Dirac structures and Poisson homogeneous spaces

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Abstract

Poisson homogeneous spaces for Poisson groupoids are classified in terms of Dirac structures for the corresponding Lie bialgebroids. Applications include Drinfel’d’s classification in the case of Poisson groups and a description of leaf spaces of foliations as homogeneous spaces of pair groupoids.

1 Introduction

Dirac structures on manifolds include closed 2-forms, Poisson structures, and foliations. They extend the flexibility of computations with such objects by permitting the passage to both submanifolds and quotients. The combination of these two operations is central to the theory of reduction in Poisson geometry. Dirac structures were introduced by Courant and Weinstein and
thoroughly investigated by Courant in [3]. Dorfman [5] used Dirac structures in the context of the formal calculus of variations for the study of completely integrable systems of partial differential equations.

Under a regularity assumption which is always satisfied on an open dense subset, a Dirac structure on a manifold $P$ is locally the same thing as a Poisson structure on the leaf space of a foliation of $P$.

An essential object in the theory of Dirac structures is a natural antisymmetric bracket operation (see Equation (7)) on the sections of $TP \oplus T^*P$ introduced by Courant. Although this Courant bracket does not satisfy the Jacobi identity, it does satisfy that identity on $\Gamma(E)$ when $E$ is a subbundle of $TP \oplus T^*P$ which is maximal isotropic for the symmetric form $(X_1 + \xi_1, X_2 + \xi_2)_+ = \frac{1}{2}((\xi_1, X_2) + (\xi_2, X_1))$ and whose sections are closed under the bracket. (For instance, we recover the usual bracket of vector fields on $TP \oplus 0$ and the zero bracket on $0 \oplus T^*P$.)

The theory of Dirac structures finds an echo in Drinfel’d’s theory of Lie bialgebras and Poisson homogeneous spaces [6][8]. A Lie bialgebra $(g, g^*)$ can be thought of as a pair of Lie algebra structures on a vector space $g$ and its dual having a common extension (which turns out to be unique) to a Lie algebra structure on $g \oplus g^*$ for which the symmetric form $(\ , \ )_+$ is ad-invariant. The Lie algebra $g \oplus g^*$ is called the double of the Lie bialgebra $(g, g^*)$.

The main result of [8] is that maximal isotropic subalgebras of the double correspond (modulo some details concerning closedness and connectedness of subgroups) to Poisson homogeneous $G$-spaces, where $G$ is the Poisson Lie group whose linearization is the given Lie bialgebra.

The similarities between the Courant bracket and the bracket on the double of a Lie bialgebra were explained in our recent paper [11], where both were exhibited as special cases in a theory of doubles of Lie algebroids. Since the bracket on sections of a Lie algebroid $A$ over $P$ satisfies the Jacobi identity, while the Courant bracket does not, it is clear that one must look beyond Lie algebroids to find these doubles. Hence we introduced in [11] a notion of Courant algebroid, in which the Lie algebroid axioms for a bracket on $\Gamma(A)$ and a bundle map $a : A \rightarrow P$ are satisfied only modulo certain “coboundary anomalies,” explicitly described in terms of a nondegenerate bilinear form on $E$ which is part of the Courant algebroid structure. When $P$ is a point, the anomalies vanish, and a Courant algebroid is just a Lie algebra with a nondegenerate ad-invariant symmetric bilinear form.

Although more explicit descriptions are available (see Section 2), we can define a Lie bialgebroid as a pair of Lie algebroid structures on a vector bundle $A$ and its dual having a common extension (which turns out to be unique) to a Courant algebroid structure on $A \oplus A^*$ with the symmetric form $(\ , \ )_+$. For the original Courant bracket, $A = TP$ with the usual bracket of vector fields and $A^* = T^*P$ with the zero bracket on 1-forms. When $P$ is a point, we recover the Lie bialgebroids. Lie bialgebroids arise as the linearizations of (possible local) Poisson groupoids $G$, in which the bracket on $A$ determines $G$, while the bracket on $A^*$ determines a compatible Poisson structure.

If $(A, A^*)$ is a Lie bialgebroid over $P$, we can now define an $(A, A^*)$ Dirac structure on $P$ to be a maximal isotropic subbundle of the Courant algebroid $L \subset A \oplus A^*$ which is closed under the bracket. Since the Lie algebroid anomalies are defined in terms of $(\ , \ )_+$, they vanish on $L$, which

\footnote{Dually, under a slightly different regularity assumption, it is the same as a smooth family of closed 2-forms on the leaves of a foliation of $P$ (generally different from the foliation in the first description).}
is therefore an ordinary Lie algebroid.

The main result of this paper, already announced in [11], is that there is a 1-1 correspondence between \((A, A^*)\) Dirac structures on \(P\) satisfying a certain regularity condition and Poisson homogeneous spaces of the form \(G/H\), where \(G\) is a Poisson groupoid whose tangent Lie bialgebroid is \((A, A^*)\), and \(H\) is a subgroupoid of \(G\) which is closed and wide, i.e. containing all the identity elements. (We will assume throughout this paper that all our groupoids are \(\alpha\)-connected; i.e. the fibres of the source map are connected). Drinfel’d’s theorem is the special case of this for \(P\) a point, while for ordinary (i.e. \((P, T^*P)\)) Dirac structures, we recover their description as Poisson structures on quotient manifolds of \(P\).

On the way to our main result, we develop several topics of independent interest. First of all, we extend to \((A, A^*)\) Dirac structures the original application of Dirac structures to Poisson reduction. A technical complication here is that we must deal with quotient spaces for which the projection is \textit{not} a Poisson mapping. This renders the streamlined methods of “coisotropic calculus” [19] inapplicable, and we need to do a number of computations by hand. Eventually, it will be useful to develop a modified coisotropic calculus to hand Dirac reductions directly. Second, we study pullbacks of Dirac structures under morphisms of Lie bialgebroids. Finally, we discuss the general notion of homogeneous spaces for groupoids.

Here is an outline of the paper. Section 2 is a review of basic definitions and properties of Lie bialgebroids, Courant algebroids, and Dirac structures (we will often omit the prefix \“(A, A^*)\”, which is still implied). In Section 3, we establish a correspondence between Dirac structures and Poisson structures on quotient manifolds. Using this correspondence, we characterize in Section 4 the Dirac structures which are invariant under Poisson actions of groups, and we prove Drinfel’d’s theorem by extending a \((\mathfrak{g}, \mathfrak{g}^*)\) Dirac structure to a left-invariant \((TG, T^*G)\) Dirac structure on \(G\). (Here, the bialgebroid has the nontrivial bracket on 1-forms coming from the Poisson structure on \(G\).) Section 5 contains a theorem about pullbacks of Dirac structures, which is used in Section 6 to extend \((A, A^*)\) Dirac structures on \(P\) to “left-invariant” \((TG, T^*G)\) Dirac structures on the Poisson groupoid \(G\). These invariant structures are then related to Poisson structures on quotients of \(P\). Section 7 establishes a characterization of Poisson actions of Poisson groupoids. Finally, in Section 8, we define Poisson homogeneous spaces and then use the results of Section 6 to prove our main theorem.

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\section{Dirac structures on a Lie bialgebroid}

The notion of Lie bialgebroids is a natural generalization of that of Lie bialgebras. Roughly speaking, a Lie bialgebroid is a pair of Lie algebroids \((A, A^*)\) satisfying a certain compatibility condition. Such a condition, providing a definition of \textit{Lie bialgebroid}, was given in [10]. We quote here an equivalent formulation from [10].
Definition 2.1 A Lie bialgebroid is a dual pair \((A, A^*)\) of vector bundles equipped with Lie algebroid structures such that the differential \(d_*\) on \(\Gamma(\wedge^* A)\) coming from the structure on \(A^*\) is a derivation of the Schouten-type bracket on \(\Gamma(\wedge^* A)\) obtained by extension of the structure on \(A\). Equivalently, \(d_*\) is a derivation for sections of \(A\), i.e.,

\[
d_*(X,Y) = [d_*(X), Y] + [X, d_*(Y)], \quad \forall X, Y \in \Gamma(A).
\]

(1)

For a Lie bialgebroid \((A, A^*)\), the base \(P\) inherits a natural Poisson structure:

\[
\{f, g\}_P := \langle df, d_*(g) \rangle, \quad \forall f, g \in C^\infty(P),
\]

(2)

where \(d_* : C^\infty(P) \rightarrow \Gamma(A)\) and \(d : C^\infty(P) \rightarrow \Gamma(A^*)\) are the usual differential operators associated to Lie algebroids \([10]\). It is easy to check the identity

\[
[\{f, g\}_P, d_*h] = d_*\{\{f, g\}_P, h\}, \quad \forall f, g, h \in C^\infty(P).
\]

(3)

Given a Lie bialgebroid \((A, A^*)\) over the base \(P\), with anchors \(a\) and \(a_*\) respectively, let \(E\) denote their vector bundle direct sum: \(E = A \oplus A^*\). On \(E\), there exist two natural nondegenerate bilinear forms, one symmetric and another antisymmetric, which are defined as follows:

\[
(X_1 + \xi_1, X_2 + \xi_2)_{\pm} = \frac{1}{2}(\langle \xi_1, X_2 \rangle \pm \langle \xi_2, X_1 \rangle).
\]

(4)

On \(\Gamma(E)\), we introduce a bracket by

\[
[e_1, e_2] = ([X_1, X_2] + L_{\xi_1}X_2 - L_{\xi_2}X_1 - d_*(e_1, e_2)_{\pm}) + ([\xi_1, \xi_2] + L_{X_1}\xi_2 - L_{X_2}\xi_1 + d(e_1, e_2)_{\pm}),
\]

(5)

where \(e_1 = X_1 + \xi_1\) and \(e_2 = X_2 + \xi_2\).

Finally, we let \(\rho : E \rightarrow TP\) be the bundle map defined by \(\rho = a + a_*\). That is,

\[
\rho(X + \xi) = a(X) + a_*(\xi), \quad \forall X \in \Gamma(A)\) and \(\xi \in \Gamma(A^*)\).
\]

(6)

When \((A, A^*)\) is a Lie bialgebra \((g, g^*)\), the bracket above reduces to the famous Lie bracket of Manin on the double \(g \oplus g^*\). On the other hand, if \(A\) is the tangent bundle Lie algebroid \(TM\) and \(A^* = T^*M\) with zero bracket, then Equation (5) takes the form:

\[
[X_1 + \xi_1, X_2 + \xi_2] = [X_1, X_2] + \{L_{X_1}\xi_2 - L_{X_2}\xi_1 + d(e_1, e_2)_{\pm}\}.
\]

(7)

This is the bracket first introduced by Courant \([3]\), then generalized to the context of the formal variational calculus by Dorfman \([3]\). In general, \(E\) together with this bracket and the bundle map \(\rho\) satisfies certain properties as outlined in the following:

**Proposition 2.2** \([11]\) Given a Lie bialgebroid \((A, A^*)\), let \(E = A \oplus A^*\). Then \(E\), with the nondegenerate symmetric bilinear form \((\cdot, \cdot)_{\pm}\), the skew-symmetric bracket \([\cdot, \cdot]\) on \(\Gamma(E)\) and the bundle map \(\rho : E \rightarrow TP\) as introduced above, satisfies the following properties:
(i). For any \( e_1, e_2, e_3 \in \Gamma(E) \), \( \left[[e_1, e_2], e_3\right] + c.p. = DT(e_1, e_2, e_3) \);

(ii). for any \( e_1, e_2 \in \Gamma(E) \), \( \rho[e_1, e_2] = [\rho e_1, \rho e_2] \);

(iii). for any \( e_1, e_2, f \in \mathcal{C}^\infty(P) \), \( [e_1, fe_2] = f[e_1, e_2] + (\rho(e_1)f)e_2 - (e_1, e_2)Df; \)

(iv). \( \rho : \mathcal{D} = 0 \), i.e., for any \( f, g \in \mathcal{C}^\infty(P) \), \( (\mathcal{D}f, \mathcal{D}g) = 0 \);

(v). for any \( e, h_1, h_2 \in \Gamma(E) \), \( \rho(e)(h_1, h_2) = \left( [e, h_1] + \mathcal{D}(e, h_1), h_2 \right) + \left( h_1, [e, h_2] + \mathcal{D}(e, h_2) \right) \),

where

\[
T(e_1, e_2, e_3) = \frac{1}{3}([e_1, e_2], e_3) + c.p.,
\]

and \( \mathcal{D} : \mathcal{C}^\infty(P) \rightarrow \Gamma(E) \) is the map: \( \mathcal{D} = d^* + d \).

Objects satisfying the above properties are called Courant algebroids in [11]. In other words, we have:

**Theorem 2.3** If \( (A, A^*) \) is a Lie bialgebroid, then \( E = A \oplus A^* \) together with \( ([\cdot, \cdot], \rho, (\cdot, \cdot)_+) \) is a Courant algebroid.

In this case, \( E \) is called the double of the Lie bialgebroid.

**Definition 2.4** Let \( E = A \oplus A^* \) be the double of a Lie bialgebroid \( (A, A^*) \). A subbundle \( L \) of \( E \) is called isotropic if it is isotropic under the symmetric bilinear form \( (\cdot, \cdot)_+ \). It is called integrable if \( \Gamma(L) \) is closed under the bracket \( [\cdot, \cdot] \). A Dirac structure, or Dirac subbundle, of the Lie bialgebroid \( (A, A^*) \) is a subbundle \( L \subset E \) which is maximally isotropic and integrable.

The following proposition follows immediately from the definition of Dirac structures, and Properties (i)-(v) of Proposition 2.2.

**Proposition 2.5** Suppose that \( L \) is an integrable isotropic subbundle of a Courant algebroid \( (E, \rho, [\cdot, \cdot], (\cdot, \cdot)_+) \). Then \( (L, \rho|_L, [\cdot, \cdot]) \) is a Lie algebroid. In particular, any Dirac subbundle itself is a Lie algebroid.

### 3 Dirac structures and Poisson reduction

Suppose that \( (A, A^*) \) is a Lie bialgebroid over the base manifold \( P \), with anchors \( a \) and \( a^* \) respectively. Let \( E = A \oplus A^* \) denote its double, and let \( L \subset E \) be a Dirac subbundle. Clearly, \( L \cap A \) is a (singular) subalgebroid of \( A \), and therefore \( \mathcal{D} = a(L \cap A) \) is an integrable (singular) distribution on \( P \). We call \( \mathcal{D} \) the characteristic distribution of \( L \). Let \( \mathcal{F} \) denote the corresponding (generally singular) foliation of \( P \).
Definition 3.1 A Dirac subbundle $L \subset E$ is called reducible if its characteristic distribution $\mathcal{D}$ induces a simple foliation. Here by a simple foliation, we mean a regular foliation $\mathcal{F}$ such that $P/\mathcal{F}$ is a nice manifold such that the projection is a submersion.

A function $f \in C^\infty(P)$ is called $L$-admissible if there is a section in $\Gamma(A)$, denoted by $Y_f$ ($Y_f$ may not be unique), such that $Y_f + df \in \Gamma(L)$. We write $C^\infty_L(P)$ for the set of all $L$-admissible functions.

Let $L \subset E$ be a reducible Dirac structure. Then $f$ is $L$-admissible iff $f$ is constant along $\mathcal{F}$, i.e.,

$$C^\infty_L(P) \cong C^\infty(P/\mathcal{F}).$$

For any $f, g \in C^\infty_L(P)$ admissible, define a bracket by

$$\{f, g\} = \rho(\varepsilon_f)g,$$

where $\varepsilon_f = Y_f + df$, which is unique up to a section of $L \cap A$.

Theorem 3.2 Suppose that $L$ is a reducible Dirac structure. The bracket (9) defines a Poisson structure on $C^\infty(P/\mathcal{F})$.

Proof. From the definition, we have

$$\{f, g\} = \langle Y_f, dg \rangle + \langle df, d^*g \rangle, \forall f, g \in C^\infty_L(P).$$

On the right hand side, the first term is skew-symmetric since $L$ is isotropic. The second term is just the Poisson bracket on $C^\infty(P)$ as defined by Equation (2). Hence, $\{\cdot, \cdot\}$ is skew-symmetric.

Next, we prove that $C^\infty_L(P)$ is closed under this bracket and the Jacobi identity holds.

Let $[e_f, e_g]^*$ denote the component of $[e_f, e_g]$ on $\Gamma(A^*)$. According to Equation (3),

$$[e_f, e_g]^* = [df, dg] + L_{Y_f}dg - L_{Y_g}df + d(\varepsilon_f, e_g)_-\varepsilon_g - d(\varepsilon_f, e_g)_+\varepsilon_g$$

$${\varepsilon_f, e_g}^* = d\{f, g\} + d < Y_f, dg >$$

This means that $\{f, g\}$ is also $L$-admissible and one can take $[e_f, e_g]$ as $e_{\{f,g\}}$. It follows that

$$\{\{f, g\}, h\} = \rho(\varepsilon_{\{f,g\}})h = \rho([e_f, e_g])h = [\rho(e_f), \rho(e_g)]h = \{f, \{g, h\}\} - \{g, \{f, h\}\}.$$

That is, $\{\cdot, \cdot\}$ defines a Poisson structure on $P/\mathcal{F}$.

Q.E.D.

In what follows, we apply the result above to a special class of Lie bialgebroids: Lie bialgebroids of Poisson manifolds. Moreover, we will prove that in this case Dirac structures, roughly speaking, are in one-one correspondence with Poisson structures on quotient spaces of the Poisson manifold.
Given a Poisson manifold \((P, \pi)\), its cotangent bundle \(T^*P\) inherits a natural Lie algebroid structure, called the cotangent Lie algebroid of the Poisson manifold \(P\) \([4]\). On the other hand, the tangent bundle \(TP\) is a Lie algebroid in an evident sense. It is known that they constitute a Lie bialgebroid \([16]\). For simplicity, we will use \((TP, T^*P; \pi)\) to denote this Lie bialgebroid.

As a special case of Theorem 3.2, therefore, any reducible Dirac structure \(L\) on its double \(TP \oplus T^*P\) will induce a Poisson structure on the quotient space \(P/\mathcal{F}\).

Conversely, suppose that \(\mathcal{D}\) is an integrable distribution on \(P\) with foliation \(\mathcal{F}\), which is simple. Assume that \(M = P/\mathcal{F}\) has a Poisson structure. Let \(J : P \rightarrow M\) denote the natural projection.

To keep under control the fact that \(J\) is not a Poisson map, we define a “difference” bracket \(\{f, g\}_1\) on \(C^\infty(M) \times C^\infty(M)\) by:

\[
\{f, g\}_1 = J^*\{f, g\} - \{J^*f, J^*g\}_P, \quad \forall f, g \in C^\infty(M).
\]  

(10)

It is easy to see, by using the Leibniz identity of Poisson brackets, that this bracket defines a skew-symmetric bilinear form on the conormal bundle \(\mathcal{D}^\perp\), which in turn induces a bundle map

\[
\Lambda : \mathcal{D}^\perp \rightarrow TP/\mathcal{D}.
\]  

(11)

Let \(pr : TP \rightarrow TP/\mathcal{D}\) be the natural projection.

Define a subbundle \(L \subset TP \oplus T^*P\) by

\[
L = \{ (\nu, \xi) | pr(\nu) = \Lambda \xi, \forall \nu \in TP, \xi \in \mathcal{D}^\perp \}.
\]  

(12)

It is clear that \(L\) is a maximal isotropic subbundle of \(TP \oplus T^*P\), and \(C^\infty_L(P) \cong C^\infty(M)\). For any \(f \in C^\infty_L(P)\), it is easy to see, from definition, that there exists a vector field \(Y_f \in \mathcal{X}(P)\) such that \(Y_f + df \in \Gamma(L)\). And in fact, \(\Gamma(L)\) is spanned by all those sections of the form \(g(Y_f + df)\), for \(f \in C^\infty_L(P)\) and \(g \in C^\infty(P)\). To prove that \(L\) is integrable, it suffices to show that the bracket is closed for those sections having the form \(Y_f + df\) according to Property (iii) of Proposition 2.2, since \(L\) is isotropic.

Given any \(f\) and \(g\) in \(C^\infty_L(P)\). Let \(e_f = Y_f + df\) be a section in \(\Gamma(L)\). Similarly, let \(e_g = Y_g + dg\) and \(e_{\{f,g\}} = Y_{\{f,g\}} + d\{f, g\} \in \Gamma(L)\). It is easy to check that

\[
\rho(e_f)g = \{f, g\}.
\]

By virtue of the Jacobi identity, we have

\[
\rho([e_f, e_g] - e_{\{f,g\}})h = 0, \quad \forall f, g, h \in C^\infty_L(P).
\]  

(13)

Since the component of \([e_f, e_g]\) on \(\Gamma(T^*P)\) is \(d\{f, g\}\) according to the proof of Theorem 3.2, \([e_f, e_g] - e_{\{f,g\}}\) is a section in \(\Gamma(TP)\). Thus Equation (13) implies that

\[
[e_f, e_g] - e_{\{f,g\}} \in \Gamma(L \cap TP) \subset \Gamma(L).
\]

Hence, \([e_f, e_g] \in \Gamma(L)\), and therefore \(L\) is integrable.

This proves the following
Theorem 3.3 Suppose that $P$ is a Poisson manifold. There is a one-one correspondence between reducible Dirac structures in the double $E = TP \oplus T^*P$ and Poisson structures on a quotient space $P/F$.

Remark (1). This is a generalization of a result of Courant [3]. As a main motivation for the introduction of Dirac structures, Courant proved that a Dirac structure on $TP \oplus T^*P$, when $P$ is equipped with the zero Poisson structure, induces a Poisson bracket on a quotient space. In fact, the first part of Theorem 3.3 can be obtained by reducing to the zero Poisson case. This can be seen as follows. If $P$ has a non-trivial Poisson structure, the double $TP \oplus T^*P$, as a Courant algebroid, is still isomorphic to the double studied by Courant. As a consequence, any Dirac structure will thus induce a Poisson structure on the quotient. However, the converse seems to be new.

(2). If the Poisson structure on the quotient $P/F$ is induced from that on $P$, i.e., the projection $J$ is a Poisson map, then it is simple to see that $L = D \oplus D^\perp$. This is called a null Dirac structure (see [11]). Thus, we have proved: a foliation $F$ on a Poisson manifold $P$ is compatible with the Poisson structure iff $L = D \oplus D^\perp$ is a Dirac subbundle of $E = TP \oplus T^*P$.

(3). The result is even interesting when $D = 0$. In this case, a Poisson structure on the quotient is simply another Poisson structure $\pi_1$ on $P$. Then, $L$ is simply the graph of the bundle map $T^*P \to TP$ induced from the bivector field $\pi_1 - \pi$.

4 Invariant Dirac structures

This section is devoted to the study of invariant Dirac structures of a Poisson group. As an application, we will give a new proof for the Drinfel’d theorem on homogeneous spaces.

Lemma 4.1 Let $P$ be a Poisson manifold with a Lie group $G$: action: $\{\varphi_k\}_{k \in G}$. Write $\varphi_k(x) = kx$ for $x \in P$. Suppose that $L \subset TP \oplus T^*P$ is a reducible Dirac subbundle. Then, $L$ is $G$-invariant iff both the characteristic distribution $D$ and the difference bracket $\{\cdot, \cdot\}_1$ defined by Equation (10) are $G$-invariant.

Proof. Firstly, Suppose that $L$ is $G$-invariant. Then, $D = L \cap TP$ is clearly $G$-invariant. For any $k \in G$ and $Y_f + df \in \Gamma(L)$,

$$(\varphi^{-1}_k \varphi_k^*)(Y_f + df) = \varphi^{-1}_k(Y_f) + d(\varphi_k^*f) \in \Gamma(L).$$

This means that $\varphi_k^*f$ is also $L$-admissible and we may take $Y_{\varphi_k^*f} = \varphi^{-1}_k(Y_f)$. Hence,

$$\{\varphi_k^*f, \varphi_k^*g\}(x) = \{Y_{\varphi_k^*f}, \varphi_k^*(dg(kx))\} = \{Y_f(kx), \varphi_k^*(dg(kx))\} = \{f, g\}_1(kx) = \varphi_k^*\{f, g\}_1(x).$$

Conversely, from the assumption, it is easy to see that the bundle map $\Lambda$ as defined by Equation (11) is $G$-invariant. Thus $L$ is $G$-invariant according to Equation (12).
Remark. We note that in general the group action does not preserve the bracket on the double $TP \oplus T^*P$ unless it preserves the Poisson structure on $P$. However, as we shall see below, in most interesting cases, we need to study a Poisson group action, which does not preserve the Poisson structure.

**Theorem 4.2** With the notation above, suppose that $G$ is a Poisson group and $P$ is a Poisson $G$-space. Then the following statements are equivalent:

(i). $L$ is $G$-invariant.

(ii). The $G$-action can be reduced to the quotient space $P/F$ such that the reduced action is also a Poisson action.

**Proof.** By definition, $\{\cdot, \cdot\}_1$ is $G$-invariant iff

$$
\varphi^*_k \{f, g\}_P - \{\varphi_k f, \varphi_k g\}_P = \varphi^*_k \{f, g\}_P - \{\varphi_k^* f, \varphi_k^* g\}, \quad \forall f, g \in C^\infty(P/F).
$$

(14)

Recall that, for a Poisson Lie group $(G, \pi_G)$, a Poisson manifold $P$ with a $G$-action is a Poisson $G$-space iff the following equality:

$$
\varphi_k^* \{f, g\}_P(x) - \{\varphi_k f, \varphi_k g\}_P(x) = \{f_x, g_x\}_G(k) \quad \text{(15)}
$$

holds for all $k \in G, x \in P$, and $f, g \in C^\infty(P)$, where the function $f_x \in C^\infty(G)$ is defined by $f_x(k) = f(kx)$.

When $L$ is $G$-invariant, its characteristic foliation $F$ is also $G$-invariant. Hence, the action can be reduced to $P/F$. Moreover, combining Equations (14) and (15), we get

$$
\varphi^*_k \{f, g\}(x) - \{\varphi_k^* f, \varphi_k^* g\}(x) = \{f_x, g_x\}_G(k), \forall k \in G, x \in P, \text{ and } f, g \in C^\infty_L(P). \quad (16)
$$

This means that $P/F$ is a Poisson $G$-space.

Conversely, if both $P$ and $P/F$ are Poisson $G$-spaces, then Equations (15) and (16) imply Equation (14), which is equivalent to $L$ being $G$-invariant.

Q.E.D.

Now, by means of the theory of Dirac structures, we are in a position to explain the Drinfel’d’s more or less mysterious theorem regarding Poisson homogeneous spaces, outlined in his short paper [8] (also see [12] for the interpretation of the associated Dirac structures of Poisson homogeneous spaces in terms of Lie algebroids).

To a Poisson Lie group $(G, \pi_G)$, there are associated two Lie bialgebroids. One is $(TG, T^*G; \pi_G)$ with the canonical Lie bialgebroid structure induced from the Poisson structure on $G$, and the other is its tangent Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$. Identifying $TG \oplus T^*G$ with the trivial vector bundle $G \times (\mathfrak{g} \oplus \mathfrak{g}^*)$ by left translations, it is clear that there is a 1-1 correspondence between maximal isotropic subspaces $L$ of $\mathfrak{g} \oplus \mathfrak{g}^*$ and left invariant maximal isotropic subbundles $\bar{L}$ of $TG \oplus T^*G$. Moreover, we have

Q.E.D.
Lemma 4.3 Given any \( e_1, e_2 \in g \oplus g^* \), let \( \bar{e}_1 \) and \( \bar{e}_2 \) be their corresponding left invariant sections in \( TG \oplus T^*G \). Then,
\[
[\bar{e}_1, \bar{e}_2] = [e_1, e_2]^-. 
\]

Proof. Assume that \( e_1 = X + \xi, e_2 = Y + \eta \in g \oplus g^* \), and \( \bar{e}_1 = \bar{X} + \bar{\xi}, \bar{e}_2 = \bar{Y} + \bar{\eta} \). Here \( \bar{X}, \bar{Y} \) and \( \bar{\xi}, \bar{\eta} \) denote the corresponding left invariant vector fields and 1-forms on \( G \) for \( X, Y \in g \) and \( \xi, \eta \in g^* \) respectively.

From the fact that \( [\bar{X}, \bar{Y}] = [X, Y]^− \) and \( [\bar{\xi}, \bar{\eta}] = [\xi, \eta]^− \) (see [20]), it follows that
\[
L_{\bar{X}} \bar{\xi} = (ad^*_X \xi)^-, L_{\bar{\xi}} \bar{X} = (ad^*_\xi X)^-. 
\]
This means that
\[
[\bar{X}, \bar{\xi}] = L_{\bar{X}} \bar{\xi} - L_{\bar{\xi}} \bar{X} + \frac{1}{2}(d^* - d) < \bar{X}, \bar{\xi}> 
= (ad^*_X \xi)^- - (ad^*_\xi X)^- 
= [X, \xi]^-.
\]
Therefore,
\[
[\bar{e}_1, \bar{e}_2] = [\bar{X}, \bar{Y}] + [\bar{X}, \bar{\eta}] + [\bar{\xi}, \bar{Y}] + [\bar{\xi}, \bar{\eta}] 
= [X, Y]^− + [X, \eta]^− + [\xi, Y]^− + [\xi, \eta]^− 
= [e_1, e_2]^−.
\]

Q.E.D.

An immediate consequence is the following:

Corollary 4.4 As above, \( L \) is a subalgebra iff \( \bar{L} \) is integrable. That is, Dirac structures of \( g \oplus g^* \) are in 1-1 correspondence with left invariant Dirac structures of \( TG \oplus T^*G \).

Proof. Sections of \( \bar{L} \) are spanned by those of the form \( f\bar{e} \), where \( f \in C^\infty(G) \). The conclusion thus follows immediately from Lemma 4.3 by using Property (iii) of Proposition 2.2.

Q.E.D.

Given a left invariant \( TG \oplus T^*G \) Dirac structure \( \bar{L} \) (for the bialgebroid associated with the Poisson structure on \( G \)), its characteristic distribution \( \bar{L} \cap TG \) is just the left translation of the subalgebra \( \mathfrak{h} = L \cap g \) of \( g \). So its quotient space is \( G/H \), where \( H \) is the connected subgroup of \( G \) with Lie algebra \( \mathfrak{h} \). On the other hand, it is well known that \( G/H \) is a nice manifold such that the projection is a submersion iff \( H \) is closed. In this situation, we call \( L \) a regular \((g, g^*)\) Dirac structure. In other words, \( L \) is regular if the left translation of \( \mathfrak{h} = L \cap g \) defines a simple foliation on \( G \). It is simple to see that \( L \) is regular iff \( \bar{L} \) is reducible. Thus, by Theorem 4.2, we obtain the following:
Theorem 4.5  Regular Dirac structures of $\mathfrak{g} \oplus \mathfrak{g}^*$ are in 1-1 correspondence with Poisson homogeneous spaces $G/H$, where $H$ is a connected closed subgroup of $G$.

Every homogeneous $G$ space $X$ is of the form $X = G/\tilde{H}$, where $\tilde{H}$ is a closed subgroup of $G$, with $H$ being its connected component at the unit. Then $D = \tilde{H}/H$ is a discrete group, and the projection $p : G/H \to X$ is a covering map with structure group $D$. Any Poisson structure on $X$ can be pulled back to $G/H$ such that $p$ is a Poisson map. Moreover, if one is a Poisson homogeneous space, so is the other. Thus, any Poisson homogeneous $G$-space $X = G/\tilde{H}$ induces a regular Dirac structure $L$ in $\mathfrak{g} \oplus \mathfrak{g}^*$. It is easy to see that $L$ is $Ad_{\tilde{H}}$-invariant.

Conversely, given a regular Dirac structure $L$ of $\mathfrak{g} \oplus \mathfrak{g}^*$, let $G/H$ be its corresponding Poisson homogeneous $G$-space. Then, the Poisson structure on $G/H$ can be reduced to a homogeneous $G$-space $X = G/\tilde{H} = (G/H)/D$ iff $L$ is $Ad_D$-invariant, or equivalently, is $Ad_{\tilde{H}}$-invariant.

Thus, we obtain the following:

Theorem 4.6 [8] Poisson homogeneous $G$-spaces bijectively correspond to pairs $(L,K)$, where $L$ is a regular Dirac structure of $\mathfrak{g} \oplus \mathfrak{g}^*$ and $K$ is a closed subgroup of $G$ with Lie algebra $L \cap \mathfrak{g}$ such that $L$ is invariant under the (adjoint,coadjoint) action of $K$.

5 Pullback of Dirac structures

This section is devoted to a discussion of pullbacks of Dirac structures. It will be used in Section 6 to extend $(A,A^*)$ Dirac structures on $P$ to “left-invariant” $(TG,T^*G)$ Dirac structures on the Poisson groupoid $G$. These invariant structures are then related to Poisson structures on quotients of $G$.

Given two vector spaces $U$, $V$, and a surjective linear map $\Phi : U \to V$, its dual $\Phi^* : V^* \to U^*$ is injective and $\Phi^*(V^*) = (\ker \Phi)^\perp$. Write

$$\Phi = \Phi \oplus (\Phi^*)^{-1} : U \oplus (\ker \Phi)^\perp \to V \oplus V^*.$$  

Clearly, $\Phi$ is a surjective linear map. Given any maximal isotropic subspace $L \subset V \oplus V^*$, we denote by $\tilde{L}$ the inverse image $\Phi^{-1}(L)$. Then, $\tilde{L}$ is a maximal isotropic subspace of $U \oplus U^*$, which is called the pullback of $L$.

Similarly suppose that $A \to P$ and $B \to Q$ are vector bundles, and $\Phi : A \to B$ is a surjective bundle map covering a map $P \to Q$. Then, given any maximal isotropic subbundle $L \subset B \oplus B^*$, we may define its pullback $\tilde{L} \subset A \oplus A^*$ as the fiberwise pullback. Then, $\tilde{L}$ is a maximal isotropic subbundle, and the restriction of $\Phi$ on $\tilde{L}$, denoted by the same symbol $\tilde{\Phi}$, is a vector bundle morphism $\tilde{L} \to L$.

Given a vector bundle morphism $\rho : E_1 \to E_2$, a section $\tilde{X}$ of $E_1$ will be called admissible if $\rho \tilde{X}$ corresponds to a section $X$ in $E_2$. In this case, $\tilde{X}$ and $X$ are said to be $\rho$-related. If $\rho$ is surjective, then $\Gamma(E_1)$ is spanned over $\mathbb{R}$ (possibly infinite sum but locally finite) by all sections of the form $f \tilde{X}$ for $f \in C^\infty(P)$ and $\tilde{X} \in \Gamma(E_1)$ admissible, by the partition of unity.

Applying the observation above to the bundle morphism $\tilde{\Phi} : \tilde{L} \to L$, it follows that

$$\Gamma(\tilde{L}) = \text{span}\{f\tilde{e} | \forall \text{ admissible } \tilde{e} \in \Gamma(\tilde{L}) \text{ and } f \in C^\infty(P)\}.  \tag{17}$$
Theorem 5.1 Let \((A,A^*),(B,B^*)\) be two Lie bialgebroids, and \(\Phi : A \to B\) a surjective bundle map. Thus the bundle map \(\bar{\Phi} = \Phi \oplus (\Phi^*)^{-1} : A \oplus (\ker\Phi)^\perp \to B \oplus B^*\) is surjective.

Given any maximal isotropic subbundle \(L \subset B \oplus B^*\), its pull back \(\bar{L} = \bar{\Phi}^{-1}(L)\) is a Dirac structure of \(A \oplus A^*\) iff \(L\) is Dirac structure. Moreover, in this case, \(\bar{\Phi} : \bar{L} \to L\) is a Lie algebroid morphism.

Proof. Since \(\Gamma(\bar{L})\) is spanned by sections of the form \(fe\), it suffices to prove the following identity:

\[
\bar{\Phi}[\bar{e}_1,\bar{e}_2] = [e_1,e_2], \quad \forall \ \text{admissible} \ \bar{e}_1, \ \bar{e}_2 \in \Gamma(\bar{L}),
\] (18)

according to Property (iii) of Proposition 2.2.

Write \(\bar{e}_1 = \bar{X} + \bar{\xi}\) and \(\bar{e}_2 = \bar{Y} + \bar{\eta}\), where \(\bar{X}\) and \(\bar{Y}\) are admissible sections of \(A\) under the map \(\Phi\), and \(\bar{\xi}\) and \(\bar{\eta}\) are admissible sections of \((\ker\Phi)^\perp\) under the map \((\Phi^*)^{-1}\). Denote by \(X, Y\), and \(\xi, \eta\) their corresponding sections in \(B\) and \(B^*\), respectively.

Since \(\Phi\) is a Lie algebroid morphism, by definition (see [9]),

\[
\Phi[X,Y] = [X,Y].\] (19)

Moreover, \(\Phi\) is a Poisson map, where \(A\) and \(B\) are equipped with the Lie-Poisson structures corresponding to the Lie algebroids \(A^*\) and \(B^*\) respectively, since \(\Phi\) is a Lie bialgebroid morphism. Thus,

\[
\Phi^*[l_\xi,l_\eta] = \{\Phi^*l_\xi, \ \Phi^*l_\eta\},
\]

where \(l_\xi\) and \(l_\eta\) are the linear functions on \(B\) corresponding to \(\xi, \ \eta \in \Gamma(B^*)\). Therefore,

\[
\Phi^*[l_{[\xi,\eta]}] = \{l_{[\xi,\eta]}\} = l_{[\bar{\xi},\bar{\eta}]}.
\]

Thus it follows that

\[
(\Phi^*)^{-1}[\xi,\eta] = [\xi,\eta].\] (20)

That is, \((\Phi^*)^{-1} : (\ker\Phi)^\perp \to B^*\) is also a Lie algebroid morphism, where \((\ker\Phi)^\perp\) is considered as a subalgebroid of \(A^*\). Now

\[
[\bar{e}_1,\bar{e}_2] = [\bar{X},\bar{Y}] + [\bar{X},\bar{\eta}] + [\bar{\xi},\bar{Y}] + [\bar{\xi},\bar{\eta}].
\]

According to Equation 18,

\[
[\bar{X},\bar{\eta}] = L_X\bar{\eta} - L_{\bar{\eta}}\bar{X} + \frac{1}{2}(d^* - d) < \bar{X},\bar{\eta} >= i_Xd\bar{\eta} - d_\eta d_\tau X - \frac{1}{2}(d_\tau - d) < \bar{X},\bar{\eta} > .
\]

Since \(\Phi\) is a Lie algebroid morphism, \(d\bar{\eta}\) is admissible and is \((\Phi^*)^{-1}\)-related to \(d\eta\). Similarly, \(d_\tau X\) is \(\Phi\)-related to \(d_\tau X\). Finally, note that \(< \bar{X},\bar{\eta} >= \varphi^* < X,\eta >\), so \(d_\tau < \bar{X},\bar{\eta} >\) and \(d_\tau < X,\eta >\) are \(\Phi\)-related while \(d < \bar{X},\bar{\eta} >\) and \(d < X,\eta >\) are \((\Phi^*)^{-1}\)-related. Hence,

\[
\bar{\Phi}[\bar{X},\bar{\eta}] = [X,\eta].
\]

Similarly, \(\bar{\Phi}[:\bar{\xi},\bar{Y}] = [\xi,\eta]\). Hence, Equality (18) follows. This concludes the proof of the theorem.

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Example 5.2 Recall that a hamiltonian operator on a Lie bialgebroid \((B,B^*)\) is a skew-symmetric two-form in \(\Gamma(\wedge^2B^*)\) satisfying the following Maurer-Cartan type equation [11]:
\[
dI + \frac{1}{2}[I,I] = 0.
\]
In particular, \(I\) is called a strong hamiltonian operator if both \(dI\) and \([I,I]\) vanish. Given a two-form \(I \in \Gamma(\wedge^2B^*)\), the graph \(L_I = \{X + I^{b}X | X \in B\}\) of its induced bundle map is a Dirac structure iff \(I\) is a hamiltonian operator.

Suppose that \(\Phi : A \rightarrow B\) is a Lie bialgebroid morphism and \(I \in \Gamma(\wedge^2B^*)\) a (strong) hamiltonian operator. \(\Phi^*\) pulls a two-form in \(\Gamma(\wedge^2B^*)\) back to a two-form in \(\Gamma(\wedge^2A^*)\). It is easy to see that \(\Phi^*I\) is then a (strong) hamiltonian operator in \(A\). Moreover, the pull-back of the corresponding Dirac structure \(L_I\) is exactly the graph \(\bar{L}_I\) of the (strong) hamiltonian operator \(\bar{I} = \Phi^*I\).

Example 5.3 Given a surjective submersion \(\varphi : M \rightarrow N\) of manifolds \(M, N\), its derivative defines a Lie algebroid morphism \(\Phi : TM \rightarrow TN\). This is also a Lie bialgebroid morphism between \((TM,T^*M)\) and \((TN,T^*N)\), where \(T^*M\) and \(T^*N\) are considered as the cotangent algebroids for the zero Poisson structure. According to Courant [3], in this case, a Dirac structure simply corresponds to a foliation on the manifold together with a family of closed 2-forms on the leaves. The pullback of the Dirac structure is just the pullback of the foliation by \(\varphi\) together with the pullback of two-forms.

Remark. A Dirac structure on a vector space is equivalent to a two-form on a subspace [3]. Thus, we could also pull back an isotropic subbundle by any bundle map, not just a surjection. Of course, the pullback might not be continuous if the map is not of constant rank. Moreover, it is not clear whether the integrability condition is preserved.

6 Left invariant Dirac structures on Poisson groupoids

To generalize Drinfel’d’s theorem on homogeneous spaces from Poisson groups to Poisson groupoids, we will first extend the notion of left invariant Dirac structure from Poisson groups to groupoids.

Let \((G \rightarrow P; \alpha, \beta)\) be a Poisson groupoid, with Lie algebroid \(A\). Here \(\Gamma(A)\) is identified with left-invariant vector fields on the groupoid. The dual bundle \(A^*\) can be naturally identified with the conormal bundle of the identity space \(P\) in the groupoid, and therefore inherits a Lie algebroid structure according to Weinstein [19]. Moreover, it was shown in [16] that \((A,A^*)\) is a Lie bialgebroid, which is called the tangent Lie bialgebroid of \(G\).

By \(T^\alpha G\), we denote the subbundle of \(TG\) consisting of all vectors tangent to \(\alpha\)-fibers. The group \(B(G)\) of bisections of \(G\) (submanifolds which project diffeomorphically to \(P\) by \(\alpha\) and \(\beta\)) acts naturally on \(G\) by left multiplication:
\[
l_Kx = K \cdot x, \quad \forall K \in B(G), \quad x \in G.
\]
As usual, the action lifts naturally to actions on $T^*G$ and $T^*G$ which leave $T^\alpha G$ and $(T^\alpha G)^\perp$ invariant.

Define a map $\Phi : T^*G \rightarrow A^*$ as follows: given any $x \in G$ and $\xi \in T^*_x G$, set $\Phi(\xi) \in A^*_p$ with $p = \beta(x)$ such that

$$<\Phi\xi, v> = <\xi, Tl_x v>, \quad \forall v \in A_p. \quad (21)$$

Then,

$$\begin{array}{ccc}
T^*G & \xrightarrow{\Phi} & A^* \\
\downarrow & & \downarrow \\
G & \xrightarrow{\beta} & P
\end{array} \quad (22)$$

is a bundle map.

**Remark.** In terms of symplectic groupoids [2], $\Phi$ is just the $\beta$-map of the cotangent groupoid $T^*G \rightrightarrows A^*$ [2]. However, our symplectic structure on $T^*G$ differs a minus sign from the one on the symplectic groupoid $T^*G \rightrightarrows A^*$. Therefore, $\Phi$ is a Poisson map.

Another interesting way to think of $\Phi$ is as the momentum map for the lifted right action of $B(G)$ on the cotangent bundle $T^*G$. Here $A^*$ is considered as a subset of $\Gamma(A)^*$ in the form of delta-distributions. It can be checked that the image of the momentum map is just $A^*$. This also indicates that $\Phi$ should be a Poisson map.

The following lemma lists some basic properties of $\Phi$.

**Lemma 6.1**

(i). $ker\Phi = (T^\alpha G)^\perp$;

(ii). for any $X \in \Gamma(A)$, $\Phi^*X$, as a section in $TG$, is exactly the left invariant tangent vector field $\bar{X}$ obtained by left translating $X$ along $\alpha$-fibers;

(iii). $\Phi$ is $B(G)$-invariant, i.e.,

$$\Phi(l_K^*\xi) = \Phi(\xi), \quad \forall K \in B(G), \text{ and } \xi \in T^*G;$$

(iv). $\Phi$ is a Lie bialgebroid morphism, where $(T^*G, TG)$ is equipped with the natural Lie bialgebroid structure associated with the Poisson structure $G$ while $(A^*, A)$ is the flipping of the tangent Lie bialgebroid $(A, A^*)$ [2].

**Proof.** The proof for (i)-(iii) is obvious, and is left for the reader. For (iv), since $\Phi$ is already known to be a Poisson map, it suffices to show that it is a Lie algebroid morphism. This is, however, quite clear since the Lie algebroid structure on $A^*$ is defined in terms of the Lie algebroid structure on $T^*G$ by identifying $A^*$ with the conormal bundle of $P$ in $G$ [19]. In fact, $T^*G$ is a LA-groupoid in terms of Mackenzie [15].

\footnotesize{
\textsuperscript{2}In order to be consistent with previous notation for Lie bialgebroid morphisms, we have flipped both Lie bialgebroids here.}

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Q.E.D.

**Proposition 6.2** A Dirac structure $\bar{L} \subset TG \oplus T^*G$ is the pullback of a Dirac structure in $A \oplus A^*$ iff

(i). $\bar{L}$ is $B(G)$-invariant, and

(ii). $(T^*G)^\perp \subset \bar{L}$.

**Proof.** Using $l_K$, $\forall K \in B(G)$, we denote the $B(G)$-action on $TG \oplus T^*G$. As in the previous section, let $\bar{\Phi}$ be the map $(\Phi^*)^{-1} \oplus \Phi : T\alpha G \oplus T^*G \rightarrow A \oplus A^*$. It is obvious that $\bar{\Phi}$ is also invariant under the $B(G)$-action, i.e.,

$$\bar{\Phi} l_K = \bar{\Phi}, \quad \forall K \in B(G).$$

Suppose that $\bar{L}$ is a Dirac structure in $TG \oplus T^*G$. If $\bar{L}$ is the pullback of a Dirac structure $L$ in $A \oplus A^*$. Then, $\bar{L} = \bar{\Phi}^{-1}(L)$. Clearly, $\bar{L}$ is $B(G)$-invariant since $\bar{\Phi}$ is invariant. Since $(T\alpha G)^\perp = \ker \bar{\Phi}$, it follows that $(T^*G)^\perp \subset \bar{L}$.

Conversely, the condition $(T\alpha G)^\perp \subset \bar{L}$ implies that $\bar{L} \subset T\alpha G \oplus T^*G$, since $\bar{L}$ is isotropic. Since $\bar{L}$ is $B(G)$-invariant, it follows that $\bar{\Phi}(\bar{L}|_K \cdot x) = \bar{\Phi}(l_K \bar{L}|_x) = \bar{\Phi}(\bar{L}|_x)$. Therefore, $\bar{\Phi}(\bar{L}|_x)$ depends only on the base point $p = \beta(x)$, and thus defines a subspace $L_p$ in $A_p \oplus A_p^*$, which is easily seen to be maximal isotropic. Thus we obtain a maximal isotropic subbundle $L$ in $A \oplus A^*$ such that $\bar{L} = \bar{\Phi}^{-1}(L)$. According to theorem 5.1 $L$ must be a Dirac subbundle.

Q.E.D.

When $G$ is a Poisson group, pullback Dirac structures in the sense above correspond exactly to left invariant Dirac structures as discussed in Section 4. For this reason, we shall call any such pullback a **left invariant Dirac structure**. Suppose that $\mathcal{F}$ is a foliation on $G$ with distribution $\mathcal{D} \subset TG$ such that $G/\mathcal{F}$ is a smooth manifold. According to Theorem 5.3, every Poisson structure on $G/\mathcal{F}$ corresponds to a Dirac structure $\bar{L} \subset TG \oplus T^*G$.

**Theorem 6.3** $\bar{L}$ is the pullback of a Dirac structure $L \subset A \oplus A^*$ iff

(i). $\mathcal{F}$ is $B(G)$-invariant;

(ii). $\{\cdot, \cdot\}_1$ is $B(G)$-invariant; and

(iii). $\mathcal{D} \subset T\alpha G$ and $\{\alpha^* f, g\}_1 = 0$, $\forall f \in C^\infty(P)$ and $g \in C^\infty(G/\mathcal{F}) \cong C_L^\infty(G)$.

**Proof.** Suppose that $\bar{L}$ is the pullback of a Dirac structure $L$ under $\Phi$. Then $\bar{L}$ is $B(G)$-invariant. Thus, $\mathcal{D} = \bar{L} \cap TG$ is also $B(G)$-invariant. According to Lemma 5.1, $\{\cdot, \cdot\}_1$ is $B(G)$-invariant. For (iii), we note that $\bar{L} \subset T\alpha G \oplus T^*G$ by Proposition 5.2 since $\bar{L}$ is isotropic. Thus it follows that $\mathcal{D} = \bar{L} \cap TG \subset T\alpha G$. Also note that $\alpha^* f$ is constant along $\alpha$-fibers, so $\mathcal{D} \subset T\alpha G$ implies that $\alpha^* f$ is admissible. Therefore, $\{\alpha^* f, g\}_1$ is well-defined. Now since $d\alpha^* f \in (T\alpha G)^\perp = \ker \Phi \subset \bar{L}$, we may choose $Y_{\alpha^* f} = 0$. Thus, $\{\alpha^* f, g\}_1 = Y_{\alpha^* f} g = 0$.
Conversely, assume that (i)-(iii) hold. So, for any \( f \in C^\infty(P) \), \( \alpha^*f \) is \( \bar{L} \)-admissible. As in Section 3, let \( Y_{\alpha^*f} \in \mathcal{X}(G) \) be any vector field that \( Y_{\alpha^*f} + d\alpha^*f \in \Gamma(\bar{L}) \). Then, \( Y_{\alpha^*f}g = \{\alpha^*f, \ g\}_1 = 0 \) for all admissible \( g \). This implies that \( Y_{\alpha^*f} \in \Gamma(D) \subset \Gamma(\bar{L}) \). Thus, \( d\alpha^*f \in \Gamma(\bar{L}) \). Since \( \ker \Phi = (T^G)^\perp \) is spanned by all such vectors, it follows that

\[
\ker \Phi \subset \bar{L}.
\]

By Lemma 4.1, (i)-(ii) imply that \( \bar{L} \) is \( B(G) \)-invariant. Thus \( \bar{L} \) is the pullback of \( L \) according to Proposition 6.2.

Q.E.D.

7 Poisson actions

Let \( G \rightrightarrows P \) be a Poisson groupoid with Poisson tensor \( \pi_G \). Suppose that \( G \) acts on a Poisson manifold \((X, \pi)\) equipped with a moment map \( J : X \rightarrow P \). Here, the action is a map

\[
m : (g, x) \mapsto g \cdot x
\]

from \( G \times P X = \{(g, x) \in G \times X | \beta(g) = J(x)\} \) to \( X \), satisfying the usual condition \( g \cdot (h \cdot x) = (gh) \cdot x \).

The action is a Poisson action if its graph \( \Omega = \{(g, x, g \cdot x) | \beta(g) = J(x)\} \) is a coisotropic submanifold of \( G \times X \times X \).

For example, consider a complete Poisson group \( H \) with dual Poisson group \( H^* \). Then \( G = HH^* \) is a symplectic groupoid over \( H^* \). Suppose that \( X \) is a Poisson \( H \)-space with an equivariant momentum map \( J : X \rightarrow H^* \). Then, \((X, J)\) is a Poisson \( G \)-space under the \( G \)-action:

\[
(g, u) \cdot x = gx, \quad g \in H, \ u \in H^* \text{ and } x \in X \text{ such that } J(x) = u.
\]

Any bisection \( K \) of the groupoid induces a diffeomorphism of \( G \) by left multiplication by \( K \). This is denoted by \( l_K \). On the other hand, we also can define a diffeomorphism of \( X \) by \( x \mapsto K \cdot x, \forall x \in X \). This is also denoted by the same symbol \( l_K \) when confusion is unlikely.

A section of the moment map is a submanifold of \( X \) to which the restriction of \( J \) is a diffeomorphism. Similarly, a local section of \( J \) is a submanifold \( \mathcal{Y} \) of \( X \) to which the restriction of \( J \) is a diffeomorphism between \( \mathcal{Y} \) and an open subset \( J(\mathcal{Y}) \) of \( P \). If \( J \) is a submersion, through any point of \( X \) there always exists a local section of \( J \). A global section \( \mathcal{Y} \) induces a map, denoted by \( r_{\mathcal{Y}} \), from \( G \) to \( X \) given by: \( g \mapsto g \mathcal{Y} \). Similarly, given any compatible \((g, x) \in G \times X \) (i.e. \( \beta(g) = J(x) \)), a local section \( \mathcal{Y} \) through the point \( x \) induces a map \( r_{\mathcal{Y}} \) from a neighborhood of \( g \) to a neighborhood of \( x \) in the same way.

The main theorem is the following:

**Theorem 7.1** Suppose that \( G \rightrightarrows P \) is a Poisson groupoid acting on a Poisson manifold equipped with a moment map \( J : X \rightarrow P \). This is a Poisson action iff
(i). For any \( f \in C^\infty(P) \),
\[
X_{J^*f}(x) = (r_x)_*X_{\alpha^*f}(u),
\]
where \( x \in X \) and \( u = J(x) \).

(ii). For any compatible \((g,x)\in G*X\),
\[
\pi(gx) = (l_K)_*\pi(x) + (r_Y)_*\pi_G(g) - (r_Y)_*(l_K)_*\pi_G(u),
\]
where \( u = \beta(g) = J(x) \), \( K \) is any local bisection of \( G \) through \( g \), and \( Y \) any local section through the point \( x \).

Remark (1). Here \( r_x \) denotes the map: \( g \rightarrow g \cdot x \) from \( \beta^{-1}(u) \) to \( X \). Since \( X_{\alpha^*f}(u) \) is tangent to \( \beta^{-1}(u) \), the right hand side of Equation (23) makes sense.

(2). When \( G \) is a Poisson group, i.e. \( P \) reduces to a point, the first condition is satisfied automatically, and the second one reduces to the usual condition of Poisson action since \( \pi_G(u) = 0 \).

Proposition 7.2 Under the hypotheses of Theorem 7.1, if the action is a Poisson action, then \( J : X \rightarrow P \) is a Poisson map.

Proof. Let \( x \in X \), \( J(x) = u \) and \( \xi, \eta \in T_uP \) be any covectors at \( u \). For any \( g \in G \) with \( \beta(g) = u \) as covectors at the point \((g,x,gx)\), both \((-\beta^*\xi, J^*\xi, 0)\) and \((-\beta^*\eta, J^*\eta, 0)\) are conormal to \( \Omega \). Therefore,
\[
\pi_G(-\beta^*\xi, -\beta^*\eta) + \pi(J^*\xi, J^*\eta) = 0,
\]
which implies that \((J_*\pi(x))(\xi, \eta) = -(\beta_*\pi_G(g))(\xi, \eta) = \pi_P(u)(\xi, \eta)\). That is \( J_*\pi = \pi_P \). In other words, \( J \) is a Poisson map.

Q.E.D.

Lemma 7.3 Let \( x \in X \) and \( J(x) = u \). Suppose that \( \delta_x \in T_xX \) and \( \delta_u \in T_uG \) such that \((\delta_u, \delta_x) \in T_{(u,x)}(G*X)\). Let \( \delta'_x = m_*(\delta_u, \delta_x) \in T_xX \). Then,
\[
\delta'_x = \delta_x + (r_x)_* (\delta_u - \epsilon_* J_* \delta_x),
\]
where \( \epsilon : P \rightarrow G \) is the inclusion of the unit space.

Proof. As a tangent vector in \( T_{(u,x)}(G*X) \), \((\delta_u, \delta_x)\) can be split into the sum of \((\epsilon_* J_* \delta_x, \delta_x)\) and \((\delta_u - \epsilon_* J_* \delta_x, 0)\). It is easy to see that \( m_*(\delta_u - \epsilon_* J_* \delta_x, 0) = (r_x)_* (\delta_u - \epsilon_* J_* \delta_x) \) and \( m_*(\epsilon_* J_* \delta_x, \delta_x) = \delta_x \). This proves the lemma.

Q.E.D.
Lemma 7.4 Suppose that \((g, x, z)\) with \(z = gx\) is any point in \(\Omega\), and \((\delta_g, \delta_x, \delta_z)\) any tangent vector of \(\Omega\) at this point. Let \(K\) be any (local) bisection of \(G\) through the point \(g\) and \(Y\) any (local) section of \(J\) through the point \(x\). Then,

\[
\delta_z = r_Y \delta_g + l_K \delta_x - l_K r_Y \epsilon_\ast J \ast \delta_x.
\] (26)

Proof. Let \(u = \beta(g) = J(x)\), and \(\delta_u = l_{K^{-1}} \delta_g\). Then, \(\delta_u\) is a tangent vector of \(G\) at \(u\). It is clear that

\[
\delta_z = m_\ast (\delta_u, \delta_x) \\
= l_K m_\ast (l_{K^{-1}} \delta_g, \delta_x) \\
= l_K (\delta_x + (r_x)_\ast (\delta_u - \epsilon_\ast J_\ast \delta_x)) \\
= l_K (\delta_x + r_Y (\delta_u - \epsilon_\ast J_\ast \delta_x)) \\
= l_K (\delta_x + r_Y (l_{K^{-1}} \delta_g - \epsilon_\ast J_\ast \delta_x)) \\
= r_Y \delta_g + l_K \delta_x - l_K r_Y \epsilon_\ast J_\ast \delta_x.
\]

Q.E.D.

Corollary 7.5 Suppose that \((g, x, z)\) \(\in\) \(\Omega\), \(K\) any (local) bisection of \(G\) through the point \(g\), and \(Y\) any (local) section of \(J\) through the point \(x\). For any \(\zeta \in T^*_Z X\), the covector \((-r_y^\ast \zeta, J_\ast \epsilon^* r_y^* l_K^\ast \zeta - l_K^* \zeta, \zeta) \in T^*_g (G \times X \times X)\) is conormal to \(\Omega\).

Lemma 7.6 Suppose that \(G \rightrightarrows P\) is a Poisson groupoid. Then for any \(f \in C^\infty(P)\) and \(u \in P\), \(X_{\alpha_\ast f}(u) - X_{\beta_\ast f}(u)\) is tangent to \(P\) and equals \(X_f(u)\).

Proof. As a covector of \(G\) at \(u\), \(\alpha^* df - \beta^* df\) is clearly conormal to the unit space \(P\). Since \(P\) is a coisotropic submanifold, it follows that \(X_{\alpha_\ast f}(u) - X_{\beta_\ast f}(u)\) is tangent to \(P\). Hence

\[
X_{\alpha_\ast f}(u) - X_{\beta_\ast f}(u) = \alpha_\ast (X_{\alpha_\ast f}(u) - X_{\beta_\ast f}(u)) = X_f(u).
\]

Q.E.D.

Proposition 7.7 Suppose that \(G \rightrightarrows P\) is a Poisson groupoid acting on a Poisson manifold \(X\) with moment map \(J : X \to P\). Suppose that the action is a Poisson action. Then, for any \(f \in C^\infty(P)\),

\[
X_{J_\ast f}(x) = (r_x)_\ast X_{\alpha_\ast f}(u), \ \forall x \in X
\]

where \(u = J(x)\).
Proof. Take any \( g \in G \) such that \( \beta(g) = J(x) \). Then \((g,x,z)\) with \( z = gx \) is in \( \Omega \). For any \( \zeta \in T^*_x X \) and \( f \in \mathcal{C}^\infty(P) \), as covectors at \((g,x,z)\), both \((-r^*_y \zeta, J^* \varepsilon^* r^*_y l^*_K \zeta - l^*_K \zeta, \zeta)\) and \((-\beta^* df, J^* df, 0)\) are conormal to \( \Omega \). Therefore,

\[
\pi_G(g)(-r^*_y \zeta, -\beta^* df) + \pi(x)(J^* \varepsilon^* r^*_y l^*_K \zeta - l^*_K \zeta, J^* df) = 0.
\]

Now

\[
\pi_G(g)(-r^*_y \zeta, -\beta^* df) = -X_{\beta^* f}(g), r^*_y \zeta > = -r_{Y_x} X_{\beta^* f}(g), \zeta > = -r_{Y_x} l^*_K X_{\beta^* f}(u), \zeta >,
\]

where the last step uses the fact: \( X_{\beta^* f}(g) = l^*_K X_{\beta^* f}(u) \).

Also,

\[
\pi(x)(J^* \varepsilon^* r^*_y l^*_K \zeta, J^* df) = (J^* \pi(x))(\varepsilon^* r^*_y l^*_K \zeta, df) = \pi_p(u)(\varepsilon^* r^*_y l^*_K \zeta) = -X_f(u)(\varepsilon^* r^*_y l^*_K \zeta) = < -l^*_K r_{Y_x} \varepsilon_\ast X_f(u), \zeta > (\text{Using Lemma 7.6}) = < -l^*_K r_{Y_x} (X_{\alpha^* f}(u) - X_{\beta^* f}(u)), \zeta >,
\]

and

\[
\pi(x)(-l^*_K \zeta, J^* df) = < X_{J^* f}(x), l^*_K \zeta > = < l^*_K X_{J^* f}(x), \zeta >.
\]

Therefore, it follows that \( < -l^*_K r_{Y_x} X_{\alpha^* f}(u), \zeta > + < l^*_K X_{J^* f}(x), \zeta > = 0 \), which implies immediately that

\[
X_{J^* f}(x) = r_{Y_x} X_{\alpha^* f}(u) = (r_x)_* X_{\alpha^* f}(u).
\]

Q.E.D.

**Proposition 7.8** Suppose that \( G \rightarrow P \) is a Poisson groupoid acting on a Poisson manifold \( X \) with moment map \( J : X \rightarrow P \). Suppose that the action is a Poisson action. Then,

\[
\pi(gx) = (l^*_K)_* \pi(x) + (r^*_Y)_* \pi_G(g) - (r^*_Y)_*(l^*_K)_* \pi_G(u), \tag{27}
\]

where \( u = \beta(g) = J(x) \), \( K \) is any bisection of \( G \) through \( g \), and \( Y \) any local section through the point \( x \).

**Lemma 7.9** For any \( u \in P \) and \( \eta \in T^*_u G \),

\[
X_\alpha^* \varepsilon_\ast \eta(u) + \varepsilon_\ast \alpha^* X_\eta(u) - \varepsilon_\ast X_\varepsilon_\ast \eta(u) = X_\eta(u). \tag{28}
\]
Proof. It is clear that $\alpha^* \epsilon^* \eta - \eta$, as a covector of $G$ at $u$, is conormal to $P$. Hence, $X_{\alpha^* \epsilon^* \eta}(u) - X_{\eta}(u)$ is tangent to $P$. It follows that

$$X_{\alpha^* \epsilon^* \eta}(u) - X_{\eta}(u) = \epsilon_x \alpha_*(X_{\alpha^* \epsilon^* \eta}(u) - X_{\eta}(u))$$

$$= \epsilon_x X_{\epsilon^* \eta}(u) - \epsilon_x \alpha_* X_{\eta}(u).$$

This completes the proof of the lemma.

Q.E.D.

Lemma 7.10

$$\pi(x)(J^* \epsilon^* r_{\gamma K}^* l_{\gamma K} \zeta - l_{\gamma K}^* \zeta, J^* \epsilon^* r_{\gamma K}^* l_{\gamma K} \eta - l_{\gamma K}^* \eta) = (l_{\gamma K}, \pi_x((\zeta, \eta) - (l_{\gamma K}, r_{\gamma K}, \pi_G(u))(\zeta, \eta)).$$

Proof.

$$\pi(x)(J^* \epsilon^* r_{\gamma K}^* l_{\gamma K} \zeta, J^* \epsilon^* r_{\gamma K}^* l_{\gamma K} \eta) = (J_x \pi(x))(\epsilon^* r_{\gamma K}^* l_{\gamma K} \zeta, \epsilon^* r_{\gamma K}^* l_{\gamma K} \eta)$$

$$= \pi_p(u)(\epsilon^* r_{\gamma K}^* l_{\gamma K} \zeta, \epsilon^* r_{\gamma K}^* l_{\gamma K} \eta)$$

$$= <X_{\epsilon^* r_{\gamma K}^* l_{\gamma K} \zeta}(u), \epsilon^* r_{\gamma K}^* l_{\gamma K} \eta>$$

$$= <\epsilon_x X_{\epsilon^* r_{\gamma K}^* l_{\gamma K} \zeta}(u), r_{\gamma K}^* l_{\gamma K} \eta>.$$
Proof of Proposition 7.8: For any $\zeta, \eta \in T^*_z X$, both $(-r^*_Y \zeta, J^* \epsilon^* r^*_Y K \zeta - l^*_K \zeta, \zeta)$ and $(-r^*_Y \eta, J^* \epsilon^* r^*_Y K \eta - l^*_K \eta, \eta)$ are conormal to $\Omega$. Therefore,

$$
\pi_G(g)(-r^*_Y \zeta, -r^*_Y \eta) + \pi(x)(J^* \epsilon^* r^*_Y K \zeta - l^*_K \zeta, J^* \epsilon^* r^*_Y K \eta - l^*_K \eta) - \pi(z)(\zeta, \eta) = 0.
$$

The conclusion follows immediately by Lemma 7.10.

Q.E.D.

Proof of Theorem 7.1: One direction has been proved by the above series of propositions. It remains to prove the other direction.

First, we note that the first condition implies that $J: X \to \mathcal{P}$ is a Poisson map. This can be seen as follows.

As a map defined on $\beta^{-1}(u)$, we have $(J \circ r)(r) = J(rx) = \alpha(r), \forall r \in \beta^{-1}(u)$. Then, it follows that $J_* X_{J^* f}(x) = J_*(r_x)_* X_{\alpha^* f}(u) = \alpha_* X_{\alpha^* f}(u) = X_f(u)$.

Given any point $(g, x, z) \in \Omega \subset G \times X \times \mathcal{X}$. The conormal space of $\Omega$ at this point is spanned by two types of vectors: $(-\beta^* df, J^* df, 0)$ for any $f \in C^\infty(\mathcal{P})$ and $(-r^*_Y \zeta, J^* \epsilon^* r^*_Y K \zeta - l^*_K \zeta, \zeta)$ for any $\zeta \in T^*_z X$. Using the same arguments as in the proof of Propositions 7.2, 7.7 and 7.8, it can be easily checked that the evaluation of the Poisson tensor on all these vectors vanish. This concludes the proof.

Q.E.D.

Remark. In the proof above, we only used the fact that Equation (24) holds for one, instead of all, such bisections. Consequently, under the rest of the assumptions of Theorem 7.1, if Equation (24) holds for any one bisection, it holds for all.

8 Poisson homogeneous spaces

The notion of homogeneous space for a groupoid action is more subtle than for groups. (This point has already been made in [1], cited in [14]). One natural candidate for such a space is $G$ acting on itself by left translations, but this action is not transitive in the usual sense, since $\beta(gx) = \beta(x)$, so that the action is transitive only on each $\beta$-fibre.

Instead, we define homogeneous $G$-spaces to be those which are isomorphic to $G/H$ for some wide (i.e. containing all the identities) subgroupoid $H$ of $G$. That is, we define $G/H$ by the equivalence relation $g \sim h \iff \exists n \in H$ such that $gn = h$, with the moment map $J([g]) = \alpha(g)$ and the action $g \cdot [h] = [gh]$. The following is an intrinsic characterization of such spaces.

Definition 8.1 A $G$-space $X$ over $P$ is homogeneous if there is a section $\sigma$ of the moment map $J$ which is saturating for the action in the sense that $G \cdot \sigma(P) = X$. The isotropy subgroupoid of the section $\sigma$ consists of those $g \in G$ for which $g \cdot \sigma(P) \subset \sigma(P)$. 

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Proposition 8.2 A $G$-space is homogeneous if and only if it is isomorphic to $G/H$ for some wide subgroupoid $H \subset G$.

Proof. It is easy to see that a $G$-space of the form $G/H$ is homogeneous, since the image of the identity section of $G$ is a saturating section of $G/H$. On the other hand, given a homogeneous space $X$ with saturating section $\sigma$, we define a map $\theta : G \rightarrow X$ by $\theta(g) = g \cdot \sigma(\beta(g))$. $\theta$ is surjective: for every $x \in X$ we have $x = g \cdot \sigma(p)$ for some $g \in G$ and $p \in P$; then $\beta(g) = J(\sigma(p)) = p$, so $x = g \cdot \sigma(\beta(g)) = \theta(g)$. On the other hand, if $\theta(g) = \theta(h)$, i.e. $g \cdot \sigma(\beta(g)) = h \cdot \sigma(\beta(h))$, we have $g^{-1}h \cdot \sigma(\beta(h)) = \sigma(\beta(g))$. Since $g^{-1}h$ can act on only one element of $\sigma(P)$, it follows that $g^{-1}h$ belongs to the isotropy groupoid $H$ of the section $\sigma$. Also, for $n \in H$, $\theta(gn) = gn \cdot \sigma(\beta(gn)) = g \cdot \sigma(p)$ for some $p$. But $\beta(g) = J(\sigma(p))$, so

$$\theta(gn) = g \cdot \sigma(\beta(g)) = \theta(g).$$

It follows that $\theta$ induces a bijection (which can be checked to be $G$-equivariant) from $G/H$ to $X$.

Q.E.D.

Let $(G \rightarrow P, \alpha, \beta)$ be a Poisson groupoid, and $H$ a connected closed subgroupoid. Suppose that $X = G/H$ is a Poisson manifold. Write $p$ as the natural projection: $G \rightarrow X$. Let $\{ \cdot, \cdot \}_1$ be the difference bracket from $C^\infty(X) \otimes C^\infty(X)$ to $C^\infty(G)$ as defined by Equation (10):

$$\{ \varphi, \psi \}_1 = p^* \{ \varphi, \psi \}_G - \{ p^* \varphi, p^* \psi \}_G, \quad \forall \varphi, \psi \in C^\infty(X).$$

$X$ is a homogeneous space of the groupoid $G$. Write $J$ as its moment map: $X \rightarrow P$. Then, $J \cdot p = \alpha$. The main theorem of this section is

Theorem 8.3 $X$ is a Poisson homogeneous space iff

(i). For any bisection $K$ of $G$,

$$\{ l_K^* \varphi, l_K^* \psi \}_1 = l_K^* \{ \varphi, \psi \}_1, \quad \forall \varphi, \psi \in C^\infty(X),$$

i.e., $\{ \cdot, \cdot \}_1$ is left invariant; and

(ii). for any $f \in C^\infty(P)$ and $\psi \in C^\infty(X)$,

$$\{ J^* f, \psi \}_1 = 0.$$

We split its proof into two propositions.

Proposition 8.4 The following statements are equivalent:

(i). For any $f \in C^\infty(P)$, $X_J f(x) = (r_x)_* X_{\alpha^* f}(u), \forall x \in X$ with $u = J(x)$.

(ii). For any $f \in C^\infty(P)$ and $\psi \in C^\infty(X)$ $\{ J^* f, \psi \}_1 = 0$. 


Proof. For any \(g \in G\), let \(x = p(g) = [gH] \in X\). Then, \(\forall f \in C^\infty(P)\),

\[
\{p^*J^*f, \ p^*\psi\}_G(g) = \{\alpha^*f, \ p^*\psi\}_G(g) \\
= X_{\alpha^*f}(g)(p^*\psi) \\
= [p_*X_{\alpha^*f}(g)]\psi \\
= [p_*r_gX_{\alpha^*f}(u)]\psi
\]

Now, it is clear that

\[
(p \circ r_g)(\gamma) = p(\gamma g) = [\gamma gH] = r_x(\gamma), \quad \forall \gamma \in \beta^{-1}(u).
\]

Hence, \(p \circ r_g = r_x\) on \(\beta^{-1}(u)\). Therefore,

\[
\{p^*J^*f, \ p^*\psi\}_G(g) = [(r_x)_*X_{\alpha^*f}(u)]\psi.
\]

On the other hand,

\[
p^*\{J^*f, \ \psi\}(g) = \{J^*f, \ \psi\}(pg) \\
= \{J^*f, \ \psi\}(x) \\
= X_{J^*f}(x)\psi.
\]

Thus, it follows immediately that \(\{J^*f, \ \psi\}_1 = 0\), \(\forall \psi \in C^\infty(X)\) is equivalent to the equation:

\[
X_{J^*f}(x) = (r_x)_*X_{\alpha^*f}(u), \ \forall x \in X.
\]

Q.E.D.

**Proposition 8.5** The following statements are equivalent:

(i). For any bisection \(K\) of \(G\),

\[
\{l_K^*\varphi, \ l_K^*\psi\}_1 = l_K^*\{\varphi, \ \psi\}_1, \quad \forall \varphi, \psi \in C^\infty(X),
\]

i.e., \(\{\cdot, \cdot\}_1\) is left invariant;

(ii). For any compatible \((g, x) \in G \ast X\),

\[
\pi(gx) = (l_K)_*\pi(x) + (r_Y)_*\pi_G(g) - (r_Y)_*(l_K)_*\pi_G(u), \quad (29)
\]

where \(u = \beta(g) = J(x), K\) is any local bisection of \(G\) through \(g\), and \(Y\) any local section through the point \(x\).

Proof. Let \(\gamma\) be any point in \(G\), \(p(\gamma) = [\gamma H] = x \in X\), and \(J(x) = \alpha(\gamma) = u \in P\). Also, let \(g = K(u) = K \cap \beta^{-1}(u) \in G\). Then, \(l_K\gamma = g\gamma\) and \(l_Kx = g \cdot x\). Thus,
\[
p^*\{l_K^*\varphi, l_K^*\psi\}(\gamma) = \{l_K^*\varphi, l_K^*\psi\}(x) = \pi(x)(l_K^*d\varphi, l_K^*d\psi) = l_K^*\pi(x)(d\varphi, d\psi),
\]

and

\[
\{p^*l_K^*\varphi, p^*l_K^*\psi\}G(\gamma) = \pi_G(\gamma)(p^*l_K^*d\varphi, p^*l_K^*d\psi) = l_K^*p^*\pi_G(\gamma)(d\varphi, d\psi).
\]

That is,

\[
\{l_K^*\varphi, l_K^*\psi\}_1(\gamma) = -l_K^*\pi(x)(d\varphi, d\psi) + l_K^*p^*\pi_G(\gamma)(d\varphi, d\psi).
\]

On the other hand,

\[
l_K^*\{\varphi, \psi\}_1(\gamma) = \{\varphi, \psi\}_1(g_\gamma) = -(p^*\{\varphi, \psi\})(g_\gamma) + \{p^*\varphi, p^*\psi\}G(g_\gamma) = -\{\varphi, \psi\}(gx) + \{p^*\varphi, p^*\psi\}G(g_\gamma) = -\pi(gx)(d\varphi, d\psi) + p^*\pi_G(g_\gamma)(d\varphi, d\psi).
\]

Therefore, \{\cdot, \cdot\}_1 is left invariant iff

\[
l_K^*\pi(x) - l_K^*p^*\pi_G(\gamma) = \pi(gx) - p^*\pi_G(g_\gamma),
\]

or

\[
p^*\pi_G(g_\gamma) - l_K^*p^*\pi_G(\gamma) = \pi(gx) - l_K^*\pi(x).
\] (30)

Since \(G\) is a Poisson groupoid, according to Theorem 2.4 in [23], we have

\[
\pi_G(g_\gamma) = l_K^*\pi_G(\gamma) + r_\mathcal{R}_*\pi_G(g) - l_K^*r_\mathcal{R}_*\pi_G(u),
\]

where \(\mathcal{R}\) is any local bisection of \(G\) through the point \(\gamma\). Hence, it follows that

\[
p^*\pi_G(g_\gamma) - l_K^*p^*\pi_G(\gamma) = p^*r_\mathcal{R}_*\pi_G(g) - l_K^*p^*r_\mathcal{R}_*\pi_G(u).
\]

Here, we have used the identity: \(p^*l_K = l_K^*p\), as both being considered as maps from \(G\) to \(X\).

Let \(\mathcal{Y} = p(\mathcal{R}) \subset X\). Then \(\mathcal{Y}\) is a local section of \(J\) through the point \(x\). It is simple to see that, as maps from \(G\) to \(X\), \(p^*r_\mathcal{R} = r_\mathcal{Y}\).
Hence,
\[ p_* r_R \pi_G(g) - l_K p_* r_R \pi_G(u) = r_Y \pi_G(g) - l_K r_Y \pi_G(u). \]

This shows that Equation (30) is equivalent to Equation (24) with \( Y = p(R). \) The conclusion thus follows from Theorem 7.1 together with the remark following its proof.

Q.E.D.

A Dirac structure \( L \) of \( A \oplus A^* \) is called regular if \( L \cap A \) is a subalgebroid of \( A \) whose left translation defines a simple foliation on \( G \). Then, \( L \) is regular iff \( \bar{L} \) is reducible. Thus, combining Theorem 6.3 and Theorem 8.3, we obtain the following main theorem, which is a generalization of Drinfel’d’s theorem in the groupoid context.

**Theorem 8.6** For a Poisson groupoid \( G \), there is an 1-1 correspondence between Poisson homogeneous spaces \( G/H \) and regular Dirac structures \( L \) of its tangent Lie bialgebroid, where \( H \) is the \( \alpha \)-connected closed subgroupoid of \( G \) corresponding to the subalgebroid \( L \cap A \).

We end this section with some examples.

**Example 8.7** Under the same hypothesis as in Theorem 8.6, if moreover \( L \) is the graph of a hamiltonian operator \( H \in \Gamma(\Lambda^2 A) \), its corresponding Poisson homogeneous space is still \( G \), but equipped with a different Poisson structure \( \pi_G + \Phi^* H \). Here \( \Phi^* H \in \Gamma(\Lambda^2 TG) \) is the pull back of \( H \) under the morphism \( \Phi : T^* G \to A^* \). This Poisson structure in fact defines a Poisson affinoid structure in terms of Weinstein [21].

**Example 8.8** For a Poisson manifold \((P, \pi)\), let \((TP, T^* P, \pi)\) be its canonical Lie bialgebroid. The corresponding Poisson groupoid is the pair groupoid \( G = P \times \bar{P} \). It is easy to see that any homogeneous \( G \)-space in this case is always of the form \( P \times P/F \), where \( F \) is a simple foliation on \( P \). The groupoid \( G \)-action is given by:
\[(x, y) \cdot (y, [z]) = (x, [z]), \quad \forall x, y, z \in P.\]

Moreover, this becomes a Poisson homogeneous \( G \)-space iff \( P \) is equipped with the original Poisson structure \( \pi \) (the Poisson structure on \( P/F \) may be arbitrary). In other words, in this case, Poisson homogeneous spaces are in 1-1 correspondence with Poisson structures on quotient manifolds of \( P \). Thus, Theorem 8.6 reduces to Theorem 3.3.

**Example 8.9** Dually, we may switch the order and consider the Lie bialgebroid \((T^* P, TP)\) for a Poisson manifold \( P \). Its corresponding Poisson groupoid \( G \), if it exists, is in fact a symplectic groupoid of \( P \) (see Theorem 5.3 in [17]). It is not difficult to see that a homogeneous space \( X \) becomes a Poisson homogeneous space iff the moment map \( J : X \to P \) is a Poisson map. Thus we obtain the following

---

2In the case of groups, this is equivalent to saying that \( L \cap A \) can be integrated to a connected closed subgroup of \( G \). However, when \( G \) is a groupoid, that \( L \cap A \) can be integrated to a connected closed subgroupoid seems not sufficient to get a simple foliation.
Corollary 8.10 Suppose that $P$ is an integrable Poisson manifold with symplectic groupoid $G$. There is a one-one correspondence between reducible Dirac structures in the double $E = T^*P \oplus TP$ and homogeneous $G$-spaces $X$ equipped with a compatible Poisson structure in the sense that the moment map $J : X \to P$ is a Poisson map.

It is worth noting that when $P$ is symplectic, for a given null Dirac structure on $E = TP \oplus T^*P$, the corresponding pair of Poisson homogeneous spaces in Example 8.8 and 8.9 correspond to a Poisson dual pair. In other words, for a symplectic manifold $P$, a null Dirac structure on $E = TP \oplus T^*P$, under a certain regularity condition, corresponds to a Poisson dual pair. It would be interesting to explore what happens for a general Dirac structure, and also even more general situation when $P$ is degenerate.

The last example is the following

Example 8.11 As in Example 8.9, let $P$ be an integrable Poisson manifold with symplectic groupoid $G$, and $E = T^*P \oplus TP$.

Assume that the Dirac structure arises from a hamiltonian operator, which is, in this case, a two form $\theta$ on $P$ satisfying the equation:

$$d\theta + \frac{1}{2} [\theta, \theta] = 0.$$

Its corresponding Poisson homogeneous space, as described in Example 8.4, is $G$ equipped with the “affine” Poisson structure $\pi_G + \Phi^* \theta$, where $\Phi : T^*G \to TP$ is the $\beta$-map of the cotangent symplectic groupoid as defined by Equation (21). In general, it is not clear whether this is still nondegnerate.

However, in the extreme case that $P$ is a zero Poisson structure, we will see that it is still symplectic.

To see this, we note that $\Phi$ fits into the following commutative diagram:

$\begin{array}{ccc}
T^*G & \xrightarrow{\Phi} & TP \\
\downarrow{\pi^\#_G} & & \downarrow{id} \\
TG & \xrightarrow{\beta_*} & TP
\end{array}$

(31)

This implies that

$$(\Phi^* \theta)^\# = -\pi^\#_G \circ (\beta^* \theta)^b \circ \pi^\#_G.$$

Therefore,

$$(\Phi^* \theta)^\# \circ \omega^b = -\pi^\#_G \circ (\beta^* \theta)^b,$$

where $\omega$ denotes the symplectic structure on $G$.  

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On the other hand, it follows from the fact that $\text{Im}(\alpha^*\theta)^b \subset T^\alpha G^\perp$ and $T^\alpha G^\perp \subset \ker \Phi$ that

$$(\Phi^*\theta)^\# \circ (\alpha^*\theta)^b = 0.$$  

Thus,

$$(\pi^\#_G + (\Phi^*\theta)^\#) \circ (\omega^b + (\alpha^*\theta)^b) = \text{id} + \pi^\#_G \circ ((\alpha^*\theta)^b - (\beta^*\theta)^b).$$  

That is, in case that $\beta^*\theta = \alpha^*\theta$, the affine Poisson structure $\pi_G + \Phi^*\theta$ is non-degenerate and the corresponding symplectic form is $\omega + \alpha^*\theta$.

Thus, when $P$ is a zero Poisson manifold, its symplectic groupoid is $T^*P$ equipped with the canonical cotangent symplectic structure. In this case, $\alpha = \beta$ and is just the natural projection from $T^*P$ to $P$. A hamiltonian operator corresponds to any closed two form $\theta$ on $P$. The homogeneous space corresponding to its induced Dirac structure is again $T^*P$, with the non-degenerate Poisson structure coming from the sum of the canonical 2-form and the pullback of $\theta$ by the projection $T^*P \rightarrow P$.

References

[1] Brown, R., Danesh-Naruie, G., and Hardy, G.P.L., Topological groupoids: II. Covering morphisms and $G$-spaces, Math. Nachr. 74 (1976), 143-156.

[2] Coste, A., Dazord, P. and Weinstein, A., Groupo"{i}des symplectiques, Publications du Département de Mathématiques de l’Université de Lyon, I, 2/A (1987), 1-65.

[3] Courant, T.J., Dirac manifolds, Trans. A.M.S. 319 (1990), 631-661.

[4] Courant, T.J., and Weinstein, A., Beyond Poisson structures, Seminaire sud-rhodanien de géométrie VIII. Travaux en Cours 27, Hermann, Paris (1988).

[5] Dorfman, I.Ya., Dirac structures and integrability of nonlinear evolution equations, Wiley, Chichester, 1993.

[6] Drinfel’d, V.G., Hamiltonian structures on Lie groups, Lie bialgebras, and the geometric meaning of the classical Yang-Baxter equations, Soviet Math. Dokl. 27, (1983), 68-71.

[7] Drinfel’d, V. G., Quasi-Hopf algebras, Leningrad Math. J. 2 (1991), 829-860.

[8] Drinfel’d, V.G., On Poisson homogeneous spaces of Poisson-Lie groups, Theor. Math. Phys. 95 (1993), 524-525.

[9] Higgins, P. J. and Mackenzie, K. C. H., Algebraic constructions in the category of Lie algebroids, J. Algebra 129 (1990), 194-230.

[10] Kosmann-Schwarzbach, Y., Exact Gerstenhaber algebras and Lie bialgebroids, Acta Appl. Math. 41 (1995), 153-165.

[11] Liu, Z.-J., Weinstein, A. and Xu, P., Manin triples for Lie bialgebroids, preprint.
[12] Lu, J.-H., Lie algebroids associated with Poisson actions, preprint.

[13] Lu, J.-H. and Weinstein, A., Poisson Lie groups, dressing transformations, and the Bruhat decomposition, *J. Diff. Geom.* 31 (1990), 501-526.

[14] Mackenzie, K., *Lie Groupoids and Lie Algebroids in Differential Geometry*, LMS Lecture Notes Series, 124, Cambridge Univ. Press, 1987.

[15] Mackenzie, K., Private communication.

[16] Mackenzie, K.C.H. and Xu, P., Lie bialgebroids and Poisson groupoids, *Duke Math. J.* (73), 1994, 415-452.

[17] Mackenzie, K.C.H. and Xu, P., Integration of Lie bialgebroids, preprint.

[18] Mikami, K., and Weinstein, A., Moments and reduction for symplectic groupoid actions, *Publ. RIMS Kyoto Univ.* 24 (1988),121-140.

[19] Weinstein, A., Coisotropic calculus and Poisson groupoids, *J. Math. Soc. Japan* 40 (1988), 705-727.

[20] Weinstein, A., Some remarks on dressing transformations, *J. Fac. Sci. Univ. Tokyo* 35 (1988), 163-167.

[21] Weinstein, A., Affine Poisson structures, *International J. of Math.* 1 (1990), 343–360.

[22] Weinstein, A. and Xu, P., Classical solutions of the quantum Yang-Baxter equations, *Commun. Math. Phys.* 148 (1992), 309-343.

[23] Xu, P., On Poisson groupoids, *International J. of Math.* 6, No. 1 (1995), 101-124.