Optimal Transmit Covariance for Ergodic MIMO Channels

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Abstract

In this paper we consider the computation of channel capacity for ergodic multiple-input multiple-output channels with additive white Gaussian noise. Two scenarios are considered. Firstly, a time-varying channel is considered in which both the transmitter and the receiver have knowledge of the channel realization. The optimal transmission strategy is water-filling over space and time. It is shown that this may be achieved in a causal, indeed instantaneous fashion. In the second scenario, only the receiver has perfect knowledge of the channel realization, while the transmitter has knowledge of the channel gain probability law. In this case we determine an optimality condition on the input covariance for ergodic Gaussian vector channels with arbitrary channel distribution under the condition that the channel gains are independent of the transmit signal. Using this optimality condition, we find an iterative algorithm for numerical computation of optimal input covariance matrices. Applications to correlated Rayleigh and Ricean channels are given.

I. INTRODUCTION

Shannon theoretic results for multiple-input multiple-output (MIMO) fading channels [4, 5] have stimulated a large amount of research activity, both in the design of practical coding strategies and in extension of the theory itself. From an information theoretic point of view, the main problem is to find the maximum possible rate of reliable transmission over t-input, r-output additive white Gaussian noise channels of the form

\[ y[k] = \sqrt{\gamma} H[k] x[k] + z[k] \]  \hspace{1cm} (1)

where \( y[k] \in \mathbb{C}^{r \times 1} \) is a complex column vector of matched filter outputs at symbol time \( k = 1, 2, \ldots, N \) and \( H[k] \in \mathbb{C}^{r \times t} \) is the corresponding matrix of complex channel coefficients. The element at row \( i \) and column \( j \) of \( H[k] \) is the complex channel coefficient from transmit element \( j \) to receive element \( i \). The vector \( x[k] \in \mathbb{C}^{t \times 1} \) is the vector of complex baseband input signals, and \( z[k] \in \mathbb{C}^{r \times 1} \) is a complex, circularly symmetric Gaussian vector with \( \mathbb{E}\{n[k]n[k]^\dagger\} = I_r \). The superscript \((\cdot)^\dagger\) means Hermitian adjoint and \( I_r \) is the \( r \times r \) identity matrix. Let \( n = \max(t, r) \) and \( m = \min(t, r) \).

Transmission occurs in codeword blocks of length \( N \) symbols. Let \( x_N \in \mathbb{C}^t \) and \( y_N \in \mathbb{C}^r \) be the column vectors resulting from stacking \( x[1], x[2], \ldots, x[N] \) resp. \( y[1], y[2], \ldots, y[N] \). Further let \( H_N \) be the block-diagonal matrix with diagonal blocks \( H[k] \).

A transmitter power constraint

\[ \frac{1}{N} \| x_N \|^2_2 \leq 1 \]  \hspace{1cm} (2)

is enforced, where \( N \) is the codeword block length. This power constraint has been explicitly written out this way to remind the reader that power constraints such as this, commonly written \( \mathbb{E}[\| x[k] \|^2] \leq 1 \) are long-term average power constraints, not deterministic per-symbol, or per-input constraints, see [6, p. 329]. Accordingly, the signal-to-noise ratio is defined as \( \gamma \). The covariance matrix of input sequences of length \( N \) is defined as the \( Nt \times Nt \) matrix

\[ Q_N = \mathbb{E}\{ x_N x_N^\dagger \} \]  \hspace{1cm} (3)

and hence the power constraint can also be written as \( \text{tr}(Q_N) \leq N \). Also define the per-symbol input covariance matrices \( Q[k] = \mathbb{E}\{ x[k] x[k]^\dagger \} \), which appear as principal sub-matrices in \( Q_N \). In the case of memoryless transmission, \( Q_N \) is a block diagonal matrix with diagonal blocks \( Q[k] \).

The power constraint (2) assumes that the power received from the collection of transmit signals at any point in space (e.g., at some imaginary point close to the transmitter) is given by the summation of the individual signal powers, ie. zero mutual coupling.

There are several possibilities for the amount of side information that the receiver or transmitter may possess regarding the channel process \( H[k] \). Perfect side information shall mean knowledge of the realizations \( H[k] \), while statistical side information refers to knowledge of the distribution from which the \( H[k] \) are selected. Perfect receiver side information will be assumed throughout the paper.

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There are several categories of channels 1 that have been investigated in the literature:

1) Channels in which $H[k]$, is a given sequence of channel matrices, known to both the transmitter and receiver.
2) Ergodic channels in which the $H[k]$, $k = 1, 2, \ldots$ are random matrices, selected independently of each other and independently of the $x[k]$, according to some matrix probability density function $p_H$, which is known at the transmitter.

The specific channel realizations are unknown at the transmitter, but are known at the receiver.

Under the assumption of additive Gaussian noise and perfect receiver side information, the optimal input distribution is Gaussian, and the main problem is therefore the determination of the capacity achieving input covariance matrix $Q_N$.

For a given input covariance, the information rate for case 1 is found via solution of the Kuhn-Tucker conditions, and results in a water-filling interpretation,

$$\psi(Q_N, H_N) = \frac{1}{N} \log \det \left( I_{Nt} + H_N Q_N H_N^{-1} \right). \tag{4}$$

The capacity is found by maximizing the information rate.

**Problem 1 (Gallager [7]):**

$$\max_{Q_N} \psi(Q_N, H_N)$$

subject to

$$\frac{1}{N} \text{tr}(Q_N) \leq 1$$

$$Q_N \geq 0$$

Note that since $\psi$ is a function of $H_N$, the optimal covariance matrix will in general be a function of $H_N$.

Telatar [4] obtained the solution of Problem 1 when $H[k] = H$ for all $k = 1, 2, \ldots$. Following Gallager [7], the solution is obtained by solution of the Kuhn-Tucker conditions, and results in a water-filling interpretation,

$$C = \sum_{\mu \lambda_i^{-1} \leq \mu} \log \mu \lambda_i, \quad \text{where } \mu \text{ is such that}$$

$$\gamma = \sum_{\mu \lambda_i^{-1} \leq \mu} \mu - \lambda_i^{-1} \tag{6}$$

and $\lambda_i, i = 1, 2, \ldots, m$ are the non-zero eigenvalues of $HH^\dagger$. The optimal transmit covariance matrix is independent of $k$ and is given by $Q[k] = Q = V^\dagger \Gamma V$, where $V$ is the matrix of right singular vectors of $H$ and $\Gamma = \text{diag}\{\max(0, \mu - 1/\lambda_i)\}$.

The information rate in the ergodic case is $\Psi = \mathbb{E}\{\psi\}$ and subject to the assumptions in case 2 above, reduces to a symbol-wise expectation with respect to $p_H$,

$$\Psi(Q, p_H) = \mathbb{E}\{\log \det (I + HQH^\dagger)\} \tag{7}$$

where $Q = Q[k]$ is $t \times t$ covariance matrix for each symbol. In this case, capacity is found via solution of

**Problem 2 (Telatar [4]):**

$$\max_Q \Psi(Q, p_H)$$

subject to

$$\text{tr}(Q) \leq 1$$

$$Q > 0$$

Since $\Psi$ is an expectation with respect to $p_H$, the optimal $Q$ will depend on $p_H$, rather than the realizations $H[k]$.

One common choice for $p_H$ is a Gaussian density. We will use the notation $\mathcal{N}_{t,r}(M, \Sigma)$ to mean a Gaussian density with $r \times t$ mean matrix $M$ and $rt \times rt$ covariance matrix $\Sigma = \mathbb{E}\{hh^\dagger\}$ where $h$ is formed by stacking the columns of the matrix into a single vector. This allows for arbitrary correlation between elements. Common special cases include i.i.d. unit variance entries, $\mathcal{N}_{t,r}(0, I)$ (corresponding to independent Rayleigh fading) and the so-called Kronecker correlation model $\mathcal{N}_{t,r}(M, R \otimes T)$. The latter model corresponds to separable transmit $T$ and receive correlation $R$, and may be generated via $M + R^{1/2}GT^{1/2}$ where $G \sim \mathcal{N}_{t,r}(0, I)$. For $H[k] \sim \mathcal{N}_{t,r}(0, I)$ Telatar showed that the optimizing $Q = I_t/t$, meaning that it is optimal to transmit independently with equal power from each antenna. Thus in that case

$$C = \mathbb{E}\{\log \det \left( I_{t} + \frac{\gamma}{t} HH^\dagger \right)\}. \tag{8}$$

Telatar also gave an expression for computation of (8), and several other expressions have subsequently been found [8–10].

Finally, Telatar considered a variation on case 1 with time-invariant $H[k] = H$ and perfect receiver side information, but only statistical transmitter side information. This requires the notion of outage probability. It was conjectured that the optimal
transmission strategy, minimizing the outage probability, is equal power signals from a subset of antennas. We do not consider outage probability in this paper.

It is clear from these results that the degree of channel knowledge at the transmitter has a significant effect on the optimal transmission strategy.

Extensions to the theory have taken several directions, for example extending the ergodic capacity results to channel matrices whose elements are no longer independent of each other. “One-ring” scatter models, resulting in single-ended correlation structure $H \sim N_t,r (0, I \otimes T)$ were considered in [11]. Bounds on capacity were obtained in that work, assuming $Q = I/t$.

Subsequently, a series of papers appeared, adopting the same single-ended correlation model. In [12] it was shown that for $H \sim N_t,r (0, I \otimes T)$ it is optimal to transmit independently on the eigenvectors of $T$. Majorization results were obtained showing that stronger modes should be allocated stronger powers, and optimal $Q$ were found using numerical optimizations.

No conditions for optimality were given. In [13], a closed-form solution for the characteristic function of the mutual information assuming $Q = I/t$ was found for the same single-ended correlation model. In [14], the special case of $t = 2$ was considered, where optimization of $Q$ could be performed, once again assuming no receiver correlation, $R = I$.

Asymptotic large systems ($r, t \rightarrow \infty$ with $r/t \rightarrow$ a constant) capacity results have been obtained in [15], for the more general case $H \sim N_t,r (0, R \otimes T)$, but under the assumption $Q = I/t$. Asymptotic results for arbitrary $Q$ were considered in [16], where the asymptotic distribution of the mutual information was found to be normal. Large-systems results have been obtained in [17], concentrating on the case where the eigenvectors of the optimal $Q$ can be identified by inspection.

Closed form solutions have been obtained for the mutual information of single-ended correlated channels [10, 18] and for $H \sim N_t,r (0, R \otimes T)$, [19, 20].

Non-zero mean multiple-input, single-output channels were considered in [21, 22]. In those papers, results were obtained for non-zero mean, in the absence of transmitter correlation, and for non-trivial transmitter correlation, with zero mean. Further results for non-zero mean channels have been presented in [23], which reports some majorization results on mutual information, with respect to the eigenvalues of the mean matrix. Exact distributions of mutual information have been obtained in for $t = 2$ or $r = 2$. Asymptotic expressions for the mutual information have been presented in [24], for arbitrary $Q$, and non-central, uncorrelated fading.

Other researchers [25–28] have examined variations on the amount of information available at transmitter and receiver.

Previous work such as [4, 7, 12, 14, 17, 22] on Gaussian vector channels focused on cases when the eigenvectors of the optimal input covariance can be easily determined by inspection of the channel statistics, and the problem becomes one of optimizing the eigenvalues of the mean matrix. Exact distributions of mutual information have been obtained in [23, 29, 30].

This paper provides general solutions of Problems 1 and 2. The latter provides a solution to [31, open problem 1 and 2], albeit not in closed form.

In Section II we extend the water-filling result to ergodic channels where the transmitter has perfect knowledge of the channel realization $H[k]$ at each symbol. In Section III we relax the degree of transmitter channel knowledge and consider the ergodic channel with arbitrary channel distribution $p_H$, such that $p_H$, but not $H[k]$ is known to the transmitter.

The semidefinite constraint $Q \geq 0$ in Problem 2 would normally make the optimization difficult. However, in several cases, the eigenvectors of the optimal $Q$ may be identified a-priori, which reduces the problem to an optimization over the space of probability vectors. In independent work, [17] has found similar results to those presented in this paper for this “diagonalizable” case. We avoid the requirement of diagonalizing $Q$. Our main result is the determination of the capacity achieving covariance for arbitrary ergodic channels. This is achieved by finding necessary and sufficient conditions for optimality, which in turn yield an iterative procedure for numerical optimization of $Q$, which finds the optimal eigenvectors in addition to the optimal eigenvalues. In each section we provide numerical examples that illustrate the application of the main results. Conclusions are drawn in Section IV. All proofs are to be found in the Appendix.

II. Perfect Transmitter Side Information

As described above, Telatar [4] solved Problem 1 for time-invariant deterministic channels. There are cases of interest however when the transmitter and receiver have perfect side information, but the channel is time-varying. One model for this case is to suppose that $H[k]$ is indeed time-varying, and that this sequence is a realization of a random process, in which each $H[k]$ is selected independently at each symbol $k$ (and independently of the $x[k]$) according to some probability law $p_H$, so the channel remains memoryless.

Subject to this model, we seek a solution to Problem 1 in which the sequence of channel matrices are generated i.i.d. according to $p_H$. It is tempting to simply average over the ordered eigenvalue density, $p_{\Lambda}(\lambda_1, \ldots, \lambda_m)$, associated with $p_H$ (see for example [32]),

$$E\{C\} = \int p(\lambda_1, \ldots, \lambda_m) \sum_{\lambda_i \leq \mu} \log \mu \lambda_i \, d\Lambda \quad (9)$$
This quantity is however in general not the capacity of the channel (1) with \( H[k] \sim p_H \). A simple counter-example suffices to show the problem.

**Example 1:** Consider a single-input single-output channel, \( r = t = 1 \), and let \( p_H(\epsilon) = p_H(1) = 1/2 \) where \( \epsilon > 0 \). Then according to (2) in which water-filling precedes averaging, the resulting information rate is \( \log (1 + \gamma \epsilon^2) / (4 + \log(1 + \gamma)) / 4 \) which as \( \epsilon \rightarrow 0 \) approaches \( \log(1 + \gamma) / 4 \).

It is obvious however that as \( \epsilon \rightarrow 0 \), the transmitter should only transmit in symbol intervals in which \( H = 1 \), resulting in the capacity \( \log(1 + \gamma) / 2 \) which is a factor of two greater than the previous approach.

The problem with (3) is that it precludes optimization of the transmit density over time as well as space. The rate (9) is maximal only under the assumption of a short-term power constraint \( \text{tr}(Q[k]) = 1 \), rather than the long-term constraint \( \text{tr}(Q_N) = N \).

The following Theorem, is proved by solving the input distribution optimization problem from first principles (see Appendix).

Theorem 1: Suppose that the channel matrices \( H[k] \) of an ergodic MIMO channel (1) are selected i.i.d. each symbol \( k \) according to a matrix density \( p_H \) which possesses an eigenvalue density \( f_\lambda \). The capacity of this channel with perfect channel knowledge at both the transmitter and the receiver is given by

\[
\frac{C}{m} = \int_{\xi^{-1}}^{\infty} \log (\xi \lambda) \, f_\lambda(\lambda) \, d\lambda \quad \text{where } \xi \text{ is such that } (10)
\]

\[
\frac{\gamma}{m} = \int_{\xi^{-1}}^{\infty} (\frac{\xi - 1}{\lambda}) \, f_\lambda(\lambda) \, d\lambda.
\]  

It is interesting to note that not only does this Theorem yield the actual capacity, as opposed to the rate given by (2), it is also easier to compute in most cases, since it is based on the distribution of an unordered eigenvalue.

Water-filling over space and time has been addressed to a limited extent in the literature. Tse and Viswanath give the result, without proof [33, Section 8.2.34]. Goldsmith also writes down the optimization problem (without solution) in [34, Equation (10.16)], and also in [26]. The correct space-time water filling approach is also implicit in [35], although no proof or discussion is offered.

Let us now examine the optimal transmit strategy in more detail. Let \( H[k] = U[k] \Lambda[k] V[k] \) be the singular value decomposition of \( H[k] \) and let \( H_N, U_N, V_N \) and \( \Lambda_N \) be the corresponding block diagonal matrices. Then the singular value decomposition of the block diagonal matrix \( H_N \) is

\[
H_N = U_N \Lambda_N V_N.
\]

This follows directly from the block-diagonal structure of \( H_N \). The fact that the singular vectors are also in block-diagonal form is important from an implementation point of view. If it had turned out that \( H_N \) had full singular vector matrices, the optimal transmission strategy would be non-causal.

The optimal transmit strategy uses a block-diagonal input covariance matrix,

\[
Q_N = \text{diag} \left\{ V^\dagger[1] \Gamma[1] V[1], \ldots, V^\dagger[N] \Gamma[N] V[N] \right\}
\]

where \( \Gamma[k] = \left( \xi I - (\Lambda[k])^{-1} \right)^+ \), using the notation \((\cdot)^+\) which replaces any negative elements with zero. The block-diagonal structure means that the input symbols are correlated only over space, and not over time. At time \( k \), the input covariance is \( Q[k] = V^\dagger[k] \Gamma[k] V[k] \). Thus the optimal transmit strategy is not only causal, but is instantaneous, i.e. memoryless over time. At time \( k \), the transmitter does not need to know any past or future values of \( H[j], j > i \) and \( j < i \) in order to construct the optimal covariance matrix.

The key thing to note from Theorem (1) is that the required power allocation is still water-filling on the eigenvalues of \( H[k] H^\dagger[k] \), but that the water level \( \xi \) is chosen to satisfy the actual average power constraint, rather than a symbol-wise power constraint. At any particular symbol time, the transmitter uses a power allocation \((\xi - 1/\lambda)^+\) for each eigenvalue \( \lambda \) of \( H[k] H^\dagger[k] \), noting that \( \xi \) is selected according to (11) rather than on a per-symbol basis, (2). This does not require any more computation that symbol-wise water filling. In fact, it is simpler, since the transmitter only needs to compute the water level \( \xi \) once. Not only does space-time water filling give a higher rate, it is in this sense easier to implement.

One possible argument against the use of space-time water-filling is that with this approach, there is a variable amount of energy transmitted at each symbol interval. In some cases that would certainly be undesirable (such as systems using constant envelope modulation).

Theorem 2: The peak-to-average power ratio resulting from space-time water-filling, (10), (11) on an ergodic channel with average power constraint \( \gamma \) and unordered eigenvalue density \( f(\lambda) \) such that \( E[1/\lambda] \) exists is upper-bounded

\[
PAPR \leq 1 + \frac{m}{\gamma} E \left[ \lambda^{-1} \right].
\]

This is a particularly simple characterization of the PAPR. The term \( mE[1/\lambda]/\gamma \) is the ratio of the average inverse eigenvalue to the average symbol energy per eigen-mode.
It is also straightforward to compute the information rate $I$ that results from adjusting the space-time water-filling solution to accommodate a peak-power limitation $\gamma_{\text{max}}$:

$$\frac{I}{m} = \int_{\xi^{-1}}^{(\xi - \gamma_{\text{max}})^{-1}} \log (\xi \lambda) \ f(\lambda) \ d\lambda$$

where $\xi$ is such that

$$\frac{\gamma}{m} = \int_{\xi^{-1}}^{(\xi - \gamma_{\text{max}})^{-1}} \left( \xi - \frac{1}{\lambda} \right) \ f(\lambda) \ d\lambda.$$

Note that this is not the same as the capacity of the peak-power constrained channel. In practice however, it may be of interest, since powers approaching $\xi$ are typically transmitted with vanishing probability. It is therefore of interest to consider the probability density function $q(\gamma)$ of the per-eigenvector transmit power, $\gamma = \xi - 1/\lambda$. The obvious transformation yields the density function.

**Theorem 3:** The probability density function $q(\gamma)$ of the energy $\gamma = \xi - 1/\lambda$ transmitted on each eigenvector according to (10), (11) is given by

$$q(\gamma) = F(\xi^{-1}) \delta(\gamma) + \frac{f(\frac{(\xi - \gamma)^{-1}}{(\xi - \gamma)^2})}{(\xi - \gamma)^2},$$

where $f(\cdot)$ is the unordered eigenvalue density, $F(\cdot)$ is the corresponding cumulative distribution and $\delta$ is the Dirac delta function. The point mass at $\gamma = 0$ corresponds to the probability of transmitting nothing on that channel (when the gain is less than $1/\xi$).

The following examples show some simple applications of the preceding space-time water-filling result.

**Example 2 (Parallel On-Off Channel):** Consider an $m$-input, $m$-output channel with eigenvalue density $(1 - p)\delta(\lambda) + p\delta(\lambda - 1)$. There are $m$ parallel channels and each channel is an independent Bernoulli random variable. With probability $p$, a channel is “on” and with probability $1 - p$ it is “off”.

Spatial water-filling yields the rate

$$E \left[ \frac{k}{2} \log \left( 1 + \frac{P}{k} \right) \right],$$

where $k \sim \text{Binomial}(m, p)$. It is straightforward to show however that the capacity is

$$C = \frac{E[k]}{2} \log \left( 1 + \frac{P}{E[k]} \right) = \frac{mp}{2} \log \left( 1 + \frac{P}{mp} \right).$$

which, as expected is strictly larger than the former rate, a fact that can be seen from Jensen’s inequality.

**Example 3 (Rayleigh, $t = r = 1$):** Consider the single-input, single-output Rayleigh fading channel. Then $f(\lambda) = e^{-\lambda}$ and $\xi$ is the solution to

$$\xi e^{-1/\xi} + \Gamma(0, \xi^{-1}) = P,$$

where $\Gamma(a, x)$ is the incomplete Gamma function [36, (8.350.2)]. Figure 1 compares the resulting capacity to the rate obtained via per-symbol water-filling. Note that in this case, the latter corresponds to the capacity when the transmitter does not know the channel realization. In other words, application of the incorrect method results in ignoring the channel knowledge at the receiver.
Example 4 (Rayleigh $t = r = 2$): Consider the two-input, two-output Rayleigh fading channel. Then $f(\lambda) = \frac{2 + (\lambda - 2) \lambda}{2e^\lambda}$ and $\xi$ is the solution to

$$e^{-1/\xi}(2\xi + 1) - 2\Gamma(0, \xi^{-1}) = P.$$ 

Figure 2 compares the resulting capacity to the rate obtained via per-symbol water-filling and to the rate obtained with $Q = PI/t$. The curves for space-time water-filling and spatial water-filling almost coincide on this figure. This is however hiding the additional gain provided by space-time water-filling at low SNR. Figure 3 shows the relative gains, compared to $Q = PI$, for space-only and space-time water-filling. Obviously, as $SNR \to \infty$, both gains approach 1, since there is asymptotically no benefit in water filling of any kind. At SNR below 0 dB, space-time water-filling yields significant benefit compared to water-filling only over space.

Example 5 (Rayleigh $t = r = 4$): Figure 4 shows the relative capacity gain over $Q = PI/t$ for a four-input, four-output system. Obviously the additional gain over spatial water-filling is decreased compared to the $t = r = 2$ case. In fact as $t, r \to \infty$, there is asymptotically no extra gain to be found by additionally water-filling over time as well as space. As the dimension increases, the eigenvalue density converges to the well-known limit law, holding on a per-symbol basis. Thus space-time water filling on Rayleigh channels is of most importance for small systems.

Figure 5 shows the peak-to-average power ratio in decibels for $t = r = 1, 2, 4$. Note that this is the exact value of the PAPR. For Rayleigh channels with finite $m$, the bound of Theorem 2 does not apply, since $E[1/\lambda]$ does not exist. From this figure, the peak-to-average power is relatively insensitive to the system dimensions for the Rayleigh channel. The particular values of PAPR are comparable with what may be experienced in an orthogonal frequency division multiplexing system.

As described earlier, the peak-to-average power ratio may be misleading, since it is conceivable that the peak power may only be transmitted infrequently. Figure 6 shows the probability density function of the power transmitted per-eigenvector for $t = r = 2$. At low SNR, the density is broad and has significant mass above the target average power $P/m$. As the SNR increases, the density converges to an impulse at $P/m$.

III. STATISTICAL TRANSMITTER SIDE INFORMATION

It is tempting to think that $Q = I/t$ is optimal when the transmitter has no knowledge about the channel, and assertions to this effect have appeared in the literature. In the complete absence of transmitter side information however (i.e. the transmitter
does not even know $p_H$), the underlying information theoretic problem is difficult to define. There are several possibilities, for instance $p_H$ may selected somehow randomly from a set of possible channel densities. Alternatively, $p_H$ could be fixed, but unknown, in the spirit of classical parameter estimation. In the absence of a thorough problem formulation and corresponding analysis, it is clear that optimality of $Q = I/t$ is at best conjecture. For example, in the case where $p_H$ is drawn randomly from a set of possible densities, it may be an outage probability that is of interest. This problem is not completely solved even when the $p_H$ are degenerate (i.e. the non-ergodic channel of Telatar), and in that case transmission on a subset of antennas is believed to be optimal. We do not consider these more difficult problems, and restrict attention to transmitter knowledge of $p_H$.

The result (8) arises from [4, Theorem 1] and holds for independent, identically distributed, circularly symmetric Gaussian channel matrix $H$, independent of transmit symbols. In general, $Q = I/t$ is not optimal, and thus provides only a lower bound to capacity. Several authors [37] have investigated the scenario of transmitting, equal power, independent Gaussian signals for various correlated central and non-central random matrix channels. Other work [38] have examined worst-case mutual

Fig. 4. Rayleigh channel, $t = r = 4$.

Fig. 5. Peak-to-Average Power Ratio $t = r = 1, 2, 4$.

Fig. 6. Per-eigenvector transmit power density, $t = r = 2$. 
information in the absence of transmitter side information, while [39] has applied game-theoretic analysis to the problem of equal power transmission, observing that (in the absence of any better option) uniform power allocation is not “so bad.”

In the previous section, we considered the optimal transmit covariance for perfect transmitter side information. We shall now relax this constraint, so the transmitter has statistical side information only, which is a well-posed information theoretic problem.

There are two main areas of interest. Firstly, in some scenarios, the eigenvectors of the optimal input covariance \( Q \) can be determined a-priori (typically by inspection). Several authors have described optimization of input covariance, by diagonalization of the transmit covariance \([12,18,40]\). In other work, [14] has outlined optimality conditions for beamforming vs MIMO diversity. Recent work [41] has also investigated the case where input and channel covariance matrices are jointly diagonalizable.

The more general case, is when the eigenvectors of the optimal input covariance structure are not apparent a-priori, and may in fact be complicated functions of \( p_H \). This is the main area of interest in this paper, and Theorem 8 (and the resulting iterative optimization procedure) is our the main result. We will begin in Section III-A by finding the optimal \( Q \) in the diagonalizable case, which results in an interesting comparison to water-filling. Section III-B extends the result to arbitrary \( p_H \).

### A. Diagonalizable Covariance

Solution of Problem 2 is in general a semidefinite program, since the maximization is over the cone of positive semidefinite hermitian matrices \( Q \geq 0 \). In certain cases however, the problem simplifies, and we can obtain convenient conditions for optimality from the Kuhn-Tucker conditions. The simplest case, case \( S \sim \mathcal{N}_{r,t}(0, I \otimes I) \) was solved in [4]. Other special cases have been solved in [12,40]. Independent work finding similar results to those described below has appeared in [17].

Suppose it can be determined that the optimal \( Q \) has the form

\[
Q = U\hat{Q}U^\dagger \quad (14)
\]

\[
\hat{Q} = \text{diag}(q_1, q_2, \ldots, q_t) \quad (15)
\]

for some fixed \( U \). For such channels, the optimization problem reduces to finding the best allocation of power to each column of \( U \).

One important example is \( H[k] \sim \mathcal{N}_{m,m}(0, R \otimes T) \), i.e. the Kronecker correlated Rayleigh channel with no line-of-sight components. In that case, it is known that \( U \) diagonalizes \( T \) and optimal transmission is independent on each eigenvector of \( T \).

In such cases, the condition \( Q > 0 \implies \hat{Q} > 0 \) allows the application of the Kuhn-Tucker conditions for maximization of a convex function over the space of probability vectors \([7, p. 87]\) to yield the following lemma.

**Lemma 1**: Consider the channel with \( H[k] \sim \mathcal{N}_{m,m}(0, R \otimes T) \). The optimal covariance \( Q \) has the form (14) and satisfies the Kuhn-Tucker conditions \([7, p. 87]\)

\[
\frac{\partial \Psi(Q)}{\partial q_i} = \mu \quad q_i > 0 \quad (16)
\]

\[
\frac{\partial \Psi(Q)}{\partial q_i} \leq \mu \quad q_i = 0 \quad (17)
\]

where \( \mu \) is a constant independent of \( q_i \), and \( q_i \) are given by (15). Thus the necessary and sufficient conditions for optimality have a particularly simple form. Differentiating \( \Psi(Q) = E_H \{ \log \det (I + HQU^\dagger) \} \) leads to the following theorem, proved in [10].

**Theorem 4 (Optimal Covariance)**: Consider the ergodic channel with \( p_H \) such that the optimal input covariance is known to be of the form (14) and satisfies the Kuhn-Tucker conditions for some fixed unitary matrix \( U \). A necessary and sufficient condition for the optimality of the diagonal \( \hat{Q} \) is

\[
E_S \left\{ \left( \left( I + S\hat{Q} \right) S \right)^{-1} \right\} = \mu \quad q_k > 0 \quad (18)
\]

\[
E_S \left\{ \left( \left( I + S\hat{Q} \right) S \right)^{-1} \right\} < \mu \quad q_k = 0 \quad (19)
\]

for \( k = 1, 2, \ldots, t \) and some constant \( \mu \). The expectation is with respect to the random matrix \( S = \gamma U^\dagger H^\dagger HU \). The notation \((A)_{ij}\) denotes element \( ij \) of \( A \).

In the case \( Q > 0 \), the condition (18) may be re-written as a fixed-point equation

\[
\hat{Q} = \nu E_S \left\{ \left( \hat{Q}^{-1} + S \right)^{-1} S \right\}, \quad (20)
\]

which suggests the following iterative procedure for numerically finding the optimal \( \hat{Q} \). Starting from an initial diagonal \( \hat{Q}^{(0)} > 0 \), compute

\[
q_k^{(i+1)} = \nu \left[ E_S \left\{ \left( \left( \hat{Q}^{(i)} \right)^{-1} + S \right)^{-1} S \right\} \right]_{kk}, \quad (21)
\]
selecting $u^{(i)}$ at each step to keep $\text{tr} \left( \hat{Q}^{(i)} \right) = \gamma$. Although there is no known closed form solution for $E_S \left\{ \left( \hat{Q}^{-1} + S \right)^{-1} S \right\}$, it may be accurately estimated using monte-carlo integration. Note that the numerical procedure may be applied to each entry $q_k = \hat{Q}_{kk}$ separately for a given $\hat{Q}^{(i)}$. Numerically, each fixed point iteration is performed once and the $t$ non-zero diagonal entries of $\hat{Q}$ are updated.

It is interesting to compare the conditions (18), (19) with the solution of Problem 1 for perfect transmitter side information. Suppose $H[r] = H$ is known at the transmitter with $HH^\dagger = USU^\dagger$ being the eigenvalues decomposition of $HH^\dagger$. The Kuhn-Tucker condition for optimality of the input covariance $Q = U^\dagger \hat{Q} U^\dagger$ can be written in the following form,

$$
\begin{align*}
\left( I + S \hat{Q} \right)^{-1} S_{kk} &= \mu & q_k > 0 \\
\left( I + S \hat{Q} \right)^{-1} S_{kk} &< \mu & q_k = 0.
\end{align*}
$$

(22)

with $\hat{Q}$ satisfying (15). Solution of these equations is straightforward and leads easily to (5) and (6).

Comparing (18) with (22) it can be seen that the only difference is the presence of the expectation in (18). Similarly for (19) and (23). This is no real surprise, and is due to the interchangability of differentiation and expectation. The result of Theorem 4 is a direct generalization of the classical water-filling result for parallel channels [42], where the transmitter has statistical side information, and the channel can be diagonalized a-priori. In the latter case however, there is no water-filling interpretation [43].

For the deterministic case, it is clear that increasing $\gamma$ can only increase the power allocated to any particular eigenvector (water-level raises). The same thing happens in the ergodic case, as demonstrated by the following theorem, proved in the Appendix.

Theorem 5: Let $\hat{Q} = \text{diag}(q_1, \ldots, q_t)$ be the eigenvalues of the optimal covariance matrix for a channel with signal-to-noise ratio $\gamma$, satisfying the conditions of Theorem 4. Then

$$
\frac{\partial q_k}{\partial \gamma} \geq 0, \quad k = 1, 2, \ldots, t.
$$

Thus a signal-to-noise ratio increase (decrease) can only increase (decrease) the power allocated to each eigenvector of the optimal covariance matrix.

Theorem 6 takes care of zero-mean Rayleigh fading channels with separable correlation structure. In the case of Ricean fading with non-zero mean, one approach is to use the following approximation by a central distribution.

Let $H \sim \mathcal{N}_{r,\ell} (M, I \otimes T)$. Then $S = HQH^\dagger$ may be approximated by a central Wishart matrix [44, p. 125]

$$
S \sim W_t (0, \Sigma)
$$

(24)

$$
\Sigma = T^{1/2} Q T^{1/2} + \frac{1}{t} M \otimes M
$$

(25)

This approximation motivates application of Theorem 4 to the Ricean case with $H[k] \sim \mathcal{N}_{r,\ell} (M, I \otimes T)$. The relation between correlation and line-of-sight (non-zero mean) has been heuristically established in MIMO channel measurement literature [45–47]. The accuracy of this approximation is investigated numerically below.

In figure 7 we have plotted the capacity and the mutual information for a channel with rank-one mean $M = \text{diag} \{ t, 0, \ldots, 0 \}$ and non-diagonal transmit covariance

$$
T = \begin{bmatrix}
1 & \tau & \tau & \cdots \\
\tau & 1 & \tau & \cdots \\
\vdots & \ddots & \ddots & \ddots
\end{bmatrix} = \tau \mathbf{1} + \text{diag} \{ \tau - 1 \}
$$

(26)

where $\mathbf{1}$ is a matrix of all ones.

The plot compares the capacity (optimal input covariance, with true probability law) with the mutual information (input covariance given by central Wishart approximation) for various SNR and numbers of transmit and receive elements. Each plot has assumed $t = r$. We note that the approximated covariance matrix is a linear combination of the transmit-end covariance $T$ and the mean, and thus approximated input covariance is a dominated by beamforming on $M$ at low SNR, and $T$ at higher SNR.

Beamforming, i.e. rank-one transmission with $\hat{Q} = \text{diag} (1, 0, 0, \ldots, 0)$ is a particularly simple strategy, which is optimal at low SNR (see Section III-D). It is interesting to consider the conditions under which beamforming is optimal.

Theorem 6: Consider an ergodic channel (11) with $H \sim \mathcal{N}_{\ell,\ell} (0, R \otimes T)$, where without loss of generality $R = \text{diag} (\rho_1, \ldots, \rho_r)$ and $T = \text{diag} (\tau_1, \ldots, \tau_r)$ with $\text{tr}(T) = t$ and $\text{tr}(R) = r$. Beamforming is optimal if and only if

$$
E \left\{ \frac{u^\dagger Ru + \gamma \tau_k u^\dagger R^2 u}{1 + \gamma \tau_k u^\dagger Ru} \right\} \geq \frac{r \tau_k}{\tau_1} \quad \text{for any } k \geq 2,
$$

(27)
where the expectation is with respect to a length $r$ Gaussian vector with i.i.d. unit variance entries, $u \sim \mathcal{N}_r(0, I)$.

The left hand side of (27) is monotonically decreasing with signal-to-noise ratio. For zero-mean Rayleigh channels, the condition (27) can be found in closed form [48]. In the appendix we give an alternate proof to that given by [48]. Our proof is simplified via use of Theorem 6.

Theorem 7 (Simon and Moustakas [48]): Consider an ergodic channel (1) with $H \sim \mathcal{N}_{t,r}(0, R \otimes T)$, where without loss of generality $R = \text{diag}(\rho_1, \ldots, \rho_r)$ and $T = \text{diag}(\tau_1, \ldots, \tau_r)$ with $\text{tr}(T) = t$ and $\text{tr}(R) = r$. Beamforming is optimal if and only if

$$\sum_{i,j=1}^r \frac{\rho_i(1 + \gamma \tau_2 \rho_i) \rho_j^{r-1}}{\prod_{k \neq j}(\rho_j - \rho_k)} \zeta_{ij} > r \gamma \tau_2$$

(28)

where

$$f(x) = e^{1/2} \Gamma(0, 1/2).$$

In the above theorem, note that $\zeta_{ii}$ is just the limit of $\zeta_{ij}$ as $\rho_i \to \rho_j$. Theorem 6 is a generalization of [49] (which was for the MISO case), and the MISO result is recovered easily from (27) via $r = 1$.

Figure 8 shows the beamforming optimality condition of Theorem 7 for a set of SNR levels $\gamma$ and a $2 \times 2$ channel, with $H \sim \mathcal{N}_{2,2}(0, R \otimes T)$ where $R = \text{diag}(\rho, 2 - \rho)$ and $T = \text{diag}(\tau, 2 - \tau)$, $1 \leq \rho, \tau \leq 2$. The plot is symmetric around the point $\rho = \tau = 1$ (and thus, only the top-left quadrant of the full $0 \leq \tau, \rho \leq 2$ plot is shown).

The lines provide the transition point from regions where beamforming is optimal (above each line) to regions where beamforming is not optimal. The plot shows the region for $1 \leq \tau, \rho \leq 2$. For $\tau = 1$, $T = I$ and for $\tau \to 2$, $T$ becomes singular, similarly for $R$: so that the top-right hand corner of the plot has highly correlated $H$, whilst the bottom left-hand corner has iid $H$.

It can be seen that for low SNR, $\gamma = -15 \text{dB}$, beamforming is almost always optimal with the transition occurring for $\tau \approx 1.03$. Note also, that the eigenvalues of $R$ have little effect on the optimality of beamforming at low SNR. As SNR increases, the region for admissible covariance matrices for optimal beamforming reduces: we require more covariance matrices with larger eigenvalue separation. The optimality of beamforming is clearly dependent upon the eigenvalues of $T$. At higher SNR, the optimality of beamforming is also dependent on $R$ (as can be seen by the $\gamma \geq 0 \text{dB}$ curves. The reason for this is that the low rank of $R$ results in an effective power loss at the receiver.
Fig. 8. Optimality of beamforming. Beamforming is optimal for a given SNR for all points \((\tau, \rho)\) above the line corresponding to that SNR value. The plot is symmetric for \(0 \leq \rho < 1\) and \(0 \leq \tau < 1\).

B. The General Case

We now wish to solve Problem 2 without the a-priori requirement of diagonal input covariance. In this case, we need to maximize \(\Psi(Q, p(S))\) over all positive definite \(Q\). In particular we do not wish to restrict ourselves to particular matrix densities such as the zero-mean Kronecker Gaussian model.

Whilst of interest in its own right, this problem arises when the input covariance structure cannot be solved by inspection. Specific examples include the non-central Gaussian random matrix channel, where the channel covariance and mean are not jointly diagonalizable, and for several random matrix channels which do not have simple (Kronecker) factorizations [29, 50].

To accommodate the positive definite constraint on \(Q\), we apply the Cholesky factorization, so the constraint becomes implicit in the final solution. By adopting this approach we force the optimization to only consider the minimum number of independent variables required for solution, \(t(t + 1)/2\) rather than \(t^2\).

Any non-negative matrix \(A\) may be written as [51]

\[ A = \Gamma^\dagger \Gamma \]  

for upper triangular matrix \(\Gamma\), with the diagonal elements \(d_{ii}\) real and non-negative. Similarly, for a given upper triangular matrix \(\Gamma\), the product \(\Gamma^\dagger \Gamma\) is positive definite. The following useful properties [44] arise from (29), \(\text{tr}(A) = \text{tr}(\Gamma^\dagger \Gamma) = \sum_{i \leq j} d_{ij}^2\) and \(\det(A) = \prod_i d_{ii}^2\).

Using (29), transform Problem 2 to

\[ \max_{\Gamma} \Psi(\Gamma^\dagger \Gamma, p_H) \]

subject to

\[ \sum_{i \leq j} d_{ij}^2 = 1 \]

\[ d_{ii} \geq 0, \ \forall i \]

The maximum \(\Psi^o\) for optimal \(d^o\), is not improved by choosing a trace less than unity, hence equality of the first constraint.

Problem 3 admits a quadratic optimization approach, using Lagrange multipliers [52]. The optimization in Problem 3 occurs on the (upper triangular) matrix \(T\) which has exactly \(t(t + 1)/2\) independent (complex) variables. This corresponds to the number of independent variables for the optimization over \(Q\) in Problem 1, since \(Q = U\hat{Q}U^\dagger\) has \(t\) independent variables in the diagonal matrix \(\hat{Q}\) and \(t(t - 1)/2\) independent variables in the unitary matrix \(U\).

In order to solve Problem 3, we produce a modified cost function \(J(\nu, \mu, \phi)\) where \(\nu = \tilde{\Gamma}\), \(\mu\) and \(\phi\) are vectors of Lagrange multipliers corresponding to equality and inequality constraints. For this we use the following:
Lemma 3 (Application Kuhn-Tucker Theorem [53]): Given a convex function \( f(\nu) \) of a vector \( \nu \), where \( \nu \) is constrained by:
\[
\sum_{i<j} \nu_{ij}^2 = 1 \quad \nu_{ii} \geq 0
\]
then
\[
\frac{\partial f(\nu)}{\partial \nu_{ij}} = 2 \mu \nu_{ij}, \quad \nu_{ii} \geq 0, \mu > 0
\] (30)
\[
\frac{\partial f(\nu)}{\partial \nu_{ii}} = 2 \mu \nu_{ii}, \quad \nu_{ii} > 0, \mu > 0
\] (31)
\[
< 0 \quad \nu_{ii} = 0
\] (32)
defines a maximum point for the function \( f(\nu) \).

Lemma 3 provides the necessary conditions for a vector \( \nu = \text{vec}(T) \) to give a capacity achieving input covariance. We now present the main result of the paper: a general condition for the capacity achieving input covariance.

Theorem 8 (Optimal Transmit Covariance): Given a MIMO channel (1) with the channel chosen ergodic according to a probability distribution \( p_H \), then the capacity achieving input is Gaussian with covariance \( Q = \Gamma^\dagger \Gamma \) where \( \Gamma \) is upper triangular, and the element \( d_{ij} \) satisfies:
\[
E_S \left\{ \text{tr} \left[ (I + S \Gamma^\dagger \Gamma)^{-1} SE^{(ij)} \right] \right\} = \begin{cases} 2 \mu d_{ij} & i \neq j, \mu > 0 \\ 2 \mu d_{ii} & \nu_{ii} > 0, \mu > 0 \\ < 0 & \nu_{ii} = 0 \end{cases}
\] (33)
\[
E_S \left\{ \text{tr} \left[ (I + S \Gamma^\dagger \Gamma)^{-1} SE^{(ii)} \right] \right\} < 0 \quad d_{ii} = 0
\] (34)
where the expectation is with respect to \( S = H^\dagger H \), the constant \( \mu \) is chosen to satisfy the power constraint and
\[
E^{(ij)} = \frac{\partial \Gamma^\dagger \Gamma}{\partial d_{ij}}
\]
\[
\left(E^{(ij)}\right)_{mn} = d_{in} \delta_{mj} + d_{im} \delta_{nj},
\]
with \( \delta_{ij} = 1 \) when \( i = j \) and zero otherwise.

The capacity of the channel is then given by application of \( \Gamma \) in \( \Psi(\Gamma^\dagger \Gamma, p(S)) \):
\[
C = E \left\{ \log \det (I_r + S \Gamma^\dagger \Gamma) \right\}
\]
Given the result of Theorem 8 we wish to numerically evaluate the optimal covariance, and hence capacity for an arbitrary multiple-input, multiple-output channel. Fortunately, the form of (33) also lends itself to a fixed-point algorithm.

If we define the matrix
\[
M = E \left\{ (I + S \Gamma^\dagger \Gamma)^{-1} S \right\}
\]
then
\[
\text{tr}(ME^{(ij)}) = \sum_k (m_{kj} + m_{jk})d_{ik} = \left[ \Gamma(M + M^\dagger) \right]_{ij}
\] (36)
The matrix \( M \) may be interpreted as a differential operator, on the function \( \Psi(\Gamma^\dagger \Gamma, p(S)) \), evaluated at a particular value of \( T \). This provides a direct fixed-point equation of projected gradient type [54]:
\[
\nu^{(k+1)} = \frac{1}{\mu} \nu^{(k)} - \nabla E_S \left\{ \Psi \left( \nu^{(k)} \right) \right\}
\] (37)
Writing this out completely gives the following algorithm

Algorithm 1 (Iterative Power Allocation):

1) Update using (35)
\[
\Gamma^{(k+1)} \rightarrow \Gamma^{(k)} (M + M^\dagger)
\] (38)
2) Scale
\[
\left[ \Gamma^{(k+1)} \right]_{ij} \rightarrow \left\{ \begin{array}{ll} \frac{1}{\mu} \left[ \Gamma^{(k+1)} \right]_{ij} & i \leq j \\ 0 & \text{otherwise} \end{array} \right.
\] (39)
with \( \mu \) constant for all \( i, j \) and chosen so that \( \text{tr}(\Gamma^\dagger \Gamma) = 1 \).

3) Repeat
We denote \( \Gamma^{(k)} \) as the triangular matrix at iteration \( k \). This algorithm may be initiated with any (upper triangular) \( \Gamma \) satisfying \( \text{tr}(\Gamma^\dagger \Gamma) = 1 \). The expectation (33) is typically intractable and may be evaluated using monte-carlo integration.
Theorem 9: Algorithm 1 converges to the optimal covariance $Q^o = \Gamma^\dagger \Gamma$.

We note that the stability of the algorithm is directly affected by the stability of the expectation in (35). In particular, at high-SNR, the off-diagonal entries of $\Gamma$ will approach zero (since $Q = \alpha I$ is optimal). In this case, the elements of $\Gamma$ may fluctuate as small movements over the Haar manifold (small changes in eigenvectors) result in large changes in the entries of $\Gamma$.

In Figure 9, we show an example of the convergence of the algorithm for several deterministic channel matrices. Each curve shows the difference between the mutual information for $Q = \Gamma^\dagger \Gamma$ vs the channel capacity $C$ for the $k$th iteration.

The example channel matrices were chosen to have common eigenvalues, but randomly chosen eigenvectors (thus each instance has the same capacity, but different optimal input covariance), with

$$S = US_oU^\dagger, S_o = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

In Figure 10, we have shown the convergence of Algorithm 1 for different matrix dimensions, correlations for $T$ and SNR values. In each plot, the channel is a non-zero mean, correlated Gaussian, $H \sim \mathcal{N}_{n,n}(M_o, I \otimes T)$. Where $M_o = \mu\mu^\dagger$ for a random vector $\mu \in \mathbb{C}^{1 \times n}$. The plots have been averaged over different values of $M_o$. Each convergence is run independently with a random seed value of $\mu$. Algorithm 1 converges to the capacity of the channel, although the convergence rate decreases for larger dimensions. As the channel dimension (and/or SNR) increases, the algorithm becomes more reliant on accurate Monte-Carlo integration, and thus individual iterations take an increasingly long time.

C. Gaussian channel, non-commuting mean and covariance

Consider a channel where

$$H = \kappa M_o + (1 - \kappa)X$$

$$X \sim \mathcal{N}_{m,m}(0, I \otimes \Sigma), 0 \leq \kappa \leq 1$$

using the notation of [44]. Further, we shall assume that the matrices $M_o$ and $\Sigma$ may not be jointly diagonalized (which is equivalent to the Hermitian matrices $M_o$ and $\Sigma$ being non-commuting [55, pp. 229]). We ask: How does the optimal covariance relate to $M_o$ and $\Sigma$ as $\kappa$ varies between 0 and 1?

For the purpose of providing graphical results we shall limit ourselves to a $2 \times 2$ case. While the numerical solution of this problem is straightforward with Algorithm 1, describing the outcome poses several problems: it is insufficient to investigate only the entries of $Q$, since the subspace over which the optimal $Q$ acts will change as $\kappa$ varies.

We note that the optimal covariance has eigenvectors which are not trivially related to the eigenvectors of the mean $M_o$ or variance $\Sigma$. Further, the eigenvectors are not given by a direct interpolation between $M_o$ and $\Sigma$, as can be seen by the superimposed the eigenvectors of $E(S)$.

Figure 11 shows the trajectory of the eigenvectors of the optimal input covariance $Q = U\hat{Q}U$ as $\kappa$ varies between 0 and 1 for $M_o = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ and $\Sigma = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$. The points are plotted by writing the columns of $U$ as two points in $\mathbb{R}^2$. The vertical axis shows the value of $\kappa$. On the plane $\kappa = 0$, the channel is zero-mean, correlated Gaussian $H \sim \mathcal{N}_{2,2}(0, \Sigma)$. It can be seen that the power allocation is divided between the eigenvectors of the covariance matrix $\Sigma$. Similarly, on the plane $\kappa = 1$, the channel is deterministic, with $H = M_o$. The optimal strategy in this case is beamforming. At each end of the plot, the singular vectors of $M_o$ and $\Sigma$ have been superimposed, for comparison with $Q$. 
Fig. 10. Converge of Algorithm 1, for various covariance matrices $T = \tau I + \text{diag}\{\tau - 1\}$ and random rank-one mean, $M_0$. Each plot is averaged over several independent choices of $M_0$. Figure 10(a) shows convergence for $5 \times 5$ matrices, Figure 10(b) shows convergence for $15 \times 15$ matrices and Figure 10(c) shows convergence for $25 \times 25$ matrices.

Fig. 11. Variation in eigenvectors of optimal $2 \times 2$ covariance matrix $Q$, with $H = \kappa \Sigma^{1/2} + (1 - \kappa)M_0$. Eigenvectors of $Q$ are shown dashed. The eigenvectors of $\Sigma$ are superimposed on the plane $\kappa = 0$ and the eigenvectors of $M_0$ are superimposed on the plane $\kappa = 1$. The eigenvectors of $E\{H^\dagger H\}$ are given as solid lines, superimposed over the dashed lines corresponding to $Q$. 

\[ C - \Psi(\Gamma) \]
D. Asymptotics

It is interesting to consider the low- and high-SNR asymptotics of the MIMO channel capacity. This has been done by many authors. Here we give a brief analysis, and in the spirit of the main result presented above, emphasize the results which hold for any $p_H$.

Consider the matrix channel (1) and define $S = HQH^\dagger$. By Taylor series expansion, (7) may be approximated near $\gamma = 0$ by

$$\Psi(Q) \approx \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\gamma^n}{n} E\{\text{tr}(S^n)\}. \quad (43)$$

where $S = H^\dagger H$. Of particular interest is the first order approximation, $\Psi(Q) \approx \gamma \text{tr} \left( Q E\{H^\dagger H\} \right)$.

**Theorem 10 (Low SNR):** Consider a matrix channel (1), with $E\{HH^\dagger\} = U\Lambda U^\dagger$, with $U$ unitary and $\Lambda$ diagonal with $\Lambda = \text{diag}\{\lambda_1, \ldots, \lambda_k\}$ and $\lambda_1 = \cdots = \lambda_k > \lambda_{k+1} \cdots > \lambda_t > 0$. For low SNR, $\gamma \lambda_1 \ll 1$ the capacity achieving distribution is $Q = U\tilde{Q}U^\dagger$ where $\tilde{Q}$ is diagonal and

$$\tilde{Q} = \text{diag} \left\{ \frac{1}{k}, \ldots, \frac{1}{k}, 0, \ldots, 0 \right\}$$

$k$ terms

and $C = \gamma k \lambda_1$.

At low SNR the transmitter only needs to know $E\{HH^\dagger\}$, regardless of the underlying $p_H$. To first order, beamforming in the direction of the largest eigenvector of $E\{HH^\dagger\}$ is optimal (assuming a unique largest eigenvalue). This aligns with well known results [14, 40].

This result must be taken with care: the approximation is for $\gamma \lambda_1 \ll 1$ so that large channel gains will necessitate a correspondingly smaller value of $\gamma$ before the expansion is accurate, see for example [14, 40].

For Ricean channels with separable correlation, a closed form result may be obtained. Suppose $H \sim \mathcal{N}_{t,r}(M, R \otimes T)$, where none of $M, R$ or $T$ are assumed to be diagonal, or jointly diagonalizable. From [44, pp. 251], $S = HH^\dagger$ is a quadratic normal form and

$$E\{HH^\dagger\} = T \text{tr}(R) + M^\dagger M. \quad (44)$$

thus

$$C(\gamma)|_{\gamma \to 0} = \gamma \lambda_1 \quad (45)$$

where $\lambda_1$ is the largest eigenvalue of $T \text{tr}(R) + M^\dagger M$. This makes it clear that the most fortuitous arrangement of $T$ and $M$ is when they share a common largest eigenvector. for $R = I$ and $r = t$, (44) is essentially the central Wishart approximation of Lemma 2. This is not coincidence, since the central Wishart approximation is found by matching the first moment of the density.

There are several special cases that result in simpler forms for $\lambda_1$.

1) In the case of identity transmit covariance $T = I_t$, $\lambda_1 = \text{tr}(R) + \lambda_1(M^\dagger M)$.
2) $M = \alpha I$. Then $\lambda_1 = \alpha^2 + \text{tr}(R)\lambda_1(T)$.
3) Weak LOS component, $T \text{tr}(R) \gg M^\dagger M$. Then $\lambda_1 = \text{tr}(R)\lambda_1(T) + \epsilon$, where $|\epsilon| \leq \lambda_1(M^\dagger M)$. Obviously if $M = 0$, $\epsilon = 0$.
4) Strong LOS component, $M^\dagger M \gg T \text{tr} R$. Then $\lambda_1 = \lambda_1(M^\dagger M) + \delta$, where $|\delta| \leq \text{tr}(R)\lambda_1(T)$. 

---

Fig. 12. Power allocation for optimal input covariance, $\tilde{q}_1$ and $\tilde{q}_2$ with $Q = U\tilde{Q}U^\dagger$.
5) For \( r = t = 2 \) it is easy to obtain a closed form solution for \( \lambda_1 \).

Turning now to the other extreme, for large \( z \), \( \log(1 + z) \to \log(z) \), and hence at high SNR,

\[
\Psi(Q) \to t \log \gamma + \log \det Q + \log \det (H^\dagger H).
\]

(46)

Care must be taken in the definition of “high” SNR. The approximation (46) is only valid when \( \gamma Q_{ii} \gg \lambda_{\min} \), ie. the high SNR, is based on high received SNR over all modes, not necessarily high transmit power.

**Theorem 11 (High SNR):** Consider a matrix channel \([1]\) with \( H \) a random variable, independent of \( Q \). Then the capacity achieving distribution is \( Q = I_t/t \) and the resulting capacity is

\[
C \to t \log \left( \frac{2}{7} \right) + E \{ \log \det (HH^\dagger) \}
\]

(47)

for any probability density function \( p_H \), provided that \( H \) is independent of \( Q \).

Theorem 11 holds regardless of the characteristics of the channel. The optimal transmit strategy at high SNR is equal power, independent white signals. This is not surprising when it is seen that for large received power, the variation in channel strength is meaningless. From a water-filling perspective, we have a very deep pool, with tiny pebbles on the bottom: allocation of power is irrelevant. The channel distribution \( p_H \) has no effect on the optimal transmission strategy, and only affects the resulting capacity via the \( E \{ \log \det (HH^\dagger) \} \) term. This is investigated in much more depth in [56, 57].

Note also that at high SNR, \( t \log(P/t) \) is asymptotic to the capacity resulting from transmitting independent data across \( t \) non-interfering AWGN channels (each channel getting \( P/t \) of the available power). The remaining term is either a capacity loss or gain over this parallel channel scenario, depending on the statistics of the channel. In the case of Wishart matrices, \( H \sim N_t(0, R \otimes I) \) \([47]\) has a known closed-form solution [23]. For numerical purposes, \( E \{ \log \det (HH^\dagger) \} \) may be obtained by Monte-Carlo methods.

IV. Conclusion

This paper has shown how to correctly compute the capacity of multiple-input multiple-output channels whose gain matrices are chosen independently each symbol interval according to a given matrix density. The optimal input density is Gaussian but is not identically distributed over time or space except in special cases.

In the case of full CSI at the transmitter, the optimal power allocation corresponds to water-pouring in space and time, and is performed instantaneously, which is an important practical consideration. At each symbol, the transmitter still performs water pouring over the channel eigenvalues at that instant, but uses a water level that results in the long-term average power constraint being satisfied. In certain circumstances, this yields a considerable gain in rate, compared to a symbol-wise water-filling, in which the transmitter uses a water level that enforces a per-symbol power constraint. The peak-to-average power ratios and entire power distribution resulting from the use of the optimal space-time water-filling strategy were also considered. For Rayleigh channels, the resulting peak-to-average power ratio can be several decibels, depending upon the average power.

We have investigated the capacity achieving input covariance in the case where the transmitter has statistical CSI. We have presented a method for calculating the optimal input covariance for arbitrary Gaussian vector channels. We have provided an iterative algorithm which converges to the optimal input covariance, by considering the covariance in terms of a Cholesky factorization. We have demonstrated the algorithm on several difficult channels, where the appropriate “diagonal” \( Q \) input cannot be readily found by inspection. Although the diagonalizing decomposition \( Q = UQU^\dagger \) always exists, we have shown that the matrix \( U \) may be non-trivially related to the pdf of the channel.

For special cases, the optimal input covariance can be a-priori diagonalized by inspection – such as for zero-mean Kronecker correlated Rayleigh channels. In such cases we gave a simpler fixed point equation that characterizes the optimal transmit covariance. This particular characterization reveals a close link between the optimality condition for deterministic channels (water filling) and that for ergodic channels.

**APPENDIX**

**Proofs**

**Proof: [Proof: Theorem 1]** The capacity is given by

\[
C = \lim_{N \to \infty} \sup_{p(x_N)} \frac{1}{N} I(x_N; y_N | H_N).
\]

(48)

For fixed \( N \) re-write the entire sequence of transmissions \([1]\) as

\[
y_N = H_N x_N + z_N.
\]

(49)
For any fixed value of \( N \), the optimal density on \( x_N \) is obtained by water-filling on the \( Nm \) eigenvalues \( \nu_1, \nu_2, \ldots, \nu_{Nm} \) of \( W_N = H_NH_N^\dagger \). Thus the optimized information rate for given \( N \) is given parametrically by

\[
C_N = \frac{1}{N} \sum_{\nu_i^{-1} \leq \xi} \log \xi \nu_i \\
P = \frac{1}{N} \sum_{\nu_i^{-1} \leq \xi} \xi - \nu_i^{-1}.
\]

(50) (51)

Now for a block diagonal matrix such as \( W_N \), the \( Nm \) eigenvalues are simply the set of all the eigenvalues of the component diagonal blocks, in this case the \( H[k]H^\dagger[k] \). As \( N \to \infty \), the distribution of the eigenvalues of \( W_N \) converges to the eigenvalue density \( p_\lambda \) associated with \( p_H \) and the summations become expectations with respect to a randomly chosen eigenvalue of \( HH^\dagger \).

**Proof:** [Proof: Theorem 2] A few observations can be made regarding the distribution of power resulting from the optimal transmit strategy. Firstly, transmit power is upper-bounded by \( m\xi \), since the instantaneous power level on each eigenvector is \( \xi - 1/\lambda_i \), and \( \lambda_i \geq 0 \). The peak-to-average power ratio (PAPR) is therefore \( m\xi/\gamma \). Now from (11),

\[
\frac{\gamma}{m} = \int_{\xi^{-1}}^\infty \left( \xi - \frac{1}{\lambda} \right) f(\lambda) d\lambda \\
\geq \int_0^\infty \left( \xi - \frac{1}{\lambda} \right) f(\lambda) d\lambda \\
= \xi - E[\lambda^{-1}].
\]

The inequality is due to the fact that the portion of the integral from 0 to \( 1/\xi \) is non-positive. Therefore \( \xi \) is upper-bounded

\[
\xi \leq \frac{\gamma}{m} + E[\lambda^{-1}].
\]

**Proof:** [Proof: Theorem 5] An optimal \( Q \) has eigenvalues with satisfy (20), and hence

\[
\frac{1}{\nu} \frac{\partial q_k}{\partial \gamma} = \frac{\partial q_k}{\partial \gamma} \left[ E \left\{ \left( (\gamma \hat{Q})^{-1} + S \right)^{-1} S \right\} \right]_{kk} \\
= \left[ (\gamma \hat{Q})^{-1} + S \right]^{-1} \gamma^{-2} \hat{Q}^{-1} \left( (\gamma \hat{Q})^{-1} + S \right)^{-1} S \right]_{kk} \\
= \left[ (\gamma I + \gamma^2 \hat{Q} S)^{-1} \left( (\gamma QS)^{-1} + I \right)^{-1} \right]_{kk} \\
= \left[ (QS)^{-1} + 2\gamma I + \gamma^2 \hat{Q} S \right]^{-1} \right]_{kk} \\
= \left[ A^{-1} \right]_{kk}
\]

where \( A = A^\dagger \geq 0 \) (since \( S \geq 0 \) and \( Q \geq 0 \) are both Hermitian). Now \( \det(A)A^{-1} = \text{adj}(A) \) and the diagonal elements of \( \text{adj}(A) \) are determinants of principal minors of \( A \geq 0 \), which are non negative [55, p. 398]. Noting that \( \partial \nu / \partial \gamma > 0 \) completes the proof.

**Proof:** [Proof: Theorem 6] Rank-one transmission with \( Q = E_{11} \) is optimal if reduction in \( q_1 \) (and corresponding increase in some other \( q_i \) results in an overall decrease in mutual information. From the Kuhn-Tucker conditions (16), (17), the condition for optimality is (see also [12, 21, 22, 31, 49])

\[
\frac{\partial \Psi}{\partial q_1} \bigg|_{Q=E_{11}} \leq \frac{\partial \Psi}{\partial q_k} \bigg|_{Q=E_{11}} \quad k \geq 2.
\]

(52)

Furthermore, we can restrict attention to \( k = 2 \) in (52). Now

\[
\frac{\partial}{\partial q_k} \Psi(Q) = E \left\{ \left( (I + SQ)^{-1} S \right)_{kk} \right\}
\]

where \( S = T^{1/2}X^T RXT^{1/2} \) with \( X \sim N_{\nu,\nu}(0, I) \).

Now \( A = I + SE_{11} \) is of the form

\[
\begin{pmatrix}
1 + S_{11} & 0_{m-1} \\
0 & \gamma I_m \end{pmatrix}
\]
where \( 0_{m-1} \) is an all-zero row vector of length \( m - 1 \) and \( b \) is a column vector of length \( m - 1 \). We need to find the inner product between row \( k \geq 2 \) of \( A^{-1} \) and the corresponding column \( k \) of \( S \). Applying the partitioned matrix inverse theorem yields

\[
A^{-1} = \begin{pmatrix} \frac{1}{1 + S_{11}} & 0_{m-1} \\ 0 & J_{m-1} \end{pmatrix}
\]

and hence for \( k > 1 \),

\[
\frac{\partial \Psi}{\partial q_k} \bigg|_{Q=E_{11}} = E\{S_{kk}\} - E\left\{ \frac{S_{1k}S_{1k}}{1 + S_{11}} \right\}
\]

(a) \( = E\{S_{kk}\} - E\left\{ \frac{|S_{1k}|^2}{1 + S_{11}} \right\} \)

(b) \( = \gamma \tau_k - E\left\{ \frac{|S_{1k}|^2}{1 + S_{11}} \right\} \)

(c) \( = \gamma \tau_k - E\left\{ \frac{\gamma^2 \tau_k (\sum_{i=1}^r \rho_i X_{1i}^* X_{ik})^2}{1 + \gamma \tau_k \sum_{i=1}^r \rho_i |X_{1i}|^2} \right\} \)

since (a) \( S = S^t \), (b) \( E\{S\} = \gamma \text{tr}(R)T \), and (c),

\[
S_{1k} = \gamma \sqrt{\frac{\tau_1 \tau_k}{2}} \sum_{i=1}^r \rho_i X_{1i}^* X_{ik}.
\]

Similarly, for \( k = 1 \)

\[
\frac{\partial \Psi}{\partial q_1} \bigg|_{Q=E_{11}} = E\left\{ \frac{S_{11}}{1 + S_{11}} \right\} - E\left\{ \frac{\gamma \tau_1 \sum_{i=1}^r \rho_i |X_{11}|^2}{1 + \gamma \tau_1 \sum_{i=1}^r \rho_i |X_{1i}|^2} \right\}
\]

Finally, the expectation with respect to the \( X_{ik} \) may be taken, which completes the proof (using the fact that the \( X_{ik} \) are independent of the \( X_{i1} \)).

Proof: [Proof: Theorem 7] We need to compute the expectation \( \frac{\partial \Psi}{\partial q_k} \) where \( W = X^tRX \), with \( X \sim \mathcal{N}_{r,2}(0, I) \). To that end, let \( u \sim \mathcal{N}_{r,1}(0, I) \) and \( v \sim \mathcal{N}_{r,1}(0, I) \) be independent Gaussian random vectors. Then \( W_{11} \sim u^tRu \) and \( W_{12} \sim u^tRv \). Noting that \( \int_0^\infty e^{-x^2} dx = 1/\sqrt{\pi} \), (which was also a key step for [48]),

\[
E = \int_0^\infty e^{-x} \mathbb{E}\left\{ \exp \left( -x \gamma \tau_1 u^tRu \right) \left( u^tRu + \gamma \tau_1 |u^tRv|^2 \right) \right\} dx
\]

\[
= \int_0^\infty e^{-x} \mathbb{E}_u \left\{ \exp \left( -x \gamma \tau_1 u^tRu \right) \left( u^tRu + \gamma \tau_1 \mathbb{E}_v \{ |u^tRv|^2 \} \right) \right\} dx
\]

\[
= \int_0^\infty e^{-x} \mathbb{E}_u \left\{ \exp \left( -x \gamma \tau_1 u^tRu \right) \left( u^tRu + \gamma \tau_1 u^tR^2u \right) \right\} dx
\]

since \( u \) and \( v \) are independent. Now define \( a_i = \gamma \tau_1 \rho_i \), let \( w_i = |u_i|^2 \) (with density \( e^{-w_i} \)). Writing out the inner products as summations and using the properties of the exponential,

\[
E = \int_0^\infty e^{-x} \mathbb{E} \left\{ \prod_{j=1}^r e^{-x_{a_j} w_j} \sum_{i=1}^r (\rho_i + \gamma \tau_1 \rho_i^2) w_i \right\} dx
\]

\[
= \int_0^\infty e^{-x} \sum_{i=1}^r (\rho_i + \gamma \tau_1 \rho_i^2) \mathbb{E} \left\{ w_i e^{-x_{a_j} w_j} \right\} \prod_{i \neq j} \mathbb{E} \left\{ e^{-x_{a_j} w_j} \right\} dx
\]

where the last line is due to the independence of the \( w_i \). Computing the expectations results in

\[
E = \sum_{i=1}^r (\rho_i + \gamma \tau_1 \rho_i^2) \int_0^\infty e^{-x} \frac{\rho_i}{1 + a_i x} \prod_{j \neq i} \frac{1}{1 + a_j x} dx
\]

\[
= \sum_{i=1}^r (\rho_i + \gamma \tau_1 \rho_i^2) \int_0^\infty e^{-x} \frac{\rho_i}{1 + a_i x} \sum_{j \neq i} \frac{a_j^{r-1}}{1 + a_j x} \prod_{k \neq j} (a_j - a_k)^{-1} dx
\]
via partial fraction expansion of the product. Exchanging the order of integration and summation and noting
\[ \int_0^{\infty} \frac{e^{-x}}{(1 + a_i x)(1 + a_j x)} \, dx = \zeta_{ij} \]
as defined in the statement of the theorem completes the proof (with a few algebraic re-arrangements).

**Proof:** We only consider entries in the upper-triangular (non-zero) part of \( d_{i<j} \). We need \( Q = \Gamma^\dagger \Gamma \) with \( \text{tr}(Q) = \sum_{i<j} (d_{ij})^2 = 1 \) and the diagonal elements of \( \Gamma > 0 \). We will minimize the negative of \( f(\nu) \) subject to
\[ k_1 = \sum_{i<j} (\nu_{ij})^2 - 1 \leq 0 \]
\[ \mu_i = -\nu_{ii} \leq 0 \]

Create a modified cost function \( J(\nu, \mu, \phi) \) to be minimized, given by
\[ J[f(\nu), \mu, \phi] = -f(\nu) + \mu k_1(\nu) + \sum_{i=1}^{t} \phi_i g_i(\nu) \]
\[ = -f(\nu) + \mu \left( \sum_{i<j} \nu_{ij}^2 - 1 \right) + \sum_{i=1}^{t} \phi_i (-\nu_{ii}) \]

We wish to find \( \min_j J[f(\nu), \mu, \phi] \). The first step is to find the conditions for the optimal point \( \nu^o \) to be a minimum. From [52, 53, 58] \( \nu^o \) must satisfy
1) \( J[f(\nu), \mu, \phi] \) is stationary at the optimal point \( \nu^o \)
2) \( \sum_i \mu_i k_i(\nu^o) = 0 \) for every constraint \( k_i(\nu) \)
3) \( \mu_i \geq 0 \) for every \( \nu_i \).
4) If \( \mu_i \neq 0 \) then constraint \( k_i(\nu) = 0 \)

From item 1,
\[ \frac{\partial L(\nu, \mu, \nu)}{\partial \nu_{ij}} = -\frac{\partial f(\nu)}{\partial \nu_{ij}} + 2 \mu \nu_{ij} - \phi_i \delta_{ij} = 0, \quad \mu, \phi_i \geq 0 \]  \hspace{1cm} (53)
where \( \delta_{ij} \) is the Kronecker Delta, \( \delta_{ij} = 1 \) for \( i = j \). Rearranging (53) gives:
\[ \frac{\partial f(\nu)}{\partial \nu_{ij}} = 2 \mu \nu_{ij}, \quad i \neq j, \mu > 0 \]  \hspace{1cm} (54)
\[ = 2 \mu \nu_{ii}, \quad \nu_{ii} > 0, \mu > 0 \]  \hspace{1cm} (55)
\[ < 0, \quad \nu_{ii} = 0 \]  \hspace{1cm} (56)

**Proof:** For a channel where \( H \) is defined by an arbitrary pdf, and the receiver has full knowledge of \( H \), whilst the transmitter has statistical knowledge, the input distribution is known to be Gaussian with certain covariance [59]. Thus it remains to find the optimal covariance \( Q^\text{opt} \) of the Gaussian input signal.

Before applying Lemma 3 we must show that \( \log \det(I + MX^\dagger XM^\dagger) \) is convex \( \cap \) on any positive definite matrix \( X \) – which implies \( \Psi(X^\dagger X, p(S)) \) is convex \( \cap \) on any positive triangular matrix as we require. Applying a variation of [55, pp.466-467].

\[ \log \det \left( I + M(\alpha A + (1 - \alpha)B)^\dagger(\alpha A + (1 - \alpha)B)M^\dagger \right) \]
\[ \geq \log \det \left( I + \alpha^2 MA^\dagger AM^\dagger + (1 - \alpha)^2 MB^\dagger BM^\dagger \right) \]
\[ = \log \det \left( \alpha I + \alpha^2 MA^\dagger AM^\dagger + (1 - \alpha)I + (1 - \alpha)^2 MB^\dagger BM^\dagger \right) \]
\[ \geq \alpha \log \det \left( I + \alpha MA^\dagger AM^\dagger \right) + (1 - \alpha) \log \det \left( I + (1 - \alpha) MB^\dagger BM^\dagger \right) \]

The result of Theorem 8 is given by applying Lemma 3 to the (convex \( \cap \) function \( f(d) = \Psi(Q = \Gamma^\dagger \Gamma, p(S)) \). The matrix \( Q \) may now be full, but remains positive semi-definite. Substituting \( X(d) = \Gamma^\dagger \Gamma \)
\[ \frac{\partial \Psi(\Gamma^\dagger \Gamma, p(S))}{\partial d_{ij}} = = E_S \left\{ \text{tr} \left[ \frac{\partial \log \det(I + SX)}{\partial X} \frac{\partial X}{\partial d_{ij}} \right] \right\} \]
\[ = E_S \left\{ \text{tr} \left[ (I + SX)^{-1} S \frac{\partial X}{\partial d_{ij}} \right] \right\} \]
\[ = E_S \left\{ \text{tr} \left[ (I + ST^\dagger \Gamma)^{-1} S \cdot \frac{\partial \Gamma^\dagger \Gamma}{\partial d_{ij}} \right] \right\} \]
Since \(d_{ij}\) and \(S\) are independent the trace, expectation and differentiation all commute, and the second line arises from application of the matrix chain rule. Observe that \(\partial f(X(t))/\partial t = \text{tr}(\partial f(X)/\partial x \cdot \partial X/\partial t)\). Define \(E^{ij}\) as the matrix of partial derivatives of \(\Gamma^\dagger \Gamma\) with respect to \(d_{ij}\). In general this matrix is full.

\[
E^{ij} = \frac{\partial (\Gamma^\dagger \Gamma)}{\partial d_{ij}} = \frac{\partial}{\partial d_{ij}} \sum_k d_{mk} d_{nj}
\]

The channel capacity is also known to be the expectation of \(\psi(Q = \Gamma^\dagger \Gamma, S = H^\dagger H)\) over \(S\), with Gaussian input [59].

Proof: The algorithm is a gradient descent algorithm on a convex problem.

Proof: The optimization may be written as

\[
C = \max_{\text{tr}(Q)=1} \sum_i E_H \{ \log(1 + \gamma \alpha_i) \}
\]

(57)

where \(\alpha_i\) is the \(i\)th largest singular value of \(S = HQH^\dagger\). Taylor expansion of (57), around \(\gamma = 0\) gives:

\[
C = \max_{\text{tr}(Q)=1} \sum_i E_H \{ \gamma \alpha_i \} = \max_{\text{tr}(Q)=1} \gamma E_H \{ \text{tr} (HQH^\dagger) \}
\]

It now remains to find the capacity achieving distribution. Note, for any Hermitian matrices \(A\) and \(B\) with eigenvalues \(a_1 \geq \cdots \geq a_n\) and \(b_1 \geq \cdots \geq b_n\),

\[
\text{tr}(AB) \leq \sum_i a_i b_i
\]

with equality if \(A\) and \(B\) are jointly diagonalizable [51]. With \(A = Q\) and \(B = E \{HH^\dagger\}\) the capacity achieving distribution diagonalizes \(E \{HH^\dagger\}\). Apply Definition 1 to give

\[
\frac{\partial I(Q, \gamma)}{\partial Q_{ii}} = \lambda_i = \mu \mid_{Q_{ii} > 0}
\]

Since we require \(\mu\) constant for all non-zero \(Q_{ii}\), the only valid solution is

\[
Q_{ii} = \begin{cases} 
1 & i = 1 \\
0 & \text{else}
\end{cases}
\]

for distinct \(\lambda_i\), which gives and substituting for (57) gives the desired result.

For \(k\) equal eigenvalues the unique solution becomes \(\mu = 1/k\), which gives the desired result.

Proof: Starting from the definition of high-SNR, note that \(I(Q, \gamma)\) is dependent on \(Q\) only through the eigenvalues of \(Q\), and not through any interaction with \(H\). Using a Lagrange-multiplier method, and differentiating\( \partial f / \partial Q_{ii}\) with respect to \(Q_{ii}\), gives:

\[
\frac{1}{Q_{ii}} = \mu \quad Q_{ii} > 0
\]

with the only solution,

\[
Q_{ii} = \frac{1}{\mu} = \frac{1}{t}
\]

Substituting in (46) gives (47).

REFERENCES

[1] L. W. Hanlen and A. J. Grant, “Optimal transmit covariance for MIMO channels with statistical transmitter side information,” in Proc. IEEE Intl. Symp. Inform. Theory, ISIT, Sept. 2005.
[2] ———, “On capacity of ergodic multiple-input multiple-output channels,” in 6th Aust. Commun. Theory Workshop, AusCTW, Brisbane, Australia, Feb. 2–4 2005, pp. 121–124.
[3] A. J. Grant, “Capacity of ergodic MIMO channels with complete transmitter channel knowledge,” in Aust. Commun. Theory Workshop, AusCTW, Brisbane, Australia, Feb. 2–4 2005, pp. 116–120.
[4] I. E. Telatar, “Capacity of multi-antenna Gaussian channels,” Euro. Trans. Telecomms., vol. 10, no. 6, pp. 585–595, Nov. 1999.
[5] G. J. Foschini and M. J. Gans, “On limits of wireless communications in a fading environment when using multiple antennas,” Wireless Personal Communications, vol. 6, pp. 311–335, 1998.
[6] T. M. Cover and J. A. Thomas, Elements of Information Theory. New York: John Wiley & Sons, 1991.
[7] R. G. Gallager, Information Theory and Reliable Communication. New York, USA: John Wiley & Sons, 1968.
[8] A. J. Grant, “Rayleigh fading multiple-antenna channels,” EURASIP J. App. Sig. Proc., Special Issue on Space-Time Coding, Part I, no. 3, pp. 316 – 329, Mar. 2002.

\(^1A = UAU^\dagger\) and \(B = UBU^\dagger\) for diagonal \(A\) and \(B\)
[49] S. A. Jafar and A. J. Goldsmith, “On optimality of beamforming for multiple antenna systems with imperfect feedback,” in Proc. IEEE Intl. Symp. Inform. Theory, ISIT, Washington DC, USA, June 2001, p. 321.

[50] H. Ozcelik, N. Czink, and E. Bonek, “What makes a good MIMO channel model?” in Proc. IEEE Vehic. Techn. Conf., VTC, 2005.

[51] R. J. Muirhead, Aspects of Multivariate Statistical Theory, ser. Wiley Series in Probability and Mathematical Statistics. New York, USA: John Wiley & Sons, Inc, 1982.

[52] S. Boyd and L. Vandenberghe, Convex Optimization. Cambridge, United Kingdom: Cambridge University Press, 2004.

[53] H. W. Kuhn and A. W. Tucker, “Nonlinear programming,” in Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, Berkeley Calif. USA, 1951, pp. 481–492.

[54] J. B. Rosen, “The gradient projection method for nonlinear programming. part II. nonlinear constraints,” J. SIAM, vol. 9, no. 4, pp. 514–532, Dec. 1961.

[55] R. A. Horn and C. R. Johnson, Matrix Analysis. Press Syndicate of the University of Cambridge, 1999.

[56] A. Lozano, A. M. Tulino, and S. Verdu, “High-SNR power offset in multi-antenna communication,” in Proc. IEEE Intl. Symp. Inform. Theory, ISIT, Chicago, USA, June 27– July 2 2004, p. 287.

[57] A. Lozano, A. M. Tulino, and A. J. Goldsmith, “High-SNR power offset in multi-antenna communication,” submitted to IEEE Trans. Inform. Theory.

[58] G. S. G. Beveridge and R. S. Schechter, Optimization: theory and practice. New York: McGraw-Hill, 1970.

[59] T. Ericson, “A Gaussian channel with slow fading,” IEEE Trans. Inform. Theory, pp. 353–355, 1970.