Algebraic Surfaces with $p_g = q = 1$, $K^2 = 4$ and Genus 3 Albanese Fibration

Songbo Ling

Abstract

In this paper, we study the Gieseker moduli space $\mathcal{M}_{1,1}^{4,3}$ of minimal surfaces with $p_g = q = 1$, $K^2 = 4$ and genus 3 Albanese fibration. Under the assumption that direct image of the canonical sheaf under the Albanese map is decomposable, we find two irreducible components of $\mathcal{M}_{1,1}^{4,3}$, one of dimension 5 and the other of dimension 4.

1 Introduction

Minimal surfaces of general type with $p_g = q = 1$ have attracted the interest of many authors (e.g. \cite{5, 8, 9, 12, 16, 23–26}). For these surfaces, one has $1 \leq K^2 \leq 9$. By results of Moishezon \cite{27}, Kodaira \cite{18} and Bombieri \cite{4}, these surfaces belong to a finite number of families.

For such a surface $S$, the Albanese map $f : S \to B := \text{Alb}(S)$ is a fibration. Since the genus $g$ of a general fibre of $f$ (cf. \cite{12} Remark 1.1) and $K_S^2$ are differentiable invariants, surfaces with different $g$ or $K^2$ belong to different connected components of the Gieseker moduli space. Hence one can study the moduli space of these surfaces according to the pair $(K^2, g)$.

The case $K^2 = 2$ has been accomplished by Catanese \cite{5} and Horikawa \cite{16} independently: these surfaces have $g = 2$ and the moduli space is irreducible; the case $K^2 = 3$ has been studied completely by Catanese-Ciliberto \cite{8, 9} and Catanese-Pignatelli \cite{12}; these surfaces have $g = 2$ or $g = 3$. Moreover, there are three irreducible connected components for surfaces with $g = 2$ and one irreducible connected component for surfaces with $g = 3$; for the case $K^2 = 4, g = 2$, Pignatelli \cite{23} found eight disjoint irreducible components of the moduli space under the assumption that the direct image of the bicanonical sheaf under the Albanese map is a direct sum of three line bundles.

For the case $K^2 = 4, g = 3$, there are some known examples (see e.g. \cite{17, 24–26}), but the moduli space is still mysterious. Now denote by $\iota$ the index of the paracanonical system (cf. \cite{8} for definition) of $S$. By a result of Barja and Zucconi \cite{2}, one has either $\iota = g - 1 = 2$ or $\iota = g = 3$ (see Lemma 2.2). By \cite{8} Theorems 1.2 and 1.4, $\iota$ is a topological invariant, hence it is a deformation invariant and surfaces with different $\iota$ belong to different connected components of the moduli space.

In this paper we study the case $K^2 = 4, g = 3, \iota = 2$. Due to technical reasons, we begin by assuming that the general Albanese fibre is hyperelliptic. We call surfaces with these properties surfaces of type $I$ and denote by $\mathcal{M}_I$ their image in $\mathcal{M}_{1,1}^{4,3}$. Our main result is the following

Theorem 1.1. $\mathcal{M}_I$ consists of two disjoint irreducible subsets $\mathcal{M}_{I_1}$ and $\mathcal{M}_{I_2}$ of dimension 4 and 3 respectively. Moreover, $\mathcal{M}_{I_1}$ is contained in a 5-dimensional irreducible component of $\mathcal{M}_{1,1}^{4,3}$ and

This work was supported by China Scholarship Council “High-level university graduate program” (No.201506010011).
The general Albanese fibre is nonhyperelliptic.

This paper is organized as follows.

In section 2, we study the relative canonical map of \( f \). We prove that every Albanese fibre of \( S \) is 2-connected. The main ingredient for that is Proposition 2.5, which gives a sufficient condition for a fibre of genus 3 to be 2-connected.

In section 3, we restrict to surfaces of type \( I \), i.e. minimal surfaces with \( p_g = q = 1, K^2 = 4, g = 3, \ell = 2 \) and hyperelliptic Albanese fibrations. Using Murakami’s structure theorem [22], we divide surfaces of type \( I \) into two types according to the order of some torsion line bundle: type \( I_1 \) and type \( I_2 \) (cf. Definition 3.6). Moreover, we show that the subspace \( \mathcal{M}_{I_1} \) of \( \mathcal{M}^{4,3}_{1,1} \) corresponding to surfaces of type \( I_1 \) and the subspace \( \mathcal{M}_{I_2} \) of \( \mathcal{M}^{4,3}_{1,1} \) corresponding to surfaces of type \( I_2 \) are two disjoint closed subset of \( \mathcal{M}^{4,3}_{1,1} \).

In section 4, we study surfaces of type \( I_1 \). We first construct a family \( M_1 \) of surfaces of type \( I_1 \) using bidouble covers of \( B^{(2)} \), the second symmetric product of an elliptic curve \( B \). Then we show that every surface of type \( I_1 \) is biholomorphic to some surface in \( M_1 \) and that \( \dim \mathcal{M}_{I_1} = 4 \). After that we study the natural deformations of the general surfaces of type \( I_1 \) and show that they give a 5-dimensional irreducible subset \( \mathcal{M}_I^1 \) of \( \mathcal{M}^{4,3}_{1,1} \). By computing \( h^1(T_S) \) for a general surface \( S \in \mathcal{M}_I \), we prove that \( \mathcal{M}_I^1 \) is an irreducible component of \( \mathcal{M}^{4,3}_{1,1} \).

In section 5, we study surfaces of type \( I_2 \). An interesting fact is that every surface of type \( I_2 \) also arises from a bidouble cover of \( B^{(2)} \), but the branch curve is in a different linear equivalence class. Using a similar method to the one of section 4, we show that \( \dim \mathcal{M}_{I_2} = 3 \) and that \( \mathcal{M}_{I_2} \) is contained in a 4-dimensional irreducible component of \( \mathcal{M}^{4,3}_{1,1} \).

Notation and conventions. Throughout this paper we work over the field \( \mathbb{C} \) of complex numbers. We usually denote by \( S \) a minimal surface of general type with \( p_g = q = 1 \) and by \( S' \) the canonical model of \( S \).

We denote by \( \Omega_S \) the sheaf of holomorphic 1-forms on \( S \), by \( T_S := \mathcal{H}om_{\mathcal{O}_S}(\Omega_S, \mathcal{O}_S) \) the tangent sheaf of \( S \) and by \( \omega_S := \wedge^2 \Omega_S \) the sheaf of holomorphic 2-forms on \( S \). \( K_S \) (or simply \( K \) if no confusion) is the canonical divisor of \( S \), i.e. \( \omega_S \cong \mathcal{O}_S(K_S) \). \( p_g := h^0(\omega_S), q := h^0(\Omega_S) \). The Albanese fibration of \( S \) is denoted by \( f : S \to B := \text{Alb}(S) \). We denote by \( g \) the genus of a general fibre of \( f \) and set \( V_n := f_*\omega_S^n \). For an elliptic curve \( B \), we denote by \( B^{(n)} \) the \( n \)-th symmetric product of \( B \) and by \( E_u(r, 1) \) (\( u \) is a point on \( B \)) the unique indecomposable rank \( r \) vector bundle over \( B \) with determinant \( O_B(u) \) (cf. [1]).

We denote by \( \mathcal{M}^{4,3}_{1,1} \) the Gieseker moduli space of surfaces of general type with \( p_g = q = 1, K^2 = 4 \) and genus 3 Albanese fibrations. For divisors, we denote linear equivalence by ‘\( \equiv \)’ and algebraic equivalence by ‘\( \sim_{alg} \)’.

2 The relative canonical map and 2-connectedness of Albanese fibrations

In this section, unless otherwise indicated, we always assume that \( S \) is a minimal surface with \( p_g = q = 1, K^2 = 4 \) and a genus 3 Albanese fibration \( f : S \to B := \text{Alb}(S) \).

First we recall the some definitions that we shall use later from [8] section 1. Let \( t \) be a point on \( B \) and set \( K \oplus t := K + f^*(t - 0) \) (where 0 is the neutral element of the elliptic curve \( B \)). Since
\( h^0(K) = p_g = 1 \) and \( h^0(K \oplus t) = 1 + h^1(K \oplus t) \) (by Riemann-Roch), by the upper semicontinuity, there is a Zariski open subset \( U \ni 0 \) of \( B \) such that for any \( t \in U \), \( h^0(K \oplus t) = 1 \). We denote by \( K_t \) the unique effective divisor in \( |K \oplus t| \) for any \( t \in U \).

We define the \textit{paracanonical incidence correspondence} to be the schematic closure \( Y \) (observe that it is a divisor) in \( S \times B \) of the set \( \{(x, t) \in S \times U | x \in K_t \} \). Let \( \pi_S : S \times B \to S \) and \( \pi_B : S \times B \to B \) be the natural projections. We define \( K_t \) as the fibre of \( \pi_B|_Y : Y \to B \) over \( t \) for any \( t \in B \setminus U \). Note that \( Y \) provides a flat family of curves on \( S \), which we denote by \( \{K_t\}_{t \in B} \) (or simply \( \{K\} \)) and call it the \textit{paracanonical system} of \( S \). The index \( \iota \) of \( \{K\} \) is the intersection number of \( Y \) with the curve \( \{x\} \times B \) for a general point \( x \in S \), which is exactly the degree of the map \( \pi_S|_Y : Y \to S \).

Now we define a rational map \( w' : S \to B^{(\iota)} \) as follows: for a general point \( x \in S \), \( w'(x) := (t_1, t_2, \ldots, t_\iota) \) such that \( (\pi_S|_Y)^{-1}(x) = \{(x, t_1), (x, t_2), \ldots, (x, t_\iota)\} \). We call \( w' \) the \textit{paracanonical map} of \( S \).

Since \( \deg V_1 = 1 \) and \( \rank V_1 = g \), \( V_1 \) has a unique decomposition into indecomposable vector bundles \( V_1 = \bigoplus_{i=1}^k W_i \) with \( \deg W_1 = 1 \) and \( \deg W_i = 0 \), \( h^0(W_i) = 0 \) \( (2 \leq i \leq k) \) (cf. [8] p. 56). Let \( w : S \to \mathbb{P}(V_1) \) be the relative canonical map of \( f \). Then we have

**Lemma 2.1** ([8] Theorem 2.3), \( \rank W_1 = 1 \) and \( \rank W_i = 1 \) \( (i = 1, 2, \cdots, k) \). Moreover, \( W_i \) \( (i = 2, \cdots, k) \) are nontrivial torsion line bundles (see [12] Remark 2.10) and we have the following commutative diagram of rational maps

\[
\begin{array}{ccc}
S & \xrightarrow{w} & \mathbb{P}(V_1) \\
\downarrow{w'} & & \downarrow{\varphi} \\
B^{(1)} & \hookrightarrow & \mathbb{P}(W_1)
\end{array}
\]

where \( \varphi \) is induced by the natural inclusion: \( W_1 \hookrightarrow V_1 \).

**Lemma 2.2.** Let \( S \) be a minimal surface with \( p_g = q = 1 \), \( K^2 = 4 \) and a genus 3 Albanese fibration. Then we have either \( \iota = 2 \) or \( \iota = 3 \).

**Proof.** Since \( K^2_{S/B} = K^2_S = 4 \) and \( \Delta(f) := \chi(\mathcal{O}_S) - (g-1)(g(B)-1) = 1 \), we see that \( \frac{K^2_S}{\Delta(f)} = 4 \). By [2] Theorem 2 and Lemma 2.1, we have either \( \iota = g-1 = 2 \) or \( \iota = g = 3 \). \( \square \)

Now we study the relative canonical map \( w \) of \( f : S \to B \).

**Lemma 2.3.** Let \( F \) be a general fibre of \( f \). Then \( |K_S + dF| \) is base point free for \( d \gg 0 \) and \( w \) is a morphism.

**Proof.** Denote by \( |m| \) the movable part of \( |K_S + dF| \) and \( \mathfrak{j} \) the fixed part of \( |K_S + dF| \). Set \( S_0 := w(S) \). Denote by \( T \) the (tautological) divisor on \( \mathbb{P}(V_1) \) such that \( \pi_s \mathcal{O}(T) = V_1 \) and by \( H \) the fibre of \( \pi : \mathbb{P}(V_1) \to B \).

For \( d \gg 0 \), let \( \xi : \mathbb{P}(V_1) \to \mathbb{P}^n \) be the holomorphic map defined by the linear system \( |T + dH| \), where \( n = h^0(T + dH) \). Let \( \psi : S \to \mathbb{P}^n \) be the rational map defined by \( |K_S + dF| \) (note that \( h^0(T + dH) = h^0(V_1(d \cdot p)) = h^0(K_S + dF) \), where \( p = \pi(H) \)). Then we have \( |w^*(T + dF)| \cong |\psi^*(K_S + dF)| \) and the following diagram

\[
\begin{array}{ccc}
S & \xrightarrow{w} & \mathbb{P}(V_1) \\
\downarrow{\psi} & & \downarrow{\xi} \\
\mathbb{P}^n & & \mathbb{P}^n
\end{array}
\]
commutes. Hence the indeterminacy points of \( w \) are exactly the base points of the movable part \(|m|\) of \(|K_S + dF|\). So we only need to show that \(|K_S + dF|\) is base point free for \( d \gg 0 \).

(i) If \( F \) is hyperelliptic, then the map \( w : S \rightarrow S_0 \) is of degree 2. Assuming \( S_0 \sim_{alg} 2T + \beta H \) for some \( \beta \), since \( K_S, m, F \) are all nef divisors, we have

\[
4 + 8d = (K_S + dF)^2 = m^2 + m_3 + (K_S + dF)_3 \geq m^2 \geq 2 \deg S_0 = 2(T + dH)^2(2T + \beta H) = 4 + 8d + 2\beta.
\]

It follows that \( \beta \leq 0 \) and that \( \beta = 0 \) if and only if \( K_S^3 = m_3 = F_3 = 0 \). Since \( K_S + dF \) is effective, big and nef, by [21] Chap.I, Lemma 4.6, \( K_S + dF \) is 1-connected. Since \( K_S + dF = m + \mathfrak{z} \), we see that \( m_3 = 0 \) if and only if \( \mathfrak{z} = 0 \). Thus \( \beta = 0 \) if and only if \(|m|\) is base point free and \( \mathfrak{z} = 0 \), i.e. \(|K_S + dF|\) is base point free. Hence it suffices to show that \( \beta \geq 0 \).

By Lemma 2.2, we have either \( \iota = 2 \) or \( \iota = 3 \). Now we discuss the two cases separately.

If \( \iota = 2 \), w.l.o.g. we can assume that \( V_1 = E_{(0)}(2, 1) \oplus N \) with \( N \) a nontrivial torsion line bundle over \( B \) (see Lemma 2.1). Note that \( H^0(2T + \beta H) \cong H^0(\pi^*O_{P(V_1)}(2T + \beta H)) \cong H^0(S^2(V_1)(\beta \cdot p)) \), where \( p = \pi(H) \) is a point on \( B \). Since \( S^2(V_1) = \mathcal{O}_B(\eta_1) \oplus \mathcal{O}_B(\eta_2) \oplus \mathcal{O}_B(\eta_3) \oplus E_{(0)}(2, 1) \oplus N \oplus N^{\otimes 2} \) (here \( \eta_1, \eta_2, \eta_3 \) are the three nontrivial 2-torsion points on \( B \)), we see that \( h^0(2T + \beta H) > 0 \) only if \( \beta \geq -1 \).

If \( \beta = -1 \), then \(|2T - H|\) is nonempty if and only if \( H = H_{\eta_i} := \pi^*\mathcal{O}_B(\eta_i) \) for \( i \in \{1, 2, 3\} \). Since \( h^0(2T - H_{\eta_i}) = h^0(S^2(V_1)(-\eta_i)) = 1, |2T - H_{\eta_i}| \) contains a unique effective divisor \( S_0 \). Note that \( S_0 \) is a cone over a curve \( C \sim_{alg} 2D - E \) lying on \( B^{(2)} \), where \( D \) (resp. \( E \)) is a section (resp. fibre) of \( B^{(2)} \rightarrow B \). Hence \( \varphi(S_0) = C \) is a curve. By Lemma 2.1, we have \( w'(S) = \varphi \circ w(S) = \varphi(S_0) = C \). On the other hand, one sees easily from the definition of the paracanonical map (cf. section 2.1) that \( w'(S) = B^{(2)} \). Hence we get a contradiction. Therefore we have \( \beta \geq 0 \).

If \( \iota = 3 \), then \( V_1 \) is indecomposable. Since \( S_0 \sim_{alg} 2T + \beta H \) and \( S_0 \) is effective, by [9] Theorem 1.13, we have \( \beta \geq 0 \).

(ii) If \( F \) is nonhyperelliptic, then the map \( w : S \rightarrow S_0 \) is birational. Assume that \( S_0 \sim_{alg} \alpha T + \beta H \) for some \( \alpha, \beta \). Since \( F \) is of genus 3, we have \( T(\alpha T + \beta H)H = \alpha = 4 \).

Since \( w \) is birational, we have

\[
4 + 8d = (K_S + dF)^2 \geq m^2 \geq (T + dH)^2(\alpha T + \beta H) = \alpha + 2\alpha \beta + \beta \geq 4 + 8d + \beta.
\]

Thus we have \( \alpha = 4 \) and \( \beta \leq 0 \). Moreover \( \beta = 0 \) if and only if \(|K_S + dF|\) is base point free. So it suffices to show that \( \beta \geq 0 \).

Recall that we have either \( \iota = 2 \) or \( \iota = 3 \).

If \( \iota = 3 \), then \( V_1 \) is indecomposable. By [9] Theorem 1.13, \(|4T + \beta H| \neq \emptyset \) if and only if \( \beta \geq -1 \).

If \( \beta = -1 \), by [9] Theorem 3.2, a general element \( S_t \) in \(|4T - H|\) is a smooth surface with ample canonical divisor. Note that \( S \rightarrow S_0 \) is the minimal resolution of \( S_0 \). Since \( S_0 \) is irreducible, \( K_{S_0} \) is Cartier, and \( K_{S_0}^2 = K_{S_0}^2 = 3 \) \( (K_{S_0} \sim_{alg} T)|_{S_0}, \) so \( K_{S_0}^2 = T^2(4T - H) = 3 \), by [19] Proposition 2.26, we have \( 4 = K_{S_0}^2 \leq K_{S_0}^2 = 3 \), a contradiction. Therefore we have \( \beta \geq 0 \).

If \( \iota = 2 \), we consider the paracanonical system of \( S \). By Lemma 2.1, we have the following commutative diagram:

\[
\begin{array}{ccc}
S & \xrightarrow{w} & S_0 \subset \mathbb{P}(V_1) \\
\downarrow \varphi & & \downarrow \varphi \\
B^{(2)} = \mathbb{P}(W_1) & & \end{array}
\]

Since \( S_0 \sim_{alg} 4T + \beta H \) and \( \varphi \) is the projection induced by the natural inclusion: \( W_1 \hookrightarrow V_1 \), we see that the degree of the map \( \varphi|_{S_0} : S_0 \rightarrow B^{(2)} \) is 4 (fibrewise, it maps a quartic plan curve to a line).
Hence \( \deg w' = 4 \). Now write \( \{ K \} = \{ M \} + Z \), where \( \{ M \} \) is the movable part of \( \{ K \} \) and \( Z \) is the fixed part of \( \{ K \} \). Note that the paracanonical map is defined by \( \{ M \} \).

Let \( \lambda : \tilde{S} \to S \) be the shortest composition of blow-ups such that the movable part \( \{ \tilde{M} \} \) of \( \lambda^* \{ M \} \) is base point free. Then we have \( 4 = K^2_S = M^2 + MZ + K_SZ \geq M^2 \geq \tilde{M}^2 = \deg(\lambda \circ \varphi|_{S_0}) \cdot 1 = 4 \).

In particular, we have \( MZ = 0 \) and \( M^2 = \tilde{M}^2 \). By the 2-connectedness of canonical divisors and the definition of \( \lambda \), we see that \( Z = 0 \) and \( \{ M \} \) is base point free, i.e. \( \{ K \} \) is base point free.

Now we consider the linear system \( |K_S + F_{t_0}|(t_0 \in B) \). For any fixed point \( x \in S \), since \( \{ K_t \}_{t \in B} \) and \( \{ F_t \}_{t \in B} \) are both base point free, if we take general \( t \in B \), then the divisor \( K_t + F_{t_0} - t \in |K_S + F_{t_0}| \) does not contain \( x \). Hence \( |K_S + F_{t_0}| \) is base point free. So \( |K_S + dF| \) is base point free for any \( d \geq 1 \). Therefore the relative canonical map is base point free.

Since the restriction map \( H^0(S, K_S + dF) \to H^0(F, K_F) \) is surjective for \( d \gg 0 \) (cf. Horikawa [15] Lemmas 1 and 2), we get the following

**Corollary 2.4.** \( |K_F| \) is base point free for any fibre \( F \) of \( f \).

Catanese-Franciosi ([10] Corollary 2.5) proved that: if \( C \) is a 2-connected curve of genus \( p_a(C) \geq 1 \) lying on a smooth algebraic surface, then \( |K_C| \) is base point free. However, the converse is not true in general, e.g. if we take \( C \) the union of two distinct smooth fibres of a genus 2 fibration, then \( |K_C| \) is base point free, but \( C \) is not even 1-connected. Now we show that the converse is true in the following case:

**Proposition 2.5.** Let \( f : S \to B \) be a relatively minimal genus 3 fibration and let \( F \) be any fibre of \( f \). If \( |K_F| \) is base point free, then \( F \) is 2-connected.

To prove Proposition 2.5, we need the following four lemmas.

**Lemma 2.6 (Zariski’s Lemma, [3] Chap. III, Lemma 8.2).** Let \( F = \sum n_iC_i \) (\( n_i > 0 \), \( C_i \) irreducible) be a fibre of the fibration \( f : S \to B \). Then we have

(i) \( C_iF = 0 \) for all \( i \);
(ii) If \( D = \sum m_iC_i \), then \( D^2 \leq 0 \), and \( D^2 = 0 \) holds if and only if \( D = rF \) for some \( r \in \mathbb{Q} \).

**Lemma 2.7 ([10] Corollary 2.5).** Let \( C \) be a curve of genus \( p_a(C) \geq 1 \) lying on a smooth algebraic surface. If \( C \) is 1-connected, then the base points of \( |K_C| \) are precisely the points \( x \) such that there exists a decomposition \( C = Y + Z \) with \( YZ = 1 \), where \( x \) is smooth for \( Y \) and \( \mathcal{O}_Y(x) \cong \mathcal{O}_Y(Z) \).

**Lemma 2.8 ([21] Chap. I, Lemmas 2.2 and 2.3).** Assume that \( D \) is a 1-connected divisor on a smooth algebraic surface and let \( D_1 \subset D \) be minimal subject to the condition \( D_1(D - D_1) = 1 \). Then \( D_1 \) is 2-connected and either

(i) \( D_1 \subset D - D_1 \) or
(ii) \( D_1 \) and \( D - D_1 \) have no common components.

**Lemma 2.9 ([21] Chap. I, Proposition 7.2).** Let \( D \) be a 2-connected divisor with \( p_a(D) = 1 \) on a smooth algebraic surface, and let \( \mathcal{L} \) be an invertible sheaf on \( D \) such that \( \deg \mathcal{L}|_C \geq 0 \) for each component \( C \) of \( D \). If \( \deg \mathcal{L}|_D = 1 \), then \( \mathcal{L} \cong \mathcal{O}_D(x) \) with \( x \) a smooth point of \( D \) and \( H^0(\mathcal{L}) \) is generated by one section vanishing only at \( x \).

Now we prove Proposition 2.5.
Proof of proposition 2.5. If $F$ is not 2-connected, then either (i) $F$ is not 1-connected, or (ii) $F$ is 1-connected, but not 2-connected. We discuss the two cases separately.

(i) If $F$ is not 1-connected, then $F$ must be a multiple fibre, i.e. $F = mF'$ with $F'$ 1-connected. Since $K_SF = mK_SF'$ is even, we see $m = 2$. Thus we have $K_SF' = 2$, $F'' = 0$ and $p_a(F'') = 2$. Since $O_{F'}(F'')$ is a nontrivial 2-torsion line bundle on $F'$, by [3] Chapter II Lemma 12.2, $h^1(\omega_{F''}(F'')) = h^0(O_{F'}(-F')) = 0$. Since $\chi(\omega_{F''}(F'')) = \chi(O_S(K_S + F')) - \chi(O_S(K_S + F')) = \frac{K_SF'}{2} = 1$, we know $h^0(\omega_{F'}|_{F'}) = h^0(\omega_{F''}(F'')) = 1$. Hence $|\omega_{F'}|$ has a base point and so does $|K_F|$, a contradiction.

(ii) Assume that $F$ is 1-connected, but not 2-connected. Let $\mathcal{D} \subset F$ realize a minimum of $K_SD$ among the subdivisors such that $D(F - D) = 1$. Let $E := F - D$. By Zariski’s Lemma, we have $D^2 = E^2 = -1$. By Lemma 2.8, $D$ is 2-connected and either

1. $D \subset E$
2. $D$ and $E$ have no common components.

In case (2), since $DE = 1$, $D$ intersects $E$ transversely in one point $x$, which must be a smooth point of both curves. Note that $O_D(x) \cong O_D(E)$. By Lemma 2.7, $x$ is a base point of $|K_F|$, a contradiction.

We study now case (1), i.e. $D \subset E$. Since $D^2 = D(F - E) = -1$ and $K_S(D + E) = 4$, we have $K_SD = 1$ and $K_SE = 3$. In particular, we have $p_a(D) = 1$. If $D$ is irreducible, we can always find a smooth point $x$ on $D$ such that $O_D(x) \cong O_D(E)$ (cf. [14] Chap. IV, Ex. 1.9). By Lemma 2.7, $x$ is a base point of $|K_F|$, a contradiction.

If $D$ is reducible, since $K_S$ is nef and $K_SD = 1$, there is a unique irreducible component $C_0$ of $D$ such that $K_SC_0 = 1$. Write $D - C_0 = \sum_{i \geq 1} m_iC_i$ with $C_i$ distinct irreducible curves, then we have $K_SC_i = 0$ for $i \geq 1$. Hence $C_i (i \geq 1)$ are $(-2)$-curves. Since $D$ is 2-connected, $DC_i = (D - C_i)C_i + C_i^2 \geq 0$. Since $-1 = D^2 = C_0D + \sum_{i \geq 1} m_iC_iD$, we have $-1 \geq C_0D = C_0^2 + C_0(D - C_0) \geq C_0^2 + 2$, thus $C_0^2 \leq -3$. Since $C_0$ is irreducible and $K_SC_0 = 1$, we get $C_0^2 = -3$ and $C_0$ is a smooth rational curve. Thus we get $C_0D = -1, C_iD = 0 (i \geq 1)$, and consequently $C_0E = 1, C_iE = 0 (i \geq 1)$.

Now let $\mathcal{L} := O_D(E)$, so that deg $\mathcal{L}|_C \geq 0$ for any component $C$ of $D$ and deg $\mathcal{L}|_D = 1$. By Lemma 2.9, we have $\mathcal{L} \cong O_D(x)$ with $x$ a smooth point of $D$. Hence $x$ is a base point of $|K_F|$ by Lemma 2.7, a contradiction.

Therefore $F$ is 2-connected.

Remark 2.10. The key point in the above proof for case (ii) is that we can find a 2-connected elliptic cycle (i.e. $K_SD = 1, D^2 = -1$) $D \subset F'$ such that $D(F - D) = 1$ and $\mathcal{L} := O_D(F - D)$ satisfies the condition of Lemma 2.6. Using a similar argument, one can get an analogous result for genus 2 fibrations, i.e.

Let $F$ be any fibre of a relatively minimal genus 2 fibration $f : S \rightarrow B$. If $|K_F|$ is base point free, then $F$ is 2-connected.

Combining Corollary 2.4 and Proposition 2.5, we get the following

Theorem 2.11. Let $S$ be a minimal surface with $p_g = q = 1, K^2 = 4$ and a genus 3 Albanese fibration. Then every Albanese fibre of $S$ is 2-connected.
3 Murakami’s structure theorem for genus 3 hyperelliptic fibrations

In this section, we use Murakami’s structure theorem for genus 3 hyperelliptic fibrations to study surfaces of type $I$. First we recall briefly the notation and idea in Murakami’s structure theorem (see [22] for details).

3.1 Murakami’s structure theorem

We first introduce the admissible 5-tuple $(B, V_1, V_2^+, \sigma, \delta)$ in Murakami’s structure theorem and then explain the structure theorem.

The 5-tuple $(B, V_1, V_2^+, \sigma, \delta)$ is defined as follows:

- $B$: any smooth curve;
- $V_1$: any locally free sheaf of rank 3 over $B$;
- $V_2^+$: any locally free sheaf of rank 5 over $B$;
- $\sigma$: any surjective morphism $S^2(V_1) \to V_2^+$;
- $\delta$: any morphism $(V_2^-)^{\otimes 2} \to A_4$. Here $V_2^-$ and $A_4$ are defined as follows: letting $L := \ker \sigma$, which gives an exact sequence $0 \to L \to S^2(V_1) \xrightarrow{\sigma} V_2^+ \to 0$.

We set $V_2^- := (\det V_1) \otimes L^{-1}$ and define $A_n$ as the cokernel of the injective morphism $L \otimes S^{n-2}(V_1) \to S^n(V_1)$ induced by the inclusion $L \to S^2(V_1)$.

Set now $A := \bigoplus_{n=0}^{\infty} A_n$ and let $S(V_1)$ be the symmetric $O_B$-algebra of $V_1$. Via the natural surjection $S(V_1) \to A$, the algebra structure of $S(V_1)$ induces a graded $O_B$-algebra structure on $A$.

Let $C := \text{Proj}(A)$, $\mathcal{R} := A \oplus (A[-2] \otimes V_2^-)$ and $X := \text{Proj}(\mathcal{R})$.

The 5-tuple $(B, V_1, V_2^+, \sigma, \delta)$ is said to be admissible if:

(i) $C$ has at most RDP’s as singularities;
(ii) $X$ has at most RDP’s as singularities.

Theorem 3.1 (Murakami’s structure theorem, cf. [22] Theorem 1). The isomorphism classes of relatively minimal genus 3 hyperelliptic fibrations with all fibres 2-connected are in one to one correspondence with the isomorphism classes of admissible 5-tuples $(B, V_1, V_2^+, \sigma, \delta)$.

More precisely (cf. [22] Propositions 1 and 2), given a relatively minimal genus 3 hyperelliptic fibration $f : S \to B$ with all fibres 2-connected and setting $V_n := f_* \omega_S^{\otimes n}$, we can define its associated 5-tuple $(B, V_1, V_2^+, \sigma, \delta)$ as follows:

- $B$ is the base curve;
- $V_1 = f_* \omega_S$;
- $V_2^-$: the hyperelliptic fibration $f$ induces an involution of $S$, which acts on $V_2 = f_* \omega_S^{\otimes 2}$. We define $V_2^+$ and $V_2^-$ to be the natural decomposition of $V_2$ into eigensheaves $V_2 = V_2^+ \oplus V_2^-$ with eigenvalues +1 and -1 respectively;
- $\sigma : S^2V_1 \to V_2^+$ is the natural morphism induced by the multiplication structure of the relative canonical algebra $\mathcal{R} = \bigoplus_{n=1}^{\infty} V_n$ of $f$;
- $\delta : (V_2^-)^{\otimes 2} \to V_4^+$ is the natural morphism induced by the multiplication of $\mathcal{R}$.

Moreover, the associated 5-tuple is admissible.
Conversely, given an admissible 5-tuple \((B, V_1, V_2^+, \sigma, \delta)\), we have the graded \(O_B\)-algebras \(S(V_1), A, R\) and varieties \(C, X\). Note that \(C \in |O_{\mathbb{P}(V_1)}(2) \otimes \pi^* L^{-1}|\) is a conic bundle determined by \(\sigma\), and \(X\) is the double cover of \(C\) with branch divisor determined by \(\delta \in \text{Hom}_B((V_2^-) \otimes, A_4) \cong H^0(C, O_C(4) \otimes (\pi|_C)^*(V_2^-)^{-2})\), where \(\pi: \mathbb{P}(V_1) = \text{Proj}(S(V_1)) \to B\) is the natural projection. Let \(\bar{f}: X \to B\) be a surface of type \(H\), and \(\eta\) that \(\deg(V_1, \eta) = 5\) and hyperelliptic Albanese fibration \(f: S \to B\) such that \(g = 3, \iota = 2\) (note that “\(V_1\) decomposable” means \(\iota = 2\) by Lemma 2.2). By Theorem 2.11, every fibre of \(f\) is 2-connected. Hence we can use Murakami’s structure theorem to study \(f\).

For later convenience we fix a group structure on \(B\), denote by 0 its neutral element and by \(\eta_1, \eta_2, \eta_3\) the three nontrivial 2-torsion points.

By Lemma 2.1, we can assume \(V_1 = E_{[0]}(2, 1) \oplus N\), where \(N\) is a nontrivial torsion line bundle over \(B\). Now we use Murakami’s structure theorem to study the order of \(N\). In the notation we introduced in section 2.4, we have:

**Lemma 3.2.** \(L \cong N^{\otimes 2}\).

*Proof.* Since \(\text{det} V_1 = N(0)\), we have \(V_2^- = (\text{det} V_1) \otimes L^{-1} = N(0) \otimes L^{-1}\). From section 2.4, we have \(\text{rank} L = \text{rank} S^2(V_1) - \text{rank} V_2^+ = 1\) and \(\text{deg} L = \frac{1}{2}(4 \text{deg} V_1 + 16(b - 1) - K_S^2) = 0\), i.e. \(L\) is a line bundle of degree 0. Hence \(V_2^-\) is a line bundle of degree 1.

Tensoring the exact sequence

\[0 \to L \to S^2(V_1) \to V_2^+ \to 0\]

with \(N^{-2}\), we get the associated cohomology long exact sequence

\[H^1(L \otimes N^{-2}) \to H^1(S^2(V_1) \otimes N^{-2}) \to H^1(V_2^+ \otimes N^{-2}) \to 0.\]

Since \(h^0(V_2^+ \otimes N^{-2}) = 5\) and \(h^0(V_2^- \otimes N^{-2}) = 1\) (as \(\text{deg}(V_2^- \otimes N^{-2}) = 1\)), we get \(h^0(V_2^+ \otimes N^{-2}) = 4\). By Riemann-Roch for vector bundles over a smooth curve (cf. [3] Chap. II, Theorem 3.1), we have \(h^1(V_2^+ \otimes N^{-2}) = h^0(V_2^+ \otimes N^{-2}) - \text{deg}(V_2^+ \otimes N^{-2}) = 0\) (note that \(\text{deg}(V_2^+ \otimes N^{-2}) = \text{deg}(V_2^+) = \text{deg}(V_2) = 4\)). Since \(h^1(S^2(V_1) \otimes N^{-2}) \geq 1\), we get \(h^1(L \otimes N^{-2}) \geq 1\). Since \(\text{deg}(L \otimes N^{-2}) = 0\), we deduce that \(L \cong N^{\otimes 2}\) (cf. [1] Theorem 5). \(\square\)

**Lemma 3.3.** The exact sequence

\[0 \to L \to S^2(V_1) = (\bigoplus_{i=1}^3 O_B(\eta_i)) \oplus E_{[0]}(2, 1) \otimes N \oplus L \to V_2^+ \to 0\]

splits.
Proof. From the proof of Lemma 3.2, we know that $L \to S^2(V_1)$ induces an isomorphism $H^1(L \otimes N^{-2}) \cong H^1(S^2(V_1) \otimes N^{-2})(\neq 0)$. Thus the composition map

$$0 \to L \to S^2(V_1) \to L$$

is nonzero (here the last map is the natural projection), hence it is an isomorphism. Therefore the above exact sequence splits.

**Remark 3.4.** As in [12] Lemma 6.14 or [23] section 1.2, since the map $L \otimes S^2(V_1) \to S^4(V_1)$ factors as

$$L \otimes S^2(V_1) \to S^2(V_1) \otimes S^2(V_1) \to S^4(V_1),$$

Lemma 3.3 implies that the exact sequence

$$0 \to L \otimes S^2(V_1) \to S^4(V_1) \to \mathcal{A}_1 \to 0$$

also splits (see [23] section 1.2 p.5 for details). Hence the branch curve $\delta \in |\mathcal{O}_{\mathcal{C}}(4) \otimes (\pi_{\mathcal{C}})^*(\det V_1 \otimes L^{-1})^{-2}|$ comes from an effective divisor in $|\mathcal{O}_{\mathcal{P}(V_1)}(4) \otimes \pi^*(\det V_1 \otimes L^{-1})^{-2}|$.

By Lemmas 3.2, 3.3 and Remark 3.4, we have

$$\det V_1 = \mathcal{O}_B(0) \otimes N,$$

$$S^2(V_1) = \left( \bigoplus_{i=1}^3 \mathcal{O}_B(\eta_i) \right) \oplus E_0(2,1) \otimes N \oplus L,$$

$$V_2^{-} = \det V_1 \otimes L^{-1} = \mathcal{O}_B(0) \otimes N^{-1}.$$

**Lemma 3.5.** $L \otimes^2 \cong N \otimes^4 \cong \mathcal{O}_B$.

**Proof.** Recall that $V_1 = E_{[0]}(2,1) \oplus N$, where $N$ is a nontrivial torsion line bundle over $B$. By Atiyah (cf. [1]), we have

$$S^4(V_1) = S^4(E_{[0]}(2,1)) \oplus (S^3(E_{[0]}(2,1)) \otimes N) \oplus (S^2(E_{[0]}(2,1) \otimes N^2) \oplus (E_{[0]}(2,1) \otimes N^3) \oplus N^4,$$

$$S^4(E_{[0]}(2,1)) = \mathcal{O}_B(2 \cdot 0) \oplus \mathcal{O}_B(2 \cdot 0) \oplus \mathcal{O}_B(\eta_1 + \eta_2) \oplus \mathcal{O}_B(\eta_2 + \eta_3) \oplus \mathcal{O}_B(\eta_3 + \eta_1),$$

$$S^3(E_{[0]}(2,1)) = E_{[0]}(2,1)(0) \oplus E_{[0]}(2,1)(0)$$

$$S^2(E_{[0]}(2,1)) = \mathcal{O}_B(\eta_1) \oplus \mathcal{O}_B(\eta_2) \oplus \mathcal{O}_B(\eta_3).$$

By [1] Lemma 15, assuming that $\mathcal{E}$ is an indecomposable vector bundle of rank $r$ and degree $d$ over an elliptic curve $B$, then $h^0(\mathcal{E}) = d$ if $d > 0$; $h^0(\mathcal{E}) = 0$ or 1 if $d = 0$. Moreover by [1] Theorem 5, if $d = 0$, then $h^0(\mathcal{E}) = 1$ if and only if det $\mathcal{E} \cong \mathcal{O}_B$.

Hence we have

$$H^0(S^4(V_1) \otimes \mathcal{O}_B(-2 \cdot 0) \otimes L) = H^0(S^4(E_{[0]}(2,1))(-2 \cdot 0) \otimes L) = H^0((\mathcal{O}_B \oplus \mathcal{O}_B \oplus N_1 \oplus N_2 \oplus N_3) \otimes L),$$

where $N_i := \mathcal{O}_B(\eta_i - 0)$ ($i = 1, 2, 3$).

If $L \otimes^2 \not\cong \mathcal{O}_B$, then $H^0(S^4(V_1) \otimes \mathcal{O}_B(-2 \cdot 0) \otimes L) = 0$ and $|\mathcal{O}_{\mathcal{P}(V_1)}(4) \oplus \pi^*(V_2^-)^{-2}| = 0$, a contradiction. Hence we have $L \otimes^2 \cong \mathcal{O}_B$ and the result follows from Lemma 3.2. 

$\square$
Using Remark 3.4 and Lemma 3.5, we can decide now the linear systems of the conic bundle $C$ and the branch divisor $\delta$ in Murakami’s structure theorem (cf. section 2.4):
\[
C \in |O_{F(V_1)}(2) \otimes \pi^*L^{-1}| \cong |O_{F(V_1)}(2)| \cong \mathbb{P}(H^0(S^2(V_1))), \delta \in |O_{F(V_1)}(4) \otimes \pi^*(V_2^{-2})||C, \text{ where} \{|O_{F(V_1)}(4) \otimes \pi^*(V_2^{-2})| \cong |O_{F(V_1)}(4) \otimes \pi^*O_B(-2 \cdot 0)| \cong \mathbb{P}(H^0(S^4(V_1) \otimes O_B(-2 \cdot 0))\}.
\]

Now we divide surfaces of type $I$ into two types according to the order of $N$.

**Definition 3.6.** Let $S$ be a surface of type $I$ and assume $V_1 = \mathcal{E}_{(0)}(2,1) \oplus N$ with $N$ a nontrivial torsion line bundle (cf. Lemma 2.1). We call $S$ of type $I_1$ if $N^{\otimes 2} \cong O_B$; we call $S$ of type $I_2$ if $N^{\otimes 2} \not\cong O_B$ and $N^{\otimes 4} \cong O_B$.

Denote by $M_{I_1}$ the the subspace of $M_{I,1}^{1,3}$ corresponding to surfaces of type $I_1$, and by $M_{I_2}$ the subspace of $M_{I,1}^{1,3}$ corresponding to surfaces of type $I_2$. Then we have $M_I = M_{I_1} \cup M_{I_2}$.

**Proposition 3.7.** $M_{I_1}$ and $M_{I_2}$ are two disjoint Zariski closed subsets of $M_{I,1}^{1,3}$.

**Proof.** Since $N$ is a torsion line bundle of order 2 for surfaces of type $I_1$, and it is a torsion line bundle of order 4 for surfaces of type $I_2$, we have $M_{I_1} \cap M_{I_2} = \emptyset$. Now we show that $M_{I_1}$ is a Zariski closed subset of $M_{I,1}^{1,3}$. By a similar argument, one can show that $M_{I_2}$ is also a Zariski closed subset of $M_{I,1}^{1,3}$.

By [11] Theorem 24, given two minimal surfaces of general type $S_1, S_2$ with their respective canonical models $S'_1, S'_2$, then $S_1$ and $S_2$ are deformation equivalent if $S'_1$ and $S'_2$ are deformation equivalent. Hence it suffices to show: if $p: S \to T$ is a smooth connected 1-parameter family of minimal surfaces such that for $0 \neq t \in T$, $S_t := p^{-1}(t)$ is a surface of type $I_1$, then $S_0 = p^{-1}(0)$ is also a surface of type $I_1$.

For $0 \neq t \in T$, a general Albanese fibre of $S_t$ is hyperelliptic of genus 3 and $V_t$ is decomposable. Since the genus of the Albanese fibration and the number of the direct summands of $V_t$ are deformation invariants, we see that a general Albanese fibre of $S_0$ is also of genus 3 and $V_1$ of $S_0$ is also decomposable. Moreover, since a general Albanese fibre of $S_t$ is hyperelliptic, a general Albanese fibre of $S_0$ is also hyperelliptic. Otherwise we would get a flat family of irreducible smooth curves $C \to T$, whose central fibre is a nonhyperelliptic curve and whose general fibre is a hyperelliptic curve, a contradiction.

Hence $S_0$ is also a surface of type $I$. Since $M_I = M_{I_1} \cup M_{I_2}$ and $\overline{M_{I_1}} \cap \overline{M_{I_2}} = \emptyset$, we conclude that $S_0$ is a surface of type $I_1$. Therefore $M_{I_1}$ is a Zariski closed subset of $M_{I,1}^{1,3}$. 

4 **Surfaces of type $I_1$**

In this section, we focus on surfaces of type $I_1$. First we show that surfaces of type $I_1$ are in one to one correspondence with some bidouble covers of $B^{(2)}$.

4.1 **Bidouble covers of $B^{(2)}$**

Recall that (cf. [6] Proposition 2.3) a smooth bidouble cover $h: S \to X$ is uniquely determined by the data of effective divisors (sometimes we also call them branch divisors) $D_1, D_2, D_3$ and divisors $L_1, L_2, L_3$ such that $D = D_1 \cup D_2 \cup D_3$ has normal crossings and
\[
2L_i \equiv D_j + D_k, \quad D_k + L_k \equiv L_i + L_j. \quad \{i, j, k\} = \{1, 2, 3\} \tag{4.1}
\]
As Manetti [20] pointed out, these facts are true in a more general situation where $X$ is smooth and $S$ is normal (in this case, each $D_i$ is still reduced, but $D$ may have other singularities except for ordinary double points).

Let $p : B^{(2)} = \{(x,y) | x \in B, y \in B, (x,y) \sim (y,x)\} \to B$ be the natural projection defined by $(x,y) \mapsto x + y$. Set $D_u := \{(u,x) | x \in B\}$ a section of $p$ and $E_u := \{(x,u-x) | x \in B\}$ a fibre of $p$. Now we construct a family of surfaces of type $I_1$ using bidouble covers of $B^{(2)}$.

**Proposition 4.1.** Let $h : S' \to X := B^{(2)}$ be a bidouble cover determined by effective divisors $D_1 \equiv 2D_0, D_2 \equiv 4D_0 - 2E_0, D_3 = 0$, and divisors $L_1 \equiv 2D_0 - E_0, L_2 \equiv D_0, L_3 \equiv 3D_0 - E_0$, such that $S'$ has at most RDP’s as singularities. Then the minimal resolution $\nu : S \to S'$ of $S'$ yields a surface $S$ of type $I_1$.

**Proof.** Let $G = (\mathbb{Z}/2\mathbb{Z})^2 = \{1, \sigma_1, \sigma_2, \sigma_3\}$ be the Galois group of the bidouble cover $h$ and let $R_i$ be the divisorial part of $Fix(\sigma_i)$. Set $R := R_1 \cup R_2 \cup R_3$. Then we have $D_i = h(R_i)$ and $K_{S'} = h^*K_X + R$.

Since $D := D_1 \cup D_2 \cup D_3 \equiv 6D_0 - 2E_0$, $K_X \equiv -2D_0 + E_0$ and $\chi(\mathcal{O}_X) = 0$, by [6] (2.21) and (2.22), we have

$$K_{S'}^2 = (2K_X + D)^2 = 4, \chi(O_{S'}) = 4\chi(O_X) + \frac{1}{2}K_XD + \frac{1}{8}(D^2 + \Sigma_iD_i^2) = 1.$$  

Moreover, for $i = 1, 2$, one has

$$h^i(O_{S'}) = h^i(h_*O_S) = h^i(O_X) + h^i(O_X(-L_1)) + h^i(O_X(-L_2)) + h^i(O_X(-L_3)) = 1.$$  

Since $S'$ has at most RDP’s as singularities and $K_{S'}$ is ample, we see that $S$ is minimal, $K_{S}^2 = K_{S'}^2 = 4$, $p_g(S) = h^2(O_S) = h^2(O_{S'}) = 1$ and $q(S) = h^1(O_S) = h^1(O_{S'}) = 1$.

The bidouble cover $h : S' \to X$ can be decomposed into two double covers $h_1 : Y \to X$ with $h_1_*\mathcal{O}_Y = \mathcal{O}_X \oplus \mathcal{O}_X(-L_2)$, and $h_2 : S' \to Y$ with branch curve $h_2^*D_2$. Note that the general fibre of $Y \to B$ is an irreducible smooth rational curve, which intersects $h_2^*D_2$ at 8 points. Hence the general fibre of $f' := p \circ h : S' \to B$ (and also the general fibre of $f := f' \circ \nu : S \to B$) is irreducible and hyperelliptic of genus 3. By the universal property of the Albanese map and of the Stein factorization, we see that $B = Alb(S)$ and $f$ is the Albanese fibration of $S$. Therefore $S$ has a genus 3 hyperelliptic Albanese fibration.

Since $V_i = f_*\omega_S = f'_*\omega_{S'}$, we have

$$h^0(V_i \otimes \mathcal{O}_B(0 - \eta_i)) = h^0(\omega_{S'} \otimes f'^*\mathcal{O}_B(0 - \eta_i)) = 2.$$  

Since $deg(V_i) = 1$, by [1] Lemma 15, $V_i$ must be decomposable. By [2] Theorem 2 and Lemma 2.1, we know that $\iota = 2$ and $V_1 = E|_0(2,1) \oplus N$ with $N$ a nontrivial torsion line bundle over $B$. Again by [1] Lemma 15, we get $N \cong \mathcal{O}_B(\eta_i - 1)$. Therefore $S$ is a surface of type $I_1$.

Denote by $M_1$ the family of minimal surfaces $S$ obtained as the minimal resolution of a bidouble cover $h : S' \to X = B^{(2)}$ as in Proposition 4.1, and by $\mathcal{M}_1$ the image of $M_1$ in $\mathcal{M}_{I_1}$. Then we have

**Lemma 4.2.** $\dim \mathcal{M}_1 = 4$.

**Proof.** The moduli space of $B^{(2)}$ has dimension 1. Since we have fixed the neutral element 0 for $B$, only a finite subgroup of $Aut(B^{(2)})$ acts on our data, and quotienting by it does not affect
the dimension. Since \( h^0(D_1) = h^0(2D_0) = h^0(S^2E_{[0]}(2, 1)) = 3 \) (cf. Lemma 3.5) and \( h^0(D_2) = h^0(-2K_X) = 2 \) (see [5] Proposition 10), we have

\[
\dim \mathcal{M}_1 = 1 + \dim |D_1| + \dim |D_2| = 1 + 2 + 1 = 4.
\]

\[\square\]

Next we show that the converse of Proposition 4.1 is also true.

(*) For the remainder of this section, we always assume that \( S \) is a surface of type \( I_1 \) and that \( S' \) is the canonical model of \( S \).

**Proposition 4.3.** \( S' \) is a bidouble cover of \( B^{(2)} \) determined by effective divisors \( D_1 \equiv 2D_0, D_2 \equiv 4D_0 - 2E_0, D_3 = 0, \) and divisors \( L_1 \equiv 2D_0 - E_{\eta}, L_2 \equiv D_0, L_3 \equiv 3D_0 - E_{\eta} \).

Since the proof is long, we divide it into three steps in the following three lemmas:

1. (Lemma 4.4) there is a finite morphism \( h : S' \to B^{(2)} \) of degree 4;
2. (Lemma 4.6) the morphism \( h \) is a bidouble cover with branch divisors \( (D_1, D_2, D_3) \) as stated above;
3. (Lemma 4.7) up to an automorphism of \( B^{(2)} \), \( L_1, L_2, L_3 \) satisfy the above linear equivalence relations.

To prove (1), we first introduce the map \( h \). Since the relative canonical map \( w : S \to \mathbb{P}(V_1) \) factors as the composition \( \nu : S \to S' \) (the map contracting \((-2)\)-curves) and \( \mu : S' \to \mathbb{P}(V_1) \). Let \( h := \varphi \circ \mu : S' \to B^{(2)} \). By Lemma 2.1, we have the following commutative diagram

\[
\begin{array}{ccc}
S' & \xrightarrow{\mu} & \mathbb{P}(V_1) \\
\downarrow{h} & & \downarrow{\varphi} \\
B^{(2)} & & \\
\end{array}
\]

where \( w = \mu \circ \nu \) and \( w' = h \circ \nu \). Since \( w \) is a morphism (cf. Lemma 2.3) and the general Albanese fibre of \( S \) is hyperelliptic, we see that \( \mu : S' \to C := \mu(S') \subset \mathbb{P}(V_1) \) is a finite double cover. Moreover \( C \) is exactly the conic bundle in Murakami’s structure theorem. Now we prove (1).

**Lemma 4.4.** The map \( h : S' \to B^{(2)} \) is a finite morphism of degree 4.

**Proof.** Since \( \mu : S' \to C \) is a finite double cover, it suffices to show that \( \varphi|_C : C \to B^{(2)} \) is also a finite double cover. To prove this, we need to study the equation of \( C \subset \mathbb{P}(V_1) \) and use the definition of \( \varphi \).

To get global relative coordinates on fibres of \( \mathbb{P}(V_1) \), first we take a unramified double cover of \( B \). Since \( N \) is a 2-torsion line bundle, we can find a unramified double cover \( \phi : \tilde{B} \to B \) such that \( \phi^*N \cong \mathcal{O}_B \) and \( \phi^*0 = \tilde{0} + \eta \) for some nontrivial 2-torsion point \( \eta \in \tilde{B} \), where \( \tilde{0} \) is the neutral element in the group structure of \( \tilde{B} \), and such that \( \phi(0) = 0 \). Moreover, by [17] Theorem 2.2 and Lemma 2.3, we have \( \phi^*E_{[0]}(2, 1) \cong \mathcal{O}_{\tilde{B}}(x) \oplus \mathcal{O}_{\tilde{B}}(x') \), where \( x, x' \) are two points on \( \tilde{B} \) such that \( \mathcal{O}_{\tilde{B}}(\phi_*(x) - 0) \cong N \) (cf. [13] Chap. 2, Proposition 27) and \( x' = x \oplus \eta \) in the group law of \( \tilde{B} \).
Set \( \tilde{E} := \phi^* (E_{[0]} (2, 1) \oplus N) \) and \( \tilde{C} := \Phi^* C \). Then we have the following commutative diagram:

\[
\begin{array}{ccc}
\tilde{C} \subset \mathbb{P}(\tilde{E}) & \xrightarrow{\varphi} & C \subset \mathbb{P}(V_1) \\
\tilde{X} := \mathbb{P}(\mathcal{O}_B(x) \oplus \mathcal{O}_B(x')) & \xrightarrow{\phi} & X = \mathbb{P}(E_{[0]} (2, 1)) \\
B & \xrightarrow{\phi} & B
\end{array}
\]

where \( \varphi : \mathbb{P}(\tilde{E}) \to \mathbb{P}(\mathcal{O}_B(x) \oplus \mathcal{O}_B(x')) \) is the natural projection induced by the injection \( \mathcal{O}_B(x) \oplus \mathcal{O}_B(x’) \to \tilde{E} = \mathcal{O}_B(x) \oplus \mathcal{O}_B(x’) + \phi^* N \).

Note that the unramified double cover \( \Phi : \tilde{E} \to \mathbb{P}(\mathcal{O}_B(x) \oplus \mathcal{O}_B(x')) \) induces an involution \( T_\eta \) on \( \mathbb{P}(\tilde{E}) \). Let \( J := \{1, T_\eta\} \) be the group generated by \( T_\eta \). Then for any divisor \( D \) on \( \mathbb{P}(V_1) \), we have \( H^0(\mathbb{P}(V_1), D) \cong H^0(\mathbb{P}(\tilde{E})/(\Phi^* D)) \) (the \( J \)-invariant part of \( H^0(\mathbb{P}(\tilde{E})/(\Phi^* D)) \)).

From the commutative diagram above, to show that \( \varphi \mid C \) is a finite double cover, it suffices to show that \( \tilde{\varphi} \mid \tilde{C} : \tilde{C} \to \mathbb{P}(\mathcal{O}_B(0) \oplus \mathcal{O}_B(\eta)) \) is a finite double cover.

Take global relative coordinates \( y_1 : \mathcal{O}_B(x) \to \tilde{E}, \ y_2 : \mathcal{O}_B(x’) \to \tilde{E}, \ y_3 : \mathcal{O}_B \to \tilde{E} \) on fibres of \( \mathbb{P}(\tilde{E}) \). In notation of section 4.2, we have \( C \in |\mathcal{O}_{\mathbb{P}(V_1)}(2)| \), hence \( \tilde{C} \) is a \( J \)-invariant divisor in \( |\mathcal{O}_{\mathbb{P}(\tilde{E})}(2)| \). Therefore the equation of \( \tilde{C} \subset \mathbb{P}(\tilde{E}) \) can be written as

\[
f_1 = a_1 y_1^2 + a_2 y_2^2 + a_3 y_3^2 + a_4 y_1 y_2 + a_5 y_1 y_3 - a_6 y_2 y_3, \quad (4.2)
\]

where \( a_1, a_2 \in H^0(\mathcal{O}_B(2x)), a_3 \in H^0(\mathcal{O}_B), a_4 \in H^0(\mathcal{O}_B(x+x’)), a_5 \in H^0(\mathcal{O}_B(x)) \) and \( a_6 \in H^0(\mathcal{O}_B(x’)) \).

Since the action of \( T_\eta \) is \( y_1 \mapsto y_2, y_2 \mapsto y_1, y_3 \mapsto -y_3 \) and \( \tilde{C} \) is \( J \)-invariant, we see that \( T_\eta^* a_1 = a_2 \) and \( T_\eta^* a_5 = a_6 \).

Since finite double cover is a local property, we can check this locally. Choose a local coordinate \( t \) for the base curve \( B \). Then \( (t, (y_1 : y_2 : y_3)) \) is a local coordinate on \( \mathbb{P}(\tilde{E}) \) and \( (t, (y_1 : y_2)) \) is a local coordinate on \( \tilde{X} \). The action of \( \tilde{\varphi} \) is locally like \( (t, (y_1 : y_2 : y_3)) \mapsto (t, (y_1 : y_2)) \). From the equation of \( \tilde{C} \), to show that \( \tilde{\varphi} \mid \tilde{C} \) is a finite double cover, it suffices to show that \( a_3 \neq 0 \).

If \( a_3 = 0 \), then \( C := \{y_1 = y_2 = 0\} \subset \tilde{C} \). Recall that the branch divisor \( \delta \) of \( u : S' \to C \) is contained in \( |\mathcal{O}_C(4) \otimes \pi^* \mathcal{O}_B(-2 \cdot 0)|_C \cong |\mathcal{O}_{\mathbb{P}(V_1)}(4) \otimes \pi^* \mathcal{O}_B(-2 \cdot 0)|_C \) and it is reduced. Hence \( \delta := \Phi^* \delta \) is a \( J \)-invariant reduced divisor in \( |\mathcal{O}_{\mathbb{P}(\tilde{E})}(4) \otimes \pi^* \mathcal{O}_B(2 \cdot 0 - 2 \eta)|_C \). Since \( 2x = 2x’ = 2\eta = 2 \cdot 0 \) and \( x + x’ = 0 + \eta \neq 2 \cdot 0 \), one sees easily that \( y_1^4, y_1^2 y_2^2, y_2^4 \) is a basis of \( H^0(\mathcal{O}_{\mathbb{P}(\tilde{E})}(4) \otimes \pi^* \mathcal{O}_B(2 \cdot 0 - 2 \eta)) \).

Let \( f_2 \) be the equation of \( \tilde{\delta} \) on \( \tilde{C} \). Then \( f_2 \) has the form \( f_2 = b_1 y_1^4 + b_2 y_1^2 y_2^2 + b_3 y_2^4 \), where \( b_1, b_2, b_3 \in \mathbb{C} \).

Note that \( C \) is contained in \( \tilde{\delta} \). Since the Jacobian matrix of \( (f_1, f_2) \) at any point of \( C \) has the form

\[
\begin{pmatrix}
\frac{\partial f_1}{\partial x} & a_5 y_3 & a_6 y_3 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

which has rank 1, \( \tilde{\delta} \) is singular along \( C \). Hence \( \tilde{\delta} \) is nonreduced, a contradiction.

Therefore \( \varphi : C \to B^{(2)} \) is a finite double cover and \( h \) is a finite morphism of degree 4. \( \Box \)

**Remark 4.5.** From that above Lemma, one sees easily that fibrewise, the composition map \( S' \to C \to B^{(2)} \) is just: a genus 3 hyperelliptic curve \( \to \) a conic curve in \( \mathbb{P}^2 \).
Now we prove (2).

**Lemma 4.6.** The morphism \( h : S' \to B^{(2)} \) is a bidouble cover with branch divisors \( D_1 \equiv 2D_0, D_2 \equiv 4D_0 - 2E_0, D_3 = 0 \).

**Proof.** As in section 4.1, we denote by \( H \) the fibre of \( \pi : \mathbb{P}(V_1) \to B \) and by \( T \) the divisor on \( \mathbb{P}(V_1) \) such that \( \pi_*\mathcal{O}(T) = V_1 \). Similarly, we denote by \( H \) the fibre of \( \tilde{\pi} : \mathbb{P}(\tilde{E}) \to \tilde{B} \) and by \( \tilde{T} \) the divisor on \( \mathbb{P}(\tilde{E}) \) such that \( \tilde{\pi}_*\mathcal{O}(\tilde{T}) = \tilde{E} \). By Lemma 4.4, the ramification divisor of \( \tilde{\phi}|_\tilde{C} \) on \( \tilde{C} \) is defined by

\[
(a_5y_1 + a_6y_2)^2 - 4a_3(a_1y_1^2 + a_2y_2^2 + a_4y_1y_2) = f_1 = 0
\]

and is linearly equivalent to \( 2\tilde{T}|_C \). Thus the ramification divisor of \( \varphi|_C \) on \( C \) is linearly equivalent to \( 2T|_C \) (which is the \( J \)-invariant part of \( 2\tilde{T}|_C \)). From the definition of \( \varphi \), we know that \( D_0 = \varphi(T) \). Hence the branch divisor of \( \varphi|_C \) is linearly equivalent to \( 2D_0 \).

Since 
\[
h^0((4T - 2H_0)|_C) = h^0(\varphi^* (4D_0 - 2E_0)|_C) = h^0(4D_0 - 2E_0) + h^0(3D_0 - 2E_0) \text{ (double cover formula)} = h^0(4D_0 - 2E_0) \text{ (cf. [9] Theorem 1.13)},
\]

we get \( |(4T - 2H_0)|_C| = |(\varphi|_C)^* (4D_0 - 2E_0)| \). Hence the branch divisor of \( \mu : S' \to C \) is invariant under the involution \( \sigma_1 \) of \( \tilde{C} \) induced by the double cover \( \varphi|_C : C \to B^{(2)} \). So \( \sigma_1 \) lifts to an involution \( \sigma_1|_{S'} \) on \( S' \). Hence we get a group \( G := \{1, \sigma_1, \sigma_2, \sigma_3 := \sigma_1 \circ \sigma_2\} \) acting effectively on \( S' \), and the quotient \( S'/G \) is nothing but \( B^{(2)} \).

Therefore \( h : S' \to B^{(2)} \) is a bidouble cover. Moreover, the three branch divisors of \( h \) are \( D_1 = h(\text{Fix}(\sigma_1)) \equiv 2D_0, D_2 = h(\text{Fix}(\sigma_2)) \equiv 4D_0 - 2E_0, D_3 = h(\text{Fix}(\sigma_3)) = 0 \).

Now we prove (3).

**Lemma 4.7.** Up to an automorphism of \( B^{(2)} \), we can assume the data \( (L_1, L_2, L_3) \) of \( h : S' \to X := B^{(2)} \) to be \( L_1 \equiv 2D_0 - E_{\eta_1}, L_2 \equiv D_0, L_3 \equiv 3D_0 - E_{\eta_1} \).

**Proof.** Since 
\[
h^1(\mathcal{O}_S) = h^1(\mathcal{O}_{S'}) = h^1(h_*\mathcal{O}_{S'}) = h^1(\mathcal{O}_X) + h^1(\mathcal{O}_X(-L_1)) + h^0(\mathcal{O}_X(-L_2)) + h^0(\mathcal{O}_X(-L_3)) = 1
\]

and \( h^1(\mathcal{O}_X) = 1 \), we see \( h^1(\mathcal{O}_X(-L_1)) = 0 \). In particular, we have \( L_1 \not\equiv -K_X \). Since \( 2L_1 \equiv D_2 + D_3 \equiv 4D_0 - 2E_0 \), we have \( L_1 \equiv 2D_0 - E_{\eta_1} \) for a nontrivial 2-torsion point \( \eta_1 \in B \). Since \( L_2 + L_3 \equiv D_1 + L_1 \equiv 4D_0 - E_{\eta_1} \), there are three choices for \( (L_2, L_3) \):

(i) \( L_2 \equiv D_0, L_3 \equiv 3D_0 - E_{\eta_1} \);
(ii) \( L_2 \equiv D_{\eta_1}, L_3 \equiv 3D_0 - E_0 \);
(iii) \( L_2 \equiv D_{\eta_1}(j \not= i), L_3 \equiv 3D_0 - E_{\eta_1} \).

Now we show that for fixed \( (D_1, D_2, D_3, L_1) \) above, the three choices (i) (ii) (iii) for \( (L_2, L_3) \) are equivalent up to an automorphism of \( X = B^{(2)} \). The automorphism \( (x, y) \mapsto (x + \eta_1, y + \eta_1) \) on \( X \) fixes fibres of \( X \to B \) and translates \( D_0 \) to \( D_{u + \eta} \). Hence it fixes \( (D_1, D_2, D_3, L_1) \) and maps \( (L_2, L_3) \) in (i) to \( (L_2, L_3) \) in (ii). Similarly, the automorphism \( (x, y) \mapsto (x + \eta_j, y + \eta_j) \) fixes \( (D_1, D_2, D_3, L_1) \) and maps \( (L_2, L_3) \) in (i) to \( (L_2, L_3) \) in (iii).

Therefore, up to an automorphism of \( B^{(2)} \), we can assume the data \( (L_1, L_2, L_3) \) of \( h \) to be \( L_1 \equiv 2D_0 - E_{\eta_1}, L_2 \equiv D_0, L_3 \equiv 3D_0 - E_{\eta_1} \).

Combining Propositions 4.1 and 4.3 together, we get the following
Theorem 4.8. If $h : S' \to B^{(2)}$ is a bidouble cover determined by branch divisors $D_1 \equiv 2D_0, D_2 \equiv 4D_0 - 2E_0, D_3 = 0$, and divisors $L_1 \equiv 2D_0 - E_\eta, L_2 \equiv D_0, L_3 \equiv 3D_0 - E_\eta$, such that $S'$ has at most RDP's as singularities, then the minimal resolution $S$ of $S'$ is a surface of type $I_1$. Conversely, if $S$ is a surface of type $I_1$, then the canonical model $S'$ of $S$ is a bidouble cover of $B^{(2)}$ (where $B = \text{Alb}(S)$) determined by the branch divisors $(D_1, D_2, D_3)$ and divisors $(L_1, L_2, L_3)$ in the respective linear equivalence classes above.

The following corollary follows easily from Lemma 4.2 and Theorem 4.8.

Corollary 4.9. $\mathcal{M}_1 = \mathcal{M}_{I_1}$. In particular, we have $\dim \mathcal{M}_{I_1} = 4$.

4.2 Natural deformations of smooth bidouble covers

Let $S$ be a general surface of type $I_1$. Then we have a smooth bidouble cover $h : S \to X = B^{(2)}$ determined by branch divisors $D_1 \equiv 2D_0, D_2 \equiv 4D_0 - 2E_0, D_3 = 0$, and divisors $L_1 \equiv 2D_0 - E_\eta, L_2 \equiv D_0, L_3 \equiv 3D_0 - E_\eta$. Since $D_3 = 0$, $h$ is a simple bidouble cover (see [7] Definition 22.4). In this subsection, we study the natural deformations of $S$. (see [7] p. 75 for details)

Let $L_1' := D_0, L_2' := 2D_0 - E_\eta$ and let $z_1, z_2$ be the fibre coordinates relative to the two summands of $V := \oplus_{j=1}^2 \mathcal{O}_X(-L_j')$. Let $x_i$ be a section of $\mathcal{O}_X(D_i)$ with $\text{div}(x_i) = D_i$ ($i = 1, 2$). By [7] p. 75, $S$ is a subvariety of $V$ defined by equations:

$$z_1^2 = x_1, \quad z_2^2 = x_2.$$  \hspace{1cm} (4.3)

and a natural deformation $Y$ of $S$ is defined by equations:

$$z_1^2 = x_1 + b_1 z_2, \quad z_2^2 = x_2 + b_2 z_1.$$ \hspace{1cm} (4.4)

with $b_1 \in H^0(\mathcal{O}_X(D_1 - L_2')), b_2 \in H^0(\mathcal{O}_X(D_2 - L_1'))$.

Note that $H^0(\mathcal{O}_X(D_1 - L_2')) = H^0(\mathcal{O}_X(D_2 - L_1')) = 1$. By [9] Theorem 1.13, we have $H^0(\mathcal{O}_X(D_2 - L_1')) = H^0(3D_0 - 2E_0) = 0$, hence we always have $b_2 = 0$.

Denote by $\mathcal{M}_1'$ the family of all surfaces arising as natural deformations of some general surface of type $I_1$, and by $\mathcal{M}_{I_1}$ the image of $\mathcal{M}_1'$ in $\mathcal{M}_{I_1,1}^3$. Let $\overline{\mathcal{M}_{I_1}}$ be the Zariski closure of $\mathcal{M}_1'$ in $\mathcal{M}_{I_1,1}^3$. Then we have

Proposition 4.10. $\dim \mathcal{M}_1' = 5$ and $\mathcal{M}_{I_1}$ is a 4-dimensional subspace of $\overline{\mathcal{M}_1'}$.

Proof. Since there is one parameter for $X = B^{(2)}$ (see Lemma 4.2). From equations (4.4), we see that $\dim \mathcal{M}_1' = 1 + \dim |D_1| + \dim |D_2| + h^0(\mathcal{O}_X(D_1 - L_2')) = 1 + 2 + 1 + 1 = 5$. \hfill \square

Remark 4.11. From equations (4.4), It is easy to see that a natural deformation $Y$ of $S$ is a bidouble cover of $X$ if and only if $b_1 = 0$ (since we always have $b_2 = 0$). By Theorem 4.8, $Y$ has a genus 3 hyperelliptic Albanese fibration if and only if $b_1 = 0$. Since $\dim \mathcal{M}_{I_1} < \dim \mathcal{M}_1'$, we see that a general surface in $\mathcal{M}_1'$ has a genus 3 nonhyperelliptic Albanese fibration.

4.3 $h^1(T_S)$ for a general surface $S$ of type $I_1$

In this section we calculate $h^1(T_S)$ for a general surface $S$ of type $I_1$. Note that for general choices of $D_1 \in |2D_0|$ and $D_2 \in |4D_0 - 2E_0|$, $D_1, D_2$ are both irreducible smooth curves and they intersect
transversally. Hence $S$ is a smooth bidouble cover of $X := B^{(2)}$ determined by effective divisors $(D_1, D_2, D_3)$ and divisors $(L_1, L_2, L_3)$ as in Theorem 4.8.

By Riemann-Roch, we have $h^0(T_S) - h^1(T_S) + h^2(T_S) = 2K_S^2 - 10\chi(O_S) = -2$. Since $h^0(T_S) = 0$, we have $h^1(T_S) = h^2(T_S) + 2 = h^0(\Omega_S \otimes \omega_S) + 2$. By [6] Theorem 2.16, we have

$$H^0(\Omega_S \otimes \omega_S) \cong h^0(\Omega_S \otimes \omega_S) \cong h^0(\Omega_S \otimes \omega_S(\omega_X(L_i))).$$

Hence to calculate $h^1(T_S)$, it suffices to calculate $h^0(\Omega_X(\log D_1, \log D_2, \log D_3))$ and $h^0(\Omega_X(\log D_1) \otimes \omega_X(L_i)) \ (i = 1, 2, 3)$.

**Lemma 4.12.** $\Omega_X = \mathcal{O}_X \oplus \omega_X$.

**Proof.** Since $p : X = B^{(2)} \to B$ is a $\mathbb{P}^1$-bundle, by [14] Chap. III, Ex. 8.4, we have the following exact sequence

$$0 \to \Omega_{X/B} \to (p^*E_{[0]})(2, 1)(-1) \to \mathcal{O}_X \to 0.$$

Since

$$\wedge^2((p^*E_{[0]})(2, 1)(-1)) = \mathcal{O}_X(-2) \otimes p^*\mathcal{O}_B(0) \cong \Omega_{X/B} \otimes \mathcal{O}_X,$$

we see $\Omega_{X/B} \cong \mathcal{O}_X(-2) \otimes p^*\mathcal{O}_B(0) \cong \omega_X$.

On the other hand, we have the exact sequence

$$0 \to p^*\omega_B \to \Omega_X \to \Omega_{X/B} \to 0$$

i.e.

$$0 \to \mathcal{O}_X \to \Omega_X \to \omega_X \to 0.$$ Since $Ext^1(\omega_X, \mathcal{O}_X) \cong H^1(\omega_X^{-1}) = H^1(\mathcal{O}_X(2D_0-E_0))$ and $h^1(\mathcal{O}_X(2D_0-E_0)) = h^1(S^2(E_{[0]}(2, 1)(-1)) = h^0(S^2(E_{[0]}(2, 1)(-1)) = 0$ (cf. proof of Lemma 3.5), we see that $\Omega_X = \mathcal{O}_X \oplus \omega_X$. \hfill $\square$

**Lemma 4.13.** $h^0(\Omega_X(\log D_1, \log D_2, \log D_3) \otimes \omega_X) = 1$.

**Proof.** Let $g : Y \to X$ be the smooth double cover with $g_*\mathcal{O}_Y = \mathcal{O}_X \oplus \mathcal{O}_X(-\epsilon)$, where $\epsilon \equiv -K_X$. Then $Y = B \times B$ (see [5] Proposition 4) and we have $\Omega_Y \cong \mathcal{O}_Y \oplus \mathcal{O}_Y$. Since (see [6] Proposition 3.1)

$$H^0(\Omega_Y \otimes \omega_Y) \cong h^0(\Omega_X(\log D_2) \otimes \omega_X) \oplus h^0(\Omega_X \otimes \omega_X(\epsilon)) \cong H^0(\Omega_X(\log D_2) \otimes \omega_X) \oplus H^0(\Omega_X),$$

$h^0(\Omega_Y \otimes \omega_Y) = h^0(\mathcal{O}_Y \oplus \mathcal{O}_Y) = 2$ and $h^0(\Omega_X) = 1$, we have $h^0(\Omega_X(\log D_2) \otimes \omega_X) = 1$. Since $\Omega_X(\log D_2) \otimes \omega_X \subset \Omega_X(\log D_1, \log D_2, \log D_3) \otimes \omega_X$, we see $h^0(\Omega_X(\log D_1, \log D_2, \log D_3) \otimes \omega_X) \geq 1$.

On the other hand, by [6] (2.12), we have the following exact sequence

$$0 \to \Omega_X \otimes \omega_X \to \Omega_X(\log D_1, \log D_2, \log D_3) \otimes \omega_X \to \mathcal{O}_D(K_X) \to 0.$$ \hfill (4.5)

Note that $h^0(\Omega_X \otimes \omega_X) = 0$. Since $K_XD_1 = -2$, we have $h^0(\mathcal{O}_D(K_X)) = 0$. Since $D_2$ is an irreducible elliptic curve in the rational pencil $\{-2K_X\}$ (cf. [5] Proposition 6), we know that $D_2|D_3 = 0$ and $(K_X + D_2)|D_3 = 0$, thus $K_X|D_3 = 0$ and $h^0(\mathcal{O}_D(K_X)) = 1$. Hence we have $h^0(\Omega_X(\log D_1, \log D_2, \log D_3) \otimes \omega_X) \leq 1$.

Therefore we get $h^0(\Omega_X(\log D_1, \log D_2, \log D_3) \otimes \omega_X) = 1$. \hfill $\square$
Lemma 4.14. $h^0(\Omega_X(logD_1) \otimes \omega_X(L_1)) = 0$.

Proof. Consider the following exact sequences

$$0 \to \mathcal{O}_X(-2D_0 + E_0 - E_{\eta_i}) \to \mathcal{O}_X(E_0 - E_{\eta_i}) \to \mathcal{O}_{D_1}(E_0 - E_{\eta_i}) \to 0$$

$$0 \to \Omega_X(E_0 - E_{\eta_i}) \to \Omega_X(logD_1) \otimes \omega_X(L_1) \to \Omega_{D_1}(E_0 - E_{\eta_i}) \to 0$$

Since $h^0(\mathcal{O}_X(-2D_0 + E_0 - E_{\eta_i})) = h^0(\mathcal{O}_X(-2D_0)(0 - \eta_i)) = 0$ (cf. [3] Chap. 1, Theorem 5.1), $h^1(\mathcal{O}_X(-2D_0 + E_0 - E_{\eta_i})) = h^1(\mathcal{O}_X(E_0)) = 0 = h^0(\mathcal{O}_X(0 - \eta_i)) = 0$, we have $h^0(\mathcal{O}_{D_1}(E_0 - E_{\eta_i})) = 0$. Since moreover $h^0(\mathcal{O}_X(E_0 - E_{\eta_i})) + h^0(\mathcal{O}_X(-2D_0 - E_{\eta_i})) = 0$, we get $h^0(\Omega_X(logD_1) \otimes \omega_X(L_1)) = 0$. □

Lemma 4.15. $h^0(\Omega_X(logD_2) \otimes \omega_X(L_2)) = 1$.

Proof. Let $g : Y \to X$ be the smooth double cover in Lemma 4.13. Since

$$H^0(\Omega_Y \otimes \omega_Y(g^*L_2)) \cong H^0(\Omega_X(logD_2) \otimes \omega_X(L_2)) \oplus H^0(\Omega_X \otimes \omega_X(\epsilon + L_2)),$$

$h^0(\Omega_X \otimes \omega_X(\epsilon + L_2)) = h^0(\mathcal{O}_X(D_0)) = h^0(\mathcal{O}_X(\mathcal{D}_0)) + h^0(\Omega_X(-D_0 + E_0)) = h^0(E_0(2,1)) = 1$, $h^0(\Omega_X(-D_0 + E_0)) = 0$ (cf. [3] Chap. 1, Theorem 5.1) and $h^0(\Omega_Y \otimes \omega_Y(g^*L_2)) = h^0(g_*(\mathcal{O}_Y \otimes \mathcal{O}_X(L_2))) = 2h^0(\mathcal{O}_X(L_2)) + 2h^0(\mathcal{O}_X(-D_0 + E_0)) = 2$, we get $h^0(\Omega_X(logD_2) \otimes \omega_X(L_2)) = 1$. □

Lemma 4.16. $h^0(\Omega_X(logD_3) \otimes \omega_X(L_3)) = 1$.

Proof. Since $D_3 = 0$ and $L_3 \equiv 3D_0 - E_{\eta_i}$, we have

$$h^0(\Omega_X(logD_3) \otimes \omega_X(L_3)) = h^0(\Omega_X(D_{\eta_i})) = h^0(\mathcal{O}_X(D_{\eta_i})) + h^0(\mathcal{O}_X(-D_0 + E_{\eta_i})) = 1.$$

□

Theorem 4.17. We have $h^1(T_S) = 5 = \dim \mathcal{M}_1$. Therefore $\mathcal{M}_1$ is an irreducible component of $\mathcal{M}_{I_1}^{4,3}$.

Proof. By Lemmas 4.13-4.16, we have $h^0(\Omega_S \otimes \omega_S) = h^0(\Omega_X(logD_1, logD_2, logD_3) + \sum_{i=1}^{3}h^0(\Omega_X(logD_i) \otimes \omega_X(L_i))) = 3$. By Reimann-Roch and Serre duality, we have $h^1(T_S) = h^2(T_S) + 2 = h^0(\Omega_S \otimes \omega_S) + 2 = 5$. By Proposition 4.10, we have $h^1(T_S) = 5 = \dim \mathcal{M}_1$. Hence $\mathcal{M}_1$ is an irreducible component of $\mathcal{M}_{I_1}^{4,3}$. □

5 Surfaces of type $I_2$

In this section, we study surfaces of type $I_2$. The method is similar to that for surfaces of type $I_1$. We omit the proof wherever it is similar to that for surfaces of type $I_1$.

5.1 Bidouble covers of $B^{(2)}$

As before, let $B^{(2)}$ be the second symmetric product of an elliptic curve $B$. Let $C_\eta := \{(x, x + \eta_i), x \in B\}$ ($i = 1, 2, 3$) be the (only) three curves homologous to $-K_X$ (see [5] proposition 7). Let $\tau$ be a point on $B$ such that $2\tau = \eta_1 + \eta_2$. 

17
Theorem 5.1. Let $h : S' \to B^{(2)}$ be a bidouble cover determined by effective divisors $D_1 \equiv 2D_0, D_2 \equiv 4D_0 - E_{\eta_1} - E_{\eta_2}, D_3 = 0$, and divisors $L_1 \equiv 2D_0 - E_\tau, L_2 \equiv D_0, L_3 \equiv 3D_0 - E_\tau$ such that $S'$ has at most RDP’s as singularities. Then the minimal resolution $S$ of $S'$ is a surface of type $I_2$. Conversely, for any surface $S$ of type $I_2$, the canonical model $S'$ of $S$ is a bidouble cover of $B^{(2)}$ where $B = \text{Alb}(S)$ determined by the effective divisors $(D_1, D_2, D_3)$ and divisors $(L_1, L_2, L_3)$ above.

Proof. The proof is similar to Theorem 4.8. Note here $h^0(4D_0 - E_{\eta_1} - E_{\eta_2}) = h^0(S^4(E_0)(2, 1)(-\eta_1 - \eta_2)) = 1$ (cf. Lemma 3.5). And $C_{\eta_1} + C_{\eta_2}$ is the unique effective divisor in $|4D_0 - E_{\eta_1} - E_{\eta_2}|$, which is the disjoint union of two smooth elliptic curves (see [5] proposition 7).

Remark 5.2. By Theorem 5.1 and using a similar calculation to Lemma 4.2, we have

$$\dim M_{I_2} = 1 + \dim |D_1| + \dim |D_2| = 1 + 2 + 0 = 3.$$

5.2 Natural deformations of smooth bidouble covers

Let $S$ be a general surface of type $I_2$. Then we have a smooth bidouble cover $h : S \to X = B^{(2)}$ determined by branch divisors $D_1 \equiv 2D_0, D_2 \equiv 4D_0 - E_{\eta_1} - E_{\eta_2}, D_3 = 0$, and divisors $L_1 \equiv 2D_0 - E_\tau, L_2 \equiv D_0, L_3 \equiv 3D_0 - E_\tau$. In this subsection, we study the natural deformations of $S$. The method is the same to that of section 4.2.

Since $D_3 = 0$, $h$ is a simple bidouble cover. Let $L'_1 := D_0, L'_2 := 2D_0 - E_\tau$ and let $z_1, z_2$ be the fibre coordinates relative to the two summands of $V := \oplus_{j=1}^3 \mathcal{O}_X(-L'_j)$. Let $x_i$ be a section of $\mathcal{O}_X(D_i)$ with $\text{div}(x_i) = D_i$ ($i = 1, 2$). By [7] p. 75, $S$ is a subvariety of $V$ defined by equations:

$$z_1^2 = x_1, \quad z_2^2 = x_2. \quad (5.1)$$

and a natural deformation $Y$ of $S$ is defined by equations

$$z_1^2 = x_1 + b_1 z_2, \quad z_2^2 = x_2 + b_2 z_1. \quad (5.2)$$

with $b_1 \in H^0(\mathcal{O}_X(D_1 - L'_2)), b_2 \in H^0(\mathcal{O}_X(D_2 - L'_1))$.

Note that $h^0(\mathcal{O}_X(D_1 - L'_2)) = h^0(E_\tau) = h^0(\mathcal{O}_B(\tau)) = 1$. By [9] Theorem 1.13, we have $H^0(\mathcal{O}_X(D_2 - L'_1)) = H^0(3D_0 - E_{\eta_1} - E_{\eta_2}) = 0$, hence we always have $b_2 = 0$.

Denote by $M'_2$ the family of all surfaces arising as natural deformations of some general surface of type $I_2$, and by $M'_2$ the image of $M'_2$ in $M^{13}_{I_2}$. Let $\overline{M}'_2$ be the Zariski closure of $M'_2$ in $M^{13}_{I_2}$. Then we have

Proposition 5.3. $\dim M'_2 = 4$ and $M'_{I_2}$ is a 3-dimensional subspace of $\overline{M}'_2$.

Proof. Since there is one parameter for $X = B^{(2)}$ (see Lemma 4.2). From equations (5.2), we see that $\dim M'_2 = 1 + \dim |D_1| + \dim |D_2| + h^0(\mathcal{O}_X(D_1 - L'_2)) = 1 + 2 + 0 + 1 = 4$.

Remark 5.4. From equations (5.2), It is easy to see that a natural deformation $Y$ of $S$ is a bidouble cover of $X$ if and only if $b_1 = 0$ (since we always have $b_2 = 0$). By Theorem 5.1, $Y$ has a genus 3 hyperelliptic Albanese fibration if and only if $b_1 = 0$. Since $\dim M_{I_2} < \dim M'_2$, we see that a general surface in $M'_2$ has a genus 3 nonhyperelliptic Albanese fibration.
5.3 $h^1(T_S)$ for a general surface $S$ of type $I_2$

Let $S$ be a general surface of type $I_2$. In this subsection we calculate $h^1(T_S)$. Note that a general surface $S$ of type $I_2$ is a smooth bidouble cover of $X = B^{(2)}$ determined by effective divisors $(D_1, D_2, D_3)$ and divisors $(L_1, L_2, L_3)$ as in Theorem 5.1.

By Riemann-Roch, we have $h^1(T_S) = h^2(T_S) + 2 = h^0(\Omega_S \otimes \omega_S) + 2$. By [6] Theorem 2.16, we have

$$H^0(\Omega_S \otimes \omega_S) \cong H^0(h_*(\Omega_S \otimes \omega_S))$$

$$= H^0(\Omega_X(\log D_1, \log D_2, \log D_3) \otimes \omega_X) \oplus \left( \bigoplus_{i=1}^3 H^0(\Omega_X(\log D_i) \otimes \omega_X(L_i)) \right).$$

**Lemma 5.5.** (1) $h^0(\Omega_X(\log D_1, \log D_2, \log D_3) \otimes \omega_X) = 0$;

(2) $h^0(\Omega_X(\log D_1) \otimes \omega_X(L_1)) = 0$;

(3) $h^0(\Omega_X(\log D_2) \otimes \omega_X(L_2)) = 1$;

(4) $h^0(\Omega_X(\log D_3) \otimes \omega_X(L_3)) = 1$.

**Proof.** (1) Consider the exact sequence

$$0 \to \Omega_X \otimes \omega_X \to \Omega_X(\log D_1, \log D_2, \log D_3) \otimes \omega_X \to \mathcal{O}_{D_1}(K_X) \oplus \mathcal{O}_{T_{\eta_1}}(K_X) \oplus \mathcal{O}_{T_{\eta_2}}(K_X) \to 0.$$ 

Since $K_X D_1 = -2$, we have $h^0(\mathcal{O}_{D_1}(K_X)) = 0$. From the exact sequence

$$0 \to \mathcal{O}_X(E_0 - E_{\eta_1}) \to \mathcal{O}_X(K_X) \to \mathcal{O}_{T_{\eta_1}}(K_X) \to 0,$$

$h^0(\mathcal{O}_X(E_0 - E_{\eta_1})) = h^1(\mathcal{O}_X(E_0 - E_{\eta_1})) = 0$ and $h^0(\mathcal{O}_X(K_X)) = 0$, we see $h^0(\mathcal{O}_{T_{\eta_1}}(K_X)) = 0$.

Similarly, we have $h^0(\mathcal{O}_{T_{\eta_2}}(K_X)) = 0$ and hence $h^0(\mathcal{O}_{D_1}(K_X) \oplus \mathcal{O}_{T_{\eta_1}}(K_X) \oplus \mathcal{O}_{T_{\eta_2}}(K_X)) = 0$. Since moreover $h^0(\Omega_X \otimes \omega_X) = 0$, we get $h^0(\Omega_X(\log D_1, \log D_2, \log D_3) \otimes \omega_X) = 0$.

(2) The proof is similar to Lemma 4.14, just replace $\eta_i$ by $\tau$.

(3) Consider the smooth double cover $g : Y \to X$ with $g_*\mathcal{O}_Y = \mathcal{O}_X \oplus \mathcal{O}_X(-\epsilon)$, where $\epsilon \equiv 2D_0 - E_\tau$. Since $K_Y \equiv g^*(K_X + \epsilon) \equiv g^*(E_0 - E_\tau)$ and $h^0(K_Y) = h^0(K_X) + h^0(K_X + \epsilon) = 0$, we have $K_Y \neq 0$ and $4K_Y \equiv 0$. Moreover, we have $q(Y) = h^1(\mathcal{O}_Y) = h^1(\mathcal{O}_X) + h^1(\mathcal{O}_X(-\epsilon)) = 1$. Thus $Y$ is a bielliptic surface and its Albanese map $\alpha_Y : Y \to B$ is a smooth map. Hence $\Omega_{Y/B}$ is a locally free sheaf and we have the following exact sequence:

$$0 \to \alpha_Y^*\omega_B \to \Omega_Y \to \Omega_{Y/B} \to 0.$$ 

Since $\omega_B \cong \mathcal{O}_B$, we have $\alpha_Y^*\omega_B \cong \omega_Y$ and thus $\Omega_{Y/B} \cong \omega_Y$. Now the above exact sequence becomes

$$0 \to \mathcal{O}_Y \to \Omega_Y \to \omega_Y \to 0.$$ 

Tensoring this exact sequence with $\omega_Y(g^*D_0)$, we get

$$0 \to \mathcal{O}_Y(g^*(D_0 + E_0 - E_\tau)) \to \Omega_Y \otimes \omega_Y(g^*D_0) \to \mathcal{O}_Y(g^*(D_0 + 2E_0 - 2E_\tau)) \to 0.$$ 

Since $h^0(\mathcal{O}_Y(g^*(D_0 + E_0 - E_\tau)) = h^0(\mathcal{O}_Y(g^*(D_0 + 2E_0 - 2E_\tau))) = 1$ and $h^1(\mathcal{O}_Y(g^*(D_0 + E_0 - E_\tau))) = 0$, we get $h^0(\Omega_Y \otimes \omega_Y(g^*D_0)) = 2$.

On the other hand, by [6] Proposition 3.1, we have $H^0(\Omega_Y \otimes \omega_Y(g^*D_0)) \cong H^0(\Omega_X(\log D_2) \otimes \omega_X(D_0)) \oplus H^0(\Omega_X \otimes \omega_X(\epsilon + D_0)).$ Since $h^0(\Omega_X \otimes \omega_X(\epsilon + D_0)) = h^0(\mathcal{O}_X(D_0 + E_0 - E_\tau)) + h^0(\omega_X(D_0 + E_0 - E_\tau)) = 1$, we get $h^0(\Omega_X(\log D_2) \otimes \omega_X(L_2)) = h^0(\Omega_X(\log D_2) \otimes \omega_X(D_0)) = 1$.

(4) $h^0(\Omega_X(\log D_3) \otimes \omega_X(L_3)) = h^0(\mathcal{O}_X(D_0 + E_0 - E_\tau)) + h^0(\omega_X(D_0 + E_0 - E_\tau)) = 1$. 

\[\square\]
Theorem 5.6. We have $h^1(T_S) = 4 = \dim \mathcal{M}'_2$. Therefore $\overline{\mathcal{M}}'_2$ is an irreducible component of $\mathcal{M}^{4,3}_{1,1}$.

Proof. By Lemma 5.5, we have $h^0(\Omega_S \otimes \omega_S) = h^0(\Omega_X(\log D_1, \log D_2, \log D_3) \otimes \omega_X) + \sum_{i=1}^{3} h^0(\Omega_X(\log D_i) \otimes \omega_X(L_i)) = 2$. By Riemann-Roch, Serre duality and Proposition 5.3, we have $h^1(T_S) = h^2(T_S) + 2 = h^0(\Omega_S \otimes \omega_S) + 2 = 4 = \dim \mathcal{M}'_2$. Hence $\overline{\mathcal{M}}'_2$ is an irreducible component of $\mathcal{M}^{4,3}_{1,1}$.

Acknowledgements. The author was sponsored by China Scholarship Council “High-level university graduate program” (No.201506010011).

The author would like to thank his advisor, Professor Fabrizio Catanese at Universität Bayreuth for suggesting this research topic, for a lot of inspiring discussion with the author and for his encouragement to the author. The author would also like to thank his domestic advisor, Professor Jinxing Cai at Peking University for his encouragement and some useful suggestions. The author is grateful to Binru Li for a lot of helpful discussion. Thanks also goes to Stephen Coughlan and Andreas Demleitner for improving the English.

References

[1] Atiyah, M. F., Vector bundles over an elliptic curve. Proc. London Math. Soc. (3) 7 (1957), 414-452.

[2] Barja, M.A.; Zucconi, F., A note on a conjecture of Xiao. J. Math. Soc. Japan 52 (2000), no. 3, 633-635.

[3] Barth, W., Hulek, K., Peters, C., Ven, A. van de., Compact complex surfaces. Second edition. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], 4. Springer-Verlag, Berlin, 2004. xii+436 pp.

[4] Bombieri, E. Canonical models of surfaces of general type. Inst. Hautes Études Sci. Publ. Math. No. 42 (1973), 171-219.

[5] Catanese, F., On a class of surfaces of general type. In Algebraic Surfaces, CIME, Liguori (1981), 269-284.

[6] Catanese, F., On the moduli space of surfaces of general type. J.Differential Geometry, 19(1984), 483-515.

[7] Catanese, F. Moduli of algebraic surfaces. Theory of moduli (Montecatini Terme, 1985), I-83, Lecture Notes in Math., 1337, Springer, Berlin, 1988.

[8] Catanese, F.; Ciliberto, C., Surfaces with $p_g = q = 1$. Problems in the theory of surfaces and their classification (Cortona,1988), 49-79, Sympos. Math., XXXII, Academic Press, London, 1991.

[9] Catanese, F.; Ciliberto, C., Symmetric products of elliptic curves and surfaces of general type with $p_g = q = 1$. J.Algebraic Geom. 2 (1993), no. 3, 389-411.

[10] Catanese, F.; Franciosi, M., Divisors of small genus on algebraic surfaces and projective embeddings. In: Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry, Ramat Gan, 1993, in: Israel Math. Conf. Proc., vol. 9, Bar-Ilan Univ., Ramat Gan, 1996, pp. 109-140.
11. Catanese, F., *A superficial working guide to deformations and moduli*. Handbook of moduli. Vol. I, 161-215, Adv. Lect. Math. (ALM), 24, Int. Press, Somerville, MA, 2013.

12. Catanese, F.; Pignatelli, R., *Low genus fibrations*. I. Ann. Sci. Ecole Norm. Sup. (4) 39 (2006), No. 6, 1011-1049.

13. Friedman, R., *Algebraic surfaces and holomorphic vector bundles*. Universitext. Springer-Verlag, New York, 1998. x+328 pp.

14. Hartshorne, R., *Algebraic geometry*. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977. xvi+496 pp.

15. Horikawa, E., *On algebraic surfaces with pencils of curves of genus 2*. In: Complex Analysis and Algebraic Geometry, Iwanami Shoten, Tokyo, 1977, pp. 79-90.

16. Horikawa E., *Algebraic surfaces of general type with small $c^2$*, V. J. Fac. Sci. Univ. Tokyo Sect. IAMath. 28 (3) (1981) 745-755.

17. Ishida, H., *Bounds for the relative Euler-Poincaré characteristic of certain hyperelliptic fibrations*. Manuscripta.math 118, 467-483(2005).

18. Kodaira, K., *Pluricanonical systems on algebraic surfaces of general type*. J. Math. Soc. Japan 20 (1968) 170-192.

19. Kollár, J.; Shepherd-Barron, N. I., *Threefolds and deformations of surface singularities*. Invent. Math. 91 (1988), no. 2, 299-338.

20. Manetti, M., *On Some Components of Moduli Space of Surfaces of General Type*. Comp. Math. 92 (1994) 285-297.

21. Mendes Lopes, M., *The relative canonical algebra for genus three fibrations*. PhD thesis, University of Warwick, 1989.

22. Murakami, M., *Notes on hyperelliptic fibrations of genus 3, I*. Preprint, arXiv:1209.6278.

23. Pignatelli, R., *Some (big) irreducible components of the moduli space of minimal surfaces of general type with $p_g = q = 1$ and $K^2 = 4$. Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. 20 (2009), no. 3, 207-226.

24. Polizzi, F., *Standard isotrivial fibrations with $p_g = q = 1$. J. Algebra 321 (2009), 1600-1631.*

25. Rito, C., *Involutions on surfaces with $p_g = q = 1$. Collect. Math., (2010), no. 1, 81-106.*

26. Rito, C., *On equations of double planes with $p_g = q = 1$. Math. Comp.,79 (2010), no. 270, 1091-1108.*

27. Šafarevič, I. R.; Averbuh, B. G.; Vainberg, Ju. R.; Žižčenko, A. B.; Manin, Ju. I.; Moišezon, B. G.; Tjurina, G. N.; Tjurin, A. N. *Algebraic surfaces. (Russian) Trudy Mat. Inst. Steklov. 75 (1965) 1-215.*
School of Mathematical Sciences, Peking University, Yiheyuan Road 5, Haidian District, Beijing 100871, People’s Republic of China
E-mail address: 1201110022@pku.edu.cn

Lehrstuhl Mathematik VIII, Universität Bayreuth, NW II, Universitätsstr. 30, 95447 Bayreuth, Germany
E-mail address: songbo.ling@uni-bayreuth.de