Scheme Independence of the Effective Hamiltonian
for $b \rightarrow s \gamma$ and $b \rightarrow s g$ Decays

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Abstract

We present a calculation of the effective weak Hamiltonian which
governs $b \rightarrow s \gamma$ and $b \rightarrow s g$ transitions in two different renormal-
ization schemes (NDR and HV). In the leading logarithmic approxi-
mation, we show that the coefficients of the effective Hamiltonian are
scheme independent only when one takes correctly into account the
scheme dependence of one- and two- loop diagrams. We demonstrate
that in NDR there are contributions which were missed in previous
calculations. These contributions are necessary to obtain scheme in-
dependent coefficients in the final results.
1 Introduction

Radiative decays of B mesons have been the subject of several investigations in the last years \cite{1-3}. With the exception of two recent papers by M. Misiak \cite{8, 9} and \cite{10} however, none of these studies has taken into account the full set of leading logarithmic corrections. Moreover calculations made in the dimensional reduction scheme (DRED) \cite{2, 3, 4} seem to differ from the results obtained in the naive dimensional regularization scheme (NDR) \cite{1, 5, 6, 8, 9}.

In refs.\cite{5} and \cite{8}, the calculation of refs.\cite{1, 6} was repeated in NDR and found in agreement with the original calculation. However the origin of the difference between the results obtained in NDR and DRED has never been clarified. In the literature it is always implicitely (or explicitely) assumed that the anomalous dimension matrix which enters in $b$ radiative decays is regularization scheme independent, as usually happens in leading logarithmic calculations. Thus, bearing in mind possible errors in the calculation, the difference between NDR and DRED has remained so far a mistery.

In this paper we have computed the anomalous dimension matrix relevant for $b \to s \gamma$ and $b \to s g$ decays in two different regularization schemes: NDR and the t’Hooft-Veltman scheme (HV) \cite{11}. We show that the values of some diagrams are indeed regularization scheme dependent. We explain the origin of the scheme dependence and show that the regularization scheme independent effective Hamiltonian is obtained only by taking into account the scheme dependence of some one loop diagrams, which appear at zero order in $\alpha_s$. The contributions of these diagrams were included in the calculation of Misiak \cite{8, 9}, but their role was not fully understood. We show that the easiest way to obtain the scheme independent result is to use the HV regularization scheme. We also briefly discuss the role of “effervescent” operators and of operators which vanish by the equations of motion. Finally we add some comments on the gauge invariance of the final result.
2 Evolution and Initial Conditions for the Coefficients

The effective Hamiltonian for $b \rightarrow s \gamma, g$ decays can be written as:

$$H_{\text{eff}} = V_{tb}V_{ts}^* \frac{G_F}{\sqrt{2}} \sum_{i=1}^{8} Q_i(\mu)C_i(\mu) \sim \vec{Q}^T(\mu)\vec{C}(\mu)$$ (1)

where $V_{ij}$ are the elements of the CKM mixing matrix. The operator basis is given by:

$$Q_1 = (\bar{s}_\alpha c_\beta)_{(V-A)}(\bar{c}_\beta b_\alpha)_{(V-A)}$$
$$Q_2 = (\bar{s}_\alpha c_\alpha)_{(V-A)}(\bar{c}_\beta b_\beta)_{(V-A)}$$
$$Q_{3,5} = (\bar{s}_\alpha b_\alpha)_{(V-A)} \sum_{q=u,d,s,\ldots} (\bar{q}_\beta q_\beta)_{(V \mp A)}$$
$$Q_{4,6} = (\bar{s}_\alpha b_\beta)_{(V-A)} \sum_{q=u,d,s,\ldots} (\bar{q}_\beta q_\alpha)_{(V \mp A)}$$
$$Q_7 = \frac{Q_d\epsilon}{16\pi^2} m_b \bar{s}_\alpha \sigma_{(V+A)}^{\mu\nu} b_\alpha F_{\mu\nu}$$
$$Q_8 = \frac{g}{16\pi^2} m_b \bar{s}_\alpha \sigma_{(V+A)}^{\mu\nu} t_A^{\alpha\beta} b_\beta G_{\mu\nu}$$ (2)

In eq.(2) the subscript $(V \mp A)$ indicates the chiral structure, $\alpha, \beta$ are colour indices, $m_b$ denotes the b quark mass, $Q_d = -\frac{1}{3}$ is the electric charge of the down-type quarks and $g$ ($\epsilon$) is the strong (electro-magnetic) coupling. Our colour matrices are normalized in such a way that $Tr(t^A t^B) = \delta^{AB}/2$.

The coefficients $\vec{C}(\mu)$ obey the renormalization group equations:

$$\left( -\frac{\partial}{\partial t} + \beta(\alpha_s) \frac{\partial}{\partial \alpha_s} - \frac{\hat{\gamma}^T(\alpha_s)}{2} \right) \vec{C}(t, \alpha_s(t)) = 0$$ (3)

where $t = \ln(M_W^2/\mu^2)$ and $\alpha_s = g^2/4\pi$. The factor of 2 in eq.(3) normalizes the anomalous dimension matrix as in refs.[12]-[17]. $\hat{\gamma}^T$ includes the contribution due to the renormalization of $m_b$ and, in the case of $Q_8$, the contribution due to the renormalization of the gluon field and of the strong coupling constant $g$, see e.g. ref.[18, 16].

The anomalous dimension matrix is “almost” triangular, in the sense that the operators of dimension six can mix with the magnetic operators $Q_7$.
and $Q_8$, whereas the dimension five operators $Q_{7,8}$ can only mix between themselves. For later convenience we introduce a reduced $6 \times 6$ anomalous dimension matrix, $\hat{\gamma}_r$, which mixes the operators $Q_1, Q_2, \ldots, Q_6$ among themselves and a 6-component column vector $\vec{C}_r(\mu) = (C_1(\mu), C_2(\mu), \ldots, C_6(\mu))$ on which $\hat{\gamma}_r$ acts. We also introduce two 6-component vectors, related to the two-loop anomalous dimension matrix, $\vec{\beta}_7 = (\gamma_{71}, \gamma_{72}, \ldots, \gamma_{76})$ and $\vec{\beta}_8 = (\gamma_{81}, \gamma_{82}, \ldots, \gamma_{86})$.

In terms of these quantities the renormalization group equations can be written as:

$$2\mu^2 \frac{d}{d\mu^2} \vec{C}_r(\mu) = \frac{\alpha_s}{4\pi} \hat{\gamma}_T \vec{C}_r(\mu)$$

$$2\mu^2 \frac{d}{d\mu^2} C_7(\mu) = \frac{\alpha_s}{4\pi} \left( \vec{\beta}_7 \cdot \vec{C}_r(\mu) + \gamma_{77} C_7(\mu) + \gamma_{87} C_8(\mu) \right)$$

$$2\mu^2 \frac{d}{d\mu^2} C_8(\mu) = \frac{\alpha_s}{4\pi} \left( \vec{\beta}_8 \cdot \vec{C}_r(\mu) + \gamma_{88} C_8(\mu) \right)$$

(4)

where $\alpha_s = \alpha_s(\mu)$ and $\mu^2 d/d\mu^2 = \mu^2 \partial/\partial \mu^2 + \beta(\alpha_s) \partial/\partial \alpha_s$. One can easily diagonalize the sub-matrix of magnetic operators and write the renormalization group equations as:

$$2\mu^2 \frac{d}{d\mu^2} \vec{C}_r(\mu) = \frac{\alpha_s}{4\pi} \hat{\gamma}_r \vec{C}_r(\mu)$$

$$2\mu^2 \frac{d}{d\mu^2} v_7(\mu) = \frac{\alpha_s}{4\pi} \gamma_{77} v_7(\mu)$$

$$2\mu^2 \frac{d}{d\mu^2} v_8(\mu) = \frac{\alpha_s}{4\pi} \gamma_{88} v_8(\mu)$$

(5)

where:

$$v_7(\mu) = C_7(\mu) + \vec{\alpha}_7 \cdot \vec{C}_r(\mu) + \frac{\gamma_{87}}{\gamma_{77} - \gamma_{88}} C_8(\mu)$$

$$v_8(\mu) = C_8(\mu) + \vec{\alpha}_8 \cdot \vec{C}_r(\mu)$$

(6)

with

$$\vec{\alpha}_7 = \left( \gamma_{77} \hat{1} - \hat{\gamma}_r \right)^{-1} \left[ \vec{\beta}_7 + \frac{\gamma_{87}}{\gamma_{77} - \gamma_{88}} \vec{\beta}_8 \right]$$

$$\vec{\alpha}_8 = \left( \gamma_{88} \hat{1} - \hat{\gamma}_r \right)^{-1} \vec{\beta}_8$$

(7)
Figure 1: One loop diagrams which mix $Q_5$ and $Q_6$ with $Q_7$ and $Q_8$. The mixing appears at order $\alpha_s^0$ and the coefficients of the mixing are regularization scheme dependent.

Contrary to the claims of refs. [1]-[5], $\vec{\beta}_7$ and $\vec{\beta}_8$ do depend on the regularization, as our calculations in NDR and HV explicitly show, see below. A regularization dependent anomalous dimension matrix in a leading logarithmic (but two loop) calculation is an exotic phenomenon and deserves some explanations. At one loop, but at order $\alpha_s^0$, the operators $Q_5$ and $Q_6$ can mix with $Q_7$ and $Q_8$ with finite coefficients, through the diagrams shown in fig.1. In the figure, the cross denotes a mass ($m_b$) insertion in the heavy quark propagator of the penguin loop. The coefficients of the mixing are regularization dependent. They vanish in HV (and DRED), but not in NDR. As a consequence, the values of all two loop diagrams contributing to $\vec{\beta}_7$ and $\vec{\beta}_8$, which have as sub-diagrams those in fig.1, will depend on the regularization. Furthermore the diagrams in fig.4, which vanish in HV, have instead to be considered in NDR.

An example of a diagram which depends on the regularization is reported in fig.2. The integration over the penguin loop, where a bare four fermion operator is inserted, will transform it in an effective $s\gamma^\mu_L D^\nu t^A G^A_{\mu\nu} b$ local vertex ($\gamma^\mu_{R,L} = \gamma^\mu (1 \pm \gamma_5)$). This effective vertex is, by the equations of motion, equivalent to $s\gamma^\mu_t t^A b \sum_q \frac{1}{2} \left( q_{\gamma_\mu L} t^A q + q_{\gamma_\mu R} t^A q \right)$, which can be written as a linear combination of $Q_3, \ldots, Q_6$. Thus, the divergent part of the coefficient of the effective vertex combines with the finite, regularization dependent, mixing coefficient of $Q_{5,6}$ with $Q_{7,8}$. This means that it will give a scheme
Figure 2: Two loop diagram contributing to the mixing of the operators $Q_{1,6}$ with $Q_7$ and $Q_8$. The value of the diagram is regularization scheme dependent. This figure explains the scheme dependence of the diagram. The integration over the penguin loop mixes the original operator $Q_i$ with $\bar{s}_L \gamma^\mu D^\nu G^A_{\mu\nu} t^A b$, denoted by a big square. By the equations of motion, this operator is equivalent to a four fermion operator, denoted by a small square in the figure, see also the text.

dependent contribution to $\tilde{\beta}_7$ and $\tilde{\beta}_8$.

We now show how, from the finite one loop coefficients and the two loop anomalous dimension matrix, which are separately scheme dependent, a physical, scheme independent, effective Hamiltonian is obtained. Let us consider the finite one loop mixing of the operators $Q_5$ and $Q_6$ with $Q_7$:

$$< s \gamma | H_{\text{eff}} | b > = C_7(\mu) < s \gamma | Q_7(\mu) | b > + C_5(\mu) < s \gamma | Q_5(\mu) | b > + C_6(\mu) < s \gamma | Q_6(\mu) | b >$$  \hspace{1cm} (8)$$

We can interpret the non vanishing of the one loop matrix elements of $Q_5$ and $Q_6$ as the effect of a mixing matrix acting at order $\alpha_s^0$, see below:

$$< s \gamma | H_{\text{eff}} | b > = \tilde{C}_7(\mu) < s \gamma | Q_7(\mu) | b >$$  \hspace{1cm} (9)$$
where $\tilde{C}_7(\mu) = C_7(\mu) + \vec{Z}_7 \cdot \vec{C}_r(\mu)$. The vector $\vec{Z}_7$ (and $\vec{Z}_8$, see below) is regularization dependent. $\vec{Z}_7$ vanishes in HV (and DRED) and

$$\vec{Z}_7 \equiv (0, 0, 0, 0, 2, 2N)$$

in NDR. Eq. (10) corresponds to a finite renormalization of the operators such that the matrix elements of $Q_5$ and $Q_6$ vanish. In this way the renormalized operators are the same in NDR and HV. Similarly one finds:

$$< sg | H_{eff} | b > = C_8(\mu) < sg | Q_8(\mu) | b > + C_5(\mu) < sg | Q_5(\mu) | b >$$

$$= \tilde{C}_8(\mu) < sg | Q_8(\mu) | b >$$

with $\tilde{C}_8(\mu) = C_8(\mu) + \vec{Z}_8 \cdot \vec{C}_r(\mu)$. $\vec{Z}_8 = 0$ in HV (and DRED) and

$$\vec{Z}_8 \equiv (0, 0, 0, 0, 2, 0)$$

in NDR.

The renormalization group equations (5) can be easily solved and combined with eqs. (9) and (11) to obtain the effective Hamiltonian expressed in terms of operators renormalized at the scale $\mu$:

$$H_{eff} \sim \vec{Q}_7^T(\mu) \cdot \vec{C}_r(\mu)$$

$$+ \left\{ v_7(\mu) + \left[ (\gamma_r - \gamma_7 \hat{1})^{-1} \left( \vec{\beta}_7 + \frac{\gamma_{87}}{\gamma_{77} - \gamma_{88}} \vec{\beta}_8 \right) + \vec{Z}_7 \right] \cdot \vec{C}_r(\mu) \right\} Q_7(\mu)$$

$$+ \left\{ v_8(\mu) + \left[ (\gamma_r - \gamma_{88} \hat{1})^{-1} \vec{\beta}_8 + \vec{Z}_8 \right] \cdot \vec{C}_r(\mu) \right\} Q_8(\mu)$$

$$= \vec{Q}_7^T(\mu) \cdot \vec{C}_r(\mu)$$

$$+ \left\{ v_7(\mu) + \frac{\gamma_{87}}{\gamma_{88} - \gamma_{77}} v_8(\mu) + \left[ (\gamma_r - \gamma_7 \hat{1})^{-1} \left( \vec{\beta}_7 + \frac{\gamma_{87}}{\gamma_{77} - \gamma_{88}} \vec{\beta}_8 \right) + \vec{Z}_7 \right] \cdot \vec{C}_r(\mu) \right\} Q_7(\mu)$$

$$+ \left\{ v_8(\mu) + \left[ (\gamma_r - \gamma_{88} \hat{1})^{-1} \vec{\beta}_8 + \vec{Z}_8 \right] \cdot \vec{C}_r(\mu) \right\} Q_8(\mu)$$

$$= \vec{Q}_7^T(\mu) \cdot \vec{C}_r(\mu)$$

$$+ \left\{ v_7(\mu) + \frac{\gamma_{87}}{\gamma_{88} - \gamma_{77}} v_8(\mu) + \left[ (\gamma_r - \gamma_7 \hat{1})^{-1} \left( \vec{\beta}_7 + \frac{\gamma_{87}}{\gamma_{77} - \gamma_{88}} \vec{\beta}_8 \right) + \vec{Z}_7 \right] \cdot \vec{C}_r(\mu) \right\} Q_7(\mu)$$

$$+ \left\{ v_8(\mu) + \left[ (\gamma_r - \gamma_{88} \hat{1})^{-1} \vec{\beta}_8 + \vec{Z}_8 \right] \cdot \vec{C}_r(\mu) \right\} Q_8(\mu)$$

$$= \vec{Q}_7^T(\mu) \cdot \vec{C}_r(\mu)$$

$$+ \left\{ v_7(\mu) + \frac{\gamma_{87}}{\gamma_{88} - \gamma_{77}} v_8(\mu) + \left[ (\gamma_r - \gamma_7 \hat{1})^{-1} \left( \vec{\beta}_7 + \frac{\gamma_{87}}{\gamma_{77} - \gamma_{88}} \vec{\beta}_8 \right) + \vec{Z}_7 \right] \cdot \vec{C}_r(\mu) \right\} Q_7(\mu)$$

$$+ \left\{ v_8(\mu) + \left[ (\gamma_r - \gamma_{88} \hat{1})^{-1} \vec{\beta}_8 + \vec{Z}_8 \right] \cdot \vec{C}_r(\mu) \right\} Q_8(\mu)$$

$$= \vec{Q}_7^T(\mu) \cdot \vec{C}_r(\mu)$$

$$+ \left\{ v_7(\mu) + \frac{\gamma_{87}}{\gamma_{88} - \gamma_{77}} v_8(\mu) + \left[ (\gamma_r - \gamma_7 \hat{1})^{-1} \left( \vec{\beta}_7 + \frac{\gamma_{87}}{\gamma_{77} - \gamma_{88}} \vec{\beta}_8 \right) + \vec{Z}_7 \right] \cdot \vec{C}_r(\mu) \right\} Q_7(\mu)$$

$$+ \left\{ v_8(\mu) + \left[ (\gamma_r - \gamma_{88} \hat{1})^{-1} \vec{\beta}_8 + \vec{Z}_8 \right] \cdot \vec{C}_r(\mu) \right\} Q_8(\mu)$$

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\( C_r(\mu), v_7(\mu) \) and \( v_8(\mu) \) are scheme independent quantities because they obey the scheme independent renormalization group equations (3). \( \tilde{\beta}_{7,8} \) and \( \tilde{Z}_{7,8} \) are scheme dependent quantities. We now prove that the combinations

\[
\tilde{\omega}_7 = (\hat{\gamma}_r - \gamma_{77})^{-1} \left( \tilde{\beta}_7 + \frac{\gamma_{87}}{\gamma_{77} - \gamma_{88}} \tilde{\beta}_8 \right) + \tilde{Z}_7 + \frac{\gamma_{87}}{\gamma_{77} - \gamma_{88}} \tilde{Z}_8
\]

\[
\tilde{\omega}_8 = (\hat{\gamma}_r - \gamma_{88})^{-1} \tilde{\beta}_8 + \tilde{Z}_8
\]

appearing in eq.(14) are instead scheme independent. Let us introduce the renormalization matrix \( \hat{Z}^a \) which gives the renormalized operators \( \tilde{Q}(\mu) \) in terms of the bare operators \( \tilde{Q}_B \), in the regularization labelled as “a”:

\[
\tilde{Q}(\mu) = (\hat{Z}^a)^{-1} \tilde{Q}_B
\]

The anomalous dimension matrix is defined in terms of \( \hat{Z}^a \) as:

\[
\hat{\gamma}^a = 2 (\hat{Z}^a(\mu))^{-1} \mu^2 \frac{d}{d\mu^2} \hat{Z}^a(\mu)
\]

If one uses two different regularization schemes, “a” and “b” say, the renormalization constants are related through the equation:

\[
\hat{Z}^a = \hat{Z}^b \hat{r}
\]

from eqs.(17) and (18) we obtain:

\[
\hat{\gamma}^a = \hat{r}^{-1} \hat{\gamma}^b \hat{r}
\]

In other cases, as for example for the operators appearing in the \( \Delta S = 1 \) weak Hamiltonian, \( \hat{r} \) differs from the identity matrix by terms of order \( \alpha_s \), i.e. \( \hat{r} = \hat{1} + frac{\alpha_s}{4\pi} \Delta r \) [14]-[19]. In the present calculation instead, because of the diagrams in fig.4, \( \hat{r} \) differs from \( \hat{1} \) by terms of order \( \alpha_s^0 \), see eqs.(9) and (11). The matrix \( \hat{r} \) can be written as:

\[
\hat{r} = \begin{pmatrix}
\hat{1}_6 & -\Delta \hat{Z} \\
0 & \hat{1}_2
\end{pmatrix}
\]

where \( \hat{1}_{6,2} \) are 6 \times 6 and 2 \times 2 identity matrices and \( \Delta \hat{Z} \) is a 6 \times 2 matrix. Its two columns are given by the difference of the vectors \( \tilde{Z}_7 \) and \( \tilde{Z}_8 \) in the
regularizations “a” and “b”, \( \Delta \hat{Z} = (\hat{Z}_7^a - \hat{Z}_7^b, \hat{Z}_8^a - \hat{Z}_8^b) \). At the order we are interested in, eq.(19) gives:

\[
\hat{\gamma}^a - \hat{\gamma}^b = \left[ \Delta \hat{Z}, \hat{\gamma}^b \right] - \Delta \hat{Z} \hat{\gamma}^b \Delta \hat{Z}
\] (21)

Eq.(21), which connects the anomalous dimension matrices in two different regularizations, holds diagram by diagram and can be used as a check of the calculation of any single diagram in different regularization schemes \([17]\). More details will be given in ref.\([21]\).

Taking into account the structure of the matrix \( \Delta \hat{Z} \) we find:

\[
\begin{align*}
\hat{\gamma}^a_r &= \hat{\gamma}^b_r = \hat{\gamma}_r \\
(\Delta \hat{Z}_7)_j &= \Delta \gamma_{7j} = \left[ \left( \gamma_{77} \hat{1} - \hat{\gamma}_r \right) \Delta \hat{Z}_7 + \gamma_{87} \Delta \hat{Z}_8 \right]_j \\
(\Delta \hat{Z}_8)_j &= \Delta \gamma_{8j} = \left[ \left( \gamma_{88} \hat{1} - \hat{\gamma}_r \right) \Delta \hat{Z}_8 \right]_j
\end{align*}
\] (22)

where \( j \) denotes the components of the vectors \( \Delta \hat{Z}_{7,8}, j = 1, \ldots, 6 \). Eq.(22) demonstrates that the combinations \( \tilde{\omega}_7 \) and \( \tilde{\omega}_8 \) appearing in eq.(13) are regularization scheme independent. Since \( \tilde{Z}_7 \) and \( \tilde{Z}_8 \) are zero in HV, in the present case \( \Delta \hat{Z} \) is simply given by \( \tilde{Z}_{7,8} \) computed in NDR (\( \Delta \hat{Z} = \hat{\gamma}^{NDR}_r - \hat{\gamma}^{HV}_r \) and \( \Delta \hat{Z} = \tilde{Z}^{NDR} - \tilde{Z}^{HV} = \tilde{Z}^{NDR} \)). We have computed the anomalous dimension in both NDR and HV \([1]\). From the one-loop diagrams in fig.4 and the two-loop diagrams in fig.4 and 5, we obtain the anomalous dimension matrices \( (25), (26) \), given below. Combining \( (25) \) and \( (26) \) with eqs.(10,12), one indeed finds the same \( \tilde{\omega}_{7,8} \) in the NDR and HV schemes.

In refs.\([1, 5, 6]\), they considered a restricted operator basis, in the NDR scheme. Our results for the two loop Feynman diagrams agree with those reported there. The anomalous dimension matrix given in these references is however regularization scheme dependent. The scheme dependence follows from the non-vanishing of the diagrams in fig.4, which spoil the equations of motion of the regularized theory:

\[
\bar{s}\gamma^\mu_L D^\nu t^A G^A_{\mu\nu} b = \bar{s}\gamma^\mu_L t^A b \sum_q \frac{1}{2} \left( \bar{q} \gamma_\mu L t^A q + \bar{q} \gamma_\mu R t^A q \right).
\] (23)

\(^3\)The reduced mixing matrix \( \hat{\gamma}_r \) has been recently computed up to the next-to-leading order in refs.\([14]-[17]\).
Figure 3: Diagrams in the reduced (a) and complete (b) bases. In (a) we insert the operator $\bar{s}\gamma^\mu_L D^\nu A^A_{\mu\nu} b$, in (b) the corresponding four fermion operators. Because of mass insertions, which spoil the equations of motion, in NDR the two diagrams have different values. On the contrary, they give the same result in the HV scheme.

Indeed, using the NDR scheme, the values of the two diagrams in fig.3 differ, because of the contributions coming from the insertion of $m_b$ in the loop propagators, see also fig.1. It is easy to show that the scheme dependence can be eliminated by enforcing the equations of motion with a suitable, finite subtraction, at order $\alpha_s^0$.

In NDR, when one uses the complete basis (2), the diagrams in fig.3 have to be computed[8]. These diagrams vanish in the HV scheme. In NDR instead they give a non-zero contribution to the anomalous dimension, precisely for the same reason that the diagrams in fig.1 do not vanish. In the final answer (14), the contribution of each of the diagrams in fig.3 is exactly cancelled by the corresponding term in $\vec{Z}_{7}$ and $\vec{Z}_{8}$, as it can be checked diagram by diagram. In fact the diagrams in fig.3 give a contribution which goes like $-\left(\gamma_T^r - \gamma_{77,88}^T\right) \vec{Z}_{7,8}$. The only exceptions are the diagrams $P_4$ and $F_4$ in fig.4 (see also fig.3). In this case, $\vec{Z}_{7,8}$ cancel the regularization dependent part of that two-loop diagram, giving the same result one would obtain in

\footnote{This diagram would be present also in the reduced operator basis used in refs.1, 5, 6.}
Figure 4: Diagrams which contribute both in the NDR and HV schemes. In the evaluation of the diagrams counter-terms and “effervescent” operators have been included, see also ref. [21].

HV. In HV, since \( \mathbf{Z}_7 \) and \( \mathbf{Z}_8 \) vanish, one obtains the regularization scheme independent result directly from the evaluation of the diagrams in fig. 4. Those calculations, which did not include the effect of \( \mathbf{Z}_7,\mathbf{Z}_8 \) in the effective Hamiltonian, are not correct because of the regularization dependence of the diagrams \( P_4 \) and \( F_4 \) in fig. 4. On the other hand, for the complete basis, it is necessary to include the contribution of the diagrams in fig. 4.

We now summarize our discussion. In HV, where \( \mathbf{Z}_{7,8} \) vanish, we have only to compute the contribution to the anomalous dimension matrix of the diagrams in fig. 4 (besides the usual one loop diagrams which mix the operators \( Q_1, \ldots, Q_6 \)). The resulting anomalous dimension matrix is regularization independent and does not require any further manipulation. In NDR, the
two loop anomalous dimension has to be computed from the diagrams in figs. 4 and 5. The result is then combined with \( \vec{Z}_{7,8} \) as in eq. (14). The final result is then identical to the result obtained in HV, as we have explicitly checked. This implies that our results for the anomalous dimension matrices in NDR and HV, (25)-(26), satisfy the general relation (21). We report below the anomalous dimension matrix in HV, to be used for the evolution of the coefficients, together with the initial conditions

\[
\begin{align*}
C_2(M_W) &= 1 \\
C_1, C_3, \ldots, C_6 (M_W) &= 0 \\
C_7(M_W) &= -3\frac{3x^3 - 2x^2}{2(1 - x)^4} \ln x - \frac{8x^3 + 5x^2 - 7x}{4(1 - x)^3} \\
C_8(M_W) &= -\frac{3x^2}{2(1 - x)^4} \ln x + \frac{x^3 - 5x^2 - 2x}{4(1 - x)^3}
\end{align*}
\]

(24)

where \( x = m_t^2/M_W^2 \).

\[
\hat{\gamma}_{ai} = \frac{\alpha_s}{4\pi} \begin{pmatrix}
-\frac{6}{N} & 6 & 0 & 0 & 0 & 0 & 0 \\
6 & -\frac{6}{N} & -\frac{2}{3N} & \frac{2}{3} & -\frac{2}{3N} & \frac{2}{3} \\
0 & 0 & -\frac{22}{3N} & \frac{22}{3} & -\frac{4}{3N} & \frac{4}{3} \\
0 & 0 & 6 & -\frac{2n_f}{3N} & -\frac{6}{N} + \frac{2n_f}{3} & -\frac{2n_f}{3N} & \frac{2n_f}{3} \\
0 & 0 & 0 & 0 & \frac{6}{N} & -6 \\
0 & 0 & -\frac{2n_f}{3N} & \frac{2n_f}{3} & -\frac{2n_f}{3N} & -12\frac{N^2-1}{2N} + \frac{2n_f}{3} \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]
\[
\hat{\gamma}_{\alpha\alpha}^{\text{HV}} = \frac{\alpha_s}{4\pi} \begin{pmatrix}
0 & 6 \\
\frac{8 N^2 - 1}{9} + \frac{12Q_u}{Q_d} \frac{N^2 - 1}{2N} & \frac{22N}{9} - \frac{58}{9N}
\end{pmatrix}
\]

where \( n_f = n_u + n_d \) is the number of active flavours, \( N \) is the number of colours, \( a = 1, \ldots, 8 \), \( i = 1, \ldots, 6 \) and \( \alpha = 7, 8 \). In the NDR scheme we obtain \( \hat{\gamma}_{\alpha\alpha}^{\text{NDR}} = \hat{\gamma}_{\alpha\alpha}^{\text{HV}} \) and:

\[
\hat{\gamma}_{\alpha\alpha}^{\text{NDR}} = \frac{\alpha_s}{4\pi} \begin{pmatrix}
0 & 6 \\
\frac{16 N^2 - 1}{9} + \frac{12Q_u}{Q_d} \frac{N^2 - 1}{2N} & \frac{22N}{9} - \frac{46}{9N}
\end{pmatrix}
\]

where \( \tilde{n}_f = n_d + \frac{Q_u}{Q_d} n_u \).
Figure 5: Diagrams which have to be considered only in the NDR scheme. In the evaluation of the diagrams counter-terms and “effervescent” operators have been included.

Let us compare our results with previous calculations.

In refs. [1]-[6] it was assumed that the values of the two loop diagrams were scheme independent. In the NDR scheme, we agree with the results of refs. [1, 5, 6] when we reduce to the basis used in those references. However the necessity of adding $\vec{Z}_{7,8}$, to obtain a scheme independent result, was missed.

In ref. [8] Misiak realized the necessity of adding $\vec{Z}_7$, even if he did not understand its role for the scheme independence of the final result. Moreover the paper contains several errors: counter-terms of current-current diagrams and “effervescent” operators were not taken into account and there are in-

\footnote{Without effervescent operators relation (21) between HV and NDR would fail.}
consistencies between different anomalous matrix elements, which can be related to the same set of diagrams. In ref. [9], some of these errors have been corrected: counter-terms of current-current diagrams and effervescent operators have been considered and included in the final result. However, our results (26) and the matrix given in eq.(21) of ref. [9] do not agree. By looking to colour factors, the disagreement seems to originate from diagrams $P_2$ and $P_3$ in fig.4 with a mass ($m_b$) insertion in the loop propagators.

The calculation done in DRED in ref. [2, 3, 4] could have lead in principle to the correct results. The authors however were biased by the idea that the two loop calculation had to be regularization scheme independent and interpreted the difference between NDR and DRED as a failure of the DRED scheme. We have demonstrated that there are not privileged schemes and that all the schemes, if correctly used, give the same results for the Wilson coefficients. We are actually performing the calculation also in the DRED scheme to show that this is indeed the case.

We were unable to understand how the calculation of ref. [10] was done. We disagree with the final result for the anomalous dimension matrix given in this reference. The results of ref. [10] differ also from the results of ref. [9].

We want to add a few comments on the operator basis and gauge invariance. In refs. [2] and [3] a redundant basis of operators which vanish (or coincide) by the equations of motion was used. In ref. [17], it was shown that this leads to an unnecessary complication and that the correct result can be obtained by working directly with a reduced basis of independent operators, as the basis in eq.(2). This remains true in the present calculation, as we have explicitly checked. As noticed already in ref. [8], it is not necessary to work in the background field gauge to obtain a gauge invariant result. We have done (for the non-abelian diagrams) the calculation in the Feynman and background gauges (see also ref. [17]) with identical final results. We plan to present a phenomenological analysis in a separate publication, together with all the details of the calculation (treatment of the effervescent operators, counter-terms, gauge invariance, contribution of any single diagram, etc.) and the results in the DRED scheme.

$^6$At this order in the perturbative expansion.
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