Mathematical model of forming screw profiles of compressor machines and pumps

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Abstract. The article presents the results of mathematical modeling of screw surfaces shaping for compressor machines and pumps. The study is based on a method of curve movable trihedron. A mathematical model of a flat gearing - the basis for a screw formation - is proposed. The model is based on geometric interpretation of plane curve trihedron motions and known in a geometric theory of plane mechanisms of the Bobillier construction. A geometric scheme of this construction was expanded due to introduction of evolutes simulating instantaneous motions of curves trihedra in a construction scheme. As a result, the mathematical model was obtained, which is more complete in comparison with the known models of flat gearing, which makes it possible to perform synthesis and analysis of profiled screws geometry. It realizes both direct and inverse problems of screws profiling with simultaneous obtaining the curvature of the desired profiles in absent ones. The proposed model can be used as a basis of automated system development for mutually enveloping surfaces screws shaping for compressor machines and pumps.

1. Formulation of the problem
Working surfaces shaping of screws of both compressor machines and pumps [1] is due to the action of geometric and kinematic laws known in a theory of gearing [2-5]. The investigation of helical gearing is based on the flat gearing theory, which deals with the problems of profiling screws, i.e. profile determination of one of the screws from the given profile of the other (direct profiling task), and the profiles definition of both screws along a given linking line (inverse profiling problem). If the direct problem was sufficiently well studied and its solutions were debugged, then a solution of the second problem is far from completion due to its insufficient knowledge. Both tasks of profiling screws are particular importance for both the theory and practical using of compressor machines and pumps, since the accuracy of the screws working surfaces largely depends on the accuracy of their profiling.

The accuracy of the same screws profiling depends on the capabilities of the mathematical model. Analysis of existing methods of profiling screws and possibilities of their realizations on a basis of modern computer technologies allows us to conclude that the mathematical model of profiling should be completed, i.e. must provide a solution to the direct and inverse problems of profiling with derivation in a simple computational form of differential geometric parameters of screw profiles; the model should be parametrically controlled, i.e. should provide the possibility of varying profiling parameters in a computer simulation of mutually enveloping screw profiles formation. The
development of the screws profiling mathematical model meeting these requirements is a goal of this work.

2. The problem solution
In geometry, the movable trihedron method proposed by Cartan [6] is known and it is used successfully in studies and construction of lines and surfaces. Using this method as a research tool in the kinematic geometry of a line curve on a plane allows you to solve problems of profiling screws.

The Frenet trihedron (FT) of the spatial curve moves along a curve and makes a complex motion, the instantaneous rotational component of which is described by the Darboux vector [7]:

\[
\vec{r}(s) = \sigma \vec{\tau} + k \vec{\beta},
\]

where \( s(T_o) \leq s \leq s(T); \ T_o \leq t \leq T; \ \vec{\rho} = \vec{\rho}(t) \) - parametric equation of the curve; \( s = s(t) \) - an arc curve length, described by a function that can be reversed, \( t = t(s); \ \vec{\tau}, \ \vec{\beta} \) - unit vectors of tangent and binormal, respectively, defining FT at a given point; \( \sigma \) and \( k \), respectively, a torsion and curvature of a curve at this point. It is supposed that FT is an absolutely rigid body. In this case, the main kinematics theorem [8] can be applied. It allows us to determine total instantaneous motion of FT. The components of this motion are instantaneous rotational motion, vector (1) of which can be transformed to the form:

\[
\vec{r}(t) = \sigma \frac{ds}{dt} \vec{\tau} + k \frac{ds}{dt} \vec{\beta},
\]

Instantaneous translational motion is described by vector

\[
\vec{r}_o(t) = \frac{ds}{dt} \vec{\tau}.
\]

An addition of the motions described by vectors (2) and (3) results in an instantaneous screw (IS) whose axis is parallel to vector \( \vec{r} \) and is removed by distance \( |\vec{\rho}_{IS}| \) representing a modulus of the vector:

\[
\vec{\rho}_{IS} = \vec{r}_o \times \vec{r} = \frac{k}{\sigma^2 + k^2} \vec{\nu},
\]

where \( \vec{\nu} \) is the unit vector of normal of the curve. The IS parameter is defined as:

\[
p = \frac{\vec{r}_o \cdot \vec{r}}{r^2} = \frac{\sigma}{\sigma^2 + k^2}.
\]

The angular position of the IS axis relative to vector \( \vec{\tau} \) is determined by

\[
\cos \psi = n \sigma; \ \sin \psi = nk; \ n = \frac{1}{\sqrt{\sigma^2 + k^2}}.
\]

The continuous one-parameter set of axes IS, formed when FT is moved along the curve, is a movable axoid (MA) in a coordinate system and fixed axoid (FA) in the XYZ coordinate system. Let us determine the equation of FA of instantaneous screw axes of FT:

\[
\vec{\rho}_M = \vec{\rho}_{IS} + \vec{d},
\]
where \( \overrightarrow{d} = d \cos \psi \overrightarrow{\tau} + d \sin \psi \overrightarrow{\beta} \); \( d = |\overrightarrow{d}| \) - parameter that determines the position of the current point of the axis of IS relative to the point of reduction on it. Taking (4) into account, the equation in generalized form is obtained:

\[
\overrightarrow{\rho}_M = l_1 \overrightarrow{\tau} + l_2 \overrightarrow{\beta} + l_3 \overrightarrow{v},
\]

where \( l_1 = d \cos \psi; l_2 = d \sin \psi; l_3 = \frac{k}{\sigma^2 + k^2} \).

Taking into account that either \( k(s) > 0 \), and \( \sigma(s) > 0 \), and \( \sigma(s) < 0 \) for a spatial curve, on the basis of equation (8), it is possible to conclude that conditions \( \sigma(s) > 0 \) and \( \sigma(s) < 0 \) correspond to two MA that are mirror-symmetric with respect to the plane of vectors \( \overrightarrow{\beta} \) and \( \overrightarrow{v} \). The equation of FA can be written in vector form

\[
\overrightarrow{\rho}_F = \overrightarrow{\rho} + \overrightarrow{\rho}_M ,
\]

where \( \overrightarrow{\rho} = x\hat{i} + y\hat{j} + z\hat{k} \) is the initial curve equation of the line. Taking into account formulas known in differential geometry for the unit vectors \( \overrightarrow{\tau}, \overrightarrow{v}, \overrightarrow{\beta} \) [7], after transformations of the expression (9), it is possible to write the equation FA in the coordinate form:

\[
x_F = x + m_1 \dot{x} \sigma + a_1, \quad y_F = y + m_1 \dot{y} \sigma + a_2, \quad z_F = z + m_1 \dot{z} \sigma + a_3,
\]

where \( m_1 = \frac{d}{\sqrt{\sigma^2 + k^2}}; m_2 = (\frac{m_1}{d})^2; a_1 = \left[ \begin{array}{c} y' \\ z' \\ \sigma \\ \sigma \\ \sigma \\ -\xi' \end{array} \right]; a_2 = \left[ \begin{array}{c} \dot{x} \\ \dot{z} \\ \dot{z} \\ a_3 = \left[ \begin{array}{c} \dot{x} \\ \dot{y} \\ \dot{z} \end{array} \right].
\]

Let us to consider a plane curve. It takes place when \( \sigma = 0, z = const \). In this case, the instantaneous angular velocity vector according to (2) takes form

\[
\vec{r}(t) = \frac{d}{dt} \overrightarrow{\beta}.
\]

The vector of the instantaneous translation velocity is determined by equation (3). Obviously that \( \overrightarrow{r}(t) \perp \overrightarrow{r}_0(t) \) and \( \rho = 0 \) on the basis of (5). From (4) it follows that \( \overrightarrow{\rho}_{IS} = \frac{1}{k} \overrightarrow{v} \). Thus, the MA of a plane curve is described in accordance with (6) and (8) by the following vector equation:

\[
\overrightarrow{\rho}_M = \overrightarrow{\beta}d + \overrightarrow{v} \frac{1}{k},
\]

where \( d \in (-\infty, \infty) \). MA is a plane of vectors \( \overrightarrow{\beta} \) and \( \overrightarrow{v} \) formed by a continuous set of axes of "pure" instantaneous rotations of the FT, parallel to the directing vector \( \overrightarrow{\beta} \) of this plane. Since \( z = const \), it follows from (10):

\[
x_F = x + \frac{\dot{x}}{k^2}; \quad y_F = y + \frac{\dot{y}}{k^2}; \quad z_F = \frac{a_3d}{k}.
\]

Previous two equations (13) coincide with known equations of the evolute of the plane curve [7] and
together with the third one they describe a FA - cylindrical surface with a directing vector $\vec{\beta}$ (Figure 1).

**Figure 1.** Axoids in motions of a plane curve trihedron.

The following assertion holds: an evolute of a plane curve is a set of instantaneous rotations point-centers of an accompanying trihedron of this curve. This statement serves as a basis for development of a geometric model of screw gear engagement.

In a gearing theory the Bobillier construction (Figure 2) and a corresponding theorem are known [2, 9]: a perspective axis (PS) of two bundles of straight lines with centers at curvature points ($O_C$ and $O_{C_1}$) of two centroids ($C$ and $C_1$), corresponding rays which pass through curvature centers ($O_f$ and $O_{f_1}$) of mutually enveloping profiles $f$ and $f_1$ (not shown) belonging to some straight lines ($O_f \in PF$, $O_{f_1} \in PF$) – this axis PS is perpendicular to common normal of two profiles ($PS \perp PF$).

**Figure 2.** Geometric scheme of a flat tooth gearing
We extend the Bobillier construction by introducing a gearing line (GL) and its evolute \( e_{GL} \) into its scheme. GL is a one-parameter set of contact points between profiles \( f \) and \( f_1 \) of wheels teeth as they pass their common normals through a gearing pole \( P \), which is a contact point of centroids \( C \) and \( C_1 \). Let GL be a curve \( C^2 \) described by parametric equations \( \bar{r} = \bar{r}(t); \alpha = \alpha(t) \), where \( t \) is a real parameter. Let us connect a radius vector \( \bar{r} \) with a trihedron \((F, \tau, \nu)\). Since \( \bar{r}\nu = 0 \) it follows that:

\[
\frac{d\bar{r}}{dt} = \frac{d}{dt}(-\bar{v}_r) = \frac{d}{dt}(-\bar{v}) \cdot r - \bar{v} \cdot dr = \bar{r} \cdot \frac{d\alpha}{dt} \cdot r - \bar{v} \cdot r' \cdot \frac{d\alpha}{dt}.
\]  

(14)

Thus, the following is obtained:

\[
d\bar{r} = \bar{r} \cdot r \cdot d\alpha - \bar{v} \cdot dr; \quad |d\bar{r}| = ds_{GL} = \sqrt{r^2 + (r_1')^2} \cdot d\alpha.
\]  

(15)

A trihedron displacement \((\bar{F}_F, \tau, \nu)\) of the varying radius vector \( \bar{r} \) along the line GL can be characterized by a dual vector \( \delta_a \) [10], constructed at point \( F \) and determining an absolute trihedron motion \((\bar{F}_F, \tau, \nu)\) of the varying radius vector \( \bar{r} \) as it moves along the line GL:

\[
\delta_a = \bar{F}_F \cdot \frac{d\alpha}{dt} + \omega(\bar{r} \cdot \frac{d\alpha}{dt} - \bar{v}_r' \cdot \frac{d\alpha}{dt}) \; \; ; \; \; \omega^2 = 0.
\]  

(16)

We now consider the motion of the trihedron \((\bar{F}_F, \tau, \nu)\) along its line GL. This motion is described by the dual vector \( \delta_{GL} \) applied at point \( F \) and determining the trihedron motion \((\bar{F}_F, \bar{\tau}_{GL}, \bar{\nu}_{GL})\) along the GL itself:

\[
\delta_{GL} = \bar{F}_F k_{GL} \cdot \frac{ds_{GL}}{dt} + \omega(\bar{r} \cdot \frac{d\alpha}{dt} - \bar{v}_r' \cdot \frac{d\alpha}{dt}).
\]  

(17)

Let us now turn to FT motion \((\bar{F}_F, \tau, \nu)\) profile \( f \) (see Figure 2). This motion is characterized by a dual vector \( \delta_f \) applied at point \( F \) and determining relative motion of the FT \((\bar{F}_F, \tau, \nu)\) along the profile \( f \):

\[
\delta_f = \bar{F}_F k \cdot \frac{ds}{dt} + \omega \cdot \bar{r} \cdot \frac{ds}{dt} \; ; \; \omega^2 = 0.
\]  

(18)

Portable motion - movement in a rigid connection with FT \((\bar{F}_P, \tau, \nu)\) along a centroid line \( C \). This motion is determined by the dual vector \( \delta_e \) that defined mobile motion of the FT \((\bar{F}_P, \tau, \nu)\):

\[
\delta_e = \bar{F}_P \frac{d\phi_e}{dt} + \omega (-\bar{r} \sin \alpha \cdot \frac{d\phi_e}{dt} - \bar{r} \cos \alpha \cdot \frac{d\phi_e}{dt}) \; ; \; \omega^2 = 0.
\]  

(19)

Let us reduce all three dual vectors (16), (18) and (19) to a single point, for example, a gearing pole \( P \). As a result of reduction, the following dual vectors are obtained:

\[
\delta_a \rightarrow \delta_{Pa} = \bar{P}_a \frac{d\alpha}{dt} + \omega (-\bar{r}_a' \cdot \frac{d\alpha}{dt}) \; ; \; \omega^2 = 0.
\]  

(20)
\[
\dot{\vec{r}} \rightarrow \vec{\delta}_r = \vec{\beta}_p k \cdot \frac{ds}{dt} + \omega \cdot (-\tau \cdot r \cdot k \cdot \frac{ds}{dt} + \tau \frac{ds}{dt}); \quad \omega^2 = 0. \quad (21)
\]

\[
\dot{\vec{e}} \rightarrow \vec{\delta}_e = \vec{\beta}_p \frac{d\varphi_e}{dt} + \omega \cdot (-\tau \sin \alpha \cdot \frac{d\varphi_e}{dt} - \bar{V} \cos \alpha \cdot \frac{d\varphi_e}{dt}); \quad \omega^2 = 0. \quad (22)
\]

Since three dual vectors sum \( \vec{\delta}_p = \vec{\delta}_r + \vec{\delta}_e \) reduced to the pole \( P \) takes place, then on the basis of (20), (21) and (22), equations are as follow:

\[
d\alpha - d\varphi_e - k \cdot ds = 0. \quad (23)
\]

\[
ds - R \sin \alpha \cdot d\varphi_e - r \cdot k \cdot ds = 0. \quad (24)
\]

\[
r\alpha' \cdot d\alpha - R \cos \alpha \cdot d\varphi_e = 0. \quad (25)
\]

By integrating equation (25), the following is obtained:

\[
\varphi_e = \frac{1}{R} \int r \alpha' \frac{d\alpha}{\cos \alpha} + C. \quad (26)
\]

The last equation can be used to determine a rotation angle of the line point GL to its position on the profile \( f \) occupying the certain angular position fixed by integration constant \( C \). A value of the rotation angle of the point from a position on the GL is determined from relation

\[
\varphi = -\frac{1}{R} \int \frac{r \alpha' \cdot d\alpha}{\cos \alpha}. \quad (27)
\]

The system of differential equations (23), (24) and (25) can be used to solve problems of analysis and synthesis of flat gearing. On the basis of formula (27), one can obtain scalar-parametric equations of the profile \( f \) in the coordinate system of the line GL:

\[
x_f = x_f(\alpha) = r \cos(\alpha + \varphi) + R \sin \varphi; \quad y_f = y_f(\alpha) = r \sin(\alpha + \varphi) + R(1 - \cos \varphi). \quad (28)
\]

The tooth profile \( f_1 \) of the second gearwheel, conjugated with the tooth profile \( f \) of the first gearwheel, can be calculated taking into account (27) and the given gear ratio.

From (26) the relations are as follows:

\[
(\varphi_e')_\alpha = \frac{d\varphi_e}{d\alpha} = \frac{r \alpha'}{R \cos \alpha} = \frac{PS}{O_{1\alpha}E}. \quad (29)
\]

From the system of equations (23) and (24), the equation is as follows:

\[
(\varphi_e')_\alpha = \frac{\rho - r}{\rho + R \sin \alpha - r}, \quad (30)
\]

where \( \rho = \frac{1}{k} \) is a radius of profile curvature \( f \). From (30) an expression for the radius of curvature follows:

\[
\rho = \frac{(\varphi_e')_\alpha \cdot R \sin \alpha + r \cdot [1 - (\varphi_e')_\alpha]}{1 - (\varphi_e')_\alpha}, \quad (31)
\]

That is, the radius of the curvature of the profile \( f \) is determined from the GL. If to take into account
(25), then from (31) the equation is as follows:

\[ \rho = r + \frac{r'_{a}}{R} \cdot R \sin \alpha \cos \alpha - r'_{a}. \]  

(32)

The result of calculating a gearwheel tooth profile according to a specified centroid \( C \) of the gearwheel \( (O_{C}, R = 100 \text{ mm}) \) and the GL representing by the circle \( R = 40 \text{ mm} \) centered on the pole straight line \( O_{C}P \) is shown in the example in Figure 3.

![Figure 3. Example of profiling by the gearing line GL.](image)

3. Findings

1. The expansion of the Bobillier construction due to the introduction of GL into it makes it possible to obtain a system of differential equations (23), (24) and (25), which is a mathematical model of a flat gearing. It allows solving the straight geometrical task (determination of the profile \( f_{1} \) of the tooth of the second gearwheel and definition of the line GL by the given profile \( f \) of the tooth of the first gearwheel) and the inverse geometrical task (the definition of the profiles \( f \) and \( f_{1} \) by the given line GL). Both tasks relate to the field of research, focused on the synthesis of planar gears with circular centroids, which include helical gears.

2. The Euler-Savary equation [9], whose geometric base is the Bobillier construction, namely:

\[ \frac{1}{\rho+c} = \frac{1}{\rho_{1}+c} = \frac{1}{d \cdot \sin \alpha}, \]

where \( \rho \) and \( \rho_{1} \) are the radii of curvatures of the profiles \( f \) and \( f_{1} \) respectively, \( d = \frac{1}{K - K_{1}} \); \( K, K_{1} \) are the curvatures of the centroid \( C \) and \( C_{1} \), respectively, with which mutually enveloping profiles \( f \) and \( f_{1} \) are connected - this equation allows us to determine the curvature of the profile \( f_{1} \) by formula (33) in its absence.
4. Conclusion
The system of differential equations (23), (24) and (25) is the mathematical model of a flat gearing with circular centroids. It summarizes known research in this direction [11, 12, 13], complements some of them on the screws shaping by cutting tool [14], and it can be used as a basis for working-out an automated system for shaping mutually enveloping screw surfaces of both compressor machines and pumps.

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