Collapse of a self-similar cylindrical scalar field with non-minimal coupling

Eoin Condron and Brien C. Nolan
School of Mathematical Sciences, Dublin City University, Glasnevin, Dublin 9, Ireland.
E-mail: eoin.condron4@mail.dcu.ie, brien.nolan@dcu.ie

Abstract. We investigate self-similar scalar field solutions to the Einstein equations in whole cylinder symmetry. Imposing self-similarity on the spacetime gives rise to a set of single variable functions describing the metric. Furthermore, it is shown that the scalar field is dependent on a single unknown function of the same variable and that the scalar field potential has exponential form. The Einstein equations then take the form of a set of ODEs. Self-similarity also gives rise to a singularity at the scaling origin. We discuss the number of degrees of freedom at an arbitrary point and prove existence and uniqueness of a 2-parameter family of solutions with a regular axis. We discuss the evolution of these solutions away from the axis toward the past null cone of the singularity, determining the maximal interval of existence in each case.

1. Introduction.

The issue of cosmic censorship is one of the principal outstanding questions in general relativity. The cosmic censorship hypothesis (CCH) states that all singularities are cut off from the external universe by an event horizon. There are, however, numerous examples of spacetimes which exhibit naked singularity formation, and the ultimate aim of this work is to determine if the class of spacetimes which give the title to this paper is numbered among them.

Self-similarity plays an important role in many theories of classical physics. In general relativity, Carr’s self-similarity hypothesis asserts that under certain physical conditions, solutions naturally evolve to a self-similar form [1], thus self-similar solutions are highly relevant to the study of gravitational collapse. The assumption of self-similarity brings about a significant simplification, reducing the field equations to ODEs, and self-similar spherically symmetric spacetimes are now well understood. Indeed, many of the spacetimes violating the CCH are self-similar.

As a departure from spherical-symmetry and in an effort to elucidate non-spherical collapse, some work has been done on cylindrical symmetry, for example [2]-[9]. This paper is the first of two, whose overall aim is to add to this body of work with a rigorous analysis of self-similar, cylindrically symmetric spacetimes coupled to a non-linear scalar field. We assume self-similarity of the first kind, and show that there exists a curvature
singularity where the homothetic Killing vector is identically zero, known as the scaling origin $O$. We study solutions with a regular axis $III$, which is necessary for regular initial data from which the field equations may be evolved. To determine whether this class of spacetimes exhibits NS formation, we aim to find the global structure of solutions and determine whether the future null cone of the singularity, which we label $N_+$, exists as part of the spacetime. In the case where $N_+$ is part of the spacetime, it corresponds to an absolute Cauchy horizon.

The fields equations have three singular points; along the axis, along the past null cone of the origin, labelled $N_-$, and along $N_+$. This gives a natural division of the problem into two stages; solutions between the axis and $N_-$, called region $I$, and between $N_-$ and $N_+$, called region $II$. The content of this paper deals with region $I$. In a follow up paper we investigate the structure of solutions which extend beyond $N_-$. The layout of the paper is as follows. Section 2 gives the formulation of the field equations and initial data along the axis, and a proof that a singularity exists at the scaling origin. We also show that the minimally coupled-case is mathematically equivalent to the vacuum case. In the general case, the system has two parameters and a free initial datum, different ranges of which give different global structures. We also summarise the main results of the paper in this section, with the proofs given in later sections. In section 3 we prove existence and uniqueness of solutions to the initial value problem formulated in section 2, using a fixed-point argument. In sections 4 and 5 we determine the global structure of solutions in region $I$. In some cases, exact solutions may be found and these are deferred to section 5. Section 6 gives the proof of the main theorem of this paper and we conclude with a brief summary of conclusions in section 7. We use units such that $8\pi G = c = 1$ throughout.

2. Self-similar cylindrically symmetric spacetimes coupled to a non-linear scalar field with a regular axis

Here we give the Einstein equations and initial conditions that correspond to self-similar cylindrically symmetric scalar field spacetimes. In section 2.1, we give the line element for whole cylinder symmetry, and write down the Einstein equations for the case of a non-linear scalar field. In section 2.2, we treat the minimally coupled case, showing that it is mathematically equivalent to the vacuum case. In section 2.3 we specialise to the case of self-similarity of the first kind, and show how this assumption reduces the EFEs to a set of ODEs. We also determine that the scalar field potential has exponential form. In section 2.4, we discuss the regular axis conditions, and show how these give rise to initial conditions for the ODEs derived in section 2.3. We then prove in section 2.5 that, except in the case of at spacetime, there is a singularity at the scaling origin. In section 2.6 we write down the initial value problem that is studied in the remainder of the paper and we conclude with a statement of the main results of the paper in section 2.7.
2.1. The Einstein equations for a cylindrically symmetric scalar field

We consider cylindrically symmetric spacetimes with whole-cylinder symmetry [2]. This class of spacetimes admits a pair of commuting, spatial Killing vectors $\xi(\theta), \xi(z)$ called the azimuthal and translational Killing vectors, respectively. Introducing double null coordinates $(u, v)$ on the lorentzian 2-spaces orthogonal to the surfaces of cylindrical symmetry, the line element may be written as:

$$ds^2 = -2e^{2\gamma+2\delta}dudv + e^{2\phi}r^2d\theta^2 + e^{-2\phi}dz^2,$$

where $r$ is the radius of cylinders, $\bar{\gamma}, \phi$ and $r$ depend on $u$ and $v$ only.

We take the matter source to be a cylindrically symmetric, self-interacting scalar field $\psi(u, v)$ with stress-energy tensor given by

$$T_{ab} = \nabla_a \psi \nabla_b \psi - \frac{1}{2}g_{ab} \nabla^c \psi \nabla_c \psi - g_{ab} V(\psi),$$

where $V(\psi)$ is the scalar field potential. The form of the line element is preserved by the coordinate transformations

$$u \to \bar{u}(u), \quad v \to \bar{v}(v), \quad z \to \lambda z,$$

for constant $\lambda$. Note that $\theta \in [0, 2\pi)$ and so transformations of the kind $\theta \to \lambda \theta$ are not allowed in general. The full set of Einstein equations for these spacetimes is

$$
\begin{align}
2r_u \bar{\gamma}_u - r_{uu} - 2r \bar{\phi}_u^2 &= r \psi_v^2, \\
r_{uv} &= re^{2\gamma+2\delta}V(\psi), \\
2r_v \bar{\gamma}_v - r_{vv} - 2r \bar{\phi}_v^2 &= r \psi_u^2, \\
2(\bar{\phi}_u \bar{\phi}_v + \bar{\gamma}_{uv}) &= -\psi_u \psi_v + e^{2\gamma+2\phi}V(\psi) \\
2(r_u \bar{\phi}_v + r_v \bar{\phi}_u + r_u \bar{\phi}_u + r_{uv} + r \bar{\phi}_{uv} + 2r \bar{\phi}_{uv}) &= -\psi_u \psi_v + e^{2\gamma+2\phi}V(\psi).
\end{align}
$$

The subscripts denote partial derivatives here. Our field $\psi$ satisfies

$$\nabla^a \nabla_a \psi - V'(\psi) = 0,$$

which implies $\nabla^a T_{ab} = 0$. Letting $g = e^{4\phi+4\gamma}r^2$ denote the metric determinant we have

$$\nabla^a \nabla_a \psi = \frac{1}{(-g)^{1/2}} \partial_a \left[(-g)^{1/2} g^{ab} \partial_b \psi\right] = \frac{1}{(-g)^{1/2}} \left[\partial_a ((-g)^{1/2} g^{uv} \partial_u \psi) + \partial_v ((-g)^{1/2} g^{vu} \partial_v \psi)\right]$$

$$= -\frac{1}{e^{2\gamma+2\phi}r} \left[\partial_a (r \psi_v) + \partial_v (r \psi_u)\right].$$

Combining this with (5) and simplifying, we arrive at

$$2r \psi_{uv} + r_v \psi_u + r_u \psi_v + re^{2\gamma+2\phi}V'(\psi) = 0,$$

which is the wave equation for the scalar field $\psi$. A useful simplification is obtained by subtracting $r(4d)$ from (4e), which gives

$$2r \bar{\phi}_{uv} + r_u \bar{\phi}_v + r_v \bar{\phi}_u + r_{uv} = 0.$$

Note that (7) can be derived from (4a)-(4e) and that from this point onward we make use of (4a)-(4e), (7) and (8) in our analysis.
2.2. The minimally coupled case

In this section we deal with the case where the scalar field potential $V$ is equal to zero. The field equations simplify greatly in this case and we show that solving the Einstein equations is effectively the same as in the vacuum case. With $V = 0$, equation (4b) gives

$$\alpha_{uv} = 0, \quad r = f(u) + g(v). \quad (9)$$

We require the absence of trapped cylinders in the initial configuration so the gradient of $r$ must be spacelike $[10]$. This reduces to the condition

$$f'(u)g'(v) < 0. \quad (10)$$

Using the coordinate freedom (3), we then set

$$r = \frac{v - u}{\sqrt{2}}. \quad (11)$$

To demonstrate equivalence to the vacuum case, we follow the example of [9] and introduce time and radial coordinates

$$T = \frac{v + u}{\sqrt{2}}, \quad X = \frac{v - u}{\sqrt{2}}. \quad (12)$$

The line element is given by

$$ds^2 = e^{2\bar{\gamma}} + 2\bar{\phi}(dX^2 - dT^2) + X^2 e^{2\bar{\phi}} d\theta^2 + e^{-2\bar{\phi}} dz^2, \quad (13)$$

the remaining field equations then reduce to

$$\bar{\gamma}_X = X \left( \bar{\phi}_T^2 + \bar{\phi}_X^2 + \frac{\psi_T^2}{2} + \frac{\psi_X^2}{2} \right), \quad (14a)$$

$$\bar{\gamma}_T = X(2\bar{\phi}_T \bar{\phi}_X + \psi_X \psi_T), \quad (14b)$$

$$\psi_{TT} - \psi_{XX} - \frac{\psi_X}{X} = 0, \quad (14c)$$

$$\bar{\phi}_{TT} - \bar{\phi}_{XX} - \frac{\bar{\phi}_X}{X} = 0. \quad (14d)$$

Given regular initial data, the linear wave equations for $\psi$ and $\bar{\phi}$ yield unique, globally hyperbolic, singularity free solutions. Solutions of $\bar{\gamma}$ may be then be obtained from (14a) and (14b). We note that this is, essentially, mathematically equivalent to the vacuum case and refer the reader to [12] and [9] (for the self-similar case) for a full treatment of the problem.

2.3. Self-similarity

We assume self-similarity of the first kind $[1]$ which is equivalent to the existence of a homothetic killing vector field $\xi$, such that

$$\mathcal{L}_\xi g_{ab} = 2g_{ab}, \quad (15)$$

where $\mathcal{L}_\xi$ denotes the Lie derivative along the vector $\xi$. We assume that $\xi$ has the form

$$\xi = \alpha(u, v) \frac{\partial}{\partial u} + \beta(u, v) \frac{\partial}{\partial v}, \quad (16)$$
Collapse of a self-similar cylindrical scalar field with non-minimal coupling

so that $\vec{\xi}$ is orthogonal to the cylinders of symmetry. Equation (15) is equivalent to

$$\nabla_\mu \xi^\nu + \nabla_\nu \xi^\mu = 2g_{\mu\nu},$$

which leads to $\alpha = \alpha(u)$ and $\beta = \beta(v)$. We then use the coordinate freedom (3) to rescale $u$ and $v$ such that $\alpha(u) = 2u$ and $\beta(v) = 2v$.

Having made this transformation, equations (15) yield

$$\bar{\gamma} = \gamma(\eta), \quad \bar{\phi} = \phi(\eta) - \log |u|^{1/2}, \quad r = |u|S(\eta),$$

where

$$\eta = \frac{v}{u},$$

is the similarity variable and $\gamma, \phi, S$ are metric functions for the self-similar metric, which is given by

$$ds^2 = -2|u|^{-2\gamma(\eta) + 2\phi(\eta)} dudv + |u|e^{2\phi(\eta)} S^2(\eta) d\theta + |u|e^{-2\phi(\eta)} dz^2.$$

The coordinate transformations that preserve this form of the metric are

$$u \rightarrow \lambda u, \quad v \rightarrow \mu v, \quad z \rightarrow \sigma z,$$

for constants $\lambda, \mu, \sigma$. The following result is stated without proof in [13] and [14], however, we found it useful to give a proof here.

**Proposition 2.1.** For a self-similar scalar field $\psi$ with energy-momentum tensor (2) and $V(\psi) \neq 0$, admitting a homothetic Killing vector $\xi$ such that (15) holds, the potential $V(\psi)$ has the exponential form

$$V(\psi) = \bar{V}_0 e^{-2\psi/k},$$

where $\bar{V}_0 \neq 0, k \neq 0$ are constants.

**Proof.** It can be shown that (15) leads to $\mathcal{L}_\xi T_{ab} = 0$ [1]. For $T_{ab}$ given by (2) we have

$$\psi_a \mathcal{L}_\xi \psi_b + \psi_b \mathcal{L}_\xi \psi_a - g_{ab}\left(\psi^c \psi_c + \frac{1}{2} \psi^c \mathcal{L}_\xi \psi_c + \frac{1}{2} \psi_c \mathcal{L}_\xi \psi^c \right) + 2V + V'(\psi) \mathcal{L}_\xi \psi = 0.$$ (23)

Now $\mathcal{L}_\xi \psi_c = \mathcal{L}_\xi g_{bc} \psi^b = 2\psi_c + g_{bc} \mathcal{L}_\xi \psi^b$, and so

$$\psi^c \mathcal{L}_\xi \psi_c = 2\psi^c \psi_c + \psi_c \mathcal{L}_\xi \psi^c.$$ (24)

Combining this with (23) and taking the trace then yields

$$-\psi^c \mathcal{L}_\xi \psi_c = 4V + 2V'(\psi) \mathcal{L}_\xi \psi.$$ (25)

Using (24) to eliminate $2V + V'(\psi) \mathcal{L}_\xi \psi$ from (23) and simplifying produces

$$\psi_a \mathcal{L}_\xi \psi_b + \psi_b \mathcal{L}_\xi \psi_a - \frac{1}{2} g_{ab} \psi^c \mathcal{L}_\xi \psi_c = 0.$$ (26)

Contracting with $\psi^a$ gives

$$\psi^c \psi_a \mathcal{L}_\xi \psi_b + \frac{1}{2} \psi_b \psi^c \mathcal{L}_\xi \psi_c = 0,$$ (27)
and contracting with $\psi^b$ gives
\begin{equation}
\frac{3}{2} \psi^b \psi_b (\psi^c \mathcal{L}_\xi \psi_c) = 0. \tag{28}
\end{equation}
In the case $\psi^c \psi_c = 0$ we have $\psi^c \mathcal{L}_\xi \psi_c = 0$, from (27), since we are assuming $\psi_b \neq 0$. It then follows from (24) that $\psi_c \mathcal{L}_\xi \psi^c = 0$. Contracting (26) with $\mathcal{L}_\xi \psi^b$ produces
\begin{equation}
\psi_a \mathcal{L}_\xi \psi^b \mathcal{L}_{\xi} \psi_b + \psi_b \mathcal{L}_\xi \psi^b \mathcal{L}_\xi \psi_a = \psi_a \mathcal{L}_\xi \psi^b \mathcal{L}_\xi \psi_b = 0, \tag{29}
\end{equation}
and we see that $\mathcal{L}_\xi \psi^b$ is null. Since it is also orthogonal to $\psi_b$, it must be parallel to it, i.e., $\mathcal{L}_\xi \psi^b = \Lambda \psi_b$ for some constant $\Lambda$. Putting this into (26) gives $2\Lambda \psi_a \psi_b = 0$, which reveals that $\Lambda$ must be zero, i.e. $\mathcal{L}_\xi \psi_b = 0$.

In the case $\psi^c \psi_c \neq 0$, we also have $\psi^c \mathcal{L}_\xi \psi_c = 0$, by (28). It follows immediately from (27) that $\mathcal{L}_\xi \psi_b = 0$ in this case also. It is straightforward to show that $\partial_b \mathcal{L}_\xi \psi = \mathcal{L}_\xi \psi_b$, so we have $\partial_b \mathcal{L}_\xi \psi = 0$, and thus $\mathcal{L}_\xi \psi = k$, for some constant $k$. Equation (25) then simplifies to $2V + kV' = 0$, which yields (22) for $k \neq 0$. Note that $k = 0$ gives $V = 0$, which has been dealt with above.

Corollary 2.1. If $\xi$ has the form (16) with $\alpha = 2u$ and $\beta = 2v$, then $\psi$ and $V(\psi)$ may be written as
\begin{equation}
\psi = F(\eta) + \frac{k}{2} \log |u|, \quad V(\psi) = \frac{\bar{V}_0 e^{-\frac{\xi}{2} F(\eta)}}{|u|}. \tag{30}
\end{equation}

Proof. In this case, $\mathcal{L}_\xi \psi = k$ reduces to
\begin{equation}
\mathcal{L}_\xi \psi = 2u \psi_a + 2v \psi_v = k, \tag{31}
\end{equation}
from which $\psi = F(\eta) + \log |u|^{k/2}$ follows. Theorem 2.1 then gives the potential $V$.

We are now in a position to formulate the field equations as a set of ODEs. In terms of $\gamma, \phi, S$ and $F$, (4a)-(4c), (7) and (8) are given by
\begin{align}
2\eta^\gamma (S - \eta S') + \eta^2 S'' + 2S \left(\eta \phi' + \frac{1}{2}\right)^2 &= -S \left(\eta F' - \frac{k}{2}\right)^2, \tag{32a} \\
\eta S'' &= -\bar{V}_0 Se^{2\gamma + 2\phi - 2F/k}, \tag{32b} \\
2S' \phi' - S'' - 2S \phi'^2 &= SF^2, \tag{32c} \\
2\eta S'' + 4\eta S \phi'' + 4\eta S' \phi' + 2S \phi' + S' &= 0, \tag{32d} \\
2\eta SF'' + 2\eta S' F' + SF' - \frac{kS'}{2} + \frac{2\bar{V}_0}{k} Se^{2\gamma + 2\phi - 2F/k} &= 0. \tag{32e}
\end{align}
Now, (32a) + $\eta^2$ (32b) simplifies to
\begin{equation}
\frac{1}{2} + 2\eta^\gamma + 2\eta \phi' = k \eta F' - \frac{k^2}{4}. \tag{33}
\end{equation}
Dividing by $\eta$ and integrating gives
\begin{equation}
2\gamma + 2\phi = k F - \left(\frac{1}{2} + \frac{k^2}{4}\right) \log |\eta| + c_1, \tag{34}
\end{equation}
for some constant $c_1$. Equation (32d) then reduces to
\[ \eta S'' = V_0 e^{(k-2/k)F} |\eta|^{-(1/2+k^2)/4} S, \] (35)
where $V_0 = \bar{V}_0 e^{c_1}$ and we have used (34) to replace $e^{2\gamma+2\phi}$. We define
\[ l = \frac{2F}{k} - \log |\eta|^{1/2}, \quad \lambda = \frac{k^2}{2} - 1, \] (36)
which gives
\[ \eta S'' = -V_0 |\eta|^{-1} e^{\lambda S}. \] (37)
Equation (32d) is exact and may be integrated to give
\[ 2S\phi' + S' = c_2 |\eta|^{-1/2}, \] (38)
for some constant $c_2$. Written in terms of $l$ and $S$, (32d) becomes
\[ \eta Sl'' + \eta S'l' + \frac{Sl'}{2} - \frac{S}{4\eta} + \frac{2V_0}{k^2|\eta|} S e^{\lambda S} = 0. \] (39)

2.4. The regular axis conditions

To ensure that the collapse ensues from an initially regular configuration we impose regular axis conditions [11] to the past of the scaling origin $(u,v) = (0,0)$. The areal radius $\rho$ and the specific length $L$ of the cylinders are given by the norms of the Killing vectors
\[ \rho = \sqrt{\xi_a(\theta) \xi_a(\theta) \xi_a(\theta) = |u|^\frac{3}{2} e^{\phi} S}, \quad L = \sqrt{\xi_a(z) \xi_a(z) \xi_a(z) = |u|^\frac{3}{2} e^{-\phi}}. \] (40)
The axis is defined by $\rho = 0$. We rule out the case $u = 0$ as this is a null hypersurface and we require the axis to be timelike. For a regular axis, the specific length $L$ must be non zero and finite, and so $\phi$ must be also be finite. A regular axis must therefore correspond to $S(\eta) = 0$. Hence, $\eta$ must be constant along the axis and a rescaling of $u$ and $v$ using the coordinate freedom (21) places the axis at $\eta = 1$.

Note that the past null cone of the origin $\mathcal{N}_-$ corresponds to $\eta = 0$ and the interval $\eta \in [0,1]$ constitutes region $\mathbf{I}$.

Further conditions for a regular axis are as follows [11]:
\[ \nabla^a \rho \nabla_a \rho = 1 + O(\rho^2), \quad \nabla^a \rho \nabla_a L = O(\rho), \quad \nabla^a L \nabla_a L = O(1), \] (41)
where the big-oh relations hold in the limit $\rho \to 0$. The first condition ensures the standard $2\pi$-periodicity of the azimuthal coordinate near the axis, while the remaining conditions ensure the absence of any curvature singularities at the axis.

**Proposition 2.2.** The regular axis conditions reduce to the following set of data:
\[ S(1) = 0, \quad S'(1) < 0, \quad \phi'(1) = -\frac{1}{4}, \quad \gamma'(1) = 0, \quad l'(1) = 0. \] (42)
Proof. Note that \( u < 0 \) to the past of the origin \((0,0)\) and, therefore, \( u < 0 \) on the axis. The equations (41) then give
\[
\lim_{\eta \to 1} 2e^{-2\gamma} (S' + S\phi')^2 = 1, \tag{43a}
\]
\[
\lim_{\eta \to 1} e^{-2\gamma} (S' + S\phi') \left( \frac{1}{2} + 2\phi' \right) = 0, \tag{43b}
\]
\[
\lim_{\eta \to 1} 2e^{-2\gamma} \left( \frac{1}{2} + \phi' \right) \phi' = L_0, \tag{43c}
\]
for some \( L_0 \in \mathbb{R} \). Equation (43a) gives
\[
\lim_{\eta \to 1} e^{-\gamma} (S' + S\phi') = \pm \frac{1}{\sqrt{2}}, \tag{43d}
\]
which may be used to simplify (43b) to
\[
\lim_{\eta \to 1} e^{-\gamma} \left( \frac{1}{2} + 2\phi' \right) = 0. \tag{43e}
\]
So we either have \( \lim_{\eta \to 1} \phi' = -1/4 \) or \( \lim_{\eta \to 1} e^{-\gamma} = 0 \). In the latter case, we must also have \( \lim_{\eta \to 1} e^{-\gamma} \phi' = 0 \). Now (38) may be used to replace \( S' + S\phi' \) with \( c_2 - S\phi' \) in (43d), and so
\[
\lim_{\eta \to 1} e^{-\gamma} (c_2 - S\phi') = \pm \frac{1}{\sqrt{2}}. \tag{43f}
\]
This clearly contradicts \( \lim_{\eta \to 1} e^{-\gamma} = \lim_{\eta \to 1} e^{-\gamma} \phi' = 0 \) since \( \lim_{\eta \to 1} S = 0 \), so we must have \( \phi'(1) = -1/4, S'(1) = c_2 \) and \( e^{-\gamma(1)} = \pm \sqrt{2}c_2 \neq 0 \).

The areal radius \( \rho \), and therefore \( S \), must increase away from the axis. Recall that \( \eta \in [0, 1] \) in region \( I \), so that \( \eta \) is decreasing away from the axis at \( \eta = 1 \). We must then have \( S'(1) < 0 \). Note that the field equations (26) are invariant under the transformation \( S \to S/(-S'(1)) \), so we may set \( S'(1) = -1 \).

It follows from finiteness of \( \gamma(1), \phi(1) \) and equations (34), (36) that \( F(1) \) and \( l(1) \) must also be finite. Equation (34) then gives \( S''(1) = 0 \) and, using this fact, (32c) gives \( \gamma'(1) = 0 \). Inserting \( \phi'(1) = -1/4 \) and \( \gamma'(1) = 0 \) into (33), we arrive at \( F'(1) = k/4 \), which is equivalent to \( l'(1) = 0 \).

Proposition 2.3. In the case \( \psi^c\psi_c = 0 \), solutions to the Einstein equation with line element and energy-momentum tensor given by (20) and (2), respectively, admit a regular axis if and only if \( k = 0 \).

Proof. first note that \( \psi^c\psi_c = 2g^{01}\psi_u\psi_v \) leads to either \( \psi_u = 0 \) or \( \psi_v = 0 \). Equation (30) then gives
\[
\psi_u = -\frac{vF'}{u^2} + \frac{k}{2u} = 0, \quad \psi_v = \frac{F'}{u} = 0, \tag{44a}
\]
\[
\Rightarrow F' = \frac{k}{2\eta}, \quad \Rightarrow F' = 0. \tag{44b}
\]
In both cases, \( F'(1) = k/4 \) holds if and only if \( k = 0 \).
Recall that $k = 0$ gives $V = 0$, which is the minimally-coupled case which has been dealt with in section 2.2.

2.5. Singular nature of the scaling origin

As an immediate consequence of the assumption of self-similarity, in the non-minimally coupled case, there exists a spacetime singularity where the homothetic Killing vector is identically zero, i.e., at $(u, v) = (0, 0)$.

**Proposition 2.4.** Let $\mathcal{T}$ denote the scalar invariant $T^{ab}T_{ab}$. Then
\[
\lim_{u \to 0} T \bigg|_{\eta = 1} = \infty.
\]

Proof. It is straightforward to show that
\[
\mathcal{T} = 2(g^{uv})^2 (T_{uu}T_{vv} + T_{uv}^2) + (g^{\theta\theta}T_{\theta\theta})^2 + (g^{zz}T_{zz})^2 \geq 2(g^{uv})^2 T_{uu}T_{vv}.
\]

Now,
\[
(g^{uv})^2 T_{uu}T_{vv} = \frac{e^{-4\gamma - 4\phi}}{u^2} \psi_u^2 \psi_v^2 = e^{-4\gamma - 4\phi} \left( \eta F' - \frac{k}{2} \right)^2 \frac{F'^2}{u^2}.
\]

Using $F'(1) = k/4$ we have
\[
\mathcal{T}(1) \geq 2e^{-4\gamma(1) - 4\phi(1)} \left( \frac{k^2}{16u} \right)^2.
\]

Since $\gamma(1), \phi(1)$ are finite and $k \neq 0$, taking the limit $u \to 0$ completes the proof. \qed

2.6. The initial value problem

Here we derive the form of the Einstein equations with which we will work for the remainder of the paper. The following choice of dependent and independent variables gives an autonomous system.

**Proposition 2.5** (Autonomous field equations). Let
\[
\tau = -\log \eta \quad R = e^{\tau/2}S.
\]

Then the interval $\eta \in [1, 0)$, i.e. region $I$ of the spacetime, corresponds to $\tau \in [0, \infty)$ and the field equations are equivalent to
\[
2\gamma + 2\phi = \frac{k^2l}{2} + \frac{\tau}{2} + c_1, \quad (50a)
\]
\[
\ddot{R} = \left( \frac{1}{4} - V_0e^{\lambda} \right) R, \quad (50b)
\]
\[
\dot{R} + \left( 2\phi - \frac{1}{2} \right) R = 1, \quad (50c)
\]
\[
\ddot{R} + \dot{R}l = \left( \frac{1}{4} - \frac{2}{k^2} V_0e^{\lambda} \right) R, \quad (50d)
\]
\frac{\dot{R}^2 - 1}{R^2} + \frac{k^2 \dot{R}}{R} + 2 V_0 e^{\lambda l} - 2 + \frac{k^2}{8} - \frac{k^2 \dot{l}^2}{2} = 0, \tag{50e}

R(0) = 0, \quad \dot{R}(0) = 1, \quad l(0) = l_0, \quad \dot{l}(0) = 0, \tag{50f}

where the overdot denotes a derivative with respect to $\tau$.

**Proof.** For a general function $f$ we have $\eta f' = -\dot{f}$ and $\eta^2 f'' = \ddot{f} + \dddot{f}$. Then for $\eta > 0$, multiplying equations (37), (38), (39) by $\eta$ and changing variables gives

\begin{align*}
\ddot{S} + \dot{S} &= -V_0 e^{\lambda l} S, \tag{51a}
2S \ddot{\phi} + \dot{S} &= -c^2 e^{-\tau/2} = e^{-\tau/2}, \tag{51b}
S \ddot{l} + \dot{S} \dot{l} + \frac{\dot{S}}{2} &= S \left( \frac{4}{4} - \frac{2 V_0 e^{\lambda l}}{k^2} \right). \tag{51c}
\end{align*}

We also have

\begin{align*}
\dot{S} &= e^{-\tau/2} \left( \dot{R} - \frac{R}{2} \right), \tag{51d}
\ddot{S} &= e^{-\tau/2} \left( \ddot{R} - \dot{R} + \frac{R}{4} \right), \tag{51e}
\end{align*}

which, when combined with the above, give (50b)-(50d). Equation (50a) comes directly from (34), (36) and the definition of $\tau$. Multiplying (32c) by $\eta^2$ and changing variables yields

\begin{equation}
2 \left( \dot{R} - \frac{R}{2} \right) \ddot{\gamma} + V_0 e^{\lambda l} R - 2 R \ddot{\phi}^2 = \frac{k^2 R}{4} \left( \dot{l} - \frac{1}{2} \right)^2, \tag{52}
\end{equation}

where we have used (36) and (37) to replace $\eta F'$ and $\eta^2 S''$, respectively. Using (50c) and the derivative of (50a) to replace $\ddot{\gamma}$ and $\ddot{\phi}$ with expressions in $R$ and $l$ then produces

\begin{equation}
\left( \dot{R} - \frac{R}{2} \right) \left( \frac{k^2 \dot{l}^2}{2} - \frac{1}{2} \frac{R}{R} + \frac{\dot{R}}{R} \right) + V_0 e^{\lambda l} R
- \frac{R}{2} \left( \frac{1}{2} \frac{R}{R} - \frac{\dot{R}}{R} + \frac{1}{2} \right)^2 = \frac{k^2 R}{4} \left( \dot{l} - \frac{1}{2} \right)^2. \tag{53}
\end{equation}

Multiplying by $2/R$ and simplifying, we arrive at (50e). Finally, the axis is at $\tau = 0$, so $R(0) = 0, l(0) = l_0$. Furthermore, $\dot{l}(0) = -l'(1) = 0$ and $\dot{S}(0) = -S'(1) = \ddot{R}(0) - R(0)/2$ which gives $\ddot{R}(0) = 1$.

Equations (50) are equivalent to the Einstein field equations with line element (20), energy-momentum tensor (2) and a regular axis. Henceforth, we study equations (50b), (50d) and (50e) to determine solutions for $R$ and $l$. $\phi$ is then obtained by integrating (50c) and once this is found, $\gamma$ is given by (50a). Note that $\tau \in [0, \infty)$ in region $I$ and $\tau \to \infty$ at $N^-$. \hfill $\square$
2.7. Summary of results

In this section we present two theorems which summarise the main results of the paper, whose proofs will follow in the following three sections.

**Theorem 2.1** (Existence and Uniqueness). Let \( k, V_0 \in \mathbb{R} \). For each \( l_0, \phi_0 \in \mathbb{R} \) there exists a unique solution of (50) on an interval \([0, \tau_\ast]\) corresponding to a spacetime with line element (20) and energy-momentum tensor (2). The spacetime admits a regular axis for \( u + v < 0 \).

**Theorem 2.2** (Global structure of solutions in the causal past of the scaling origin \( \mathcal{O} \)). For each \( k, V_0, l_0 \in \mathbb{R} \), let \([0, \tau_M)\) be the maximal interval of existence for the unique solution of (50). The global structure of the solution is given by one of the following cases:

Case 1. If \( k^2 > 2, V_0 < 0 \), then \( \tau_M < \infty \) and the hypersurface \( \tau = \tau_M \) corresponds to radial null infinity, with the Ricci scalar decaying to zero there.

Case 2. If \( k^2 = 2 \) and \( V_0 < 0 \), then \( \tau_M = +\infty \) and \( N_- \) corresponds to radial null infinity, with the Ricci scalar decaying to zero there.

Case 3. If

\[
(i) \quad k^2 < 2 \quad \text{and} \quad V_0 < 0, \\
(ii) \quad k^2 > 2, V_0 > 0 \quad \text{and} \quad V_0 e^{\lambda l_0} < k^2/8,
\]

then \( \tau_M = \infty \), the radius of the cylinders is non zero and finite on \( N_- \), which is reached in finite affine time along radial null rays. Hence, \( N_- \) exists as part of the spacetime.

Case 4. If

\[
(i) \quad k^2 < 2, V_0 > 0 \quad \text{and} \quad V_0 e^{\lambda l_0} > k^2/8, \\
(ii) \quad k^2 > 2 \quad \text{and} \quad V_0 > 0, \\
(iii) \quad k^2 = 2 \quad \text{and} \quad V_0 > 1/4,
\]

then \( \tau_M < \infty \) and there is a singularity at \( \tau = \tau_M \), with the radius of the cylinders of symmetry equal to zero there and which is reached by outgoing radial null rays in finite affine time.

Case 5. If

\[
(i) \quad k^2 = 2 \quad \text{and} \quad 0 < V_0 \leq 1/4, \\
(ii) \quad k^2 < 2 \quad \text{and} \quad V_0 e^{\lambda l_0} = k^2/8,
\]

then \( \tau_M = +\infty \) and there is a curvature singularity on \( N_- \), with the radius of the cylinders of symmetry equal to zero there and which is reached by outgoing radial null rays in finite affine time.
3. Existence and uniqueness of solutions with a regular axis

In this section we present a series of results which culminate in the proof of Theorem 2.1. The axis is a singular point of (50d) and so existence and uniqueness of a solution is not guaranteed. However, using a fixed-point argument, we can prove that for a given initial data set, a unique solution to (50) exists on an interval \([0, \tau^*]\), for some \(\tau^* > 0\).

Note that on the axis, \(\dot{R}\) is zero to first order only, that is, \(\dot{R}(0) = \frac{dR}{d\tau}|_{\tau=0} \neq 0\). It is convenient to work with a new variable \(x = R/\tau\), which is non-zero on the axis, and to look for solutions which are \(C^2\). We use a first-order reduction and write the system as a set of integral equations according to the following results:

**Lemma 3.1.** Let \(x = R/\tau\). Then initial data for \(x, \dot{x}\) corresponding to a regular axis are given by

\[
x(0) = 1, \quad \dot{x}(0) = 0.
\] (54)
Proof. Using Taylor’s theorem about \( \tau = 0 \) with the assumption that \( R \in C^2 \), we write \( R \) and thus \( x \) as

\[
R(\tau) = \tau + \ddot{R}(\hat{\tau}(\tau)) \frac{\tau^2}{2}, \tag{55a}
\]

\[
x(\tau) = 1 + \ddot{x}(\hat{\tau}(\tau)) \frac{\tau}{2}, \tag{55b}
\]

for some \( \hat{\tau} \in [0, \tau] \). Evaluating \( \dot{x}(0) \) from first principles, we find

\[
\dot{x}(0) = \lim_{\tau \to 0} \frac{x(\tau) - x(0)}{\tau} = \lim_{\tau \to 0} \frac{\ddot{R}(\hat{\tau})}{2\tau} = \lim_{\tau \to 0} \frac{\dddot{R}(\hat{\tau}(\tau))}{2}. \tag{56}
\]

Now, \( \hat{\tau} \in [0, \tau] \) goes to zero in the limit \( \tau \to 0 \) and so \( \dot{x}(0) = \dddot{R}(0)/2 = 0 \), from (50b).

Lemma 3.2. Let \( x_1 = x, x_2 = \dot{x}, x_3 = l, x_4 = \dot{l} \). Then (50b), (50d) and (50e) are equivalent to the following set of integral equations:

\[
x_1 = 1 + \int_0^\tau x_2(t) dt, \tag{57a}
\]

\[
x_2 = \int_0^\tau \frac{t^2}{\tau} x_1(t) \alpha(x_3(t)) dt, \tag{57b}
\]

\[
x_3 = l_0 + \int_0^\tau x_4(t) dt, \tag{57c}
\]

\[
x_4 = \int_0^\tau tx_1(t) \frac{\beta(x_3(t))}{\tau x_1(\tau)} dt, \tag{57d}
\]

where

\[
\alpha(x_3(t)) = \frac{1}{4} - V_0 e^{\lambda x_3(t)}, \quad \beta(x_3(t)) = \frac{1}{4} - \frac{2}{k^2} V_0 e^{\lambda x_3(t)}. \tag{57e}
\]

Proof. From (50b) and \( R = \tau x \), we have

\[
\tau \ddot{R} = \tau^2 \ddot{x} + 2\tau \dot{x} = \tau^2 x \left( \frac{1}{4} - V_0 e^{\lambda t} \right) \tag{58}
\]

which can be integrated to give

\[
\dot{x} = \frac{1}{\tau^2} \int_0^\tau t^2 x(t) \left( \frac{1}{4} - V_0 e^{\lambda t(t)} \right) dt.
\]

Equation (50d) may also be written in the integral form

\[
i = \frac{1}{R} \int_0^\tau R \left( \frac{1}{4} - \frac{2}{k^2} V_0 e^{\lambda t} \right) dt, \tag{59}
\]

\[
= \int_0^\tau \frac{tx_1(t)}{\tau x_1(\tau)} \beta(x_3(t)) dt.
\]

Equations (57a) and (57c) follow immediately from the definitions.
A solution of (57) corresponds to a fixed point of the the mapping $T : \vec{x} \rightarrow T(\vec{x}) = \vec{y}$ where

$$y_1 = 1 + \int_0^\tau x_2(t) dt,$$

$$y_2 = \int_0^\tau \frac{t^2}{\tau^2} x_1(t) \alpha(x_3(t)) dt,$$

$$y_3 = l_0 + \int_0^\tau x_4(t) dt,$$

$$y_4 = \int_0^\tau \frac{tx_1(t)}{\tau x_1(\tau)} \beta(x_3(t)) dt. \tag{60d}$$

We aim to use Banach’s fixed point theorem (the contraction mapping principle)\cite{15} to show that $T$ has a unique fixed point. We begin by defining the space $\chi$ in which $\vec{x}$ lies, which we require to be a closed subset of a Banach space. Let $E = C^0([0, \tau_*], \mathbb{R}^4)$, with the norm of a vector $\vec{x}$ given by

$$||\vec{x}||_E = \sup_{\tau \in [0, \tau_\ast]} |\vec{x}(\tau)| = \sup_{\tau \in [0, \tau_\ast]} \max_{1 \leq i \leq 4} |x_i(\tau)|. \tag{61}$$

$E$ is therefore a Banach space \cite{15}. Let

$$\chi(\tau_*, b, B) = \{ \vec{x} \in C^0([0, \tau_*], \mathbb{R}^4) : \vec{x}(0) = \vec{x}_0, \sup_{\tau \in [0, \tau_*]} ||\vec{x} - \vec{x}_0|| \leq B, \inf_{\tau \in [0, \tau_*]} x_1(\tau) \geq b > 0 \}, \tag{62}$$

where $\vec{x}(0) = (1, 0, l_0, 0)^T$, and $b < 1$. Then $\chi$ is a closed subset of $E$, and is therefore also a Banach space. We wish to show that it is possible to choose $\tau_*, b$ and $B$ such that $T$ is a contraction mapping on $\chi$, i.e. $T$ maps $\chi$ into itself and that there is a number $0 < \kappa < 1$ such that for any vectors $\vec{x}_1, \vec{x}_2 \in \chi$,

$$||\vec{y}_1 - \vec{y}_2|| \leq \kappa ||\vec{x}_1 - \vec{x}_2||.$$  \tag{63}

The following four results verify that $T\vec{x} = \vec{y} \in \chi$.

**Lemma 3.3.** The image $T\vec{x} = \vec{y}$ of $\vec{x} \in \chi$, has the same initial data as $\vec{x}$. That is, $\vec{y}(0) = \vec{x}_0$.

**Proof.** It is straightforward to show that the integral components in (60) equal zero at $\tau = 0$ by using the weighted mean value theorem for integrals. \hfill \square

**Lemma 3.4.** Let

$$M_1 = \max \left\{ B, (B + 1) \left( \frac{1}{4} + \delta \right), \frac{(B + 1)}{b} \left( \frac{1}{4} + \frac{2\delta}{k^2} \right) \right\}. \tag{64}$$

If $\tau_* \leq B/M_1$, then

$$\sup_{\tau \in [0, \tau_*]} ||\vec{x} - \vec{x}_0|| \leq B, \quad \Rightarrow \quad \sup_{\tau \in [0, \tau_*]} ||\vec{y} - \vec{y}_0|| \leq B, \tag{65}$$
Proof. We first note the following inequalities, which hold on \([0, \tau_*)\):
\[
\begin{align*}
b & \leq x_1 \leq B + 1, \quad x_2 \leq B, \quad x_3 \leq B + |l_0|, \quad x_4 \leq B, \quad (66a) \\
|\alpha(x_3)| &= \left| \frac{1}{4} - V_0 e^{\lambda x_3} \right| \leq \frac{1}{4} + \delta, \quad (66b) \\
|\beta(x_3)| &= \left| \frac{1}{4} - \frac{2}{k^2} V_0 e^{\lambda x_3} \right| \leq \frac{1}{4} + \frac{2\delta}{k^2}, \quad (66c)
\end{align*}
\]
where \(\delta = |V_0| e^{[\lambda(B+|l_0|)}\). Now,
\[
\sup_{\tau \in [0, \tau_*]} ||\vec{g} - \vec{g}_0|| = \sup_{\tau \in [0, \tau_*]} \max(A),
\]
where
\[
A = \left\{ \left| \int_0^\tau x_2(t) dt \right|, \left| \int_0^\tau \frac{t^2}{\tau^2} x_1(t) \alpha(x_3(t)) dt \right|, \left| \int_0^\tau x_4(t) dt \right|, \left| \int_0^\tau \frac{tx_1(t)}{\tau x_1(t)} \beta(x_3(t)) dt \right| \right\}.
\]
We derive a bound for each element of \(A\) as follows:
\[
\left| \int_0^\tau x_2(t) dt \right| \leq \int_0^\tau |x_2(t)| dt \leq \int_0^\tau Bdt = B\tau. \quad (69)
\]
\[
\left| \int_0^\tau \frac{t^2}{\tau^2} x_1(t) \alpha(x_3(t)) dt \right| \leq \int_0^\tau |x_1(t)\alpha(x_3(t))| dt, \\
\quad \leq \int_0^\tau (B + 1) \left( \frac{1}{4} + \delta \right) dt, \\
\quad = (B + 1) \left( \frac{1}{4} + \delta \right) \tau. \quad (70)
\]
\[
\left| \int_0^\tau x_4(t) dt \right| \leq \int_0^\tau |x_4(t)| dt \leq \int_0^\tau Bdt = B\tau. \quad (71)
\]
\[
\left| \int_0^\tau \frac{tx_1(t)}{\tau x_1(t)} \beta(x_3(t)) dt \right| \leq \int_0^\tau \left| \frac{x_1(t)}{x_1(\tau)} \beta(x_3(t)) \right| dt, \\
\quad \leq \int_0^\tau \frac{(B + 1)}{b} \left( \frac{1}{4} + \frac{2\delta}{k^2} \right) dt \\
\quad = \frac{B + 1}{b} \left( \frac{1}{4} + \frac{2\delta}{k^2} \right) \tau. \quad (72)
\]
We then have \(\max(A) \leq M_1 \tau\) and so \(\sup_{\tau \in [0, \tau_*]} \max(A) \leq M_1 \tau_*\). We choose \(\tau_* \leq B/M_1\) and (63) is satisfied.

**Lemma 3.5.** If \(\tau_* \leq (1 - b)/B\), then
\[
\inf_{\tau \in [0, \tau_*]} x_1 \geq b > 0 \implies \inf_{\tau \in [0, \tau_*]} y_1 \geq b > 0. \quad (73)
\]
Proof. It follows from (69) that that
\[
\inf_{\tau \in [0, \tau_*]} \int_0^\tau x_2(t) dt \geq -B \tau_*.
\] (74)
Hence,
\[
\inf_{\tau \in [0, \tau_*]} y_1 = 1 + \inf_{\tau \in [0, \tau_*]} \int_0^\tau x_2(t) dt \geq 1 - B \tau_*.
\] (75)
We then choose \( \tau_* \leq \frac{(1 - b)}{B} \) so that (73) is satisfied. Note that \( x_1(0) = 1 > b \) and so the upper bound on \( \tau_* \) is strictly positive. 

**Proposition 3.1.** For a given \( \tilde{x} \in \chi(\tau_*, B, b) \), with \( \tau_* \leq \min\{b/M_1, (1 - b)/B\} \), we have \( T \tilde{x} = \tilde{y} \in \chi \), where \( T \) is defined by (37).

**Proof.** The proof follows immediately from the three preceding lemmas. 

**Proposition 3.2.** Let
\[
\tilde{M} = \max \left\{ \frac{1}{4} + \delta, \lambda (B + 1) \delta \right\},
\] (76a)
\[
\bar{M} = \max \left\{ \frac{1}{4} + \frac{2\delta}{k^2}, \frac{2\lambda}{k^2} (B + 1) \delta \right\},
\] (76b)
\[
M_2 = \max \left\{ 1, \tilde{M}, (B + 1) \left( \frac{1}{4} + \frac{2\delta}{k^2} \right) \frac{1}{b^2} + \frac{1}{b} \bar{M} \right\}.
\] (76c)

The mapping
\[
T : \chi \rightarrow \chi \tilde{x}; \mapsto T \tilde{x} = \tilde{y},
\] (77)
with \( \chi \) defined by (62), with \( \tau_* < \min\{b/M_1, (1 - b)/B, 1/M_2\} \), is a contraction mapping, i.e., there exists a number \( 0 < \kappa < 1 \) such that
\[
||T \tilde{x}(1) - T \tilde{x}(2)||_\chi = ||\tilde{y}(1) - \tilde{y}(2)||_\chi \leq \kappa ||\tilde{x}(1) - \tilde{x}(2)||_\chi,
\] (78)
for any \( \tilde{x}(1), \tilde{x}(2) \) in \( \chi \).

**Proof.** Recall
\[
||y(1) - y(2)||_\chi = \sup_{\tau \in [0, \tau_*]} \max_{1 \leq i \leq 4} |y_i^{(1)} - y_i^{(2)}|.
\] (79)
We show that for each \( i, 1 \leq i \leq 4 \), we have \( |y_i^{(1)} - y_i^{(2)}| \leq a \sigma \tau^* \) where \( a \) is some constant and \( \sigma = ||\tilde{x}(1) - \tilde{x}(2)||_\chi \). Then by choosing an appropriate value for \( \tau_* \), we show that \( T \) is a contraction on the interval \([0, \tau_*]\). We have
\[
|y_1^{(1)} - y_1^{(2)}| = \left| \int_0^\tau x_2^{(1)}(t) - x_2^{(2)}(t) dt \right| \leq \int_0^\tau \left| x_2^{(1)}(t) - x_2^{(2)}(t) \right| dt,
\] (80)
\[
\leq \int_0^\tau |x_2^{(1)}(t) - x_2^{(2)}(t)| dt \leq \int_0^\tau \sigma dt = \sigma \tau.
\]
Using (83) and (85) we find
\[ |y_2^{(1)} - y_2^{(2)}| = \left| \int_0^\tau (\alpha^{(1)}(t)x_1^{(1)}(t) - \alpha^{(2)}(t)x_1^{(2)}(t)) \frac{t^2}{\tau^2} dt \right|, \]
\[ \leq \int_0^\tau |(\alpha^{(1)}(t)x_1^{(1)}(t) - \alpha^{(2)}(t)x_1^{(2)}(t))| dt, \quad (81) \]
where \( \alpha^{(j)} = \alpha(x_3^{(j)}) \). Let \( p^{(i)} = (x_1^{(i)}, x_3^{(i)})^T, f(p^{(i)}) = \alpha(x_3^{(i)})x_1^{(i)} \) for \( i = 1, 2 \).

Then by the mean value theorem, there exists some point \( \hat{p} = (\hat{x}_1, \hat{x}_3) \) on the line segment joining \( p^{(1)} \) to \( p^{(2)} \) such that
\[ \alpha^{(1)}x_1^{(1)} - \alpha^{(2)}x_1^{(2)} = f(\hat{p}^{(1)}) - f(\hat{p}^{(2)}) = \nabla f(\hat{p}) \cdot (p^{(1)} - p^{(2)}). \quad (82) \]

Then by the Cauchy-Schwarz inequality we have
\[ \left| \alpha^{(1)}x_1^{(1)} - \alpha^{(2)}x_1^{(2)} \right| \leq \left| \nabla f(\hat{p}) \right| |p^{(1)} - p^{(2)}|. \quad (83) \]

Note that \( f \) is differentiable everywhere and thus satisfies the hypotheses of the mean value theorem. We have
\[ \nabla f(\hat{p}) = \left( \frac{1}{4} - V_0e^{\lambda\hat{x}_4}, -\lambda V_0e^{\lambda\hat{x}_4}\hat{x}_1 \right)^T, \quad (84) \]

Using the inequalities \( \hat{x}_1 \leq B + 1, \hat{x}_3 \leq B + |l_0| \), we find
\[ \left| \nabla f(\hat{p}) \right| \leq \max \left\{ \frac{1}{4} + \delta, \lambda(B + 1)\delta \right\} = \tilde{M}. \quad (85) \]

Using (83) and (85) we find \( |\alpha^{(1)}x_1^{(1)} - \alpha^{(2)}x_1^{(2)}| \leq \tilde{M}|p^{(1)} - p^{(2)}| \). Clearly \( |p^{(1)} - p^{(2)}| \leq \sigma = ||x^{(1)} - x^{(2)}||_\chi \), and so
\[ \int_0^\tau |y_2^{(1)} - y_2^{(2)}| dt = \int_0^\tau \left| (\alpha^{(1)}(t)x_1^{(1)}(t) - \alpha^{(2)}(t)x_1^{(2)}(t)) \right| dt \]
\[ \leq \int_0^\tau \sigma \tilde{M} dt = \sigma \tilde{M} \tau. \quad (86) \]

Similarly, we have
\[ |y_3^{(1)} - y_3^{(2)}| = \int_0^\tau |x_4^{(1)}(t) - x_4^{(2)}(t)| dt \leq \int_0^\tau |x_4^{(1)}(t) - x_4^{(2)}(t)| dt \]
\[ \leq \int_0^\tau \sigma dt = \sigma \tau. \quad (87) \]

Finally,
\[ |y_4^{(1)} - y_4^{(2)}| = \int_0^\tau \left( \frac{\beta^{(1)}x_1^{(1)}(t)}{x_1^{(1)}(\tau)} - \frac{\beta^{(2)}x_1^{(2)}(t)}{x_1^{(2)}(\tau)} \right) \frac{t}{\tau} dt \]
\[ \leq \int_0^\tau \left| \frac{\beta^{(1)}x_1^{(1)}(t)}{x_1^{(1)}(\tau)} - \frac{\beta^{(2)}x_1^{(2)}(t)}{x_1^{(2)}(\tau)} \right| dt \]
\[ = \int_0^\tau \left| \beta^{(1)}x_1^{(1)}(t) \left( \frac{x_1^{(2)}(\tau) - x_1^{(1)}(\tau)}{x_1^{(1)}(\tau)x_1^{(2)}(\tau)} \right) + \frac{\beta^{(1)}x_1^{(1)}(t) - \beta^{(2)}x_1^{(2)}(t)}{x_1^{(1)}(\tau)} \right| dt \]
\[ \leq \int_0^\tau \left| \beta^{(1)}x_1^{(1)}(t) \right| \frac{\sigma}{b^2} dt + \int_0^\tau \frac{1}{b} \left| \beta^{(1)}x_1^{(1)}(t) - \beta^{(2)}x_1^{(2)}(t) \right| dt = I_1 + I_2, \quad (88) \]
using $1/b \geq 1/x_1$ and $|x_1^{(2)} - x_1^{(1)}| \leq \sigma$. Using the mean value theorem and the Cauchy-Schwarz inequality again here we find $I_2 \leq \bar{M}\sigma/\tau$, where

$$
\bar{M} = \max \left\{ \frac{1}{4} \frac{1 + \frac{2}{k^2} \delta}{ \frac{2}{k^2} \lambda(B + 1)\delta} \right\}.
$$

Using the bounds defined by (60) we find

$$
I_1 \leq (B + 1) \left( \frac{1}{4} + \frac{2}{k^2} \delta \right) \frac{\sigma}{b^2} \tau.
$$

So we have

$$
|y_4^{(1)} - y_4^{(2)}| \leq \left[ (B + 1) \left( \frac{1}{4} + \frac{2}{k^2} \delta \right) \frac{1}{b^2} + \frac{1}{b} \bar{M} \right] \sigma \tau.
$$

Gathering these bounds we find that $\sup_{\tau \in [0, \tau^*]} \max_{1 \leq i \leq 4} |y_i - y_i^2| \leq M_2\tau_\sigma$, where

$$
M_2 = \max \left\{ 1, \bar{M}, (B + 1) \left( \frac{1}{4} + \frac{2}{k^2} \delta \right) \frac{1}{b^2} + \frac{1}{b} \bar{M} \right\}.
$$

For $\tau_* < 1/M_2$, $||\vec{y}^1 - \vec{y}^2||_\chi \leq \kappa ||\vec{x}^1 - \vec{x}^2||_\chi$ where $0 < \kappa < 1$.

**Proposition 3.3.** For $\tau_*$ sufficiently small, the mapping $T$ defined above has a unique fixed point on $[0, \tau_*]$.

**Proof.** Given any constants $B$ and $b$, let $m = \min \{B/M_1, (1-b)/B, 1/M_2\}$. For $\tau_* < m$, then, Propositions 3.1 and 3.2 hold, so $T$ is a contractive mapping from a closed subset $\chi$ of a Banach space $E$, into itself. Using Banach’s fixed point theorem completes the proof.

In light of this theorem, we know that there is a unique $\vec{x}$ on $[0, \tau_*]$, such that $T\vec{x} = \vec{x}$. Combining this with (60) shows that there is a unique $\vec{x}$ such that (57) holds, hence (50) has a unique solution on some interval $[0, \tau_*]$.

**Proposition 3.4.** (50b) and (50d) subject to (50f) have a unique solution on $[0, \tau_*]$, for some $\tau_* > 0$.

**Proof.** Proposition 3.3 shows that we have a unique solution for $l$ and $x = R\tau$ on $[0, \tau_*]$, so we have a unique solution for $R$ and $l$.

**Proof of Theorem 2.2**

**Proof.** Using Proposition 3.4 we have a unique solution for $R$ and $l$ on some interval $[0, \tau_*]$. Integrating (50a) we have

$$
\phi = \phi_0 + \frac{\tau}{4} + \int_0^\tau \frac{1 - \dot{R}}{R} dt,
$$

which gives a unique solution for $\phi$. Equation (50a) then gives a unique solution for $\gamma$. 

4. Evolution of solutions.

In this section we determine the global structure of solutions in the region bounded by the axis and $N_-$, with the exception of a few special cases, which are deferred until the following section. An alternative set of variables proves useful:

$$u_1 = \dot{R}/R, \quad u_2 = |V_0|e^{\lambda l}, \quad u_3 = \dot{l},$$

(93)

They satisfy

$$\dot{u}_1 = \frac{1}{4} - \epsilon u_2 - u_1^2,$$

(94a)

$$\dot{u}_2 = \lambda u_2 u_3,$$

(94b)

$$\dot{u}_3 = \frac{1}{4} - \frac{2u_2}{k^2} - u_1 u_3,$$

(94c)

$$u_2(0) = |V_0|e^{\lambda l_0} > 0, \quad u_3(0) = 0,$$

(94d)

where $\epsilon = \text{sgn}(V_0)$ and $\lambda = k^2/2 - 1$. Note that $u_1$ is not defined on $\tau \leq 0$ and that $\lim_{\tau \to 0^+} u_1 = \infty$. Note also that $u_2 > 0$ by definition.

Using results from section 3, there exists $\tau_M$ such that $\vec{u} = (u_1, u_2, u_3)^T$ has a unique solution on $(0, \tau_M)$. The following standard result proves useful in determining the maximal interval of existence in each case. (see, for example, [16])

**Theorem 4.1.** Let $\Psi_a(t)$ be the unique solution of the DE $x' = f(x)$, where $f \in C^1(\mathbb{R}^n)$, which satisfies $x(0) = a$, and let $(t_{\text{min}}, t_{\text{max}})$ be the maximal interval of existence on which $\Psi_a(t)$ is defined. If $t_{\text{max}}$ is finite, then

$$\lim_{t \to t_{\text{max}}} ||\Psi_a(t)|| = +\infty.$$  

(95)

This result may be adapted to our system by defining $(0, \tau_M)$ as the maximal interval of existence for the unique solution $\vec{u}(\tau)$ of (94). It follows from Theorem 4.1 that if the components of the solution $u_1, u_2$ and $u_3$ satisfy finite lower and upper bounds for all $\tau \in (0, \tau_M)$, then we have $\tau_M = \infty$. Furthermore, if $\tau_M$ is finite then we have $\lim_{\tau \to \tau_M^+} |u_i| = +\infty$ for at least one $i \in \{1, 2, 3\}$.

The system has three parameters $\{k^2, V_0, l_0\}$. The qualitative picture of solutions depends primarily on the signs of $V_0$ and $\lambda = k^2/2 - 1$ and so we devote a subsection to each of the four permutations.

4.1. $V_0 < 0, \lambda > 0$

In this case we find that radial null infinity exists along a hypersurface corresponding to a finite value of $\tau$. Note that $\epsilon = -1$ and $k^2 > 2$ here.

**Lemma 4.1.** If $V_0 < 0, \lambda > 0$ then $u_3 > 0, \dot{u}_3 > 0$ and $u_1 > u_3$ for $\tau \in (0, \tau_M)$.

**Proof.** Consider

$$\dot{u}_1 - \dot{u}_3 = -\epsilon \frac{2\lambda}{k^2} u_2 - u_1(u_1 - u_3) = \frac{2\lambda}{k^2} u_2 - u_1(u_1 - u_3).$$

(96)
Lemma 4.3. If \( \beta V_0 < 0, \lambda > 0, \) then there exists \( \tau_1 \in (0, \tau_0) \) such that \( u_i(\tau) = k_2 u_2(\tau) \) and \( u_i < k_2 \) for all \( \tau \in (\tau_1, \tau_0) \).

\( u_2(\tau) = \frac{u_2^2(\tau)}{2 \lambda} < \frac{u_2^2(\tau)}{2 \lambda} \) for all \( \tau \in (\tau_1, \tau_0) \).

Suppose there exists \( \tau \) such that \( u_i(\tau) = u_i^2(\tau) = \frac{1}{2} \lambda u_2(\tau) \). Then, using (90) and (91), we have

\[
\frac{d}{d\tau} \left[ \frac{(1/2) u_i^2(\tau)}{u_2(\tau)} \right] = -\frac{2(1/4 + u_2^2)}{2 \lambda u_2^2(\tau)} < 0.
\]

Therefore, \( u_i(\tau) \) is decreasing and bounded above by \( \frac{1}{2} \lambda u_2(\tau) \) for all \( \tau \in (\tau_1, \tau_0) \).

Lemma 4.2. Let \( \hat{\lambda} = A + \sqrt{A^2 + 4b/4} \) if \( \beta V_0 < 0 \) and \( \lambda > 0 \), then for all \( \tau \in (0, \tau_0) \), we have

\[
i_1 = \int_0^\tau \left(1 + \frac{2a}{2b} \right) Rdr.
\]

Proof: first note that \( u_i > (1/4 + u_2^2)^{1/2} \hat{\lambda} \) on some initial interval, where the second inequality holds due to \( \Lambda > 1 \). The preceding lemma tells us that \( u_2 > 0 \) for all \( \tau \in (0, \tau_0) \). Since \( u_i < k_2 \), the \( u_i \) are from which it follows that \( u_i > 0 \). The \( u_i \) are decreasing. By inspection of (97), we have

\[
u_2 > 0 \quad \text{for all} \quad \tau \in (0, \tau_0).
\]

Thus, \( u_i(\tau) \) is monotonically decreasing and bounded above by \( \frac{1}{2} \lambda u_2(\tau) \) for all \( \tau \in (\tau_1, \tau_0) \).

\( u_2(\tau) = (1/4 + u_2^2)^{1/2} \) for all \( \tau \in (\tau_1, \tau_0) \). Now consider

\[
i_1 = \int_0^\tau \left(1 + \frac{2a}{2b} \right) Rdr.
\]

which is clearly positive for \( R > 0, \epsilon = -1 \). Since \( u_3(0) = 0 \) and \( \epsilon = -1 \) shows that \( u_3 \) cannot cross \( 1/2 \) from above and so \( u_i > k_2 \) for all \( \tau \in (\tau_0, \tau_0) \). Hence, Equation (90) may be integrated to give

\[
i_1 = \int_0^\tau \left(1 + \frac{2a}{2b} \right) Rdr.
\]

for \( \epsilon = -1 \). Since \( u_2 > 0, \lambda > 0 \), it is clear that \( u_i - u_3 \) cannot cross zero from above, for \( \epsilon = -1 \) and for all \( \tau \in (0, \tau_0) \).
Proof. Consider
\[ \dot{u}_1 - k^2 \ddot{u}_3 = -\frac{k^2}{4} u_2 - u_1 (u_1 - k^2 u_3) \leq -\frac{1}{4} u_2, \] (101)
provided \( u_1 > k^2 u_3 \), which holds initially. If \( \tau_M \) is infinite, then the result must follow. If not, then by Theorem 4.1, and since \( u_2^{1/2} / \lambda < u_1 < (1/4 + u_2)^{1/2} \) and \( u_3 < u_1 \) for \( \tau \in (\tau_1, \tau_M) \), we must have \( \lim_{\tau \to \tau_M^-} u_2 = \lim_{\tau \to \tau_M^-} u_1 = \infty \). We know from Lemma 4.2 that \( u_1 > u_2^{1/2} / \lambda \) for all \( 0 < \tau < \tau_M \), which gives \( \dot{u}_1 < 1/4 + \lambda u_2/2 \), since \( 1 - 1/\lambda^2 = \lambda/2 \). We then have
\[ \lim_{\tau \to \tau_M} \int_{\tau_1}^{\tau} \frac{\lambda u_2}{2} \, d\tau' > \lim_{\tau \to \tau_M} \left( u_1 - u_1(\tau_1) - \frac{\tau - \tau_1}{4} \right) = \infty. \] (102)
Suppose then that \( u_1 > k^2 u_3 \) for all \( \tau \in (0, \tau_M) \). Integrating (101) and taking the limit gives \( \lim_{\tau \to \tau_M^-} (u_1 - k^2 u_3) = -\infty \), so we have a contradiction. \( \square \)

Lemma 4.4. If \( V_0 < 0 \) and \( \lambda > 0 \), then \( \tau_M < \infty \) and for \( i = \{1, 2, 3\} \),
\[ \lim_{\tau \to \tau_M} u_i = \infty. \] (103)

Proof. Using the previous lemma,
\[ \dot{u}_2 > \frac{\lambda u_1 u_2}{k^2} > \frac{\lambda u_2^{3/2}}{k^2 \lambda}, \] (104)
for \( \tau \in (\tau_2, \tau_M) \). Integrating over \([\tau_2, \tau]\) and rearranging we find
\[ u_2 > \left( \frac{1}{u_2^{1/2}(\tau_2)} - \frac{\lambda (\tau - \tau_2)}{2 k^2 \lambda} \right)^{-2}, \] (105)
so we have \( \tau_M \leq k^2 \lambda u_2^{-1/2}(\tau_2) / \lambda + \tau_2 \) and \( \lim_{\tau \to \tau_M} u_2 = \infty \). It follows directly from Lemma 4.2 that \( \lim_{\tau \to \tau_M} u_1 \infty \). Using Lemmas 4.1 and 4.3 complete the proof. \( \square \)

Proposition 4.1. For \( V_0 < 0 \), \( \lambda > 0 \), the surface corresponding to \( \tau = \tau_M \) represents future null infinity and the Ricci scalar decays to zero there.

Proof. We aim to show that along outgoing radial null geodesics, an infinite amount of affine parameter time is required to reach the surface \( \tau = \tau_M \). These geodesics correspond to the lines \( u = u_0 \) where \( u_0 \) is constant. We look for solutions to the equation for null geodesics, which reduces to
\[ \ddot{v} + (2 \tilde{\sigma}_v + 2 \tilde{\phi}_v) \dot{v} = 0, \] (106)
where here the dot denotes differentiation with respect to an affine parameter \( \mu \), which is chosen such that \( \dot{v} > 0 \) and \( \mu(\tau = 0) = 0 \). Dividing by \( \dot{v} \) and integrating, we find
\[ e^{2\gamma + 2\phi} \dot{v} = \frac{1}{|u_0|} e^{2\gamma + 2\phi} \dot{v} = C, \] (107)
with \( C > 0 \). Substituting \( 2\gamma + 2\phi \) using (50a) gives
\[ \frac{1}{|u_0|} e^{(k^2 l + r)/2} \dot{v} = C. \] (108)
We also have \( v = u_0 \sigma = u_0 e^{-\tau} \), and thus \( dv = -u_0 e^{-\tau} d\tau \), along the geodesics. Integrating then leads to
\[
\frac{1}{|u_0|} \int_{\tau_0}^{\tau} e^{(k^{2l}+\tau)'/2} dv' = \int_{0}^{\tau} e^{(k^{2l}-\tau)'/2} d\tau' = C \mu. \tag{109}
\]
Clearly \( e^{k^{2l}/2} = |V_0|^{-1} e^{u_2} > u_2 \) holds for \( \tau \) sufficiently close to \( \tau_M \). Using (109) then gives \( \lim_{\tau \to \tau_M} \mu = \infty \). This confirms that the surface \( \tau = \tau_M \) corresponds to radial null infinity.

To demonstrate the decay of the Ricci scalar, which we label \( R \), it is convenient to consider the trace of the energy-momentum tensor:
\[
g^{ab}T_{ab} = -2g^{01} \phi \psi_v - 4V = 2|u|e^{-2\gamma-2\phi} \left( -\frac{\eta F'}{u} + \frac{k}{2u} \right) \frac{F'}{u} - \frac{4V_0 e^{-2F/k}}{|u|}
\]
\[
= \frac{2e^{-2\gamma-2\phi}}{|u| |\eta|} \left[ \left( \frac{k\eta l'}{2} + \frac{k}{4} \right) \left( \frac{k\eta l'}{2} + \frac{k}{4} \right) - 2V_0 e^M \right]
\]
\[
= \frac{e^{-k^{2l}/2+\tau/2-c_1}}{|u|} \left( \frac{k^2}{2} \left( \frac{1}{4} - i^2 \right) e^{-k^{2l}/2} - 4V_0 e^{-l} \right) = -R. \tag{110}
\]

Note that \( 1/4 - i^2 < 0 \) approaching \( \tau_M \). Using \( i = u_3 < (1/4 + u_2)^{1/2} = (1/4 - V_0 e^M)^{1/2} \) we have \( V_0 e^{-l} < e^{-k^{2l}/2(1/4 - i^2)} \). Hence, both terms in the bracket decay to zero as \( \tau \to \tau_M \). Since \( \tau_M < \infty \), it follows that \( \lim_{\tau \to \tau_M} R = 0 \).

4.2. \( V_0 < 0, \lambda < 0 \)

Here we find that the surface \( \mathcal{N} \) corresponds to a fixed-point of the system, is regular and is reached by radial null rays in finite affine time. These are some of the solutions which may be extended into region \( \Pi \).

**Lemma 4.5.** If \( V_0 < 0 \) and \( \lambda < 0 \), then \( \tau_M = +\infty \) and
\[
\lim_{\tau \to \infty} (u_1, u_2, u_3) = \left( \frac{1}{2}, 0, \frac{1}{2} \right). \tag{111}
\]

**Proof.** We have seen in the proof of Lemma 4.1 that if \( V_0 < 0 \), then \( u_3 > 0 \) for all \( \tau \in (0, \tau_M) \). For \( \lambda < 0 \), equation (94b) then tells us that \( u_2 \) is monotonically decreasing on \( (0, \tau_M) \). Equation (94a) then tells us that \( u_1 \) cannot cross \( (1/4 + u_2)^{1/2} \) from above and is therefore decreasing and bounded below by this term for all \( \tau \in (0, \tau_M) \). For any \( \tau_* \in (0, \tau_M) \) then, \( \dot{u}_3 > 1/4 - u_1(\tau_*)u_3 \) for all \( \tau \in (\tau_*, \tau_M) \). Hence, \( \dot{u}_3 > 0 \) for \( u_3 < 1/4 u_1(\tau_*) \) and so we must have \( u_3 > \bar{C}_1 = \min\{u_3(\tau_*), 1/4 u_1(\tau_*)\} \) for all \( \tau \in (\tau_*, \tau_M) \). Indeed, we can say that \( u_3 > C_1 \) for any \( C_1 \leq \bar{C}_1 \) and, without loss of generality, we choose \( C_1 < 1/2|\lambda| \), such that \( \lambda C_1 > -1/2 \).

It follows from (94b) that \( u_2 < C_2 e^{\lambda C_1 \tau} \) for \( t \in (\tau_*, \tau_M) \) and constant \( C_2 > 0 \). Combining this with \( u_1 > (1/4 + u_2)^{1/2} > 1/2 \) we have
\[
\dot{u}_1 < \frac{1}{4} + C_2 e^{\lambda C_1 \tau} - \frac{u_1}{2}. \tag{112}
\]
Integrating leads to
\[ u_1 < \frac{1}{2} + C_3 e^{\lambda C_1 \tau}, \]  
(113)
for constant \( C_3 > 0 \), where we have used \( \lambda C_1 > -1/2 \) to absorb the \( e^{-\tau/2} \) term. Using \( u_1 > 1/2 \) and \( u_2 < C_2 e^{\lambda C_1 \tau} \) we find
\[ \dot{u}_3 < \frac{1}{4} + \frac{1}{k^2} C_2 e^{\lambda C_1 \tau} - \frac{u_3}{2}, \]  
(114)
which leads to
\[ u_3 < \frac{1}{2} + C_4 e^{\lambda C_1 \tau}, \]  
(115)
for constant \( C_4 \). We have thus far proven that \( 1/2 < u_1 < u_1(\tau_*) \), \( 0 < u_2 < u_2(0) \) and \( 0 < u_3 < 1/2 + C_4 e^{\lambda C_1 \tau} \) for \( \tau \in (\tau_*, \tau_M) \). It then follows from Theorem 4.1 that \( \tau_M = +\infty \). Equation (113) and \( u_1 > 1/2 \) then tell us that \( \lim_{\tau \to \infty} u_1 = 1/2 \). Using (94c), (113) and (115) we have
\[ \dot{u}_3 > \frac{1}{4} - \left( \frac{1}{2} + C_3 e^{\lambda C_1 \tau} \right) u_3 > \frac{1}{4} - \frac{u_3}{2} - C_3 e^{\lambda C_1 \tau} \left( \frac{1}{2} + C_4 e^{\lambda C_1 \tau} \right), \]  
(116)
from which it follows that
\[ u_3 > \frac{1}{2} - C_5 e^{\lambda C_1 \tau}, \]  
(117)
for some \( C_5 > 0 \). Combining (115) and (117) we find
\[ \lim_{\tau \to \infty} \left( \frac{1}{2} - C_5 e^{\lambda C_1 \tau} \right) < \lim_{\tau \to \infty} u_3 < \lim_{\tau \to \infty} \left( \frac{1}{2} + C_4 e^{\lambda C_1 \tau} \right), \]  
(118)
which yields \( \lim_{\tau \to \infty} u_3 = 1/2 \). Integrating (115) over \([\tau_*, \tau]\) and taking the limit gives
\[ \lim_{\tau \to \infty} u_2 = \lim_{\tau \to \infty} u_2(\tau_*) \exp \left( \lambda \int_{\tau_*}^{\tau} u_3 \, d\tau \right) \]  
\[ < \lim_{\tau \to \infty} \exp \left( \lambda \int_{\tau_*}^{\tau} \frac{1}{2} - C_5 e^{\lambda C_1 \tau} \, d\tau \right) = 0. \]  
(119)
\[ \square \]

**Proposition 4.2.** If \( V_0 < 0 \) and \( \lambda < 0 \), then
\[ 0 < \lim_{\tau \to \infty} S < \infty, \]  
(120)
and outgoing radial null rays reach \( \mathcal{N}_- \) in finite parameter time.

**Proof.** Recall \( u_1 = \dot{R}/R \). Integrating (113) over \((\tau_*, \tau)\) and taking the exponential we have
\[ \exp \left( \int_{\tau_*}^{\tau} u_1 \, d\tau \right) = \frac{R}{R(\tau_*)} < C_3 e^{\tau/2}, \]  
(121)
where we have absorbed the exponential term into \( \tilde{C}_3 \). Recall that \( S = e^{-\tau/2} R \) and so we have \( S < R(\tau_*) \tilde{C}_3 \) for all \( \tau \in [0, \infty) \). We also have \( u_1 - 1/2 = \dot{S}/S > 0 \) for all \( \tau \in [0, \infty) \), so \( S \) is monotonically increasing. Equation (??) immediately follows.
Recall $u_3 = \dot{\lambda}$. Integrating (113) we find that $l < \tau/2 + C_6$ for some constant $C_6$ and $\tau \in (\tau_*, \tau_M)$, where $\tau_*$ comes from the previous lemma. Combining this with (109) we have

$$C \mu < \int_0^{\tau_*} e^{(k^2l-\tau')/2} d\tau' + \int_{\tau_*}^{\infty} e^{\lambda(\tau'/2 + k^2C_6)/2} d\tau' = C_7, \quad (122)$$

for some constant $C_7 > 0$.

4.3. $V_0 > 0, \lambda < 0$

There are two subcases here, distinguished the sign of $u_2(0) - k^2/8$. When negative, the solutions have a similar structure to those outlined in the previous section. In the positive case, we have a finite interval of existence and a singularity at $\tau_M$. We deal with the case $u_2(0) = k^2/8$ in section 5.2.

**Lemma 4.6.** If $V_0 > 0, \lambda < 0$ and $u_2(0) < k^2/8$, then $\tau_M = \infty$ and

$$\lim_{\tau \to \infty} (u_1, u_2, u_3) = \left( \frac{1}{2}, 0, \frac{1}{2} \right). \quad (123)$$

**Proof.** Using equation (97), with $\epsilon = 1$, and $u_2(0) < k^2/8$, we must have $u_3$ initially positive, since $R$ is initially positive.

Since $u_3$ cannot cross zero from above while $u_2 < k^2/8$ and $u_3 > 0$ gives $\dot{u}_2 < 0$, we have $\dot{u}_2 < 0, u_2 < k^2/8$ and $u_3 > 0$ for all $\tau \in (0, \tau_M)$. Given any $\tau_* \in (0, \tau_M)$ such that $u_1(\tau_*) > 1/2$, then $u_1 < u_1(\tau_*)$ for $\tau \in (\tau_*, \tau_M)$, since $\dot{u}_1 < 0$ for $u_1 > 1/2$. So we have $\dot{u}_3 > 1/4 - 2u_2(\tau_*)/k^2 - u_1(\tau_*)u_3$ where $1/4 - 2u_2(\tau_*)/k^2 > 0$. Similarly to the proof of Lemma 4.5, we have $u_3 > C_1$ and $u_2 < C_2 e^{\lambda C_1 \tau}$ for $\tau \in (\tau_*, \tau_M)$, where $\lambda C_1 > -1/2$.

Now, either $u_1 > 1/2$ for all $\tau \in (0, \tau_M)$, or there exists $\tau_* \in (\tau_*, \tau_M)$ such that $u_1(\tau_*) = 1/2$. At $u_1 = 1/2$ we have $\dot{u}_1 = -u_2 < 0$ and so in this case we have $u_1 < 1/2$ for all $\tau \in (\tau_*, \tau_M)$. In the former case, $\dot{u}_1 < 1/4 - u_1/2$ for all $\tau \in (\tau_*, \tau_M)$, from which it we find that

$$\frac{1}{2} < u_1 < \frac{1}{2} + C_8 e^{-\tau/2}, \quad \tau_* < \tau < \tau_M, \quad (124)$$

for some constant $C_8$. In the latter case, $\dot{u}_1 > 1/4 - C_2 e^{\lambda C_1 \tau} - u_1/2$ for all $\tau \in (\tau_*, \tau_M)$, from which we find that

$$\frac{1}{2} - C_9 e^{\lambda C_1 \tau} < u_1 < \frac{1}{2}, \quad \tau_* < \tau < \tau_M, \quad (125)$$

where we have absorbed the $e^{-\tau/2}$ term. Using (124) and (125) in a similar fashion to the proof of Lemma 4.5, we have

$$\frac{1}{2} - C_{10} e^{\lambda C_1 \tau} < u_3 < \frac{1}{2} + C_{11} e^{\lambda C_1 \tau}, \quad \tilde{\tau}_* < \tau < \tau_M, \quad (126)$$

where $\tilde{\tau}_* = \tau_*$ or $\tau_*$ depending on the case. The remainder of the proof is similar to that of Lemma 4.5. \qed
Proposition 4.3. If $V_0 > 0$, $\lambda < 0$ and $u_2(0) < k^2/8$, then

$$0 < \lim_{\tau \to \infty} S < \infty,$$

and outgoing radial null rays reach $N_-$ in finite parameter time.

Proof. Similarly to the proof of Proposition 4.2, integrating (124) or (125) gives finite, non-zero upper and lower bounds for $S$. The remainder of the proof is identical. □

Lemma 4.7. For $V_0 > 0$, $\lambda < 0$ and $u_2(0) > k^2/8$, we have $\tau_M < \infty$ and

$$\lim_{\tau \to \tau_M} u_1 = -\infty, \quad \lim_{\tau \to \tau_M} u_3 = -\infty, \quad \lim_{\tau \to \tau_M} u_2 = +\infty.$$  (128)

Proof. In this case we have $u_2 > k^2/8, u_3 < 0$ on some initial interval, using equation (97). Indeed, $u_3$ cannot cross zero from below while $u_2 > k^2/8$ and since $u_2$ is increasing for $u_3 < 0$, these conditions hold for all $\tau \in (0, \tau_M)$. We then have

$$\dot{u}_1 < -\lambda/4 - u_1^2,$$  (129)

for all $\tau \in (0, \tau_M)$. For $u_1 > \sqrt{|\lambda|/2}$, this may be integrated over $(0, \tau)$ to give

$$u_1 < m \coth m\tau,$$  (130)

where $m = \sqrt{|\lambda|/2}$ and we have used $\lim_{\tau \to 0^+} u_1 = \infty$. Note that (130) automatically holds for $u_1 < m$ also, since $\coth m\tau > 1$ for all $\tau$.

Combining (130) with (94c) gives

$$\dot{u}_3 < -b - (m \coth m\tau)u_3,$$  (131)

where $b = 2u_2(0)/k^2 - 1/4 > 0$. This may be integrated to give

$$u_3 < -\frac{b(\cosh m\tau - 1)}{m \sinh m\tau},$$  (132)

$$\lambda \int_0^\tau u_3 \, d\tau' > -\frac{2b}{m^2} \log \left( \cosh \left( \frac{m\tau}{2} \right) \right) = 8b \log \left( \cosh \left( \frac{m\tau}{2} \right) \right).$$  (133)

We then arrive at

$$u_2 = u_2(0) \exp \left[ \int_0^\tau u_3 \, d\tau' \right] > \cosh^{8b} \left( \frac{m\tau}{2} \right).$$  (134)

It follows that $\tau_M$ must be finite since, otherwise, we would have $\tau_\ast \in (0, \tau_M)$ such that $u_2 > 1/2$ for all $\tau > \tau_\ast$, which gives $\dot{u}_1 < -1/4 - u_1^2$ and thus $\lim_{\tau \to \tau_M} u_1 = -\infty$ for $\tau_M$ finite, using (94a) with $\epsilon = 1$. So we must have $\lim_{\tau \to \tau_M} |u_i| = \infty$ for at least one of the $u_i$. Now consider $X = u_1 - k^2u_3/2$, which satisfies

$$\dot{X} = \frac{\lambda}{4} - u_1 X > -u_1 X.$$  (135)

Note that $X$ is initially positive and at $X = 0$ we have $\dot{X} > 0$, so $X > 0$ for $\tau \in (0, \tau_M)$. Furthermore, if $u_1$ is bounded below for $\tau \in (0, \tau_M)$, then $X$ is bounded above, using (135), which in turn gives a lower bound for $u_3$. This gives an upper bound on $u_2$, which contradicts $\lim_{\tau \to \tau_M^-} ||\vec{u}(\tau)|| = \infty$. Hence, we must have $\lim_{\tau \to \tau_M^-} u_1 = -\infty$ and, since
Collapse of a self-similar cylindrical scalar field with non-minimal coupling

\( X > 0, \lim_{\tau \to \tau_M^-} u_3 = -\infty. \)

To complete the proof, we assume \( \lim_{\tau \to \tau_M^-} u_2 = B + 1/4 < \infty \) for some constant \( B \), and then arrive at a contradiction. Note that \( u_2 \) is monotone and so the limit must exist. The assumption gives \( u_2 < B + 1/4 \), and thus \( u_1 > -B - u_1^2 \), for all \( \tau \in (0, \tau_M) \). Dividing by \( u_1 \) we have \( u_1/u_1 < -B/u_1 - u_1 \) for \( u_1 < 0 \). Choosing \( \tau_0 \) such that \( u_1(\tau_0) < 0 \) and integrating over \([\tau_0, \tau]\) gives

\[ u_1 > u_1(\tau_0) \exp \left[ \int_{\tau_0}^{\tau} \frac{-B}{u_1} - u_1 \, d\tau' \right]. \]

(136)

Since \( \lim_{\tau \to \tau_M^-} u_1 = -\infty \), it must follow that

\[ \lim_{\tau \to \tau_M^-} \int_{\tau_0}^{\tau} u_1 \, d\tau' = -\infty. \]

(137)

Now, \( u_1 > k^2 u_3/2 \) gives \( \dot{u}_2 > 2\lambda u_2 u_1/k^2 \). Dividing by \( u_2 \), integrating and using the above, we find \( \lim_{\tau \to \tau_M^-} u_2 = \infty \), which is our contradiction. \( \square \)

**Proposition 4.4.** If \( V_0 > 0, \lambda < 0 \), and \( u_2(0) > k^2/8 \) then \( \tau_M < \infty \) and there is a singularity at \( \tau_M \).

**Proof.** In terms of \( u_2, u_3 \), with \( V_0 > 0 \), the Ricci scalar is given by

\[ \mathcal{R} = \frac{e^{\tau/2 - c_1}}{|u|} \left( \frac{k^2}{2} \left( u_3^2 - \frac{1}{4} \right) u_2^{-k^2/2\lambda} + 4u_2^{-1/\lambda} \right). \]

(138)

Using the previous lemma we have \( \tau_M < \infty \), \( \lim_{\tau \to \tau_M^-} u_2 = \infty \) and \( \lim_{\tau \to \tau_M^-} u_3 = -\infty \). For \( \lambda < 0 \) then, \( \lim_{\tau \to \tau_M^-} \mathcal{R} = \infty. \)

\[ 4.4. \] \( V_0 > 0, \lambda > 0. \)

Similarly to the previous section, we have two different pictures depending on the sign of \( u_2(0) - k^2/8 \). When positive, \( u_3 \) is initially negative, and vice-versa. Hence, \( u_2 \) either starts above \( k^2/8 \) and is decreasing, or vice-versa. The case \( u_2(0) = k^2/8 \) is dealt with in the next section. The following results show that in all cases, the maximal interval of existence of solutions is finite and there is a singularity at \( \tau_M \).

**Lemma 4.8.** For \( V_0 > 0, \lambda > 0 \), suppose that \( \lim_{\tau \to \tau_M^-} u_3 = -\infty. \) Then \( \lim_{\tau \to \tau_M^-} \mathcal{R} = \infty. \)

**Proof.** If \( \lim_{\tau \to \tau_M^-} u_3 = -\infty \) and \( \lambda > 0 \), then \( \lim_{\tau \to \tau_M^-} u_2^{-k^2/2\lambda} \neq 0 \). Using (138) then gives the result. \( \square \)

**Lemma 4.9.** For \( V_0 > 0, \lambda > 0 \), suppose that \( \tau_M < \infty \) and \( \lim_{\tau \to \tau_M^-} u_1 = -\infty, \lim_{\tau \to \tau_M^-} u_3 = +\infty. \) Then for any \( \tau_* < \tau_M \) such that \( u_3 > 0 \) for all \( \tau \in (\tau_*, \tau_M) \) we have

\[ \lim_{\tau \to \tau_M^-} \int_{\tau_*}^{\tau} u_1 \, d\tau = -\infty. \]

(139)
Lemma 4.10. For \( V_0 > 0, \lambda > 0 \), suppose that \( \tau_M < \infty \) and \( \lim_{\tau \to \tau_M^-} u_1 = -\infty, \lim_{\tau \to \tau_M^-} u_3 = +\infty. \) Then
\[
\lim_{\tau \to \tau_M^-} u_2 = +\infty, \quad -\infty < \lim_{\tau \to \tau_M^-} \frac{u_1}{u_3} = L < -k^2 - \frac{1}{4}.
\]

Proof. First we define \( q_1 = u_1/u_3 \), which satisfies
\[
\dot{q}_1 = \frac{1}{u_3} \left( \frac{1}{4} - u_2 \right) - q_1 \left( \frac{1}{4} - \frac{2u_2}{k^2} \right).
\]

Now, suppose \( u_2 \) is bounded above for all \( \tau \in (0, \tau_M) \). Then it straight-forward to show that \( q_1 \) must be bounded below, i.e., there exists some \( L_* < 0 \) such that \( u_1/u_3 > L_* \), for \( \tau \in (0, \tau_M) \). We then have \( \dot{u}_2/u_2 > \lambda u_1/L_* \) for \( \tau \in (0, \tau_M) \). Integrating and using Lemma 4.9 then shows that \( \lim_{\tau \to \tau_M^-} u_2 = +\infty \), and so \( u_2 \) is unbounded. Given that \( u_2 \) is monotone increasing for \( u_3 > 0 \), which obtains for \( \tau \) sufficiently close to \( \tau_M \), we must have \( \lim_{\tau \to \tau_M^-} u_2 = +\infty \).

It is clear from (143) that \( q_1 \) is negative if \( u_2 > k^2/8 > 1/4, u_1 < 0 \) and \( u_3 > 0 \), which all hold for \( \tau \) sufficiently close to \( \tau_M \). Thus, \( q_1 \) is monotone decreasing for \( \tau \) sufficiently close to \( \tau_M \) and the limit \( \lim_{\tau \to \infty} q_1 = L < 0 \) exists. We now prove by contradiction that \( L \) is finite. Assuming \( \lim_{\tau \to \tau_M^-} q_1 = -\infty \), there must exist some \( \tau_* \in (0, \tau_M) \) such that \( q_1 < -2\lambda \) for \( \tau \in (\tau_*, \tau_M) \). Now define \( q_2 = u_2/u_1 \), which satisfies
\[
\frac{\dot{q}_2}{q_2} = \lambda u_3 - \frac{1}{4u_1} + \frac{u_2}{u_1} + u_1 < -\frac{1}{4u_1} + \frac{u_1}{2},
\]

for \( u_1 < 0, u_2 > 1/4, q_1 < -2\lambda \). Integrating and taking the limit \( \tau \to \tau_* \) we have
\[
\lim_{\tau \to \tau_M^-} q_2 \geq q_2(\tau_*) \lim_{\tau \to \tau_M^-} \exp \left( \frac{1}{2} \int_{\tau_*}^\tau \frac{u_1 - \frac{1}{2u_1} d\tau'}{u_1} \right) = 0,
\]

where we’ve used Lemma 4.9, \( \lim_{\tau \to \tau_M^-} u_1 = 0 \) and \( q_2 < 0 \). It follows that \( \lim_{\tau \to \tau_M^-} q_2 = 0 \). Defining \( q_3 = u_2/u_3 \) we have
\[
\frac{\dot{q}_3}{q_3} = \lambda u_3 - \frac{1}{4u_3} + \frac{2u_2}{k^2 u_3} + u_1.
\]

Since \( \lim_{\tau \to \tau_M^-} q_2 = 0 \) we can choose \( \tau_* \) such that we also have
\[
\frac{2u_2}{k^2 u_3} = \frac{2q_2u_1}{k^2 u_3} < -\frac{u_1}{4},
\]

Proof. Note that for \( u_3 > 0 \) and \( \epsilon = 1 \) we have \( \dot{u}_3/u_3 < 1/4u_3 - u_1 \). Integrating then gives
\[
u_3 < u_3(\tau_*) \exp \left( \int_{\tau_*}^\tau \frac{1}{4u_3} - u_1 d\tau' \right),
\]

(140)

\[
\lim_{\tau \to \tau_M^-} \exp \left( -\int_{\tau_*}^\tau u_1 d\tau' \right) > \lim_{\tau \to \tau_M^-} \frac{u_3}{u_3(\tau_*)} \exp \left( -\int_{\tau_*}^\tau \frac{1}{4u_3} d\tau' \right) = +\infty,
\]

(141)
since \( \lim_{\tau \to \tau_M^-} 1/4u_3 = 0 \). The result immediately follows.

Lemma 4.9 then shows that \( \lim_{\tau \to \tau_M^-} u_1 = -\infty, \lim_{\tau \to \tau_M^-} u_3 = +\infty. \). Then
\[
\lim_{\tau \to \tau_M^-} u_2 = +\infty, \quad -\infty < \lim_{\tau \to \tau_M^-} \frac{u_1}{u_3} = L < -k^2 - \frac{1}{4}.
\]

Proof. First we define \( q_1 = u_1/u_3 \), which satisfies
\[
\dot{q}_1 = \frac{1}{u_3} \left( \frac{1}{4} - u_2 \right) - q_1 \left( \frac{1}{4} - \frac{2u_2}{k^2} \right).
\]

(143)
and thus
\[ \frac{\dot{q}_3}{q_3} < \frac{u_1}{4} - \frac{1}{4u_3}, \] (148)
for \( \tau \in (\tau_*, \tau_M) \), where we have again used \( \lambda u_3 < -u_1/2 \). Integrating and using Lemma 4.9 then shows that \( \lim_{\tau \to \tau_M} q_3 = 0 \).

However, it follows from \( u_1 < 0, u_3 > 0 \) and (143) that
\[ \dot{q}_1 > -q_3 + \frac{2q_3}{k^2}q_1. \] (149)

It is clear that if \( \lim_{\tau \to \tau_M} q_3 = 0 \), then \( \lim_{\tau \to \tau_M} q_1 \) is finite and so we have a contradiction. Therefore, \( L \) must be finite.

To estimate \( L \), we divide (50) across by \( u_3^2 \) which, in the case \( V_0 > 0 \), gives
\[ \left(1 - \frac{1}{R^2}\right) q_1^2 + \frac{2u_2}{u_3^2} + k^2q_1 + \frac{2 + k^2}{8u_3^2} - \frac{k^2}{2} = 0, \] (150)
in terms of \( R, u_1, u_2, u_3 \). Let \( q_4 = u_2/u_3^2 \). To determine the limiting behaviour of \( q_4 \), we consider its derivative, which may be written as
\[ \dot{q}_4 = \frac{u_2}{u_3} \left( \lambda + \frac{4q_4}{k^2} + 2q_1 \right) = q_2 Y, \] (151)
where \( Y = \lambda + 4q_4/k^2 + 2q_1 \). Recall that \( q_1 \) is monotone decreasing and \( u_3 > 0 \) sufficiently close to \( \tau_M \), say, on an interval \((\tau_0, \tau_M)\). Suppose there exists \( \tau_1 \in (\tau_0, \tau_M) \) such that \( \dot{q}_4(\tau_1) = 0 \). Then we must have \( Y(\tau_1) = 0 \), since \( u_2(\tau_1)/u_3(\tau_1) > 0 \). It is easily shown that this gives \( \dot{q}_4(\tau_1) = q_2(\tau_1)Y(\tau_1) \). Moreover, since \( Y = 4q_4/k^2 + 2\dot{q}_4 \), we have \( \dot{Y}(\tau_1) = 2\dot{q}_4(\tau_1) < 0 \), and thus \( \dot{q}_4(\tau_1) < 0 \). So \( q_4 \) can only cross zero in \((\tau_0, \tau_M)\) with negative slope, i.e., it may only change sign once on \((\tau_0, \tau_M)\). Therefore, \( q_4 \) must be monotone close to \( \tau_M \), i.e., \( \lim_{\tau \to \tau_M} q_4 \) exists. Suppose \( \lim_{\tau \to \tau_M} q_4 \neq 0 \). Then, since \( q_4 \) is positive, there must exist some \( \delta > 0 \) such that \( u_2 > \epsilon u_3^2 > 1/4 \) for \( \tau = \tau_M - \delta < \delta \).

Using this, \( u_1/u_3 > L \) and (143) produces
\[ \dot{q}_1 < -\frac{L}{4u_3} + \frac{2\epsilon u_1}{k^2}, \] (152)
for \( \tau \in (\tau_M - \delta, \tau_M) \). It follows that
\[ \lim_{\tau \to \tau_M} q_1 < \lim_{\tau \to \tau_M} \left( q_1(\tau_M - \delta) + \int_{\tau_M - \delta}^{\tau} \left( \frac{2\epsilon u_1}{k^2} - \frac{L}{4u_3} \right) d\tau' \right) = -\infty, \] (153)
using Lemma 4.9 and \( \lim_{\tau \to \tau_M} L/u_3 = 0 \). This contradicts the fact the \( L \) is finite and so we have \( \lim_{\tau \to \tau_M} q_4 = 0 \). Taking the limit of (150) then yields
\[ \omega L^2 + k^2L - \frac{k^2}{2} = 0, \] (154)
where \( \omega = \lim_{\tau \to \tau_M} (1 - \dot{R}^{-2}) \). Note that \( \omega \leq 0 \) gives \( L \geq 1/2 \) which contradicts \( L < 0 \). We then have
\[
L = -\frac{k^2}{2\omega} - \sqrt{\frac{k^4}{4\omega^2} + \frac{k^2}{2\omega}},
\tag{155}
\]
since the upper root of (155) is positive and, therefore, not allowed. Clearly \( \omega < 1 \), so \( L < -k^2/2 - \sqrt{k^4/4 + k^2/2} < -k^2 - 1/4 \), if \( k^2 > 1/4 \).

**Lemma 4.11.** For \( V_0 > 0 \), \( \lambda > 0 \), suppose that \( \tau_M < \infty \) and
\[
\lim_{\tau \to \tau_M} u_1 = -\infty, \quad \lim_{\tau \to \tau_M} u_3 = +\infty.
\tag{156}
\]
Then \( \lim_{\tau \to \tau_M} R = \infty \) and \( \lim_{\tau \to \tau_M} \mu < +\infty \), i.e. there exists a singularity at \( \tau = \tau_M \) which is reached by outgoing null rays in finite affine time.

**Proof.** Lemma 4.10 and \( \lambda > 0 \) give \( \lim_{\tau \to \tau_M} u_2 = \lim_{\tau \to \tau_M} l = \infty \). Using equation (110) we have
\[
\lim_{\tau \to \tau_M} R = \frac{k^2e^{\tau_M/2-c_1}}{2|u|} \lim_{\tau \to \tau_M} \left( e^{-k^2l/4}u_3 \right)^2.
\tag{157}
\]
Define \( Z = e^{-k^2l/4}u_1 \), which satisfies
\[
\dot{Z} = e^{-k^2l/4} \left( \frac{1}{4} - u_2 - u_1^2 - k^2u_1u_3 \right) < (-u_1 - k^2u_3) Z,
\tag{158}
\]
for \( u_2 > 1/4 \). Using Lemma 4.10, we may choose some \( \tau_* \in (0, \tau_M) \) such that \( u_1/u_3 < -k^2 - 1/8 \) and \( Z < 0 \) for all \( \tau \in (\tau_*, \tau_M) \). We then have \( \dot{Z}/Z = u_3/8 > -u_1/8L \) for all \( \tau \in (\tau_*, \tau_M) \). Integrating and using Lemma 4.9 then proves \( \lim_{\tau \to \tau_M} Z = -\infty \). Hence, \( \lim_{\tau \to \tau_M} e^{-k^2l/4}u_3 = L^{-1} \lim_{\tau \to \tau_M} Z = +\infty \), which gives \( \lim_{\tau \to \tau_M} R = +\infty \).

**Lemma 4.12.** Suppose there exists \( \tau_0 \in (0, \tau_M) \) such that \( u_1(\tau_0) \leq -1/2 \). Then \( \tau_M < \infty \) and \( \lim_{\tau \to \tau_M} u_1 = -\infty \).

**Proof.** We define a new variable \( \bar{u}_1 = u_1 + 1/2 \), which satisfies
\[
\dot{\bar{u}}_1 = \bar{u}_1 - u_2 - \bar{u}_1^2 < \bar{u}_1 - \bar{u}_1^2.
\tag{159}
\]
It is clear that if \( \bar{u}_1(\tau_0) \leq 0 \) then there exists some \( \tau > \tau_0 \) such that \( \lim_{\tau \to \tau_1^-} \bar{u}_1 = -\infty \). Then we must have \( \tau_M \leq \tau_1 \) and, using Theorem 4.1, \( \lim_{\tau \to \tau_M^-} |u_i| = +\infty \) for some \( i \). Suppose that \( \lim_{\tau \to \tau_M^-} u_i = -\infty \). It is clear from (94B) that \( u_2 \) is finite provided \( u_3 \) and \( \tau \) are finite and so we must have \( \lim_{\tau \to \tau_M^-} |u_3| = \infty \). If \( \lim_{\tau \to \tau_M^-} u_3 = +\infty \), it follows from (97) and the fact that \( u_2 > 0 \) and \( 0 < R < R(\tau_0) \) that \( \lim_{\tau \to \tau_M^-} R = 0 \). Note that \( R < R(\tau_0) \) follows from \( u_1 < 0 \) here. Note also that
\[
R = R(\tau_0) \exp \left( \int_{\tau_0}^{\tau} u_1 \, d\tau' \right),
\tag{160}
\]
and so \( \lim_{\tau \to \tau_M^-} R = 0 \) implies that
\[
\lim_{\tau \to \tau_M^-} \int_{\tau_0}^{\tau} u_1 \, d\tau' = -\infty,
\tag{161}
\]
from which it must follow that \( \lim_{\tau \to \tau_M} u_1 = -\infty \). If \( \lim_{\tau \to \tau_M} u_3 = -\infty \) then we have either \( \lim_{\tau \to \tau_M} R = 0 \) or

\[
\lim_{\tau \to \tau_M} \int_0^\tau u_2 \, d\tau' = +\infty,
\]

where we have used (97) and fact that \( R \) is bounded above again. It follows immediately from (94a) and (162) that \( \lim_{\tau \to \tau_M} u_1 = -\infty \) in this case also. Hence \( \tau_1 = \tau_M \) and the proof is complete.

**Lemma 4.13.** If \( V_0 > 0, \lambda > 0 \) and \( u_2(0) > k^2/8 \), then there exists \( \tau_0 \in (0, \tau_M) \) such that \( u_1(\tau_0) = 0 \) and \( u_2 > k^2/8 \) for all \( \tau \in [0, \tau_0] \).

**Proof.** Recall that \( u_2(0) > k^2/8 \) gives \( u_3 < 0, \hat{u}_3 < 0 \) on some initial interval. Differentiating (94c) gives

\[
\dot{u}_3 = \left( \frac{2u_2}{k^2} - \frac{1}{4} + u_1^2 \right) u_3 - u_1 \hat{u}_3.
\]

At \( \dot{u}_3 = 0 \) we have \( \ddot{u}_3 = (u_1 - u_3)u_1u_3 \), which is negative for \( u_1 > 0 > u_3 \), and so \( \dot{u}_3 < 0 \) holds while \( u_1 > 0 \). We then have \( 0 < u_2 < u_2(0) \) and \( \dot{u}_3 > 1/4 - 2u_2(0)/k^2 \) for \( u_1 > 0 \). Hence, the \( u_i \) are all bounded above and below for \( u_1 > 0 \), and so either \( \tau_M = \infty \), or there exists \( \tau_0 \) such that \( u_1(\tau_0) = 0 \). Consider (133) with \( u_1 > 0 \) and \( u_3 < 0 \), which give \( \dot{X} < -\lambda/4 - u_3^2 \). It is obvious that \( X \) or \( u_1 \) must cross zero in finite \( \tau \). However, \( u_1 < X \) if \( u_3 < 0 \) and so there must exist \( \tau_0 \) such that \( u_1(\tau_0) = 0 \). We then have \( \dot{u}_3(\tau_0) = 1/4 - 2u_2(\tau_0)/k^2 < 0 \), from which \( u_2(\tau_0) > k^2/8 \) immediately follows. The fact that \( \dot{u}_2 = \lambda u_2 u_3 < 0 \) for \( \tau \in (0, \tau_0) \) completes the proof.

**Lemma 4.14.** If \( V_0 > 0, \lambda > 0, u_2(0) > k^2/8 \), then \( u_1 < 0, u_3 < 0, \dot{u}_3 < 0, \ddot{u}_3 < 0 \) for all \( \tau \in (\tau_0, \tau_M) \), where \( u_1(\tau_0) = 0 \).

**Proof.** If \( u_2 \geq k^2/8 \) then \( \dot{u}_1 \leq -\lambda/4 - u_1^2 < 0 \). If \( u_2 < k^2/8 \) and \( \dot{u}_3 < 0 \), then by (94c) we have \( u_1u_3 > 0 \). For \( \tau \in (\tau_0, \tau_M) \) then, \( u_1 < 0 \) holds while \( u_3 < 0, \dot{u}_3 < 0 \) hold. Using the preceding lemma and (50c), we have \( \dot{R} < -\lambda R/4 \) for all \( \tau \in [0, \tau_0] \). This may be integrated to give \( R < m^{-1} \sin m\tau \leq m^{-1} \), which gives \( R^{-2} > m^2 = \lambda/4 \), for \( \tau \in [0, \tau_0] \). Now, (50c) may be rearranged to give

\[
\frac{1}{R^2} - \frac{\lambda}{4} + \lambda u_1^2 + \frac{k^2}{2} (u_1 - u_3)^2 = k^2 \left( \frac{2u_2}{k^2} - \frac{1}{4} + u_1^2 \right).
\]

Then, for \( R^{-2} > \lambda/4, \lambda > 0 \) we must have

\[
\frac{2u_2}{k^2} - \frac{1}{4} + u_1^2 > 0.
\]

We see from equation (163) that this inequality, along with \( u_1 < 0, u_3 < 0, \dot{u}_3 < 0 \), gives \( \dot{u}_3 < 0 \), which preserves \( \dot{u}_3 < 0 \). Since \( R^{-2} > \lambda/4 \) holds for \( u_1 = \dot{R}/R < 0 \), we must have \( u_1, u_3, \dot{u}_3, \ddot{u}_3 \) negative for \( \tau \in [\tau_0, \tau_M] \).

\( \square \)
Lemma 4.15. If $V_0 > 0$, $\lambda > 0$ and $u_2(0) > k^2/8$, then $\tau_M < \infty$ and $\lim_{\tau \to \tau_M^-} u_1 = \lim_{\tau \to \tau_M^-} u_3 = -\infty$.

Proof. We know that there exists $\tau_0$ such that $\dot{u}_3 < 0, \ddot{u}_3 < 0$ for all $\tau \in (\tau_0, \tau_M)$. Supposing that $\tau_M = +\infty$, then we must have $\lim_{\tau \to \tau_M^-} u_3 = -\infty$. We must also have $u_1 > -1/2$ for all $\tau \in (0, \tau_M)$, by Lemma 4.12. We also know from the preceding proof that $R^{-2} > \lambda/4$ for all $\tau \in (0, \tau_M)$. Equation (161) then gives

$$\frac{k^2}{2}(u_1 - u_3)^2 < k^2 \left( \frac{2u_2}{k^2} - \frac{1}{4} + u_1^2 \right).$$

If $u_1 > -1/2$, then the lefthand side blows up at $\tau_M$ which, given that $u_2$ is finite, is a clear contradiction. Hence, $u_1$ crosses $-1/2$ at some finite $\tau$ and $\lim_{\tau \to \tau_M^-} u_1 = -\infty$ for $\tau_M$ finite. Dividing (94a) by $u_1$, integrating and taking the limit $\tau \to \tau_M$ we find

$$\lim_{\tau \to \tau_M^-} \int_{\tau_*}^\tau \frac{1 - 4u_2}{4u_1} - u_1 \, d\tau' = \lim_{\tau \to \tau_M^-} \log \frac{u_1}{u_1(\tau_*)} = \infty,$$

where $\tau_*$ is chosen such that $u_1 < 0$ for $\tau \in [\tau_*, \tau_M)$. Since $\lim_{\tau \to \tau_M^-} (1 - 4u_2)/4u_1 = 0$, it follows that

$$\lim_{\tau \to \tau_M^-} \int_{\tau_*}^\tau u_1 \, d\tau' = -\infty.$$

Integrating $\dot{u}_3/u_3$ and taking the limit we find

$$\lim_{\tau \to \tau_M^-} \log \frac{u_3}{u_3(\tau_*)} = \lim_{\tau \to \tau_M^-} \int_{\tau_*}^\tau \frac{k^2 - 8u_2}{4k^2 u_3} - u_1 \, d\tau' = +\infty,$$

using the fact that $(k^2 - 8u_2)/4k^2u_3$ is bounded for $\tau \in (0, \tau_M)$. The result immediately follows. \hfill \Box

Note that it follows from this result and Lemma 4.8 that there is a singularity at $\tau = \tau_M$.

Lemma 4.16. If $V_0 > 0$, $\lambda > 0$ and $u_2(0) < k^2/8$, then $\tau_M > \pi/2m$ and $u_1(\pi/2m) > 0, u_2(\pi/2m) < k^2/8$, $R(\pi/2m) > m^{-1}$ and $k^2u_3/2 > u_1$ for all $\tau \in [\pi/2m, \tau_M)$.

Proof. For $u_2 < k^2/8$ we have $\dot{u}_1 > -\lambda/4 - u_1^2$. Integrating over $(0, \tau)$ gives

$$u_1 > m\cot(m\tau),$$

where we have used $\lim_{\tau \to 0^+} u_1 = \infty$. While $u_3 > 0$ we have $u_2 > u_2(0)$ which, combined with the above, gives

$$\dot{u}_3 < \frac{1}{4} - \frac{2u_2(0)}{k^2} - (m\cot m\tau)u_3.$$

Integrating over $(0, \tau)$ gives

$$u_3 < \frac{b(1 - \cos m\tau)}{m \sin m\tau} = \frac{b \sin(m\tau/2)}{m \cos(m\tau/2)}.$$
where $b = 1/4 - 2u_2(0)/k^2$. Integrating again we find
\[
\lambda \int_0^\tau u_3 \, d\tau' < -\frac{2\lambda b}{m^2} \log \left[ \cos \left( \frac{m \tau}{2} \right) \right] = -8b \log \left[ \cos \left( \frac{m \tau}{2} \right) \right],
\]
and so using $\dot{u}_2 = \lambda u_2 u_3$,
\[
u_2 < u_2(0) \cos^{-8b} \left( \frac{m \tau}{2} \right).
\]
Note that $u_3$ cannot cross zero from above if $u_2 < k^2/8$. The bounds $u_3 > 0, (170), (171)$ and (174) therefore hold, and solutions exist, as long as $u_2 < k^2/8$ holds. Assuming $\tau_M > \pi/2m$ we have $u_2(\pi/2m) < 2^b u_2(0)$. Letting $z = 8u_2(0)/k^2 < 1$, and using $4b = 1 - 8u_2(0)/k^2 = 1 - z$, we have
\[
\frac{8u_2(\pi/2m)}{k^2} < 2^{1-z} z \leq 1,
\]
for all $z \leq 1$, which is equivalent to $u_2(\pi/2m) < k^2/8$. Our assumption is then validated. So we have $u_1(\pi/2m) > 0$ from (170), and it is straight-forward to show that $R > m^{-1} \sin m \tau$ on $[0, \pi/2m]$, which gives $R(\pi/2m) > m^{-1}$.

Recall $X = u_1 - k^2 u_3/2$, which satisfies
\[
\dot{X} < -\frac{\lambda}{4} X^2,
\]
provided $u_3 > 0, X \geq 0$, using (135). Integrating over $(0, \tau)$ we find $X < m \cot m \tau$. Since $\cot m \tau = 0$ at $\tau = \pi/2m$ and $u_3 > 0$ for $\tau \in (0, \pi/2m)$, there must exist $\tau_* \in (0, \pi/2m)$ such that $X(\tau_*) = 0$. Note also that $X$ cannot cross zero from below if $\lambda > 0$ and so $X < 0$ for $\tau \in (\tau_*, \tau_M)$.

**Lemma 4.17.** If $V_0 > 0, \lambda > 0$ and $u_2(0) < k^2/8$, then there exists $\tau_0 \in (0, \tau_M)$ such that $u_1(\tau_0) = 0, u_3(\tau_0) > 0$ and $R(\tau_0) > m^{-1}$.

**Proof.** Suppose $u_1 > 0$ for all $\tau \in (0, \tau_M)$. Then $u_3 > 0$ for all $\tau \in (0, \tau_M)$, using the previous lemma. We then have $\dot{u}_3 < 1/4$ for all $\tau \in (0, \tau_M)$, which gives a finite upper bound on $u_3$, and thus $u_2$, for finite $\tau$. Hence, $\tau_M = +\infty$. If $u_2 \leq 1/4$ we have
\[
\dot{u}_3 \geq \frac{\lambda}{2k^2} - u_1 u_3 > \frac{\lambda}{2k^2} - u_1(\tau_*) u_3,
\]
for any $\tau_* \in (0, \tau_M)$. It follows that $u_3 > u_m = \min\{u_3(\tau_*), \lambda/2k^2 u_1(\tau_*)\}$ for $u_2 \leq 1/4$ and $\tau \in (\tau_*, \tau_M)$. This gives $\dot{u}_2 > \lambda u_m u_2$, and so there must exist $\tau_{**} \in (\tau_*, \tau_M)$ such that $u_2 > 1/4$ for all $\tau \in (\tau_{**}, \tau_M)$. By inspection of (94a), there must then exist $\tau_0 \in (\tau_{**}, \tau_M)$ such that $u_1(\tau_0) = 0$. Using Lemma 4.16, we have $\tau_0 > \pi/2m$ and thus $u_3(\tau_0) > 0$ and $R(\tau_0) > R(\pi/2m) > m^{-1}$.

**Lemma 4.18.** For $V_0 > 0, \lambda > 0$, suppose there exists $\tau_0 \in (0, \tau_M)$ such that $u_1(\tau_0) = 0$ and $u_2(\tau_0) > k^2/8$. Then $\tau_M = +\infty$ and $\lim_{\tau \to \tau_M} u_1 = -\infty, \lim_{\tau \to \tau_M} u_3 = +\infty$. 
Proof. Equation (150) may also be written as
\[
u_{1}^2 - \frac{1}{R^2} + \frac{\lambda}{4} - \frac{k^2 u_{2}^2}{2} = k^2 \left( \frac{1}{4} - \frac{2u_{2}}{k^2} - u_{1}u_{3} \right) = k^2 \dot{u}_{3}.
\]

Now define
\[
\Gamma = u_{1}^2 - \frac{1}{R^2} + \frac{\lambda}{4},
\]
which satisfies
\[
\dot{\Gamma} = 2u_{1}\dot{u}_{1} + \frac{2\dot{R}}{R^3} = 2u_{1}\left( \frac{1}{4} - u_{2} - u_{1}^2 + \frac{1}{R^2} \right)
= 2u_{1}\left( \frac{k^2}{2} - u_{2} - \Gamma \right).
\]

If \(\Gamma > 0, u_{2} > k^2/8\) and \(u_{1} < 0\), then \(\dot{\Gamma} > 0\). From the hypothesis we have \(\Gamma(\tau_{0}) > 0\) and as long as \(u_{2} > k^2/8\) holds we have \(\dot{u}_{1} < -\lambda/4 - u_{1}^2\). Equation (178) tells us that \(\dot{u}_{3} > 0\) if \(u_{3} < \sqrt{2\Gamma/k^2}\) and so we have \(u_{3} > 0\), which gives \(u_{2} > k^2/8\), while \(\Gamma > 0\).

Hence, \(\Gamma > 0, u_{2} > k^2/8, u_{1} < 0\) and \(u_{3} > 0\) hold for all \(\tau \in (\tau_{0}, \tau_{M})\). We then have \(\dot{u}_{1} < -\lambda/4 - u_{1}^2\) for \(\tau \in (\tau_{0}, \tau_{M})\), \(\tau_{M} < +\infty\) and \(\lim_{\tau \to \tau_{M}^{-}} u_{1} = \lim_{\tau \to \tau_{M}^{-}} X = -\infty\).

Integrating \(\dot{X}/X = -\lambda/4X - u_{1}\), then shows that
\[
\lim_{\tau \to \tau_{M}^{-}} \int_{\tau_{*}}^{\tau} u_{1} = -\infty,
\]
where \(\tau_{*}\) is chosen such that \(u_{1}(\tau_{*}) < 0\). We can use this to show \(\lim_{\tau \to \tau_{M}^{-}} \Gamma = +\infty\) by integrating (180). Since \(\dot{u}_{3} < 0\) for \(u_{3} > \sqrt{2\Gamma/k^2}\), we must have \(u_{3} < \sqrt{2\Gamma/k^2}\) and \(u_{3} > 0\), for \(\tau\) sufficiently close to \(\tau_{M}\) and so \(\lim_{\tau \to \tau_{M}^{-}} u_{3}\) must exist. Now suppose \(\lim_{\tau \to \tau_{M}^{-}} u_{3} < +\infty\). We then have \(\lim_{\tau \to \tau_{M}^{-}} u_{2} < +\infty\) and
\[
\lim_{\tau \to \tau_{M}^{-}} \log \left( \frac{u_{3}}{u_{3}(\tau_{*})} \right) = \lim_{\tau \to \tau_{M}^{-}} \int_{\tau_{*}}^{\tau} \left( \frac{1}{4u_{3}} - \frac{2u_{2}}{k^2 u_{3}} - u_{1} \right) d\tau' = +\infty,
\]
since \(1/4u_{3} - 2u_{2}/k^2 u_{3}\) is bounded above and below under the assumption. Hence, we have \(\lim_{\tau \to \tau_{M}^{-}} u_{3} = +\infty\), by contradiction.

\[\square\]

**Lemma 4.19.** For \(V_{0} > 0, \lambda > 0\), suppose there exists \(\tau_{0} \in (0, \tau_{M})\) such that \(u_{1}(\tau_{0}) = 0, u_{2}(\tau_{0}) < k^2/8\). Then \(\tau_{M} < +\infty\) and \(\lim_{\tau \to \tau_{M}^{-}} u_{1} = -\infty, \lim_{\tau \to \tau_{M}^{-}} u_{3} = +\infty\).

**Proof.** Lemma 4.17 tells us that if \(u_{1} < 0, u_{2} < k^2/8\) and \(u_{3} > 0\), then \(\dot{u}_{3} > 0\), which together give \(\dot{u}_{2} > 0, \ddot{u}_{2} > 0\). While \(|u_{1}|\) and \(u_{2}\) remain bounded above, \(u_{3}\) remains bounded above also, so either there exists \(\tau_{*} \in (\tau_{0}, \tau_{M})\) such that \(u_{2}(\tau_{*}) = k^2/8, u_{1}(\tau_{*}) < 0\), or we have \(\lim_{\tau \to \tau_{M}^{-}} u_{1} = -\infty\) for \(\tau_{M} < +\infty\). Suppose the former is true. Then we have \(\dot{u}_{3}(\tau_{*}) > 0\). It follows from (163) that \(u_{3}\) cannot cross zero from above if \(u_{2} \geq k^2/8\) and \(u_{3} > 0\). Hence, \(u_{2} > k^2/8, u_{3} > 0, u_{3} > 0\) hold for all \(\tau \in (\tau_{*}, \tau_{M})\). We then have \(\dot{u}_{1} < -\lambda/4 - u_{1}^2\), from which it follows that \(\lim_{\tau \to \tau_{M}^{-}} u_{1} = -\infty\) for \(\tau_{M} < +\infty\) in this case also. Given that, in both cases, \(\dot{u}_{3} > 0\) for all \(\tau \in [0, \tau_{M})\), then \(\lim_{\tau \to \tau_{M}^{-}} u_{3}\) must exist. A similar argument
to one given in the preceding lemma gives \( \lim_{\tau \to \tau_M} u_3 = +\infty \). Using Lemma 4.10 then gives \( \lim_{\tau \to \tau_M} u_2 = +\infty \) and so such a \( \tau_* \) does exist after all.

**Lemma 4.20.** For \( V_0 > 0, \lambda > 0 \), suppose there exists \( \tau_0 \in (0, \tau_M) \) such that \( u_1(\tau_0) = 0, u_2(\tau_0) = k^2/8 \). Then \( \tau_M < +\infty \) and \( \lim_{\tau \to \tau_M^-} u_1 = -\infty, \lim_{\tau \to \tau_M^-} u_3 = +\infty \).

**Proof.** At \( \tau_0 \) we have \( \dot{u}_3 = \ddot{u}_3 = 0 \), and it is not hard to check that at the third derivative of \( u_3 \) reduces to \(-\dot{u}_1(\tau_0)u_3(\tau_0)^2 > 0 \). A similar argument to the one given above then shows that \( u_2 > k^2/8, \dot{u}_3 > 0 \) obtain for \( \tau \in (\tau_0, \tau_M) \) and the rest follows in a similar fashion.

**Proposition 4.5.** For \( V_0 > 0, \lambda > 0, u_2(0) \neq k^2/8 \) we have \( \tau_M < \infty \) and there is a curvature singularity at \( \tau_M \), which is reached by outgoing null rays in finite affine time.

**Proof.** For \( u_2(0) > k^2/8 \), Lemma 4.15 tells us that \( \lim_{\tau \to \tau_M^-} u_1 = \lim_{\tau \to \tau_M^-} u_3 = -\infty \). Lemma 4.8 then tells us that \( \lim_{\tau \to \tau_M^-} R = \infty \) in this case. Clearly \( u_2 \) is bounded above for all \( \tau \in [0, \tau_M) \) in this case and by inspection of \( \text{(184)} \) we see that \( \lim_{\tau \to \tau_M^-} \mu < +\infty \).

In the case \( u_2(0) < k^2/8 \), Lemma 4.17 shows that there exists \( \tau_0 \in (0, \tau_M) \) such that \( u_1(\tau_0) = 0 \). Depending on the sign on \( u_2(\tau_0) - k^2/8 \), one of the preceding three lemmas shows that \( \lim_{\tau \to \tau_M^-} u_1 = -\infty \) and \( \lim_{\tau \to \tau_M^-} = +\infty \). Lemma 4.11 then gives \( \lim_{\tau \to \tau_M^-} R = +\infty \). To show that \( \lim_{\tau \to \tau_M^-} \mu < +\infty \) in this case, we recall from Lemma 4.11 that \( e^{-k^2}u_1 = Z < Z(\tau_*) \) for some \( \tau_* \in (0, \tau_M) \). This gives \( e^{k^2} < u_1/Z(\tau_*) \), which in turn gives \( e^{k^2} < (u_1/Z(\tau_*))^{1/2} \). Now let \( p = (-u_1)^{1/2} \) and consider

\[
\dot{p} = \frac{1}{2p} \left( -\frac{1}{4} + u_2 + u_1^2 \right) > \frac{p^3}{2},
\]

for \( u_2 > 1/4 \). Dividing by \( p^2 \) and integrating we have

\[
\int_{\tau_*}^{\tau} \frac{\dot{p}}{p^2} \, d\tau' = \frac{1}{p(\tau_*)} - \frac{1}{p} > \int_{\tau_*}^{\tau} \frac{p}{2} \, d\tau'.
\]

Using equation \( \text{(109)} \) we have

\[
C\mu = \int_{0}^{\tau} e^{k^2/2-\tau'/2} \, d\tau' < \int_{0}^{\tau_*} e^{k^2/2} \, d\tau' + \int_{\tau_*}^{\tau} \left( \frac{u_1}{Z(\tau_*)} \right)^{1/2} \, d\tau' = \int_{0}^{\tau_*} e^{k^2/2} \, d\tau' + \int_{\tau_*}^{\tau} \left( \frac{p}{-Z(\tau_*)} \right)^{1/2} \, d\tau'.
\]

Taking the limit and using \( \text{(184)} \) we find that \( \lim_{\tau \to \tau_M^-} \mu < +\infty \).

5. Exact solutions

5.1. \( k^2 = 2 \)

In this case we have \( \lambda = 0 \) which gives us constant potential \( V = V_0 \). We then have

\[
\ddot{R} = \frac{d}{d\tau} R\dot{\ell} = \left( \frac{1}{4} - V_0 \right) R.
\]

(186)
Proposition 5.1. If $k^2 = 2$ and $0 < V_0 \leq 1/4$, then there is a curvature singularity along $N_-$ which is reached in finite affine time.

Proof. In the case $V_0 < 1/4$, solutions of (186) in terms of $S$ are given by

$$S = v^{-1}e^{-\tau/2} \sinh v\tau, \quad l = l_0 + \log \left[\frac{1}{2}(1 + \cosh v\tau)\right],$$

(187)

where $v = \sqrt{1/4 - V_0}$. We also have

$$\lim_{\tau \to \infty} \dot{l} = \lim_{\tau \to \infty} \frac{v \sinh v\tau}{1 + \cosh v\tau} = v. \quad (188)$$

For $k^2 = 2$ we then have

$$\lim_{\tau \to \infty} R = \lim_{\tau \to \infty} \frac{e^{\tau/2} - l}{|u|} \left(\frac{1}{4} - \dot{l}^2 + 4V_0\right) = \lim_{\tau \to \infty} \frac{10V_0 e^{\tau/2 - l_0}}{|u|(1 + \cosh v\tau)}. \quad (189)$$

The solution to the geodesic equation (109) reduces to

$$\frac{1}{2} \int_0^\tau e^{l_0 - \tau'/2} (1 + \cosh v\tau') d\tau' = C\mu. \quad (190)$$

Note that $V_0 > 0$ gives $v < 1/2$ for which

$$\lim_{\tau \to \infty} S = 0, \quad \lim_{\tau \to \infty} R = \infty, \quad \lim_{\tau \to \infty} \mu < \infty. \quad (191)$$

For $V_0 = 1/4$ we have

$$S = \tau e^{-\tau/2}, \quad l = l_0, \quad R = \frac{5e^{\tau/2 - l_0}}{4|u|}, \quad (192)$$

which give

$$\lim_{\tau \to \infty} S = 0, \quad \lim_{\tau \to \infty} R = \infty, \quad \lim_{\tau \to \infty} \mu < \infty. \quad (193)$$

\[ \square \]

Proposition 5.2. If $k^2 = 2$, $V_0 < 0$, then $N_-$ corresponds to radial null infinity and the Ricci scalar decays to zero there.

Proof. In the case $v > 1/2 (V_0 < 0)$, (187), (189) and (190) tell us that

$$\lim_{\tau \to \infty} S = \infty, \quad \lim_{\tau \to \infty} R = 0, \quad \lim_{\tau \to \infty} \mu = \infty. \quad (194)$$

We remind the reader that we are not considering the case $V_0 = 0 (v = 1/2)$.

\[ \square \]

Proposition 5.3. If $k^2 = 2$, $V_0 > 1/4$, there exists a curvature singularity along $\tau = \pi/\bar{v}$ where $\bar{v} = \sqrt{V_0 - 1/4}$. 

Proof. In the case \( V_0 > 1/4 \), solutions to (186) are given by
\[
S = \bar{v}^{-1}e^{-\tau/2} \sin \bar{v}\tau, \quad l = l_0 + \log \left[ \frac{1}{2}(1 + \cos \bar{v}\tau) \right],
\]
where \( \bar{v} = \sqrt{V_0 - 1/4} \). At \( \bar{v}\tau = \pi \) we have \( S = 0 \) and
\[
\lim_{\tau \to \pi/\bar{v}} l = -\infty, \quad \lim_{\tau \to \pi/\bar{v}} \dot{l} = -\lim_{\tau \to \pi/\bar{v}} \bar{v} \tan \left( \frac{\bar{v}\tau}{2} \right) = -\infty,
\]
which give \( \lim_{\tau \to \pi/\bar{v}} R = \infty \). \( \square \)

5.2. \( V_0e^{\lambda_0} = k^2/8 \)

Lemma 5.1. \( u_3 \) is monotone in a neighbourhood of the axis.

Proof. Note that there exists \( \tau_1 \in (0, \tau_M) \) such that \( u_1 > u_3 \) and \( u_1 > 0 \) hold for \( \tau \in (0, \tau_1) \). Suppose there exists \( \tau_0 \in (0, \tau_1) \) with \( u_3(\tau_0) = 0 \). We then have \( \ddot{u}_3(\tau_0) = (u_1(\tau_0) - u_3(\tau_0))u_1(\tau_0)u_2(\tau_0) \), which has the same sign as \( u_3(\tau_0) \), since \( u_1(\tau_0) > u_3(\tau_0), u_1(\tau_0) > 0 \). So either \( u_3(\tau_0) < 0 \) and is a local max, or \( u_3(\tau_0) > 0 \) and is a local min. Since \( u_3(0) = 0 \), in the former case we must then have \( \tau_* \in (0, \tau_0) \) such that \( u_3(\tau_*) < 0 \) is a local min, which is contradiction. Similarly for the latter case. Hence, \( u_3 \) is monotone on \( (0, \tau_1) \). \( \square \)

Lemma 5.2. If \( u_2(0) = k^2/8 \), then \( u_2 = k^2/8 \) and \( u_3 = 0 \) for all \( \tau \in [0, \tau_M) \).

Proof. First note that \( u_2 = k^2/8, u_3 = 0 \) is an invariant manifold of the system (97) with \( \epsilon = -1 \). The system (97) is not defined at \( \tau = 0 \) and so we must show that there exists \( \tau_0 > 0 \) such that \( u_2(\tau_0) = k^2/8, u_3(\tau_0) = 0 \). Using the preceding result, \( u_3 \) is monotone and, since \( u_3(0) = 0 \), has the same sign while \( u_1 > u_3 \) and \( u_1 > 0 \) hold. There must therefore exist \( \tau_1 \) such that \( u_2 \) is monotone on \( [0, \tau_1] \). It follows that \( u_2 - k^2/8 \) has the same sign on \( (0, \tau_1) \). Suppose that \( u_2 - k^2/8 > 0 \) on \( (0, \tau_1) \). We can choose \( \tau_1 \) such that \( R > 0 \) on \( (0, \tau_1) \). Then, using (97) and \( R > 0, \dot{l} = u_3 \) must be negative on \( (0, \tau_1) \), which is a contradiction. A similar argument rules out \( u_2 - k^2/8 < 0 \) on \( (0, \tau_1) \), so we have \( u_2 - k^2/8 = 0 \) for all \( \tau \in (0, \tau_1) \). If \( u_2 \) is constant on \( (0, \tau_1) \) then \( u_3 = 0 \) must also hold there. \( \square \)

Proposition 5.4. Recall \( m = \sqrt{\lambda}/2 \). If \( V_0e^{\lambda_0} = k^2/8 \) and \( \lambda < 0 \) then there is a singularity at \( \tau = \infty \), which is reached by radial null rays in finite affine time. If \( V_0e^{\lambda_0} = k^2/8 \) and \( \lambda > 0 \) then there is a singularity at \( \tau = \pi/m \), which is reached by radial null rays in finite affine time.

Proof. Using the preceding result, we have \( u_2 = k^2/8, u_3 = 0 \), and thus \( \ddot{R} = -\lambda R/4 \), for all \( \tau \in (0, \tau_M) \). The solutions in terms of \( S \) are
\[
S = \begin{cases} 
  m^{-1}e^{-\tau/2} \sin m\tau, & \text{if } \lambda > 0, \\
  m^{-1}e^{-\tau/2} \sinh m\tau, & \text{if } \lambda < 0.
\end{cases}
\]
Note that the case $\lambda = 0, u_2(0) = k^2/8$ is precisely the case $k^2 = 2, V_0 = 1/4$ covered in proposition 5.1. If $\lambda < 0$ then we clearly have $\tau_M = +\infty$. In this case we also have $m = 1/2 - k^2/4 < 1/2$ and so $\lim_{\tau \to \infty} S = 0$. Using $\dot{l} = 0, V_0 e^\lambda = k^2/8$ and (110) the Ricci scalar reduces to

$$R = \frac{3k^2 e^{-k^2 l_0/2 + \tau/2 - c_1}}{8|u|},$$

and it immediately apparent that $\lim_{\tau \to \infty} R = +\infty$.

In the case $\lambda > 0$ we have $S(\pi/m) = 0$. In the cases studied thus far, surfaces characterised by $S = 0$, other than the regular axis, have been singular, which was demonstrated by an infinite curvature invariant. In this case, however, it is clear from (198) above that $R$ is finite if $\tau$ is finite, and one can check that this is the case for other invariants such as $T = T_{ab}T^{ab}$ and the Kretschmann scalar $R_{abcd}R^{abcd}$. However, the specific length of the cylinders $L$ limits to zero as $\tau \to \pi/m$, which violates the regular axis conditions. This may be seen solving (50c) for $\phi$ given the solutions for $R = e^{\tau/2}S$ given above, which yields

$$e^\phi = \frac{e^{\phi_0 + \tau/4}}{\cos(m\tau/2)}.$$  \hfill (199)

Recalling that $L = |u| e^{-\phi}$, we have $\lim_{\tau \to \pi/m} L = 0$. We speculate that we have a non-scalar curvature spacetime singularity at $\tau = \pi/m$ in this case. The solution to the geodesic equation (109) with $l = l_0$ shows that $\mu$ is finite for all $\tau > 0$ in both cases.

6. Proof of Theorem 2.3

In this section we gather the results from the two previous sections which give the proof of Theorem 2.3.

Proof of Theorem 2.3

Proof. The proof of cases 1 and 2 are given by Propositions 4.1 and 5.2, respectively. For case 3, part (i) is given by Proposition 4.2 and part (ii) is given by Proposition 4.3. Case 4 part (i) is given by Propositions 4.4, part (ii) is given by Propositions 4.5 and 5.4, and part (iii) by Proposition 5.3. Case 5 is proven by Propositions 5.1 and 5.4. \qed

7. Conclusions and further work

We have determined the global structure of solutions in the causal past of the singularity at $O$ for all values of the parameters $V_0$ and $k$ and the initial datum $l_0$. For $k^2 \geq 2$, the spacetime terminates either on or before the surface $N^-$. For $k^2 < 2$, solutions exist on $\mathcal{N}^-$, which are regular, and may be extended into region II. In a follow up paper, we investigate the evolution of these solutions with a view to answering the question of cosmic censorship relative to this class of spacetimes.
8. Acknowledgments

BN acknowledges gratefully fruitful discussions with Hideki Maeda, who suggested this problem, and who shared his preliminary calculations with the authors. This project was funded by the Irish Research Council for Science, Engineering and Technology, grant number P07650.

References

[1] Carr B. J. and Coley A. A., Self-similarity in general relativity. Class. Quant. Grav. 16 081502 (1999)
[2] Apostolatos T A and Thorne K S Rotation halts cylindrical, relativistic collapse. Phys. Rev. D 46 2435 (1992)
[3] Shapiro S.L. and Teukolsky S.A., Gravitational collapse of rotating spheroids and the formation of naked singularities. Phys. Rev. D 45, 2006 (1992)
[4] Echeverria F. Gravitational collapse of an infinite, cylindrical dust shell Phys. Rev. D 47 2271-2282 (1993)
[5] Letelier P. S. and Wang A. and Singularities formed by the focusing of cylindrical null fluids. Phys. Rev. D 49 064006 (1994)
[6] Nolan B. C. Naked singularities in cylindrical collapse of counterrotating dust shells. Phys. Rev. D 65 104006 (2002)
[7] Wang A., Critical collapse of a cylindrically symmetric scalar field in four-dimensional Einstein's theory of gravity. Phys. Rev. D 68 064006 (2003)
[8] Nolan B. C. and Nolan L. V. On isotropic cylindrically symmetric stellar models. Class. Quant. Grav. 21 3693 (2004)
[9] Harada T., Nakao K. and Nolan B. Einstein-Rosen waves and the self-similarity hypothesis in cylindrical symmetry. Phys. Rev. D40 024025 (2009)
[10] Thorne K. in (ed.) Klaudner J. R. Non-spherical collapse - a short review. Magic Without Magic: John Archibald Wheeler (W. H. Freeman and Company, 1972)
[11] Hayward S. A. Gravitational waves, black holes and cosmic strings in cylindrical symmetry. Class. Quantum Grav. 17 1749 (2000)
[12] Ashtekar A., Bičákov J., Schmidt B. Asymptotic structure of symmetry-reduced general relativity. Phys. Rev. D 55, 669-686 (1997)
[13] Kyo M., Harada T. Maeda H. Asymptotically Friedmann self-similar scalar field solutions with potential Phys. Rev. D 77 124036 (2008)
[14] Wainwright J., Ellis G.F.R., Dynamical Systems in Cosmology (Cambridge University Press, Cambridge, England, 1997)
[15] Debnath L. and Mikusinski P. Introduction to Hilbert spaces with applications (Academic Press, San Diego, 1999)
[16] Tavakol R. in (ed.) Wainwright J. and Ellis G.F.R., Introduction to dynamical systems. Dynamical systems in cosmology (Cambridge University Press, Cambridge, 1997)