A New Computational Approach for Solving Linear Bilevel Programs Based on Parameter-Free Disjunctive Decomposition

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Abstract

Linear bilevel programs (linear BLPs) have been widely used in computational mathematics and optimization in several applications. Single-level reformulation for linear BLPs replaces the lower-level linear program with its Karush-Kuhn-Tucker optimality conditions and linearizes the complementary slackness conditions using the big-M technique. Although the approach is straightforward, it requires finding the big-M whose computation is recently shown to be NP-hard. This paper presents a disjunctive-based decomposition algorithm which does not need finding the big-Ms whereas guaranteeing that obtained solution is optimal. Our experience shows promising performance of our algorithm.

Keywords: Big-M technique, bilevel optimization, disjunctive-based decomposition, linear bilevel program, linear programming.

1 Introduction

The single-level reformulation approach for solving linear bilevel programs (linear BLPs) requires finding disjunctive parameters (big-M parameters) that do not cut off
any bilevel-optimal solution. Heuristic techniques are often used to find the big-M parameters [1] although it is well-known that these techniques might fail. Recent papers [2], [3], [4], [5], and [6] [7], [8], and [9] show the computational challenges of finding the correct big-M parameters (and solving BLPs) which is shown to be strongly NP-hard [10]. This in turn makes finding the global optimal solution of linear BLPs quite challenging. Accordingly, we propose a disjunctive-based decomposition (DBD) solution algorithm for solving the linear BLPs which (1) does not require finding big-M parameters and thus (2) it is computationally advantageous. Accordingly, using our proposed DBD algorithm: no heuristic technique is needed to find big-M parameters and the obtained solution is guaranteed to be optimal to the given linear BLP under reasonable assumptions. Table 1 compares our proposed DBD algorithm with the relevant solution algorithms in the literature.

Table 1 A comparative study based on the existing literature

| Solution algorithm | [4] | [5] | [11] | [3] | [6] | DBD |
|--------------------|-----|-----|------|-----|-----|-----|
| Require tuning disjunctive parameters | No  | No  | Yes  | Yes | Yes | No  |
| Find global optimal solution | No  | No  | No   | Yes | Yes | Yes |

The rest of this paper is organized as follows. Section 2 explains the proposed DBD algorithm to solve a general linear BLP. Section 3 gives an illustrative example. Section 4 presents several general linear BLPs and Section 5 contains large-scale case studies. Section 6 concludes the paper.

2 The Proposed Disjunctive-Based Decomposition (DBD) Algorithm

The general linear BLP is given in (1) with matrices $A \in \mathbb{R}^{(n_u \times n_x)}$, $B \in \mathbb{R}^{(n_y \times 1)}$, $C \in \mathbb{R}^{(n_u \times n_x)}$, $D \in \mathbb{R}^{(n_y \times n_x)}$, $E \in \mathbb{R}^{(n_u \times 1)}$, $H \in \mathbb{R}^{(n_l \times n_x)}$, $J \in \mathbb{R}^{(n_l \times n_y)}$, and $N \in \mathbb{R}^{(n_l \times 1)}$.

Minimize $f(x,y) = A^\top x + B^\top y$  \hspace{1cm} (1a)

Subject to: $Cx + Dy \leq E$  \hspace{1cm} (1b)

$x \geq 0$  \hspace{1cm} (1c)

$x \in X$  \hspace{1cm} (1d)

$y \in S_1(x)$  \hspace{1cm} (1e)

$S_1(x) := \arg \min_y g(x,y) = G^\top y$  \hspace{1cm} (1f)

Subject to: $Hx + Jy \leq N : \lambda$  \hspace{1cm} (1g)

$y \geq 0$  \hspace{1cm} (1h)

$y \in Y$  \hspace{1cm} (1i)
The sets $X \subseteq \mathbb{R}^{(n_x \times 1)}$ and $Y \subseteq \mathbb{R}^{(n_y \times 1)}$ are polyhedral sets. Since the lower-level problem $(\min_y \{G^\top y | Hx + Jy \leq N, y \geq 0, y \in Y\})$ is a linear programming (LP) problem, the Karush-Kuhn-Tucker conditions are necessary and sufficient and the linear BLP (1) can be reformulated as the following single-level equivalent model SL.

$$\text{SL} := \text{Minimize } A^\top x + B^\top y$$

Subject to:

1. $C x + D y \leq E$ (2b)
2. $0 \leq (N - Hx - Jy) \perp \lambda \geq 0$ (2c)
3. $0 \leq (G + J^\top \lambda) \perp y \geq 0$ (2d)
4. $x \geq 0$ (2e)
5. $\lambda \geq 0$ (2f)

where $\lambda \in \mathbb{R}^{(n_l \times 1)}$ is the vector of Lagrange multipliers for $Hx + Jy \leq N$. Accordingly, if $(x, y, \lambda)$ is a global optimal solution of SL problem, then $(x, y)$ is a global optimal solution of the original linear BLP (1). We assume that the upper- and lower-level variables $x$ and $y$ are bounded, consistent with practical real-life problems, and that disjunctive parameters $\{M_1, M_2, M_3, M_4\}$ for the lower-level complementarity conditions exist and they are sufficiently large.

**Definition 1:** A set of positive scalars $\{M_1, M_2, M_3, M_4\}$ is **LP-correct** if the vector of solutions $y$ to the lower-level problem and vector of solutions $\lambda$ to its dual problem are the same as vectors of solutions $\{y, \lambda\}$ to problem (3e)-(3h).

The complementary slackness condition in SL (designated by $\perp$) have disjunctive property. Therefore it can be linearized with the big-M technique [12] with LP-correct big-M parameters $\{M_1, M_2, M_3, M_4\}$. This results an equivalent problem for SL formulated in (3).

$$\text{Minimize } f(x, y) = A^\top x + B^\top y$$

Subject to:

1. $C x + D y \leq E : \mu_1$ (3b)
2. $Hx + Jy \leq N : \mu_2$ (3c)
3. $-J^\top \lambda \leq G : \mu_3$ (3d)
4. $N - Hx - Jy \leq M_1(e_1 - u_1) : \nu_1$ (3e)
5. $\lambda \leq M_2 u_1 : \gamma_1$ (3f)
6. $G + J^\top \lambda \leq M_3(e_2 - u_2) : \nu_2$ (3g)
7. $y \leq M_4 u_2 : \gamma_2$ (3h)
8. $x \in X, y \in Y$ (3i)
9. $x \geq 0, y \geq 0, \lambda \geq 0$ (3j)

Where $u_1 \in \{0, 1\}^{(n_l \times 1)}$ and $u_2 \in \{0, 1\}^{(n_y \times 1)}$ are vectors of binary variables, the parameters $\{M_1, \ldots, M_4\}$ are LP-correct, and $e_1 \in \mathbb{R}^{(n_l \times 1)}$ and $e_2 \in \mathbb{R}^{(n_y \times 1)}$ are...
vectors of ones. In the proposed DBD approach, (3) is replaced by a subproblem (SP) in (5) and a master problem (MP) in (6) where MP and SP do not depend on any disjunctive parameter (big-M). In effect, the role of SP is to find an optimal solution for a given \( \boldsymbol{u} \). The goal of MP is to find an optimal value for a relaxation of (3) with feasibility/optimality cuts added to MP iteratively. The procedure then alternates between SP which provides an upper bound (UB) for the optimal objective value to (3) and MP which provides a lower bound (LB) for that, in order to solve (3). From this perspective, our approach has some similarities to the Benders decomposition [13].

Next, we fix the binary variables \( \boldsymbol{u}_1 = \hat{\boldsymbol{u}}_1 \) and \( \boldsymbol{u}_2 = \hat{\boldsymbol{u}}_2 \) so that (3) is an LP with its dual program as (4).

\[
\text{DP} := \text{Maximize} \quad (-\mu_1^T E - \mu_2^T N - \mu_3^T G + \nu_1^T (N - M_1 (e_1 - \hat{\boldsymbol{u}}_1)) - \\
\quad \gamma_1^T M_2 \hat{\boldsymbol{u}}_1 + \nu_2^T (G - M_3 (e_2 - \hat{\boldsymbol{u}}_2)) - \gamma_3^T M_4 \hat{\boldsymbol{u}}_2)
\]

Subject to:
\[\begin{align*}
-\mu_1^T C + (\nu_1^T - \mu_1^T) H & \leq A^T : \boldsymbol{x} \\
-\mu_2^T D + (\nu_2^T - \mu_2^T) J - \gamma_2^T & \leq B^T : \boldsymbol{y} \\
-\mu_3^T J^T - \gamma_1^T - \nu_2^T J^T & \leq : \lambda \\
\mu_1, \mu_2, \nu_1, \nu_2, \gamma_1, \gamma_2 & \geq 0
\end{align*}\]  

The DP is equivalent to the SP below as it is shown in Proposition 1. The SP is solved in Step 1 of the proposed DBD algorithm in Fig. 1. It shows that the dependence on the disjunctive parameters can be dropped under certain circumstances (Proposition 1).

\[
\text{SP}:= \text{Maximize} \quad h(\mu, \nu)
\]

Subject to:
\[\begin{align*}
-\mu_1^T C + (\nu_1^T - \mu_1^T) H & \leq A^T : \boldsymbol{x} \\
-\mu_2^T D + (\nu_2^T - \mu_2^T) J - \gamma_2^T & \leq B^T : \boldsymbol{y} \\
-\mu_3^T J^T - \gamma_1^T - \nu_2^T J^T & \leq : \lambda \\
\nu_1^T (e_1 - \hat{\boldsymbol{u}}_1) = \nu_2^T (e_2 - \hat{\boldsymbol{u}}_2) = 0 \\
\gamma_1^T \hat{\boldsymbol{u}}_1 = \gamma_2^T \hat{\boldsymbol{u}}_2 = 0 \\
\mu_1, \mu_2, \nu_1, \nu_2, \gamma_1, \gamma_2 & \geq 0
\end{align*}\]  

**Proposition 1.** Suppose LP-correct parameters \( \{M_1, M_2, M_3, M_4\} \) exist. Furthermore, assume that these parameters are large enough so that: i) \( (N - H \boldsymbol{x} - J \gamma)_j < M_1 \), \( \forall \boldsymbol{x}, \gamma, j \), and ii) \( (G + J^T \lambda)_j < M_3 \), \( \forall \lambda, j \) (element \( j \) of each vector). Then, the terms \( \nu_1^T (e_1 - \hat{\boldsymbol{u}}_1), \gamma_1^T \hat{\boldsymbol{u}}_1, \nu_2^T (e_2 - \hat{\boldsymbol{u}}_2), \) and \( \gamma_2^T \hat{\boldsymbol{u}}_2 \) in the objective function of the DP are zero at an optimal point of (3). This means that the DP and the SP have the same solution set.

**Proof.** If the binary variable \( (\hat{\boldsymbol{u}}_1)_j = 0 \), then row \( j \) of constraints (3c) \((N - H \boldsymbol{x} - J \gamma)_j \leq M_1\) is not binding and the corresponding Lagrange multiplier
Step 3: \( UB - LB > 0 \)

In the case of infeasibility of the SP, an alternative always-feasible version is formulated and solved employing artificial variables in the objective function and infeasible constraint(s) [11]. Observe that SP provides an upper bound to the optimal solution of the mixed-integer LP (MILP) problem in (3) and it does not have any big-M parameters.

In Step 2, the MP (6) is solved using the optimal values \( V_k \) of the SP and extreme points \( k \in \{0, \ldots, K\} \) and extreme rays \( l \in \{1, \ldots, L\} \) found by solving the SP in previous iterations. If the SP has an extreme point, then index of the binary variables

\[ (\nu_1)_j = 0. \] Accordingly the term \( (\nu_1)_j(e_1 - \hat{u}_1)_j \) is zero. Otherwise, the binary variable is \( (\hat{u}_1)_j = 1 \) and it is clear that \( (\nu_1)_j(e_1 - \hat{u}_1)_j \) is zero. Therefore, we have \( \nu_1^T(e_1 - \hat{u}_1) = 0 \). The same arguments are valid for the other three terms in Proposition 1. Hence, these terms can be removed from the objective function of DP and represented as constraints (5e)-(5f) in the SP.

Figure 1: The proposed DBD algorithm
corresponding to $\nu_1 > 0$ and $\nu_2 > 0$ are stored in the set $\Omega_k$ and the set $\Omega_k'$ stores index of the binary variables corresponding to $\gamma_1 > 0$ and $\gamma_2 > 0$. If the SP has an extreme ray, then the set $\Psi_l$ stores indices of the binary variables corresponding to $\nu_1 > 0$ and $\nu_2 > 0$ and the set $\Psi_l'$ stores indices of the binary variable corresponding to $\gamma_1 > 0$ and $\gamma_2 > 0$. $|\Omega_k|$, $|\Omega_k'|$, $|\Psi_l|$, and $|\Psi_l'|$ are the cardinality of these sets and (6e) is used to enforce the tree search algorithm. Lastly, in the objective function (6a), $V_k$ should be re-ordered such that $V_k$ is non-decreasing in $k$.

**MP:** Minimize $\sum_{k=0}^{K} V_k v_k$ \hspace{1cm} (6a)

Subject to: $\sum_{i \in \Omega_k} u_i + \sum_{i \in \Omega_k'} u_i \leq |\Omega_k| + |\Omega_k'| - 1 + \sum_{k \geq k'} v_{k'}; \quad \forall k \geq 1$ \hspace{1cm} (6b)

$\sum_{i \in \Psi_l} u_i + \sum_{i \in \Psi_l'} (1 - u_i) \leq |\Psi_l| + |\Psi_l'| - 1; \quad \forall l \geq 1$ \hspace{1cm} (6c)

$\sum_{k=0}^{K} v_k = 1$ \hspace{1cm} (6d)

$u_i, v_k \in \{0, 1\}$ \hspace{1cm} (6e)

As it is proved in [14], the MP provides a lower bound for the optimal solution of the MILP problem (3). In Step 3, if $UB - LB = 0$ then the DBD algorithm terminates. Otherwise, the next iteration starts from the new values obtained in Step 2. Proposed DBD algorithm is detailed in Fig. 1 and applied to a simple example in Section 3 to clearly explain the process.

### 3 Illustrative example

In this section, the proposed DBD algorithm for illustrative example is implemented in Julia 1.4.0 [15] and LP problems are solved with GNU linear programming kit (GLPK) 0.12.1 [16]. Detailed operation of our proposed DBD algorithm is shown using the illustrative linear BLP (7) below:

Minimize $f(x, y) = 0.01x - y$ \hspace{1cm} (7a)

Subject to: $0 \leq x \leq 1$ \hspace{1cm} (7b)

\[ y \in S_2(x) \] \hspace{1cm} (7c)

\[ S_2(x) := \arg \min_{y \in \mathbb{R}^2} g(x, y) = y \] \hspace{1cm} (7d)

Subject to: $0.01y - x \geq -0.5$ \hspace{1cm} (7e)

\[ y + x \geq 1 \] \hspace{1cm} (7f)
\(\lambda_1\) and \(\lambda_2\) are Lagrange variables of constraints 0.01\(y - x \geq -0.5\) and \(y + x \geq 1\), respectively. The single-level MILP reformulation of (7) is derived in (8).

\[
\text{Minimize } f(x, y) = 0.01x - y \quad (8a)
\]

\[
\begin{align*}
\text{Subject to: } & -x \geq -1 : \mu_1 \\
& 1 - 0.01\lambda_1 - \lambda_2 \geq 0 : \mu_2 \\
& 0.01y - x \geq -0.5 : \mu_3 \\
& y + x \geq 1 : \mu_4 \\
& x - 0.01y \geq 0.5 - M_1(1 - u_1) : \nu_1 \\
& -x - y \geq 1 - M_2(1 - u_2) : \nu_2 \\
& 0.01\lambda_1 + \lambda_2 \geq 1 - M_5(1 - u_3) : \nu_3 \\
& -\lambda_1 \geq -M_3 u_1 : \gamma_1 \\
& -\lambda_2 \geq -M_4 u_2 : \gamma_2 \\
& -y \geq -M_6 u_3 : \gamma_3 \\
& x, y, \lambda_i, \lambda_j, u_i, u_j, u_i \in \mathbb{R}^+ \\
& u_1, u_2, u_3 \in \{0, 1\}
\end{align*}
\]

The SP for a given \(\hat{u} = [\hat{u}_1, \hat{u}_2, \hat{u}_3]^T\) is derived in SP-1 below where \(\mu = [\mu_1, \mu_2, \mu_3, \mu_4]^T, \nu = [\nu_1, \nu_2, \nu_3]^T\), and \(\gamma = [\gamma_1, \gamma_2, \gamma_3]^T\) are vectors of Lagrange multipliers.

**SP-1:**

\[
\begin{align*}
\text{Maximize } & -\mu_1 - \mu_2 - 0.5\mu_3 + \mu_4 + 0.5\nu_1 - \nu_2 + \nu_3 \\
\text{Subject to: } & -\mu_1 - \mu_3 + \mu_4 + \nu_1 - \nu_2 \leq 0.01 : x \\
& 0.01\mu_3 + \mu_4 - 0.01\nu_1 - \nu_2 - \gamma_3 \leq -1 : y \\
& -0.01\mu_2 + 0.01\nu_1 - \gamma_1 \leq 0 : \lambda_1 \\
& -\mu_2 + \nu_3 - \gamma_2 \leq 0 : \lambda_2 \\
& \nu_1(1 - \hat{u}_1) = \nu_2(1 - \hat{u}_2) = \nu_3(1 - \hat{u}_3) = 0 \\
& \gamma_1\hat{u}_1 = \gamma_2\hat{u}_2 = \gamma_3\hat{u}_3 = 0 \\
& \mu_1, \mu_2, \mu_3, \mu_4, \nu_1, \nu_2, \nu_3, \gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}^+ 
\end{align*}
\]

SP-1 is solved for all combinations of \(\hat{u}_1\) to \(\hat{u}_3\) as shown in Table 2 only for reference. The linear BLP problem (7) is solved in four iterations as follows:

**Initialization:** Initialize the proposed DBD algorithm with \(UB = 10^4, LB = -10^4, u = [0, 0, 0]^T, K = L = 0,\) and \(V_0 = -10^4\).

**Iteration 1:** The SP-1 is solved in Step 1 which is unbounded. This results in \(L \leftarrow L + 1 = 1\) and \(\gamma_3 = 1.01 > 0\) from row 1 of Table 2. Index sets are \(\Psi_L = \emptyset\) and \(\Psi'_L = \{3\}\) with cardinalities \(|\Psi_L| = 0\) and \(|\Psi'_L| = 1\). In Step 2 employing (6c), we
Table 2  SP-1 solutions for all combinations of \{ˆu_1, ˆu_2, ˆu_3\}

| Row | ˆu_1 | ˆu_2 | ˆu_3 | ˆμ_1 | ˆμ_2 | ˆμ_3 | ˆν_1 | ˆν_2 | ˆγ_3 | ˆγ_2 | ˆγ_1 | ˆV_K | Status |
|-----|------|------|------|------|------|------|------|------|------|------|------|-------|--------|
| 1   | 0    | 0    | 0    | 0    | 0    | 0.01 | 0    | 0    | 0    | 0    | 1.01 | -∞    | UNB    |
| 2   | 0    | 0    | 1    | ×    | ×    | ×    | ×    | ×    | ×    | ×    | 1.01 | -∞    | INF    |
| 3   | 0    | 1    | 0    | 0    | 0    | 0.01 | 0    | 0    | 0    | 0    | 1.01 | -∞    | UNB    |
| 4   | 0    | 1    | 1    | 0    | 0    | 0    | 1    | 0    | 0    | 0    | 0    | -1    | OPT    |
| 5   | 1    | 0    | 0    | 0    | 0    | 0    | 0.01 | 0    | 0    | 0    | 0    | 1.01 | -∞    | UNB    |
| 6   | 1    | 0    | 1    | 0.99 | 0    | 0    | 100  | 0    | 0    | 0    | 0    | -49.99| OPT    |
| 7   | 1    | 1    | 0    | 0    | 0.01 | 0    | 0    | 0    | 0    | 0    | 1.01 | -∞    | UNB    |
| 8   | 1    | 1    | 1    | 0    | 0    | 0.99 | 0    | 0    | 0    | 0    | -0.49| OPT    |

UNB: Unbounded, INF: Infeasible, OPT: Optimal

generate the constraint \(1 \leq u_3\) and formulate the MP-1 in (10).

\[
\text{MP-1: Minimize } V_0 v_0 \\
\text{Subject to: } 1 \leq u_3 \\
v_0 = 1 \\
u_3, v_0 \in \{0, 1\}
\]

\(u_1\) and \(u_2\) are free and initialized by zero. Optimal solutions are \(v_0 = 1\) and \(\hat{u} = [0, 0, 1]^\top\). Step 3 is to go to the next iteration since \(LB = V_0 v_0 = -10^4\) and \(UB - LB > 0\).

**Iteration 2:** The problem SP-1 is infeasible with \(\hat{u} = [0, 0, 1]^\top\). Therefore, we add the term \((-s - t)\) to the objective function (9a) and change the infeasible constraint (9c) to \(0.01 \mu_3 + \mu_4 - 0.01 \nu_4 - \nu_2 - \gamma_3 + s - t \leq -1\). In this case, SP-1 becomes feasible but unbounded. This results in \(L \leftarrow L + 1 = 2\), \(\nu_3 = 100 > 0\), \(\gamma_1 = 1 > 0\), and \(\gamma_2 = 100 > 0\). Index sets are \(\Psi_L = \{3\}\) and \(\Psi'_L = \{1, 2\}\) with cardinalities \(|\Psi_L| = 2\) and \(|\Psi'_L| = 1\). In Step 2 employing (6c), the constraint \(u_3 \leq u_1 + u_2\) is generated in MP-2.

\[
\text{MP-2: Minimize } V_0 v_0 \\
\text{Subject to: } 1 \leq u_3 \\
v_0 = 1 \\
u_3 \leq u_1 + u_2 \\
u_1, u_2, u_3, v_0 \in \{0, 1\}
\]

Optimal solutions are \(v_0 = 1\) and \(\hat{u} = [1, 0, 1]^\top\). Then, Step 3 is to go to the next iteration since \(LB = V_0 v_0 = -10^4\) and \(UB - LB > 0\).
Iteration 3: First, we solve the SP-1 with \( \hat{u} = [1, 0, 1]^T \). It is bounded where \( v_1 = 100 > 0 \), \( K \leftarrow K + 1 = 1 \), and \( V_K = -49.99 \). Then, the \( UB \) is updated to \( UB \leftarrow \min\{UB = 10^4, V_K = -49.499\} = -49.99 \). Index sets are \( \Omega_K = \{1\} \) and \( \Omega'_K = \{\} \) with cardinalities \( |\Omega_K| = 1 \) and \( |\Omega'_K| = 0 \). In Step 2 employing (6b), the constraint \( u_1 \leq v_1 \) is formulated in MP-3. Indices are not changed since \( V_k \)s are already non-decreasing in \( k \) (\( V_0 \leq V_1 \)).

\[ \text{MP-3: Minimize } V_0v_0 - 49.99v_1 \]
\[ \text{Subject to: } v_0 + v_1 = 1 \]
\[ 1 \leq u_3 \leq u_1 + u_2 \]
\[ u_1 \leq v_1 \]
\[ u_1, u_2, u_3, v_0, v_1 \in \{0, 1\} \]

Optimal solutions are \( v_0 = 1, v_1 = 0 \), and \( \hat{u} = [0, 1, 1]^T \) which result in \( LB = V_0v_0 + V_1v_1 = -10^4 \). We go to the next iteration since \( UB - LB > 0 \).

Iteration 4: The SP-1 is solved with \( \hat{u} = [0, 1, 1]^T \) which is bounded so \( K \leftarrow K + 1 = 2 \), \( V_K = -1 \), \( u_2 = 1 > 0 \), and \( UB \leftarrow \min\{UB = -49.99, V_K = -1\} = -49.99 \). The index sets are \( \Omega_K = \{2\} \) and \( \Omega'_K = \{\} \) with cardinalities \( |\Omega_K| = 1 \) and \( |\Omega'_K| = 0 \). The constraint \( u_2 \leq v_2 \) is generated using (6b). \( V_k \)s are non-decreasing in \( k \) which results in MP-4.

\[ \text{MP-4: Minimize } V_0v_0 - 49.99v_1 - v_2 \]
\[ \text{Subject to: } v_0 + v_1 = 1 \]
\[ 1 \leq u_3 \leq u_1 + u_2 \]
\[ u_1 \leq v_1 + v_2 \]
\[ u_2 \leq v_2 \]
\[ u_1, u_2, u_3, v_0, v_1, v_2 \in \{0, 1\} \]

Optimal solutions are \( v_0 = v_2 = 0, v_1 = 1 \), and \( \hat{u} = [1, 0, 1]^T \). This results in \( LB = -49.99 \) and \( UB - LB = 0 \). Therefore, there is no need for more iterations and \( \hat{u} = [1, 0, 1]^T \) is optimal. The optimal solutions of MILP problem (8) (or the bilevel-optimal solutions of linear BLP (7)) are \( \hat{u} = [1, 0, 1]^T, x = 1 \), and \( y = 50 \). Note that in iterations 1 to 4, we did not have to find the LP-correct big-M parameters at all.

4 General case studies

Numerical results for eight general case studies are presented in Table 3, Table 4, Table 5, and Table 6. Number of variables and constraints in the upper- and lower-level problems are shown in Table 3. Values of the parameter matrices \( A \) to \( N \) are available in the corresponding references.

These problems are solved with both the big-M algorithm and our proposed DBD algorithm. Optimal values of objective functions (\( \hat{f}(x,y) \) and \( \hat{g}(x,y) \)) and optimal
Table 3  General case studies

| Row | Problem | \(n_x\) | \(n_y\) | \(n_u\) | \(n_l\) | Reference |
|-----|---------|---------|---------|--------|--------|-----------|
| 1   | bard1   | 2       | 3       | 2      | 6      | [17]      |
| 2   | bard2   | 2       | 2       | 2      | 5      | [17]      |
| 3   | candler | 2       | 6       | 2      | 6      | [18]      |
| 4   | anan    | 1       | 1       | 1      | 6      | [19]      |
| 5   | bard3   | 1       | 2       | 1      | 5      | [20]      |
| 6   | bard4   | 1       | 2       | 1      | 5      | [21]      |
| 7   | clark   | 1       | 2       | 1      | 5      | [22]      |
| 8   | colson  | 2       | 3       | 1      | 3      | [23]      |

\(n_x\): Number of variables in the upper-level problem, \(n_y\): Number of variables in the lower-level problem, \(n_u\): Number of constraints in the upper-level problem, \(n_l\): Number of constraints in the lower-level problem.

The solutions of the upper- and lower-level decision variables \((\hat{x} \text{ and } \hat{y})\) are shown in Table 4. As we can see for these small case studies, both the big-M algorithm and our proposed DBD algorithm achieve the same results.

Table 4  Solutions

| Row | Problem | \(\hat{f}(x, y)\) | \(\hat{g}(x, y)\) | \(\hat{x}\) | \(\hat{y}\) |
|-----|---------|------------------|------------------|-----------|-----------|
| 1   | bard1   | -26              | 3.2              | [0,0.9]   | [0,0.6,0.4] |
| 2   | bard2   | -3.25            | -4               | [2,0]     | [1.5,0]   |
| 3   | candler | -29.2            | 3.2              | [0,0.9]   | [0.06,0.4,0,0,0] |
| 4   | anan    | -49              | 17               | [16]      | [11]      |
| 5   | bard3   | -1               | 0                | [1]       | [0.0]     |
| 6   | bard4   | -2               | -1               | [0]       | [0.1]     |
| 7   | clark   | -13              | -4               | [5]       | [4.2]     |
| 8   | colson  | -14.6            | 0.3              | [0,0.65]  | [0,0.3,0] |

The vector of binary variables \(u\) is initialized using an arbitrary binary vector in the proposed DBD algorithm in Fig. 1. To study sensitivity of the proposed DBD algorithm to the initial binary vector \(\hat{u}\), eight problems listed in Table 3 are solved with different initial binary vectors. This requires solving each problem up to \(2^{(n_y+n_l)}\) times. For instance, the problem candler should be solved up to \(2^{6+6} = 4,096\) times. To compare results of different problems, 100 random initial binary vectors are selected for each problem. Number of iterations in the proposed DBD algorithm for different initial points \(\hat{u}\) is demonstrated in Table 5 and Fig. 2. These calculations are performed on a computer with 2.4 GHz 8-Core processor and memory of 32 GB. Solution time is elapsed real time or wall-time in seconds.

Number of iterations in the big-M algorithm and the proposed DBD algorithm are shown in Table 6 for problems bard1 to colson. The big-M algorithm requires tuning the big-M parameter for each problem. We observe two issues in tuning the big-M parameters: First, some problems may become infeasible for some big-M parameters. For example, problems bard1, anan, bard3, and bard4 become infeasible with big-M parameters 5, 10, 5, and 5, respectively. Second, sub-optimal solutions are found with sub-optimal big-M parameters. For instance, employing the big-M algorithm, objective
**Table 5** Solution time (wall-time) and number of iterations for different initial points \( \hat{u} \)

| Row | Problem | Iterations | Time (seconds) |
|-----|---------|------------|----------------|
|     |         | Min | Median | Mean | Max | Min | Median | Mean | Max |
| 1   | bard1   | 7   | 8     | 8.7  | 12  | 0.67 | 0.79   | 0.84  | 1.13 |
| 2   | bard2   | 1   | 6     | 7.2  | 11  | 0.18 | 0.62   | 0.73  | 1.1  |
| 3   | candler | 7   | 11    | 10.99| 20  | 0.73 | 1.08   | 1.1   | 1.91 |
| 4   | anan    | 1   | 5     | 5.46 | 8   | 0.18 | 0.57   | 0.61  | 0.91 |
| 5   | bard3   | 1   | 3     | 2.65 | 4   | 0.19 | 0.37   | 0.35  | 0.51 |
| 6   | bard4   | 1   | 6     | 5.72 | 8   | 0.17 | 0.6    | 0.58  | 0.84 |
| 7   | clark   | 1   | 12    | 10.78| 14  | 0.17 | 1.15   | 1.06  | 1.45 |
| 8   | colson  | 1   | 4     | 3.79 | 7   | 0.17 | 0.43   | 0.42  | 0.77 |

![Figure 2](chart.png)

**Figure 2** Distribution of solution time and number of iterations for different initial points

The function value for problem *bard2* with big-M parameter 6 is 1.75 instead of optimal objective value which is \(-3.25\). In addition, using large values for the big-M parameters do not guarantee finding optimal solution. For instance, the big-M parameter equal to \(5 \times 10^5\) in problem *bard1* leads to objective function value of zero while the optimal objective function value is \(-26\) which is found with \(M=10\). These eight problems in Table 6 are small enough that we can tune the big-M parameter and find the optimal solution (reported in Table 6). Although both big-M and proposed DBD algorithms find the optimal solutions, the proposed DBD algorithm still has the advantage of being big-M free. Number of iterations is reported in Table 6 for both algorithms in problems *bard1* to *colson*. As we can see, the number of iterations in the proposed DBD algorithm is decreased significantly as compared to the one in the big-M algorithm. We have observed a reduction between 60% and 93% in number of iterations for all problems except the problem *clark* where number of iterations are close (10 and 12 iterations).
Table 6 Number of iterations in the big-M algorithm and the proposed DBD algorithm

| Row | Problem | Big-M algorithm | Tuned big-M | DBD |
|-----|---------|-----------------|-------------|-----|
|     |         | Iterations     | Iterations | %   | \( f(x, y) \) |
| 1   | bard1   | 5 Inf. Inf.     | 10 25      | 8   | 68             | -26 |
| 2   | bard2   | 6 13 1.75      | 10 15      | 6   | 60             | -3.25 |
| 3   | candler | 5 28 -23       | 20 29      | 11  | 62             | -29.2 |
| 4   | anan    | 10 Inf. Inf.   | 50 21      | 5   | 76             | -49 |
| 5   | bard3   | 5 Inf. Inf.    | 10 14      | 3   | 79             | -1  |
| 6   | bard4   | 5 Inf. Inf.    | 10 15      | 6   | 60             | -2  |
| 7   | clark   | 10 31 -11      | 100 10     | 12  | -20            | -13 |
| 8   | colson  | 10 59 -14.6    | 20 57      | 4   | 93             | -14.6 |

Inf.: Infeasible.

The convergence graph of the proposed DBD algorithm for problem bard1 is shown in Fig. 3. The UB and LB values are shown for iterations 1 to 12. The UB and LB are initialized with 100 and -100, respectively. UB is decreased to -26 at iteration 9 and LB is increased to -26 at iteration 12 where the proposed DBD algorithm stops.

Figure 3 The convergence graph of the proposed DBD algorithm

5 Large-scale case studies

Performance of the proposed approach in large-scale case studies is investigated in this section. First, linear BLPs are defined with random numbers. Then, these problems are solved both with the big-M algorithm [12] as well as our proposed DBD algorithm.

5.1 Case-study parameters

Four large-scale case studies (Case1 to Case4) are defined with \( n_x = n_y = 25, 50, 75, \) and 100, respectively to study the effect of problem size on performance of the big-M and the DBD algorithms. Number of constraints are \( n_u = n_l = 15, 25, 50, \) and 75,
Table 7  The large-scale case studies

| Problem | $n_x$ | $n_y$ | $n_u$ | $n_l$ | Average number of Iterations | Average optimal value |
|---------|-------|-------|-------|-------|-------------------------------|----------------------|
|         |       |       |       |       | big-M | DBD | % | big-M | DBD | % |       |       |       |     |
| Case 1  | 25    | 25    | 15    | 15    | 840   | 287 | 65.9 | 2.4912 | 2.6007 | 4.39 |
| Case 2  | 50    | 50    | 25    | 25    | 8,947 | 371 | 95.85 | 2.4619 | 2.5022 | 1.64 |
| Case 3  | 75    | 75    | 50    | 50    | 457,002 | 457 | 99.9 | 4.6533 | 4.7974 | 3.09 |
| Case 4  | 100   | 100   | 75    | 75    | 714,711 | 688 | 99.9 | 11.6428 | 11.6735 | 0.26 |

respectively. The proposed DBD algorithm does not have any disjunctive parameter (such as the big-M). The big-M parameter is set to 100 in all case studies for the big-M algorithm. The parameter matrices are defined with random numbers as shown in (14). Normal($Mean$, $Std.$) is a function which gives a random number from normal distribution with the given mean ($Mean$) and standard deviation ($Std.$). Uniform($Min$, $Max$) is a function which gives a random number from the uniform distribution with the given minimum and maximum values ($Min$ and $Max$).

\[
A = [\text{Uniform}(0, 1)]^{(n_x \times 1)}; \quad B = [\text{Uniform}(0, 1)]^{(n_y \times 1)}; \quad (14a)
\]
\[
C = [\text{Normal}(0, 1)]^{(n_u \times n_u)}; \quad D = [0]^{(n_y \times n_u)}; \quad (14b)
\]
\[
E = [\text{Normal}(0, 1)]^{(n_x \times 1)}; \quad G = [\text{Uniform}(0, 1)]^{(n_y \times 1)}; \quad (14c)
\]
\[
H = [\text{Normal}(0, 1)]^{(n_x \times n_l)}; \quad J = [\text{Normal}(0, 1)]^{(n_y \times n_l)}; \quad (14d)
\]
\[
N = [\text{Normal}(0, 1)]^{(n_l \times 1)}; \quad (14e)
\]

The matrix $D$ is set to zero to avoid infeasible problems. Uniform distribution with values between zero and one are employed in $A$, $B$, and $G$ in the objective functions to avoid unbounded problems. Other parameters are determined from a normal distribution with mean and standard deviation equal to zero and one, respectively. Since all parameters ($A$ to $N$) are obtained using random numbers, one hundred problems are defined for each case to mitigate effect of the random values. This allows us to focus mainly on the performance of our proposed DBD algorithm for different problem sizes.

All problems in Section 4 and Section 5 are modeled in GAMS 25.1.3 [23] and solved with CPLEX 12.8.0.0 solver [24]. The optimality gap for CPLEX solver (optCR option) is set to zero. The branch and cut algorithm of the CPLEX solver is used for MIP problems. Pre-solving option of the CPLEX solver are disabled to have a fair comparison between results of the CPLEX solver and the proposed DBD algorithm [24]. All simulations for the large-scale case studies are performed using Tegner processing systems available at KTH PDC Center for High Performance Computing (HPC).

Average number of iterations and average of optimal objective values for Case 1 to Case 4 are shown in Table 7. This tables demonstrates significant reduction in number of iterations in our proposed DBD algorithm. For Case 1, the number of iterations in the DBD algorithm is decreased by 65% compared with the big-M algorithm. This is while, the big-M algorithm finds a sub-optimal solution with optimality gap of 4.39% (the DBD algorithm finds the optimal solution with zero optimality gaps in all cases). For Case 2, the DBD algorithm finds the optimal solution in 371 iterations as compared
to the 8947 iterations from the big-M algorithm (a reduction of 95.85%). Also, the big-M algorithm solution has an optimality gap of 1.64%. For Case 3 and Case 4, the reduction in the number of iterations is significant (99.9%). The big-M algorithm finds sub-optimal solutions with optimality gaps of 3.09% and 0.26%, respectively. As mentioned before, the proposed DBD algorithm finds the optimal solutions for both Case 3 and Case 4 with zero duality gap.

6 Conclusion

This letter proposes a disjunctive-based decomposition (DBD) approach to solve linear bilevel programs (linear BLPs). The proposed DBD algorithm does not need selecting LP-correct big-M parameters which is a hard (if not impossible) process. Accordingly, our obtained solutions are guaranteed to be bilevel optimal without checking the LP-correctness of the big-M parameters as long as the conditions of the Proposition 1 are met. Effectiveness of our proposed DBD algorithm is shown through 12 different case studies. All these studies confirm promising performance of our proposed DBD algorithm. Also, different applications of the proposed DBD algorithm are reported in [25] and [26] addressing electricity network investment problem and in [27] addressing optimal bidding problem of a hydropower generator.

Declarations

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References

References

[1] Bard, J.F.: Practical Bilevel Optimization: Algorithms and Applications. Springer, ??? (1998). https://doi.org/10.1007/978-1-4757-2836-1

[2] Sinha, A., Malo, P., Deb, K.: A review on bilevel optimization: From classical to evolutionary approaches and applications. IEEE Transactions on Evolutionary Computation 22(2), 276–295 (2018) https://doi.org/10.1109/TEVC.2017.2712906

[3] Davarikia, H., Barati, M.: A tri-level programming model for attack-resilient control of power grids. Journal of Modern Power Systems and Clean Energy 6(5), 918–929 (2018) https://doi.org/10.1007/s40565-018-0436-y
[4] Kovács, A.: Bilevel programming approach to demand response management with day-ahead tariff. Journal of Modern Power Systems and Clean Energy 7(6), 1632–1643 (2019) https://doi.org/10.1007/s40565-019-0569-7

[5] Xu, F., Guo, Q., Sun, H., Zhang, B., Jia, L.: A two-level hierarchical discrete-device control method for power networks with integrated wind farms. Journal of Modern Power Systems and Clean Energy 7(1), 88–98 (2019) https://doi.org/10.1007/s40565-018-0417-1

[6] Zeng, B., An, Y.: Solving bilevel mixed integer program by reformulations and decomposition. Optimization online, 1–34 (2014)

[7] Kleinert, T., Labbé, M., Plein, F., Schmidt, M.: There’s No Free Lunch: On the Hardness of Choosing a Correct Big-M in Bilevel Optimization (2019). preprint

[8] Pineda, S., Morales, J.M.: Solving linear bilevel problems using big-m's: Not all that glitters is gold. IEEE Transactions on Power Systems 34(3), 2469–2471 (2019) https://doi.org/10.1109/TPWRS.2019.2892607

[9] Pineda, S., Bylling, H., Morales, J.: Efficiently solving linear bilevel programming problems using off-the-shelf optimization software. Optimization and Engineering 19(1), 187–211 (2018)

[10] Hansen, P., Jaumard, B., Savard, G.: New branch-and-bound rules for linear bilevel programming. SIAM Journal on Scientific and Statistical Computing 13(5), 1194–1217 (1992) https://doi.org/10.1137/0913069

[11] Conejo, A.J., Castillo, E., Minguez, R., García-Bertrand, R.: Decomposition Techniques in Mathematical Programming: Engineering and Science Applications, pp. 88–89. Springer, ??? (2006)

[12] Fortuny-Amat, José, McCarl, B.: A representation and economic interpretation of a two-level programming problem. Journal of the operational Research Society 32(9), 783–792 (1981)

[13] Benders, J.F.: Partitioning procedures for solving mixed-variables programming problems. Numerische Mathematik 4, 238–252 (1962)

[14] Bean, J.C.: A bender’s approach to disjunctive programming (1992)

[15] Bezanson, J., Edelman, A., Karpinski, S., Shah, V.B.: Julia: A fresh approach to numerical computing. SIAM Review 59(1), 65–98 (2017) https://doi.org/10.1137/141000671

[16] Makhorin, A.: GLPK (GNU Linear Programming Kit). https://www.gnu.org/software/glpk

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[17] Bard, J.F., Falk, J.E.: An explicit solution to the multi-level programming problem. Computers & Operations Research 9(1), 77–100 (1982) https://doi.org/10.1016/0305-0548(82)90007-7

[18] Candler, W., Townsley, R.: A linear two-level programming problem. Computers & Operations Research 9(1), 59–76 (1982) https://doi.org/10.1016/0305-0548(82)90006-5

[19] Anandalingam, G., White, D.J.: A solution method for the linear static stackelberg problem using penalty functions. IEEE Transactions on Automatic Control 35(10), 1170–1173 (1990) https://doi.org/10.1109/9.58565

[20] Bard, J.F.: Some properties of the bilevel programming problem. Journal of optimization theory and applications 68(2), 371–378 (1991)

[21] Clark, P.A., Westerberg, A.W.: Bilevel programming for steady-state chemical process design—i. fundamentals and algorithms. Computers & Chemical Engineering 14(1), 87–97 (1990) https://doi.org/10.1016/0098-1354(90)87007-C

[22] Colson, B.: BIPA (bilevel programming with approximation methods): Software guide and test problems (2002)

[23] Corporation, G.D.: General Algebraic Modeling System (GAMS) Release 25.1.3. Washington, DC, USA (2021). http://www.gams.com/

[24] IBM ILOG: CPLEX Optimizer 12.8 (2018)

[25] Khastieva, D., Mohammadi, S., Hesamzadeh, M.R., Bunn, D.: Optimal transmission investment with regulated incentives based upon forward considerations of firm and intermittent resources with batteries. IEEE Transactions on Power Systems 36(5), 4420–4434 (2021) https://doi.org/10.1109/TPWRS.2021.3068052

[26] Khastieva, D., Hesamzadeh, M.R., Vogelsang, I., Rosellon, J.: Transmission network investment using incentive regulation: A disjunctive programming approach. Networks and Spatial Economics 20(4), 1029–1068 (2020)

[27] Moiseeva, E., Hesamzadeh, M.R.: Strategic bidding of a hydropower producer under uncertainty: Modified benders approach. IEEE Transactions on Power Systems 33(1), 861–873 (2018) https://doi.org/10.1109/TPWRS.2017.2696058