Abstract

We consider for the first time the solutions of Klein-Gordon equation in gravitational field of massive point source in GR. We examine numerically the basic bounded quantum state and the next few states in the discrete spectrum for different values of the orbital momentum. A novel feature of the solutions under consideration is the essential dependence if their physical properties on the gravitational mass defect of the point source. Such mass defect was not introduced up to recently. Its variation yields a repulsion or an attraction of the quantum levels up to their quasi-crossing.

1 Introduction

The correct solutions of the Einstein equations for the gravitational field of point particle with bare mechanical mass $M_0$ were found recently [1]. It turns out that these fundamental solutions are described by mathematical distributions and own the necessary jumps in the first derivatives of the metric, needed to satisfy the Einstein equations at the place of the point source. The new solutions form a two-parameters family of metrics on singular manifolds $\mathbb{M}^{(1,3)}\{g_{\mu\nu}\}$. They are defined by the bare mass $M_0$ and by the Keplerian mass $M$, or equivalently, by the Keplerian mass $M$ and the ratio of the masses: $\varrho = M/M_0 \in (0, 1)$. This ratio describes the gravitational mass defect of the point particle.

The mathematical and the physical properties of the new solutions are essentially different in comparison with the well known other spherically symmetric static solutions to the Einstein equations with different type of singularities at the center of the symmetry, which is surrounded by an empty space. The previously known solutions were often considered as a solutions for describing of single point particle in GR. In the most of the known cases this turns to be incorrect [1]. Therefore the new solutions call for reconsideration of many of commonly accepted notions and their physical interpretation.

Here we are starting the study of the GR wave equations in the gravitational field of massive point. In the present article we shall consider as an example only the simplest GR wave equation – the Klein-Gordon one. We shall present a numerical study of this well known equation which describes (anti)particles of zero spin [3].

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The reader can find the basic results of the analysis of this equation in the vacuum Schwarzschild metric, when described in the commonly used Hilbert gauge or in other gauges, for example, in the recent review article [4], in [5], and in the large amount of the references therein. A discussion of different radial gauges and corresponding references one can find in [1].

2 The Formulation of the Problem

Making use of the natural radial variable of the problem: \( r \in (0, \infty) \) [1], one can represent the new regular solutions for the gravitational field outside of the massive point source in the form:

\[
\begin{align*}
\text{ds}^2 &= e^{2\varphi_G} \left[ dt^2 - \frac{dr^2}{N_G(r)^4} - \rho(r)^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right].
\end{align*}
\]

Here

\[
\varphi_G(r; M, M_0) := -\frac{G_N M}{r + G_N M/\ln(M_0/M)}
\]

is the modified Newton potential, the coefficient \( N_G(r) = (2\varphi_G)^{-1}(e^{2\varphi_G} - 1) \), the Hilbert luminosity variable is

\[
\rho(r) = \frac{2G_N M}{1 - e^{2\varphi_G}} = \frac{r + G_N M/\ln(M_0/M)}{N_G(r)},
\]

and \( G_N \) is the Newton gravitational constant.

In Hilbert gauge, outside of the source, the solution has a standard form:

\[
\begin{align*}
g_{tt}(\rho) &= 1 - \rho_G/\rho, \quad g_{\rho\rho}(\rho) = -1/g_{tt}(\rho),
\end{align*}
\]

where \( \rho_G = 2G_N M \) is the Schwarzschild radius. We have to stress that the presence of the matter source forces us to consider this form of the solution only on the physical interval of the luminosity variable \( \rho \in (\rho_0, \infty) \), where

\[
\rho_0 = 2G_N M/(1 - g^2) \geq \rho_G.
\]

One can find in [1] the physical and the mathematical justification of this cutting procedure of the admissible interval of luminosity variable \( \rho \) for the discovered by Hilbert, Droste and Weyl [2] form of the solution (2.4). Here we shall stress that such unusual cutting is a ultimate consequence of the existence of a matter source of the gravitational field. After all, it permit us to satisfy the physical requirement not to allow an infinite luminosity of the point sources in correspondence to the real observations. In the articles [1] it was shown that in GR we have a natural cutting parameter for the classical divergences. It originates from the drastic change of the space-time geometry due to the infinite mass-energy density of the point particle. It turns out that in presence of matter source all phenomena, related with the event horizon, together with the very horizon, belong to the non-physical domain of the variables. Therefore, in accord to Dirac’s intuition [6], all such notions ”should not be taken into account in any physical theory” of matter.
The luminosity variable $\rho$ is an exclusive one, because the results, expressed by making use of it, are \textit{locally} invariant under radial gauge transformations. From computational and from physical point of view a more preferable variable is: $g = g_{tt} = 1 - \rho_G/\rho$. Then

$$ds^2 = g dt^2 - \rho_G^2 \left[ \frac{dg^2}{g(1-g)^4} + \frac{d\theta^2 + \sin^2 \theta d\phi^2}{(1-g)^2} \right].$$

(2.6)

In these variables the 4D D’Alembert operator in the curved space-time $\mathbb{R}^{1,3}(g_{\mu\nu})$, created by a massive point, acquires the form

$$\Box = g^{-1} \partial_t^2 - \rho_G^{-2} \left[ (1-g)^4 \partial_g \left( g \partial_g \right) + (1-g)^2 \Delta_{\theta \phi} \right]$$

(2.7)

where $\Delta_{\theta \phi} = \sin^{-1} \theta \partial_{\theta} (\sin \theta \partial_{\theta}) + \sin^{-2} \phi \partial_{\phi}^2$.

The Klein-Gordon equation for test particles of mass $m$ in space-time $\mathbb{R}^{1,3}(g_{\mu\nu})$ has a standard form:

$$\Box \Phi + m^2 \Phi = 0.$$  

(2.8)

As a result of the spherical symmetry the pseudo-Reimannian manifold $\mathbb{R}^{1,3}(g_{\mu\nu})$, created by the point particle, has a group of motion $SO(3)$. Nontrivial are the quantities and equations only on the quotient space $\mathbb{R}^{1,1} = \mathbb{R}^{1,3}/SO(3)$, with natural global coordinates $t$ and $g$. The reduction of Eq. (2.6) on the quotient space $\mathbb{R}^{1,1}$ can be reached by the substitution $\Phi(t, g, \theta, \phi) = \Psi_t(t, g)Y_{l_1 l_2}(\theta, \phi)$, where $Y_{l_1 l}(\theta, \phi)$ are the standard spherical functions: $\Delta_{\theta \phi}Y_{l_1 l}(\theta, \phi) = -(l+1)Y_{l_1 l}(\theta, \phi), \ l = 0, 1, 2, ..., l_z = -l, ..., 0, ..., l$.

For the time-dependent radial wave function $\Psi_t(t, g)$ on the manifold $\mathbb{R}^{1,1}$ we have the partial differential equation:

$$g^{-1} \partial_t^2 \Psi_t - \rho_G^{-2} \left\{ (1-g) \partial_g \left( g \partial_g \Psi_t \right) + \left[ l(l+1)(1-g)^2 + m^2 \right] \Psi_t \right\} = 0.$$  

(2.9)

The space-time $\mathbb{R}^{1,3}(g_{\mu\nu})$ of the problem at hand is invariant under translations with respect to the time variable $t$, as well. This symmetry of $\mathbb{R}^{1,3}(g_{\mu\nu})$ is related to the translation group $T(1): t \rightarrow t + \text{const}$ and allows the total reduction of the problem to one dimensional one using the anzatz $\Psi_t(t, g) = e^{-iEt}R_t(g)$. For the time-independent radial wave function $R_t(g)$ on the quotient space $\mathbb{R}^{1}(1) = \mathbb{R}^{1,1}/T(1)$ we obtain an ordinary differential equation:

$$\frac{d^2 R_t}{dg^2} + \frac{1}{g} \frac{dR_t}{dg} + \left[ \frac{\varepsilon^2}{g^2 (1-g)^4} - \frac{\mu^2}{g(1-g)^4} - \frac{l(l+1)}{g(1-g)^2} \right] R_t = 0.$$  

(2.10)

Here $\varepsilon = \rho_G E$ and $\mu = \rho_G m$ are, correspondingly, the dimensionless total energy and mass of the spinless test particles. We are using the units $c = \hbar = 1$. The dimensionless orbital momentum in our units is $l = L/m \rho_G$, where $L$ is the dimensionfull one.

The motion of the Klein-Gordon particles with a fixed orbital momentum $l$ can be treated as a relativistic motion in a dimensionless potential

$$v_t(g) = (1-g) \left[ l(l+1)g(1-g) - \mu^2 \right], \ g \in [g_0, 1],$$  

(2.11)

which is shown (for $\mu = 1$ and several different values $l \geq 0$) in Fig. 2. The point $g = 0$ corresponds to the event horizon and is placed in the nonphysical domain $g \in [0, g_0)$,
Figure 1: The potential $v_l(g)$ for $l = 0, 1, 2, 3, 4$ and $\mu = 1$. The dashed line corresponds to the value $l_{\text{crit}}^{v} = \frac{1}{2} \sqrt{1 + 12\mu^2} - \frac{1}{2}$ for which the maximum of the function $v_l(g)$ arise.

where $g_0 = g^2 > 0$. The point $g = 1$ corresponds to the physical infinity (with respect to the variables $r$, or $\rho$). There the space-time $M_{(1,3)}\{g_{\mu\nu}\}$ is flat. Writing down the Eq. (2.10) in the form

$$(1 - g)^4 \left( \frac{d}{dg} \right)^2 R_l + [\lambda - v_l(g)] R_l = 0$$

we see that the points $g = 0 \quad g = 1$ are singular points of this equation. The second one is a non-regular singular point. Here $\lambda = \varepsilon^2 - \mu^2$.

The substitutions:

$$u = \ln \left( \frac{g}{1 - g} \right) + \frac{1}{(1 - g)}, \quad (2.12)$$

introduce “the tortoise coordinate” $u$ and the corresponding radial wave function $P_l(u)$, and lead to the following standard (Schrödinger like) form of the Eq. (2.10):

$$\frac{d^2 P_l}{du^2} + [\lambda - w_l(u)] P_l = 0. \quad (2.13)$$

The bounded quantum states can be found as a solutions of (2.13) under the boundary conditions

$$P_l(u_0) = 0, \quad P_l(\infty) = 0, \quad (2.14)$$

together with the usual $L_2$-normalization

$$\int_{u_0}^{\infty} P_l^2(u) du - 1 = 0. \quad (2.15)$$

The quantity

$$w_l(g) = v_l(g) + g(1 - g)^3 = (1 - g) \left[ l(l + 1)g(1 - g) + g(1 - g)^2 - \mu^2 \right] \quad (2.16)$$
is the extended dimensionless potential of the problem and depends implicitly on the variable \( u \) via the solution (2.12) of the Cauchy problem

\[
\frac{d g}{du} = g (1 - g)^2, \quad g(u_0) = g^2. \tag{2.17}
\]

An essential novel feature of our approach to the problem is the correct fixing of the physical domain of the variables: the potentials \( v_l \) and \( w_l \) must be considered only on the interval \([u_0, \infty)\), where \( u_0 = u_0(\varrho^2) \). Thus the nonphysical infinitely deep potential well at \( \varrho \to 0 \), and the event horizon at \( \varrho = \varrho_G (\Rightarrow u_G = -\infty) \), are excluded from our consideration.

Figure 2: The radial wave function \( P_{0,l}(u, \varrho^2) \) for different values of the quantum number \( l = 2, 3, 4, 5, 6, 7, 8 \).

As a completely new phenomenon, a discrete spectrum comes into being in the problem at hand. We denote by \( \varepsilon_{n,l} = \sqrt{\lambda_{n,l} + \mu^2} \) the discrete values of the energy \( \varepsilon \) of the bounded states. The principle quantum number \( n = 0, 1, \ldots \) determines the number of the zeros of the radial wave function \( P_{n,l}(u) \) in the interval \( u \in (u_0, \infty) \) and of the function \( R_{n,l}(g) = (1 - g)P_{n,l}(u(g)) \) in the interval \( g \in (g_0, 1) \).

3 Basic Numerical Results

For the numerical study of the solutions of the Sturm-Liouville problem (2.13) – (2.15) on some finite interval \([u_0, u_\infty]\) we use the continuous analog of Newton method [7]. The corresponding linear boundary problems at each iteration are solved by means of spline-collocation scheme of fourth order of approximation. The convergence of the calculated eigenfunctions and corresponding eigenvalues \( \lambda_{n,l} \) for large enough values of \( u_\infty \) is proved on a set of embedded intervals. As a result of numerical experiments we have obtained a number of new properties of the bounded quantum states of the problem at hand, a part of which were unexpected to some extend.

The basic results for the radial wave function \( P_{n,l}(u) \) for different values of the orbital quantum number \( l \) an \( n = 0 \) are shown in Fig. 2.
Figure 3: The radial wave function \( P_{02}(u; \varrho^2) \) for different values of the two variables \( u \) and the squared mass ratio \( \varrho^2 \).

Figure 4: The radial functions \( P_{02}(u; \varrho^2) \) for different values of the gravitational mass defect of the source.

In the Fig. 3 using as an example the wave function \( P_{02}(u; \varrho^2) \), we show the typical dependence of the wave functions of the bounded states both on the variables \( u \) and on the squared mass ratio \( \varrho^2 \).

The radial functions \( P_{02}(u; \varrho^2) \) for different values of the gravitational mass defect \( \varrho \) are shown in Fig. 4. As seen, depending on the values of the squared mass ratio \( \varrho^2 \) of the point source, the wave function of the test particle in the point particle’s gravitational field is located in the inner potential well (for small \( \varrho^2 \) – the curves 1 and 2 in Fig. 4), or in the exterior potential well (for big \( \varrho^2 \) – the curve 4 in Fig. 4). In the narrow intermediate domain of values of \( \varrho^2 \) a transition regime take place – the curve 3 in Fig. 4. Actually, this transition depends continuously on the variable \( \varrho^2 \), but it seems that it is a jump-like, because the transition happens in a quite smaller scales with comparison to the other much more smooth changes of the corresponding quantities. (See, for example, the dependence of the eigenvalues on the variable \( \varrho^2 \)). These phenomena are described in
a more transparent way in the 3D Fig. 3.

In our approach to the problem the basic new physical phenomena are related with the gravitational mass defect of the point source. Such mass defect was not considered and studied until now, because for the standard Hilbert form of the Schwarzschild solution "the bare rest-mass density is never even introduced" [8] correctly.

The dependence of the first four eigenvalues \( \lambda_{0l,1l,2l,3l} \) on the squared mass ratio \( \nu^2 \) is shown in Fig. 5 for \( l = 2 \). A typical steep-like dependence of the discrete eigenvalues \( \lambda_{nl}(\nu^2) \) on the variable \( \nu^2 \) is seen. For each value of the principle quantum number \( n \) the number of the jumps of the energy levels \( \varepsilon_{nl}(\nu^2) = \sqrt{\lambda_{nl}(\nu^2)} + \mu^2 \) depends on the number of the maxima of the corresponding wave function which are moved from the inner well to the exterior one during the transition wave function which develops with the increase of the values of the mass ratio \( \nu^2 \).

Figure 5: The dependence of the first four eigenvalues \( \lambda_{0l,1l,2l,3l} \) on the mass ratio \( \nu^2 \).

It’s clear that such nontrivial behavior of the energy levels \( \varepsilon_{nl}(\nu^2) \) of the Klein-Gordon equation in a gravitational field of a point particle is due to the existence of two finite potential wells: 1) An inner one which has a size of order of the Schwarzschild radius and in general case of macroscopic orbital momenta is very deep; and 2) An exterior one, which is extremely shallow in comparison with the inner well. The normal world
with almost Newton gravity "lives" in the exterior well. The both wells are separated typically by a huge potential barrier which looks like a centrifugal barrier from outside, and as a suspensory barrier of the type of a spherical potential wall from inside. Our calculations describe the quantum penetration under this barrier. The quantum result depends strongly on the mass defect of the point source.

The obtained in this article behavior of a quantum test particles in the gravitational field of point source of gravity seems to us to be much more physical then the one in the wide spread models of black holes. Clearly, in contrast to such space-time holes with nonphysical infinitely deep well in them, in our case the finite inner well plays the role of a trap for the test particles. It is not excluded that this way one may be able to construct a model of very compact matter objects with an arbitrary large mass and a size of order of Schwarzschild radius. It's possible that such type of objects may explain the observed in the Nature compact dark objects. They may be the final product of the stelar evolution, instead of the quite formal and sophisticated constructions like black holes which correspond to empty space solutions of Einstein equations. Of course, at present these possibilities are an open problem which deserves further careful study.

![Figure 6: The attraction and the repulsion of the quantum energy levels for different values of the gravitational mass defect of the point source.](image)

It is easy to observe one more astonishing phenomena in the discrete spectrum of a test quantum particle in gravitational field of point source. It is related, too, with the mass defect of the source. If one puts in the same figure the functional dependence of all squared discrete energy levels $\varepsilon_{n,l}^2$ on the squared mass ratio $\varrho^2 = g_{tt} |_{r=0}$ (see in [1]), one can observe a repulsion and an attraction (up to a quasi-crossing) of these levels, as shown in Fig. 6. Such type of behavior of quantum discrete levels is well known in the laser physics, in the neutrino oscillations and in other branches of quantum physics. To the best of our knowledge we are observing such behavior of quantum levels in the fundamental gravitational physics for the first time.
4 Conclusions

In the present-days traditional approach to the motion of test particles in the Schwarzschild gravitational field, inside the event horizon there exist a nonphysical infinitely deep potential well. Everything which somehow can be placed inside this well will fall unavoidable to its center $\rho = 0$ for a finite time. It is well known that this center is a physically inadmissible geometrical singularity. To hide such undesired singularity one must use a unprovable mathematical hypothesis like the cosmic censorship one. Today we have enough examples which show that such hypotheses can not be correct at least in its original formulation, see, for example [9] and the references therein.

In the article [1] the principal role of the matter point source of gravity was stressed. In GR the matter point source presents a natural cutting factor for the physical values of the luminosity variable $\rho \in [\rho_0, \infty)$, where $\rho_0 > \rho_G$ (2.5). This is because the infinite mass density of the matter point changes drastically the geometry of the space-time around it. In full accord with Dirac’s suggestion [6] this cutting places all nonphysical phenomena, together with the very event horizon in the nonphysical domain of the variables and yields a large number of new interesting phenomena.

In particular, as shown in the present article, the described cutting yields an interesting novel discrete spectra for Klein-Gordon test particles in the gravitational field of massive point. Mathematically this is caused by the universal change of the boundary conditions for the corresponding wave equation, due to the presence of the matter source of gravity and depends on its mass defect.

A real discrete spectrum does not exist in the case of black hole solution, just because of the presence of a hole in the space-time, which absorbs everything around it. Our consideration gives a unique possibility for a direct experimental test of the existence of space-time holes. If one will find a discrete spectra of corresponding stationary waves, propagating around some compact dark object, one will have indisputable evidence that there is no space-time hole inside such object. If, in contrast, one will find only a decaying quasi-normal modes with complex eigenvalues (see [4] and the references herein), this will indicate that we are observing indeed a black hole.

We have obtained similar results for electromagnetic and gravitational waves in the gravitational field of point particle[10]. For them the properly modified quasi-normal modes turn to depend on gravitational mass defect, too. The further study of these phenomena is an important physical problem and will give us more practical physical criteria for an experimental distinguishing of the two complete different hypothetical objects: 1) the space-time black holes and 2) the possible new very compact dark objects made of real matter. Present days astrophysical observations are still not able do make a difference between them. The only real observational fact is that we are seen very compact and very massive objects, which show up only due to their strong gravitational field. An actual theoretical problem is to find a convincing model for description of these already observed compact dark objects and the criteria for the experimental justification of such model. We hope that the present article is a new step in this direction.

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