Universal cocycle Invariants for singular knots and links

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October 10, 2019

Abstract

Given a biquandle \((X, S)\), a function \(\tau\) with certain compatibility and a pair of non commutative cocyles \(f, h : X \times X \to G\) with values in a non necessarily commutative group \(G\), we give an invariant for singular knots / links. Given \((X, S, \tau)\), we also define a universal group \(U_{fh}^{\text{nc}}(X)\) and universal functions governing all 2-cocycles in \(X\), and exhibit examples of computations. When the target group is abelian, a notion of abelian cocycle pair is given and the “state sum” is defined for singular knots/links. Computations generalizing linking number for singular knots are given. As for virtual knots, a “self-linking number” may be defined for singular knots.

Keywords: Singular Knots, Cocycle Invariants, Non Commutative Cocycles.
Mathematical Subject Classification 2010: 57M25, 57M27.

Introduction and preliminaries

Following methods from [FG2] and [CES]-[CEGS] we define two types of cocycles and corresponding invariants for singular knots.

A singular knot is a smooth map \(f : S^1 \to \mathbb{R}^3\) whose image eventually has singularities: a finite number of double points, with transversal crossings. A singular link is the union of singular knots, that is generically disjoint: the intersection is finite and transversal. Singular links/knots may be considered to be equivalence classes of planar singular knot diagrams under the equivalence relation generated by the three (classical) Reidemeister moves and the singular (RV and RIV) Reidemeister moves depicted in Figure 1. A singular crossing is a crossing where two strands are fused together. An orientation of each circle induces an orientation on each component of the link. All links and knots considered in this work will be oriented ones.

After [BEHY], it is known that only 3 oriented singular Reidemeister moves are sufficient in order to generate them all. In [BEHY] the basic singular movements are called \(\Omega4a, \Omega4e\) and \(\Omega5a\) and correspond to our \(oRIVb, oRIVA\) and (equivalent to) \(oRV\) respectively (see figures 3, 3 and 4).

Remark 1. In Figure 1 the bottom left figure represents both of the following movements:
In the case of classical knot theory, coloring arcs (of projections) of knots “unchanged” by Reidemeister moves gave rise to the definition of quandles, painting semiarcs led to the definition of biquandles. In Section 1, we generalize these definitions to color (label) singular links/knots. The rest of the work is organized as follows:

In Section 1, after introducing the notion of singular pair, that is the natural notion for labeling (planar diagrams of) singular knots or links, the main result is Theorem [16] that gives a full characterization of singular pairs when one uses a bialexander switch \((x, y) \mapsto (sy, tx + (1 - st)y)\) at classical crossings, under the assumption that \((1 - st)\) is a unit. A consequence

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is that in some cases, the singular switch companion to the bialexander is necessarily a linear map (see Theorem 18). We end Section 1 with other types of general examples.

In Section 2 we introduce weights at crossings and the notion of non-abelian 2-cocycle pairs. A non commutative invariant is deduced (Theorem 29) from the data of a pair of cocycles \( f, h \). Similarly to [FG1] and [FG2], given a set of labels \( X \) and \( S, \tau : X \times X \rightarrow X \times X \) a singular pair, we introduce a universal group \( U_{nc}^{fh}(X) \) factorizing all 2-cocycles in \( X \). We compute this group for some particular cases, and the knot/link invariant that it produces. In particular, we comment the possible generalizations of the notion of linking number to singular knots and links that may be non trivial for singular knots.

In Section 3 we consider a more classical situation when the target group is abelian. A notion of abelian cocycle pair is given and the state sum is defined for singular knots/links. As in the non commutative case, a commutative group can be constructed that work as universal target for abelian 2-cocycle pairs and so, the state sum really depends on the choice of a singular pair \( (S, \tau) \), and a choice of cocycle pair is not needed, because there is always the universal one.

1 Singular pairs

First recall the set theoretical Yang-Baxter equation and the notion of biquandle.

**Definition 2.** A set theoretical solution of the Yang-Baxter equation is a pair \( (X, S) \) where \( S : X \times X \rightarrow X \times X \) is a bijection satisfying

\[
(Id \times S)(S \times Id)(Id \times S) = (S \times Id)(Id \times S)(S \times Id)
\]

Notation: \( S(x, y) = (S^1(x, y), S^2(x, y)) \) and \( S^{-1}(x, y) = \overline{S}(x, y) \). A solution \( (X, S) \) is called non-degenerate, or birack if in addition:

1. (left invertibility) for any \( x, z \in X \) there exists a unique \( y \) such that \( S^1(x, y) = z \),
2. (right invertibility) for any \( y, t \in X \) there exists a unique \( x \) such that \( S^2(x, y) = t \).

A birack is called biquandle if, given \( x_0 \in X \), there exists a unique \( y_0 \in X \) such that \( S(x_0, y_0) = (x_0, y_0) \). In other words, if there exists a bijective map \( s : X \rightarrow X \) such that

\[
\{(x, y) : S(x, y) = (x, y)\} = \{(x, s(x)) : x \in X\}
\]

See Lemma 0.3 in [FG1] for biquandle equivalent conditions.

Recall that a set \( X \) with a binary operation \( \triangleleft : X \times X \rightarrow X \) is called a rack if

- \( -\triangleleft x : X \rightarrow X \) is a bijection \( \forall x \in X \) and
- \( (x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z) \forall x, y, z \in X \).

If \( X \) also verifies that \( x \triangleleft x = x \) then \( X \) is called a quandle.

It is clear that \( (X, \triangleleft) \) is a rack if and only if

\[
S_\triangleleft(x, y) := (y, x \triangleleft y)
\]

is a non-degenerate set theoretical solution of the YBeq (i.e. a birack). Given \( (X, \triangleleft) \), a rack, the birack \( (X, S_\triangleleft) \) is a biquandle if and only if \( (X, \triangleleft) \) is a quandle.

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**Definition 3.** Let \((X, S)\) be a biquandle and \(\tau : X \times X \to X \times X\) be a bijective map that verifies left and right invertibility. The pair \((X, S, \tau)\) is called singular pair if

\[
\tau \circ S = S \circ \tau \quad \text{(coming from RV)},
\]

\[
(S \times 1)(1 \times S)(\tau \times 1) = (1 \times \tau)(S \times 1)(1 \times S) \quad \text{(coming from RIVb)},
\]

\[
(1 \times S)(S \times 1)(1 \times \tau) = (\tau \times 1)(1 \times S)(S \times 1) \quad \text{(coming from RIVa)}
\]

simultaneously holds.

Equations (2) and (3) are different, take for example \(S = \text{bialexander}(3, 1, -1)\), one can compute the total amount of functions \(\tau\) such that verifies (1) and (2), it is different from the total amount of functions that verifies (1) and (3). On the other hand, if \(S\) is involutive, equation (2) is equivalent to equation (3).

**Remark 4.** Given \((X, S)\) a biquandle and a bijective function \(\tau\), previous definition written in elements \((x, y, z) \in X^3\) gives:

\[
\tau^1(S(x, y)) = S^1(\tau(x, y)) \quad \text{(coming from RV)}
\]

\[
\tau^2(S(x, y)) = S^2(\tau(x, y)) \quad \text{(coming from RV)}
\]

\[
\tau^1(S^1(x, y), S^1(S^2(x, y), z)) = S^1\left(x, \tau^1(y, z)\right) \quad \text{(coming from RIVa)}
\]

\[
\tau^2\left(S^1(x, y), S^1(S^2(x, y), z)\right) = S^1\left(S^2(x, \tau^1(y, z)), \tau^2(y, z)\right) \quad \text{(coming from RIVa)}
\]

\[
S^1\left(x, S^1(y, z)\right) = S^1\left(\tau^1(x, y), S^1(\tau^2(x, y), z)\right) \quad \text{(coming from RIVb)}
\]

\[
\tau^1\left(S^2(x, S^1(y, z)), S^2(y, z)\right) = S^2\left(\tau^1(x, y), S^1(\tau^2(x, y), z)\right) \quad \text{(coming from RIVb)}
\]

\[
\tau^2\left(S^2(x, S^1(y, z)), S^2(y, z)\right) = S^2\left(\tau^2(x, y), z\right) \quad \text{(coming from RIVb)}.
\]

**Remark 5.** The conditions for \((X, S, \tau)\) to be a singular pair are precisely the compatibility of the set of colorings with the oriented Reidemeister moves \((\text{RI}, \text{RII}, \text{RIII}, \text{RIVa}, \text{RIVb}\) and \(\text{RV})\). In other words, given \((X, S, \tau)\) a classical pair, the number of colorings of a link (or a knot) using \((X, S, \tau)\) is an invariant of singular links/knots.

**Example 6.** Given a biquandle \((X, S)\), then \((X, S, S)\) and \((X, S, S^{-1})\) are singular pairs.

**Remark 7.** For some biquandles these are the only possible singular pairs, see Section [1.2](#). Corollary 18 and the examples after it.

**Example 8.** If \(X = \{0, 1\}\) and \(S = \text{flip}\) then there are, up to isomorphism, 2 different singular pairs: \((X, \text{flip}, \text{flip})\) and \((X, \text{flip}, i_2)\) where \(i_2\) is the nontrivial (involutive) biquandle of size 2 given by \(i_2(x, y) = (y - 1, x + 1)\), for \(x, y \in \{0, 1\} \cong \mathbb{Z}/2\mathbb{Z}\).

**Example 9.** If \(S = \text{flip}\), then equations (2) and (3) are trivially satisfied. The remaining condition:

\[
\tau^1(y, x) = \tau^2(x, y)
\]

**Remark 10.** Given a map \(\tau : X \times X \to X \times X\), the left invertibility means that, for any \(x\), the map \(\tau^1(x, -) : X \to X\) is bijective, so, to give \(\tau^1\) is the same as giving a list of permutations \(\{\tau^1(x, -)\}_{x \in X}\). Similar considerations for right invertibility: \(\tau^2(-, y) : X \to X\) is bijective for any fixed \(y \in X\). If \(#X = n\), then the set of maps \(X \times X \to X \times X\) satisfying left and right invertibility condition is of cardinal \((n!)^n\), because \(\tau^1\) is determined by a list of \(n\)
permutations, and the same for \( \tau^2 \). Nevertheless, conditions of right and left invertibility do not imply bijectivity. If we consider examples as above, the restriction \( \tau^1(y, x) = \tau^2(x, y) \) means precisely that the list of permutations determining \( \tau^1 \) is the same list as the one determining \( \tau^2 \). For small \( n \), one can make the list of all left and right invertible maps verifying \( \tau^1(y, x) = \tau^2(x, y) \) (it has \((n!)^n\) members) and check that very few give a bijective \( \tau \). The following table illustrates the situation:

| #X | left – right invertibles | isoclasses | also bijective | isoclasses |
|----|--------------------------|------------|---------------|------------|
| 2  | 4                        | 3          | 2             | 2          |
| 3  | 216                      | 44         | 24            | 7          |
| 4  | 331176                   | 14022      | 3360          | 169        |

### 1.1 Colorings

Let \((X, S, \tau)\) be a singular pair. Let \( L \) be a singular oriented link diagram on the plane. A coloring of \( L \) by \( X \) is a rule that assigns an element of \( X \) to each semi-arc of \( L \), in such a way that for every classical crossing (figure on the left corresponds to a positive crossing and figure on the right to a negative one)

\[
\begin{array}{ccc}
x & \searrow & y \\
\downarrow & & \downarrow \\
z & \nearrow & t
\end{array}
\quad
\begin{array}{ccc}
x & \nearrow & y \\
\downarrow & & \downarrow \\
z & \searrow & t
\end{array}
\]

where \((z, t) = S(x, y)\) and in case of a singular crossing

\[
\begin{array}{ccc}
x & \searrow & y \\
\downarrow & & \downarrow \\
z & \nearrow & t
\end{array}
\quad
\begin{array}{ccc}
x & \nearrow & y \\
\downarrow & & \downarrow \\
z & \searrow & t
\end{array}
\]

where \((z, t) = \tau(x, y)\).

Call \( \text{Col}_X(L) \) the set of all possible colorings of \( L \) by the singular pair \((X, S, \tau)\).

Using analogous methods to the ones in [HN], the authors define singular semiquandle to color flat virtual knots and virtual singular semiquandles to color virtual flat singular knots.

**Example 11.** (Singquandle from [CEHN]) Let \((X, S)\) be an involutive quandle (that is, \( S(x, y) = (y, x \triangleleft y) \) where \((X, \triangleleft)\) is a quandle satisfying \((x \triangleleft y) \triangleleft y = x \forall x, y \in X\) and a map \( \tau(x, y) = (R_1(x, y), R_2(x, y)) \) with \( R_1, R_2 : X \times X \rightarrow X \). It is straightforward to check that equations [1][2] and [3] leave:

\[
\begin{align*}
(y \triangleleft z) \triangleleft R_2(x, z) &= (y \triangleleft x) \triangleleft R_1(x, z) \quad \text{coming from RIVa} \\
R_1(x, y) &= R_2(y \triangleleft x, x) \quad \text{coming from RV} \\
R_2(x, y) &= R_1(y \triangleleft x, x) \triangleleft R_2(y \triangleleft x, x) \quad \text{coming from RV} \\
R_1(x \triangleleft y, z) \triangleleft y &= R_1(x, z \triangleleft y) \quad \text{coming from RIVb} \\
R_2(x \triangleleft y, z) &= R_2(x, z \triangleleft y) \triangleleft y \quad \text{coming from RIVb}
\end{align*}
\]

which is called singquandle in [CEHN].
Example 12. If \((X, \triangleleft)\) is a quandle where the operation \(\triangleleft\) is a trivial, that is \(x \triangleleft y = x\) for all \(x, y \in X\) (hence \(S = \text{flip}\)), then the axioms of Definition 3 reduce to the condition \(\tau^1(x, y) = \tau^2(y, x)\). Same as in Example 4.3 in [CEHN].

Remark 13. Unlike the virtual case (see [FG2]), where for any biquandle \((X, S)\), the pair \((X, S, \text{flip})\) may be used to color a virtual knot or link, it is not always true that \((X, S, \text{flip})\) is a singular pair. Moreover, to impose \(\tau\)-flip is a serious condition on a biquandle \(S\) in order to get \((X, S, \text{flip})\) a singular pair. A simple evaluation shows the following:

Lemma 14. \((X, S, \tau = \text{flip})\) is a singular pair if and only if \(S\) simultaneously satisfies:

\[
\begin{align*}
S^1(x, y) &= S^2(y, x), \\
S^1(S^2(x, y), z) &= S^1(x, z), \\
S^2(S^2(x, y), z) &= S^2(S^2(x, z), y), \\
S^1(x, S^1(y, z)) &= S^1(y, S^1(x, z)), \\
S^2(y, z) &= S^2(y, S^1(x, z)).
\end{align*}
\]

In particular, if \((X, \triangleleft)\) is a quandle and \(S(x, y) = (y, x \triangleleft y)\), then the first condition of the previous lemma says \(y = y \triangleleft x\) for all \(x, y\), that is true, if and only if \(S = \text{flip}\). Nevertheless, there are nontrivial biquandle examples:

Example 15. Let \(s : X \to X\) be an involutive bijection, that is, a bijective map with \(s^2 = \text{Id}_X\). If one defines \(S(x, y) = (sy, sx)\), then \((X, S, \text{flip})\) is a singular pair.

1.2 Singular pairs for Bialexander switch

We will study singular pairs for the \(S = \text{bialexander switch}\), under the general assumption that \((1 - st)\) is a unit. We point out some cases where the only possible singular pairs are \((S, S)\) and \((S, S^{-1})\). The main result of this section is the following:

Theorem 16. Let \(S(x, y) = S_{s,t}(x, y) = (sy, tx + (1 - st)y)\) and assume also that \((1 - st)\) is a unit. Then \(\tau : X \times X \to X \times X\) gives a singular pair \((S, \tau)\) if and only if

\[
\begin{align*}
\tau(\lambda x, \lambda y) &= \lambda \tau(x, y) \quad &\text{for } \lambda = s, t, -1 \\
\tau(x, y) &= \tau(0, y - x/s) + (x, x/s) \\
\tau(x, y) &= \tau(x - sy, 0) + (sy, y) \\
t\tau^1(0, x) &= s\tau^2(x, 0)
\end{align*}
\]

holds for all \(x, y \in X\). Moreover, \(\tau\) is fully determined by \(\tau^1(0, x), (x \in X)\).

Before the proof let us consider some consequences and examples.

Remark 17. Keeping notations and hypothesis as in Theorem 16 denote \(\varphi : X \to X\) the map given by

\[
\varphi(x) := \frac{1}{s} \tau^1(0, x)
\]

Because of the non-degenerate assumption for \(\tau\), we know \(\varphi\) must be a bijective map. The last equation of Proposition 16 says that \(\tau^2(x, 0) = t\varphi(x)\), and combining the second and third equation of the same proposition, we get that necessarily \(\tau\) is of the form

\[
\tau(x, y) = (x + \varphi(sy - x), y - t\varphi(sy - x)) =: \tau_\varphi(x, y)
\]

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Finnally, condition $\tau(\lambda x, \lambda y) = \lambda(x, y)$ for $\lambda = s, t, -1$ is equivalent to $\varphi(\lambda(x)) = \lambda\varphi(x)$ for $\lambda = s, t, -1$.

It is not difficult to see that a map $\tau$ as above gives a singular pair for the bialexander switch, under the only hypothesis that $\varphi(\lambda x) = \lambda\varphi(x)$ for $\lambda = s, t, -1$. The proposition above shows that when $(1 - st)$ is a unit, every $\tau$ is of the form $\tau_\varphi$. Also, even in the case when $(1 - st)$ is not necessarily a unit, it is not hard from the computational point of view to find all possible bijections $\varphi$ commuting with multiplication by $\lambda (\lambda = s, t, -1)$, and then check if $\tau_\varphi$ is bijective or not. In other words, this formula gives a way to find a big family of singular pairs.

**Theorem 18.** Let $X = K$ be a finite field, $s, t \in K^\times$ with $t \neq s^{-1}$ and assume that $\{ -1, s, t \}$ generate $K^\times$ as multiplicative group. If $S$ is the bialexander switch associated to $s$ and $t$ and $\tau$ gives a singular pair for $S$ then

- $\tau : K^2 \to K^2$ is necessarily a linear map.
- Denoting $a := \tau^1(0, 1) \in K^\times$, we have
  $$\tau^1(0, y) = ay, \ \tau^2(sx, 0) = atx$$

  and
  $$\tau(x, y) = \tau_a(x, y) = \left( ay + \left(1 - \frac{a}{s}\right)x, \ \frac{at}{s}x + (1 - at)y \right)$$

- The map $\tau_a$ is bijective if and only if $(st+1)a \neq s$. In particular, the map $\tau$ is necessarily linear and there are at most $|K| - 1$ singular pairs for $S_{s,t}$.

**Proof.** (of Theorem 18 using Proposition 16). Since $\tau(0, \lambda x) = \lambda\tau(0, x)$ for $\lambda = s, t, -1$ and $\{ -1, s, t \} = K^\times$ if follows that $\tau(0, \lambda x) = \lambda\tau(0, x)$ for $\lambda \in K^\times$. But also $\tau(0, -x) = -\tau(0, -x)$ implies $\tau(0, 0) = (0, 0)$, so $\tau(0, x) = x\tau(0, 1)$ for all $x \in K$. That is, $\tau(0, -) : K \to K \times K$ is $K$-linear, so, by the second condition of Proposition 16, $\tau$ is $K$-linear: it is given by a matrix. Denoting $(a, b) := \tau(0, 1)$, by the second condition of Proposition 16 we have

  $$\tau(x, y) = \tau(0, y-x/s) + (x, x/s) = (y-x/s)(a, b) + (x, x/s) = \left( \left(1 - \frac{a}{s}\right)x + ay, \ \frac{1-b}{s}x + by \right)$$

The condition $t\tau^1(0, x) = s\tau^2(x, 0)$, for $x = 1$ gives $ta = 1 - b$.

Since $\tau$ is a linear map, the bijectivity condition is controlled by the determinant:

$$\det\begin{pmatrix} 1 - \frac{a}{s} & a \\ \frac{at}{s} & 1 - at \end{pmatrix} = 1 - \frac{a}{s} - at = \frac{s - a - ast}{s} = \frac{s - a(1 + st)}{s}$$

We see that it is different from zero if and only if $s \neq a(1 + st)$.

**Remark 19.** $a = s$ corresponds to $\tau_a = S$ and $a = \frac{1}{t}$ corresponds to $\tau_a = S^{-1}$. But also if $\tau$ is of the form $\tau(x, y) = (ay + bx, cx + dy)$ and $\tau$ satisfies YBeq, then necessarily $b = 0$ or $d = 0$, so the only singular pairs of the form $\tau_a$ that also satisfies YBeq are $S$ and $S^{-1}$.
Remark 20. Let \( S(x, y) = S_{s,t}(x, y) = (sy, tx + (1 - st)y) \) be as in Proposition 16 that is, assuming that \((1 - st)\) is a unit, and assume that \( X \) is a module over a commutative ring containing \( s \) and \( t \) as units. If an affine map \( \tau \)

\[
\tau_{a,b,c,a',b',c'}(x, y) = (ax + by + c, a'x + b'y + c')
\]

(with \( a, b, c, a', b', c' \) elements of the ring) gives a singular pair, then the condition \( \tau(\lambda x, \lambda y) = \lambda(x, y) \) for \( \lambda = s, t, -1 \) implies \( c = 0 = c' \), so \( \tau \) is linear and the same argument of the proof of Corollary 18 applies, we have \( \tau \) is necessarily of the form

\[
\tau(x, y) = \left( ay + \left(1 - \frac{a}{s}\right)x, \frac{at}{s}x + (1 - at)y \right)
\]

Example 21. \( K = \mathbb{F}_3 = \mathbb{Z}/3\mathbb{Z}, s, t \in \{\pm 1\}, s \neq t \), then the only singular pairs are \((S, S)\) and \((S, S^{-1})\), because in this case we know there are at most \(|K| - 1 = 2\) pairs.

Example 22. \( K = \mathbb{F}_4 \), the field with 4 elements, \( \alpha \in \mathbb{F}_4 \) with \( \alpha^2 = \alpha + 1 \). Assume \( s = \alpha \) and \( t \neq \alpha + 1 = s^{-1} \). Notice that \( 1 + st = 1 - st \) is a unit, so the condition \((st + 1)a \neq s\) is equivalent to \( a \neq \frac{s}{st + 1} \). That is, \( a \in \mathbb{F}_4 \setminus \{0, \frac{s}{st + 1}\} \); we have only 2 possibilities for \( a \), so again in this case the only singular pairs are \((S, S)\) and \((S, S^{-1})\).

Proof. (of Proposition 16) Equations of singular pairs for \( S(x, y) = (sy, tx + (1 - st)y) \) are

\[
\begin{align*}
\tau^1(sy, tx + (1 - st)y) &= st^2(x, y) \\
\tau^2(sy, tx + (1 - st)y) &= tr^1(x, y) + (1 - st)t^2(x, y) \\
\tau^1(sy, sz) &= s^2(y, z) \\
\tau^2(sy, sz) &= s^2(y, z) \\
t(tx + (1 - st)y) + (1 - st)z &= t(tx + (1 - st)y)(y, z) + (1 - st)t^2(y, z) \\
s^2z &= s^2z \\
\tau^1(tx + (1 - st)sz, ty + (1 - st)z) &= tr^1(x, y) + (1 - st)sz \\
\tau^2(tx + (1 - st)sz, ty + (1 - st)z) &= tr^2(x, y) + (1 - st)z.
\end{align*}
\]

Equation 22 is trivial. One can arrange also equations 19 and 20 into a single equation

\[
\tau(sx, sy) = s\tau(x, y)
\]

After cancelling \( t^2x \), equation 21 becomes independent of \( x \):

\[
t(1 - st)y + (1 - st)z = t(1 - st)\tau^1(y, z) + (1 - st)t^2(y, z)
\]

but since \((1 - st)\) is a unit, this is equivalent to

\[
ty + z = t\tau^1(y, z) + \tau^2(y, z)
\]

Eq. 23 and 24 may also be written in the form

\[
\tau(tx + (1 - st)sz, ty + (1 - st)z) = t\tau^1(x, y) + (1 - st)(sz, z)
\]
Again under the assumption \((1 - st)\) being a unit, it is equivalent to
\[
\tau(tx + sz, ty + z) = t\tau(x, y) + (sz, z)
\]
Notice that for \(z = 0\) we get \(\tau(tx, ty) = t\tau(x, y)\), so it is easy to see that the above equation is equivalent to the following two:
\[
\tau(x + sz, y + z) = \tau(x, y) + (sz, z) \quad \text{and} \quad \tau(tx, ty) = t\tau(x, y)
\]
We write again the set of equations under these simplifications:
\[
\begin{align*}
\tau_1'(sy, tx - sty) + sy &= s\tau^2(0, y) \quad \text{(25)} \\
\tau_2'(sy, tx - (1 - st)y) &= t\tau_1'(x, y) + (1 - st)\tau^2(x, y) \quad \text{(26)} \\
\tau_2'(sx, sy) &= s\tau(x, y) \quad \text{(27)} \\
x + y &= t\tau_1'(x, y) + \tau^2(x, y) \quad \text{(28)} \\
\tau(x + sz, y + z) &= \tau(x, y) + (sz, z) \quad \text{(29)} \\
\tau(tx, ty) &= t\tau(x, y). \quad \text{(30)}
\end{align*}
\]
Notice that eq. \(25\) says that \(\tau^2\) is determined by \(\tau^1\) (or vice versa).
Equation \(29\) for \(z = -y\) or \(z = -\frac{x}{s}\) gives respectively
\[
\begin{align*}
\tau(x, y) &= \tau(x - sy, 0) + (sy, y) \quad \text{(31)} \\
\tau(x, y) &= \tau(0, y - \frac{x}{s}) + (x, \frac{x}{s}) \quad \text{(32)}
\end{align*}
\]
So, \(\tau\) is determined by \(\tau(-, 0)\) (and hence by \(\tau^1(-, 0)\)).
Equations \(25\) and \(28\) for \(x = 0\) gives
\[
\begin{align*}
\tau_1'(sy, (1 - st)y) &= s\tau^2(0, y) \quad \text{(33)} \\
y &= t\tau_1'(0, y) + \tau^2(0, y) \quad \text{(34)}
\end{align*}
\]
and writing \(x\) instead of \(y\) one gets
\[
\begin{align*}
\tau_1'(sx, (1 - st)x) &= s\tau^2(0, x) \quad \text{(35)} \\
x &= t\tau_1'(0, x) + \tau^2(0, x) \quad \text{(36)}
\end{align*}
\]
Equation \(35\) together with \(32\) gives
\[
s\tau^2(0, x) = \tau_1'(sx, (1 - st)x) = \tau^1(0, -stx) + sx
\]
hence
\[
\tau^2(0, x) = t\tau_1'(0, -x) + x
\]
Notice that Eq. \(36\) is \(x = t\tau_1'(0, x) + \tau^2(0, x)\). We conclude \(\tau^1(0, -x) = -\tau^1(0, x)\) and consequently (use equations \(31\) and \(32\)) \(\tau(-x, -y) = -\tau(x, y)\).
Now (using \(31\) and \(32\)) we write equations \(25\) and \(30\) in terms of \(\tau^1(0, *)\) and \(\tau^2(*, 0)\):
\[
\begin{align*}
\tau_1'(0, tx - sty) + sy &= s(\tau^2(x - sy, 0) + y) \quad \text{(37)} \\
\tau^2(-stx + st^2y, 0) + tx + (1 - st)y &= t(\tau^1(0, y - x/s) + x) + (1 - st)(\tau^2(x - sy, 0) + y) \quad \text{(38)} \\
x + y &= t(\tau^1(0, y - x/s) + x) + \tau^2(x - sy, 0) + y \quad \text{(39)}
\end{align*}
\]
In equation 38 one can simplify \(tx\) and \((1-st)y\). There are also easy simplifications in equations 37 and 39. Using also that \(\tau\) is \(s\) and \(t\) homogeneous, we get

\[
\tau^1(0, tx - sty) = s\tau^2(x - sy, 0) \tag{40}
\]
\[
st\tau^2(-x + sy, 0) = t\tau^1(0, y - x/s) + (1-st)\tau^2(x - sy, 0) \tag{41}
\]
\[
0 = t\tau^1(0, y - x/s) + \tau^2(x - sy, 0) \tag{42}
\]

Equations 41 and 42 are actually the same equation, so we only have

\[
t\tau^1(0, x - sy) = s\tau^2(x, 0) \tag{43}
\]
\[
0 = t\tau^1(0, y - x/s) + \tau^2(x - sy, 0) \tag{44}
\]

or equivalently

\[
t\tau^1(0, x) = s\tau^2(x, 0) \tag{46}
\]
\[
t\tau^1(0, -x) = -s\tau^2(x, 0) \tag{47}
\]

But in the presence of \(\tau(-x, -y) = -\tau(x, y)\) this is also the same condition. We finally conclude that, in the presence of equations 31 and 32, together with \(\tau(\lambda x, \lambda y) = \lambda\tau(x, y)\) \((\lambda = -1, s, t)\), the full set of equations for singular pairs is equivalent to the single equation 46 and the claim of the proposition follows.

### 1.3 Computer examples

A computer can give (just by brute force checking) all singular pairs \((X, S, \tau)\) for a given biquandle \((X, S)\) of cardinal 2 or 3, and with some time and memory, for \(n = 4\). For instance, if \(X = \{1, 2, 3\}\) and \(S = \text{flip}\), then there are (up to isomorphism) 7 singular pairs \((X, \text{flip}, \tau)\). Five of them are such that \(\tau\) satisfies YBeq, four of them are actually biquandles, and none of them are given by quandles or racks, except the trivial one. The two solutions that do not satisfy the YBeq are given by

\[
\tau_1:
\begin{align*}
(1, 1) &\mapsto (1, 1) \\
(2, 2) &\mapsto (3, 3) \mapsto (2, 2) \\
(1, 2) &\mapsto (3, 1) \mapsto (3, 2) \mapsto (1, 2) \\
(1, 3) &\mapsto (2, 3) \mapsto (2, 1) \mapsto (1, 3)
\end{align*}
\]

\[
\tau_2:
\begin{align*}
(1, 1) &\mapsto (2, 2) \mapsto (3, 3) \mapsto (1, 1) \\
(1, 2) &\mapsto (1, 2) \\
(1, 3) &\mapsto (3, 2) \mapsto (3, 1) \mapsto (2, 3) \mapsto (1, 3) \\
(2, 1) &\mapsto (2, 1)
\end{align*}
\]

Given a general biquandle \((X, S)\) with small cardinality, computational algorithms can find all possible \(\tau\) such that \((X, S, \tau)\) is a singular pair. Using Remark 17 one can find a family of singular pairs with bigger cardinal for bi-Alexander switch, and thanks to Theorem 18 we
know they are all of this type when \((1 - st)\) is a unit. We call \(\tau_\varphi\) a switch as in Remark 17.

In http://mate.dm.uba.ar/~mfarinat/papers/GAP/singular/ one can find several examples implemented in GAP [GAP2019]. We include here a table with some of the total amounts (as usual, we denote \(D_n\) the Alexander switch with \(s = 1, t = -1\)):

| \(n\) | \# of s. pairs | \# of isoclasses |
|-------|----------------|------------------|
| 2     | 2              | 2                |
| 3     | 24             | 7                |
| 4     | 3360           | 169              |

| \(n\) | \(I_n \) | \(I_n\) | \# of isoclasses of s. pairs for \(D_n\) of type \(\tau_\varphi\) |
|-------|---------|---------|----------------|------------------|
| 3     | 4       | 4       | 5              | 6                |
| 4     | 7       | 9       | 10             | 11               |
| 5     | 16      | 776     | 20             | 3904             |

### 1.4 Examples of colorings

**Example 23.** The singular trefoil (right) and its mirror image (left), see Figure 5.

![Figure 5: Singular trefoil and its mirror image](image)

Consider \((X, S)\) a biquandle and the virtual pair \(p := (S, S)\). It is clear that for the trefoil with negative crossings, the number of colorings equal the cardinal of the set of pairs \(\{(x, y) \in X \times X : S(S^{-1})^2(x, y) = (x, y)\}\) while on the other one, the set of colorings of the semiarcps are in 1-1 correspondence with \(\{(x, y) : S^3(x, y) = (x, y)\}\). What actually happens is that using this virtual pair, the set of colorings of the singular knot correspond to the set of colorings of the classical knot obtained from replacing the singular crossings by classical positive ones. We see that in the first case the number of colorings is just the cardinal of \(X\), while in the second case, the number of colorings may be different. For example, with the dihedral quandle of size 3 we get nontrivial colorings of the usual trefoil.

If one uses the virtual pair \((S, S^{-1})\) then the roles of trivial or nontrivial colorings are interchanged.

**Example 24.** The singular Hopf link

Consider \(X = \{1, 2\}, S = \text{flip} \) and \(\tau = i_2\), that is, \(i_2(x, y) = (y + 1, x - 1) \mod 2\). Then the set of colorings of the singular Hopf link (see Figure 6) is empty.
2 Weights and non commutative cocycles

Let \((X, S, \tau)\) be a singular pair, \(H\) be a group and \(f, h : X \times X \to H\) two maps. Given a coloring on a singular knot or link, we decorate the crossing with elements of \(H\) (that we call Boltzmann Weights) in the following way: at a positive classical crossing \(\gamma\), let \(x_\gamma, y_\gamma\) be the color on the incoming arcs. The Boltzmann weight at \(\gamma\) is \(B_{f,h}(\gamma, C) = f(x_\gamma, y_\gamma)\). At a negative classical crossing \(\gamma\), denote \(S(x_\gamma, y_\gamma)\) the colors on the incoming arcs. The Boltzmann weight at \(\gamma\) is \(B_{f,h}(\gamma, C) = f(x_\gamma, y_\gamma)^{-1}\).

At a singular crossing \(\gamma\), let \(x_\gamma, y_\gamma\) be the color on the incoming arcs. The Boltzmann weight at \(\gamma\) is \(B_{f,h}(\gamma, C) = h(x_\gamma, y_\gamma)\).

Depending on the way we choose to multiply weights, we arrive at two different notions of cocycles. One is adapted for maps into general groups (e.g. non necessarily commutative) and generalizes the non commutative cocycles for classical links given in [FG1], and the second is only valid for maps with values into commutative groups, its is a generalization of the state-sum procedure for biquandles (see [CEGS]). We call respectively non-abelian 2-cocycles and abelian 2-cocycles.

2.1 Non abelian 2-cocycle pair

Let \(L\) be a link, \((X, S, \tau)\) a singular pair and \(C \in Col_X(L)\) be a coloring of \(L\) by \(X\). Call \((K_1, \ldots, K_r)\) the components of \(L\) and let \((b_1, \ldots, b_r)\) be a set of base points on the components \((K_1, \ldots, K_r)\). Let \(\gamma^{(i)}\), for \(i = 1, \ldots, r\), be the (ordered) set of crossings \(\gamma\) such that \(\gamma\) is a classical crossings where the under-arc belongs to component \(i\) or \(\gamma\) is a virtual crossing of...
component $i$. The order of the set $\gamma^{(i)}$ is given by the orientation of the component starting at the base point.

Notice that choosing a different base-point leads to a different product, but with the same cyclic order, so its conjugacy class is the same. The following definition gives the precise conditions on the maps $f$ and $h$ in order to make this conjugacy classes an invariant of links/knots.

**Definition 25.** The pair $(f, h)$ is called non commutative 2-cocycle if the maps $f$ and $h$ verify

\[(f1) \quad f(x, y) f(S^2(x, y), z) = f(x, S^4(y, z)) f(S^2(x, S^4(y, z)), S^2(y, z)) \] (due to RIII),

\[(f2) \quad f(S^4(x, y), S^4(S^2(x, y), z)) = f(y, z) \] (due to RIII),

\[(f3) \quad f(x, s(x)) = 1 \] (due to RI),

\[(f4) \quad f(x, y) f(S^2(x, y), z) = f(x, \tau_1(y, z)) f(S^2(x, \tau_1(y, z)), \tau_2(y, z)) \] (due to RIVa),

$h$ satisfies

\[(h1) \quad h(S^4(x, y), S^4(S^2(x, y), z)) = h(y, z) \] (due to RIVa),

and compatibility conditions

\[(c1) \quad f(x, S^4(y, z)) h(S^2(x, S^4(y, z)), S^2(y, z)) = h(x, y) f(\tau^2(x, y), z) \] (due to RIVb),

\[(c2) \quad f(y, z) h(S^2(x, S^4(y, z)), S^2(y, z)) = h(x, y) f(\tau_1(x, y), S^4(\tau^2(x, y), z)) \] (due to RIVb),

\[(c3) \quad h(x, y) = f(x, y) h(S(x, y)) \] (due to RV),

\[(c4) \quad h(S(x, y)) = h(x, y) f(\tau(x, y)) \] (due to RV).

**Remark 26.** If $f, h$ is a cocycle pair, then condition (c3) implies

\[f(x, y) = h(x, y) h(S(x, y))^{-1}.\]

In particular, $f$ is determined by $h$, so for instance if $h \equiv 1$ then $f \equiv 1$ as well. One may think of $h$ as a kind of "square root" of $f$. One may also keep the formula $f(x, y) = h(x, y) h(S(x, y))^{-1}$ and write all others in terms only of $h$. For some equations this is not particularly enlightening and we will continue writing everything in terms of $h$ and $f$. However, there are two simplifications:

**Lemma 27.** Equation (c3) implies (f3) and equation (c3) together with (h1) imply (f2).

**Proof.** We know that $S(x, s(x)) = (x, s(x))$, so (c3) for $(x, y) = (x, s(x))$ gives

\[h(x, s(x)) = f(x, s(x)) h(x, s(x)) \]

and so $f(x, s(x)) = 1$. For the other implication, it is convenient to observe first that

\[h(S^4(x, y), S^4(S^2(x, y), z)) = h(y, z)\]

is equivalent to

\[h(S(S^4(x, y), S^4(S^2(x, y), z))) = h(S(y, z))\]
This is because, for a fixed $x$, the map

$$X \times X \to X \times X$$

$$(y, z) \mapsto (S^1(x, y), S^1(S^2(x, y), z))$$

is $S$-invariant. A diagrammatic proof is the following:

\[ \begin{array}{c}
S^1(x, y) \quad S^1(S^2(x, y), z) \\
S^1(x, y') \quad S^1(S^2(x, y'), z')
\end{array} \quad \leftrightarrow \quad \begin{array}{c}
S^1(x, y') \quad S^1(S^2(x, y'), z')
\end{array} \]

**Remark 28.** If $f, h$ is a cocycle pair, using conditions (c4) and (c3) we get

$$f(\tau(x, y)) = h(x, y)^{-1} f(x, y)^{-1} h(x, y) = h(x, y)^{-1} h(S(x, y)).$$

In particular, if $H$ is abelian, $f(\tau(x, y)) = f(x, y)^{-1}$.

**Theorem 29.** Let $f, h : X \times X \to H$ be a n.c. 2-cocycle pair. For any (oriented) diagram of a link $L$ using the singular pair $(X, S, \tau)$, each Reidemeister move establish a bijection between the set of colorings of the diagram of the link $L$ before and after the Reidemeister move is applied. For each coloring of $L$, the (conjugacy class of the) product of weights defined above is invariant under Reidemeister moves.

**Proof.** For invariance under classical Reidemeister moves see [FG1]. For invariance under $RIVb$, we show a particular orientation (see Figure 7).

There will be no product of Boltzmann weights due to the horizontal line. When traveling the semiarc labeled by $x$ in each diagram the product of weights will leave both sides of equation (c1) in Definition 25. When traveling the semiarc labeled by $y$ in each diagram the product of weights will leave both sides of equation (c2) in Definition 25. The remaining orientations will leave equivalent equations.

For invariance under $RIVa$, we show a particular orientation (see Figure 8).

When traveling the semiarc labeled by $x$ in each diagram the product of weights will leave both sides of equation (f4) in Definition 25. When traveling the semiarc labeled by $y$ or $z$ in each diagram the product of weights will leave both sides of equation (h1) in Definition 25. The remaining orientations will leave equivalent equations.

For invariance under RV: when traveling the semiarc labeled by $x, y$ (see Figure 9) in each diagram the product of weights will leave both sides of equation (c3) and (c4) in Definition 25 respectively. The remaining orientations will leave equivalent equations.
2.2 Universal noncommutative 2-cocycle pair

Given a singular pair \((X, S, \tau)\) we shall define a group together with a universal n.c. 2-cocycle pair in the following way:

**Definition 30.** Let \(U_{fh}^{nc} = U_{fh}^{nc}(X, S, \tau)\) be the group freely generated by symbols \((x, y)_f\) and \((x, y)_h\) with relations

- \((x, y)_f(S^2(x, y), z)_f = (x, S^1(y, z))_f(S^2(x, S^1(y, z)), S^2(y, z))_f\)
- \((x, y)_f(S^2(x, y), z)_f = (x, \tau^1(y, z))_f(S^2(x, \tau^1(y, z)), \tau^2(y, z))_f\)
- \((x, S^1(y, z), S^2(x, S^1(y, z)))_h = (y, z)_h\)
- \((y, z)_f(S^2(x, S^1(y, z)), S^2(y, z))_h = (x, y)_h(\tau^1(x, y), S^1(\tau^2(x, y), z))_f\)
- \((x, y)_h = (x, y)_f(S(x, y), h)\)
- \((S(x, y))_h = (x, y)_h(\tau(x, y))_f\)

Denote \(f_{xy}\) and \(h_{xy}\) the class in \(U_{nc}^{fh}\) of \((x, y)_f\) and \((x, y)_h\) respectively. We also define \(\pi_f, \pi_h : X \times X \to U_{nc}^{fh}\) by

\[
\pi_f, \pi_h : X \times X \to U_{nc}^{fh}, \quad \pi_f(x, y) := f_{xy}, \quad \pi_h(x, y) := h_{xy}
\]

The following is immediate from the definitions:

**Theorem 31.** Let \((X, S, \tau)\) be a singular pair:
The pair of maps \( \pi_f, \pi_h : X \times X \to U^{fh}_{nc} \) is a noncommutative 2-cocycle pair.

Let \( H \) be a group and \( f, h : X \times X \to H \) a noncommutative 2-cocycle pair, then there exists a unique group homomorphism \( \rho : U^{fh}_{nc} \to H \) such that \( f = \rho \circ \pi_f \) and \( h = \rho \circ \pi_h \).

Remark 32. \( U^{fh}_{nc} \) is functorial. That is, if \( \phi : (X, S, \tau) \to (Y, S', \tau') \) is a morphism of virtual pairs, namely \( \phi \) satisfy

\[
(\phi \times \phi)S(x_1, x_2) = S'((\phi x_1, \phi x_2), \quad (\phi \times \phi)\tau(x_1, x_2) = \tau'((\phi x_1, \phi x_2)
\]

then, \( \phi \) induces a (unique) group homomorphism \( U^{fh}_{nc}(X) \to U^{fh}_{nc}(Y) \) satisfying

\[
f_{x_1x_2} \mapsto f_{\phi x_1\phi x_2}, \quad h_{x_1x_2} \mapsto h_{\phi x_1\phi x_2}
\]
2.3 Some particular cases

2.3.1 Non abelian cocycles with trivial $f$

If $f \equiv 1$ then it is straightforward to see that $(f, h)$ is a 2-cocycle pair if and only if $h$ satisfies

\begin{align*}
(h1) & \quad h(S^1(x, y), S^1(S^2(x, y), z)) = h(y, z) \\
(c1) & \quad h(S^2(x, S^1(y, z)), S^2(y, z)) = h(x, y) \text{ (c2 is the same equation)}, \\
(c3) & \quad h(x, y) = h(S(x, y)) \text{ (c4 is the same equation)},
\end{align*}

Notice that there is no condition involving $h$ together with $\tau$. If in addition $S(x, y) = (y, x \triangleleft y)$ then (h1) is trivial and the others are

\begin{align*}
(c1) & \quad h(x \triangleleft z, y \triangleleft z) = h(x, y) \\
(c3) & \quad h(x, y) = h(y, x \triangleleft y)
\end{align*}

\textbf{Example 33}. Take a singular link with two connected components. Consider $S = \tau = \text{flip}$, $f \equiv 1$ and $h$ such that $h(x, y) = h(y, x)$ (as in $[12]$). Take for example $X = \{1, 2\}$, so, any map $h : X \times X \to H$ with trivial $f$ (giving a 2-cocycle) factors through

$$h : X \times X \to \text{Free}(a, b, c)$$

where $h(1, 1) = a$, $h(2, 2) = b$, $h(1, 2) = h(2, 1) = c$. For a coloring with color 1 in one connected component and color 2 the second component, the invariant gives products of $a$’s and $c$’s for the first component (in some cyclic order) and products of $b$’s and $c$’s (in a given cyclic order) for the second connected component. If one considers the abelianization, the exponent of $a$ gives twice the number of singular self intersections of the first component, the exponent of $b$ gives twice the number of singular self intersection of the second component, and the exponent of $c$ gives the number of singular intersections of the first component with the second one.

2.3.2 A particular virtual pair: $(X, S, S)$ with $S^2 = \text{Id}$

\textbf{Example 34}. Take $S : X \times X \to X \times X$ an involutive (i.e. $S^2 = \text{Id}$) solution of the set theoretical Yang-Baxter equation and consider the virtual pair $(X, S, S)$. Denote $U := U_{nc}^f$ and $U_{ab} = U/[U, U]$ its abelianization. Then $U_{ab} \cong \text{Free}\{(X \times X)/\sim\}_{ab}$ the free abelian group generated by $(X \times X)/\sim$ (written multiplicatively), where $\sim$ is the equivalent relation given by

$$(y, z) \sim (S^1(x, y), S^1(S^2(x, y), z))$$

Moreover, $h : X \times X \to U_{ab}$ is given by

$$h(x, y) = \overline{(x, y)}$$

and

$$f(x, y) = \overline{(x, y)} \cdot \overline{S(x, y)}^{-1}$$
Proof. First note that, when $S^2 = \text{Id}$, the equivalence relation

$$(y, z) \sim (S^1(x, y), S^1(S^2(x, y), z)) \forall x$$

is the same as

$$(S^2(x, S^1(y, z)), S^2(y, z)) \sim (x, y) \forall z.$$ 

Diagrammatically

represents $(\text{Id} \times S)(S \times \text{Id})(x, y, z)$. Applying $(S \times \text{Id})(\text{Id} \times S)$ and using twice that $S^2 = \text{Id}$ leads to

Using this, and replacing $\tau$ by $S$, we get that the cocycle conditions are

(f1) $f(x, y)f(S^2(x, y), z) = f(x, S^1(y, z))f(S^2(x, S^1(y, z)), S^2(y, z)),$

(f4) $f(x, y)f(S^2(x, y), z) = f(x, S^1(y, z))f(S^2(x, S^1(y, z)), S^2(y, z)),$

(h1) $h(S^1(x, y), S^1(S^2(x, y), z)) = h(y, z),$

(c1) $f(x, S^1(y, z))h(x, y) = h(x, y)f(S^2(x, y), z),$

(c2) $f(y, z)h(x, y) = h(x, y)f(y, z),$

(c3) $h(x, y) = f(x, y)h(S(x, y)),$

(c4) $h(S(x, y)) = h(x, y)f(S(x, y))$

Again from $S^2 = \text{Id}$, condition (c4) is equivalent to

$$h(x, y) = h(S(x, y))f(x, y)$$

and because we assume the group is commutative, this is the same as (c3), giving the formula $f(x, y) = h(x, y)h(S(x, y))^{-1}$. 

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Equation (c2) is trivial due to commutativity. Equation (c1) gives the condition \( f(x, S^1(y, z)) = f(S^2(x, y), z) \). But the relation

\[
(y, z) \sim (S^1(x, y), S^1(S^2(x, y), z))
\]

implies

\[
(x, S^1(y, z)) \sim (S^2(x, y), z)
\]

So, the proof finishes by noticing that, if \( h \) satisfy

\[
h(S^1(x, y), S^1(S^2(x, y), z)) = h(y, z) = h(S^2(x, S^1(y, z)), S^2(y, z))
\]

then also

\[
h(x, S^1(y, z)) = h(S^2(x, y), z)
\]

and moreover, if \( f(x, y) = h(x, y)h(S(x, y))^{-1} \), then \( f \) also satisfies

\[
f(S^1(x, y), S^1(S^2(x, y), z)) = f(y, z) = f(S^2(x, S^1(y, z)), S^2(y, z))
\]

\[
f(x, S^1(y, z)) = f(S^2(x, y), z)
\]

and hence (f1), (f4) and c(3) are consequences of (c3). We conclude that the value of \( h \) in \((x, y)\) depends only in the equivalence class of \((x, y)\) under the relation \( \sim \), necessarily \( f(x, y) = h(x, y)h(S(x, y))^{-1} \), and no other condition is needed for the pair \((f, h)\) in order to have a 2-cocycle pair.

**Example 35.** As sub examples one can consider \( X = \{1, 2\} \) and \( S = \tau = \text{flip} \), in this case the relation \( \sim \) is just the identity, \( U_{ab} \) is the free abelian group on generators \( a = (1, 1), b = (1, 2), c = (2, 1), d = (2, 2) \), and the values of \( f \) and \( h \) are given by

\[
\begin{array}{c|cccc}
(x, y) & (1, 1) & (1, 2) & (2, 1) & (2, 2) \\
\hline
f & 1 & bc^{-1} & cb^{-1} & 1 \\
\hline
h & a & c & b & d
\end{array}
\]

If one instead considers \( X = \{1, 2\} \) with \( S = \tau \) given by \( S(x, y) = (y - 1, x + 1) \text{ Mod } 2 \):

\[
S(1, 2) = (1, 2), \quad S(2, 1) = (2, 1)
\]

\[
S(1, 1) = (2, 2), \quad S(2, 2) = (1, 1)
\]

then the equivalence classes on \( X \times X \) are \((1, 1) \sim (2, 2), (1, 2) \sim (2, 1)\). The group \( U_{ab} \) is the free group on two generators \( a = (1, 1) \) and \( b = (1, 2) \). The table with the values of \( f \) and \( h \) is

\[
\begin{array}{c|cccc}
(x, y) & (1, 1) & (1, 2) & (2, 1) & (2, 2) \\
\hline
f & 1 & 1 & 1 & 1 \\
\hline
h & a & b & b & a
\end{array}
\]
2.3.3 Particular examples: $S = \text{flip}$, $\tau(x,y) = (y+1,x-1)$, $x,y \in \mathbb{Z}/2\mathbb{Z}$: extending linking number

If $S = \text{flip}$ then necessarily $\tau^1(y,x) = \tau^2(x,y)$ and equations for the generators of $U := U_{nc}$ are

- $(x,y)f(x,z)f = (x,z)f(x,y)f$
- $(x,y)f(x,z)f = (x,z+1)f(x,y+1)f$
- $(x,z)f(x,y)h = (x,y)h(x+1,z)f$
- $(y,z)f(x,y)h = (x,y)h(y+1,z)f$
- $(x,y)h = (x,y)f(y,x)h$
- $(y,x)h = (x,y)h(y+1,x+1)f$.

If we further consider the abelianization $U/[U,U]$ we get for free the first equation; also, from the third and fourth one can cancel $(x,y)h$, getting

- $(x,y)f(x,z)f = (x,z+1)f(x,y+1)f$
- $(x,z) = (x+1,z)f$
- $(x,y)h = (x,y)f(y,x)h$
- $(y,x)h = (x,y)h(y+1,x+1)f$.

From the third equation above we get $(x,y)h(y,x)h^{-1} = (x,y)f$. In particular $(x,x)f = 1$, but also from $(x,z)f = (x+1,z)f$ we necessarily get $f \equiv 1$. So, (together with $f \equiv 1$) we get the single condition

$$ (x,y)h = (y,x)h. $$

We finally obtain that the abelianization of the universal group associated to $S = \text{flip}$ and $\tau = i_2$ is the free abelian group on 3 generators $a, b, c$, where $f \equiv 1$ and

$$ h(1,1) = a, \ h(1,2) = b = h(2,1), \ h(2,2) = c $$

![Figure 10: Singular trefoil knot](image)
Example 36. Take the singular trefoil knot and paint it as shown in Figure 10 with \((\mathbb{Z}/2\mathbb{Z}, S, \tau)\) as in subsection 2.3.3. The computation of the invariant gives \(\{b^2\}\) which can be understood as a "self-linking number" similar to the case of virtual knots [FG2].

Example 37. Take the singular links shown in Figure 11. The invariant will distinguish both links. Call singular hopf link the link on the right, it has \(\{cabb^2, cabb^2\}\) as invariant for the coloring shown. The invariant for the other link, for all possible colorings, are \(\{b^2, b^2\}\) (twice) and \(\{(ca)^2, (ea)^2\}\) (twice).

![Figure 11: 4 singular crossing links](image)

3 Abelian 2-cocycle pairs and state-sum

In this section we assume \(H\) is an abelian group. One may ignore that fact and consider cocycle pairs as in previous sections, or one can make a different definition of 2-cocycle pair that works only in the commutative case. We will perform this second option in this section.

Definition 38. Let \((X, S, \tau)\) be a singular pair. A pair of functions \(f, h : X \times X \to H\) is an abelian 2-cocycle pair if: the pair \(f, S\) satisfies:

\[
(f_1') \quad f(x, y)f(S^2(x, y), z)f(S^1(x, y), S^1(S^2(x, y), z)) = f(x, S^3(y, z))f(S^2(x, S^1(y, z)), S^2(y, z))f(y, z) \quad \text{(due to RMIII)}
\]

\[
(f_2') \quad f(x, s(x)) = 1 \quad \text{(due to RMI)}
\]

and compatibility conditions between \(f, h, \tau, S\):

\[
(c_1') \quad h(y, z)f(x, \tau^1(y, z))f(S^2(x, \tau^1(y, z)), \tau^2(y, z)) = f(x, y)f(S^2(x, y), z)h(S^1(x, y), S^1(S^2(x, y), z)) \quad \text{(due to RIVa)}
\]

\[
(c_2') \quad f(y, z)f(x, S^1(y, z))h(S^2(x, S^1(y, z)), S^2(y, z)) = h(x, y)f(\tau^1(x, y), \tau^2(x, y), z)f(\tau^1(x, y), S^1(\tau^2(x, y), z)) \quad \text{(due to RIVb)}
\]

\[
(c_3') \quad f(x, y)h(S(x, y)) = h(x, y)f(\tau(x, y)) \quad \text{(due to RV)}
\]

are satisfied for any \(x, y, z \in X\).

Remark 39. The invariant given by non commutative cocycles were computed by traveling along the connected components of the oriented link. Since one always meet a crossing twice, in the non commutative case one has to choose if one considers the situation of going over or under arc. One of these choices gives the definition of the previous section. In the commutative
case, one can make the simultaneous (i.e. without caring about the order) multiplication over all crossings, and so there is no choice of over or under arc. The notion of abelian cocycle pair is more symmetric, and general properties of cocycle pairs are very different, for instance, for non commutative cocycles, \( f \) is determined by \( h \), but for abelian cocycles, \( h \equiv 1 \) does not imply \( f \equiv 1 \).

**Example 40.** If \((X,S)\) is a biquandle and \( f : X \times X \to H \) is a biquandle (abelian) 2-cocycle of type I, that is \( f \) satisfies

\[
(f'') \quad f(x, y) f(S^2(y, x), z) f(S(x, y), S(S^2(y, x), z)) = f(x, y) f(S^2(x, y), S(y, z)) f(y, z)
\]

then \((X, S, S)\) is a singular pair, and \( (f, h) = (f, f) \) is an abelian 2-cocycle pair. Moreover, if \( \tilde{f} : X \times X \to G \) is a non commutative 2-cocycle and \( f \) is the composition of \( \tilde{f} \) with the canonical projection \( G \to H := G/[G,G] \), then \( f \) is a commutative 2-cocycle, and one can construct abelian 2-cocycles pairs in that way.

**Example 41.** If \((X, S)\) is a biquandle and \( f : X \times X \to H \) is an abelian 2-cocycle such that \( f \circ S = f \), then \((X, S, S^{-1})\) is a singular pair, and \( h(x, y) = f(x, y)^{-1} \) gives an abelian 2-cocycle pair.

**Remark 42.** Unlike the non commutative cocycle definition, \( h \) does not determine \( f \). In particular, \( h \equiv 1 \) does not imply \( f \equiv 1 \).

For instance, if we force \( h \equiv 1 \), then \( f \) must satisfy

\[
(f'') \quad f(x, y) f(S^2(y, x), z) f(S(x, y), S(S^2(y, x), z)) = f(x, y) f(S^2(x, y), S(y, z)) f(y, z)
\]

\[
(f'') \quad f(x, s(x)) = 1
\]

\[
(c1') \quad f(x, \tau^1(y, z)) f(S^2(x, \tau^1(y, z)), \tau^2(y, z)) = f(x, y) f(S^2(x, y), z)
\]

\[
(c2') \quad f(y, z) f(x, S^1(y, z)) = f(\tau^2(x, y), z) f(\tau^1(x, y), S^1(\tau^2(x, y), z))
\]

\[
(c3') \quad f(x, y) = f(\tau(x, y))
\]

**Example 43.** For \((X, S, S)\) and \( f \) satisfying \((f'')\) and \( f \circ S = f \), then \((f, h \equiv 1)\) is an abelian cocycle.

Another special situation is the following:

**Example 44.** If \( S = \text{flip} \) (recall that necessarily \( \tau^1(x, y) = \tau^2(y, x) \) in this case), then the cocycle equations are

\[
(f'') \quad f(x, x) = 1,
\]

\[
(c1') \quad f(x, \tau^1(y, z)) f(x, \tau^2(y, z)) = f(x, y) f(x, z),
\]

\[
(c2') \quad f(y, z) f(x, z) = f(\tau^2(x, y), z) f(\tau^1(x, y), z),
\]

\[
(c3') \quad f(x, y) h(y, x) = h(x, y) f(\tau(x, y)).
\]
If in addition $f \equiv 1$ then the single condition for $h$ is $h(x, y) = h(y, x)$. Also, if $\tau=\text{flip}$, then conditions (c1') and (c2') are trivial, the remaining conditions are

$$f(x, x) = 1, \text{ and } f(x, y)h(x, y) = f(y, x)h(y, x).$$

Let $L$ be a singular link, $(X, S, \tau)$ a singular pair, $C \in \text{Col}_X(L)$ be a coloring of $L$ by $X$ and $f, h : X \times X \to H$ an abelian 2-cocycle pair. Consider the product of all Boltzmann weights associated to every crossing (classical or singular) in $L$:

$$P_C = \prod_{\gamma} B_{f, h}(\gamma, C) \in H$$

where $B_{f, h}(\gamma, C)$ is defined in the same way as in Subsection 2.

The partition function, or state-sum (associated with $f, h$) is the expression

$$\Phi_{f, h}(L) = \sum_{C \in \text{Col}_X} P_C \in \mathbb{Z}[H]$$

**Theorem 45.** The partition function defines an invariant of singular links.

**Remark 46.** From a biquandle $(X, S, \tau)$ and an abelian 2-cocycle $f : X \times X \to H$ one can extend the state-sum invariant for classical knots or links to singular ones by considering the singular pairs and cocycle pairs as in examples 40 and 41.

**Proof.** We will check the product is invariant under Reidemeister moves. Same as before, we will show one orientation for each Reidemeister move, the remaining orientations will give equivalent equations. For classical Reidemeister moves see [FG1] (for quandles see [CKLS]). For singular Reidemeister moves, start with oRIVa: the product of Boltzmann weights associated to the crossings shown on the top diagram of Figure 8 is

$$f(x, y) f(S^2(x, y), z) h(S^1(x, y), S^1(S^2(x, y), z))$$

and the product of Boltzmann weights associated to the crossings shown on the bottom diagram of Figure 8 is

$$h(y, z) f(x, \tau^1(y, z)) f(S^2(x, \tau^1(y, z)), \tau^2(y, z)).$$

The equality of these two elements is given by (c1') in Definition 38.

The product of Boltzmann weights associated to the crossings shown on the top diagram of Figure 7 is

$$f(y, z) f(x, S^1(y, z)) h(S^2(x, S^1(y, z)), S^2(y, z))$$

and the product of Boltzmann weights associated to the crossings shown on the bottom diagram of Figure 7 is

$$h(x, y) f(\tau^2(x, y), z) f(\tau^1(x, y), S^1(\tau^2(x, y), z)).$$

The equality of these two elements is given by (c2') in Definition 38.

The product of Boltzmann weights associated to the crossings shown on the left diagram of Figure 9 is

$$h(x, y) f(\tau(x, y))$$

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and the product of Boltzmann weights associated to the crossings shown on the right diagram of Figure 9 is

\[ f(x, y)h(S(x, y)). \]

The equality of these two elements is given by (c3’) in Definition 38.

**Remark 47.** Let \( \text{Ab}^{fh} = \text{Ab}^{fh}(X, S, \tau) \) be the abelian (multiplicative) group freely generated by symbols \((x, y)_f\) and \((x, y)_h\) with relations

\[
\begin{align*}
(\text{f1'}) \quad (x, y)_f (S^2(x, y), z)_f (S^4(x, y), S^4(S^2(x, y), z))_f &= (x, S^4(y, z))_f (S^2(x, S^4(y, z)), S^2(y, z))_f (y, z), \\
(\text{f2'}) \quad (x, s(x))_f &= 1 \\
(\text{c1'}) \quad (y, z)_h (x, \tau^1(y, z))_f (S^2(x, \tau^1(y, z)), \tau^2(y, z))_f &= (x, y)_f (S^2(x, y), z)_f (S^1(x, y), S^1(S^2(x, y), z))_h \\
(\text{c2'}) \quad (y, z)_f (x, S^1(y, z))_f (S^2(x, S^1(y, z)), S^2(y, z))_h &= (x, y)_h (\tau^2(x, y), z)_f (\tau^1(x, y), S^1(\tau^2(x, y), z))_f \\
(\text{c3'}) \quad (x, y)_f (S(x, y))_h &= (x, y)_h (\tau(x, y))_f
\end{align*}
\]

Denote \( f_{xy} \) and \( h_{xy} \) the class in \( \text{Ab}^{fh} \) of \((x, y)_f\) and \((x, y)_h\) respectively. We also define \( \pi_f, \pi_h : X \times X \to \text{Ab}^{fh} \) by

\[
\begin{align*}
\pi_f(x, y) &= f_{xy}, \\
\pi_h(x, y) &= h_{xy}
\end{align*}
\]

The following is immediate from the definitions:

**Theorem 48.** Let \((X, S, \tau)\) be a singular pair:

- The pair of maps \( \pi_f, \pi_h : X \times X \to \text{Ab}^{fh} \) is an abelian 2-cocycle pair.

- Let \( H \) be an abelian group and \( f, h : X \times X \to H \) an abelian 2-cocycle pair, then there exists a unique group homomorphism \( \rho : \text{Ab}^{fh} \to H \) such that \( f = \rho \circ \pi_f \) and \( h = \rho \circ \pi_h \)

\[
\begin{array}{ccc}
X \times X & \xrightarrow{\pi_f} & H \\
\downarrow{\pi_h} & & \downarrow{\pi_h} \\
\text{Ab}^{fh} & \xrightarrow{\rho} & \text{Ab}^{fh}
\end{array}
\]

**Remark 49.** \( \text{Ab}^{fh} \) is functorial. That is, if \( \phi : (X, S, \tau) \to (Y, S', \tau') \) is a morphism of virtual pairs, namely \( \phi \) satisfy

\[
(\phi \times \phi)S(x_1, x_2) = S'((\phi x_1, \phi x_2), \quad (\phi \times \phi)\tau(x_1, x_2) = \tau'(\phi x_1, \phi x_2)
\]

then, \( \phi \) induces a (unique) group homomorphism \( \text{Ab}^{fh}(X) \to \text{Ab}^{fh}(Y) \) satisfying

\[
f_{x_1, x_2} \mapsto f_{\phi x_1, \phi x_2} \quad h_{x_1, x_2} \mapsto h_{\phi x_1, \phi x_2}
\]

**Remark 50.** In general, \( \text{Ab}^{fh} \) is not the abelianization of \( \text{U}_{nc}^{fh} \).

**Question 51.** If \( (f, h) \) is a non commutative 2-cocycle and \( H \) is abelian, then \( (f, h) \) is a commutative 2-cocycle? In virtual case, \( H \) needed to have no 2-torsion.
Polynomial interpretation

An easy observation is that if $X$ is a finite set, then for any singular pair on $X$, the 2-cocycle pairs always takes values into finitely generated abelian group, simply because the universal target group $Ab^{fh}$ is finitely generated, or because one may eventually change the target group $H$ by the subgroup generated by the images of the cocycles $f$ and $h$.

If $H$ is a finitely generated abelian group, then it is necessarily of the form

$$H \cong T \oplus \mathbb{Z}^n$$

where $T$ is abelian and finite and $n \in \mathbb{N}$ is the rank of $H$. As a consequence, the state-sum invariant takes values into a Laurent Polynomial algebra, because of the following isomorphisms of the group algebra:

$$\mathbb{Z}[H] \cong \mathbb{Z}[T \oplus \mathbb{Z}^n] \cong \mathbb{Z}[T] \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}^n] \cong \mathbb{Z}[T] \otimes_{\mathbb{Z}} \mathbb{Z}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}] \cong \mathbb{Z}[T][z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$$

In this way, via the state-sum invariant, any singular pair on a finite set $X$ associates, to any singular knot or link, a Laurent polynomial, with the same number of variables as the rank of $Ab^{fh}$.

**Abelian cocycles with $S = \text{flip}$**

We will look at abelian cocycles when $S = \text{flip}$, in order to get generalizations of the linking number. For $S = \text{flip}$, the relations in the universal group are equivalent to

$$(f2') \ (x, x)_f = 1$$

$$(c1') \ (x, \tau^1(y, z)) f(x, \tau^2(y, z)) f = (x, y)_f(x, z)_f$$

$$(c2') \ (y, z)_f(x, z)_f = (\tau^2(x, y), z)_f(\tau^1(x, y), z)_f$$

$$(c3') \ (x, y)_f(y, x)_h = (x, y)_h(\tau(x, y))_f$$

Notice that if $\tau = \text{flip}$, then equations (c1’) and (c2’) are trivially satisfied. Now consider the case $\tau(x, y) = (sy, sx)$ where $s : X \to X$ is a bijection satisfying $s^2 = \text{Id}$. The relations are

$$(f2') \ (x, x)_f = 1$$

$$(c1') \ (x, sz)_f(x, sy)_f = (x, y)_f(x, z)_f$$

$$(c2') \ (y, z)_f(x, z)_f = (sx, z)_f(sy, z)_f$$

$$(c3') \ (x, y)_f(y, x)_h = (x, y)_h(sy, sx)_f$$

for $y = sx$, notice $\tau(x, sx) = (x, sx)$ and (c3’) gives

$$(sx, x)_h = (x, sx)_h$$

Also, for $x = y = z$, (c1’) and (c2’) gives

$$(sx, x)_f^2 = 1 = (x, sx)_f^2$$

We have almost done for the following
Example 52. For $X = \{1, 2\}$ and $s$ the transposition $1 \leftrightarrow 2$, the universal group $Ab^f h$ is given by the generators

$$u_1 = (1, 2)_f, \ u_2 = (2, 1)_f, \ a = (1, 1)_h, \ b = (1, 2)_h = (2, 1)_h, \ c = (2, 2)_h$$

with relations $u_1^2 = 1 = u_2^2$ (and $(1, 1)_f = 1 = (2, 2)_f$). That is, is isomorphic to

$$U = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}^3$$

The group ring $\mathbb{Z}[U]$ is isomorphic to an extension of Laurent polynomial in 3 variables:

$$\mathbb{Z}[U] \cong \mathbb{Z}[u_1, u_2, a^{\pm 1}, b^{\pm 1}, c^{\pm 1}]/(u_1^2 = 1 = u_2^2)$$

Example 53. Take as above $X = \{1, 2\}$ and $s$ the transposition $1 \leftrightarrow 2$. The state sum invariant for the two singular links shown in Figure 11 gives:

- $4ab^2c$ for the link on the right,
- $2a^2c^2 + 2b^4$ for the link on the left.

It worth noticing that the state-sum invariant for $X = \{1, 2\}$ but $(S, \tau) = (\text{flip}, \text{flip})$ does not distinguish these links.

References

[BEHY] K. Bataineh, M. Elhamdadi, M. Hajij and W. Youmans. Generating sets of Reidemeister moves of oriented singular links and quandles, Journal of Knot Theory and its Ramifications, Volume 27, Number 14 (2018). Doi 10.1142/S0218216518500645.

[CEGS] J. Carter, M. Elhamdadi, M. Graña and M. Saito. Cocycle knot invariants from quandle modules and generalized quandle homology. Osaka Journal of Mathematics 42, No. 3, 499-541 (2005).

[CES] J. Carter, M. Elhamdadi and M. Saito. Homology Theory for the Set-Theoretic Yang-Baxter Equation and Knot Invariants from Generalizations of Quandles. Fund. Math. 184 (2004), 31-54.

[CJKLS] J. Carter, D. Jelsovsky, S. Kamada, L. Langford and M. Saito. Quandle cohomology and state-sum invariants of knotted curves and surfaces. Transactions of the American Mathematical Society. Volume 355, Number 10, Pages 3947-3989. S 0002-9947(03)03046-0

[CEHN] I. Churchill, M. Elhamdadi, M. Hajij, and S. Nelson. Singular knots and involutive quandles. Journal of Knot Theory and its Ramifications, Volume 26, Number 14 (2017). Doi 10.1142/S0218216517500997

[FG1] M. Farinati and J. García Galofre. Link and knot invariants from non-abelian Yang-Baxter 2-cocycles. Journal of Knot Theory and Its Ramifications Volume 25, Number 13 (2016), doi: 10.1142/S021821651650070X.
[FG2] M. Farinati and J. García Galofre. *Virtual link and knot invariants from non-abelian Yang-Baxter 2-cocycle pairs*. Osaka Journal of Mathematics. Volume 56, Number 3 (2019), 525-547.

[GAP2019] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.10.2; 2019. (https://www.gap-system.org)

[HN] A. Henrich and S. Nelson. *Semiquandles and flat virtual knots*. Pacific Journal of Mathematics. Vol. 248, No. 1, 2010.