Abstract

In this paper we study the nonabelian, gauge invariant and asymptotically free quantum gauge theory with a mass parameter introduced in [8]. We develop the Feynman diagram technique, calculate the mass and coupling constant renormalizations and the effective action at the one–loop order. Using the BRST technique we also prove that the theory is renormalizable within the dimensional regularization framework.

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1 Introduction

In paper [8] we introduced a new classical gauge invariant nonlocal lagrangian generating a local quantum field theory which is reduced to several copies of the massive vector field when the coupling constant vanishes. This unexpected phenomenon is similar to that for the Faddeev-Popov determinant in case of the quantized Yang-Mills field. Recall that being a priori nonlocal quantity the Faddeev-Popov determinant can be made local by introducing additional anticommuting ghost fields and applying a formula for Gaussian integrals over Grassmann variables. Similarly, in case of the lagrangian suggested in [8] one can introduce extra ghost fields and make the expression for the generating function of the Green functions local using tricks with Gaussian integrals.

In this paper we develop the Feynman diagram technique, calculate the mass and coupling constant renormalizations and the effective action at the one–loop order for the theory introduced in [8]. Using the BRST technique we also prove that the theory is renormalizable within the dimensional regularization framework.

2 Preliminaries

First we fix the notation used throughout this paper. Let $G$ be a compact simple Lie group, $\mathfrak{g}$ its Lie algebra with the commutator denoted by $[\cdot,\cdot]$. Denote by $\text{tr}$ the Killing form on $\mathfrak{g}$ with the opposite sign. Recall that $\text{tr}$ is a nondegenerate invariant under the adjoint action scalar product on $\mathfrak{g}$. Let $t^a$, $a = 1, \ldots, \dim \mathfrak{g}$ be a linear basis of $\mathfrak{g}$ normalized in such a way that
\[ \text{tr}(t^a, t^b) = \delta^{ab}. \] Note that for technical reasons the normalizations of scalar products in this paper are slightly different from [8] and from the standard ones (see [5]).

Since the tensor formulas that will appear in this paper are even more complicated than in the case of the Yang-Mills field we shall frequently use the invariant tensor notation and a shortened notation for scalar products of tensors. Denote by \( \Omega^k(g) \) the space of \( g \)-valued differential \( k \)-forms on the standard Minkowski space equipped with the metric \( g_{\mu\nu}, g_{00} = 1, g_{ii} = -1 \) for \( i = 1, 2, 3 \), and \( g_{ij} = 0 \) for \( i \neq j \).

We shall use the scalar product \( <\cdot, \cdot> \) on the space \( \Omega^k(g) \) defined by

\[ <\omega, \omega'> = \int \text{tr}(\omega \wedge *, \omega' \wedge *), \omega, \omega' \in \Omega^k(g), \quad (1) \]

where \( * \) stands for the Hodge star operation associated to the metric on the Minkowski space, and we evaluate the Killing form on the values of \( \omega_1 \) and \( *\omega_2 \) and also take their exterior product.

Let \( A \in \Omega^1(g) \) be the \( g \)-valued gauge field (connection one-form on the Minkowski space),

\[ A = A_\mu dx^\mu = A^a_\mu t^a dx^\mu, \]

and \( F \in \Omega^2(g) \) the strength tensor (curvature) of \( A \),

\[ F = dA - \frac{g}{2}[A \wedge A]. \]

Here \( g \) is a coupling constant and as usual we denote by \([A \wedge A]\) the operation which takes the exterior product of \( g \)-valued 1-forms and the commutator of their values in \( g \). In terms of components we have

\[ F = F_{\mu\nu} dx^\mu \wedge dx^\nu = \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu - g[A_\mu, A_\nu]) dx^\mu \wedge dx^\nu. \]

We shall also need the covariant derivative operator \( d_A : \Omega^k(g) \to \Omega^{k+1}(g) \) associated to the connection \( A \),

\[ d_A \omega = d\omega - g[A \wedge, \omega]. \]

The operator \( d'_A \) adjoint to \( d_A \) with respect to scalar product (1) is equal to \((-1)^{(4-k)+1} d_A * \).

The gauge group of \( G \)-valued functions \( g(x) \) defined on the Minkowski space acts on the gauge field \( A \) by

\[ A \mapsto \frac{1}{g} dgg^{-1} + gAg^{-1}. \quad (2) \]

This action generates transformation laws for the covariant derivative and the strength tensor,

\[ d_A \mapsto gd_A g^{-1}, \]
\[ F \mapsto gFg^{-1}. \quad (3) \]

In formula (3) we assume that the gauge group acts on tensor fields according to the representation of the group \( G \) induced by that of the Lie algebra \( g \), and the r.h.s. of (3) should be regarded as the composition of operators.

We shall also need a covariant rough d’Alambert operator \( \Box_A \) associated to the gauge field \( A \),

\[ \Box_A = D_\mu D^\mu, \quad D_\mu = \partial_\mu - gA_\mu, \]

Here and thereafter we use the standard convention about summations and lowering tensor indexes with the help of the standard metric on the Minkowski space.
The covariant d’Alambert operator can be applied to any tensor field defined on the Minkowski space and taking values in a representation space of the Lie algebra $\mathfrak{g}$, the $\mathfrak{g}$-valued gauge field $A$ acts on the tensor field according to that representation. Note that the operator $\Box_A$ is scalar, i.e. it does not change types of tensors.

Formula (3) implies that the covariant d’Alambert operator is transformed under gauge action (2) as follows

$$\Box_A \mapsto g \Box_A g^{-1}.$$  \hspace{1cm} (5)

For the zero connection $A = 0$ we simply write $d_0 = d$ and $\Box_0 = \Box$, the usual d’Alambert operator.

Using the tensor notation the “localized” action for the massive gauge field $A$ introduced in [8] can be defined by the formula

$$S' = -\frac{1}{2} < F, F > + \frac{1}{32} < \Box_A \Phi, \Phi > - \frac{m^2}{4} < \Phi, F > + \sum_{i=1}^{3} < d_A \eta_i, d_A \eta_i >.$$  \hspace{1cm} (6)

Here $\Phi \in \Omega^2(\mathfrak{g})$ is a bosonic ghost field in the adjoint representation of $\mathfrak{g}$; $\eta_i, \overline{\eta}_i$, $i = 1, 2, 3$ are pairs of anticommuting scalar ghost fields in the adjoint representation of $\mathfrak{g}$. In formula (6) $g$ should be regarded as a coupling constant and $m$ is a mass parameter. From (3), (4) and (5) it follows that action (6) is invariant under gauge transformations (2).

In formula (6) we use normalizations slightly different from those introduced in [8]. For instance, the coefficient in front of the last term in (6) is different from that which we used in [8]. But this coefficient is not important for the study of renormalizability of the theory.

To define the Green functions corresponding to the gauge invariant action $S'$ we have to add to action (6) a gauge fixing term $S_{gf}$. As it was observed in [8] the most convenient choice of the gauge fixing term is

$$S_{gf} = -\frac{\lambda}{2} < d^* A, d^* A > - \frac{m^2}{2} < \Box^{-1} d^* A, d^* A > + < d \overline{\eta}, d A \eta >,$$

where $\eta, \overline{\eta}$ is a pair of anticommuting scalar ghost fields in the adjoint representation of $\mathfrak{g}$, and we introduced dimensionless parameter $\lambda$ to consider different gauge fixing conditions (in [8] we only discussed the case when $\lambda = 1$). In this paper we shall also assume that all inverse d’Alambert operators are of Feynman type.

The total action $S = S' + S_{gf}$ that should be used in the definition of the Green functions takes the form

$$S = -\frac{1}{2} < F, F > + \frac{1}{32} < \Box_A \Phi, \Phi > - \frac{m}{4} < \Phi, F > + < d \overline{\eta}, d A \eta > + \sum_{i=1}^{3} < d_A \eta_i, d_A \eta_i > - \frac{\lambda}{2} < d^* A, d^* A > - \frac{m^2}{2} < \Box^{-1} d^* A, d^* A >.$$  \hspace{1cm} (7)

Now consider the expression for the generating function $Z(J)$ of the Green functions for the theory with action (7) via a Feynman path integral,

$$Z(J) = \int \mathcal{D}(A)\mathcal{D}(\Phi)\mathcal{D}(\eta)\mathcal{D}(\overline{\eta}) \prod_{i=1}^{3} \mathcal{D}(\eta_i)\mathcal{D}(\overline{\eta}_i) \exp \{ i(S + < J, A >) \},$$  \hspace{1cm} (8)

where $J$ is the source for $A$. Here and thereafter we assume that the measure in Feynman path integrals is suitably normalized. We also assume that the Feynman path integral over $\Phi$
in (8) is only taken with respect to the linearly independent components $\Phi_{\mu \nu}$, $\mu < \nu$ of the skew-symmetric tensor $\Phi_{\mu \nu}$.

Observe that in the r.h.s. of formula (8) all the integrals over the ghost fields are Gaussian. The Gaussian integrals can be explicitly evaluated (see [8] for details). This yields

$$Z(J) = \int D(A) \det(d^*d_A|_{\Omega^0(g)}) \exp\{i(-\frac{1}{2} < F, F > - \frac{m^2}{2} < \Box^{-1}F, F > +$$

$$+ < J, A > - \frac{\lambda}{2} < d^*A, d^*A > - \frac{\lambda}{2} < d^*A, d^*A >)}.$$  

The r.h.s. of (9) looks like the generating function of the Green functions for the Yang-Mills theory with an extra nonlocal term, $-\frac{m^2}{2} < \Box^{-1}F, F >$, and in a generalized gauge (see [3], Ch 3, §3). Although the action $S_m$ that appears in formula (9),

$$S_m = -\frac{1}{2} < F, F > - \frac{m^2}{2} < \Box^{-1}F, F > - \frac{\lambda}{2} < d^*A, d^*A > -$$

$$- \frac{m^2}{2} < \Box^{-1}d^*A, d^*A >,$$

is not local the generating function $Z(J)$ for this action is equal to that for action (7). It is the phenomenon that was observed in [8].

The purpose of this paper is to study generating function (8) in detail. In the next section we shall discuss the Feynman diagram technique for action (7).

### 3 Feynman diagram technique

In order to develop perturbation theory for generating function (8) we have to introduce additional sources $I \in \Omega^2(g)$, $\xi, \bar{\xi}, \xi^i, \bar{\xi}^i$, $i = 1, 2, 3$ for the ghost fields $\Phi, \eta, \bar{\eta}, \eta_i, \bar{\eta}_i$, respectively. We also need the full generating function $Z(J, I, \xi, \bar{\xi}, \xi^i, \bar{\xi}^i)$ for the Green functions,

$$Z(J, I, \xi, \bar{\xi}, \xi^i, \bar{\xi}^i) = \int D(A) D(\Phi) D(\eta) D(\bar{\eta}) \prod_{i=1}^3 D(\eta_i) D(\bar{\eta}_i) \times$$

$$\times \exp\{i(S + < J, A > + \frac{1}{4} < I, \Phi > + < \xi, \bar{\xi} > + < \xi, \bar{\eta} > +$$

$$+ \sum_{i=1}^3 (< \bar{\xi}^i, \eta_i > + < \xi^i, \bar{\eta}_i >))\}.$$  

Recall that the connected Green functions are defined as the coefficients of the generating series $G(J, I, \xi, \bar{\xi}, \xi^i, \bar{\xi}^i)$ related to $Z(J, I, \xi, \bar{\xi}, \xi^i, \bar{\xi}^i)$ by

$$Z(J, I, \xi, \bar{\xi}, \xi^i, \bar{\xi}^i) = \exp(G(J, I, \xi, \bar{\xi}, \xi^i, \bar{\xi}^i)).$$

We shall first calculate the free propagators for the theory with generating function (8). The free propagators are the connected Green functions corresponding to the free generating function $Z_0(J, I, \xi, \bar{\xi}, \xi^i, \bar{\xi}^i)$,

$$Z_0(J, I, \xi, \bar{\xi}, \xi^i, \bar{\xi}^i) = \int D(A) D(\Phi) D(\eta) D(\bar{\eta}) \prod_{i=1}^3 D(\eta_i) D(\bar{\eta}_i) \times$$

$$\times \exp\{i(S_0 + < J, A > + \frac{1}{4} < I, \Phi > + < \xi, \bar{\xi} > + < \xi, \bar{\eta} > +$$

$$+ \sum_{i=1}^3 (< \bar{\xi}^i, \eta_i > + < \xi^i, \bar{\eta}_i >))\},$$
where
\[ S_0 = -\frac{1}{2} \phi^+ dA, dA \phi + \frac{1}{4} \phi^+ \square \phi > + \frac{m}{4} \phi^+ dA > + < d\eta^+, d\eta > + \]
\[ + \sum_{i=1}^{3} < d\eta^i, d\eta > - \lambda \phi^+ d^* A, d^* A > - \frac{m^2}{2} \phi^+ \square^{-1} d^* A, d^* A > \]

is the unperturbed action corresponding to (7).

Feynman path integral (12) is Gaussian, and we can explicitly evaluate it. First we integrate over the ghost fields in (12). This yields
\[ Z_0(J, I, \xi, \xi^i, \overline{\xi}^i) = \int \mathcal{D}(A) \exp\{i(\frac{1}{2} \phi^+ (\square + m^2 + (1 - \lambda)dd^*)^{-1} J, J > - \]
\[ - \frac{1}{2} \phi^+ \square^{-1} + m^2 \phi^+ \square^{-1} d^* (\square + m^2 + (1 - \lambda)dd^*)^{-1} I, I > - \]
\[ - m \phi^+ \square^{-1} + m^2 + (1 - \lambda)dd^*)^{-1} \square^{-1} d^* I, J > + \]
\[ + < \square^{-1} \overline{\xi}, \xi > + \sum_{i=1}^{3} < \square^{-1} \overline{\xi}^i, \xi^i > \}. \]

Now using the Hodge formula, \( dd^* + d^* d = -\square \), in the exponent in (14) and integrating over \( A \) we finally obtain
\[ Z_0(J, I, \xi, \xi^i, \overline{\xi}^i) = \exp\{i(\frac{1}{2} \phi^+ (\square + m^2 + (1 - \lambda)dd^*)^{-1} J, J > - \]
\[ - \frac{1}{2} \phi^+ \square^{-1} + m \phi^+ \square^{-1} d^* (\square + m^2 + (1 - \lambda)dd^*)^{-1} I, I > - \]
\[ - m \phi^+ \square^{-1} + m^2 + (1 - \lambda)dd^*)^{-1} \square^{-1} d^* I, J > + \]
\[ + < \square^{-1} \overline{\xi}, \xi > + \sum_{i=1}^{3} < \square^{-1} \overline{\xi}^i, \xi^i > \}, \]

where all the inverse d’Alambert operators are of Feynman type. We see that the free propagator for the gauge field \( A \) coincides with that for a massive field. When \( \lambda = 1 \) it is reduced to the propagator for the free massive vector field, \( i(\square + m^2)^{-1} \). The fact crucial for the mass generation is that free action (13) contains a cross term, \( -\frac{m}{4} \phi^+ \phi, dA > \), which generates mass in the free propagator of the gauge field. Note that the free propagator is also not diagonal. It contains the cross term \( < m(\square + m^2 + (1 - \lambda)dd^*)^{-1} \square^{-1} d^* I, J > \).

We shall develop perturbation theory in the momentum space for the Fourier transforms of the Green functions. We normalize the Fourier transform as follows
\[ \hat{f}(k) = \frac{1}{(2\pi)^4} \int e^{-ikx} f(x) d^4x. \]

The Fourier transforms of the propagators arising from (15), along with the corresponding diagrammatic symbols, are summarized in the following table:
\[ I^\mu_\alpha_{\gamma} \rightarrow k \quad \nu < \delta \rightarrow b \quad I^\nu_\delta \quad i(\Box^{-1} + m^2\Box^{-2}dd^*\Box + m^2 + (1 - \lambda)dd^*)^{-1} = \]
\[ = -\frac{4i\delta^{ab}}{k^2 + i\varepsilon} \left[ g_{\mu\nu}g_{\gamma\delta} + \frac{m^2}{(k^2 + i\varepsilon)(k^2 - m^2 + i\varepsilon)} (g_{\mu\nu}k_{\gamma\delta} - g_{\gamma\nu}k_{\mu\delta} - g_{\mu\delta}k_{\gamma\nu} + g_{\gamma\delta}k_{\mu\nu}) \right] \]

\[ J^\mu_\alpha \rightarrow k \quad \nu < \gamma \quad b \rightarrow \nu \rightarrow \gamma \rightarrow b \quad I^\nu_\gamma \quad im(\Box + m^2 + (1 - \lambda)dd^*)^{-1}\Box^{-1}d^* = \]
\[ = -\frac{2m\delta^{ab}}{(k^2 + i\varepsilon)(k^2 - m^2 + i\varepsilon)} (g_{\mu\nu}k_{\gamma} - g_{\mu\gamma}k_{\nu}) \]

\[ \xi_a \rightarrow k \quad \rightarrow \xi_b \quad -i\Box^{-1} = \frac{i\delta^{ab}}{k^2 + i\varepsilon} \]

\[ \xi_i a \rightarrow k \rightarrow \xi_i b \quad -i\Box^{-1} = \frac{i\delta^{ab}}{k^2 + i\varepsilon} \]

In the diagrams above we indicate the sources coupled to the corresponding propagators in formula (15). We assume that in coordinate formulas containing summations over indexes of skew-symmetric (2,0)-type tensors all summations are only taken over the indexes corresponding to the independent components of the skew-symmetric tensors; for instance, in formulas containing summations over the indexes of the source \( I^\mu_\alpha \) we take sums over \( \mu < \nu \). This gives some extra numerical factors in the formulas for the propagators. For instance, we have

\[ \langle \omega, \omega' \rangle = 4 \int \sum_{\mu<\nu} \text{tr}(\omega_{\mu\nu}, \omega'_{\mu\nu})d^4x, \; \omega, \omega' \in \Omega^2(\mathfrak{g}). \]

In the formulas for the Fourier transforms of the propagators given above we also omit, as usual, the delta functions of the total external momenta.

The perturbation theory for generating series (8) is based on the following formula

\[ Z(J) = \exp \left\{ iV \left( \frac{\delta}{i\delta J}, \frac{\delta}{i\delta \Phi}, \frac{\delta}{i\delta \eta}, \frac{\delta}{i\delta \xi}, \frac{\delta}{i\delta \zeta}, \frac{\delta}{i\delta \xi}, \frac{\delta}{i\delta \zeta} \right) \right\} \bigg|_{J=\xi=\zeta=\xi'=\zeta'=0} Z_0(J, I, \xi, \zeta, \xi', \zeta'), \]

where

\[ V(A, \Phi, \eta, \bar{\eta}, \eta_i, \bar{\eta}_i) = S(A, \Phi, \eta, \bar{\eta}, \eta_i, \bar{\eta}_i) - S_0(A, \Phi, \eta, \bar{\eta}, \eta_i, \bar{\eta}_i) \]

is the perturbation of the free action \( S_0 \), and in formula (17) all variational derivatives with respect to the Grassmann variables \( \bar{\xi}, \xi, (\xi, \xi_i) \) are right (left), respectively.

To complete our study of the perturbation theory based on formula (17) we have to calculate the contributions of the vertices corresponding to the terms of the perturbation \( iV \). These terms and their contributions evaluated in the momentum representation, along with the corresponding diagrammatic symbols, are listed below.
\[ \frac{ig}{2} < d A, [A \wedge, A] >, \]
\[ g C^{abc} (2 \pi)^4 \delta(p + q + r)(g_{\mu\nu}(p - q)_{\rho} + g_{\nu\rho}(q - r)_{\mu} + g_{\rho\mu}(r - p)_{\nu}) \]

\[ -\frac{ig^2}{8} < [A \wedge, A], [A \wedge, A] >, \]
\[ -ig^2 (2 \pi)^4 \delta(p + q + r + s)(C^{reb}C^{eced} (g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) + C^{ead}C^{reb} (g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma})) \]

\[ -\frac{ig}{16} < \partial^{\mu} \Phi, [A_{\mu}, \Phi] >, \]
\[ \frac{g}{4} (2 \pi)^4 C^{abc} \delta(p + q + r)(p_{\mu} - q_{\mu})(g_{\alpha\beta}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}) \]

\[ -\frac{ig^2}{42} < [A_{\mu}, \Phi], [A^\mu, \Phi] >, \]
\[ -\frac{ig^2}{42} (2 \pi)^4 \delta(p + q + r + s)g_{\mu\nu}(C^{reb}C^{ced} + C^{eced}C^{reb}) \times \]
\[ (g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}) \]
In the coordinate formulas above $C^{abc}$ are the structure constants of the Lie algebra $g$, $[t^a, t^b] = C^{abc} t^c$. In the diagrams containing ghost lines we assume that the variables $\eta, \eta_i$ correspond to the outgoing ghost lines. All the coordinate formulas include the corresponding combinatorial factors. Each diagram constructed with the help of vertices and lines given above gives a contribution to the generating function $Z(J)$. To calculate the contribution of a connected diagram one has to integrate the expression corresponding to the diagram with the measure $\frac{d^4k}{(2\pi)^4}$ over all internal momenta and multiply the result by $(-1)^l$, where $l$ in the number of the $\eta$– and $\eta_i$–ghost loops and $S$ is the order of the symmetry group of the diagram.
4 Effective action at the one-loop order and one-loop renormalization

In order to calculate the Green functions at the one-loop order we shall use a formula expressing the effective action at the one-loop order via a Gaussian Feynman path integral (see [5], Sect. 12.3). We recall that the effective action \( \Gamma(A) \) is defined as the Legendre transform of the generating function \( G(J) = \ln Z(J) \),

\[
i\Gamma(A) = G(J) - i < J, A >, \quad A = \frac{\delta G}{\delta J}.
\]

At the lowest order the effective action coincides with the classical action of the theory. The first quantum one-loop correction \( \Gamma^1(A) \) can be represented as follows

\[
i\Gamma^1(A) = \ln \left\{ \int \mathcal{D}(A') \mathcal{D}(\Phi') \mathcal{D}(\eta') \prod_{i=1}^3 \mathcal{D}(\eta_i') \mathcal{D}(\pi_i) \exp \{ i S^2(A; A', \Phi', \eta', \pi_i, \eta_i') \} \right\},
\]

where \( S^2(A; A', \Phi', \eta', \pi_i, \eta_i') \) is the quadratic term in the Taylor expansion of the classical action \( S(A + A', \Phi, \eta, \pi_i, \eta_i') \), with respect to the primed variables at point \( (A, 0, 0, 0, 0) \).

From formula (7) using the definition of the curvature we deduce that

\[
S^2 = \frac{g}{2} \langle \star [F(A) \wedge, A'] , A' \rangle + \frac{1}{2} \langle d_A \star d_A A' , A' \rangle + \sum_{i=1}^3 \langle \pi_i , \Box_A \eta_i' \rangle - \frac{\lambda}{2} \langle dd^* A' , A' \rangle - \frac{m^2}{2} \langle d\Box^{-1} d^* A' , A' \rangle.
\]

Now Gaussian integrals in formula (20) can be easily evaluated. This leads to the following expression for \( \Gamma^1(A) \)

\[
i\Gamma^1(A) = \ln \det (\mathbf{-\Box}^{-1} d^* d_A)_{|\Omega^0(g)} \left( \det \left[ (\mathbf{\Box} + m^2 + (1 - \lambda)dd^*)^{-1} \times \right] \right)
\]

\[
\times (g \star [F(A) \wedge, \cdot] - d^*_A d_A - m^2 d^*_A \Box_A^{-1} d_A - \lambda dd^* - m^2 d\Box^{-1} d^*)_{|\Omega^0(g)} \right)^{-\frac{1}{2}}.
\]

As usual we normalize the one-loop effective action in such a way that \( \Gamma^1(A) = 0 \) when \( g = 0 \). Recalling that \( \ln \det X = \text{tr} \ln X \) for any operator \( X \) we can rewrite formula (22) in a form suitable for calculations,

\[
i\Gamma^1(A) = \text{tr} \ln \det (\mathbf{-\Box}^{-1} d^* d_A)_{|\Omega^0(g)} - \frac{1}{2} \ln \left[ (\mathbf{\Box} + m^2 + (1 - \lambda)dd^*)^{-1} \times \right]
\]

\[
\times (g \star [F(A) \wedge, \cdot] - d^*_A d_A - m^2 d^*_A \Box_A^{-1} d_A - \lambda dd^* - m^2 d\Box^{-1} d^*)_{|\Omega^0(g)} \right].
\]

Formula (23) is the most explicit general expression for the effective action at the one-loop order. The first term in the r.h.s. of (23) is the contribution related to the Faddeev-Popov determinant.

Now using formula (23) we shall calculate the one-loop correction \( \Gamma^1 \) to the two-point function for the gauge field \( A \). In practical calculations we shall use the momentum representation.

First we apply the Wick rotation to the time components of all momenta and of the gauge field entering formula (23),

\[
k^0 \mapsto ik^0, \quad A_0(k) \mapsto iA_0(k).
\]
We shall use dimensional regularization to preserve gauge invariance, so that all the integrals with respect to momenta are taken over \( \mathbb{R}^d \) with respect to the measure \( \frac{dt}{(2\pi)^d} \). In the regularized expressions evaluated in \( \mathbb{R}^d \) the coupling constant \( g \) should be replaced with \( g\mu^{\frac{d}{2}} \), where \( \mu \) is an arbitrary mass scale, so that \( g \) remains dimensionless in \( \mathbb{R}^d \).

The contribution \((\Gamma^1_2)_{\text{ghost}}\) to the two-point function related to the Faddeev-Popov determinant

\[
\text{tr}\{\ln(-\Box^{-1}d^*d_A)|\Omega_{(g)}\} = \text{tr}\{\ln(I-g\Box^{-1}\partial_\mu[A^{\mu},\cdot]|)|\Omega_{(g)}\}
\]

in the r.h.s. of (23) is standard and coincides with that for the pure Yang-Mills field. Indeed, expanding the r.h.s. of the last formula with respect to \( g \) we obtain

\[
i(\Gamma^1_2)_{\text{ghost}} = -\frac{g^2}{2}\text{tr}\left(\Box^{-1}\partial_\mu \circ [A^{\mu},\cdot]| |\partial_\nu \circ [A^{\nu},\cdot]| |\Omega_{(g)}\right),
\]

where the symbol \( \circ \) stands for the composition for operators. The trace in formula (24) can be evaluated using the momentum representation, with the conventions about the Wick rotation and the dimensional regularization introduced above,

\[
i(\Gamma^1_2)_{\text{ghost}} = \frac{g^2\mu^\varepsilon}{2}C\int \frac{(p+q) \cdot \tilde{A}^\mu(p) q \cdot \tilde{A}^\mu(-p)}{(p+q)^2q^2} \frac{d^4p}{(2\pi)^d} \frac{d^4q}{(2\pi)^d},
\]

where \( \cdot \) denotes the standard scalar product of vectors in \( d \)-dimensional Euclidean space \( \mathbb{R}^d \), \( q^2 = q \cdot q \), etc., and the constant \( C \) is defined with the help of the structure constants of the Lie algebra \( \mathfrak{g} \).

\[
C_{\delta^{ab}} = \sum_{c,d} C^{acd}C^{bcd}.
\]

The contribution \((\Gamma_2)_A\) to the two-point function related to the second term

\[
\text{tr}\left\{-\frac{1}{2}\ln|\Box + m^2 + (1 - \lambda)d^*d| - d^*_A d_A - m^2d^*_A \Box^{-1}d_A - \lambda dd^* - m^2d\Box^{-1}d^*| |\Omega_{(g)}\right\}
\]

in the r.h.s. of (23) can be calculated in a similar way. For simplicity we shall only consider the case when \( \lambda = 1 \). Expanding the r.h.s. of (26) with respect to \( g \) and using the resolvent formula for the d’Alambert operator \( \Box^{-1} \),

\[
\Box^{-1} = (I - g\Box^{-1}(\partial_\mu \circ [A^{\mu},\cdot]| |\partial_\nu \circ [A^{\nu},\cdot]| |\Omega_{(g)})
\]

we deduce that

\[
i(\Gamma^1_2)_A = \frac{g^2}{2}\text{tr}\left(\frac{1}{2}X_1^2 - X_2\right)|\Omega_{(g)}|,
\]

where the operators \( X_1 \) and \( X_2 \) are defined by

\[
X_1 = (\Box + m^2)^{-1}(d^* \circ [A\wedge,\cdot]| |[A\wedge,\cdot]| | d - m^2d\Box^{-1}(\partial_\mu \circ [A^{\mu},\cdot]| |[A^{\mu},\cdot]| | \partial_\nu) \Box^{-1}d + m^2[A\wedge,\cdot]| d + m^2d^*\Box^{-1} \circ [A\wedge,\cdot]| | + \frac{1}{2} * [s(dA)\wedge,\cdot]| |
\]

\[
X_2 = (\Box + m^2)^{-1}(-[A\wedge,\cdot]| |A\wedge,\cdot]| | + m^2d^*\Box^{-1} \circ [A\wedge,\cdot]| | [A\wedge,\cdot]| | d - m^2d^*\Box^{-1}(\partial_\mu \circ [A^{\mu},\cdot]| | + A^{\mu},\cdot]| | \partial_\nu) \Box^{-1}d + m^2[A\wedge,\cdot]| | d + \frac{1}{2} * [s(dA)\wedge,\cdot]| |
\]

\[
+ m^2d^*\Box^{-1}(\partial_\mu \circ [A^{\mu},\cdot]| | + A^{\mu},\cdot]| | \partial_\nu) \Box^{-1}d + m^2[A\wedge,\cdot]| | d - m^2[A\wedge,\cdot]| | - \frac{1}{2} * [s(A\wedge,\cdot)| |
\]

\[
- \frac{1}{2} * [s(A\wedge,\cdot)| |
\]
In formulas (28), (29) the superscript * for operators always denotes the operator adjoint with respect to the scalar product on $\Omega^2(g)$.  

Now we have to calculate the traces of the operators $X_1^2$ and $X_2$ using the dimensional regularization as above. After tedious algebra we get

$$\text{tr} X_1^2 = -C \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \{ 2(d-1) \frac{(p+q) \cdot \hat{A}^a(p) q \cdot \hat{A}^a(-p)}{(p+q)^2 q^2} + \]

$$

$$+ m^4 \frac{(p+2q) \cdot \hat{A}^a(p) (p+2q) \cdot \hat{A}^a(-p) ((d-2)((p+q) \cdot q)^2 + (p+q)^2 q^2)}{(p+q)^4 q^4((p+q)^2 + m^2)(q^2 + m^2)} - \]

$$- 4m^2 \frac{(p+2q) \cdot \hat{A}^a(p)((d-2)q \cdot \hat{A}^a(-p) (p+q) - q \cdot q^2(p+q) \cdot \hat{A}^a(-p))}{q^4(p+q)^2((p+q)^2 + m^2)} + \]

$$+ 2 \frac{(q^2 + m^2)}{q^4((p+q)^2 + m^2)} \frac{(d-2)q \cdot \hat{A}^a(-p) q \cdot \hat{A}^a(p) + q^2 \hat{A}^a(-p) \cdot \hat{A}^a(p)}{q^4(p+q)^2((p+q)^2 + m^2)} - \]

$$- 2m^2 \frac{(p+2q) \cdot \hat{A}^a(p)(p \cdot \hat{A}^a(-p)p^2 - p \cdot q \cdot \hat{A}^a(-p))}{(p+q)^2 q^2((p+q)^2 + m^2)} + \]

$$+ 4 \frac{p \cdot \hat{A}^a(p) q \cdot \hat{A}^a(-p) - p \cdot q \cdot \hat{A}^a(p) \cdot \hat{A}^a(-p)}{q^2((p+q)^2 + m^2)} - \]

$$- 2\frac{p \cdot \hat{A}^a(p) p \cdot \hat{A}^a(-p) - p^2 \hat{A}^a(p) \cdot \hat{A}^a(-p)}{(p+q)^2 q^2(q^2 + m^2)} \}

$$\text{tr} X_2 = C(d-1) \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \{ m^2(p+q)^2 - m^2 q^2 - q^2(p+q)^2}{(p+q)^2 q^2((p+q)^2 + m^2)} \hat{A}^a(p) \cdot \hat{A}^a(-p) - \]

$$- m^2 \frac{p \cdot \hat{A}^a(-p) (2q+p) \cdot \hat{A}^a(p)}{(p+q)^2 q^2(q^2 + m^2)} \}

As a function of $\varepsilon$, $\Gamma_1^2$ has a simple pole singularity when $\varepsilon = 4-d \to 0$. To find the counterterm corresponding to the function $\Gamma_1^2$ we have to calculate its residue at point $\varepsilon = 0$ using formulas (25), (27), (30) and (31).

Note that we use the minimal subtraction procedure to regularize the effective action. Therefore the divergent part of $\Gamma_1^2$, that we use to find the counterterms, is the principal part of the Laurent series of $\Gamma_1^2(\varepsilon)$ at point $\varepsilon = 0$.

The integrals over $q$ in (25), (30) and (31) can be evaluated with the help of the following formulas

$$\int \frac{d^d q}{(2\pi)^d} \frac{1}{((p+q)^2 + m_1^2)^n(q^2 + m_2^2)^p} = \]

$$\frac{1}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(n+p - \frac{d}{2})}{\Gamma(n) \Gamma(p)} \int_0^1 da \alpha^{-1}(1-\alpha)^{p-1}(\alpha(1-\alpha)p^2 + \alpha m_1^2 + (1-\alpha)m_2^2)^{\frac{n-p}{2}} \]

$$\int \frac{d^d q}{(2\pi)^d} \frac{q \mu}{((p+q)^2 + m_1^2)^n(q^2 + m_2^2)^p} = \]

$$\frac{p \mu}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(n+p - \frac{d}{2})}{\Gamma(n) \Gamma(p)} \int_0^1 da \alpha^{-1}(1-\alpha)^{p-1}(\alpha(1-\alpha)p^2 + \alpha m_1^2 + (1-\alpha)m_2^2)^{\frac{n-p}{2}} \]

\text{(32)
\[ \int \frac{d^4q}{(2\pi)^d} \frac{q_\mu q_\nu}{(p-q)^2 + m^2_n(q^2 + m^2_n)^p} = \]
\[ = \frac{1}{(4\pi)^d} \left\{ \frac{\Gamma(n+p-d)}{\Gamma(n)} \Gamma(p) \int_0^1 d\alpha \alpha^{n+1}(1-\alpha)^{p-1}(\alpha(1-\alpha)p^2 + am^2_1 + \right. \]
\[ + (1-\alpha)m^2_2 \right\}^{d-n-p} p_\mu p_\nu + \]
\[ + \frac{\Gamma(n+p-d)}{\Gamma(n)} \Gamma(p) \int_0^1 d\alpha \alpha^{n-1}(1-\alpha)^{p-1}(\alpha(1-\alpha)p^2 + am^2_1 + \right. \]
\[ + (1-\alpha)m^2_2 \right\}^{d-1-n-p} \delta_{\mu\nu} \}, \]

where \( \Gamma \) is the Euler gamma function.

Finally, recalling that for \( \varepsilon \to 0 \) the principal part of the Laurent series of the function \( \Gamma(-n+\varepsilon) \), \( n = 0, 1, 2, \ldots \) is equal to \( (-1)^n/n\varepsilon \), we obtain an expression for the divergent part \( \Gamma_2^{1,\text{div}} \) of the function \( \Gamma_2^1 \),

\[ i(\Gamma_2^{1,\text{div}}) = -\frac{Cg^2}{(4\pi)^2\varepsilon} \int \frac{d^4p}{(2\pi)^d} \frac{5}{3}(p^2 \hat{A}^0(p) \cdot \hat{A}^0(-p) - p \cdot \hat{A}^0(p) p \cdot \hat{A}^0(-p)) + \]
\[ + 2m^2 \hat{A}^0(p) \cdot \hat{A}^0(-p)). \]

Returning to the Minkowski space by the inverse Wick rotation we can rewrite the last formula in the configuration space as follows

\[ (\Gamma_2^{1,\text{div}}) = \frac{Cg^2}{(4\pi)^2\varepsilon} \left( \frac{5}{3} < dA, dA > + 2m^2 < A, A > \right). \] (35)

The divergent term (35) can be eliminated by adding a counterterm \( S_2^1 \) to nonlocal action (10),

\[ S_2^1 = -\frac{Cg^2}{(4\pi)^2\varepsilon} \left( \frac{5}{3} < dA, dA > + 2m^2 < A, A > \right). \] (36)

This is, of course, equivalent to a modification of the coefficients of local action (7). Actually only the coefficients in the expression for the free action (13) have to be renormalized, and the expression for the renormalized free action \( (S_0)^1_2 \) takes the form

\[ (S_0)^1_2 = \frac{1}{2} Z_A < dA, dA > + \frac{1}{32} < \Box \Phi, \Phi > - \frac{m^2}{4} Z_m < \Phi, dA > + < d\bar{\eta}, d\eta > + \]
\[ + \sum_{i=1}^3 < d\bar{\eta}, d\eta > - \frac{1}{2} < d^* A, d^* A > - \frac{m^2}{2} Z_m < d^{-1} d^* A, d^* A >, \]

where

\[ Z_A = 1 + \frac{g^2}{16\pi^2} \frac{5}{3} C_2 \varepsilon, \quad Z_m = \sqrt{1 - \frac{g^2}{16\pi^2} C_4 \varepsilon}. \] (38)

By rescaling field \( A \) in formula (37), \( A^0 = Z_A^{1/2} A \), we obtain

\[ (S_0)^1_2 = \frac{1}{2} < dA^0, dA^0 > + \frac{1}{32} < \Box \Phi, \Phi > - \frac{m^2}{4} Z^0_m < \Phi, dA^0 > + < d\bar{\eta}, d\eta > + \]
\[ + \sum_{i=1}^3 < d\bar{\eta}, d\eta > - \frac{Z^{-1}_A}{2} < d^* A^0, d^* A^0 > - \frac{m^2}{2} Z^{-1}_m < d^{-1} d^* A^0, d^* A^0 >. \] (39)

12
Now from formulas (38) and (39) we can find the mass renormalization,

\[ m_0 = m_\text{bare} Z^{-\frac{1}{2}} = m \left( 1 - \frac{g^2}{16\pi^2} \frac{11}{6} C^2 \varepsilon \right), \tag{40} \]

where \( m_0 \) is the bare mass.

Note that since the possible renormalization constant of the ghost field \( \Phi \) is canceled in the Gaussian integral in formula (12) the renormalization of the wave function of \( \Phi \) is not required for calculating the mass renormalization.

To calculate the one-loop coupling constant renormalization one has to find the divergent part of the three-point function \( \Gamma^3 \) at the one-loop order. Looking at formulas (28) and (29) one should expect that technically this problem is extremely complicated.

In the next section we shall prove the universality of the coupling constant renormalization and Slavnov-Taylor identities for the theory with generating function (8). Using the universality of the coupling constant renormalization one can also find the one-loop coupling constant renormalization with the help of the one-loop corrections to the three-point function represented by diagram (19). This calculation is standard and is completely similar to the case of the pure Yang-Mills field (see [3], Ch. 4, §1). We only note that in case of the dimensional regularization the mass in the propagator for the gauge field does not add new divergent terms to the quantum corrections. The coupling constant renormalization computed in this way coincides with that for the pure Yang-Mills field,

\[ g_0 = g \left( 1 - \frac{g^2}{16\pi^2} \frac{11}{6} C^2 \varepsilon \right), \tag{41} \]

where \( g_0 \) is the bare coupling constant.

One can also observe that the coupling constant renormalization in the theory with generating function (8) calculated within the dimensional regularization framework coincides with the coupling constant renormalization for the pure Yang-Mills theory to all orders of perturbation theory. Indeed, by the results of [4] in the case of dimensional regularization the coupling constant renormalization is independent of the mass for dimensional reasons. Therefore the coupling constant renormalization in the theory with generating function (8) is the same as in the massless case, i.e. in case of the Yang-Mills field.

From formulas (38), (40) and (41) one can calculate the corresponding renormalization group coefficients at the one-loop order. An elementary renormalization group analysis of the theory with generating function (8) can be found in [8].

## 5 Renormalization

To complete our investigation of the theory with generating function (8) we have to prove its renormalizability to all orders of perturbation theory.

By simple dimensional counting all the integrals that enter the perturbative expressions for the Green functions can be dimensionally regularized. Indeed, the degree of divergence \( \omega(G) \) of any diagram \( G \) that appears in the perturbative expansion for generating function (8) can be computed as follows:

\[ \omega(G) = 4 - E - \delta + \sum (-1), \tag{42} \]

where \( E \) is the number of the external lines in the diagram \( G \), \( \delta \) is the total power of external momenta factorized from the corresponding integral, and the sum in the last term is taken over parts (16) of the free propagator which appear as internal lines in \( G \) and over vertexes (18). From formula (42) we deduce that divergences may only appear in the Green functions with \( E = 2, 3, 4, \) and the theory is renormalizable.
We have to prove yet that the gauge invariance of the theory is preserved by the renormalization. As usual this is achieved with the help of the corresponding Slavnov-Taylor identities that we are going to derive now.

The proof of renormalizability of the theory with generating function (8) is completely analogous to that for the Yang-Mills field (see [5], Sect. 12.4). In the proof we shall use the BRST technique. We shall consider the case of arbitrary $g$-valued gauge fixing condition $F(A)$ linear in $A$, $F(A) = \phi_{ab}^\mu A^\mu_{\mu}$. This situation is slightly more general than the one discussed above for action (7), and the new total action including the gauge fixing term takes the form

$$S_F = -\frac{1}{2} < F, F > + \frac{1}{32} < \Box A, \Phi > - \frac{m}{4} < \Phi, F > - < \Phi, \mathcal{M} \eta > +$$

$$+ \sum_{i=1}^{3} < d_A \bar{\eta}_i, d_A \eta_i > - \frac{\lambda}{2} < F(A), F(A) >,$$

where $\mathcal{M}$ is the Faddeev-Popov operator for the gauge fixing condition $F(A)$.

The BRST transformation $s$ can be defined similarly to the case of the Yang-Mills field ([5], Sect. 12.4.2),

$$sA = d_A \eta, \quad s\Phi = g[\eta, \Phi], \quad s\eta = g[\eta, \eta],$$

$$s\eta_i = -\frac{g}{2} C^{abc} \eta^b \eta^c, \quad s\bar{\eta}_i = g[\eta, \bar{\eta}_i].$$

If we introduce the ghost number $gh$ for the fields as follows

$$ghA = gh\Phi = gh\eta_i = gh\bar{\eta}_i = 0,$$

$$gh\eta = -1, \quad gh\bar{\eta} = 1,$$

then $s$ becomes a right superderivation of the algebra of the fields, and

$$s^2 A = s^2 \Phi = s^2 \eta = s^2 \eta_i = s^2 \bar{\eta}_i = 0, \quad s^3 \bar{\eta} = 0.$$

We also define three ghost numbers $gh'_i$, $i = 1, 2, 3$ by

$$gh'_i A = gh'_i \Phi = gh'_i \eta = gh'_i \bar{\eta} = 0,$$

$$gh'_i \eta_j = -\delta_{ij}, \quad gh'_i \bar{\eta}_j = \delta_{ij}.$$

We shall find the Slavnov-Taylor identities for generating function $G(J, I, \xi, \xi^i, \xi^i, K, L, N, P^i, \bar{P}^i)$,

$$\exp G(J, I, \xi, \xi^i, \xi^i, K, L, N, P^i, \bar{P}^i) = \int D(A) D(\Phi) D(\eta) D(\bar{\eta}) \prod_{i=1}^{3} D(\eta_i) D(\bar{\eta}_i) \times$$

$$\times \exp \{ i(S_F + < J, A > + \frac{1}{4} < I, \Phi > + < \xi, \eta > + < \xi, \bar{\eta} > +$$

$$+ \sum_{i=1}^{3} (< \xi^i, \eta_i > + < \xi^i, \bar{\eta}_i >) + < K, sA > - < L, s\eta > + \frac{1}{4} < N, s\Phi > +$$

$$+ \sum_{i=1}^{3} (< P^i, s\eta_i > + < \bar{P}^i, s\bar{\eta}_i >) \},$$

14
where $K, L, N, P^i, P^{P_j}$ are the sources coupled to $sA, s\eta, s\Phi, s\eta_i$ and $s\eta_{ji}$, respectively. The generating function $G(J, I, \xi, \xi^i, \xi^{P_i}, K, L, N, P^i, P^{P_j})$ is reduced to $G(J)$ when all the sources, except for $J$, are equal to zero. The new sources have the following ghost numbers

\[ ghK = ghN = ghP^i = ghP^{P_j} = 1, \quad ghL = 2, \]
\[ gh_iK = gh_iN = gh_iP^j = \delta_{ij}, \quad gh_iP^{P_j} = -\delta_{ij}. \]

We shall also need the dimensions of the fields and of the sources,

\[ \dim A = \dim \Phi = \dim \eta = \dim \eta_i = \dim \eta_{ji} = 1, \]
\[ \dim K = \dim L = \dim N = \dim P_i = \dim P_{P_i} = 2, \]
\[ \dim J = \dim I = \dim \xi = \dim \bar{\xi} = \dim \xi^i = \dim \bar{\xi}^i = 3, \]

the dimension of the mass $m$ being equal to 1, and the coupling constant $g$ is dimensionless.

To derive the Slavnov-Taylor identities we have to apply the BRST transformation (44) to the variables of integration in the r.h.s. of formula (45). Observe that both action (43) and the measure in the Feynman path integral (43) are BRST-invariant. The former statement can be checked directly, and the latter one is true since the BRST transformation is nilpotent. Therefore the BRST coordinate change in (43) yields the following identity

\[
\int D(A)D(\Phi)D(\eta)D(\bar{\eta}) \prod_{i=1}^{3} D(\eta_i)D(\bar{\eta}_i)(< J, sA > + \frac{1}{4} < I, s\Phi > + < \bar{\xi}, s\eta > - \sum_{i=1}^{3} (< \xi^i, s\eta_i > + < s\eta_i, \xi^i >)) \exp \{ i(S_F + < J, A > + \frac{1}{4} < I, \Phi > + < \bar{\xi}, \eta > + < \xi, \eta > + \sum_{i=1}^{3} (< \bar{\xi}, \eta_i > + < \xi^i, \bar{\eta}_i >) + < K, sA > - < L, s\eta > + \frac{1}{4} < N, s\Phi > + \sum_{i=1}^{3} (< P^i, s\eta_i > + < P^{P_i}, s\eta_{P_i} >)) \} = 0.
\]

This identity can be rewritten as

\[
< J, \frac{\delta}{i\partial K} > + \frac{1}{4} < I, \frac{\delta}{i\partial N} > - < \bar{\xi}, \frac{\delta}{i\partial L} > - \lambda < \xi, F(\frac{\delta}{i\partial J}) > + \sum_{i=1}^{3} (< \bar{\xi}, \frac{\delta}{i\partial P^i} > - < \xi^i, \frac{\delta}{i\partial P^{P_i}} >)) \exp G(J, I, \xi, \xi^i, \xi^{P_i}, K, L, N, P^i, P^{P_j}) = 0,
\]

which is, in turn, equivalent to

\[
< J, \frac{\delta}{i\partial K} > + \frac{1}{4} < I, \frac{\delta}{i\partial N} > - < \bar{\xi}, \frac{\delta}{i\partial L} > - \lambda < \xi, F(\frac{\delta}{i\partial J}) > + \sum_{i=1}^{3} (< \bar{\xi}, \frac{\delta}{i\partial P^i} > - < \xi^i, \frac{\delta}{i\partial P^{P_i}} >))G(J, I, \xi, \xi^i, \xi^{P_i}, K, L, N, P^i, P^{P_j}) = 0
\]

due to linearity of $F$. 

15
The Slavnov-Taylor identities (46) are supplemented with another identity following from the fact that the Feynman path integral in the r.h.s. of formula (45) is invariant under translations of the ghost variable of integration, 

\[ \eta \mapsto \eta + \delta \eta. \]

From this fact we infer that

\[ \int D(A)D(\Phi)D(\eta)D(\bar{\eta})(-M\eta + \xi) \exp\{i(S_F + < J, A > + \frac{1}{4} < I, \Phi > + \\
+ < \xi, \eta > + < J, \bar{\eta} > + \sum_{i=1}^{3} (< \xi^i, \eta_i > + < \xi^i, \bar{\eta}_i >) + < K, sA > - < L, s\eta > + \\
+ \frac{1}{4} < N, s\Phi > + \sum_{i=1}^{3} (< P^i, s\eta_i > + < P^i, s\bar{\eta}_i >) \} = 0, \]

or, since \( M\eta = sF = \phi^\mu a_b(sA)_b^\mu \cdot \phi \) and

\[ (\xi - \phi \cdot \delta_{i0} K) \exp G(J, I, \xi, \xi^i, \bar{\eta}, K, L, N, P^i, \bar{P}^i) = 0. \]

For connected functions the last identity gives

\[ \phi \cdot \delta_{i0} K G(J, I, \xi, \xi^i, \bar{\eta}, K, L, N, P^i, \bar{P}^i) = \xi. \tag{47} \]

In the proof of renormalizability it is more convenient to use the proper functions instead of the Green functions, and instead of the generating function \( G(J, I, \xi, \xi^i, \bar{\eta}, K, L, N, P^i, \bar{P}^i) \) we shall consider its Legendre transform \( \Gamma(A, \Phi, \eta, \bar{\eta}, \eta_i, \bar{\eta}_i, K, L, N, P^i, \bar{P}^i) \),

\[ \Gamma(A, \Phi, \eta, \bar{\eta}, \eta_i, \bar{\eta}_i, K, L, N, P^i, \bar{P}^i) = -iG(J, I, \xi, \xi^i, \bar{\eta}, K, L, N, P^i, \bar{P}^i) - \]

\[ - < J, A > + \frac{1}{4} < I, \Phi > + < \xi, \eta > + < J, \bar{\eta} > + \sum_{i=1}^{3} (< \xi^i, \eta_i > + < \xi^i, \bar{\eta}_i >), \]

where

\[ A = \frac{\delta G}{i\delta J}, \quad \eta = \frac{\delta G}{i\delta \xi}, \quad \bar{\eta} = -\frac{\delta G}{i\delta \xi}, \quad \Phi = \frac{\delta G}{i\delta I}, \quad \eta_i = \frac{\delta G}{i\delta \xi^i}, \quad \bar{\eta}_i = -\frac{\delta G}{i\delta \xi^i}, \tag{49} \]

and the variables \( K, L, N, P^i, \bar{P}^i \) are passive.

Since

\[ J = -\frac{\delta \Gamma}{\delta A}, \quad \xi = -\frac{\delta \Gamma}{\delta \eta}, \quad \bar{\xi} = -\frac{\delta \Gamma}{\delta \bar{\eta}}, \quad I = -\frac{\delta \Gamma}{\delta \Phi}, \quad \xi^i = -\frac{\delta \Gamma}{\delta \eta_i}, \quad \bar{\xi}^i = -\frac{\delta \Gamma}{\delta \bar{\eta}_i} \]

and

\[ \frac{\delta G}{i\delta K} = \frac{\delta G}{i\delta \xi}, \quad \frac{\delta G}{i\delta L} = \frac{\delta G}{i\delta \eta}, \quad \frac{\delta G}{i\delta N} = \frac{\delta G}{i\delta \bar{\xi}}, \quad \frac{\delta G}{i\delta P^i} = \frac{\delta G}{i\delta \xi^i} \]

identities (46), (47) can be expressed in terms of the function \( \Gamma \) as

\[ < \frac{\delta \Gamma}{\delta A} > + \frac{1}{4} < \frac{\delta \Gamma}{\delta \Phi} > + < \frac{\delta \Gamma}{\delta \eta} > - \lambda < \frac{\delta \Gamma}{\delta \bar{\eta}} > = 0, \tag{50} \]

where \( \lambda = \frac{\delta \Gamma}{\delta \bar{\eta}} \) and the variables \( A, \Phi, \eta, \bar{\eta}, \eta_i, \bar{\eta}_i, K, L, N, P^i, \bar{P}^i \) are passive.

Since

\[ J = -\frac{\delta \Gamma}{\delta A}, \quad \xi = -\frac{\delta \Gamma}{\delta \eta}, \quad \bar{\xi} = -\frac{\delta \Gamma}{\delta \bar{\eta}}, \quad I = -\frac{\delta \Gamma}{\delta \Phi}, \quad \xi^i = -\frac{\delta \Gamma}{\delta \eta_i}, \quad \bar{\xi}^i = -\frac{\delta \Gamma}{\delta \bar{\eta}_i} \]

and

\[ \frac{\delta G}{i\delta K} = \frac{\delta G}{i\delta \xi}, \quad \frac{\delta G}{i\delta L} = \frac{\delta G}{i\delta \eta}, \quad \frac{\delta G}{i\delta N} = \frac{\delta G}{i\delta \bar{\xi}}, \quad \frac{\delta G}{i\delta P^i} = \frac{\delta G}{i\delta \xi^i} \]

identities (46), (47) can be expressed in terms of the function \( \Gamma \) as

\[ < \frac{\delta \Gamma}{\delta A} > + \frac{1}{4} < \frac{\delta \Gamma}{\delta \Phi} > + < \frac{\delta \Gamma}{\delta \eta} > - \lambda < \frac{\delta \Gamma}{\delta \bar{\eta}} > = 0, \tag{50} \]

where \( \lambda = \frac{\delta \Gamma}{\delta \bar{\eta}} \) and the variables \( A, \Phi, \eta, \bar{\eta}, \eta_i, \bar{\eta}_i, K, L, N, P^i, \bar{P}^i \) are passive.
Formulas (50), (51) take a slightly simpler form for the modified effective action $\tilde{\Gamma}$ defined by

$$\tilde{\Gamma} = \Gamma + \frac{\lambda}{2} < \mathcal{F}, \mathcal{F}>.$$ 

Indeed, from (50), (51) we have

$$< \frac{\delta \tilde{\Gamma}}{\delta A}, \frac{\delta \tilde{\Gamma}}{\delta K} > + \frac{1}{4} < \frac{\delta \tilde{\Gamma}}{\delta \Phi}, \frac{\delta \tilde{\Gamma}}{\delta N} > + < \frac{\delta \tilde{\Gamma}}{\delta \eta}, \frac{\delta \tilde{\Gamma}}{\delta L} > +$$

$$+ \sum_{i=1}^{3} (- < \frac{\delta \tilde{\Gamma}}{\delta \eta_i}, \frac{\delta \tilde{\Gamma}}{\delta P^i} > + < \frac{\delta \tilde{\Gamma}}{\delta \eta_i}, \frac{\delta \tilde{\Gamma}}{\delta P^i} > ) = 0,$$

$$\phi \cdot \delta \frac{\delta \tilde{\Gamma}}{\delta K} + \delta \frac{\delta \tilde{\Gamma}}{\delta \eta} = 0.$$ 

For the operation in the r.h.s. of formula (52) we shall use the compact notation

$$\Gamma_1 * \Gamma_2 = < \frac{\delta \Gamma_1}{\delta A}, \frac{\delta \Gamma_2}{\delta K} > + \frac{1}{4} < \frac{\delta \Gamma_1}{\delta \Phi}, \frac{\delta \Gamma_2}{\delta N} > + < \frac{\delta \Gamma_1}{\delta \eta}, \frac{\delta \Gamma_2}{\delta L} > +$$

$$+ \sum_{i=1}^{3} (- < \frac{\delta \Gamma_1}{\delta \eta_i}, \frac{\delta \Gamma_2}{\delta P^i} > + < \frac{\delta \Gamma_1}{\delta \eta_i}, \frac{\delta \Gamma_2}{\delta P^i} > ).$$ 

Now we can prove that the theory with generating function (8) is renormalizable. First recall that by simple dimensional counting the integrals contained in the perturbative loop expansion

$$\tilde{\Gamma} = \tilde{\Gamma}^0 + \tilde{\Gamma}^1 + \tilde{\Gamma}^2 + \ldots$$

for the generating function $\tilde{\Gamma}$ can be dimensionally regularized. Assume for a moment that this is done, i.e. the Wick rotation is performed and all the integrals are taken over $\mathbb{R}^d$ as in the one-loop case discussed in the previous section. When the dimension $d$ of the space $\mathbb{R}^d$ goes to 4 each term in expansion (54) acquires a divergent term corresponding to a pole in variable $\varepsilon = 4 - d$ as $\varepsilon \to 0$ (minimal subtraction). The divergent terms $\Gamma_{div}^n$ can be determined recursively from the identity

$$\tilde{\Gamma}_{reg}^n = \tilde{\Gamma}_{R}^n + \tilde{\Gamma}_{div}^n,$$

where $\tilde{\Gamma}_{reg}^n$ is computed taking into account all lower-order counterterms in the perturbation expansion, and $\tilde{\Gamma}_{R}^n$ is the renormalized effective action at the n-th order.

At the zeroth order we obviously have

$$\tilde{\Gamma}^0 = - \frac{1}{2} < F, F > + \frac{1}{32} < \Box_A \Phi, \Phi > - \frac{m}{4} < \Phi, F > - < \mathcal{M}, \mathcal{M} > +$$

$$+ \sum_{i=1}^{3} < d_A \eta_i, d_A \eta_i > + < K, sA > - < L, s\eta > + \frac{1}{4} < N, s\Phi > +$$

$$+ \sum_{i=1}^{3} ( < P^i, s\eta_i > + < \overrightarrow{P^i}, s\eta_i > ) =$$
\[
= -\frac{1}{2} < F, F > + \frac{1}{32} < \Box A \Phi, \Phi > - \frac{m}{4} < \Phi, F > + \frac{1}{4} < (K - \nabla \cdot \phi), sA > + \\
+ \frac{1}{3} \sum_{i=1}^{3} < d_A \eta_i, d_A \eta_i > - \frac{1}{4} < L, s\eta > + \frac{1}{4} < N, s\Phi > + \\
+ \frac{1}{3} \sum_{i=1}^{3} ( < P^i, s\eta_i > + < \overline{F}_i, s\overline{\eta}_i > ),
\]

where \( \overline{\eta} \cdot \phi^\mu = \overline{\eta}^a \phi^\mu_{ab} \).

From (52) we infer by induction over \( n \) that every divergent part \( \tilde{\Gamma}_{\text{div}}^n \) must satisfy the following equation (see [5], Sect. 12.4.3 for details):

\[
\tilde{\Gamma}^0 * \tilde{\Gamma}_{\text{div}}^n + \tilde{\Gamma}_{\text{div}}^n * \tilde{\Gamma}^0 = 0.
\]

Equation (56) allows to find the most general form of the divergent terms.

If we assume, by the choice of gauge, that the symmetry of \( \tilde{\Gamma}_{\text{div}}^n \) under constant gauge transformations is unbroken then simple power counting and the ghost number conservation restrict the form of \( \tilde{\Gamma}_{\text{div}}^n \) to

\[
\tilde{\Gamma}_{\text{div}}^n = \mathcal{L}^n + \alpha < (K - \overline{\eta} \cdot \phi), sA > + (\alpha - \beta) < (K - \overline{\eta} \cdot \phi), [A, \eta] > + \\
+ \gamma < L, s\eta > + \delta \frac{1}{4} < N, s\Phi > + \sum_{i=1}^{3} ( \varepsilon_i < P^i, s\eta_i > + \overline{\varepsilon}_i < \overline{P}^i, s\overline{\eta}_i > ),
\]

where \( \alpha, \beta, \gamma, \delta, \varepsilon_i, \overline{\varepsilon}_i \) are numerical coefficients, and \( \mathcal{L}^n \) is a functional of degree four depending on \( A, \Phi, \eta_i, \overline{\eta}_i \) and having zero ghost numbers.

Substituting expression (57) into (56) we get

\[
\beta = \delta = \varepsilon_i = \overline{\varepsilon}_i = -\gamma,
\]

\[
(\alpha - \beta) < \frac{\delta \mathcal{L}}{\delta A} d\eta > + < \frac{\delta \mathcal{L}^n}{\delta A}, sA > + \frac{1}{4} < \frac{\delta \mathcal{L}^n}{\delta \Phi}, s\Phi > - \\
- < \frac{\delta \mathcal{L}^n}{\delta \eta_i}, s\eta_i > - < \frac{\delta \mathcal{L}^n}{\delta \overline{\eta}_i}, s\overline{\eta}_i > = 0,
\]

where

\[
\mathcal{L} = -\frac{1}{2} < F, F > + \frac{1}{32} < \Box A \Phi, \Phi > - \frac{m}{4} < \Phi, F > + \sum_{i=1}^{3} < d_A \eta_i, d_A \eta_i > .
\]

The general solution to equation (59) is

\[
\mathcal{L}^n = (\beta - \alpha)< \frac{\delta \mathcal{L}}{\delta A}, A > + \frac{1}{4} < \frac{\delta \mathcal{L}}{\delta \Phi}, \Phi > - < \frac{\delta \mathcal{L}}{\delta \eta_i}, \eta_i > - < \frac{\delta \mathcal{L}}{\delta \overline{\eta}_i}, \overline{\eta}_i > + \mathcal{L}_{\text{inv}},
\]

where \( \mathcal{L}_{\text{inv}} \) is an arbitrary gauge invariant functional of degree four depending on \( A, \Phi, \eta_i, \overline{\eta}_i \) and having zero ghost numbers.

We actually do not need to consider the most general expression for \( \mathcal{L}_{\text{inv}} \). Some terms in this expression are a priori not required for the renormalization. From formula (7) it follows that for \( m = 0 \) the spatial components of the ghost tensor field \( \Phi \) do not interact with each other and with the other ghost fields. Taking into account the ghost numbers conservation we infer that the counterterms containing interactions of different components of \( \Phi \) and of anticommuting
ghosts $\eta_i, \bar{\eta}_i$ for different indexes i should appear with multiples $m$ or $n$. The latter possibility is not realized in case of the minimal subtraction. Note also that the ghosts $\eta_i, \bar{\eta}_i$ symmetrically appear in formula (7). Therefore they must symmetrically appear in the expression for the counterterms.

These rules restrict the expression for $\mathcal{L}_{\text{inv}}$ required for the renormalization to

$$
\mathcal{L}_{\text{inv}} = -\frac{a_0}{2} < F, F > + \frac{a_1}{32} < \Box A \Phi, \Phi > - \frac{a_3 m^4}{4} < \Phi, F > + \\
+ \sum_{i=1}^{3} a_2 < d_A \eta_i, d_A \eta_i > + \sum_{i=1}^{3} a_4 m_\eta^2 < \eta_i, \eta_i > + a_5 \frac{m_\Phi^2}{4} < \Phi, \Phi >,
$$

where $a_0, a_1, a_2, a_3, a_4, a_5$ are numerical coefficients, and four new mass parameters $m_\Phi, m_\eta$ are, of course, proportional to $m$.

Substituting (60) and (61) into (57), using relations (58) and the homogeneity property of the functional $\mathcal{L}_{\text{inv}}$ with respect to coupling constants and fields we finally obtain the following expression for the most general form of the counterterms (compare with [5], Sect 12.4.3., formula 12-165)

$$
\tilde{\Gamma}_n^{\text{div}} = \{ (\beta - \alpha + \frac{a_0}{2}) < A, \frac{\delta}{\delta A} > + \frac{1}{4} (\beta - \alpha + \frac{a_1}{2}) < \Phi, \frac{\delta}{\delta \Phi} > + (a_3 - \frac{a_0 + a_1}{2}) m \frac{\partial}{\partial m} + \\
+ \frac{a_0 - a_1}{8} < N, \frac{\delta}{\delta N} > + \alpha < \eta, \frac{\delta}{\delta \eta} > + \sum_{i=1}^{3} (a_0 - a_2) < P_i, \frac{\delta}{\delta P_i} > + < \bar{\eta}_i, \frac{\delta}{\delta \bar{\eta}_i} > + (a_4 - a_2 - 2(\beta - \alpha)) m_\eta^2 \frac{\partial}{\partial m_\eta^2} + \\
+ (a_5 - a_1 - 2(\beta - \alpha)) m_\Phi^2 \frac{\partial}{\partial m_\Phi^2} - a_0 g \frac{\partial}{\partial g} + (\beta - 2\alpha + \frac{a_0}{2}) < L, \frac{\delta}{\delta L} > \} (\tilde{\Gamma}^0)',
$$

where

$$
(\tilde{\Gamma}^0)' = -\frac{1}{2} < F, F > + \frac{1}{32} < \Box A \Phi, \Phi > - \frac{m}{4} < \Phi, F > + \sum_{i=1}^{3} < d_A \eta_i, d_A \eta_i > - \\
- < \bar{\eta}_i, M \eta_i > + \sum_{i=1}^{3} m_\eta^2 < \eta_i, \eta_i > + \frac{m_\Phi^2}{4} < \Phi, \Phi > + < K, s A > - < L, s \eta_i > + \\
+ \frac{1}{4} < N, s \Phi > + \sum_{i=1}^{3} < P_i, s \eta_i > + < \bar{\eta}_i, s \bar{\eta}_i >).
$$

Formula (62) tells that the divergent part of the effective action at the n-th order has the same form as action (63) with renormalized coupling constants, masses and fields. Comparing formula (63) with (55) we see that a priori new counterterms have to be added to the original action (55) to cancel all the divergences. The only difference between expressions (63) and (55) is that the ghost fields acquire masses.

The renormalized generating function $G_R(J)$ is obtained by taking the inverse Legendre transform of the renormalized effective action $\Gamma_R(A, \Phi, \eta, \bar{\eta}_i, \bar{\eta}_i, K, L, N, P^i, \bar{P}^i)$ and by putting all the arguments, except for $J$, equal to zero. Recalling the definition (48), (49) of the Legendre transform we get

$$
G_R(J, I, \xi, \bar{\xi}, \xi^i, \bar{\xi}^i, K, L, N, P^i, \bar{P}^i) = i \Gamma_R(A, \Phi, \eta, \bar{\eta}_i, \bar{\eta}_i, K, L, N, P^i, \bar{P}^i) + \\
+ i( < J, A > - \frac{1}{4} < I, \Phi > - < \bar{\xi}, \eta > - < \xi, \bar{\eta} > - \sum_{i=1}^{3} < \bar{\xi}^i, \eta_i > + < \xi^i, \bar{\eta}_i > ).
$$
$G_R(J) = G_R(J, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$.  

This completes the perturbative study of the theory with generating function (8).

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