ON THE COLLAPSE OF THE LOCAL RAYLEIGH CONDITION FOR THE HYDROSTATIC EULER EQUATIONS AND THE FINITE TIME BLOW-UP FOR THE SEMI-LAGRANGIAN EQUATIONS

VICTOR CAÑULEF-AGUILAR

Abstract. In this paper we study the propagation of the local Rayleigh condition for the two-dimensional hydrostatic Euler equation in the framework of the local well-posedness result by Masmoudi and Wong [MW12]. We show under certain assumptions that such solutions will develop singularities or collapse the local Rayleigh condition. In addition, we find necessary conditions for the global solvability. Finally, we establish the finite time blow-up of solutions to the semi-lagrangian equations introduced by Brenier in [Bre99] for certain class of initial data.

1. Introduction

We consider the two-dimensional hydrostatic Euler equation in a periodic channel Ω = T × (0, 1):

\[
\begin{aligned}
    & u_t + uu_x + vu_y + P_x = 0, \\
    & u_x + v_y = 0, \\
    & P_y = 0, \\
    & v|_{y=0,1} = 0, \\
    & u|_{t=0} = u_0,
\end{aligned}
\]

(1.1)

where \((u, v)\) is the velocity field, \(P\) is the scalar pressure and \(u\) satisfies the local Rayleigh condition (i.e. \(\partial_y^2 u > 0\)). Equations (1.1) are derived from the two-dimensional incompressible Euler equation by means of the hydrostatic approximation. The mathematical justification of the formal limit has been established in several articles under the local Rayleigh condition (see for instance [Bre03], [Gre99] and [MW12]). On the other hand, if the initial condition does not satisfy the local Rayleigh condition, the convergence may not hold, as shown in [Gre00], [Gre99]. Furthermore, the ill-posedness of the linearization of (1.1) around certain shear flows was proved in [Ren09], as well as the ill-posedness of (1.1) around certain shear flows, which was proved in [HKN10].

The local existence of solutions to (1.1) has been proven in [KTVZ10] and [KTVZ11] in the analytic setting. Nevertheless, the finite time blow-up of solutions to (1.1) was established in [KW12] (under the assumption \(u_0(x_0, y, t) \equiv C\), for \(y \in [0, 1]\), which is incompatible with the local Rayleigh condition) and [CINT12] (under the assumption of parity, which is again incompatible with the local Rayleigh condition).
condition. See also \[EE97\], where the finite time blow-up of certain solutions to the unsteady Prandtl equations is proved by similar methods).

Equivalently, we may consider the vorticity formulation of the hydrostatic Euler equations \[1.1\]

\[
\begin{aligned}
\omega_t + u\omega_x + v\omega_y &= 0, \\
(u, v) &= \nabla^\perp \mathcal{A} \omega, \\
\omega|_{t=0} &= \omega_0,
\end{aligned}
\]

where the vorticity \(\omega\) is defined by

\[\omega = u_y.\]

The local well-posedness of \[1.2\] was proved in \[MWT2\] in Sobolev spaces \(H^s\), with \(s \geq 4\), for initial vorticity profiles that satisfy the local Rayleigh condition (i.e. \(\partial_y \omega_0 > 0\)). In this article we address the propagation in time of the local Rayleigh condition; we get lower bounds for certain functionals that quantify in some sense the validity of the local Rayleigh condition (see \[1.3\] and \[1.4\]). Under certain assumptions, we will prove that the above functionals cannot remain bounded, which implies the collapse of the local Rayleigh condition or the formation of singularities (see Theorem 1 which is proved in Section 4). One of the main achievements of this work, is the derivation of certain identities which are satisfied by every solution to \[1.2\], as long as the solution exists (see Proposition 4 in Section 3). In addition, we derive some necessary conditions for the global solvability in the framework of the local well-posedness result by Masmoudi and Wong \[MWT2\] (see Theorem 2 and Section 5). Despite Theorem 1 does not guarantees that the solution remains in \(H^1(\mathbb{T} \times (0,1))\) as long as the local Rayleigh condition holds, we can control the \(H^1\) norm and the validity of the local Rayleigh condition under additional assumptions, which is presented in Section 6.

The second main result of this article deals with the finite time blow-up of smooth solutions to the semilagrangian equations, which are derived from the hydrostatic Euler equations \[1.2\] under certain assumptions. More precisely, if \(\omega_0\) satisfies the local Rayleigh condition and

\[
\begin{aligned}
\omega_0(x, 0) &\equiv k, \\
\omega_0(x, 1) &\equiv k + 1,
\end{aligned}
\]

for certain constant \(k\), then problem \[1.2\] is equivalent to the following problem in \(\Omega = \mathbb{T} \times (0,1)\):

\[
\begin{aligned}
v_t + \partial_x \left( \frac{v^2}{2} + P \right) &= 0, \\
\partial_t h_a + \partial_x (vh_a) &= 0, \\
\int_0^1 h_a da &= 1, \\
\partial_a P &= 0, \\
(\nu, h_a)|_{t=0} &= (\nu_0, \partial_a h_0),
\end{aligned}
\]
where \( v \) and \( h_a \) are defined by

\[
\begin{align*}
\omega(x, h(x, t, a), t) &= k + a, \quad \text{for } a \in [0, 1], \\
v(x, t, a) &= u(x, h(x, t, a), t), \quad \text{for } a \in [0, 1],
\end{align*}
\]

in this case we assume that \( \partial_y h_0 \) is positive, which is equivalent to the validity of the local Rayleigh condition. Equations (1.6) are known as the semilagrangian equations, which were introduced by Brenier in [Bre99]. In the same article, the local existence of solutions to (1.2) was proved in the class \( C \) (which is defined as the set of \( C^1 \)-functions in \( T \times (0, 1) \) that satisfy the local Rayleigh condition, (1.4) and (1.5)). The semilagrangian equations (1.6) have a natural extension to higher dimensions, namely

\[
\begin{align*}
\partial_t v + v \cdot \nabla v + \nabla P &= 0, \\
\partial_t h_a + \nabla \cdot (v h_a) &= 0, \\
\int_0^1 h_a da &= 1, \\
\partial_a P &= 0, \\
\partial_i v_j - \partial_j v_i &= 0, \quad \text{for } i, j \leq d,
\end{align*}
\]

where \( v : T^d \times [0, 1] \to \mathbb{R}^d \) is the semilagrangian velocity, \( P \) is the scalar pressure and \( h_a \) is a positive function in \( T^d \times [0, 1] \). Let us point out that the semilagrangian equations (1.6) establish a vertical change of coordinates by means of the level curves of the vorticity \( \omega \), where the injectivity follows by the local Rayleigh condition. Hence, the extension of the semilagrangian equations to higher dimensions have no clear physical meaning.

A necessary condition for the global solvability of (1.9) was obtained in [Bre99], which is given by:

\[
\int_0^1 \left[ \int_{T^d} |v(x', t, a)dx'| \right]^2 \int_T h_a(x, t, a)dxdy = \int_{T \times (0,1)} |v|^2(x, t, a)h_a(x, t, a)dxdy,
\]

where both sides are time independent. Theorem 3 establishes the finite time blow-up of solutions to the semilagrangian equations (1.9) for certain class of initial data, which is proved in Section 7. In the following subsections, we will state the main results of this article.

1.1. Collapse of the local Rayleigh condition or singularity formation.

**Theorem 1.** Assume that the initial vorticity \( \omega_0 \in H^4(T \times (0,1)) \) satisfies the local Rayleigh condition (i.e. \( \partial_y \omega_0 > 0 \)) and \( \partial_x \omega_0 \) is not identically 0. Let \( \omega \in C([0,T];H^4(T \times (0,1))) \) be the solution to the problem (1.2) that satisfies the local Rayleigh condition in \([0,T]\). Set

\[
E_1(t) = \int_{T \times (0,1)} \frac{\omega x}{\omega} dxdy = \int_{T \times (0,1)} \left( \frac{\omega x}{\omega} - u_x \right) dxdy
\]

and
Remark 3. It is worth mentioning that the set of initial conditions that satisfy the assumptions of Theorem 4 is nonempty. Indeed, if we choose \( \omega_0(x, y) = 2y - \)
\( \sin(2\pi x - y) \), then \( \partial_x \omega_0 = -2\pi \cos(2\pi x - y) \) and \( \partial_y \omega_0 = 2 + \cos(2\pi x - y) \), which yields

\[
E_1(0) = \int_{T \times (0,1)} \frac{(2y - \sin(2\pi x - y)) \cdot -2\pi \cos(2\pi x - y)}{2 + \cos(2\pi x - y)} \, dx \, dy \\
= -2\pi \int_{T \times (0,1)} (2y - \sin(2\pi x - y)) \, dx \, dy + 4\pi \int_{T \times (0,1)} \frac{(2y - \sin(2\pi x - y))}{2 + \cos(2\pi x - y)} \, dx \, dy \\
= -2\pi + \int_{T \times (0,1)} \frac{4\pi \cdot 2y}{2 + \cos(2\pi x - y)} \, dx \, dy \\
= -2\pi + \frac{4\pi}{\sqrt{3}} \int_{0}^{1} 2y \, dy \\
= 2\pi \left( \frac{2}{\sqrt{3}} - 1 \right) > 0.
\]

**Remark 4.** Suppose that \( \omega_0(x, y) \in H^4(T \times (0,1)) \) satisfies the local Rayleigh condition, \( \partial_y \omega_0 \) is not identically 0 but \( E_1(0) < 0 \), then \( \tilde{\omega}_0(x, y) = \omega_0(-x, y) \) satisfies \( E_1(0) > 0 \) (the same applies for \( E_2 \)). Hence, the assumptions of Theorem 1 are not so restrictive.

1.2. Necessary conditions for global solvability.

**Theorem 2.** Assume that the initial vorticity \( \omega_0 \in H^4(T \times (0,1)) \) satisfies the local Rayleigh condition (i.e. \( \partial_x \omega_0 > 0 \)). Let \( \omega \in C([0,T]; H^4(T \times (0,1))) \) be the solution to the problem \( 1.2 \) that satisfies the local Rayleigh condition in \( [0,T] \). Let \( E_1 \) and \( E_2 \) be defined by (1.11) and (1.12) respectively. Assume further that \( T \) can be chosen arbitrarily large, then we have:

\[
\int_{0}^{\infty} \int_{T \times (0,1)} \left| P_x + u \left( u_x - \frac{\omega_x}{\omega_y} \right) \right|^2 \, dx \, dy \, dt = -E_2(0),
\]

(1.17)

\[
\int_{0}^{\infty} \int_{T \times (0,1)} \left| u_x - \frac{\omega_x}{\omega_y} \right|^2 \, dx \, dy \, dt = -E_1(0),
\]

(1.18)

\[
\int_{0}^{\infty} \int_{T} P_x^2 \, dx \, dt \leq -2E_2(0) - \frac{9}{2} \left\| \omega_0 \right\|_{\infty}^2 E_1(0),
\]

(1.19)

\[
\int_{0}^{\infty} |E_1(t)| \, dt \leq \int_{T \times (0,1)} \log \left( \frac{2 |\omega_0|_{\infty}}{\partial_y \omega_0} \right) \, dx \, dy,
\]

(1.20)

where \( E_1(0), E_2(0) < 0 \).

**Remark 5.** Let us observe that in the semilagrangian formulation (1.6), the above energies are equal to (see Proposition 2):
\[ E_1(T) = - \int_{T \times (0,1)} v_x h_a dx \, da = E_1(0) + \int_0^T \int_{T \times (0,1)} v^2 h_a \, dx \, da dt, \]
\[ E_2(T) = - \int_{T \times (0,1)} v^2 v_x h_a dx \, da = E_2(0) + \int_0^T \int_{T \times (0,1)} v^2 h_a \, dx \, da dt, \]

thanks to the change of coordinates \( y = h(x, t, a) \). The above identities are a particular case of (7.2) and (7.3), when \( d = 1 \).

1.3. Finite time blow-up of the Semi-lagrangian equations.

**Theorem 3.** Let \((v, h_a)\) be a smooth solution to (1.9). Set

\[ E_1(t) = - \int_{T^d \times (0,1)} (\nabla \cdot v) h_a dx \, da \]

and

\[ E_2(t) = \int_{T^d \times (0,1)} (v \cdot v_t) h_a dx \, da. \]

Assume that \( E_1(0) > 0 \) or \( E_2 > 0 \), then the solution blows up in finite time. Moreover, as long as the solution exists, we have the following estimates:

\[ d \log \left( \frac{d}{E_1(0)} \frac{1}{E_1(t) - t} \right) \leq \int_{T^d \times (0,1)} \log(h_a(\tau)) h_a(\tau) dx \, da \big|_{\tau = 0} = t, \quad \text{if } E_1(0) > 0, \]

\[ \frac{d}{E_1(0)} - t \leq E_1(t) = E_1(0) + \int_0^t \int_{T^d \times (0,1)} |\nabla v|^2 h_a dx \, da dt, \quad \text{if } E_1(0) > 0, \]

\[ \frac{\|v\|^2}{E_2(0)} - t \leq E_2(t) = E_2(0) + \int_0^t \int_{T^d \times (0,1)} |\partial_t v|^2 h_a dx \, da dt, \quad \text{if } E_2(0) > 0, \]

where \( \|v\|^2 = \int_{T^d \times (0,1)} |v|^2 h_a dx \, da. \)

The following corollary is a direct consequence of Theorem 3, Jensen’s inequality and Lemma 7 (applied to the probability measure \( \mu = h_a dx \, da \)).

**Corollary 1.** Let \((v, h_a)\) be a smooth solution to (1.9) in \([0, T]\). Let \( E_1 \) be defined by (1.21). Suppose that \( E_1(0) > 0 \), then

\[
\exp \left( \int_{T^d \times (0,1)} \partial_a h_0 \log(\partial_a h_0) \, dx \, da \right) \left( \frac{d}{E_1(0)} \frac{1}{E_1(t) - t} \right)^d
\leq \exp \left( \int_{T^d \times (0,1)} h_a(T) \log(h_a(T)) \, dx \, da \right)
\leq \left( \int_{T^d \times (0,1)} h_a(T)^{1+p} \, dx \, da \right)^{1/p},
\]
for every \( p \in (0, \infty) \). Moreover

\[
\exp \left( \int_{T^d \times (0,1)} h_a \log(h_a) dx da \right) = \lim_{p \to 0+} \left( \int_{T^d \times (0,1)} h_a^{1+p} dx da \right)^{\frac{1}{p}}.
\]

**Remark 6.** Corollary 1 gives a lower bound for the \( L^p(T^d \times (0,1)) \) norm of \( h_a \) for \( p > 1 \), which cannot be extended to \( p = 1 \), because

\[
\int_{T^d \times (0,1)} h_a^1 dx da = 1.
\]

**Remark 7.** The first equation in (1.9), is equivalent to:

\[
\partial_t v + \nabla \left( \frac{|v|^2}{2} + P \right) = 0,
\]

thanks to the curl free condition \( \partial_i v_j = \partial_j v_i \).

**Remark 8.** In particular, smooth solutions to (1.9) satisfy the following equations in \( T^d \):

\[
\begin{align*}
\partial_t \int_0^1 v h_a da + \nabla \cdot \int_0^1 v \otimes v h_a da + \nabla P &= 0, \\
\int_0^1 h_a da &= 0, \\
\nabla \cdot \int_0^1 v h_a da &= 0, \\
\partial_i v_j - \partial_j v_i &= 0, \quad \text{for } i, j \leq d,
\end{align*}
\]

which can be seen as an averaged version of the incompressible Euler equations in \( T^d \)

\[
\begin{align*}
\partial_t v + \nabla \cdot (v \otimes v) + \nabla P &= 0, \\
\nabla \cdot v &= 0, \\
v|_{t=0} &= v_0.
\end{align*}
\]

## 2. Preliminaries

In this section we will present some elementary results concerning equations (1.1), (1.2) and (1.9) that will be used throughout this article.

### 2.1. Hydrostatic Euler equations.

First, let us point out that any solution to (1.1) satisfies \( \int_0^1 u(x, y, t) dy = k \) (see Proposition 11). Moreover, \( \tilde{u}(x, y, t) = u(x + kt, y, t) - k \) solves (1.1) with a slightly different initial data and \( \int_0^1 \tilde{u}(x, y, t) dy = 0 \). Hence, without loss of generality, we may assume that the solution to (1.1) satisfies

\[
\int_0^1 u(x, y, t) dy \equiv 0.
\]

The following proposition summarizes some properties that will be used in the rest of the article.
Proposition 1. Assume that the initial vorticity $\omega_0 \in H^4(T \times (0, 1))$ satisfies the local Rayleigh condition (i.e. $\partial_y \omega_0 > 0$). Let $\omega \in C([0,T]; H^4(T \times (0,1)))$ be the solution to the problem (1.2) that satisfies the local Rayleigh condition in $[0,T]$. Then we have

\begin{align*}
(2.2) \quad & A(\omega) = (1-y) \int_0^y \int_0^z \omega(x,s,t)dsdz + y \int_y^1 (1-z)\omega(x,z,t)dz,
(2.3) \quad & u(x,y,t) = \int_0^1 \int_0^z \omega(x,s,t)dsdz - \int_y^1 \omega(x,z,t)dz,
(2.4) \quad & v(x,y,t) = (1-y) \int_0^y \omega(x,z,t)dz + y \int_y^1 (1-z)\omega(x,z,t)dz,
(2.5) \quad & \partial_x \int_0^1 udy = 0,
(2.6) \quad & \partial_t \int_0^1 udy = 0,
(2.7) \quad & P_x(x,t) = -\partial_x \left( \int_0^1 u^2(x,y,t)dy \right),
(2.8) \quad & \partial_t \|u\|^2_2 = 0,
(2.9) \quad & \|u\|_{\infty} \leq \frac{3}{2} \|\omega_0\|_{\infty}.
\end{align*}

Proof. By (2.1), we may assume that the stream function $A(\omega)$ solves

\begin{align*}
(2.10) \quad & \begin{cases} 
-\partial_y^2 A(\omega) = \omega, \\
A(\omega)|_{y=0,1} = 0,
\end{cases}
\end{align*}

which implies

\begin{align*}
A(\omega) &= -\int_0^y \int_0^z \omega(x,s,t)dsdz + y \int_y^1 \int_0^z \omega(x,s,t)dsdz \\
&= (1-y) \int_0^y \omega(x,z,t)dz + y \int_y^1 (1-z)\omega(x,z,t)dz,
\end{align*}

from which we get (2.2). Next, (2.3) and (2.4) follow directly from (2.2). The incompressibility condition and the boundary value in (1.1) implies (2.5). Next, by
(2.5), (11), the $x$-periodicity and integration by parts,
\[
\partial_t \int_0^1 u dy = \partial_t \int_{T \times (0,1)} u dx dy
\]
\[
= - \int_{T \times (0,1)} \partial_x \left( \frac{u^2}{2} + P \right) + v \omega dx dy
\]
\[
= - \int_{T \times (0,1)} u u_x dx dy
\]
\[
= - \int_{T \times (0,1)} \partial_x \left( \frac{u^2}{2} \right) dx dy
\]
\[
= 0,
\]
which proves (2.6). Let us prove (2.7). By (1.1), (2.6) and integration by parts, we have
\[
0 = \int_0^1 (u_t + uu_x + vu_y + P_x) dy
\]
\[
= \int_0^1 (uu_x + vu_y) dy + P_x
\]
\[
= 2 \int_0^1 uu_x dy + P_x,
\]
which implies (2.7). By (1.1), the $x$-periodicity and integration by parts
\[
\partial_t \int_{T \times (0,1)} u^2 dx dy = 2 \int_{T \times (0,1)} uu_t dx dy
\]
\[
= -2 \int_{T \times (0,1)} (u^2 u_x + vu u_y + u P_x) dx dy
\]
\[
= -3 \int_{T \times (0,1)} u^2 u_x dx dy + 2 \int_T \left( \int_0^1 u_x dy \right) P(x,t) dx
\]
\[
= 0,
\]
from which we get (2.8). Applying (2.3), we get
\[
\|u\|_\infty \leq \|\omega\|_\infty \int_0^1 zdz + \|\omega\|_\infty
\]
\[
= \frac{3}{2} \|\omega\|_\infty
\]
\[
= \frac{3}{2} \|\omega_0\|_\infty.
\]
This completes the proof of Proposition 1. □

2.2. Semilagrangian equations. The following proposition summarizes how the change of variable works between the hydrostatic Euler equations (1.2) and the semilagrangian equations (1.6).
Proposition 2. Let \( \omega \) be a smooth solution to (1.2) that satisfies the local Rayleigh condition, (1.4) and (1.5) in \([0, T]\). Let \((v, h_a)\) be a smooth solution to (1.6) in \([0, T]\). Assume that \( \omega, h, u \) and \( v \) satisfy (1.7) and (1.8) in \([0, T]\), then

\[
\begin{align*}
(2.12) \quad h_a(x, t, a) &= \frac{1}{\omega_y(x, h(x, t, a), t)}, \\
(2.13) \quad h_x(x, t, a) &= -\frac{\omega_x}{\omega_y}(x, h(x, t, a), t), \\
(2.14) \quad v_a(x, t, a) &= (k + a)h_a, \\
(2.15) \quad v_x(x, t, a) &= \left( u_x - \frac{\omega \omega_y}{\omega_y} \right)(x, h(x, t, a), t), \\
(2.16) \quad v(x, t, a) &= -\frac{k + 1}{2} + (k + a)h - \frac{1}{2} \int_0^1 h^2 db + \int_a^1 hdb, \\
(2.17) \quad \mathcal{A}(\omega)(x, h(x, t, a), t) &= -v(x, t, a)h(x, t, a) + \frac{1}{2}(k + a)h^2 - \frac{1}{2} \int_0^a h^2(x, t, b)db, \end{align*}
\]

where \( 0 \leq t \leq T \).

Proof. First observe that (2.12), (2.13), (2.14) and (2.15) follow directly by (1.7) and (1.8). Applying (2.3) and the change of variable \( z = h(x, t, b) \)

\[
u(x, h, t) = \int_0^1 z \omega(x, z, t)dz - \int_{h(x, t, a)}^1 \omega(x, z, t)dz = \int_0^1 (k + b)h db - \int_a^1 (k + b)h db = -\frac{k + 1}{2} - \frac{1}{2} \int_0^1 h^2 db + (k + a)h + \int_a^1 hdb,
\]

which proves (2.16). Next, by (1.2), we have \(-\mathcal{A}(\omega)(x, y, t) = \int_0^h u(x, z, t)dz\), which implies
\[-\mathcal{A}(\omega)(x, t, a) = \int_0^{h(x, t, a)} u(x, z, t) \, dz \]

\[-= \int_0^a v(x, t, b) h_b(x, t, b) \, db \]

\[-= v(x, t, a) h(x, t, a) - \int_0^a (k + b) h h_b \, db \]

\[-= v(x, t, a) h(x, t, a) - \frac{1}{2} (k + a) h^2 + \frac{1}{2} \int_0^a h^2(x, t, b) \, db, \]

which gives \((2.17)\). Let us prove \((2.18)\). First observe that

\[h(x, t, a) = \int_0^a h_b(x, t, b) \, db,\]

because \(h(x, 0, t) \equiv 0\). Thus, \((2.18)\) follows by integrating the second equation of \((1.6)\) in \(b \in [0, a]\). Finally, by \((2.18)\)

\[v_t = u_t(x, h, t) + (k + a) h_t \]

\[-= -u u_x(x, h, t) - v(x, h, t)(k + a) - P_x(x, t) + (k + a)(v(x, h, t) - u(x, h, t) h_x) \]

\[-= -u u_x(x, h, t) - P_x(x, t) - (k + a) u(x, h, t) h_x,\]

which leads to \((2.19)\) thanks to \((2.13)\). This concludes the proof of Proposition 2. \(\square\)

**Proposition 3.** Let \((v, h_a)\) be a smooth solution to \((1.9)\), then

\[\partial_t \int_{\mathbb{T}^d \times (0, 1)} |v|^2 h_a \, dx \, da = 0,\]

\[P(x, t) = (-\Delta)^{-1} \left(\nabla \cdot \int_0^1 (v \otimes v) h_b \, db\right).\]

**Proof.** First observe that

\[\partial_t \int_0^1 h_a \, da = -\nabla \cdot \int_0^1 v h_a \, da \]

\[= 0.\]

By \((1.27)\), \((2.22)\) and integration by parts

\[\partial_t \int_{\mathbb{T}^d \times (0, 1)} |v|^2 h_a \, dx \, da = \int_{\mathbb{T}^d \times (0, 1)} 2v \cdot v_t h_a + |v|^2 \partial_t h_a \, dx \, da \]

\[= \int_{\mathbb{T}^d \times (0, 1)} -v \cdot \nabla (|v|^2 + 2P) h_a - \nabla \cdot (v h_a) |v|^2 \, dx \, da \]

\[= \int_{\mathbb{T}^d \times (0, 1)} -\nabla \cdot (v |v|^2 h_a) + 2P \nabla \cdot (v h_a) \, dx \, da \]

\[= 2 \int_{\mathbb{T}^d} P \nabla \cdot \left(\int_0^1 v h_a \, da\right) \, dx \, da \]

\[= 0,\]

which proves \((2.20)\). Now, let us prove \((2.21)\). First observe that
\[
\partial_t \int_0^1 \mathbf{v} h_a da = - \int_0^1 \left( \nabla \left( \frac{|\mathbf{v}|^2}{2} + P \right) h_a + \mathbf{v} \nabla \cdot (\mathbf{v} h_a) \right) da
\]
\[
= - \nabla P - \nabla \cdot \int_0^1 (\mathbf{v} \otimes \mathbf{v}) h_a da,
\]

thanks to \( \partial_i v_j = \partial_j v_i \). Applying divergence, we get:

\[
0 = - \Delta P - \nabla \cdot \int_0^1 (\mathbf{v} \otimes \mathbf{v}) h_a da,
\]
which implies (2.21). This concludes the proof of Proposition 3.

\[\square\]

3. Monotonicity and lower bounds for \( E_1 \) and \( E_2 \)

Throughout this section we will work in the framework of the local well-posedness result by Masmoudi and Wong [MW12], namely, we will assume that \( \omega_0 \in H^4(\mathbb{T} \times (0,1)) \) satisfies the local Rayleigh condition. We will show the monotonicity of \( E_1 \) and \( E_2 \), as well as the validity of certain lower bounds that will be useful for proving Theorem 1.

The following two lemmas are elementary, so the proof will be omitted.

**Lemma 1.** Assume that \( \omega_0 \in H^4(\mathbb{T} \times (0,1)) \) satisfies the local Rayleigh condition. Let \( \omega \in C([0,T]; H^4(\mathbb{T} \times (0,1))) \) be the solution to (1.2) that satisfies the local Rayleigh condition in \([0,T]\). Denote by \( D_t = \partial_t + u \partial_x + v \partial_y \) the material derivative, then:

\[
\partial_t \int_{\mathbb{T} \times (0,1)} f(x,y,t) dx dy = \int_{\mathbb{T} \times (0,1)} D_t f(x,y,t) dx dy,
\]

\[
D_t(f g) = f D_t g + g D_t f,
\]

\[
D_t \left( \frac{f}{g} \right) = \frac{g D_t f - f D_t g}{g^2}, \quad \text{if } g \text{ is strictly positive},
\]

\[
D_t (\log(g)) = \frac{D_t g}{g}, \quad \text{if } g \text{ is strictly positive},
\]

where \( 0 \leq t \leq T \).

**Lemma 2.** Assume that \( \omega_0 \in H^4(\mathbb{T} \times (0,1)) \) satisfies the local Rayleigh condition. Let \( \omega \in C([0,T]; H^4(\mathbb{T} \times (0,1))) \) be the solution to (1.2) that satisfies the local Rayleigh condition in \([0,T]\). Denote by \( D_t = \partial_t + u \partial_x + v \partial_y \) the material derivative, then:

\[
D_t(u) = - P_x,
\]

\[
D_t(\omega) = 0,
\]

\[
D_t(u_x) = - u_x^2 - v_x \omega - P_{xx},
\]

\[
D_t(P_x) = P_{xt} + u P_{xx},
\]

\[
D_t(\omega_x) = - u_x \omega_x - v_x \omega_y,
\]

\[
D_t(\omega_y) = u_x \omega_y - \omega \omega_x,
\]
where \(0 \leq t \leq T\).

Lemma 3. Assume that \(\omega_0 \in H^4(\mathbb{T} \times (0,1))\) satisfies the local Rayleigh condition. Let \(\omega \in C([0,T]; H^4(\mathbb{T} \times (0,1)))\) be the solution to (1.2) that satisfies the local Rayleigh condition in \([0,T]\). Denote by \(D_t = \partial_t + u \partial_x + v \partial_y\) the material derivative, then:

\[
D_t \left( \log \left( \frac{1}{\omega_y} \right) \right) = \frac{u_x \omega_x - u \omega_y}{\omega_y} = \frac{\omega_x}{\omega_y} - u_x,
\]

(3.11)

\[
D_t \left( \frac{\omega_x}{\omega_y} - u_x \right) = \left| \frac{\omega_x}{\omega_y} - u_x \right|^2 + P_{xx},
\]

(3.12)

\[
D_t \left( u^2 \left( \frac{\omega_x}{\omega_y} - u_x \right) - u P_x \right) = \left| u \left( \frac{\omega_x}{\omega_y} - u_x \right) - P_x \right|^2 - u P_{xt},
\]

(3.13)

for \(0 \leq t < T\).

Proof. By Lemmas 1 and 2,

\[
D_t \left( \log \left( \frac{1}{\omega_y} \right) \right) = \frac{u_x \omega_x - u \omega_y}{\omega_y} = \frac{\omega_x}{\omega_y} - u_x,
\]

which gives (3.11). Next, by Lemmas 1 and 2,

\[
D_t \left( \frac{\omega_x}{\omega_y} - u_x \right) = \left| \frac{\omega_x}{\omega_y} - u_x \right|^2 - \omega v_x - u_x^2.
\]

(3.14)

Thus, by (3.14) and (3.7)

\[
D_t \left( \frac{\omega_x}{\omega_y} - u_x \right) = \left| \frac{\omega_x}{\omega_y} - u_x \right|^2 + P_{xx},
\]

from which we get (3.12). Finally, by Lemmas 1 and 2 and (3.12)

\[
D_t \left( u^2 \left( \frac{\omega_x}{\omega_y} - u_x \right) - u P_x \right) = u^2 \left( \left| \frac{\omega_x}{\omega_y} - u_x \right|^2 + P_{xx} \right) - 2 u P_x \left( \frac{\omega_x}{\omega_y} - u_x \right) + P_x^2 - u^2 P_{xx} - u P_{xt}
\]

\[
= \left| u \left( \frac{\omega_x}{\omega_y} - u_x \right) - P_x \right|^2 - u P_{xt},
\]

which implies (3.13). This completes the proof of Lemma 3. \(\Box\)

Proposition 4. Assume that \(\omega_0 \in H^4(\mathbb{T} \times (0,1))\) satisfies the local Rayleigh condition. Let \(\omega \in C([0,T]; H^4(\mathbb{T} \times (0,1)))\) be the solution to (1.2) that satisfies the local Rayleigh condition in \([0,T]\). Set
\[ E_1(t) = \int_{T \times (0,1)} \left( \frac{\omega_x}{\omega_y} - u_x \right) dxdy = \int_{T \times (0,1)} \frac{\omega_x}{\omega_y} dxdy \]

and

\[ E_2(t) = \int_{T \times (0,1)} \left( u^2 \left( \frac{\omega_x}{\omega_y} - u_x \right) - uP_x \right) dxdy = \int_{T \times (0,1)} u^2 \frac{\omega_x}{\omega_y} dxdy. \]

Then

\[ E_1 = \partial_t \int_{T \times (0,1)} \log \left( \frac{1}{\omega_y} \right) dxdy, \]

\[ \partial_t E_1 = \int_{T \times (0,1)} \left| u_x - \frac{\omega_x}{\omega_y} \right|^2 dxdy, \]

\[ \partial_t E_2 = \int_{T \times (0,1)} \left| P_x + u \left( u_x - \frac{\omega_x}{\omega_y} \right) \right|^2 dxdy, \]

where \( 0 \leq t < T \).

**Proof.** By Lemma 1, (3.11) and the \( x \)-periodicity

\[ \partial_t \int_{T \times (0,1)} \log \left( \frac{1}{\omega_y} \right) dxdy = \int_{T \times (0,1)} \left( \frac{\omega_x}{\omega_y} - u_x \right) dxdy, \]

which proves (3.17). Next, by Lemma 1, (3.12) and the \( x \)-periodicity

\[ \partial_t \int_{T \times (0,1)} \left( \frac{\omega_x}{\omega_y} - u_x \right) dxdy = \int_{T \times (0,1)} \left| \frac{\omega_x}{\omega_y} - u_x \right|^2 + P_{xx} dxdy \]

\[ = \int_{T \times (0,1)} \left| \frac{\omega_x}{\omega_y} - u_x \right|^2 dxdy, \]

which gives (3.18). Finally, by Lemma 1 and (3.13)

\[ \partial_t E_2 = \int_{T \times (0,1)} \left| P_x + u \left( u_x - \frac{\omega_x}{\omega_y} \right) \right|^2 - uP_{xt} dxdy \]

\[ = \int_{T \times (0,1)} \left| P_x + u \left( u_x - \frac{\omega_x}{\omega_y} \right) \right|^2 + u_x P_t dxdy, \]

from which we get (3.19), thanks to (2.5). This concludes the proof of Proposition 4.

---

**Lemma 4.** Assume that \( \omega_0 \in H^4(T \times (0,1)) \) satisfies the local Rayleigh condition. Let \( \omega \in C([0,T]; H^4(T \times (0,1))) \) be the solution to (1.2) that satisfies the local Rayleigh condition in \([0,T]\). If \( \int_{T \times (0,1)} \left| u_x - \frac{\omega_x}{\omega_y} \right|^2 dxdy = 0 \) (at any time) then \( \omega \) is stationary and independent of \( x \).
Proof. Assume that $\frac{\omega \partial_x u}{\omega_y} - u_x = 0$ at time $t = t_0$. Then,

$$\partial_y \left( \frac{u_x}{\omega} \right) = \frac{\omega \omega_x - U_x \omega_y}{\omega^2} = \frac{\omega_y}{\omega^2} \left( \frac{\omega \omega_x}{\omega_y} - u_x \right) = 0 \quad \text{if } \omega \neq 0.$$  

Let us first assume that $\omega$ does not vanish. By (3.25), there exists $y_0 \in [0, 1]$ such that $u_x(x, y_0(x), t_0) = \frac{u_x(x, y_0(x), t_0)}{\omega(x, y_0(x), t_0)} = 0$. By (3.21), $u_x \equiv 0$ at time $t = t_0$. Thus, $\omega$ is $x$-independent at $t = t_0$. Moreover, $\omega(y, t_0)$ and $\omega(x, y, t_0 + t)$ satisfy (1.2) with the same initial condition, since the solution is unique, $\omega(x, y, t_0 + t) \equiv \omega(y, t_0)$ (see [MW12]).

Conversely, assume that $\omega$ vanishes in some points. Applying (3.21), we have

$$u_x(x, y, t_0) = \begin{cases} f(x, t_0) \omega(x, y, t_0) + c(x), & \text{if } \omega > 0, \\ \tilde{f}(x, t_0) \omega(x, y, t_0) + \tilde{c}(x), & \text{if } \omega < 0, \\ 0, & \text{if } \omega = 0. \end{cases}$$

Since $u_x$ is continuous, $c \equiv \tilde{c} \equiv 0$. Moreover,

$$\omega_x = \begin{cases} f(x, t_0) \omega_y, & \text{if } \omega > 0, \\ \tilde{f}(x, t_0) \omega_y, & \text{if } \omega < 0, \\ 0, & \text{if } \omega = 0. \end{cases}$$

Therefore, the continuity of $\frac{\omega_x}{\omega_y}$ yields $f(x, t_0) = \tilde{f}(x, t_0)$. Thus, $u_x = f(x, t_0) \omega$ and $\omega_x = f(x, t_0) \omega_y$. Integrating in $y \in [0, 1]$, we get

$$\int_0^1 u_x dy = 0 = \int_0^1 f(x, t_0) \omega dy = f(x, t_0) u_{|y=1} - f(x, t_0) u_{|y=0},$$

Therefore, the continuity of $\frac{\omega_x}{\omega_y}$ yields $f(x, t_0) = \tilde{f}(x, t_0)$. Thus, $u_x = f(x, t_0) \omega$ and $\omega_x = f(x, t_0) \omega_y$. Integrating in $y \in [0, 1]$, we get

$$\int_0^1 u_x dy = 0 = \int_0^1 f(x, t_0) \omega dy = \int_0^1 \omega_x dy = \partial_x u_{|y=1} - \partial_x u_{|y=0}.\]$$

Applying (3.24), (3.25) and the positivity of $\omega_y$, we get

$$\partial_x \left( |u(1) - u(0)|^2 \right) = 2u_{|y=1} \partial_x u_{|y=0} = 0.$$  

Finally, if $f(x_0, t_0) \neq 0$, (3.24) implies $u(x_0, t_0)|_{y=1} = 0$. Moreover, by (3.20), $u(x, y, t_0)|_{y=0} = 0$, which contradicts (3.25). Thus, $f \equiv u_x \equiv 0$ at time $t = t_0$. Furthermore, $\omega(y, t_0)$ and $\omega(x, y, t_0 + t)$ satisfy (1.2) with the same initial data. Then, by uniqueness $\omega(x, y, t_0 + t) \equiv \omega(y, t_0)$ (see [MW12]). This completes the proof of Lemma 4. 

\[\square\]

Corollary 2. Let $\omega \in H^4(\mathbb{T} \times (0, 1))$ be a stationary solution to (1.2) that satisfies the local Rayleigh condition. Then, $\omega$ is $x$-independent. Conversely, if $\omega$ is an $x$-independent solution to (1.2), then $\omega$ is stationary.

Proof. Let $\omega$ be a stationary solution, then by (3.18):

$$\partial_t E_1 = \int_{\mathbb{T} \times (0, 1)} \left| \frac{\omega \omega_x}{\omega_y} - u_x \right|^2 dxdy = 0.$$
By Lemma 4, \( \omega \) is \( x \)-independent. On the other hand, if \( \omega \) is an \( x \)-independent solution to \( \text{(1.2)} \), then

\[
\omega_t = -u \omega_x - A(\omega_x) \omega_y \equiv 0,
\]

where \( A(\omega) \) is defined by \( \text{(2.2)} \), which concludes the proof of Corollary 2.

**Lemma 5.** Assume that \( \omega_0 \in H^4(\mathbb{T} \times (0,1)) \) satisfies the local Rayleigh condition. Let \( \omega \in C([0,T]; H^4(\mathbb{T} \times (0,1))) \) be the solution to \( \text{(1.2)} \) that satisfies the local Rayleigh condition in \( [0,T] \). If \( \int_{\mathbb{T} \times (0,1)} \left| P_x + u \left( u_x - \frac{\omega u}{\omega_y} \right) \right|^2 \, dx \, dy = 0 \) (at any time) then \( \omega \) is stationary and independent of \( x \).

**Proof.** Assume that \( P_x + u \left( u_x - \frac{\omega u}{\omega_y} \right) \equiv 0 \) at time \( t = t_0 \). Then,

\[
\text{(3.27)} \quad P_x + u \left( u_x - \frac{\omega u}{\omega_y} \right) = P_x + uu_x + \frac{\omega uu_y}{\omega_y} - \frac{\omega uu_y}{\omega_y} - \frac{\omega uu_x}{\omega_y} = \frac{\omega u_t}{\omega_y} - u_t.
\]

Furthermore,

\[
\text{(3.28)} \quad \partial_y \left( \frac{u_t}{\partial_y} \right) = \frac{\omega u_t}{\omega_y} - \frac{u_t \omega_y}{\omega_y} = \frac{\omega u_t}{\omega_y} - u_t \equiv 0, \quad \text{if } \omega \neq 0.
\]

Let us first assume that \( \omega \) does not vanishes. By \( \text{(2.6)} \), there exists \( y_0 \in [0,1] \) such that \( u_t(x, y_0(x), t_0) = \frac{u_t(x, y_0(x), t_0)}{\omega(x, y_0(x), t_0)} = 0 \). By \( \text{(3.28)} \), \( u_t \equiv 0 \) at time \( t = t_0 \). Moreover, \( \omega(x, y, t_0) \) and \( \omega(x, y, t_0 + t) \) satisfy \( \text{(1.2)} \) with the same initial condition. Since the solution is unique, \( \omega(x, y, t_0 + t) \equiv \omega(x, y, t_0) \). Furthermore, by Corollary 2, \( \omega \) is \( x \)-independent.

Conversely, assume that \( \omega \) vanishes in some points. Applying \( \text{(3.28)} \), we have

\[
\text{(3.29)} \quad u_t(x, y, t_0) = \begin{cases} f(x, t_0) \omega(y, t_0) + c(x), & \text{if } \omega > 0, \\ \tilde{f}(x, t_0) \omega(y, t_0) + \tilde{c}(x), & \text{if } \omega < 0, \\ 0, & \text{if } \omega = 0. \end{cases}
\]

Since \( u_t \) is continuous, \( c \equiv \tilde{c} \equiv 0 \). Moreover,

\[
\text{(3.30)} \quad \omega_t = \begin{cases} f(x, t_0) \omega_y, & \text{if } \omega > 0, \\ \tilde{f}(x, t_0) \omega_y, & \text{if } \omega < 0, \\ 0, & \text{if } \omega = 0. \end{cases}
\]

Therefore, the continuity of \( \frac{\omega}{\omega_y} \) yields \( f(x, t_0) = \tilde{f}(x, t_0) \). Thus, \( u_t = f(x, t_0) \omega \) and \( \omega_t = f(x, t_0) \omega_y \). Integrating in \( y \in [0,1] \), we get

\[
\text{(3.31)} \quad \int_0^1 u_t \, dy = 0 = f(x, t_0) \int_0^1 \omega \, dy = f(x, t_0) u|_{y=1}^{|y=0},
\]

\[
\text{(3.32)} \quad \int_0^1 \omega_t \, dy = f(x, t_0) \int_0^1 \omega_y \, dy = \partial_x u|_{y=1}^{|y=0} = \frac{1}{2} \partial_x (u^2(0) - u^2(1)).
\]

Applying \( \text{(3.31)}, \text{(3.32)} \) and the positivity of \( \omega_y \), we get
\[
(3.33) \quad \partial_t \left( |u(1) - u(0)|^2 \right) = 2u_{y=0}^{y=1} \partial_y u_{y=0}^{y=1} = 2u_{y=0}^{y=1} \partial_x (u^2(0) - u^2(1)) = 0.
\]
Multiplying by \(u(1) + u(0)\), we obtain
\[
(3.34) \quad \partial_x \left( |u^2(0) - u^2(1)|^2 \right) = 0.
\]

Finally, if \(f(x, t_0) \neq 0\), \((3.31)\) implies \(u(x, y, t_0)_{y=0}^{y=1} = 0\). Moreover, by \((3.34)\) and \((3.32)\), \(u(x, y, t_0)_{y=0}^{y=1} \equiv 0\), which contradicts \((3.32)\). Thus, \(f \equiv u_t \equiv 0\) at time \(t = t_0\). Furthermore, \(\omega(x, y, t_0)\) and \(\omega(x, y, t_0 + t)\) satisfy \((1.2)\) with the same initial data. Then, by uniqueness \(\omega(x, y, t_0 + t) \equiv \omega(x, y, t_0)\) (see [MW12]). Moreover, since \(\omega\) is stationary, \(\partial_t E_1 = 0\). Applying Lemma 4, we deduce that \(\omega\) is \(x\)-independent.

This concludes the proof of Lemma 5.

Lemma 6. Let \(f : [0, L] \to \mathbb{R}\) be a \(C^1\) function and \(C > 0\). Assume that \(\partial_t f \geq Cf^2\) and \(f(0) > 0\). Then there exists a positive time \(T^*\) such that \(f \to \infty\) as \(t \to T^*\).

Moreover, \(T^* \leq \frac{1}{Cf(0)}\) and \(f\) satisfies:
\[
(3.35) \quad f(t) \geq \frac{1}{C} \frac{1}{f(0) - t}, \quad \text{for } t < T^*.
\]

Proof. Since \(f\) is positive we have:
\[
\frac{\partial f}{f^2} \geq C.
\]
Integrating the above we get:
\[
\int_0^t \frac{f_s}{f^2} ds = \frac{1}{f(0)} - \frac{1}{f(t)} \geq C \int_0^t ds = Ct.
\]
From which we obtain:
\[
f(t) \geq \frac{1}{f(0) - Ct},
\]
which concludes the proof.

Proposition 5. Assume that \(\omega_0 \in H^4(\mathbb{T} \times (0, 1))\) satisfies the local Rayleigh condition and \(\partial_x \omega_0\) is not identically 0. Let \(\omega \in C([0, T]; H^4(\mathbb{T} \times (0, 1)))\) be the solution to \((1.2)\) that satisfies the local Rayleigh condition in \([0, T]\). Then, \(E_1\) satisfies:
\[
(3.36) \quad E_1(t) \geq \frac{1}{E_1(0) - t}, \quad \text{if } E_1(0) > 0,
\]
where \(0 \leq t \leq T\).

Furthermore, if \(E_1(0) = 0\) there exists a positive time \(t'\) (sufficiently small) such that \(E_1(t') > 0\) and
\[
E_1(t + t') \geq \frac{1}{E_1(t') - t},
\]
where \(0 < t' \leq T\).
where $0 \leq t \leq T$.

Proof. By (3.18) and Hölder’s inequality,

$$
\partial_t E_1 = \int_{T \times (0,1)} \left| u_x - \frac{\omega_x}{\omega_y} \right|^2 \, dx \, dy \\
\geq \left( \int_{T \times (0,1)} \left( u_x - \frac{\omega_x}{\omega_y} \right) \, dx \, dy \right)^2
$$

(3.38)

If $E_1(0) > 0$ the result follows by applying Lemma 6. On the other hand, if $E_1(0) = 0$, Lemma 4 implies

$$
\int_{T \times (0,1)} \left| \frac{\omega_x}{\omega_y} - u_x \right|^2 \, dx \, dy > 0,
$$

at every time. Therefore, for every positive time $t'$ we have

$$
\int_0^{t'} \int_{T \times (0,1)} \left| \frac{\omega_x}{\omega_y} - u_x \right|^2 \, dx \, dy \, dt = \int_0^{t'} \partial_t E_1(t) \, dt = E_1(t') - E_1(0) = E_1(t') > 0.
$$

Choose $t'$ such that $\omega \in C([0, T + t']; H^4(T \times (0, 1)))$ satisfies the local Rayleigh condition in $[0, T + t']$. Applying (3.36) to $\omega(t + t')$ we get

$$
E_1(t' + t) \geq \frac{1}{E_1(t') - t},
$$

for $0 \leq t \leq T$, which completes the proof of Proposition 6. \hfill \Box

Proposition 6. Assume that $\omega_0 \in H^4(T \times (0, 1))$ satisfies the local Rayleigh condition and $\partial_x \omega_0$ is not identically 0. Let $\omega \in C([0, T]; H^4(T \times (0, 1)))$ be the solution to (1.2) that satisfies the local Rayleigh condition in $[0, T]$. Then, $E_2$ satisfies:

(3.39) $$
E_2(t) \geq \frac{\|u\|_2^2}{E_2(0)} - t, \quad \text{if } E_2(0) > 0,
$$

where $0 \leq t \leq T$. Furthermore, if $E_2(0) = 0$ there exists a positive time $t'$ (sufficiently small) such that $E_2(t') > 0$ and

(3.40) $$
E_2(t' + t) \geq \frac{\|u\|_2^2}{E_2(t')} - t,
$$

where $0 \leq t \leq T$.

Proof. By (3.19), Hölder’s inequality, (2.5) and the $x$-periodicity:
\[ \|u\|^2_2 \cdot \partial_t E_2(t) \]
\[ = \left( \int_{T \times (0,1)} u^2 \, dx \, dy \right) \cdot \int_{T \times (0,1)} \left| P_x + u \left( u_x - \frac{\omega \omega_x}{\omega_y} \right) \right|^2 \, dx \, dy \]
\[ \geq \left( \int_{T \times (0,1)} u \left( P_x + u \left( u_x - \frac{\omega \omega_x}{\omega_y} \right) \right) \, dx \, dy \right)^2 \]
\[ = \left( \int_T \left( \int_0^1 u\, dy \right) P_x \, dx + \int_{T \times (0,1)} \partial_x \left( \frac{u^3}{3} \right) \, dx \, dy - \int_{T \times (0,1)} u^2 \frac{\omega \omega_x}{\omega_y} \, dx \, dy \right)^2 \]
\[ = \left( \int_{T \times (0,1)} u^2 \frac{\omega \omega_x}{\omega_y} \, dx \, dy \right)^2 \]
\[ = E_2^2(t). \]

Thus
\[ \partial_t E_2 \geq \|u\|^{-2}_2 E_2^2, \]
where \(\|u\|_2^2\) is time independent thanks to (2.13). If \(E_2(0) > 0\), Lemma 4 yields
\[ E_2(t) \geq \frac{\|u\|_2^2}{E_2(0) - t}, \]
where \(0 \leq t \leq T\). On the other hand, if \(E_2(0) = 0\), Lemma 5 implies
\[ \int_{T \times (0,1)} \left| P_x + u \left( u_x - \frac{\omega \omega_x}{\omega_y} \right) \right|^2 \, dx \, dy > 0, \]
at every time, from which we get
\[ \int_0^{t'} \int_{T \times (0,1)} \left| P_x + u \left( u_x - \frac{\omega \omega_x}{\omega_y} \right) \right|^2 \, dx \, dy \, d\tau = \int_0^{t'} \partial_\tau E_2 \, d\tau = E_2(t') > 0, \]
for every \(t' > 0\). Choose \(t'\) such that \(\omega \in C([0, T + t'); H^4(T \times (0, 1)))\) satisfies the local Rayleigh condition in \([0, T + t']\). Applying (3.39) to \(\omega(t + t')\) we get
\[ E_2(t' + t) \geq \frac{\|u\|_2^2}{E_2(t') - t}, \]
for \(0 \leq t \leq T\), which concludes the proof of Proposition 5.

4. Proof of Theorem 4: Collapse of the local Rayleigh condition or singularity formation

Proof. Let us proceed by contradiction. Suppose that \(\|\omega(t)\|_{H^4(T \times (0, 1))}\) and \(\frac{1}{\|\omega(\cdot)\|_{L^\infty(T \times (0, 1))}}\)
remain bounded for every time \(t > 0\). Assume first that \(E_1(0) \geq 0\). By Proposition
\[ E_1(\tau)_{\tau=t}^{\tau=0} \]
\[ = \int_0^t \int_{T \times (0,1)} \left| u_x - \frac{\omega \omega_y}{\omega_y} \right|^2 \, dx \, dy \, d\tau \]
\[ \leq 2 \int_0^t \int_{T \times (0,1)} u_x^2 + \left( \frac{\omega \omega_y}{\omega_y} \right)^2 \, dx \, dy \, d\tau \]
\[ \leq 2 \int_0^t \int_{T \times (0,1)} \left( \frac{\omega_x}{\omega^2} + \| \omega_0 \|_\infty^2 \left( \frac{\omega_x}{\omega_y} \right)^2 \right) \, dx \, dy \, d\tau \]
(4.1)
\[ \leq \left( \frac{2}{\pi^2} + 2\| \omega_0 \|^2_\infty \right) \int_0^t \int_{T \times (0,1)} \omega_x^2 \left( 1 + \frac{1}{\omega_y^2} \right) \, dx \, dy \, d\tau \]

which yields a contradiction because \( t \) is arbitrary and \( E_1 \) blows up in finite time.

Next, if \( E_2(0) \geq 0 \), by Proposition 3.18 \( E_2 \) blows up in finite time. Furthermore, by (3.19)

\[ E_2(\tau)_{\tau=t}^{\tau=0} = \int_0^t \int_{T \times (0,1)} \left| P_x + u \left( u_x - \frac{\omega \omega_y}{\omega_y} \right) \right|^2 \, dx \, dy \, d\tau \]
\[ \leq 2 \int_0^t \int_{T \times (0,1)} \left( P_x^2 + u^2 \left| u_x - \frac{\omega \omega_y}{\omega_y} \right|^2 \right) \, dx \, dy \, d\tau. \]

By (2.17), Hölder’s inequality, Poincare’s inequality and (2.9)

\[ \int_T P_x^2 \, dx = 4 \int_T \left( \int_0^1 u_x \, dy \right)^2 \, dx \]
\[ \leq 4 \int_T \int_0^1 u^2 \, dy \int_0^1 u_x^2 \, dy \, dx \]
\[ \leq 4 \left( \frac{3}{2} \| \omega_0 \|^2_\infty \right)^2 \frac{1}{\pi^2} \int_{T \times (0,1)} \omega_x^2 \, dx \, dy \]
\[ = \left( \frac{3}{\pi} \| \omega_0 \|^2_\infty \right)^2 \int_{T \times (0,1)} \omega_x^2 \, dx \, dy. \]

Applying (1.11) and (2.9), we get

\[ E_2(\tau)_{\tau=t}^{\tau=0} \]
(4.2)
\[ \leq \left( 2 \left( \frac{3}{\pi} \| \omega \|_\infty \right)^2 + 2 \left( \frac{2}{\pi^2} + 2\| \omega_0 \|^2_\infty \right) \right) \int_0^t \int_{T \times (0,1)} \omega_x^2 \left( 1 + \frac{1}{\omega_y^2} \right) \, dx \, dy \, d\tau \]
\[ \leq \left( 6 \| \omega_0 \|_\infty + \frac{4}{\pi^2} \right) \int_0^t \int_{T \times (0,1)} \| \omega_x(\cdot) \|_{L^2(T \times (0,1))}^2 \left( 1 + \frac{1}{\omega_y(\cdot)} \right)^2 \| \omega_y(\cdot) \|_{L^\infty(T \times (0,1))} \, d\tau, \]

which again yields a contradiction because \( t \) is arbitrary and \( E_2 \) blows up in finite time. From now on, we will assume that \( \| \omega(t) \|_{H^s(T \times (0,1))} \) and \( \frac{1}{\omega_y(t)} \| L^\infty(T \times (0,1)) \)
remain bounded, so we can apply the local well-posedness result by Masmoudi and Wong [MW12]. First let us prove (1.13). Assume that $E_1(0) > 0$. By Proposition 5

$$\frac{1}{E_1(0) - \tau} \leq E_1(\tau).$$

Applying (3.17) and integrating in $\tau \in [0, t]$, we get:

$$\log \left( \frac{1}{E_1(0)} - \frac{1}{E_1(0) - t} \right) = \int_0^t \frac{1}{E_1(\tau) - \tau} d\tau \leq \int_0^t E_1(\tau) d\tau = \int_0^t \int_{\mathbb{T} \times (0,1)} \log \left( \frac{1}{\omega_y(\tau)} \right) dxdy + \int_0^t \int_{\mathbb{T} \times (0,1)} \log \left( \frac{\partial y \omega_0}{\omega_y(t)} \right) dxdy,$$

which completes the proof of (1.13). Let us now deduce the estimate (1.14). Assume that $E_2(0) > 0$. By Proposition 6

$$\left( \frac{\|u\|_2^2}{E_2(0) - \tau} \right)^2 \leq E_2^2(t).$$

Moreover, by Hölder’s inequality and (3.18),

$$E_2^2(t) = \left( \int_{\mathbb{T} \times (0,1)} u^2 \left( \frac{\omega \omega_x}{\omega_y} - u_x \right) dxdy \right)^2 \leq \|u\|^4 \int_{\mathbb{T} \times (0,1)} \left| \frac{\omega \omega_x}{\omega_y} - u_x \right|^2 dxdy = \|u\|^4 \partial_t E_1.$$

Therefore, by (4.3), (4.4) and (2.9),
\[ \frac{\|u\|_2^2}{E_2(0)} - t - E_2(0)\|u\|_2^2 = \int_0^t \left( \frac{\|u\|_2^2}{E_2(0) - \tau} \right)^2 d\tau \leq \int_0^t E_2(\tau) d\tau \leq \int_0^t \|u\|_2^2 \partial_\tau E_1(\tau) d\tau \leq \left( \frac{3}{2} \|\omega_0\|_\infty \right)^4 (E_1(t) - E_1(0)) \leq \left( \frac{3}{2} \|\omega_0\|_\infty \right)^4 (E_1(t) + |E_1(0)|). \]

Thus
\[
\frac{1}{\|u\|_2^2/E_2(0) - t} \leq \left( \frac{E_2(0)}{\|u\|_2^2} + \left( \frac{3}{2} \|\omega_0\|_\infty \right)^4 |E_1(0)| \right) + \left( \frac{3}{2} \|\omega_0\|_\infty \right)^4 E_1(t).
\]

Integrating in time, we get
\[
\log \left( \frac{\|u\|_2^2}{E_2(0)} - t \right) \leq t C_1(\omega_0) + C_2(\omega_0) \int_{\Omega \times (0,1)} \log \left( \frac{\partial_y \omega_0}{\omega_y(t)} \right) dxdy,
\]
which concludes the proof of (1.14). Finally, Propositions 5 and 6, (4.1) and (4.2) lead to (1.15) and (1.16). This completes the proof of Theorem 1.

5. **Proof of Theorem 2** Necessary conditions for global solvability

In this section we will prove Theorem 2 which establishes necessary conditions for the global solvability of (1.1). The following proposition will be useful for proving Theorem 2.

**Proposition 7.** Assume that \( \omega_0 \in H^4(\Omega \times (0,1)) \) satisfies the local Rayleigh condition. Let \( \omega \in C([0,T]; H^4(\Omega \times (0,1))) \) be the solution to (1.2) that satisfies the local Rayleigh condition in \([0,T]\). Let \( E_1 \) and \( E_2 \) be defined by (1.11) and (1.12) respectively. Then, we have:

\[
\int_0^T \int_\Omega P_2^2 \omega \omega_y dxdydt \leq 2E_2 |_{t=0}^{\infty} + \frac{9}{2} \|\omega_0\|^2_\infty E_1 |_{t=0}^{\infty}.
\]

**Proof.** By (3.19), (3.18) and (2.9)
\[
\int_0^T \int T \, dx \, dt = \int_0^T \int T \times (0,1) \left| P_x + u \left( \frac{\omega_x}{\omega_y} - u_x \right) \right|^2 \, dx \, dy \, dt
\]
\[
\leq 2 \int_0^T \int T \times (0,1) \left| P_x + u \left( \frac{\omega_x}{\omega_y} - u_x \right) \right|^2 \, dx \, dy \, dt
\]
\[
\leq 2 \int_0^T \partial_t E_2(t) \, dt + 2 \left( \frac{3}{2} \| \omega_0 \|_\infty \right)^2 \int_0^T \int T \times (0,1) \left| \frac{\omega_x}{\omega_y} - u_x \right|^2 \, dx \, dy \, dt
\]
\[
= 2 \int_0^T \partial_t E_2(t) \, dt + 2 \left( \frac{3}{2} \| \omega_0 \|_\infty \right)^2 \int_0^T \partial_t E_1(t) \, dt
\]
\[
= 2 E_2 \big|_{t=0} + \frac{9}{2} \| \omega_0 \|_\infty E_1 \big|_{t=0},
\]

which concludes the proof of Proposition 7.

Now we are ready to prove Theorem 2.

Proof of Theorem 2. By Theorem 1, \( E_1(t) \) and \( E_2(t) \) remain negative, for all \( t > 0 \). Moreover, by (3.18) and (3.19), \( E_1(t) \) and \( E_2(t) \) are convergent (since they are monotone and bounded). Furthermore, by (3.38) and (3.41),

(5.2) \[
\int_0^t E_1^2(\tau) \, d\tau \leq E_1 \big|_{\tau=0}.
\]

(5.3) \[
\int_0^t E_2^2(\tau) \, d\tau \leq \| u \|_2^2 E_2 \big|_{\tau=0}.
\]

Which implies that \( E_1(t), E_2(t) \to 0 \), as \( t \to \infty \) (otherwise (5.2) and (5.3) would growth arbitrarily, which is a contradiction). Thus,

(5.4) \[
E_1 \big|_{\tau=0}^t = \int_0^t \int T \times (0,1) \left| \frac{\omega_x}{\omega_y} - u_x \right|^2 \, dx \, dy \, d\tau \to -E_1(0), \quad \text{as} \, \, \, t \to \infty
\]

and

(5.5) \[
E_2 \big|_{\tau=0}^t = \int_0^t \int T \times (0,1) \left| P_x + u \left( \frac{\omega_x}{\omega_y} - u_x \right) \right|^2 \, dx \, dy \, d\tau \to -E_2(0), \quad \text{as} \, \, \, t \to \infty,
\]

which proves (1.17) and (1.18). Next, Proposition 7 (5.4) and (5.5) lead to (1.19). Finally, by Jensen’s inequality

(5.6) \[
\exp \left( \int T \times (0,1) \log (\omega_y(t)) \, dx \, dy \right) \leq \int T \times (0,1) \omega_y \, dx \, dy
\]

\[
\leq 2 \| \omega_0 \|_\infty.
\]

Thus, (3.17) and (5.6) implies
\[
\int_0^T |E_1(t)| dt = -\int_0^T E_1(t) dt \\
= \int_{T \times (0,1)} \log \left( \frac{\omega_y(T)}{\partial_y \omega_0} \right) \, dx dy \\
\leq \int_{T \times (0,1)} \log \left( \frac{2 \|\omega_0\|_{\infty}}{\partial_y \omega_0} \right) \, dx dy,
\]
which gives (1.20) because \(T\) is arbitrary. This completes the proof of Theorem 2. \(\square\)

6. Propagation of the local Rayleigh condition and \(H^1\) control

It is worth mentioning that Theorem 1 does not give any information about what happens first; the collapse of the local Rayleigh condition or the formation of singularities, even if the lower bounds (1.13) and (1.14) hold, Theorem 1 does not guarantees that \(\|\omega\|_{H^4(T \times (0,1))}\) remains bounded before the collapse of the local Rayleigh condition. Nevertheless, if we assume that the vorticity \(\omega\) does not vanishes, then we can control the \(L^\infty([0,T]; L^\infty(T \times (0,1)))\) norm of \(\omega\) and the \(L^\infty([0,T]; L^\infty(T \times (0,1)))\) norm of \(\frac{1}{\omega_y}\) by means of (6.1)

\[
M(T) = \int_0^T \left| \frac{\omega_x}{\omega_y} - u_x \right|_\infty \, dt,
\]
which is the main result of this section. The next proposition will be useful for proving Theorem 4.

**Proposition 8.** Assume that \(\omega_0 \in H^4(T \times (0,1))\) satisfies the local Rayleigh condition. Let \(\omega \in C([0,T]; H^4(T \times (0,1)))\) be the solution to (1.2) that satisfies the local Rayleigh condition in \([0,T]\). Assume further that \(|\omega_0| > 0\) in \(T \times (0,1)\). Then, we have:

(6.2) \[
|u_x|_\infty \leq \left( \frac{\|\omega_0\|_{\infty}}{\min|\omega_0|} + 1 \right) \left| \frac{\omega_x}{\omega_y} \right|_\infty - u_x \right|_\infty,
\]

(6.3) \[
\int_{T \times (0,1)} \frac{\omega_y^2}{\omega_y} \, dx dy|_{t=T} \leq \int_{T \times (0,1)} \frac{\partial_x \omega_y^2}{\partial_y \omega_y} \, dx dy \exp \left( C(\omega_0) \int_0^T \left| \frac{\omega_x}{\omega_y} - u_x \right|_\infty \, dt \right),
\]
where \(C(\omega_0) = 3 + \frac{2\|\omega_0\|_{\infty}}{\min|\omega_0|}\).

**Proof.** Let \(x\) be any point in \(T\). By (2.5), there exists \(y_0(x) \in [0,1]\) such that \(u_x(x, y_0(x), t) = 0\). The positivity of \(|\omega|\) yields

\[
\frac{u_x}{\omega} (x, y, t) = \int_{y_0(x)}^{y} \frac{\omega \omega_x - u_x \omega_y}{\omega^2} \, dy = \int_{y_0(x)}^{y} \frac{\omega \omega_x - u_x \omega_y \omega}{\omega} \, dy,
\]
from which we get:
|u_x|(x, y, t) \leq |\omega|(x, y, t) \left| \frac{\omega_x \omega_y - u_x \omega_y}{\omega_y} \right| \int_{y_0(x)}^y \frac{\omega_y}{\omega_y} dy

= \left| \frac{\omega_x \omega_y - u_x \omega_y}{\omega_y} \right| \omega_y = \left| \frac{\omega_x \omega_y - u_x \omega_y}{\omega_y} \right| \omega_y

\leq \left( \frac{\|\omega_0\|_{\infty}}{\min|\omega_0|} + 1 \right) \left| \frac{\omega_x \omega_y - u_x \omega_y}{\omega_y} \right| \omega_y \infty

\int_{y_0(x)}^y \frac{\omega_y}{\omega_y} dx dy = \left( \frac{\omega_x \omega_y - u_x \omega_y}{\omega_y} \right) \omega_y = 2u_x \omega_x \omega_y - 2u_x \omega_x \omega_y + \omega_x \omega_y^3

integrating in \mathbb{T} \times (0, 1), we get

(6.4) \quad \partial_t \int_{\mathbb{T} \times (0, 1)} \frac{\omega_x^2}{\omega_y} dx dy = \int_{\mathbb{T} \times (0, 1)} \frac{\omega_x^2}{\omega_y} \left( \frac{\omega_x \omega_y}{\omega_y} - 3u_x \right) dx dy,

where we have used that

\int_{\mathbb{T} \times (0, 1)} \omega_x \omega_x dx dy = \int_{\mathbb{T} \times (0, 1)} u_x u_x dx dy = 0,

thanks to integration by parts and the x-periodicity. Thus, by (6.4) and (6.2)

\partial_t \int_{\mathbb{T} \times (0, 1)} \frac{\omega_x^2}{\omega_y} dx dy = \int_{\mathbb{T} \times (0, 1)} \frac{\omega_x^2}{\omega_y} \left( \frac{\omega_x \omega_y}{\omega_y} - 3u_x \right) dx dy

\leq \left( \frac{\|\omega_0\|_{\infty}}{\min|\omega_0|} + 2 \left| \frac{u_x}{\omega_y} \right| \infty \right) \int_{\mathbb{T} \times (0, 1)} \frac{\omega_x^2}{\omega_y} dx dy

\leq \left( 3 + 2 \frac{\|\omega_0\|_{\infty}}{\min|\omega_0|} \right) \left| \frac{\omega_x \omega_y - u_x \omega_y}{\omega_y} \right| \omega_y \infty \int_{\mathbb{T} \times (0, 1)} \frac{\omega_x^2}{\omega_y} dx dy,

By applying Grönwall’s inequality, we get (6.3), which completes the proof of Proposition 8.

\square

Let us state the main result of this section.

**Theorem 4.** Assume that \( \omega_0 \in H^4(\mathbb{T} \times (0, 1)) \) satisfies the local Rayleigh condition. Let \( \omega \in C([0, T]; H^4(\mathbb{T} \times (0, 1))) \) be the solution to (1.2) that satisfies the local Rayleigh condition in \([0, T] \). Let \( E_1 \) and \( E_2 \) be defined by (1.11) and (1.12) respectively. Assume further that \( |\omega_0| > 0 \) in \( \mathbb{T} \times (0, 1) \). Then, we have:
\begin{align}
&\left\| \frac{1}{\omega_y(t)} \right\|_{L^\infty(T \times (0,1))} \leq \left\| \frac{1}{\partial_y \omega_0} \right\|_{L^\infty(T \times (0,1))} \exp(M(t)), \text{ for } 0 \leq t \leq T, \\
&\left\| \omega_y(t) \right\|_{L^\infty(T \times (0,1))} \leq \left\| \partial_y \omega_0 \right\|_{L^\infty(T \times (0,1))} \exp(M(t)), \text{ for } 0 \leq t \leq T, \\
&\left\| \frac{\partial_x \omega_0}{\sqrt{\partial_y \omega_0}} \right\|_{L^2(T \times (0,1))} \leq \left\| \frac{\partial_x \omega_0}{\sqrt{\partial_y \omega_0}} \right\|_{L^2(T \times (0,1))} \exp \left( \frac{C(\omega_0)}{2} M(t) \right), \text{ for } 0 \leq t \leq T, \\
&E_1(T) \leq \left\| \frac{\partial_x \omega_0}{\sqrt{\partial_y \omega_0}} \right\|_{L^2(T \times (0,1))} \left\| \omega_0 \right\|_{\infty} \left\| \frac{1}{\partial_y \omega_0} \right\|_{L^\infty(T \times (0,1))} \exp \left( \hat{C}(\omega_0) M(T) \right), \\
&E_2(T) \leq \frac{9}{4} \left\| \frac{\partial_x \omega_0}{\sqrt{\partial_y \omega_0}} \right\|_{L^2(T \times (0,1))} \left\| \omega_0 \right\|_{\infty}^3 \left\| \frac{1}{\partial_y \omega_0} \right\|_{L^\infty(T \times (0,1))} \exp \left( \hat{C}(\omega_0) M(T) \right),
\end{align}

where
\begin{align}
M(t) &= \int_0^t \left| \frac{\omega_x}{\omega_y} - u_x \right|_{L^\infty(T \times (0,1))} d\tau, \\
C(\omega_0) &= 3 + \frac{2\|\omega_0\|_{L^\infty(0,1)}}{\min|\omega_0|} \text{ and } \hat{C}(\omega_0) = \frac{1}{2} + \frac{C(\omega_0)}{2}.
\end{align}

**Proof.** Let \( \alpha \) be any point in \( T \times (0,1) \). By (3.11)

\[
|\log(\omega_y(X(\alpha, \tau), Y(\alpha, \tau), \tau))|_{t=0}^{t=T} = \int_0^t \partial_\tau \log(\omega_y(X(\alpha, \tau), Y(\alpha, \tau), \tau)) d\tau
\]

\[
= \int_0^t \left( \frac{\omega_x \omega_y}{\omega_y^2} - u_x \right) (X(\alpha, \tau), Y(\alpha, \tau), \tau) d\tau
\]

\[
\leq \int_0^t \left( \frac{\omega_x}{\omega_y} - u_x \right) L^\infty(T \times (0,1)) d\tau,
\]

where \( (X(\alpha, \tau), Y(\alpha, \tau)) \) is the characteristic curve starting at \( \alpha \). Thus, applying exponential we get (6.5) and (6.6). From (6.3) we get (6.7). By Hölder’s inequality, (6.3) and (6.4)

\[
\int_{T \times (0,1)} |\frac{\omega_x}{\omega_y}| dxdy \leq \left\| \frac{\omega_x}{\omega_y} \right\|_{L^2(T \times (0,1))} \left\| \omega \right\|_{L^2(T \times (0,1))}
\]

\[
\leq \left\| \omega_0 \right\|_{L^\infty(T \times (0,1))} \left\| \frac{1}{\partial_y \omega_0} \right\|_{L^\infty(T \times (0,1))} \left\| \frac{\partial_x \omega_0}{\sqrt{\partial_y \omega_0}} \right\|_{L^2(T \times (0,1))} \exp \left( \frac{M(t)}{2} + \frac{C(\omega_0) M(t)}{4} \right),
\]

(6.11)
which yields (6.8) and (6.9) thanks to (2.9). This concludes the proof of Theorem 4.

7. Proof of Theorem 3: Finite time blow-up of the Semi-lagrangian equations

The aim of this section is to prove Theorem 3, which establishes the finite time blow-up of solutions to the semilagrangian equations (1.9) for certain class of initial data. The following proposition will be useful for proving Theorem 3.

Proposition 9. Let \((v, h_a)\) be a smooth solution to (1.9). Let \(E_1\) and \(E_2\) be defined by (1.21) and (1.22) respectively. Then, we have:

\[
E_1 = \partial_t \int_{T^d \times (0, 1)} \log(h_a) h_a \, dx \, da,
\]

\[
\partial_t E_1 = \int_{T^d \times (0, 1)} |\nabla v|^2 h_a \, dx \, da,
\]

\[
\partial_t E_2 = \int_{T^d \times (0, 1)} |\partial_t v|^2 h_a \, dx \, da.
\]

Proof. By (1.9) and the \(x\)-periodicity

\[
\partial_t \int_{T^d \times (0, 1)} \log(h_a) h_a \, dx \, da = \int_{T^d \times (0, 1)} \frac{\partial h_a}{h_a} h_a + \log(h_a) \partial h_a \, dx \, da
\]

\[
= \int_{T^d \times (0, 1)} -\nabla \cdot (v h_a) - \log(h_a) \nabla \cdot (v h_a) \, dx \, da
\]

\[
= \int_{T^d \times (0, 1)} \nabla (\log(h_a)) \cdot (v h_a) \, dx \, da
\]

\[
= \int_{T^d \times (0, 1)} \nabla(h_a) \cdot v \, dx \, da
\]

\[
= \int_{T^d \times (0, 1)} (\nabla \cdot v) h_a \, dx \, da,
\]

which proves (7.1). Next, let us compute the time derivative of \(E_1\):

\[
\partial_t E_1(t)
\]

\[
= \int_{T^d \times (0, 1)} \nabla \cdot \nabla \left( \frac{|v|^2}{2} + P \right) h_a + \nabla \cdot v \nabla \cdot (v h_a) \, dx \, da
\]

\[
= \int_{T^d \times (0, 1)} (v \cdot \Delta v) h_a + (\nabla v \cdot \nabla v) h_a + \nabla \cdot v \nabla \cdot (v h_a) \, dx \, da + \int_{T^d} \Delta P \int_0^1 h_a \, dx \, da
\]

\[
= \int_{T^d \times (0, 1)} (v \cdot \Delta v) h_a + |\nabla v|^2 h_a + \nabla \cdot (\nabla v) h_a - \nabla (\nabla \cdot v) \cdot h_a \, dx \, da
\]

\[
= \int_{T^d \times (0, 1)} (v \cdot \Delta v) h_a + |\nabla v|^2 h_a - (\Delta v \cdot v) h_a \, dx \, da
\]

\[
= \int_{T^d \times (0, 1)} |\nabla v|^2 h_a \, dx \, da,
\]
where we have used the identity $\nabla(\nabla \cdot \mathbf{v}) = \Delta \mathbf{v}$, which follows by the curl free condition $\partial_i v_j = \partial_j v_i$. Thus, we proved (7.2). Finally, let us compute the time derivative of $E_2$:

$$
\partial_t E_2(t) = \int_{T^d \times (0,1)} \mathbf{v}_t \cdot \mathbf{v}_t h_a + \mathbf{v} \cdot \mathbf{v}_{tt} h_a - \mathbf{v} \cdot \nabla \cdot (\mathbf{v} h_a) dx \, da.
$$

The second term is equal to:

$$
\int_{T^d \times (0,1)} \mathbf{v} \cdot \mathbf{v}_{tt} h_a dx \, da = - \int_{T^d \times (0,1)} \mathbf{v} \cdot \nabla (\mathbf{v} + P_t) h_a dx \, da
$$

$$
= \int_{T^d} P_t \nabla \cdot \int_0^1 \mathbf{v} h_a dx \, da + \int_{T^d \times (0,1)} \mathbf{v} \cdot \mathbf{v}_t \nabla \cdot (\mathbf{v} h_a) dx \, da
$$

$$
= \int_{T^d \times (0,1)} \mathbf{v} \cdot \mathbf{v}_t \nabla \cdot (\mathbf{v} h_a) dx \, da,
$$

thanks to (2.22), which leads to (7.3). Thus the proof of Proposition 9 is now complete. $\square$

Now we can prove Theorem 3.

**Proof of Theorem 3.** First assume that (1.24) holds. Integrating in time and applying (7.1), we get:

$$
d \log \left( \frac{d}{E_1(0)} \frac{1}{E_1(0) - t} \right) = \int_0^t \frac{d}{E_1(0) - \tau} d\tau
$$

$$
\leq \int_0^t E_1(\tau) d\tau
$$

$$
= \int_{T^d \times (0,1)} \log(\mathbf{h}_a(\tau) h_a(\tau)) dx \, da \bigg|_{\tau=0}^{\tau=t},
$$

which implies (1.23). Next, by (7.2) and Hölder’s inequality

$$
\partial_t E_1 = \int_{T^d \times (0,1)} |\nabla \mathbf{v}|^2 h_a dx \, da
$$

$$
\geq \frac{1}{d} \int_{T^d \times (0,1)} |\nabla \cdot \mathbf{v}|^2 h_a dx \, da
$$

$$
\geq \frac{1}{d} \left( \int_{T^d \times (0,1)} |\nabla \cdot \mathbf{v}| h_a dx \, da \right)^2
$$

$$
(7.5)
$$

$$
\geq \frac{1}{d} E_1^2.
$$
Thus, Lemma 6, (7.2) and (7.5) lead to (1.24). Finally, by (7.3) and Hölder’s inequality
\[
\int_{\mathbb{T}^d \times (0,1)} |\mathbf{v}|^2 h_a dx da \cdot \partial_t E_2 \geq \left( \int_{\mathbb{T}^d \times (0,1)} |\mathbf{v}| h_a dx da \right)^2
\]
(7.6)
Thus, Lemma 6, (7.6), (7.3) and (2.20) lead to (1.25). This concludes the proof of Theorem 3.

8. Appendix. Integral lemma

The aim of this appendix is to prove the integral identity concerning the integral of the logarithm of a function.

Lemma 7. Let \((\Omega, \mu)\) be a measure space and let \(f \in L^1(\Omega)\) be a positive function in \(\Omega\). Assume that \(\mu(\Omega) = 1\), then
\[
\exp \left( \int_\Omega \log(f) d\mu \right) = \lim_{p \to 0} \left( \int_\Omega f^p d\mu \right)^{\frac{1}{p}}.
\]
Proof. Let us proceed as in [Gra14], let \(p_n\) be any positive sequence converging to 0. By Jensen’s inequality,
\[
\exp \left( \int_\Omega \log(f) d\mu \right) \leq \left( \int_\Omega f^p d\mu \right)^{\frac{1}{p}}, \quad \forall p > 0,
\]
which implies
\[
\exp \left( \int_\Omega \log(f) d\mu \right) \leq \liminf_{p \to 0} \left( \int_\Omega f^{p_n} d\mu \right)^{\frac{1}{p_n}}.
\]
Since \(\log t \leq t - 1\), for \(t > 0\), we conclude that
\[
\log \left( \int_\Omega f^{p_n} d\mu \right) \leq \int_\Omega (f^{p_n} - 1) d\mu.
\]
Multiplying (8.3) by \(\frac{1}{p_n}\) and applying exponential, we get
\[
\left( \int_\Omega f^{p_n} d\mu \right)^{\frac{1}{p_n}} \leq \exp \left( \frac{1}{p_n} \int_\Omega (f^{p_n} - 1) d\mu \right),
\]
which implies
\[
\limsup \left( \int_\Omega f^{p_n} d\mu \right)^{\frac{1}{p_n}} \leq \limsup \exp \left( \frac{1}{p_n} \int_\Omega (f^{p_n} - 1) d\mu \right).
\]
Now, let us compute \(\limsup \exp \left( \frac{1}{p_n} \int_\Omega (f^{p_n} - 1) d\mu \right)\). Assume that \(q_n \to 0\), then
\[
\frac{1}{q_n} \int_\Omega (t^{q_n} - 1) d\mu \log(t), \quad \forall t > 0.
\]
Let \(h_n : \Omega \to \mathbb{R}\) be a sequence of positive functions defined by
\[ h_n(x) = \frac{1}{q_0} (f^{q_0} - 1) - \frac{1}{q_n} (f^{q_n} - 1). \]

By (8.5),
\[ h_n(x) \nearrow \frac{1}{q_0} (f^{q_0} - 1) - \log(f). \]

Applying Lebesgue’s monotone convergence theorem, we get
\[ \int_\Omega h_n d\mu \nearrow \int_\Omega \frac{1}{q_0} (f^{q_0} - 1) - \log(f) d\mu. \]

Thus,
\[ \int_\Omega \frac{1}{q_n} (f^{q_n} - 1) d\mu \searrow \int_\Omega \log(f) d\mu. \]

Therefore,
\[ \limsup \exp \left( \frac{1}{p_n} \int_\Omega (f^{p_n} - 1) d\mu \right) = \exp \left( \int_\Omega \log(f) d\mu \right). \]

Finally, by (8.2), (8.4) and (8.6)
\[ \limsup \left( \int_\Omega f^{p_n} d\mu \right)^{\frac{1}{p_n}} \leq \exp \left( \int_\Omega \log(f) d\mu \right) \leq \liminf \left( \int_\Omega f^{p_n} d\mu \right)^{\frac{1}{p_n}}. \]

This concludes the proof of Lemma 7. □

ACKNOWLEDGEMENTS

The author thanks Yann Brenier for many interesting discussions (in particular for mentioning the possible extension to higher dimensions of the blow-up result) and for his hospitality during the author’s stay at Laboratoire de Mathématiques d’Orsay.

This work is part of the grant SEV-2015-0554-17-4 funded by: MCIN/AEI/10.13039/501100011033.

REFERENCES

[Bre99] Yann Brenier. Homogeneous hydrostatic flows with convex velocity profiles. Nonlinearity, 12:495, 01 1999.
[Bre03] Yann Brenier. Remarks on the derivation of the hydrostatic euler equations. Bulletin Des Sciences Mathématiques - BULL SCI MATH, 127:585–595, 09 2003.
[CINT12] Chongsheng Cao, Slim Ibrahim, Kenji Nakanishi, and Edriss Titi. Finite-time blowup for the inviscid primitive equations of oceanic and atmospheric dynamics. Communications in Mathematical Physics, 337, 10 2012.
[EE97] Weinan E and Bjorn Engquist. Blowup of solutions of the unsteady prandtl’s equation. Communications on Pure and Applied Mathematics, 50(12):1287–1293, 1997.
[Gra14] Loukas Grafakos. Classical Fourier Analysis, volume 249. 01 2014.
[Gre99] Emmanuel Grenier. On the derivation of homogeneous hydrostatic equations. Mathematical Modelling and Numerical Analysis, 33:965–970, 09 1999.
[Gre00] Emmanuel Grenier. On the nonlinear instability of euler and prandtl equations. Communications on Pure and Applied Mathematics, 53:1067 – 1091, 09 2000.
[HKN16] Daniel Han-Kwan and Tuan Nguyen. Ill-posedness of the hydrostatic euler and singular vlasov equations. Archive for Rational Mechanics and Analysis, 221, 09 2016.
[KTVZ10] Igor Kukavica, Roger Temam, Vlad Vicol, and Mohammed Ziane. Existence and uniqueness of solutions for the hydrostatic euler equations on a bounded domain with analytic data. *Comptes Rendus Mathematique - C R MATH*, 348:639–645, 06 2010.

[KTVZ11] Igor Kukavica, Roger Temam, Vlad Vicol, and Mohammed Ziane. Local existence and uniqueness for the hydrostatic euler equations on a bounded domain. *Journal of Differential Equations*, 250:1719–1746, 02 2011.

[KW12] Tak Kwong Wong. Blowup of solutions of the hydrostatic euler equations. *Proc. Amer. Math. Soc.*, 143, 11 2012.

[MW12] Nader Masmoudi and Tak Kwong Wong. On the hydrostatic euler equations. *Archive for Rational Mechanics and Analysis*, 204(1):231–271, Apr 2012.

[Ren09] Michael Renardy. Ill-posedness of the hydrostatic euler and navier-stokes equations. *Arch. Ration. Mech. Anal.*, 194:877–886, 12 2009.