A characterization of cellular motivic spectra

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Abstract

Let \( \alpha : \mathcal{C} \to \mathcal{D} \) be a symmetric monoidal functor from a stable presentable symmetric monoidal category \( \mathcal{C} \) compactly generated by the tensor unit to a stable presentable symmetric monoidal category \( \mathcal{D} \) with compact tensor unit. Let \( \beta : \mathcal{D} \to \mathcal{C} \) be a right adjoint of \( \alpha \) and \( X : \mathcal{B} \to \mathcal{D} \) a symmetric monoidal functor starting at a small rigid symmetric monoidal category \( \mathcal{B} \).

We construct a symmetric monoidal equivalence between modules in the category of functors \( \mathcal{B} \to \mathcal{C} \) over the \( E_\infty \)-algebra \( \beta \circ X \) and the full subcategory of \( \mathcal{D} \) compactly generated by the essential image of \( X \).

Especially for every motivic \( E_\infty \)-ring spectrum \( A \) we obtain a symmetric monoidal equivalence between the category of cellular motivic \( A \)-module spectra and modules in the category of functors \( QS \) to spectra over some \( E_\infty \)-algebra, where \( QS \) denotes the 0th space of the sphere spectrum.

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1 Introduction

In motivic homotopy theory cellular motivic spectra, i.e. those motivic spectra that can be built from the smash powers of \( \mathbb{P}^1 \) by iterately taking coproducts, shifts and cofibers, play a prominent role as virtually all motivic spectra representing cohomology theories are cellular and cellular motivic spectra are appreciated for their good behaviour.
Examples of cellular motivic spectra are $\text{MGL}$, $\text{KGL}$, $\text{KQ}$ representing algebraic cobordism, $K$-theory and hermitian $K$-theory and also the motivic cohomology spectrum $\text{MZ}$ if one inverts the characteristic of each residue field of the base scheme, which is prime.

In stable homotopy theory it is a well-known fact that a map of spectra is an equivalence if it induces an isomorphism on all stable homotopy groups.

The analogous statement, where one replaces spectra by motivic spectra and stable homotopy groups by bigraded stable homotopy groups, fails unless one expects the motivic spectra to be cellular.

Consequently it is a goal of stable motivic homotopy theory to have a good understanding of the category of cellular motivic spectra.

More generally one considers $X$-cellular motivic $A$-module spectra for some $E_\infty$-ring spectrum $A$ and some tensorinvertible motivic $A$-modul spectrum $X$, i.e. those $A$-module spectra that can be built from the tensorpowers of $X$ by iterately taking coproducts, shifts and cofibers, which we simply call cellular motivic $A$-module spectra if $X$ is the free $A$-module on $P^1$.

In [3] Spitzweck shows (in the setting of model categories) that the category of cellular motivic $\text{MGL}$- respectively $\text{MZ}$-module spectra are canonically symmetric monoidal equivalent to the category of modules in $\mathbb{Z}$-graded spectra over some $E_\infty$-algebra.

Denote $QS$ the $0$th space of the sphere spectrum with the structure of a grouplike $E_\infty$-space coming from its infinite loop space structure.

$QS$ is the free grouplike $E_\infty$-space generated by a point so that $\pi_0(QS) = \mathbb{Z}$ and evaluation at $1 \in QS$ induces an equivalence $E_\infty(\text{Cat}_\infty)(QS, \text{Pic}(\mathcal{D})) \simeq \text{Pic}(\mathcal{D})^2$ between the maximal subspace in $\mathcal{D}$ consisting of the tensorinvertible objects and the space of symmetric monoidal functors $QS \to \text{Pic}(\mathcal{D})$.

Consequently the free $\text{MGL}$ respectively $\text{MZ}$-module $P^1 \wedge \text{MGL}$ respectively $P^1 \wedge \text{MZ}$ on the tensorinvertible motivic spectrum $P^1$ corresponds to a symmetric monoidal functor $\Psi$ from $QS$ to the category of motivic $\text{MGL}$- respectively $\text{MZ}$-modules.

To construct the symmetric monoidal equivalence between the category of cellular motivic $\text{MGL}$- respectively $\text{MZ}$-module spectra and the category of modules in $\mathbb{Z}$-graded spectra, Spitzweck needs to show that $\Psi$ extends to a symmetric monoidal functor from $\mathbb{Z}$ to the category of motivic $\text{MGL}$- respectively $\text{MZ}$-modules along the canonical map $QS \to \pi_0(QS) = \mathbb{Z}$ of grouplike $E_\infty$-spaces.

Having this presentation of the category of cellular motivic $\text{MGL}$-module spectra he is able to define the motivic fundamental group, a differential graded hopf algebra, whose category of perfect representations is the category of compact cellular motivic $\text{MGL}$-module spectra if the base scheme is nice enough.

Inspired by his results we give a presentation of the category of $X$-cellular motivic $A$-module spectra for an arbitrary motivic $E_\infty$-ring spectrum $A$ and an arbitrary tensorinvertible $A$-module $X$ generalizing Spitzweck’s result.

Moreover to characterize the category of $X$-cellular motivic $A$-module spectra we don’t need to require that the symmetric monoidal functor $\Phi$ from $QS$ to motivic $A$-module spectra corresponding to the tensorinvertible $A$-module $X$ extends to $\mathbb{Z}$ but get a more complicated presentation of $X$-cellular motivic $A$-module spectra as modules over some $E_\infty$-algebra in $QS$-graded spectra (instead of $\mathbb{Z}$-graded spectra).

But if $\Phi$ extends to $\mathbb{Z}$, as it does for $A = \text{MGL}$ and $X = P^1 \wedge \text{MGL}$ respectively
$A = MZ$ and $X = P^1 \wedge MZ$, we get a presentation of X-cellular motivic $A$-module spectra as modules over some $E_\infty$-algebra in $\mathbb{Z}$-graded spectra.

More generally, if $\Phi$ extends to higher connected covers $T$ of $QS$, we get a presentation of X-cellular motivic $A$-module spectra as $E_\infty$-modules in $T$-graded spectra.

Given a stable presentable symmetric monoidal category $\mathcal{D}$ with compact tensorunit and a dualizable object $X$ of $\mathcal{D}$ we define the category $\text{Cell}_X(\mathcal{D})$ of $X$-cellular objects of $\mathcal{D}$ to be the full subcategory of $\mathcal{D}$ compactly generated by the tensorpowers $X^\otimes n \otimes (X^\vee)^\otimes m$, $n, m \in \mathbb{N}$ (Definition 3.6).

If we choose $\mathcal{D}$ to be the category of motivic $A$-module spectra for some $E_\infty$-ring spectrum $A$ and $X$ to be $A \wedge P^1$, the category $\text{Cell}_X(\mathcal{D})$ coincides with the category of cellular motivic $A$-module spectra.

Especially for $A = MR$ for some commutative ring $R$, the category $\text{Cell}_X(\mathcal{D})$ coincides with the category of Tate-motives with $R$-coefficients.

As $\text{Cell}_X(\mathcal{D})$ is closed under the tensorproduct of $\mathcal{D}$, the symmetric monoidal structure of $\mathcal{D}$ restricts to $\text{Cell}_X(\mathcal{D})$.

We construct a canonical symmetric monoidal equivalence between $\text{Cell}_X(\mathcal{D})$ and modules over some $E_\infty$-algebra in the Day-convolution symmetric monoidal structure on functors from $QS$ to the category of spectra. (Theorem 3.5)

Moreover we give relative versions of this result:

- We may replace the category of spectra by any stable presentable symmetric monoidal category $\mathcal{C}$ compactly generated by the tensorunit if we assume that $\mathcal{D}$ receives a symmetric monoidal functor $\alpha : \mathcal{C} \to \mathcal{D}$ from $\mathcal{C}$ that admits a right adjoint $\beta : \mathcal{D} \to \mathcal{C}$.

  An example for $\alpha : \mathcal{C} \to \mathcal{D}$ is the canonical functor from the category of $H(R)$-module spectra for some commutative ring $R$ to the category of motivic $H(R)$-module spectra, where $H(R)$ denotes the Eilenberg-Mac Lane spectrum of $R$.

- We may replace the tensorinvertible object $X$ of $\mathcal{D}$ (corresponding to a symmetric monoidal functor $QS \to \mathcal{D}$ by the universal property of $QS$) by an arbitrary symmetric monoidal functor $X : \mathcal{B} \to \mathcal{D}$ starting at a small grouplike symmetric monoidal category $\mathcal{B}$.

  One could take $\mathcal{B}$ to be the direct sum of copies of $QS$ indexed by some set $I$, in which case a symmetric monoidal functor $\mathcal{B} \to \mathcal{D}$ corresponds to an $I$-indexed family of tensorinvertible objects of $\mathcal{D}$ and $\text{Cell}_X(\mathcal{D})$ denotes the full subcategory of $\mathcal{D}$ compactly generated by the tensorpowers of the elements of the $I$-indexed family.

  In this more general situation we define $\text{Cell}_X(\mathcal{D})$ to be the full subcategory of $\mathcal{D}$ compactly generated by the essential image of $X$. (Definition 3.6)

- For $0 < k \in \mathbb{N} \cup \{ \infty \}$ denote $E_k$ the $k$-th little cubes operad.

  We may assume that $\mathcal{C}$ is only a $k+1$-monoidal presentable category instead of symmetric monoidal one, $\mathcal{D}$ is only an $E_k$-algebra in presentable left $\mathcal{C}$-modules, which we call a presentable $k$-monoidal left $\mathcal{C}$-module, instead of
a presentable symmetric monoidal left \( \mathcal{C} \)-modul (corresponding to a left adjoint symmetric monoidal functor \( \mathcal{C} \to \mathcal{D} \)) and that \( \mathcal{B} \) and the functor \( X : \mathcal{B} \to \mathcal{D} \) are only \( k \)-monoidal to obtain a canonical \( k - 1 \)-monoidal equivalence between the \( k \)-monoidal category \( \text{Cell}_X(\mathcal{D}) \) and the \( k - 1 \)-monoidal category of modules over some \( E_k \)-algebra in the \( k \)-monoidal category of functors \( \mathcal{B} \to \mathcal{C} \) endowed with the Day-convolution \( k \)-monoidal structure.

We give an explicite description of the \( E_k \)-algebra, over which the modules are taken:

\( E_k \)-algebras in the Day-convolution \( k \)-monoidal structure on \( \text{Fun}(\mathcal{B}, \mathcal{C}) \) correspond to lax \( k \)-monoidal functors \( \mathcal{B} \to \mathcal{C} \).

Being a composition of lax \( k \)-monoidal functors the functor \( \beta \circ X : \mathcal{B} \to \mathcal{D} \to \mathcal{C} \) has the structure of a lax \( k \)-monoidal functor corresponding to an \( E_k \)-algebra structure on \( \beta \circ X \).

This \( E_k \)-algebra structure on \( \beta \circ X \) is exactly the \( E_k \)-algebra structure, over which the modules are taken.

So all together we obtain the following statement (theorem 3.5):

**Theorem 1.1.** Let \( 0 < k \in \mathbb{N} \cup \{ \infty \} \) and \( \mathcal{C} \) a stable presentable \( k + 1 \)-monoidal category compactly generated by the tensorunit.

Let \( \mathcal{D} \) be a stable presentable \( k \)-monoidal left \( \mathcal{C} \)-modul compactly generated by the tensorunit and \( X : \mathcal{B} \to \text{Pic}(\mathcal{D}) \subset \mathcal{D} \) a \( k \)-monoidal functor starting at a small rigid \( k \)-monoidal category \( \mathcal{B} \).

Denote \( \alpha : \mathcal{C} \to \mathcal{D} \) the unique \( k \)-monoidal left \( \mathcal{C} \)-linear functor with right adjoint \( \beta : \mathcal{D} \to \mathcal{C} \).

There is a canonical \( k - 1 \)-monoidal equivalence between \( \text{Cell}_X(\mathcal{D}) \) and the category of left modules \( \text{Mod}_{\beta \circ X}(\text{Fun}(\mathcal{B}, \mathcal{C})) \) in the category \( \text{Fun}(\mathcal{B}, \mathcal{C}) \) of functors \( \mathcal{B} \to \mathcal{C} \) over the \( E_k \)-algebra \( \beta \circ X : \mathcal{B} \to \mathcal{D} \to \mathcal{C} \).

For \( k = n = \infty \) we obtain corollary 3.5.

**Corollary 1.2.** Let \( \alpha : \mathcal{C} \to \mathcal{D} \) be a symmetric monoidal functor from a stable presentable symmetric monoidal category \( \mathcal{C} \) compactly generated by the tensorunit to a stable presentable symmetric monoidal category \( \mathcal{D} \) with compact tensorunit. Let \( \beta : \mathcal{D} \to \mathcal{C} \) be a right adjoint of \( \alpha \) and \( X : \mathcal{B} \to \mathcal{D} \) a symmetric monoidal functor starting at a small rigid symmetric monoidal category \( \mathcal{B} \).

There is a canonical symmetric monoidal equivalence between \( \text{Cell}_X(\mathcal{D}) \) and the category of modules \( \text{Mod}_{\beta \circ X}(\text{Fun}(\mathcal{B}, \mathcal{C})) \) in the category \( \text{Fun}(\mathcal{B}, \mathcal{C}) \) of functors \( \mathcal{B} \to \mathcal{C} \) over the \( E_k \)-algebra \( \beta \circ X : \mathcal{B} \to \mathcal{D} \to \mathcal{C} \).

Taking \( \mathcal{B} = \text{Bord} \) theorem 3.5 provides for every object \( X \) of \( \text{Pic}(\mathcal{D}) \) a canonical symmetric monoidal equivalence

\[ \text{Cell}_X(\mathcal{D}) \simeq \text{Mod}_{\beta \circ X}(\text{Fun}(\text{Bord}, \mathcal{C})). \]

Taking \( \mathcal{B} = \text{QS} \) theorem 3.5 provides for every object \( X \) of \( \text{Pic}(\mathcal{D}) \) a canonical symmetric monoidal equivalence

\[ \text{Cell}_X(\mathcal{D}) \simeq \text{Mod}_{\beta \circ X}(\text{Fun}(\text{QS}, \mathcal{C})). \]
Especially one can take $C$ to be the category of spectra $\text{Sp}$.

More generally one can take $C$ to be the category $\text{Mod}_A(\text{Sp})$ of $A$-module spectra over an $E_\infty$-ring spectrum $A$ and $D$ to be an $A$-linear presentable symmetric monoidal category with compact tensor unit.

For $B = \text{Pic}(D)$ and $X : B \to \text{Pic}(D)$ the identity we obtain a canonical symmetric monoidal equivalence

$$\text{Cell}_{\text{id}}(\text{Pic}(D)) \simeq \text{Mod}_{[\text{id}](\text{Pic}(D))}(\text{Fun}(\text{Pic}(D), C)),$$

where $\text{Cell}_{\text{id}}(\text{Pic}(D))$ is the full subcategory of $D$ compactly generated by $\text{Pic}(D)$.

Applied to stable motivic homotopy theory theorem 3.5 yields the following corollaries:

Denote $M_{\text{sp}}$ the category of motivic spectra. For $D = \text{Mod}_A(M_{\text{sp}})$ the category of motivic $A$-module spectra over some motivic $E_\infty$-ring spectrum $A$ and $X = A \wedge P^1$ we obtain a canonical symmetric monoidal equivalence

$$\text{Cell}_{A \wedge P^1}(\text{Mod}_A(M_{\text{sp}})) \simeq \text{Mod}_{[A, A \wedge P^1, -]_{\text{Sp}}}(\text{Fun}(\text{QS}, \text{Sp})).$$

where $\text{Cell}_{A}(\text{Mod}_A)$ is the category of cellular motivic $A$-module spectra and $[-, -]_{\text{Sp}}$ the spectral hom.

Especially for $A = MZ$ we have a canonical symmetric monoidal equivalence

$$\text{Cell}_{MZ \wedge P^1}(\text{Mod}_{MZ}(M_{\text{sp}})) \simeq \text{Mod}_{[MZ, MZ \wedge P^1, -]_{\text{Sp}}}(\text{Fun}(\text{QS}, \text{Sp})).$$

where $\text{Cell}_{MZ \wedge P^1}(\text{Mod}_{MZ})$ is the category of Tate motives.

Let $X \in \text{Mod}_{H(Z)}(M_{\text{sp}})$ be the motive associated to a smooth projective variety.

Then $X$ is a dualizable and we obtain a canonical symmetric monoidal equivalence

$$\text{Cell}_X(\text{Mod}_{H(Z)}(M_{\text{sp}})) \simeq \text{Mod}_{\beta \circ X}(\text{Fun}(\text{Bord}, \text{Sp})).$$

Let $D$ be the category of chain complexes of quasicoherent sheaves of $\mathcal{O}_S$-modules over a scheme $S$ and $\alpha : \text{Ch} \to D$ the symmetric monoidal functor from chain complexes of abelian groups to $D$ left adjoint to the forgetful functor $\beta : D \to \text{Ch}$ induced by the unique map of schemes $S \to \text{Spec}(Z)$.

If we choose $S$ nice enough, $D$ is compactly generated by a dualizable object $X$ so that we have $\text{Cell}_X(D) = D$.

So we obtain a canonical symmetric monoidal equivalence

$$D \simeq \text{Mod}_{\beta \circ X}(\text{Fun}(\text{Bord}, \text{Ch})).$$

Let $A$ be an $E_\infty$-ring spectrum, e.g. the Eilenberg-Mac Lane spectrum of a commutative ring $R$.

The canonical left adjoint symmetric monoidal functor $\text{Sp} \to M_{\text{sp}}$ from spectra to motivic spectra induces a left adjoint symmetric monoidal functor $\text{Mod}_A(\text{Sp}) \to \text{Mod}_A(M_{\text{sp}})$, which we can take for $\alpha : C \to D$. 


Choosing $X = A \wedge \mathbb{P}^1$ we obtain a canonical symmetric monoidal equivalence

$$\text{Cell}_{A \wedge \mathbb{P}^1} (\text{Mod}_A (\text{Msp})) \simeq \text{Mod}_{[A, A \wedge \mathbb{P}^1, (-)]_A} (\text{Fun}(\mathbb{Q}S, \text{Mod}_A (\text{Sp}))),$$

where $[-,-]_A$ denotes the $A$-linear hom.

1.1 Overview

This article is organized in the following way:

After some remarks about conventions and notations we start with the chapter “Cellular objects”, where we define and study the category $\text{Cell}_X (\mathcal{D})$ of cellular objects (Definition 3.6) of a cocomplete stable $k$-monoidal category with compact tensorunit $\mathcal{D}$ and a $k$-monoidal functor $X : \mathcal{B} \to \mathcal{D}$ starting at a small grouplike $k$-monoidal category $\mathcal{B}$ with $0 < k \in \mathbb{N} \cup \{\infty\}$.

Main statement of this chapter (proposition 2.6) is that for every stable presentable $k$-monoidal category $\mathcal{C}$ which is compactly generated by the tensorunit the "generalized Yoneda-embedding" $\mathcal{K} : \mathcal{B}^{op} \subset \text{Fun}(\mathcal{B}, \mathcal{S}) \to \text{Fun}(\mathcal{B}, \mathcal{C})$ is the "generic tensorinvertible object with respect to which cellular objects are taken" in the sense that the category $\text{Cell}_X (\text{Fun}(\mathcal{B}, \mathcal{C}))$ of $X$-cellular objects is whole of $\text{Fun}(\mathcal{B}, \mathcal{C})$.

In the following chapter "Presenting cellular objects as modules" we prove in proposition 3.3 combined with remark 3.4 that given a left adjoint $k$-monoidal functor $\phi : \mathcal{C} \to \mathcal{D}$ between cocomplete stable $k$-monoidal categories, an $E_k$-algebra $A$ and a map $f : \phi(A) \to B$ of $E_k$-algebras in $\mathcal{D}$ the $k-1$-monoidal functor $\theta : \text{Mod}_A (\mathcal{C}) \to \text{Mod}_{\phi(A)} (\mathcal{D}) \to \text{Mod}_B (\mathcal{D})$ on left modules induced by $\phi$ and tensoring up along $f$ is fully faithful if $f : \phi(A) \to B$ is adjoint to an equivalence and $\mathcal{C}$ is compactly generated by a set of dualizable objects that are sent to compact objects by $\phi$.

Moreover we show that $\theta$ induces an equivalence to the full subcategory of $\text{Mod}_B (\mathcal{D})$ compactly generated by the free $B$-modules on the images under $\phi$ of the compact generators in $\mathcal{C}$.

Combining both propositions 2.6 and 3.3 we are able to show theorem 3.5 using a universal property of the Day convolution $k$-monoidal structure on $\text{Fun}(\mathcal{B}, \mathcal{C})$ (remark 2.4).

In the appendix we prove lemma 4.1, a standard lemma about adjunctions between cocomplete stable categories, used in the proof of proposition 3.3 and we show proposition 4.2 which gives equivalent conditions under which a set of compact objects in a cocomplete stable category generates the category under forming iterately colimits, used for the definition of the category of cellular objects.

Moreover we show lemma 4.3 from which we deduce that for every $k$-monoidal category $\mathcal{C}$ for $0 < k \leq \infty$ there is a free rigid $k$-monoidal category on $\mathcal{C}$. 
1.2 Notation and Terminology

Category always means ∞-category.

Moreover by category we always mean locally small ∞-category unless stated differently.

By a cocomplete monoidal category or a presentable monoidal category we always mean a cocomplete respectively presentable category endowed with a monoidal structure such that the tensor product preserves small colimits in each component.

Given a category \( C \) and objects \( X, Y \) of \( C \) denote \( \mathcal{C}(X, Y) \) the space of maps from \( X \) to \( Y \) in \( C \) and \( \mathcal{C}^\sim \) the maximal subspace in \( \mathcal{C} \).

For two categories \( A, B \) denote \( \text{Fun}(A, B) \) the category of functors.

If \( D \) is a category with zero object, denote \( \Sigma : D \to D \) the suspension and \( \Omega : D \to D \) its right adjoint.

Denote \( \mathcal{S} \) the category of spaces, \( \text{Sp} \) the category of spectra and \( \text{QS} \) the 0th space of the sphere spectrum with its canonical structure of a grouplike \( E_{\infty} \)-space coming from its infinite loop space structure.

Denote \( \text{Bord} \) the free rigid symmetric monoidal category on a point.

Recall that an object of a category \( C \) is called \( \kappa \)-compact for a regular cardinal \( \kappa \) if its image under the Yoneda-embedding \( C^{\text{op}} \to \text{Fun}(C, \mathcal{S}) \) preserves \( \kappa \)-filtered colimits. For \( \kappa = \omega \) we say compact for \( \kappa \)-compact.

Denote \( C^\kappa \) the full subcategory of \( C \) spanned by the \( \kappa \)-compact objects of \( C \).

Given a monoidal category \( D \) denote \( 1_D \) the (essentially unique) tensorunit of \( D \).

If \( D \) is monoidal closed, denote \( [-,-] : D^{\text{op}} \times D \to D \) and \( [\mathcal{D}] : D^{\text{op}} \times D \to D \) the internal homs that are determined by the existence of natural equivalences \( D(Y, [X, Z]) \simeq D(X \otimes Y, Z) \simeq D(X, [Y, Z]) \) for objects \( X, Y, Z \) of \( D \).

Every presentable monoidal category is monoidal closed by the adjoint functor theorem.

1.2.1 Dualizable and tensorinvertible objects

Let \( \mathcal{C} \) be a k-monoidal category.

Given objects \( X, Y \) of \( \mathcal{C} \) we call \( Y \) a left dual of \( X \) if there are morphisms \( 1_C \to X \otimes Y \) and \( Y \otimes X \to 1_C \) related by the triangular identities or equivalently if \( Y \otimes - : \mathcal{C} \to \mathcal{C} \) is left adjoint to \( X \otimes - : \mathcal{C} \to \mathcal{C} \) and \( - \otimes Y : \mathcal{C} \to \mathcal{C} \) is right adjoint to \( - \otimes X : \mathcal{C} \to \mathcal{C} \).

\[ \mathcal{C}(Y \otimes A, B) \simeq \mathcal{C}(A, X \otimes B) \]
\[ \mathcal{C}(A \otimes X, B) \simeq \mathcal{C}(A, B \otimes Y) \]

Dually, we call \( Y \) a right dual of \( X \) if there are morphisms \( 1_C \to Y \otimes X \) and \( X \otimes Y \to 1_C \) related by the triangular identities or equivalently if \( Y \otimes - : \mathcal{C} \to \mathcal{C} \) is right adjoint to \( X \otimes - : \mathcal{C} \to \mathcal{C} \) and \( - \otimes Y : \mathcal{C} \to \mathcal{C} \) is left adjoint to \( - \otimes X : \mathcal{C} \to \mathcal{C} \).

Given objects \( X, Y \) of \( \mathcal{C} \) we call \( Y \) a dual of \( X \) if \( Y \) is both a left and right dual of \( X \).

By symmetry of the definition \( Y \) is a dual of \( X \) if and only if \( X \) is a dual of \( Y \).
As right respectively left adjoints of a given functor are unique, left duals, right duals and duals of an object are unique if they exist. We write \( X^\vee \) for the dual of an object \( X \) of \( \mathcal{C} \) and have a canonical equivalence \( (X^\vee)^\vee \cong X \).

We call an object \( X \) of \( \mathcal{C} \) left dualizable, right dualizable respectively dualizable if \( X \) admits a left dual, right dual respectively dual.

Given objects \( X, Y \) of \( \mathcal{C} \) we call \( Y \) a left respectively a right inverse of \( X \) if there is an equivalence \( Y \otimes X \cong 1_\mathcal{C} \) respectively \( X \otimes Y \cong 1_\mathcal{C} \).

We call \( Y \) an inverse of \( X \) if \( Y \) is both a left inverse and right inverse of \( X \) or equivalently if \( Y \otimes - : \mathcal{C} \to \mathcal{C} \) is inverse to \( Y \otimes - : \mathcal{C} \to \mathcal{C} \) and \( - \otimes X : \mathcal{C} \to \mathcal{C} \) is inverse to \( - \otimes Y : \mathcal{C} \to \mathcal{C} \).

We call an object \( X \) of \( \mathcal{C} \) left tensorinvertible, right tensorinvertible respectively tensorinvertible if \( X \) admits a left inverse, right inverse respectively inverse.

An object of \( \mathcal{C} \) is tensorinvertible if and only if it is both left tensorinvertible and right tensorinvertible.

Every inverse of an object \( X \) of \( \mathcal{C} \) is a dual of \( X \) and so every tensorinvertible object of \( \mathcal{C} \) is dualizable.

Denote \( \text{Rig}(\mathcal{C}) \) the full subcategory of \( \mathcal{C} \) spanned by the dualizable objects of \( \mathcal{C} \) and \( \text{Pic}(\mathcal{C}) \subset \text{Rig}(\mathcal{C}) \) the full subcategory of \( \mathcal{C} \) spanned by the tensorinvertible objects of \( \mathcal{C} \).

Let \( \kappa \in \mathbb{N} \cup \{\infty\} \). We call a \( \kappa \)-monoidal category \( \mathcal{C} \) rigid if every object of \( \mathcal{C} \) is dualizable, i.e. \( \text{Rig}(\mathcal{C}) = \mathcal{C} \).

As \( \text{Rig}(\mathcal{C}) \) and \( \text{Pic}(\mathcal{C}) \) contain the tensorunit of \( \mathcal{C} \) and are closed under the tensorproduct of \( \mathcal{C} \), the \( \kappa \)-monoidal structure on \( \mathcal{C} \) restricts to the full subcategories \( \text{Rig}(\mathcal{C}) \) and \( \text{Pic}(\mathcal{C}) \) so that \( \text{Pic}(\mathcal{C}) \) and \( \text{Rig}(\mathcal{C}) \) are rigid \( \kappa \)-monoidal categories and the full subcategory inclusions \( \text{Pic}(\mathcal{C}) \subset \text{Rig}(\mathcal{C}) \subset \mathcal{C} \) are \( \kappa \)-monoidal functors.

Left dualizable, right dualizable, dualizable, left tensorinvertible, right tensorinvertible and thus tensorinvertible objects are preserved by every monoidal functor between monoidal categories.

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2 Cellular objects

Before we define the category of cellular objects, we need some terminology.

2.1 Compactly generated stable categories

Let \( \mathcal{D} \) be a cocomplete stable category, \( \mathcal{C} \) a full stable subcategory of \( \mathcal{D} \) closed under small colimits and containing a set \( \mathcal{E} \) of \( \kappa \)-compact objects of \( \mathcal{C} \) for a
We say that $C$ is $\kappa$-compactly generated by $E$ or that $E$ is a set of $\kappa$-compact generators for $C$ if $C$ is the smallest full stable subcategory of $D$ containing $E$ and closed under small colimits.

If $\kappa = \omega$ we call $E$ a set of compact generators for $C$ or say that $C$ is compactly generated by $E$.

$C$ is $\kappa$-compactly generated by $E$ if and only if $C$ is the only full stable subcategory of $D$ containing $E$ and closed under small colimits.

Therefore proposition 4.2 implies that $C$ is presentable (to be more specific $\kappa$-accessible) if $C$ is $\kappa$-compactly generated by $E$.

For every cocomplete stable category $D$ and every set $E$ of $\kappa$-compact objects of $D$ for a regular cardinal $\kappa$, there exists a unique full stable subcategory $C$ of $D$ containing $E$ and closed under small colimits and $\kappa$-compactly generated by $E$, which is given by the intersection of all full stable subcategories of $D$ containing $E$ and closed under small colimits.

We call $C$ the full subcategory of $D$ $\kappa$-compactly generated by $E$ and drop $\kappa$ if $\kappa = \omega$.

Moreover we have the following functoriality:

Given stable cocomplete categories $C$, $D$ and a set $E$ of $\kappa$-compact objects of $C$ every functor $\phi : C \to D$ between stable cocomplete categories that preserves small colimits and sends objects of $E$ to $\kappa$-compact objects of $D$ restricts to a functor from the full subcategory of $C$ $\kappa$-compactly generated by $E$ to the full subcategory of $D$ $\kappa$-compactly generated by $\phi(E)$.

This follows from the fact that the full subcategory of $C$ spanned by the objects that are sent to the full subcategory of $D$ $\kappa$-compactly generated by $\phi(E)$ is a full stable subcategory of $C$ containing $E$ and closed under small colimits and consequently has to contain the full subcategory of $C$ $\kappa$-compactly generated by $E$.

Remark 2.1. Let $F : C \rightleftarrows D : G$ be an adjunction between cocomplete stable categories $C$, $D$ such that $G$ preserves $\kappa$-filtered colimits and reflects equivalences.

If $C$ is $\kappa$-compactly generated by a set $E$ of $\kappa$-compact objects of $C$ for a regular cardinal $\kappa$, then $D$ is $\kappa$-compactly generated by the set $F(E)$.

This follows from characterization 1. of proposition 4.2.

Remark 2.2. Let $k \in \mathbb{N} \cup \{\infty\}$ and $D$ be a $k$-monoidal cocomplete stable category and $E$ a set of $\kappa$-compact objects of $D$ for a regular cardinal $\kappa$ such that $E$ contains the tensorunit of $D$ and is closed under the tensor product of $D$.

Then the full subcategory $C$ of $D$ $\kappa$-compactly generated by $E$ is a $k$-monoidal full subcategory of $D$.

For this it is enough to see that $C$ is closed under the tensor product of $D$.

This follows from the fact that the full subcategories of $D$ spanned by \{X \in D | X \otimes Y \in C \text{ for all } Y \in E\} and \{Y \in D | X \otimes Y \in C \text{ for all } X \in C\} are full stable subcategories of $D$ closed under small colimits.

Thus the condition that $E$ is closed under the tensor product of $D$ implies that $C \subseteq \{X \in D | X \otimes Y \in C \text{ for all } Y \in E\}$ and so $C \subseteq \{Y \in D | X \otimes Y \in C \text{ for all } X \in C\}$. 
2.2 Definition and properties of cellular objects

Let $\mathcal{D}$ be a stable cocomplete $k$-monoidal category with $\kappa$-compact tensorunit and $X : \mathcal{B} \to \text{Rig}(\mathcal{D})$ a $k$-monoidal functor starting at a small rigid $k$-monoidal category $\mathcal{B}$.

As the tensorunit of $\mathcal{D}$ is $\kappa$-compact, every dualizable object of $\mathcal{D}$ is $\kappa$-compact so that we have $X(\mathcal{B}) \subset \text{Rig}(\mathcal{D}) \subset \mathcal{D}^\kappa$.

So the essential image $X(\mathcal{B})$ of $X$ is $\kappa$-small.

Thus there is a $\kappa$-small set of representatives of equivalence classes of objects in the essential image of $X$ and we can make the following definition:

**Definition 2.3.** Denote $\text{Cell}_X(\mathcal{D}) \subset \mathcal{D}$ the full subcategory of $\mathcal{D}$ compactly generated by (a system of representatives of equivalence classes of objects in) the essential image of $X$.

We call $\text{Cell}_X(\mathcal{D})$ the category of $X$-cellular objects of $\mathcal{D}$.

Denote $\text{Bord}$ the free rigid symmetric monoidal category on a point so that we have a canonical equivalence $E_\infty(\text{Cat}_\infty)(\text{Bord}, \text{Rig}(\mathcal{D})) \simeq \text{Rig}(\mathcal{D})$ between the space of symmetric monoidal functors $\text{Bord} \to \text{Rig}(\mathcal{D})$ and the full subspace of $\mathcal{D}^\kappa$ spanned by the dualizable objects.

We have a canonical isomorphism of commutative monoids $\pi_0(\text{Bord}) \cong \mathbb{N} \times \mathbb{N}$, where $\mathbb{N}$ is endowed with the sum.

If $\mathcal{B} = \text{Bord}$ every object $X$ of $\text{Rig}(\mathcal{D})$ corresponds to a symmetric monoidal functor $\bar{X} : \text{Bord} \to \text{Rig}(\mathcal{D}) \subset \mathcal{D}$ that sends $(n, m) \in \text{Bord}$ to $X^\otimes n \otimes (X^\vee)^\otimes m$.

In this case we set $\text{Cell}_X(\mathcal{D}) := \text{Cell}_{\bar{X}}(\mathcal{D})$

so that $\text{Cell}_X(\mathcal{D})$ is the full subcategory of $\mathcal{D}$ compactly generated by the set $(X^\otimes n \otimes (X^\vee)^\otimes m \mid n, m \in \mathbb{N}) \subset \mathcal{D}$.

Denote $QS$ the 0th space of the sphere spectrum with the structure of a grouplike $E_\infty$-space coming from its infinite loop space structure.

$QS$ is the free grouplike $E_\infty$-space generated by a point so that evaluation at $1 \in QS$ induces an equivalence $E_\infty(\text{Cat}_\infty)(QS, \text{Pic}(\mathcal{D})) \simeq \text{Pic}(\mathcal{D})^\kappa$ between the maximal subspace in $\mathcal{D}$ consisting of the tensorinvertible objects and the space of symmetric monoidal functors $QS \to \text{Pic}(\mathcal{D})$.

The universal property of $QS$ yields a canonical isomorphism of abelian groups $\pi_0(QS) \cong \mathbb{Z}$.

If $\mathcal{B} = QS$ every object $X$ of $\text{Pic}(\mathcal{D})$ corresponds to a symmetric monoidal functor $X^\otimes(-) : QS \to \text{Pic}(\mathcal{D}) \subset \mathcal{D}$ that sends $n \in QS$ to $X^\otimes n$.

In this case we set $\text{Cell}_X(\mathcal{D}) := \text{Cell}_{X^\otimes(-)}(\mathcal{D})$

so that $\text{Cell}_X(\mathcal{D})$ is the full subcategory of $\mathcal{D}$ compactly generated by the set $(X^\otimes n \mid n \in \mathbb{Z}) \subset \mathcal{D}$ consisting of all tensorpowers of $X$.

By remark 2.2 with the essential image of $X$ also $\text{Cell}_X(\mathcal{D})$ is a $k$-monoidal full subcategory of $\mathcal{D}$.

Thus $\text{Cell}_X(\mathcal{D})$ is a $k$-monoidal full stable subcategory of $\mathcal{D}$ closed under small colimits. Moreover $\text{Cell}_X(\mathcal{D})$ is presentable by proposition 4.2.
If $\mathcal{D}$ is a stable presentable $k$-monoidal category with compact tensor unit, then $\mathcal{D}^\omega$ and thus $\text{Pic}(\mathcal{D}) \subset \mathcal{D}^\omega$ is a small category so that we can choose $X : \mathcal{B} \to \text{Pic}(\mathcal{D})$ to be the identity.

Given a $k$-monoidal functor $X : \mathcal{B} \to \text{Pic}(\mathcal{D})$ starting at a small grouplike $k$-monoidal category $\mathcal{B}$ every $k$-monoidal small colimits preserving functor $\phi : \mathcal{C} \to \mathcal{D}$ between stable cocomplete $k$-monoidal categories with compact tensor unit restricts to a $k$-monoidal functor $\text{Cell}_X(\mathcal{C}) \to \text{Cell}_{\phi \circ X}(\mathcal{D})$.

2.3 generic cellular objects

In this section we show that for every rigid $k$-monoidal category $\mathcal{B}$ and stable presentable $k$+1-monoidal category $\mathcal{C}$ compactly generated by the tensorunit for $0 < k \in \mathbb{N} \cup \{\infty\}$ the category of $K$-cellular objects is whole of $\text{Fun}(\mathcal{B}, \mathcal{C})$ (proposition 2.6).

To make this statement precise, we have to introduce the Day convolution $k$-monoidal structure on $\text{Fun}(\mathcal{B}, \mathcal{C})$:

Let $k \in \mathbb{N} \cup \{\infty\}$, $\mathcal{B}$ a small $k$-monoidal category and $\mathcal{C}$ a presentable $k$-monoidal category.

The Day-convolution makes $\text{Fun}(\mathcal{B}, \mathcal{C})$ to a presentable $k$-monoidal category.

We have a $k$-monoidal functor $\mathcal{C} \to \text{Fun}(\mathcal{B}, \mathcal{C})$ that is left adjoint to evaluation at the tensorunit of $\mathcal{B}$ and a $k$-monoidal functor $\mathcal{K} : \mathcal{B}^{op} \subset \text{Fun}(\mathcal{B}, \mathcal{C}) \to \text{Fun}(\mathcal{B}, \mathcal{C})$ induced by the Yoneda embedding of $\mathcal{B}^{op}$ and the colimits preserving functor $\delta : \mathcal{B} \to \mathcal{C}$ corresponding to the tensorunit of $\mathcal{C}$.

Let $0 < k \in \mathbb{N} \cup \{\infty\}$.

Denote $\text{Pr}_{k}$ the large category of presentable categories with morphisms the left adjoint functors, $E_k$ the k-th little cubes operad and $\text{Pr}_k = \text{Alg}_{E_k}(\text{Pr}_{k}^{L})$ the large category of presentable $k$-monoidal categories.

$\text{Pr}_{k}$ carries a closed symmetric monoidal structure determined by the property that its internal hom of two presentable categories $\mathcal{A}, \mathcal{B}$ is the presentable category of small colimits preserving functors $\mathcal{A} \to \mathcal{B}$.

The symmetric monoidal structure on $\text{Pr}_{k}^{L}$ lifts to a symmetric monoidal structure on $\text{Pr}_{k}$.

We have a canonical equivalence $\text{Pr}_{k+1} = \text{Alg}(\text{Pr}_k)$ between the category of presentable $k + 1$-monoidal categories and the category of associative algebras in the category of presentable $k$-monoidal categories.

Let $\mathcal{C}$ be a presentable $k + 1$-monoidal category corresponding to an associative algebra in the category of presentable $k$-monoidal categories.

We call a left-modul respectively right-modul in $\text{Pr}_{k}$ over the associative algebra corresponding to $\mathcal{C}$ a presentable $k$-monoidal left respectively right $\mathcal{C}$-modul.

We have canonical equivalences $\text{LMod}_{\mathcal{C}}(\text{Pr}_k) \cong \text{Alg}_{E_k}(\text{LMod}_{\mathcal{C}}(\text{Pr}_{k}^{L}))$ and $\text{RMod}_{\mathcal{C}}(\text{Pr}_k) \cong \text{Alg}_{E_k}(\text{RMod}_{\mathcal{C}}(\text{Pr}_{k}^{L}))$.

Let $\mathcal{C}$ be a presentable $k + 1$-monoidal category and $\mathcal{D}$ a presentable $k$-monoidal left $\mathcal{C}$-modul.

Then the action map $\mathcal{C} \otimes \mathcal{D} \to \mathcal{D}$ in $\text{Pr}_k$ corresponds to a $k$-monoidal functor $\mathcal{C} \times \mathcal{D} \to \mathcal{D}$ that preserves small colimits in each variable.

Remark 2.4. Let $k \in \mathbb{N} \cup \{\infty\}$ and $\mathcal{C}$ a presentable $k + 1$-monoidal category.
Then Fun(\mathcal{B}, \mathcal{C}) endowed with the Day-convolution is a presentable k-monoidal left \mathcal{C}-module, where the \mathcal{C}-module structure is the levelwise one.

The Day-convolution on Fun(\mathcal{B}, \mathcal{C}) is characterized by the following universal property:

For every presentable k-monoidal left \mathcal{C}-modal \mathcal{D} composition with \mathcal{K} : \mathcal{B}^{op} \to Fun(\mathcal{B}, \mathcal{C}) induces an equivalence

\text{Fun}_{k}^{\mathcal{B}, \mathcal{C}}(Fun(\mathcal{B}, \mathcal{C}), \mathcal{D}) \simeq \text{Fun}_{k}^{\mathcal{B}, \mathcal{C}}(\mathcal{B}^{op}, \mathcal{D})

between the category of k-monoidal left \mathcal{C}-linear functors Fun(\mathcal{B}, \mathcal{C}) \to \mathcal{D} that admit a right adjoint and the category of k-monoidal functors \mathcal{B}^{op} \to \mathcal{D}.

So every k-monoidal functor \psi : \mathcal{B}^{op} \to \mathcal{D} uniquely extends to a k-monoidal left \mathcal{C}-linear functor \phi : Fun(\mathcal{B}, \mathcal{C}) \to \mathcal{D} that admits a right adjoint \gamma : \mathcal{D} \to Fun(\mathcal{B}, \mathcal{C}).

The functor \gamma : \mathcal{D} \to Fun(\mathcal{B}, \mathcal{C}) is adjoint to the functor \mathcal{B} \times \mathcal{D} \xrightarrow{\psi^{op} \times \mathcal{D}} \mathcal{D}^{op} \times \mathcal{D} \xrightarrow{[\cdot, \cdot]_{\mathcal{B}}} \mathcal{C},

where the functor \([\cdot, \cdot]_{\mathcal{B}} : \mathcal{D}^{op} \times \mathcal{D} \to \mathcal{C}\) denotes the \mathcal{C}-hom of \mathcal{D} (determined by the existence of a natural equivalence \mathcal{D}(X \otimes Y, Z) \simeq \mathcal{C}(X, [Y, Z]_{\mathcal{B}}) for objects X of \mathcal{C} and Y, Z of \mathcal{D}).

Remark 2.5. The functor \gamma : \mathcal{D} \to Fun(\mathcal{B}, \mathcal{C}) obtains the structure of a lax k-monoidal functor from its k-monoidal left adjoint \phi.

By the universal property of the Day-convolution the structure of a lax k-monoidal functor on \gamma corresponds to the structure of a lax k-monoidal functor on the composition \mathcal{B} \times \mathcal{D} \xrightarrow{\psi^{op} \times \mathcal{D}} \mathcal{D}^{op} \times \mathcal{D} \xrightarrow{[\cdot, \cdot]_{\mathcal{B}}} \mathcal{C}.

This structure is exactly the structure, the functor \mathcal{B} \times \mathcal{D} \xrightarrow{\psi^{op} \times \mathcal{D}} \mathcal{D}^{op} \times \mathcal{D} \xrightarrow{[\cdot, \cdot]_{\mathcal{B}}} \mathcal{C}

obtains as a composition of the lax k-monoidal functors \psi : \mathcal{B}^{op} \to \mathcal{D}

and \([\cdot, \cdot]_{\mathcal{B}} : \mathcal{D}^{op} \times \mathcal{D} \to \mathcal{C}.

The structure of a lax k-monoidal functor on \([\cdot, \cdot]_{\mathcal{B}} : \mathcal{D}^{op} \times \mathcal{D} \to \mathcal{C}\) is induced from the k-monoidal action map \mathcal{C} \times \mathcal{D} \to \mathcal{D} of the k-monoidal left \mathcal{C}-modal \mathcal{D}

as both functors are related by an adjunction of two variables.

If the tensorunit of \mathcal{C} is compact, the tensorunit of Fun(\mathcal{B}, \mathcal{C}) is compact, too, being the image of the tensorunit of \mathcal{C} under the k-monoidal functor \mathcal{C} \to Fun(\mathcal{B}, \mathcal{C}) that is left adjoint to the colimits preserving functor Fun(\mathcal{B}, \mathcal{C}) \to \mathcal{C}

that evaluates at the tensorunit of \mathcal{B}.

Let k \in \mathbb{N} \cup \{\infty\}.

Let \mathcal{B} be a small rigid k-monoidal category, \mathcal{C} a stable presentable k + 1-monoidal category, \mathcal{D} a stable presentable k-monoidal left \mathcal{C}-modul and \psi : \mathcal{B}^{op} \to \mathcal{D} a k-monoidal functor.

By remark 2.3 there is a unique colimits preserving k-monoidal left \mathcal{C}-linear functor \phi : Fun(\mathcal{B}, \mathcal{C}) \to \mathcal{D} with \phi \circ \mathcal{K} \simeq \psi.

If we assume that the tensorunits of \mathcal{C} and \mathcal{D} are compact, \phi : Fun(\mathcal{B}, \mathcal{C}) \to \mathcal{D}

restricts to a k-monoidal functor Cell_{\mathcal{X}}(Fun(\mathcal{B}, \mathcal{C})) \to Cell_{\phi \circ \mathcal{K}}(\mathcal{D}) = Cell_{\psi}(\mathcal{D}).

The following proposition asserts that Cell_{\mathcal{X}}(Fun(\mathcal{B}, \mathcal{C})) = Fun(\mathcal{B}, \mathcal{C}) if \mathcal{C} is compactly generated by its tensorunit.

In this case we end up with a k-monoidal functor Fun(\mathcal{B}, \mathcal{C}) \to Cell_{\psi}(\mathcal{D}).
Proposition 2.6. Let $k \in \mathbb{N} \cup \{\infty\}$.

Let $\mathcal{C}$ be a stable presentable $k$-monoidal category which is compactly generated by the tensorunit and $\mathcal{B}$ a small rigid $k$-monoidal category.

Then $\text{Fun}(\mathcal{B}, \mathcal{C})$ is compactly generated by the essential image of $\mathcal{K}$, in other words

$$\text{Cell}_{\mathcal{K}}(\text{Fun}(\mathcal{B}, \mathcal{C})) = \text{Fun}(\mathcal{B}, \mathcal{C}).$$

Proof. Denote $(-) \otimes 1\mathcal{C} : \mathcal{S} \to \mathcal{C}$ the colimits preserving functor corresponding to the tensorunit of $\mathcal{C}$.

Thus for every object $Z \in \mathcal{B}$ we can write $K(Z) = \mathcal{B}(Z, -) \otimes 1\mathcal{C}$ and using the Yoneda-lemma we obtain a canonical equivalence

$$\text{Fun}(\mathcal{B}, \mathcal{C})(K(Z), F) = \text{Fun}(\mathcal{B}, \mathcal{C})(\mathcal{B}(Z, -) \otimes 1\mathcal{C}, F) = \text{Fun}(\mathcal{B}, 1\mathcal{C})(\mathcal{B}(Z, -), \mathcal{C}(1\mathcal{C}, F(-)))$$

for every $Z \in \mathcal{B}$ and $F \in \text{Fun}(\mathcal{B}, \mathcal{C})$.

We have to show that a morphism $f : U \to V$ of $\text{Fun}(\mathcal{B}, \mathcal{C})$ is an equivalence if

$$\alpha : \text{Fun}(\mathcal{B}, \mathcal{C})(\Sigma^n(K(Z)), f) : \text{Fun}(\mathcal{B}, \mathcal{C})(\Sigma^n(K(Z)), U) \to \text{Fun}(\mathcal{B}, \mathcal{C})(\Sigma^n(K(Z)), V)$$

is an equivalence for all $Z \in \mathcal{B}$ and $n \in \mathbb{Z}$.

$\alpha$ is equivalent to

$$\mathcal{C}(1\mathcal{C}, \Omega^n(f(Z))) : \mathcal{C}(1\mathcal{C}, \Omega^n(U(Z))) \to \mathcal{C}(1\mathcal{C}, \Omega^n(V(Z)))$$

and thus equivalent to

$$\beta : \mathcal{C}(\Sigma^n(1\mathcal{C}), f(Z)) : \mathcal{C}(\Sigma^n(1\mathcal{C}), U(Z)) \to \mathcal{C}(\Sigma^n(1\mathcal{C}), V(Z)),$$

where we use the canonical equivalence from above and the fact that evaluation at $Z$ is an exact functor.

Therefore we have to see that a morphism $f : U \to V$ in $\text{Fun}(\mathcal{B}, \mathcal{C})$ is an equivalence if $\beta$ is an equivalence for all $Z \in \mathcal{B}$ and $n \in \mathbb{Z}$.

This follows from the assumption that $\mathcal{C}$ is compactly generated by the tensorunit.

$\square$
3 Presenting cellular objects as modules

Recall for the following lemma that for every monoidal category $\mathcal{D}$ that admits geometric realizations of simplicial objects that are preserved by the tensor product in each component and for every map $f : A \to B$ of associative algebras of $\mathcal{D}$ the forgetful functor $\text{Mod}_B(\mathcal{D}) \to \text{Mod}_A(\mathcal{D})$ admits a left adjoint $B \otimes_A - : \text{Mod}_A(\mathcal{D}) \to \text{Mod}_B(\mathcal{D})$. [prop. 4.6.2.17.]

The following lemma 3.1 is a main ingredient in the proof of proposition 3.3, from which we deduce theorem 3.5:

**Lemma 3.1.** Let $\phi : \mathcal{C} \to \mathcal{D}$ be a monoidal functor that admits a right adjoint $\gamma : \mathcal{D} \to \mathcal{C}$.

Assume that $\mathcal{D}$ admits geometric realizations of simplicial objects and that the tensor product of $\mathcal{D}$ preserves geometric realizations of simplicial objects in each component.

Let $A$ be an associative algebra of $\mathcal{C}$, $B$ a map of associative algebras of $\mathcal{D}$ and $g : A \to \gamma(B)$ its adjoint map.

Denote $\Phi : \text{Mod}_A(\mathcal{C}) \to \text{Mod}_\phi(A)(\mathcal{D})$ the induced functor on left modules and $\Gamma$ its right adjoint.

Let $\eta$ be the unit of the induced adjunction on left modules

$$\text{Mod}_A(\mathcal{C}) \xrightarrow{\Phi} \text{Mod}_\phi(A)(\mathcal{D}) \xrightarrow{B \otimes_A -, \gamma} \text{Mod}_B(\mathcal{D})$$

Then $g : A \to \gamma(B)$ is an equivalence if and only if $\eta_X$ is an equivalence for every free $A$-module $X$ on a right-dualizable object of $\mathcal{C}$.

**Proof.** Let $Y \in \mathcal{C}$ be right-dualizable and $Y^\vee \in \mathcal{C}$ a right dual of $Y$.

As right duals are preserved by monoidal functors, $\phi(Y^\vee)$ is a right dual of $\phi(Y)$. So the functors $- \otimes Y^\vee : \mathcal{C} \to \mathcal{C}$ is left adjoint to $- \otimes Y : \mathcal{C} \to \mathcal{C}$ and $- \otimes \phi(Y^\vee) : \mathcal{D} \to \mathcal{D}$ is left adjoint to $- \otimes \phi(Y) : \mathcal{D} \to \mathcal{D}$.

The structure of a monoidal functor on $\phi : \mathcal{C} \to \mathcal{D}$ provides an equivalence

$$\alpha : \phi(-) \otimes \phi(Y^\vee) \simeq \phi(- \otimes Y^\vee)$$

in $\text{Fun}(\mathcal{C}, \mathcal{D})$ which induces an equivalence $\beta : \gamma(-) \otimes Y \simeq \gamma(- \otimes \phi(Y))$ in $\text{Fun}(\mathcal{D}, \mathcal{C})$ between the right adjoints.

$\beta$ is uniquely characterized by the property that the square

\[
\begin{array}{ccc}
\text{id}_{\mathcal{C}} & \rightarrow & (- \otimes Y) \circ \gamma \circ \phi \circ (- \otimes Y^\vee) \\
\downarrow & \simeq & \downarrow \\
\gamma \circ (- \otimes \phi(Y)) \circ (- \otimes \phi(Y^\vee)) & \circ \phi & \gamma \circ (- \otimes \phi(Y)) \circ \phi \circ (- \otimes Y^\vee)
\end{array}
\]

in $\text{Fun}(\mathcal{C}, \mathcal{C})$ commutes.

The canonical morphism $\gamma(-) \otimes Y \to \gamma(-) \otimes \phi(Y) \simeq \gamma(- \otimes \phi(Y))$ in $\text{Fun}(\mathcal{D}, \mathcal{C})$ also makes the last diagram commute and is thus homotopic to $\beta$. 

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Given a monoidal category $\mathcal{E}$ and a map $\varphi : \mathcal{E} \to \mathcal{E}'$ of associative algebras in $\mathcal{E}$ denote $V_\mathcal{E} : \text{Mod}_\mathcal{E}(\mathcal{E}) \to \mathcal{E}$ and $\varphi^* : \text{Mod}_{\mathcal{E}'}(\mathcal{E}) \to \text{Mod}_\mathcal{E}(\mathcal{E})$ the forgetful functors and $F_\mathcal{E} : \mathcal{E} \to \text{Mod}_\mathcal{E}(\mathcal{E})$, $\mathcal{E}' \otimes_{\mathcal{E}} - : \text{Mod}_\mathcal{E}(\mathcal{E}) \to \text{Mod}_{\mathcal{E}'}(\mathcal{E})$ their left adjoints.

The unit $\eta_{A\otimes Y}$ of the induced adjunction on left modules

$$
\begin{array}{cccc}
\text{Mod}_A(\mathcal{C}) & \overset{\Phi}{\longrightarrow} & \text{Mod}_{\phi_\mathcal{A}}(\mathcal{D}) & \overset{\Phi}{\longrightarrow} & \text{Mod}_B(\mathcal{D})
\end{array}
$$

is given by $A \otimes Y \to \Gamma \Phi(A \otimes Y) \to \Gamma \Phi^*(B \otimes_{\phi_\mathcal{A}} \Phi(A \otimes Y))$.

Thus the statement of the lemma follows from the commutativity of the following diagram in $\mathcal{C}$, whose top horizontal morphism is the image of $\eta_{A\otimes Y}$ in $\mathcal{C}$ under the forgetful functor and whose bottom horizontal morphism is $g \otimes Y$:

The commutativity of the top left square of the diagram says that the unit of the adjunction $\Phi : \text{Mod}_A(\mathcal{C}) \cong \text{Mod}_{\phi_\mathcal{A}}(\mathcal{D}) : \Gamma$ lifts the unit of the adjunction $\phi : \mathcal{C} \cong \mathcal{D} : \gamma$ along the forgetful functors.

This follows from the fact that the forgetful functors $V_A : \text{Mod}_A(\mathcal{C}) \to \mathcal{C}$ and $V_{\phi_\mathcal{A}} : \text{Mod}_{\phi_\mathcal{A}}(\mathcal{D}) \to \mathcal{D}$ are part of a map of adjunctions from $\Phi : \text{Mod}_A(\mathcal{C}) \cong \text{Mod}_{\phi_\mathcal{A}}(\mathcal{D}) : \Gamma$ to $\phi : \mathcal{C} \cong \mathcal{D} : \gamma$ by construction of the induced adjunction on left modules.

The bottom left square of the diagram commutes as the unit $\text{id}_\mathcal{C} \to \gamma \circ \phi$ of the adjunction $\phi : \mathcal{C} \cong \mathcal{D} : \gamma$ gets a monoidal natural transformation when we give $\gamma$ the structure of a lax monoidal functor from its monoidal left adjoint $\phi$.

It remains to check that the only two squares with five knots commute, which are the square in the middle of the diagram and the right middle square, as all other squares commute by naturality.

To do so, it is helpful to describe the free functor more explicitily.

The free functor $F_A : \mathcal{C} \to \text{Mod}_A(\mathcal{C})$ can be constructed in the following way:

Recall that the category of left $A$-modules $\text{Mod}_A(\mathcal{C})$ is a right modul over $\mathcal{C}$ in the expected way so that we have an action map $\text{Mod}_A(\mathcal{C}) \times \mathcal{C} \to \text{Mod}_A(\mathcal{C})$ over the tensor product functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$. [1][const. 4.8.3.24.]
Especially we get a functor \( F_A : \mathcal{C} \cong (A) \times \mathcal{C} \to \mathsf{Mod}_A(\mathcal{C}) \times \mathcal{C} \to \mathsf{Mod}_A(\mathcal{C}) \) with \( V_A \circ F_A \cong A \otimes - \) and a natural transformation \( \rho : \mathsf{id}_C \cong 1_C \otimes - \to A \otimes - \cong V_A F_A. \)

For every object \( X \) of \( \mathcal{C} \) the left action map \( \mu_X : A \otimes (A \otimes X) \cong A \otimes V_A F_A(X) \to V_A F_A(X) \cong A \otimes X \) of \( F_A(X) \) is given by the canonical map \( A \otimes (A \otimes X) \cong (A \otimes A) \otimes X \to A \otimes X \) induced by the multiplication map \( A \otimes A \to A \) of the associative algebra \( A \). Therefore the composition \( A \otimes X \xrightarrow{A \otimes \phi(X)} A \otimes V_A F_A(X) \xrightarrow{\mu_X} V_A F_A(X) \) is an equivalence so that \( \rho : \mathsf{id}_C \cong 1_C \otimes - \to A \otimes - \cong V_A F_A \) exhibits \( F_A : \mathcal{C} \to \mathsf{Mod}_A(\mathcal{C}) \) as left adjoint to \( V_A : \mathsf{Mod}_A(\mathcal{C}) \to \mathcal{C}. \) [1][prop. 4.2.4.2.]

For a morphism \( \varphi : A \to A' \) of associative algebras in \( \mathcal{C} \) the forgetful functor \( \varphi^* : \mathsf{Mod}_A(\mathcal{C}) \to \mathsf{Mod}_{A'}(\mathcal{C}) \) is a map of right \( \mathcal{C} \)-modules [1][constr. 4.8.3.24.] so that we obtain a canonical equivalence \( F_{\varphi^*(A)} \cong \varphi^* F_{A'} \) of functors \( \mathcal{C} \to \mathsf{Mod}_A(\mathcal{C}) \) lifting the identity natural transformation of \( \mathcal{C} \).

\( \varphi : A \to A' \) corresponds to a map \( A \to \phi^*(A) \) of left \( A \)-modules in \( \mathcal{C} \) that yields a functor \( \Delta^1 \times \mathcal{C} \to \mathsf{Mod}_A(\mathcal{C}) \times \mathcal{C} \to \mathsf{Mod}_A(\mathcal{C}) \) corresponding to a natural transformation \( \theta : F_A \to F_{\phi^*(A)} \cong \varphi^* F_{A'} \) of functors \( \mathcal{C} \to \mathsf{Mod}_A(\mathcal{C}) \) lifting the natural transformation \( \varphi \otimes - : A \otimes - \to A' \otimes - \) of functors \( \mathcal{C} \to \mathcal{C} \).

For every object \( X \) of \( \mathcal{C} \) and every left \( A' \)-module \( M \) of \( \mathcal{C} \) the canonical map \( \mathsf{Mod}_{A'}(\mathcal{C})(F_A(X), M) \to \mathsf{Mod}_A(\mathcal{C})(\phi^*(F_A(X)), \phi^*(M)) \to \mathsf{Mod}_A(\mathcal{C})(F_A(X), \phi^*(M)) \cong \mathcal{C}(X, V_A(M)) \) is homotopic to the canonical equivalence \( \mathsf{Mod}_{A'}(\mathcal{C})(F_{A'}(X), M) \to \mathcal{C}(X, V_{A'}(M)) \).

Thus \( \theta : F_A \to \varphi^* F_{A'} \) is adjoint to an equivalence \( \sigma : A' \otimes_A F_A(-) \to F_{A'}, \) and the natural transformation \( A \otimes - \cong V_A F_A \to V_A \varphi^* A' \otimes_A F_A(-) \cong V_A A' \otimes_A F_{A'}(-) \xrightarrow{\varphi \otimes \sigma} V_A F_{A'} \cong A' \otimes - \) is homotopic to the natural transformation \( \varphi \otimes - : A \otimes - \to A' \otimes - \).

This implies the commutativity of the right middle square.

The functor \( \mathsf{Mod}_A(\mathcal{C}) \to \mathsf{Mod}_{A'}(\mathcal{C}) \) induced by \( \phi : \mathcal{C} \to \mathcal{D} \) is a map of right \( \mathcal{C} \)-modules, where \( \mathcal{C} \) acts on \( \mathsf{Mod}_{A'}(\mathcal{C}) \) via \( \phi \). [1][constr. 4.8.3.24.]

Thus we obtain a canonical equivalence \( F_{\phi(A)} \circ \phi \to \Phi \circ F_A \) of functors \( \mathcal{C} \to \mathsf{Mod}_{A'}(\mathcal{C}) \) lifting the canonical equivalence \( \phi(A) \otimes \phi(-) \to \phi(A \otimes -) \) of functors \( \mathcal{C} \to \mathcal{D} ). \)

This shows the commutativity of the middle square.

\[ \square \]

Remark 3.2. Lemma [1][constr. 4.8.3.24.] has the following dual version, where one replaces left modules by right modules and right-dualizable objects by left-dualizable ones:

Let \( \psi : \mathcal{D} \to \mathcal{C} \) be a monoidal functor that admits a right adjoint \( \gamma : \mathcal{D} \to \mathcal{C} \).

Assume that \( \mathcal{D} \) admits geometric realizations of simplicial objects and that the tensor product of \( \mathcal{D} \) preserves geometric realizations of simplicial objects in each component.

Let \( A \) be an associative algebra of \( \mathcal{C} \), \( B \) an associative algebra of \( \mathcal{D} \), \( f : \phi(A) \to B \) a map of associative algebras of \( \mathcal{D} \) and \( g : A \to \gamma(B) \) its adjoint map.

Denote \( \Phi : \mathsf{Mod}_A(\mathcal{C}) \to \mathsf{Mod}_{\phi(A)}(\mathcal{D}) \) the induced functor on right modules and \( \Gamma \) its right adjoint.

Let \( \eta \) be the unit of the induced adjunction on right modules

\[ \mathsf{Mod}_A(\mathcal{C}) \xrightarrow{\Phi} \mathsf{Mod}_{\phi(A)}(\mathcal{D}) \xrightarrow{- \otimes B \circ \eta(A)} \mathsf{Mod}_B(\mathcal{D}) . \]
Then \( g : A \to \gamma(B) \) is an equivalence if and only if \( \eta_X \) is an equivalence for every free \( A \)-module \( X \) on a left-dualizable object of \( \mathcal{E} \).

Proposition 3.2, Lemma 3.11, and Lemma 3.1 combine to imply the following corollary:

**Proposition 3.3.** Let \( \phi : \mathcal{E} \to \mathcal{D} \) be a monoidal functor between cocomplete stable monoidal categories and \( \gamma : \mathcal{D} \to \mathcal{E} \) a right adjoint of \( \phi \).

Let \( A \) be an associative algebra of \( \mathcal{E} \) and \( B \) an associative algebra of \( \mathcal{D} \) and \( f : \phi(A) \to B \) a map of associative algebras of \( \mathcal{D} \) whose adjoint map \( g : A \to \gamma(B) \) is an equivalence.

Denote \( \Phi : \text{Mod}_A(\mathcal{E}) \to \text{Mod}_{\phi(A)}(\mathcal{D}) \) the induced functor on left modules and \( \Gamma \) its right adjoint.

Let \( \mathcal{E} \) be compactly generated by a set \( \mathcal{E} \subset \text{Pic}(\mathcal{E}) \) of right-dualizable objects such that \( \phi \) sends every object of \( \mathcal{E} \) to a compact one.

Then

\[
(B \otimes -) \circ \Phi : \text{Mod}_A(\mathcal{E}) \to \text{Mod}_{\phi(A)}(\mathcal{D}) \to \text{Mod}_B(\mathcal{D}).
\]

is fully faithful and its essential image is compactly generated by the free \( B \)-modules on the images under \( \phi \) of the compact generators in \( \mathcal{E} \).

**Proof.** As \( \mathcal{E} \) is compactly generated by \( \mathcal{E} \), Remark 2.1 implies that \( \text{Mod}_A(\mathcal{E}) \) is compactly generated by the set of free \( A \)-modules on objects of \( \mathcal{E} \).

Moreover as \( \phi : \mathcal{E} \to \mathcal{D} \) sends every object of \( \mathcal{E} \) to a compact object of \( \mathcal{D} \), the induced functor \((B \otimes -) \circ \Phi : \text{Mod}_A(\mathcal{E}) \to \text{Mod}_{\phi(A)}(\mathcal{D}) \to \text{Mod}_B(\mathcal{D})\) sends every free \( A \)-module on an object of \( \mathcal{E} \) to a compact object of \( \text{Mod}_B(\mathcal{D}) \).

Denote \( \eta \) the unit of the induced adjunction \((B \otimes -) \circ \Phi : \text{Mod}_A(\mathcal{E}) \to \text{Mod}_{\phi(A)}(\mathcal{D}) \to \text{Mod}_B(\mathcal{D}) \) on left modules.

By Lemma 5.1 the assumptions that \( g : A \to \gamma(B) \) is an equivalence and that \( \mathcal{E} \subset \text{Rig}(\mathcal{E}) \) guarantee that \( \eta_{\mathcal{E} \to X} \) is an equivalence for every free \( A \)-module \( X \) on an object of \( \mathcal{E} \) and every natural \( n \in Z \).

So the statement follows from Lemma 5.1.

\( \square \)

**Remark 3.4.** Given a \( k \)-monoidal category \( \mathcal{E} \) and an \( E_k \)-algebra \( A \) of \( \mathcal{E} \) for some \( 0 < k \in \mathbb{N} \cup \{ \infty \} \) the category \( \text{Mod}_A(\mathcal{E}) \) of left \( A \)-modules in \( \mathcal{E} \) is a \( k-1 \) monoidal category.

Let \( \phi : \mathcal{E} \to \mathcal{D} \) be a \( k \)-monoidal functor between \( k \)-monoidal categories, \( A \) be an \( E_k \)-algebra of \( \mathcal{E} \) and \( f : \phi(A) \to B \) a map of \( E_k \)-algebras of \( \mathcal{D} \).

Then the induced functors \( \Phi : \text{Mod}_A(\mathcal{E}) \to \text{Mod}_{\phi(A)}(\mathcal{D}) \) and \((B \otimes -) \circ \Phi : \text{Mod}_A(\mathcal{E}) \to \text{Mod}_{\phi(A)}(\mathcal{D}) \to \text{Mod}_B(\mathcal{D})\) are \( k-1 \) monoidal.

Thus if we assume the monoidal functor \( \phi : \mathcal{E} \to \mathcal{D} \) in corollary 3.2 to be \( k \)-monoidal, \( A \) to be an \( E_k \)-algebra of \( \mathcal{E} \) and \( f : \phi(A) \to B \) to be a map of \( E_k \)-algebras of \( \mathcal{D} \), then the functor \((B \otimes -) \circ \Phi : \text{Mod}_A(\mathcal{E}) \to \text{Mod}_{\phi(A)}(\mathcal{D}) \to \text{Mod}_B(\mathcal{D})\) is \( k-1 \) monoidal and induces a \( k-1 \) monoidal equivalence from \( \text{Mod}_A(\mathcal{E}) \) to the full \( k-1 \) monoidal subcategory of \( \text{Mod}_B(\mathcal{D}) \) compactly generated by the free \( B \)-modules on the images under \( \phi \) of the compact generators in \( \mathcal{E} \).
Before we use corollary 3.3 and proposition 2.6 to prove theorem 3.5 we need some preparations:

Let \( k \in \mathbb{N} \cup \{\infty\} \) and \( \mathcal{D} \) be a closed k-monoidal category so that we have functors \([-,-]_l : \mathcal{D}^{op} \times \mathcal{D} \to \mathcal{D} \) and \([-,-]_r : \mathcal{D}^{op} \times \mathcal{D} \to \mathcal{D} \) determined by the existence of natural equivalences \( \mathcal{D}(Y,[X,Z]) \simeq \mathcal{D}(X \otimes Y,Z) \simeq \mathcal{D}(X,[Y,Z]) \) for objects \( X,Y,Z \) of \( \mathcal{D} \).

Choosing \( Z = 1_\mathcal{D} \) in the last equivalence we obtain a natural equivalence \( \mathcal{D}^{op}([X,1_\mathcal{D}],[Y]) \simeq \mathcal{D}(Y,1_\mathcal{D}) \simeq \mathcal{D}(X,[Y,1_\mathcal{D}]) \) that exhibits the functor \([-,1_\mathcal{D}]_r : \mathcal{D}^{op} \to \mathcal{D} \) as right adjoint to the functor \([-,1_\mathcal{D}]_l : \mathcal{D} \to \mathcal{D}^{op} \).

We will show in the following that both functors \([-,1_\mathcal{D}]_r : \mathcal{D}^{op} \to \mathcal{D} \) and \([-,1_\mathcal{D}]_l : \mathcal{D}^{op} \to \mathcal{D} \) restrict to equivalences \( \text{Rig}(\mathcal{D})^{op} \to \text{Rig}(\mathcal{D}) \):

Given an object \( X \) of \( \text{Rig}(\mathcal{D}) \) with dual \( X^\vee \) we have canonical morphisms \( \alpha_X : X^\vee \to [X,1_\mathcal{D}]_r \) and \( \beta_X : X^\vee \to [X,1_\mathcal{D}]_l \) in \( \mathcal{D} \) adjoint to the counits \( X^\vee \otimes X \to 1_\mathcal{D} \) and \( X \otimes X^\vee \to 1_\mathcal{D} \).

By the Yoneda-lemma for every object \( Y \) of \( \mathcal{D} \) the induced maps \( \mathcal{D}(Y,X^\vee) \to \mathcal{D}(Y,[1_\mathcal{D},X]) \simeq \mathcal{D}(Y \otimes X,1_\mathcal{D}) \) respectively \( \mathcal{D}(Y,X^\vee) \to \mathcal{D}(Y,[X,1_\mathcal{D}]) \simeq \mathcal{D}(X \otimes Y,1_\mathcal{D}) \) are homotopic to the canonical equivalences \( \mathcal{D}(Y,X^\vee) \simeq \mathcal{D}(Y \otimes X,1_\mathcal{D}) \) respectively \( \mathcal{D}(Y,X^\vee) \simeq \mathcal{D}(X \otimes Y,1_\mathcal{D}) \).

So by Yoneda both morphisms \( \alpha_X : X^\vee \to [X,1_\mathcal{D}]_r \) and \( \beta_X : X^\vee \to [X,1_\mathcal{D}]_l \) are equivalences.

Moreover the equivalence \( \alpha_X \otimes X : X \to [X^\vee \otimes X,1_\mathcal{D}]_r \) is adjoint to the equivalence \( \beta_X \otimes X : X \to [X^\vee \otimes X,1_\mathcal{D}]_l \) as both correspond to the counit \( X \otimes X^\vee \to 1_\mathcal{D} \).

Consequently the adjunction \( [-,1_\mathcal{D}]_l^{op} : \mathcal{D} \to \mathcal{D}^{op} \) restricts to an adjunction \( \text{Rig}(\mathcal{D}) \to \text{Rig}(\mathcal{D})^{op} \), whose unit and counit are equivalences.

Moreover the functor \([-,1_\mathcal{D}]_r : \mathcal{D}^{op} \to \mathcal{D} \) is lax k-monoidal and thus restricts to a k-monoidal equivalence \( \text{Rig}(\mathcal{D})^{op} \to \text{Rig}(\mathcal{D}) \) that is inverse to the functor \([-,1_\mathcal{D}]_l^{op} : \text{Rig}(\mathcal{D}) \to \text{Rig}(\mathcal{D})^{op} \).

Now we are ready to deduce theorem 3.5 from proposition 3.3 and proposition 2.6

**Theorem 3.5.** Let \( 0 < k \in \mathbb{N} \cup \{\infty\} \) and \( \mathcal{C} \) a stable presentable k + 1-monoidal category compactly generated by the tensorunit.

Let \( \mathcal{D} \) be a stable presentable k-monoidal left \( \mathcal{C} \)-modul compactly generated by the tensorunit and \( X : \mathcal{B} \to \text{Pic}(\mathcal{D}) \subset \mathcal{D} \) a k-monoidal functor starting at a small rigid k-monoidal category \( \mathcal{B} \).

Denote \( \alpha : \mathcal{C} \to \mathcal{D} \) the unique k-monoidal left \( \mathcal{C} \)-linear functor with right adjoint \( \beta : \mathcal{D} \to \mathcal{C} \).

The lax k-monoidal functor \( \beta \circ X : \mathcal{B} \to \mathcal{D} \to \mathcal{C} \) corresponds to a \( E_k \)-algebra in the Day-convolution k-monoidal structure on \( \text{Fun}(\mathcal{B},\mathcal{C}) \).

There is a canonical k + 1 monoidal equivalence

\[
\text{Cell}_X(\mathcal{D}) \to \text{Mod}_{\beta \circ X}(\text{Fun}(\mathcal{B},\mathcal{C})).
\]

**Proof.** The lax k-monoidal functor \( [-,1_\mathcal{D}]_r : \mathcal{D}^{op} \to \mathcal{D} \) restricts to a k-monoidal equivalence \( \text{Rig}(\mathcal{D})^{op} \to \text{Rig}(\mathcal{D}) \) that is inverse to the functor \( [-,1_\mathcal{D}]_l^{op} : \text{Rig}(\mathcal{D}) \to \text{Rig}(\mathcal{D})^{op} \).
The k-monoidal functor \( Y := [-, 1_D] \circ X^{op} : B^{op} \to \text{Rig}(D) \subset D \) uniquely extends to a k-monoidal left \( C \)-linear functor \( \phi : \text{Fun}(B, C) \to D \) that admits a right adjoint \( \gamma \).

By remark 3.6, the \( E \)-algebra \( \gamma(1_D) \simeq \beta \circ [-, 1_D] \circ Y^{op} \simeq \beta \circ X \) in the Day-convolution on \( \text{Fun}(B, C) \) corresponds to the lax k-monoidal functor \( \beta \circ [-, 1_D] \circ Y^{op} \simeq \beta \circ X \).

Moreover the k-monoidal functors \( X : B \to \text{Rig}(D) \subset D \) and \( Y = [-, 1_D] \circ X^{op} : B^{op} \to \text{Rig}(D) \subset D \) have the same essential image as for every object \( Z \in B \) we have an equivalence \( Y(Z) = [X(Z), 1_D] \simeq X(Z)^y = X(Z') \).

Thus the full subcategories \( \text{Cell}_X(D) \) and \( \text{Cell}_Y(D) \) of \( D \) coincide.

By proposition 2.5 \( \text{Fun}(B, C) \) is compactly generated by the essential image of \( X : B^{op} \to \text{Fun}(B, C) \).

Thus we can apply proposition 3.6 to the k-monoidal functor \( \phi : \text{Fun}(B, C) \to D \), the identity \( g := \text{id}_{\gamma(1_D)} \) of \( \gamma(1_D) \simeq \beta \circ X \) and \( E := \) the essential image of \( X : B^{op} \to \text{Fun}(B, C) \) to deduce that the functor \( (1_D) \otimes (\gamma(1_D)) \circ \Phi : \text{Mod}_{\gamma(1_D)}(\text{Fun}(B, C)) \to \text{Mod}_{\gamma(1_D)}(\text{Fun}(B, C)) \simeq D \) induces a k-1 monoidal equivalence \( \text{Mod}_{\beta\circ X}(\text{Fun}(B, C)) \to \text{Cell}_{\beta\circ X}(D) \simeq \text{Cell}_Y(D) \) with inverse the k-1 monoidal equivalence \( \text{Cell}_X(D) \subset D = \text{Mod}_{1_{B^{op}}} \to \text{Mod}_{1_{B^{op}}} \text{(Fun}(B, C)) \) induced by \( \gamma \).

\[ \text{Remark 3.6. Assume that the conditions of theorem 3.6 are satisfied.} \]

Denote \( X^\circ : B^\circ \subset B \to \text{Pic}(D) \subset D \) the restriction of \( X \) along the k-monoidal subcategory inclusion of the maximal subspace of \( B \) into \( B \).

We can apply 3.5 to \( X^\circ \) to obtain a canonical k-1 monoidal equivalence \( \text{Cell}_X(D) = \text{Cell}_X(D) \to \text{Mod}_{\beta\circ X}(\text{Fun}(B, C)) \)

and therefore may always reduce to the case that \( B \) is a grouplike \( E_\infty \)-space.

Taking \( C \) to be the category \( \text{Sp} \) of spectra we obtain the following corollary:

**Corollary 3.7.** Let \( 0 < k \in \mathbb{N} \cup \{\infty\} \) and \( D \) be a stable presentable k-1 monoidal category with compact tensorunit and \( X : B \to \text{Pic}(D) \) a k-monoidal functor starting at a small rigid k-monoidal category \( B \).

Denote \( [-, -] : B^{op} \times D \to \text{Sp} \) the spectral hom of \( D \).

There is a canonical k-1 monoidal equivalence \( \text{Cell}_X(D) \to \text{Mod}_{[1_D, -]_X}(\text{Fun}(B, \text{Sp})) \).

More generally we can take \( C \) to be the r-monoidal category \( \text{Mod}_A(\text{Sp}) \) of \( A \)-module spectra for an \( E_{r+1} \)-ring spectrum \( A \) and some \( 0 < r \in \mathbb{N} \cup \{\infty\} \) we get the following corollary:

**Corollary 3.8.** Let \( 0 < k \in \mathbb{N} \cup \{\infty\}, A \) be an \( E_{k+2} \)-ring spectrum and \( D \) be a \( A \)-linear presentable k-monoidal category, i.e. a presentable k-monoidal left \( \text{Mod}_A(\text{Sp}) \)-modul. Assume that the tensorunit of \( D \) is compact.

Let \( \beta : D \to \text{Mod}_A(\text{Sp}) \) be a right adjoint of the unique k-monoidal left \( A \)-linear functor \( \alpha : \text{Mod}_A(\text{Sp}) \to D \).
Let $X : B \to \text{Pic}(D)$ be a $k$-monoidal functor starting at a small rigid $k$-monoidal category $B$.

Denote $[-, -]_A : \text{D}^{op} \times \text{D} \to \text{Mod}_A(\text{Sp})$ the $A$-linear hom of $D$.

There is a canonical $k-1$-monoidal equivalence

$$\text{Cell}_X(D) \to \text{Mod}_{[-, -]_A \ast X}(\text{Fun}(B, \text{Mod}_A(\text{Sp}))).$$

For $k = n = \infty$ we get the corollary:

**Corollary 3.9.** Let $\alpha : C \to D$ be a symmetric monoidal functor from a stable presentable symmetric monoidal category $C$ compactly generated by the tensorunit to a stable presentable symmetric monoidal category $D$ with compact tensorunit. Let $\beta : D \to C$ be a right adjoint of $\alpha$ and $X : B \to D$ a symmetric monoidal functor starting at a small rigid symmetric monoidal category $B$.

There is a canonical symmetric monoidal equivalence

$$\text{Cell}_X(D) \to \text{Mod}_{\beta \ast X}(\text{Fun}(B, C)).$$

For $B = QS$ a symmetric monoidal functor $B \to \text{Pic}(D)$ corresponds to an object of $\text{Pic}(D)$.

So we obtain the following corollary:

**Corollary 3.10.** Let $\alpha : C \to D$ be a symmetric monoidal functor from a stable presentable symmetric monoidal category $C$ compactly generated by the tensorunit to a stable presentable symmetric monoidal category $D$ with compact tensorunit and $\beta : D \to C$ be a right adjoint of $\alpha : C \to D$.

For every object $X \in \text{Pic}(D)$ corresponding to a symmetric monoidal functor $X^{\otimes (-)} : QS \to \text{Pic}(D) \subset D$ there is a canonical symmetric monoidal equivalence

$$\text{Cell}_X(D) \simeq \text{Mod}_{\beta \otimes X^{\otimes -}}(\text{Fun}(QS, C)),$$

where $\text{Cell}_X(D)$ denotes the full subcategory of $D$ compactly generated by the set $\{X^{\otimes n} \mid n \in \mathbb{Z}\} \subset D$.

As we have full subcategory inclusions $\text{Pic}(D) \subset \text{Rig}(D) \subset D^\omega$ if the tensorunit of $D$ is compact, $\text{Pic}(D), \text{Rig}(D)$ are essentially small.

Thus we can take $B = \text{Pic}(D)$ or $B = \text{Rig}(D)$ and $X : B \to \text{Rig}(D)$ to be the inclusion respectively identity.

This leads to the following corollary:

**Corollary 3.11.** Let $0 < k \in \mathbb{N} \cup \{\infty\}$ and $\mathcal{C}$ a stable presentable $k+1$-monoidal category compactly generated by the tensorunit.

Let $D$ be a stable presentable $k$-monoidal left $\mathcal{C}$-module compactly generated by the tensorunit and $\alpha : \mathcal{C} \to D$ the unique $k$-monoidal left $\mathcal{C}$-linear functor with right adjoint $\beta : D \to \mathcal{C}$.

There are $k-1$-monoidal equivalences

$$\text{Cell}_{id_{\text{Pic}(D)}}(D) \simeq \text{Mod}_{\beta \otimes \text{Pic}(D)}(\text{Fun}(\text{Pic}(D), \mathcal{C})).$$
and
\[ \text{Cell}_{\text{Rig}(\mathcal{D})}(\mathcal{D}) \cong \text{Mod}_{\beta_{\text{Rig}(\mathcal{D})}}(\text{Fun}(\text{Rig}(\mathcal{D}), \mathcal{C})) \]
respectively
\[ \text{Cell}_{\text{Pic}(\mathcal{D})}(\mathcal{D}) \cong \text{Mod}_{\beta_{\text{Pic}(\mathcal{D})}}(\text{Fun}(\text{Pic}(\mathcal{D})^\times, \mathcal{C})) \]
by remark \ref{remark3.6}, where \( \text{Cell}_{\text{Pic}(\mathcal{D})}(\mathcal{D}) \) and \( \text{Cell}_{\text{Rig}(\mathcal{D})}(\mathcal{D}) \) are the full subcategories of \( \mathcal{D} \) compactly generated by \( \text{Pic}(\mathcal{D}) \) respectively \( \text{Rig}(\mathcal{D}) \).
4 Appendix

In the appendix we show lemma 4.1, proposition 4.2 and lemma 4.3.

Lemma 4.1 is an important ingredient in the proof of proposition 3.3 from which we deduce theorem 3.5.

Lemma 4.1 is a standard lemma about adjunctions between cocomplete stable categories which we show for completeness.

Proposition 4.2 gives equivalent conditions under which a set of compact objects in a cocomplete stable category generates the category under forming iteratively colimits and is used for the definition of the category of cellular objects.

From lemma 4.3 we deduce that for every k-monoidal category $C$ for $0 < k \leq \infty$ there is a free rigid k-monoidal category on $C$.

Lemma 4.1. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between cocomplete stable categories, $G : \mathcal{D} \to \mathcal{C}$ a right adjoint of $F$ and $\mathcal{E}$ a set of $\kappa$-compact generators for $\mathcal{C}$.

a) Then the following conditions are equivalent:

1. $G : \mathcal{D} \to \mathcal{C}$ preserves $\kappa$-filtered colimits.
2. $F$ sends $\kappa$-compact objects to $\kappa$-compact objects.
3. $F$ sends every $\kappa$-compact generator of $\mathcal{E}$ to a $\kappa$-compact object.

b) Let $\eta : \text{id} \to G \circ F$ be the unit of the adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$.

If the adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ satisfies the equivalent conditions of a) for $\kappa = \omega$ and

$$
\eta_{\Sigma^n}(X) : \Sigma^n(X) \to G(F(\Sigma^n(X)))
$$

is an equivalence for all $X \in \mathcal{E}$ and $n \in \mathbb{Z}$, then $\eta : \text{id} \to G \circ F$ is an equivalence (equivalently $F : \mathcal{C} \to \mathcal{D}$ is fully faithful) and the essential image of $F$ is compactly generated by $F(\mathcal{E})$.

Proof. a)

If $G : \mathcal{D} \to \mathcal{C}$ preserves $\kappa$-filtered colimits, the equivalence $\mathcal{D}(F(Y), -) \simeq \mathcal{C}(Y, -) \circ G$ implies that $F(Y)$ is $\kappa$-compact if $Y$ is.

Trivially, 2. implies 3.

Let 3. be satisfied and let $H : J \to \mathcal{D}$ be a diagram in $\mathcal{D}$ indexed by a $\kappa$-filtered category $J$.

We have to prove that the canonical morphism $\text{colim}(G \circ H) \to G(\text{colim}(H))$ is an equivalence.

As $\mathcal{C}$ is $\kappa$-compactly generated by $\mathcal{E}$, this is true if and only if

$$
\mathcal{C}(\Sigma^n(X), \text{colim}(G \circ H)) \simeq \mathcal{C}(\Sigma^n(X), G(\text{colim}(H)))
$$

is an equivalence for all $X \in \mathcal{E}$ and $n \in \mathbb{Z}$.

With $X$ also $\Sigma^n(X)$ is $\kappa$-compact and we have to see that the composition

$$
\text{colim}(\mathcal{C}(\Sigma^n(X), -) \circ G \circ H) \simeq \mathcal{C}(\Sigma^n(X), \text{colim}(G \circ H)) \to \mathcal{C}(\Sigma^n(X), G(\text{colim}(H)))
$$

is an equivalence.
By adjunction this map is equivalent to the canonical map
\[
\text{colim}(\mathcal{D}(F(\Sigma^n(X)), -) \circ H) \to \mathcal{D}(F(\Sigma^n(X)), \text{colim}(H)).
\]

Condition 3. guarantees that \(F(X)\) is \(\kappa\)-compact. Therefore \(F(\Sigma^n(X)) \simeq \Sigma^n(F(X))\) is also \(\kappa\)-compact so that the last map is an equivalence.

b) Being a right adjoint between stable categories \(G : \mathcal{D} \to \mathcal{C}\) is an exact functor.

If \(G : \mathcal{D} \to \mathcal{C}\) preserves filtered colimits, it preserves small colimits because a functor between cocomplete categories preserves small colimits if and only if it preserves finite colimits and filtered colimits.

Hence \(\eta : \text{id} \to G \circ F\) is a natural transformation between colimits preserving functors.

Consider the full subcategory \(\mathcal{C}\) of \(\mathcal{C}\) spanned by all \(X \in \mathcal{C}\), for which \(\eta_{\Sigma^n(X)} : \Sigma^n(X) \to \mathcal{C}(F(\Sigma^n(X)))\) is an equivalence for all \(n \in \mathcal{Z}\). We want to see that \(\mathcal{C} = \mathcal{C}\).

By assumption \(\mathcal{E} \subset \mathcal{C}\). As \(\mathcal{C}\) is compactly generated by \(\mathcal{E}\), one has \(\mathcal{C} = \mathcal{E}\) if \(\mathcal{C}\)
is a stable full subcategory of \(\mathcal{C}\) closed under small colimits or equivalently if \(\mathcal{C}\) is closed under all suspensions and small colimits.

The first property follows from the definition of \(\mathcal{C}\), the second property from the fact that \(\eta : \text{id} \to G \circ F\) is a natural transformation between colimit preserving functors.

It remains to check that the essential image \(F(\mathcal{E})\) of \(F\) is compactly generated by \(F(\mathcal{E})\).

By assumption every object of \(F(\mathcal{E})\) is compact in \(\mathcal{D}\) and thus in particular compact in \(F(\mathcal{C})\) because the full subcategory inclusion \(F(\mathcal{C}) \subset \mathcal{D}\) preserves small colimits.

Therefore it is enough to see that the set \(\{Z \in F(\mathcal{E}) \mid \mathcal{D}(\Sigma^n(F(X)), Z)\) is contractible for all \(X \in \mathcal{E}\) and \(n \in \mathcal{Z}\}\) consists of zero objects.

The equivalence
\[
\mathcal{C}(\Sigma^n(X), Y) \simeq \mathcal{C}(\Sigma^n(X), G(F(Y))) \simeq \mathcal{D}(\Sigma^n(F(X)), F(Y)) \simeq \mathcal{D}(\Sigma^n(F(X)), F(Y))
\]
for \(X, Y \in \mathcal{E}\) implies that \(F(\mathcal{E}^\perp) = F(\mathcal{E}^\perp)\), where \(\mathcal{E}^\perp = \{Y \in \mathcal{E} \mid \mathcal{C}(\Sigma^n(X), Y)\) is contractible for all \(X \in \mathcal{E}\) and \(n \in \mathcal{Z}\}\).

Thus \(F(\mathcal{E}^\perp) = F(\mathcal{E}^\perp)\) consists of zero objects if \(\mathcal{E}^\perp\) does.

**Proposition 4.2.** Let \(\mathcal{C}\) be a cocomplete stable category, \(\kappa\) a regular cardinal and \(\mathcal{E}\) a set of \(\kappa\)-compact objects of \(\mathcal{C}\).

Then the following conditions are equivalent:

1. A morphism \(f : U \to V\) of \(\mathcal{C}\) is an equivalence if \(\mathcal{C}(\Sigma^n(X), f) : \mathcal{C}(\Sigma^n(X), U) \to \mathcal{C}(\Sigma^n(X), V)\) is an equivalence for all \(X \in \mathcal{E}\) and \(n \in \mathcal{Z}\).

2. The set \(\mathcal{E}^\perp = \{Y \in \mathcal{E} \mid \mathcal{C}(\Sigma^n(X), Y)\) is contractible for all \(X \in \mathcal{E}\) and \(n \in \mathcal{Z}\}\) consists of zero objects.
3. Let \((C_\lambda)_{\lambda \leq \kappa}\) be the inductively defined increasing sequence of full subcategories of \(\mathcal{C}\) with

- \(C_0\) the full subcategory of \(\mathcal{C}\) spanned by all retracts of suspensions of objects of \(\mathcal{E}\),
- \(C_{\lambda+1}\) the full subcategory of \(\mathcal{C}\) spanned by all retracts of colimits of \(\kappa\)-small diagrams in \(C_\lambda\) for \(\lambda < \kappa\) and \(C_\lambda = \bigcup_{\alpha < \lambda} C_\alpha\) for every limit ordinal \(\lambda \leq \kappa\).

Then \(C_\kappa\) is essentially small and the full subcategory inclusion \(C_\kappa \subset \mathcal{C}\) extends to an equivalence \(\text{Ind}_\kappa(C_\kappa) \simeq \mathcal{C}\).

4. \(\mathcal{C}\) is the only stable full subcategory of \(\mathcal{C}\) containing \(\mathcal{E}\) and closed under small colimits.

In particular \(\mathcal{C}\) is \(\kappa\)-accessible if and only if one of the equivalent conditions of the proposition holds.

**Proof.** We start by showing that 1. implies 2.:

Let \(Y \in \mathcal{E}\) be an object.

Given an object \(X \in \mathcal{E}\) and a natural \(n \in \mathbb{Z}\) the unique morphism \(f : Y \to 0\) to the zero object \(0\) of \(\mathcal{C}\) induces an equivalence \(\mathcal{E}(\Sigma^n(X), f) : \mathcal{E}(\Sigma^n(X), Y) \to \mathcal{E}(\Sigma^n(X), 0)\) being a morphism between contractible spaces by the choice of \(Y\).

If 1. is satisfied, one concludes that \(f : Y \to 0\) is an equivalence.

We continue by showing that 3. follows from 2.:

Let \((C_\lambda)_{\lambda \leq \kappa}\) be defined as in the statement of the proposition.

Using induction it follows that every \(C_\lambda\) for \(\lambda \leq \kappa\) is closed under arbitrary suspensions and retracts and consists of \(\kappa\)-compact objects of \(\mathcal{C}\):

- \(C_0\) is closed under arbitrary suspensions and retracts by definition and consists of \(\kappa\)-compact objects because \(\mathcal{E}\) consists of \(\kappa\)-compact objects, \(\kappa\)-compact objects are preserved by equivalences and the full subcategory of \(\mathcal{C}\) spanned by \(\kappa\)-compact objects is closed under retracts.

For limit ordinals \(\lambda \leq \kappa\) it trivially follows that \(C_\lambda = \bigcup_{\alpha < \lambda} C_\alpha\) is closed under arbitrary suspensions and retracts and consists of \(\kappa\)-compact objects of \(\mathcal{C}\) if \(C_\alpha\) does for all \(\alpha < \lambda\).

By definition \(C_{\lambda+1}\) is closed under retracts for every \(\lambda < \kappa\).

If \(\lambda < \kappa\) and \(C_\lambda\) is closed under arbitrary suspensions, then arbitrary suspensions of retracts of colimits of \(\kappa\)-small diagrams in \(C_\lambda\) are retracts of colimits of the suspended \(\kappa\)-small diagrams which lie in \(C_\lambda\). So \(C_{\lambda+1}\) is closed under arbitrary suspensions.

If \(\lambda < \kappa\) and \(C_\lambda\) consists of \(\kappa\)-compact objects of \(\mathcal{C}\), then every retract of a colimit of a \(\kappa\)-small diagram in \(C_\lambda\) is \(\kappa\)-compact because the full subcategory of \(\mathcal{C}\) spanned by the \(\kappa\)-compact objects is closed under \(\kappa\)-small colimits and retracts. \[\text{[cor. 5.3.4.15]}\] So \(C_{\lambda+1}\) consists of \(\kappa\)-compact objects of \(\mathcal{C}\).

As \(\kappa\) is a regular cardinal, \(C_\kappa\) is closed under \(\kappa\)-small colimits and especially under finite colimits.

Consequently \(C_\kappa\) is a full stable subcategory of \(\mathcal{C}\).

Because \(\mathcal{C}\) is locally small and \(\mathcal{E}\) is a set, \(C_\kappa \subset \mathcal{C}\) is locally small and has a set of equivalence classes of objects and is thus essentially small.

As \(\mathcal{C}\) is cocomplete, there exists a unique extension \(F : \text{Ind}_\kappa(C_\kappa) \to \mathcal{C}\) of \(C_\kappa \subset \mathcal{C}\) preserving \(\kappa\)-filtered colimits, which is fully faithful because \(C_\kappa \subset \mathcal{C}\) is
fully faithful and \( \mathcal{C}_\kappa \) consists of \( \kappa \)-compact objects of \( \mathcal{C} \). [2][prop. 5.3.5.10., prop. 5.3.5.11.]

Because \( \mathcal{C}_\kappa \) admits \( \kappa \)-small colimits, \( \text{Ind}_\kappa(\mathcal{C}_\kappa) \subset \text{Fun}(\mathcal{C}_\kappa^{\text{op}}, \mathcal{S}) \) coincides with the full subcategory spanned by the functors \( \mathcal{C}_\kappa^{\text{op}} \to \mathcal{S} \) that preserve \( \kappa \)-small limits. [2][cor. 5.3.5.4.]

Therefore \( \text{Ind}_\kappa(\mathcal{C}_\kappa) \) is an accessible localisation of \( \text{Fun}(\mathcal{C}_\kappa^{\text{op}}, \mathcal{S}) \) and thus is presentable. Moreover with \( \mathcal{C}_\kappa \) also \( \text{Ind}_\kappa(\mathcal{C}_\kappa) \) is stable:

The Yoneda-embedding \( y : \mathcal{C}_\kappa \to \text{Ind}_\kappa(\mathcal{C}_\kappa) \) from \( \mathcal{C}_\kappa \) to the full subcategory of \( \text{Fun}(\mathcal{C}_\kappa^{\text{op}}, \mathcal{S}) \) spanned by the functors \( \mathcal{C}_\kappa^{\text{op}} \to \mathcal{S} \) that preserve \( \kappa \)-small limits preserves small limits and \( \kappa \)-small colimits.

Thus the image of the zero object of \( \mathcal{C}_\kappa \) under the Yoneda-embedding is a zero object of \( \text{Ind}_\kappa(\mathcal{C}_\kappa) \) and the canonical natural transformation \( \Sigma_{\text{Ind}_\kappa(\mathcal{C}_\kappa)} \circ y \to y \circ \Sigma_{\mathcal{C}_\kappa} \) is an equivalence, where \( \Sigma_{\text{Ind}_\kappa(\mathcal{C}_\kappa)} : \text{Ind}_\kappa(\mathcal{C}_\kappa) \to \text{Ind}_\kappa(\mathcal{C}_\kappa) \) and \( \Sigma_{\mathcal{C}_\kappa} : \mathcal{C}_\kappa \to \mathcal{C}_\kappa \) denote the corresponding suspension functors.

Therefore \( \Sigma_{\text{Ind}_\kappa(\mathcal{C}_\kappa)} \) is the unique \( \kappa \)-filtered colimits preserving extension of \( y \circ \Sigma_{\mathcal{C}_\kappa} \) to \( \text{Ind}_\kappa(\mathcal{C}_\kappa) \) and is thus an equivalence as \( \Sigma_{\mathcal{C}_\kappa} \) is.

As next we will show that \( F : \text{Ind}_\kappa(\mathcal{C}_\kappa) \to \mathcal{C} \) preserves small colimits, where we use that \( \mathcal{C}_\kappa \) is closed under retracts, \( \mathcal{C} \) is cocomplete and the full subcategory inclusion \( \mathcal{C}_\kappa \subset \mathcal{C} \) preserves \( \kappa \)-small colimits:

As \( \mathcal{C}_\kappa \) is closed under retracts and \( \mathcal{C} \) is idempotent complete being a cocomplete category, \( \mathcal{C}_\kappa \) is also idempotent complete.

Thus the canonical idempotent completion functor \( \mathcal{C}_\kappa \to \text{Ind}_\kappa(\mathcal{C}_\kappa)^\kappa \) induced by \( \mathcal{C}_\kappa \to \text{Ind}_\kappa(\mathcal{C}_\kappa) \) to the full subcategory \( \text{Ind}_\kappa(\mathcal{C}_\kappa)^\kappa \) of \( \text{Ind}_\kappa(\mathcal{C}_\kappa) \) spanned by the \( \kappa \)-compact objects is an equivalence.

Consequently the restriction of \( F : \text{Ind}_\kappa(\mathcal{C}_\kappa) \to \mathcal{C} \) to \( \text{Ind}_\kappa(\mathcal{C}_\kappa)^\kappa \) preserves \( \kappa \)-small colimits so that \( F : \text{Ind}_\kappa(\mathcal{C}_\kappa) \to \mathcal{C} \) preserves small colimits by [2][prop. 5.5.1.9.].

As \( \text{Ind}_\kappa(\mathcal{C}_\kappa) \) is presentable, the adjoint functor theorem [2][cor. 5.5.2.9] implies that \( F : \text{Ind}_\kappa(\mathcal{C}_\kappa) \to \mathcal{C} \) has a right adjoint \( G : \mathcal{C} \to \text{Ind}_\kappa(\mathcal{C}_\kappa) \) which is an exact functor because \( \text{Ind}_\kappa(\mathcal{C}_\kappa) \) and \( \mathcal{C} \) are stable.

In the following we will see how condition 2. guarantees that \( F : \text{Ind}_\kappa(\mathcal{C}_\kappa) \to \mathcal{C} \) is an equivalence:

Being fully faithful \( F : \text{Ind}_\kappa(\mathcal{C}_\kappa) \to \mathcal{C} \) is an equivalence if and only if its right adjoint \( G : \mathcal{C} \to \text{Ind}_\kappa(\mathcal{C}_\kappa) \) reflects equivalences.

Let \( \phi : X \to Y \) be a morphism of \( \mathcal{C} \), whose image \( G(\phi) : G(X) \to G(Y) \) is an equivalence.

To see that \( \phi : X \to Y \) is an equivalence it is enough to check that the cofiber \( \text{cof}(\phi) \) of \( \phi : X \to Y \) in \( \mathcal{C} \) is a zero object of \( \mathcal{C} \).

But \( G(\text{cof}(\phi)) \cong \text{cof}(G(\phi)) \) is a zero object of \( \mathcal{C} \) as \( G(\phi) : G(X) \to G(Y) \) is an equivalence, so that

\[
\mathcal{C}(\Sigma^n(X), \text{cof}(\phi)) \cong \mathcal{C}(\Sigma^n(F(X)), \text{cof}(\phi)) \cong \mathcal{C}(\Sigma^n(X), G(\text{cof}(\phi)))
\]

is contractible for all \( X \in \mathcal{C} \) and \( n \in \mathbb{Z} \).

So 2. implies that \( \text{cof}(\phi) \) is a zero object of \( \mathcal{C} \).

As next we show that 3. implies 4.:
Let $3$. be satisfied and let $D$ be a stable full subcategory of $E$ containing $E$ and closed under small colimits and retracts.

Being closed under arbitrary suspensions and retracts $D$ contains $E_0$.

Using that $D$ is closed under small colimits and retracts it follows by induction that $D$ contains all $E_1$, for $\lambda \leq \kappa$ and especially contains $E_\kappa$.

As $D$ is cococomplete and locally small, there exists an extension $F' : \text{Ind}_n(E_\kappa) \to D$ of $E_\kappa \subset D$ preserving $\kappa$-filtered colimits.\cite[sec. 5.3.5.10.]{article}

The functors $F' : \text{Ind}_n(E_\kappa) \to D \subset E$ and $F : \text{Ind}_n(E_\kappa) \to E$ are equivalent because both functors preserve $\kappa$-filtered colimits and their restrictions to $E_\kappa$ coincide.

Thus $F' \circ F^{-1} : E \to \text{Ind}_n(E_\kappa) \to D \subset E$ is the identity so that $D = E$.

We complete the proof by verifying that $1.$ follows from $4.$

Let $f : U \to V$ be a morphism in $E$.

Consider the full subcategory $\bar{E}$ of $E$ spanned by all $X \in E$ having the property that for all $n \in \mathbb{Z}$ the induced map $E(\Sigma^n(X), f) : E(\Sigma^n(X), U) \to E(\Sigma^n(X), V)$ is an equivalence.

By definition $\bar{E}$ is closed under arbitrary suspensions.

Moreover $\bar{E}$ is closed under colimits:

Let $\Phi : K \to \bar{E}$ be a diagram in $\bar{E}$. Then $\bar{E}(\cdot, f) \circ \Sigma^n \circ \Phi : \bar{E}(\cdot, U) \circ \Sigma^n \circ \Phi \to \bar{E}(\cdot, V) \circ \Sigma^n \circ \Phi$ is an equivalence for all $n \in \mathbb{Z}$ and thus the induced map on limits $\lim(\bar{E}(\cdot, f) \circ \Sigma^n \circ \Phi) : \lim(\bar{E}(\cdot, U) \circ \Sigma^n \circ \Phi) \to \lim(\bar{E}(\cdot, V) \circ \Sigma^n \circ \Phi)$ is an equivalence for all $n \in \mathbb{Z}$.

The equivalence $\lim(\bar{E}(\cdot, f) \circ \Sigma^n \circ \Phi) \simeq \bar{E}(\colim(\Sigma^n \circ \Phi), f) = \bar{E}(\Sigma^n(\colim(\Phi)), f)$ in $\text{Fun}(\Delta^1, \bar{E})$ implies that $\colim(\Phi) \in \bar{E}$.

If $E(\Sigma^n(X), f) : E(\Sigma^n(X), U) \to E(\Sigma^n(X), V)$ is an equivalence for all $X \in E$ and $n \in \mathbb{Z}$, then $E \subset \bar{E}$ and one concludes that $\bar{E} = E$ assuming $4$. So by Yoneda $f : U \to V$ is an equivalence.

For $0 < k \leq \infty$ denote $E_k$ the $k$-th little cubes operad.

$\text{Mon}_{E_k}(\text{Cat}_{\alpha}(\kappa))$ the category of $k$-monoidal categories, $\text{Mon}_{E_k}(\text{Cat}_{\alpha}(\kappa))^{\text{rig}}$ the full subcategory spanned by the $k$-monoidal categories that consist of dualizable objects and $\text{Mon}_{E_k}(\text{Cat}_{\alpha}(\kappa))^{\text{pic}}$ the full subcategory spanned by the $k$-monoidal categories that consist of tensorinvertible objects.

In the following we show the existence of left adjoints of the full subcategory inclusions $\text{Mon}_{E_k}(\text{Cat}_{\alpha}(\kappa))^{\text{rig}} \subset \text{Mon}_{E_k}(\text{Cat}_{\alpha}(\kappa))$ and $\text{Mon}_{E_k}(\text{Cat}_{\alpha}(\kappa))^{\text{pic}} \subset \text{Mon}_{E_k}(\text{Cat}_{\alpha}(\kappa))$.

We deduce the existence of these left adjoints from the following lemma.\cite[4.8]{article}

**Lemma 4.3.** Let $D$ be a presentable category and $E \subset D$ be a full subcategory such that the full subcategory inclusion $\iota : E \subset D$ admits a right adjoint $G : D \to E$.

Then the following conditions are equivalent:

1. $\iota : E \subset D$ admits a left adjoint $F : D \to E$ such that the composition $\iota \circ F : D \to E \subset D$ is an accessible functor.

In this case $E$ is presentable.
2. \( \mathcal{C} \) is closed under small limits in \( \mathcal{D} \) and \( G: \mathcal{D} \to \mathcal{C} \) is an accessible functor. (equivalently \( 2 \circ G: \mathcal{D} \to \mathcal{C} \subset \mathcal{D} \) is an accessible functor that preserves small limits.)

Proof. \( \mathcal{C} \) is bicomplete being a colocalization of a bicomplete category.

If 1. holds, the full subcategory inclusion \( \iota: \mathcal{C} \subset \mathcal{D} \) preserves small limits being a right adjoint so that \( \mathcal{C} \) is closed under small limits in \( \mathcal{D} \).

If 1. holds, \( \mathcal{C} \subset \mathcal{D} \) is an accessible localization so that \( \mathcal{C} \) is accessible.

Hence \( G: \mathcal{D} \to \mathcal{C} \) is an accessible functor being a right adjoint functor between accessible categories.

Assume that 2. holds.

The composition \( R = \iota \circ G: \mathcal{D} \to \mathcal{C} \subset \mathcal{D} \) is an accessible functor that preserves small limits.

As \( \mathcal{D} \) is presentable, \( R: \mathcal{D} \to \mathcal{D} \) admits a left adjoint \( T: \mathcal{D} \to \mathcal{D} \) by the adjoint functor theorem.

Denote \( \varepsilon: R = \iota \circ G \to \text{id}_\mathcal{D} \) the counit.

As \( \iota: \mathcal{C} \subset \mathcal{D} \) is fully faithful, the unit \( \text{id}_\mathcal{C} \to G \circ \iota \) is an equivalence so that the composition \( G \circ \varepsilon: G \circ \iota \circ G \to G \) and thus \( R \circ \varepsilon: R \circ \iota \to R \) is an equivalence by the triangular identities.

Let \( X, Y \) be objects of \( \mathcal{D} \). The map \( \mathcal{D}(\iota(X), \iota(Y)) \to \mathcal{D}(\iota(X), \iota(Y)) \) induced by the counit \( \varepsilon(Y): R(Y) \to Y \) is equivalent to the map \( \mathcal{D}(\iota(X), \iota(Y)) \to \mathcal{D}(\iota(X), \iota(Y)) \to \mathcal{D}(\iota(X), \iota(Y)) \) induced by the counit \( R(\varepsilon(Y)): R \circ \iota \circ G \to Y \) and thus is an equivalence.

Thus \( T(X) \) is a colocal object of \( \mathcal{D} \) and so belongs to \( \mathcal{C} \).

Consequently \( T: \mathcal{D} \to \mathcal{D} \) admits a factorization \( F: \mathcal{D} \to \mathcal{C} \subset \mathcal{D} \).

As \( T: \mathcal{D} \to \mathcal{D} \) is left adjoint to \( R: \mathcal{D} \to \mathcal{D} \), the functor \( F: \mathcal{D} \to \mathcal{C} \) is left adjoint to \( \iota \circ R \circ \iota: \mathcal{C} \to \mathcal{D} \).

\( \Box \)

We apply lemma 4.13 to the following situation:

Let \( 0 < k \leq \infty \).

We take \( \mathcal{D} = \text{Mon}_{E_k}(\text{Cat}_\infty(k)) \) to be the category of \( k \)-monoidal categories and \( \mathcal{C} = \text{Mon}_{E_k}(\text{Cat}_\infty(k))^{\text{ris}} \) to be the full subcategory of rigid \( k \)-monoidal categories.

Denote \( \text{Rig}_k: \text{Mon}_{E_k}(\text{Cat}_\infty(k)) \to \text{Mon}_{E_k}(\text{Cat}_\infty(k))^{\text{ris}} \) the full subfunctor of the identity with \( \text{Rig}(B) \subset B \) the \( k \)-monoidal subcategory spanned by the dualizable objects of \( B \) for every \( k \)-monoidal category \( B \).

For every rigid \( k \)-monoidal category \( A \) composition with the \( k \)-monoidal subcategory inclusion \( \text{Rig}_k(B) \subset B \) yields an equivalence \( \text{Fun}^{k}(A, \text{Rig}_k(B)) \to \text{Fun}^{k}(A, B) \).

Consequently \( \text{Rig}_k: \text{Mon}_{E_k}(\text{Cat}_\infty(k)) \to \text{Mon}_{E_k}(\text{Cat}_\infty(k))^{\text{ris}} \) is right adjoint to the full subcategory inclusion \( \text{Mon}_{E_k}(\text{Cat}_\infty(k))^{\text{ris}} \subset \text{Mon}_{E_k}(\text{Cat}_\infty(k)) \).

Moreover \( \text{Rig}_k: \text{Mon}_{E_k}(\text{Cat}_\infty(k)) \to \text{Mon}_{E_k}(\text{Cat}_\infty(k))^{\text{ris}} \subset \text{Mon}_{E_k}(\text{Cat}_\infty(k)) \) preserves small limits and small filtered colimits:

As the forgetful functor \( \text{Mon}_{E_k}(\text{Cat}_\infty(k)) \to \text{Cat}_\infty(k) \) reflects small limits and small filtered colimits, it is enough to check that the composition \( \text{Mon}_{E_k}(\text{Cat}_\infty(k)) \to \text{Mon}_{E_k}(\text{Cat}_\infty(k))^{\text{ris}} \subset \text{Mon}_{E_k}(\text{Cat}_\infty(k)) \to \text{Cat}_\infty(k) \) of \( \text{Rig}_k \) and the forgetful functor \( \text{Mon}_{E_k}(\text{Cat}_\infty(k)) \to \text{Cat}_\infty(k) \) preserves small limits and small filtered colimits.
This composition is equivalent to the composition $\text{Mon}_E(\text{Cat}_\infty(\kappa)) \to \text{Mon}(\text{Cat}_\infty(\kappa)) \to \text{Mon}(\text{Cat}_\infty(\kappa))^{\text{rig}} \subset \text{Mon}(\text{Cat}_\infty(\kappa)) \to \text{Cat}_\infty(\kappa)$ of the forgetful functor $\text{Mon}_E(\text{Cat}_\infty(\kappa)) \to \text{Mon}(\text{Cat}_\infty(\kappa))$, the functor $\text{Rig}_1 : \text{Mon}(\text{Cat}_\infty(\kappa)) \to \text{Mon}(\text{Cat}_\infty(\kappa))^{\text{rig}} \subset \text{Mon}(\text{Cat}_\infty(\kappa))$ and the forgetful functor $\text{Mon}(\text{Cat}_\infty(\kappa)) \to \text{Cat}_\infty(\kappa)$.

As the forgetful functor $\text{Mon}_E(\text{Cat}_\infty(\kappa)) \to \text{Mon}(\text{Cat}_\infty(\kappa))$ preserves small limits and small filtered colimits, we can reduce to the case $k = 1$.

Given a small limit diagram $\alpha : K^\circ \to \text{Mon}(\text{Cat}_\infty(\kappa))$ and a small filtered colimit diagram $\beta : K^\circ \to \text{Mon}(\text{Cat}_\infty(\kappa))$ the induced functors $\text{lim}(\text{Rig}_1 \circ \alpha) \to \text{lim}(\alpha)$ and $\text{colim}(\text{Rig}_1 \circ \beta) \to \text{colim}(\beta)$ are fully faithful and thus also the comparison functors $\text{Rig}_1(\text{lim}(\alpha)) \to \text{lim}(\text{Rig}_1 \circ \alpha)$ and $\text{colim}(\text{Rig}_1 \circ \beta) \to \text{Rig}_1(\text{colim}(\beta))$ are.

So the comparison functors are equivalences if and only if they are essentially surjective.

Being a functor between complete categories $\text{Rig}_1$ preserves small limits if and only if it preserves arbitrary small products and pullbacks.

The case of arbitrary small products is clear and the case of pullbacks and small filtered colimits follows from the uniqueness of duals of a given object.

So we can take $G : \mathcal{D} \to \mathcal{C}$ to be $\text{Rig}_k$ to obtain a free rigid $k$-monoidal category functor $\mathcal{F}_k^{\text{rig}} : \text{Mon}_E(\text{Cat}_\infty(\kappa)) \to \text{Mon}_E(\text{Cat}_\infty(\kappa))^{\text{rig}}$ with $\mathcal{F}_k^{\text{rig}}(*) \simeq \text{Bord}$.

Similarly one can take $\mathcal{D} = \text{Mon}_E(\text{Cat}_\infty(\kappa))$ to be the category of $k$-monoidal categories and $\mathcal{C} = \text{Mon}_E(\text{Cat}_\infty(\kappa))^{\text{pic}}$ to be the full subcategory of $k$-monoidal categories that consist of tensorinvertible objects.

Denote $\text{Pic}_k : \text{Mon}_E(\text{Cat}_\infty(\kappa)) \to \text{Mon}_E(\text{Cat}_\infty(\kappa))^{\text{pic}}$ the full subfunctor of the identity with $\text{Pic}_k(\mathcal{B}) \subset \mathcal{B}$ the $k$-monoidal subcategory spanned by the tensorinvertible objects of $\mathcal{B}$ for every $k$-monoidal category $\mathcal{B}$.

For every $A \in \text{Mon}_E(\text{Cat}_\infty(\kappa))^{\text{pic}}$ composition with the $k$-monoidal subcategory inclusion $\text{Pic}_k(\mathcal{B}) \subset \mathcal{B}$ yields an equivalence $\text{Fun}^{\otimes,k}(A, \text{Pic}_k(\mathcal{B})) \to \text{Fun}^{\otimes,k}(A, \mathcal{B})$.

Consequently $\text{Pic}_k : \text{Mon}_E(\text{Cat}_\infty(\kappa)) \to \text{Mon}_E(\text{Cat}_\infty(\kappa))^{\text{pic}}$ is right adjoint to the full subcategory inclusion $\text{Mon}_E(\text{Cat}_\infty(\kappa))^{\text{pic}} \subset \text{Mon}_E(\text{Cat}_\infty(\kappa))$.

By a similar argument one shows that $\text{Pic}_k : \text{Mon}_E(\text{Cat}_\infty(\kappa)) \to \text{Mon}_E(\text{Cat}_\infty(\kappa))^{\text{pic}} \subset \text{Mon}_E(\text{Cat}_\infty(\kappa))$ preserves small limits and small filtered colimits.

Taking $G : \mathcal{D} \to \mathcal{C}$ to be $\text{Pic}_k$ we get a free functor $\mathcal{F}_k^{\text{pic}} : \text{Mon}_E(\text{Cat}_\infty(\kappa)) \to \text{Mon}_E(\text{Cat}_\infty(\kappa))^{\text{pic}}$ with $\mathcal{F}_k^{\text{pic}}(*) \simeq \text{QS}$.

References

[1] Jacob Lurie. Higher Algebra. available at http://www.math.harvard.edu/~lurie/.

[2] Jacob Lurie. Higher topos theory, volume 170 of Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2009.

[3] Markus Spitzweck. Periodizable motivic ring spectra. arXiv:0907.1510.