Conservative and radiative dynamics of spinning bodies at third post-Minkowskian order using worldline quantum field theory

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Using the spinning worldline quantum field theory formalism we calculate the quadratic-in-spin momentum impulse $\Delta p_i^\mu$ and spin kick $\Delta a_i^\mu$ from a scattering of two arbitrarily oriented spinning massive bodies (black holes or neutron stars) in a weak gravitational background up to third post-Minkowskian (PM) order ($G^3$). Two-loop Feynman integrals are performed in the potential region, yielding conservative results. For spins aligned to the orbital angular momentum we find a conservative scattering angle that is fully consistent with state-of-the-art post-Newtonian results. Using the 2PM radiated angular momentum previously obtained by Pfeffer, Steinhoff and the present authors we generalize the angle to include radiation-reaction effects, in which case it avoids divergences in the high-energy limit.

Recent detections by the LIGO and Virgo collaborations of gravitational waves emitted by binary black hole and neutron star mergers [1–5] have driven demand for high-precision gravitational waveform templates. In the early stage these inspirals typically run over many cycles, making them difficult to model using numerical techniques [6–8]; yet, as the gravitational field is weak this regime is well tackled using perturbation theory. Often this is done in a post-Newtonian (PN) expansion in both $G$ (Newton’s constant) and $c$ (the speed of light); however, methods involving the post-Minkowskian (PM) expansion in $G$ are gaining prominence.

The crucial insight driving this shift is that bound orbits are closely related to unbound scattering events, the latter more naturally handled in the PM expansion. A well-studied approach to the bound problem in gravity is reverse engineering a gravitational potential from scattering data [9–15], which can in turn be used to describe bound orbits. More recent techniques such as the Bound-to-Boundary (B2B) correspondence directly relate bound with unbound observables [16–18]; scattering observables may also be used as direct input for an effective one-body description of the bound dynamics [19–22], also of spinning black holes or neutron stars [23–28].

To this end an enormous effort is now underway to apply techniques used to calculate scattering amplitudes in quantum field theory (QFT) to the bound-state problem in gravity. The technologies involved for both constructing integrands and performing loop integrals are well honed [29–33], and gauge-invariant scattering observables can now be obtained directly [34–37] without introducing a gravitational potential. Some impressive results have been achieved: at 3PM (two-loop) order [38–42] including radiation-reaction corrections [43–49], tidal effects [50–52] and most recently also at 4PM order [53, 54]. A closely related approach is heavy-particle EFT [55–60].

However, QFT-based methods suffer a drawback: the need to suppress terms that ultimately disappear in the classical limit. While the classical limit is now well understood in the non-spinning case as a soft limit [20, 34, 35, 61–63], the situation is further complicated by the need to re-interpret quantized spin degrees of freedom in a classical setting [64–67]. Nevertheless, these obstacles have been successfully overcome at 2PM order [26, 68, 69] up to quartic order in spin [70]; other studies of higher-spin amplitudes in this context have been done [71–81].

In this regard the worldline EFT framework is more economical [82–87], avoiding quantum corrections from the outset. Partial results for the gravitational potential are now available up to 6PN order [88–93]; in the PM expansion recent progress has closely followed the QFT program [94–98] including at 4PM order [99, 100]. To handle spin a local co-rotating frame is often introduced [101–103]: quadratic-in-spin results are available up to 5PN (N^2LO) [104–112] and 2PM orders [113] — until now at 4PM order the former have remained unchecked.

The recently developed worldline QFT (WQFT) formalism [114–117] innovates over these approaches by quantizing worldline degrees of freedom. This leads to a highly streamlined PM setup wherein classical scattering observables are directly computed as sums of tree-level Feynman diagrams. The use of an $\mathcal{N} = 2$ supersymmetric extension to the point-particle action to encapsulate spin degrees of freedom [116, 117] circumvents the need for a local co-rotating frame. Recent work on the WQFT has included the double copy [118] and applications to light bending [119]; other closely related approaches involve directly solving the classical equations of motion [120] and Wilson line operators [121].

In this Letter we realize the spinning WQFT’s full potential with a state-of-the-art calculation: deriving the quadratic-in-spin conservative momentum impulse $\Delta p_i^\mu$ and spin kick $\Delta a_i^\mu$ in a scattering encounter between massive bodies at 3PM order, including finite-size effects. Specializing to aligned spins yields the conservative scattering angle $\theta_{\text{cons}}$, which we generalize to include dissipa-
Spinning WQFT formalism. — The dynamics of Kerr black holes with masses $m_i$ and positions $x_i^{(r)}(\tau)$ on a curved $D$-dimensional background metric $g_{\mu\nu}$ are described up to quadratic order in spin by the $N = 2$ supersymmetric worldline action [125, 126]

$$\frac{S^{(i)}}{m_i} = - \int d\tau \left[ \frac{1}{2} g_{\mu\nu} \ddot{b}^\mu_i \ddot{b}^\nu_i + i \ddot{\psi}_i \slashed{D}_\tau \ddot{\psi}_i^\dagger + \frac{1}{2} R_{abcd} \ddot{\psi}_i^a \ddot{\psi}_i^b \ddot{\psi}_i^c \ddot{\psi}_i^d \right].$$

(1)

The complex Grassmannian-valued vectors $\psi_i^\dagger(\tau)$, defined in a local frame $e^a_\mu$ with $g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$ and $D_\tau^a = \psi_i^a + \dot{x}_i^a \omega_{ab} \psi_i^b$, encode spin degrees of freedom (we use the mostly minus metric). The spin tensors $S^\mu_{\nu\rho}$ and Pauli-Lubanski spin vectors $a_i^\mu$ are composite fields:

$$S^\mu_{\nu\rho}(\tau) = -2 i e^a_\mu e^b_\nu \tilde{z}^{(a)}_{\rho}(\psi_i^b), \quad a_i^\mu(\tau) = \frac{1}{2 m_i} g^a_{\nu\rho} S^{\mu a}_{\nu\rho} p_i^a,$$

(2)

where $p_{i,\mu} = m_i g_{\mu\nu} \dot{x}_i^\nu$ (referred to as $\pi_{i,\mu}$ in Ref. [117]).

Reparametrization invariance in $\tau$ and U(1) symmetry on the Grassmann vectors respectively imply conservation of $p_i^\mu$ and $\psi_i \cdot \dot{\psi}_i$. Global $N = 2$ supersymmetry provides two additional fermionic charges: $p_i \cdot \psi_i$ and $\dot{p}_i \cdot \psi_i$, which when set to zero together imply the Tulczyjew-Dixon spin-supplementary condition (SSC) $p_{i,\mu} S^\mu_{\nu\rho} = 0$ [127, 128]. The action (1) extends naturally to include finite-size objects like neutron stars by also including

$$S_E^{(i)} = - m_i C_E \int d\tau R_{\mu\nu\rho\sigma} \ddot{\psi}_i^{\dagger} \slashed{D}_\tau \ddot{\psi}_i + \frac{1}{2} g^a_{\nu\rho} S^{\mu a}_{\nu\rho} p_i^a,$$

(3)

with projector $P_{ab} := \eta_{ab} - c_{ab} c_{bc} \ddot{x}_i^{\dagger} / \dot{x}_i^2$ and Wilson coefficients $C_E$, where $C_E = 0$ for black holes. The projector ensures supersymmetry for terms up to $O(S^2)$: enough to maintain the SSC and preserve lengths of the spin vectors.

The WQFT’s distinguishing feature is quantization of both bulk and worldline degrees of freedom. In a weak gravitational field with $k = \sqrt{32 \pi G}$ we expand $g_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa h_{\mu\nu}(x)$ with the vielbein $e^a_\mu = \eta^{\mu\nu} (\eta_{\mu\nu} + \frac{2}{3} h_{\mu\nu} - \frac{1}{3} \eta_{\mu\nu} h^{\rho\nu} + \cdots)$. Thereafter we no longer distinguish between spacetime $\mu, \nu, \ldots$ and local frame $a, b, \ldots$. Indices.

The worldline fields are similarly expanded around their background values:

$$x_i^\mu(\tau) = b_i^\mu + \psi_i^\mu \tau + z_i^\mu(\tau), \quad \psi_i^\dagger(\tau) = \Psi_i^\dagger + \psi_i^\dagger(\tau), \quad S_i^\mu(\tau) = S_i^{\mu\nu}(\tau), \quad a_i^\mu(\tau) = a_i^{\mu0} + a_i^{\mu}(\tau),$$

(4)

where $S_i^{\mu\nu} = -2 i \tilde{z}_i^{(\mu)}(\psi_i^\nu)$ and $a_i^{\mu0} = \frac{1}{2} e_{\mu\nu\rho\sigma} S^{\mu\nu}_{\rho\sigma}$. Vanishing of the supercharges implies $\psi_i = \bar{\psi}_i = 0$, so $\psi_i^\dagger S_i^{\mu\nu} = 0$; using $\tau$-reparametrization invariance on each worldline we fix $b_i = 0$ where $b^\mu = b_0^\mu = b_i^0$. We also define the Lorentz factor $\gamma = v_1 \cdot v_2$ and the relative velocity $v = \sqrt{1 - \gamma} / \gamma$.

The WQFT is defined by a path integral, with physical observables calculated as operator expectation values:

$$(\mathcal{O}) := \int \mathcal{D}[h_{\mu\nu}, z_i^{\mu}, \psi_i^{\dagger\mu}] e^{i (S_{EH} + S_{sl} + \sum_{i=1}^{2} S_i^{(i)} + S_E^{(i)})} \mathcal{O}.$$
FIG. 1: The ten types of diagrams contributing to the $m_1 m_2^2$ components of $\Delta p_{1}^{(3)\mu}$ and the $m_1^2$ components of $\Delta \psi_{1}^{(3)\mu}$, involving $T^{(1;\pm)}$-type integrals (12). In the test-body limit $m_1 \ll m_2$ these are the only surviving contributions. All graphs should be considered trees — the dotted lines represent the worldlines on which energy is conserved, instead of momentum.

Diagrammatically this amounts to drawing all tree-level diagrams with a single cut external $z_i^\mu$ or $\psi_i^\mu$ line. The diagrams required to calculate both $\Delta p_{1}^{(3)\mu}$ and $\Delta \psi_{1}^{(3)\mu}$ are divided into three categories, the first two of which are illustrated schematically in Figs. 1 and 2. As the diagrams involved in $\Delta p_{1}^{(3)\mu}$ and $\Delta \psi_{1}^{(3)\mu}$ differ only by the cut outgoing line we display them together. For additional brevity we use only solid lines to represent propagating worldline modes $z_i^\mu$, $\psi_i^\mu$ and $\bar{\psi}_i^\mu$; however, it should be assumed that each internal worldline mode could be of all three types (with expressions adjusted accordingly). The third set of diagrams (not drawn) consists simply of mirrored versions of the graphs in Fig. 1 through a horizontal plane, but with the external cut line still on the first (upper) worldline. For the impulse we avoid calculating these contributions directly, instead making use of momentum conservation $\Delta p_{2}^{(3)\mu} = -\Delta p_{1}^{(3)\mu}$ (for conservative scattering).

We assemble expressions using the WQFT Feynman rules in $D = 4 - 2\epsilon$ spacetime dimensions, with the later intention of recovering four-dimensional results in the $\epsilon \to 0$ limit. Each retarded graviton (6) and worldline (7) propagator points toward the outgoing line: from cause to effect. As diagrams belonging to each of the three categories carry common overall factors of the masses $m_1^2 m_2^2$ the categories themselves are separately gauge invariant. This helpfully breaks the calculation up into gauge-invariant subcomponents. Diagrams in Fig. 1 carry the maximum allowed power of $m_2$, and represent the test-body limit $m_1 \ll m_2$. Integrals are performed over the energies (on the worldlines $\int_\omega$) or momenta (in the bulk $\int_\nu$) of all internal lines.

The integrals involved in both $\Delta p_{1}^{(3)\mu}$ and $\Delta \psi_{1}^{(3)\mu}$ are Fourier transforms of two-loop Feynman integrals:

$$\int_q e^{i q \cdot \delta(q \cdot v_1)\delta(q \cdot v_2)}|q|^{\alpha} T^{(1;\pm)}_{\nu_1,\nu_2,\ldots,\nu_7}, \quad \alpha = 1, 2, 3,$$

where $\delta(\omega) := 2\pi \delta(\omega)$, $q^\mu$ is the total momentum exchanged from the second to the first worldline and $\alpha$ is an arbitrary power of $|q| := \sqrt{-q \cdot q}$. The two-loop integral families are

$$T^{(1,2;\pm)}_{\nu_1,\ldots,\nu_7} [\ell^1_1 \ldots \ell^1_\ell \ell^2_1 \ldots \ell^2_\ell \ldots \ell^m_m]$$

(12)

$$:= \int_{\ell_1,\ell_2} \delta(\ell_1 \cdot v_2)\delta(\ell_2 \cdot v_2)\delta(\ell_1 \cdot v_1)\delta(\ell_2 \cdot v_1) \frac{D\ell^1_1 \ldots D\ell^1_\ell D\ell^2_1 \ldots D\ell^2_\ell \ldots D\ell^m_m}{D\ell^1_1 D\ell^1_\ell D\ell^2_1 D\ell^2_\ell \ldots D\ell^m_m},$$

$$D_1 = \ell_1 \cdot v_1 + i0^+, \quad D_2 = \pm(\ell_2 \cdot v_1 + i0^+, \quad D_3 = \ell_1^2, \quad D_4 = \ell_2^2, \quad D_5 = (\ell_1 + \ell_2 - q)^2, \quad D_6 = (\ell_1 - q)^2, \quad D_7 = (\ell_2 - q)^2,$$

and

$$T^{(2;\pm)}_{\nu_1,\ldots,\nu_7} = T^{(1;\pm)}_{\nu_1,\ldots,\nu_7} \big|_{v_1=v_2=0}.$$ Each pair ($\pm$) is associated with one of the three categories of diagrams. To achieve these representations one must first integrate on the energies carried by any internal deflection $z_i^\mu$ or spin $\psi_i^\mu$, $\bar{\psi}_i^\mu$ nodes on the worldlines.

As two-loop integrals of this kind are now well studied — see e.g. Refs. [39, 46, 132, 133] — we relegate
full details of how to perform them to Appendix A. The $I_{n_1,\ldots,n_r}$ integrals — associated with the test-body diagrams in Fig. 1 — are more straightforward, being naturally evaluated in the rest frame $v_2^\mu = (1, 0)$. The more involved $I_{n_1,\ldots,n_r}$ integrals — associated with the diagrams in Fig. 2 — contain the arcosh$\gamma$ function. To fix boundary conditions we adopt the potential region of integration, which ignores radiation-reaction contributions and may be interpreted as a resummation of the terms arising from a conservative PN expansion $\frac{v_1}{\gamma v} \ll 1$. We have therefore excluded certain graphs from Fig. 2 — the so-called “mushroom graphs” — which integrate to zero within this regime.

Our final results for $\Delta p_1^{(3)\mu}$ and $\Delta a_1^{(3)\mu}$ are presented partially in Appendix B, and in full in an ancillary file attached to the arXiv submission of this Letter. They have the schematic form

$$\Delta p_1^{(3)\mu} = \sum_{s=0}^2 \sum_{n=1}^{m_2 m_2} \left[ c_{(s)\mu}^{(0)} \text{arcosh}\gamma + \frac{m_2}{m_2} \right] \left( \frac{m_1}{m_2} \right)^{n-2} d_{(s)\mu}^{(1)}$$

(13a)

$$\Delta a_1^{(3)\mu} = \sum_{s=0}^2 \sum_{n=1}^{m_2 m_2} \left[ a_{(s)\mu}^{(0)} \text{arcosh}\gamma + \frac{m_2}{m_2} \right] \left( \frac{m_1}{m_2} \right)^{n-2} a_{(s)\mu}^{(1)}$$

(13b)

The coefficients $c_{(s)\mu}^{(0)}$ and $d_{(s)\mu}^{(1)}$ are rational functions of $v_1^\mu$, the initial spin vectors $a_{(s)\mu}^{(0)}$ and the unit-normalized impact parameter $b^\mu := b^\mu |b|$, where $|b| := \sqrt{-b \cdot b}$. We have performed several consistency checks. Firstly, all poles in $\epsilon = 2 - \frac{D}{2}$ arising from the dimensionally regularized two-loop integrals (12) are seen to cancel, thus ensuring finiteness of our results in the limit $D \rightarrow 4$. Secondly, conservation of $p_1^2$, $\psi_1 \cdot \psi_1$ and the fermionic supercharge $p_1 \cdot \psi_1$ between initial and final states implies a set of consistency requirements:

$$0 = m_1 v_1 \cdot \Delta p_1^{(3)} + \Delta p_1^{(1)} \cdot \Delta p_1^{(2)},$$

(14)

$$0 = \Psi_1 \cdot \Delta p_1^{(3)} + \Delta p_1^{(1)} \cdot \Delta p_1^{(2)} + \Delta p_1^{(2)} \cdot \Delta p_1^{(1)},$$

$$0 = m_1 v_1 \cdot \Delta a_1^{(3)} + \Delta a_1^{(1)} \cdot \Delta a_1^{(2)} + \Delta a_1^{(2)} \cdot \Delta a_1^{(1)}.$$ 

All three of these checks are highly nontrivial: for instance, the third compares parts of $\Delta p_1^{(3)\mu}$ containing arcosh$\gamma$ with $\Delta p_1^{(3)\mu}$ at different orders in spin.

Scattering angle. — We now specialize to spin vectors $a_1^\mu$ aligned with the orbital angular momentum: $a_1^\mu = s_1 l^\mu$, where $l^\mu := c^{\nu \rho \sigma} b^{\nu} e^{\rho} e^{\sigma} / (\gamma v)$, confining the motion to a plane. The conservative part of the scattering angle is then given by (see e.g. Ref. [94]):

$$\sin \left( \frac{\theta_{cons}}{2} \right) = \frac{\Delta p_1}{2p_\infty}. \quad \text{(15)}$$

with the full scattering angle (including radiative corrections) given by $\theta = \theta_{cons} + \theta_{rad}$. The center-of-mass momentum is $p_\infty = \mu v / \Gamma$, where $\mu = m_1 m_2 / M$ is the symmetric mass, $M = m_1 + m_2$ is the total mass and $\Gamma = E / M = \sqrt{1 + 2v / (\sqrt{1 - v^2})}$, $E$ being the total energy. We decompose the scattering angle as

$$\frac{\theta}{\Gamma} = \sum_n \left( \frac{GM}{|b|} \right)^n \theta^{(n)}, \quad \theta^{(n)} = \sum_m \theta^{(n,m)} \frac{|b|^m}{M}, \quad \text{(16)}$$

with $n$ and $m$ counting the PM and spin orders respectively. At 3PM order using our results

$$\theta_{cons}^{(3)} = \frac{2 s_1}{\gamma^2 - 1} \left[ \Delta \theta_{cons}^{(3)} + \frac{\Delta \theta_{cons}^{(1)}}{\Delta \theta_{cons}^{(2)}} \right]$$

where we have defined $\delta = (m_2 - m_1) / M$ as well as $s_\pm = s_1 \pm s_2$ and $s_{E,\pm}^2 = C_E s_1^2 / C_E s_2^2$. We have checked $\theta_{cons}^{(3)}$ both in the test-body limit $\nu \rightarrow 0$ and up to 4PN order ($N^2LO$) for comparable masses against Refs. [26, 134].

1 We thank Mohammed Khalil for providing us with an extension of the 4PN scattering angle to include finite-size $C_E e$ coefficients.
the complete quadratic-in-spin conservative dynamics of compact binaries at 4PN order [107, 108] together with recent work in the worldline EFT formalism [112].

As explained by Bini and Damour [122–124], the conservative scattering angle is generalized to include radiation using the linear response relation:

\[
\theta_{\text{rad}} = -\frac{1}{2} \frac{\partial \theta_{\text{cons}}}{\partial E} E_{\text{rad}} - \frac{1}{2} \frac{\partial \theta_{\text{cons}}}{\partial J} J_{\text{rad}}. \tag{18}
\]

Here \( J \) is the total angular momentum in the center-of-mass frame: the derivative is equivalent to one with respect to the orbital angular momentum \( L = p_\infty \, |b| \).

It has recently been clarified [135] that Eq. (18) applies only using an “intrinsic” gauge choice with respect to BMS symmetry, where in the radiated angular momentum \( J_{\text{rad}} \) begins at \( \mathcal{O}(G^2) \). With \( E_{\text{rad}} \) starting at \( \mathcal{O}(G^3) \) to deduce \( \theta_{\text{rad}}^{(3)} \) we need only \( J_{\text{rad}}^{(2)} \), which was provided by Plefka, Steinhoff and the present authors for arbitrary spin orientations in Ref. [116]. For aligned spins,

\[
\frac{J_{\text{rad}}^{(2)}}{L} = \left( 1 + \frac{2v s_+}{|b| (1 + v^2)} + \frac{s_+^2 - s_{E,+}^2}{|b|^2} \right)
\times \frac{4m_1 m_2 (2\gamma^2 - 1)}{|b|^2 \sqrt{\gamma - 1}} \left( -\frac{4}{3} + \frac{1}{v^2} + \frac{(3v^2 - 1)}{v^3} \arccosh \gamma \right). \tag{19}
\]

This yields the radiative part of the scattering angle:

\[
\theta_{\text{rad}}^{(3)} = \frac{4\nu (2\gamma^2 - 1)^2}{(\gamma^2 - 1)^{3/2}} \left[ \frac{8}{3} + \frac{1}{v^2} + \frac{(3v^2 - 1)}{v^3} \arccosh \gamma \right]
\times \left[ 1 + \frac{6\gamma^2 v}{(2\gamma^2 - 1) |b|} s_+ + 4 \left( \frac{6\gamma^4 - 6\gamma^2 + 1}{(2\gamma^2 - 1)^2} \frac{s_+^2}{|b|^2} - \frac{s_{E,+}^2}{|b|^2} \right) \right]. \tag{20}
\]

The non-spinning part of \( \theta_{\text{rad}}^{(3)} \) has also been confirmed without reference to Eq. (18) — see e.g. Refs. [43, 44].

A key criterion of \( \theta_{\text{rad}}^{(3)} \) is that the total scattering angle should remain finite in the high-energy limit. We write \( \theta(E, \nu, |b|, \gamma, s_+) \) in terms of the energy, symmetric mass ratio, impact parameter, Lorentz factor and spin magnitudes and let \( \gamma \to \infty \), in which case the individual masses are negligible. In this limit:

\[
\theta = \frac{4}{|b|} \frac{GE}{\left( 1 + \frac{s_+}{|b|} + \frac{s_+^2 - s_{E,+}^2}{|b|^2} \right)} + \frac{32}{3} \left( \frac{GE}{|b|} \right)^3 \left[ 1 + \frac{3s_+}{|b|} \right]
+ \frac{3}{20} \frac{41s_+^2 + s_+^2 - 16s_{E,+}^2}{|b|^2} + \mathcal{O}(G^4, \gamma^{-1/2}). \tag{21}
\]

While we know of no spinning extension to Amati, Ciafaloni and Veneziano’s result [136] to compare with in the high-energy limit, we do see that a logarithmic divergence appearing in the conservative part of the angle (17) is canceled by the radiative correction (20).

Discussion. — We conclude with a brief discussion of bound observables. Using the B2B dictionary [16–18] one may, for instance, recover the aligned-spin periastron advance \( \Delta \Phi \) from our scattering angle:

\[
\Delta \Phi = \theta(E, L, m_i, s_i) + \theta(E, -L, m_i, -s_i). \tag{22}
\]

Similarly one may relate the unbound and bound radial actions, from which the scattering angle and periastron advance are respectively given by a derivative with respect to \( L \). At 3PM order \( \theta^{(3)} \) cancels in Eq. (22); nevertheless, from \( \theta^{(3)} \) one may reconstruct the leading-PN parts of \( \theta^{(4)} \) and \( \theta^{(6)} \) (and similarly for the radial action) [134]. This suffices for a comparison with bound quadratic-in-spin results at \( N^2\text{LO} \): for example, we have reproduced the quadratic-in-spin \( N^2\text{LO} \) binding energy for circular orbits [107, 108] as was also very recently done in Ref. [112].

For arbitrarily aligned spins there is currently no extension of the B2B map (22). An alternative would therefore be to make an ansatz for a conservative two-body Hamiltonian — for example, building on that used at 2PM order [68, 69] and solve Hamilton’s equations for comparison with \( \Delta q_\mu^{(3)} \) and \( \Delta a_\mu^{(3)} \), thus extending those results to 3PM. On the other hand, we are hopeful that direct maps between unbound and bound gauge-invariant observables for arbitrary spins will be discovered in the near future. In that spirit, all information is captured by the impulse and spin kick.

There remains much work to be done: for example, extending \( \Delta q_\mu^{(3)} \) and \( \Delta a_\mu^{(3)} \) to incorporate radiation-reaction effects, as we have already done for the scattering angle \( \theta_{\text{rad}}^{(3)} \) (20). This requires us to upgrade our two-loop master integrals to account for the retarded pole displacement on the graviton propagator (6) and restore the mushroom graphs to Fig. 2. We are also interested in the eikonal phase, which was computed in Ref. [117] at 2PM order as the free energy of the WQFT, and captures both the impulse and spin kick. Nevertheless, for the time being we believe that we have effectively casced the spinning WQFT’s utility and efficiency.

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Appendix A: Two-loop integration

In this Appendix we outline the steps required to perform integrals of the kind appearing in Eq. (11). All scalar two-loop integrals \( \mathcal{I}^{(i);\pm}_{n_1,...,n_7} \) (those without free indices) are functions only of \(|q|\), \( \gamma \) and the dimensional regularization parameter \( \epsilon = 2 - \frac{D}{2} \). The \(|q|\) dependence is easily established as an overall factor, \( |q| \) being the only dimensional scale. The \( \mathcal{I}^{(i);\pm}_{n_1,...,n_7} \) integrals further factorize into functions of \( \gamma \) and \( \epsilon \); one sees this by working in the rest frame of the second body \( v_2^\mu = (1, 0) \), wherein the two \( \delta \)-functions are naturally resolved as \( \delta_1^0 = \delta_0^1 = 0 \).

1. Tensor reduction. — Two-loop tensor integrals \( \mathcal{I}^{(i);\pm}_{n_1,...,n_7} \) in arbitrary \( D \) dimensions are decomposed onto bases consisting of the vectors \( v_i^\mu \) and \( q^\mu \) appearing in their expressions, plus the metric. We find it convenient to introduce

\[
    w_1^\mu = \frac{\gamma v_1^\mu - v_2^\mu}{\gamma^2 - 1}, \quad w_2^\mu = \frac{\gamma v_2^\mu - v_1^\mu}{\gamma^2 - 1}, \quad P^{\mu\nu} = \eta^{\mu\nu} - w_1^\mu v_1^\nu - w_2^\mu v_2^\nu + \frac{q^\mu q^\nu}{|q|^2},
\]

(A1)

where by design dual vectors \( w_1^\mu \) satisfy \( v_1 \cdot w_1 = \delta_{ij} \); \( P^{\mu\nu} \) is the metric of the \((D - 3)\)-dimensional space orthogonal to \( v_i^\mu \) and \( q^\mu \). Then, for example

\[
    \mathcal{I}^{(i);\pm}_{n_1,...,n_7}[\ell_1^\mu] = \mathcal{I}^{(i);\pm}_{n_1,...,n_7}[\ell_1 \cdot v_1] w_1^\mu - |q|^{-2} \mathcal{I}^{(i);\pm}_{n_1,...,n_7}[\ell_1 \cdot q] q^\mu, \quad (A2a)
\]

\[
    \mathcal{I}^{(i);\pm}_{n_1,...,n_7}[\ell_1 \ell_2^\mu] = \frac{1}{2} \mathcal{I}^{(i);\pm}_{n_1,...,n_7}[\ell_1 \cdot P \cdot \ell_2] P^{\mu\nu} + \mathcal{I}^{(i);\pm}_{n_1,...,n_7}[(\ell_1 \cdot v_1)^2] w_1^\mu w_1^\nu - |q|^{-2} \mathcal{I}^{(i);\pm}_{n_1,...,n_7}[\ell_1 \cdot v_1 \ell_1 \cdot q] w_1^\mu q^\nu + |q|^{-4} \mathcal{I}^{(i);\pm}_{n_1,...,n_7}[(\ell_1 \cdot q)^2] q^\mu q^\nu. \quad (A2b)
\]

The resulting integrals are straightforwardly reduced to scalar-type:

\[
    \mathcal{I}^{(i);\pm}_{n_1,n_2,...,n_7}[\ell_1 \cdot q] = \frac{1}{2} \mathcal{I}^{(i);\pm}_{n_1,n_2,n_3+1,n_4,n_5,n_6,n_7 - \ell_1 \cdot v_1, n_3+1,n_4,n_5,n_6,n_7 - \ell_1 \cdot v_2, n_3+1,n_4,n_5,n_6,n_7 - \ell_1 \cdot q} \mathcal{I}^{(i);\pm}_{n_1,n_2,n_3,n_4,n_5,n_6,n_7}. \quad (A3)
\]

The highest-rank integrals we encountered in this project had five free indices: \( \mathcal{I}^{(i);\pm}_{n_1,n_2,...,n_7}[\ell_1 \ell_2 \ell_3 \ell_4 \ell_5] \).

2. Integration-by-parts relations. — Our next task is to find linear identities satisfied by the scalar integrals, and thus establish a minimal basis — the so-called master integrals. We do this separately for each of the six integral families \( \mathcal{I}^{(i);\pm}_{n_1,...,n_7} \). These linear identities are generated by integration-by-parts relations (IBPs):

\[
    0 = \int_{\ell_1, \ell_2} \frac{\partial}{\partial t_{12}^\mu} \left[ k_{12}^\mu (\ell_1 \cdot v_1 + i0^+)^{n_1}(\ell_2 \cdot v_1 + i0^+)^{n_2}(\ell_2 \cdot v_2,1)^{n_3}(\ell_1 + \ell_2 - q)^{n_4}(\ell_1 - q)^{n_5}(\ell_2 - q)^{n_6} \right], \quad (A4)
\]

where \( k_{12}^\mu = \{ q^\mu, v_1^\mu, v_2^\mu, \ell_1^\mu, \ell_2^\mu \} \). Following the reverse unitarity approach of Refs. [43, 44] we have generalized the definition of the \( \delta \)-functions to include derivatives:

\[
    \frac{\delta(\nu)}{(\omega + i0^+)^{n+1}} = \frac{i}{(\omega - i0^+)^{n+1}}, \quad (A5)
\]

From the perspective of IBPs the \( \delta \)-functions may thus also be regarded as (cut) propagators, so here the integral families have nine propagators instead of seven. The IBPs are linear relationships of the form

\[
    0 = \sum_j \alpha_j \mathcal{I}^{(i);\pm}_{n_1+1,b_1,j,n_2+b_2,j,...,n_9+b_9}, \quad (A6)
\]

where \( b_{j,k} = \{-1, 0, 1\} \) on a case-by-case basis and \( \alpha_j \) are generically functions of \( \gamma \), \( |q| \) and \( \epsilon = 2 - \frac{D}{2} \). The two new indices \( n_8, n_9 \) represent the propagators associated with the delta functions — in general, we always choose masters with \( n_8 = n_9 = 0 \). The IBPs separate into two categories depending on whether \( n_1 + n_2 \) is even or odd. The integral families also enjoy symmetry relations:

\[
    \mathcal{I}^{(i);\pm}_{n_1,n_2,n_3,n_4,n_5,n_6,n_7} = \mathcal{I}^{(i);\pm}_{n_2,n_1,n_3,n_4,n_5,n_6,n_7}, \quad \mathcal{I}^{(i);\pm}_{n_1,n_2,n_3,n_4,n_5,n_6,n_7} = \mathcal{I}^{(i);\pm}_{n_3,n_1,n_2,n_4,n_5,n_6,n_7}, \quad (A7)
\]

where the first is due to \( \ell_1 \leftrightarrow \pm \ell_2 \) symmetries and the second due to shifts in \( \ell_1 \) by \( q \). Using the Laporta algorithm [137, 138], implemented in publicly-available packages such as Fire [139], LiteRed [140, 141] and KIRA [142, 143], we reduce to minimal bases: two bases for each family, with even or odd \( n_1 + n_2 \).
3. Insertion of master integrals. — We now provide expressions for the master integrals, beginning with the simpler \( I^{(1;±)} \) families to all orders in \( \epsilon = 2 - \frac{1}{2} \)

\[
I^{(1;±)}_{0,0,1,1,1,0,0} = -(4\pi)^{-3+2\epsilon} \frac{\Gamma^3(\frac{1}{2} - \epsilon)\Gamma(2\epsilon)}{\Gamma(\frac{1}{2} - 3\epsilon)}, \tag{A8a}
\]

\[
I^{(1;±)}_{1,0,1,1,1,0,0} = (4\pi)^{-\frac{3}{2}+2\epsilon} \frac{i}{\sqrt{\gamma^2 - 1}} \frac{\Gamma(\frac{1}{2} - 2\epsilon)\Gamma^2(\frac{1}{2} - \epsilon)\Gamma(-\epsilon)\Gamma(\frac{1}{2} + 2\epsilon)}{\Gamma(\frac{1}{2} - 3\epsilon)\Gamma(1 - 2\epsilon)}, \tag{A8b}
\]

\[
I^{(1;±)}_{1,1,1,1,1,0,0} = 2I^{(1;±)}_{1,1,1,1,1,1,0,0} = (4\pi)^{-2+2\epsilon} \frac{\Gamma^3(-\epsilon)\Gamma(1 + 2\epsilon)}{3(\gamma^2 - 1)\Gamma(-3\epsilon)}, \tag{A8c}
\]

where we have set \(|q| = 1\). As promised the dependence on \( \gamma \) and \( \epsilon \) factorizes. These results are well-established, and may be found in e.g. Ref. [132]. The \( I^{(2;±)} \) master integrals do not factorize, and we provide them only up to the order in \( \epsilon \) to which they are required. For \( n_1 + n_2 \) even:

\[
I^{(2;±)}_{0,0,0,1,1,0,1} = 0, \tag{A9a}
\]

\[
I^{(2;±)}_{0,0,1,1,0,1,1} = (4\pi)^{-3+2\epsilon} \frac{\Gamma^4(\frac{1}{2} - \epsilon)\Gamma^2(\frac{1}{2} + \epsilon)}{\Gamma^2(1 - 2\epsilon)}, \tag{A9b}
\]

\[
I^{(2;±)}_{0,0,1,1,1,0,0} = -(4\pi)^{-2+2\epsilon} e^{-2\gamma\epsilon} \frac{\text{arccosh} \gamma}{4\epsilon \sqrt{\gamma^2 - 1}} + O(\epsilon^0), \tag{A9c}
\]

\[
I^{(2;±)}_{0,0,2,1,1,0,0} = -(4\pi)^{-2+2\epsilon} e^{-2\gamma\epsilon} \frac{(1 - 2\epsilon)\gamma \sqrt{\gamma^2 - 1} + 2\epsilon(\gamma^2 - 1)\text{arccosh} \gamma}{2\sqrt{\gamma^2 - 1}} + O(\epsilon^2), \tag{A9d}
\]

\[
I^{(2;±)}_{0,0,1,1,2,0,0} = -(4\pi)^{-2+2\epsilon} e^{-2\gamma\epsilon} \frac{\text{arccosh} \gamma}{2\sqrt{\gamma^2 - 1}} + O(\epsilon), \tag{A9e}
\]

\[
I^{(2;±)}_{0,0,1,1,1,1,1} = (4\pi)^{-2+2\epsilon} e^{-2\gamma\epsilon} \frac{\text{arccosh} \gamma + \epsilon(\text{arccosh}^2 \gamma + Li_2)}{2\epsilon \sqrt{\gamma^2 - 1}} + O(\epsilon), \tag{A9f}
\]

\[
I^{(2;±)}_{0,0,1,1,2,1,1} = (4\pi)^{-2+2\epsilon} e^{-2\gamma\epsilon} \frac{(1 + 5\epsilon)\gamma \sqrt{\gamma^2 - 1} - (1 + \epsilon + 2\gamma^2)\text{arccosh} \gamma - \epsilon(\text{arccosh}^2 \gamma + Li_2)}{2\sqrt{\gamma^2 - 1}} + O(\epsilon^2), \tag{A9g}
\]

\[
I^{(2;±)}_{1,1,1,1,1,0,0} = \frac{1}{2} I^{(2;±)}_{1,1,1,1,1,1,0,0} = (4\pi)^{-2+2\epsilon} e^{-2\gamma\epsilon} \frac{1}{2\epsilon^2(\gamma^2 - 1)} + O(\epsilon^{-1}), \tag{A9h}
\]

where the first two are known to all orders in \( \epsilon \) — the integral (A9b) is a product of one-loop integrals. The dilogarithm appearing in the integrals (A9f) and (A9g) is \( Li_2(2 - 2\gamma^2 + 2\gamma \sqrt{\gamma^2 - 1}) \): this dilogarithm and \( \text{arccosh}^2 \gamma \) cancel from all of our final results between these two integrals. In the non-spinning part of \( \Delta p_1^{(3)\mu} \) (B1) these eight master integrals are associated with terms proportional to the impact parameter \( b^\mu \). For \( n_1 + n_2 \) odd:

\[
I^{(2;±)}_{1,0,1,0,1,1,0} = 0, \tag{A10a}
\]

\[
I^{(2;±)}_{1,0,1,0,1,1,0} = (2\pi)^{-1+2\epsilon} e^{-2\gamma\epsilon} \frac{i}{32\epsilon \sqrt{\gamma^2 - 1}} + O(\epsilon^0), \tag{A10b}
\]

\[
I^{(2;±)}_{1,0,1,1,0,0,0} = -(2\pi)^{-1+2\epsilon} e^{-2\gamma\epsilon} \frac{i}{32\epsilon \sqrt{\gamma^2 - 1}} + O(\epsilon^0), \tag{A10c}
\]

\[
I^{(2;±)}_{1,0,1,1,2,0,0} = (2\pi)^{-1+2\epsilon} e^{-2\gamma\epsilon} \frac{i(1 + 4\epsilon - 8\epsilon^2)}{16\sqrt{\gamma^2 - 1}} + O(\epsilon^3), \tag{A10d}
\]

\[
I^{(2;±)}_{1,0,1,1,1,1,1} = 4^{-1-3\epsilon} \pi^{-1+2\epsilon} e^{-2\gamma\epsilon} \frac{i(-1 + 6\epsilon)}{8\sqrt{\gamma^2 - 1}} + O(\epsilon^2), \tag{A10e}
\]

which are associated with terms proportional to the velocities \( u_i^\mu \).

To derive these expressions for the master integrals we set up systems of differential equations (DEs). Using publicly available tools such as Fuchsia [144] and epsilon [145] one may find linear transformations to canonical bases \( F(x, \epsilon) \) of master integrals that obey

\[
\frac{dF}{dx} = \epsilon M(x) F, \tag{A11}
\]
where \( \mathcal{M}(x) \) is a matrix depending only on \( x = \gamma - \sqrt{\gamma^2 - 1} \). The use of \( x \) rather than \( \gamma \) in the DEs was proposed in Ref. [132]: \( \mathcal{M}(x) \) contains only poles in \( \{ x, 1 + x, 1 - x \} \), i.e. the symbol alphabet.\(^2\) The essential property of Eq. (A11) is factorization of the \( \epsilon \)-dependence, which enables a straightforward solution to these DEs for \( \bar{F} \) as Laurent series expansions in \( \epsilon = 2 - \frac{D}{2} \). Integration constants are fixed in the potential region by comparison with the static limit \( v \to 0 \), i.e. \( \gamma \to 1 \). To leading order in \( v \) the \( J^{(1; \pm)} \) and \( J^{(2; \mp)} \) scalar integral families reduce to the same expression:

\[
\int_{\ell_1, \ell_2} \frac{1}{(\ell_1 \cdot \mathbf{v} + i0^+)^{n_1}(\pm \ell_2 \cdot \mathbf{v} + i0^+)^{n_2}(-\ell_1^2)^{n_3}(-\ell_2^2)^{n_4}(-\ell_1 \cdot \ell_2 - q_1)^{n_5}(-\ell_1 - q_2)^{n_6}(-\ell_2 - q_2)^{n_7}} + O(v^{1-n_1-n_2}).
\]

We therefore fix boundary conditions on the \( J^{(2; \pm)} \) integrals by equating them with the \( J^{(1; \mp)} \) integrals: expressions for the masters in this case having already been given (A8).

4. Fourier transform. — Finally we perform the Fourier transform \( q \)-integrals from Eq. (11). These generically take the form

\[
I_v^{(D)}[q^{\mu_1} q^{\mu_2} \cdots q^{\mu_n}] := \int_q e^{iq \cdot b} \delta(q \cdot v_1) \delta(q \cdot v_2)|q|^{\nu_1} q^{\mu_1} q^{\mu_2} \cdots q^{\mu_n}.
\]

(A13)

The scalar integral is well-known — see e.g. Ref. [114]:

\[
I_v^{(D)} = \frac{2^\nu}{\pi^{(D-2)/2}} \frac{\Gamma\left(\frac{D-2+\nu}{2}\right)}{\Gamma\left(-\frac{\nu}{2}\right)} \frac{1}{(b \cdot P_{12} \cdot b)^{-\frac{D-2+\nu}{2}}},
\]

(A14)

where \( P_{12}^{\mu\nu} := \eta^{\mu\nu} - w_{1i} v_i^\mu - w_{2i} v_i^\nu \) projects to the \((D-2)\)-dimensional space orthogonal to \( v_i^\mu \). The generalization to higher-rank integrals follows easily by taking derivatives with respect to \( b_i^\mu \):

\[
I_v^{(D)}[q^{\mu_1} q^{\mu_2} \cdots q^{\mu_n}] = (-i)^n \frac{\partial^n I_v^{(D)}}{\partial b_{\mu_1} \partial b_{\mu_2} \cdots \partial b_{\mu_n}}.
\]

(A15)

One should avoid imposing \( b \cdot v_i = 0 \) until after these derivatives have been taken — hence our use of the projector \( P_{12}^{\mu\nu} \).

Appendix B: Results

In this Appendix we collect our results for the 3PM conservative momentum impulse \( \Delta p^{(3)\mu}_1 \) and spin kick \( \Delta a^{(3)\mu}_1 \), respectively providing the coefficients \( c_i^{(e)\mu} \) and \( d_i^{(e)\mu} \) in Eq. (13). At leading (zeroth) order the momentum impulse is well-known (see e.g. Ref. [95]):

\[
c_0^{(0)\mu} = -\hat{b}_\mu \frac{8(4\gamma^4 - 12\gamma^2 - 3)}{(\gamma^2 - 1)}, \quad c_1^{(0)\mu} = \hat{b}_\mu \frac{2(16\gamma^6 - 32\gamma^4 + 16\gamma^2 - 1)}{(\gamma^2 - 1)^{5/2}} + (\gamma v_2^\mu - v_1^\mu) \frac{3\pi(5\gamma^4 - 1)(2\gamma^2 - 1)}{2(\gamma^2 - 1)^2}, \quad c_2^{(0)\mu} = \hat{b}_\mu \frac{4\gamma(20\gamma^6 - 90\gamma^4 + 120\gamma^2 - 53)}{3(\gamma^2 - 1)^{5/2}} + (1 + \gamma)(v_2^\mu - v_1^\mu) \frac{3\pi(5\gamma^2 - 1)(2\gamma^2 - 1)}{2(\gamma^2 - 1)^2}, \quad c_3^{(0)\mu} = -c_1^{(0)\mu} |_{1 \rightarrow 2}, \quad c_4^{(0)\mu} = -c_0^{(0)\mu} |_{1 \rightarrow 2},
\]

and of course there is no spin kick. The linear-in-spin coefficients of the impulse are

\[
c_0^{(1)\mu} = \frac{16\gamma (\gamma^2 - 6)(2\gamma^2 + 1)(3\hat{b}_\mu \cdot a_+ + \hat{p}^\mu \hat{b} \cdot a_+)}{(\gamma^2 - 1)^{3/2}}, \quad (B2a)
\]

\(^2\) We used PolyLogTools [146] for manipulation of polylogarithms.
\[ c^{(1)\mu} = \hat{b}^\mu \left( \frac{-32\gamma (3\gamma^2 - 1) l \cdot a_1 - 4\gamma (40\gamma^4 - 52\gamma^2 + 13) l \cdot a_2}{(\gamma^2 - 1)^2} \right) \]
\[ + \mu^\mu \left( \frac{-4\gamma (8\gamma^4 - 12\gamma^2 + 3) \hat{b} \cdot a_1 - 3\pi (10\gamma^4 - 9\gamma^2 + 1) a_1 \cdot v_2 + \pi \gamma (35\gamma^4 - 40\gamma^2 + 9) a_2 \cdot v_1}{(\gamma^2 - 1)^{5/2}} \right) \]
\[ + (v_1^\mu - \gamma v_2^\mu) \left( \frac{3\pi \gamma (3\gamma^2 - 1) (5\gamma^2 - 4) l \cdot a_1}{(\gamma^2 - 1)^{5/2}} + \frac{\pi \gamma (55\gamma^4 - 62\gamma^2 + 15) l \cdot a_2}{(\gamma^2 - 1)^{5/2}} \right), \]
\[ c^{(2)\mu} = -\hat{b}^\mu \left( \frac{4 (20\gamma^6 - 140\gamma^4 + 80\gamma^2 + 41) l \cdot a_1}{(\gamma^2 - 1)^2} + \mu^\mu \left( \frac{\pi (60\gamma^5 + 35\gamma^4 - 69\gamma^3 - 30\gamma^2 + 15\gamma + 3) a_1 \cdot (v_1 - v_2)}{2 (\gamma^2 - 1)^{5/2}} \right) \]
\[ - \frac{8 (10\gamma^6 - 78\gamma^4 + 45\gamma^2 + 20) \hat{b} \cdot a_2}{(\gamma^2 - 1)^2} \right) + (v_1^\mu l \cdot a_1 - v_2^\mu l \cdot a_2) \pi \gamma (55\gamma^5 + 55\gamma^4 - 51\gamma^3 - 62\gamma^2 + 12\gamma + 15)}{(\gamma^2 - 1)^{5/2}} \]
\[ + (v_1^\mu l \cdot a_2 - v_2^\mu l \cdot a_1) \pi \gamma (55\gamma^5 + 45\gamma^4 - 62\gamma^3 - 51\gamma^2 + 15\gamma + 12)}{(\gamma^2 - 1)^{5/2}}, \]
\[ c^{(3)\mu} = -c^{(1)\mu} \bigg|_{\gamma \to v}, \]

where \( a_1^\mu = a_1^\mu + \gamma a_2^\mu \) and \( l^\mu := e^\mu_{\nu\rho\sigma} \hat{b}^\nu e^\rho^\sigma v_2^\mu / (\gamma v) \) and for brevity we omit the additional subscripts on \( a_1^\mu \). The coefficients of the spin kick are

\[ d^{(1)\mu}_0 = (v_2^\mu \hat{b} \cdot a_1 - \hat{b}^\mu a_1 \cdot v_2) \frac{16\gamma (\gamma^2 - 6) (2\gamma^2 + 1)}{(\gamma^2 - 1)^2} + v_1^\mu \hat{b} \cdot a_1 \frac{24 (2\gamma^4 + 7\gamma^2 + 1)}{(\gamma^2 - 1)^2}, \]
\[ d^{(1)\mu}_1 = \hat{b}^\mu \left( \frac{3\pi \hat{b} \cdot a_1}{2} + \frac{8\gamma (4\gamma^2 - 1) a_1 \cdot v_2}{(\gamma^2 - 1)^{3/2}} \right) + v_1^\mu \left( \frac{2 (16\gamma^4 - 14\gamma^2 + 1) \hat{b} \cdot a_1 - 3\pi \gamma (5\gamma^2 - 2) a_1 \cdot v_2}{(\gamma^2 - 1)^{5/2}} \right), \]
\[ d^{(2)\mu}_2 = \hat{b}^\mu \left( \frac{\pi (10\gamma^4 - 3\gamma^2 - 3) \hat{b} \cdot a_1}{2(\gamma^2 - 1)} \right) + v_1^\mu \left( \frac{4 (20\gamma^6 - 132\gamma^4 + 72\gamma^2 + 43) a_1 \cdot v_2}{3(\gamma^2 - 1)^{5/2}} \right) \]
\[ + v_2^\mu \left( \frac{-4\gamma (10\gamma^4 + 20\gamma^2 - 33) \hat{b} \cdot a_1}{(\gamma^2 - 1)^{3/2}} - \frac{\pi (30\gamma^4 + 35\gamma^3 - 21\gamma^2 - 15\gamma + 3) a_1 \cdot v_2}{2(\gamma^2 - 1)^2} \right), \]
\[ d^{(3)\mu}_3 = \hat{b}^\mu a_1 \cdot v_2 \frac{4\gamma (16\gamma^4 - 20\gamma^2 + 5)}{(\gamma^2 - 1)^{3/2}} - v_1^\mu a_1 \cdot v_2 \frac{4\gamma (16\gamma^4 - 20\gamma^2 + 5)}{(\gamma^2 - 1)^{5/2}} \]
\[ + v_1^\mu \left( \frac{2 (2\gamma^2 + 1) (8\gamma^4 - 8\gamma^2 + 1) \hat{b} \cdot a_1}{(\gamma^2 - 1)^{5/2}} - \frac{3\pi (2\gamma^2 - 1) (5\gamma^2 - 1) a_1 \cdot v_2}{2(\gamma^2 - 1)^2} \right). \]

Our quadratic-in-spin results are of a similar nature, but due to their considerable length we restrict ourselves to only including spin on the first body, and with \( C_{E,1} = 0 \) (a Kerr black hole scattering off a Schwarzschild black hole). For our full results — including spins on both bodies, and with finite-size \( C_{E,1} \) coefficients allowing for a generalization to neutron stars — we refer the interested reader to the ancillary file attached to the arXiv submission of this Letter.

The components of the momentum impulse are

\[ c^{(2)\mu}_0 = \hat{b}^\mu \left( \frac{64\gamma^2 (4\gamma^6 - 36\gamma^4 + \gamma^2 + 6) (\hat{b} \cdot a_1)^2}{(\gamma^2 - 1)^3} + \frac{64 (3\gamma^8 - 35\gamma^6 + 9\gamma^4 + 42\gamma^2 + 6) (a_1 \cdot v_2)^2}{(\gamma^2 - 1)^4} \right) \]
\[ + \frac{192 (6\gamma^6 - 8\gamma^4 - 7\gamma^2 - 1) a_1^2}{(\gamma^2 - 1)^2} - \mu^\mu \frac{32\gamma^2 (4\gamma^6 - 36\gamma^4 + \gamma^2 + 6) \hat{b} \cdot a_1 l \cdot a_1}{(\gamma^2 - 1)^3}, \]
\[ c^{(2)\mu}_1 = \hat{b}^\mu \left( \frac{-4 (64\gamma^6 - 108\gamma^4 + 45\gamma^2 - 2) (\hat{b} \cdot a_1)^2}{(\gamma^2 - 1)^{3/2}} - \frac{3\pi \gamma (5\gamma^2 - 2) \hat{b} \cdot a_1 a_1 \cdot v_2}{(\gamma^2 - 1)^2} \right) \]
Finally, the coefficients of the quadratic-in-spin part of the 3PM spin kick are

\[
\begin{align*}
\epsilon_{1}^{(2)} & = \hat{\mu} \left( -8 \gamma (200 \gamma^{6} - 2444 \gamma^{4} + 353 \gamma^{2} + 376) \hat{b} \cdot a_{1} l \cdot a_{1} \\
& + \pi (960 \gamma^{5} - 295 \gamma^{4} - 1217 \gamma^{2} + 327 \gamma^{2} + 285 \gamma - 36) a_{1} \cdot v_{2} l \cdot a_{1} \right)
\end{align*}
\]

\[
\begin{align*}
\epsilon_{2}^{(2)} & = \hat{\mu} \left( -8 \gamma (200 \gamma^{6} - 2444 \gamma^{4} + 353 \gamma^{2} + 376) \hat{b} \cdot a_{1} l \cdot a_{1} \\
& + \pi (960 \gamma^{5} - 295 \gamma^{4} - 1217 \gamma^{2} + 327 \gamma^{2} + 285 \gamma - 36) a_{1} \cdot v_{2} l \cdot a_{1} \right)
\end{align*}
\]

\[
\begin{align*}
\epsilon_{3}^{(2)} & = \hat{\mu} \left( -8 \gamma (200 \gamma^{6} - 2444 \gamma^{4} + 353 \gamma^{2} + 376) \hat{b} \cdot a_{1} l \cdot a_{1} \\
& + \pi (960 \gamma^{5} - 295 \gamma^{4} - 1217 \gamma^{2} + 327 \gamma^{2} + 285 \gamma - 36) a_{1} \cdot v_{2} l \cdot a_{1} \right)
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& + \pi (960 \gamma^{5} - 295 \gamma^{4} - 1217 \gamma^{2} + 327 \gamma^{2} + 285 \gamma - 36) a_{1} \cdot v_{2} l \cdot a_{1} \right)
\end{align*}
\]
\[ \begin{align*}
\tilde{d}_1^{(2)\mu} &= \mu^\mu \left( -\frac{3\pi\gamma}{\gamma^2 - 1} \hat{b} \cdot a_1 l \cdot a_1 - \frac{2 (48\gamma^6 - 68\gamma^4 + 22\gamma^2 + 1) l \cdot a_1 a_1 \cdot v_2} {(\gamma^2 - 1)^{3/2}} \right) \\
&+ \mu^\mu \left( -\frac{3\pi\gamma}{2\sqrt{\gamma^2 - 1}} \frac{\hat{b} \cdot a_1 l \cdot a_1}{\gamma^2 - 1} - \frac{2 (16\gamma^6 - 36\gamma^4 + 18\gamma^2 - 1) \hat{b} \cdot a_1 a_1 \cdot v_2} {(\gamma^2 - 1)^3} - \frac{3\pi\gamma}{\gamma^2 - 1} \frac{5\gamma^2 - 2}{(\gamma^2 - 1)^{5/2}} (a_1 \cdot v_2)^2 \right) \\
&+ \mu^\mu \left( \frac{3\pi}{\gamma^2 - 1} \frac{(30\gamma^4 - 29\gamma^2 + 3) l \cdot a_1 a_1 \cdot v_2}{(\gamma^2 - 1)^{5/2}} - \frac{8\gamma (16\gamma^2 - 7) \hat{b} \cdot a_1 l \cdot a_1}{(\gamma^2 - 1)^2} \right) \\
&+ \mu^\mu \left( \frac{8 (24\gamma^4 - 16\gamma^2 + 1) \hat{b} \cdot a_1 l \cdot a_1}{(\gamma^2 - 1)^2} - \frac{9\pi\gamma (10\gamma^2 - 3) l \cdot a_1 a_1 \cdot v_2}{2 (\gamma^2 - 1)^{3/2}} \right), \\
\tilde{d}_2^{(2)\mu} &= \mu^\mu \left( \frac{\pi (65\gamma^3 - 3\gamma - 6) \hat{b} \cdot a_1 l \cdot a_1}{4\gamma^2 - 1} - \frac{4\gamma (216\gamma^6 - 4220\gamma^4 + 2492\gamma^2 + 4557) l \cdot a_1 a_1 \cdot v_2}{15(\gamma^2 - 1)^3} \right) \\
&+ \mu^\mu \left( -\frac{\pi (10\gamma^5 - 3\gamma^3 + 3\gamma^2 - 3\gamma - 3) (\hat{b} \cdot a_1)^2}{2 (\gamma^2 - 1)^{3/2}} - \frac{4\gamma (184\gamma^6 - 188\gamma^4 + 2416\gamma^2 - 3625) \hat{b} \cdot a_1 a_1 \cdot v_2}{15(\gamma^2 - 1)^3} \right) \\
&+ \mu^\mu \left( \frac{\pi (60\gamma^4 + 35\gamma^3 - 39\gamma^2 - 15\gamma + 3) (a_1 \cdot v_2)^2}{2 (\gamma^2 - 1)^{5/2}} \right), \\
\tilde{d}_3^{(2)\mu} &= \mu^\mu \left( \frac{\pi (10\gamma^4 - 3\gamma^3 - 3\gamma^2 - 3\gamma) \hat{b} \cdot a_1 l \cdot a_1}{2 (\gamma^2 - 1)^{3/2}} - \frac{2 (160\gamma^6 - 264\gamma^4 + 116\gamma^2 - 9) l \cdot a_1 a_1 \cdot v_2}{(\gamma^2 - 1)^3} \right) \\
&+ \mu^\mu \left( -\frac{\pi (10\gamma^4 - 3\gamma^3 - 3\gamma) (\hat{b} \cdot a_1)^2}{2 (\gamma^2 - 1)^{3/2}} + \frac{2 (8\gamma^4 - 4\gamma^2 - 1) \hat{b} \cdot a_1 a_1 \cdot v_2}{(\gamma^2 - 1)^3} - \frac{\pi (70\gamma^4 - 45\gamma^2 + 3) (a_1 \cdot v_2)^2}{2 (\gamma^2 - 1)^{5/2}} \right) \\
&+ \mu^\mu \left( \frac{\pi\gamma (110\gamma^4 - 159\gamma^2 + 45) l \cdot a_1 a_1 \cdot v_2}{2 (\gamma^2 - 1)^{5/2}} - \frac{16\gamma (2\gamma^2 - 1) (4\gamma^2 + 3) \hat{b} \cdot a_1 l \cdot a_1}{(\gamma^2 - 1)^2} \right) \\
&+ \mu^\mu \left( \frac{16 (20\gamma^4 - 14\gamma^2 + 1) \hat{b} \cdot a_1 l \cdot a_1}{(\gamma^2 - 1)^2} + \frac{\pi (70\gamma^4 - 45\gamma^2 + 3) l \cdot a_1 a_1 \cdot v_2}{2 (\gamma^2 - 1)^{5/2}} \right). 
\end{align*}\]
J. M. Henn and J. C. Plefka, Nucl. Phys. B 877 (2013) 177 [1304.7263].

D. Neill and I. Z. Rothstein, Classical Space-Times from the S Matrix, Nucl. Phys. B 877 (2013) 177 [1304.7263].

V. Vaidya, Gravitational spin Hamiltonians from the S matrix, Phys. Rev. D91 (2015) 024017 [1410.5348].

T. Damour, High-energy gravitational scattering and the general relativistic two-body problem, Phys. Rev. D97 (2018) 044035 [1710.10599].

N. E. J. Bjerrum-Bohr, A. Cristofoli and P. H. Damgaard, Post-Minkowskian Scattering Angle in Einstein Gravity, JHEP 08 (2020) 038 [1910.09366].

A. Cristofoli, P. H. Damgaard, F. Di Vecchia and C. Heissenberg, Second-order Post-Minkowskian scattering in arbitrary dimensions, JHEP 07 (2020) 122 [2003.10274].

Kālin, Gregor and Porto, Rafael A., From Boundary Data to Bound States, JHEP 01 (2020) 072 [1910.03098].

Kālin, Gregor and Porto, Rafael A., From boundary data to bound states. Part II. Scattering angle to dynamical invariants (with twist), JHEP 02 (2020) 120 [1911.09130].

G. Cho, G. Kālin and R. A. Porto, From Boundary Data to Bound States III: Radiative Effects, 2112.03976.

A. Buonanno and T. Damour, Effective one-body approach to general relativistic two-body dynamics, Phys. Rev. D 59 (1999) 084006 [gr-qc/9811091].

T. Damour, Classical and quantum scattering in post-Minkowskian gravity, Phys. Rev. D102 (2020) 024060 [1912.02139].

A. Antonelli, A. Buonanno, J. Steinhoff, M. van de Meent and J. Vines, Energetics of two-body Hamiltonians in post-Minkowskian gravity, Phys. Rev. D99 (2019) 104004 [1901.07102].

P. H. Damgaard and P. Vanhove, Remodeling the effective one-body formalism in post-Minkowskian gravity, Phys. Rev. D 104 (2021) 104029 [2108.11248].

J. Vines and J. Steinhoff, Spin-multipole effects in binary black holes and the test-body limit, Phys. Rev. D 97 (2018) 064010 [1606.08832].

J. Vines, D. Kunst, J. Steinhoff and T. Hinderer, Canonical Hamiltonian for an extended test body in curved spacetime: To quadratic order in spin, Phys. Rev. D 93 (2016) 103008 [1601.07529].

J. Vines, Scattering of two spinning black holes in post-Minkowskian gravity, to all orders in spin, and effective-one-body mappings, Class. Quant. Grav. 35 (2018) 084002 [1709.06016].

J. Vines, J. Steinhoff and A. Buonanno, Spinning-black-hole scattering and the test-black-hole limit at second post-Minkowskian order, Phys. Rev. D 99 (2019) 064054 [1812.00956].

D. Bini and T. Damour, Gravitational spin-orbit coupling in binary systems, post-Minkowskian approximation and effective one-body theory, Phys. Rev. D96 (2017) 104038 [1709.00590].

D. Bini and T. Damour, Gravitational spin-orbit coupling in binary systems at the second post-Minkowskian approximation, Phys. Rev. D98 (2018) 044036 [1805.10809].

L. J. Dixon, Calculating scattering amplitudes efficiently, in Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 95): QCD and Beyond, 1, 1996, hep-ph/9601359.

H. Elvang and Y.-t. Huang, Scattering Amplitudes, 1308.1697.

J. M. Henn and J. C. Plefka, Scattering Amplitudes in Gauge Theories, vol. 883. Springer, Berlin, 2014, 10.1007/978-3-642-5022-6.

Z. Bern, J. J. Carrasco, M. Chiodaroli, H. Johansson and R. Roiban, The Duality Between Color and Kinematics and its Applications, 1909.01358.

S. Weinzierl, Feynman Integrals, 1, 2022, 2201.03593.

D. A. Kosower, B. Maybee and D. O’Connell, Amplitudes, Observables, and Classical Scattering, JHEP 02 (2019) 137 [1811.10950].

B. Maybee, D. O’Connell and J. Vines, Observables and amplitudes for spinning particles and black holes, JHEP 12 (2019) 156 [1906.09260].

A. Cristofoli, R. Gonzo, D. A. Kosower and D. O’Connell, Waveforms from Amplitudes, 2107.10193.

A. Cristofoli, R. Gonzo, N. Moynhian, D. O’Connell, A. Ross, M. Sergola et al., The Uncertainty Principle and Classical Amplitudes, 2112.07556.

Z. Bern, C. Cheung, R. Roiban, C.-H. Shen, M. P. Solon and M. Zeng, Scattering Amplitudes and the Conservative Hamiltonian for Binary Systems at Third Post-Minkowskian Order, Phys. Rev. Lett. 122 (2019) 201603 [1901.04424].

Z. Bern, C. Cheung, R. Roiban, C.-H. Shen, M. P. Solon and M. Zeng, Black Hole Binary Dynamics from the Double Copy and Effective Theory, JHEP 10 (2019) 206 [1908.01493].

Z. Bern, H. Ita, J. Parra-Martinez and M. S. Ruf, Universality in the classical limit of massless gravitational scattering, Phys. Rev. Lett. 125 (2020) 031601 [2002.02459].

C. Cheung and M. P. Solon, Classical gravitational scattering at $O(G^3)$ from Feynman diagrams, JHEP 06 (2020) 144 [2003.08351].
[47] C. Heissenberg, Infrared divergences and the eikonal exponentiation, *Phys. Rev. D* **104** (2021) 046016 [2105.04594].

[48] N. E. J. Bjerrum-Bohr, P. H. Damgaard, L. Planté and P. Vanhove, The Amplitude for Classical Gravitational Scattering at Third Post-Minkowskian Order, *JHEP* **08** (2017) 172 [2105.05218].

[49] P. H. Damgaard, L. Plante and P. Vanhove, On an exponential representation of the gravitational S-matrix, *JHEP* **11** (2021) 213 [2107.12891].

[50] C. Cheung and M. P. Solon, Tidal Effects in the Post-Minkowskian Expansion, *Phys. Rev. Lett.* **125** (2020) 191601 [2006.06665].

[51] Z. Bern, J. Parra-Martinez, R. Roiban, E. Sawyer and C.-H. Shen, Leading Nonlinear Tidal Effects and Scattering Amplitudes, *JHEP* **05** (2021) 188 [2010.08559].

[52] M. Accettulli Huber, A. Brandhuber, S. De Angelis and G. Travaglini, From amplitudes to gravitational radiation with cubic interactions and tidal effects, *Phys. Rev. D* **103** (2021) 045015 [2012.08548].

[53] Z. Bern, J. Parra-Martinez, R. Roiban, M. S. Ruf, C.-H. Shen, M. P. Solon et al., Scattering Amplitudes and Conservative Binary Dynamics at O(G^4), *Phys. Rev. Lett.* **126** (2021) 171601 [2101.07254].

[54] Z. Bern, J. Parra-Martinez, R. Roiban, M. S. Ruf, C.-H. Shen, M. P. Solon et al., Scattering Amplitudes, the Tail Effect, and Conservative Binary Dynamics at O(G^4), 2112.10750.

[55] P. H. Damgaard, K. Haddad and A. Helset, Heavy Black Hole Effective Theory, *JHEP* **11** (2019) 070 [1908.10308].

[56] R. Aoude, K. Haddad and A. Helset, On-shell heavy particle effective theories, *JHEP* **05** (2020) 051 [2001.09164].

[57] A. Brandhuber, G. Chen, G. Travaglini and C. Wen, A new gauge-invariant double copy for heavy-mass effective theory, *JHEP* **07** (2021) 047 [2104.11206].

[58] A. Brandhuber, G. Chen, G. Travaglini and C. Wen, Classical gravitational scattering from a gauge-invariant double copy, *JHEP* **10** (2021) 118 [2108.04216].

[59] R. Aoude, K. Haddad and A. Helset, Tidal effects for spinning particles, *JHEP* **03** (2021) 097 [2012.05256].

[60] K. Haddad, Exponentiation of the leading eikonal phase with spin, *Phys. Rev. D* **105** (2022) 026004 [2109.04427].

[61] N. J. Bjerrum-Bohr, P. H. Damgaard, G. Festuccia, L. Planté and P. Vanhove, General Relativity from Scattering Amplitudes, *Phys. Rev. Lett.* **121** (2018) 171601 [1806.04920].

[62] P. Di Vecchia, A. Luna, S. G. Naculich, R. Russo, G. Veneziano and C. D. White, A tale of two exponentiations in N = 8 supergravity, *Phys. Lett. B798* (2019) 134927 [1908.05603].

[63] P. Di Vecchia, S. G. Naculich, R. Russo, G. Veneziano and C. D. White, A tale of two exponentiations in N = 8 supergravity at subleading level, *JHEP* **03** (2020) 173 [1911.11716].

[64] A. Guevara, A. Ochirov and J. Vines, Scattering of Spinning Black Holes from Exponentiated Soft Factors, *JHEP* **09** (2019) 056 [1812.06895].

[65] Y. F. Bautista and A. Guevara, From Scattering Amplitudes to Classical Physics: Universality, Double Copy and Soft Theorems, 1903.12419.

[66] A. Guevara, A. Ochirov and J. Vines, Black-hole scattering with general spin directions from minimal-coupling amplitudes, *Phys. Rev. D* **100** (2019) 104024 [1906.10071].

[67] R. Aoude and A. Ochirov, Classical observables from coherent-spin amplitudes, *JHEP* **10** (2021) 008 [2108.01649].

[68] Z. Bern, A. Luna, R. Roiban, C.-H. Shen and M. Zeng, Spinning black hole binary dynamics, scattering amplitudes, and effective field theory, *Phys. Rev. D* **104** (2021) 065014 [2005.03071].

[69] D. Kosmopoulos and A. Luna, Quadratic-in-spin Hamiltonian at O(G^2) from scattering amplitudes, *JHEP* **07** (2021) 037 [2102.10137].

[70] W.-M. Chen, M.-Z. Chung, Y.-t. Huang and J.-W. Kim, The 2PM Hamiltonian for binary Kerr to quartic in spin, 2111.13639.

[71] A. Guevara, Holomorphic Classical Limit for Spin Effects in Gravitational and Electromagnetic Scattering, *JHEP* **04** (2019) 033 [1708.02314].

[72] M.-Z. Chung, Y.-T. Huang, J.-W. Kim and S. Lee, The simplest massive S-matrix: from minimal coupling to Black Holes, *JHEP* **04** (2019) 156 [1812.08752].

[73] M.-Z. Chung, Y.-T. Huang and J.-W. Kim, Classical potential for general spinning bodies, *JHEP* **09** (2020) 074 [1908.08463].

[74] N. Arkani-Hamed, Y.-t. Huang and D. O’Connell, Kerr black holes as elementary particles, *JHEP* **01** (2020) 046 [1906.10100].

[75] A. Guevara, B. Maybee, A. Ochirov, D. O’Connell and J. Vines, A worksheet for Kerr, *JHEP* **03** (2021) 201 [2012.11570].

[76] Y. F. Bautista and N. Siemonsen, Post-Newtonian waveforms from spinning scattering amplitudes, *JHEP* **01** (2022) 006 [2110.12537].

[77] M. Ciafaloni, L. H. Johansson and P. Pichini, Compton black-hole scattering for s ≤ 5/2, *JHEP* **02** (2022) 156 [2107.14779].

[78] Y. F. Bautista, A. Guevara, C. Kavanagh and J. Vines, From Scattering in Black Hole Backgrounds to Higher-Spin Amplitudes: Part I, 2107.10179.

[79] C. Crawford, A. Guevara, N. Miller and A. Strominger, Black Holes in Klein Space, 2112.03954.

[80] A. Guevara, Reconstructing Classical Spacetimes from the S-Matrix in Twistor Space, 2112.05111.

[81] T. Adamo, A. Cristofoli and P. Tourkine, Eikonal amplitudes from curved backgrounds, 2112.09113.

[82] W. D. Goldberger and I. Z. Rothstein, An Effective field theory of gravity for extended objects, *Phys. Rev. D* **73** (2006) 104029 [hep-th/0409156].

[83] W. D. Goldberger and I. Z. Rothstein, Towers of Gravitational Theories, *Gen. Rel. Grav.* **38** (2006) 1537 [hep-th/0605238].

[84] W. D. Goldberger and A. Ross, Gravitational radiative corrections from effective field theory, *Phys. Rev. D* **81** (2010) 124015 [0912.4254].

[85] R. A. Porto, The effective field theorist’s approach to gravitational dynamics, *Phys. Rept.* **633** (2016) 1 [1601.04914].

[86] M. Levi, Effective Field Theories of Post-Newtonian Gravity: A comprehensive review, *Rept. Prog. Phys.* **83** (2020) 075901 [1807.01699].
[87] W. D. Goldberger, J. Li and S. G. Prabhu, Spinning particles, axion radiation, and the classical double copy, Phys. Rev. D 97 (2018) 105018 [1712.09250].

[88] Blümlein, J. and Maier, A. and Marquard, P. and Schäfer, G., Testing binary dynamics in gravity at the sixth post-Newtonian level, Phys. Lett. B807 (2020) 135496 [2003.07145].

[89] D. Bini, T. Damour and A. Geralico, Sixth post-Newtonian local-in-time dynamics of binary systems, Phys. Rev. D102 (2020) 024061 [2004.05407].

[90] D. Bini, T. Damour and A. Geralico, Sixth post-Newtonian nonlocal-in-time dynamics of binary systems, Phys. Rev. D102 (2020) 084047 [2007.11239].

[91] J. Blümlein, A. Maier, P. Marquard and G. Schäfer, The fifth-order post-Newtonian Hamiltonian dynamics of two-body systems from an effective field theory approach: potential contributions, Nucl. Phys. B 965 (2021) 115352 [2010.13672].

[92] S. Foffa, R. Sturani and W. J. Torres Bobadilla, Efficient resummation of high post-Newtonian contributions to the binding energy, JHEP 02 (2021) 165 [2010.13738].

[93] J. Blümlein, A. Maier, P. Marquard and G. Schäfer, The 6th post-Newtonian potential term at \(O(G_4)\), Phys. Lett. B 816 (2021) 136260 [2101.08630].

[94] G. Kälin and R. A. Porto, Post-Minkowskian Effective Field Theory for Conservative Binary Dynamics, JHEP 11 (2020) 106 [2006.01184].

[95] Kälin, Gregor and Liu, Zhengwen and Porto, Rafael A., Conservative Dynamics of Binary Systems to Third Post-Minkowskian Order from the Effective Field Theory Approach, Phys. Rev. Lett. 125 (2020) 261103 [2007.04977].

[96] G. Kälin, Z. Liu and R. A. Porto, Conservative Tidal Effects in Compact Binary Systems to Next-to-Leading Post-Minkowskian Order, Phys. Rev. D 102 (2020) 124025 [2008.06047].

[97] S. Mougiakakos, M. M. Riva and F. Vernizzi, Gravitational Bremsstrahlung in the post-Minkowskian effective field theory, Phys. Rev. D 104 (2021) 024041 [2102.08339].

[98] M. M. Riva and F. Vernizzi, Radiated momentum in the post-Minkowskian worldline approach via reverse unitarity, JHEP 11 (2021) 228 [2110.10140].

[99] C. Diapa, G. Kälin, Z. Liu and R. A. Porto, Dynamics of Binary Systems to Fourth Post-Minkowskian Order from the Effective Field Theory Approach, 2106.08276.

[100] C. Diapa, G. Kälin, Z. Liu and R. A. Porto, Conservative Dynamics of Binary Systems at Fourth Post-Minkowskian Order in the Large-eccentricity Expansion, 2112.11296.

[101] R. A. Porto, Post-Newtonian corrections to the motion of spinning bodies in NRG, Phys. Rev. D 73 (2006) 104031 [gr-qc/0511061].

[102] M. Levi and J. Steinhoff, Spinning gravitating objects in the effective field theory in the post-Newtonian scheme, JHEP 09 (2015) 219 [1501.04956].

[103] W. D. Goldberger, J. Li and I. Z. Rothstein, Non-conservative effects on spinning black holes from world-line effective field theory, JHEP 06 (2021) 053 [2012.14869].

[104] J. Hartung and J. Steinhoff, Next-to-next-to-leading order post-Newtonian spin(1)-spin(2) Hamiltonian for self-gravitating binaries, Annalen Phys. 825 (2011) 919 [1107.4294].

[105] M. Levi, Binary dynamics from spin1-spin2 coupling at fourth post-Newtonian order, Phys. Rev. D 85 (2012) 064043 [1107.4322].

[106] M. Levi and J. Steinhoff, Equivalence of ADM Hamiltonian and Effective Field Theory approaches at next-to-next-to-leading order spin1-spin2 coupling of binary inspirals, JCAP 12 (2014) 003 [1408.5762].

[107] M. Levi and J. Steinhoff, Next-to-next-to-leading gravitational spin-squared potential via the effective field theory for spinning objects in the post-Newtonian scheme, JCAP 01 (2016) 008 [1506.05794].

[108] M. Levi and J. Steinhoff, Complete conservative dynamics for inspiralling compact binaries with spins at the fourth post-Newtonian order, JCAP 09 (2021) 029 [1607.04262].

[109] M. Levi, A. J. Mcleod and M. Von Hippel, \(N^3\)LO gravitational spin-orbit coupling at order \(G^4\), JHEP 07 (2021) 115 [2003.02827].

[110] M. Levi, A. J. Mcleod and M. Von Hippel, \(N^3\)LO gravitational quadratic-in-spin interactions at \(G^4\), JHEP 07 (2021) 116 [2003.07980].

[111] J.-W. Kim, M. Levi and Z. Yin, Quadratic-in-spin interactions at the fifth post-Newtonian order probe new physics, 2112.01509.

[112] G. Cho, R. A. Porto and Z. Yang, Gravitational radiation from inspiralling compact objects: Spin effects to fourth Post-Newtonian order, 2201.05138.

[113] Z. Liu, R. A. Porto and Z. Yang, Spin Effects in the Effective Field Theory Approach to Post-Minkowskian Conservative Dynamics, JHEP 06 (2021) 012 [2102.10089].

[114] G. Mogull, J. Plefka and J. Steinhoff, Classical black hole scattering from a worldline quantum field theory, JHEP 02 (2021) 048 [2010.02865].

[115] G. U. Jakobsen, G. Mogull, J. Plefka and J. Steinhoff, Classical Gravitational Bremsstrahlung from a Worldline Quantum Field Theory, Phys. Rev. Lett. 126 (2021) 210103 [2101.12688].

[116] G. U. Jakobsen, G. Mogull, J. Plefka and J. Steinhoff, Gravitational Bremsstrahlung and Hidden Supersymmetry of Spinning Bodies, Phys. Rev. Lett. 128 (2022) 011101 [2106.10256].

[117] G. U. Jakobsen, G. Mogull, J. Plefka and J. Steinhoff, SUSY in the sky with gravitons, JHEP 01 (2022) 027 [2109.04465].

[118] C. Shi and J. Plefka, Classical double copy of worldline quantum field theory, Phys. Rev. D 105 (2022) 026007 [2109.10345].

[119] F. Bastianelli, F. Comberiati and L. de la Cruz, Light bending from eikonal in worldline quantum field theory, JHEP 02 (2022) 209 [2112.05013].

[120] M. V. S. Saketh, J. Vines, J. Steinhoff and A. Buonanno, Conservative and radiative dynamics in classical relativistic scattering and bound systems, Phys. Rev. Res. 4 (2022) 013127 [2109.05994].

[121] D. Bonocore, A. Kulesza and J. Pirsch, Classical and quantum gravitational scattering with Generalized Wilson Lines, JHEP 03 (2022) 147 [2112.02009].

[122] D. Bini and T. Damour, Gravitational radiation reaction along general orbits in the effective one-body formalism, Phys. Rev. D86 (2012) 124012.
[120.2834].

[123] T. Damour, Radiative contribution to classical gravitational scattering at the third order in G, Phys. Rev. D 102 (2020) 124008 [2010.01641].

[124] D. Bini, T. Damour and A. Geralico, Radiative contributions to gravitational scattering, Phys. Rev. D 104 (2021) 084031 [2107.08896].

[125] F. Bastianelli, P. Benincasa and S. Giombi, Worldline approach to vector and antisymmetric tensor fields, JHEP 04 (2005) 010 [hep-th/0503155].

[126] F. Bastianelli, P. Benincasa and S. Giombi, Worldline approach to vector and antisymmetric tensor fields. II., JHEP 10 (2005) 114 [hep-th/0510010].

[127] W. Tulczjew, Equations of motion of rotating bodies in general relativity theory, Acta Phys. Polon. 18 (1959) 37.

[128] J. Steinhoff, Spin and quadrupole contributions to the motion of astrophysical binaries, Fund. Theor. Phys. 179 (2015) 615 [1412.3251].

[129] M. Mathisson, Neue Mechanik materieller Systeme, Acta Phys. Polon. 6 (1937) 163.

[130] A. Papapetrou, Spinning test particles in general relativity. I., Proc. Roy. Soc. Lond. A 209 (1951) 248.

[131] W. G. Dixon, Dynamics of extended bodies in general relativity. I. Momentum and angular momentum, Proc. Roy. Soc. Lond. A 314 (1970) 499.

[132] J. Parra-Martinez, M. S. Ruf and M. Zeng, Extremal black hole scattering at O(G^3): graviton dominance, eikonal exponentiation, and differential equations, JHEP 11 (2020) 023 [2005.04236].

[133] N. E. J. Bjerrum-Bohr, P. H. Damgaard, L. Planè and P. Vanhove, Classical gravity from loop amplitudes, Phys. Rev. D 104 (2021) 026009 [2104.04510].

[134] A. Antonelli, C. Kavanagh, M. Khalil, J. Steinhoff and J. Vines, Gravitational spin-orbit and aligned spin$_1$-spin$_2$ couplings through third-subleading post-Newtonian orders, Phys. Rev. D 102 (2020) 124024 [2010.02018].

[135] G. Veneziano and G. A. Vilkovisky, Angular momentum loss in gravitational scattering, radiation reaction, and the Bondi gauge ambiguity, 2201.11607.

[136] D. Amati, M. Ciafaloni and G. Veneziano, Higher Order Gravitational Deflection and Soft Bremsstrahlung in Planckian Energy Superstring Collisions, Nucl. Phys. B347 (1990) 550.

[137] S. Laporta and E. Remiddi, The Analytical value of the electron (g-2) at order alpha**3 in QED, Phys. Lett. B 379 (1996) 283 [hep-ph/9602417].

[138] S. Laporta, High precision calculation of multiloop Feynman integrals by difference equations, Int. J. Mod. Phys. A 15 (2000) 5087 [hep-ph/0102033].

[139] A. V. Smirnov and F. S. Chuharev, FIRE6: Feynman Integral REduction with Modular Arithmetic, Comput. Phys. Commun. 247 (2020) 106877 [1901.07808].

[140] R. N. Lee, Presenting LiteRed: a tool for the Loop InTEgrals REDuction, 1212.2685.

[141] R. N. Lee, LiteRed 1.4: a powerful tool for reduction of multiloop integrals, J. Phys. Conf. Ser. 523 (2014) 012059 [1310.1145].

[142] P. Maierhöfer, J. Usovitsch and P. Uwer, Kira—A Feynman integral reduction program, Comput. Phys. Commun. 230 (2018) 99 [1705.05610].

[143] J. Klappert, F. Lange, P. Maierhöfer and J. Usovitsch, Integral reduction with Kira 2.0 and finite field methods, Comput. Phys. Commun. 266 (2021) 108024 [2008.06494].

[144] O. Gituliar and V. Magerya, Fuchsia: a tool for reducing differential equations for Feynman master integrals to epsilon form, Comput. Phys. Commun. 219 (2017) 329 [1701.04269].

[145] M. Prausa, epsilon: A tool to find a canonical basis of master integrals, Comput. Phys. Commun. 219 (2017) 361 [1701.00725].

[146] C. Duhr and F. Dulat, PolyLogTools — polylogs for the masses, JHEP 08 (2019) 135 [1904.07279].