Mathematical-physical modeling for analytical calculation of multilevel pulse-width modulations

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Abstract. Currently there is a great interest from the electronic physical point of view in reducing losses and optimizing the harmonic content of power converters. Especially of the multilevel ones that have great advantages since they can work with low commutations. That is why, this paper proposes and verifies a methodology for the development of equations for the analytical calculation of the effective value and the total harmonic distortion of step and Pulse width modulations type multilevel. The equations are obtained by analysis in the frequency domain using Fourier theory. A generalized mathematical expression is obtained as a function of the switching angles, which is theoretically verified by comparing the calculated with the expression and the MATLAB fast Fourier transform algorithm. These equations allow the modification of the upper limit of calculation of the series, a characteristic that can be used to determine a vector of commutation angles with the minimum value of harmonic distortion for a range of harmonics.

1. Introduction
Nowadays the multilevel power converters (MLC) have had an unprecedented boom. They have become a subject of wide development from the scientific and industrial point of view. Within this development, a field of great interest has been to obtain converters with high efficiency and with a low harmonic content. This has implied manipulating the techniques with which the switching angles of the converter's physical devices are controlled [1]. The common technique that has been used to determine the switching angles of power inverters is mainly based on the comparison of a modulating signal with a triangular carrier signal [1,2]. There are different alternatives to modify the total harmonic distortion (THD) value, mainly to reduce it. These techniques such as the amplitude and the frequency variations of the carrier signal [3,4], even the use of multiple carrier signals for multilevel inverters. This brings a high degree of inaccuracy as the switching options are limited to the interaction of the carrier and modulating signals. Nevertheless, those techniques have been tried unsuccessfully [5] to establish an analytical solution of the modulation ratio and the carrier signals using advanced techniques that provide the lowest THD value [6,7]. But with the doubt of which is the harmonic content that must be eliminated, the problem approached based on the effects of the specific harmonic content in electric motors [8-10]. The need to calculate the switching angles in real time is raised, due to the practical variations of the system [11]. Similarly, the problem of calculating a limited solution when using Fourier arises and a study of the phenomenon in time domain was proposed [12,13]. The optimization has been carried out with the use of evolutionary techniques such as genetic algorithms, with variations to the phenomenon such as: unequal voltage levels [14,15], other approaches are the use of vector spaces [16], predictive methods.
[17], solutions with other theorems such as Parseval's [18,19], Fourier 1-D spectral analysis and power series [20]. Based on the search for the generalization of the THD and root mean square (RMS) voltage calculations, in terms of the switching angles, in such a way that it facilitates obtaining optimal values [21-24]. Other previous works focus on the specific calculation of the THD value in terms of the switching angles for the optimization of cases [25,26]. Even directly in the calculation of the THD value in terms of the line voltages that feed the three-phase motors [27,28]. Despite all these developments and work in the area, there has not been a complete solution, so the generation of these modulations directly and in a digital form comes as an excellent option, through an analytical formulation that includes optimization techniques for minimizing the THD using selective harmonics elimination (SHE) techniques. Several techniques try to reduce the THD value in limited specific harmonics by determining the mathematical forms that contribute to very low harmonics content [7]. Which are those that have a severe impact due to their greater amplitude. Therefore, this paper presents the mathematical physical modeling of these modulations since it is very important to achieve the analytical solution that describes the relationship of the switching angles with the THD applicable to any multilevel modulation of the step and pulse-width modulation (PWM) types. In order to optimize to obtain the reduction of the THD and increase the efficiency.

2. Multilevel step modulation

The step modulation is made up by a single switching angle in each step with a ladder shape without sub-pulses per voltage level. Figure 1 shows an example of 5-level step modulation. The definition of its steps in terms of switching angles. These switching angles are presented according to the symmetry with the first quarter wave. Therefore, the switching angles are in terms of the angles of the first quarter wave. The Fourier series for periodic waveforms is presented in Equation (1) [29], where \( n \) is the harmonic number, \( w_0 \) is the fundamental frequency of the waveform, \( t \) is time and \( a_0/2 \) is the DC value or the constant component, as shown by Equation (1).

\[
v(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n w_0 t + b_n \sin n w_0 t).
\]  

For symmetric waveforms like the one in Figure 1, where the positive part (first half of the waveform) is equal to the negative (second half), the direct current (DC) component to be zero \( a_0 = 0 \) and as the symmetry of the waveform is odd, making the coefficient \( a_n = 0 \). Therefore, the Fourier series will only be in terms of the coefficient \( b_n \), Equation (2) [29] related to the sine function as presented in Equation (3) [29].

\[
b_n = \frac{1}{\pi} \int_{0}^{2\pi} v(t) \sin(nwt) \, dt,
\]

\[
v(t) = \sum_{n=1}^{\infty} b_n \sin(nwt).
\]

In the expanded form of the Fourier series in terms of the fundamental component and the harmonics, without the direct current component it is expressed by Equation (4) [29].

\[
v(t) = c_1 \sin(wt + \varphi_1) + c_2 \sin(2wt + \varphi_2) + \cdots + c_n \sin(nwt + \varphi_n),
\]

where the peak magnitude of the harmonic \( n \) is defined by the value of the coefficient \( b_n \) for this case. Where THD can be calculated by Equation (5).

\[
\text{THD} = \frac{\sqrt{\sum_{n=1}^{\infty} c_n^2}}{c_1} \times 100%.
\]
2.1. level step modulation
The step modulation which consists of five voltage levels, it is shown in Figure 1. It is found in terms of the switching angles \(\alpha_1\) and \(\alpha_2\). The calculation of the coefficient \(b_n\) is carried out by means of Equation (6). Where Vdc voltage is value of the step, \(n\) is the harmonic number.

\[
b_n = \frac{2V_{dc}}{\pi n} \left[ 1 - (-1)^n \right] \left[ \cos(n\alpha_1) + \cos(n\alpha_2) \right], \tag{6}
\]

but \(b_n = 0\) for the \(n\) even, so the expression in Equation (7) can be simplified. Additionally, it represents the peak magnitude of harmonic \(n\).

\[
c_n = b_n = \frac{4V_{dc}}{\pi n} \cos(n\alpha_1) + \cos(n\alpha_2). \tag{7}
\]

2.2. 7-step modulation
A step modulation of seven levels, in terms of the angles \(\alpha_1\), \(\alpha_2\) and \(\alpha_3\), with the proposed symmetry characteristics, the calculation of the coefficient \(b_n\) for the odd \(n\) is determined by Equation (8).

\[
b_n = \frac{2V_{dc}}{\pi n} \left[ 1 - (-1)^n \right] \left[ \cos(n\alpha_1) + \cos(n\alpha_2) + \cos(n\alpha_3) \right], \tag{8}
\]

but \(b_n = 0\) for the \(n\) even and the expression in Equation (9) can be simplified.

\[
c_n = b_n = \frac{4V_{dc}}{\pi n} \cos(n\alpha_1) + \cos(n\alpha_2) + \cos(n\alpha_3). \tag{9}
\]

2.3. Generalization of the step modulation
To generalize the equation, the variable \(k\) is defined as the number of steps, in this way the peak magnitude of each harmonic is given as shown by Equation (10).

\[
c_n = \frac{4V_{dc}}{\pi n} \left[ \cos(n\alpha_1) + \cos(n\alpha_2) + \cos(n\alpha_3) + \cdots + \cos n\alpha_{k-1} \right] = \frac{4V_{dc}}{\pi n} \sum_{i=1}^{k-1} \cos n\alpha_i. \tag{10}
\]

The voltage over time in terms of the \(k\) steps is represented by Equation (11).

\[
v(t) = \sum_{n=1}^{\text{hmax}} \left[ \frac{4V_{dc}}{\pi n} \cdot \sum_{i=1}^{k-2} \cos n\alpha_i \right] \text{sen}(nwt). \tag{11}
\]

The THD value is calculated by Equation (12) and the RMS voltage by Equation (13), for odd \(n\).

\[
\text{THD} = \sqrt{\frac{\sum_{n=2}^{\text{hmax}} \left( \frac{1}{n} \sum_{i=1}^{k-1} \cos n\alpha_i \right)^2}{\sum_{i=1}^{k-1} \cos n\alpha_i}} \times 100, \quad n = 3, 5, 7, \ldots, \tag{12}
\]

\[
VRMS = \sqrt{\sum_{n=1}^{\text{hmax}} \left( \frac{1}{2} \frac{4V_{dc}}{\pi n} \sum_{i=1}^{k-1} \cos n\alpha_i \right)^2}, \quad n = 3, 5, 7, \ldots, \tag{13}
\]

3. Multilevel step-pulse wide modulations
The step-PWM modulation is shaped as a ladder, however this has sub-pulses per voltage level, because it has more than one switching angle per step. One example of waveform is presented in Figure 2. Where the first half of the waveform is equal to the second half of the waveform but negative. Resulting that
the direct current (DC) component or the constant value is zero \( a_0 = 0 \) and the odd symmetry makes the coefficient \( a_0 = 0 \). In Figure 2, only the angles of the first quarter waveform are mentioned. Because, like the cases previously analyzed, the quarter wave symmetry is fulfilled.

\[ b_n = \frac{2v_{dc}}{\pi n} \left[ 1 - (-1)^n \right] \left\{ \cos(n\alpha_{11}) - \cos(n\alpha_{12}) + \cos(n\alpha_{13}) - \cos(n\alpha_{14}) + \cos(n\alpha_{15}) + \cos(n\alpha_{21}) - \cos(n\alpha_{22}) + \cos(n\alpha_{23}) - \cos(n\alpha_{24}) \right\}. \]  

(14)

But \( b_n = 0 \) for the \( n \) even and the expression in Equation (15) can be simplified. Additionally, it represents the peak magnitude of harmonic \( n \).

\[ b_n = \frac{4v_{dc}}{\pi n} \left[ \sum_{j=1}^{5} (-1)^{j-1} \cos(n\alpha_{1j}) + \sum_{j=1}^{4} (-1)^{j-1} \cos(n\alpha_{2j}) \right]. \]  

(15)

Defining \( i \) as the number of the step, \( j \) as the number of the switching angle in step \( i \), \( \alpha_{ij} \) as the on or off angle at each step \( i \) and angle \( j \). Finally, defining \( L \) as a vector that contains the information of the number of switching angles per step, which allows to simplify the expression in Equation (16). In this case \( L = [5, 4] \) where \( L_1 = 5 \) and \( L_2 = 4 \). The number of switching angles for step \( i \) in the first quarter waveform is considered.

\[ c_n = b_n = \frac{4v_{dc}}{\pi n} \left[ \sum_{j=1}^{5} (-1)^{j-1} \cos(n\alpha_{1j}) \right]. \]  

(16)

3.2. 7-level Step- pulse-width modulation

The step-PWM modulation of seven voltage levels is found in terms of the switching angles \( \alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{15} \) for the first step, \( \alpha_{21}, \alpha_{22}, \alpha_{23}, \alpha_{24} \) for the second step and \( \alpha_{31}, \alpha_{32}, \alpha_{33} \) for the third step. The calculation of the coefficient \( b_n \) is represented in Equation (17). Where the \( n \) even does not exist by quarter wave form symmetry, as explained above.

\[ b_n = \frac{4v_{dc}}{\pi n} \left[ \sum_{j=1}^{5} (-1)^{j-1} \cos(n\alpha_{1j}) + \sum_{j=1}^{4} (-1)^{j-1} \cos(n\alpha_{2j}) + \sum_{j=1}^{3} (-1)^{j-1} \cos(n\alpha_{3j}) \right]. \]  

(17)

In this case \( L = [5, 4, 3] \) where \( L_i \) is the number of switching angles for step \( i \) in the first quarter of the waveform (Equation (18)).

\[ b_n = c_n = \frac{4v_{dc}}{\pi n} \left[ \sum_{j=1}^{3} (-1)^{j-1} \cos(n\alpha_{1j}) \right]. \]  

(18)
3.3. Generalization of step-pulse-width modulation
To generalize the equation, the variable k is defined as the number of steps and the vector \( L_i \) that indicates the number of switching angles for step i. The voltage as a function of time, in terms of the k steps is represented by Equation (19); where n = 1, 3, 5, ...

\[
v(t) = \sum_{n=1}^{\infty} \frac{4v_{dc}}{\pi n} \left[ \frac{k-1}{2} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} (-1)^{j-1} \cos \alpha_{ij} \right] \sin(nwt). \tag{19}
\]

Therefore, The THD value by Equation (20), for n = 3, 5, 7, ...

\[
\text{THD} = \sqrt{\frac{\sum_{n=2}^{v_{\text{max}}} \left( \frac{k-1}{2} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} (-1)^{j-1} \cos \alpha_{ij} \right)^2}{\sum_{i=1}^{\infty} \cos \alpha_i}} \times 100. \tag{20}
\]

The RMS voltage by Equation (21), for n = 1, 3, 5, ...

\[
\text{VRMS} = \sqrt{\sum_{n=1}^{v_{\text{max}}} \left( \frac{k-1}{2} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} (-1)^{j-1} \cos \alpha_{ij} \right)^2}. \tag{21}
\]

4. Validation
The expressions are validated or theoretically verified by comparing the value of the THD calculated with the expression and the result given by the fast Fourier transform (FFT) algorithm of the MATLAB® computational tool. For this, two essential tests were performed. One for Step modulation and the other for Step PWM modulation.

4.1. Step modulations
To verify the equations, a test modulation of 11 step is defined based upon the switching angles according vector \( L = [5.744 \ 16.544 \ 28.832 \ 42.096 \ 59.607] \). Figure 3 details the waveform for the switching angles and the Figure 4 presents the harmonic spectrum for the waveform. This spectrum was calculated with the magnitude expression \( C_n \) of the generalized equation of step modulation. The THD value calculated with Equation (12) and defining an upper limit at harmonic 50 is 6.127%. and evaluated up to harmonic 80 is 6.537%.

Figure 3. Waveform of the 11-level step modulation.

Figure 4. Calculated harmonic spectrum (up to the 80th harmonic).
by the MATLAB algorithm. It is observed that the results given by the MATLAB® FFT algorithm and those calculated are practically the same. Thus, verifying the validity of the spectrum calculated by the proposed expression.

4.2. Step-pulse width modulation

The modulation to be used in this validation was defined by the vector \( L = [3.95, 5.86, 8.72, 19.18, 22.15, 24.77, 39.24, 41.69, 53.94, 60.21, 64.57, 66.55, 69.68] \). In this they are divided into 3 angles for the first step. Additionally, there are 3, 4 and 5 angles for steps 2, 3 and 4 of the first quarter of the waveform and Figure 7 shows the waveform for the switching angles.

Figure 5. Results of the THD value using the Simulink® block FFT.

Figure 6. Results of the harmonic spectrum using the Simulink® block FFT.

Figure 7. Multilevel PWM modulation waveform.

Figure 8 presents the harmonic spectrum calculated using the magnitude expression \( C_n \) of the generalized equation for step-PWM modulation. This shows the effect of an optimization process. The magnitudes of the first 40 harmonics are low as a result of the applied algorithm, which was obtained through the mathematical expression developed in this work. The THD value calculated with Equation (20) for modulation up to the 40th harmonic is 1.601%. While the THD value evaluated up to the 50th harmonic was 6.615%. These coincide with the results of the MATLAB® FFT algorithm as shown in Figure 9 and the Figure 10 that presents the harmonic spectrum calculated by the MATLAB® algorithm. Verifying the coincidence with the spectrum calculated by the expression obtained in this work.

Figure 8. Calculated harmonic spectrum (up to the 50th harmonic).

Figure 9. Multilevel PWM modulation waveform.

Figure 10. Calculated harmonic spectrum by MATLAB (up to the 50th harmonic).

5. Conclusions

The coincidence of the results of the FFT algorithm and the analytical expressions proposed complete verify the validity of obtained expressions. With the additional advantage that this expression requires only the switching angles and a series evaluation index by setting the upper limit of the summation. The values for the THD obtained by the FFT algorithm and the proposed analytical expression are the same. This is because the harmonic spectra are also the same, because the calculations of the magnitudes of the harmonics for the two methodologies give the same results. The harmonic spectrum for PWM
multilevel modulation optimized with the use of the obtained equation and a genetic algorithm, taking the forty harmonics as an upper bound. Thus, verifying the hypothesis that indicates that the analytical expressions are valid to optimize the modulation waveforms. Taking a complete set of harmonics defined by the criteria of the application and represented in the limits of the summation. For the THD minimization it is appropriate to use the analytical equation presented for PWM multilevel modulations as an optimization function. This is because the magnitudes of the harmonics within the selected calculation limits are relatively low. Unlike with step modulation which has limitations with the suppression of specific harmonics, since the reduction effects are not as drastic for the harmonics present in the same range.

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