Relaxation time of a CFT plasma at finite coupling

Alex Buchel\textsuperscript{1,2} and Miguel Paulos\textsuperscript{3}

\textsuperscript{1}Department of Applied Mathematics
University of Western Ontario
London, Ontario N6A 5B7, Canada

\textsuperscript{2}Perimeter Institute for Theoretical Physics
Waterloo, Ontario N2J 2W9, Canada

\textsuperscript{3}Department of Applied Mathematics and
Theoretical Physics, Cambridge CB3 0WA, U.K.

Abstract

Following recent formulation of second order relativistic viscous hydrodynamics for conformal fluids, we compute finite coupling corrections to the relaxation time of $\mathcal{N} = 4$ supersymmetric Yang-Mills plasma. The result is expected to be universal for any strongly coupled conformal gauge theory plasma in four dimensions.

June 2008
1 Introduction and Summary

To a large extent motivated and guided by the gauge theory/string theory correspondence of Maldacena [1–3], Baier et. al [4] and Bhattacharyya et. al [5] recently formulated second order relativistic viscous hydrodynamics of conformal fluids. It was found that Mueller-Israel-Stewart (MIS) theory [6, 7], often used in hydrodynamic simulations of strongly coupled quark-gluon plasma (sQCD) produced in heavy ion collisions at RHIC, does not properly account for all second order viscous corrections. The framework of this new viscous hydrodynamics was applied to bulk physics analysis at RHIC in [8].

In this paper we continue the program of computing the transport coefficients of strongly coupled gauge theories from the dual string theory holographic description. The motivation to do so, even though we lack a tractable string theory dual to QCD, is the amazing regularity of transport coefficients for a large class of gauge theory plasma at strong ’t Hooft coupling [9–16], and the fact that sQGP appears to exhibit near-conformal dynamics. Specifically, we concentrate on finite ’t Hooft coupling corrections to two second order transport coefficients of a CFT plasma — a relaxation time $\tau\Pi$ and $\kappa$. These transport coefficients are the simplest to compute as they can be extracted from the equilibrium correlation functions. While we perform the computations in the string theory dual to $\mathcal{N} = 4$ $SU(N_c)$ superconformal Yang-Mills (SYM) theory, following the arguments in [16], the results are expected to hold for any conformal gauge theory plasma in four dimensions, and should provide a reasonable estimate for sQGP.

Following [4, 5], the stress energy tensor $T^{\mu\nu}$ of a viscous conformal fluid to second order in derivatives of the four-velocity $u^{\mu}$ takes form

$$T^{\mu\nu} = \epsilon u^{\mu}u^{\nu} + P (g^{\mu\nu} + u^{\mu}u^{\nu}) + \Pi^{\mu\nu},$$

(1.1)

where $\epsilon$, $P$ are the energy density and the pressure, and the dissipative part $\Pi^{\mu\nu}$ is a sum of the first ($\Pi_1$) and the second order ($\Pi_2$) terms:

$$\Pi^{\mu\nu} = \Pi_1^{\mu\nu} (\nabla_\alpha u_\beta; \{\eta\}) + \Pi_2^{\mu\nu} (\nabla_\gamma \nabla_\alpha u_\beta; \{\eta, \tau, \kappa, \lambda_1, \lambda_2, \lambda_3\}) .$$

(1.2)

The four-velocity dependence of the dissipative part of the stress-energy tensor is uniquely fixed by conformal invariance up to a single phenomenological coefficient.

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1 The coefficient $\lambda_1$ is necessary to describe the Bjorken expansion [18] of plasma and its finite coupling corrections will be discussed elsewhere.

2 A more detailed discussion is given in [17].
to the first order in the derivative expansion, and up to five additional coefficients to the second order in the derivative expansion. In the former case, a single coefficient is a shear viscosity $\eta$, while the latter includes a relaxation time $\tau_\Pi$, three coefficients $\lambda_i$ (describing viscous terms bilinear in four-velocities), and a $\kappa$-term (describing viscous hydrodynamics in curved backgrounds). In the linear regime, and in Minkowski space-time, i.e., setting $\lambda_i = \kappa = 0$, the new viscous hydrodynamics of [4, 5] reduces to MIS theory [6, 7]. A crucial observation of [4, 5] was that the MIS regime of a strongly coupled four-dimensional conformal gauge theory plasma is simply inconsistent. In a specific example of $\mathcal{N} = 4$ SYM, and at infinite 't Hooft coupling $\lambda \equiv g^2_{YM} N_c \to \infty$, it was found that [19, 4, 5]

$$\begin{align*}
\frac{\eta}{s} &= \frac{1}{4\pi}, \\
\tau_\Pi &= \frac{2 - \ln 2}{2\pi T}, \\
\kappa &= \frac{\eta}{\pi T}, \\
\lambda_1 &= \frac{\eta}{2\pi T}, \\
\lambda_2 &= -\frac{\eta}{\pi T}, \\
\lambda_3 &= 0,
\end{align*}$$

(1.3)

where $s$ is the entropy density, and $T$ is the temperature.

The finite coupling corrections were computed only for the shear viscosity. It was found in [20–23] that

$$\frac{\eta}{s} = \frac{1}{4\pi} \left( 1 + \frac{120}{8} \zeta(3) \lambda^{-3/2} + \cdots \right).$$

(1.4)

In this paper, extending analysis of [20, 21], we find

$$\begin{align*}
\tau_\Pi T &= \frac{2 - \ln 2}{2\pi} + \frac{375}{32\pi} \zeta(3) \lambda^{-3/2} + \cdots, \\
\kappa &= \frac{\eta}{\pi T} \left( 1 - \frac{145}{8} \zeta(3) \lambda^{-3/2} + \cdots \right).
\end{align*}$$

(1.5)

(1.6)

The computations are quite technical, so we present only relevant steps and for the details refer the reader to previous work on the subject: [3, 4, 20, 21, 23]. In the next section we describe $\mathcal{O}(\alpha'^3)$ near-extremal D3 brane geometry [24, 25], primarily to set-up our notation. In section 3, following [20, 23], we compute $\{\tau_\Pi, \kappa\}$ from the retarded correlation function of the stress energy tensor. In section 4, following [21, 23], we compute the dispersion relation of a sound quasinormal mode up to third order in momentum, and confirm the value of $\tau_\Pi$ obtained in section 3.

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This is in fact universal to all four dimensional CFT gauge theories with a string theory dual.
2 Background geometry

The background black brane geometry dual to a strongly coupled $\mathcal{N} = 4$ SYM plasma at finite 't Hooft coupling was found in [24, 25]. The ten dimensional background takes the form

$$ds^2_{10} = e^{-\frac{10}{3}\nu} ds^2_5 + e^{2\nu} d\Omega^2_5,$$

$$F_5 = \mathcal{F}_5 + \star \mathcal{F}_5, \quad \mathcal{F}_5 = -4dvol_{S_5}$$

(2.1)

where $d\Omega^2_5$ is a volume element of a round five-sphere, and

$$ds^2_5 \equiv g_{\mu\nu} dx^\mu dx^\nu = \frac{r_0^2}{u} e^{c(u)} \left( -f e^{a(u)} dt^2 + dx^2 \right) + \frac{du^2}{4u^2 f} e^{b(u)}.$$  

(2.2)

Here $f(u) = 1 - u^2$, $r_0$ is the parameter of non-extremality of the black brane geometry, and we set the “AdS radius” $L$ to one. To leading order in $\frac{1}{8\zeta(3)\alpha'^3}$, functions $a, b, c, \nu$ were found in [24, 25]

$$a(u) = -15\gamma (5u^2 + 5u^4 - 3u^6),$$

$$b(u) = 15\gamma (5u^2 + 5u^4 - 19u^6),$$

$$c(u) = 0,$$

$$\nu(u) = \frac{15\gamma}{32} u^4 (1 + u^2).$$

(2.4)

The Hawking temperature corresponding to the metric (2.1) is

$$T \equiv T_0 \left( 1 + 15\gamma \right) = \frac{r_0}{\pi} \left( 1 + 15\gamma \right).$$

(2.5)

These corrections were found assuming the only relevant term at order $\gamma$ is $C^4$ [28]. In [29] the full set of $\gamma$ corrections were computed including five-form terms, and there it was found that the black D3-brane solution receives corrections only from the $C^4$ term. In [17] it was further shown that the full spectrum of quasi-normal modes in this background is also unaffected by the five-form terms, and therefore we are justified in this paper in working just with the $C^4$ correction.

The string tension $\alpha'$, or more precisely $\alpha'/L^2$, is identified with $\lambda^{-1/2}$ of the $\mathcal{N} = 4$ SYM.

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4 The string tension $\alpha'$, or more precisely $\alpha'/L^2$, is identified with $\lambda^{-1/2}$ of the $\mathcal{N} = 4$ SYM.
3 Relaxation time from the Kubo formula

To obtain retarded correlation function of the boundary stress energy tensor, we study scalar perturbations of the background geometry (2.2) (see [3, 4]):

\[ g_{\mu\nu} \rightarrow g_{\mu\nu} + h_{xy}(u, x). \] (3.1)

It will be convenient to introduce a field \( \varphi(u, x) \),

\[ \varphi(u, x) = \frac{u}{r^2_0} h_{xy}(u, x), \] (3.2)

and use the Fourier decomposition

\[ \varphi(u, x) = \int \frac{d^4k}{(2\pi)^4} e^{-i\omega t + i k \cdot x} \varphi_k(u). \] (3.3)

Finally, we introduce

\[ m \equiv \frac{\omega}{2\pi T_0}, \quad k \equiv \frac{k}{2\pi T_0}. \] (3.4)

3.1 The effective action

The effective action to order \( \mathcal{O}(\gamma) \) for \( \varphi_k(u) \) takes form [20]:

\[ S_{\text{eff}} = \frac{N_c^2}{8\pi^2} \int \frac{d^4k}{(2\pi)^4} \int_0^1 du \left[ A \varphi_k'' \varphi_{-k} + B \varphi_k' \varphi_{-k}' + C \varphi_k' \varphi_{-k} \right. \]
\[ \left. + D \varphi_k \varphi_{-k} + E \varphi_k'' \varphi_{-k}' + F \varphi_k' \varphi_{-k}' \right]. \] (3.5)

The coefficients \( A, B, C, D, E, F \) are even functions of the momentum. They are given explicitly in appendix A.

Variation of \( S_{\text{eff}} \) leads to

\[ \delta S_{\text{eff}} = \frac{N_c^2}{8\pi^2} \int \frac{d^4k}{(2\pi)^4} \left[ \int_0^1 du (EOM) \delta \varphi_{-k} \right. \]
\[ \left. + \left( B_1 \delta \varphi_{-k} + B_2 \delta \varphi_{-k}' \right) \right]_0^1, \] (3.6)

where the coefficients of the boundary term are given by

\[ B_1 = -(A \varphi_k)' + 2B \varphi_k' + C \varphi_k - 2(E \varphi_k'')' + F \varphi_k'' - (F \varphi_k')', \] (3.7)
\[ B_2 = A \varphi_k + F \varphi_k' + 2E \varphi_k''. \] (3.8)
and EOM denotes the left hand side of the Euler-Lagrange equation

\[ A\phi''_k + C\phi'_k + 2D\phi_k - \frac{d}{du} (2B\phi'_k + C\phi + F\phi''_k) + \frac{d^2}{du^2} (A\phi''_k + 2E\phi''_k + F\phi'_k) = 0. \quad (3.9) \]

In order to have a well-defined variational principle, one has to add a generalized Gibbons-Hawking boundary term to the action \((3.5)\). As explained in \([20]\), this should be done perturbatively in \(\gamma\). Specifically, if we rewrite \((3.9)\) in the form

\[ \phi''_k + p_1\phi'_k + p_0\phi_k = O(\gamma), \quad (3.10) \]

with all \(\gamma\)-dependent terms exiled to the right, the generalized Gibbons-Hawking term \(K_{\text{gen}}\) rendering the variation of \((3.6)\), takes form

\[ K_{\text{gen}} = -A\phi''_k \phi'_k - \frac{F}{2} \phi'_k \phi'_k + E p_1 \phi'_k \phi'_k + 2E p_0 \phi'\phi' + 2 \phi''\phi'. \quad (3.11) \]

Notice that \(K_{\text{gen}}\) differs from the standard (supergravity) Gibbons-Hawking term \(K_{\text{std}}\):

\[ K_{\text{std}} = -A\phi''_k \phi'_k + K_1 \phi\phi' + K_2 \phi^2, \quad (3.12) \]

where the coefficients \(K_1 \propto O(\gamma)\) and \(K_2 \propto O(\gamma^0)\) are given in appendix A.

The bulk action \((3.5)\) can be rewritten in the form

\[ S_{\text{eff}} = \frac{N^2}{8\pi^2} \int \frac{d^4k}{(2\pi)^4} \int_0^1 du \left( \partial_\mu B + \frac{1}{2} [EOM] \right), \quad (3.13) \]

where

\[ B = -\frac{A'}{2} \phi''_k \phi'_k + B \phi'_k \phi''_k + C \phi''_k \phi' - E \phi''_k \phi' - E \phi'_k \phi''_k - E \phi''_k \phi' - E \phi''_k \phi' - E \phi''_k \phi' - E \phi''_k \phi' - E \phi''_k \phi' - E \phi''_k \phi' - E \phi''_k \phi' - E \phi''_k \phi'. \quad (3.14) \]

Thus on-shell, it reduces to the sum of two boundary term: the horizon contribution \((\text{as } u \to 1)\) and the boundary contribution \((\text{as } u \to 0)\). In computing the two-point retarded correlation function of the boundary stress-energy tensor, the horizon contribution must be discarded \([26]\); the boundary contribution is divergent as \(u = \epsilon \to 0\) and must be supplemented by the counterterm action \([27]\):

\[ S_{\text{ct}} = -\frac{3N^2}{4\pi^2} \int_{u=\epsilon} d^4x \sqrt{-\gamma} \left( 1 + \frac{1}{2} P - \frac{1}{12} (P_{kl} P_{kl} - P^2) \ln \epsilon \right), \quad (3.15) \]

where \(\gamma_{ij}\) is the metric induced at the \(u = \epsilon\) boundary, and

\[ P = \gamma^{ij} P_{ij}, \quad P_{ij} = \frac{1}{2} \left( R_{ij} - \frac{1}{6} R \gamma_{ij} \right). \quad (3.16) \]
We can further rewrite (3.15) as
\[ S_{ct} = -\frac{N_c^2}{8\pi^2} \int \frac{d^4 k}{(2\pi)^4} \left( T + \mathcal{O}(w^4, w^2 k^2, k^4) \right) \varphi_k \varphi_{-k}, \] (3.17)
where \( T \) is given in appendix A.

We would like to emphasize that the counterterm action (3.15) is computed assuming the standard Gibbons-Hawking term (3.12). In particular, it takes into account the divergent as \( u = \epsilon \rightarrow 0 \) boundary contribution coming from (the supergravity part of) \( K_2 \) term in (3.12). Thus, even though at the level of the effective action (3.5) the variational principle is well defined just with \( K_{gen} \), in order to obtain finite retarded correlation functions with the counterterm action (3.15), we need to supplement \( K_{gen} \) with the supergravity part of \( K_2 \varphi_k \varphi_{-k} \):
\[ K_{gen} \rightarrow K_{gen} + K_2 \varphi_k \varphi_{-k}. \] (3.18)

Altogether, the total renormalized boundary action takes the form
\[ S_{tot} (\epsilon) = -\frac{N_c^2}{8\pi^2} \int \frac{d^4 k}{(2\pi)^4} \left. F_k \right|_{u=\epsilon}, \] (3.19)
where
\[ F_k = \frac{N_c^2}{8\pi^2} \left[ (B - A) \varphi_k \varphi_{-k} + \frac{1}{2} (C - A' + K_2 + T) \varphi_k \varphi_{-k} - E' \varphi''_k \varphi_{-k} + E \varphi''_k \varphi'_{-k} \right. \]
\[ - E' \varphi''_k \varphi_{-k} - \frac{F'}{2} \varphi_k \varphi_{-k} + E p_1 \varphi'_k \varphi'_{-k} + 2 E p_0 \varphi'_k \varphi'_{-k} \]. (3.20)

\(^5\)Note that the conformal anomaly term in (3.15) does not contribute to the retarded correlation functions at order \( \mathcal{O}(m^2, k^2) \).

\(^6\)As observed in [20], the \( \mathcal{O}(\gamma) \) parts of the coefficients in (3.11), (3.12) (and also (3.17)) will not contribute in the limit \( u = \epsilon \rightarrow 0 \).
3.2 The solution for $\varphi_k$

We now turn to finding the solution to the equation of motion (3.9), or rather its equivalent to order $O(\gamma)$, (3.10):

$$\varphi''_k - \frac{u^2 + 1}{uf} \varphi'_k + \frac{k^2 u^2 + w^2 - k^2}{uf^2} \varphi_k = -\frac{1}{4} \gamma \left[ 3171 u^4 + 3840 k^2 u^3 + 2306 u^2 
- 600 \right] u \varphi'_k + \frac{u}{f^2} \left( 600 w^2 - 300 k^2 + 50 u + (3456 k^2 - 2856 w^2) u^2 + 768 u^3 k^4 \right) \varphi_k \right).$$

(3.21)

The incoming wave boundary condition for $\varphi_k$ at the horizon implies [23]

$$\varphi_k \propto (1 - u^2)^{-\frac{im(1 - 15\gamma)}{2}}, \quad u \to 1_-.$$  

(3.22)

Thus we represent the solution to (3.21) perturbatively in both $\gamma$ and $(w, k^2)$ as

$$\varphi_k = (1 - u^2)^{-\frac{im(1 - 15\gamma)}{2}} \left( G_{k}^{(0)}(u) \right) + \gamma G_{k}^{(1)}(u) + O(w^3, w k^2) \right), \quad (3.23)$$

where

$$G_{k}^{(i)}(u) \equiv G_{k}^{(i)}(w, q^2; u) = \lambda^0 G_{k}^{(i)}(\lambda w, \lambda^2 k^2; u), \quad (3.24)$$

and

$$\lim_{u \to 1_-} G_{k}^{(0)}(u) = 1, \quad \lim_{u \to 1_-} G_{k}^{(1)}(u) = \lim_{u \to 1_-} G_{k}^{(2)}(u) = 0. \quad (3.25)$$

Explicitly, we find:

$$G_{k}^{(0,0)} = 1, \quad G_{k}^{(0,1)} = -\frac{25}{16} u^6 - \frac{25}{16} u^4 + \frac{25}{8}, \quad (3.26)$$

$$G_{k}^{(1,0)} = 0, \quad G_{k}^{(1,1)} = \frac{43}{2} u^6 + \frac{195}{2} u^2 + \frac{135}{2} u^4 - \frac{373}{2}. \quad (3.27)$$
\[ G_k^{(2,0)} = \frac{1}{4} \ln^2 \left( \frac{u}{2} + \frac{1}{2} \right) + \ln \left( \frac{u}{2} + \frac{1}{2} \right) - \frac{1}{2} \text{dilog} \left( \frac{u}{2} + \frac{1}{2} \right) - \frac{k^2}{\bar{w}^2} \ln \left( \frac{u}{2} + \frac{1}{2} \right), \]

\[ G_k^{(2,1)} = \left( \frac{25}{32} u^4 + \frac{25}{32} u^6 + \frac{215}{16} \right) \left( \text{dilog} \left( \frac{u}{2} + \frac{1}{2} \right) - \frac{1}{2} \ln^2 (1 + u) \right) + \left( \frac{215}{16} \ln 2 + \frac{2445}{8} \right) \]

\[ - \frac{195}{2} u^2 + \left( \frac{25}{32} \ln 2 - \frac{1105}{16} \right) u^4 + \left( - \frac{369}{16} + \frac{25}{32} \ln 2 \right) u^6 \ln (1 + u) \]

\[ - \frac{2445}{8} \ln 2 + \frac{2885}{6} - \frac{215}{32} \ln^2 2 - \frac{605}{2} u + \left( \frac{195}{2} \ln 2 - \frac{195}{2} \right) u^2 - \frac{40}{3} u^3 \]

\[ + \left( - \frac{135}{2} - \frac{25}{64} \ln^2 2 + \frac{1105}{16} \ln 2 \right) u^4 + \frac{43}{2} u^5 \]

\[ + \left( - \frac{43}{2} - \frac{25}{64} \ln^2 2 + \frac{369}{16} \ln 2 \right) u^6 + \frac{k^2}{\bar{w}^2} \left( \frac{1375}{8} + \frac{25}{16} u^6 + \frac{25}{16} u^4 \right) \ln (1 + u) \]

\[ + \frac{821}{6} - \frac{1375}{8} \ln 2 - 175 u + \frac{195}{2} u^2 - \frac{250}{3} u^3 + \left( \frac{135}{2} - \frac{25}{16} \ln 2 \right) u^4 - 65 u^5 \]

\[ + \left( \frac{43}{2} - \frac{25}{16} \ln 2 \right) u^6 \right). \]

(3.28)

### 3.3 Coupling constant correction to relaxation time

Having found the solution for a gravitational perturbation, we can compute the correlation function \( G_{xy,xy}(\omega, q) \) by applying the Minkowski AdS/CFT prescription \([26]\)

\[ G^R_{xy,xy}(\omega, q) = \lim_{u \to 0} \frac{2 \Phi_q}{|q|^2}. \]  

(3.29)

Explicitly we find

\[ G^R_{xy,xy}(\omega, q) = \frac{\pi^2 N_c^2 T^4 (1 + 15 \gamma)}{4} \left( \frac{1}{2} - i \hat{w} \left[ 1 + 120 \gamma \right] + \left[ - \hat{q}^2 + \hat{w}^2 - \hat{w}^2 \ln 2 \right] \right. \]

\[ + \gamma \left( -120 \hat{w}^2 \ln 2 + 25 q^2 + \frac{905}{2} \hat{w}^2 \right) \right) + O(\hat{w}^3, \hat{q}^2) + O(\gamma^2), \]

(3.30)

where we used (2.5) to reintroduce the temperature, and denoted

\[ \hat{\omega} \equiv \frac{\omega}{2\pi T}, \quad \hat{q} \equiv \frac{q}{2\pi T}. \]

(3.31)

In the hydrodynamic limit the retarded correlation function \( G^R_{xy,xy}(\omega, q) \) takes form \([4]\]

\[ G^R_{xy,xy}(\omega, q) = P - i \eta \omega + \eta \gamma \omega^2 - \frac{k}{2} (\omega^2 + q^2) + O(\omega^3, \omega q^2). \]

(3.32)
Comparing (3.30) and (3.32) we conclude

\[
P = \frac{\pi^2 N_c T^4}{8} \left( 1 + 15\gamma + \mathcal{O}(\gamma^2) \right), \quad \frac{\eta}{s} = \frac{1}{4\pi} \left( 1 + 120\gamma + \mathcal{O}(\gamma^2) \right),
\]

\[
\tau T = \frac{2 - \ln 2}{2\pi} + \frac{375}{4\pi} \gamma + \mathcal{O}(\gamma^2), \quad \kappa = \frac{\eta}{\pi T} \left( 1 - 145\gamma + \mathcal{O}(\gamma^2) \right).
\]

(3.33)

Eq. (3.33) is our main result.

4 Relaxation time from the sound pole

The equation of motion for the sound quasinormal mode has been obtained in [21]. We represent it here in an equivalent form, to order \( \mathcal{O}(\gamma) \):

\[
0 = Z''_{\text{sound}} + \frac{3w^2 + (3x^2 - 2)q^2}{x(3w^2 - (x^2 + 2)q^2)} Z'_{\text{sound}}
\]

\[
+ \frac{3w^4 - (4x^2(1 - x^2)^{3/2} + 2w^2(2x^2 + 1))q^2 + x^2(x^2 + 2)q^4}{(3w^2 - (x^2 + 2)q^2)x^2(1 - x^2)^{3/2}} Z_{\text{sound}}
\]

\[
+ J_{\text{sound}}[Z_{\text{sound}}] + \mathcal{O}(\gamma),
\]

(4.1)

where \( Z_{\text{sound}} = Z_{\text{sound}}(x) \), \( x = \sqrt{1 - u^2} \) and \( (w, q) \) are the dimensionless momenta reduced with respect to \( 2\pi T_0 \), (3.4). The order \( \mathcal{O}(\gamma) \) the source term \( J_{\text{sound}} \) is given by

\[
J_{\text{sound}} = -\frac{\gamma}{4x^2 \left( 3w^2 - (x^2 + 2)q^2 \right)^{3}} \left[ Z'_{\text{sound}} x^3 J_{\text{sound},0} - Z_{\text{sound}} J_{\text{sound},1} \right],
\]

(4.2)

\[
J_{\text{sound},0} = 256q^2(1 - x^2)^2(3w^2 - (x^2 + 2)q^2) \left( 15w^4 - 30w^2(5x^2 + 2)q^2 
\right.
\]

\[
+ (116x^2 + 44 + 35x^4)q^4 \bigg) + \left( 27w^6(3171x^4 + 4877 - 8648x^2) 
\right.
\]

\[
+ 27w^4(20291x^2 - 12394 - 9790x^4 + 3293x^6)q^2 - 9w^2(-24153x^4 + 54172x^2 
\right.
\]

\[
- 31188 + 51661x^8 - 47492x^6)q^4 + (322145x^6 + 220694x^8 - 77416 
\right.
\]

\[
+ 89709x^{10} - 528064x^4 + 147364x^2)q^6 \bigg) (1 - x^2)^{1/2},
\]

(4.3)
The sound quasinormal mode and its spectrum to the leading and the first sub-leading order in the hydrodynamic approximation was found in \([21, 23]\):

\[
J_{\text{sound},1} = 648w^8(-5 - 59x^2 + 89x^4) - 36w^6(-3989x^4 - 180 + 7003x^6 - 1259x^2)q^2 \\
+ 12w^4q^4(-40299x^4 + 6242x^2 - 360 + 159x^6 + 40333x^8) - 12w^2q^6(37927x^8 \\
- 80 + 8900x^2 - 30499x^6 + 17099x^4 - 29972x^4) + 4x^2(x^2 + 2)(5811x^8 \\
+ 18043x^6 - 14991x^4 - 12192x^2 + 4004)q^8 + \left( -675x^2w^6(9x^4 - 16x^2 + 5) \\
+ 9x^2w^4q^2(8190 + 87357x^6 + 12989x^2 - 111386x^4) + q^4((-79980x^8 \\
- 820917x^{10} + 1316595x^6 - 284748x^4 - 83700x^2)w^2 + 20736x^2(1 - x^2)w^6) \\
\right. \\
\left. + q^6(x^2(131316x^2 + 134283x^{10} + 24040 + 410574x^8 - 430838x^4 - 292777x^6) \\
+ (20736x^6 + 20736x^6 - 41472x^2)w^4) + (27648w^2x^2 - 20736x^6w^2 \\
- 6912x^8w^2)q^8 - 768q^{10}x^2(1 - x^2)(x^2 + 2)^3 \right) (1 - x^2)^{1/2}.
\]

(4.4)

The incoming wave boundary condition at the horizon implies the following perturbative expansion in the hydrodynamic limit [23]

\[
\begin{align*}
Z_{\text{sound}} &= x^{-i m(1 - 15\gamma)} \left( z_{\text{sound}}^{(0)} + i q z_{\text{sound}}^{(1)} + q^2 z_{\text{sound}}^{(2)} + O(w^3, wq^2) \right), \\
z_{\text{sound}}^{(i)} &= z_{\text{sound},0}^{(i)} + \gamma z_{\text{sound},1}^{(i)} + O(\gamma^2),
\end{align*}
\]  
(4.5)

where

\[
z_{\text{sound}}^{(i)}(x) \equiv z_{\text{sound}}^{(i)}(w, q^2; x) = \lambda^0 z_{\text{sound}}^{(i)}(\lambda w, \lambda^2 q^2; x),
\]

and

\[
\lim_{x \to 0^+} z_{\text{sound}}^{(0)}(x) = 1, \quad \lim_{x \to 0^+} z_{\text{sound}}^{(1)}(x) = \lim_{x \to 0^+} z_{\text{sound}}^{(2)}(x) = 0.
\]

(4.7)

The dispersion relation for the sound quasinormal modes is obtained by imposing a Dirichlet condition on \(Z_{\text{sound}}\) at the boundary:

\[
\lim_{x \to 1^-} Z_{\text{sound}}(x) = 0 \iff w \equiv w(q).
\]

(4.8)

4.1 Sound quasinormal spectrum to \(O(q^2)\)

The sound quasinormal mode and its spectrum to the leading and the first sub-leading order in the hydrodynamic approximation was found in \([21, 23]\):

\[
\begin{align*}
z_{\text{sound},0}^{(0)} &= \frac{3w^2 + (x^2 - 2)q^2}{3w^2 - 2q^2}, \\
z_{\text{sound},0}^{(1)} &= \frac{2wqx^2}{3w^2 - 2q^2}.
\end{align*}
\]

(4.9)
\[ z_{\text{sound,1}}^{(0)} = \frac{5x^2}{16(3\omega^2 - 2q^2)^2} \left( q^4 \left( 2404 + 446x^2 - 4164x^4 + 2006x^6 \right) - 3\omega^2q^2 \left( 1588 + 183x^2 - 2072x^4 + 1003x^6 \right) + 45\omega^4 \left( 5 - 4x^2 + x^4 \right) \right), \]

\[ z_{\text{sound,1}}^{(1)} = \frac{w x^2}{8q(3\omega^2 - 2q^2)^2} \left( q^4 \left( -13344 + 5846x^2 - 4520x^4 + 1734x^6 \right) - 3\omega^2q^2 \left( -9744 + 5035x^2 - 2604x^4 + 867x^6 \right) - 36\omega^4 \left( 594 - 264x^2 + 43x^4 \right) \right) + \frac{30wq^2x^4}{2q^2 - 3\omega^2} \]

\[ (4.10) \]

\[ w(q) = \frac{1}{\sqrt{3}} q - iq^2 \left( \frac{1}{3} + \frac{105}{3} \gamma \right) + \mathcal{O}(q^3, \gamma^2). \]  

\[ (4.11) \]

### 4.2 Sound quasinormal spectrum to $\mathcal{O}(q^3)$

Here, we extend the analysis of the previous section to the next order in the hydrodynamic approximation. Since our ultimate goal is to determine the dispersion relation \( \mathcal{I} = (4.8) \) to order $\mathcal{O}(q^3)$, it is sufficient to use the zeroth order dispersion relation, \textit{i.e.}, to set $w = \frac{1}{\sqrt{3}} q$. This drastically simplifies the hydrodynamic equations for $z_{\text{sound,0}}^{(2)}$ and $z_{\text{sound,1}}^{(2)}$. We find it more convenient to solve the resulting equations using the $u = \sqrt{1 - x^2}$ variable.

Explicitly we find

\[ z_{\text{sound,0}}^{(2)}(u) = \frac{u^2}{6} \left[ \frac{1}{2} \ln^2 \left( \frac{u}{1 + u} \right) - \text{dilog} \left( \frac{u}{1 + u} \right) \right] + \frac{2}{3} \ln \left( \frac{u}{1 + u} \right) + \frac{2}{3} (u + 2)(1 - u), \]

\[ (4.12) \]

\[ z_{\text{sound,1}}^{(2)}(u) = u^2 C_2 + \left( 2 + \frac{u^2}{2} \ln(1 - u^2) \right) C_1 + \frac{u^2}{96} I_1(u) \]

\[ - \frac{1}{24} \left( 1 + \frac{u^2}{4} \ln(1 - u^2) \right) I_2(u), \]

\[ (4.13) \]

where

\[ I_1(u) = - \int_0^u dt \frac{4 + t^2 \ln(1 - t^2)}{(2 - t^2)^3(t + 1)} \times I(t), \]

\[ (4.14) \]

\[ I_2(u) = \int_0^u dt \frac{t^2}{(2 - t^2)^3(t + 1)} \times I(t), \]

\[ (4.15) \]
\[ \mathcal{I}(t) = \left\{ 10t^3(t + 1)(9027t^6 - 43808t^4 + 58722t^2 \\
- 23100) \left( \text{dilog} \left( \frac{t}{2} + \frac{1}{2} \right) - \frac{1}{2} \ln^2(1 + t) \right) + 2(t + 1)t(10000 + (376440 \\
- 115500 \ln 2)t^2 + (293610 \ln 2 - 1104128)t^4 + (-219040 \ln 2 + 847972)t^6 \\
+ (45135 \ln 2 - 179418)t^8) \ln(1 + t) - 184400 + (-20000 \ln 2 - 12568)t \\
+ (-20000 \ln 2 + 1912)t^2 + (543780 - 752880 \ln 2 + 115500 \ln^2 2)t^3 \\
+ (1143490 + 115500 \ln^2 2 - 752880 \ln 2)t^4 + (-293610 \ln^2 2 - 1403332 \\
+ 2208256 \ln 2)t^5 + (-2629768 + 2208256 \ln 2 - 293610 \ln^2 2)t^6 \\
+ (219040 \ln^2 2 - 1695944 \ln 2 + 946434)t^7 + (1729061 + 219040 \ln^2 2 \\
- 1695944 \ln 2)t^8 + (-45135 \ln^2 2 + 358836 \ln 2 - 188378)t^9 + (-340799 \\
+ 358836 \ln 2 - 45135 \ln^2 2)t^{10} \right\}, \] 

(4.16)

and the integration constants \( \mathcal{C}_i \) are tuned to satisfy the horizon boundary condition:

\[ \lim_{u \to 1^-} z^{(2)}_{\text{sound,1}}(u) = 0. \] 

(4.17)

The latter is achieved provided

\[ \mathcal{C}_1 = \frac{1}{48} \mathcal{I}_2(1), \quad \mathcal{C}_2 = -\frac{1}{96} \mathcal{I}_1(1). \] 

(4.18)

If we denote

\[ \lim_{u \to 0^+} z^{(2)}_{\text{sound,1}}(u) = 2\mathcal{C}_1 \equiv z^{(2)}_{1,0}, \] 

(4.19)

the Dirichlet boundary condition (4.18) will lead to the following dispersion relation for the sound quasinormal mode

\[ w(q) = \frac{1}{\sqrt{3}}q - iq^2 \left( \frac{1}{3} + \frac{105}{3} \gamma \right) + q^3 \left( \frac{3 - 2 \ln 2}{6\sqrt{3}} \right) \\
+ \frac{1}{24\sqrt{3}} \left( -2758 + 12z^{(2)}_{1,0} + 1705 \ln 2 \right) \gamma + \mathcal{O}(q^4, \gamma^2). \] 

(4.20)

We were unable to evaluate (4.18) analytically; numerically, we find

\[ z^{(2)}_{1,0} = 264.7598406. \] 

(4.21)
4.3 Relaxation time from the sound quasinormal spectrum

Second order relativistic hydrodynamics of conformal fluids implies the following dispersion relation for the sound mode [4]

\[ \omega = c_s q - i \Gamma q^2 + \frac{\Gamma}{c_s} \left( c_s^2 \tau_{\Pi} - \frac{\Gamma}{2} \right) k^3 + \mathcal{O}(k^4), \]  

where

\[ \Gamma = \frac{2 \eta}{3 s T}. \]  

Comparing (4.20) and (4.22) we find

\[ c_s = \frac{1}{\sqrt{3}} + 0 \cdot \gamma + \mathcal{O}(\gamma^2), \quad \Gamma T = \frac{1}{6 \pi} \left( 1 + 120 \gamma \right) + \mathcal{O}(\gamma^2), \]  

in agreement with the conformal equation of state at order \( \mathcal{O}(\gamma) \), as well as in agreement with the ratio \( \frac{\eta}{s} \) as given by (3.33). Additionally, we compute

\[ \tau_{\Pi} T = \frac{2 - \ln 2}{2 \pi} + \frac{1}{16 \pi} \left( 2425 \ln 2 - 3358 + 12 z_{1,0}^{(2)} \right) \gamma + \mathcal{O}(\gamma^2). \]  

A required agreement between (3.33) and (4.25) provides a prediction for \( z_{1,0}^{(2)} \)

\[ z_{1,0}^{(2)} \bigg|_{\text{prediction}} = \frac{2429}{6} - \frac{2425}{12} \ln 2, \]  

which is in excellent agreement with the actually numerical result (4.21). Thus we have a highly nontrivial check on our analysis.

Acknowledgments

We would like to thank Rob Myers, Aninda Sinha and Sam Vazquez for valuable discussions. AB research at Perimeter Institute is supported in part by the Government of Canada through NSERC and by the Province of Ontario through MRI. AB gratefully acknowledges further support by an NSERC Discovery grant and support through the Early Researcher Award program by the Province of Ontario. MP work is supported by the Portuguese Fundacao para a Ciencia e Tecnologia, grant SFRH/BD/23438/2005. MP also gratefully acknowledges Perimeter Institute for its hospitality.
A Coefficients of the effective action, the Gibbons-Hawking term, and the counterterm

\[ A = \frac{r_0^4}{2u} \left[ 8f(u) + \gamma u^2 \left( -600 + 25u^2 + (-44\beta u^2 + 44\kappa^2)u^3 + 1760u^4 - 44\kappa^2u^5 \\
- 1185u^6 \right) \right]. \]  \tag{A.1}

\[ B = \frac{r_0^4}{8u} \left[ 24f(u) + \gamma u^2 \left( -1800 + 179u^2 - (768\kappa^2 + 768\beta u^2)u^3 + 5424u^4 + 768\kappa^2u^5 \\
- 3387u^6 \right) \right]. \]  \tag{A.2}

\[ C = -\frac{r_0^4}{4u^2f} \left[ 8f(u)(3 + u^2) + \gamma u^2 \left( 600 - 825u^2 + (-104\kappa^2 + 104\beta u^2)u^3 - 16945u^4 \\
+ (384\kappa^2 + 872\beta u^2)u^5 + 34005u^6 - 280\kappa^2u^7 - 16835u^8 \right) \right]. \]  \tag{A.3}

\[ D = \frac{r_0^4}{8u^3f^2} \left[ 16f(u)^2 + 8uf(u)\beta^2 + \gamma u^3 \left( 600\beta^2 + 250u + (-25\kappa^2 + 25\beta^2)u^2 \\
+ (944\kappa^2\beta^2 + 296\beta^4 + 296\kappa^4 - 2120)u^3 + (-2400\beta^2 + 825\kappa^2)u^4 + (1570 \\
- 592\kappa^4 - 944\kappa^2\beta^2)u^5 + (1007\beta^2 - 1575\kappa^2)u^6 + (296\kappa^4 + 2220)u^7 + 775\kappa^2u^8 \\
- 1920u^9 \right) \right]. \]  \tag{A.4}

\[ E = \gamma 37 r_0^4u^5f(u)^2. \]  \tag{A.5}

\[ F = \gamma 2r_0^4u^4f(u) \left( 11 - 37u^2 \right). \]  \tag{A.6}

\[ K_1 = -\gamma r_0^4u^3 \left( -25 + (44\beta u^2 - 44\kappa^2)u + 160u^2 + 44\kappa^2u^3 - 135u^4 \right). \]  \tag{A.7}

\[ K_2 = -\frac{4r_0^4}{u^2} \left[ u^2 - 2 + 15\gamma u^2 \left( 2u^6 - 8u^4 + 5 \right) \right]. \]  \tag{A.8}
\[ T = -\frac{r_0^4}{2u^2\sqrt{1-u^2}} \left[ 6 + (-2w^2 + 2k^2)u - 6u^2 - 2k^2u^3 ight.
\]
\[ \left. + 15\gamma u^2 \left( 3u^4 - 5u^2 - 5 \right) \left( 3 + (w^2 + k^2)u - 3u^2 - k^2u^3 \right) \right]. \]  

(A.9)

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