Wedge-shaped surfaces with constant length generators in architectural design

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Abstract. The characteristic feature of ruled surfaces is that they are easy to describe and manufacture; therefore, they are widely used in the design of various architectural forms and building structures. Ruled surfaces are formed when a rectilinear generator moves along the guide curves. The shape of the surface does not only substantially depend on the shape of the guides, but also the law of the generator movement. Three conditions should be specified to select a surface from the whole variety of three-dimensional space lines. The article considers the surfaces distinguished from a linear congruence with \( u, v \) axes, subjecting only to two geometric conditions, namely: the conditions of the intersection of the rectilinear surface generators with the congruence axes. The third condition is metric: the length of the segments of all the generators enclosed between \( u, v \) axes should be the same. The design parameters of a surface with constant length generators are selected based on the theorem proved in the article. We recommend using the proposed solution in the manufacture of temporary or mobile T-tent structures. We presented examples of portable structures (tent circus, indoor stage for public events).

Keywords Ruled algebraic surfaces, Architectural design, Surface generators.

1. Introduction
The characteristic feature of ruled surfaces is that they are easy to describe and manufacture; therefore, they are widely used in the design of various architectural forms and building structures [1-4]. Ruled surfaces are formed when a rectilinear generator moves (slides) along some predetermined curves (guides). Moreover, the shape of the surface does not only substantially depend on the shape of the guides, but also the law of the generator movement [5-10].

If three guides are indicated, there is no need to establish the law of the generator movement. Let, for example, a rectilinear generator slide along a second-order guide curve \( r \) and two guide straight lines \( u, v \). In this case, we obtain a biaxial surface with \( u, v \) axes [11,12]. The surface frame is formed by the straight lines and second-order curves (Figure 1). In this case, the ruled surface is unambiguously distinguished from the four-parameter set of three-dimensional space lines. The unambiguosity is violated only on the lines \( u, v \), since two rectilinear generators pass through any point of these lines. Such surfaces are called two-sheeted.

Theorem 1 is valid for ruled surfaces: if the guides of the ruled surface \( \varphi \) are algebraic curves \( m, n, r \), the surface \( \varphi (m, n, r) \) has an order equal to twice the product of the orders of the guide curves [13-15]. In particular, Figure 1 shows a fourth-order algebraic surface.

If only two guides are given (for example, the straight line \( u \) and the curve \( r \)), we should specify an additional geometric condition, to which the generator movement is subjected. For example, you can
require that all the generators remain parallel to a certain predetermined plane during the movement. We obtain a surface called a “straight wedge”. A straight wedge is a fourth-order biaxial algebraic surface with \( u, v_\infty \) axes, where \( v_\infty \) is an improper line (Figure 2).

![Figure 1. A biaxial surface.](image1)

![Figure 2. A wedge-shaped surface.](image2)

2. Problem statement

If \( u, v \) axes are specified for a wedge-shaped biaxial surface, the surface is not completely defined. Indeed, in three-dimensional space, there is a two-parameter set of lines intersecting the specified lines \( u, v \). To single out a ruled surface from this set (a one-parameter set of lines), we should specify an additional positional or metric condition [16-18].

We can specify the requirement of the generator length constancy as a metric condition [19]. The load-bearing frame of such a surface contains rectilinear beams or rods of one size, which simplifies the installation of the frame. We obtain the task: to make a geometric algorithm for building a ruled biaxial surface with constant length generators.

3. A ruled biaxial surface with constant length generators

Let us consider the surface \( \Theta \) formed by the movement of the segment \( MN \) of a fixed length \( |MN| = \text{const} \). The endpoints \( M, N \) of the segment slide along the skew lines \( m, n \). Let us show that the surface \( \Theta \) can be singled out from a hyperbolic linear congruence with \( m, n \) axes by immersing a guiding ellipse into the congruence body, the eccentricity of which is completely defined by the angle between the congruence axes. A hyperbolic linear congruence is a two-parameter set of lines intersecting two given skew lines.

First, let us prove the theorem on the constant length segment movement.

**Theorem 2.** If the ends of the segment \( MN \) slide along the skew lines \( m, n \), an arbitrary point \( A \) of the segment \( MN \) describes an ellipse lying in a plane parallel to the straight lines \( m, n \).

**Proof.** Let us make an orthogonal projection of the spatial configuration \( \{a, m, n, A, M, N\} \) onto the plane \( \Pi \) parallel to the straight lines \( m, n \). The movable segment \( MN \) (of a fixed length) is projected onto the segment \( M'N' \) which length is \( |M'N'| = \sqrt{(MN^2 - d^2)} \), where \( d \) is the shortest distance between the lines \( m, n \) (Figure 3, left).

The slope angle of the straight line \( a \) to the plane \( \Pi \) during its movement remains unchanged, consequently, point \( A \) fixed on the straight line \( a=MN \) moves in the plane parallel to the plane \( \Pi \). The projection of the trajectory of point \( A \) onto the plane \( \Pi \) is congruent with the trajectory itself; therefore it is enough to show that the theorem is valid for the case when the lines \( m, n \) are not skew but intersected (Figure 3, right).
If \( m, n \) axes are mutually perpendicular and intersect at point \( O \), the trajectory of the middle \( S \) of the segment is the circle \( r \) with center \( O \) and radius \( OS = 0.5 \) \( MN \). Let us mark an arbitrary point \( A \) on the segment \( MN \). We draw a chord \( AA' \) parallel to the base ON in the isosceles triangle \( OSN \). When the segment \( MN \) moves, point \( A' \) describes a circle \( r' \) with center \( O \) and radius \( OA' = AN \) (Figure 4, left).

Let us introduce a rectangular coordinate system with \( x, y \) axes coinciding with the straight lines \( m, n \). The length of the chord \( AA' \) is proportional to \( y_N \) coordinate of point \( N \): \(|AA'| = k y_N \), where \( k = AS/NS \).

If \( y_N = 0 \), the triangle \( MON \) degenerates into the segment \( MN \) aligned with \( x \) axis. In this case, points \( A, A' \) coincide with point \( A_m \), where \( O A_m = O A' = AN \). If \( x_M = 0 \), points \( A, A' \) occupy the positions \( A_n, A'_n \), respectively; \( y_N \) coordinate and the length of the chord \( AA' \) take the maximum values \( y_N = |MN|, |AA'| = 2AS \).

We obtain a perspective affine (related) transformation with the affinity axis \( x = m \), the affinity direction \( y = n \), and the corresponding straight lines \( AA_n, A'A'_n \), intersecting at point \( V \) on the affinity axis. In this transformation, the circle \( r' \) is “stretched” into the ellipse \( e \), which is a trajectory of point \( A \) (Figure 4, left). The theorem is proved for the case of mutually perpendicular \( m, n \) axes.

Let us turn one of the mutually perpendicular straight lines \( m, n \) (for example, the straight line \( n \)) by an arbitrary angle \( \alpha \) to the position \( n' \) (Figure 4, right). We obtain a shift transformation with \( x = m \) axis. In this transformation, the segment \( MN \), resting on the mutually perpendicular straight lines \( m, n \) with its endpoints, is shifted parallel to itself to the position \( M'N' \). Point \( A \) incident to \( MN \) shifts in the direction of \( m \) axis by the distance \( AA' = k y_A \), where \( k = tg \alpha / (1 - NA/NM) \).

This transformation is affine; therefore, the ellipse \( e' \) corresponds to the ellipse \( e \) [20]. The ellipse \( e' \) is a trajectory of point \( A' \) lying on the straight line, the fixed points \( M'N' \) of which slide along the straight lines \( m, n' \) intersecting at an arbitrary angle. The theorem is completely proved.
Based on the proved theorem, we form an algorithm to build a ruled surface with constant length generators. Let the segment $|MN|=\Delta$ slide with its endpoints along the intersecting guides $m, n$, the angle between which is equal to $\alpha$.

According to the theorem, the middle $A$ of the segment $MN$ moves in the median plane $\Sigma$ along the throat ellipse $e$, which values of the principal axes $\delta_1, \delta_2$ are determined by the angle $\alpha$, the distance $d$ between the straight lines $m, n$, as well as the length $\Delta$ of the generating segment $MN$:

$$\delta_1 = 2AO = \frac{\alpha}{2} \sqrt{(\Delta^2 - d^2)}$$  \hspace{1cm} (1)

$$\delta_2 = 2AO = \frac{\alpha}{2} \sqrt{(\Delta^2 - d^2)}$$  \hspace{1cm} (2)

It follows that the ratio of the main axes $\delta_1/\delta_2=\tan^2(\alpha/2)$ of the ellipse $e$ is determined only by the value of the angle $\alpha$ between the skew guides $m, n$.

Thus, the throat ellipse $e$ with the ratio of the main axis $\delta_1/\delta_2$equal to $\tan^2(\alpha/2)$, located symmetrically to the guides $m, n$, singles out a fourth-order algebraic ruled surface with the constant length generators $MN$ from a hyperbolic linear congruence

$$|MN| = \Delta = \sqrt{(d^2 + \delta_1 \delta_2)}$$  \hspace{1cm} (3)

According to Theorem 2, this surface carries a family of ellipses, which planes are parallel to the median plane $\Sigma$. In particular, if $\alpha=\pi/2$, we obtain a twice symmetrical ruled surface with a throat circle and mutually perpendicular axes (Figure 5).

4. Projects of architectural structures

The project of a modern tent circus. A tent circus is a modular construction consisting of masts (rods) and a canvas stretched on them. Instead of the traditional spherical dome, we propose to use a fourth-order ruled algebraic surface. The frame of the proposed structure consists of masts of the same length $l$, which ensures an easy assembly and disassembly of the dome. Horizontal stiffening rings consist of the arcs of ellipses with a variable eccentricity (Figure 6). A general view of the structure is shown in Figure 7.

![Figure 5. A biaxial surface.](image)

![Figure 6. A ruled dome (drawing).](image)

The proposed design of the dome allows us to increase the inner space of the circus room without changing the traditional size of the arena of 42 feet (13 meters), which is identical throughout the world.

Let us calculate the length $l$ of the rods. Assume that $D$ is the diameter of the basement of the circus building, and $H$ is the height of the ruled dome. Substituting in (2) $l=\Delta/2, \delta_1=\delta_2=D, d=2H$, we obtain
The performances of aerialists and tightrope walkers can become more spectacular and effective due to the increased dome volume.

The project of a stage for public events. The stage design should provide for its easy assembly and disassembly [21]. We propose to use a frame consisting of rods of the same length and a set of ribs (transverse force elements), the shape of which varies from a semicircle to a straight line (Figure 8). The general view of the stage is shown in Fig. 9. The dimensions of the basic frame elements are determined by the formulas (1) ... (4).

$$l = 0.5 \sqrt{4H^2 + D^2}$$  \hspace{1cm} (4)

For example, if $D=25$ meters, $H=15$ meters, according to formula (4), we obtain $l=19.5$ meters.

5. Conclusion
Wedge-shaped surfaces singled out from the hyperbolic linear congruence with $u, v$ axes are distinguished by their simplicity and variety of shapes. The known methods of building algebraic surfaces are based on immersing a certain algebraic curve into the congruence body [22-26].

The article proposes a new method for building fourth-order ruled algebraic surfaces. Instead of a second-order guide curve, we propose to use an additional metric condition: constant length of the segments of the generators enclosed between $u, v$ axes. The fulfillment of this condition allows us to obtain surfaces, the load-bearing frame of which is formed by rods of the same length and second-order curves, which significantly simplifies the assembly and disassembly of the frame. We recommended using the proposed solution in the manufacture of temporary or mobile T-tent structures.

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Acknowledgments

The work was supported by Act 211 Government of the Russian Federation, contract No 02.A03.21.0011.