if given any two vertices $v$ and $w$ in $E$, there is a path from $v$ to $w$. Now if the graphs $E$ and $F$ are irreducible, the main theorem of Franks [17] gives that $A_E$ is flow equivalent to $A_F$. Thus $F$ can be obtained from $E$ by a finite sequence of in/out-splitting and expansion of graphs (see [28, 8]). Each of these transformation preserve the Morita equivalence ([8]) and thus $C^*(E)$ is Morita equivalent to $C^*(F)$.

Our result now allows us to extend this fact from finite irreducible graphs to finite graphs without sinks (see the Figure 1.4).

\textbf{Proposition 8.1.} Let $E$ and $F$ be finite graphs with no sinks. If $E$ and $F$ are shift equivalent, then the $C^*$-algebras $C^*(E)$ and $C^*(F)$ are Morita equivalent.

\textbf{Proof.} Since $E$ and $F$ are shift equivalent, we have an isomorphism of Krieger’s dimension groups

$$(\Delta_E, \Delta^+_E, \delta_E) \cong (\Delta_F, \Delta^+_F, \delta_F).$$
But since Krieger's dimension group for the graph $E$ coincides with the graded Grothendieck group $K^G_0 (L(E))$ ([22, Lemma 11]), we obtain an order-preserving $\mathbb{Z}[x, x^{-1}]$-module isomorphism $K^G_0 (L(E)) \cong K^G_0 (L(F))$. By Theorem 7.3, the filtered $K$-theory of $L(E)$ and $L(F)$ are isomorphic. By Theorem 6.2 the filtered $K$-theory of the corresponding graph $C^*$-algebras are also isomorphic. Now the main theorem of [15] gives that the $C^*$-algebras $C^*(E)$ and $C^*(F)$ are Morita equivalent.

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References

[1] G. Abrams, G. Aranda Pino, The Leavitt path algebra of a graph, J. Algebra 293 (2) (2005), 319–334.
[2] G. Abrams, P. Ara, M. Siles Molina, Leavitt path algebras, Lecture Notes in Mathematics, vol. 2191, Springer Verlag, 2017.
[3] P. Ara, M. Brustenga, G. Cortiñas, $K$-theory of Leavitt path algebras, Münster J. Math. 2 (2009), 5–33.
[4] P. Ara, K.R. Goodearl, Ideals in graph algebras, J. Reine Angew. Math. 669 (2012), 165–224.
[5] P. Ara, M.A. Moreno, E. Pardo, Nonstable $K$-theory for graph algebras, Algebr. Represent. Theory 10 (2) (2007), 157–178.
[6] P. Ara, E. Pardo, Towards a $K$-theoretic characterization of graded isomorphisms between Leavitt path algebras, J. K-Theory, 14 (2014), no. 2, 203–245.
[7] P. Ara, R. Hazrat, H. Li, A. Sims, Graded Steinberg algebras and their representations, Algebr. Number theory, 12-1 (2018), 131–172.
[8] T. Bates, D. Pask, Flow equivalence of graph algebras, Ergodic Theory Dynam. Systems 24 (2004), no. 2, 367–382.
[9] M. Brustenga i Bort, Algebras associated a un buirac, Ph.D. Thesis, Universitat Autònoma de Barcelona, 2007.
[10] T. M. Carlsen, S. Eilers, M. Tomforde, Index maps in the $K$-theory of graph algebras, J. K-theory 9 (2012), 385–406.
[11] G. Cortiñas, Algebraic v. topological $K$-theory: a friendly match. In Topics in algebraic and topological $K$-theory, volume 2008 of Lecture Notes in Math., pages 103–165. Springer, Berlin, 2011.
[12] G. Cortiñas, D. Montero, Algebraic bivariant $K$-theory and Leavitt path algebras, arXiv:1806.09204v2.
[13] S. Eilers, G. Restorff, E. Ruiz, On graph $C^*$-algebras with a linear ideal lattice, Bull. Malays. Math. Sci. Soc. 33 (2) (2010), no. 2, 233–241.
[14] S. Eilers, G. Restorff, E. Ruiz, A.P.W. Sørensen, The complete classification of unital graph $C^*$-algebras: Geometric and strong, arXiv:1611.07120.
[15] S. Eilers, G. Restorff, E. Ruiz, A.P.W. Sørensen, Geometric classification of graph $C^*$-algebras over finite graphs, Canadian Journal of Mathematics, 70 (2018), 294–353.
[16] S. Eilers, G. Restorff, E. Ruiz, A.P.W. Sørensen, Filtered $K$-theory for graph algebras, 2016 MATRIX Annals, MATRIX Book Series 1, 229–249.
[17] J. Franks, Flow equivalence of subshifts of finite type, Ergodic Theory Dynam. Systems 4 (1984) 53–66.
[18] J. Gabe, E. Ruiz, M. Tomforde, T. Whalen, $K$-theory for Leavitt path algebras: Computation and classification, J. Algebra 433 (2015), 35–72.
[19] K.R. Goodearl, von Neumann regular rings, 2nd ed., Krieger Publishing Co., Malabar, FL, 1991.
[20] R. Hazrat, The graded Grothendieck group and the classification of Leavitt path algebras, Math. Annalen 355 (2013), 273–325.
[21] R. Hazrat, A note on the isomorphism conjectures for Leavitt path algebras, J. Algebra 375 (2013), 33–40.
[22] R. Hazrat, The dynamics of Leavitt path algebras, J. Algebra 384 (2013), 242–266.
[23] R. Hazrat, Graded rings and graded Grothendieck groups, London Math. Society Lecture Note Series, Cambridge University Press, 2016.
[24] R. Hazrat, H. Li, Homology of etale groupoids, a graded approach, arXiv:1806.03398.
[25] W. Krieger, On dimension functions and topological Markov chains, Invent. Math. 56 (1980), 239–250.
[26] T. Y. Lam, M.K. Siu, $K_0$ and $K_1$—an introduction to algebraic $K$-theory, Amer. Math. Monthly 82 (1975), 329–364.
[27] D. Lind, B. Marcus. An introduction to symbolic dynamics and coding, Cambridge University Press, 1995.
[28] W. Parry, D. Sullivan, A topological invariant of flows on 1-dimensional spaces, Topology 14 (1975), 297–299.
[29] I. Raeburn, W. Szymanski, Cuntz–Krieger algebras of infinite graphs and matrices, Trans. Amer. Math. Soc. 356 (2004), 39–59.
[30] K.M. Rangaswamy, The theory of prime ideals of Leavitt path algebras over arbitrary graphs, J. Algebra 375 (2013), 73–96.
[31] G. Restorff, Classification of Cuntz–Krieger algebras up to stable isomorphism, J. Reine Angew. Math. 598 (2006), 185–210.
[32] M. Rørdam, Classification of extensions of certain $C^*$-algebras by their six term exact sequences in $K$-theory, Math. Annalen 308 (1997), no. 1, 93–117.
[33] M. Rørdam, Classification of nuclear, simple $C^*$-algebras, Encyclopaedia of Mathematical Sciences, vol. 126, Springer, Berlin, 2001.
[34] M. Rørdam, Structure and classification of $C^*$-algebras, International Congress of Mathematicians. Vol. II, 1581–1598, Eur. Math. Soc., Zürich, 2006.
[35] J. Rosenberg, Algebraic $K$-theory and its applications, Graduate Text in Math. 147, Springer-Verlag, New York, 1994.
[36] E. Ruiz, M. Tomforde, Ideal-related $K$-theory for Leavitt path algebras and graph $C^*$-algebras, Indiana Univ. Math. J. 62 (5) (2013), 1587–1620.
[37] E. Ruiz, M. Tomforde, Ideals in graph algebras, Algebr. Represent. Theor. 17 (2014), 849–861.
[38] J. Spielberg, Semiprojectivity for certain purely infinite $C^*$-algebras, Trans. Amer. Math. Soc. 361 (6) (2009), 2805–2830.
[39] M. Tomforde, Uniqueness theorems and ideal structure for Leavitt path algebras, J. Algebra 318 (2007), 270–299.
[40] M. Tomforde, Classification of graph algebras: a selective survey, Operator algebras and applications—the Abel Symposium 2015, 303–325, Abel Symp., 12, Springer, 2017.

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