LOW REGULARITY CONSERVATION LAWS FOR INTEGRABLE PDE

Rowan Killip · Monica Vişan · Xiaoyi Zhang

Abstract. We present a general method for obtaining conservation laws for integrable PDE at negative regularity and exhibit its application to KdV, NLS, and mKdV. Our method works uniformly for these problems posed both on the line and on the circle.

1 Introduction

The original goal of this work was to obtain low-regularity conservation laws for the Korteweg–de Vries equation

$$\frac{d}{dt}q = -q''' + 6qq'$$. (KdV)

However, the method we developed to address this problem turns out to be of more general validity, as we shall demonstrate by applying it to the cubic NLS and mKdV equations. All three equations can be posed both on the real line $\mathbb{R}$ and on the circle $\mathbb{R}/\mathbb{Z}$. (The latter case is equivalent to that of spatially periodic initial data on the whole line.) The methods we use here apply equally well in both settings and correspondingly, we shall be treating them in parallel.

Naturally, the existence of conservation laws must be predicated on the existence of solutions. Conversely, it is difficult to construct solutions without control on the growth of crucial norms. As is usual in the study of PDE, our approach here is to only consider solutions that are smooth and rapidly decaying, but prove results that are uniform in low-regularity norms. The fact that Schwartz-space initial data lead to unique global solutions to (KdV) that remain in Schwartz class has been known for some time; see, for example, [BS75, Kat83, Sjo67, Sjo70, Tem69]. (Recall that Schwartz space on $\mathbb{R}/\mathbb{Z}$ is coincident with $C^\infty(\mathbb{R}/\mathbb{Z})$; on the line, it is comprised of those $C^\infty(\mathbb{R})$ functions that decay faster than any polynomial as $|x| \to \infty$.)

It has been known since [MGK68] that (KdV) admits infinitely many conservation laws. The first three are

$$\int q(t, x) \, dx, \quad \int q(t, x)^2 \, dx, \quad \text{and} \quad \int \frac{1}{2} q'(t, x)^2 + q(t, x)^3 \, dx.$$
We can regard this original family of conservation laws as ordered: Each conserved quantity is a polynomial in $q$ and its derivatives that is scaling homogeneous. Thus, they can be ordered by scaling, or equivalently, by the highest order derivative that appears. The exact form of these conservation laws is rather delicate; in particular, they need not be sign-definite. Nonetheless, they can be used (cf. \[ Lax75, \S 3\]) to show that for an integer $s \geq 0$, the $H^s$-norm of the solution admits a global in time bound depending only on the corresponding norm of the initial data. For well-posedness questions, such global bounds are of greater significance than the particular conservation laws that begot them; correspondingly, our presentation will emphasize such bounds, beginning with the following result.

**Theorem 1.1.** Fix $-1 \leq s < 1$ and let $q$ be a Schwartz solution to (KdV) either on $\mathbb{R}$ or on $\mathbb{R}/\mathbb{Z}$. Then

$$
\|q(0)\|_{H^s} (1 + \|q(0)\|^{2}_{H^s})^{-\frac{|s|}{1+2|s|}} \lesssim \|q(t)\|_{H^s} \lesssim \|q(0)\|_{H^s} (1 + \|q(0)\|^{2}_{H^s})^{\frac{|s|}{1+2|s|}}.
$$

A number of instances of Theorem 1.1 result have appeared before. In the torus case, this result is completely subsumed by [KMT05]. On the real line, the case $s = -1$ was treated in [BK15]. There is also the work [KT], contemporaneous with our own, which covers the full range $s \geq -1$ in the line case. We claim two particular merits for our method here: (i) it is much simpler, and (ii) it works uniformly on both the line and the circle. Our methods also allow us to obtain a priori bounds in Besov spaces (see Section 3 for the definition). Specifically, we shall prove the following:

**Theorem 1.2.** Fix parameters $r$ and $s$ conforming to either of the following restrictions: $s = -1$ and $1 \leq r \leq 2$, or $-1 < s < 1$ and $1 \leq r \leq \infty$. If $q$ is a Schwartz solution to (KdV) either on $\mathbb{R}$ or on $\mathbb{R}/\mathbb{Z}$, then

$$
\|q(0)\|_{B^s_{r,2}} (1 + \|q(0)\|^{2}_{B^s_{r,2}})^{-\frac{3}{2}} \lesssim \|q(t)\|_{B^s_{r,2}} \lesssim \|q(0)\|_{B^s_{r,2}} (1 + \|q(0)\|^{2}_{B^s_{r,2}}).
$$

As a rather obvious corollary of this result, we see that local well-posedness in any of these Besov spaces can be immediately upgraded to global well-posedness. We note two particular applications of this: First, our result provides a simpler alternative to [CKS03], which extended the local well-posedness results of [CKST04,Guo09,KPV96,Kis09] to global well-posedness. The range of Sobolev spaces so obtained ($s \geq -\frac{1}{2}$ on the circle and $s \geq -\frac{3}{4}$ on the line) were shown in [CCT03] to be sharp for analytic well-posedness; that is, the data-to-solution map cannot be analytic on any larger $H^s$ space.

The bounds shown in [CKS03] are not uniform in time; indeed, to transfer local well-posedness to global well-posedness, one need only show that the norm does not blow up in finite time. We also wish to draw attention to the paper [Liu15], which proves a priori bounds in $H^{-4/5}(\mathbb{R})$ locally in time.

As a second application, we note that Theorem 1.2 extends to global-in-time the analytic local well-posedness result of Koch [Koc14, Theorem 6.6] in the $B^{-3/4,2}_{\infty}(\mathbb{R})$