INVASIVE SPEED FOR A COMPETITION-DIFFUSION SYSTEM WITH THREE SPECIES

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Abstract. Competition stems from the fact that resources are limited. When multiple competitive species are involved with spatial diffusion, the dynamics becomes even complex and challenging. In this paper, we investigate the invasive speed to a diffusive three species competition system of Lotka-Volterra type. We first show that multiple species share a common spreading speed when initial data are compactly supported. By transforming the competitive system into a cooperative system, the determinacy of the invasive speed is studied by the upper-lower solution method. In our work, for linearly predicting the invasive speed, we concentrate on finding upper solutions only, and don’t care about the existence of lower solutions. Similarly, for nonlinear selection of the spreading speed, we focus only on the construction of lower solutions with fast decay rate. This greatly develops and simplifies the ideas of past references in this topic.

1. Introduction. One of the most significant mathematical models in population biology describing the competition between \( N \) species population is the Lotka-Volterra type system

\[
 u_i^t = D_i u_{xx}^i + r_i u_i^i \left( 1 - \sum_{j=1}^{N} b_{ij} u_j^j \right), \quad i = 1, 2, \cdots, N, \quad t \geq 0, \quad x \in (-\infty, \infty),
\]

where \( D_i, r_i, b_{ij} > 0 \) for \( i, j = 1, 2, \cdots, N \). This model indicates that with the existence of only single species, the growth of its population always follows the classical logistical growth law. When competition for the resource among \( N \) (> 1) species is involved, the dynamics becomes extremely complicated due to various

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growth and competition strengths of individuals. If initial data are not homogeneous, diffusion phenomena result in spatial invasion. In order to obtain a comprehensive understanding of the invasion behaviors, traveling wave solutions of (1) have always been an interesting and stimulating topic in the propagation dynamics. When \( N = 2 \), spreading dynamics has been extensively investigated recently (see, [1, 2, 3, 5, 11, 13, 14, 15, 17, 18, 21]). For the model with \( N \geq 3 \), the study of dynamics becomes more challenging due to the higher phase space dimension. Chen et al. [7] observed by numerical simulations that when the invading species is not too small, the density changes of other two species under competitive interaction occurred easily. Numerical results in [6] further indicated that two strongly competing species could result in a coexistence equilibrium, accompanied with the existence of multiple semi-trivial equilibria. Hou and Li [12] investigated the existence of traveling wave solutions for a three species competition-cooperation system by upper-lower solution method. They obtained the monotonicity, uniqueness and asymptotics for the wave solution. By using singular perturbation method, Kan and Mimura [16] obtained stable spatially-inhomogenous positive equilibrium solutions of one-dimensional system under Neumann boundary condition. When one exotic competing species invades the native system of two strongly competing species, the problem of competitive exclusion or competitor-mediated coexistence was discussed in [20]. As to the spatial invasion of the species in unbounded domain, Guo et al. [10] studied the minimal wave speed determinacy and obtained linear selection results, with an idea of studying the corresponding discrete lattice dynamical systems in [20].

In this paper, we will thoroughly study the determinacy mechanism of the minimal wave speed of traveling wave fronts to the following three species competition system of Lotka-Volterra type

\[
\begin{cases}
\tilde{u}_t = D_1 \tilde{u}_{xx} + r_1 \tilde{u}(1 - \tilde{u} - b_{12} \tilde{v} - b_{13} \tilde{w}), \\
\tilde{v}_t = D_2 \tilde{v}_{xx} + r_2 \tilde{v}(1 - b_{21} \tilde{u} - \tilde{v}), \\
\tilde{w}_t = D_3 \tilde{w}_{xx} + r_3 \tilde{w}(1 - b_{31} \tilde{u} - \tilde{w}).
\end{cases}
\]  

(2)

Here \( \tilde{u}, \tilde{v} \) and \( \tilde{w} \) are the population densities of three species respectively; \( b_{ij} \) and \( b_{ij} (j = 2, 3) \) are the competition coefficients, reflecting the competitive strength between the first species and the other two species; \( r_i \) is the growth rate and \( D_i \) is the diffusion coefficient of species \( i \) (\( i = 1, 2, 3 \)).

To proceed, we first make the following transformations

\[
\sqrt{\frac{r_1}{D_1}} x \to x, \quad r_1 t \to t, \quad \frac{D_2}{D_1} = d_1, \quad \frac{D_3}{D_1} = d_2, \quad \frac{r_2}{r_1} = \alpha, \quad \frac{r_3}{r_1} = \beta,
\]

(3)

to obtain a non-dimensional system

\[
\begin{cases}
\tilde{u}_t = \tilde{u}_{xx} + \tilde{u}(1 - \tilde{u} - b_{12} \tilde{v} - b_{13} \tilde{w}), \\
\tilde{v}_t = d_1 \tilde{v}_{xx} + \alpha \tilde{v}(1 - b_{21} \tilde{u} - \tilde{v}), \\
\tilde{w}_t = d_2 \tilde{w}_{xx} + \beta \tilde{w}(1 - b_{31} \tilde{u} - \tilde{w}).
\end{cases}
\]  

(4)

A change of variables \( \tilde{u} = u, \tilde{v} = 1 - v, \tilde{w} = 1 - w \) further transforms (4) into a cooperative system

\[
\begin{cases}
u_t = u_{xx} + u(1 - u - b_{12} + b_{12} v - b_{13} + b_{13} w), \\
v_t = d_1 v_{xx} + \alpha (1 - v)(b_{21} u - v), \\
w_t = d_2 w_{xx} + \beta (1 - w)(b_{31} u - w).
\end{cases}
\]  

(5)
It is easy to see that (5) has at least five equilibria in the region \((u, v, w)|0 \leq u \leq 1, 0 \leq v \leq 1, 0 \leq w \leq 1\). We denote them by \(e_0 = (0, 0, 0), e_1 = (1, 1, 1), e_2 = (0, 1, 1), e_3 = (0, 1, 0), e_4 = (0, 0, 1)\) respectively.

Qualitative properties of the equilibrium points \(e_i (i = 0, 1, \cdots, 4)\) will affect the existence of the traveling waves of system (5). Thus, throughout this paper, we shall always assume that

\[
\begin{align*}
    b_{12} + b_{13} &< 1, \quad b_{21} > 1, \quad b_{31} > 1, \\
\end{align*}
\]

which implies that the original species \(\hat{\nu}\) and \(\hat{\nu}_\nu\) are weak competitors to the species \(\nu\). In other words, the species \(\hat{\nu}\) and \(\hat{\nu}_\nu\) shall lose the competition during the evolution. As a result, for system (5), we are interested in traveling wave solution connecting the equilibria \(e_1 = (1, 1, 1)\) and \(e_0 = (0, 0, 0)\) in a special form

\[
(u, v, w)(x, t) = (U, V, W)(z), \quad z = x - ct,
\]

where \(c\) is called the wave speed and \((U, V, W)\) is called the wave profile. Substituting \((U, V, W)(z)\) into (5) leads to the following wave profile problem

\[
\begin{align*}
    (U'' + cU' + U(1 - b_{12} - b_{13} - U + b_{12}V + b_{13}W)) &= 0, \\
    (d_1V'' + cV' + \alpha(1 - V)(b_{21}U - V)) &= 0, \\
    (d_2W'' + cW' + \beta(1 - W)(b_{31}U - W)) &= 0, \\
    (U, V, W)(-\infty) &= e_1, \quad (U, V, W)(+\infty) = e_0.
\end{align*}
\]

Alternatively, for system (5), when the initial value \((u_0(x), v_0(x), w_0(x))\) is monotonically decreasing in \(x\) with half compact support at positive infinity, it is interesting to study the existence of constant \(c^*_\nu\) for each species, called the asymptotic speed of spreading (or spreading speed for short), such that

\[
\lim_{t \to \infty, x > (c^*_\nu + \epsilon)t} s(x, t) = 0
\]

and

\[
\lim_{t \to \infty, x < (c^*_\nu - \epsilon)t} s(x, t) \neq 0,
\]

where \(s(x, t) = u(x, t), v(x, t), w(x, t),\) and \(\epsilon\) is any sufficiently small positive constant. Our first result will show that there exists a common spreading speed \(c^\ast\) for all species of (5) and it further follows that system (7) has a traveling wave solution if and only if \(c \geq c^\ast\). In other words, the single spreading speed \(c^\ast\) is exactly the minimal traveling wave speed \(c_{\text{min}}\) of (7). The exact formula for this speed \(c_{\text{min}}\) is difficult to find. To estimate it, we can linearize the system (7) at \((0, 0, 0)\) to have a linear speed \(c_0 = \sqrt{1 - b_{12} - b_{13}}\) (for a detailed analysis, the reader is referred to Section 2). This speed \(c_0\) can be thought as the corresponding spreading speed of the linearized system. Usually, when the nonlinear system is bounded by the linearized system, the minimal speed \(c_{\text{min}}\) is equal to \(c_0\). However, in some cases, the minimal speed is determined by the nonlinear system itself and it may be bigger than \(c_0\). As such, we say that the minimal wave speed is linearly selected (or determined) if \(c_{\text{min}} = c_0\) and nonlinearly selected if \(c_{\text{min}} > c_0\).

The purpose of this paper is to study the speed selection mechanism directly by virtue of upper-lower solution method, which is totally different from the ideas employed in [10]. The main novelty of our paper lies in that we greatly develop the theory of invasive speed determinacy for multiple species where fast and slow spreadings for different species are possible to exist. We first show that for the strong-weak-weak species competition, a single spreading speed exists and all species...
Spreading speed, traveling waves and their decay behaviours near $e_0$. Let the initial data $(u_0(x), v_0(x), w_0(x))$ be bounded and continuous. The solution of (5) can define a semiflow $Q_t(u_0(x), v_0(x), w_0(x))$. For any fixed $t$, it is easy to know that $Q_t$ is continuous, monotone and compact. At $t = 1$, we set $P = Q_1$. We will define the spreading speed of the operator $P$ so that it is same as the spreading speed of the system (5).

Let $C = C(\mathbb{R}, \mathbb{R}^3)$ be the set of bounded and continuous functions from $\mathbb{R}$ to $\mathbb{R}^3$ with $|| \cdot ||_C = || \cdot ||_{\infty}$. Any vectors in $\bar{C} = \mathbb{R}^3$ can be regarded as a function in $C$. $\bar{C}$ is equipped with the maximum norm and the positive cone is defined by $C_+ = \{ \phi \in C : \phi \geq 0 \}$. The space $C$ is also equipped with the compact open topology in the sense that $v^n \to v$ in $C$ means that the sequence $v^n(x)$ converges to $v(x)$ uniformly for $x$ in every compact set. A bounded subset $C_\omega$ is defined by $C_\omega = \{ \phi \in C : 0 \leq \phi \leq \omega \}$ where $\omega = (1, 1, 1) \gg 0$. To define the spreading speed, let $\lambda \in C_\omega$, with $0 \ll \lambda \ll \omega$, and assume $\phi = (\phi^1, \phi^2, \phi^3) \in C_\omega$ has the properties

(K1) $\phi^i(\cdot)$ is a non-increasing function for any $1 \leq i \leq 3$;

(K2) $\phi^i(x) = 0$ for any $x \geq 0$, $1 \leq i \leq 3$;

(K3) $\phi(-\infty) = \lambda$.

Given a real number $c$, define an operator $R_c$ and a sequence of functions $\{a_n\}_{n=0}^\infty$ as

$$R_c[a](\theta, s) = \max\{\phi(\theta, s), T_{-c}[P[a]](\theta, s)\};$$

$$a_0(c; \theta, s) = \phi(\theta, s), \quad a_{n+1}(c; \theta, s) = R_c[a_n(c; \cdot)](\theta, s),$$

where $T$ is the translation operator satisfying $T_y(u(x)) = u(x - y)$.

Thus, by the idea of Lui [19], it follows the following results:

(a) $a_n \leq a_{n+1} \leq \omega$;

(b) $\lim_{n \to \infty} a_n(c; s) = a(c; s)$ pointwise;

(c) $a(c; s)$ is non-increasing in $c$ and $s$;

(d) $a(c; \cdot, -\infty) = \omega$, $a(c; \cdot, +\infty)$ exists in $C_\omega$ and is a fixed point of $P$. 

The rest of this paper is organized as follows. We study the local behavior of the wave profile near the equilibrium point $e_0$ in Section 2. Based on our choice of the upper and the lower solutions, we derive general criteria for the linear and nonlinear speed selection in Section 3. We further find explicit conditions for the speed selection in Section 4. The results of numerical simulation are shown in Section 5. Conclusions are presented in Section 6.
Using the above properties, two spreading speeds \( c^* \) and \( c^*_+ \) of the operator \( P \) can be defined by

\[
c^* := \sup \{ c : a(c; \cdot, +\infty) = \omega \},
\]

and

\[
c^*_+ := \sup \{ c : a(c; \cdot, +\infty) \neq 0 \}.
\]

If \( c^* = c^*_+ \), we say there exists a single spreading speed. By Theorem 4.2(2) in [8] we have the following theorem.

**Theorem 2.1.** There exists a single spreading speed \( c^* \) for (5) and system (7) has positive traveling waves if and only if \( c \geq c^* \).

**Proof.** Since for \( c > 0 \), there is no traveling wave for (7) connecting \( e_i, 2 \leq i \leq 4 \), and \( e_0 \), by Theorem 4.2(2) in [8], this means there exists a single spreading speed, and traveling waves of (7) exist if and only if \( c \geq c^* \).

For the traveling waves, to investigate the speed determinacy, we first need to analyze the local behavior for the positive wave profile of (7) near the equilibrium point \( e_0 \). By linearizing the system around \( e_0 \), we obtain the following constant-coefficient system

\[
\begin{align*}
(U'') + cU' + (1 - b_{12} - b_{13})U &= 0, \\
d_1V'' + cV' + \alpha(b_{21}U - V) &= 0, \\
d_2W'' + cW' + \beta(b_{31}U - W) &= 0.
\end{align*}
\]

To find positive solutions for the above system, let \( (U, V, W)(z) = (\xi_1, \xi_2, \xi_3)e^{-\mu z} \) for some positive constants \( \xi_1, \xi_2, \xi_3 \) and \( \mu \), and we will obtain characteristic equations of the system. Substituting it into (13), we get

\[
A(\mu) \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},
\]

where \( A(\mu) \) is a 3 \times 3 matrix given by

\[
A(\mu) = \begin{bmatrix} \Gamma_1(\mu) & 0 & 0 \\ \alpha b_{21} & \Gamma_2(\mu) & 0 \\ \beta b_{31} & 0 & \Gamma_3(\mu) \end{bmatrix},
\]

with \( \Gamma_1(\mu) = \mu^2 - c\mu + 1 - b_{12} - b_{13}, \Gamma_2(\mu) = d_1\mu^2 - c\mu - \alpha \) and \( \Gamma_3(\mu) = d_2\mu^2 - c\mu - \beta \).

The first equation is decoupled. If \( \Gamma_1(\mu) = 0 \), then \( \mu \) equals one of the following values

\[
\mu_1(c) = \frac{1}{2} \left[ c - \sqrt{c^2 - 4(1 - b_{12} - b_{13})} \right],
\]

\[
\mu_2(c) = \frac{1}{2} \left[ c + \sqrt{c^2 - 4(1 - b_{12} - b_{13})} \right].
\]

To make \( \mu_1(c) \) and \( \mu_2(c) \) real so that the solution \( U \) is positive, the speed \( c \) has to satisfy

\[
c \geq c_0 = 2\sqrt{1 - b_{12} - b_{13}},
\]

where \( c_0 \) is called the linear speed of the system. For \( c > c_0 \), the traveling wave \( U \) should satisfy

\[
U \sim C_1e^{-\mu_1(c)z} + C_2e^{-\mu_2(c)z}
\]

as \( z \to \infty \), for constant \( C_1 > 0 \), or \( C_1 = 0 \) with \( C_2 > 0 \).
When \( \Gamma_2(\mu) = 0 \), we get a positive solution of \( \mu \) with
\[
\mu_3(c) = \frac{1}{2d_1} \left( c + \sqrt{c^2 + 4d_1^2} \right) > 0,
\]
and \( \Gamma_3(\mu) = 0 \) has a positive root
\[
\mu_4(c) = \frac{1}{2d_2} \left( c + \sqrt{c^2 + 4d_2^2} \right) > 0.
\]

The behaviors of \( V \) and \( W \) are related to all \( \mu_i(c), 1 \leq i \leq 4 \), and can be solved from (13). For instance, when \( \mu_i(c), 1 \leq i \leq 4 \), are not equal, we can have
\[
\begin{pmatrix}
U(z) \\
V(z) \\
W(z)
\end{pmatrix} \sim C_1 \begin{pmatrix}
1 & -\frac{\alpha b_{21}}{\Gamma_2(\mu_2)} & -\frac{\alpha b_{21}}{\beta b_{21}} \\
-\frac{\Gamma_2(\mu_1)}{\beta b_{21}} & -\frac{\beta b_{21}}{\beta b_{21}} & -\frac{\beta b_{21}}{\beta b_{21}} \\
0 & 0 & 0
\end{pmatrix} e^{-\mu_1 z}
+ C_3 \begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix} e^{-\mu_3 z} + C_4 \begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix} e^{-\mu_4 z},
\]
for constants \( C_1 > 0, C_3 > 0, C_4 > 0, \) or \( C_1 = 0 \) with \( C_2 > 0, C_3 > 0, C_4 > 0 \).

3. **The speed selection mechanism.** In this section, the speed selection mechanism of the system (7) is studied by using the upper-lower solution method. According to the ideas in [4], the existence of solutions \( V(z) \) and \( W(z) \) will first be proved for a given function \( U(z) \) which is continuous and satisfies \( U(-\infty) = h \) and \( U(\infty) = 0 \).

**Lemma 3.1.** For a given continuous and non-increasing function \( U(z) \) satisfying \( U(-\infty) = h > 0 \) and \( U(\infty) = 0 \), there exist two non-increasing functions \( V(z) \) and \( W(z) \) satisfying the problem
\[
\begin{align*}
d_1 V'' + c V' + \alpha(1 - V)(b_{21} U - V) &= 0, \\
d_2 W'' + c W' + \beta(1 - W)(b_{31} U - W) &= 0, \\
(V, W)(-\infty) &= \left( \min\{1, h b_{21}\}, \min\{1, h b_{31}\} \right), \quad (V, W)(+\infty) = (0, 0),
\end{align*}
\]

**Proof.** The first equation in (22) is independent of \( W \) and the second equation is independent of \( V \), as such, the idea of the proof is similar to [4] and we leave it to the interested readers.

From Lemma 3.1, if we denote \( V \) and \( W \) as functions of \( U \) respectively, that is \( V(z) = V(U(z)) \), \( W(z) = W(U(z)) \), then we can have the monotonic properties below, which play important roles in the study of nonlinear selection.

**Lemma 3.2.** The function \( W(U) \) (resp. \( V(U) \)) is monotonically non-decreasing with respect to \( U \) in the sense that \( W(U_1) \geq W(U_2) \) (resp. \( V(U_1) \geq V(U_2) \)) if \( U_1 \geq U_2 \).

**Proof.** We first re-express (22) as the following equivalent form
\[
\begin{align*}
d_1 V'' + c V' - \kappa V &= -G(U, V), \\
d_2 W'' + c W' - \kappa W &= -H(U, V),
\end{align*}
\]
where
\[
G(U, V) = \kappa V + \alpha(1 - V)(b_{21} U - V), \quad H(U, W) = \kappa W + \beta(1 - W)(b_{31} U - W)
\]
and $\kappa$ is chosen sufficiently large such that $G(U, V)$ is monotone in $V$ and $H(U, W)$ is monotone in $W$ respectively. Moreover, it is easy to see that $G(U, V)$ and $H(U, W)$ are also monotone in $U$. An application of variation-of-parameters on the system (23) leads to its integral form

$$
\begin{align*}
V(z) & := I_1(U,V), \\
W(z) & := I_2(U,W),
\end{align*}
$$

(25)

where

$$
\begin{align*}
I_1(U,V) & := \frac{1}{d_1(\mu_1^+ - \mu_1^-)} \left\{ \int_{-\infty}^{z} e^{\mu_1^+(z-s)} G(U,V)(s)ds + \int_{z}^{\infty} e^{\mu_1^-(z-s)} G(U,V)(s)ds \right\}, \\
I_2(U,W) & := \frac{1}{d_2(\mu_2^+ - \mu_2^-)} \left\{ \int_{-\infty}^{z} e^{\mu_2^+(z-s)} H(U,V)(s)ds + \int_{z}^{\infty} e^{\mu_2^-(z-s)} H(U,V)(s)ds \right\},
\end{align*}
$$

and $\mu_i^+, i = 1, 2$ are given by

$$
\mu_i^+ = -c + \sqrt{c^2 + 4\kappa} > 0, \quad \mu_i^- = \frac{-c - \sqrt{c^2 + 4\kappa}}{2d_i} < 0.
$$

By employing the framework of upper-lower solution method (see for example [2]), one can define an iteration scheme as

$$
\begin{align*}
V_{n+1}(z) & = I_1(U,V_n), \\
W_{n+1}(z) & = I_2(U,W_n).
\end{align*}
$$

(26)

Indeed, we can choose the initial function $(V_0, W_0) = (0, 0)$ as a lower solution. As for the upper solution, we define a pair of functions as follows

$$
\mathcal{V}(z) = \min\{1, 1 - \hat{v}(-z)\}, \quad \mathcal{W}(z) = \min\{1, 1 - \hat{w}(-z)\},
$$

where $\hat{v}(t)$ and $\hat{w}(t)$ are respectively the bistable waves of

$$
d_1\hat{v}''(t) + \hat{c}_1\hat{v}'(t) + f_1(\hat{v}(t)) = 0,
$$

and

$$
d_2\hat{w}''(t) + \hat{c}_2\hat{w}'(t) + f_2(\hat{w}(t)) = 0,
$$

connecting $1 - \epsilon_1$ to $-\epsilon_1$ and $1 - \epsilon_2$ to $-\epsilon_2$. Here,

$$
f_1(\hat{v}) = \begin{cases} 
\alpha \hat{v}(1 - \epsilon_1 - \hat{v}), & \hat{v} \geq 0, \\
\alpha \hat{v}(\epsilon_1 + \hat{v}), & \hat{v} < 0,
\end{cases}
$$

and

$$
f_2(\hat{w}) = \begin{cases} 
\beta \hat{w}(1 - \epsilon_2 - \hat{w}), & \hat{w} \geq 0, \\
\beta \hat{w}(\epsilon_2 + \hat{w}), & \hat{w} < 0,
\end{cases}
$$

and $0 < \epsilon_1, \epsilon_2 \ll 1$ are two small numbers. The existences of $\hat{v}(t)$ and $\hat{w}(t)$ are referred to [9, Theorem 3.5]. The verification of $(\mathcal{V}, \mathcal{W})(z)$ to be an upper solution can be obtained directly. By virtue of the Helly’s lemma, such defined sequences $\{V_n(U)\}_{n=0}^{\infty}$ and $\{W_n(U)\}_{n=0}^{\infty}$ converge to two functions $V(U)$ and $W(U)$. Moreover, $V(z)$ and $W(z)$ are nonincreasing in $z \in \mathbb{R}$. Next, we have to show that $V(z)$ and $W(z)$ satisfy the boundary conditions $(V, W)(-\infty) = (1, 1)$ and $(V, W)(\infty) = (0, 0)$. The existences of the limits $V(\pm \infty)$ and $W(\pm \infty)$ result from the monotonicity of $V(z)$ and $W(z)$ as well as the inequalities $0 \leq V(z) \leq \mathcal{V}(z) \leq 1$ and $0 \leq W(z) \leq \mathcal{W}(z) \leq 1, \forall z \in \mathbb{R}$. In view of (23) and (24), we have

$$
(1 - V(\infty))V(\infty) = 0, \quad (1 - V(-\infty))(b_{21}h - V(-\infty)) = 0, \quad (27)
$$
and
\[(1 - W(\infty))W(\infty) = 0, \quad (1 - W(-\infty))(b_{31}h - W(-\infty)) = 0.\] (28)
Noticing that \(V(\infty) = \epsilon_1\), it enables us to derive from the first equation of (26) that \(V(\infty) = 0\). In addition, whenever \(b_{21}h = 1\) or \(b_{21}h \neq 1\), one can always infer from the second equation of (27) that \(V(-\infty) = 1\). The boundary conditions \(W(\infty) = 0\) and \(W(-\infty) = 1\) follow from a similar discussion depending on (28).

Next, we are going to show the monotonicity of \(V(U)\). Assume \(U_1 \geq U_2\) for all \(z \in \mathbb{R}\). Then by (26), we can get two sequences \(\{V_n(U_1)\}_{n=0}^{\infty}\) and \(\{V_n(U_2)\}_{n=0}^{\infty}\). As a matter of fact that \(G(U, V)\) is monotonically increasing in \(U\), it gives \(V_i(U_1) \geq V_i(U_2), \forall i \geq 1\). By taking the limit, we have \(V(U_1) = \lim_{n \to \infty} V_n(U_1) \geq \lim_{n \to \infty} V_n(U_2) = V(U_2)\). Thus, the monotonicity of \(V(U)\) in \(U\) is proved. One can verify the monotonicity of \(W(U)\) by a similar argument. The proof is complete. \(\Box\)

**Remark 1.** Under the condition \(b_{12} = b_{13} = 0\), system (7) reduces to
\[
\begin{align*}
U'' + cU' + U(1 - U) &= 0, \\
2V'' + cV' + \alpha(1 - V)(b_{21}U - V) &= 0, \\
2W'' + cW' + \beta(1 - W)(b_{31}U - W) &= 0, \\
(U, V, W)(-\infty) &= e_1, \quad (U, V, W)(+\infty) = e_0.
\end{align*}
\] (29)
It can be checked that the linear speed selection is realized from Lemma 3.1. Indeed, it is easy to know that the first equation of Eq. (29) is the well-known Fisher equation. The traveling wave solution of the Fisher equation exists for all \(c \geq 2\). Therefore, by Lemma 3.1, the solutions \(V(z), W(z)\) also exist.

**Theorem 3.3 (Linear Selection).** For \(c = c_0\), assume that there exists a continuous, monotone and positive upper solution \((\overline{U}, \overline{V}, \overline{W})\) satisfying
\[
1 \geq \lim_{z \to -\infty} \overline{U}(z) > 0, \quad \lim_{z \to \infty} \overline{U}(z) = 0.
\] (30)
Then the spreading speed is linearly selected.

**Proof.** Here we can use the definition (12) to prove our result. We can let \(\phi(-\infty)\) be small so that the upper solution \((\overline{U}, \overline{V}, \overline{W})\) (or a shift of \((\overline{U}, \overline{V}, \overline{W})\) if needed) satisfies
\[
a_0(c_0; x) \leq (\overline{U}, \overline{V}, \overline{W})(x)
\] (31)
for all \(x \in (-\infty, \infty)\). From (10) and (11), by induction, it follows that
\[
a_{n+1}(c_0; x) \leq (\overline{U}, \overline{V}, \overline{W})(x), \quad n \geq 0.
\]
Thus \(a(c_0; -\infty) = 0\). By (12), we have \(c^* \leq c_0\), that is \(c_{\min} \leq c_0\). By (18), the proof is complete. \(\Box\)

Next, we will try to construct a suitable upper solution to the \(U\)-equation. Assume that \(c = c_0\). Define a continuous monotonic function
\[
\overline{U}(z) = \frac{1}{1 + A e^{\mu_1(c_0)z}},
\] (32)
where \(A\) is a constant and \(\mu_1(c_0)\) is defined in (16). Direct calculation shows \(\overline{U}' = -\mu_1 U(1 - \overline{U})\) and \(\overline{U}'' = -\mu_1^2(U - \overline{U}^2)(2\overline{U} - 1)\). By Lemma 3.1, we let \(V(z)\)}
(\(\overline{W}(z)\)) be the solution of the \(V\) (or \(W\)) equation when \(U(z) = \overline{U}(z)\). Substituting \((\overline{U}, \overline{V}, \overline{W})(z)\) into the \(U\)-equation, the left-hand side becomes

\[
\overline{U}(1 - \overline{U}) \left[ \mu_1^2 - c \mu_1 + 1 - b_{12} - b_{13} + \overline{U} \left( -2\mu_1^2 + \frac{b_{12}\overline{V} + b_{13}\overline{W} - (b_{12} + b_{13})\overline{U}}{\overline{U}(1 - \overline{U})} \right) \right].
\]

(33)

Note that, \(\mu_1(c_0) = \sqrt{1 - b_{12} - b_{13}}\). We can conclude that \((\overline{U}, \overline{V}, \overline{W})(z)\) is an upper solution to the system (7) if

\[-2(1 - b_{12} - b_{13}) + J(z) \leq 0, \quad (34)\]

where \(J(z) = \frac{1}{\mu_1(c_0)^2} \left[ b_{12}\overline{V} + b_{13}\overline{W} - (b_{12} + b_{13})\overline{U} \right].\)

This leads to the following linear speed selection result.

**Theorem 3.4.** The speed selection of the system (7) is linearly realized when (34) is satisfied for \(c = c_0\), with the choice of \(\overline{U}(z)\) being given by (32).

Now, we are ready to study the nonlinear selection. To this end, we intend to construct a lower solution \(U(z)\) which has the faster decay rate. The following theorem provides a justification.

**Theorem 3.5.** For \(c_1 > c_0\), assume that we can find \((\overline{U}, \overline{V}, \overline{W})(x - c_1t) \geq 0\) as a lower solution to the partial differential system

\[
\begin{aligned}
    u_t &= u_{xx} + u(1 - b_{12} - b_{13} - u + b_{12}v + b_{13}w), \\
    v_t &= d_1v_{xx} + \alpha(1 - v)(b_{21}u - v), \\
    w_t &= d_2w_{xx} + \beta(1 - w)(b_{31}u - w).
\end{aligned}
\]

(35)

In addition, also assume that \(\overline{U}(z_1)\), \(z_1 = x - c_1t\), is monotonic and satisfies \(\lim \sup \overline{U}(z_1) < 1\), \(\overline{U}(z_1) \sim e^{-\mu_2z_1}\) as \(z_1 \to \infty\), where \(\mu_2\) is defined in (17). Then the traveling wave solutions of system (7) do not exist for \(c \in [c_0, c_1)\).

**Proof.** We argue by contradiction. For the system (35), assume there does exist a monotone traveling wave solution \((U, V, W)(x - ct)\) subject to the initial data

\[
u(x, 0) = U(x), \quad v(x, 0) = V(x), \quad w(x, 0) = W(x),
\]

for some \(c \in (c_0, c_1)\). Due to the decay behaviors of the initial data, we can obtain by comparison principle that

\[
U(x - c_1t) \leq U(x - ct), \quad V(x - c_1t) \leq V(x - ct), \quad W(x - c_1t) \leq W(x - ct),
\]

(36)

hold for all \((x, t) \in (0, \infty) \times (0, \infty)\). Indeed, we first conclude that \(U(x) \leq U(x)\) (by shifting if necessary). From Lemma 3.2, it follows that \((\overline{U}, \overline{V}, \overline{W})(x) \leq (U, V, W)(x)\). Since \((\overline{U}, \overline{V}, \overline{W})(x - c_1t)\) is a lower solution to the system (7) with the initial data \((\overline{U}, \overline{V}, \overline{W})(x)\). By comparison, (36) holds true. However, one can fix \(z_1 = x - c_1t\) so that \(\overline{U}(z_1) > 0\). Thus, as \(t \to \infty\), we infer from the first inequality of (36) that

\[
\overline{U}(z_1) = U(x - c_1t) \leq U(z_1 + (c_1 - c)t) \to 0,
\]

(37)

which gives a contradiction. As for the case \(c = c_0\), the above analysis is also valid, since \(c_0\) in this case equals the minimal wave speed and thus we can always have a speed \(c\) in \((c_0, c_1)\). As such, the proof is complete. \(\Box\)
Next we are ready to construct a typical lower solution to study the nonlinear selection in terms of Lemma 3.6. Define
\[ U(z) = \frac{k}{1 + Be^{\mu_2 z}}, \]
where \( B \) is a positive constant and \( 0 < k < 1 \). Let \( V_1 \) and \( W_1 \) be the corresponding solutions of the \( V \)-equation and \( W \)-equation respectively with \( U(z) = U_1(z) \). Substituting this pair of functions into the \( U \)-equation in (7), the left-hand side becomes
\[ U_1(1 - \frac{U_1}{k}) \cdot \left[ \mu_2^2 - c\mu_2 + 1 - b_{12} - b_{13} + \frac{U_1}{k} (b_{12} + b_{13} - 1 + k) \right]. \]
From this, it is easy to see that the pair \( (U_1, V_1, W_1) \) is a lower solution to the system (7) if
\[ -2\mu_2^2 + J_1(z) > 0, \]
where
\[ J_1(z) = \frac{b_{21} V + b_{31} W - U_1 (b_{12} + b_{13} - 1 + k)}{(1 - \frac{U_1}{k})^2 k}. \]

As such, we have the following theorem.

**Theorem 3.6. (Nonlinear Selection).** The minimal wave speed of the system (7) is nonlinearly realized if (40) is satisfied for some \( c > c_0 \) with the choice of \( U(z) \) being given by (38).

4. Explicit conditions on the speed selection mechanism. With \( U(z) \) and \( U_1(z) \) defined in (32) and (38) respectively, we can further derive some specific conditions for linear/nonlinear speed selection by reconstructing upper or lower solutions to the system (7). We provide new results other than those in [10]. Particularly, the nonlinear selection results obtained here are not studied in [10].

For the sake of convenience, in Theorems 4.1 and 4.3, we assume hereafter that \( b_{21} < b_{31} \) and denote \( \mu_0 := \mu_1(c_0) \). From the viewpoint of biology, it means the intrinsic growth rate of species \( v \) is greater than that of species \( w \). The conclusions obtained in Theorems 4.1 and 4.3 under the case \( b_{21} \geq b_{31} \) still hold but with a slight modification of the proof.

**Theorem 4.1.** Assume \( 0 < d_1, d_2 \leq 2 \). If
\[ -2(1 - b_{12} - b_{13}) + b_{12} b_{21} + b_{13} b_{31} \leq 0, \]
then the minimal wave speed of (7) is linearly selected.

**Proof.** Let \( c = c_0 \). According to the structure of \( J(z) \) defined in (34), we redefine
\[ \overline{V}(z) = \min \{1, b_{21} \overline{U}(z) \} = \begin{cases} 1, & z \leq z_2, \\ b_{21} \overline{U}(z), & z > z_2, \end{cases} \]
and
\[ \overline{W}(z) = \min \{1, b_{31} \overline{U}(z) \} = \begin{cases} 1, & z \leq z_3, \\ b_{31} \overline{U}(z), & z > z_3, \end{cases} \]
where $z_2$ and $z_3$ satisfy $b_{21}U(z_2) = 1$ and $b_{31}U(z_3) = 1$ respectively. Moreover, the assumption $b_{21} < b_{31}$ combined with the monotonicity of the function $U(z)$ gives that $z_2 < z_3$.

For the $V$–equation, when $z \leq z_2$, it follows that $d_1\nabla'' + c_0\nabla' + \alpha(1 - \nabla)(b_{21}U - \nabla) = 0$ and when $z > z_2$, it can be inferred from the assumption $d_1 \leq 2$ that

$$
\left. d_1\nabla'' + c_0\nabla' + \alpha(1 - \nabla)(b_{21}U - \nabla) = b_{21}U(1 - U)(d_1\mu_0^2 - c_0\mu_0 - 2d_1\mu_0^2U) \leq 0. \right) 
$$

(45)

A similar inequality can be also proved for the $W$–equation but under another condition $d_2 \leq 2$. Hence, we are left to the $U$–equation. To proceed, we have to distinguish with three cases.

1. When $z \leq z_2$, it is easy to check that $\frac{1}{b_{21}} \leq \frac{U}{U} \leq 1$. Then

$$
J(z) = \frac{b_{12} + b_{13}}{U} \leq b_{21}(b_{12} + b_{13}). 
$$

(46)

2. When $z > z_3$, it follows that $V(z) = b_{21}U(z)$ and $W(z) = b_{31}U(z)$. Therefore, $J(z)$ becomes

$$
J(z) = \frac{1}{U(1 - U)} \left[ b_{12}V + b_{13}W - (b_{12} + b_{13})U \right] \leq \frac{b_{12}b_{21} + b_{13}b_{31} - b_{12} - b_{13}}{1 - \frac{U}{b_{21}}}. 
$$

(47)

3. When $z_2 < z \leq z_3$, we have $V(z) = b_{21}U(z)$ and $W(z) = 1$. Then $J(z)$ can be rewritten as

$$
J(z) = \frac{1}{U(1 - U)} \left[ b_{12}V + b_{13}W - (b_{12} + b_{13})U \right] = \frac{b_{12}b_{21} - b_{12}}{1 - \frac{U}{b_{21}}} + \frac{b_{13}}{U}. 
$$

(48)

It is easy to see that $\frac{1}{b_{21}} < \frac{U}{U} < \frac{1}{b_{21}}$, which leads to

$$
J(z) \leq b_{12}b_{21} + b_{13}b_{31}. 
$$

(49)

By a careful calculation, we find the maximum of the right sides of (46), (47) and (49) is $b_{12}b_{21} + b_{13}b_{31}$. As a result, (34) is satisfied provided that

$$
-2(1 - b_{12} - b_{13}) + b_{12}b_{21} + b_{13}b_{31} \leq 0.
$$

The proof is complete.

\[\square\]

**Theorem 4.2.** If

$$
0 \leq d_1 < 2, \quad b_{12} < \frac{1 - b_{13}}{2}, \quad \alpha < \frac{(2 - d_1)(1 - b_{12} - b_{13})^2}{b_{12}b_{21} - (1 - b_{12} - b_{13})}, 
$$

(50)

and

$$
0 \leq d_2 < 2, \quad b_{13} < \frac{1 - b_{12}}{2}, \quad \beta < \frac{(2 - d_2)(1 - b_{12} - b_{13})^2}{b_{13}b_{31} - (1 - b_{12} - b_{13})}, 
$$

(51)

are satisfied, then the minimal wave speed of (7) is linearly selected.

**Proof.** As before we construct an upper solution at the speed $c = c_0$. We choose the forms of $V(z)$ and $W(z)$ as

$$
V(z) = \min \left\{ 1, \frac{(1 - b_{12} - b_{13})U}{b_{12}} \right\} = \begin{cases} 1, & \text{if } z \leq z_4, \\ \frac{(1 - b_{12} - b_{13})U}{b_{12}}, & \text{if } z > z_4, \end{cases} 
$$

(52)
\[ W(z) = \min \left\{ 1, \frac{(1 - b_{12} - b_{13})U}{b_{13}} \right\} = \begin{cases} 1, & z \leq z_5, \\ \frac{(1 - b_{12} - b_{13})U}{b_{13}}, & z > z_5, \end{cases} \quad (53) \]

where \( z_4 \) and \( z_5 \) are the unique roots of \((1 - b_{12} - b_{13})U(z) = b_{12} \) and \((1 - b_{12} - b_{13})U(z) = b_{13} \) respectively. Without loss of generality, we assume that \( b_{12} < b_{13} \) for clarity. The argument for the case \( b_{12} \geq b_{13} \) is similar. The assumption \( b_{12} < b_{13} \) combined with \( b_{12} + b_{13} < 1 \) (see (6)) implies that \( z_4 > z_5 \). Next, we shall prove this theorem for three cases.

(1). When \( z \leq z_5 \), it is easy to see that
\[ d_1 V'' + c_0 V' + \alpha(1 - V)(b_{21}U - V) = 0 \]
and \( d_2 U'' + c_0 U' + \beta(1 - W)(b_{31}U - W) = 0 \). Meanwhile, according to (52) and (53), we have \( \frac{1}{U} \leq \frac{1 - b_{12} - b_{13}}{b_{13}} \). Hence it follows from \( b_{12} < b_{13} \) that
\[ -2(1 - b_{12} - b_{13}) + J(z) = -2(1 - b_{12} - b_{13}) + \frac{b_{12} + b_{13}}{U} < 0. \quad (54) \]

(2). When \( z > z_4 \), we have
\[ d_1 V'' + c_0 V' + \alpha(1 - V)(b_{21}U - V) = \frac{1 - b_{12} - b_{13}}{b_{12}} U^2 \left[ -d_1 \mu_0^2 (2U - 1) - c_0 \mu_0 \right] + \alpha U \left( 1 - \frac{1 - b_{12} - b_{13}}{b_{12}} U \right) \left( b_{21} - \frac{1 - b_{12} - b_{13}}{b_{12}} \right). \quad (55) \]
By the second inequality of (50), we have \( 1 - \frac{1 - b_{12} - b_{13}}{b_{12}} U \leq 1 - U \). Hence,
\[ d_1 V'' + c_0 V' + \alpha(1 - V)(b_{21}U - V) \leq \frac{1 - b_{12} - b_{13}}{b_{12}} U (1 - U) \left[ d_1 \mu_0^2 c_0 \mu_0 - 2d_1 \mu_0^2 U + \alpha \left( \frac{b_{12}b_{21}}{1 - b_{12} - b_{13}} - 1 \right) \right]. \quad (56) \]
By noting \( \mu_0 = \sqrt{1 - b_{12} - b_{13}} \) and (50), we obtain
\[ d_1 V'' + c_0 V' + \alpha(1 - V)(b_{21}U - V) \leq 0. \quad (57) \]

The next focus is on the \( W \)-equation. Similar to the discussion for the \( V \)-equation, the condition (51) ensures
\[ d_2 W'' + c_0 W' + \beta(1 - W)(b_{31}U - W) \leq 0. \quad (58) \]
As for the \( U \)-equation, we need to verify (34). In fact, it is easy to see that
\[ J(z) = \frac{1}{U(1 - U)} \left[ b_{12}V + b_{13}W - (b_{12} + b_{13})U \right] = \frac{2 - 3b_{12} - 3b_{13}}{1 - U}. \quad (59) \]
In view of (52) and (53), we obtain
\[ U \leq \frac{1}{2} \frac{(b_{12} + b_{13})}{1 - b_{12} - b_{13}}. \]
Hence, it follows that
\[ J(z) \leq 2(1 - b_{12} - b_{13}). \quad (60) \]
Then the inequality (34) holds obviously.

(3). When \( z_4 < z < z_5 \), according to the choices of \( V(z) \) and \( W(z) \), we have \( V(z) = 1 \) and \( W(z) = \frac{1 - b_{12} - b_{13}}{b_{13}} U \). Obviously, \( d_1 V'' + c_0 V' + \alpha(1 - V)(b_{21}U - V) = 0 \). As a matter of fact that \( W(z) \) possesses the same expression for the two cases
z > z_4 and z_5 < z < z_4, thus (58) still holds. Last, we turn to the $U-$equation. Substituting $\mathcal{V}(z)$ and $\mathcal{W}(z)$ into $J(z)$ yields

$$J(z) = \frac{1 - b_{12} - 2b_{13}}{1 - \mathcal{U}} + \frac{b_{12}}{\mathcal{U}}. \quad (61)$$

From (52) and (53), we know that $\frac{b_{12}}{1 - b_{12} - b_{13}} \leq \mathcal{U} \leq \frac{b_{13}}{1 - b_{12} - b_{13}}$. Hence, we get

$$J(z) \leq 2(1 - b_{12} - b_{13}), \quad (62)$$

which further implies the validity of (34).

Summing up the above analysis, one can infer that $(\mathcal{U}, \mathcal{V}, \mathcal{W})$ is an upper solution of (7). Therefore, the linear selection is realized by virtue of Theorem 3.4.

\textbf{Theorem 4.3.} For $0 < d_1 < 2 + \frac{\alpha}{1 - b_{12} - b_{13}}$ and $0 < d_2 < 2 + \frac{\beta}{1 - b_{12} - b_{13}}$, the minimal wave speed is linearly selected if

$$\frac{\alpha - (d_1 - 2)(1 - b_{12} - b_{13})}{\alpha b_{21}} > \max \left\{ \frac{d_1 - 2}{2d_1}, \frac{b_{12} + b_{13}}{2(1 - b_{12} - b_{13})} \right\}, \quad (63)$$

and

$$\frac{\beta - (d_2 - 2)(1 - b_{12} - b_{13})}{\beta b_{31}} > \max \left\{ \frac{d_2 - 2}{2d_2}, \frac{b_{12} + b_{13}}{2(1 - b_{12} - b_{13})} \right\}. \quad (64)$$

\textbf{Proof.} Let $c = c_0$. We choose

$$\mathcal{V}(z) = \min \left\{ 1, b_{21}k\mathcal{U} \right\} = \begin{cases} 1, & z \leq z_6, \\ b_{21}k\mathcal{U}, & z > z_6, \end{cases} \quad (65)$$

and

$$\mathcal{W}(z) = \min \left\{ 1, b_{31}k\mathcal{U} \right\} = \begin{cases} 1, & z \leq z_7, \\ b_{31}k\mathcal{U}, & z > z_7, \end{cases} \quad (66)$$

where $k$ is to be determined later. Since $b_{21} \leq b_{31}$, we have $z_6 \leq z_7$. To proceed, we firstly concentrate on the treatment of $\mathcal{V}-$equation and $\mathcal{W}-$equation.

When $z \leq z_6$, we get $d_1\mathcal{V}' + c_0\mathcal{V} + \alpha(1 - \mathcal{V})(b_{21}\mathcal{U} - \mathcal{V}) = 0$ and $d_2\mathcal{W}' + c_0\mathcal{W}' + \beta(1 - \mathcal{W})(b_{31}\mathcal{U} - \mathcal{W}) = 0$. While for $z > z_6$, it follows that $\mathcal{V} = b_{21}k\mathcal{U}$. Then we have

$$d_1\mathcal{V}' + c_0\mathcal{V} + \alpha(1 - \mathcal{V})(b_{21}\mathcal{U} - \mathcal{V}) = b_{21}\mathcal{U} \left[ k(1 - \mathcal{U})(d_1\mu_0^2 - c_0\mu_0 - 2d_1\mu_0^2\mathcal{U}) + \alpha(1 - b_{21}k\mathcal{U})(1 - k) \right] \quad (67)$$

$$= b_{21}\mathcal{U}(\mathcal{F}(\mathcal{U}) := k(1 - \mathcal{U})(d_1\mu_0^2 - c_0\mu_0 - 2d_1\mu_0^2\mathcal{U}) + \alpha(1 - b_{21}k\mathcal{U})(1 - k).$$

where $\mathcal{F}(\mathcal{U}) = k(1 - \mathcal{U})(d_1\mu_0^2 - c_0\mu_0 - 2d_1\mu_0^2\mathcal{U}) + \alpha(1 - b_{21}k\mathcal{U})(1 - k)$.

By a straightforward calculation, one can see that $\mathcal{F}''(\mathcal{U}) = 4d_1\mu_0^2k > 0$. Therefore, $\mathcal{F}$ is concave upward for $\mathcal{U} \in \left[ 0, \frac{1}{b_{21}k} \right]$. In order to prove $\mathcal{F}''(\mathcal{U}) < 0$, we only need to check that $\mathcal{F}(0) < 0$ and $\mathcal{F}'\left( \frac{1}{b_{21}k} \right) < 0$. Indeed, by a direct substitution, as $c = c_0$, we obtain

$$\mathcal{F}(0) = k(d_1\mu_0^2 - c_0\mu_0) + \alpha(1 - k) \quad (68)$$

$$= k \left[ d_1(1 - b_{12} - b_{13}) - 2\sqrt{1 - b_{12} - b_{13}} \sqrt{1 - b_{12} - b_{13}} \right] + \alpha(1 - k)$$

$$= k(d_1 - 2)(1 - b_{12} - b_{13}) + \alpha - \alpha k.$$
By choosing
\[ k = \frac{\alpha}{\alpha - (d_1 - 2)(1 - b_{12} - b_{13})} + \eta_1, \]
(69) one can obtain that \( F(0) < 0 \), where \( \eta_1 \) is a suitable positive number.

Next we need prove the other inequality \( F \left( \frac{1}{b_{21}} \right) < 0 \). By (63), we can find some number \( k \) such that
\[ \frac{1}{b_{21}k} > \max \left\{ \frac{d_1 - 2}{2d_1}, \frac{b_{12} + b_{13}}{2(1 - b_{12} - b_{13})} \right\}, \]
(70) for a sufficiently small positive constant \( \eta_1 \). By a direct substitution, we have
\[ F \left( \frac{1}{b_{21}} \right) = k \left( 1 - \frac{1}{b_{21}k} \right) (1 - b_{12} - b_{13}) \left[ d_1 - 2 - \frac{2d_1}{b_{21}k} \right]. \]
(71) Therefore, by virtue of (70), we have \( F \left( \frac{1}{b_{21}} \right) < 0 \). This means that
\[ d_1 W'' + c_0 V' + \alpha(1 - V)(b_{21} U - V) \leq 0, \]
for all \( z \in (-\infty, \infty) \).

Considering the \( W \)-equation for two cases \( z \leq z_7 \) and \( z > z_7 \), we can infer from (64) that
\[ d_3 W'' + c_0 W' + \beta(1 - W)(b_{31} U - W) \leq 0, \]
(72) for all \( z \in (-\infty, \infty) \). Thus we omit it for convenience.

To complete the proof, it is remained to prove the inequality (34), corresponding to the \( U \)-equation, by decomposing the range of values of \( z \) into three regions.

(1). When \( z \leq z_6 \), we have
\[ J(z) = \frac{1}{U(1 - U)} \left[ b_{12}V + b_{13}W - (b_{12} + b_{13})U \right] = \frac{b_{12} + b_{13}}{U}. \]
(73) Then the inequality \(-2(1 - b_{12} - b_{13}) + J(z) < 0 \) is true on the basis of (70).

(2). When \( z > z_7 \), we get
\[ J(z) = \frac{b_{12}(b_{21}k - 1) + b_{13}(b_{31}k - 1)}{1 - U}. \]
When \( b_{21} \leq b_{31} \), it follows that
\[ -2(1 - b_{12} - b_{13}) + J(z) = -2(1 - b_{12} - b_{13}) + \frac{b_{31}k}{b_{31}k - 1} \left[ b_{12}b_{31}k + b_{13}b_{31}k - (b_{12} + b_{13}) \right]. \]
(74) Then the inequality \(-2(1 - b_{12} - b_{13}) + J(z) < 0 \) holds if \( \frac{1}{b_{31}k} > \frac{b_{12} + b_{13}}{2(1 - b_{12} - b_{13})} \).

(3). When \( z_6 < z \leq z_7 \), it follows that \( b_{21}kU < 1 \leq b_{31}kU \). As a result, \( J(z) \) can be rewritten as
\[ J(z) = \frac{1}{1 - U} (b_{12}b_{21}k - b_{12}) + \frac{b_{13}}{U} \leq (b_{12}b_{21} + b_{13}b_{31})k. \]
By using (70), we simplify (34) as
\[ -2(1 - b_{12} - b_{13}) + J(z) \leq -2(1 - b_{12} - b_{13}) + (b_{12}b_{21} + b_{13}b_{31})k \leq 0. \]
(75) Equation (73) together with (74) and (75) gives
\[ U'' + c_0 U' + U(1 - b_{12} - b_{13} - U + b_{12}V + b_{13}W) \leq 0. \]
(76) Gathering (71), (72) and (76), we know that \((\bar{U}, \bar{V}, \bar{W})\) is an upper solution of (7). Hence, by Theorem 3.4, the proof is accomplished.
Taking a view of Theorem 3.5 and by constructing smooth lower solution, we are able to study the nonlinear selection of the system (7). We define a lower solution in the form of

\[ U_1(z) = k + B e^{\mu_2 z}, \quad V_1(z) = W_1(z) = \frac{U_1}{k}. \]  

(77)

Here \( B \) is a positive constant and \( 0 < k < 1 \). Then we have the following theorem.

**Theorem 4.4.** The minimal speed of the system (7) is nonlinearly selected if

\[
\max \left\{ \frac{(d_1 + 2)(1 - b_{12} - b_{13}) + \alpha}{\alpha b_{21}}, \frac{(d_2 + 2)(1 - b_{12} - b_{13}) + \beta}{\beta b_{31}} \right\} < 1 - 2(1 - b_{12} - b_{13}).
\]

(78)

**Proof.** By means of the relationship

\[ U_1'(z) = -\mu_2 U_1 \left( 1 - \frac{U_1}{k} \right), \]

(79)

the left hand side of the \( U \)-equation becomes

\[ U_1 \left( 1 - \frac{U_1}{k} \right) \frac{U_1}{k} \left( -2\mu_2^2 + \frac{1 - k}{\frac{U_1}{k}} \right) > U_1 \left( 1 - \frac{U_1}{k} \right) \frac{U_1}{k} (-2\mu_2^2 + 1 - k) > 0. \]

(80)

The last inequality of (80) is true if the undetermined constant in (77) is chosen to satisfy

\[ k < 1 - 2(1 - b_{12} - b_{13}). \]

(81)

Furthermore, if we impose an additional condition on \( k \) as follows

\[ k > \max \left\{ \frac{(d_1 + 2)(1 - b_{12} - b_{13}) + \alpha}{\alpha b_{21}}, \frac{(d_2 + 2)(1 - b_{12} - b_{13}) + \beta}{\beta b_{31}} \right\}, \]

(82)

then an substitution of \((U_1, V_1)\) enables us to estimate the left side of the \( V \)-equation and the \( W \)-equation as

\[ \frac{U_1}{k} (1 - \frac{U_1}{k}) \left[ d_1 \mu_2^2 - c \mu_2 - \alpha + b_{21} \alpha k - 2d_1 \mu_2^2 \frac{U_1}{k} \right] > \frac{U_1}{k} (1 - \frac{U_1}{k}) \left[ d_1 \mu_2^2 - c \mu_2 - \alpha + b_{21} \alpha k - 2d_1 \mu_2^2 \right] > 0, \]

(83)

and

\[ \frac{U_1}{k} (1 - \frac{U_1}{k}) \left[ d_2 \mu_2^2 - c \mu_2 - \beta + b_{31} \beta k - 2d_2 \mu_2^2 \frac{U_1}{k} \right] > \frac{U_1}{k} (1 - \frac{U_1}{k}) \left[ d_2 \mu_2^2 - c \mu_2 - \beta + b_{31} \beta k - 2d_2 \mu_2^2 \right] > 0, \]

(84)

respectively, where we have made use of the fact that \( \mu_2 \sim \sqrt{1 - b_{12} - b_{13}} \) for sufficiently small positive \( \epsilon_1 \) in \( c = c_0 + \epsilon_1 \). Clearly, condition (78) enables to select a constant \( k \) such that it satisfies (81) and (82). Then we can directly infer from (80), (83) and (84) that (77) is a lower solution to the wave profile system (7). The proof is complete. \( \square \)
5. Numerical simulation. In order to illustrate our theoretical results, we consider numerical simulations of the waves for the cooperative system (5). We recall that $e_0 = (0, 0, 0)$ and $e_1 = (1, 1, 1)$ are the equilibrium points under the condition (6), where $e_0$ is unstable and $e_1$ is stable. Therefore, three species shall coexist eventually in this competing process, i.e. $u$, $v$ and $w$ should move to the same direction.

In this section, we simulate the propagation speed of the traveling waves by choosing the following piecewise-constant wave-like initial data

$$ (U, V, W)(x, 0) = \begin{cases} 
1, & x \leq 0, \\
0, & x > 0.
\end{cases} \quad (85) $$

We will use the numerical spreading speed $c_{\text{num}}$ of the solution, derived from the level sets, subject to the initial values (85), to approximate the minimal wave speed $c_{\text{min}}$ (see e.g. [8]), i.e., $c_{\text{num}} \approx c_{\text{min}}$.

According to Theorem 4.2, if we choose the parameters as follows: $b_{12} = b_{13} = \frac{1}{4}$, $d_1 = d_2 = 1$, $\alpha = \beta = \frac{1}{2}$, $b_{21} = b_{31} = 8$, which satisfy (50), then the minimal wave speed is expected to be linearly selected, i.e. $c_{\text{min}} = c_0 = 2\sqrt{T - b_{12} - b_{13}} = \sqrt{2}$. The solution $u(x, t)$ is shown by numerical simulation. In order to investigate the speed $c_{\text{num}}$ numerically, we adopt the formula $c_{\text{num}} = \frac{\Delta s}{\Delta t}$, where $\Delta s = s_2 - s_1$ and $\Delta t = t_2 - t_1$. Therefore, taking the cross-section of the solution at $t_1 = 16$, $t_2 = 36$, $t_3 = 56$ (see Fig. 1(a)), and calculating the displacement at the function value $u(x, t) = \frac{1}{2}$, we derive the approximate speed $c_{\text{num}} = 1.419$. Due to the absolute error between $c_{\text{num}}$ and $c_0$ is to $10^{-3}$, it is reasonable to believe that the minimal wave speed is linearly selected.

![Figure 1](image1.png)

(a) $b_{12} = b_{13} = \frac{1}{4}$, $d_1 = d_2 = 1$, $\alpha = \frac{1}{2}$, $b_{21} = b_{31} = 8$  
(b) $b_{12} = \frac{1}{4}$, $b_{13} = \frac{1}{2}$, $d_1 = d_2 = 4$, $\alpha = \beta = 1$, $b_{21} = b_{31} = 8$

**Figure 1.** The solution $u(x, t)$ at $t = 16, 36, 56$ for two sets of parameters.

According to Theorem 4.4, if we choose another set of the parameters as follows: $b_{12} = \frac{1}{4}$, $b_{13} = \frac{1}{2}$, $d_1 = d_2 = 4$, $\alpha = \beta = 1$, $b_{21} = b_{31} = 8$, which satisfy the inequalities (78), then the minimal wave speed must be nonlinearly selected, i.e. $c_{\text{min}} > c_0 = 2\sqrt{T - b_{12} - b_{13}} = 1$. Again, extracting the cross-section of the solution at $t_4 = 16$, $t_5 = 36$, $t_6 = 56$ (see Fig. 1(b)), and calculating the displacement at the function value $u(x, t) = \frac{1}{2}$, we calculate the approximate speed $c = 1.338 > 1$.

The above simulations show that the numerical results are highly in agreement with our theoretical results. What should point out that the the results of Theorems 4.1 and 4.3 can be justified by a similar manner; thus we omit them for simplicity.
6. Conclusions. Linear and nonlinear selection of the minimal speed of traveling wave solutions for a diffusive three species Lotka-Volterra system are investigated by using the upper-lower solution method. We constructed upper-lower solutions and gave sufficient conditions for linear or nonlinear selection of the minimal speed. Indeed, the results can be extended to study the invasion speed for multiple-species competitive models.

Due to the model we considered involves three species, our analysis is rather complicated than the classical 2-component Lotka-Volterra system. The conditions of linear and nonlinear selection in 2-component Lotka-Volterra system [1, 2] can be covered by the results in this paper by chosen appropriate parameters. By studying the linear determinacy of the corresponding approximated lattice dynamical systems and applying the discrete Fourier transform, Guo et al. [10] showed that the minimal speed of traveling wave solutions of (5) is linearly selected provided that

\[(d_i, \alpha, \beta, b_{11}) \in B^1_i \bigcup B^2_i, i = 2, 3,\]

where

\[B^1_i := \{d_i \in (0, 2], b_{11}(b_{12} + b_{13}) \leq 1, r_i > 0, i = 2, 3\},\]

and

\[B^2_i := \left\{d_i \in (0, 2), b_{11}(b_{12} + b_{13}) > 1, 0 < r_i < \frac{1 - b_{12} - b_{13}}{b_{11}(b_{12} + b_{13}) - 1}, i = 2, 3 \right\}.

We intend to compare our results with the above conditions one by one. Firstly, it is easy to see that \(d_i, i = 2, 3\), appeared in (87) and (88) are required to be not greater than 2. However, we established a condition so that the linear selection is still realized when \(d_i, i = 2, 3\), are greater than 2 (see Theorem 4.4). Secondly, even under the same restrictive condition \(d_i \in (0, 2), i = 2, 3\), our results could be better than (88). For instance, thanks to \(1 < \frac{b_{12}b_{31}}{1 - b_{12} - b_{13}} < b_{31}(b_{12} + b_{13})\) and \(1 < \frac{b_{12}b_{31}}{1 - b_{12} - b_{13}} < b_{31}(b_{12} + b_{13})\), we are able to conclude that the domains of \(r_1\) and \(r_2\) are the subsets of the domains of \(\alpha\) and \(\beta\) respectively. In other words, our results in Theorem 4.2 improve the ones in [10]. Besides, we provide a sufficient condition for the nonlinear selection, which is not discussed in [10].

In summary, our paper improved some of the results explored in past references and also provided some new results, such as Theorems 4.3 and 4.4 on linear and nonlinear selection of the minimal speed. Illustrations of these results were also demonstrated by numerical simulations. In addition, the techniques and framework in the present paper can be extended to the \(N\)-species model but with a great difficulty in calculation.

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