1. Introduction. The phenomena of collective motions [5, 12, 14, 15, 16, 17, 33, 35, 36, 37, 38] such as flocking of birds, schooling of fish, swarming of bacteria and self-driven particles are often observed in complex biological systems. Among many models [10, 11, 37, 38] for investigating the above mentioned systems, the one introduced by Cucker and Smale [11], referred to the C-S model hereafter, has quickly attracted much attention due to the fact that a smoothly weighted communication function is introduced. Consequently, the mathematical theory on flocking can not only be obtained but also brings about the development of some subjects in other disciplines [38] such as robotics and control of unmanned vehicles [31, 32, 33], sensor networks [4], economic and sociology [11]. In the theoretical analysis of flocking, it is well-known that the coupling network topology plays an important role. For the
symmetric case [11], the flocking analysis relies heavily on the conservation of momentum, a property following from the symmetric structure of underlying network. Shen [34] was the first one to study the flocking dynamics in C-S model under an asymmetric coupling topology, in which a hierarchical leadership (HL) is introduced so that agents can be ordered in such a way that lower-rank agents are led and only led by some agents of higher-ranks. Such hierarchical leadership is further studied in [7]. A more general topology named rooted leadership (RL) was proposed by Li and Xue [26], for which a unique leader is required. Dong and Qiu [18] later proposed a general coupling network having a spanning tree for which the symmetric and asymmetric network topologies mentioned above are contained as special cases. Recently, switching directed interaction network topologies were proposed by Cucker and Dong [13]. The main difference between the network topologies proposed in [26] and [18] lies on the fact that the latter allows the network to have multiple leaders. It is worthwhile to mention that nowadays the C-S model has extended to various directions, for instance, to explore the time dependent networks [24, 25, 26], to consider the effects of stochastic noise and time delay [1, 3, 19, 29], to avoid collision [2, 8, 9, 23, 30], to examine infinitely many agents modeled as kinetic and fluid models [6, 14, 15, 16, 17, 20, 21, 32, 22].

Motivated by the remarks in [34], we imagine the following scenario. When a flock of birds is at rest, the birds first sensing the approach of an unexpected pedestrian or predators often first take flight with their own intrinsic acceleration. Such scenario highlights the need for taking into account the effect of multiple leaders and agents having their own intrinsic dynamics on flocking models. Shen [34] was the first group to address the flocking problem with non-identical agents. In particular, the author considered a flocking model with only a leader having a free-will acceleration under the HL. Moreover, the assumption that the rate of leader’s free-will acceleration tends to zero rapidly enough is given to ensure the finite flocking velocity of the group. Li and Xue [27] then studied the emergent behavior of a discrete-time C-S model under the RL with all agents having free-will accelerations, where the relatively free-will accelerations are absolutely summable. Li and Yang [28] later investigated the flocking behavior of a discrete-time C-S model under the HL with each agent having identical acceleration satisfying globally Lipschitz condition in the velocity variable. The drawback of the work of [28] is that in proving the flocking dynamics of their models, they did not show that each of the agents has a finite velocity eventually. In fact, a necessary condition for a continuous/discrete-time C-S model to have flocking agents with a finite velocity is that the accelerations of the agents must be summable/integrable.

The purpose of the paper is to investigate the flocking behavior of the discrete-time Cucker-Smale (C-S) model under general interaction network topologies with agents having their free-will accelerations. We make use of the techniques developed in [11, 28] to prove theoretically that if the free-will accelerations of agents are summable, then, for any given initial conditions, the solution achieves flocking with a finite moving speed by suitably choosing the time step as well as the communication rate of the system or the strength of the interaction between agents. Specifically, if the communication rate $\beta$ of the system is subcritical, i.e., $\beta$ is less than a critical value $\beta_c$, then flocking holds for any initial conditions regardless of the strength of the interaction between agents. While, if the communication rate is critical ($\beta = \beta_c$) or supercritical ($\beta > \beta_c$), then flocking can only be achieved by making the strength of the interaction large enough. In particular, for classical C-S model, i.e.,
agents having no free-will accelerations, for any initial conditions, flocking can still be achieved by increasing the strength of communication between agents even for communication rates being critical or supercritical. We also present some numerical simulations to support our obtained theoretical results.

The paper is organized as follows. In Section 2, we give a description of our model and provide some needed mathematical preliminaries. In Section 3, we state the main results of the paper. In Section 4, we illustrate some numerical simulations to support our obtained results in Section 3. Some concluding remarks are given in Section 5.

2. Models and mathematical preliminary. In this section, we shall introduce our flocking model based on the C-S ones [11] and provide some necessary preliminaries for obtaining our main results. We begin with giving a detail description of the coupling network topology under consideration. That is, a network has a spanning tree.

Definition 2.1 (see e.g., Example 2.1). Consider a network consisting of \( n \) vertices, labeled as \( 1, 2, \ldots, n \). Then, for any two distinct vertices \( i \) and \( j \), we give the precise definition of the following terms.

(i) vertex \( i \) can directly reach vertex \( j \) if there exists a directed edge from vertex \( i \) to vertex \( j \).

(ii) vertex \( i \) can reach vertex \( j \) (by \( K \) steps) if there exists a path (of length \( K \)) from vertex \( i \) to vertex \( j \). That is, there exists a finite sequence \( \{n_k\}_{k=0}^{K} \) with \( n_0 = i \) and \( n_K = j \) such that vertex \( n_k \) can directly reach vertex \( n_{k+1} \) for any \( k = 0, \ldots, K - 1 \). Moreover, the length \( d(i, j) \) of a shortest path from vertex \( i \) to vertex \( j \) is defined to be

\[
d(i, j) := \min\{K \in \mathbb{N} : \text{vertex } i \text{ can reach vertex } j \text{ by } K \text{ steps} \} \geq 1
\]

provided that vertex \( i \) can reach vertex \( j \).

Furthermore, we define, for each \( i = 1, 2, \ldots, n \),

(iii) \( \mathcal{L}(i) \) to the collection of vertex \( i \) and vertices that can directly reach vertex \( i \).

Denote

\[
\ell := \max_{1 \leq i \leq n} \mathcal{L}(i). \tag{1}
\]

(iv) \( \mathcal{N}(i) \) to the collection of vertices that can reach vertex \( i \).

Definition 2.2 (see e.g., [18], [39] and Example 2.1). A coupling network consisting of \( n \) vertices is said to have a spanning tree if there exists an \( i \), called a root or a leader, such that \( i \in \mathcal{N}(j) \) for any \( j \in S_i := \{1, 2, \ldots, n\} - \{i\} \). Moreover, for such the network, we define the height \( h \) of the network by

\[
h := \min_{i \in \mathcal{R}} h_i, \tag{2}
\]

where \( \mathcal{R} \) is the set of roots and

\[
h_i := \max_{j \neq i} d(i, j), \text{ for } i \in \mathcal{R},
\]

is the height of the network with respect to the root \( i \). The cardinality of the set of roots for which \( h_i = h \) is denoted by

\[
r := \#\{i \in \mathcal{R} : h_i = h\}. \tag{3}
\]
Fig. 1. The network consisting of 9 vertices has a spanning tree.
For this network, $\mathcal{R} = \{1, 2, 3\}$, $n = 9$, $\ell = 3$, $h = 3$ and $r = 2$.

**Remark 1.** For a network having a spanning tree, a vertex $i \in \mathcal{R}$ is called a leader in the sense that all other vertices can receive signals from it. Moreover, this network topology allows the number of leaders greater than 1. Hence such coupling network topology is more general than RL [26], which requires a unique leader.

**Example 2.1.** Consider a network consisting of 9 vertices with its coupling structure being illustrated in Fig. 1. Then, by Definition 2.1, $L(1) = \{1, 3\}$, $L(2) = \{1, 2\}$, $L(3) = \{2, 3\}$, $\mathcal{N}(1) = \mathcal{N}(2) = \mathcal{N}(3) = \{1, 2, 3\}$ and $\ell = 3$. Moreover, by Definition 2.2, we have that this network has a spanning tree since, for each $i = 1, 2, 3$, $i \in \mathcal{N}(j)$ for any $j \in S_{-i}$. In fact, it is easy to verify that $\mathcal{R} = \{1, 2, 3\}$, $h_1 = 4$ and $h_2 = h_3 = 3$. So $h = 3$ and $r = 2$.

We shall assume, from hereafter, that the coupling network for the flocking model under consideration has a spanning tree. We are now in a position to describe our flocking model. Given a flock of $n$ agents, labeled as $i = 1, 2, \ldots, n$, the motion of the C-S flocking model reads as follows.

$$x_i(t+1) = x_i(t) + \varepsilon v_i(t),$$

$$v_i(t+1) = v_i(t) + \varepsilon \sum_{j \in L(i)} \psi(\|x_i(t) - x_j(t)\|_\infty)(v_j(t) - v_i(t)) + \varepsilon f_i(t),$$

(4)

where

$$x_i(t) = (x_{1i}(t), x_{2i}(t), \ldots, x_{mi}(t))^\top \in \mathbb{R}^m,$$

$$v_i(t) = (v_{1i}(t), v_{2i}(t), \ldots, v_{mi}(t))^\top \in \mathbb{R}^m,$$

$$f_i(t) = (f_{1i}(t), f_{2i}(t), \ldots, f_{mi}(t))^\top \in \mathbb{R}^m,$$

(5)

are, respectively, the position, velocity and free-will/intrinsic acceleration of agent $i$ at time $t$, $t \in \mathbb{N} \cup \{0\}$, $\varepsilon$ is the time step and $\psi : [0, \infty) \rightarrow (0, \infty)$ is the weighted function given by

$$\psi(y) := \frac{\kappa}{(1+y^2)^{\beta}}.$$  

(6)

Here $\kappa > 0$ and $\beta \geq 0$ represent, respectively, the strength of the interaction and the communication rate between agents. Note that $\psi$ is a decreasing function, which indicates that the strength of the influence decreases as the group disperses.
Definition 2.3. Let \((x_i(t), v_i(t))_{i=1,...,n}\) be a solution of (4). Then we say that this solution achieve flocking provided that the following three conditions hold.

(F1) Velocity alignment: \(\lim_{t \to \infty} \|v_i(t) - v_j(t)\|_\infty = 0\) for all \(1 \leq i < j \leq n\).

(F2) Forming a group: \(\sup_{t \in \mathbb{N}} \|x_i(t) - x_j(t)\|_\infty < \infty\) for all \(1 \leq i < j \leq n\).

(F3) Having a finite velocity: \(\sup_{t \in \mathbb{N}} \|v_i(t)\|_\infty < \infty\) for all \(1 \leq i \leq n\).

Note that if \(f_i(t) \equiv c\), a nonzero constant vector, then we may get some sufficient conditions on the parameters and initial conditions so that (F1) and (F2) are satisfied. However, the speed of each agent tends to infinity as \(t\) goes to infinity.

It should also be mentioned that condition (F1) implies condition (F3) provided that model (4) has no free-will accelerations and are under the HL or RL networks.

To study the flocking model (4), we permute the state variables in the following way. Let, for \(j = 1, 2, \ldots, m\),

\[
\hat{x}_j := (x_{j1}, x_{j2}, \ldots, x_{jn})^\top \in \mathbb{R}^n, \quad \hat{v}_j := (v_{j1}, v_{j2}, \ldots, v_{jn})^\top \in \mathbb{R}^n, \quad (7)
\]

and

\[
\hat{x} := (\hat{x}_1^\top, \hat{x}_2^\top, \ldots, \hat{x}_m^\top)^\top \in \mathbb{R}^{mn}, \quad \hat{v} := (\hat{v}_1^\top, \hat{v}_2^\top, \ldots, \hat{v}_m^\top)^\top \in \mathbb{R}^{mn}. \quad (8)
\]

In fact, \(\hat{x}_j, j = 1, 2, \ldots, m\), has its \(i\)-th component equal to the \(j\)-th component of the position vector of agent \(i\). For \(j = 1, 2, \ldots, m\), \(\hat{v}_j\) is defined similarly. Then (4) canbe reformulated as

\[
\hat{x}(t + 1) = \hat{x}(t) + \hat{v}(t), \quad (9a)
\]

\[
\hat{v}(t + 1) = (I_m \otimes P_t)\hat{v}(t) + \varepsilon\hat{f}(t), \quad (9b)
\]

where \(I_m \in \mathbb{R}^{m \times m}\) is the identity matrix, \(\otimes\) is the kroncker product, \(P_t \in \mathbb{R}^{n \times n}\) is a time-depending matrix defined by

\[
P_t := D_t + \varepsilon\tilde{\Psi}_t \quad (10a)
\]

with \(D_t \in \mathbb{R}^{n \times n}\) being a diagonal matrix of its \(i\)-th diagonal element

\[
1 - \varepsilon \sum_{j \in \mathcal{L}(i)} \psi(\|x_i(t) - x_j(t)\|_\infty) = 1 - \varepsilon \sum_{j \in \mathcal{L}(i)} \psi_{ij}(t), \quad (10b)
\]

and

\[
\tilde{\Psi}_t = (\tilde{\psi}_{ij}(t)) \in \mathbb{R}^{n \times n} \text{ defined by}
\]

\[
\tilde{\psi}_{ij}(t) := \begin{cases} 
\psi_{ij}(t) & \text{if } j \in \mathcal{L}(i), \\
0 & \text{otherwise},
\end{cases} \quad (10c)
\]

and

\[
\hat{f}(t) := (f_{j1}(t), f_{j2}(t), \ldots, f_{jn}(t))^\top \in \mathbb{R}^n. \quad (11)
\]

with \(f_j(t) := (f_{j1}(t), f_{j2}(t), \ldots, f_{jn}(t))^\top \in \mathbb{R}^n\). Such rearrangement would allow us to be in a position to directly apply Propositions 1–4, stated in the following.

Notation. Let \(A\) be a matrix. We shall denote \(A \geq 0\) if \(A\) is a nonnegative matrix, and \(A \geq B\) if \(A - B \geq 0\).

Definition 2.4. Let \(A = (a_{ij}) \in \mathbb{R}^{n \times n}\) be a nonnegative matrix. Then the ergodicity coefficient of \(A\) is defined to be

\[
\chi(A) := \min_{i,j} \sum_{k=1}^n \min\{a_{ik}, a_{jk}\}.
\]
In particular, if $A$ has each of its row sums being $c$, then we define the contraction factor of $A$ to be
\[ \delta(A) := c - \chi(A). \]

**Proposition 1.** Let $A_s$ and $B_s$, $s = 1, \ldots, \tau$, be nonnegative matrices of size $n \times n$. Suppose that $A_s \leq B_s$ for all $s = 1, \ldots, \tau$. Then
\[ A_1 A_2 \cdots A_\tau \leq B_1 B_2 \cdots B_\tau \]
and
\[ \chi(A_1 A_2 \cdots A_\tau) \leq \chi(B_1 B_2 \cdots B_\tau). \]

The proof of Proposition 1 is trivial and hence omitted.

**Proposition 2** (Theorem 6.14 of [39]). Let $A \in \mathbb{R}^{n \times n}$ be a nonnegative matrix with each of its row sums being $c$. Then, for any $x = (x_1, x_2, \ldots, x_n)^\top \in \mathbb{R}^n$,
\[ \max_{1 \leq i \leq n} y_i - \min_{1 \leq i \leq n} y_i \leq \delta(A) \left( \max_{1 \leq i \leq n} x_i - \min_{1 \leq i \leq n} x_i \right), \]
where $(y_1, y_2, \ldots, y_n)^\top := Ax$.

**Proposition 3** (Theorem 6.15 of [39]). Let $A_i \in \mathbb{R}^{n \times n}, i = 1, \ldots, \tau$, be nonnegative matrices with each of their row sums being 1. Then so is $A_1 \cdots A_\tau$. Moreover,
\[ \delta(A_1 \cdots A_\tau) \left( = 1 - \chi(A_1 \cdots A_\tau) \right) \leq \delta(A_1) \cdots \delta(A_\tau). \]

**Proposition 4** (Proposition 1 of [18]). Let $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ be a nonnegative matrix with all diagonal entries being positive and suppose that the network associated with $A$ has a spanning tree. Then
\[ \chi(A^h) \geq r a_h, \]
where $a := \min\{a_{ij} : a_{ij} > 0\}$, and $h$ and $r$ are defined as in (2) and (3), respectively.

3. **Main results.** The main results of the paper are contained in this section. We begin with providing some notations and establishing the needed estimates.

**Notation.** For each vector $\tilde{w} \in \mathbb{R}^{mn}$ of the form
\[ \tilde{w} = (\tilde{w}_1, \ldots, \tilde{w}_m)^\top, \]
where $\tilde{w}_j = (w_{j1}, \ldots, w_{jn})^\top \in \mathbb{R}^n$, $j = 1, \ldots, m$,
we define the operators $\tilde{\tau}_j, j = 1, \ldots, m,$ and $\tau$ by
\[ \tilde{w}_j := \max_{1 \leq i \leq n} w_{ji} - \min_{1 \leq i \leq n} w_{ji}, \quad j = 1, \ldots, m, \]
\[ \tilde{w} := \max_{1 \leq j \leq m} \tilde{w}_j. \quad (12) \]

In fact, we shall apply operators $\tilde{\tau}_j, j = 1, \ldots, m$, and $\tau$ to $\tilde{x}, \tilde{v}$, and $\tilde{f}$, defined in (8) and (11), respectively, for our subsequent analysis. Note that $\overline{w}_j, j = 1, \ldots, m,$ denotes the largest difference between the $j$-th components of the position vectors of all agents. Similar meanings for $\overline{f}_j$ and $\overline{x}_j$.

**Lemma 3.1.** Let $\tilde{x}_j, j = 1, \ldots, m$, and $\overline{x}$ be defined as in (8) and (12), respectively. Then
\[ \max_{1 \leq A, B \leq n} \|x_A - x_B\|_\infty = \overline{x}. \quad (13) \]
Here $x_A$ and $x_B$ are defined as in (5).
Then, by Proposition 3, we have that

$$
\max_{1 \leq A,B \leq n} \|x_A - x_B\|_\infty = \max_{1 \leq A,B \leq n} \left[ \max_{1 \leq j \leq m} (x_{jA} - x_{jB}) \right] = \max_{1 \leq j \leq m} \left[ \max_{1 \leq A,B \leq n} (x_{jA} - x_{jB}) \right] = \max_{1 \leq j \leq m} \left[ \max_{1 \leq i \leq n} x_{ji} - \min_{1 \leq i \leq n} x_{ji} \right] = \max_{1 \leq j \leq m} x_j = x.
$$

We have just completed the proof of the lemma.

To prove that a solution \((x_i(t), v_i(t))_{i=1,...,n}\) of C-S models is to achieve flocking, the main step is to show that the relative positions \(|x_A(t) - x_B(t)|\) for any two agents \(A\) and \(B\) have a uniform bound that is independent of the time \(t\), or equivalently, by (13),

$$
\overline{F}_t := \max_{0 \leq s \leq t} \overline{F}(s) = \max_{0 \leq s \leq t} \max_{1 \leq A,B \leq n} \|x_A(s) - x_B(s)\|_\infty, \quad t \in \mathbb{N} \cup \{0\}, \tag{14}
$$

has a time independent upper bound. Note also that, for any \(1 \leq A, B \leq n\) and \(0 \leq s \leq t\),

$$
\psi_{AB}(s) = \psi(\|x_A(s) - x_B(s)\|_\infty) \geq \frac{\kappa}{(1 + \overline{F}_t)^d} =: \delta_t
$$

since \(\psi\) defined in (6) is a decreasing function.

We next apply a bootstrap argument to show that \(\overline{F}_t\) is bounded by a time independent quantity under certain conditions. To this end, we need the following proposition.

**Lemma 3.2.** For any \(t \in \mathbb{N}\) and \(1 \leq s \leq t\),

$$
\delta(P_{t-1} \cdot P_{t-2} \cdots P_{t-s}) \leq (1 - r(\varepsilon \delta_t)^h)^{\left\lfloor \frac{s}{h} \right\rfloor} =: (\alpha_t)^{\left\lfloor \frac{s}{h} \right\rfloor}, \tag{16}
$$

provided that

$$
\kappa < \frac{1}{\ell \varepsilon}.
$$

Here \(\ell, h, r \text{ and } \delta_t\) are defined as in (1), (2), (3) and (15), respectively, and \(\lfloor x \rceil\) denotes the largest integer that is less than or equal to \(x\).

**Proof.** Let \(t \in \mathbb{N}\) be arbitrary given. Then, by (10) and (17), we have that \(P_t \geq 0\). Moreover, for the case of \(1 \leq s < h\), since \(\frac{s}{h} = 0\), we have that \((\alpha_t)^{\left\lfloor \frac{s}{h} \right\rfloor} = 1\). It implies (16) holds. While, for the case of \(s \geq h\) (so \(t \geq h\)), represent \(s = a_1 h + a_2\), where \(a_1 = \left\lfloor \frac{s}{h} \right\rfloor\) and \(0 \leq a_2 < h\). We then group \(P_{t-1} \cdot P_{t-2} \cdots P_{t-s}\) as follows.

$$
(P_{t-1} \cdots P_{t-h}) (P_{t-h-1} \cdots P_{t-2h}) \cdots (P_{t-a_1 h-1} \cdots P_{t-a_1 h}) P_{t-a_1 h-1} \cdots P_{t-a_1 h - a_2}.
$$

Then, by Proposition 3, we have that

$$
\delta(P_{t-1} \cdot P_{t-2} \cdots P_{t-s}) \leq \prod_{i=1}^{a_1} \delta(P_{t-(i-1)h-1} \cdots P_{t-ih}) \prod_{j=1}^{a_2} \delta(P_{t-a_1 h-j}) \leq \prod_{k=1}^{a_1} \delta(P_{t-(k-1)h-1} \cdots P_{t-kh}) \tag{18}
$$

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since $P_{t-a_1h-j}$, $j = 1, \ldots, a_2$, has each of its row sums being 1 and so $\delta(P_{t-a_1h-1}) \leq 1$. Moreover, by (10) and (15), it is easy to see that, for any $\tau = 1, \ldots, s$,

$$P_{t-\tau} \geq \varepsilon M,$$

where $M = (m_{ij}) \in \mathbb{R}^{n \times n}$ is defined by

$$m_{ij} := \begin{cases} d_i & \text{if } j \in \mathcal{L}(i), \\ 0 & \text{otherwise.} \end{cases}$$

It follows, by Propositions 1 and 4, that, for $k = 1, \ldots, a_1$,

$$\delta(P_{t-(k-1)h-1} \cdots P_{t-kh}) = 1 - \chi(P_{t-(k-1)h-1} \cdots P_{t-kh}) \leq 1 - \chi(\varepsilon M)^h \leq 1 - r(\varepsilon d_i)^h.$$  \hfill (19)

The assertion (16) of the lemma now follows directly by (18) and (19).

The main results in this paper are stated as follows.

**Theorem 3.3.** Consider the discrete-time C-S flocking model (4) under a coupling network having a spanning tree. Let $\ell > 0$, $h > 0$ and $r > 0$ be parameters defined in (1), (2) and (3), respectively, and $\kappa > 0$, $\beta \geq 0$, and $\varepsilon > 0$ be parameters given in (4). For any given initial conditions $(x_i(0), v_i(0))_{i=1,\ldots,n}$, define

$$a := \overline{v}(0) + \varepsilon f_{\infty},$$

$$b := \overline{x}(0).$$  \hfill (20a)

where

$$\overline{f}_{\infty} := \max_{1 \leq j \leq m} f_{j,\infty} \text{ and } \overline{f}_{j,\infty} := \sum_{\tau=0}^{\infty} f_{j}(\tau).$$  \hfill (20b)

Suppose that the following three assumptions (A1)–(A3) hold.

(A1) Time step $\varepsilon$ satisfies the inequality $\kappa < \frac{1}{\varepsilon}$.

(A2) Each acceleration $f_i$, $i = 1, \ldots, n$, of agent $i$ satisfies

$$\left| \sum_{\tau=0}^{\infty} f_{ji}(\tau) \right| < \infty.$$  \hfill (21)

(A3) The communication rate $\beta$ and the strength $\kappa$ of the interaction satisfy either one of the following three conditions.

(A3-i) $\beta$ is subcritical, i.e., $\beta < 1/(2h)$.

(A3-ii) $\beta$ is critical, i.e., $\beta = 1/(2h)$, and

$$\kappa > \kappa_1 := \left( \frac{ha}{r^h h - 1} \right)^{\frac{1}{h}}.$$  \hfill (22)

(A3-iii) $\beta$ is supercritical, i.e., $\beta > 1/(2h)$, and

$$\kappa > \kappa_2 := \left( \frac{2h^2 a}{r^h h - 1} \right)^{\frac{1}{h}} \left( \frac{\beta h(1+b+2ca) + \sqrt{\beta^2 h^2(1+b+2ca)^2 + 2\beta h(2\beta h-1)r^2 a^2}}{2\beta h - 1} \right)^{\frac{2\beta h-1}{h}}.$$  \hfill (23)

Then the corresponding solution $(x_i(t), v_i(t))_{i=1,\ldots,n}$ achieves flocking.
Proof. Let \( t \in \mathbb{N} \) and the initial conditions \((x_i(0), v_i(0))_{i=1,\ldots,n}\) be arbitrary given. Then using (9a) and (9b), we arrive at the following. For \( 1 \leq j \leq m \),

\[
\tilde{x}_j(t) = \tilde{x}_j(0) + \varepsilon \sum_{\tau=0}^{t-1} \tilde{v}_j(\tau),
\]

\[
\tilde{v}_j(t) = P_{t-1} \ldots P_1 P_0 \tilde{v}_j(0) + \varepsilon \left[ P_{t-1} \ldots P_1 \tilde{f}_j(0) + P_{t-1} \ldots P_2 \tilde{f}_j(1) + \ldots + P_{t-1} \tilde{f}_j(t-2) + \tilde{f}_j(t-1) \right].
\]

Using condition (A1), Lemma 3.1 and Lemma 3.2 on (9b) inductively that, for \( 1 \leq \tau \leq t \),

\[
\tilde{v}_j(\tau) \leq \alpha_t^j \tilde{v}_j(0) + \varepsilon \left( \sum_{s=0}^{\tau-1} \alpha_t^j \tilde{f}_j(\tau-1-s) \right).
\]

Consequently,

\[
\sum_{\tau=0}^{t-1} \tilde{v}_j(\tau) = \sum_{\tau=0}^{t-1} \left( \alpha_t^j \tilde{v}_j(0) \right) + \varepsilon \sum_{\tau=0}^{t-1} \sum_{s=0}^{\tau-1} \left( \alpha_t^j \tilde{f}_j(\tau-1-s) \right)
\]

\[
\leq h \tilde{v}_j(0) \sum_{\tau=0}^{\infty} \alpha_t^j + \varepsilon h \left( \sum_{s=0}^{\infty} \alpha_t^j \right) \left( \sum_{\tau=0}^{\infty} \tilde{f}_j(\tau) \right)
\]

\[
= \left( \frac{h}{1-\alpha_t} \right) \tilde{v}_j(0) + \left( \frac{\varepsilon h}{1-\alpha_t} \right) \tilde{f}_{j,\infty}.
\]

Note that it is easy to see that \( \tilde{f}_{j,\infty}, j = 1,\ldots,m \), is finite by condition (A2). Hence, for any \( 1 \leq \tau \leq t \),

\[
\tilde{x}_j(\tau) \leq \tilde{x}_j(0) + \varepsilon \sum_{s=0}^{\tau-1} \tilde{v}_j(s)
\]

\[
\leq \tilde{x}_j(0) + \varepsilon h \left( \tilde{v}_j(0) + \varepsilon \tilde{f}_{j,\infty} \right),
\]

which implies directly that, via the formulas of \( d_t \) and \( \alpha_t \) given, respectively, in (15) and (16),

\[
\tilde{x}(\tau) \leq \tilde{x}(0) + \varepsilon h \left( \tilde{x}(0) + \varepsilon \tilde{f}_{\infty} \right).
\]

\[
= \tilde{x}(0) + \varepsilon h \left( 1 + \tilde{x}^2 \right)^{\beta h} \left( \tilde{v}(0) + \varepsilon \tilde{f}_{\infty} \right)
\]

\[
= b + ac \left( 1 + \tilde{x}^2 \right)^{\beta h},
\]

where \( a \) and \( b \) are defined as in (20a) and

\[
c := \frac{\varepsilon h}{r(\varepsilon k)^{\beta h}}.
\]

Because the above inequality holds for all \( 1 \leq \tau \leq t \) (and arbitrary \( t \in \mathbb{N} \)), we get that

\[
\tilde{x}_t \leq b + ac \left( 1 + \tilde{x}_t^2 \right)^{\beta h}.
\]

In the following, we shall show first that under assumption (A3), the inequality given in (27) would imply that \( \sup_{t \in \mathbb{N}} \tilde{x}_t \) is finite, and, so, condition (F2) required for
flocking holds. To this end, we introduce a transformation, which is first appeared in [11]. Set

\[ z_t := (1 + \mathbf{x}_t^2)^{\frac{1}{2}}. \]  

(28a)

It follows, from (27), that, for any \( t \in \mathbb{N} \), \( z_t \leq 1 + \mathbf{x}_t \leq (1 + b) + acz_t^{2\beta h} \), or equivalently,

\[ G(z_t) \leq 0, \]  

(28b)

where

\[ G(z) := z - acz^{2\beta h} - (1 + b). \]  

(28c)

It is clear that proving \( \sup_{t \in \mathbb{N}} \mathbf{x}_t \) is finite is equivalent to proving \( \sup_{t \in \mathbb{N}} \mathbf{z}_t \) is finite.

To prove the latter claim, we break the communication rate \( \beta \) into the following three cases: (I) \( \beta < 1/(2h) \), (II) \( \beta = 1/(2h) \), (III) \( \beta > 1/(2h) \).

**Case (I):** \( \beta < 1/(2h) \) (\( \beta \) is subcritical). In this case, it can be verified easily that \( G(z) \) has a unique positive root \( z^* > 1 \) such that \( G(z^*) < 0 \) on \( [0, z^*) \) and \( G(z) > 0 \) on \( (z^*, \infty) \). Therefore, by (28b),

\[ z_t \leq z^*, \ \forall t \in \mathbb{N}. \]

So (F2) holds.

We next show that condition (F1) required for flocking holds, or, equivalently, \( \lim_{t \to \infty} \mathbf{u}(t) = 0 \). Note that, via the formulas of \( d_t \) and \( \alpha_t \) given, respectively, in (15) and (16),

\[ \alpha_t = 1 - r(\varepsilon d_t)^h = 1 - r \left( \varepsilon \frac{\kappa}{z_t^2} \right)^h \leq 1 - r \left( \varepsilon \frac{\kappa}{z^*_2} \right)^h =: \alpha. \]  

(29)

It follows, by choosing \( \tau = t \) in (24), that

\[ \mathbf{v}_j(t) \leq \sum_{s=0}^{t-1} \alpha_{t-s} \mathbf{f}_j(t-s) \]

\[ \leq \sum_{s=0}^{t-1} \alpha_{t-s} \mathbf{f}_j(t-s) \]

\[ = \alpha_{t-s} \mathbf{f}_j(t-s) + h \left( \mathbf{f}_j(t-s) \right) \]

\[ \leq \alpha_{t-s} \mathbf{f}_j(t-s) + \sum_{s=t-(\frac{1}{2})}^{t-1} \mathbf{f}_j(s). \]

So

\[ \mathbf{v}(t) \leq \sum_{s=t-(\frac{1}{2})}^{t-1} \mathbf{f}_j(s). \]  

(30)

The first two terms on the right hand side of the above inequality converges to 0 exponentially as \( t \) tends to infinity, while the third term converges to 0 as \( t \) tends to infinity is guaranteed by (21). So (F1) holds.

Finally, we show that condition (F3), \( \sup_{t \in \mathbb{N}} \| \mathbf{u}_i(t) \|_\infty < \infty, i = 1, \ldots, n \), holds as well. Indeed, by applying the triangular inequality to (4), we have, via Theorem 3.1,
that
\[ \|v_i(t)\|_{\infty} \leq \|v_i(0)\|_{\infty} + \epsilon \sum_{s=0}^{t-1} \|v_i(s+1) - v_i(s)\|_{\infty} \]
\[ \leq \|v_i(0)\|_{\infty} + \epsilon \sum_{s=0}^{t-1} \|\sum_{j \in \mathcal{L}(i)} \psi(\|x_i(s) - x_j(s)\|_{\infty})(v_j(s) - v_i(s)) + \epsilon f_i(s)\|_{\infty} \]
\[ \leq \|v_i(0)\|_{\infty} + \epsilon \sum_{s=0}^{t-1} \|v_i(s)\|_{\infty} + \epsilon \sum_{s=0}^{t-1} \|f_i(s)\|_{\infty} \]
Thus, by (29), (24) and the similar argument derived in (25), we have that
\[ \|v_i(t)\|_{\infty} \leq \|v_i(0)\|_{\infty} + \epsilon \kappa L \left( \left( \frac{h}{1 - \alpha} \right) \|v(0)\| + \left( \frac{\epsilon h}{1 - \alpha} \right) F_{\infty} \right) + \epsilon \sum_{s=1}^{t-1} \|f_i(s)\|_{\infty} \]
Consequently, condition (F3) holds by (21). Hence, the proof of the theorem for this case has just completed.

**Case (II):** $\beta = 1/(2h)$ ($\beta$ is critical). In this case, $G(z) = (1 - ac)z - (1 + b)$. Here $ac < 1$ by condition (22). Hence, $G(z)$ has a unique positive root $z^* = \frac{1+b}{1-ac}$ such that $G(z) < 0$ on $[0, z^*)$ and $G(z) > 0$ on $(z^*, \infty)$. Therefore, by (28b), $z_t \leq z^*$ for all $t \in \mathbb{N}$. So (F2) holds. Then, by the similar arguments as given in Case (I), we have that model (4) achieves flocking.

**Case (III):** $\beta > 1/(2h)$ ($\beta$ is supercritical). In this case, it is easy to see that $G(0) < 0$, $G(1) < 0$, $\lim_{z \to \infty} G(z) = -\infty$, $G'(0) > 0$ and $G''(z) < 0$ on $[0, \infty)$. It implies that $G(z)$ has a unique critical point $z_*$ on $[0, \infty)$, which is a local maximum. In fact, by direct computation, $z_*$ satisfies $ac z_0^{2\beta h} = \frac{z_*}{2\beta h}$ and so
\[ z_* = \left( \frac{1}{2\beta hac} \right)^{\frac{1}{2\beta h - 1}}. \]
Moreover, it can be easily verified that
\[ G(z_*) > 0 \] (31)
if and only if $z_* > \frac{2\beta h(1+b)}{2\beta h - 1}$, or equivalently,
\[ \kappa > \kappa_3, \] (32)
where
\[ \kappa_3 := \left( \frac{2\beta h^2 a}{c^2 h - 1} \right)^{1/2} \left( \frac{2\beta h(1+b)}{2\beta h - 1} \right)^{2\beta h - 1}. \]
Hence the inequality in (31) is guaranteed by condition (23) since $\kappa_2 > \kappa_3$, which can be checked easily. Consequently, $G(z)$ has exactly two positive roots $z_t$ and $z_r$ satisfying
\[ 1 < z_t < z_* < z_r \] (33a)
such that
\[ G(z) \begin{cases} < 0 & \text{if } z \in (0, z_t) \cup (z_r, \infty), \\ = 0 & \text{if } z = z_t \text{ or } z_r, \\ > 0 & \text{if } z \in (z_t, z_r); \end{cases} \] (33b)
see Fig. 2 for the graph of $G(z)$. 
Next, we aim to show that \( z_t \leq z_\ell, \forall t \in \mathbb{N} \cup \{0\}. \) (34)

Indeed, for \( t = 0, \) by (28a) and (20a),
\[
z_0 = (1 + x_0^2)^{\frac{1}{2}} = (1 + b^2)^{\frac{1}{2}} < 1 + b = G(z_\ell) - G(0) \leq z_\ell,
\]
by the mean value theorem and the fact that \( G'(z) = 1 - (2\beta h a c)z^{2\beta h - 1} < 1 \) on \([0, \infty)\). To show that (34) also hold true for \( t \in \mathbb{N}, \) we assume to the contrary that \( t_0 \in \mathbb{N} \) is the first time such that
\[
z_t \begin{cases}
\leq z_\ell & \text{if } t \leq t_0 - 1, \\
> z_\ell & \text{if } t = t_0.
\end{cases} \tag{35}
\]
Then, by (28b) and (33b), we have that \( z_t \geq z_\tau. \) It follows that
\[
\mathcal{F}_{t_0}^2 - \mathcal{F}_{t_0-1}^2 = z_{t_0}^2 - z_{t_0-1}^2 \geq z_\tau^2 - z_{t_0}^2 \geq (z_\tau - z_\ell)z_\tau \\
\geq G(z_\tau)z_\tau,
\]
by the mean value theorem and the fact that \( G'(z) < 1 \) on \([0, \infty)\). On the other hand, we derive, by (24), that, for any \( t \in \mathbb{N}, \)
\[
\mathcal{V}(t) \leq \mathcal{V}(0) + \varepsilon \mathcal{F}_\infty.
\]
It implies that
\[
\mathcal{F}_{t_0} - \mathcal{F}_{t_0-1} \leq \varepsilon \mathcal{V}_{t_0-1} \leq \varepsilon (\mathcal{V}(0) + \varepsilon \mathcal{F}_\infty) = \varepsilon a.
\]
Hence,
\[
\mathcal{F}_{t_0}^2 - \mathcal{F}_{t_0-1}^2 = (\mathcal{F}_{t_0} - \mathcal{F}_{t_0-1})^2 + 2 (\mathcal{F}_{t_0} - \mathcal{F}_{t_0-1}) \mathcal{F}_{t_0-1} \\
\leq \varepsilon^2 a^2 + 2\varepsilon a (z_\tau^2 - 1)^{\frac{1}{2}} < \varepsilon^2 a^2 + 2\varepsilon az_\tau. \tag{37}
\]
It then follows from (36) and (37) that
\[
G(z_\tau)z_\tau \leq \varepsilon^2 a^2 + 2\varepsilon az_\tau,
\]
a contradiction to the assumption that \( \kappa > \kappa_2, \) which can be directly verified. Therefore, (34) and hence (F2) holds. Consequently, by the similar arguments as given in Case (I), we have that model (4) achieves flocking.

The proof of Theorem 3.3 is now complete. \( \square \)
Remark 2. (I) If the accelerations $f_i$, $i = 1, \ldots, n$, of agent $i$ satisfies

$$\mathcal{F}_i(t) \leq O(t^{-p}), \quad \text{where } p > 1.$$  

(38)

Then it is easy to see, via (30), that $v(t)$ approaches 0 at the rate no less than $O(t^{-p+1})$. While, if $f_i(t) \leq c e^{-\gamma t}$, then $v(t)$ approaches 0 exponentially. (II) It is easy to verify that $\kappa_2 > \kappa_1$ and $\lim_{2^h \to 1^+} \kappa_2 = \kappa_1$. (III) For the strength $\kappa$ of the interaction to fulfill (17) and (22) or (17) and (23), the order of magnitude with respect to the time step $\epsilon$ has to satisfy the following.

$$O\left(\frac{1}{\epsilon^{1-\kappa}}\right) < \kappa < O\left(\frac{1}{\epsilon}\right).$$

That is to say that for critical or supercritical case, the flocking assumptions can be met by choosing suitable $\epsilon$ and $\kappa$ provided that all other parameters of the system are given. In particular, our results indicate that, for any initial conditions, the corresponding solution can reach flocking by choosing suitable $\epsilon$ and $\kappa$. On the other hand, if $\kappa$ is fixed, to achieve flocking, the initial conditions need to be suitably chosen.

4. Numerical simulations. In this section, we provide some numerical simulation results for model (4) to support our main analytical theorem provided in Theorem 3.3. In the following, we assume that the coupling network under consideration is given in Fig. 1 and so $(\ell, h, r) = (3, 3, 2)$, the intrinsic accelerations $f_i$ of agent $i = 1, \ldots, 9$, are set to be $f_i(t) = \frac{\sin((t+1)^{\epsilon+1})}{(t+1)^{\epsilon+1}}(1, 1, 1) \in \mathbb{R}^3$ and the time step is chosen to be $\epsilon = 0.001$. Then we first consider three cases by picking the values of $(\beta, \kappa)$ to be $(1/10, 10)$, $(1/6, 120)$ and $(1/3, 200)$, which represent, respectively, subcritical, critical and supercritical cases, while the initial positions and velocities of agents are chosen randomly so that they satisfy $a \approx 0.8932$, $b \approx 0.8104$. Here $a, b$ are defined in (20a). Note that, by direct computations, via (22) and (23), $\kappa_1 \approx 110.2427$ and $\kappa_2 \approx 195.08032$. Consequently, from Theorem 3.3, we can expect the corresponding solution achieves flocking for these three cases. That is conditions (F1), (F2) and (F3) hold; see e.g., Figs. 3–5. Finally, we consider the case by picking $(\beta, \kappa) = (1, 0.1)$, which represent the supercritical case, while the initial positions and velocities of agents are chosen randomly so that they satisfy $a \approx 0.2872$, $b \approx 0.2850$. Then, by direct computations, $\kappa_2 \approx 282.6553 (\kappa > 0.1)$. We then see from Fig. 6 that the corresponding solution in this case does not achieve flocking. Numerical simulations also suggest that our sufficient condition (23) is not sharp.
Fig. 3. (Subcritical) Numerical simulation for model Eq. (4) with the network given in Fig. 1. This simulation result shows the solution achieves flocking. Here parameters in Eq. (4) are chosen as $\beta = 1/10$, $\kappa = 10$ and $\varepsilon = 0.001$. The initial conditions are randomly chosen and they satisfy $a \approx 0.8932$, $b \approx 0.8104$. Here $a, b$ are defined in (20a).

Fig. 4. (Critical) Numerical simulation for model Eq. (4) with the network given in Fig. 1. This simulation result shows the solution achieves flocking. Here parameters in Eq. (4) are chosen as $\beta = 1/6$, $\kappa = 120$ and $\varepsilon = 0.001$. The initial conditions are randomly chosen and they satisfy $a \approx 0.8932$, $b \approx 0.8104$. Here $a, b$ are defined in (20a).
(A) The graph of $\max_{1 \leq i \leq 9} \|x_i(t) - x_1(t)\|_{\infty}$

Fig. 5. (Supercritical) Numerical simulation for model Eq. (4) with the network given in Fig. 1. This simulation result shows the solution achieves flocking. Here parameters in Eq. (4) are chosen as $\beta = 1/3$, $\kappa = 200$ and $\varepsilon = 0.001$. The initial conditions are randomly chosen and they satisfy $a \approx 0.8932, b \approx 0.8104$. Here $a, b$ are defined in (20a).

(B) The graph of $\max_{1 \leq i \leq 9} \|v_i(t) - v_1(t)\|_{\infty}$

(C) The graph of $\|v_1(t)\|_{\infty}$

Fig. 6. Numerical simulation for model Eq. (4) under the network provided in Fig. 1. This simulation result shows the solution does not achieve flocking. Here parameters in Eq. (4) are chosen as $\beta = 1$ (supercritical), $\kappa = 0.1$ and $\varepsilon = 0.001$. The initial conditions are randomly chosen and they satisfy $a \approx 0.2872, b \approx 0.2850$. Here $a, b$ are defined in (20a). Such set of parameters and initial conditions do not satisfy the sufficient condition (23).
5. **Conclusion.** In this paper, we consider a discrete-time C-S model with intrinsic accelerations under the interaction network having a spanning tree. Then we prove analytically, under the assumption that the accelerations are summable, that the model achieves flocking with a finite moving speed by suitable choosing the time step and the communication rate of the model or the strength of the interaction between agents. Numerical simulations are also provided to support our main results.

We conclude this paper by mentioning some possible future work. It is of great interest to assume that the free-will acceleration of each agent satisfies only the Lipschitz condition, which, in turn, would create more complex motions mimicking the real world situations. It is also worthwhile to study the problem of collision free of the models.

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*E-mail address: yjhuang@math.nctu.edu.tw*  
*E-mail address: o831606.am05g@nctu.edu.tw*  
*E-mail address: jjuan@math.nctu.edu.tw*  
*E-mail address: yhliang@nuk.edu.tw*