A constrained variational model for radial symmetry breaking

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Abstract

A simple constrained minimization problem with an integral constraint describes a symmetry breaking of a circular front around a point source. As a single control parameter, the total flux $\phi$ from the source, is varied, apparently polygonal solutions with an arbitrary number of corners $m$ are shown to bifurcate from the circular solution. Our asymptotic analysis shows that the branches with $m \geq 3$ bifurcate supercritically at $\phi = \pi(m^2 + 2)$ and continue as $\phi \to \infty$ whereas those with $m = 1$ or 2 bifurcate subcritically and are terminated at $\phi = 3\pi$ and $4\pi$, respectively. The second variation can be evaluated directly for the circular state which is proven to be the minimizing solution only up to $\phi = 3\pi$.

1. Introduction

In this article we consider minimization of the circumference of a closed curve $y(\theta)$ in the polar coordinate on the plane:

Minimize $\int F(y, y')d\theta$ where $F = \sqrt{y^2 + y'^2}$

subject to an integral constraint involving a prescribed function $G(y)$ and a control parameter $\phi(>0)$:

$\int G(y)d\theta = \phi$ where $G(y) = \frac{1}{1 - y}$

under $C^2$-perturbations. We restrict $y$ to be within $0 < y < 1$ for all $\theta$.

As shown in Fig. 1.1, the problem describes a situation when a point source of "fluid" is located at the origin and is surrounded by a deformable front described by $r = y(\theta)$. A barrier zone resides between this front and the outer boundary $r = 1$ in such a way that the flux per unit angle $q$ monotonically increases with $y$ as specified...
Figure 1.1. Schematic of a barrier zone around a point source of fluid at the origin. The inner boundary of the zone is a deformable front \( r = y(\theta) \) whereas the outer boundary is a circle \( r = 1 \). The local flux \( G(y(\theta)) \) can become non-uniform, but the total flux \( \phi \) is a given constant.

by \( G \). (The function \( G \) denotes the local flux.) In other words more fluid tends to pass where \( y \) is large so that the barrier is thinner and weaker. The total flux from the source is set to be \( \phi \), and we ask which function \( y \) minimizes the circumference under the situation.

Despite the simple appearance of the problem, the answer turns out to be non-trivial. It is known that the solution up to the first variation is the expected “circular” solution, for which \( y \) is constant \( (y \equiv 1 - 2\pi\phi^{-1}) \), only when \( \phi \) is sufficiently small. As \( \phi \) becomes larger, however, modulated solutions which often appear in the polar coordinates as polygons with an arbitrary number of corners become solutions.

This problem was originally proposed as a heuristic model for intriguing observations of the polygonal hydraulic jumps in a free surface flow by Ellegaard et al. [7, 8]. In their mostly experimental work they described the model and showed a few examples of the computed “polygonal” solutions. The model was then studied by Hansen et al. [9] who analyzed it further by a change of variable. They also considered a modified model by replacing (1.1) with a more physically relevant equation. The parameter space for this model was thoroughly studied by Martens [10].

Aristoff, Bush, Hosoi and coworkers [3, 1, 4, 5] performed careful and thorough experiments, and interpreted the phenomenon of non-circular jumps to be due to surface tension. A model including the capillary effect was proposed and studied by Martens et al.[11]. In a related but different context Putkaradze and Dimon [13] considered whether non-circular solutions exist in the two-dimensional Navier-Stokes equation without a free surface, and found families of exact solutions.

There are many other systems where radial symmetry becomes broken as a control parameter is varied such as cellular flames [2], fluid sheets [3], spinning drops [12], and lasers [6]. The simple model (1.1, 1.2) provides one concrete scenario for appearance of apparently polygonal solutions from a circular shape. In this article it is our aim...
to come back to the original simple model and present an analysis up to the second variation.

In particular, several questions remained open about the constrained minimization model.

(Q1) Exactly how are branches of the polygonal solutions bifurcate from the circular solution? How does the amplitude of modulations depend on the bifurcation parameter?

(Q2) How are the branches terminated? Do they disappear at a finite $\phi$ or continue up to arbitrarily large $\phi$ values?

(Q3) Can we analyze the second variation of the problem so that we can conclude that the solutions to the first variation are indeed minimizing the circumference or not?

In Sec. 2 we review the previous work, and present a comprehensive analysis of the model within the first variation. The corresponding Euler-Lagrange equation is written in Sec. 2.1, and the solutions we look for are specified in Sec. 2.2. In Sec. 2.3, we find that solutions may exist only in a part of the parameter plane. In Sec. 2.4 we actually compute the bifurcation and the $\phi$-dependence of solutions as in Fig. 2.3. Actual solutions are shown in Sec. 2.5.

In Sec. 3 we answer the questions (Q1) and (Q2) above. Following a branch of any polygonal solution numerically requires higher and higher accuracy and it, thus, becomes harder away from the bifurcation point. How the branch can be continued must be studied analytically. Careful evaluations of the limiting values of two integrals in Sec. 3.1 turn out to be necessary, especially for $m = 1$ and 2. On the other hand non-circular solutions just after a bifurcation can be approximated perturbatively by a weakly nonlinear analysis in Sec. 3.2.

Section 4 discusses the question (Q3) above. We find that the model can also be treated by the direct method. The first variation is shown to be equivalent to the Euler-Lagrange equation. An explicit expression for the second variation of the model is obtained in Sec. 4.1. We evaluate it analytically for the circular state in Sec. 4.2, and obtain the following theorem:

**Theorem 1.1.** Let $\phi \in \mathbb{R}$ be $\phi > 2\pi$. Then, the constant function $y \equiv 1 - 2\pi \phi^{-1}$ satisfies the constraint (1.2) and locally minimizes the functional (1.1) under $C^2$-perturbations if and only if $\phi < 3\pi$.

Finally, conclusions are stated in Sec. 5.

2. Analysis of the first variation

In this section we review the previous work [7, 8, 9], and present a comprehensive analysis of the model within the first variation.

2.1. Euler-Lagrange equation

Since both $F$ and $G$ do not depend explicitly on $\theta$, the Euler-Lagrange equation to the variational problem can be integrated once to yield

$$y' \frac{\partial \tilde{F}}{\partial y'} - \tilde{F} = -c$$
where \( \tilde{F} = F - \lambda G \), \( \lambda \) is a Lagrange multiplier, and \( c \) is an integration constant. Plugging \( F = \sqrt{y'^2 + y^2} \) into the equation and solving it for \( y' \), the equation can be cast into the form

\[
y'^2 + V(y) = 0
\]  \hspace{1cm} \text{(2.1)}

with a restriction

\[
c + \lambda G = \frac{c(1 - y) + \lambda}{1 - y} > 0 \quad \text{for all} \ \theta
\]  \hspace{1cm} \text{(2.2)}

where

\[
V(y) = y^2 W(y) \quad \text{and} \quad W(y) = 1 - \frac{y^2}{(c + \lambda G)^2} = 1 - \frac{y^2(1 - y)^2}{(c(1 - y) + \lambda)^2}.
\]  \hspace{1cm} \text{(2.3)}

It is convenient to write \( W \) as

\[
W(y) = -L(y)/S(y)^2
\]  \hspace{1cm} \text{(2.4)}

where

\[
S(y) = c(1 - y) + \lambda
\]
\[
L(y) = y^2(1 - y)^2 - S(y)^2 = P(y)Q(y)
\]
\[
P(y) = y(1 - y) - S(y) = -y^2 + (1 + c)y - (c + \lambda)
\]
\[
Q(y) = y(1 - y) + S(y) = -y^2 + (1 - c)y + (c + \lambda)
\]

Because \( y \) is restricted within \( y \in (0, 1) \), Eq. (2.2) is equivalent to \( S(y) > 0 \) for all \( \theta \).

### 2.2. Classification of potentials and solutions

We note that Eq. (2.1) describes a Hamiltonian dynamics of a fictitious single particle, with zero total energy, in a potential \( V \) which depends on two parameters \( c \) and \( \lambda \). Thus, it is crucial to study dependence of zeroes of \( V \) with respect to the parameters. Apart from the trivial \( y = 0 \), zeroes of \( V \) are obtained as zeroes of \( W \), or the quartic \( L \).

It is easy to see that: \( L(0) = -S^2(0) = -(c + \lambda)^2 < 0 \), \( L(1) = -S^2(1) = -\lambda^2 < 0 \), and \( L \sim y^4 \) as \( |y| \to \infty \). They together imply there is one root \( y < 0 \) and another root \( y > 1 \). Therefore, there may be either zero, one (double), or two real roots of \( L \) within \( y \in (0, 1) \) for a given set of parameter values \( (c, \lambda) \).

When there is no root of \( W \) in \( y \in (0, 1) \), there is no zero-energy trajectory of (2.1) within \( y \in (0, 1) \), hence no solution to the variational problem.

When \( W \) has a double root \( y^* \in (0, 1) \), the zero-energy solution to (2.1) is the constant function \( y(\theta) \equiv y^* \), or the “circular” state in the polar coordinates. The corresponding trajectory on the \( (y, y') \)-phase plane is simply a fixed point. In this case, the constraint (1.2) becomes

\[
\phi = \frac{2\pi}{1 - y^*}.
\]  \hspace{1cm} \text{(2.5)}

The final case when \( W \) has two distinct roots in \( y \in (0, 1) \) is illustrated in Fig. 2.1. Denote the roots by \( y_1 \) and \( y_2 \) with \( y_1 < y_2 \). There is a zero-energy trajectory on the
Figure 2.1. When the potential function $V$ has two roots in $y \in (0, 1)$ as shown in (a), the trajectory with zero total energy is the thick closed loop in the phase plane (b), corresponding to a non-circular solution in the polar coordinates (c), provided that the period of oscillation $T$ in the phase plane satisfies the commensurability condition (2.6). (In (c), for example, $m = 3$.) Such a non-circular state may appear as a polygon depending on the control parameter $\phi$.

phase plane which oscillates between the roots. This periodic trajectory corresponds to modulations in $y(\theta)$. In this case, not only the constraint (1.2) must be satisfied but also the fundamental period of the trajectory $T$ must be commensurate with the geometry, i.e.

$$mT = 2\pi, \quad m = 1, 2, \ldots$$

(2.6)

Even though the potential and the solution are smooth, they may take highly non-uniform shapes such that the solution appears as a polygon with sharp corners. By following the periodic trajectory $m$ rounds, we may only obtain a regular “polygon” with $m$ corners, in agreement with the observation.

2.3. Parameter regime

Hansen et al. [9] noticed that using $(y_1, y_2)$ as the independent parameters makes the expressions much more transparent. This presentation turns out to be useful in our analysis of the limiting $\phi$-values in Sec. 3.1. We mostly reproduce their work in this and the next sections for completeness and because the thesis is not easily accessible.

We begin by analyzing roots of the quartic $L$ and determining regions in the parameter plane $(c, \lambda)$ where we have solutions. We first concentrate on the half plane $\lambda > 0$. Four roots of $L$ are two roots of $P$

$$y = P_\pm = (1 + c)/2 \pm \sqrt{(1 - c)^2/4 - \lambda}$$

and two roots of $Q$

$$y = Q_\pm = (1 - c)/2 \pm \sqrt{(1 + c)^2/4 + \lambda}.$$ 

Clearly, $Q_\pm$ are always real when $\lambda > 0$. Moreover, $Q_+ > 1$ is guaranteed since $Q(1) = \lambda > 0$ and $Q \to -\infty$ as $y \to \infty$. Thus, it is necessary to have real $P_\pm$ in order
to have a solution. The discriminant condition $D_P = (1 + c)^2 - 4(c + \lambda) > 0$ translates into a condition for possible solutions

$$0 < \lambda < (1 - c)^2/4.$$ 

This region is further divided into four zones according to the values of three roots $P_\pm$ and $Q_-$. It is straightforward to show that:

1. $Q_- < 0 < P_- < P_+ < 1$ when $\max(0, -c) < \lambda$ and $-1 < c < 1$.
2. $P_- < 0 < Q_- < P_+ < 1$ when $\lambda < -c$.
3. $P_- < 0, Q_- < 0$, hence no solution, when $-c < \lambda$ and $c < -1$.
4. $Q_- < 0, P_+ > 1$, hence no solution, when $c > 1$.

In the second zone it is clear that $S(y)$ changes its sign and violates (2.2) across $y_s = 1 + \lambda/c$. Thus, the first zone is the only region in which a zero-energy periodic trajectory as in Fig. 2.1(c) may be found. Because the potential $V$ changes continuously with the parameters, a zero-energy fixed point must be observed on one or more of the borders of this region which we’ll determine soon.

Now, consider the other half plane $\lambda < 0$. It is useful to consider the following symmetries of the functions; as the parameters $(c, \lambda)$ is changed to $(-c, -\lambda)$ while $y$ is held fixed, we have $S(y; -c, -\lambda) = -S(y; c, \lambda)$, $P(y; -c, -\lambda) = Q(y; c, \lambda)$, and $Q(y; -c, -\lambda) = P(y; c, \lambda)$ while $L$, $W$, and $V$ are invariant. Therefore, there exists a zero-energy solution for $(c, \lambda)$ if and only if there is one for $(-c, -\lambda)$. However, in the symmetric counterpart to the zone 1 considered above, we have $S(y) < 0$, and no solution here satisfies (2.2). Similarly, the counterpart to the zone 2 corresponds to alternating signs of $S$ and is also rejected. Consequently, there is no region for solutions on this half plane.

Therefore, we consider the zone 1, the only zone which may have solutions, in the rest of this section.

### 2.4. Computation of solutions

In zone 1 two roots of $V$ within $y \in (0, 1)$ are $y_1 = P_-$ and $y_2 = P_+$. They serve as turning points for a zero-energy periodic orbit. It is, in fact, more convenient to use $y_{1,2}$ as independent parameters instead of $(c, \lambda)$. Since they are roots of $P(y)$, the parameter sets are related to each other by

$$\begin{cases} 
y_1 + y_2 = 1 + c \\
y_1y_2 = c + \lambda
\end{cases} \quad (2.7)$$

Computing solutions is carried out in a standard manner for one degree of freedom Hamiltonian systems. We integrate the Euler-Lagrange equation (2.1) for a half period $T/2$ in which $y \geq 0$. Without loss of generality we assume $y = y_1$ when $\theta = 0$, and $y = y_2$ when $\theta = T/2$. Within this interval Eq. (2.1) can be integrated by

$$\int_0^\theta d\theta = \int_{y_1}^y \frac{d\eta}{\sqrt{-V(\eta)}} = \int_{y_1}^y \frac{S(\eta)d\eta}{\eta\sqrt{L(\eta)}}$$

using (2.2–2.4) where $0 \leq \theta \leq T/2$ and $y_1 \leq y \leq y_2$. Since $L(y) = (y_2 - y)(y - y_1)Q(y)$ vanishes at $y_1$ and $y_2$, we transform the variable from $y$ to $s$ as

$$y = y_1 + \frac{1}{2}(1 - \cos s)(y_2 - y_1) \quad (2.8)$$
to remove the singularities. Now the equation becomes

$$\theta(s; y_1, y_2) = \int_0^s \frac{S(y(s'))}{y(s') \sqrt{Q(y(s'))}} ds'$$

where \(0 \leq s \leq \pi\). This integrand diverges at \(s = 0\) and/or \(\pi\) only when \(y_{1,2}\) approach two of the borders of zone 1 as we’ll see in Sec. 3.1. It is easy to integrate it numerically except near the singularities where a high precision numerics is necessary.

Equations (2.8, 2.9) provide the parametric expression of the solution, but a periodic solution must also satisfy the compatibility condition (2.6) where \(\theta(\pi) = T/2\). This is achieved by defining

$$m(y_1, y_2) = \frac{\pi}{\theta(\pi; y_1, y_2)}$$

(2.10)

and seeking the parameter set \((y_1, y_2)\) such that \(m\) takes an integer value 1, 2, ..., 

Once a solution is found, the corresponding value of \(\phi\) is computed by (1.2). It can be evaluated in a similar manner which transforms (1.2) into

$$\phi(y_1, y_2) = 2mI \quad \text{where} \quad I = \int_0^{\pi} \frac{S(y(s))G(y(s))}{y(s) \sqrt{Q(y(s))}} ds$$

(2.11)

Here, the factor \(2m\) resulted from the relation (2.6).

To compute polygons with \(m\) corners for a given positive integer \(m\) it is easy to start from a circular solution, then continue a branch on the parameter plane. (We shall call this branch the \(m\)-th branch.) All the circular solutions live on the diagonal \(y_1 = y_2\) which corresponds to one of the borders of the zone \(\lambda = (1 - c)^2/4\). Here, \(c = 2y - 1\), \(\lambda = (1 - y)^2\), \(\theta(\pi) = \pi \sqrt{(1 - y)/(2y)}\), and \(I = \pi \sqrt{2y(1 - y)}\). Therefore, as \(y\) increases from 0 to 1, \(m = \sqrt{2y/(1 - y)}\) increases monotonically from 0 to \(\infty\) whereas \(\phi = 2\pi/(1 - y)\) increases monotonically from 2\(\pi\) to \(\infty\). Conversely, for a given \(m\), the \(m\)-th branch can be started at

$$y = m^2/(m^2 + 2).$$

(2.12)

The corresponding value of \(\phi\) given by (2.5) is \(\phi = \pi(m^2 + 2)\).

2.5. Computed results

Figure 2.2 shows the results of continuation of the \(m\)-th branches for \(m = 1\) to 5 in solid curves. Starting from a point on the diagonal of this parameter plane, a squeezing algorithm was employed to accurately find the level curves of \(m\). Each curve for \(m = 3, 4, \ldots\) appears to converge to a point on the \(y_2 = 1\) line while curves for \(m = 1, 2\) appear to approach the corner \((0,1)\). However, numerics becomes increasingly harder near the borders \(y_1 = 0\) or \(y_2 = 1\) as the integrand starts to rapidly change near \(s = 0\) or \(\pi\).

In Fig. 2.2 the \(\phi\)-contours are also computed for \(\phi = 10, 20, \ldots, 50\) and shown as dashed curves. They appear qualitatively similar to the \(m\)-contours, and their limiting behavior near the borders is again hard to resolve numerically.
Figure 2.2. Contour curves for $m = 1, \ldots, 5$ (solid) and $\phi = 10, \ldots, 50$ (dashed) on the parameter plane. The diagonal $y_1 = y_2$ denotes the circular states, and each of the $m$-th branches emanates from it at $y = m^2/(m^2 + 2)$. It is hard to resolve the curves numerically near borders $y_1 = 0$ or $y_2 = 1$. In Sec. 3.1 it is shown that each $m$-contour terminates at $(y_1, y_2) = (\cos(\pi/m), 1)$ for $m = 3, 4, \ldots$ and at $(0, 1)$ for $m = 1, 2$ while all $\phi$-contours converge to $(0, 1)$. The two sets of contours are generally transverse to each other so that $\phi$ varies along an $m$-contour.

The two sets of curves are generally transversal to each other, so $\phi$ value changes along each $m$-contour curve. This is more explicitly shown in the bifurcation diagram Fig. 2.3. Here, $y_{1,2}$ values (the minimum and maximum radii of $y$) for each solution along the $m$-th branch are plotted as functions of the control parameter $\phi$. The circular state is represented by a single curve since $y_1 = y_2$. The curve is given by $y = 1 - 2\pi/\phi$, the inverse of (2.5), and exists for any $\phi \geq 2\pi$. Each $m$-th branch emanates from this curve when $y = m^2/(m^2 + 2)$, or $\phi = \pi(m^2 + 2)$. Then, the difference between $y_1$ and $y_2$ quickly widens as $\phi$ is varied.

Numerically, the branches $m = 1$ and 2 are found to bifurcate subcritically with respect to $\phi$ whereas the branches with higher $m$ values bifurcate supercritically. Along each branch $y_2$ quickly approaches unity as $\phi$ is varied away from the bifurcation point, so it was not easy to follow the branches beyond what is shown in the figure. Asymptotic analysis is clearly necessary to determine how the branches end, which will be the issue of the next section.

Five points are selected along the $m = 3$ branch, and the corresponding solutions on the phase plane as well as in the polar coordinates are shown in Fig. 2.4. The solutions far away from the diagonal in the parameter plane indeed appear as regular triangles. The corresponding trajectories on the phase portrait are strongly deformed.

Solutions in the polar coordinates along contours for $m = 1, 2, 4, 5$ are shown in Fig. 2.5. The solutions for $m = 1, 2$ approach needle-like segments. Consequently, $y_1$ for these solutions converges to 0 as can be seen in Fig. 2.3. In contrast the solution for $m = 3$ and higher seems to approach a regular polygon whose corner is unit distance
Figure 2.3. The minimum and maximum radii, $y_1$ and $y_2$, of the solutions to the Euler-Lagrange equation are shown as functions of the control parameter $\phi$. The curve $y = 1 - 2\pi/\phi$ originating from $(2\pi, 0)$ denotes the branch of the circular state. Each $m$-th branch emanates from this curve at $\phi = \pi(m^2 + 2) \approx 9.4, 18.8, 34.6, 56.5, 84.8, \ldots$. The first five branches are shown in (a), and the $m = 1$ and $m = 2$ branches are enlarged in (b) and (c), respectively. The branching is subcritical for $m = 1, 2$ and supercritical for $m = 3$ and higher. In Sec. 3.1 it is shown that the branches for $m = 1$ and $2$ terminate at $\phi = 3\pi$ and $4\pi$, respectively. The branches for higher $m$ continue to $\phi \to \infty$, indicating many coexisting solutions when $\phi$ is large. It becomes increasingly hard to resolve the branches numerically as $y_2 \to 1$; only computed parts of the branches are shown.

Figure 2.4. (a) Trajectories corresponding to five different solutions along the $m = 3$ branch in Fig. 2.2 shown in the phase plane. A trajectory just after the bifurcation point makes a small closed loop whereas it is strongly deformed away from the bifurcation. (b) Corresponding solutions to (a) shown in the polar coordinates. As $\phi$ is increased, the solution deforms from circular into a regular triangle with apparently sharp corners.
Figure 2.5. Solutions for $m = 1, 2, 4, 5$ in the polar coordinates. Just as in Fig. 2.4(b) five solutions are shown along each of the $m$-th branches. They are nearly circular just after the bifurcation point, but approach regular $m$-gons for $m = 3, 4, \ldots$ and line segments for $m = 1, 2$. 
away from the center. If so, the limiting values of \( y_1 \) would be \( \cos(\pi/m) \), the distance from the center to the sides of a regular polygon with \( m \) corners. This explains the limiting values of \( y_1 \) observed in Fig. 2.3, and will be proven to be so in Sec. 3.1.2.

3. Asymptotic analysis along the polygonal branches

In this section we use asymptotic methods to answer the questions (Q1) and (Q2) in Sec. 1.

3.1. Limiting \( \phi \) values

The two integrals \( \theta(\pi) \) and \( I \) defined in (2.9,2.11) show intricate asymptotic behaviors near the borders \( y_1 = 0 \) or \( y_2 = 1 \). We analyze them in order to determine the limiting \( \phi \) values along the \( m \)-contours.

3.1.1. When \( y_1 \to 0 \) while \( y_2 = \) fixed

In this limit we have \( c = -(1 - y_2) - y_1 \) and \( \lambda = (1 - y_2) - y_1(1 - y_2) \). If we simply took only the leading order term in \( y_1 \), both integrands of \( \theta(\pi) \) and \( I \) would be proportional to \( 1/s \) near \( s = 0 \), making the integrals divergent. Thus, we consider an interval \((0,s_0)\) with a positive constant \( s_0 \) being so small that the terms in the integrands can be approximated by their expansions. By taking up to \( O(y_1) \), we have \( y \sim y_2 s^2/4 + y_1, \ S(y) \sim y_2 (1 - y_2) s^2/4 + y_1, \ Q(y) \sim y_2 (1 - y_2) s^2/4 + 2 y_1 \). Then, the integrals can be carried out explicitly. Their leading order terms do not depend on the choice of \( s_0 \). Since the integrands are otherwise regular, the integrals over \( s \in (s_0, \pi) \) contribute only at \( O(1) \). As a result, the leading order term of the integrals over \((0,\pi)\) are identical to the local integrals over \((0,s_0)\). They are found to be \( \theta(\pi) \sim \{(1 - y_2)/\sqrt{y_2(2 - y_2)}\} \log(1/y_1) \) and \( I \sim \theta(\pi) \). This means \( m = \pi/\theta(\pi) \sim \{\pi \sqrt{y_2(2 - y_2)/(1 - y_2)}\} \log^{-1}(1/y_1) \to 0 \) and \( \phi \sim 2\pi \) as \( y_1 \to 0 \).

3.1.2. When \( y_2 \to 1 \) while \( y_1 = \) fixed

On the line we have \( c = y_1, \ \lambda = 0, \ y = (1 + y_1)/2 - (1 - y_1) \cos(s)/2, \ S = y_1 (1 - y), \) and \( Q = (1 - y)(y + y_1) \). The first integral can be carried out explicitly, and \( \theta(\pi) = 2 \arctan(1 - y)/\arctan(1 + y) \), or \( m = \pi/\{2 \arctan(1 - y)/\arctan(1 + y)\} \). Thus, \( m \) increases monotonically from 2 to \( \infty \) as \( y_1 \) varies from 0 to 1 on this line. Conversely, for a given \( m \geq 2 \), we solve this relation for \( y_1 \) and find where the \( m \)-th branch terminates to be \( y_1 = 2/(1 + \tan^2(\pi/2/m)) - 1 = \cos(\pi/m) \). This confirms the limiting values guessed in Sec. 2.5 from geometrical consideration.

In contrast the second integral \( I \) diverges since \( G = 1/(1 - y) \) is singular at \( s = \pi \). In a similar manner to the previous section we consider an interval \((\pi - s_0, \pi)\) with a small \( s_0 > 0 \), and approximate the terms in the integrand as \((1 - y_2) \to 0 \) by \( y \sim 1, \ S \sim y_1 (1 - y_1)(\pi - s)^2/4 + (1 - y_2), \ Q \sim (1 - y_1^2)(\pi - s)^2/4 + 2(1 - y_2), \) and \( G \sim 1/(1 - y_1)(\pi - s)^2/4 + (1 - y_2) \). Then, the integral can be carried out explicitly, and we find the leading order term independent of \( s_0 \) to be \( I \sim (y_1/\sqrt{1 - y_1^2}) \log(1/(1 - y_2)) \). This means that \( \phi \sim (2m y_1/\sqrt{1 - y_1^2}) \log(1/(1 - y_2)) \to \infty \) in the limit. From this we conclude that the \( m \)-th branch for \( m = 3 \) or higher continues up to arbitrarily large values of \( \phi \).
3.1.3. Near the corner \((y_1, y_2) = (0,1)\)

It is clear from the previous two sections that both \(m\) and \(\phi\) are singular at the corner, and this leads to an ambiguity as to the limiting \(\phi\) values for the \(m = 1, 2\) branches. Taking distinguished limits can be carried out in a similar manner except that now both neighborhoods of \(s = 0\) and \(s = \pi\) may contribute to the leading order.

Along straight rays \(y_1 = \epsilon \cos \sigma, y_2 = 1 - \epsilon \sin \sigma\) for some \(0 < \sigma < \pi/2\), we can evaluate the local integrals in the limit of \(\epsilon \to +0\). It is straightforward to calculate them as

\[
\int_{s_0}^{s_1} \frac{S}{\sqrt{Q}} ds \sim \frac{1}{2} + (\sin \sigma) \epsilon \log(1/\epsilon) + O(\epsilon), \quad \int_{s_0}^{\pi - s_1} \frac{S}{\sqrt{Q}} ds \sim (\sin \sigma) \epsilon \log(1/\epsilon) + O(\epsilon), \quad \int_{s_0}^{\pi - s_1} \frac{S G}{Q} ds \sim \frac{\pi}{2}, \quad \text{and} \quad \int_{\pi - s_0}^{\pi} \frac{S G}{Q} ds \sim \frac{\pi}{2}.
\]

Observing in addition that integrals over \((s_0, \pi - s_0)\) are of \(O(\epsilon)\) for both integrals, we obtain approximations up to \(O(\epsilon)\) by summing the local integrals: \(m \sim 2/(1 + (4/\pi)(\sin \sigma) \epsilon \log(1/\epsilon))\) and \(\phi \sim 4\pi\). Thus, \(m\) approaches 2 from below while the limiting value of \(\phi\) is \(4\pi\) regardless of \(\sigma\). The singular behavior near the corner is still not sufficiently resolved.

3.1.4. Along an exponential curve near \(y_1 = 0\)

We notice from the analysis of Sec. 3.1.1 that level curves of \(m\) near the \(y_2\) axis locally look like \(y_1 \sim \exp(-\pi \sqrt{y_2(2 - y_2)(1 - y_2)^{-1} m^{-1}})\) indicating an exponentially small layer near the axis where \(m\) and \(\phi\) vary rapidly.

Therefore, we take distinguished limits along a family of curves \(y_1 = a \exp(-\sigma/(1 - y_2))\) as \(y_2 \to 1\) for \(a > 0\) and \(\sigma > 0\). As expected, an exponentially small term in the integrands becomes important near \(s = 0\). It is straightforward to show \(m \sim 2 \pi/((\pi + 2 \sigma))\) and \(\phi \sim 4\pi(\pi + \sigma)/(\pi + 2 \sigma)\) in this limit, independent of \(a\). Thus, as \(\sigma\) increases from 0 to \(\infty\), \(m\) decreases monotonically from 2 to 0, and \(\phi\) decreases monotonically from \(4\pi\) to \(2\pi\). This fills the gaps in \(m\) and \(\phi\) near the \(y_2\) axis. In particular, the \(m = 1\) contour is achieved when \(\sigma = \pi/2\), and this corresponds to the limiting value of \(\phi = 3\pi\).

3.1.5. Along an exponential curve near \(y_2 = 1\)

Finally, we consider similar distinguished limits along a family of curves \(y_2 = 1 - a \exp(-\sigma/y_1)\) as \(y_1 \to 0\) for \(a > 0\) and \(\sigma > 0\). Similar local approximations of the integrals near \(s = 0\) and \(s = \pi\) lead to \(m \sim 2 + O(y_1)\) and \(\phi \sim 4\pi(\pi + \sigma)\). Now, \(\phi\) is well resolved as it increases monotonically from \(4\pi\) to \(\infty\) as \(\sigma\) increases from 0 to \(\infty\).

However, \(m\) is found to be constant at this order, and we cannot identify which value of \(\phi\) corresponds to the \(m = 2\) contour curve. Thus, the leading correction at \(O(y_1)\) needs to be resolved. We note that not only neighborhoods of the end points but the whole interval \((0, \pi)\) contribute at \(O(y_1)\).

The leading order contribution \(m \sim 2\) in this case comes solely from the neighborhood of \(s = 0\). Here, terms in the integrand can be locally approximated as before as \(y \sim s^2/4 + y_1, S \sim y_1, Q \sim s^2/4 + 2y_1\). We subtract this part from the integrand and consider

\[
J = \int_0^\pi \left[ \frac{S}{\sqrt{Q}} - (8y_1)/\{(s^2 + 4y_1)^{\sqrt{s^2 + 8y_1}}\} \right] ds.
\]

The new integrand can be expanded in powers of \(y_1\) from which we take only the leading order terms:

\[
J = y_1 \int_0^\pi \frac{2(1 + \cos \sigma)}{(1 - \cos \sigma) \sqrt{\sin^2 \sigma}} ds + O(y_1^2) = y_1(4/\pi^2 - 2/3) + O(y_1^2).
\]

Because

\[
\int_0^\pi 8y_1/\{(s^2 + 4y_1)^{\sqrt{s^2 + 8y_1}}\} ds = 2 \arctan(\sqrt{s^2 + 8y_1}/y_1) = \pi/2 - 4y_1/\pi^2 + O(y_1^2),
\]

we obtain \(m = \pi\{\pi/2 - 2y_1/3 + O(y_1^2)\} + \pi/2 - 4y_1/\pi^2 + O(y_1^2)\). Therefore, as \(y_1 \to 0\) along the family of curves, \(m \to 2\) from above regardless of \(a\) or \(\sigma\).

Together with what we found in Sec. 3.1.3, the \(m = 2\) contour has now been
shown to approach the corner \((0,1)\) along a curve which converges to the line \(y_2 = 1\) faster than any linear function \(y_2 = 1 - (\tan \sigma)y_1\) for \(0 < \sigma < \pi/2\) but slower than any exponentials \(y_2 = 1 - \exp(-\sigma/y_1)\) for \(0 < \sigma < \infty\). Although the precise functional form of the level curve is still unclear, this information suffices for our purpose. Since the limiting value \(\phi \sim 4\pi\) is identical for all the linear functions as well as the exponential function in the limit \(\sigma \to 0\), the limiting value for the \(m = 2\) level curve must also be \(\phi = 4\pi\).

### 3.2. Approximate polygonal solution just after bifurcation

We now turn to solutions just after a bifurcation, and approximate them using a standard weakly nonlinear perturbation. Exactly at the bifurcation point \(\phi^* = \pi(m^2 + 2)\) of the \(m\)-th branch, we have

\[
y^* = \frac{m^2}{m^2 + 2}, \quad c^* = 2y^* - 1 = \frac{m^2 - 2}{m^2 + 2}, \quad \lambda^* = (1 - y^*)^2 = \frac{4}{(m^2 + 2)^2}.
\]

We perturb away from the point using a small parameter \(\delta > 0\) defined to be

\[
\sqrt{\delta} = \frac{y_2 - y_1}{2}
\]

and pose

\[
c = c^* + \delta c_1 + \delta^2 c_2 + \ldots
\]

Since \(y_1 = P_-\) and \(y_2 = P_+\), we need

\[
\delta = \frac{(1-c)^2}{4} - \lambda, \quad \text{or}, \quad \lambda = \frac{(1-c^*)^2}{4} - \delta \left\{1 + c_1(1-c^*)\right\} + \ldots
\]

and

\[
y_{1,2} = \frac{1+c^*}{2} \mp \sqrt{\delta} + \delta \frac{c_1}{2} + \ldots
\]

The modulation of \(y\) is approximated as

\[
y(s) = y_1 + \frac{1 - \cos s}{2}(y_2 - y_1) = \frac{1+c^*}{2} - \sqrt{\delta} \cos s + \delta \frac{c_1}{2} + \ldots
\]

Then, by substitution, it is straightforward to approximate the integrand of \(\theta(s)\) as

\[
\frac{S}{y\sqrt{Q}} = \frac{1}{m} + \sqrt{\delta} \frac{2\cos s}{1-c^2} - \delta \left\{\frac{c_1 + 3}{1-c^2} + \frac{3c^2 - 4c^* + 5}{(1-c^2)^2} \cos^2 s\right\} + \ldots
\]

Integrating this expression over \(s \in (0,\pi)\) yields the following condition for \(c_1\) in order to satisfy the constraint (2.10) for an integer \(m\) up to \(O(\delta)\).

\[
c_1 = -\frac{m^2}{8} + \frac{3}{2m^2} - 1
\]

Using this, we can similarly approximate the other integral \(\phi = \phi^* + \delta \phi_1 + \ldots\) where

\[
\phi_1 = \frac{\pi(m^2 + 2)(m^4 - 8m^2 + 4)}{32m^2}.
\]
We find that $\phi_1 < 0$ for $m = 1$ or 2, so these branches emanate subcritically as $\phi$ increases. In contrast, $\phi_1 > 0$ when $m \geq 3$, so these branches emanate supercritically. As expected, the amplitude of oscillation in $y$ in terms of the control parameter grows like $\sqrt{\delta} \approx \phi_1^{-1/2}(\phi - \phi^*)^{1/2}$ a little away from the bifurcation point. Finally, the solution is approximated parametrically by (3.2) and

$$\theta(s) = \int_0^s \frac{S(y(s'))}{y(s')\sqrt{Q(y(s'))}} ds' = \frac{s}{m} + \sqrt{\delta} \frac{(m^2 + 2)^2}{4m^3} \sin s - \delta \frac{(m^2 + 2)(m^4 - 16m^2 - 12)}{128m^5} \sin(2s) + \ldots (3.3)$$

4. Direct method and the second variation

In this section we answer the question (Q3) in Sec. 1.

4.1. Derivation of the second variation

Having now obtained the solutions to the Euler-Lagrange equation, we revisit the original problem (1.1,1.2) and study whether they indeed minimize the circumference (in which case we call the solution to be “stable”) or not (“unstable”) by finding the second variation directly. We express a solution as $Y(\theta)$, and denote a perturbed solution by $y(\theta)$. Admissible solutions to the minimization problem must satisfy the constraint (1.2), so we relate $y$ to $Y$ by

$$\frac{1}{1-y} = \frac{1}{1-Y} + p(\theta)$$

where

$$p(\theta) = \epsilon \sum_{\ell=-\infty}^{\infty} \alpha_{\ell} e^{i\ell \theta}.$$  

(4.2)

Here, the perturbation must fulfill $\frac{1}{2\pi} \oint p d\theta = \alpha_0 = 0$ as well as $\alpha_{-\ell} = \alpha_{\ell}^*$ for $\ell \geq 1$.

We take the limit $\epsilon \to 0$. Since the perturbation $p = O(\epsilon)$, we obtain $y = \{Y + p(1-Y)\}/\{1 + p(1-Y)\} = Y + p(1-Y)^2 - p^2(1-Y)^3 + O(\epsilon^3)$. From this $y'$ can be calculated, and it is easy to show

$$\sqrt{y'^2 + y^2} = \sqrt{Y'^2 + Y'^2} \left\{1 + h_1 + \frac{1}{2}(h_2 - h_1^2) + O(\epsilon^3)\right\}$$

(4.3)

where

$$h_1(\theta) = \frac{p\left\{Y(1-Y)^2 - 2Y'^2(1-Y)\right\} + pY'(1-Y)^2}{Y^2 + Y'^2} = O(\epsilon)$$

$$h_2(\theta) = \frac{p^2 \left\{(1-3Y)(1-Y)^3 + 10Y'^2(1-Y)^2\right\} - 8pp'Y'(1-Y)^3 + p^2(1-Y)^4}{Y^2 + Y'^2} = O(\epsilon^2)$$

Therefore, the first variation of the functional (1.1) is

$$\oint h_1 \sqrt{Y'^2 + Y'^2} d\theta = \oint p(\theta) \left\{\frac{Y(1-Y)^2 - 2Y'^2(1-Y)}{\sqrt{Y'^2 + Y'^2}} - \frac{d}{d\theta} \left(\frac{Y'(1-Y)^2}{\sqrt{Y'^2 + Y'^2}}\right)\right\} d\theta$$
where we have integrated by parts once. This must vanish for all admissible functions satisfying $\oint p d\theta = 0$. Therefore, the terms inside the curly brackets need not vanish, but must remain constant.

$$\frac{d}{d\theta} \left( \frac{Y'(1-Y)^2}{\sqrt{Y^2 + Y'^2}} \right) - \frac{Y(1-Y)^2 - 2Y'^2(1-Y)}{\sqrt{Y^2 + Y'^2}} = -\lambda \quad (4.4)$$

Rewriting this equation recovers the Euler-Lagrange equation (2.1) which we have already analyzed. Then, the second variation of the functional (1.1) is given by

$$V_2 = \frac{1}{2} \oint (h_2 - h_1^2) \sqrt{Y'^2 + Y''^2} d\theta. \quad (4.5)$$

### 4.2. The circular states

Now, we are ready to prove Theorem 1.1.

**Proof.** The second variation can be analytically treated for the circular state for which $Y$ is constant. For this case we obtain

$$V_2 = \frac{(1-Y)^3}{2Y} \left\{ (1-Y) \oint p^2 d\theta - 2Y \oint p^2 d\theta \right\}$$

$$= \frac{2\pi\epsilon^2(1-Y)^3}{Y} \left\{ (1-Y) \sum_{k=1}^{\infty} k^2 |\alpha_k|^2 - 2Y \sum_{k=1}^{\infty} |\alpha_k|^2 \right\}. \quad (4.6)$$

Clearly, as $Y$ varies from 0 to 1, its sign changes from positive to negative for any nonzero perturbation $p$. Correspondingly, the basic state is stable against the chosen $p$ iff

$$Y \leq \frac{\sum_{k=1}^{\infty} k^2 |\alpha_k|^2}{\sum_{k=1}^{\infty} k^2 |\alpha_k|^2 + 2 \sum_{k=1}^{\infty} |\alpha_k|^2} = \frac{\sum_{k=1}^{\infty} k^2 |\alpha_k|^2}{\sum_{k=1}^{\infty} (k^2 + 2)|\alpha_k|^2} = Y_c$$

When $p$ consists only of one mode $k = m$, the critical value is $Y_c = m^2/(m^2 + 2)$, coinciding (2.12). When $p$ consists of modes $k = m$ and higher, the critical value becomes

$$Y_c = \frac{m^2 \sum_{k=m}^{\infty} \frac{k^2}{m^2} |\alpha_k|^2}{(m^2 + 2) \sum_{k=m}^{\infty} \frac{k^2 + 2}{m^2 + 2} |\alpha_k|^2} = \frac{m^2}{m^2 + 2} \left\{ 1 + \frac{2 \sum_{k=m}^{\infty} (k^2 - m^2)|\alpha_k|^2}{m^2 \sum_{k=m}^{\infty} (k^2 + 2)|\alpha_k|^2} \right\} \geq \frac{m^2}{m^2 + 2}$$

which shows that interactions from the additional higher modes always shift $Y_c$ to a higher value. Consequently, the circular solution is stable against a perturbation including mode $m$ and higher up to $Y_c = m^2/(m^2 + 2)$. It follows that, as $Y$ (or $\phi$) is increased from 0 (or $2\pi$, respectively), the circular solution first loses its stability at $Y_c = \frac{1}{3}$ (or $\phi_c = 3\pi$) for $p$ consisting purely of $k = 1$, and remain unstable for all higher $Y$ or $\phi$ values. \qed

### 5. Conclusions and discussions

In this article a simple but nontrivial variational problem with an integral constraint is considered that is motivated by the experimental observations of the polygonal hydraulic jumps. The model is analytically tractable to great extent, and includes
only one parameter $\phi$, expressing the total flux. Possible solutions are summarized in Fig. 2.3 where each $m$-th branch emanates from the circular state at $\phi_c = \pi(m^2 + 2)$. The amplitude of the modulation increases in proportion to $\sqrt{\phi - \phi_c}$ near the bifurcation point. The branches for $m = 1$ and 2 emanate subcritically, and are terminated at $\phi = 3\pi$ and $4\pi$, respectively. The branches for $m \geq 3$ emanate supercritically, and continue for arbitrarily large $\phi$ values, leading to coexisting polygonal solutions when $\phi$ is large.

The circular state is “stable,” i.e. minimizes the circumference, at least locally, when $\phi < 3\pi$, and is unstable when $\phi > 3\pi$. Study on the stability of the polygonal states remains still open, but may be possible just after a bifurcation based on the expansions in our weakly nonlinear analysis.

We note that no solution to the Euler-Lagrange equation with the constraint exists in the intervals: $3\pi < \phi < 4\pi$ between the $m = 1$ and $m = 2$ branches, and $6\pi < \phi < 11\pi$ between the $m = 2$ and $m = 3$ branches. It is probable that the minimizing solution must be sought among less smooth functions than the ones we assumed throughout this article, but how to find it is not immediately clear.

Study of the effect of generalizing the flux function $G(y)$ is also open. Finally, a problem of minimizing the area of a two-dimensional surface around a point source in the three-dimensional space under a similar flux constraint could be of interest which may show spherical symmetry breaking.

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