A GERBE OBSTRUCTION TO QUANTIZATION OF FERMIONS
ON ODD DIMENSIONAL MANIFOLDS WITH BOUNDARY

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ABSTRACT We consider the canonical quantization of fermions on an odd dimensional manifold with boundary, with respect to a family of elliptic hermitean boundary conditions for the Dirac hamiltonian. We show that there is a topological obstruction to a smooth quantization as a function of the boundary conditions. The obstruction is given in terms of a gerbe and its Dixmier-Douady class is evaluated.

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0. INTRODUCTION

In this paper we study the Hamiltonian quantization of massless fermions on a compact odd-dimensional manifold $X$ with boundary $Y = \partial X$. Field theories on manifolds with boundary arise in several situations including gravitation on odd dimensional anti-de Sitter spacetimes [W]. Our aim is to investigate topological obstructions (or anomalies) arising from non-trivial topology in the boundary conditions. We assume that a Riemannian metric and spin structure is given on $X$. The Dirac field might also be coupled to a Yang-Mills potential but in this paper we shall concentrate only on the problem of the dependence of the canonical quantization on the boundary conditions.
The obstruction to canonical quantization arises in the following way. We have a family of hermitean Dirac hamiltonians in a 1-particle space $H$. Let $G$ be the parameter space for the family. In quantization the 1-particle space $H$ is replaced by a Fock space $\mathcal{F}_g$ (with $g \in G$) and the 1-particle hamiltonian $D_g$ by a second quantized hamiltonian $\hat{D}_g$. If the hamiltonians do not have any zero modes this does not cause any problems. The Fock space associated to the parameter $g$ is defined as the representation space for the algebra of canonical anticommutation relations (CAR) with a vacuum vector ('Dirac sea') corresponding to the polarization of the 1-particle space $H = H_+(g) \oplus H_-(g)$ to positive and negative eigenmodes of the hamiltonian $D_g$.

In case of zero modes the above construction is not continuous as a function of the parameter $g$ since $H_{\pm}(g)$ is not a continuous function of $g$. This is a familiar problem in the hamiltonian quantization of chiral fermions and its resolution is related to the appearance of Schwinger terms in the gauge current algebra which form an algebraic obstruction to gauge invariant quantization of chiral fermions. Topologically the obstruction can be understood in terms of the structure of the space $\mathcal{A}/G$, the space of vector potentials modulo gauge transformations.

In the case at hand the space $\mathcal{A}/G$ is replaced by the parameter space $G$ for boundary conditions. The spectrum of the hamiltonian $D_g$ depends on the boundary conditions and in the case of massless fermions zero modes will occur.

It turns out that one can avoid a detailed analysis of the zero mode set of the family of hamiltonians (which in general is a complicated task). Instead, the problem of constructing representations of the CAR algebra can be rewritten as the problem of prolonging a principal bundle $\mathcal{P}$ over $G$, with structure group $U_{res}$, to a principal bundle with the structure group $\hat{U}_{res}$. Here $U_{res}$ is the group of implementable Bogoliubov automorphisms of the CAR algebra and $\hat{U}_{res}$ is its central extension. The bundle of Fock spaces can be viewed as an associated bundle to the prolongation, through a representation of $\hat{U}_{res}$ in a Fock space.

The topological obstruction to the bundle prolongation is a three-cohomology class on the base $G$, the Dixmier-Douady class. In the case of chiral fermions this has been discussed before in [CMM1]. In the present paper we shall compute the DD class on the parameter space $G$ and we show that it indeed gives a nonzero obstruction to a continuous second quantization of fermions.
It turns out that the parameter space $G$ can be naturally identified as the group of unitary operators $g$ in a fixed Hilbert space such that $g - 1$ is a trace class operator. The cohomology of this group is known. It has one generator in each odd dimension and no generators in even dimensions. An explicit representation for the odd generators is given by the Wess-Zumino-Witten type differential forms $\text{tr} (dgg^{-1})^{2k+1}$ on the group manifold. We are interested only on the form with $k = 1$ and we shall prove that this (when properly normalized) is the Dixmier-Douady class for the bundle prolongation.

The plan of the paper is as follows. In section 1 we shall describe the parametrization of self-adjoint elliptic boundary conditions for Dirac operators in odd dimensions and recall some facts about canonical quantization. In section 2 we shall go through some technicalities related to local triviality of the relevant bundles and explain in more detail the relation between canonical quantization and the principal bundle $P$. In section 3 we first explain general aspects of gerbes and their DD class and we give two different constructions for a universal $U_{res}$ bundle (bundle with contractible total space) over the parameter space $G$. These will be needed in section 4, where we finally compute the DD class for the bundle prolongation and the associated gerbe.

1. BOUNDARY CONDITIONS AND FOCK SPACES

We consider Dirac operators on an odd dimensional compact manifold $X$ with boundary $Y$. As an additional technical assumption we require that the metric becomes a product metric near the boundary and that the normal component of the gauge potential $A$ and the normal derivatives of $A$ vanish at the boundary. Then near the boundary the Dirac equation can be written as

$$i c_t \partial_t \psi = - h_Y \psi,$$

where $h_Y$ is the Dirac operator on the boundary, $t$ is a local coordinate in the normal direction and $c_t$ denotes Clifford multiplication by the unit normal in the $t$ direction. The operator $h_Y$ anticommutes with Clifford multiplication by $c_t$.

In order that the Dirac operator in the bulk $X$ becomes an elliptic Fredholm operator we use Atiyah-Patodi-Singer type boundary conditions. These elliptic
boundary conditions are labelled by projection operators $P$ on the boundary Hilbert space $L^2(Y, S)$ (where $S$ denotes the combined spin and gauge vector bundle) such that the difference $P - P_+$ is a trace-class operator. Here $P_+$ is the projection on to the spectral subspace on which $h_Y$ is positive. The boundary condition is then written as

$$P\psi|_Y = 0.$$  

The precise estimate on $P - P_+$ is actually not very critical in the following. Often one requires that $P - P_+$ is a smoothing operator. This has advantages when studying the analytical properties of Dirac determinants. However, for analyzing topological properties of the family of Dirac operators parametrized by such boundary conditions it turns out to be more convenient to work in the trace-class setting.

In order that $D_P$ be hermitean we must have

$$Pc_t = c_t(1 - P).$$

We shall use the parametrization of the boundary conditions studied in [S]: For each boundary projection $P$ above we think of $P(L^2(Y, S))$ as the graph of an unitary operator

$$T : S^+ \rightarrow S^-,$$

where $S^\pm$ are the eigenspaces of the chirality operator $c_t$ on the boundary. If $T_0$ is the unitary operator corresponding to the projection $P_+$ then $K = TT_0^{-1}$ should differ from the identity operator in $L^2(Y, S^-)$ by a trace-class operator.

Denote by $G$ the group of all unitaries $g$ in $L^2(Y, S^-)$ such that $g - 1$ is a trace-class operator. Thus $G$ is the parameter space of the elliptic hermitean boundary conditions for the family of Dirac operators $D_P$ and henceforth we write $D_g, g \in G$ to denote elements in this family.

For each $g \in G$ we would like to produce a fermionic Fock space $\mathcal{F}_g$ carrying an irreducible quasi-free representation of the canonical anticommutation relations (CAR) and a compatible action of the second quantized Dirac operator $\hat{D}_g$. Here quasi-free means that there is a vacuum vector $|0 > \in \mathcal{F}_g$ such that

$$a(v)|0 > = 0 = a^*(u)|0 >$$

for all $v \in H_+$ and $u \in H_-$ where $H_+ \oplus H_-$ is the polarization of the 1-particle Hilbert space $H = L^2(X, S)$ to positive and negative frequencies with respect to
the Dirac operator $D_g$. The CAR algebra is generated by the relations

$$a^*(u)a(u') + a(u')a^*(u) = \langle u', u \rangle$$

for $u, u' \in H$ with all other anticommutators zero and the Hilbert space inner product $\langle \cdot, \cdot \rangle$ is antilinear in the first argument.

Note that in general one cannot expect that there would be a continuous choice $|0\rangle = |0\rangle_g$ of vacuum vectors parametrized by the boundary conditions $g \in G$; the vacuum line bundle (in case it is well-defined) could be nontrivial. In fact, we shall show that there is a more serious problem: There is a topological obstruction to the above quantization of fermions parametrized smoothly by elements of $G$.

The obstruction appears as follows. In order to define the quasi-free representation of the CAR algebra we need a polarization $H = H_+(g) \oplus H_-(g)$ of the 1-particle space $H$. This polarization should be a continuous function of the boundary condition $g \in G$. Furthermore, each Fock space $\mathcal{F}_g$ defined by the polarization should contain the vacuum vector for the Dirac operator $D_g = ic_t \partial_t + h_{Y,g}$, that is, the vacuum defined by the splitting to positive and negative spectral subspaces of $D_g$. This requirement is equivalent to the condition that the plane $H_+(g)$ lies in the infinite-dimensional Grassmannian $Gr_g$ consisting of all closed subspaces $W \subset H$ such that the difference $P_g - Q_g$ is Hilbert-Schmidt; here $P_g$ is the orthogonal projection onto $H_+(g)$ and $Q_g$ is the orthogonal projection onto the positive spectral subspace of $D_g$.

We have now a bundle of infinite-dimensional Grassmannians $\{Gr_g| g \in G\}$ (modelled by the ideal of Hilbert-Schmidt operators) over the base $G$. We still need to show that this bundle is defined in terms of local trivializations and smooth transition functions. Once this is done we will show that the potential obstruction to quantization is the topological nontriviality of this bundle.

**Proposition 1.** The bundle $Gr$ is smoothly locally trivial.

The proof is given in appendix 1.

2. THE OBSTRUCTION TO CANONICAL QUANTIZATION

A closer examination reveals that the discussion in Section 1 is not quite accurate...
in the sense that often one would be satisfied with a determination of the CAR
algebra representation without an explicit choice of the vacuum vector.

To explain this let us denote by $U_{res}$ the group of unitaries $T$ in a complex
polarized Hilbert space $H = H_+ \oplus H_-$ such that the commutator $[P_+, T]$ is Hilbert-
Schmidt; here $P_+$ is the orthogonal projection onto $H_+$. This group acts naturally
and transitively on the Grassmannian $Gr(H_+)$ consisting of closed subspaces $W \subset
H$ such that $P_W - P_+$ is Hilbert-Schmidt, where $P_W$ is the orthogonal projection
onto $W$.

Over $Gr(H_+)$ there is a canonical complex line bundle $DET$. When the projection
from $W$ to $H_+$ has Fredholm index equal to zero, the fiber at $W$ is the set
of equivalence classes $[q, \lambda]$, where $q : H_+ \rightarrow W$ is an isomorphism such that
$P_+ q - id_{H_+}$ is trace-class and $\lambda \in \mathbb{C}$. The equivalence is defined by $(q, \lambda) \sim
(q t^{-1}, \lambda \det t)$, where $t : H_+ \rightarrow H_+$ is an isomorphism with $t - 1$ trace-class; for
details, see [PrSe].

A central extension $\hat{U}_{res}$ of $U_{res}$ acts in the total space of $DET$, [PrSe]. This
extension acts unitarily in the fermionic Fock space corresponding to the given
polarization $H = H_+ \oplus H_-$.

If a representation of the CAR is given with respect to a polarization $H = H_+ \oplus H_-$ then the set of (normalized) Fock vacua is a $\hat{U}_{res}$ orbit through some
fixed vacuum $|H_+ >$ defined by the polarization. The orbit of $H_+$ under the $U_{res}$
action is the infinite-dimensional Grassmannian $Gr(H_+)$ and the $\hat{U}_{res}$ orbit in the
Fock space is the set of vectors of unit length in the canonical determinant bundle $DET$ over $Gr(H_+)$.

In the case of a family of Grassmannians, the construction of the family of CAR
representations can now be formulated as the problem of prolonging the Grassmann-
nian bundle to a bundle with fiber equivalent to the determinant bundle $DET$.
There is an illuminating alternative formulation of the prolongation problem which
we shall now describe.

The Grassmannian $Gr(H_+)$ is a homogeneous space $U_{res}/(U(H_+) \times U(H_-))$. The
topology of the block diagonal unitary group $U(H_+) \times U(H_-)$ is trivial by Kuiper’s
theorem. Thus $U_{res}$ contracts to $Gr(H_+)$. It follows that the prolongation of a $U_{res}$
bundle over some base manifold to a $\hat{U}_{res}$ bundle is equivalent to the problem of
prolonging the associated Grassmann bundle to a bundle with model fiber $DET$. 
The relevance of the $\hat{U}_{\text{res}}$ bundle in Fock space quantization is immediate: selecting a model Fock space with a $\hat{U}_{\text{res}}$ action, one can construct a bundle of Fock spaces as an associated vector bundle to a given $\hat{U}_{\text{res}}$ bundle.

To close the circle, we note that starting from the given Grassmannian bundle over the parameter space $G$ one constructs a natural $U_{\text{res}}$ bundle $\mathcal{P}$ such that the Grassmannian bundle is recovered as an associated bundle. If $H = H_+ \oplus H_-$ is any fixed polarization then the fiber of $\mathcal{P}$ at $g \in G$ consists of all unitaries $h$ in $H$ such that $h \cdot H_+ \in \text{Gr}_g$.

Instead of looking at the specific construction of the $U_{\text{res}}$ bundle over the family of boundary conditions $G$ we can extend this to another universal construction over the space of all (bounded) self-adjoint Fredholm operators $\mathcal{F}_*$ such that the essential spectrum is neither negative nor positive. This means that the spectral subspaces both on the negative and positive side of the real axis are infinite-dimensional. The topology in $\mathcal{F}_*$ is defined by the operator norm. The fact that Dirac operators are unbounded need not bother us since we shall be really interested only on the spectral resolutions defined by the sign operators $(D - \lambda)/|D - \lambda|$, which are bounded.

The space $\mathcal{F}_*$ retracts onto the subspace $\hat{\mathcal{F}}_*$ consisting of operators with essential spectrum at $\pm 1$, [AS]. Thus it is sufficient to study the $U_{\text{res}}$ bundle $\mathcal{P}'$ over $\hat{\mathcal{F}}_*$. The fiber $\mathcal{P}'_A$ at $A \in \hat{\mathcal{F}}_*$ is defined as the set of unitary operators $g$ in $H_+ \oplus H_-$ such that $g \cdot H_+ \in \text{Gr}(W_A)$ where $W_A$ is the positive spectral subspace of the operator $A$.

The proof of the local triviality of the bundle $\mathcal{P}'$ is a slight extension of the argument which was used in the proof of Proposition 1. First, we can choose an operator norm continuous (smooth) section $F : U(H)/(U(H_+) \times U(H_-)) \to U(H)$ (again by Kuiper’s theorem). Defining $U_{\lambda}$, for $1 > \lambda > -1$, as the open set in $\hat{\mathcal{F}}_*$ consisting of operators $A$ such that $\lambda \neq \text{Spec}(A)$ then $g_{\lambda}(A) = F((A - \lambda)/|A - \lambda|)$ is a local section of $\mathcal{P}'$ with local norm continuous transition functions $g_{\lambda \mu} = g_{\lambda}^{-1} g_{\mu}$. The Hilbert-Schmidt norm continuity (and smoothness) of the off-diagonal blocks follows as in Proposition 1 since, by our assumption about the essential spectrum of $A$, the spectral subspaces corresponding to the open intervals $]\mu, \lambda[$ are finite-dimensional for $-1 < \mu < \lambda < 1$. 7
3. A UNIVERSAL CONSTRUCTION

In this section $G$ denotes the group of unitary operators $g$ in $H$ such that $g - 1$ is trace-class.

Let $\mathcal{P}$ be a locally trivial principal $U_{res}$ bundle over $G$. The Lie algebra of $\hat{U}_{res}$ is defined by the standard 2-cocycle

$$c(X, Y) = \frac{1}{4} \text{tr} \epsilon[\epsilon, X][\epsilon, Y].$$

Let $(U_\alpha)$ be a family of open contractible sets covering the base $G$. Let $\mathcal{L} \to U_{res}$ be the complex line bundle associated to the circle bundle $1 \to S^1 \to \hat{U}_{res} \to U_{res} \to 1$. Let $\varphi_\alpha : U_\alpha \to \mathcal{P}$ be a family of local trivializations of the bundle $\mathcal{P}$. Let $g_{\alpha\beta}$ be the corresponding family of $U_{res}$ valued transition functions. We define a family of local line bundles over the open sets $U_{\alpha\beta} = U_\alpha \cap U_\beta$ by pull-back, $\mathcal{L}_{\alpha\beta} = g_{\alpha\beta}^* \mathcal{L}$.

Since $\hat{U}_{res}$ is a group we have a natural isomorphism

$$\mathcal{L}_g \otimes \mathcal{L}_f \cong \mathcal{L}_{gf}$$

for all $g, f \in U_{res}$. This gives a family of isomorphisms

$$\mathcal{L}_{\alpha\beta} \otimes \mathcal{L}_{\beta\gamma} = \mathcal{L}_{\alpha\gamma}$$

over the common intersections $U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cap U_\gamma$. Thus we have a bundle gerbe $\mathcal{Q}$ over the base $G$, [CMM1,CMM2]. The product structure gives also a natural isomorphism $\mathcal{L}_g \cong \mathcal{L}_{g^{-1}}$ and therefore an isomorphism $\mathcal{L}_{\alpha\beta} \cong \mathcal{L}_{\beta\alpha}^{-1}$. Combining this with (3) we obtain a natural trivialization $f_{\alpha\beta\gamma}$ of the product bundle $\mathcal{L}_{\alpha\beta} \otimes \mathcal{L}_{\beta\gamma} \otimes \mathcal{L}_{\gamma\alpha}$ over $U_{\alpha\beta\gamma}$.

The family $\{f_{\alpha\beta\gamma}\}$ of trivializations (local $S^1$ valued functions) satisfies the cocycle condition

$$f_{\beta\gamma\delta} f_{\alpha\gamma\delta}^{-1} f_{\alpha\beta\delta} f_{\alpha\beta\gamma}^{-1} = 1$$

on the intersections of four open sets.

Because of the relations (3) the local curvature forms $\omega_{\alpha\beta} = g_{\alpha\beta}^* c$ satisfy the relations

$$(4a) \quad [\omega_{\alpha\beta}] + [\omega_{\beta\gamma}] + [\omega_{\gamma\alpha}] = 0$$
in de Rham cohomology $H^2(U_{\alpha\beta\gamma})$ on the base. Note that these equations do not hold on the level of differential forms. However, this can be corrected by adding an exact 2-form $d\theta_{\alpha\beta}$ to each of the closed forms $\omega_{\alpha\beta}$; the modified forms $\omega_{\alpha\beta}$ satisfy then
\begin{equation}
\omega_{\alpha\beta} + \omega_{\beta\gamma} + \omega_{\gamma\alpha} = 0.
\end{equation}

Actually, because of the given local trivializations $f_{\alpha\beta\gamma}$ on triple intersections we have the consistency condition
\begin{equation}
\nabla_{\alpha\beta\gamma} f_{\alpha\beta\gamma} = 0,
\end{equation}

where $\nabla_{\alpha\beta\gamma}$ is the connection on the trivial bundle $L_{\alpha\beta} \otimes L_{\beta\gamma} \otimes L_{\gamma\alpha}$ composed from the connections on the factors $L_{\alpha\beta}$ with curvature forms $\omega_{\alpha\beta}$. In fact, (5) implies (4b): If $A_{\alpha\beta}$ is a local potential, $dA_{\alpha\beta} = \omega_{\alpha\beta}$, then (5) can be written as
\begin{equation}
df_{\alpha\beta\gamma} + (A_{\alpha\beta} + A_{\beta\gamma} + A_{\gamma\alpha}) f_{\alpha\beta\gamma} = 0,
\end{equation}

and multiplying by $f_{\alpha\beta\gamma}^{-1}$ and then taking the exterior derivative gives the cocycle relations (4b) for the forms $\omega_{\alpha\beta}$.

If $(\rho_{\alpha})$ is a partition of unity on $G$ subordinate to the covering $(U_{\alpha})$ then we can produce a closed 3-form on $G$ in the usual way. First, we have a closed 3-form $\omega_{\alpha}$ on $U_{\alpha}$,
\begin{equation}
\omega_{\alpha} = d \sum_{\beta} \omega_{\alpha\beta} \rho_{\beta} = \sum_{\beta} \omega_{\alpha\beta} d\rho_{\beta}.
\end{equation}

Since $\omega_{\alpha} - \omega_{\beta} = 0$ on $U_{\alpha\beta}$ they can be pasted together to give the closed 3-form $\omega_{3}$ on $G$. This is the de Rham form of the Dixmier-Douady (DD) class of the bundle gerbe. Of course, this description disregards all potential torsion information. However, for our purposes the differential form picture is quite sufficient since we going to show that there is already on this level an obstruction to quantization.

Unfortunately the existence of a partition of unity on the infinite-dimensional manifold $G$ does not appear to be known. However, in order to define the DD class as an element of the dual of the 3-homology classes it is sufficient to use the partition of unity on the (singular) homology 3-cycles and to pull-back the forms $\omega_{\alpha\beta}$ down to the embedded 3-cycles and then proceed as above. An alternative solution would be to replace $G$ by the group of unitaries differing from the identity.
by a Hilbert-Schmidt operator (which is a Hilbert manifold) where we would have a partition of unity.

Note however that whichever method we use to define the DD class we can normalize it so that its integral around closed 3-cycles is $2\pi$ times an integer.

In the case when the gerbe is coming from a principal $U_{res}$ bundle $\mathcal{P}$ over $G$ there is another method to construct the Dixmier-Douady class which gives directly the integrals of $\omega_3$ over 3-cycles in $G$. The homology cycles can be generated by mappings from $S^3$ to $G$, so we shall restrict to this case.

Map the 3-disk $D^3$ onto the sphere $S^3$ such that the boundary is mapped to one point $g \in S^3$. Pulling back the bundle $\mathcal{P}$ to $D^3$ leads to a trivial $U_{res}$ bundle over $D^3$. In this trivialization the boundary $S^2 \subset D^3$ is mapped to the fiber $\mathcal{P}_g$ over $g$. Selecting a base point $x$ in the fiber we can identify $\mathcal{P}_g \simeq U_{res}$. The integral of the curvature form $c$ over $S^2$ gives then the integral of $\omega_3$ over $S^3 \subset G$. Note that the result does not depend on the choice of the base point $x$ since a different choice $x'$ is related by a right translation $x' = x \cdot q$ by an element $q \in U_{res}$. The cohomology class $[c]$ is invariant under right (and left) translations on the group manifold. One can check that this construction gives the same result as the one starting from the cocycle of 2-forms $\omega_{\alpha\beta}$. Namely, selecting a local trivialization of the bundle $\mathcal{P}$ at $g \in G$ such that $g$ is mapped to the point $x$ in the fiber, we observe that the map $S^2 \to U_{res}$ above is just the transition function from the local trivialization over $D^3$ to the local trivialization around $g$. The rest follows from Stokes’ theorem, using the fact that $\omega_{\alpha\beta} = d^{-1}\omega_\alpha - d^{-1}\omega_\beta$ on the overlap and that $\omega_\alpha = \omega_\beta = \omega_3$ on $U_{\alpha\beta}$.

**Universal $U_{res}$ bundle over $G$.** Let $\mathcal{P}$ be the space of smooth paths (parametrized by $0 \leq t \leq 2\pi$) in $G$ with initial point $1 \in G$. We also require that the derivatives of $g(t)$ vanish at the end points. To each $g \in \mathcal{P}$ there corresponds a vector potential $A$ on the circle $S^1$, with values in the Lie algebra $\mathfrak{g}$ of $G$, $A(t) = g(t)^{-1}g'(t)$. The group $\Omega G$ of based loops at $1 \in G$ acts freely from the right on $\mathcal{P}$ through gauge transformations $A^g = g^{-1}Ag + g^{-1}g'$ and the set of orbits is $\mathcal{P}/\Omega G = G$. Since clearly $\mathcal{P}$ is contractible, $\mathcal{P}$ is the universal $\Omega G$ bundle over $G$. The local triviality of the path fibration is obtained by the exponential mapping; locally, near the unit element, the trivialization is $(g, h) \mapsto \tilde{h}$, where $g \in G$, $h(t)$ is a based loop at $1$, and $\tilde{h}(t) = \exp(tZ)h(t)$ with $\exp(2\pi Z) = g$.

By Bott periodicity, $\Omega G$ is homotopy equivalent with $U_{res}$. Actually, we can
define a group homomorphism \( j : \Omega G \to U_{\text{res}} \) which is a homotopy equivalence, see appendix 2 for proofs. The group \( G \) acts, by definition, as the unitary group of 1 + trace-class operators in a complex Hilbert space \( H \). Let \( \mathcal{H} = L^2(S^1, H) \) and let \( \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \) be the polarization to nonnegative and negative Fourier components. Then each \( g \in \Omega G \) acts naturally in \( \mathcal{H} \), by pointwise multiplication, and this action gives the promised homomorphism to \( U_{\text{res}}(\mathcal{H}_+ \oplus \mathcal{H}_-) \). The homomorphism can be used to define an associated bundle \( \tilde{P} = P \times_{\Omega G} U_{\text{res}} \) with fiber \( U_{\text{res}} \). This latter bundle is then a universal \( U_{\text{res}} \) bundle over \( G \), see Proposition 2.

**Remark.** There is an alternative construction of the universal \( U_{\text{res}} \) bundle over \( G \) which makes its appearance in quantum field theory (QFT) more transparent. For each gauge potential \( A(t) = g(t)^{-1}dg(t), g \in P \), let \( W_A \) be the set of unitary operators \( T \) in \( \mathcal{H} \) such that the projections onto the subspaces \( T \cdot \mathcal{H}_+ \) and \( \mathcal{H}_+(D_A) \) differ by a Hilbert-Schmidt operator; here \( \mathcal{H}_+(D_A) \) is the positive frequency subspace for the Dirac operator \( D_A \). It is easy to see that if both \( T, T' \in W_A \) then \( T' = Th \) for some \( h \in U_{\text{res}} \). Consequently, one can view \( W_A \) as a fiber in a principal \( U_{\text{res}} \) bundle over \( P \) [CMM2]. The base is contractible, thus there exists a global section \( A \mapsto T_A \). Actually, we can construct explicitly a global section by setting

\[
T_A = T_{f^{-1}df} = f,
\]

that is, \( T_A \) is the multiplication operator by the function \( f(t) \) on the interval \([0, 2\pi] \).

We can define a \( U_{\text{res}} \) valued 1-cocycle for the natural \( \Omega G \) action on \( P \) by

\[
S(A; g) = T_A^{-1}T_{A^g}.
\]

In the case of the choice \( T_A = f \) above, we have simply \( S(A; g) = g \). This cocycle defines the same associated \( U_{\text{res}} \) bundle \( \tilde{P} \) over \( G \) as above, by the equivalence relation \( (A, T) \simeq (A^g, TS(A; g)) \) for \( g \in \Omega G \) and \( T \in W_A \).

**Proposition 2.** The total space of the bundle \( \tilde{P} \) is contractible and thus \( \tilde{P} \to G \) is the universal \( U_{\text{res}} \) bundle over \( G \).

**Proof.** By a well known theorem in homotopy theory [G] a space with the homotopy type of a CW complex is contractible if it is weakly homotopy equivalent to a point. All of the spaces we consider are Banach manifolds and have the homotopy type that of a CW complex. We want to compare the homotopy groups of \( P \), which are
all trivial, with the homotopy groups $\pi_i(\tilde{\mathcal{P}})$. From the appendix 2 we know that the embedding $j$ of $\Omega G$ to $U_{res}$ is a homotopy equivalence. This embedding extends to an embedding of principal bundles $\tilde{j} : \mathcal{P} \rightarrow \tilde{\mathcal{P}}$ by mapping $p \in \mathcal{P}$ to the equivalence class of the pair $(p, e) \in \mathcal{P} \times U_{res}$ in $\tilde{\mathcal{P}}$; $e$ is the identity element in $U_{res}$. On each fibre this map reduces to $j$. This means we have a commutative diagram

$$
\pi_i(\Omega G) \xrightarrow{j^*} \pi_i(U_{res}) \xleftarrow{} \pi_{i+1}(G)
$$

where the up arrows represent the connecting maps for the homotopy long exact sequences for the locally trivial fibrations $\mathcal{P}$ and $\tilde{\mathcal{P}}$ respectively.

Since $\pi_i(P) = 0$ for all $i$, by the homotopy exact sequence

$$
\ldots \pi_{i+1}(G) \rightarrow \pi_i(\Omega G) \rightarrow \pi_i(P) \rightarrow \pi_i(G) \rightarrow \ldots
$$

the connecting map $k : \pi_{i+1}(G) \rightarrow \pi_i(\Omega G)$ is an isomorphism (both groups are known to be 0 when $i$ is odd and equal to $\mathbb{Z}$ when $i$ is even (these facts rely on results of Palais, see [Q] for a discussion). From the commutative diagram above it follows that the corresponding map for the fibering $U_{res} \rightarrow \tilde{\mathcal{P}} \rightarrow G$ is also an isomorphism and from the homotopy long exact sequence for this fibering we conclude that $\pi_i(\tilde{\mathcal{P}}) = 0$. Moreover this shows that $\tilde{j}$ is a weak homotopy equivalence from which we deduce that $\tilde{\mathcal{P}}$ is a contractible.

4. THE DIXMIER DOUADY CLASS

Next we want to relate the Dixmier-Douady class $\omega_3 \in H^3(G, \mathbb{Z})$ of the various gerbes over $G$ to the natural curvature 2-form on the group $\Omega G$. Recall that this curvature form on $\Omega G$ is the homogeneous 2-form

$$
\omega_2 = \frac{1}{2\pi} \int_{S^1} \text{tr} (g^{-1} dg) \frac{d}{dt}(g^{-1} dg).
$$

We start from the universal $\Omega G$ bundle $\mathcal{P}$ over $G$ and its bundle gerbe as described in Section 2.

**Proposition 3.** The cohomology class $[\omega_3]$ is represented by the closed 3-form
\[
\theta_3 = \frac{1}{12\pi} \text{tr}(g^{-1}dg)^3.
\]

**Proof.** We have to show that the integral of \(\theta_3\) over any closed 3-cycle on \(G\) agrees with the integral of \(\omega_3\). We can generate 3-cycles by 3-spheres, so we take as the 3-cycle a (differentiable) mapping \(\chi\) of \(S^3\) into \(G\). Since \(G\) is connected, we can assume the image under \(\chi\) of \(S^3\) is such that the poles of \(S^3\) are mapped to the unit element in \(G\).

We cut the 3-cycle \(S^3\) along the equator to two disks \(B_+\) and \(B_-\). Over these disks the pullback under \(\chi\) of the principal bundle \(\mathcal{P} \to G\) is trivial; choose a pair of local trivializations \(\psi_{\pm}\) over \(B_{\pm}\). Concretely, the local trivializations can be defined as follows. For each \(x \in S^2\) on the equator we have path \(t \mapsto \phi_{\pm}(x)(t) \in S^3\) by connecting the point \(x\) by a segment of a great circle to the to the 'north pole'. Similarly, we obtain a path \(\phi_{-}(x)(t)\) by connecting \(x\) by a great circle to the 'south pole'. We choose the parametrization such that \(t = 2\pi\) corresponds to the point \(x\) and \(t = 0\) corresponds to either of the poles (which is mapped to the unit element in \(G\)). Setting \(\psi_{\pm} = \chi \circ \phi_{\pm}\) the transition function on the equator (with values in \(\Omega G\)) is then given by \(\psi_{\pm}(x,t) = \psi_{-}(x)(t)g(x,t)\) for some \(g(x,.) \in \Omega G\). On the other hand, \((x,t) \mapsto \psi_{\pm}(x,t)\) is a parametrization of points on \(\chi(B_{\pm}) \subset G\) and therefore, by a simple calculation,

\[
12\pi \int_{S^3} \theta_3 = \sum_{\alpha=\pm} \int_{B_{\alpha}} \text{tr}(\psi_{\alpha}^{-1}d\psi_{\alpha})^3
\]

\[
= \int_{S^2 \times S^1} \text{tr}(g^{-1}dg)^3 + 3 \int_{S^2} \text{tr} \psi_{-}^{-1}d\psi_{-} \wedge dgg^{-1}.
\]

The second term on the right vanishes since \(g(x,2\pi) = 1\) on \(S^2\). The first term on the right is

\[
\int_{S^2 \times S^1} \text{tr}(g^{-1}dg)^3 = 12\pi \int_{S^2} \omega_2.
\]

On the other hand, the integral on the right is equal to the integral of \(\omega_3\) over \(S^3\); this follows, by Stokes’ theorem, directly from the construction of the DD class.
\( \omega_3 \) from the family of local 2-forms \( \eta_\alpha \) such that \( \eta_\alpha - \eta_\beta = \omega_{\alpha \beta} \) and \( d\eta_\alpha = \omega_\alpha = \omega_3|_{U_\alpha} \); recall that the class \([\omega_{\alpha \beta}]\) is given by the pull-back of \( \omega_2 \) with respect to the transition function \( g : U_{\alpha \beta} \to \Omega G \).

Thus indeed

\[
(9) \quad \int_{S^3} \theta_3 = \int_{S^3} \omega_3
\]

and so \( \omega_3 \) and \( \theta_3 \) represent the same cohomology class.

The DD class is an obstruction to writing the line bundles \( \mathcal{L}_{\alpha \beta} \) as tensor products \( \mathcal{L}_\alpha \otimes \mathcal{L}_\beta^* \) of local line bundles over the open sets \( U_\alpha \) on the base, that is, an obstruction to a trivialization of the gerbe \( Q \).

We use the following theorem, taken from \([K], \text{Theorem 3.17}\) in a slightly reformulated form:

**Theorem 1.** \([K]\). Let \( M \) be a compact space and \([M,G]\) the set of homotopy classes of maps from \( M \) to \( G \). Then \( K^1(M) \) is isomorphic with \([M,G]\). The group structure in the latter group is given by pointwise multiplication of maps.

On the other hand, we have seen that \( P \) can be viewed as an universal \( U_{\text{res}} \) bundle (or, what is essentially the same, as an \( \Omega G \) bundle) over \( G \). Thus we can construct \( U_{\text{res}} \) bundles over any compact space \( M \) by pulling back this universal bundle via a map from \( M \) to \( G \).

The index theorem in \([MP]\), Proposition 12, tells us that for a compact subset \( M \subset G \ K^1(M) \) is realised by homotopy classes of maps into the family of odd-dimensional Dirac operators parametrized by the boundary conditions \( M \). By the above theorem we can conclude that the universal \( U_{\text{res}} \) bundle over \( G \) can be identified as a (universal) \( U_{\text{res}} \) bundle over the family of Dirac operators (identified topologically as the family \( G \) of boundary conditions).

In the case of the universal \( U_{\text{res}} \) bundle over \( G \) we already know the DD class of the bundle gerbe \( Q \) is given by the generator \((7) \) (divided by \( 2\pi \)) of \( H^3(G,\mathbb{Z}) \). Thus this is also the (nontrivial) obstruction to a trivialization of the gerbe over the space of Dirac operators parametrized by the boundary conditions \( G \).

As a conclusion we obtain our main result:

**Theorem 2.** There is an obstruction to a prolongation of the \( U_{\text{res}} \) bundle \( P \) (as defined in Section 2) over a compact submanifold \( M \subset G \) of hermitean elliptic
boundary conditions, that is, an obstruction to the construction of the bundle of fermionic Fock spaces for the Dirac operators parametrized by \( M \). The Dixmier-Douady class of the obstruction is given by the restriction of the de Rham class \( \theta_3 \) to \( M \).

In particular, the obstruction is nontrivial when \( M = U(N) \) is any finite-dimensional subgroup of \( G \) with \( N \geq 2 \).

**Remark 1.** We could interpret our family of Dirac operators parametrised by \( G \) as an element of \( K^1(G) \) if the latter were defined as homotopy classes of maps into the self adjoint Fredholm operators. As \( G \) is not compact this is problematic. For our purposes it is enough to work with compact subsets.

**Remark 2.** In the case of an odd-dimensional manifold without boundary there is a similar obstruction to gauge invariant quantization, related to Schwinger terms in current algebra, [CMM1, CMM2]. Recently the case of gravitational Schwinger terms was discussed in the same formalism, [EM].

**APPENDIX 1: PROOF OF THE PROPOSITION 1**

Let \( T_0 \) be any fixed boundary condition, with \( T_0 : S^+ \to S^- \) a unitary map. If \( T \) is another boundary condition then the graph of \( T \) is obtained from the graph of \( T_0 \) by the unitary transformation \((u_+, u_-) \mapsto (u_+, g \cdot u_-) \) with \( g = TT_0^{-1} \in G \) and \( u_{\pm} \in S^\pm \). In a small open neighborhood \( U \) of \( 1 \in G \) we can choose in a smooth way, for any \( g \in U \), a smooth path \( g(t) \) such that \( g(0) = 1 \) and \( g(1) = g \); this is achieved for example by writing \( g = \exp(Z) \) and setting \( g(t) = \exp(tZ) \).

Near the boundary \( Y \) we may think of the \( L^2 \) functions on \( X \) as functions \( f(t, y) \) on \([0, 1] \times Y \) (where \( t \) is a parameter in the normal direction at the boundary) and we can extend the action of \( g \) in the boundary Hilbert space \( L^2(Y, S) \) to an action on \( H = L^2(X, S) \) by setting

\[
(R(g)f)(t, x) = (f_+(t, x), g(t) \cdot f_-(t, x))
\]

in the tubular neighborhood \([0, 1] \times Y \) and \( R(g) \) acts as an identity on \( f \) outside of this neighborhood. The map \( g \mapsto R(g) \) is smooth with respect to the \( L^1 \) norm in \( G \) and the operator norm in the algebra of bounded operators in \( L^2(X, S) \). This
follows from the smoothness of the embedding of trace-class operators to the algebra of bounded operators and from the smoothness of the exponential mapping.

Clearly $R(g)$ is an unitary operator in $L^2(X, S)$ and it maps the domain $\text{dom}(D_g)$ onto the domain $\text{dom}(D_1)$ of the reference Dirac operator. Thus $D_g$ is unitarily equivalent to the Dirac operator $R(g)D_gR(g)^{-1}$ in the fixed reference domain $\text{dom}(D_1)$.

Next we choose a smooth mapping from the space of polarizations $\epsilon$ in $H$ to the unitary group $U(H)$ such that $\epsilon = F(\epsilon)\epsilon_0F(\epsilon)^{-1}$ where $\epsilon_0$ is the fixed reference polarization given by the sign of the Dirac operator $D_1$. This mapping exists because the space of polarizations $U(H)/(U(H_+) \times U(H_-))$ is a contractible Banach manifold (with respect to the operator norm) by Kuiper’s theorem.

With these tools we can write an explicit local trivialization of the Grassmannian bundle $Gr$ over $G$. Near the unit element in $G$ the Hilbert-Schmidt Grassmannians are parametrized as

$$(g, \epsilon) \mapsto F(\epsilon_g)\epsilon F(\epsilon_g)^{-1},$$

where $\epsilon \in Gr_1$ and $\epsilon_g = R(g)\frac{D_g}{|D_g|}R(g)^{-1}$. We have assumed that zero does not belong to the spectrum of $D_g$; otherwise, we replace $D_g$ by $D_g - \lambda$ for some real number $\lambda$ in the neighborhood of $g = 1$.

Because of the potential non-triviality of the kernel of the operator $D_g$ we cannot have a global trivialization of the bundle. However, for each real number $\lambda$ the trivialization described above is well-defined in the open set $G_\lambda$ consisting of those elements $g$ for which $\lambda \notin \text{Spec}(D_g)$. The transition function on the overlap $G_\lambda \cap G_\mu$ is then given by

$$\epsilon \mapsto F(\epsilon_g(\lambda))^{-1}F(\epsilon_g(\mu))\epsilon F(\epsilon_g(\mu))^{-1}F(\epsilon_g(\lambda)),$$

where $\epsilon_g(\lambda)$ is defined as $\epsilon_g$ above but with the shifted operator $D_g - \lambda$. By the construction, the transition function $h_{\mu\lambda}(g) = F(\epsilon_g(\mu))^{-1}F(\epsilon_g(\lambda))$ satisfies

$$[\epsilon_0, h_{\mu\lambda}] = F(\epsilon_g(\mu))^{-1}\Delta_{\mu\lambda}F(\epsilon_g(\mu))$$

with $\Delta_{\mu\lambda} = \epsilon_g(\mu) - \epsilon_g(\lambda)$. Now on the overlap $G_\lambda \cap G_\mu$ the difference $\Delta_{\mu\lambda}$ has constant finite rank and therefore also $[\epsilon_0, h_{\mu\lambda}]$ has constant finite rank. A norm continuous mapping to operators of constant finite rank is continuous also with
respect to the Hilbert-Schmidt norm (or with respect to any $L_p$ norm) which proves the continuity of the $U_{res}$ valued transition functions. The same argument can then be used for the derivatives of the transition function.

**APPENDIX 2: PROOF OF THE HOMOTOPY EQUIVALENCE $j$**

We define a system of closed $n$ forms ($n = 2, 4, 6, \ldots$) on the Hilbert-Schmidt Grassmannian $Gr(H_+)$ for the polarization $H = H_+ \oplus H_-$, by

\[ \omega_n = a_n \text{tr} F (dF)^n, \]

where $F$ is the grading operator associated to $W \in Gr(H_+)$, that is, $F$ restricted to $W$ is $+1$ and the restriction to $W^\perp$ is $-1$. Note that since $F^2 = 1$, the differentials $dF$ anticommute with $F$. Thus for odd $n$ the form $\omega_n$ vanishes identically; $a_n$ is a normalization coefficient given by

\[ a_n = -\left(\frac{1}{2\pi i}\right)^j \frac{(j-1)!}{(2j-1)!}, \text{ with } n = 2j. \]

With this normalization the form $\omega_n$ is the generator in $H^n(Gr(H_+), \mathbb{Z})$.

Since $U_{res}/(U(H_+) \times U(H_-)) = Gr(H_+)$ and the diagonal subgroup is contractible, the natural projection $p : U_{res} \to Gr(H_+)$ can be used to pull back the generator $\omega_n$ to the generator $\phi_n$ in $H^n(U_{res}, \mathbb{Z})$. The projection can be written as $p(g) = g \epsilon g^{-1} = F_g$, where $\epsilon$ is the grading associated to $W = H_+$. It follows that the pull-back of $dF$ is $[\theta, F_g]$, where $\theta = dgg^{-1}$ is the Maurer-Cartan 1-form on $U_{res}$, and so

\[ \phi_n = a_n \text{tr} F_g [\theta, F_g]^n. \]

The homotopy type of both $\Omega G$ and $U_{res}$ is known. The homotopy groups vanish in odd dimensions whereas in even dimensions the homotopy groups are all isomorphic with $\mathbb{Z}, [C], [PrSe], [Q]$. A generator $x_n$ of the homotopy group $\pi_n(U_{res})$ when paired with the generator $\phi_n$ of $H^n(U_{res}, \mathbb{Z})$ gives $< x_n, \phi_n > = 1$. Thus the only thing we need to check to prove that the embedding $j : \Omega G \to U_{res}$ is a homotopy equivalence is to show that $< j(y_n), \phi_n >= 1$ for all even $n$, where $y_n$ is the generator of $\pi_n(\Omega G)$. 

17
The odd generators in $H^*(G, \mathbb{Z})$ are given by the differential forms

\[(A-3) \quad \psi_{2j-1} = a_{2j} \text{tr} \theta^{2j-1}.\]

From this one obtains by transgression the even generators in $H^*(\Omega G, \mathbb{Z})$,

\[(A-4) \quad \psi'_{2j} = (2j + 1)a_{2j+2} \int_{S^1} \text{tr} (g'(t)g(t)^{-1})\theta^{2j}.\]

We shall show that $\psi'_n$ is in the same cohomology class as the restriction of $\phi_n$ to the subgroup $\Omega G \subset U_{res}$. Provided that this is the case, we have $1 = < y_n, \psi'_n > = < j(y_n), \phi_n >$ and thus indeed $j(y_n)$ is the generator of $\pi_n(U_{res})$, and consequently $j$ is a homotopy equivalence.

Note that $[\theta, F_g] = g[g^{-1} dg, \epsilon]g^{-1}$ and therefore

\[(A-5) \quad \phi_n = a_n \text{tr} \epsilon[\theta_L, \epsilon]^n,\]

where $\theta_L = g^{-1} dg$. The proof of the equivalence of $\phi_n$ and $\psi'_n$ is through a standard trick using the Cartan homotopy method. We set $\theta_L = \theta_0 + \theta_1$ where $\theta_0$ commutes with $\epsilon$ and $\theta_1$ anticommutes with $\epsilon$. Define $G_t = -t\theta_1^2 + (1-t)\theta_0^2$ for $0 \leq t \leq 1$. Then by $d\theta_L = -\theta_L^2$ we get

\[(A-6) \quad dG_t = -\frac{1}{t} [\theta_0, G_t] \quad \frac{d}{dt} G_t = -(\theta_0^2 + \theta_1^2) = d\theta_0.\]

Thus we obtain

\[(A-7) \quad \text{tr} \epsilon G_1^n - \text{tr} \epsilon G_0^n = \int_0^1 \text{tr} \epsilon \frac{d}{dt} G_t^n dt = -n \int_0^1 \text{tr} \epsilon (\theta_0^2 + \theta_1^2) G_t^{n-1} dt = nd \int_0^1 \text{tr} \epsilon \theta_0 G_t^{n-1} dt.\]

Actually, $\epsilon G_0^n$ is not trace-class. However, this operator is an even wedge power of $\theta_0$ and by the permutation properties of the wedge product it is a commutator of some zero order pseudodifferential operators on the circle. Such an operator is always conditionally trace-class: this means that the trace is evaluated by first taking the trace over matrix indices, then integrating the product symbol over the circle and finally integrating over the momentum variable from $-\Lambda$ to $\Lambda$ and
taking the limit $\Lambda \to \infty$. But for conditionally trace-class operators $\text{tr}[A, B] \neq 0$, in general.

In the present case we can write

\begin{equation}
\text{tr} \epsilon G_0^n = \frac{1}{2} \text{tr} [\theta_0, \epsilon \theta_0^{2n-1}]
\end{equation}

where we have used the fact that $\theta_0$ commutes with $\epsilon$. If $a(x, p)$ and $b(x, p)$ are the symbol functions of a pair of zero order PSDO’s $A$, $B$ on the circle, then the conditional trace of the commutator $[A, B]$ is given by the expression

\begin{equation}
\text{tr}[A, B] = -\frac{i}{2\pi} \lim_{\Lambda \to \infty} \int_{p=-\Lambda}^{\Lambda} dp \frac{\partial}{\partial p} \int_x dx \text{tr} \left( \frac{\partial a}{\partial x}(x, p)b(x, p) - a(x, p) \frac{\partial b}{\partial x}(x, p) \right).
\end{equation}

Since in the expression (A-8) above the momentum variable is contained only in $\epsilon = p/|p|$ and the derivative of this symbol is twice the delta function in momentum space, we obtain

\begin{equation}
\text{tr} \epsilon G_0^n = \frac{1}{2} -\frac{i}{2\pi} \int_x \text{tr} \theta^{2n-1} \frac{d}{dx} \theta dx.
\end{equation}

Note that in the case $n = 1$ this is just the curvature on a loop group arising from the canonical central extension of the loop algebra.

By integration in parts, we can conclude that the integral of (A-10) over any compact $2n$ manifold $M_{2n}$ in $G$ is equal to the integral

\begin{equation}
-\frac{i}{4\pi} \int_{M_{2n}} \int_{S^1} dx \text{tr} \theta^{2n}(g^{-1} \frac{d}{dx} g) = -\frac{i}{4(2n+1)\pi} \int_{S^1 \times M_{2n}} \text{tr} \theta^{2n+1}
\end{equation}

and therefore this last expression under the integral sign represents the same cohomology class as (A-10), and therefore after a multiplication by the normalization coefficient $a_{2n}$, through (A-7), the same class as (A-4). Note that the coefficient in front of the integral on the right-hand side of the equation is equal to the ratio $a_{2n+2}/a_{2n}$ which is necessary to recover the correct normalization for $\theta^{2n+1}$. This proves that $\langle j(y_n), \phi_n \rangle = \langle j(y_n), \psi_n' \rangle = 1$.

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