Topological Landau-Ginzburg models on a world-sheet foam

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Abstract

We define topological Landau-Ginzburg models on a world-sheet foam, that is, on a collection of 2-dimensional surfaces whose boundaries are sewn together along the edges of a graph. We use matrix factorizations in order to formulate the boundary conditions at these edges and produce a formula for the correlators. Finally, we present the gluing formulas, which correspond to various ways in which the pieces of a world-sheet foam can be joined together.

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1. Introduction

It is always easier to define a quantum field theory on a closed manifold: there is no need to formulate the boundary conditions for the fields in the path integral. If the boundary exists, then one might limit oneself to the easiest case of the Neumann boundary conditions. The last decade showed, however, that the world of boundary conditions may be even more interesting and diverse than the world of the ‘bulk’ QFTs. A bulk 2-dimensional QFT yields an algebra, boundary conditions are objects of a category, and all morphisms between two objects form a module over the bulk algebra. Now it turns out that the existence of a rather general class of boundary conditions may change the very nature of the world-sheet manifold: instead of being just a surface with boundary, it may become a ‘foam’, that is, a version of a 2-dimensional $CW$-complex endowed with a complex structure, if needed.

Although the QFTs themselves do not require the presence of a world-sheet foam, a foam appeared in the paper [4] as a necessary element in the categorification of the SU(3) HOMFLY polynomial. The paper [6] interpreted that part of the categorification as a 2-dimensional topological A-model defined on a world-sheet foam. Each 2-dimensional connected component $\Sigma_i$ of the foam carries its own topological $\sigma$-model, whose target space is the complex Grassmannian $\text{Gr}_{m,n}$, while the boundary condition at an edge of the seam graph is specified by selecting a Lagrangian submanifold in the cross-product of the Grassmannians assigned to the components $\Sigma_i$ bounding the edge.

The paper [6] described the general setup of a QFT on a world-sheet foam, using topological A-models as an illustration. Topological A-models, however, are notorious for their complexity even on the usual smooth surfaces, and the paper [6] presented neither an accurate description of the Hilbert spaces corresponding to the seam graph vertices, nor a complete formula for the partition function.

In this paper we give a detailed description of a topological Landau-Ginzburg model on a world-sheet foam. Each component $\Sigma_i$ of the foam carries its own fields $\phi_i = \phi_{i,1}, \ldots, \phi_{i,m_i}$ and its own potential $W_i(\phi_i)$, while each edge of the seam graph carries a matrix factorization of the sum of potentials of the bounding components $\Sigma_i$, in the spirit of [3]. We describe the Hilbert spaces of the vertices of the seam graph and also provide a formula for the correlators, which generalizes the formulas of Vafa [7] and Kapustin-Li [3]. Finally, we present the gluing formulas for joining ‘space-like’ and even ‘time-like’ boundary components of the world-sheet foam.

This paper is closely related to our categorification [5] of the SU($N$) HOMFLY polynomial. Although we do not use world-sheet foams explicitly in [5], the construction of the graded vector spaces associated to 3-valent graphs in [5] is a particular case of the definition of an
operator space $H_\gamma$ related to a decorated local graph $\gamma$, as described in subsection 5.2. We refer the reader to [5] for a detailed discussion of matrix factorizations.

2. A topological LG theory on a world-sheet with a boundary

According to [6], any QFT defined on a 2-dimensional surface with boundary can be transferred to a world-sheet foam. Hence we begin by reviewing the topological LG theory $\mathcal{T}$ on a surface with boundary, as presented in [2] (see also references therein). We assume for simplicity that the target space of $\mathcal{T}$ is a flat $\mathbb{C}^m$ and there are no gauge fields. Then the bulk theory is characterized by a (polynomial) super-potential $W \in \mathbb{C}[\phi]$, $\phi = \phi_1, \ldots, \phi_m$, and we denote this Landau-Ginzburg theory by $\mathcal{T} = (\phi; W)$.

2.1. The bulk Lagrangian. The topological LG theory $\mathcal{T}$ contains bosonic fields $\phi$ and $\bar{\phi}$ as well as the fermionic fields $\eta^i$, $\theta_i$, $\rho^i_z$ and $\rho^i_{\bar{z}}$. The bulk Lagrangian of the theory is

$$L_\mathcal{T} = \frac{1}{2} \left( \partial_z \phi^i \partial_{\bar{z}} \bar{\phi}^i + \partial_{\bar{z}} \bar{\phi}^i \partial_z \phi^i - \rho^i_z \partial_{\bar{z}} \eta^i - \rho^i_{\bar{z}} \partial_z \bar{\eta}^i \right)$$

$$- 2i \left( \theta_i \left( \partial_z \rho^i_{\bar{z}} - \partial_{\bar{z}} \rho^i_z \right) + \partial_i \partial_j W \rho^i_z \rho^j_{\bar{z}} \right)$$

$$+ \frac{1}{4} \left( \partial_i \partial_j W \theta_i \eta^j - \partial_i W \partial_j W \right).$$

(2.1)

Each line in this expression is invariant under the topological BRST transformation $Q$

$$\delta_Q \phi^i = 0 \quad \delta_Q \bar{\phi}^i = \eta^i$$

$$\delta_Q \eta^i = 0 \quad \delta_Q \theta_i = \partial_i W$$

$$\delta_Q \rho^i_z = \partial_z \phi^i \quad \delta_Q \rho^i_{\bar{z}} = \partial_{\bar{z}} \phi^i,$$

(2.2)

except that the second line generates the boundary Warner term: for a world-sheet $\Sigma$ with a boundary $\partial \Sigma$ the BRST variation of the action is

$$\delta_Q \int_{\Sigma} L_\mathcal{T} = \int_{\partial \Sigma} \partial_i W (\rho^i_z dz + \rho^i_{\bar{z}} d\bar{z}).$$

(2.3)

Note that if we treat $(\rho^i_z, \rho^i_{\bar{z}})$ as a 1-form on $\Sigma$, then the Lagrangian (2.1) can be written without a reference to the complex structure of the world-sheet. The only remnant of that complex structure would be the orientation that it induces on $\Sigma$. Let $L_\mathcal{T}$ denote the Lagrangian (2.1) in which we conjugated the complex structure or, equivalently, reversed the orientation of the world-sheet. It is easy to see that this change can be compensated by switching two signs: the sign of the field $\theta_i$ (that is, $\theta_i$ is a ‘pseudo-scalar’) and the sign of the super-potential $W$. Thus,

$$L_{(\eta; W)} = L_{(\phi; -W)}.$$

(2.4)
2.2. **The boundary Wilson line.** M. Kontsevich suggested that the Warner term (2.3) could be compensated by putting the appropriate Wilson lines at the boundary components of $\Sigma$. This idea was implemented in papers [3], [1] and [2]. We will follow the approach of Lazaroiu [2] as the most suitable for our purposes.

The linear space for a LG Wilson line is provided by a matrix factorization of the superpotential $W$. According to [3], a matrix factorization of $W$ is a triple $(M, D, W)$, where $M$ is a finite-dimensional $\mathbb{Z}_2$-graded free $\mathbb{C}[\phi]$ module, $M = M^0 \oplus M^1$ ($\text{rank } M^0 = \text{rank } M^1$), while the twisted differential $D$ is an operator $D \in \text{End}(M)$, such that $\deg D = 1$ and

$$D^2 = W \text{Id}. \quad (2.5)$$

Simply saying, a matrix factorization is described by a $2n \times 2n$-dimensional matrix with polynomial entries

$$D = \begin{pmatrix} 0 & F \\ G & 0 \end{pmatrix}, \quad \text{such that} \quad FG = GF = W \text{Id}. \quad (2.6)$$

Lazaroiu introduces a connection

$$A_T = \begin{pmatrix} W \text{Id} \\ (\rho^i \partial_i F - \rho^i \partial_i G) \end{pmatrix} \begin{pmatrix} (\rho^i \partial_i F + \rho^i \partial_i G) \partial_i F \\ W \text{Id} \end{pmatrix} \quad (2.7)$$

acting on $M$. Suppose that $\partial \Sigma$ is a union of disjoint circles:

$$\partial \Sigma = \bigsqcup_k C_k. \quad (2.8)$$

To each circle $C_k$ we assign a matrix factorization $(M_k, D_k, W)$ of $W$. Then Lazaroiu proves that the path integrand

$$\exp \left( \int_{\Sigma} L_T \right) \prod_k W_{C_k}, \quad W_{C_k} = \text{STr}_{M_k} \exp \oint_{C_k} A, \quad (2.9)$$

with the orientation of $C_k$ induced by the orientation of $\Sigma$, is invariant under the topological BRST transformation (2.2).

An orientation of a boundary component $C_k$ can be reversed without affecting its supertrace, if we replace the associated matrix factorization $(M_k, D_k, W)$ with its dual. First, observe that the matrix factorizations can be tensored:

$$(M_1, D_1, W_1) \otimes (M_2, D_2, W_2) = (M_1 \otimes M_2, D_1 + D_2, W_1 + W_2). \quad (2.10)$$

Then we define the dual matrix factorization as

$$(M, D, W)^* = (M^*, D^*, -W). \quad (2.11)$$
where the module $M^*$ is the dual of $M$ over $\mathbb{C}[\phi]$ and

$$D^* = \begin{pmatrix} 0 & G^* \\ -F^* & 0 \end{pmatrix},$$

(2.12)

where $F^*$ and $G^*$ are the dual maps (transposed matrices) of $F$ and $G$. The choice (2.12) guarantees that the natural pairing map $M^* \otimes M \xrightarrow{f} \mathbb{C}[\phi]$ satisfies the property $f D = 0$ and thus ‘commutes’ with $D$.

2.3. Boundary operators. In order to simplify our notations, whenever we work with multiple matrix factorizations, we will denote all their twisted differentials by $D$, if it is clear on which particular module that $D$ is acting. Also, let $[\cdot, \cdot]_s$ denote the super-commutator:

$$[A, B]_s = AB - (-1)^{\deg A \deg B} BA.$$  

(2.13)

The super-traces of (2.9) have an obvious generalization. Let $t \in [0, 1]$ parameterize a boundary circle $C$. The values $0 = t_0 \leq t_1 \leq \cdots \leq t_n = 1$ split $C$ into $n$ segments. We can assign any matrix factorizations $(M_j, D_j, W_j)$ to the segments $[t_{j-1}, t_j]$. To each value $t_j$ we assign an operator $O_j \in \text{Hom}(M_j, M_{j+1})$, which super-commutes with the twisted differential $D$: $[D, O_j]_s = 0$. Then the Wilson line contribution $W_C$ can be replaced in the integrand (2.9) by

$$W_C = \text{Str}_{M_1} \left( \text{Pexp} \int_{[t_0, t_1]} A \right) O_1 \cdots \left( \text{Pexp} \int_{[t_{n-1}, t_n]} A \right) O_n,$$  

(2.14)

while still maintaining the BRST invariance.

Following [3], let us give a more precise description of the space of the operators $O_j$. For two matrix factorizations $(M_0, D_0, W)$ and $(M_1, D_1, W)$ consider the $\mathbb{Z}_2$-graded module $\text{Hom}(M_0, M_1)$ of $\mathbb{C}[\phi]$-linear maps between the modules. Define a differential $d$ on this module by

$$dO = [D, O]_s,$$  

(2.15)

where $O \in \text{Hom}(M_0, M_1)$. It turns out that $d^2 = 0$, and $d$ describes the BRST action on $\text{Hom}(M_0, M_1)$. The space $H_P$ of operators at the junction $P$ of two segments carrying the modules $M_0$ and $M_1$ can be presented as

$$H_P = \text{Ext}(M_0, M_1) = \ker d / \text{im} d.$$  

(2.16)

An equivalent presentation of the operator space comes from the dual matrix factorization. Namely, consider the tensor product of matrix factorizations

$$(M_1, D_1, W) \otimes (M_0, D_0, W)^* = (M_1 \otimes M_0^*, D, 0), \quad D = D_1 + D_0^*.$$  

(2.17)
Since $D^2 = 0$, we can take
\[ H_P = \ker D / \text{im } D, \tag{2.18} \]
because the cohomology of $D$ is canonically isomorphic to $\text{Ext}(M_1, M_0)$ in view of the canonical isomorphism
\[ \text{Hom}(M_0, M_1) = M_1 \otimes M_0^*, \tag{2.19} \]
and the fact that $d$ corresponds to $D$.

3. A topological LG theory on a world-sheet foam

3.1. The world-sheet foam. Let us recall the definition of a world-sheet foam given in [6]. Let $\Gamma$ be a graph such that every vertex has adjacent edges. $\Gamma$ is allowed to contain disjoint circles. A cycle on $\Gamma$ is defined to be either a disjoint circle or a cyclicly ordered finite sequence of edges, such that the beginning of the next edge corresponds to the end of the previous edge. Let $\Sigma$ be an orientable and possibly disconnected smooth 2-dimensional surface, its boundary $\partial \Sigma$ being a union of disjoint circles. A world-sheet foam $(\Sigma, \Gamma)$ is a union $\Gamma \cup \Sigma$, in which the boundary circles of $\Sigma$ are glued to some cycles on $\Gamma$ in such a way that every edge of $\Gamma$ is glued to at least one circle of $\partial \Sigma$.

Defining a topological LG theory on a world-sheet foam $(\Sigma, \Gamma)$ involves three steps: first, we assign bulk theories to oriented connected components of $\Sigma$; second, we assign appropriate boundary condition to the oriented seam edges, and third, we choose the operators at the seam vertices. The first two steps comprise a decoration of the world-sheet foam.

3.2. Bulk theories on the 2-dimensional connected components. The first step is simple. Suppose that the 2-dimensional surface $\Sigma$ is a union of $N_\Sigma$ disjoint components $\Sigma = \bigsqcup_{i=1}^{N_\Sigma} \Sigma_i$. Then to every oriented component $\Sigma_i$ we assign its own topological LG theory $T_i = (\phi_i; W_i)$ with its own target space $\mathbb{C}^{m_i}$, bosonic fields $\phi_i$, fermionic fields and a superpotential $W_i$ in such a way that if $\Sigma_i$ and $\Sigma_i$ represent the same component of $\Sigma$ with opposite orientations, then $\Sigma_i$ should be assigned the conjugated theory $\bar{T}_i$.

3.3. Boundary conditions at the seam graph edges. Next, we have to formulate the boundary conditions at the seam edges, which would be compatible with the BRST-invariance of the actions of Landau-Ginzburg theories sitting on the adjacent surfaces $\Sigma_i$. According to [6], these boundary conditions are just particular cases of an ordinary boundary condition for a Landau-Ginzburg theory defined on a smooth surface with a boundary.

Let us orient an edge $e$ of $\Gamma$ and let $\mathcal{I}_e$ be the set of indices $i$, such that $e$ bounds $\Sigma_i$. We orient all $\Sigma_i$ ($i \in \mathcal{I}_e$) in such a way that their orientations are compatible with the orientation
If \( P \) is an inner point of \( e \), then a small neighborhood of \( P \) looks like a union of upper half-planes \( \mathbb{H}^+_i \subset \mathbb{C} \) glued along the common real line, the point \( P \) being the origin of that real line. If \( \mathbb{H}^+ \) is a ‘standard’ upper half-plane, then we can identify all \( \mathbb{H}^+_i \) analytically (preserving orientation) with it. Thus, if every \( \mathbb{H}^+_i \) carries a QFT \( \mathcal{T}_i \) with the Lagrangian \( L_{\mathcal{T}_i} \), then formulating a boundary condition for them at \( e \) is equivalent to formulating it for the combined theory \( \mathcal{T}_e \) with the Lagrangian \( L_{\mathcal{T}_e} = \sum_{i \in \mathcal{I}_e} L_{\mathcal{T}_i} \). In case of the topological LG theories this means that to every seam edge \( e \) we assign the theory

\[
\mathcal{T}_e = (\phi_e; W_e), \quad \text{where} \quad \phi_e = (\phi | i \in \mathcal{I}_e), \quad W_e = \sum_{i \in \mathcal{I}_e} W_i. \tag{3.20}
\]

Then to every oriented edge \( e \) of the seam graph \( \Gamma \) we associate a matrix factorization \((M_e, D_e, W_e)\) in such a way that if two oriented edges \( e \) and \( e^* \) represent the same edge with opposite orientations, then \((M_{e^*}, D_{e^*}, W_{e^*}) = (M_e, D_e, W_e)^*\). Also, we assign to \( e \) the Lazaroiu connection \( A_e = A_{\mathcal{T}_e} \). If the matrix factorization \((M_e, D_e, W_e)\) does not factor into a tensor product of matrix factorizations of individual super-potentials \( W_i \), then the boundaries of the components \( \Sigma_i \) can not be ‘unglued’ at the edge \( e \).

It is important to note that the construction of a matrix factorization associated to a seam edge must be local. It may happen that because of the global structure of the world-sheet foam \((\Sigma, \Gamma)\), some of the strips that bound an edge \( e \) come from the same world-sheet component \( \Sigma_i \). In this case, we first treat their theories \( \mathcal{T}_i \) as different, that is, they share the same dimension of the target space \( m_i \) and the same super-potential \( W_i \), yet their target spaces and fields are considered distinct. After we pick a matrix factorization \((M_e, D_e, W_e)\), we impose a condition that the fields coming from the different strips of the same world-sheet component are the same.

### 3.4. Operators at the seam graph vertices

Let \( v \) be a seam graph vertex and let \( \mathcal{I}_v \) be the set of seam edges adjacent to \( v \). We orient these edges away from \( v \) and consider the factorization

\[
(M_v, D_v, W_v) = \bigotimes_{e \in \mathcal{I}_v} (M_e, D_e, W_e). \tag{3.21}
\]

Obviously,

\[
M_v = \bigotimes_{e \in \mathcal{I}_v} M_e, \quad W_v = 0, \tag{3.22}
\]

the latter equation follows, since for every component \( \Sigma_i \) attached to \( v \) there are two (or, more generally, an even number of) bounding edges, which are attached to \( v \) in such a way that \( \Sigma_i \) contributes an equal number of \( W_i \) and \(-W_i\) to \( W_v \).

Since \( W_v = 0 \), then \( D_v^2 = 0 \) and we can consider its cohomology

\[
H_v = \ker D_v / \text{im } D_v. \tag{3.23}
\]
The space \( H_v \) is \( \mathbb{Z}_2 \)-graded: \( H_v = H_v^0 \oplus H_v^1 \). \( D_v \) plays the role of the BRST operator at \( v \), so we take \( H_v \) as the space of operators at the vertex \( v \). In other words, to every vertex \( v \) we associate an element

\[
O_v \in \ker D_v, \tag{3.24}
\]

and the BRST-invariance of the path integral will guarantee that the correlators depend on \( O_v \) only modulo \( \text{im} \, D_v \).

3.5. Wilson network. For a world-sheet foam (\( \Sigma, \Gamma \)), the analog of the Wilson lines at the boundary components of the world-sheet \( \Sigma \) is the Wilson network formed by the seam graph \( \Gamma \). Its contribution is a generalization of the super-trace (2.14) and it is expressed through multiple contractions between the pairs of dual modules in a big tensor product

\[
\left( \bigotimes_{v \in \mathcal{V}} M_v \right) \otimes \left( \bigotimes_{e \in \mathcal{E}} (M_e \otimes M_e^*) \right), \tag{3.25}
\]

where \( \mathcal{V} \) and \( \mathcal{E} \) are the sets of all vertices and of all edges of \( \Gamma \). If we substitute the formulas (3.22) for \( M_v \), then all the elementary modules \( M_e \) and \( M_e^* \) can be grouped in pairs of mutually dual modules: \( M_e \) (or \( M_e^* \)) coming from \( M_e \otimes M_e^* \) and the dual module \( M_e^* \) (or \( M_e \)) coming from the tensor product expression (3.22) for \( M_v \), where \( v \) is the beginning (or the end point) of the oriented edge \( e \). Performing contractions within each pair, we get the map

\[
\left( \bigotimes_{v \in \mathcal{V}} M_v \right) \otimes \left( \bigotimes_{e \in \mathcal{E}} M_e \otimes M_e^* \right) \xrightarrow{f_{\Gamma}} \mathbb{C}[\phi_{\Sigma}], \tag{3.26}
\]

where

\[
\phi_{\Sigma} = \phi_1, \ldots, \phi_{N_{\Sigma}}, \tag{3.27}
\]

and \( N_{\Sigma} \) is the number of connected components \( \Sigma_i \) of \( \Sigma \). If we consider the Lazaroiu connection holonomy along an edge \( e \) to be the element of \( M_e \otimes M_e^* \), then the Wilson network contribution \( W_{\Gamma} \) can be expressed as the contraction map (3.26) applied to the tensor product of all the Lazaroiu holonomies and all the vertex operators \( O_v \):

\[
W_{\Gamma} = f_{\Gamma} \left( \left( \bigotimes_{v \in \mathcal{V}} O_v \right) \otimes \left( \bigotimes_{e \in \mathcal{E}} \text{Pexp} \int_e A_e \right) \right). \tag{3.28}
\]

Clearly, this expression is a generalization of the super-trace (2.14), if we assume that the seam graph \( \Gamma \) in the latter case consists of a disjoint union of cycles formed by the seam edges \([t_{j-1}, t_j]\).
3.6. **Operators at the internal points of the seam edges.** The network structure of the boundary Wilson lines of the world-sheet foam forces us to always include the seam vertex operators in any correlator on \((\Sigma, \Gamma)\). Moreover, note that the operator spaces \(H_v\) generally do not contain canonical elements, so there is no special choice for \(O_v\).

In addition to the required operators at the seam vertices, the correlators may also include the optional operators at the internal points of the seam edges \(e\) and at the internal points of the world-sheet components \(\Sigma_i\).

Let \(P\) be an internal point of a seam edge \(e\). In order to describe its space of local operators, we can simply declare it to be a new seam vertex, thus breaking the edge \(e\) into two consecutive edges. Then the space of the operators can be presented either in the form (2.16) or in the form (2.18), in which we substitute \(M_0 = M_1 = M_e\). Note that the space \(\text{Ext}(M_e, M_e)\) has a canonical element, which is the identity map.

3.7. **Jacobi algebra and the local operators of the bulk.** The description of the operator spaces for the internal points of the components \(\Sigma_i\) comes from the topological LG theories on closed surfaces. Namely, for a topological LG theory \((\phi; W)\)

\[
H_P = \mathbb{C}[\phi]/\partial W,
\]

where \(\partial W\) is the ideal of \(\mathbb{C}[\phi]\) generated by the first partial derivatives \(\partial_i W\) (indeed, \(H_P\) must be the cohomology of the BRST operator acting according to (2.2) on the algebra of the fields). The \(\mathbb{C}[\phi]\)-module \(H_P\) has an algebra structure and is called the Jacobi algebra of \(W\), so we will also denote it as \(J_W\).

The operator product expansion arguments show that for any world-sheet foam component \(\Sigma_i\), whose boundary passes through a seam vertex \(v\), the operator space \(H_v\) must be a module over the Jacobi algebras \(J_{W_i}\). The space \(H_v\) by its definition (3.23) is already a module over \(\mathbb{C}[\phi_i]\), so we have to check that if \(O_v \in \ker D_v\), then

\[
(\partial W_i/\partial \phi_{i,j}) O_v \in \text{im } D_v.
\]

(3.30)

The proof is based on a general relation

\[
\{\partial_j D, D\} = \partial_j W,
\]

(3.31)

which follows from eq. (2.5) by taking the derivative \(\partial_j\) of both sides. Now let \(e_0\) be an adjacent edge of \(v\), which is a part of the boundary \(\partial \Sigma_i\). Suppose that all edges adjacent to \(v\) go out of \(v\). Then \(D_v = \sum_{e \in I_v} D_e\). Since \(\{D_e, D_{e_0}\} = 0\) for all \(e \neq e_0\), eq.(3.31) implies that

\[
\{\partial_j D_e, D_v\} = \{\partial_j D_e, D_e\} = \partial_j W_e = \partial_j W_i,
\]

(3.32)
the latter equality following from the last equation of (3.20). Thus
\[ \partial_j W_e O_v = \{ \partial_j D_e, D_v \} O_v = (\partial_j D_e) D_v O_v + D_v (\partial_j D_e) O_v. \]
(3.33)
Since \( O_v \in \ker D_v \), the first term in the r.h.s. of this formula is zero. The second term belongs to \( \text{im} D_v \), and this proves our assertion.

3.8. Topological Landau-Ginzburg path integral. Finally, we combine all the data into the path integral, which represents the correlator of the operators \( O_v \) at the seam graph vertices and the operators \( O_P \) at the internal points of \( \Sigma \). The exponential part of the integrand is simply \( \exp (\sum_i L_{T_i}) \), whereas the operators \( O_P \) and the Wilson network contribution \( W_{\Gamma} \) provide the pre-exponential factors:
\[
\langle \prod_{P \in P} O_P \prod_{v \in V} O_v \rangle_{(\Sigma, \Gamma)} = \int \exp \left( \sum_i L_{T_i} \right) \left( \prod_P O_P \right) W_{\Gamma} \mathcal{D}\phi_{\Sigma} \mathcal{D}\eta_{\Sigma} \mathcal{D}\theta_{\Sigma} \mathcal{D}\rho_{\Sigma},
\]
(3.34)
where \((\text{field})_{\Sigma}\) means all fields of the given type from all the components \( \Sigma_i \).

3.9. An example of a topological LG theory on a world-sheet foam. Let us now consider a specific example of a topological LG theory which can be put on a world-sheet foam. In other words, we are going to present a set of topological LG theories \( T_i = (\phi_i; W_i) \) and some matrix factorizations of the sums of their super-potentials which do not factor into the tensor products of matrix factorizations of the individual super-potentials \( W_i \). This example is inspired by the construction of [4] and, following [8], we suggest that it is the mirror image of the world-sheet foam theory presented in [6]. Also, the constructions of our paper [5] are based on a particular case of the matrix factorizations described here.

Let us fix a positive integer \( N \) and a complex parameter \( a \). Following [8], for \( 1 \leq m \leq N-1 \) we consider the polynomial
\[
c_m(\phi_m; t) = 1 + \sum_{j=1}^m \phi_{m,j} t^j, \quad \phi_m = (\phi_{m,j} \mid 1 \leq j \leq m)
\]
(3.35)
and the expansion of its logarithm in power series of \( t \):
\[
\ln c_m(\phi_m; t) = \sum_{j=1}^{\infty} c_{m,j}(\phi_m) t^j.
\]
(3.36)
We set
\[
W_m(\phi_m) = (-1)^{N+1} c_{m,N+1}(\phi_m) - a \phi_{m,1},
\]
(3.37)
thus defining the topological LG theory \( T_m = (\phi_m; W_m) \) with the target space \( \mathbb{C}^m \). Its Jacobi algebra coincides with the quantum cohomology algebra of the complex Grassmannian \( \text{Gr}_{m,N} \).
The matrix factorizations that we are going to use belong to a special class sometimes called the ‘Koszul factorizations’. Here is the general construction. Suppose that a super-potential $W(\phi)$ factors over $\mathbb{C}[\phi]$

$$W = pq, \quad p, q \in \mathbb{C}[\phi].$$  

Then there exists a $(1|1)$-dimensional matrix factorization of $W$ with the twisted differential

$$D = \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix}.$$  

We denote this matrix factorization by $(p; q)$. If $W(\phi)$ can be presented as a sum of products

$$W = \sum_{j=1}^{n} p_j q_j, \quad p, q \in \mathbb{C}[\phi],$$

then there is a $(2^{n-1}|2^{n-1})$-dimensional factorization of $W$

$$(p; q) = \bigotimes_{j=1}^{n} (p_j; q_j).$$

Now we come back to the potentials (3.37). For a list of integer numbers $m$ such that

$$\sum_{i} m_i = N,$$

we are going to construct a Koszul matrix factorization of the super-potential

$$W_m = \sum_{i} W_{m_i}$$

by presenting it in the form (3.40). Consider the polynomial $W_N(\tilde{p})$ as defined by eqs.(3.35)--(3.37), in which the variables $\phi$ are replaced by the variables $\tilde{p} = (p_j \mid 1 \leq j \leq N)$. The equation

$$\prod_{i} c_{m_i}(\phi_{m_i}; t) = 1 + \sum_{j=1}^{N} p_j(\phi) t^j$$

defines $\tilde{p}$ as polynomial functions of all variables $\phi = (\phi_{m_i} \mid i)$. In particular,

$$p_1 = \sum_{i} \phi_{m_i,1},$$

and it is easy to verify that

$$W_m(\phi) = W_N(\tilde{p}(\phi)).$$

Consider again the polynomial $W_N(\tilde{p})$. If we assign degrees to the variables $\tilde{p}$ as $\text{deg} p_j = j$, then $W_N$ is a homogeneous polynomial of degree $N + 1$. Therefore, each monomial of $W_N(\tilde{p})$
is proportional to at least one variable \( p = (p_j | 1 \leq j \leq (N + 1)/2) \) and we can present \( W_N \) as a sum of products

\[
W_N(\tilde{p}) = \sum_{1 \leq j \leq \frac{N+1}{2}} p_j \, r_j(\tilde{p}).
\]

If we recall that eq. (3.44) turns \( \tilde{p} \) into polynomials of \( \phi \) and define the new polynomials \( q(\phi) \) as

\[
q_j(\phi) = r_j(\tilde{p}(\phi)),
\]

then, according to eq. (3.46),

\[
W_m(\phi) = \sum_{1 \leq j \leq \frac{N+1}{2}} p_j(\phi) \, q_j(\phi),
\]

and the super-potential \( W_m \) has a matrix factorization \((p; q)\). Although the presentation of \( W_N \) as a sum of products (3.47) is not unique, all these presentations lead to isomorphic Koszul matrix factorizations. We expect them to be the mirror images of the special Lagrangian submanifolds introduced in [4] and [6]: a Lagrangian submanifold of the cross-product of the complex Grassmannians \( \text{Gr}_{m_i},N \) is defined by the condition that the subspaces \( \mathbb{C}^{m_i} \subset \mathbb{C}^N \) provide an orthogonal decomposition of \( \mathbb{C}^N \).

It is interesting to note a resemblance between these topological LG theories and the representation theory of \( \text{SU}(N) \). If \( V \) denotes the fundamental \( N \)-dimensional representation of \( \text{SU}(N) \), then the critical points of a super-potential \( W_m \) correspond to the weights of the fundamental representation \( \wedge^m V \), a matrix factorization \((p; q)\) corresponds to the invariant element in the tensor product \( \otimes_i \wedge^{m_i} V \) and for a local graph \( \gamma \) the dimension of the space \( H_\gamma \) equals the result of the contraction of the Clebsch-Gordan tensors placed at its vertices. This correspondence is at the heart of the categorification construction of [5].

### 4. Formulas for the correlators

4.1. **Correlators on a closed surface.** The formula for the correlator of a topological LG theory on a closed surface was derived by Vafa in [7]. Let us define a ‘Frobenius trace’ map \( \mathbb{C}[\phi] \xrightarrow{\text{Tr}_W} \mathbb{C} \) by the formula

\[
\text{Tr}_W(O) = \frac{1}{(2\pi i)^m} \oint O(\phi) \, d\phi_1 \cdots d\phi_m \frac{\partial_1 W \cdots \partial_m W}{\partial_1 \cdots \partial_m},
\]

where the variables \( \phi \) are integrated over the contours which encircle all critical points of \( W \). Let \( P \) be a finite set of punctures (marked points) on a closed surface \( \Sigma \) of genus \( g \).
According to [7], the correlator of the operators \( O_P \in \mathbb{C}[\phi] \) placed at the punctures \( P \in \mathcal{P} \), is
\[
\langle \prod_{P \in \mathcal{P}} O_P \rangle_\Sigma = \mathrm{Tr}_W \left( (\det \partial_i \partial_j W)^g \prod_{P \in \mathcal{P}} O_P(\phi) \right). \tag{4.51}
\]
This formula indicates that the Frobenius trace (4.50) computes the correlator of the operators on a sphere \( S^2 \), while the factors \( \det \partial_i \partial_j W \) represent the ‘effective contributions’ of the handles: if we imagine that \( \Sigma \) is a sphere with \( g \) tori attached to it by thin tubes, then these tori can be equivalently replaced by the operators \( \det \partial_i \partial_j W \) placed at the points where the tubes join the sphere.

An important property of the trace (4.50) is that it annihilates the ideal \( \partial W \): for any \( O \in \mathbb{C}[\phi] \)
\[
\mathrm{Tr}_W (\partial_i W O) = 0. \tag{4.52}
\]
This is consistent with the fact that the space of local operators is the quotient (3.29).

4.2. Correlators on a surface with a boundary. Suppose that the boundary of the world-sheet \( \Sigma \) of genus \( g \) is a union of disjoint circles (2.8), each circle \( C_k \) is split into \( n_k \) segments \( e_{k,i} \), a matrix factorization \((M_{k,i}, D_{k,i}, W)\) is assigned to each segment and an operator \( O_{k,i} \) is placed at the junction of the segments \( e_{k,i} \) and \( e_{k,i+1} \). We also place some operators \( O_P, P \in \mathcal{P} \) at the internal points of \( \Sigma \). In order to write a simple expression for the resulting correlator, we have to introduce a general notation. For a matrix factorization \((M, D, W)\), Kapustin and Li [3] define an operator \( \partial D^\wedge \in \mathrm{End}(M) \) by the formula
\[
\partial D^\wedge = \frac{1}{m!} \sum_{\sigma \in S_m} (-1)^{\text{sign}(\sigma)} \partial_{\sigma(1)} D \cdots \partial_{\sigma(m)} D, \tag{4.53}
\]
where \( S_m \) is the symmetric group of \( m \) elements, \( m \) being the dimension of the target space of the topological LG theory \((\phi; W)\). Now to a circle \( C_k \), which is a part of the boundary \( \partial \Sigma \), we associate an element \( O_{C_k} \in \mathbb{C}[\phi] \) defined by the formula
\[
O_{C_k} = \mathrm{STr}_M \partial D^\wedge_{k,1} O_{k,1} \cdots O_{k,n_k}. \tag{4.54}
\]
Note the similarity between the expressions (4.54) and (2.14): the former is obtained from the latter by replacing all holonomies with the identity operators except the first one, which is replaced by \( \partial D^\wedge_{k,1} \).

A. Kapustin and Y. Li [3] derived the formula for the correlator of the boundary and bulk operators:
\[
\langle \prod_{P \in \mathcal{P}} O_P \prod_{k,i} O_{k,i} \rangle_\Sigma = \mathrm{Tr}_W \left( (\det \partial_i \partial_j W)^g \prod_{P \in \mathcal{P}} O_P(\phi) \prod_k O_{C_k} \right). \tag{4.55}
\]
By comparing this formula with eq. (4.51), we see that the factors $O_{C_k}$ represent the boundary state operators corresponding to the boundary components $C_k$: the correlator does not change if $C_k$ is contracted to a point and the operator $O_{C_k}$ is placed at that point.

Following [3], let us verify directly (without using the path integral arguments) that the correlator formula (4.55) satisfies two properties related to the BRST invariance. First of all, if one of the boundary operators $O_{k,l}$ is BRST-exact ($O_{k,l} = [D, O'_{k,l}]_s$), then the correlator (4.55) is zero, since we can move the operator $D$ around the super-trace expression (4.54): all the operators $O$ commute with $D$, while $\{\partial_j D, D\} = \partial_j W$ and a term proportional to $\partial_j W$ is annihilated by the Frobenius trace.

Second, we could insert the operator $\partial D^\wedge$ at any place in the product of the operators $O_{k,1} \cdots O_{k,n}$ in eq. (4.54): the r.h.s. of eq. (4.55) would not change. Indeed, if for some value of $l$

$$O_{k,l} = [\partial_j D, O'_{k,l}]_s, \quad \text{while} \quad [D, O'_{k,l}]_s = 0,$$

then the r.h.s. of eq. (4.55) is zero (take the derivative $\partial_j$ of the second equation of (4.56) and use the already established fact that $D$-commutators annihilate the r.h.s. of eq. (4.55)).

4.3. Correlators on a world-sheet foam. Now we consider a correlator on a world-sheet foam $(\Sigma, \Gamma)$. First, we present a formula and then comment on its derivation.

We define an operator $O_{\Gamma} \in \mathbb{C}[\phi_\Sigma]$ which is the analog of $O_{C_k}$ from eq. (4.54) and which represents the boundary state operator contribution of the Wilson network. For each connected component $C_{i,j}$ of the boundary $\partial \Sigma$, we choose a seam edge among the edges to which $C_{i,j}$ is glued, and we denote that edge as $e_{i,j}$. Then we introduce the operators (4.53)\n
$$\partial D^\wedge_{i,j} \in \text{End}(M_{e_{i,j}}), \quad \partial D^\wedge_{i,j} = \sum_{\sigma \in S_{m_i}} (-1)^{\text{sign}(\sigma)} \partial_{\phi_{i,\sigma(1)}D e_{i,j}} \cdots \partial_{\phi_{i,\sigma(m_i)}D e_{i,j}},$$

where $\phi_{i,1}, \ldots, \phi_{i,m_i}$ are the bosonic fields of the topological LG theory $(\phi_i; W_i)$ assigned to the connected component $\Sigma_i$ of $\Sigma$.

To every seam edge $e$ we assign an operator

$$O_e = \prod_{(i,j): e = e_{i,j}} \partial D^\wedge_{i,j}$$

(if $e = e_{i,j}$ for more than one combination $(i, j)$, then we choose any order of the operators in the product (4.58); if $e$ never appears as $i, j$, then the corresponding operator $O_e$ is the identity). Similarly to eq. (3.28), we define

$$O_{\Gamma} = f_{\Gamma} \left( \bigotimes_{v \in V} O_v \right) \otimes \left( \bigotimes_{e \in E} O_e \right)$$
and the correlator (3.34) is expressed as

$$\langle \prod_{P \in \mathcal{P}} O_P \prod_{v \in \mathcal{V}} O_v \rangle_{(\Sigma, \Gamma)} = \text{Tr}_{\Sigma} \left( O_{\mathcal{P}} \prod_{P \in \mathcal{P}} \prod_{i=1}^{N_{\Sigma}} \left( \text{det} \partial_j \partial_k W_i \right) g^{(\Sigma_i)} \right),$$

(4.60)

where

$$\text{Tr}_{\Sigma} : \mathbb{C}[\phi_{\Sigma}] \rightarrow \mathbb{C}, \quad \text{Tr}_{\Sigma} = \text{Tr}_{W_1} \cdots \text{Tr}_{W_{N_{\Sigma}}}$$

(cf. eq. (4.55)).

Now let us comment on the derivation of this correlator formula. Note that the original Vafa’s formula (4.51) was derived in the assumption that the critical points of the super-potential $W$ are non-degenerate. Then the BRST-invariance of the theory guarantees that the correlator is a sum of the contributions of the individual critical points and at each point the super-potential $W$ can be replaced by its quadratic part.

Kapustin and Li derived their formula under the same assumption, although it is harder to justify in their case: $W$ can be perturbed in order to make its critical points non-degenerate, but it is not clear whether matrix factorizations can be deformed together with it.

Kapustin and Li derived the correlator for the disk world-sheet. This is sufficient in order to establish the state operator corresponding to the boundary, and their general formula would follow from Vafa’s formula (4.51). We use the same approach. Namely, it would be sufficient to derive eq.(4.60) under the assumption that the surface $\Sigma$ is a union of disjoint disks. Then the computation of the path integral (3.34) that leads to eq.(4.60) is exactly the same as in [3]. The only minor novelty is that it may happen that a seam edge $e$ is assigned to two (or more) different disks $\Sigma_i$ and $\Sigma_j$ ($i \neq j$). Then it bears their operators $\partial D_i^\wedge$ and $\partial D_j^\wedge$ (we left only one index in their notation, since $\Sigma_i$ and $\Sigma_j$ have only one boundary component). The path integration over the fermionic fields leaves the derivatives $\partial_{\phi_{i,k}} D_e$ ($1 \leq k \leq m_i$) and $\partial_{\phi_{j,l}} D_e$ ($1 \leq l \leq m_j$), which enter in the expressions (4.57), intermixed. However, they can still be pulled apart into the operators $\partial D_i^\wedge$ and $\partial D_j^\wedge$, because $\partial_{\phi_{i,k}} D_e$ and $\partial_{\phi_{j,l}} D_e$ anti-commute up to a BRST-closed operator. Indeed, since the super-potential $W_e$ is a sum (3.20) of the individual super-potentials, each depending on its own set of fields, then $\partial_{\phi_{i,k}} \partial_{\phi_{j,l}} W_e = 0$. Hence, if we apply $\partial_{\phi_{i,k}} \partial_{\phi_{j,l}}$ to both sides of the relation $D_e^2 = W_e$, then we find that

$$\{ \partial_{\phi_{i,k}} D_e, \partial_{\phi_{j,l}} D_e \} = -\{ D_e, \partial_{\phi_{i,k}} \partial_{\phi_{j,l}} D_e \}.$$

(4.62)
5. **Gluing Formulas**

5.1. **Gluing of a 2-dimensional world-sheet.** The gluing property of the correlators is an important feature of general QFTs and of topological theories, in particular. Let us quickly review the gluing rules of a 2-dimensional topological QFT.

For a point \( P \in \Sigma \), let \( \gamma_P \) denote the intersection between \( \Sigma \) and a small sphere centered at \( P \). We will call \( \gamma_P \) the local space section of \( P \). In the context of a string theory, \( \gamma_P \) is simply called a string. If \( P \) is an internal point of \( \Sigma \), then \( \gamma_P \) is a circle (closed string), and if \( P \) is a point at the boundary \( \partial \Sigma \), then \( \gamma_P \) is a segment (half-circle, or open string). The endpoints of the segment come from the intersection of the small sphere and the boundary \( \partial \Sigma \), so they are 'decorated' with the TQFT boundary conditions at \( \partial \Sigma \). If \( P \) is a point at the junction of two different boundary conditions, then the decorations at its endpoints are also different. The segment is oriented, its orientation being induced by the orientation of \( \Sigma \). For a decorated local space section \( \gamma \) we define its dual \( \gamma^* \) to be the same as \( \gamma \) but with the opposite orientation. Obviously, a circle is self-dual.

The space of the TQFT states corresponding to \( \gamma_P \) coincides with the space \( H_P \) of the local operators that can be inserted at \( P \).

Suppose that for two points \( P_1, P_2 \in \Sigma \), their decorated local space sections are dual to each other: \( \gamma^*_1 = \gamma_2 \). Then the spaces \( H_1 \) and \( H_2 \) are also dual. In order to define the duality pairing between them, we consider the world-sheet \( S_{(1,2)} = U_1 \# U_2 \) constructed by gluing together small neighborhoods \( U_1 \) and \( U_2 \) of \( P_1 \) and \( P_2 \) over the boundaries \( \gamma_1 \sim \gamma_2 \) identified with opposite orientations. If \( \gamma_1 \) is a circle, then \( S_{(1,2)} \) is a 2-sphere, and if \( \gamma_1 \) is a segment, then \( S_{(1,2)} \) is a disk. The pairing is defined by the correlator on \( S_{(1,2)} \):

\[
(O_1, O_2) = \langle O_1 O_2 \rangle_{S_{(1,2)}}, \quad O_1 \in H_1, \ O_2 \in H_2. \tag{5.63}
\]

As a result, there is a canonical dual element

\[
I_{1,2} \in H_1^* \otimes H_2^*, \tag{5.64}
\]

defined by the relation

\[
(I_{1,2}, O_1 \otimes O_2) = (O_1, O_2). \tag{5.65}
\]

We will need the inverse element

\[
I_{1,2}^{-1} \in H_1 \otimes H_2. \tag{5.66}
\]

Let us cut small neighborhoods \( U_1 \) and \( U_2 \) from the world-sheet \( \Sigma \) and glue (that is, identify) the boundaries \( \gamma_1 \) and \( \gamma_2 \) of the cuts in such a way that their orientations are
opposite. Denote the resulting oriented manifold as $\Sigma'$, then according to the gluing property of a TQFT,

$$\langle \prod_{P \in P} O_P \rangle_{\Sigma'} = \langle I_{1,2}^{-1} \prod_{P \in P} O_P \rangle_{\Sigma}. \quad (5.67)$$

A more ‘pedestrian’ way to formulate the same property is to introduce a basis of operators $O_j \in H_1$ and a dual basis $O_j^* \in H_2$ so that $(O_j, O_j^*) = \delta_{j,j'}$. Then

$$\langle \prod_{P \in P} O_P \rangle_{\Sigma'} = \sum_j \langle O_j O_j^* \prod_{P \in P} O_P \rangle_{\Sigma}. \quad (5.68)$$

5.2. **Complete space gluing.** Let us check how the general gluing formula (5.67) works for a topological LG theory on a world-sheet foam.

Let $P$ be a point on a world-sheet foam $(\Sigma, \Gamma)$. We call its local space section a local graph and denote it as $\gamma_P$. If $P$ is an internal point of $\Sigma$, then $\gamma_P$ is a circle. If $P$ is an internal point of a seam edge $e$, then $\gamma_P$ is a graph with two vertices connected by multiple edges, each edge corresponding to a strip of a component $\Sigma_i$ attached to $e$. If $P$ is a seam vertex $v$, then $\gamma_P$ is a graph, whose vertices correspond to the seam edges adjacent to $v$ and whose edges correspond to the strips of the components $\Sigma_i$, which pass through $v$. In fact, the vertex-edge and edge-surface correspondence between $\gamma_P$ and $(\Sigma, \Gamma)$ works for all three types of points $P$. The orientation of the components $\Sigma_i$ induces the orientation of the corresponding edges of $\gamma_P$. A small neighborhood $U_P$ of $P$ in $(\Sigma, \Gamma)$ can be restored from its local graph $\gamma_P$, because $U_P$ is the cone of $\gamma_P$:

$$U_P = C\gamma_P. \quad (5.69)$$

Generally speaking, a local graph $\gamma$ is just a graph. A decorated local graph means the following. To an oriented edge $\epsilon$ of $\gamma$ we assign a topological LG theory $(\phi_\epsilon; W_\epsilon)$ in such a way that if $\epsilon$ and $\epsilon^*$ represent the same edge with opposite orientations, then they are assigned the conjugated theories. To a vertex $\nu$ of $\gamma$ we associate a matrix factorization $(M_\nu, D_\nu, W_\nu)$, such that

$$W_\nu = \sum_{\epsilon \in \Upsilon_\nu} W_\epsilon, \quad (5.70)$$

where $\Upsilon_\nu$ is the set of edges of $\gamma$, which are attached to $\nu$ (we assume that they are oriented away from $\nu$).

For a decorated local graph $\gamma$ consider the matrix factorization $(M_\gamma, D_\gamma, W_\gamma)$, which is the tensor product of all the matrix factorizations of its vertices

$$(M_\gamma, D_\gamma, W_\gamma) = \bigotimes_{\nu} (M_\nu, D_\nu, W_\nu). \quad (5.71)$$
Obviously

\[ W_\gamma = 0, \quad (5.72) \]

so \( D_\gamma^2 = 0 \) and we denote its cohomology as

\[ H_\gamma = \ker D_\gamma / \operatorname{im} D_\gamma. \quad (5.73) \]

If a world-sheet foam \((\Sigma, \Gamma)\) is decorated, then for any \( P \in (\Sigma, \Gamma) \) its local graph \( \gamma_P \) is also decorated: a topological LG theory of an edge \( \epsilon \) is the theory of the corresponding component \( \Sigma_\epsilon \) and a matrix factorization of a vertex \( \nu \) is the matrix factorization of the corresponding seam edge, if it is oriented away from \( P \), or the conjugated matrix factorization otherwise. Then it is easy to see that

\[ H_{\gamma_P} = H_P, \quad (5.74) \]

that is, the space of operators at a point \( P \) is determined by its decorated local graph \( \gamma_P \).

For a decorated local graph \( \gamma \) we define its dual graph \( \gamma^* \) to be the same graph as \( \gamma \), except that it is decorated with the conjugate topological LG theories and with the dual matrix factorizations.

Consider a suspension \( \Sigma_\gamma \) of a local graph \( \gamma \): by definition it is constructed by gluing together the cones \( C_\gamma \) and \( C_{\gamma^*} \) along their common boundary \( \gamma \). \( \Sigma_\gamma \) has a structure of a world-sheet foam: its seam graph consists of two vertices \( v_1 \) and \( v_2 \), which are the vertices of the cones, the suspensions of the vertices of \( \gamma \) form the edges that connect \( v_1 \) and \( v_2 \), and the 2-dimensional components \( \Sigma_i \) are the suspensions of the edges of \( \gamma \). Obviously, \( \gamma_{\epsilon_1} = \gamma \) and \( \gamma_{\epsilon_2} = \gamma^* \).

If a local graph \( \gamma \) is decorated, then a topological LG theory is defined on its suspension \( \Sigma_\gamma \). The correlators of this theory provide a pairing between the spaces \( H_\gamma \) and \( H_{\gamma^*} \)

\[ (O, O') = \langle O O' \rangle_{\Sigma_\gamma}. \quad (5.75) \]

This pairing determines an inverse canonical element \( I_{\gamma, \gamma^*}^{-1} \in H_\gamma \otimes H_{\gamma^*} \).

Suppose that for two points \( P_1, P_2 \in (\Sigma, \Gamma) \) their local graphs are dual: \( \gamma^*_1 = \gamma_2 \). Then we can cut their small neighborhoods from the world-sheet foam \((\Sigma, \Gamma)\) and glue the cut borders together, thus forming a new world-sheet foam \((\Sigma', \Gamma')\). If a topological LG theory is defined on \((\Sigma', \Gamma')\), then it induces a topological LG theory on \((\Sigma, \Gamma)\) and the correlators of both theories are related by the gluing formula

\[ \langle \prod_{P \in \mathcal{P}'} O_P \prod_{\nu \in \mathcal{V}'} O_{\nu} \rangle_{(\Sigma', \Gamma')} = \langle I_{1,2}^{-1} \prod_{P \in \mathcal{P}'} O_P \prod_{\nu \in \mathcal{V}'} O_{\nu} \rangle_{(\Sigma, \Gamma)}, \quad (5.76) \]
where $P'$ and $V'$ are the punctures and seam vertices of $(\Sigma', \Gamma')$. Thus the gluing property of a topological LG theory on a world-sheet foam is very similar to the gluing property (5.67) on a usual world-sheet.

### 5.3. A topological LG theory on a world-sheet foam as a 2-category.

The graph structure of world-sheet foam local space sections permits more complicated types of gluing than those described in the previous subsection, when two dual local space sections are glued together. These new types of gluing can be arranged into the mathematical structure known as a 2-category.

The usual 1-category structure of a TQFT on a 2-dimensional world-sheet comes from the composition property of transition amplitudes. Consider a world-sheet $\Sigma$ with a finite puncture set $\mathcal{P}$ and two special punctures $P_1$ and $P_2$ with local graphs $\gamma_1$, $\gamma_2$. If we choose the operators at the punctures of $P_1$, then the correlator on $\Sigma$ defines a transition amplitude

$$H_{\gamma_1} \xrightarrow{A[\gamma_1, \gamma_2]} H_{\gamma_2}$$

by the formula

$$I_{\Sigma_1^2}(A[\gamma_1, \gamma_2]) = \langle O_1 O_2 \prod_{P \in \mathcal{P}} O_P \rangle_{\Sigma_1^2} \quad \text{for any} \quad O_1 \in H_{\gamma_1}, \ O_2 \in H_{\gamma_2}. \quad (5.78)$$

Let $\Sigma_{12}$ denote the result of cutting small neighborhoods of $P_1$ and $P_2$ from $\Sigma_{12}$. If we have another surface $\Sigma'$ with special punctures $P_3$ and $P_4$ such that $\gamma_2' = \gamma_3$, then we can glue the boundary components $\gamma_2$ and $\gamma_3$ of $\Sigma_{12}$ and $\Sigma_{34}'$ together to form a new world-sheet with boundary $\Sigma''_{14}$. The transition amplitude of $\Sigma''_{14}$ is given by the composition of transition amplitudes

$$A[\gamma_1, \gamma_4'] = A[\gamma_3, \gamma_4'] A[\gamma_1, \gamma_2'], \quad (5.79)$$

and this formula corresponds to the gluing formula (5.67) formulated for $P_2$ and $P_3$.

The composition property (5.79) extends verbatim to the world-sheet foams. In the foam case, however, there is an important generalization. Namely, the surface or the world-sheet foam $\Sigma_{12}$ presented a cobordism between two closed space section (be it 1-manifolds or graphs). Now we are going to consider cobordisms between the spaces that have boundaries.

Let us take a local graph $\gamma$ and make cuts across some of its edges, so that $\gamma$ splits into two disconnected partial local graphs $\alpha_1$ and $\alpha_2$: $\gamma = \alpha_1 \# \alpha_2$. The partial local graphs have special univalent vertices at the cuts: we call them boundary vertices, and their adjacent edges are called legs. We think of the boundary vertices as the boundary of a partial local graph.

The partial local graphs inherit the decorations of $\gamma$, except that their boundary vertices are not assigned matrix factorizations. To a decorated partial local graph $\alpha$ we associate a
matrix factorization \((M_\alpha, D_\alpha, W_\alpha)\) which is the tensor product of all the matrix factorizations of its non-boundary vertices. However this time instead of eq.\((5.72)\) we have
\[
W_\alpha = \sum_{\epsilon \in L_\alpha} W_\epsilon, \tag{5.80}
\]
where \(L_\alpha\) is the set of legs of \(\alpha\), and we assume that the legs are oriented away from the boundary vertices.

Let \(\phi_\alpha = (\phi_\epsilon | \epsilon \in L_\alpha)\) be the list of all variables of the legs of a partial local graph \(\alpha\), and let \(R_\alpha = \mathbb{C}[\phi_\alpha]\) be their polynomial ring. Obviously,
\[
R_\alpha = \bigotimes_{\epsilon \in L_\alpha} \mathbb{C}[\phi_\epsilon]. \tag{5.81}
\]
Since the super-potential \(W_\alpha\) of eq.\((5.80)\) depends only on the ‘external’ variables \(\phi_\alpha\), from now on we will consider \((M_\alpha, D_\alpha, W_\alpha)\) to be a matrix factorization over the ring \(R_\alpha\). However this poses a problem: if we ignore the ‘internal’ variables of \(\alpha\), then the module \(M_\alpha\) is infinite-dimensional as a module over \(R_\alpha\). Indeed, the multiplication by the powers of an internal variable now produces an infinite sequence of linearly-independent elements of \(M_\alpha\).

In order to resolve this problem, we can contract \(M_\alpha\) homotopically to a finite-dimensional \(R_\alpha\)-module. Here is the relevant definition: two matrix factorizations \((M_i, D_i, W_i), i = 1, 2\) are considered \(\text{homotopically equivalent}\) over the polynomial ring \(R \ni W\), if there exist two \(R\)-linear maps \(f_{12}, f_{21}\)
\[
M_1 \xrightarrow{f_{12}} M_2 \xrightarrow{f_{21}} M_1 \tag{5.82}
\]
commuting with the twisted differential \(D\), such that the compositions \(f_{21}f_{12} \in \text{End}_R(M_1)\) and \(f_{12}f_{21} \in \text{End}_R(M_2)\) are BRST-equivalent to the identity maps. We showed in [5] that under some mild assumptions an infinite rank matrix factorization is homotopically equivalent to a finite rank one.

If a decorated local graph is split: \(\gamma = \alpha_1 \# \alpha_2\), then according to eq.\((5.71)\)
\[
M_\gamma = M_{\alpha_1} \otimes_R M_{\alpha_2}, \quad R = R_{\alpha_1} = R_{\alpha_2}. \tag{5.83}
\]
and we used the notation \(\otimes_R\) in order to emphasize that the tensor product is taken over the polynomial ring of all leg variables of \(\alpha_1\) (or, equivalently, \(\alpha_2\)). If we replace the \(R\)-modules \(M_{\alpha_1}\) and \(M_{\alpha_2}\) in eq.\((5.83)\) by their finite-dimensional homotopic equivalents, then the module \(M_\gamma\) will change, but its \(D\) cohomology \(H_\gamma\) will stay the same. Therefore we will use the same notation \(M_\alpha\) for the whole homotopy equivalence class of \(R_\alpha\)-modules related to a decorated partial local graph \(\alpha\).

Let \((\Sigma, \Gamma)\) be a decorated world-sheet foam with a puncture set \(\mathcal{P}\). Let us pick a vertex \(v \in \mathcal{V}\) with a local graph \(\gamma\) and denote \(\mathcal{V}' = \mathcal{V} \setminus \{v\}\). A choice of the operators at the
punctures of \( \mathcal{P} \) and at the vertices of \( \mathcal{V}' \) determines an element \( A[\gamma] \in H_{\gamma^*} \) by the formula

\[
I_{\gamma,\gamma^*}(O, A[\gamma]) = \langle O \prod_{P \in \mathcal{P}} O_P \prod_{v' \in \mathcal{V}'} O_{v'} \rangle_{(\Sigma, \Gamma)} \quad \text{for any} \quad O \in H_{\gamma}.
\]

(5.84)

Suppose that \( \gamma \) splits: \( \gamma = \alpha_1 \# \alpha_2 \). Then, in view of eq.(5.83),

\[
M_{\gamma^*} = M_{\alpha_1^*} \otimes_R M_{\alpha_2^*} = \text{Hom}_R(M_{\alpha_1}, M_{\alpha_2}),
\]

(5.85)

and hence

\[
H_{\gamma^*} = \text{Ext}(M_{\alpha_1}, M_{\alpha_2})
\]

(5.86)

(cf. the definition (2.16)). The latter isomorphism allows us to translate the element \( A[\gamma] \) of eq.(5.84) into a transition amplitude \( A[\alpha_1, \alpha_2^*] \in \text{Ext}(M_{\alpha_1}, M_{\alpha_2}) \). This transition amplitude is the analog of the amplitude (5.77): \( A[\gamma_1, \gamma_2^*] \) describes the transition between two closed space sections, while \( A[\alpha_1, \alpha_2^*] \) describes the transition between two space sections with boundary.

The distinction between the open and closed space transitions has a topological and an algebraic manifestation. Topologically, if we cut small neighborhoods of the punctures \( P_1 \) and \( P_2 \), then the remainder \( (\Sigma, \Gamma)_{12} \) has two disconnected boundary components: the ‘in’ space \( \gamma_1 \) and the ‘out’ space \( \gamma_2 \). The foam topology is more complicated. If we cut out a small neighborhood of the seam vertex \( v \), then the boundary of the remainder \( (\Sigma, \Gamma)_v \) is obviously the local graph \( \gamma \). If we cut \( \gamma \) just into \( \alpha_1 \) and \( \alpha_2 \), then these partial local graphs would have common points. However, since we think of \( \alpha_1 \) and \( \alpha_2 \) as space sections corresponding to different values of ‘time’, we would like them to be completely separated. Therefore, rather than slicing the edges that connect \( \alpha_1 \) and \( \alpha_2 \), we cut out finite length segments from them. These segments form the time-like section. Thus the boundary of \( (\Sigma, \Gamma)_v \) consists of three, rather than two, pieces: two ‘space-like’ ones (the ‘in’ space \( \alpha_1 \) and the ‘out’ space \( \alpha_2^* \)) as well as the time-like section. The algebraic consequence of the presence of a time-like section in the boundary of \( (\Sigma, \Gamma)_v \) is that the ‘in’ and ‘out’ spaces of states are not just linear spaces over \( \mathbb{C} \), but rather \( R \)-modules, and the transition amplitude \( A[\alpha_1, \alpha_2^*] \) is \( R \)-linear.

Since the world-sheet foam of the open space transition \( (\Sigma, \Gamma)_v \) has two types of boundary, the corresponding transition amplitude \( A[\alpha_1, \alpha_2^*] \) satisfies two gluing relations. First of all, there is the gluing associated to the composition of transitions, which is similar to eq.(5.79). Consider two world-sheet foams \( (\Sigma_j, \Gamma_j) \) \((j = 1, 2)\) with marked seam vertices \( v_j \). Suppose that their local graphs \( \gamma_j \) can be split \( \gamma_j = \alpha_j \# \alpha_j' \) in such a way that \( \alpha_j^* = \alpha_j' \). Then we can glue \( (\Sigma_1, \Gamma_1)_{v_1} \) and \( (\Sigma_2, \Gamma_2)_{v_2} \) along these matching partial local graphs. Its resulting transition amplitude \( A[\alpha_1, \alpha_2^*] \) should be the composition of the elementary ones:

\[
A[\alpha_1, \alpha_2^*] = A[\alpha_2, (\alpha_2')^*] A[\alpha_1, (\alpha_1')^*].
\]

(5.87)
One can also glue the world-sheet foams along the time-like sections. Let us describe the corresponding cutting of a world-sheet foam $(\Sigma, \Gamma)$. Suppose that it has two marked vertices $v_j$ ($j = 1, 2$), whose local graphs $\gamma_j$ are split: $\gamma_j = \alpha_j' \# \alpha_j''$, and both $\gamma_1$ and $\gamma_2$ have $m$ slice points. Let $p_{j,k}$ ($1 \leq k \leq m$) denote the points at which the legs of $\alpha_j'$ and $\alpha_j''$ are joined. Suppose that for all $k$, the points $p_{1,k}$ and $p_{2,k}$ belong to the same connected component $\Sigma_{i(k)}$ of $\Sigma$, and we can choose nonintersecting curves $c_k$ which lie on $\Sigma_{i(k)}$ and join the points $p_{1,k}$ and $p_{2,k}$.

Finally, suppose that if we make the cuts along all curves $c_k$, then $(\Sigma, \Gamma)_{v_1,v_2}$ splits into two disconnected pieces $(\Sigma, \Gamma)_{v_1,v_2}'$ and $(\Sigma, \Gamma)_{v_1,v_2}''$, in such a way that $(\Sigma, \Gamma)_{v_1,v_2}'$ is bound by $\alpha_1', \alpha_2'$ and the curves $c_k$, while $(\Sigma, \Gamma)_{v_1,v_2}''$ is bound by $\alpha_1'', \alpha_2''$ and the curves $c_k$. Then $(\Sigma, \Gamma)_{v_1,v_2}'$ and $(\Sigma, \Gamma)_{v_1,v_2}''$ produce their own transition amplitudes $A[\alpha_1', (\alpha_2')^*]$ and $A[\alpha_1'', (\alpha_2'')^*]$. Their tensor product

$$M_{\alpha_1'} \otimes_R M_{\alpha_2'} A[\alpha_1', (\alpha_2')^*] \otimes_R A[\alpha_1'', (\alpha_2'')^*] \rightarrow M_{(\alpha_1')} \otimes_R M_{(\alpha_2')}$$

commutes with the twisted differential $D$ and therefore, in view of (5.83), it defines a map from $H_{\gamma_1}$ to $H_{\gamma_2}$. Thus, the gluing of two world-sheet foams $(\Sigma, \Gamma)_{v_1,v_2}'$ and $(\Sigma, \Gamma)_{v_1,v_2}''$ along their time-like sections produces the formula

$$A[\gamma_1, \gamma_2] = A[\alpha_1', (\alpha_2')^*] \otimes_R A[\alpha_1'', (\alpha_2'')^*],$$

relating two open-space transition amplitudes to one closed-space transition amplitude. In this gluing formula the usual composition (5.79) and (5.87) is replaced by the tensor product over an appropriate ring.

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