On Scattering for Small Data of 2+1 Dimensional Equivariant Einstein-Wave Map System

Benjamin Dodson and Nishanth Gudapati

Abstract. We consider the Cauchy problem of 2+1 equivariant wave maps coupled to Einstein’s equations of general relativity and prove that two separate (nonlinear) subclasses of the system disperse to their corresponding linearized equations in the large. Global asymptotic behaviour of 2+1 Einstein-wave map system is relevant because the system occurs naturally in 3+1 vacuum Einstein’s equations.

Mathematics Subject Classification (2010). Primary 35L70; Secondary 83C05.

Keywords. Einstein’s equations, critical wave maps, dispersive PDE.

1. Background and Introduction

Let \((M, g)\) be a regular, globally hyperbolic, spatially asymptotically flat, rotationally symmetric 2 + 1 dimensional Lorentzian spacetime and \((N, h)\) be a surface of revolution with the generating function \(f\), then the Einstein-equivariant wave map system is defined as follows

\[
E_{\mu\nu} = T_{\mu\nu} \tag{1a}
\]

\[
\Box_g(u)u = \frac{k^2 f_u(u) f(u)}{r^2}, \tag{1b}
\]

where \(E\) is the Einstein tensor of \((M, g)\),

\[
T_{\mu\nu} := \langle \partial_\mu U, \partial_\nu U \rangle_h - \frac{1}{2} g_{\mu\nu} \langle \partial_\sigma U, \partial_\sigma U \rangle_h \tag{2}
\]

is the energy-momentum tensor of the equivariant wave map \(U : (M, g) \to (N, h), U := (u, k\theta)\), \(\Box_g\) is the covariant wave operator defined on \((M, g)\), \(r\) is the area-radius function on \((M, g)\), \(f_u(u)\) is the derivative of \(f(u)\) with

(BD) is supported by NSF by grant no. DMS-1500424 and (NG) is supported by Deutsche Forschungsgemeinschaft (DFG) Postdoctoral Fellowship GU1513/1-1 to Yale University.
B. Dodson and N. Gudapati

respect to $u$. $k$ is the homotopy degree of the equivariant map which shall henceforth be assumed to be 1. Furthermore, we assume that $f$ is a smooth, odd function such that $f(0) = 0$ and $f_u(0) = 1$, which for instance is admitted by a metric on the hyperbolic 2-plane $N = \mathbb{H}^2$. In particular, it follows that

$$f(u)f_u(u) = u + u^3\zeta(u).$$

for some smooth function $\zeta$.

Let us assume that the metric $g$ of $M$ can be represented in the following form in null coordinate system $(\xi, \eta, \theta)$

$$ds_g^2 = -e^{2Z(\xi, \eta)}(d\xi d\eta) + r^2(\xi, \eta)d\theta^2,$$

for some function $Z(\xi, \eta)$ and the radius function $r(\xi, \eta)$ with

$$r = 0 \quad \text{and} \quad Z = 0 \quad \text{on the axis} \quad \Gamma \quad \text{of} \quad M.$$

Furthermore, we introduce the coordinate functions $T$ and $R$ such that

$$T : = \frac{1}{2}(\xi + \eta) \quad \text{and} \quad R : = \frac{1}{2}(\xi - \eta)$$

so that $R = 0$ and $T = \xi = \eta$ on $\Gamma$. Further suppose that

$$\partial_R r = 1 \quad \text{and} \quad \partial_R Z = 0 \quad \text{on the axis} \quad \Gamma \quad \text{of} \quad M.$$

Consequently,

$$ds_g^2 = e^{2Z(T, R)}(-dT^2 + dR^2) + r^2(T, R)d\theta^2.$$

As calculated in [6] (cf. Section 3.5) the Einstein tensor

$$E_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R_g$$

in null coordinates is given by

$$E_{\xi\xi} = r^{-1}(2\partial_\xi Z \partial_\xi r - \partial^2_\xi r),$$

$$E_{\xi\eta} = r^{-1}\partial_\xi \partial_\eta r,$$

$$E_{\eta\eta} = r^{-1}(2\partial_\eta Z \partial_\eta r - \partial^2_\eta r),$$

$$E_{\theta\theta} = -4r^2 e^{-2Z} \partial_\xi \partial_\eta Z,$$

$$E_{\xi\theta} = 0 \quad \text{and}$$

$$E_{\eta\theta} = 0.$$

The components of $T_{\mu\nu}$ are

$$T_{\xi\xi} = \partial_\xi u \partial_\xi u,$$

$$T_{\eta\eta} = \partial_\eta u \partial_\eta u,$$

$$T_{\xi\eta} = \frac{e^{2Z} f^2(u)}{4 \sqrt{r^2}},$$

$$T_{\theta\theta} = \frac{r^2}{2} e^{-2Z} \left(4\partial_\eta u \partial_\xi u + e^{2Z} \frac{f^2(u)}{r^2}\right).$$
Furthermore, noting that

\[ \sqrt{-g} = \frac{1}{2} re^{2Z(T,R)}, \]

the wave operator in null coordinates can be expressed as

\[ \square_g u = \frac{1}{\sqrt{-g}} \partial^\nu (\sqrt{-g} \partial_\nu u) \]

\[ = -2e^{-2Z} r^{-1} \left( \partial_\eta (r \partial_\xi u) + \partial_\xi (r \partial_\eta u) \right). \]  

(5)

Therefore, from (1b)

\[ -2e^{-2Z} r^{-1} \left( \partial_\eta (r \partial_\xi u) + \partial_\xi (r \partial_\eta u) \right) = \frac{f_u(u)f(u)}{r^2} \]  

(6)

Consequently, the equivariant Einstein-wave map system can be represented as follows

\[ r^{-1} (2 \partial_\xi Z \partial_\xi r - \partial_\xi^2 r) = \partial_\xi u \partial_\xi u \]  

(7a)

\[ r^{-1} \partial_\xi^2 r = \frac{e^{2Z} f^2(u)}{4} \]  

(7b)

\[ r^{-1} (2 \partial_\eta Z \partial_\eta r - \partial_\eta^2 r) = \partial_\eta u \partial_\eta u, \]  

(7c)

\[ -4r^2 e^{-2Z} \partial_\xi^2 Z \frac{r^2}{2} \left( 4e^{-2Z} \partial_\eta u \partial_\xi u + \frac{f^2(u)}{r^2} \right) \]  

(7d)

\[ \square_g(u) = \frac{f_u(u)f(u)}{r^2}. \]  

(7e)

**Proposition 1.1.** If we define the function \( V \) such that \( RV := u \), then the following statements hold

1. \[ \partial_\xi^2 u = R \partial_\xi^2 V - \frac{1}{2} \partial R V. \]  

(8)

2. \[ 4^{+1} \square V = -4 \partial_\xi^2 V + \frac{3}{R} (\partial_\xi - \partial_\eta) V. \]  

(9)

3. The wave maps equation \( 7e \) reduces to

\[ 4^{+1} \square V = \left( e^{2Z} - 1 \right) + \left( \frac{r}{R} \partial_\eta r + \frac{1}{2} \right) - \left( \frac{r}{R} \partial_\xi r - \frac{1}{2} \right) \frac{V}{r^2} \]

\[ + 2 \partial_\xi V \partial_\eta \log \left( \frac{r}{R} \right) + 2 \partial_\eta V \partial_\xi \log \left( \frac{r}{R} \right) \]

\[ + e^{2Z} \frac{\zeta(u)}{r^2} R^2 V^3 \]  

(10)

where \( 4^{+1} \square \) is the wave operator on \( \mathbb{R}^{4+1} \).
Proof. The proofs of 1. and 2. immediately follow from the definitions. Although we shall work in the \((T, R, \theta)\) coordinates later, the proof of 3. shall be most elegant in double null coordinates. Recall:

\[
\begin{align*}
\partial_\xi &= \frac{1}{2} (\partial_T + \partial_R) \\
\partial_\eta &= \frac{1}{2} (\partial_T - \partial_R)
\end{align*}
\]

Therefore for \(u = RV\) we have,

\[
\begin{align*}
\partial_\xi u &= R \partial_\xi V + \frac{V}{2} \\
\partial_\eta u &= R \partial_\eta V - \frac{V}{2}
\end{align*}
\]

Now consider the quantity

\[
\left( \partial_\eta (r \partial_\xi u) + \partial_\xi (r \partial_\eta u) \right) = 2r \partial_\xi \partial_\eta u + \partial_\xi u \partial_\eta r + \partial_\eta u \partial_\xi r \\
= 2rR \partial_\xi^2 V - r \partial_R V + R (\partial_\xi V \partial_\eta r + \partial_\eta V \partial_\xi r) \\
+ \frac{V}{2} (\partial_\eta r - \partial_\xi r).
\]

Based on our assumptions on the target manifold \((N, h)\) we have,

\[
\begin{align*}
\frac{f_u(u) f(u)}{r^2} &= \frac{u}{r^2} + \frac{\zeta(u) u^3}{r^2} = \frac{R}{r^2} V + \frac{\zeta(u) R^3}{r^2} V^3
\end{align*}
\]

for a smooth function \(\zeta\). Therefore, the equation consecutively transforms as follows

\[
-2r^{-1} \left( 2rR \partial_\xi^2 V - r \partial_R V + R (\partial_\xi V \partial_\eta r + \partial_\eta V \partial_\xi r) + \frac{V}{2} (\partial_\eta r - \partial_\xi r) \right) = e^{2Z} \frac{f_u(u) f(u)}{r^2},
\]
\[-4\partial^2_{\xi \eta} V + \frac{3}{R} \partial_R V = \left( \frac{V}{r^2} + \frac{\zeta(u)}{r^2} R^2 V^3 \right) e^{2Z} + \frac{2}{r} (\partial_\xi V \partial_\eta r + \partial_\eta V \partial_\xi r) + \frac{V}{r^2} (\partial_\eta r - \partial_\xi r) + \frac{1}{R} \partial_R V \]

\[4^{1+1} \Box V = \left( \frac{e^{2Z}}{r} + \frac{1}{R} (\partial_\eta r - \partial_\xi r) \right) \frac{V}{r} + \left( \frac{2}{r} (\partial_\xi V \partial_\eta r + \partial_\eta V \partial_\xi r) + \frac{1}{R} \partial_R V \right) \]

\[4^{1+1} \Box V = \left( \frac{e^{2Z}}{r} + \frac{1}{R} (\partial_\eta r - \partial_\xi r) \right) \frac{V}{r} + 2 \partial_\xi V \partial_\eta \log \left( \frac{r}{R} \right) + 2 \partial_\eta V \partial_\xi \log \left( \frac{r}{R} \right) + e^{2Z} \frac{\zeta(u)}{r^2} R^2 V^3 \]

\[4^{1+1} \Box V = \left( e^{2Z} - 1 \right) + \left( \frac{r}{R} \partial_\eta r + \frac{1}{2} \right) - \left( \frac{r}{R} \partial_\xi r - \frac{1}{2} \right) \right) \frac{V}{r^2} + 2 \partial_\xi V \partial_\eta \log \left( \frac{r}{R} \right) + 2 \partial_\eta V \partial_\xi \log \left( \frac{r}{R} \right) + e^{2Z} \frac{\zeta(u)}{r^2} R^2 V^3. \quad (17) \]

Thus, \([10]\) is a nonlinear wave equation in the Minkowski space \(\mathbb{R}^{4+1}\) which contains a critical power \([1\) for a smooth function \(\zeta\) (cf. flat equivariant wave maps version \([13]\) and \((t,r,\theta)\) coordinate version \([6]\) ).

Without loss of generality, consider the 2+1 splitting of \(M\) such that \(\Sigma_0\) is the \(T = 0\) level set. The unit normal of \(\Sigma_T\) hypersurfaces for the \(\Sigma_T \hookrightarrow M\) embedding is \(N := e^{-Z} \partial_T\), so that \(g(N,N) = -1\).

In order to have well-posed initial value problem for Einstein’s equations, the initial data needs to satisfy the following constraint equations.

\[E(N,N) = T(N,N) \quad (18a)\]

\[E(N,e_i) = T(N,e_i) \quad (18b)\]

on \(\Sigma_0\), for \(i = 1, 2\).

Let us define the following quantities

\[u|_{\Sigma_0} = u_0, \quad \partial_T u|_{\Sigma_0} = u_1, \quad (19a)\]

\[Z|_{\Sigma_0} = Z_0, \quad \partial_T Z|_{\Sigma_0} = Z_1, \quad (19b)\]

\[r|_{\Sigma_0} = r_0, \quad \partial_T r|_{\Sigma_0} = r_1, \quad (19c)\]

\(^1\)In general the equation \(n+1 \Box u = u|u|^{p-1}, u : \mathbb{R}^{n+1} \rightarrow \mathbb{R}\) is critical for \(p = 1 + \frac{4}{n-2}\) and \(n > 2\). This is because for this case the scaling symmetry of the energy matches exactly with that of the equation.
with \( r_1|_\Gamma = 0 \) and \( \partial_R r_0|_\Gamma = 1 \). Typically, the initial constraint equations are represented in terms of the 5-tuple \((\Sigma_0, q_0, K_0, u_0, u_1)\)\(^2\) which is directly related to (19). We are interested in the global behavior of the initial value problem of (1) with initial data \((\Sigma_0, q_0, K_0, u_0, u_1)\). Furthermore, we assume that the initial data is asymptotically flat as defined in [1].

Define the energy as follows

\[
E(t) := \int_{\Sigma_t} \mathbf{T}(\mathcal{N}, \mathcal{N}) \bar{\mu}_q = \int_{\Sigma_t} e \bar{\mu}_q
\]

and let \( E_0 := E(0) \).

As a consequence of the Hardy’s inequality and the aforementioned assumptions on the function \( f \), the following estimates hold

\[
E_0 \geq \| u_1 \|_{L^2(\mathbb{R}^2)} + \| u_0 \|_{\dot{H}^1(\mathbb{R}^2)} \geq \| V_1 \|_{L^2(\mathbb{R}^4)} + \| V_0 \|_{\dot{H}^1(\mathbb{R}^4)}.
\]

(20) (21)

We are now in a position to present the Cauchy problem for the equivariant Einstein-wave map system:

\[
\begin{align*}
\mathbf{E}_{\mu\nu} &= \mathbf{T}_{\mu\nu} \quad \text{on } M \\
\Box_g u &= f(u) f(u) \quad \text{on } M
\end{align*}
\]

(22)

with the regular, compactly supported equivariant initial data set

\((\Sigma_0, q_0, K, u_0, u_1)\)

satisfying the constraint equations on \( \Sigma_0 \). Immediately, we have the following theorem.

**Theorem 1.2.** Let \((\Sigma_0, q_0, K, u_0, u_1)\) be smooth, compactly supported, equivariant initial data satisfying the constraint equations (18), then there exists a regular, equivariant, globally hyperbolic maximal future development \((M, g, u)\) satisfying (22).

Theorem 1.2 is a classic result of Choquet-Bruhat and Geroch [3]. This beautiful, seminal theorem in mathematical general relativity allows us to speak about the future of the initial data but does not shed light on the global structure of \((M, g, u)\). As a consequence of the final result of [1] (cf. Theorem 1.8) the following statement holds

**Theorem 1.3 (Global regularity of equivariant Einstein-wave maps).** Let \( E_0 < \epsilon^2 \) for \( \epsilon \) sufficiently small and let \((M, g, u)\) be the maximal Cauchy development of an asymptotically flat, compactly supported, regular Cauchy data set for the \( 2 + 1 \) equivariant Einstein-wave map problem (22) with target \((N, h)\) satisfying

\[
\int_0^s f(s') ds' \rightarrow \infty \quad \text{for } s \rightarrow \infty.
\]

(23)

Then \((M, g, u)\) is regular and causally geodesically complete.

\(^2\)where \( q_0 \) is the metric of \( \Sigma_0 \) and \( K_0 \) is a symmetric 2-tensor
Actually, as a consequence of the Theorem 5.1 in [1] (also Theorem 1.3.1 in [6]), Theorem 1.8 in [1] also holds without the smallness restriction on the initial energy, with the following additional condition on the target manifold \((N, h)\)

\[f_s(s)f(s)s + f^2(s) > 0 \quad \text{for} \quad s > 0. \tag{24}\]

Theorem 1.8 in [1] carried forward the program initiated in [6] to understand global behavior of the 2+1 wave maps coupled to Einstein’s equations. The motivation to study 2+1 Einstein-wave map system comes from the fact that the system arises naturally in 3+1 vacuum Einstein’s equations with one isometry group (see [6] for a detailed discussion). In the current work we carry the program further by addressing Open Problem 2 listed in [6] concerning the global asymptotic behavior of the 2+1 self-gravitating wave maps. In the general context of the initial value problem of general relativity, the question of global asymptotic behavior is a subtle yet important question. Indeed, a comprehensive understanding of the asymptotic behaviour even in our special case shall be useful in understanding the asymptotics of more general Einstein’s equations.

In precise terms, in the current work we prove that globally regular solutions of two subclasses of the system (1) exhibit scattering as \(T \to \infty\). These two subclasses are classified as Problem I and Problem II below.

**Problem I**

Consider a function \(v\) such that

\[
4+1 \Box v = F(v) \quad \text{on} \quad \mathbb{R}^{4+1}, \quad v_0 = v(0, x) \quad \text{and} \quad v_1 = \partial_T v(0, x) \quad \text{on} \quad \mathbb{R}^4
\]

(25)

with

\[F(v) = \left( e^{2Z} - 1 + \left( \frac{r}{R} \partial_\eta r + \frac{1}{2} \right) - \left( \frac{r}{R} \partial_\xi r - \frac{1}{2} \right) \right) \frac{v}{r^2} + e^{2Z} \frac{R^2}{r^2} v^3 \zeta(Rv)\]

and coupled to the equations (1a) with \(u = Rv\). It may be noted that the wave equation (25) is a partially linearized version of the fully nonlinear wave maps equation (10) where the linearization is applied only to the higher order terms. A special case of (25) is the equation

\[
4+1 \Box v = \left( e^{2Z} - 1 \right) \frac{v}{r^2} + e^{2Z} v^3 \zeta(Rv) \tag{26}\]

which corresponds to the linearization of the equation (7b) (implies \(r \equiv R\) with the boundary conditions on the axis \(\Gamma\)). Let

\[E(v) = \|v_0\|_{H^1(\mathbb{R}^4)} + \|v_1\|_{L^2(\mathbb{R}^4)}, \tag{27}\]

we prove scattering for \(v\) as follows

\footnote{about the trivial solution \(Z \equiv 0, r \equiv R, V \equiv 0\)}
Theorem 1.4. Suppose $E(v) < \epsilon^2$ for $\epsilon$ sufficiently small, then any globally regular solution $v$ of (25) with
\[ e^{2Z} - 1, \quad \left| \frac{R}{r} - 1 \right|, \quad \left| \text{Re}(T, R) \right| \leq E(v) \] (28)
scatters forward in time i.e., converges to a solution of its linearized equation
\[ 4+1 \square v_\infty = 0 \] (29)
in the energy topology as $T \to \infty$.

Equivalently, scattering backwards in time can be proven using time reversal. It should be noted that the assumptions (28) are consistent with the results proven for the fully nonlinear system (1) in [6, 1]. The proof of Theorem 1.4 is based on an argument that the linear part of the equation (25) dominates the nonlinear part in the large. This argument in turn is based on the construction of function spaces $X$ and $Y$ (to be formally defined later), such that $X$ contains a solution to the free wave equation, if $v$ lies in $X$ then the nonlinearity lies in $Y$, and finally that if the nonlinearity lies in $Y$, then $v$ lies in $X$. The result then follows by the contraction mapping principle. Our function space $X$ exploits the endpoint Strichartz estimates, Morawetz estimates, and radial symmetry of the problem. We are able to show that if $v$ lies in this space, then the nonlinearity can be split into a term lying in $\| \cdot \|_{L^1_t L^2_x}$ and a term lying in a space that is dual to the Morawetz estimates. This implies that if the nonlinearity lies in $Y$, then $v$ lies in $X$. The details are schematically illustrated below.

Theorem 1.1. If $v$ is a radial solution to the equation
\[
4+1 \square v = F(v) \quad \text{on } \mathbb{R}^{4+1} \quad \begin{cases} v_0 = v(0, x) \quad & \text{and} \quad v_1 = \partial_T v(0, x) \quad & \text{on } \mathbb{R}^4 \end{cases} \] (30)
then
\[ \| v \|_X \leq \| v_0 \|_{\dot{H}^1(\mathbb{R}^4)} + \| v_1 \|_{L^2(\mathbb{R}^4)} + \| F \|_Y. \] (31)

Theorem 1.1 uses the endpoint Strichartz estimates of Keel and Tao [7] and Morawetz estimates.

Lemma 1.5 (First Morawetz Estimate). Suppose $v$ solves the linear wave equation
\[
4+1 \square v = 0 \quad \text{on } \mathbb{R}^{4+1} \quad \begin{cases} v_0 = v(0, x) \quad & \text{and} \quad v_1 = \partial_T v(0, x) \quad & \text{on } \mathbb{R}^4 \end{cases} \] (32)
then
\[ \int_{\mathbb{R}} \int_{\mathbb{R}^4} \frac{1}{|x|^3} v^2 dx dt \leq \| v_0 \|_{\dot{H}^1(\mathbb{R}^4)} + \| v_1 \|_{L^2(\mathbb{R}^4)} \] (33)

Lemma 1.6 (Second Morawetz Estimate). Suppose $v$ solves the linear wave equation
\[
4+1 \square v = 0 \quad \text{on } \mathbb{R}^{4+1} \quad \begin{cases} v_0 = v(0, x) \quad & \text{and} \quad v_1 = \partial_T v(0, x) \quad & \text{on } \mathbb{R}^4 \end{cases} \] (34)
then for a fixed $\rho > 0$,

$$\left( \sup_{\rho} \frac{1}{\rho^{1/2}} \left\| \nabla v \right\|_{L^{2}_{T,x}(\mathbb{R} \times \{|x| \leq \rho\})} \right) + \left( \sup_{\rho} \frac{1}{\rho^{1/2}} \left\| \partial_{T} v \right\|_{L^{2}_{T,x}(\mathbb{R} \times \{|x| \leq \rho\})} \right) \leq \left\| v_{0} \right\|_{H^{1}(\mathbb{R}^{4})} + \left\| v_{1} \right\|_{L^{2}(\mathbb{R}^{4})}.$$  \hfill (35)

We prove the Morawetz estimates using the vector fields method: If $\mathbf{T}$ is the energy-momentum tensor of the linear wave equation for $v : \mathbb{R}^{4+1} \to \mathbb{R}$, then we construct momenta or ‘currents’

$$J_{\mathbf{X}} = \mathbf{T}(\mathbf{X})$$

for suitable choices of Morawetz multipliers $\mathbf{X} = \mathbf{3}(R) \partial_{R}$. The undesirable bulk terms in the divergence of $J_{\mathbf{X}}$ are corrected using the lower-order momentum

$$J_{\nu}^{\nu}[v] = \kappa v \nabla^{\nu} v - \frac{1}{2} v^{2} \nabla^{\nu} \kappa$$

for suitable choices of $\kappa$.

Equivalent Morawetz estimates can be established for inhomogeneous and nonlinear wave equations following a similar procedure.

Formally, the function spaces $X$ and $Y$ are defined as follows

**Definition 1.7 (Function spaces).** Suppose $\phi(x)$ is a smooth, compactly supported, radial, decreasing function with $\phi(x) = 1$ when $|x| \leq 1$ and $\phi(x)$ is supported on $|x| \leq 2$. Then let $P_{N}$ be the Littlewood - Paley Fourier multiplier such that if $\mathcal{F}$ is a Fourier transform and $f$ is an $L^{1}$ function,

$$\mathcal{F}(P_{N}f)(\xi) = [\phi\left(\frac{\xi}{2}\right) - \phi(\xi)] \hat{f}(\xi),$$ \hfill (36)

then let
\[
\|v\|_{L^2_N}^2 = \sum_N \left\| P_N v \right\|_{L^2_{T,x} (\mathbb{R} \times \mathbb{R}^4)}^2 + \sum_N \left\| |x|^{1/4} P_N v \right\|_{L^2_{T,x} L^6_{x} (\mathbb{R} \times \mathbb{R}^4)}^2 + \sum_N N^2 \left\| P_N v \right\|_{L^\infty_{T,x} L^2_{x} (\mathbb{R} \times \mathbb{R}^4)}^2 + \sum_N \left( \sup_{\rho > 0} \rho^{-1/2} \left\| P_N \partial_T v \right\|_{L^2_{T,x} (\mathbb{R} \times \{|x| \leq \rho\})} \right)^2 + \sum_N \left( \sup_{\rho > 0} \rho^{-1/2} \left\| P_N \nabla_x v \right\|_{L^2_{T,x} (\mathbb{R} \times \{|x| \leq \rho\})} \right)^2 + \sum_N N^2 \left( \sup_{\rho > 0} \rho^{-1/2} \left\| P_N v \right\|_{L^2_{T,x} (\mathbb{R} \times \{|x| \leq \rho\})} \right)^2 + \sum_N \left\| |x|^{-3/2} P_N v \right\|_{L^2_{T,x} (\mathbb{R} \times \mathbb{R}^4)}^2 + \sum_N N^{-2} \left\| P_N \partial_T v \right\|_{L^2_{T,x} L^6_{x} (\mathbb{R} \times \mathbb{R}^4)}^2.
\] (37)

Suppose \( F = F_1 + F_2 \)

\[
\|F\|_{Y}^2 := \inf_{F_1, F_2 = F} \left\{ \sum_N \left( \sum_j 2^{j/2} \| P_N F_j \|_{L^2_{T,x} (\mathbb{R} \times \{|x| \leq 2^j \}}) \right)^2 \right. \] (38)

Finally, we prove the following theorem which controls the nonlinearity. The proof uses the structure of the nonlinearity in [25] in a conveniently modified form using the coupled equations [14].

**Theorem 1.8.** The nonlinear wave equation

\[
4^+1 \Box v = F(v) \quad \text{on } \mathbb{R}^{4+1} \\
v_0 = v(0, x) \quad \text{and} \quad v_1 = \partial_T v(0, x) \quad \text{on } \mathbb{R}^4
\] (38)

with \( F(v) \) as in [25], has a solution with \( \|v\|_{L^2_{T,x} L^6_{x}} < \infty \) for \( E(v) < \epsilon^2, \epsilon \) sufficiently small.

**Problem II**

Consider a function \( \tilde{v} \) such that

\[
4^+1 \Box \tilde{v} = \tilde{F}(\tilde{v}) \quad \text{on } \mathbb{R}^{4+1} \\
\tilde{v}_0 = \tilde{v}(0, x) \quad \text{and} \quad \tilde{v}_1 = \partial_T \tilde{v}(0, x) \quad \text{on } \mathbb{R}^4
\] (39)
where
\[
\tilde{F} (\tilde{v}) = \left( \frac{1}{r} \partial_\eta r + \frac{1}{2R} \right) \partial_\xi \tilde{v} + \left( \frac{1}{r} \partial_\xi r - \frac{1}{2R} \right) \partial_\eta \tilde{v} \\
+ \left( \left( \frac{r}{R} \partial_\eta r + \frac{1}{2} \right) - \left( \frac{r}{R} \partial_\xi r - \frac{1}{2} \right) \right) \frac{\tilde{v}}{r^2} \\
+ \frac{R^2}{r^2} \tilde{v}^3 \zeta (R\tilde{v})
\]  
(40)

and \(\tilde{v}\) is coupled to Einstein’s equations (11) with \(u = R\tilde{v}\). It may be noted again that the wave equation (39) is the original wave maps equation (10) with (7d) linearized (implies \(Z \equiv 0\) due to the boundary conditions on the axis \(\Gamma\)). Define the energy,
\[
\tilde{E}(\tilde{v}) = \| \tilde{v}_0 \|_{H^1 (\mathbb{R}^4)} + \| \tilde{v}_1 \|_{L^2 (\mathbb{R}^4)} + \frac{1}{2} \| \tilde{v}_0 \|_{L^4 (\mathbb{R}^4)} ,
\]  
(41)

We prove scattering for (39) as follows

**Theorem 1.9.** Suppose \(\tilde{E}(\tilde{v}) < \epsilon^2\) for \(\epsilon\) sufficiently small, then any globally regular solution \(\tilde{v}\) of (25) with
\[
| R \frac{r}{r} - 1 |, \quad | R \tilde{v}(T,R) | \leq \tilde{E}(\tilde{v})
\]  
(42)

scatters forward in time i.e., converges to a solution of its linearized equation
\[
^{4+1} \Box \tilde{v}_\infty = 0
\]  
(43)

in the energy topology as \(T \to \infty\).

The proof is based on the following (nonlinear) Morawetz estimate for small data

**Lemma 1.10.** For any globally regular solution \(\tilde{v}\) of (39)
\[
\int_{\mathbb{R}^{4+1}} \frac{\tilde{v}^2}{|x|^4} \tilde{\mu}_\delta \leq \tilde{E}(\tilde{v}).
\]  
(44)

This estimate directly implies
\[
\int \int | \partial_\eta r | | \partial_\xi \tilde{v} | d\xi d\eta < \infty,
\]  
(45)

which then implies
\[
\int \sup_\xi | \partial_\eta r - \frac{1}{2} | d\eta < \infty, \quad \text{and} \quad \int \sup_\eta | \partial_\xi r + \frac{1}{2} | d\xi < \infty.
\]  
(46)

This fact implies that the contribution of the nonlinearity \(\tilde{F}(\tilde{v})\) at large times is quite small in the energy norm, implying scattering.

We would like to remark that the wave map field \(u\) is the crucial field in the system (11) that drives all important geometric aspects of the evolution
of the system and also the corresponding 3+1 Einstein’s equations. For instance, the field $u$ was the central object of study in both non-concentration and small data arguments. Furthermore, in principle the wave map field $u$ also represents the nonlinear asymptotic effects of the system (e.g. nonlinear memory effect). In this regard, Theorem 1.8 implies that the ‘soul’ of the system is asymptotically linear in linear approximation of either $r$ or $Z$.

Wave maps are natural geometric generalizations of harmonic maps on one hand and linear wave equations on the other, and have been popular in the analysis and PDE community due to the nice structure and the applications in several models in mathematical physics. Thus, there exist several deep and diverse results in the literature, focusing mainly on $\mathbb{R}^{n+1}$. In the following we discuss a few of these results, we refer the reader to [17, 11, 19] for instance, for detailed surveys on the study of wave maps.

Christodoulou, Tahvildar-Zadeh and Shatah published a pioneering series of works in early 90s on equivariant and spherically symmetric wave maps on $\mathbb{R}^{2+1}$ in which they proved global existence and asymptotics for these wave maps [5, 4, 13, 12]. Subsequently, it was observed in [2] that spherically symmetric wave maps $U : \mathbb{R}^{2+1} \rightarrow \mathbb{H}^2$ can be correlated to $G_2$-symmetric 3+1 dimensional spacetimes, which eventually led to a proof of strong cosmic censorship for these spacetimes. In this context, we would like to emphasize that the nonzero homotopy degree in our case prevents us from reducing our system to flat space wave maps like in [2]. Thus, we are forced to deal with the coupling with Einstein’s equations. A detailed discussion of the occurrence of 2+1 wave maps in 3+1 spacetimes in general relativity and further sub-cases can be found in [6].

Global existence for general wave maps was studied by Tao through a series of works [18]. Global existence for wave maps $U : \mathbb{R}^{2+1} \rightarrow \mathbb{H}^2$ for small data was proved in [9]. Global existence and scattering for semilinear wave equations with power nonlinearity was proved in the classic paper of Kenig and Merle [8]. Global existence and scattering for wave maps $\mathbb{R}^{n+1} \rightarrow M, n = 2, 3$ was proved in [20]. Concentration compactness for these wave maps was established in [10]. Likewise, large data wave maps for more general targets were studied in [15, 16].

**Notation**

We shall use the Einstein’s summation convention throughout. Inconsequential constants in the estimates are scaled to 1 to avoid cluttering up the notation. For a scalar function like $v$, we shall use the notation $\partial_T v$ and $v_T$ equivalently for partial derivatives.

## 2. Scattering for Problem I

### 2.1. Morawetz Estimates

Firstly, let us start with the following linear wave equation
\begin{align}
4+1 \Box v &= 0 \quad \text{on } \mathbb{R}^{4+1} \\
v_0 = v(0, x) \quad \text{and} \quad v_1 = \partial_T v(0, x) \quad \text{on } \mathbb{R}^4
\end{align}
\tag{47}

Denote by $\mathbf{T}$ the energy momentum tensor of $v$

$$
\mathbf{T}_{\mu\nu}(v) := \nabla_{\mu} v \nabla_{\nu} v - \frac{1}{2} \tilde{g}_{\mu\nu} \nabla^{\sigma} v \nabla_{\sigma} v,
$$
\tag{48}

where $\tilde{g}$ is the metric on the Minkowski space $\mathbb{R}^{4+1}$. If we define $\mathbf{\mathbf{\mathcal{L}}} := \frac{1}{2} \nabla^{\sigma} v \nabla_{\sigma} v$,

$$
\mathbf{T}_{\mu\nu}(v) = \nabla_{\mu} v \nabla_{\nu} v - \tilde{g}_{\mu\nu} \mathbf{\mathbf{\mathcal{L}}}.
$$
\tag{49}

We shall prove the desired Morawetz estimates for (47) using the vector fields method. Recall that the vector fields method is based on finding suitable spacetime multiplier vectors $\mathbf{x}$ such that the corresponding momentum or ‘current’

$$
J_{\mu}^{\nu} = \mathbf{T}_{\mu}^{\nu} \mathbf{x}^\mu
$$

has desirable properties in view of the divergence theorem. The divergence of $J_{\mathbf{x}}$ is given by

$$
\nabla_{\nu} J_{\mathbf{x}}^{\nu} = \frac{1}{2} \mathbf{T}^{\mu\nu} (x) \pi_{\mu\nu},
$$
\tag{50}

where we have used the fact that the energy-momentum tensor is divergence free $\nabla_{\nu} \mathbf{T}_{\mu}^{\nu} = 0$ which is a consequence of the equation (47). The tensor $(x) \pi_{\mu\nu}$ is called the deformation tensor, formally defined as

$$(x) \pi_{\mu\nu} := L_{\mathbf{x}} \tilde{g}_{\mu\nu}$$

where $L_{\mathbf{x}}$ is the Lie derivative in the direction of $\mathbf{x}$. For the sake of brevity, let us further define $\tilde{e} := \mathbf{T}(\partial_T, \partial_T)$ and $\tilde{m} := \mathbf{T}(\partial_T, \partial_R)$.

Firstly note the multiplier $\mathbf{x} = \partial_T$ has the current

$$
J_{\partial_T} = -\tilde{e} \partial_T + \tilde{m} \partial_R,
$$
\tag{51}

which is divergence-free in view of the fact that $\partial_T$ is a Killing vector of $\tilde{g}$, so the ‘deformation’ is zero

$$(\partial_T) \pi_{\mu\nu} = 0$$

i.e.,

$$
\nabla_{\nu} J_{\mathbf{x}}^{\nu} = \frac{1}{2} (\partial_T) \pi_{\mu\nu} \mathbf{T}^{\mu\nu} = 0.
$$
\tag{52}

If we use this fact on the domain enclosed by two Cauchy surfaces $\Sigma_\tau$ and $\Sigma_s$, $s > \tau$ we have from the divergence theorem

$$
0 = \int \nabla_{\nu} J_{\mathbf{x}}^{\nu} = \int_{\Sigma_\tau} (\partial_T, J_{\partial_T}) \tilde{\mu} \tilde{q} - \int_{\Sigma_s} (\partial_T, J_{\partial_T}) \tilde{\mu} \tilde{q}.
$$
\tag{53}

Thus we have deduced the conservation law formally, if we impose $s = T$ and $\tau = 0$
\[ \|v(T)\|_{H^1(\mathbb{R}^4)} + \|\partial_T v(T)\|_{L^2(\mathbb{R}^4)} = \|v_0\|_{H^1(\mathbb{R}^4)} + \|v_1\|_{L^2(\mathbb{R}^4)}. \] (54)

Now consider a Morawetz multiplier vector \( \mathcal{X} := \mathcal{F}(R) \partial_R \) so that the corresponding momentum is given by

\[ J_\mathcal{X} = \mathcal{T}(\mathcal{X}) = \mathcal{F}(R)(-\tilde{m}\partial_T + \tilde{e}\partial_R) \] (55)

and its divergence

\[ \nabla_\nu J^\nu_\mathcal{X} = \frac{1}{2} \tilde{T}^{\mu\nu}(\mathcal{X}) \pi_{\mu\nu}, \] (56)

where the non-zero terms of deformation tensor \( \pi \) are given by

\[ \begin{align*}
(\mathcal{X})_{\pi_{RR}} &= 2g_{RR}\partial_R \mathcal{F}(R), \\
(\mathcal{X})_{\pi_{\theta\theta}} &= 2Rg_{\theta\theta}\mathcal{F}(R), \\
(\mathcal{X})_{\pi_{\phi\phi}} &= 2Rg_{\phi\phi}\mathcal{F}(R), \\
(\mathcal{X})_{\pi_{\psi\psi}} &= 2Rg_{\psi\psi}\mathcal{F}(R).
\end{align*} \]

Consequently a calculation shows that (56) can be represented as

\[ \nabla_\nu J^\nu_\mathcal{X} = \left(-\frac{6\mathcal{F}(R)}{R}\mathcal{L} + \tilde{e}\partial_R \mathcal{F}(R) \right) \] (57)

Now define the following lower-order momentum vector

\[ J^\nu_1[v] := \kappa v \nabla^\nu v - \frac{1}{2} v^2 \nabla^\nu \kappa. \] (58)

Its divergence is

\[ \begin{align*}
\nabla_\nu J^\nu_1 &= \kappa v \Box v + \kappa \nabla^\nu v \nabla_\nu v + v \nabla^\nu v \nabla_\nu \kappa - \left(\Box \kappa\right) \frac{v^2}{2} - v \nabla^\nu \kappa \nabla_\nu v \\
&= \kappa \nabla^\nu v \nabla_\nu v - \left(\Box \kappa\right) \frac{v^2}{2} \\
&= 2\kappa \mathcal{L} - \left(\Box \kappa\right) \frac{v^2}{2}.
\end{align*} \] (59)

It may noted that the momentum or ‘current’ \( J_1 \) has been constructed to neutralize the undesirable terms in the divergence formula (57) while the price to pay are the lower order terms in the spacetime integrals and boundary terms which can be handled, for instance, using the Hardy’s inequality. We shall precisely do this in the following.

**Lemma 2.1 (First Morawetz Estimate).** Suppose \( v \) solves the linear wave equation

\[ \begin{align*}
\Box^{4+1} v &= 0 \quad \text{on } \mathbb{R}^{4+1} \\
v_0 &= v(0, x) \quad \text{and } v_1 = \partial_T v(0, x) \quad \text{on } \mathbb{R}^4
\end{align*} \] (60)

then

\[ \int_R \int_{\mathbb{R}^4} \frac{1}{|x|^3} v^2 dx dt \leq \|v_0\|_{H^1(\mathbb{R}^4)} + \|v_1\|_{L^2(\mathbb{R}^4)}. \] (61)
Proof. We shall prove the theorem for a radial function \( v \), the proof is essentially the same in the general case. Consider the choice of \( \kappa = \frac{1}{|x|} \), then
\[
\square \kappa = \partial^2_R \kappa + \frac{3}{R} \partial_R \kappa = -\frac{1}{R^3}.
\]  
(62)
Now consider the case when \( \mathfrak{F}(R) = \frac{1}{3} \), so that \( \mathfrak{X}_1 = \frac{1}{3} \partial_R \) then
\[
J_{\mathfrak{X}_1} = \frac{1}{3} (-\tilde{m} \partial_T + \tilde{e} \partial_R)
\]  
(63)
and
\[
J_1 = -\left( \frac{1}{R} v \partial_T v \right) \partial_T + \left( \frac{1}{R} v \partial_R v + \frac{v^2}{R^2} \right) \partial_R.
\]  
(64)
The divergences of \( J_{\mathfrak{X}_1} \) and \( J_1 \) are given by
\[
\nabla_\nu J^\nu_{\mathfrak{X}_1} = -\frac{2}{R} \tilde{\kappa}
\]  
(65)
\[
\nabla_\nu J^\nu_1 = \frac{2}{R} \tilde{\kappa} + \frac{v^2}{2R^3}
\]  
(66)
Now consider the sum vector \( J^\nu_S := J^\nu_{\mathfrak{X}_1} + J^\nu_1 \), then it follows that
\[
\nabla_\nu J^\nu_S = \nabla_\nu J^\nu_{\mathfrak{X}_1} + \nabla_\nu J^\nu_1
\]  
(67)
\[
= \frac{1}{2} \frac{v^2}{R^3}
\]  
(68)
Let us now apply the Stokes’ theorem between \( \hat{\Sigma}_0 \) and \( \hat{\Sigma}_T \) Cauchy surfaces,
\[
\frac{1}{2} \int \frac{v^2}{R^3} \hat{\mu}_g = \int \nabla_\nu J^\nu_S \hat{\mu}_g = \int_{\hat{\Sigma}_0} \langle \partial_T, J_S \rangle \hat{\mu}_g - \int_{\hat{\Sigma}_T} \langle \partial_T, J_S \rangle \hat{\mu}_q
\]  
(69)
Now let us calculate the boundary terms
\[
\tilde{g}_{\mu\nu} J^\mu_S (\partial_T)^\nu = \tilde{g}_{\mu\nu} J^\mu_{\mathfrak{X}} (\partial_T)^\nu + \tilde{g}_{\mu\nu} J^\mu_1 (\partial_T)^\nu
\]  
(70)
Note that
\[
\tilde{g}_{\mu\nu} J^\mu_{\mathfrak{X}} (\partial_T)^\nu = \tilde{g}_{TT} J^T_{\mathfrak{X}} (\partial_T)^T = \mathfrak{F}(R) \tilde{m}
\]  
(71)
and
\[
\tilde{g}_{\mu\nu} J^\mu_1 (\partial_T)^\nu = \tilde{g}_{TT} J^T_1 (\partial_T)^T = -\frac{1}{|x|} v \partial_T v.
\]  
(72)
From the dominant energy condition and Hardy’s inequality it follows that
\[
\int_{[0,T]} \int_{\mathbb{R}^4} \frac{1}{|x|^3} |v|^2 \hat{\mu}_g \leq \|v_0\|_{H^1(\mathbb{R}^4)} + \|v_1\|_{L^2(\mathbb{R}^4)}.
\]  
(73)
As the right is independent of $T$, taking the limit $T \to \infty$ and time reversal, it follows that

$$\int_{\mathbb{R}} \int_{\mathbb{R}^4} \frac{1}{|x|^3} |v|^2 \tilde{\mu}_{\tilde{g}} \leq \|v_0\|_{H^1(\mathbb{R}^4)} + \|v_1\|_{L^2(\mathbb{R}^4)}. \quad (74)$$

□

**Lemma 2.2 (Second Morawetz Estimate).** Suppose $v$ solves the linear wave equation

$$\begin{align*}
4 + 1 \Box v &= 0 & \text{on } \mathbb{R}^{4+1} \\
v_0 &= v(0, x) & \text{and } v_1 = \partial_T v(0, x) & \text{on } \mathbb{R}^4
\end{align*} \quad (75)$$

then for a fixed $\rho > 0$,

$$\begin{align*}
&\left( \sup_{\rho} \frac{1}{\rho^{1/2}} \| \nabla v \|_{L^2_{T,x}(\mathbb{R} \times \{ |x| \leq \rho \})} \right) + \left( \sup_{\rho} \frac{1}{\rho^{1/2}} \| \partial_T v \|_{L^2_{T,x}(\mathbb{R} \times \{ |x| \leq \rho \})} \right) \\
&\leq \|v_0\|_{H^1(\mathbb{R}^4)} + \|v_1\|_{L^2(\mathbb{R}^4)}. \quad (76)
\end{align*}$$

**Proof:** Let $\chi \in C_0^\infty(\mathbb{R}^4)$ be a positive, radially symmetric function such that $\chi(x) = 1$, $|x| \leq 1$, $\chi(x) = 1$ for $|x| > 2$, so that

$$\psi(R) := \frac{d}{dR} (R \cdot \chi(R)) \geq 0, \quad \forall R \in (0, \infty). \quad (77)$$

Notice that $\psi(R) \geq 0$ is supported on $R \leq 2$ and $\psi(R) = 1$ for $R \leq 1$.

Consider the Morawetz multiplier $\chi_2 = \tilde{\chi}(\rho) \partial_R$ for $\tilde{\chi}(R) = \frac{1}{3} R \cdot \chi(\frac{x}{\rho})$. Then

$$\nabla \nu J^\nu_{\chi_2} = \left( -2 \chi \left( \frac{x}{\rho} \right) \mathcal{L} + \psi \left( \frac{x}{\rho} \right) \tilde{e} \right) \quad (79)$$

Consider the lower order momentum

$$J^\nu_{\kappa_2}[v] := \kappa_2 v \nabla^\nu v - v^2 \nabla^\nu \kappa_2 \quad (80)$$

with $\kappa_2 = \chi(\frac{x}{\rho})$. Consequently,

$$\nabla \nu J^\nu_{\kappa_2} = \kappa \nabla^\nu v \nabla^\nu v - \left( \Box \kappa \right) \frac{v^2}{2} \quad (81)$$

$$= 2 \chi \left( \frac{x}{\rho} \right) \mathcal{L} - \Delta \chi \left( \frac{x}{\rho} \right) \frac{v^2}{2}. \quad (82)$$

The divergence of the sum is then

$$\nabla \nu J^\nu_{\chi_2} = \nabla \nu J^\nu_{\chi_2} + \nabla \nu J^\nu_{\kappa_2} \quad (83)$$

$$= \psi \left( \frac{x}{\rho} \right) \tilde{e} - \Delta \chi \left( \frac{x}{\rho} \right) \frac{v^2}{2}. \quad (84)$$

Let us now use the divergence theorem between two Cauchy surfaces

$$\int \nabla \nu J^\nu_{\chi_2} \tilde{\mu}_{\tilde{g}} = \int_{\Sigma_0} \left\langle \partial_T, J_{\chi_2} \right\rangle \tilde{\mu}_{\tilde{g}} - \int_{\Sigma_T} \left\langle \partial_T, J_{\chi_2} \right\rangle \tilde{\mu}_{\tilde{g}} \quad (85)$$
Moreover,
\[
\int \nabla \nu J^\nu_2 \bar{\mu}_g \leq \frac{1}{2} \int \psi \left( \frac{R}{\rho} \right) (\partial_T v)^2 \bar{\mu}_g + \frac{1}{2} \int \psi \left( \frac{R}{\rho} \right) |\nabla_x v|^2 \bar{\mu}_g \tag{86}
\]
\[
+ c \int_{\mathbb{R}} \int_{|x| > \rho} \rho \frac{1}{|x|^3} |v|^2 \bar{\mu}_g. \tag{87}
\]
By previous lemma it follows that
\[
\int_{\mathbb{R}} \int_{|x| > \rho} \rho \frac{1}{|x|^3} |v|^2 \bar{\mu}_g \leq \rho \left( \| v_0 \|_{\dot{H}^1(\mathbb{R})} + \| v_1 \|_{L^2(\mathbb{R})} \right) \tag{88}
\]
By the Hardy's theorem and the dominant energy condition, the boundary terms can be estimated by the initial energy. Therefore,
\[
\int_{\mathbb{R}} \int_{|x| < \rho} (\partial_T v)^2 + |\nabla_x v|^2 \bar{\mu}_g \leq \rho \left( \| v_0 \|_{\dot{H}^1(\mathbb{R})} + \| v_1 \|_{L^2(\mathbb{R})} \right). \tag{89}
\]
The result of the theorem now follows.

**Theorem 2.3.** Suppose that \( v \) is a solution to the inhomogeneous wave equation
\[
\begin{align*}
4+1 \Box v &= F \\
0 = v(0, x) \quad \text{and} \quad v_1 = \partial_T v(0, x) & \quad \text{on } \mathbb{R}^{4+1}
\end{align*}
\]  
Then
\[
\| v \|_{L^\infty_t \dot{H}^1(\mathbb{R} \times \mathbb{R}^4)} + \| v_t \|_{L^\infty_t L^2_x(\mathbb{R} \times \mathbb{R}^4)} \leq \| v_0 \|_{\dot{H}^1(\mathbb{R})} + \| v_1 \|_{L^2(\mathbb{R})} + \left( \sum_j 2^{j/2} \| F \|_{L^2_{t,x}(\mathbb{R} \times \{ x : 2^j \leq |x| \leq 2^{j+1} \})} \right), \tag{90}
\]
and
\[
\| |x|^{-3/2} v \|_{L^2_{t,x}(\mathbb{R} \times \mathbb{R}^4)} + \left( \sup_{\rho > 0} \rho^{-1/2} \| \nabla v \|_{L^2_{t,x}(\mathbb{R} \times \{ x : |x| \leq \rho \})} \right) \leq \| v_0 \|_{\dot{H}^1(\mathbb{R})} + \| v_1 \|_{L^2(\mathbb{R})} + \left( \sum_j 2^{j/2} \| F \|_{L^2_{t,x}(\mathbb{R} \times \{ x : 2^j \leq |x| \leq 2^{j+1} \})} \right), \tag{91}
\]
Proof: We start with (90), which is the dual of (16). If \( f \in L^2(\mathbb{R}^4) \) then
\begin{align*}
\langle f, \nabla \int_0^t \frac{\sin((t-\tau)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(\tau) d\tau \rangle \\
= \int_0^t \langle \nabla \frac{\sin((t-\tau)\sqrt{-\Delta})}{\sqrt{-\Delta}} f, F(\tau) \rangle d\tau \\
\leq \left( \sup_j 2^{-j/2} \| \nabla \frac{\sin((t-\tau)\sqrt{-\Delta})}{\sqrt{-\Delta}} f \|_{L_t^2_x(\mathbb{R} \times \{ x : 2^j \leq |x| \leq 2^{j+1} \})} \right) \\
\cdot \left( \sum_j 2^{j/2} \| F \|_{L_t^2_x(\mathbb{R} \times \{ x : 2^j \leq |x| \leq 2^{j+1} \})} \right) \\
\leq \| f \|_{L^2(\mathbb{R}^4)} \left( \sum_j 2^{j/2} \| F \|_{L_t^2_x(\mathbb{R} \times \{ x : 2^j \leq |x| \leq 2^{j+1} \})} \right). \tag{93}
\end{align*}

An identical computation also proves

\begin{align*}
\langle f, \partial_t \int_0^t \frac{\sin((t-\tau)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(\tau) d\tau \rangle \\
= \langle f, \int_0^t \cos((t-\tau)\sqrt{-\Delta}) F(\tau) d\tau \rangle \\
= \int \langle \int_t^\infty \cos((t-\tau)\sqrt{-\Delta}) f dt, F(\tau) \rangle d\tau \\
\leq \| f \|_{L^2(\mathbb{R}^4)} \left( \sum_j 2^{j/2} \| F \|_{L_t^2_x(\mathbb{R} \times \{ x : 2^j \leq |x| \leq 2^{j+1} \})} \right). \tag{94}
\end{align*}

(92) is proved in a similar way as on the Morawetz estimates in Lemmas 2.1 and 2.2 but adjusting the identities and estimates for \( \Box v = F \).

\begin{align*}
\int \frac{1}{|x|^3} |v(t, x)|^2 dx dt \leq \| v_0 \|_{H^1(\mathbb{R}^4)}^2 + \| v_1 \|_{L^2(\mathbb{R}^4)}^2 + \int |\nabla v(t, x)||F(t, x)| dx dt \\
+ \int \frac{1}{|x|} |v(t, x)||F(t, x)| dx dt \tag{95}
\end{align*}

by (91).
\[
\int \int \frac{1}{|x|^3} |v(t, x)|^2 dx dt \leq \|v_0\|^2_{H^1(\mathbb{R}^4)} + \|v_1\|^2_{L^2_x(\mathbb{R}^4)} \\
+ \left( \int \int \frac{1}{|x|^3} |v(t, x)|^2 dx dt \right)^{1/2} \cdot \left( \sum_j 2^{j/2} \|F\|_{L^2_{t,x}(\mathbb{R} \times \{x: 2^j \leq |x| \leq 2^{j+1}\})} \right) \\
+ \left( \sup_j 2^{-j/2} \|\nabla v\|_{L^2_{t,x}(\mathbb{R} \times \{x: 2^j \leq |x| \leq 2^{j+1}\})} \right) \cdot \left( \sum_j 2^{j/2} \|F\|_{L^2_{t,x}(\mathbb{R} \times \{x: 2^j \leq |x| \leq 2^{j+1}\})} \right) \cdot \left( \int \int \frac{1}{|x|^3} |u(t, x)|^2 dx dt \right)^{1/2} \cdot \left( \sum_j 2^{j/2} \|F\|_{L^2_{t,x}(\mathbb{R} \times \{x: 2^j \leq |x| \leq 2^{j+1}\})} \right).
\]

(96)

As in Lemma 2.2, again this time using \( \Box v = F \),

\[
\frac{1}{\rho} \int \int \chi(\frac{x}{\rho}) [v_t(t, x)^2 + |\nabla v(t, x)|^2] dx dt + C \int \int \frac{1}{|x|^3} |v(t, x)|^2 dx dt \\
\leq \|v_0\|^2_{H^1(\mathbb{R}^4)} + \|v_1\|^2_{L^2_x(\mathbb{R}^4)} \\
+ \int |\nabla v(t, x)||F(t, x)| dx + \int \frac{1}{|x|} |v(t, x)||F(t, x)| dx.
\]

(98)

where \( \chi(x) \) is as in Lemma 2.2. Consequently,

\[
\frac{1}{\rho} \int \int \int_{|x| \leq \rho} [|\nabla v(t, x)|^2 + |v_t(t, x)|^2] dx dt \\
\leq \|v_0\|^2_{H^1(\mathbb{R}^4)} + \|v_1\|^2_{L^2_x(\mathbb{R}^4)} + \int \int \frac{1}{|x|^3} |v(t, x)|^2 dx dt \\
+ \left( \sum_j 2^{j/2} \|F\|_{L^2_{t,x}(\mathbb{R} \times \{x: 2^j \leq |x| \leq 2^{j+1}\})} \right)^2 \\
+ \left( \sup_j 2^{-j/2} \|\nabla v\|_{L^2_{t,x}(\mathbb{R} \times \{x: 2^j \leq |x| \leq 2^{j+1}\})} \right) \cdot \left( \sum_j 2^{j/2} \|F\|_{L^2_{t,x}(\mathbb{R} \times \{x: 2^j \leq |x| \leq 2^{j+1}\})} \right) \\
+ \left( \int \int \frac{1}{|x|^3} |u(t, x)|^2 dx dt \right)^{1/2} \cdot \left( \sum_j 2^{j/2} \|F\|_{L^2_{t,x}(\mathbb{R} \times \{x: 2^j \leq |x| \leq 2^{j+1}\})} \right). \]

(99)
Combining (96) and (99) proves (92).

2.2. Strichartz Estimates

Theorem 2.4 (Endpoint Strichartz estimate). Suppose that $v$ solves the inhomogeneous wave equation

$$4^+ \Box v = F_1 + F_2 \quad \text{on } \mathbb{R}^{4+1}$$

and $v_0 = v(0, x) \in \dot{H}^1(\mathbb{R}^4)$ and $v_1 = \partial_T v(0, x) \in L^2(\mathbb{R}^4)$.

Then,

$$\|v\|_{L_t^2 L_x^8(\mathbb{R} \times \mathbb{R}^4)} \leq \|v_0\|_{\dot{H}^1(\mathbb{R}^4)} + \|v_1\|_{L^2(\mathbb{R}^4)} + \|\nabla F_1\|_{L_t^3 L_x^{8/7}(\mathbb{R} \times \mathbb{R}^4)} + \|F_2\|_{L_t^4 L_x^2(\mathbb{R} \times \mathbb{R}^4)}. \quad (101)$$

Proof: See Keel and Tao [7]. □

Theorem 2.5 (Radially symmetric Strichartz estimate). Suppose $v$ solves the wave equation

$$4^+ \Box v = 0 \quad \text{on } \mathbb{R}^{4+1}$$

with $v_0$ and $v_1$ radially symmetric. Then

$$\left\| |x|^{1/2} \right\|_{L_t^2 L_x^\infty(\mathbb{R} \times \mathbb{R}^4)} \leq \|v_0\|_{\dot{H}^1(\mathbb{R}^4)} + \|v_1\|_{L^2(\mathbb{R}^4)}. \quad (103)$$

Proof: To prove this for $|x| >> T$ we only need to use Hardy’s inequality, finite propagation speed, and the Sobolev embedding theorem. Suppose $|x| \geq 32T$ and make a partition of unity.

Let $\phi \in C_0^\infty(\mathbb{R}^4)$ be a radial, decreasing function, with $\phi(x) = 1$ for $|x| \leq 1$ and $\phi(x)$ is supported on $|x| \leq 2$. Then let

$$\chi(x) = \phi\left(\frac{x}{2}\right) - \phi(x). \quad (104)$$

If $x \neq 0$,

$$1 = \sum_{j \in \mathbb{Z}} \chi(2^j x), \quad (105)$$

and for each $k \in \mathbb{Z}$ let

$$\chi_k(x) = \chi(2^{-k} x),$$

$$\tilde{\chi}_k(x) = \chi(2^{-k+2} x) + \chi(2^{-k+1} x) + \chi(2^{-k} x) + \chi(2^{-k-1} x) + \chi(2^{-k-2} x). \quad (106)$$

Then by finite propagation speed, for $0 \leq T \leq 2^{k-4}$ and $2^k \leq |x| \leq 2^{k+1}$,
\[
\chi_k(x)v(T, x) = \cos(T\sqrt{-\Delta})\tilde{\chi}_k(x)v_0(x) + \frac{\sin(T\sqrt{-\Delta})}{\sqrt{-\Delta}}\tilde{\chi}_k(x)v_1(x). \tag{107}
\]

Then by the radial Sobolev embedding theorem and conservation of energy,

\[
\left\| |x|\chi_k(x)v(T, x) \right\|_{L^p_x L^\infty_r([0, 2^{k-1}] \times \{x: 2^k \leq |x| \leq 2^{k+1}\})} \leq \chi_k v_0 \left\| \tilde{\chi}_k v_1 \right\|_{\dot{H}^1(\mathbb{R}^4)} \tag{108}
\]

and by Hölder’s inequality

\[
\left\| |x|^{1/2} v(T, x) \right\|_{L^p_x L^\infty_r([0, 2^{k-1}] \times \{x: 2^k \leq |x| \leq 2^{k+1}\})} \leq \sum_k 2^k \left\| |x|^{1/2} \chi_k(x)v(T, x) \right\|^2_{L^\infty_r} \tag{109}
\]

Then by Hardy’s inequality and (107),

\[
\left\| |x|^{1/2} v(T, x) \right\|^2_{L^p_x L^\infty_r([0, 2^{k-1}] \times \{x: |x| \geq 32T\})} \leq \sum_k 2^k \left\| |x|^{1/2} \chi_k(x)v(T, x) \right\|^2_{L^\infty_r} \tag{110}
\]

Remark: This estimate is not necessarily sharp in this region.

Now consider \( |x| \leq 32T \) and suppose \( v_0 = 0 \) and \( v_1 = g \in L^2(\mathbb{R}^4) \). Without loss of generality suppose \( T > 0 \). Then by the fundamental solution to the wave equation (see for example Sogge [14]),

\[
u(T, x) = 3T \int_{|y| < 1} \frac{g(x + Ty)}{(1 - |y|^2)^{1/2}} dy + T^2 \int_{|y| < 1} \frac{y \cdot (\nabla g)(x + Ty)}{(1 - |y|^2)^{1/2}} dy. \tag{113} \]
Suppose $\omega$ is the surface area of the unit sphere $S^3 \subset \mathbb{R}^4$. If $g$ is radial then $v$ is radial, so

$$v(T, x) = \frac{1}{\omega|x|^3} \int_{\partial B(0,|x|)} v(T, z)\,d\sigma(z). \tag{114}$$

g radial implies that $g(y_1, y_2, y_3, y_4) = g(|y_1|, (y_2^2 + y_3^2 + y_4^2)^{1/2})$. For any $y \in \mathbb{R}^4$, $|y| < 1$ let $R$ be the rotation matrix that rotates $y \in \mathbb{R}^4$ to the vector $|y|e_1$, where $e_1 = (1, 0, 0, 0)$. $g$ radial implies

$$\begin{align*}
&g(x + Ty)\,d\sigma(x)\,dy \\
&= \frac{3T}{\omega|x|^3} \int_{|y|<1} \frac{1}{(1 - |y|^2)^{1/2}} \int_{\partial B(0,|x|)} g(R(x + Ty))\,d\sigma(x)\,dy \\
&= 3T \int_0^1 \frac{y_1^3}{(1 - y_1^2)^{1/2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g(Ty_1 + |x|\sin(\theta), |x|\cos(\theta)) \cos(\theta)^2\,d\theta\,dy_1. \tag{115}
\end{align*}$$

Making a change of variables,

$$\begin{align*}
&= 3T \int_0^1 \frac{y_1^3}{(1 - y_1^2)^{1/2}} \int_{-1}^1 g(Ty_1 + |x|v, |x|(1 - v^2)^{1/2})(1 - v^2)^{1/2}\,dv\,dy_1. \tag{116}
\end{align*}$$

Choosing

$$s^2 = (Ty_1 + |x|v)^2 + |x|^2(1 - v^2) = T^2y_1^2 + |x|^2 + 2T|x|y_1v, \tag{117}$$

$$sds = T|x|y_1\,dv,$$

$$\begin{align*}
[116] &= 3\int_0^1 \frac{y_1^2}{(1 - y_1^2)^{1/2}} \int_{|x| - Ty_1}^{||x| + Ty_1}\! g(s)(1 - v(s)^2)^{1/2}\,ds\,dy_1. \tag{118}
\end{align*}$$

Then since $|x|^{1/2}(1 - v^2)^{1/2} \leq s^{1/2}$,

$$|x|^{1/2} \leq \int_0^1 \frac{y_1^2}{(1 - y_1^2)^{1/2}} \mathcal{M}(g(s)s^{3/2})(Ty_1), \tag{119}$$

where $\mathcal{M}$ is the Maximal function in one dimension,

$$\mathcal{M}f(x) = \sup_{R>0} \frac{1}{R} \int_{x-R}^{x+R} f(t)\,dt. \tag{120}$$

It is a well-known fact (see for example [21]) that for any $1 < p \leq \infty$,

$$\|\mathcal{M}f\|_{L^p(\mathbb{R})} \lesssim_p \|f\|_{L^p(\mathbb{R})}. \tag{121}$$

Then $g \in L^2(\mathbb{R}^4)$ radial implies $g(s)s^{3/2} \in L^2([0, \infty))$, so by a change of variables $\|\mathcal{M}(g(s)s^{3/2})(Ty_1)\|_{L^2_+(\mathbb{R})} \leq \frac{1}{\sqrt{y_1}}$, so
\[ \| T^2 \|_{L^2_x} \leq \int_0^1 \frac{y_1^{3/2}}{(1 - y_1^2)^{1/2}} \| g \|_{L^2_x} \, dy_1 \leq \| g \|_{L^2_x}. \]  

(122)

Next we compute
\[
T^2 \int_{|y| < 1} \frac{y \cdot (\nabla g)(x + Ty)}{(1 - |y|^2)^{1/2}} \, dy.
\]

(123)

It will be convenient to split this integral into two pieces,
\[
= T^2 \int_{\sup(\frac{s}{T}, 1 - \frac{|s|^2}{T^2}) \leq |y| \leq 1} \frac{y \cdot (\nabla g)(x + Ty)}{(1 - |y|^2)^{1/2}} \, dy
\]
\[ + T^2 \int_{|y| \leq \sup(\frac{s}{T}, 1 - \frac{|s|^2}{T^2})} \frac{y \cdot (\nabla g)(x + Ty)}{(1 - |y|^2)^{1/2}} \, dy. \]

(124)

(125)

Since \( g \) is radially symmetric, making the change of variables (117),
\[
\frac{1}{\omega |x|^3} \int_{\partial B(0,|x|)} T^2 \int_{\sup(\frac{s}{T}, 1 - \frac{|s|^2}{T^2}) \leq |y| \leq 1} \frac{y \cdot (\nabla g)(x + Ty)}{(1 - |y|^2)^{1/2}} \, dy \, d\sigma(x)
\]
\[ = \frac{T}{|x|} \int_{\sup(\frac{s}{T}, 1 - \frac{|s|^2}{T^2})} \frac{y_1^3}{(1 - y_1^2)^{1/2}} g(s)(|x|v(s) + Ty_1)(1 - v(s)^2)^{1/2} \, dy \, dsdy_1. \]

(126)

(127)

Since (117) implies \((1 - v^2) = 0\) when \( s = |x| + Ty_1 \) or \(|x| - Ty_1 |\), integrating by parts,
\[
\frac{1}{|x|} \int_{\sup(\frac{s}{T}, 1 - \frac{|s|^2}{T^2})} \frac{y_1^3}{(1 - y_1^2)^{1/2}} g(s)s(1 - v(s)^2)^{1/2} \, dy \, dsdy_1
\]
\[ + \frac{1}{|x|^2} \int_{\sup(\frac{s}{T}, 1 - \frac{|s|^2}{T^2})} \frac{y_1^3}{(1 - y_1^2)^{1/2}} g(s)s(|x|v(s) + Ty_1)v(s)(1 - v(s)^2)^{-1} \, dy \, dsdy_1. \]

(128)

(129)

Again since \(|x| (1 - v(s)^2)^{1/2} \leq s \) and \( g(s)s^{3/2} \in L^2(\mathbb{R}) \),
\[
|x|^{1/2} \leq \int_{\sup(\frac{s}{T}, 1 - \frac{|s|^2}{T^2})} \frac{y_1^3}{(1 - y_1^2)^{1/2}} M(g)(Ty_1) \, dy_1,
\]

(130)

so by a change of variables,
\[
\left\| |x|^{1/2} \right\|_{L^2_x L^\infty_x(\mathbb{R} \times \mathbb{R}^4)} \leq \| g \|_{L^2_x(\mathbb{R}^4)}.
\]

(131)

Now take (129),
\[
v(s) = \frac{s^2 - |x|^2 - T^2 y_1^2}{2T|x|y_1},
\]

(132)
Now consider the second term in (124). Integrating by parts, and the estimate of the first term in (124) is complete. Changing the order of integration, since \( y \leq |x| \), implies

\[
\frac{1}{(1 - v)^{1/2}} = \frac{(2T|x|y_1)^{1/2}}{(s^2 - (|x| - Ty_1)^2)^{1/2}},
\]

and

\[
\frac{1}{(1 + v)^{1/2}} = \frac{(2T|x|y_1)^{1/2}}{(s^2 - (|x| + Ty_1)^2)^{1/2}}.
\]

Therefore, for any \( s \) lying in \( [\sup(|x| - T, T - 2|x|), |x| + Ty_1] \), \( |v| \leq 1 \), which implies

\[
\frac{v}{(1 - v^2)^{1/2}} \leq \frac{1}{(1 - v)^{1/2}} + \frac{1}{(1 + v)^{1/2}}.
\]

Now since \( ||x + Ty_1| \leq s \), \( g(s)s(|x|v(s) + Ty_1)^{1/2} \in L^2(\mathbb{R}) \). Therefore, changing the order of integration, since \( y_1 \geq 1 - \frac{|x|}{2T} \),

\[
|x|^{1/2} \leq \frac{1}{|x|^{1/2}} \left( \sup_{R > 0} \frac{1}{R} \int_{T - R}^{T + R} g(s)s^{3/2}|ds| \right).
\]

Since

\[
\int_{\sup(\frac{2}{7}, 1 - \frac{|x|}{2T})}^{\frac{y_1^3}{1 - y_1^{2/3}}} \left( \frac{(2T|x|y_1)^{1/2}}{(s^2 - (|x| - Ty_1)^2)^{1/2}} + \frac{(2T|x|y_1)^{1/2}}{(s^2 - (|x| + Ty_1)^2)^{1/2}} \right) dy_1
\]

\[
\leq \frac{|x|^{1/2}}{s^{1/2}},
\]

\[
|x|^{1/2} \leq \mathcal{M}(g)(T),
\]

and the estimate of the first term in (124) is complete.

Now consider the second term in (124). Integrating by parts,

\[
T^2 \int_{|y| < \sup(\frac{2}{7}, 1 - \frac{|x|}{2T})} \frac{y \cdot (\nabla g)(x + Ty)}{(1 - |y|^2)^{1/2}} dy
\]

\[
= T \cdot \sup \left( \frac{5}{6} \right) - \frac{1}{2T} \int_{|y| = \sup(\frac{2}{7}, 1 - \frac{|x|}{2T})} \frac{g(x + Ty)}{(1 - |y|^2)^{1/2}} dy
\]

\[
- 4T \int_{|y| < \sup(\frac{2}{7}, 1 - \frac{|x|}{2T})} \frac{g(x + Ty)}{(1 - |y|^2)^{1/2}} dy + T \int_{|y| < \sup(\frac{2}{7}, 1 - \frac{|x|}{2T})} \frac{g(x + Ty)|y|^2}{(1 - |y|^2)^{3/2}} dy
\]
As in (118), since \( g \) is radially symmetric, \( |x| \leq 32T \), \( g(s)s^{3/2} \in L^2([0, \infty)) \), \( 1 - v^2 \leq 1 \), and (117),

\[
|x|^{1/2} \leq \frac{T^{1/2}}{|x|} \int_{|x|-|x|-T}^{T} g(s)(1-v^2)^{1/2}ds \leq \mathcal{M}(g)(T). \tag{140}
\]

Also,

\[
|x|^{1/2} \leq \int_{0}^{\sup\left(\frac{x}{|x|-|x|-T} + \frac{x}{|x|-|x|-T}\right)} \frac{y_1^3}{(1-y_1^2)^3/2} \frac{1}{|x|^{1/2}} \int_{|x|-T}^{T+y_1} g(s)(1-v^2)^{1/2}dsdy_1
\]

\[
\leq \int_{0}^{\sup\left(\frac{x}{|x|-|x|-T} + \frac{x}{|x|-|x|-T}\right)} \frac{y_1^3}{(1-y_1^2)^3/2} \text{inf}\left(\frac{|x|^{1/2}}{T^{1/2}y_1^{1/2}}, 1\right)\mathcal{M}(g)(T)dy_1.
\]

Indeed, since \( |x|^{1/2}(1-v^2) \leq s^{1/2} \) and \( g(s)s^{3/2} \in L^2(\mathbb{R}) \), for any \( y_1 \),

\[
\frac{y_1^3}{(1-y_1^2)^3/2} \frac{1}{|x|^{1/2}} \int_{|x|-T}^{T+y_1} g(s)(1-v^2)^{1/2}dsdy_1 \leq \frac{y_1^3}{(1-y_1^2)^3/2} \mathcal{M}(g)(T)dy_1.
\]

Next, when \( Ty_1 >> |x|, (1-v^2) \leq 1 \) implies \( |g(s)(1-v^2)^{1/2}| \leq \frac{1}{T^{1/2}y_1^{1/2}}|g(s)s^{3/2}| \) for all \( s \in [Ty_1 - |x|, Ty_1 + |x|] \), which implies

\[
\frac{y_1^3}{(1-y_1^2)^3/2} \frac{1}{|x|^{1/2}} \int_{|x|-T}^{T+y_1} g(s)(1-v^2)^{1/2}dsdy_1 \leq \frac{y_1^3}{(1-y_1^2)^3/2} \mathcal{M}(g)(T)dy_1.
\]

This completes the proof of the theorem when \( v_0 = 0 \). Now suppose \( v_0 = f \in \dot{H}^1(\mathbb{R}^4) \) is radial and \( v_1 = 0 \). Then

\[
v(T, x) = 5T \int_{|y|<1} \frac{y \cdot (\nabla f)(x + Ty)}{(1-|y|^2)^{1/2}}dy + T^2 \int_{|y|<1} \frac{y_jy_k(\partial_j \partial_k f)(x + Ty)}{(1-|y|^2)^{1/2}}dy
\]

\[
+ 3 \int_{|y|<1} \frac{f(x + Ty)}{(1-|y|^2)^{1/2}}dy.
\]

Because \( f \in \dot{H}^1(\mathbb{R}^4) \), (146) can be estimated in exactly the same manner as (113). This leaves only (147). Since \( f \) is radial, making a change of variables,
Theorem 2.6. Suppose that \( v \) is a solution to the wave equation
\[
\Box^{4+1} v = F(v) \quad \text{on } \mathbb{R}^{4+1}
\]
with initial conditions \( v_0 = v(0, x) \) and \( v_1 = \partial_T v(0, x) \) on \( \mathbb{R}^4 \).

Then it follows that
\[
\left\| v \right\|_{L^2_TL^6_x(\mathbb{R} \times \mathbb{R}^4)} + \left\| |x|^{1/4} v \right\|_{L^2_TL^6_x(\mathbb{R} \times \mathbb{R}^4)} \leq \left\| v_0 \right\|_{\dot{H}^1(\mathbb{R}^4)} + \left\| v_1 \right\|_{L^2(\mathbb{R}^4)} + \left( \sum_j 2^{j/2} \left\| F \right\|_{L^2_{T,x}(\mathbb{R} \times \{x: 2^j \leq |x| \leq 2^{j+1} \})} \right) \tag{152}
\]

Proof: It suffices to prove (152) for \( F \in L^2_{T,x}(\mathbb{R} \times \mathbb{R}^4) \) when \( F \) is supported on \( \frac{\rho}{2} \leq |x| \leq \rho \) with bounds independent of \( \rho \). By finite propagation speed, \( \frac{\sin((T-\tau)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(\tau) \) is supported on \( |x| \leq \rho + |T-\tau| \).

For \( |x| < |T-\tau| - 3\rho \), observe that by the fundamental solution of the wave equation (see for example Sogge[14])
\[
\frac{\sin(T-\tau)\sqrt{-\Delta}}{\sqrt{-\Delta}} F(\tau) = 3(T-\tau) \int_{|y|<1} \frac{F(\tau, x + (T-\tau)y)}{(1-|y|^2)^{1/2}} dy + (T-\tau)^2 \int_{|y|<1} \nabla F(\tau, x + (T-\tau)y) \cdot y dy. \tag{153}
\]
If \( |x| < |T-\tau| - 3\rho \) and \( F(\tau, z) \) is supported on \( |z| \leq \rho \), then for \( |T-\tau| > 3\rho \), \( |y| < 1 - \frac{2\rho}{|T-\tau|} \). Then by Hölder’s inequality and change of variables,
Interpolating (154), (155), and (156), imply that (157) is identically zero. However, since we are in even dimensions, if we were in odd dimensions the sharp Huygens principle would
Remark:
Meanwhile, by the Sobolev embedding theorem and Hölder’s inequality,
Also, integrating by parts,
Interpolating (154), (155), and (156),
Remark: If we were in odd dimensions the sharp Huygens principle would imply that (157) is identically zero. However, since we are in even dimensions, (157) is nonzero.
Next, by finite propagation speed and interpolating (154), (155), and (157),
Therefore, for any \( l \in \mathbb{Z}, l \geq 0 \), if \( \tau \in [(l-1)\rho, l\rho], |t - \tau| > \rho, \)
Replacing Theorem 2

Now we abuse notation and let $\lfloor T - \rho \rfloor = \lfloor \frac{T - \rho}{\rho} \rfloor \rho$, where $|x|$ is the integer part of $x$. Then because the sets $\{(t, x) : (l - 1)\rho \leq t - |x| \leq (l + 1)\rho\}$ are pairwise disjoint for any two $l_1, l_2 \in \mathbb{Z}$,

\[
\left\| \int_0^{[T - \rho]} \frac{\sin((T - \tau)\sqrt{-1})}{\sqrt{-1}} F(\tau) d\tau \right\|_{L^2_x(L^6_t(\mathbb{R} \times \mathbb{R}^4))} \leq \left\| \int_0^{[T - \rho]} \frac{\rho^{9/16}}{|T - \tau|^{17/16}} \|F(\tau)\|_{L^2_x(L^6_t(\mathbb{R}^4))} d\tau \right\|_{L^2_t(L^2_x(\mathbb{R}^4))} + \left( \sum_{l \in \mathbb{Z}} \left\| \int_{l\rho}^{(l+1)\rho} \frac{\sin((T - \tau)\sqrt{-1})}{\sqrt{-1}} F(\tau) d\tau \right\|_{L^2_x(L^6_t(\mathbb{R} \times \mathbb{R}^4))}^2 \right)^{1/2}.
\]  

(159)

Then by Theorem 2.3, Hölder’s inequality, and Young’s inequality,

\[
\leq \rho^{1/2} \left\| F \right\|_{L^2_{T,x}(\mathbb{R} \times \mathbb{R}^4)} + \left( \sum_{l \in \mathbb{Z}} \|F(\tau)\|_{L^2_x(L^6_t(\mathbb{R} \times \mathbb{R}^4))}^2 \right)^{1/2}.
\]

(160)

Finally, by Theorem 2.3 and Young’s inequality,

\[
\left\| \int_{[T - \rho]}^T \frac{\sin((T - \tau)\sqrt{-1})}{\sqrt{-1}} F(\tau) d\tau \right\|_{L^2_t(L^6_x(\mathbb{R} \times \mathbb{R}^4))} \leq \rho^{1/2} \left\| F \right\|_{L^2_{T,x}(\mathbb{R} \times \mathbb{R}^4)}.
\]

(162)

Therefore, Theorem 24, (160), (161), and (162) combine to prove

\[
\|v\|_{L^2_x(L^6_t(\mathbb{R} \times \mathbb{R}^4))} \leq \|v_0\|_{H^1(\mathbb{R}^4)} + \|v_1\|_{L^2(\mathbb{R}^4)} + \rho^{1/2} \left\| F \right\|_{L^2_{T,x}(\mathbb{R} \times \mathbb{R}^4)}.
\]

(163)

Replacing Theorem 2.4 with Theorem 2.5 in (160) - (162), proves
and thus completes the proof of Theorem 2.6 □

2.4. A Function Space

We will use the function space

Definition 2.7 (Function spaces). If $P_N$ is a Littlewood - Paley operator then let

$$
\|v\|_X^2 = \sum_N \|P_N v\|_{L_2^p L_2^q(\mathbb{R} \times \mathbb{R}^4)}^2 + \sum_N \|x^{1/4} P_N v\|_{L_2^p L_2^q(\mathbb{R} \times \mathbb{R}^4)}^2 + \sum_N N^2 \|P_N v\|_{L_2^p L_2^q(\mathbb{R} \times \mathbb{R}^4)}^2
$$

$$
+ \sum_N \left( \sup_{\rho>0} \rho^{-1/2} \|P_N \partial_T v\|_{L_2^p L_2^q(\mathbb{R} \times \{x:|x|<\rho\})} \right)^2
$$

$$
+ \sum_N \left( \sup_{\rho>0} \rho^{-1/2} \|P_N \nabla_x v\|_{L_2^p L_2^q(\mathbb{R} \times \{x:|x|<\rho\})} \right)^2
$$

$$
+ \sum_N N^2 \left( \sup_{\rho>0} \rho^{-1/2} \|P_N v\|_{L_2^p L_2^q(\mathbb{R} \times \{x:|x|<\rho\})} \right)^2
$$

$$
+ \sum_N \|x^{-3/2} P_N v\|_{L_2^p L_2^q(\mathbb{R} \times \mathbb{R}^4)}^2 + \sum_N N^{-2} \|P_N \partial_T v\|_{L_2^p L_2^q(\mathbb{R} \times \mathbb{R}^4)}^2.
$$

(165)

We also define the norm

$$
\|F\|_Y^2 = \inf_{F_1+F_2=F} \left( \|F_1\|_{L_2^p L_2^q(\mathbb{R} \times \mathbb{R}^4)}^2 + \sum_N \left( \sum_j 2^{j/2} \|P_N F_2\|_{L_2^p L_2^q(\mathbb{R} \times \{2^j \leq |x| \leq 2^{j+1}\})} \right)^2 \right).
$$

(166)

Lemma 2.8.

$$
\left( \sup_{\rho>0} \rho^{-1/2} \|\nabla v\|_{L_2^p L_2^q(\mathbb{R} \times \{x:|x|<\rho\})} \right) + \left( \sup_{\rho>0} \rho^{-1/2} \|\partial_T v\|_{L_2^p L_2^q(\mathbb{R} \times \{x:|x|<\rho\})} \right)
\leq \|v\|_X.
$$

(167)

Proof: Fix $\rho > 0$. Letting

$$
\tilde{P}_N = P_N + P_N + P_{2N},
$$

(168)
\[ \phi \left( \frac{x}{\rho} \right) \nabla P_N v = \tilde{P}_N \left( \phi \left( \frac{x}{\rho} \right) \nabla P_N v \right) + \left( P_{\leq \frac{N}{\lambda}} \phi \left( \frac{x}{\rho} \right) \right) \nabla P_N v. \] (169)

By Bernstein’s inequality,

\[ \| P_{\leq \frac{N}{\lambda}} \phi \left( \frac{x}{\rho} \right) \|_{L^p_t L^{8/3}_x(\mathbb{R} \times \mathbb{R}^4)} \leq \inf (N^{-2} \rho^{-2}, 1), \] (170)

so

\[ \rho^{-1/2} \left\| \sum_N \left( P_{\leq \frac{N}{\lambda}} \phi \left( \frac{x}{\rho} \right) \right) \nabla P_N v \right\|_{L^2_t L^2_x(\mathbb{R} \times \mathbb{R}^4)} \leq \rho^{-1/2} \sum_N \inf (N^{-2} \rho^{-2}, 1) N \| P_N v \|_{L^2_t L^8_x(\mathbb{R} \times \mathbb{R}^4)} \leq \rho^{-3/2} \| v \|_X. \] (171)

Meanwhile, since the \( \tilde{P}_N \) are finitely overlapping,

\[ \rho^{-1} \left\| \sum_N \tilde{P}_N \left( \phi \left( \frac{x}{\rho} \right) \nabla P_N v \right) \right\|^2_{L^2_t L^2_x(\mathbb{R} \times \mathbb{R}^4)} \leq \rho^{-1} \sum_N \left\| \phi \left( \frac{x}{\rho} \right) \nabla P_N v \right\|^2_{L^2_t L^2_x(\mathbb{R} \times \mathbb{R}^4)} \leq \| v \|^2_X, \] (172)

which proves

\[ \left( \sup_{\rho > 0} \rho^{-1/2} \left\| \nabla v \right\|_{L^2_t L^2_x(\mathbb{R} \times \{ x : |x| \leq \rho \})} \right) \leq \| v \|_X. \] (173)

The proof of

\[ \left( \sup_{\rho > 0} \rho^{-1/2} \left\| \partial_T v \right\|_{L^2_t L^2_x(\mathbb{R} \times \{ x : |x| \leq \rho \})} \right) \leq \| v \|_X. \] (174)

is similar. \( \square \)

**Lemma 2.9.**

\[ \left\| |x|^{1/4} v \right\|_{L^2_t L^4_x(\mathbb{R} \times \mathbb{R}^4)} \leq \| v \|_X, \] (175)

**Proof:** By the Littlewood - Paley theorem

\[ \left\| |x|^{1/4} v \right\|^2_{L^2_t L^4_x(\mathbb{R} \times \mathbb{R}^4)} \leq \sum_N \left\| P_N (|x|^{1/4} v) \right\|^2_{L^2_t L^4_x(\mathbb{R} \times \mathbb{R}^4)}. \] (167)

By Hölder’s inequality

\[ \left\| |x|^{1/4} \phi (N x) (P_{\leq N} v) \right\|_{L^2_t L^4_x} \leq N^{-1/2} \sum_{M \leq N} \left\| P_M v \right\|_{L^2_t L^4_x} \leq \left( \frac{M}{N} \right)^{1/2} \sum_{M \leq N} \left\| P_M v \right\|_{L^2_t L^4_x}, \] (177)
By the Sobolev embedding theorem $\dot{H}^{7/4}(\mathbb{R}^4) \subset L^{16}(\mathbb{R}^4)$,
\[
\left\| P_N(\left| x \right|^{1/4}\phi(Nx)(P_{> N} v)) \right\|_{L^2_{\rho}L^{16}_{\rho}} \leq N^{3/2} \sum_{M \geq N} \left\| P_M v \right\|_{L^2_\rho L^2_{\rho}(\left| x \right| \leq \frac{x}{N})} \leq \sum_{M \geq N} \left( \frac{N}{M} \right) M \left( \sup_{\rho > 0} \rho^{-1/2} \left\| P_M v \right\|_{L^2_\rho L^2_{\rho}(\left| x \right| \leq \rho)} \right). 
\] (178)

Next,
\[
\left\| \left| x \right|^{1/4}(1 - \phi(Nx))(\tilde{P}_N v) \right\|_{L^2_{\rho}L^{16}_{\rho}} \leq \sum_{N \leq M \leq 32N} \left\| \left| x \right|^{1/4}P_M v \right\|_{L^2_\rho L^2_{\rho}(\left| x \right| \leq \rho)}. 
\] (179)

Next, by Bernstein’s inequality, letting $\chi(x) = \phi(\frac{x}{2}) - \phi(x)$,
\[
\sum_{j \geq 0} \left\| P_N \left( \left| x \right|^{1/4}\chi(2^{-j}N) x)(P_{> 32N} v) \right) \right\|_{L^2_{\rho}L^{16}_{\rho}} \leq \sum_{j \geq 0} N^{3/4} \left\| \nabla \left( \left| x \right|^{1/4}\chi(2^{-j}N) x\right) \right\|_{L^\infty_\rho L^\infty_\rho} \left\| P_{> 32N} v \right\|_{L^2_\rho L^2_{\rho}(\left| x \right| \leq \frac{32}{N})} \leq \sum_{j \geq 0} 2^{-3j/4}N^{3/2} \sum_{M \geq 32N} \frac{2j/2}{MN^{1/2}} \cdot \sup_{\rho > 0} \left( \rho^{-1/2} M \left\| P_M v \right\|_{L^2_\rho L^2_{\rho}(\left| x \right| \leq \rho)} \right) 
\] (180)

Finally, by Hölder’s inequality and Sobolev embedding,
\[
\sum_{j \geq 0} \left\| P_N \left( \left| x \right|^{1/4}\chi(2^{-j}N x)(P_{\leq \frac{N}{32}} v) \right) \right\|_{L^2_{\rho}L^{16}_{\rho}} \leq \sum_{j \geq 0} \frac{1}{N} \left\| \nabla \left( \left| x \right|^{1/4}\chi(2^{-j}N x)\right) \right\|_{L^\infty_\rho L^\infty_\rho} \left\| P_{\leq \frac{N}{32}} v \right\|_{L^2_\rho L^2_{\rho}(\left| x \right| \leq \frac{32}{N})} \leq \sum_{j \geq 0} 2^{-3j/4}N^{-1/4} \sum_{M \leq \frac{N}{32}} \frac{2j/4}{N^{1/4}} \left\| P_M v \right\|_{L^2_\rho L^\infty_\rho} \leq \sum_{M \leq \frac{N}{32}} \left( \frac{M}{N} \right)^{1/2} \left\| P_M v \right\|_{L^2_\rho L^2_{\rho}}. 
\] (181)

Combining (177) - (181), by Young’s inequality and (165),
\[
\sum_{N} \left\| P_N(\left| x \right|^{1/4}v) \right\|_{L^2_{\rho}L^{16}_{\rho}(\mathbb{R} \times \mathbb{R}^4)} \leq \left\| v \right\|_{X}^2. 
\] (182)

□

**Theorem 2.10.** If $v$ is a radial solution to the equation
\[
\begin{align*}
\Box v &= F(v) \quad &\text{on } \mathbb{R}^{4+1} \\
v_0 &= v(0, x) &\text{and } v_1 = \partial_T v(0, x) &\text{on } \mathbb{R}^4
\end{align*}
\] (183)
then
\[ \|v\|_X \leq \|v_0\|_{\dot{H}^1(\mathbb{R}^4)} + \|v_1\|_{L^2(\mathbb{R}^4)} + \|F\|_Y. \]  
(184)

**Proof:** By Theorem 2.6 it only remains to prove

\[ \sum_N N^2 \sup_{\rho > 0} \rho^{-1/2} \left\| P_N v \right\|^2_{L_{T,x}^2(\mathbb{R} \times \mathbb{R}^4)} \leq \|v_0\|_{\dot{H}^1(\mathbb{R}^4)} + \|v_1\|_{L^2(\mathbb{R}^4)} + \|F\|_Y. \]  
(185)

and

\[ \sum_N N^{-2} \left\| P_N \partial_T v \right\|^2_{L_{T,x}^2(\mathbb{R} \times \mathbb{R}^4)} \leq \|v_0\|_{\dot{H}^1(\mathbb{R}^4)} + \|v_1\|_{L^2(\mathbb{R}^4)} + \|F\|_Y. \]  
(186)

We start with (186). Fix \( N \). Suppose \( \phi \in C_0^\infty(\mathbb{R}^4) \) is a positive radial function, \( \phi(x) = 1 \) for \( |x| \leq 1 \), \( \phi(x) = 0 \) for \( |x| > 1 \). If \( \rho \geq \frac{1}{N} \), we take the commutator

\[ \rho^{-1/2} N \phi \left( \frac{x}{\rho} \right) P_N v = \rho^{-1/2} N \phi \left( \frac{x}{\rho} \right) P_N \tilde{P}_N v 
= \rho^{-1/2} N P_N \left( \phi \left( \frac{x}{\rho} \right) \tilde{P}_N v \right) + \rho^{-1/2} N \left[ \phi \left( \frac{x}{\rho} \right), P_N \right] \tilde{P}_N v. \]  
(187)

Then by Bernstein’s inequality

\[ \rho^{-1/2} \left\| N P_N \left( \phi \left( \frac{x}{\rho} \right) \tilde{P}_N v \right) \right\|_{L_{T,x}^2(\mathbb{R} \times \mathbb{R}^4)} \leq \rho^{-1/2} \left\| \nabla \left( \phi \left( \frac{x}{\rho} \right) \tilde{P}_N v \right) \right\|_{L_{T,x}^2(\mathbb{R} \times \mathbb{R}^4)} \leq \rho^{-3/2} \left\| \phi' \left( \frac{x}{\rho} \right) \tilde{P}_N v \right\|_{L_{T,x}^2(\mathbb{R} \times \mathbb{R}^4)} + \rho^{-1/2} \left\| \phi \left( \frac{x}{\rho} \right) \nabla \tilde{P}_N v \right\|_{L_{T,x}^2(\mathbb{R} \times \mathbb{R}^4)} \leq \left\| \tilde{P}_N v \right\|_{L_{T,x}^2(\mathbb{R} \times \mathbb{R}^4)} + \rho^{-1/2} \left\| \phi \left( \frac{x}{\rho} \right) \nabla \tilde{P}_N v \right\|_{L_{T,x}^2(\mathbb{R} \times \mathbb{R}^4)}. \]  
(188)

Now compute the commutator

\[ \rho^{-1/2} N [P_N, \phi \left( \frac{x}{\rho} \right)] \tilde{P}_N v = \rho^{-1/2} N \int N^4 K(N(x-y)) \phi \left( \frac{x}{\rho} \right) \phi \left( \frac{y}{\rho} \right) (\tilde{P}_N v)(y)dy, \]  
(189)

where \( K(\cdot) \) is the kernel of the Littlewood - Paley projection \( P_1 \). By the fundamental theorem of calculus, \( |\phi \left( \frac{x}{\rho} \right) - \phi \left( \frac{y}{\rho} \right)| \leq \frac{|x-y|}{\rho} \), so by Hölder’s inequality, because \( K(\cdot) \) is rapidly decreasing for \( |x| \geq 1 \),

\[ \|\rho^{-1/2} N \int N^4 K(N(x-y)) \phi \left( \frac{x}{\rho} \right) \left( \frac{y}{\rho} \right) (\tilde{P}_N v)(y)dy\|_{L_{T}^2(|x| \leq 10\rho)} \leq \|\tilde{P}_N v\|_{L_{T,x}^2}. \]  
(190)
When $|x| \geq 10\rho$, by the support of $\phi$ and the fact that $K(\cdot)$ is rapidly decreasing for $|x| \geq 1$,

$$
\rho^{-1/2}N \int N^4 K(N(x-y))[\phi(x/\rho) - \phi(y/\rho)](\tilde{P}_Nv)(y)dy
= -\rho^{-1/2}N \int N^4 K(N(x-y))\phi(y/\rho)(\tilde{P}_Nv)(y)dy
\leq \rho^{-1/2} \frac{N}{N^{10}|x|^{10}} \int N^4 K(N(x-y))\phi(y/\rho)(\tilde{P}_Nv)(y)dy. 
$$

(191)

Therefore, if $\rho \geq \frac{1}{N}$,

$$
N\rho^{-1/2} \left\| \phi \left( \frac{x}{\rho} \right), P_N \right\|_{L^2_{T,x}(\mathbb{R} \times \mathbb{R}^4)} \leq \left\| \tilde{P}_Nv \right\|_{L^2_{T,x}(\mathbb{R} \times \mathbb{R}^4)}. 
$$

(192)

On the other hand, if $\rho \leq \frac{1}{N}$, then simply apply Holder’s inequality,

$$
\rho^{-1/2}N \left\| \phi \left( \frac{x}{\rho} \right) P_Nv \right\|_{L^2_{T,x}(\mathbb{R} \times \mathbb{R}^4)} \leq \left\| \tilde{P}_Nv \right\|_{L^2_{T,x}(\mathbb{R} \times \mathbb{R}^4)}. 
$$

(193)

Combining (188), (192), and (193),

$$
\sum_N \left( \sup_{\rho > 0} \rho^{-1/2} \left\| P_Nv \right\|_{L^2_{T,x}(\mathbb{R} \times \{x:|x| \leq \rho\})} \right)^2
\leq \sum_N \left( \sup_{\rho > 0} \rho^{-1/2} \left\| \nabla P_Nv \right\|_{L^2_{T,x}(\mathbb{R} \times \{x:|x| \leq \rho\})} \right)^2 + \sum_N \left\| P_Nv \right\|_{L^2_{T,x}(\mathbb{R} \times \mathbb{R}^4)}^2.
$$

(194)

The proof of (186) is straightforward. Applying (153), the Huygens principle, Theorem 2.4 and 2.5

$$
\left\| P_N \partial_Tv \right\|_{L^2_T L^2(\mathbb{R} \times \mathbb{R}^4)}
\leq N \left\| P_Nv_0 \right\|_{H^1(\mathbb{R}^4)} + N \left\| P_Nv_1 \right\|_{L^2(\mathbb{R}^4)} + \left\| \nabla P_N F \right\|_{Y+L^2_T L^6_{y/7}^3},
$$

(195)

where if $V$ and $W$ are Banach spaces,

$$
\left\| f \right\|_{V+W} = \inf_{f=f_1+f_2} \left\| f_1 \right\|_V + \left\| f_2 \right\|_W.
$$

By Holder’s inequality,
Lemma 2.12.
\[
\left\| \nabla P_N(\phi(Nx)\tilde{P}_N F) \right\|_{L^2_{t,x} L^{8/7}_x(\mathbb{R} \times \mathbb{R}^4)} \leq N \sum_j 2^{j/2} \left\| \tilde{P}_N F \right\|_{L^2_{t,x} \{x : 2^j \leq |x| \leq 2^{j+1}\}}.
\]
(196)

Meanwhile, by Holder's inequality and the fact that the Littlewood - Paley kernel is rapidly decreasing,
\[
\left\| (1 - \phi(Nx)) \nabla P_N((1 - \phi(Nx))\tilde{P}_N F) \right\|_Y \\
+ \left\| \phi(Nx) \nabla P_N((1 - \phi(Nx))\tilde{P}_N F) \right\|_{L^2_{t,x} L^{8/7}_x(\mathbb{R} \times \mathbb{R}^4)} \\
\leq N \sum_j 2^{j/2} \left\| \tilde{P}_N F \right\|_{L^2_{t,x} \{x : 2^j \leq |x| \leq 2^{j+1}\}}.
\]
(197)

Since \( \tilde{P}_N \) has finite overlap, this completes the proof of (186). □

2.5. Scattering for Small Data

Theorem 2.11 (Scattering). The nonlinear wave equation
\[
4+1 \Box v = F(v) \quad \text{on } \mathbb{R}^{4+1} \\
v_0 = v(0, x) \quad \text{and} \quad v_1 = \partial_T v(0, x) \quad \text{on } \mathbb{R}^4
\]
(198)
with
\[
F(v) = \left( e^{2Z} - 1 + \left( \frac{r}{R} \partial_r r + \frac{1}{2} \right) - \left( \frac{r}{R} \partial_r r - \frac{1}{2} \right) \right) \frac{v}{r^2} + e^{2Z} \frac{R^2}{r^2} v^3 \zeta(Rv)
\]
and
\[
|e^{2Z} - 1|, \quad \left| \frac{R}{r} - 1 \right|, \quad |Rv(T, R)| \leq E(v)
\]
(199)
has a solution with \( \|v\|_{L^2_t L^6_x(\mathbb{R} \times \mathbb{R}^4)} < \infty \) for energy \( E(v) \) sufficiently small.

Proof: By Theorem 2.10 and (199), it suffices to prove
\[
\left\| e^{2Z} - 1 + \left( \partial_r r + \frac{1}{2} \right) - \left( \partial_r r - \frac{1}{2} \right) \right\| \frac{v}{r^2} + e^{2Z} \frac{R^2}{r^2} \zeta(Rv)v^3 \right\|_Y \\
\leq \|v\|_X E(v)^{1/2} + E(v)^3 + c(E(v)) \|v\|_X,
\]
for some quantity \( c(E(v)) \to 0 \) as \( E(v) \to 0 \). Indeed, then
\[
\|v\|_X \leq E(v)^{1/2} + c(E(v)) \|v\|_X + E(v)^{1/2} \|v\|_X^2 + E(v)^{3/2},
\]
(201)
so for \( E(v) \) sufficiently small, \( \|v\|_X \leq E(v)^{1/2} \). The proof of (200) will occupy the remainder of the paper.

Lemma 2.12.
\[
\left\| e^{2Z} \frac{R^2}{r^2} v^3 \zeta(Rv) \right\|_{L^1_t L^6_x(\mathbb{R} \times \mathbb{R}^4)} \leq \|v\|_X^2 E(v)^{1/2}.
\]
(202)
Proof: This is straightforward. From (6.1), for $E(v)$ sufficiently small,

$$\left| e^{2Z} - 1 \right|, \quad \left| \frac{R}{r} - 1 \right|, \quad \sup_{T,R} \left| Rv(T,R) \right| \leq E(v), \quad (203)$$

so

$$\left\| e^{2Z} \frac{R^2}{r^2} v^3(Rv) \right\|_{L^1_T L^2_R(\mathbb{R} \times \mathbb{R}^4)} \leq \left\| v \right\|^2_{L^2_T L^1_R} \left\| v \right\|_{L^\infty_T L^4_R} \leq \left\| v \right\|^2_{X} E(v)^{1/2}, \quad (204)$$

Next,

**Lemma 2.13.**

$$\left\| \left( \partial_\eta r + \frac{1}{2} \right) \frac{v}{R^2} \right\|_{L^1_T L^2_R(\mathbb{R} \times \mathbb{R}^4)} \leq \left\| v \right\|^2_{X} E(v), \quad (205)$$

and

$$\left\| \left( \partial_\xi r - \frac{1}{2} \right) \frac{v}{R^2} \right\|_{L^1_T L^2_R(\mathbb{R} \times \mathbb{R}^4)} \leq \left\| v \right\|^2_{X} E(v). \quad (206)$$

**Proof:** First take (205). By the fundamental theorem of calculus,

$$\left. \partial_\eta r \right|_{R=0} = -\frac{1}{2}, \quad \partial_\xi \left( \partial_\eta + \frac{1}{2} \right) r = \frac{e^{2Z} f^2(Rv)}{4}, \quad (207)$$

$Z \sim 1$, $f(0) = 0$, $f'(0) = 1$, $|Rv| \leq E(v)^{1/2}$, $\xi = T + R$, $\eta = T - R$, then

$$\left( \partial_\xi r + \frac{1}{2} \right) \frac{v(T,R)}{R^2} \leq \frac{1}{R^2} \left( \int_0^R v(T - R + s, s)^2 \cdot sds \right) v(T,R). \quad (208)$$

Making a change of variables $s = \lambda R$, $0 \leq \lambda \leq 1$,

$$\left\| \left( \partial_\eta r + \frac{1}{2} \right) \frac{v(T,R)}{R^2} \right\|_{L^1_T L^2_R(\mathbb{R} \times \mathbb{R}^4)}$$

$$\leq \left\| \left( \int_0^1 v(T + (\lambda - 1)R, \lambda R)^2 v \ d\lambda \right) v(T,R) \right\|_{L^1_T L^2_R(\mathbb{R} \times \mathbb{R}^4)}$$

$$\leq \left\| R^{1/4} v(T,R) \right\|_{L^2_T L^{16}_R(\mathbb{R} \times \mathbb{R}^4)} \cdot \left( \int_0^1 \left\| \frac{1}{R^{1/4}} v(T + (\lambda - 1)R, \lambda R)^2 \right\|_{L^2_T L^{16/7}_R(\mathbb{R} \times \mathbb{R}^4)} \lambda \ d\lambda \right). \quad (209)$$

Doing a change of variables,
Plugging (212) and (211) into (209), the expansion
\[ \| \frac{1}{R^{1/4}} v(T + (\lambda - 1)R, \lambda R) \|_{L^4_{T,R}(\mathbb{R} \times \mathbb{R}^4)}^4 \]

\[ = \int \int v(T + (\lambda - 1)R, \lambda R)^4 R^2 dRdT = \lambda^{-3} \int \int v(T, R)^4 R^2 dRdT. \tag{211} \]

By Hardy’s inequality and the Sobolev embedding theorem,
\[ \lambda^{-3} \| \frac{1}{R^{1/4}} v(T, R) \|_{L^4_{T,R}(\mathbb{R} \times \mathbb{R}^4)}^4 \leq \lambda^{-3} \left( \frac{1}{R} \right) v(T, R) \| v(T, R) \|_{L^\infty_{T,R}(\mathbb{R} \times \mathbb{R}^4)} \| v(T, R) \|_{L^4_{T,R}(\mathbb{R} \times \mathbb{R}^4)}^2 \]
\[ \leq E(v) \| v \|_X^2. \tag{212} \]

Meanwhile, by the Sobolev embedding theorem and interpolation,
\[ \left\| v(T + (\lambda - 1)R, \lambda R) \right\|_{L^{4,16/3}_{T,R}(\mathbb{R} \times \mathbb{R}^4)}^4 \]
\[ \leq \left\| \partial_R \right\|_{L^{4,16/3}_{T,R}(\mathbb{R} \times \mathbb{R}^4)}\left\| \frac{v(T + (\lambda - 1)R, \lambda R)}{R} \right\|_{L^4_{T,R}(\mathbb{R} \times \mathbb{R}^4)}^4 \]
\[ \leq \left\| \partial_R v(T + (\lambda - 1)R, \lambda R) \right\|_{L^\infty_{T,R}(\mathbb{R} \times \mathbb{R}^4)} \left\| v(T, R) \right\|_{L^4_{T,R}(\mathbb{R} \times \mathbb{R}^4)}^3 \]
\[ \leq \lambda^{-4} \left( \left\| \partial_T v(T, R) \right\|_{L^\infty_{T,R}} + \left\| \partial_R v(T, R) \right\|_{L^\infty_{T,R}} \right) \left\| v(T, R) \right\|_{L^4_{T,R}}^2 \]
\[ \leq \lambda^{-4} (T, R) \left\| v(T, R) \right\|_{X}^2 E(v). \tag{215} \]

Plugging (212) and (211) into (209),
\[ \left\| \left( \partial_r + \frac{1}{2} \right) \frac{v}{R^2} \right\|_{L^1_{T}L^4_{R}(\mathbb{R} \times \mathbb{R}^4)} \leq \| v \|_X^2 E(v) \left( \int_0^1 \lambda^{-3/4} d\lambda \right) \leq \| v \|_X^2 E(v)^{1/2}. \tag{216} \]

This takes care of (203). The proof of (206) is almost identical, this time integrating \( \partial_\eta \partial_\eta r \) with respect to \( \eta \) and utilizing (207), and \( \partial_\eta r \big|_{R=0} = \frac{1}{2}. \)

To compute
\[ (e^{2Z} - 1) \frac{v}{R^2}, \tag{217} \]
we will use the following ‘mass-aspect’ function,
\[ m := 1 + 4e^{-2Z} \partial_\xi r \partial_\eta r. \tag{218} \]
\[ |e^{2Z} - 1| \leq c(E(v)) \] small implies that \( |1 - e^{-2Z}| \) is small, so we can make the expansion.
\[ e^{2Z} = \frac{1}{1 - (1 - e^{-2Z})} = \sum_{n=0}^{\infty} (1 - e^{-2Z})^n, \quad (219) \]

and

\[ e^{2Z} - 1 = e^{2Z} (1 - e^{-2Z}) = \sum_{n=0}^{\infty} (1 - e^{-2Z})^{n+1}. \quad (220) \]

The sums converge exponentially, so we will confine our computations to the leading order term in (220), and estimate

\[ \left( 1 - e^{-2Z} \right) \frac{v}{R^2}. \quad (221) \]

\[ \left( 1 - e^{-2Z} \right) \frac{v}{R^2} = \left( 1 + 4e^{-2Z} \partial_\eta r \partial_\xi r \right) \frac{v}{R^2} - 4e^{-2Z} \left( \partial_\eta r \partial_\xi r + \frac{1}{4} \right) \frac{v}{R^2}. \quad (222) \]

Now since

\[ \partial_\eta r \partial_\xi r + \frac{1}{4} = \partial_\eta r \left( \partial_\xi r - \frac{1}{2} \right) + \frac{1}{2} \left( \partial_\eta r + \frac{1}{2} \right), \quad (223) \]

lemma 2.13 implies

\[ \left\| \left( \partial_\eta r \partial_\xi r + \frac{1}{4} \right) \frac{v}{R^2} \right\|_{L^1_t L^2_{x}(\mathbb{R} \times \mathbb{R}^4)} \leq \|v\|_X^2 E(v) \frac{1}{2}, \quad (224) \]

so it only remains to compute

\[ \sum_{j} 2^{j/2} \left\| m \frac{v}{R^2} \right\|_{L^2_{t,x}(\mathbb{R} \times \{ x : 2^j \leq |x| \leq 2^{j+1} \})}. \quad (225) \]

By (222) and \( |1 - e^{2Z}|, |\partial_\eta r + \frac{1}{2}|, |\partial_\xi r - \frac{1}{2}| \leq c(E(v)) \),

\[ \sup_{T,R} |m| \leq c(E(\omega)). \quad (226) \]

Now make a spatial partition of unity. Suppose \( \phi(x) \in C_0^\infty(\mathbb{R}^4) \) is a radial, decreasing function, \( \phi(x) = 1 \) for \( |x| \leq 1 \), and \( \phi(x) \) is supported on \( |x| \leq 2 \). Then let

\[ \chi(x) = \phi\left( \frac{x}{2} \right) - \phi(x). \quad (227) \]

For any \( x \neq 0 \),

\[ \sum_{j \in \mathbb{Z}} \chi(2^{-j} x) = 1. \quad (228) \]

Combining (228) with the Littlewood - Paley decomposition,
\[ PN \left( \frac{m \cdot v}{R^2} \right) = \sum_M \sum_{j \in \mathbb{Z}} PN \left( \chi(2^{-j} \rho) m \frac{P_M v}{R^2} \right) \]

\[ = \sum_M PN(\phi(NR) \frac{P_M v}{R^2}) + \sum_M \sum_{2^j \geq \frac{N}{M}} PN \left( \chi(2^{-j} \rho) m \frac{P_M v}{R^2} \right). \tag{229} \]

Since the Littlewood-Paley convolution kernel is rapidly decreasing,

\[ \sum_{2^j \geq \frac{N}{M}} \sum_k 2^{k/2} \left\| PN(\chi(2^{-j} R) \frac{P_M v}{R^2}) \right\|_{L^2_{T,x}(\mathbb{R} \times \{ x: 2^k \leq |x| \leq 2^{k+1} \})} \]

\[ \leq \sum_{2^j \geq \frac{N}{M}} \sum_{k \geq j} 2^{k/2} 2^{-5|j-k|} \left\| PN(\chi(2^{-j} R) \frac{P_M v}{R^2}) \right\|_{L^2_{T,x}(\mathbb{R} \times \mathbb{R}^d)} \]

\[ + \sum_{2^j \geq \frac{N}{M}} \sum_{k \leq j} 2^{k/2} \left\| PN(\chi(2^{-j} R) \frac{P_M v}{R^2}) \right\|_{L^2_{T,x}(\mathbb{R} \times \mathbb{R}^d)} \]

\[ \leq \sum_{2^j \geq \frac{N}{M}} 2^{j/2} \left\| PN(\chi(2^{-j} R) \frac{P_M v}{R^2}) \right\|_{L^2_{T,x}(\mathbb{R} \times \mathbb{R}^d)}, \tag{230} \]

and

\[ \sum_k 2^{k/2} \left\| PN(\phi(NR) \frac{P_M v}{R^2}) \right\|_{L^2_{T,x}(\mathbb{R} \times \{ x: 2^k \leq |x| \leq 2^{k+1} \})} \]

\[ \leq \sum_{2^k \leq \frac{N}{M}} 2^{k/2} \left\| PN(\phi(NR) \frac{P_M v}{R^2}) \right\|_{L^2_{T,x}(\mathbb{R} \times \mathbb{R}^d)} \]

\[ + \sum_{2^k \geq \frac{N}{M}} 2^{k/2} \left( \frac{2^{-k}}{N} \right)^5 \left\| PN(\phi(NR) \frac{P_M v}{R^2}) \right\|_{L^2_{T,x}(\mathbb{R} \times \mathbb{R}^d)} \]

\[ \leq N^{-1/2} \left\| PN(\phi(NR) \frac{P_M v}{R^2}) \right\|_{L^2_{T,x}(\mathbb{R} \times \mathbb{R}^d)}. \tag{231} \]

For each \( N \) we will consider four cases, \( M \geq N \) on the support of \( \phi(N \rho) \), \( M \geq N \) and \( R \geq \frac{1}{N} \), \( M \leq N \) on the support of \( \phi(NR) \), and \( M \leq N \) and \( R \geq \frac{1}{N} \). We start with \( M \geq N \) and \( R \geq \frac{1}{N} \). By (226),
\[
\left\| \sum_{M \geq N} \sum_{2^j \geq N} P_N(\chi(2^{-j}R)m \frac{P_M v}{R^2}) \right\|_Y
\leq \sum_{M \geq N} \sum_{2^j \geq N} 2^{j/2} c(E(v)) \left\| \chi(2^{-j}R) \frac{P_M v}{R^2} \right\|_{L^2_{T,x}(\mathbb{R} \times \mathbb{R}^4)}
\leq \sum_{M \geq N} \sum_{2^j \geq N} \frac{2^{-j}}{M} c(E(v)) \left( \sup_{\rho > 0} \rho^{-1/2} \cdot M \left\| P_M v \right\|_{L^2_{T,x}(\mathbb{R} \times \{x: |x| \leq \rho\})} \right) \tag{232}
\leq \sum_{M \geq N} \frac{N}{M} c(E(v)) \left( \sup_{\rho > 0} \rho^{-1/2} \cdot M \left\| P_M v \right\|_{L^2_{T,x}(\mathbb{R} \times \{x: |x| \leq \rho\})} \right) \tag{233}
\leq \sum_{M \geq N} \frac{N}{M} c(E(v)) \left( \sup_{\rho > 0} \rho^{-1/2} \cdot M \left\| P_M v \right\|_{L^2_{T,x}(\mathbb{R} \times \{x: |x| \leq \rho\})} \right) \tag{234}
\]

Then by Young’s inequality and (165), since we are summing \( M \) and \( N \) over the dyadic integers \( M = 2^k \), \( N = 2^l \) for \( k, l \in \mathbb{Z} \),

\[
\sum_N \left( \sum_{M \geq N} \frac{N}{M} c(E(v)) \left( \sup_{\rho > 0} \rho^{-1/2} \cdot M \left\| P_M v \right\|_{L^2_{T,x}(\mathbb{R} \times \{x: |x| \leq \rho\})} \right) \right)^2
\leq c(E(v))^2 \|v\|_{X}^2. \tag{235}
\]

For \( M \geq N \) on the support of \( \phi(NR) \), the Sobolev embedding theorem and Hölder’s inequality imply that

\[
\sum_{M \geq N} N^{-1/2} \left\| P_N \left( \phi(NR)m \frac{P_M v}{R^2} \right) \right\|_{L^2_{T,x}(\mathbb{R} \times \mathbb{R}^4)} \leq N^{-1/2} N^2 \sum_{M \geq N} \left\| \phi(RN)m \frac{P_M v}{R^2} \right\|_{L^2_T L^1_x(\mathbb{R} \times \mathbb{R}^4)} \tag{236}
\]

\[
\leq c(E(v)) N^{3/2} \sum_{M \geq N} \| P_M v \|_{L^2_{T,x}(\mathbb{R} \times \{x: |x| \leq \alpha\})} \| P_M v \|_{L^{11/6}_{T,x}(\mathbb{R} \times \{x: |x| \leq \alpha\})} \| L^\infty_T L^2_x(\mathbb{R} \times \{x: |x| \leq \alpha\}) \| \tag{237}
\]

\[
\leq c(E(v)) \sum_{M \geq N} \frac{N}{M} \left( \sup_{\rho > 0} \rho^{-1/2} \cdot M \left\| P_M v \right\|_{L^2_{T,x}(\mathbb{R} \times \{x: |x| \leq \rho\})} \right). \tag{238}
\]

Again by Young’s inequality,

\[
\sum_N \left( \sum_{M \geq N} \frac{N}{M} c(E(v)) \left( \sup_{\rho > 0} \rho^{-1/2} \cdot M \left\| P_M v \right\|_{L^2_{T,x}(\mathbb{R} \times \{x: |x| \leq \rho\})} \right) \right)^2 \leq c(E(v))^2 \|v\|_{X}^2. \tag{240}
\]

Likewise, for \( M \leq N \) and \( R \) on the support of \( \phi(NR) \), the Sobolev embedding theorem and Hölder’s inequality imply
\[
\sum_{M \leq N} N^{-1/2} \left\| P_N \left( \phi(RN) \frac{P_M v}{R^2} \right) \right\|_{L^2_T L^8_x(\mathbb{R} \times \mathbb{R}^4)} \label{eq:241}
\]
\[
\leq \sum_{M \leq N} c(E(v)) \left\| \phi(RN) \frac{P_M v}{R^2} \right\|_{L^2_T L^{8/5}_x(\mathbb{R} \times \mathbb{R}^4)} \label{eq:242}
\]
\[
\leq \sum_{M \leq N} N^{-1/2} c(E(v)) \left\| P_M v \right\|_{L^2_T L^8_x(\mathbb{R} \times \mathbb{R}^4)} \label{eq:243}
\]
\[
\leq \sum_{M \leq N} \frac{M^{1/2}}{N^{1/2}} c(E(v)) \left\| P_M v \right\|_{L^2_T L^8_x(\mathbb{R} \times \mathbb{R}^4)} \label{eq:244}
\]
and
\[
\sum_N \left( \sum_{M \leq N} c(E(v)) \frac{M^{1/2}}{N^{1/2}} \left\| P_M v \right\|_{L^2_T L^8_x(\mathbb{R} \times \mathbb{R}^4)} \right)^2 \leq c(E(v))^2 \|v\|_X^2. \label{eq:245}
\]

Finally suppose \( R \geq \frac{1}{N} \) and \( M \leq N \). By Hölder’s inequality and the Sobolev embedding theorem,
\[
\sum_{M \leq N} \sum_{\frac{1}{2} \leq 2^j \leq M^{-1/4} N^{-3/4}} 2^{j/2} \left\| P_N \left( \chi(2^{-j} R) \frac{P_M v}{R^2} \right) \right\|_{L^2_T L^8_x(\mathbb{R} \times \mathbb{R}^4)} \label{eq:246}
\]
\[
\leq \sum_{M \leq N} \sum_{\frac{1}{2} \leq 2^j \leq M^{-1/4} N^{-3/4}} \left\| P_M v \right\|_{L^2_T L^{8/5}_x(\mathbb{R} \times \mathbb{R}^4)} 2^{j/2} c(E(v)) \label{eq:247}
\]
\[
\leq c(E(v)) \sum_{M \leq N} \frac{M^{1/8}}{N^{1/8}} \left\| P_M v \right\|_{L^2_T L^8_x(\mathbb{R} \times \mathbb{R}^4)} \label{eq:248}
\]
and by Young’s inequality,
\[
\sum_N \left( \sum_{M \leq N} \frac{M^{1/8}}{N^{1/8}} \left\| P_M v \right\|_{L^2_T L^8_x(\mathbb{R} \times \mathbb{R}^4)} c(E(v)) \right)^2 \leq c(E(v))^2 \|v\|_X^2. \label{eq:249}
\]

It only remains to compute
\[
\sum_{M \leq N} \sum_{2^j \geq M^{-1/4} N^{-3/4}} 2^{j/2} \left\| P_N \left( \frac{m P_M v}{\rho^2} \right) \right\|_{L^2_T L^8_x(\mathbb{R} \times \mathbb{R}^4)} \label{eq:250}
\]
To compute this we will use Bernstein’s inequality, which by the product rule will make use of a derivative of \( m \). By Einstein’s equations (7), and \(|\Omega^2| \leq 1,\)
\[
\left| \partial_R m \right| \leq \frac{f(u)^2}{4r} + r(\partial u)^2, \label{eq:251}
\]
where \((\partial u)^2\) is shorthand for \((\partial_T u)^2 + |\nabla_x u|^2\), \( u = Rv \). By Bernstein’s inequality, the product rule, and \( \rho \sim R \),
\[
2^{j/2} \left\| P_N \left( m\chi(2^{-j} \rho) \frac{P_M v}{R^2} \right) \right\|_{L^2_{T,x}(\mathbb{R} \times \mathbb{R}^4)} \\
\leq \frac{2^{j/2}}{N} \left\| \nabla P_N \left( m\chi(2^{-j} \rho) \frac{P_M v}{R^2} \right) \right\|_{L^2_{T,x}(\mathbb{R} \times \mathbb{R}^4)} \\
\leq \frac{2^{j/2}}{N} \left\| P_N \left( m\chi(2^{-j} \rho) \nabla P_M v \right) \right\|_{L^2_{T,x}(\mathbb{R} \times \mathbb{R}^4)} \\
+ \frac{2^{j/2}}{N} \left\| P_N \left( m\chi(2^{-j} \rho) \frac{P_M v}{R^2} \right) \right\|_{L^2_{T,x}(\mathbb{R} \times \mathbb{R}^4)} \\
+ \frac{2^{-j/2}}{N} \left\| P_N \left( m\chi'(2^{-j} \rho) \frac{P_M v}{R^2} \right) \right\|_{L^2_{T,x}(\mathbb{R} \times \mathbb{R}^4)} \\
+ \frac{2^{j/2}}{N} \left\| P_N \left( (\partial_R m)\chi(2^{-j} \rho) \frac{P_M v}{R^2} \right) \right\|_{L^2_{T,x}(\mathbb{R} \times \mathbb{R}^4)}. 
\]

First take (253).

\[
(253) \leq c(E(v)) 2^{-2j} \cdot \frac{2^{j/2}}{N} \left\| \nabla P_M v \right\|_{L^2_{T,x}(\mathbb{R} \times \{2^j \leq |x| \leq 2^{j+1}\})} \\
\leq c(E(v)) \frac{2^{-j}}{N} \left( \sup \rho^{-1/2} \left\| \nabla P_M v \right\|_{L^2_{T,x}(\mathbb{R} \times \{x:|x| \leq \rho\})} \right). 
\]

and by Young’s inequality,

\[
\sum_N (c(E(v)) \sum_{M \leq N} \frac{M^{1/4}}{N^{1/4}} \left( \sup \rho^{-1/2} \left\| \nabla P_M v \right\|_{L^2_{T,x}(\{x:|x| \leq \rho\})} \right)^2 \leq c(E(v))^2 \left\| v \right\|_{X}^2. 
\]

Next take (261). This time, by Hölder’s inequality,

\[
\frac{2^{j/2}}{N} \left\| m\chi(2^{-j} \rho) \frac{P_M v}{R^3} \right\|_{L^2_{T,x}(\mathbb{R} \times \mathbb{R}^4)} \\
\leq \frac{2^{-j/2}}{N} c(E(v)) \left\| P_M v \right\|_{L^p_{T,x} L^\infty_x(\mathbb{R} \times \mathbb{R}^4)} \leq \frac{2^{-j/2} M^{1/2}}{N} c(E(v)) \left\| P_M v \right\|_{L^2_p L^{8}_x(\mathbb{R} \times \mathbb{R}^4)}.
\]
\[
\sum_N \left( \sum_{M \leq N} \sum_{2^j \geq CM^{-1/4}N^{-3/4}} \frac{2^{-j/2}M^{1/2}}{N} c(E(v)) \left\| P_M v \right\|_{L^2_T L^8_x(\mathbb{R} \times \mathbb{R}^4)}^2 \right) \leq c(E(v))^2 \| v \|_X^2.
\]

(262)

The estimate of (265) is virtually identical to the estimate of (254).

Finally, we turn our attention to (256). \( f(0) = 0, f'(0) = 1, Rv \) uniformly bounded implies that the first term in (251),

\[
\frac{f(u)^2}{r} \leq \frac{c(E(v))}{R},
\]

(263)

and then

\[
\frac{2^{j/2}}{N} \left\| \frac{f(u)^2}{r} \chi(2^{-j} \rho) \frac{P_M v}{R^2} \right\|_{L^2_{T,x}(\mathbb{R} \times \mathbb{R}^4)}
\]

(264)

can be computed in exactly the same manner as (254) or (255).

Now we compute

\[
\frac{2^{j/2}}{N} \left\| P_N \left( \chi(2^{-j} \rho) \frac{P_M v}{R^2} r(\partial u)^2 \right) \right\|_{L^2_{T,x}(\mathbb{R} \times \mathbb{R}^4)}.
\]

(265)

By the radial Sobolev embedding theorem and lemma 2.8

\[
\frac{2^{j/2}}{N} \left\| (1 - \phi(2^{-j+10} \rho)) P_N \left( \chi(2^{-j} \rho) r^3(\partial v)^2 \frac{P_M v}{R^2} \right) \right\|_{L^2_{T,x}(\mathbb{R} \times \mathbb{R}^4)}
\]

(266)

\[
\leq \frac{2^{j/2}}{N} \left\| (1 - \phi(2^{-j+10} \rho)) P_N \left( \chi(2^{-j} \rho)(\partial v)^2 \frac{R P_M v}{R^2} \right) \right\|_{L^2_{T,x}(\mathbb{R} \times \mathbb{R}^4)}
\]

(267)

\[
\leq \frac{2^{-j}}{N^{1/2}} \left\| \chi(2^{-j} \rho)(\partial v)^2 \frac{R P_M v}{R^2} \right\|_{L^2_{T,x}(\mathbb{R} \times \mathbb{R}^4)}
\]

(268)

\[
\leq \frac{2^{-j/2}}{N^{1/2}} E(v)^{1/2} \left( \sup_{\rho > 0} \rho^{-1/2} \left\| \partial v \right\|_{L^2_{T,x}(\mathbb{R} \times \{x: |x| \leq \rho\})} \right) \left\| R P_M v \right\|_{L^\infty_{T,x}(\mathbb{R} \times \mathbb{R}^4)}
\]

(269)

\[
\leq \frac{2^{-j/2}}{N^{1/2}} E(v)^{1/2} \left\| v \right\|_X \left\| R P_M v \right\|_{L^\infty_{T,x}(\mathbb{R} \times \mathbb{R}^4)}.
\]

(270)

Therefore by Young’s inequality,
Now if $x \in \text{supp}(\phi(2^{-j+10}\rho))$ and $y \in \text{supp}(\chi(2^{-j}\rho))$, $|x-y| \sim 2^j$. Therefore, since the Littlewood - Paley projection kernel is rapidly decreasing,

Now if $x \in \text{supp}(\phi(2^{-j+10}\rho))$ and $y \in \text{supp}(\chi(2^{-j}\rho))$, $|x-y| \sim 2^j$. Therefore, since the Littlewood - Paley projection kernel is rapidly decreasing,
\[ \tilde{E}(\tilde{v}) := \|\tilde{v}_0\|_{H^1(\mathbb{R}^4)} + \|\tilde{v}_1\|_{L^2(\mathbb{R}^4)} + \frac{1}{2} \|\tilde{v}_0\|_{L^4(\mathbb{R}^4)}, \]  

(278)

Suppose

\[ \left| \frac{R}{r} - 1 \right| \leq \tilde{E}(\tilde{v}) \quad \text{and} \quad |R\tilde{v}| \leq \tilde{E}(\tilde{v}). \]

Recall that the equation \(276\) is a partially linearized equation of the original wave maps equation obtained by the linearization of the wave equation \(7d\) for \(Z\) (which implies \(Z \equiv 0\)).

Firstly we prove the following nonlinear Morawetz estimate for small energy.

**Lemma 3.1.** For any global solution \(\tilde{v}\) of \((276)\) such that

\[ \left| \frac{R}{r} - 1 \right| \leq \tilde{E}(\tilde{v}) \quad \text{and} \quad |R\tilde{v}| \leq \tilde{E}(\tilde{v}), \]

\[ \int_{\mathbb{R}^4} \frac{\tilde{v}^2}{|x|^3} \, d\tilde{g} \leq \tilde{E}(\tilde{v}) \]  

(279)

for \(\tilde{E}(\tilde{v}) < \epsilon^2\), \(\epsilon\) sufficiently small.

**Proof.** We shall use the estimates \( |\frac{R}{r} - 1|, |R\tilde{v}| \leq \tilde{E}(\tilde{v}) \) throughout. Define the Morawetz quantity

\[ M(T) := \int \tilde{v}_R \tilde{v}_T R^3 \, dR + \frac{3}{2} \int \tilde{v}_T \tilde{v} R^2 \, dR. \]  

(280)

Taking the time derivative,

\[ \frac{d}{dt}(M(t)) = \int \partial R(\tilde{v}_T) \tilde{v}_T R^3 \, dR + \frac{3}{2} \int \tilde{v}_T^2 R^2 \, dR + \int \tilde{v}_R \tilde{v}_T R^3 \, dR \]

\[ + \frac{3}{2} \int \tilde{v}_T \tilde{v} R^2 \, dR. \]  

(281)

Integrating by parts,

\[ \int \partial R(\tilde{v}_T) \tilde{v}_T R^3 \, dR + \frac{3}{2} \int \tilde{v}_T^2 R^2 \, dR = 0. \]  

(282)

Now using \((276)\) to split \(\tilde{v}_{TT}\),
\[
\int \tilde{v}_R(\tilde{v}_{RR} + \frac{3}{R} \tilde{v}_R)R^3dR + \frac{3}{2} \int (\tilde{v}_{RR} + \frac{3}{R} \tilde{v}_R)\tilde{v}R^2dR
\]  
(283)

\[
= -\frac{3}{2} \int \tilde{v}_R^2 R^2dR + 3 \int \tilde{v}_R^2 R^2dR - \frac{3}{2} \int \tilde{v}_R^2 R^2dR
\]

\[
- 3 \int \tilde{v}_R u RdR + \frac{9}{2} \int \tilde{v}_R \tilde{v} RdR
\]

(284)

\[
= -\frac{3}{4} \int \tilde{v}^2 dR.
\]

Therefore,

\[
\int \int \tilde{v}^2 dRdT = \frac{4}{3} (M(T) - M(0)) + \int \int F(\tilde{v}) \tilde{v}_R R^3 dRdT + \int \int F(\tilde{v}) \tilde{v} R^2 dRdT.
\]

(286)

By Hardy's inequality and conservation of energy,

\[
|M(T) - M(0)| \leq \tilde{E}(\tilde{v}).
\]

(287)

First consider

\[
\int \int \tilde{F}(\tilde{v}) \tilde{v}_R R^3 dRdT.
\]

(288)

Making a change of variables

\[
\int \int \frac{1}{R} (\partial_\eta r - \frac{1}{2}) (\partial_\xi \tilde{v})(\partial_R \tilde{v}) R^3dRdT = \int \int \frac{1}{R} (\partial_\eta r - \frac{1}{2}) (\partial_\xi \tilde{v})(\partial_R \tilde{v}) R^3 d\eta d\xi
\]

\[
\leq \int (\sup_{R>0} \frac{1}{R} \int_0^R f^2(\tilde{v}) sds)(\int (\partial \tilde{v})^2 d\xi) d\eta
\]

\[
\leq \tilde{E}(\tilde{v}) \int \int \tilde{v}^2 dRdT. 
\]

(289)

Similarly,

\[
\int \int \frac{1}{R} (\partial_\xi r + \frac{1}{2}) (\partial_\eta \tilde{v})(\partial_R \tilde{v}) R^3dRdT \leq \tilde{E}(\tilde{v}) \int \int \tilde{v}^2 dRdT.
\]

(290)

Next,

\[
\int \int (\partial_\eta r + \frac{1}{2}) \tilde{v}_R R^3dRdT \leq \int \int \frac{1}{R} (\partial_\eta r + \frac{1}{2}) \tilde{v}_R^2 R^3dRdT
\]

\[
+ \int \int (\partial_\eta r + \frac{1}{2}) \tilde{v}^2 dRdT
\]

\[
\leq \tilde{E}(\tilde{v}) \int \int \tilde{v}^2 dRdT + \epsilon \int \int \tilde{v}^2 dRdT.
\]

(291)

Likewise,
\[
\int \int (\partial_{\xi} r - \frac{1}{2}) \frac{\tilde{v}}{R^2} \tilde{v} R^3 dRdT \leq \tilde{E}(\tilde{v}) \int \int \tilde{v}^2 dRdT + \epsilon \int \int \tilde{v}^2 dRdT. \tag{292}
\]
Expanding \(\zeta(R\tilde{v}) = c_1 + c_2(R\tilde{v}) + \cdots\), then integrating by parts
\[
c_1 \int \int \frac{R^2}{r^2} \tilde{v}^3 \tilde{v} R^3 dRdT = \frac{c_1}{4} \int \int \frac{R^5}{r^2} \partial_R (\tilde{v}^4) dRdT - \frac{5c_1}{4} \int \int \frac{R^4}{r^2} \tilde{v}^4 dRdT - \frac{c_1}{2} \int \int \frac{R^5}{r^3} (\partial_{RR} \tilde{v}) \tilde{v}^4 dRdT. \tag{293}
\]
Then by the radial Sobolev embedding theorem, \(R\tilde{v} \leq \epsilon\). Therefore,
\[
293 \lesssim \epsilon^2 \int \int \tilde{v}^2 dRdT. \tag{294}
\]
Now we turn to
\[
\int \int \tilde{F}(\tilde{v}) \tilde{v} R^2 dRdT. \tag{295}
\]
First,
\[
\int \int \frac{1}{R} (\partial_\eta r - \frac{1}{2}) (\partial_\xi \tilde{v}) \tilde{v} R^2 dRdT \\
\leq \int \int \frac{1}{R} (\partial_\eta r - \frac{1}{2}) (\partial_\xi \tilde{v})^2 R^3 dRdT + \int \int (\partial_\eta r - \frac{1}{2}) \tilde{v}^2 dRdT \tag{296}
\]
\[
\leq \tilde{E}(\tilde{v}) \int \int \tilde{v}^2 dRdT + \epsilon \int \int \tilde{v}^2 dRdT.
\]
Similarly,
\[
\int \int \frac{1}{R} (\partial_\xi r + \frac{1}{2}) (\partial_\eta \tilde{v}) \tilde{v} R^2 dRdT \leq \tilde{E}(\tilde{v}) \int \int \tilde{v}^2 dRdT + \epsilon \int \int \tilde{v}^2 dRdT. \tag{297}
\]
Therefore, by the fundamental theorem of calculus,
\[
\int \int \tilde{v}^2 dRdT \leq \tilde{E}(\tilde{v}) + \epsilon \int \int \tilde{v}^2 dRdT + \tilde{E}(\tilde{v}) \int \int \tilde{v}^2 dRdT. \tag{298}
\]
Therefore, for \(\tilde{E}(\tilde{v})\) sufficiently small,
\[
\int \int \tilde{v}^2 dRdT \leq E(\tilde{v}). \tag{299}
\]
Now then, it is necessary to prove scattering.

**Theorem 3.2.** The globally regular solution to (276) scatters forward and backward in time.
Proof: To prove scattering it suffices to show that the solution to

\[ \begin{align*}
\Box^{4+1} \bar{v} &= F(\bar{v}) & \text{on } \mathbb{R}^{4+1} \\
\bar{v}(T_0, x) &= 0 \quad \text{and} \quad \partial_T \bar{v}(T_0, x) = 0 & \text{on } \mathbb{R}^4
\end{align*} \]  

(300)

where \( F(\bar{v}) \) is

\[ \begin{align*}
F(\bar{v}) &= \left( \frac{1}{r} \partial_\eta r + \frac{1}{2R} \right) (\partial_\xi \bar{v}) + \left( \frac{1}{R} \partial_\xi r - \frac{1}{2} \right) (\partial_\eta \bar{v}) \\
&\quad + \left( \frac{r}{R} \partial_\eta r + \frac{1}{2} - \frac{r}{R} \partial_\xi r - \frac{1}{2} \right) \frac{\bar{v}}{r^2} + \frac{R^2}{r^2} \bar{v}^3(\bar{R} \bar{v}) \\
&\quad + \frac{1}{2} (\partial_\eta r + \frac{1}{2}) (\partial_\xi \bar{v}) \cdot \bar{v}_T dRdT \\
&\quad + \frac{1}{2} (\partial_\xi r - \frac{1}{2}) (\partial_\eta \bar{v}) \cdot \bar{v}_T dRdT
\end{align*} \]  

(301)

has a solution with

\[ \sup_{T \geq T_0} \left[ \| \bar{v}(T) \|_{\dot{H}^1} + \| \bar{v}_T(T) \|_{L^2} \right] \to 0 \]

as \( T_0 \to \infty \). Then

\[ \bar{v}(T) = \bar{v}(T) + S(T - T_0)(\bar{v}(T_0), \bar{v}_T(T_0)) = \bar{v}(T) + w(T), \]

(303)

where \( S(t)(\bar{v}_0, \bar{v}_1) \) is the solution to the wave equation \( \Box w = 0 \) with initial data \( (\bar{v}_0, \bar{v}_1) \). In particular, this implies \( \bar{E}(\bar{v}) \leq \tilde{E}(\bar{v}) \), where

\[ \begin{align*}
\bar{E}(\bar{v}) &= \| \bar{v}_T \|_{L^2}^2 + \| \nabla \bar{v} \|_{L^2}^2 + \frac{1}{2} \| \bar{v} \|_{L^4}^4 \\
&\quad + \frac{1}{2} (\partial_\eta r + \frac{1}{2}) \frac{\bar{v}_3}{\bar{v}} R^2 \bar{v}^2 dRdT \\
&\quad + \frac{1}{2} (\partial_\xi r - \frac{1}{2}) \frac{\bar{v}}{\bar{v}} R^2 \bar{v}_T dRdT
\end{align*} \]  

(304)

Now compute

\[ \frac{d}{dT} \left[ \frac{1}{2} \langle \bar{v}_T, \bar{v}_T \rangle + \frac{1}{2} \langle \nabla \bar{v}, \nabla \bar{v} \rangle \right] = -\langle \bar{v}_T, \tilde{F}(\bar{v}) \rangle. \]  

(305)

Then as in (289) and (290), using \( \left| \frac{R}{r} - 1 \right| \leq \tilde{E}(\bar{v}) \), by the dominated convergence theorem,

\[ \lim_{T_0 \to \infty} \int_{T_0}^\infty \int \frac{1}{R} \left( \partial_\eta - \frac{1}{2} \right) \left( \partial_\xi \bar{v} \right) \cdot \bar{v}_T dRdT = 0, \]  

(306)

and

\[ \lim_{T_0 \to \infty} \int_{T_0}^\infty \int \frac{1}{R} \left( \partial_\xi + \frac{1}{2} \right) \left( \partial_\eta \bar{v} \right) \cdot \bar{v}_T dRdT = 0. \]  

(307)

As in (291) and (292),

\[ \lim_{T_0 \to \infty} \int_{T_0}^\infty \int \left( \partial_\eta r + \frac{1}{2} \right) \frac{\bar{v}}{R^2} \bar{v}_T dRdT = 0, \]  

(308)

\[ \lim_{T_0 \to \infty} \int_{T_0}^\infty \int \left( \partial_\xi r - \frac{1}{2} \right) \frac{\bar{v}}{R^2} \bar{v}_T dRdT = 0. \]  

(309)

To compute

\[ \int \int \frac{R^2}{r^2} \bar{v}^3(\bar{R} \bar{v}) R^3 \bar{v}_T dRdT, \]

(310)
again expand out $\zeta(R\tilde{v})$. Then

$$
c_1 \int \frac{R^2}{r^2} \tilde{v}^3 \tilde{v}_T R^3 dR = \frac{d}{dT} \left( \frac{c_1}{4} \int \frac{R^2}{r^2} \tilde{v}^4 R^3 dR \right) + \frac{c_1}{2} \int \frac{R^2}{r^3} \tilde{v}^4 R^3 (\partial_T r) dR.
$$

(311)

Now by the radial Sobolev embedding theorem, combined with the Morawetz estimates,

$$
\int_{T \geq T_0} \int \frac{R^2}{r^3} \tilde{v}^4 R^3 (\partial_T r) dRdT \rightarrow 0 \quad (312)
$$

as $T_0 \rightarrow \infty$. Next, using the radial Strichartz estimate

$$
\left\| |x|^{1/2} S(t)(\tilde{v}_0, \tilde{v}_1) \right\|_{L^2_x L^\infty_t} \leq \|\tilde{v}_0\|_{H^1} + \|\tilde{v}_1\|_{L^2}, \quad (313)
$$

so

$$
\int_{T \geq T_0} \int \frac{R^2}{r^2} \tilde{v}^2 w \tilde{v}_T R^3 dRdT \leq \left( \int_{T \geq T_0} \int \tilde{v}^2 dRdT \right)^{1/2} \cdot \|R^{1/2} w\|_{L^2_T L^\infty_x} \|\tilde{v}_T\|_{L^\infty_T L^2_x} \|R\tilde{v}\|_{L^\infty_T L^\infty_x}. \quad (314)
$$

If

$$
E(\tilde{v}(T)) = \|\tilde{v}_T\|_{L^2}^2 + \|\nabla \tilde{v}(T)\|_{L^2}^2 + \frac{1}{2}\|\tilde{v}(T)\|_{L^4}^4, \quad (315)
$$

then the Sobolev embedding theorem implies that for small energy,

$$
(314) \leq \left( \sup_{T \geq T_0} E(\tilde{v}(T)) \right)^{1/2} \tilde{E}(\tilde{v}) \left( \int_{T \geq T_0} \int \tilde{v}^2 dRdT \right)^{1/2}. \quad (316)
$$

Also,

$$
\int_{T \geq T_0} \int \frac{R^2}{r^2} \tilde{v} \cdot w \cdot \tilde{v}_T R^3 dRdT \leq \left( \int_{T \geq T_0} \int \tilde{v}^2 dRdT \right)^{1/2} \cdot \|R^{1/2} w\|_{L^2_T L^\infty_x} \|\tilde{v}_T\|_{L^\infty_T L^2_x} \|R\tilde{v}\|_{L^\infty_T L^\infty_x} \leq \tilde{E}(\tilde{v}) \left( \sup_{T \geq T_0} E(\tilde{v}(T)) \right). \quad (317)
$$

Finally,
\[ \int_{T \geq T_0} \int \frac{R^2}{r^2} w \bar{v}^2 \bar{v}_T R^3 dRdT \leq \left\| |x|^{1/2} w \right\|_{L_T^2 L_x^\infty} \left\| \bar{v}_T \right\|_{L_T^\infty L_x^2} \]
\[ \cdot \left( \int \bar{v}^2 + w^2 dRdT \right)^{1/2} \left\| |x| \bar{v} \right\|_{L_T^\infty L_x^2} \] (318)
\[ \leq \tilde{E}(\bar{v}) \left( \sup_{T \geq T_0} E(\bar{v})(T) \right) . \]

Therefore we have proved
\[ \sup_{T \geq T_0} E(\bar{v}(T)) \leq \epsilon \left( \sup_{T \geq T_0} E(\bar{v}(T)) \right) + \left( \int_{T \geq T_0} \int \bar{v}^2 dRdT \right) . \] (319)

Since
\[ \int_{T \geq T_0} \int \bar{v}^2 dRdT \to 0 \] (320)
as \( T_0 \to \infty \), we have scattering. □

**Acknowledgements**

The authors are grateful to Jalal Shatah for answering their questions related to his work. The authors are also thankful to the referee for the thoughtful comments and for suggesting an error in a previous work.

**References**

[1] L. Andersson, N. Gudapati, and J. Szeftel. Global regularity for 2+1 dimensional Einstein-wave map system. arXiv, 1501.00616, 2015.

[2] B. K. Berger, P.T. Chruściel, and V. Moncrief. On “asymptotically flat” space-times with \( G_2 \)– invariant Cauchy surfaces. Ann. Phys., 237:322–354, 1995.

[3] Y. Choquet-Bruhat and R. Geroch. Global aspects of the Cauchy problem in general relativity. Comm. Math. Phys., 14:329–335, 1969.

[4] D. Christodoulou and A.S. Tahvildar Zadeh. On the asymptotic behavior of spherically symmetric wave maps. Duke Math. J., 71(1):31–69, 1993.

[5] D. Christodoulou and A.S. Tahvildar Zadeh. On the regularity of spherically symmetric wave maps. Comm. Pure Appl. Math., 46(7):1041–1091, 1993.

[6] N. Gudapati. The Cauchy problem for energy critical self-gravitating wave maps. Dissertation (FU Berlin), 2013.

[7] M. Keel and T. Tao. Endpoint Strichartz estimates. Amer. J. Math., 120:955–980, 1998.

[8] C. Kennig and F. Merle. Global well posedness, scattering and blow-up for the energy-critical focusing nonlinear wave equation. Acta. Math, 201(2):147–212, 2008.

[9] J. Krieger. Global regularity of wave maps from \( \mathbb{R}^{2+1} \) to \( \mathbb{H}^2 \). Comm. Math. Phys., 250(4):507–580, 2004.
[10] J. Krieger and W. Schlag. Concentration compactness for critical wave maps. *EMS Monographs in Mathematics*, 2012.

[11] J. Shatah and M. Struwe. Geometric wave equations. *AMS*, 2000.

[12] J. Shatah and A.S. Tahvildar Zadeh. Regularity of harmonic maps from the Minkowski space into rotationally symmetric manifolds. *Comm. Pure Appl. Math.*, 45:1041–1091, 1992.

[13] J. Shatah and A.S. Tahvildar Zadeh. Cauchy problem for equivariant wave maps. *Comm. Pure Appl. Math.*, 47(5):719–754, 1994.

[14] C. Sogge. Lectures on nonlinear wave equation. *International Press*, 2nd Edition, 2008.

[15] J. Sterbenz and D. Tataru. Energy dispersed large data wave maps in 2+1 dimensions. *Comm. Math. Phys.*, 298:139–230, 2010.

[16] J. Sterbenz and D. Tataru. Regularity of wave maps in dimension 2+1. *Comm. Math. Phys.*, 298:231–264, 2010.

[17] M. Struwe. Wave maps with and without symmetries. *Clay lecture notes*, 2008.

[18] T. Tao. Global regularity of wave maps i–vii. *arXiv e-prints*.

[19] T. Tao. Nonlinear dispersive equations. *CBMS*, 106, 2006.

[20] D. Tataru. On global existence and scattering for the wave maps equation. *Amer. J. Math.*, 123(3):385–423, 2001.

[21] M. Taylor. Measure Theory and Integration. *American Mathematical Society*, Graduate Studies in Mathematics, 76, 2006.

Benjamin Dodson
Department of Mathematics, Johns Hopkins University, 3400 N. Charles Street, Baltimore, MD-21218, USA
e-mail: dodson@math.jhu.edu

Nishanth Gudapati
Department of Mathematics, Yale University, 10 Hillhouse Avenue, New Haven, CT-06511, USA
e-mail: nishanth.gudapati@yale.edu