A GEOMETRIC APPROACH TO INEQUALITIES FOR THE HILBERT–SCHMIDT NORM

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ABSTRACT. We define angle $\Theta_{X,Y}$ between non-zero Hilbert–Schmidt operators $X$ and $Y$ by $\cos \Theta_{X,Y} = \frac{\text{Re} \text{Tr}(Y^*X)}{\|X\|_2 \|Y\|_2}$, and give some of its essentially properties. It is shown, among other things, that $|\cos \Theta_{X,Y}| \leq \min \left\{ \sqrt{\cos \Theta_{|X^*|,|Y^*|}}, \sqrt{\cos \Theta_{|X|,|Y|}} \right\}$. It enables us to provide alternative proof of some well-known inequalities for the Hilbert–Schmidt norm. In particular, we apply this inequality to prove Lee’s conjecture [Linear Algebra Appl. 433 (2010), no. 3, 580–584] as follows

$$\|X + Y\|_2 \leq \sqrt{\frac{\sqrt{2} + 1}{2}} \|X| + |Y|\|_2.$$ 

A numerical example is presented to show the constant $\sqrt{\frac{\sqrt{2} + 1}{2}}$ is smallest possible. Other related inequalities for the Hilbert–Schmidt norm are also considered.

1. Introduction and preliminaries

Let $\mathcal{B}(\mathcal{H})$ denote the $C^*$-algebra of all bounded linear operators acting on a Hilbert space $(\mathcal{H}, [\cdot, \cdot])$. For every $X \in \mathcal{B}(\mathcal{H})$, let $|X|$ denote the square root of $X^*X$, that is, $|X| = (X^*X)^{1/2}$. Let $X = U|X|$ be the polar decomposition of $X$, where $U$ is some partial isometry. The polar decomposition satisfies

$$U^*X = |X|, \quad U^*U|X| = |X|, \quad U^*UX = X, \quad X^* = |X|^* = |X^*|,$$

where $\{Xe_i, Y e_i\}$ is any orthonormal basis of $\mathcal{H}$ and $\text{Tr}(\cdot)$ is the natural trace on $\mathcal{C}_1(\mathcal{H})$. Three principal properties of the trace are that it is a linear functional and, for every $X$ and $Y$, we have $\text{Tr}(X^*) = \overline{\text{Tr}(X)}$ and $\text{Tr}(XY^*) = \text{Tr}(YX^*)$. The Hilbert–Schmidt norm of $X \in \mathcal{C}_2(\mathcal{H})$ is given by $\|X\|_2 = \sqrt{\text{Tr}(X^*X)}$. One more fact that will be needed the sequel is that if $X \in \mathcal{C}_2(\mathcal{H})$, then

$$\|X\|_2 = \|X^*\|_2 = \|X\|_2 = \|X\|_2.$$
The reader is referred to [9, 11] for further properties of the Hilbert–Schmidt class.

In Section 2, we define the angle $\Theta_{X,Y}$ between non-zero Hilbert–Schmidt operators $X, Y$ and present some of its essentially properties. We obtain the cosine theorem for Hilbert–Schmidt operators and propose some geometric results. In particular, we state an extension of the Fuglede–Putnam theorem modulo the Hilbert–Schmidt class. We also present a triangle inequality for angles in the Hilbert–Schmidt class. Moreover, we prove that $|\langle X, Y \rangle|^2 \leq \langle |X^*|, |Y^*| \rangle \langle |X|, |Y| \rangle$ and then we apply it to obtain $\cos^2 \Theta_{X,Y} \leq \cos \Theta_{|X^*|,|Y^*|} \cos \Theta_{|X|,|Y|}$.

In Section 3, by using the results in Section 2, we provide alternative proof of some well-known inequalities for the Hilbert–Schmidt norm. For example we present a considerably briefer proof of an extension of the Araki–Yamagami inequality [1] obtained by F. Kittaneh [5]. In particular, we prove Lee’s conjecture [8, p. 584] on the sum of the square roots of operators. Some related inequalities and numerical examples are also presented.

2. ANGLE BETWEEN TWO HILBERT–SCHMIDT OPERATORS

Let $X, Y \in C_2(\mathcal{H})$. Since $C_2(\mathcal{H})$ is a Hilbert space with the inner product (1.2), by the Cauchy–Schwarz inequality, we have

$$-\|X\|_2\|Y\|_2 \leq -|\langle X, Y \rangle| \leq \text{Re}\langle X, Y \rangle \leq |\langle X, Y \rangle| \leq \|X\|_2\|Y\|_2.$$  (2.1)

Therefore, when $X$ and $Y$ are non-zero operators, the inequality (2.1) implies

$$-1 \leq \frac{\text{Re}\langle X, Y \rangle}{\|X\|_2\|Y\|_2} \leq 1.$$

This motivates (see also [2, 13]) defining the angle between $X$ and $Y$ as follows.

**Definition 2.1.** For non-zero operators $X, Y \in C_2(\mathcal{H})$, the angle $\Theta_{X,Y}$ between $X$ and $Y$ is defined by

$$\cos \Theta_{X,Y} = \frac{\text{Re}\langle X, Y \rangle}{\|X\|_2\|Y\|_2}, \quad 0 \leq \Theta_{X,Y} \leq \pi.$$

**Remark 2.2.** Let $X, Y \in C_2(\mathcal{H}) \setminus \{0\}$.

(i) Since $0 \leq \Theta_{X,Y} \leq \pi$, we have $\sin \Theta_{X,Y} = \sqrt{1 - \cos^2 \Theta_{X,Y}}$.

(ii) If $X$ and $Y$ are positive, then $\text{Re}\langle X, Y \rangle \geq 0$ and hence $0 \leq \Theta_{X,Y} \leq \frac{\pi}{2}$.

(iii) One can see that $\cos \Theta_{\gamma X,\gamma Y} = \cos \Theta_{X,Y}$ for all $\gamma \in \mathbb{C} \setminus \{0\}$.

(iv) For every $\alpha, \beta \in \mathbb{R} \setminus \{0\}$, it is easy to check that

$$\cos \Theta_{\alpha X,\beta Y} = \begin{cases} 
\cos \Theta_{X,Y} & \alpha \beta > 0 \\
-\cos \Theta_{X,Y} & \alpha \beta < 0.
\end{cases}$$

**Definition 2.3.** Let $X$ and $Y$ be non-zero Hilbert–Schmidt operators on $\mathcal{H}$.

(i) $X$ is called weak orthogonal to $Y$, denoted by $X \perp_w Y$, if $\cos \Theta_{X,Y} = 0$.

(ii) $X$ is called weak parallel to $Y$, denoted by $X \parallel_w Y$, if $\sin \Theta_{X,Y} = 0$.

**Example 2.4.** Let us recall that by [9, p. 66] we have

$$\text{Tr}(a \otimes b) = [a, b] \quad \text{and} \quad \|a \otimes b\|_2 = \|a\|\|b\|,$$
for all \(a, b \in \mathcal{H}\). Here, \(a \otimes b\) denotes the rank one operator in \(\mathcal{B}(\mathcal{H})\) defined by \((a \otimes b)c := [c, a]\) for all \(c \in \mathcal{H}\). Now, let \(x, y, z \in \mathcal{H} \setminus \{0\}\). Put \(X = x \otimes z\) and \(Y = y \otimes z\). A simple calculation shows that \(\langle X, Y \rangle = \|z\|^2\langle x, y \rangle\). Thus,

\[
\cos \Theta_{X,Y} = \frac{\text{Re}\langle x, y \rangle}{\|x\|\|y\|}.
\]

In particular, \(x \otimes z \perp_{w} y \otimes z\) if and only if \(x \perp_{w} y\) (i.e., satisfies \(\text{Re}\langle x, y \rangle = 0\)).

First we obtain the cosine theorem for Hilbert–Schmidt operators.

**Theorem 2.5.** If \(X\) and \(Y\) are non-zero Hilbert–Schmidt operators on \(\mathcal{H}\), then

\[
\|X \pm Y\|_2^2 = \|X\|_2^2 + \|Y\|_2^2 + 2\|X\|_2\|Y\|_2 \cos \Theta_{X,Y}.
\]

**Proof.** By (1.2) and Definition 2.3, we have

\[
\|X \pm Y\|_2^2 = \langle X \pm Y, X \pm Y \rangle
\]

\[
= \langle X, X \rangle + \langle Y, Y \rangle \pm \langle X, Y \rangle \pm \langle Y, X \rangle
\]

\[
= \|X\|_2^2 + \|Y\|_2^2 \pm 2\text{Re}\langle X, Y \rangle
\]

\[
= \|X\|_2^2 + \|Y\|_2^2 \pm 2\|X\|_2\|Y\|_2 \cos \Theta_{X,Y}.
\]

\[\square\]

As an immediate consequence of Theorem 2.5, we get the Pythagorean relation for Hilbert–Schmidt operators.

**Corollary 2.6.** If \(X\) and \(Y\) are non-zero Hilbert–Schmidt operators on \(\mathcal{H}\), then \(X \perp_{w} Y\) if and only if \(\|X + Y\|_2^2 = \|X\|_2^2 + \|Y\|_2^2\).

The following result is another immediate consequence of Theorem 2.5.

**Corollary 2.7.** If \(X\) and \(Y\) are non-zero Hilbert–Schmidt operators on \(\mathcal{H}\), then

\(X \parallel_{w} Y\) if and only if \(\|X + Y\|_2 = \|X\|_2 \pm \|Y\|_2\).

Let us recall the Fuglede–Putnam theorem modulo the Hilbert–Schmidt class due to G. Weiss [12, Theorem 1]: if \(X\) and \(Y\) are normal operators and \(Z\) is an operator on \(\mathcal{H}\), then \(\|XZ - ZY\|_2 = \|X^*Z - ZY^*\|_2\). In the following theorem, we present an extension of the Fuglede–Putnam theorem modulo the Hilbert–Schmidt class which provided by Kittaneh in [6]. Our proof is very different from that in [6, Theorem 5].

**Theorem 2.8.** If \(X, Y\) and \(Z\) are operators on \(\mathcal{H}\), then

\[
\|XZ - ZY\|_2^2 + \|X^*Z\|_2^2 + \|ZY^*\|_2^2 = \|XZ\|_2^2 + \|ZY\|_2^2 + \|X^*Z - ZY^*\|_2^2.
\]

**Proof.** We may assume that \(XZ, ZY, X^*Z, ZY^* \neq 0\) otherwise the desired equality trivially holds. By (1.2) and Definition 2.3 we have

\[
\|XZ\|_2\|ZY\|_2 \cos \Theta_{XZ,YZ} = \text{Re}\langle XZ, ZY \rangle = \text{ReTr}(Y^*Z^*XZ)
\]

\[
= \text{ReTr}(Z^*X^*YZ) = \text{ReTr}(Z^*X^*ZY)
\]

\[
= \text{ReTr}(Y^*X^*Z) = \text{Re}\langle X^*Z, ZY^* \rangle
\]

\[
= \|X^*Z\|_2\|ZY^*\|_2 \cos \Theta_{X^*Z, ZY^*}.
\]
and hence
\[ \| XZ \|_2^2 \| ZY \|_2 \cos \Theta_{XZ,ZY} = \| X^*Z \|_2^2 \| ZY^* \|_2 \cos \Theta_{X^*Z,ZY^*}. \tag{2.2} \]

So, by (2.2) and Theorem 2.5, we have
\[
\| XZ - ZY \|_2^2 + \| X^*Z \|_2^2 + \| ZY^* \|_2^2 \\
= \| XZ \|_2^2 + \| ZY \|_2^2 - 2\| XZ \|_2 \| ZY \|_2 \cos \Theta_{XZ,ZY} + \| X^*Z \|_2^2 + \| ZY^* \|_2^2 \\
= \| XZ \|_2^2 + \| ZY \|_2^2 - 2\| X^*Z \|_2 \| ZY^* \|_2 \cos \Theta_{X^*Z,ZY^*} + \| X^*Z \|_2^2 + \| ZY^* \|_2^2 \\
= \| XZ \|_2^2 + \| ZY \|_2^2 + \| XZ - ZY \|_2^2.
\]

\[ \square \]

Remark 2.9. Let \( X \) and \( Y \) be normal operators and let \( Z \) be a operator on \( \mathcal{H} \).
By (1.2) we have
\[
\| X^*Z \|_2^2 + \| ZY^* \|_2^2 = \langle X^*Z, X^*Z \rangle + \langle ZY^*, ZY^* \rangle \\
= \text{Tr} \left( Z^*XX^*Z \right) + \text{Tr} \left( Y^*ZY^* \right) \\
= \text{Tr} \left( Z^*XX^*Z \right) + \text{Tr} \left( Z^*ZY^*Y \right) \\
= \text{Tr} \left( Z^*XX^*Z \right) + \text{Tr} \left( Z^*ZY^*Y \right) \\
= \text{Tr} \left( Z^*XX^*Z \right) + \text{Tr} \left( Y^*Z^*ZY \right) \\
= \langle XZ, XZ \rangle + \langle ZY, ZY \rangle = \| XZ \|_2^2 + \| ZY \|_2^2.
\]

Therefore if in Theorem 2.8, \( X \) and \( Y \) are assumed to be normal operators, then we retain the Fuglede-Putnam theorem modulo the Hilbert-Schmidt class.

In the following result we present a triangle inequality for angles in the Hilbert–Schmidt class.

Theorem 2.10. If \( X, Y \) and \( Z \) are non-zero Hilbert–Schmidt operators on \( \mathcal{H} \), then
\[
\sin \Theta_{X,Y} \leq \sin \Theta_{X,Z} + \sin \Theta_{Z,Y}.
\]

Proof. First note that, since \( C_2(\mathcal{H}) \) is a Hilbert space, by [10] we have
\[
\Theta_{X,Y} \leq \Theta_{X,Z} + \Theta_{Z,Y}. \tag{2.3}
\]
If \( \text{Re} \langle X, Z \rangle \leq 0 \), we replace \( X \) by \( -X \); if \( \text{Re} \langle Z, Y \rangle \leq 0 \), we replace \( Y \) by \( -Y \).
Therefore, we may assume that \( 0 \leq \Theta_{X,Z}, \Theta_{Z,Y} \leq \pi \) and \( 0 \leq \Theta_{X,Y} \leq \pi \).
Now, we consider two cases.

Case 1. \( 0 \leq \Theta_{X,Z} + \Theta_{Z,Y} \leq \frac{\pi}{2} \).
Since \( \sin \theta \) is a increasing function of \( \theta \) in the interval \([0, \frac{\pi}{2}]\), by the inequality (2.3), we have
\[
\sin \Theta_{X,Y} \leq \sin \left( \Theta_{X,Z} + \Theta_{Z,Y} \right) = \sin \Theta_{X,Z} \cos \Theta_{X,Z} + \cos \Theta_{Z,Y} \sin \Theta_{Z,Y},
\]
and hence \( \sin \Theta_{X,Y} \leq \sin \Theta_{X,Z} + \sin \Theta_{Z,Y} \).
Case 2. $\frac{\pi}{2} < \Theta_{X,Z} + \Theta_{Z,Y} \leq \pi$. Then $0 \leq \frac{\pi}{2} - \Theta_{Z,Y} < \Theta_{X,Z} \leq \frac{\pi}{2}$. Hence,

$$\sin \Theta_{X,Y} \leq 1 \leq \sqrt{2} \sin \left(\frac{\pi}{4} + \Theta_{Z,Y}\right) = \cos \Theta_{Z,Y} + \sin \Theta_{Z,Y} = \sin\left(\frac{\pi}{2} - \Theta_{Z,Y}\right) + \sin \Theta_{Z,Y} \leq \sin \Theta_{X,Z} + \sin \Theta_{Z,Y},$$

and the proof is completed. \qed

As consequences of Theorem 2.10, we have the following results.

**Corollary 2.11.** Let $X, Y, Z \in C_2(\mathcal{H}) \setminus \{0\}$. If $X \parallel Y$ and $Z \parallel Y$, then $X \parallel Y$.

**Proof.** Suppose that $X \parallel Y$ and $Z \parallel Y$. Then $\Theta_{X,Y} = \Theta_{Z,Y} = 0$. Hence, by Theorem 2.10, we obtain $\sin \Theta_{X,Y} = 0$, or equivalently, $X \parallel Y$. \qed

**Corollary 2.12.** Let $X, Y, Z \in C_2(\mathcal{H}) \setminus \{0\}$. If $X \perp Y$ and $Z \parallel Y$, then $X \perp Y$.

**Proof.** Suppose that $X \perp Y$ and $Z \parallel Y$. Then $\cos \Theta_{X,Y} = 0$ and $\Theta_{Z,Y} = 0$. Hence $\sin \Theta_{X,Y} = 1$ and $\sin \Theta_{Z,Y} = 0$. Now, by Theorem 2.10, we get $\sin \Theta_{X,Y} = 1$, or equivalently, $\cos \Theta_{X,Z} = 0$. Thus $X \perp Y$. \qed

We now state an interesting inequality based on some ideas of [4, Lemma 4.1].

**Theorem 2.13.** If $X$ and $Y$ are Hilbert-Schmidt operators on $\mathcal{H}$, then

$$|\langle X, Y \rangle|^2 \leq |\langle |X|, |Y| \rangle|^2. \quad (2.4)$$

Moreover, the inequality in (2.4) becomes an equality if and only if $\zeta Y^*X$ is positive for some scalar $\zeta$.

**Proof.** Let $X = U|X|$ and $Y = V|Y|$ be the polar decompositions of $X$ and $Y$, respectively. By (1.1), (1.2) and the Cauchy-Schwarz inequality, we have

$$|\langle X, Y \rangle|^2 = \left|\text{Tr}(|Y|V^*U|X|)\right|^2$$

$$= \left|\text{Tr}(|X|^{1/2}|Y|V^*U|X|^{1/2})\right|^2$$

$$= \left|\langle |Y|^{1/2}V^*U|X|^{1/2}, |Y|^{1/2}|X|^{1/2} \rangle\right|^2$$

$$\leq \left\||Y|^{1/2}V^*U|X|^{1/2}\right\|^2 \left\||Y|^{1/2}|X|^{1/2}\right\|^2$$

$$= \text{Tr}\left(|X|^{1/2}U^*Y|V^*U|X|^{1/2}\right)\text{Tr}\left(|X|^{1/2}|Y|X|^{1/2}\right)$$

$$= \text{Tr}\left(|Y|V^*U|X|U^*\right)\text{Tr}\left(|Y|X|\right)$$

$$= \text{Tr}\left(|Y^*||X^*|\right)\text{Tr}\left(|Y||X|\right)$$

$$= \text{Tr}\left(|Y^*|^2|X^*|\right)\text{Tr}\left(|Y|^2|X|\right) = \langle |X|^2, |X| \rangle = \langle |X|, |Y| \rangle.$$
The inequality in (2.4) becomes an equality if and only if
\[ |Y|^{1/2}|X|^{1/2} = \zeta|Y|^{1/2}V^*U|X|^{1/2} \]
for some \( \zeta \in \mathbb{C} \), and hence \( |Y||X| = \zeta|Y^*U|X| \). So, by (1.1), we obtain \( |Y||X| = \zeta Y^*X \). Therefore, the inequality in (2.4) becomes an equality if and only if \( \zeta Y^*X \) is positive for some \( \zeta \in \mathbb{C} \).

Here we present one of the main results of this paper. In fact, the following theorem enables us to provide alternative proof of some well-known inequalities for the Hilbert–Schmidt norm. In particular, we will use the following theorem to prove Lee’s conjecture [8, p. 584] on the sum of the square roots of operators.

**Theorem 2.14.** For non-zero Hilbert–Schmidt operators \( X \) and \( Y \) on \( \mathcal{H} \) the following properties hold.

1. \( \cos^2 \Theta_{X,Y} \leq \cos \Theta_{|X^*||Y^*|} \cos \Theta_{|X||Y|} \).
2. \( |\cos \Theta_{X,Y}| \leq \min \left\{ \sqrt{\cos \Theta_{|X^*||Y^*|} \cos \Theta_{|X||Y|}} \right\} \).
3. \( \sin^2 \Theta_{|X^*||Y^*|} + \sin^2 \Theta_{|X||Y|} \leq 2 \sin^2 \Theta_{X,Y} \).

**Proof.** (i) By (1.3), (2.1), Definition 2.3 and Theorem 2.13 we have
\[
\cos^2 \Theta_{X,Y} = \left( \frac{\text{Re}\langle X, Y \rangle}{\|X\|_2 \|Y\|_2} \right)^2 \\
\leq \frac{|\langle X, Y \rangle|^2}{\|X\|_2^2 \|Y\|_2^2} \\
\leq \frac{\langle |X^*||Y^*| \rangle \langle |X||Y| \rangle}{\|X^*\|_2 \|Y^*\|_2 \|X\|_2 \|Y\|_2} \\
= \frac{\text{Re}\langle |X^*||Y^*| \rangle}{\|X^*\|_2 \|Y^*\|_2} \frac{\text{Re}\langle |X||Y| \rangle}{\|X\|_2 \|Y\|_2} \\
= \cos \Theta_{|X^*||Y^*|} \cos \Theta_{|X||Y|}.
\]

(ii) The proof follows immediately from (i).

(iii) By the arithmetic-geometric mean inequality and (i) we have
\[
\sin^2 \Theta_{|X^*||Y^*|} + \sin^2 \Theta_{|X||Y|} = 2 - \left( \cos^2 \Theta_{|X^*||Y^*|} + \cos^2 \Theta_{|X||Y|} \right) \\
\leq 2 - 2 \cos \Theta_{|X^*||Y^*|} \cos \Theta_{|X||Y|} \\
\leq 2 - 2 \cos^2 \Theta_{X,Y} = 2 \sin^2 \Theta_{X,Y}.
\]

As consequences of the preceding theorem, we have the following results.

**Corollary 2.15.** Let \( X, Y \in \mathcal{C}_2(\mathcal{H}) \setminus \{0\} \). If \( |X^*| \perp_w |Y^*| \) or \( |X| \perp_w |Y| \), then \( X \perp_w Y \).

**Proof.** Let \( |X^*| \perp_w |Y^*| \) or \( |X| \perp_w |Y| \). Then \( \cos \Theta_{|X^*||Y^*|} \cos \Theta_{|X||Y|} = 0 \). Hence, by Theorem 2.14 (i), we obtain \( \cos \Theta_{X,Y} = 0 \), or equivalently, \( X \perp_w Y \).
Corollary 2.16. Let $X, Y \in C_2(H) \setminus \{0\}$. If $X ∥_w Y$, then $|X^*| ∥_w |Y^*|$ and $|X| ∥_w |Y|$. 

Proof. Suppose that $X ∥_w Y$. Then $\sin \Theta_{X,Y} = 0$. So, by Theorem 2.14 (iii), we obtain $\sin \Theta_{|X^*|,|Y^*|} = \sin \Theta_{|X|,|Y|} = 0$. Therefore, $|X^*| ∥_w |Y^*|$ and $|X| ∥_w |Y|$. □

3. Inequalities for the Hilbert–Schmidt norm

In this section, by using Theorems 2.5 and 2.14, we provide alternative proof of some well-known inequalities for the Hilbert–Schmidt norm. First we present a considerably briefer proof of an extension of the Araki–Yamagami inequality [1] obtained by F. Kittaneh [5, Theorem 2].

Theorem 3.1. If $X$ and $Y$ are operators on $H$, then

$$\| |X^*| - |Y^*| \|^2_2 + \| |X| - |Y| \|^2_2 \leq 2 \| X - Y \|^2_2.$$ 

Proof. Since the desired inequality trivially holds when $X = 0$ or $Y = 0$, we may assume $X, Y \neq 0$. By (1.3), Theorem 2.5, Theorem 2.14(i) and the arithmetic-geometric mean inequality we have

$$\| |X^*| - |Y^*| \|^2_2 + \| |X| - |Y| \|^2_2 = \| |X^*| \|^2_2 + \| |Y^*| \|^2_2 - 2 \| |X^*| \| \| |Y^*| \| \cos \Theta_{|X^*|,|Y^*|}$$

$$\quad + \| |X| \|^2_2 + \| |Y| \|^2_2 - 2 \| |X| \| \| |Y| \| \cos \Theta_{|X|,|Y|}$$

$$= 2 \| X \|^2_2 + 2 \| Y \|^2_2 - 2 \| X \| \| Y \| \left( \cos \Theta_{|X^*|,|Y^*|} + \cos \Theta_{|X|,|Y|} \right)$$

$$\leq 2 \| X \|^2_2 + 2 \| Y \|^2_2 - 4 \| X \| \| Y \| \sqrt{\cos \Theta_{|X^*|,|Y^*|} \cos \Theta_{|X|,|Y|}}$$

$$\leq 2 \| X \|^2_2 + 2 \| Y \|^2_2 - 4 \| X \| \| Y \| \cos \Theta_{X,Y}$$

$$\leq 2 \| X \|^2_2 + 2 \| Y \|^2_2 - 4 \| X \| \| Y \| \cos \Theta_{X,Y} = 2 \| X - Y \|^2_2.$$ □

As an immediate consequence of Theorem 3.1, we get the Araki–Yamagami inequality [1, Theorem 1].

Corollary 3.2. If $X$ and $Y$ are operators on $H$, then

$$\| |X| - |Y| \|^2_2 \leq \sqrt{2} \| X - Y \|^2_2.$$ 

Remark 3.3. In [1], H. Araki and S. Yamagami remarked that $\sqrt{2}$ is the best possible coefficient for a general $X$ and $Y$. Now let $X$ and $Y$ be normal operators. Since $X$ and $Y$ are normal operators, the spectral theorem (see [9]) implies that $|X^*| = |X|$, $|Y^*| = |Y|$ and hence, by Theorem 3.1 we obtain

$$\| |X| - |Y| \|^2_2 \leq \| X - Y \|^2_2.$$ 

Therefore, if $X$ and $Y$ are restricted to be normal, then the best coefficient in the Araki–Yamagami inequality is 1 instead of $\sqrt{2}$.

The following result is a special case of [8, Theorem 2.1].

Theorem 3.4. If $X$ and $Y$ are operators on $H$, then

$$\| X + Y \|^2_2 \leq \| |X^*| + |Y^*| \|_2 \| |X| + |Y| \|_2.$$
Proof. We may assume that \( X, Y \neq 0 \) otherwise the desired inequality trivially holds. By (1.3), Theorem 2.5, Theorem 2.14(i) and the arithmetic-geometric mean inequality we have

\[
\| |X| + |Y| \|_2^2 \| |X| + |Y| \|_2^2 = \left( \| |X|^*| \|_2^2 + \| |Y|^*| \|_2^2 + 2\| |X|^*| \|_2 \| |Y|^*| \|_2 \cos \Theta_{|X|^*,|Y|^*} \right) \\
\times \left( \| |X| \|_2^2 + \| |Y| \|_2^2 + 2\| |X| \|_2 \| |Y| \|_2 \cos \Theta_{|X|,|Y|} \right) \\
= \left( \| |X| \|_2^2 + \| |Y| \|_2^2 + 2\| |X| \|_2 \| |X|^*| \|_2 \cos \Theta_{|X|^*,|Y|^*} \right) \\
\times \left( \| |X|^*| \|_2^2 + \| |Y|^*| \|_2^2 + 2\| |X|^*| \|_2 \| |Y|^*| \|_2 \cos \Theta_{|X|^*,|Y|^*} \right) \\
= \left( \| |X| \|_2^2 + \| |Y| \|_2^2 \right) \left( \| |X|^*| \|_2^2 + \| |Y|^*| \|_2^2 \right) \left( \cos \Theta_{|X|^*,|Y|^*} + \cos \Theta_{|X|,|Y|} \right) \\
\geq \left( \| |X| \|_2^2 + \| |Y| \|_2^2 \right) \left( \| |X|^*| \|_2^2 + \| |Y|^*| \|_2^2 \right) \cos \Theta_{X,Y} \\
+ 4\| |X| \|_2 \| |Y| \|_2 \left( \| |X|^*| \|_2^2 + \| |Y|^*| \|_2^2 \right) \cos \Theta_{X,Y} \\
\geq \left( \| |X| \|_2^2 + \| |Y| \|_2^2 \right) \left( \| |X|^*| \|_2^2 + \| |Y|^*| \|_2^2 \right) \cos \Theta_{X,Y} \\
+ 4\| |X| \|_2 \| |Y| \|_2 \left( \| |X|^*| \|_2^2 + \| |Y|^*| \|_2^2 \right) \cos \Theta_{X,Y} \\
= \left( \| |X| \|_2^2 + \| |Y| \|_2^2 + 2\| |X| \|_2 \| |Y| \|_2 \cos \Theta_{X,Y} \right)^2 = \| X + Y \|_2^4.
\]

\[\square\]

Next, we provide alternative proof of an inequality for the Hilbert–Schmidt norm due to F. Kittaneh [7, Theorem 2.1].

**Theorem 3.5.** If \( X \) and \( Y \) are operators on \( \mathcal{H} \), then

\[
\| |X| - |Y| \|_2^2 \leq \| X + Y \|_2 \| X - Y \|_2.
\]

**Proof.** Since the desired inequality trivially holds when \( X = 0 \) or \( Y = 0 \), we may assume \( X, Y \neq 0 \). By the arithmetic-geometric mean inequality we have

\[
\| X \|_2 \| Y \|_2 \left( 1 + \cos \Theta_{|X|,|Y|} \right) \leq 2 \| X \|_2 \| Y \|_2 \leq \| X \|_2^2 + \| Y \|_2^2.
\]

Then, since \( \cos \Theta_{|X|,|Y|} \geq 0 \), we obtain

\[
\| X \|_2 \| Y \|_2 \left( \cos \Theta_{|X|,|Y|} + \cos^2 \Theta_{|X|,|Y|} \right) \leq \cos \Theta_{|X|,|Y|} \left( \| X \|_2^2 + \| Y \|_2^2 \right),
\]

and so

\[
\| X \|_2 \| Y \|_2 \cos \Theta_{|X|,|Y|} \leq \cos \Theta_{|X|,|Y|} \left( \| X \|_2^2 + \| Y \|_2^2 \right) - \| X \|_2 \| Y \|_2 \cos^2 \Theta_{|X|,|Y|}. \tag{3.1}
\]
Therefore by (1.3), Theorem 2.5, Theorem 2.14(ii) and (3.1) we have
\[
\|X + Y\|_2^2 \|X - Y\|_2^2 = \left(\|X\|_2^2 + \|Y\|_2^2\right)^2 - 4\|X\|_2^2\|Y\|_2^2 \cos^2 \Theta_{X,Y}
\geq \left(\|X\|_2^2 + \|Y\|_2^2\right)^2 - 4\|X\|_2\|Y\|_2 \left(\|X\|_2^2 + \|Y\|_2^2\right) \cos \Theta_{X,Y}
\geq \left(\|X\|_2^2 + \|Y\|_2^2\right)^2 - 4\|X\|_2\|Y\|_2 \left(\|X\|_2^2 + \|Y\|_2^2\right) \cos \Theta_{X,Y} + 4\|X\|_2^2\|Y\|_2^2 \cos^2 \Theta_{X,Y}
= \left(\|X\|_2^2 + \|Y\|_2^2 - 2\|X\|_2\|Y\|_2 \cos \Theta_{X,Y}\right)^2
= \left(\|X\|_2^2 + \|Y\|_2^2 - 2\|X\|_2\|Y\|_2 \cos \Theta_{X,Y}\right)^2 = \|X\| + \|Y\|_2^4.
\]
\[\Box\]

The following result may be stated as well.

**Theorem 3.6.** If $X$ and $Y$ are operators on $\mathcal{H}$, then
\[
\|X^* + Y^*\|_2 \leq \sqrt{2}\|X + Y\|_2.
\]

**Proof.** We may assume that $X, Y \neq 0$ otherwise the desired inequality trivially holds. Since $-2 \leq \cos \Theta_{X^*|Y^*} - 2 \cos \Theta_{X|Y} \leq 1$, by the arithmetic-geometric mean inequality we get
\[
2\|X\|_2\|Y\|_2 \left(\cos \Theta_{X^*|Y^*} - 2 \cos \Theta_{X|Y}\right) \leq 2\|X\|_2\|Y\|_2 \leq \|X\|_2^2 + \|Y\|_2^2.
\]
Hence
\[
2\|X\|_2\|Y\|_2 \cos \Theta_{X^*|Y^*} \leq \|X\|_2^2 + \|Y\|_2^2 + 4\|X\|_2\|Y\|_2 \cos \Theta_{X|Y}.
\] (3.2)

So, by (1.3), Theorem 2.5 and (3.2) we have
\[
\|X^* + Y^*\|_2^2 = \|X^*\|_2^2 + \|Y^*\|_2^2 + 2\|X^*\|_2\|Y^*\|_2 \cos \Theta_{X^*|Y^*}
= \|X\|_2^2 + \|Y\|_2^2 + 2\|X\|_2\|Y\|_2 \cos \Theta_{X|Y}
\leq 2\|X\|_2^2 + 2\|Y\|_2^2 + 4\|X\|_2\|Y\|_2 \cos \Theta_{X|Y}
= 2\left(\|X\|_2^2 + \|Y\|_2^2 + 2\|X\|_2\|Y\|_2 \cos \Theta_{X|Y}\right) = 2\|X\| + \|Y\|_2^2.
\]
\[\Box\]

In [8, p. 584], E.-Y. Lee conjectured that for arbitrary $n$-by-$n$ matrices $A$ and $B$, the inequality
\[
\|A + B\|_2 \leq \sqrt{\frac{\sqrt{2} + 1}{2}} \|A\| + \|B\|_2
\]
holds. We end this section by a proof of Lee’s conjecture for operators.

**Theorem 3.7.** If $X$ and $Y$ are operators on $\mathcal{H}$, then
\[
\|X + Y\|_2 \leq \sqrt{\frac{\sqrt{2} + 1}{2}} \|X\| + \|Y\|_2.
\]
Proof. Since the desired inequality trivially holds when $X = 0$ or $Y = 0$, we may assume $X, Y \neq 0$. By (1.3), Theorem 2.5, Theorem 2.14(ii) we have

$$\frac{\sqrt{2}+1}{2} \parallel X \parallel + \parallel Y \parallel^2 = \frac{\sqrt{2}+1}{2} \left( \parallel X \parallel^2 + \parallel Y \parallel^2 + 2\parallel X \parallel \parallel Y \parallel \cos \Theta_{X,Y} \right)$$

$$= \frac{\sqrt{2}+1}{2} \left( \parallel X \parallel^2 + \parallel Y \parallel^2 + 2\parallel X \parallel \parallel Y \parallel \cos \Theta_{X,Y} \right)$$

$$\geq \frac{\sqrt{2}+1}{2} \left( \parallel X \parallel^2 + \parallel Y \parallel^2 + 2\parallel X \parallel \parallel Y \parallel \cos^2 \Theta_{X,Y} \right)$$

$$= \parallel X \parallel^2 + \parallel Y \parallel^2 + 2\parallel X \parallel \parallel Y \parallel \cos \Theta_{X,Y} + \frac{\sqrt{2}-1}{2} \left( \parallel X \parallel^2 + \parallel Y \parallel^2 \right)$$

$$+ \parallel X \parallel \parallel Y \parallel \left( \sqrt{2} + 1 \right) \cos \Theta_{X,Y} - 2 \cos \Theta_{X,Y}$$

$$= \parallel X + Y \parallel^2 + \frac{\sqrt{2}-1}{2} \left( \parallel X \parallel - \parallel Y \parallel \right)^2$$

$$+ \parallel X \parallel \parallel Y \parallel \left( \sqrt{2} - 1 \right) \cos \Theta_{X,Y} - 2 \cos \Theta_{X,Y}$$

$$\geq \parallel X + Y \parallel^2.$$

Remark 3.8. Suppose $\mathcal{M}_2(\mathbb{C})$ is the algebra of all complex $2 \times 2$ matrices. Let $\text{Det}(A)$ denote the determinant of $A \in \mathcal{M}_2(\mathbb{C})$. Recall (see e.g. [3, p. 460]) that for $A \in \mathcal{M}_2(\mathbb{C})$ we have

$$|A| = \frac{1}{\sqrt{\text{Tr}(A^*A) + 2\sqrt{\text{Det}(A^*A)}}} \left( \sqrt{\text{Det}(A^*A)} I + A^*A \right).$$

Now, let $X = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$, $Y = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $Z = \begin{bmatrix} 0 & 0 \\ 1 - \sqrt{2} & \sqrt{8} - 2 \end{bmatrix}$. Then simple computations show that $|X| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $|X|^* = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $|Y| = |Y|^* = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and

$$|Z| = \begin{bmatrix} 3 - \sqrt{8} & -\sqrt{200 - 14} \\ -\sqrt{200 - 14} & \sqrt{8} - 2 \end{bmatrix}.$$
Hence the inequality in Theorem 3.6 is sharp. In addition, we have

$$\|X + Z\|_2 = \left\| \begin{bmatrix} 0 & 0 \\ -\sqrt{2} & \sqrt{8} - 2 \end{bmatrix} \right\|_2 = \sqrt{\text{Tr} \left( \begin{bmatrix} 2 & -\sqrt{32} - 4 \\ -\sqrt{32} - 4 & \sqrt{8} - 2 \end{bmatrix} \right)} = \sqrt{8}$$

and

$$\sqrt{\frac{\sqrt{2} + 1}{2}} \| |X| + |Z| \|_2 = \sqrt{\frac{\sqrt{2} + 1}{2}} \left\| \begin{bmatrix} 4 - \sqrt{8} \\ -\sqrt{200} - 14 \\ \sqrt{8} - 2 \end{bmatrix} \right\|_2 = \sqrt{\frac{\sqrt{2} + 1}{2}} \sqrt{\text{Tr} \left( \begin{bmatrix} 10 - \sqrt{72} \\ -\sqrt{3200} - 56 \\ \sqrt{8} - 2 \end{bmatrix} \right)} = \sqrt{\frac{\sqrt{2} + 1}{2}} \sqrt{2\sqrt{8} - 4} = \sqrt{8}.$$ 

So, the inequality in Theorem 3.7 is also sharp.

References

1. H. Araki and S. Yamagami, *An inequality for the Hilbert–Schmidt norm*, Commun. Math. Phys. 81 (1981), 89–96.
2. T. Bottazzi, C. Conde, M. S. Moslehian, P. Wójcik and A. Zamani, *Orthogonality and parallelism of operators on various Banach spaces*, J. Aust. Math. Soc. 106 (2019), 160–183.
3. L. P. Franca, *An algorithm to compute the square root of a $3 \times 3$ positive definite matrix*, Computers Math. Appl. 18 (1989), no. 5, 459-466.
4. M. Kennedy and P. Skoufranis, *Thompsons theorem for $II_1$ factors*, Trans. Amer. Math. Soc. 369 (2017), 1495–1511.
5. F. Kittaneh, *Inequalities for the Schatten $p$-norm. III*, Commun. Math. Phys. 104 (1986), 307–310.
6. F. Kittaneh, *Inequalities for the Schatten $p$-norm. IV*, Commun. Math. Phys. 106 (1986), 581–585.
7. F. Kittaneh and H. Kosaki, *Inequalities for the Schatten $p$-norm. V*, Publ. RIMS, Kyoto Univ. 23 (1986), 433–443.
8. E.-Y. Lee, *Rotfel’d type inequalities for norms*, Linear Algebra Appl. 433 (2010), no. 3, 580–584.
9. G. J. Murphy, *C*-Algebras and Operator Theory*, Academic Press, New York, 1990.
10. D. K. Rao, *A triangle inequality for angles in a Hilbert space*, Rev. Colombiana Mat. 10 (1976), 95–97.
11. B. Simon, *Trace ideals and their applications*, Cambridge University Press, Cambridge, 1997.
12. G. Weiss, *The Fuglede commutativity theorem modulo the Hilbert–Schmidt class and generating functions for matrix operators. II*, J. Oper. Theory 5 (1981), 3–16.
13. A. Zamani, M. S. Moslehian and M. Frank, *Angle preserving mappings*, Z. Anal. Anwend. 34 (2015), 485–500.

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