NEW CASES OF DIFFERENTIAL RIGIDITY FOR NON-GENERIC PARTIALLY HYPERBOLIC ACTIONS

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ABSTRACT. We prove the locally differentiable rigidity of generic partially hyperbolic abelian algebraic high-rank actions on compact homogeneous spaces obtained from split symplectic Lie groups. We also gave a non-generic action rigidity example on compact homogeneous spaces obtained from $SL(2n, \mathbb{R})$ or $SL(2n, \mathbb{C})$. The conclusions are based on geometric Katok-Damjanovic way and progress towards computations of the generating relations in these groups.

1. INTRODUCTION

Let $G$ be a $\mathbb{R}$-semisimple Lie group with real rank greater than 2, $\mathfrak{h} = \mathbb{R}^k$ its split Cartan subalgebra and $\Gamma$ be a cocompact lattice in $G$. Let $A$ be a maximal split Cartan subgroup of $G$ with Lie algebra $\mathfrak{h}$, and $K$ be the compact part of the centralizer of $A$ which intersects with $A$ trivially. Let $\Phi$ be the root system of $G$ with respect to $\mathfrak{h}$. Every root $r \in \Phi$ defines a Lyapunov hyperplane $H_r = \ker r$. A 2-dimensional plane in $\mathfrak{h}$ is said to be in general position if it intersects each two distinct Lyapunov hyperplanes along distinct lines. Let $\mathfrak{s} \subseteq \mathfrak{h}$ and $S$ the connected subgroup in $G$ with Lie algebra $\mathfrak{s}$. The action $\alpha_{0,S}$ of $S$ by left translations on $X := G/\Gamma$ is generic if it contains a lattice contained in a 2-plane in general position.

A. Katok and R. Spatzier proved the locally differentiable rigidity of the Anosov full Cartan actions (also called Weyl chamber flow) on $K \backslash G/\Gamma$ by a harmonic analysis method. Later A. Katok and Damjanovic [3, 4] proved the locally differentiable rigidity of partially hyperbolic generic actions on $X$ if $G$ is simple with $\Phi$ of nonsymplectic type with combination of geometric methods and K-theory. The natural difficulty of symplectic type is related to infinite types of reducible Lyapunov-foliation cycles resulted by infinitely different homotopic classes. For quasi-split groups, $(BC)_n$-type root systems have the same infiniteness problem although the generating relations are available[5]. In this paper, we proved locally differentiable rigidity of split symplectic Lie groups which has been left open in [4]. In fact, we
can extend the generic action rigidity results to quasi-split groups (see remark [17]).

A necessary condition for applicability of the Damjanovic–Katok geometric method (although not for local rigidity) is that contracting distributions of various action elements and their brackets of all orders generate the tangent space to the phase space. Generic restrictions for Cartan actions satisfy that condition. Naturally one may look at non-generic restrictions of Cartan actions.

**Example 1.1.** In $SL(4, \mathbb{R})$, consider the plane $P$ given by the equation $t_2 + 2t_3 + 3t_4 = 0$. It is not in general position since the intersection of Lyapunov hyperplanes obtained by roots $L_1 - L_4$ with $P$ is the same as that of $L_2 - L_3$. Though two unipotents $v_{14}$ and $v_{23}$ are not both stable for any element of the action, but consider

$$ [v_{14}(t), v_{23}(s)] = [[v_{13}(t), v_{34}(1)], v_{23}(s)]. $$

$v_{34}$ and $v_{23}$ are in the stable foliation of $(5, -8, 1, 2)$, $v_{13}$ and $v_{23}$ in the stable foliation of $(0, -1, 2, -1)$.

Hence we still get cocycle rigidity for action $\alpha_{0,P}$ (more details can be found in [2]). Generally speaking, for cocycle rigidity, “generic” is not necessary since we have enough elements to trivialize all Lyapunov-cycles if $P$ intersects each two distinct Lyapunov hyperplanes defined by simple roots along distinct lines. It is based on the fact that Lie brackets of simple roots and their inverse can generate the whole root system. But for differential rigidity, more arguments are needed since usually the coarse Lyapunov spaces are changed.

In this paper, we obtain an important example of locally differentiable rigidity of non-generic actions on compact homogeneous spaces obtained from $SL(2n, \mathbb{R})$ and $SL(2n, \mathbb{C})$. In this example, the plane intersects Lyapunov hyperplanes defined by simple roots along same lines, which disables the geometric Katok-Damjanovic method. We gave new generating relations adapted to the dynamical systems. These results are of independent interest and have widen classes of locally rigid actions and the method can be applied to other non-generic actions.

To prove Theorem 1 and 2 we make sufficient progress towards the computations of generating relations of $Sp(2n, \mathbb{R})$, $SL(2n, \mathbb{R})$ and $SL(2n, \mathbb{C})$ where $n \geq 2$. Generating relations are available for these groups[17], however, they do not provide sufficient information adapted to the dynamical systems and need to be supplemented by more detailed calculations.
Results for non-generic differential rigidity in other high rank groups and more general conditions admitting differential rigidity phenomenons will appear in a separate paper.

I'd like to thank my advisor Anatole Katok, who introduced me to the non-generic problem and constantly encouraged me.

1.1. The main results.

**Theorem 1.** Let $\alpha_{0,S}$ be a high rank generic restriction of the action of a maximal split Cartan subgroup on $Sp(2n,\mathbb{R})/\Gamma$ where $n \geq 2$. If $\tilde{\alpha}$ is $C^\infty$ action sufficiently $C^2$-close to $\alpha_{0,S}$, then there exists a $C^\infty$ diffeomorphism $h : X \to X$ such that $h^{-1}\tilde{\alpha}(a, h(x)) = \alpha_{0,\tilde{S}}$ where $\tilde{S}$ is isomorphic and close to $S$ in a maximal split Cartan subgroup of $Sp(2n,\mathbb{R})$.

Note that for $Sp(2n,\mathbb{R})$ generating relations are available in [17], but to get enough information for reducible classes, further calculations are carried out on the Schur multiplier. The proof resembles Theorem 2 in [24] on dealing with infinite homotopic classes.

**Remark 1.1.** By [5], any quasi-split simple groups of non-$\text{(BC)}_n$ type and non-symplectic type, are subject to following generating relations:

1. $c(s, tz) = c(s, t)c(s, z)$,
2. $c(tz, s) = c(t, s)c(z, s)$,
3. $c(s, 1-s) = 1$.

Thus for full maximal Cartan actions of these groups, rigidity are abstained by using almost the same manners as in [4]. For $(\text{BC})_n$-type are groups $SU(m+1, m)$; for symplectic type are $Sp(2n,\mathbb{R})$ and $SU(m, m)$. Rigidity of $SU(m+1, m)$ and $SU(m, m)$ are solved in [24]; rigidity of $Sp(2n,\mathbb{R})$ was proved in Theorem 1. Hence we in fact have locally differential rigidity for high rank quasi-split Lie groups.

For a more general case when the actions are not generic, for example, if $G = SL(2n,\mathbb{R})(SL(2n, \mathbb{C}))(n \geq 2)$ and $\Gamma$ a cocompact lattice in $G$. Let

$$D_+ = \exp \mathbb{D}_+ = \{ \text{diag}(\exp t_1, \ldots, \exp t_n, \exp(-t_1), \ldots, \exp(-t_n)) : (t_1, \ldots, t_n) \in \mathbb{R}^n \}.$$ 

Let the root system with respect to $D_+$ be $\Phi$.

**Theorem 2** (Differential rigidity of non-generic actions). Let $G = SL(2n,\mathbb{R})(SL(2n, \mathbb{C}))(n \geq 2)$. If $\tilde{\alpha}$ is $C^\infty$ action sufficiently $C^2$-close to $\alpha_{0,S}$ where $\tilde{S}$ contains a lattice contained in a generic $2$-plane in $\mathbb{D}_+$ with respect to $\Phi$. Then there exists a $C^\infty$ diffeomorphism $h : X \to X$ such that $h^{-1}\tilde{\alpha}(a, h(x)) = \alpha_{0,\tilde{S}}$ where $\tilde{S}$ is isomorphic and...
close to $S$ in the centralizer of a maximal split Cartan subgroup of $SL(2n, \mathbb{R})(SL(2n, \mathbb{C}))$.

**Remark 1.2.** For any 2-plane $P$ in $D_+$ it is not generic with respect to the root system defined by the maximal Cartan subgroups (that is $\{L_k - L_\ell\}_{k \neq \ell}$) since for any different indices $i, j$, Lyapunov hyperplanes $H_{i,j}$ and $H_{i+n,j+n}$ intersect $P$ on the same lines. $\Phi$ is different from the usual roots systems of special linear groups. It behaves similarly to that of symplectic groups instead.

1.2. **Generating relations and Steinberg symbols.** In this section we state two theorems which play a crucial role in proofs of Theorem 1 and 2. The proof of those theorems are given in Section 4 and 5 which comprise the algebraic part of the paper.

We use $e_{k, \ell}$ to denote the $2n \times 2n$ matrix in with the $(k, \ell)$ entry equal to 1, and all other entries equal to 0. Let

\[
\begin{align*}
    f_{L_i + L_j} &= (e_{i,j+n} - e_{j,i+n})_{i < j}, \\
    f_{L_i - L_j} &= (e_{i,j} - e_{j+i,n+i+n})_{i \neq j}, \\
    f_{-L_i - L_j} &= (e_{j+n,i} - e_{i+n,j})_{i < j}, \\
    f_{2L_i} &= e_{i,i+n}, \\
    f_{-2L_i} &= e_{i+n,i+n}.
\end{align*}
\]

Let exp be the exponentiation map for matrices. For $t \in \mathbb{R}$ we write

\[x_r(t) = \exp(tf_r)\quad \text{for } 0 \neq r = \pm L_i \pm L_j.\]

Then we have following results

**Theorem 3.** $Sp(2n, \mathbb{R}), n \geq 2$ is generated by $x_r(a)$, where $0 \neq r \in \Phi = \{\pm L_i \pm L_j\} (0 \leq i, j \leq n)$ subject to the relations:

1. \[x_r(a)x_r(b) = F_r(a + b),\]
2. \[[x_r(a), x_p(b)] = \prod_{ir+jp \in \Phi, i,j > 0} x_{ir+jp}(g_{irjp}(a,b)), r + p \neq 0\]
3. \[[x_r(a), x_p(b)] = \text{id}, \quad 0 \neq r + p \notin \pm L_i \pm L_j,\]

here $a, b \in \mathbb{R}^*$ and $g_{irjp}$ are functions of $a, b$ depending only on the structure of $Sp(2n, \mathbb{R})$;

4. \[h_{L_1 - L_2}(a)h_{L_1 - L_2}(b) = h_{L_1 - L_2}(ab),\]

where $h_{L_1 - L_2}(t) = x_{L_1 - L_2}(t)x_{L_2 - L_1}(-t^{-1})x_{L_1 - L_2}(t)x_{L_1 - L_2}(-1)x_{L_2 - L_1}(1)x_{L_1 - L_2}(-1)$ for each $t \in \mathbb{R}^*$;

5. \[h_{2L_2}(-1)h_{2L_2}(-1) = \text{id},\]

where \[h_{2L_2}(-1) = \left(x_{2L_2}(-1)x_{2L_2}(1)x_{2L_2}(-1)\right)^2.\]
Now we state the theorem about new generating relations in $SL(2n, \mathbb{K})$, \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \), \( n \geq 2 \). Let
\[
\begin{align*}
    f_{L_i+L_j}(t_1, t_2) &= (t_1e_{i,j} + t_2e_{j,i})_{i<j}, \\
    f_{L_i-L_j}(t_1, t_2) &= (t_1e_{i,j} + t_2e_{j,i})_{i\neq j}, \\
    f_{-L_i-L_j}(t_1, t_2) &= (t_1e_{i,j} + t_2e_{j,i})_{i<j}, \\
    f_{-L_i}(t) &= te_{i,i+n}, \\
    f_{-2L_i}(t) &= t e_{i+n,n}.
\end{align*}
\]
For \((t_1, t_2) \in \mathbb{K}^2, t \in \mathbb{K}\) we write
\[
\begin{align*}
    x_p(t) &= \exp(tf_p) \text{ for } p = \pm 2L_i, \\
    x_r(t_1, t_2) &= \exp(f_r(t_1, t_2)) \text{ for } r = \pm L_i \pm L_j, i \neq j.
\end{align*}
\]
Since \( \mathbb{K} \) is embedded in \( \mathbb{K}^2 \) in a obvious way, there is no confusion if we write \( x_r(t, 0) = x_r(t) \) for \( r = \pm 2L_i \). On the other hand, if we write \( x_r(a) \) where \( a \in \mathbb{K}^2 \), then \( a = (t, 0) \) for some \( t \in \mathbb{K} \).

**Theorem 4.** $SL(2n, \mathbb{K})$ (\( \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \)), \( n \geq 2 \) is generated by \( x_r(a)(a \in \mathbb{K}^2) \), where \( 0 \neq r \in \Phi = \{ \pm L_i \pm L_j \} \) subject to the relations:
\[
\begin{align*}
    (1.6) \quad x_r(a)x_r(b) &= x_r(a+b), \\
    (1.7) \quad [x_r(a), x_p(b)] &= \prod_{ir+jp \in \Phi, i,j > 0} x_{ir+jp}(g_{ijpr}(a,b)), r + p \neq 0 \\
    (1.8) \quad [x_r(a), x_p(b)] &= \text{id}, \quad 0 \neq r + p \notin \Phi
\end{align*}
\]
here \( a, b \in \mathbb{K}^* \) and \( g_{ijpr} \) are functions of \( a, b \) depending only on the structure of $SL(2n, \mathbb{R})(SL(2n, \mathbb{C}))$;
\[
\begin{align*}
    (1.9) \quad h_{L_1-L_2}(t_1, 0)h_{L_1-L_2}(t_2, 0) &= h_{L_1-L_2}(t_1 \cdot t_2, 0),
\end{align*}
\]
where
\[
\begin{align*}
    h_{L_1-L_2}(t, 0) &= x_{L_1-L_2}(t, 0)x_{L_2-L_1}(-t^{-1}, 0)x_{L_1-L_2}(t, 0) \\
    &\quad \cdot x_{L_1-L_2}(-1, 0)x_{L_2-L_1}(1, 0)x_{L_1-L_2}(-1, 0)
\end{align*}
\]
for each \( t \in \mathbb{K}^* \).

1.3. **Proof of Theorem 1** Details for Cartan action \( \alpha_{0,S} \) can be found in [4]. \( \alpha_{0,S} \) can be lifted to a \( S \)-action \( \tilde{\alpha}_{0,S} \) on \( \widetilde{Sp}(2n, \mathbb{R}) \) where \( \widetilde{Sp}(2n, \mathbb{R}) \) is the universal cover of \( Sp(2n, \mathbb{R}) \). We denote the new action by \( \tilde{\alpha}_{0,S} \) and the projection from \( \widetilde{Sp}(2n, \mathbb{R}) \) to \( Sp(2n, \mathbb{R}) \) by \( p \). Following the proof-line of Theorem 2 in [4], we just need to show the following 2 things: 1. Reducibility of closed lifted cycles defined by relations (1.1) to (1.3) in the universal cover. 2. Trivialization of any homomorphism from \( p^{-1}(\Gamma) \) to \( \mathbb{R}^n \). The second one is clear by Margulis normal subgroup theorem [15]. We show the proof of the first one.
Relations of the type (1.1), (1.2) and (1.3) are contained in a leaf of the stable manifold for some element of $\alpha_{0, S}$, hence they also lie in a stable leaf of the stable manifold for some element of $\tilde{\alpha}_{0, S}$.

For relation (1.4) follow exactly the same way as in Milnor’s proof in [19], Theorem A1 or in [3], combined with (4.6) in proof of Lemma 4.4, we can show that they are contractible and after an allowable substitution, it is reducible.

For relation (1.5), notice $h_{2L_n}(-1) = \text{diag}(1, \ldots, -1, 1, \ldots, -1)$, thus homotopy classes of $(h_{2L_n}(-1)h_{2L_n}(-1))^k (k \in \mathbb{Z})$ generate the fundamental group of $Sp(2n, \mathbb{R})$ which is isomorphic to $\mathbb{Z}$. Hence we don’t need to consider this relation in $\tilde{Sp}(2n, \mathbb{R})$.

Hence we finished the proof.

2. Local differential rigidity of non-generic actions

2.1. Non-generic Cartan actions on $SL(2n, \mathbb{R})/\Gamma$ and $SL(2n, \mathbb{C})/\Gamma$.

We consider Lie groups $G = SL(2n, K)$, $K = \mathbb{R}$ or $\mathbb{C}$, $n \geq 2$. Its Lie algebra is the set of traceless matrices. Let

\[ D_+ = \exp \mathbb{D}_+ = \{ \text{diag} (\exp t_1, \ldots, \exp t_n, \exp (-t_1), \ldots, \exp (-t_n)) : (t_1, \ldots, t_n) \in \mathbb{R}^n \}. \]

Let $\alpha_0$ be left translations of $D_+$ on $G/\Gamma$. Let $\Phi$ be the root system with respect to $D_+$. The roots are $\pm L_i \pm L_j (i < j) \leq n$ with dimensions 2 and $\pm 2L_i (1 \leq i \leq n)$ with dimension 1. The set of positive roots $\Phi^+$ and the corresponding set of simple roots $\Delta$ are

\[ \Phi^+ = \{ L_i - L_j \}_{i < j} \cup \{ L_i + L_j \}_{i < j} \cup \{ 2L_i \}_{i}, \]

\[ \Delta = \{ L_i - L_{i+1} \}_{i} \cup \{ 2L_n \}. \]

For $1 \leq i \neq j \leq n$ the hyperplanes in $\mathbb{D}_+$ defined by

\[ \mathbb{H}_{i-j} = \{ (t_1, \ldots, t_n) \in \mathbb{D}_+ : t_i = t_j \}, \]

\[ \mathbb{H}_{i+j} = \{ (t_1, \ldots, t_n) \in \mathbb{D}_+ : t_i + t_j = 0 \} \quad \text{and} \quad \]

\[ \mathbb{H}_i = \{ (t_1, \ldots, t_n) \in \mathbb{D}_+ : t_i = 0 \} \]

are Lyapunov hyperplanes for the action $\alpha_0$, i.e. kernels of Lyapunov exponents of $\alpha_0$. Elements of $\mathbb{D}_+ \setminus \bigcup \mathbb{H}_r$ (where $r = i \pm j, i$) are regular elements of the action. Connected components of the set of regular elements are Weyl chambers.

The smallest non-trivial intersections of stable foliations of various elements of the action $\alpha_0$ are Lyapunov foliations.
The corresponding root spaces are
\[ g_{L_i + L_j} = (\mathbb{K}e_{i,j+n} + \mathbb{K}e_{j,i+n})_{i<j}, \]
\[ g_{L_i - L_j} = (\mathbb{K}e_{i,j+n} + \mathbb{K}e_{i+n,j})_{i<j}, \]
\[ g_{-2L_i} = \mathbb{K}e_{i,i+n}, \]
\[ g_{-2L_i} = \mathbb{K}e_{i,n,n}. \]

For \( t_1, t_2 \in \mathbb{K} \), let
\[ f_{L_i + L_j}(t_1, t_2) = (t_1e_{i,j+n} + t_2e_{j,i+n})_{i<j}, \]
\[ f_{L_i - L_j}(t_1, t_2) = (t_1e_{i,j} + t_2e_{j+i+n})_{i<j}, \]
\[ f_{2L_i}(t_1) = t_1e_{i,i+n}. \]

For \( (t_1, t_2) \in \mathbb{K}^2, t \in \mathbb{K} \) we write
\[ x_\rho(t) = \exp(tf_\rho) \quad \text{for } \rho = \pm 2L_i, \]
\[ x_r(t_1, t_2) = \exp(f_r(t_1, t_2)) \quad \text{for } r = \pm L_i \pm L_j. \]

We define foliations \( F_r \) for \( r = \pm L_i \pm L_j(i \neq j) \), and \( F_\rho \) for \( \rho = \pm 2L_i \) for which the leaf through \( x \)
\[ F_r(x) = x_r(t_1, t_2)x, \quad F_\rho(x) = x_\rho(t)x \]
(2.1)
consists of all left multiples of \( x \) by matrices of the form \( x_r(t_1, t_2) \) or \( x_\rho(t) \).

The foliations \( F_r \) and \( F_\rho \) are invariant under \( \alpha_0 \). In fact, let \( t = (t_1, t_2, \ldots, t_n) \in \mathbb{D}_+ \), for \( \forall a_1, a_2 \in \mathbb{K} \) we have Lie bracket relations
\[ [t, f_r(a_1, a_2)] = r(t)f_r(a_1, a_2), \quad [t, f_\rho(a_1)] = \rho(t)f_\rho(a_1) \]
where \( r(t) = \pm t_i \pm t_j \) if \( r = \pm L_i \pm L_j(i \neq j) \); \( r(t) = \pm 2t_i \) if \( \rho = \pm 2L_i \).

Using the basic identity for any square matrices \( X, Y \):
\[ \exp X \exp Y = \exp(e^tY) \exp X, \quad \text{if } [X, Y] = sY, \]
it follows
(2.2)
\[ \alpha_0(t) \exp(f_r(a_1, a_2))x = \exp(e^{r(t)}f_r(a_1, a_2))\alpha_0(t)x, \]
(2.3)
\[ \alpha_0(t) \exp(f_\rho(a_1))x = \exp(e^{\rho(t)}f_\rho(a_1))\alpha_0(t)x. \]

Hence the leaf \( F_r(x) \) is mapped into \( F_r(\alpha_0(t)x) \) and \( F_\rho(x) \) is mapped into \( F_\rho(\alpha_0(t)x) \). Consequently the foliation \( F_r \) and \( F_\rho \) are contracted (corr. expanded or neutral) under \( t \) if \( r(t) < 0 \) (corr. \( r(t) > 0 \) or \( r(t) = 0 \)). If the foliation \( F_r \) and \( F_\rho \) are neutral under \( \alpha_0(t) \), it is in fact isometric under \( \alpha_0(t) \). The leaves of the orbit foliation is \( \mathcal{O}(x) = \{ \alpha_0(t)x : t \in \mathbb{D}_+ \} \).

The tangent vectors to the leaves in (2.1) for various \( r \) and \( \rho \) together with their length one Lie brackets form a basis of the tangent space at every \( x \in X \).
Let $S \subset \mathbb{D}_+$ be a closed subgroup which contains a lattice $\mathbb{L}$ in a plane in general position and let $S = \exp S$. One can naturally think of $S$ as the image of an injective homomorphism $i_0 : \mathbb{Z}^k \times \mathbb{R}^\ell \to \mathbb{D}_+$ (where $k + \ell \geq 2$).

The action $\alpha_{0,S}$ of $S$ by left translations on $G/\Gamma$ is given by

$$\alpha_{0,S}(a, x) = i_0(a) \cdot x, \quad x \in G/\Gamma.$$  \hspace{1cm} (2.4)

If $\mathbb{P}$ is a generic 2-plane with respect to $\Phi$ then the foliations $F_r$ and $F_\rho$ are also Lyapunov foliations for $\alpha_{0,\mathbb{P}}$. The leaves of $F_r$ and $F_\rho$ are intersections of the leaves of stable manifolds of the action by different elements of $\mathbb{P}$. The same holds for the action by any regular lattice in $\mathbb{P}$ and thus for any generic restriction $\alpha_{0,S}$ with respect to $\Phi$.

If $\mathbb{K} = \mathbb{R}$, the neutral foliation for a generic restriction $\alpha_{0,S}$ is given by

$$\mathcal{N}_0(x) = \{D \cdot x : x \in SL(2n, \mathbb{R})/\Gamma\}$$

where $D$ is the set of diagonal matrices in $SL(2n, \mathbb{R})$ with positive entries; if $\mathbb{K} = \mathbb{C}$, the neutral foliation is given by

$$\mathcal{N}_0(x) = \{DT \cdot x : x \in SL(2n, \mathbb{C})/\Gamma\}$$

where $T$ is the set of diagonal matrices in $SL(2n, \mathbb{C})$ whose entries are of absolute value 1. Thus $T$ is isomorphic to $\mathbb{T}^{2n-1}$. Let $D_G = D$ if $G = SL(2n, \mathbb{R})$; $D_G = DT$ if $G = SL(2n, \mathbb{C})$.

### 2.2. Preliminaries of cocycles.

Let $\alpha : A \times M \to M$ be an action of a topological group $A$ on a compact Riemannian manifold $M$ by diffeomorphisms. For a topological group $Y$ a $Y$-valued cocycle (or an one-cocycle) over $\alpha$ is a continuous function $\beta : A \times M \to Y$ satisfying:

$$\beta(ab, x) = \beta(a, \alpha(b, x))\beta(b, x)$$  \hspace{1cm} (2.5)

for any $a, b \in A$. A cocycle is cohomologous to a constant cocycle (cocycle not depending on $x$) if there exists a homomorphism $s : A \to Y$ and a continuous transfer map $H : M \to Y$ such that for all $a \in A$

$$\beta(a, x) = H(\alpha(a, x))s(a)H(x)^{-1}$$  \hspace{1cm} (2.6)

In particular, a cocycle is a coboundary if it is cohomologous to the trivial cocycle $\pi(a) = id_Y$, $a \in A$, i.e. if for all $a \in A$ the following equation holds:

$$\beta(a, x) = H(\alpha(a, x))H(x)^{-1}.$$  \hspace{1cm} (2.7)

For more detailed information on cocycles adapted to the present setting see \cite{3}.
2.3. Paths and cycles for a collection of foliations. In this section we recall some notation and results from [3]. Let $T_1, \ldots, T_r$ be a collection of mutually transversal continuous foliations on a compact manifold $M$ with smooth simply connected leaves.

For $N \in \mathbb{N}$ and $j_k \in \{1, \ldots, r\}, k \in \{1, \ldots, N - 1\}$ an ordered set of points $p(j_1, \ldots, j_{N-1}): x_1, \ldots, x_N \in M$ is called an $T$-path of length $N$ if for every $k \in \{1, \ldots, N - 1\}, x_{i+1} \in T_{j_k}(x_k)$. A closed $T$-path (i.e., when $x_N = x_1$) is a $T$-cycle.

A $T$-cycle $p(j_1, \ldots, j_{N-1}): x_1, \ldots, x_N = x_1 \in M$ is called stable for the $A$ action $\alpha$ if there exists a regular element $\alpha \in A$ such that the whole cycle $p$ is contained in a leaf of the stable foliations for the map $\alpha(a, \cdot)$, i.e., if

$$\bigcap_{k=1}^{N} \{ a : \chi_{j_k}(a) < 0 \} \neq \emptyset.$$ 

**Definition 2.1.** Let $p(j_1, \ldots, j_{N-1}): x_1, \ldots, x_N$ and $p_n(j_1, \ldots, j_{N-1}): x_1^n, \ldots, x_N^n$ be two $T$-paths. Then $p = \lim_{n \to \infty} p_n$ if for all $k \in \{1, \ldots, N\}$

$$x_k = \lim_{n \to \infty} x_k^n.$$ 

Limits of $T$-cycles are defined similarly.

Two $T$-cycles, $p(j_1, \ldots, j_{N+1}): x_1, \ldots, x_k, y, x_k, \ldots, x_N = x_1$ and $p(j_1, \ldots, j_{N-1}): x_1, \ldots, x_k, x_{k+1}, \ldots, x_N$ are said to be conjugate if $y \in T_i(x_k)$ for some $i \in \{1, \ldots, r\}$. For $T$-cycles, $p(j_1, \ldots, j_{N-1}): x_1, \ldots, x_N = x_1$ and $p'(j'_1, \ldots, j'_{K-1}): x_1 = x'_1, \ldots, x'_K = x_1$ define their composition or concatenation $p * p'$ by

$$p * p'(j_1, \ldots, j_{N-1}, j'_1, \ldots, j'_K) : x_1, \ldots, x_N, x'_1, \ldots, x'_K = x_1.$$ 

Let $\mathcal{AS}_T^x(\alpha)$ denote the collection of stable $T$-cycles. Let $\mathcal{AS}_T(\alpha)$ denote the collection of $T$-cycles which contains $\mathcal{AS}_T^x(\alpha)$ and is closed under conjugation, concatenation of cycles, and under the limitation procedure defined above. $\mathcal{AS}_T^x(\alpha)$ denotes the subset of $\mathcal{AS}_T(\alpha)$ which contain point $x$.

A path $p : x_1, \ldots, x_N$ reduces to a path $p' : x_1, x'_2, \ldots, x'_k, \ldots, x'_N$ via an $\alpha$-allowable $T$-substitution if the $T$-cycle

$$p * p' : x_1, \ldots, x_N, x'_1, \ldots, x'_N$$

obtained by concatenation of $p$ and $p'$ is in the collection $\mathcal{AS}_T(\alpha)$.

Two $T$-cycle $c_1$ and $c_2$ are $\alpha$-equivalent if $c_1$ reduces to $c_2$ via a finite sequence of $\alpha$-allowable $T$-substitutions. A $T$-cycle we call $\alpha$-reducible if it is in $\mathcal{AS}_T(\alpha)$. 
Definition 2.2. For \( N \in \mathbb{N} \) and \( j_k \in \{1, \ldots, r\}, k \in \{1, \ldots, N\} \) an ordered set of points \( p(j_1, \ldots, j_N) : x_1, \ldots, x_N, x_{N+1} = x_1 \in M \) is called an \( \mathcal{T} \)-cycle of length \( N \) if for every \( k \in \{1, \ldots, N\}, x_{i+1} \in T_{j_k}(x_k) \). A \( \mathcal{T} \)-cycle which consists of a single point is a trivial \( \mathcal{T} \)-cycle.

Definition 2.3. Foliations \( \mathcal{T}_1, \ldots, \mathcal{T}_r \) are locally transitive if there exists \( N \in \mathbb{N} \) such that for any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for every \( x \in M \) and for every \( y \in B_X(x, \delta) \) (where \( B_M(x, \delta) \) is a \( \delta \) ball in \( M \)) there is a \( \mathcal{T} \)-path \( p(j_1, \ldots, j_{N-1}) : x = x_1, x_2, \ldots, x_{N-1}, x_N = y \) in the ball \( B_M(x, \varepsilon) \) such that \( x_{k+1} \in T_{j_k}(x_k) \) and \( d_{T_{j_k}(x_k)}(x_{k+1}, x_k) < 2\varepsilon \).

In other words, any two sufficiently close points can be connected by a \( \mathcal{T} \)-path of not more than \( N \) pieces of a given bounded length. Here, for a submanifold \( Y \) in \( M \), \( d_Y(x, y) \) denotes the infimum of lengths of smooth curves in \( Y \) connecting \( x \) and \( y \).

2.4. Cocycle rigidity for \( \alpha_{0, S} \). The purpose of this section is to describe a geometric method for proving cocycle rigidity for this action following \([2, 3]\).

Proposition 2.1. Any small \( D_G \)-valued Hölder cocycle over \( \alpha_{0, S} \) on \( G/\Gamma \) is cohomologous to a constant cocycle via a Hölder transfer function.

Any \( D_G \)-valued \( C^\infty \) small cocycle over the generic restriction of the split Cartan action on \( G/\Gamma \) is cohomologous to a constant cocycle via a \( C^\infty \) transfer function.

For a cocycle \( \beta : S \times G/\Gamma \to DT \) over \( \alpha_0 \), we define \( DT \)-valued potential of \( \beta \) as

\[
\begin{cases}
P^\gamma_a(y, x) = \lim_{n \to +\infty} \beta(na, y)^{-1}\beta(na, x), & \gamma(a) < 0 \\
P^\gamma_a(y, x) = \lim_{n \to -\infty} \beta(na, y)^{-1}\beta(na, x), & \gamma(a) > 0
\end{cases}
\]

where \( \gamma \in \Phi \) and \( a \in S \). Now for any \( F \)-cycle \( c : x_1, \ldots, x_{N+1} = x_1 \) on \( M \), we can define the corresponding periodic cycle functional:

\[
PCF(c)(\beta) = \prod_{i=1}^{N} P^\gamma_{a_i}(x_i, x_{i+1})(\beta)
\]

where \( \gamma_i \in \Phi \).

Two essential properties of the PCF which are crucial for our purpose are that PCF is continuous and that it is invariant under the operation of moving cycles around by elements of the action \( \alpha_{0, S} \). We now state an important proposition which is the base of our further proof.
Proposition 2.2. (Proposition 4. [2]) Let $\alpha$ be an $R^k$ action by diffeomorphisms on a compact Riemannian manifold $M$ such that a dense set of elements of $R^k$ acts normally hyperbolically with respect to an invariant foliation. If the foliations $F_1, \ldots, F_r$ are locally transitive and if $\beta$ is a Hölder cocycle over the action $\alpha$ such that $F(C)(\beta) = 0$ for any cycle $C$ then: $\beta$ is cohomologous to a constant cocycle via a continuous map $h : M \to Y$.

2.5. Proof of Proposition 2.1. At first we show the cocycle rigidity for Hölder cocycles. The invariant foliations that we considered are $F_r$ and $F_\rho$ where $r = \pm L_i \pm L_j (i \neq j)$, and $\rho = \pm 2L_i$. Notice that those foliations are smooth and their Lie brackets at length one generate the whole tangent space. This implies that this system of foliations is locally $1/2 - \text{Hölder}$ transitive [11, Section 4, Proposition 1]. Every such cycle represents a relation in the group. The word represented by this cycle can be written as a product of conjugates of basic relations in Theorem 4.

Relations of the type (1.6)-(1.8) are contained in a leaf of the stable manifold for some element of $\alpha_{0,s}$.

For relation (1.9), if $K = \mathbb{R}$, if doubled, follow exactly the same way as in Milnor’s proof in [19, Theorem A1] or in [3], we can show that they are contractible and reducible; if $K = \mathbb{C}$, then they are contractible and reducible. Hence we finished the proof. Finally, to cancel conjugations one notices that canceling $F_r(t_1, t_2)F_r(t_1, t_2)^{-1} = \text{id}$ or $F_\rho(t_1)F_\rho(t_1)^{-1} = \text{id}$ are also an allowed substitution and each conjugation can be canceled inductively using that.

Thus, the value of the periodic cycle functional for any Hölder cocycle $\beta$ depends only on the element of $\Gamma$ this cycle represents. Furthermore, these values provide a homomorphism $p$ from $\Gamma$ to $D_G$. The restriction of $p$ on $D$ is trivial by Margulis normal subgroup theorem[15]. Notice $T$ is abelian, thus order of $p(\Gamma)$ is bonded by $[\Gamma : [\Gamma, \Gamma]]$ which is finite[15, 4’ Theorem]. By smallness of the cocycle, restriction of $p$ on $T$ vanishes.

Hence all periodic cycle functionals vanish on $\beta$. Now Proposition 2.2 implies that $\beta$ is cohomologous to a constant cocycle via a Hölder transfer function.

Now consider the case of $C^\infty$ cocycles. Notice that the transfer function $H$ constructed using periodic cycle functionals is $C^\infty$ along the stable foliations of various elements of the action. Now a general result stating that in case the smooth distributions along with their Lie brackets generate the tangent space at any point of a manifold a function smooth along corresponding foliations is necessarily smooth
(see [14] for a detailed discussion and references to proofs), implies that the transfer map $H$ is $C^\infty$.

3. Proof of Theorems [4]

The neutral foliation for a generic restriction $\alpha_{0,S}$ is a smooth foliation, we may use the Hirsch-Pugh-Shub structural stability theorem [8, Chapter 6]. Namely if $\tilde{\alpha}_S$ is a sufficiently $C^1$-small perturbation of $\alpha_{0,S}$ then for all elements $a \in A$ which are regular for $\alpha_{0,S}$ and sufficiently away from non-regular ones (denote this set by $\overline{A}$) are also regular for $\tilde{\alpha}_S$. The central distribution is the same for any $a \in \overline{A}$ and is uniquely integrable to an $\tilde{\alpha}_S(a, \cdot)$-invariant foliation which we denote by $\mathcal{N}$. Moreover, there is a Hölder homeomorphism $\tilde{h}$ of $G/\Gamma$, $C^0$ close to the $id_X$, which maps leaves of $\mathcal{N}_0$ to leaves of $\mathcal{N}$: $\tilde{h}\mathcal{N}_0 = \mathcal{N}$. This homeomorphism is uniquely defined in the transverse direction, i.e. up to a homeomorphism preserving $\mathcal{N}$. Furthermore, $\tilde{h}$ can be chosen smooth and $C^1$ close to the identity along the leaves of $\mathcal{N}_0$ although we will not use the latter fact. Clearly the leaves of the foliation $\mathcal{N}_0$ are preserved by every $a \in \overline{A}$. The action $\alpha_S$ is Hölder but it is smooth and $C^1$-close to $\alpha_{0,S}$ along the leaves of the neutral foliation $\mathcal{N}_0$.

Let us define an action $\alpha_S$ of $S$ on $G/\Gamma$ as the conjugate of $\tilde{\alpha}_S$ by the map $\tilde{h}$ obtained from the Hirsch-Pugh-Shub stability theorem:

$$\alpha_S := \tilde{h}^{-1} \circ \tilde{\alpha}_S \circ \tilde{h}$$

Since the action $\alpha_S$ is a $C^0$ small perturbation of $\alpha_{0,S}$ along the leaves of the neutral foliation of $\alpha_{0,S}$ whose leaves are $\{D_G \cdot x : x \in X\}$, we have that $\alpha_S$ is given by a map $\beta : (\mathbb{Z}^k \times \mathbb{R}^l) \times G/\Gamma \to D_G$ by

$$\alpha_S(a, x) = \beta(a, x) \cdot \alpha_{0,S}(a, x)$$

(3.1)

for $a \in \mathbb{Z}^k \times \mathbb{R}^l$ and $x \in G/\Gamma$. Notice that since $\alpha_S$ is a small perturbation of the action by left translations $\alpha_{0,S}$, it can be lifted to a $S$-action $\overline{\alpha}_S$ on $G$ commuting with the right $\Gamma$ action on $G$, and $\beta$ is lifted to a cocycle $\overline{\beta}$ over $\overline{\alpha}_S$ (for more details see [16, example 2.3]). In particular we have:

$$\overline{\beta}(ab, x) = \beta(a, \overline{\alpha}_S(b, x))\overline{\beta}(b, x).$$

It follows that since $\alpha_S$ is Hölder, $\overline{\beta}(a, x)$ is small Hölder cocycle over the action $\overline{\alpha}_S$, due to the smallness of the perturbation.

Let $U : U_1, \ldots, U_r$ denote the invariant unipotent foliations for the lifted action $\overline{\alpha}_{0,S}$ of $\alpha_{0,S}$ on $G$ which projects to invariant Lyapunov foliations for $\alpha_{0,S}$; and let $T : T_1, \ldots, T_r$ denote invariant Lyapunov foliations for lifted $\overline{\alpha}_S$ which projects to invariant Lyapunov foliations.
for $\alpha_S$. Notice that the latter foliations have only Hölder leaves but we are justified in calling them Lyapunov foliations since they are images of Lyapunov foliations for a smooth perturbed action under a Hölder conjugacy. Denote the neutral foliation $N_0$ on $G$ by $N_0$. An immediate corollary of the result of Brin and Pesin [1] on persistence of local transitivity of stable and unstable foliations of a partially hyperbolic diffeomorphisms and the fact that the collection of homogeneous Lyapunov foliations $U : U_1, \ldots, U_r$ is locally transitive and $T : T_1, \ldots, T_r$ is transitive and they are leafwise $C^0$ close. Following the proof line closely with only trivial modifications from those of [Section 6.2, 6.2 and 6.4 [3]], and [Section 5.3.5.4, [4]], we can show $U$-cycles and $T$-cycles project to each other along the neutral foliations (precise definitions are in [Section 6.2, [3]]), which implies:

**Proposition 3.1.** The lifted cocycle for the perturbed action $\tilde{\alpha}_S$ is cohomologous to a constant cocycle.

The value of the periodic cycle functional for Hölder cocycle $\beta$ over $\tilde{\alpha}_S$ or its Hölder conjugate $\alpha_S$ depends only on the element of $\Gamma$ this cycle represents. Using the same trick as in proof of Proposition [2.1] we can show every homomorphism from $\Gamma$ to $D_G$ is trivial.

$\beta$ is cohomologous to a small constant cocycle $g : \mathbb{Z}^k \times \mathbb{R}^\ell \to D_G$ via a continuous transfer map $H : G/\Gamma \to D_G$ which can be chosen close to identity in $C^0$ topology if the perturbation $\tilde{\alpha}_S$ small in $C^2$ topology.

Let us consider the map $h' : H^{-1}(x) \cdot x$. We have from the cocycle equation (3.1) and the cohomology equation (2.6)

$$h'(\alpha_S(a, x)) = \alpha_{0, \tilde{\alpha}}(a, h'(x))$$

where $\alpha_{0, \tilde{\alpha}}(a, x) := i(a) \cdot x$, where $i(a) := g(a)i_0(a), a \in A$ and $i_0$ is as in (2.7). Since the map $h'$ is $C^0$ close to the identity it is surjective and thus the action $\alpha_S$ is semi-conjugate to the standard perturbation $\alpha_{0, \tilde{\alpha}}$ of $\alpha_{0, S}$, i.e. $\alpha_{0, \tilde{\alpha}}$ is a factor of $\alpha_S$. It is enough to prove that $h'$ is injective. By simple transitivity of $U$-holonomy group and the fact that there is no non-trivial element in $DT$ such that all its powers are small [Section 7.1 [3]] we have:

**Proposition 3.2.** (Section 6.1 [3]) The map $h'$ is a homeomorphism and hence provides a topological conjugacy between $\alpha_S$ and $\alpha_{0, \tilde{\alpha}}$. 

Now by letting $h := h'h^{-1}$ we have

$$h \circ \tilde{\alpha}_S h^{-1} = \alpha_{0, \tilde{\alpha}}$$

thus there is a topological conjugacy between $\tilde{\alpha}_S$ and $\alpha_{0, \tilde{\alpha}}$. The smoothness of this homeomorphism follows as in [13], [3] or [16], by the general
Katok-Spatzier theory of non-stationary normal forms for partially hyperbolic abelian actions.

4. Proof of Theorem 3

4.1. Basic settings in \( Sp(2n, \mathbb{R}) \). Let \( Q \) be a non-degenerate standard skew-symmetric bilinear form on \( \mathbb{R}^{2n} \). Take \( Q \) to be the bilinear form given, in terms of a basis \( e_1, \ldots, e_{2n} \) for \( \mathbb{R}^{2n} \), by \( Q(e_i, e_{i+n}) = 1 \), \( Q(e_{i+n}, e_i) = -1 \) and \( Q(e_i, e_j) = 0 \) otherwise.

Using this base, the Lie algebra \( sp(2n, \mathbb{R}) \) of \( Sp(2n, \mathbb{R}) \) can be represented as \( 2n \times 2n \) matrices

\[
\begin{pmatrix}
A_1 & A_2 \\
A_3 & A_4
\end{pmatrix},
\]

where \( A_1, A_2, A_3, A_4 \) are \( n \times n \) matrices satisfying \( A_1 = -A_1^T \) and \( A_2 \) and \( A_3 \) are symmetric.

We denote by \( S \) the set of \( 2n \times 2n \) diagonal matrices in \( Sp(2n, \mathbb{R}) \) with positive entries. Let \( \Phi \) be the root system with respect to \( S \). The roots are \( \pm L_i \pm L_j \) \((i < j \leq n)\) and \( \pm 2L_i \) \((1 \leq i \leq n)\). The set of positive roots \( \Phi^+ \) and the corresponding set of simple roots \( \Delta \) are

\[
\Phi^+ = \{ L_i - L_j \}_{i<j} \cup \{ L_i + L_j \}_{i<j} \cup \{ 2L_i \}_{i}, \\
\Delta = \{ L_i - L_{i+1} \}_{i} \cup \{ 2L_n \}.
\]

Let \( 1 \leq i, j \leq n, i \neq j \) be two distinct indices and let \( \exp \) be the exponentiation map for matrices.

The corresponding root spaces are

\[
\begin{align*}
g_{L_i+L_j} &= \mathbb{R}(e_{i,j+n} + e_{j,i+n})_{i<j}, & g_{L_i-L_j} &= \mathbb{R}(e_{i,j} - e_{j+n,i+n})_{i\neq j}, \\
g_{-L_i-L_j} &= \mathbb{R}(e_{j+n,i} + e_{i+n,j})_{i<j}, & g_{2L_i} &= \mathbb{R}e_{i,i+n}, \\
g_{-2L_i} &= \mathbb{R}e_{i+n,n}.
\end{align*}
\]

Let

\[
\begin{align*}
f_{L_i+L_j} &= (e_{i,j+n} + e_{j,i+n})_{i<j}, & f_{L_i-L_j} &= (e_{i,j} - e_{j+n,i+n})_{i\neq j}, \\
f_{-L_i-L_j} &= (e_{j+n,i} + e_{i+n,j})_{i<j}, & f_{2L_i} &= e_{i,i+n}, \\
f_{-2L_i} &= e_{i+n,n}.
\end{align*}
\]

For \( t \in \mathbb{R} \) we write

\[
x_r(t) = \exp(t f_r) \in U_r^r \quad \text{for } r \in \Phi.
\]

Let

\[
w_r(t) = x_r(t)x_{-r}(-t^{-1})x_r(t), \quad t \in \mathbb{R}^*
\]
where
\[ x_i = x_r(t)\forall i, \quad y_i = x_{-r}(t^{-1})\forall i. \]

Correspondingly, we define
\[ h_r(t) = w_r(t)w_r(1)^{-1}, \quad t \in \mathbb{R}^*, r \in \Phi. \]

4.2. Relations in universal central extension. For \( \gamma, \beta \in \Phi \) such that \( \gamma \neq -\beta \), it is known that
\[ [x_\gamma, x_\beta] \subset \prod_{\chi=i\gamma+j\beta, i,j \geq 1} x_\chi. \]

This clearly gives rise to numbers \( g_{ijpr} \) satisfying
\( (4.1) \quad x_r(a)x_r(b) = x_r(a+b) \)
\( (4.2) \quad [x_r(a), x_p(b)] = \prod_{ir+jp \in \Phi, i,j > 0} x_{ir+jp}(g_{ijpr}a^ib^j), r + p \neq 0, \)
\( (4.3) \quad [x_r(a), x_p(b)] = \text{id}, \quad 0 \neq r + p \notin \Phi. \)

If \( \tilde{G} \) is the group defined by relations (4.1)-(4.3), and if \( \pi_1 \) is the natural homomorphism from \( \tilde{G} \) to \( Sp(2n,\mathbb{R}) \), then \( (\pi_1, \tilde{G}) \) is a universal central extension of \( Sp(2n,\mathbb{R}) \). (For the proof of this and other elementary properties of a universal central extension, one may refer to [23], Section 7.) We write for \( x_p(t) \), the corresponding element in \( \tilde{G} \) by \( \tilde{x}_p(t) \). Then \( \tilde{w}_p(u), \tilde{h}_p(u), u \in \mathbb{R}^* \) are obviously defined elements of \( \tilde{G} \).

**Lemma 4.1.** If \( a, t_1 \in \mathbb{R}^* \), the following hold in \( \tilde{G} \) (and hence in \( G \) too).

1. \( \tilde{w}_{2L_n}(a)\tilde{w}_{L_{n-1}-L_n}(t_1)\tilde{w}_{2L_n}(a)^{-1} = \tilde{w}_{L_{n-1}+L_n}(-at_1), \)
2. \( \tilde{w}_{2L_n}(a)\tilde{w}_{L_{n-1}+L_n}(t_1)\tilde{w}_{2L_n}(a)^{-1} = \tilde{w}_{L_{n-1}-L_n}(a^{-1}t_1), \)
3. \( \tilde{w}_{L_{n-1}-L_n}(t_1)\tilde{w}_{2L_n}(a)\tilde{w}_{L_{n-1}-L_n}(t_1)^{-1} = \tilde{w}_{2L_{n-1}}(at_1^2), \)
4. \( \tilde{w}_{L_{n-1}-L_n}(t_1)\tilde{w}_{2L_{n-1}}(a)\tilde{w}_{L_{n-1}-L_n}(t_1)^{-1} = \tilde{w}_{2L_n}(at_1^{-2}). \)

Hence,

5. \( \tilde{h}_{L_{n-1}-L_n}(t_1)\tilde{w}_{2L_n}(a)\tilde{h}_{L_{n-1}-L_n}(t_1)^{-1} = \tilde{w}_{2L_n}(at_1^{-2}), \)
6. \( \tilde{w}_{2L_n}(a)\tilde{h}_{L_{n-1}-L_n}(t_1)\tilde{w}_{2L_n}(a)^{-1} = \tilde{h}_{L_{n-1}+L_n}(-at_1)\tilde{h}_{L_{n-1}+L_n}(-a)^{-1}. \)
Lemma 4.2. For \(\gamma \in \Phi\), denote by \(\tilde{H}_\gamma\) the subgroup generated by \(\tilde{h}_\gamma(t)(t \in \mathbb{R}^*)\), Let \(\tilde{H}\) be the subgroup generated by \(\{\tilde{H}_\alpha, \alpha \in \Phi\}\).

1. \(\ker(\pi_1) \cap \tilde{H}_{L_1-L_2} = \{\prod_i \tilde{h}_{L_1-L_2}(t_i) \mid \prod_i t_i = 1\}\).
2. \(\ker(\pi_1) \cap \tilde{H}_r = \ker(\pi_1) \cap \tilde{H}_{L_1-L_2}, \quad \text{for } r = \pm L_i \pm L_j (i \neq j)\).
3. \(\ker(\pi_1) = \prod_{\gamma \in \Delta} (\ker(\pi_1) \cap \tilde{H}_\gamma)\).

Proof. (1). Notice \(\pi_1(\tilde{h}_{L_1-L_2}(t)) = \text{diag}(t_1, (t^{-1})_2, (t^{-1})_{1+n}, t_{2+n})\). Thus (1) is clear.

It follows from (1) that \(\ker(\pi_1) \cap \tilde{H}_{L_1-L_2}\) is generated by elements
\[
\tilde{h}_{L_1-L_2}(t_1)\tilde{h}_{L_1-L_2}(t_2)\tilde{h}_{L_1-L_2}(t_1t_2)^{-1}, \quad \text{where } t_1, t_2 \in \mathbb{R}^*.
\]

(2) We can prove similarly that \(\ker(\pi_1) \cap \tilde{H}_r, (r = \pm L_i \pm L_j)\) is generated by elements \(\tilde{h}_r(t_1)\tilde{h}_r(t_2)\tilde{h}_r(t_1t_2)^{-1}\). Since these simple roots belong to the same orbit under the Weyl group, an argument similar to one in \([18, \text{Lemma } 8.2]\) shows that \(\ker(\pi_1) \cap \tilde{H}_r \subseteq \ker(\pi_1) \cap \tilde{H}_{L_1-L_2}\) for all roots \(r = \pm L_i \pm L_j\). This proves (2).

(3) By \([23, \text{p.40}]\), we have \(\ker(\pi_1) \subseteq \tilde{H}\). Using the method similar to that in the proof of \([22, \text{7.7}]\), we have \(\tilde{H} = \prod_{\gamma \in \Delta} \tilde{H}_\gamma\). Using the simple connectedness of \(Sp(2n, \mathbb{C})\) over \(\mathbb{C}\) \([5, \text{p.24}]\), we get (3).

For \(t_1, t_2 \in \mathbb{R}^*\), we define:
\[
\{t_1, t_2\} = \tilde{h}_{L_1-L_2}(t_1)\tilde{h}_{L_1-L_2}(t_2)\tilde{h}_{L_1-L_2}(t_1t_2)^{-1}.
\]

Now in exactly the same manner as the proof in the appendix of \([18]\), we prove that these \(\{t_1, t_2\}\)’s satisfy the conditions

Lemma 4.3.

\[
\begin{align*}
\{t_1, t_2\} &= \{t_2, t_1\}^{-1}, \quad \forall t_1, t_2 \in \mathbb{R}^*, \\
\{t_1, t_2 \cdot t_3\} &= \{t_1, t_2\} \cdot \{t_1, t_3\}, \quad \forall t_1, t_2, t_3 \in \mathbb{R}^*, \\
\{t_1 \cdot t_2, t_3\} &= \{t_1, t_3\} \cdot \{t_2, t_3\}, \quad \forall t_1, t_2, t_3 \in \mathbb{R}^*, \\
\{t, 1 - t\} &= 1, \quad \forall t \in \mathbb{R}^*, t \neq 1, \\
\{t, -t\} &= 1, \quad \forall t \in \mathbb{R}^*.
\end{align*}
\]
Let \( \tilde{H}_0 \) denote the cyclic group generated by \( \tilde{h}_{2L_n}(-1) \tilde{h}_{2L_n}(-1) \) and \( \tilde{H}_c \) denote the cyclic group generated by \( \tilde{h}_{2L_n}(-1) \). To prove Theorem 3, it is equivalent to prove:

**Proposition 4.1.** \( \ker(\pi_1) = (\ker(\pi_1) \cap \tilde{H}_{L_1-L_2}) \cdot \tilde{H}_0. \)

The proof of this proposition relies on the following results.

**Lemma 4.4.**

(i) \( \tilde{H}_{2L_n} \subseteq \tilde{H}_{L_{n-1}-L_n} \cdot \tilde{H}_{L_{n+1}-L_n} \cdot \tilde{H}_c. \)

(ii) \( \ker(\pi_1) \cap \tilde{H}_{2L_n} \subseteq (\ker(\pi_1) \cap \tilde{H}_{L_1-L_2}) \cdot \tilde{H}_0. \)

**Proof.** (i) Using Lemma 4.4 for \( \forall t \in \mathbb{R}^* \), let \( z^2 = |t| \) we have

\[
\tilde{h}_{2L_n}(t) = \tilde{w}_{2L_n}(t) = \tilde{h}_{L_{n-1}-L_n}(z^{-1}) \tilde{w}_{2L_n}(t) \tilde{h}_{L_{n-1}-L_n}(z^{-1})^{-1} \tilde{w}_{2L_n}(-1).
\]

If \( t > 0 \) we have

\[
\tilde{h}_{L_{n-1}-L_n}(z^{-1}) \tilde{w}_{2L_n}(t) \tilde{h}_{L_{n-1}-L_n}(z^{-1})^{-1} \tilde{w}_{2L_n}(-1)
= \tilde{h}_{L_{n-1}-L_n}(z^{-1})(\tilde{w}_{2L_n}(1) \tilde{h}_{L_{n-1}-L_n}(z^{-1})^{-1} \tilde{w}_{2L_n}(-1))
= \tilde{h}_{L_{n-1}-L_n}(z^{-1}) \tilde{h}_{L_{n-1}+L_n}(-1) \tilde{h}_{L_{n-1}+L_n}(-1)^{-1}.
\]

If \( t < 0 \) we have

\[
\tilde{h}_{L_{n-1}-L_n}(z^{-1}) \tilde{w}_{2L_n}(t) \tilde{h}_{L_{n-1}-L_n}(z^{-1})^{-1} \tilde{w}_{2L_n}(-1)
= \tilde{h}_{L_{n-1}-L_n}(z^{-1}) \tilde{w}_{2L_n}(-1) \tilde{h}_{L_{n-1}-L_n}(z^{-1})^{-1} \tilde{w}_{2L_n}(-1)
= \tilde{h}_{L_{n-1}-L_n}(z^{-1}) \tilde{h}_{L_{n-1}+L_n}(z^{-1})^{-1} \tilde{w}_{2L_n}(-1).
\]

Especially, if \( t = 1, z = -1 \) we have

\begin{equation}
(4.4) \quad e = \tilde{h}_{L_{n-1}-L_n}(-1) \tilde{h}_{L_{n-1}+L_n}(-1).
\end{equation}

Especially, if \( t = -1, z = -1 \) we have

\begin{equation}
(4.5) \quad e = \tilde{h}_{L_{n-1}-L_n}(-1) \tilde{h}_{L_{n-1}+L_n}(-1)^{-1}.
\end{equation}

Thus we get

\begin{equation}
(4.6) \quad e = (\tilde{h}_{L_{n-1}-L_n}(-1))^2.
\end{equation}

Hence we proved (i).

(ii) By Lemma 5.3 and (i), any \( h \in \tilde{H}_{2L_n} \) can be written as

\[
h = \tilde{h}_{L_{n-1}-L_n}(t_1) \tilde{h}_{L_{n-1}+L_n}(t_2) h_1 h_2
\]

where \( t_1, t_2 \in \mathbb{R}^* \), \( h_1 \in \ker(\pi_1) \cap \tilde{H}_{L_1-L_2} \) and \( h_2 \in \tilde{H}_c. \)

If \( \pi_1(h) = I_{2n} \), we have \( t_1 = t_2 = \pm 1 \), and \( \pi_1(h_2) = I_{2n} \). If \( t_1 = t_2 = 1 \), we have \( h = h_1 h_2 \). If \( t_1 = t_2 = -1 \), by (4.4) we still have \( h = h_1 h_2 \).
Notice \( \pi_1(\tilde{h}_{2n}(-1)) = \text{diag}((-1)_n, (-1)_{2n}) \), it follows \( h_2 = (\tilde{h}_{2n}(-1))^{2k}, k \in \mathbb{Z} \). Hence we proved (ii). \( \square \)

4.3. Proof of Proposition 4.1. By (3) of Lemma 4.2

\[ \ker \pi_1 = \prod_{\alpha \in \Delta} (\ker \pi_1 \cap \tilde{H}_\alpha), \]

where \( \Delta = \{ L_i - L_{i+1} \}_i \cup \{2L_n\} \).

By (ii) of Lemma 4.4 we have

\[ \ker \pi_1 \subseteq (\ker \pi_1 \cap \tilde{H}_{L_1-L_2}) \cdot \tilde{H}_0. \]

The inverse inclusion is obvious. Hence we finished the proof.

5. Proof of Theorem 4.1.

5.1. Basic settings. We follow some notations in Section 4. We consider Lie groups \( G = SL(2n, \mathbb{K}), \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \), \( n \geq 2 \). Its Lie algebra is the set of traceless matrices. Let

\[ D_+ = \{ \text{diag}(\exp t_1, \ldots, \exp t_n, \exp(-t_1), \ldots, \exp(-t_n)): (t_1, \ldots, t_n) \in \mathbb{R}^n \}. \]

Let \( \Phi \) be the root system with respect to \( D_+ \). The roots are \( \pm L_i \pm L_j (i < j \leq n) \) with dimensions 2 and \( \pm 2L_i (1 \leq i \leq n) \) with dimension 1. The set of positive roots \( \Phi^+ \) and the corresponding set of simple roots \( \Delta \) are

\[ \Phi^+ = \{ L_i - L_j \}_{i<j} \cup \{ L_i + L_j \}_{i<j} \cup \{2L_i\}_i, \]
\[ \Delta = \{ L_i - L_{i+1} \}_i \cup \{2L_n\}. \]

Let \( 1 \leq i, j \leq n, i \neq j \) be two distinct indices and let \( \exp \) be the exponentiation map for matrices.

The corresponding root spaces are

\[ \mathfrak{g}_{L_i + L_j} = (\mathbb{R}e_{i,j+n} + \mathbb{R}e_{j,i+n})_{i<j}, \quad \mathfrak{g}_{L_i - L_j} = (\mathbb{R}e_{i,j} + \mathbb{R}e_{j+n,i+n})_{i \neq j}, \]
\[ \mathfrak{g}_{-L_i - L_j} = (\mathbb{R}e_{j+n,i} + \mathbb{R}e_{i,j+n})_{i<j}, \quad \mathfrak{g}_{2L_i} = \mathbb{R}e_{i,i+n}, \]
\[ \mathfrak{g}_{-2L_i} = \mathbb{R}e_{i+n,n}. \]

Let

\[ f^1_{L_i + L_j}(t) = (te_{i,j+n})_{i<j}, \quad f^2_{L_i + L_j}(t) = (te_{i,j+n})_{i<j}, \]
\[ f^1_{L_i - L_j}(t) = (te_{i,j})_{i \neq j}, \quad f^2_{L_i - L_j}(t) = (te_{j+n,i+n})_{i \neq j}, \]
\[ f^1_{-L_i - L_j}(t) = (te_{j+n,i})_{i<j}, \quad f^2_{-L_i - L_j}(t) = (te_{i+n,j})_{i<j}, \]
\[ f_{2L_i}(t) = te_{i,i+n}, \quad f_{-2L_i}(t) = te_{i+n,n}. \]
For \((t_1, t_2) \in \mathbb{K}^2, t \in \mathbb{K}\) we write
\[x_\rho(t) = \exp(tf_\rho) \quad \text{for } \rho = \pm 2L_i,\]
\[x_r(t_1, t_2) = \exp(f_r^1(t_1)) \exp(f_r^2(t_2)), \quad \text{for } r = \pm L_i \pm L_j.\]

Let
\[w_\rho(t) = x_\rho(t) x_{-\rho}(-t^{-1}) x_\rho(t), \quad t \in \mathbb{K}^*, \rho = \pm 2L_i.\]

For \(r = \pm L_i \pm L_j (i \neq j), (t_1, t_2) \in \mathbb{K}^* \times \mathbb{K}^*,\) let
\[w_r(t_1, t_2) = x_r(t_1, t_2) x_{-r}(-t_1^{-1}, -t_2^{-1}) x_r(t_1, t_2);\]
for \(t \in \mathbb{K}^*\) let
\[w_r(t, 0) = x_r(t, 0) x_{-r}(-t^{-1}, 0) x_r(t, 0),\]
\[w_r(0, t) = x_r(0, t) x_{-r}(0, -t^{-1}) x_r(0, t).\]

Correspondingly, we define
\[h_\rho(t) = w_\rho(t) w_\rho(1)^{-1}, \quad t \in \mathbb{K}^*, \rho = \pm 2L_i,\]
for \(r = \pm L_i \pm L_j (i \neq j), (t_1, t_2) \in \mathbb{K}^* \times \mathbb{K}^*,\) let
\[h_r(t_1, t_2) = w_r(t_1, t_2) w_r(-1, -1);\]
for \(t \in \mathbb{K}^*\) let
\[h_r(t, 0) = w_r(t, 0) w_r(-1, 0),\]
\[h_r(0, t) = w_r(0, t) w_r(0, -1).\]

Let us write \(p(\pi)\) the permutation matrix corresponding to the permutation \(\pi,\) that is, the \(i, j\) entry of \(p(\pi)\) is 1 if \(i = \pi(j)\) and zeros otherwise. With these notations we have:
\[w_{L_i-L_j}(t_1, t_2) = p(\pi) \text{diag} \left( (-t_1^{-1})_i, (t_1)_j, (t_2)_{i+n}, (-t_2^{-1})_{j+n} \right), \quad (t_1, t_2) \in \mathbb{R}^* \times \mathbb{R}^*\]
where \(\pi\) only permutes \((i, j)\) and \((i+n, j+n)\) while fixes other numbers.
\[w_{L_i-L_j}(t_1, 0) = p(\pi) \text{diag} \left( (-t_1^{-1})_i, (t_1)_j \right), \quad t_1 \in \mathbb{R}^*\]
where \(\pi\) only permutes \((i, j)\) while fixes other numbers.
\[w_{L_i-L_j}(0, t_2) = p(\pi) \text{diag} \left( (t_2)_{i+n}, (-t_2^{-1})_{j+n} \right), \quad t_2 \in \mathbb{R}^*\]
where \(\pi\) only permutes \((i+n, j+n)\) while fixes other numbers.
\[w_{L_i+L_j}(t_1, t_2) = p(\pi) \text{diag} \left( (-t_1^{-1})_i, (-t_1^{-1})_j, (t_2)_{i+n}, (t_1)_{j+n} \right), \quad (t_1, t_2) \in \mathbb{R}^* \times \mathbb{R}^*\]
where \(\pi\) only permutes \((i, j+n)\) and \((j, i+n)\) while fixes other numbers.
\[w_{L_i+L_j}(0, t_2) = p(\pi) \text{diag} \left( (-t_2^{-1})_j, (t_2)_{i+n} \right), \quad t_2 \in \mathbb{R}^*\]
where \(\pi\) only permutes \((j, i+n)\) while fixes other numbers.
\[w_{L_i+L_j}(t_1, 0) = p(\pi) \text{diag} \left( (-t_1^{-1})_i, (t_1)_{j+n} \right), \quad t_1 \in \mathbb{R}^*\]
where \( \pi \) only permutes \((i, j + n)\) while fixes other numbers.

\[
w_{2L_i}(t) = p(\pi) \text{diag}\((-t^{-1})_i, t_{i+n}), \quad \text{for } a \in \mathbb{R}^*,
\]

where \( \pi \) only permutes \((i, i + n)\) while fixes other numbers. Let \( W_0 \) be the set composed of all permutations stated above. Then \( W_0 \) is just \( S_{2n} \), the permutation group on \( 2n \) elements.

The root system is not stable under \( W_0 \). For example, if \( w \in W_0 \) permutes \( i \) and \( j \) only, then \( w(L_i - L_j) \) is not a root for any \( \ell \leq n \). If we consider the restricted roots, that is let \( \gamma^\delta, \gamma \in \Phi, \delta = 1, 2 \) be the root restricted on root space \( f_\gamma^\delta \), then restricted root system are stable under \( W_0 \).

We denote by \( x_\gamma(\gamma \in \Phi) \) the subgroup generated by \( x_\gamma(t), t \in \mathbb{K} \) or \( t \in \mathbb{K}^2 \). We can construct the extension as was done in Section 4.2 with respect to \( \Phi \). We still get \( \tilde{G} \) and a well-defined homomorphism \( \pi_1 : \tilde{G} \to SL(2n, \mathbb{K}) \). We can also define elements \( \tilde{x}_r(t_1, t_2), \tilde{x}_\rho(t), \tilde{w}_r(t_1, t_2), \tilde{w}_\rho(t), \tilde{h}_r(t_1, t_2), \tilde{h}_\rho(t), \tilde{x}_\gamma \) etc. as was done in Section 4.2

**Remark 5.1.** Notice now we don’t know if \((\tilde{G}, \pi_1)\) is central or not, not to mention universal central or not (in fact, we can prove it is). But \( \pi_1 \) is surjective since \( x_\gamma \) and their Lie brackets generate the the whole group.

It is clear certain relations hold both in \( \tilde{G} \) and \( SL(2n, \mathbb{K}) \). To simplify notation, we write for \( f \in x_\gamma(\gamma \in \Phi) \) the corresponding element in \( \tilde{G} \) by \( \tilde{x}_\gamma(f) \). The notation coincides with the former one. We have

**Lemma 5.1.** If \( \gamma, \beta = \pm L_i \pm L_j(i \neq j), (u_1, u_2) \in \mathbb{K}^2 \setminus 0, u, v_1, v_2 \in \mathbb{K}^* \) then

\[
\begin{align*}
1 \quad &\tilde{w}_r(u_1, u_2)\tilde{x}_\rho(v)\tilde{w}_r(u)^{-1} \\
&= \tilde{x}_{w_\gamma(\beta)}(w_\gamma(u_1, u_2)x_\rho(v)w_\gamma(u_1, u_2)^{-1}). \\
2 \quad &\tilde{w}_{2L_i}(u)\tilde{x}_\rho(v)\tilde{w}_{2L_i}(u)^{-1} \\
&= \tilde{x}_{w_2L_i(\beta)}(w_{2L_i}(u)x_\rho(v)w_{2L_i}(u)^{-1}).
\end{align*}
\]

**Proof.** It is easily proved by computations using [23, p.40] and 1.10–1.12 in [5]. \( \square \)

**Lemma 5.2.** \( \tilde{w}_\gamma(t_1, t_2)(t_1, t_2 \in \mathbb{K}^*) \) with \( \gamma = \pm L_i \pm L_j(i \neq j) \) are generated by \( \tilde{w}_\beta(t, 0), \tilde{w}_\beta(0, t) \) and \( \tilde{w}_{2L_i}(t) \) where \( \beta = L_i - L_j(i < j), t \in \mathbb{K}^* \).
Proof. For \( a \in \mathbb{K}^* \), \( t_1, t_2 \in \mathbb{K}^* \), keep using Lemma 5.1 we have

\[
\tilde{w}_{L_i+L_j}(a, 0)\tilde{w}_{L_i-L_j}(t_1, t_2)\tilde{w}_{L_i+L_j}(a, 0)^{-1}
= \tilde{w}_{L_i+L_j}(a, 0)\tilde{x}_{L_i-L_j}(t_1, t_2)\tilde{x}_{L_i-L_j}(-t_1^{-1}, -t_2^{-1})\tilde{x}_{L_i-L_j}(t_1, t_2)\tilde{w}_{L_i+L_j}(a, 0)^{-1}
= (-a^{-1}t_1)\tilde{x}_{2L_j}(at_2)\tilde{x}_{2L_j}(at_1^{-1})\tilde{x}_{-2L_i}(-a^{-1}t_2^{-1})\tilde{x}_{-2L_i}(-a^{-1}t_1)\tilde{x}_{2L_i}(at_2)
= (-a^{-1}t_1)\tilde{x}_{2L_j}(at_1^{-1})\tilde{x}_{2L_i}(at_2)\tilde{x}_{-2L_i}(-a^{-1}t_2^{-1})\tilde{x}_{-2L_i}(-a^{-1}t_1)
= (-a^{-1}t_1)\tilde{x}_{2L_j}(at_1^{-1})\tilde{x}_{-2L_i}(-a^{-1}t_1)\tilde{x}_{2L_i}(at_2)
= (-a^{-1}t_1)\tilde{x}_{2L_i}(at_2).
\]

Similarly, for \( t \in \mathbb{K}^* \) keep using Lemma 5.1 we have

\[
\tilde{w}_{2L_i}(a)\tilde{w}_{L_i+L_j}(t_1, t_2)\tilde{w}_{2L_i}(a)^{-1} = \tilde{w}_{L_i-L_i}(t_2a^{-1}, -t_1a^{-1}),
\]

\[
\tilde{w}_{2L_i}(a)\tilde{w}_{L_i+L_j}(t, 0)\tilde{w}_{2L_i}(a)^{-1} = \tilde{w}_{L_i-L_i}(0, -ta^{-1}),
\]

for \( \gamma = \pm L_i \pm L_j (i \neq j) \) we have

\[
\tilde{w}_{\gamma}(t_1, t_2) = \tilde{w}_{-\gamma}(-t_1^{-1}, -t_2^{-1}), \quad \tilde{w}_{\gamma}(t, 0) = \tilde{w}_{-\gamma}(-t^{-1}, 0),
\]

\[
\tilde{w}_{\gamma}(0, t) = \tilde{w}_{-\gamma}(0, -t^{-1}), \quad \tilde{w}_{2L_i}(t) = \tilde{w}_{-2L_i}(-t^{-1}).
\]

Hence we get the conclusion. \( \square \)

Using a method similar to that in the proof of [22, 7.7], we have

Corollary 5.1. Let \( \tilde{W} \) be the subgroup of \( \tilde{G} \) generated by \( \{\tilde{w}_\gamma(u_1, u_2), \gamma \in \Phi, (u_1, u_2) \in \mathbb{K}^2 \setminus 0 \} \). Then \( \tilde{W} \) is generated by \( \tilde{w}_{L_i-L_i+1}(u, 0), \tilde{w}_{L_i-L_i+1}(0, u) \) and \( \tilde{w}_{2L_n}(u) \) where \( u \in \mathbb{K}^* \).

Lemma 5.3. For \( \gamma \in \Phi \), denote by \( \tilde{H}_\gamma \) the subgroup generated by \( \tilde{h}_\alpha(v_1, v_2), (v_1, v_2) \in \mathbb{K}^2 \setminus 0 \); \( \tilde{H}_1 \) the subgroup generated by \( \tilde{h}_\alpha(v, 0) \) and \( \tilde{H}_2 \) the subgroup generated by \( \tilde{h}_\alpha(0, v), v \in \mathbb{K}^* \). Let \( \tilde{H} \) be the subgroup generated by \( \{\tilde{H}_\gamma, \gamma \in \Phi \} \). Then

1. \( \tilde{H}^\delta, \gamma \in \Phi, \delta = 1, 2 \) is normal in \( \tilde{H} \), and \( \tilde{H} \) is normal in \( \tilde{W} \).
2. \( \tilde{H} \) normalizes each \( \tilde{x}_\gamma \), and hence \( \tilde{x}^+ \) which is generated by \( \tilde{x}_\beta (\beta \in \Phi^+) \).
3. \( \tilde{H} = \left( \prod_{\beta=L_i-L_i+1}^{\beta} (\tilde{H}_1 \tilde{H}_2)^3 \right) \cdot \tilde{H}_{2L_n} \).

Proof. The statements (i) and (ii) are clear from Lemma 5.1. For (iii) we use Corollary 5.1 and a method similar to that in the proof of [22, 7.7]. \( \square \)

An important step towards proof of Theorem 4 is
Proposition 5.1. \( \ker(\pi_1) \subseteq Z(\tilde{G}) \subseteq \tilde{H} \).

Proof. Step 1, we prove \( \tilde{W}\tilde{x}^+\tilde{W} \subseteq \tilde{x}^+\tilde{W}\tilde{x}^+ \).

Denote by \( \tilde{w}_{L_i-L_{i+1}} \) the subgroup generated by \( \tilde{w}_{L_i-L_{i+1}}(u, 0) \), \( \tilde{w}_{L_i-L_{i+1}}^2 \) the subgroup generated by \( \tilde{w}_{L_i-L_{i+1}}(0, u) \) and \( \tilde{w}_{2L_i} \) the subgroup generated by \( \tilde{w}_{2L_i}(u) \) where \( u \in \mathbb{K}^* \). Then by Corollary 5.1 it is enough to prove for any \( w \in \tilde{W} \)

\[
 w\tilde{x}^+\tilde{w}_{L_i-L_{i+1}}^\delta \subseteq \tilde{x}^+\tilde{W}\tilde{x}^+ \\
 w\tilde{x}^+\tilde{w}_{2L_i} \subseteq \tilde{x}^+\tilde{W}\tilde{x}^+, \ \delta = 1, 2.
\]

We write \( \tilde{x}^+ = \tilde{x}_{L_i-L_{i+1}}^1 \tilde{x}' \) where \( \tilde{x}' = \prod \tilde{x}_\beta \tilde{x}_{L_i-L_{i+1}}^2, \ \beta \in \Phi^+, \beta \neq L_i-L_{i+1} \).

If \( w\tilde{x}_{L_i-L_{i+1}}^1 w^{-1} \subseteq \tilde{x}^+ \), we have

\[
 w\tilde{x}^+w_{L_i-L_{i+1}}^1 = w\tilde{x}_{L_i-L_{i+1}}^1 \tilde{x}' w_{L_i-L_{i+1}}^1 \\
 \subseteq w\tilde{x}_{L_i-L_{i+1}}^1 \tilde{x}_{L_i-L_{i+1}}^1 \tilde{x}^+ \\
 \subseteq (w\tilde{x}_{L_i-L_{i+1}}^1 w^{-1})w\tilde{x}_{L_i-L_{i+1}}^1 \tilde{x}^+ \\
 \subseteq \tilde{x}^+\tilde{W}\tilde{x}^+.
\]

If \( w\tilde{x}_{L_i-L_{i+1}}^1 w^{-1} \subseteq \tilde{x}^- \), for any \( v \in \tilde{x}_{L_i-L_{i+1}}^1 \), there is \( u \in \tilde{x}_{L_i-L_{i+1}} \) such that \( vuw = \tilde{x}' \in \tilde{x}_{L_i-L_{i+1}}^1 \), and we have

\[
 wuvw_{L_i-L_{i+1}}^1 = wuv'w^{-1}u^{-1}w_{L_i-L_{i+1}}^1 \\
 \subseteq wuv'w^{-1}u^{-1}w_{L_i-L_{i+1}}^1 \tilde{x}^+ \\
 \subseteq w\tilde{x}_{L_i-L_{i+1}}^1 ww'w_{L_i-L_{i+1}}^1 \tilde{x}^+ \\
 \subseteq \tilde{x}^+ww'w_{L_i-L_{i+1}}^1 \tilde{x}^+.
\]

It follows that

\[
 w\tilde{x}^+w_{L_i-L_{i+1}}^1 \subseteq \tilde{x}^+\tilde{W}\tilde{x}^+.
\]

The proof of \( w\tilde{x}^+w_{L_i-L_{i+1}}^2 \subseteq \tilde{x}^+\tilde{W}\tilde{x}^+ \) and \( w\tilde{x}^+w_{2L_i} \subseteq \tilde{x}^+\tilde{W}\tilde{x}^+ \) follows the same manner. Thus we finished the first step.

Step 2, we prove \( \ker(\pi_1) \subseteq \tilde{H} \).

Since \( \tilde{G} \) is generated by \( \tilde{x}^+ \) and \( \tilde{W} \) and \( \tilde{x}^+ \cdot \tilde{x}^+ \subseteq \tilde{x}^+ \), by conclusion of Step 1, we have \( \tilde{G} = \tilde{x}^+\tilde{W}\tilde{x}^+ \). If \( \pi_1(x_1w'x_2) = e \), where \( x_1, x_2 \in \tilde{x}^+ \) and \( w' \in \tilde{W} \), one has \( \pi_1(w') = \pi_1(x_2^{-1}x_1^{-1}) \). Since \( \pi_1(x_2^{-1}x_1^{-1}) \) is of the following form

\[
 \begin{pmatrix}
 A_1 & A_2 \\
 0 & A_3
 \end{pmatrix},
\]

where \( A_1, A_2, A_3 \) are \( n \times n \) matrices with \( A_1 \) unipotent upper triangular and \( A_3 \) unipotent lower triangular. It follows immediately that \( \pi(w') = \)
Lemma 5.4. Let \( \pi \) be an arbitrary representation of \( G \) and \( \tilde{\pi} \) its universal central extension. If \( \pi(w') = e \), then by similar arguments given by Steinberg [23, p.31 Theorem 10], or by similar explicit calculations in [21, p.186], we can show that \( \tilde{\pi} \) is in the kernel of \( \pi_1 \).

We now consider the conditions under which \( \tilde{h} \in \tilde{H} \) is in the kernel of \( \pi_1 \).

**Lemma 5.4.**

(1) \( \ker(\pi_1) \cap \tilde{H}_r^\delta = \ker(\pi_1) \cap \tilde{H}_{L_1-L_2}^1, \quad r = \pm L_i \pm L_j (i \neq j), \delta = 1, 2, \)

(2) \( \ker(\pi_1) \cap \tilde{H}_{2L_n} = \ker(\pi_1) \cap \tilde{H}_{L_1-L_2}^1, \)

(3) \( \ker(\pi_1) = \ker(\pi_1) \cap \tilde{H}_{L_1-L_2}^1. \)

**Proof.** (1) and (2) Since these simple roots belong to the same orbit under the Weyl group, an argument similar to one in [18, Lemma 8.2], shows that \( \ker(\pi_1) \cap \tilde{H}_r^\delta \subseteq \ker(\pi_1) \cap \tilde{H}_{L_1-L_2}^1 \) for all roots \( r = \pm L_i \pm L_j (i \neq j) \) and \( \ker(\pi_1) \cap \tilde{H}_{2L_n} = \ker(\pi_1) \cap \tilde{H}_{L_1-L_2}^1. \) This proves (1) and (2).

(3) For any \( h \in \tilde{H} \), by Lemma 5.3, \( h \) can be written as

\[
\begin{align*}
  h &= h_1^1 h_2^1 \ldots h_{n-1}^1 h_2^2 \ldots h_{n-1}^2 h_0
\end{align*}
\]

where \( h_i^\delta \in \tilde{H}_{L_i-L_{i+1}}^\delta (i \leq n-1), \delta = 1, 2 \) and \( h_0 \in \tilde{H}_{2L_n}. \)

If \( \pi_1(h) = e \), notice

\[
\begin{align*}
  \pi_1(\tilde{H}_{L_i-L_{i+1}}^1) &= \text{diag} \left( a_{i, (a^{-1})_{i+1}}, a \in \mathbb{K}^* \right) \\
  \pi_1(\tilde{H}_{L_i-L_{i+1}}^2) &= \text{diag} \left( a_{i+n, (a^{-1})_{i+n+1}}, a \in \mathbb{K}^* \right),
\end{align*}
\]

we have \( \pi_1(h_i^\delta) = e \) for \( 0 \leq i \leq n-1, \delta = 1, 2 \) and \( \pi_1(h_0) = e. \) By (1) and (2) we get the conclusion. \( \square \)

For \( t_1, t_2 \in \mathbb{R}^*, \) we define:

\[
\{t_1, t_2\} = \tilde{h}_{L_1-L_2}(t_1, 0)\tilde{h}_{L_1-L_2}(t_2, 0)\tilde{h}_{L_1-L_2}(t_1 \cdot t_2, 0)^{-1}.
\]
Now in exactly the same manner as the proof in the appendix of [18], we prove that these \( \{t_1, t_2\} \)'s, \( \delta = 1, 2 \) satisfy the conditions

\[
\begin{align*}
\{t_1, t_2\} & = \{t_2, t_1\}^{-1} \quad \forall t_1, t_2 \in K^*, \\
\{t_1, t_2 \cdot t_3\} & = \{t_1, t_2\} \cdot \{t_1, t_3\} \quad \forall t_1, t_2, t_3 \in K^*, \\
\{t_1 \cdot t_2, t_3\} & = \{t_1, t_3\} \cdot \{t_2, t_3\} \quad \forall t_1, t_2, t_3 \in K^*, \\
\{t, 1 - t\} & = 1 \quad \forall t \in K^*, t \neq 1, \\
\{t, -t\} & = 1 \quad \forall t \in K^*.
\end{align*}
\]

Hence we also define a symbol on \( K^* \).

5.2. **Proof of Theorem** [4]. Notice for \( t \in K^* \)

\[
\pi_1(\tilde{h}_{L_1-L_2}(t, 0)) = \text{diag} \left( t_1, (t_1^{-1})_2 \right).
\]

Then

\[
\ker(\pi_1) \cap \tilde{H}_{L_1-L_2} = \left\{ \prod_i \tilde{h}_{L_1-L_2}(t_i, 0) \mid \text{with } \prod_i t_i = 1 \right\}.
\]

It follows that \( \ker(\pi_1) \cap \tilde{H}_{L_1-L_2} \) is generated by elements

\[
\tilde{h}_{L_1-L_2}(t_1, 0)\tilde{h}_{L_1-L_2}(t_2, 0)\tilde{h}_{L_1-L_2}(t_1t_2, 0)^{-1}, \text{ where } t_1, t_2 \in K^*.
\]

Hence it is a immediate result by (3) of Lemma 5.4.

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