Stochastic Heat Equation Driven by Fractional Noise and Local Time

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Abstract

The aim of this paper is to study the \( d \)-dimensional stochastic heat equation with a multiplicative Gaussian noise which is white in space and it has the covariance of a fractional Brownian motion with Hurst parameter \( H \in (0, 1) \) in time. Two types of equations are considered. First we consider the equation in the Itô-Skorohod sense, and later in the Stratonovich sense. An explicit chaos development for the solution is obtained. On the other hand, the moments of the solution are expressed in terms of the exponential moments of some weighted intersection local time of the Brownian motion.

1 Introduction

This paper deals with the \( d \)-dimensional stochastic heat equation

\[
\frac{\partial u}{\partial t} = \frac{1}{2} \Delta u + u \diamond \frac{\partial^2 W^H}{\partial t \partial x}
\]

(1.1)
driven by a Gaussian noise \( W^H \) which is a white noise in the spatial variable and a fractional Brownian motion with Hurst parameter \( H \in (0, 1) \) in the time variable (see (2.1) in the next section for a precise definition of this noise). The initial condition \( u_0 \) is a bounded continuous function on \( \mathbb{R}^d \), and the solution will be a random field \( \{u_{t,x}, t \geq 0, x \in \mathbb{R}^d\} \). The symbol \( \diamond \) in Equation (1.1) denotes the Wick product. For \( H = \frac{1}{2} \), \( \frac{\partial^2 W^H}{\partial t \partial x} \) is a space-time white noise, and in this case, Equation (1.1) coincides with the stochastic heat equation considered by Walsh (see [17]). We know that in this case the solution exists only in dimension one \( (d = 1) \).

There has been some recent interest in studying stochastic partial differential equations driven by a fractional noise. Linear stochastic evolution equations in

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a Hilbert space driven by an additive cylindrical fBm with Hurst parameter $H$ were studied by Duncan et al. in [3] in the case $H \in (\frac{1}{2}, 1)$ and by Tindel et al. in [15] in the general case, where they provide necessary and sufficient conditions for the existence and uniqueness of an evolution solution. In particular, the heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{2} \Delta u + \frac{\partial^2 W^H}{\partial t \partial x}$$

on $\mathbb{R}^d$ has a unique solution if and only if $H > \frac{d}{2}$. The same result holds when one adds to the above equation a nonlinearity of the form $b(t, x, u)$, where $b$ satisfies the usual linear growth and Lipschitz conditions in the variable $u$, uniformly with respect to $(t, x)$ (see Maslowski and Nualart in [9]).

The stochastic heat equation on $[0, \infty) \times \mathbb{R}^d$ with a multiplicative fractional white noise of Hurst parameter $H = (H_0, H_1, \ldots, H_d)$ has been studied by Hu in [6] under the conditions $\frac{1}{2} < H_i < 1$ for $i = 0, \ldots, d$ and $\sum_{i=0}^d H_i < d - \frac{d^2}{2H_0 - 1}$.

The main purpose of this paper is to find conditions on $H$ and $d$ for the solution to Equation (1.1) to exist as a real-valued stochastic process, and to relate the moments of the solution to the exponential moments of weighted intersection local times. This relation is based on Feynman-Kac’s formula applied to a regularization of Equation (1.1). In order to illustrate this fact, consider the particular case $d = 1$ and $H = \frac{1}{2}$. It is known that there is no Feynman-Kac’s formula for the solution of the one-dimensional stochastic heat equation driven by a space-time white noise. Nevertheless, using an approximation of the solution by regularizing the noise we can establish the following formula for the moments:

$$E[u_{t,x}^k] = E^B \left[ \prod_{j=1}^k u_0(x + B^i_t) \exp \left( \sum_{i,j=1, i<j}^k \int_0^t \delta_0(B^i_s - B^j_s)ds \right) \right], \quad (1.2)$$

for all $k \geq 2$, where $B^i_t$ is a $k$-dimensional Brownian motion independent of the space-time white noise $W^\frac{H}{2}$. In the case $H > \frac{1}{2}$ and $d \geq 1$, a similar formula holds but $\int_0^t \delta_0(B^i_s - B^j_s)ds$ has to be replaced by the weighted intersection local time

$$L_t = H(2H - 1) \int_0^t \int_0^t |s - r|^{2H-2} \delta_0(B^i_s - B^j_r)dsdt, \quad (1.3)$$

where $\{B^j, j \geq 1\}$ are independent $d$-dimensional Brownian motions (see Theorem 5.3).

The solution of Equation (1.1) has a formal Wiener chaos expansion $u_{t,x} = \sum_{n=0}^\infty f_n(f_n(\cdot, t, x))$. Then, for the existence of a real-valued square integrable solution we need

$$\sum_{n=0}^\infty n! \|f_n(\cdot, t, x)\|_{\mathcal{H}_d^\otimes n}^2 < \infty, \quad (1.4)$$

where $\mathcal{H}_d$ is the Hilbert space associated with the covariance of the noise $W^H$. It turns out that, if $H > \frac{1}{2}$, the asymptotic behavior of the norms $\|f_n(\cdot, t, x)\|_{\mathcal{H}_d^\otimes n}$
is similar to the behavior of the \(n\)th moment of the random variable \(L_t\) defined in (1.3). More precisely, if \(u_0\) is a constant \(K\), for all \(n \geq 1\) we have

\[(n!)^2 \|f_n(\cdot, t, x)\|_{\mathcal{H}^d_0}^2 = K^2 E(L_t^n).\]

These facts lead to the following results:

i) If \(d = 1\) and \(H > \frac{1}{2}\), the series (1.4) converges, and there exists a solution to Equation (1.1) which has moments of all orders that can be expressed in terms of the exponential moments of the weighted intersection local times \(L_t\). In the case \(H = \frac{1}{2}\) we just need the local time of a one-dimensional standard Brownian motion (see (1.2)).

ii) If \(H > \frac{1}{2}\) and \(d < 4H\), the norms \(\|f_n(\cdot, t, x)\|_{\mathcal{H}^d_0}\) are finite and \(E(L_t^n) < \infty\) for all \(n\). In the particular case \(d = 2\), the series (1.4) converges if \(t\) is small enough, and the solution exists in a small time interval. Similarly, if \(d = 2\) the random variable \(L_t\) satisfies \(E(\exp \lambda L_t) < \infty\) if \(\lambda\) and \(t\) are small enough.

iii) If \(d = 1\) and \(\frac{3}{8} < H < \frac{1}{2}\), the norms \(\|f_n(\cdot, t, x)\|_{\mathcal{H}^d_0}\) are finite and \(E(L_t^n) < \infty\) for all \(n\).

A natural problem is to investigate what happens if we replace the Wick product by the ordinary product in Equation (1.1), that is, we consider the equation

\[\frac{\partial u}{\partial t} = \frac{1}{2} \Delta u + u \frac{\partial^2 W^H}{\partial t \partial x} .\]  

(1.5)

In terms of the mild formulation, the Wick product leads to the use of Itô-Skorohod stochastic integrals, whereas the ordinary product requires the use of Stratonovich integrals. For this reason, if we use the ordinary product we must assume \(d = 1\) and \(H > \frac{1}{2}\). In this case we show that the solution exists and its moments can be computed in terms of exponential moments of weighted intersection local times and weighted self-intersection local times in the case \(H > \frac{1}{2}\).

The paper is organized as follows. Section 2 contains some preliminaries on the fractional noise \(W^H\) and the Skorohod integral with respect to it. In Section 3 we present the results on the moments of the weighted intersection local times assuming \(H \geq \frac{1}{4}\). Section 4 is devoted to study the Wiener chaos expansion of the solution to Equation (1.1). The case \(H < \frac{1}{2}\) is more involved because it requires the use of fractional derivatives. We show here that if \(\frac{1}{4} < H < \frac{1}{2}\), the norms \(\|f_n(\cdot, t, x)\|_{\mathcal{H}^d_0}\) are finite and they are related to the moments of a fractional derivative of the intersection local time. We derive the formulas for the moments of the solution in the case \(H \geq \frac{1}{4}\) in Section 5. Finally, Section 6 deals with equations defined using ordinary product and Stratonovich integrals.
2 Preliminaries

Suppose that $W^H = \{W^H(t, A), t \geq 0, A \in \mathcal{B}(\mathbb{R}^d), |A| < \infty\}$, where $\mathcal{B}(\mathbb{R}^d)$ is the Borel $\sigma$-algebra of $\mathbb{R}^d$, is a zero mean Gaussian family of random variables with the covariance function

$$E(W^H(t, A)W^H(s, B)) = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H})|A \cap B|, \quad (2.1)$$

defined in a complete probability space $(\Omega, \mathcal{F}, P)$, where $H \in (0, 1)$, and $|A|$ denotes the Lebesgue measure of $A$. Thus, for each Borel set $A$ with finite Lebesgue measure, $\{W^H(t, A), t \geq 0\}$ is a fractional Brownian motion (fBm) with Hurst parameter $H$ and variance $t^{2H}|A|$, and the fractional Brownian motions corresponding to disjoint sets are independent.

Then, the multiplicative noise $\frac{\partial^2 W^H}{\partial t \partial x}$ appearing in Equation (1.1) is the formal derivative of the random measure $W^H(t, A)$:

$$W^H(t, A) = \int_A \int_0^t \frac{\partial^2 W^H}{\partial s \partial x} ds dx.$$ 

We know that there is an integral representation of the form

$$W^H(t, A) = \int_0^t \int_A K_H(t, s)W(ds, dx),$$

where $W$ is a space-time white noise, and the square integrable kernel $K_H$ is given by

$$K_H(t, s) = c_H s^{1-H} \int_s^t (u-s)^{H-\frac{1}{2}} u^{H-\frac{3}{2}} du,$$

for some constant $c_H$. We will set $K_H(t, s) = 0$ if $s > t$.

Denote by $\mathcal{E}$ the space of step functions on $\mathbb{R}_+$. Let $\mathcal{H}$ be the closure of $\mathcal{E}$ with respect to the inner product induced by

$$\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}} = K_H(t, s).$$

The operator $K_H^*: \mathcal{E} \rightarrow L^2(\mathbb{R}_+)$ defined by $K_H^*(1_{[0,t]})(s) = K_H(t, s)$ provides a linear isometry between $\mathcal{H}$ and $L^2(\mathbb{R}_+)$.

The mapping $1_{[0,t]} \times A \rightarrow W^H(t, A)$ extends to a linear isometry between the tensor product $\mathcal{H} \otimes L^2(\mathbb{R}^d)$, denoted by $\mathcal{H}_d$, and the Gaussian space spanned by $W^H$. We will denote this isometry by $W^H$. Then, for each $\varphi \in \mathcal{H}_d$ we have

$$W^H(\varphi) = \int_0^\infty \int_{\mathbb{R}^d} (K_H^* \otimes I) \varphi(t, x) W(dt, dx).$$

We will make use of the notation $W^H(\varphi) = \int_0^\infty \int_{\mathbb{R}^d} \varphi dW^H$. 

If $H = \frac{1}{2}$, then $\mathcal{H} = L^2(\mathbb{R}_+)$, and the operator $K_H^*$ is the identity. In this case, we have $\mathcal{H}_d = L^2(\mathbb{R}_+ \times \mathbb{R}^d)$. 

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Suppose now that \( H > \frac{1}{2} \). The operator \( K_H^* \) can be expressed as a fractional integral operator composed with power functions (see [11]). More precisely, for any function \( \varphi \in \mathcal{E} \) with support included in the time interval \([0,T]\) we have
\[
(K_H^* \varphi) (t) = c'_H t^{\frac{1}{2} - H} I^{H - \frac{1}{2}}_{T-} \left( \varphi(s)s^{H - \frac{1}{2}} \right) (t),
\]
where \( I^{H - \frac{1}{2}}_{T-} \) is the right-sided fractional integral operator defined by
\[
I^{H - \frac{1}{2}}_{T-} f(t) = \frac{1}{(H - \frac{1}{2})!} \int_t^T (s - t)^{H - \frac{1}{2}} f(s) ds.
\]
In this case the space \( \mathcal{H} \) is not a space of functions (see [14]) because it contains distributions. Denote by \( |\mathcal{H}| \) the space of measurable functions on \([0,T]\) such that
\[
\int_0^\infty \int_0^\infty |r - u|^{2H - 2} |\varphi_r||\varphi_u| dr du < \infty.
\]
Then, \( |\mathcal{H}| \subset \mathcal{H} \) and the inner product in the space \( \mathcal{H} \) can be expressed in the following form for \( \varphi, \psi \in |\mathcal{H}| \)
\[
\langle \varphi, \psi \rangle_{\mathcal{H}} = \int_0^\infty \int_0^\infty \phi(r,u) \varphi_r \varphi_u dr du,
\]
where \( \phi(s,t) = H(2H - 1)|t - s|^{2H - 2} \).

Using Hölder and Hardy-Littlewood inequalities, one can show (see [10]) that
\[
\| \varphi \|_{\mathcal{H}^d} \leq \beta_H \| \varphi \|_{L^\infty \mathcal{H}^d(\mathbb{R}^d)} ,
\]
and this easily implies that
\[
\| \varphi \|_{\mathcal{H}^d_\gamma} \leq \beta^n_H \| \varphi \|_{L^\infty \mathcal{H}^d_\gamma(\mathbb{R}^n)} .
\]

If \( H < \frac{1}{2} \), the operator \( K_H^* \) can be expressed as a fractional derivative operator composed with power functions (see [11]). More precisely, for any function \( \varphi \in \mathcal{E} \) with support included in the time interval \([0,T]\) we have
\[
(K_H^* \varphi) (t) = c''_H t^{\frac{1}{2} - H} D^{\frac{1}{2} - H}_{T-} \left( \varphi(s)s^{H - \frac{1}{2}} \right) (t),
\]
where \( D^{\frac{1}{2} - H}_{T-} \) is the right-sided fractional derivative operator defined by
\[
D^{\frac{1}{2} - H}_{T-} f(t) = \frac{1}{(H + \frac{1}{2})!} \frac{f(t)}{(T - t)^{\frac{1}{2} - H}} - \left( \frac{1}{2} - H \right) \int_t^T \frac{f(s) - f(t)}{(s - t)^{H - \frac{1}{2}}} ds.
\]
Moreover, for any \( \gamma > \frac{1}{2} - H \) and any \( T > 0 \) we have \( C^\gamma([0,T]) \subset \mathcal{H} = I^{\frac{1}{2} - H}_{T-} (L^2([0,T])). \)
If $\varphi$ is a function with support on $[0, T]$, we can express the operator $K_H^*$ in the following form:

$$K_H^* \varphi(t) = K_H(T, t) \varphi(t) + \int_t^T [\varphi(s) - \varphi(t)] \frac{\partial K_H}{\partial s}(s, t) ds. \quad (2.5)$$

We are going to use the following notation for the operator $K_H^*$:

$$K_H^* \varphi = \int_{[0, T]} \varphi(t) K_H^*(dt, r). \quad (2.6)$$

Notice that if $H > \frac{1}{2}$, the kernel $K_H$ vanishes at the diagonal and we have

$$K_H^*(dt, r) = \frac{\partial K_H}{\partial t}(t, r) 1_{[r, T]}(t) dt.$$ 

Let us now present some preliminaries on the Skorohod integral and the Wick product. The $n$th Wiener chaos, denoted by $H_n$, is defined as the closed linear span of the random variables of the form $H_n(W^H(\varphi))$, where $\varphi$ is an element of $H_d$ with norm one and $H_n$ is the $n$th Hermite polynomial. We denote by $I_n$ the linear isometry between $H_d^\otimes n$ (equipped with the modified norm $\sqrt{n!} \| \cdot \|_{H_d^\otimes n}$) and the $n$th Wiener chaos $H_n$, given by $I_n(\varphi^\otimes n) = n! H_n(W^H(\varphi))$, for any $\varphi \in H_d$ with $\| \varphi \|_{H_d} = 1$. Any square integrable random variable, which is measurable with respect to the $\sigma$-field generated by $W^H$, has an orthogonal Wiener chaos expansion of the form

$$F = E(F) + \sum_{n=1}^{\infty} I_n(f_n),$$

where $f_n$ are symmetric elements of $H_d^\otimes n$, uniquely determined by $F$.

Consider a random field $u = \{u_{t, x}, t \geq 0, x \in \mathbb{R}^d\}$ such that $E(u_{t, x}^2) < \infty$ for all $t, x$. Then, $u$ has a Wiener chaos expansion of the form

$$u_{t, x} = E(u_{t, x}) + \sum_{n=1}^{\infty} I_n(f_n(\cdot, t, x)), \quad (2.7)$$

where the series converges in $L^2(\Omega)$.

**Definition 2.1** We say the random field $u$ satisfying (2.7) is Skorohod integrable if $E(u) \in H_d$, for all $n \geq 1$, $f_n \in H_d^\otimes (n+1)$, and the series

$$W^H(E(u)) + \sum_{n=1}^{\infty} I_{n+1}(\tilde{f}_n)$$

converges in $L^2(\Omega)$, where $\tilde{f}_n$ denotes the symmetrization of $f_n$. We will denote the sum of this series by $\delta(u) = \int_0^\infty \int_{\mathbb{R}^d} u dW^H$. 


The Skorohod integral coincides with the adjoint of the derivative operator. That is, if we define the space $D^{1,2}$ as the closure of the set of smooth and cylindrical random variables of the form

$$F = f(W^H(h_1), \ldots, W^H(h_n)),$$

$h_i \in \mathcal{H}_d$, $f \in C^\infty_p(\mathbb{R}^n)$ ($f$ and all its partial derivatives have polynomial growth) under the norm

$$\|DF\|_{1,2} = \sqrt{E(F^2) + E(\|DF\|_{\mathcal{H}_d}^2)},$$

where

$$DF = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(W^H(h_1), \ldots, W^H(h_n))h_j,$$

then, the following duality formula holds

$$E(\delta(u)F) = E\left(\langle DF, u \rangle_{\mathcal{H}_d}\right),$$

(2.8)

for any $F \in D^{1,2}$ and any Skorohod integrable process $u$.

If $F \in D^{1,2}$ and $h$ is a function which belongs to $\mathcal{H}_d$, then $Fh$ is Skorohod integrable and, by definition, the Wick product equals to the Skorohod integral of $Fh$:

$$\delta(Fh) = F \circ W^H(h).$$

(2.9)

This formula justifies the use of the Wick product in the formulation of Equation (1.1).

Finally, let us remark that in the case $H = \frac{1}{2}$, if $u_{t,x}$ is an adapted stochastic process such that $E\left(\int_0^\infty \int_{\mathbb{R}^d} u_{t,x}^2 dt dx\right) < \infty$, then $u$ is Skorohod integrable and $\delta(u)$ coincides with the Itô stochastic integral:

$$\delta(u) = \int_0^\infty \int_{\mathbb{R}^d} u_{t,x} W(dt, dx).$$

### 3 Weighted intersection local times for standard Brownian motions

In this section we will introduce different kinds of weighted intersection local times which are relevant in computing the moments of the solutions of stochastic heat equations with multiplicative fractional noise.

Suppose first that $B^1$ and $B^2$ are independent $d$-dimensional standard Brownian motions. Consider a nonnegative measurable function $\eta(s, t)$ on $\mathbb{R}_+^2$. We are interested in the weighted intersection local time formally defined by

$$I = \int_0^T \int_0^T \eta(s, t)\delta_0(B^1_t - B^2_t)dsdt.$$

(3.1)

We will make use of the following conditions on the weight $\eta$:
where $s$

Using the Fourier transform of the heat kernel we can write

$$
\Psi
$$

Let us compute the $k$

$$
\xi
$$

where $p$

heat kernel

$\alpha$

Then, if $I$

limit

$I$

in $L$

for all $q > p$

Then the following result holds.

Clearly, C2) is stronger than C1). We will denote by $p_t(x)$ the $d$-dimensional heat kernel $p_t(x) = (2\pi t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{2t}}$. Consider the approximation of the intersection local time (3.1) defined by

$$
I_{\varepsilon} = \int_0^T \int_0^T \eta(s, t) p_{\varepsilon}(B_s^1 - B_t^1) ds dt. \quad (3.2)
$$

Let us compute the $k$th moment of $I_{\varepsilon}$, where $k \geq 1$ is an integer. We can write

$$
E(I_{\varepsilon}^k) = \int_{[0, T]^{2k}} \prod_{i=1}^k \eta(s_i, t_i) \psi_{\varepsilon}(s, t) ds dt, \quad (3.3)
$$

where $s = (s_1, \ldots, s_k)$, $t = (t_1, \ldots, t_k)$ and

$$
\psi_{\varepsilon}(s, t) = E(p_{\varepsilon}(B_{s_1}^1 - B_{t_1}^1) \cdots p_{\varepsilon}(B_{s_k}^1 - B_{t_k}^1)). \quad (3.4)
$$

Using the Fourier transform of the heat kernel we can write

$$
\psi_{\varepsilon}(s, t) = \frac{1}{(2\pi)^{kd}} \left( \int_{\mathbb{R}^k} E \left( \exp \left( \sum_{j=1}^k \left( i \left( \xi_j, b_{s_j}^1 - b_{t_j}^1 \right) - \frac{\varepsilon}{2} |\xi_j|^2 \right) \right) \right) d\xi \right)^d
= \frac{1}{(2\pi)^{kd}} \left( \int_{\mathbb{R}^k} e^{-\frac{1}{2} \sum_{j=1}^k \xi_j \operatorname{Cov}(b_{s_j}^1 - b_{t_j}^1, b_{s_j}^1 - b_{t_j}^1)} e^{-\frac{\varepsilon}{2} |\xi|^2} d\xi \right)^d, \quad (3.5)
$$

where $\xi = (\xi_1, \ldots, \xi_k)$ and $b_i^1$, $i = 1, 2$, are independent one-dimensional Brownian motions. Then $\psi_{\varepsilon}(s, t) \leq \psi(s, t)$, where

$$
\psi(s, t) = (2\pi)^{-\frac{kd}{2}} [\det(s_j \wedge s_l + t_j \wedge t_l)]^{-\frac{d}{2}}. \quad (3.6)
$$

Set

$$
\alpha_k = \int_{[0, T]^{2k}} \prod_{i=1}^k \eta(s_i, t_i) \psi(s, t) ds dt. \quad (3.7)
$$

Then, if $\alpha_k < \infty$ for all $k \geq 1$, the family $I_{\varepsilon}$ converges in $L^p$, for all $p \geq 2$, to a limit $I$ and $E(I_{\varepsilon}^k) = \alpha_k$. In fact,

$$
\lim_{\varepsilon, i \downarrow 0} E(I_{\varepsilon} I_\delta) = \alpha_2,
$$

so $I_{\varepsilon}$ converges in $L^2$, and the convergence in $L^p$ follows from the boundedness in $L^q$ for $q > p$. Then the following result holds.
Proposition 3.1 Suppose that C1) holds and $d = 1$. Then, for all $\lambda > 0$ the random variable defined in (3.2) satisfies

$$\sup_{\epsilon > 0} E \left( \exp \left( \lambda I_\epsilon \right) \right) \leq 1 + \Phi \left( \frac{\sqrt{T}}{2} \| \eta \|_{1,T} \lambda \right), \quad (3.8)$$

where $\Phi(x) = \sum_{k=1}^{\infty} \frac{x^k}{k! \Gamma(k+1)}$. Also, $I_\epsilon$ converges in $L^p$ for all $p \geq 2$, and the limit, denoted by $I$, satisfies the estimate (3.8).

Proof The term $\psi(s, t)$ defined in (3.6) can be estimated using Cauchy-Schwarz inequality:

$$\psi(s, t) \leq (2\pi)^{-\frac{k}{2}} \left[ \det (s_j \wedge s_l) \right]^{\frac{1}{k}} \left[ \det (t_j \wedge t_l) \right]^{\frac{1}{k}}$$

$$= 2^{-k} \pi^{-\frac{k}{2}} \left[ |\beta(s)\beta(t)|^{-\frac{1}{2}} \right], \quad (3.9)$$

where for any element $(s_1, \ldots, s_k) \in (0, \infty)^k$ with $s_i \neq s_j$ if $i \neq j$, we denote by $\sigma$ the permutation of its coordinates such that $s_{\sigma(1)} < \cdots < s_{\sigma(n)}$ and $\beta(s) = s_{\sigma(1)}(s_{\sigma(2)} - s_{\sigma(1)}) \cdots (s_{\sigma(k)} - s_{\sigma(k-1)})$. Therefore, from (3.9) and (3.7) we obtain

$$\alpha_k \leq 2^{-k} \pi^{-\frac{k}{2}} \left( \int_{[0,T]^2k} \prod_{i=1}^{k} \eta(s_i, t_i) [\beta(s)][\beta(t)]^{-\frac{1}{2}} ds dt \right)^{\frac{1}{2}} \quad (3.10)$$

Applying again Cauchy-Schwarz inequality yields

$$\alpha_k \leq 2^{-k} \pi^{-\frac{k}{2}} \left( \int_{[0,T]^2k} \prod_{i=1}^{k} \eta(s_i, t_i) [\beta(s)][\beta(t)]^{-\frac{1}{2}} ds dt \right)^{\frac{1}{2}}$$

$$\times \left( \int_{[0,T]^2k} \prod_{i=1}^{k} \eta(s_i, t_i) [\beta(t)]^{-\frac{1}{2}} ds dt \right)^{\frac{1}{2}}$$

$$\leq \left( 2^{-1} \pi^{-\frac{k}{2}} \| \eta \|_{1,T} \right) \left( \frac{k!}{\Gamma(\frac{k+1}{2})} \right)$$

$$= k! 2^{-k} T^{\frac{1}{2}} \| \eta \|_{1,T} \frac{1}{\Gamma(\frac{k+1}{2})}, \quad (3.11)$$

where $T_k = \{ s = (s_1, \ldots, s_k) : 0 < s_1 < \cdots < s_k < T \}$, which implies the estimate (3.8). \hfill \blacksquare

This result can be extended to the case of a $d$-dimensional Brownian motion under the stronger condition C2):

Proposition 3.2 Suppose that C2) holds and $d < 4H$. Then, $\lim_{\epsilon \to 0} I_\epsilon = I$, exists in $L^p$, for all $p \geq 2$. Moreover, if $d = 2$ and $\lambda < \lambda_0(T)$, where

$$\lambda_0(T) = \frac{H(2H - 1)4\pi}{\gamma_T^2 \beta_H} T^{-\frac{2H}{2H} - 2H} \Gamma \left( 1 - \frac{d}{4H} \right) \Gamma \left( \frac{k+1}{2} \right)^{-2H}, \quad (3.12)$$

...
and \( \beta_H \) is the constant appearing in the inequality (2.3), then
\[
\sup_{\varepsilon > 0} E \left( \exp \left( \lambda |I_{\varepsilon} - E(I_{\varepsilon})| \right) \right) < \infty, \tag{3.13}
\]
and \( I \) satisfies \( E(\exp(\lambda I)) < \infty \).

**Proof.** As in the proof of Proposition 3.1 using condition C2) and inequality (2.4) we obtain the estimates
\[
\alpha_k \leq \gamma_T 2^{-dk} \pi^{-d/2} \frac{\beta_H}{H(2H-1)} \int_{[0,T]^k} \prod_{i=1}^k (|t_i - s_i|^{2H-2} |\beta(s)\beta(t)|^{-\frac{d}{2}}) \, ds \, dt.
\]
\[
\leq \gamma_T 2^{-dk} \pi^{-d/2} \alpha_H \int_{[0,T]^k} \left( \frac{|\beta(s)|^{-d/2H}}{\Gamma(k(1 - d/2H) + 1)^2H} \right) \, ds.
\]
\[
= (\gamma_T \beta_H 2^{-d-2} |\pi|^{-1}) \Gamma(1 - \frac{d}{2H}) \frac{2^{2H} \Gamma(1 - \frac{d}{2H})}{\Gamma(k(1 - \frac{d}{4H}) + 1)^2H} \right) \frac{\beta_H}{H(2H-1)} \, ds.
\]
where \( \alpha_H = \frac{\beta_H}{H(2H-1)} \). This allows us to conclude the proof. \( \blacksquare \)

If \( d = 2 \) and \( \eta(s,t) = 1 \) it is known that the intersection local time \( \int_0^T \int_0^T \delta_0(B_s - B_t) \, ds \, dt \) exists and it has finite exponential moments up to a critical exponent \( \lambda_0 \) (see Le Gall [7] and Bass and Chen [1]).

Consider now a one-dimensional standard Brownian motion \( B \), and the weighted self-intersection local time
\[
I = \int_0^T \int_0^T \eta(s,t) \delta_0(B_s - B_t) \, ds \, dt.
\]
As before, set
\[
I_{\varepsilon} = \int_0^T \int_0^T \eta(s,t) \delta_\varepsilon(B_s - B_t) \, ds \, dt.
\]

**Proposition 3.3** Suppose that C2) holds. If \( H > \frac{1}{2} \), then we have
\[
\sup_{\varepsilon > 0} E \left( \exp \left( \lambda |I_{\varepsilon} - E(I_{\varepsilon})| \right) \right) < \infty, \tag{3.14}
\]
for all \( \lambda > 0 \). Moreover, the normalized local time \( I - E(I) \) exists as a limit in \( L^p \) of \( I_{\varepsilon} - E(I_{\varepsilon}) \), for all \( p \geq 2 \), and it has exponential moments of all orders.

If \( H > \frac{3}{4} \), then we have for all \( \lambda > 0 \)
\[
\sup_{\varepsilon > 0} E \left( \exp \left( \lambda |I_{\varepsilon}| \right) \right) < \infty, \tag{3.15}
\]
for all \( \lambda > 0 \), and the local time \( I \) exists as a limit in \( L^p \) of \( I_{\varepsilon} \), for all \( p \geq 2 \), and it is exponentially integrable.
Proof. We will follow the ideas of Le Gall in [7]. Suppose first that $H > \frac{1}{2}$ and let us show (3.14). To simplify the proof we assume $T = 1$. It suffices to show these results for

$$J_\varepsilon := \int_0^1 \int_0^t \eta(s, t) p_\varepsilon (B_s - B_t) ds dt.$$ 

Denote, for $n \geq 1$, and $1 \leq k \leq 2^{n-1}$

$$A_{n,k} = \left[ \frac{2k - 2}{2^n}, \frac{2k - 1}{2^n} \right] \times \left[ \frac{2k - 1}{2^n}, \frac{2k}{2^n} \right].$$

Set

$$\alpha^\varepsilon_{n,k} = \int_{A_{n,k}} \eta(s, t) p_\varepsilon (B_s - B_t) ds dt$$

and

$$\bar{\alpha}^\varepsilon_{n,k} = \alpha^\varepsilon_{n,k} - E (\alpha^\varepsilon_{n,k}).$$

Notice that the random variables $\alpha^\varepsilon_{n,k}$, $1 \leq k \leq 2^{n-1}$, are independent. We have

$$J_\varepsilon = \sum_{n=1}^{\infty} \sum_{k=1}^{2^{n-1}} \alpha^\varepsilon_{n,k};$$

and

$$J_\varepsilon - E (J_\varepsilon) = \sum_{n=1}^{\infty} \sum_{k=1}^{2^{n-1}} \bar{\alpha}^\varepsilon_{n,k}.$$

We can write

$$\alpha^\varepsilon_{n,k} = 2^{-2n} \int_0^1 \int_0^1 \eta \left( \frac{2k - 1}{2^n} - \frac{s}{2^n}, \frac{2k - 1}{2^n} + \frac{t}{2^n} \right)$$

$$\times p_\varepsilon (B_{\frac{2k - 1}{2^n} - \frac{s}{2^n}} - B_{\frac{2k - 1}{2^n} + \frac{t}{2^n}}) ds dt$$

$$\leq \gamma_1 2^{-2n -(2H-2)n} \int_0^1 \int_0^1 |t + s|^{2H-2} p_\varepsilon (B_{\frac{2k - 1}{2^n} - \frac{s}{2^n}} - B_{\frac{2k - 1}{2^n} + \frac{s}{2^n}}) ds dt,$$

which has the same distribution as

$$\beta^\varepsilon_{n,k} = \gamma_1 2^{-2n -(2H-2)n} \int_0^T \int_0^T |t + s|^{2H-2} p_{2^n} (B^1_s - B^2_t) ds dt,$$

where $B^1$ and $B^2$ are independent one-dimensional Brownian motions. Hence, using the estimate (3.11), we obtain

$$E (\exp (\lambda (\bar{\alpha}^\varepsilon_{n,k}))) = 1 + \sum_{j=2}^{\infty} \frac{\lambda^j}{j!} E \left( (\bar{\alpha}^\varepsilon_{n,k})^j \right)$$

$$\leq 1 + \sum_{j=2}^{\infty} \frac{(2\lambda)^j}{j!} E \left( (\bar{\beta}^\varepsilon_{n,k})^j \right)$$

$$\leq 1 + \sum_{j=2}^{\infty} \frac{\left( C_\lambda 2^{-\frac{1}{2}n -(2H-2)n} \right)^j}{\Gamma\left( \frac{j+1}{2} \right)}.$$
for some constant $C_T$. Hence,

$$E\left(\exp\left(\lambda \left(\bar{\alpha}_{n,k}\right)\right)\right) \leq 1 + c_\lambda 2^{-3n-2(2H-2)n} \lambda^2,$$

for some function $c_\lambda$.

Fix $a > 0$ such that $a < 2(2H-1)$. For any $N \geq 2$ define

$$b_N = \prod_{j=2}^{N} (1 - 2^{-a(j-1)}),$$

and notice that $\lim_{N \to \infty} b_N = b_\infty > 0$. Then, by H"{o}lder's inequality, for all $N \geq 2$ we have

$$E\left[\exp\left(\lambda b_N \sum_{n=1}^{N-1} \sum_{k=1}^{2^{n-1}} \bar{\alpha}_{n,k}\right)\right] \leq \left\{ E \left[\exp\left(\frac{\lambda b_N}{1 - 2^{-a(N-1)}} \sum_{n=1}^{N-1} \sum_{k=1}^{2^{n-1}} \bar{\alpha}_{n,k}\right)\right]\right\} \left\{ E \left[\exp\left(\lambda b_N 2^{a(N-1)} \sum_{k=1}^{2^{N-1}} \bar{\alpha}_{N,k}\right)\right]\right\} \left\{ E \left[\exp\left(\lambda b_{N-1} \sum_{n=1}^{N-1} \sum_{k=1}^{2^{n-1}} \bar{\alpha}_{n,k}\right)\right]\right\} \left\{ E \left[\exp\left(\lambda b_N 2^{a(N-1)} \bar{\alpha}_{N,k}\right)\right]\right\}. $$

Using (3.16), the second factor in the above expression can be dominated by

$$\left\{ E \left[\exp\left(\lambda b_N 2^{a(N-1)} \bar{\alpha}_{N,k}\right)\right]\right\}^{2(1-a)/(N-1)} \leq \left(1 + c_\lambda 2^{2a(N-1)} 2^{-3N-2(2H-2)N}\right)^{2(1-a)/(N-1)} \leq \exp\left(\kappa c_\lambda 2^{2(a-2-2H-2)(N-1)}\right),$$

where $\kappa = b_2^2 2^{-a-1}$. Thus by induction we have

$$E\left[\exp\left(\lambda b_N \sum_{n=1}^{N} \sum_{k=1}^{2^{n-1}} \bar{\alpha}_{n,k}\right)\right] \leq \exp\left\{ \sum_{n=2}^{N} \kappa c_\lambda \lambda 2^{(a-2-2(2H-2))n} \right\} \times E\left(\exp\bar{\alpha}_{1,1}\right) \leq \exp(\kappa c_\lambda \lambda^2 (1 - 2^{a+2-4H})^{-1}) \times E\left(\exp(\bar{\alpha}_{1,1})\right) < \infty,$$
because \( a < 2(2H - 1) \). By Fatou lemma we see that
\[
\sup_{\varepsilon > 0} E \left( \exp \left( \lambda b_\varepsilon \left( J_\varepsilon - E (J_\varepsilon) \right) \right) \right) < \infty,
\]
and (3.14) follows.

On the other hand, one can easily show that
\[
\lim_{\varepsilon, \delta \to 0} E((J_\varepsilon - E(J_\varepsilon))(J_\delta - E(J_\delta))) = \frac{1}{2\pi} \int_{s < t < 1, s' < t'} \eta(s, t)\eta(s', t')
\times \left[ \left( \det \begin{bmatrix} t - s & [s, t] \cap [s', t'] \\ [s, t] \cap [s', t'] & t' - s' \end{bmatrix} \right)^{-\frac{1}{2}} - ((t - s)(t' - s'))^{-\frac{1}{2}} \right] ds dt ds' dt' < \infty,
\]
which implies the convergence of \( I_\varepsilon \) in \( L^2 \). The convergence in \( L^p \) for \( p \geq 2 \) and the estimate (3.14) follow immediately.

The proof of the inequality (3.15) is similar. The estimate (3.16) is replaced by
\[
E \left( \exp \left( \lambda \left( \alpha_n, k \right) \right) \right) \leq 1 + d_\lambda 2^{-3n - 2(2H - 2)n} \lambda,
\]
for a suitable function \( d_\lambda \), and we obtain
\[
E \left[ \exp \left( \lambda b_N \sum_{n=1}^{N} \sum_{k=1}^{2^{n-1}} \alpha_{n,k} \right) \right]
\leq \exp \left\{ \sum_{n=2}^{N} \sqrt{k_\lambda} \lambda 2^{-\frac{5}{2} - 2H} n \right\} E (\exp (\alpha_{1,1}))
\leq \exp (\sqrt{k_\lambda} \lambda^2 (1 - 2^{-\frac{5}{2} - 2H} n - 1)) E (\exp (\alpha_{1,1})) < \infty,
\]
because \( H > \frac{3}{4} \). By Fatou lemma we see that
\[
\sup_{\varepsilon > 0} E \left( \exp \left( \lambda b_\varepsilon \left( J_\varepsilon - E (J_\varepsilon) \right) \right) \right) < \infty,
\]
which implies (3.15). The convergence in \( L^p \) of \( I_\varepsilon \) is proved as usual. ■

Notice that condition \( H > \frac{3}{4} \) cannot be improved because
\[
E \left( \int_0^T \int_0^T |t - s|^{-\frac{1}{2}} b_0(B_s - B_t) ds dt \right) = \frac{1}{\sqrt{2\pi}} \int_0^T \int_0^T |t - s|^{-1} ds dt < \infty.
\]

4 Stochastic heat equation in the Itô-Skorohod sense

In this section we study the stochastic partial differential equation (1.1) on \( \mathbb{R}^d \), where \( W^H \) is a zero mean Gaussian family of random variables with the
covariance function (2.1), defined on a complete probability space \((\Omega, \mathcal{F}, P)\), and the initial condition \(u_0\) belongs to \(C_b(\mathbb{R}^d)\). First we give the definition of a solution using the Skorohod integral, which corresponds formally to the Wick product appearing in Equation (1.1).

For any \(t \geq 0\), we denote by \(\mathcal{F}_t\) the \(\sigma\)-field generated by the random variables \(\{W(s, A), 0 \leq s \leq t, A \in \mathcal{B}(\mathbb{R}^d), |A| < \infty\}\) and the \(P\)-null sets. A random field \(u = \{u_{t,x}, t \geq 0, x \in \mathbb{R}\}\) is adapted if for any \((t, x)\), \(u_{t,x}\) is \(\mathcal{F}_t\)-measurable.

For any bounded Borel function \(\varphi\) on \(\mathbb{R}\) we write \(p_t \varphi(x) = \int_{\mathbb{R}^d} p_t(x - y) \varphi(y) dy\).

**Definition 4.1** An adapted random field \(u = \{u_{t,x}, t \geq 0, x \in \mathbb{R}^d\}\) such that \(E(u_{t,x}^2) < \infty\) for all \((t, x)\) is a solution to Equation (1.1) if for any \((t, x)\) \(\in [0, \infty) \times \mathbb{R}^d\), the process \(\{p_{t-s}(x - y) u_{s,y} 1_{[0,t]}(s), s \geq 0, y \in \mathbb{R}^d\}\) is Skorohod integrable, and the following equation holds

\[
\begin{align*}
    u_{t,x} &= p_t u_0(x) + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x - y) u_{s,y} \delta W_{t,y}^H.
\end{align*}
\]

(4.1)

The fact that Equation (1.1) contains a multiplicative Gaussian noise allows us to find recursively an explicit expression for the Wiener chaos expansion of the solution. This approach has extensively used in the literature. For instance, we refer to the papers by Hu [6], Buckdahn and Nualart [2], Nualart and Zakai [13], Nualart and Rozovskii [12], and Tudor [16], among others.

Suppose that \(u = \{u_{t,x}, t \geq 0, x \in \mathbb{R}^d\}\) is a solution to Equation (1.1). Then, for any fixed \((t, x)\), the random variable \(u_{t,x}\) admits the following Wiener chaos expansion

\[
\begin{align*}
    u_{t,x} &= \sum_{n=0}^{\infty} I_n(f_n(\cdot, t, x)),
\end{align*}
\]

(4.2)

where for each \((t, x)\), \(f_n(\cdot, t, x)\) is a symmetric element in \(\mathcal{H}_d^{\otimes n}\). To find the explicit form of \(f_n\) we substitute (4.2) in the Skorohod integral appearing in (4.1) we obtain

\[
\begin{align*}
    \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x - y) u_{s,y} \delta W_{t,y}^H &= \sum_{n=0}^{\infty} \int_0^t \int_{\mathbb{R}^d} I_n(p_{t-s}(x - y) f_n(\cdot, s, y)) \delta W_{t,y}^H
    \end{align*}
\]

\[
= \sum_{n=0}^{\infty} I_{n+1}(p_{t-s}(x - y) \tilde{f}_n(\cdot, s, y)).
\]

Here, \((p_{t-s}(x - y) \tilde{f}_n(\cdot, s, y)\) denotes the symmetrization of the function

\[
p_{t-s}(x - y) f_n(s_1, x_1; \ldots; s_n, x_n; s, y)
\]
in the variables $(s_1, x_1), \ldots, (s_n, x_n), (s, y)$, that is,

\[ p_{t-s}(x-y)f_n(\cdot, s, y) = \frac{1}{n+1}[p_{t-s}(x-y)f_n(s_1, x_1, \ldots, s_n, x_n, s, y) + \sum_{j=1}^{n} p_{t-s_j}(x-y_j) \times f_n(s_1, x_1, \ldots, s_{j-1}, x_{j-1}, s, y, s_{j+1}, x_{j+1}, \ldots, s_n, y, s_j, y_j)]. \]

Thus, Equation (4.1) is equivalent to say that $f_0(t, x) = p_t u_0(x)$, and

\[ f_{n+1}(\cdot, t, x) = p_{t-s}(x-y)f_n(\cdot, s, y) \tag{4.3} \]

for all $n \geq 0$. Notice that, the adaptability property of the random field $u$ implies that $f_n(s_1, x_1, \ldots, s_n, x_n, t, x) = 0$ if $s_j > t$ for some $j$.

This leads to the following formula for the kernels $f_n$, for $n \geq 1$

\[ f_n(s_1, x_1, \ldots, s_n, x_n, t, x) = \frac{1}{n!} \times p_{t-s_{\sigma(n)}}(x-x_{\sigma(n)}) \cdots p_{s_{\sigma(2)}-s_{\sigma(1)}}(x_{\sigma(2)}-x_{\sigma(1)}) u_0(x_{\sigma(1)}), \tag{4.4} \]

where $\sigma$ denotes the permutation of $\{1, 2, \ldots, n\}$ such that $0 < s_{\sigma(1)} < \cdots < s_{\sigma(n)} < t$. This implies that there is a unique solution to Equation (4.1), and the kernels of its chaos expansion are given by (4.4). In order to show the existence of a solution, it suffices to check that the kernels defined in (4.4) determine an adapted random field satisfying the conditions of Definition 4.1. This is equivalent to show that for all $(t, x)$ we have

\[ \sum_{n=1}^{\infty} n! \|f_n(\cdot, t, x)\|^2_{H^d_1} < \infty. \tag{4.5} \]

It is easy to show that (4.5) holds if $H = \frac{1}{2}$ and $d = 1$. In fact, we have, assuming $|u_0| \leq K$, and with the notation $x = (x_1, \ldots, x_n)$, and $s = (s_1, \ldots, s_n)$:

\[
\|f_n(\cdot, t, x)\|^2_{H^d_1} = \frac{1}{(n!)^2} \int_{[0,t]^n} \int_{\mathbb{R}^n} p_{t-s_{\sigma(n)}}(x-x_{\sigma(n)})^2 \cdots p_{s_{\sigma(2)}-s_{\sigma(1)}}(x_{\sigma(2)}-x_{\sigma(1)})^2 \\
\times p_{s_{\sigma(1)}} u_0(x_{\sigma(1)})^2 \, dx \, ds \\
\leq K^2 (4\pi)^{-\frac{d}{2}} (n!)^2 \int_{[0,t]^n} \prod_{j=1}^{n} (s_{\sigma(j+1)} - s_{\sigma(j)})^{-\frac{d}{2}} \, ds \\
= \frac{K^2 (4\pi)^{-\frac{d}{2}}}{n!} \int_{T_n} \prod_{j=1}^{n} (s_{j+1} - s_j)^{-\frac{d}{2}} \, ds,
\]

where $T_n = \{(s_1, \ldots, s_n) \in [0,t]^n : 0 < s_1 < \cdots < s_n < t\}$ and by convention $s_{n+1} = t$. Hence,

\[ \|f_n(\cdot, t, x)\|^2_{H^d_1} \leq \frac{K^2 2^{-n} t^\frac{d}{2}}{n! \Gamma(n+1)}.
\]
which implies (4.5). On the other hand, if \( H = \frac{1}{2} \) and \( d \geq 2 \), these norms are infinite.

Notice that if \( u_0 = 1 \), then \((n!)^2 \| f_n(\cdot, t, x) \|_{H^{\otimes n}}^2\) coincides with the moment of order \( n \) of the local time at zero of the one-dimensional Brownian motion with variance \( 2t \), that is,

\[
(n!)^2 \| f_n(\cdot, t, x) \|_{H^{\otimes n}}^2 = E \left[ \left( \int_0^t \delta_0(B_{2s}) \, ds \right)^n \right].
\]

To handle the case \( H > \frac{1}{2} \), we need the following technical lemma.

**Lemma 4.2** Set

\[
g_s(x_1, \ldots, x_n) = p_{t-s}(x-x_\sigma(1)) \cdots p_{s(2)-s_\sigma(1)}(x_\sigma(2) - x_\sigma(1)). \tag{4.6}
\]

Then,

\[
\langle g_s, g_t \rangle_{L^2(\mathbb{R}^d)} = \psi(s, t),
\]

where \( \psi(s, t) \) is defined in (3.4).

**Proof** By Plancherel’s identity

\[
\langle g_s, g_t \rangle_{L^2(\mathbb{R}^d)} = (2\pi)^{-dn} \langle \mathcal{F}g_s, \mathcal{F}g_t \rangle_{L^2(\mathbb{R}^d)},
\]

where \( \mathcal{F} \) denotes the Fourier transform, given by

\[
\mathcal{F}g_s(\xi_1, \ldots, \xi_n) = (2\pi)^{-dn} \prod_{j=1}^n (s_\sigma(j+1) - s_\sigma(j))^{-\frac{d}{2}}
\]

\[
\times \int_{\mathbb{R}^d} \prod_{j=1}^n \exp \left( i \langle \xi_j, x \rangle - \frac{|x_\sigma(j+1) - x_\sigma(j)|^2}{2(s_\sigma(j+1) - s_\sigma(j))} \right) \, dx,
\]

with the convention \( x_{n+1} = x \) and \( s_{n+1} = t \). Making the change of variables \( u_j = x_\sigma(j+1) - x_\sigma(j) \) if \( 1 \leq j \leq n-1 \), and \( u_n = x - x_\sigma(n) \), we obtain

\[
\mathcal{F}g_s(\xi_1, \ldots, \xi_n) = (2\pi)^{-dn} \prod_{j=0}^n (s_\sigma(j+1) - s_\sigma(j))^{-\frac{d}{2}}
\]

\[
\times \int_{\mathbb{R}^d} \prod_{j=1}^n \exp \left( i \langle \xi_j, x - u_n - \cdots - u_j \rangle - \frac{|u_j|^2}{2(s_\sigma(j+1) - s_\sigma(j))} \right) \, du
\]

\[
= E \left[ \prod_{j=1}^n \exp \left( i \langle \xi_\sigma(j), x - B_t - B_{s_\sigma(j)} \rangle \right) \right]
\]

\[
= E \left[ \prod_{j=1}^n \exp \left( i \langle \xi_j, x - B_t - B_{s_j} \rangle \right) \right].
\]
As a consequence,
\[
\langle g_s, g_t \rangle_{L^2(\mathbb{R}^d)} = (2\pi)^{-nd} \int_{\mathbb{R}^d} E \left( \prod_{j=1}^n \exp \left( i \left\langle \xi_j, B_{1,s}^j - B_{2,t}^j \right\rangle \right) \right) d\xi,
\]
which implies the desired result. ■

In the case \( H > \frac{1}{2} \), and assuming that \( u_0 = 1 \), the next proposition shows that the norm \((n!)^2 \| f_n(\cdot, t, x) \|_{\mathcal{H}_d^{\otimes n}}^2 \) coincides with the \( n \)th moment of the intersection local time of two independent \( d \)-dimensional Brownian motions with weight \( \phi(t, s) \).

**Proposition 4.3** Suppose that \( H > \frac{1}{2} \) and \( d < 4H \). Then, for all \( n \geq 1 \)
\[
(n!)^2 \| f_n(\cdot, t, x) \|_{\mathcal{H}_d^{\otimes n}}^2 \leq \| u_0 \|_{\infty}^2 E \left( \left( \int_0^t \int_0^t \phi(s, r) \delta_0(B_s^1 - B_r^2) dsdr \right)^n \right) < \infty,
\]
with equality if \( u_0 \) is constant. Moreover, we have:

1. If \( d = 1 \), there exists a unique solution to Equation (4.1).
2. If \( d = 2 \), then there exists a unique solution in an interval \([0, T]\) provided \( T < T_0 \), where
\[
T_0 = \left( \beta_H \Gamma(1 - \frac{d}{4H})^{2H} \right)^{-1/(2H-1)}.
\]

**Proof** We have
\[
(n!)^2 \| f_n(\cdot, t, x) \|_{\mathcal{H}_d^{\otimes n}}^2 \leq \| u_0 \|_{\infty}^2 \int_{[0,1]^n} \prod_{j=1}^k \phi(s_j, t_j) \langle g_s, g_t \rangle_{L^2(\mathbb{R}^d)} dsdt,
\]
where \( g_s \) is defined in (4.6). Then the results follow easily from from Lemma 4.2 and Proposition 3.2. ■

In the two-dimensional case and assuming \( H > \frac{1}{2} \), the solution would exists in any interval \([0, T]\) as a distribution in the Watanabe space \( \mathcal{D}^{\alpha,2} \) for any \( \alpha > 0 \) (see [18]).

### 4.1 Case \( H < \frac{1}{2} \) and \( d = 1 \)

We know that in this case, the norm in the space \( \mathcal{H} \) is defined in terms of fractional derivatives. The aim of this section is to show that \( \| f_n(\cdot, t, x) \|_{\mathcal{H}_d^{\otimes n}}^2 \) is related to the \( n \)th moment of a fractional derivative of the self-intersection local time of two independent one-dimensional Brownian motions, and these moments are finite for all \( n \geq 1 \), provided \( \frac{3}{8} < H < \frac{1}{2} \).
Consider the operator \((K_H^*)^\otimes 2\) on functions of two variables defined as the action of the operator \(K_H^*\) on each coordinate. That is, using the notation \(\otimes\) we have
\[
(K_H^*)^\otimes 2 f(r_1, r_2) = K_H(T, r_1)K_H(T, r_2)f(r_1, r_2)
\]
\[+K_H(T, r_1)\int_{r_2}^t \frac{\partial K_H}{\partial s}(s, r_2) (f(r_1, s) - f(r_1, r_2)) \, ds
\]
\[+K_H(T, r_2)\int_{r_1}^t \frac{\partial K_H}{\partial s}(v, r_1) (f(v, r_2) - f(r_1, r_2)) \, dv
\]
\[+\int_{r_1}^t \int_{r_2}^r \frac{\partial K_H}{\partial s}(s, r_2) \frac{\partial K_H}{\partial v}(v, r_1) (f(v, s) - f(r_1, s) - f(v, r_2) - f(r_1, r_2)) \, ds \, dv.
\]
Suppose that \(f(s, t)\) is a continuous function on \([0, T]^2\). Define the Hölder norms
\[
\|f\|_{1,\gamma} = \sup \left\{ \frac{|f(s_1, t) - f(s_2, t)|}{|s_1 - s_2|^\gamma}, s_1, s_2, t \in T, s_1 \neq s_2 \right\},
\]
\[
\|f\|_{2,\gamma} = \sup \left\{ \frac{|f(s_1, t_1) - f(s_2, t_2)|}{|t_1 - t_2|^\gamma}, t_1, t_2, s \in T, t_1 \neq t_2 \right\}
\]
and
\[
\|f\|_{1,2,\gamma} = \sup \frac{|f(s_1, t_1) - f(s_1, t_2) - f(s_2, t_1) + f(s_2, t_2)|}{|s_1 - s_2|^\gamma|t_1 - t_2|^\gamma},
\]
where the supremum is taken in the set \(\{t_1, t_2, s_1, s_2 \in T, s_1 \neq s_2, t_1 \neq t_2\}\). Set
\[
\|f\|_{0,\gamma} = \|f\|_{1,\gamma} + \|f\|_{2,\gamma} + \|f\|_{1,2,\gamma}
\]
Then, \((K_H^*)^\otimes 2 f\) is well defined if \(\|f\|_{0,\gamma} < \infty\) for some \(\gamma > \frac{1}{2} - H\). As a consequence, if \(B^1\) and \(B^2\) are two independent one-dimensional Brownian motions, the following random variable is well defined for all \(\varepsilon > 0\)
\[
J_\varepsilon = \int_0^T (K_H^*)^\otimes 2 p_\varepsilon(B^1 - B^2)(r, r) \, dr. \quad (4.10)
\]
The next theorem asserts that \(J_\varepsilon\) converges in \(L^p\) for all \(p \geq 2\) to a fractional derivative of the intersection local time of \(B^1\) and \(B^2\).

**Proposition 4.4** Suppose that \(\frac{3}{8} < H < \frac{1}{2}\). Then, for any integer \(k \geq 1\) and, \(T > 0\) we have \(E(J_\varepsilon^k) \geq 0\) and
\[
\sup_{\varepsilon > 0} E(J_\varepsilon^k) < \infty.
\]
Moreover, for all \(p \geq 2\), \(J_\varepsilon\) converges in \(L^p\) as \(\varepsilon\) tends to zero to a random variable denoted by
\[
\int_0^T (K_H^*)^\otimes 2 \delta_0(B^1 - B^2)(r, r) \, dr.
\]
Proof  Fix $k \geq 1$. Let us compute the moment of order $k$ of $J_{\epsilon}$. We can write
\[
E(J_{\epsilon}^k) = \int_{[0,T]^{2k}} E \left( \prod_{i=1}^{k} (K_H^*)^{\otimes 2} p_{2\epsilon} (B^1 - B^2)(r_i, r_i) \right) dr. \tag{4.11}
\]
Using the expression (2.6) for the operator $K_H^*$, and the notation (3.4) yields
\[
E(J_{\epsilon}^k) = \int_{[0,T]^{2k}} \psi_\epsilon (s, t) \prod_{i=1}^{k} K_H^*(ds_i, r_i)K_H^*(dt_i, r_i) dr. \tag{4.12}
\]
As a consequence, using (3.5) we obtain
\[
E(J_{\epsilon}^k) = (2\pi)^{-k} \int_{[0,T]^k} \int_{\mathbb{R}^k} \int_{[0,T]^{2k}} e^{-\sum_{i=1}^{k} \xi_i \xi_i \text{Cov}(B_{i,j}^1 - B_{i,j}^2, B_{i,j}^1 - B_{i,j}^2)}
\times \prod_{i=1}^{k} K_H^*(ds_i, r_i)K_H^*(dt_i, r_i)e^{-\frac{\epsilon^2}{2} \sum_{i=1}^{k} \xi_i^2} d\xi dr
\leq (2\pi)^{-k} \int_{[0,T]^k} \int_{\mathbb{R}^k} \left( \int_{[0,T]^{2k}} e^{-\frac{\epsilon^2 \text{Var}(\sum_{i=1}^{k} \xi_i B_{i,j})}{} \prod_{i=1}^{k} K_H^*(dt_i, r_i)} \right)^2 d\xi dr.
\]
Then, it suffices to show that for each $k$ the following quantity is finite
\[
\int_{T_k} \int_{T_k} \prod_{j=1}^{k} [s_j - s_{j-1} + t_j - t_{j-1}]^{-\frac{1}{2}} k \prod_{i=1}^{k} K_H^*(ds_i, r_i) \prod_{i=1}^{k} K_H^*(dt_i, r_i) dr, \tag{4.13}
\]
where $T_k = \{0 < t_1 < \cdots < t_k < T\}$. Fix a constant $a > 0$. We are going to compute
\[
\int_{T_k} \prod_{j=1}^{k} [t_j - t_{j-1} + a]^{-2} k \prod_{i=1}^{k} K_H^*(dt_i, r_i).
\]
To do this we need some notation. Let $\Delta_j$ and $I_j$ be the operators defined on a function $f(t_1, \ldots, t_k)$ by
\[
\Delta_j f = f - f|_{t_j=r_j},
\]
and
\[
I_j f = f|_{t_j=r_j}.
\]
The operator $K_H^*(dt_i, r_i)$ is the sum of two components (see (2.5)), and it suffices to consider only the second one because the first one is easy to control. In this way we need to estimate the following term
\[
\int_{T_k} \left[ \int_{[0,T]^k} \Delta_1 \cdots \Delta_k \left( \prod_{j=1}^{k} t_j^{H-\frac{1}{2}} [t_j - t_{j-1} + a]^{-\frac{1}{2}} 1_{\{t_{j-1} < t_j\}} \right) \right]^2 dr.
\]

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Because \( t_j^{H - \frac{1}{2}} r_j^{\frac{1}{2} - H} \leq 1 \), we can disregard the factors \( r_j^{\frac{1}{2} - H} \) and \( t_j^{H - \frac{1}{2}} \). Using the rule

\[
\Delta_j(FG) = F(t_j)G(t_j) - F(r_j)G(r_j)
\]

\[
= [F(t_j) - F(r_j)]G(t_j) + F(t_j) [G(t_j) - G(r_j)]
\]

we obtain

\[
\Delta_1 \cdots \Delta_k \left( \prod_{i=1}^{k} [t_j - t_{j-1} + a]^{-\frac{1}{2}} 1_{\{t_{j-1} < t_j\}} \right)
\]

\[
= \sum_{S} \prod_{j=1}^{k} S_j \left( [t_j - t_{j-1} + a]^{-\frac{1}{2}} 1_{\{t_{j-1} < t_j\}} \right),
\]

where \( S_j \) is an operator of the form:

\( II_j, I \Delta_j, \Delta_{j-1} I_j, \Delta_{j-1} \Delta_j, \)

and for each \( j \), \( \Delta_j \) must appear only once in the product \( \prod_{j=1}^{k} S_j \). Let us estimate each one of the possible four terms. Fix \( \varepsilon > 0 \) such that \( H - \frac{3}{8} > 2\varepsilon \).

1. Term \( II_j \):

\[
II_j \left( [t_j - t_{j-1} + a]^{-\frac{1}{2}} 1_{\{t_{j-1} < t_j\}} \right) = [r_j - t_{j-1} + a]^{-\frac{1}{2}} 1_{\{t_{j-1} < r_j\}}.
\]

2. Term \( I \Delta_j \):

\[
|I \Delta_j \left( [t_j - t_{j-1} + a]^{-\frac{1}{2}} 1_{\{t_{j-1} < t_j\}} \right)|
\]

\[
= \left| [t_j - t_{j-1} + a]^{-\frac{1}{2}} 1_{\{t_{j-1} < t_j\}} - [r_j - t_{j-1} + a]^{-\frac{1}{2}} 1_{\{t_{j-1} < r_j\}} \right|
\]

\[
\leq C [t_j - r_j]^{H - \frac{3}{8} + \varepsilon} [r_j - t_{j-1} + a]^{H - 1 - \varepsilon} 1_{\{t_{j-1} < r_j\}}
\]

\[
+C [t_j - t_{j-1} + a]^{-\frac{1}{2}} 1_{\{r_j < t_{j-1}\}}.
\]

3. Term \( \Delta_{j-1} I \):

\[
|\Delta_{j-1} I \left( [t_j - t_{j-1} + a]^{-\frac{1}{2}} 1_{\{t_{j-1} < t_j\}} \right)|
\]

\[
= \left| [t_j - t_{j-1} + a]^{-\frac{1}{2}} 1_{\{t_{j-1} < t_j\}} - [t_j - r_{j-1} + a]^{-\frac{1}{2}} 1_{\{r_{j-1} < t_j\}} \right|
\]

\[
\leq C [t_{j-1} - r_{j-1}]^{\frac{1}{2} - H + \varepsilon} [t_j - t_{j-1} + a]^{H - 1 - \varepsilon} 1_{\{r_{j-1} < t_{j-1} < t_j\}}.
\]
4. Term $\Delta_{j-1}\Delta_j$: 

$$
\left| \Delta_{j-1}\Delta_j \left( [t_j - t_{j-1} + a]^{-\frac{k}{2}} 1_{(t_{j-1} < t_j)} \right) \right|
$$

$$
= \left| [t_j - t_{j-1} + a]^{-\frac{k}{2}} 1_{(t_{j-1} < t_j)} - [r_j - t_{j-1} + a]^{-\frac{k}{2}} 1_{(t_{j-1} < r_j)} 
- [t_j - r_{j-1} + a]^{-\frac{k}{2}} 1_{(r_{j-1} < t_j)} + [r_j - r_{j-1} + a]^{-\frac{k}{2}} 1_{(r_{j-1} < r_j)} \right|
$$

$$
\leq C [t_j - r_j]^{\frac{k}{2} - H + \varepsilon} [t_{j-1} - r_{j-1}]^{\frac{k}{2} - H + \varepsilon} [r_j - t_{j-1} + a]^{2H - \frac{k}{2} - 2\varepsilon} 1_{(t_{j-1} < r_j < t_j)} 
+ C [t_{j-1} - r_{j-1}]^{\frac{k}{2} - H + \varepsilon} [t_j - t_{j-1} + a]^{H - 1 - \varepsilon} 1_{(r_j < t_{j-1} < t_j)} 
+ C [r_j - r_{j-1} + a]^{-\frac{k}{2}} 1_{(r_{j-1} < r_j < t_{j-1} < t_j)}.
$$

If we replace the constant $a$ by $s_j - s_{j-1}$ and we treat the term $s_j - s_{j-1}$ in the same way, using the inequality

$$(a + b)^{-\alpha} \leq a^{-\frac{k}{2}b^{-\frac{k}{2}}} ,$$

we obtain the same estimates as if we had started with

$$
\int_{T_k} \left( \int_{T_k} \left[ t_j - t_{j-1} \right]^{-\frac{k}{4}} \prod_{j=1}^{k} K_H(\mathrm{d}t_j, r_j) \right)^2 \mathrm{d}r,
$$

instead of (4.13). As a consequence, it suffices to control the following integral

$$
\int_{T_k} \left( \int_{T_k} \prod_{j=1}^{k} A_{j}^{a,b}(t, r) \mathrm{d}t \right)^2 \mathrm{d}r, \quad (4.14)
$$

where $a, b \in \{0, 1\}$, and $A_{j}$ has one of the following forms

$$
A_{j}^{0,0} = [t_{j-1} - r_{j-1}]^{\frac{k}{4}} 1_{(t_{j-1} < r_j)},
A_{j}^{0,1} = [t_j - r_j]^{-1+\varepsilon} [t_{j-1} - r_{j-1}]^{H - \frac{k}{4} - \varepsilon} 1_{(t_{j-1} < r_j)},
A_{j}^{0,2} = [t_{j-1} - r_{j-1}]^{-\frac{k}{4}} [t_j - r_j]^{H - \frac{k}{2}} 1_{(r_{j-1} < t_{j-1})},
A_{j}^{1,0} = [t_{j-1} - r_{j-1}]^{-1+\varepsilon} [t_j - r_j]^{H - \frac{k}{4} - \varepsilon} 1_{(t_{j-1} < r_{j-1} < t_j)},
A_{j}^{1,1} = [t_{j-1} - r_{j-1}]^{-1+\varepsilon} [t_{j-1} - r_{j-1}]^{-1+\varepsilon} [t_j - t_{j-1}]^{2H - \frac{k}{4} - 2\varepsilon} 1_{(t_{j-1} < r_j < t_j)},
A_{j}^{1,2} = [t_{j-1} - r_{j-1}]^{-1+\varepsilon} [t_j - t_{j-1}]^{H - \frac{k}{4} - \varepsilon} [t_j - r_j]^{H - \frac{k}{4}} 1_{(r_j < t_{j-1} < t_j)},
A_{j}^{1,3} = [r_j - r_{j-1}]^{-\frac{k}{4}} [t_j - r_j]^{H - \frac{k}{4}} [t_{j-1} - r_{j-1}]^{H - \frac{k}{4}} 1_{(r_{j-1} < r_j < t_{j-1} < t_j)},
$$

and with the convention that any term of the form $A_{j}^{0,1}$ or $A_{j}^{1,1}$ must be followed by $A_{j}^{0,0}$ or $A_{j}^{0,1}$ and any term of the form $A_{j}^{0,0}$ or $A_{j}^{1,0}$ must be followed by

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$A_j^{1.0}$ or $A_j^{1.1}$. It is not difficult to check that the integral (4.14) is finite. For instance, for a product of the form $A_j^{0.0} A_j^{1.1}$ we get

$$
\int_{\{r_{j-1} < t_{j-1} < r_j < t_j\}} [r_{j-1} - t_{j-2}]^{-\frac{1}{2}} [t_{j-1} - r_{j-1}]^{-1+\varepsilon} [r_j - t_{j-1}]^{2H-\frac{1}{2}-2\varepsilon} \times [t_j - r_j]^{-1+\varepsilon} dt_{j-1}
$$

$$
= [r_{j-1} - t_{j-2}]^{-\frac{1}{2}} [r_j - r_{j-1}]^{2H-\frac{1}{2}-\varepsilon} [t_j - r_j]^{-1+\varepsilon},
$$

and the integral in the variable $r_j$ of the square of this expression will be finite because $4H - \frac{5}{2} - 2\varepsilon > -1$.

So, we have proved that $\sup_\varepsilon E(J_k^{\varepsilon}) < \infty$ for all $k$. Notice that all these moments are positive. It holds that $\lim_{\varepsilon, \delta \downarrow 0} E(J_\varepsilon J_\delta)$ exists, and this implies the convergence in $L^2$, and also in $L^p$, for all $p \geq 2$.

On the other hand, if the initial condition of Equation (1.1) is a constant $K$, then for all $n \geq 1$ we have

$$(n!)^2 \|f_n(\cdot, t, x)\|^2_{H_t^{1.0} \otimes \mathbb{F}_n} = K^2 E \left[ \left( \int_0^T (K_H^*)^{\otimes 2} \delta_0(B^1 - B^2)(r, r) dr \right)^n \right] < \infty,$$

provided $H \in (\frac{3}{8}, \frac{1}{2})$. In fact, by Lemma 4.2 we have

$$(n!)^2 \|f_n(\cdot, t, x)\|^2_{H_t^{1.0} \otimes \mathbb{F}_n} = K^2 \int_{[0, t]^{2n}} (\sigma_s, \sigma_t)_{L^2(\mathbb{R}^n)} \prod_{i=1}^n K_H^*(dt_i, r_i) \times \prod_{i=1}^n K_H^*(ds_i, r_i) ds dt$$

$$= K^2 \int_{[0, t]^{2n}} \psi(s, t) \prod_{i=1}^n K_H^*(dt_i, r_i) \prod_{i=1}^n K_H^*(ds_i, r_i) ds dt,$$

and it suffices to apply the above proposition.

However, we do not know the rate of convergence of the sequence $\|f_n(\cdot, t, x)\|^2_{H_t^{1.0} \otimes \mathbb{F}_n}$ as $n$ tends to infinity, and for this reason we are not able to show the existence of a solution to Equation (1.1) in this case.

### 5 Moments of the solution

In this section we introduce an approximation of the Gaussian noise $W_H^t$ by means of an approximation of the identity. In the space variable we choose the heat kernel to define this approximation and in the time variable we choose a rectangular kernel. In this way, for any $\varepsilon > 0$ and $\delta > 0$ we set

$$W_{t, x}^{\varepsilon, \delta} = \int_0^t \int_{\mathbb{R}^d} \varphi_\delta(t - s) p_\varepsilon(x - y) dW_{s, y}^H, \quad (5.1)$$
where
\[ \varphi_{\delta}(t) = \frac{1}{\delta}1_{[0,\delta]}(t). \]

Now we consider the approximation of Equation (1.1) defined by

\[ \frac{\partial u_{\varepsilon,\delta}}{\partial t} = \frac{1}{2} \Delta u_{\varepsilon,\delta} + u_{\varepsilon,\delta} \circ \dot{W}_{\varepsilon,\delta}. \tag{5.2} \]

We recall that the Wick product \( u_{\varepsilon,\delta} \circ \dot{W}_{\varepsilon,\delta} \) is well defined as a square integrable random variable provided the random variable \( u_{\varepsilon,\delta} \) belongs to the space \( D_{1,2} \) (see (2.9)), and in this case we have

\[ u_{\varepsilon,\delta} \circ \dot{W}_{\varepsilon,\delta} = \int_0^t \int_{\mathbb{R}^d} \varphi_{\delta}(s-r)p_{\varepsilon}(y-z)u_{\varepsilon,\delta} \delta W_{r,z}. \tag{5.3} \]

The mild or evolution version of Equation (5.2) will be

\[ u_{\varepsilon,\delta}(t,x) = p_t u_0(y) + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y)u_{\varepsilon,\delta} \circ \dot{W}_{\varepsilon,\delta} \, ds \, dy. \tag{5.4} \]

Substituting (5.3) into (5.4), and formally applying Fubini’s theorem yields

\[ u_{\varepsilon,\delta}(t,x) = p_t u_0(y) + \int_0^t \int_{\mathbb{R}^d} \left( \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y) \varphi_{\delta}(s-r)p_{\varepsilon}(y-z) \, ds \, dy \right) \delta W_{r,z}. \tag{5.5} \]

This leads to the following definition.

**Definition 5.1** An adapted random field \( u_{\varepsilon,\delta} = \{u_{\varepsilon,\delta}(t), t \geq 0, x \in \mathbb{R}^d\} \) is a mild solution to Equation (5.2) if for each \((r, z) \in \mathbb{R}_+ \times \mathbb{R}^d\) the integral

\[ Y_{r,z}^t = \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y) \varphi_{\delta}(s-r)p_{\varepsilon}(y-z)u_{\varepsilon,\delta} \delta W_{r,z} \]

exists and \( Y_{r,z}^t \) is a Skorohod integrable process such that (5.5) holds for each \((t, x)\).

The above definition is equivalent to saying that \( u_{\varepsilon,\delta} \in L^2(\Omega) \), and for any random variable \( F \in D_{1,2} \), we have

\[
E(Fu_{\varepsilon,\delta}) = E(F)p_t u_0(y) \\
+ \left< \left( \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y) \varphi_{\delta}(s-r)p_{\varepsilon}(y-z)u_{\varepsilon,\delta} \, ds \, dy \right), DF \right>_{\mathcal{H}_d}.
\]

Our aim is to construct a solution of Equation (5.2) using a suitable version of Feynman-Kac’s formula. Suppose that \( B = \{B_t, t \geq 0\} \) is a \( d \)-dimensional Brownian motion starting at 0, independent of \( W \). Set

\[
\int_0^t W_{t-s,x+B_s} \, ds = \int_0^t \int_{\mathbb{R}^d} \varphi_{\delta}(t-s-r)p_{\varepsilon}(B_s + x-y) \, dW_{r,z} \, ds \\
= \int_0^t \int_{\mathbb{R}^d} A_{r,z} \, dW_{r,z},
\]

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\[ A_{r,y}^{\varepsilon,\delta} = \int_0^t \varphi_\delta(t - s - r)p_\varepsilon(B_s + x - y)ds. \]  

(5.6)

Define

\[ u_{t,x}^{\varepsilon,\delta} = E \left( u_0(x + B_t) \exp \left( \int_0^t \int_{\mathbb{R}^d} A_{r,y}^{\varepsilon,\delta} dW_{r,y}^H - \frac{1}{2} \alpha^{\varepsilon,\delta} \right) \right), \]

(5.7)

where \( \alpha^{\varepsilon,\delta} = ||A^{\varepsilon,\delta}||^2_{H_d}. \)

**Proposition 5.2** The random field \( u_{t,x}^{\varepsilon,\delta} \) given by (5.7) is a solution to Equation (5.2).

**Proof** The proof is based on the notion of S transform from white noise analysis (see [5]). For any element \( \varphi \in H_1 \) we define

\[ S_{t,x}(\varphi) = E \left( u_{t,x}^{\varepsilon,\delta} F_\varphi \right), \]

where

\[ F_\varphi = \exp \left( W^H(\varphi) - \frac{1}{2} ||\varphi||^2_{H_d} \right). \]

From (5.7) we have

\[
\begin{align*}
S_{t,x}(\varphi) &= E \left( u_0(x + B_t) \exp \left( W^H(A^{\varepsilon,\delta} + \varphi) - \frac{1}{2} \alpha^{\varepsilon,\delta} - \frac{1}{2} ||\varphi||^2_{H_d} \right) \right) \\
&= E \left( u_0(x + B_t) \exp \left( \langle A^{\varepsilon,\delta}, \varphi \rangle_{H_d} \right) \right) \\
&= E \left( u_0(x + B_t) \exp \left( \int_0^t \langle \varphi_\delta(t - s - \cdot)p_\varepsilon(B_s + x - \cdot), \varphi \rangle_{H_d} ds \right) \right).
\end{align*}
\]

By the classical Feynman-Kac’s formula, \( S_{t,x}(\varphi) \) satisfies the heat equation with potential \( V(t, x) = \langle \varphi_\delta(t - \cdot)p_\varepsilon(x - \cdot), \varphi \rangle_{H_d} \), that is,

\[
\frac{\partial S_{t,x}(\varphi)}{\partial t} = \frac{1}{2} \Delta S_{t,x}(\varphi) + S_{t,x}(\varphi) \langle \varphi_\delta(t - \cdot)p_\varepsilon(x - \cdot), \varphi \rangle_{H_d}.
\]

As a consequence,

\[ S_{t,x}(\varphi) = p_t u_0(x) + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x - y)S_{s,y}(\varphi) \langle \varphi_\delta(s - \cdot)p_\varepsilon(y - \cdot), \varphi \rangle_{H_d} ds dy. \]

Notice that \( DF_\varphi = \varphi F_\varphi \). Hence, for any exponential random variable of this form we have

\[
E( u_{t,x}^{\varepsilon,\delta} F_\varphi ) = p_t u_0(x) + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x - y) E \left( u_{t,x}^{\varepsilon,\delta} \langle \varphi_\delta(s - \cdot)p_\varepsilon(y - \cdot), DF_\varphi \rangle_{H_d} \right),
\]

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and we conclude by the duality relationship between the Skorohod integral and the derivative operator. ■

The next theorem says that the random variables $u_{t,x}^{\epsilon,\delta}$ have moments of all orders, uniformly bounded in $\epsilon$ and $\delta$, and converge to the solution to Equation (1.1) as $\delta$ and $\epsilon$ tend to zero. Moreover, it provides an expression for the moments of the solution to Equation (1.1).

**Theorem 5.3** Suppose that $H \geq \frac{1}{2}$ and $d = 1$. Then, for any integer $k \geq 1$ we have

$$\sup_{\epsilon,\delta} \mathbb{E} \left[ \left| u_{t,x}^{\epsilon,\delta} \right|^k \right] < \infty,$$

(5.8)

and the limit $\lim_{\epsilon \downarrow 0} \lim_{\delta \downarrow 0} u_{t,x}^{\epsilon,\delta}$ exists in $L^p$, for all $p \geq 1$, and it coincides with the solution $u_{t,x}$ of Equation (1.1). Furthermore, if $U_0^B(t,x) = \prod_{j=1}^k u_0(x+B_t^{j})$, where $B^j$ are independent $d$-dimensional standard Brownian motions starting at 0, independent of $W^H$. Moreover, (5.10) holds for all $1 \leq k \leq M - 1$.

**Proof** Fix an integer $k \geq 2$. Suppose that $B^j = \{ B^j_t, t \geq 0 \}, i = 1, \ldots, k$ are independent $d$-dimensional standard Brownian motions starting at 0, independent of $W^H$. Then, using (5.7) we have

$$E \left[ \left( u_{t,x}^{\epsilon,\delta} \right)^k \right] = E \left( \prod_{j=1}^k E^B \left[ u_0(x+B_t^j) \exp \left( \int_0^t \int_{\mathbb{R}} A_{r,y}^{\epsilon,\delta,B^j} dW_{r,y}^H - \frac{1}{2} \alpha_{r,y}^{\epsilon,\delta,B^j} \right) \right] \right),$$

where $A_{r,y}^{\epsilon,\delta,B^j}$ and $\alpha_{r,y}^{\epsilon,\delta,B^j}$ are computed using the Brownian motion $B^j$. Therefore,

$$E \left[ \left( u_{t,x}^{\epsilon,\delta} \right)^k \right] = E^B \left[ \exp \left( \frac{1}{2} \left\| \sum_{j=1}^k A_{y}^{\epsilon,\delta,B} \right\|_{H^d}^2 - \frac{1}{2} \sum_{j=1}^k \alpha_{y}^{\epsilon,\delta,B} \right) \prod_{j=1}^k u_0(x+B_t^j) \right]$$

$$= E^B \left[ \exp \left( \sum_{i<j} \left\langle A_{y}^{\epsilon,\delta,B^i}, A_{y}^{\epsilon,\delta,B^j} \right\rangle_{H^d} \right) \prod_{j=1}^k u_0(x+B_t^j) \right].$$
That is, the correction term \( \frac{1}{2} \alpha^{\varepsilon, \delta} \) in (5.7) due to the Wick product produces a cancellation of the diagonal elements in the square norm of \( \sum_{j=1}^{k} A^{\varepsilon, B_j} \). The next step is to compute the scalar product \( \langle A^{\varepsilon, B_i}, A^{\varepsilon, B_j} \rangle_{\mathcal{H}_d} \) for \( i \neq j \). We consider two cases.

**Case 1.** Suppose first that \( H = \frac{1}{2} \) and \( d = 1 \). In this case we have

\[
\langle A^{\varepsilon, B_i}, A^{\varepsilon, B_j} \rangle_{\mathcal{H}_1} = \int_{\mathbb{R}} \int_{0}^{t} \int_{0}^{t} \int_{0}^{t} \varphi_\delta(t - s_1 - r) p_\varepsilon(B_{s_1}^i + x - y) \times \varphi_\delta(t - s_2 - r) p_\varepsilon(B_{s_2}^j + x - y) ds_1 ds_2 dr dy
\]

\[
= \int_{0}^{t} \int_{0}^{t} \varphi_\delta(t - s_1 - r) \varphi_\delta(t - s_2 - r) \times p_{2\varepsilon}(B_{s_1}^i - B_{s_2}^j) ds_1 ds_2 dr.
\]

We have

\[
\int_{0}^{t} \varphi_\delta(t - s_1 - r) \varphi_\delta(t - s_2 - r) dr
\]

\[
= \delta^{-2} [ (t - s_1) \wedge (t - s_2) - (t - s_1 - \delta)^+ \vee (t - s_2 - \delta)^+]^+
\]

\[
= \eta_\delta(s_1, s_2).
\]

It is easy to check that \( \eta_\delta \) is a symmetric function on \([0, t]^2\) such that for any continuous function \( g \) on \([0, t]^2\),

\[
\lim_{\delta \downarrow 0} \int_{0}^{t} \int_{0}^{t} \eta_\delta(s_1, s_2) g(s_1, s_2) ds_1 ds_2 = \int_{0}^{t} g(s, s) ds.
\]

As a consequence the following limit holds almost surely

\[
\lim_{\delta \downarrow 0} \int_{0}^{t} \varphi_\delta(t - s_1 - r) \varphi_\delta(t - s_2 - r) dr
\]

\[
= \eta_\delta(s_1, s_2).
\]

and by the properties of the local time of the one-dimensional Brownian motion we obtain that, almost surely.

\[
\lim \lim_{\varepsilon \downarrow 0, \delta \downarrow 0} \langle A^{\varepsilon, B_i}, A^{\varepsilon, B_j} \rangle_{\mathcal{H}_1} = \int_{0}^{t} \delta_0(B_{s}^i - B_{s}^j) ds.
\]

The function \( \eta_\delta \) satisfies

\[
\sup_{0 \leq r \leq t} \int_{0}^{t} \eta_\delta(s, r) ds \leq 1,
\]

and, as a consequence, the estimate (3.8) implies that for all \( \lambda > 0 \)

\[
\sup_{\varepsilon, \delta} E^{B} \left[ \lambda \exp \langle A^{\varepsilon, B_i}, A^{\varepsilon, B_j} \rangle_{\mathcal{H}_1} \right] < \infty.
\]
Hence \(5.8\) holds and \(\lim_{t \to 0} \lim_{x \to 0} u_{t,x}^{\varepsilon,\delta} := v_{t,x}\) exists in \(L^p\), for all \(p \geq 1\). Moreover, \(E(v_{t,x}^{k})\) equals to the right-hand side of Equation \(5.9\). Finally, Equation \(5.3\) and the duality relationship \(4.8\) imply that for any random variable \(F \in \mathbb{D}^{1,2}\) with zero mean we have

\[
E\left( F u_{t,x}^{\varepsilon,\delta} \right) = E\left( \left\langle DF, \int_0^t \int_{-\infty}^{\infty} \left( \int_0^t p_{t-s}(x-y) \varphi_\delta(s-\cdot) p_{\varepsilon}(y-\cdot) u_{s,y}^{\varepsilon,\delta} dsdy \right) \right\rangle_{\mathcal{H}_1} \right),
\]

and letting \(\delta\) and \(\varepsilon\) tend to zero we get

\[
E\left( F u_{t,x} \right) = E\left( \left\langle DF, \int_0^t \int_{-\infty}^{\infty} \left( \int_0^t p_{t-s}(x-y) \varphi_\delta(s-\cdot) p_\varepsilon(y-\cdot) v_{s,y} dsdy \right) \right\rangle_{\mathcal{H}_1} \right),
\]

which implies that the process \(v\) is the solution of Equation \(1.1\), and by the uniqueness \(v_{t,x} = u_{t,x}\).

**Case 2.** Consider now the case \(H > \frac{1}{2}\) and \(d = 2\). We have

\[
\left\langle A^{\varepsilon,\delta,B}, A^{\varepsilon,\delta,B} \right\rangle_{\mathcal{H}_d} = \int_{\mathbb{R}^2} \int_0^t \int_0^t \int_0^t \varphi_\delta(t - s_1 - r_1) p_\varepsilon(B_{s_1}^i + x - y) \times \varphi_\delta(t - s_2 - r_2) p_\varepsilon(B_{s_2}^j + x - y) ds_1 ds_2 \phi(r_1, r_2) dr_1 dr_2 dy
\]

\[= \int_0^t \int_0^t \int_0^t \varphi_\delta(t - s_1 - r_1) \varphi_\delta(t - s_2 - r_2) \times p_\varepsilon(B_{s_1}^i - B_{s_2}^j) ds_1 ds_2 \phi(r_1, r_2) dr_1 dr_2.
\]

This scalar product can be written in the following form

\[
\left\langle A^{\varepsilon,\delta,B}, A^{\varepsilon,\delta,B} \right\rangle_{\mathcal{H}_d} = \int_0^t \int_0^t \eta_s(s_1 - s_2) p_\varepsilon(B_{s_1}^i - B_{s_2}^j) ds_1 ds_2,
\]

where

\[
\eta_s(s_1, s_2) = \int_0^t \int_0^t \varphi_\delta(t - s_1 - r_1) \varphi_\delta(t - s_2 - r_2) \phi(r_1, r_2) dr_1 dr_2. \quad (5.11)
\]

We claim that there exists a constant \(\gamma\) such that

\[
\eta_s(s_1, s_2) \leq \gamma |s_1 - s_2|^{2H-2}. \quad (5.12)
\]

In fact, if \(|s_2 - s_1| = s\) we have

\[
\eta_s(s_1, s_2) \leq H(2H - 1)\delta^{-2} \int_{s-\delta}^{s+\delta} \int_s^{s+\delta} |u - v|^{2H-2} dudv
\]

\[= \frac{1}{2\delta^2} \left[ (s + \delta)^{2H} - (s - \delta)^{2H} - 2s^{2H} \right].
\]

Then

\[
H\delta^{-2} \int_s^{s+\delta} \left( y^{2H-1} - (y - \delta)^{2H-1} \right) dy \leq H\delta^{2H-2} \leq H2^{2-2H}s^{2H-2},
\]

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if $s \leq 2\delta$. On the other hand, if $s \geq 2\delta$, we have
\[
\frac{1}{2\delta^2} [(s + \delta)^{2H} - (s - \delta)^{2H} - 2s^{2H}] \leq \frac{H}{\delta} [s^{2H-1} - (s - \delta)^{2H-1}]
\]
\[
\leq H(2H - 1)(s - \delta)^{2H-2}
\]
\[
\leq H(2H - 1)2^{2-2H}s^{2H-2}.
\]

It is easy to check that for any continuous function $g$ on $[0, t]^2$,
\[
\lim_{\delta \downarrow 0} \int_0^t \int_0^t \eta_\delta(s_1, s_2)g(s_1, s_2)ds_1ds_2 = \int_0^t \int_0^t \phi(s_1, s_2)g(s_1, s_2)ds_1ds_2.
\]

As a consequence, the following limit holds almost surely
\[
\lim_{\varepsilon \downarrow 0} \lim_{\delta \downarrow 0} \mathbb{E}_{\varepsilon, \delta} \left[ \mathbb{E} \left[ \exp \left( \lambda \langle A^{\varepsilon, \delta, B^1}, A^{\varepsilon, \delta, B^2} \rangle_{\mathcal{H}_d} \right) \right] \right] < \infty,
\]
(5.13)
if $\lambda < \lambda_0(t)$, where $\lambda_0(t)$ is defined in (5.12) with $\gamma_T$ replaced by $\gamma$.

Hence, for any integer $k \geq 2$, if $t < t_0(k)$, where $\frac{k(k-1)}{2} = \lambda_0(t_0(k))$, then (5.8) holds because
\[
E \left[ \left( u^{\varepsilon, \delta}_{t,x} \right)^k \right] \leq \|u_0\|^k \left( E \left[ \exp \left( \frac{k(k-1)}{2} \langle A^{\varepsilon, \delta, B^1}, A^{\varepsilon, \delta, B^2} \rangle_{\mathcal{H}_d} \right) \right] \right)^{2\epsilon(3-\gamma)}.
\]
Finally, if $t < t_0(M)$ and $M \geq 3$, the limit $\lim_{\varepsilon \downarrow 0} \lim_{\delta \downarrow 0} u^{\varepsilon, \delta}_{t,x} := v_{t,x}$ exists in $L^p$, for all $2 \leq p < M$ and it is equal to the right-hand side of Equation (5.10). As in the case $H = \frac{1}{2}$ we show that $v_{t,x} = u_{t,x}$. □

6 Pathwise heat equation

In this section, we consider the one-dimensional stochastic partial differential equation
\[
\frac{\partial u}{\partial \tau} = \frac{1}{2}\Delta u + u \dot{W}^H_{t,x},
\]
(6.1)
where the product between the solution $u$ and the noise $\dot{W}^H_{t,x}$ is now an ordinary product. We first introduce a notion of solution using the Stratonovich integral and a weak formulation of the mild solution. Given a random field $v = \{v_{t,x}, t \geq 0, x \in \mathbb{R}\}$ such that $\int_0^T \int_{\mathbb{R}} |v_{t,x}| \, dx \, dt < \infty$ a.s. for all $T > 0$, the Stratonovich integral
\[
\int_0^T \int_{\mathbb{R}} v_{t,x} \, dW^H_{t,x}
\]
is defined as the following limit in probability if it exists

$$
\lim_{\epsilon \downarrow 0} \lim_{\delta \downarrow 0} \int_{0}^{T} \int_{\mathbb{R}} u_{t,x} W_{t,x}^{\epsilon,\delta} dx dt,
$$

where $W_{t,x}^{\epsilon,\delta}$ is the approximation of the noise $W^{H}$ introduced in (5.1).

**Definition 6.1** A random field $u = \{u_{t,x}, t \geq 0, x \in \mathbb{R}\}$ is a weak solution to Equation (6.1) if for any $C^\infty$ function $\varphi$ with compact support on $\mathbb{R}$, we have

$$
\int_{\mathbb{R}} u_{t,x} \varphi(x) dx = \int_{\mathbb{R}} u_{0}(x) \varphi(x) dx + \int_{0}^{t} \int_{\mathbb{R}} u_{s,x} \varphi''(x) dx ds + \int_{0}^{t} \int_{\mathbb{R}} u_{s,x} \varphi(x) dW_{s,x}^{H}.
$$

Consider the approximating stochastic heat equation

$$
\frac{\partial u^{\epsilon,\delta}}{\partial t} = \frac{1}{2} \Delta u^{\epsilon,\delta} + u^{\epsilon,\delta} W_{t,x}^{\epsilon,\delta}.
$$

**Theorem 6.2** Suppose that $H > \frac{3}{4}$. For any $p \geq 2$, the limit

$$
\lim_{\epsilon \downarrow 0} \lim_{\delta \downarrow 0} u_{t,x}^{\epsilon,\delta} = u_{t,x}
$$

exists in $L^{p}$, and defines a weak solution to Equation (6.2) in the sense of Definition 6.1. Furthermore, for any positive integer $k$

$$
E\left(u_{t,x}^{k}\right) = E\left[\exp\left(\sum_{i,j=1}^{k} \int_{0}^{t} \int_{0}^{t} \phi(s_{1},s_{2}) d(B_{s_{1}}^{i} - B_{s_{2}}^{j}) ds_{1} ds_{2}\right)\right],
$$

where $U_{0}^{\delta}(t,x)$ has been defined in Theorem (5.3).

**Proof** By Feynman-Kac’s formula we can write

$$
u_{t,x}^{\epsilon,\delta} = E\left[u_{0}(x + B_{t}) \exp\left(\int_{0}^{t} A_{s,y}^{\epsilon,\delta} dW_{s,y}^{H}\right)\right],
$$

where $A_{r,y}^{\epsilon,\delta}$ has been defined in (5.6). We will first show that for all $k \geq 1$

$$
\sup_{\delta,\epsilon} E\left[u_{t,x}^{\epsilon,\delta}\right] < \infty.
$$

Suppose that $B^{i} = \{B_{t}^{i}, t \geq 0\}, i = 1, \ldots, k$ are independent standard Brownian motions starting at 0, independent of $W^{H}$. Then, we have, as in the proof of Theorem 5.3

$$
E\left(\left(u_{t,x}^{\epsilon,\delta}\right)^{k}\right) = E\left[\exp\left(\frac{1}{2} \sum_{i,j=1}^{k} \left(A_{r,y}^{\epsilon,\delta,B^{i}} A_{r,y}^{\epsilon,\delta,B^{j}}\right)\right) U_{0}^{\delta}(t,x)\right].
$$
Notice that

$$\left\langle A^{\varepsilon, \delta} B^i, A^{\varepsilon, \delta} B^j \right\rangle_{H_1} = \int_0^t \int_0^t \eta_0(s_1, s_2) p_{2\varepsilon}(B^i_{s_1} - B^j_{s_2}) ds_1 ds_2,$$

where $\eta_0(s_1, s_2)$ satisfies (5.12). As a consequence, the inequalities (3.8) and (3.15) imply that for all $\lambda > 0$, and all $i, j$ we have

$$\sup_{\varepsilon, \delta} E \left( \exp \lambda \left\langle A^{\varepsilon, \delta} B^i, A^{\varepsilon, \delta} B^j \right\rangle_{H_1} \right) < \infty.$$ 

Thus, (6.4) holds, and

$$\lim_{\varepsilon \downarrow 0} \lim_{\delta \downarrow 0} E \left( \left\{ u_{t,x}^{\varepsilon, \delta} \right\}_{H_1}^k \right) = E^B \exp \left( \frac{1}{2} \sum_{i,j=1}^k \int_0^t \int_0^t \phi(s_1, s_2) \delta_0(B^i_{s_1} - B^j_{s_2}) ds_1 ds_2 \right).$$

In a similar way we can show that the limit $\lim_{\varepsilon, \varepsilon' \downarrow 0} \lim_{\delta, \delta' \downarrow 0} E \left( u_{t,x}^{\varepsilon, \delta} u_{t,x}^{\varepsilon', \delta'} \right)$ exists. Therefore, the iterated limit $\lim_{\varepsilon \downarrow 0} \lim_{\delta \downarrow 0} E \left( u_{t,x}^{\varepsilon, \delta} \right)$ exists in $L^2$.

Finally we need to show that

$$\lim_{\varepsilon \downarrow 0} \lim_{\delta \downarrow 0} \left( \int_0^t \int_{\mathbb{R}} u_{s,x} \varphi(x) dW^H_{s,x} - \int_0^t \int_{\mathbb{R}} u_{s,x}^{\varepsilon, \delta} \varphi(x) d\tilde{W}_{s,x}^H dsdx \right) = 0,$$

in probability. We know that $\int_0^t \int_{\mathbb{R}} u_{s,x} \varphi(x) d\tilde{W}_{s,x}^H dsdx$ converges in $L^2$ to some random variable $G$. Hence, if

$$B_{s,\delta} = \int_0^t \int_{\mathbb{R}} (u_{s,x}^{\varepsilon, \delta} - u_{s,x}) \varphi(x) d\tilde{W}_{s,x}^H dsdx$$

(6.5)

converges in $L^2$ to zero, $u_{s,x} \varphi(x)$ will be Stratonovich integrable and

$$\int_0^t \int_{\mathbb{R}} u_{s,x} \varphi(x) dW_{s,x}^H dsdx = G.$$

The convergence to zero of $B_{s,\delta}$ is done as follows. First we remark that $B_{s,\delta} = \delta(\phi^{\varepsilon, \delta})$, where

$$\phi^{\varepsilon, \delta} = \int_0^t \int_{\mathbb{R}} (u_{s,x}^{\varepsilon, \delta} - u_{s,x}) \varphi(x) \varphi_s(s-r) p_x(s-z) dsdx.$$

Then, from the properties of the divergence operator, it suffices to show that

$$\lim_{\varepsilon \downarrow 0} \lim_{\delta \downarrow 0} E \left( \|D\phi^{\varepsilon, \delta}\|_{H_1 \otimes H_1}^2 \right) = 0.$$ 

(6.6)
It is clear that \( \lim_{\epsilon \downarrow 0} \lim_{\delta \downarrow 0} E \left( \| \phi^{\epsilon, \delta} \|_{\mathcal{H}_1}^2 \right) = 0 \). On the other hand,
\[
D \left( \phi^{\epsilon, \delta}_{r, z} \right) = \int_0^t \int_{\mathbb{R}} \left( D \left( u^{\epsilon, \delta}_{s, x} \right) - D \left( u_{s, x} \right) \right) \varphi(x) \varphi_{\delta}(s-r)p_{\epsilon}(x-z) ds dx,
\]
and
\[
D \left( u^{\epsilon, \delta}_{s, x} \right) = E^B \left\{ u_0(x + B_t) \exp \left( \int_0^t \int_{\mathbb{R}} A^{\epsilon, \delta}_{s, y} dW^H_{r, y} \right) A^{\epsilon, \delta} \right\}.
\]
Then, as before we can show that
\[
\lim_{\epsilon, \epsilon' \downarrow 0} \lim_{\delta, \delta' \downarrow 0} E \left( \langle D \left( u^{\epsilon, \delta}_{s, x} \right), D \left( u^{\epsilon', \delta'}_{s, x} \right) \rangle_{\mathcal{H}_1}^2 \right)
\]
\[
= E^B \left[ u_0(x + B^1_t)u_0(x + B^2_t) \exp \left( \sum_{i, j = 1}^2 \int_0^t \int_0^t \phi(s_1, s_2) \delta_0(B^i_{s_1} - B^j_{s_2}) ds_1 ds_2 \right) \right.
\]
\[
\times \left. \delta_0(B^1_{t_1} + x - x_1) \cdots \delta_0(B^1_{t_n} + x - x_n) \right].
\]
This implies that \( u^{\epsilon, \delta}_{s, x} \) converges in the space \( \mathbb{D}^{1,2} \) to \( u_{s, x} \) as \( \delta \downarrow 0 \) and \( \epsilon \downarrow 0 \). Actually, the limit is in the norm of the space \( \mathbb{D}^{1,2}(\mathcal{H}_1) \). Then, (6.6) follows easily.

Since the solution is square integrable it admits a Wiener-Itô chaos expansion. The explicit form of the Wiener chaos coefficients are given below.

**Theorem 6.3** The solution to (6.1) is given by
\[
\sum_{n=0}^{\infty} I_n(f_n(\cdot, t, x)) \quad (6.7)
\]
where
\[
f_n(t_1, x_1, \ldots, t_n, x_n, t, x)
\]
\[
= E^B \left[ u_0(x + B_t) \exp \left( \frac{1}{2} \int_0^t \int_0^t \phi(s_1, s_2) \delta_0(B_{s_1} - B_{s_2}) ds_1 ds_2 \right) \right.
\]
\[
\times \left. \delta_0(B_{t_1} + x - x_1) \cdots \delta_0(B_{t_n} + x - x_n) \right].
\]

**Proof** From the Feynman-Kac formula it follows that
\[
u^{\epsilon, \delta}_{t, x} = E^B \left( u_0(x + B_t) \exp \left( \int_{\mathbb{R}^2} A^{\epsilon, \delta}_{r, y} dW^H_{r, y} \right) \right)
\]
\[
= E^B \left\{ u_0(x + B_t) \exp \left( \frac{1}{2} \| A^{\epsilon, \delta} \|_{\mathcal{H}_1}^2 \right) \exp \left( \int_0^t \int_{\mathbb{R}} A^{\epsilon, \delta}_{r, y} dW^H_{r, y} \right) \right\}
\]
\[
= \sum_{n=0}^{\infty} I_n(f^{\epsilon, \delta}_n(t, x)),
\]
where
\[ f^{\varepsilon,\delta}_n(t_1, x_1, \ldots, t_n, x_n, t, x) = E^B \left[ u_0(x + B_t) \exp \left( \frac{1}{2} \| A^{\varepsilon,\delta} \|_{H^1}^2 \right) A^{\varepsilon,\delta}_{t_1, x_1} \cdots A^{\varepsilon,\delta}_{t_n, x_n} \right]. \]

Letting \( \delta \) and \( \varepsilon \) go to 0, we obtain the chaos expansion of \( u_{t,x} \).

Consider the stochastic partial differential equation (6.1) and its approximation (6.2). The initial condition is \( u_0(x) \). We shall study the strict positivity of the solution. In particular we shall show that \( E \left[ |u_t(x)|^{-p} \right] < \infty \).

**Theorem 6.4** Let \( H > 3/4 \). If \( E \left[ |u_0(B_t)| \right] > 0 \), then for any \( 0 < p < \infty \), we have that
\[ E \left[ |u_t,x|^{-p} \right] < \infty \]
and moreover,
\[ E \left[ |u_t(x)|^{-p} \right] \leq (E|u_0(x + B_t)|)^{-p-1} E^B \left[ |u_0(x + B_t)| \right] \times \exp \left( \frac{p^2}{2} \int_0^t \int_0^t \delta(B_{s_1} - B_{s_2}) \phi(s_1, s_2) ds_1 ds_2 \right). \] (6.10)

**Proof** Denote \( \kappa_p = (E^B (|u_0(x + B_t)|))^{-p-1} \). Then, Jensen’s inequality applied to the equality \( u^{\varepsilon,\delta}_{t,x} = E^B \left\{ u_0(x + B_t) \exp \left( \int_0^t \int_R A^{\varepsilon,\delta}_{r,y} dW_{r,y} \right) \right\} \) implies that
\[ |u^{\varepsilon,\delta}_{t,x}|^{-p} \leq \kappa_p E^B \left\{ |u_0(x + B_t)| \exp \left( -p \int_0^t \int_R A^{\varepsilon,\delta}_{r,y} dW_{r,y} \right) \right\}. \]

Therefore
\[ E \left[ |u^{\varepsilon,\delta}_{t,x}|^{-p} \right] \leq \kappa_p E^B \left\{ |u_0(x + B_t)| E \left[ \exp \left( -p \int_0^t \int_R A^{\varepsilon,\delta}_{r,y} dW_{r,y} \right) \right] \right\} \]
\[ = \kappa_p E^B \left\{ |u_0(x + B_t)| E \left[ \exp \left( \frac{p^2}{2} \| A^{\varepsilon,\delta} \|_{H^1}^2 \right) \right] \right\}, \]
and we can conclude as in the proof of Theorem 6.2.

Using the theory of rough path analysis (see [5]) and \( p \)-variation estimates, Gubinelli, Lejay and Tindel [4] have proved that for \( H > \frac{3}{4} \), the equation
\[ \frac{\partial u}{\partial t} = \frac{1}{2} \Delta u + \sigma(u) \dot{W}_t^{H} \]
had a unique mild solution up to a random explosion time \( T > 0 \), provided \( \sigma \in C^2_b(\mathbb{R}) \). In this sense, the restriction \( H > \frac{3}{4} \), that we found in the case \( \sigma(x) = x \) is natural, and in this particular case, using chaos expansion and Feynman-Kac’s formula we have been able to show the existence of a solution for all times.

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