Zero-point energies and the multiplicative anomaly.

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For the case of a relativistic scalar field at finite temperature with a chemical potential, we calculate an exact expression for the one-loop effective action using the full fourth order determinant and ζ-function regularisation. We find that it agrees with the exact expression for the factored operator and thus there appears to be no multiplicative anomaly. The appearance of the anomaly for the fourth order operator in the high temperature limit is explained and we show that the multiplicative anomaly can be calculated as the difference between two ζ-regularised zero-point energies. This difference is a result of using a charge operator in the Hamiltonian which has not been normal ordered.

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I. INTRODUCTION

Renormalisation and regularisation techniques form a vital part of the physicist’s arsenal when performing calculations in quantum field theory. The technique of ζ-function regularisation [1–3] is a well established method for obtaining finite results in quantum field theoretical calculations and has shown itself to be a very elegant and powerful method, particularly in curved spacetimes and spacetimes with a non-trivial topology.

Calculations involving Feynman path integrals typically involve the determinant of a differential operator. This determinant is an infinite product and has to be regularised in some way. It has been shown recently that results obtained from ζ-function regularised determinants can be ambiguous - the source of this ambiguity is known as a multiplicative anomaly. The multiplicative anomaly is essentially the difference between different zeta regularised factorisations of a determinant. For example, in calculating a determinant one may wish to factorise it in order to make calculations easier. Normally one would write det(AB) = det(A)det(B), but in the case of infinite matrices this relation is not always correct after regularisation. The multiplicative anomaly in a D dimensional spacetime is defined as,

\[ a_D = \ln \det(AB) - \ln \det(A) - \ln \det(B) \quad (1.1) \]

The relevance of the multiplicative anomaly for physics was first brought to light by Elizalde, Vanzo and Zerbini [4,5] and they showed its connection with the Wodzicki residue. The high temperature limit of the one loop effective action for a charged scalar field with chemical potential was considered by Elizalde, Filippi, Vanzo and Zerbini [6], and the resulting anomaly was found to depend on the chemical potential. It appeared therefore that an extra term, previously overlooked, might be present in the effective action, a term which could not be removed by renormalisation. (This idea received criticism from Evans [7] and Dowker [8]. Elizalde, Filippi, Vanzo and Zerbini responded to these criticisms in [9,10].)

It seemed that there were many different expressions for the effective action, one for each way in which the determinant can be factorised. So, given these varying expressions, each one differing from another by a corresponding multiplicative anomaly, how can we know which one (if any) is correct? The present authors concluded in [11] that the ambiguity associated in choosing different factorisations could only be resolved by making comparisons with calculations performed using canonical methods. In this paper we shall consider two factorisations; the full fourth order operator:

\[ \Gamma_A = \frac{1}{2} \ln \det \left( \left( -\Box + m^2 - \mu^2 \right)^2 - 4\mu^2 \frac{\partial^2}{\partial t^2} \right) \]  

and the ‘standard’ factorisation of two second order operators (eg. as used in [12]):

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\[
\Gamma_B = -\frac{1}{2} \ln \det t^2 \left(-\Box + m^2 - \mu^2 + 2i\mu \frac{\partial}{\partial t}\right) - \frac{1}{2} \ln \det t^2 \left(-\Box + m^2 - \mu^2 - 2i\mu \frac{\partial}{\partial t}\right). \tag{1.3}
\]

In [11], an exact expression for the effective action could only be calculated for the B factorisation, and this agreed with the standard, well known thermodynamical expression - a sum over the zero-point energies and the thermal contributions of Bose-Einstein sums for particles and anti-particles. The high temperature limits of the effective action for both cases were calculated and it was found that \( \Gamma_B \) agreed with the result obtained by Haber and Weldon [13] who did not use path integrals or \( \zeta \)-function regularisation. \( \Gamma_A \) differed from \( \Gamma_B \) by an amount exactly equal to the multiplicative anomaly calculated in [11]. There does not seem to be an \textit{a priori} way of determining which factorisation will yield the correct physics; an ‘objective’ comparison with canonical methods needs to be performed. Clearly this is a problem if calculations using \( \zeta \)-function regularisation need to be made on a system where the canonical answer is not known.

Since the high temperature expansions of \( \Gamma_A \) and \( \Gamma_B \) differ, it might be thought that there would be some discrepancy between their exact expressions also. In Sec. II we shall calculate the exact effective action \( \Gamma_A \), and show that it actually gives the correct result, in complete agreement with \( \Gamma_B \). Is the multiplicative anomaly therefore just an artefact of the high temperature expansion? In Sec. III we postulate that the multiplicative anomaly arises from the zeta-regularised zero-point energies and is not a thermal phenomenon. It arises in factorisations where the charge operator \( Q \) has not been normal ordered. The multiplicative anomaly is calculated explicitly as the difference between \( \zeta \)-regularised zero-point energies with and without a chemical potential. We also consider the interacting case. Both calculations give multiplicative anomalies which agree with those calculated in [11]. In Sec. IV we shall draw some conclusions.

II. THE EXACT EFFECTIVE ACTION \( \Gamma_A \)

We shall use the notation and conventions of [11], now setting \( e = 1 \). We are working with a relativistic, non-interacting, charged scalar field with a chemical potential. The action is

\[
S[\phi] = \int_0^\beta dt \int d\sigma_x \left\{ \frac{1}{2} (\dot{\phi}_1 - i\mu \phi_2)^2 + \frac{1}{2} (\dot{\phi}_2 + i\mu \phi_1)^2 + \frac{1}{2} |\nabla \phi_1|^2 + \frac{1}{2} |\nabla \phi_2|^2 + \frac{1}{2} m^2 (\phi_1^2 + \phi_2^2) \right\}. \tag{2.1}
\]

We expand about a constant background field with \( \phi_1 = \phi \) and \( \phi_2 = 0 \). \( \Gamma_A \) is defined formally as

\[
\Gamma_A = \frac{1}{2} \ln \det t^2 \left(t^2 S_{ij}[\phi]\right)
= \frac{1}{2} \ln \det t^4 \left(-\Box + m^2 - \mu^2\right)^2 - 4\mu^2 \frac{\partial^2}{\partial t^2} \tag{2.2}
\]

Using \( \zeta \)-function regularisation [3], we can define:

\[
\Gamma_A = -\frac{1}{2} \zeta_A(0) + \frac{1}{2} \zeta_A(0) \ln t^4 \tag{2.3}
\]

with

\[
\zeta_A(s) = \sum_n \sum_{j=-\infty}^\infty \left[ (\omega_j^2 - \mu^2 + E_n^2)^2 + 4\mu^2 \omega_j^2 \right]^{-s}. \tag{2.4}
\]

The \( \omega_j \) are the Matsubara frequencies for scalars, \( \omega_j = 2\pi j / \beta \) and \( E_n^2 = \sigma_n + m^2 \). \( \sigma_n \) are the eigenvalues of \( -\nabla^2 \) on the spatial part of the manifold. We shall apply the Abel-Plana summation formula,

\[
\sum_{j=-\infty}^{\infty} f(j) = \int_{-\infty}^{+\infty} f(x) dx + \int_{-\infty+ic}^{+\infty+ic} dz \left(e^{-2\pi i z} - 1\right)^{-1} \left[f(z) + f(-z)\right]. \tag{2.5}
\]

to evaluate \( \zeta_A(s) \). Let us label the two integrals arising on the right hand side of (2.4) \( Q(s) \) and \( R(s) \) respectively after using (2.5) to perform the sum over \( j \). Then,
\[ Q(s) = \sum_n \int_{-\infty}^{\infty} \left[ \left( \frac{2\pi x}{\beta} \right)^2 - \mu^2 + E_n^2 \right]^{-s} dx \]
\[ = \sum_n \int_{-\infty}^{\infty} \left[ ax^4 + bx^2 + c \right]^{-s} dx \]  
(2.6)

where \( a = (2\pi/\beta)^4, b = 2(2\pi/\beta)^2(E_n^2 + \mu^2) \) and \( c = (E_n^2 - \mu^2)^2 \). Making a simple substitution and using the identity
\[ \alpha^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty dt \, t^{s-1} \, e^{-\alpha t} \]  
(2.7)
gives us
\[ Q(s) = \sum_n \frac{1}{2\Gamma(s)} \sqrt{\frac{b}{a}} \int_0^\infty dt \, e^{-t(\frac{c}{8a})} \, t^{s-1} \, e^{-(ax^2+bx)t} \]  
(2.8)

Evaluating the integral over \( x \) first results in
\[ Q(s) = \sum_n \frac{1}{2\Gamma(s)} \sqrt{\frac{b}{a}} \int_0^\infty dt \, e^{-t(\frac{c}{8a})} \, t^{s-1} K_{\frac{1}{4}} \left( \frac{b^2t}{8a} \right) \]  
(2.9)

where \( K_{\frac{1}{4}} \) is a modified Bessel function. Performing the integral over \( t \) we get,
\[ Q(s) = \sum_n \frac{1}{2\Gamma(s)} \sqrt{\frac{b}{a}} e^{\pi^2/4a} \left( \frac{b^2}{4a} \right)^{\frac{1}{4}} \Gamma(s + \frac{1}{4}) \Gamma(s - \frac{1}{4}) 2F_1 \left( s + \frac{1}{4}, \frac{3}{4}; s + \frac{1}{2}; 1 - \frac{b^2}{4ac} \right) \]
\[ = \sum_n \sqrt{\frac{\pi}{2}} \left( \frac{\beta}{2\pi} \right) \left( \frac{E_n^2 + \mu^2}{E_n^2 - \mu^2} \right)^{\frac{1}{4}} \Gamma(s - \frac{1}{4}) \Gamma(\frac{1}{4}) \left( 1 + \frac{4\mu^2 E_n^2 t}{(E_n^2 - \mu^2)^2} \right)^{-\frac{1}{2}} \cos \left[ \frac{1}{2} \arctan \left( \frac{4\mu^2 E_n^2}{(E_n^2 - \mu^2)^2} \right) \right] \]
\[ = -2\beta \sum_n (E_n^2 + \mu^2)^{\frac{1}{2}} \cos \left[ \frac{1}{2} \arctan \left( \frac{2\mu E_n}{E_n^2 - \mu^2} \right) \right] \]  
(2.10)

This last result involves the sum of zero-point energies. Turning now to the contour integral in (2.13), we can write
\[ R(s) = 2 \sum_n \int_{-\infty+ie}^{\infty+ie} (e^{-2\pi i z} - 1)^{-1} \left[ \left( \frac{2\pi z}{\beta} \right)^2 - \mu^2 + E_n^2 \right]^{2} + 4\mu^2 \left( \frac{2\pi z}{\beta} \right)^2 \]  
(2.12)

since \( f(z) \) is even. There are poles at all integers on the real axis, and branch points where the expression in square brackets in the above equation is equal to zero, namely at \( z = \pm i(\beta/2\pi)(E_n + \mu), \pm i(\beta/2\pi)(E_n - \mu) \). By taking branch cuts between the poles in each half of the complex plane, and closing the contour in the upper half plane, it can be shown that
\[ R(s) = -2s \sum_n \ln \left[ \left( 1 - e^{-\beta(E_n - \mu)} \right) \left( 1 - e^{-\beta(E_n + \mu)} \right) \right] \]  
(2.13)

Hence \( R(0) = 0 \) and
\[ R'(0) = -2s \sum_n \ln \left[ \left( 1 - e^{-\beta(E_n - \mu)} \right) \left( 1 - e^{-\beta(E_n + \mu)} \right) \right] \]  
(2.14)
Therefore $\zeta_A(0) = Q(0) + R(0) = 0$ and so

$$\Gamma_A = -\frac{1}{2} \zeta_A'(0) = \sum_n \left\{ \beta E_n + \ln \left[ \left( 1 - e^{-\beta(E_n - \mu)} \right) \left( 1 - e^{-\beta(E_n + \mu)} \right) \right] \right\}$$

(2.15)

which is the same result as was obtained for $\Gamma_B$ in [11]. A simpler, but perhaps less elegant way of evaluating (2.4) is presented in the appendix.

So we have a paradox - the exact expressions for $\Gamma_A$ and $\Gamma_B$ agree, while their high temperature expansions differ by the multiplicative anomaly. The resolution of this paradox lies in the fact that the sums over the energy levels (integrals over $k$ when $\sigma_n = k^2$) have not yet been performed in the exact expressions we are considering. In the high temperature expansions, the $\zeta$-functions were expanded in powers of $\mu$, the chemical potential, and the sums over the energy levels and Matsubara frequencies were then performed. As we shall see in the next section, the multiplicative anomaly arises from the chemical potential being present in the zero-point energy contributions. This is due to a lack of normal ordering in the charge operator.

III. NORMAL ORDERING AND THE MULTIPLICATIVE ANOMALY

In this section we shall show that the multiplicative anomaly stems from the zero-point energy contribution to the effective action.

A. Non-interacting model

There are two ways to write down the zero-point energies for the system described by (2.1); with or without a chemical potential. One can write $(\beta/2) \sum_n (E_n \pm \mu)$ for particles and anti-particles ($-\mu$ and $+\mu$ respectively) or one can simply write $\beta \sum_n E_n$ (which was derived in our exact expressions for $\Gamma_A$ and $\Gamma_B$). The multiplicative anomaly is the difference between these $\zeta$-regularised zero-point energies. Let us define

$$I_{\pm} = \frac{\beta}{2} \sum_n (E_n \pm \mu) \quad (3.1)$$

$$J = \beta \sum_n E_n. \quad (3.2)$$

Formally of course, there is no difference between the zero-point energies in the two cases: $I_+ + I_- - J = 0$. But if we regularise first then the anomaly appears. Define

$$I_{\pm}(s) = \frac{\beta}{2} \sum_n (E_n \pm \mu)^{-s} \quad (3.3)$$

$$J(s) = \beta \sum_n E_n^{-s}. \quad (3.4)$$

Then, we claim that the multiplicative anomaly is

$$a_4 = I_+(1) + I_-(1) - J(1). \quad (3.5)$$

Let us calculate $I_{\pm}(s)$ with $E_n^2 = k^2 + m^2$:

$$I_{\pm}(s) = \frac{\beta}{2} \frac{4\pi V}{(2\pi)^3} \int_0^\infty dk \frac{k^2}{k^{2 + 2s} \pm \mu} \quad (3.6)$$

We can binomially expand the square bracket in powers of $\mu$ up to $O(\mu^4)$. We do not need to consider higher order terms for reasons which will become apparent in due course. After performing the integrals we have,
\[ I_\pm(s) = \frac{\sqrt{\pi} \beta V}{16 \pi^2} \left\{ \frac{\Gamma\left(\frac{s}{2} - \frac{3}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} (m^2)^{\frac{s-1}{2}} \mp s \mu \frac{\Gamma\left(\frac{s}{2} - 1\right)}{\Gamma\left(\frac{s}{2} + \frac{1}{2}\right)} (m^2)^{\frac{s+1}{2}} + s(s+1) \frac{\mu^2 \Gamma\left(\frac{s}{2} - \frac{1}{2}\right)}{2 \Gamma\left(\frac{s}{2} + 1\right)} (m^2)^{\frac{s-1}{2}} \right\} . \] (3.7)

The terms with odd powers of \( \mu \) cancel when we write down an expression for \( I_+(s) + I_-(s) \). In the \( \mu^2 \) and \( \mu^4 \) terms, the \( \Gamma \)-function in the numerator is divergent at \( s = -1 \), but can be analytically continued to cancel away the factor of \( (s+1) \) multiplying each term. Thus,

\[ I_+(s) + I_-(s) = \frac{\sqrt{\pi} \beta V}{8 \pi^2} \left\{ \frac{\Gamma\left(\frac{s}{2} - \frac{3}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} (m^2)^{\frac{s-1}{2}} + 2s \mu^2 \frac{\Gamma\left(\frac{s}{2} + \frac{3}{2}\right)}{(s-1)\Gamma\left(\frac{s}{2} + 1\right)} (m^2)^{\frac{s+1}{2}} + s(s+2)(s+3) \frac{\mu^4 \Gamma\left(\frac{s}{2} + \frac{1}{2}\right)}{12 \Gamma\left(\frac{s}{2} + 2\right)} (m^2)^{\frac{s+1}{2}} \right\} . \] (3.8)

All even, higher order terms in \( \mu \) have analytic \( \Gamma \)-functions in the numerator, and so the \( (s+1) \) ensures they are all zero at \( s = -1 \). This is why we were able to stop expanding at fourth order in \( \mu \).

Turning now to \( J(s) \), we see it is exactly the first term of (3.5):

\[ J(s) = \frac{4 \pi \beta V}{(2\pi)^3} \int_0^\infty dk \ k^2 \ (k^2 + m^2)^{-\frac{s}{2}} \]

\[ = \frac{\sqrt{\pi} \beta V \Gamma\left(\frac{s}{2} - \frac{3}{2}\right)}{8 \pi^2 \Gamma\left(\frac{s}{2}\right)} (m^2)^{\frac{s-1}{2}} \] \quad (3.9)

Using (3.3), it is now a straightforward matter to show that the multiplicative anomaly is

\[ a_4 = \frac{\beta V}{8 \pi^2 \mu^2} \left( m^2 - \frac{\mu^2}{3} \right) \] \quad (3.10)

in agreement with (3).

This calculation sheds some light on why the high temperature expansions of \( \Gamma_A \) and \( \Gamma_B \) differ, and the exact expressions agree. In the high temperature situation, the integrations over \( k \) were carried out after the expansions, and for a reason which is not clear to us at the present, the chemical potentials in the zero-point energies of the A-factorisation were not able to cancel. So the zero-point energies were of the form \( (\beta/2) \sum_n(E_n \pm \mu) \). In the B-factorisation, the energy levels were just \( \beta \sum_n E_n \) and so there was no anomaly. In the exact expressions for \( \Gamma_A \) and \( \Gamma_B \), the integrals have not even been performed, and so the \( +\mu \) and \( -\mu \) simply disappear, leaving no trace of a discrepancy.

Although simply by looking at the A factorisation (2.3), (2.4), we cannot say whether or not it will produce a multiplicative anomaly in the high temperature expansion, given that we know it does produce an anomaly, we can say something about why it does. The canonical energy levels for our system are derived from the Hamiltonian operator \( H \),

\[ H = \sum_n E_n (a_n^{\dagger} a_n + \frac{1}{2}) + b_n^{\dagger} b_n + \frac{1}{2} \] \quad (3.11)

where \( a_n^{\dagger}, a_n \) (\( b_n^{\dagger}, b_n \)) are the creation and annihilation operators for particles (anti-particles). For a system of charged fields with a chemical potential, the full Hamiltonian (which is the argument of the exponential in the partition function) is

\[ \tilde{H} = H - \mu : Q : \] \quad (3.12)

where \( : Q : \) is the normal ordered charge operator

\[ : Q := \sum_n (a_n^{\dagger} a_n - b_n^{\dagger} b_n) . \] \quad (3.13)

Note that we have to normal order by hand. There is no good mathematical reason why we normal order, we just like to have an uncharged vacuum.
\[ \langle 0 | : Q : | 0 \rangle = 0 . \]  

(3.14)

It now becomes clear why we have two different expressions for the energy levels (3.1), (3.2). They correspond to the eigenvalues of \( H \) and \( \bar{H} \) respectively in the case when the charge operator \( Q \) is not normal ordered. So to avoid having an anomaly we need to ensure that both \( H \) and \( \bar{H} \) have the same eigenvalues - we need to normal order \( Q \).

This implies that the A factorisation, which gives rise to an anomaly, is not normal ordered. This is a symptom of using Feynman path integrals, indeed Bernard was aware of this in 1974 - the last sentence in section II of his seminal paper \[15\] reads, ‘... the functional-integral formalism never does normal ordering for us.’

**B. The interacting case**

The multiplicative anomaly can also be calculated in the interacting case, and can again be seen to be the difference of \( \zeta \)-regularised zero-point energies.

Let us define

\[
X_{\pm} = \frac{\beta}{2} \sum_{n} \left[ E_{n}^{2} + \frac{\lambda \phi^{2}}{3} + \mu^{2} \pm \left( 4 \mu^{2} \left( E_{n}^{2} + \frac{\lambda \phi^{2}}{3} \right) + \frac{\lambda^{2} \phi^{4}}{36} \right)^{\frac{1}{2}} \right]^{\frac{\pm}{2}} \tag{3.15}
\]

\[
Y_{\pm} = \frac{\beta}{2} \sum_{n} \left( E_{n}^{2} + \frac{\lambda \phi^{2}}{2} \right)^{\frac{\pm}{2}} + \frac{\beta}{2} \sum_{n} \left( E_{n}^{2} + \frac{\lambda \phi^{2}}{6} \right)^{\frac{\pm}{2}} \tag{3.16}
\]

in an analogous way to \( I_{\pm} \) and \( J \) in the non-interacting case. (See for example \[16\] for a full derivation of the energy levels). Then we have

\[
X_{\pm}(s) = \frac{\beta}{2} \sum_{n} \left[ E_{n}^{2} + \frac{\lambda \phi^{2}}{3} + \mu^{2} \pm \left( 4 \mu^{2} \left( E_{n}^{2} + \frac{\lambda \phi^{2}}{3} \right) + \frac{\lambda^{2} \phi^{4}}{36} \right)^{\frac{1}{2}} \right]^{-\frac{\pm}{2}} \tag{3.17}
\]

\[
Y(s) = \frac{\beta}{2} \sum_{n} \left( E_{n}^{2} + \frac{\lambda \phi^{2}}{2} \right)^{-\frac{\pm}{2}} + \frac{\beta}{2} \sum_{n} \left( E_{n}^{2} + \frac{\lambda \phi^{2}}{6} \right)^{-\frac{\pm}{2}} \tag{3.18}
\]

To evaluate \( X_{\pm}(s) \), we expand the square root inside the square bracket in powers of \( \lambda \) up to \( O(\lambda^{2}) \):

\[
X_{\pm}(s) = \frac{\beta}{2} \sum_{n} \left[ a_{0} + a_{1} \lambda + a_{2} \lambda^{2} \right]^{-\frac{\pm}{2}} \tag{3.19}
\]

We note that we can switch from \( X_{+}(s) \) to \( X_{-}(s) \) by letting \( \mu \to -\mu \). Therefore we shall work with \( X_{+}(s) \) for simplicity, and let \( \mu \to -\mu \) to write down \( X_{-}(s) \) at the end of the calculation. \( X_{+}(s) \) can be written in the form,

\[
X_{+}(s) = \frac{\beta}{2} \sum_{n} \left[ a_{0} + a_{1} \lambda + a_{2} \lambda^{2} \right]^{-\frac{\pm}{2}} = \frac{\beta}{2} \sum_{n} a_{0}^{-\frac{\pm}{2}} \left[ 1 + a_{0}^{-1} a_{1} \lambda + a_{0}^{-1} a_{2} \lambda^{2} \right]^{-\frac{\pm}{2}} = \frac{\beta}{2} \sum_{n} \left( a_{0}^{-\frac{\pm}{2}} - \frac{1}{2} \frac{\phi^{2}}{3} a_{0}^{-\frac{\pm}{2}} a_{1} \lambda - \frac{1}{2} \frac{\phi^{4}}{36} a_{0}^{-\frac{\pm}{2}} a_{2} \lambda^{2} + \frac{1}{8} s(s + 2) a_{0}^{-\frac{\pm}{2}} - 2 a_{1} \lambda^{2} \right) \tag{3.20}
\]

where

\[
a_{0} = \left( E_{n} + \mu \right)^{2} \]
\[
a_{1} = \frac{\phi^{2}}{3} \left( 1 + \mu E_{n}^{-1} \right) \]
\[
a_{2} = \frac{\phi^{4}}{36} \left( \frac{E_{n}^{-1}}{4 \mu} - \mu E_{n}^{-3} \right) . \tag{3.21}
\]

It is then a straightforward (if a little tedious) matter to expand each term of (3.20) and integrate over \( k \), as was done in the non-interacting case.
After the dust settles, we find

\[
X_+(s) + X_-(s) = \frac{\sqrt{\pi \beta V}}{8\pi^2} \left\{ \frac{\Gamma\left(\frac{s}{2} - \frac{3}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} (m^2)^{\frac{s}{2} - \frac{3}{2}} + 2s\mu^2 \frac{\Gamma\left(\frac{s}{2} + \frac{1}{2}\right)}{(s-1)\Gamma\left(\frac{s}{2}\right)} (m^2)^{\frac{s}{2} - \frac{1}{2}} \right. \\
\quad + s(s+2)(s+3) \frac{\mu^4}{12} \frac{\Gamma\left(\frac{s}{2} + \frac{3}{2}\right)}{\Gamma\left(\frac{s}{2} + 2\right)} (m^2)^{\frac{s}{2} - \frac{3}{2}} - \frac{\sqrt{\pi \beta V} \lambda \phi^2}{48\pi^2} \left. \int \frac{s(s+2)(s+3) \frac{\mu^4}{12} \frac{\Gamma\left(\frac{s}{2} + \frac{3}{2}\right)}{\Gamma\left(\frac{s}{2} + 2\right)} (m^2)^{\frac{s}{2} - \frac{3}{2}}}{8\pi^2 s(s+2)(s+3) \frac{\mu^4}{12} \frac{\Gamma\left(\frac{s}{2} + \frac{3}{2}\right)}{\Gamma\left(\frac{s}{2} + 2\right)} (m^2)^{\frac{s}{2} - \frac{3}{2}}} \right) \right\}
\]

(3.22)

By expanding (3.18) to order \( \lambda^2 \), it can easily be shown that

\[
Y(s) = \frac{\sqrt{\pi \beta V}}{8\pi^2} \left\{ \frac{\Gamma\left(\frac{s}{2} - \frac{1}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} (m^2)^{\frac{s}{2} - \frac{1}{2}} - s \frac{\lambda \phi^2}{6} \frac{\Gamma\left(\frac{s}{2} - \frac{1}{2}\right)}{\Gamma\left(\frac{s}{2} + 1\right)} (m^2)^{\frac{s}{2} - \frac{1}{2}} + \frac{5}{288} s(s+2) \lambda^2 \phi^4 \frac{\Gamma\left(\frac{s}{2} + \frac{1}{2}\right)}{\Gamma\left(\frac{s}{2} + 2\right)} (m^2)^{\frac{s}{2} - \frac{3}{2}} \right\}
\]

(3.23)

and hence,

\[
a_4 = X_+(-1) + X_-(-1) - Y(-1) = \frac{\beta V}{8\pi^2 \mu^2} \left( m^2 + \frac{\lambda \phi^2}{3} - \frac{\beta V}{3} \right).
\]

(3.24)

This agrees with the result in [17] except for a term in \( \lambda^2 \phi^4 \). This is of no consequence however, since any term proportional to the background field \( \phi \) (but not the chemical potential) may be added to the effective action without changing the physics of the system. All such terms can be harmlessly absorbed by renormalisation.

This section has shown that the multiplicative anomaly has its roots in the manipulation of infinite sums - the non-interacting case in particular demonstrates how the anomaly can appear in relatively simple situations. Elizalde showed the existence of the multiplicative anomaly in possibly the simplest of all cases - infinite, diagonal matrices with real numbers [17]. The first worked example in [17] is striking in its similarity to the calculation presented above in the non-interacting model.

IV. CONCLUSIONS

We have demonstrated that the multiplicative anomaly originates in the zero-point energies of fields and is a consequence of shifting the energies by a constant amount. When regularisation is performed, these shifts (+\( \mu \) and −\( \mu \), for example) are unable to cancel and result in a multiplicative anomaly. It seems therefore that in order to perform anomaly-free calculations one must resist the temptation to integrate over the momentum until the very end, after say, a high temperature expansion has been written down.

It should be borne in mind that the functional integral approach to quantum field theory is not as complete as the canonical one. As was mentioned in Sec. III, path integrals completely neglect normal ordering. Coleman discusses the merits of path integrals at length in his Erice lectures [18] and echoes the comments of Bernard; the functional integral approach does not normal order.

From inspection of equations (3.1) and (3.2) it is tempting to conclude that \( \zeta\)-function regularisation may be to blame for the multiplicative anomaly (as opposed to functional integration). Certainly it does not seem likely that an anomaly would survive if say, dimensional regularisation were used to calculate the difference between (3.1) and (3.2). But it should be remembered that these equations were written down almost naively, to demonstrate the source of the anomaly; as was mentioned above, one should wait until the last possible moment before performing the integration over \( k \), after everything that can cancel has done so. Nevertheless this does not seem very satisfactory. Why is there a difference between (3.3) and (3.4)? The mathematical properties of the \( \zeta\)-function are rigorous and well defined - to negatively criticise the whole subject of \( \zeta\)-function regularisation is a step not to be taken lightly. The paper by Elizalde [17] provides some very interesting mathematical examples of multiplicative anomalies derived from infinite matrices, sometimes using nothing more than Riemann’s \( \zeta\)-function.

These mathematical peculiarities aside, it has been clearly demonstrated in this paper that the problem associated with the multiplicative anomaly can be removed by considering a Hamiltonian with a normal ordered charge operator. Normal ordering is in some ways an artificial procedure that physicists perform to make the theory more physical - it is not prescribed by the theory and there is no mathematical reason why it is done. Using canonical techniques it is easy to see how to normal order, unfortunately it is not so obvious in the functional integral approach - equation
is not normal ordered, but (1.3) is. An important step in understanding the multiplicative anomaly would be finding an a priori method of knowing which factorisations lead to anomalies and which do not, without having to compare with canonical calculations.

The problem of the multiplicative anomaly appears to be quite deeply rooted in the mathematics of infinite, divergent series.

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**APPENDIX: AN ALTERNATIVE METHOD OF CALCULATING $\Gamma_A^{(1)}$**

The method presented here for calculating (3.5) is simpler than that given in the main text, but has the disadvantage that the $\zeta$-function can only be evaluated at $s = 0$. The method in the main text can be used for an arbitrary value of $s$ (although the integral in (2.10) may have to evaluated numerically).

We can re-write (2.4) as,

$$
\zeta_A(s; a, b) = \left( \frac{2\pi}{\beta} \right)^{-4s} \sum_n \sum_{j=-\infty}^{\infty} (j^2 + a^2)^{-s} (j^2 + b^2)^{-s}
$$  \hspace{1cm} (A1)

where $a = \beta/2\pi(E_n + \mu)$ and $b = \beta/2\pi(E_n - \mu)$. Differentiating with respect to $(a^2)$ and $(b^2)$:

$$
\frac{\partial}{\partial(a^2)} \zeta_A(s; a, b) = -sf(s; a, b)
$$

$$
\frac{\partial}{\partial(b^2)} \zeta_A(s; a, b) = -sg(s; a, b)
$$  \hspace{1cm} (A2)

where

$$
f(s; a, b) = \left( \frac{2\pi}{\beta} \right)^{-4s} \sum_n \sum_{j=-\infty}^{\infty} (j^2 + a^2)^{-s-1} (j^2 + b^2)^{-s}
$$

$$
g(s; a, b) = \left( \frac{2\pi}{\beta} \right)^{-4s} \sum_n \sum_{j=-\infty}^{\infty} (j^2 + a^2)^{-s} (j^2 + b^2)^{-s-1}.
$$  \hspace{1cm} (A3)

Clearly,

$$
\frac{\partial}{\partial(a^2)} \zeta_A(0; a, b) = 0
$$

$$
\frac{\partial}{\partial(b^2)} \zeta_A(0; a, b) = 0
$$  \hspace{1cm} (A4)

and so we can conclude

$$
\zeta_A(0; a, b) = C
$$  \hspace{1cm} (A5)

where $C$ is a constant independent of $a$ and $b$. We can set $a = b$ in order to evaluate the left hand side of (A5) and determine the constant $C$. Using the Plana formula to calculate $\zeta_A(s; a, a)$ is straightforward:

$$
\sum_{j=-\infty}^{\infty} (j^2 + a^2)^{-2s} = \int_{-\infty}^{\infty} dj (j^2 + a^2)^{-2s} + 2 \int_{-\infty+i\epsilon}^{\infty+i\epsilon} (e^{-2\pi iz} - 1)^{-1} (z^2 + a^2)^{-2s} dz
$$  \hspace{1cm} (A6)

and so

$$
\zeta_A(s; a, a) = \left( \frac{2\pi}{\beta} \right)^{-4s} \sum_n \left[ \sqrt{\pi} \frac{\Gamma(2s - \frac{1}{2})}{\Gamma(2s)} (a^2)^{\frac{1}{2} - 2s} - 4s \ln (1 - e^{-2\pi a}) \right]
$$  \hspace{1cm} (A7)
(See the appendix of [11] for a more thorough evaluation of a similar sum.) Consequently \( \zeta_A(0; a, a) = C = 0 \). Next we need to evaluate \( \zeta_A'(0; a, b) \),

\[
\frac{\partial}{\partial (a^2)} \zeta_A'(0; a, b) = -f(0; a, b) \\
\frac{\partial}{\partial (b^2)} \zeta_A'(0; a, b) = -g(0; a, b)
\]  

(A8)

The functions \( f(s; a, b) \) and \( g(s; a, b) \) can easily be calculated at \( s = 0 \):

\[
\frac{\partial}{\partial (a^2)} \zeta_A'(0; a, b) = -\sum_n \frac{\pi}{a} \coth(\pi a) \\
\frac{\partial}{\partial (b^2)} \zeta_A'(0; a, b) = -\sum_n \frac{\pi}{b} \coth(\pi b)
\]  

(A9)

Integrating,

\[
\zeta_A'(0; a, b) = \sum_n \left\{ -2 \ln \left[ \sinh(\pi a) \right] - 2 \ln \left[ \sinh(\pi b) \right] \right\} + \mathcal{K}.
\]  

(A10)

Again there is a constant \( \mathcal{K} \), independent of \( a \) and \( b \). We can evaluate it in the same way as before,

\[
\mathcal{K} = \zeta_A'(0; a, a) + 4 \sum_n \ln \left[ \sinh(\pi a) \right].
\]  

(A11)

Using (A7) we see that \( \mathcal{K} = -4 \ln 2 \). Writing the \( \sinh \)s in (A10) as exponentials we arrive at

\[
\zeta_A'(0; a, b) = \sum_n \left\{ -2\pi a - 2\pi b - 2 \ln \left( 1 - e^{-2\pi a} \right) - 2 \ln \left( 1 - e^{-2\pi b} \right) \right\}
\]  

(A12)

Substituting in the values of \( a \) and \( b \),

\[
\zeta_A'(0) = \sum_n \left\{ -\beta(E_n + \mu) - \beta(E_n - \mu) - 2 \ln \left( 1 - e^{-\beta(E_n+\mu)} \right) - 2 \ln \left( 1 - e^{-\beta(E_n-\mu)} \right) \right\}
\]  

(A13)

and so finally we have,

\[
\Gamma_A^{(1)} = -\frac{1}{2} \zeta_A'(0) = \sum_n \left\{ \beta E_n + \ln \left( 1 - e^{-\beta(E_n+\mu)} \right) + \ln \left( 1 - e^{-\beta(E_n-\mu)} \right) \right\}.
\]  

(A14)

It is interesting to see that the zero-point energies in (A13) are written initially with positive and negative chemical potentials.

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