RANDOM MOTION ON FINITE RINGS, I: COMMUTATIVE RINGS

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Abstract. We consider irreversible Markov chains on finite commutative rings randomly generated using both addition and multiplication. We restrict ourselves to the case where the addition is uniformly random and multiplication is arbitrary. We first prove formulas for eigenvalues and multiplicities of the transition matrices of these chains using the character theory of finite abelian groups. The examples of principal ideal rings (such as $\mathbb{Z}_n$) and finite chain rings (such as $\mathbb{Z}_{p^k}$) are particularly illuminating and are treated separately. We then prove a recursive formula for the stationary probabilities for any ring. For the special case where multiplication is also uniformly random, we obtain a full description of the stationary distribution for finite chain rings, and prove that the mixing time is uniformly bounded for the rings $\mathbb{Z}_{p^k}$.

1. Introduction

Random walks on general groups are an extremely well-studied subject, and even those on finite groups have been explored in great detail, with some results appearing as early as the 1950s [Goo51]. The subject acquired a life of its own starting with the work of Diaconis and Shashahani [DS81], where probabilistic questions were answered by appealing to the representation theory of the symmetric group. See [Dia88, SC04] for generalisations in this direction.

A concept more general than a random walk is a Markov chain, wherein the probability of being in a future state depends on the past only through the present state. A random walk is then a Markov chain which has the additional property of reversibility (see Definition 4.2). In parallel with random walks on groups, there has been a growing interest in Markov chains on finite semigroups and monoids, such as the Markov chain on the symmetric group known as the Tsetlin library [Tse63, Hen72]. A far-reaching generalisation of the latter on
hyberplane arrangements \cite{BHR99} led to a systematic study of Markov chains on monoids known as left-regular bands \cite{Bro00}. This has since been extended to a more general class known as $R$-trivial monoids \cite{ASST15, Ste16}.

In a similar vein, Markov chains on finite dimensional vector spaces over finite fields generated by affine random transformations have also been studied; see for example \cite{Hil93, HM08, Asc09b, Asc09a}.

In this work, we study Markov chains on finite commutative rings generated simultaneously by both addition and multiplication operations as follows. At each step, we choose either to add or multiply the current state with an element of the ring according to a coin toss. The addition is done according to the uniform distribution on the ring, and multiplication according to an arbitrary distribution. Although we will mostly work on rings with identity, results for rings without identity can also be deduced similarly; see Remark 3.6. We will be interested in the stationary distribution of these chains and their convergence here. Results about Markov chains on noncommutative finite rings will appear in a subsequent work \cite{AS16}.

The plan of the article is as follows. We will give the basic definitions and summarise the main results in Section 2. In Section 3, we will prove a general formula for the eigenvalues (and multiplicities) of the transition matrix of the chain. This is related to the Markov chains on semigroups stated above; see the discussion after Proposition 2.2. In Section 4, we will prove the formula for the stationary distribution for general rings. Lastly, we will prove the formula for the mixing time for the finite chain ring $\mathbb{Z}_{p^k}$ in Section 5. We will end with related open questions in Section 6.

2. Definitions and summary of results

Let $R$ be a finite commutative ring with identity and let $|R|$ denote its cardinality. We will define a discrete-time Markov chain $(X_n)_{n \in \mathbb{Z}_+}$ with state space $R$ which uses the ring structure on it. The informal description of the chain is as follows. Suppose we are at a certain state $r \in R$ at some time. At the next time step, we toss a biased coin with Heads probability $\alpha$. If the coin lands Heads, we pick a uniformly random element of $R$ and add it to $r$. If it lands Tails, we pick an element from $R$ according to an arbitrary distribution and multiply it to $r$.

To describe the transition probabilities of this chain more formally, we will define a probability distribution on the product space

$$S_R = \{ (\star, r) \mid \star \in \{\times, +\}, \ r \in R \}$$
as follows. The marginal distribution on \( \star \) is given by

\[
\mathbb{P}(\star) = \begin{cases} 
\alpha & \text{if } \star = +, \\
1 - \alpha & \text{if } \star = \times,
\end{cases}
\]

where \( \alpha \in (0, 1) \) and the conditional distribution on \( R \) is

\[
\mathbb{P}(X = r | \star) = \begin{cases} 
\frac{1}{|R|} & \text{if } \star = +, \\
\beta_r & \text{if } \star = \times,
\end{cases}
\]

where \( \beta_r \in [0, 1] \) for each \( r \) and \( \sum_r \beta_r = 1 \). Let \((\star, r)\) be sampled from this distribution. We then have the following Markov chain \((X_n)_{n \in \mathbb{Z}_+}\) on the state space \( R \) given by

\[
X_{n+1} = X_n \star r.
\]

We will also consider this Markov chain where multiplication is also performed in a uniformly random manner. To distinguish the two, we will denote the latter by \((X_n^{(u)})_{n \in \mathbb{Z}_+}\). That is to say,

\[
X_{n+1}^{(u)} = X_n^{(u)} \star r,
\]

where \( \star \) is still chosen according to (2.1), but \( r \) is also chosen uniformly in \( R \) independent of \( \star \). In other words, the distribution here on \( S_R \) is a product distribution of Bernoulli(\( \alpha \)) on \{+, \times\} and the uniform distribution on \( R \).

Unless explicitly specified, we will be talking about the chain \((X_n)_{n \in \mathbb{Z}_+}\). For \( a, b \in R \), we will denote the probability of making a single-step transition from \( a \) to \( b \) in \( R \) by \( \mathbb{P}(a \rightarrow b) \). Let \( M_R = (\mathbb{P}(a \rightarrow b))_{a, b \in R} \) be the transition matrix of \((X_n)_{n \in \mathbb{Z}_+}\) using some ordering of \( R \). Thus, \( M_R \) is a row-stochastic matrix, that is, a matrix of nonnegative entries whose rows sum to 1. More precisely, let \( \mathbb{1}_m \) be the column vector of size \( m \) consisting of all 1’s and consider the matrix \( B_R = (\beta_{a, b})_{a, b \in R} \) with \( \beta_{a, b} = \sum_{x, y \in R} \beta_{x, y} \). Then

\[
M_R = \frac{\alpha}{|R|} \mathbb{1}_{|R|} \mathbb{1}_{|R|}^t + (1 - \alpha) B_R.
\]

Roughly, a Markov chain is said to be irreducible if there is a positive probability to get from any state in the chain to any other state in the future. An irreducible Markov chain is said to be aperiodic if the greatest common divisor of the set of return times to any state is 1. See [LPW09] for the precise definitions. Since each entry of \( M_R \) is nonzero, we immediately have the following result.

**Proposition 2.1.** The Markov chain \((X_n)_{n \in \mathbb{Z}_+}\) is irreducible and aperiodic.
By standard theory (see, for example, [LPW09, Theorem 4.9]), it follows that \((X_n)_{n \in \mathbb{Z}_+}\) has a unique stationary distribution (see Definition 4.1) denoted by \(\pi\). The stationary probability of an element \(x \in R\) will be denoted by \(\pi(x)\). We will consider \(\pi\) as a row-vector ordered in the same basis as for \(M_R\).

We are going to be interested in the eigenvalues of \(M_R\) and the following result tells us that we only need to consider the semigroup action on \(R\) by multiplication. Since the \(\beta_r's\) are nonnegative and sum to one, \(B_R\) has the largest eigenvalue 1 by the Perron-Frobenius theorem.

**Proposition 2.2** ([DZ07, Corollary 3.1]). Let \(\lambda_1 = 1 \geq \lambda_2 \geq \cdots \geq \lambda_{|R|}\) be the eigenvalues of \(B_R\) counted with multiplicity. Then the eigenvalues of \(M_R\) are \(\lambda_1 = 1 > (1 - \alpha)\lambda_2 \geq (1 - \alpha)\lambda_3 \geq \cdots \geq (1 - \alpha)\lambda_{|R|}\) counted with multiplicity.

In view of the above proposition, to determine eigenvalues and their multiplicities it is sufficient to consider the random walk on the semigroup \(R\) under multiplication. It is well known that eigenvalues of \(B_R\) are the same as that of the operator of the semigroup algebra \(\mathbb{C}[R]\) obtained by multiplying on the left by \(\sum_{x \in R} \beta_x x\) (see [Bro00, Section 7]). The commutativity of \(R\) implies that \(\mathbb{C}[R]\) is a basic monoid algebra. Basic semigroup algebras have already been studied by Steinberg [Ste08, Ste16]. For example, Steinberg [Ste16, Proposition 12.10] proves that eigenvalues can be determined using the fact that \(\mathbb{C}[R]\) projects onto a commutative inverse monoid algebra. However, in this article we approach the problem differently. In particular, we explore the ring structure of \(R\) which enables us to give an easy description of the eigenvalues, their multiplicities, the stationary distribution and the mixing time.

We now write down the main results. Let \(R\) be a finite commutative ring with identity. The group of invertible elements of \(R\) is denoted by \(U_R\). For \(a \in R\), let \(I_a\) denote the principal ideal generated by \(a\). Let \(\phi\) be a fixed set of generators of distinct principal ideals of \(R\).

Let \(\text{ann}(a) = \{x \in R \mid xa = 0\}\) be the annihilator of \(a\) in \(R\). For \(a \in R\), let \(Q_a = R/\text{ann}(a)\) be the quotient ring, \(U_a := U_{Q_a}\) be its unit group and \(f_a : R \to Q_a\) be the natural projection map. Define \(F_a = f_a^{-1}(U_a)\).

The set of characters of \(U_R\), that is, the group of homomorphisms from \(U_R\) to \(\mathbb{C}^\times\), is denoted by \(\widehat{U_R}\). For \(a \in \phi\), let \(\Sigma_a\) be the set of all characters of \(U_R\) that are pulled back from those of \(U_a\), that is

\[\Sigma_a = \{\chi \in \widehat{U_R} \mid \chi((1 + \text{ann}(a)) \cap U_R) = 1\} .\]
By definition of $F_a$, for every $x \in F_a$, there exists a unit $u \in U(R)$ such that $f_a(x) = f_a(u)$. Further if $u_1$ and $u_2$ are two such units then for $\chi \in \Sigma_a$, we have $\chi(u_1) = \chi(u_2)$ (see Proposition 3.3). This is the context in which we require units associated with $x \in F_a$. Henceforth for $x \in F_a$ we fix a unit, denoted $u_a(x)$, such that $f_a(u_a(x)) = f_a(x)$.

We are now in a position to describe the spectrum of the transition matrix.

**Theorem 2.3.** For every $\chi \in \Sigma_a$, we obtain an eigenvalue $\lambda_\chi$ of $B_R$ given by,

$$
\lambda_\chi = \sum_{x \in F_a} \beta_x \chi(u_a(x)).
$$

Conversely, every eigenvalue of $B_R$ is of the form $\lambda_\chi$ for some $\chi \in \Sigma_a$ for some $a \in R$. The algebraic multiplicity, $m(\lambda_\chi)$ of $\lambda_\chi$ for $\chi \in \Sigma_a$, is given by

$$
m(\lambda_\chi) = |\{b \in \phi \mid F_b = F_a \text{ and } \chi \in \Sigma_b\}|.
$$

The background and proof of Theorem 2.3 will be presented in Section 3. From this, the results for principal ideal rings in Corollary 3.10 and finite chain rings in Corollary 3.14 will follow.

We now describe probabilistic aspects of this chain. The first result is about the stationary distribution. For $x, y \in R$ such that $I_x \subseteq I_y$, denote $U_{y,x}$ as the subgroup $((1 + \text{ann}(x)) \cap U_R)/((1 + \text{ann}(y)) \cap U_R)$ of $U_y$. Recall that $\beta_{a,b} = \sum_{a \cdot x = b} \beta_x$.

**Theorem 2.4.** Let $R$ be a finite ring. The stationary distribution $\pi(x)$ for $x \in R$ of the chain $(X_n)_{n \in \mathbb{Z}^+}$ is given by

$$
\pi(x) = \frac{\alpha}{|R|} + (1 - \alpha) \sum_{y \in \phi, I_x \subseteq I_y} \frac{|U_y|}{|U_x|} \left( \sum_{u \in U_y / U_{y,x}} \beta_{f_y^{-1}(u)y,x} \right) \pi(y) \cdot

\quad \frac{1 - (1 - \alpha) \left( \sum_{r \in F_x} \beta_r \right)}{1 - (1 - \alpha) \left( \sum_{r \in F_x} \beta_r \right)}.
$$

We also obtain the the stationary distribution of $(X_n^{(u)})_{n \in \mathbb{Z}^+}$ as a corollary.

**Corollary 2.5.** Let $R$ be a finite ring. The stationary distribution $\pi(x)$ for $x \in R$ of the chain $(X_n^{(u)})_{n \in \mathbb{Z}^+}$ is given by

$$
\pi(x) = \frac{\alpha}{|R|} + \frac{1 - \alpha}{|R|} \sum_{y \in \phi, I_x \subseteq I_y} |U_y| |\text{ann}(y)| \pi(y) \cdot

\quad \frac{1 - \left( \frac{1 - \alpha}{|R|} \right) |U_x| |\text{ann}(x)|}{1 - \left( \frac{1 - \alpha}{|R|} \right) |U_x| |\text{ann}(x)|}.
$$
Computationally, Theorem 2.4 can be used recursively by going upwards along the poset of principal ideals (see Section 3). The lowest element of this poset is the set of units, and their stationary probability is given by Corollary 4.4. Theorem 2.4 and Corollary 2.5 will be proved in Section 4. Even for \((X_n^{(u)})_{n \in \mathbb{Z}_+}\), the stationary probabilities seem to be complicated for general rings. However, they become simpler for local rings (given in Corollary 4.6) and are completely described for finite chain rings in Theorem 4.8.

The mixing time for a Markov chain gives an estimate of the speed of convergence of the chain to its stationary distribution. See Section 5 for the precise definitions. We study the mixing times for finite chain rings. For simplicity, we work out the details for the ring \(\mathbb{Z}_{p^k}\), where we prove the following result.

Let the error parameter \(\epsilon_{p,k}\) be defined as

\[
\epsilon_{p,k} = \frac{1}{2(1 + (p - 1)\alpha)^k} < \frac{1}{2}
\]

(which gets smaller as \(p\) and \(k\) get larger).

**Theorem 2.6.** The mixing time of the chain \((X_n^{(u)})_{n \in \mathbb{Z}_+}\) for the ring \(\mathbb{Z}_{p^k}\) is bounded above by the absolute constant

\[
t_{mix}(\epsilon_{p,k}) \leq \frac{\log(1/4)}{\log((1-\alpha)/2)} = \log((1-\alpha)/2) \cdot \left(\frac{1}{4}\right).
\]

The following example of the ring \(\mathbb{Z}_8\) should serve to illustrate the main results described here.
Example 2.7. Let $R = \mathbb{Z}_8$. We will denote elements of the ring with bars to avoid confusion and order the elements using the natural increasing order on the integers $\{\bar{0}, \ldots, \bar{7}\}$. One can check that the multiplicative part $B_R$ of the transition matrix is given by

$$
\begin{pmatrix}
\beta_0 + \beta_1 + \beta_2 + \beta_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
+\beta_4 + \beta_5 + \beta_6 + \beta_7 & \beta_0 & \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 & \beta_7 \\
\beta_0 + \beta_4 & 0 & \beta_1 + \beta_5 & 0 & \beta_2 + \beta_6 & 0 & \beta_3 + \beta_7 & 0 \\
\beta_0 & \beta_3 & \beta_6 & \beta_1 & \beta_4 & \beta_7 & \beta_2 & \beta_5 & \beta_0 \\
\beta_0 + \beta_2 & 0 & 0 & \beta_1 + \beta_3 & 0 & 0 & 0 \\
+\beta_4 + \beta_6 & \beta_5 & \beta_2 & \beta_7 & \beta_4 & \beta_1 & \beta_6 & \beta_3 \\
\beta_0 + \beta_4 & 0 & \beta_3 + \beta_7 & 0 & \beta_2 + \beta_6 & 0 & \beta_1 + \beta_5 & 0 \\
\beta_0 & \beta_7 & \beta_6 & \beta_5 & \beta_4 & \beta_3 & \beta_2 & \beta_1 & \beta_0 \\
\end{pmatrix},
$$

and $M_R$ by $(2.5)$. The graph of multiplicative transitions is drawn in Figure 1, where each transition of a particular value has been drawn in a distinct colour.

The eigenvalues of $B_R$ are given by Theorem 2.3. Since $R$ is a finite chain ring, we can appeal directly to Corollary 3.14. Other than the trivial eigenvalue $1$ with multiplicity one, given by the table

| Eigenvalue         | Multiplicity |
|--------------------|--------------|
| $\beta_1 + \beta_3 - \beta_5 - \beta_7$ | 1            |
| $\beta_1 - \beta_3 + \beta_5 - \beta_7$ | 2            |
| $\beta_1 - \beta_3 - \beta_5 + \beta_7$ | 1            |
| $\beta_1 + \beta_3 + \beta_5 + \beta_7$ | 3            |

In the special case when $\beta_i = 1/8$ for all $i$, we get eigenvalues $1/2$ with multiplicity three and $0$ with multiplicity four. This can also be seen from Corollary 3.15. This shows that the relaxation time of the
The stationary probabilities are given by
\[\pi(\bar{0}) = \frac{1}{(1 + \alpha)^3},\]
\[\pi(\bar{4}) = \frac{\alpha}{(1 + \alpha)^3},\]
\[\pi(\bar{2}) = \pi(\bar{6}) = \frac{\alpha}{2(1 + \alpha)^2},\]
\[\pi(\bar{1}) = \pi(\bar{3}) = \pi(\bar{5}) = \pi(\bar{7}) = \frac{\alpha}{4(1 + \alpha)}.
\]

The mixing time is given by Theorem 2.6 with \(\epsilon_{2,3} = \frac{1}{2(1 + \alpha)^3}\).

Figure 1. The multiplication action of elements in \(Z_8\). Elements are grouped according to the largest principal ideals they belong to. Transitions with different probabilities are shown in different colours; see (2.6) for the values.

3. Eigenvalues and Multiplicities

The eigenvalues of the transition matrix of a Markov chain give important information about the rate of convergence of the chain to its stationary distribution. Suppose \(M\) is the transition matrix for a Markov chain \((Y_n)_{n \in \mathbb{Z}_+}\) on the finite state space \(\Omega\). The eigenvalues of \(M\) will have their real parts bounded in absolute value by
1. Let us order them in weakly decreasing order of their real parts: 
   \[1 = \lambda_1 \geq \Re(\lambda_2) \geq \cdots \geq \Re(\lambda_{|\Omega|}) \geq -1.\]

**Definition 3.1.** The spectral gap is given by \(\gamma = 1 - \Re(\lambda_2)\) and the absolute spectral gap, by \(\gamma_* = 1 - \max(|\Re(\lambda_2)|, |\Re(\lambda_{|\Omega|})|)\). The relaxation time is given by \(t_{rel} = 1/\gamma_*\).

The relaxation time is a rough estimate of the time to convergence to the stationary distribution. The mixing time is a more precise estimate, which will be discussed in Section 5. In this section we prove Theorem 2.3 that describes the eigenvalues of \(M_R\) and deduce its corollaries for principal ideal rings and finite chain rings.

Let \(R\) be a finite commutative ring with identity. Recall that for \(a \in R\), \(I_a\) denotes the principal ideal generated by \(a\) and \(\phi\) is the fixed set of generators of distinct principal ideals of \(R\). Moreover, we have an equivalence relation for \(a, b \in R\) whenever \(I_a = I_b\). We denote the set of equivalence classes under this relation by \(\Phi\). The set \(\Phi\) has a natural poset structure, where \(a < b\) if \(I_b \subsetneq I_a\). See Figure 2 for an illustration for the Galois ring \(\mathbb{Z}_4[t]/\langle t^2 \rangle\), which is not a principal ideal ring. In general, this poset is not a lattice.

From the definitions of \(I_a\) and \(\phi\) it is clear that
\[R = \bigcup_{a \in \phi} I_a.\]

For a given ideal \(I\) of \(R\), we consider the set of all ideals \(J\) such that \(J \subsetneq I\) and the set,
\[S_I = I \setminus \bigcup_{J \subseteq I} J.\]

It is easy to see that for an ideal \(I\) of \(R\), the set \(S_I\) is non-empty if and only if \(I\) is a principal ideal. When \(I\) is a principal ideal, the set \(S_I\) is precisely the set of generators of \(I\). Whenever \(I = I_a\) for some \(a \in R\), we write \(S_I\) by \(S_a\). Therefore, we obtain
\[R = \bigcup_{a \in \phi} S_a.\]

Recall that \(U_a\) denotes the group of units of the quotient ring \(Q_a = R/\text{ann}(a)\). For \(a = 0\), the ring \(Q_0 = U_0\) denotes the zero ring. Further \(F_a = f_a^{-1}(U_a)\), where \(f_a\) is the natural projection map of \(R\) onto \(Q_a\).

**Lemma 3.2.** For any element \(x \in R\), the following are equivalent.

1. \(xS_a \subseteq S_a.\)
2. \(f_a(x) \in U_a.\)
3. \(x \in F_a\)
Proof. For $a = 0$, the result is true by definition. For nonzero $a$, $xS_a \subseteq S_a$ if and only if there exists $y \in R$ such that $yxa = a$. Now $yxa = a$ if and only if $yx \in 1 + \text{ann}(a)$. This is equivalent to the fact $f_a(y)f_a(x) = 1$, which in turn is equivalent to $x \in F_a$. Therefore the result follows.

Proposition 3.3. For any $a \in R$, the following are true.

(1) There exists a 1-1 correspondence between $S_a$ and $U_a$ given by $h_a : xa \mapsto f_a(x)$.

(2) For every $x \in F_a$, there exists $u_a(x) \in U_R$ such that

\[
(3.1) \quad xz = u_a(x)z \forall z \in S_a.
\]

Further $u_a(x)$ above is unique in the sense that if $y \in U_R$ satisfies (3.1) then

\[
\chi(y) = \chi(u_a(x)) \forall \chi \in \Sigma_a.
\]

Proof. For $a = 0$, (1) is true by definition and for (2) we take $u_0(x) = 1$ and the rest follows easily. From now on, we assume $a \neq 0$. By Lemma 3.2, $xa \in S_a$ implies $f_a(x) \in U_a$. Therefore $h_a$ maps $S_a$ to $U_a$.

![Figure 2. The Hasse diagram of $\Phi$ of $\mathbb{Z}_4[t]/\langle t^2 \rangle$.](image-url)
Lemma 3.5. For $U$ and (2) follow. (3) follows from the fact that $u$ to see that if $\mathcal{S}$
there is no ambiguity, we will write elements of $\Sigma$ operator “left multiplication by $P$
Finally, (4) follows from the definitions of $P$
Proof. Let $r \in P_i \cap P_j$, which implies $ru_i y = ru_j y = x$. Then $(1 - u_i u_j^{-1}) x = 0$
by $u_i u_j^{-1} \in U_{y,x}$ and $r \in P_i$, then $r \in P_j$. From this (1)
and (2) follow. (3) follows from the fact that $U_{y,x}$ is a subgroup of $U_y$. Finally, (4) follows from the definitions of $P_i$
and $\text{ann}(y)$. 

Remark 3.4. From Proposition 3.3, the elements of $S_a$ can be written
as $ua$ such that $u \in U_R \cap F_a$ with the property that $ua = u'a$ if and only
if $f_a(u) = f_a(u') \in U_a$. From now on, to simplify notation, whenever
there is no ambiguity, we will write elements of $S_a$ by $ua$ for $u \in U_a$.

Lemma 3.5. For $x \in R$ and $y \in \phi$ and $y_i = u_i y \in S_y$, consider the
sets:

$$P_i = \{ r \in R \mid ry_i = x \}.$$ 

Then the following are true.

1. Either $P_i = P_j$ or $P_i \cap P_j = \emptyset$.
2. $P_i = P_j$ if and only if $u_i u_j^{-1} \in U_{y,x} \subseteq U_y$.
3. The relation $y_i \sim y_j$ if and only if $P_i = P_j$ partitions $S_y$
into $|U_x|$ classes of size $|U_y|/|U_x|$.
4. $|P_i| = |\text{ann}(y)|$ for all $i$.

Proof. For a given set $T$, we denote $\mathbb{C}[T]$ as the formal vector space with basis elements parametrized by $T$. In case $T$ is a group (resp. semigroup), we extend the multiplication to $\mathbb{C}[T]$ and obtain a group algebra (resp. semigroup algebra). We consider $\mathbb{C}[R]$ as a semigroup algebra with multiplication inherited from that of $R$. As mentioned in the discussion after Proposition 2.2, the eigenvalues of $B_R$ are same as that of operator “left multiplication by $\sum_{x \in R} b_x x$” in the regular representation of the semigroup algebra $\mathbb{C}[R]$. We will use this equivalence to prove Theorem 2.3.
Proof of Theorem 2.3. We order the $S_a$'s such that if $a < b$ in $\phi$, then $S_a$ occurs before $S_b$. Thus, in this ordering, $S_1$ is the first and $S_0$ is the last. By Proposition 3.3 for $x \in R$ and $at \in S_a$ we have,

$$x(at) = \begin{cases} 
    u_a(x)at \in S_a & \text{if} \ x \in F_a \\
    xat \in I_b \subseteq I_a & \text{if} \ x \notin F_a.
\end{cases}$$

By definition, the set $S_1$ coincides with the group of units of $R$ and therefore there exists a basis say $B_1 = \{v_1, v_2, \ldots, v_{|U_R|}\}$ of the group algebra $\mathbb{C}[S_1]$ such that $v_i$ are eigenvectors under the regular action of $S_1$. This implies that for each $1 \leq i \leq |U_R|$, there exists $\chi_i \in \hat{S}_1$ such that

$$u v_i = \chi_i(u) v_i, \ \forall \ u \in U_R.$$

We choose a maximal linearly independent subset of $\{av_1, av_2, \ldots, av_{|U_R|}\}$ as a subset of $\mathbb{C}[R]$. We denote this by $B_a$. This set is our required basis of the vector space $\mathbb{C}[S_a]$ for each $a \in \phi$. For any $u \in S_1$ and $av_i \in B_a$, we have

$$u a v_i = a u v_i = \chi_i(u) a v_i.$$  

Note that for any $(1 + \alpha) \in (1 + \text{ann}(a)) \cap U_R$ we have

$$a v_i = (1 + \alpha) a v_i = \chi_i(1 + \alpha) a v_i$$

implying that by considering the action of $U_R$ on $\mathbb{C}[S_a]$ given by (3.3), we obtain only those characters of $U_R$ that belong to $\Sigma_a$. Thus, combining equations (3.2) and (3.3) and the above discussion we obtain that

$$\sum_{x \in R} \beta_x x(a v_i) = \sum_{x \in F_a} \beta_x \chi_i(u_a(x)) a v_i + C$$

where $\chi \in \Sigma_a$, $C \in \mathbb{C}[I_a \setminus S_a]$ and therefore the former belongs to $\sum_{b > a} \mathbb{C}[S_b]$. Thus all eigenvalues of $B_R$ are of the form $\sum_{x \in F_a} \beta_x \chi(u_a(x))$ for some $a \in \phi$ and $\chi \in \Sigma_a$.

Further we observe that by Proposition 3.3, the set $S_a$ is in bijection with $U_a$. Thus the action of $U_R$ on $S_a$ can in fact be viewed as regular action of $U_a$ on itself. This implies that every character $\chi \in \Sigma_a$ occurs in the decomposition of $\mathbb{C}[S_a]$ as a $U_R$-space and that too exactly once. Therefore for $\chi \in \Sigma_a$ and for generic values of $\beta_x$ the algebraic multiplicity of $\lambda_\chi$ is equal to the cardinality of $b \in \phi$ such that $\lambda_\chi$ occurs in the decomposition of $\mathbb{C}[S_b]$. From the above proof, it follows that $\lambda_\chi$ occurs in the decomposition of $\mathbb{C}[S_b]$ if and only if $F_a = F_b$ and $\chi \in \Sigma_b$. This justifies the statement about the algebraic multiplicity.

Remark 3.6. Consider the Markov chain $(X_n)_{n\in \mathbb{Z}_+}$ on a finite commutative ring $R$ without identity. Proposition 2.2 is still valid and so
is the fact that eigenvalues of $B_R$ are same as those of operator of $\mathbb{C}[R]$ described as left multiplication by $\sum_{x \in R} \beta_x$. Let $m$ be the characteristic of $R$. Consider $\tilde{R} = R \times \mathbb{Z}_m$ as set with addition coordinate wise and multiplication as follows.

$$(x, a)(y, b) = (bx + ay, ab)$$

Then $\tilde{R}$ is a finite commutative ring with identity called the Dorroh extension of $R$ (see [Dor32]). The ring $R$ embeds into $\tilde{R}$ as an ideal. We consider the given probability distribution $\{\beta_x\}_{x \in R}$ as a probability distribution on $\tilde{R}$ with its support on $R$. By restricting thus action of $\sum_{x \in R} \beta_x$ on ideal $\mathbb{C}[R]$, we can extract the eigenvalues and multiplicities for transition matrix $B_R$ and therefore that of $M_R$.

**Corollary 3.7.** The sum $\sum_{x \in R} \beta_x = 1$ is an eigenvalue of $B_R$ and it occurs with multiplicity one.

**Proof.** The set $\Sigma_0$ consists of only the trivial character of $U_R$ and therefore we obtain that sum $\sum_{x \in R} \beta_x$ is an eigenvalue of $B_R$. Further $F_a = R$ if and only if $a = 0$. This implies our multiplicity result. □

**Corollary 3.8.** If $\beta_x = \frac{1}{|R|}$ for all $x \in R$, then the following are true.

1. All eigenvalues of $B_R$ are rational.
2. Any nonzero eigenvalue of $B_R$ is equal to $|F_a|/|R|$ for some $a \in \phi$.
3. The number of nonzero eigenvalues of $B_R$ is equal to the number of distinct principal ideals of $R$.

**Proof.** It is clear that (2) implies (1). For (2), let $W \subseteq U_R$ be the set of distinct coset representatives of $(1+\text{ann}(a)) \cap U_R$ in $U_R$. Then for every $x \in F_a$, there exists a unique $w \in W$ such that $u_a(x)w^{-1} \in 1 + \text{ann}(a)$. Then $\chi \in \Sigma_a$ implies $\chi(u_a(x)) = \chi(w)$. This gives that

$$\frac{1}{|R|} \sum_{x \in F_a} \chi(u_a(x)) = \frac{|\text{ann}(a)|}{|R|} \sum_{w \in W} \chi(w)$$

Further $1+\text{ann}(a)$ is in the kernel of $\chi$, and therefore $\chi$ can be viewed as character of $U_a$ satisfying $\chi(w) = \chi(f_a(w))$. The fact that $W$ consists of coset representatives gives that $f_a(w_1) \neq f_a(w_2)$ for $w_1, w_2 \in W$ whenever $w_1 \neq w_2$. Thus $\sum_{w \in W} \chi(w) = \sum_{y \in U_a} \chi(y)$ for a character $\chi$ of $U_a$. By Schur’s lemma, we have

$$\sum_{y \in U_a} \chi(y) = \begin{cases} |U_a| & \text{if } \chi = 1_U_a \\ 0 & \text{if } \chi \neq 1_U_a \end{cases}$$
Now (2) follows by observing that $|F_a| = |U_a||\text{ann}(a)|$. For (3) observe that for each $a \in \phi$, we will have exactly one nonzero eigenvalue given by $|F_a|/|R|$. \hfill \square

3.1. Principal Ideal Rings. Now we specialize to the case where $R$ is a principal ideal ring (PIR), i.e. the rings with the property that every ideal is principal. Due to their simpler ideal structure, Theorem 2.3 specializes considerably. By the structure theory of finite commutative rings $R \cong \prod_{i=1}^{r} R_i$, where each $R_i$ is a principal ideal local ring \cite{McD74}, Chapter 6, or equivalently a finite chain ring (see Section 3.2). For $x \in R$, by $x_i$ we mean the $i$’th coordinate of $x$. The element $x = (x_1, x_2, \ldots, x_r) \in R$ is also denoted by $\prod_{i=1}^{r} x_i$. Let $m_i$ be the unique maximal ideal of $R_i$ with a fixed generator $\pi_i$. We set $(m_i)^0 = R_i$. Let $k_i$ be the smallest positive integer such that $m_i^{k_i-1} \neq 0$ and $m_i^{k_i} = 0$. The set $\phi$ can be identified with the set of elements \{$(\pi_1^{a_1}, \pi_2^{a_2}, \ldots, \pi_r^{a_r}) | 0 \leq a_i \leq k_i$\}. In view of this, every ideal of $R$ is of the form $\prod_{i=1}^{r} (m_i)^{a_i}$, with $0 \leq a_i \leq k_i$ for all $1 \leq i \leq r$, generated by $a = \prod_{i=1}^{r} (\pi_i)^{a_i}$. For any $a = \prod_{i=1}^{r} \pi_i^{a_i} \in \phi$, let $s(a) = \{i \in \{1, \ldots, r\} | a_i \neq k_i\}$ denote the support of $a$. Then we denote $I_a$ by $\prod_{i \in s(a)} m_i^{a_i}$. For $T \subseteq \{1, \ldots, r\}$, define $R_T$, a subset of $R$, by a set consisting of $x \in R$ such that $x_i \in U_{R_i}$ for $i \in T$. Then $R_{s(a)} = F_a$. Further, for any $x \in F_a$, the associated unit $u_a(x)$ can be easily defined by the following.

\[
\begin{cases}
(u_a(x))_i = 1, & \text{for all } i \not\in s(a), \\
(u_a(x))_i = x_i, & \text{for all } i \in s(a).
\end{cases}
\]

The following definition is important for us.

**Definition 3.9.** For a commutative ring $R$ with identity and $\chi \in \hat{U}_R$, we say that the ideal $I$ is a conductor of $\chi$, denoted $\text{cond}(\chi)$, if $I$ is the largest ideal of $R$ such that

$$\chi((1 + I) \cap U_R) = 1.$$ 

For principal ideal rings, we obtain the following result.

**Corollary 3.10.** Let $R$ be a PIR of the form $R \cong \prod_{i=1}^{r} R_i$. For every $\chi \in \Sigma_a$, there exists an eigenvalue $\lambda_{\chi}$ of $B_R$ given by,

$$\lambda_{\chi} = \sum_{x \in R_{s(a)}} \beta_x \chi(u_a(x)),$$
and conversely every eigenvalue of \( B_R \) is of the form \( \lambda_\chi \) for some \( \chi \in \Sigma_a \) for some \( a \in R \). For generic values of \( \beta_x \) and for character \( \chi \in \Sigma_a \) such that

\[
\text{cond}(\chi) = \prod_{i=1}^{r} m_i^{b_i},
\]

the algebraic multiplicity of \( \lambda_\chi \) is \( \prod_{i \in s(a)} (k_i - b_i) \).

**Proof.** The result about eigenvalues is given by Theorem 2.3. For the algebraic multiplicity, we observe that if \( \chi \in \Sigma_a \) and \( \text{cond}(\chi) = I_b = \prod_{i=1}^{r} m_i^{b_i} \) then by definition of conductor \( \text{ann}(a) \subseteq I_b \). This in particular implies that \( b_i < k_i \) for all \( i \in s(a) \) and \( b_i = k_i \) for all \( i \notin s(a) \). Therefore, by Theorem 2.3, the algebraic multiplicity of \( \lambda_\chi \) is the same as the cardinality of \( c \in \phi \) such that \( \text{ann}(c) \subseteq I_b \) and \( s(c) = s(a) \). Therefore \( I_c = \prod_{i \in s(a)} m_i^{r_i} \) such that \( b_i \leq k_i - r_i < k_i \) for all \( i \in s(a) \). This justifies the result about algebraic multiplicity. \( \square \)

The following corollary is a direct consequence of Corollary 3.8.

**Corollary 3.11.** Let \( \beta_x = 1/|R| \) for all \( x \in R \). Then the distinct eigenvalues of \( B_R \) are given as follows: for each \( \chi \in \widehat{U}_a \) we have the eigenvalue

\[
\lambda_\chi = \begin{cases} 
\frac{|R_{s(a)}|}{|R|} & \text{if } \chi = 1u_a \\
0 & \text{if } \chi \neq 1u_a
\end{cases}
\]

Now we specialize to the principal ideal ring \( R = \mathbb{Z}_m \).

**Corollary 3.12.** Let \( R = \mathbb{Z}_m \) with \( m = p_1^{e_1}p_2^{e_2} \cdots p_r^{e_r} \) where \( p_i \)'s are distinct primes \( (p_1 < \cdots < p_r) \) and further suppose that \( \beta_x = 1/m \) for all \( x \in \mathbb{Z}_m \). Then the following are true.

1. The eigenvalues of \( B_R \) are given by

\[
\prod_{i \in T}(1 - 1/p_i) \quad \text{for } \varnothing \neq T \subseteq \{1, \ldots, r\}
\]

2. The second largest eigenvalue of \( B_R \) is \( (1 - 1/p_r) \).

3. The algebraic multiplicity of eigenvalue \( \prod_{i \in T}(1 - 1/p_i) \) is \( \prod_{i \in T} e_i \).

**Proof.** From Corollary 3.11 and by the facts that \( \mathbb{Z}_m \cong \prod_{i=1}^{k} \mathbb{Z}_{p_i} \), and \( |U_{Z_{p^e}}| = (p - 1)p^{e-1} \), we obtain (1) and (2). For (3), we note that if \( I_a \) and \( I_b \) are ideals of \( \mathbb{Z}_m \) such that \( s(a) \neq s(b) \) then \( |R_{s(a)}| \neq |R_{s(b)}| \). Now result follows by Corollaries 3.10 and 3.11. \( \square \)

**Remark 3.13.** Consider the chain \( (X_n^{(a)})_{n \in \mathbb{Z}_+} \) on \( R = \mathbb{Z}_m \) with \( m = p_1^{e_1}p_2^{e_2} \cdots p_r^{e_r} \) where \( p_i \)'s are distinct primes \( (p_1 < \cdots < p_r) \) and where
the multiplication distribution is uniform. By Corollary 3.12, the spectral gap of the chain is $1/p_r$ and the relaxation time is $p_r$.

3.2. Finite Chain Rings. In this subsection, we give eigenvalues and their algebraic as well as geometric multiplicities of the transition matrix of $B_R$ for finite chain rings $R$. Apart from this we also fix some terminology for these rings that we use in Section 4.

A commutative ring $R$ with identity is called finite chain ring if the set of its ideals form a chain under inclusion [CL73, CD73, BF02]. Let $R$ be a finite chain ring with maximal ideal $m$ such that the cardinality of the residue field $R/m$ is $q$ and the index of nilpotency of $m$ is $k$, i.e. $k$ is such that $m^{k-1} \neq 0$ but $m^k = 0$. Then the size of the ring $R$ and its unit group $U_R$ are given by $q^k$ and $(q-1)q^{k-1}$ respectively. We also note that a ring $R$ is a finite chain ring if and only if it is a finite principal ideal local ring. Hence there exists $\pi \in R$ such that $m = (\pi)$ as an ideal.

Corollary 3.14. Let $R$ be a finite chain ring with length $k$. Let $\mathcal{E}_R$ be the set of eigenvalues of $B_R$. Then $\mathcal{E}_R \setminus 1$ is in one to one correspondence with $\hat{U}_R$, with bijection from $\hat{U}_R$ to $\mathcal{E}_R \setminus 1$ given by

$$\chi \mapsto \lambda_\chi = \sum_{x \in U_R} \chi(x)\beta_x.$$ 

Further, for generic values of $\beta_x$, the geometric multiplicity of $\lambda_\chi$ is one and the algebraic multiplicity of $\lambda_\chi$ is $k - e$ where $e$ is such that $\text{cond}(\chi) = m^e$.

Proof. The result about the bijective correspondence between $\mathcal{E}_R \setminus 1$ and $\hat{U}_R$ and their algebraic multiplicity follows from Corollary 3.10. To prove the result of geometric multiplicity, we follow the notations of the proof of Theorem 2.3. Let $\chi$ has conductor $(\pi^e)$. This means that $\text{Ker}(\chi) = 1 + m^e$. Let $v \in \mathbb{C}[S_1]$ be the unique (upto scalar multiplication) vector such that

$$wv = \chi(u)v \ \forall \ u \in U_R.$$ 

Then by the definition of conductor, we have $\pi^{k-e}v = 0$ and $\pi^{k-e-1}v \neq 0$. Therefore we get that the space generated by $\{\pi^iv\}_{0 \leq i \leq k-e-1}$, say $W$, is the generalized $\lambda_\chi$-eigenspace of dimension $k - e$. We prove that restriction of $(B_R - \lambda_\chi I)|_W$ has index of nilpotency equal to $k - e$. This will prove that geometric multiplicity is equal to one. For this observe that $((B_R - \lambda_\chi I)|_W)^{k-e-1}(v)$ is a scalar multiple of $\pi^{k-e-1}v$ with the scalar being some combination of $\beta_x$. As $\beta_x$’s are generic, this scalar must be nonzero and therefore we have that the index of nilpotency is in fact $k - e$. This proves the result about geometric multiplicity. □
Corollary 3.15. Let $R$ be a finite chain ring with length $k$. When $\beta_x = 1/|R|$ for all $x \in R$, we have exactly three distinct eigenvalues given by $1$, $|U_R|/|R|$, and $0$ with multiplicities one, $k$ and $|R| - (k + 1)$ respectively.

Proof. The result follows from Corollary 3.11 and the observation that in case $R$ is finite chain ring, it has $k$ nonzero ideals and for any nonzero ideal $I_a$ of $R$, we have $R_{a(a)} = U_R$.

Now we discuss an example of $\mathbb{Z}_9$ to make above ideas clear.

Example 3.16. We write the elements of $R = \mathbb{Z}_9$ by $\{0, \ldots, 8\}$, where it is understood that addition and multiplication is modulo $9$. Then $U_R = \{1, 2, 4, 5, 7, 8\}$. Note that $U_R$ is a cyclic group of order $6$ generated by $2$. Let $\zeta$ be the sixth primitive root of unity. Define $\chi_i: U_R \to \mathbb{C}^\times$ by $\chi_i(2) = (\zeta^i)$ for $1 \leq i \leq 6$. Then $\chi_i$'s form a complete set of distinct characters of $U_R$. Here $S_0 = U_R$, $S_1 = \{3, 6\}$, $S_2 = \{0\}$. For $B_0$, we consider the following vectors in $\mathbb{C}[S_0]$.

\[
\begin{align*}
v_1 &= \bar{2} + \zeta^5 \bar{4} + \zeta \bar{8} + \bar{7} + \zeta^5 \bar{5} + \zeta \bar{1} \\
v_2 &= \bar{2} + \zeta^4 \bar{4} + \zeta^2 \bar{8} + \bar{7} + \zeta^4 \bar{5} + \zeta^2 \bar{1} \\
v_3 &= \bar{2} + \zeta^3 \bar{4} + \bar{8} + \zeta^3 \bar{7} + \bar{5} + \zeta^3 \bar{1} \\
v_4 &= \bar{2} + \zeta^2 \bar{4} + \zeta^4 \bar{8} + \bar{7} + \zeta^2 \bar{5} + \zeta^4 \bar{1} \\
v_5 &= \bar{2} + \zeta \bar{4} + \zeta^2 \bar{8} + \zeta^3 \bar{7} + \zeta^4 \bar{5} + \zeta^5 \bar{1} \\
v_6 &= \bar{2} + \bar{4} + \bar{8} + \bar{7} + \bar{5} + \bar{1}
\end{align*}
\]

Then it is easy to see that for $u \in U_R$, we have

\[w_i = \chi_i(u)v_i \; \forall \; u \in U_R \; \text{and} \; 1 \leq i \leq 6.\]

Since $\chi_i$'s are distinct characters, so the set $\{v_i\}_{1 \leq i \leq 6}$ clearly form an eigenbasis of $C[S_0]$ under the action of $U_R$. For $B_1$, observe that $3v_1$, $3v_2$, $3v_4$ and $3v_6$ are all scalar multiples of each other and $3v_3$, $3v_5$ are linearly dependent. So it is clear that $v_1 = 3v_3$ and $v_2 = 3v_6$ form required the basis of $C[S_1]$ and we obtain,

\[\bar{2}(w_1) = w_1 \; ; \; \bar{2}(w_2) = -w_2 = \chi^3 w_2.\]

Thus the only characters of $U_R$ obtained by its action on $\mathbb{C}[S_1]$ are $\chi_3$ and $\chi_6$. These are precisely the characters with conductor (3). Hence the eigenvalues $\sum_{x \in U_R} \beta_x \chi_i(x)$ for $i = 3, 6$ appear with multiplicity two and the eigenvalues $\sum_{x \in U_R} \beta_x \chi_i(x)$ for $i = 1, 2, 4, 5$ appear with multiplicity one. At last by action of $B_R$ on $\mathbb{C}[S_2]$ we obtain eigenvalue $\sum_{x \in R} \beta_x = 1$ and this clearly occurs with multiplicity one.
4. The stationary distribution

In this section, we will prove the general results for the stationary distributions of \((X_n)_{n \in \mathbb{Z}^+}\) (Theorem 2.4) and \((X_n^{(u)})_{n \in \mathbb{Z}^+}\) (Corollary 2.5). We will also write down an explicit expression for the stationary probability of units in both chains in Corollary 4.4 and Corollary 4.5 respectively. We will also deduce the formula for local rings for the chain \((X_n^{(u)})_{n \in \mathbb{Z}^+}\) in Corollary 4.6. We will give the complete formula for finite chain rings in Section 4.1. We first begin with the relevant definitions.

More details can be found, for example, in [LPW09]. Let \((Y_n)_{n \in \mathbb{Z}^+}\) be a discrete time Markov chain on the space \(\Omega\) with transition matrix \(M\).

Definition 4.1. The stationary distribution of the Markov chain \((Y_n)_{n \in \mathbb{Z}^+}\) is the row-vector \(\pi\) satisfying \(\pi M = \pi\) whose entries sum to 1.

Definition 4.2. A Markov chain \((Y_n)_{n \in \mathbb{Z}^+}\) is said to be reversible if, for any two states \(x, y \in \Omega\), its stationary distribution \(\pi\) satisfies

\[
\pi(x) P(x \to y) = \pi(y) P(y \to x).
\]

Proposition 4.3. Let \(R\) be a ring and \(I\) be an ideal in \(R\). For \(a, b \in S_I\), the stationary probabilities of the chain \((X_n)_{n \in \mathbb{Z}^+}\) satisfy \(\pi(a) = \pi(b)\).

Proof. This follows from the existence of an automorphism \(u \in U_R\) from Remark 3.4 which takes \(a \mapsto b = ua\). Then, for any ideal \(J\) and any \(x \in S_J\), there exists a \(y \in S_J\) (for example, \(y = ux\)) such that \(\beta_{x,a} = \beta_{y,b}\). \(\square\)

Proof of Theorem 2.4. By the uniqueness of the stationary distribution (see Proposition 2.1), it suffices to solve the so-called master equation,

\[
(4.1) \quad \pi(x) = \sum_{y \in R} P(y \to x) \pi(y).
\]

Every element \(y\) in \(R\) can make a transition to \(x\) by the addition of \(x - y\) with probability \(\alpha/n\). This is the unique transition by addition. We now split the above sum on the right hand side in two parts according to whether \(y\) can make a multiplicative transition to \(x\) or not. Let \(R_{y,x} = \{r \in R \mid yr = x\}\). If \(I_y \cap I_x \neq I_x\), then there is no such transition and if \(I_x \subseteq I_y\), there is one transition for each element in \(R_{y,x}\). This gives

\[
\pi(x) = \sum_{y \in R \atop I_x \subseteq I_y} \left(\frac{\alpha}{|R|} + (1 - \alpha) \beta_{y,x}\right) \pi(y) + \sum_{y \in R \atop I_y \cap I_x \neq I_x} \frac{\alpha}{|R|} \pi(y).
\]
Combining the first term from the first sum and the second sum gives

\[ \pi(x) = \alpha \frac{1}{|R|} + (1 - \alpha) \sum_{y \in R, I_x \subseteq I_y} \beta_{y,x} \pi(y). \]

We now split the second sum according to whether \( I_y \) equals \( I_x \) or not. Then, using Proposition 4.3, we obtain

\[ \pi(x) = \alpha + (1 - \alpha) \left( \pi(x) \sum_{x' \in R, I_x = I_{x'}} \beta_{x',x} + \sum_{y \in R, I_x \subsetneq I_y} \beta_{y,x} \pi(y) \right). \]

By Lemma 3.5, parts (1) and (2), we can restrict the \( y \)-sum to be over \( \phi \) and collect coset representatives in \( U_y/U_{y,x} \) to account for all the terms. By part (3), the number of times each representative occurs is \( |U_y|/|U_x| \), leading to the identity

\[ \sum_{y \in R, I_x \subseteq I_y} \beta_{y,x} = \sum_{y \in \phi, I_x \subseteq I_y} \frac{|U_y|}{|U_x|} \sum_{u \in U_y/U_{y,x}} \beta_{f_y^{-1}(u)y,x}. \]

When \( y \in S_x \), \( U_{y,x} \) is trivial and the sets \( P_i \) in Lemma 3.5 are disjoint and form a partition of \( F_x \), giving

\[ \sum_{x' \in R, I_x = I_{x'}} \beta_{x',x} = \sum_{r \in F_x} \beta_r. \]

Combining these elements and simplifying leads to the desired result.

\[ \square \]

**Proof of Corollary 2.5.** From Lemma 3.5 parts (3) and (4), when \( \beta_x = 1/|R| \) for all \( x \in R \), we obtain Theorem 2.4

\[ \sum_{u \in U_y/U_{y,x}} \beta_{f_y^{-1}(u)y,x} = |\text{ann}(y)||U_x|. \]

Finally, from the definition of \( F_x \), it is clear that \( |F_x| = |\text{ann}(x)||U_x| \), completing the proof.

\[ \square \]

Theorem 2.4 and Corollary 2.5 can be used to calculate the stationary probability of \( x \in R \) using the poset of principal ideals. The difficulty in the calculation depends on the height of \( I_x \) in this poset. The easiest stationary probabilities to calculate are those of units, while the hardest is that for the zero element.
Corollary 4.4. The stationary probability of \( x \in U_R \) in the chain \((X_n)_{n \in \mathbb{Z}^+}\) is given by
\[
\pi(x) = \frac{\alpha}{|R| \left( \sum_{y \not\in U_R} \beta_y + \alpha \sum_{y \in U_R} \beta_y \right)}.
\]

Proof. Since \( I_x = R \), the sum in the numerator of Theorem 2.4 is empty and \( F_x = U_R \). \( \square \)

The following corollary is then immediate.

Corollary 4.5. The stationary probability of \( x \in U_R \) in the chain \((X_n^{(u)})_{n \in \mathbb{Z}^+}\) is given by
\[
\pi(x) = \frac{\alpha}{|R| - |U_R|(1 - \alpha)}.
\]

For local rings, Corollary 2.5 simplifies to the following.

Corollary 4.6. Let \( R \) be a finite local ring. Then the stationary probability \( \pi(x) \) for \( x \in R \) in the chain \((X_n^{(u)})_{n \in \mathbb{Z}^+}\) is given by
\[
\pi(x) = \frac{\alpha}{|R| + (1 - \alpha)|U_R|} \sum_{y \in \phi, I_x \subset I_y} \pi(y) \left( 1 - \left( \frac{(1 - \alpha)|U_R|}{|R|} \right) \right).
\]

Proof. For a local ring,
\[
|U_x| = \frac{|U_R|}{|1 + \text{ann}(x)| \cap |U_R|} = \frac{|U_R|}{|\text{ann}(x)|}.
\]
which implies \(|\text{ann}(x)||U_x| = |U_R|\) for all \( x \in R \). \( \square \)

Remark 4.7. Although the stationary distribution has a simple product structure, note that the Markov chains \((X_n)_{n \in \mathbb{Z}^+}\) and \((X_n^{(u)})_{n \in \mathbb{Z}^+}\) are not reversible (see Definition 4.2). We illustrate this by comparing the stationary probabilities of the entries 0 and 1 in a finite chain ring for \((X_n^{(u)})_{n \in \mathbb{Z}^+}\). Using Corollary 4.5, the ratio of the transitions between 1 and 0 are given by
\[
\frac{\mathbb{P}(0 \to 1)}{\mathbb{P}(1 \to 0)} = \frac{\alpha/|R|}{\alpha/|R| + (1 - \alpha)\beta_0} = \frac{\alpha}{\alpha + |R|(1 - \alpha)\beta_0}.
\]
but this is not equal to the ratio \( \pi(1)/\pi(0) \).
4.1. **Finite chain rings.** It turns out that the stationary distribution can be described completely in the case of finite chain rings. We refer to Section 3.2 for terminology on finite chain rings. The poset of ideals of $R$ is a chain of height $k$. Every nonzero element $x$ in $R$ belongs to some $S_i$ for $0 \leq i \leq k$.

**Theorem 4.8.** The stationary distribution $\pi(x)$ for $x \in R$ in the chain $(X_n(u))_{n \in \mathbb{Z}^+}$ is given by

\[
\pi(x) = \begin{cases} 
\frac{\alpha}{q^{k-1}} (1 + (q - 1) \alpha)^{i+1}, & \text{if } x \in S_i \text{ with } i < k, \\
\frac{1}{(1 + (q - 1) \alpha)^k}, & \text{if } x = 0.
\end{cases}
\]

**Proof.** Since finite chain rings are also local, we use Corollary 4.6. In this case, $\phi$ can be identified with $\{0, \ldots, k\}$ with 0 corresponding to units and $k$ to the zero element. For $i, j \in \phi$, $I_i \subsetneq I_j$ if and only if the corresponding integers satisfy $j < i$. The case $i = 0$ is already covered by Corollary 4.3. We prove the other cases for $i \leq k - 1$ by induction. We obtain, for $x \in S_i$,

\[
\pi(x) = \frac{\alpha + (1 - \alpha) u \sum_{j<i, y \in S_j} \pi(y)}{q^{k-1} (1 + (q - 1) \alpha)},
\]

\[
= \pi(1) + (1 - \alpha)(q - 1) \sum_{j=0}^{i-1} \frac{\alpha}{q^{k-j-1} (1 + (q - 1) \alpha)^{j+1}},
\]

by the induction assumption. This is now a geometric series, which is easily summed to obtain the desired result. The case of $\pi(0)$ can be then explicitly evaluated again using Corollary 4.6. \qed

5. **Mixing Time**

As described in Section 4, irreducible and aperiodic Markov chains converge to their unique stationary distribution. In this section, we will be interested in the speed of this convergence. It is well-known (see, for example [LPW09, Theorem 4.9]) that the convergence is exponentially fast. But we would like to know how the constant in the exponent scales with the size of the ring. Throughout this Section, we will be focussing on the chain $(X_n(u))_{n \in \mathbb{Z}^+}$. Computing the mixing time even for these chains over general rings seems to be a difficult problem. We will focus on the case of finite chain rings. For concreteness, we will stick to the specific rings $\mathbb{Z}_{p^k}$, where $p$ is prime and $k$ a positive integer, although these can be extended to general finite chain rings without too much difficulty.
We begin with the relevant definitions. Define a natural metric on the space of probability distributions on $\Omega$ as follows.

**Definition 5.1.** The total variation distance between two probability distributions $\mu$ and $\nu$ on $\Omega$ is given by

$$||\mu - \nu||_{TV} = \frac{1}{2} \sum_{x \in \Omega} |\mu(x) - \nu(x)|.$$ 

Suppose we start the Markov chain at some $x \in \Omega$. Then we obtain for each $n \in \mathbb{N}$, a probability distribution on $\Omega$ simply by evolving the chain, which we call $M^n(x, \cdot)$. We will denote the distance at time $n$ between this distribution, maximised over $x$, and $\pi$ by

$$(5.1) \quad d(n) = \max_{x \in \Omega} ||M^n(x, \cdot) - \pi(\cdot)||_{TV}.$$ 

Fix an $\epsilon < 1/2$ for technical reasons.

**Definition 5.2.** The mixing time of a Markov chain $(Y_n)_{n \in \mathbb{Z}^+}$ with stationary distribution $\pi$ is given by

$$t_{\text{mix}}(\epsilon) = \min \{ n \mid d(n) \leq \epsilon \},$$

where $d(n)$ is defined in (5.1).

Roughly speaking, the mixing time is at least as large as the relaxation time (see Definition 3.1. The precise apriori bounds for reversible chains are given in [LPW09, Theorems 12.3 and 12.4].

For reversible Markov chains (see Definition 4.2), there are an abundance of techniques to compute the mixing time [AF02, LPW09]. As we have shown (see Remark 4.7), $(X_n)_{n \in \mathbb{Z}^+}$ is not reversible. However, we will use the knowledge of the spectrum of $M_R$ from Section 3 to estimate the mixing time in special cases.

We will now prove that $(X^{(n)}_n)_{n \in \mathbb{Z}^+}$ on the integer ring $\mathbb{Z}_{p^k}$ mixes in finite time. Just as in Example 3.16, the elements of $\mathbb{Z}_{p^k}$ will be denoted by $\{\bar{x} \mid x \in \{0, \ldots, p^k - 1\}\}$. From Corollary 3.15 we know that $M_{p^k}$ has only three distinct eigenvalues, given by $1, (1 - \alpha)(p - 1)/p, 0$ with algebraic multiplicities $1, k, p^k - k - 1$ respectively. Let $\pi$ be the vector encoding the stationary distribution of the Markov chain, see Definition 4.1.

**Proposition 5.3.** Let $\beta_x = 1/p^k$ for all $x \in \mathbb{Z}_{p^k}$. The vector $\sigma$ given by

$$\sigma_t = \begin{cases} -(p - 1) & t = 0, \\ 1 & p^{-1}|t|, \\ 0, & \text{otherwise}. \end{cases}$$
is an eigenvector of $M_{p^k}$ corresponding to the second-largest eigenvalue $(1 - \alpha)(p - 1)/p$.

**Proof.** From Proposition 2.2 it suffices to show that $\sigma$ is an eigenvector of $B_{p^k}$ with eigenvalue $(p - 1)/p$. We write $\sigma$ and $B_{p^k}$ as elements in $\mathbb{C}[\mathbb{Z}_{p^k}]$ as

$$\sigma = -(p - 1)0 + \sum_{i=1}^{p-1} p^{k-1}i, \quad B_{p^k} = \frac{1}{p^k} \sum_{j=0}^{p^k-1} j.$$

It is then an easy exercise to check that

$$B_{p^k} \cdot \sigma = \frac{p - 1}{p} \sigma.$$

Lastly, noting that $\sigma \cdot 1_{p^k} = 0$ completes the proof. \qed

We will compute the mixing time for $(X_n^{(u)})_{n \in \mathbb{Z}_+}$ starting with a special initial condition. To specify the result, we will need some terminology. We say that a random variable $X$ is distributed according to the censored geometric distribution with parameters $(n, u)$ if $X$ is defined on $\{0, \ldots, n - 1\}$ and has probability mass function given by

$$P(X = i) = \begin{cases} u^i (1 - u) & i < n - 1, \\ u^{n-1} & i = n - 1, \end{cases}$$

where $0 \leq u \leq 1$. Let the initial state $X_0$ of the Markov chain be distributed according to the censored geometric distribution with parameters $(k, 1/(1 + (p - 1)\alpha))$ on the set $\{p^i| i = 0, \ldots, k - 1\}$ in the natural way. That is, the vector $V$ representing the initial configuration is given by

$$V_t = \begin{cases} \frac{(p - 1)\alpha}{(1 + (p - 1)\alpha)^{i+1}} & t = p^i, i \neq k - 1, \\ \frac{1}{(1 + (p - 1)\alpha)^{k-1}} & t = p^{k-1}, \\ 0, & \text{otherwise}. \end{cases}$$

The following statement shows that the vector $V$ belongs to the eigenspace of $M_{p^k}$.

**Proposition 5.4.** Let $\beta_x = 1/p^k$ for all $x \in \mathbb{Z}_{p^k}$. The initial distribution satisfies

$$V = \pi + \frac{1}{(p - 1)(1 + (p - 1)\alpha)^k} \sigma + \tau,$$

where $\tau$ is an eigenvector of $M_{p^k}$ with eigenvalue 0.
Proof. Let \( \tau \) be defined by
\[
\tau = V - \pi - \frac{1}{(p - 1)(1 + (p - 1)\alpha)^k} \sigma.
\]
Since every eigenvector of \( M_p^k \) other than \( \pi \) has row sum equal to zero, it follows that \( \tau \cdot 1_p = 0 \). Hence, it suffices to show that \( \tau \cdot B_p^k = 0 \). From (2.5), we have
\[
\pi \cdot B_p^k = \frac{1}{1 - \alpha} \left( \pi - \frac{\alpha}{p^k} \mathbb{1}_{p^k}^{tr} \right),
\]
from which it suffices to check that
\[
V \cdot B_p^k + \frac{\alpha}{p^k(1 - \alpha)} \mathbb{1}_{p^k}^{tr} - \frac{1}{p(1 + (p - 1)\alpha)^k} \sigma = \frac{1}{1 - \alpha} \pi.
\]
Since the entries of \( V \) are supported at the powers of \( p \) and those of \( \sigma \) are supported at multiples of \( p^{k-1} \) (including zero), we will need to consider three cases: elements in \( S_j \) for \( 0 \leq j < k - 1 \), in \( S_{k-1} \) and the zero element. We will go over the computation in the first case and leave the other two cases to the reader.

We will perform this computation in the semigroup algebra, exactly as in the proof of Proposition 5.3. From Proposition 4.3, it suffices to look at the coefficient of \( \bar{p}^j \) for \( 0 \leq j \leq k - 1 \). For the first term in the left hand side of (5.2), note that every term of the form \( p^i \) when multiplied by \( p^j \) gives \( p^{i+j} \). The third term does not contribute. Hence, the left hand side of (5.2) simplifies to
\[
\sum_{i=0}^{j} \frac{p^j(p - 1)\alpha}{p^k(1 + (p - 1)\alpha)^{i+1}} + \frac{\alpha}{p^k(1 - \alpha)^{j+1}}.
\]
The sum can be easily computed and after some simplification, we obtain
\[
\frac{\alpha}{(1 - \alpha)p^{k-j-1}(1 + (p - 1)\alpha)^{j+1}},
\]
which can be seen to be equal to the right hand side of (5.2) by comparing with Theorem 4.8. The computations in the other cases involve \( \sigma \) and are similar. \( \square \)

To compute the mixing time, we will choose the total variation distance \( d_n = ||VM_p^n - \pi|| \) and the error parameter to be
\[
\epsilon_{p,k} = \frac{1}{2(1 + (p - 1)\alpha)^k} < \frac{1}{2}.
\]
Notice that the error gets smaller as \( p \) and \( k \) get larger.
Proof of Theorem 2.6. Using Proposition 5.4,

\[ VM^n = \pi M^n + \frac{1}{(p-1)(1 + (p-1)\alpha)^k} \sigma M^n + \tau M^n, \]

\[ = \pi + \frac{1}{(p-1)(1 + (p-1)\alpha)^k} \left( \frac{(1 - \alpha)(p-1)}{p} \right)^n \sigma. \]

Using Proposition 5.3 it is immediate that \(|\sigma| = 2(p-1)|. Thus, the total variation distance is

\[ d_n = \frac{2}{(1 + (p-1)\alpha)^k} \left( \frac{(1 - \alpha)(p-1)}{p} \right)^n. \]

The exact expression for the mixing time is then

\[ t_{\text{mix}}(\epsilon_{p,k}) = \frac{\log(1/4)}{\log \left( \frac{(p-1)(1-\alpha)}{p} \right)}, \]

and the absolute bound follows from the fact that \((p-1)/p \geq 1/2. \]

6. Open Questions

In this work, we have studied algebraic and probabilistic properties of a natural Markov chain on a finite commutative ring, where the addition probabilities are uniformly random and the multiplication probabilities are arbitrary. A natural generalisation of these results would be to cases where the addition probabilities are more general. Preliminary experimentation suggests that a result of the flavour of Theorem 2.3 will not hold if the addition probabilities are also completely arbitrary, but one can conceivably come up with an addition distribution that is not uniform where the eigenvalues continue to be simple functions of the parameters.

With regard to the stationary distribution for \( (X^n)_{n \in \mathbb{Z}_+} \), several questions remain unanswered. In particular, one can consider the least common denominator of the stationary probabilities, informally called the partition function. For instance, the partition function for the finite chain rings studied in Section 4.1 is given, using Theorem 4.8, by

\[ q^{k-1}(1 + (q - 1)\alpha)^k. \]

In all the cases that we have looked at, the partition function factorises completely in terms of factors linear in \( \alpha \). Why this factorisation happens is an open question. A natural class of rings for which more refined results should be available are the integer rings \( \mathbb{Z}_m \). In the case of squarefree integers, we have the following empirical observation. Suppose \( m = p_1 \cdots p_k \), where \( p_i \)'s are primes. For \( S \subset \{1, \ldots, k\}, \)
let \( m_S = \prod_{i \in S} p_i \) and \( u_S = \prod_{i \in S} (p_i - 1) \). Then the partition function for \((X_n^{(u)})_{n \in \mathbb{Z}_+}\) on \( \mathbb{Z}_m \) seems to be

\[
\prod_{\emptyset \neq S \subseteq \{1, \ldots, k\}} (m_S - u_S + u_S \alpha).
\]

The transition graph of \((X_n)_{n \in \mathbb{Z}_+}\) on any ring \( R \) is always a complete graph on \(|R|\) vertices simply because rings are groups under addition. This suggests that these chains mix very fast, i.e. in \( o(|R|) \) time. We have shown this in strong sense for the restricted setting of the finite chain rings \( \mathbb{Z}_{p^k} \) in Theorem 2.6, but the above argument suggests that such a result should be true in much more generality. It is natural to ask the following question: for which families of rings does \((X_n)_{n \in \mathbb{Z}_+}\) or even \((X_n^{(u)})_{n \in \mathbb{Z}_+}\) mix in uniformly bounded time (i.e. independent of the size of the ring)?

Lastly, analogous results about these Markov chains on noncommutative rings will appear in a future work [AS16].

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