LOCALIZATION FOR ONE-DIMENSIONAL TWO-PARTICLE RANDOM SCHRÖDINGER OPERATORS WITH POISSON POTENTIAL

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Abstract. We prove the complete spectral and the strong dynamical Anderson localization in a two-particle random Schrödinger operators with the Poisson potential. The results apply with sufficiently weak interaction between the particle system.

1. Introduction, assumptions and the main results

1.1. Introduction. In this work we consider a system of two-particle Anderson model with a Poisson potential in the continuous one-dimensional space and prove the localization results (Anderson spectral localization and the strong dynamical localization ) for a sufficiently weakly interacting particle system. The novelty of this problem is that for the Poisson potential, we have a lack of monotonicity in the random parameter. A property which was successfully used in proofs of localization for Anderson-type models [8, 18, 20].

This difficulty was earlier overcome in the works by Stolz [21, 22]. Recall that in [19], the authors studied the spectra of random operators and almost periodic operators. We can find in the books by Carmona et al. [6,9] some materials on spectral theory of random Schrödinger operators for one and higher dimensional models.

The theory of multi-particle models such as two-particle Anderson models is relatively recent and constitute a new direction in the spectral theory of random schrödinger operators [1, 5].

In our earlier work [14] in multi-particle Anderson models in one dimension, we prove the complete spectral and strong dynamical localization for the weakly interacting multi-particle system. While the continuous version of the work can be found in [14].

Let now discuss, on the structure of the paper: in the next Section, we present the model and state the assumptions and the main results. Section 2 is devoted to the initial length scale estimates of the multi-particle multi-scale analysis. In Section 3 we prove the initial length scales estimates of the multi-scale analysis. In Section 4 we prove the multi-scale induction step of the multi-scale analysis. In the last Section, Section 6 we prove the main results on spectral localization Theorem 1.1 and dynamical localization Theorem 1.2.

1.2. The model and the assumptions. The two-particle one-dimensional Anderson model with a Poisson random potential is given by the Schrödinger Hamiltonian

\[ H^{(2)}_h(\omega) = -\Delta + V(x, \omega) + hU(x), \quad x = (x_1, x_2) \in \mathbb{R}^2, \]

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acting on $L^2(\mathbb{R}^2)$, where

$$V(x,\omega) = \sum f(x - X_i(\omega)), $$

with $f \in L^2(\mathbb{R})$ and where $\{X_i(\omega)\}$ is a finite set of points $X_i(\omega) \in \mathbb{R}$ so that $V$ is a random variable relative to some probability space $(\Omega, \mathcal{B}, \mathbb{P})$ and acts on $L^2(\mathbb{R}^2)$ as a multiplication operator by the function $V(x)$. Also $U$ is the interaction potential between the two-particle and acts on $L^2(\mathbb{R}^2)$ as a multiplication operator by the function $U(x)$.

Set $\Omega = \mathbb{R}^{DZ^d}$ and $\mathcal{B} = \otimes B(\mathbb{R})$ where $B(\mathbb{R})$ is the Borel sigma-algebra on $\mathbb{R}$. Let $\mu$ be a probability measure on $\mathbb{R}$ and define $\mathbb{P} = \otimes_{Z^d} \mu$ on $\Omega$.

(P) Log-Hölder continuity condition. The random potential $V : \mathbb{Z}^d \times \Omega \to \mathbb{R}$ is i.i.d. and the corresponding probability distribution function $F_V$ is log-Hölder continuous: More precisely

$$s(F_V, \varepsilon) := \sup_{a \in \mathbb{R}} |F_V(a + \varepsilon) - F_V(a)| \leq \frac{C}{|\ln \varepsilon|^{2A}}$$

for some $C \in (0, \infty)$ and $A \in (\frac{3}{2} \times 4^N + 9Nd, \infty)$.

Further, the single-site potential $f$ is non negative and compactly supported.

(I) Short-range interaction. The interaction potential $U$ is bounded and there exists $r_0 \in \mathbb{N}$ such that

$$U(x_1, x_2) = 0 \ \text{if} \ |x_1 - x_2| \geq r_0$$

1.3. The results.

Theorem 1.1. Let $d = 1$. Under assumption (I) and (P) there exists $h^* \in (0, \infty)$ such that for any $h \leq |h^*|$ the Hamiltonian $H_h^{(N)}$ with interaction of amplitude $|h|$ exhibits complete Anderson localization, i.e., with $\mathbb{P}$-probability one, the spectrum of $H_h^{(N)}$ is pure point and each eigenfunction $\Psi$ is exponentially decaying fast at infinity:

$$\|\chi_x \cdot \Psi\| \leq Ce^{-c|x|}$$

for some positive constants $c, C$.

Theorem 1.2. Under assumptions (I) and (P) there exists $h^*, s^* \in (0, \infty)$ such that for any $h \leq |h|$ and any $s \leq |s^*|$ and any compact domain $K \subset \mathbb{R}^{Nd}$ we have that the quantity

$$\mathbb{E} \left[ \sup_{t \geq 0} \left\| X(t) - 2 e^{iH^{(N)}(\omega)} P_I(H^{(N)}(\omega)) 1_K \right\|_{L^2(\mathbb{R}^{Nd})} \right]$$

is finite, where $\langle X(\Psi)(x) \rangle := \langle x | \Psi(x) \rangle$. $P_I(H^{(N)}(\omega))$ is the spectral projection onto the interval $I$ and $1_K$ is the characteristic function of the set $K$.

2. The multi-particle multi-scale analysis scheme

2.1. Geometric facts. According to the general structure of the multi-scale analysis we work with rectangular domains. For $u = (u_1, \ldots, u_n) \in \mathbb{Z}^{nd}$, we denote by $C_L^{(n)}(u)$ the $n$-particle cube, i.e.,

$$C_L^{(n)}(u) = \{ x \in \mathbb{R}^{nd} : |x - u| \leq L \},$$

and given $\{L_i : i = 1, \ldots, n\}$, we define the rectangle

$$(2.1) \quad C^{(n)}(u) = \prod_{i=1}^{n} C_{L_i}^{(1)}(u_i),$$
where $C_L^{(i)}(u_i)$ are the cubes of side length $2L_i$, center at points $u_i \in \mathbb{Z}^d$. We also define
\[
C_L^{(n,\text{int})}(u) := C_L^{(n)}(u), \quad C_L^{(n,\text{out})}(u) := C_L^{(n)}(u) \setminus C_L^{(n)}(u), \quad u \in \mathbb{Z}^d
\]
and introduce the characteristic functions:
\[
1_{x}^{(n,\text{int})} := 1_{C_L^{(n,\text{int})}(x)}, \quad 1_{x}^{(n,\text{out})} := 1_{C_L^{(n,\text{out})}(x)}.
\]
The volume of the cube $C_L^{(n)}(u)$ is $|C_L^{(n)}(u)| = (2L)^n$. We denote the restriction of the Hamiltonian $H$ to $C^{(n)}(u)$ by
\[
H^{(n)}(u) = H^{(n)}|_{C^{(n)}(u)}
\]
with dirichlet boundary conditions.

We denote the spectrum of $H^{(n)}_{C^{(n)}(u)}$ by $\sigma(H^{(n)}_{C^{(n)}(u)})$ and its resolvent by
\[
G^{(n)}_{C^{(n)}(u)}(E) := \left(H^{(n)}_{C^{(n)}(u)} - E\right)^{-1}, \quad E \in \mathbb{R} \setminus \sigma(H^{(n)}_{C^{(n)}(u)}).
\]

Let $m$ be a positive constant and consider $E \in \mathbb{R}$. A cube $C_L^{(n)}(u) \subset \mathbb{R}^d$, $1 \leq n \leq N$ will be called $(E,m)$-non-singular $(\text{E,N-S})$ if $E \notin \sigma(H^{(n)}_{C_L^{(n)}(u)})$ and
\[
\|1_x^{(n,\text{out})}G^{(n)}_{C_L^{(n)}(x)}(E)1_x^{(n,\text{int})}\| \leq e^{-\gamma(m,L,n)L},
\]
where
\[
\gamma(m,L,n) = m(1 + L^{-1/8})N^{-n+1}.
\]
Otherwise, it is called $(E,m)$-singular $(\text{E,S})$.

Let us introduce the following:

**Definition 2.1.** Let $n \geq 1$, $E \in \mathbb{R}$ and $\alpha = 3/2$.

A) A cube $C_L^{(n)}(v) \subset \mathbb{R}^d$ is called $E$-resonant (E-R) if
\[
\text{dist} \left[E, \sigma(H^{(n)}_{C_L^{(n)}(v)})\right] \leq e^{-L^{1/2}},
\]
otherwise, it is called $E$-non-resonant (E-NR).

B) A cube $C_L^{(n)}(v) \subset \mathbb{R}^d$ is called $E$-completely non-resonant (E-CNR), if it does not contain any E-R cube of size $\geq L^{1/\alpha}$. In particular $C_L^{(n)}(v)$ is itself E-NR.

We will also make use of the following notion.

**Definition 2.2.** A cube $C_L^{(i)}(x)$ is $J$-separable from $C_L^{(i)}(y)$ if there exists a non empty subset $J \subset \{1, \ldots, n\}$ such that
\[
\left(\bigcup_{j \in J} C_L^{(i)}(x_j)\right) \cap \left(\bigcup_{j \notin J} C_L^{(i)}(x_j) \cup \bigcup_{j=1}^n C_L^{(i)}(y_j)\right) = \emptyset.
\]
A pair $(C_L^{(i)}(x), C_L^{(i)}(y))$ is separable if $|x - y| \geq 7NL$ and if one of the cube is $J$-separable from the other.

**Lemma 2.1.** Let $L \geq 1$. 

A) For any \( x \in \mathbb{Z}^n \), there exists a collection of \( n \)-particle cubes \( C_{2nL}^{(n)}(x^{(\ell)}) \) with \( \ell = 1, \ldots, \kappa(n) \), \( \kappa(n) = n^a \), \( x^{(\ell)} \in \mathbb{Z}^n \) such that if \( y \in \mathbb{Z}^n \) satisfies 
\[
|y - x| \geq 7NL
\]
then the cubes \( C_{L}^{(n)}(x) \) and \( C_{L}^{(n)}(y) \) are separable.

B) Let \( C_{L}^{(n)}(y) \subset \mathbb{R}^n \) be an \( n \)-particle cube. Any cube \( C_{L}^{(n)}(x) \) with 
\[
|y - x| \geq \max_{1 \leq i,j \leq n} |y_i - y_j| + 5NL
\]
is \( J \)-separable from \( C_{L}^{(n)}(y) \) for some \( J \subset \{1, \ldots, n\} \).

Proof. See the appendix Section 7

2.2. The multi-particle Wegner estimates. In our earlier work [15] as well in other previous papers in the multi-particle localization theory [3, 5] the notion of separability was crucial in order to prove the Wegner estimates for pairs of multi-particle cubes via the Stollmann’s Lemma. It is Plain (cf. [15] Section 4.1) that sufficiently distant pairs of fully interactive cubes have disjoint projections and this fact combined with independence is used in that case to bound the probability of an intersection of events relative to those projections. We state below the Wegner estimates directly in a form suitable to the multi-particle multi-scale analysis using assumption (P).

**Theorem 2.1.** Assume that the random potential satisfies assumption (P), then

A) For any \( E \in \mathbb{R} \)
\[
\mathbb{P}\left\{ C_{L}^{(n)}(x) \text{ is not } E\text{-CNR} \right\} \leq L^{-p4^{N-n}}.
\]

B) \[
\mathbb{P}\left\{ \exists E \in \mathbb{R} \text{ neither } C_{L}^{(n)}(x) \text{ nor } C_{L}^{(n)}(y) \text{ is } E\text{-CNR} \right\} \leq L^{-p4^{N-n}}
\]
where \( p \geq 6Nd \), depends only on the fixed number of particles \( N \) and the configuration dimension \( d \).

Proof. See the article [3,10].

We also give the Combes-Thomas estimates in

**Theorem 2.2.** Let \( H = -\Delta + W \) be a Schrödinger operator on \( L^2(\mathbb{R}^D) \), \( E \in \mathbb{R} \) and \( E_0 = \inf \sigma(H) \). Set \( \eta = \text{dist}(E, \sigma(H)) \). If \( E \) is less than \( E_0 \), then for any \( \gamma \in (0, 1) \), we have that 
\[
\|1_x(H - E)^{-1}1_y\| \leq \frac{1}{(1 - \gamma^2)\eta^3} e^{-\gamma \sqrt{\eta}|x-y|},
\]
for all \( x,y \in \mathbb{R}^D \).

Proof. See the proof of Theorem 1 in [16].

We define the mass \( m \) depending on the parameters \( N, \gamma \) and the initial length scale \( L \) in the following way:
\[
m := \frac{2^{-N} \gamma L^{-1/4}}{3\sqrt{2}}.
\]
We recall below the geometric resolvent and the eigenfunction decay inequalities.
Theorem 2.3 (Geometric resolvent inequality (GRI)). For a given bounded \( I_0 \subset \mathbb{R} \). There is a positive constant \( C_{\text{geom}} \) such that for \( C_L^{(n)}(x) \subset C_L^{(n)}(u) \), \( A \subset C_L^{(n,int)}(x) \), \( B \subset C_L^{(n)}(u) \setminus C_L^{(n)}(x) \) and \( E \in I_0 \), the following inequality holds true:

\[
\|1_B G^{(n)}_{C_L^{(n)}(u)}(E)1_A\| \leq C_{\text{geom}} \cdot \|1_B G^{(n)}_{C_L^{(n)}(u)}(E)1_{C_L^{(n,int)}(x)}\|.
\]

\[
\|1_{C_L^{(n,\text{out})}(x)}(E) G^{(n)}_{C_L^{(n)}(x)}(E)1_A\| \leq \|1_{C_L^{(n,\text{out})}(x)}(E) G^{(n)}_{C_L^{(n)}(x)}(E)1_A\|.
\]

Proof. See [17], Lemma 2.5.4.

Theorem 2.4 (Eigenfunctions decay inequality (EDI)). For every \( E \in \mathbb{R} \), \( C_L^{(n)}(x) \subset \mathbb{R}^d \) and every polynomially bounded function \( \Psi \in L^2(\mathbb{R}^d) \):

\[
\|1_{C_L^{(n)}(x)}(E) \cdot \Psi\| \leq C \cdot \|1_{C_L^{(n,\text{out})}(x)} G^{(n)}_{C_L^{(n)}(x)}(E)1_{C_L^{(n,\text{int})}(x)}\| \cdot \|1_{C_L^{(n,\text{out})}(x)} \cdot \Psi\|.
\]

Proof. See Section 2.5 and Proposition 3.3.1. in [17].

3. The initial bounds of the multi-particle multi-scale analysis

3.1. The fixed energy MSA bound for the n-particle system without interaction. We begin with the well known single-particle exponential localization for the eigenfunctions and for one-dimensional Anderson models in the continuum proved in the paper by Damanik et al. [12]. Let \( H_{C_L^{(1)}(x)}^{(1)}(\omega) \) be the restriction of the single-particle Hamiltonian into the cube \( C_L^{(1)}(x) \) and denote by \( \{\lambda_j, \phi_j\}_{j \geq 0} \), its eigenvalues and corresponding eigenfunctions. We have the following namely the single-particle exponential localization for the eigenfunctions in any cube.

Theorem 3.1 (Single-particle localization). There exists a constant \( \tilde{\mu} \in (0, \infty) \) such that for every generalized eigenfunctions \( \varphi \) of the single-particle Hamiltonian \( H_{C_L^{(1)}(x)}^{(1)}(\omega) \) we have:

\[
\mathbb{E} \left[ \|1_{C_L^{(1,\text{out})}(x)} \cdot \varphi \cdot 1_{C_L^{(1,\text{int})}(x)}\| \right] \leq e^{-\tilde{\mu}L}.
\]

Proof. We refer to the book by Stollmann [17].

The main result of this subsection is Theorem 3.2 given below. The proof of Theorem 3.2 relies on an auxiliary statement Lemma 4.3. We need to introduce first \( \{\lambda_j^{(i)}, \psi_j^{(i)} : j_i \geq 1\} \) the eigenvalues and the corresponding eigenfunctions of \( H_{C_L^{(1)}(x)}^{(1)}(\omega), i = 1, \ldots, n \). Then the eigenvalues of the non-interacting multi-particle random Hamiltonians \( H_{C_L^{(n)}(u)}^{(n)}(\omega) \) are written as sums:

\[
E_{j_1 \cdots j_n} = \sum_{j=1}^n \lambda_j^{(i)} = \lambda_1^{(i)} + \cdots + \lambda_n^{(i)}
\]

while the corresponding eigenfunctions \( \Psi_{j_1 \cdots j_n} \) can be chosen as tensor products

\[
\Psi_{j_1 \cdots j_n} = \psi_1^{(i)} \otimes \cdots \otimes \psi_j^{(n)}.
\]

The eigenfunctions of finite volume Hamiltonians are assumed normalized.

Theorem 3.2. Let \( 1 \leq n \leq N \) and \( I_0 \subset \mathbb{R} \) a bounded interval. there exists \( m^* \in (0, \infty) \) such that for any cube \( C_L^{(n)}(u) \) and all \( E \in I_0 \),

\[
\mathbb{P} \left\{ C_L^{(n)}(u) \text{ is } (E, m^*, 0)\text{-S} \right\} \leq \frac{1}{2} L_0^{-2p^*4^N-n}
\]

with \( L_0 \) large enough and \( p^* \in (6Nd, \infty) \).
The proof of Theorem 3.2 relies on the following auxiliary statement.

**Lemma 3.1.** Let be given \( N \geq n \geq 2 \) and \( m^* \in (0, \infty) \) a cube \( C_{L^0}^{(n)}(u) \) and \( E \in \mathbb{R} \). Suppose that \( C_{L^0}^{(n)}(u) \) is \( E-NR \), and for any operator \( H_{L^0}^{(1)}(u) \) all its eigenfunctions \( \psi_{j_0} \) satisfy

\[
1_{C_{L^0}^{(1)}(u)} \leq 2\gamma(m^*, L_0, n)L_0
\]

Then \( C_{L^0}^{(n)}(u) \) is \( (E, m^*, L_0) \)-NS provided that \( L_0 \geq L_0(m^*, N, d) \)

**Proof.** We choose the multi-particle eigenfunctions as tensor products of those of the single-particle Hamiltonians \( H_{L^0}^{(1)}(u) \), \( i = 1, \ldots, n \), i.e., \( \Psi_j = \varphi_j^{(1)} \otimes \cdots \otimes \varphi_j^{(n)} \) corresponding to the eigenvalues \( E_j = \lambda_{j_0}^{(1)} + \cdots + \lambda_{j_0}^{(1)} \). Now we have that

\[
G_{C_{L^0}^{(1)}}^{(n)}(E) = \sum_{j_0} \varphi_j^{(1)} \otimes \cdots \otimes \varphi_j^{(n)} \varphi_j^{(1)} \otimes \cdots \otimes \varphi_j^{(n)} \varphi_j^{(1)} \otimes \cdots \otimes \varphi_j^{(n)} \varphi_j^{(1)} \otimes \cdots \otimes \varphi_j^{(n)} (E - \lambda_{j_0}^{(1)})
\]

where \( \lambda_{j_0}^{(1)} = \sum_{1 \leq i \leq n - 1} \lambda_i \) so that

\[
1_{C_{L^0}^{(n,m,r)}}(u) G_{C_{L^0}^{(n)}}^{(n)}(E) 1_{C_{L^0}^{(n,m,r)}}(u) \leq 1_{C_{L^0}^{(n)}}(u) G_{C_{L^0}^{(n)}}^{(n)}(E) 1_{C_{L^0}^{(n)}}(u)
\]

By the Weyl’s law there exists \( E^* \in (0, \infty) \) such that \( \lambda_j \geq E^* \) for all \( j \geq j^* = C_{Weyl}[C_{L^0}^{(1)}(u)] \). Therefore, we divide the above sum into two parts as follows

\[
1_{C_{L^0}^{(n,m,r)}}(u) G_{C_{L^0}^{(n)}}^{(n)}(E) 1_{C_{L^0}^{(n,m,r)}}(u) \leq \sum_{j \leq j^*} \sum_{j \geq j^* + 1} \sum_{j \leq j^*} \sum_{j \geq j^* + 1}
\]

Since

\[
\| 1_{C_{L^0}^{(1)}}(u) \otimes 1_{C_{L^0}^{(n,m,r)}}(E - \lambda_{j_0}^{(1)}) \| \leq \| 1_{C_{L^0}^{(1)}}(u) \| \cdot \| \varphi_j^{(1)} \| \cdot e^{L^{1/2}} \leq e^{-2 \gamma(m^*, L_0, n)L_0^{1/2}}
\]

for \( L \in (L^*, N, d, C_{Weyl}, \infty) \) large enough where we used the hypotheses on the exponential decay of the eigenvalues of the single-particle Hamiltonian. Thus, the infinite sum can be made as small as an exponential decay provided that the length \( L_0 \) is large enough,

\[
\sum_{j \geq j^* + 1} \| 1_{C_{L^0}^{(1)}}(u) \otimes 1_{C_{L^0}^{(n,m,r)}}(E - \lambda_{j_0}^{(1)}) \| \leq \frac{1}{2} e^{-\gamma(m^*, L_0, n)L_0}
\]

while the finite can be bounded by

\[
n \cdot C_{Weyl} \cdot |C_{L^0}^{(1)}(u)| \cdot e^{-\gamma(m^*, n, L_0)L} e^{L/2} \leq \frac{1}{2} e^{-\gamma(m, L, n)L},
\]

which proves the Lemma.

Now we turn to the proof of Theorem 4.1.
Proof of Theorem 3.3. Recall that by the single-particle Anderson localization theory there exists $\bar{\mu} \in (0, \infty)$ such that such that we have the following decay bound on the exponential decay of the eigenfunctions: for all $u \in \mathbb{Z}^d$

\[
\|1_{C_L^{(1)}(u)} \cdot \Psi\| \leq e^{-\bar{\mu}|x|}.
\]

(3.1)

Set $m^* = 2^{-N-1}\bar{\mu}$ and introduce the events

$\mathcal{N} := \{\exists i = 1, \ldots, n : \exists \lambda_j \in \sigma(H^{(1)}_{C_L^{(1)}}(u_i)) : \|1_{C_L^{(1)}(u_i)} \cdot \varphi_j(u_i)\| \geq e^{-2\gamma(m^*, L_0, n) L_0}\}$

$\mathcal{R} := \{C_L^{(n)}(u) \text{ is E-NR}\}$

Then by Lemma 3.1, Eqn (3.1) and Theorem 2.1 (A), we have:

\[
\mathbb{P}\left\{C_L^{(n)}(u) \text{ is } (E, m^*, 0) - S\right\} \leq \mathbb{P}\{\mathcal{N}\} + \mathbb{P}\{\mathcal{R}\}
\]

\[
\leq \sum_{i=1}^n \sum_{j=0}^n \mathbb{E}\left[\|1_{C_L^{(1)}(u_i)} \cdot \varphi_j\|\right] + \mathbb{P}\{\mathcal{R}\}
\]

\[
\sum_{i=1}^n \sum_{j=0}^n e^{(-\bar{\mu}_i + 2\gamma(m^*, L_0, n)) L_0} + L_0^{-4N^p}.
\]

Since $2\gamma(m^*, n, L) \leq 2^{N+1}m^* \leq \bar{\mu}_1$ and $\bar{\mu}_1 - 2\gamma(m, L, n) \in (0, \infty)$. Using the Weyl’s law, we can divide the infinite sum above into two sums. Namely there exists a positive $E^*$ arbitrarily large such that $\lambda_j^{(1)} \geq E^*$ for $j \geq j^* = C_L^{(1)}(u_i)$ which yields

\[
\sum_{j \geq 0} e^{(-\bar{\mu}_j + 2\gamma(m^*, L_0, n)) L_0} = \left(\sum_{j \leq j^*} + \sum_{j \geq j^*+1}\right) e^{(-\bar{\mu}_j + 2\gamma(m^*, L_0, n)) L_0}.
\]

Above, the infinite sum can be made small than any polynomial power law provided that $L_0$ is large enough. We have

\[
\sum_{i} \left(\sum_{j \leq j^*} + \sum_{j \geq j^*+1}\right) e^{(-\bar{\mu}_j + 2\gamma(m^*, L_0, n)) L_0} \leq \frac{1}{2} L_0^{-3p^{4^{-N^p}}} + 1 \frac{1}{3} L_0^{-3p^{4^{-N^p}}} + L_0^{-4p^N} \leq L_0^{-3p^{4^{-N^p}}}.
\]

We state and give here the proof of some important results from the paper [14] which use the fact that we are in the weakly interacting regime. The positive constant $m^*$ is the one from theorem 3.3.

3.2. The fixed energy MSA bound for weakly interacting multi-particle systems. Now we derive the required initial estimate from its counterparts established for non-interacting systems.

Theorem 3.3. $1 \leq n \leq N$. Suppose that the Hamiltonians $H_{0}^{(n)}(\omega)$ (without inter-particle interaction) fulfills the following condition for all $E \in I$ and all $u \in \mathbb{Z}^d$

\[
(3.2) \quad \mathbb{P}\left\{C_{L_0}^{(n)}(u) \text{ is } (E, m^*, 0) - S\right\} \leq \frac{1}{2} L_0^{-2p*4^{-N^p}} \quad \text{with } p^* \in (6Nd, +\infty).
\]

Then there exists $h^* \in (0, \infty)$ such that for all $h \in (-h^*, h^*)$ the Hamiltonian $H_{h}^{(n)}(\omega)$ with interaction of amplitude $|h|$ satisfies a similar bound. There exist some $p \in (6Nd, +\infty)$, $m \in (0, +\infty)$ such that for all $E \in I$ and all $u \in \mathbb{Z}^d$

\[
\mathbb{P}\{C_{L_0}^{(n)}(u) \text{ is } (E, m, h) - S\} \leq \frac{1}{2} L_0^{-2p^N N^p}.
\]
By Theorem 2.1 applied to Hamiltonians 

\[ H^{(n)}_{C_L} (u), \quad h \in \mathbb{R}, \]

By definition a cube \( C_L^{(n)}(u) \) is \((E, m^*, 0)\)-NS iff

\[ \|1^{(n, out)}_u C^{(n)}_{C_L}(u)^{(n, int)} \| \leq e^{-\gamma(m, L, n)} L. \]

Therefore there exists sufficiently small positive \( \epsilon \) such that

\[ (3.3) \quad \|1^{(n, out)}_u C^{(n)}_{C_L}(u)^{(n, int)} \| \leq e^{-\gamma(m, L, n)} L - \epsilon \]

where \( m = m^*/2 \in (0, \infty) \). Since by assumption \( p^* \in (6Nd, \infty) \) there exists \( p \in (6Nd, p^*) \) and \( \tau \in (0, \infty) \) such that \( L_0^{-2p4^{N-n}} \tau \geq L_0^{-2p^*4^{N-n}} \). With such values \( p \) and \( \tau \) inequality (3.3) implies

\[ (3.4) \quad P \{ C_L^{(n)}(u) \text{ is } (E, m^*, 0)-S \} \leq \frac{1}{2} L_0^{-2p4^{N-n}} - \frac{1}{2}. \]

Next, it follows from the first resolvent identity that

\[ \| G^{(n)}_{C_L^{(n)}(u), 0}(E) - G^{(n)}_{C_L^{(n)}(u), h}(E) \| \leq \|h\| U \| \| G^{(n)}_{C_L^{(n)}(u), 0}(E) \| \| G^{(n)}_{C_L^{(n)}(u)}(E) \| \| G^{(n)}_{C_L^{(n)}(u), h}(E) \|. \]

By Theorem 2.1 applied to Hamiltonians \( H^{(n)}_{C_L^{(n)}(u), 0} \) and \( H^{(n)}_{C_L^{(n)}(u), h} \) for any \( \tau \in (0, \infty) \) there is \( B(\tau) \in (0, +\infty) \) such that

\[ P \{ \| G^{(n)}_{C_L^{(n)}(u), 0}(E) \| \geq B(\tau) \} \leq \frac{\tau}{4} \]

\[ P \{ \| G^{(n)}_{C_L^{(n)}(u), h}(E) \| \geq B(\tau) \} \leq \frac{\tau}{4}. \]

Therefore

\[ P \{ \| G^{(n)}_{C_L^{(n)}(u), 0}(E) \| - \| G^{(n)}_{C_L^{(n)}(u), h}(E) \| \geq \|h\| U \| B^2(\tau) \}
\]

\[ P \{ \| G^{(n)}_{C_L^{(n)}(u), 0}(E) \| \geq B(\tau) \} + P \{ \| G^{(n)}_{C_L^{(n)}(u), h}(E) \| \geq B(\tau) \}
\]

\[ 2 \cdot \frac{\tau}{4} \]

Set \( h^* := \frac{\epsilon}{2} U \| B(\tau) \|^2 \in (0, +\infty) \). We see that if \( |h| \leq h^* \), then \( |h| \times \| U \| \times B(\tau)^2 \leq \frac{\epsilon}{4} \). Hence,

\[ (3.5) \quad P \{ \| G^{(n)}_{C_L^{(n)}(u), 0}(E) \| - \| G^{(n)}_{C_L^{(n)}(u), h}(E) \| \geq \frac{\epsilon}{4} \} \leq \frac{\tau}{2}. \]

Combining (3.3), (3.4) and (3.5) we obtain that for all \( E \in I \)

\[ P \{ C_L^{(n)}(u) \text{ is } (E, m, h)-S \}
\]

\[ + P \{ \| G^{(n)}_{C_L^{(n)}(u), 0}(E) \| - \| G^{(n)}_{C_L^{(n)}(u), h}(E) \| \geq \frac{\epsilon}{2} \}
\]

\[ \leq \left( \frac{1}{2} L_0^{-2p4^{N-n}} - \frac{\tau}{2} \right) + \frac{\tau}{2} = \frac{1}{2} L_0^{-2p4^{N-n}} \]

\[ \square \]
3.3. The variable energy multi-scale analysis bounds for the weakly interacting multi-particle systems. Here, we deduce from the fixed energy bound, the variable energy initial multi-scale analysis bound for the weakly interacting multi-particle system. We will prove localization in each compact interval acting multi-particle systems.

We will prove localization in each compact interval acting multi-particle systems. Let

\[ E \]

with \( \delta = \frac{1}{2}2^{2L_0^{4/3}}(e^{-m_1L_0} - e^{-m_2L_0}) \) where \( m_1 \in (0, m) \) by definition. Set

\[ I_0 := [E - \delta, E_0 + \delta]. \]

The result on the variable energy multi-scale analysis is given below in Theorem 3.4.

**Theorem 3.4.** Let \( 1 \leq n \leq N \). For any \( u \in \mathbb{Z}^d \) we have

\[ \mathbb{P}\{ \exists E \in I_0: C_L^{(n)}(u) \text{ is } (E, m_1)\text{-S} \leq L_0^{-2p4^{N-n}} \}

for some \( m_1 \in (0, \infty) \).

**Proof.** Let \( E_0 \in I \). By the resolvent equation

\[ G^{(n)}_{L_0}(u, h)(E) = \frac{1}{G^{(n)}_{L_0}(u, h)(E) + (E - E_0)G^{(n)}_{L_0}(u, h)(E)}(E)G^{(n)}_{L_0}(u, h)(E_0) \]

If \( \text{dist}(E, \sigma(H^{(n)}_{L_0}(u, h)(\omega))) \geq e^{-L_0^{4/3}} \text{ and } |E - E_0| \leq \frac{1}{2}e^{-L_0^{4/3}}, \text{ then } \text{dist}(E, \sigma(H^{(n)}_{L_0}(u, h))) \geq \frac{1}{2}e^{-L_0^{4/3}}. \]

If in addition \( C_L^{(n)}(u) \) is \( (E_0, m, h)\text{-NS} \) then

\[ \|1_{x}^{(n, \text{out})}G^{(n)}_{C_L^{(n)}}(u)(E)1_{x}^{(n, \text{int})}\| \leq e^{-m(1+L_0^{1/8}4^{N-n+1})L_0 + 2|E_0 - E|e^{2L_0^{4/3}}}. \]

Therefore, for \( m_1 = \frac{m}{2} \), if we put

\[ \delta = \frac{1}{2}2^{2L_0^{4/3}}(e^{-m_1(1+L_0^{1/8})4^{N-n+1}L_0} - e^{-m(1+L_0^{1/8})4^{N-n+1}}) \]

we have that

\[ \mathbb{P}\{ \exists E \in I_0: C_L^{(n)}(u) \text{ is } (E, m_1, h)\text{-S} \}

\[ \leq \mathbb{P}\{ C_L^{(n)}(u) \text{ is } (E, m, h)\text{-S} \}

\[ + \mathbb{P}\{ \text{dist}(E_0, \sigma(H^{(n)}_{C_L^{(n)}}(u, h))) \leq e^{-L_0^{4/3}} \}\]

\[ \leq \frac{1}{2}L_0^{-2p4^{N-n}} + L_0^{-p4^{N}} \leq L_0^{-2p4^{N-n}}. \]

We used Theorem ?? to bound the first term and the Wegner estimates Theorem ?? to bound the other term. \( \square \)

Below, we develop the induction step of the multi-scale and for the reader convenience, we also give the proof of some important results.

4. Multi-scale induction

In the rest of the paper, we assume that \( n \geq 2 \) and \( I_0 \) is the interval of the previous Section. Recall the following facts from [15]. Consider a cube \( C_L^{(n)}(u) \) with \( u = (u_1, \ldots, u_n) \in (\mathbb{Z}^d)^n \). We have

\[ Iu = \{u_1, \ldots, n\} \]

and

\[ \Pi C_L^{(n)}(u) = C_L^{(1)}(u_1) \cup \cdots \cup C_L^{(n)}(u_n) \]
Definition 4.1. Let $L_0 \geq 3$ be a constant and $\alpha = 3/2$. We define the sequence \{L_k, k \geq 1\} recursively as follows
\[ L_k = [L_{k-1}] + 1, \quad \text{for all } k \geq 1. \]

Let $m \in (0, \infty)$ be a positive constant, we also introduce the following property, namely the multi-scale analysis bounds at any scale length $L_k$ and for any pair of separable cubes $C^{(n)}_L(u) C^{(n)}_L(v)$

\[ (\text{DS}, k, n, N). \]

\[ \mathbb{P}\left\{ \exists E \in I_0 \ C^{(n)}_L(u) C^{(n)}_L(v) \text{ are } (E, m)-S \right\} \leq L^{-2p} n^{-n} \]

where $p \in (6Nd, \infty)$.

In both the single-particle and the multi-particle system, given the results of the multi-particle multi-scale analysis property (DS, k, n, N) above, one can deduce the localization results see for example the papers [7, 11] for those concerning the single-particle case and [5, 15] for multi-particle systems. We have the following:

Theorem 4.1. For any $n' \in (1, n)$ assume that property (DS, k', n, N) holds true for all $k \geq 0$ then there exists a positive constant $\tilde{\mu} \in (0, \infty)$ such that for cube $C^{(n')}_{L_k}(u')$

\[ (4.1) \]

\[ \mathbb{E} \left[ \|1_{C^{(n')}_{L_k'}(w')} G^{(n')}_{C^{(n')}_{L_k'}(w')} (E) 1_{C^{(n')}_{L_k'}(w')} \| \right] \leq e^{-\tilde{\mu} L} \]

Definition 4.2 (partially/fully interactive). An $n$-particle cube $C^{(n)}_L(u) \subset \mathbb{Z}^n$ is called fully interactive (FI) if

\[ (4.2) \]

\[ \text{diam } \Pi u := \max_{i \neq j} |u_i - u_j| \leq n(2L + r_0), \]

and partially interactive (PI) otherwise.

The following simple statement clarifies the notion of PI cubes

Lemma 4.1. If a cube $C^{(n)}_L(u)$ is PI then there exists a subset $J \subset \{1, \ldots, n\}$ with $1 \leq \text{card } J \leq n - 1$ such that

\[ \text{dist} \left( \Pi_{\bar{J}} C^{(n)}_L(u), \Pi_{J^c} C^{(n)}_L(u) \right) \geq r_0 \]

Proof. See the proof in the appendix Section 7 \[ \square \]

If a cube $C^{(n)}_L(u)$ is PI then by Lemma 4.1, we can write it as

\[ (4.3) \]

\[ C^{(n)}_L(u) = C^{(n')}_{L_k}(u') \times C^{(n'')}_{L_k}(u'') \]

with

\[ (4.4) \]

\[ \text{dist} (\Pi C^{(n')}_{L_k}(u'), \Pi C^{(n'')}_{L_k}(u'')) \geq r_0 \]

where $u' = u_J = (u_j : j \in J) u'' = u_{J^c} = (u_j : j \in J^c)$ $n' = \text{card } J$ and $n'' = \text{card } J^c$. Throughout, when we write a PI cube $C^{(n)}_L(u)$ in the form (4.3), we implicitly assume that the projections satisfy (4.4). Let $C^{(n')}_{L_k}(u') \times C^{(n'')}_{L_k}(u'')$ be the decomposition of the PI cube $C^{(n)}_L(u)$ and $\{\lambda_i, \varphi_i\}$ and $\{\mu_j, \phi_j\}$ be the eigenvalues and corresponding eigenfunctions of $H^{(n')}_{C^{(n')}_{L_k}(w')}$ and $H^{(n'')}_{C^{(n'')}_{L_k}(w'')}$ respectively. Next, we can choose the eigenfunctions $\Psi_{ij}$ as tensor product

\[ \Psi_{ij} = \varphi_i \otimes \phi_j \]

The eigenfunctions appearing in subsequent argument and calculations will be assume normalized.

Now we turn to geometrical property of FI cubes
Lemma 4.2. Let \( n \geq 1, L \geq 2r_0 \) and consider two FI cubes \( C_L^{(n)}(x) \) and \( C_L^{(n)}(y) \) with \( |x - y| \geq 7nL \). Then
\[
\Pi C_L^{(n)}(x) \cap \Pi C_L^{(n)}(y) = \emptyset
\]
Proof. See the proof in the Appendix Section \( \square \).

Given an \( n \)-particle cube \( C_L^{(n)}(u) \) and \( E \in \mathbb{R} \), we denote by
- \( M_{\text{FI}}^{\text{sep}}(C_L^{(n)}(u), E) \) the maximal number of pairwise separable \( (E, m) \)-singular PI cubes \( C_L^{(n)}(u^{(j)}) \subset C_L^{(n)}(u) \);
- by \( M_{\text{FI}}(C_L^{(n)}(u), E) \), the maximal number of (not necessary separable) \( (E, m) \)-singular PI-cubes \( C_L^{(n)}(u^{(j)}) \) contain in \( C_L^{(n)}(u) \) with \( u^{(j)}, u' \in \mathbb{Z}^n \) and \( |u^{(j)} - u'_{(j)}| \geq 7NL_k \) for all \( j \neq j' \);
- \( M_{\text{FI}}(C_L^{(n)}(u), E) \) the maximal number of \( (E, m) \)-singular FI cubes \( C_L^{(n)}(u^{(j)}) \subset C_L^{(n)}(u) \) with \( |u^{(j)} - u'_{(j)}| \geq 7NL_k \) for all \( j \neq j' \);
- \( M_{\text{FI}}(C_L^{(n)}(u), E) \) the maximal number of pairwise separable \( (E, m) \)-singular cube \( C_L^{(n)}(u^{(j)}) \subset C_L^{(n)}(u) \);

Clearly,
\[
M_{\text{FI}}(C_L^{(n)}(u), E) + M_{\text{FI}}(C_L^{(n)}(u), E) \geq M(C_L^{(n)}(u), E).
\]

4.1. Pairs of partially interactive cubes. Let \( C_L^{(n)}(u_{k+1}) = C_L^{(n)}(u') \times C_L^{(n)}(u'') \) be a PI-cube. We also write \( x = (x', x'') \) for any point \( x \in C_L^{(n)}(u) \), in the same way as \( (u', u'') \). So the corresponding Hamiltonian \( H_{C_L^{(n)}(u)}^{(n)} \) is written in the form:
\[
H_{C_L^{(n)}(u)}^{(n)} \Psi(x) = (-\Delta \Psi)(x) + [U(x') + V(x', \omega) + U(x'') + V(x'', \omega)] \Psi(x)
\]
or in compact form:
\[
H_{C_L^{(n)}(u)}^{(n)} = H_{C_L^{(n)}(u')}^{(n')} \otimes I + I \otimes H_{C_L^{(n)}(u'')}^{(n'')}
\]

Definition 4.3. Let \( n \geq 2 \) and \( C_L^{(n)}(u') \times C_L^{(n)}(u'') \) be the decomposition of the PI cube \( C_L^{(n)}(u) \). Then \( C_L^{(n)}(u) \) is called

\footnote{Note that by Lemma ??; two FI cubes \( C_L^{(n)}(u^{(j)}) \) and \( C_L^{(n)}(u'_{(j')}) \) with \( |u^{(j)} - u'_{(j')}| \geq 7NL_k \) are automatically separable.}
(i) \( m \)-left-localized if for any normalized eigenfunction \( \varphi^{(n')} \) of the restricted Hamiltonian \( H^{(n')}_{C_{m}^{L}(\omega)} \), we have
\[
\|1_{C^{(n')}_{m} \cap (\omega')} \varphi^{(n')} \| \leq e^{-2\gamma(m,L_{k},n')L_{k}}
\]
otherwise it is called \( m \)-non-left-localized,

(ii) \( m \)-right-localized if for any normalized eigenfunction \( \varphi^{(n'')} \) of the restricted Hamiltonian \( H^{(n'')}_{C_{m}^{L}(\omega)} \), we have
\[
\|1_{C^{(n'')}_{m} \cap (\omega'')} \varphi^{(n'')} \| \leq e^{-2\gamma(m,L_{k},n'')L_{k}}
\]
otherwise it is called \( m \)-non-right-localized,

(iii) \( m \)-localized if it is \( m \)-left-localized and \( m \)-right-localized. Otherwise it is called \( m \)-non-localized

**Lemma 4.3.** Let \( E \in I \) and \( C_{L_{k}}^{(n)}(u) \) be a PI cube. Assume that \( C_{L_{k}}^{(n)}(u) \) is \( E \)-NR and \( m \)-localized. Then the cube \( C_{L_{k}}^{(n)}(u) \) is \( (E,m) \)-NS.

**Proof.** We proceed as in Lemma 3.1 \( \square \)

Now, before proving the main results of this Subsection concerning the probability of two PI cubes to be singular at the same energy we need first to estimate the one for a non-localized cube given in the statement below

**Lemma 4.4.** Let \( C_{L_{k}}^{(n)}(u) \) be a PI cube. Then
\[
P\{C_{L_{k}}^{(n)}(u) \text{ is } m \text{-non-localized}\} \leq L_{k}^{-4p4^{n}-n}.
\]

**Proof.** The proof combines the ideas of Theorem 3.2 in the multi-particle systems without interaction and the induction assertion of localization given in Theorem 4.1 \( \square \)

Now, we state the main result of this Subsection, i.e., the probability bound of two PI cubes to be singular at the same energy belonging to the compact interval \( I_{0} \) introduced at the beogning of the Section.

**Theorem 4.2.** Let \( 2 \leq n \leq N \). There exists \( L_{1}^{*} = L_{1}^{*}(N,d) \in (0,\infty) \) such that if \( L_{0} \geq L_{1}^{*} \) and if for \( k \geq 0 \) \( (DS,k,n',N) \) holds true for any \( n' \in (1,n) \) then \( (DS,k+1,n,N) \) holds for any pair of separable PI cubes \( C_{L^{k+1}}^{(n)}(x) \) and \( C_{L^{k+1}}^{(n')}(y) \).

**Proof.** Let \( C_{L_{k+1}}^{(n)}(x) \) and \( C_{L_{k+1}}^{(n')}^{(n')} \) be two separable PI cubes. Consider the events:
\[
B_{k+1} = \{ \exists E \in I_{0} : C_{L_{k+1}}^{(n)}(x) C_{L_{k+1}}^{(n)}(y) \text{ are } (E,m)-S\},
\]
\[
R = \{ \exists E \in I_{0} : C_{L_{k+1}}^{(n)}(x) \text{ and } C_{L_{k+1}}^{(n')} \text{ are } E-R\},
\]
\[
N_{x} = \{ C_{L_{k+1}}^{(n)}(x) \text{ is } m \text{-non-localized}\},
\]
\[
N_{y} = \{ C_{L_{k+1}}^{(n')} \text{ is } m \text{-non-localized}\}
\]

If \( \omega \in B_{k+1} \setminus R \) then \( \forall E \in I_{0}, C_{L_{k+1}}^{(n)}(x) \) or \( C_{L_{k+1}}^{(n')} \) is \( E \)-NR, then it must be \( m \)-non-localized: otherwise it would have been \( (E,m) \)-NS by Lemma 4.3. Similarly if \( C_{L_{k+1}}^{(n)} \) is \( E \)-NR, then it must be \( m \)-non-localized. This implies that
\[
B_{k+1} \subset R \cup N_{x} \cup N_{y}
\]
Therefore, using Theorem 4.1 and Lemma 4.4 we have
\[ \mathbb{P}(B_{k+1}) \leq \mathbb{P}(R + \mathbb{P}(\mathcal{N}_k) + \mathbb{P}(N) \]
\[ \leq \frac{1}{2} L_{k+1} - 2^{4p4n} - n ] \]
Finally
\[ \mathbb{P}(B_{k+1}) \leq L_{k+1} - 2^{4p4n} - n ] \]
which proves the result. \( \square \)

For subsequent calculations and proofs we give the following two Lemmas:

**Lemma 4.5.** If \( M(C^{(n)}_{L_k+1}(u), E) \geq \kappa(n) + 2 \) with \( \kappa(n) = n^2 \), then \( M^{\text{sep}}(C^{(n)}_{L_k+1}(u), E) \geq 2 \). Similarly if \( M_{\text{PI}}(C^{(n)}_{L_k+1}(u), E) \geq \kappa(n) + 2 \) then \( M^{\text{sep}}(C^{(n)}_{L_k+1}(u), E) \geq 2 \).

**Proof.** See the appendix Section \( 7 \). \( \square \)

**Lemma 4.6.** With the above notations, assume that \( (DS,k-1,n',N) \) holds true for all \( n' \in [1, n] \) then
\[ \mathbb{P}(M_{\text{PI}}(C^{(n)}_{L_k+1}(u), I) \geq \kappa(n) + 2) \leq \frac{3^{2nd}}{2} L_{k+1} 2^{4p4n} - n ) \]

**Proof.** See the appendix Section \( 7 \). \( \square \)

4.2. Pairs of fully interactive cubes. Our aim now is to prove \( (DS,k-1,n,N) \) for a pair of fully interactive cubes \( C^{(n)}_{L_k+1}(x) \) and \( C^{(n)}_{L_k+1}(y) \). We adapt to the continuum a very crucial and hard result obtained in the paper \( 15 \) and which generalized to multi-particle systems some previous work by von Dreifus and Klein \( 17 \) on the lattice and Stollmann \( 27 \) in the continuum for single particle models.

**Lemma 4.7.** Let \( J = \kappa(n) + 5 \) with \( \kappa(n) = n^2 \) and \( E \in \mathbb{R} \). Suppose that
i) \( C^{(n)}_{L_k+1}(x) \) is \( E \)-CNR.
ii) \( M(C^{(n)}_{L_k+1}(x), E) \leq J \).

Then there exists \( \tilde{L}_2^2(J,N,d) \geq 0 \) such that if \( L_0 \geq \tilde{L}_2^2(J,N,d) \) we have that \( C^{(n)}_{L_k+1}(x) \) is \( (E,m) \)-NS.

**Proof.** Since \( M(C^{(n)}_{L_k+1}(x), E) \leq J \), there exist at most \( J \) cubes of side length \( 2L_k \) contained in \( C^{(n)}_{L_k+1}(x) \) that are \( (E,m)-S \) with centers at distance \( \geq 7NL_k \).

Therefore, we can find \( x_i \in C^{(n)}_{L_k+1}(x) \cap \Gamma_x \) with \( \Gamma_x = x + \frac{4}{7} 2^{nd} \).
\[ \text{dist}(x_i, \partial C^{(n)}_{L_k+1}(x)) \geq 2L_k, \quad i = 1, \ldots, r \leq J \]

such that, if \( x_0 \in C^{(n)}_{L_k+1}(x) \setminus \bigcup_{i=1}^{r} C^{(n)}_{2L_k}(x_i) \), then the cube \( C^{(n)}_{L_k}(x_0) \) is \( (E,m) \)-NS.

We do an induction procedure in \( C^{(n,int)}_{L_k+1}(x) \) and start with \( x_0 \in C^{(n,int)}_{L_k+1}(x) \). We estimate \( ||1^{C^{(n,int)}}_{L_k+1}(x) G^{(n)}_{L_k+1}(E) 1^{C^{(n,int)}}_{L_k+1}(x_0) || \). Suppose that \( x_0, \ldots x_r \) have been chosen for \( \ell \geq 0 \) We have two cases

- **case a)** \( C^{(n)}_{L_k}(x_0) \) is \( (E,m) \)-NS
In this case, we apply the (GRI) Theorem 2.3 and obtain
\[
\| C_{L_{k+1}}^{(n)}(x) G_{L_{k+1}}^{(n)}(x) (E) 1_{C_{L_{k+1}}^{(n,int)}}(x_0) \| 
\leq C_{geom} \| C_{L_{k+1}}^{(n)}(x) G_{L_{k+1}}^{(n)}(x) (E) 1_{C_{L_{k+1}}^{(n,out)}}(x_0) \|.
\]
\[
\| C_{L_{k+1}}^{(n)}(x) G_{L_{k+1}}^{(n)}(x) (E) 1_{C_{L_{k+1}}^{(n,int)}}(x_0) \| 
\leq C_{geom} \| C_{L_{k+1}}^{(n)}(x) G_{L_{k+1}}^{(n)}(x) (E) 1_{C_{L_{k+1}}^{(n,out)}}(x_0) \|. 
\]
We replace in the above analysis \( x \) with \( x_\ell \) and we get
\[
\| C_{L_{k+1}}^{(n)}(x_\ell) G_{L_{k+1}}^{(n)}(x_\ell) (E) 1_{C_{L_{k+1}}^{(n,int)}}(x_\ell) \| 
\leq 3^n \| C_{L_{k+1}}^{(n)}(x_\ell) G_{L_{k+1}}^{(n)}(x_\ell) (E) 1_{C_{L_{k+1}}^{(n,int)}}(x_\ell) \|, 
\]
where \( x_{\ell+1} \) is chosen in such a way that the norm in the right hand side in the above equation is maximal. Observe that \( |x_\ell - x_{\ell+1}| = L_k/3 \). We therefore obtain
\[
\| C_{L_{k+1}}^{(n)}(x) G_{L_{k+1}}^{(n)}(x) (E) 1_{C_{L_{k+1}}^{(n,int)}}(x_\ell) \| 
\leq C_{geom} 3^n e^{-\gamma(m,L_k,n)L_k} \| C_{L_{k+1}}^{(n)}(x) G_{L_{k+1}}^{(n)}(x) (E) 1_{C_{L_{k+1}}^{(n,int)}}(x_\ell) \| 
\leq \delta e^{\| C_{L_{k+1}}^{(n)}(x) G_{L_{k+1}}^{(n)}(x) (E) 1_{C_{L_{k+1}}^{(n,int)}}(x_\ell) \|}
\]
with \( \delta = 3^n C_{geom} e^{-\gamma(m,L_k,n)L_k} \).

**case (b)** \( C_{L_k}^{(n)}(x_\ell) \) is \((E,m)-S\). Thus, there exists \( i_0 = 1, \ldots, r \) such that \( C_{L_k}^{(n)}(x_\ell) \subset C_{2L_k}^{(n)}(x_{i_0}) \). We apply again the (GRI) this time with \( x \) \( C_{L_k}^{(n)}(x_\ell) \) and and obtain
\[
\| C_{L_{k+1}}^{(n)}(x) G_{L_{k+1}}^{(n)}(x) (E) 1_{C_{L_{k+1}}^{(n)}}(x_{i_0}) \| 
\leq C_{geom} \| C_{L_{k+1}}^{(n)}(x) G_{L_{k+1}}^{(n)}(x) (E) 1_{C_{L_{k+1}}^{(n)}}(x_{i_0}) \| 
\times \| C_{L_k}^{(n)}(x_{i_0}) G_{L_k}^{(n)}(x_{i_0}) (E) 1_{C_{L_k}^{(n)}}(x_{i_0}) \| 
\leq C_{geom} \| (2L_k)^{1/2} \| C_{L_{k+1}}^{(n)}(x) G_{L_{k+1}}^{(n)}(x) (E) 1_{C_{L_{k+1}}^{(n)}}(x_{i_0}) \|.
\]
We have almost everywhere
\[
1_{C_{L_{k+1}}^{(n)}}(x_{i_0}) \sum_{\tilde{x} \in C_{2L_k}^{(n)}}^{C_{L_{k+1}}^{(n)}} 1_{C_{L_k}^{(n)}}(x)
\]
Hence, by choosing \( \tilde{x} \) is such a way that the right hand side is maximal, we get
\[
\| C_{L_{k+1}}^{(n)}(x) G_{L_{k+1}}^{(n)}(x) (E) 1_{C_{L_{k+1}}^{(n)}}(x_{i_0}) \| 
\leq 6^n \| C_{L_{k+1}}^{(n)}(x) G_{L_{k+1}}^{(n)}(x) (E) 1_{C_{L_{k+1}}^{(n)}}(\tilde{x}) \|.
\]
Since \( C_{L_k}^{(n)}(\tilde{x}) \not\subset C_{2L_k}^{(n)}(x_{i_0}) \), \( \tilde{x} \in C_{2L_k}^{(n)}(x_{i_0}) \) and the cubes \( C_{2L_k}^{(n)}(x_{i_0}) \) are disjoint, we obtain that
\[
C_{L_k}^{(n)}(\tilde{x}) \not\subset \bigcup_{i=1}^{r} C_{2L_k}^{(n)}(x_{i_0}).
\]
so that the cube $C_{L_k}^{(n)}(\tilde{x})$ must be $(E,m)$-NS. We therefore perform a new step as in case (a) and obtain

$$\ldots \leq 6^{nd}3^{nd}C_{geom}e^{-\gamma(m,L_k,n)L_k} \cdot \|1_{C_{L_k+1}^{(n,out)}}(x) G_{C_{L_k+1}^{(n)}}(x) (E) 1_{C_{L_k+1}^{(n,int)}}(x_{f+1})\|,$$

with $x_{f+1} \in \Gamma_x$ and $|\tilde{x} - x_{f+1}| = L_k/3$.

Summarizing, we get $\|1_{C_{L_k+1}^{(n,out)}}(x) G_{C_{L_k+1}^{(n)}}(x) (E) 1_{C_{L_k+1}^{(n,int)}}(x_{f+1})\| \leq \delta_0 \|1_{C_{L_k+1}^{(n,out)}}(x) G_{C_{L_k+1}^{(n)}}(x) (E) 1_{C_{L_k+1}^{(n,int)}}(x_{f+1})\|$, with $\delta_0 = 18^{nd}C_{geom}^{e(2L_k)^{1/2}}e^{-\gamma(m,L_k,n)L_k}$.

After $\ell$ iterations with $n_+$ steps of case (a) and $n_0$ steps of case (b), we obtain

$$\|1_{C_{L_k+1}^{(n,out)}}(x) G_{C_{L_k+1}^{(n)}}(x) (E) 1_{C_{L_k+1}^{(n,int)}}(x_{f+1})\| \leq (\delta_0)^{n_0} \times \|1_{C_{L_k+1}^{(n,out)}}(x) G_{C_{L_k+1}^{(n)}}(x) (E) 1_{C_{L_k+1}^{(n,int)}}(x_{f+1})\|.$$

Now since $\gamma(m,L_k,n) \geq m$ we have that

$$\delta_+ \leq 3^{nd}C_{geom}e^{-mL_k}.$$

So $\delta_+$ can be made arbitrarily small if $L_0$ and hence $L_k$ is large enough. We also have for $\delta_0$

$$\delta_0 = 18^{nd}C_{geom}^{e(2L_k)^{1/2}}e^{-\gamma(m,L_k,n)L_k} \leq 18^{nd}C_{geom}^{e2L_k^{1/2}}e^{-\gamma(m,L_k,n)L_k} \leq 18^{nd}C_{geom}^{e2L_k^{1/2}-mL_k} \leq \frac{1}{2}.$$

For large $L_0$ hence $L_k$. Using the (GRI), we can iterate if $C_{L_k+1}^{(n,out)}(x) \cap C_{L_k}^{(n)}(x) = \emptyset$. Thus, we can have at least $n_+$ steps of case (a) with

$$n_+ \cdot \frac{L_k}{3} + \sum_{i=1}^{r} 2L_k \geq \frac{L_{k+1}}{3} - \frac{L_k}{3},$$

until the induction eventually stop. Since $r \leq J$, we can bound $n_+$ from below:

$$n_+ \cdot \frac{L_k}{3} \geq \frac{L_{k+1}}{3} - \frac{L_k}{3} - r(L_k) \geq \frac{L_{k+1}}{3} - \frac{L_k}{3} - 2J L_k$$

Which yields

$$n_+ \geq \frac{L_{k+1}}{L_k} - 1 - 6J \geq \frac{L_{k+1}}{L_k} - 7J$$

Therefore

$$\|1_{C_{L_k+1}^{(n,out)}}(x) G_{C_{L_k+1}^{(n)}}(x) (E) 1_{C_{L_k+1}^{(n,int)}}(x_{f+1})\| \leq \delta_+^{n_+} \cdot \|G_{C_{L_k+1}^{(n)}}(x) (E)\|$$

Finally, by $E$-non-resonance of $C_{L_k+1}^{(n)}(x)$ and since we can cover $C_{L_k+1}^{(n,int)}(x)$ by $\left(\frac{L_{k+1}}{L_k}\right)^{nd}$ small cubes $C_{L_k}^{(n,int)}(y)$, equation (4.6) with $y$ instead of $x_0$,
yields
\[
\left\| \mathbf{1}_{C_{k_{k+1}}^{(m,\text{out})}(x)} \mathbf{G}_{C_{k_{k+1}}^{(n)}}^{(n)}(x) \mathbf{1}_{C_{k_{k+1}}^{(m,\text{int})}(x)} \right\| \leq \left( \frac{L_{k+1}}{L_k} \right)^{\frac{2}{3}} e^{\frac{7}{2} J L_k} \mathcal{C}_{\text{geom}} \cdot e^{-\gamma(m, L_k, n)} L_{k+1}^{\frac{7}{4} J} \left[ e^{\frac{1}{2}} \right] \left( 1 + \delta_{k+1}^{1/2} \right)
\]
\[
\leq \left( \frac{L_{k+1}}{L_k} \right) \cdot \left[ 3^{\text{nd}} \cdot \mathcal{C}_{\text{geom}} \cdot e^{-\gamma(m, L_k, n)} \right] L_{k+1}^{\frac{7}{4} J} \left[ e^{\frac{1}{2}} \right] \left( 1 + \delta_{k+1}^{1/2} \right)
\]
\[
\leq L_{k+1}^{\frac{7}{4} J} \left( \frac{L_{k+1}^{1/3} e^{\frac{1}{2}} J \gamma(m, L_k, n)}{L_k} \right) \left[ e^{\frac{1}{2}} \right] \left( 1 + \delta_{k+1}^{1/2} \right)
\]
\[
\leq L_{k+1}^{\frac{7}{4} J} \left( \frac{L_{k+1}^{1/3} e^{\frac{1}{2}} J \gamma(m, L_k, n)}{L_k} \right) \left[ e^{\frac{1}{2}} \right] \left( 1 + \delta_{k+1}^{1/2} \right)
\]
\[
\leq e^{-\gamma(m, L_k, n) L_k - \frac{1}{L_{k+1}} \ln \left( 2^{\gamma(m, L_k, n)} \right)}
\]

where
\[
m' = \frac{1}{L_{k+1}} \left[ n + \gamma(m, L_k, n) L_k - n_n \ln \left( 2^{\gamma(m, L_k, n)} \right) \right] - \frac{1}{L_{k+1}}
\]

with
\[
L_{k+1} L_k^{-1} - 7 J \leq n_n \leq L_{k+1} L_k^{-1}
\]

we obtain
\[
m' \geq \gamma(m, n, L_k) - \gamma(m, L_k, n) \frac{7 J L_k}{L_{k+1}}
\]
\[
- \frac{1}{L_{k+1}} \ln \left( 2^{\gamma(m, L_k, n)} \right) L_k^{-1/2} \leq \frac{1}{L_{k+1}}
\]
\[
\geq \gamma(m, L_k, n) - \gamma(m, L_k, n) \frac{7 J L_k^{-1/2}}{L_k}
\]
\[
- L_k^{-1}(\ln(2^{\gamma(m, L_k, n)})) - (nd - 1) \ln(L_k) - L_k^{-3/4}
\]
\[
\geq \gamma(m, L_k, n) \left[ 1 - (7 J + \ln(2^{\gamma(m, L_k, n)} + N d)) L_k^{-1/2} \right]
\]

if \( L_0 \geq L_2^2(J, N, d) \) for some \( L_2^2(J, N, d) \geq 0 \) large enough. Since \( \gamma(m, L_k, n) = m(1 + L_k^{-1/8})^N - n + 1 \)

\[
\frac{\gamma(m, L_k, n)}{\gamma(m, L_{k+1}, n)} = \left( \frac{1 + L_k^{-1/8}}{1 + L_k^{-3/16}} \right)^{N-n+1} \geq 1 + L_k^{-1/8} \left( 1 + L_k^{-3/16} \right)^{N-n+1}
\]

Therefore we can compute
\[
\frac{\gamma(m, L_k, n)}{\gamma(m, L_{k+1}, n)} \left( 1 - (7 J + \ln(2^{\gamma(m, L_k, n)} + N d)) L_k^{-1/2} \right)
\]
\[
\frac{1 + L_k^{-1/8}}{1 + L_k^{-3/16}} \left( 1 - (7 J + \ln(2^{\gamma(m, L_k, n)} + N d)) L_k^{-1/2} \right) \geq 1
\]

provided \( L_0 \geq L_2^2 \) for some large enough \( L_2^2(J, N, d) \geq 0 \). Finally, we obtain that \( m' \geq \gamma(m, L_{k+1}, n) \). This proves the result.
Lemma 4.8. Given \( k \geq 0 \), assume that property \((DS,k,n,N)\) holds true for all pairs of separable FI cubes. Then for any \( \ell \geq 1 \)

\[
\mathbb{P}\left\{ M_{\text{Ft}}(C_{L_{k+1}}^{(n)}(u), I) \geq 2\ell \right\} \leq C(n, N, d, \ell)L_k^{2d+n}L_k^{-2p4^{N-n}}
\]

Proof. See the proof in the appendix Section 7.

Theorem 4.3. Let \( 1 \leq n \leq N \). There exists \( L_2 = L_2(N, d) \geq 0 \) such that if \( L_0 \geq L_2 \) and if for \( k \geq 0 \)

(i) \((DS,k-1,n',N)\) for all \( n' \in [1,n] \) holds true,

(ii) \((DS,k,n,N)\) holds true for all pairs of FI cubes

then \((DS,k+1,n,N)\) holds true for any pairs of separable FI cubes \( C_{L_{k+1}}^{(n)}(x) \) and \( C_{L_{k+1}}^{(n)}(y) \).

Above we use the convention \((DS,-1,n,N)\) means no assumption.

Proof. Consider a pair of separable FI cubes \( C_{L_{k+1}}^{(n)}(x) \) and \( C_{L_{k+1}}^{(n)}(y) \) and set \( J = \kappa(n) + 5 \). Define

\[
B_{k+1} = \left\{ \exists E \in I_0 : C_{L_{k+1}}^{(n)}(x) \text{ and } C_{L_{k+1}}^{(n)}(y) \text{ are } (E, m)\text{-S} \right\}
\]

\[
\Sigma = \left\{ \exists E \in I_0 : \text{ neither } C_{L_{k+1}}^{(n)}(x) \text{ nor } C_{L_{k+1}}^{(n)}(y) \text{ is } E\text{-CNR} \right\}
\]

\[
S_x = \left\{ \exists E \in I_0 : M(C_{L_{k+1}}^{(n)}(x); E) \geq J + 1 \right\}
\]

\[
S_y = \left\{ \exists E \in I_0 : M(C_{L_{k+1}}^{(n)}(y); E) \geq J + 1 \right\}
\]

Let \( \omega \in B_{k+1} \). If \( \omega \notin \Sigma \cup S_x \), then \( \forall E \in I_0 \) either \( C_{L_{k+1}}^{(n)}(x) \) or \( C_{L_{k+1}}^{(n)}(y) \) is \( E\text{-CNR} \) and \( M(C_{L_{k+1}}^{(n)}(x), E) \leq J \). The cube \( C_{L_{k+1}}^{(n)}(x) \) cannot be \( E\text{-CNR} \); indeed, by Lemma [4.7] it would be \( (E, m)\text{-NS} \). So the cube \( C_{L_{k+1}}^{(n)}(y) \) is \( E\text{-CNR} \) and \( (E, m)\text{-S} \). This implies again by Lemma [4.7] that

\[
M(C_{L_{k+1}}^{(n)}(y), E) \geq J + 1.
\]

Therefore \( \omega \in S_y \), so that \( B_{k+1} \subset \Sigma \cup S_x \cup S_y \), hence

\[
\mathbb{P}\{B_{k+1}\} \leq \mathbb{P}\{\Sigma\} + \mathbb{P}\{S_x\} + \mathbb{P}\{S_y\},
\]

and \( \mathbb{P}\{\Sigma\} \leq L_{k+1}^{-4}p \) By Theorem [2.1]. Now let us estimate \( \mathbb{P}\{S_x\} \) and similarly \( \mathbb{P}\{S_y\} \). Since

\[
M_{\text{Ft}}(C_{L_{k+1}}^{(n)}(x), E) + M_{\text{Ft}}(C_{L_{k+1}}^{(n)}(y), E) \geq M(C_{L_{k+1}}^{(n)}(x), E),
\]

the inequality \( M(C_{L_{k+1}}^{(n)}(x), E) \geq \kappa(n) + 6 \) implies that either \( M_{\text{Ft}}(C_{L_{k+1}}^{(n)}(x), E) \geq \kappa(n) + 2 \) or, \( M_{\text{Ft}}(C_{L_{k+1}}^{(n)}(x), E) \geq 4 \). Therefore, by Lemma [4.6] and Lemma [4.8] with
Therefore, by Lemma 4.7

\[ M_{C}^{(n)}(x, E) \geq \kappa(n) + 2 \]

and

\[ \exists E \in I : \text{M}(C_{L_{k+1}}^{(n)}(x, E) \geq \kappa(n) + 2 \}

\[ \geq 3^{2n}2^{L_{k+1}}(L_{k}^{-4N^{p}} + L_{k}^{-4N^{p}4^{N-n}}) + C'(n, N, d) L_{k+1}^{4dn} \]

\[ \leq C''(n, N, d) \left( L_{k+1}^{-4N^{p}+2nd} + L_{k+1}^{-4N-n+2nd} + L_{k+1}^{-4N-n+4nd} \right) \]

Thus

\[ \leq C''(n, N, d) L_{k+1}^{-4N-n+4nd} \]

\[ \leq \frac{1}{4}L_{k+1}^{-2p4^{N-n}} \]

where we used that \( \alpha = 3/2 \), \( p \geq 4\alpha N \) = 6N. Finally

\[ \mathbb{P}(B_{k+1}) \leq L_{k+1}^{-4N^{p}} + \frac{1}{2}L_{k+1}^{-2p4^{N-n}} \leq L_{k+1}^{-2p4^{N-n}}. \]

\[ \square \]

4.3. Mixed pairs of cubes. Finally, it remains only to derive (DS, k+1, n, N) in case (III) i.e., for pairs of n-particle cubes where one is PI while the other is FI.

**Theorem 4.4.** Let \( 1 \leq n \leq N \). There exists \( L_{3}^{(n)} = L_{3}^{(n)}(N, d) \geq 0 \) such that if \( L_{0} \geq L_{3}^{(n)}(N, d) \) and if for \( k \geq 0 \)

(i) (DS, k, n', N) holds true all \( n' \in [1, n] \);

(ii) (DS, k, n', N) holds true for all \( n' \in [1, n] \) and

(iii) (DS, k, n, N) holds true for all pairs of FI cubes

then (DS, k+1, n, N) holds true for any pair of separable cubes \( C_{L_{k+1}}^{(n)}(x) \) and \( C_{L_{k+1}}^{(n)}(y) \) where one is PI while the other is FI.

**Proof.** Consider a pair of separable n-particle cubes \( C_{L_{k+1}}^{(n)}(x) \), \( C_{L_{k+1}}^{(n)}(y) \) and suppose that \( C_{L_{k+1}}^{(n)}(x) \) is PI while \( C_{L_{k+1}}^{(n)}(y) \) is FI. Set \( J = \kappa(n) + 5 \) and introduce the events

\[ B_{k+1} = \{ 3E \in I_{0} : C_{L_{k+1}}^{(n)}(x) \text{ and } C_{L_{k+1}}^{(n)}(y) \text{ are } (E, m)-S \} \]

\[ \Sigma = \{ 3E \in I_{0} : \text{neither } C_{L_{k+1}}^{(n)}(x) \text{ nor } C_{L_{k+1}}^{(n)}(y) \text{ is E-CNR} \} \]

\[ T_{x} = \{ C_{L_{k+1}}^{(n)}(x) \text{ is } (E, m)-T \} \]

\[ S_{y} = \{ 3E \in I_{0} : M(C_{L_{k+1}}^{(n)}(y), E) \geq J + 1 \} \]

Let \( \omega \in B_{k+1} \setminus (\Sigma \cup T_{x} \cup S_{y}) \) then, for all \( E \in I_{0} \) either \( C_{L_{k+1}}^{(n)}(x) \) is E-CNR or \( C_{L_{k+1}}^{(n)}(y) \) is E-CNR and \( C_{L_{k+1}}^{(n)}(x) \) is E, m)-NT. The cube \( C_{L_{k+1}}^{(n)}(x) \) cannot be E-CNR. Indeed by Lemma 4.3 it would have been (E, m)-NS. Thus the cube \( C_{L_{k+1}}^{(n)}(y) \) is E-CNR, so by Lemma 4.7 \( M(C_{L_{k+1}}^{(n)}(y), E) \geq J + 1 \) otherwise \( C_{L_{k+1}}^{(n)}(y) \) would be (E, m)-NS. Therefore \( \omega \in S_{y} \). Consequently,

\[ B_{k+1} \subset \Sigma \cup T_{x} \cup S_{y}. \]
Recall that the probabilities $P\{T_x\}$ and $P\{S_y\}$ have already been estimated in Sections 4.1 and 4.2. We therefore obtain

$$P\{B_{k+1}\} \leq P\{T_x\} + P\{S_y\} \leq L_{k+1}^{-2p4^{N-n}} + \frac{1}{4}L_{k+1}^{-2p4^{N-n}} \leq L_{k+1}^{-2p4^{N-n}}$$

5. Conclusion: The multi-particle multi-scale analysis

**Theorem 5.1.** Let $1 \leq n \leq N$ and $H^{(n)}(\omega) = -\Delta + \sum_{j=1}^{n} V(x_j, \omega) + U$, where $U, V$ satisfy (I) and (P) respectively. There exists a positive $m$ such that for any $p \geq 6Nd$ property ($\mathbf{DS.k,n,N}$) holds true for all $k \geq 0$ provided $L_0$ is large enough.

**Proof.** We prove that for each $n = 1, \ldots, N$, property ($\mathbf{DS.k,n,N}$) is valid. To do so, we use an induction on the number of particles $n' = 1, \ldots, n$. For $n = 1$ the property holds true for all $k \geq 0$ by the single-particle localization theory [17]. Now suppose that for all $n' \in [1, n)$ ($\mathbf{DS.k,n',N}$) holds true for all $k \geq 0$, we aim to prove that ($\mathbf{DS.k,n,N}$) holds true for all $k \geq 0$. For $k = 0$, the property is valid using Theorem 4.1. Next, suppose that ($\mathbf{DS.k',n,N}$) holds true for all $k' \in (0, k)$, then by combining this last assumption with ($\mathbf{DS.k,n',N}$) above, one can conclude that:

(i) ($\mathbf{DS.k,n,N}$) holds true for all $k \geq 0$ and for all pairs of PI cubes using Theorem 4.2.

(ii) ($\mathbf{DS.k,n,N}$) holds true for all $k \geq 0$ and for all pairs of FI cubes using Theorem 4.3.

(iii) ($\mathbf{DS.k,n,N}$) holds true for all $k \geq 0$ and for all pairs of MI cubes using Theorem 4.4.

Hence, Theorem 5.1 is proven.

6. Proofs of the Results

6.1. Proof of Theorem 4.1. Using the multi-particle multi-scale analysis bounds in the continuum property ($\mathbf{DS.k,n,N}$), we extend to multi-particle systems the strategy of Stollmann [17].

For $x_0 \in \mathbb{Z}^{Nd}$ and an integer $k \geq 0$, using the notations of lemma 2.1,

$$R(x_0) := \max_{1 \leq \ell \leq \kappa(N)} |x_0 - x_{(\ell)}|; \quad b_k(x_0) := 7N + R(x_0)L_k^{-1},$$

$$M_k(x_0) := \bigcup_{\ell=1}^{\kappa(N)} C_{7N L_k}^{(N)}(x_{(\ell)})$$

and define

$$A_{k+1}(x_0) := C_{b_k L_k}^{(N)}(x_0) \setminus C_{b_k L_k}^{(N)}(x_0),$$

where the positive parameter $b$ is to be chosen later. We can easily check that

$$M_k(x_0) \subset C_{b_k L_k}^{(N)}(x_0).$$

Moreover, if $x \in A_{k+1}(x_0)$, then the cubes $C_{L_k}^{(N)}(x)$ and $C_{L_k}^{(n)}(x_0)$ are separable by Lemma 2.1. Now, also define

$$\Omega_k(x_0) := \{ \exists E \in I_0 \text{ and } x \in A_{k+1}(x_0) \cap \Gamma_k : C_{L_k}^{(n)}(x) \text{ and } C_{L_k}^{(n)}(x_0) \text{ are } (E,m)-S \},$$
with $\Gamma_k := x_0 + \frac{L_k}{2} \mathbb{Z}^d$. Now property (DS,k,N,N) combined with the cardinality of $A_{k+1}(x_0) \cap \Gamma_k$ imply

$$\mathbb{P}\{\Omega_k(x_0)\} \leq (2bbk+1L_{k+1})^N L_k^{-2p} \leq (2bbk+1)^N L_k^{-2p+\alpha N_d}.$$ 

Since, $p \geq (\alpha N_d + 1)/2$ (in fact $p \geq 6N_d$), we get $\sum_{k=0}^{\infty} \mathbb{P}\{\Omega_k(x_0)\}$ is finite. Thus, setting

$$\Omega_\infty := \{\forall x_0 \in \mathbb{Z}^d, \Omega_k(x_0) \text{ occurs finitely many times}\},$$

by the Borel Cantelli Lemma and the countability of $\mathbb{Z}^d$ we have that $\mathbb{P}\{\Omega_\infty\} = 1$. Therefore it suffices to pick $\omega \in \Omega_\infty$ and prove the exponential decay of any nonzero eigenfunction $\Psi$ of $H^{(N)}(\omega)$.

Let $\Psi$ be a polynomially bounded eigenfunction satisfying (EDI) (see Theorem 2.4). Let $x_0 \in \mathbb{Z}^d$ with positive $\|1_{C_1(x_0)}\Psi\|$ (if there is no such $x_0$, we are done.) The cube $C^{(N)}_{L_k}(x_0)$ cannot be $(E,m)$-NS for infinitely many $k$. Indeed, given an integer $k \geq 0$, if $C^{(N)}_{L_k}(x_0)$ is $(E,m)$-NS then by (EDI) and the polynomial bound on $\Psi$, we get

$$\|1_{C_1^{(N)}(x_0)}\Psi\| \leq \|1_{C_1^{(N,\omega)}(x_0)}G^{(N)}_{\{x_0\}}(E)1_{C_1^{(N,\omega)}(x_0)}\| \cdot \|1_{C_1^{(N,\omega)}(x_0)}\Psi\|$$

and the last term tends to 0 as $L_k$ tends to infinity in contradiction with the choice of $x_0$. So there is an integer $k_1 = k_1(\omega,E,x_0)$ finite such that for all $k \geq k_1$ the cube $C^{(N)}_{L_k}(x_0)$ is $(E,m)$-S. At the same time, since $\omega \in \Omega_\infty$, there exists $k_2 = k_2(\omega,x_0)$ such that if $k \geq k_2$ $\Omega_k(x_0)$ does not occur. We conclude that for all $k \geq \max\{k_1,k_2\}$, for all $x \in A_{k+1}(x_0) \cap \Gamma_k$, $C^{(N)}_{L_k}(x)$ is $(E,m)$-NS. Let $\rho \in (0,1)$ and choose positive $b$ such that

$$b \geq \frac{1 + \rho}{1 - \rho},$$

so that

$$\tilde{A}_{k+1} := C^{(N)}_{\frac{bL_k}{1 - \rho}}(x_0) \setminus C^{(N)}_{\frac{bL_k}{1 + \rho}}(x_0) \subset A_{k+1}(x_0),$$

for $x \in \tilde{A}_{k+1}(x_0)$.

(1) Since, $|x - x_0| \geq \frac{bL_k}{1 - \rho}$,

$$\text{dist}(x, \partial C^{(N)}_{bL_k}(x_0)) \geq |x - x_0| - bL_k \geq |x - x_0| - (1 - \rho)|x - x_0| = \rho|x - x_0|,$$

(2) Since $|x - x_0| \leq \frac{bL_k}{1 + \rho}$,

$$\text{dist}(x, \partial C^{(N)}_{bL_k+1}(x_0)) \geq bL_k+1 - |x - x_0| \geq (1 + \rho)|x - x_0| - |x - x_0| = \rho|x - x_0|.$$ 

Thus,

$$\text{dist}(x, \partial A_{k+1}(x_0)) \geq \rho|x - x_0|.$$ 

Now, setting $k_3 = \max\{k_1,k_2\}$, the assumption linking $b$ and $\rho$ implies that

$$\bigcup_{k \geq k_3} \tilde{A}_{k+1}(x_0) = \mathbb{R}^d \setminus C^{(N)}_{\frac{bL_k}{1 - \rho}}(x_0).$$
because \( \frac{b_{k+1} L_{k+1}}{1 + \rho} \geq \frac{b_k L_k}{1 + \rho} \). Let \( k \geq k_3 \), recall that this implies that all the cubes with centers in \( A_{k+1}(x_0) \cap \Gamma_k \) and side length \( 2L_k \) are \((E, m)\)-NS. Thus, for any \( x \in A_{k+1}(x_0) \), we choose \( x_1 \in A_{k+1}(x_0) \) such that \( x \in C_{L_k}(x_1) \). Therefore

\[
\| C^{(N)}_1(x) \| \leq \| C^{(N, \text{int})}_1(x_1) \| \leq C \cdot e^{-mL_k} \cdot \| C^{(N, \text{int})}_1(x_1) \|.
\]

Up to a set of Lebesgue measure zero, we can cover \( C_{L_k}(x_1) \) by at most \( 3^{N_d} \) cubes

\[
C_{L_k}(x_1), \quad x \in \Gamma_k, \quad |x - x_1| = \frac{L_k}{3}.
\]

By choosing \( x_2 \) which gives a maximal norm, we get

\[
\| C^{(N, \text{int})}_1(x_1) \| \leq 3^{N_d} \cdot \| C^{(N, \text{int})}_1(x_2) \|,
\]

so that

\[
\| C^{(N)}_1(x) \| \leq 3^{N_d} \cdot e^{-mL_k} \cdot \| C^{(N, \text{int})}_1(x_2) \|.
\]

Thus, by an induction procedure, we find a sequence \( x_1, x_2, \ldots, x_n \) in \( \Gamma_k \cap A_{k+1}(x_0) \) with the boundary

\[
\| C^{(N)}_1(x) \| \leq (C \cdot 3^{N_d} \exp(-mL_k))^n \cdot \| C^{(N, \text{int})}_1(x_n) \|.
\]

Since \( |x_i - x_{i+1}| = L_k/3 \) and \( \text{dist}(x, \partial A_{k+1}) \geq \rho \cdot |x - x_0| \), we can iterate at least \( \rho \cdot |x - x_0| \cdot 3/L_k \) times until we reach the boundary of \( A_{k+1}(x_0) \). Next, using the polynomial bound on \( \Psi \), we obtain:

\[
\| C^{(N)}_1(x) \| \leq (C \cdot 3^{N_d} \exp(-mL_k))^n \cdot \| C^{(N, \text{int})}_1(x_n) \| \cdot L_k^{N_d}.
\]

We can conclude that given \( \rho' \) with \( \rho' \in (0, 1) \), we can find \( k_4 \geq k_3 \) such that if \( k \geq k_4 \), then

\[
\| C^{(N)}_1(x) \| \leq e^{-\rho'/m|x - x_0|},
\]

if \( |x - x_0| \geq \frac{b_k L_k}{1 + \rho} \). This completes the proof of the exponential localization in the max-norm.

6.2. Proof of Theorem 1.2. For the proof of the multi-particle dynamical localization given the multi-particle multi-scale analysis in the continuum, we refer to the paper by Boutet de Monvel et al. [4].

7. Appendix

7.1. proof of Lemma 2.1. (A) Consider positive \( L \), \( \emptyset \neq J \subset \{1, \ldots, n\} \) and \( y \in \mathbb{Z}^n \). \( \{y_j\}_{j \in J} \) is called an \( L \)-cluster if the union

\[
\bigcup_{j \in J} C_1^{(1)}(y_j),
\]

cannot be decomposed into two non-empty disjoint subsets. Next, given two configurations \( x, y \in \mathbb{Z}^n \), we proceed as follows:

1. We decompose the vector \( y \) into maximal \( L \)-clusters \( \Gamma_1, \ldots, \Gamma_M \) (each of diameter \( \leq 2nL \)) with \( M \leq n \)

2. Each position \( y_i \) corresponds to exactly one cluster \( \Gamma_j, j = j(i) \in \{1, \ldots, M\} \).
(3) If there exists \( j \in \{1, \ldots, M\} \) such that \( \Gamma_j \cap \Pi \mathcal{C}^{(n)}_{L_k}(x) = \emptyset \), then the cubes \( \mathcal{C}^{(n)}_{L_k}(y) \) and \( \mathcal{C}^{(n)}_{L_k}(x) \) are separable.

(4) If (3) is wrong, then for all \( k = 1, \ldots, M \) \( \Gamma_k \cap \Pi \mathcal{C}^{(n)}_{L_k}(x) \neq \emptyset \). Thus for all \( k = 1, \ldots, M \), \( \exists \gamma_k \neq 1, \ldots, n \) such that \( \Gamma_k \cap \mathcal{C}^{(1)}_{L_k}(x_j) \neq \emptyset \). Now for any \( j = 1, \ldots, n \) there exists \( k = 1, \ldots, M \) such \( y_j \in \Gamma_k \). Therefore for such \( k \), by hypothesis there exists \( i = 1, \ldots, n \) such that \( \gamma_k \cap \mathcal{C}^{(1)}_{L_k}(x_i) \neq \emptyset \). Next let \( z \in \Gamma_k \cap \mathcal{C}^{(1)}_{L_k}(x_i) \) so that \(|z - x_i| \leq L\). We have that

\[
|y_j - x_i| \leq |y_j - z| + |z - x_i| \\
\leq 2nL - L + L = 2nL
\]

since \( y_j \in \Gamma_k \).

Notice that above we have the bound \(|y_j - z| \leq 2nL - L\) because \( y_j \) is a center of the \( L \)-cluster \( \Gamma_k \). Hence for all \( j = 1, \ldots, n \) \( y_j \) must belong to one of the cubes \( C^{(2nL)}_{L_k}(x_i) \) for the \( n \)-positions \( (y_1, \ldots, y_n) \). Set \( \kappa(n) = n^n \). For any choice of at most \( \kappa(n) \) possibilities: \( y = (y_1, \ldots, y_n) \) must belong to the cartesian product of \( n \) cubes of side length \( 2L \) i.e., an \( nd \)-dimensional cube of size \( 2nL \), the assertion then follows.

(B) Set \( R(y) = \max_{1 \leq i, j \leq n} |y_i - y_j| + 5nL \) and consider a cube \( \mathcal{C}^{(n)}_{L_k}(x) \) with \(|y - x| \geq R(y)\). Then there exist \( i_0 \in \{1, \ldots, n\} \) such that \(|y_{i_0} - x_{i_0}| \geq R(y)\).

Consider the maximal connected component \( \Lambda_x := \bigcup_{i \in J} \mathcal{C}^{(1)}_{L_k}(x_i) \) of the union \( \bigcup_{i} \mathcal{C}^{(1)}_{L_k}(x_i) \) containing \( x_{i_0} \). Its diameter bis bounded by \( 2nL \). We have

\[
\text{dist}(\Lambda_x; \Pi \mathcal{C}^{(n)}_{L_k}(y)) = \min_{u,v} |u - v|
\]

now, since

\[
|x_{i_0} - y_{i_0}| \leq |x_{i_0} - u| + |u - v| + |v - y_{i_0}|
\]

then

\[
\text{dist}(\Lambda_x; \Pi \mathcal{C}^{(n)}_{L_k}(y)) = \min_{u,v} |u - v| - \text{diam}(\Lambda_x) - \max_{v,y_{i_0}} |v - y_{i_0}|
\]

Recall that \( \text{diam}(\Lambda_x) \leq 2nL \) and

\[
\max_{v,y_{i_0}} |v - y_{i_0}| \leq \max_{v} |v - y_j| + \max_{y_{i_0}} |y_j - y_{i_0}|
\]

for some \( j = 1, \ldots, n \) such that \( v \in \mathcal{C}^{(1)}_{L_k}(y_j) \). Finally, we get

\[
\text{dist}(\Lambda_x, \Pi \mathcal{C}^{(n)}_{L_k}(y)) \geq R(y) - \text{diam}(\Lambda_x) - (2L + \text{diam}(\Pi y)),
\]

and the latter quantity is strictly positive. This implies that \( \mathcal{C}^{(n)}_{L_k}(x) \) is \( J \)-separable from \( \mathcal{C}^{(n)}_{L_k}(y) \).

7.2 Proof of Lemma 4.1. Set \( R := 2L + r_0 \) and assume that \( \text{diam} \Pi u = \max_{i,j} |u_i - u_j| \geq nR \). If the union of cubes \( \mathcal{C}^{(1)}_{R/2}(u_i), i = 1, \ldots, n \) were not decomposable into two (or more) disjoint groups, then, it would be connected hence its diameter would be bounded by \( n(2R/2) = nR \) hence \( \text{diam} \Pi u \leq nR \) which contradicts the hypothesis. Therefore, there exists an index subset \( J \subset \{1, \ldots, n\} \) such that \(|u_{j_1} - u_{j_2}| \geq 2(R/2)\) for all \( j_1 \in J \) and \( j_2 \in J^c \), this implies that
\[
\text{dist} \left( \Pi_J C_L^{(n)}(u), \Pi_J C_L^{(n)}(u) \right) = \min_{j_1, j_2 \in J} \text{dist} \left( C_L^{(1)}(u_{j_1}), C_L^{(1)}(u_{j_2}) \right) \geq \min_{j_1, j_2 \in J} |u_{j_1} - u_{j_2}| - 2L \geq r_0.
\]

### 7.3. Proof of Lemma 4.5

If for some positive \( R \)
\[
R \leq |x - y| = \max_{1 \leq j \leq n} |x_j - y_j|
\]
then there exists \( 1 \leq j_0 \leq n \) such that \( |x_{j_0} - y_{j_0}| \geq R \). Since both cubes are fully interactive,
\[
|x_{j_0} - x_i| \leq \Pi x \leq n(2L + r_0),
\]
\[
|y_{j_0} - y_j| \leq \Pi y \leq n(2L + r_0).
\]

By the triangle inequality, for any \( 1 \leq i, j \leq n \) and \( R \geq 7nL \geq 6nL + 2nr_0 \), we have
\[
|x_i - y_j| \geq |x_{j_0} - y_{j_0}| - |x_{j_0} - x_i| - |y_{j_0} - y_j| \geq 6nL + 2nr_0 - 2n(2L + r_0) = 2nL.
\]

Therefore, for any \( 1 \leq i, j \leq n \),
\[
\min_{i,j} \text{dist} \left( C_L^{(1)}(x_i), C_L^{(1)}(y_j) \right) \geq \min_{i,j} |x_i - y_j| - 2L \geq 2(n - 1)L.
\]

which proves the claim.

### 7.4. Proof of Lemma 4.5

Assume that \( M_{\text{sep}}(C_L^{(n)}(u), E) \) is less than 2 (i.e., there is no pair of separable cubes of radius \( L_k \) in \( C_L^{(n)}(u) \)) but \( M(C_L^{(n)}(u), E) \geq \kappa(n) + 2 \). Then \( C_L^{(n)}(u) \) must contain at least \( \kappa(n) + 2 \) cubes \( C_L^{(n)}(v_i), 0 \leq i \leq \kappa(n) + 1 \) which are not separable but satisfy \( |v_i - v_{i'}| \geq 7nL_k \) for all \( i \neq i' \). On the other hand, by Lemma 2.3 there are at most \( \kappa(n) \) cubes \( C_L^{(n)}(v_i) \), such that any cube \( C_L^{(n)}(x) \) with \( x \notin \bigcup_i C_L^{(n)}(y_i) \), is separable from \( C_L^{(n)}(v_0) \). Hence \( v_i \in \bigcup_j C_L^{(n)}(y_j) \) for all \( i = 1, \ldots, \kappa(n) + 1 \). But since for all \( i \neq i' \) \( |v_i - v_{i'}| \geq 7nL_k \) there must be at most one center \( v_i \) per cube \( C_L^{(n)}(y_j) \), \( 1 \leq j \leq \kappa(n) \). Hence we come to a contradiction
\[
\kappa(n) + 1 \leq \kappa(n).
\]

The same analysis holds true if we consider only PI cubes.

### 7.5. Proof of Lemma 4.6

Suppose that \( M_{\text{PI}}(C_L^{(n)}(u), I) \geq \kappa(n) + 2 \), then by Lemma 4.14 \( M_{\text{PI}}(C_L^{(n)}(u), I) \geq 2 \). i.e., there are at least two separable \((E,m)\)-S PI cubes \( C_{L_k}^{(n)}(u^{(j)}) \), \( C_{L_k}^{(n)}(u^{(j')}) \) inside \( C_L^{(n)}(u) \). The number of possible pairs of centers \( \{u^{(j)}, u^{(j')})\} \) such that
\[
C_{L_k}^{(n)}(u^{(j)}), C_{L_k}^{(n)}(u^{(j')}) \subset C_L^{(n)}(u)
\]
is bounded by \( \frac{3^{2n}}{2} L_{k+1}^{2n} \). Then, setting
\[
B_k = \{ \exists E \in I, C_{L_k}^{(n)}(u^{(j)}), C_{L_k}^{(n)}(u^{(j')}) \text{ are } (E,m)-S \}
\]
\[
\mathbb{P} \left\{ M_{\text{PI}}(C_L^{(n)}(u), I) \geq 2 \right\} \leq \frac{3^{2n}}{2} L_{k+1}^{2n} \times \mathbb{P} \{ B_k \}
\]
with \( \mathbb{P} \{ B_k \} \leq L_k^{4p^4} + L_k^{4p^4} \).
7.6. Proof of Lemma 4.8. Suppose there exist 2\ell pairwise separable fully interactive cubes \( C_{L_k}^{(n)}(u^{(j)}) \subset C_{L_{k+1}}^{(n)}(u) \), 1 \leq j \leq 2\ell. Then by Lemma 4.8 for any pair \( C_{L_k}^{(n)}(u^{(2i-1)}) \), \( C_{L_k}^{(n)}(u^{(2i)}) \) the corresponding random Hamiltonians \( H_{C_{L_k}^{(n)}(u^{(2i-1)})} \) and \( H_{C_{L_k}^{(n)}(u^{(2i)})} \) are independent and so are their spectra and their Green functions. For \( i = 1, \ldots, \ell \), we consider the events:

\[
A_i = \left\{ \exists E \in I : C_{L_k}^{(n)}(u^{(2i-1)}) \text{ and } C_{L_k}^{(n)}(u^{(2i)}) \text{ are } (E, m) - S \right\}. 
\]

then by assumption (DS,k,n,N), we have for \( i = 1, \ldots, \ell \)

\[
P\{A_i\} \leq L_k^{-2pd^{n-\alpha}}.
\]

and by independence of the events \( A_1, \ldots, A_\ell \)

\[
P\left\{ \bigcap_{1 \leq i \leq \ell} A_i \right\} = \prod_{i=1}^{\ell} P\{A_i\} \leq \left(L_k^{-2pd^{n-\alpha}}\right)^\ell.
\]

To complete the proof, note that the total number of different families of 2\ell cubes \( C_{L_k}^{(n)}(u^{(j)}) \subset C_{L_{k+1}}^{(n)}(u) \), 1 \leq j \leq 2\ell is bounded by

\[
\frac{1}{(2\ell)!} \left| C_{L_{k+1}}^{(n)}(u) \right|^{2\ell} \leq C(n, N, d, \ell)L_k^{2\ell d n\alpha}
\]

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