**$l_p$-NORMS OF FOURIER COEFFICIENTS OF POWERS OF A BLASCHKE FACTOR**

**By**

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**Abstract.** We determine the asymptotic behavior of the $l_p$-norms of the sequence of Taylor coefficients of $b^n$, where

$$b(z) = \frac{z - \lambda}{1 - \overline{\lambda} z}, \quad |\lambda| < 1,$$

is an automorphism of the unit disk, $p \in [1, \infty]$, and $n$ is large. It is known that in the parameter range $p \in [1, 2]$ a sharp upper bound

$$\|b^n\|_{l^p} \leq c_p n^{\frac{2-p}{2p}}$$

holds. In this article we find that this estimate is valid even when $p \in [1, 4)$. We prove that

$$\|b^n\|_{l^4} \leq c_4 \left( \frac{\log n}{n} \right)^{\frac{1}{4}}$$

and for $p \in (4, \infty]$ that

$$\|b^n\|_{l^p} \leq c_p n^{\frac{1-p}{4p}}.$$  

The upper bounds are shown to be asymptotically sharp as $n$ tends to $\infty$. As a direct application we prove the sharpness of existing upper estimates on analytic capacities in Beurling–Sobolev spaces. Our investigation is also motivated by a question of J. J. Schäffer about optimal estimates for norms of inverse matrices.

1 **Introduction**

1.1 **Notation.** We denote by $\mathbb{D} = \{ z : |z| < 1 \}$ the open unit disk in the complex plane and by $\partial \mathbb{D}$ its boundary. For a given $\lambda \in \mathbb{D}$ we denote by

$$b(z) = b(z) = \frac{z - \lambda}{1 - \overline{\lambda} z}$$

the elementary Blaschke factor corresponding to $\lambda$. For any $n$ we have that $b^n$ is a holomorphic function $\mathbb{D} \to \mathbb{D}$ and $|b(z)| = 1$ is equivalent to $z \in \partial \mathbb{D}$. It is well known that such functions possess a natural identification with their boundary.

*The work is supported by Russian Science Foundation grant 14-41-00010.*
behavior on $\partial \mathbb{D}$ and their Taylor- and Fourier-coefficients can be identified [NN2, Section 3.4]. We will use these terms interchangeably in what follows. Let $b^n = \sum_{k \geq 0} \hat{b}^n(k) z^k$ denote the Taylor expansion of $b^n$. We write

$$\| b^n \|_{lp} := \| \hat{b}^n \|_{lp} := \sum_{k \geq 0} |\hat{b}^n(k)|^p$$

for the usual $l_p$-norm of the sequence of Taylor coefficients of $b^n$. In the limit of large $p$ we set $\| b^n \|_{l\infty} := \sup_k |\hat{b}^n(k)|$. This article is devoted to the study of the asymptotic behavior of the norm

$$\| b^n \|_{lp} \quad \text{for } p \in [1, \infty]$$

in the limit of large $n$.

### 1.2 History of the problem.

The asymptotic behavior of $\| b^n \|_{lp}$ appears to be relatively well-studied in the literature. The discussion was probably initiated by J.-P. Kahane in [KJP, Theorem 1] generalizing a theorem by Z. K. Leibenson [LZ]. The latter is a special case of a theorem [RW, Theorem 4.1.3] about homomorphisms of group algebras due to P. T. Cohen. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a continuous and $2\pi$-periodic function. A. Beurling and H. Helson [BeHe] proved that if $\| e^{in\varphi} \|_{l1} = O(1)$, then $\varphi$ is affine. Kahane proved that

1. if $\varphi$ is affine then $\| e^{in\varphi} \|_{l1} = O(\log(|n|))$, [KJP, Theorem III], and
2. if $\varphi$ is analytic and non-constant then $\| e^{in\varphi} \|_{l1} \sim \sqrt{|n|}$, [KJP, Theorem V].

Writing $b(e^{it})$ as $e^{i\varphi(t)}$ for $t \in (-\pi, \pi]$, point (2) entails that $\| b^n \|_{l1} \sim c_1 \sqrt{n}$, $n \to \infty$. The correct prefactor $c_1$ with $\| b^n \|_{l1}/\sqrt{n} \to c_1$ as $n$ tends to $\infty$ was computed in [GD] but the proof was carried out more precisely in [BaHl]. The discussion of $l^A_p$-norms for $p \neq 1$ occurred in [BS], where the asymptotic behavior

$$\| b^n \|_{lp} \sim n^{2-2p} \quad \text{for } p \in [1, 2]$$

is derived. The discussion of [BS] is motivated by investigating the boundedness of the compositions operator $(\text{comp}_b(f)) (z) := f(b(z))$. To assess if $\text{comp}_b$ is a bounded linear operator from one Banach space of analytic functions into another it is often enough to know the asymptotic behavior of $\| b^n \|$. For example, the closed graph theorem shows that $\text{comp}_b$ is bounded from $l^A_1$ to $l^A_p$ iff the norms $\| b^n \|_{lp}$ are uniformly bounded. In this article we extend the mentioned result of [BS] to the whole interval $p \in [1, \infty]$ and apply our findings to

1. derive estimates to analytic capacities in the $l^A_p$-spaces, see Section 1.3.2, and
2. to introduce a constructive method to approach Schäffer’s question about optimal estimates for norms of inverses of power-bounded matrices, see Section 1.3.3.
1.3 New results.

1.3.1 Asymptotic analysis of $\|b^n\|_{l^p}$ . To formulate our analysis we use some standard notation from asymptotic analysis. For two positive functions $f, g : \mathbb{C} \to \mathbb{R}^+$ we say that $f$ is dominated by $g$, denoted by $f \lesssim g$, if there is a constant $c > 0$ such that $f \leq cg$. We say that $f$ and $g$ are comparable, denoted by $f \asymp g$, if both $f \lesssim g$ and $g \lesssim f$.

The $\|b^n\|_{l^p}$ norms only depend on the absolute values $|\hat{b}(k)|$. As a consequence we have

$$
\left\| \left( \frac{z - \lambda}{1 - \lambda z} \right)^n \right\|_{l^p} = \left\| \left( \frac{z - |\lambda|}{1 - |\lambda| z} \right)^n \right\|_{l^p}
$$

so that we can assume $\lambda \in (0, 1)$. Our main result is as follows.

**Theorem 1.** Let $b = b_\lambda$ with $\lambda \in (0, 1)$ and $n \geq 1$. For the $l^p$-norms of the Taylor coefficients of $b^n$ we have the following asymptotic behavior as $n$ tends to $\infty$:

$$
\|b^n\|_{l^p} \asymp \begin{cases} 
  n^{\frac{2-p}{2}} & \text{if } p \in (1, 4), \\
  (\frac{\log n}{n})^{\frac{1}{4}} & \text{if } p = 4, \\
  n^{\frac{1-p}{2}} & \text{if } p \in (4, \infty],
\end{cases}
$$

where implicit prefactors depend on $p$ and $\lambda$.

Our proof of Theorem 1 is based on a detailed analysis of the asymptotic growth of the Taylor coefficients $\hat{b}(k)$ with respect to both $k$ and $n$. As it turns out this analysis is delicate. Holomorphy of $b^n$ implies that for any fixed $n$ the coefficients $\hat{b}(k)$ decay exponentially when $k$ grows large. Similarly, it is not difficult to see that at any fixed $k$ the coefficient $\hat{b}(k)$ decays exponentially with $n$; see Proposition 3 below. The interesting behavior, which is relevant for determining the norms $\|b^n\|_{l^p}$, therefore occurs when $k = k(n)$ is a sequence. As $n$ grows large the region of values $k$ that provide the dominating contribution to $\|b^n\|_{l^p}$ can change. For instance, in case of $\|b^n\|_{l^\infty} = \sup_{k \geq 0} |\hat{b}(k)|$ we can guess that the supremum will be achieved on a coefficient whose index $k$ depends on $n$. The question is now, what is actually the right sequence $k = k(n)$ such that $\|b^n\|_{l^\infty}$ is achieved.

More generally, for our exercise it is crucial to identify, for each $p$, which values of $k$ provide the dominating contribution to $\|b^n\|_{l^p}$. We therefore decompose the set of values for $k$ into $n$-depending “intervals” and show that the regions of $k$ that provide the dominating contributions to $\|b^n\|_{l^p}$ depend on $p$ and $\lambda \in (0, 1)$.

This fact is one of the main findings of the article at hand and was not observed in preceding publications. In this fact lies also the reason for the structure of the asymptotic behavior provided in Theorem 1. Depending on whether $p \in (1, 4)$ or
$p \in (4, \infty)$ the dominating contribution stems from different regions of $k$ resulting in the differing asymptotics. The dependence on $\lambda$ can be described in terms of the “critical” values $a_0 = \frac{1-\lambda}{1+\lambda}$ and $a_0^{-1}$, which will be stationary points for the expansion of integrals in our asymptotic analysis. For now a simple way of understanding their critical nature is to view them as values that identify the slowest decay for $\hat{b}^n(k)$. At $k = \lfloor a_0 n \rfloor$ and $k = \lfloor a_0^{-1} n \rfloor$ we observe the slowest decay of $\hat{b}^n(k(n))$ when $n$ grows large. A summary of order-of-magnitude decay rates that asymptotically bound $|\hat{b}^n(k)|$ is provided in Figure 1, depending on the region where $k$ belongs.

| Values of $k(n)$ in interval | Decay of $\hat{b}^n(k)$ | Region |
|-------------------------------|-------------------------|--------|
| $[0, a_0 n]$                  | exponential             | I      |
| $(a_0 n, a_0 n - n^{1/3}]$   | $\frac{1}{a_0 n - k}$  | II     |
| $[a_0 n - n^{1/3}, a_0 n + n^{1/3}]$ | $\frac{1}{n^{1/3}}$ | III    |
| $[a_0 n + n^{1/3}, a_0^{-1} n - n^{1/3}]$ | $n^{1/3}(\frac{1}{a_0 n^{1/3}-(a_0^{-1} n^{1/3})})$ | IV     |
| $[a_0^{-1} n - n^{1/3}, a_0^{-1} n + n^{1/3}]$ | $\frac{1}{n^{1/3}}$ | V      |
| $[a_0^{-1} n + n^{1/3}, a^{-1} n)$ | $\frac{1}{k-a_0^{-1} n}$ | VI     |
| $[a^{-1} n, \infty)$         | exponential             | VII    |

Figure 1. Illustration of order-of-magnitude upper estimates on $|\hat{b}^n(k)|$ as $n$ gets large, as a function of $k = k(n)$. We set $a_0 = \frac{1-\lambda}{1+\lambda}$ and fix arbitrary $\alpha \in (0, a_0)$.

In this article we split the discussion of $\hat{b}^n(k)$ and $\|b^n\|_{L_p}$ conceptually in the derivation of upper and lower estimates. In Section 2 we derive upper estimates $|\hat{b}^n(k)|$ and compute the resulting upper estimates for $\|b^n\|_{L_p}$. In Section 3 we prove the asymptotic sharpness of our upper estimates. Our proof of upper bounds on $\hat{b}^n(k)$ will be based on a well-known van der Corput type estimate, Lemma 4. It turns out that in the interval $p \in [1, 4]$ sharpness follows from a simple application of Hölder’s inequality. The proof of sharpness in the range $p \in [4, \infty)$, however, requires new methods. A core step will be the introduction and development of the so-called uniform method of stationary phase [BV, CFU], which we employ to derive an asymptotic expansion of $\hat{b}^n(k)$ when $k$ is near to $a_0^{-1} n$. This method will provide the sharpness of our upper bound when $p \in [4, \infty]$.

1.3.2 Application to analytic capacities. Let $Hol(\mathbb{D})$ denote the space of holomorphic functions $f = \sum_{k \geq 0} \hat{f}(k)z^k$ over $\mathbb{D}$. For a Banach space $X \subset Hol(\mathbb{D})$ and a finite sequence $\sigma = (\lambda_1, \lambda_2, \ldots, \lambda_n) \subset \mathbb{D}^n$ with $\lambda_i \neq 0$ for all $i = 1, \ldots, n$, N. K. Nikolski [NN] introduced the $X$-zero capacity of $\sigma$ (at $z = 0$) as

$$\text{cap}_X(\sigma) = \inf\{ \|f\|_X : f(0) = 1, f|\sigma = 0 \}.$$
Here $f|\sigma = 0$ means that $f(\lambda) = 0$ for all $\lambda \in \sigma$ taking into account possible multiplicities. As explained in [NN] the $X$-zero capacities are closely related to
(1) the so-called uniqueness problem for $X$ in complex analysis, and
(2) to condition numbers and norms of inverses in operator theory.

[NN, Theorem 3.30] shows upper estimates for $\text{cap}_X(\sigma)$ where $X = l_A^p(w)$ is a Beurling–Sobolev space

$$l_A^p(w) := \left\{ f = \sum_{k \geq 0} \hat{f}(k)z^k \in \text{Hol}(\mathbb{D}) : \|f\|^p_{l_A^p(w)} := \left( \sum_{k \geq 0} |\hat{f}(k)|^p w_k^p \right)^{1/p} < \infty \right\},$$

where the weight $w = (w_k)_{k \geq 1}$ satisfies $w_k > 0, \forall k \geq 0$, $\lim_k w_k^{1/k} = 1$ and

$$0 < \inf_k \frac{w_{k+1}}{w_k} \leq \sup_k \frac{w_{k+1}}{w_k} < \infty.$$

In case all weights $w_k = 1$ we recover $\|f\|^p_{l_A^p(w)} = \|f\|^p_{l_p}$ and $l_A^p(1) = l_p^A$. In case of polynomial weights $w_k = k^\beta, \beta \geq 0$ for all $k \geq 1$ and $w_0 = 1$, Nikolski proves [NN, Theorem 3.30] that if $p \in [2, \infty)$, then

$$\text{cap}_{l_A^p(w)}(\sigma) \leq \frac{n^\beta}{|\prod_{i=1}^n \lambda_i|}$$

and if $p \in [1, 2)$ then $\text{cap}_{l_A^p(w)}(\sigma) \leq \frac{a_1 n^{2/p} + a_2 n^{1/p}}{|\prod_{i=1}^n \lambda_i|}$ where $a_1, a_2 > 0$ only depend on $p$ and $\beta$. The asymptotic sharpness of these bounds remains open but an application of Hölder’s inequality and Theorem 1 gives a simple method to derive lower bounds. For simplicity, assume $\beta = 0$ (when $\beta \neq 0$ the formulas get rather cumbersome) and let the one-point sequence of cardinality $n$ be $\sigma_{\lambda,n} := (\lambda, \lambda, \ldots, \lambda) \in \mathbb{D}^n$.

**Proposition 2.**

(1) If $p \in [1, 4/3)$ then

$$\frac{n^{1/p}}{|\lambda|^p} \lesssim \text{cap}_{l_p^A}(\sigma_{\lambda,n}) \lesssim \frac{n^{2/p}}{|\lambda|^p},$$

(2) if $p = 4/3$ then

$$\frac{1}{|\lambda|^p (\log n)^{1/4}} \lesssim \text{cap}_{l_p^A}(\sigma_{\lambda,n}) \lesssim \frac{1}{|\lambda|^p},$$

(3) if $p \in (4/3, 4)$ then

$$\text{cap}_{l_p^A}(\sigma_{\lambda,n}) \asymp \frac{n^{2/p}}{|\lambda|^p},$$

(4) if $p = 4$ then

$$\frac{1}{|\lambda|^p} \frac{1}{n^{1/4}} \lesssim \text{cap}_{l_p^A}(\sigma_{\lambda,n}) \lesssim \frac{1}{|\lambda|^p} \left( \frac{\log(n)}{n} \right)^{1/4},$$
(5) and if \( p \in (4, \infty] \) then
\[
\frac{n^{\frac{2-p}{p}}}{|\lambda|^p} \lesssim \text{cap}_p(\sigma_{\lambda,n}) \lesssim \frac{n^{\frac{1-p}{p}}}{|\lambda|^p}.
\]

The proof of Proposition 2 is provided in Section 4.

1.3.3 Motivation in matrix analysis: Schäffer’s question. Theorem 1 is also motivated by a line of research that aims at Schäffer’s question about the norm of inverses of matrices [SJ, GMP]. The latter was initiated by studies of B. L. Van der Waerden [SJ] and J. J. Schäffer [SJ]: let \( M_D \) be the set of complex \( D \times D \) matrices and let \( \|T\| \) denote any Banach norm of \( T \in M_D \). What is the best \( S(D) \) so that
\[
|\det T|\|T^{-1}\| \leq S(D)\|T\|^{D-1}
\]
holds for any invertible \( T \in M_D \)? Schäffer [SJ, Theorem 3.8] proved that
\[
S(D) \leq \sqrt{eD},
\]
and he conjectured that \( S(D) \) is bounded. This claim was refuted by lower estimates on \( S(D) \) derived in contributions by E. Gluskin, M. Meyer and A. Pajor [GMP] as well as J. Bourgain [GMP] that are based on probabilistic methods. The currently best lower estimates seems to be due to H. Queffelec [QH], where a number theoretic method is employed to demonstrate that
\[
\sqrt{D}(1 - \Theta(1/D)) \leq S(D).
\]
The common point in the mentioned lower bounds is that they relate Schäffer’s problem to estimates for sums of powers of complex numbers. In essence this boils down to the so-called Turan power-sum problem, which is a well-studied problem in number theory; see, e.g., [TP, MH, AJ, AJ2]. The mentioned lower bounds on \( S(D) \) have two drawbacks. First, they are not constructive in the sense that only the existence of matrices with certain \( S(D) \) is established but an explicit representation is not clear (see [QH] for details). Second, the lower bounds on \( S(D) \) do not reach Schäffer’s upper bound and an attempt to improve the known methods leads into deep number theory. To approach Schäffer’s question we have constructed an explicit sequence of operators \( T = T(D) \) acting, respectively, on a \( D \)-dimensional Banach space, whose spectrum is concentrated at a given point \( \lambda \in \mathbb{D} \) and with
\[
|\lambda|^D\|T^{-1}\| \gtrsim \frac{1}{\|bD\|_{l_\infty}}\|T\|^{D-1}, \quad b = b_\lambda.
\]
Our discussion of $l^p_{N}$-norms of $b^n$ was also motivated by this expression and the assertion $\|b^n\|_p \approx c_p n^{\frac{2p}{p+1}}$ for $p \in [1, 2]$ in [BS]. If this bound could be extrapolated to $p = \infty$ then our sequence of operators would satisfy $S(D) \gtrsim D^{1/2}$, which would be a constructive proof for the estimate of [QH]. However, a direct application of Theorem 1 for $p = \infty$ yields $S(D) \gtrsim D^{1/3}$. This is not as good as the best lower bounds derived in [QH] but it is sufficient to refute Schäffer’s claim that $S$ is bounded, and our method is constructive in that we present an explicit class of matrices $T(D)$ with fixed spectrum in $\mathbb{D}$. In addition, it leads to well-defined expressions in $D$ for which good estimates entail respective lower bounds for Schäffer’s question. The details, particularly regarding the construction of $T$, are going to appear in a forthcoming publication [SZ2].

Acknowledgments. We are grateful to Professor Jean-Pierre Kahane, Professor Nikolai Kapitonovich Nikolski and to the referee for valuable comments on an earlier version of the manuscript.

2 Upper estimates

To prove the upper bounds in Theorem 1 we estimate the norm of the $k$-th Taylor coefficient of the $n$-th power of $b_\lambda$. Summing the individual coefficients will provide the desired bounds for Theorem 1.

Proposition 3. Let $b = b_\lambda$ with $\lambda \in (0, 1)$ and $n \geq 1$. Set $a_0 := \frac{1-\lambda}{1+\lambda}$ and choose a fixed $\alpha \in (0, a_0)$. In the following we consider sequences $k = k(n)$ and all assertions are meant to hold for large enough $n$:

1. If $k/n \leq \alpha$, then $|\hat{b}^\alpha(k)|$ decays exponentially as $n$ tends to $\infty$. Similarly, if $k/n \geq \alpha^{-1}$, then $|\hat{b}^\alpha(k)|$ decays exponentially as $n$ tends to $\infty$.

2. If $k/n \in (\alpha, a_0 - n^{-2/3}) \cup (a_0^{-1} + n^{-2/3}, \alpha^{-1})$, then

\[ |\hat{b}^\alpha(k)| \lesssim 4 \max \left\{ \left| \frac{1}{a_0 n - k} \right|, \left| \frac{1}{a_0^{-1} n - k} \right| \right\}. \]

3. If $k/n \in [a_0 - n^{-2/3}, a_0 + n^{-2/3}) \cup (a_0^{-1} - n^{-2/3}, a_0^{-1} + n^{-2/3}]$, then

\[ |\hat{b}^\alpha(k)| \lesssim \frac{1}{n^{1/3}}. \]

4. If $k/n \in (a_0 + n^{-2/3}, a_0^{-1} - n^{-2/3})$, then

\[ |\hat{b}^\alpha(k)| \lesssim \max \left\{ \left| \frac{1}{n^{1/2}(a_0 - k)^{1/4}} \right|, \left| \frac{1}{n^{1/2}(a_0^{-1} - k)^{1/4}} \right| \right\}. \]
We begin with a well-known lemma due to van der Corput. It will be the key ingredient for the upper estimates of Proposition 3.

**Lemma 4.** Let \( g \) be a continuously differentiable real function on \([a, b] \subset \mathbb{R}\), such that \( g \) and \( g' \) are monotone and \( g' \) does not vanish on \([a, b] \). Then

\[
\left| \int_{a}^{b} e^{ig(t)} \, dt \right| \leq \frac{2}{|g'(a)|} + \frac{2}{|g'(b)|}.
\]

**Proof.** Integration by parts shows that

\[
\int_{a}^{b} e^{ig(t)} \, dt = \int_{a}^{b} e^{ig(t)} \frac{g'(t)}{g'(t)} \, dt = \left[ e^{ig(t)} \frac{1}{g'(t)} \right]_{a}^{b} - \int_{a}^{b} e^{ig(t)} \left( \frac{1}{g'(t)} \right)' \, dt.
\]

This provides the rough upper estimate

\[
\left| \int_{a}^{b} e^{ig(t)} \, dt \right| \leq \frac{1}{|g'(a)|} + \frac{1}{|g'(b)|} + \int_{a}^{b} \frac{|g''(t)|}{(g'(t))^2} \, dt.
\]

Since \( g' \) is monotone on \([a, b] \) we have either \( g'' \geq 0 \) or \( g'' \leq 0 \) on \([a, b] \) and consequently

\[
\int_{a}^{b} \frac{|g''(t)|}{(g'(t))^2} \, dt = \int_{a}^{b} \left( \frac{1}{g'(t)} \right)' \, dt \leq \frac{1}{|g'(a)|} + \frac{1}{|g'(b)|}.
\]

\[\square\]

To apply the lemma we rewrite \( \hat{b}^{n}(k) \) in a convenient way. First \( b(e^{it}) \in \partial \mathbb{D} \) for any \( t \in (-\pi, \pi] \) and there exists a real valued and continuously differentiable function \( f_{\lambda} \) so that

\[
b(e^{it}) = e^{\lambda f_{\lambda}(t)}, \quad \forall t \in (-\pi, \pi].
\]

Deriving the above equality with respect to \( t \) we find

\[
ie^{it} \frac{1 - \lambda^2}{(1 - \lambda e^{it})^2} = if_{\lambda}'(t) b(e^{it})
\]

which shows that

\[
f_{\lambda}'(t) = \frac{1 - \lambda^2}{|1 - \lambda e^{it}|^2} = \frac{1 - \lambda^2}{1 + \lambda^2 - 2\lambda \cos(t)}, \quad \forall t \in (-\pi, \pi].
\]

For the Taylor coefficient with \( n \geq 1 \) and \( k \geq 0 \) we can write

\[
\hat{b}^{n}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} b^{n}(e^{it}) e^{-ikt} \, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(nf_{\lambda}(t) - kt)} \, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ig(t)} \, dt
\]

\[\frac{1}{\pi} \Re \left\{ \int_{0}^{\pi} e^{ig(t)} \, dt \right\},\]
where
\[ g(t) = nf_i(t) - kt, \quad t \in [0, \pi]. \]
Computing derivatives we find that \( g'(0) = na_0^{−1} - k, \ g'(\pi) = na_0 - k \) and
\[
g''(t) = -\frac{2\lambda n(1 - \lambda^2) \sin(t)}{(1 + \lambda^2 - 2\lambda\cos(t))^2}. \]
This implies that \( g' \) is strictly decreasing on \((0, \pi)\) with
\[
g'(0) = na_0^{−1} - k > g'(t) > na_0 - k = g'(\pi). \]

**Proof of Proposition 3.** For simplicity we focus on the case where \( k \) is closer to \( na_0 \) than to \( na_0^{−1} \). The discussion in the alternative case is identical.

1. This is a direct application of [SZ, Theorem 2, point (3)]. We recapitulate the main steps for completeness. It is well known [GJ] that for \( z, w \in \mathbb{D} \) we have upper and lower bounds on the elementary Blaschke factor as
\[
\frac{|z| - |w|}{1 - |z||w|} \leq \frac{|z - w|}{1 - |z||w|} \leq \frac{|z| + |w|}{1 + |z||w|}.
\]
Fourier coefficients can be expressed using the usual contour integral
\[
|\hat{b}^n(k)| = \left| \frac{1}{2\pi} \int_{|z|=1} z^{-k-1} \left( \frac{z - \lambda}{1 - \lambda z} \right)^n dz \right|.
\]
For the magnitude of the integral we find that
\[
(2.1) \quad |\hat{b}^n(k)| \leq \max_{|z|=s} \frac{|b^n(z)|}{|z|^k} = \begin{cases} \frac{b^n(s)}{s^k}, & s \in (1, 1/\lambda), \\ \frac{(\alpha + i\gamma)^n}{s^\alpha}, & s \in (0, 1). \end{cases}
\]
If \( k/n \geq \alpha^{-1} \), then there exists \( s^* \in (1, 1/\lambda) \) such that \( \frac{b(s^*)}{s^k} \leq \frac{b(s^*)}{s^{\alpha^{-1}}} < 1 \) [SZ].
If \( k/n \leq \alpha \), then there exists \( s^* \in (\lambda, 1) \) such that \( \frac{b(s^*)}{s^k} \leq \frac{b(s^*)}{s^{\alpha^{-1}}} < 1 \) [SZ].

2. If \( k/n \in (a, a_0 - n^{-2/3}) \) then \( g'(\pi) > 0 \). In particular, \( g' > 0 \) on \([0, \pi]\) and \( g \) is strictly increasing while \( g' \) is decreasing on this interval. Applying Lemma 4 we get
\[
|\hat{b}^n(k)| \leq \frac{2}{n\alpha_0 - k} + \frac{2}{na_0^{−1} - k} \leq \frac{4}{na_0 - k}.
\]

3. If \( k/n \in [a_0 - n^{-2/3}, a_0 + n^{-2/3}] \), then \( g'(\pi) = a_0n - k \) might be positive or negative depending on the choice of \( k \). We fix a constant \( c_1 > 0 \) (independent of \( n \)) whose exact value is to be specified later. We split the integral
\[
\int_0^\pi e^{ig(t)} dt = \int_0^{\pi - c_1n^{-1/3}} e^{ig(t)} dt + \int_{\pi - c_1n^{-1/3}}^\pi e^{ig(t)} dt
\]
and notice that
\[ \left| \int_{\pi - c_1n^{-1/3}}^{\pi} e^{ig(t)} \, dt \right| \leq c_1 n^{-1/3}. \]

To estimate the second integral we intend to apply the van der Corput-type Lemma 4, which requires a lower estimate on \( g'(t) \) for \( t \in [0, \pi - c_1n^{-1/3}] \).

To achieve this we expand the function \( f'_\lambda \) in a neighborhood of \( \pi \), which provides
\[ f'_\lambda(\pi - u) = a_0 + \frac{\lambda(1 - \lambda)}{(1 + \lambda)^3} u^2 + O(u^3) \]
as \( u \) tends to 0. Hence for the decreasing function \( g' \) we find that for large \( n \) and \( t \in [0, \pi - c_1n^{-1/3}] \) we have
\[ g'(t) \geq g'(\pi - c_1n^{-1/3}) = nf'_\lambda(\pi - c_1n^{-1/3}) - k \]
\[ \geq a_0n + c_1^2 \frac{\lambda(1 - \lambda)}{(1 + \lambda)^3} n^{1/3} + O(n^{-1/3}) - k \]
\[ \geq c_1^2 \frac{\lambda(1 - \lambda)}{(1 + \lambda)^3} n^{1/3} - n^{1/3} + O(n^{-1/3}) \]
\[ \geq n^{1/3}, \]
where we made use of the assumption \( k \leq a_0n + n^{1/3} \) and have chosen appropriate \( c_1 = c_1(\lambda) > 0 \) for the last inequality. Applying Lemma 4 on \([0, \pi - c_1n^{-1/3}]\) we obtain
\[ \left| \int_{0}^{\pi - c_1n^{-1/3}} e^{ig(t)} \, dt \right| \leq 4n^{-1/3}. \]

(4) If \( k/n \in (a_0 + n^{-2/3}, a_0 - n^{-2/3}) \), then the equation \( g'(t) = 0 \) has exactly one solution \( \varphi_+ \) on \((0, \pi)\); see [SZ]. Direct computation shows that
\[ |g''(\varphi_+)| = k \left( \frac{k}{n} - a_0 \right)^{1/2} \left( a_0^{-1} - \frac{k}{n} \right)^{1/2}. \]

We choose \( \delta = \delta(n) > 0 \) whose exact value is to be specified later. We split the integral
\[ \int_{0}^{\pi} e^{ig(t)} \, dt = \int_{0}^{\varphi_+ - \delta} e^{ig(t)} \, dt + \int_{\varphi_+ - \delta}^{\varphi_+ + \delta} e^{ig(t)} \, dt + \int_{\varphi_+ + \delta}^{\pi} e^{ig(t)} \, dt \]
and notice that
\[ \left| \int_{\varphi_+ - \delta}^{\varphi_+ + \delta} e^{ig(t)} \, dt \right| \leq 2\delta. \]

The remaining integrals are treated via Lemma 4. Since \( g' \) is decreasing on \((0, \pi)\) and \( g'(0) = a_0n - k \) we have
\[ \left| \int_{0}^{\varphi_+ - \delta} e^{ig(t)} \, dt \right| \leq \frac{1}{|g'(0)|} + \frac{1}{|g'(\varphi_+ - \delta)|} = \frac{1}{n(a_0^{-1} - k)} + \frac{1}{|g'(\varphi_+ - \delta)|}. \]
As always we assume that $k$ is closer to $\alpha_0 n$ so that $1/|n\alpha_0^{-1} - k| \leq 1/|n\alpha_0 - k|$ and we seek a suitable lower bound for $|g'(\varphi_+ - \delta)|$. This is achieved as follows. First we use the mean-value theorem for integrals to see that there is $s = s(n) \in (\varphi_+ - \delta, \varphi_+)$ with

$$|g'(\varphi_+ - \delta)| = \left| \int_{\varphi_+ - \delta}^{\varphi_+} g''(t) \, dt \right| = |g''(s)| \delta.$$

By the mean-value theorem for differentiation there exists also

$$u = u(n) \in (s, \varphi_+)$$

such that

$$\frac{|g''(\varphi_+) - g''(s)|}{|g''(\varphi_+)|} = \frac{(\varphi_+ - s)|g''(u)|}{|g''(\varphi_+)|} \lesssim \frac{\delta n}{k(\frac{k}{n} - \alpha_0)^{1/2}(\alpha_0^{-1} - \frac{k}{n})^{1/2}} \lesssim \frac{\delta}{(\frac{k}{n} - \alpha_0)^{1/2}}.$$

For the last inequality we use that $k/n \in (\alpha_0 + n^{-2/3}, \alpha_0^{-1} - n^{-2/3})$ is bounded from below. We also made use of the assumption that $k/n$ is closer to $\alpha_0$ than $\alpha_0^{-1}$, which implies that $(\frac{k}{n} - \alpha_0)^{-1/2}$ is bounded by a constant. In particular, assuming $\delta = \frac{1}{2(\frac{k}{n} - \alpha_0)^{1/2} n^{1/2}}$ we have $\delta \leq \frac{1}{2n^{1/3}}$ and

$$|g''(s)| \gtrsim |g''(\varphi_+)| \cdot \left| 1 - \frac{|g''(\varphi_+) - g''(s)|}{|g''(\varphi_+)|} \right| \gtrsim |g''(\varphi_+)|.$$

In summary we find

$$\left| \int_{0}^{\varphi_+ - \delta} e^{ig(t)} \, dt \right| \lesssim \frac{1}{k - \alpha_0 n} + \frac{1}{\delta n|g''(\varphi_+)|} \lesssim \frac{1}{k - \alpha_0 n} + \frac{1}{\delta n(\frac{k}{n} - \alpha_0)^{1/2}} \lesssim \frac{1}{k - \alpha_0 n} + \frac{1}{n^{1/2}(\frac{k}{n} - \alpha_0)^{1/4}}.$$

A similar reasoning applies to $\int_{\varphi_+ + \delta}^{\pi} e^{ig(t)} \, dt$. We obtain in total

$$\left| \int_{0}^{\pi} e^{ig(t)} \, dt \right| \lesssim \delta + \frac{1}{k - \alpha_0 n} + \frac{1}{n^{1/2}(\frac{k}{n} - \alpha_0)^{1/4}} \lesssim \frac{1}{k - \alpha_0 n} + \frac{1}{n^{1/2}(\frac{k}{n} - \alpha_0)^{1/4}} \lesssim \frac{1}{n^{1/2}(\frac{k}{n} - \alpha_0)^{1/4}} \left( 1 + \frac{1}{n^{1/2}(\frac{k}{n} - \alpha_0)^{3/4}} \right) \lesssim \frac{1}{n^{1/2}(\frac{k}{n} - \alpha_0)^{1/4}} \left( 1 + \frac{1}{n^{1/2}(\frac{k}{n} - \alpha_0)^{3/4}} \right) \lesssim \frac{1}{n^{1/2}(\frac{k}{n} - \alpha_0)^{1/4}}$$

which completes the proof.
Proof of upper bound in Theorem 1. We set \( \beta = \frac{a_0 + a_0^{-1}}{2} \) and split the sum
\[
\| b_n \|_p = \sum_{k \leq \beta n} |\hat{b}_n(k)|^p + \sum_{k > \beta n} |\hat{b}_n(k)|^p.
\]
For the proof we focus on the second sum, i.e., we assume that \( k \) is closer to \( a_0^{-1}n \) than to \( a_0 n \). (This is for completeness of the exposition and complementary to the proof of Proposition 3, where we focus on \( k \) closer to \( a_0 n \).) The discussion of the first sum is identical. Let \( \alpha < a_0 \). We split the sum over \( k > \beta n \) according to the regions of Proposition 3:
\[
\sum_{k > \beta n} |\hat{b}_n(k)|^p = \left( \sum_{\beta n < k \leq a_0^{-1}n - n^{1/3}} + \sum_{a_0^{-1}n - n^{1/3} < k \leq a_0^{-1}n + n^{1/3}} + \sum_{a_0^{-1}n + n^{1/3} < k \leq \alpha^{-1}n} + \sum_{\alpha^{-1}n < k} \right) |\hat{b}_n(k)|^p.
\]
We make use of the respective estimates of Proposition 3 to bound the individual sums.

- We begin by the “large values of \( k \)”, where coefficients decay exponentially. We shall prove
\[
\sum_{\alpha^{-1}n < k} |\hat{b}_n(k)|^p = O(n^{1/3} e^{-p(\alpha^{-1} - a_0^{-1})n^{2/3}}).
\]
Using the first estimate in (2.1) we find
\[
\sum_{\alpha^{-1}n < k} |\hat{b}_n(k)|^p \leq |b(s)|^{pn} \frac{s^{-pN_1}}{1 - s^{-p}}
\]
where \( N_1 = [\alpha^{-1}n] + 1 \). We choose the radius \( s = s_n = 1 + \frac{1}{n^{1/3}} \), which gives
\[
\frac{b(s_n)}{s_n^{\alpha_0^{-1}}} = 1 + \frac{\lambda(1 + \lambda)}{3(1 - \lambda)^3} + O\left(\frac{1}{n^{4/3}}\right).
\]
Moreover
\[
s_n^{-p} = 1 - \frac{p}{n^{1/3}} + O\left(\frac{1}{n^{2/3}}\right).
\]
In total we get
\[
\sum_{\alpha^{-1}n < k} |\hat{b}_n(k)|^p \leq \left( \frac{b(s_n)}{s_n^{\alpha_0^{-1}}} \right)^{pn} s_n^{-p(\alpha^{-1} - a_0^{-1})n} \frac{s_n^{-pN_1}}{1 - s_n^{-p}} \leq n^{1/3} e^{\frac{p\lambda(1 + \lambda)}{3(1 - \lambda)^3} \frac{s_n^{-p(\alpha^{-1} - a_0^{-1})n}}{p + O\left(\frac{1}{n^{1/3}}\right)}}
\]
\[
= O(n^{1/3} e^{-p(\alpha^{-1} - a_0^{-1})n^{2/3}}).
\]
We estimate
\[ \sum_{a_{0}^{-1}n+n^{1/3} < k \leq a_{0}^{-1}n} |\hat{b}(k)|^{p} \leq \frac{c^{p}}{n^{p-1}} \sum_{a_{0}^{-1}n+n^{1/3} < k \leq a_{0}^{-1}n} \frac{1}{(\frac{k}{n}-a_{0}^{-1})^{p}}. \]

We set \( f(t) = \frac{1}{(t-a_{0}^{-1})^{p}} \) and bound the Riemann sum
\[ \frac{1}{n} \sum_{a_{0}^{-1}n+n^{1/3} < k \leq a_{0}^{-1}n} \int_{a_{0}^{-1}n+n^{-2/3}}^{a_{0}^{-1}n} f(t) \, dt \leq \int_{n^{-2/3}}^{a_{0}^{-1}n} \frac{1}{u^{p}} \, du. \]

We find for \( p = 1 \)
\[ \sum_{a_{0}^{-1}n+n^{1/3} < k \leq a_{0}^{-1}n} |\hat{b}(k)| \lesssim \log n, \]
and for \( p \neq 1 \)
\[ \sum_{a_{0}^{-1}n+n^{1/3} < k \leq a_{0}^{-1}n} |\hat{b}(k)|^{p} \lesssim \frac{1}{n^{p-1}} \frac{n^{2p-2}}{n^{p-1}} = \frac{1}{1 - \frac{1}{n^{p-3}}}. \]

We estimate
\[ \sum_{a_{0}^{-1}n^{-1/3} < k \leq a_{0}^{-1}n+n^{1/3}} |\hat{b}(k)|^{p} = \sum_{|k/n-a_{0}^{-1}| < \frac{n^{1/3}}{p^{3/2}}} |\hat{b}(k)|^{p} \leq \frac{c^{p}}{n^{p/3}} \sum_{|k/n-a_{0}^{-1}| < \frac{n^{1/3}}{p^{3/2}}} 1 \lesssim \frac{1}{n^{p-3}}. \]

We estimate
\[ \sum_{\beta n < k \leq a_{0}^{-1}n-n^{1/3}} |\hat{b}(k)|^{p} \leq \frac{c^{p}}{n^{p/2-1}} \sum_{\beta n < k \leq a_{0}^{-1}n-n^{1/3}} \frac{1}{(\alpha_{0}^{-1} - \beta)^{p/4}}. \]

We set \( g(t) = \frac{1}{(\alpha_{0}^{-1} - t)^{p/4}} \) and we bound the Riemann sum
\[ \frac{1}{n} \sum_{\beta < k/n \leq a_{0}^{-1}n-2^{1/3}} g\left(\frac{k}{n}\right) \lesssim \int_{\beta}^{a_{0}^{-1}n-2^{1/3}} g(t) \, dt \lesssim \int_{n^{-2/3}}^{a_{0}^{-1}n-\beta} \frac{1}{u^{p/4}} \, du, \]

where \( a_{0}^{-1} - \beta = a_{0}^{-1} - \frac{a_{0}^{-1}+a_{0}^{-1}}{2} = \frac{2a_{0}^{-1}}{2} > 0 \). Evaluating the integral gives
the following upper estimates. If \( p = 4 \) then
\[ \sum_{\beta n < k \leq a_{0}^{-1}n-n^{1/3}} |\hat{b}(k)|^{4} \lesssim \frac{\log n}{n}, \]
if $p > 4$ then
\[ \sum_{\beta n < k \leq a_6^i n - n^{1/3}} |\hat{b}^n(k)|^p \lesssim \frac{1}{p - 4} \frac{n^{\frac{p-4}{p}}}{n^{p/2-1}} = \frac{1}{p - 4} \frac{1}{n^{\frac{p-4}{p}}}, \]
and if $p < 4$ then
\[ \sum_{\beta n < k \leq a_6^i n - n^{1/3}} |\hat{b}^n(k)|^p \lesssim \frac{1}{4 - p} \frac{1}{n^{p/2-1}}. \]
\[ \square \]

3 Lower estimates

Before going into the details of a technical discussion of sharpness in Theorem 1 we summarize some known facts and provide a simple argument for sharpness in the interval $p \in (4/3, 4)$. The discussion of sharpness in the interval $p \in (4, \infty)$ is built on an expansion of the Taylor coefficients of $b^n$ in terms of the Airy function. The most complicated case turns out to be the boundary case $p = 4$, which is treated separately in the end. We notice that the upper bound in Theorem 1 is sharp for $p = \infty$ as a consequence of [SZ, Theorem 2, point (2)]. In that reference the behavior of the largest coefficient, $\sup_{k \geq 0} |\hat{b}^n(k)|$, is analyzed and it is shown that to first order we have $\sup_{k \geq 0} |\hat{b}^n(k)| \sim n^{-1/3}$. For the case $p = 1$ the limit $\lim_{n \to \infty} \|b^n\|_{L^1_\mu} / \sqrt{n}$ is computed in [GD, BaH]. The case $p = 2$ is trivial and a direct consequence of Plancherel’s theorem
\[ \|b^n\|_{L^2} = \sum_{k \geq 0} |\hat{b}^n(k)|^2 = (b^n, b^n)_{L^2(\partial D)} = 1, \]
where the usual scalar product $(\cdot, \cdot)_{L^2(\partial D)}$ on $L^2(\partial D)$ coincides with the Cauchy duality $(\cdot, \cdot)_{L^2(\partial D)}$
\[ (3.1) \quad \langle f, g \rangle = \sum_{j \geq 0} \hat{f}(j) \overline{\hat{g}(j)}, \]
whenever the sum makes sense for two holomorphic functions $f = \sum_k \hat{f}(k)z^k$ and $g = \sum_k \hat{g}(k)z^k$ on $\mathbb{D}$, and the last step makes use of the fact that $b^n(z) = 1/b^n(z)$ for $z \in \mathbb{C}$. Lower estimates are derived in the whole range $p \in [1, 2]$ in [BS], which imply that the theorem is sharp in this region. The asymptotic statement of Theorem 1 is actually sharp for $p \in [1, 4)$. This is a consequence of the upper estimate in Theorem 1.

**Corollary 5.** For $p \in [1, 4)$ we have
\[ \|b^n\|_{L^p_\mu} \asymp c_p n^{\frac{2-p}{2p}}. \]
Here is an elementary proof which holds curiously for \( p \in \left( \frac{4}{3}, 4 \right) \) only. (The rest of the interval is covered already in [BS] and there is no need to reproduce the discussion.)

**Proof of Corollary 5.** Let \( p' \) denote the Hölder conjugate of \( p \), i.e., \( \frac{1}{p} + \frac{1}{p'} = 1 \). By assumption \( \frac{1}{p} > 1 - \frac{3}{4} \) and \( p' < 4 \). It follows from the upper estimates in Theorem 1 (proved in the previous section) that

\[
\|b^n\|_{l_p'} \leq Cp' n^{\frac{2-p'}{2p'}} = C_p n^{\frac{p-2}{2p'}}.
\]

A straightforward application of Hölder’s inequality gives

\[
1 = \|b^n\|_{l_2}^2 \leq \|b^n\|_{l_p'} \|b^n\|_{l_{p'}}.
\]

We conclude that

\[
\|b^n\|_{l_p'} \geq \frac{1}{\|b^n\|_{l_{p'}}} \geq \tilde{C}_p n^{\frac{2-p'}{2p'}}
\]

with \( \tilde{C}_p = \frac{1}{C_p} \).

For our discussion of sharpness we can henceforth assume that \( p \in [4, \infty] \). We have already seen in Proposition 3 point (1) that if \( k = k(n) \) is a sequence with \( k/n \leq a \) or \( k/n \geq a^{-1} \) for some \( a \in (0, a_0 = \frac{1-\lambda}{1+\lambda}) \), then \( |\hat{b}^n(k)| \) decays exponentially as \( n \to \infty \). This means that for any \( p \in [1, \infty] \) the main contribution in the \( l_p \)-norms of \( \hat{b}^n \) is due to a critical range of \( k \) with \( k \in [a_0 n, a_0^{-1} n] \). In this section we compute an asymptotic expansion of \( \hat{b}^n(k) \) as \( k \) and \( n \) tend simultaneously to \( \infty \) and \( k \) approaches the right boundary of \([a_0 n, a_0^{-1} n]\) from inside:

\[
\lim_{n \to \infty} \left( a_0^{-1} - \frac{k}{n} \right) = 0^+.
\]

In this region the asymptotic behavior of \( \hat{b}^n(k) \) can be written in terms of the Airy function \( Ai(x) \). For real arguments the latter can be defined as an improper Riemann integral

\[
Ai(x) = \frac{1}{\pi} \int_0^\infty \cos \left( \frac{t^3}{3} + xt \right) dt.
\]

For us the most interesting will be the oscillatory behavior of \( Ai \) for large negative arguments. We have the asymptotic approximation

\[
Ai(-x) \sim \frac{1}{x^{1/4} \sqrt{\pi}} \cos \left( \frac{2}{3} x^{3/2} - \frac{\pi}{4} \right), \quad x \to +\infty.
\]

**Proposition 6** (Asymptotic expansion of \( \hat{b}^n(k) \) for \( k \) in a left neighborhood of \( a_0^{-1} n \)). Let \( b = b_\lambda \) with \( \lambda \in (0, 1) \) and \( n \geq 1 \). Consider sequences \( k = k(n) \) with
$k \in [a_0 n, a_0^{-1} n]$ such that $\lim_{n \to \infty} \frac{k}{n} = a_0^{-1}$. For the Taylor coefficients of $b^n$ we have the following asymptotic expansion as $n \to \infty$:

$$
\hat{b}^n(k) = \frac{(1 - \lambda)^{1/4}}{(\lambda(1 + \lambda))^{1/12}} \frac{\sqrt{2}}{\sqrt{\frac{k}{n} - \frac{1}{\lambda} - \frac{1}{\lambda + 1}}^{1/4}} \frac{\text{Ai}(\lambda n^{2/3} \gamma^2)}{n^{1/3}} (1 + O(n^{-1/3})) ,
$$

where $\gamma^2 = \frac{1 - \lambda}{(\lambda(1 + \lambda))^{1/12}} (a_0^{-1} - \frac{k}{n}) + o((a_0^{-1} - \frac{k}{n}))$.

The slowest decay of $\hat{b}^n(k)$ occurs at the boundary $k = a_0^{-1} n$. Here the supremum $|\hat{b}^n(k)|$ is attained and corresponds to the $l_\infty^A$-norm. In this situation we can recover some of the findings of [SZ, Proposition 4]. We find that the boundary behavior as $n$ gets large (at $k = a_0^{-1} n$) is

$$
\hat{b}^n(k) \sim \frac{(1 - \lambda)^{1/4}}{(\lambda(1 + \lambda))^{1/12}} \frac{\sqrt{2}}{\sqrt{\frac{k}{n} - \frac{1}{\lambda} - \frac{1}{\lambda + 1}}^{1/4}} \frac{\text{Ai}(0)}{n^{1/3}},
$$

which proves (as already shown in [SZ]) that $\|b^n\|_{l_\infty^A} \sim n^{-1/3}$. This can be extended to the whole interval $p \in (4, \infty]$. We find the corresponding corollary to Proposition 6 and Proposition 3.

**Corollary 7.** For $p \in (4, \infty]$,

$$
\|b^n\|_{l_p^A} \sim n^{\frac{1-p}{3p}} .
$$

**Proof of Corollary 7.** We consider $p \in (4, \infty)$ as the case $p = \infty$ is clear from above. Since $\lim_{n \to \infty} (a_0^{-1} - \frac{k}{n}) = 0$ the prefactor in Proposition 6 is comparable to a positive constant

$$
\frac{(1 - \lambda)^{1/4}}{(\lambda(1 + \lambda))^{1/12}} \frac{\sqrt{2}}{\sqrt{\frac{k}{n} - \frac{1}{\lambda} - \frac{1}{\lambda + 1}}^{1/4}} \times 1.
$$

We consider the set $I_n$ of integers in $[a_0^{-1} n - cn^{1/3}, a_0^{-1} n]$,

$$
I_n = \mathbb{N} \cap [a_0^{-1} n - cn^{1/3}, a_0^{-1} n],
$$

where the constant $c > 0$ is chosen such that $n^{2/3} \gamma^2 < 2$ for $k \in I_n$ and $n$ is large enough. Explicitly this condition reads as

$$
n^{2/3} \gamma^2 \sim \frac{1 - \lambda}{(\lambda(1 + \lambda))^{1/3}(a_0^{-1} - \frac{k}{n})n^{2/3}} \leq c \frac{1 - \lambda}{(\lambda(1 + \lambda))^{1/3}} < 2,
$$

$$
\text{Ai}(\lambda n^{2/3} \gamma^2)
$$

where $\gamma^2 = \frac{1 - \lambda}{(\lambda(1 + \lambda))^{1/12}} (a_0^{-1} - \frac{k}{n}) + o((a_0^{-1} - \frac{k}{n}))$. The slowest decay of $\hat{b}^n(k)$ occurs at the boundary $k = a_0^{-1} n$. Here the supremum $|\hat{b}^n(k)|$ is attained and corresponds to the $l_\infty^A$-norm. In this situation we can recover some of the findings of [SZ, Proposition 4]. We find that the boundary behavior as $n$ gets large (at $k = a_0^{-1} n$) is

$$
\hat{b}^n(k) \sim \frac{(1 - \lambda)^{1/4}}{(\lambda(1 + \lambda))^{1/12}} \frac{\sqrt{2}}{\sqrt{\frac{k}{n} - \frac{1}{\lambda} - \frac{1}{\lambda + 1}}^{1/4}} \frac{\text{Ai}(0)}{n^{1/3}},
$$

which proves (as already shown in [SZ]) that $\|b^n\|_{l_\infty^A} \sim n^{-1/3}$. This can be extended to the whole interval $p \in (4, \infty]$. We find the corresponding corollary to Proposition 6 and Proposition 3.

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$$
\frac{(1 - \lambda)^{1/4}}{(\lambda(1 + \lambda))^{1/12}} \frac{\sqrt{2}}{\sqrt{\frac{k}{n} - \frac{1}{\lambda} - \frac{1}{\lambda + 1}}^{1/4}} \times 1.
$$

We consider the set $I_n$ of integers in $[a_0^{-1} n - cn^{1/3}, a_0^{-1} n]$,

$$
I_n = \mathbb{N} \cap [a_0^{-1} n - cn^{1/3}, a_0^{-1} n],
$$

where the constant $c > 0$ is chosen such that $n^{2/3} \gamma^2 < 2$ for $k \in I_n$ and $n$ is large enough. Explicitly this condition reads as

$$
n^{2/3} \gamma^2 \sim \frac{1 - \lambda}{(\lambda(1 + \lambda))^{1/3}(a_0^{-1} - \frac{k}{n})n^{2/3}} \leq c \frac{1 - \lambda}{(\lambda(1 + \lambda))^{1/3}} < 2,
$$

$$
\text{Ai}(\lambda n^{2/3} \gamma^2)
$$
i.e., we can choose $c < 2\frac{(1+\hat{\lambda})^{1/3}}{1-\hat{\lambda}}$. This choice of $c$ ensures that for $k \in I_n$ the quantity $-n^{2/3} \gamma^2$ lies in the compact interval $[-2, 0]$ on which the Airy function takes values that are separated from 0,

$$|Ai(-n^{2/3} \gamma^2)| \geq \min_{\xi \in [-2,0]} |Ai(\xi)| > 0.$$ 

(The first negative zero of the Airy function occurs at approximately $-2.33811$.) In other words, from Proposition 6 we have for sufficiently large $n$ and all $k \in I_n$ an estimate of the form

$$|\hat{b}^n(k)| \geq \frac{K_1}{n^{1/3}}(1 + \mathcal{O}(n^{-1/3}))$$

with a constant $K_1 > 0$. Therefore we have

$$\|b^p\|_p^p \gtrsim \frac{\#I_n}{n^{p/3}} \gtrsim \frac{n^{1/3}}{n^{p/3}}.$$ 

Notice that for $p < 4$ we have that $\frac{2-p}{2p} > \frac{1-p}{3p}$ such that the argument given above does not reach the lower bound of Corollary 5 for the interval $p \in [1, 4)$. Since Proposition 6 provides the exact asymptotic behavior, we can conclude that the dominant contribution to the $l_p$ norms of $\hat{b}^n$ does not come from the interval $I_n$ when $p \in [1, 4)$. Instead, for $p \geq 4$ estimates are achieved in regions $III$ and $V$ of Table 1. As we will see in this situation the main contribution to

$$\hat{b}^n(k) = \overline{\hat{b}^n(k)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} b^{-n}(e^{i\theta})e^{ik\theta} d\theta$$

comes from a small interval around $\varphi = 0$; see [SZ, Proposition 4, point 2]). For technical convenience we focus our analysis on the integral representation of $\overline{\hat{b}^n(k)}$, which is the same as $\hat{b}^n(k)$ as $\hat{\lambda}$ is real. We fix $\varepsilon \in (0, \pi)$ and split $\hat{b}^n(k)$ as

$$\hat{b}^n(k) = \frac{1}{2\pi} \int_{-\varepsilon}^{\varepsilon} b^{-n}(z)z^k \bigg|_{z=e^{i\varphi}} d\varphi + \frac{1}{2i\pi} \int_{\varepsilon \mathbb{D} \setminus e_{\varepsilon}} b^{-n}(z)z^{k-1} dz,$$

where $e_{\varepsilon} = \{ z = e^{i\varphi} | \varphi \in (-\varepsilon, \varepsilon) \}$. We write the integrals in a way that is convenient for asymptotic analysis. We introduce a function $h_{\alpha}$ with $\alpha \in \mathbb{R}^+$ and

$$h_{\alpha}(z) = -i \log \left( \frac{z^\alpha(1-\hat{\lambda}z)}{z-\hat{\lambda}} \right) = \text{Arg} \left( \frac{z^\alpha(1-\hat{\lambda}z)}{z-\hat{\lambda}} \right),$$

where log denotes the principal branch of the complex logarithm. We have

$$\frac{1}{2i\pi} \int_{\varepsilon \mathbb{D} \setminus e_{\varepsilon}} b^{-n}(z)z^{k-1} dz = \frac{1}{\pi} \text{Re} \left\{ \int_{\varepsilon}^{\varepsilon} z^k \bigg|_{z=e^{i\varphi}}^\varphi d\varphi \right\}$$

$$= \frac{1}{\pi} \text{Re} \left\{ \int_{\varepsilon}^{\varepsilon} e^{ih_{\alpha}(z)} \bigg|_{z=e^{i\varphi}} d\varphi \right\}.$$ 

To prove Proposition 6 we proceed by the following steps:
(1) We prove that 
\[ \frac{1}{\pi} \int_{\varepsilon}^{\pi} b^n(z) z^{-k-1} d\phi = O\left(\frac{1}{n}\right), \]
where we make use of a van der Corput type lemma; see Lemma 8 below.

(2) We compute an asymptotic expansion for 
\[ \frac{1}{\pi} \int_{-\varepsilon}^{\varepsilon} b^{-n}(z) z^k |_{z=e^{i\phi}} d\phi \]
relying on the so-called uniform method of stationary phase [BV, Section 2.3, p. 41]. The technical core of the latter will be a locally one-to-one cubic transformation of the integrand’s argument following the methods of [CFU].

**Lemma 8.** Let \( k(n) \) be a sequence that approaches \( \alpha_0^{-1} n \) from the left, i.e., \( \alpha_0^{-1} - \frac{k}{n} \to 0^+ \). Given fixed \( \varepsilon \in (0, \pi) \) we have as \( n \to \infty \) that
\[ \hat{b}^n(k) = \frac{1}{2\pi} \int_{-\varepsilon}^{\varepsilon} e^{inh_{k/n}(\varepsilon)} |_{z=e^{i\phi}} d\phi + O\left(\frac{1}{n}\right). \]

**Proof.** We define a function \( \tilde{h}_d \) on \((-\pi, \pi)\) by setting \( \tilde{h}_d(\varphi) := h_d(e^{i\varphi}) \). Deriving \( h \) with respect to \( \varphi \) we find
\[ \frac{\partial \tilde{h}_{k/n}}{\partial \varphi}(\varphi) = \frac{k}{n} - \frac{1 - \lambda^2}{1 + \lambda^2 - 2\lambda \cos \varphi}, \]
For \( \varphi \in [\varepsilon, \pi) \) we have
\[ n \frac{\partial \tilde{h}_{k/n}}{\partial \varphi}(\varphi) \geq n \left( \frac{k}{n} - \frac{1 - \lambda^2}{1 + \lambda^2 - 2\lambda \cos \varepsilon} \right) = n \left( \alpha_0^{-1} - \frac{1 - \lambda^2}{1 + \lambda^2 - 2\lambda \cos \varepsilon} - \varepsilon_n \right) \]
\[ = n \left( \frac{2\lambda(1 + \lambda)(1 - \cos \varepsilon)}{(1 + \lambda^2 - 2\lambda \cos \varepsilon)(1 - \lambda)} - \varepsilon_n \right), \]
where
\[ \varepsilon_n := \alpha_0^{-1} - \frac{k}{n} \to 0^+ \]
as \( n \) tends to \( \infty \). For \( n \) sufficiently large
\[ n \frac{\partial \tilde{h}_{k/n}}{\partial \varphi}(\varphi) \geq \text{constant} \ast n \]
for any \( \varphi \in [\varepsilon, \pi) \). Therefore an application of Lemma 4 provides
\[ \int_{-\varepsilon}^{\varepsilon} e^{inh_{k/n}(\varepsilon)} |_{z=e^{i\phi}} d\phi = O\left(\frac{1}{n}\right). \]

The next step is to compute an asymptotic expansion of
\[ J_n(k) = \frac{1}{2\pi} \int_{-\varepsilon}^{\varepsilon} e^{inh_{k/n}(\varepsilon)} |_{z=e^{i\phi}} d\phi. \]
We will see that \( J_n(k) \) is well suited for an application of the uniform method of stationary phase [BV, Section 2.3, p. 41], which is based in turn on a locally
one-to-one cubic transformation of $h_a$, which is described in [CFU] and [WR, p. 366], [BIHa, p. 369]. In order to apply the result from [CFU] we perform a locally one-to-one cubic transformation to the function $h_a$. First we notice that both $\alpha_0$ and $\alpha_0^{-1}$ are critical values in the sense that for $a \notin \{\alpha_0, \alpha_0^{-1}\}$ the function $h_a$ has two distinct saddle points $z_+$ and $z_-$, of rank 1. However, if $a \in \{\alpha_0, \alpha_0^{-1}\}$ the points $z_+$ and $z_-$ merge to a single saddle point $z_0$ (respectively $\tilde{z}_0$) of rank 2. For notational convenience we shall write the function $h_a$ with an additional argument instead of the index, $h_a(z, a)$. To be precise the conditions for the mentioned saddle points read

\begin{align}
\frac{\partial h}{\partial z}(z_+, a) = \frac{\partial h}{\partial z}(z_-, a) = 0, \quad \frac{\partial^2 h}{\partial z^2}(z_+, a) \neq 0
\end{align}

for $a \notin \{\alpha_0, \alpha_0^{-1}\}$ for saddle points $z_+$, $z_-$ of rank one and

\begin{align}
\frac{\partial h}{\partial z}(z_0, a_0) = \frac{\partial^2 h}{\partial z^2}(z_0, a_0) = 0, \quad \frac{\partial^3 h}{\partial z^3}(z_0, a_0) \neq 0,
\end{align}

respectively

\begin{align}
\frac{\partial h}{\partial z}(\tilde{z}_0, a_0^{-1}) = \frac{\partial^2 h}{\partial z^2}(\tilde{z}_0, a_0^{-1}) = 0, \quad \frac{\partial^3 h}{\partial z^3}(\tilde{z}_0, a_0^{-1}) \neq 0
\end{align}

for saddle points of rank 2. Computing derivatives we find

\begin{align}
\frac{i}{\partial z} = -\frac{1}{z-\lambda} + a \frac{1}{z} - \frac{\lambda}{z-\lambda}, \\
\frac{i}{\partial z^2} = -\frac{a}{z-\lambda} - \frac{\lambda^2}{z^2} - \frac{1 - \lambda z}{z-\lambda}, \\
\frac{i}{\partial z^3} = \frac{2a}{(z-\lambda)z^3} - \frac{2\lambda^2}{z^3} - \frac{2}{(z-\lambda)^3}.
\end{align}

The function $h(z, a)$ has a stationary point if and only if $\frac{\partial h}{\partial z} = 0$, i.e., iff

\begin{align}
a = 1 + \frac{\lambda}{z-\lambda} + \frac{\lambda z}{1-\lambda z}.
\end{align}

Solving the latter for $z$ gives

\begin{align}
z_\pm = \frac{a(1 + \lambda^2) - (1 - \lambda^2)}{2\lambda a} \pm i \sqrt{1 - \left(\frac{a(1 + \lambda^2) - (1 - \lambda^2)}{2\lambda a}\right)^2} \in \partial \mathbb{D}
\end{align}

and we write $z_\pm = e^{i\varphi_\pm}$ with $\varphi_+ \in [0, \pi]$ and $\varphi_- \in (-\pi, 0]$. Observe that $\varphi_+ = \varphi_+(a)$, $\varphi_- = -\varphi_+$ and

\begin{align}
\cos^2 \varphi_+ = \frac{a(1 + \lambda^2) - (1 - \lambda^2)}{2\lambda a}.
\end{align}
We distinguish the two cases (1) $a \in (a_0, a_0^{-1})$ and (2) $a \in \{a_0, a_0^{-1}\}$, which are characterized by the presence of a stationary point of order one ($\frac{\partial h}{\partial \zeta}(z_\pm, a) = 0$ but $\frac{\partial^2 h}{\partial \zeta^2}(z_\pm, a) \neq 0$) in Case (1) and of order two ($\frac{\partial h}{\partial \zeta}(z_\pm, a) = 0$ but $\frac{\partial^2 h}{\partial \zeta^2}(z_\pm, a) \neq 0$) in Case (2).

**Case (1)** If $a \in (a_0, a_0^{-1})$ then the zeros $z_+ = e^{i\varphi_+}$ and $z_- = e^{i\varphi_-}$ of $\frac{\partial h}{\partial \zeta}$ are distinct points located on $\partial \mathbb{D}$ with $\varphi_+ \in [0, \pi]$ and $\varphi_- \in (-\pi, 0]$. Plugging in we see that

$$
(3.4) \quad i \frac{\partial^2 h}{\partial \zeta^2} \bigg|_{\zeta = z_\pm} = \frac{(1 - \lambda^2)(1 - z_\pm^2)\lambda}{z_\pm(z_\pm - \lambda)^2(1 - \lambda z_\pm)^2}.
$$

**Case (2)** If $a \in \{a_0, a_0^{-1}\}$, then $\frac{\partial h}{\partial \zeta}$ has a unique zero. If $a = a_0^{-1}$, then $z_+ = z_- = 1 = \tilde{z}_0$ and

$$
 h(1, a_0^{-1}) = \frac{\partial h}{\partial \zeta}(1, a_0^{-1}) = \frac{\partial^2 h}{\partial \zeta^2}(1, a_0^{-1}) = 0,
$$

with

$$
 i \frac{\partial^3 h}{\partial \zeta^3}(1, a_0^{-1}) = -\frac{2\lambda(1 + \lambda)}{(1 - \lambda)^3} \neq 0.
$$

If $a = a_0$, then $z_+ = z_- = -1 = z_0$ and

$$
 i \frac{\partial h}{\partial \zeta}(-1, a_0) = i \frac{\partial^2 h}{\partial \zeta^2}(-1, a_0) = 0, \quad i \frac{\partial^3 h}{\partial \zeta^3}(-1, a_0) = -\frac{2\lambda(1 - \lambda)}{(1 + \lambda)^3} \neq 0.
$$

The contour of integration $\mathcal{C}_e$ in $J_e(k)$ is chosen so that it is located in a neighborhood of $\tilde{z}_0 = 1$. This leads to considering the right boundary $a_0^{-1}$ so that $z_+$ and $z_-$ lie in $\mathcal{C}_e$ for some $\varepsilon > 0$ and $a_0$ close to $a_0^{-1}$; (3.3) and (3.2) are clearly satisfied.

We are now ready to perform the one-to-one cubic transformation of [CFU]. The proposition below is from [CFU] stated in the formulation of [WR, Theorem 1, p. 368].

**Proposition 9** ([CFU]). For a near $a_0^{-1}$, the cubic transformation

$$
 h(z, a) = \frac{t^3}{3} - \gamma^2 t
$$

with

$$
 \gamma^2 = \frac{(a_0^{-1} - a)(1 - \lambda)}{(\lambda(1 + \lambda))^{1/3}} + o((a_0^{-1} - a))
$$

has exactly one branch $t = \tau(z, a)$ that can be expanded into a power series in $z$ with coefficients that are continuous in $a$. On this branch the points $z = z_\pm$ correspond to $t = \pm \gamma$. The mapping of $z$ to $t$ is one-to-one on a small neighborhood $\mathbb{D}_\pm$ of $z = 1$ containing $z_+$ and $z_-$. 
We prove Proposition 9 in the appendix for completeness. With the developed theory we are ready to conclude the proof of Proposition 6. We will apply the uniform method of stationary phase [BV, Section 2.3, pp. 41–44] to
\[
J_n(k) = \frac{1}{2\pi} \int_{-\epsilon}^{\epsilon} e^{i\Omega(z_+)} \left|_{z=i\epsilon}^{z=\epsilon} \right. \text{d}\varphi
\]
making use of the above one-to-one cubic transformation of \(h\).

**Proof of Proposition 6.** We will rely on Lemma 8. Differentiating \(\tilde{h}\) with respect to \(\varphi\) we have
\[
\frac{\partial^2 \tilde{h}}{\partial \varphi^2} = \frac{\partial}{\partial \varphi} \left( \frac{\partial h}{\partial z} \frac{\partial z}{\partial \varphi} \right) = \frac{\partial^2 h}{\partial z^2} \left( \frac{\partial z}{\partial \varphi} \right)^2 + \frac{\partial h}{\partial z} \frac{\partial^2 z}{\partial \varphi^2}.
\]
Computation shows that
\[
\frac{\partial^2 \tilde{h}}{\partial \varphi^2} (\varphi_+, a) = \frac{2 \Im(z_+)(1 - \lambda^2)\lambda}{|1 - \lambda z_+|^4} > 0
\]
and
\[
\frac{\partial^2 \tilde{h}}{\partial \varphi^2} (\varphi_-, a) = - \frac{2 \Im(z_+)(1 - \lambda^2)\lambda}{|1 - \lambda z_+|^4} < 0
\]
which shows that \(J_n(k)\) is perfectly suited for applying the approach in [BV, Section 2.3, p. 41] (with \(x_1 = \varphi_-\) and \(x_2 = \varphi_+\)). Observe that
\[
|1 - \lambda z_\pm| = \sqrt{\frac{1 - \lambda^2}{\alpha}}
\]
and
\[
2 \Im(z_+) = 2 \sqrt{1 - \left( \frac{a(1 + \lambda^2) - (1 - \lambda^2)}{2\lambda a} \right)^2}
\]
\[
= \sqrt{(1 - \lambda^2)(a(1 + \lambda) - (1 - \lambda))(1 + \lambda) - a(1 - \lambda))}
\]
\[
= \frac{1 - \lambda^2}{\lambda a} \sqrt{(a - a_0)(a_0^{-1} - a)}.
\]
This gives
\[
\frac{\partial^2 \tilde{h}}{\partial \varphi^2} (\varphi_\pm, a) = \pm a \sqrt{(a - a_0)(a_0^{-1} - a)}.
\]
A straightforward application of [BV, formula (2.36), p. 43] gives
\[
J_n(k) = \left( \frac{Ai(-n^{2/3} \gamma^2)}{n^{1/3}} A_0 + \frac{Ai'(-n^{2/3} \gamma^2)}{n^{2/3}} A_1 \right) (1 + O(n^{-1/3}))
\]
where $A_0$ and $A_1$ are given (see [BV, formula (2.36c), p. 44]) by

$$A_0 = \frac{\sqrt{7}}{\sqrt{2}} \left( \frac{1}{\sqrt{2\lambda^2(\varphi_+ - a)}} + \frac{1}{\sqrt{2\lambda^2(\varphi_+ - a)}} \right) = \frac{\sqrt{2\gamma}}{\sqrt{a(a - a_0)^{1/4}(a_0^1 - a)^{1/4}}}$$

and

$$A_1 = \frac{(1 - \lambda)^{1/4}}{\sqrt{2}} \left( \frac{1}{\sqrt{2\lambda^2(\varphi_+ - a)}} - \frac{1}{\sqrt{2\lambda^2(\varphi_+ - a)}} \right) = 0.$$  

This yields

$$J_n(k) = \frac{(1 - \lambda)^{1/4}}{\sqrt{2}} \left( \frac{\sqrt{2}}{(\lambda(1 + \lambda))^{1/12}} \sqrt{a(a - a_0)^{1/4}} \right) \frac{\text{Ai}(n^{2/3} \gamma^2)^2}{n^{1/3}} (1 + O(n^{-1/3}))$$

and the result follows. □

We conclude our analysis with the discussion of the lower bound for $p = 4$.

**Proposition 10.** For $p = 4$ we have

$$\|b^n\|_{l_4^n} \asymp \left( \frac{\log n}{n} \right)^{1/4}.$$  

**Proof.** The upper bound $\|b^n\|_{l_4^n} \lesssim \left( \frac{\log n}{n} \right)^{1/4}$ is shown in Section 2. The proof of the lower bound will be concluded in four steps.

**Step 1. Application of Theorem 6 and summation over a suitable range of $k$.**

Given $\lambda \in (0, 1)$ we recall the notation we finally choose: $a_0 = \frac{1 - k}{1 + k}$. We define $I_n$ to be the following set of integers:

$$I_n = \mathbb{Z} \cap [a_0^{-1} n - n^{3/4}, a_0^{-1} n - \sqrt{n}].$$

A direct application of Theorem 6 gives

$$\sum_{k \in I_n} |\hat{b}(k)|^4 \asymp \sum_{k \in I_n} \frac{1}{(\lambda)^2(\frac{k}{n} - a_0)} \frac{(\text{Ai}(-n^{2/3} \gamma^2))^4}{n^{4/3}}$$

where $\gamma^2 = \frac{1 - k}{(\lambda(1 + \lambda))^{1/12}}(a_0^{-1} - k)$. Plugging in the oscillatory behavior of $\text{Ai}$,

$$\text{Ai}(-x) = \frac{1}{x^{1/4}\sqrt{\pi}} \cos \left( \frac{2}{3} x^{3/2} - \frac{\pi}{4} \right) + O(x^{-7/4}), \quad x \to +\infty,$$

we find that

$$\sum_{k \in I_n} |\hat{b}(k)|^4 \gtrsim \frac{1}{n^{4/3}} \sum_{k \in I_n} \frac{1}{(\lambda)^2(\frac{k}{n} - a_0)} \frac{1}{n^{2/3} \gamma^2 \pi^2} \cos^4 \left( \frac{2}{3} n \gamma^3 - \frac{\pi}{4} \right)$$

$$\gtrsim \frac{1}{n^2} \sum_{k \in I_n} \gamma^2 \cos^4 \left( \frac{2}{3} n \gamma^3 - \frac{\pi}{4} \right).$$
because \( \lim_{n \to \infty} \frac{k}{n} = \alpha_0^{-1} \) and \( \lambda \) is fixed in \((0,1)\). Plugging in the value of \( \gamma \) this yields

\[
\sum_{k \in I_n} |\hat{\phi}(k)|^4 \geq \frac{1}{n^2} \sum_{k \in I_n} \left( \alpha_0^{-1} - \frac{k}{n} \right) \times \cos^4 \left( \frac{2}{3} n \left( 1 - \lambda \right)^{3/2} \left( \alpha_0^{-1} - \frac{k}{n} \right)^{3/2} + \mathcal{O} \left( \left( \alpha_0^{-1} - \frac{k}{n} \right)^{5/2} \right) - \frac{\pi}{4} \right) = \frac{1}{n^2} \sum_{k \in I_n} \left( \alpha_0^{-1} - \frac{k}{n} \right) \cos^4 \left( \pi \frac{s_{nk} - \gamma}{4} \right)
\]

where \( s_{nk} = \langle n \varphi(\frac{k}{n}) \rangle \), \( \langle s \rangle \) denoting the fractional part of \( s \) and

\[
\varphi(t) \sim \frac{2}{3 \pi} \frac{(1 - \lambda)^{3/2}}{(\lambda(1 + \lambda))^{1/2}} \left( \alpha_0^{-1} - t \right)^{3/2}.
\]

**Step 2. Equidistribution of \( s_{nk} \).**

Given \( j \neq 0 \) in \( \mathbb{Z} \) and \( k \) in \( I_n \) we consider

\[
A_k := \sum_{l \in I_n, l \leq k} \exp(2\pi i j s_{nl}) = \sum_{l \in I_n, l \leq k} \exp(2\pi i j n \varphi(\frac{l}{n})).
\]

To estimate the above sum we use one of van der Corput’s lemma [ZA, Ch. 5, Lemma 4.6]: if \( f''(x) \geq \mu > 0 \) or \( f''(x) \leq -\mu < 0 \) on \([a, b]\), then

\[
\left| \sum_{a < k \leq b} \exp(2\pi i f(k)) \right| \leq (|f'(b) - f'(a)| + 2) \left( \frac{4}{\sqrt{\mu}} + A \right),
\]

where \( A \) is an absolute constant. We shall apply this lemma to the case \( f(x) = j n \varphi(\frac{x}{n}) \), \( b = \alpha_0^{-1} n - \sqrt{n} \) and \( a = \alpha_0^{-1} n - n^{3/4} \). We have \( f'(x) = j \varphi'(\frac{x}{n}) \) and \( f''(x) = \frac{j}{n} \varphi''(\frac{x}{n}) \) where \( \varphi'(t) \sim -\frac{1}{\pi} \frac{(1 - \lambda)^{3/2}}{(\lambda(1 + \lambda))^{1/2}} (\alpha_0^{-1} - t)^{1/2} \) and \( \varphi''(t) \sim \frac{1}{2\pi} \frac{(1 - \lambda)^{3/2}}{(\lambda(1 + \lambda))^{1/2}} (\alpha_0^{-1} - t)^{-1/2} \). In particular, for \( k \in I_n \) we have \( \frac{1}{\sqrt{n}} \leq \alpha_0^{-1} - \frac{k}{n} \leq \frac{1}{n^{3/4}} \), so that \( f''(a) = o(1), f''(b) = o(1) \) as \( n \) tends to \( \infty \) and

\[
f''(x) \sim \frac{j}{n^{7/8}}, \quad x \in [a, b], j \geq 1,
\]

\[
f''(x) \sim -\frac{|j|}{n^{7/8}}, \quad x \in [a, b], j \leq -1.
\]

For \( n \) large enough we obtain that for any \( k \in I_n \)

\[
|A_k| \lesssim n^{7/16}.
\]
Step 3. Approximation by trigonometric polynomials and Abel’s transformation.

We reproduce and adapt the proof of [GD, Lemma 3]. We define
\[ g(x) = \cos^4(\pi(x - \frac{1}{4})). \]

Our aim is to prove that there exists a limit:
\[
\lim_{n \to \infty} \frac{1}{n \log n} \sum_{k \in I_n} \frac{g(s_{nk})}{(a_0^{-1} - \frac{k}{n})} = \int_0^1 g(x) \, dx.
\]

We observe that \( g \) is continuous and 1-periodic on the real line. In particular, according to Fejér’s Theorem, for any \( \epsilon > 0 \) there is a trigonometric polynomial \( p(x) = \sum_{|j| \leq N} c_j \exp(2\pi ijx) \) such that for \( x \in [0, 1] \), \( -\epsilon \leq g(x) - p(x) \leq \epsilon \).

We put \( M = \sum_{k \in I_n} \frac{1}{(a_0^{-1} - \frac{k}{n})} \). Using the monotonicity and positivity of \( t \mapsto \frac{1}{a_0^{-1} - t} \) we compare \( M \) with an integral to obtain \( M \sim n \to \infty n^{-1/4} n \int_{a_0^{-1} - n^{-1/4}}^{a_0^{-1} - n^{-1/2}} \frac{dt}{(a_0^{-1} - t)} \). After a change of variable we find \( M \sim n \to \infty n \int_{n^{-1/4}}^{n^{-1/2}} \frac{du}{u} \), that is to say
\[
M \sim n \to \infty n \log n \frac{4}{n}.
\]

We now write
\[
M^{-1} \sum_{k \in I_n} \frac{p(s_{nk})}{(a_0^{-1} - \frac{k}{n})} = \sum_{|j| \leq N} c_j M^{-1} \sum_{k \in I_n} \frac{1}{(a_0^{-1} - \frac{k}{n})} \exp(2\pi ijs_{nk}),
\]
and by applying Abel’s summation formula we get for \( j \neq 0 \)
\[
\sum_{k \in I_n} \frac{1}{(a_0^{-1} - \frac{k}{n})} \exp(2\pi ijs_{nk}) = \sum_{k \in I_n} \frac{1}{(a_0^{-1} - \frac{k}{n})} (A_k - A_{k-1})
= \left[ \frac{A_{a_0^{-1}n - \sqrt{n}}}{(a_0^{-1} - \frac{a_0^{-1}n - \sqrt{n} + 1}{n})} - \frac{A_{a_0^{-1}n - n^{3/4}}}{(a_0^{-1} - \frac{a_0^{-1}n - n^{3/4} + 1}{n})} \right]
- \sum_{k \in I_n} A_k \left( \frac{1}{(a_0^{-1} - \frac{k+1}{n})} - \frac{1}{(a_0^{-1} - \frac{k}{n})} \right).
\]
This yields
\[
M^{-1} \sum_{k \in I_n} \frac{1}{\alpha_0 - \frac{k}{n}} \exp(2\pi ij s_{nk})
\]
\[
= O\left(\frac{n^{1/4}}{n \log(n)}\right) + O\left(\frac{n^{1/4}n^{7/16}}{n \log(n)}\right) + O\left(\frac{n^{7/16}}{n \log(n)}\right) \sum_{k \in I_n} \left(\frac{1}{\alpha_0 - \frac{k+1}{n}} - \frac{1}{\alpha_0 - \frac{k}{n}}\right)
\]
\[
= O\left(\frac{1}{n^{1/16} \log(n)}\right) + O\left(\frac{n^{7/16}}{n \log(n)}\right) \left[ \frac{1}{\alpha_0 - \frac{[\alpha_0^{-1}n - \sqrt{n}]^2+1}{n}} - \frac{1}{\alpha_0 - \frac{[\alpha_0^{-1}n - n^{7/4}]^2+1}{n}} \right]
\]
\[
= O\left(\frac{1}{n^{1/16} \log(n)}\right).
\]
In particular
\[
\lim_{n \to \infty} M^{-1} \sum_{k \in I_n} \frac{p(s_{nk})}{\alpha_0^{-1} - \frac{k}{n}} = c_0 = \int_0^1 p(x) dx,
\]
but by construction on the interval \( [0, 1] \) \( p - \epsilon \leq g \leq \epsilon + p \) and therefore
\[
M^{-1} \sum_{k \in I_n} \frac{p(s_{nk})}{\alpha_0^{-1} - \frac{k}{n}} - \epsilon \leq M^{-1} \sum_{k \in I_n} \frac{g(s_{nk})}{\alpha_0^{-1} - \frac{k}{n}} \leq M^{-1} \sum_{k \in I_n} \frac{p(s_{nk})}{\alpha_0^{-1} - \frac{k}{n}} + \epsilon.
\]
Passing over to the limit as \( n \) tends to \( \infty \) we get
\[
\int_0^1 p(x) dx - \epsilon \leq \limsup_{n \to \infty} M^{-1} \sum_{k \in I_n} \frac{g(s_{nk})}{\alpha_0^{-1} - \frac{k}{n}} \leq \int_0^1 p(x) dx + \epsilon.
\]
Substracting \( \int_0^1 g(x) dx \) this yields
\[
\limsup_{n \to \infty} M^{-1} \sum_{k \in I_n} \frac{g(s_{nk})}{\alpha_0^{-1} - \frac{k}{n}} - \int_0^1 g(x) dx \leq \int_0^1 (p(x) - g(x)) dx + \epsilon \leq 2\epsilon
\]
on the one hand and
\[
-2\epsilon \leq \int_0^1 (p(x) - g(x)) dx - \epsilon \leq \limsup_{n \to \infty} M^{-1} \sum_{k \in I_n} \frac{g(s_{nk})}{\alpha_0^{-1} - \frac{k}{n}} - \int_0^1 g(x) dx
\]
on the other hand. Since \( \epsilon \) is arbitrarily small we obtain
\[
\limsup_{n \to \infty} M^{-1} \sum_{k \in I_n} \frac{g(s_{nk})}{\alpha_0^{-1} - \frac{k}{n}} = \int_0^1 g(x) dx
\]
and in the same way
\[
\liminf_{n \to \infty} M^{-1} \sum_{k \in I_n} \frac{g(s_{nk})}{\alpha_0^{-1} - \frac{k}{n}} = \int_0^1 g(x) dx.
\]
The result follows.
Step 4. Conclusion

With the result from Step 3 and going back to Step 1

\[
\sum_{k \in I_n} |\hat{b}_n(k)|^4 \gtrsim \frac{1}{n^2} \sum_{k \in I_n} \frac{1}{(a_0^{-1} - \frac{k}{n})^4} \cos^4 \left( \pi s_{nk} - \frac{\pi}{4} \right)
\]

\[
= \frac{1}{n^2} \sum_{k \in I_n} \frac{g(s_{nk})}{(a_0^{-1} - \frac{k}{n})^4} \gtrsim \frac{\log n}{n},
\]

which completes the proof. \(\square\)

4 Proof of Proposition 2

Proof of Proposition 2. Due to rotation invariance of \(l_A^p\) we have

\[
\text{cap}_{l_A^p}(\sigma_{\lambda,n}) = \text{cap}_{l_A^p}(\sigma_{|\lambda|,n}) = \inf \{ \| f \|_{l_A^p} : f(0) = 1, \ f^{(j)}(|\lambda|) = 0, \ 0 \leq j < n \}
\]

and therefore assume without loss of generality that \(\lambda \in (0, 1)\).

We first prove the upper bounds in (1), (2), (3), (4), (5). A direct application of Theorem 1 with the test function \(f = b_n^{\lambda}/(-|\lambda|)^n\) shows that

\[
\text{cap}_{l_A^p}(\sigma_{\lambda,n}) \leq \frac{1}{|\lambda|^n} \begin{cases} 
  c_p n^{\frac{2-p}{2p}} & \text{if } p \in [1, 4), \\
  c_4 (\log n)^{\frac{1}{4}} & \text{if } p = 4, \\
  c_p n^{-\frac{1-p}{2p}} & \text{if } p \in (4, \infty].
\end{cases}
\]

We prove the lower bounds in (1), (2), (3), (4), (5). Let \(f = \sum_k \hat{f}(k)z^k\) and \(g = \sum_k \hat{g}(k)z^k\) be holomorphic functions on \(\mathbb{D}\). A simple way to lower bound the norm of a function in \(l_A^p\) is to apply Hölder’s inequality in the form

\[
|\langle f, g \rangle| \leq \| f \|_{l_A^p} \| g \|_{l_A^q}
\]

where \(\langle \cdot, \cdot \rangle\) is the Cauchy duality defined in (3.1) and \(1/p + 1/q = 1\). In other words, the dual space \((l_A^p)^*\) of \(l_A^p\) with respect to Cauchy duality is simply \(l_A^q\). Suppose now that \(f \in l_A^p\) is given with \(f^{(j)}(|\lambda|) = 0, \ 0 \leq j < n \) and \(f(0) = 1\). We put \(b = b_{|\lambda|}\). Since \(l_A^p\) enjoys the division property, i.e.,

\[
[f \in l_A^p \text{ and } f(w) = 0 \text{ for some } w \in \mathbb{D}] \implies g : z \mapsto \frac{f(z)}{z - w} \in l_A^p,
\]

we have \(f/b^n \in l_A^p\) and

\[
\langle f, b^n \rangle = \langle f/b^n, 1 \rangle = \frac{f(0)}{(-|\lambda|)^n}.
\]
Applying (4.1) we obtain

$$\|f\|_{L^p} \geq \frac{1}{|\lambda|^n \|b^n\|_{L^q}}.$$  

Taking the infimum over all $f \in L^1_p$ with $f^{(j)}(|\lambda|) = 0$, $0 \leq j < n$ and $f(0) = 1$ this yields

$$\text{cap}_p(\sigma_{\lambda,n}) \geq \frac{1}{|\lambda|^n \|b^n\|_{L^q}}.$$  

We first assume $p > 4/3$. Then we have $1/q > 1 - 3/4$ and therefore $q < 4$, which yields $|\lambda|^n \text{cap}_p(\sigma_{\lambda,n}) \gtrsim n^{\frac{4-q}{2}}$ since $\|b^n\|_{L^q} \lesssim n^{\frac{2-q}{2}}$. If $p = 4/3$ then $q = 4$ and $|\lambda|^n \text{cap}_p(\sigma_{\lambda,n}) \gtrsim (\frac{n}{\text{log} n})^{\frac{4}{3}}$ since $\|b^n\|_{L^q} \lesssim (\frac{\text{log} n}{n})^{\frac{4}{3}}$. Finally if $p \in [1, 4/3)$ then $|\lambda|^n \text{cap}_p(\sigma_{\lambda,n}) \gtrsim n^{\frac{4}{3p}}$ since $\|b^n\|_{L^q} \lesssim n^{\frac{4}{3p}}$.  

5 Appendix

Proof of Proposition 9. Following [CFU] let us define $z(t)$ by the equation

$$h(z, a) = \frac{t^3}{3} - \gamma^2 t + \rho$$

where $\gamma = \gamma(a)$ and $\rho = \rho(a)$ are to be determined. This transformation is shown in [CFU] to be locally one-to-one and analytic for all $a$ in a neighborhood of $a_0^{-1}$. Taking derivatives with respect to $t$ we obtain

$$z'(t) = \frac{t^2 - \gamma^2}{\frac{\partial h}{\partial z}(z(t), a)}.$$  

Since $t \mapsto z(t)$ should be a conformal map we require that $z'$ is finite and nonzero. We see from the above equality that difficulties can only arise when $z = z_{\pm}$ and when $t = \pm \gamma$. We change variables such that we have $t = \pm \gamma$ when $z = z_{\pm}$. More precisely it follows from [CFU] that (see [WR, Theorem 1, p. 368]):

1. the parameters $\gamma = \gamma(a)$ and $\rho = \rho(a)$ can be explicitly determined so that the transformation (5.1) has exactly one branch $t = t(z, a)$ which can be expanded into a power series in $z$, with coefficients which are continuous in $a$ for $a$ near $a_0^{-1}$,

2. on this branch the points $z = z_{\pm}$ correspond to $t = \pm \gamma$ respectively,

3. for $a$ near $a_0^{-1}$ the correspondence $t$ to $z$ is locally one-to-one, from a neighborhood of 0 onto a neighborhood of 1; the two simple saddle points $z_+ = z(t_+)$ and $z_- = z(t_-)$ for $z \mapsto f(z, a)$ (resp. $t_+ = \gamma$ and $t_- = -\gamma$ for $t \mapsto \frac{t^3}{3} - \gamma^2 t + \rho$) coalesce to a single saddle point of order 2 when $z_+ = z_- = 1 = z_0$ or equivalently when $t_+ = t_- = \gamma = 0$.  

\[ l_p\text{-NORMS OF FOURIER COEFFICIENTS 27 } \]
Determination of $\gamma$ and $\rho$. It follows from (5.1) that
\[ h(z(t_+), a) = h(z(t_-), a) = -\frac{2\gamma^3}{3} + \rho \]
and
\[ h(z(t_-), a) = h(z(-\gamma), a) = \frac{2\gamma^3}{3} + \rho, \]
so that
\[ \gamma^3 = 3 \left( h(z_-, a) - h(z_+, a) \right) \]
and
\[ \rho = \frac{1}{2} \left( h(z_+, a) + h(z_-, a) \right). \]
Since $a < a_0^{-1} z_+$, and $z_-$ belong to the unit circle and $z_+ = \overline{z_-}$. In particular,
\[ \frac{z^a(1 - \lambda z_+)}{z_+ - \lambda} = e^{ih(z_+, a)} \]
implies
\[ \frac{z^a(1 - \lambda z_-)}{z_- - \lambda} = e^{-ih(z_+, a)}, \]
which gives us
\[ \gamma^3 = -\frac{3}{2} h(z_+, a), \quad \rho = 0. \]
Substituting the value of $z_+$ and using numerical plotting it is a triviality to see that since $a < a_0^{-1}$ we have $\gamma^3 > 0$. Expanding into a power series in $(a_0^{-1} - a)$ as $a$ approaches $a_0^{-1}$ we obtain that
\[ \gamma^3 = \frac{(1 - \lambda)^{3/2}}{\sqrt{\lambda(1 + \lambda)}}(a_0^{-1} - a)^{3/2} + o((a_0^{-1} - a)^{3/2}). \]
We observe that $\gamma$ is not uniquely determined by the above equality. Indeed when $z_+ \neq z_-$ it defines three values of $\gamma$. We discuss below this ambiguity and compute $\arg \gamma$ following the method developed in [BIHa]. First we write
\[ \hat{\gamma}^a(k) = \frac{1}{2i\pi} \oint_{\partial D} \exp(\text{inh}(z, a)) \frac{dz}{z}, \quad a = k/n. \]
The authors in [BIHa] compute $\arg \gamma$ by developing contours $C_j, j = 1, 2, 3$ such that $C_1$ (resp. $C_2, C_3$) starts at the point at infinity with argument $-2\pi/3$ (resp. $2\pi/3, 0$) and ends at the point at infinity with argument $2\pi/3$ (resp. $0, -2\pi/3$); see [BIHa, Figure 2.5, p. 51]. For each of the three possible choices of $\gamma$ the regular branch of $z(t)$ promised by [CFU] maps $\partial D \cap \mathbb{D}_\pm$ onto a contour asymptotically equivalent to one of $C_1 \cap \mathbb{D}_\pm, C_2 \cap \mathbb{D}_\pm$ or $C_3 \cap \mathbb{D}_\pm$. The authors in [BIHa] show it is correct
to choose the determination of $\gamma$ leading to an image contour asymptotically equivalent to $C_1 \cap \hat{D}_\pm$. Furthermore, they show that this choice of $\gamma$ is that satisfying the equation

$$\frac{\arg \Delta z}{2} + \frac{1}{2} \arg \left( \frac{1}{i} \frac{\partial h}{\partial z}(z_0, a) \right) - \frac{2\pi}{3} < \arg \gamma < \frac{\arg \Delta z}{2} + \frac{1}{2} \arg \left( \frac{1}{i} \frac{\partial h}{\partial z}(z_0, a) \right) - \frac{\pi}{3} \mod \pi,$$

where $z = z_0 = 1$ is the preimage of $t = 0$ and $\Delta z$ is an increment directed from $z = z_0 = 1$ to the contour $\partial \mathbb{D}$. In light of the fact that $z_0 = 1$, $\Delta z$ will be a negative real increment, so $\arg \Delta z = \pi$. Moreover, direct computation shows that $\frac{\partial h}{\partial z}(1, a) = i(a_0^{-1} - a)$. Substituting into the above equation we find that since $a < a_0^{-1}$,

$$\frac{\pi}{2} + 0 - \frac{2\pi}{3} < \arg \gamma < \frac{\pi}{2} + 0 - \frac{\pi}{3} \mod \pi,$$

so the argument of $\gamma$ is within $\frac{\pi}{6}$ or the argument of a real number and $\gamma^3 \in \mathbb{R}_+$ leads to $\gamma \in \mathbb{R}_+$. We conclude that

$$\gamma = \frac{(1 - \lambda)^{1/2}}{(\lambda(1 + \lambda))^{1/6}}(a_0^{-1} - a)^{1/2} + o((a_0^{-1} - a)^{1/2}).$$

□

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(Received May 20, 2017 and in revised form September 1, 2017)