Emergence of Non-Abelian Moore-Read state in double-layer bosonic Fractional quantum Hall system

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Identifying and understanding interacting systems that can host non-Abelian topological phases with fractionalized quasiparticles have attracted intense attentions in the past twenty years. Theoretically, it is possible to realize a rich variety of such states by coupling two Abelian fractional quantum Hall (FQH) states together through gapping out part of the low energy degrees of freedom. So far, there are some indications, but no robust example has been established in bilayer systems for realizing the non-Abelian state in the past. Here, we present a phase diagram of a double-layer bosonic FQH system based on the exact diagonalization and density-matrix renormalization group (DMRG) calculations, which demonstrate a potential regime with the emergence of the non-Abelian bosonic Moore-Read state. We start from the Abelian phase with fourfold topological degeneracies on torus geometry when the two layers are weakly coupled. With the increase of interlayer tunneling, we find an intermediate regime with a threefold groundstate degeneracy and a finite fractional drag Hall conductance.

We find the different topological sectors in consistent with Moore-Read state by inserting different fluxes in adiabatic DMRG study. We also extract the modular matrix, which supports the emergence of the non-Abelian Ising anyon quasiparticle in this system.

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I. INTRODUCTION

The topological states of matter with emergent fractionalized quasiparticles have attracted the intense attentions in the past two decades. The statistics of fractionalized quasiparticles fall into two broad categories: Abelian and non-Abelian. Interchanging two Abelian quasiparticles changes the groundstate by a non-trivial phase factor, whereas interchanging two non-Abelian quasiparticles rotates the groundstates within a set of degenerated groundstate manifold and the final state will depend on the order of operations being carried out. The non-Abelian quasiparticles are fundamentally important for our understanding of the emerging physics, which also have potential application for the fault-tolerant topological quantum computation. So far, the most promising platform to investigate the fractional statistics is the fractional quantum Hall (FQH) systems, where most of the observed FQH states carry Abelian quasiparticles. The prominent candidates which may host non-Abelian quasiparticles include the Moore-Read (MR) Pfaffian state at the filling factor $\nu = 5/2$ and the Read-Rezayi (RR) state at $\nu = 12/5$, which are the FQH states for electron systems subject to strong magnetic field.

FQH states are also observed in the double-layer systems, which can be described in terms of the two-component Halperin states. Interestingly, the non-Abelian FQH states may also be realized in the double-layer FQH systems through tuning interlayer tunneling and interactions. In such double-layer systems, the Halperin wavefunction upon symmetrization over the layer index indeed shows the characteristic features predicted for the non-Abelian states, including the counting of edge excitations and the quasihole states. In the theoretical considerations, the non-Abelian state may be induced by increasing the interlayer coupling, which can gap out the low energy degrees of freedom that are antisymmetric about the layer inversion. However, there are no strong evidence that this mechanism has been realized in physical systems. The numerical studies based on exact diagonalization (ED) on the $\nu = 1/2$ FQH state for electron systems in double-layer find a large wavefunction overlap between the ground state and the non-Abelian state, which is consistent with the symmetrization mechanism of the Halperin wavefunction. However, the obtained energy spectrum has redundant low-energy excitations without a robust energy gap or the groundstate degeneracy on torus geometry, which are not consistent with a topological ordered state. Very recently, a total filling number $\nu = 1$ double-layer bosonic system on a square lattice model with topological flat bands has been studied variationally based on the path construction and Gutzwiller projected wavefunction. The non-Abelian MR state has been identified by using the topological spin and chiral central charge. However, it remains an open question whether this non-Abelian state is indeed the ground state of the microscopic Hamiltonian. As the possible MR state in this double-layer system appears weak, the accurate simulations for large systems and the systematic numerical characterization of the topological features are highly desired to pin down the nature of the intermediate phase in these systems.

In this paper, we study a bosonic double-layer system with each layer in the $\nu = 1/2$ Laughlin state in the decoupled limit using ED and density-matrix renormalization group (DMRG) calculations. With tuning the tunneling and interaction, we find that the fourfold degenerate ground states in the decoupled limit split to form three lowest-energy states, which are symmetric with respect to the layer inversion. These low energy states are separated from the higher energy spectrum by a finite gap in an intermediate parameter regime. To identify the nature of the intermediate phase, we design and perform different flux insertion simulations, which can iden-
tify total Hall and drag Hall conductances. We find that the intermediate phase is characterized by the quantized charge Hall conductance and non-zero drag Hall conductance, which is distinguished from the Abelian phase with the quantized charge Hall conductance and zero drag Hall conductance. The flux insertion studies indicate that with growing interlayer tunneling, the system evolves from the two-component Abelian phase to a one-component phase, which is consistent with a non-Abelian state. Furthermore, we calculate the modular $S$-matrix using ED on finite-size clusters and the obtained $S$-matrix fully matches that of the MR state.

The remaining of the paper is organized as following: In Sec. II, we introduce the double-layer lattice model built from the topological flat-band (TFB) model. In Sec. III, we present our phase diagram determined by the energy spectrum, charge Hall conductance (Chern number) and drag Hall conductance. In Sec. IV, we present the details of numerical results for the quantum phase diagram. We show the evolution of the energy spectrum with tunneling $t_{\perp}$ obtained from ED calculations. The robust groundstate degeneracy is confirmed by inserting charge flux and spin flux. We also study the response of the ground state to the charge flux and drag flux in the possible non-Abelian region, based on the newly developed adiabatical adiabatical algorithm. Furthermore, the modular matrix is calculated to support the emergence of non-Abelian Ising anyon in the possible non-Abelian regime. In Sec. V, we discuss the topological trivial phase in our phase diagram. Finally, in Sec. VI, we summarize our main results and discuss open questions.

II. THEORETICAL MODEL

We consider a double-layer system composed from two single layer topological flat band model, which can be generally written as:

$$
\mathcal{H} = \mathcal{H}_t + \mathcal{H}_\perp + \mathcal{H}_t + \mathcal{H}_U,
$$

$$
\mathcal{H}_{\perp(\downarrow)} = -\sum_{(rr')} \left[ J_{rr'} e^{i\phi_{rr'}} b_{r\uparrow(\downarrow)}^\dagger b_{r'\uparrow(\downarrow)} + \text{H.c.} \right],
$$

$$
\mathcal{H}_t = t_{\perp} \sum_{r} \left[ b_{r\downarrow}^\dagger b_{r\uparrow} + \text{H.c.} \right],
$$

$$
\mathcal{H}_U = U_{\perp} \sum_{r} n_{r\downarrow} n_{r\uparrow},
$$

where $\mathcal{H}_{\perp(\downarrow)}$ denotes the hopping terms in the top layer (bottom layer), $\mathcal{H}_t$ describes the interlayer tunneling and $\mathcal{H}_U$ is the interlayer interaction. $b_{r\uparrow(\downarrow)}^\dagger$ (s = $\uparrow$ or $\downarrow$) creates (annihilates) a hard-core boson at site $r$. We consider the TFB model on the square lattice and select the phase factor $\phi_{rr'}$ corresponding to half flux quanta per plaquette. The intralayer hopping terms in $\mathcal{H}_{\perp(\downarrow)}$ include the nearest-neighbor coupling $J_{rr'} = 1.0$ (energy scale), the next-nearest-neighbor coupling $J_{(rr'\downarrow)} = 0.2941$, and the next-next-nearest-neighbor coupling $J_{(rr'\downarrow)} = -0.2061$, which give a TFB in each layer with the flatness ratio around 28. We set the interlayer tunneling $t_{\perp} \geq 0$ and the interlayer coupling $U_{\perp} \geq 0$. It should be noticed that $t_{\perp} < 0$ does not change the results as it can be related with Hamiltonian Eq. (1) through a $\pi$ rotation around the $z$ axis for a single layer. Such a double-layer model can realize the FQH effect for hardcore bosons or interacting fermions without a magnetic field due to the nontrivial Chern number ($C = 1$) carried by the topological band and the reduced kinetic energy.

We consider a finite size system with $2 \times N_x \times N_y$ sites ($N_x \times N_y$ sites for each layer) and the total filling factor of the lower TFB $\nu = N_p/N_s = (N_p + N_p^\perp)/N_s = 1$, where $N_p$ is the total number of hardcore bosons and $N_s$ is the number of single-particle states of the lower TFB. In the absence of tunneling, the system reduces to the decoupled two layers with each layer at the $\nu = 1/2$ filling. While the ED calculations on torus geometry are limited to the system with 40 sites (we only show the results of the 32-site system in paper), DMRG allows us to study the systems up to 256 sites on the cylinder which have the periodic boundary in the $y$ direction and the open boundary in the $x$ direction. We keep up to $M = 3600$ states in DMRG calculations, which give accurate results.

III. PHASE DIAGRAM

Our main results are summarized as the phase diagram in the $t_{\perp} - U_{\perp}$ plane as shown in Fig. 1. We find that the Abelian FQH state of single layer ($t_{\perp} = 0$) is stable against the weak interlayer tunneling. In the double-layer system, this Abelian phase is characterized by the robust fourfold degenerate ground states, the quantized total charge Chern number $C^c = 1$, and the vanishing-small drag Hall conductance. With the increase of interlayer tunneling, the energy of the single antisymmetric ground state splits from the other three symmetrized ground states on finite-size system. The higher

![FIG. 1: (Color online) Quantum phase diagram of the double-layer system for the Hamiltonian Eq. (1) with changing the interlayer tunneling $t_{\perp}$ and interaction $U_{\perp}$. The blue color filled region represents the topological Abelian phase with the fourfold degenerate ground states and the topological Chern number $C^c = 1$. The green colored squares indicate an intermediate topological order phase with threefold degeneracy and $C^c = 1$, which is identified as the one-component non-Abelian Moore-Read state by the finite fractional drag Hall conductance $\sim 0.5$ and the modular $S$-matrix. The blue dots represent a solid phase that breaks lattice symmetries.](image-url)
energy spectrum has a gap from the threefold symmetrized
ground states in an intermediate \( t_\perp \) region (for \( U_\perp = 0, 0.2 \leq t_\perp \lesssim 0.3 \)). In both the Abelian phase and the inte-
mediate region, we find that the charge Chern number is
always quantized as \( C^c = 1 \); however, the drag Hall con-
ductance jumps from the vanishing-small value in the Abelian
phase to the almost saturated value in the intermediate region,
indicating that the system evolves from a two-component to a
one-component FQH state. Interestingly, using the threefold
ground states, we extract the modular \( S \)-matrix, which sup-
ports a non-Abelian MR state with the emergent Ising anyon
quasiparticle and the corresponding fusion rule. In the larger
tunneling regime (\( t_\perp > 0.3 \)), the system becomes a topologi-
cally trivial state with vanishing Chern numbers \( (C^c = 0) \).
In the large \( U_\perp \) region, we find a charge density wave state
that breaks lattice symmetry.

IV. TOPOLOGICAL NONTRIVIAL QUANTUM HALL
PHASES

A. Energy spectrum on torus

We first use ED to study the evolution of energy spectrum
with tunneling \( t_\perp \) at \( U_\perp = 0 \) on the \( 2 \times 4 \times 4 \) torus system. At
\( t_\perp = 0 \), both layers are the bosonic \( \nu = 1/2 \) Laughlin state
with twofold degenerate ground states. Thus, the energy spec-
trum of the whole system has a fourfold degeneracy separated
from higher energy levels by a robust gap. As the double layer
system has a layer inversion symmetry, the fourfold ground
states can be classified into two groups: the symmetric group
with three states \( E_{S1}, E_{S2}, E_{S3} \) shown as open circles
and the antisymmetric group with a single state \( E_{AS} \)
represented by blue stars as shown in Fig. 2. By increasing the in-
terlayer tunneling, the groundstate degeneracy is lifted gradu-
ally. The energy of the antisymmetric state grows rapidly and
merges into the high energy continuum at \( t_\perp \approx 0.25 \). With
further increasing \( t_\perp \), two of the three symmetric states also
merge into high energy continuum at \( t_\perp \approx 0.36 \). In the inter-
mediate regime \( 0.20 \leq t_\perp \lesssim 0.36 \), the three symmetric states
which have close energy are separated from the antisymmet-
ric state with a higher energy by a finite gap. Although there
is a finite splitting between the lowest three states due to the
finite size effect, the low-energy spectrum of the intermediate
region implies a possible fourfold degenerate ground states
protected by a gap in the thermodynamic limit. In the pres-
ence of interlayer interaction, we observe the similar results
as shown in Fig. 2(b) for \( U_\perp = 0.5 \). Thus, the phase region
with \( t_\perp \approx 0.25 \), which has the maximum finite-size gap, may
be the proper phase regime for observing the possible non-
Abelian Moore-Read state.

B. Flux insertion based on ED

We check whether the threefold degeneracy is robust by
considering the response of the low-energy spectrum to the
flux insertion. To induce the flux, we impose a twisted boundary
condition in the \( \hat{y} \)-direction: \( (r + N_\hat{y} \tilde{y}) | \Psi_{\theta_y} \rangle =
| \langle e^{i \theta_y \sigma_y} | \Psi_{\theta_y} \rangle \), where \( \Psi_{\theta_y} \) is the many-body state with
boundary phase \( \theta_y \) and Pauli matrix \( \sigma_y \) acts on the spin
degrees of freedom. The twisted boundary is equivalent to
threading a flux in the hole of a torus along the \( \hat{x} \)-direction. In
the double-layer system, we introduce two kinds of bound-
ary conditions: charge flux \( (\sigma_0) \) and spin flux \( (\sigma_y) \).
In the charge and spin flux, the boundary phases in the top and
bottom layers have the same \( (\theta_1 = \theta_0) \) and the opposite
(\( \theta_1 = -\theta_0 \)) signs, respectively. For the topological states,
the degenerate ground states should remain gapped and no level
crossing with the higher energy levels in the charge flux inser-
tion. Therefore, the charge flux insertion can be used to iden-
tify the near degenerate ground states from the low-energy
levels. Furthermore, it is expected that a two-component
double-layer system will have the similar responses to the
charge and spin fluxes, while a coupled one-component sys-
tem has the different responses.

Fig. 3 shows the ED results of the evolution of low-energy

![Fig. 2](image-url) (Color online) Energy spectrum evolution with the interlayer
tunneling \( t_\perp \) on \( 2 \times 4 \times 4 \) torus geometry using ED calculation for
(a) \( U_\perp = 0.0 \) and (b) \( U_\perp = 0.5 \). Among the four near degenerating
ground states, the three symmetric (S1,S2,S3) states are labeled by
the black, green, and red circles, while the anti-symmetric (AS) state
is labeled by the blue stars.

![Fig. 3](image-url) (Color online) Energy spectrum evolution with (a-b) inserting
a charge flux \( \theta_1^y = \theta_0^y \) and (c-d) a spin flux \( \theta_1^y = -\theta_0^y \) for
\( (t_\perp, U_\perp) = (0.1, 0.5) \) and \( (t_\perp, U_\perp) = (0.25, 0.5) \) on the \( 2 \times 4 \times 4 \)
torus. The lowest five energy levels are shown.
Theoretically, the drag Hall conductance and its connection to the topological Chern number matrix have been established before. Conventionally, one obtains such topological Chern invariants based on ED calculations. Very recently, the flux insertion has been introduced in the large-scale DMRG simulation on cylinder systems, which can be used to detect different Hall conductances.

Here, we find that by applying both the charge and drag fluxes in DMRG calculations, we can access all the topological sectors in the double-layer system. This argument can be easily understood in the decoupled limit. Starting from the ground state without any flux, we insert the charge flux by adiabatically increasing the twist boundary phase in the closed boundary along the y axis. By inserting $2\pi$ flux, the ground state of each layer evolves into a new topological sector with a fractional 1/2 charged quasiparticle being pumped from one edge of cylinder to the other one. Then, by adding the drag flux in either layer, the system would evolve to the other two sectors, which has one more pumped charge 1/2 quasiparticle in the layer with drag flux. Therefore, we obtain the four topological degenerate ground states in the Abelian phase. Qualitatively, this picture in the decoupled limit applies to the whole Abelian phase as long as the drag Hall conductance is vanishing small, i.e., the flux in one layer cannot effectively drag the Hall conductance of the other layer. When the system has a transition from two components to one component, the drag Hall conductance would jump to a finite value. In the coupled one-component system, the drag flux applied to either one of the two layers will evolve the system to the same topological sector with one fractional charged quasiparticle pumped, thus we only obtain three topological sectors.

As shown in Fig. 4(a), after the insertion of one charge flux quantum ($\theta_y^\uparrow = \theta_y^\downarrow = 0 \rightarrow 2\pi$), a net quantized charge (boson number) accumulates at the left edge. The charge accumulation is equivalent to a net charge transfer from the right edge to the left edge. According to the fundamental correspondence between edge transfer and bulk Chern number, we find a quantized charge Chern number $C^\uparrow = 1$ for this Abelian FQH state.
The quantized charge transfer also persists in the possible non-Abelian region. As shown in Fig. 4(b), the net charge transfer corresponds to a quantized charge Chern number \( C^c = 1 \), which indicates the topological nontrivial nature of the possible non-Abelian phase. Nevertheless, the charge flux cannot distinguish the Abelian phase from the possible non-Abelian phase since both of them share the same \( C^c = 1 \). To further identify the phase transition between the Abelian phase and the possible non-Abelian phase, we consider the effects of the drag flux.

By threading a drag flux quantum in the top layer (\( \theta_y^+ = 0 \rightarrow 2\pi, \theta_y^0 = 0 \)), we observe the boson accumulations in the bottom layer\(^{36,49}\) as shown in Fig. 4(c). By calculating the drag Hall conductance as a function of \( t_\perp \) as displayed in Fig. 4(d), we find a strong enhancement of drag Hall conductance at \( t_\perp \approx 0.15 \), which coincides with the phase boundary for the disappearance of the Abelian phase identified from ED energy spectrum. Within the regime \( 0.25 \lesssim t_\perp \lesssim 0.30 \), the drag Hall conductance of each layer approaches the saturated value 0.25 (therefore a total half-charge 0.5 is pumped), which is consistent with an effective one-component system. Based on the above results, we determine the Abelian phase (blue color), the one-component non-Abelian phase (green squares) as shown in Fig. 1. The trivial topological feature of the solid phase is also revealed by the zero charge Chern number.

To further identify the possible non-Abelian state, we investigate the entanglement spectra of each topological sector. Physically, the different topological ground states on a cylinder are expected to have the different well-defined anyonic flux through the cylinder. Thus, the cylinder system with the charge flux or the drag flux corresponds to the other two topologically distinct ground states besides the vacuum state. To explicitly demonstrate the ground states with different anyonic flux on cylinder geometry, we bipartite the cylinder into two halves, and observe entanglement spectrum\(^{53}\) to distinguish the different topological sectors. As shown in Fig. 5, we show the entanglement spectrum for the vacuum ground state in Fig. 5(a), the new ground state obtained by inserting a charge flux quantum in Fig. 5(b), and the ground state obtained by inserting a drag flux in Fig. 5(c). These three ground states are anticipated to have one-to-one correspondence with identity, fermion, and Ising anyon sectors, respectively. We also calculate the momentum dependence of the entanglement spectra in each quantum number sector with different relative boson number \( \Delta N \), and obtain the counting of the leading eigenvalues in the entanglement spectra\(^{54}\). The obtained results are similar to those of coupled two Laughlin \( \nu = 1/2 \) states. Due to the calculation limit, \( N_y = 8 \) (16 lattice sites in the \( \hat{y} \) direction) is the largest width we can reach convergence in our DMRG calculations, which gives four momentum quantum numbers \( K = 0, \pi/2, \pi, 3\pi/2 \) in each \( \Delta N \) sector. Although we observe that a very small entanglement gap opens up in \( K = \pi \) and \( 3\pi/2 \) sectors between the expected counting for non-Abelian MR state and the other part of entanglement spectrum, we cannot determine if the gap will survive in the thermodynamic limit or it is a finite size effect. Since all other results support the non-Abelian QHE state, we believe this result is due to the finite size effect and we leave this part for the future study.

### D. Modular matrix

From the above observations of DMRG, we find that the intermediate phase region appears as a one-component topological nontrivial phase. Here we calculate the modular \( S^- \)-matrix using the near degenerate threefold states in the ED energy spectrum to further investigate the nature of the possible non-Abelian phase\(^4\). The modular \( S^- \)-matrix encodes the information of quasiparticle statistics including quantum dimension and fusion rules\(^{56-58}\), which have been successfully used to identify various Abelian and non-Abelian topological orders\(^{54,59-64}\). To calculate the modular matrix, we follow the method based on the minimal entangled states (MESS)\(^{60}\). The MESs are the eigenstates of the Wilson loop operators with a definite type of quasiparticle\(^{59}\). Thus, the modular transformations on the MESs give rise to the modular matrix\(^{60}\).

Here we show the results at \( t_\perp = 0.25 \) and \( U_\perp = 0.0 \) as an example. We denote the three lowest-energy states in ED spectrum as \( |\xi^j_1\rangle \) \((j = 1, 2, 3)\), from which we can form the general superposition states as,

\[
|\Psi(c_1, c_2, \phi_2, \phi_3)\rangle = c_1 |\xi^1_1\rangle + c_2 e^{i\phi_2} |\xi^2_1\rangle + c_3 e^{i\phi_3} |\xi^3_1\rangle,
\]

where \( c_1, c_2, c_3, \phi_2, \phi_3 \) are real superposition parameters. For each state \( |\Psi\rangle \), we construct the reduced density matrix and obtain the corresponding entanglement entropy. To find the MESs, we optimize the superposition parameters to find the minimum entanglement entropy. As shown in Fig. 6(a), we show the entropy profile of \( |\Psi\rangle \) with the optimized parameters \((\phi_2^*, \phi_3^*)\) for the middle cut along the \( x \)-direction. We find the first global MES \( |\Xi^1_1\rangle \) with the entropy \( S \sim 3.10 \) at the position pointed by the red arrow. The second MES \( |\Xi^2_1\rangle \) (blue arrow) and the third MES \( |\Xi^3_1\rangle \) (green arrow) can be determined in the state space orthogonal to \( |\Xi^j_1\rangle \), as shown in Fig. 6(b). Finally, we can obtain the modular matrix \( S^- = \langle \Xi^I_1 | \Xi^J_1 \rangle \) extracted from the overlap between the MESs for two noncon-
where $S^{CS}$ represents the theoretical prediction from the $SU(2)_2$ Chern-Simons theory. $S^{CS}$ determines the quasiparticle quantum dimension as $d_{\Psi} = 1$, $d_{\chi} = 1$, $d_{\sigma} = \sqrt{2}$ and non-trivial fusion rule as $\sigma \times \sigma = \mathbb{I} + \psi$, where $\mathbb{I}$ represents the identity particle, $\psi$ the fermion-type quasiparticle, $\sigma$ the Ising anyon quasiparticle. Thus, the numerical extracted modular $S$–matrix identifies the intermediate topological phase with threefold ground state degeneracy as the non-Abelian MR state with the emergence of the Ising anyon quasiparticles satisfying the non-Abelian fusion rule ($S_{33} \approx 0$).

Generally speaking, to uniquely determine a topological order, one needs both the modular $S$ and $U$ matrices. From modular $U$ matrix, one can access the chiral central charge $c$ and topological spin of each quasiparticle, which distinguish the non-Abelian MR state from the double-layer Abelian Laughlin states. For example, the chiral central charge of non-Abelian MR state is $c = 3/2$, while the double-layer Laughlin state has $c = 2$. Unfortunately, in the current lattice model, the MES route can not give the $S$ and $U$ matrix together since there is no $\pi/3$ rotation symmetry here. Recently, we note that a general method to extract the quasiparticle statistics in modular $U$ matrix has been proposed. It is interesting to further apply the method to our model using DMRG on cylinder geometry, which we leave for future study.

VI. SUMMARY AND DISCUSSION

We have studied a bosonic double-layer system on a square lattice using ED and DMRG calculations. Through the studies of the energy spectrum, the flux insertion on cylinder, and the modular matrix, we find numerical evidences for a non-Abelian Moore-Read state emerging from the bilayer Halperin states through gapping out the interlayer anti-symmetry state. Although this practically powerful route to a variety of non-Abelian quantum states has been introduced theoretically for decades based on parton construction and field theory, there were limited numerical evidences to support the realization of non-Abelian state in microscopic systems. Our numerical calculations rely on the insertion of charge and drag fluxes, which allow us to detect the quantum phase transition from a two-component topological state to a one-component state characterized by the onset of the finite drag Hall conductance. In particular, the fractionalized drag Hall conductance 0.5 indicates a half-charge boson accumulation that is consistent with the Ising anyon quasiparticle of the MR state. In combining with the modular matrix simulation for the quasiparticle statistics, we identify the nature of the intermediate $t_\perp$ (with threefold near degeneracy) phase as the non-Abelian Moore-Read state, although this state is relatively weak and the entanglement spectrum does not show a robust entanglement spectrum gap for the counting associated with the non-Abelian state. We leave this challenge issue for future studies. It would be particularly interesting to study the possibility of realizing the non-Abelian phase from coupled bilayer Halperin states in fermionic systems, which will be investigated in the future work.

Note added. Upon finalizing the manuscript we noticed several preprints focusing on double-layer $\nu = 1/3 + 1/3$ fermionic systems, which have found the interesting results for a possible non-Abelian state.

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