A General Method for Robust Learning from Batches

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Abstract
In many applications, data is collected in batches, some of which are corrupt or even adversarial. Recent work derived optimal robust algorithms for estimating discrete distributions in this setting. We consider a general framework of robust learning from batches, and determine the limits of both classification and distribution estimation over arbitrary, including continuous, domains. Building on these results, we derive the first robust agnostic computationaly-efficient learning algorithms for piecewise-interval classification, and for piecewise-polynomial, monotone, log-concave, and gaussian-mixture distribution estimation.

1 Introduction
1.1 Motivation

In many learning applications, some samples are inadvertently or maliciously corrupted. A simple and intuitive example shows that this erroneous data limits the extent to which a distribution can be learned, even with infinitely many samples. Consider $p$ that could be one of two possible binary distributions: $(1, 0)$ and $(1-\beta, \beta)$. Given any number of samples from $p$, an adversary who observes a $1-\beta$ fraction of the samples and can determine the rest, could use the observed samples to learn $p$, and set the remaining samples to make the distribution always appear to be $(1-\beta, \beta)$. Even with arbitrarily many samples, any estimator for $p$ fails to decide which $p$ is in effect, hence incurs a total-variation (TV) distance $\geq \beta/2$, that we call the adversarial lower bound.

The example may seem to suggest the pessimistic conclusion that if an adversary can corrupt a $\beta$ fraction of the data, a TV-loss of $\geq \beta/2$ is inevitable. Fortunately, that is not necessarily so.

In the following applications, and many others, data is collected in batches, most of which are genuine, but some possibly corrupted. Data may be gathered by sensors, each providing a large amount of data, and some sensors may be faulty. The word frequency of an author may be estimated from several large texts, some of which are mis-attributed. Or user preferences may be learned by querying several users, but some users may intentionally bias their feedback. Interestingly, for data arriving in batches, even when a $\beta$-fraction of which are corrupted, more can be said.

Recently, [QV17] formalized the problem for discrete domains. They considered estimating a distribution $p$ over $[k]$ in TV-distance when the samples are provided in batches of size $\geq n$. A total of $m$ batches are provided, of which a fraction $\leq \beta$ may be arbitrarily and adversarially corrupted, while in every other batch $b$ the samples are drawn according a distribution $p_b$ satisfying $||p_b-p||_{TV} \leq \eta$, allowing for the possibility that slightly different distributions generate samples in each batch.

For $\beta < 1/900$, they derived an estimation algorithm that approximates any $p$ over a discrete domain to TV-distance $\epsilon = O(\eta + \beta/\sqrt{n})$, surprisingly, much lower than the individual samples limit of $\Theta(\eta + \beta)$. They also derived a matching lower bound, showing that even for binary distributions, for any number $m$ of batches, and hence for general discrete distributions, the lowest achievable total variation distance is $\geq \eta + \frac{\beta}{2\sqrt{2m}}$. We refer to this result as the adversarial batch lower bound.
Their estimator requires $O\left(\frac{d+k}{n^2}\right)$ batches of samples, or equivalently $O\left(\frac{d+k}{n^2}\right)$ samples in total, which is not always optimal. It also runs in time exponential in the domain size, rendering it impractical.

Recently, [CLM19] reduced the exponential time complexity. Allowing quasi-polynomially many samples, they derived an estimator that achieves TV distance $\epsilon = O(\eta + \beta/\sqrt{(\ln 1/\beta)/n})$ and runs in quasi-polynomial time. When a sufficiently larger distance is permitted, their estimator has polynomial time and sample complexities. Concurrently, [JO19] derived a polynomial-time, hence computationally efficient, estimator, that achieves the same $O(\eta + \beta/\sqrt{(\ln 1/\beta)/n})$ TV distance, and for domain size $k$ uses the optimal $O(k/\epsilon^2)$ samples.

When learning general distributions in TV-distance, the sample complexity's linear dependence on the domain size is inevitable even when all samples are genuine. Hence, learning general distributions over large discrete, let alone continuous domains, is infeasible. To circumvent this difficulty, [CLM19] considered robust batch learning of structured discrete distributions, and studied the class of $t$-piecewise degree-$d$ polynomials over the discrete set $[k] = \{1, \ldots, k\}$.

They first reduced the noise with respect to an $F_k$ distance described later, and used existing methods on this cleaned data to estimate the distribution. This allowed them to construct an estimator that approximates these distributions with number of batches $m$ that grows only poly-logarithmically in the domain size $k$. Yet this number still grows with $k$, and is quasi-polynomial in other parameters $t$, $d$, batch size $n$, and $1/\beta$. Additionally, its computational complexity is quasi-polynomial in these parameters and the domain size $k$. Part of our paper generalizes and improves this technique.

The above results suffer several setbacks. While for general distributions there are sample-optimal polynomial-time algorithms, for structured distributions existing algorithms have suboptimal quasi-polynomial sample and time complexity. Furthermore both their sample- and time-complexities grow to infinity in the domain size, making them impractical for many complex applications, and essentially impossible for the many practical applications with continuous domains such as $\mathbb{R}$ or $\mathbb{R}^d$.

This leaves several natural questions. For sample efficiency, can distributions over non-discrete spaces, be estimated in to the adversarial batch lower bound using finitely many samples, and if so, what is their sample complexity? For computational efficiency, are there estimators whose computational complexity is independent of the domain size, and can their run time be polynomial rather than quasi-polynomial in the other parameters. More broadly, can similar robustness results be derived for other important learning scenarios, such as classification? And most importantly, is there a more general theory of robust learning from batches?

### 1.2 Summary of techniques and contributions

To answer these questions, we first briefly foray into VC theory. Consider estimation of an unknown target distribution $p$ to a small $F$-distance, where $F$ is a family of subsets with finite VC-dimension. Without adversarial batches, the empirical distribution of samples from $p$ estimates it to a small $F$-distance. When some of the batches are adversarial, the empirical distribution could be far from $p$. We construct an algorithm that "cleans" the batches and returns a sub-collection of batches whose empirical distribution approximates $p$ to near optimal $F$-distance.

While the algorithm is near sample optimal, as expected from the setting’s broad generality, for some subset families, the it is necessarily not computationally efficient. We then consider the natural and important family $F_k$ of all unions of at most $k$ intervals in $\mathbb{R}$. We provide a computationally efficient algorithm that estimates distributions to near-optimal $F_k$ distance and requires only a small factor more samples than the best possible.

Building on these techniques, we return to estimation in total variation (TV) distance. We consider the family of distributions whose Yatracos Class [Yat85] has finite VC dimension. This family consists of both discrete and continuous distributions, and includes piecewise polynomials, Gaussians in one or more dimensions, and arguably most practical distribution families. We provide a nearly-tight upper bound on the TV-distance to which these distributions can be learned robustly from batches.

Here too, the algorithms’ broad generality makes them computationally inefficient some distribution classes. For one-dimensional $t$-piecewise degree-$d$ polynomials, we derive a polynomial-time algorithm whose sample complexity has optimal linear dependence on $td$ and moderate dependence.
on other parameters. This is the first efficient algorithm for robust learning of general continuous distributions from batches.

The general formulation also allows us to extend robust distribution-estimation results to other learning tasks. We apply this framework to derive the first robust classification results, where the goal is to minimize the excess risk in comparison to the best hypothesis, in the presence of adversarial batches. We obtain tight upper bounds on the excess risk and number of samples required to achieve it for general binary classification problems. We then apply the results to derive a computationally efficient algorithm for hypotheses consisting of \( k \) one-dimensional intervals using only \( \mathcal{O}(k) \) samples.

The rest of the paper is organized as follows. Section 2 describes the paper’s main technical results and their applications to distribution estimation and classification. Section 3 introduces basic notation and techniques. Section 4 recounts basic tools from VC theory used to derive the results. Section 5 derives a framework for robust distribution estimation in \( \mathcal{F} \)-distance from corrupt and adversarial sample batches, and obtains upper bounds on the estimation accuracy and sample complexity. Finally, section 6, develops computationally efficient algorithms for learning in \( \mathcal{F}_k \) distance.

### 1.3 General related work

The current results extend several long lines of work on estimating structured distributions, including \([O'B16,Dia16,AM18]\). The results also relate to classical robust-statistics work \([Tuk60,Hu92]\). There has also been significant recent work leading to practical distribution learning algorithms that are robust to adversarial contamination of the data. For example, \([DKK+16,LRV16]\) presented algorithms for learning the mean and covariance matrix of high-dimensional sub-gaussian and other distributions with bounded fourth moments in presence of the adversarial samples. Their estimation guarantees are typically in terms of \( L_2 \), and do not yield the \( L_1 \)-distance results required for discrete distributions.

The work was extended in \([CSV17]\) to the case when more than half of the samples are adversarial. Their algorithm returns a small set of candidate distributions one of which is a good approximate of the underlying distribution. For more extensive survey on robust learning algorithms in the continuous setting, see \([SCV17,DKK+19]\).

Another motivation for this work derives from the practical federated-learning problem, where information arrives in batches \([MMR+16,MR17]\).

### 2 Results

We consider learning from batches of samples, when a \( \beta \)-fraction of batches are adversarial.

More precisely, \( B \) is a collection of \( m \) batches, composed of two unknown sub-collections. A good sub-collection \( B_G \subseteq B \) of \( \geq (1 - \beta)m \) good batches, where each batch \( b \) consists of \( n \) independent samples, all distributed according to the same distribution \( p_b \) satisfying \( \|p_b - p\|_{TV} \leq \eta \). And an adversarial sub-collection \( B_A = B \setminus B_G \) of the remaining \( \leq \beta m \) batches, each consisting of the same number \( n \) of arbitrary \( \Omega \) elements, that for simplicity we call samples as well. Note that the adversarial samples may be chosen in any way, including after observing the the good samples.

Section 2.1 of \([JO19]\) shows that for discrete domains, results for the special case \( \eta = 0 \), where all batch distributions \( p_b \) are the target distribution \( p \), can be easily extended to the general \( \eta > 0 \) case. The same can be shown for our more general result, hence for simplicity we assume that \( \eta = 0 \).

The next subsection describes our main technical results for learning in \( \mathcal{F} \) distance. The subsections thereafter derive applications of these results for learning distributions in total variation distance and for binary classification.

#### 2.1 Estimating distributions in \( \mathcal{F} \) distance

Let \( \mathcal{F} \) be a family of subsets of a domain \( \Omega \). The \( \mathcal{F} \)-distance between two distributions \( p \) and \( q \) over \( \Omega \) is the largest difference between the probabilities \( p \) and \( q \) assign to any subset in \( \mathcal{F} \),

\[
\|p - q\|_{\mathcal{F}} \triangleq \sup_{S \in \mathcal{F}} |p(S) - q(S)|.
\]
The $\mathcal{F}$-distance clearly generalizes the total-variation and $L_1$ distances. For the collection $\Sigma$ of all subsets of $\Omega$, $\|p - q\|_{\Sigma} = \|p - q\|_{TV} = \frac{1}{2}\|p - q\|_1$.

Our goal is to use samples generated by a target distribution $p$ to approximate it to a small $\mathcal{F}$-distance. For general families $\mathcal{F}$, this goal cannot be accomplished even with just good batches. Let $\mathcal{F} = \Sigma$ be the collection of all subsets of the real interval domain $\Omega = [0, 1]$. For any total number $t$ of samples, with high probability, it is impossible to distinguish the uniform distribution over $[0, 1]$ from a uniform discrete distribution over a random collection of $\gg t^2$ elements in $[0, 1]$. Hence any estimator must incur TV-distance 1 for some distribution.

This difficulty is addressed by Vapnik-Chervonenkis (VC) Theory. The collection $\mathcal{F}$ shatters a subset $S \subseteq \Omega$ if every subset of $S$ is the intersection of $S$ with a subset in $\mathcal{F}$. The VC-dimension $\text{VC-dimension} \ F \ F \ F \ F$.

The sample complexity in both the theorems are independent of the domain and depends linearly on $\text{VC-dimension} \ F \ F \ F \ F$.

For any $\delta > 0$, with probability $1 - \delta$,

$$\|p - \tilde{p}_t\|_F \leq O\left(\sqrt{\frac{V_F + \log 1/\delta}{t}}\right).$$

It can also be shown that $\tilde{p}_t$ achieves the lowest possible $\mathcal{F}$-distance, that we call the information-theoretic limit.

In the adversarial-batch scenario, a fraction $\beta$ of the batches may be corrupted. It is easy to see that for any number $m$ of batches, however large, the adversary can cause $\tilde{p}_t$ to approximate $p$ to $\mathcal{F}$-distance $\geq \beta$, namely $\|p - \tilde{p}_t\|_F \geq \beta$.

Let $\tilde{p}_{B'}$ be the empirical distribution induced by the samples in a collection $B' \subseteq B$. Our first result states that if $\mathcal{F}$ has a finite VC-dimension $V_F$, then $\tilde{p}_t$ estimates $p$ well in $\mathcal{F}$ distance. For all $\delta > 0$, with probability $1 - \delta$,

$$\|p - \tilde{p}_t\|_F \leq O\left(\sqrt{\frac{V_F + \log 1/\delta}{t}}\right).$$

The $\mathcal{F}$-distance bound matches the adversarial limit up to a small $O((\log 1/\beta))$ factor. The bound on the number $m$ of batches required to achieve this bound is also tight up to a logarithmic factor.

The theorem applies to all families with finite VC dimension, and like most other results of this generality, it is necessarily non-constructive in nature. Yet it provides a road map for constructing efficient algorithms for many specific natural problems. In Section 6 we use this approach to derive a polynomial-time algorithm that learns distributions with respect to one of the most important and practical VC classes, where $\Omega = \mathbb{R}$, and $\mathcal{F} = \mathcal{F}_k$ is the collection of all unions of at most $k$ intervals.

**Theorem 2.** For any $n, \beta \leq 0.4, \delta > 0, k > 0$, and $m \geq O\left(\frac{k \log (n/\beta) + \log 1/\delta}{\beta} \cdot \sqrt{n}\right)$, there is an algorithm that runs in time polynomial in all parameters, and with probability $\geq 1 - \delta$ returns a sub-collection $B' \subseteq B$, such that $|B' \cap B_G| \geq (1 - \frac{\beta}{6})|B_G|$ and

$$\|\tilde{p}_{B'} - p\|_F \leq O\left(\beta \sqrt{\frac{\log (1/\beta)}{n}}\right).$$

The sample complexity in both the theorems are independent of the domain and depends linearly on the VC dimension of the family $\mathcal{F}$. 


2.2 Approximating distributions in total-variation distance

Our ultimate objective is to estimate the target distribution in total variation (TV) distance, one of the most common measures in distribution estimation. In this and the next subsection, we follow a framework developed in [DL01], see also [Dia16].

The sample complexity of estimating distributions in TV-distance grows with the domain size, becoming infeasible for large discrete domains and impossible for continuous domains. A natural approach to address this intractability is to assume that the underlying distribution belongs to, or is near, a structured class $\mathcal{P}$ of distributions.

Let $\text{opt}_\mathcal{P}(p) \triangleq \inf_{q \in \mathcal{P}} ||p - q||_{TV}$ be the TV-distance of $p$ from the closest distribution in $\mathcal{P}$. For example, for $p \in \mathcal{P}$, $\text{opt}_\mathcal{P}(p) = 0$. Given $\epsilon, \delta > 0$, we try to use samples from $p$ to find an estimate $\hat{p}$ such that, with probability $\geq 1 - \delta$,

$$||p - \hat{p}||_{TV} \leq \alpha \cdot \text{opt}_\mathcal{P}(p) + \epsilon$$

for a universal constant $\alpha \geq 1$, namely, to approximate $p$ about as well as the closest distribution in $\mathcal{P}$.

Following [DL01], we utilize a connection between distribution estimation and VC dimension. Let $\mathcal{P}$ be a class of distributions over $\Omega$. The Yatracos class [Yat85] of $\mathcal{P}$ is the family of $\Omega$ subsets

$$\mathcal{Y}(\mathcal{P}) \triangleq \{\{\omega \in \Omega : p(\omega) \geq q(\omega)\} : p, q \in \mathcal{P}\}.$$

It is easy to verify that for distributions $p, q \in \mathcal{P}$,

$$||p - q||_{TV} = ||p - q||_{\mathcal{Y}(\mathcal{P})}.$$  

The Yatracos minimizer of a distribution $p$ is its closest distribution, by $\mathcal{Y}(\mathcal{P})$-distance, in $\mathcal{P}$,

$$\psi_\mathcal{P}(p) = \arg \min_{q \in \mathcal{P}} ||q - p||_{\mathcal{Y}(\mathcal{P})},$$

where ties are broken arbitrarily. Using this definition and equations, and a sequence of triangle inequalities, Theorem 6.3 in [DL01] shows that, for any distributions $p, p'$, and any class $\mathcal{P}$,

$$||p - \psi_\mathcal{P}(p')||_{TV} \leq 3 \cdot \text{opt}_\mathcal{P}(p) + 4||p - p'||_{\mathcal{Y}(\mathcal{P})}. \quad (1)$$

Therefore, given a distribution that approximates $p$ in $\mathcal{Y}(\mathcal{P})$-distance, it is possible to find a distribution in $\mathcal{P}$ approximating $p$ in TV-distance. In particular, when $p \in \mathcal{P}$, the opt term is zero.

If the Yatracos class $\mathcal{Y}(\mathcal{P})$ has finite VC dimension, the VC Uniform deviation inequality ensures that for the empirical distribution $p'$ of i.i.d. samples from $p$, $||p' - p||_{\mathcal{Y}(\mathcal{P})}$ decreases to zero, and can be used to approximate $p$ in TV-distance. This general method has lead to many sample- and computationally-efficient algorithms for estimating structured distributions in TV-distance.

However, as discussed earlier, with a $\beta$-fraction of adversarial batches, the empirical distribution of all samples can be at a $\mathcal{Y}(\mathcal{P})$-distance as large as $\Theta(\beta)$ from $p$, leading to a large TV-distance.

Yet Theorem 1 shows that data can be "cleaned" to remove outlier batches and retain batches whose empirical distribution approximates $p$ to a much smaller $\mathcal{Y}(\mathcal{P})$-distance of $O(\beta \sqrt{(\log 1/\beta)n})$. Combined with Equation (1), we obtain a much better approximation of $p$ in total variation distance.

**Theorem 3.** For a distribution class $\mathcal{P}$ with Yatracos Class of finite VC dimension $v$, for any $n$, $\beta \leq 0.4$, $\delta > 0$, and $m \geq O\left(\frac{v \log(n/\beta) + \log 1/\delta}{\beta^2}\right)$, there is an algorithm that with probability $\geq 1 - \delta$ returns a distribution $p' \in \mathcal{P}$ such that

$$||p - p'||_{TV} \leq 3 \cdot \text{opt}_\mathcal{P}(p) + O\left(\beta \sqrt{\frac{\log(1/\beta)}{n}}\right).$$

The estimation error achieved in the theorem for TV-distance matches the lower to a small logarithmic factor of $O(\sqrt{\log(1/\beta)})$, and is valid for any class $\mathcal{P}$ with finite VC Dimensional Yatracos Class.

Moreover, the upper bound on the number of samples (or batches) required by the algorithm to estimate $p$ to the above distance matches a similar general upper bound obtained for non adversarial
setting to a log factor. This results for the first time shows that it is possible to learn a wide variety of distributions robustly using batches, even over continuous domains.

Theorem 5. Let \( p \) be any distribution over \( \mathbb{R} \). For any \( n, \beta \leq 0.4, t, d, \delta > 0 \), and \( m \geq \mathcal{O}\left(\frac{td \log(n/\beta) + \log 1/\delta}{\beta^3}\cdot \sqrt{n}\right) \), there is a polynomial time algorithm that with probability \( \geq 1 - \delta \) returns a distribution \( \hat{p} \in \mathcal{P}_{t,d} \) such that

\[
||p - \hat{p}||_{TV} \leq \mathcal{O}(opt_{\mathcal{P}_{t,d}}(p)) + \mathcal{O}\left(\beta \frac{\log(1/\beta)}{n}\right).
\]

Next we provide a polynomial-time algorithm for estimating \( p \) to the same \( \mathcal{O}(\beta \sqrt{\log(1/\beta)/n}) \) TV-distance, but with an extra \( \mathcal{O}(\sqrt{n}/\beta) \) factor in sample complexity.

Theorem 2 provides a polynomial time algorithm that returns a sub-collection \( B' \subseteq B \) of batches whose empirical distribution \( \hat{p}_{B'} \) is close to \( p \) in \( \mathcal{F}_{2t,d} \)-distance. [ADLS17] provides a polynomial time algorithm that for any distribution \( q \) returns a distribution in \( \hat{p} \in \mathcal{P}_{t,d} \) minimizing \( ||\hat{p} - q||_{\mathcal{F}_{2t,d}} \) to an additive error. Then Equation (1) and Theorem 2 yield the following result.

Theorem 5. Let \( p \) be any distribution over \( \mathbb{R} \). For any \( n, \beta \leq 0.4, t, d, \delta > 0 \), and \( m \geq \mathcal{O}\left(\frac{td \log(n/\beta) + \log 1/\delta}{\beta^3}\cdot \sqrt{n}\right) \), there is a polynomial time algorithm that with probability \( \geq 1 - \delta \) returns a distribution \( \hat{p}' \in \mathcal{P}_{t,d} \) such that

\[
||p - \hat{p}'||_{TV} \leq \mathcal{O}(opt_{\mathcal{P}_{t,d}}(p)) + \mathcal{O}\left(\beta \frac{\log(1/\beta)}{n}\right).
\]

2.3 Learning univariate structured distributions

We apply the general results in the last two subsections to estimate distributions over the real line. We start with one of the most studied, and important, distribution families, the class of piecewise-polynomial distributions, and then observe that it can be generalized to even broader classes.

A distribution \( p \) over \([a, b]\) is \( t \)-piecewise, degree-\( d \), if there is a partition of \([a, b]\) into \( t \) intervals \( I_1, \ldots, I_t \), and degree-\( d \) polynomials \( r_1, \ldots, r_t \) such that \( \forall j, x \in I_j, p(x) = r_j(x) \). The definition extends naturally to discrete distributions over \([k]\) = \{1, \ldots, k\}.

Let \( \mathcal{P}_{t,d} \) denote the collection of all \( t \)-piece-wise degree \( d \) distributions. \( \mathcal{P}_{t,d} \) is interesting in its own right, as it contains important distribution classes such as histograms. In addition, it approximates other important distribution classes, such as monotone, log-concave, Gaussians, and their mixtures, arbitrarily well, e.g., [ADLS17].

Note that for any two distributions \( p, q \in \mathcal{P}_{t,d} \), the difference \( p - q \) is a \( 2t \)-piecewise degree-\( d \) polynomial, hence every set in the Yatracos class of \( \mathcal{P}_{t,d} \),

\[
\{x \in \mathbb{R} : p(x) \geq q(x)\} = \{x \in \mathbb{R} : p(x) - q(x) \geq 0\}
\]

is the union of at most \( 2t \cdot d \) intervals in \( \mathbb{R} \). Therefore, \( \mathcal{Y}(\mathcal{P}_{t,d}) \subseteq \mathcal{F}_{2t,d} \). And since \( V_{\mathcal{F}_k} = \mathcal{O}(k) \) for any \( k \), \( \mathcal{Y}(\mathcal{P}_{t,d}) \) has VC dimension \( \mathcal{O}(td) \).

Theorem 3 can then be applied to show that any target distribution \( p \) can be estimated by a distribution in \( \mathcal{P}_{t,d} \) to a TV-distance that is within a small \( \sqrt{\log(1/\beta)} \) factor from adversarial lower bound, using a number of samples, and hence batches, that is within a logarithmic factor from the information-theoretic lower bound [CDSS14].

Corollary 4. Let \( p \) be distribution over \( \mathbb{R} \). For any \( n, \beta \leq 0.4, t, d, \delta > 0 \), and \( m \geq \mathcal{O}\left(\frac{td \log(n/\beta) + \log 1/\delta}{\beta^3}\right) \), there is an algorithm that with probability \( \geq 1 - \delta \) returns a distribution \( \hat{p}' \in \mathcal{P}_{t,d} \) such that

\[
||p - \hat{p}'||_{TV} \leq 3 \cdot opt_{\mathcal{P}_{t,d}}(p) + \mathcal{O}\left(\beta \frac{\log(1/\beta)}{n}\right).
\]
2.4 Binary classification

The framework developed in this paper extends beyond distribution estimation. Here we describe its application to Binary classification. Consider a family $\mathcal{H} : \Omega \rightarrow \{0, 1\}$ of Boolean functions, and a distribution $p$ over $\Omega \times \{0, 1\}$. Let $(X, Y) \sim p$, where $X \in \Omega$ and $Y \in \{0, 1\}$. The loss of hypothesis $h \in \mathcal{H}$ for distribution $p$ is

$$r_p(h) = \Pr_{(X,Y) \sim p}[h(X) \neq Y].$$

The optimal classifier for distribution $p$ is

$$h^*(p) = \arg\min_{h \in \mathcal{H}} r_p(h),$$

and the optimal loss is

$$r_p^*(\mathcal{H}) = r_p(h^*(p)).$$

The goal is to return a hypothesis $h \in \mathcal{H}$ whose loss $r_p(h)$ is close to the optimal loss $r_p^*(\mathcal{H})$. Consider the following natural extension of VC-dimension from families of subsets to families of Boolean functions. For a boolean-function family $\mathcal{H}$, define the family

$$\mathcal{F}_\mathcal{H} = \{(\omega \in \Omega : h(\omega) = z), y : h \in \mathcal{H}, y, z \in \{0, 1\}\}$$

of subsets of $\Omega \times \{0, 1\}$, and let the VC dimension of $\mathcal{H}$ be $\mathcal{V}_\mathcal{H} = \mathcal{V}_{\mathcal{F}_\mathcal{H}}$.

The largest difference between the loss of a classifier for two distributions $p$ and $q$ over $\omega \times \{0, 1\}$ is related to their $\mathcal{F}_\mathcal{H}$-distance,

$$\sup_{h \in \mathcal{H}} |r_p(h) - r_q(h)| = \sup_{h \in \mathcal{H}} \left|\Pr_{(X,Y) \sim p}[h(X) \neq Y] - \Pr_{(X,Y) \sim q}[h(X) \neq Y]\right|$$

$$\leq \sup_{h \in \mathcal{H}} \sum_{y \in \{0, 1\}} \left|\Pr_{(X,Y) \sim p}(h(X) = \bar{y}, Y = y) - \Pr_{(X,Y) \sim q}(h(X) = \bar{y}, Y = y)\right|$$

$$\leq 2||p - q||_{\mathcal{F}_\mathcal{H}}.$$  \hspace{1cm} (2)

The next simple lemma, proved in the appendix, upper bounds the excess loss of the optimal classifier in $\mathcal{H}$ for a distribution $q$ for another distribution $p$ in terms of $\mathcal{F}_\mathcal{H}$ distance between the distributions.

**Lemma 6.** For any two distributions $p$ and $q$ and hypothesis class $\mathcal{H}$,

$$r_p(h^*(q)) - r_p^*(\mathcal{H}) \leq 4||p - q||_{\mathcal{F}_\mathcal{H}}.$$

When $q$ is the empirical distribution of non-adversarial i.i.d. samples from $p$, $h^*(q)$ is called the empirical risk minimizer, and the excess loss of the empirical risk minimizer in the above equation goes to zero if VC dimension of $\mathcal{H}$ is finite.

Yet as discussed earlier, when a $\beta$-fractions of the batches, and hence samples, are chosen by an adversary, the empirical distribution of all samples can be at a large $\mathcal{F}_\mathcal{H}$-distance $O(\beta)$ from $p$, leading to an excess classification loss up to $O(\beta)$ for the empirical-risk minimizer.

Theorem 1 states that the collection of batches can be "cleaned" to obtain a sub-collection whose empirical distribution has a lower $\mathcal{F}_\mathcal{H}$-distance from $p$. The above lemma then implies that the optimal classifier for the empirical distribution of the cleaner batches will have a small excess risk for $p$ as well. The resulting non-constructive algorithm has excess risk and sample complexity that are optimal to a logarithmic factor.

**Theorem 7.** For any $\mathcal{H}$, $n$, $\beta \leq 0.4$, $\delta > 0$, and $m \geq O\left(\frac{\mathcal{V}_\mathcal{H} \log(n/\beta)+\log 1/\delta}{\beta^2}\right)$, there is an algorithm that with probability $\geq 1 - \delta$ returns a sub-collection $B' \subseteq B$ such that $|B' \cap B_G| \geq (1 - \frac{\beta}{n})|B_G|$ and

$$r_p(h^*(\bar{p}_{B'})) - r_p^*(\mathcal{H}) \leq O\left(\beta \sqrt{\frac{\log(1/\beta)}{n}}\right).$$
To derive a computationally efficient algorithm, we focus on the following class of binary functions. For \( k \geq 0 \) let \( \mathcal{H}_k \) denote the collection of all binary functions over \( \mathbb{R} \) whose decision region, namely values mapping to 1, consists of at most \( k \)-intervals. The VC dimension of \( \mathcal{F}_{\mathcal{H}_k} \) is clearly \( O(k) \).

Theorem 2 describes a polynomial time algorithm that returns a cleaner data w.r.t. \( \mathcal{F}_{\mathcal{H}_k} \) distance. From Lemma 6, the hypothesis that minimizes the loss for the empirical distribution of this cleaner data will have a small excess loss. Furthermore, \([\text{Maa94]}\) derived a polynomial time algorithm to find the hypothesis \( h \in \mathcal{H}_k \) that minimizes the loss for a given empirical distribution. Combining these results, we obtain a computationally efficient classifier in \( \mathcal{H}_k \) that achieves the excess loss in the above theorem.

**Theorem 8.** For any \( \mathcal{H} = \mathcal{H}_k, n, \beta \leq 0.4, \delta > 0, \) and \( m \geq O\left(\frac{k \log(n/\beta) + \log(1/\delta)}{\beta^2} \cdot \sqrt{n}\right) \), there is a polynomial time algorithm that with probability \( \geq 1 - \delta \) returns a sub-collection \( B' \subseteq B \) such that \( |B' \cap B_G| \geq (1 - \frac{\delta}{6})|B_G| \) and

\[
\mathbb{P}\left(h^*(\tilde{\mu}_{B'}) - h^*(\tilde{\mu}_B) \leq O\left(\beta \frac{\log(1/\beta)}{n}\right)\right).
\]

### 3 Preliminaries

We introduce terminology that helps describe the approach and results. Some of the work builds on results in \([\text{JO19}]\), and we keep the notation consistent.

Recall that \( B, B_G, \) and \( B_A \) are the collections of all-, good-, and adversarial-batches. Let \( B' \subseteq B, B'_G \subseteq B_G, \) and \( B'_A \subseteq B_A \), denote sub-collections of all-, good-, and bad-batches. We also let \( S \) denote a subset of the Borel \( \sigma \)-field \( \Sigma \).

Let \( X^b_1, X^b_2, ..., X^b_n \) denote the \( n \) samples in a batch \( b \), and let \( 1_S \) denote the indicator random variable for a subset \( S \subseteq \Sigma \). Every batch \( b \in B \) induces an empirical measure \( \tilde{\mu}_b \) over the domain \( \Omega \), where for each \( S \subseteq \Sigma \),

\[
\tilde{\mu}_b(S) \triangleq \frac{1}{n} \sum_{i \in [n]} 1_S(X^b_i).
\]

Similarly, any sub-collection \( B' \subseteq B \) of batches induces an empirical measure \( \tilde{\mu}_{B'} \) defined by

\[
\tilde{\mu}_{B'}(S) \triangleq \frac{1}{|B'|} \sum_{b \in B'} \frac{1}{n} \sum_{i \in [n]} 1_S(X^b_i) = \frac{1}{|B'|} \sum_{b \in B'} \tilde{\mu}_b(S).
\]

We use two different symbols to denote empirical distribution defined by single batch and a sub-collection of batches to make them easily distinguishable. Note that \( \tilde{\mu}_{B'} \) is the mean of the empirical measures \( \tilde{\mu}_b \) defined by the batches \( b \in B' \).

Recall that \( n \) is the batch size. For \( r \in [0, 1] \), let \( V(r) \triangleq \frac{r(1-r)}{n} \), the variance of a Binomial \((r, n)\) random variable. Observe that

\[
\forall r, s \in [0, 1], V(r) \leq \frac{1}{4n} \quad \text{and} \quad |V(r) - V(s)| \leq \frac{|r - s|}{n}, \tag{3}
\]

where the second property follows as

\[
|r(1-r) - s(1-s)| = |r - s| \cdot |1 - (r+s)| \leq |r - s|.
\]

For \( b \in B_G \), the random variables \( 1_S(X^b_i) \) for \( i \in [n] \) are distributed i.i.d. Bernoulli \((p(S))\), and since \( \tilde{\mu}_b(S) \) is their average,

\[
E[\tilde{\mu}_b(S)] = p(S) \quad \text{and} \quad \text{Var}[\tilde{\mu}_b(S)] = E[(\tilde{\mu}_b(S) - p(S))^2] = V(p(S)).
\]

For batch collection \( B' \subseteq B \) and subset \( S \subseteq \Sigma \), the empirical probability \( \tilde{\mu}_b(S) \) of \( S \) will vary with the batch \( b \in B' \). The empirical variance of these empirical probabilities is

\[
\mathbb{V}_B'(S) \triangleq \frac{1}{|B'|} \sum_{b \in B'} (\tilde{\mu}_b(S) - \tilde{\mu}_{B'}(S))^2.
\]
4 Vapnik-Chervonenkis (VC) theory

We recall some basic concepts and results in VC theory, and derive some of their simple consequences that we use later in deriving our main results.

The VC shatter coefficient of \( F \) is

\[
S_F(t) \overset{\text{def}}{=} \sup_{x_1, x_2, \ldots, x_t \in \Omega} |\{x_1, x_2, \ldots, x_t\} \cap S : S \in F|,
\]

the largest number of subsets of \( t \) elements in \( \Omega \) obtained by intersections with subsets in \( F \). The VC dimension of \( F \) is

\[
V_F \overset{\text{def}}{=} \sup\{t : S_F(t) = 2^t\},
\]

the largest number of \( \Omega \) elements that are "fully shattered" by \( F \). The following Lemma \cite{DL01} bounds the Shatter coefficient for a VC family of subsets.

Lemma 9 \cite{DL01}. For all \( t \geq V_F \), \( S_F(t) \leq \left( \frac{4 \epsilon}{t} \right)^{V_F} \).

Next we state the VC-inequality for relative deviation \cite{VC74, AST93}.

Theorem 10. Let \( p \) be a distribution over \( (\Omega, \Sigma) \), and \( F \) be a VC-family of subsets of \( \Omega \) and \( \bar{p}_t \) denote the empirical distribution from \( t \) i.i.d samples from \( p \). Then for any \( \epsilon > 0 \), with probability \( \geq 1 - 8S_F(2t)e^{-t\epsilon^2/4} \),

\[
\sup_{S \in F} \max \left\{ \frac{\bar{p}_t(S) - p(S)}{\sqrt{p(S)}}, \frac{p(S) - \bar{p}_t(S)}{\sqrt{p(S)}} \right\} \leq \epsilon.
\]

Another important ingredient commonly used in VC Theory is the concept of covering number that reflects the smallest number of subsets that approximate each subset in the collection.

Let \( p \) be any probability measure over \( (\Omega, \Sigma) \) and \( F \subseteq \Sigma \) be a family of subsets. A collection of subsets \( C \subseteq \Sigma \) is an \( \epsilon \)-cover of \( F \) if for any \( S \in F \), there exists a \( S' \in C \) with \( p(S \Delta S') \leq \epsilon \). The \( \epsilon \)-covering number of \( F \) is

\[
N(F, p, \epsilon) \overset{\text{def}}{=} \inf \{|C| : C \text{ is an } \epsilon \text{-cover of } F\}.
\]

If \( C \subseteq F \) is an \( \epsilon \)-cover of \( F \), then \( C \) is \( \epsilon \)-self cover of \( F \).

The \( \epsilon \)-self-covering number is

\[
N^*(F, p, \epsilon) \overset{\text{def}}{=} \inf \{|C| : C \text{ is an } \epsilon \text{-self-cover of } F\}.
\]

Clearly, \( N^*(F, p, \epsilon) \geq N(F, p, \epsilon) \). The next lemma establishes a reverse relation.

Lemma 11. For any \( \epsilon \geq 0 \), \( N^*(F, p, \epsilon) \leq N(F, p, \epsilon/2) \).

Proof. If \( N(F, p, \epsilon/2) = \infty \), the lemma clearly holds. Otherwise, let \( C \) be an \( \epsilon/2 \)-cover of size \( N(F, p, \epsilon/2) \). We construct an \( \epsilon \)-self-cover of equal or smaller size.

For every subset \( S_C \in C \), there is a subset \( S = f(S_C) \in F \) with \( p(S_C \Delta f(S_C)) \leq \epsilon/2 \). Otherwise, \( S_C \) could be removed from \( C \) to obtain a strictly smaller \( \epsilon/2 \) cover, which is impossible.

The collection \( \{ f(S_C) : S_C \in C \} \subseteq F \) has size \( \leq |C| \), and it is an \( \epsilon \)-self-cover of \( F \) because for any \( S \in F \), there is an \( S_C \in C \) with \( p(S \Delta S_C) \leq \epsilon/2 \), and by the triangle inequality, \( p(S \Delta f(S_C)) \leq \epsilon \).

Let \( N_{F, \epsilon} \overset{\text{def}}{=} \sup_{p} N(F, p, \epsilon) \) and \( N_{F}^* \overset{\text{def}}{=} \sup_{p} N^*(F, p, \epsilon) \) be the largest covering numbers under any distribution.

The next theorem bounds the covering number of \( F \) in terms of its VC-dimension.

Theorem 12 \cite{VW96}. There exists a universal constant \( c \) such that for any \( \epsilon > 0 \), and any family \( F \) with VC dimension \( V_F \),

\[
N_{F, \epsilon} \leq cV_F \left( \frac{4e}{\epsilon} \right)^{V_F}.
\]
Combining the theorem and Lemma 11, we obtain the following corollary.

**Corollary 13.**

\[ N_{\beta, \epsilon}^p \leq cV_F \left( \frac{8\epsilon}{\epsilon} \right)^{V_F} \]

For any distribution \( p \) and family \( F \), let \( C^*(F, p, \epsilon) \) be any minimal-size \( \epsilon \)-self-cover for \( F \) of size \( \leq N_{\beta, \epsilon}^p \).

5 A framework for distribution estimation from corrupted sample batches

We develop a general framework to learn \( p \) in \( F \) distance and derive Theorem 1. Recall that the \( F \) distance between two distributions \( p \) and \( q \) is

\[ ||p - q||_F = \sup_{S \in F} |p(S) - q(S)|. \]

The algorithms presented enhance the algorithm of [JO19], developed for \( F = 2^\Omega \) of a discrete domain \( \Omega = [k] \), to any VC-family \( F \) of subsets of any sample space \( \Omega \). We retain the part of the analysis and notation that are common in our enhanced algorithm and the one presented in [JO19].

At a high level, we remove the adversarial, or "outlier" batches, and return a sub-collection \( B' \subseteq B \) of batches whose empirical distribution \( \bar{p}_{B'} \) is close to \( p \) in \( F \) distance. The uniform deviation inequality in VC theory states that the sub-collection \( B_G \) of good batches has empirical distribution \( \bar{p}_{B_G} \) that approximates \( p \) in \( F \) distance, thereby ensuring the existence of such a sub-collection.

The family \( F \) can be potentially uncountable, hence learning a distribution to a given \( F \) distance may entail simultaneously satisfying infinitely many constraints. To decrease the constraints to a finite number, Corollary 13 shows that for any distribution and any \( \epsilon > 0 \), there exists a finite \( \epsilon \)-cover of \( F \) w.r.t. this distribution.

Our goal therefore is to find an \( \epsilon \)-cover \( C \) of \( F \) w.r.t. an appropriate distribution such that if for some sub-collection \( B' \) the empirical distribution \( \bar{p}_{B'} \) approximates \( p \) in \( C \)-distance it would also approximate \( p \) in \( F \)-distance. The two natural distribution choices are the target distribution \( p \) and empirical distribution from its samples. Yet the distribution \( p \) is unknown to us, and its samples provided in the collection \( B \) of batches are corrupted by an adversary.

The next theorem overcomes this challenge by showing that although the collection \( B \) includes adversarial batches, for small enough \( \epsilon \), for any \( \epsilon \)-cover \( C \) of \( F \) w.r.t. the empirical distribution \( \bar{p}_B \), a small \( C \)-distance \( ||\bar{p}_{B'} - p||_C \), between \( p \) and the empirical distribution induced by a sub-collection \( B' \subseteq B \) would imply a small \( F \)-distance \( ||\bar{p}_{B'} - p||_F \), between the two distributions.

Note that the theorem allows the \( \epsilon \)-cover \( C \) of \( F \) to include sets in the subset family \( F' \) containing \( F \).

**Theorem 14.** For \( m \geq \mathcal{O} \left( \frac{V_F \log(n/\beta) + \log(1/\delta)}{\beta^2} \right) \) and \( \epsilon \leq \frac{\beta}{\sqrt{n}} \), let \( C \subseteq F' \) be an \( \epsilon \)-cover of family \( F \) w.r.t. the empirical distribution \( \bar{p}_B \). Then with probability \( \geq 1 - \delta \), for any sub-collection of batches \( B' \subseteq B \) of size \( |B'| \geq m/2 \),

\[ ||\bar{p}_{B'} - p||_F \leq ||\bar{p}_{B'} - p||_C + \frac{5\beta}{\sqrt{n}}. \]

**Proof.** Consider any batch sub-collection \( B' \subseteq B \). For every \( S, S' \subseteq \Omega \), by the triangle inequality,

\[ |\bar{p}_{B'}(S) - p(S)| = \left| \left( \bar{p}_{B'}(S') + \bar{p}_{B'}(S \setminus S') - \bar{p}_{B'}(S' \setminus S) \right) - \left( p(S') + p(S \setminus S') - p(S' \setminus S) \right) \right| \]

\[ \leq |\bar{p}_{B'}(S') - p(S')| + |\bar{p}_{B'}(S \setminus S') + \bar{p}_{B'}(S' \setminus S) + p(S \setminus S') + p(S' \setminus S)| \]

\[ = |\bar{p}_{B'}(S') - p(S')| + |\bar{p}_{B'}(S \triangle S') + p(S \triangle S')|. \]

Since \( C \) is an \( \epsilon \)-cover w.r.t. \( \bar{p}_B \), for every \( S \in \mathcal{F} \) there is an \( S' \in C \) such that \( \bar{p}_B(S \triangle S') \leq \epsilon \). For such pairs, we bound the second term on the right in the above equation.

\[ \bar{p}_{B'}(S \triangle S') = \frac{1}{B' |n|} \sum_{b \in B'} \sum_{i \in [n]} 1_{S \triangle S'}(X_i^b) \]
We start with the following observation. Consider a subset $S$. The following discussion develops some notation and intuitions that leads to these properties.

To quantify this effect, for a subset $S \in \mathcal{F}$, the preceding discussion shows that the corruption score of most good batches for a fixed subset $S$ is zero, and that adversarial batches that may significantly change the overall mean of empirical probabilities have high corruption score.

Choosing $B' = B_G$ in the above equation and using $B_G = (1 - \beta)m \geq m/2$ gives,

$$\bar{p}_{B_G}(S \triangle S') < 2\epsilon.$$  \hfill (6)

Then

$$p(S \triangle S') \leq |p(S \triangle S') - \bar{p}_{B_G}(S \triangle S')| + \bar{p}_{B_G}(S \triangle S')$$

\begin{align*}
&\leq (a) \sup_{S, S' \in \mathcal{F}} |p(S \triangle S') - \bar{p}_{B_G}(S \triangle S')| + 2\epsilon \\
&\leq (b) 2\epsilon + \beta \sqrt{\frac{n}{m}}.
\end{align*}

with probability $\geq 1 - \delta$, here (a) used equation (6) and (b) follows from Lemma 22. Combining equations (4), (5) and the above equation completes the proof.

The above theorem reduces the problem of estimating in $\mathcal{F}$ distance to finding a sub-collection $B' \subseteq B$ of at least $m/2$ batches such that for an $\epsilon$-cover $C$ of $\mathcal{F}$ w.r.t. distribution $\bar{p}_B$, the distance $\|\bar{p}_{B'} - p\|_C$ is small. If we choose a finite $\epsilon$-cover $C$ of $\mathcal{F}$, the theorem would ensure that the number of constrains is finite.

To find a sub-collection of batches as suggested above, we show that with high probability, certain concentration properties hold for all subsets in $\mathcal{F}'$. Note that the cover $C$ is chosen after seeing the samples in $B$, but since $C \subseteq \mathcal{F}'$, the results also hold for all subsets in $C$.

The following discussion develops some notation and intuitions that leads to these properties.

We start with the following observation. Consider a subset $S \in \mathcal{F}'$. For every good batch $b \in B_G$, $\bar{\mu}_b(S)$ has a sub-gaussian distribution $\text{subG}(p(S), \frac{1}{\sqrt{n}})$ with variance $V(p(S))$. Therefore, most of the good batches $b \in B_G$ assign the empirical probability $\bar{\mu}_b(S) \in p(S) \pm O(1/\sqrt{n})$. Moreover, the empirical mean and variance of $\bar{\mu}_b(S)$ over $b \in B_G$ converges to the expected values $p(S)$ and $V(p(S))$, respectively.

In addition to the good batches, the collection $B$ of batches also includes an adversarial sub-collection $B_A$ of batches that constitute up to a $\beta$-fraction of $B$. If the difference between $p(S)$ and the average of $\bar{\mu}_b(S)$ over all adversarial batches $b \in B_A$ is $\leq O(\frac{1}{\sqrt{n}})$, namely comparable to the standard deviation of $\bar{\mu}_b(S)$ for the good batches $b \in B_G$, then the adversarial batches can change the overall mean of empirical probabilities $\bar{\mu}_b(S)$ by at most $O(\frac{1}{\sqrt{n}})$, which is within our tolerance. Hence, the mean of $\bar{\mu}_b(S)$ will deviate significantly from $p(S)$ only in the presence of a large number of adversarial batches $b \in B_A$ whose empirical probability $\bar{\mu}_b(S)$ differs from $p(S)$ by $\gg O(\frac{1}{\sqrt{n}})$.

To quantify this effect, for a subset $S \in \mathcal{F}'$ let

$$\text{med}(\bar{\mu}(S)) \triangleq \text{median}\{\bar{\mu}_b(S) : b \in B\}$$

be the median empirical probability of $S$ over all batches. Property 1 shows that $\text{med}(\bar{\mu}(S))$ is a good approximation of $p(S)$. Define the corruption score of batch $b$ for $S$ to be

$$\psi_b(S) \triangleq \begin{cases} 
0 & \text{if } |\bar{\mu}_b(S) - \text{med}(\bar{\mu}(S))| \leq 3\sqrt{\frac{\ln(6e/\beta)}{n}}, \\
(\bar{\mu}_b(S) - \text{med}(\bar{\mu}(S)))^2 & \text{otherwise}.
\end{cases}$$

The preceding discussion shows that the corruption score of most good batches for a fixed subset $S$ is zero, and that adversarial batches that may significantly change the overall mean of empirical probabilities have high corruption score.
The corruption score of a sub-collection \( B' \) for a subset \( S \) is the sum of the corruption score of its batches,

\[
\psi_{B'}(S) \triangleq \sum_{b \in B'} \psi_b(S).
\]

A high corruption score of \( B' \) for a subset \( S \) indicates that \( B' \) has many batches \( b \) with large difference \( |\hat{\mu}_b(S) - \text{med}(\hat{\mu}(S))| \). Finally, the corruption score of a sub-collection \( B' \) for a family of subsets \( \mathcal{F}' \subseteq \mathcal{F} \) is the largest corruption score of any \( S \in \mathcal{F}' \),

\[
\psi_{B'}(\mathcal{F}') \triangleq \max_{S \in \mathcal{F}'} \psi_{B'}(S).
\]

Note that removing batches from a sub-collection reduces its corruption. We can simply make corruption zero by removing all batches, but we would lose all the information as well. As described later in this section, the algorithm reduces the corruption below a threshold by removing a few batches while not sacrificing too many good batches in the process.

Recall that \( B \) is a collection of \( m \) batches, each containing \( n \) samples, and that a sub-collection \( B' \subseteq B \) consists of \( (1-\beta)m \) good batches where all samples are drawn from the target distribution \( p \). We show that regardless of the samples in adversarial batches, with high probability, \( B \) satisfies the following three concentration properties.

1. For all \( S \in \mathcal{F}' \), the median of the estimates \( \{ \hat{\mu}_b(S) : b \in B \} \) approximates \( p(S) \) well,

\[
|\text{med}(\hat{\mu}(S)) - p(S)| \leq \sqrt{\ln(6)/n}.
\]

2. For every sub-collection \( B'_G \subseteq B_G \) containing a large portion of the good batches, \( |B'_G| \geq (1-\beta/6)|B_G| \), and for all \( S \in \mathcal{F}' \), the empirical mean and variance of \( \hat{\mu}_b(S) \) estimate \( p(S) \) and \( \text{V}(p(S)) \) well,

\[
|\bar{p}_{B'_G}(S) - p(S)| \leq \frac{\beta}{2} \sqrt{\frac{\ln(6e/\beta)}{n}},
\]

and

\[
\left| \frac{1}{|B'_G|} \sum_{b \in B'_G} (\hat{\mu}_b(S) - p(S))^2 - \text{V}(p(S)) \right| \leq \frac{6\beta \ln(6e/\beta)}{n}.
\]

3. The corruption score of the collection \( B_G \) of good batches for family \( \mathcal{F}' \) is small,

\[
\psi_{B_G}(\mathcal{F}') \leq \frac{\beta m \ln(6e/\beta)}{n} \triangleq \kappa_G.
\]

**Lemma 15.** Let \( \mathcal{F}' \) have finite VC dimension and \( m \geq \mathcal{O}(\frac{\sqrt{\text{V}(p(S))} \log(n/\beta)}{\beta^2}) \). With probability \( \geq 1 - \delta \) the three essential properties hold.

These properties extend the same properties in [JO19] (Section 2) from subsets of discrete domains to families of subset with finite VC dimension in any euclidean space.

To prove that properties hold with high probability, we first show that for an appropriately chosen epsilon, they hold for all subsets in a minimal-size \( \epsilon \)-cover of \( \mathcal{F}' \) w.r.t. the target distribution \( p \). Since the cover has finite size, a proof similar to the one in [JO19], for discrete domains, shows that the properties hold for all subsets in the cover. This uses the observation that for \( b \in B_G \), \( \hat{\mu}_b(S) \) has a sub-gaussian distribution subG\((p(S), \frac{1}{m})\), and variance \( \text{V}(p(S)) \). We then use Lemma 22 to extend the properties from subsets in the cover to all subsets in class \( \mathcal{F}' \). The proof is in Appendix A.

The remainder of this section assumes that the properties in the above lemma hold.

For any \( \mathcal{C} \in \mathcal{F}' \) conditions 1, 2 and 3 holds for all subsets in \( \mathcal{C} \) w.h.p.

Next a simple adaptation of the Batch Deletion algorithm in [JO19] is used to find a sub collection of batches \( B' \) such that \( ||\bar{p}_{B'} - p||_c \) is small.

For any \( \mathcal{C} \subseteq \mathcal{F}' \), the next Lemma bounds \( \mathcal{C} \)-distance of empirical distribution \( \bar{p}_{B'} \) in terms of the corruption of \( B' \) for sub-family \( \mathcal{C} \).
Lemma 16. Suppose Properties 1-3 hold. Then for any $B'$ such that $|B' \cap B_G| \geq (1 - \frac{2}{e})|B_G|$ and any family $C \subseteq \mathcal{F}$ such that $\psi_{B'}(C) \leq t \cdot \kappa_G$, for some $t \geq 0$, then

$$||\hat{\mu}_{B'} - \mu||_C \leq (5 + 1.5\sqrt{T})\beta \sqrt{\frac{\ln(6e/\beta)}{n}}.$$ 

The proof of the above lemma is the same as the proof of a similar Lemma 4 in [JO19], hence we only give a high level idea here. For any sub-collection $B'$ retaining a major portion of good batches, from Property 2, the mean of $\mu_b$ of the good batches $B' \cap B_G$ approximates $p$. Then showing that a small corruption score of $B'$ w.r.t. all subsets $S \subseteq C$ imply that the adversarial batches $B' \cap B_A$ have limited effect on $\hat{\mu}_{B'}(S)$ proves the above lemma.

Next we describe the Batch Deletion Algorithm in [JO19]. Given a sub-collection $B'$ and any subset $S \in \mathcal{F}$, the algorithm successively removes batches from $B'$, invoking Property 3 to ensure that each batch removed is adversarial with probability $\geq 0.95$. The algorithm stops when the sub-collection’s corruption score w.r.t. $S$ is at most $20\kappa_G$.

Algorithm 1 Batch Deletion

1: **Input:** Sub-Collection $B'$ of Batches, subset $S \subseteq \mathcal{F}$, med=med($\hat{\mu}(S)$), and $\kappa_G$
2: **Output:** A smaller sub-collection $B'$ of batches
3: **Comment:** The terms $\kappa_G$, $\psi_b(S)$, and $\psi_{B'}(S)$ used below are defined earlier in this section, and computing $\psi_b(S)$ and $\psi_{B'}(S)$ require med($\hat{\mu}(S)$) as input.
4: while $\psi_{B'}(S) \geq 20\kappa_G$ do
5: Select a single batch $b \in B'$ where batch $b$ is selected with probability $\frac{\psi_b(S)}{\psi_{B'}(S)}$;
6: $B' \leftarrow \{B' \setminus b\}$;
7: end while
8: return $(B')$;

Given any finite $C \subseteq \mathcal{F}$, the next algorithm 2 uses Batch Deletion to successively update $B$ and decrease the corruption score for each subset $S \in \mathcal{F}$.

Since each batch removed is adversarial with probability $\geq 0.95$ and the number of adversarial batches $\leq \beta m$, the the final sub-collection returned by the algorithm retains a large fraction of good batches.

Algorithm 2

1: **Input:** Collection $B$ of Batches, finite subset family $C \subseteq \mathcal{F}$, adversarial batches fraction $\beta$
2: **Output:** A sub-collection $B'$ of batches
3: **Comment:** The terms $\kappa_G$, $\psi_{B'}(S)$, and med($\hat{\mu}(S)$) used below are defined earlier in this section
4: $B' = B$;
5: for $S \in C$ do
6: if $\psi_{B'}(S) \geq 25\kappa_G$ then
7: med $\leftarrow$ med($\hat{\mu}(S)$);
8: $B' \leftarrow$ Batch Deletion($B'$, $S$, med);
9: end if
10: end for
11: return $(B')$;

The next lemma characterizes the algorithm’s performance. The proof of the lemma is immediate from the above discussion.

Lemma 17. Suppose Properties 1, 2 and 3 hold. Let $C \subseteq \mathcal{F}$ be a finite family of subsets. Then algorithm 2 returns a sub-collection of batches $B'$ such that with probability $\geq 1 - e^{O(\beta m)}$, $|B' \cap B_G| \geq (1 - \frac{2}{e})|B_G|$ and $\psi_{B'}(C) \leq 20\kappa_G$.

Next choose $\mathcal{F}' = \mathcal{F}$, and $C$ to be the $\epsilon$-self-cover of $\mathcal{F}$. The above Lemma, Theorem 14, Lemma 16, and Lemma 15 imply the following theorem that derives the upper bounds for robust distribution estimation from batches.
Theorem 18 (Theorem 1 restated). For any given $\beta \leq 0.4$, $\delta > 0$, $n$, $F$, and $m \geq O\left(\frac{\log(n/\beta) + \log 1/\delta}{\beta^2}\right)$, there is a non-constructive algorithm that with probability $\geq 1 - \delta$ returns a sub-collection of batches $B'$ such that $\|\bar{p}_{B'} - p\|_F \leq O\left(\beta \sqrt{\frac{\log(1/\beta)}{m/n}}\right)$.

6 Computationally efficient algorithm for $F_k$ distance

For discrete domains $\Omega = [\ell]$ and $F' = 2^\Omega$, where properties 1, 2, and 3 hold for all subsets of $D \in F'$, [JO19] derived a method that finds high corruption subsets in $F'$ in time polynomial in the domain size $\ell$. Then instead of brute force search over all $2^{|\ell|}$ subsets as in algorithm 2, they found the subsets with high corruption score efficiently and use the Batch Deletion procedure for these subsets. This lead to a computationally efficient algorithm for learning discrete distributions $p$.

To obtain a computationally efficient algorithm for learning in $F_k$ distance over $\Omega = \mathbb{R}$, and derive Theorem 2, we first reduce this problem to that of robust learning distributions over discrete domains in total variation distance and use the algorithm in [JO19].

For $\ell > 0$, let $\mathcal{I}_\ell$ be the collection of all interval partitions $I \triangleq \{I_1, \ldots, I_\ell\}$ of $\mathbb{R}$. For $I \in \mathcal{I}_\ell$, let $I^{-1} : \mathbb{R} \rightarrow [\ell]$ map any $x \in \mathbb{R}$ to the unique $j$ such that $x \in I_j$. The mapping $I^{-1}$ converts every continuous distribution $q$ over $\Omega = \mathbb{R}$ to the discrete distribution $q'$ over $\Omega = [\ell]$, where $q'(j) = q(I_j)$ for each $j \in [\ell]$. Given samples from $q$ the mapping $I^{-1}$ can be used to simulate samples from the distribution $q'$.

For a subset $D \subseteq [\ell]$ and a partition $I \in \mathcal{I}_\ell$, let

$$S'_D = \bigcup_{j \in D} I_j,$$

be the union of $I$ intervals corresponding to elements of $D$. It follows that for any $I \in \mathcal{I}_\ell$, distribution $q$ over $\mathbb{R}$, and $D \subseteq [\ell]$,

$$q(S'_D) = q^I(D).$$

For $I \in \mathcal{I}_\ell$, define the collection of intervals

$$S(I) \triangleq \{ S'_D : D \in 2^{[\ell]} \}$$

to be the family of all possible unions of intervals in $I$. Observe that $\forall I \in \mathcal{I}_\ell$ $S(I) \subseteq F_\ell$.

The next theorem describes a simple modification of a polynomial-time algorithm in [JO19], that for any $I \in \mathcal{I}_\ell$ returns a sub-collection $B^*$ of batches whose empirical distribution estimates $p$ to a small $S(I)$-distance.

Theorem 19. If Properties 1, 2, and 3 in Lemma 15 hold for $F' = F_\ell$, then for any given partition $I \in \mathcal{I}_\ell$, there is an algorithm that runs in time polynomial in partition size $\ell$, number of batches $m$, and batch-size $n$, and with probability $\geq 1 - e^{-O(\beta m)}$ returns a sub-collection of batches $B^* \subseteq B$ such that $B^* \cap B_G \geq (1 - \beta/6)|B_G|$ and

$$\|p - \bar{p}_{B^*}\|_{S(I)} \leq 100\beta \sqrt{\frac{\ln(1/\beta)}{m/n}}.$$

Proof. Suppose Properties 1–3 hold for all subsets in $F_\ell$. Since $F_\ell \supseteq S(I)$ for all $I \in \mathcal{I}_\ell$, these properties hold for all subsets in $S(I)$. For any partition $I \in \mathcal{I}_\ell$, the one-to-one correspondence $I^{-1}$ maps samples in $\mathbb{R}$ to $[\ell]$, and subsets in $S(I)$ to subsets in $2^{[\ell]}$. This implies that the three properties hold also for the transformed distribution $p'$ and the batches of discretized samples for all subsets of $2^{[\ell]}$.

Recall that $\bar{p}_{B'}$ denotes the empirical distribution induced by a sub-collection $B'$, therefore $\bar{p}'_{B'}$ denotes the empirical distribution induced by a sub-collection $B'$ over the transformed domain $[\ell]$. 

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Since these properties hold, Theorem 9 in [JO19] implies that algorithm 2 therein runs in time polynomial in the domain size $\ell$, the number of batches $m$, and the batch-size $n$, and with probability $\geq 1 - e^{-O(\beta m)}$ returns a sub-collection of batches $B^* \subseteq B$ such that $B^* \cap B_G \geq (1 - \beta/6)|B_G|$ and

$$||p^I - \bar{p}_{B^*}||_{TV} \leq 100\beta\sqrt{\frac{\ln(1/\beta)}{n}}.$$ 

Next we show that a pair of distributions $q_1$ and $q_2$ over the reals is close in $S(I)$-distance iff $q_1^I$ and $q_2^I$ are close in total variation distance. For every distribution pair $q_1, q_2$ over $\mathbb{R}$, 

$$||q_1 - q_2||_{S(I)} = \max_{S \in S(I)} |q_1(S) - q_2(S)| = \max_{S_D \in S(I)} |q_1(S^I_D) - q_2(S^I_D)| = \max_{D \in 2^\mathbb{R}} |q_1^I(D) - q_2^I(D)| = ||q_1^I - q_2^I||_{TV}.$$ 

Therefore the empirical distribution of the sub-collection $B^*$ of samples over the original domain $\mathbb{R}$ estimates $p$ in $S(I)$-distance,

$$||p - \bar{p}_{B^*}||_{S(I)} \leq 100\beta\sqrt{\frac{\ln(1/\beta)}{n}}. \quad \square$$

Next, we construct $I^* \in I_\ell$ such that $S(I^*)$ is a $\frac{2k}{\ell}$-cover of $\mathcal{F}_k$ w.r.t. the empirical measure $\bar{p}_B$.

Recall that $B$ is a collection of $m$ batches and each batch has $n$ samples. Let $s = n \cdot m$ and let $x^s = x_1, x_2, \ldots, x_s \in \mathbb{R}$ be the samples of $B$ arranged in non-decreasing order. And recall that the points $x^s$ induce an empirical measure $\bar{p}_B$ over $\mathbb{R}$, where for $S \subseteq \mathbb{R}$,

$$\bar{p}_B(S) = |\{i : x_i \in S\}|/s.$$ 

Let $\Delta \overset{\text{def}}{=} \frac{1}{s}$, and for simplicity assume that it is an integer. Construct the $\ell$-partition $I^* \overset{\text{def}}{=} \{I_1^*, \ldots, I_k^*\}$ of $\mathbb{R}$, where

$$I_j^* \overset{\text{def}}{=} \begin{cases} (-\infty, x_\Delta] & j = 1, \\ (x_{(j-1)\Delta}, x_j\Delta] & 2 \leq j < \ell, \\ (x_{(\ell-1)\Delta}, \infty] & j = \ell. \end{cases}$$

We show that $S(I^*)$ is an $2k/\ell$-cover of $\mathcal{F}_k$ w.r.t. the empirical measure $\bar{p}_B$ of points $x^s_i$.

**Lemma 20.** For any $k$, and $\ell$, $S(I^*)$ is an $\frac{2k}{\ell}$-cover of $\mathcal{F}_k$ w.r.t. $\bar{p}_B$.

**Proof.** Any set $S \in \mathcal{F}_k$ is a union of $k$ real intervals $I_1 \cup I_2 \cup \ldots \cup I_k$. Let $S^* \subseteq \mathbb{R}$ be the union of all $P_j$-intervals that are fully contained in one of the intervals $I_1, \ldots, I_k$. By definition, $S^* \in S(I^*)$, and we show that $\bar{p}_B(S \Delta S^*) \leq 2k/\ell$. By construction, $S^* \subseteq S$, hence,

$$\bar{p}_B(S \Delta S^*) = \bar{p}_B(S \setminus S^*) = \sum_{j=1}^k \bar{p}_B(I_j \setminus S^*) = \sum_{j=1}^k \left|\{x_i \in I_j \setminus S^*\}\right|/s \leq \sum_{j=1}^k 2 \cdot \frac{\Delta}{s} = \frac{2k}{\ell},$$

where the inequality follows as each $I_j \setminus S^*$ contains at most $\Delta$ points and the left and right.

Next choose $\ell = \frac{2k\sqrt{\pi}}{\beta}$ then the lemma implies that the corresponding $S(I^*)$ is an $\frac{1}{\sqrt{n}}$ cover. Combining Theorems 14 and 19, and the Lemma, we get the following theorem that implies learning in $\mathcal{F}_k$ distance.

**Theorem 21 (Theorem 2 restated).** For any given $\beta \leq 0.4$, $\delta > 0$, $n$, $k > 0$, and $m \geq \mathcal{O}\left(\frac{k\log(n/\beta) + \log 1/\delta}{\beta^2 \cdot \sqrt{n}}\right)$, there is an algorithm that runs in time polynomial in all parameters, and with probability $\geq 1 - \delta$ returns a sub-collection of batches $B'$ such that $|B' \cap B_G| \geq (1 - \frac{\delta}{\beta})|B_G|$ and

$$||\bar{p}_{B'} - p||_{\mathcal{F}_k} \leq \mathcal{O}\left(\beta \sqrt{\frac{\log(1/\beta)}{n}}\right).$$
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We prove the Lemma without the constants stated in the properties here for simplicity of the presentation. The constant stated can be obtained with a more careful calculations.

In this section, we show that the properties 1-3 hold when the family $F'$ has a finite VC-dimension.

The proof of a similar Lemma [JO19] establish that for to show that the properties 1-3 can be shown to hold for a subset $S$ if the following conditions are satisfied for $S$.

For any $\beta \in (0, 0.4], 

A.1 Proof of Lemma 15

We prove the Lemma without the constants stated in the properties here for simplicity of the presentation. The constant stated can be obtained with a more careful calculations.

In this section, we show that the properties 1-3 hold when the family $F'$ has a finite VC-dimension.

The proof of a similar Lemma [JO19] establish that for to show that the properties 1-3 can be shown to hold for a subset $S$ if the following conditions are satisfied for $S$.

For any $\beta \in (0, 0.4],

A Properties of the Collection of Good Batches

Lemma 22. Let $F$ be a VC family of subsets of $\Omega$. Then for any $\delta > 0$ and $|B_G| \geq \mathcal{O}(\frac{V_F \log(n/\beta) + \log(1/\delta)}{\beta^2})$, with probability $\geq 1 - \delta$,

$$\sup_{S,S' \in F} \max \left\{ \frac{\tilde{p}_{B_G}(S \triangle S') - p(S \triangle S')}{\sqrt{\tilde{p}_{B_G}(S \triangle S')}}, \frac{p(S \triangle S') - \tilde{p}_{B_G}(S \triangle S')}{\sqrt{p(S \triangle S')}} \right\} \leq \frac{\beta}{\sqrt{n}}.$$

Proof. Consider the collection of symmetric differences of subsets in $F$,

$$F_\triangle \triangleq \{S \triangle S': S, S' \in F\}.$$

The next auxiliary lemma bounds the shatter coefficient of $F_\triangle$.

Lemma 23. For $t \geq V_F$, $S_{F_\triangle}(t) \leq \left(\frac{t e}{V_F}\right)^{2V_F}$.

Proof. For $t \geq V_F$ and $x_1, x_2, \ldots, x_t \in \Omega$, let

$$F(x_1^t) = \{(x_1, x_2, \ldots, x_t) \cap S: S \in F\}.$$

Note that $S_F(t) = \max_{x_1, \ldots, x_t} |F(x_1^t)|$.

From the definition of shatter coefficient $|F(x_1^t)| \leq S_F(t)$. Then

$$|F_\triangle(x_1^t)| = |\{(x_1, \ldots, x_t) \triangle (x_1', \ldots, x_t'): S, S' \in F(x_1^t)\}| \leq (S_F(t))^2 \leq (\frac{t e}{V_F})^{2V_F}.$$

Recall that the sub-collection of good batches has $n|B_G|$ samples. Then applying Theorem 10 for family of subsets $F_\triangle$, and using Lemma 23, for $|B_G| \geq \mathcal{O}(\frac{V_F \log(n/\beta) + \log(1/\delta)}{\beta^2})$, with probability $\geq 1 - \delta,

$$\sup_{S \in F_\triangle} \max \left\{ \frac{\tilde{p}_{B_G}(S) - p(S)}{\sqrt{\tilde{p}_{B_G}(S)}}, \sup_{S \in F} \frac{p(S) - \tilde{p}_{B_G}(S)}{\sqrt{p(S)}} \right\} \leq \frac{\beta}{\sqrt{n}}.$$
1. For all $B_G' \subseteq B_G$, such that $|B_G'| \geq (1 - \beta/6)|B_G|$

\[
|\bar{\mu}_{B_G'}(S) - p(S)| \leq O\left(\beta \frac{\ln(1/\beta)}{n}\right),
\]

(7)

\[
\left|\frac{1}{|B_G'|} \sum_{b \in B_G'} (\bar{\mu}_b(S) - p(S))^2 - V(p(S'))\right| \leq O\left(\frac{\beta \ln(\frac{1}{\beta})}{n}\right).
\]

(8)

2. \[
\left|\{b \in B_G : |\bar{\mu}_b(S) - p(S)| \geq O\left(\sqrt{\frac{\ln(1/\beta)}{n}}\right)\}\right| \leq O(1) \cdot |B_G| \beta.
\]

(9)

3. For all $B_G' \subseteq B_G$, such that $|B_G'| \leq O(\beta)|B_G|$

\[
\sum_{b \in B_G' \setminus \{S\}} (\bar{\mu}_b(S) - p(S))^2 < O\left(\beta|B_G| \frac{\ln(1/\beta)}{n}\right),
\]

(10)

They also showed that above conditions hold for all subsets in a fixed finite collection of subsets $\mathcal{C}$, with probability $\geq 1 - \delta$, if $|B_G| \geq O\left(\frac{\log|\mathcal{C}| + \log 1/\delta}{\beta^2 \ln(1/\beta)}\right)$.

But this doesn’t give the result for subsets in a general VC class $\mathcal{F}'$ as it may have uncountable subsets.

From Corollary 13, there exist a minimal-self $\epsilon$-cover $C^*$ of $\mathcal{F}'$ w.r.t. distribution $p$ of size $O(V_{\mathcal{F}'}(\frac{4\epsilon}{p^2})^{V_{\mathcal{F}'}^*})$. Fix $\epsilon = O(\frac{4\epsilon}{p^2})$.

Therefore, for $|B_G| \geq O\left(\frac{V_{\mathcal{F}'} \log(n/\beta) + \log 1/\beta}{\beta^2 \ln(1/\beta)}\right)$, the above properties hold for all subsets in $C^*$.

To complete the proof, we show if the above conditions hold for all subsets in $C^*$, they also hold for all subsets in $\mathcal{F}'$. For subset $S \in \mathcal{F}'$ choose $S' \in C^*$ such that $p(S \setminus S') \leq \epsilon$. Existence of such a subset $S' \in C^*$ is guaranteed for all $S \in \mathcal{F}'$ as $C^*$ is an $\epsilon$-cover w.r.t. $p$.

Note that for any subset $S, S' \in \mathcal{F}'$ with $p(S \setminus S') \leq O(\frac{4\epsilon}{p^2})$, Lemma 22 implies

\[
\bar{\mu}_{B_G}(S \Delta S') \leq O\left(\frac{4\epsilon^2}{n}\right) = O(\epsilon).
\]

(11)

Then for any batch $b \in B$

\[
\bar{\mu}_b(S) - p(S) = \left(\bar{\mu}_b(S') + \bar{\mu}_b(S \setminus S') - \bar{\mu}_b(S' \setminus S)\right) - \left(p(S') + p(S \setminus S') - p(S' \setminus S)\right)
\]

\[= \left(\bar{\mu}_b(S') - p(S')\right) + \left(\bar{\mu}_b(S \setminus S') + p(S \setminus S') - p(S' \setminus S)\right) - \left(p(S \setminus S') - p(S' \setminus S)\right)\]

From the above equation we get

\[
\left|\left(\bar{\mu}_b(S) - p(S)\right) - \left(\bar{\mu}_b(S') - p(S')\right)\right| \leq \bar{\mu}_b(S \setminus S') + \bar{\mu}_b(S' \setminus S) + p(S \setminus S') + p(S' \setminus S) - \bar{\mu}_b(S \Delta S') + O(\epsilon).
\]

(12)

Next we generalise condition (7) to any subset $S \in \mathcal{F}'$.

\[
|\bar{\mu}_{B_G'}(S) - p(S)| = \left|\frac{1}{|B_G'|} \sum_{b \in B_G'} \bar{\mu}_b(S) - p(S)\right| \leq \left|\frac{1}{|B_G'|} \sum_{b \in B_G'} \left(\bar{\mu}_b(S) - p(S)\right)\right| \leq \left|\frac{1}{|B_G'|} \sum_{b \in B_G'} \left(\bar{\mu}_b(S') - p(S')\right)\right| + \left|\frac{1}{|B_G'|} \sum_{b \in B_G'} \left(\bar{\mu}_b(S \Delta S') + O(\epsilon)\right)\right|
\]

\[
\leq \left|\frac{1}{|B_G'|} \sum_{b \in B_G'} \bar{\mu}_b(S') - p(S')\right| + \left|\frac{1}{|B_G'|} \sum_{b \in B_G'} \bar{\mu}_b(S \Delta S') + O(\epsilon)\right|
\]

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\[ \leq |\bar{p}_{B_G}(S') - p(S')| + \frac{|B_G|}{|B_G|} \bar{p}_{B_G}(S \triangle S') + O(\epsilon) \]
\[ \leq O\left( \beta \sqrt{\frac{\ln(1/\beta)}{n}} \right) + \frac{1}{(1 - \beta/6)} \cdot O(\epsilon) + O(\epsilon) \]
\[ \leq O\left( \beta \sqrt{\frac{\ln(1/\beta)}{n}} \right), \]

here (a) uses (12).

Next we generalise condition (8) to subsets \( S \in \mathcal{F}' \). From equation (12) we get

\[(\bar{\mu}_b(S) - p(S))^2 \leq \left( |\bar{\mu}_b(S') - p(S')| + (\bar{\mu}_b(S \triangle S') + O(\epsilon)) \right)^2\]
\[= (\bar{\mu}_b(S') - p(S'))^2 + 2|\bar{\mu}_b(S') - p(S')|(\bar{\mu}_b(S \triangle S') + O(\epsilon)) + (\bar{\mu}_b(S \triangle S') + O(\epsilon))^2.\]

Therefore,
\[\sum_{b \in B_G'} (\bar{\mu}_b(S) - p(S))^2 - \sum_{b \in B_G'} (\bar{\mu}_b(S') - p(S'))^2 \leq \sum_{b \in B_G'} 2|\bar{\mu}_b(S') - p(S')|(\bar{\mu}_b(S \triangle S') + O(\epsilon)) + \sum_{b \in B_G'} (\bar{\mu}_b(S \triangle S') + O(\epsilon))^2\]
\[\leq 2 \left( \sum_{b \in B_G'} (\bar{\mu}_b(S') - p(S'))^2 \right) \sqrt{\sum_{b \in B_G'} (\bar{\mu}_b(S \triangle S') + O(\epsilon))^2} + \sum_{b \in B_G'} (\bar{\mu}_b(S \triangle S') + O(\epsilon))^2,\]

here the last inequality follows from Cauchy-Schwarz inequality. Next, we bound the last terms on the right in above expression.
\[\sum_{b \in B_G'} (\bar{\mu}_b(S \triangle S') + O(\epsilon))^2 \leq \sum_{b \in B_G'} (\bar{\mu}_b(S \triangle S') + O(\epsilon))(1 + O(\epsilon))\]
\[\leq 2 \cdot \left( |B_G'| O(\epsilon) + \sum_{b \in B_G} (\bar{\mu}_b(S \triangle S')) \right)\]
\[\leq 2|B_G'| \left( O(\epsilon) + \frac{|B_G|}{|B_G|} \bar{p}_{B_G}(S \triangle S') \right)\]
\[\leq |B_G'| O(\epsilon).\]

Also,
\[\sum_{b \in B_G} (\bar{\mu}_b(S') - p(S'))^2 \leq |B_G'| \left( O\left( \frac{\beta \ln(1/\beta)}{n} \right) + V(p(S')) \right)\]
\[\leq |B_G'| O\left( \frac{1}{n} \right),\]

here we used equation (3) and the fact that \( \beta \ln(\epsilon/\beta) = O(1) \). Combining the above three equations we get
\[\sum_{b \in B_G'} (\bar{\mu}_b(S) - p(S))^2 - \sum_{b \in B_G'} (\bar{\mu}_b(S') - p(S'))^2 \leq 2 \sqrt{|B_G'| O\left( \frac{1}{n} \right)} \sqrt{|B_G'| O(\epsilon)} + |B_G'| O(\epsilon) < |B_G'| O\left( \sqrt{\frac{\epsilon}{n}} \right).\]

Similarly, one can prove the other direction of the inequality to get the following
\[\left| \sum_{b \in B_G} (\bar{\mu}_b(S) - p(S))^2 - \sum_{b \in B_G} (\bar{\mu}_b(S') - p(S'))^2 \right| < |B_G'| O\left( \sqrt{\frac{\epsilon}{n}} \right).\]
And from (3) we get
\[ |V(p(S)) - V(p(S'))| \leq \frac{|p(S) - p(S')|}{n} \leq \frac{|p(S \Delta S')|}{n} \leq O\left(\frac{\epsilon}{n}\right). \]

From the above two equations we get
\[
\left| \frac{1}{|B_G'|} \sum_{b \in B_G'} (\tilde{\mu}_b(S) - p(S))^2 - V(p(S)) \right|
\leq \left| \frac{1}{|B_G|} \sum_{b \in B_G} (\tilde{\mu}_b(S') - p(S'))^2 - V(p(S')) \right| + O\left(\sqrt{\frac{\epsilon}{n}}\right) + O\left(\frac{\epsilon}{n}\right)
\leq O\left(\frac{\beta \ln(\frac{1}{\nu})}{n}\right) + O\left(\sqrt{\frac{\epsilon}{n}}\right) + O\left(\frac{\epsilon}{n}\right)
\leq O\left(\frac{\beta \ln(\frac{1}{\nu})}{n}\right),
\]
where inequality (a) uses equation (8), (b) uses \( \epsilon \leq O\left(\frac{s^2 \ln(\frac{1}{\nu})}{n}\right) \).

This completes the proof of the extension of condition (8) to subsets \( S \in \mathcal{F}' \) and in a similar fashion condition (10) can be extended.

Next, we extend condition (9) to subsets \( S \in \mathcal{F}' \).
\[
\left| \{ b \in B_G : |\tilde{\mu}_b(S) - p(S)| \geq t \} \right|
\leq \left| \{ b \in B_G : |\tilde{\mu}_b(S') - p(S')| + \tilde{\mu}_b(S \Delta S') + O(\epsilon) \geq t \} \right|
\leq \left| \{ b \in B_G : |\tilde{\mu}_b(S') - p(S')| \geq \frac{2}{3} \cdot t \} \right| + \left| \{ b \in B_G : |\tilde{\mu}_b(S \Delta S') \geq \frac{t}{3} - O(\epsilon) \} \right|
\leq \left| \{ b \in B_G : |\tilde{\mu}_b(S') - p(S')| \geq \frac{2}{3} \cdot t \} \right| + \frac{\sum_{b \in B_G} \tilde{\mu}_b(S \Delta S')}{\frac{t}{3} - O(\epsilon)}
\leq \left| \{ b \in B_G : |\tilde{\mu}_b(S') - p(S')| \geq \frac{2}{3} \cdot t \} \right| + |B_G| \frac{O(\epsilon)}{\frac{t}{3} - O(\epsilon)}
\leq \left| \{ b \in B_G : |\tilde{\mu}_b(S') - p(S')| \geq \frac{2}{3} \cdot t \} \right| + |B_G| \frac{O(\epsilon)}{t - O(\epsilon)},
\]
Choosing \( t = O\left(\sqrt{\frac{\ln(1/\beta)}{n}}\right) \) in the above equation extends condition (9) to subsets \( S \in \mathcal{F}' \).

**B Proof of Lemma 6**

*Proof.*
\[
r_p(h^*(q)) - r_p^*(\mathcal{H})
= r_p(h^*(q)) - r_p(h^*(p))
= r_p(h^*(q)) - r_q(h^*(q)) + r_q(h^*(q)) - r_q(h^*(p)) + r_q(h^*(p)) - r_p(h^*(p))
\leq r_q(h^*(q)) - r_q(h^*(p)) + 2 \sup_{h \in \mathcal{H}} |r_q(h) - r_p(h)|
\leq 2 \sup_{h \in \mathcal{H}} |r_q(h) - r_p(h)|
\leq 4\|p - q\|_{\mathcal{F}_K},
\]
here the last inequality uses (2).