FREE COMMUTING INVOLUTIONS ON CLOSED TWO-DIMENSIONAL SURFACES

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Abstract. We consider the function $f(g)$ that assigns to an orientable surface $M$ of genus $g$ the maximal number of free commuting independent involutions on $M$. We show that the surface of minimal genus $g$ with $f(g) = n$ is a real moment-angle complex $R_K$, where $K$ is the boundary of an $(n+2)$-gon. The genus is given by the formula $g = 1 + 2^{n-1}(n-2)$.

1. Introduction

One of the main objects of study in toric topology is the moment-angle-complex $Z_K$, which is a cell complex with a torus action constructed from a simplicial complex $K$, see [1]. Along with the moment-angle complex $Z_K$, its real analog $R_K$ is considered and has many interesting properties. If $K$ is a simplicial subdivision of an $(n-1)$-dimensional sphere with $m$ vertices, then $Z_K$ is an $(m+n)$-dimensional (closed) manifold, and $R_K$ is an $n$-dimensional manifold. In particular, if $K$ is the boundary of an $m$-gon, then $R_K$ is an orientable closed two-dimensional manifold (a surface). Given a positive integer $g$, let $f(g)$ be the maximal number of free commuting independent involutions on a closed orientable surface $M_g$ of genus $g$ (that is, $f(g)$ is the largest possible $n$ such that $(\mathbb{Z}_2)^n$ acts on $M_g$ freely). In this paper, it is proved that the function $f(g)$ attains a local maximum on the surfaces $R_K$, see Fig. 1. In other words, for any integer $n$, the surface of minimum genus $g$ that supports a free action of $(\mathbb{Z}_2)^n$ has the form $R_K$, where $K$ is an $(n+2)$-gon. The genus is given by the formula $g = 1 + 2^{n-1}(n-2)$.

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2. Basic notions and preliminary statements

A moment-angle complex is a special case of the following construction:

Construction 2.1 (polyhedral product). Let $K$ be a simplicial complex on the set $[m] = \{1, 2, \ldots, m\}$ and let

$$(X, A) = \{(X_1, A_1), \ldots, (X_m, A_m)\}$$

be a collection of $m$ pairs of topological spaces, $A_i \subset X_i$. For each subset $I \subset [m]$ denote

$$(X, A)^I = \{ (x_1, \ldots, x_m) \in \prod_{j=1}^m X_j : x_j \in A_j \text{ for } j \notin I \}$$

and define the polyhedral product of $(X, A)$ corresponding to a simplicial complex $K$ by

$$(X, A)^K = \bigcup_{I \in K} (X, A)^I = \bigcup_{I \in K} \left( \prod_{i \in I} X_i \times \prod_{i \notin I} A_i \right).$$

Here the union is a subset of $\prod_{j=1}^m X_j$.

In the case when all pairs $(X_i, A_i)$ are the same, i.e. $X_i = X$ and $A_i = A$ for $i = 1, \ldots, m$, we use the notation $(X, A)^K$ for $(X, A)^K$.

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If \((X, A) = (D^2, S^1)\) then \(\mathcal{Z}_K = (D^2, S^1)^K\) is called the moment-angle complex. We will consider its real analogue:
\[
\mathcal{R}_K = ([−1, 1], \{-1, 1\})^K,
\]
where \([-1, 1] = D^1\) is the line segment, and \([-1, 1] = \partial[-1, 1]\) is the pair of points.

If \(K\) is a simplicial decomposition of \((n-1)\)-dimensional sphere, then \(\mathcal{R}_K\) is an \(n\)-dimensional manifold with the action of group \((\mathbb{Z}_2)^m\) (see [1] Theorem 4.1.7). In other words, we have \(m\) commuting involutions on \(\mathcal{R}_K\). Of these \(m\) involutions no more than \(m-n\) act freely. For completeness, below we present a proof of these facts for \(K\) being the boundary of an \(m\)-gon. The surfaces \(\mathcal{R}_K\) corresponding to \(m\)-gons appear as “regular topological skew polyhedra” in the 1937 work of Coxeter [2].

**Proposition 2.2.** Let \(K\) be the boundary of an \(m\)-gon. Then \(\mathcal{R}_K\) is an orientable closed surface of genus
\[
g = 1 + 2^{m-3}(m-4).
\]

**Proof.** Consider the cube \(I^m = [−1, 1]^m\) with the standard structure of cubical complex. Its two-dimensional skeleton consists of faces of the form
\[
\{(ε_1, \ldots, ε_{i-1}, x_i, ε_{i+1}, \ldots, ε_j, x_j, ε_{j+1}, \ldots, ε_m): x_i, x_j \in [−1, 1]\},
\]
where \(ε_1, \ldots, ε_m = ±1\).

By definition, \(\mathcal{R}_K\) is a cubical subcomplex of \(I^m\). Indeed, \(\mathcal{R}_K\) is a union of two-dimensional faces of cube of the following form:
\[
\{(ε_1, \ldots, ε_{i-1}, x_i, ε_{i+1}, \ldots, ε_j, x_j, ε_{j+1}, \ldots, ε_m): x_i, x_{i+1} \in [−1, 1]\}, \text{ where } ε_1, \ldots, ε_m = ±1
\]
(here and below we consider subscripts modulo \(m\), i.e., \(m + 1 ≡ 1\)). Each one-dimensional face \(\{(ε_1, \ldots, x_i, \ldots, ε_m): x_i \in [−1, 1]\}\) is the boundary of precisely two squares, namely \((x_i, x_{i+1}) \in [−1, 1]^m\) and \((x_{i-1}, x_i) \in [−1, 1]^m\). Moreover, the intersection of two squares is either empty, or a vertex, or a one-dimensional face (since this is valid for the whole cube and one-dimensional faces of \(\mathcal{R}_K\) form the whole one-dimensional skeleton of the cube). Each vertex is contained in precisely \(m\) squares. This implies that \(\mathcal{R}_K\) is a closed two-dimensional orientable manifold.

The surface \(\mathcal{R}_K\) is glued of \(F = m2^{m-2}\) squares. Each square has 4 edges, and each edge is contained in two 2 squares. Therefore the number of edges is \(E = 4F/2 = 2F\). All vertices of the cube are vertices of our surface, therefore the number of vertices is \(V = 2^m\). Thus, the Euler characteristic is
\[
\chi(\mathcal{R}_K) = V - E + F = V - 2F + F = V - F = 2^{m-2}(4 - m),
\]
which immediately implies the formula for genus of the surface.

**Example 2.3.** Let \(K\) be the boundary of a triangle. Then \(\mathcal{R}_K\) is the boundary of a 3-dimensional cube and therefore is homeomorphic to a sphere, i.e., the genus is 0.

**Lemma 2.4.** If \(K\) is the boundary of an \(m\)-gon, then there is a free action of \(\mathbb{Z}_2^{m-2}\) on \(\mathcal{R}_K\).

**Proof.** We describe a freely acting subgroup \(\mathbb{Z}_2^{m-2} \subset \mathbb{Z}_2^m\) by explicitly defining its generators. Let \(ψ_i\) be the involution sending \(x_i\) to \(−x_i\) and fixing the other coordinates. The involutions \(ψ_i, i = 1, \ldots, m\), commute and therefore generate a \(\mathbb{Z}_2^m\)-action. However, this action is not free as \(ψ_i\) fixes the points whose \(i\)-th coordinate is zero. Consider the composition \(ψ_i ∘ ψ_j\), where \(|i−j| > 1\) (it corresponds to a diagonal in the polygon \(K\)). The set of fixed points of the involution \(ψ_i ∘ ψ_j\) is the set with coordinates \(x_i = x_j = 0\), but \(\mathcal{R}_K\) does not contain such points, since \(x_i, x_j\) are not consecutive. If \(m = 2k\), then consider the \(m - 2\) involutions
\[
ψ_1 ∘ ψ_3, ψ_3 ∘ ψ_5, \ldots, ψ_{2k-3} ∘ ψ_{2k-1} \text{ and } ψ_2 ∘ ψ_4, ψ_4 ∘ ψ_6, \ldots, ψ_{2k-2} ∘ ψ_{2k}.
\]
These
involutions commute pairwise and generate a free action of $\mathbb{Z}_2^{m-2}$. Similarly, for an odd $m = 2k + 1$ consider the $m - 2$ involutions $\psi_1 \circ \psi_3, \psi_3 \circ \psi_5, \ldots, \psi_{2k-3} \circ \psi_{2k-1}, \psi_2 \circ \psi_4, \psi_4 \circ \psi_6, \ldots, \psi_{2k-2} \circ \psi_{2k}$ and $\psi_1 \circ \psi_{2k} \circ \psi_{2k+1}$.

**Remark.** For even $m$, each element of the freely acting group $\mathbb{Z}_2^{m-2}$ defined above preserves the orientation of $R_K$, since it is a composition of an even number of elementary involutions $\psi_i$. For odd $m$, the involution $\psi_1 \circ \psi_{2k} \circ \psi_{2k+1}$ reverses the orientation of $R_K$.

Next, consider an arbitrary closed two-dimensional surface $M$. We ask the following question: find the maximal $n$ such that there is a free action of $\mathbb{Z}_2^n$ on $M$. Obviously, $n$ depends only on the genus $h$ of the surface $M$. (The Euler characteristic of an orientable surface of genus $h$ is $2 - 2h$, and the Euler characteristic of a nonorientable surface of genus $h$ is $2 - h$.) Define the function

$$f(h) = \max\{n : \text{there is an action of } \mathbb{Z}_2^n \text{ on a surface of genus } h\}.$$ 

Let $f(h) = n$. Then $B = M/\mathbb{Z}_2^n$ is a closed two-dimensional manifold with Euler characteristic $\chi(B) = \chi(M)/2^n$.

**Proposition 2.5.** Let $g$ be the genus of $B = M/\mathbb{Z}_2^n$. If the surface $B$ is orientable, then $n \leq 2g$. If $B$ is nonorientable, then $n \leq g$.

**Proof.** Consider the case of orientable $B$. The fundamental group of $B$ is

$$G = \pi_1(B) = \langle a_1, b_1, \ldots, a_g, b_g \mid a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1} = 1 \rangle.$$ 

The fundamental group $\pi_1(M) = H$ of the manifold $M$ is a normal subgroup of the group $G$ (since the action of $\mathbb{Z}_2^n$ is free) and

$$G/H \approx \mathbb{Z}_2^n.$$ 

Therefore the square of each coset is 1,

$$[a_i]^2 = [b_j]^2 = 1,$$

for all $i, j = 1, \ldots, g$. Moreover, all cosets (i.e., elements of $G/H \approx \mathbb{Z}_2^n$) commute. The quotient group $G/H$ is generated by the cosets $[a_1], [b_1], \ldots, [a_g], [b_g]$. Since they commute and their squares equal the unit, we can compose at most $2^g$ words of them. Therefore the order of the group $G/H \approx \mathbb{Z}_2^n$ is at most $2^{2g}$. Hence, $n \leq 2g$.

For a nonorientable surface $B$ the argument is similar: the fundamental group

$$G = \pi_1(B) = \langle a_1, \ldots, a_g \mid a_1^2 \ldots a_g^2 = 1 \rangle$$

has $g$ generators and the order of the quotient $G/H$ is at most $2^g$. \hfill $\square$

**Proposition 2.6.**

a) For any orientable surface $B$ of genus $g$ and an integer $n \leq 2g$, there exists an orientable surface $M$ with a free action of $\mathbb{Z}_2^n$ such that $B = M/\mathbb{Z}_2^n$.

b) For any nonorientable surface $B$ of genus $g$ and an integer $n \leq g$, there exists a surface $M$ with a free action of $\mathbb{Z}_2^n$ such that $B = M/\mathbb{Z}_2^n$.

**Proof.** Let $B$ be orientable. Consider the subgroup $P$ in the fundamental group $\pi(B) = G$ generated by the squares of all elements

$$P = \langle g^2, g \in G \rangle,$$

and consider its normalizer subgroup $H = GPG^{-1}$. Then $H$ is a normal subgroup and $H \neq G$, since $H$ contains only elements with even number of letters. The relations $[a_i]^2 = [b_j]^2 = [a_ib_ja_ib_j] = 1$ in the quotient group $G/H$ imply that $G/H \cong \mathbb{Z}_2^{2g}$. Hence there exists a regular covering of $B$, which gives a free action of $G/H \cong \mathbb{Z}_2^{2g}$ (see [3]). Now add one generator to $P$: $P_1 = \langle P, a_1 \rangle$ and consider $H_1 = GP_1G^{-1}$. Then $H_1$ is a proper normal subgroup of $G$ (since in each of its
elements the generator \(b_i\) occurs even number of times), and the quotient group is \(G/H_1 \cong \mathbb{Z}_2^{2g-1}\). The corresponding covering of \(B\) is regular and gives a free action of \(\mathbb{Z}_2^{2g-1}\). Continuing this process, we add to \(P\) the other generators \(a_i, b_i\), and obtain regular coverings of \(B\) corresponding to free actions of \(\mathbb{Z}_2^{2g-k}\) for all \(k = 1, 2, ..., 2g - 1\).

The nonorientable case is considered similarly. \(\square\)

3. Main results

Let \(M_g\) be a surface of genus \(g\).

**Proposition 3.1.** Let \(n\) be the maximal integer such that

\[
\chi(M_g) = a \cdot 2^n \quad \text{with} \quad n \leq 2 - a, \quad a \leq 1.
\]

Then \(f(g) = n\) if \(a\) is even, and \(n - 1 \leq f(g) \leq n\) if \(a\) is odd.

**Proof.** Let \(\mathbb{Z}_2^k\) act freely on a surface \(M_g\). Then \(\chi(M_g) = a' \cdot 2^k\) where \(a' = \chi(M_g/\mathbb{Z}_2^k)\), and \(k \leq 2 - a'\) by Proposition 2.5. By definition, \(n\) is the maximum of such \(k\), hence \(f(g) \leq n\). Now we estimate \(f(g)\) from below. Consider the two cases:

*Case 1: \(a\) is even.* Consider an orientable surface \(B\) with Euler characteristic \(a\) and genus \(2 - 2a\). Since \(n \leq 2 - 2a\), by Proposition 2.6 there exists an orientable surface \(M\) such that \(\mathbb{Z}_2^k\) acts freely on \(M\) and \(M = B/\mathbb{Z}_2^k\). The Euler characteristic is \(\chi(M) = a \cdot 2^n = \chi(M_g)\), therefore \(M\) has genus \(g\) and \(f(g) \geq n\).

*Case 2: \(a\) is odd.* Write \(\chi(M) = 2a \cdot 2^{n-1}\). Consider an orientable surface \(B\) of Euler characteristic \(2a\), i.e., of genus \(\frac{2 - 2a}{2}\). We have \(n - 1 \leq 2 - 2a\), since \(n \leq 2 - a\) and \(a \leq 1\). By Proposition 2.6 there exists an orientable surface \(M\) such that \(\mathbb{Z}_2^{n-1}\) acts freely on \(M\) and \(M = B/\mathbb{Z}_2^{n-1}\). The Euler characteristic is \(\chi(M) = 2a \cdot 2^{n-1} = \chi(M_g)\), therefore \(M\) has genus \(g\) and \(f(g) \geq n - 1\). \(\square\)

Define \(H(g)\) as the inverse function to \(g(x) = 1 + 2^{x-1}(x - 2)\), that is,

\[
H(g) = \frac{W \left( \frac{1}{2} (g - 1) \ln 2 \right)}{\ln 2} + 2,
\]

where \(W\) denotes the Lambert function (i.e., the function inverse to \(xe^x\)).

**Theorem 3.2.** Theorem \(f(g) \leq H(g)\), and an equality is attained on real moment-angle manifolds and only on them.

**Proof.** Substituting \(\chi(M_g) = 2 - 2g\) and \(a = 2 - b\) in \(3.1\) we obtain

\[
g = 1 + 2^{n-1}(b - 2), \quad \text{where} \quad n \leq b, \quad b \geq 1.
\]

By Proposition 3.1 for the maximal \(n\) satisfying the conditions above we have \(f(g) \leq n\). On the other hand we have

\[
g(n) = 1 + 2^{n-1}(n - 2) \leq 1 + 2^{n-1}(b - 2) = g,
\]

or equivalently \(n \leq H(g)\). This implies the required inequality \(f(g) \leq H(g)\). An equality is attained if and only if the genus \(g\) can be written as \(g = 1 + 2^{n-1}(n - 2)\). Compairing this expession with the formula from Proposition 2.2 we obtain that \(M \cong K\), where \(K\) is the boundary of \((n + 2)\)-gon. \(\square\)

The values of the function \(f(g)\) are shown in Fig. 1, where the dashed line is the graph of \(H(g)\).
Figure 1. The graph of the function $f(g)$, where $g$ is the genus of a surface.

References

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