Real “Units Imaginary”
in Kähler’s Quantum Mechanics∗

Jose G. Vargas†

Abstract

Inspired by a similar, more general treatment by Kähler, we obtain the spin operator by pulling to the Cartesian coordinate system the azimuthal partial derivative of differential forms. At this point, no unit imaginary enters the picture, regardless of whether those forms are over the real or the complex field. Hence, the operator is to be viewed as a real operator. Also a view of Lie differentiation as a pull-back emerges, thus avoiding concepts such as flows of vector fields for its definition.

Enter Quantum Mechanics based on the Kähler calculus. Independently of the unit imaginary in the phase factor, the proper values of the spin part of angular momentum emerge as imaginary because of the idempotent defining the ideal associated with cylindrical symmetry. Thus the unit imaginary has to be introduced by hand as a factor in the angular momentum operator —and as a result also in its orbital part— for it to have real proper values. This is a concept of real operator opposite to that of the previous paragraph. Kähler stops short of stating the antithesis in this pair of concepts, both of them implicit in his work.

A solution to this antithesis lies in viewing units imaginary in those idempotents as being the real quantities of square −1 in rotation operators of real tangent Clifford algebra. In so doing, one expands the calculus, and launches in principle a geometrization of quantum mechanics, whether by design or not.

∗Dedicated to Professor A. Rueda, in recognition for the impact he had on my professional development.
†138 Promontory Rd., Columbia, SC 29209, USA. josegvargas@earthlink.net
1 Introduction

In the standard treatment of angular momentum in the literature, the orbital one comes first, through replacement with quantum mechanical operators for $r$ and $p$ in $r \times p$. One then invokes internal degrees of freedom to attach spin to it, which is a little bit awkward since spin has to do with rotations in configuration space, not in some internal space. In addition, cylindrical symmetry is handled by in the not too transparent way of ignoring part of what is conserved in the spherical case. This is not too transparent since we have conservation for all directions at the same time in the spherical case, but its manifestation is curtailed by non-commutativity of operators for conservation of angular momentum in different directions. Ideally, one should not resort to $r \times p$, which does not pertain to just cylindrical symmetry. Finally, the unit imaginary makes explicit appearance through the $p$ operator in the standard treatment. This appearance is totally unnecessary in Kähler’s approach to quantum mechanics through his calculus, though the unit imaginary still becomes necessary in the applications. We end these considerations on angular momentum by quoting from E. Kähler [1]:

“The spin of the electron will be interpreted as the necessity of representing its state not by a wave function but by a wave differential form” (translated from the German original).

Kähler’s treatment of angular momentum is the result of his extension of the concept of Lie derivative from the ring of functions to the ring of differential forms. But, in his case as in Cartan’s, the concept of Lie derivative of a differential form is a consequence of what it is for a function ($0$–form), not a matter of an ad hoc definition. In fact, Cartan did not even define what is now considered to be Lie differentiation of differential forms, but did formulate a famous theorem, which can be used to compute it.

In standard quantum mechanics, linear momentum and particles come to the fore from the start. Thus, concepts like linear momentum are foundational in Dirac’s approach. In Kähler’sKähler, they certainly remain relevant, but as concomitant or derived issues, not as foundational ones. In fact, Kähler did not even directly deal with square-integrability, and the position operator was not even mentioned in his work.

We obtain Kähler’s expression for angular momentum starting which the action of a partial derivative on a differential form. He obtained it in the
general framework of what Cartan called \[2\] infinitesimal transformations, which he cleverly transformed into partial derivatives \[3\]. But this is a distraction that obscures the humble nature of the Lie derivative of a differential form. It is simply a pull-back to a different coordinate system of a partial derivative of a differential form. This simplicity is best appreciated when, searching the web, one realizes the confusion that surrounds the concept of Lie differentiation.

Except for the difference just mentioned, our argument is totally Kähler’s up to this point, the unit imaginary having not yet appeared. It is at the point of obtaining solutions to equations with cylindrical symmetry that he required the presence of something like the unit imaginary. But Kähler did not try to relate expressions without the unit imaginary to expressions with it. In dealing with this issue, we find that the geometric objects of square \(-1\) that suggest themselves as alternatives for the unit imaginary in the treatment of rotations are different in principle from those in the idempotents that define related ideals. The solution to this leads to what we consider the beginning of a process of geometrization of quantum mechanics.

In sections 2 and 3, we shall simplify Kähler’s treatment of spin in order to target it directly without reference to the general subject of Lie differentiation. Section 4 shows to non-experts how one deals with rotations in Clifford algebra. This knowledge is used to infer in section 5 the case by case replacement of the unit imaginary in real quantities in phase shifts, rotations and energy and angular momentum operators.

2 Lie differentiations of differential forms as pull-backs of their partial differentiations

We begin by reversing an argument by Kähler in 1960 \[1\]. Basically, he started with an operator \(\zeta^j(x)\frac{\partial}{\partial x^j}\) and converted it into a partial derivative \(\partial/\partial y^n\) in some appropriately chosen coordinate system \((y^i)\). We are interested in a simpler problem. We have some partial derivative \(\partial/\partial \phi\) acting on a differential form and we want to pull that action to another coordinate system. Once we do that, we shall specialize to when \(\phi\) is the azimuthal coordinate in 3D-Euclidean space.

Give any (usually local) coordinate system \((z^i)\) on a manifold, each \(dz^i\) is an exact differential form whose evaluation on curves between points \(A\) and
By yields the curve-independent difference $z_B^i - z_A^i$. We shall assume that (if it were not obvious from the meaning of ) $\partial(dz^m)/\partial z^l$ equals zero for all $l$ and $m$ independently of whether $l$ equals $m$ or not.

Let $\phi$ be a coordinate in some coordinate system $(y^j)$, and let $(x^i)$ be some overlapping coordinate system. As just stated, $\partial \phi/\partial y^l = 0$ and, therefore,

$$\frac{\partial(dx^i)}{\partial \phi} = \frac{\partial x^i}{\partial \phi} \frac{dy^l}{\partial y^l} = \frac{\partial x^i}{\partial \phi} \left( \frac{\partial x^i}{\partial \phi} \right) dy^l = d \frac{\partial x^i}{\partial \phi} = d \zeta^i, \quad (1)$$

where we have defined

$$\zeta^i \equiv \frac{\partial x^i}{\partial \phi}, \quad (2)$$

and where we use throughout Einstein’s summation convention over repeated indices, one up and one down. Thus, for an arbitrary 1–form, we have

$$\frac{\partial(u_i dx^i)}{\partial \phi} = u_{i;\phi} dx^i + u_i d\zeta^i = \zeta^j \frac{\partial u_i}{\partial x^j} dx^i + d\zeta^i u_i. \quad (3)$$

Let $u$ denote now any differential form in the exterior or in the Clifford algebras of differential forms built upon the n-dimensional module spanned by the $dx^i$. Those algebras are themselves modules of dimension $2^n$. Hence, we can span $u$ as

$$u = u_\Lambda dx^\Lambda, \quad \Lambda = 1, 2, ..., 2^n \quad (4)$$

where $dx^\Lambda$ represents the different elements of a basis of differential forms in the $2^n$-dimensional module. For simplicity, each element of the basis is taken to be of definite grade, going from 0 to $n$. Easy calculations show that, instead of (3), we would now have

$$\frac{\partial u}{\partial \phi} = \zeta^j \frac{\partial u_\Lambda}{\partial x^j} dx^\Lambda + d\zeta^i \wedge e_i u, \quad (5)$$

where $e_i u$ is defined by

$$u = dx^i \wedge e_i u + u', \quad (6)$$

and that no term in $u'$ contains the factor $dx^i$. If $u$ were just a scalar function $f$, the second term in (5) would disappear, the only non-vanishing $dx^\Lambda$ is the unity and $u_\Lambda$ is $f$ for that $dx^\Lambda$.

Warning: Lie operators as defined by their action on differential forms do not constitute a vector space. This can be seen as follows. Let the system
\((y^i)\) coincide with the system \((x^i)\). Make \(\phi\) be the last coordinate. Then \(\zeta^i=(0,0,\ldots,1)\). Equation (5) reduces to
\[
\frac{\partial u}{\partial x^n} = \frac{\partial u_\Lambda}{\partial x^n} dx^\Lambda.
\] (7)

If we were dealing with a vector space, the Lie operator \(\zeta^j \partial / \partial x^j\) acting on \(u\) would yield only the first of the two terms on the right hand side of (5).

In order to minimize overlooking these facts, we shall use the notation \(\chi\) for the pull back of the operator \(\frac{\partial}{\partial \phi}\) that acts on differential forms in the \(x\) coordinate system. This addresses a deficiency in notation of (5). But, we shall also avoid using the symbol \(\frac{\partial u}{\partial x^j}\) for \(\frac{\partial u_\Lambda}{\partial x^\Lambda} dx^\Lambda\), which is inimical to the result just obtained.

Assume \(\phi\) were the azimuthal coordinate in 3-D Euclidean space. Then \(\zeta^j=(-y,x,0)\) and
\[
\frac{\partial u}{\partial \phi} = x \frac{\partial u}{\partial y} - y \frac{\partial u}{\partial x} - dy \wedge e_1u + dx \wedge e_2u,
\] (8)
where the subscripts 1 and 2 stand for the \(x\) and \(y\) coordinate respectively. The terms \(-dy \wedge e_1u + dx \wedge e_2u\) should not be ignored. They constitute the action of the spin operator, as we shall show further below.

## 3 Spin for cylindrical symmetry

We associate with \(\chi\) a differential 1-form \(\tau = \zeta_i dx^i\). For the azimuthal coordinate, \(\tau = -ydx + xdy\), \(d\tau = 2dxdy\). (9)

Define differential forms \(u_0\) to \(u_3\) that do not contain \(dx\) and \(dy\) by
\[
u = dxdyu_0 + dxu_1 + dyu_2 + u_3.
\] (10)

One readily gets that
\[
d\tau u = -2u_0 - dyu_1 + dxu_2 + dxdyu_3,
\] (11)
and
\[
ud\tau = -2u_0 + dyu_1 - dxu_2 + dxdyu_3
\] (12)
Hence
\[ \frac{1}{2}(d\tau u - ud\tau) = -dyu_1 + dxu_2. \] (13)

On the other hand
\[ -dy \wedge e_1 u = -dy \wedge u_1, \quad dx \wedge e_2 u = dx \wedge du_2, \] (14)

which then yields
\[ \frac{\partial u}{\partial \phi} = x \frac{\partial u}{\partial y} - y \frac{\partial u}{\partial x} + \frac{1}{2} d\tau u - \frac{1}{2} ud\tau. \] (15)

We define
\[ w_1 = dydz, \quad w_2 = dzdx, \quad w_3 = dxdy, \] (16)

and rewrite (15) as
\[ \chi_3 = x^1 \frac{\partial u}{\partial x^2} - x^2 \frac{\partial u}{\partial x^1} + \frac{1}{2} w_3 u - \frac{1}{2} uw_3. \] (17)

We have not used spherical symmetry, and we have not invoked the unit imaginary for any purpose. It simply happens that \((dxdy)^2 = -1\).

For spherical symmetry, we would construct, in addition, an operator \(K\) defined by \((K+1)u \equiv \sum_{i=1}^{3} \chi_i u \vee w_i\) and show that \(\sum_{i=1}^{3} \chi_i^2 u = -K(K+1)u\). The commutation relations are \(\chi_i \chi_j - \chi_j \chi_i = -\chi_k\). Again, there is no need to invoke the unit imaginary for anything.

## 4 Units imaginary in tangent Clifford algebra

### 4.1 Units imaginary in Kähler’s algebra

So far, we have dealt with Kähler’s algebra, i.e. Clifford algebra of differential forms. He writes solutions with rotational symmetry around the \(z\) axis in configuration space as
\[ u = p e^{im\phi} \tau^\pm, \quad \tau^\pm \equiv \frac{1}{2}(1 \pm idxdy) = \frac{1}{2}(1 \pm iw), \] (18)

where, as he states, \(p\) is a genuine meridian differential. A meridian differential form whose pull-back to the \((\rho, \phi, z)\) coordinate system does not depend
on $d\rho$ and $dz$. They are said to be genuine (our translation from the German “reines”) if, in addition, they do not depend on $(\rho, z)$. In other words, all the dependence on $\phi$ and $d\phi$ is in $e^{\imath m\phi\tau^\pm}$. It is clear that we can replace $\rho$, $z$, $d\rho$ and $dz$ with $r$, $\theta$, $dr$ and $d\theta$ in what has just been said.

The unit imaginary is not present in (17), but it can be introduced there so that proper values of angular momentum for (18) be real. Kähler introduces the unit imaginary in the exponential, which he did not need appear in (17). He does not state. We have to assume that the argument is the standard one in quantum mechanics.

Also, the factor of $dxdy$ has to be of square minus one for $\tau^\pm$ to be idempotents. The algebra can be decomposed into two complementary such ideals

$$Cl^3 = Cl^3\frac{1}{2}(1 + idxdy) + \frac{1}{2}(1 - idxdy),$$

(19)

where $Cl^3$ denotes in this case the Kähler in 3-D Euclidean space. Idempotents, and specially the primitive ones, play a very large role in Kähler’s approach to quantum mechanics and, less explicitly so, in the Dirac theory.

There are different real alternatives —whose square is minus one— to the unit imaginary. They suggest themselves when one uses Clifford algebra throughout, i.e. for the valuedness of differential forms. Kähler considered tensor-valued ones, when not simply scalar-valued. Let us start by recalling basic features of tangent Clifford algebra.

### 4.2 The true tangent Euclidean algebra

Clifford algebra is the true Euclidean algebra. The usual vector algebra in $E^3$ is a corruption of Clifford algebra, as it is not available in dimensions other than three and seven (though these two are of different nature). Exterior products exist in any number $n$ of dimensions, regardless of whether the vector space upon which we build the exterior algebra is Euclidean (i.e. endowed with a vector product) or not. Their combination with the dot product yields the Clifford product if the vector space is Euclidean and at least one of the two factors is a vector.

The vector product is a combination of the exterior product and Hodge duality. The exterior product of two vectors is of grade two, like planes, with which they are intimately related. Hodge duality assigns to each object of grade 2 in a Clifford algebra an object of grade $n-2$ in the same algebra. In $E^3$, the duality operation on that exterior product yields an object of grade
one, which is the vector to which we refer to as vector product. For that reason, a concept of vector product in that sense only exists in dimension three.

Rotations using Clifford algebra hold the key to understanding our replacements for the unit imaginary.

### 4.3 Rotations in the tangent Clifford algebra

In a plane, let \( u \) and \( u' \) be each other’s reflected vector with respect to the unit vector \( t \). Then, clearly, \( tu' = ut \) where juxtaposition represents Clifford product. Let \( n \) be a unit normal to \( t \). By a theorem of elementary geometry relating reflections with respect to perpendicular directions, we have

\[
u' = t^{-1}ut = -n^{-1}un, \tag{20}\]

Thus \( u' = -n^{-1}un \) applies for the reflection with respect to a plane which has \( n \) as a normal unit vector.

By another theorem of elementary geometry, the product of reflections with respect to two planes yields a rotation. Let \( n_1 \) and \( n_2 \) be the respective normals. The result of rotating \( u \) in this way yields

\[
u' = n_2^{-1}n_1^{-1}un_1n_2 = (n_1n_2)^{-1}u(n_1n_2), \tag{21}\]

the angle \( \phi \) being double the angle between the planes, oriented from the first to the second plane.

Let \( N \) be the unit bivector \((N^2 = -1)\) in the plane of \( n_1 \) and \( n_2 \). We then have

\[
n_1n_2 = n_1 \cdot n_2 + n_1 \wedge n_2 = \cos \frac{\phi}{2} + \sin \left(\frac{\phi}{2}N\right) = e^{\frac{\phi}{2}N}. \tag{22}\]

So, finally,

\[
u' = e^{-\frac{\phi}{2}N}ue^{\frac{\phi}{2}N}. \tag{23}\]

Assume that, instead of tensor-valuedness of differential forms (as Kähler considered), we were interested in valuedness in a tangent Clifford algebra. Any member of the algebra is a sum of products of vectors. Call any one such term \( U (= ab...m) \). Its rotation is given by

\[
U' = (A^{-1}aA)(A^{-1}bA)...(A^{-1}mA) = (A^{-1}ab...mA) = A^{-1}UA, \tag{24}\]

where \( A \equiv e^{\frac{\phi}{2}N} \). And if \( U \) is a member of a left ideal, we have

\[
U' = e^{-\frac{\phi}{2}N}U. \tag{25}\]
### 4.4 Non-scalar-valued differential forms

An example of differential form that is not scalar-valued is the energy-momentum tensor, which is a disguised form of a vector-valued differential 3-form [4]. It is a 3-form because it is to be integrated on a 3-volume in spacetime. It is vector-valued because so is the result of the integration.

To make the discussion as clear as possible, consider a surface density of a vector quantity at each point of a surface in 3-D Euclidean space. Let us denote it as \( j(u, v)du \wedge dv \), where \( u \) and \( v \) are parameters on the surface. If we integrate on a very small piece of surface (appropriately chosen to fit the parametrization), we can approximate the integral by \( j(u_0, v_0)\Delta u \Delta v \). Its rotation would be given by \( e^{-\frac{\phi}{2}N}j(u_0, v_0)e^{\frac{\phi}{2}N}\Delta u \Delta v \) where \( N \) is the unit bivector at the point of the surface with parameters \( (u_0, v_0) \). It is clear that, correspondingly, there is the following expression for the rotation of the vector-valued differential forms:

\[
e^{-\frac{\phi}{2}N}j(u, v)e^{\frac{\phi}{2}N}du \wedge dv.
\]  

(26)

The angle of rotation is of course \( \phi \). We should have present the dissociation of the 3-D vector space where the rotation takes place from the 2-D parameter space. This space is in turn a proxy for a 2-D submanifold of the 3-D Euclidean space, which does not change by the rotation. In addition, one has to express \( e^{-\frac{\phi}{2}N}j(u, v)e^{\frac{\phi}{2}N} \) in terms of a fixed basis of vectors, for it does not make any sense to sum components computed on different bases at different points of the surface.

### 4.5 Kähler’s inverse problem and the issue of the unit imaginary in Lie differentiation and in quantum mechanics

At least for quantum mechanical physicists, the unit imaginary is inextricably attached to Lie operators. Such is the case with those for angular momentum, energy and momentum, which are Lie operators. This is so not because of Lie theory itself, but because of its use in quantum mechanics.

In order to understand this, let us give a bird’s-eye view of some not yet considered Lie operator theory in Kähler’s calculus. In 1960, Kähler solved what we shall call the inverse problem in Lie differentiation [3]. Given an infinitesimal transformation, obtain a coordinate system in which the pullback becomes simply a partial differentiation. It completes the argument
that allows us to view the Lie derivative of a differential form as a pull-back of a partial derivative. Lie derivatives of differential transformations $\alpha$ are infinitesimal transformations of $\alpha$, transformations which in turn are actions of partial derivatives in appropriate coordinate systems. Of course, all of it up to pull-backs, which is a redundant statement since that is what a coordinate transformation is (It is in 1962, not in 1960, that Kähler referred to the action of an infinitesimal transformation as a Lie derivative [II]). Hence, Lie theory is the relating of partial derivatives of differential forms in different coordinate systems.

Together with the case we made in our book on differential geometry [3], differential forms and their derivatives are thus shown to be the mother and father of all concepts in calculus, analysis and differential geometry. Kähler’s approach to quantum mechanics is precisely based on this view, thus on the Kähler calculus. But, in this calculus, the unit imaginary has nothing to do with the theory of the Lie derivative. The introduction of this unit in Kähler’s calculus takes place at a later stage, the stage of finding differential forms that are solutions of differential systems with some symmetry property and that also belong to ideals defined by that property. For those purposes, it can be replaced with anything as square minus one, like, for example, any of the $w_i$’s.

To conclude, the standard unit imaginary has nothing to do with Lie theory proper, but with the idiosyncrasy of the Dirac equation in context of solutions that belongs to ideals in a Clifford algebra, and not simply in the algebra itself.

5 First geometrization of quantum mechanics

5.1 Angular momentum’s action on spin idempotents

Kähler proposed the form (18) for spatial factors of solutions of equations with rotational symmetry around the $z$ axis. Although (18) owes its form to the fact that (17) is what it is, Kähler does not relate the two. Had he gone into it, he would have found certain issues with whether the spin of the electron is given its value of $1/2$, if he had analyzed this issue from the perspective of Eq. (17) (see next subsection). We know that it is $1/2$. But that is in classical treatments, where the electron is, except for spin, a point particle. But the hydrogen atom treated with the Kähler equation yields
exactly the right solutions, and thus includes the right spin.

In addition for the detailed, exhaustive treatment of the issue of spin, this author brings to attention from his retrospective perspective that one must distinguish the arenas for classical systems (or at least classically treated systems) and quantum systems (or at least quantum-mechanically treated systems). Both topics will be the subject of future papers.

We shall now limit consideration of the interconnection of (17) and (18) to the geometrization of the unit imaginary that those equations together suggest.

5.2 The unit imaginary in the spin part of angular momentum

The relation of the spin operator to active rotations is subtle, since, as we said, the operator acts on differential forms, and rotations act on the structure where those forms take their value. In addition, one must distinguish between the form of the rotation of members of the tangent algebra and of members of the ideals defined by idempotents in the same algebra that are viewed as not being affected by those rotations.\[5\].

Consider the spin action

$$\frac{1}{2} w_3 u - \frac{1}{2} u w_3,$$

(27)

when $u$ is given by (18). If $p$ commutes with $w_3$, we have

$$\frac{1}{2} w_3 p e^{i \phi \tau} \pm \frac{1}{2} p e^{i \phi \tau} \mp w_3 = 0.$$  

(28)

And, if $p$ anticommutes with $w_3$,

$$\frac{1}{2} w_3 u - \frac{1}{2} u w_3 = -w_3 u = -p e^{i \phi \tau} w_3 \frac{1}{2} (1 \pm iw_3),$$

(29)

but

$$w_3 (1 \pm iw_3) = w_3 \mp i = \mp i (1 \pm iw_3).$$

(30)

Hence, proper values are imaginary:

$$\frac{1}{2} w_3 u - \frac{1}{2} u w_3 = \mp i u.$$  

(31)
The absence of a factor of $1/2$ on the right hand side of (31) is the issue we had in mind in the first paragraph of the previous subsection.

In order to have real proper values, $w_3$ in $\chi_3$ must be multiplied by some real quantity of square $-1$. From this perspective, $\partial/\partial \phi$ (of which (17) is a pull-back) is not real. And that real quantity must be the same that should replace $i$ in $e^{im\phi}$. Notice that having a real “unit imaginary” in this phase factor is dictated in last instance by the desire to have a real unit imaginary in the spin part of the angular momentum operator, not in the orbital part. It was a result of the interplay of $w_3$ and $\tau^\pm$, not $e^{im\phi}$.

### 5.3 Geometrization of angular momentum

In the Kähler calculus, the issue of the presence or not of the unit imaginary in the angular momentum operator can then be formulated as follows. We need, so it seems, (constant) idempotent as factors in solutions of differential form equations having cylindrical or spherical symmetry. These idempotents generate ideals of the type $Cl_3\tau^\pm$ where $Cl_3$ is the Kähler algebra in 3D (i.e. the Clifford algebra of differential forms for that number of dimensions). The “unit imaginary” by which $\chi_3$ is to be multiplied must be the same as in $\tau^\pm$, which can be expected to be the same as in the rotation operator in the $(x, y)$ plane, i.e. $a_1a_2$, i.e. the $N$ in Eq. (26). Thus, the angular momentum operator should thus be

$$\chi_3' = a_1a_2\chi_3,$$

(32)

and corresponding idempotents being

$$\frac{1}{2}(1 \pm dx a_1 dy a_2),$$

(33)

becoming

$$u = pe^{m\phi}a_1a_2\frac{1}{2}(1 \pm dx a_1 dy a_2).$$

(34)

$\chi_3'$ so defined is a real quantity. It no longer is the pull-back of $\partial/\partial \phi$, but of $a_1a_2\partial/\partial \phi$. In this way, we have resolved the antithetical situation about the two meanings of an operator being real. But the replacement of $\partial/\partial \phi$ with $a_1a_2\partial/\partial \phi$ requires a revision of analysis. The pullback of he orbital part of $a_1a_2\partial/\partial \phi$ to Cartesian coordinates will be written as follows:

$$a_1a_2\partial/\partial \phi = (xa_1)\frac{\partial}{a_2\partial y} - (ya_2)\frac{\partial}{a_1\partial x}.$$

(35)
This is a more geometric view of partial derivatives. We leave for other purposes the development of this approach to analysis, or of calculus if you will.

5.4 Other geometrizations of units imaginary

Kähler wrote proper functions (meaning proper differential forms) pertaining to a specific proper value of energy as

$$u = e^{-iEt} \epsilon^{\pm}, \quad \epsilon^{\pm} \equiv \frac{1}{2}(1 \mp idt). \quad (36)$$

Following his choice of signature, (-1,1,1,1), we can replace the unit imaginary with the unit vector $e_0$ in the time direction (Warning: in order to be consistent with not yet published results obtained by this author, let us advance that $t$ should be replaced with propertime viewed as a fifth dimension, and $e_0$ with the unit vector tangent to the $\tau$ coordinate lines; but we avoid doing so at this time to avoid confusing readers). Hence, instead of (36), we have

$$u = pe^{-a_0 E t} \frac{1}{2}(1 \mp dt a_0), \quad (37)$$

where $p$ depends only on the Cartesian coordinates and their differentials. The application of the energy operator to this expression does not yield additional terms since the $d\zeta_i$’s are null.

In 1961, Kähler wrote solutions for given values of proper energy (rest mass) and proper angular momentum (spin) as

$$u = e^{im\phi - iEt} p \vee \tau^\pm \vee \epsilon^*, \quad (38)$$

where, again, $p$ only depends on $(\rho, z, d\rho, dz)$ or $(r, \theta, dr, d\theta)$. The geometric form of these solutions would be given by

$$u = e^{im\phi_1 e_2 - E t e_0} \vee p \vee \tau^\pm \vee \epsilon^*, \quad (39)$$

with $\tau^\pm$ and $\epsilon^\pm$ now replaced with

$$\epsilon^\pm = \frac{1}{2}(1 \mp e_0 dt), \quad \tau^\pm = \frac{1}{2}(1 \pm e_1 e_2 dxdy), \quad (40)$$

Space translation symmetry does not have the same character as time translation symmetry. No quantum mechanical system has time translational
symmetry. A beam of particles approaches having such symmetry, but as pertaining to a semiclassical treatment where a composite of particles is treated as if it were a simple system to which the translational symmetry applied. We may still use the representation

$$u = pe^{-a_1 p_x} \frac{1}{2} (1 \pm dx a_1),$$  

but we may not expect that all considerations pertaining to time translation and rotational symmetry will apply to it. The difficulties associated with position operators in quantum mechanics are well known and need not be repeated here.

6 Concluding remarks

We have simplified Kähler’s demonstrating that spin is an overlooked term in the expression for the pull-back to another coordinate system of a partial derivative of a differential form. We have then replaced the unit imaginary with real geometric quantities. This forces one to develop such geometrization in the future, if those replacements are to be taken seriously.

As a test for our ideas, consider the hydrogen atom. Replacing $i$ with $e^0$ does not alter computations. A harder testing might consist in replacing the scalar-valued differential forms with Clifford-valued ones through the replacement $dx^\mu$ goes to $dx^\mu a_\mu$. But this hard testing might be spurious, as we proceed to discuss.

In Eq. (34), $dxdy$ is accompanied by $a_1 a_2$, but the other $a_1 a_2$ in the same expression is not so accompanied. Thus, say, $dx$ would not need to go to $dx^\mu a_\mu$ if it were the result of the product $a_1 (dx a_1)$ in the formulation of a specific Kähler equation. To complicate the argument, or rather enrich the realm of possibilities, consider that the pair of equations by Kähler and Dirac for the H atom yield the same result, although the position of the unit imaginary is not equivalent in those equations.

There are other issues. The concept of a particle in classical physics is nothing like in quantum physics, where it is not a point but a sophisticated system, even if the case of an electron. With the benefit of hindsight provided by his not yet published research, this author wishes to submit that pure classical and quantum systems will take overlapping sectors of a Kaluza-Klein type 5-D space. Of course, present day treatments are not pure, or else
the quantum mechanical system of an electron would have already yielded its mass. As for the appearance of the electron in classical treatments, we use properties like its mass and its magnetic moment whose nature is such that one cannot even expect that they will have a classical formulation in the traditional sense.

The Lorentz transformations belong to the classical sector. Similarities with this sector and the hybrid nature of the analyses let us obtain results from this symmetry even in the quantum sector, say, the obtaining of the fine structure of the hydrogen atom. The Lorentz symmetry is the manifestation in the spacetime subsector of a symmetry of the geometric structure, i.e. of the 5-D space. The manifestation in the space-propertime subspace of the same symmetry is $U(1) \times SU(2)$. As for $SU(3)$, it would be too complicated to be more specific about it at this point. Suffice to say that these two symmetries of the quantum sector are as intertwined as intertwined are $dx$ and $a_1$ in $dx a_1$; $U(1) \times SU(2)$ and $SU(3)$ belong directly and respectively to the tangent (i.e. valuedness) algebra and to the Kähler algebra, i.e. of differential forms.

Acknowledgments. Financial support from PST Associates is acknowledged and deeply appreciated.

References

[1] Kähler, E.: Der innere Differentialkalkül. Rendiconti di Matematica, 21, 425-523 (1962).
[2] Cartan, É.: Leçons sur les invariants intégraux. Hermann, Paris (1922).
[3] Kähler, E.: Innerer und äußerer Differentialkalkül. Abh. Dtsch. Akad. Wiss. Berlin, Kl. Math. Phys. Tech. 4, 1-32 (1960).
[4] Vargas, J.G.: Differential Forms for Cartan-Klein Geometries: Erlangen Program with Moving Frames, Abramis, London (2012).
[5] Vargas, J. G.: “The Foundations of Quantum Mechanics and the Evolution of the Cartan-Kähler Calculus”. Found. Phys. 38, 610-647 (1997).
[6] Kähler, E.: Die Dirac Gleichung. Abh. Dtsch. Akad. Wiss. Berlin, Kl. Math. Phys. Tech. 1, 1-38 (1961).