Energy, angular momentum, superenergy and angular supermomentum in conformal frames

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Abstract

We find the rules of the conformal transformation for the energetic quantities such as the Einstein energy-momentum complex, the Bergmann-Thomson angular momentum complex, the superenergy tensor, and the angular supermomentum tensor of gravitation and matter.

We show that the conformal transformation rules for the matter parts of both the Einstein complex and the Bergmann-Thomson complex are fairly simple, while the transformation rules for their gravitational parts are more complicated. We also find that the transformational rules of the superenergy tensor of matter and the superenergy tensor of gravity are quite complicated except for the case of a pure gravity. In such a special case the superenergy density as well as the sum of the superenergy density and the matter energy density are invariants of the conformal transformation. Besides, in that case, a conformal invariant is also the Bel-Robinson tensor which is a part of the superenergy tensor. As for the angular supermomentum tensor of gravity - it emerges that its transformational rule even for a pure gravity is quite complicated but this is not the case for the angular supermomentum tensor of matter.

Having investigated some technicalities of the conformal transformations, we also find the conformal transformation rule for the curvature invariants and, in particular, for the Gauss-Bonnet invariant in a spacetime of arbitrary dimension.

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I. INTRODUCTION

It is widely known that there exists a problem of the energy-momentum of gravitational field in general relativity. Since the gravitational field may locally vanish, then one is always able to find the frame in which the energy-momentum of the gravitational field vanishes in this frame, while it may not vanish in the other frames. In fact, the physical objects which describe this situation are not tensors, and they are called the gravitational field pseudotensors. They form the energy-momentum complexes which are the sums of the obvious energy-momentum tensors of matter and appropriate pseudotensors. The choice of the gravitational field pseudotensor is not unique so that many different definitions of the pseudotensors have been proposed. To our knowledge, the most frequently used are the energy-momentum complexes of Einstein [1], Landau-Lifshitz [2], Møller [3], Papapetrou [4], Bergmann-Thomson [5], Weinberg [6], and Bak-Cangemi-Jackiw [7]. Among them only Landau-lifshitz, Weinberg, and Bak-Cangemi-Jackiw pseudotensors are symmetric. On the other hand, only the Einstein complex is canonical. Bearing in mind the formalism of field theory, it is obvious that there also exist the angular momentum complexes. Among them the Bergmann-Thomson angular momentum complex [5] and the Landau-Lifshitz angular momentum complex [2] are most widely used.

The arbitrariness of the choice of pseudotensors and the fact that they usually give different results for the same spacetime inspired some authors [8–10] to define quantities which describe the generalized energy-momentum content of the gravitational field in a tensorial way. These quantities are called gravitational superenergy tensors [8] and gravitational angular supermomentum tensors [9] or super\(^{(k)}\)-energy tensors [10].

The canonical superenergy tensors and the canonical angular supermomentum tensors have successfully been calculated for plane, plane-fronted and cylindrical gravitational waves, Friedmann universes, Schwarzschild, and Kerr spacetimes [8, 9].

By use of the superenergy and angular supermomentum tensors one can also prove that a real gravitational wave with \( R_{iklm} \neq 0 \) possesses and carries positive-definite superenergy and angular supermomentum [8, 9, 11].

In Refs. [12, 13], the properties of the Einstein and Bergmann-Thomson complexes as well as the superenergy and supermomentum tensors for Gödel spacetime were studied. It was shown that the former were sensitive to a particular choice of coordinates while the
latter were coordinate-independent. Some interesting properties of superenergy and supermomentum were found. For example, the relation between the positivity of the superenergy and causality violation.

In this paper we are going to discuss another interesting problem - the sensitivity of the complexes, superenergy and supermomentum to the conformal transformations of the metric tensor [14]. Conformal transformations are interesting characteristics of the scalar-tensor theories of gravity [15–18], including its conformally invariant version [19–22]. The point is that these theories can be represented in the two conformally related frames: the Jordan frame in which the scalar field is non-minimally coupled to the metric tensor, and in the Einstein frame in which it is minimally coupled to the metric tensor. It is most striking that the scalar-tensor theories of gravity are the low-energy limits of the currently considered as the most fundamental unification of interactions theory such as superstring theory [23–26]. It has been shown that some physical processes such as the universe inflation and density perturbations look different in conformally related frames [25, 26]. This is the main motivation why we find interesting to investigate the problem of energy-momentum in the context of conformal transformations. Inspired by similar ideas, in Ref. [27] it was shown that Arnowitt-Deser-Misner (ADM) masses were invariant in different conformal frames for asymptotically AdS and flat spacetimes as long as the conformal factor goes to unity at infinity.

Our paper is organized as follows. In Section II we give basic review of the idea of the conformal transformations of the metric tensor and discuss the transformational properties of the geometric quantities such as, for example, the Gauss-Bonnet invariant in $D$–spacetime dimensions. In Section III we discuss the conformal transformation of the Einstein energy-momentum complex while in the Section IV the conformal transformation of the Bergmann-Thomson angular momentum complex. In Section V we discuss the conformal transformation of the superenergy tensors and in Section VI the conformal transformations of the angular supermomentum tensors. In Section VII we give our conclusions.
II. BASIC PROPERTIES OF CONFORMAL TRANSFORMATIONS

Consider a spacetime \((\mathcal{M}, g_{ab})\), where \(\mathcal{M}\) is a smooth \(n\)–dimensional manifold and \(g_{ab}\) is a Lorentzian metric on \(M\). The following conformal transformation

\[
\tilde{g}_{ab}(x) = \Omega^2(x)g_{ab}(x),
\]

where \(\Omega\) is a smooth, non-vanishing function of the spacetime point is a point-dependent rescaling of the metric and is called a conformal factor. It must be a twice-differentiable function of coordinates \(x^k\) and lie in the range \(0 < \Omega < \infty\) \((a, b, k, l = 0, 1, 2, \ldots D)\). The conformal transformations shrink or stretch the distances between the two points described by the same coordinate system \(x^a\) on the manifold \(\mathcal{M}\), but they preserve the angles between vectors (in particular null vectors which define light cones) which leads to a conservation of the (global) causal structure of the manifold [14]. If we take \(\Omega = \text{const.}\) we deal with the so-called scale transformations [16]. In fact, conformal transformations are localized scale transformations \(\Omega = \Omega(x)\).

On the other hand, the coordinate transformations \(x^a \rightarrow \tilde{x}^a\) only change coordinates and do not change geometry so that they are entirely different from conformal transformations [14]. This is crucial since conformal transformations lead to a different physics [16]. Since this is usually related to a different coupling of a physical field to gravity, we will be talking about different frames in which the physics is studied (see also Refs. [20, 21] for a slightly different view).

In \(D\) spacetime dimensions the determinant of the metric \(g = \det g_{ab}\) transforms as

\[
\sqrt{-\tilde{g}} = \Omega^D \sqrt{-g}.
\]

It is obvious from (II.1) that the following relations for the inverse metrics and the spacetime intervals hold

\[
\tilde{g}^{ab} = \Omega^{-2}g^{ab},
\]

\[
ds^2 = \Omega^2ds^2.
\]

Finally, the notion of conformal flatness means that

\[
\tilde{g}_{ab}\Omega^{-2}(x) = \eta_{ab}.
\]
where $\eta_{ab}$ is the flat Minkowski metric.

The application of (II.1) to the Christoffel connection coefficients gives [14]

$$\tilde{\Gamma}_a^c = \Gamma_a^c + \frac{1}{\Omega} \left[ g^a_c \Omega_b + g_b^c \Omega_a - g_{ab} g^{cd} \Omega_d \right], \quad \tilde{\Gamma}_b^a = \Gamma_b^a + D \frac{\Omega_a}{\Omega}, \quad (II.6)$$

$$\Gamma_a^c = \tilde{\Gamma}_a^c - \frac{1}{\Omega} \left( g_a^c \Omega_b + g_b^c \Omega_a - g_{ab} g^{cd} \Omega_d \right), \quad \tilde{\Gamma}_b^a = \Gamma_b^a - D \frac{\Omega_a}{\Omega}. \quad (II.7)$$

The Riemann tensors, Ricci tensors, and Ricci scalars in the two related frames $g_{ab}$ and $\tilde{g}_{ab}$ transform as [36]

$$\tilde{R}_{bcd} = R_{bcd} + \frac{1}{\Omega} \left[ \delta_d^a \Omega_{bc} - \delta_c^a \Omega_{bd} + g_{bc} \Omega^{\alpha}_{;d} - g_{bd} \Omega^{\alpha}_{;c} \right], \quad (II.8)$$

$$R_{bcd} = \tilde{R}_{bcd} - \frac{1}{\Omega} \left[ \delta_d^a \Omega_{bc} - \delta_c^a \Omega_{bd} + \tilde{g}_{bc} \Omega^{\alpha}_{;d} - \tilde{g}_{bd} \Omega^{\alpha}_{;c} \right], \quad (II.9)$$

$$\tilde{R}_{ab} = R_{ab} + \frac{1}{\Omega^2} \left( 2(D - 2) \Omega_\alpha \Omega_{,ab} - (D - 3) \Omega_\alpha \Omega^{,ab} - \frac{2}{\Omega} \left( (D - 2) \Omega_{;ab} + g_{ab} \Box \Omega \right) \right), \quad (II.10)$$

$$R_{ab} = \tilde{R}_{ab} - \frac{1}{\Omega^2} (D - 1) \tilde{g}_{ab} \Omega_\alpha \Omega^{,c} + \frac{1}{\Omega} \left[ (D - 2) \Omega_{;ab} + \tilde{g}_{ab} \Box \tilde{\Omega} \right], \quad (II.11)$$

$$\tilde{R} = \Omega^{-2} \left[ R - 2(D - 1) \frac{\Box \Omega}{\Omega} - (D - 1) (D - 4) \frac{g_{ab} \Omega_\alpha \Omega_b}{\Omega^2} \right], \quad (II.12)$$

$$R = \Omega^2 \left[ \tilde{R} + 2(D - 1) \frac{\Box \tilde{\Omega}}{\tilde{\Omega}} - D(D - 1) \tilde{g}_{ab} \Omega_\alpha \Omega_b \frac{\Omega}{\Omega^2} \right], \quad (II.13)$$

and the appropriate d’Alambertian operators change under (II.1) as

$$\Box \phi = \Omega^{-2} \left[ \Box \tilde{\phi} + (D - 2) g^{ab} \Omega_a \phi_b \right], \quad (II.14)$$

$$\Box \tilde{\phi} = \Omega^2 \left[ \Box \phi - (D - 2) \tilde{g}^{ab} \Omega_a \phi_b \right]. \quad (II.15)$$

In these formulas the d’Alembertian $\Box$ taken with respect to the metric $\tilde{g}_{ab}$ is different from $\Box$ which is taken with respect to a conformally rescaled metric $g_{ab}$. Same refers to the covariant derivatives $\tilde{;}$ and $;$ in (II.8)-(II.11).

For the Einstein tensor we have

$$\tilde{G}_{ab} = G_{ab} + \frac{D - 2}{2 \Omega^2} \left[ 4 \Omega_\alpha \Omega_{,ab} + (D - 5) \Omega_\alpha \Omega^{,ab} - \frac{D - 2}{\Omega} [\Omega_{;ab} - g_{ab} \Box \Omega] \right], \quad (II.16)$$

$$G_{ab} = \tilde{G}_{ab} - \frac{D - 2}{2 \Omega^2} (D - 1) \Omega_\alpha \Omega^{,ab} \tilde{g}_{ab} + \frac{D - 2}{\Omega} [\Omega_{;ab} - \tilde{g}_{ab} \Box \tilde{\Omega}], \quad (II.17)$$
An important feature of the conformal transformations is that they preserve Weyl-conformal curvature tensor \((D \geq 3)\)

\[
C_{abcd} = R_{abcd} + \frac{2}{D-2} \left( g_{a[d}R_{c]b} + g_{b[c}R_{d]a} \right) + \frac{2}{(D-1)(D-2)} R g_{a[c}g_{d]b} , \tag{II.18}
\]

which means that we have (note that one index is raised)

\[
\tilde{C}_{abcd} = C_{abcd} \tag{II.19}
\]

under (II.1). Using this property (II.19) and the rules (II.1)-(II.3) one can easily conclude that the Weyl Lagrangian [28]

\[
\tilde{L}_w = -\alpha \sqrt{-\tilde{g}} \tilde{C}_{abcd} \tilde{C}^{abcd} = -\alpha \sqrt{-g} C_{abcd} C^{abcd} = L_w \tag{II.20}
\]

is an invariant of the conformal transformation (II.1).

In further considerations it would be useful to know the conformal transformation rules for the widely applied curvature invariants which are given by

\[
\tilde{R}^2 = \Omega^{-4} \left[ R^2 + 4(D-1)^2\Omega^{-2} (\Box \Omega)^2 + (D-1)^2(D-4)^2\Omega^{-4} g^{ab}\Omega_{,a}\Omega_{,b} g^{cd}\Omega_{,c}\Omega_{,d} \right. \\
- 4(D-1)R\Omega^{-1}\Box \Omega - 2R(D-1)(D-4)\Omega^{-2} g^{ab}\Omega_{,a}\Omega_{,b} \\
+ \left. 4(D-1)^2(D-4)\Omega^{-3}\Box \Omega g^{ab}\Omega_{,a}\Omega_{,b} \right] , \tag{II.21}
\]

\[
\tilde{R}_{ab}\tilde{R}^{ab} = \Omega^{-4} \left\{ R_{ab}R^{ab} - 2\Omega^{-1} [(D-2)R_{ab}\Omega^{,ab} + R\Box \Omega] \right. \\
+ \left. \Omega^{-2} \left[ 4(D-2)R_{ab}\Omega^a\Omega^b - 2(D-3)R\Omega_{,e}\Omega^e + (D-2)^2\Omega_{,ab}\Omega^{ab} + (3D-4) (\Box \Omega)^2 \right] \right. \\
- \left. \left[ (D-2)^2\Omega_{,ab}\Omega^a\Omega^b - (D^2 - 5D + 5)\Box \Omega\Omega_{,e}\Omega^e \right] \right. \\
+ \left. \Omega^{-4}(D-1)(D^2 - 5D + 8) (\Omega_{,a}\Omega^{a})^2 \right\} , \tag{II.22}
\]

\[
\tilde{R}_{abcd}\tilde{R}^{abcd} = \Omega^{-4} \left\{ R_{abcd}R^{abcd} - 8\Omega^{-1}R_{bc}\Omega^{bc} + 4\Omega^{-2} [(\Box \Omega)^2 + (D-2)\Omega_{,bc}\Omega^{bc} - R\Omega_{,b}\Omega^b} \\
+ \left. 4R_{bc}\Omega^b\Omega^c \right] + 8\Omega^{-3} \left[ (D-3)\Box \Omega\Omega_{,e}\Omega^e - 2(D-2)\Omega_{,bc}\Omega^{bc}\Omega^e \right] \right. \\
+ \left. 2\Omega^{-4}D(D-1) (\Omega_{,a}\Omega^{a})^2 \right\} . \tag{II.23}
\]

In fact out of these curvature invariants one forms the well-known Gauss-Bonnet term which is one of the Euler (or Lovelock) densities [29, 30]. Its conformal transformation (II.1)
reads as

\[ R_{GB} \equiv \tilde{R}_{abcd} \tilde{R}^{abcd} - 4 \tilde{R}_{ab} \tilde{R}^{ab} + \tilde{R}^2 = \Omega^{-4} \left\{ R_{GB} + 4(D - 3)\Omega^{-1} \left[ 2R_{ab}\Omega^{ab} - R\square\Omega \right] + 2(D - 3)\Omega^{-2} \left[ 2(D - 2) \left( (\square\Omega)^2 - \Omega_{,ab}\Omega^{,ab} \right) - 8R_{ab}\Omega^{a}\Omega^{b} - (D - 6)R\Omega_{a}\Omega^{a} \right] + 4(D - 2)(D - 3)\Omega^{-3} \left[ (D - 5)\square\Omega,_{a}\Omega^{a} + 4\Omega_{,ab}\Omega^{a}\Omega^{b} \right] + (D - 1)(D - 2)(D - 3)(D - 8)\Omega^{-4} (\Omega_{a}\Omega^{a})^2 \right\} . \quad (II.24) \]

The inverse transformation is given by

\[ R_{GB} \equiv R_{abcd} R^{abcd} - 4 R_{ab} R^{ab} + R^2 = \Omega^4 \left\{ \hat{R}_{GB} - 4(D - 3)\Omega^{-1} \left[ 2\hat{R}_{ab}\Omega^{ab} - \hat{R} \hat{\square} \Omega \right] + 2(D - 2)(D - 3)\Omega^{-2} \left[ 2 \left( \hat{\square} \hat{\Omega} \right)^2 - 2\hat{\Omega}_{,ab}\hat{\Omega}^{,ab} - \hat{R}\hat{\Omega}_{,a}\hat{\Omega}^{a} \right] \right\} . \quad (II.25) \]

So far we have considered only geometrical part. For the matter part we usually consider matter lagrangian of the form

\[ \tilde{S}_m = \int \sqrt{-\tilde{g}} \Omega^{-D} d^D x \mathcal{L}_m = \int \sqrt{-g} d^D x \mathcal{L}_m = S , \quad (II.26) \]

where the conformal transformation (II.1) have been used [24]. Then, the energy-momentum tensor of matter in one conformal frame reads as

\[ \tilde{T}^{ab} = \frac{2}{\sqrt{-\tilde{g}}} \frac{\delta}{\delta \tilde{g}_{ab}} \left( \sqrt{-\tilde{g}} \Omega^{-D} \mathcal{L}_m \right) = \Omega^{-D} \frac{2}{\sqrt{-g}} \frac{\partial g_{cd}}{\partial \tilde{g}_{ab}} \frac{\delta}{\delta g_{cd}} \left( \sqrt{-g} \mathcal{L}_m \right) , \quad (II.27) \]

which under (II.1) transforms as

\[ \tilde{T}^{ab} = \Omega^{-D-2}T^{ab} , \quad \tilde{T}^a_b = \Omega^{-D}T^a_b \]

\[ \tilde{T} = \Omega^{-D}T . \quad (II.28) \]

For the matter in the form of the perfect fluid with the four-velocity \( v^a \) \( (v_a v^b = -1) \), the energy density \( \varrho \) and the pressure \( p \)

\[ T^{ab} = (\varrho + p) v^a v^b + p g^{ab} , \quad (II.30) \]

the conformal transformation gives

\[ \tilde{T}^{ab} = (\tilde{\varrho} + \tilde{p}) \tilde{v}^a \tilde{v}^b + \tilde{p} \tilde{g}^{ab} , \quad (II.31) \]
where
\[ T^{ab} = \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g_{ab}} \left( \sqrt{-g} L_m \right) \], \hspace{1cm} (II.32)

and
\[ \tilde{\nu}^a = \frac{dx^a}{ds} = \frac{1}{\Omega} \frac{dx^a}{ds} = \Omega^{-1} \nu^a \]. \hspace{1cm} (II.33)

Therefore, the relation between the pressure and the energy density in the conformally related frames reads as
\[ \tilde{\varrho} = \Omega^{-D} \varrho \], \hspace{1cm} (II.34)
\[ \tilde{p} = \Omega^{-D} p \]. \hspace{1cm} (II.35)

It is easy to note that the imposition of the conservation law in the first frame
\[ T^{ab}_{;b} = 0 \], \hspace{1cm} (II.36)
gives in the conformally related frame
\[ \tilde{T}^{ab}_{;\tilde{b}} = -\frac{\Omega^a}{\Omega} \tilde{T} \]. \hspace{1cm} (II.37)

From (II.37) it appears obvious that the conformally transformed energy-momentum tensor is conserved only, if the trace of it vanishes (\( \tilde{T} = 0 \)) [6, 16, 19, 22]. For example, in the case of barotropic fluid with
\[ p = (\gamma - 1) \varrho \quad \gamma = \text{const.} \], \hspace{1cm} (II.38)
it is conserved only for the radiation-type fluid \( p = [1/(D - 1)] \varrho \).

Similar considerations are also true if we first impose the conservation law in the second frame
\[ \tilde{T}^{ab}_{;\tilde{b}} = 0 \], \hspace{1cm} (II.39)
which gives in the conformally related frame (no tildes)
\[ T^{ab}_{;b} = \frac{\Omega^a}{\Omega} T \]. \hspace{1cm} (II.40)

Finally, it follows from (II.29) that that vanishing of the trace of the energy-momentum tensor in one frame necessarily requires its vanishing in the second frame, i.e., if \( T = 0 \) in one frame, then \( \tilde{T} = 0 \) in the second frame and vice versa. This means only the traceless type of matter fulfills the requirement of energy conservation.
III. EINSTEIN ENERGY-MOMENTUM COMPLEX IN CONFORMAL FRAMES

Following our earlier work [12, 13] we consider the canonical double-index energy-momentum complex \[1–3\]
\[ E^k_i = \sqrt{-g} (T^k_i + E^k_i), \] (III.1)
where \( T^k_i \) is a symmetric energy-momentum tensor of matter which appears on the right-hand side of the Einstein equations and \( g \) is the determinant of the metric tensor. In fact, its definition results from the rewritten Einstein equations [3]
\[ \sqrt{-g} (T^k_i + E^k_i) = F_{U^k_i,l}, \] (III.2)
where
\[ F_{U^k_i,l} = -F_{U^k_i,l} = \alpha g^{ia} \sqrt{-g} \left[ (g_{ka} g_{lb} - g_{la} g_{kb}) \right]. \] (III.3)
are Freud’s superpotentials and
\[ E^k_i = \alpha \left\{ \delta^{k}_{i} g^{ms} (\Gamma_{mr}^{l} \Gamma_{sl}^{r} - \Gamma_{ms}^{r} \Gamma_{rl}^{l}) + g^{ms,i} \left[ \frac{1}{2} (\Gamma_{tp}^{l} g^{tp} - \Gamma_{it}^{l} g^{kt}) g_{ms} - \frac{1}{2} (\delta^{k}_{l} \Gamma_{ml}^{i} + \delta^{k}_{m} \Gamma_{ml}^{l}) \right] \right\} \] (III.4)
is the Einstein’s gravitational energy-momentum pseudotensor (\( \alpha = c^4/16\pi G \)). It is useful to apply the property of metricity
\[ g^{ms,i} = g^{ms,i} - \Gamma_{ia}^{m} g^{as} - \Gamma_{ia}^{s} g^{ma} = 0, \] (III.5)
to (III.4) and reduce it to a simpler form, i.e.,
\[ E^k_i = \alpha \left\{ \delta^{k}_{i} g^{ms} (\Gamma_{mr}^{l} \Gamma_{sl}^{r} - \Gamma_{ms}^{r} \Gamma_{rl}^{l}) + \Gamma_{ia}^{m} (\Gamma_{tp}^{l} g^{tp} - \Gamma_{it}^{l} g^{kt}) + \Gamma_{ia}^{l} (\Gamma_{ms}^{m} g^{as} + \Gamma_{ia}^{s} g^{am}) - \Gamma_{ms}^{l} (\Gamma_{ia}^{m} g^{as} + \Gamma_{ia}^{s} g^{am}) \right\}. \] (III.6)

Now, we apply the conformal transformation (II.1) to the Einstein pseudotensor (III.6). We assume that we start with the Einstein pseudotensor in the conformal frame (with tildes) and express it in the conformal frame (no tildes) as below
\[ E^k_i = \Omega^{-2} \left\{ t^k_i + \Omega^{-1} (D - 2) \left[ \delta^k_i (\Gamma_{tl}^{i} \Omega^t - g^{ms} \Gamma_{ms} \Omega^{r}) + \Omega_{is} (g^{tp} \Gamma_{tl}^{k} - g^{kt} \Gamma_{tl}^{i}) \right] + (g^{ak} \Gamma_{ia}^{m} \Omega_{m} + \Gamma_{ia}^{k} \Omega^{a}) - 2 \Gamma_{ia}^{k} \Omega^{k} \right) + \Omega^{-2} (D - 1) (D - 2) \left[ \delta^k_i (\Omega, \Omega^t - 2 \Omega_{i} \Omega^{k}) \right] \right\}. \] (III.7)
Note that in $D = 2$ the rule of transformation is very simple: $E_{\tilde{t}_i}^{\tilde{k}} = \Omega^{-2}t_i^k$, which seem to reflect the fact of the conformal flatness of all the two-dimensional manifolds. On the other hand, it is not so simple in the case of a flat space where all the Christoffel connection coefficients vanish. In a general case it is clear that the transformed Einstein pseudotensor would differ from the starting one. This seems to be natural conclusion in the context of its sensitivity to a change of coordinates (this is why we call it "pseudotensor") despite we do not change those coordinates here at all (cf. Section II).

From (II.2) and (III.7) it follows that

\[ \sqrt{-\tilde{g}_{\tilde{E}\tilde{t}_i}^{\tilde{k}}} = \Omega^D \sqrt{-g_{E_t}^k} \]

\[ = \Omega^{D-2} \sqrt{-g} \left\{ t_i^k + \Omega^{-1}(D-2)[\delta_i^k (\Gamma_r^l \Omega^r - g^{ms} \Gamma_{ms}^r \Omega_r) + \Omega_\lambda (g^{lp} \Gamma^k_{tp} - g^{kt} \Gamma^l_{tl}) + (g^{ak} \Gamma^m_{ia} \Omega_m + \Gamma^k_{ia} \Omega^a) - 2 \Gamma^a_{ia} \Omega^k] + \Omega^{-2}(D-1)(D-2)[\delta_i^k \Omega_r \Omega^r - 2 \Omega_\lambda \Omega^k] \right\} \]

under conformal transformation (II.1). On the other hand, the transformation rule of the material part of the Einstein complex (III.2) reads as (cf. (II.2) and (II.28))

\[ \sqrt{-\tilde{g}_{\tilde{T}_i}^{\tilde{k}}} = \sqrt{-g_{T_i}^k} , \]

i.e., the quantity $\sqrt{-g T_i^k}$ is an invariant of the conformal transformation (II.1). Besides, if the initial metric is Minkowskian, then the transformational rule (III.7) simplifies since the terms which contain $\Gamma_i^{kl}$ vanish.

IV. BERGMANN-THOMSON ANGULAR MOMENTUM COMPLEX IN CONFORMAL FRAMES

The Bergmann-Thomson angular momentum complex

\[ B_T M^{ijk} = -B_T M^{jki} = -B_T M^{jik} \]

consists of the sum of the material part

\[ m M^{ijk} = \sqrt{-g}(x^j T^{jk} - x^j T^{ik}), \]

and the gravitational part

\[ g M^{ijk} = \sqrt{-g}(x^i B_T t^{jk} - x^j B_T t^{ik}) + \frac{\alpha}{\sqrt{-g}} \left[ (-(g^{kl} g^{ij} - g^{ki} g^{jl})) \right]_i. \]
where
\[ \sqrt{-g} \mathcal{E}_k^{jk} := \sqrt{-g} g^{ji} E^i_k + g^{ij} F U_i^{[kl]} \]  (IV.13)
is the Bergmann-Thomson energy-momentum pseudotensor of the gravitational field [5] (see also [31]).

As a consequence of the local energy-momentum conservation law
\[ E_{k}^{K}_{ki} = 0, \]  (IV.14)
which immediately follows from (III.2), the angular momentum complex satisfies local conservation laws
\[ \mathcal{B}_T^{ijk}_{,k} = 0. \]  (IV.15)

Under the conformal transformation (II.1), the material part \( m M^{ijk} \) and the gravitational part \( g M^{ijk} \) of the complex transform as
\[ m \tilde{M}^{ijk} = \Omega^{-2} m M^{ijk}, \]  (IV.16)
\[ g \tilde{M}^{ijk} = g M^{ijk} + \Omega^{-1} \left\{ 2\alpha \sqrt{-g} (x^i g^{jil} - x^j g^{ili}) P_i^k + \Omega \left[ 4\alpha (x^j U_i^{jk} - x^i U_j^{ij}) g_{ia} U^{[kl]}_{ab} - 2 (x^i F U_j^{[kb]} + x^j F U_i^{[kb]}) \right] \right\} + \alpha \Omega^{-2} \left\{ 6 \sqrt{-g} (x^i g^{jil}) - x^i g^{jil} Q_{,k}^k + 8 \Omega_{,a} \Omega_{,i} [(-g) (x^j g^{kij} - x^i g^{kji}) g^{tb} - (x^i g^{tij} - x^j g^{tij}) g^{kb}] \right\}, \]  (IV.17)
where
\[ P_i^k := \delta_i^k (\Gamma^l_{rl} g^{dq} \Omega_{,d} - g^{ms} \Gamma_{ms}^l \Omega_{,l}) + \Omega_{,i} (\Gamma^k_{tp} g^{ip} - \Gamma^l_{tl} g^{kt}) - 2 \Gamma^m_{im} g^{kp} \Omega_{,p} + \Gamma^k_{ia} g^{a} \Omega_{,s} + \Gamma^m_{ia} g^{ak} \Omega_{,m}, \]  (IV.18)
and
\[ Q_i^k := \delta_i^k g^{ms} \Omega_{,m} \Omega_{,s} - 2 g^{kp} \Omega_{,p}, \]
\[ U^{[kl]}_{ab} := \frac{1}{\sqrt{-g}} \left[ (-g) (g^{ka} g^{kb} - g^{ta} g^{tb}) \right], \]
\[ F U_i^{[km]} = \frac{\alpha}{\sqrt{-g}} g_{ia} ((-g) (g^{ka} g^{mb} - g^{ma} g^{kb}))_b, \]
\[ F U_i^{[kb]} := g^t_{F i} U_t^{[kb]}. \]  (IV.19)
As we can see from above formulas the conformal transformation rule (IV.16) for the matter part of the Bergmann-Thomson angular momentum complex is fairly simple. Let us also mention that the transformation (IV.16) holds in any dimension of spacetime, while the transformation (IV.17) holds only in \( D = 4 \) dimensions. Besides, if the initial metric is Minkowskian, then the transformational rule (IV.17) further simplifies since the terms which contain \( \Gamma^i_{kl}, g_{ik,l} \) and \( g^{ik}_{,l} \) vanish. In this case also \( g M^{jk}_{ij} = 0 \).

V. SUPERENERGY TENSORS IN CONFORMAL FRAMES

Following [8] one is able to introduce the canonical superenergy tensor. The definition of the superenergy tensor \( S^b_a(P) \), which can be applied to an arbitrary gravitational as well as matter field is [12]

\[
S^b_a(P) = S^b_a(P) := \lim_{\Omega \to P} \frac{\int_{\Omega} \left[ T^{(b)}_{(a)}(y) - T_{(a)}^{(b)}(P) \right] d\Omega}{1/2 \int_{\Omega} \sigma(P;y) d\Omega},
\]

where

\[
T^{(b)}_{(a)}(y) := T^k_i(y) e^{(a)}_i(y) e^{(b)}_k(y),
\]

\[
T_{(a)}^{(b)}(P) := T^k_i(P) e^{(a)}_i(P) e^{(b)}_k(P) = T^{ab}(P)
\]

are the tetrad components of a tensor or a pseudotensor field \( T^k_i(y) \) which describe an energy-momentum, \( y \) is the collection of normal coordinates \( \text{NC}(P) \) at a given point \( P \), \( \sigma(P,y) \) is the world-function, \( e^{(a)}_i(y) \), \( e^{(b)}_k(y) \) denote an orthonormal tetrad field and its dual, respectively, \( e^{(a)}_i(P) = \delta^i_a \), \( e^{(a)}_k(P) = \delta^k_a \), \( e^i_{(a)}(y) e^{(b)}_k(y) = \delta^b_a \), and they are parallel propagated along geodesics through \( P \). At \( P \) the tetrad and normal components of an object are equal. We apply this and omit tetrad brackets for indices of any quantity attached to the point \( P \); for example, we write \( T^{ab}(P) \) instead of \( T^{(a)(b)}(P) \) and so on.

Firstly, the symmetric superenergy tensor of matter is given by (from now on we will also interchangeably use \( \nabla \) to mark the covariant differentiation) [12]

\[
mS^b_a(P) = \delta^m_l \nabla_l \nabla_m T^b_a.
\]

In terms of the four-velocity of an observer taken as comoving \( v^l = \delta^l_0 \), \( v^l v_l = -1 \), a more convenient covariant form of (V.3) is

\[
mS^b_a(P; v^l) = h^{lm} \nabla_l \nabla_m T^b_a,
\]

\[\text{(V.4)}\]
where
\[ h^{lm} \equiv 2v^l v^m + g^{lm} = h^{ml}. \] (V.5)

Secondly, the canonical superenergy tensor of the gravitational field reads as
\[ g^b_a(P; v^l) = h^{lm} W^b_a l_m, \] (V.6)

where
\[
W^b_a l_m = \frac{2\alpha}{9} [B^b_{alm} + P^b_{alm} \\
- \frac{1}{2} \delta^b a R^{ijk}_m (R_{ijkl} + R_{ikjl}) + 2 \delta^b a R_{(l|g} R^{a|}_m] \\
- 3 R_a(l|R^b_{ikm}) + 2 R^b_{ag(l|R_g^a|m)}].
\] (V.7)

and
\[
B^b_{alm} := 2 R^b_{ik} (l|R_{aik|m}) - \frac{1}{2} \delta^b a R^{ijk}_l R_{ikjm}.
\] (V.8)

In vacuum, the gravitational superenergy tensor (V.6) reduces to a simpler form:
\[ g^b_a(P; v^l) = \frac{8\alpha}{9} h^{lm} [C^{ljk}(l|R_{aik|m}) - \frac{1}{2} \delta^b a C^{ljk}(l|R_{ikp|m})] . \] (V.10)

It is symmetric and the quadratic form \( g^b_a(P; v^l)v^a v^b \) is positive-definite.

It is suggested [8, 12] that the superenergy tensor \( g^b_a(P; v^l) \) should be taken as a quantity which may serve as the energy-momentum tensor for the gravitational field. Its advantage is that it is a conserved quantity in vacuum. The disadvantage is that the superenergy tensors \( g^b_a(P; v^l) \) and \( m^b_a(P; v^l) \) have the dimension: [the dimension of the components of an energy-momentum tensor (or pseudotensor)] \times m^{-2}. This means it is rather that their flux gives the appropriate energy-momentum tensors or pseudotensors. However, in some other approach one is able to introduce average relative energy and angular momentum tensors which have proper dimension [32]. In fact, these new tensors differ from the superenergy and angular supermomentum tensors by a constant dimensional factor of \((\text{length})^2\).

Now, we consider the conformal transformations of the superenergy tensors. For the matter superenergy tensor one has
\[
m^b_i = \tilde{h}^{lm} \tilde{\nabla}_l \tilde{\nabla}_m \tilde{\nabla}_i. \] (V.11)
\[\begin{align*}
&= \Omega^{-6} \mathcal{S}_a^b + \Omega^{-2} h^{lm}_{\alpha\beta} \left[ 2 \left( \Omega^{-4}\right)_{l,m} T_i^k (\Omega^{-4})_{,m} - (\Omega^{-4})_{,m} T_i^k \right] \\
&+ \Omega^{-3} h^{lm}_{\alpha\beta} \left[ 2 \nabla^k_i (\Omega^{-4})_{l,m} - \nabla^p_{lm} T_i^k (\Omega^{-4})_{,p} - \nabla^p_{lt} T_i^k (\Omega^{-4})_{,t} \right] \\
&+ \Omega^{-9} h^{lm}_{\alpha\beta} \left[ (P^k_{lm} T_i^t)_{,l} - (P^t_{ml} T_i^k)_{,t} \right] + \Omega^{-7} h^{lm}_{\alpha\beta} \left[ F^k_{tp}(\nabla^p_{lm} T_i^t - \nabla^t_{lm} T_p^k) \right] \\
&- \Gamma^p_{lm}(\nabla^k_{ip} T_i^t - \nabla^t_{ip} T_i^k) - \Gamma^p_{lt}(\nabla^k_{mp} T_p^t - \nabla^t_{mp} T_p^k) - \nabla^p_{lm}(\nabla^k_{tp} T_i^t - \nabla^t_{tp} T_i^k) - \nabla^p_{lt}(\nabla^k_{mp} T_p^t - \nabla^t_{mp} T_p^k) \right] ,
\end{align*}\]

where

\[P^a_{bc} = P^a_{cb} \equiv \Omega^{-1} D^a_{bc} ,\]  

and

\[D^a_{bc} \equiv \delta^a_b \Omega_{c} - \delta^a_c \Omega_{b} - g_{bc} g^{ad} \Omega_{d} .\] 

For the gravitational superenergy tensor one has

\[g^{\tilde{S}^a_b} = \Omega^{-4} g^{\tilde{S}^a_b} + \frac{2\alpha}{9} h^{lm}_{\alpha\beta} \left[ \left( g^{i[b} \Omega^{j]}_l \right) - \delta^{[b}_l \Omega^{j]}_l \right] + \left( g^{i[b} \Omega^{k]}_l - \delta^{[b}_l \Omega^{k]}_l \right) \left( g_{[a]} \Omega_{[m]} \right) \]

\[+ \frac{\delta^{[b}_l \Omega^{j]}_l - \delta^{[i}_l \Omega^{j]}_l}{2} (g_{[a]} \Omega_{[m]} + g_{[m]} \Omega_{[a]}) \left( g_{[m]} \Omega_{[a]} \right) \]

\[+ \frac{\alpha}{9} \Omega^{-1} h^{lm}_{\alpha\beta} \left( \Gamma_{\alpha\beta} \Omega_{\gamma} \right)_{,mc} g^{\beta \gamma} + 6 \left( \Omega^{-1} \right)_{,m} \left( \Omega^{-1} \right)_{,b} \]

\[+ \left( g^{i[b} \Omega^{j]}_l - \delta^{[b}_l \Omega^{j]}_l \right) R_{a(lk)m} - \frac{\delta^a_l}{2} (g^{i[b} \Omega^{j]}_l - \delta^{[i}_l \Omega^{j]}_l) R_{a(kj)m} \]

\[+ \frac{1}{8} \left( R_{m}^{b} g^{ab} + \delta_{a}^{b} R_{m}^{g} - 2 \delta_{a}^{i} R_{m}^{b} \right) R_{a(lk)m} \]

\[+ \frac{4\alpha}{3} \Omega^{-3} h^{lm}_{\alpha\beta} \left( \Gamma_{\alpha\beta} \Omega_{\gamma} \right)_{,cm} R_{c(lm)} + \left( \Omega^{-1} \right)_{,m} \left( R_{a(lm)} - \left( \Omega^{-1} \right)_{,m} R_{b(a)} \right) \]

\[+ \frac{1}{3} \left( \Omega^{-1} \right)_{,mc} \left( R_{a(lm)} - \left( R_{a(lm)} + R_{a(m)} \right) \right) \]

\[+ \frac{\alpha}{36} \Omega^{-4} (\Omega^2)_{,rs} g^{rs} \left[ h^{ab} \Omega_{a} + h^{ag} (g_{la} \Omega_{a} - \delta^{a} \Omega_{gl}) \right] \]

\[+ g_{al} \left( \Omega^{-1} \right)_{,m} (\Omega^2)_{,rs} g^{rs} - \frac{8\alpha}{9} \Omega^{-5} h^{lm}_{\alpha\beta} \left( \Omega^{-1} \right)_{,m} \left( \Omega^{-1} \right)_{,ml} \]

\[+ g_{al} (\Omega^{-1} \left( \Omega^{-1} \right)_{,m}) (\Omega^2)_{,rs} g^{rs} - \frac{4\alpha}{9} \Omega^{-6} h^{lm}_{\alpha\beta} (\Omega^2)_{,rs} g^{rs} \left( \delta^a_b R_{lm} \right) \]

\[+ \frac{3}{4} \left( R_{al} R_{m}^{b} + g_{al} R_{m}^{b} \right) + \frac{1}{4} R_{ma(t)} \]

\[+ \frac{\alpha}{9} \Omega^{-8} h^{lm}_{\alpha\beta} (\Omega^2)_{,rs} g^{rs} \left( \delta^a_b g_{lm} + \frac{3}{2} g_{lm} \right) ,\]

where

\[\Omega^a_{b} = 4 \Omega^{-1} (\Omega^{-1})_{,bc} g^{ae} - 2 (\Omega^{-1})_{,c} (\Omega^{-1})_{,d} g^{cd} \delta^a_b .\]
Bearing in mind (II.19), we have for a pure gravitational field that
\[ \tilde{C}_{ilm} = \tilde{g}_{it} \tilde{C}_{ilm}^t = \Omega^2 C_{ilm}^t, \quad (V.16) \]
\[ \tilde{C}^{k|lm} = \Omega^{-4} C^{k|lm}. \quad (V.17) \]
and so
\[ \tilde{B}_{iab}^k = \tilde{C}_{a}^{k|lm} \tilde{C}_{ilm}^t + \tilde{C}_{b}^{k|lm} \tilde{C}_{ilm}^t - \frac{1}{2} \delta_{i}^{k} \tilde{C}_{a}^{d|mn} \tilde{C}_{ilmn}^t \]
\[ = \Omega^{-2} \left( C^{k|lm} \tilde{C}_{ilm}^t + C^{k|lm} \tilde{C}_{ilm}^t - \frac{1}{2} \delta_{i}^{k} C^{d|mn} C_{ilmn}^t \right), \quad (V.18) \]
which means that the four times covariant form of the Bel-Robinson tensor (V.8) for the pure gravitational field is an invariant of the conformal transformation, i.e.,
\[ \tilde{B}_{kiab} = B_{kiab}, \quad (V.19) \]
Similarly, for a pure gravity, from (V.9) and from the formulas given in Section II, one can easily obtain that
\[ g^{\tilde{S}}_{a}(P; \tilde{v}^l) = \Omega^{-D} g_S^{b}(P; v^l), \quad (V.20) \]
from which we have that
\[ \sqrt{-g} g^{\tilde{S}}_{a}(P; \tilde{v}^l) = \sqrt{-g} S^{b}_{a}(P; v^l), \quad (V.21) \]
i.e., the tensorial density \( \sqrt{-g} S^{b}_{a}(P; v^l) \) is an invariant of the conformal transformation (II.1). For a conformally flat spacetime and for a pure gravitational field we have that
\[ g^{\tilde{S}}_{a} = \sqrt{-g} g^{\tilde{S}}_{a} = g S^{b}_{a} = \sqrt{-g} S^{b}_{a} = 0, \quad (V.22) \]
which is, for example, the case of the Friedman universes.

VI. ANGULAR SUPERMOMENTUM TENSORS IN CONFORMAL FRAMES

In this Section we further extend the notion of superenergy onto the angular momentum which has been introduced in Ref. [9].

The canonical angular supermomentum tensors can be defined in analogy to the canonical superenergy tensors as
\[ S^{(a)(b)(c)} (P) = S^{abc} (P) := \lim_{\Omega \rightarrow P} \frac{\int \left[ M^{(a)(b)(c)} (y) - M^{(a)(b)(c)} (P) \right] d\Omega}{1/2 \int \sigma (P; y) d\Omega}, \quad (VI.23) \]
where
\[ M^{(a)(b)(c)}(y) := M^{ijkl}(y)e_i^{(a)}(y)e_j^{(b)}(y)e_k^{(c)}(y), \] (VI.24)
\[ M^{(a)(b)(c)}(P) := M^{ijkl}(P)e_i^{(a)}(P)e_j^{(b)}(P)e_k^{(c)}(P) = M^{ijkl}(P)\delta^a_i \delta^b_j \delta^c_k \] (VI.25)
are the physical (or tetrad) components of the field \( M^{ijkl}(y) = -M^{jikl}(y) \) which describe angular momentum densities. As in (V.1)-(V.2), \( e_i^{(a)}(y), \ e_j^{(b)}(y) \) denote orthonormal bases such that \( e^i_{(a)}(P) = \delta^i_a \) and its dual are parallel propagated along geodesics through \( P \) and \( \Omega \) is a sufficiently small ball with centre at \( P \). As in Section V we apply the fact that at \( P \) the tetrad and normal components of an object are equal and so we again omit tetrad brackets for indices of any quantity attached to the point \( P \).

For matter as \( M^{ijkl}(y) \) we take
\[ m M^{ijkl}(y) = \sqrt{-g}(y^i T^{kl} - y^k T^{il}), \] (VI.26)
where \( T^{ik} \) are the components of a symmetric energy-momentum tensor of matter and \( y^i \) denote the normal coordinates. The formula (VI.26) gives the total angular momentum densities, orbital and spinorial because the dynamical energy-momentum tensor of matter \( T^{ik} \) comes from the canonical energy-momentum tensor by using the Belinfante-Rosenfeld symmetrization procedure and, therefore, includes the spin of matter [5]. Note that the normal coordinates \( y^i \) form the components of the local radius-vector \( \vec{y} \) with respect to the origin \( P \). In consequence, the components of the \( m M^{ijkl}(y) \) form a tensor density.

For the gravitational field we take the gravitational angular momentum pseudotensor of Bergmann and Thomson (IV.12) to construct
\[ g M^{ijkl}(y) = g_U^{ijkl}(y) - F \ U^{[kl]}(y) + \sqrt{|g|}(y^i_{BT} t^{kl} - y^k_{BT} t^{il}), \] (VI.27)
where \( F_U^{ijkl} := g^{im} U_{ni}^{[kl]} \) are von Freud superpotentials (III.3).

In fact, the Bergmann-Thomson pseudotensor can be interpreted as the sum of the spinorial part
\[ S^{ijkl} := g_U^{ijkl} - F \ U^{[kl]} \] (VI.28)
and the orbital part
\[ O^{ijkl} := \sqrt{-g}(y^i_{BT} t^{kl} - y^k_{BT} t^{il}) \] (VI.29)
of the gravitational angular momentum densities.

Substitution of (VI.26) and (VI.27) (expanded up to third order) into (VI.23) gives the canonical angular supermomentum tensors for matter and gravitation, respectively [9],

$$m S^{abc}(P; v^l) = 2 [h^{ap} \nabla_p T^{bc} - h^{bp} \nabla_p T^{ac}], \quad (V I.30)$$

$$ g S^{abc}(P; v^l) = \alpha h^{pl} [(g^{ac} g^{br} - g^{bc} g^{ar}) \nabla (t R_{pr})$$

$$+ 2 g^{ar} \nabla (t R_{p}^{(b)} c_\rho) - 2 g^{br} \nabla (t R_{p}^{(a)} c_\rho)$$

$$+ \frac{2}{3} g^{bc} (\nabla R^{a} R_{t \rho} - \nabla \{p R_{t}^{a}\})$$

$$- \frac{2}{3} g^{ac} (\nabla R^{b} R_{t \rho} - \nabla (p R_{t}^{b}))]. \quad (V I.31)$$

Both these tensors are antisymmetric in the first two indices $S^{abc} = - S^{bac}$. In vacuum, the gravitational canonical angular supermomentum tensor (VI.31) simplifies to

$$g S^{abc}(P; v^l) = 2 \alpha h^{pl} [g^{ar} \nabla (t R_{p}^{(b)} c_\rho) - g^{br} \nabla (t R_{p}^{(a)} c_\rho)]. \quad (V I.32)$$

Note that the orbital part $O^{kl} = \sqrt{-g} (g_{BT}^{kl} - g_{BT}^{k} t^{i})$ gives no contribution to $g S^{abc}(P; v^l)$. Only the spinorial part $S^{kl} = \Omega^{k} U^{[k]} l_{-} - \Omega^{k} U^{l_{-}} [k]$ contributes. Also, the canonical angular supermomentum tensor $g S^{abc}(P; v^l)$ and $m S^{abc}(P; v^l)$ of gravitation and matter do not require the introduction of the notion of a radius vector.

After some algebra, one may show that there are the following transformational rules for the angular supermomentum tensors of matter $m S^{kl}(P; v^a)$ and for pure gravitation (which is composed of the Weyl tensor only) $g S^{kl}(P; v^a)$ under conformal transformation (II.1):

$$m \tilde{S}^{kl}(P; \tilde{v}^a) = \Omega^{-8} m S^{kl}(P; v^a) + 4 \Omega^{-8} [h^{pl} (P^{k}_{pr} T^{rl})$$

$$+ P^{l}_{pr} T^{(k)} (r)] - 2 \Omega^{-9} h^{pl} \Omega T^{(k)} l_{-}; \quad (V I.33)$$

$$g \tilde{S}^{kl}(P; \tilde{v}^a) = \Omega^{-6} g S^{kl}(P; v^a) + 2 \alpha \Omega^{-6} h^{pl} \{g^{ib} [P^{l}_{(t|s} C_{|b|}^{[s]} k] C_{|b|}^{[s]} k)$$

$$+ P^{b}_{(t|s} C_{|b|}^{[s]} k] C_{|b|}^{[s]} k] + P^{b}_{(t|s} C_{|b|}^{[s]} k] C_{|b|}^{[s]} k])$$

$$- \frac{1}{2} [g^{bb} [P^{l}_{(t|s} C_{|b|}^{[s]} i] C_{|b|}^{[s]} i]$$

$$+ P^{b}_{(t|s} C_{|b|}^{[s]} i] C_{|b|}^{[s]} i])$$

$$- 4 \Omega^{-7} h^{pl} [g^{ib} \Omega^{l_{-}} C_{b}^{(k)} - g^{bb} \Omega^{l_{-}} C_{b}^{(k)} - g^{bb} \Omega^{l_{-}} C_{b}^{(i)} - g^{bb} \Omega^{l_{-}} C_{b}^{(i)}] \quad (V I.34)$$
with $h^{lm}$ and $P_{bc}^{a}$ given by (V.5) and (V.12) and $C_{bcd}^{a}$ being the components of the Weyl conformal curvature tensor (II.18). It is obvious that in a conformally flat spacetime, one has

$$g \tilde{S}^{ikl} = g S^{ikl} = 0.$$  \hspace{1cm} (VI.35)

VII. CONCLUSION

In this paper we have analyzed the rules of the conformal transformations of the energetic and superenergetic quantities which were proposed in general relativity and can be applied to some extended theories of gravity in which physics is studied in different conformal frames.

In particular, we have found the rules of the conformal transformation for the energetic quantities such as the Einstein energy-momentum complex, the Bergmann-Thomson angular momentum complex, the superenergy tensor, and the angular supermomentum tensor of gravitation and matter.

We have shown that the conformal transformation rules for the matter parts of both the Einstein complex and the Bergmann-Thomson complex are fairly simple (Eqs. (III.9) and (IV.16)), while the transformation rules for their gravitational parts are more complicated (Eqs. (III.7) and (IV.17)). We have also found that the transformational rules of the superenergy tensor of matter (V.11) and the superenergy tensor of gravity (V.14) are quite complicated, except for the case of a pure gravity (V.20). In such a special case the superenergy density as well as the sum of the matter energy density and the superenergy density are invariants of the conformal transformation, i.e.

$$\sqrt{-\tilde{g}} \left( \tilde{T}_{i}^{k} + g \tilde{S}_{i}^{k} \right) = \sqrt{-g} \left( T_{i}^{k} + g S_{i}^{k} \right).$$

Besides, in that case (of a pure gravity), a conformal invariant is also the Bel-Robinson tensor

$$\tilde{B}_{kiab} = B_{kiab},$$

which is a part of the superenergy tensor. As for the angular supermomentum tensor of gravity - it emerges that its transformational rule (VI.34), even for a pure gravity, is quite complicated. This, however, is not the case for the angular supermomentum tensor of matter (Eq. (VI.33)).
Some other remarks from our investigations are as follows. The conformal transformation rule of the Einstein pseudotensor (III.7) vastly simplifies in \( D = 2 \) dimensional spacetime. This seems to reflect the fact that all the two-dimensional manifolds are conformally flat. On the other hand, both a pure gravity superenergy tensor and a pure gravity angular supermomentum tensor vanish in all conformally flat spacetimes.

Because the superenergetic quantities are constructed of some combinations of the geometric quantities, we have studied the rules of their transformations from one conformal frame to another. In particular, we have derived the rules of the conformal transformation for the curvature invariants \( R^2, R_{ab}R^{ab}, R_{abcd}R^{abcd} \) (Eqs. (II.21)-(II.23)) and the Gauss-Bonnet invariant \( R_{GB} = R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2 \) (Eqs. (II.24)-(II.25)) in an arbitrary spacetime dimension.

All the rules we found would be applied to the discussion of the conformal transformation of energetic and superenergetic quantities in some special models of spacetime [33]. Especially, these rules should be very effective to calculate energetic and superenergetic quantities in conformally flat spacetimes.

Quite recently, the form of the energy-momentum complexes within the framework of extended gravity \( f(R) \) theories have been studied [34]. In fact, it is very common to use conformal frames (Jordan and Einstein) in presentation of these theories [35] so that our results of Sections III and IV can be applied to study the sensitivity of the energy-momentum complexes to conformal transformations in these more general theories of gravity. On the other hand, since the complexes are sensitive to coordinate transformations, then it would be much better to study the superenergy and the angular supermomentum (which are covariant) in \( f(R) \) theories of gravity with the stress onto the problem of their sensitivity to conformal transformations.

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[36] We use the sign convention (+...+), the Riemann tensor convention $R^a_{bcd} = \Gamma^a_{bd,c} - \Gamma^a_{bc,d} + \Gamma^a_{ce} \Gamma^e_{bd} - \Gamma^a_{de} \Gamma^e_{cb}$, and the Ricci tensor is $R_{bd} = R^a_{bad}$. 