Stationary and static stellar dynamic models with axial symmetry

Gerhard Rein
Department of Mathematics, Indiana University
Bloomington, IN 47405, U.S.A.
On leave from:
Mathematisches Institut der Universität München
Theresienstr. 39, 80333 München, Germany

Abstract

We study the existence of stationary solutions of the Vlasov-
Poisson system with finite radius and finite mass in the stellar dynamics case. So far, the existence of such solutions is only known under the assumption of spherical symmetry. Using the implicit function theorem we show that certain stationary, spherically symmetric solutions can be embedded in one parameter families of stationary, axially symmetric solutions with finite radius and finite mass. In general, these new steady states have non-vanishing average velocity field, but they can also be constructed such that their velocity field does vanish, in which case they are called static.

1 Introduction

Large stellar systems such as galaxies or globular clusters are often described by a density function \( f = f(t,x,v) \geq 0 \) on phase space; \( t \in \mathbb{R} \) denotes time and \( x,v \in \mathbb{R}^3 \) denote position and velocity respectively. Under the assumption that the mass points in the ensemble, i.e., the stars, interact only by the gravitational potential which they create collectively and that in particular collisions are negligible, the time evolution of the ensemble is described by
the following nonlinear system of partial differential equations, known as the Vlasov-Poisson system:

$$\partial_t f + v \cdot \nabla_x f - \nabla_x U \cdot \nabla_v f = 0,$$

$$\triangle U = 4\pi \rho, \quad \rho(t,x) = \int f(t,x,v) dv.$$  

Here $U = U(t,x)$ denotes the gravitational potential of the ensemble and $\rho = \rho(t,x)$ denotes its spatial mass density. For simplicity we assume that all particles have the same mass, equal to unity, and set the gravitational constant equal to unity as well.

In the present investigation we are interested in the existence and properties of solutions of this system, which are independent of time. Such solutions are usually called stationary. If they have the additional property that their average velocity field vanishes, i.e., $\int vf(x,v) dv/\rho(x) = 0$, $x \in \mathbb{R}^3$, we shall call them static. If $U$ is independent of time, the energy

$$E(x,v) := \frac{1}{2} v^2 + U(x)$$

of a particle with coordinates $(x,v) \in \mathbb{R}^6$ is constant along solutions of the characteristic system

$$\dot{x} = v, \quad \dot{v} = -\nabla U(x)$$

of the Vlasov equation. Therefore, the ansatz

$$f(x,v) = \Phi(E)$$

automatically satisfies the Vlasov equation and reduces the problem of finding a stationary solution of the Vlasov-Poisson system to solving the semi-linear elliptic problem

$$\triangle U = 4\pi h_\Phi(U)$$

where $h_\Phi$ is obtained by inserting the ansatz for $f$ into the definition of $\rho$. If other invariants of the characteristic flow are known—such as the modulus of the angular momentum $|x \times v|$ in case of spherical symmetry—then $\Phi$ can depend on these additional invariants as well and the right hand side of (1.2) can become explicitly $x$-dependent. The main difficulty with this approach is to show that a solution of (1.2)—once its existence is established—leads to a stationary model with physically reasonable properties, in particular,
with finite mass and finite radius, i. e., $\rho$ is compactly supported. In [3] this program was carried out under the assumption of spherical symmetry, where it can also be shown that the distribution function $f$ must be of the form $f(x,v) = \Phi(E,|x \times v|)$ for some $\Phi$. Spherically symmetric stationary solutions are automatically static. To the best of our knowledge, the existence of static or even stationary solutions of the Vlasov-Poisson system in the stellar dynamics case with finite radius and finite mass and without spherical symmetry is open. This is interesting because for a selfgravitating fluid it is known that every static solution must be spherically symmetric, cf. [13].

In the present paper we show that this is not so for a selfgravitating ensemble as described by the Vlasov-Poisson system. In fact, we will show that every static solution $(f_0,\rho_0,U_0)$ in a certain subclass of the spherically symmetric ones obtained in [3] is embedded in continuous one-parameter families $(f_\gamma,\rho_\gamma,U_\gamma)$ of stationary solutions with axial symmetry which coincide with $(f_0,\rho_0,U_0)$ for $\gamma = 0$, have finite radius and finite mass, and are not spherically symmetric for $\gamma \neq 0$. Families of static as well as families of stationary but not static such solutions are obtained for the same spherically symmetric steady state. For the precise statement of our result we refer to the next section. The basic idea of the proof is the following: Assuming that $U$ is axially symmetric, i. e., $U(Ax) = U(x)$ for every $x \in \mathbb{R}^3$ and every rotation $A$ around, say, the $x_3$-axis, the quantity

$$P(x,v) := x_1v_2 - x_2v_1,$$

that is the $x_3$-component of the angular momentum of the particle with coordinates $(x,v) \in \mathbb{R}^6$, is conserved along characteristics. We make the ansatz

$$f(x,v) = \Phi(E,\gamma P)$$

with $\Phi$ such that $\gamma = 0$ leads to one of the spherically symmetric solutions with finite radius and finite mass obtained in [3]. Then we transform the problem (1.2) to the problem of finding zeros of an operator $T(\gamma,\cdot)$ where for $\gamma = 0$ we know a zero, namely $U_0$, and we can try to prove our result by applying the implicit function theorem. The central idea which makes this approach work is to look for $U_\gamma$ as a deformation of $U_0$, i. e., $U_\gamma(x) = U_0(g(x))$ for some diffeomorphism $g$ on $\mathbb{R}^3$, and to formulate the problem of finding zeros of $T$ over the space of such deformations instead of the space of potentials. Whereas the original problem (1.2) had to be solved on $\mathbb{R}^3$, it turns out
that one needs to know the deformation only on a compact neighbourhood of the support of the original solution \((f_0, \rho_0, U_0)\), and this provides useful compactness properties. In particular, this deformation approach is essential in proving that the derivative of \(T\) at \(U_0\) is an isomorphism. Finite radius and finite mass of the resulting stationary solutions are then immediate consequences of the corresponding properties of \((f_0, \rho_0, U_0)\).

The approach which we explained above has been used by Lichtenstein for proving the existence of slowly rotating Newtonian stars, as described by selfgravitating fluid balls, cf. [12, 13]. A translation of Lichtenstein’s approach into modern mathematical language and the framework of the implicit function theorem is due to Heilig, cf. [9], and the present paper owes much to that investigation. Our approach is analogous to [9] but the actual proofs are different, so that we decided to give a self-contained presentation of the arguments for the present case of the Vlasov-Poisson system. Our paper proceeds as follows: In the next section we formulate our result and the general framework for its proof. In particular, we define the Banach spaces which serve as domain and range for the operator \(T(\gamma, \cdot)\), introduce the deformation mappings and show how our result is obtained from the implicit function theorem. The continuous Fréchet-differentiability of \(T\) with respect to the second argument and the fact that at zero this derivative is an isomorphism are then established in Sections 3 and 4 respectively.

We conclude this introduction with some references to the now quite extensive literature on the Vlasov-Poisson system. Global existence of classical solutions has been established in [13], cf. also [14, 20, 21] and the review article [18]. For the plasma physics case, where the sign of the source term in the Poisson equation is reversed, the existence of stationary solutions, say, on bounded domains or with a fixed ion background or external force field, is much easier to obtain, cf. for example [16]. Moreover, there are now several results on the stability properties of such stationary plasmas, cf. for example [8, 17]. The stability question for the stellar dynamics case is much harder, and preliminary results can be found in [4, 22], cf. also [19]. Coming back to the topic of the present paper we mention that in [2] families of stationary solutions of the stellar dynamic Vlasov-Poisson system with axial symmetry were obtained, but these models have infinite mass and infinite radius.
2 The main result

In this section we give the precise formulation of our result and show how it is obtained from the implicit function theorem, postponing the rather technical verification of the assumptions of the latter to the last two sections. We hope that most of our notation and terminology is self-explaining, but the following needs to be introduced: The closed ball in $\mathbb{R}^3$ with center 0 and radius $R > 0$ is denoted by
\[ B_R := \{ x \in \mathbb{R}^3 \mid |x| \leq R \}, \]
and
\[ \dot{B}_R := B_R \setminus \{0\}; \]
$|x|$ denotes the Euclidean norm of $x \in \mathbb{R}^3$. Also, let
\[ S_1 := \{ x \in \mathbb{R}^3 \mid |x| = 1 \}, \]
denote the unit sphere in $\mathbb{R}^3$, and denote the line segment joining two points $x, x' \in \mathbb{R}^3$ by
\[ \overline{x, x'} := \{ \lambda x + (1 - \lambda)x' \mid \lambda \in [0, 1] \}. \]
The set of transformations which are to leave our solutions invariant is
\[ S := \{ A \in O(3) \mid A \text{ is a rotation around the } x_3\text{-axis} \]
\[ \text{or the mapping } \mathbb{R}^3 \ni (x_1, x_2, x_3) \mapsto (x_1, x_2, -x_3) \}; \]
note that in addition to axial symmetry we require reflection symmetry with respect to the plane \( \{x_3 = 0\} \). Let
\[ C_S(B_R) := \{ f \in C(B_R) \mid f(Ax) = f(x), \ A \in S, \ x \in B_R \}. \]
Clearly,
\[ \nabla f(0) = 0, \ f \in C^1(B_R) \cap C_S(B_R), \]
and this is the reason for introducing the extra reflection symmetry.

For the phase space distribution function $f$ of our stationary solution we make the ansatz
\[ f(x, v) := \phi(E)\psi(\gamma P), \ x, v \in \mathbb{R}^3 \]
where $\gamma \in \mathbb{R}$ and $U$ is assumed to be axially symmetric; the quantities $E$ and $P$ were defined in (1.1) and (1.3). Throughout our investigation, $\phi : \mathbb{R} \to [0, \infty]$ and $\psi : \mathbb{R} \to [0, \infty]$ will satisfy the following assumptions:
(φ1) \( \phi \in L^p_{loc}(\mathbb{R}) \) for some \( p > 2 \), and there exists a constant \( E_0 \in \mathbb{R} \) such that 
\( \phi(E) = 0 \) for \( E \geq E_0 \) a. e., and \( \phi(E) > 0 \) for \( E < E_0 \) a. e.

(φ2) The ansatz \( f_0(x,v) = \phi(E) \) leads to a nontrivial, static solution 
\( (f_0,\rho_0,U_0) \) of the Vlasov-Poisson system, which is spherically symmetric, i. e., \( \rho_0 \) and \( U_0 \) depend only on \( |x| \), and such that \( \rho_0 \in C^1_c(\mathbb{R}^3) \) with 
\( \text{supp} \rho_0 = B_1 \) and \( U_0 \in C^2(\mathbb{R}^3) \) with 
\( \lim_{|x| \to \infty} U_0(x) = 0 \).

(ψ) \( \psi \in C^2(\mathbb{R}) \) with \( \psi'(0) = 0 \) and \( \psi(P) = 1 \Leftrightarrow P = 0 \).

**Remark:** For \( E_0 \in \mathbb{R} \) and \( -\frac{1}{2} < \mu < \frac{7}{2} \) the function 
\( \phi(E) := \begin{cases} (E_0 - E)^\mu, & E < E_0, \\ 0, & E \geq E_0 \end{cases} \)

obviously satisfies (φ1) and leads to a spherically symmetric steady state 
\( (f_0,\rho_0,U_0) \) with finite radius and finite mass, cf. [3, Thm. 5.4]. The solution has the required regularity, without loss of generality we can assume that 
\( \text{supp} \rho_0 = B_1 \), and since \( \lim_{|x| \to \infty} U_0(x) \) exists we can take this limit to be zero by redefining \( E_0 \) accordingly. Thus a large class of the so-called polytropic steady states satisfies the assumptions (φ1) and (φ2).

**Theorem:** There exists a constant \( \gamma_0 > 0 \) such that for every \( \gamma \in ] - \gamma_0, \gamma_0 [ \) there exists a nontrivial stationary solution \( (f^\gamma,\rho^\gamma,U^\gamma) \) of the Vlasov-Poisson system with the following properties:

(i) \( f^\gamma(x,v) = \phi(E)\psi(\gamma P), \ x,v \in \mathbb{R}^3. \)

(ii) \( (f^0,\rho^0,U^0) = (f_0,\rho_0,U_0), \ (f^\gamma,\rho^\gamma,U^\gamma) \) is axially symmetric for \( |\gamma| < \gamma_0, \)

more precisely, for all \( x,v \in \mathbb{R}^3 \) and \( A \in S \) we have 
\( f^\gamma(Ax,Av) = f^\gamma(x,v), \ \rho^\gamma(Ax) = \rho^\gamma(x), \ U^\gamma(Ax) = U^\gamma(x), \)

and \( (f^\gamma,\rho^\gamma,U^\gamma) \) is not spherically symmetric for \( \gamma \neq 0 \), i. e., the above identities fail if \( S \) is replaced by \( SO(3) \).

(iii) \( \rho^\gamma \in C^1_c(\mathbb{R}^3) \) and \( U^\gamma \in C^2(\mathbb{R}^3) \) for \( |\gamma| < \gamma_0. \)

(iv) The mappings \( ] - \gamma_0, \gamma_0 [ \ni \gamma \mapsto \rho^\gamma \) and \( ] - \gamma_0, \gamma_0 [ \ni \gamma \mapsto U^\gamma \) are continuous with respect to the norms \( \| \cdot \|_{1,\infty} \) or \( \| \cdot \|_{2,\infty} \) respectively.
The question whether the axially symmetric steady states obtained above are static or not is addressed at the end of this section. In order to prove the above theorem we first deduce the semilinear elliptic problem (1.2) introduced in the introduction:

**Lemma 2.1** Let \( f \) be given by \( f(x,v) := \phi(E)\psi(\gamma P) \) for some potential \( U : \mathbb{R}^3 \to \mathbb{R} \). Then

\[
\rho(x) = h(\gamma, r(x), U(x)), \ x \in \mathbb{R}^3,
\]

where

\[
r(x) := \sqrt{x_1^2 + x_2^2}, \ x \in \mathbb{R}^3,
\]

and

\[
h(\gamma, r, u) := \begin{cases} 
2\pi \int_{u}^{E_0} \phi(E) \int_{-\sqrt{2(E-u)}}^{\sqrt{2(E-u)}} \psi(\gamma rs) ds dE, & u < E_0, \ \gamma \in \mathbb{R}, \ r \geq 0, \\
0, & u \geq E_0, \ \gamma \in \mathbb{R}, \ r \geq 0.
\end{cases}
\]

Moreover, \( h \in C^1(\mathbb{R} \times [0,\infty [ \times \mathbb{R}), \ \partial_\gamma h \in C^1(\mathbb{R} \times [0,\infty [ \times \mathbb{R}), \) and for every bounded subset \( B \subset \mathbb{R} \times [0,\infty [ \times \mathbb{R} \) there exist constants \( C > 0 \) and \( \nu \in ]0,1[ \) with

\[
|\partial_\gamma h(\gamma, r, u)| \leq Cr,
\]

\[
|h(\gamma, r, u) - h(\gamma', r, u')| \leq C(|\gamma - \gamma'|r + |u - u'|),
\]

\[
|\partial_u h(\gamma, r, u) - \partial_u h(\gamma', r, u')| \leq C(|\gamma - \gamma'| + |u - u'|^\nu)
\]

for all \((\gamma, r, u), (\gamma', r, u') \in B\).

**Proof:** The formula for \( h \) follows by introducing cylindrical coordinates with respect to \((-x_2, x_1, 0)/r(x)\) in velocity space; if \( r(x) = 0 \) then \( \psi(\gamma P) = 1 \) and one can use spherical coordinates. The function \( h \) is easily seen to be continuously differentiable with

\[
\partial_\gamma h(\gamma, r, u) = 2\pi r \int_{u}^{E_0} \phi(E) \int_{-\sqrt{2(E-u)}}^{\sqrt{2(E-u)}} s\psi'(\gamma rs) ds dE,
\]

\[
\partial_r h(\gamma, r, u) = 2\pi \int_{u}^{E_0} \phi(E) \int_{-\sqrt{2(E-u)}}^{\sqrt{2(E-u)}} s\psi'(\gamma rs) ds dE,
\]

\[
\partial_u h(\gamma, r, u) = -2\pi \int_{u}^{E_0} \left( \psi(\gamma r \sqrt{2(E-u)}) + \psi(-\gamma r \sqrt{2(E-u)}) \right) \frac{\phi(E) dE}{\sqrt{2(E-u)}}
\]

where
for $u < E_0$. The assumptions on $\psi$ imply that $|\psi'(P)| \leq C|P|$ on bounded sets containing 0, which yields the estimate for $\partial_u h$. The second estimate is straightforward. Since $\partial_u h$ is continuously differentiable with respect to $\gamma$, it is locally Lipschitz with respect to $\gamma$. As to the asserted Hölder continuity of $\partial_u h$ with respect to $u$, take $(\gamma, r, u), (\gamma, r, u') \in B, B \subset \mathbb{R} \times [0, \infty] \times \mathbb{R}$ bounded, assume $u \leq u'$, and let $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$
|\partial_u h(\gamma, r, u) - \partial_u h(\gamma, r, u')| \leq C \int_u^{u'} \frac{\phi(E)}{\sqrt{E-u}} dE
$$

$$
+ C \int_{u'}^{E_0} \frac{\phi(E)}{\sqrt{E-u}} dE
$$

$$
+ C \left| \frac{1}{\sqrt{E-u}} - \frac{1}{\sqrt{E-u'}} \right| \phi(E) dE
$$

$$
\leq C |u-u'|^{\frac{1}{q} - \frac{1}{2}} + C |u-u'|^{\frac{1}{2}} + C |u-u'|^{\frac{1}{q} - \frac{1}{2}},
$$

where we have used Hölder’s inequality for the first and the last term. Since $\frac{1}{q} - \frac{1}{2} > 0$ this completes the proof.

We note that the above estimates would simplify if we assumed that $\phi$ is Hölder continuous, but this would exclude the polytropes with $-\frac{1}{2} < \mu \leq 0$.

Next we collect some further properties of the spherically symmetric steady state $(f_0, \rho_0, U_0)$:

**Lemma 2.2** The spherically symmetric steady state $(f_0, \rho_0, U_0)$ has the following additional properties:

(a) The function $h(0, \cdot, \cdot)$ does not depend on the variable $r(x)$; we will write it as $h_0 = h_0(u)$ for simplicity.

(b) The potential $U_0$ is given by

$$
U_0(x) = -\int \frac{\rho_0(y)}{|x-y|} dy = -\frac{4\pi}{|x|} \int_0^{\frac{|x|}{s}} s^2 \rho_0(s) ds - 4\pi \int_{|x|}^{\infty} s \rho_0(s) ds, \ x \in \mathbb{R}^3,
$$

and

$$
U'_0(|x|) = \frac{4\pi}{|x|^2} \int_0^{\frac{|x|}{s}} s^2 \rho_0(s) ds, \ x \in \mathbb{R}^3.
$$

(c) $\rho_0$ is decreasing with $\rho_0(0) > 0, U_0''(0) > 0$, for every $R > 0$ there exists $C > 0$ such that $U_0'(r) \geq Cr, r \in [0, R]$, and $U_0(1) = E_0$.  

(d) \( \rho_0' \) is Hölder continuous, and \( U_0' \in C^2(\mathbb{R}^3) \) where \( \mathbb{R}^3 := \mathbb{R}^3 \setminus \{0\} \).

We identify \( \rho_0 \) and \( U_0 \) as functions of \( |x| \) with \( \rho_0 \) and \( U_0 \) as functions of \( x \); the derivative with respect to \( |x| \) is denoted by ‘.

Proof: The assertion in (a) is obvious from Lemma 2.1. Since we require that \( \lim_{|x| \to \infty} U_0(x) = 0 \), the assertion in (b) holds by uniqueness. Since \( h_0 \) is decreasing and \( U_0 \) is increasing we find that \( \rho_0 \) is also decreasing, and since the steady state \( (f_0,\rho_0,U_0) \) is assumed to be nontrivial we must have \( \rho_0(0) > 0 \). Thus actually \( U_0'(r) > 0, \ r > 0, \) and since \( U_0''(0) = \frac{4\pi}{3} \rho_0(0) > 0 \) this implies the estimate on \( U_0' \) from below. The assertion that \( U_0(1) = E_0 \) follows from the form of \( h_0 \) and the fact that by assumption \( \text{supp} \rho_0 = B_1 \). The regularity of \( U_0' \) follows from the formula above and the fact that \( \rho_0 \in C^1_c(\mathbb{R}^3) \). Finally, the Hölder continuity of \( \rho_0'(r) = \partial_u h_0(U_0(r)) U_0'(r) \) follows from Lemma 2.1.

We want to find solutions of the equation

\[ \triangle U = 4\pi h(\gamma, r(x), U), \]  

\[ (2.1) \]

and the central idea is to reformulate this as a problem of finding zeros of an operator \( T \) which acts not on the space of potentials directly but on deformations of the given spherically symmetric potential \( U_0 \). We now define the Banach spaces which will serve as domain and range of \( T \):

\[ X := \{ f \in C_S(B_3) | f(0) = 0, f \in C^1(\bar{B}_3), \exists C > 0: |\nabla f(x)| \leq C, x \in \bar{B}_3, \]  

\[ \forall x \in S_1: \lim_{t \to 0} \nabla f(tx) =: \nabla f(0x) \text{ exists, uniformly in } x \in S_1 \}, \]

which we equip with the norm

\[ ||f||_X := \sup_{x \in B_3} |\nabla f(x)|, \ f \in X, \]

and

\[ Y := \{ f \in C_S(B_3) | f(0) = 0, f \in C^1(B_3), \exists C > 0: |\nabla f(x)| \leq C|x|, x \in B_3, \]  

\[ \forall x \in S_1: \lim_{t \to 0} \nabla f(tx) t =: \nabla f(0x) \text{ exists, uniformly in } x \in S_1 \}, \]  

\[ 9 \]
which we equip with the norm 

$$
\|f\|_Y := \sup_{x \in B_3} \frac{|\nabla f(x)|}{|x|}, \quad f \in Y.
$$

For $f \in X$ the function $\nabla f(0 \cdot)$, being defined as the uniform limit of functions in $C(S_1)$, is itself in $C(S_1)$. Furthermore, since $f(0) = 0$,

$$
|f(x)| \leq \|f\|_X |x|, \; x \in B_3, \; f \in X,
$$

and the norm $\|\cdot\|_X$ is equivalent to the norm

$$
|||f|||_X := \sup_{x \in B_3} \left( \frac{|f(x)|}{|x|} + |\nabla f(x)| \right) + \sup_{x \in S_1} |\nabla f(0x)|.
$$

It easily follows that $(X, |||\cdot|||_X)$ is a Banach space. For $f \in Y$ note first that

$$
\frac{f(tx)}{t^2} := \lim_{t \searrow 0} \frac{f(tx)}{t^2} = \frac{1}{2} \nabla f(0x) \cdot x
$$

uniformly in $x \in S_1$. We have

$$
\nabla f(0 \cdot), \; \frac{f(0 \cdot)}{0^2} \in C(S_1), \; |f(x)| \leq \|f\|_Y |x|^2, \; x \in B_3,
$$

and the norm $\|\cdot\|_Y$ is equivalent to the norm

$$
|||f|||_Y := \sup_{x \in B_3} \left( \frac{|f(x)|}{|x|^2} + \frac{|\nabla f(x)|}{|x|} \right) + \sup_{x \in S_1} \left( \left| \frac{f(0x)}{0^2} \right| + \frac{|\nabla f(0x)|}{0} \right),
$$

from which it follows that $(Y, |||\cdot|||_Y)$ is a Banach space.

Using the elements in the Banach space $X$ we can deform spherically symmetric sets, e.g., the level sets of the given, spherically symmetric static solution, into axially symmetric sets in the following way:

**Lemma 2.3** For $\zeta \in X$ define

$$
g_\zeta : B_3 \to \mathbb{R}^3, \; g_\zeta(x) := x + \zeta(x) \frac{x}{|x|}, \; x \in \bar{B}_3, \; g_\zeta(0) := 0.
$$

Then there exists $r > 0$ such that for all

$$
\zeta \in \Omega := \{ \zeta \in X \mid ||\zeta||_X < r \}
$$

the following holds:
(a) \(g_\zeta : B_3 \to B_{3,\zeta} := g_\zeta(B_3)\) is a homeomorphism, \(g_\zeta : \hat{B}_3 \to \hat{B}_{3,\zeta}\) is a \(C^1\)-diffeomorphism whose Jacobian satisfies the estimate

\[|Dg_\zeta(x) - id| < \frac{1}{2}, \ x \in \hat{B}_3,\]

and for every \(x \in S_1\) the restriction

\[g_\zeta : [0, 3x] \ni y \mapsto g_\zeta(y) \in [0, |g_\zeta(3x)|]x\]

is one-to-one, onto, and preserves the natural ordering of points on the line segment \([0, 3x]\).

(b) \(\frac{1}{2}|x| \leq |g_\zeta(x)| \leq \frac{3}{2}|x|, \ x \in B_3,\) and \(g_\zeta(B_1) \subset \hat{B}_2, \ B_2 \subset g_\zeta(B_3) \subset B_4.\)

(c) \(g_\zeta(Ax) = Ag_\zeta(x), \ x \in B_3,\) and \(g_\zeta^{-1}(Ax) = Ag_\zeta^{-1}(x), \ x \in B_{3,\zeta}, \ A \in S.\)

(d) \(|Dg_\zeta^{-1}(x) - id| < \frac{1}{2}, \ x \in \hat{B}_{3,\zeta},\) and there exists a constant \(C > 0\) such that for all \(\zeta, \zeta' \in \Omega,\)

\[\frac{1}{|x|}|g_\zeta(x) - g_\zeta'(x)| + |Dg_\zeta(x) - Dg_\zeta'(x)| \leq C\|\zeta - \zeta'\|_X, \ x \in \hat{B}_3,\]

and

\[|g_\zeta^{-1}(x) - g_\zeta'^{-1}(x)| \leq C\|\zeta - \zeta'\|_X|x|, \ x \in B_2.\]

**Proof:** On \(\hat{B}_3\) we have for \(i, j = 1, 2, 3,\)

\[\partial_{x_i}g_{\zeta,j}(x) = \delta_{ij} + \partial_{x_i} \zeta(x) \frac{x_j}{|x|} + \frac{\zeta(x)}{|x|} \left(\delta_{ij} - \frac{x_i x_j}{|x|^2}\right)\] \hspace{1cm} (2.2)

which implies that

\[|Dg_\zeta(x) - id| \leq 3\|\zeta\|_X < \frac{1}{2}, \ x \in \hat{B}_3,\]

provided \(\|\zeta\|_X < 1/6.\) Using the inverse function theorem we obtain the first two assertions in (a). Since \(g_\zeta(y) \in [0, \infty]y\) for every \(y \in B_3\) the remaining assertion in (a) follows. The assertions in (b) are obvious, provided \(r > 0\) is chosen sufficiently small, and so are the assertions in (c). As to (d), the first
assertion follows by choosing \( r \) still smaller, since \( Dg^{-1}_\zeta(x) = (Dg_\zeta)^{-1}(g_\zeta^{-1}(x)) \)

The estimate for \( g_\zeta - g_\zeta' \) is immediate from the definition of \( g_\zeta \). The estimate for \( Dg_\zeta - Dg_\zeta' \) follows from (2.2). Finally, \( x \in \bar{B}_2 \) implies that \( x \in g_\zeta(B_3) \cap g_\zeta'(B_3) \), and there exists \( y \in \bar{B}_3 \) such that \( x = g_\zeta'(y) \). Thus

\[
|g_\zeta^{-1}(x) - g_\zeta'^{-1}(x)| = |g_\zeta^{-1}(g_\zeta'(y)) - y| = |g_\zeta^{-1}(g_\zeta'(y)) - g_\zeta^{-1}(g_\zeta(y))| \\
\leq 2|g_\zeta(y) - g_\zeta'(y)| \leq 2\|\zeta - \zeta'\|_X |y| \leq 4\|\zeta - \zeta'\|_X |x|
\]

by the mean value theorem, the estimate for \( Dg_\zeta^{-1} \) which we already established and the fact that \( g_\zeta(y), g_\zeta'(y) \subset g_\zeta(\bar{B}_3) \).

We want to find solutions of the reduced problem (2.1) of the form

\[
U(x) = U_\zeta(x) := U_0(g_\zeta^{-1}(x)), x \in B_{3,\zeta},
\]

for some \( \zeta \in \Omega \). Of course \( U \) will have to be defined on all of \( \mathbb{R}^3 \), but this will be easy once we have it on \( B_{3,\zeta} \). Using the fundamental solution of the Poisson equation we integrate (2.1) and transform our problem to that of solving the equation

\[
U_0(x) + \int_{B_{3,\zeta}} \frac{h(\gamma, r(y), U_0(g_\zeta^{-1}(y)))}{|g_\zeta(x) - y|} \, dy = 0, x \in B_3;
\]

observe that \( g_\zeta \) is invertible. It turns out that we can avoid the dependence of the domain of integration on \( \zeta \), and also that the operator above is not quite the right thing yet. We are now in the position to give the proof of the theorem:

**Proof of the Theorem:** For \( \zeta \in \Omega \) and \( \gamma \in \mathbb{R} \) we define

\[
T(\gamma, \zeta)(x) := U_0(x) + \int_{B_2} \frac{h(\gamma, r(y), U_0(g_\zeta^{-1}(y)))}{|g_\zeta(x) - y|} \, dy \\
- U_0(0) - \int_{B_2} \frac{h(\gamma, r(y), U_0(g_\zeta^{-1}(y)))}{|y|} \, dy, x \in B_3.
\]

(2.3)

Assume we already know that this defines a continuous operator \( T : [-1,1[ \times \Omega \to Y \), that \( T \) is continuously Fréchet differentiable with respect to \( \zeta \), and that

\[
\partial_\zeta T(0,0) : X \to Y
\]
is an isomorphism; that all this is indeed the case is shown in Sections 3 and 4, cf. Proposition 3.1 and Proposition 4.1. Note that by definition of $Y$ we must have $T(\gamma, \zeta)(0) = 0$ which is the reason for subtracting the constant term in the definition of $T$. By assumption (\phi 2) we know that $T(0,0) = 0$; note that $g_0 = id$ and that $\text{supp} \rho_0 = \text{supp} h_0 \circ U_0 = B_1 \subset B_2$. By the implicit function theorem there exists a constant $\gamma_0 \in ]0,1[$ and a continuous mapping 

$$\gamma \mapsto \zeta \in \Omega$$

such that 

$$T(\gamma, \zeta) = 0, \gamma \in ]-\gamma_0, \gamma_0[$$

and $\zeta^0 = 0$, cf. [3, Theorem 15.1]. Let $\zeta = \zeta^\gamma$ for some $\gamma \in ]-\gamma_0, \gamma_0[$ and define 

$$\rho_\zeta(x) := h(\gamma, r(x), U_0(g_\zeta^{-1}(x))), x \in B_2.$$ (2.4) 

Then $\rho_\zeta \in C_s(B_2)$; at the moment the differentiability at $x = 0$ is not yet obvious. By Lemma 2.1 $\rho_\zeta(x) > 0$ if and only if $U_0(g_\zeta^{-1}(x)) < E_0$ which by Lemma 2.2 is equivalent to $|g_\zeta^{-1}(x)| < 1$. Therefore, by Lemma 2.3 

$$\text{supp} \rho_\zeta = g_\zeta(B_1) \subset \mathring{B}_2,$$

and we can extend $\rho_\zeta$ by 0 to all of $\mathbb{R}^3$, obtaining an element of $C_c(\mathbb{R}^3)$ with $\text{supp} \rho_\zeta \subset \mathring{B}_2$ which for the moment need not satisfy (2.4) everywhere. The equation $T(\gamma, \zeta) = 0$ can now be written as 

$$U_0(x) = -\int_{B_2} \frac{\rho_\zeta(y)}{|g_\zeta(x) - y|} dy + C, x \in B_3,$$

or 

$$U_0(g_\zeta^{-1}(x)) = -\int_{B_2} \frac{\rho_\zeta(y)}{|x - y|} dy + C, x \in B_3, \zeta,$$

where 

$$C := U_0(0) + \int_{B_2} \frac{\rho_\zeta(y)}{|y|} dy.$$ 

Now define 

$$U_\zeta(x) = -\int_{\mathbb{R}^3} \frac{\rho_\zeta(y)}{|x - y|} dy + C, x \in \mathbb{R}^3.$$
Clearly, \( U_\zeta \in C^1(\mathbb{R}^3) \) with

\[
U_\zeta(x) = U_0(g_\zeta^{-1}(x)), \quad x \in B_2 \subset B_{3,\zeta}. \tag{2.5}
\]

This immediately implies that \( \rho_\zeta \in C^1(\mathbb{R}^3) \) and \( U_\zeta \in C^2_b(\mathbb{R}^3) \) with \( \Delta U_\zeta = 4\pi \rho_\zeta \) on \( \mathbb{R}^3 \). On the other hand,

\[
\Delta U_\zeta(x) = 4\pi h(\gamma, r(x), U_\zeta(x)), \quad x \in B_2 \subset B_{3,\zeta}. \tag{2.6}
\]

If fact the latter equation holds on all of \( \mathbb{R}^3 \). To see this we have to show that \( \rho_\zeta(x) = h(\gamma, r(x), U_\zeta(x)), \quad x \in \mathbb{R}^3 \), that is, \( U_\zeta(x) > E_0 \) for \( x \in \mathbb{R}^3 \setminus g_\zeta(B_1) \). We know that

\[
\lim_{|x| \to \infty} U_\zeta(x) = C, \quad U_\zeta(x) = E_0, \quad x \in \partial g_\zeta(B_1), \quad U_\zeta(x) > E_0, \quad x \in B_2 \setminus g_\zeta(B_1),
\]

where we used the identity \( \Delta U_\zeta = 0 \) for \( x \in \mathbb{R}^3 \setminus g_\zeta(B_1) \), and the fact that \( U_0 \) is strictly increasing as a function of \( |x| \) with \( U_0(1) = E_0 \). The assumption \( C \leq E_0 \) would contradict the maximum principle. Thus, \( C > E_0 \), and again by the maximum principle, \( U_\zeta > E_0 \) on \( \mathbb{R}^3 \setminus g_\zeta(B_1) \). Therefore, \( \Delta U_\zeta(x) = 0 \), \( x \in \mathbb{R}^3 \setminus g_\zeta(B_1) \),

and

\[
\lim_{|x| \to \infty} U_\zeta(x) = C, \quad U_\zeta(x) = E_0, \quad x \in \partial g_\zeta(B_1), \quad U_\zeta(x) > E_0, \quad x \in B_2 \setminus g_\zeta(B_1),
\]

then the assertions (i)–(iii) of the theorem are established, except for the assertion that the solution is not spherically symmetric for \( \gamma \neq 0 \). To see the latter, choose \( x \in \mathbb{R}^3 \) with \( \rho^\gamma(x) > 0, \quad x_1 \neq 0, \quad x_2 = x_3 = 0 \). There must then exist \( \eta \neq 0 \) such that \( \frac{1}{2}\eta^2 + U^\gamma(x) < E_0 \). Let \( v = (0, 0, \eta) \) and \( v' = (0, \eta, 0) \). Then there exists a rotation \( A \) around the \( x_1 \)-axis such that \( Av = v' \), and clearly, \( Ax = x \). But since \( E(x, v) = E(x, v') \) and \( P(x, v) = 0 \neq x_1\eta = P(x, v') \) we have

\[
f^\gamma(x, v) = \phi(E(x, v))\psi(\gamma P(x, v)) = \phi(E(x, v)) = \phi(E(x, v')) \neq \phi(E(x, v'))\psi(\gamma P(x, v')) = f^\gamma(x, v'),
\]

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provided $\gamma \neq 0$. The continuity properties asserted in (iv) follow from the fact that $\zeta_\gamma$ depends continuously on $\gamma$ with respect to the norm $\| \cdot \|_X$. First the estimate

$$|U^\gamma(x) - U'^\gamma(x)| \leq \|U'_0\|_\infty \|g_{\zeta_\gamma}(x) - g'_{\zeta_\gamma}(x)| \leq C \|\zeta_\gamma - \zeta'_\gamma\|_X, \quad x \in B_2,$$

and the relation $\rho^\gamma(x) = h(\gamma, r(x), U^\gamma(x))$ imply that $\rho^\gamma$ depends continuously on $\gamma$ with respect to $\| \cdot \|_\infty$. Since

$$U^\gamma(x) = -\int_{B_2} \frac{\rho^\gamma(y)}{|x-y|} dy + U_0(0) + \int_{B_2} \frac{\rho^\gamma(y)}{|y|} dy, \quad x \in \mathbb{R}^3,$$

this implies that $U^\gamma$ depends continuously on $\gamma$ with respect to $\| \cdot \|_{1,\infty}$. Differentiating the above formula for $\rho^\gamma$ we obtain the asserted continuity of $\rho^\gamma$ with respect to $\| \cdot \|_{1,\infty}$ and thus also of $U^\gamma$ with respect to $\| \cdot \|_{2,\infty}$, and the proof of the theorem is complete. 

**Remark:**

(a) For fixed $\psi$ the family $(f^\gamma, \rho^\gamma, U^\gamma)$ is in a neighbourhood of $\gamma = 0$ unique. However, different functions $\psi$ give different families of stationary solutions in which $(f_0, \rho_0, U_0)$ is embedded.

(b) The mass current density is given by

$$j^\gamma(x) := \int v f^\gamma(x,v) dv = 2\pi \int_{U^\gamma(x)} \phi(E) \sqrt{\frac{2(E-U^\gamma(x))}{2(E-U^\gamma(x))}} s \psi(\gamma r(x)s) ds dE e_t(x), \quad x \in \mathbb{R}^3,$$

where

$$e_t(x) := \frac{1}{r(x)}(-x_2, x_1, 0)$$

denotes the unit vector field tangent to the orbits of points under counterclockwise rotations around the $x_3$-axis; note that the integral above vanishes on the $x_3$-axis, where $e_t$ is not defined. Now the average velocity $j^\gamma/\rho^\gamma$ of the steady state vanishes identically if $\psi$ is an even function, in which case we obtain a family of static, axially symmetric solutions. In general, for example if $\psi(-P) < 1 < \psi(P)$, $P > 0$, the average velocity does not vanish, and we obtain a stationary stellar
system which rotates around the $x_3$-axis. It is also easy to see that the average velocity vanishes at the boundary of the support of $\rho^\gamma$. As already mentioned in the introduction, static solutions of the Euler-Poisson system are necessarily spherically symmetric. As we now see, a corresponding result does not hold for the Vlasov-Poisson system.

(c) It would require very little additional effort to show that $T$ is continuously differentiable also with respect to $\gamma$. This would imply that the family $(f^\gamma, \rho^\gamma, U^\gamma)$ depends on $\gamma$ in a differentiable way with respect to the appropriate norms, cf. [3, Cor. 15.1].

(d) If in our ansatz $f$ depends only on the particle energy $E$ then the right hand side of (1.2) does not explicitly depend on $x$. One can then apply a result by GIDAS, NI, and NIRENBERG, cf. [3, Theorem 4], to conclude that under mild regularity assumptions a corresponding steady state with finite radius and finite mass must be spherically symmetric with respect to some point in $\mathbb{R}^3$. Therefore, it is necessary to include further invariants in the ansatz in order to obtain stationary models which are not spherically symmetric.

3 The Fréchet-differentiability of $T$

The aim of this section is to prove the following result:

**Proposition 3.1** The mapping $T:]-1,1[ \times \Omega \to Y$ defined in (2.3) is continuous and continuously Fréchet-differentiable with respect to $\zeta$ with Fréchet derivative

$$[\partial_\zeta T(\gamma, \zeta)\xi](x) = -\int_{B_2} \left( \frac{1}{|g_\zeta(x)-y|} - \frac{1}{|y|} \right) \partial_u h(\gamma, r(y), U_\zeta(y)) \nabla U_\zeta(y) \cdot \frac{g_{\zeta}^{-1}(y)}{|g_{\zeta}^{-1}(y)|} \xi(g_{\zeta}^{-1}(y)) \, dy$$

$$-\int_{B_2} \frac{g_\zeta(x)-y}{|g_\zeta(x)-y|^3} h(\gamma, r(y), U_\zeta(y)) \, dy \cdot \frac{x}{|x|} \xi(x), \; x \in B_3,$$

where $\gamma \in ]-1,1[,$ $\zeta \in \Omega,$ $\xi \in X,$ and $U_\zeta(y) := U_0(g_{\zeta}^{-1}(y)),$ $y \in B_2$.

In order to prove this result we need more information on the elements of the space $X$ and the deformation mappings constructed in Lemma 2.3.
Lemma 3.2 Let $\zeta \in \Omega$. Then the following holds:

(a) $|\zeta(x) - \zeta(x')| \leq \|\zeta\|_X |x-x'|$, $x,x' \in B_3$.

(b) For $x \in S_1$ the mapping $[0,3] \ni t \mapsto \zeta(tx)$ is continuously differentiable, and $\lim_{t \searrow 0} \frac{\zeta(tx)}{t} = \nabla \zeta(0x) \cdot x$, uniformly in $x \in S_1$.

(c) The mapping $[0,3] \times S_1 \ni (t,x) \mapsto \nabla \zeta(tx)$ is uniformly continuous.

(d) For $x \in S_1$ the limits $\lim_{t \searrow 0} g_\zeta(tx) = : g_\zeta(0x)$ and $\lim_{t \searrow 0} Dg_\zeta(tx) = : Dg_\zeta(0x)$ exist, uniformly in $x \in S_1$.

Proof: The assertion in (a) follows easily by distinguishing the cases $0 \in \overline{x,x'}$ and $0 \notin \overline{x,x'}$. As to (b),

$$\frac{d}{dt} \zeta(tx) = \nabla \zeta(tx) \cdot x \to \nabla \zeta(0x) \cdot x, \ t \searrow 0,$$

by definition of the space $X$, and the rest follows. The assertion in (c) follows from the fact that $\nabla \zeta \in C(\hat{B}_3)$ and $\nabla \zeta(tx) \to \nabla \zeta(0x)$ uniformly in $x \in S_1$ as $t \searrow 0$. The assertions in (d) are easy consequences of the definitions of $g_\zeta$ and the space $X$, together with (2.2).

Next we establish some estimates for the spatial density induced by a deformation of the potential $U_0$:

Lemma 3.3 For $\gamma \in [-1,1]$ and $\zeta \in \Omega$ let

$$\rho_{\gamma,\zeta}(x) := h(\gamma,r(x),U_0(g_\zeta^{-1}(x))), \ x \in B_2.$$

Then the following holds:

(a) $\rho_{\gamma,\zeta} \in C_S(B_2) \cap C^1(B_2)$ with $\text{supp} \rho_{\gamma,\zeta} \subset \hat{B}_2$, and there exists a constant $C > 0$ such that for all $\gamma \in [-1,1]$ and $\zeta \in \Omega$,

$$|\nabla \rho_{\gamma,\zeta}(x)| \leq C|x|, \ x \in B_2.$$

(b) There exists a constant $C > 0$ such that for all $\gamma,\gamma' \in [-1,1]$ and $\zeta,\zeta' \in \Omega$,

$$|\rho_{\gamma,\zeta}(x) - \rho_{\gamma',\zeta'}(x)| \leq C\left(|\gamma - \gamma'| + \|\zeta - \zeta'|_X\right)|x|, \ x \in B_2.$$
Proof: Lemma 2.3 and Lemma 2.1 imply that \( \rho = \rho_{\gamma, \zeta} \in C_S(B_2) \cap C^1(\hat{B}_2) \).
For \( x \in \hat{B}_2 \) we have
\[
\nabla \rho_{\gamma, \zeta}(x) = \partial_s h(\gamma, r(x), U_0(g^{-1}_\zeta(x))) \nabla r(x) \\
+ \partial_h(\gamma, r(x), U_0(g^{-1}_\zeta(x))) \nabla U_0(g^{-1}_\zeta(x)) \cdot D g^{-1}_\zeta(x),
\]
and Lemma 2.1, the fact that \( U_0 \in C^2(\mathbb{R}^3) \) with \( \nabla U_0(0) = 0 \), and Lemma 2.3 imply the estimate
\[
|\nabla \rho(x)| \leq C|x| + C|\nabla U_0(g^{-1}_\zeta(x))| \leq C|x| + C|g^{-1}_\zeta(x)| \leq C|x|, \ x \in \hat{B}_2;
\]
note that the range of \( U_0 \) is bounded. Since \( x \not\in g_\zeta(B_1) \) implies \( U_0(g^{-1}_\zeta(x)) > E_0 \) and thus \( \rho(x) = 0 \), the assertion on the support of \( \rho \) follows by Lemma 2.3 (b). The assertion in (b) is immediate from Lemma 2.1 and Lemma 2.3 (d).

We shall need the following assertions on Newtonian potentials:

**Lemma 3.4** Let \( \sigma \in C_S(B_2) \) be such that
\[
c_\sigma := \sup_{x \in \hat{B}_2} \frac{\sigma(x)}{|x|} < \infty
\]
and define
\[
V_\sigma(x) := -\int_{B_2} \frac{\sigma(y)}{|x-y|} dy, \ x \in \mathbb{R}^3.
\]
Then \( V_\sigma \in C^1(\mathbb{R}^3) \), and there exists \( C > 0 \) such that for all \( \sigma \) as above the following estimates hold:

(a) \(|\nabla V_\sigma(x)| \leq C c_\sigma |x|, \ x \in \mathbb{R}^3\),

(b) \(|\nabla V_\sigma(g_\zeta(x)) - \nabla V_\sigma(g_\zeta'(x))| \leq C c_\sigma \|\zeta - \zeta'\|^{1/2} |x|, \ x \in B_3, \ \zeta, \zeta' \in \Omega\).

**Proof:** For \( \sigma \in C_S(B_2) \) we have \( \nabla V_\sigma(0) = 0 \) and thus
\[
|\nabla V_\sigma(x)| \leq \left| \int_{B_2} \left( \frac{x-y}{|x-y|^3} + \frac{y}{|y|^3} \right) \sigma(y) dy \right|, \ x \in \mathbb{R}^3.
\]
Let \( x \neq 0 \) and \( r := 2|x| \). We obtain the estimate
\[
|\nabla V_\sigma(x)| \leq c_\sigma \int_{B_2 \setminus B_r} \frac{x-y}{|x-y|^3} \frac{y}{|y|^3} |y| dy + c_\sigma \int_{B_2 \cap B_r} \left( \frac{1}{|x-y|^2} + \frac{1}{|y|^2} \right) |y| dy
\]
\[
=: I_1 + I_2.
\]
For almost every $y \in B_2$ there exists $\tau \in [0, 1]$ such that
\[
\left| \frac{x - y}{|x - y|^3} + \frac{y}{|y|^3} \right| \leq |x| \frac{4}{|\tau x - y|^3}
\]
and since for $|y| \geq r$,
\[
|\tau x - y| \geq |y| - |x| = |y| - \frac{r}{2} \geq \frac{|y|}{2},
\]
we can estimate the first term as
\[
I_1 \leq C c_\sigma |x| \int_{B_2 \setminus \{y\}} \frac{1}{|y|^2} dy = C c_\sigma |x|;
\]
constants denoted by $C$ may change their value from line to line or even within one and the same line. For the second term we have
\[
I_2 \leq 2c_\sigma \left( \int_{B_r} \frac{1}{|x - y|^2} dy + \int_{B_{r_2}} \frac{1}{|y|^2} dy \right) \leq 4c_\sigma \int_{B_r} \frac{1}{|y|^2} dy = C c_\sigma r = C c_\sigma |x|,
\]
and the proof of part (a) is complete. As to (b) we have
\[
|\nabla V_\sigma(g_\zeta(x)) - \nabla V_\sigma(g_\zeta'(x))| \leq c_\sigma \int_{B_2} \left| \frac{g_\zeta(x) - y}{|g_\zeta(x) - y|^3} - \frac{g_\zeta'(x) - y}{|g_\zeta'(x) - y|^3} \right| |y| dy.
\]
Let $x \in \hat{B}_3$ and $\delta := \|\zeta - \zeta'\|_X < 1, r_1 := 2\delta|x|$, and $r_2 := 4|x| > r_1$; recall that we chose the radius of the set $\Omega$ less than $1/3$. We split the integral above into three parts, $I_1$, $I_2$, and $I_3$, according to the decomposition
\[
B_2 = \left( B_2 \setminus B_{r_2} \right) \cup \left( (B_2 \cap B_{r_2}) \setminus B_{r_1}(g_\zeta(x)) \right) \cup \left( B_2 \cap B_{r_1}(g_\zeta(x)) \right).
\]
As to $I_1$ we find for almost every $y \in B_2$ a $\tau$ between $\zeta(x)$ and $\zeta'(x)$ such that
\[
\left| \frac{g_\zeta(x) - y}{|g_\zeta(x) - y|^3} - \frac{g_\zeta'(x) - y}{|g_\zeta'(x) - y|^3} \right| \leq \frac{C}{|x + \tau \frac{x}{|x|} - y|^3} |\zeta(x) - \zeta'(x)|;
\]
note that both $g_\zeta(x)$ and $g_\zeta'(x)$ lie on the line $\mathbb{R}x$. Since
\[
|\zeta(x) - \zeta'(x)| \leq \|\zeta - \zeta'\|_X |x| = \delta|x|,
\]

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and for \(|y| \geq r_2\),

\[
\left| x + \frac{x}{|x|} - y \right| = \left| y - g_\zeta(x) + (\zeta(x) - \tau) \frac{x}{|x|} \right| \geq |y| - |g_\zeta(x)| - |\zeta(x) - \zeta'(x)| \\
\geq |y| - \frac{3}{2}|x| - \delta|x| \geq |y| - \frac{5}{2}|x| = |y| - \frac{5}{8}r_2 \geq \frac{3}{8}|y|,
\]

we find the estimate

\[
I_1 \leq Cc_\sigma \|\zeta - \zeta'\|_X |x| \int_{B_2} \frac{dy}{|y|^2} = Cc_\sigma \|\zeta - \zeta'\|_X |x|.
\]

To estimate the second term \(I_2\) we start like for \(I_1\), but for \(y \notin B_{r_1}(g_\zeta(x))\) obtain the estimate

\[
\left| x + \frac{x}{|x|} - y \right| = \left| y - g_\zeta(x) + (\zeta(x) - \tau) \frac{x}{|x|} \right| \geq |y - g_\zeta(x)| - |\zeta(x) - \zeta'(x)| \\
\geq |y - g_\zeta(x)| - \delta|x| \geq \frac{1}{2}|y - g_\zeta(x)|.
\]

On the other hand for \(y \in B_{r_2}\) we have

\[
|y - g_\zeta(x)| \leq |y| + \frac{3}{2}|x| \leq r_2 + \frac{3}{8}r_2 \leq 2r_2,
\]

and

\[
I_2 \leq Cc_\sigma \delta |x| \int_{B_{2r_2}(g_\zeta(x)) \setminus B_{r_1}(g_\zeta(x))} \frac{1}{|g_\zeta(x) - y|^3} dy = Cc_\sigma \delta |x| \frac{4\pi}{r_1} \ln \frac{2r_2}{r_1} \\
= Cc_\sigma \delta |x| \ln \frac{4}{\delta} \leq Cc_\sigma \delta^{1/2} |x| = Cc_\sigma \|\zeta - \zeta'\|_X^{1/2} |x|.
\]

As to the third term we have

\[
I_3 \leq 2c_\sigma \left( \int_{B_{r_1}(g_\zeta(x))} \frac{dy}{|g_\zeta(x) - y|^2} + \int_{B_{r_1}(g_\zeta(x))} \frac{dy}{|g_\zeta'(x) - y|^2} \right) \\
\leq 4c_\sigma \int_{B_{r_1}(g_\zeta(x))} \frac{dy}{|g_\zeta(x) - y|^2} = Cc_\sigma r_1 = Cc_\sigma \delta |x| = Cc_\sigma \|\zeta - \zeta'\|_X |x|,
\]

and the proof of part (b) is complete.

We are now ready to prove part of the assertion in Proposition 3.1, namely:
Assertion 1: For $\gamma \in [-1,1]$ and $\zeta \in \Omega$ we have $T(\gamma, \zeta) \in Y$, and the mapping
$T : [-1,1] \times \Omega \to Y$ is continuous.

Proof: Let

$$V_{\gamma, \zeta}(x) := -\int_{B_2} \frac{\rho_{\gamma, \zeta}(y)}{|x-y|} dy, \ x \in \mathbb{R}^3, \ (\gamma, \zeta) \in [-1,1] \times \Omega.$$ 

The assertions in Lemma 3.3 (a) imply that $V_{\gamma, \zeta} \in C^2(\mathbb{R}^3)$ with $\nabla V_{\gamma, \zeta}(0) = 0$. Since

$$T(\gamma, \zeta)(x) = U_0(x) - V_{\gamma, \zeta}(g_\zeta(x)) - U_0(0) + V_{\gamma, \zeta}(0), \ x \in B_3,$$

cf. (2.3), we have $T(\gamma, \zeta)(0) = 0$ and $T(\gamma, \zeta) \in C^1(B_3) \cap C_S(B_3)$. While we show that $T(\gamma, \zeta) \in Y$ for $(\gamma, \zeta) \in [-1,1] \times \Omega$ the arguments $\gamma$ and $\zeta$ remain fixed, and we write $V = V_{\gamma, \zeta}$. From

$$\nabla T(\gamma, \zeta)(x) = \nabla U_0(x) - \nabla (g_\zeta(x)) D g_\zeta(x), \ x \in B_3,$$

we obtain the estimate

$$|\nabla T(\gamma, \zeta)(x)| \leq \|D^2 U_0\|_\infty |x| + 2\|D^2 V\|_\infty |g_\zeta(x)| \leq C|x|$$

with some constant $C$ which depends on $U_0$ and $V$ but not on $x$. In particular, this shows that $T(\gamma, \zeta) \in C^1(B_3)$. Now fix $x \in S_1$. Since any point on the line segment $0, g_\zeta(tx)$ can be written in the form $g_\zeta(\tau x)$ with $\tau \in [0, t]$ we have

$$\frac{\partial_x T(\gamma, \zeta)(tx)}{t} = \frac{\partial_x U_0(tx)}{t} - \frac{1}{t} \nabla (g_\zeta(tx)) \cdot \partial_x g_\zeta(tx) = \frac{\partial_x U_0(tx)}{t} - \frac{1}{t} \left( D^2 V(\gamma_\tau(x)) g_\zeta(tx) \right) \cdot \partial_x g_\zeta(tx) \rightarrow \nabla \partial_x U_0(0) \cdot x - \left( D^2 V(0) \frac{g_\zeta(0x)}{0} \right) \cdot \partial_x g_\zeta(0x)$$

as $t \downarrow 0$, uniformly in $x \in S_1$, by Lemma 3.3 (d). This shows that $T(\gamma, \zeta) \in Y$.

To show that $T$ is continuous we fix $(\gamma', \zeta') \in [-1,1] \times \Omega$. Constants denoted by $C$ may depend on $(\gamma', \zeta')$ but not on $(\gamma, \zeta) \in [-1,1] \times \Omega$ or $x \in B_3$. Restoring the subscript of $V$ we have

$$\|T(\gamma, \zeta) - T(\gamma', \zeta')\|_Y = \sup_{x \in B_3} \frac{1}{|x|} \left| \nabla V_{\gamma, \zeta}(g_\zeta(x)) D g_\zeta(x) - \nabla V_{\gamma', \zeta'}(g_{\zeta'}(x)) D g_{\zeta'}(x) \right|$$

$$\leq \sup_{x \in B_3} \frac{1}{|x|} (I_1 + I_2 + I_3),$$

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where for $x \in \hat{B}_3$,

\[
I_1 := |Dg_\zeta(x)||\nabla V_{\gamma,\zeta}(g_\zeta(x)) - \nabla V_{\gamma',\zeta'}(g_\zeta(x))|,
I_2 := |Dg_\zeta(x)||\nabla V_{\gamma',\zeta'}(g_\zeta(x)) - \nabla V_{\gamma',\zeta'}(g_{\zeta'}(x))|,
I_3 := |\nabla V_{\gamma',\zeta'}(g_{\zeta'}(x))||Dg_\zeta(x) - Dg_{\zeta'}(x)|.
\]

Using Lemma 3.3 (b) and Lemma 3.4 (a) with $\sigma = \rho_{\gamma,\zeta} - \rho_{\gamma',\zeta'}$ we find

\[
|\nabla V_{\gamma,\zeta}(g_\zeta(x)) - \nabla V_{\gamma',\zeta'}(g_\zeta(x))| \leq C\left(|\gamma - \gamma'| + \|\zeta - \zeta'||_X\right)|g_\zeta(x)|
\]

and thus by Lemma 2.3,

\[
I_1 \leq C\left(|\gamma - \gamma'| + \|\zeta - \zeta'||_X\right)|x|, \ x \in B_3.
\]

Since $V_{\gamma',\zeta'} \in C^2(\mathbb{R}^3)$ with $\nabla V_{\gamma',\zeta'}(0) = 0$ we have by Lemma 2.3 (d),

\[
I_2 \leq C|g_\zeta(x) - g_{\zeta'}(x)| \leq C\|\zeta - \zeta'||_X|x|, \ x \in B_3,
\]

and

\[
I_3 \leq \|D^2V_{\gamma',\zeta'}\|_\infty|g_{\zeta'}(x)||\zeta - \zeta'||_X \leq C\|\zeta - \zeta'||_X|x|, \ x \in B_3,
\]

and the proof of Assertion 1 is complete. \hfill \bullet

To deal with the differentiability of $T$ we have to investigate the integrand of the first term in the formula for $\partial_\xi T$, cf. Proposition 3.1.

**Lemma 3.5** For $\gamma \in [-1,1]$, $\zeta \in \Omega$, and $\xi \in X$ define

\[
\sigma_{\gamma,\zeta,\xi}(x) := \partial_u h(\gamma, r(x), U_\zeta(x))\nabla U_\zeta(x) \cdot \frac{g_\zeta^{-1}(x)}{|g_\zeta^{-1}(x)|}\xi(g_\zeta^{-1}(x)), \ x \in B_2,
\]

where we recall that $U_\zeta(x) = U_0(g_\zeta^{-1}(x))$, $x \in B_2$. Then $\sigma_{\gamma,\zeta,\xi} \in C_\Sigma(B_2)$, and there exists $C > 0$ such that for every $\gamma \in [-1,1]$, $\zeta \in \Omega$, and $\xi \in X$,

\[
|\sigma_{\gamma,\zeta,\xi}(x)| \leq C\|\zeta||_X|x|, \ x \in B_2.
\]

Moreover, if we fix $(\gamma',\zeta') \in [-1,1] \times \Omega$ there exists for each $\epsilon > 0$ a $\delta > 0$ such that for all $(\gamma,\zeta) \in [-1,1] \times \Omega$ with $|\gamma - \gamma'| + \|\zeta - \zeta'||_X < \delta$, and $\xi \in X$,

\[
|\sigma_{\gamma,\zeta,\xi}(x) - \sigma_{\gamma',\zeta',\xi}(x)| \leq \epsilon\|\zeta||_X|x|, \ x \in B_2.
\]

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**Proof:** Since the range of $U_0$ and thus also of $U_\zeta$ is bounded, the first factor in $\sigma_{\gamma,\zeta,\xi}$ is bounded, uniformly in $\gamma$ and $\zeta$, and the same is clearly true for the second and third factor. Together with

$$|\xi(g_\zeta^{-1}(x))| \leq \|\xi\|_X g_\zeta^{-1}(x) \leq 2\|\xi\|_X |x|, \ x \in B_2, \quad (3.1)$$

the estimate for $\sigma_{\gamma,\zeta,\xi}$ follows. The continuity of $\sigma_{\gamma,\zeta,\xi}$ on $\dot{B}_2$ is clear, and at $x=0$ it follows from the estimate above. The symmetry follows from the corresponding properties of $U_0$, $g_\zeta$, and $\xi$. In the following $C$ denotes a constant which may depend on $U_0$ and $(\gamma', \zeta')$ but not on $\gamma, \zeta, \xi$, or $x$. Making excessive use of the triangle inequality we find that

$$|\sigma_{\gamma,\zeta,\xi}(x) - \sigma_{\gamma',\zeta',\xi}(x)|$$

$$\leq C|\partial_u h(\gamma, r(x), U_\zeta(x)) - \partial_u h(\gamma', r(x), U_{\zeta'}(x))| |\xi(g_\zeta^{-1}(x))|$$

$$+ C|\nabla U_\zeta(x) - \nabla U_{\zeta'}(x)||\xi(g_\zeta^{-1}(x))|$$

$$+ C\left|\frac{g_\zeta^{-1}(x)}{|g_\zeta^{-1}(x)|} - \frac{g_{\zeta'}^{-1}(x)}{|g_{\zeta'}^{-1}(x)|}\right| |\xi(g_\zeta^{-1}(x))|$$

$$+ C|\xi(g_\zeta^{-1}(x)) - \xi(g_{\zeta'}^{-1}(x))|$$

$$=: I_1 + I_2 + I_3 + I_4, \ x \in \dot{B}_2.$$  

Now the estimate (3.1) together with the properties of the function $h$ stated in Lemma 2.1 imply that

$$I_1 \leq C\left(|\gamma - \gamma'| + |U_0(g_\zeta^{-1}(x)) - U_0(g_{\zeta'}^{-1}(x))|\right)\|\xi\|_X |x|$$

$$\leq C\left(|\gamma - \gamma' + g_\zeta^{-1}(x) - g_{\zeta'}^{-1}(x)|\right)\|\xi\|_X |x|$$

$$\leq C\left(|\gamma - \gamma' + \|\zeta - \zeta'\|_X\right)\|\xi\|_X |x|, \ x \in B_2.$$  

The crucial estimate is the one for $I_2$: it is at this point that we need the limit condition in the definition of the Banach space $X$ and its consequences. First note that

$$I_2 \leq C|\nabla U_0(g_\zeta^{-1}(x)) - \nabla U_0(g_{\zeta'}^{-1}(x))|\|\xi\|_X |x|$$

$$+ C|Dg_\zeta^{-1}(x) - Dg_{\zeta'}^{-1}(x)|\|\xi\|_X |x|$$

$$\leq C\|\zeta - \zeta'\|_X \|\xi\|_X |x| + C|Dg_\zeta^{-1}(x) - Dg_{\zeta'}^{-1}(x)|\|\xi\|_X |x|.$$
Now with $z := g_{\zeta}^{-1}(x)$ and $z' := g_{\zeta'}^{-1}(x)$,

$$|Dg_{\zeta}^{-1}(x) - Dg_{\zeta'}^{-1}(x)| = |(Dg_{\zeta})^{-1}(z) - (Dg_{\zeta'})^{-1}(z')|$$

$$\leq C|Dg_{\zeta}(z) - Dg_{\zeta'}(z')|$$

$$\leq C|Dg_{\zeta}(z) - Dg_{\zeta'}(z)| + C|Dg_{\zeta'}(z) - Dg_{\zeta'}(z')|$$

$$\leq C\|\zeta - \zeta\|_X + C|Dg_{\zeta'}(z) - Dg_{\zeta'}(z')|,$$

and it remains to estimate the last term in the line above. From (2.2) we get the estimate

$$|Dg_{\zeta'}(z) - Dg_{\zeta'}(z')| \leq C|\nabla \zeta'(z) - \nabla \zeta'(z')|$$

$$+ C|\nabla \zeta'(z')| \left| \frac{z}{|z|} - \frac{z'}{|z'|} \right| + \frac{C}{|z|} |\zeta'(z) - \zeta'(z')|$$

$$+ C|\zeta'(z')| \left( \left| \frac{1}{|z|} - \frac{1}{|z'|} \right| + \max_{i,j=1,2,3} \left| \frac{z_i z_j}{|z|^2} - \frac{z_i' z_j'}{|z'|^2} \right| \right)$$

$$=: J_1 + J_2 + J_3 + J_4.$$

With $\bar{x} := x/|x|$ there exist $s, s' > 0$ such that $z = g_{\zeta}^{-1}(x) = s\bar{x}$ and $z' = g_{\zeta'}^{-1}(x) = s'\bar{x}$ so that $s = |z|$, $s' = |z'|$, and

$$|s - s'| = ||z| - |z'|| \leq |g_{\zeta}^{-1}(x) - g_{\zeta'}^{-1}(x)| \leq C\|\zeta - \zeta\|_X, \ x \in \dot{B}_2.$$

Now given $\epsilon > 0$ we can choose $\delta > 0$ according to Lemma 3.2 (c) such that $\|\zeta - \zeta\|_X < \delta$ implies

$$J_1 = C|\nabla \zeta'(s\bar{x}) - \nabla \zeta'(s'\bar{x})| < \epsilon, \ x \in \dot{B}_2.$$

Using Lemma 2.3 (d) and Lemma 3.2 we obtain

$$J_2 \leq C \left( \frac{1}{|z|} + \frac{1}{|z'|} \right) |z - z'| \leq \frac{C}{|x|} \|\zeta - \zeta\|_X |x| \leq C\|\zeta - \zeta\|_X,$$

$$J_3 \leq \frac{C}{|z|} |z - z'| \leq C\|\zeta - \zeta\|_X,$$

and

$$J_4 \leq C|z'| \left( \frac{1}{|z|^2} + \frac{1}{|z'|^2} \right) |z - z'| \leq C\|\zeta - \zeta\|_X,$$
so that finally
\[ I_2 \leq C \| \zeta - \zeta' \|_X \| \xi \|_X |x| + C \epsilon \| \xi \|_X |x|, \quad x \in \hat{B}_2, \]
provided \( \| \zeta - \zeta' \|_X < \delta \). The remaining terms \( I_3 \) and \( I_4 \) are much easier to estimate:
\[ I_3 \leq C \left( \frac{1}{|g_{\zeta}^{-1}(x)|} + \frac{1}{|g_{\zeta'}^{-1}(x)|} \right) |g_{\zeta}^{-1}(x) - g_{\zeta'}^{-1}(x)| \| \xi \|_X |x| \]
\[ \leq C \| \zeta - \zeta' \|_X \| \xi \|_X |x|, \quad x \in \hat{B}_2, \]
and
\[ I_4 \leq C \| \xi \|_X |g_{\zeta}^{-1}(x) - g_{\zeta'}^{-1}(x)| \leq C \| \xi \|_X \| \zeta - \zeta' \|_X |x|, \quad x \in \hat{B}_2, \]
and the proof of Lemma 3.3 is complete.

To continue with the proof of Proposition 3.1 we denote for fixed \((\gamma, \zeta) \in ]-1,1[ \times \Omega\) by \(L\xi\) the right hand side of the definition of \(\partial_x T(\gamma, \zeta)\xi\), \(\xi \in X\).

We now show:

**Assertion 2:** \(L \in \mathcal{L}(X, Y)\) is a bounded, linear operator, and for all \(\xi \in X\),
\[
\lim_{t \to 0} \frac{T(\gamma, \zeta + t\xi) - T(\gamma, \zeta)}{t} = L\xi
\]
with respect to \(\| \cdot \|_Y\).

**Proof:** It is convenient to introduce the auxiliary Banach space
\[
\mathcal{Y} := \{ f \in C_S(B_3) \mid f(0) = 0, \ f \in C^1(B_3), \ \exists C > 0 : | \nabla f(x) | \leq C |x|, \ x \in B_3 \},
\]
which we equip with the norm \(\| \cdot \|_Y\); clearly, \(Y\) is a closed subspace of \(\mathcal{Y}\).

Since we already know that \(T\) maps \(X\) into \(Y\) it is then sufficient to show that \(L \in \mathcal{L}(X, \mathcal{Y})\) and that the asserted convergence holds. To see the former define
\[
V_\xi(x) := - \int_{B_2} \frac{1}{|x-y|} \sigma_{\gamma, \zeta} \xi(y) dy, \ x \in \mathbb{R}^3,
\]
and
\[
W(x) := - \int_{B_2} \frac{1}{|x-y|} \rho_{\gamma, \zeta} dy, \ x \in \mathbb{R}^3.
\]
Then $V_{\xi} \in C^1(\mathbb{R}^3)$, $W \in C^2(\mathbb{R}^3)$, and we can write
\[
(L\xi)(x) = V_{\xi}(g_{\xi}(x)) - V_{\xi}(0) - \nabla W(g_{\xi}(x)) \cdot \frac{x}{|x|} \xi(x), \ x \in B_3.
\]
This implies that for $\xi \in X$ we have $L\xi \in C^1(\dot{B}_3)$, $(L\xi)(0) = 0$, and
\[
(\nabla L\xi)(x) = \nabla V_{\xi}(g_{\xi}(x))Dg_{\xi}(x) - D^2W(g_{\xi}(x))Dg_{\xi}(x) \frac{x}{|x|} \xi(x)

- \nabla W(g_{\xi}(x))D\left(\frac{x}{|x|}\right)\xi(x) - \nabla W(g_{\xi}(x)) \cdot \frac{x}{|x|} \nabla \xi(x), \ x \in \dot{B}_3.
\]
Using Lemma 3.3 and Lemma 3.4 (a) we obtain the estimate
\[
|\nabla L\xi(x)| \leq C\|\xi\|_X |g_{\xi}(x)| + C|\xi(x)| + C\|D^2W\|_{\infty} |g_{\xi}(x)| \left(\frac{|\xi(x)|}{|x|} + |\nabla \xi(x)|\right)

\leq C\|\xi\|_X |x|, \ x \in \dot{B}_3.
\]
In particular, this implies that $L\xi$ is differentiable also at $x = 0$, and
\[
\|L\xi\|_Y \leq C\|\xi\|_X, \ \xi \in X.
\]
The symmetry of $L\xi$ follows easily from the corresponding properties of $V_{\xi}$, $W$, $\zeta$, and $\xi$. In order to show that $L\xi$ is indeed the Gateaux derivative of $T$ at $(\gamma, \zeta)$ in the direction of $\xi$ we choose $t_0 > 0$ such that $\zeta + t_0 \xi \in \Omega$ for $|t| < t_0$. Although this is in conflict with earlier notation it is convenient to abbreviate
\[
g_t(x) = g_{\zeta + t\xi}(x) = x + (\zeta(x) + t\xi(x)) \frac{x}{|x|}, \ x \in B_3, \ t \in [-t_0, t_0[.
\]
Then for each $x \in \dot{B}_2$ the mapping $]-t_0, t_0[ \ni t \mapsto g_t^{-1}(x)$ is continuously differentiable with
\[
\frac{d}{dt}g_t^{-1}(x) = -(Dg_t)^{-1}(g_t^{-1}(x)) \xi(g_t^{-1}(x)) \frac{g_t^{-1}(x)}{|g_t^{-1}(x)|}.
\]
To see this, define for fixed $z \in \dot{B}_2$ the mapping
\[
G(t, x) := g_t(x) - z, \ t \in [-t_0, t_0[, \ x \in \dot{B}_3.
\]
Since \( G(t, g_t^{-1}(z)) = 0, t \in [-t_0, t_0] \), the asserted regularity of \( g_t^{-1} \) with respect to \( t \) follows from the regularity of \( G \), the fact that \( \partial_x G(t, x) = Dg_t(x) \) is invertible, and the implicit function theorem. If we now differentiate the identity \( x = g_t(g_t^{-1}(x)) \) with respect to \( t \) we obtain the formula for \( \frac{d}{dt} g_t^{-1}(x) \).

It will also be convenient to abbreviate

\[
\rho_t(x) := \rho_{\gamma, \xi}(x), \quad \sigma_t(x) := \sigma_{\gamma, \xi}(x), \quad t \in [-t_0, t_0], \quad x \in B_2,
\]

and define

\[
F(t, x) := \int_{B_2} \left( \frac{1}{|x - y|} - \frac{1}{|y|} \right) \rho_t(y) dy, \quad x \in \mathbb{R}^3, \quad t \in [-t_0, t_0].
\]

Then except for \( \partial_t^2 F \) all derivatives of \( F \) up to second order exist and are continuous on \([-t_0, t_0] \times \mathbb{R}^3 \), and

\[
\partial_t F(t, x) = -\int_{B_2} \left( \frac{1}{|x - y|} - \frac{1}{|y|} \right) \sigma_t(y) dy,
\]

\[
\nabla F(t, x) = -\int_{B_2} \frac{x - y}{|x - y|^3} \rho_t(y) dy.
\]

These results follow easily from the fact that \( \rho_t \in C^1_c(\mathbb{R}^3) \) and

\[
\frac{d}{dt} \rho_t(y) = \partial_u h(\gamma, r(y), U_0(g_t^{-1}(y))) \nabla U_0(g_t^{-1}(y)) \cdot \frac{d}{dt} g_t^{-1}(x)
\]

\[
= -\partial_u h(\gamma, r(y), U_0(g_t^{-1}(y))) \nabla U_0(g_t^{-1}(y)) (Dg_t)^{-1}(g_t^{-1}(y))
\]

\[
\times \frac{g_t^{-1}(y)}{|g_t^{-1}(y)|} \xi(g_t^{-1}(y))
\]

\[
= -\partial_u h(\gamma, r(y), U_0(g_t^{-1}(y))) \nabla (U_0(g_t^{-1}(y))) \cdot \frac{g_t^{-1}(y)}{|g_t^{-1}(y)|} \xi(g_t^{-1}(y))
\]

\[
= -\sigma_t(y).
\]

Now

\[
\frac{T(\gamma, \zeta + t\xi)(x) - T(\gamma, \zeta)(x)}{t} = \frac{F(t, g_t(x)) - F(0, g_t(x))}{t} + \frac{F(0, g_t(x)) - F(0, g_0(x))}{t}, \quad t \in [-t_0, t_0], \quad x \in B_3,
\]
and
\[(L\xi)(x) = \partial_t F(0,g_0(x)) + \nabla F(0,g_0(x)) \cdot \frac{x}{|x|} \xi(x), \ x \in B_3;\]
one should be careful to note that here \(g_0 = g_{\xi+\xi_0} = g_\xi\). We claim that as \(t \to 0\),
\[
\frac{F(t,g_t(x)) - F(0,g_t(x))}{t} \to \partial_t F(0,g_0(x)) \quad (3.2)
\]
and
\[
\frac{F(0,g_t(x)) - F(0,g_0(x))}{t} \to \nabla F(0,g_0(x)) \cdot \frac{x}{|x|} \xi(x), \quad (3.3)
\]
where both limits are understood with respect to the norm \(\| \cdot \|_Y\). This would then prove that \(L\) is the Gateaux differential of \(T\) at \((\gamma,\zeta)\). As to (3.2) we observe that
\[
\left| \nabla F(t,g_t(x)) - F(0,g_t(x)) \right| - \left| \nabla F(0,g_0(x)) \right| \leq |Dg_t(x)| + |\nabla \partial_t F(0,g_0(x)) - \nabla \partial_t F(0,g_t(x))| |Dg_t(x)| + |\nabla \partial_t F(0,g_0(x)) - \nabla \partial_t F(0,g_0(x))| |Dg_t(x)| - Dg_0(x)| =: I_1 + I_2 + I_3.
\]

Let \(\epsilon > 0\). For every \(z \in \mathbb{R}^3\) there exists \(\tau\) between 0 and \(t\) such that
\[
\left| \nabla F(t,z) - \nabla F(0,z) \right| - \left| \nabla \partial_t F(\tau,z) - \nabla \partial_t F(0,z) \right| = \left| \nabla \begin{align*}
& \int_{B_2} \frac{1}{|z-y|} (\sigma_\tau(y) - \sigma_0(y)) dy \\
& \left(\nabla \partial_t F(\tau,z) - \nabla \partial_t F(0,z)\right) \end{align*} \right|
\]
and using Lemma 3.4 (a), the latter integral can be estimated by \(C \epsilon \| \xi \|_X \| z \|\), provided
\[|\sigma_\tau(y) - \sigma_0(y)| \leq \epsilon \| \xi \|_X \| y \|, \ y \in B_2.\]

Lemma 3.5 therefore implies that for \(\delta > 0\) sufficiently small we have
\[I_1 \leq C \epsilon |g_t(x)| \leq C \epsilon |x|, \ x \in B_3,\]

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provided $|t|<\delta$. Note that $C$ depends on $\zeta$ and $\xi$, but not on $t$ or $x$. Again by Lemma 3.3 and Lemma 3.4 (b) we find the estimate

$$I_2 \leq C \|\xi\|_X \|\zeta + t\xi - \zeta\|^{1/2}_X \leq C |t|^{1/2} |x|, \quad x \in B_3,$$

and by Lemma 3.3, Lemma 3.4 (a), and Lemma 2.3 (d) we conclude that

$$I_3 \leq C \|\xi\|_X |g_0(x)| \|\zeta + t\xi - \zeta\| \leq C |t| |x|, \quad x \in B_3.$$

This proves the convergence in (3.2) with respect to $\|\cdot\|_Y$. As to (3.3) we observe that for every $x \in B_3$,

$$\frac{F(0,g_t(x)) - F(0,g_0(x))}{t} = \frac{d}{dt} F(0,g_t(x))_{|t=\tau} = \nabla F(0,g_{\tau}(x)) \cdot \frac{x}{|x|} \xi(x)$$

where $\tau$ lies between 0 and $t \in ]-t_0,t_0[$. Therefore,

$$\left| \nabla \frac{F(0,g_t(x)) - F(0,g_0(x))}{t} - \nabla \left( \nabla F(0,g_0(x)) \cdot \frac{x}{|x|} \xi(x) \right) \right|$$

$$= \left| \nabla \left( \nabla F(0,g_{\tau}(x)) - \nabla F(0,g_0(x)) \right) \cdot \frac{x}{|x|} \xi(x) \right|$$

$$\leq |D^2 F(0,g_{\tau}(x)) - D^2 F(0,g_0(x))| C |x|$$

$$+ |D^2 F(0,g_0(x))| |D g_{\tau}(x) - D g_0(x)| C |x|$$

$$+ \left| \nabla F(0,g_{\tau}(x)) - \nabla F(0,g_0(x)) \right| D \left( \frac{\xi(x)}{|x|} \right)$$

$$\leq C |t| |x| + C |D^2 F(0,g_{\tau}(x)) - D^2 F(0,g_0(x))| |x|, \quad x \in B_3.$$

Since $D^2 F(0,\cdot)$ is uniformly continuous on $B_4$, which contains $g_{\tau}(x)$ for $x \in B_3$ and $\tau \in ]-t_0,t_0[$, cf. Lemma 2.3 (b), and

$$|g_{\tau}(x) - g_0(x)| \leq \|\xi\|_X |\tau| \leq C |t|, \quad x \in B_3,$$

we obtain the convergence in (3.3) with respect to the norm $\|\cdot\|_Y$. This completes the proof of Assertion 2.

Since a continuous Gateaux derivative is a Fréchet derivative the proof of Proposition 3.1 will be complete, once we show:

**Assertion 3:** The mapping $]-1,1[ \times \Omega \ni (\gamma,\zeta) \mapsto \partial_\zeta T(\gamma,\zeta) \in \mathcal{L}(X,Y)$ is continuous.
Proof: Fix \((\gamma', \zeta') \in ]-1, 1[ \times \Omega\) and let \((\gamma, \zeta) \in ]-1, 1[ \times \Omega\) and \(\xi \in X\) with \(\|\xi\|_X = 1\). Since
\[
[\partial_\zeta T(\gamma, \zeta)\xi](x) - [\partial_\zeta T(\gamma', \zeta')\xi](x)
= - \int_{B_2} \left[ \left( \frac{1}{|g_\zeta(x) - y|} - \frac{1}{|y|} \right) \sigma_\gamma, \zeta, \xi(y) - \left( \frac{1}{|g_{\zeta'}(x) - y|} - \frac{1}{|y|} \right) \sigma_{\gamma', \zeta', \xi}(y) \right] dy
- \int_{B_2} \left[ \frac{g_\zeta(x) - y}{|g_\zeta(x) - y|^3} \rho_{\gamma, \zeta}(y) - \frac{g_{\zeta'}(x) - y}{|g_{\zeta'}(x) - y|^3} \rho_{\gamma', \zeta'}(y) \right] dy \cdot \frac{x}{|x|} \xi(x), \ x \in B_3,
\]
we find
\[
\left| \nabla \left( [\partial_\zeta T(\gamma, \zeta)\xi](x) - [\partial_\zeta T(\gamma', \zeta')\xi](x) \right) \right| \leq I_1 + I_2 + I_3 + I_4 + I_5 + I_6,
\]
where
\[
I_1 := \left| \nabla \int_{B_2} \left( \frac{1}{|g_\zeta(x) - y|} - \frac{1}{|y|} \right) \left( \sigma_\gamma, \zeta, \xi(y) - \sigma_{\gamma', \zeta', \xi}(y) \right) dy \right|,
I_2 := \left| \nabla \int_{B_2} \left( \frac{1}{|g_\zeta(x) - y|} - \frac{1}{|g_{\zeta'}(x) - y|} \right) \sigma_{\gamma', \zeta', \xi}(y) dy \right|,
I_3 := D \int_{B_2} \frac{g_\zeta(x) - y}{|g_\zeta(x) - y|^3} (\rho_{\gamma, \zeta}(y) - \rho_{\gamma', \zeta'}(y)) dy \left| \xi(x) \right|,
I_4 := D \int_{B_2} \frac{g_\zeta(x) - y}{|g_\zeta(x) - y|^3} (\rho_{\gamma', \zeta'}(y)) dy \left| \xi(x) \right|,
I_5 := \int_{B_2} \frac{g_\zeta(x) - y}{|g_\zeta(x) - y|^3} (\rho_{\gamma, \zeta'}(y) - \rho_{\gamma', \zeta'}(y)) dy \left| D \left( \xi(x) \frac{x}{|x|} \right) \right|,
I_6 := \int_{B_2} \frac{g_\zeta(x) - y}{|g_\zeta(x) - y|^3} (\rho_{\gamma', \zeta'}(y)) dy \left| D \left( \xi(x) \frac{x}{|x|} \right) \right|.
\]
Given \(\epsilon > 0\) we choose \(\delta > 0\) so that the second estimate in Lemma 3.3 holds for \(|\gamma - \gamma'| + \|\zeta - \zeta'\|_X < \delta\). Then Lemma 3.4 (a) implies, with \(z = g_\zeta(x)\), the estimate
\[
I_1 \leq \left| \nabla \int_{B_2} \frac{1}{|z - y|} \left( \sigma_\gamma, \zeta, \xi(y) - \sigma_{\gamma', \zeta', \xi}(y) \right) dy D g_\zeta(x) \right| \leq C \epsilon |z| \leq C \epsilon |x|, \ x \in B_3.
\]
Defining
\[
V(x) := \int_{B_2} \frac{1}{|x - y|} \sigma_{\gamma', \zeta', \xi}(y) dy, \ x \in \mathbb{R}^3,
\]
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Lemma 2.3. In order to estimate the remaining terms we define

\[ I_2 = \left| \nabla V(\zeta(x)) D\zeta(x) - \nabla V(\zeta'(x)) D\zeta'(x) \right| \]

\[ \leq C \left| \nabla V(\zeta(x)) - \nabla V(\zeta'(x)) \right| + \left| \nabla V(\zeta'(x)) \right| \left| D\zeta(x) - D\zeta'(x) \right| \]

\[ \leq C \left\| \zeta - \zeta' \right\|_{X}^{1/2} |x| + C \left| g_3(x) \right| \left\| \zeta - \zeta' \right\|_{X} \leq C \left\| \zeta - \zeta' \right\|_{X}^{1/2} |x|, \quad x \in B_3, \]

where we have used the first estimate in Lemma 3.3, Lemma 3.4, and Lemma 2.3. In order to estimate the remaining terms we define

\[ V_{\gamma, \zeta}(x) := \int_{B_2} \frac{1}{|x-y|} \rho_{\gamma, \zeta}(y) dy, \quad x \in \mathbb{R}^3. \]

Then \( V_{\gamma, \zeta} \in C^2(\mathbb{R}^3) \) and

\[ I_3 = \left| D^2 V_{\gamma, \zeta}(\zeta(x)) - D^2 V_{\gamma, \zeta}(\zeta'(x)) \right| \left| Dg_3(x) \right| \left| \xi(x) \right| \]

\[ \leq C \left| D^2 V_{\gamma, \zeta}(\zeta(x)) \right| \left| Dg_3(x) \right| \left| \zeta(x) \right| \]

\[ \leq C \left( |\gamma - \gamma'| + \left\| \zeta - \zeta' \right\|_{X} \right)^{3/4} |x|, \quad x \in B_3, \]

where we have used Lemma 3.3 and [1, Lemma 1]. Next we have

\[ I_4 = \left| D^2 V_{\gamma, \zeta}(\zeta(x)) Dg_3(x) - D^2 V_{\gamma, \zeta}(\zeta'(x)) Dg_3'(x) \right| \left| \xi(x) \right| \]

\[ \leq C \left| D^2 V_{\gamma, \zeta}(\zeta(x)) \right| \left| Dg_3(x) \right| \left| \zeta(x) \right| \]

\[ \leq \epsilon |x| + C |x| \left\| \zeta - \zeta' \right\|_{X}, \quad x \in B_3, \]

provided \( \left\| \zeta - \zeta' \right\|_{X} \) is small enough, where we have used the fact that \( D^2 V_{\gamma, \zeta} \) is uniformly continuous on \( B_4 \ni g_3(x), g_3'(x) \) and \( |g_3(x) - g_3'(x)| \leq \left\| \zeta - \zeta' \right\|_{X}. \)

By Lemma 3.3 (b) and Lemma 3.4 (a) for \( \sigma = \rho_{\gamma, \zeta} - \rho_{\gamma, \zeta}' \) we obtain, with \( z = g_3(x) \),

\[ I_5 \leq C \left| \nabla \int_{B_2} \frac{1}{|z-y|} \left( \rho_{\gamma, \zeta}(y) - \rho_{\gamma, \zeta}'(y) \right) dy \right| \leq C \left( |\gamma - \gamma'| + \left\| \zeta - \zeta' \right\| \right) |x|, \quad x \in B_3. \]

Finally,

\[ I_6 \leq C \left| \nabla V_{\gamma, \zeta}(\zeta(x)) - \nabla V_{\gamma, \zeta}(\zeta'(x)) \right| \leq C \left\| D^2 V_{\gamma, \zeta} \right\|_{\infty} \left| g_3(x) - g_3'(x) \right| \]

\[ \leq C \left\| \zeta - \zeta' \right\|_{X} |x|, \quad x \in B_3. \]
We have shown that for every $\epsilon > 0$ there exists $\delta > 0$, depending on $(\gamma', \zeta')$, such that for all $(\gamma, \zeta) \in [-1,1] \times \Omega$ with $|\gamma - \gamma'| + \|\zeta - \zeta'\|_X < \delta$ and all $\xi \in X$ with $\|\xi\|_X = 1$ we have

$$\|\partial_\zeta T(\gamma, \zeta)\xi - \partial_\zeta T(\gamma', \zeta')\xi\|_Y \leq \epsilon,$$

and the proof of Proposition 3.1 is complete.

4 \ $\partial_\zeta T(0,0)$ is an isomorphism

The aim of this section is to prove the following result:

**Proposition 4.1** The mapping $\partial_\zeta T(0,0) : X \to Y$ is a linear isomorphism.

Let us abbreviate $L_0 \xi := \partial_\zeta T(0,0)\xi$ for $\xi \in X$. In order to prove the result above we rewrite $L_0 \xi$: Observe first that $g_0 = id$, and therefore $U_0$, defined in Proposition 3.1, coincides with the potential $U_0$ of the spherically symmetric steady state we started with, if $\zeta = 0$. In particular, $\rho_0(|x|) = h_0(U_0(|x|)) = h(0, r(x), U_0(|x|))$ for $x \in \mathbb{R}^3$, and

$$\rho_0(|x|) = \partial_u h(0, r(x), U_0(|x|))U_0'(|x|)\nabla U_0(x) \cdot \frac{x}{|x|}, \quad x \in \mathbb{R}^3.$$

Therefore,

$$(L_0 \xi)(x) = -\int_{B_2} \left( \frac{1}{|x-y|} - \frac{1}{|y|} \right) \rho_0'(|y|)\xi(y) dy - \int_{B_2} \frac{x-y}{|x-y|^3}\rho_0(|y|) dy \cdot \frac{x}{|x|}\xi(x)$$

$$= -U_0'(|x|)\xi(x) - \int_{B_2} \left( \frac{1}{|x-y|} - \frac{1}{|y|} \right) \rho_0'(|y|)\xi(y) dy, \quad x \in B_3, \ \xi \in X.$$

Now let

$$(K \xi)(x) := -\frac{1}{U_0'(|x|)} \int_{B_2} \left( \frac{1}{|x-y|} - \frac{1}{|y|} \right) \rho_0'(|y|)\xi(y) dy, \quad x \in B_3, \ \xi \in C_S(B_3).$$

Then we can write

$$(L_0 \xi)(x) = -U_0'(|x|)[(id - K)\xi](x), \quad x \in B_3, \ \xi \in X. \quad (4.1)$$

As a first step towards proving Proposition 4.1 we show:
**Assertion 1:** The linear operator $K : C_S(B_3) \to C_S(B_3)$ is compact, where $C_S(B_3)$ is equipped with the supremum norm $\| \cdot \|_\infty$.

**Proof:** For $\xi \in C_S(B_3)$ let

$$V_\xi(x) := -\int_{B_2} \frac{1}{|x-y|} \rho'_0(|y|) \xi(y) \, dy, \ x \in \mathbb{R}^3. \quad (4.2)$$

Then $V_\xi \in C^1(\mathbb{R}^3)$, $\nabla V_\xi(0) = 0$, and

$$(K \xi)(x) = \frac{1}{U'_0(|x|)} (V_\xi(x) - V_\xi(0)), \ x \in B_3.$$  

Using Lemma 2.2 (c) we obtain the estimate

$$|(K \xi)(x)| \leq \frac{1}{C|x|} \| \nabla V_\xi \|_\infty |x| \leq C \| \xi \|_\infty, \ x \in B_3,$$

where the constant $C$ depends on $\rho_0$ and $U_0$, but not on $\xi$ or $x$. Thus $K$ maps bounded sets into bounded sets. We claim that $K \xi$ is Hölder continuous with exponent $1/2$, uniformly on bounded sets in $C_S(B_3)$. Let $M > 0$ and assume $\| \xi \|_\infty \leq M$. In the following, constants denoted by $C$ depend on $\rho_0$, $U_0$, and $M$, but not on $\xi$ itself. There exists a constant $C > 0$ such that

$$|\nabla V_\xi(x) - \nabla V_\xi(x')| \leq C \| \rho'_0 \|_\infty |x - x'|^{1/2}, \ x, x' \in B_3, \quad (4.3)$$

cf. [7, Probl. 4.8]. Since $\nabla V_\xi(0) = 0$, (4.3) implies

$$|\nabla V_\xi(x)| \leq C |x|^{1/2}, \ x \in B_3.$$  

Now let $x, x' \in B_3$ and $|x| \leq |x'|$. Then

$$|(K \xi)(x) - (K \xi)(x')| \leq \frac{1}{U'_0(|x|)} - \frac{1}{U'_0(|x'|)} |V_\xi(x) - V_\xi(0)|$$

$$+ \frac{1}{U'_0(|x'|)} |V_\xi(x) - V_\xi(x')| =: I_1 + I_2,$$

and we obtain for some $z \in B_3$ with $|z| \leq |x'|$ the estimates

$$I_1 \leq C \frac{|U'_0(|x|) - U'_0(|x'|)|}{|x| |x'|} |\nabla V_\xi(z)| |x| \leq C |x - x'|^{1/2} \frac{(|x| + |x'|)^{1/2}}{|x'|} |z|^{1/2}$$

$$\leq C |x - x'|^{1/2},$$
and
\[ I_2 \leq \frac{C}{|x'|} |\nabla V_\xi(z)||x-x'| \leq \frac{C}{|x'|} |z|^{1/2} |x-x'| \leq C|x-x'|^{1/2} \]
so that
\[ |(K\xi)(x)-(K\xi)(x')| \leq C|x-x'|^{1/2}, \ x,x' \in \hat{B}_3. \]

Also,
\[ |(K\xi)(x)| \leq C|\nabla V_\xi(z)| \leq C|x|^{1/2}, \ x \in \hat{B}_3, \]
and we have shown that \( K \) maps bounded subsets of \( C_S(B_3) \) into bounded and equicontinuous subsets of \( C_S(B_3) \). Thus \( K \) is compact by the Arzela-Ascoli theorem, and the proof of Assertion 1 is complete.

As second step in the proof of Proposition 4.1 we show:

**Assertion 2:** \( \text{id} - K : C_S(B_3) \to C_S(B_3) \) is one-to-one and onto.

**Proof:** Since \( K \) is compact is suffices to show that \( \text{id} - K \) is one-to-one. Let \( \xi \in C_S(B_3) \) with \( \xi - K\xi = 0 \). In order to show that \( \xi = 0 \) we expand \( \xi \) into spherical harmonics \( Y_{lm} \), \( l \in \mathbb{N}_0, m = -l, \ldots, l \), where we use the notation of [11, Ch. 3] concerning the latter. Denote by \((r, \theta, \phi)\) and \((s, \tau, \psi)\) the polar coordinates of a point \( x \) or \( y \in B_3 \) respectively. For \( l \in \mathbb{N}_0 \) and \( m = -l, \ldots, l \) we define
\[ \xi_{lm}(r) := \frac{1}{r^2} \int_{|x|=r} Y_{lm}^*(\theta, \phi) \xi(x) dS_x. \]

Using the expansion
\[ \frac{1}{|x-y|} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \frac{r_<}{r_>^{l+1}} Y_{lm}^*(\tau, \psi) Y_{lm}(\theta, \phi), \]
where \( r_< := \min(r, s) \) and \( r_> := \max(r, s) \), cf. [11, Eqn. (3.70)], we find that
\[
\xi_{lm}(r) = -\frac{1}{r^2 U_0'(r)} \int_{B_3} \int_{|x|=r} \left( \frac{1}{|x-y|} - \frac{1}{|y|} \right) Y_{lm}^*(\theta, \phi) dS_x \rho_0'(s) \xi(y) dy \\
= -\frac{4\pi}{2l+1} \frac{1}{U_0'(r)} \int_0^3 \rho_0'(s) \left( \frac{r_<}{r_>^{l+1}} - \frac{l}{s^{l+1}} \right) \int_{|y|=s} Y_{lm}^*(\tau, \psi) \xi(y) dS_y ds \\
= -\frac{4\pi}{2l+1} \frac{1}{U_0'(r)} \int_0^3 \rho_0'(s) \left( \frac{r_<}{r_>^{l+1}} - \frac{l}{s^{l+1}} \right) s^2 \xi_{lm}(s) ds, \ r \in [0, 3].
\]
This implies that

$$
\xi_{00}(r) = -\frac{4\pi}{rU_0'(r)} \int_0^r \rho_0'(s) s(s-r) \xi_{00}(s) \, ds.
$$

Clearly, $\xi_{00}$ vanishes in the limit at $r = 0$. Let $R \geq 0$ be such that $\xi_{00}$ vanishes on $[0, R]$. Then for $r \in [R, 3]$,

$$
|\xi_{00}(r)| \leq \frac{4\pi}{rU_0'(r)} \|\rho_0\|_\infty \sup_{0 \leq s \leq r} |\xi_{00}(s)| \int_r^R s(r-s) \, ds \leq C(r-R) \sup_{0 \leq s \leq r} |\xi_{00}(s)|.
$$

This implies that $\xi_{00}$ vanishes in a right neighborhood of $R$ and thus on the whole interval $[0, 3]$. Up to multiplicative constants the spherical harmonics for $l = 1$ are given by $\sin \theta e^{\pm i \phi}$ and $\cos \theta$, and the fact that $\xi \in C_S$ implies that $\xi_{1-1} = \xi_{10} = \xi_{11} = 0$. Let $l \geq 2$. Then

$$
\xi_{lm}(r) = -\frac{4\pi}{2l+1} \frac{1}{U_0'(r)} \left( \int_0^r \rho_0'(s) s^l \xi_{lm}(s) \, ds + \int_r^3 \rho_0'(s) \frac{s^{l-1}}{s^{l+1}} s^2 \xi_{lm}(s) \, ds \right),
$$

and

$$
|\xi_{lm}(r)| \leq \frac{4\pi}{2l+1} \frac{1}{U_0'(r)} \|\xi_{lm}\|_\infty \left( \frac{1}{r^2} \int_0^r (-\rho_0')(s) s^{l-1} \frac{s}{s^{l-1}} s^3 \, ds + r \int_r^3 (-\rho_0')(s) s^{l-1} \, ds \right)
\leq \frac{4\pi}{2l+1} \frac{1}{U_0'(r)} \|\xi_{lm}\|_\infty \left( \frac{1}{r^2} \int_0^r (-\rho_0')(s) s^3 \, ds + r \int_r^3 (-\rho_0')(s) \, ds \right)
= \frac{4\pi}{2l+1} \frac{1}{U_0'(r)} \|\xi_{lm}\|_\infty \left( -\frac{1}{r^2} r^3 \rho_0(r) + \frac{3}{r^2} \int_0^r \rho_0(s) s^2 \, ds + r \rho_0(r) \, ds \right)
= \frac{3}{2l+1} \|\xi_{lm}\|_\infty,
$$

and since $2l+1 > 3$ this implies that $\xi_{lm}$ vanishes for $l \geq 2$ as well. We have shown that $id - K$ is one-to-one as claimed, and Assertion 2 is therefore established.

It is clear that $L_0 : X \to Y$ is now one-to-one as well: just observe (4.1) and the fact that $U_0'(r) > 0$ for $r > 0$. It remains to show:

**Assertion 3:** $L_0 : X \to Y$ is onto.

**Proof:** Let $g \in Y$ and define $q := g/U_0'$. We claim that $q \in X$. To see this we first observe that $q \in C^1(\bar{B}_3) \cap C_S(\bar{B}_3)$, and

$$
|\nabla q(x)| \leq \frac{|\nabla g(x)|}{U_0'(|x|)} + |g(x)| \left| \frac{U_0''(|x|)}{U_0'(|x|)^3} \frac{x}{|x|} \right| \leq C \left( \frac{|\nabla g(x)|}{|x|} + \frac{|g(x)|}{|x|^3} \right) \leq 2C \|g\|_Y.
$$
By definition of $Y$ and since $U_0 \in C^2([0, \infty[)$ with $U_0''(0) > 0$ we have that for every $x \in S_1$,

$$\nabla q(tx) = \frac{\nabla g(tx)}{t} \frac{t}{U'_0(t)} - \frac{g(tx)}{t^2} \frac{U''_0(t)}{U'_0(t)}\left(\frac{t}{U'_0(t)}\right)^2 x$$

$$\to \nabla g(0x) \frac{1}{U'_0(0)} - \frac{g(0x)}{0^2} \frac{U''_0(0)}{0}^2 x$$

as $t \searrow 0$, uniformly in $x \in S_1$.

Since $X \subset C^S(B_3)$ there exists by Assertion 2 an element $\xi \in C^S(B_3)$ such that

$$\xi - K\xi = -q = -\frac{g}{U'_0}.$$ 

This implies that $L_0\xi = g$ and thus that $L_0$ is onto, provided $\xi \in X$. To see the latter we observe that $\xi = K\xi + q$ is Hölder continuous since $K\xi$ is Hölder continuous. As above we conclude that $V_\xi \in C^2(\mathbb{R}^3)$ and thus $K\xi \in C^2(\hat{B}_3)$.

Denoting by $H_{V_\xi}$ the Hessian of $V_\xi$ we obtain for each $x \in \hat{B}_3$ a point $z \in 0, x$ such that

$$|\nabla (K\xi)(x)| \leq \left| \frac{U''_0(|x|)}{U'_0(|x|)^2} \right| |V_\xi(x) - V_\xi(0)| + \frac{1}{|U'_0(|x|)|} |\nabla V_\xi(x)|$$

$$\leq \frac{C}{|x|^2} \left| \langle H_{V_\xi}(z)x, x \rangle \right| + \frac{C}{|x|} |\nabla V_\xi(x)| \leq C \|D^2 V_\xi\|_{\infty}.$$ 

Finally, for $x \in S_1$ we have

$$\nabla (K\xi)(tx) = -\frac{U''_0(t)}{U'_0(t)^2} x(V_\xi(tx) - V_\xi(0)) + \frac{1}{U'_0(t)} \nabla V_\xi(tx)$$

$$= -U''_0(t) \left(\frac{t}{U'_0(t)}\right)^2 x \frac{1}{t^2} \frac{1}{2} \langle H_{V_\xi}(\tau x)tx, tx \rangle + \frac{t}{U'_0(t)} \nabla V_\xi(tx)$$

$$\to -\frac{1}{2U''_0(0)} \langle H_{V_\xi}(0)x, x \rangle x + \frac{1}{U''_0(0)} D^2 V_\xi(0)x$$

as $t \searrow 0$, uniformly in $x \in S_1$. We have shown that $K\xi \in X$, and since $q \in X$ as seen above this implies that $\xi \in X$. This completes the proof that $L_0$ is onto and thus also the proof of Proposition [4.1].

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