THE RELATIVISTIC AND NON RELATIVISTIC QUARK-ANTIQUAQRK BOUND STATE PROBLEM IN A WILSON LOOP CONTEXT

N. BRAMBILLA and G. M. PROSPERI
Dipartimento di Fisica dell’ Università di Milano and I.N.F.N.
Via Celoria 16, 20133 Milano

The paper is on the line of the path integral technique developed in preceding articles. Taking advantage of a semirelativistic and a full relativistic representation of the quark propagator in an external field we present an unified derivation of the semirelativistic potential and of a Bethe–Salpeter like equation for the quark-antiquark system. We consider three different models for the evaluation of the Wilson loop: the Modified Area Law model (MAL), the Stochastic Vacuum Model (SVM) and the Dual QCD (DQCD). We compare the corresponding potentials and show that they all agree at the short and the long distances. In the case of the Bethe–Salpeter equation we treat explicitly only the MAL model and give an explicit expression for the kernel. Then we show that an effective mass operator can be obtained which agrees with the MAL potential in the semirelativistic limit. In the light quark mass limit this mass operator produces straight Regge trajectories with Nambu–Goto slope in agreement with the data. Finally we briefly discuss the mass independence of the hyperfine splitting in the heavy–light case.

1 Introduction

In preceding papers\footnote{1} we have shown how, using certain path integral representations of the quark-antiquark gauge invariant Green function $G(x_1, x_2; y_1, y_2)$ is possible to derive a semirelativistic potential for heavy quarks or a Bethe-Salpeter like ($BS$) equation for light and heavy quarks in terms of the Wilson loop integral $W$ and its functional derivatives. Such representations are the consequence of corresponding representations for the quark propagator in an external gauge field.

We have used two different path integral representations for the propagator. The first one is expressed in terms of a $\frac{1}{m}$ expansion and quark paths in ordinary phase space.\footnote{2} This is appropriate for the derivation of the heavy quark potential. The second one is expressed in terms of covariant space-time quark paths and it is useful for full relativistic developments.

Since up to now it is not possible to evaluate the Wilson loop analytically from first principles, to obtain explicit expressions one has to rely on models and/or lattice simulations. In this paper we shall discuss three different models: the Modified Area Law model (MAL), in which $i \ln W$ is written as the sum of a perturbative term, an area term and a perimeter term; the Stochastic Vacuum Model (SVM), which is based on a cumulant expansion;
the dual QCD (DQCD)\(^6\), which introduces effective fields and an effective lagrangian which should simulate what is believed to be the real mechanism for the formation of the confining flux tube. For each of these models a different potential can be obtained. We shall compare such potentials and show that they agree for large and small distances, but differ in the intermediate range.

Up today an explicit BS equation can be obtained only in the MAL case. Such equation is written in terms of a second order Green function \(H(x_1, x_2; y_1, y_2)\) to which \(G^{gi}(x_1, x_2; y_1, y_2)\) is related for large \(x^0_i - y^0_j(j = 1, 2)\) simply by the application of Dirac type differential operator. The BS kernel \(I(x_1, y_1; x_2, y_2)\) is expressed in principle as an expansion in the strong coupling constant \(\alpha_s\) and the string tension \(\sigma\) (more precisely the quantity \(\sigma a^2\), \(a\) being the typical radius of a bound state). In principle the method could be applied also to SVM and DQCD models but the actual calculations would be very complicated and have not been performed. In practice even the MAL kernel has been worked out only at the lowest order. We show that starting from the MAL BS equation a squared \((M^2)\) or a linear \((M)\) relativistic effective mass operator involving certain relativistic potential \(U\) or \(V\) can be obtained in the instantaneous limit. Spectrum calculation using such expressions are in progress. Obviously, in the semirelativistic limit \(V\) reproduces the MAL potential. On the contrary, if one retains full relativistic kinematic but neglects the spin dependent terms, \(M\) coincides with the hamiltonian of the Relativistic Flux Tube Model, discussed in Ref.\(^{12}\), up to ordering prescription and in the limit of vanishing quark masses gives Regge trajectories with slope \(\alpha' = \frac{1}{16\pi\sigma}\) as in Nambu-Goto string which are in agreement with phenomenology \((\sigma \sim 0.18\text{GeV}^2, \alpha' \sim 0.88)\). Furthermore, if the spin dependent terms are kept but the limit of heavy–light quarks is considered, spin symmetries similar to those discussed by Kaidalov\(^{16}\) are obtained. In Sec. 2 we give the basic equation for the definition of the potential and BS equation; in Secs. 3 and 4 we review the three mentioned models for the evaluation of the Wilson loop and compare the form of the corresponding potentials; in Sect. 5 we report the BS kernel in the MAL case and discuss the mass operator. The present paper is mainly based on\(^7\),\(^8\),\(^9\),\(^3\) but it is in part original in the presentation and contains some new results.

2 Definition of the semirelativistic potential and BS equation

In the usual functional integral formulation, after integration on the fermionic variables, the gauge invariant quark-antiquark Green function can be written

\[
G^{gi}(x_1, x_2; y_1, y_2) = \frac{1}{3} \text{Tr}(U(x_2, x_1)S^{(1)}(x_1, y_1|A)U(y_1, y_2)C^{-1}S^{(2)}(y_2, x_2|A)C). \tag{1}
\]
Here $C$ is the charge conjugation matrix, $U(a,b)$ denotes the Schwinger string

$$U(a,b) = P \exp \left\{ ig \int_a^b dx^\mu A_\mu(x) \right\} \quad (2)$$

(with $A_\mu(x) = \frac{1}{2} \lambda^a A^a_\mu(x)$ and $P$ the path ordering operator over the color matrices), $S(x,y;A)$ is the quark propagator in an external field

$$i \gamma^\mu D_\mu - m) S(x,y;A) = \delta^4(x-y) \quad (3)$$

($D_\mu = \partial_\mu - igA_\mu(x)$) and the notation

$$\langle f[A] \rangle = \left\langle \frac{DA M_f(A) f[A]}{DA M_f(A) e^{i S_{YM}[A]}} \right\rangle \quad (4)$$

is used, $M_f(A)$ being the fermionic determinant and $S_{YM}[A]$ the pure Yang–Mills action.

For a closed curve $\Gamma$ we set across the paper (Wilson loop correlator)

$$W(\Gamma) = \frac{1}{3} \langle \text{Tr} P \exp \left\{ \oint_\Gamma dx^\mu A_\mu(x) \right\} \rangle .$$

We are interested in loops formed by a line $\Gamma_1$ joining $y_1$ to $x_1$ (the quark path), another line $\Gamma_2$ joining $x_2$ to $y_2$ (the reverse antiquark path) and two straight lines connecting $x_1$ with $x_2$ and $y_2$ with $y_1$. In term of these the semirelativistic quark–antiquark potential is defined by

$$\int_{y_0}^{x_0} dt V_{QQ} = i \log W(\Gamma) - \sum_{j=1}^2 \frac{g}{2m_j} \int_{\Gamma_j} dx^\mu \left\{ \sigma_j^l \langle \hat{F}_{l\mu}(x) \rangle - \frac{1}{2m_j} \sigma_j^l \varepsilon^{lkr} p_j^k \langle F_{\mu r}(x) \rangle \right\}$$

$$- \frac{1}{8m_j} \langle D^\mu F_{\mu r}(x) \rangle - \frac{1}{2} \sum_{j,j'=1}^2 \frac{ig^2}{4m_j m_{j'}} T_\sigma \int_{\Gamma_j} dx^\mu \int_{\Gamma_{j'}} dx'^\sigma \sigma_j^l \sigma_{j'}^k$$

$$\times \left( \langle \hat{F}_{l\mu}(x) \hat{F}_{k\sigma}(x') \rangle - \langle \hat{F}_{l\mu}(x) \rangle \langle \hat{F}_{k\sigma}(x') \rangle \right) . \quad (5)$$

with $y_1^0 = y_2^0$, $x_1^0 = x_2^0 = x_0$, $T_\sigma$ time ordering prescription on the spin matrices, $\hat{F}_{\mu \nu} = \frac{1}{2} \epsilon_{\mu \nu \rho \sigma} F^{\rho \sigma}$ and

$$\langle f[A] \rangle = \left\langle \frac{DA e^{i S_{YM}(A)} \text{Tr} P \left\{ f(A) \exp \left\{ ig \oint_a^b dx^\mu A_\mu(x) \right\} \right\}}{DA e^{i S_{YM}(A)} \text{Tr} P \exp \left\{ ig \oint_a^b dx^\mu A_\mu(x) \right\}} \right\rangle . \quad (6)$$

Eq. (5) is obtained solving Eq. (3) for $x_0 > y_0$ by a Foldy–Wouthysen transformation and path integral technique and comparing the resulting expression for
$G^{g_i}$ with the two particle Schrödinger equation path representation. Notice that if $\delta S^{\mu\nu}(z) = \frac{1}{2}(dz^\mu \delta z^\nu - dz^\nu \delta z^\mu)$ denotes the surface swept by the quark path as a consequence of an infinitesimal variation $z(\tau) \to z(\tau) + \delta z(\tau)$ one has

$$\delta \{ P \exp[i \int_y^x dz^\mu A_\mu(z)] \} = ig P \int \delta S^{\mu\nu}(z) F_{\mu\nu}(z) \exp \{ ig \int_y^x dz^\mu A_\mu(z') \}. $$

From this it follows immediately

$$g \langle \langle F_{\mu\nu}(z_j) \rangle \rangle = (-1)^{j+1} \frac{\delta i \log(W(\Gamma))}{\delta S^{\mu\nu}(z_j)}, \quad (j = 1, 2) \quad (7)$$

$$g^2 (\langle \langle F_{\mu\nu}(z_1) F_{\lambda\rho}(z_2) \rangle \rangle - \langle \langle F_{\mu\nu}(z_1) \rangle \rangle \langle \langle F_{\lambda\rho}(z_2) \rangle \rangle) = -ig \frac{\delta}{\delta S^{\lambda\rho}(z_2)} \langle \langle F_{\mu\nu}(z_1) \rangle \rangle. \quad (8)$$

etc. and so everything in the right hand side of Eq. 5 can be expressed in terms of $W$. Notice, however, that, both through the expression of $i \ln W$ (e.g. cf. 19) and the spin dependent terms, the right hand side of 5 contains integrals on two separate times $t_1$ and $t_2$, while, by assumption, the left hand side contains one single time $t$. One can dispose of the above situation by integrating explicitly the relative time $\tau = t_1 - t_2$ after setting

$$z_j(t_j) = z_j(t) + \frac{1}{2}(-1)^{j+1} \tau \dot{z}_j(t) + \frac{1}{8} \tau^2 \ddot{z}_j(t) + \ldots \quad (9)$$

keeping only the appropriate order term and eliminating the second time derivatives by partial integration (notice that $\dot{z}_j^0 = 1; \ddot{z}_j = O(\dot{z}_j^2)$).

On the contrary, if one sets

$$S(x, y; A) = (i\gamma^\mu D_\mu + m)\Delta(x, y; A), \quad (10)$$

Eq. 3 gives

$$(D_\mu D^\mu + m^2 - \frac{1}{2}\sigma^{\mu\nu} F_{\mu\nu})\Delta(x, y; A) = -\delta^4(x - y). \quad (11)$$

Then, for large time intervals one can write

$$G^{g_i}(x_1, x_2; y_1, y_2) = (i\gamma^\mu \partial_\mu + m_1)(i\gamma^\nu \partial_\nu + m_2)H(x_1, x_2; y_1, y_2), \quad (12)$$

with

$$H(x_1, x_2; y_1, y_2) = \frac{1}{3} \Gamma(U(x_2, x_1) \Delta^{(1)}(x_1, y_1; A) U(y_1, y_2) \Delta^{(2)}(y_2, x_2; A)) \quad (13)$$
where the tilde denotes transposition of the color matrices. Furthermore, let us assume that $i\ln W(\Gamma)$ can be set in the general form

$$i\ln W = \frac{\lambda}{2} \sum_{i,j} \int_0^{s_i} d\tau_i \int_0^{s_j} d\tau_j' E(z_i, z_i', \dot{z}_i, \dot{z}_i') + \ldots , \quad (14)$$

where the dots stand for $\lambda^2$ terms involving similar fourth order integrals, $\lambda^3$ terms involving six order integrals and so on ($\lambda$ being a parameter that has been introduced for convenience) and $E$ is an homogeneous function of degree 0 in $\dot{z}_1$, $\dot{z}_2$. Then, using for $\Delta(x, y; A)$ the covariant path integral Feynman–Schwinger representation, one can obtain the inhomogeneous BS equation

$$H(x_1, x_2; y_1, y_2) = H^{(1)}(x_1 - y_1) H^{(2)}(x_2 - y_2) - i \int d^4 \xi_1 d^4 \xi_2 d^4 \eta_1 d^4 \eta_2$$. \quad (15)

with the kernel $I$ expressed as expansion in $\lambda$ and $H(x - y) = \langle \Delta(x, y; A) \rangle$. At the first order in $\lambda$ one has

$$I(\xi_1, \xi_2, \eta_1, \eta_2) = -4 \int \frac{d^4k_1 d^4k_2}{(2\pi)^8} R(\frac{\xi_1 + \eta_1}{2}, \frac{\xi_2 + \eta_2}{2}, k_1, k_2) \exp \left\{ -i[(\xi_1 - \eta_1)k_1 + (\xi_2 - \eta_2)k_2] \right\} , \quad (16)$$

with $R$ defined by

$$S_0^{s_1} S_0^{s_2} \int d\tau_1 \int d\tau_2 E(z_1, z_2, p_1, p_2) (S_0^{s_1} S_0^{s_2})^{-1} = \int_0^{s_1} d\tau_1 \int_0^{s_2} d\tau_2 R(z_1, z_2, p_1, p_2) \quad (17)$$

having set $S_0^T = T_s \exp \left[ -\frac{1}{4} \int_0^t d\tau \sigma^\mu{}^\nu \frac{\partial}{\partial S_0^\mu} (\tau) \right]$.

3 Models for Wilson Loop Evaluation

3.1 Modified Area Law Model

As mentioned, MAL model consists in assuming $i\ln W$ to be the sum of its perturbative value, area and perimeter term

$$i\log(W(\Gamma)) = i\log(W(\Gamma))_{\text{pert}} + \sigma S_{\text{min}} + \frac{1}{2} CP , \quad (18)$$

where obviously $S_{\text{min}}$ denotes the minimal area enclosed by the loop $\Gamma$ and $P$ its length as suggested by the Wilson strong coupling limit, while $\sigma$ and
C are in practice treated as independent adjustable parameters (which must however agree with the lattice simulations). At lowest order one has

\[ i(\ln W)_{\text{pert}} = \frac{4}{3} g^2 \int_{y^{10}}^{x^{10}} dt_1 \int_{y^{20}}^{x^{20}} dt_2 \dot{z}_1^\mu \dot{z}_2^\nu D_{\mu\nu}(z_1 - z_2). \]  

(19)

Furthermore

\[ S_{\text{min}} = \min \int_{t_i}^{t_f} dt \int_0^1 d\alpha \left[\left(\frac{\partial u^\mu}{\partial t} \frac{\partial u_\mu}{\partial t}\right) + \left(\frac{\partial u^\mu}{\partial \alpha} \frac{\partial u_\mu}{\partial \alpha}\right)\right]^2 \frac{1}{2}, \]

(20)

\[ P = \sum_j \int_{y^{j0}}^{x^{j0}} dt_j [\dot{z}_j^\mu \dot{z}_j^\nu]^\frac{1}{2}, \]

(21)

where \( u^\mu = u^\mu(t, \alpha) \) with \( y^0 < t < x^0, \) \( 0 < \alpha < 1 \) is the equation of an arbitrary surface enclosed by \( \Gamma \) and satisfying therefore boundary conditions

\[ u^\mu(t, 0) = z_2^\mu(t), \quad u^\mu(t, 1) = z_1^\mu(t). \]  

(22)

In practice an equal time straight line approximation is commonly adopted for \( S_{\text{min}}. \) This amounts to setting

\[ u^\mu(t, \alpha) = t, \quad u(t, \alpha) = \alpha z_1(t) + (1 - \alpha) z_2(t), \]

(23)

\[ S_{\text{min}} = \int_{t_i}^{t_f} dt \int_0^1 d\alpha [1 - (\alpha \dot{z}_{1T} + (1 - \alpha) \dot{z}_{2T})^2]^\frac{1}{2} = \]

(24)

\[ = \int_0^{s_1} \, ds_1 \int_0^{s_2} \, ds_2 \delta(z_1^0 - z_2^0) |z_1 - z_2| \int_0^1 \{ z_1^0 z_2^0 - [\alpha z_{1T} z_2^0 + (1 - \alpha) z_{2T} z_2^0]^2 \}^\frac{1}{2} \]

with \( z_j^h = (\delta^{hk} - \eta^{jk}) z_j^k \) (obviously in the first step \( \dot{z}_j = \frac{dz_j}{dt} \), in the second one \( \dot{z}_j^h = \frac{dz_j^h}{dt} \)). This equation it is exact to order \( 1/m^2 \) and for particular geometries. The second step is given in terms of a covariant parametrization of the quark and the antiquark paths and is useful in the derivation of the BS equation; however it is not Lorentz invariant and it is assumed to be true in the center of mass frame.

3.2 Stochastic Vacuum Model

Using the non abelian Stokes theorem and cumulant expansions, one can write

\[ \langle W(\Gamma) \rangle = \left\langle P \exp \left( ig \int_S d\sigma^{\mu\nu}(u) F_{\mu\nu}(u, x_0) \right) \right\rangle \]  

(25)
\[ = \exp \sum_{j=1}^{\infty} \frac{(ig)^j}{j!} \int_S dS^{\mu_1 \nu_1} (u_1) \ldots \int_S dS^{\mu_j \nu_j} (u_j) \langle F_{\mu_1 \nu_1} (u_1, x_0) \ldots F_{\mu_j \nu_j} (u_j, x_0) \rangle_{\text{cum}} \] 

where \( F_{\mu \nu} (u, x_0) = U(x_0, u) F_{\mu \nu} U(u, x_0) \) and the cumulants \( \langle \ldots \rangle_{\text{cum}} \) are defined by

\[ \langle F(1) \rangle_{\text{cum}} = \langle F(1) \rangle, \quad \langle F(1) F(2) \rangle_{\text{cum}} = \langle F(1) F(2) \rangle - \langle F(1) \rangle \langle F(2) \rangle, \ldots \] 

(27)

\( S \) is an arbitrary surface enclosed by \( \Gamma \), \( x_0 \) a reference point on \( S \), \( P_S \) an ordering prescription on \( S \) and the \( U(u, x_0) \) are the Schwinger strings.

The basic approximation consists in assuming that the second cumulant is dominant and actually independent of \( x_0 \). Then, since the first cumulant vanishes trivially, one can write

\[ \log \langle W(\Gamma) \rangle = -g^2 \int_S dS^{\mu \nu} (u) \int_S dS^{\lambda \rho} (v) \langle F_{\mu \nu} (u, x_0) F_{\lambda \rho} (v, x_0) \rangle_{\text{cum}}. \] 

(28)

and, taking into account Lorentz invariance,

\[ g^2 \langle F_{\mu \nu}(u, x_0) F_{\lambda \rho}(v, x_0) \rangle_{\text{cum}} = g^2 \langle F_{\mu \nu}(u, x_0) F_{\lambda \rho}(v, x_0) \rangle 
= \beta \left\{ (\delta_{\mu \lambda} \delta_{\nu \rho} - \delta_{\mu \rho} \delta_{\nu \lambda}) D((u - v)^2) + \frac{1}{2} \frac{\partial}{\partial u_{\mu}}((u - v)_{\lambda} \delta_{\nu \rho} - (u - v)_{\rho} \delta_{\nu \lambda}) + \frac{\partial}{\partial u_{\nu}}((u - v)_{\rho} \delta_{\mu \lambda} - (u - v)_{\lambda} \delta_{\mu \rho}) D_1((u - v)^2) \right\} \] 

(29)

where \( \beta \equiv \frac{g^2}{36} \frac{\langle \text{Tr} F_{\mu \nu}(0) F_{\mu \nu}(0) \rangle}{D(0)+D_1(0)} \) and \( D \) and \( D_1 \) are unknown functions. In the euclidean space, at the lowest order in perturbation theory one finds

\[ D_{\text{pert}}(x^2) = 0 \quad D_{\text{1-pert}}^1(x^2) = \frac{16 \alpha_s}{3 \pi} \frac{1}{x^2}, \] 

which is supposed to give the behaviour of the functions for \( x \to 0 \). A good parametrization for the long range region seems to be

\[ \begin{align*}
\beta D_{\text{LR}}(x^2) &= d e^{-\delta |x|}, \quad \delta = (1 \pm 0.1) \text{ GeV}, \quad d = 0.073 \text{ GeV}^4, \\
\beta D_1^{\text{LR}}(x^2) &= d_1 e^{-\delta_1 |x|}, \quad \delta_1 = (1 \pm 0.1) \text{ GeV}, \quad d_1 = 0.0254 \text{ GeV}^4,
\end{align*} \] 

(30)

where the value of gluonic condensate and lattice simulations have been combined.
3.3 Dual QCD

Dual QCD is an effective theory described by a lagrangian density $L_{\text{eff}}$ in which the fundamental variables are an octet of dual potential $C^\mu$ coupled to a classical quark source and to three octets of scalar fields $B_i$ carrying "magnetic" color charge. The "monopole" fields $B_i$ develop nonvanishing vacuum expectation values $B_{0i}$ that give rise to a dual Meissner effect and provide a concrete realization of the Mandelstam-t’Hooft picture of confinement. We set

$$W_{\text{eff}}(\Gamma) = \frac{\int D\bar{C}_\mu D\phi DB_3 e^{i \int dx [L_{\text{eff}}(G_{\mu\nu}^S) + L_{\text{GF}}]}}{\int D\bar{C}_\mu D\phi DB_3 e^{i \int dx [L_{\text{eff}}(G_{\mu\nu}^S=0) + L_{\text{GF}}]}}.$$  

(31)

where

$$L_{\text{eff}} = 2\text{Tr} \left[ -\frac{1}{4} G_{\mu\nu}^S G_{\mu\nu} + \frac{1}{2} (D_\mu B_i)^2 \right] - W(B_i),$$

(32)

$$D_\mu B_i = \partial_\mu B_i - ig_M [C_\mu, B_i],$$

(33)

$$G_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu - ig_M [C_\mu, C_\nu] + G_{\mu\nu}^S,$$  

(34)

$L_{\text{GF}}$ is the gauge fixing term, $g_M = \frac{2\pi}{g}$ is the magnetic coupling constant and

$$G_{\mu\nu}^S(x) = g \frac{\lambda_8}{\sqrt{3}} \varepsilon_{\mu\nu\rho\sigma} \int_0^1 d\alpha \frac{\partial u^\rho}{\partial \alpha} \frac{\partial u^\sigma}{\partial t} \delta(x - u(\alpha, t)),$$

(35)

$u^\mu = u^\mu(t, \alpha)$ being the surface swept by the Dirac string connecting the quark and the antiquark. The Higgs potential $W(B_i)$ (which we do not give explicitly) has a minimum at non zero values $B_{0i}$ of the form

$$B_1 = B_0 \lambda_7, \quad B_2 = -B_0 \lambda_5, \quad B_3 = B_0 \lambda_2.$$  

(36)

In our present context the basic assumption is that for large $\Gamma$

$$W(\Gamma) \simeq W_{\text{eff}}.$$  

(37)

To calculate explicitly $W_{\text{eff}}$ even for restrict class of quantum fluctuation is not a trivial task. However, in the so called classical approximation this quantity is given by the expression $\text{exp} \{ i \int d^4x L_{\text{GF}}(G_{\mu\nu}^S) \}$ evaluated for a classical solution of the field equations of the form

$$\bar{C}^\mu = \frac{\bar{C}^\mu \lambda_8}{\sqrt{3}}$$

$$\bar{B}_1 = B \lambda_7 \quad \bar{B}_2 = -B \lambda_5 \quad \bar{B}_3 = B' \lambda_2$$  

(38)
where $\bar{C}^\mu$, $\bar{B}$ and $\bar{B}'$ are appropriate solutions of the equations

$$
\partial^\mu (\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu) = -\partial^\mu \bar{G}^S_{\mu\nu} - 6g^2 \bar{C}_\mu \bar{B}^2
$$

$$
(\partial_\mu + ig \bar{C}_\mu)^2 \bar{B} = -\frac{1}{4} \frac{\delta U}{\delta \bar{B}} \quad \partial^2 \bar{B}' = -\frac{1}{4} \frac{\delta U}{\delta \bar{B}'}
$$

(39)

satisfying the asymptotic conditions $\bar{C}^\mu \to 0$, $\bar{B}, \bar{B}' \to B_0$. Then Eqs. (39) can be solved numerically and analytically parametrized.

4 Semirelativistic Potential

On the basis of general invariance principle the quark-antiquark potential up to order $1/m^2$ can be written as

$$
V_{QQ} = V_0(r) + V_{VD}(r, p_1, p_2) + V_{SD}(r, p_1, p_2, \sigma_1, \sigma_2)
$$

(40)

with

$$
V_{VD}(r) = \frac{1}{m_1 m_2} \left\{ p_1 \cdot p_2 V_b(r) + \left( \frac{1}{3} p_1 \cdot p_2 - \frac{p_1 \cdot r}{r^2} \right) V_c(r) \right\}_{\text{Weyl}}
$$

$$
+ \sum_{j=1}^2 \frac{1}{m_j^2} \left\{ p_j^2 V_d(r) + \left( \frac{1}{3} p_j^2 - \frac{p_j \cdot r}{r^2} \right) V_e(r) \right\}_{\text{Weyl}},
$$

(41)

and

$$
V_{SD} = \frac{1}{8} \left( \frac{1}{m_1^2} + \frac{1}{m_2^2} \right) [V_0(r) + V_a(r)]
$$

$$
+ \left( \frac{1}{4m_1^2} L_1 \cdot \sigma_1 - \frac{1}{4m_2^2} L_2 \cdot \sigma_2 \right) \frac{1}{r} \frac{d}{dr} [V_0(r) + 2V_1(r)]
$$

$$
+ \frac{1}{2m_1 m_2} (L_1 \cdot \sigma_2 - L_2 \cdot \sigma_1) \frac{1}{r} \frac{d}{dr} V_2(r) + \frac{1}{4m_1 m_2} \left( \sigma_1 \cdot r \sigma_2 \cdot r - \frac{1}{3} \sigma_1 \cdot \sigma_2 \right) V_3(r)
$$

$$
+ \frac{1}{12m_1 m_2} \sigma_1 \cdot \sigma_2 V_4(r),
$$

(42)

in terms of functions of $r$ alone.

The various functions $V_0(r), \ldots V_4(r)$ can be written in terms of momenta of fields. Such expressions are very useful for lattice simulations.

Finally, one can work out a potential in explicit form for any of the models considered in Sec. 3. In Tab. 1 for the MAL model we report the complete expression, while for SVM and DQCD we report only the long range behaviour.
Table 1: Complete MAL potential and long distance SVM and DQCD potentials

| V   | MAL                              | SVM                              | DQCD                             |
|-----|----------------------------------|----------------------------------|----------------------------------|
| V₀  | $-\frac{1}{3}\frac{\sigma r}{\delta^2} + \sigma r + C$ | $\sigma_2 r + \frac{1}{2}C_2^{(1)} - C_2$ | $\sigma r - 0.646\sqrt{\sigma \alpha_s}$ |
| $\Delta V_a$ | 0                               | Self-energy terms                |                                  |
| $V_1'$ | $\sigma$                        | $-\sigma_2 + \frac{C_2 r}{\delta}$ | $-\sigma + \frac{0.681}{\sqrt{\sigma \alpha_s}}$ |
| $V_2'$ | $\frac{4\alpha_s r}{\delta}$   | 0                               |                                  |
| $V_3$ | $32\pi\alpha_s \delta^3 (r)$   | 0                               |                                  |
| $V_4$ | $\frac{8\alpha_s r}{\delta} - \frac{1}{9}\sigma r$ | $-\frac{1}{9}\sigma_2 r - \frac{2}{3}D_2 - \frac{1}{3}E_2$ | $-0.097\sigma r - 0.226\sqrt{\sigma \alpha_s}$ |
| $V_5$ | $-\frac{4\alpha_s r}{\delta} - \frac{1}{6}\sigma r$ | $-\frac{1}{6}\sigma_2 r - \frac{2}{3}D_2 - \frac{1}{3}E_2$ | $-0.146\sigma r - 0.516\sqrt{\sigma \alpha_s}$ |
| $V_6$ | $-\frac{1}{6}\sigma r - \frac{1}{4}C$ | $-\frac{1}{4}\sigma_2 r + \frac{1}{6}C_2 - \frac{1}{3}C_2^{(1)} + \frac{1}{3}D_2 - \frac{1}{3}E_2$ | $-0.118\sigma r + 0.275\sqrt{\sigma \alpha_s}$ |
| $V_7$ | $-\frac{1}{6}\sigma r$           | $-\frac{1}{6}\sigma_2 r + \frac{1}{3}D_2 - \frac{1}{3}E_2$ | $-0.177\sigma r + 0.258\sqrt{\sigma \alpha_s}$ |

(we omit exponentially vanishing terms). In terms of the parametrization of Eq. 30 the constants occurring in the second column are given by

\[
\begin{align*}
\sigma_2 &= \frac{\pi d}{\delta^2} \\
C_2 &= \frac{4d}{\delta^2} \\
C_2^{(1)} &= \frac{4d_1}{\delta_1^2} \\
D_2 &= \frac{3\pi d}{\delta^3} \\
E_2 &= \frac{32d}{\delta^3}
\end{align*}
\]

(43)

Complete expressions even for SVM and DQCD can be found in [3] and [4]. Notice that the short range behaviour must agree for the three potentials by construction. In the long range region, as one can see, there is essentially agreement among the leading terms. Some minor discrepancies in the DQCD case are possibly due to numerical inaccuracy. On the contrary the discrepancies in the subleading terms seem to be more significant. Notice however the complete similarities between the SVM and the DQCD columns for the first six lines of the table. Obviously the three models definitely differ at intermediate ranges.

For what concerns the numerical values of the constants one may recall that
from heavy quarkonia fitting typical values are \( \sigma \simeq 0.18 \text{GeV}^2 \) and \( \alpha_s \simeq 0.37 \). On the other side if from lattice simulation we assume [cf. Eq.30] the marginal values \( \delta = \delta_1 = 1.1, \) we find \( \sigma \simeq 0.19 \text{GeV}^2, \quad C_2 \simeq 0.22 \text{GeV}, \quad C_2^{(1)} \simeq 0.08 \text{GeV}, \quad D_2 \simeq 0.47, \quad E_2 \simeq 1.45 \text{GeV}^{-1}. \) In this case the agreement between SVM and DQCD is therefore remarkable for the first six lines even from the numerical point of view. Finally, notice that in the MAL case the spectrum is very little sensible to the values of the constant \( C. \) In fact due to the combined form of \( V_0 \) and \( V_2 \) a variation of \( C \) can be reabsorbed in redefinition of the quark masses \( m \to m + \delta \). This is not true in the SVM and DQCD cases due to the form of \( V_1 \) and \( V_2 \).

5 Bethe-Salpeter kernel and effective mass operator

Let us begin to notice that in the MAL case, neglecting the perimeter term and taking into account Eqs. 14 and 24, Eq. 19 can turn out to be of the general form of Eq. 14 with only the first term. Additional terms would be necessary if we want to include higher order perturbation terms in Eq. 19 or replace the simple MAL non perturbative part of the Wilson loop by more complicate expressions, like that of Eq. 37. Of the terms in \( \lambda \) the \( i = j \) terms correspond to self–energies of the quark or of the antiquark, the \( i \neq j \) terms to the interaction between the quark and the antiquark.

Introducing Eqs. 19, 24 in 17,16 one finds explicitly in the momentum space (after factorizing the conservation delta \((2\pi)^4 \delta^4(p_1 + p_2 - p_1' - p_2')\))

\[
\tilde{I}(p_1, p_2; p_1', p_2') = \tilde{I}_{\text{pert}} + \tilde{I}_{\text{conf}} = 16\pi^2 \alpha_s \{ D_{\rho\sigma}(Q)q_1^{\rho}q_2^{\sigma} - \\
\frac{i}{4} \sigma_{\mu\nu}^1 (\delta_{\mu}^\rho Q_{\nu} - \delta_{\nu}^\rho Q_{\mu})q_1^{\rho} D_{\rho\sigma}(Q) + \frac{4}{4} \sigma_{\mu\nu}^2 (\delta_{\mu}^\rho Q_{\nu} - \delta_{\nu}^\rho Q_{\mu})q_1^{\rho} D_{\rho\sigma}(Q) + \\
+ \frac{1}{16} \sigma_{\mu\nu}^{11} (\delta_{\mu}^\rho Q_{\nu} - \delta_{\nu}^\rho Q_{\mu})q_1^{\rho} D_{\rho\sigma}(Q) \}
\]

\[
+ \int d^3r e^{iQr} J(r, q_1, q_2), \quad (44)
\]

with

\[
J(r, q_1, q_2) = \frac{2\pi r}{q_{10} + q_{20}} \left[ q_{20} \sqrt{q_{10}^2 - q_{1T}^2} + q_{10} \sqrt{q_{20}^2 - q_{2T}^2} + \\
+ \frac{q_{10}q_{20}}{|q_T|} (\arcsin \frac{|q_{1T}|}{q_{10}} + \arcsin \frac{|q_{2T}|}{q_{20}}) \right] + \frac{2\sigma r^k}{r} \left( \frac{\sigma_{12}^{12} q_{20} q_{10}}{q_{10}^2 - q_{1T}^2} + \frac{\sigma_{22}^{12} q_{10} q_{20}}{q_{20}^2 - q_{2T}^2} \right) + \ldots
\]

In such equations \( \alpha_s = \frac{2}{\pi} \) denotes the strong coupling constant and \( D_{\rho\sigma}(Q) \) the free gluon propagator; furthermore we set \( q_1 = \frac{p_1 + p_2}{2}, q_2 = \frac{p_2 + p_2'}{2}, Q = \)
having kept only first order terms in $U$.

From this one can also derive the more conventional operator $M$ to 1 on the energy shell ($\Phi$ with $\eta$ with $(\text{consisting in setting } H_2 = \eta_2 P_B - k) \tilde{I}(k, k'; P_B) \Phi(k')$, (46)

with $\eta_j = -\frac{m_j}{(m_1 + m_2)}; P_B = (m_B, 0)$, $m_B$ being the mass of the bound state and $\Phi_B$ an appropriate wave function. In the instantaneous approximation (consisting in setting $H_2 = -\frac{i}{2} H_2^0$ and $k_0 = k_0' = \eta_2 \frac{w_1 + w_1'}{2} - \eta_1 \frac{w_2 + w_2'}{2}$) $\tilde{I}(k, k'; P)$ with $w_j = \sqrt{m_j^2 + k^2}, w_j' = \sqrt{m_j^2 + k''}$ the residual variables $k_0$ and $k_0'$ can be integrated explicitly and one is left with the eigenvalue equation for an effective mass squared operator $M^2 = M_0^2 + U$ with $M_0 = w_1 + w_2$ and

$$\langle k|U|k' \rangle = \frac{1}{(2\pi)^3} \sqrt{\frac{w_1 w_2}{w_1' w_2'}} \tilde{I}_{\text{inst}}(k, k') \sqrt{\frac{w_1 + w_2}{w_1' + w_2'}}.$$ (47)

From this one can also derive the more conventional operator $M = M_0 + V$, where

$$\langle k|V|k' \rangle = \frac{1}{(2\pi)^3} \frac{1}{4 \sqrt{w_1 w_2 w_1' w_2'}} \tilde{I}_{\text{inst}}(k, k') + \ldots$$ (48)

having kept only first order terms in $U$ and neglected kinematical factors equal to 1 on the energy shell ($w_1 + w_2 = w_1' + w_2'$).

From Eqs. (44–45) one obtains explicitly

$$\langle k|U|k' \rangle = \sqrt{\frac{(w_1 + w_2)(w_1' + w_2')}{w_1 w_2 w_1' w_2'}} \left\{ -\frac{4}{3} \alpha \frac{1}{Q^2} \left[ q_{10} q_{20} + \mathbf{q}^2 + \frac{(\mathbf{Q} \cdot \mathbf{q})^2}{Q^2} \right] 
+ \frac{i}{2Q^2} \mathbf{k} \times \mathbf{k}' \cdot (\sigma_1 + \sigma_2) + \frac{1}{2Q^2} \left[ q_{20} (\alpha_1 \cdot \mathbf{Q}) - q_{10} (\alpha_2 \cdot \mathbf{Q}) \right] 
+ \frac{1}{6} \sigma_1 \cdot \sigma_2 + \frac{1}{4} \left( \frac{1}{3} \sigma_1 \cdot \mathbf{Q} \cdot \sigma_2 - \frac{\mathbf{Q} \cdot \sigma_1 (\mathbf{Q} \cdot \sigma_2)}{Q^2} \right) + \frac{1}{4Q^2} (\alpha_1 \cdot \mathbf{Q}) (\alpha_2 \cdot \mathbf{Q}) \right\} (49)
+ \frac{1}{(2\pi)^3} \int d^3 r e^{i \mathbf{r} \cdot \mathbf{k}} J_{\text{inst}}(\mathbf{r}, \mathbf{q}_1, q_{10}, q_{20}) \right\}$$
\[
J^{\text{inst}}(r, q, q_{10}, q_{20}) = -\frac{\sigma r}{q_{10} + q_{20}} \left[ q_{20}^2 \sqrt{q_{10}^2 - q_T^2} + q_{10}^2 \sqrt{q_{20}^2 - q_T^2} \right] + \\
+ \frac{q_{10}^2 q_{20}^2}{q_T^2} \left( \arcsin \frac{|q_T|}{q_{10}} + \arcsin \frac{|q_T|}{q_{20}} \right) - i \frac{q_{20}}{r \sqrt{q_{10}^2 - q_T^2}} (r \times q \cdot \sigma_1 + iq_{10}(r \cdot \alpha_1)) + \\
\frac{q_{10}}{\sqrt{q_{20}^2 - q_T^2}} (r \times q \cdot \sigma_2 - iq_{20}(r \cdot \alpha_2)) \right] + \\
\frac{q_{10}^2 q_{20}^2}{q_T^2} \left( \arcsin \frac{|q_T|}{q_{10}} + \arcsin \frac{|q_T|}{q_{20}} \right) - i \frac{q_{20}}{r \sqrt{q_{10}^2 - q_T^2}} (r \times q \cdot \sigma_1 + iq_{10}(r \cdot \alpha_1)) + \\
\frac{q_{10}}{\sqrt{q_{20}^2 - q_T^2}} (r \times q \cdot \sigma_2 - iq_{20}(r \cdot \alpha_2)) \right)
\]

(50)

Here \( \alpha_j^k \) denote the usual Dirac matrices \( \gamma_j^0 \gamma_j^k \), \( \sigma_j^k \) the \( 4 \times 4 \) Pauli matrices \( \sigma_j^k = \begin{pmatrix} 0 & \sigma_j^k \\ \sigma_j^k & 0 \end{pmatrix} \) and obviously \( q = \frac{k + k'}{2}, \quad Q = k - k', \quad q_{j0} = \frac{w_j + w_j'}{2} \). Notice that, due to the terms in \( \alpha_j^k \), such \( U \) is hermitian only with reference to the undefined metric operator \( \gamma_0^0 \).

Due to Eq. (48) the potential \( V \) can be obtained from \( U \) as given by Eqs. (49-50) simply by the kinematical replacement \( \sqrt{\frac{w_1 w_2 (w_1' + w_2')}{w_1 w_2' w_1' w_2'}} \rightarrow \frac{1}{2\sqrt{w_1 w_2 w_1' w_2'}} \).

Notice that, if in the resulting expression we perform an \( 1/m \) expansion and make an appropriate Foldy-Wouthuysen transformation to eliminate the terms in \( \alpha_j^k \), we reobtain the MAL potential as given in Tab. 1.

On the contrary, if we keep only the long range terms in \( V \), neglect the spin dependent part and set for simplicity \( m_1 = m_2 = m \), we can write in an operatorial form

\[
M = 2 \sqrt{m^2 + q^2} + \frac{\sigma r}{2} \left( \sqrt{m^2 + q^2} \right) \arcsin \frac{|q_T|}{\sqrt{m^2 + q^2}} + \sqrt{m^2 + q_T^2} \quad \left( q_L = q - q_T \right)
\]

(51)

which the ordering implied by the definition referring to Eq. (50) has to be understood. Such expression is identical to the hamiltonian of the Relativistic Flux Tube Model [12, 13] up to the order \( \sigma a^2 \). Two different limits of (51) are of interest. The first one is the limit of small angular momentum. i.e. small transversal momentum \( (q_T^2 = \frac{q_T^2}{m^2} \leq 1) \). In this case we have (strictly for \( s \) waves)

\[
M = \sqrt{m^2 + q^2} + \sigma r
\]

(52)

This result justifies the use that of Eq. (52) has been done in the study of light meson spectrum. The second limit is for large angular momentum or transversal momentum (negligible \( q_T^2 \)) and is

\[
M = 2 \sqrt{m^2 + q^2} + \frac{\sigma r}{2} \left( \sqrt{m^2 + q^2} \right) \arcsin \frac{|q|}{\sqrt{m^2 + q^2}} + \frac{m}{\sqrt{m^2 + q^2}} \quad \left( q_T = 0 \right)
\]

(53)
or under the additional assumption of negligible $m$

$$M = 2|q| + \frac{\pi}{4}\sigma r$$  \quad (54)

Eq. 54 is very important for an understanding of the Regge trajectory properties. In fact it is well known that such hamiltonian produces asymptotically straight trajectories with

$$m_l^2 \to 8\frac{\pi}{2}\sigma l = 2\pi\sigma l$$  \quad (55)

$m_l$ being obviously the mass of a given radial quantum number bound state as a function of the angular momentum $l$. Notice that Eq. 55 corresponds to the slope $\alpha' = \frac{dl}{dm} \to \frac{1}{2\sigma\pi}$ which is identical to that of the Nambu-Goto string model. Notice that for $\sigma = 0.18$ we find $\alpha' = 0.88$ in very good agreement with the experimental slope of the $\rho$ trajectory. Had we used the naive Eq. 52, even for large $l$, we would have find $\alpha' = 1/8\sigma = 0.69$. As mentioned such results are in perfect agreement with those of Refs. 13.

Finally, if we keep all terms in Eq. 51 but assume $m_2 \gg m_1$, we can discuss the light–heavy quark symmetry much on the lines followed by Kaidalov. 13

E. g. let us consider the quadratic hyperfine separation between triplet and singlet $\delta m^2 = m_1^2 - m_2^2$ in the $(uQ)$ states. Empirically this quantity is nearly independent of $Q$ ($\sim 0.55$ MeV$^2$). In fact the hyperfine splitting term in $U$ (cf. 49) depends on the quark masses only through the kinematical factor occurring in it. Now for $m_2 \gg m_1$ such factor reduces to $\sqrt{\frac{1}{m_1 m_2}}$ and any dependence on $m_2$ disappears.

Acknowledgments

We gratefully acknowledge discussions with Baker, Dosch, Dubin, Kaidalov, Simonov.

References

1. A. Barchielli, E. Montaldi and G. M. Prosperi, Nucl. Phys. B 296, 625 (1988) A. Barchielli, N. Brambilla and G. M. Prosperi, Nuovo Cimento 103 A, 59 (1990);
2. N. Brambilla and G. M. Prosperi, in Proceedings of “Quark Confinement and the Hadron Spectrum”, eds. N. Brambilla and G. M. Prosperi, p.113, (World Scientific, Singapore, 1995);
3. N. Brambilla, E. Montaldi and G. M. Prosperi, Phys. Rev. D54 3506 (1996)
4. M.A. Peskin, in Proceedings of the 11th SLAC Inst., SLAC Rep. n.207, 151 ed. by P. Mc. Donough (1993);
5. K.D. Born, E. Laermann, R. Sommer, T.F. Walsh and P.M. Zerwas, Phys. Lett. B 329 325 (1994); 332 (1994); G. Bali, these Proceedings;
6. M. Baker, J. Ball and F. Zachariasen, Phys. Rev. D 51, 1968 (1995);
7. M. Baker, J. S. Ball, N. Brambilla, G. M. Prosperi and F. Zachariasen, Phys. Rev. D54 2829 (1996);
8. H. G. Dosch, Phys. Lett. B 190, 177 (1987); H. G. Dosch and Yu. A. Simonov, Phys. Lett. B 205, 339 (1988);
9. N. Brambilla and A. Vairo, ‘Heavy quarkonia: Wilson area law ⋅⋅⋅’, IFUM 533/FT (June 1996); hep-ph/9606344, to appear in Phys. Rev. D
10. A. Yu. Dubin, A. B. Kaidalov and Yu. A. Simonov, Phys. Lett. B 323, 41 (1994);
11. W. Lucha, F. F. Schöberl and D. Gromes, Phys. Rep. 200, 127 (1991) and refs. therein;
12. M. G. Olsson, this proceedings and references therein;
13. N. Brambilla and G. M. Prosperi, Phys. Rev. D 47, 2107 (1993);
14. A. Di Giacomo and H. Panagopoulos Phys. Lett. B 285, 133 (1992);
15. Yu. A. Simonov, Nucl. Phys. B324 67 (1989); Yu. A. Simonov and J. Tjon, Ann. Phys. (N.Y.) 228 1 (1993)
16. A. Kaidalov, talk at this Conference and review in preparation