Primordial black hole tower: Dark matter, earth-mass, and LIGO black holes

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We investigate a possibility of primordial black hole (PBH) formation with a hierarchical mass spectrum in multiple phases of inflation. As an example, we find that one can simultaneously realize a mass spectrum which has recently attracted a lot of attention, stellar-mass PBHs ($\sim O(10)M_{\odot}$) as a possible source of binary black holes detected by LIGO/VIRGO collaboration, asteroid-mass ($\sim O(10^{-12})M_{\odot}$) as a main component of dark matter, and earth-mass ($\sim O(10^{-5})M_{\odot}$) as a source of ultrashort-timescale events in OGLE microlensing data [1]. The recent refined swampland conjecture may support these multi-phase inflationary scenario with hierarchical mass PBHs as a transition signal of each inflationary phase.

I. INTRODUCTION

The epochal success of first direct detection of gravitational waves (GWs) GW150914 by the LIGO/Virgo collaboration [2] opens up a quite new research field of the Universe. Their subsequent observations so far reveals the ubiquitous $\sim O(10)M_{\odot}$ black holes (BHs) more massive than one thought. For the origin of such a massive stellar BH $\sim O(10)M_{\odot}$, one might have recourse to primordial ones. Several cosmological scenarios predict BH formation in advance of ordinary stars, which is called primordial black holes (PBHs). For example, an order-unity overdense region can collapse to a BH by self-gravitation soon after its horizon reentry during the radiation-dominated era [3–5].

Right after GW150914, the possibility of PBHs to be GW sources were discussed [6, 7] and it was suggested that even PBHs comprising only sub-percent of total dark matters (DMs) can form binaries during the radiation-dominated era and explain the estimated merger rate [8]. Making up all DMs by PBH itself is also an interesting scenario other than GW. Recently several authors revisited the constraints by gravitational lensing events, one of the main constraining schemes on PBH abundance particularly for light mass [9, 10]. They showed that the finite size of luminous sources or the wave property of photons significantly reduces the lensing sensitivity and now windows for all DMs are open on around asteroid-mass $\sim 10^{-6} - 10^{-13}M_{\odot}$ and $\sim 10^{-13} - 10^{-11}M_{\odot}$. Furthermore, Ref. [1] has recently showed that earth-mass $\sim 10^{-5}M_{\odot}$ PBHs comprising $O(1)\%$ of total DMs can explain ultrashort-timescale microlensing events which are reported in 5-years observations of stars in Galactic bulge by the Optical Gravitational Lensing Experiment (OGLE) [11]. Thus, it might be interesting to investigate the possibility of generating such a hierarchical PBH mass spectrum in the early Universe.

One of the widely studied sources of the overdense region collapsing to a BH in the early Universe is large primordial curvature perturbations, which could be realized by inflation like as those on the cosmic microwave background (CMB) scale. However recent CMB observations indicate their amplitudes $\sim 10^{-5}$ [12] which are far from order-unity, and moreover standard single-field slow-roll inflation models predict almost scale-invariant perturbations in general. Thus in order to realize large primordial perturbations to form PBHs, one needs much amplification on smaller scales by considering extended inflationary models (see e.g. a recent review article [13]). Among such inflationary models, double-phase inflation has attracted attention recently [9, 14–17], which can yield PBH formation with nearly monochromatic mass spectrum as a transition signal between the two inflationary phases. From this viewpoint, the hierarchy in preferred PBH masses would suggest multi-phase (more than double) inflation. This possibility is important also in the context of string theory. According to the refined swampland conjecture [18, 19], even quasi-stable de Sitter vacua might not be consistent with UV-complete effective field theories, implying each continuous phase of inflation can not last so long, compared to required e-folds $\sim 50–60$ for our whole observable universe. A series of short inflationary phases instead can account for total e-folds, leaving some signals of phase transitions. In fact our model satisfies the second condition of the refined swampland conjecture except in the CMB phase, for which we do not assume any specific model in this paper.

We investigate in this paper a PBH formation scenario with a hierarchical mass spectrum in multi-phase inflation. In Sec. II, we review the estimation of the PBH abundance as well as the current observational constraints. The resultant PBH mass spectrum in our model is also shown in Fig. 1 first. Our multi-phase inflationary model is described in detail in Sec. III. As its testability,
our model leaves stochastic GWs by the second-order effect of large primordial scalar perturbations with sufficient amplitude to be detected by future observations as discussed in Sec. IV. Sec. V is devoted to the conclusion as well as the discussion about the swampland conjecture. We adopt the natural unit $c = \hbar = 1$ throughout this paper.

II. HIERARCHICAL PRIMORDIAL BLACK HOLE SPECTRUM

A. Primordial black holes as dark matters and progenitors of LIGO GW/OGLE ultrashort-timescale microlensing events

An extreme astrophysical object, black hole (BH), has been playing a key role not only in astrophysics, but also in gravitational theory, quantum physics, particle physics and cosmology. While it forms as a remnant of an explosive death of a massive star in general, several cosmological scenarios can also predict abundant BH formation in the early Universe before the ordinary star formation, called primordial black hole (PBH). For a PBH formed in the radiation-dominated era, its mass is roughly given by the horizon mass at its formation time:

$$M_H \sim \frac{t}{G} \sim M_\odot \left(\frac{t}{10^{-5} s}\right),$$  \hbox{(1)}

where $G \simeq 6.7 \times 10^{-39} \text{GeV}^{-2}$ is the Newtonian constant of gravitation and $M_\odot \simeq 2 \times 10^{33} \text{g}$ is the solar mass. It shows that the possible mass of PBH spans a very wide range, from the Planck mass $\simeq 2 \times 10^{-5} \text{g}$ as the extremely light end to e.g. $\sim 10^9 M_\odot$ for seeds of supermassive BHs at galactic cores [20, 21]. Particularly if a BH lighter than the sun is found, it should be a primordial one instead of the ordinary remnant of star.

While ones lighter than $\sim 10^{15} \text{g}$ have evaporated by now, more massive PBHs can survive against the Hawking radiation and play a role of the main/sub component of dark matters (DMs). Their abundance has been constrained by many kinds of observations represented by gravitational lensing events, depending on their mass. The current conservative constraints are summarized in Fig. 1. It should be noted that PBHs of $\sim 10^{-10} - 10^{-11} M_\odot$ had been thought to be constrained by non-detection of femtolensing events of gamma-ray bursts [22] nor microlensing against the Andromeda galaxy [23]. However it was recently reported that the finiteness of the luminous source size and the wave property of photons significantly reduce the lensing efficiency [9, 10, 23], and the windows for $\sim 10^{-16} - 10^{-14} M_\odot$ and $\sim 10^{-13} - 10^{-11} M_\odot$ are open now. Therefore PBHs can comprise all DMs in this range.

As other interesting regions, it has been suggested that PBHs can form their binaries by the gravitational attracting force between them even during the radiation-dominated era and the merger rate estimated by LIGO/Virgo gravitational events can be explained if the PBH fraction to total DM is $\sim 0.1\%$ [8] as indicated by a red point with error bars for GW150914 event [33] in Fig. 1.1 LIGO/VIRGO collaboration reported that they detected ten GW events from binary BH mergers during the first and second observing runs [34], and masses of observed BHs $\sim 30 - 50 M_\odot$ are slightly more massive than previously found stellar ones. Thus, it would suggest that PBHs can be a candidate of these events instead of the astrophysical ones. In addition to this range, recently, Ref. [1] investigated the possibility that earth-mass PBHs can explain ultrashort-timescale microlensing events which has been reported in 5-years observations of million stars in the Galactic bulge fields by the Optical Gravitational Lensing Experiment (OGLE) collaboration, without introducing free-floating planets [11]. Ref. [1] showed that the existence

FIG. 1. The predicted PBH mass spectrum is shown by the black lines. The lines with half shades represent the current observational constraints: extra-galactic gamma-ray by the Hawking radiation ($E\gamma$) [24], non-destruction of white dwarfs in our local galaxy (WD) [25], Subaru HSC microlensing (HSC) [23], Kepler milli/microlensing (Kepler) [26], EROS/MACHO microlensing (EROS/MACHO) [27], dynamical heating of ultra-faint dwarf galaxies (UDF) [28], and the most conservative accretion constraints by CMB (CMB) [29–32]. The red region is the inferred PBH abundance by the OGLE ultrashort-timescale microlensing events [1], while the point with error bars corresponds with GW150914 [8].

\footnote{Though the merger rate is updated by including all ten binary-BH events with slightly varied masses [34], the relation between such a merger rate estimated by more than single events and the extended PBH mass function has not been clarified yet. Thus we focus only on the single event GW150914 with the approximation of the monochromatic mass function as a simplified indicator, leaving such a problem for future works. The error in the PBH abundance corresponds with the range of the merger rate $2 - 53 \text{Gpc}^{-3} \text{yr}^{-1}$, while that in the PBH mass is given by the high end of the massive one $(36 + 5) M_\odot$ and the low end of the lighter one $(29 - 4) M_\odot$ respectively [33].}
of PBHs of $\sim 1\%$ fraction to DM can consistently explain the OGLE ultrashort-timescale microlensing events (indicated by the red region in Fig. 1).

**B. Mass spectrum of primordial black holes**

In order to make a theoretical estimation for the required PBH mass spectrum as discussed above, we need to specify the formation scenario of PBH. Amongst several scenarios like as the collapse of topological defects (see e.g. Ref. [35]) or the gravitational growth of density perturbations enhanced by disappearance of fluid pressure (see e.g. Ref. [36]), we focus on a widely-studied one that order-unity overdense Hubble patches crunch into BHs during the radiation-dominated era. As discussed in a pioneer work by Carr and Hawking [4], a sufficiently overdense Hubble patch is expected to be decoupled from the background expansion beyond the Jeans scale and collapsing directly into a BH soon after its horizon reentry. The threshold for collapse was first briefly estimated by Carr [5] as $\delta_{th} \sim w$ at the horizon reentry on the uniform Hubble slice where $w = p/\rho = 1/3$ is the equation of state for radiation fluid, verified by several numerical simulations as $\delta_{th} \simeq 0.4$ [37, 38] on the comoving slice, and theoretically also refined by Harada, Yoo and Kohri [39] as $\delta_{th} = \frac{3(1+w)}{5(1+3w)} \sin^2 \frac{\pi}{3} \frac{w}{T_w} \simeq 0.4$. Though these values slightly depend on the profile of density perturbations (see e.g. Ref. [40]), we adopt $\delta_{th} = 0.4$ as a fiducial value in this paper.

Based on the Press-Schechter approach which is a conventional way to evaluate the mass spectrum of collapsed objects, with an assumption that the primordial density perturbations $\delta$ on comoving slice follow the Gaussian distribution for simplicity (see e.g. Refs. [41–43] for the non-Gaussian effect due to the non-linear relation between the density and curvature perturbations), the formation probability $\beta(R)$ of collapsed objects (PBH) on some coarse-graining comoving scale $R$ is given by

$$\beta(R) = \int_{\delta_{th}}^{\infty} \frac{1}{\sqrt{2\pi}\sigma^2_{\delta}(R)} \exp \left[ -\frac{\delta^2}{2\sigma^2_{\delta}(R)} \right] d\delta.$$  

(2)

Here the variance $\sigma^2_{\delta}(R)$ is related with the initial power spectrum of the conserved curvature perturbations on comoving slice, $P_{\chi}(k)$, as

$$\sigma^2_{\delta}(R) = \int d\log k \tilde{W}^2(kR) \frac{16}{81} (kR)^4 T^2(k, \eta = R^{-1}) P_{\chi}(k),$$  

(3)

where $T(k, \eta)$ represents the scalar transfer function during the radiation-dominated era given by

$$T(k, \eta) = 3 \frac{\sin(k\eta/\sqrt{3}) - (k\eta/\sqrt{3}) \cos(k\eta/\sqrt{3})}{(k\eta/\sqrt{3})^3}.$$  

(4)

For the window function $\tilde{W}$, we adopt the real-space top-hat one in Fourier space:

$$\tilde{W}(kR) = 3 \frac{\sin(kR - kR \cos kR)}{(kR)^3}.$$  

(5)

The mass of formed PBH is related with the coarse-graining scale $R$ by the horizon mass at its horizon reentry $aH = R^{-1}$ as

$$M(R) = \gamma \rho_0 \frac{4\pi}{3} R^3 |_{aH = R^{-1}},$$

$$\simeq \frac{\gamma M_{\text{eq}}}{\sqrt{2}} \left( \frac{g_*(T_R)}{g_{\text{eq}}(T_R)} \right)^{1/6} (R_{\text{eq}})^2,$$

$$\simeq 10^{20} \gamma \left( \frac{g_*(T_R)}{106.75} \right)^{-1/6} \left( \frac{R}{6.4 \times 10^{-14} \text{Mpc}} \right)^2,$$  

(6)

where $M_{\text{eq}} = \frac{8\pi}{15} \rho_0^2 / a_{\text{eq}} k_{\text{eq}}^2$ is the horizon mass at the matter-radiation equality with the current radiation density $\rho_0^2 \simeq 7.84 \times 10^{-34} \text{g cm}^{-3}$ as well as the comoving horizon scale $k_{\text{eq}} \simeq 0.07 \Omega_m h^2 \text{Mpc}^{-1}$ and the scale factor $a_{\text{eq}} \simeq 2.4 \times 10^4 \Omega_m h^2 - 1$ at the equality [44]. $\Omega_m h^2 \simeq 0.143$ [12] is the current density parameter for the matter components. $g_*$ denotes the effective degrees of freedom for energy density ($g_{\text{eq}} = 3.83$ at the equality), assumed to be almost equal to those for entropy density. It is a function of the fluid temperature (see e.g. Ref. [12]), while the coarse-graining scale $R$ and the temperature $T_R$ can be related by [45]

$$(R_{\text{eq}})^{-1} \simeq 2(\sqrt{2} - 1) \left( \frac{g_*}{g_{\text{eq}}(T_R)} \right)^{1/6} T_R / T_{\text{eq}},$$  

(7)

with the temperature $T_{\text{eq}} = \frac{2.725K}{a_{\text{eq}}}$ at the equality. $\gamma$ is a numerical parameter representing the mass efficiency of collapse. In this paper $\gamma = 1$ is adopted for simplicity though there are several works addressing this value (e.g. Ref. [38]).

After their formation, PBHs behave as non-relativistic matters and therefore their current fraction to total cold DMs is given by

$$f_{\text{PBH}}(M) = \frac{\Omega_{\text{PBH}}(M) h^2}{\Omega_{\text{DM}} h^2} = \frac{T_R}{T_{\text{eq}}} \frac{\Omega_m h^2}{\Omega_{\text{DM}} h^2} \gamma \beta(R)$$

$$\simeq \gamma^2 \left( \frac{7.2 \times 10^{-16}}{0.12} \right) \left( \frac{g_{\text{eq}(T_R)}}{106.75} \right)^{-1/6} \left( \frac{M}{10^{20} \text{g}} \right)^{-1/2},$$  

(8)

$\Omega_{\text{DM}} h^2 \simeq 0.12$ [12] is the current density parameter of total DMs. As mentioned, the estimation procedure of the PBH abundance involves several uncertainties in the model parameters, the choice of the window function, and so on. Given the PBH abundance fixed, such uncertainties are inherited by the required primordial perturbations and affect model testability described in Sec. IV.
as discussed in Ref. [46]. The refined procedure in the peak theory has been also proposed recently beyond the Press-Schechter approach [47]. We address these issues in Appendix A.

One finds from these equations that sizable amount of PBHs requires the significant amplification of the primordial curvature perturbations as $\mathcal{P}_R \simeq 10^{-2}$ on the corresponding scale (e.g., $k \sim R^{-1} \sim 10^{12}$ Mpc$^{-1}$ for DM-PBH with $\sim 10^{20}$ g) compared to $\mathcal{P}_R \simeq 2 \times 10^{-9}$ on the cosmic microwave background (CMB) scale $k_* \sim 0.05$ Mpc$^{-1}$ [12]. However both the scale dependence of $\mathcal{P}_R$, i.e. $n_s - 1 = \frac{d \log \mathcal{P}_R}{d \log k}$, and its running $\alpha = \frac{d n_s}{d \log k}$ are slow-roll suppressed in the simplest single-field slow-roll inflation, prohibiting such an amplification. This difficulty has been reported even beyond the slow-roll approximation [48, 49]. On the other hand, if one allows two/multiple phases of inflation during the last 50–60 e-folds, the PBH scale can be free from the CMB scale constraint and such an amplification can be realized much more easily. In particular, Refs. [9, 14–17] show that sharp peaks of the curvaton power spectrum can appear on the scales corresponding with the transitions between inflationary phases. From this viewpoint, each interesting mass region mentioned above may correspond with such an transition time of some multi-phase inflation. In the next section, a concrete multi-phase model is shown which realizes the three peaks of the PBH mass spectrum simultaneously as shown in Fig. 1.

### III. MULTI-PHASE INFLATION

In this section we consider a quadruple-phase inflationary model to realize three peaks in the PBH mass spectrum at each phase transition. Note that we do not specify the model responsible for the CMB scale nor the reheating mechanism but simply assume that they consistently success. Each inflationary phase is governed by a different scalar inflaton $\phi_i$, ($i = 1, 2, 3, 4$). As the energy scales for later phases should be lower than those for earlier phases and that on the CMB scale is already constrained from above as $\rho_{\text{inf,CMB}} \lesssim (1.7 \times 10^{16}$ GeV)$^4$ by the non-detection of primordial tensor modes [50], it is natural to assume a low-energy small-field model represented by e.g. hilltop inflation:

\[
V_{\text{hill}, i}(\phi_i) = \left( v_i^2 - g_i \frac{\phi_i^n}{M_{\text{Pl}}^2} \right)^2 - \frac{1}{2} \kappa_i v_i^4 \phi_i^2 - \varepsilon_i v_i^4 \phi_i \frac{\phi_i}{M_{\text{Pl}}^2},
\]

\[
n \geq 3,
\]

for the phases responsible for PBHs. Here a dimensionful parameter $v_i$ determines the energy scale of phase-$i$ while $g_i$, $\kappa_i$, and $\varepsilon_i$ are dimensionless parameters controlling the duration of the phase, the shape of the scalar power spectrum, and so on. $M_{\text{Pl}} = (8\pi G)^{-1/2} \simeq 2.4 \times 10^{18}$ GeV is the reduced Planck mass. We included the linear and quadratic terms as corrections to the simple wine-bottle potential.

In addition to this hilltop potential, we consider the following Planck-suppressed couplings between inflatons as stabilizers, the total potential being given by [9, 14–17]

\[
V(\phi) = \sum_{i=1,2,3,4} V_{\text{hill}, i}(\phi_i) + \frac{1}{2} \sum_{i \neq j} c_{ij} V_{\text{hill}, i}(\phi_i) \frac{\phi_j^2}{M_{\text{Pl}}^2}.
\]

Here we briefly review the dynamics of inflatons given by this potential. Let the energy scales be sufficiently hierarchical $v_1 \gg v_2 \gg v_3 \gg v_4$. Then the first inflationary phase is governed by $\phi_1$. During this phase, the high enough potential energy of $\phi_1$, $V_{\text{hill}, 1} \simeq v_1^4$, stabilizes all other inflatons $\phi_2$, $\phi_3$, and $\phi_4$ to their origins through the couplings $\sum_{j \neq 1} \frac{1}{2} c_{1j} V_{\text{hill}, 1} \frac{\phi_j^2}{M_{\text{Pl}}^4}$. After phase-1 inflation, $\phi_1$ rapidly oscillates around its potential minimum and loses its energy. When the second energy scale $v_2^3$ becomes non-negligible compared to $\phi_1$’s energy, $\phi_2$ cannot be stabilized any longer and the second phase of inflation begins. The same mechanism works for subsequent phases. Without the linear term in the hilltop potential Eq. (9), the inflatons are stabilized completely at the origin and the dynamics after the onset of the phase is stochastic, dominated by the quantum diffusion. Recalling that $V_{\text{hill}, i} \sim v_i^4$ at the beginning of phase-$(i+1)$, the linear term shifts the minimum to $\phi_{i+1} = \phi_{*i+1} = (\varepsilon_{i+1}/c_{i+1, i}) M_{\text{Pl}}$ to make the background dynamics deterministic.

After the stabilizer decays out, the potential tilt around this minimum reads $V'_{\text{hill}, i+1}(\phi_{*i+1}) = -\varepsilon_{i+1} v_{i+1}^4/M_{\text{Pl}}^4$ and therefore the corresponding amplitude of the curvature power spectrum can be estimated as

\[
\mathcal{P}_R(k_{*i+1}) \simeq \frac{1}{12\pi^2 M_{\text{Pl}}^2} \frac{V_{\text{hill}, i+1}(\phi_{*i+1})}{V_{\text{hill}, i+1}(\phi_{*i+1})} \sim \frac{1}{12\pi^2 M_{\text{Pl}}^4} \frac{v_{i+1}^4}{\varepsilon_{i+1}}.
\]

Hence $\varepsilon_{i+1} \sim v_{i+1}^2/M_{\text{Pl}}^2$ could realize an enough power $\mathcal{P}_R \sim \mathcal{O}(10^{-2})$ for PBH formation as can be seen. Apart from this peak, the $k$-dependence of the power spectrum is given by $\mathcal{P}_R \propto k^{3-2\epsilon_{i+1}}$ where the effective mass squared of $\phi_{i+1}$ determines $\nu = \sqrt{\frac{9}{4} - \frac{m_{\text{eff}, i+1}^2}{3H^2}}$. During the transition between $i$ and $i + 1$ phases, $\phi_1$ should be sufficiently massive as $m_{\text{eff}, i+1}^2/3H^2 \simeq \varepsilon_{i+1} c_{i+1, i}^2 \gtrsim 9/4$ to be well-stabilized, giving rise to the blue-tilted spectrum $\mathcal{P}_R \propto k^3$ for $k < k_{*i+1}$. On the other hand, the $\phi_{i+1}$’s mass after the onset of phase-$(i + 1)$ is given by $m_{\text{eff}, i+1}^2/3H^2 \simeq -\kappa_{i+1}$ and therefore sufficiently large positive $\kappa_{i+1}$ can realize red-tilted spectrum for $k > k_{*i+1}$. In this way, the power spectrum can have a sharp peak on the onset scale $k_{*i+1}$, and so does the corresponding PBH mass function [51, 52].

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2 We note that in Refs. [9, 15, 17] the sharp peak is realized mainly
itive $\kappa_{i+1}$ also benefit us at the point of phase duration. For the negligibly small first slow-roll parameter $\epsilon_H = -\dot{H}/H^2$ as we are considering, its time-evolution during the inflationary phase is controlled by the second derivative of the potential:

$$\frac{\dot{H}}{H\epsilon_H} \simeq -2M_{Pl}^2 \frac{V''_{\text{hill},i+1}}{V_{\text{hill},i+1}} \simeq 2\kappa_{i+1}. \quad (12)$$

Therefore larger $\kappa_{i+1}$ shortens the duration of the inflationary phase-$i+1$, given the initial value of $\epsilon_H \simeq M_{Pl}^2/2 \left( \frac{V''_{\text{hill},i+1}}{V_{\text{hill},i+1}} \right)^2 \simeq \epsilon_{i+1}^2$ and the end of inflation $\epsilon_H = 1$. Multiple peaks in the PBH mass can be realized then.

One finally takes account of the resonance condition at the end of hilltop inflation. In general small-field inflation tends to bring about the resonance amplification of primordial perturbations soon after the end of inflation (see e.g. Ref. [53]), which easily breaks the validity of the linear approximation of perturbations. For a robust calculation, we avoid such a resonance in this paper. The resonance condition of hilltop inflation has been investigated in detail in Ref. [17], and according to this work, roughly $n \leq 3$ and $\phi_{\text{min},i} \simeq (v_i^2 M_{Pl}^{-2}/g_i)^{1/n} \gtrsim 0.1 M_{Pl}$ are favored. Respecting this condition, we consider the following concrete set of parameters:

$$\begin{align*}
n & = 3, \\
 & \left\{ \frac{v_i}{M_{Pl}} = (10^{-5}, 10^{-6}, 10^{-7}, 10^{-8}), \\
 & \frac{\epsilon_i M_{Pl}^2}{v_i^2} = (2, 4.1, 5.9, 4.73), \\
 & \kappa_i = (2, 4.6, 5, 5), \\
 & (v_i^2 M_{Pl}^{-2}/g_i)^{1/n} = 0.1 M_{Pl}, \quad \text{for all } i, \\
 & c_{ij} = 10, \quad \text{for all } i \text{ and } j, \end{align*} \quad (13)$$

and the full numerical result of the linear perturbation for these parameters is shown in Fig. 2, solving the Mukhanov-Sasaki equation. Two dotted lines are indicators of $k^3$ and $k^{3-2\nu}$ where $\nu = \sqrt{9/4 + \kappa_3}$ as discussed above. The numerical result is well consistent with these lines, indicating that there is no resonance amplification and the calculation of perturbations is well controlled. For this power spectrum, the PBH mass spectrum shown in Fig. 1 is obtained.

by the cancellation of the Hubble-induced mass during the oscillation phase due to the kinetic coupling, while it is simply a red-tilded one due to the inflaton’s mass in this paper like as Refs. [16, 51, 52]. If one adopts the Press-Schechter approach with the Gaussian window which is relatively inefficient, the required primordial perturbations become large and the peak of the power spectrum should be highly sharp with use of such a complicated mechanism in order to avoid the current PTA constraints on the stochastic GWs. However the real-space top-hat window used in this paper or the refined peak-theory approach [47] can reduce the accompanying GWs and be consistent with the current constraints without employing that mechanism.

FIG. 2. The power spectrum of the curvature perturbations for the parameters (13) obtained by solving the Mukhanov-Sasaki equation. Note that the normalization of the wavenumber $k$ is artificially fixed by assuming a suitable cosmic evolution after phase-4. Red-shaded regions show current upper bounds on the power spectrum: CMB $\mu$-distortion due to the Silk damping ($\mu$-dist) [54, 55] and the effect on n-p ratio during big-bang nucleosynthesis (BBN) [56] (see also Refs. [57, 58]).

IV. TESTABILITY

As it has been discussed by many authors [45, 59–65], the inflationary PBH formation often leaves sizable stochastic GWs other than the binary PBH coalescence as a double-check evidence. Our model is also not an exception as we investigate in this section.

In the scenario of the collapsing radiation overdensity, abundant PBH formation requires the significant amplification of the primordial scalar perturbation as $P_R \sim 10^{-2}$ under its Gaussian ansatz (see e.g. Refs. [66–71] for the non-Gaussian effect on the required amplitude). Even though the scalar and tensor perturbations are decoupled at the linear level, such a large amplitude of perturbations means the breakdown of the linear approximation so that the tensor mode (i.e. stochastic GW) induced by the scalar perturbations becomes non-negligible. Leaving the details below, one roughly obtains the following relation between the current density of induced GWs and the amplitude of the scalar perturbations: [15]

$$\Omega_{GW} h^2 \sim 10^{-9} \left( \frac{\theta_*}{10.75} \right)^{-1/3} \left( \frac{\Omega_* h^2}{4.2 \times 10^{-5}} \right) \left( \frac{P_R}{10^{-2}} \right)^2, \quad (14)$$

where $\Omega_* h^2 \sim 4.2 \times 10^{-5}$ is the current radiative density parameter. Because the current PTA constraints reach $\Omega_{GW} h^2 \sim 10^{-9}$ [72–74] and future GW-detection projects will go further, it is required to predict the amplitude of the induced stochastic GWs in detail.
Taking the conformal Newtonian gauge:
\[ ds^2 = -a^2(1 + 2\Phi)dt^2 + a^2 [(1 - 2\Psi)\delta_{ij} + h_{ij}] dx^i dx^j, \]
\[ \partial_i h^i_j = h^i_i = 0, \]
the Einstein equation at the linear order in \( h \) but the second order in \( \Phi \) and \( \Psi \) can be summarized by [15]
\[ h''_{ij} + 2h'_{ij} - \nabla^2 h_{ij} = -4\tilde{T}_{ij;kl}S_{kl}, \quad \mathcal{H} = \frac{a'}{a}. \]
(16)
with the source term
\[ S_{ij} = 4\Psi\partial_i\partial_j\Psi + 2\partial_i\Psi\partial_j\Psi - \frac{4}{3(1 + w)}\partial_i \left( \frac{\Psi'}{\mathcal{H}} + \Psi \right) \partial_j \left( \Psi' + \Psi \right). \]
(17)
Here \( \tilde{T}_{ij;kl} \) represents the traceless-transverse projection operator satisfying
\[ \tilde{T}_{ij;kl} \tilde{T}_{kl;mn} = \tilde{T}_{ij;mn}, \quad \partial_i \tilde{T}_{ij;kl} \mathcal{O}_{kl} = \tilde{T}_{ii;kl} \mathcal{O}_{kl} = 0. \]
(18)

\[ T^2(s, t, x \rightarrow \infty) \simeq \frac{288(-5 + s^2 + t(2 + t))^2}{x^2(1 - s + t)^6(1 + s + t)^6} \left\{ \frac{\pi^2}{4} (-5 + s^2 + t(2 + t))^2 \Theta(t - \sqrt{3} + 1) \\
+ \left( -(1 - s + t)(1 + s + t) + \frac{1}{2}(-5 + s^2 + t(2 + t)) \log \left[ \frac{-2 + t(2 + t)}{3 - s^2} \right] \right)^2 \right\}, \]
(20)
where \( \Theta \) is the Heaviside step function. The energy density of GWs per logarithmic \( k \)-bin is related to the power spectrum by
\[ \Omega_{\text{GW}}(\eta, k) = \frac{1}{24} \left( \frac{k}{aH} \right)^2 \mathcal{P}_h(\eta, k). \]
(21)
Making use of the relation \( aH = \eta^{-1} \) during the radiation-dominated era, one finds that the time dependence in \( \Omega_{\text{GW}} \) is cancelled soon after its horizon reentry. After this freezeout time \( \eta_0 \), the density ratio of GWs to the background radiation is almost constant up to the current time \( \eta \).

V. DISCUSSION AND CONCLUSIONS

In this paper, we discussed the PBH formation with the hierarchical mass function in a multi-phase inflationary model. Once one allows the multiple phase for the last 50–60 e-folds, the power spectrum of the primordial curvature perturbation can easily have peaks on the transition scales, as we showed by considering a simple model: four single-field hilltop inflations (9) coupled by the natural Planck-suppressed terms (10). With the parameters (13) chosen to avoid the resonance am-
energy effective field theory denoted by the effective action
\[ S = \int d^4 x \sqrt{-g} \left[ \frac{1}{2} M_{Pl}^2 R - \frac{1}{2} g^{\mu \nu} G_{IJ}(\phi) \partial_\mu \phi^I \partial_\nu \phi^J - V(\phi) \right], \]

must satisfy either
\[ |\nabla V| M_{Pl} \geq c V, \]

or
\[ \min(\nabla_I \nabla_J V) M_{Pl}^2 \leq -c' V, \]
at any field-space point for some universal constants \( c, c' > 0 \) of order unity. Here \( \nabla_I \) is the covariant derivative along with the field-space metric \( G_{IJ} \), \( |\nabla V| = \sqrt{G^{IJ} \nabla_I V \nabla_J V} \) is the invariant norm of the potential tilt, and \( \min(\nabla_I \nabla_J V) \) is the minimum eigenvalue of the Hessian \( \nabla_I \nabla_J V \) in an orthonormal frame.

In terms of inflation, this conjecture claims that any single continuous inflationary phase cannot last so long. That is, if it is true, one needs multiple phases of inflation to explain sufficient e-folds \( \sim 50–60 \) for our observable universe in total, as we proposed. In fact our model always satisfies the second condition during any inflation phase as
\[ -\frac{\min(\nabla_I \nabla_J V)}{V} \simeq \kappa_i > 1, \quad \text{for all phase-i}, \]
while the first condition is satisfied apart from the inflationary trajectory. The existence of large negative eigenvalue in the Hessian matrix generally causes a negligibly short (less than 1 e-fold) inflationary phase unless the initial value of the first slow-roll parameter \( \epsilon_H \) is significantly small (see Eq. (12)) so that the corresponding amplitude of the power spectrum is large. In other words, the PBH formation on the onset scale is relatively natural for a non-negligibly continuing phase in terms of the above conjecture.

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Appendix A: Uncertainties in PBH abundance prediction

Even though the statistics of primordial curvature perturbations are completely fixed, the analysis of PBH formation the fully non-linear process requires some approximations. That causes theoretical uncertainties in
the prediction of the PBH abundance. For example, the Press-Schechter approach characterizes overdensities simply by the universal threshold value and the typical window function. However of course profiles of overdensities are not uniform, varying the corresponding threshold value as $\delta_{\text{th}} \sim 0.4-0.6$ [38]. Moreover, while it naively assumes the one-to-one correspondence between the coarse-graining scale $R$ and the PBH mass $M(R)$ with a single parameter $\gamma = M(R)/(\rho^{3/2}H^{-3})|_{aH=R^{-1}}$ (6), it is known that the resultant PBH mass depends on the shape of the density profile and the excess density (see e.g. Refs. [38, 79–83]). Given the PBH mass spectrum (65) of Ref. [47] about to the three peaks otherwise the profile is contaminated by other wavelength modes. Since the tails of our peaks decay only in the one-to-one correspondence between the coarse-graining scale $R$, subhorizon modes do not contribute to the variance needs not to be introduced. However this formulation can be described in the high peak limit. According to the peak theory [84], the radial profile of such a high peak of Gaussian curvature perturbations can be characterized stochastically determined by the curvature of the profile in principle and therefore any specific window function is different from that in the main body (2) by the factor $\gamma$.

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3 Note that the coarse-grained volume slightly depends on the choice of the window function as $V(R) = 4\pi R^3/3$ for the real-space top-hat while $V(R) = (2\pi)^3/2 R^4$ for the Gaussian window. This gives rise to the difference in the corresponding PBH mass.

4 Note that the sign of the definition of the curvature perturbation is opposite from Ref. [47].

5 Note that the definition of $\beta$ is different from that in the main body (2) by the factor $\gamma$. 

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of our curvature power spectrum. Let us briefly review this estimation below.

Thanks to the high peak limit, the PBH formation is mainly contributed by the peak scale mode $k_c$ determined by

$$k_c = \frac{\sigma_1}{\sigma_0}, \quad \sigma_n^2 = \int \frac{dk}{k}k^{2n}P(R)(k).$$

(A2)

For this mode, the profile of the peak curvature perturbation is simply given by the two-point function

$$\psi(r) = \frac{1}{\sigma_0^2} \int \frac{dk}{k} \frac{\sin kr}{kr} P(R)(k),$$

(A3)

as

$$R(r) = \mu \psi(r), \quad \mu = R(r = 0).$$

(A4)

If the maximal value of the compaction for this profile exceeds some threshold, the corresponding peak is assumed to collapse.

The compaction function is defined by

$$C(r) = \frac{\delta M}{R},$$

(A5)

where $\delta M$ represents the excess of the Misner-Sharp mass from the expected one by the background universe and $R$ is the areal radius. It can be simplified as

$$C(r) = \frac{1}{3} \left[ 1 - (1 + rR')^2 \right],$$

(A6)

and therefore the maximally compact radius $r_m$ can be found by the condition

$$C'(r_m) = 0, \quad \iff \quad R' + rR''|_{r=r_m} = 0.$$  

(A7)

The threshold value for the compaction is estimated as $C_{\text{th}} \approx 0.267$. Therefore the PBH formation criterion is given by

$$C(r_m) \geq C_{\text{th}}, \quad \iff \quad \mu \geq \mu_c = \frac{\sqrt{1-3C_{\text{th}}}-1}{r_m \psi'(r_m)}.$$  

(A8)

With use of this threshold, the estimated PBH density ratio $\beta$ to the background at its formation time is given by the Gaussian distribution with the phase-space volume factor $\mu_c^3/\sigma_0^3$ as

$$\beta \sim \gamma \frac{\mu_c^3}{\sigma_0^3} G^3 \psi(r_m) \exp \left[ -\frac{\mu_c^2}{2\sigma_0^2} \right].$$  

(A9)
Here the factor $e^{3\mu_+\psi(r_m)}$ comes from the ratio of the physical volume with the areal radius $R$ and the background value corresponding with the coordinate radius $r_m$. The PBH mass is given by the horizon mass (6) as

$$M = \gamma \rho \frac{4\pi}{3} H^{-3} \bigg|_{H=R^{-1}}$$

$$\simeq 10^{20} \gamma \left( \frac{g_s}{106.75} \right)^{-1/6} \left( \frac{r_m e^{\mu_+\psi(r_m)}}{6.4 \times 10^{-14} \text{Mpc}} \right)^2 \text{g}.$$  

(A10)

and the current fraction to total DMs (8) reads

$$f_{\text{PBH}} \simeq \gamma \frac{\beta}{7.2 \times 10^{-16}} \left( \frac{\Omega_{\text{DM}} h^2}{0.12} \right)^{-1} \left( \frac{g_s}{106.75} \right)^{-\frac{1}{4}} \left( \frac{M_{\text{PBH}}}{10^{20} \text{g}} \right)^{-\frac{1}{4}}.$$  

(A11)

We then plot the corresponding secondary GW spectrum with $\gamma = 1$ by the black dashed line in Fig. 3, fixing the three peaks of the PBH mass function. Because this improved estimation procedure is more efficient, the resultant GW density becomes lower than that for the Press-Schechter approach with the Gaussian window. Consequently it is marginally consistent with the current PTA constraints.

[1] Hiroko Niikura, Masahiro Takada, Shuichiro Yokoyama, Takahiro Sumi, and Shogo Masaki, “Earth-mass black holes? - Constraints on primordial black holes with 5-years OGLE microlensing events,” (2019), arXiv:1901.07120 [astro-ph.CO].

[2] B. P. Abbott et al. (LIGO Scientific, Virgo), “Observation of Gravitational Waves from a Binary Black Hole Merger,” Phys. Rev. Lett. 116, 061102 (2016), arXiv:1602.03837 [gr-qc].

[3] S. Hawking, “Gravitationally collapsed objects of very low mass,” Mon. Not. Roy. Astron. Soc. 152, 75 (1971).

[4] Bernard J. Carr and S. W. Hawking, “Black holes in the early Universe,” Mon. Not. Roy. Astron. Soc. 168, 399–415 (1974).

[5] Bernard J. Carr, “The Primordial black hole mass spectrum,” Astrophys. J. 201, 1–19 (1975).

[6] Simeon Bird, Ilias Cholis, Julian B. Muñoz, Yacine Ali-Haimoud, Marc Kamionkowski, Ely D. Kovetz, Alvise Raccanelli, and Adam G. Riess, “Did LIGO detect dark matter?” Phys. Rev. Lett. 116, 201301 (2016), arXiv:1603.00464 [astro-ph.CO].

[7] Sebastien Clesse and Juan García-Bellido, “The clustering of massive Primordial Black Holes as Dark Matter: measuring their mass distribution with Advanced LIGO,” Phys. Dark Univ. 15, 142–147 (2017), arXiv:1603.05234 [astro-ph.CO].

[8] Misao Sasaki, Teruaki Suyama, Takahiro Tanaka, and Shuichiro Yokoyama, “Primordial Black Hole Scenario for the Gravitational-Wave Event GW150914,” Phys. Rev. Lett. 117, 061101 (2016), [erratum: Phys. Rev. Lett.121,no.5,059901(2018)], arXiv:1603.08338 [astro-ph.CO].

[9] Keisuke Inomata, Masahiro Kawasaki, Kohei Mukaida, Yuichiro Tada, and Tsutomu T. Yanagida, “Inflationary primordial black holes for the LIGO gravitational wave events and pulsar timing array experiments,” Phys. Rev. D95, 123510 (2017), arXiv:1611.06130 [astro-ph.CO].

[10] Keisuke Inomata, Masahiro Kawasaki, Kohei Mukaida, Yuichiro Tada, and Tsutomu T. Yanagida, “Inflationary Primordial Black Holes as All Dark Matter,” Phys. Rev. D96, 043504 (2017), arXiv:1701.02544 [astro-ph.CO].

[11] Keisuke Inomata, Masahiro Kawasaki, Kohei Mukaida, Yuichiro Tada, and Tsutomu T. Yanagida, “C(10)MΩ primordial black holes and string axion dark matter,” Phys. Rev. D96, 123527 (2017), arXiv:1709.07865 [astro-ph.CO].

[12] Georges Obied, Hirosi Ooguri, Lev Spodyneiko, and Cumrun Vafa, “De Sitter Space and the Swampland,” (2018), arXiv:1806.08362 [hep-th].

[13] Hirosi Ooguri, Eran Palti, Gary Shiu, and Cumrun Vafa, “Distance and De Sitter Conjectures on the Swampland,” Phys. Lett. B788, 180–184 (2019), arXiv:1810.05506 [hep-th].

[14] Rachel Bean and Joao Magueijo, “Could supermassive black holes be quintessential primordial black holes?” Phys. Rev. D66, 063505 (2002), arXiv:astro-ph/0204486
A. Barnacka, J. F. Glicenstein, and R. Moderski, “New constraints on primordial black hole abundance from femtolensing of gamma-ray bursts,” Phys. Rev. D 86, 043001 (2012), arXiv:1204.2056 [astro-ph.CO].

Hiroki Niikura et al., “Microlensing constraints on primordial black holes with the Subaru/HSC Andromeda observation,” (2017), arXiv:1701.02151 [astro-ph.CO].

B. J. Carr, Kazunori Kohri, Yuuiti Sendouda, and Jun’ichi Yokoyama, “New cosmological constraints on primordial black holes,” Phys. Rev. D81, 104019 (2010), arXiv:0912.5297 [astro-ph.CO].

Peter W. Graham, Surjeet Rajendran, and Jaime Varela, “Dark Matter Triggers of Supernovae,” Phys. Rev. D92, 063007 (2015), arXiv:1505.04444 [hep-ph].

Kim Griest, Agnieszka M. Cieplak, and Matthew J. Lehner, “New Limits on Primordial Black Hole Dark Matter from an Analysis of Kepler Source Microlensing Data,” Phys. Rev. Lett. 111, 181302 (2013).

P. Tisserand et al. (EROS-2), “Limits on the Macho Content of the Galactic Halo from the EROS-2 Survey of the Magellanic Clouds,” Astron. Astrophys. 469, 387–404 (2007), arXiv:astro-ph/0607207 [astro-ph].

Timothy D. Brandt, “Constraints on MACHO Dark Matter from Compact Stellar Systems in Ultra-Faint Dwarf Galaxies,” Astrophys. J. 824, L31 (2016), arXiv:1605.03665 [astro-ph.GA].

Yacine Ali-Haïmoud and Marc Kamionkowski, “Cosmic microwave background limits on accreting primordial black holes,” Phys. Rev. D95, 043534 (2017), arXiv:1612.05644 [astro-ph.CO].

Daniel Aloni, Kfir Blum, and Raphael Flauger, “Cosmic microwave background constraints on primordial black hole dark matter,” JCAP 1705, 017 (2017), arXiv:1612.06811 [astro-ph.CO].

Benjamin Horowitz, “Revisiting Primordial Black Holes Constraints from Ionization History,” (2016), arXiv:1612.07264 [astro-ph.CO].

Vivian Poulin, Pasquale D. Serpico, Francesca Calore, Sebastien Clesse, and Kazunori Kohri, “CMB bounds on disk-accreting massive primordial black holes,” Phys. Rev. D96, 083524 (2017), arXiv:1707.04206 [astro-ph.CO].

B. P. Abbott et al. (LIGO Scientific, Virgo), “The Rate of Binary Black Hole Mergers Inferred from Advanced LIGO Observations Surrounding GW150914,” Astrophys. J. 833, L1 (2016), arXiv:1602.03842 [astro-ph.HE].

B. P. Abbott et al. (LIGO Scientific, Virgo), “GWTC-1: A Gravitational-Wave Transient Catalog of Compact Binary Mergers Observed by LIGO and Virgo during the First and Second Observing Runs,” (2018), arXiv:1811.12907 [astro-ph.HE].

Heling Deng, Jaume Garriga, and Alexander Vilenkin, “Primordial black hole and wormhole formation by domain walls,” JCAP 1704, 050 (2017), arXiv:1612.03753 [gr-qc].

Karsten Jedamzik, “Primordial black hole formation during the QCD epoch,” Phys. Rev. D55, 5871–5875 (1997), arXiv:astro-ph/9605152 [astro-ph].

Masaru Shibata and Misao Sasaki, “Black hole formation in the Friedmann universe: Formulation and computation in numerical relativity,” Phys. Rev. D60, 084002 (1999), arXiv:gr-qc/9905064 [gr-qc].

Ilià Musco, John C. Miller, and Luciano Rezzolla, “Computations of primordial black hole formation,” Class. Quant. Grav. 22, 1405–1424 (2005), arXiv:gr-qc/0412063 [gr-qc].

Tomohiro Harada, Chul-Moon Yoo, and Kazunori Kohri, “Threshold of primordial black hole formation,” Phys. Rev. D88, 084051 (2013), Erratum: Phys. Rev.D89,no.2,029903(2014), arXiv:1309.4201 [astro-ph.CO].

Tomohiro Nakama, Tomohiro Harada, A. G. Polnarev, and Jun’ichi Yokoyama, “Identifying the most crucial parameters of the initial curvature profile for primordial black hole formation,” JCAP 1401, 037 (2014), arXiv:1310.3007 [gr-qc].

Masahiro Kawasaki and Hiromasa Nakatsuka, “Effect of nonlinearity between density and curvature perturbations on the primordial black hole formation,” (2019), arXiv:1903.02994 [astro-ph.CO].

V. De Luca, G. Franciolini, A. Kehagias, M. Peloso, A. Riotto, and C. Únal, “The Ineludible Non-Gaussianity of the Primordial Black Hole Abundance,” (2019), arXiv:1904.00970 [astro-ph.CO].

Sam Young, Ilià Musco, and Christian T. Byrne, “Primordial black hole formation and abundance: contribution from the non-linear relation between the density and curvature perturbation,” (2019), arXiv:1904.00984 [astro-ph.CO].

Amandeep S. Josan, Anne M. Green, and Karim A. Malik, “Generalised constraints on the curvature perturbation from primordial black holes,” Phys. Rev. D79, 103520 (2009), arXiv:0903.3184 [astro-ph.CO].

Keisuke Inomata and Tomohiro Nakama, “Gravitational waves induced by scalar perturbations as probes of the small-scale primordial spectrum,” Phys. Rev. D99, 043511 (2019), arXiv:1812.00674 [astro-ph.CO].

Kenta Ando, Keisuke Inomata, and Masahiro Kawasaki, “Primordial black holes and uncertainties in the choice of the window function,” Phys. Rev. D97, 103528 (2018), arXiv:1802.06393 [astro-ph.CO].

Chul-Moon Yoo, Tomohiro Harada, Jaume Garriga, and Kazunori Kohri, “Primordial black hole abundance from random Gaussian curvature perturbations and a local density threshold,” PTEP 2018, 123 (2018), arXiv:1805.03946 [astro-ph.CO].

Hayato Motohashi and Wayne Hu, “Primordial Black Holes and Slow-Roll Violation,” Phys. Rev. D96, 063503 (2017), arXiv:1706.06784 [astro-ph.CO].

Samuel Passaglia, Wayne Hu, and Hayato Motohashi, “Primordial Black Holes and Local Non-Gaussianity in Canonical Inflation,” (2018), arXiv:1812.08243 [astro-ph.CO].

Y. Akrami et al. (Planck), “Planck 2018 results. X. Constraints on inflation,” (2018), arXiv:1807.06211 [astro-ph.CO].

M. Kawasaki, N. Sugiyama, and T. Yanagida, “Primordial black hole formation in a double inflation model in supergravity,” Phys. Rev. D57, 6050–6056 (1998), arXiv:hep-ph/9710259 [hep-ph].

M. Kawasaki and T. Yanagida, “Primordial black hole formation in supergravity,” Phys. Rev. D59, 043512
(1999), arXiv:hep-ph/9807544 [hep-ph].

[53] Philippe Brax, Jean-Francois Dufaux, and Sophie Mariadassou, “Preheating after Small-Field Inflation,” Phys. Rev. D83, 103510 (2011), arXiv:1012.4656 [hep-th].

[54] Jens Chluba, Adrienne L. Ericcek, and Ido Ben-Dayan, “Probing the inflation: Small-scale power spectrum constraints from measurements of the CMB energy spectrum,” Astrophys. J. 758, 76 (2012), arXiv:1203.2681 [astro-ph.CO].

[55] Kazunori Kohri, Tomohiro Nakama, and Teruaki Suyama, “Testing scenarios of primordial black holes being the seeds of supermassive black holes by ultracompact minihalos and CMB \( \mu \)-distortions,” Phys. Rev. D90, 083514 (2014), arXiv:1405.5999 [astro-ph.CO].

[56] Keisuke Inomata, Masahiro Kawasaki, and Yuichiro Tada, “Revisiting constraints on small scale perturbations from big-bang nucleosynthesis,” Phys. Rev. D94, 043527 (2016), arXiv:1605.04646 [astro-ph.CO].

[57] Donghui Jeong, Josef Pradler, Jens Chluba, and Marc Kamionkowski, “Silk damping at a redshift of a billion: a new limit on small-scale adiabatic perturbations,” Phys. Rev. Lett. 113, 061301 (2014), arXiv:1403.3697 [astro-ph.CO].

[58] Tomohiro Nakama, Teruaki Suyama, and Jun’ichi Yokoyama, “Reheating the Universe Once More: The Dissipation of Acoustic Waves as a Novel Probe of Primordial Inhomogeneities on Even Smaller Scales,” Phys. Rev. Lett. 113, 061302 (2014), arXiv:1403.5407 [astro-ph.CO].

[59] Kishore N. Ananda, Chris Clarkson, and David Wands, “The Cosmological gravitational wave background from primordial density perturbations,” Phys. Rev. D75, 123518 (2007), arXiv:gr-qc/0612013 [gr-qc].

[60] Daniel Baumann, Paul J. Steinhardt, Keitaro Takahashi, and Kiyotomo Ichiki, “Gravitational Wave Spectrum Induced by Primordial Scalar Perturbations,” Phys. Rev. D76, 084019 (2007), arXiv:hep-th/0703290 [hep-th].

[61] Ryo Saito and Jun’ichi Yokoyama, “Gravitational wave backgrounds as a probe of the primordial black hole abundance,” Phys. Rev. Lett. 102, 161101 (2009), [Erratum: Phys. Rev. Lett.107,069901(2011)], arXiv:0812.4339 [astro-ph].

[62] Edgar Bugaev and Peter Klimai, “Induced gravitational wave background and primordial black holes,” Phys. Rev. D81, 023517 (2010), arXiv:0908.0664 [astro-ph.CO].

[63] Ryo Saito and Jun’ichi Yokoyama, “Gravitational-Wave Constraints on the Abundance of Primordial Black Holes,” Prog. Theor. Phys. 123, 867–886 (2010), [Erratum: Prog. Theor. Phys.126,351(2011)], arXiv:0912.5317 [astro-ph.CO].

[64] Edgar Bugaev and Peter Klimai, “Constraints on the induced gravitational wave background from primordial black holes,” Phys. Rev. D83, 083521 (2011), arXiv:1012.4697 [astro-ph.CO].

[65] Christian T. Byrnes, Philippa S. Cole, and Subodh P. Patil, “Steepest growth of the power spectrum and primordial black holes,” (2018), arXiv:1811.11158 [astro-ph.CO].

[66] Sam Young and Christian T. Byrnes, “Primordial black holes in non-Gaussian regimes,” JCAP 1308, 052 (2013), arXiv:1307.4995 [astro-ph.CO].

[67] Tomohiro Nakama, Teruaki Suyama, and Jun’ichi Yokoyama, “Supermassive black holes formed by direct collapse of inflationary perturbations,” Phys. Rev. D94, 103522 (2016), arXiv:1609.02245 [gr-qc].

[68] Tomohiro Nakama, Joseph Silk, and Marc Kamionkowski, “Stochastic gravitational waves associated with the formation of primordial black holes,” Phys. Rev. D95, 043511 (2017), arXiv:1612.06264 [astro-ph.CO].

[69] Rong-gen Cai, Shi Pi, and Misao Sasaki, “Gravitational Waves Induced by non-Gaussian Scalar Perturbations,” (2018), arXiv:1810.11000 [astro-ph.CO].

[70] N. Bartolo, V. De Luca, G. Franciolini, M. Peloso, D. Racco, and A. Riotto, “Testing Primordial Black Holes as Dark Matter through LISA,” (2018), arXiv:1810.12224 [astro-ph.CO].

[71] Caner Unal, “Imprints of Primordial Non-Gaussianity on Gravitational Wave Spectrum,” Phys. Rev. D99, 043513 (2019), arXiv:1811.09151 [astro-ph.CO].

[72] L. Lentati et al., “European Pulsar Timing Array Limits On An Isotropic Stochastic Gravitational-Wave Background,” Mon. Not. Roy. Astron. Soc. 453, 2576–2598 (2015), arXiv:1504.03692 [astro-ph.CO].

[73] R. M. Shannon et al., “Gravitational waves from binary supermassive black holes missing in pulsar observations,” Science 349, 1522–1525 (2015), arXiv:1509.07320 [astro-ph.CO].

[74] K. Aggarwal et al., “The NANOGrav 11-Year Data Set: Limits on Gravitational Waves from Individual Supermassive Black Hole Binaries,” (2018), arXiv:1812.11585 [astro-ph.GA].

[75] Kazunori Kohri and Takahiro Terada, “Semianalytic calculation of gravitational wave spectrum nonlinearly induced from primordial curvature perturbations,” Phys. Rev. D97, 123532 (2018), arXiv:1804.08577 [gr-qc].

[76] C. J. Moore, R. H. Cole, and C. P. L. Berry, “Gravitational-wave sensitivity curves,” Class. Quant. Grav. 32, 015014 (2015), arXiv:1408.0740 [gr-qc].

[77] B. P. Abbott et al. (KAGRA, LIGO Scientific, VIRGO), “Prospects for Observing and Localizing Gravitational-Wave Transients with Advanced LIGO, Advanced Virgo and KAGRA,” Living Rev. Rel. 21, 3 (2018), arXiv:1304.0670 [gr-qc].

[78] B. S. Sathyaprakash and B. F. Schutz, “Physics, Astrophysics and Cosmology with Gravitational Waves,” Liv Rev. Rel. 12, 2 (2009), arXiv:0903.0338 [gr-qc].

[79] Matthew W. Choptuik, “Universality and scaling in gravitational collapse of a massless scalar field,” Phys. Rev. Lett. 70, 9–12 (1993).

[80] Jens C. Niemeyer, “Numerical investigation of the threshold for primordial black hole formation,” in Sources and detection of dark matter in the universe. Proceedings, 3rd International Symposium, and Workshop on Primordial Black Holes and Hawking Radiation, Marina del Rey, USA, February 17-20, 1998, (1998) arXiv:astro-ph/9806043 [astro-ph].

[81] Ilia Musco, John C. Miller, and Alexander G. Polnarev, “Primordial black hole formation in the radiative era: Investigation of the critical nature of the collapse,” Class. Quant. Grav. 26, 235001 (2009), arXiv:0811.1452 [gr-qc].

[82] Ilia Musco and John C. Miller, “Primordial black hole formation in the early universe: critical behaviour and self-similarit,” Class. Quant. Grav. 30, 145009 (2013), arXiv:1201.2379 [gr-qc].

[83] Florian Kühnel, Cornelius Rampf, and Marit Sandstad, “Effects of Critical Collapse on Primordial Black-Hole Mass Spectra,” Eur. Phys. J. C76, 93 (2016), arXiv:1502.05903 [astro-ph.CO].
[84] James M. Bardeen, J. R. Bond, Nick Kaiser, and A. S. Szalay, “The Statistics of Peaks of Gaussian Random Fields,” Astrophys. J. 304, 15-61 (1986).