Abstract—This paper deals with adaptive radar detection of a subspace signal competing with two sources of interference. The former is Gaussian with unknown covariance matrix and accounts for the joint presence of clutter plus thermal noise. The latter is structured as a subspace signal and models coherent pulsed jammers impinging on the radar antenna. The problem is solved via the Principle of Invariance which is based on the identification of a suitable group of transformations leaving the considered hypothesis testing problem invariant. A maximal invariant statistic, which completely characterizes the class of invariant decision rules and significantly compresses the original data domain, as well as its statistical characterization are determined. Thus, the existence of the optimum invariant detector is addressed together with the design of practically implementable invariant decision rules. At the analysis stage, the performance of some receivers belonging to the new invariant class is established through the use of analytic expressions.

Index Terms—Adaptive Radar Detection, Constant False Alarm Rate, Invariance, Maximal Invariants, Subspace Model, Coherent Interference.

I. INTRODUCTION

Adaptive radar detection of targets embedded in Gaussian interference is an active research field, which has enjoyed the effort of world class scientists. As a result, in the last decades, a plethora of adaptive or nonadaptive detection strategies have been proposed to face with realistic applications. Generally speaking, the existing architectures can be classified into two macro-classes which differ in the level of a priori knowledge about the target response. The former assumes perfectly known target signature at the design stage (rank-1 receivers) [1]–[6] and references therein], whereas the latter takes into account possible uncertainties in the nominal steering vector [7]–[9]. As a matter of fact, in situations of practical interest, several error sources (such as array calibration uncertainties, beampointing errors, multipath, etc.) [10] can give rise to mismatched signals which can degrade the detection performances of the architectures belonging to the first class [7], [11], [12].

The subspace detection represents an effective analytic tool incorporating the partial knowledge of the target response in the detector design and, hence, mitigating the performance degradation due to steering vector errors [13]–[21]. The main idea of the subspace approach is to constrain the target steering vector to belong to a suitable subspace of the observation space. By doing so, it is possible to capture the energy of the potentially distorted wavefront of a mainlobe target. As a result, architectures exploiting this trick are capable of declaring the presence of targets whose signature is significantly different from the nominal one. It is worth pointing out that the subspace idea can be also exploited to model coherent interfering signals impinging on the radar antenna, whose directions of arrival have been estimated within some uncertainty. For instance, in [13], [14] the authors devise adaptive decision schemes to reveal extended targets when the interference comprises a random contribution (referring to clutter and thermal noise) and a structured unwanted component.

In [17], [22], and [14] the subspace detection has been dealt with the invariance theory in hypothesis testing problems [23]–[25]. By doing so, the authors focus on decision rules exhibiting some natural symmetries implying important practical properties such as the Constant False Alarm Rate (CFAR) behavior. Besides, the use of invariance leads to a data reduction because all invariant tests can be expressed in terms of a statistic, called maximal invariant, which organizes the original data into equivalence classes. Also the parameter space is usually compressed after reduction by invariance and the dependence on the original set of parameters becomes embodied into a maximal invariant in the parameter space (induced maximal invariant). Further examples of the application of such theory to radar detection problems can be also found in [26]–[33].

It is now important to observe that some of the just mentioned papers do not assume the presence of structured interference [17], [22], [26]–[33], while others exploit the Principle of Invariance to solve the nonadaptive subspace detection problem when a coherent interference source illuminates the radar antenna [14]. The possibility to address the adaptive subspace detection problem in the joint presence of random and subspace structured interference via invariance, to the authors best knowledge, has not yet been considered in open literature. In this respect, the scope of this paper is to fill this gap. To this end, this work is focused on adaptive detection of a subspace signal competing with two interference sources. The former is a completely random component, modeled as a Gaussian vector with unknown covariance matrix, and represents the returns from clutter and thermal noise. The latter is a subspace structured signal (with unknown location parameters).
and accounts for the presence of (possible) multiple pulsed coherent jammers impinging on the radar antenna from some directions. The problem is analytically formulated as a binary hypothesis test and the Principle of Invariance is exploited at the design stage to concentrate the attention on radar detectors enjoying some desirable practical features. More specifically, a suitable group of transformation leaving the considered testing problem unaltered is established. Hence, the practical importance of the resulting group action is explained as a mean to impose the CFAR property with respect to the clutter plus noise covariance matrix and the jammer location parameters. Otherwise stated, all the receivers sharing invariance with respect to the devised group of transformations exhibit the mentioned CFAR features. Due to the primary importance of characterizing the family of invariant detectors, a maximal invariant statistic [23], [24], organizing the data into orbits and reducing the size of the observation space, is determined together with its statistical characterization. All the decision rules in the invariance family can be cast as functions of a maximal invariant. The problem of synthesizing the Most Powerful Invariant (MPI) detector [23] is also addressed and a discussion on the existence of the Uniformly MPI (UMPI) test is provided. Interestingly, the UMPI detector does not exist but for the case where the dimension of the useful signal plus jammer subspace equals the size of the observation space. In this specific situation, the energy detector turns out to be UMPI and coincides with the GLRT and the Locally MPI Detector (LMPID). When the UMPI test does not exist, the Conditional MPI Detector (CMPID), obtained as the optimum detector conditioned on the ancillary part of the maximal invariant is designed [23], showing that it is statistically equivalent to the GLRT. At the analysis stage, we investigate the performances of some practically implementable invariant detectors providing analytical equations for the computation of the false alarm and the detection probabilities. The performance loss with respect to the clairvoyant MPI test is finally assessed showing that it is an acceptable value in some instances of practical interest.

The paper is organized as follows. The next section contains the definitions used in the mathematical derivations, while Section III is devoted to problem formulation and derivation of the maximal invariant statistic. In Section IV, we provide the statistical distribution of the maximal invariant, whereas in Section V, invariant decision schemes are devised and assessed. Some concluding remarks and hints for future work are given in Section VI. Finally, Appendices A and B contain analytical derivations of the presented results.

II. PROBLEM FORMULATION AND MAXIMAL INVARIANTS

In this section, we describe the detection problem at hand and introduce the Principle of Invariance that allows a compression of data dimensionality. Assume that a sensing system collects data from $N \geq 2$ channels (spatial and/or temporal). The returns from the cell under test, after pre-processing, are properly sampled and organized to form a $N \times r$ dimensional vector, $r$ say. We want to test whether or not $r$ contains useful target echoes assuming the presence of an interfering signal.

The target signature is modeled as a vector in a known subspace, $H$ say, where $H \in \mathbb{C}^{N \times r}$, $r \geq 1$, is a full-column-rank matrix. In other words, the useful echoes can be expressed as a linear combination of the columns of $H$, i.e., $Hp$ with $p \in \mathbb{C}^{1 \times r}$. On the other hand, the interference component consists of two contributions. The former is representative of the clutter echoes and thermal noise, while the latter accounts for possible coherent sources impinging on the receive antenna from directions different to that where the radar system is steered. More precisely, the structured interferer signal is assumed to belong to a known subspace, $J$ say, with $J \in \mathbb{C}^{N \times 1}$, $t \geq 1$, a full-column-rank matrix, and, hence, it is a linear combination, $q \in \mathbb{C}^{1 \times 1}$ say, of the columns of $J$ with $m = t + r \leq N$. The above model (with reference to Doppler processing) comes in handy to deal with scenarios where the presence of one or multiple coherent
pulsed jammers from standoff platforms attempt to protect a target located in the mainlobe of the radar antenna. We further assume that the matrix \( [J \, H] \in \mathbb{C}^{N \times m} \) is full-column-rank, namely that the columns of \( J \) are linearly independent of those of \( H \). Finally, as customary, we suppose that a set of \( K \) secondary data, \( r_k \in \mathbb{C}^{N \times 1}, k = 1, \ldots, K, \) are iid complex normal random vectors with zero mean and unknown vectors and those of \( K \) are not orthogonal but only linearly independent of those of \( J \). Two important remarks are now in order. First, it is worth pointing out that the case \( n \) is another nonsingular upper triangular matrix \( M_0 \in \mathbb{C}^{m \times N} \). Let us now recast (1) in canonical form [13]. To this end, we exploit the QR-decomposition of the partitioned matrix \([J \, H]\) given by

\[
[J \, H] = QR,
\]

where \( Q \in \mathbb{C}^{N \times m} \) is a slice of unitary matrix (or a unitary matrix if \( m = N \)), i.e., \( Q^1Q = I_m \) (or \( Q^1Q = QQ^1 = I_N \) if \( m = N \)), and \( R \in \mathbb{C}^{m \times m} \) is a nonsingular upper triangular matrix. Moreover, observe that

\[
Q = [Q_j \, Q_0], \quad R = \begin{bmatrix} R_j & R_0 \\ I_t & 0 \end{bmatrix},
\]

where \( Q_j \in \mathbb{C}^{N \times t} \) and \( R_j \in \mathbb{C}^{t \times t} \) come from the QR-decomposition of \( J \), namely, \( J = Q_jR_j \) with \( Q_j \) such that \( Q_j^1Q_j = I_t \) and \( R_j \) a nonsingular upper triangular matrix. \( R_0 \in \mathbb{C}^{t \times r} \), \( R_1 \in \mathbb{C}^{r \times r} \) is another nonsingular upper triangular matrix. Equalities (3) are almost evident consequences of the Gram-Schmidt procedure. Now, let us define a unitary matrix \( U \in \mathbb{C}^{N \times N} \) which rotates the columns of \( Q \) onto the first \( m \) elementary vectors of the standard basis of \( \mathbb{C}^{N \times 1} \), i.e.,

\[
UQ = \begin{bmatrix} I_t & 0 \\ 0 & I_r \end{bmatrix} = [E_t \, E_r],
\]

where \( E_t = [I_t, 0 ; 0, 0]^T \) and \( E_t = [I_r, 0]^T \). Thus, under \( H_1 \), primary and secondary data can be transformed as follows

\[
z = Ur = U (J + n_0)
\]

\[
= U \left( [R_q \, R_p] q + n_0 \right)
\]

\[
= [E_t \, E_r] (\theta_1 + n) = E_t \, \theta_1 + E_r \, \theta_2 + n,
\]

where \( \theta_1 = R_j q \in \mathbb{C}^{t \times 1} \). Gathering the above results, we can rewrite problem (1) in canonical form

\[
H_0 : \begin{cases} z = E_t \theta_1 + n, \\ z_k = n_k, \end{cases} \quad k = 1, \ldots, K,
\]

\[
H_1 : \begin{cases} z = E_t \theta_1 + E_r \theta_2 + n, \\ z_k = n_k, \end{cases} \quad k = 1, \ldots, K,
\]

where \( n, n_k, k = 1, \ldots, K \) are iid complex normal random vectors with zero mean and covariance matrix \( M = UM_0U^T \). Two important remarks are now in order. First, notice that \( \langle H \rangle \) receives part of the interference energy and that \( \langle J \rangle \) includes useful signal components since the columns of \( J \) and those of \( H \) are not orthogonal but only linearly independent. This reciprocity becomes more evident after applying the transformations which lead to the canonical form. Second, in problem (1) the relevant parameter to decide for the presence of a target is \( p \). Otherwise stated, if the \( H_1 \) hypothesis holds true, then \( ||p|| > 0 \), while \( ||p|| = 0 \) under the disturbance-only hypothesis \( H_0 \). As a consequence, since \( R_1 \) is nonsingular, problem (2) is equivalent to

\[
H_0 : \| \theta_2 \| = 0,
\]

\[
H_1 : \| \theta_2 \| > 0,
\]

which partitions the parameter space, \( \Theta \), say, as

\[
\Theta = \{ 0 \} \cup \{ \theta_2 \in \mathbb{C}^{r \times 1} : \| \theta_2 \| > 0 \}.
\]

In the following, we can look for decision rules sharing some invariance with respect to those parameters (nuisance parameters, in this case \( M, \theta_1 \) under \( H_1 \), and \( \theta_1 \) under \( H_0 \)) which are irrelevant to the specific decision problem. To this end, we exploit the Principle of Invariance [22], whose main idea consists in finding transformations that properly cluster data without altering

1. the formal structure of the hypothesis testing problem given by \( H_0 : \| \theta_2 \| = 0, \quad H_1 : \| \theta_2 \| > 0; \)
2. the Gaussian model assumption under the two hypotheses;
The subspace containing the useful signal components. Before introducing the transformation group that fulfills the above requirements, let us define \( Z = [z_1 \ldots z_K] \).
\[
z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}, \quad S = Z z^T = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix},
\]
where \( z_1 \in \mathbb{C}^{t \times 1}, z_2 \in \mathbb{C}^{r \times 1}, z_3 \in \mathbb{C}^{(N-m) \times 1} \), and \( S_{ij} \), \((i,j) \in \{1,2,3\} \times \{1,2,3\}\), is a submatrix whose dimensions can be obtained replacing 1, 2, and 3 with \( t, r, \) and \( N-m \), respectively. Moreover, Fisher-Neyman factorization theorem ensures that deciding from the raw data \((z, Z^T)\) is tantamount to deciding from a sufficient statistic \((z, S)\).

Now, denote by \( G \mathcal{L}(N) \) the linear group of the \( N \times N \) nonsingular matrices and introduce the following sets
\[
\mathcal{G} = \{ G \in \mathbb{N} | G_{11} G_{22} G_{33} = 0 \} \in \mathcal{L}(N).
\]
and
\[
\mathcal{F} = \left\{ f = \begin{bmatrix} f_{11} \\ 0 \end{bmatrix} \in \mathbb{C}^{N \times 1} : f_{11} \in \mathbb{C}^{t \times 1} \right\}.
\]
It follows that the set \( \mathcal{G} \times \mathcal{F} \) together with the operation \( \circ \) defined by
\[
(G_1, f_1) \circ (G_2, f_2) = (G_2 G_1, G_2 f_1 + f_2),
\]
form a group, \( \mathcal{L} \) say, that leaves the hypothesis testing problem \((9)\) invariant under the action \( \circ \) defined by
\[
l(z, S) = (Gz + f, GSG^T), \quad \forall (G, f) \in \mathcal{L}.
\]
The proof of the above statement is given in Appendix A. Moreover, it is important to point out that \( \mathcal{L} \) preserves the family of distributions and, at the same time, includes those transformations which are relevant from the practical point of view as they allow to claim the CFAR property (with respect to \( M \) and \( q \)) as a consequence of the invariance.

Summarizing, we have identified a group of transformations \( \mathcal{L} \) which leaves unaltered the decision problem under consideration. As a consequence, it is reasonable to find decision rules that are invariant under \( \mathcal{L} \). Toward this goal, the Principle of Invariance is invoked because it allows to construct statistics that organize data into distinguishable equivalence classes. Such functions of data are called maximal invariant statistics and, given the group of transformations, every invariant test may be written as a function of a maximal invariant \((24)\).

The following proposition provides the expression of a maximal invariant for the problem at hand. To this end, it is necessary to distinguish between \( m < N \) and \( m = N \); the latter equality implies that the “3-components”, namely \( z_3, S_{13}, S_{23}, S_{33}, S_{31}, S_{32}, G_{13}, G_{23}, \) and \( G_{33} \), are no longer present in the partitioned data matrices and vectors.

**Proposition 1.** A maximal invariant statistic with respect to \( L \) for problem \((9)\) is given by
\[
t_1(z, S) = \begin{bmatrix} z_1^2 S_{23}^{-1} z_3 \\ z_2^2 S_{33}^{-1} z_3 \\ z_3^2 S_{22}^{-1} z_2 \end{bmatrix},
\]
where \( z_3 = z - S_{23} S_{33}^{-1} z_3 \) and \( S_{23} = S_{22} - S_{21} S_{33}^{-1} S_{32} \).

**Proof:** See Appendix B

Some remarks are now in order.

- Observe that when \( m < N \) the maximal invariant statistic is a 2-dimensional vector where the second component represents an ancillary part, whose distribution does not depend on which hypothesis is in force.
- It is important to compare \((17)\) with the maximal invariant derived in \((17)\) assuming the absence of subspace structured interference. Interestingly, they share a similar structure/dimensionality but for the presence of a projection operation (onto the orthogonal complement of the rotated \( J \)) which amounts to cleaning the received data vectors from the structured interference.
- Finally, note that (see Theorem 6.2.1 of \((23)\)) every invariant test may be written as a function of \((17)\) (see Figure I for a schematic representation).

**III. DISTRIBUTION OF THE MAXIMAL INVARIANTS**

In this section, we provide the statistical characterization of the maximal invariant for the case \( m < N \) and, then, we give a corollary referring to \( m = N \). To this end, firstly we show that the maximal invariant can be written as a function of whitened random vectors and matrices and then we find a suitable stochastic representation by means of one-to-one transformations. Finally, the probability density functions (pdfs) in the invariant domain will be used in Section IV to construct the Likelihood Ratio Test (LRT) of \( H_0 \) versus \( H_1 \), also referred to as the MPI Detector (MPID).

Let us invoke the invariance principle and consider the following transformation \((W, f_0) \in \mathcal{L} \)
\[
w = Vz + f_0 = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix},
\]
\[
S_0 = VSV^T = \begin{bmatrix} S_{011} & S_{012} & S_{013} \\ S_{021} & S_{022} & S_{023} \\ S_{031} & S_{032} & S_{033} \end{bmatrix},
\]
where
\[
V = \begin{bmatrix} V_{11} & V_{12} & V_{13} \\ 0 & V_{22} & -M_{22} M_{23} M_{33}^{-1} \\ 0 & 0 & M_{33}^{-1/2} \end{bmatrix}.
\]
In ([20]), \( V_{22} = (M_{22} - M_{23}M_{33}^{-1}M_{32})^{-1/2} \), \( V_{ij}, i = 1, j = 1, 2, 3, \) are matrices of proper dimensions, while \( M_{22} \in \mathbb{C}^{r \times r}, \) \( M_{23} \in \mathbb{C}^{r \times (N-m)}, \) and \( M_{33} \in \mathbb{C}^{(N-m) \times (N-m)} \) are obtained partitioning \( M \) in the same way as \( S. \)

Now, observe that under \( H_1, i = 0, 1, \)

\[
\begin{bmatrix}
  w_2 \\
  w_3 \\
\end{bmatrix} = V_2 \begin{bmatrix}
  z_2 \\
  z_3 \\
\end{bmatrix} \sim \mathcal{CN}(iV_2E, \theta_2, I_{N-t})
\]

\[
\begin{bmatrix}
  S_{022} \\
  S_{032} \\
  S_{033} \\
\end{bmatrix} = V_2 \begin{bmatrix}
  S_{22} \\
  S_{32} \\
  S_{33} \\
\end{bmatrix} V_2^\dagger \sim \mathcal{CW}_{N-t}(K, I_{N-t}),
\]

where

\[
V_2 = \begin{bmatrix}
  V_{22} & -M_{22}M_{33}^{-1}M_{32}^{1/2} \\
  0 & M_{33}^{-1/2} \\
\end{bmatrix}.
\]

Thus, we can recast \( t_1(z, S) \) in terms of whitened quantities, more precisely, as

\[
m_1 = (z_2 - S_{23}S_{33}^{-1}z_3)\dagger(S_{22} - S_{23}S_{33}^{-1}S_{32})^{-1}
\times (z_2 - S_{23}S_{33}^{-1}z_3) \\
= (w_2 - S_{023}S_{33}^{-1}w_3)\dagger(S_{022} - S_{023}S_{33}^{-1}S_{032})^{-1}
\times (w_2 - S_{023}S_{33}^{-1}w_3),
\]

\[
m_2 = z_3S_{33}^{-1}z_3 = w_3S_{033}^{-1}w_3.
\]

Finally, apply a one-to-one transformation to (17) to obtain the following stochastic representation for the maximal invariant

\[
t_2(z, S) = t_2(w, S_0) = \begin{bmatrix}
  1 \\
  1 + m_1 \\
  1 + m_2 \\
\end{bmatrix}
= \begin{bmatrix}
  p_1 \\
  p_1 \\
  p_2 \\
\end{bmatrix}.
\]

Summarizing, we have all the elements to provide the probability density functions (pdfs) of the maximal invariant under both hypotheses.

**Proposition 2.** The joint pdf of \( p_1 \) and \( p_2 \) under the \( H_1 \) hypothesis, \( i = 0, 1, \) and assuming \( m < N \) has the following expression

\[
f_{p_1, p_2}(x; y; H_1) = f_\beta(x; K - (N - t) + 1, r, 0)
\times f_\beta(y; K - (N - t) + r + 1, N - t - r, 0)e^{-i \text{SINR} xy}
\times \sum_{k=0}^{K-(N-t)+1} \binom{K-(N-t)+1}{k}(K-(N-t)+1)^k
\times \frac{(r - 1)!}{(r + k - 1)!}[\text{SINR} y(1-x)]^k,
\]

where \((x, y) \in \{0 1\} \times \{0 1\} \) and \( \text{SINR} = \theta_2^{\dagger}(M_{22} - M_{23}M_{33}^{-1}M_{32})^{-1}\theta_2 \)

is the Signal-to-Noise-plus-Interference Ratio (SINR), and \( f_\beta(z; n, m, \delta) \) is the pdf of a [7], [13]

- a random variable following a complex noncentral beta distribution with \( n, m \) complex degrees of freedom and noncentrality parameter \( \delta, \) if \( \delta > 0; \)

- a random variable following a complex central beta distribution with \( n, m \) complex degrees of freedom, if \( \delta = 0. \)

**Proof:** See Appendix [C] ■

**Corollary 1.** In the case \( m = N, \) the pdf of \( p_3 = 1/(1 + z_2^2S_{22}^{-1}z_2) \) under the \( H_1 \) hypothesis, \( i = 0, 1, \) is given by

\[
f_{p_3}(x; H_1) = f_\beta(x; K - r + 1, r, 0)e^{-i \text{SINR} x}
\times \sum_{k=0}^{K-r+1} \binom{K-r+1}{k}(r-1)!(r+k-1)!\text{SINR} (1-x)^k,
\]

where \( x \in \{0 1\} \) and \( \text{SINR} = \theta_2^{\dagger}M_{22}^{-1}\theta_2. \)

**Proof:** It is an application of the results contained in the proof of Proposition [2] ■

As final remark, the induced maximal invariant can be easily obtained observing that the pdf of \( t_2(z, S) \) depends on the unknown parameters only through the SINR.

**IV. DESIGN AND ANALYSIS OF IN Variant Tests**

The goal of this section is twofold. First of all, we derive invariant decision schemes based on the LRT and its variants. In particular, we focus on the MPID, the LMPID (which is derived under the low SINR regime), and the Conditional MPID (CMPID), which are investigated for the case where the Uniformly MPID (UMPID) does not exist. Second, we consider some sub-optimum receivers providing the expression of their decision statistics as functions of the maximal invariant.

**A. LRT-based Decision Schemes in the Invariant Domain**

In the sequel, we consider the cases \( m < N \) and \( m = N. \)

1) Case \( m < N: \) As starting point of our analysis, we devise the MPID that, according to the Neyman-Pearson criterion, is given by the LRT after data compression. Specifically, the MPID has the following expression

\[
t_{\text{MPID}} = f_{p_1, p_2}(x, y; H_1)
\frac{f_{p_1, p_2}(x, y; H_0)}{f_{p_1, p_2}(x, y; H_0)}
\times \sum_{k=0}^{K-(N-t)+1} \binom{K-(N-t)+1}{k}(K-(N-t)+1)
\times \frac{(r - 1)!}{(r + k - 1)!}[\text{SINR} p_2(1-p_1)]^k 
\]

where \( \eta \) is the threshold set according to the desired Probability of False Alarm \( (P_f). \) It is important to observe that receiver (30) is clairvoyant and, as a consequence, it is not of great practical interest in radar applications since it requires the knowledge of the induced maximal invariant. Nevertheless, it still represents a noteworthy benchmark, i.e., an upper bound to the detection performance achievable by

\[\text{Observe that } p_3 \text{ obeys the complex beta distribution with } K - r + 1, r \text{ complex degrees of freedom.}\]

\[\text{Hereafter, we use symbol } \eta \text{ to denote the generic detection threshold.}\]
any invariant receiver. Before proceeding with the exposition, it is worth pointing out that decision scheme (30) is formally analogous to that found in [17], where the optimum detector is devised assuming a subspace structured signal embedded in random interference only (no subspace interference). This is perfectly consistent with the observation that when an additive structured interference is present, the MPID (30) is obtained after projecting pre-processed data in canonical form onto the orthogonal complement of the interference subspace whose contribution is thus removed.

Since in many applications the performances become critical for low SINR values, the LMPID in the limit of zero SINR could be of interest. Following the lead of (34) and (35), the LMPID is given by

$$t_{LMPID} = \frac{\delta f_{p_1, p_2}(p_1, p_2; H_1)}{\delta \text{SINR}} \bigg|_{\text{SINR}=0} \frac{H_1}{H_0} \geq \eta. \quad (31)$$

The following proposition provides the expression of $t_{LMPID}$.

**Proposition 3.** The LMPID is the following decision rule

$$t_{LMPID} = \frac{K - (N - t) + 1}{r} p_2(1 - p_1) - p_1 p_2 \frac{H_1}{H_0} \geq \eta. \quad (32)$$

**Proof:** See Appendix [13].

As concluding part of this section, we provide the expression of the CLRT, which is the LRT derived under the assumption that the ancillary part of the maximal invariant is assigned [23]. The main idea behind this approach relies on the fact that the ancillary part does not carry any information on the parameter and, hence, can be considered given. Moreover, it would be of interest to ascertain the existence of a CUMPID, since the UMPID does not exist for the problem at hand when $m < N$.

The test statistic of the CMPID, $t_{CMPID}$, has the same expression as the MPID (see equation (30)) with the difference that in this case $p_2$ is a constant. Now, observe that $t_{CMPID}$ is a nonincreasing function of $p_1$. As a matter of fact, let us evaluate those values of $p_1 \in (0, 1]$ such that the first derivative of $t_{CMPID}$ with respect to $p_1$ is less than zero, namely

$$\frac{\delta t_{CMPID}}{\delta p_1} = -\text{SINR} p_2 e^{-\text{SINR} p_1 p_2}$$

\[ \times \sum_{k=0}^{K-(N-t)+1} \left( \frac{K - (N - t) + 1}{r} \right)^{(r-1)!(r+k-1)!(K-r+k)!} \left( \frac{K - (N - t) + 1}{r} \right)^{-(r-1)!(r+k-1)!(K-r+k)!} \left( \text{SINR} p_2 (1 - p_1) \right)^k - \text{SINR} p_2 e^{-\text{SINR} p_1 p_2} \]

\[ \times \sum_{k=1}^{K-(N-t)+1} \left( \frac{K - (N - t) + 1}{r} \right)^{(r-1)!(r+k-1)!(K-r+k)!} \left( \text{SINR} p_2 (1 - p_1) \right)^{k-1} \leq 0. \quad (33) \]

The above inequality holds true $\forall p_1 \in (0, 1]$ and, hence, $t_{CMPID}$ is a nonincreasing function of $p_1$. As a consequence of the Karlin-Rubin Theorem, the test

$$H_0 \quad p_1 \geq \eta \quad H_1 \quad \frac{p_1}{p_1} \frac{H_1}{H_0} \geq \eta. \quad (34)$$

is UMPI conditionally to $p_2$. Moreover, we can consider the following equivalent form for test (34)

$$t_{CLRT} = \frac{1 - p_1}{p_1} \frac{H_1}{H_0} \geq \eta. \quad (35)$$

where, as we will show in the next subsection, $t_{CLRT}$ coincides with the GLRT in the original data space.

2) Case $m=N$: Recall that when $m = N$ the maximal invariant statistic obeys the complex beta distribution with $K - r + 1, r$ complex degrees of freedom and, hence, the LRT is given by

$$t_{LRT} = \frac{f_{p_3}(x; H_1)}{f_{p_3}(x; H_0)} \bigg|_{x=p_3} = e^{-\text{SINR} p_3} \sum_{k=0}^{K-r+1} \left( \frac{K-r+1}{K-r+k} \right) \times \frac{(r-1)!}{(r+k-1)!} \left( \text{SINR} (1-p_3) \right)^k \geq \eta. \quad (36)$$

Now, following the same steps used to prove that the GLRT is conditionally UMPI, it is easy to show that the test

$$H_0 \quad p_3 \geq \eta \quad H_1 \quad \frac{p_3}{p_3} \frac{H_3}{H_0} \geq \eta. \quad (37)$$

is UMPI and statistically equivalent to the Energy Detector (ED) given by

$$t_{ED} = \frac{1 - p_3}{p_3} = z_2^1 s_{22}^{-1} z_2 \frac{H_1}{H_0} \geq \eta. \quad (38)$$

Finally, the following corollary provides the expression of the LMPID when $m = N$.

**Corollary 2.** The expression of the LMPID in the case $m = N$ is

$$H_0 \quad \frac{K - r + 1}{r} (1 - p_3) - p_3 \frac{H_1}{H_0} \geq \eta. \quad (39)$$

**Proof:** The above result can be easily obtained following the line of reasoning of the proof of Proposition [3].

It is clear that the LMPID for $m = N$ is statistically equivalent to the ED.

B. Receivers Obtained as Functions of the Maximal Invariant

Several invariant decision rules can be obtained exploiting different rationales based on either solid theoretical paradigms or heuristic criteria. The basic observation is the property that any invariant test can be written as a function of the maximal invariant. Thus, it is possible to devise invariant architectures choosing proper functions of the maximal invariant. Specifically, we focus on the GLRT, which under some mild technical conditions always leads to an invariant detector [36], and the
It is not difficult to show that to show that it follows that both receivers are invariant with respect to unknown covariance matrix of the disturbance and, hence, they ensure the CFAR property with respect to the

\[ P \sim \text{CFR}, K = (N-t) + 1 \), \]

while under \( H_1 \), \( t_{\text{GLRT}} \sim CF_{r,K = (N-t)}(0) \) with \( \delta^2 = p_2 \text{SINR} \). It follows that the \( P_{fa} \) can be expressed as

\[ P_{fa}^\text{GLRT}(\eta) = 1 - F(\eta; r, K - (N-t) + 1, 0) \]

\[ = \frac{1}{(1+\eta)^{K-N+t}} \sum_{l=0}^{r-1} \left( r + K - N + t \right)^l \eta^l, \]

where \( F(x; n, m, \delta), x \geq 0 \), is the Cumulative Distribution Function (CDF) of a random variable ruled by the complex central (noncentral) F-distribution with \( n, m \) complex degrees of freedom if \( \delta = 0 \) (\( \delta > 0 \)). Following the same line of reasoning as for the \( P_{fa} \), it is not difficult to show that \( P_d \) is given by

\[ P_d^\text{GLRT}(\eta, \text{SINR}) = 1 - \int_0^1 F(\eta; r, K - (N-t) + 1, A) \]

\[ \times f_\beta(u; K - (N-t) + r + 1, N - t - r, 0) du, \]

where \( \delta^2(u) = u \text{SINR} \).

As to the 2S-GLRT, observe that it can be recast as \( t_{\text{2S-GLRT}} / p_2 \), and, hence, \( P_{fa} \) and \( P_d \) can be easily derived relying on the above results, more precisely

\[ P_{fa}^\text{2S-GLRT}(\eta) = 1 - \int_0^1 F(\eta u; r, K - (N-t) + 1, 0) \]

\[ \times f_\beta(u; K - (N-t) + r + 1, N - t - r, 0) du, \]

\( P_d^\text{2S-GLRT}(\eta, \text{SINR}) = 1 - \int_0^1 F(\eta u; r, K - (N-t) + 1, A) \)

\[ \times f_\beta(u; K - (N-t) + r + 1, N - t - r, 0) du. \]

In order to provide the expressions of \( P_{fa} \) and \( P_d \) for the LMPID, let \( a = |K - (N-t) + 1| / r \) and observe that

\[ t_{\text{LMPID}} = a p_2 (1 - p_1) - p_1 p_2 = p_1 p_2 \left[ a \left( \frac{1}{p_1} - 1 \right) - 1 \right] \]

\[ = p_1 p_2 \left[ a t_{\text{GLRT}} - 1 \right] = \frac{p_2}{t_{\text{GLRT}} + 1} (a t_{\text{GLRT}} - 1). \]

1) Case \( m < N \): Let us focus on the GLRT and observe that under the \( H_0 \) hypothesis, \( t_{\text{GLRT}} \sim \text{CFR}_{r,K = (N-t)}(0) \), while under \( H_1 \), \( t_{\text{GLRT}} \sim CF_{r,K = (N-t)}(0) \) with \( \delta^2 = p_2 \text{SINR} \). It follows that the \( P_{fa} \) can be expressed as

\[ t_{\text{GLRT}} = \frac{1 + z_j S^{-1/2} P_{S^{-1/2} E_i} S^{-1/2} z}{1 + z_j S^{-1/2} P_{S^{-1/2} E_m} S^{-1/2} z}, \]

where \( P_A = A(A^\dagger A)^{-1} A^\dagger \) is the projection matrix onto the subspace spanned by the columns of \( A \in C^{N \times M} \), \( P_A^\perp = I_N - P_A \), and \( E_m = [E_i E_r] \). It is tedious but not difficult to show that

\[ z_j S^{-1/2} P_{S^{-1/2} E_i} S^{-1/2} z = 1 - p_1 p_2, \]

\[ z_j S^{-1/2} P_{S^{-1/2} E_m} S^{-1/2} z = 1 - p_2. \]

Sumarizing the statistic of the GLRT can be recast as

\[ t_{\text{GLRT}} = \frac{1}{p_1}, \]

which is clearly statistically equivalent to the left-hand side of \( P_{fa} \).

On the other hand, the two-step GLRT (2S-GLRT) is given by

\[ t_{\text{2S-GLRT}} = z_j S^{-1/2} P_{S^{-1/2} E_i} S^{-1/2} z \]

\[ = z_j S^{-1/2} P_{S^{-1/2} E_m} S^{-1/2} z = \frac{1 - p_1}{p_1 p_2}. \]

It follows that both receivers are invariant with respect to \( \mathcal{L} \) and, hence, they ensure the CFAR property with respect to the unknown covariance matrix of the disturbance and \( q \).

2) Case \( m = N \): In this case, the GLRT and the 2S-GLRT share the same test statistic which has the following expression

\[ t_{\text{GLRT}} = t_{\text{2S-GLRT}} = z_j S^{-1/2} P_{S^{-1/2} E_i} S^{-1/2} z. \]

It is not difficult to show that

\[ z_j S^{-1/2} P_{S^{-1/2} E_i} S^{-1/2} z = z_j S^{-1/2} z = 1 - p_1 \]

\[ = \frac{1}{p_3}, \]

where \( z_j S^{-1/2} P_{S^{-1/2} E_m} S^{-1/2} z = \frac{1 - p_1}{p_3} \), \( \) which is the statistic of the ED. It follows that, under the assumption \( m = N \), the GLRT, the 2S-GLRT, and the LMPID coincide with the ED, which, in turn, is UMPI.

C. \( P_{fa} \) and \( P_d \) of the GLRT, the 2S-GLRT, the LMPID, and ED

This subsection is aimed at the derivation of closed-form expressions for the \( P_{fa} \) and the Probability of detection (\( P_d \)) of the GLRT, the 2S-GLRT, and the LMPID. To this end, we deal with equivalent decision statistics and distinguish between \( m < N \) and \( m = N \).
This implies that \( P_{fa} \) can be written as:

\[
P_{fa}^{\text{LMPID}}(\eta) = \begin{cases} \frac{p_2}{p_2 + \eta} (a_i - \eta) > \eta; H_0 \\ \frac{p_2}{p_2 - \eta} (a_i + \eta) < a; H_0 \end{cases}
\]

\[
P_{fa}^{\text{LMPID}}(\eta) = \begin{cases} \frac{p_2}{p_2 + \eta} (a_i - \eta) > \eta; H_0 \\ \frac{p_2}{p_2 - \eta} (a_i + \eta) < a; H_0 \end{cases}
\]

The expression of \( P_d \) is formally analogous to that of the \( P_{fa} \) but for the presence of the noncentrality parameter in the distribution of \( \tilde{t}_{\text{GLRT}} \), namely

\[
P_d^{\text{LMPID}}(\eta, \text{SINR}) = \begin{cases} \int_{\eta/a}^{1} \left[ 1 - F \left( \frac{u + \eta}{ua - \eta}; r, K - (N - t) + 1, \delta(u) \right) \right] f_\beta(u; K - (N - t) + r + 1, N - t - r, 0) du \\ + \int_{0}^{\eta/a} F \left( \frac{u + \eta}{ua - \eta}; r, K - (N - t) + 1, \delta(u) \right) f_\beta(u; K - (N - t) + r + 1, N - t - r, 0) du \\ 0 \leq \eta < a, \end{cases}
\]

\[
1) \text{Case } m = 2N: \text{ Under the assumption } m = 2N, \text{ the } 2S-\text{GLRT, the } 2S-\text{GLRT, and the LMPID are statistically equivalent to the ED. For this reason, the object of this section is the statistical characterization of the ED. Specifically, it is easy to show that } t_{\text{fa}} \text{ is ruled by}
\]

\[
\text{• the complex central F-distribution with } r, K + r + 1 \text{ complex degrees of freedom under } H_0;
\]

\[
\text{• the complex noncentral F-distribution with } r, K + r + 1 \text{ degrees of freedom and noncentrality parameter } \delta \text{ with } \delta^2 = \theta_1^2 M^{-1}_{22} \theta_2 \text{ under } H_1.
\]

It follows that

\[
P_{fa}^\text{LMPID}(\eta) = \frac{1}{(1 + \eta)^n} \sum_{l=0}^{r-1} \binom{K}{l} \eta^l
\]

\[
P_d^{\text{LMPID}}(\eta, \text{SINR}) = 1 - F(\eta; r, K - r + 1, \delta).
\]

\[
D. \text{ Illustrative Examples}
\]

Figures 4 and 5 display \( P_d \) versus the SINR for the considered decision schemes. The curves have been obtained by means the above closed-form formulas which have been computed by standard numerical routines but for the MPID, whose performance has been obtained via standard Monte Carlo counting techniques. More precisely, the thresholds necessary to ensure a preassigned value of \( P_{fa} \) have been evaluated exploiting 1000/\( P_{fa} \) independent trials, while the \( P_d \) values are estimated over 5000 independent trials. As to the disturbance, it is modeled as an exponentially-correlated Gaussian vector with covariance matrix \( M = \sigma_c^2 I_N + \sigma_n^2 M_c \), where \( \sigma_n^2 > 0 \) is the thermal noise power, \( \sigma_c^2 > 0 \) is the clutter power, and the \((i, j)\)-th element of \( M_c \) is given by \( 0.95^{|i-j|} \). The clutter-to-noise ratio \( \sigma_c^2/\sigma_n^2 \) and the interferer-to-noise ratio \( ||\theta_1||^2/\sigma_n^2 \) are both set to 30 dB with \( \sigma_n^2 = 1 \). Finally, all numerical examples assume \( P_{fa} = 10^{-4} \).

Figures 2 and 3 show the performances of the considered detectors assuming \( N = 8, K = 12 \), and different values of \((r, t)\). Inspection of the figures highlights that the GLRT, the 2S-GLRT, and the MPID share practically the same performance, while the LMPID performs poorer. In particular,
the latter exhibits a loss of about 3 dB at $P_d = 0.9$ in Figure 2 and does not achieve $P_d = 0.9$ for the SNR values considered in Figure 3. Moreover, the GLRT and the MPID are slightly superior to the 2S-GLRT. Finally, it is worth observing that for high values of $r$, the $P_d$ curves of the considered detectors move toward high values of SINR, namely there is a performance degradation with respect to the case where $r$ is low; this is strictly tied to the increase of the number of unknown parameters to be estimated. In Figures 4 and 5, $P_d$ versus SINR is plotted assuming $N = 8$ and $K = 16$. Inspection of the figures highlights that the above hierarchy to the previous case the used for estimation purposes decreases the loss of the 2S-
keeps unaltered, while the increased number of secondary data used for estimation purposes decreases the loss of the 2S-GLRT with respect to the GLRT. Finally, notice that contrary to the previous case the $P_d$ of the LMPID can be higher than 0.9 for SINR values greater than 16 dB when $r = 2$ and 18 dB when $r = 4$.

V. CONCLUSIONS

In this paper we have considered adaptive radar detection of a subspace signal embedded in two sources of interference: a Gaussian component with unknown covariance matrix modeling the joint effect of clutter plus receiver noise and a subspace structured part accounting for the effect of coherent jammers impinging on the radar antenna. We have formulated the problem as a binary hypothesis test and have exploited the theory of invariance to characterize adaptive detectors enjoying some relevant symmetries. In this context the main achieved technical results are:

- The identification of a suitable group of transformations leaving the considered hypothesis test invariant and the interpretation of the group action as a mean to force at the design stage the CFAR property with respect to clutter plus noise covariance matrix and jammer parameters.
- The design and the statistical characterization of a maximal invariant statistic which organizes the radar data in equivalence classes and significantly realizes a compression of the original observation domain.
- The synthesis of the optimum invariant detector and the discussion on the existence of the UMID test. In this respect, we have shown that the UMID detector in general does not exists but for the case where the useful signal plus jammer subspace size completely fills the dimension of the observation domain.
- The synthesis of the LMPID, the CUMPID, and the proof that the latter test coincides with the GLRT.
- The development of analytic expressions for the performance assessment of some practically implementable invariant detectors.

Possible future research tracks might concern the possibility to deal with a partially homogeneous Gaussian interference through the additional invariance under a common scaling of the secondary data, as well as the eventualty to force some structure in the covariance matrix of the Gaussian interference (as for instance persymmetry [37, 38]). Last but not least, it would be interesting the extension of the entire framework to a non-homogeneous scenario where the training data may exhibit different power levels.

APPENDIX A

INVARINENCE OF PROBLEM (5) WITH RESPECT TO THE GROUP $L$

Let $(G, f) \in L$ and observe that, under $H_1$, $x = Gz + f$ obeys the complex normal distribution with covariance matrix $GMG^\dagger$ and mean

$G(E_t \theta_{11} + E_t \theta_{2}) + f = \begin{bmatrix} G_{11} \theta_{11} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} G_{12} \theta_{2} \\ G_{22} \theta_{2} \\ 0 \end{bmatrix} + \begin{bmatrix} f_{11} \\ 0 \\ 0 \end{bmatrix} = E_t \theta_{10} + E_t \theta_{2}$, \quad \text{(57)}$

where $\theta_1 = G_{11} \theta_{11} + G_{12} \theta_{2} + f_{11}$ and $\theta_2 = G_{22} \theta_{2}$. Moreover, $x$ is independent of $x_k = Gz_k$, $k = 1, \ldots, K$, which are iid complex normal vectors with zero mean and covariance matrix $GMG^\dagger$. Now, observe that $\|\theta_2\| > 0$ if and only if $\|\theta_2\| > 0$ (since $G_{22}$ is nonsingular).

On the other hand, when $H_0$ is in force, $x$ is distributed as under $H_1$ but for the mean, which is given by

$GE_t \theta_{10} + f = \begin{bmatrix} G_{11} \theta_{10} + f_{11} \\ 0 \\ 0 \end{bmatrix} = E_t \tilde{\theta}_{10}$, \quad \text{(58)}$

where $\tilde{\theta}_{10} = G_{11} \theta_{10} + f_{11}$, whereas $x_k$, $k = 1, \ldots, K$, are still iid complex normal random vectors with zero mean and covariance matrix $GMG^\dagger$.

Thus, we have shown that the original partition of the parameter space, data distribution, and the structure of the subspace containing the useful signal components are preserved after the trasformation $(G, f)$.

APPENDIX B

DERIVATION OF THE MAXIMAL INVARIANT STATISTICS

This appendix is devoted to the proof of Proposition 7. In particular, we focus on the more challenging case $2 \leq m < N$ since $m = N$ can be obtained according to the same proof method.

Before proceeding, it is worth recalling that a statistic $t(z, S)$ is said to be maximal invariant statistic with respect to a group of transformations $L$ if and only if $t(\tilde{z}, \tilde{S}) = t(z, S)$, $\forall l \in L$, \quad \text{(59)}$

and $t(z, S) = t(\tilde{z}, \tilde{S}) \Rightarrow \exists l \in L : (z, S) = l(\tilde{z}, \tilde{S})$. \quad \text{(60)}$

In order to prove (59), we consider the following vector

$z_{23} = \begin{bmatrix} z_2 \\ z_3 \end{bmatrix}, \quad S_2 = \begin{bmatrix} S_{22} & S_{23} \\ S_{32} & S_{33} \end{bmatrix}$, \quad \text{(62)}$

and

$z_{23}^\dagger S_2^{-1} z_{23} = z_{23}^\dagger S_{23}^{\dagger} z_{23} = z_{23}^\dagger S_{23}^{\dagger} z_{23} + z_{2,3} S_{2,3}^{-1} z_{2,3}$. \quad \text{(63)}$
Moreover, let us partition the matrix \( G \) as follows

\[
G = \begin{bmatrix} G_1 & G_2 \\ G_3 & 0 \end{bmatrix},
\]

(64)

where \( G_1 = G_{11} \in \mathbb{C}^{t \times t} \), \( G_2 = [G_{12} G_{13}] \in \mathbb{C}^{t \times (N-t)} \), and

\[
G_3 = \begin{bmatrix} G_{22} & G_{23} \\ 0 & G_{33} \end{bmatrix} \in \mathbb{C}^{(N-t) \times (N-t)}.
\]

(65)

Now, let \((\bar{z}, \bar{S}) = l(z, S)\) and observe that

\[
\bar{z} = Gz + f = \begin{bmatrix} G_1z_1 + G_2z_{23} + f_{11} \\ G_3z_{23} \end{bmatrix}
\]

and

\[
\bar{S} = GSG^{\dagger} = \begin{bmatrix} A_1 & A_2 \\ A_3 & G_3S_2G_3^{\dagger} \end{bmatrix},
\]

(67)

where \( A_i, i = 1, 2, 3 \), are matrices of proper size and

\[
G_3S_2G_3^{\dagger} = \begin{bmatrix} B_1 & B_2 \\ B_3 & G_3S_3S_3G_3^{\dagger} \end{bmatrix}
\]

(68)

with, in turn, \( B_i, i = 1, 2, 3 \), matrices of proper dimensions. It follows that

\[
\begin{bmatrix} \bar{z}_{23} \bar{S}_{23}^{-1} \bar{z}_{23} \\ \bar{z}_3 \bar{S}_3^{-1} \bar{z}_3 \end{bmatrix} = \begin{bmatrix} \bar{z}_{23} \bar{G}_3 \bar{G}_3^\dagger \bar{S}_2 \bar{S}_2^{-1} \bar{G}_3 \bar{G}_3^\dagger \bar{z}_{23} \\ \bar{z}_3 \bar{G}_3 \bar{G}_3^\dagger \bar{S}_3 \bar{S}_3^{-1} \bar{G}_3 \bar{G}_3^\dagger \bar{z}_3 \end{bmatrix}
\]

\[
= \begin{bmatrix} \bar{z}_{23} \bar{S}_3^{-1} \bar{z}_{23} \\ \bar{z}_3 \bar{S}_3^{-1} \bar{z}_3 \end{bmatrix}
\]

(69)

and, hence, that

\[
z_{23}S_{23}^{-1}z_{23} = \bar{z}_{23} \bar{S}_{23}^{-1} \bar{z}_{23}.
\]

(70)

The proof of the second property (60) adopts the following rationale: first, we find the submatrix \( G_3 \) and then we construct \( f_{11} \) and then the remaining blocks of \( G \).

Assume that there exist \((z, S)\) and \((\bar{z}, \bar{S})\) such that

\[
t(z, S) = \begin{bmatrix} \bar{z}_{23} \bar{S}_3^{-1} \bar{z}_{23} \\ \bar{z}_3 \bar{S}_3^{-1} \bar{z}_3 \end{bmatrix} = \begin{bmatrix} \bar{z}_{23} \bar{S}_3^{-1} \bar{z}_{23} \\ \bar{z}_3 \bar{S}_3^{-1} \bar{z}_3 \end{bmatrix} = t(\bar{z}, \bar{S}).
\]

(71)

The above equalities can be recast using the Euclidean norm of a vector, more precisely

\[
\|y_{23}\|^2 = \|\bar{y}_{23}\|^2,
\]

(72)

\[
\|y_3\|^2 = \|\bar{y}_3\|^2,
\]

(73)

where \( y_{23} = S_{23}^{-1/2}z_{23} \), \( \bar{y}_{23} = \bar{S}_{23}^{-1/2}\bar{z}_{23} \), \( y_3 = S_{33}^{-1/2}z_3 \), and \( \bar{y}_3 = \bar{S}_{33}^{-1/2}\bar{z}_3 \). As a consequence, there exist \( U_3 \in \mathbb{C}^{(N-m) \times (N-m)} \) and \( U_3 \in \mathbb{C}^{r \times r} \) unitary matrices such that

\[
y_{23} = U_3 y_{23} \quad \text{and} \quad y_3 = U_3 y_3.
\]

(74)

\footnote{We partition \( \bar{z} \) and \( \bar{S} \) according to the same rule used for \( z \) and \( S \). For this reason in the sequel we omit the definition of the submatrices and subvectors of \( \bar{z} \) and \( \bar{S} \).}

Moreover, partition \( S_2^{-1} \) and \( \tilde{S}_2^{-1} \) as follows

\[
S_2^{-1} = \begin{bmatrix} I_r & S_3 S_2 \ I_{N-m} & 0 \\ S_3^{-1} S_2 \ 0 & S_3^{-1} \end{bmatrix}
\]

\[
P^{-1}
\]

(75)

\[
\tilde{S}_2^{-1} = \begin{bmatrix} I_r & \tilde{S}_3 \tilde{S}_2 \ I_{N-m} & 0 \\ \tilde{S}_3^{-1} \tilde{S}_2 \ 0 & \tilde{S}_3^{-1} \end{bmatrix}
\]

\[
P^{-1}
\]

(76)

and note that

\[
WP z_{23} = \begin{bmatrix} S_2^{-1/2} z_{23} \\ S_3^{-1/2} z_3 \end{bmatrix} = \begin{bmatrix} y_{23} \\ y_3 \end{bmatrix},
\]

(77)

\[
WP z_{23} = \begin{bmatrix} S_2^{-1/2} z_{23} \\ S_3^{-1/2} z_3 \end{bmatrix} = \begin{bmatrix} y_{23} \\ y_3 \end{bmatrix}.
\]

(78)

Gathering the above results yields

\[
WP z_{23} = \begin{bmatrix} U_{23} & 0 \\ 0 & U_3 \end{bmatrix} WP z_{23}
\]

\[
\Rightarrow z_{23} = (WP)^{-1} U_1 WP \bar{z}_{23},
\]

(79)

where \( D_3 = (WP)^{-1} U_1 WP \) is an upper block-triangular matrix with the same structure as \( G_3 \) in (65). Besides, from equations (75) and (76) it follows that

\[
WPS_2 P W = I_{N-m} = U_1 U_1^\dagger
\]

\[
\Rightarrow U_1^\dagger WP S_2 (U_1 WP)^\dagger = I_{N-m} = WP S_2 (WP)^\dagger
\]

\[
\Rightarrow S_2 = (WP)^{-1} U_1 WP S_2 (WP)^\dagger U_1^\dagger (WP)^{-1}
\]

\[
= D_3 S_2 D_3^\dagger.
\]

(80)

So far, we have constructed the block \( G_3 = D_3 \) of \( G \), it still remains to find the other blocks of \( G \). To this end, let \( S_3 = [S_{12} S_{13}] \), \( S_1 = S_{11} \) and write (67) as

\[
G^{-1} S = \bar{S} G^{\dagger}
\]

\[
\Rightarrow \begin{bmatrix} G_1^{-1} & -G_1^{-1} G_2 G_3^{-1} \\ 0 & G_3^{-1} \end{bmatrix}
\]

\[
\times \begin{bmatrix} S_1 & S_3 \\ S_1^\dagger & S_3^\dagger \end{bmatrix} = \begin{bmatrix} \tilde{S}_1 & \tilde{S}_3 \\ \tilde{S}_1^\dagger & \tilde{S}_3^\dagger \end{bmatrix} \begin{bmatrix} G_1^{-1} & 0 \\ G_1^{-1} S_3^\dagger & G_3^{-1} \end{bmatrix}
\]

\[
\Rightarrow \begin{bmatrix} G_1^{-1} S_1 - G_1^{-1} G_2 G_3^{-1} S_3^\dagger \\ G_3^{-1} S_3 \end{bmatrix} = \begin{bmatrix} S_1 G_1^{-1} + S_3 G_2^\dagger & S_3 G_3^\dagger \\ S_1 G_1^{-1} + S_3 G_2^\dagger & S_3 G_3^\dagger \end{bmatrix}.
\]

(87)
The last equation is equivalent to the following system of equations

$$\begin{align*}
G_1^{-1}S_1 - G_1^{-1}G_2G_3^{-1}S_3^t &= S_1G_1^t + S_3G_2^t, \\
G_1^{-1}S_3 - G_1^{-1}G_2G_3^{-1}S_2 &= S_3G_3^t, \\
G_3^{-1}S_3^t &= S_3^tG_1^t + S_2G_2^t.
\end{align*}$$

(88)

Replacing the following equality\(^{13}\)

$$G_3^{-1}S_2 = S_2G_3^t$$

(89)

into the second equation of the above system yields

$$\begin{align*}
G_1^{-1}S_3 - G_1^{-1}G_2G_3^{-1}S_3^t &= S_3G_3^t, \\
\Rightarrow S_3 - G_2S_2G_3^{-1} &= G_1S_3G_3^t, \\
\Rightarrow (G_3^{-1}S_3^t)^t &= G_1S_3 + G_2S_2, \\
\Rightarrow G_3^{-1}S_3^t &= S_3^tG_1^t + S_2G_2^t.
\end{align*}$$

(90-93)

Thus, the second matrix equation of (88) is redundant and can be neglected. From the third equation it stems that

$$G_3^t = S_2^{-1}\left(G_3^{-1}S_3^t - S_3^tG_1^t\right)$$

(94)

which, replaced in the first equation of (88), leads to

$$\begin{align*}
G_1^{-1}S_1 - G_1^{-1}\left[S_3(G_3^t)^t - G_1S_3^t \right]S_2^{-1}G_3^{-1}S_3^t &= S_1G_1^t + S_3S_2^{-1}G_3^{-1}S_3^t, \\
\Rightarrow S_1 - S_3S_2^{-1}S_3^t + G_1S_3^tS_2^{-1}G_3^{-1}S_3^t &= G_1S_3G_3^t + G_1S_3S_2^{-1}S_3^{-1}S_3^t - G_1S_3S_2^{-1}S_3^t, \\
\Rightarrow G_1(S_1 - S_3S_2^{-1}S_3^t)G_3^t &= S_1 - S_3S_2^{-1}S_3^t.
\end{align*}$$

(95-96)

(97)

The solution of the last equation has the following expression

$$G_1 = (S_1 - S_3S_2^{-1}S_3^t)^{1/2}(S_1 - S_3S_2^{-1}S_3^t)^{-1/2}$$

(98)

and it is such that det$(G_1) \neq 0$. Replacing $G_1$ in (94) with (98) yields the expression of $G_2$.

Finally, $f_{11}$ can be evaluated as follows

$$\begin{align*}
f = z - G\tilde{z} &= \left[\begin{array}{c}
z_1 \\
z_{23}
\end{array}\right] - \left[\begin{array}{c}
G_1 \\
G_2 \\
G_3
\end{array}\right]\left[\begin{array}{c}
\tilde{z}_1 \\
\tilde{z}_{23}
\end{array}\right] \\
= \left[\begin{array}{c}
z_1 - G_1\tilde{z}_1 - G_2\tilde{z}_{23} \\
0
\end{array}\right],
\end{align*}$$

(99)

(100)

where the last equality comes from (80).

**APPENDIX C**

**Statistical Characterization of the Maximal Invariant**

The joint pdf of $p_1$ and $p_2$ can be computed exploiting the following equality

$$f_{p_1,p_2}(x, y) = f_{p_1|p_2}(x|p_2 = y)f_{p_2}(y),$$

(101)

where $f_{p_1|p_2}(x|p_2 = y)$ is the conditional pdf of $p_1$ given $p_2$ and $f_{p_2}(y)$ is the marginal pdf of $p_2$.

\(^{10}\)Observe that it comes from (83).

Let us focus on $f_{p_2}(y)$ and observe that from (21) it stems that $w_3 \sim CN_{N-m}(0, I_{N-m})$, while by Theorem A.11 of $\mathcal{S}_{033} \sim CW_{N-m}(K, I_{N-m})$. Now, recast $w_3^tS_{033}w_3$ as

$$\tilde{p}_2 = \frac{w_3^t w_3}{w_3^t w_3} = \frac{a}{b}$$

(102)

where $a \sim C\beta_{X_{N-m}}(0)$ and, by Theorem A.13 of $\mathcal{S}_{033}$, $b \sim C\beta_{X_{(N-m)+1}}(0)$ is statistically independent of $a$. It follows that $\tilde{p}_2$ is ruled by the complex central F-distribution with $N - m, K - (N - m) + 1$ complex degrees of freedom and, hence, $p_2 = 1/(1 + \tilde{p}_2) \sim C\beta_{K-(N-m)+1,N-m}(0)$. Note that $p_2$ is an ancillary statistic and its distribution is the one and the same under both hypotheses.

The distribution of $p_1$ can be obtained assuming that the “3-components”, namely the entries of the considered random vectors sharing the subscript equal to 3, are no longer random variables but are assigned. In such a case $\tilde{p}_2$ is deterministic and under $H_1$

- $d = (w_2 - S_{023}S_{033}w_3)/\sqrt{1 + w_3^tS_{033}w_3} \sim$ $CN_{K,K-(N-t)+1}(\delta, 1)$,
- $X = (S_{022} - S_{033}S_{033}S_{0332}) \sim CW_{K-(N-t)+1,R}$

(103)

Again, exploiting Theorem A.13 of $\mathcal{S}_{033}$ leads to

$$\tilde{p}_1 = -\frac{d^t \tilde{d}}{d^tX^{-1}d} \sim C\beta_{K-(N-t)+1}(\delta),$$

(104)

where

$$\delta^2 = \theta_2^{1/2}(M_{22} - M_{23}M_{33}^{-1}M_{32})^{1/2}$$

(105)

is the noncentrality parameter. As a consequence, given $p_2, p_1 = 1/(1 + \tilde{p}_1) \sim C\beta_{K-(N-t)+1,R}(\delta)$ with $\delta^2 = \text{SINR}p_2$. It is clear that under $H_0$ and given $p_2, p_1 \sim C\beta_{K-(N-t)+1,R}(0)$, which does not depend on $p_2$.

Gathering the above results, the joint pdf of $p_1$ and $p_2$ can be written as

$$f_{p_1,p_2}(x, y; H_1) = f_\beta(x; K - (N - t) + 1, r, 0) \times f_\beta(y; K - (N - t) + r + 1, N - t - r, 0) \times e^{-i \text{SINR} \sum_{k=0}^{K-(N-t)+1} (K - (N - t) + 1) \cdot (r - 1)! / (r + k - 1)!(\text{SINR} \gamma(1 - x))^k,}$$

(106)

under $H_1$, and

$$f_{p_1,p_2}(x, y; H_0) = f_\beta(x; K - (N - t) + 1, r, 0) \times f_\beta(y; K - (N - t) + r + 1, N - t - r, 0),$$

(107)

under $H_0$, where

$$f_\beta(x; n, m, 0) = (n + m + 1)! / (n - 1)! (m - 1)! x^{-1} (1 - x)^{m-1}. $$

(107)
APPENDIX D
DERIVATION OF THE LMPID

It is not difficult to show that

$$t_{\text{LMPID}} = \frac{\delta}{\Delta \text{SNR}} \left[ e^{-\text{SNR}} p_1 p_2 \sum_{k=0}^{K-(N-t)+1} \binom{K-(N-t)+1}{k} \right]$$

$$\times \frac{(r-1)!}{(r+k-1)!} \left[ p_2(1-p_1) \right]^k \text{SNR}=0$$

$$= -p_1 p_2 e^{-\text{SNR}} p_1 p_2 \sum_{k=0}^{K-(N-t)+1} \binom{K-(N-t)+1}{k}$$

$$\times \frac{(r-1)!}{(r+k-1)!} \left[ p_2(1-p_1) \right]^k + e^{-\text{SNR}} p_1 p_2$$

$$\times \frac{K-(N-t)+1}{r} \left[ p_2(1-p_1) \right]^k \text{SNR}=0$$

$$= K-(N-t)+1 \quad p_2(1-p_1) - p_1 p_2.$$  (108)

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\[(z, Z_s)\]  
\[(z, S)\]  
Computation of \[t_1(z, S)\]  
\[
\begin{align*}
\text{Invariant test} \\
\Phi(t_1(z, S)) \begin{cases} 
H_1 & \geq \eta \\
H_0 & \end{cases} 
\end{align*}
\[
\Phi(\cdot) \text{ generic function}
\]

Fig. 1. Block diagram of the transformations that lead to a generic invariant test.

\[P_d \text{ versus SINR for the GLRT, the 2S-GLRT, the LMPID, and the MPID assuming } N = 8, K = 12, r = 4, t = 2, \text{ and } P_{fa} = 10^{-4}.\]

Fig. 2. \[P_d \text{ versus SINR for the GLRT, the 2S-GLRT, the LMPID, and the MPID assuming } N = 8, K = 12, r = 4, t = 2, \text{ and } P_{fa} = 10^{-4}.\]

Fig. 3. \[P_d \text{ versus SINR for the GLRT, the 2S-GLRT, the LMPID, and the MPID assuming } N = 8, K = 12, r = 2, t = 4, \text{ and } P_{fa} = 10^{-4}.\]

Fig. 4. \[P_d \text{ versus SINR for the GLRT, the 2S-GLRT, the LMPID, and the MPID assuming } N = 8, K = 16, r = 4, t = 2, \text{ and } P_{fa} = 10^{-4}.\]

Fig. 5. \[P_d \text{ versus SINR for the GLRT, the 2S-GLRT, the LMPID, and the MPID assuming } N = 8, K = 16, r = 4, t = 2, \text{ and } P_{fa} = 10^{-4}.\]
The graph shows the relationship between SINR (dB) and the probability of detection ($P_d$). The x-axis represents SINR in dB, ranging from 5 to 20, and the y-axis represents $P_d$ ranging from 0 to 1.

The graph includes the following methods:
- GLRT
- 2S-GLRT
- LMPID
- ED
- MPID

The curves indicate how each method performs across different SINR values, with GLRT and 2S-GLRT showing similar trends, while LMPID, ED, and MPID exhibit slightly different behaviors. The probability of detection increases as SINR increases for all methods.