RIEMANNIAN MANIFOLDS WITH UNIFORMLY BOUNDED EIGENFUNCTIONS

JOHN A. TOTH AND STEVE ZELDITCH

Abstract. The standard eigenfunctions $\phi_\lambda = e^{i\langle \lambda, x \rangle}$ on flat tori $\mathbb{R}^n/L$ have $L^\infty$-norms bounded independently of the eigenvalue. In the case of irrational flat tori, it follows that $L^2$-normalized eigenfunctions have uniformly bounded $L^\infty$-norms. Similar bases exist on other flat manifolds. Does this property characterize flat manifolds? We give an affirmative answer for compact Riemannian manifolds with completely integrable geodesic flows.

0. Introduction

This paper is concerned with the relation between the dynamics of the geodesic flow $G^t$ on the unit sphere bundle $S^*M$ of a compact Riemannian manifold $(M, g)$ and the growth rate of the $L^\infty$-norms of its $L^2$-normalized $\Delta$-eigenfunctions (or 'modes') $\{\phi_\lambda\}$. Let $V_\lambda := \{\phi : \Delta \phi = \lambda \phi\}$ denote the $\lambda$-eigenspace for $\lambda \in Sp(\Delta)$ and define

$$L^\infty(\lambda, g) = \sup_{\phi \in V_\lambda, ||\phi||_{L^2} = 1} ||\phi||_{L^\infty}, \quad \ell^\infty(\lambda, g) = \inf_{ONB(\phi_j) \in V_\lambda} (\sup_{j=1, \ldots, \dim V_\lambda} ||\phi_j||_{L^\infty}).$$

The universal bound

$$L^\infty(\lambda, g) = O(\lambda^{\frac{4-n}{2}})$$

holds for any $(M, g)$ in consequence of the local Weyl law [Ho IV]

$$N(T, x) := \sum_{j : \lambda_j \leq T} |\phi_{\lambda_j}(x)|^2 = \frac{1}{(2\pi)^n} \text{vol}(M, g)T^{\frac{n}{2}} + R(T, x), \quad R(T, x) = O(T^{\frac{n-1}{2}}).$$

It is attained in the case of the standard $(S^n, can)$ (by the zonal spherical harmonics) but is far off in the case of irrational flat tori $(T^n, ds^2)$ where $L^\infty(\lambda, g) = O(1)$. These cases represent the extremes, and the problem arises of characterizing the manifolds with extremal growth rates of $L^\infty$-norms of eigenfunctions.

In this article, we are interested in the case of minimal growth:

- Problem: Determine the $(M, g)$ for which $\ell^\infty(\lambda, g) = O(1)$ and those for which $L^\infty(\lambda, g) = O(1)$.

The same kind of problem may be posed in the more general setting of semi-classical Schroedinger operators $\hbar^2 \Delta + V$. The eigenvalue problem $(\hbar^2 \Delta + V)\phi_j = E_j(\hbar)\phi_j$ now depends on $\hbar$, and we are interested in the behaviour of eigenfunctions $\phi_j$ in the semiclassical limit $\hbar \to 0$. The spectrum becomes dense around each regular value $E$ of the classical Hamiltonian $H(x, \xi) = |\xi|^2_g + V(x)$ on $T^*M$, and for any $0 < \delta < 1$, the asymptotics of spectral data from an interval $[E - c\hbar^{1-\delta}, E + c\hbar^{1+\delta}]$ around $E$ will reflect the dynamics of the classical Hamiltonian flow $\Phi^E_t$ on the energy surface $X_E = \{H(x, \xi) = E\}$. We fix $E$ and $0 < \delta < 1$,
and consider the eigenvalues $E_j(h) \in [E - ch^{1-\delta}, E + ch^{1-\delta}]$. Denote by $V_{E_j(h)}$ the eigenspace of eigenvalue $E_j(h)$ and put:

$$L^\infty(h, E_j(h); g, V) = \sup_{\phi \in V_{E_j(h)}} ||\phi||_{L^\infty}, \quad \ell^\infty(h, E_j(h); g, V) = \inf_{ONB(\phi_j) \in V_{E_j(h)}} \left( \sup_{j=1, \ldots, \dim V_{E_j(h)}} ||\phi||_{L^\infty} \right),$$

and pose the analogous questions:

- **Problem:** Determine the $(M, g, V)$ for which there exists a regular energy level $E$ such that $\ell^\infty(h, E_j(h); g, V) = O(1)$ and the $(M, g, V)$ for which $L^\infty(h, E_j(h); g, V) = O(1)$ as $h \to 0$ with $E_j(h) \in [E - ch^{1-\delta}, E + ch^{1-\delta}]$ for some $c > 0$.

The problem on Laplace operators is the same as the problem on Schrödinger operators in the case $V = 0$, for any value of $E > 0$.

The problems about $\ell^\infty$ asks which Laplacians or Schrödinger operators possess an ONBE (orthonormal basis of eigenfunctions) of minimal growth. The problems about $L^\infty$ ask which ones have the property that every ONBE has minimal growth. Obviously, the distinction between $\ell^\infty$ and $L^\infty$ only arises when the spectrum of $\Delta$ is multiple. At the opposite extreme, one may ask which $(M, g)$ possess eigenfunctions which achieve the maximal rate of growth, but we will not discuss that problem here. One may also pose quantitative problems of giving upper and lower bounds on $\ell^\infty(\lambda, g)$, $L^\infty(\lambda, g)$, and their $L^p$-analogues, under various dynamical hypotheses. Some results on such quantitative problems will be given in a subsequent article [12].

The known connections between $h^2 \Delta + V$-eigenfunctions and the dynamics of $\Phi_t^L$ are not strong enough at present to answer these questions in the general setting of compact Riemannian manifolds. If, however, the systems are assumed to be completely integrable geodesic flows then much more can be said.

Let us assume in fact that $\Delta$ is quantum completely integrable in the (well-known) sense that there exist $P_1, \ldots, P_n \in \Psi^1(M) \ (n = \dim M)$ satisfying

- $[P_i, P_j] = 0$;
- $dp_1 \wedge dp_2 \wedge \cdots \wedge dp_n \neq 0$ on a dense open set $\Omega \subset T^*M - 0$ of ‘finite complexity’ (see below);
- $\sqrt{\Delta} = K(P_1, \ldots, P_n)$ for some polyhomogeneous function $K$ on $\mathbb{R}^n - 0$.

Here, $\Psi^m(M)$ is the space of $m$th order pseudodifferential operators over $M$, and $p_k = \sigma_{p_k}$ is the principal symbol of $P_k$. Since $\sigma_{[P_i, P_j]} = \{p_i, p_j\}$ (the Poisson bracket), it follows that the $p_j$’s generate a homogeneous Hamiltonian action $\Phi_t$ of $t \in \mathbb{R}$ on $T^*M - 0$ with moment map

$$\mathcal{P} : T^*M - 0 \to \mathbb{R}^n, \quad \mathcal{P} = (p_1, \ldots, p_n).$$

We denote the image $\mathcal{P}(T^*M - 0)$ by $B$, by $B_{reg}$ (resp. $B_{sing}$) the regular values (resp. singular values) of the moment map. By ‘finite complexity’ we mean the following: for each $b = (b^{(1)}, \ldots, b^{(n)}) \in B$, let $m_{cl}(b)$ denote the number of $\mathbb{R}^n$-orbits of the joint flow $\Phi_t$ on the level set $\mathcal{P}^{-1}(b)$. Then

**Finite complexity condition:** $\exists M : m_{cl}(b) < M \ (\forall b \in B). \quad (3)$

When $b \in B_{reg}$, then $\mathcal{P}^{-1}(b)$ is the union of $m_{cl}(b)$ isolated Lagrangean tori. If $b \in B_{sing}$, then $\mathcal{P}^{-1}(b)$ consists of a finite number of connected components, each of which is a finite union of orbits. These orbits may be Lagrangean tori, singular compact tori (ie. compact tori of dimension $< n$), or non-compact orbits consisting of cylinders or planes.

We will also make the following assumption on the quantum level:

**Bounded eigenvalue multiplicity:** $\exists M' : m(\lambda) \leq M' \ (\forall \lambda; \ m(\lambda) = \dim V_\lambda). \quad (4)$

With this assumption, $L^\infty$ is bounded by a constant times $\ell^\infty$, so all ONBE’s are uniformly bounded if and only if one is. Without assumption (4), it is simple to construct an ONBE which is not uniformly bounded.
We will recall the construction in §4, and discuss some open problems in which the bounded eigenvalue multiplicity is dropped.

The Hamiltonian \( |\xi|_g = \sqrt{\sum_{i,j=1}^{n} g^{ij}(x) \xi^i \xi^j} \) is then given by \( |\xi|_g = K(p_1, \ldots, p_n) \) where \( K \) is the homogeneous term of order 1 of \( K \). Hence the geodesic flow commutes with a Hamiltonian \( \mathbb{R}^n \)-action, i.e. it is completely integrable. We will assume throughout the following properness assumption:

Our main result is the following rigidity theorem:

**Theorem 0.1.** Suppose that \( \Delta \) is a quantum completely integrable Laplacian on a compact Riemannian manifold \( (M, g) \), and suppose that the corresponding moment map satisfies \( \mathbb{R}^n \). Then:

(a) If \( L^\infty(\lambda, g) = O(1) \) then \( (M, g) \) is flat.

(b) If \( \ell^\infty(\lambda, g) = O(1) \) and if \( \mathbb{R}^n \) holds, then \( (M, g) \) is flat.

More generally, suppose that \( h^2 \Delta + V \) is a quantum completely integrable Schrödinger operator, and that the corresponding moment map \( P \) is proper and satisfies \( \mathbb{R}^n \). Assume there exists an energy level \( E \) such that:

(a) \( L^\infty(h, E_j(h); g, V; ) = O(1) \) as \( h \to 0 \);

(b) \( \ell^\infty(h, E_j(h); g, V) = O(1) \) as \( h \to 0 \), and \( \mathbb{R}^n \) holds.

Then: \( E > \max V \) and \( (M, (E - V)g) \) is flat. If (a) (or (b)) holds for all energy levels \( E \) in an interval \( E_1 < E < E_2 \), then \( (M, g) \) is flat and \( V \) is constant.

As mentioned above, (a)-(b) are equivalent so we only consider (a) henceforth.

We recall that flat manifolds are manifolds carrying a flat metric. By the Bieberbach theorems ([W], Theorems 3.3.1 - 3.3.2), a flat manifold \( (M, g) \) may be expressed as the quotient \( M = \mathbb{R}^n / \Gamma \) of \( \mathbb{R}^n \) by a discrete (crystallographic) subgroup of Euclidean motions \( \Gamma \subset \mathbb{E}(n) \). The subgroup \( \Gamma^* := \Gamma \cap \mathbb{R}^n \) is normal and of finite index in \( \Gamma \) so there exists a flat torus \( T^n = \mathbb{R}^n / \Gamma^* \) and a finite normal Riemannian cover \( \pi : T^n \to M \) with deck transformation group \( G = \Gamma / \Gamma^* \). For each \( n > 0 \), there are only finitely many affine equivalence classes of flat compact connected \( (M, g) \) of dimension \( n \) (affinely equivalent same fundamental group), and in low dimensions they have been classified (cf. [W]). The eigenfunctions \( \phi_\lambda \) of \( \Delta_g \) on \( (M, g) \) may be lifted to \( G \)-invariant eigenfunctions \( \pi^* \phi_\lambda \) on \( T^n \) and hence the eigenspace \( E_\lambda(M, g) \) may be identified with the \( G \)-invariant eigenspace \( E_\lambda(T^n, g_T)^G \). The latter eigenfunctions may be written as sums of exponential functions.

Let us outline the proof of the Theorem 0.1 in the simplest case of *toric* integrable systems (see §1 for background), and then explain what more is involved in the case of general integrable systems. By definition, the geodesic flow \( G_g^t : T^* M \to T^* M \) of a compact Riemannian manifold \( (M, g) \) is toric integrable if it commutes with a Hamiltonian action of the \( n \)-torus \( \mathbb{R}^n / \mathbb{Z}^n \). Equivalently, if there exist global action variables \( \{ (I_j, \theta_j) : j = 1, \ldots, n \} \) for the geodesic flow, i.e. functions of \( (p_1, \ldots, p_n) \) whose Hamilton flows are \( 2\pi \)-periodic. The level sets \( T_1 := \mathcal{I}^{-1}(I) \) of the moment map

\[
\mathcal{I} = (I_1, \ldots, I_n) : T^* M - 0 \to \mathbb{R}^n
\]

are then orbits \( \mathbb{R}^n / \mathbb{Z}^n \cdot (x_0, \xi_0) \) of the torus action and hence are tori. The image \( B \) of \( T^n - 0 \) under \( \mathcal{I} \) is a convex polyhedral cone and \( \mathcal{I} \) is a Lagrangean torus bundle over its interior. Such moment maps \( \mathcal{I} \) are the cotangent bundle analogues of toric varieties in algebraic geometry.

In the toric case, it is always possible to quantize the action variables as first order pseudodifferential *action operators* \( \hat{I}_j \) which commute with \( \Delta \). The actions define a (projective) action of \( \mathbb{R}^n / \mathbb{Z}^n \) by Fourier integral operators, or equivalently, the joint spectrum \( \text{Sp}(\hat{I}_1, \ldots, \hat{I}_1) \) is contained in an (off-centered) lattice \( \mathbb{Z}^n + \mu \). The joint eigenfunctions

\[
(\hat{I}_1, \ldots, \hat{I}_n) \phi_\lambda = \lambda \phi_\lambda \quad \lambda \in \mathbb{R}^n
\]
are therefore quantizations of the invariant Lagrangean torii $T_\lambda$ with integral actions $\lambda \in \mathbb{Z}^n + \mu$. In particular, eigenfunctions $\{\phi_\lambda\}$ localize on the invariant tori in the semiclassical limit in the sense that for any zeroth order pseudodifferential operator $A$ (with symbol $\sigma_A$),

$$ (A\phi_{k\lambda}, \phi_{k\lambda}) = \int_{T_\lambda} \sigma_A d\mu_\lambda + O(k^{-1}), \quad (5) $$

where $d\mu_\lambda$ is the normalized Lebesgue (probability) measure on $T_\lambda$. Hence, $|\phi_\lambda(x)|^2$ measures the density of the natural projection $\pi_\lambda: T_\lambda \to M$ at $x$.

The proof of Theorem (0.1) in the toric case is based on the following simple Lemmas. First we have:

Suppose that $G^t$ is toric integrable and that $L^\infty(M, g) = 0(1)$. Then every invariant torus $T_\lambda$ has a non-singular projection to $M$.

The proof uses the fact that for any invariant torus $T_I$, there exists a sequence of joint eigenfunctions $\{\phi_\lambda\}$ of the quantum torus action which localizes on $T_I$. Uniform boundedness of the eigenfunctions then implies regular projection of the tori.

The second ingredient in the proof of the main theorem in the case of toric integrable systems is the following purely geometric statement which follows from the recently proved Hopf conjecture (cf. [BI] [CK]).

Suppose that $(M, g)$ is a compact Riemannian manifold with toric integrable geodesic flow, and suppose that all the invariant torii project regularly to $M$. Then $(M, g)$ is a flat manifold.

By ‘projecting regularly’ we mean that the projection has no singular values, hence (in view of the dimensions) is a covering map.

The proof of Theorem (1.1) in the case of general Hamiltonian $\mathbb{R}^n$ actions is basically similar, but there are some new complications to handle. Geometrically, the new features are that the fibers $P^{-1}(b)$ may have several components (‘geometric multiplicity’), that there may exist non-compact orbits (e.g. embedded cylinders), and that there may exist singular orbits lying over the interior of the image of $T^*M - 0$ under $P$. Analytically, the main new feature is that modes need not localize on individual components of $P^{-1}(b)$.

What does localize on individual torii are quasimodes, i.e. semiclassical Lagrangean distributions which approximately solve the eigenvalue problem. In the toric case, modes and quasimodes are the same but this is not the case in general. As originally stressed by Arnold [A], and as is evident from simple examples such as the symmetric double well potential, eigenfunctions may be linear combinations of quasi-modes with very close quasi-eigenvalues and in the classical limit their mass concentrates in some way on the union of the components. How the mass is distributed involves the question whether the tori are resonant or not, and whether or not there is tunnelling between torii. We will discuss such relations between modes and quasimodes in detail in [TZ], where we prove (among other things) that quasimodes have uniformly bounded sup norms when modes do and where we determine precisely how modes blow up around singular orbits. In this paper, we take a softer approach via quantum limits of eigenfunctions and semiclassical trace formulae.

We close with some acknowledgements. We thank Bruce Kleiner for pointing out the paper [4], Leonid Polterovich for helpful comments on [BP], and Francois Lalonde for helpful comments on an earlier version of the paper. We would especially like to thank the referee of this paper for pointing out that one of our original (non-degeneracy) hypotheses could be removed from the proof of Theorem (0.1), and for several other corrections and improvements. To clarify the ingredients in the proof, we cut the original manuscript (which appeared on the lanl archive as math-ph/0002038) into two parts, the present qualitative one and the subsequent quantitative one ([TZ]).

1. Background

1.1. Completely integrable systems. By a completely integrable system on $T^*M$ we mean a set of $n$ independent, $C^\infty$ functions $p_1, \ldots, p_n$, on $T^*M$ satisfying:
The associated moment map is defined by
\[ \mathcal{P} = (p_1, \ldots, p_n) : T^* M \to B \subset \mathbb{R}^n. \] (6)
We refer to to the set \( B \) as the ‘image of the moment map.’ The Hamiltonians generate an action of \( \mathbb{R}^n \) defined by
\[ \Phi_t = \exp t_1 \Xi_{p_1} \circ \exp t_2 \Xi_{p_2} \cdots \circ \exp t_n \Xi_{p_n}. \]
We often denote \( \Phi_t \)-orbits by \( \mathbb{R}^n \cdot (x, \xi) \). The isotropy group of \( (x, \xi) \) will be denoted by \( \mathcal{I}_{(x, \xi)} \). When \( \mathbb{R}^n \cdot (x, \xi) \) is a compact Lagrangean orbit, then \( \mathcal{I}_{(x, \xi)} \) is a lattice of full rank in \( \mathbb{R}^n \), and is known as the ‘period lattice’, since it consists of the ‘times’ \( T \in \mathbb{R}^n \) such that \( \Phi_T |_{\Lambda^{(j)}(b)} = \text{Id} \).

We will need the following:

**Definition 1.1.** We say that:

- \( b \in B_{\text{sing}} \) if \( \mathcal{P}^{-1}(b) \) is a singular level of the moment map, i.e. if there exists a point \( (x, \xi) \in \mathcal{P}^{-1}(b) \) with \( dp_1 \wedge \cdots \wedge dp_n(x, \xi) = 0 \). Such a point \( (x, \xi) \) is called a singular point of \( \mathcal{P} \).
- a connected component of \( \mathcal{P}^{-1}(b) \) \( (b \in B_{\text{sing}}) \) is a singular component if it contains a singular point ;
- an orbit \( \mathbb{R}^n \cdot (x, \xi) \) of \( \Phi_t \) is singular if it is non-Lagrangean, i.e. has dimension \( < n \);
- \( b \in B_{\text{reg}} \) and that \( \mathcal{P}^{-1}(b) \) is a regular level if all points \( (x, \xi) \in \mathcal{P}^{-1}(b) \) are regular, i.e. if \( dp_1 \wedge \cdots \wedge dp_n(x, \xi) \neq 0 \).
- a component of \( \mathcal{P}^{-1}(b) \) \( (b \in B_{\text{sing}} \cup B_{\text{reg}}) \) is regular if it contains no singular points.

By the Liouville-Arnold theorem [AM], the orbits of the joint flow \( \Phi_t \) are diffeomorphic to \( \mathbb{R}^k \times T^m \) for some \( (k, m), k + m \leq n \). By the properness assumption on \( \mathcal{P} \), a regular level has the form
\[ \mathcal{P}^{-1}(b) = \Lambda^{(1)}(b) \cup \cdots \cup \Lambda^{(m_{\text{cl}})}(b), \quad (b \in B_{\text{reg}}) \] (7)
where each \( \Lambda^{(i)}(b) \simeq T^n \) is an \( n \)-dimensional Lagrangian torus. The classical (or geometric) multiplicity function \( m_{\text{cl}}(b) = \# \mathcal{P}^{-1}(b) \), i.e. the number of orbits on the level set \( \mathcal{P}^{-1}(b) \), is constant on connected components of \( B_{\text{reg}} \) and the moment map \( \mathcal{P} \) is a fibration over each component with fiber \( \{ \} \). In sufficiently small neighbourhoods \( \Omega^{(i)}(b) \) of each component torus, \( \Lambda^{(i)}(b) \), the Liouville-Arnold theorem also gives the existence of local action-angle variables \( (I_1^{(i)}, \ldots, I_n^{(i)}, \theta_1^{(i)}, \ldots, \theta_n^{(i)}) \) in terms of which the joint flow of \( \Xi_{p_1}, \ldots, \Xi_{p_n} \) is linearized [AM]. For convenience, we henceforth normalize the action variables \( I_1^{(i)}, \ldots, I_n^{(i)} \) so that \( I_j^{(i)} = 0; \quad j = 1, \ldots, n \) on the torus \( \Lambda^{(i)}(b) \).

When \( b \in B_{\text{reg}} \), the Lagrangean tori \( \Lambda^{(j)}(b) \) of \( \mathcal{P}^{-1}(b) \) carry two natural measures, which we take some care to distinguish.

**Definition 1.2.** We define:

- Lebesgue measure \( d\mu^{(j)}_b = (2\pi)^{-n} d\theta_1 \wedge \cdots \wedge d\theta_n \) on \( \Lambda^{(j)}(b) \), as the normalized (mass one) \( \Phi_t \)-invariant measure on this orbit;
- The Liouville measure \( d\omega^{(j)}_b \) on \( \Lambda^{(j)}(b) \), as the surface measure induced by the moment map \( \mathcal{P} \), i.e.
\[ d\omega^{(j)}_b = \frac{dV}{dp_1 \wedge \cdots \wedge dp_n} \]
where \( dV \) is the symplectic volume measure on \( T^* M \). By the Liouville mass of \( \Lambda^{(j)}(b) \) we mean the integral
\[ \omega^{(j)}(b) := \int_{\Lambda^{(j)}(b)} d\omega^{(j)}_b. \]
The Liouville mass of a compact Lagrangean orbit $\Lambda^{(j)}(b)$ has a simple dynamical interpretation: it is the Euclidean volume of the fundamental domain of the common period lattice $I^{(j)}_b = I^{(j)}_{(x, \xi)}$ of points $(x, \xi) \in \Lambda^{(j)}(b)$, i.e.

$$\omega^{(j)}(b) = Vol(\mathbb{R}^n/I^{(j)}_b).$$  \hspace{1cm} (8)

Indeed, by writing Liouville measure in local action-angle variables, we see that

$$d\omega^{(j)}_b = \det(T^b_\ell(b))d\mu^{(j)}_b, \quad \text{where} \quad T^b_\ell = \frac{\partial I_k}{\partial p_\ell}.$$  \hspace{1cm} (9)

It is clear from the definition of the action-angle variables that $I^{(j)}_b$ is the Euclidean volume of the fundamental domain of the common period lattice.

We now turn to singular levels. When $b \in B_{\text{sing}}$ we first decompose

$$\mathcal{P}^{-1}(b) = \bigcup_{j=1}^{p} \Gamma^{(j)}_{\text{sing}}(b)$$

the singular level into connected components $\Gamma^{(j)}_{\text{sing}}(b)$ and then decompose

$$\Gamma^{(j)}_{\text{sing}}(b) = \bigcup_{k=1}^{n} \mathbb{R}^n \cdot (x_k, \xi_k)$$  \hspace{1cm} (11)

each component into orbits. Both decompositions can take a variety of forms. The regular components $\Gamma^{(j)}_{\text{sing}}(b)$ must be Lagrangean tori by the properness assumption. A singular component consists of finitely many orbits by the finite complexity assumption. The orbit $\mathbb{R}^n \cdot (x, \xi)$ of a singular point is necessarily singular, hence has the form $\mathbb{R}^k \times T^m$ for some $(k, m)$ with $k + m < n$. Regular points may also occur on a singular component, whose orbits are Lagrangean and can take any one of the forms $\mathbb{R}^k \times T^m$ for some $(k, m)$ with $k + m = n$.

We will need the following result in the proof of Theorem 0.1:

**Proposition 1.3.** A singular component $\Gamma^{(j)}_{\text{sing}}(b) \subset \mathcal{P}^{-1}(b)$ (with $b \in B_{\text{sing}}$) must contain a compact singular orbit $\mathbb{R}^n \cdot (x, \xi) \simeq T^k, k < n$.

**Proof:** It follows by a standard averaging argument \cite{M2} that the set $\mathcal{M}^{(j)}_{\text{sing}}$ of invariant probability measures supported on $\Gamma^{(j)}_{\text{sing}}$ is non-empty: for any probability measure $\mu$ supported on $\Gamma^{(j)}_{\text{sing}}$, the set of weak* limit points of the set of finite time averages $\mu_T = \frac{1}{\text{vol}(I^{(j)}_{\text{sing}})} \int_{|t| \leq T} \mu(\Phi_t) \mu_0 \, dt$ gives at least one non-trivial element of $\mathcal{M}^{(j)}_{\text{sing}}$. Since $\Gamma^{(j)}_{\text{sing}}$ consists of only finitely many orbits, any invariant measure in $\mathcal{M}^{(j)}_{\text{sing}}$ is a finite sum of (ergodic) measures, each supported on just one orbit. The non-compact orbits $\mathbb{R}^k \times T^m$ obviously cannot carry invariant probability measures; hence, at least one orbit must be compact. \hfill $\Box$

We will need a further result on Hamiltonian $\mathbb{R}^n$-actions $\Phi_t$. We define a non-zero period of $\Phi_t$ to be a time $T \in I^{(j)}_b - \{0\}$ for some $(b, j)$, and denote the set of periods by $\mathcal{T}$.

**Proposition 1.4.** There exists a constant $C > 0$, which depends on the Riemannian manifold $(M, g)$, such that $\inf_{\{T \in \mathcal{T}\}} |T| \geq C$.

**Proof:** In the case of a Hamiltonian flow with Hamilton vector field $\mathbf{X}$, this is a case of Yorke’s theorem \cite{Y}. In fact, $C = \frac{2\pi}{L}$ where $L = \|d\mathbf{X}\|_\infty$. In the case of $\mathbb{R}^n$ actions, we can apply Yorke’s theorem to any one parameter subgroup. \hfill $\Box$

### 1.2. Hamiltonian torus actions

In special cases (see \cite{Y} for the geometric conditions), the Hamiltonian $\mathbb{R}^n$ action descends to the Hamiltonian action of the torus $\mathbb{R}^n/\mathbb{Z}^n$ on $T^*M$. Such Hamiltonian torus actions are the cotangent space analogues of toric varieties in algebraic geometry. In this case, there exist generators

$$I := (I_1, \ldots, I_n) : T^*M \rightarrow B \subset \mathbb{R}^n.$$
of of the Hamiltonian $\mathbb{R}^n$ action so that each $I_j$ generates a $2\pi$-periodic Hamiltonian flow. The components $I_j$ are called global action variables and $I$ is called a toric moment map. In the toric case, $B$ is a convex polyhedral cone, $B_{reg}$ is simply the interior of $B$, $B_{sing} = \partial B$ (its boundary) and $m_{cl}(b) \equiv 1$. Since tori are now labelled by actions, we write $T_I := \mathcal{I}^{-1}(I)$. Singular orbits $\mathbb{R}^n \cdot (x, \xi)$ are obviously compact non-Lagrangian tori, and singular levels consist of just one singular orbit.

Examples:
(i) $M = \mathbb{R}^n/\mathbb{Z}^n$, $I_j = \xi_j$, the usual linear coordinates on $T^*(\mathbb{R}^n/\mathbb{Z}^n)$.
(ii) $M = \mathbb{S}^2$, $I_1 = p_0$, $I_2 = |\xi|_0$, where $p_0(x, \xi) = \xi(\frac{\partial}{\partial \theta})$ (the infinitesimal generator of rotations around the $z$-axis), and where $|\xi|_0$ is the length function of the standard metric.

1.3. Riemannian manifolds with completely integrable geodesic flow. Now suppose that $g$ is a Riemannian metric on $M$ and let $H(x, \xi) = |\xi|_g$ denote the associated length function on covectors. The Hamilton flow $G_t$ of $H$ on $T^*M - 0$ is homogeneous of degree 1 with respect to the natural $\mathbb{R}^+$ action, and will be referred to as the geodesic flow. It leaves invariant the cosphere bundles $S^*M_E = \{ H = E \}$ and the flows $G^E_t$ on $S^*M_E$ are all equivalent under dilation $(x, \xi) \rightarrow E(x, \xi)$ to $G^1_t$.

The geodesic flow $G_t$ will be called integrable if it commutes with a homogeneous Hamiltonian action of $\mathbb{R}^n$. We may then put $H = p_1$. It is called toric integrable if it commutes with a homogeneous Hamiltonian action of $\mathbb{R}^n/\mathbb{Z}^n$. Because $m_{cl}(b) \equiv 1$ in this case, there exists a homogeneous function $K$ on $B$ such that $H = K(I)$.

Examples: The following is a short list of examples:
(i) $M = \mathbb{R}^n/\mathbb{Z}^n$ and $g$ is flat. Then $(M, g)$ is toric integrable.
(ii) $M = \mathbb{S}^2$ and $g$ is a rotationally invariant metric. If $g$ is of 'simple type' (e.g. convex), then $(M, g)$ is toric integrable [CV1].
(iii) $M = \mathbb{S}^2$ and $g$ is the metric for which $(S^2, g)$ is an ellipsoid.
(iv) $M = \mathbb{R}^2/\mathbb{Z}^2$ and $g$ is a Liouville metric (cf. [B.K.S, KMS]).
(v) Bi-invariant metrics on compact Lie groups. Geodesic flow on $SO(3)$ is known as the Euler top.

1.4. Manifolds without conjugate points. A Riemannian manifold $(M, g)$ is said to be without conjugate points if there exists a unique geodesic between each two points of its universal Riemannian cover $(\hat{M}, \hat{g})$, or equivalently if every exponential map $\exp_x : T_xM \rightarrow M$ is non-singular. We will need the following geometric theorems on manifolds without conjugate points.

Theorem 1.5. [M] Let $(M, g)$ be a compact Riemannian manifold with (co)-geodesic flow $G^t : T^*M - 0 \rightarrow T^*M - 0$. Suppose that $G^t$ preserves a (non-singular) Lagrangean foliation $\mathcal{L}$ of $T^*M - 0$, i.e. suppose that $G^t L = L$ for all leaves $L$ of $\mathcal{L}$. Then $(M, g)$ has no conjugate points.

The Hopf conjecture on tori without conjugate points was proved by Burago-Ivanov:

Theorem 1.6. [B] Suppose that $g$ is a metric on the $n$-torus $T^n$ without conjugate points. Then $g$ is flat.

1.5. Integrable Newtonian flows on cotangent bundles. We will also consider Newtonian flows, i.e. flows of classical Hamiltonians $H(x, \xi) = \frac{1}{2}|\xi|^2 + V(x)$ on cotangent bundles $T^*M$. Such Hamiltonians and their flows $G_t$ are no longer homogeneous. The invariant energy surfaces $X_E = \{ H = E \}$ and the restricted flows $G^E_t$ of $G_t$ to $X_E$ may change drastically with $E$. In particular, it may be completely integrable for some values of $E$ and not others.

Examples:
(i) The spherical pendulum: $M = \mathbb{S}^2$, $H = |\xi|^2 + \cos \phi$; $|\xi|^2$ corresponds to the round metric and $\phi$ is the azimuthal angle.
(ii) The C. Neumann oscillator on \( T^*\mathbb{S}^n \), \( H = |\xi|^2 + \sum_{j=1}^n \alpha_j x_j^2 \) on \( T^*\mathbb{S}^n \). Here \( 0 < \alpha_1 < ... < \alpha_n \) are constants, \((x_1,...,x_n)\) are Cartesian coordinates on \( \mathbb{R}^{n+1} \) and \( |\xi|^2 \) corresponds to the usual round metric.

(iii) The Kowalevsky and Chaplygin tops \( \mathbb{H}_k \).

We note that in the non-homogeneous case, the joint flows \( \Phi_t^E \) on each energy level are distinct systems, and may be integrable for only some values of \( E \). An interesting case is the Chaplygin top \( \mathbb{H}_k \), which is integrable only when the angular momentum integral is set equal to zero.

1.6. Rigidity theorems for Newtonian flows. We will need a generalization of Mane’s rigidity theorem to Newtonian flows on tori. The following combines some ideas of Bialy-Polterovich \( [BP] \) and Knauf \( [K] \) to give a rigidity result when \( M \) is a torus and \( H \) is completely integrable with only compact regular orbits. In fact, it is more general:

**Proposition 1.7.** Suppose that \( g \) is a metric and \( V(x) \) is a potential on the \( n \)-torus \( \mathbb{T}^n \) such that the Hamiltonian flow \( G^E_t \) of \( H(x,\xi) \) on \( X_E \) preserves a \( C^1 \) Lagrangean foliation by tori which project regularly to \( \mathbb{T}^n \). Then \( E > \max_V \) and \( (E-V)g \) is a flat metric.

**Proof:** By ( \( [K] \), Theorem 2) no such invariant foliation exists unless \( E > \max_V \), so we may assume this is the case. The Jacobi metric \( (E-V)g \) is then a well-defined metric on \( \mathbb{T}^n \). We denote by \( |\xi|^2_{J,E} \) the associated homogeneous Hamiltonian (length squared of a covector). Since the sets \( \{ H = E \} \) and \( \{ |\xi|^2_{J,E} = 1 \} \) are the same, the latter carries a Lagrangean foliation by tori which project regularly to \( \mathbb{T}^n \). Since the geodesic flow \( G^t_{J,E} \) of \( (E-V)g \) on \( \{ |\xi|^2_{J,E} = 1 \} \) coincides (up to a time re-parametrization) with \( G^E_t \), this foliation is invariant under \( G^t_{J,E} \).

Now let \( D_r : T^*M \to T^*M \) be the dilation \( D_r(x,\xi) = (x,r\xi) \). Then \( D_r : \{ |\xi|^2_{J,E} = 1 \} \to \{ |\xi|^2_{J,E} = r^2 \} \) intertwines the geodesic flows on these sphere bundles (up to constant time reparametrization). Since \( D_r \) is conformally symplectic it also carries the invariant Lagrangean torus foliation of \( \{ |\xi|^2_{J,E} = 1 \} \) to an invariant Lagrangean torus foliation of \( \{ |\xi|^2_{J,E} = r^2 \} \). It follows that \( T^*M \) carries a Lagrangean torus foliation invariant under the geodesic flow of the Jacobi metric. By Mane’s theorem, the geodesic flow has no conjugate points and so by Burago-Ivanov’s theorem, \( (E-V)g \) must be flat.

**Corollary 1.8.** With the same notation as above, suppose that there exists an interval \( [E_0 - \epsilon, E_0 + \epsilon] \) such that, for all \( E \in [E_0 - \epsilon, E_0 + \epsilon] \), \( G^E_t \) preserves a Lagrangean foliation by tori which project regularly to \( \mathbb{T}^n \). Then: \( g \) is flat and \( V \) is constant.

**Proof:** The assumption implies that \( (E-V)g \) is flat for all \( E \) in the interval. Let \( R_E \) denote the curvature tensor of \( (E-V)g \). It is clearly a real analytic function of \( E \). Since \( R_E \equiv 0 \) in \( [E_0 - \epsilon, E_0 + \epsilon] \), it must vanish identically. Therefore the Newton’s flow \( \Phi_t \) on \( T^*\mathbb{T}^n \) has no conjugate points. By Remark 1.C and Theorem 1.B of \( [BP] \), it follows that \( g \) is flat and \( V \) is constant.

1.7. Semiclassical quantum integrable systems: semiclassical calculus. We now provide the necessary background on quantum integrable systems. Since we wish to include quantizations of possibly inhomogeneous Hamiltonians, the proper framework is that of semiclassical pseudodifferential operators.

First, we introduce symbols. On a given open \( U \subset \mathbb{R}^n \), we say that \( a(x,\xi;\hbar) \in C^\infty(U \times \mathbb{R}^n) \) is in the symbol class \( S^{m,k}(U \times \mathbb{R}^n) \), provided

\[
|\partial_x^\alpha \partial_\xi^\beta a(x,\xi;\hbar)| \leq C_{\alpha,\beta} \hbar^{-m}(1 + |\xi|)^{k-|\beta|}.
\]

We say that \( a \in S^{m,k}_{cl}(U \times \mathbb{R}^n) \) provided there exists an asymptotic expansion:

\[
a(x,\xi;\hbar) \sim \hbar^{-m} \sum_{j=0}^\infty a_j(x,\xi)\hbar^j,
\]
with \(a_j(x, \xi) \in S^{0,k-j}(U \times \mathbb{R}^n)\). The associated \(\hbar\)-quantization by \(Op_\hbar(a)\) is defined locally by the standard formula:

\[
Op_\hbar(a)(x,y) = (2\pi \hbar)^{-n} \int_{\mathbb{R}^n} e^{i(x-y)\xi/\hbar} a(x, \xi; \hbar) \, d\xi.
\]

By using a partition of unity, one constructs a corresponding class, \(Op_\hbar(S^{m,k})\), of properly-supported \(\hbar\)-pseudodifferential operators acting globally on \(C^\infty(M)\); as is well known, it is independent of the choice of partition of unity. Given \(a \in S^{m_1,k_1}\) and \(b \in S^{m_2,k_2}\), the composition is given by \(Op_\hbar(a) \circ Op_\hbar(b) = Op_\hbar(c) + O(\hbar^\infty)\) in \(L^2(M)\) where locally,

\[
c(x, \xi; \hbar) \sim \hbar^{-(m_1+m_2)} \sum_{|\alpha| = 0}^{\infty} \frac{(-i\hbar)^{|\alpha|}}{\alpha!} (\partial_{\xi}^\alpha a) \cdot (\partial_{x}^\alpha b).
\]

**Definition 1.9.** We say that the operators \(P_j^\hbar \in Op_\hbar(S^{m,k}_{cl})\), \(j = 1, \ldots, n\), generate a semiclassical quantum completely integrable system on \(M\) if for each \(\hbar\),

\[
\bullet \sum_{j=1}^{n} P_j^{\hbar*} P_j^\hbar \text{ is jointly elliptic on } T^* M,
\]

\[
\bullet |P_i^\hbar, P_j^\hbar| = 0; \ \forall 1 \leq i, j \leq n,
\]

and the respective semiclassical principal symbols \(p_1, \ldots, p_n\) generate a classical integrable system on \(T^* M\) with \(dp_1 \wedge dp_2 \wedge \cdots \wedge dp_n \neq 0\) on a dense open subset of \(T^* M\). We also assume that the finiteness condition \([\mathcal{I}]\) is satisfied.

1.7.1. **Examples.** The basic examples we have in mind are where \(P_i^\hbar = \hbar^2 \Delta + V \in Op_\hbar(S^{0,2}_{cl})\) is a Schrödinger operator over a compact manifold \(M\). Examples include:

- Quantum integrable Laplacians \(\Delta\) such as Laplacians of Liouville metrics on the sphere or torus \([B.K.S]\), or of the ellipsoid \([T3]\).
- Toric integrable Laplacians such as the flat Laplacian on \(\mathbb{T}^n\), or Laplacians for surfaces of revolution of ‘simple type’ (see below and \([CV1]\)).
- The quantum spherical pendulum \(\hbar^2 \Delta + \cos \phi\): \(M = S^2\), \(\Delta\) is the standard Laplacian, \(V = \cos \phi\) where \(\phi\) is the azimuthal angle. The commuting operator is \(\hbar \frac{\partial}{\partial \phi}\), the generator of rotations around the \(z\)-axis.
- The C. Neumann oscillator on \(S^n\). Here the quantum Hamiltonian is the Schrödinger operator \(\hbar^2 \Delta + \sum_{j=1}^{n} \alpha_j x_j^2\) acting on \(C^\infty(S^n)\). Here, \(\Delta\) is the spherical, constant curvature Laplacian and the potential is the one described above. For the quantized C. Neumann system, one can construct quantum integrals that are all second-order, real-analytic, semiclassical partial differential operators on the sphere \([I3]\).
- The quantized Euler, Lagrange and Kowalevsky tops. The Euler and Lagrange cases are classical \([Hd]\), while the quantum Kowalevsky top was shown to be QCI recently by Heckman \([Hd]\). Here, the integrals are semiclassical differential operators in the enveloping algebra of \(so(3) \triangleright \mathbb{R}^3\) defined as follows: Let \(E_1, E_2, E_3\) be the standard Pauli basis of \(so(3, \mathbb{R})\) and \(L_1, L_2, L_3\) be the corresponding left-invariant vector fields defined by:

\[
L_i(f)(x) := \frac{d}{dt}\{f(x \exp tE_i)\}_{t=0}.
\]

Fix a unit vector \(e \in \mathbb{R}^3\) and define the \(C^\infty\) functions on \(SO(3)\) by

\[
Q_i(x) := \langle xe_i, e \rangle.
\]

Then, the space of operators generated by \(Q_1, Q_2, Q_3, L_1, L_2, L_3\) can be identified with \(so(3) \triangleright \mathbb{R}^3\). Two of the quantum integrals are the quantized energy Schrödinger operator, \(P_1 := \frac{1}{2}\hbar^2 (L_1^2 + L_2^2 + 2L_3^2) - Q_1\) and the quantized momentum operator, \(P_2 = \hbar \sum_{j=1}^{3} Q_j L_j\). In analogy with the classical case, the third quantum integral is a fourth-order partial differential operator defined as follows: Put \(K := \hbar^2 (L_1 + iL_2)^2 + 4(Q_1 + iQ_2)\). Then, in terms of \(K\), \(P_3 = KK^* + K^*K - 8\hbar^4(L_1^2 + L_2^2)\).
Homogeneous quantum completely integrable systems are the special case where $\hbar$ occurs with the same power in each term and where the usual homogeneous symbols of the operators are all of order one, e.g. \( \hbar \sqrt{\Delta} \) or \( \hbar \sqrt{\Delta + V} \). In this case, one could remove $\hbar$ and use the homogeneous symbolic calculus. However, it is often more convenient to convert homogeneous systems $P_1, \ldots, P_n$ into semiclassical ones by introducing a semiclassical parameter $\hbar$ (with values in some sequence \( \{ \hbar_k; k = 1, 2, 3, \ldots \} \) with $\hbar_k \to 0$) and semiclassically scaling the $P_j$'s:

\[
P_j^\hbar := \hbar P_j; \quad j = 1, 2, \ldots, n.
\]

When $P_1 = \sqrt{\Delta}, P_2, \ldots, P_n$ are classical pseudodifferential operators of order one, then $P_j^\hbar := \hbar P_j \in Op(S^0_{cl})$ generate the semiclassical quantum integrable system in the sense of Definition 1.9.

1.8. Quantum torus actions. (see [GS] for many details on this case). Classical torus actions can always be quantized and produce the simplest examples of toric quantum integrable systems. The classical actions \( \{ I_j \} \) can be quantized as commuting pseudodifferential operators \( \hat{I}_1, \ldots, \hat{I}_n \) whose joint spectrum

\[
Sp(\hat{I}_1, \ldots, \hat{I}_n) = \Lambda \subset (Z^n + \nu) \cap B
\]

is a lattice (translated by a Maslov index). The simplest case is that of the torus, where \( \hat{I}_j = \frac{\partial}{\partial \theta_j} \) (with \( \theta_j \) denoting the usual angular coordinates). The operators \( \sqrt{\Delta + 1/4} \frac{\partial}{\partial \theta} \) on \( S^2 \) provide another example. Less obviously, any convex surface of revolution has a toric integrable Laplacian (cf. [CV.1]).

Just as the classical multiplicity $m_{\nu I}(b) = 1$ in the toric case, so also the multiplicity $m(\lambda)$ of the joint eigenvalues is 1 for $|\lambda|$ sufficiently large [CV.1]. Hence up to a finite dimensional subspace, there is a unique (up to unit scalars) orthonormal basis of joint eigenfunctions

\[
\hat{I}_j \phi_\lambda = \lambda_j \phi_\lambda, \quad \lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda.
\]

1.9. Joint eigenvalue ladders. In the next section, we will study the localization of sequences of eigenfunctions on level sets of the moment map. To obtain sequences which localize on a given level $P^{-1}(b)$ it is necessary to choose the corresponding joint eigenvalues to tend in an appropriate sense to $b$. Roughly speaking, such joint eigenvalues form an ‘eigenvalue ladder’.

The term comes from the toric case, where the joint spectrum $\Lambda$ of the action operators is a semi-lattice (i.e. the set of lattice points in a cone) We define ladders (or rays) in a direction $\lambda$ by:

\[
N_\lambda = \{ k\lambda + \nu, k = 0, 1, 2, \ldots \} \subset \Lambda.
\]

In the case of quantizations of torus and other Hamiltonian compact group actions, semiclassical limits are essentially the same as limits along ladders (cf. [GS] [CV1]).

In the $\mathbb{R}^n$ case, there is usually no optimal choice of the generators $P_j$, and their joint spectrum is quite far from a lattice. We therefore define a homogeneous ladder of eigenvalues in the direction $b = (b^{(1)}, b^{(2)}, \ldots, b^{(n)}) \in \mathbb{R}^n$ to be a sequence satisfying

\[
\{ \lambda_k := (\lambda_k^{(1)}, \ldots, \lambda_k^{(n)}), \forall j = 1, \ldots, n, \lim_{k \to \infty} \frac{\lambda_k}{|\lambda_k|} = b \},
\]

where $|\lambda_k| := \sqrt{|\lambda_k^{(1)}|^2 + \cdots + |\lambda_k^{(n)}|^2}$.

Finally, we introduce a notion of semiclassical ladders: We fix $0 < \delta < 1, b = (b^{(1)}, b^{(2)}, \ldots, b^{(n)}) \in \mathbb{R}^n$, and define the set

\[
L_{b, \delta}(h) := \{ b_j(h) := (b_j^{(1)}(h), b_j^{(2)}(h), \ldots, b_j^{(n)}(h)) \in Spec(P_1, \ldots, P_n); |b_j(h) - b| \leq \hbar^{1-\delta} \}.
\]

Here, $b_j^{(1)}(h) = E_j(h)$. Taking a sequence $\hbar \to 0$, the joint eigenvalues in $L_{b, \delta}(h)$ form a sequence tending to $b$ which is the analogue of a homogeneous ladder.
2. Localization on Tori

One of the main inputs in the proof of the Theorem is the localization of a ladder of joint eigenfunctions of a quantum completely integrable system in a regular direction \( b \in B_{\text{reg}} \) on the level set \( \mathcal{P}^{-1}(b) \) of the moment map. In this section, we prove the relevant localization results. We first consider toric systems, where level sets are regular and connected and eigenfunctions necessarily localize on individual tori. In the general \( \mathbb{R}^n \) case, ladders of eigenfunctions localize on the possibly disconnected level set \( \mathcal{P}^{-1}(b) \), and it is a complicated problem to determine how the limit eigenfunction mass (or ‘charge’) is distributed among the components. To deal with this problem, we define a notion of the charge of a component, and prove that every compact component of \( \mathcal{P}^{-1}(b) \) is charged by some sequence of eigenfunctions. This result will play an important role in the proof of the Theorem.

2.1. Toric integrable systems. Let \( A \in \Psi^0(M) \) denote any zeroth order pseudodifferential operator and \( d\mu_L \) denote Lebesgue measure on the Lagrangian torus \( T_\lambda \). In the toric case we have the following localization theorem:

**Proposition 2.1.** \([Z]\)** For any ladder \( \{k\lambda + \nu : k = 0, 1, 2, \ldots \} \) of joint eigenvalues, we have:

\[
\langle A\phi_{k\lambda}, \phi_{k\lambda} \rangle = \int_{T_\lambda} \sigma_{\lambda} d\mu_{\lambda} + O(k^{-1}).
\]

We thus have:

**Corollary 2.2.** For any invariant torus \( T_\lambda \subset S^*M \), there exists a ladder \( \{\phi_{k\lambda}, k = 0, 1, 2, \ldots \} \) of eigenfunctions localizing on \( T_\lambda \).

2.2. \( \mathbb{R}^n \)-integrable systems. The proper generalization of the toric localization result Proposition \([Z]\) to \( \mathbb{R}^n \) actions says that ladders of joint eigenfunctions localize on level sets of the moment map rather than on individual tori. This result is more or less a folk theorem in the physics literature (see \([E, Be, Be2]\)), and the rigorous result is in principle known to experts. However, we were unable to find the result in the literature, so we sketch the proof here. It uses some material on quantum Birkhoff normal forms from \([CV2]\).

Let \( b \) be a regular value of the moment map \( \mathcal{P} \), let

\[
\mathcal{P}^{-1}(b) = \Lambda^{(1)}(b) \cup \cdots \cup \Lambda^{(m-1)}(b),
\]

where the \( \Lambda^{(i)}(b) \); \( i = 1, \ldots, m \) are \( n \)-dimensional Lagrangian tori, and \( d\mu_{\Lambda^{(i)}(b)} \) denote the normalize Lesbegue measure on the torus \( \Lambda^{(i)}(b) \). Let \( b_j(h) \in L_{b,\delta}(h) \) and define

\[
c_l(h; b_j(h)) := \langle Op_h(\chi_{l}) \phi_{b_j(h)}, \phi_{b_j(h)} \rangle; \quad l = 1, \ldots, m_{\text{el}}(b). \tag{16}
\]

We recall that \( \chi_l \) is cutoff function which is equal to 1 in the neighbourhood \( \Omega^{(l)}(b) \) of the torus \( \Lambda^{(l)}(b) \) and vanishes on \( \cup_{k \neq l} \Omega^{(k)}(b) \).

**Proposition 2.3.** Let \( b \in B_{\text{reg}} \), and let \( \{\phi_{b_j(b)}\} \) be a sequence of \( L^2 \)-normalized joint eigenfunctions of \( P_1, \ldots, P_n \) with joint eigenvalues in the ladder \( L_{b,\delta}(h) \) of \([13]\). Then, for any \( a \in S^{0,-\infty} \), we have that as \( h \to 0 \):

\[
\langle Op_h(a) \phi_{b_j(b)}, \phi_{b_j(b)} \rangle = \sum_{j=1}^{m} c_l(h; b_j(h)) \int_{\Lambda^{(j)}(b)} a d\mu_{\Lambda^{(j)}(b)} + O(h^{1-\delta}).
\]

Here, \( d\mu_{\Lambda^{(j)}(b)} \) denotes Lebesgue measure on \( \Lambda^{(j)}(b) \).

**Proof:** Let \( \mathcal{L}^{(l)} \) be the pullback of the Maslov line bundle over \( \Lambda^{(l)} \) to the affine torus given by \( I_1^{(l)} = \cdots = I_n^{(l)} = 0 \) and \( \Omega^{(l)} \) be a sufficiently small neighbourhood of \( \Lambda^{(l)} \) on which there exist action-angle variables \((\theta^{(l)}, I^{(l)})\). According to the quantum Birkhoff normal form (QBNF) construction \([CV2]\), for \( l = 1, \ldots, k \) and
where the proposition follows.

I normalized the action variables so that

\[\phi(\Omega)\]

We conjugate this equation to Birkhoff normal form (17) and use the fact that the microlocal solutions of

\[\text{is one-dimensional. Indeed, such solutions are the same as solutions of}
\]

Now let \(\chi_l(x, \xi) \in C_0^\infty(T^*M); l = 1, \ldots, m_d(b)\) be a cutoff function which is identically equal to one on the neighbourhood \(\Omega^{(l)}(b)\) and vanishes on \(\Omega^{(k)}(b)\) for \(k \neq l\). For \(h\) sufficiently small, we then have

\[\langle Op_h(a) \phi_{b_i}(h), \phi_{b_j}(h) \rangle = \sum_{l=1}^{m_d(b)} \langle Op_h(a) \circ Op_h(\chi_l) \phi_{b_i}(h), \phi_{b_j}(h) \rangle + \mathcal{O}(h^\infty).\]

It follows by (11), the semiclassical Egorov theorem and a Taylor expansion about the Lagrangian torus \(I^{(l)} = 0\) that:

\[\langle Op_h(a) \circ Op_h(\chi_l) \phi_{b_i}(h), \phi_{b_j}(h) \rangle = c_l(h; b_j(h))(Op_h(a) \circ Op_h(\chi_l)U^{(l)}_{b_i,h}(e^{i(n_j+\pi\gamma/4)\theta}), U^{(l)}_{b_i,h}(e^{i(n_j+\pi\gamma/4)\theta}))
\]

\[= c_l(h; b_j(h))(U^{(l)*}_{b_i,h} Op_h(a) \circ Op_h(\chi_l)U^{(l)}_{b_i,h} e^{i(n_j+\pi\gamma/4)\theta}, e^{i(n_j+\pi\gamma/4)\theta})
\]

\[= (2\pi)^{-n} c_l(h; b_j(h)) \left(\int_{\Lambda^{(l)}(h)} a \, d\mu + e(h)\right) + \mathcal{O}(h),
\]

where \(e(h) = \langle Op_h(r)u_h, u_h \rangle\) for some function \(r \in C_0^\infty(T^n \times D_1)\) satisfying \(r(\theta, I) = \mathcal{O}(|I|)\) (recall, we have normalized the action variables so that \(I^{(l)} = 0\) on the torus \(\Lambda^{(l)}(b)\)). Here, \(u_h(\theta) = \exp[i(m_1\theta_1 + \ldots + m_n\theta_n)]\) with \(m_j(h) = \mathcal{O}(h^{1-\delta})\).

An integration by parts in the \(I_1, \ldots, I_n\) variables shows that:

\[\langle Op_h(r)u_h, u_h \rangle = \mathcal{O}(h^{1-\delta})\]

and the proposition follows.

### 2.3. Charge of compact Lagrangean orbits.

We now investigate the coefficients \(c_j(h)\) in Proposition 2.3 for ‘ladders’ of eigenfunctions. Our purpose is to show that there exist ladders for which the limit as \(h \to 0\) of \(c_j(h)\) is bounded below by a positive geometric constant. It is convenient at this point to introduce the language of quantum limits.
2.3.1. Quantum limits. Let \((P_1, \ldots, P_n)\) denote a quantum integrable system, with classical integrable flow \(\Phi_t\). Fix \(E\) and let \(M^E_f\) denote the set of invariant probability measures for \(\Phi^E_t\) on \(X_E\). For instance, \(M^E_f\) includes the orbital averaging measures \(\mu_z\), defined by

\[
\int_{X_E} f d\mu_z = \lim_{T \to \infty} \frac{1}{T} \int_{\max |t_i| \leq T} f(\Phi_t(z)) \, dt.
\]

In the case of compact (torus) orbits, \(\mu_z\) is the Lebesgue probability measure on the orbit of \(z\).

By the set \(Q_E\) of ‘quantum limit’ measures of the quantum integrable system at energy level \(E\), we mean the set of weak* limits of \(\mu_z\) as \(h \to 0\): \(\mu \in Q_E\) if for each \(\mu_z\) there exists \(\mu \to \mu_z\) weak*. Fix \(E \in \mathcal{E}\), we mean the semiclassical Egorov theorem that \(Q_E \subset M^E_f\). When \(d\mu\) equals Lebesgue probability measure on an orbit, we say that the sequence \(\{\phi_{bj}(\ell)\}\) localizes on the orbit. For background, terminology and references in a closely related context, we refer to [22].

We now consider quantum limits of eigenfunctions corresponding to a ladder of joint eigenvalues. Put:

\[
\mathbf{V}_{b,\delta}(h) = \{\phi_{bj}(h) : bj(h) \in \mathbf{L}_{b,\delta}(h)\}
\]

(22)

There are many possible weak* limit points of the set \(\cup_{\ell \in [0,h]} \mathbf{V}_{b,\delta}(h)\). We say:

**Definition 2.4.** For \(b \in B_{reg}\), a ladder of eigenfunctions is a sequence \(\mathcal{E}_b := \{\phi_{bj}(h)\}\) of joint eigenfunctions with the following properties:

- \(bj(h) \in \mathbf{L}_{b,\delta}(h)\) as \(h \to 0\) forms an eigenvalue ladder;
- \(d\Phi_{bj}(h)\) has a unique weak limit \(d\Phi_{\mathcal{E}_b}\) as \(h \to 0\).

For a ladder of eigenfunctions, \(\lim_{h \to 0} c_\ell(h; bj(h))\) exists for each \(\ell\) in Proposition (21).

**Definition 2.5.** Given \(b \in B_{reg}\), we say that the ladder \(\mathcal{E}_b = \{\phi_{bj}(h)\}\) gives charge \(c_\ell(\mathcal{E}_b) := \lim_{h \to 0} c_\ell(h; bj(h))\) to the component torus \(\Lambda^{(\ell)}(b)\), and that it charges \(\Lambda^{(\ell)}(b)\) if \(c_\ell(\mathcal{E}_b) > 0\).

The limit in Definition (22) above clearly depends on the ladder \(\mathcal{E}_b\). For instance, there could be sequences of joint eigenfunctions localizing on each single component of \(\mathcal{P}^{-1}(b)\). To obtain an invariant of the Lagrangean orbits which is independent of the ladder, we say:

**Definition 2.6.** The charge \(c(\Lambda^{(\ell)}(b))\) of a component torus \(\Lambda^{(\ell)}(b) \subset \mathcal{P}^{-1}(b)\) is defined by the formula:

\[
c(\Lambda^{(\ell)}(b)) = \sup_{\mathcal{E}_b} c_\ell(\mathcal{E}_b)
\]

where \(c_\ell\) is the coefficient in the sum of Proposition (20).

A useful formula for the charge is:

**Proposition 2.7.** \(c(\Lambda^{(\ell)}(b)) = \lim_{h \to 0} \sup_{\phi_{bj}(h) \in \mathcal{V}_h(h)} \langle Op_h(\chi_l)\phi_{bj}(h), \phi_{bj}(h) \rangle\).

Proof:

(i) \(\geq\): By definition, \(c_\ell(h; bj(h)) = \langle Op_h(\chi_l)\phi_{bj}(h), \phi_{bj}(h) \rangle\) where \(\chi_l\) is a cutoff to \(\Omega_l\). Since \(\mathcal{V}_{b,\delta}(h)\) is a finite set for each \(h\), there exists \(\phi_{\max}(h) \in \mathcal{V}_{b,\delta}(h)\) such that \(\langle Op_h(\chi_l)\phi_{\max}(h), \phi_{\max}(h) \rangle = \max_{\phi_{bj}(h) \in \mathcal{V}_{b,\delta}(h)} \langle Op_h(\chi_l)\phi_{bj}(h), \phi_{bj}(h) \rangle\). We form the sequence \(\{\phi_{\max}(h)\}_{h \in (0,\delta)}\) and then choose a sub-ladder \(\mathcal{E}^{\max}_b\) with a unique quantum limit. Then

\[
c(\Lambda^{(\ell)}(b)) \geq \lim_{h \to 0} \langle Op_h(\chi_l)\phi_{\max}(h), \phi_{\max}(h) \rangle
\]

\[
\geq \lim_{h \to 0} \sup_{\phi_{bj}(h) \in \mathcal{V}(h)} \langle Op_h(\chi_l)\phi_{bj}(h), \phi_{bj}(h) \rangle.
\]

(ii) \(\leq\): It is clear that for each ladder \(\mathcal{E}_b\) we have

\[
c_\ell(\mathcal{E}_b) \leq \lim_{h \to 0} \sup_{\phi_{bj}(h) \in \mathcal{V}(h)} \langle Op_h(\chi_l)\phi_{bj}(h), \phi_{bj}(h) \rangle.
\]
Therefore the same holds after taking the supremum over $E_b$.

The following lemma is the main result of this section:

**Lemma 2.8.** Let $\omega^{(l)}(b)$ denote Liouville measure of the Lagrangian torus $\Lambda^{(l)}(b)$; $l = 1, \ldots, m_{cl}(b)$. Then, for all $(b, l) \in B_{reg} \times \{1, \ldots, m_{cl}(b)\}$ we have that

$$c(\Lambda^{(l)}(b)) \geq \frac{\omega^{(l)}(b)}{\sum_{j=1}^{m_{cl}(b)} \omega^{(j)}(b)}.$$

**Proof:** Fix $\zeta \in S(\mathbb{R}^n)$ with $\zeta \geq 0$, $\zeta \in C_0^\infty(\mathbb{R}^n)$ and $\zeta(0) = 1$. Assume moreover that $0 \in \mathbb{R}^n$ is the only point of intersection of supp $\zeta$ with the joint periods of the joint flow $\Phi_t$. Let $K$ be a fixed compact neighbourhood of $b = (b^{(1)}, \ldots, b^{(n)})$ and $a \in S^{0, -\infty}$. Consider the localized semiclassical trace:

$$Tr_a(\zeta) := \sum_{b \in K} \langle \text{Op}_b(a)\phi_{b\ell}(h), \phi_{b\ell}(h) \rangle \zeta \left( \frac{b\ell(h) - b}{\hbar} \right).$$

(23)

The localized semiclassical trace formula for commuting operators $[\text{Ch}]$ implies that for any $a \in S^{0, -\infty}$ and $\zeta \in S(\mathbb{R}^n)$ as above,

$$Tr_a(\zeta) = (2\pi)^{-n} \int_{\mathbb{R}^{n-1}} a \, d\omega^{(l)}(b) + O(h).$$

(24)

So, in particular putting $a(x, \xi) = \chi(x, \xi)$, we have that:

$$Tr_{\chi_l}(\zeta) = (2\pi)^{-n} \int_{\Lambda^{(l)}(b)} \chi_l d\omega^{(l)}(b) + O(h) = (2\pi)^{-n} \omega^{(l)}(b) + O(h),$$

(25)

since $\chi_l = 1$ on the torus, $\Lambda^{(l)}(b)$. On the other hand, since $\zeta \in S(\mathbb{R}^n)$, it follows that

$$Tr_{\chi_l}(\zeta) = \sum_{\{b \in K \mid \chi_l(b) \neq 0\}} \langle \text{Op}_b(\chi_l)\phi_{b\ell}(h), \phi_{b\ell}(h) \rangle \zeta \left( \frac{b\ell(h) - b}{\hbar} \right) + O(h\tilde{\infty}).$$

(26)

Thus, by the definition [2.6] of the charge $c(\Lambda^{(l)}(b))$ and the fact that $\zeta \geq 0$, we have that:

$$|Tr_{\chi_l}(\zeta)| \leq (2\pi)^{-n} \left( \max_{\{b \in L_b(h) \mid \chi_l(b) \neq 0\}} \langle \text{Op}_b(\chi_l)\phi_{b\ell}(h), \phi_{b\ell}(h) \rangle \right) \sum_{\{b \in L_{b,\ell}(h) \mid \chi_l(b) \neq 0\}} \zeta \left( \frac{b\ell(h) - b}{\hbar} \right) + O(h\tilde{\infty}).$$

(27)

Next, by applying the trace formula once again we get that:

$$\sum_{\{b \in L_{b,\ell}(h) \mid \chi_l(b) \neq 0\}} \zeta \left( \frac{b\ell(h) - b}{\hbar} \right) = (2\pi)^{-n} \sum_{j=1}^{m_{cl}(b)} \omega^{(j)}(b) + O(h).$$

(28)

Substituting (28) in (27) yields the estimate

$$|Tr_{\chi_l}(\zeta)| \leq (2\pi)^{-n} \max_{\{b \in L_{b,\ell}(h) \mid \chi_l(b) \neq 0\}} \langle \text{Op}_b(\chi_l)\phi_{b\ell}(h), \phi_{b\ell}(h) \rangle \cdot \left( \sum_{j=1}^{m_{cl}(b)} \omega^{(j)}(b) \right) + O(h).$$

(29)

The lemma then follows by combining (24) and (25) and letting $h \to 0$.

This yields a generalization of Corollary [2.2]:

**Corollary 2.9.** For any $b \in B_{reg}$, and for any $1 \leq \ell \leq m_{cl}(b)$, there exists a ladder $E_b^{(\ell)} = \{b\ell(h)\}$ such that $c(\Lambda^{(l)}(b)) \geq \frac{\omega^{(l)}(b)}{\sum_{j=1}^{m_{cl}(b)} \omega^{(j)}(b)}$.

Thus, every regular torus orbit is charged by some ladder. This follows from Lemma [2.8], Proposition [2.7] and Proposition [2.3].
2.3.2. Charge of compact singular orbits. Our next step is to prove that some compact singular orbits are also charged. To be precise, we have so far only defined the notion of charge for regular levels of the moment map (Definition 2.3). The analogous definition in the case of a singular value $b_s \in B_s$ is as follows. Let $\mathcal{P}^{-1}(b_s) = \cup_{j=1}^{r} \Gamma^{(j)}(b_s)$ be the decomposition (11) into connected components.

**Definition 2.10.** When $b_s \in B_{\text{sing}}$, we define an eigenfunction ladder $\mathcal{E}_{b_s}$ to be a sequence of joint eigenfunctions with joint eigenvalues satisfying $b_j(r) - b_s = O(1)$ as $h \to 0$ and with unique limit measure $d\Phi_{\mathcal{E}_{b_s}}$. We say that $\mathcal{E}_{b_s}$ gives charge $\int_{\Gamma^{(j)}(b_s)} \Phi_{\mathcal{E}_{b_s}}$ to the component $\Gamma^{(j)}(b_s)$. Similarly, we say that it gives charge $\int_{\Lambda^{(j)}(b_s)} \Phi_{\mathcal{E}_{b_s}}$ to any orbit $\Lambda^{(j)}(b_s)$ on $\Gamma^{(j)}(b_s)$ (see 2.10). Finally, the charge $c(\Gamma^{(j)}(b_s))$, resp. $c(\Lambda^{(j)}(b_s))$ of a component, resp. an orbit on the component, is the supremum of the same over all ladders $\mathcal{E}_{b_s}$.

We then have:

**Lemma 2.11.** Let $b_s \in B_{\text{sing}}$, and let $\{\Gamma^{(j)}(b_s)\}$ denote the singular components of $\mathcal{P}^{-1}(b_s)$. Then, there exists $j$ such that $c(\Gamma^{(j)}(b_s)) > 0$. Further, there exists a compact singular orbit $\Lambda^{(j)}(b_s) \subset \Gamma^{(j)}(b_s)$ such that $c(\Lambda^{(j)}(b_s)) > 0$.

**Proof:** Let $U_{\text{sing}}$ be a $\Phi_\mathcal{L}$-invariant neighbourhood of $\cup_{j=1}^{r} \Gamma^{(j)}(b_s)$. Let $\{b_n\} \subset B_{\text{reg}}$ be a sequence of regular points such that $b_n \to b_s$. For each $j$ and sufficiently large $n$, there exists at least one component $\Lambda^{(j)}(b_n)$ of $\mathcal{P}^{-1}(b_n)$ such that $\Lambda^{(j)}(b_n) \subset U_{\text{sing}}$. By Lemma (2.10), $\Lambda^{(j)}(b_n)$ is charged by an amount $\geq \sum_{j=1}^{r} \omega^{(j)}(b_n)$.

We now break up the discussion into two cases:

**Case 1:** All $\mathbb{R}^n$-orbits of $\cup_{j=1}^{r} \Gamma^{(j)}(b_s)$ are compact

In this case, we just need a positive lower bound for the quotient $\frac{\omega^{(j)}(b_n)}{\sum_{j=1}^{r} \omega^{(j)}(b_n)}$ as $n \to \infty$. A lower bound for the numerator is given by the minimal period of Yorke’s theorem (Proposition 1.4). Since all orbits (including the limit) are compact, the masses in the denominator have uniform upper bounds. Indeed, by [5], the masses are the co-volumes of the period lattices of $\Lambda^{(j)}(b_n)$. Since the period vectors generating the lattices are uniformly bounded as $n \to \infty$, the volumes are also uniformly bounded above. Hence the denominator is bounded above, and therefore the quotient is bounded below by a positive constant.

**Case 2:** There exists a non-compact orbit in $\cup_{j=1}^{r} \Gamma^{(j)}(b_s)$

In this case, the denominator will tend to infinity, so we need a better lower bound on the numerator. We claim that there exists $\ell$ such that $\Lambda^{(\ell)}(b_n) \subset U_{\text{sing}}$ and $c(\Lambda^{(\ell)}(b_n)) \geq \frac{1}{m_{\mathcal{L}}(b_n)}$. To prove this, it suffices to find $\ell$ such that

$$\frac{\omega^{(\ell)}(b_n)}{\sum_{j: \Lambda^{(j)}(b_n) \subset U_{\text{sing}}} \omega^{(j)}(b_n)} \geq \frac{1}{\# \{j: \Lambda^{(j)}(b_n) \subset U_{\text{sing}}\}}.$$ (30)

The natural candidate is to choose $\ell$ such that

$$\omega^{(\ell)}(b_n) = \max_{\{j: \Lambda^{(j)}(b_n) \subset U_{\text{sing}}\}} \omega^{(j)}(b_n).$$ (31)

We now prove that this choice of $\ell$ satisfies (30).

We write

$$\sum_{j=1}^{m_{\mathcal{L}}(b_n)} \omega^{(j)}(b_n) = \sum_{j: \Lambda^{(j)}(b_n) \subset U_{\text{sing}}} \omega^{(j)}(b_n) + \sum_{j: \Lambda^{(j)}(b_n) \cap U_{\text{sing}} = \emptyset} \omega^{(j)}(b_n).$$

The second term is bounded above by a constant $C$ independent of $n$. The first term tends to infinity since $\cup_{j=1}^{r} \Gamma^{(j)}(b_s)$ contains a non-compact orbit. Indeed, at least one vector of the period lattice of $\Lambda^{(j)}(b_n)$ must
tend to infinity as \( n \to \infty \) since the limit orbit is non-compact. It follows that the set of period lattices \( \Xi_b^{(j)} \) is non-compact in the manifold of lattices of full rank of \( \mathbb{R}^n \). Now according to Mahler’s theorem, any set
\[
\{ \Gamma \subset \mathbb{R}^n \mid \| \gamma \| \geq C, \quad (\gamma \in \Gamma - \{0\}) \}, \quad \text{and} \quad \text{Vol}(\mathbb{R}^n / \Gamma) \leq K
\]
is compact. By Yorke’s theorem (loc. cit.), the minimal period stays bounded below, so non-compactness of the lattices forces some volume \( \omega^{(\ell)}(b_n) \to \infty \) as \( n \to \infty \).

It follows that when a non-compact orbit exists in \( \mathcal{P}^{-1}(b_s) \), then for each \( \ell \),
\[
\frac{\omega^{(\ell)}(b_n)}{\sum_{j=1}^{m_s(b_n)} \omega^{(j)}(b_n)} = \frac{\omega^{(\ell)}(b_n)}{\sum_{j: \Lambda^{(j)}(b_n) \subset U_{\text{sing}}} \omega^{(j)}(b_n)} + o(1) \quad \text{as} \quad n \to \infty.
\]
Then (34) follows if we select \( \ell \) as in (33). \( \square \)

We now complete the proof of Lemma (2.11). By the finite complexity condition, we have found \( \Lambda^{(j_n)}(b_n) \subset U_{\text{sing}} \) such that \( c(\Lambda^{(j_n)}(b_n)) \geq c := \frac{1}{M} > 0 \). Further, for each \( n \), there exists a ladder \( \mathcal{E}_{b_n} \) which gives charge \( \geq c \) to \( \Lambda^{(j_n)}(b_n) \subset \mathcal{P}^{-1} \cap U_{\text{sing}} \). Let \( d\Phi_{\mathcal{E}_{b_n}} \) denote the unique weak limit measure of the ladder. Then let \( \nu \) denote any weak* limit of the sequence \( \{d\Phi_{\mathcal{E}_{b_n}}\} \). It follows that \( \nu \) is an invariant probability measure supported on \( \cup_{j=1}^{\ell} \Gamma^{(j)}_{b_n} \). Indeed, its support must be contained in the set of limit points of the sequence of orbits \( \{\Lambda^{(j_n)}(b_n)\} \), hence in \( \mathcal{P}^{-1}(b_s) \cap U_{\text{sing}} \). Since \( Q \) is closed in the weak* topology (since it is a set of limit points), it follows further that \( \nu \in Q \). Hence there exists a ladder \( \mathcal{E}_{b_s} \) such that \( \Phi_{\mathcal{E}_{b_s}} \to \nu \), and which charges \( \cup_{j=1}^{\ell} \Gamma^{(j)}_{b_n} \) by an amount \( c > 0 \). This proves the first part of the lemma. The second statement is an immediate consequence of Proposition (1.3): There must exist at least one compact singular orbit \( \Lambda^{(j)}_{b_n} \subset \cup_{j=1}^{\ell} \Gamma^{(j)}_{b_n} \). Since \( \nu \) is an invariant probability measure, it must be supported on union of the compact singular orbits, hence must charge at least one such orbit. \( \square \)

3. Proof of the Theorem

We break up the proofs into three steps. Step 1 is to show that the uniform boundedness assumption implies that all regular tori project without singularities to the base. Step 2 is to show that there are no singular tori. Step 3 is a geometric argument showing that any completely integrable system with no singular tori and with all tori projecting regularly to the base is flat.

3.1. Step 1: regular tori project regularly. We first consider the simplest case of toric systems:

3.1.1. Toric integrable systems.

**Proposition 3.1.** Suppose that \( (M, g) \) is toric integrable and that \( L^\infty(E, g) = O(1) \). Then every orbit of the torus action has a non-singular projection to \( M \). In particular, the orbit foliation is a non-singular Lagrangean foliation.

**Proof:** The assumption implies that the joint eigenfunctions \( \{\phi_\lambda\} \) of the quantum torus action have uniformly bounded sup-norms.

By Proposition (2.1), for every invariant torus \( T_\lambda \), there exists a ladder \( \{k\lambda, k = 1, 2, \ldots\} \) of joint eigenvalues such that for all \( V \in C^\infty(M) \) we have
\[
\lim_{k \to \infty} \int_M V(x)|\phi_{k\lambda}(x)|^2 dvol = \int_M V \pi_{\lambda*} d\mu_{\lambda}.
\]
If we have ||\( k \lambda \)|| \( \leq C \) for all \( (k, \lambda) \), then
\[
| \int_M V(x)|\phi_{k\lambda}(x)|^2 dvol | \leq C ||V||_{L^1} \quad (\forall k)
\]
and hence
\[
\lim_{k \to \infty} | \int_M V(x)|\phi_{k\lambda}(x)|^2 dvol | \leq C ||V||_{L^1}.
\]
Therefore
\[
| \int_M V \pi_* d\mu_\lambda | \leq C \| V \|_{L^1},
\] (32)
which implies that \( \pi_* d\mu_\lambda \) is a continuous linear functional on \( L^1(M) \), hence belongs to \( L^\infty(M) \). That is, we may write \( \pi_* d\mu_\lambda = f_\lambda dvol \), with \( \| f_\lambda \|_\infty \leq C \). If \( \pi_\lambda \) had a singular value, it is easy to check that \( \pi_* d\mu_\lambda \) would blow up there. Hence, \( \pi_\lambda \) is a non-singular projection. \( \square \)

Now we turn to the general case:

3.1.2. \( \mathbb{R}^n \) actions.

**Proposition 3.2.** All regular tori project diffeomorphically to the base.

**Proof:** Since by Lemma (2.3) a regular torus \( \Lambda^{(t)}(b) \) has charge \( c(\Lambda^{(t)}(b)) \geq \sum_{i=1}^{m_{\omega^{(t)}(b)}} \omega^{(t)}(b) > 0 \), it follows by Corollary (2.9) that there exists a joint eigenfunctions \( \{ \phi_{b_j(h)} \} \subset \mathcal{E}_b \) with the property that:
\[
(V \phi_{b_j(h)}, \phi_{b_j(h)}) = \sum_{l=1}^{m_{\omega^{(t)}(b)}} c_l(\mathcal{E}_b) \int_{\Lambda^{(t)}(b)} V d\mu_{\Lambda^{(t)}(b)} + o(1),
\]
where \( c_l(\mathcal{E}_b) \geq \sum_{j=1}^{m_{\omega^{(t)}(b)}} \omega^{(t)}(b) > 0 \) and \( c_k(\mathcal{E}_b) \geq 0 \) for \( k \neq l \). Thus, we have (as in the toric case) that
\[
c_l(\mathcal{E}_b) \int_M V \pi_* d\mu_{\Lambda^{(t)}(b)} \leq C \| V \|_{L^1},
\]
where we can take \( C = c_l(\mathcal{E}_b) \cdot L^\infty(h, b_j(h); g, V) \). Since \( c_l(\mathcal{E}_b) > 0 \) we can cancel it to find that the torus projects regularly. \( \square \)

As an immediate of Proposition (3.2) we have:

**Corollary 3.3.** Let \( \{ \pi_* d\mu_\lambda \} \) denote the set of projections to \( M \) of normalized Lebesgue measures on compact Lagrangean tori \( \Lambda \subset X_\mathcal{P} \). Then, under the assumptions of Theorem (0.1), the family is uniformly bounded as linear functionals on \( L^1(M) \).

3.2. Non-existence of singular levels. We have:

**Lemma 3.4.** Under the assumptions of Theorem (0.1), \( \mathcal{P} \) has no singular levels; all orbits are Lagrangean.

**Proof:** Existence of a compact singular orbit contradicts the uniform boundedness of eigenfunctions assumption. Indeed, it follows from Lemma (2.11) that, for any \( V \in C^\infty(M) \), there exist a compact, singular orbit \( \Lambda^{(t)}_{\text{sing}} \) and \( L^2 \)-normalized joint eigenfunctions \( \{ \phi_{b_j(h)} \} \) such that for some \( c(\Lambda^{(t)}_{\text{sing}}) > 0 \),
\[
c(\Lambda^{(t)}_{\text{sing}}) \int_{\Lambda^{(t)}_{\text{sing}}} V \pi_* d\mu_\lambda \geq C \| V \|_{L^1(M)}.
\] (33)
However, the estimate in (33) cannot hold since by definition, compact singular orbits have dimension \( \dim \Lambda^{(t)}_{\text{sing}} < n \). Therefore, there cannot exist singular levels of the moment map \( \mathcal{P} \). \( \square \)

3.3. Completion of proof of Theorem. We first complete the proof of Theorem (1.1) for general metrics with quantum completely integrable Laplacians. Subsequently we take up the case of Schroedinger operators.

The first step is to consider projections of regular Lagrangean tori. By Proposition (3.2), the assumption of uniformly bounded eigenfunctions then applies to show that all Lagrangean torus orbits must project regularly to \( M \). Furthermore, by Lemma (3.4) we know that the under the finite geometric multiplicity condition (3) and uniform boundedness condition on the eigenfunctions, there do not exist any singular leaves of the moment map. Consequently, the proof of Theorem (1.1) in the case of Laplacians is a direct consequence of the following:
Lemma 3.5. Suppose that the geodesic flow $G^t$ of $(M, g)$ commutes with a Hamiltonian $\mathbb{R}^n$ action. Suppose that there are no singular levels of the moment map, and suppose that each regular Lagrangean orbit $\mathbb{R}^n \cdot (x, \xi)$ has a non-singular projection to $M$. Then $(M, g)$ is a flat manifold.

Proof: We will give two proofs of the lemma.

First Proof:

The first proof uses Mane’s theorem (1.5): Since the foliation by orbits has no singular leaves, Mane’s theorem implies that $(M, g)$ has no conjugate points. Since each leaf is compact, it must be a torus which covers $M$. Thus, there exists a cover $p : T^n \to M$. Lift the metric to $p^* g$ on $T^n$. The lifted metric must have no conjugate points since the universal covering metric is the same. By the Burago-Ivanov theorem (1.4), the metric is flat.

In the second proof, we do not use Mane’s theorem, and directly relate the condition on torus projections to non-existence of conjugate points.

Second Proof:

As above, let $\pi : T^* M - 0 \to M$ denote the natural projection and let $\pi_I = \pi|_{T_I}$. Since each $\pi_I : T_I \to M$ is non-singular, and $\dim T_I = \dim M$, $\pi_I$ must be a covering map.

3.3.1. Case 1: $M$ is a torus. Let us first assume that $M$ is a torus, i.e. diffeomorphic to $\mathbb{R}^n/\mathbb{Z}^n$; we make no assumptions on the metric.

From the fact that $p_I$ is a covering map, it follows by a result of Lalonde-Sikorav (LS) that the degree of $\pi_I : T_I \to M$ equals 1 for all $I$. Since $\pi_I$ is a diffeomorphism, there are well-defined inverse maps

$$\tilde{\pi}^{-1}_I : M \to T_I$$

with $K(I) = 1$. They define sections of $\pi : S^* M \to M$ and hence are given by graphs of 1-forms $\alpha_I : M \to S^* M$. Thus, $|\alpha_I(x)| \equiv 1$ where $|\cdot|$ is the co-metric. We have $\tilde{\pi}^{-1}_I \ast \alpha = \alpha_I \ast \alpha$, where $\alpha$ is the canonical 1-form. Since the tori $T_I$ are Lagrangean, and since $d\alpha = \omega$, the 1-forms are closed, i.e. $d\alpha_I = 0$.  

Now let $p : \tilde{M} \to M$ denote the universal cover of $M$ and let $\mathbb{Z}^n$ denote the deck transformation group, with generators $\alpha_1, \ldots, \alpha_n$. The metric $g$ lifts to a $\mathbb{Z}^n$-periodic metric $\tilde{g}$ on $\tilde{M}$. We note that the corresponding geodesic flow $\tilde{G}^t$ is also completely integrable. Indeed, the cover $p$ induces the universal cover $p_I : T^* \tilde{M} \to T^* M$ whose deck transformation group we continue to denote by $\mathbb{Z}^n$. Then $\tilde{G}^t$ commutes with the $T^n$-action on $T^n M - 0$ generated by the lifted action integrals $\tilde{I}_j = p_I^* I_j$. The invariant tori $\tilde{T}_I$ therefore lift to $G^t$-invariant level sets $\tilde{T}_I$ of $(\tilde{I}_1, \ldots, \tilde{I}_n)$.

Furthermore, the 1-forms $\alpha_I$ lift to $\mathbb{Z}^n$-invariant closed 1-forms $\tilde{\alpha}_I$ on $\tilde{M}$. They are exact $\tilde{M}$ and hence have the form $dB_I$ for some ‘potential’ $B_I \in C^\infty(\tilde{M})$. The gradient $\nabla B_I$ is then a $\mathbb{Z}^n$-invariant vector field on $\tilde{M}$. Since $|dB_I| \equiv 1$ we have $|\nabla B_I| \equiv 1$. We now claim that the integral curves of $\nabla B_I$ are lifts of geodesics on $T_I$.

To see this, we recall that the generator $\Xi_H$ of the geodesic flow lies tangent to each torus $T_I$. Hence for each $I$ it projects from $T_I$ to a non-singular vector field $\pi_{I*} \Xi_H = \Xi_I$ on $M$. We have

$$\langle \nabla B_I, \Xi_I \rangle = dB_I(\Xi_I) = \alpha_I(\Xi_I) = \langle \alpha_I|_{T_I}, \Xi_H|_{T_I} \rangle = 1$$

since $\Xi_H$ is a contact vector field for $(S^* M, \alpha)$. Since $|\nabla B_I| = 1$ it follows that $\nabla B_I = \Xi_I$. This relation holds for the lifts to $\tilde{M}$ and hence the integral curves of $\nabla B_I$ are the lifts of the geodesics on $T_I$.

We now claim that $\tilde{g}$ has no conjugate points, i.e. that each geodesic of $\tilde{g}$ on $\tilde{M}$ is length minimizing between each two points on it. This follows by a well-known argument: Let $\tilde{x}$ be any point of $\tilde{M}$, let $\tilde{v} \in S^2 \tilde{M}$, and let $\gamma_{\tilde{v}}$ be the geodesic of $\tilde{g}$ in the direction $\tilde{v}$. To see that $\gamma_{\tilde{v}}$ is length minimizing between $\tilde{x}$ and any other point $\gamma_{\tilde{v}}(t_o)$, we project it to $S^* \tilde{M}$. The image lies in one of the (possibly singular) invariant
tori $T_I$ and by the above, $\gamma_0$ is an integral curve of $\nabla B_I$. If it is not length minimizing to $\gamma_0(t_o)$, then there exists $s_o < t_o$ and a second geodesic $\alpha$ with $\alpha(0) = \hat{x}, \alpha(s_o) = \gamma_0(t_o)$. This leads to a contradiction since

$$B_I(\alpha(s_o)) = \int_0^{s_o} \langle \nabla B_I, \alpha'(s) \rangle ds = \int_0^{t_o} \langle \nabla B_I, \gamma_0'(s) \rangle ds = t_o > s_o$$

but

$$t_o = \left| \int_0^{s_o} \langle \nabla B_I, \alpha'(s) \rangle ds \right| \leq s_o$$

as $|\nabla B_I| = 1$. Therefore, $(T^n, g)$ is a torus without conjugate points. Theorem A then follows in this case from the recent proof by Burago-Ivanov [3] of the Hopf conjecture that a metric on $T^n$ with no conjugate points is flat.

3.3.2. The general case. We now consider the general case where $M$ is only covered by a torus $T^n$ (namely $T_I$ for each $I$). We denote by $p : T^n \to M$ a fixed d-fold covering map. For notational clarity we denote the metric on $M$ by $g_M$. By Lemma [3], there is a Hamiltonian torus action on $T^* M - 0$ with the property that every orbit projects non-singularly to $M$.

Let $g_T = p^* g_M$ be the metric induced on $T^n$ by the cover. We claim that $g_T$ is a flat metric. Since $p : (T^n, g_T) \to (M, g)$ is a Riemannian cover, this will imply that $g_M$ is a flat metric and conclude the proof of (a).

To prove $g_T$ is flat, we lift the torus foliation of $T^* M - 0$ to $T^* T^n - 0$. Given a metric $g$ on a manifold $X$ we denote by $\tilde{g} : TX \to T^* X$ the induced bundle map $\tilde{g}(X) = g(X, \cdot)$. We also consider the bundle map: $dp : TT^n \to TM$. Since $dp_x$ is a fiber-isomorphism for each $x \in T^n$, $p$ is a d-fold covering map. It follows that

$$F : T^* (T^n) \to T^* M, \quad F := \tilde{g}_M d\rho \tilde{g}_T^{-1}$$

is also a d-fold covering map. Let $\mathcal{T}$ denote the foliation of $T^* M - 0$ by orbits of the torus action. We define $F^{-1} \mathcal{T}$ to be the foliation of $T^* T^n - 0$ whose leaves are given by $\tilde{T}_I := F^{-1} T_I$ where $\{T_I\}$ are the leaves of $\mathcal{T}$. (The associated involutive distribution of the $n$-planes $\tilde{T}_{x,\nu} \subset T_{x,\nu} T^* T^n - 0$ is defined by $dF(\tilde{T}_{x,\nu}) = T_{F(I(x, \nu))}\tilde{T}_I(\tilde{F}(x, \nu))$. This foliation could also defined as orbits of the commuting Hamiltonians $F^* I_j$ on $T^* T^n - 0$. Each of the leaves is compact, hence a torus. We note that $F : \tilde{T}_I \to T_I$ is always a smooth covering map.

We then have the commutative diagrams:

$$\begin{array}{ccc}
\tilde{T}_I & \overset{F}{\to} & T_I \\
\pi \downarrow & & \downarrow \pi \\
T^n & \overset{p}{\to} & M
\end{array} \quad (34)$$

We claim that the map $\pi : \tilde{T}_I \to T^n$ is non-singular. If not, the map $\pi \circ F : \tilde{T}_I \to M$ would be singular. But as observed above, it is a covering map. It further follows by the result of [LS] that $\pi : \tilde{T}_I \to T^n$ has degree one, hence is a diffeomorphism.

We have now reduced to the previous case of the torus: the metric $g_T$ must be a flat metric, hence $g_M$ must be flat. This completes the second proof of Theorem [0.1] in the case of torus actions. \[ \square \]

3.4. Proof of Theorem [0.1] for Schrödinger operators. We now consider the case of semiclassical Schrödinger operators $\hbar^2 \Delta + V$. Our proof in the homogeneous case (i.e. $V = 0$) was based on the use of semiclassical pseudodifferential operators, so it generalizes with little change.

Proof:

We fix an energy level $E$ and consider eigenvalues of $\hbar^2 \Delta + V$ lying in $[E - C\hbar^{1-\delta}, E + C\hbar^{1-\delta}]$ for some fixed $C > 0$. The eigenfunctions we consider are the joint eigenfunctions of $P_1, \ldots, P_n$ with joint eigenvalues $(E_j(h) = b_j^{(1)}(h), \ldots, b_j^{(n)}(h))$ respectively, satisfying $b_j^{(1)}(h) \in [E - C\hbar^{1-\delta}, E + C\hbar^{1-\delta}]$ for some $0 < \delta < 1$. We recall that $b = (b^{(1)} = E, b^{(2)}, \ldots, b^{(n)})$ and $E$ corresponds to the energy shell $X_E$ of the classical Hamiltonian
1/2|ξ|^2 + V corresponding to the quantum Hamiltonian $P_1 = \hbar^2 \Delta + V$. By assumption, the eigenfunctions corresponding to these joint eigenvalues are uniformly bounded independently of $\hbar \leq \hbar_0$.

By Proposition (3.2), it follows that all Lagrangian torus orbits of $\Phi^E_t$ on $X_E$ project regularly to the base. Indeed, the proof that the torus $\Lambda^{(j)}(b)$ projects regularly only involves trace formula and quantum limits over joint eigenvalues in the set $\{(E_j(\hbar) = b_j^{(1)}(\hbar), b_j^{(2)}(\hbar), ..., b_j^{(n)}(\hbar)) : |b_j(\hbar) - b| \leq \hbar^{1-\delta}\}$. Hence our assumption on uniform boundedness of the eigenfunctions of $P_1 = \hbar^2 \Delta + V$ with eigenvalues in the interval $[E - c\hbar^{1-\delta}, E + c\hbar^{1-\delta}]$ is sufficient to obtain the result of Proposition (3.2) for the tori on the energy shell $X_E$.

Hence, by a simple covering space argument, we can without loss of generality assume that the base manifold is a torus. By Lemma (3.4), there are no singular levels of the moment map $P|_{X_E}$. Hence $X_E$ has a smooth Lagrangian foliation invariant under $\Phi^E_t$. By Proposition (1.7), we must have that $E > V_{\max}$ and the Jacobi metric $(E - V)g$ is flat.

If we additionally assume that the sup norms are bounded independently of $\hbar$ and $E$ in some interval $[E_0 - \epsilon, E_0 + \epsilon]$, then the Jacobi metrics $(E - V)g$ are flat for all $E$ in this interval, and it follows by Corollary (1.8) that $g$ is flat and $V$ is constant.

4. Problems and Conjectures

We conclude with some problems conjectures on integrable systems and their eigenfunctions.

4.1. Symplectic geometry of toric integrable systems. Some of the ideas of this paper are relevant to purely geometric problems.

**Conjecture 4.1.** Suppose that $g$ is a metric on $\mathbb{R}^n/\mathbb{Z}^n$ which is toric integrable. Then $g$ is flat.

This would follow from the solution of the Hopf conjecture and from

**Conjecture 4.2.** Up to symplectic equivalence, the only homogeneous Hamiltonian torus action on $T^*(\mathbb{R}^n/\mathbb{Z}^n)$ is the standard one $(\Phi_t(x, \xi) = (x + t\xi, \xi))$.

Indeed, the geodesic flow of $(\mathbb{R}^n/\mathbb{Z}^n, g)$ would preserve the Lagrangian foliation defined by orbits of $\Phi_t$ and hence by Mane’s theorem $g$ would have no conjugate points.

Since the time of the original submission of this article, these conjectures have been proved by E. Lerman and N. Shirokova [LS].

4.2. Eigenfunctions. We assume throughout that the Laplacian or Schroedinger operator was quantum completely integrable. It is natural to ask if the hypothesis can be weakened to classical integrability.

**Conjecture 4.3.** Suppose that $(M, g)$ is a compact Riemannian manifold with completely integrable geodesic flow. Suppose that $\Delta_g + V$ is a Schroedinger operator on $(M, g)$ all of whose ONBE’s have uniformly bounded sup norms. Then $(M, g)$ is flat.

Without the assumption of quantum complete integrability, it is not even known whether eigenfunctions localize on level sets of the classical moment map.

There are also interesting problems in the converse direction. We will explain the difficulty of the next conjecture when we come to multiplicities.

**Conjecture 4.4.** Suppose that $\Delta_g + V$ is a Schroedinger operator on a flat manifold $(M, g)$. Then for generic $V$, are the eigenfunctions uniformly bounded?

We further note that all the questions about sup norms are equally reasonable in the non-compact case.
4.3. KAM and classically non-integrable systems. We now consider the extent to which even classical complete integrability can be dropped. It is plausible that sup norm blow-up occurs whenever there exists a stable elliptic orbit of the geodesic flow. In that case one can construct quasimodes associated to the orbit which do blow up. The relation between modes and quasimodes can be quite complicated in general, but it is plausible that there should exist a sequence of modes which also blows up. KAM systems always contain such stable elliptic orbits. We plan to consider these issues in a future article.

4.4. Multiplicities and sup norms. There are (well-known) relations between eigenvalue multiplicities and sup-norm blow up of eigenfunction. If there exists a sequence of eigenvalues of unbounded multiplicity, then there exists an ONBE with unbounded sup norms. Indeed, for each \( x \), consider the eigenfunction \( \Pi_E(x, \cdot) \) where \( \Pi_E \) is the orthogonal projection onto the eigenspace \( V_E \). Then \( \Pi_E(x, \cdot) \) has \( L^2 \)-norm equal to \( \sqrt{\Pi_E(x, x)} \). So the normalized eigenfunction is \( \phi_E^*(\cdot) := \Pi_N(x, \cdot)/\sqrt{\Pi_N(x, x)} \). It is well-known and easy to see (by the Schwartz inequality) that \( \phi_E^*(\cdot) \) has its maximum at \( x \), where it equals \( \sqrt{\Pi_N(x, x)} \). Since \( \int_M \Pi_N(x, x) \, dvol(x) = m(E) \) (with \( dvol(x) \) the volume form), there must exist \( x \) so that \( \Pi_N(x, x) \geq m(E) \). Hence \( \|\phi_E^*(\cdot)\|_{\infty} \geq \sqrt{m(E)} \). When \( (M, g) \) is a rational torus, \( L^\infty(\lambda, M, g) \) therefore grows at a polynomial rate while \( \ell^\infty(\lambda, M, g) \) stays bounded.

For instance, on a flat torus \( \mathbb{R}^n/L \), an ONBE of the standard Laplacian \( \Delta_0 \) is given by the exponentials \( e^{i(\lambda, x)} \), with \( \lambda \in \Lambda := L^* \), the dual lattice to \( L \). The associated eigenvalue is \( E = |\lambda|^2 \) and its multiplicity \( m(E) \) is the number of lattice points of \( \Lambda \) on the sphere of radius \( \sqrt{E} \). Counting this number is a well-known problem in number theory when the lattice is rational. When \( L = \mathbb{Z}^n \), for instance, the multiplicity function \( m(E) \) has logarithmic growth for \( n = 2 \), and polynomial growth in higher dimensions.

Under a perturbation by a potential \( \epsilon V \), there exists a smoothly varying orthonormal basis of eigenfunctions (sometimes called the Kato-Rellich basis). It is possible that for some potential \( V \) on \( \mathbb{R}^n/\mathbb{Z}^n \), the Kato-Rellich basis for the perturbation \( \Delta_0 + \epsilon V \) may be a smooth deformation of the eigenfunctions just described with high sup norms. If so, it is then possible that even if the multiplicity is broken and all eigenvalues become simple, the eigenfunctions can still have unbounded sup norms. Conjecture 4.4 states that such potentials should be sparse. It would be of some interest to understand if there exist any potentials for which sup norm blow up occurs.

The most extreme case of multiplicity is that of course that on the standard sphere \( (S^2, g_0) \). At this time of writing, it remains an open problem whether \( \ell^\infty(\lambda, g) = O(1) \) on the standard sphere. The best result to date is the upper bound of VanderKam [V], that for a “random” ONBE of eigenfunctions \( \{\phi_\lambda\} \) the sup-norms satisfy \( \|\phi_\lambda\|_\infty/\|\phi_\lambda\|_{L^2} = \mathcal{O}(\sqrt{\log \lambda}) \), i.e. \( \ell^\infty(\lambda, S^2, g_0) = \mathcal{O}(\sqrt{\log \lambda}) \). Our methods do not apply to this problem.

4.5. Quantitative problems. Can one weaken the hypothesis of uniform boundedness of eigenfunctions in \( L^\infty \) in the rigidity results? It is plausible that our rigidity results holds as long as \( L^\infty(\lambda, M, g) \) lies below some threshold. One may ask the same question for the analogous \( L^p \) quantities \( L^p(\lambda, M, g) \). In \([12]\) (see also [T1][T2]), we analyse sup norm blow-up of eigenfunctions near singular levels (among other things). We also study some cases of sup norm blow up near singular projections of regular levels. To obtain a threshold of some generality one needs to estimate the minimal blow up corresponding to the possible types of singular behaviour.

References

[AM] R. Abraham and J.E. Marsden, Foundations of mechanics, second edition, Benjamin/Cummings (1978).
[Be] M. V. Berry, Regular and irregular semiclassical wavefunctions, J. Phys. A 10 (1977), no. 12, 2083–2091.
[Be2] M. V. Berry, Semi-classical mechanics in phase space: a study of Wigner’s function. Philos. Trans. Roy. Soc. London Ser. A 287 (1977), 237–271.
[BP] M. Bialy and L. Polterovich, Hopf-type rigidity for Newton equations, Math. Res. Lett. 2 (1995), 695-700.
[B.K.S] P. Bleher, D. Kosygin and Ya.G. Sinai, Distribution of energy levels of a quantum free particle on a Liouville surface and trace formulae, Comm.Math.Phys. 179 (1995), 375-403.
[BI] D. Burago and S. Ivanov, Riemannian tori without conjugate points are flat, GAFA 4:3 (1994), 259-269.

[Ch] A. M. Charbonnel, Comportement semi-classique du spectre conjoint d’opérateurs pseudo-différentiel qui commutent, Asympt. Anal. 1 (1988), 227-261.

[CP] Y. Colin de Verdière and B. Parisse, Équilibre instable en régime semi-classique I: concentration microlocale, Comm. in P.D.E. 19 (1994), 1535-1563.

[CV1] Y. Colin de Verdière, Spectre conjoint d’opérateurs pseudo-différentiels qui commutent II: Le cas intégrable, Math. Zeit. 171 (1980), 51-75.

[CV2] Y. Colin de Verdière, Quasi-modes sur les variétés riemanniennes, Invent. Math. 43 (1977), 15-52.

[CV3] Y. Colin de Verdière, Sur le spectre des opérateurs elliptiques à bicaractéristiques toutes périodiques. (French) Comment. Math. Helv. 54 (1979), 508-522.

[CK] C. Croke and B. Kleiner, On tori without conjugate points, Invent. Math. 120 (1995), 241-257.

[D] J. J. Duistermaat, On global action-angle coordinates, Comm. Pure Appl. Math. 33 (1980), 687-706.

[E] A. Einstein, Ver. Deut. Phys. Ges. 19 (82-92).

[GS] V. Guillemin and S. Sternberg, Homogeneous quantization and multiplicities of group representations, J. Fun. Anal. 47(1982), 344-380.

[GS2] V. Guillemin and S. Sternberg, Geometric Asymptotics, AMS Surveys 14, AMS (1977).

[He] G. J. Heckman, Quantum integrability for the Kovalevsky top. Indag. Math. (N.S.) 9 (1998), 359–365.

[H] B. Helffer, Semiclassical analysis for the Schrödinger operator and applications, Lecture Notes vol. 1336, Springer-Verlag (1988).

[JZ] D. Jakobson and S. Zelditch, Classical limits of eigenfunctions for some completely integrable systems, in Emerging Applications of Number Theory, IMA vol. 109, Springer, New York (1999).

[K] A. Knauf, Closed orbits and reverse KAM theory, Nonlinearity 3 (1990), 961-973.

[KMS] D. Kosygin, A. Minasov, Ya. G. Sinai, Statistical properties of the spectra of Laplace-Beltrami operators on Liouville surfaces. (Russian) Uspekhi Mat. Nauk 48 (1993), no. 4(292), 3-130; translation in Russian Math. Surveys 48 (1993), no. 4, 1–142

[LS] F. Lalonde, J. C. Sikorav, Sous-variétés lagrangiennes et lagrangiennes exactes des fibrés cotangents, Comment. Math. Helv. 66 (1991), 18-33.

[L1] E. Lerman, A convexity theorem for torus actions on contact manifolds [math.SG/0012017].

[L2] E. Lerman, Contact toric manifolds (in preparation).

[LS] E. Lerman and N. Shirokova, Toric integrable geodesic flows (e-math.DG/0011139.

[M] R. Mane, On a theorem of Klingenberg. Dynamical systems and bifurcation theory (Rio de Janeiro, 1985), 319–345, Pitman Res. Notes Math. Ser., 160, Longman Sci. Tech., Harlow, 1987.

[M2] R. Mane, Ergodic theory and differentiable dynamics, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), Springer-Verlag, Berlin-New York, 1987.

[T1] J. A. Toth, Eigenfunction localization in the quantized rigid body, J. Diff. Geom. 43(4)(1996), 844-858.

[T2] J. A. Toth, On the quantum expected values of integrable metric forms, J. Diff. Geom. 52(2)(1999), 327-374.

[T3] J. A. Toth, Various quantum mechanical aspects of quadratic forms, J. Func. Anal. 130 (1995), 1-42.

[TZ] J. A. Toth and S. Zelditch, Estimates of eigenfunctions in the completely integrable case (preprint, 2000).

[V] J. VanderKam, $L^{\infty}$ norms and quantum ergodicity on the sphere. Internat. Math. Res. Notices (1997), 329–347; Correction to: "$L^{\infty}$ norms and quantum ergodicity on the sphere", Internat. Math. Res. Notices (1998), 65.

[W] J. A. Wolf, Spaces of constant curvature. Fifth edition. Publish or Perish, Inc., Houston, Tex., 1984.

[Y] J. A. Yorke, Periods of periodic solutions and the Lipschitz constant. Proc. Amer. Math. Soc. 22 (1969), 509-512.

[Z1] S. Zelditch, Quantum transition amplitudes for classically ergodic or completely integrable systems, J. Fun. Anal. 94 (1990), 415-436.