Hierarchic distributed stabilization of a class of three-dimensional formations for underactuated agents

Mohamed I. El-Hawwary

1 Department of Electrical Power Engineering, Faculty of Engineering, Cairo University, Giza, Egypt
2 Division of Decision and Control System, KTH Royal Institute of Technology, Stockholm, Sweden

Abstract
The paper presents a distributed hierarchic control design solving a formations problem in three dimension. The agents are modelled as thrust-underactuated rigid bodies. The class of formations addressed encompasses path following of closed convex paths with shape, size, orientation, relative displacements and relative agents’ on-path angular positions all seen as formation parameters. This problem can be seen as a generalization of circular formation problems which acquired particular attention in the field of multi-agents, and the agents model used is one that usually models certain unmanned aerial vehicles and other autonomous vehicles. The solution relies on reduction-based set stabilization where the problem is broken down into simpler nested sub-problems. The results illustrate how addressing complex control problems using hierarchical set stabilization is natural, simplifies the solution, and provides useful properties. The results are illustrated via simulations.

1 INTRODUCTION

Formation control continues to receive wide attention in control theory and practice. Recent advances in computation, communications and sensor technologies provide the capability to build teams of relatively small and inexpensive robots. These can perform cooperative tasks in air, on land, underwater or in space; and often rely on formation control to fulfill these tasks. The span of applications for such systems is quite diverse [1]. Examples include: environmental monitoring and exploration, surveillance, search and rescue, reconnaissance, space interferometry, and mobile and reconfigurable sensors [1].

Numerous formation control methods have been introduced in literature [2]. This paper utilizes an approach that has not received particular interest [3, 4]. The approach is hierarchical in nature and relies on theories of set stabilization and reduction [5]. Simply put, the desired formation behaviour is first expressed as a goal set in the agents state space. A hierarchy then ensues by breaking down the problem into \( l \in \mathbb{Z}^+ \) sub-problems of stabilizing nested sets \( \Gamma_1 \supset \cdots \supset \Gamma_l \), with \( \Gamma_l \) encoding the formation (goal set), and sub-problem \( i \) is that of stabilizing \( \Gamma_i \) relative to \( \Gamma_{i-1} \) (\( \Gamma_0 \) being the state space). The motivation and contribution aspects of the paper can be seen as threefold.

First, the paper illustrates the potential benefit of hierarchical set stabilization in formation control problems. Formations are usually given by certain relations between the agents; in addition the agents are required to attain certain dynamics while in formation. Being the generalization of equilibrium stabilization set stabilization presents a natural framework to consolidate several specifications into one: the goal set. Set stabilization also naturally prompts a hierarchic approach where subsets of the specifications provide nested sets containing the goal set. Hierarchy is favorable in solving control problems from a number of perspectives. From a design perspective, with the appropriate tools, a complex problem can be solved more easily by breaking it down into simpler decoupled sub-problems. Individual sub-problems can be of independent interest. From a practical perspective, if, with initial design, the system satisfies specifications that are higher in the hierarchy, then a hierarchic design would stabilize the system to lower specifications without violating the higher ones.

Hierarchy has been used in different contexts in formation control, see, for instance, [6]. See also reduction-based results such as [7, 8]. Combining hierarchy and set stabilization, as done here, helps to elucidate the stability analysis, especially for goal behaviours expressed by unbounded sets, a common instance in formation control.
Second, the formation problem addressed here can be seen as a generalization of circular formations problems which received significant attention in literature. The breadth of applications is quite wide [9]. Examples include patrolling and surveillance [10, 11], sensing and data collection [12], target tracking and source seeking [13], search and rescue [14], environmental monitoring [15], and aerospace applications [16, 17]. The generalization presented here provides certain advantages which makes the problem more oriented to the practical setting. Refer to Section 2.1 for a discussion on the problem potential. In addition, the hierarchic solution approach provides extra advantages on its own. Refer to the discussion at the end of Section 3.

Numerous methods and different instances of the circular formation problem can be found in literature. Common approaches include optimization methods [18], virtual structure control [], model predictive control [20], artificial potential functions [21], leader follower approach [22], and game theory [23]. Cyclic pursuit was studied in [24], Lyapunov guidance vector field was used in [25] for circular orbit stabilization for unmanned aerial vehicles (UAVs), and in [26] circular formations were stabilized using modified Kuramoto model. The problem addressed here is generalized to include several formations: co-path and different paths, co-formation plane and different planes, and uniform and non-uniform formations. The approach further breaks the problem down into sub-problems of potential independent interest. Refer to Sections 2.1 and 3.

Regarding the control algorithm used, the main difference between it and previous methods lies more in the conceptual design (hierarchic set stabilizing) than in the particulars of the controllers. The main focus here is on the breakdown of the problem to simpler sub-problems that can be solved relatively simpler than by using a direct approach, and that could allow for the use different methods to solve the particular sub-problems.

The approach of the paper has been used to solve for formation control in [3] and [4]. Those results can be seen as precursors to this work. In [3], a circular formation problem was solved for dynamic unicycles using hierarchical set stabilization. A group of unicycles were required to follow a common circular path with prescribed radius, and desired on-path separations. The problem was solved using three hierarchical sub-problems. In [4], a circular formation problem was solved for steered kinematic particles in three-dimensional space. The particles did not necessarily need to follow a common path. The formation was determined by the spacings between the different paths, and the relative angular positions of the particles on their paths. These aspects are shared with the problem addressed here. The solution in [4] broke down the problem into three sub-problems where the last is similar to that of [3]. Although the problem addressed here shares a number of aspects with that of [4], it is considered more amenable to practical implementation where the agents are dynamic and underactuated, the paths are not necessarily circular and the communication graph is directed (in [4] it was undirected). The results here, as well as in [3] and [4], utilize the globally stabilizing feedbacks for path following of strictly convex paths developed in [27], and the reduction theorem developed in [3] is used to consolidate the sub-problems solutions to achieve asymptotic stabilization of the goal set.

The third contribution of the paper concerns the agents model used. Different models have been considered in literature when studying multi-agent formations. These include mobile robots and UAVs [6, 10, 11, 13, 18], spacecrafts and satellites [21, 28, 29], and surface and underwater vehicles [30]. The agents model used in this paper can be used for underwater, and aerial or space flying robots. However, that model would usually be accompanied with force fields, and/or with coupled degrees of actuation. While such forces and couplings are not considered here it is expected that the hierarchy could allow to address such challenges. Refer to Section 7 for a discussion on how to extend the current results to address more practical agent models with the previously mentioned limitations.

A particular point of interest of the model used here is its underactuations. Handling actuation limitations is a significant challenge for multi-agent systems and problems. The paper shows how the approach presented can be beneficial in addressing this challenge. Instead of developing a particular solution for the formation problem in question, underactuation is addressed by controlling the attitudes to an orientation where the unactuated degrees are not needed. This is possible by achieving a decoupling (stemming from the hierarchy) between the underactuation aspect of the problem and the formation ones. The result is a relatively simple solution that can be utilized in other problems, and that can incorporate available algorithms for attitude control. Underactuated rigid bodies in ℝ^3 has been used as model for many autonomous flying vehicles. For instance, quadrotor helicopter and vertical take-off and landing (VTOL) aircraft can be modelled as degree-two underactuated rigid bodies with one thrust and three torques. Control for such autonomous systems has received considerable attention [31–34]. The design in this paper utilizes the general idea of using attitude stabilization to acquire desired thrust orientation. However, the problem here is more involved with different specifications, and information exchange constraints.

Notation. For scalars a_1, ..., a_n, diag(a_1, ..., a_n) denotes the diagonal matrix with entries a_i. If A, B are two matrices, diag(A, B) denotes the block diagonal matrix with blocks A, B, and A ⊗ B denotes their Kronecker product. For vectors a, b, col(a,b) denotes the concatenation of a and b vertically. The index set {1, ..., n} is denoted by n, and the n-vector of ones is denoted by 1. For x ∈ ℝ, x mod 2π denotes its value modulo 2π, and if x, y ∈ ℝ then x + y mod 2π states that x ∈ [y + 2πk, kZ]. Similarly, if x, y ∈ ℝ^n, then x + y mod 2π states that x_i = y_i mod 2π, i ∈ n.

sat(ℝ) will be used to denote the class of C^1 saturation functions φ : ℝ → ℝ such that for all y ∈ ℝ, φ(y) ≥ 0, |φ(y)| < 1, and φ(y) = 0 only when y = 0. Without loss of generality, it will be assumed here that φ(0) = 1. For x ∈ ℝ^3, x^∞ denotes its skew symmetric matrix representation, where x^∞ y = x × y for any y ∈ ℝ^3. For a dynamical system Σ : ẍ = f(x), ̇Φ(x_0, y) denotes the solution with initial condition x_0 at t = 0. Finally, for a closed set Γ, ∥·∥ and ∥·∥_Γ denote the point to set distance, and for ε > 0, B_ε(Γ) := {y ∈ A : ∥y∥_Γ < ε}. 
2 | PROBLEM FORMULATION

A system of \( n \geq 2 \) rigid bodies in three-dimensional Euclidean space is considered. Figure 1 depicts agent \( i \).

Let \( B_i = \text{span}(f_1^i, f_2^i, f_3^i) \) denote the body frame in inertial frame \( I = \text{span}(e_1, e_2, e_3) \) (the standard basis). It is assumed here that each rigid body is provided with two thrusts \( f_1^i, f_2^i \) in the directions \( b_1^i, b_2^i \). The states of agent \( i \) are given as: \( \chi_i, \dot{\chi}_i \in \mathbb{R}^3 \) the centre of mass and linear velocity in \( I, B_i = [b_1^i, b_2^i, b_3^i] \in \text{SO}(3) \) the attitude matrix, and \( \omega_i \in \mathbb{R}^3 \) the angular velocity in \( B_i \). The state vector of agent \( i \) is given by \( \chi_i = (\chi_i^1, \chi_i^2, \chi_i^3, \omega_i^1, \omega_i^2, \omega_i^3) \in \mathcal{X}_i := \mathbb{R}^3 \times \text{SO}(3) \times \mathbb{R}^3 \), and the collective state vector is given by \( \chi = (\chi_1, \chi_2) \in \mathcal{X} := \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \). The system model is given by

\[
\dot{\chi}_i = \frac{1}{m_i} B_i f_i, \quad B_i = B_i(\omega_i) \times, \quad i \in \mathbb{n}, \tag{1}
\]

where \( f_i := [f_1^i, f_2^i, 0]^T \) is input thrust, and \( m_i \) is the mass of rigid body \( i \). Let \( x = (x_1, \ldots, x_n), \quad \dot{x} = (\dot{x}_1, \ldots, \dot{x}_n), B = (B_1, \ldots, B_n) \) and \( b_i = (b_1^i, \ldots, b_3^i) \).

**Assumption 1.** The angular dynamics of (1) are neglected, that is, it is assumed that desired angular velocities \( \omega_i \) can be achieved instantaneously.

**Remark 1.** Due to hierarchic nature of the approach, Assumption 1 can be relaxed to: desired angular velocities \( \omega_i \) can be achieved exponentially, regardless of the form of torques used or their degree of actuation. By this, and on account of available literature on angular dynamics stabilization, the instantaneous assumption is used.

**Assumption 2.** The information exchange between the rigid bodies is modelled by a directed sensor graph \( \mathcal{G} \) where

- agent \( i \) has access to its orientation \( B_i \); its linear and angular velocities \( \dot{x}_i, \omega_i \); and
- relative positions, orientations, linear and angular velocities expressed in \( B_j \) of agents visible to it per \( \mathcal{G} \).

\[ \mathcal{G} \] will be considered to be static and balanced, with a globally reachable node \([35]\]. \( N(i) \) will be used to denote the set of nodes connected to \( i \) according to \( \mathcal{G} \). \( L \) will denote the Laplacian of \( \mathcal{G} \) with \( L_i \) its \( i \)th row, and \( L_{(3)} := L \otimes I_3, L_{(3)} = L_i \otimes I_3 \).

The following lemma will be used.

**Lemma 1.** If \( L \) is the Laplacian of an \( n \)-node digraph with a globally reachable node, then for sufficiently small \( k > 0 \),

\[
\begin{bmatrix}
0_{n \times n} & L_{(3)} \\
-kL & -L_{\text{diag}}
\end{bmatrix}
\]

has eigenvalues with negative real parts except for a simple eigenvalue at 0.

The proof is provided in Appendix A.1.

Part of the formation problem addressed here entails path following of a closed convex path. Consider a smooth Jordan curve \( \mathcal{C} \in \mathbb{R}^2 \) that is strictly convex. From lemma 3.1 in \([27]\), there exists a regular parametrization \( \sigma = (\sigma_1, \sigma_2) : S^1 \to \mathbb{R}^2 \) of \( \mathcal{C} \) such that, for each \( \Theta \in S^1 \), the angle of the tangent vector \( \sigma'(\Theta) \) is \( \Theta \pmod{2\pi} \), refer to Figure 2(a). Using a frame \( P \) with axes \( p_1, p_2, p_3 \), a strictly convex path in \( \mathbb{R}^3 \) can be given by

\[
\left\{ y \in \mathbb{R}^3 : y = y_i + P [\sigma_1(\Theta) \sigma_2(\Theta) 0]^T, \Theta \in S^1 \right\},
\]

where \( P = [p_1, p_2, p_3] \), and \( y_i \in \mathbb{R}^3 \) denotes the path centre. The frame \( P \) determines the orientation of the path in \( \mathbb{R}^3 \) where \( p_1, p_2 \) span the path plane, refer to Figure 2(b). This path will be denoted \( (\sigma, P) \). Now, suppose that agent \( i \) is following the path with forward heading \( \dot{b}_1^i \), and let

\[
\dot{b}_1^i = \angle(\dot{b}_1^i, p_1) \quad \text{and} \quad p = [p_1 \cdots p_3]^T.
\]

By this, \( \dot{b}_1^i \) equals the angle of the tangent to the path at \( x_i \), and \( p \) determines the on-path angular position of rigid body \( i \). Let \( b_1^i = [b_1^i \cdots b_1^i]^T \) and \( p = [p_1 \cdots p_3]^T \).

2.1 | Problem statement

For system (1) of \( n \) rigid bodies with digraph \( \mathcal{G} \) modelling the information exchange between them. The Convex-Path-Following.
FIGURE 3 Agents i and i + 1 in formation

Formations problem (C_{x}PFF) is defined as the distributed stabilization problem of the goal set

\[
\Pi = \{ \chi \in \mathcal{X} : L_{(3)}(\chi - \alpha) = 0, L_{(1)}(\chi - \beta) = 0 \mod 2\pi, \]

\[
|p_{3} \cdot b_{i}'| = 1, x'_{i} = \frac{\tilde{u}(\chi')b_{i}' + u_{2} b_{i}'_{2} + u_{3} b_{i}'_{3}}{\|\sigma'(b_{i}')\|} p_{3}, i \in \mathbb{N},
\]

where \(\alpha \in \mathbb{R}^{n}, \beta \in \mathbb{R}^{n}, \tilde{u} : \mathcal{X} \to \mathbb{R}^{+} \) a C^1 function, and

\[
\chi_{i}(\chi) = \begin{bmatrix} x_{i}^{1}(\chi')^{T} \cdots x_{i}^{n}(\chi')^{T} \end{bmatrix}^{T},
\]

\[
x'_{i}(\chi') = x'_{i} - P \begin{bmatrix} \sigma_{1}(b_{i}') \sigma_{2}(b_{i}') \end{bmatrix}^{T}.
\]

This problem is a combination of the following specifications.

(i) Convex path following: in C_{x}PFF each agent follows, the same or different, strictly convex path \((\sigma, P)\) with stationary centre (dependent on initial conditions). The path plane orientation is determined by \(p_{3}\), that is, if the agent heading is in this plane then \(p_{3} \cdot b_{i}' = 0\). Also, while path following, the heading should be in the direction of the linear velocity, that is, \(x'_{i} \cdot b_{i}'\) equals the forward speed. Let this be \(\tilde{u}(\chi') > 0\). By proposition 3.4 in [27], in order for the agent to stay on the path, the angular velocity on the path, that is, the component of \(u'\) orthogonal to the path should be \(\tilde{u}(\chi') > 0\). By this, specification (i) can thus be seen as making the following set

\[
C = \{ \chi : p_{3} \cdot b_{i}' = 0, u_{2} b_{i}'_{2} + u_{3} b_{i}'_{3} = \frac{\tilde{u}(\chi')}{\|\sigma'(b_{i}')\|} p_{3}, \]

\[
x'_{i} = \frac{\tilde{u}(\chi')b_{i}'}{\|\sigma'(b_{i}')\|} \}
\]
asymptotically stable.

(ii) Path centres sub-formation: in formation, the paths of specification (i) satisfy desired relative spacial relations given by desired relative displacements \(q_{i}, i = 1, \ldots, n - 1\) of the centres \(x'_{i}(\chi')\), where \(x_{i} - x_{i+1} = q_{i}, i = 1, \ldots, n - 1\). Refer to Figure 3. As \(G\) has a globally reachable node, by lemma 2 in [35], \(\ker L_{3}\) span 1. Using this, specification (ii) is equivalent to asymptotically stabilizing

\[
F = \{ \chi \in C : L_{(3)}(\chi - \alpha) = 0 \}
\]

where \(\alpha_{i} = 0, \alpha_{j} = \sum_{j=i}^{n-1} q_{j}, i = 1, \ldots, n - 1, \) and \(\alpha = [\alpha_{1} \cdots \alpha_{n-1}]^{T}\).

(iii) Angular positions sub-formation: the agents attain on the paths of specification (ii), desired relative on-path angular positions given by desired differences of \(p_{1}\) (relative on-path angular positions), as defined in (3). This can be alternatively given by desired differences of the heading angles \(b'_{i}\). Let \(p_{j} - p_{j+1} = \theta_{j} \mod 2\pi, i = 1, \ldots, n - 1\), for desired angles \(\theta_{1}, \ldots, \theta_{n-1}\), refer to Figure 3. Specification (iii) can be seen as asymptotically stabilizing

\[
A = \{ \chi \in F : L_{(1)}(\chi - \beta) = 0 \mod 2\pi, \}
\]

where \(\beta_{i} = 0, \beta_{j} = \sum_{j=i}^{n-1} \theta_{j}, i = 1, \ldots, n - 1, \) and \(\beta = [\beta_{1} \cdots \beta_{n-1}]^{T}\).

Remark 2. Note that \(A\) in (6) strictly contains \(\Pi\) in (4). \(\Pi\) is the largest controlled invariant subset of \(A\) for (1).

It is suggested here that C_{x}PFF can be of particular interest in applications of multi-agent systems. First, it is easy to imagine a situation where a team of autonomous robots is required to surround a compact region by following a common path, an instance of C_{x}PFF. Applications include wild fire monitoring, search and rescue missions, and pollutants containment using flying agents. Also, in environmental monitoring in air or underwater, near surface planetary exploration, and military applications. Although a compact region can be contoured by a circular path, yet, the path class addressed in C_{x}PFF, which is more general, could allow for better coverage. The path shape and orientation parameters are incorporated as control parameters which can be changed online. This helps to adapt the formation to changing situations, for example, spreading fires or pollutants, as well as different terrain inclinations. Also, of the control parameters are the relative path angular positions, and so uniform as well as non-uniform distributions are achievable. The ordering of the agents can be controlled as well, which can be useful if different agents perform different tasks, for example, monitoring, sampling and taking action. In addition, addressing the problem as stabilizing a goal set has an added advantage. Suppose a certain instance of C_{x}PFF is achieved, since the design will render the goal set stable, the previous parameters can be changed on-line in small increments without the agents deviating much, during the transition, from previous states.

Second, although the case where each agent follows a different path, whether on the same or on different planes, is not
familiar, at least to the author, yet one can imagine a situation
where each agent is required to monitor a certain region while
maintaining contact with the rest. The design presented would
allow for completely arbitrary paths configurations by setting
the control parameters. In this case, the on-path angular posi-
tions aspect is a part of controlling the agents relative positions.
This can be useful in applications in the field of sampling and
remote sensing, for example, in localization.

2.2 Approach and tools

C לפפ is solved using a hierarchy of five steps of stabilizing
nested sets $Γ_1 ⊃ Γ_2 ⊃ \cdots ⊃ Γ_k$, where $Γ_k$ equals the goal set
$Γ$. For $i \in \{1, \ldots, 5\}$, step $i$ of the hierarchy involves designing
distributed feedbacks to asymptotically stabilize $Γ_i$ relative to
$Γ_{i-1}$ ($Γ_0 := Y$). This notion is defined as follows.

Definition 1. For a dynamical system $\dot{x} = f(x)$ with state
space $X$, let $Γ_1, Γ_2, Γ_1 ⊂ Γ_2 \subset Y$, be closed positively sets.
$Γ_1$ is asymptotically stable relative to $Γ_2$ if it is stable and
attractivity to $Γ_2$. $Γ_1$ is stable relative to $Γ_2$ if for any $δ > 0$
there exists a neighbourhood $N(Γ_1)$ such that $f(Γ_1) \cap Γ_2 \subset B_δ(Γ_1)$.
$Γ_1$ is attractive relative to $Γ_2$ if there exists a neighbourhood
$N(Γ_1)$ such that $\|f(Γ_1)\|_0 \to 0$ for all $x_0 \in N(Γ_1) \cap Γ_2$.

The following result, adopted from [5] gives conditions for
when the previous implies that $Γ_1$ is asymptotically stable.

Corollary 1. For a dynamical system $\dot{x} = f(x)$ with state
space $X$, a domain in $R^n$. Let $l > 1$ be some constant, and $X \supset Γ_1 \supset \cdots ⊃ Γ_l$, with $Γ_l$ non-compact, closed positively invariant sets for $Γ$. If for $i = 1, \ldots, l$
• $Γ_i$ is asymptotically stable relative to $Γ_{i-1}$, and
• for some $k \in \{1, \ldots, l\}$, $Σ$ is locally uniformly bounded (LUB) near
$Γ_k$,
then $Γ_k, \ldots, Γ_l$ are asymptotically stable for $Σ$. The first condition
is also necessary.

Showing the second condition can be a technical chal-
lenge, refer to [3]. The following result shows that this sim-
plifies in the case where each step of the hierarchy is solved
for exponential stability. In such case, the property ‘migrates’
up the hierarchy, from the restriction on $Γ_k$ to the state
space.

Corollary 2. For a dynamical system $Σ$ and $Γ_1 \supset \cdots ⊃ Γ_l$ as in
Corollary 1. Suppose that there exists a diffeomorphism $x \mapsto z$ such that

$Σ$ can be written as

\[
\begin{align*}
\dot{z}_i &= f_i(z_{i+1}, z_i) \\
\dot{z}_2 &= f_2(z_{i+1}, z_i) + g_1(z_{i+1}, z_i) z_i \\
& \vdots \\
\dot{z}_k &= f_k(z_{i+1}, z_k) + g_{k-1}(z_{i+1}, z_k) z_{k-1} \\
\dot{z}_{k+1} &= f_{k+1}(z_{i+1}, z_{k+1})
\end{align*}
\]

where $z = col(z_1, \ldots, z_{k+1})$, $z_i := \text{col}(z_1, \ldots, z_i)$, all vector fields
are locally Lipschitz, and such that $Γ_j = \{X : z' = 0\}$, $j \in \{1, \ldots, k\}$.
$Σ$ is LUB near $Γ_j$ if

a. for $j \in \{1, \ldots, k\}$, the origin $z_j = 0$ of $\dot{z}_j = f_j(z_{j+1}(t), z_j)$ is
exponentially stable;

b. for $j \in \{1, \ldots, k-1\}$, $g_j(z_{j+1}(t), z_j)$ satisfies

\[\|g_j(z_{j+1}(t), z_j(z')) \leq c_1 \|z'\| \|z_{j+1}(t)\| + c_2 \|z'\|\]

for some $c_1, c_2 > 0$;

c. and the solution of $\dot{z}_j = f_j(z_{j+1}(t), z_j(t))$ is uniformly bounded
if $z_j(t) \to 0$ exponentially.

In addition, $Γ_j, j \in \{1, \ldots, k\}$ is asymptotically stable.

The proof follows from developments as in Lemma 9.4 in [36].

3 SOLUTION HIERARCHY

C לפפ is broken down into five hierarchical steps H1−H5.
H1—Feedback-linearizing-thrusts orientation stabilization.

Suppose (1) is fully-actuated, that is, $f'_2$ can take any value. In
this case, the feedback linearizing thrusts

\[f' = m_i \dot{b}_i \left[ u_i(\chi) \dot{b}_i + n_i(\chi) \ddot{b}_i - k'_i(\chi)(\dot{x} - n_i(\chi) \dot{b}_i) \right], \]

(7)

with $k'_i(\chi) > 0$ and $n_i(\chi)$ some $C^1$ design functions, render the
set $\{X \in X : \dot{x} = n_i(\chi) \dot{b}_i, i \in \mathbf{n}\}$ asymptotically stable, that is,
the agents linear velocities are stabilized to vectors with ampli-
tudes $n_i(\chi)$ and directions $\dot{b}_i$.

In this step, it is required to stabilize the orientations of the
rigid bodies to a direction making $f'_2 = 0$ for the feedback
linearizing thrusts (7). This can be achieved by stabilizing the set

\[Γ_1 = \{X \in X : f'_2(\chi) = 0, i \in \mathbf{n}\}, \]

(8)

where

\[f'_2(\chi) = m_i \dot{b}_i \left[ u_i(\chi) \dot{b}_i + n_i(\chi) \ddot{b}_i - k'_i(\chi)(\dot{x} - n_i(\chi) \dot{b}_i) \right], \]

(9)
H2—Linear velocity stabilization.

Design the feedbacks $f_1'(\chi), f_2'(\chi), i \in n$, to obtain desired linear velocities on $\Gamma_1$, that is, stabilize the set

$$\Gamma_2 = \{ \chi \in \Gamma_1 : \dot{x}' = n_i(\chi)\dot{b}_i, i \in n \}$$

(10)

with $n_i(\chi)$ as in $H1$. Note that the dynamics on $\Gamma_2$ are that of steered kinematic particles.

H3—Vertical formation stabilization.

For the kinematic motion on $\Gamma_2$, make the agents approach planes orthogonal to $p_3$, with separations consistent with the desired formation. This corresponds to stabilizing the set

$$\Gamma_3 = \{ \chi \in \Gamma_2 : L_3(s_3(\chi) - \alpha_3) = 0, p_3 \cdot \dot{b}_i = 0, i \in n \},$$

(11)

where $s_3(\chi) = [x'_3 \ldots x'_5]^T$, $x'_3 = x' \cdot p_3$, $i \in n$, (the vector of projection of $x'$ on $p_3$), and $\alpha_3 = [\alpha_i \cdot p_3 \ldots \alpha_i \cdot p_3]^T$ (the vector of projection of $\alpha_i$ on $p_3$).

H4—Horizontal formation stabilization.

On the planes of $H3$, make the agents approach the formation in specification (ii) of $C_{PF}$: This corresponds to stabilizing the set $\{ \chi \in \Gamma_3 : L_3(s_3(\chi) - \alpha_3) = 0 \}$. Consider the centres of rotation in (5). Define their projections orthogonal to $p_3$ by $x'_3 = x'_3 - (x'_3 \cdot p_3)p_3$, and denote $x_3(\chi) = [(x'_3)^T \ldots (x'_5)^T]^T$. Also, define the projection of $\alpha$ orthogonal to $p_3$ by $\alpha_p = [\alpha_1^T \ldots \alpha_3^T]^T$, where $\alpha_i = \alpha_i - (\alpha_i \cdot p_3)p_3$. Using this, $H4$ requires stabilizing the set

$$\Gamma_4 = \{ \chi \in \Gamma_3 : L_3(s_3(\chi) - \alpha_3) = 0 \}.$$ 

(12)

H5—Angular positions stabilization.

While in the formation of $H4$, make the agents acquire the desired relative on-path angular positions, with the required speed specification. This corresponds to stabilizing

$$\Gamma_5 = \{ \chi \in \Gamma_4 : L_4(\theta - \beta) = 0 \mod 2\pi, \xi' = \tilde{n}_i(\chi')\dot{b}_i, i \in n \}.$$ 

(13)

This hierarchy provides extra problem features which can be of interest. As the sets expressing the different sub-problems are nested, and are to be rendered stable by control design, variations in certain formation aspects can be made while maintaining others. For example, as the design will render the agents path invariant, the formation on the path(s) can be changed without the agents leaving it. For instance, for a single path formation, the agents could be made to come close together for communication purposes, or to focus on certain regions. Also, if an agent is lost from a formation, the rest can be made to redistribute without leaving the path. Another useful property is that the design renders the path planes invariant, so the agents can be made to follow different formations, for example, a path with different shape, individual paths on the same plane, or different paths on the previous planes. This means that the agents would not lose altitude, relative altitudes or inclination while changing formation.

Finally, the underactuation is handled at the first step of the hierarchy where on $\Gamma_2$ the system is seen as fully actuated. This implies that the specifications of the hierarchy can be changed without the need to revisit this step. Also, it is worth noting that on $\Gamma_3$ the dynamics are that of kinematic unicycles, and on account of the hierarchy available solutions for different problem for that model can be implemented, more or less directly.

4 | CONTROL DESIGN

In this section, the solution of the previous five-step hierarchy is solved in five consecutive steps. Refer to Figure 4 for an overview.

4.1 | Solution of $H1$

The solution of this step uses the functions $n_i(\chi), w_i^2(\chi), w_j^3(\chi)$ to be designed in Sections 4.3, 4.4, and 4.5. Those forms will be used here directly, but on account of the hierarchy one can follow the developments in those sections first. Consider

$$f_j^i = m_i(\dot{b}_i)^T [\tilde{n}_i(\chi')\dot{b}_i + n_i(\chi')\dot{b}_i - k_j^i(\dot{x}' - n_i(\chi')\dot{b}_i)],$$

(14)

with $n_i(\chi), w_i^2(\chi), w_j^3(\chi)$ as defined in (19), (21), (22), and

$$k_j^i = \frac{\mu_i(\chi)|\zeta_j^i||\xi_j^i|}{\sqrt{\epsilon_i + (\dot{b}_j^i \cdot \dot{x}')^2 + (\dot{b}_j^i \cdot \xi')^2}},$$

(15)

where $\epsilon_i > 0$ and $\zeta_j^i, \xi_j^i$ are defined in (A.4). In Appendix A.2, it is shown that $u_1(\chi), w_i^2(\chi), w_j^3(\chi)$ of Sections 4.3, 4.4 and 4.5, with (14) and (15) render $\Gamma_1$ asymptotically stable.
4.2 | Solution of $H_2$

Consider the diffeomorphism
\[
\dot{\chi} = \tilde{\chi} - n_i(\chi) h_i, \quad e' = B_i^T \dot{\chi},
\]
and the candidate Lyapunov functions
\[
W_i = \frac{1}{2} e' \cdot e', \quad i \in \mathbf{n}.
\]
With (14), the time derivative of $W_i$ along the system dynamics on $\Gamma_2$ takes the form $\dot{W_i} = -2k_i \Gamma_i W_i$. This implies that the set $\Gamma_2$ is asymptotically stable relative to $\Gamma_1$.

4.3 | Solution of $H_3$

The objective in this step is to design $n_i(\chi)$, $u'(\chi)$ to asymptotically stabilize $\Gamma_3$ in (11) relative to $\Gamma_2$ for the closed-loop system (1), (14). Those dynamics restricted to $\Gamma_2$ takes the form
\[
\dot{\chi} = n_i(\chi) h_i, \quad \dot{\chi} = B_i(u'(\chi))^\top, \quad i \in \mathbf{n}.
\]
Let
\[
\tilde{u}_i = \tilde{\chi} + \tilde{\chi} + \tilde{\chi} + \tilde{\chi}, \quad i \in \mathbf{n}.
\]
Pick $u'(\chi), \tilde{\chi}, \tilde{\chi}, \tilde{\chi}$ such that $\Gamma_3$ is invariant for
\[
\dot{\chi} = n_i(\chi) h_i, \quad \dot{\chi} = B_i(\tilde{\chi})^\top, \quad i \in \mathbf{n},
\]
where $\tilde{\chi} = [u_i(\chi) \tilde{\chi} + \tilde{\chi} + \tilde{\chi} + \tilde{\chi}]^\top$. Let
\[
u_i = \delta(\chi_i)(v + \tilde{\chi} + \tilde{\chi} + \tilde{\chi})
\]
\[
\tilde{u}_i = \left(\begin{array}{c}
\tilde{\chi} \\
\tilde{\chi} \\
\tilde{\chi} \\
\tilde{\chi}
\end{array}\right)
\]
\[
\tilde{u}_i = \left(\begin{array}{c}
\tilde{\chi} \\
\tilde{\chi} \\
\tilde{\chi} \\
\tilde{\chi}
\end{array}\right)
\]
where $k, \tilde{k} > 0$, and
\[
\psi_i(\chi) = \varphi \left(\sum_{j \in \mathbf{n}} B_j^T \left(\chi' - \chi'\right) \cdot B_j^T \varphi - (\chi - \chi') \cdot \varphi \right).
\]
with $\varphi \in \mathbf{sat}(\mathbf{R})$. In Appendix A.3 it shown that this renders $\Gamma_3$ asymptotically stable relative to $\Gamma_2$, where $\tilde{u}_i(\chi) = \tilde{u}_i(\chi) = 0$ on $\Gamma_3$.

4.4 | Solution of $H_4$

The objective of this step is to design $\delta_i(\chi)$ of (21) such that $\Gamma_4$ in (12) is asymptotically stable for (20), and $\delta_i(\chi) = 0$ on $\Gamma_4$. From the solution of $H_3$,
\[
u_i(\chi) = \delta_i(\chi_i)(v + \tilde{\chi} + \tilde{\chi} + \tilde{\chi})
\]
\[
\tilde{u}_i = \left(\begin{array}{c}
\tilde{\chi} \\
\tilde{\chi} \\
\tilde{\chi} \\
\tilde{\chi}
\end{array}\right)
\]
\[
\tilde{u}_i = \left(\begin{array}{c}
\tilde{\chi} \\
\tilde{\chi} \\
\tilde{\chi} \\
\tilde{\chi}
\end{array}\right)
\]
on $\Gamma_3$. So, on $\Gamma_4$, when $\delta_i(\chi) = 0$, each agent follows a stationary path as required. Therefore, $\Gamma_4$ is invariant for
\[
\dot{\chi} = u'(\chi) h_i, \quad \dot{\chi} = B_i(\tilde{\chi})^\top, \quad i \in \mathbf{n},
\]
where $\tilde{\chi} = [u_i(\chi) \tilde{\chi} + \tilde{\chi} + \tilde{\chi} + \tilde{\chi}]^\top$. Let
\[
\delta_i(\chi) = \varphi \left(\sum_{j \in \mathbf{n}} (B_j^T \chi' - \chi')^\top \varphi - (\chi - \chi')^\top \varphi \right).
\]
with $\varphi \in \mathbf{sat}(\mathbf{R})$. In Appendix A.4, it is shown that this choice solves for the required asymptotic stability.

4.5 | Solution of $H_5$

The objective in this step is to design $\tilde{\chi}(\chi_i), \tilde{\chi}(\chi_i)$ in (21) to asymptotically stabilize $\Gamma_5$ in (13) for (24). From the previous step, on $\Gamma_4$ all agents follow stationary convex paths as in specification (i). Their motion, therefore, can be completely characterized by $\tilde{\chi}$. On $\Gamma_4$, $u_i(\chi) = \delta_i(\chi_i)(v + \tilde{\chi} + \tilde{\chi} + \tilde{\chi})$, and $\tilde{u}_i(\chi) h_i = \tilde{u}_i(\chi) h_i = \tilde{u}_i(\chi) h_i = \tilde{u}_i(\chi) h_i$, and from (A.16)
\[
\dot{b}_i = \frac{\delta_i(\chi_i)}{\varphi(\chi_i)}(v + \tilde{\chi} + \tilde{\chi} + \tilde{\chi}).
\]
Using this
\[
\tilde{\psi}_i = \frac{\sigma_i(\chi_i) - \sigma_i(\chi_i) \sigma_i(\chi_i)}{\|\sigma(\chi_i)\|^2} \left(\frac{\sigma_i(\chi_i)}{\|\sigma(\chi_i)\|} (v + \tilde{\chi} + \tilde{\chi} + \tilde{\chi}) \right).
\]
By choosing
\[ \hat{\sigma}(\hat{b}_1) = \frac{\|\sigma(\hat{b}_1')\|}{\tau (\sin b'_1 \cos p_1 - \cos b'_1 \sin p_1)} \] (27)
with \( \tau > 0 \), it follows that \( \hat{p}_1 = \frac{1}{\tau} (r + \hat{b}_1 (h_1)) \). To stabilize \( \Gamma_1 \) for (24), one needs to design \( C_1 \) feedbacks \( \hat{u}_i (h_1) \) that asymptotically stabilize \( \{ b_1 : L_i(p - \beta) = 0 \text{ mod } 2\pi \} \) for (26), and in addition, on that set must hold that \( \hat{b}_1 (h_1) = 0 \). Let
\[
\hat{u}_i (h_1) = -\sin \left( \sum_{j \in N(i)} \tan^{-1} \left( \frac{\sigma_2(\zeta(e_1, p'_1))}{\sigma_1(\zeta(e_1, p'_1))} \right) \right)
\]
and
\[
-\tan^{-1} \left( \frac{\sigma_2(\zeta(B_i^T b'_1, p'_1))}{\sigma_1(\zeta(B_i^T b'_1, p'_1))} \right) - (\beta_i - \beta_j)
\]
where \( p'_1 = B_i^T p_1 \). Using the fact that
\[
\sum_{j \in N(i)} \tan^{-1} \left( \frac{\sigma_2(\zeta(e_1, p'_1))}{\sigma_1(\zeta(e_1, p'_1))} \right) - \tan^{-1} \left( \frac{\sigma_2(\zeta(B_i^T b'_1, p'_1))}{\sigma_1(\zeta(B_i^T b'_1, p'_1))} \right)
\]
and \( (\beta_i - \beta_j) = L_i (p - \beta_j) \), the result follows from Proposition V.4 in [3].

5 | SOLUTION OF \( C_\text{PFF} \)

This section provides the solution of \( C_\text{PFF} \) for (1), in addition to two corollaries for fully-actuated, and degree-two underactuation with \( f'_i = 0 \).

5.1 | Degree-one underactuation

The next result follows from the design in Section 4.

**Theorem 1.** Consider the \( n \) rigid bodies in (1) under Assumptions 1 and 2. For \( i \in n \) and sufficiently small \( k, \bar{k}, \bar{\kappa} \), the feedbacks (14), where \( u_i (\chi), u'_i (\chi), u'_i (\chi) \) and \( u'_i (\chi) \) are given in (19), (A.6), (21) and (22), respectively, with \( \Psi_i (\chi), \delta_i (\chi) \) and \( \hat{u}_i (h_1) \) as in (23), (25) and (28), solve \( C_\text{PFF} \) for fully actuated agents, rendering \( \Gamma' \) asymptotically stable, where \( \Gamma' \) equals \( \Gamma \) with \( |p_3 \cdot \hat{b}_3| = 1 \) replaced by \( |p_3 \cdot \hat{b}_3| = 0 \).

**Corollary 3.** Under the assumptions of Theorem 1, the feedbacks (7), with constant \( k_i f_i \), where the functions \( u_i (\chi), u'_i (\chi) \) and \( u'_i (\chi) \) are given in (19), (21), and (22), respectively, with \( \Psi_i (\chi), \delta_i (\chi) \) and \( \hat{u}_i (h_1) \) as in (23), (25) and (28), and \( u'_i (\chi) \) any \( C_1 \) uniformly bounded functions, solve \( C_\text{PFF} \) for fully actuated agents, rendering \( \Gamma' \) asymptotically stable, where \( \Gamma' \) equals \( \Gamma \) with \( |p_3 \cdot \hat{b}_3| = 0 \) in \( \Gamma \).

**Remark 3.** The requirement \( p_3 \cdot \hat{b}_3 = 0 \) in \( \Gamma \) is not part of the problem description but rather is due to the underactuation. When the system is fully-actuated, that requirement is not necessary which renders \( u'_i \) a degree of freedom. Due to the hierarchic nature of the approach, \( u'_i \) can be stabilized separately to desired \( u'_i (\chi) \).

5.3 | Degree-two underactuation

Suppose that for system (1), \( f'_i = f'_i = 0 \). In this case, \( C_\text{PFF} \) can be solved only if the desired forward speed is constant. With constant speed, specification (iii) (desired relative on-path positions) cannot be satisfied unless the path is circular. Thus, for this case \( C_\text{PFF} \) can be solved if

1. the desired path is circular, \( \hat{u} (\chi) = v (v > 0 \text{ desired speed}, \text{ and } \|\sigma'(\hat{b}_i')\| = r (r > 0 \text{ desired path radius}) \);
II. or for a convex path with \( u_i(\chi) = v \), and without specification (iii) (desired on-path angular positions).

Note that in the following development \( u_i \) in (21) is set equal to \( v \) in only the first equation for case I, and in the first and second for case II. In those cases, the problem can be solved using four hierarchical steps. Steps 1–3 are \( H1–H3 \), and step 4 is, for case I \( H4 \) and \( H5 \) solved simultaneously, and only \( H4 \) for case II. The design for steps 1–3 is the same as in Sections 4.1–4.3, however, showing the result for steps 2–3 is different.

Consider

\[
\begin{equation}
\begin{align*}
\dot{f}^2_i &= \nu LD^2_i k_i'(\chi)(\frac{\dot{\chi}}{2} - v l_i') \quad (i \in n),
\end{align*}
\end{equation}
\]

with \( k_i'(\chi), w_i'(\chi), w_i''(\chi), w_i'''(\chi) \) as in Theorem 1, the diffeomorphism

\[
\frac{\dot{\chi}}{\chi} = \frac{\dot{\chi}}{\dot{\chi}}, \quad \frac{\epsilon'}{\epsilon'} = B_i^T \frac{\dot{\chi}}{\dot{\chi}} \quad (0)
\]

and the functions \( W_i \) in (17). The time derivative of \( W_i \) along the system dynamics on \( \Gamma_1 \) takes the form

\[
\begin{equation}
\begin{align*}
\dot{W}_i &= -k_i'(\chi)(\frac{\dot{\chi}}{2} + \frac{\epsilon'}{\epsilon'}) + \epsilon_i'(w_i''(\chi)v + k_i'(\chi)(\frac{\dot{\chi}}{2} - v l_i')).
\end{align*}
\end{equation}
\]

On \( \Gamma_1 \), that time derivative takes the form \( \dot{W}_i = -k_i'(\chi)(\frac{\dot{\chi}}{2} + \frac{\epsilon'}{\epsilon'}) \). By Barbala’s lemma [36], this implies that \( \epsilon_i'(t) = \epsilon_i'(t) \rightarrow 0 \). Using the facts: \( W_i \) is positive definite, \( \epsilon_i'(t) = \epsilon_i'(t) \rightarrow 0 \). On \( \Gamma_2 \), this development in Section 4.2 directly applies showing that \( \Gamma_2 = \{ \chi \in \Gamma_1 : \chi = \alpha_0, i \in n \} \) is asymptotically stable relative to \( \Gamma_1 \). On \( \Gamma_3 \), the dynamics is (18) with \( u_i(\chi) = v_i' \). Using this, the development in Section 4.2 directly applies showing that \( \Gamma_3 = \{ \chi \in \Gamma_2 : L_\chi = \alpha_0, p_3 \cdot \dot{\epsilon}_1 = 0, i \in n \} \) is asymptotically stable relative to \( \Gamma_2 \). Note that on \( \Gamma_4 \), \( w_i''(\chi) \) is negative. By choosing \( k_i \chi < s/2 \), \( p_3 \cdot \dot{\epsilon}_1 = 0 \) on \( \Gamma_3 \), and so \( w_i''(\chi) \) is bounded away from zero. Since, on \( \Gamma_4, \epsilon_i' = 0 \) and \( \dot{\epsilon}_i = w_i''(\chi) \epsilon_i \), this is only possible if \( \dot{\epsilon}_1 = 0 \). Consequently, \( \Gamma_4, \Gamma_5 \) coincide with \( \Gamma_2, \Gamma_3 \), and the solutions of \( H2 \) and \( H3 \) follows.

For case II, \( H4 \) follows as in Section 4.4. For \( H4 \) and \( H5 \), for case I, the development is a technical modification of that provided in Sections 4.4 and 4.5, and is omitted here.

**Corollary 4.** For system (1) with \( f_i' = 0 \), the feedbacks of Theorem 1 solve \( C_3 \) PPF for the previous cases I and II.

**Remark 4.** While this result is local as that of Theorem 1. Simulations have shown that the latter is more robust to large initial conditions. Still, the result might be useful in situations when applying the degree-one underactuated feedback, and the second degree of actuation is lost close to the formation.

### 6 | SIMULATIONS

Consider five rigid bodies with \( G \) having a cyclic Laplacian

\[
L = \begin{bmatrix}
1 & -1 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 \\
-1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

#### 6.1 | Formation 1

The path in Figure 5 is an ellipse with major axis of length 2, and minor axis of length 1. The path orientation is such that \( P = I \), where the major axis points in the \( p_1 \) direction.

The formation is given by \( q_1 = \cdots = q_4 = \left[ \frac{1}{\sqrt{3}} \frac{1}{\sqrt{3}} \frac{1}{\sqrt{3}} \right]^T \) (the separation between the path centres), and \( \theta_1 = \cdots = \theta_4 = 0 \) (the on-path relative angular positions). Therefore, in this formation the agents should follow individual paths, neither coplanar nor concentric, belonging to horizontal planes, that is, parallel to the \( e_1 - e_1 \) plane, and have synchronized headings.

The stabilizing feedbacks used in this simulation are those of degree-one underactuation (Theorem 1). Note that while the agents follow their respective paths, they are orientated such that their forward directions \( \dot{\epsilon}_i \)'s (marked with blue) are tangent to the paths and pointing in the direction of motion, and the unactuated directions \( \dot{\epsilon}_i \)'s (marked with red) are orthogonal to the path planes.

#### 6.2 | Formation 2

The path in Figure 6 is an ellipse similar to that of Figure 5. The orientation of the path is given by \( p_1 = [1 0 0]^T \), \( p_2 = [0 \cos 30^\circ - \sin 30^\circ]^T \), and \( p_3 = [0 \sin 30^\circ \cos 30^\circ]^T \).
The formation is such that the centres of the agents’ paths form a uniform pentagon in the $p_1 - p_2$ plane. This is given by $q_1 = T[5 \cos 72^\circ \ 5 \sin 72^\circ \ 0]^T$, $q_2 = T[5 \ 0 \ 0]^T$, $q_3 = T[5 \cos 288^\circ \ 5 \sin 288^\circ \ 0]^T$, $q_4 = T[5 \cos 216^\circ \ 5 \sin 216^\circ \ 0]^T$, where

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos -30^\circ & -\sin -30^\circ \\ 0 & \sin -30^\circ & \cos -30^\circ \end{bmatrix}.$$ 

This gives co-planar paths on a plane that is a $-30^\circ$ tilt of the horizontal around $\mathbf{e}_1$. In this case too, it is required to synchronize the agents’ headings, that is, $\theta_1 = \cdots = \theta_4 = 0$.

The stabilizing feedbacks used in this simulation are those of degree-one underactuation (Theorem 1). Same as in Formation 1, while the agents follow their respective paths they are oriented such that their forward directions $b_i^1$’s (marked with blue) are tangent to the paths, pointing in the direction of motion, and the unactuated directions $b_i^3$’s (marked with red) are orthogonal to the path planes.

### 6.3 Formation 3

The simulations presented in Figures 7, 8, and 9 are for a circular formation with radius $r = 2$, and orientation given by $p_1 = \left[ \frac{1}{\sqrt{2}} \ 0 \ -\frac{1}{\sqrt{2}} \right]^T$, $p_2 = [0 \ 1 \ 0]^T$, $p_3 = \left[ \frac{1}{\sqrt{2}} \ 0 \ \frac{1}{\sqrt{2}} \right]^T$.

The agents are required to follow a common path, that is, $q_1 = \cdots = q_4 = [0 \ 0 \ 0]^T$, and the formation required is uniform, that is, $\theta_1 = \cdots = \theta_4 = \frac{2\pi}{5}$.

The simulation in Figure 7 is for degree-one underactuated feedbacks. Notice that the agents’ orientations are stabilized to forward directions tangent to the path in the direction of motion, and unactuated directions orthogonal to the path plane as in the previous formations.
Practical agents hierarchy

The simulation in Figure 8 starts as in Figure 7, then after 20 s, $f_1$ is set to zero.

The simulation in Figure 9 is for fully-actuated feedbacks. Notice that in this case the unactuated direction $b_3(t)$ does not approach the $p_3$ direction, but instead stabilizes at arbitrary directions. Here, $w_1$ is set to zero.

7 PRACTICAL AGENTS

As mentioned previously in the introduction, although the agents model used here is more relevant to actual aerospace or underwater agent models, yet in order to use it in actual applications extra aspects need to be addressed such as force fields and coupled degrees of actuation. It is conjectured here that the hierarchic approach could facilitate this task.

One way is as following. Instead of carrying out a new design, it is sufficient to design those agents actuator inputs to provide desired forces

$$[f_1(\chi), f_2(\chi), 0]^\top + d'(\chi),$$

where $d'(\chi) : \mathbb{R} \to \mathbb{R}^3$ is a term added to address the force fields, and angular velocities $w'(\chi)$, or torques used to achieve them. This can be done in two hierarchic steps:

1. Design actuator inputs to achieve desired actuator outputs.
2. Design actuator outputs to achieve desired forces.

Refer to Figure 10. The second step involves addressing the couplings between the actuation degrees.

Figure 11 shows two examples of practical agents that can be partially modelled by (1). The first is fixed wing UAV. This system usually has one force from a propeller or a different kind engine, and another force generated by the movement of the airfoil. The second system is an underactuated satellite from [29]. While satellite thrusts can be more or less than two, usually two orthogonal thrusts are needed for orbital maneuvers. One for orbital plane stabilization, and the second for stabilizing desired orbit on that plane. The work in [29] was motivated by this where only two thrusts were used to design continuous feedbacks for formation stabilization (keeping). If the satellite is already on the desired orbital plane, with the correct orientation, then only one thrust is used.

Regarding the force fields, both of the agents types above would experience gravitational force, however the way to deal with it near the earth surface, that is for a UAV, is different than that for a spacecraft. For the former, this force needs to be cancelled by the vehicle actuators. For the latter, this force cannot be cancelled, the controller should taking into account that this force shapes the dynamics of the vehicles. In [29], circular formations were addressed, and despite that it is not feasible to use the approach discussed here, that is, the two hierarchies of Figures 4 and 10, yet it was still possible to design another hierarchy to solve the problem. In addition, to gravity, UAVs would experience air resistance. To apply the previous approach, one needs to determine the function $d(\chi)$ which is in this case the weight of the vehicle plus an air drag function. This can be challenging to specify as air drag depends on the shape and orientation of the vehicle, as well as other factors.

Regarding actuator inputs design, the first step above is usually very simple. It amounts to designing the input of an electric motor, a pneumatic valve, or a thruster... etc. to achieve a desired actuator output. The second step is the main challenge for agents like the fixed wing UAV. In that system, the second force results from the forward motion, the shape, including the angles of the control surfaces, and orientation of the vehicles. Also, the angular velocity is a function of the forward force and the angles of the control surfaces. For the satellite, the situation is simpler because usually the thrusters are fixed in certain body directions, and there are several methods for achieving desired angular velocities.

7.1 Example

Considered a quadrotor as shown in Figure 12, in earth force field, and consider the model used in [37]

$$m\ddot{x} = \bar{f}b_3 - mg_3, \quad \dot{B} = B(w)^X, \quad f\dot{w} = \tau - (w)^Xf_w;$$

\[
(31)
\]
where \( \vec{f} \in \mathbb{R} \) is the total thrust from the rotors, \( g \) is the gravity constant, \( \tau \in \mathbb{R}^3 \) is the applied torque in body frame, parametrized by \( B \), and \( f \) is the inertia matrix with respect to \( B \).

Suppose that a certain feedback \( f(\chi) \) has been designed to achieve a certain behaviour as in the previous development. This can be used for (31) by applying the following:

- Design \( \tau \) to achieve orientation such that \( b_j \) points in the direction of \( f(\chi) + mg \hat{e}_3 \).
- Set \( \vec{f} = \| f(\chi) + mg \hat{e}_3 \| \).

Figure 13 shows a simulation for stabilizing (31) to a circular orbit centred at \([1\ 1\ 1]\), with its plane orthogonal to \([1\ 0\ 1]\) and with radius value of 1. In this case, it is impossible to have, while on the path, the forward direction \( b_j \) pointing in the direction of linear velocity, unless the path is horizontal. In the simulation, \( b_j \) is stabilized to the path plane. Note also that in order for this design to work, \( f(\chi) + mg \hat{e}_3 \) should be well defined at least in a neighbourhood of the desired goal set.

**Remark 5.** Model (31) is a simplified model for a quadrotor flying vehicle. One point of simplification is the decoupled actuation degrees. Generally, the torques depend on the rotor thrusts, however, the two can be easily decoupled [38].

## 8 Conclusion

The paper presented a hierarchy solving a formation problem for underactuated agents in \( \mathbb{R}^3 \), which generalizes circular formations. The problem provides certain advantages for applications of multi-agents, and the hierarchic set stabilizing approach used complements these advantages. The agents model used is close to how several practical agents can be modelled. Several questions need to be addressed to facilitate the implementation of the results in practical applications. For instance, how force fields, actuator dynamics/couplings can be handled, and how to find/extend the region of attraction of the local result.

## References

1. Murray, R.M.: Recent research in cooperative control of multivehicle systems. J. Dyn. Syst., Meas. Contr. 129(5), 571–583 (2007)
2. Oh, K.K., et al.: A survey of multi-agent formation control. Automatica 53, 424–440 (2015)
3. El-Hawwary, M.I., Maggiore, M.: Distributed circular formation stabilization for dynamic unicycles. IEEE Trans. Autom. Control 58(1), 149–162 (2013)
4. El-Hawwary, M.I.: Three-dimensional circular formations via set stabilization. Automatica 54, 374–381 (2015)
5. El-Hawwary, M.I., Maggiore, M.: Reduction theorems for stability of closed sets with application to backstepping control design. Automatica 49(1), 214–222 (2013)
6. Kwon, J.W., Chwa, D.: Hierarchical formation control based on a vector field method for wheeled mobile robots. IEEE Trans. Rob. 28(6), 1335–1345 (2012)
7. Hardlmann, H., et al.: Symmetry and reduction for coordinated rigid bodies. J. Dyn. Syst. Meas. Contr. 129, 662–677 (2007)
8. Lee, D.J., Li, P.Y.: Passive decomposition approach to formation and maneuver control of multiple rigid bodies. J. Dyn. Syst. Meas. Contr. 10(28), 1–13 (2013)
9. Corteis, J., et al.: Coverage control for mobile sensing networks. IEEE Trans. Rob. Autom. 20(2), 243–255 (2004)
10. Shaferman, V., Shima, T.: Unmanned aerial vehicles cooperative tracking of moving ground target in urban environments. J. Guid. Control Dyn. 31(5), 1360–1371 (2008)
11. Bhandari, S., et al.: Search and rescue using unmanned aerial vehicles. In: Proceeding of AIAA Infotech@ Aerospace, p. 1458 (2015)
12. Leonard, N.E., et al.: Collective motion, sensor networks and ocean sampling. Proc. IEEE 95(1), 48–47 (2007)
13. Scharf, D.P., et al.: A survey of spacecraft formation flying guidance and control. Part II: Control. In: Proc. of the American Control Conference, Vol. 4, Boston, MA, USA, (2004)
14. Ramirez, J.L., et al.: Distributed control of spacecraft formations via cyclic pursuit: Theory and experiments. J. Guid. Control Dyn. 33(5), 1655–1669 (2010)
15. Qiu, J., et al.: Circular formation algorithms for multiple nonholonomic mobile robots: An optimization-based approach. IEEE Trans. Ind. Electron. 66(5), 3693–3701 (2019)
16. Li, N.H.M., et al.: Formation UAV flight control using virtual structure and motion synchronization. In: Proceeding of the 2008 American Control Conference, Seattle, WA, USA (2008)
17. Hafez, A.T., et al.: Solving multi-UAV dynamic encirclement via model predictive control. IEEE Trans. Control Syst. Technol. 23(6), 2251–2265 (2015)
18. Renevey, S., Spencer, D.A.: Establishment and control of spacecraft formations using artificial potential functions. Acta Astronaut. 162, 314–326 (2019)
19. Sadraey, M.: Multi-vehicle circular formation flight in an unknown time-varying flow-field. In: Proceeding of the 2013 International Conference on Unmanned Aircraft Systems (ICUAS), Atlanta, GA, USA (2013)
20. Goto, T., et al.: Potential game theoretic attitude coordination on the circle: Synchronization and balanced circular formation. In: Proceeding of the IEEE International Symposium on Intelligent Control, Yokohama, Japan, (2010)
24. Marshall, J.A., et al.: Formations of vehicles in cyclic pursuit. IEEE Trans. Autom. Control 49, 1963–1974 (2004)
25. Summers, T.H., et al.: Coordinated standoff tracking of moving targets: Control laws and information architectures. J. Guid. Control Dyn. 32(1), 56–69 (2009)
26. Strogatz, S.H.: From Kuramoto to Crawford: Exploring the onset of synchronization in populations of coupled oscillators. Physica D 143(1), 1–20 (2000)
27. El-Hawwary, M.I., Maggiore, M.: Case studies on passivity-based stabilisation of closed sets. Int. J. Control 84(2), 336–350 (2011)
28. Scharf, D.P., et al.: A survey of spacecraft formation flying guidance and control. Part I: Guidance. In: Proc. of the American Control Conference, Vol. 2, Denver, CO, USA (2003)
29. El-Hawwary, M.I., Scherpen, J.M.A.: Distributed stabilization of circular orbit formations for underactuated satellites. In: 8th International Workshop on Satellite Constellations and Formation Flying (IWSCFF15), Delft, The Netherlands (2015)
30. Bucknall, Y.L.R.: Path planning algorithm for unmanned surface vehicles and its application to VTOL drones. IEEE Trans. Autom. Control 54(8), 1837–1853 (2009)
31. El-Hawwary, M.I., Scherpen, J.M.A.: Hierarchical distributed stablization of circular orbit formations for underactuated satellites. In: 8th International Workshop on Satellite Constellations and Formation Flying (IWSCFF15), Delft, The Netherlands (2015)
32. Abdessameud, A., Tayebi, A.: Formation control of VTOL unmanned aerial vehicles on a balanced graph. Automatica 48, 2971–2977 (2012)
33. Lee, D.: Distributed backstepping control of multiple thrust-propelled aerial vehicles with communication delays. Automatica 47, 2383–2394 (2011)
34. Abdessameud, A., Tayebi, A.: Global trajectory tracking control of VTOL–UAVs without linear velocity measurements. Automatica 46(6), 1053–1059 (2010)
35. El-Hawwary, M.I., Scherpen, J.M.A.: Distributed stabilization of circular orbit formations for underactuated satellites. In: 8th International Workshop on Satellite Constellations and Formation Flying (IWSCFF15), Delft, The Netherlands (2015)
36. Bucknall, Y.L.R.: Path planning algorithm for unmanned surface vehicles and its application to VTOL drones. IEEE Trans. Autom. Control 54(8), 1837–1853 (2009)
37. Mercado1, D.A., et al. Quadrotors flight formation control using a leader-follower approach. In: Proceeding of the 2013 European Control Conference (ECC13), Zürich, Switzerland (2013)
38. Tayebi, A., McGilvray, S.: Attitude stabilization of a VTOL quadrotor aircraft. IEEE Trans. Control Syst. Technol. 14(3), 562–571 (2006)
39. Bhatia, R.: Matrix Analysis. Springer-Verlag, New York (1996)
40. Varga, R.S.: Gershgorin and His Circles. Vol. 36 of Springer Series in Computational Mathematics. Springer, New York (2004)

How to cite this article: El-Hawwary MI. Hierarchical distributed stabilization of a class of three-dimensional formations for underactuated agents. IET Control Theory Appl. 2021;15:472–488. https://doi.org/10.1049/cth2.12057

APPENDIX A

A.1 | Proof of Lemma 1

By the Gershgorin circle theorem [39], and Lemma 2 in [35], is a simple eigenvalue at 0, and all its other eigenvalues have positive real parts. The eigenvalues of (2) are the roots of the polynomial

\[
|sI_{\omega x} + kL| = \bar{s}(s + 1 + kP_{\omega x} + kL),
\]

Since \(L\) and \(I_{\omega x}\) commute, this is equal to the roots of \(|s(s + 1)I_{\omega x} + kL|\). From the above, the eigenvalues of \(-kL\), which are the roots of \(\bar{s}I_{\omega x} + kL\), have negative real parts except for one at the origin. Therefore

\[
|sI_{\omega x} + kL| = \bar{s}(s + 1 + \cdots + (s + kP_{\omega x} + kL)),
\]

where \(p_1, \ldots, p_{n-1}\) are the non-zero eigenvalues of \(L\). By substituting \(s(s + 1)\) for \(\bar{s}\) in the above equation, it follows that the eigenvalues of (2) are the roots of the polynomial

\[
|s(s + 1)I_{\omega x} + kL| = s(s + 1)(s + 1 + kP_{\omega x} + kL),
\]

(A.1)

If \(p_1, \ldots, p_{n-1}\) are real, then the result follows for any \(k \geq 0\). Now, suppose that some eigenvalue of \(L\), \(\rho_i\), is a complex number. By the properties of \(L\), and the Gershgorin circle theorem, \(\rho_i\) lies in a disc, in the complex plane, centred at some \(r_i \in \mathbb{R}^+\), with radius \(r_i\). \(r_i\) is a diagonal entry of \(L\). The roots of the term \(s^2 + s + k\rho_i\) by choosing \(k < \frac{1}{8r_i}\), one gets \(1 - 4k\rho_i < 1\), and hence the roots of \(s^2 + s + k\rho_i\) lie in the open left half of the complex plane. Therefore, by choosing

\[
k < \min_{\rho_i \in \mathbb{R}^+} \frac{1}{8r_i},
\]

\(\epsilon(i)\) being the set of indexes where \(\rho_i\) is complex, all the roots of (A.1) lie in the open left half of the complex plane except for the one at zero.

A.2 | H1 details

Since \(\dot{y}_i = u_2(\chi)\dot{b}_2 - u_2(\chi)\dot{b}_2\), it follows from (9) that

\[
f_3(\chi) = 0 \Rightarrow u_2(\chi)u_2(\chi) + k_3(\chi)\dot{b}_2 \cdot \dot{\chi} = 0.
\]

(A.2)

Consider \(\psi_i(\chi), \delta_i(\chi), \tilde{u}_i(h_i)\) rewritten here as

\[
\psi_i(\chi) = \varphi(L_i^{\chi}(\chi_{\alpha} - \alpha)), \delta_i(\chi) = \varphi(L_i^{\chi}(\chi_{\alpha} - \alpha)), \tilde{u}_i(h_i) = -\sin(L_i(\chi - \beta)).
\]

(A.3)

These forms are equivalent to the ones in (23), (25) and (28), respectively, but are more compact. In the latter forms, the agents sensed information are expressed by variables in its body frame. Using these functions, \(u_2(\chi)\) can be rewritten as \(u_2(\chi) = \xi^{'i} \cdot \xi^{i}\), where

\[
\begin{align*}
\xi^{'i} &= [p_1 \cdot b_2^{'i} p_3 \cdot b_3^{'i}]^T = [\xi^{'i}] \angle \Theta_2^{'i}, \\
\xi^i &= \left[ \frac{1}{\|\varphi(L_i^{\chi})(\chi_{\alpha} - \alpha)\|} \left( u_i(\chi) + k\varphi(L_i^{\chi}(\chi_{\alpha} - \alpha))\dot{b}_2 \right) \right] \\
&= [\tilde{k} \left( p_3 \cdot b_1^i + k \frac{\varphi(L_i^{\chi}(\chi_{\alpha} - \alpha))}{\|\varphi(L_i^{\chi})(\chi_{\alpha} - \alpha)\|} \right)] (A.4)
\end{align*}
\]
From this, (A.2) is satisfied if $\Theta_{\xi_i} - \Theta_{\xi_i} = \cos^{-1}(-k_j^f(\chi)\xi_i^f)\xi_i^f$, provided that this angle is well defined. To this end, consider the candidate Lyapunov function

$$\Theta = \frac{1}{2} \sum_{i=1}^{n} (\Theta_{\xi_i} - \Theta_{\xi_i} - \Theta_{\xi_i})^2,$$

(A.5)

where $\Theta_{\xi_i} = \cos^{-1}(-k_j^f(\chi)\xi_i^f)\xi_i^f$. Note that by the choice (15), this angle is well defined. Since $\xi_i^f, \xi_i^f$ do not appear in $\xi_i^f$, it follows that $\Theta_{\xi_i} - \Theta_{\xi_i}$ is not a function of $\xi_i^f$. Taking the time derivative of $\Theta_{\xi_i} - \Theta_{\xi_i}$ along (1) gives

$${\dot{\Theta}}_{\xi_i} = -w_i^f + (p_3 \cdot \xi_i^f)\xi_i^f,$$

where $\kappa_i(\chi) = \sqrt{(n(\chi)\xi_i^f||\xi_i^f||^2 - (k_j^f(\chi)\xi_i^f)^2)}$ and

$$\kappa_j(\chi) = \sqrt{(n(\chi)\xi_i^f||\xi_i^f||^2 - (k_j^f(\chi)\xi_i^f)^2)}.$$ 

It is straightforward to show that $n_i(\chi)\xi_i^f||\xi_i^f||^2$ is not a function of $w_i^f$. By setting

$$w_i^f(\chi) = \frac{-\cos \Theta_{\xi_i}(\chi)\xi_i^f||\xi_i^f|| - k_j^f(\chi)\xi_i^f}{\kappa_i(\chi) - k_j^f(\chi)\xi_i^f} + \kappa_j(\chi)\xi_i^f,$$

(A.6)

with some constant $k_{\alpha_i} > 0$, the time derivative of $\Theta$ in (A.5) along (1) becomes

$$\dot{\Theta} = -k_{\alpha_i} \sum_{i=1}^{n} (\Theta_{\xi_i} - \Theta_{\xi_i} - \Theta_{\xi_i}) \sin(\Theta_{\xi_i} - \Theta_{\xi_i} - \Theta_{\xi_i}).$$

(A.7)

By this, and the fact that $\xi_i^f, \xi_i^f$ are uniformly bounded, see (A.4), it follows that for initial conditions in

$$T = \{\chi \in \mathcal{X} : |\Theta_{\xi_i}(\chi) - \Theta_{\xi_i}(\chi) - \Theta_{\xi_i}(\chi)| \leq \varepsilon_0 < \pi, i \in n\},$$

the dynamics $\dot{y}, \dot{\varphi}$ takes the form

$$\dot{y} = P^{-1} \sum_{j \in \mathcal{N}(f)} (x_j^f - x_j^f) \cdot p_j,$$

(A.10)

Using (21), (22) and the fact that

$$\sum_{j \in \mathcal{N}(f)} (x_j^f - x_j^f) \cdot p_j - (\alpha_j - \alpha_j) \cdot p_j = L^\top (\varsigma - \varsigma),$$

the dynamics $\dot{y}, \dot{\varphi}$ takes the form

$$\dot{y} = P^{-1} \Sigma(h) (rL_{\alpha_n} + kU(h_l)) P_{\Sigma},$$

(A.11)

where $\Sigma(h) = \text{diag}(1 - (p_3 \cdot b_1^f)^2, \ldots, 1 - (p_3 \cdot b_f^f)^2)$, $S(h_l) = \text{diag}(\sigma(\mathbf{b}_f^f), \ldots, \sigma(\mathbf{b}_f^f))$, and $U(h_l) = \text{diag}(\mathbf{a}_1^f, \ldots, \mathbf{a}_f^f)$. Consider a change of coordinates $\mathbf{v} \mapsto \tilde{\mathbf{v}}$, $\tilde{\mathbf{v}} = P^{-1} \Sigma(h_l) \mathbf{v}$. By the assumption on $\mathbf{b}_f^f$, refer to (21), this transformation is
well defined for all time. Substituting this in \((\ref{eq:11})\) gives
\[
j = v^T \ddot{z} + \frac{1}{\delta} P \dot{S}^{-1} \dot{h} \dot{U} \dot{h} \dot{S}^{-1} \dot{h} P \ddot{y},
\]
\[
\ddot{\psi} = -\frac{1}{\delta} P \dot{S}^{-1} \dot{h} \dot{A} \dot{\psi} \dot{S}^{-1} \dot{h} \left( P \ddot{y} + \frac{1}{\delta} \dot{S}^{-1} \dot{h} \dot{\varphi} (L \dot{y}) \right)
\]
\[= - P^{-1} S (h) \dot{\psi}^{-1} (h) P \ddot{y},
\]
\[\text{(A.12)}
\]
Note that by \((\ref{eq:21})\) and \((\ref{eq:22})\), \(u_s^\xi (\chi)\) and \(u_s^\psi (\chi)\) are bounded, and hence so is \(\dot{\psi}^{-1} (h)\). Now, consider the partition \(\gamma = [\ddot{z} \quad \dot{\psi}]^T\) where \(\ddot{z} \in \mathbb{R}, \dot{\psi} \in \mathbb{R}^{n-1}\). Using this in \((\ref{eq:12})\) gives
\[
\dot{y} = v^T \ddot{z} + \frac{1}{\delta} \ddot{D} (h) \ddot{z}, \dot{\psi} = v^T [\dddot{\psi}_1 \cdots \dddot{\psi}_d]^T + \ddot{D} (h) \ddot{\psi},
\]
\[
\ddot{\psi} = -\frac{1}{\delta} P \dot{S}^{-1} \dot{h} \dot{A} \dot{\psi} \dot{S}^{-1} \dot{h} \left( P \ddot{y} + \frac{1}{\delta} \dot{S}^{-1} \dot{h} \dot{\varphi} (L \dot{y}) \right)
\]
\[= - P^{-1} S (h) \dot{\psi}^{-1} (h) P \ddot{y},
\]
\[\text{(A.13)}
\]
with \(\ddot{D} : \{\text{SO}(3)\}^n \rightarrow \mathbb{R} \times \mathbb{R}^n, \ddot{\psi} : \{\text{SO}(3)\}^n \rightarrow \mathbb{R}^{n-1} \times \mathbb{R}^n\) where \([\dddot{y}^T (h) \quad \ddot{\psi}^T (h)]^T = P^{-1} \dot{S} (h) P\), and
\[
[D^T (h) \quad D^T (h)]^T = P^{-1} \dot{S} (h) U (h) P.
\]
Note that since \(L\) is balanced, \(L^T\) is a Laplacian of a digraph which has a globally reachable node as well. From this, and the fact that the first column of \(P\) lies in ker \(L\), \(P^{-1} L P = \text{diag}(0, M)\), for some matrix \(M \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n}\), whose eigenvalues have strictly positive real parts, and so \(L P = P \{0 \quad \ddot{y}^T M\}^T\). Since \(P \dot{y}^T \{0_{1 \times n-1}\}^T \in \text{ker} \, L, \Gamma_3\) can be rewritten as \(\{\chi \in \mathbb{R}^2 : \ddot{y} = 0, \ddot{\psi} = 0\}\), and so, \(\Gamma_3\) is asymptotically stable for \((\ref{eq:18})\) if the origin of the \((\dot{y}, \ddot{\psi})\) sub-system of \((\ref{eq:13})\) is so. This sub-system can be viewed as a perturbed system with perturbation \(\ddot{\psi} (\chi) = [\dddot{\psi}^T (h) \quad \ddot{\psi}^T (h) \quad \ddot{\psi}^T (h) \quad \ddot{\psi}^T (h)]^T \ddot{\psi} (h)\). As \(\ddot{\psi} (\chi)\) satisfies \(\|\ddot{\psi} (\chi)\| \leq \frac{\ddot{\psi} (\chi)}{\left\| v^T \dddot{\psi} (h) \right\|}\), for some \(\epsilon > 0\), then, by Lemma 9.1 in [36], \(\ddot{\psi}\) could be chosen small enough such that the origin of the \((\ddot{y}, \ddot{\psi})\) sub-system of \((\ref{eq:13})\) is exponentially stable if that of the nominal system is so. The linearization of that nominal system around \((\ddot{y}, \ddot{\psi}) = 0\) takes the form
\[
[\ddot{y}] = v^T [\dddot{\psi}_1 \cdots \dddot{\psi}_d]^T,
\]
\[
[\ddot{\psi}] = -\frac{1}{\delta} \dddot{\psi} - \frac{1}{\delta} \dddot{\psi} P [0 \quad \dddot{y}^T M]^T - P^{-1} S (h) \dot{\psi}^{-1} (h) P \ddot{\psi},
\]
\[\text{(A.14)}
\]
By the boundedness of \(S (h)\) and \(\dot{\psi}^{-1} (h)\), it can be shown that the origin of this system is asymptotically stable if that of \([\ddot{y}] = v^T [\dddot{\psi}_1 \cdots \dddot{\psi}_d]^T, \ddot{\psi} = -\frac{1}{\delta} \dddot{\psi} - \frac{1}{\delta} \dddot{\psi} P [0 \quad \dddot{y}^T M]^T\) is so, and \(\ddot{\psi}\) is large enough. By Lemma 1, the eigenvalues of
\[
\left[
\begin{array}{cc}
0_{\delta \times \delta} & v L_{\delta \times \delta} \\
-\delta k P^{-1} L P & -\delta k L_{\delta \times \delta}
\end{array}
\right]
\]
have negative real parts except for one at the origin, and the result follows. Thus, the origin of the \((\dot{y}, \ddot{\psi})\) sub-system of \((\ref{eq:13})\) is asymptotically (in fact exponentially) stable, for sufficiently small \(\delta\). Note that, from \((\ref{eq:13})\), and the uniform boundedness of \(A (\psi), U (h), S (h), \text{if} \ddot{\psi} (\text{and so} \psi (t))\) converges to zero exponentially then, \(\ddot{\psi}\) has a finite limit. Thus, for all initial conditions in a neighbourhood of \(\Gamma_3\), the solutions \(y(t), \psi(t)\) converge to points on \(\Gamma_3\), i.e. for all such initial conditions the agents approach fixed planes.

**A.4 | H4 details**

The following development is restricted to \(\Gamma_3\).

\[
\dot{\chi} = \dot{\psi}^2 = \dot{\psi}^2 - \dot{\psi}_1 (b_1^i) p_1 - \dot{\psi}_2 (b_1^i) p_2
\]

\[
= \dot{\psi}^2 - (\dot{\psi}_1 (b_1^i) p_1 + \dot{\psi}_2 (b_1^i) p_2) b_1^i,
\]

\[\text{(A.15)}
\]

where

\[
\left[\begin{array}{c}
\bar{b}_1^i = (p_3 \cdot b_1^i) b_2^i - (p_3 \cdot b_2^i) b_1^i
\end{array}\right],
\]

which is a unit vector where \(p_3 \times b_1^i = -b_1^i\). Also, \(\dot{\chi} = \dot{\chi}_p = \dot{\psi}^2\) and \(b_1^i = \tan^{-1} \frac{\bar{b}_1^i}{\bar{b}_1^i}.\) Using this one gets

\[
\dot{\bar{b}}_1^i = \dot{\psi}_1^2 \cos b_1^i - \dot{\psi}_2^i \sin b_1^i
\]

\[\text{(A.16)}
\]

Since

\[
\dot{\psi}_1 (b_1^i) p_1 + \dot{\psi}_2 (b_1^i) p_2 = \|\dot{\psi}_1 (b_1^i)\| (\cos b_1^i p_1 + \sin b_1^i p_2)
\]

\[
= \|\dot{\psi}_1 (b_1^i)\| \bar{b}_1^i = -\|\dot{\psi}_1 (b_1^i)\| p_3 \times \bar{b}_1^i
\]

Using this, \((\ref{eq:15})\) and the fact that on \(\Gamma_3, \dot{\chi} = u^2 (\chi) \bar{b}_1^i,\)

\[
\dot{\chi}_p = \dot{\chi} + \|\dot{\psi}_1\| \|\dot{\psi}_2\| p_3 \times \bar{b}_1^i
\]

\[= \dot{\psi}^2 - (\frac{\dot{\psi}_1 (b_1^i)}{\|\dot{\psi}_1 (b_1^i)\|} \frac{\ddot{\psi}_1 (b_1^i)}{\|\dot{\psi}_1 (b_1^i)\|} + \frac{\ddot{\psi}_2 (b_1^i)}{\|\dot{\psi}_1 (b_1^i)\|}) \bar{b}_1^i
\]

\[= - \bar{b}_1^i \ddot{\psi}_1 (b_1^i) \bar{b}_1^i.
\]
By (25), and using the fact that
\[
\sum_{j \in \mathbb{N} \setminus \{0\}} (x^j_p - x^j_\alpha - \bar \alpha_i + \bar \alpha_j)^T \bar b_i^j = (I_{i(3)} (x_p - \alpha_p))^T \bar b_i^j,
\]
it follows that
\[
\dot x^j_p = -\dot \kappa \varphi (I_{i(3)} (x_p - \alpha_p))^T \bar b_i^j \bar b_i^j \quad (A.17)
\]
on \Gamma_4. Let
\[
\eta_j = \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_n \end{bmatrix}^T, \quad \lambda = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}^T
\]
\[
\eta_j = (I_{i(3)} (x_p - \alpha_p))^T \bar b_i^j, \quad \lambda_j = (I_{i(3)} (x_p - \alpha_p))^T \bar b_i^j,
\]
where \( \bar b_1^j = (p_1 \cdot \bar b_1^j) \bar b_1^j - (p_3 \cdot \bar b_1^j) \bar b_2^j \). Note that \( \bar b_1^j \cdot \bar b_1^j = 0 \), and \( \|\bar b_1^j\| = 1 \), that is, \( \bar b_1^j, \bar b_1^j \) are orthonormal. From this, \( \eta_j = \lambda_j = 0 \) if and only if \( I_{i(3)} (x_p - \alpha_p) = 0 \). Hence, the asymptotic stability of \( \Gamma_4 \) for (20) is equivalent to the asymptotic stability of
\[
\dot \eta_j = -\dot \kappa \begin{bmatrix} \varphi (\eta_1) b_1^j \\ \vdots \\ \varphi (\eta_n) b_1^j \end{bmatrix}^T \bar b_i^j + (I_{i(3)} (x_p - \alpha_p))^T \bar b_i^j \quad (A.18)
\]
Using (A.15) and (A.18), \( \bar b_i^j \) and \( \bar b_i^j \) take the forms
\[
\bar b_i^j = \sigma (\bar b_1^j) \begin{bmatrix} x + \dot \kappa \bar u (b_1^j) \bar b_1^j \\ \|\sigma' (\bar b_1^j)\| \end{bmatrix} - \begin{bmatrix} \dot \kappa \varphi (\eta_1) \\ \|\sigma' (\bar b_1^j)\| \end{bmatrix} \bar b_i^j,
\]
\[
\bar b_i^j = -\sigma (\bar b_1^j) \begin{bmatrix} x + \dot \kappa \bar u (b_1^j) \bar b_1^j \\ \|\sigma' (\bar b_1^j)\| \end{bmatrix} - \begin{bmatrix} \dot \kappa \varphi (\eta_1) \\ \|\sigma' (\bar b_1^j)\| \end{bmatrix} \bar b_i^j.
\]
By this, (A.19) can be rewritten as
\[
\dot \eta_j = -\dot \kappa C (b_1^j) \varphi (\eta) + S_1 (b_1^j) (i \omega_{\text{os}} + \dot \kappa U (b_1^j)) \bar \lambda + \dot \kappa S_2 (b_1^j) \varphi (\eta) \bar \lambda_1 \cdots \varphi (\eta) \bar \lambda_n^T,
\]
\[
\dot \lambda_j = -\dot \kappa E (b_1^j) \varphi (\eta) - S_1 (b_1^j) (i \omega_{\text{os}} + \dot \kappa U (b_1^j)) \eta - \dot \kappa S_2 (b_1^j) \varphi (\eta) \eta_1 \cdots \varphi (\eta) \eta_n^T,
\]
System (A.20) can be viewed as a time-varying system where the time variation is brought about by \( b_1^j(t) \), and whose average is
\[
\eta_{av} = -\bar \kappa C \varphi (\eta_{av}) + S_1 (i \omega_{\text{os}} + \bar \kappa U) \bar \lambda_{av}
\]
\[
+ \dot \kappa S_2 [\varphi (\eta_{av}) \bar \lambda_{av}] \cdots \varphi (\eta_{av}) \bar \lambda_{av}]^T,
\]
\[
\dot \lambda_{av} = -\bar \kappa E [\varphi (\eta_{av}) - S_1 (i \omega_{\text{os}} + \bar \kappa U) \eta_{av}] \eta_{av}^T
\]
\[
- \dot \kappa S_2 [\varphi (\eta_{av}) \eta_1 \cdots \varphi (\eta_{av}) \eta_n]^T,
\]
where \( C, E, \bar \lambda, S_1, \) and \( S_2 \) denote the averages of \( C (b_1^j(t)) \), \( E (b_1^j(t)) \), \( \bar \lambda (b_1^j(t)) \), and \( S_2 (b_1^j(t)) \). Since \( i \omega_{\text{os}} + \bar \kappa U \leq 0 \), for all \( t \geq 0 \), then, from the properties of \( L_i \) and the Gershgorin circle theorem [40], \( C + \bar \lambda^T \) is either positive definite or positive semi-definite with a single eigenvalue at zero. By Tausky's theorem [40], the latter happens only when all the off-diagonal element of \( C \) has magnitude one. For this, the following development addresses the case where \( C = L_i \). The development for \( C \neq L_i \) is almost similar. Consider the candidate Lyapunov function \( V = \frac{1}{2} \eta_{av}^T \bar \lambda_{av} + \frac{1}{2} \lambda_{av}^T \bar \lambda_{av} \). Using the fact that \( C = L_i \), \( E = 0 \), the time derivative of \( V \) along (A.22) takes the form
\[
\dot V = -\dot \kappa \eta_{av}^T I \eta_{av} \quad (A.23)
\]
This implies that all solutions of (A.22) bounded. The zero level set of \( V \) takes the form \( V^{-1}(0) = \{ \eta_{av} : \lambda_{av} \in \ker L_i \} \). Suppose that (A.22) is initialized on the largest invariant subset of \( V^{-1}(0) \). If this is the case, then
\[
\begin{cases}
S_i^j (x + \bar \kappa U) + \bar \kappa S_j^i \varphi (\eta_{av}) \bar \lambda_{av} \bar \lambda_{av} = \frac{1}{2} \varphi (\eta_{av}) \bar \lambda_{av} \bar \lambda_{av},
S_i^{j+1} (x + \bar \kappa U + \bar \kappa S_j^i + \bar \kappa S_j^i \varphi (\eta_{av}) \bar \lambda_{av} \bar \lambda_{av} + \bar \kappa S_j^i \varphi (\eta_{av}) \bar \lambda_{av} \bar \lambda_{av})
\end{cases}
\]
for all \( i \in \{1, \ldots, n-1\}, \frac{1}{2} \geq 0 \), \( S_i^j, U \), denote the \( n \)th diagonal elements of \( S_i^j, U \), respectively. From (A.22) and the fact that on \( V^{-1}(0), \eta_{av} (t) = \lambda_{av} (t) \), \( i \in \{1, \ldots, n-1\} \),
\[
\begin{cases}
S_i^{j+1} (x + \bar \kappa U + \bar \kappa S_j^i + \bar \kappa S_j^i \varphi (\eta_{av}) \bar \lambda_{av} \bar \lambda_{av} + \bar \kappa S_j^i \varphi (\eta_{av}) \bar \lambda_{av} \bar \lambda_{av})
\end{cases}
\]
The previous two equalities imply \( \lambda_{av} \bar \lambda_{av} = \lambda_{av} \bar \lambda_{av} + 1 \), which is satisfied only if \( \lambda_{av} \bar \lambda_{av} = \pm 1 \), \( i \in \{1, \ldots, n-1\} \). Since \( S_i^j, S_j^i \) are positive, and by the properties of \( U \), the equalities are satisfied only if \( \lambda_{av} \bar \lambda_{av} = \pm 1 \), and so the largest invariant subset of \( V^{-1}(0) \) is contained in \( \{ \eta_{av} : \lambda_{av} \bar \lambda_{av} \in \ker L_i \} \). Since \( i \omega_{\text{os}} + \bar \kappa U \leq 1 \), the average of \( (b_1^j(t))^T (b_1^j(t)) \) being one implies \( (b_1^j(t))^T (b_1^j(t)) \rightarrow \pm 1 \). Also, since all solutions of (A.22) are bounded, the average of \( L_i (x_p (t) - \alpha_p) \) takes the form
\[
L_i (x_p (t) - \alpha_p) = 1 \otimes \tilde \sigma, \quad \text{for some } \tilde \sigma \in \mathbb{R}^2.
\]
As \( 1 \in \ker L_i \), it follows that
\[
L_i (x_p (t) - \alpha_p) = 0, \quad \text{that is, } (x_p - \alpha_p) \in \ker L_i, \quad \text{and so}
\]
Theorem 1 proof sketch

For the linearization of \((A.22)\) around its origin. By \((A.23)\), the result follows. Note that, by \((A.17)\), if \(\dot{\eta}(t) \rightarrow 0\) exponentially then \(x_j'(t) \rightarrow 0\) a constant.

\(\tilde{\delta} = 0\). Thus, the largest invariant subset of \(\dot{V}'(0)\) is \(\{\eta_{av}, \lambda_{av} : \eta_{av} = \lambda_{av} = 0\}\), and from LaSalle’s invariance principle, the origin of \((A.22)\) is asymptotically stable. That origin is also exponentially stable as the previous development applies for the linearization of \((A.22)\) around its origin. By \((A.23)\), the real parts of the poles of the latter system are linear in \(\tilde{\delta}\).

For the case where \(C + C^\top\) is positive definite, the linearization of \((A.22)\) takes the form

\[
\begin{bmatrix}
\dot{\eta}_{av} \\
\dot{\lambda}_{av}
\end{bmatrix} =
\begin{bmatrix}
-kC & D \\
-kE - D & 0_{p \times p}
\end{bmatrix}
\begin{bmatrix}
\eta_{av} \\
\lambda_{av}
\end{bmatrix},
\]

where \(D\) is the diagonal matrix \(S_j(d_{i\ell_k} + \hat{k}U)\). The matrix \(\begin{bmatrix}
-kC & D \\
-kE - D & 0_{p \times p}
\end{bmatrix}\) is Hurwitz. This can be shown using a Lyapunov function \(\eta_{av} \cdot \eta_{av} + \lambda_{av} \cdot \lambda_{av}\), and LaSalle’s invariance principle. For sufficiently small \(\hat{k}\), \((A.24)\) is asymptotically stable. From the averaging theorem, there exists \(\hat{k} > 0\) such that for all \(\hat{k} \in (0, \hat{k}^*)\) the origin of \((A.20)\) is exponentially stable, and the result follows. Note that, by \((A.17)\), if \(\dot{\eta}(t) \rightarrow 0\) exponentially then \(x_j'(t) \rightarrow 0\) a constant.

A.5 Theorem 1 proof sketch

The design in Section 4 provides the following: \(\Gamma_1\) is asymptotically stable, and \(\Gamma_j\) is asymptotically stable relative to \(\Gamma_{j-1}\), \(j \in \{2, \ldots, 5\}\) for the closed-loop system. By Corollary 1, \(\Gamma_4\) and \(\Gamma_5\) are asymptotically stable if the closed-loop system is LUB near \(\Gamma_4\). To show this, it suffices to show that the conditions of Corollary 2 are satisfied for the closed-loop system with \(k = 4\). Consider the coordinate change \(\chi \mapsto \zeta := \text{col}(\zeta_1, \ldots, \zeta_5)\) where

\[
\begin{align*}
\zeta_3 &= \text{col}(w_2(\chi)u(\chi) + k_0(\chi)b_j x_j', i \in \mathbb{n}), \\
\zeta_2 &= \text{col}(\ell^j(\chi), i \in \mathbb{n}),
\end{align*}
\]

By Corollary 1, \(\Gamma_3\) is exponentially stable, and the corollary is also satisfied. This shows that the LUB property applies, and so \(\Gamma_4\) and \(\Gamma_5\) are asymptotically stable for the closed-loop system. In addition, for initial conditions in a neighbourhood of \(\Gamma_4\) the agents approach fixed planes. The non-restricted dynamics \((\tilde{\gamma}, \tilde{\nu})\) as the restricted ones with exponentially decaying perturbations, and since \(\tilde{\gamma}(\chi) = 0\) on \(\Gamma_3\), the agents approach fixed planes.