Research Article

Some $\delta$-Tempered Fractional Hermite–Hadamard Inequalities Involving Harmonically Convex Functions and Applications

Nousheen Akhtar and Muhammad Uzair Awan

Department of Mathematics, Government College University, Faisalabad, Pakistan

Correspondence should be addressed to Muhammad Uzair Awan; awan.uzair@gmail.com

Received 23 June 2021; Accepted 26 July 2021; Published 9 August 2021

Academic Editor: Ji Gao

Copyright © 2021 Nousheen Akhtar and Muhammad Uzair Awan. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The main objective of this paper is to obtain some new $\delta$-tempered fractional versions of Hermite–Hadamard’s inequality using the class of harmonic convex functions. In order to show the significance of the main results, we also discuss some interesting applications.

1. Introduction and Preliminaries

A function $f: I \subset \mathbb{R} \longrightarrow \mathbb{R}$ is said to be convex if

$$f((1 - \tau)v_1 + \tau v_2) \leq (1 - \tau)f(v_1) + \tau f(v_2), \quad \forall v_1, v_2 \in I, \tau \in [0, 1].$$

(1)

In recent years, several new extensions of classical convexity have been proposed in the literature. Iscan [1] introduced the notion of harmonically convex functions as follows.

A function $f: I \subset (0, \infty) \longrightarrow \mathbb{R}$ is said to be harmonically convex if

$$f\left(\frac{v_1v_2}{\tau v_1 + (1 - \tau)v_2}\right) \leq (1 - \tau)f(v_1) + \tau f(v_2), \quad \forall v_1, v_2 \in I, \tau \in [0, 1].$$

(2)

Hermite–Hadamard’s inequality is one of the most studied results pertaining to convexity property of the functions. This result of Hermite and Hadamard reads as follows.

Let $f: I = [v_1, v_2] \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function; then,

$$f\left(\frac{v_1 + v_2}{2}\right) \leq \frac{1}{v_2 - v_1} \int_{v_1}^{v_2} f(x)dx \leq \frac{f(v_1) + f(v_2)}{2}. \quad (3)$$

Iscan [1] extended the classical version of Hermite–Hadamard’s inequality using the harmonic convexity property of the function.

Let $f: I = [v_1, v_2] \subset (0, \infty) \longrightarrow \mathbb{R}$ be a harmonically convex function; then,

$$f\left(\frac{2v_1v_2}{v_1 + v_2}\right) \leq \frac{v_1v_2}{v_2 - v_1} \int_{v_1}^{v_2} f(x)dx \leq \frac{f(v_1) + f(v_2)}{2}. \quad (4)$$

The interrelation between theory of convex functions and theory of integral inequalities has attracted several inequality experts and as result several new versions of classical results have been obtained in the literature. For example, Sarikaya et al. [2] used the concepts of fractional calculus in obtaining the fractional analogue of Hermite–Hadamard’s inequality. This idea attracted several researchers and a result number of new refined fractional analogues of classical inequalities have been obtained in the literature. For example, Gurbuz et al. [3] obtained some new refinements of integral inequalities using fractional integral operators of positive real order. İşcan and Wu [4] obtained fractional analogue of Hermite–Hadamard’s inequality using the concept of harmonically convex functions Awan et al. [5] obtained conformable fractional Hermite–Hadamard’s inequality using the harmonic convexity property of the functions. İftikhar et al. [6] obtained some local fractional Newton’s type inequalities via generalized harmonic convex
functions. Recently, Sanli et al. [7] obtained some more new fractional Hermite–Hadamard type of inequalities using the harmonic convexity property of the functions.

In recent years, the classical concepts of fractional calculus have been extended and generalized using novel and innovative ideas. For instance, Meerschaert et al. [8] introduced the concepts where power laws are tempered by an exponential factor and showed that this exponential tempering has both mathematical and practical advantages. This inspired Mohammed et al. [9], and they obtained new generalizations of Hermite–Hadamard’s inequality using tempered fractional integrals. Mubeen [10] and Sarikaya and Karaca [11] introduced the notion of tempered fractional integrals. Mohammed et al. [9], and they obtained new fractional analogues of Hermite–Hadamard’s inequality using \( \delta \)-fractional calculus, Lei et al. [12] showed that this exponential tempered fractional integral is the well-known Gamma function.

The concept of \( \delta \)-Riemann–Liouville fractional integral is defined as follows: let \( \mathcal{F} \) be piecewise continuous on \( I^* = (0, \infty) \) and integrable on any finite subinterval of \( I = [0, \infty) \). Then, for \( \lambda > 0 \), we consider \( \delta \)-Riemann–Liouville fractional integral of \( \mathcal{F} \) of order \( \alpha \)

\[
\tau I_{\gamma_2}^{\alpha \lambda} \mathcal{F}(x) = \frac{1}{\Gamma(\alpha)} \int_{\gamma_2}^{x} (\mu - x)^{\alpha-1} \exp(-\mu - x) \mathcal{F}(\mu) d\mu, \quad x \in [\gamma_1, \gamma_2],
\]

where

\[
\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha-1} dx,
\]

is the well-known Gamma function.

If \( \delta \to 1 \), then \( \delta \)-Riemann–Liouville fractional integrals reduce to classical Riemann–Liouville fractional integral.

In [8], the authors have described a new variation on the fractional calculus as follows.

**Definition 2.** Let \( \mathcal{F} \in L[\gamma_1, \gamma_2] \) with \( \lambda \geq 0 \) and \( \alpha > 0 \). Then, right- and left-tempered fractional integrals are defined as

\[
\tau I_{\gamma_1}^{\alpha \lambda} \mathcal{F}(x) = \frac{1}{\Gamma(\alpha)} \int_{\gamma_1}^{x} (\mu - \mu)^{\alpha-1} \exp(-\mu - \mu) \mathcal{F}(\mu) d\mu, \quad x \in [\gamma_1, \gamma_2],
\]

We now introduce the \( \delta \)-tempered fractional integrals.

**Definition 3.** Let \( \mathcal{F} \in L[\gamma_1, \gamma_2] \) with \( \lambda \geq 0, \alpha > 0 \) and \( \delta \geq 1 \). Then, right- and left-tempered \( \delta \)-fractional integrals are defined as

\[
\tau I_{\gamma_1}^{\alpha \lambda \delta} \mathcal{F}(x) = \frac{1}{\delta \Gamma(\delta)} \int_{\gamma_1}^{x} (\mu - \mu)^{\alpha-1} \exp(-\mu - \mu) \mathcal{F}(\mu) d\mu, \quad x \in [\gamma_1, \gamma_2],
\]

\[
\tau I_{\gamma_2}^{\alpha \lambda \delta} \mathcal{F}(x) = \frac{1}{\delta \Gamma(\delta)} \int_{\gamma_2}^{x} (\mu - x)^{\alpha-1} \exp(-\mu - x) \mathcal{F}(\mu) d\mu, \quad x \in [\gamma_1, \gamma_2].
\]

**Definition 4.** For the real numbers \( \alpha > 0, \lambda \geq 1 \) with \( \delta \geq 1 \), we define the \( \lambda \)-incomplete \( \delta \)-gamma function by

\[
\delta \gamma_\lambda(\alpha, x) = \int_0^x \mu^{\alpha-1} e^{-(\lambda \mu)^{\delta}} d\mu.
\]
If $\lambda = 0$, it reduces the incomplete $\delta$-gamma function:
\[
\delta\gamma(\alpha, x) = \int_0^x \mu^{\alpha - 1} e^{-\mu/x} d\mu. \tag{11}
\]

**Remark 1.** For the real numbers $\alpha > 0, \lambda \geq 1$ with $\delta \geq 1$, we have
\[
\begin{align*}
(1) & \quad \delta\gamma_{\lambda}(v_2 - v_1)(\alpha, 1) = \int_0^1 \mu^{\alpha - 1} e^{-(\lambda(v_2 - v_1)/\mu)^{\delta}/\mu} d\mu = 1/\nu_2 - v_1) \\
(2) & \quad \int_0^1 \delta\gamma_{\lambda}(v_2 - v_1)(\alpha, x) dx = \delta\gamma_{\lambda}(\alpha, v_2 - v_1)/(v_2 - v_1)^{\alpha - 1} - \\
& \quad \delta\gamma_{\lambda}(\alpha + 1, v_2 - v_1)/(v_2 - v_1)^{\alpha - 1}.
\end{align*}
\]

**Proof.**
(1) The proof is straightforward, by using the change of variable technique $x = (v_2 - v_1)/\mu$.

(2) To prove this, we use definition of $\lambda$-incomplete $\delta$-gamma function:
\[
\int_0^1 \delta\gamma_{\lambda}(v_2 - v_1)(\alpha, x) dx = \int_0^1 \int_0^x \mu^{\alpha - 1} e^{-(\lambda(v_2 - v_1)/\mu)^{\delta}/\mu} d\mu dx.
\]

By changing the order of integration, we obtain
\[
\int_0^1 \delta\gamma_{\lambda}(v_2 - v_1)(\alpha, x) dx = \int_0^1 \int_0^x \mu^{\alpha - 1} e^{-(\lambda(v_2 - v_1)/\mu)^{\delta}/\mu} d\mu dx.
\]

Using Remark 1 (1), we have
\[
\int_0^1 \delta\gamma_{\lambda}(v_2 - v_1)(\alpha, x) dx = \frac{\delta\gamma_{\lambda}(\alpha, v_2 - v_1)}{(v_2 - v_1)^{\alpha - 1}} - \frac{\delta\gamma_{\lambda}(\alpha + 1, v_2 - v_1)}{(v_2 - v_1)^{\alpha - 1}}.
\]

This completes the proof. \qed

**2. Results and Discussion**

In this section, we discuss our main results.

\[
f\left(\frac{2v_1v_2}{v_1 + v_2}\right) \leq \left(\frac{v_1v_2}{v_2 - v_1}\right)^{\alpha} \frac{\delta\Gamma_\delta(\alpha)}{2\delta\gamma_{\lambda}(v_2 - v_1)(\alpha, 1)} \left[f_{\mu}^{\mu \lambda v_1 v_2} \left(\frac{1}{v_2}\right) + f_{\mu}^{\mu \lambda v_1 v_2} \left(\frac{1}{v_1}\right)\right] \leq \frac{f(v_1) + f(v_2)}{2},
\]

where $g(x) = 1/x$ and for all $\alpha > 0$ and $\lambda \geq 1$.

**Proof.** Since $f$ is a harmonic convex function, then
\[
f\left(\frac{2xy}{x + y}\right) \leq \left(\frac{f(x) + f(y)}{2}\right).
\]

This implies
\[
2f\left(\frac{2v_1v_2}{v_1 + v_2}\right) \int_0^1 r^{\alpha - 1} e^{-(\lambda(v_2 - v_1)/r)^{\delta}/r} dr
\]
\[
\leq \int_0^1 r^{\alpha - 1} e^{-(\lambda(v_2 - v_1)/r)^{\delta}/r} f\left(\frac{v_1v_2}{(1 - r)v_1 + rv_2}\right) dr + \int_0^1 r^{\alpha - 1} e^{-(\lambda(v_2 - v_1)/r)^{\delta}/r} f\left(\frac{v_1v_2}{rv_1 + (1 - r)v_2}\right) dr.
\]

**2.1. A New Version of Hermite–Hadamard's Inequality.**

We now derive a new $\delta$-tempered fractional Hermite–Hadamard inequality via harmonically convex function.

**Theorem 1.** Let $f: [v_1, v_2] \to \mathbb{R}$ be an harmonically convex function on $[v_1, v_2]$ with $v_1 < v_2$; then,

\[
f\left(\frac{2v_1v_2}{v_1 + v_2}\right) \leq f\left(\frac{v_1v_2}{(1 - r)v_1 + rv_2}\right) + f\left(\frac{v_1v_2}{rv_1 + (1 - r)v_2}\right),
\]

Multiplying both sides of the above inequality by $\tau^{\alpha - 1} e^{-(\lambda(v_2 - v_1)/r)^{\delta}/r}$ and integrating with respect to $\tau$ on $[0, 1]$, we have

\[
2f\left(\frac{2v_1v_2}{v_1 + v_2}\right) \int_0^1 \tau^{\alpha - 1} e^{-(\lambda(v_2 - v_1)/\tau)^{\delta}/\tau} d\tau
\]
\[
\leq \int_0^1 \tau^{\alpha - 1} e^{-(\lambda(v_2 - v_1)/\tau)^{\delta}/\tau} f\left(\frac{v_1v_2}{(1 - \tau)v_1 + \tau v_2}\right) d\tau + \int_0^1 \tau^{\alpha - 1} e^{-(\lambda(v_2 - v_1)/\tau)^{\delta}/\tau} f\left(\frac{v_1v_2}{\tau v_1 + (1 - \tau)v_2}\right) d\tau.
\]
This implies
\[
2^{\delta y_{(v_2-v_1)}}(v_2-v_1)^{a_1} \int_{v_2}^{v_1} \left( x - \frac{1}{v_2} \right)^{a_1} e^{-\left( \lambda y_{v_2} \left( x - \frac{1}{v_2} \right) \right)^{\delta} f\left( \frac{1}{x} \right) dx} \]
\[
+ \frac{(v_2-v_1)^{a_1}}{v_2-v_1} \int_{v_2}^{v_1} \left( 1 - x \right)^{a_1} e^{-\left( \lambda y_{v_2} \left( 1 - x \right) \right)^{\delta} f\left( \frac{1}{x} \right) dx}. \tag{19}
\]
Thus, we have
\[
f\left( \frac{v_1 v_2}{v_1 + v_2} \right) \leq \frac{(v_1 v_2)^{a_1}}{2(v_2-v_1)^{a_1} \delta y_{(v_2-v_1)}}(a,1)
\[
\times \left[ \tau \int_{0}^{1} \left( \lambda y_{v_2} \right)^{\delta} f\left( \frac{v_1 v_2}{v_1 + v_2} \right) d\tau + \int_{0}^{1} \tau^{-1} e^{-\left( \lambda y_{v_2} \right)^{\delta} f\left( \frac{v_1 v_2}{v_1 + v_2} \right) d\tau} \right]. \tag{20}
\]
This implies
\[
\int_{0}^{1} \tau^{-1} e^{-\left( \lambda y_{v_2} \right)^{\delta} f\left( \frac{v_1 v_2}{v_1 + v_2} \right) d\tau} \leq \left[ f\left( v_1 \right) + f\left( v_2 \right) \right] \int_{0}^{1} \tau^{-1} e^{-\left( \lambda y_{v_2} \right)^{\delta} f\left( \frac{v_1 v_2}{v_1 + v_2} \right) d\tau}. \tag{24}
\]
Also,
\[
f\left( \frac{v_1 v_2}{v_1 + v_2} \right) \leq \tau f\left( v_1 \right) + \tau f\left( v_2 \right). \tag{21}
\]
Adding (21) and (22), we have
\[
f\left( \frac{v_1 v_2}{v_1 + v_2} \right) + f\left( \frac{v_1 v_2}{v_1 + v_2} \right) \leq f\left( v_1 \right) + f\left( v_2 \right). \tag{23}
\]
Multiplying the above inequality by \( \tau^{-1} e^{-\left( \lambda y_{v_2} \right)^{\delta} f\left( \frac{v_1 v_2}{v_1 + v_2} \right) d\tau} \)
and integrating with respect to \( \tau \) on \([0,1], \) we have
\[
f\left( \frac{v_1 v_2}{v_1 + v_2} \right) \leq \frac{f\left( v_1 \right) + f\left( v_2 \right)}{2}. \tag{26}
\]
Combining (20) and (26), we get the required inequality. \( \square \)
Remark 2

(1) For \( \alpha = 1 = k \) and \( \lambda = 0 \) in (15), we obtain the classical Hermite–Hadamard’s type inequality for the notion of harmonic convexity

(2) For \( k = 1 \) and \( \lambda = 0 \) in 2.1, we obtain the fractional version of Hermite–Hadamard’s type inequality for the notion of harmonic convexity

\[
f(\gamma_1) + f(\gamma_2) - \frac{\gamma_1 \gamma_2}{2} \left( \frac{\gamma_2 - \gamma_1}{\gamma_2 - \gamma_1} \right)^{\alpha} \frac{\delta \Gamma_{\alpha}(\alpha)}{\delta \gamma_k(\gamma_2 - \gamma_1)(\alpha)} \left[ \tau^{\alpha \lambda \gamma_k \gamma_2} \frac{\partial}{\partial \gamma_k(\gamma_2 - \gamma_1)} \right] f^\alpha(g(\frac{1}{\gamma_2})) + \tau^{\alpha \lambda \gamma_k \gamma_2} \frac{\partial}{\partial \gamma_k(\gamma_2 - \gamma_1)} f^\alpha(g(\frac{1}{\gamma_1})) \]

\[
= \frac{\gamma_1 \gamma_2 (\gamma_2 - \gamma_1)}{2 \delta \gamma_k(\gamma_2 - \gamma_1)(\alpha, \lambda)} \int_0^1 \left( \frac{\delta \gamma_k(\gamma_2 - \gamma_1)(\alpha, \lambda) - \delta \gamma_k(\gamma_2 - \gamma_1)(\alpha, \lambda) + \delta \gamma_k(\gamma_2 - \gamma_1)(\alpha, \lambda)}{(1 - \tau) \gamma_1 + \tau \gamma_2} \right) f' \left( \frac{\gamma_1 \gamma_2}{(1 - \tau) \gamma_1 + \tau \gamma_2} \right) d\tau,
\]

for all \( \alpha > 0, \lambda \geq 0 \).

\[
I = \frac{\gamma_1 \gamma_2 (\gamma_2 - \gamma_1)}{2 \delta \gamma_k(\gamma_2 - \gamma_1)(\alpha, \lambda)} \int_0^1 \delta \gamma_k(\gamma_2 - \gamma_1)(\alpha, 1 - \tau) \left( \frac{\gamma_1 \gamma_2}{(1 - \tau) \gamma_1 + \tau \gamma_2} \right) f' \left( \frac{\gamma_1 \gamma_2}{(1 - \tau) \gamma_1 + \tau \gamma_2} \right) d\tau
\]

\]

2.2. New Auxiliary Results. We now derive two new auxiliary results. These results will play a significant role in the development of our next results.

Lemma 1. Let \( f : [\gamma_1, \gamma_2] \rightarrow \mathbb{R} \) be \( L^1 \) function; then,

\[
I = \frac{\gamma_1 \gamma_2 (\gamma_2 - \gamma_1)}{2 \delta \gamma_k(\gamma_2 - \gamma_1)(\alpha, \lambda)} \int_0^1 \delta \gamma_k(\gamma_2 - \gamma_1)(\alpha, 1 - \tau) \left( \frac{\gamma_1 \gamma_2}{(1 - \tau) \gamma_1 + \tau \gamma_2} \right) f' \left( \frac{\gamma_1 \gamma_2}{(1 - \tau) \gamma_1 + \tau \gamma_2} \right) d\tau
\]

\[
= \frac{\gamma_1 \gamma_2 (\gamma_2 - \gamma_1)}{2 \delta \gamma_k(\gamma_2 - \gamma_1)(\alpha, \lambda)} \left[ I_1 - I_2 \right].
\]

Now,

\[
I_1 = \int_0^1 \frac{\delta \gamma_k(\gamma_2 - \gamma_1)(\alpha, 1 - \tau)}{(1 - \tau) \gamma_1 + \tau \gamma_2} f' \left( \frac{\gamma_1 \gamma_2}{(1 - \tau) \gamma_1 + \tau \gamma_2} \right) d\tau
\]

\[
= \delta \gamma_k(\gamma_2 - \gamma_1)(\alpha, 1) f(\gamma_2) - \int_0^1 (1 - \tau)^{\alpha - 1} \delta \left( \frac{\lambda (\gamma_2 - \gamma_1)(1 - \tau)}{(1 - \tau) \gamma_1 + \tau \gamma_2} \right) f' \left( \frac{\gamma_1 \gamma_2}{(1 - \tau) \gamma_1 + \tau \gamma_2} \right) d\tau
\]

\[
= \delta \gamma_k(\gamma_2 - \gamma_1)(\alpha, 1) f(\gamma_2) - \frac{\gamma_1 \gamma_2}{\gamma_2 - \gamma_1} \int_{1/\gamma_1}^{1/\gamma_2} \left( \frac{1}{x} - x \right)^{\alpha - 1} \delta \left( \frac{\lambda (\gamma_2 - \gamma_1)(1/\gamma_1 - x)}{(1/\gamma_1 - x)} \right) f' \left( \frac{1}{x} \right) dx
\]

Similarly,
\[
I_2 = \int_0^1 \left[ \delta Y_\lambda(\gamma - \gamma) (\alpha, \tau) \right] f' \left( \frac{\nu_1 \nu_2}{(1 - \tau) \nu_1 + \nu_2} \right) d\tau
\]

\[
= -\delta Y_\lambda(\gamma - \gamma) (\alpha, 1) f(\nu_1) + \int_0^1 \tau^{a - 1} e^{(\lambda(\gamma - \gamma))} f\left( \frac{\nu_1 \nu_2}{(1 - \tau) \nu_1 + \nu_2} \right) d\tau
\]

\[
= -\delta Y_\lambda(\gamma - \gamma) (\alpha, 1) f(\nu_2) + \left( \frac{\nu_1 \nu_2}{\nu_1 - \nu_2} \right) \int_{1/\nu_1}^{1/\nu_2} \left( x - \frac{1}{\nu_2} \right)^{a - 1} e^{(\lambda(\gamma - \gamma))(x - (1/\nu_1))} f\left( \frac{1}{x} \right) dx
\]

\[
= -\delta Y_\lambda(\gamma - \gamma) (\alpha, 1) f(\nu_2) + \left( \frac{\nu_1 \nu_2}{\nu_1 - \nu_2} \right) \delta \Gamma_\delta(\alpha) f^a \delta Y_\lambda(\gamma - \gamma) (\alpha, 1) f(\nu_2).
\]

Substituting the values of \( I_1 \) and \( I_2 \) in (28), we get the required result.

\[ \square \]

**Lemma 2.** Let \( f : [\nu_1, \nu_2] \rightarrow \mathbb{R} \) be \( L^1 \) function; then,

\[
\frac{2^{a - 1} (\nu_1 \nu_2)^a}{\delta Y_\lambda(\alpha, \nu_2 - \nu_1)} \left[ \int_0^{2 \nu_1 \nu_2} \left( \nu_1 \nu_2/2 \nu_1 \right) \right] f^a g \left( \frac{1}{\nu_1} \right) + \int_0^{2 \nu_1 \nu_2} \left( \nu_1 \nu_2/2 \nu_1 \right) f^a g \left( \frac{1}{\nu_1} \right) - f \left( \frac{2 \nu_1 \nu_2}{\nu_1 + \nu_2} \right)
\]

\[
= \frac{(\nu_2 - \nu_1)^a}{2 \delta Y_\lambda(\alpha, \nu_2 - \nu_1)} \left[ \int_0^{2 \nu_1 \nu_2} \left( \nu_2 - \nu_1 \right) \delta Y_\lambda(\gamma - \gamma) (\alpha, \tau) \right] f' \left( \frac{2 \nu_1 \nu_2}{(2 - \tau) \nu_1 + \nu_2} \right) d\tau
\]

where \( a > 0, \lambda \geq 1 \).

**Proof.** Consider

\[
I = \frac{(\nu_2 - \nu_1)^a}{2 \delta Y_\lambda(\alpha, \nu_2 - \nu_1)} \left[ \int_0^{2 \nu_1 \nu_2} \left( \nu_2 - \nu_1 \right) \delta Y_\lambda(\gamma - \gamma) (\alpha, \tau) \right] f' \left( \frac{2 \nu_1 \nu_2}{(2 - \tau) \nu_1 + \nu_2} \right) d\tau
\]

\[
- \int_0^{2 \nu_1 \nu_2} \left( \nu_2 - \nu_1 \right) \delta Y_\lambda(\gamma - \gamma) (\alpha, \tau) \right] f' \left( \frac{2 \nu_1 \nu_2}{(2 - \tau) \nu_1 + \nu_2} \right) d\tau
\]

\[
= \frac{(\nu_2 - \nu_1)^a}{2 \delta Y_\lambda(\alpha, \nu_2 - \nu_1)} \left[ I_1 - I_2 \right].
\]

Using Remark 1 (1), we have

\[
I_1 = \int_0^{2 \nu_1 \nu_2} \left( \nu_2 - \nu_1 \right) \delta Y_\lambda(\gamma - \gamma) (\alpha, \tau) \right] f' \left( \frac{2 \nu_1 \nu_2}{(2 - \tau) \nu_1 + \nu_2} \right) d\tau
\]

\[
= -\delta Y_\lambda(\gamma - \gamma) (\alpha, 1) f \left( \frac{2 \nu_1 \nu_2}{\nu_1 + \nu_2} \right) + \int_0^{1} \tau^{a - 1} e^{(\lambda(\gamma - \gamma))} f\left( \frac{2 \nu_1 \nu_2}{(2 - \tau) \nu_1 + \nu_2} \right) d\tau
\]

\[
= \delta Y_\lambda(\gamma - \gamma) (\alpha, 1) f \left( \frac{2 \nu_1 \nu_2}{\nu_1 + \nu_2} \right) + \int_0^{1} \tau^{a - 1} e^{(\lambda(\gamma - \gamma))} f\left( \frac{2 \nu_1 \nu_2}{(2 - \tau) \nu_1 + \nu_2} \right) d\tau
\]

\[
= -\delta Y_\lambda(\gamma - \gamma) (\alpha, 1) f \left( \frac{2 \nu_1 \nu_2}{\nu_1 + \nu_2} \right) + \int_0^{1} \tau^{a - 1} e^{(\lambda(\gamma - \gamma))} f\left( \frac{2 \nu_1 \nu_2}{(2 - \tau) \nu_1 + \nu_2} \right) d\tau
\]

\[
= -\delta Y_\lambda(\gamma - \gamma) (\alpha, 1) f \left( \frac{2 \nu_1 \nu_2}{\nu_1 + \nu_2} \right) + \int_0^{1} \tau^{a - 1} e^{(\lambda(\gamma - \gamma))} f\left( \frac{2 \nu_1 \nu_2}{(2 - \tau) \nu_1 + \nu_2} \right) d\tau
\]

\[
= -\delta Y_\lambda(\gamma - \gamma) (\alpha, 1) f \left( \frac{2 \nu_1 \nu_2}{\nu_1 + \nu_2} \right) + \int_0^{1} \tau^{a - 1} e^{(\lambda(\gamma - \gamma))} f\left( \frac{2 \nu_1 \nu_2}{(2 - \tau) \nu_1 + \nu_2} \right) d\tau
\]
Similarly,

\[ I_2 = \int_0^1 \frac{2 \nu_1 \nu_2 (v_2 - v_1) \delta \gamma_k (v_2 - v_1) (\alpha, \tau)}{[\tau v_1 + (2 - \tau)v_2]^2} f' \left( \frac{2 \nu_1 \nu_2}{\tau v_1 + (2 - \tau)v_2} \right) d\tau \]

\[ = \delta \gamma_k (v_2 - v_1) (\alpha, 1) f \left( \frac{2 \nu_1 \nu_2}{\nu_1 + v_2} \right) - \int_0^1 \left( \frac{\nu_1 \nu_2}{v_2 - v_1} \right)^{\alpha - 1} e^{-\frac{(\lambda (v_2 - v_1) \tau)^{\delta}}{\delta}} f \left( \frac{2 \nu_1 \nu_2}{\tau v_1 + (2 - \tau)v_2} \right) d\tau \]

\[ = \delta \gamma_k (v_2 - v_1) (\alpha, 1) f \left( \frac{2 \nu_1 \nu_2}{\nu_1 + v_2} \right) - \frac{(2 \nu_1 \nu_2)^{\alpha}}{(v_2 - v_1)^{\alpha}} \int_0^{1/v_1} \left( \frac{1}{v_1} - x \right)^{\alpha - 1} e^{-2 \gamma_k 2/2} \Gamma(\alpha, \nu_1 + v_1, x) f \left( \frac{1}{x} \right) dx \]

\[ = \delta \gamma_k (\alpha, 1) f^a g \left( \frac{2 \nu_1 \nu_2}{\nu_1 + v_2} \right) - \frac{(2 \nu_1 \nu_2)^{\alpha}}{(v_2 - v_1)^{\alpha}} \int_0^{\nu_1 + v_1, v_2} \frac{\Gamma(\alpha, \nu_1 + v_1, x)}{\nu_1 + v_2} f^a g \left( \frac{1}{x} \right). \]

Substituting the values of \( I_1 \) and \( I_2 \) in (32) and using Remark 1 (1), we get the required result.

**Theorem 2.** Let \( f : [\nu_1, \nu_2] \rightarrow \mathbb{R} \) be \( L^1 \) function. If \( |f|^p, q \geq 1 \), is harmonically convex on \([\nu_1, \nu_2]\) with \( \nu_1 < \nu_2 \), then

\[ f(\nu_1) + f(\nu_2) \leq \frac{\nu_1 \nu_2}{2 \delta \gamma_k (\nu_2 - \nu_1)(\alpha, 1)} \left( \int_0^1 \left( \frac{\delta \gamma_k (\nu_2 - \nu_1)(\alpha, 1 - \tau) - \delta \gamma_k (\nu_2 - \nu_1)(\alpha, \tau)}{(1 - \tau)\nu_1 + \tau v_2} \right)^p d\tau \right)^{1/p} \left( \left| f^p (\nu_1)^q + |f^p (\nu_2)|^q \right| \right)^{1/q} \]

**Proof.** Using Lemma 1, the harmonic convexity of \(|f|^p\), and Hölder's inequality, we have

\[ f(\nu_1) + f(\nu_2) \leq \frac{\nu_1 \nu_2}{2 \delta \gamma_k (\nu_2 - \nu_1)(\alpha, 1)} \left( \int_0^1 \left( \frac{\delta \gamma_k (\nu_2 - \nu_1)(\alpha, 1 - \tau) - \delta \gamma_k (\nu_2 - \nu_1)(\alpha, \tau)}{(1 - \tau)\nu_1 + \tau v_2} \right)^p d\tau \right)^{1/p} \]

\[ \times \left( \int_0^1 \left| f \left( \frac{\nu_1 \nu_2}{(1 - \tau)\nu_1 + \tau v_2} \right) \right|^q d\tau \right)^{1/q} \]
Proof. Since

\[
\frac{\partial f_{\lambda, \nu_2 - \nu_1}}{\partial \alpha} (\alpha, 1) \left( \int_0^1 \left| \frac{\delta f_{\lambda, \nu_2 - \nu_1}}{\partial \nu_2} \left( \alpha, 1 - \tau \right) - \delta f_{\lambda, \nu_2 - \nu_1} \left( \alpha, \tau \right) \right|^p \frac{d\tau}{(1 - \tau) \nu_2 + \nu_1 \tau^2} \right)^{1/p} \times \left( \int_0^1 \left| f' (\nu_1) \right|^q + (1 - \tau) f' (\nu_2) \right|^{1/q} d\tau \right)^{1/p} = \frac{\nu_1 \nu_2 (\nu_2 - \nu_1)}{2 \delta f_{\lambda, \nu_2 - \nu_1}} (\alpha, 1) \left( \int_0^1 \left| \frac{\delta f_{\lambda, \nu_2 - \nu_1}}{\partial \nu_2} \left( \alpha, 1 - \tau \right) - \delta f_{\lambda, \nu_2 - \nu_1} \left( \alpha, \tau \right) \right|^p \frac{d\tau}{(1 - \tau) \nu_2 + \nu_1 \tau^2} \right)^{1/p} \right) \left( \left| f' (\nu_1) \right|^q + \left| f' (\nu_2) \right|^q \right)^{1/q}.
\]

This completes our proof.

\[ \square \]

**Theorem 3.** Let \( f : [\nu_1, \nu_2] \rightarrow \mathbb{R} \) be an harmonically convex function on \([\nu_1, \nu_2]\) with \( \nu_1 < \nu_2 \); then, we have

\[
f \left( \frac{2 \nu_1 \nu_2}{\nu_1 + \nu_2} \right) \leq \left( \frac{\nu_1 \nu_2}{\nu_2 - \nu_1} \right)^{\alpha} 2^{\alpha - 1} \delta f_\lambda (\alpha) \left( \int f_{\nu_1, \nu_2} (\nu_1, \nu_2) \frac{d\tau}{(2 - \tau) \nu_1 + \nu_2 \tau^2} \right) \leq f (\nu_1) + f (\nu_2),
\]

for all \( \alpha > 0, \lambda \geq 1 \).

**Proof.** Since \( f \) is an harmonically convex function, then we have

\[
f \left( \frac{2 \nu_1 \nu_2}{\nu_1 + \nu_2} \right) \leq \frac{f (x) + f (y)}{2}.
\]

This implies

\[
2 f \left( \frac{2 \nu_1 \nu_2}{\nu_1 + \nu_2} \right) \int_0^1 \tau^{\alpha - 1} e^{-\left( \lambda (\nu_2 - \nu_1) \right) \frac{\tau}{\delta}} d\tau \leq \int_0^1 \tau^{\alpha - 1} e^{-\left( \lambda (\nu_2 - \nu_1) \right) \frac{\tau}{\delta}} d\tau \cdot f \left( \frac{2 \nu_1 \nu_2}{(2 - \tau) \nu_1 + \nu_2 \tau^2} \right) d\tau + \int_0^1 \tau^{\alpha - 1} e^{-\left( \lambda (\nu_2 - \nu_1) \right) \frac{\tau}{\delta}} f \left( \frac{2 \nu_1 \nu_2}{\nu_1 (2 - \tau) \nu_2} \right) d\tau.
\]

Thus, we have

\[
2 \delta f_{\lambda, \nu_2 - \nu_1} (\alpha, 1) \left( \frac{2 \nu_1 \nu_2}{\nu_1 + \nu_2} \right) \leq \left( \frac{\nu_1 \nu_2}{\nu_2 - \nu_1} \right)^{\alpha} \int \left( x - \frac{1}{\nu_2} \right)^{\alpha - 1} e^{-\left( \lambda \nu_1 \nu_2 (x - \nu_2) \right) \frac{\tau}{\delta}} f \left( \frac{1}{x} \right) dx + \left( \frac{\nu_1 \nu_2}{\nu_2 - \nu_1} \right)^{\alpha} \int \left( \frac{1}{\nu_2} - x \right)^{\alpha - 1} e^{-\left( \lambda \nu_1 \nu_2 (\frac{1}{\nu_2} - x) \right) \frac{\tau}{\delta}} f \left( \frac{1}{x} \right) dx.
\]

\[ \square \]
Using Remark 1 (1), we have

\[ f\left(\frac{2v_1v_2}{v_1 + v_2}\right) \leq \frac{2^{a-1}(v_1v_2)^a}{\delta y_3(a, v_2 - v_1)} \left[ f^{a, 2\lambda v_1v_2}_{(v_1 + 2\lambda v_2, v_1, v_2)} \Delta f + f^{a, 2\lambda v_1v_2}_{(v_1 + 2\lambda v_2, v_1, v_2)} \Delta f \right]g\left(\frac{1}{v_2}\right) + f^{a, 2\lambda v_1v_2}_{(v_1 + 2\lambda v_2, v_1, v_2)} \Delta f \right] g\left(\frac{1}{v_1}\right) \]  \hspace{1cm} (42)

Also,

\[ f\left(\frac{2v_1v_2}{(2 - \tau)v_1 + \tau v_2}\right) + f\left(\frac{2v_1v_2}{\tau v_1 + (2 - \tau)v_2}\right) \leq f(v_1) + f(v_2) \]  \hspace{1cm} (43)

Multiplying the above inequality by \( \tau^{a-1}e^{-\lambda(\tau_2 - \tau_1)} \) and integrating with respect to \( \tau \) on \([0, 1]\), we have

\[ \int_{0}^{1} \tau^{a-1}e^{-\lambda(\tau_2 - \tau_1)}f\left(\frac{2v_1v_2}{(2 - \tau)v_1 + \tau v_2}\right) d\tau + \int_{0}^{1} \tau^{a-1}e^{-\lambda(\tau_2 - \tau_1)}f\left(\frac{2v_1v_2}{\tau v_1 + (2 - \tau)v_2}\right) d\tau \leq [f(v_1) + f(v_2)] \int_{0}^{1} \tau^{a-1}e^{-\lambda(\tau_2 - \tau_1)} \]  \hspace{1cm} (44)

Combining (42) and (46), we get the required inequality (37).

**Theorem 4.** If \( f: [v_1, v_2] \rightarrow \mathbb{R} \) is \( L^1 \) function, using the harmonic convexity property of \( |f'|q, q \geq 1 \), on \([v_1, v_2]\) with \( v_1 < v_2 \), we have

\[ \frac{2^{a-1}(v_1v_2)^a}{\delta y_3(a, v_2 - v_1)} \left[ f^{a, 2\lambda v_1v_2}_{(v_1 + 2\lambda v_2, v_1, v_2)} \Delta f + f^{a, 2\lambda v_1v_2}_{(v_1 + 2\lambda v_2, v_1, v_2)} \Delta f \right] g\left(\frac{1}{v_2}\right) + f^{a, 2\lambda v_1v_2}_{(v_1 + 2\lambda v_2, v_1, v_2)} \Delta f \right] g\left(\frac{1}{v_1}\right) \]  \hspace{1cm} (47)

**Proof.** Using Lemma 2 and the harmonic convexity property of \( |f'| \), we have

\[ \frac{2^{a-1}(v_1v_2)^a}{\delta y_3(a, v_2 - v_1)} \left[ f^{a, 2\lambda v_1v_2}_{(v_1 + 2\lambda v_2, v_1, v_2)} \Delta f + f^{a, 2\lambda v_1v_2}_{(v_1 + 2\lambda v_2, v_1, v_2)} \Delta f \right] g\left(\frac{1}{v_2}\right) + f^{a, 2\lambda v_1v_2}_{(v_1 + 2\lambda v_2, v_1, v_2)} \Delta f \right] g\left(\frac{1}{v_1}\right) \]
\[ \leq \frac{(v_2 - v_1)^a}{2\delta y_1(a, v_2 - v_1)} \left[ \int_0^1 \frac{2v_1 v_2 (v_2 - v_1) \delta y_{\lambda}(v_2 - v_1, 2 \tau)}{\tau} \right] \left( \frac{2v_1 v_2}{(2 - \tau)v_1 + \tau v_2} \right) \frac{d\tau}{\tau} \left( \frac{2v_1 v_2}{(2 - \tau)v_1 + \tau v_2} \right) \right] \]

\[ \leq \frac{(v_2 - v_1)^a}{2\delta y_1(a, v_2 - v_1)} \left[ \int_0^1 \frac{2v_1 v_2 (v_2 - v_1) \delta y_{\lambda}(v_2 - v_1, \tau)}{(2 - \tau)v_1 + \tau v_2} \right] \left( \int \frac{\frac{2v_1 v_2}{(2 - \tau)v_1 + \tau v_2}}{\tau} \left( \int f'\left( \frac{2v_1 v_2}{(2 - \tau)v_1 + \tau v_2} \right) \right) \frac{d\tau}{\tau} \right] \]

Changing the order of integration, we have

\[ = \frac{(v_2 - v_1)^a}{2\delta y_1(a, v_2 - v_1)} \left[ \frac{v_1 v_2}{v_2 - v_1} \ln \left( \frac{a y (2 - y) v_2}{(2 - y)v_1 + y b} \right) - \frac{4v_1^2 v_2 (1 - y)}{(2 - y)v_1 + y b} \right] \delta y_{\lambda}(v_2 - v_1, \tau) \frac{d\tau}{\tau} \]

This completes our proof.

\[ \square \]

**Theorem 5.** Let \( f : [v_1, v_2] \rightarrow \mathbb{R} \) be a \( L^1 \) function. If \( |f'|^q, q \geq 1 \), is harmonically convex on \( [v_1, v_2] \) with \( v_1 < v_2 \), then we have

\[ \frac{2^a}{\delta y_1(a, v_2 - v_1)} \left[ f' + \frac{f}{v_1 v_2} \left( \frac{v_1 v_2}{(2 - y)v_1 + y b} \right) \right] \left( \frac{1}{v_2} \right) + \left( M_1(a, \lambda) \right)^{1 - \frac{1}{q}} \left( M_2(a, \lambda) \right)^{f' \left( v_1 \right)^q} + \left( M_3(a, \lambda) \right)^{f' \left( v_2 \right)^q} \]

\[ \leq \frac{(v_2 - v_1)^a}{2\delta y_1(a, v_2 - v_1)} \left[ (M_1(a, \lambda))^{1 - \frac{1}{q}} \left( M_2(a, \lambda) \right)^{f' \left( v_1 \right)^q} + \left( M_3(a, \lambda) \right)^{f' \left( v_2 \right)^q} \right] \]

where

\[ M_1(a, \lambda) = \delta y_{\lambda}(v_2 - v_1, a, 1) \left( \frac{2v_1 v_2}{(2 - y)v_1 + y b} - \frac{2v_1 v_2}{v_1 + v_2} \right), \]

\[ M_2(a, \lambda) = \delta y_{\lambda}(v_2 - v_1, a, 1) \left( \frac{2v_1^2 v_2}{v_2 - v_1} \left( \frac{1}{v_2} - \frac{1}{(2 - y)v_1 + y b} \right) + \frac{v_1 v_2}{v_2 - v_1} \ln \left( \frac{v_1 + v_2}{(2 - y)v_1 + y b} \right) \right), \]

\[ M_3(a, \lambda) = \delta y_{\lambda}(v_2 - v_1, a, 1) \left( \frac{2v_1^2 v_2}{v_2 - v_1} \left( \frac{1}{(2 - y)v_1 + y b} - \frac{1}{v_1 + v_2} \right) - \frac{v_1 v_2}{v_2 - v_1} \ln \left( \frac{v_1 + v_2}{(2 - y)v_1 + y b} \right) \right). \]
\[ M_4(\alpha, \lambda) = \delta Y_4(\nu_2-\nu_1)(\alpha, \lambda) \left( \frac{2\nu_1\nu_2}{\nu_1 + \nu_2}, \frac{2\nu_1\nu_2}{\nu_1 + (2-y)\nu_2} \right), \]
\[ M_5(\alpha, \lambda) = \delta Y_5(\nu_2-\nu_1)(\alpha, \lambda) \left[ \frac{2\nu_1^2\nu_2}{\nu_2 - \nu_1} \left( \frac{1}{\nu_2 + (2-y)\nu_2} - \frac{1}{\nu_1 + \nu_2} \right) - \frac{\nu_1\nu_2}{\nu_2 - \nu_1} \ln \left( \frac{\nu_1 + \nu_2}{\nu_2 - \nu_1} \right) \right], \]
\[ M_6(\alpha, \lambda) = \delta Y_6(\nu_2-\nu_1)(\alpha, \lambda) \left[ \frac{2\nu_1\nu_2^2}{\nu_2 - \nu_1} \left( \frac{1}{\nu_1 + \nu_2} - \frac{1}{\nu_1 + (2-y)\nu_2} \right) + \frac{\nu_1\nu_2}{\nu_2 - \nu_1} \ln \left( \frac{\nu_1 + \nu_2}{\nu_2 - \nu_1} \right) \right]. \]
Changing the order of integration and after simple calculation, we get the required result. \( \square \)

\[
\left| \frac{f(v_1) + f(v_2)}{2} - \frac{1}{v_2 - v_1} \int_{v_1}^{v_2} f(x) \, dx \right| \leq \frac{v_1 v_2}{2} (v_2 - v_1)
\]

\[
\times \left( \frac{v_1^{-2p}}{2(p + 1)} \left[ {}_2F_1 \left( 2p, 1, p + 2; \frac{v_1 - v_2}{2v_1} \right) + {}_2F_1 \left( 2p, p + 1, p + 2; \frac{v_1 - v_2}{v_1 + v_2} \right) \right] \right)^{1/p} \left( \frac{f'(v_1)^p + f'(v_2)^p}{2} \right)^{1/q}.
\]

3. Applications

In this section, we investigate some applications to special means and special functions.

3.1. Applications to Special Means. We now recall some special means for two nonnegative numbers.

1. The arithmetic mean: \( A(v_1, v_2) = \frac{v_1 + v_2}{2} \)
2. The geometric mean: \( G(v_1, v_2) = \sqrt{v_1 v_2} \)
3. The harmonic mean: \( H(v_1, v_2) = \frac{2v_1 v_2}{v_1 + v_2} \)
4. The logarithmic mean: \( L(v_1, v_2) = \frac{v_1 - v_2}{\ln v_2 - \ln v_1} \)
5. The \( p \)-logarithmic mean: \( L_p(v_1, v_2) = \left( \frac{v_1^{p+1} - v_1^{p+1}/(p+1)(v_2 - v_1)}{p} \right), p \in \mathbb{R} \setminus \{-1, 0\} \)

Proposition 1. Let \( 0 < v_1 < v_2 \); then, we have

\[
|A(v_1, v_2) - L^{-1}(v_1, v_2)| \leq G^2(v_1, v_2) \frac{(v_2 - v_1)}{2}
\]

\[
\times \left( \frac{v_1^{-2p}}{2(p + 1)} \left[ {}_2F_1 \left( 2p, 1, p + 2; \frac{v_1 - v_2}{2v_1} \right) + {}_2F_1 \left( 2p, p + 1, p + 2; \frac{v_1 - v_2}{v_1 + v_2} \right) \right] \right)^{1/p}.
\]

Proposition 2. Let \( 0 < v_1 < v_2 \); then, we have

\[
A \left( v_1^{n+2}, v_2^{n+2} \right) - G^2(v_1, v_2)L_p \left( v_1, v_2 \right) \leq G^2(v_1, v_2)(n + 2) \frac{(v_2 - v_1)}{2}
\]

\[
\times \left( \frac{v_1^{-2p}}{2(p + 1)} \left[ {}_2F_1 \left( 2p, 1, p + 2; \frac{v_1 - v_2}{2v_1} \right) + {}_2F_1 \left( 2p, p + 1, p + 2; \frac{v_1 - v_2}{v_1 + v_2} \right) \right] \right)^{1/p}
\]

\[
A^{1/q} \left( v_1 \left( n+1 \right), v_2 \left( n+1 \right) \right).
\]

Proposition 3. For \( v > 1, v_1, v_2 \in \mathbb{R} \) with \( 0 < v_1 < v_2 \), then we have

\[
J_v = \sum_{n=0}^{\infty} \frac{(z/2)^{v+2n}}{n! \Gamma(v + n + 1)}.
\]

\[
J'_v = \frac{z}{2(v + 1)}J_{v+1}(z).
\]
\[
\frac{|f_r(v_2) - f_r(v_1)|}{v_2 - v_1} \leq \frac{\psi_q^2 f_{r+1}(v_1) + \psi_q^2 f_{r+1}(v_2)}{4(v + 1)} \tag{57}
\]

For a particular case, we have
\[
\frac{\cosh(v_2) - \cosh(v_1)}{v_2 - v_1} \leq \frac{\psi_q^2 \sinh(v_1) + \psi_q^2 \sinh(v_2)}{2} \tag{58}
\]

**Proof.** The proof is obtained by taking \( x \to x^2 f_r(x) \) and using it in Corollary 1. For the second inequality, we use relation \( f_{1/2}(z) = \cosh(z) \) and \( f_{1/2}(z) = \sin(z)/z \). \( \square \)

### 3.3 \( q \)-Digamma Functions

Let \( 0 < q < 1 \). Then, \( q \)-digamma function \( \psi_q \) is defined as
\[
\psi_q(x) = -\ln(1 - q) + \ln \sum_{u=0}^{\infty} \frac{q^{1-u}}{1 - q^{1-u}}
\]

\[
= -\ln(1 - q) + \ln q \sum_{u=0}^{\infty} \frac{q^{1-u}}{1 - q^{1-u}} \tag{59}
\]

For \( q > 1 \) and \( z > 1 \), the \( q \)-digamma function \( \psi_q \) is defined as
\[
\psi_q(x) = -\ln(1 - q) + \ln \left[ x - \frac{1}{2} - \sum_{u=0}^{\infty} \frac{q^{-(u+q)}}{1 - q^{-(u+q)}} \right]
\]

\[
= -\ln(1 - q) + \ln \left[ x - \frac{1}{2} - \sum_{u=1}^{\infty} \frac{q^{1-u}}{1 - q^{1-u}} \right] \tag{60}
\]

**Proposition 4.** For \( v_1, v_2 \in \mathbb{R} \), such that \( v_1 < v_2 \),
\[
H^2(v_1, v_2) \psi_q^2 \left( \frac{2v_1 v_2}{v_1 + v_2} \right) \leq \psi_q(\psi_q(v_1) - \psi_q(v_1)) \leq \frac{v_1^2 \psi_q(v_1) + v_2^2 \psi_q(v_2)}{2} \tag{61}
\]

**Proof.** The proof is straightforward, by considering \( f(x) = x^2 \psi_q(x) \) and using Remark 2. \( \square \)

**Proposition 5.** For \( v_1, v_2 \in \mathbb{R} \), such that \( v_1 < v_2 \),
\[
\left[ \frac{v_1^2 \psi_q(v_1) + v_2^2 \psi_q(v_2)}{2} - \psi_q(\psi_q(v_1) - \psi_q(v_1)) \right] \leq \frac{v_1 v_2 (v_2 - v_1)}{2}
\]

\[
\times \left( \frac{v_1^{2p}}{2(p + 1)} \left[ \text{E}_1 \left( 2p, 1, p + 2; \frac{v_1 - v_1}{2v_2} \right) + \text{E}_1 \left( 2p, p + 1, p + 2; \frac{v_1 - v_2}{v_1 + v_2} \right) \right]^{1/p} \right)^{1/q} \tag{62}
\]

**References**

[1] İ. İscan, “Hermite-Hadamard type inequalities for harmonically convex functions,” *Hacettepe Journal of Mathematics and Statistics*, vol. 43, no. 6, pp. 935–942, Article ID 279158, 2014.

[2] M. Z. Sarıkaya, E. Set, H. Yaldız, and N. Başak, “Hermite-Hadamard’s inequalities for fractional integrals and related fractional inequalities,” *Mathematical and Computer Modelling*, vol. 57, no. 9-10, pp. 2403–2407, 2013.

[3] M. Gürbüz, Y. Taşdan, and E. Set, “Some inequalities obtained by fractional integrals of positive real orders,” *Journal of Inequalities and Applications*, vol. 2020, no. 1, Article ID 152, 2020.

[4] İ. İscan and S. Wu, “Hermite-Hadamard type inequalities for harmonically convex functions via fractional integrals,” *Applied Mathematics and Computation*, vol. 238, pp. 237–244, 2014.

[5] M. U. Awan, M. A. Noor, M. V. Mihai et al., “Inequalities via harmonic convex functions: conformable fractional calculus approach,” *Journal of Mathematical Inequalities*, vol. 12, no. 1, pp. 143–153, 2018.
[6] S. Iftikhar, S. Erden, P. Kumam et al., “Local fractional Newton’s inequalities involving generalized harmonic convex functions,” *Advances in Difference Equations*, vol. 2020, no. 1, Article ID 185, 2020.

[7] Z. Sanli, M. Kunt, and T. Koroglu, “New Riemann-Liouville fractional Hermite-Hadamard type inequalities for harmonically convex functions,” *Arabian Journal of Mathematics*, vol. 9, pp. 431–441, 2020.

[8] M. M. Meerschaert, F. Sabzikar, and J. Chen, “Tempered fractional calculus,” *Journal of Computational Physics*, vol. 293, pp. 14–28, 2015.

[9] P. O. Mohammed, M. Z. Sarikaya, and D. Baleanu, “On the generalized Hermite-Hadamard inequalities via the tempered fractional integrals,” *Symmetry*, vol. 12, no. 4, p. 595, 2020.

[10] S. Mubeen and G. M. Habibullah, “k-fractional integrals and application,” *International Journal of Mathematics and Mathematical Science*, vol. 7, no. 1–4, pp. 89–94, 2012.

[11] M. Z. Sarikaya and A. Karaca, “On the k-Riemann-Liouville fractional integral and applications,” *International Journal of Statistics and Mathematics*, vol. 1, no. 3, pp. 33–43, 2014.

[12] H. Lei, G. Hu, Z.-J. Cao et al., “Hadamard -k fractional inequalities of Fejer type for GA-s-convex mappings and applications,” *Journal of Inequalities and Applications*, vol. 2019, Article ID 264, 2019.

[13] C. Luo, B. Yu, Y. Zhang et al., “Certain bounds related to multi-parameterized k-fractional integral inequalities and their applications,” *IEEE Access*, vol. 7, pp. 124662–124673, 2019.

[14] M. U. Awan, M. A. Noor, M. V. Mihai et al., “On bounds involving -k appell’s hypergeometric functions,” *Journal of Inequalities and Applications*, vol. 2017, Article ID 118, 2017.

[15] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, Amsterdam, Netherlands, 2006.