Distributed Nash Equilibrium Seeking for Games in Systems with Bounded Control Inputs

Maojiao Ye

Abstract—Noticing that physical limitations are ubiquitous in practical engineering systems, this paper considers Nash equilibrium seeking for games in systems where the control inputs are bounded. More specifically, first-order integrator-type systems with bounded control inputs are firstly considered and two saturated control strategies are designed to seek for the Nash equilibrium of the game. Then, we further consider the Nash equilibrium seeking problem for games in second-order integrator-type systems. As this problem has rarely been investigated, we firstly propose a centralized seeking strategy without considering the boundedness of the control inputs, followed by a distributed counterpart. By further adapting a saturation function into the Nash equilibrium seeking strategy, a new seeking strategy is then designed for the considered second-order systems with bounded controls. In the proposed distributed strategies, consensus protocols are included for information sharing and the saturation functions are utilized to construct bounded control inputs. The convergence results are established through conducting Lyapunov stability analysis. Lastly, by considering the connectivity control of mobile sensor networks, the proposed methods are numerically verified.

Index Terms—Nash equilibrium seeking; bounded control inputs; first-order integrator-type systems; second-order integrator-type systems.

I. INTRODUCTION

Games are attracting growing interests from researchers in the multi-agent communities for the analysis of multi-agent systems in recent years [1]. For example, consensus was accomplished by utilizing cooperative game theory in [2]. Differential games were applied to solve distributed optimal tracking control of multi-agent systems with external disturbance in [3]. The works in [4]-[6] linked games to cooperative control and optimization of multi-agent systems, respectively. In [7], the consensus analysis for a class of hybrid multi-agent systems was conducted based on a non-cooperative game. These works motivate the consideration of the physical constraints in multi-agent systems for Nash equilibrium seeking problems. The concerned constraints include but are not limited to communication issues among the agents, input saturation and system dynamics. Recent years witnessed the efforts made by researchers to accommodate the communication issues for games in distributed networks (see, e.g., [25]). However, actuator limitations and system dynamics have rarely been investigated.

As many engineering systems are subject to actuator limitations (e.g., robotic manipulators [8], spacecraft [9], hard disk drive servo systems [10], just to name a few), the boundedness of control inputs appears to be a problem that is both practically and theoretically concerned. The study for systems with bounded control inputs has a rich history. For example, input-saturated linear systems were considered in [11] based on an anti-windup design. Backstepping approaches were employed for developing robust adaptive control strategies to accommodate uncertain nonlinear systems subject to input saturation [12]. Two-player zero-sum games with non-quadratic payoffs were employed to solve the $H_{\infty}$ control of systems with bounded control inputs in [13]. Moreover, with the development of multi-agent systems, consensus of input-saturated multi-agent systems has attracted a lot of attention. The authors in [14] dealt with leader-following consensus of linear multi-agent systems with input saturation. Global consensus of saturated discrete-time systems was addressed in [15]. Optimal consensus for multi-agent systems with bounded control inputs was investigated in [16]-[17]. However, games in systems with bounded controls still remain to be solved. Moreover, most of the existing Nash equilibrium seeking strategies are designed for games in quasi-static systems or systems with first-order integrator-type dynamics. Velocity-actuated vehicles are typical examples that are of first-order integrator-type dynamics. However, acceleration-actuated vehicles are second-order integrator-type systems that are beyond the scope of first-order integrator-type systems.

Inspired by the above observations, we intend to design Nash equilibrium seeking strategies for games in both first-order and second-order integrator-type systems in which the controls are bounded. The considered problem is challenging as the saturation function would introduce high nonlinearity into the closed-loop system. Moreover, the analysis of second-order systems is more complex compared with the first-order systems studied by the existing literature. In summary, compared with the existing works, this paper contributes in the following aspects:

1) Distributed Nash equilibrium seeking for games in systems with bounded control inputs is considered in this paper. First-order integrator-type systems are firstly considered, in which both the saturated gradient play and a distributed strategy are investigated. Then, second-order integrator-type systems are explored. A centralized algorithm is firstly proposed without considering the boundedness of the control inputs, fol-
and \( \min\{a, b\} = b \) if \( a > b \). In addition, the graph related definitions follow those in [24] and are omitted directly in this context.

II. Problem Formulation

Consider a game with \( N \) players whose dynamics are governed by

\[
x^n_i = u_i,
\]

where \( x_i \in \mathbb{R} \) is the action of player \( i \) and \( u_i \in \mathbb{R} \) is the control input that satisfies \( |u_i| \leq U \). Moreover, \( x^n_i \) denotes the \( n \)-th order time derivative of \( x_i \) and in the subsequent section, \( n = 1 \) and \( n = 2 \) will be investigated successively. Let \( f_i(x) \), where \( x = [x_1, x_2, \cdots, x_N]^T \), be the cost function of player \( i \) and \( \{1, 2, \cdots, N\} \) denotes the set of \( N \) players. Given that the Nash equilibrium of the game exits, this paper aims to design the bounded controls to seek for the Nash equilibrium \( x^* = (x^*_1, x^*_2, \cdots, x^*_N) \) on which

\[
f_i(x^n_i, x^n_{-i}) \leq f_i(x_i, x^*),
\]

for \( x_i \in \mathbb{R}, i \in \{1, 2, \cdots, N\} \) and \( x_{-i} = [x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_N]^T \) under the following conditions.

**Assumption 1:** The players’ cost functions are \( C^2 \) functions. Moreover, for each \( i \in \{1, 2, \cdots, N\} \), \( \frac{\partial f_i(x)}{\partial x_i} \) is globally Lipschitz for \( x \in \mathbb{R}^N \).

**Assumption 2:** The players are equipped with a communication graph \( \mathcal{G} \), which is undirected and connected.

**Assumption 3:** [24][27] There exists a positive constant \( m \) such that

\[
(x - z)^T(\bar{P}(x) - \bar{P}(z)) \geq m||x - z||^2,
\]

for all \( x, z \in \mathbb{R}^N \). Note that in (3), \( \bar{P}(x) = \left[ \frac{\partial f_1(x)}{\partial x_1}, \frac{\partial f_2(x)}{\partial x_2}, \cdots, \frac{\partial f_N(x)}{\partial x_N} \right]^T \).

**Remark 1:** Let \( g_i(x) = \frac{\partial f_i(x)}{\partial x_i} \), then,

\[
\begin{align*}
g_i(x_1, x_2, \cdots, x_N) - g_i(z_1, z_2, \cdots, z_N) \\
g_i(x_1, x_2, \cdots, x_N) - g_i(z_1, z_2, \cdots, x_N) \\
+ g_i(z_1, x_2, \cdots, x_N) - g_i(z_1, z_2, \cdots, x_N) \\
+ \cdots \\
+ g_i(z_1, z_2, \cdots, z_{N-1}, x_N) - g_i(z_1, z_2, \cdots, z_N).
\end{align*}
\]

By the mean value theorem,

\[
g_i(x_1, x_2, \cdots, x_N) - g_i(z_1, z_2, \cdots, z_N) = \frac{\partial g_i}{\partial x_1}(\bar{s}_1, x_2, \cdots, x_N)(x_1 - z_1) + \frac{\partial g_i}{\partial x_2}(z_1, \bar{s}_2, \cdots, x_N)(x_2 - z_2) + \cdots + \frac{\partial g_i}{\partial x_N}(z_1, z_2, \cdots, z_{N-1}, \bar{s}_N)(x_N - z_N),
\]

where \( \bar{s}_i \in [x_i, z_i] \) for all \( i \in \{1, 2, \cdots, N\} \). Hence,

\[
(x - z)^T(\bar{P}(x) - \bar{P}(z)) = (x - z)^TH_1(s)(x - z),
\]

where \( H_1(s) \) is a diagonal matrix whose diagonal elements are \( h_1, h_2, \cdots, h_N \), and \( \lambda_{\min}(Q) \) denotes the minimum eigenvalue of \( Q \). Moreover, \( \otimes \) is the Kronecker product. The notation \( \min\{a, b\} = a \) if \( a \leq b \) and \( \min\{a, b\} = b \) if \( a > b \). In addition, the graph related definitions follow those in [24] and are omitted directly in this context.
where \( H_1(s) \) is defined as a matrix in which the element on the \( i \)th row and \( j \)th column is \( h_{ij}(s_i) = \frac{\partial^2 f_i}{\partial x_i \partial x_j} (z_1, z_2, \cdots, z_{j-1}, \bar{s}_j, x_{j+1}, \cdots, x_N) \). Hence, Assumption 3 is satisfied if and only if there exists a positive constant \( m \) such that \( H_1(s) + H_1^T(s) \geq 2mI \). Consider a special case in which \( x_i = \bar{s}_i \), \( \forall i \in \{1, 2, \cdots, N\} \), then,

\[
H^T(x) + H(x) \geq 2mI,\tag{7}
\]

for \( x \in \mathbb{R}^N \), where

\[
H(x) = \begin{bmatrix}
\frac{\partial f_1(x)}{\partial x_1} & \frac{\partial f_2(x)}{\partial x_2} & \cdots & \frac{\partial f_N(x)}{\partial x_N} \\
\frac{\partial f_1(x)}{\partial x_2} & \frac{\partial f_2(x)}{\partial x_2} & \cdots & \frac{\partial f_N(x)}{\partial x_N} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_1(x)}{\partial x_N} & \frac{\partial f_2(x)}{\partial x_N} & \cdots & \frac{\partial f_N(x)}{\partial x_N}
\end{bmatrix}.	ag{8}
\]

**Assumption 4:** The elements in \( H(x) \) are bounded for \( x \in \mathbb{R}^N \).

**Remark 2:** The global Lipschitz of \( \frac{\partial f_i(x)}{\partial x_i} \) for \( i \in \{1, 2, \cdots, N\} \) (in Assumption 3) and Assumption 4 are utilized for the development of global convergence results, and without these conditions, weaker convergence results can be obtained.

### III. Main Results

In the following, Nash equilibrium seeking for games in which the players are of first-order integrator-type dynamics and second-order integrator-type dynamics will be successively investigated.

**A. First-order integrator-type systems**

In this section, we consider games in which the players’ actions are governed by

\[
\dot{x}_i = u_i, \quad i \in \{1, 2, \cdots, N\}.\tag{9}
\]

In the subsequent sections, saturated gradient play will be firstly considered, followed by a distributed seeking strategy.

1) **Saturated gradient play:** To seek for the Nash equilibrium of the game, we suppose that the players update their actions according to the saturated gradient play given by

\[
\dot{x}_i = -\rho_U \left( \frac{\partial f_i(x)}{\partial x_i} \right),\tag{10}
\]

where \( i \in \{1, 2, \cdots, N\} \), and \( \rho_U(\eta_i) = \text{sgn}(\eta_i) \min\{\eta_i, \bar{U}\} \).

**Theorem 1:** The Nash equilibrium of the game is globally asymptotically stable under (10) given that Assumptions 1 and 3 are satisfied.

**Proof:** Let

\[
V(\vec{P}(x)) = \sum_{i=1}^{N} \int_0^t \rho_U(\dot{x}_i) dt,\tag{11}
\]

be the Lyapunov candidate function. Then, it can be easily verified that the Lyapunov candidate function is positive definite and radially unbounded.

Moreover,

\[
\dot{V} = \sum_{i=1}^{N} \rho_U \left( \frac{\partial f_i(x)}{\partial x_i} \right)^T H(x) \rho_U \left( \frac{\partial f_i(x)}{\partial x_i} \right) \cdot \vec{P} \cdot \vec{P}^T.	ag{12}
\]

By Assumption 3 \( H(x) \geq 2mI \). Therefore,

\[
\dot{V} \leq -m \left| \rho_U \left( \frac{\partial f_i(x)}{\partial x_i} \right) \right|^2.	ag{13}
\]

Hence, \( \left| \rho_U \left( \frac{\partial f_i(x)}{\partial x_i} \right) \right| \to 0 \) for all \( i \in \{1, 2, \cdots, N\} \), which indicates that \( |x_i - x_j| \to 0 \) as \( t \to 0 \).

**Remark 3:** Note that the global Lipschitz condition in Assumption 1 is not required to establish the convergence result in Theorem 1 though we directly suppose that Assumption 1 is satisfied in the statement of Theorem 1 for convenience.

In Theorem 1, the convergence property of the saturated gradient play in (10) is investigated. In the following, we investigate the problem in distributed networks.

2) **Consensus-based distributed Nash equilibrium seeking:** To achieve Nash equilibrium seeking in distributed networks, we suppose that the players can communicate with each other via a local communication graph \( \mathcal{G} \). Then, the Nash equilibrium seeking strategy can be designed as

\[
\dot{x}_i = -\rho_U \left( \frac{\partial f_i(x)}{\partial x_i} \right),
\]

\[
\dot{y}_{ij} = -\theta_{ij} \left( \sum_{k=1}^{N} a_{ik}(y_{ij} - y_{kj}) + a_{ij}(y_{ij} - x_j) \right),\tag{14}
\]

for \( i, j \in \{1, 2, \cdots, N\} \) and \( \theta_{ij} = \theta_{bij} \), where \( \theta \) is a positive parameter to be determined and \( \theta_{ij} \) is a fixed positive constant for each \( i, j \in \{1, 2, \cdots, N\} \). Moreover, \( y_i = [y_{i1}, y_{i2}, \cdots, y_{iN}]^T \) stands for player \( i \)'s local estimate on \( x \) and \( \frac{\partial f_i}{\partial x_i}(x_i) \) is defined as \( \frac{\partial f_i}{\partial x_i}(y_i) = \frac{\partial f_i}{\partial x_i}(x = y) \).

Furthermore, \( a_{ij} \) is the element on the \( i \)th row and \( j \)th column of the adjacency matrix of the communication graph \( \mathcal{G} \).

Then, the concatenated-vector form of (14) is

\[
\dot{\vec{y}} = -\theta(\mathcal{L} \otimes I_N \otimes x + A)(y - I_N \otimes x),\tag{15}
\]

where \( y = [y_{ij}]_{vec}, \theta = \text{diag}([\theta_{ij}]), \mathcal{L} \) is the Laplacian matrix of \( \mathcal{G}, A = \text{diag}([a_{ij}]) \) and \( I_{N \times N} \) is an \( N \times N \) dimensional identity matrix.

**Remark 4:** The seeking strategy in (14) is adapted from the seeking strategy in [24] in which the saturation function is included to ensure that \( |\dot{x}_i| \leq U \).

The following is a supportive lemma for further facilitation of the closed-loop system analysis in (14).

**Lemma 1:** For all \( \eta_1, \eta_2 \in \mathbb{R} \),

\[
|\rho_U(\eta_1) - \rho_U(\eta_2)| \leq |\eta_1 - \eta_2|.\tag{16}
\]
Proof: Noticing that $|\rho_\nu(\eta_1) - \rho_\nu(\eta_2)|$ can be regarded as the distance between $\rho_\nu(\eta_1)$ and $\rho_\nu(\eta_2)$ and $|\eta_1 - \eta_2|$ is the distance between $\eta_1$ and $\eta_2$, the conclusion can be easily derived.

The following theorem establishes the stability result for the seeking strategy in (14).

**Theorem 2:** Suppose that Assumptions [1][4] are satisfied, and the players update their actions according to (14). Then, there exists a $\theta^*$ such that for each $\theta \in (0^*, \infty)$, the Nash equilibrium is globally asymptotically stable.

**Proof:** Define the Lyapunov candidate function as

$$V = \sum_{i=1}^{N} \int_{0}^{\rho_\nu(t)} \rho_\nu(t) \, dt + (y - 1_N \otimes x)^T \mathcal{P}(y - 1_N \otimes x)$$

where $\mathcal{P}$ is a symmetric positive definite matrix that satisfies [23]

$$\mathcal{P}\Theta(L \otimes I_{N \times N} + A) + (L \otimes I_{N \times N} + A)\Theta\mathcal{P} = Q$$

where $Q$ is a symmetric positive definite matrix.

Then,

$$\dot{V} = \left[ \rho_\nu \left( \frac{\partial f_i(x)}{\partial x_i} \right) \right]^T \mathcal{H}(x) \left[ \rho_\nu \left( \frac{\partial f_i(x)}{\partial x_i} \right) \right] - l_i \|y - 1_N \otimes x\|^2 - \lambda_{\min}(Q)\theta \|y - 1_N \otimes x\|^2.$$  

Therefore,

$$\dot{V} \leq \left[ \rho_\nu \left( \frac{\partial f_i(x)}{\partial x_i} \right) \right]^T \mathcal{H}(x) \left[ \rho_\nu \left( \frac{\partial f_i(x)}{\partial x_i} \right) \right] - \lambda_{\min}(Q)\theta \|y - 1_N \otimes x\|^2.$$

By Lemma [1] and Assumption [1]

$$\left| \rho_\nu \left( \frac{\partial f_i(x)}{\partial x_i} \right) - \rho_\nu \left( \frac{\partial f_i(x)}{\partial x_i} \right) \right| \leq \frac{1}{\rho_\nu} |\frac{\partial f_i(x)}{\partial x_i} - \frac{\partial f_i(x)}{\partial x_i} (y_i)| \leq \bar{t}_i |x - y_i|,$$  

for some positive constant $\bar{t}_i$.

Hence, there are positive constants $l_1, l_2, l_3$ such that

$$\dot{V} \leq -m \left| \left[ \rho_\nu \left( \frac{\partial f_i(x)}{\partial x_i} \right) \right] \right|^2 - \lambda_{\min}(Q)\theta \|y - 1_N \otimes x\|^2 + l_1 \left| \left[ \rho_\nu \left( \frac{\partial f_i(x)}{\partial x_i} \right) \right] \right| \|y - 1_N \otimes x\|^2 + l_2 \|y - 1_N \otimes x\|^2 + l_3 \|y - 1_N \otimes x\|^2 \left| \left[ \rho_\nu \left( \frac{\partial f_i(x)}{\partial x_i} \right) \right] \right|,$$

by further noticing that the elements in $H(x)$ are bounded according to Assumption [4].

Therefore,

$$\dot{V} \leq - \left( m - \frac{l_1}{2\epsilon_1} - \frac{l_3}{2\epsilon_2} \right) \left| \left[ \rho_\nu \left( \frac{\partial f_i(x)}{\partial x_i} \right) \right] \right|^2 - \left( \lambda_{\min}(Q)\theta - l_2 - \frac{l_1\epsilon_1}{2} - \frac{l_3\epsilon_2}{2} \right) \|y - 1_N \otimes x\|^2$$

in which $\epsilon_1, \epsilon_2$ are positive constants that can be arbitrarily chosen. Choose $\epsilon_1, \epsilon_2$ such that $m - \frac{l_1}{2\epsilon_1} - \frac{l_3}{2\epsilon_2} > 0$ and for fixed $\epsilon_1, \epsilon_2$, choose $\theta$ such that $\lambda_{\min}(Q)\theta - l_2 - \frac{l_1\epsilon_1}{2} - \frac{l_3\epsilon_2}{2} > 0$. If this is the case, there exists a positive constant $l_4$ such that

$$\dot{V} \leq -l_4\|x\|^2,$$

where $x = \left[ \left[ \rho_\nu \left( \frac{\partial f_i(x)}{\partial x_i} \right) \right] \right]^T (y - 1_N \otimes x)^T$. By further noticing that $V$ is positive definite and radially unbounded, we can conclude that $\|x\| \to 0$ as $t \to \infty$, which indicates that $y \to 1_N \otimes x \to 1_N \otimes x^*$ as $t \to \infty$. To this end, we arrive at the conclusion.

**B. Second-order integrator-type systems**

In this section, we consider Nash equilibrium seeking for games in second-order integrator-type systems in which player $i$’s action is governed by

$$\dot{x}_i = \nu_i$$

$$\nu_i = u_i,$$

for $i \in \{1, 2, \ldots, N\}$. More specifically, in Section [III-B.1], a centralized algorithm will be proposed without considering the boundedness of the control inputs. Moreover, the problem is reconsidered under distributed networks in Section [III-B.3]. Lastly, the boundedness of the control inputs will be addressed in Section [III-B.3].

1) Centralized Nash equilibrium seeking without considering the boundedness of the control inputs: Let the Nash equilibrium seeking strategy be

$$\dot{x} = \nu$$

$$\dot{\nu} = -\beta (\frac{\partial f_i(x)}{\partial x_i}) - H(x)\nu,$$

where $\nu = [\nu_i]_{vec}$ and $\alpha, \beta$ are positive control gains to be determined.

Then, the following result can be obtained.
**Theorem 3:** Suppose that Assumptions 1-3 are satisfied and the players update their actions according to (26). Then, there exists a positive constant $\alpha^*$ such that for each $\alpha \in (0, \alpha^*)$, there exists a positive constant $\beta^*(\alpha)$ such that for each $\beta \in (0, \beta^*)$, the Nash equilibrium is globally asymptotically stable under (26).

**Proof:** Define the Lyapunov candidate function as

$$V = \nu^T \nu + \frac{1}{2} \left[ \frac{\partial f_i(x)}{\partial x_i} \right]_{vec}^T \left[ \frac{\partial f_i(x)}{\partial x_i} \right]_{vec} + \nu^T \left[ \frac{\partial f_i(x)}{\partial x_i} \right]_{vec}. \tag{27}$$

Then,

$$V = \frac{1}{6} \left[ \left[ \frac{\partial f_i(x)}{\partial x_i} \right]_{vec} \right]^2 + \frac{1}{4} ||\nu||^2 + \frac{1}{2} \frac{\nu^T \nu}{\nu_i}(\nu)_{vec}$$

and it can be easily concluded that the Lyapunov candidate function is positive definite and radially unbounded. Moreover,

$$\dot{V} = 2\nu^T \left( -\beta \nu - \alpha \left[ \frac{\partial f_i(x)}{\partial x_i} \right]_{vec} - H(x) \nu \right)$$

$$+ \nu^T H(x) \nu + \nu^T \nu$$

$$\leq - (2\beta + m) ||\nu||^2 - \alpha \left[ \frac{\partial f_i(x)}{\partial x_i} \right]_{vec}$$

$$- \left( 2\alpha + \beta \right) ||\nu|| \left[ \frac{\partial f_i(x)}{\partial x_i} \right]_{vec}$$

$$\leq - 2\alpha + \beta + 2\alpha + \beta$$

$$\leq 2\alpha + \beta$$

where $\epsilon_1$ is a positive constant that can be arbitrarily chosen. Let

$$\frac{2\alpha + \beta}{2\beta + m} < \epsilon_1 < \frac{2\alpha + \beta}{2\alpha + 2\sqrt{am}}.$$ 

Then, $\dot{V}$ is negative definite. Hence, the conclusion can be drawn with $\alpha^* = m$ and $\beta^* = 2\alpha + 2\sqrt{am}$. \qed

**Remark 5:** Note that similar to Theorem 1 the global Lipschitz condition in Assumption 1 is not required in the development of Theorem 3 though we directly suppose that Assumption 1 is satisfied for statement convenience. The seeking strategy in (26) achieves the Nash equilibrium seeking in a centralized fashion. In the following, we consider Nash equilibrium seeking in distributed networks.

2) **Distributed Nash equilibrium seeking without considering the boundedness of control inputs:** Suppose that in the considered game, each player $i, i \in \{1, 2, \cdots, N\}$ updates their own action according to

$$\dot{x}_i = \nu_i$$

$$\dot{\nu}_i = -(x_i - \nu_i) - (\nu_i - \hat{\nu}_i)$$

$$\dot{z}_i = -\bar{K}_i \frac{\partial f_i(y_i)}{\partial x_i}(\nu)_{vec}$$

$$\dot{\nu}_{ij} = -\theta_{ij} \sum_{k=1}^N a_{ik}(y_{ij} - y_{kj}) + a_{ij}(y_{ij} - z_{ij}), \tag{31}$$

where $j \in \{1, 2, \cdots, N\}$ and $z_{ij}, \nu_{ij}$ are auxiliary variables. Moreover, $\bar{K}_i = \theta_i K_i, \theta_{ij} = \theta_i \theta_j$ in which $\theta, \theta_1$ are positive parameters to be determined and $K_i, \theta_{ij}$ are fixed positive constants.

The concatenated vector form of (31) is

$$\dot{x} = \nu$$

$$\dot{\nu} = -(x - z) - (\nu - \bar{\nu})$$

$$\dot{z} = -\bar{K} \left[ \frac{\partial f_i(y_i)}{\partial x_i} \right]_{vec}$$

$$\dot{\nu}_{ij} = -\theta_{ij} \sum_{k=1}^N a_{ik}(y_{ij} - y_{kj}) + a_{ij}(y_{ij} - z_{ij}), \tag{32}$$

where $\bar{K} = diag\{\bar{K}_i\}, \Theta = diag\{\theta_{ij}\}$ and $z = [z_{ij}]_{vec}$.

The following theorem establishes the stability of the equilibrium in (32).

**Theorem 4:** Suppose that Assumptions 1-4 are satisfied and the players update their actions according to (32). Then, there exists a positive constant $\theta^*$ such that for each $\theta \in (\theta^*, \infty)$, there exists a positive constant $\theta^*_1(\theta)$ such that for each $\theta_1 \in (0, \theta^*_1(\theta))$, the Nash equilibrium is globally asymptotically stable.

**Proof:** Consider

$$V(\eta) = \frac{1}{2} \left[ (z - x^*)^T K^{-1} (z - x^*) + (y - 1_N \otimes z)^T \mathcal{P} (y - 1_N \otimes z) + \frac{1}{2} (x - z)^T (x - z) + \frac{1}{2} (\nu - \bar{\nu})^T (\nu - \bar{\nu}) \right]$$

where $\mathcal{P}$ is defined in the proof of Theorem 2 $\eta = ([z - x^*]^T, (y - 1_N \otimes z)^T, (\nu - \bar{\nu})^T)^T$ and $K = diag\{\bar{K}_i\}$ as the Lyapunov candidate function.

$$\dot{V} \leq - \theta_1 (z - x^*)^T \left[ \frac{\partial f_i(z)}{\partial x_i} \right]_{vec}$$

$$- \lambda_{min}(Q) \theta_1 ||y - 1_N \otimes z||^2$$

$$- ||\nu - \bar{\nu}||^2$$

$$- \theta_1 (z - x^*)^T \left[ \frac{\partial f_i(z)}{\partial x_i} - \frac{\partial f_i(y_i)}{\partial x_i} \right]_{vec}$$

$$- 2 (y - 1_N \otimes z)^T \mathcal{P} \bar{1}_N \otimes \bar{z} - (\nu - \bar{\nu})^T \bar{z}.$$ \tag{34}

By Assumption 3

$$- (z - x^*)^T \left[ \frac{\partial f_i(z)}{\partial x_i} \right]_{vec} \leq -m ||z - x^*||^2. \tag{35}$$

Moreover, by Assumption 1 there exist positive constants $\tilde{l}_{i1}, \tilde{l}_{i2}$ such that

$$||\frac{\partial f_i(z)}{\partial x_i} - \frac{\partial f_i(y_i)}{\partial x_i}|| \leq \tilde{l}_{i1} ||y - 1_N \otimes z||, \tag{36}$$
and
\[
\left\| \frac{\partial f_i}{\partial x_i}(y_i) \right\| = \left\| \frac{\partial f_i}{\partial x_i}(y_i) - \frac{\partial f_i}{\partial x_i}(z) + \frac{\partial f_i}{\partial x_i}(z) - \frac{\partial f_i}{\partial x_i}(x^*) \right\| \\
\leq l_2 \|y_i - z\| + l_2 \|z - x^*\|.
\]
(37)

In addition,
\[
\dot{z} = -\theta^2 K \tilde{H}(y) \tilde{\Theta}(\mathcal{L} \otimes I_{N \times N} + A)(y - 1_N \otimes z),
\]
(38)

where \( \tilde{H}(y) = \begin{bmatrix} \tilde{h}_{11} & \tilde{h}_{12} & \cdots & \tilde{h}_{1N} \\
\tilde{h}_{21} & \tilde{h}_{22} & \cdots & \tilde{h}_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{h}_{N1} & \tilde{h}_{N2} & \cdots & \tilde{h}_{NN} \end{bmatrix} \) and \( \tilde{h}_{ij} \in \mathbb{R}^{1 \times N} \). Moreover, \( \tilde{h}_{ij} = \left[ \frac{\partial^2 f_i}{\partial x_i \partial x_j}(y_i), \frac{\partial^2 f_i}{\partial x_i \partial x_j}(y_i), \cdots, \frac{\partial^2 f_i}{\partial x_i \partial x_j}(y_i) \right] \). By further noticing that \( \tilde{H}(y) \) is bounded according to Assumption 4, there exist positive constants \( l_1, l_2, l_3 \) and \( l_4 \) such that
\[
\dot{V} \leq -\theta_1 \eta_1 \| z - x^* \|^2 - \lambda_{\min}(Q) \theta_1 \| y - 1_N \otimes z \|^2 - \| \nu - \dot{z} \|^2 + \theta_1 l_2 \| y - 1_N \otimes z \|^2 + \theta_1 \| z - x^* \|^2 \| y - 1_N \otimes z \| + \theta^2_1 l_4 \| \nu - \dot{z} \| \| y - 1_N \otimes z \|. 
\]
(39)

Define \( A_1 = \begin{bmatrix} \frac{m}{4} \lambda_{\min}(Q) \theta_1 - l_2 \\
\frac{m}{4} \lambda_{\min}(Q) \theta_1 - l_2 \end{bmatrix} \), and choose \( \theta > \frac{1}{\lambda_{\min}(Q)} \), then,
\[
\dot{V} \leq -\theta_1 \lambda_{\min}(A_1) \| E_1 \|^2 - \| \nu - \dot{z} \|^2 + \theta^2_1 l_4 \| \nu - \dot{z} \| \| y - 1_N \otimes z \|,
\]
(40)

where \( \lambda_{\min}(A_1) > 0 \) and \( E_1 = [(z - x^*)^T, (y - 1_N \otimes z)^T]^T \).

Moreover, define \( A_2 = \begin{bmatrix} \frac{1}{2} \lambda_{\min}(A_2) - \frac{\theta_1 l_4}{\theta^2_1} & 0 \\
0 & \theta^2_1 l_4 \end{bmatrix} \). Then,
\[
\lambda_{\min}(A_2) > 0 \text{ given that } \theta_1 < \left( \frac{\lambda_{\min}(A_2)}{\theta^2_1 l_4} \right)^{-\frac{1}{2}}. \text{ If this is the case,}
\]
\[
\dot{V} \leq -\lambda_{\min}(A_2) \| E \|^2,
\]
(41)

where \( E = [(z - x^*)^T, (y - 1_N \otimes z)^T, (\nu - \dot{z})^T]^T \). If this is the case, we have \( z = x^*, y = 1_N \otimes z, \nu = \dot{z} \) at \( V = 0 \), which indicates that
\[
\dot{x} = \nu, \dot{\nu} = -(x - z), \dot{z} = 0_N, \dot{\dot{y}} = 0_N^2.
\]
(42)

Recalling that \( \nu = \dot{z} \) at \( V = 0 \), we have \( \nu = 0 \). Hence, \( \dot{x} = 0 \) and \( x = C_1, z = C_2 \) at \( V = 0 \), where \( C_1, C_2 \) are constant vectors. Therefore,
\[
\dot{\nu} = -C_1 + C_2
\]
(43)

at \( \dot{V} = 0 \). Recalling that \( \nu = 0 \), we can get that \( C_1 = C_2 \), i.e., \( x = z \). Hence, \( \dot{V} \) is negative definite under the given conditions and the conclusion can be derived by utilizing the LaSalle Invariant Principle.

The strategy in [32] addressed the Nash equilibrium seeking problem for games in second-order integrator-type systems without considering the boundedness of the controls. In the upcoming section, the seeking strategy in [32] will be adapted for systems where the controls are bounded.

3) Distributed Nash equilibrium with bounded control inputs: Let the Nash equilibrium seeking strategy be
\[
\begin{align*}
\dot{x} &= \nu \\
\dot{\nu} &= -\rho G ((x - z) + (\nu - \dot{z})) \\
\dot{z} &= -K \left( \frac{\partial f_i}{\partial x_i}(y_i) \right)_{vec} \\
\dot{\nu} &= -\theta(\mathcal{L} \otimes I_{N \times N} + A)(y - 1_N \otimes z).
\end{align*}
\]
(44)

Then, the following result can be derived.

**Theorem 5:** Suppose that Assumptions [13] are satisfied. Then, for any positive constant \( \Delta \), there exists a positive constant \( \theta^* \) such that for each \( \theta \in (\theta^*, \infty) \), there exists a positive constant \( \theta^*_1(\Delta, \theta) \) such that for each \( \theta_1 \in (0, \theta^*_1) \), \( x \) generated by (44) converges asymptotically to \( \mathbf{x}^* \) given that \( (\| (\nu(t) - \dot{z}(0)) \|^2, (x(0) - z(0))^T, (y(0) - 1_N \otimes z(0))^T, (z(0) - x^*)^T) \| \leq \Delta \).

**Proof:** Define the Lyapunov candidate function as
\[
V(\eta) = \frac{1}{2} (z - x^*)^T K^{-1} (z - x^*) + \rho \eta_1 y_1^T \mathcal{P}(y - 1_N \otimes z) \rho_1^T \mathcal{P}(y - 1_N \otimes z) + \sum_{i=1}^{N} \int_{t_i}^{t_{i+1}} \rho G(t) dt + \sum_{i=1}^{N} \int_{t_i}^{t_{i+1}} \rho G(t) dt + (\nu - \dot{z})^T (\nu - \dot{z}),
\]
(45)

where \( \mathcal{P} \) is defined in the proof of Theorem 1 and \( \eta = [(z - x^*)^T, (y - 1_N \otimes z)^T, (\nu - \dot{z})^T]^T \). Then, it can be easily derived that the Lyapunov candidate function is positive definite and radially unbounded. Moreover, following the analysis in the proof of Theorem 4, it can be derived that there are positive constants \( l_1, l_2 \) such that
\[
\dot{V} \leq -\theta_1 \eta_1 \| z - x^* \|^2 - \lambda_{\min}(Q) \theta_1 \| y - 1_N \otimes z \|^2 + \theta_1 l_1 \| z - x^* \|^2 \| y - 1_N \otimes z \| + \rho G (x - z + \nu - \dot{z})^T (\nu - \dot{z}) - \rho G (x - z + \nu - \dot{z})^T (\nu - \dot{z}) - \rho G (x - z + \nu - \dot{z})^T (\nu - \dot{z} - \dot{z}).
\]
(46)

Since
\[
- (\nu - \dot{z})^T (\rho G (x - z + \nu - \dot{z}) - \rho G (x - z)) \leq 0,
\]
we have
\[
\dot{V} \leq -\theta_1 \eta_1 \| z - x^* \|^2 - \lambda_{\min}(Q) \theta_1 \| y - 1_N \otimes z \|^2 + \theta_1 l_1 \| z - x^* \|^2 \| y - 1_N \otimes z \| + \theta_1 l_2 \| y - 1_N \otimes z \|^2 - \rho G (x - z + \nu - \dot{z})^T (\nu - \dot{z}) - \rho G (x - z + \nu - \dot{z})^T (\nu - \dot{z} - \dot{z}).
\]
(47)
By further following the proof of Theorem 3, we can conclude that by choosing \( \theta > \frac{4m\lambda_{\min}(Q)}{\sum_{k=1}^{m}l_k^2} \), there exist positive constants \( l_4, l_5 \) such that
\[
\dot{V} \leq -\theta_1\lambda_{\min}(A_1)||E_1||^2
+ \rho_{\theta}(x-z+\nu-\dot{z})^T\rho_{\theta}(x-z+\nu-\dot{z})
+ l_4\theta_1^2||\nu-\dot{z}||||y-1_N \otimes z|| + l_5\theta_1^2||\nu-\dot{z}||||y-1_N \otimes z||
\]
(49)
where \( A_1 = \left[ \begin{array}{cc}
m & -\frac{1}{m} \\
\frac{1}{m} & \lambda_{\min}(Q)\theta - l_2 \end{array} \right] \), \( E_1 = [(z-x^*)^T, (y-1_N \otimes z)^T]^T \) and \( l_4, l_5 \) are positive constants such that
\[
||\rho_{\theta}(x-z+\nu-\dot{z})|| \leq l_4\theta_1^2||\rho_{\theta}(x-z+\nu-\dot{z})||||y-1_N \otimes z||, \tag{50}
\]
and
\[
||\nu-\dot{z}|| \leq l_5\theta_1^2||\nu-\dot{z}||||y-1_N \otimes z||. \tag{51}
\]

To facilitate the subsequent analysis, define
\[
W(\eta) = \lambda_{\min}(A_1)||E_1||^2 + \rho_{\theta}(x-z+\nu-\dot{z})^T\rho_{\theta}(x-z+\nu-\dot{z}). \tag{52}
\]
Then, \( W(\eta) \geq 0 \). Moreover, \( W(\eta) = 0 \) if and only if \( ||E_1|| = 0 \) and \( ||\rho_{\theta}(x-z+\nu-\dot{z})|| = 0 \). By further noticing that (44) is satisfied, we have
\[
\dot{\nu} = 0 \\
\dot{y} = 0 \\
\dot{z} = 0, \tag{53}
\]
at \( W(\eta) = 0 \). If this is the case, \( \nu = C_1 \), where \( C_1 \) is a positive constant vector. Moreover, for \( W(\eta) = 0 \),
\[
x = x^* - C_1 \\
x = C_1, \tag{54}
\]
by which we can derive that \( C_1 = 0 \) and \( x = x^* \). Hence, \( W(\eta) = 0 \) if and only if \( ||\eta|| = 0 \). Therefore, \( W(\eta) \) is positive definite and there exists a class \( \mathcal{K} \) function \( \gamma \) such that
\[
\gamma(||\eta||) \leq W(\eta). \tag{55}
\]
Hence,
\[
\dot{V} \leq -\theta_1\gamma(||\eta||) + l_4\theta_1^2||\nu-\dot{z}||||y-1_N \otimes z|| + l_5\theta_1^2||\nu-\dot{z}||||y-1_N \otimes z||, \tag{56}
\]
given that \( \theta_1 < 1 \).

Therefore, for \( \eta \) that belongs to any compact set that contains the origin, there exists a positive constant \( l_6 \) such that
\[
\dot{V} \leq -\theta_1\gamma(||\eta||) + \theta_1^2l_6, \tag{57}
\]
which indicates that
\[
\dot{V} \leq -\theta_1\gamma(||\eta||), \forall ||\eta|| \geq \gamma^{-1}(2\theta_1 l_6). \tag{58}
\]
Recalling that \( V \) is positive definite, there exist \( \gamma_1, \gamma_2 \in \mathcal{K} \) such that
\[
\gamma_1(||\eta||) \leq V(\eta) \leq \gamma_2(||\eta||). \tag{59}
\]
Therefore, there exists a positive constant \( T_1 \) such that \( ||\eta|| \leq \gamma_1^{-1}(\gamma_2^{-1}(2\theta_1 l_6)) \) for all \( t \geq T_1 \). Choosing \( \theta_1 \) to be sufficiently small such that \( \gamma_2^{-1}(\gamma_2^{-1}(2\theta_1 l_6)) < \bar{U} \). Then, for \( t \geq T_1 \), the system in (44) is reduced to the system in (52). Following the result in Theorem 3, the conclusion can be derived.

Remark 6: The theoretical results presented in this section are established for \( x_i \in \mathbb{R} \). However, they can be easily extended to the case in which \( x_i \in \mathbb{R}^p \) and \( p \geq 2 \) is a positive integer.

IV. SIMULATION STUDIES: CONNECTIVITY CONTROL OF MOBILE SENSOR NETWORKS

This section verifies the effectiveness of the proposed seeking strategies in a mobile sensor network in which \( x_i \in \mathbb{R}^2 \) (denoted as \( x_{1i} \) and \( x_{2i} \), respectively). To highlight the applications of the proposed methods, we consider the connectivity control for a network of 3 mobile sensors in which the sensors’ objective functions are given by [19]
\[
f_i(x_i, x_{-i}) = x_i^T r_{ii} x_i + x_i^T p_i + q_i + \sum_{j \in N_i} m_{ij} ||x_i - x_j||^2, \tag{60}
\]
where \( r_{ii} \in \mathbb{R}^{2 \times 2}, p_i \in \mathbb{R}^{2 \times 1}, q_i \in \mathbb{R}, m_{ij} \in \mathbb{R} \) are constant matrices, vectors or parameters and \( N_i \) denotes the neighboring set of player \( i \). In the subsequent simulations, we consider Example 1 of [19] in which \( i = 3, r_{ii} \) for \( i \in \{1, 2, 3\} \) are identity matrices, and \( m_{ij} \) except \( m_{13} = m_{31} = 0 \). Moreover, \( p_1 = [2, -2], p_2 = [-2, 2], p_3 = [-4, 2], q_1 = 3 \) for \( i \in \{1, 2\} \) and \( q_3 = 6 \). In addition, the game admits a unique Nash equilibrium at \( x^* = [-0.125, 0.75, 0.75, 0.5, 1.375, -0.25]^T \) [19].

In the following, velocity-actuated vehicles and force-actuated vehicles will be simulated, successively.

A. Velocity-actuated vehicles

In this section, we consider velocity-actuated vehicles, whose dynamics can be described as
\[
\dot{x}_i = u_i, \tag{61}
\]
where \( x_i = [x_{1i}, x_{2i}]^T \) denotes the position of sensor \( i, u_i = [u_{i1}, u_{i2}]^T \in \mathbb{R}^2, u_{ij} \) for \( i \in \{1, 2, 3\}, j \in \{1, 2\} \) denotes the velocity of sensor \( i \) and satisfies \( |u_{ij}| \leq U \). The saturated gradient play given in (10) and the distributed method shown in (14) will be verified, successively.

1) Saturated gradient play: In this section, we suppose that the mobile sensors can communicate with each other via the communication graph depicted in Fig. 1(a).

```
1 2 3
|   |   |
\|---\|---\|
\  3  1  2
```

Fig. 1: The communication graph among the sensors.
Moreover, we suppose that the sensors update their actions according to the saturated gradient play given in (10). With \( x(0) = [10, 0, 0, 5, 0, 0]^T \) and \( \bar{U} = 5 \), the trajectories of the sensors’ positions generated by the saturated gradient play are depicted in Fig. 2. The figure shows that the sensors’ positions would converge to the Nash equilibrium of the game asymptotically. Moreover, the control inputs are illustrated in Fig. 3 from which we can see that the control inputs are bounded by the given value. Hence, by the presented results, Theorem 1 is numerically verified.

![Fig. 2: The trajectories of the sensors’ positions generated by the saturated gradient play in (10).](image1)

![Fig. 3: The control inputs generated by the saturated gradient play in (10).](image2)

Let \( x(0) = [10, 0, 0, 5, 0, 0]^T \), \( \bar{U} = 5 \), and \( y_{ij}(0) = 10 \). Driven by the method in (14), the trajectories of the sensors’ positions are plotted in Fig. 4 and the control inputs are illustrated in Fig. 5. Fig. 4 demonstrates that the trajectories of the sensors’ positions would converge to the Nash equilibrium. Moreover, the control inputs stay in the bounded region as shown in Fig. 5. The simulation results verify the theoretical result in Theorem 2.

![Fig. 4: The trajectories of the players’ actions generated by the saturated gradient play in (14).](image3)

![Fig. 5: The control inputs generated by the saturated gradient play in (14).](image4)

**B. Force-actuated vehicles**

In this section, we suppose that the agents are force-actuated vehicles whose dynamics can be described by

\[
\dot{x}_i = \nu_i \quad \nu_i = u_i,
\]

where \( x_i = [x_{i1}, x_{i2}]^T \in \mathbb{R}^2 \) is vector containing the positions of sensor \( i \), \( \nu_i = [\nu_{i1}, \nu_{i2}]^T \in \mathbb{R}^2 \) is the vector containing the velocities of sensor \( i \) and \( u_i = [u_{i1}, u_{i2}]^T \in \mathbb{R}^2 \) is the vector containing the control inputs that satisfy \( |u_{ij}| \leq \bar{U} \), for all \( i \in \{1, 2, 3\}, j \in \{1, 2\} \).
1) Centralized Nash equilibrium seeking: In this section, we suppose that the sensors adopt the proposed strategy in (26) to update their positions and all the variables in (26) are initialized at zero. By setting $\alpha = 1$, $\beta = 1$, the trajectories of the sensors’ positions are depicted in Fig. 6. From the figure, we can see that the trajectories of the sensors’ positions would converge to the Nash equilibrium of the game. Moreover, the plots of the sensors’ velocities are given in Fig. 7, which demonstrates that the sensors’ velocities converge to zero as their positions converge to the Nash equilibrium. The simulation results are in line with the theoretical results in Theorem 3.

![Fig. 6: The trajectories of the sensors’ positions generated by (26).](image)

![Fig. 7: The trajectories of the sensors’ velocities generated by (26).](image)

2) Distributed Nash equilibrium without input saturation: In this section, we suppose that the sensors update their actions according to (32). Moreover, $z(0) = [-10, 0, 0, 5, 0, 0]^T$ and all the other variables are initialized at zero. Under the communication graph depicted in Fig. 1 (b), the simulation results generated by the method in (32) are shown in Figs. 8-9. Fig. 8 depicts the trajectories of the sensors’ positions and Fig. 9 shows the trajectories of the sensors’ velocities. From these figures, we can see that the sensors’ positions converge to the Nash equilibrium of the game, which numerically verifies the theoretical results in Theorem 4.

![Fig. 8: The trajectories of the sensors’ positions generated by (32).](image)

![Fig. 9: The trajectories of the sensors’ velocities generated by (32).](image)

3) Distributed Nash equilibrium with bounded control inputs: In this section, we suppose that the sensors update their positions according to (44), in which $U = 5$ and the other variables are the same as those in Section IV-B.2. Under the communication graph depicted in Fig. 1 (b), the simulation results are given in Figs. 10-12. Fig. 10 depicts the trajectories of the sensors’ positions, which shows that the sensors’ positions asymptotically converge to the Nash equilibrium of the game. Fig. 11 demonstrates that the velocities converge to zero as the sensors’ positions converge to the Nash equilibrium. Moreover, as plotted in Fig. 12, the control inputs are bounded by the given value. The simulation results are in line with the theoretical results in Theorem 5.

![Fig. 10: The trajectories of the sensors’ positions generated by (44).](image)

![Fig. 11: The trajectories of the sensors’ velocities generated by (44).](image)

![Fig. 12: The control inputs bounded by the given value.](image)

V. Conclusions

This paper considers Nash equilibrium seeking for games in systems where the control inputs are bounded. More
adapted to distributed networks. Based on the Lyapunov stability analysis, the convergence properties of the designed algorithms are analytically investigated. It is shown that the proposed seeking strategies would enable the players’ actions to converge to the Nash equilibrium under the given conditions. Lastly, the proposed methods are applied for the connectivity control of sensor networks.

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