The action of $\mathsf{GT}$-shadows on child’s drawings

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Abstract

$\mathsf{GT}$-shadows [8] are tantalizing objects that can be thought of as approximations of elements of the mysterious Grothendieck-Teichmueller group $\widehat{\mathsf{GT}}$ introduced by V. Drinfeld in 1990. $\mathsf{GT}$-shadows form a groupoid $\mathsf{GTSh}$ whose objects are finite index subgroups of the pure braid group $\mathbb{P}B_4$, that are normal in $B_4$. The goal of this paper is to describe the action of $\mathsf{GT}$-shadows on Grothendieck’s child’s drawings and show that this action agrees with that of $\widehat{\mathsf{GT}}$. We discuss the hierarchy of orbits of child’s drawings with respect to the actions of $\mathsf{GTSh}$, $\widehat{\mathsf{GT}}$, and the absolute Galois group $G_\mathbb{Q}$ of rationals. We prove that the monodromy group and the passport of a child’s drawing are invariant with respect to the action of the subgroupoid $\mathsf{GTSh}^\flat$ of charming $\mathsf{GT}$-shadows. We use the action of $\mathsf{GT}$-shadows on child’s drawings to prove that every Abelian child’s drawing admits a Belyi pair defined over $\mathbb{Q}$. Finally, we describe selected examples of non-Abelian child’s drawings.

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1 Introduction

The profinite version $\hat{\text{GT}}$ of the Grothendieck-Teichmueller group [6, Section 4] [27], [29] connects topology to number theory in a fascinating way. $\hat{\text{GT}}$ receives an injective homomorphism [21], [29, Section 3.2] from the absolute Galois group $G\mathbb{Q}$ of rationals. It acts on Grothendieck’s child’s drawings and this action is compatible with the embedding $G\mathbb{Q} \hookrightarrow \hat{\text{GT}}$ and the standard action of $G\mathbb{Q}$.

Just like $G\mathbb{Q}$, the group $\hat{\text{GT}}$ is a rather intractable object. For example, currently, we know explicitly only two elements of $\hat{\text{GT}}$: the identity element and the element that comes from the complex conjugation. The author also believes that, for an arbitrary child’s drawing $D$, tools of modern mathematics do not allow us to say much about the orbit $\hat{\text{GT}}(D)$.

Let us denote by $B_n$ (resp. $\text{PB}_n$) the Artin braid group (resp. the pure braid group) on $n$ strands. We denote by $x_{ij}$, $1 \leq i < j \leq n$ the standard generators of $\text{PB}_n$ and recall that the elements $x_{12}, x_{23}$ generate a free subgroup of $\text{PB}_3$. In this paper, we tacitly identify the free group $F_2$ on two generators with $\langle x_{12}, x_{23} \rangle \leq \text{PB}_3$.

$\hat{\text{GT}}$ can be defined as the group of continuous automorphisms of the profinite completion $\hat{\text{PaB}}$ of the operad $\text{PaB}$ of parenthesized braids [3], [8, Appendix A], [12, Chapter 6], [35]. $\text{PaB}$ is an operad in the category of groupoids and it is “assembled” from the braid groups $(B_n)_{n \geq 2}$. Objects of the groupoid $\text{PaB}(n)$ are completely parenthesized sequences of number $1, 2, \ldots, n$ in which each number appears exactly once. For example, $\text{Ob}(\text{PaB}(3))$ has 12 objects: $(1, 2)3$, $1(2, 3)$, $(2, 1)3$, $2(1, 3)$, $\ldots$.

The isomorphisms $\alpha \in \text{PaB}((1, 2)3, 1(2, 3))$ and $\beta \in \text{PaB}((1, 2), (2, 1))$ shown in figure 1.1 play an important role for $\text{PaB}$.

![Diagram](https://via.placeholder.com/150)

Fig. 1.1: The isomorphisms $\alpha$ and $\beta$

Due to [12, Theorem 6.2.4], $\text{PaB}$ is generated by $\alpha$ and $\beta$ (as the operad in the category of groupoids) and any relation involving $\alpha$ and $\beta$ is a consequence of the two hexagon relations and the pentagon relation (see (A.13), (A.14) and (A.15) in [8, Appendix A]).

Since $\alpha$ and $\beta$ are topological generators of $\hat{\text{PaB}}$, every $\hat{T} \in \hat{\text{GT}}$ is uniquely determined by the values $\hat{T}(\alpha) \in \hat{\text{PaB}}((1, 2)3, 1(2, 3))$ and $\hat{T}(\beta) \in \hat{\text{PaB}}((1, 2), (2, 1))$. In addition, since the automorphism group of $(1, 2)3$ (resp. $(1, 2)$) in $\text{PaB}$ is $\text{PB}_3$ (resp. $\text{PB}_2 \cong \mathbb{Z}$), every $\hat{T} \in \hat{\text{GT}}$ is uniquely determined by a pair $(\hat{m}, \hat{f}) \in \mathbb{Z} \times \text{PB}_3$ via the equations

$$\hat{T}(\beta) = \beta \circ x_{12}^\hat{m}, \quad \hat{T}(\alpha) = \alpha \circ \hat{f}.$$  

Using the hexagon relations and the pentagon relation, one can show that $\hat{f} \in \hat{\mathbb{F}}_2$. In fact, one can show that $\hat{f}$ belongs to the topological closure of the commutator subgroup $[\hat{\mathbb{F}}_2, \hat{\mathbb{F}}_2]$.

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1 We tacitly assume that automorphisms of $\hat{\text{PaB}}$ act trivially on the set of objects of $\hat{\text{PaB}}(n)$ for every $n$. 


It is easy to see [Remark 1.1] that the action of $\hat{\mathcal{G}}\mathcal{T}$ on $\widehat{\mathcal{P}}\mathcal{B}_3 \cong \text{Aut}((1,2)3)$ descends to the action on $\widehat{F}_2 \leq \mathcal{P}B_3$ and it is given (on the topological generators of $\widehat{F}_2$) by the formulas
\begin{equation}
\hat{T}_{\widehat{F}_2}(x) := x^{2\hat{m}+1}, \quad \hat{T}_{\widehat{F}_2}(y) := \hat{f}^{-1}y^{2\hat{m}+1}\hat{f},
\end{equation}
where $(\hat{m}, \hat{f})$ is the pair in $\mathbb{Z} \times \hat{F}_2$ corresponding to $\hat{T}$.

Let $S_d$ be the symmetric group of degree $d$ and $\text{Hom}_{\text{tran}}(\widehat{\mathcal{F}}_2, S_d)$ be the set of continuous group homomorphisms $\psi : \widehat{F}_2 \to S_d$ for which the subgroup $\psi(\widehat{F}_2)$ acts transitively on the set $\{1, 2, \ldots, d\}$. Here $S_d$ is considered with the discrete topology.

The set $\text{Hom}_{\text{tran}}(\widehat{\mathcal{F}}_2, S_d)$ carries the obvious action of $S_d$ by conjugation and a child's drawing of degree $d$ can be defined as an orbit of the $S_d$-action on $\text{Hom}_{\text{tran}}(\widehat{\mathcal{F}}_2, S_d)$. In these terms, it is easy to define the (right) action of $\hat{\mathcal{G}}\mathcal{T}$ on child's drawings: given a child's drawing $[\psi]$ represented by a continuous group homomorphism $\psi : \widehat{F}_2 \to S_d$ and $\hat{T} \in \hat{\mathcal{G}}\mathcal{T}$ the child's drawing $[\psi]^{\hat{T}}$ is represented by the continuous group homomorphism $\psi \circ \hat{T}_{\widehat{F}_2} : \widehat{F}_2 \to S_d$.

Let us denote by $\text{NFI}_{\mathcal{P}B_4}(B_4)$ the poset of finite index normal subgroups $N \trianglelefteq B_4$ satisfying the condition $N \leq \mathcal{P}B_4$. In paper [8], the authors introduced the concept of a $\hat{\mathcal{G}}\mathcal{T}$-shadow. Loosely speaking, a $\hat{\mathcal{G}}\mathcal{T}$-shadow is an onto morphism of (truncated) operads $\mathcal{P}B^{\leq 4} \to \mathcal{P}B^{\leq 4}/\sim_N$, where $\sim_N$ is an equivalence relation on the set of morphisms of $\mathcal{P}B^{\leq 4}$ that comes from an element $N$ of the poset $\text{NFI}_{\mathcal{P}B_4}(B_4)$.

In paper [8], it was proved that $\hat{\mathcal{G}}\mathcal{T}$-shadows form a groupoid $\hat{\mathcal{G}}\mathcal{T}\text{Sh}$ whose objects are elements of $\text{NFI}_{\mathcal{P}B_4}(B_4)$. For every $K, N \in \text{NFI}_{\mathcal{P}B_4}(B_4)$, the set $\hat{\mathcal{G}}\mathcal{T}\text{Sh}(K, N)$ may be identified with the set of isomorphisms of (truncated) operads $\mathcal{P}B^{\leq 4}/\sim_K \cong \mathcal{P}B^{\leq 4}/\sim_N$.

In this paper, we use the poset $\text{NFI}_{\mathcal{P}B_4}(B_4)$ and the set of all child's drawings to form a category $\text{Dessin}$. We show that $\hat{\mathcal{G}}\mathcal{T}$ acts naturally on the poset $\text{NFI}_{\mathcal{P}B_4}(B_4)$ and the action of $\hat{\mathcal{G}}\mathcal{T}$ on child's drawings gives us a cofunctor
\begin{equation}
\mathcal{A} : \hat{\mathcal{G}}\mathcal{T}_{\text{NFI}} \to \text{Dessin},
\end{equation}
from the corresponding transformation groupoid $\hat{\mathcal{G}}\mathcal{T}_{\text{NFI}}$ to the category $\text{Dessin}$.

We show (see Theorem 3.1) that $\hat{\mathcal{G}}\mathcal{T}$-shadows act on child’s drawings in the sense that we have a natural cofunctor
\begin{equation}
\mathcal{A}^{\text{sh}} : \hat{\mathcal{G}}\mathcal{T}\text{Sh} \to \text{Dessin}.
\end{equation}

Recall [8 Section 2.4] that, for every $\hat{T} \in \hat{\mathcal{G}}\mathcal{T}$ and $N \in \text{NFI}_{\mathcal{P}B_4}(B_4)$, we can produce a $\hat{\mathcal{G}}\mathcal{T}$-shadow $T_N$ with the target $N$, and $T_N$ may be viewed as an approximation of $\hat{T}$. Using this passage from elements of $\hat{\mathcal{G}}\mathcal{T}$ to $\hat{\mathcal{G}}\mathcal{T}$-shadows we define a functor
\begin{equation}
\mathcal{PR} : \hat{\mathcal{G}}\mathcal{T}_{\text{NFI}} \to \hat{\mathcal{G}}\mathcal{T}\text{Sh},
\end{equation}

3
We prove (see Theorem 3.2) that the action of GT-shadows on child’s drawings is compatible with the action of \(\hat{\Gamma}\) in the sense that the functors
\[
A \quad \text{and} \quad A^{sh} \circ \mathcal{P} \mathcal{R}
\]
are equal in the strict sense.

Using Theorem 3.2 and the compatibility of the \(G_{\mathbb{Q}}\)-action and the \(\hat{\Gamma}\)-action on child’s drawings, we deduce several interesting corollaries:

- Corollary 3.9 describes the hierarchy of orbits of the \(GTSh\)-action, \(\hat{\Gamma}\)-action and \(G_{\mathbb{Q}}\)-action on the set of child’s drawings.
- Corollary 3.12 may be viewed as a version of [18, Proposition 14]; this statement gives us a useful bound on the degree of the field of moduli of a child’s drawing.
- Corollary 3.13 tells us that, using the poset \(NFI_{PB_4}(B_4)\), we can produce many examples of Galois child’s drawings that admit Belyi pairs defined over \(\mathbb{Q}\).

Using consequences of the hexagon relations for GT-shadows, we show (see Theorem 3.16) that the passport of a child’s drawing is invariant with respect to the action of (charming) GT-shadows.

In this paper, we also prove some basic facts about Abelian child’s drawings and show (see Corollary 4.8 and [20, Corollary 3.5]) that every Abelian child’s drawing admits a Belyi pair defined over \(\mathbb{Q}\).

Finally, we describe selected examples of non-Abelian child’s drawings whose \(GTSh\)-orbits were computed using software package [7]. Whenever possible, the results were compared to \(G_{\mathbb{Q}}\)-orbits from database [25]. It is amazing to see that, if a \(GTSh\)-orbit and the \(G_{\mathbb{Q}}\)-orbit of a child’s drawing can be computed then these orbits coincide.

**Organization of the paper**

Section 2 is devoted to the background material. It contains a brief reminder of the groupoid \(GTSh\) and its link to \(\hat{\Gamma}\). In this section, we also recall child’s drawings, introduce the category \(\text{Dessin}\) and define the action of \(\hat{\Gamma}\) on child’s drawings in terms of a cofunctor from a certain transformation groupoid \(\hat{\Gamma}_{NFI}\) to \(\text{Dessin}\).

In Section 3, we define the action of GT-shadows on child’s drawings, describe the relationship between orbits of a child’s drawing with respect to different actions, and prove that the monodromy group and the passport of a child’s drawing are invariant with respect to the action of (charming) GT-shadows.

Section 4 is devoted to various properties of Abelian child’s drawings. In this section, we prove that every Abelian child’s drawing admits a Belyi pair defined over \(\mathbb{Q}\).

In Section 5, we describe selected examples of non-Abelian child’s drawings whose \(GTSh\)-orbits were computed.

In Appendix A, we prove that charming GT-shadows satisfy versions of relations (4.3) and (4.4) from [6, Section 4].

In Appendix B, we give an outline of the package \(GT\) for working with GT-shadows and their action on child’s drawings. For more details, please see the documentation [7] for this package.
Remark 1.1 We should remark that the notation $\hat{\text{GT}}$ is a bit misleading: the hat over \text{"GT} does not mean that $\hat{\text{GT}}$ is a profinite completion of some \text{"well known group GT}". However, please see [8, Theorem 3.8].

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Memorial note. Andrey Loginov (1977-2016) was an experimental physicist, and his work helped to push frontiers of knowledge related to the fundamental questions about elementary particles. There were several moments in my life when he played for me a role of an older brother. Life connected us in bizarre ways by cities scattered around the globe: Berdychiv, Chicago, Dolgoprudny, Moscow and Tomsk. It is very sad that I did not have a chance to say a proper goodbye to him...

1.1 Notational conventions

For a set $X$ with an equivalence relation and $a \in X$ we will denote by $[a]$ the equivalence class represented by the element $a$.

The notation $B_n$ (resp. $\text{PB}_n$) is reserved for the Artin braid group on $n$ strands (resp. the pure braid group on $n$ strands). The standard generators of $B_n$ are denoted by $\sigma_1, \ldots, \sigma_{n-1}$ and the standard generators of $\text{PB}_n$ are denote by $x_{ij}$ for $1 \leq i < j \leq n$. We set $c := x_{23}x_{12}x_{13} \in \text{PB}_3$ and recall that $c$ generates the center $\mathcal{Z}(\text{PB}_3)$ of $\text{PB}_3$ (as well as the center $\mathcal{Z}(B_3)$ of $B_3$).

The free group $F_2$ on two generators is tacitly identified with the subgroup $\langle x_{12}, x_{23} \rangle \leq \text{PB}_3$. Occasionally, we denote the standard generators of $F_2$ by $x$ and $y$, i.e. $x := x_{12}$ and $y := x_{23}$.

$S_d$ denotes the group of bijections $\{1, 2, \ldots, d\} \xrightarrow{\cong} \{1, 2, \ldots, d\}$ (i.e. the symmetric group of degree $d$). A subgroup $H \leq S_d$ is called\textbf{ transitive} if its standard action on $\{1, 2, \ldots, d\}$ is transitive. Every finite group is tacitly considered with the discrete topology. For a group $G$, the notation $[G, G]$ is reserved for the commutator subgroup of $G$. For a normal subgroup $H \trianglelefteq G$ of finite index, we denote by $\text{NFI}_H(G)$ the poset of finite index normal subgroups $N$ in $G$ such that $N \leq H$. The notation $\text{NFI}(G)$ is reserved for the poset of finite index normal subgroups of $G$. For $N \in \text{NFI}(G)$, $\mathcal{P}_N$ denotes the standard projection $\mathcal{P}_N : G \to G/N, \quad \mathcal{P}_N(g) := gN$. Moreover, $\hat{\mathcal{P}}_N$ denotes the standard continuous group homomorphism from the profinite completion $\hat{G}$ of $G$ to the finite group $G/N$. For a group $G$ and $g \in G$, the notation $\mathcal{Z}_G(g)$
is reserved for the centralizer of \( g \) in \( G \). If a group \( G \) is residually finite (e.g. \( G = \mathbb{F}_n, \mathbb{P}B_n \) or \( G = B_n \)), then we tacitly identify \( G \) with its image in the profinite completion \( \hat{G} \).

For objects \( a, b \) of a category \( \mathcal{C} \), \( \mathcal{C}(a, b) \) often denotes the set of morphisms from \( a \) to \( b \). For a groupoid \( \mathcal{G} \), the notation \( \gamma \in \mathcal{G} \) means that \( \gamma \) is a morphism of this groupoid. We assume that all functors are covariant and the word “cofunctor” means a contravariant functor.

We will freely use the language of operads [9, Section 3], [12, Chapter 1], [23], [24], [32]. In this paper, we encounter operads in the category of sets and in the category of (topological) groupoids. The category of topological groupoids is treated in the “strict sense”. For example, the associativity axioms for the elementary insertions\(^2\) \( \circ_i \) (for operads in the category of groupoids) are satisfied “on the nose”.

For an integer \( q \geq 1 \), a \( q \)-truncated operad in the category of groupoids is a collection of groupoids \( \{ \mathcal{G}(n) \}_{1 \leq n \leq q} \) such that

- For every \( 1 \leq n \leq q \), the groupoid \( \mathcal{G}(n) \) is equipped with an action of \( S_n \).
- For every triple of integers \( i, n, m \) such that \( 1 \leq i \leq n, n, m, n + m - 1 \leq q \) we have functors
  \[ \circ_i : \mathcal{G}(n) \times \mathcal{G}(m) \to \mathcal{G}(n + m - 1). \] (1.4)
- The axioms of the operad for \( \{ \mathcal{G}(n) \}_{1 \leq n \leq q} \) are satisfied in the cases where all the arities are \( \leq q \).

For every operad \( \mathcal{O} \) and for every integer \( q \geq 1 \), the disjoint union

\[ \mathcal{O}^{\leq q} := \bigsqcup_{n=0}^{q} \mathcal{O}(n) \]

is clearly a \( q \)-truncated operad.

In this paper, we mostly consider 4-truncated operads. So, we will simply call them truncated operads.

The operad \( \mathbb{P}A \mathbb{B} \) of parenthesized braids, its truncation \( \mathbb{P}A \mathbb{B}^{\leq 4} \) and its completion \( \hat{\mathbb{P}A \mathbb{B}}^{\leq 4} \) play an important role in this paper. See [3], [8, Appendix A], [12, Chapter 6], [35] for more details about these objects.

2 Background material

2.1 Reminder of GT-shadows

In this section, we review the groupoid \( \mathbb{G}T \mathbb{S}h \) whose objects are elements of the poset \( \mathcal{NFI}_{\mathbb{P}B_4}(B_4) \) and whose morphisms are called GT-shadows.

Recall [8, Section 2.2] that every element \( \mathbf{N} \in \mathcal{NFI}_{\mathbb{P}B_4}(B_4) \) gives us a compatible equivalence relation \( \sim_N \) on the truncation

\[ \mathbb{P}A \mathbb{B}^{\leq 4} := \mathbb{P}A \mathbb{B}(1) \sqcup \mathbb{P}A \mathbb{B}(2) \sqcup \mathbb{P}A \mathbb{B}(3) \sqcup \mathbb{P}A \mathbb{B}(4) \] (2.1)

of the operad \( \mathbb{P}A \mathbb{B} \), i.e.

- \( \sim_N \) is an equivalence relation on the set of morphisms of \( \mathbb{P}A \mathbb{B}(1) \sqcup \mathbb{P}A \mathbb{B}(2) \sqcup \mathbb{P}A \mathbb{B}(3) \sqcup \mathbb{P}A \mathbb{B}(4) \);

\(^2\)In the literature, elementary insertions are sometimes called partial compositions.
• ∼ is compatible with the actions of the symmetric groups $S_2, S_3, S_4$;
• ∼ is compatible with the composition of morphisms and the operadic insertions;
• finally, the quotient operad $\mathbb{P}A\mathbb{B}^{\leq 4}/\sim$ is finite.

More precisely, given $N \in NFI_{PB_4}(B_4)$, we produce $N_{PB_3} \in NFI_{PB_3}(B_3)$ and $N_{PB_2} \in NFI_{PB_2}(B_2)$. (Since $PB_2$ is the infinite cyclic group and $B_2$ is Abelian, $N_{PB_2}$ is uniquely determined by its index $N_{ord} := |PB_2 : N_{PB_2}|$.) Then $N$ (resp. $N_{PB_3}$, $N_{PB_2}$) gives us an equivalence relation on $\mathbb{P}A\mathbb{B}(4)$ (resp. $\mathbb{P}A\mathbb{B}(3)$, $\mathbb{P}A\mathbb{B}(2)$). Due to [8, Proposition 2.4], these equivalence relations on $\mathbb{P}A\mathbb{B}(4)$, $\mathbb{P}A\mathbb{B}(3)$ and $\mathbb{P}A\mathbb{B}(2)$ assemble into a compatible equivalence relation $\sim_N$ on (2.1).

For $N \in NFI_{PB_4}(B_4)$, we denote by $\mathcal{P}_N$ the standard projection

$$\mathcal{P}_N : \mathbb{P}A\mathbb{B}^{\leq 4} \to \mathbb{P}A\mathbb{B}^{\leq 4}/\sim_N.$$  (2.2)

Similarly, $\hat{\mathcal{P}}_N$ denotes the standard continuous (onto) map of truncated operads

$$\hat{\mathcal{P}}_N : \hat{\mathbb{P}A}\mathbb{B}^{\leq 4} \to \mathbb{P}A\mathbb{B}^{\leq 4}/\sim_N,$$  (2.3)

where $\hat{\mathbb{P}A}\mathbb{B}^{\leq 4}$ is the profinite completion of $\mathbb{P}A\mathbb{B}^{\leq 4}$.

We set

$$N_{F_2} := N_{PB_3} \cap F_2,$$  (2.4)

where $F_2$ is (tacitly) identified with the subgroup $\langle x_{12}, x_{23} \rangle \leq PB_3$.

A GT-pair with the target $N$ is a morphism of truncated operads $\mathbb{P}A\mathbb{B}^{\leq 4} \to \mathbb{P}A\mathbb{B}^{\leq 4}/\sim_N$ and such morphisms are in bijection with pairs

$$(m,f_{PB_3}) \in \{0,1, \ldots, N_{ord} - 1\} \times PB_3/N_{PB_3}$$  (2.5)

satisfying the hexagon relations

$$\sigma_1 x_{12}^m f^{-1} \sigma_2 x_{23}^m f_{PB_3} = f^{-1} \sigma_1 \sigma_2 x_{12}^{-m} c^m N_{PB_3},$$  (2.6)

$$f^{-1} \sigma_2 x_{23}^m \sigma_1 x_{12}^m N_{PB_3} = \sigma_2 \sigma_1 x_{23}^{-m} c^m f N_{PB_3}$$  (2.7)

and the pentagon relation

$$\varphi_{234}(f) \varphi_{1,2,3,4}(f) \varphi_{123}(f) N = \varphi_{1,2,3,4}(f) \varphi_{12,3,4}(f) N.$$  (2.8)

Both sides of (2.6) and (2.7) are elements of $B_3/N_{PB_3}$ and both sides of (2.8) are elements of $PB_4/N$. Explicit formulas for the group homomorphisms $\varphi_{234}$, $\varphi_{1,2,3,4}$, $\varphi_{123}$, $\varphi_{1,2,3,4}$ and $\varphi_{12,3,4}$ from $PB_3$ to $PB_4$ are given in [8, Appendix A.4, (A.18)].

Just as in [8], we tacitly identify morphisms $\mathbb{P}A\mathbb{B}^{\leq 4} \to \mathbb{P}A\mathbb{B}^{\leq 4}/\sim_N$ with pairs (2.5) satisfying (2.6), (2.7) and (2.8).

It is convenient to represent GT-pairs by tuples $(m,f) \in \mathbb{Z} \times PB_3$ and we denote by $[m,f]$ the GT-pair represented by the tuple $(m,f)$. For a GT-pair $[m,f]$ with the target $N$, we denote by

$$T_{m,f} : \mathbb{P}A\mathbb{B}^{\leq 4} \to \mathbb{P}A\mathbb{B}^{\leq 4}/\sim_N$$  (2.9)

the corresponding morphism of (truncated) operads.

\footnote{Note that $\mathbb{P}A\mathbb{B}(1)$ is the groupoid with exactly one object and exactly one (identity) morphism.}
It is clear that, for every $GT$-pair $[m, f]$ with the target $N$, $T_{m,f}$ gives us the following group homomorphisms

$$T_{m,f}^{PB_3} : PB_3 \to PB_3/N, \quad T_{m,f}^{PB_2} : PB_2 \to PB_2/NPB_2.$$ 

For more details, see [8, Corollary 2.7].

A $GT$-shadow with the target $N$ is an onto morphism of truncated operads $PaB^{\leq 4} \to PaB^{\leq 4}/\sim_N$ and such morphisms are in bijection with pairs $(2.5)$ that satisfy the following conditions:

- $(m, f)$ obeys relations $(2.6)$, $(2.7)$, $(2.8)$,
- $2m + 1$ represents a unit in the ring $\mathbb{Z}/N_{ord}\mathbb{Z}$, and
- the group homomorphism $T_{m,f}^{PB_3} : PB_3 \to PB_3/NPB_3$ is onto.

We will identify $GT$-shadows with isomorphisms (2.10). (See (2.21) below for the explicit formula of the composition of $PB_3$.

Recall [8, Corollary 2.8] that, for every $[m, f] \in GT(N)$, the homomorphism $T_{m,f}^{PB_3} : PB_3 \to PB_3/NPB_3$ is given by the explicit formulas:

$$T_{m,f}^{PB_3} (x_{12}) = x_{12}^{2m+1}NPB_3, \quad T_{m,f}^{PB_3} (x_{23}) = f^{-1}x_{23}^{2m+1}fNPB_3, \quad T_{m,f}^{PB_3} (c) = c^{2m+1}NPB_3. \quad (2.12)$$

Since $PB_3 \cong \langle x_{12}, x_{23} \rangle \times \langle c \rangle$, the restriction of $T_{m,f}^{PB_3}$ to $F_2 = \langle x_{12}, x_{23} \rangle$ gives us a group homomorphism

$$T_{m,f}^{F_2} : F_2 \to F_2/NF_2 \quad (2.13)$$

with

$$T_{m,f}^{F_2} (x) := x^{2m+1}NF_2, \quad T_{m,f}^{F_2} (y) := f^{-1}y^{2m+1}fNF_2. \quad (2.14)$$

Let us prove the following useful statement:
Proposition 2.1 For every $N \in \text{NFI}_{PB_4}(B_4)$ and every $[m, f] \in \text{GT}(N)$, the group homomorphism (2.13) is onto.

Proof. Due to [8, Proposition 2.3], $N_{\text{ord}}$ is the least common multiple of the orders of the elements $x_{12}N_{PB_3}$, $x_{23}N_{PB_3}$ and $cN_{PB_3}$ in $PB_3/N_{PB_3}$.

Therefore, since $2m + 1$ is coprime with $N_{\text{ord}}$, there exist $q_1, q_2 \in \mathbb{Z}$ such that

$$T_{m,f}^F(x_{12}^{q_1}) = x_{12}N_F, \quad T_{m,f}^F(x_{23}^{q_2}) = f^{-1}x_{23}fN_F.$$  \hspace{1cm} (2.15)

Since the homomorphism $T_{PB_3}^{PB_3}$ is onto, there exists $w \in PB_3$ such that

$$T_{PB_3}^{PB_3}(w) = fN_{PB_3}$$  \hspace{1cm} (2.16)

where $w_1 \in F_2$. Therefore, using (2.16), we get

$$T_{m,f}^F(w_1x_{23}^{q_2}w_1^{-1}) = T_{m,f}^{PB_3}(w_1c^kx_{23}^{q_2}c^{-k}w_1^{-1}) = T_{m,f}^{PB_3}(w_2x_{23}^{q_2}w_1^{-1}) = f^{-1}x_{23}f^{-1}N_{PB_3} = x_{23}N_{PB_3}.$$

Thus, $T_{m,f}^F(w_1x_{23}^{q_2}w_1^{-1}) = x_{23}N_F$.

Since both generators $x_{12}N_F$ and $x_{23}N_F$ of $F_2/N_F$ belong to $T_{m,f}^F(F_2)$, we proved that $T_{m,f}^F$ is indeed onto. \hspace{1cm} □

2.1.1 $\hat{\text{GT}}$ versus the groupoid $\text{GTSh}$

Just as in [8], we denote by $I$ the standard morphism $\text{PaB}^{\leq 4} \to \overset{\sim}{\text{PaB}^{\leq 4}}$.

Let $\hat{T} \in \hat{\text{GT}}$ and $N \in \text{NFI}_{PB_4}(B_4)$. It was shown in [8, Section 2.4] that the formula

$$T_N := \hat{P}_N \circ \hat{T} \circ I : \text{PaB}^{\leq 4} \to \text{PaB}^{\leq 4}/\sim_N$$  \hspace{1cm} (2.17)

defines a GT-shadow with the target $N$.

Let us prove that

Proposition 2.2 The assignment

$$N^\hat{T} := \ker(PB_4 \xrightarrow{T_{PB_4}^{PB_4}} PB_4/N)$$  \hspace{1cm} (2.18)

defines a right action of $\hat{\text{GT}}$ on the set $\text{NFI}_{PB_4}(B_4)$. Moreover, the assignment $\hat{T} \mapsto T_N$ defines a functor $\mathcal{P} \mathcal{R}$ from the corresponding transformation groupoid to $\text{GTSh}$.

Proof. It is clear that, if $\hat{T}$ is the identity element of $\hat{\text{GT}}$, then

$$T_N = \mathcal{P}_N : \text{PaB}^{\leq 4} \to \text{PaB}^{\leq 4}/\sim_N.$$
For every \( \hat{T} \in \hat{\mathbb{GT}} \), we have the following commutative diagram of maps of truncated operads:

\[
\begin{array}{c}
\text{PaB} \leq 4 \\
\downarrow \hat{p}_{n,T} \\
\text{PaB}^4/ \sim_{N^n} \leq 4 \\
\end{array}
\xrightarrow{\hat{T}}
\begin{array}{c}
\text{PaB} \leq 4 \\
\downarrow \hat{p}_n \\
\text{PaB}^4/ \sim_{N^n} \leq 4 \\
\end{array}
\]

(2.19)

Let \( \tilde{T}_1, \tilde{T}_2 \in \hat{\mathbb{GT}}, \, N, N^\circ \in \text{NFI}_{\text{PB}_4}(B_4), \, N^{(1)} := N^{\tilde{T}_1} \) and \( N^s := (N^{(1)})^{\tilde{T}_2} \). Combining the corresponding commutative diagrams for \( \tilde{T}_1 \) and \( \tilde{T}_2 \), we get the commutative diagram:

\[
\begin{array}{c}
\text{PaB} \leq 4 \\
\downarrow \hat{p}_{n,T} \\
\text{PaB}^4/ \sim_{N^n} \leq 4 \\
\end{array}
\xrightarrow{\tilde{T}_2}
\begin{array}{c}
\text{PaB} \leq 4 \\
\downarrow \hat{p}_{n^{(1)}} \\
\text{PaB}^4/ \sim_{N^{(1)}} \leq 4 \\
\end{array}
\xrightarrow{\tilde{T}_1}
\begin{array}{c}
\text{PaB} \leq 4 \\
\downarrow \hat{p}_n \\
\text{PaB}^4/ \sim_{N^n} \leq 4 \\
\end{array}
\]

(2.20)

where \( \hat{T} := \tilde{T}_1 \circ \tilde{T}_2 \) and \( T_N \) is the corresponding \( \hat{\mathbb{GT}} \)-shadow with the target \( N \).

Thus

\[
(N^{\tilde{T}_1^{T_2}}) = N^{\tilde{T}_1 \circ \tilde{T}_2}.
\]

We proved that formula (2.18) defines a (right) action of \( \hat{\mathbb{GT}} \) on \( \text{NFI}_{\text{PB}_4}(B_4) \).

Let us denote by \( \hat{\mathbb{GT}}_{\text{NFI}} \) the corresponding transformation groupoid. The set of objects of \( \hat{\mathbb{GT}}_{\text{NFI}} \) is \( \text{NFI}_{\text{PB}_4}(B_4) \) and, for \( N^{(1)}, N^{(2)} \in \text{NFI}_{\text{PB}_4}(B_4) \), the set of morphisms \( \hat{\mathbb{GT}}_{\text{NFI}}(N^{(1)}, N^{(2)}) \) consists of elements \( \hat{T} \in \hat{\mathbb{GT}} \) such that \( (N^{(2)})^\hat{T} = N^{(1)} \).

The diagram in (2.20) tells us that, if \( \hat{T} := \tilde{T}_1 \circ \tilde{T}_2 \), then

\[
T_{N^{(1)}}^\text{isom} = T_{1,N}^{\text{isom}} \circ T_{2,N^{(1)}}^{\text{isom}}.
\]

Thus the assignment \( \hat{T} \mapsto T_N \) indeed defines a functor \( \mathcal{P} \mathcal{R} \) from \( \hat{\mathbb{GT}}_{\text{NFI}} \) to \( \mathbb{GTSh} \). (On the level of objects, the functor \( \mathcal{P} \mathcal{R} \) operates as the identity map.) □

Recall [8, Section 2.6] that a \( \hat{\mathbb{GT}} \)-shadow \( [m, f] \in \hat{\mathbb{GT}}(N) \) is called genuine if there exists \( \hat{T} \in \hat{\mathbb{GT}} \) such that \( T_{m,f} = T_N \), i.e. \( [m, f] \) comes from an element of \( \hat{\mathbb{GT}} \). If such \( \hat{T} \in \hat{\mathbb{GT}} \) does not exist then the \( \hat{\mathbb{GT}} \)-shadow \( [m, f] \) is called fake.

It was shown in [8, Section 2] that genuine \( \hat{\mathbb{GT}} \)-shadows satisfy additional conditions.

A \( \hat{\mathbb{GT}} \)-shadow is called practical, if it can be represented by a pair \( (m, f) \) where \( f \in F_2 \). Since every genuine \( \hat{\mathbb{GT}} \)-shadow is practical, in this paper, we assume that all \( \hat{\mathbb{GT}} \)-shadows are practical. In particular, \( \hat{\mathbb{GT}}(N) \) denotes the set of all practical \( \hat{\mathbb{GT}} \)-shadows with the target \( N \). Furthermore, we will use the same notation \( \mathbb{GTSh} \) for the (sub)groupoid of a practical \( \hat{\mathbb{GT}} \)-shadows.

\footnote{Recall that \( F_2 \) is identified with the subgroup \( \langle x_{12}, x_{23} \rangle \leq PB_3 \).}

\footnote{See, for example, [8, Proposition 2.20].}
If \([m_1, f_1] \in \text{GTSh}(\mathbb{N}^{(2)}, \mathbb{N}^{(1)})\), \([m_2, f_2] \in \text{GTSh}(\mathbb{N}^{(3)}, \mathbb{N}^{(2)})\) and
\[
m := 2m_1m_2 + m_1 + m_2, \]
\[
f(x, y) := f_1(x, y)f_2(x^{2m_1+1}, f_1(x, y)^{-1}y^{2m_1+1}f_1(x, y)),
\]
then the pair \((m, f)\) represents a GT-shadow in \(\text{GTSh}(\mathbb{N}^{(3)}, \mathbb{N}^{(1)})\) and \([m, f] := [m_1, f_1] \circ [m_2, f_2]\), i.e. formula (2.21) defines the composition of (practical) GT-shadows. For more details, see [8, Remark 2.15].

It was proved in [8, Proposition 2.20] that, for every genuine GT-shadow \([m, f] \in \text{GT}(\mathbb{N})\), the coset \(f\mathbb{N}_{F_2}\) belongs to the commutator subgroup \([\mathbb{F}_2/\mathbb{N}_{F_2}, \mathbb{F}_2/\mathbb{N}_{F_2}]\).

\[
f\mathbb{N}_{F_2} \in [\mathbb{F}_2/\mathbb{N}_{F_2}, \mathbb{F}_2/\mathbb{N}_{F_2}].
\]

GT-shadows satisfying this additional property are called charming.

Just as in [8], the notation \(\text{GT}^\Diamond(\mathbb{N})\) is reserved for the subset of charming GT-shadows in \(\text{GT}(\mathbb{N})\). Due to [8, Proposition 2.22], charming GT-shadows form a subgroupoid \(\text{GTSh}^\Diamond\) of \(\text{GTSh}\).

**Remark 2.3** Due to Proposition 2.11, the condition about the homomorphism \(T_{m,f}^{F_2}\) in [8, Definition 2.19] is redundant. In other words, a GT-shadow \([m, f] \in \text{GT}(\mathbb{N})\) is charming if and only if condition (2.22) is satisfied.

**Remark 2.4** Since, for every \(\hat{T} \in \hat{\text{GT}}\) and \(N \in \text{NFI}_{PB_4}(B_4)\), the GT-shadow \(T_N \in \text{GT}(\mathbb{N})\) is charming, the functor \(\mathcal{P} \mathcal{R} : \hat{\text{GT}}_{\text{NFI}} \rightarrow \text{GTSh}\) from Proposition 2.2 lands in the subgroupoid \(\text{GTSh}^\Diamond\).

Recall that the groupoid \(\text{GTSh}^\Diamond\) is highly disconnected. Indeed, if \(K, N \in \text{NFI}_{PB_4}(B_4)\) have different indices in \(PB_4\), then \(\text{GTSh}^\Diamond(K, N)\) is empty. Just as in [8], \(\text{GTSh}^\Diamond_{\text{conn}}(N)\) denotes the connected component of \(N\) in the groupoid \(\text{GTSh}^\Diamond\). Elements \(N \in \text{NFI}_{PB_4}(B_4)\) for which the connected component \(\text{GTSh}^\Diamond_{\text{conn}}(N)\) has exactly one object are called isolated and they play a special role:

- For every isolated element \(N \in \text{NFI}_{PB_4}(B_4)\), \(\text{GT}^\Diamond(\mathbb{N})\) is a finite group.
- Due to [8, Proposition 3.3], the subposet \(\text{NFI}_{\text{isolated}}_{PB_4}(B_4)\) of isolated elements in \(\text{NFI}_{PB_4}(B_4)\) is coinitial: for every \(N \in \text{NFI}_{PB_4}(B_4)\), there exists \(K \in \text{NFI}_{\text{isolated}}_{PB_4}(B_4)\) such that \(K \leq N\).
- Due to [8, Proposition 3.6], \(\text{NFI}_{\text{isolated}}_{PB_4}(B_4)\) is closed under taking finite intersections.
- Due to [8, Proposition 3.7], the assignment \(N \rightarrow \text{GT}^\Diamond(\mathbb{N})\) upgrades to a functor \(\mathcal{M} \mathcal{L}\) from the poset \(\text{NFI}_{\text{isolated}}_{PB_4}(B_4)\) to the category of finite groups.
- Finally, due to [8, Theorem 3.8], the limit of \(\mathcal{M} \mathcal{L}\) is isomorphic to \(\hat{\text{GT}}\).
2.2 Permutation pairs, permutation triples and child’s drawings

A permutation pair of degree \(d\) is a pair \(c = (c_1, c_2) \in S_d \times S_d\) for which the subgroup \(\langle c_1, c_2 \rangle \leq S_d\) is transitive. We consider permutation pairs with the action of \(S_d\) by conjugation

\[
h(c) := (hc_1h^{-1}, hc_2h^{-1}), \quad h \in S_d.
\]

Recall that a child’s drawing of degree \(d\) is a conjugacy class \([c]\) of a permutation pair \(c = (c_1, c_2)\). Let us denote by \(\text{ct}\) the standard map from \(S_d\) to the set \(\mathcal{P}_d\) of partitions of \(d\):

\[
\text{ct}(h) \text{ is the cycle structure of } h.
\]

The (conjugacy class of the) permutation group \(\langle c_1, c_2 \rangle \leq S_d\) is called the monodromy group of the child’s drawing \([c]\).

In the literature\(^{6}\), child’s drawings (of degree \(d\)) are often represented by permutation triples, i.e. elements \((g_1, g_2, g_3)\) in \((S_d)^3\) such that

- \(g_1g_2g_3 = \text{id}\) and
- the subgroup \(\langle g_1, g_2, g_3 \rangle \leq S_d\) is transitive.

The assignment

\[
(c_1, c_2) \mapsto (c_1, c_2, c_2^{-1}c_1^{-1})
\]

(2.23)

gives us an obvious bijection from the set of permutation pairs (of degree \(d\)) to the set of permutation triples (of degree \(d\)). \(S_d\) acts on permutation triples by conjugation

\[
h(g_1, g_2, g_3) := (hg_1h^{-1}, hg_2h^{-1}, hg_3h^{-1})
\]

and this action is compatible with bijection (2.23).

The triple of partitions \((\text{ct}(c_1), \text{ct}(c_2), \text{ct}(c_2^{-1}c_1^{-1}))\) is called the passport of a child’s drawing \([c]\).

2.3 Representing child’s drawings by group homomorphisms

It is convenient to represent a child’s drawing \([c]\) of degree \(d\) by the group homomorphism \(\psi : F_2 \to S_d\):

\[
\psi(x) := c_1, \quad \psi(y) := c_2.
\]

So we denote by

\[
\text{Hom}_{\text{tran}}(F_2, S_d)
\]

(2.24)

the set of group homomorphisms \(\psi : F_2 \to S_d\) for which \(\psi(F_2)\) is transitive.

Two homomorphisms \(\psi, \tilde{\psi} \in \text{Hom}_{\text{tran}}(F_2, S_d)\) represent the same child’s drawing if and only if \(\exists h \in S_d\) such that

\[
\tilde{\psi}(w) = h\psi(w)h^{-1}, \quad \forall w \in F_2.
\]

In other words, the set of child’s drawing of degree \(d\) can be identified with the set of orbits

\[
\text{Hom}_{\text{tran}}(F_2, S_d)/S_d
\]

(2.25)

---

\(^{6}\)This list of references is far from complete.
of the $S_d$-action on $\text{Hom}_{\text{tran}}(F_2, S_d)$ by conjugation.

Note that $\ker(\psi)$ depends only on the child’s drawing $[\psi]$ but not on a particular choice of a representative $\psi \in \text{Hom}_{\text{tran}}(F_2, S_d)$.

Child’s drawings can also be represented by continuous group homomorphisms from $\hat{F}_2$ to $S_d$. The goal of the following proposition is to recall this equivalent description:

**Proposition 2.5** Let $\text{Hom}(\hat{F}_2, S_d)$ be the set of continuous group homomorphisms $\hat{F}_2 \to S_d$ and

$$\text{Hom}(\hat{F}_2, S_d)_{\text{tran}} := \{ \hat{\psi} \in \text{Hom}(\hat{F}_2, S_d) \mid \hat{\psi}(\hat{F}_2) \text{ is a transitive subgroup of } S_d \}.$$ 

The assignment

$$\hat{\psi} \mapsto \hat{\psi}|_{F_2}$$

(2.26)

gives us a bijection from $\text{Hom}(\hat{F}_2, S_d)$ (resp. from $\text{Hom}_{\text{tran}}(\hat{F}_2, S_d)$) to $\text{Hom}(F_2, S_d)$ (resp. to $\text{Hom}_{\text{tran}}(F_2, S_d)$). This assignment is compatible with the action of $S_d$ by conjugation and it gives us a bijection between the set

$$\text{Hom}_{\text{tran}}(\hat{F}_2, S_d)_{S_d}$$

(2.27)

of orbits of the $S_d$-action and the set of child’s drawings of degree $d$.

**Proof.** We will prove this proposition by constructing the inverse of the assignment in (2.26).

Let $\psi \in \text{Hom}(F_2, S_d)$ and $K := \ker(\psi)$. The homomorphism $\psi : F_2 \to S_d$ factors as follows

$$\psi = \psi_K \circ \mathcal{P}_K,$$

where the homomorphism $\psi_K : F_2/K \to S_d$ is defined by the formula

$$\psi_K(wK) := \psi(w).$$

(2.28)

We denote by $\hat{\psi} \in \text{Hom}(\hat{F}_2, S_d)$ defined by (2.28) the continuous homomorphism $\hat{\psi} : \hat{F}_2 \to S_d$ defined by the formula

$$\hat{\psi} := \psi_K \circ \hat{\mathcal{P}}_K.$$

(2.29)

We claim that the assignment

$$\psi \mapsto \hat{\psi}$$

gives us the inverse of the map from $\text{Hom}(\hat{F}_2, S_d)$ to $\text{Hom}(F_2, S_d)$ defined in (2.26).

Indeed, $\hat{\psi}|_{F_2}$ clearly coincides with $\psi$. Thus it remains to show that, if

$$\hat{\psi} = \hat{\varphi}|_{F_2}$$

(2.30)

for $\varphi \in \text{Hom}(\hat{F}_2, S_d)$, then $\hat{\psi}$ coincides with $\hat{\varphi}$.

Let $K := \ker(\psi)$ and $\psi_K$ be the homomorphism $F_2/K \to S_d$ defined by $\psi$ in (2.28), where $\psi$ is defined in (2.30). As above, we set $\hat{\psi} := \psi_K \circ \hat{\mathcal{P}}_K$ and observe that

$$\hat{\psi}|_{F_2} = \hat{\varphi}|_{F_2}.$$
Since \( F_2 \) is dense in \( \hat{F}_2 \), \( S_d \) is Hausdorff and \( \hat{\psi}, \hat{\varphi} \) are continuous, we conclude that \( \hat{\psi} = \hat{\varphi} \).

It is easy to see that, for every \( \hat{\psi} \in \text{Hom}(\hat{F}_2, S_d) \), \( \psi(F_2) = \hat{\psi}(\hat{F}_2) \). Moreover the assignments (2.26) and \( \psi \mapsto \hat{\psi} \) are compatible with the (adjoint) action of \( S_d \). Thus the remaining statements of the proposition are obvious.

**Remark 2.6** In the above proof, we showed that every group homomorphism \( \psi : F_2 \to S_d \) extends uniquely to a continuous group homomorphism \( \hat{\psi} : \hat{F}_2 \to S_d \). Equation (2.29) is an explicit definition of this extension. See also [26, Lemma 1.1.16, (b)].

**Remark 2.7** Depending on the context, we will represent a child’s drawing of degree \( d \) by a group homomorphism \( \psi : F_2 \to S_d \) or by a continuous group homomorphism \( \hat{\psi} : \hat{F}_2 \to S_d \). Of course, due to Proposition 2.5, we have

\[
[\hat{\psi}] = [\hat{\psi}|_{F_2}]
\]

for every \( \hat{\psi} \in \text{Hom}_{\text{tran}}(\hat{F}_2, S_d) \).

**Remark 2.8** There is a natural bijection between the set \( \text{Hom}_{\text{tran}}(F_2, S_d)_{S_d} \) and the set of conjugacy classes of index \( d \) subgroups \( H \) in \( F_2 \). This bijection sends \( [\psi] \in \text{Hom}_{\text{tran}}(F_2, S_d)_{S_d} \) to the conjugacy class (in \( F_2 \)) of the subgroup

\[
H_\psi := \{ h \in F_2 \mid \psi(h)(1) = 1 \}.
\]

The inverse of this correspondence operates as follows: given a subgroup \( H \leq F_2 \) of index \( d \), we choose a bijection between the set \( F_2/H \) of left cosets of \( H \) and the set \( \{1, 2, \ldots, d\} \); then the canonical action of \( F_2 \) on \( F_2/H \) gives us a desired group homomorphism \( \psi : F_2 \to S_d \). Thus every child’s drawing of degree \( d \) can be represented by an index \( d \) subgroup in \( F_2 \).

**Remark 2.9** Recall [19, Section 1.3] that conjugacy classes of index \( d \) subgroups in \( F_2 \) are in bijection with connected degree \( d \) coverings of \( \mathbb{C}P^1 \setminus \{0, 1, \infty\} \). We say that a child’s drawing \( [H] \) represented by a finite index subgroup \( H \leq F_2 \) is Galois if the corresponding covering of \( \mathbb{C}P^1 \setminus \{0, 1, \infty\} \) is Galois. Due to [19, Proposition 1.39], the child’s drawing \( [H] \) is Galois if and only if \( H \) is a normal subgroup of \( F_2 \).

Recall that a Belyi pair is a pair \( (X, \gamma) \), where \( X \) is a smooth projective curve defined over \( \overline{\mathbb{Q}} \) and \( \gamma : X \to \mathbb{P}^1_{\overline{\mathbb{Q}}} \) is a morphism of curves unramified outside of the set \( \{0, 1, \infty\} \).

Using [33, Proposition 4.5.13] and [33, Theorem 4.6.10], one can show\(^8\) that isomorphism classes of connected degree \( d \) coverings of \( \mathbb{C}P^1 \setminus \{0, 1, \infty\} \) are in bijection with isomorphism classes of degree \( d \) Belyi pairs. Thus child’s drawings can be also represented by Belyi pairs. The standard action of \( G_Q \) on child’s drawings is often defined in terms of Belyi pairs.

For more details, we refer the reader to books [13], [22], blog [14], and survey [31]. The curious reader may also try to “surf through” the database of Belyi pairs [25].

---

\(^8\)Since the subgroup \( \psi(F_2) \leq S_d \) acts transitively on \( \{1, 2, \ldots, d\} \), the subgroup \( H_{\psi,i} := \{ h \in F_2 \mid \psi(h)(i) = i \} \leq F_2 \) is conjugate to \( H_{\psi} \) in \( F_2 \) for every \( i \in \{1, 2, \ldots, d\} \).

\(^9\)Some mathematicians also like to cite [10].
2.4 The action of $\hat{G}\mathcal{T}$ on child’s drawings and the functor $A : \hat{G}\mathcal{T}_{\text{NFI}} \to \text{Dessin}$

The goal of the following proposition is to recall the action of $\hat{G}\mathcal{T}$ on child’s drawings.

**Proposition 2.10** Let $\hat{T} \in \hat{G}\mathcal{T}$ and $\hat{T}_F$ be the corresponding continuous automorphism of $\hat{F}_2 \leq \hat{PB}_3 = \text{PaB}((12)3,1(23))$. Then the formula

$$\hat{\psi}^\hat{T} := \hat{\psi} \circ \hat{T}_F$$

(2.31)

defines a right action of $\hat{G}\mathcal{T}$ on the set $\text{Hom}(\hat{F}_2,S_d)$. This action descends to the action of $\hat{G}\mathcal{T}$ on $\text{Hom}_{\text{tran}}(\hat{F}_2,S_d)$ and on $\text{Hom}_{\text{tran}}(\hat{F}_2,S_d)s_d$.

**Proof.** It is obvious that, for $\hat{T}^{(1)}, \hat{T}^{(2)} \in \hat{G}\mathcal{T}$,

$$(\hat{T}^{(1)} \circ \hat{T}^{(2)})_F = \hat{T}^{(1)}_F \circ \hat{T}^{(2)}_F.$$  

Thus the first statement of the proposition is obvious.

Since $\hat{T}_F$ is an isomorphism $\hat{F}_2 \cong \hat{F}_2$, the subgroups $\hat{\psi} \circ \hat{T}_F(\hat{F}_2) \leq S_d$ and $\hat{\psi}(\hat{F}_2) \leq S_d$ coincide. Moreover, the resulting action of $\hat{G}\mathcal{T}$ on $\text{Hom}_{\text{tran}}(\hat{F}_2,S_d)$ clearly commutes with the action of $S_d$. Hence the remaining two statements of the proposition are also obvious. \(\square\)

**On the Ihara embedding.** Recall [21] that $G_Q$ injects into the group $\hat{G}\mathcal{T}$

$$G_Q \to \hat{G}\mathcal{T},$$

(2.32)

and the standard action of $G_Q$ on child’s drawings agrees with homomorphism [2.32] and the above action of $\hat{G}\mathcal{T}$ on child’s drawings. For details, please see [21, Proposition 1.6], [21, Theorem 1.7] and [29, Section 3.2]. We will call homomorphism [2.32] the *Ihara embedding*.

For our purposes, it is convenient to describe the action of $\hat{G}\mathcal{T}$ on child’s drawing in terms of a functor from the transformation groupoid $\hat{G}\mathcal{T}_{\text{NFI}}$ to certain category assembled from child’s drawings and elements of the poset $\text{NFI}_{\text{PB}_4}(B_4)$.

**Definition 2.11** Let $N \in \text{NFI}_{\text{PB}_4}(B_4)$ and $[\psi]$ be a child’s drawing of degree $d$ represented by a homomorphism $\psi : F_2 \to S_d$. We say that a child’s drawing $[\psi]$ is *subordinate* to $N$ if

$$N_{F_2} \leq \ker(\psi).$$

(2.33)

If $[\psi]$ is subordinate to $N$, then we say that $N$ *dominates* $[\psi]$.

It is easy to see that condition (2.33) does not depend on the choice of representing homomorphism $\psi$. Moreover, if a child’s drawing is represented by a continuous homomorphism $\hat{\psi} : \hat{F}_2 \to S_d$ then $[\hat{\psi}]$ is subordinate to $N$ if and only if $N_{F_2} \leq \ker(\hat{\psi}|_{F_2})$. We denote by $\text{Dessin}(N)$ the set of child’s drawings subordinate to $N$.

It is clear that, if $K \leq N$ ($K, N \in \text{NFI}_{\text{PB}_4}(B_4)$) and child’s drawing $[\psi]$ is subordinate to $N$ then $[\psi]$ is also subordinate to $K$. In other words, if $K \leq N$, then $\text{Dessin}(N) \subset \text{Dessin}(K)$.

Let us introduce the category $\text{Dessin}$ whose objects are elements of $\text{NFI}_{\text{PB}_4}(B_4)$. For $N^{(1)}, N^{(2)} \in \text{NFI}_{\text{PB}_4}(B_4)$, morphisms from $N^{(1)}$ to $N^{(2)}$ are functions from $\text{Dessin}(N^{(1)})$ to $\text{Dessin}(N^{(2)})$. 

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Proposition 2.12 For every \( \hat{T} \in \hat{G}T \) and \([\hat{\psi}] \in \text{Dessin}(N)\), the child’s drawing represented by the homomorphism
\[
\hat{\psi} \circ \hat{T}|_{F_2} : F_2 \to S_d
\]
is subordinate to \( N^\hat{T} \). Moreover, the formulas
\[
\mathcal{A}(N) := N, \quad \mathcal{A}(\hat{T})([\hat{\psi}]) := [\hat{\psi} \circ \hat{T}]
\]
define a cofunctor
\[
\mathcal{A} : \hat{G}T\text{NFI} \to \text{Dessin}.
\]

Proof. Identifying \( F_2 \) and \( \hat{F}_2 \) with the corresponding subgroups of \( \hat{P}B_3 = \text{PaB}((1, 2)3, (1, 2)3) \) and using (2.17) we see that \( N_{F_2} \) is the kernel of the homomorphism
\[
\hat{P}_{N_{F_2}} \circ \hat{T}|_{F_2} : F_2 \to F_2/N_{F_2}.
\]

Let \( \psi := \hat{\psi}|_{F_2} \). Since \( N_{F_2} \leq \ker(\psi) \), \( \psi \) induces the group homomorphism
\[
\hat{\psi} : F_2/N_{F_2} \to S_d, \quad \hat{\psi}(wN_{F_2}) := \psi(w).
\]

Composing \( \hat{\psi} \) with \( \hat{P}_{N_{F_2}} \) we get a continuous group homomorphism \( \hat{\psi} \circ \hat{P}_{N_{F_2}} : \hat{F}_2 \to S_d \). Since \( \hat{\psi} \circ \hat{P}_{N_{F_2}}|_{F_2} = \hat{\psi}|_{F_2} \), \( F_2 \) is dense in \( \hat{F}_2 \) and \( S_d \) is Hausdorff, we conclude that
\[
\hat{\psi} = \hat{\psi} \circ \hat{P}_{N_{F_2}}.
\]

Hence
\[
\hat{\psi} \circ \hat{T}|_{F_2} = \hat{\psi} \circ (\hat{P}_{N_{F_2}} \circ \hat{T}|_{F_2}).
\]

(2.35) implies that
\[
N_{F_2}^\hat{T} \leq \ker(\hat{\psi} \circ \hat{T}|_{F_2}).
\]

The first statement of the proposition is proved.

Using the statement we just proved and Proposition 2.10 we see that, for all \( N^{(1)}, N^{(2)} \in \text{NFI}_{PB_3}(B_4) \), the second formula in (2.34) defines a map
\[
\hat{G}T\text{NFI}(N^{(1)}, N^{(2)}) \to \text{Dessin}(N^{(2)}, N^{(1)}).
\]

Moreover, since \( \hat{\psi} \circ \text{id}_{F_2} = \hat{\psi} \) and
\[
\hat{\psi} \circ (\hat{T}^{(1)} \circ \hat{T}^{(2)}))|_{F_2} = (\hat{\psi} \circ \hat{T}^{(1)}|_{F_2}) \circ (\hat{\psi} \circ \hat{T}^{(2)}|_{F_2}), \quad \forall \hat{T}^{(1)}, \hat{T}^{(2)} \in \hat{G}T, \hat{\psi} \in \text{Hom}(\hat{F}_2, S_d),
\]
we conclude that the formulas in (2.34) indeed define a cofunctor \( \mathcal{A} \) from the transformation groupoid \( \hat{G}T\text{NFI} \) to the category \( \text{Dessin} \). \( \square \)

Using the Ihara embedding \( G_Q \hookrightarrow \hat{G}T \) and the action of \( \hat{G}T \) on \( \text{NFI}_{PB_3}(B_4) \), we get the action of \( G_Q \) on \( \text{NFI}_{PB_3}(B_4) \). We denote by \( G_{Q,\text{NFI}} \) the corresponding transformation groupoid and by \( \text{Ih} \) the natural functor
\[
\text{Ih} : G_{Q,\text{NFI}} \to \hat{G}T\text{NFI}
\]
coming from the Ihara embedding.

Composing \( \text{Ih} \) with the cofunctor \( \mathcal{A} : \hat{G}T\text{NFI} \to \text{Dessin} \), we get a cofunctor from the groupoid \( G_{Q,\text{NFI}} \) to the category \( \text{Dessin} \). We denote this cofunctor by \( \mathcal{A}^Q \):
\[
\mathcal{A}^Q := \mathcal{A} \circ \text{Ih} : G_{Q,\text{NFI}} \to \text{Dessin}.
\]

(2.37)
The action of GT-shadows on child’s drawings

It is convenient to introduce the action of GT-shadows on child’s drawings as a cofunctor from the groupoid GTSh to the category Dessin:

**Theorem 3.1** Let $N^{(1)}, N^{(2)} \in \text{NFI}_{PB_3}(B_4)$, $[m, f] \in \text{GTSh}(N^{(2)}, N^{(1)})$ and $[\psi] \in \text{Dessin}(N^{(1)})$ be a child’s drawing represented by a homomorphism $\psi : F_2 \to S_d$. Let $\psi^{(m,f)} : F_2 \to S_d$ be the homomorphism defined by the formulas

$$
\psi^{(m,f)}(x) := \psi(x^{2m+1}), \quad \psi^{(m,f)}(y) := \psi(f^{-1}y^{2m+1}f).
$$

Then

- $\psi^{(m,f)}$ does not depend on the choice of the pair $(m, f)$ representing the GT-shadow $[m, f]$;
- $\psi^{(m,f)}$ represents a child’s drawing of degree $d$ subordinate to $N^{(2)}$.

Moreover, the formulas

$$
\mathcal{A}^{sh}(N) := N, \quad \mathcal{A}^{sh}([m, f])([\psi]) := [\psi^{(m,f)}]
$$

define a cofunctor $\mathcal{A}^{sh} : \text{GTSh} \to \text{Dessin}$.

**Proof.** Since $N_{F_2}^{(1)} \leq \ker(\psi)$ the formula

$$
\tilde{\psi}(wN_{F_2}^{(1)}) := \psi(w)
$$

defines a group homomorphism $\tilde{\psi} : F_2/N_{F_2}^{(1)} \to S_d$ and $\tilde{\psi}(F_2/N_{F_2}^{(1)}) = \psi(F_2)$. In particular, the subgroup $\tilde{\psi}(F_2/N_{F_2}^{(1)})$ is transitive.

Due to [2,13],

$$
\psi^{(m,f)} = \tilde{\psi} \circ T_{m,f}^F.
$$

Since the homomorphism $T_{m,f}^F : F_2 \to F_2/N_{F_2}^{(1)}$ does not depend on the choice of the representative $(m, f)$ of the GT-shadow $[m, f]$, the first statement of the proposition is proved.

Since $T_{m,f}^F : F_2 \to F_2/N_{F_2}^{(1)}$ is onto $\psi^{(m,f)}(F_2) = \tilde{\psi}(F_2/N_{F_2}^{(1)})$, hence $\psi^{(m,f)}(F_2)$ is a transitive subgroup of $S_d$.

We know that $\ker(T_{m,f}^F) = N_{F_2}^{(2)}$. Hence [3,3] implies that

$$
N_{F_2}^{(2)} \leq \ker(\psi^{(m,f)}).
$$

We proved the second statement of the proposition.

It is easy to see that $[\psi^{(m,f)}]$ does not depend on the representative $\psi$ of the child’s drawing $[\psi]$. Thus the formula

$$
\mathcal{A}^{sh}([m, f])([\psi]) := [\psi^{(m,f)}]
$$

defines a map $\text{GTSh}(N^{(2)}, N^{(1)}) \to \text{Dessin}(N^{(1)}, N^{(2)})$.

It is also easy to see that, for every $[\psi] \in \text{Dessin}(N^{(1)})$, $[\psi^{(0,1_{F_2})}] = [\psi]$.

Thus it remains to show that, for all $[m_1, f_1] \in \text{GTSh}(N^{(2)}, N^{(1)})$, $[m_2, f_2] \in \text{GT}(N^{(2)})$, and $[\psi] \in \text{Dessin}(N^{(1)})$,

$$
\mathcal{A}^{sh}([m_1, f_1] \circ [m_2, f_2])([\psi]) = [(\psi^{(m_1,f_1)})^{(m_2,f_2)}].
$$

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Equation (3.4) can be verified directly using (2.21).

Here is another way to prove (3.4). Let $[m, f] := [m_1, f_1] \circ [m_2, f_2]$. Since the composition of GT-shadows is defined via the identification of elements of $\text{GT}(N)$ with isomorphisms (2.10), we have

$$T_{m,f} = T_{m_1,f_1} \circ T_{m_2,f_2}.$$  

Hence

$$T_{F_2}^{m,f} = T_{F_2}^{m_1,f_1} \circ T_{F_2}^{m_2,f_2}. \quad (3.5)$$

Since $\psi^{(m,f)} = \tilde{\psi} \circ T_{m,f}^{F_2}$, identity (3.5) implies that

$$\psi^{(m,f)} = (\tilde{\psi} \circ T_{m_1,f_1}^{F_2,\text{isom}}) \circ T_{m_2,f_2}^{F_2}. \quad (3.6)$$

Since $\tilde{\psi} \circ T_{m_1,f_1}^{F_2,\text{isom}} \circ P_{N_{F_2}}^{(2)} = \psi^{(m_1,f_1)}$, identity (3.6) implies the desired equation in (3.4). \(\square\)

The action of GT-shadows on child’s drawings is compatible with the action of $\hat{\text{GT}}$ in the following sense:

**Theorem 3.2** Let $\mathcal{P} \mathcal{R}$ be the natural functor from $\hat{\text{GT}}_{NFI}$ to $\text{GTSh}^{\heartsuit}$ introduced in the proof of Proposition 2.2 and let $\mathcal{A}$ be the cofunctor from $\hat{\text{GT}}_{NFI}$ to Dessin defined in Proposition 2.12. The diagram of (co)functors

$$
\begin{array}{ccc}
\hat{\text{GT}}_{NFI} & \xrightarrow{\mathcal{P} \mathcal{R}} & \text{GTSh}^{\heartsuit} \\
\downarrow \mathcal{A} & & \downarrow \mathcal{A}^{sh} \\
\text{Dessin} & & \\
\end{array}
$$

(3.7)

commutes “on the nose”.

**Proof.** Since all three functors operate as the identity map on the level of objects, we need to show that, for every $N \in \text{NFI}_{PB}^4(B_4)$, $\hat{T} \in \hat{\text{GT}}$ and $[\psi] \in \text{Dessin}(N)$ we have

$$\hat{\psi} \circ \hat{T}_{F_2} = \tilde{\psi} \circ T_{N_{F_2}}^{F_2}, \quad (3.8)$$

where $\hat{\psi} : \hat{F}_2 \to S_d$ is the continuous group homomorphism that extends $\psi : F_2 \to S_d$, and $\tilde{\psi}$ is the group homomorphism $F_2/N_{F_2} \to S_d$ defined by the formula

$$\tilde{\psi}(wN_{F_2}) := \psi(w).$$

Due to the relation defining the GT-shadow $T_N : \text{PaB}^{\leq 4} \to \text{PaB}^{\leq 4}/\sim_N$ in terms of $\hat{T} \in \hat{\text{GT}}$ (see equation (2.17)), the diagram

$$
\begin{array}{ccc}
\hat{F}_2 & \xrightarrow{T_{F_2}} & \hat{F}_2 \\
\downarrow \hat{P}_{F_2} & & \downarrow \hat{P}_{F_2} \\
F_2 & \xrightarrow{T_{F_2}^{N_{F_2}}} & F_2/N_{F_2} \\
\end{array}
$$

(3.9)
commutes.

Using equation (2.29) from the proof of Proposition 2.5 and the relation \( N_{F_2} \leq \ker(\psi) \), it is easy to see that the diagram

\[
\begin{array}{ccc}
\hat{F}_{2} & \xrightarrow{\hat{\psi}} & S_d \\
\uparrow \hat{\psi} & & \\
F_2/N_{F_2} & \xrightarrow{\psi} & S_d
\end{array}
\]

(3.10)

commutes.

Putting (3.9) and (3.10) together, we get the following commutative diagram

\[
\begin{array}{ccc}
\hat{F}_{2} & \xrightarrow{\hat{\psi}} & S_d \\
\uparrow \hat{\psi} & & \\
F_2/N_{F_2} & \xrightarrow{\psi} & S_d
\end{array}
\]

Thus equation (3.8) is proved and Theorem 3.2 follows. \( \square \)

Let us prove that

**Proposition 3.3** For every child’s drawing \( D \), there exists \( N \in NFI_{PB_4}(B_4) \) that dominates \( D \), i.e. \( D \in Dessin(N) \). In fact, for every child’s drawing \( D \), there are infinitely many elements \( K \in NFI_{PB_4}(B_4) \) such that \( D \in Dessin(K) \).

**Proof.** Let \( \psi \) be a group homomorphism from \( F_2 \) to \( S_d \) that represents a child’s drawing \( D \).

The following formulas

\[
\varphi(x_{12}) := \psi(x), \quad \varphi(x_{23}) := \psi(y), \quad \varphi(x_{13}) := \psi(x^{-1}y^{-1}), \quad \varphi(x_{14}) = \varphi(x_{24}) = \varphi(x_{34}) := 1_{S_d}
\]

(3.11)

define a group homomorphism \( \varphi : PB_4 \to S_d \).

Indeed, since \( x_{13} = x_{12}^{-1}x_{23}^{-1}c \), the first three equations in (3.11) define a group homomorphism from \( PB_3 \) to \( S_d \). Since the elements \( x_{14}, x_{24}, x_{34} \) generate a free subgroup of \( PB_4 \) and \( PB_4 \) is isomorphic to the semi-direct product \( PB_3 \rtimes \langle x_{14}, x_{24}, x_{34} \rangle \), the formulas in (3.11) define a group homomorphism \( \varphi : PB_4 \to S_d \).

It is clear that

\[
\ker(\varphi|_{PB_3}) \cap F_2 = \ker(\psi).
\]

(3.12)

Unfortunately, in general, \( \ker(\varphi) \) is not normal in \( B_4 \).

We denote by \( N \) the normal core of \( \ker(\varphi) \) in \( B_4 \). Since \( \ker(\varphi) \) has finite index in \( B_4 \), so does \( N \). In addition, \( N \leq PB_3 \). Thus \( N \in NFI_{PB_4}(B_4) \).

Using the definition of \( N_{PB_3} \) (see equation (2.4) in [8, Section 2.2]) and the inclusion \( N \leq \ker(\varphi) \), we conclude that

\[
N_{PB_3} \leq \ker(\varphi|_{PB_3}).
\]

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Combining this observation with identity (3.12), we conclude that
\[ N \cap F_2 \leq \ker(\psi). \]

Thus \( N \) dominates the child’s drawing \([\psi]\).

To prove the second statement, let us show that, for every \( K \in NFI_{PB_4}(B_4) \) that dominates \([\psi]\), there exists \( \tilde{K} \in NFI_{PB_4}(B_4) \) such that

- \( \tilde{K} \) dominates \([\psi]\) and
- \( \tilde{K} \) is properly contained in \( K \).

Since \( K \) is non-trivial, there exists a non-identity element \( w \in K \). Since \( B_4 \) is residually finite, there exists \( H \in NFI(B_4) \) such that \( w \notin H \). We set
\[ \tilde{K} := H \cap K. \]

It is clear that \( \tilde{K} \in NFI_{PB_4}(B_4) \) and \( \tilde{K} \) is properly contained in \( K \).

Since \( \tilde{K} \leq K \), \( \tilde{K} \) also dominates \([\psi]\). \[\square\]

**Remark 3.4** Although paper [18] uses a more restrictive assumption on the analogue of \( N \in NFI_{PB_4}(B_4) \), a discussion that is very similar to the proof of Proposition 3.3 can be found on page 225 of [18].

**Remark 3.5** Using Belyi pairs, we can construct a natural analogue \( \text{Dessin}_Q \) of the category \( \text{Dessin} \). Objects of \( \text{Dessin}_Q \) are number fields \( K \supset \mathbb{Q} \); for a number field \( K \), we denote by \( \text{Dessin}_Q(K) \) the set of child’s drawings that admit a Belyi pair defined over \( K \); finally, for number fields \( E, K \), the set of morphisms from \( E \) to \( K \) is the set of all maps from \( \text{Dessin}_Q(E) \) to \( \text{Dessin}_Q(K) \). The natural analogue of the groupoid \( \tilde{G}_NFI \) is the transformation groupoid \( G_{numfields}^Q \): objects of \( G_{numfields}^Q \) are number fields and morphisms from \( K \) to \( E \) are elements of \( g \in G_Q \) such that \( g(K) = E \). Clearly, the action of \( G_Q \) on child’s drawings (defined via Belyi pairs) is nothing but a cofunctor from \( G_{numfields}^Q \) to the category \( \text{Dessin}_Q \).

In the same vein, the relation of subordinance from Definition 2.11 is analogous to a natural relationship between child’s drawings and finite Galois extensions of \( \mathbb{Q} \). One can say that a child’s drawing \( D \) is **subordinate** to a finite Galois extension \( E \supset \mathbb{Q} \) if \( D \) can be represented by a Belyi pair \((X, \gamma)\) defined over an intermediate field of the extension \( E \supset \mathbb{Q} \). The direct analogue of the first statement of Proposition 3.3 for this relation is obvious because every finite field extension \( K \supset \mathbb{Q} \) is contained in a finite Galois extension of \( \mathbb{Q} \). The direct analogue of the second statement of Proposition 3.3 is also obvious since the field extension \( \mathbb{Q} \supset \mathbb{Q} \) is infinite.

There may be further interesting parallels between the pairs of categories
\[(\tilde{G}_NFI, \text{Dessin}) \quad \text{and} \quad (G_{numfields}^Q, \text{Dessin}_Q)\]
and it is tempting to explore these parallels.

It is relatively easy to see that, if a child’s drawing \([\psi]\) is Galois, then so is \([\psi]^g\) for every \( g \in G_Q \). The corresponding version of this statement for the action of \( \tilde{G}_T \)-shadows requires a proof:

\[10\] I came up with the category \( \text{Dessin}_Q \) thanks to a suggestion of a diligent referee.
Proposition 3.6 Let $N \in \text{NFI}_{PB_4}(B_4)$, $[\psi] \in \text{Dessin}(N)$ and $[m, f] \in \text{GT}(N)$. If the child’s drawing $[\psi]$ is Galois then so is $[\psi]^{[m,f]}$.

Proof. Let $\psi : F_2 \to S_d$ be a homomorphism that represents the child’s drawing $[\psi]$. Note that $[\psi]$ is Galois if and only if the order of the subgroup $\psi(F_2) \leq S_d$ is $d$. (This is equivalent to the statement that the stabilizer of 1 in $F_2$ coincides with its normal core.)

Let us denote by $\tilde{\psi}$ the homomorphism from $F_2/N_{F_2} \to S_d$ defined by the formula

$$\tilde{\psi}(wN_{F_2}) = \psi(w).$$

The child’s drawing $[\psi]^{[m,f]}$ is represented by the homomorphism

$$\tilde{\psi} \circ T_{m,f}^F : F_2 \to S_d.$$

Since $T_{m,f}^F : F_2 \to F_2/N_{F_2}$ is onto (see Proposition 2.1) and $\tilde{\psi}(F_2/N_{F_2}) = \psi(F_2)$,

$$\tilde{\psi} \circ T_{m,f}^F(F_2) = \psi(F_2). \quad (3.13)$$

Thus the order of the subgroup $\tilde{\psi} \circ T_{m,f}^F(F_2) \leq S_d$ also coincides with the degree $d$. Hence the child’s drawing $[\psi]^{[m,f]}$ is Galois.

The following proposition shows that there is a large supply of Galois child’s drawings whose $\text{GTSh}^\bigcirc$-orbits are singletons:

Proposition 3.7 For every $N \in \text{NFI}_{PB_4}(B_4)$, the Galois child’s drawing $D_N$ represented by $N_{F_2}$ is subordinate to $N$. Moreover, if $N$ is isolated, then the orbit $\text{GT}^\bigcirc(N)(D_N)$ is a singleton.

Proof. We set $d := |F_2 : N_{F_2}|$ and choose a bijection between the set $F_2/N_{F_2}$ of left cosets and $\{1, 2, \ldots, d\}$. Then the standard (left) action of $F_2$ on $F_2/N_{F_2}$ gives us a group homomorphism

$$\psi : F_2 \to S_d$$

which represents the child’s drawing $D_N$.

Since $N_{F_2}$ is normal in $F_2$, $\ker(\psi) = N_{F_2}$. Thus $D_N$ is subordinate to $N$.

For the second statement, we assume that $N$ is isolated.

Let $\psi$ be the homomorphism $F_2/N_{F_2} \to S_d$ defined by the formula

$$\tilde{\psi}(wN_{F_2}) := \psi(w)$$

and let $[m, f] \in \text{GT}^\bigcirc(N)$.

The (new) child’s drawing $D_N^{[m,f]}$ is represented by the homomorphism

$$\psi^{[m,f]} := \tilde{\psi} \circ T_{m,f}^F : F_2 \to S_d.$$ 

Since $\ker(\psi) = N_{F_2}$, the homomorphism $\tilde{\psi}$ is injective.

Since $N$ is isolated, the kernel of the morphism $T_{m,f} : \text{PaB}^{\leq 4} \to \text{PaB}^{\leq 4}/\sim_N$ is the compatible equivalence relation corresponding to $N$. Hence $\ker(T_{PB_4}^{PB_3}) = N_{PB_3}$ and therefore $\ker(T_{m,f}^{F_2}) = N_{F_2}$.

Combining this observation with the injectivity of $\tilde{\psi}$, we conclude that

$$\ker(\psi^{[m,f]}) = N_{F_2}.$$
Due to Proposition 3.6, the child’s drawing \(\psi(m,f)\) is Galois.

To show that \(\psi(m,f)\) is represented by the subgroup \(\ker(\psi(m,f)) = N_{F_2}\), we consider the action of \(F_2\) on \(\{1, 2, \ldots, d\}\) corresponding to the homomorphism \(\psi(m,f) : F_2 \to S_d\). Since \(\psi(m,f)\) is Galois the stabilizer \(\text{Stab}_{F_2}(j)\) of \(j\) coincides with the kernel of \(\psi(m,f)\) for every \(1 \leq j \leq d\).

Thus \(\psi(m,f)\) is represented by the same subgroup \(N_{F_2}\) and the proposition is proved. \(\Box\)

3.1 The hierarchy of orbits

Combining Theorem 3.2 with the definition of the cofunctor \(\mathcal{A}^{\mathcal{Q}}\) (see (2.37)), we get the following statement:

**Corollary 3.8** The diagram of (co)functors

\[
\begin{array}{c}
G_{Q,NFI} \\
\text{lh} \quad \widehat{GT}_{NFI} \quad \mathcal{P} \quad GTSh^{\mathcal{Q}} \\
\mathcal{A}^{\mathcal{Q}} \quad \mathcal{A}^h \\
\text{Dessin}
\end{array}
\]

(3.14)

commutes “on the nose”. \(\Box\)

Let \(K, N \in \text{NFI}_{PB_4}(B_4)\) and \(K \leq N\). Recall [8 Section 3.2] that, if a pair \((m, f) \in \mathbb{Z} \times F_2\) represents a charming GT-shadow with the target \(K\), then the same pair \((m, f)\) also represents a charming GT-shadow with the target \(N\). Hence, for every pair \(K, N \in \text{NFI}_{PB_4}(B_4)\) with \(K \leq N\), we have a natural map

\[P_{K,N} : \text{GT}^{\mathcal{Q}}(K) \to \text{GT}^{\mathcal{Q}}(N).\] (3.15)

Let \(\psi\) be a homomorphism \(F_2 \to S_d\) that represents a child’s drawing subordinate to \(N\). Since \(K \leq N\), \([\psi]\) is also subordinate to \(K\).

Let \([m, f] \in \text{GT}^{\mathcal{Q}}(K)\). Since \([\psi]^{[m,f]}\) depends only on the residue class of \(m\) modulo \(\text{lcm}(\text{ord}(x \ker(\psi)), \text{ord}(y \ker(\psi)))\) and the coset \(f \ker(\psi) \in F_2 / \ker(\psi)\), we have

\[[\psi]^{[m,f]} = [\psi]^{P_{K,N}([m,f])}.\] (3.16)

Combining this observation with Corollary 3.8 we get the following statement:

**Corollary 3.9** Let \(K, N\) be elements of \(\text{NFI}_{PB_4}(B_4)\) that dominate a child’s drawing \(D\) and \(K \leq N\). Then we have the following hierarchy of orbits:

\[\text{GT}^{\mathcal{Q}}(N)(D) \supset \text{GT}^{\mathcal{Q}}(K)(D) \supset \widehat{GT}(D) \supset G_{Q}(D).\] (3.17)

\(\Box\)

**Remark 3.10** There may be examples of pairs \(K, N \in \text{NFI}_{PB_4}(B_4)\) with \(K \leq N\) such that the natural map \(P_{K,N}\) is not onto. So, in principle, there may be examples of \(K, N \in \text{NFI}_{PB_4}(B_4)\) with \(K \leq N\) and \(D \in \text{Dessin}(N)\) for which the inclusion

\[\text{GT}^{\mathcal{Q}}(N)(D) \supset \text{GT}^{\mathcal{Q}}(K)(D)\]
is proper. At the time of writing, the author did not find any examples of this kind. In fact, for all examples in which both orbits $\text{GT}^\triangledown(N)(D)$ and $G_Q(D)$ can be computed, we have $\text{GT}^\triangledown(N)(D) = G_Q(D)$.

**Remark 3.11** Since the orbit $\hat{\text{GT}}(D)$ is finite and $\hat{\text{GT}}$ is the limit of the functor that sends $N \in \text{NFI}_{\text{PB}_4}(B_4)$ to the finite group $\text{GT}^\triangledown(N)$ (see [8, Theorem 3.8]), for every child’s drawing $D$, there exists $N \in \text{NFI}_{\text{PB}_4}(B_4)$ such that

- $D \in \text{Dessin}(N)$, i.e. $N$ dominates $D$, and
- $\hat{\text{GT}}(D) = \text{GT}^\triangledown(N)(D)$.

See also, the Corollary and the Problem on page 227 of [18].

Let $D$ be a child’s drawing and $H_D$ be the stabilizer of $D$ in $G_Q$. Recall that the **field of moduli** of a child’s drawing $D$ is the (finite) extension $M_D \supset Q$ corresponding to the (closed) subgroup $H_D \leq G_Q$ (via the Galois correspondence). Since $[M_D : Q] = |G_Q : H_D|$, the orbit-stabilizer theorem and Corollary [3.9] imply the following version of [18, Proposition 14]:

**Corollary 3.12** For every child’s drawing $D$ and every $N \in \text{NFI}_{\text{PB}_4}(B_4)$ that dominates $D$, we have

$$[M_D : Q] \leq |\text{GT}^\triangledown(N)(D)|.$$  \hspace{1cm} (3.18)

Let us also recall [4, Proposition 2.5], [30, Corollary on page 2] that, if the child’s drawing $D$ is Galois then it admits a Belyi pair $(X, \gamma)$ defined over its field of moduli $M_D$. Combining this observation with Corollary (3.9), we conclude that if $D$ is a Galois child’s drawing and $\text{GT}^\triangledown(N)(D)$ is a singleton (for some $N$ that dominates $D$), then $D$ admits a Belyi pair $(X, \gamma)$ defined over $Q$.

A large supply of examples of such Galois child’s drawings comes from isolated elements of $\text{NFI}_{\text{PB}_4}(B_4)$. Indeed, combining [4, Proposition 2.5], [30, Corollary on page 2] with Proposition [3.7] and Corollary [3.9], we get the following statement:

**Corollary 3.13** For every isolated element $N \in \text{NFI}_{\text{PB}_4}(B_4)$, the child’s drawing represented by the subgroup $N_{F_2} \trianglelefteq F_2$ admits a Belyi pair defined over $Q$. \hspace{1cm} □

**Remark 3.14** Section 4 of [8] presents the basic information about 35 selected elements $N^{(0)}, N^{(1)}, \ldots, N^{(34)}$ (3.19) of the poset $\text{NFI}_{\text{PB}_4}(B_4)$. More information about these elements can be found in [7]. According to Table 1 (on page 40) in [8], 27 of these 35 elements are isolated. Due to Corollary 3.13 for every isolated element $N$ in list (3.19), the child’s drawing represented by $N_{F_2}$ admits a Belyi pair defined over $Q$. Moreover, for many of these isolated elements, the degrees of the corresponding child’s drawings are quite large. For example, the degree of the child’s drawing represented by $N^{(34)}_{F_2}$ is $20575296 = 2^6 \cdot 3^8 \cdot 7^2$. 

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Remark 3.15 Proposition 3.7, Remark 3.14 and examples considered in Section 5 indicate that the size of the orbit \( \text{GT}^{\triangleright}(\mathcal{N})(\mathcal{D}) \) (for some \( \mathcal{N} \) dominating a child’s drawing \( \mathcal{D} \)) may be significantly smaller than the number of the child’s drawings with the same passport as \( \mathcal{D} \). Thus the bound on the degree of the field of moduli (3.18) has more practical value than the one given in [31, Proposition 7.1].

3.2 The monodromy group and the passport are invariant with respect to the action of \( \text{GT} \)-shadows

It is easy to see that the degree and the monodromy group of a child’s drawing are invariant with respect to the action of \( \text{GT} \)-shadows. (See, for example, (3.13) in the proof of Proposition 3.6).

Let us prove that the passport of a child’s drawing is invariant with respect to the action of charming \( \text{GT} \)-shadows.

**Theorem 3.16** Let \( \mathcal{N} \in \text{NFI}_{\text{PB}}(\mathcal{B}_4) \) and \( \psi \) be a homomorphism \( \mathbb{F}_2 \to S_d \) that represents \( [\psi] \in \text{Dessin}(\mathcal{N}) \). Then, for every charming \( \text{GT} \)-shadow \([m, f]\) in \( \text{GT}(\mathcal{N}) \), the child’s drawing \([\psi][m, f]\) has the same passport as \([\psi]\).

**Proof.** We set

\[
\begin{align*}
x &:= x_{12}, \\
y &:= x_{23}, \\
z &:= (xy)^{-1}, \\
w &:= (yx)^{-1},
\end{align*}
\]

and

\[
\begin{align*}
g_x := \psi(x),

 g_y := \psi(y), \\
g_z := \psi(z).
\end{align*}
\]

Due to (3.1), the child’s drawing \([\psi][m, f]\) is represented by the triple:

\[
(g_x^{2m+1}, \psi(f)^{-1}g_y^{2m+1}\psi(f), \psi(f)^{-1}g_y^{-2m-1}\psi(f)g_x^{-2m-1}).
\] (3.20)

The passport of the child’s drawing \([\psi]\) is the triple of partitions \((\text{ct}(g_x), \text{ct}(g_y), \text{ct}(g_z))\).

Thus our goal is to show that

\[
\text{ct}(g_x^{2m+1}) = \text{ct}(g_x), \\
\text{ct}(\psi(f)^{-1}g_y^{2m+1}\psi(f)) = \text{ct}(g_y),
\] (3.21)

and

\[
\text{ct}(\psi(f)^{-1}g_y^{-2m-1}\psi(f)g_x^{-2m-1}) = \text{ct}(g_z).
\] (3.22)

It is clear that the second equation in (3.21) is equivalent to \( \text{ct}(g_y^{2m+1}) = \text{ct}(g_y) \). Thus equations (3.21) are consequences of the following simple fact about permutations: if

\[
\gcd(q, \text{ord}(h)) = 1,
\]

then the permutations \( h^q \) and \( h \) have the same cycle structure.

The integer \( 2m+1 \) is coprime with the orders of \( g_x := \psi(x) \) and \( g_y := \psi(y) \) because \( 2m+1 \) is coprime with the orders of \( x_{12}\mathbb{F}_2 \) and \( x_{23}\mathbb{F}_2 \).

The proof of equation (3.22) requires more work.

Since the \( \text{GT} \)-shadow \([m, f]\) is charming, we may assume, without loss of generality, that \( f \in [\mathbb{F}_2, \mathbb{F}_2] \). Hence Proposition A.3 from Appendix A implies that the pair \((m, f)\) satisfies relations (A.9) and (A.10).
Conjugating (A.10) by \(x^{-m}\), we get
\[
f(z, x)z^m f(y, z)y^m f(x, y)x^m \in \mathbb{N}_{F_2}.
\] (3.23)

Furthermore, conjugating (3.23) by \(\theta := \sigma_1 \sigma_2 \sigma_1\), we get
\[
f(w, y)w^m f(x, w)x^m f(y, x)y^m \in \mathbb{N}_{F_2}.
\] (3.24)

Equation (A.10) implies that
\[
y^m f(x, y) \mathbb{N}_{F_2} = f(y, z)^{-1}z^m f(z, x)^{-1}x^{-m} \mathbb{N}_{F_2}
\] (3.25)

Equation (3.24) implies that
\[
f(y, x)y^m \mathbb{N}_{F_2} = x^{-m} f(x, w)^{-1}w^{-m} f(w, y)^{-1} \mathbb{N}_{F_2}
\] (3.26)

To prove equation (3.22), we need to show that the permutations
\[
\psi(xy) \quad \text{and} \quad \psi(x^{2m+1}y^{-m+1}f)
\] (3.27)

have the same cycle structure.

We have
\[
x^{2m+1} f^{-1} y^{2m+1} f \mathbb{N}_{F_2} = x^{2m+1} f(y, x)y^{2m+1} f(x, y) \mathbb{N}_{F_2} = x^{2m+1}(f(y, x)y^m)y(y^m f(x, y)) \mathbb{N}_{F_2}.
\]

Applying (3.25) to \(y^m f(x, y)\) and (3.26) to \(f(y, x)y^m\) we get\(^\text{11}\)
\[
x^{2m+1} f^{-1} y^{2m+1} f \mathbb{N}_{F_2} = x^{2m+1} f(x, w)^{-1}w^{-m} f(w, y)^{-1} y f(y, z)^{-1}z^m f(z, x)^{-1}x^{-m} \mathbb{N}_{F_2}
\]
\[
\sim x f(x, w)^{-1} W^{-m} f(w, y)^{-1} y f(y, z)^{-1}z^m f(z, x)^{-1} \mathbb{N}_{F_2}
\]
\[
= x f(x, w)^{-1} W^{-m} f(w, y) y f(y, z)z^{-m} f(z, x) \mathbb{N}_{F_2}.
\]

Applying the obvious relations \(xwx^{-1} = z, yzy^{-1} = w\) to \(x f(x, w)^{-1} W^{-m} f(w, y) y f(y, z)z^{-m} f(z, x) \mathbb{N}_{F_2}\), we see that, up to conjugation in \(F_2/\mathbb{N}_{F_2}\),
\[
x^{2m+1} f^{-1} y^{2m+1} f \mathbb{N}_{F_2} \sim f(x, z)^{-1} z^{-m} x f(y, z)^{-1}z^m f(x, z) \mathbb{N}_{F_2} =
\]
\[
f(x, z)^{-1} z^{-m} x f(y, z)^{-1}z^m f(x, z) \mathbb{N}_{F_2} \sim (xy)^{2m+1} \mathbb{N}_{F_2}.
\]

Hence the permutations \(\psi(x^{2m+1} f^{-1} y^{2m+1} f)\) and \(\psi(xy)^{2m+1}\) have the same cycle structure.

To prove that the permutations in (3.27) have the same cycle structure, it remains to show that the cycle structure of \(\psi(xy)^{2m+1}\) coincides with the cycle structure of \(\psi(xy)\). It suffices to show that the order of \(xy\mathbb{N}_{PB_3}\) divides \(N_{ord}\).

Since \(xy\mathbb{N}_{PB_3} = z^{-1} \mathbb{N}_{PB_3}\), we need to show that the order of \(z \mathbb{N}_{PB_3}\) divides \(N_{ord}\).

Since \(x_{23}^{N_{ord}} \in \mathbb{N}_{PB_3}\) and \(\sigma_1 x_{23}^{N_{ord}} = cz_{23}^{N_{ord}} x_{12}^{N_{ord}} = cz\), we have
\[
c_{N_{ord}} z_{N_{ord}} = (cz)^{N_{ord}} \in \mathbb{N}_{PB_3},
\]

Combining this observation with \(c_{N_{ord}} \in \mathbb{N}_{PB_3}\), we conclude that \(z_{N_{ord}} \mathbb{N}_{PB_3} = 1\). Hence the order of \(z \mathbb{N}_{PB_3}\) divides \(N_{ord}\).

Since the permutations in (3.27) have the same cycle structure, (3.22) is proved. \(\Box\)

\(^{11}\)In these calculations, \(\sim\) means “conjugate in \(F_2/\mathbb{N}_{F_2}\)”.\]
4 Abelian child’s drawings

Recall that the monodromy group of a child’s drawing \([\psi]\) of degree \(d\) is defined up to conjugation in \(S_d\). It is clear that the following definition does not depend on the choice of the representative of the monodromy group of a child’s drawing.

**Definition 4.1** A child’s drawing \(D\) is called **Abelian** if its monodromy group is Abelian.

For example, for every \(d \geq 1\), the pair \(((1,2,\ldots,d),(1,2,\ldots,d))\) represents an Abelian child’s drawing of degree \(d\).

Let us prove that

**Proposition 4.2** Every Abelian child’s drawing \([g_1,g_2]\) is Galois. In particular, the order of the monodromy group \(G := \langle g_1, g_2 \rangle\) of \([g_1,g_2]\) coincides with the degree of \([g_1,g_2]\).

**Proof.** Let \(\psi : F_2 \to S_d\) be the group homomorphism corresponding to \((g_1, g_2)\), \(K := \ker(\psi)\) and \(H\) be the stabilizer of 1 in \(F_2\). Note that, since the action of \(F_2\) on \(\{1,2,\ldots,d\}\) is transitive, \(d = |G : H|\).

Since the quotient \(F_2/K \cong G\) is Abelian, every subgroup \(N \leq F_2\) that contains \(K\) is normal in \(F_2\). In particular, \(H\) is normal in \(F_2\). Thus the child’s drawing \([\psi]\) is Galois.

Since \(K\) is the normal core of \(H\) in \(F_2\) and \(H\) is normal, we conclude that \(H = K\). Thus the order of the monodromy group of \([g_1,g_2]\) is \(|G : H| = d|\). \(\square\)

Using the basic properties of group actions, we can prove the following useful properties of Abelian child’s drawings:

**Proposition 4.3** Let \([g_1,g_2]\) be an Abelian child’s drawing of degree \(d\). Then

a) For every \(i \in \{1,2\}\), the permutation \(g_i\) is a product of disjoint cycles of the same length.

b) If \(g_1\) (resp. \(g_2\)) is a cycle of length \(d\) then \(g_2 \in \langle g_1 \rangle\) (resp. \(g_1 \in \langle g_2 \rangle\)).

**Proof.** As above, we set \(G := \langle g_1, g_2 \rangle\).

For a), it suffices to prove that \(g_1\) is a product of disjoint cycles of the same length.

Let \(H := \langle g_1 \rangle\) and

\[
\bigcup_{t=1}^{r} O_t
\]

be the partition of the set \(\{1,2,\ldots,d\}\) into the orbits of the action of \(H_1\). If \(r = 1\), i.e. \(g_1\) is a cycle of length \(d\), then there is nothing to prove. So we consider the case when the number of orbits \(r\) is \(\geq 2\). Our goal is to prove that all orbits have the same size.

Since \(g_2\) commutes with \(g_1\), the subgroup \(H_2 := \langle g_2 \rangle\) acts on the set of orbits \(\{O_1, \ldots, O_r\}\). Furthermore, since the permutation group \(G\) is transitive, \(H_2\) acts on \(\{O_1, \ldots, O_r\}\) transitively.

Therefore, for two distinct orbits \(O\) and \(\tilde{O}\), there exists \(k \in \mathbb{Z}_{\geq 1}\) such that \(g_2^k(i) \in \tilde{O}\) for every \(i \in O\), i.e. \(g_2^k \mid_{O}\) is a map

\[
O \to \tilde{O}.
\] (4.1)

Since \(g_2^{-k} \mid_{\tilde{O}}\) is the inverse of (4.1), we conclude that \(O\) and \(\tilde{O}\) have the same size. Thus the first statement is proved.
The second statement is a particular case of one of the exercises in [10, Section 4.3]. Here is a proof of this statement.
Without loss of generality, we may assume that \( g_1 = (1,2,\ldots,d) \).
If \( g_2(1) = 1 \) then, for every \( k \in \{2,3,\ldots,d\} \), we have
\[
g_2(k) = g_2(g_1^{k-1}(1)) = g_1^{k-1}(g_2(1)) = g_1^{k-1}(1) = k.
\]
Thus, in this case, \( g_2 = \text{id} \).
If \( g_2(1) = k > 1 \) then the permutation \( h := g_1^{1-k}g_2 \) satisfies these two properties:
- \( h \) commutes with \( g_1 \) and
- \( h(1) = 1 \).
Using the previous argument, we conclude that \( h = \text{id} \) and hence \( g_2 \in \langle g_1 \rangle \). \( \Box \)

**Remark 4.4** Assume that we are in the set-up of Proposition 4.3 and \( \{1,2,\ldots,d\} \) partitions into \( r \geq 2 \) orbits of the \( \langle g_1 \rangle \)-action. Although, \( \langle g_2 \rangle \) acts transitively on the set of these orbits, in general, \( g_2 \) is not a product of cycles of length \( r \). For example, the permutations
\[
g_1 = (1,2,3,4)(5,6,7,8)(9,10,11,12), \quad g_2 = (1,6,12,3,8,10)(2,7,9,4,5,11)
\]
generate an Abelian and transitive subgroup of \( S_{12} \).

**Remark 4.5** Since every Abelian child’s drawing is Galois, Proposition 4.3 can be deduced from the properties of Galois branched coverings (see, for example, [33, Proposition 3.2.10]). Of course, it is strange to use Riemann’s existence theorem [17] for a statement that can be proved using basic properties of group actions.

### 4.1 GTSh-orbit of an Abelian child’s drawing is a singleton

Let us prove the following auxiliary statement:

**Proposition 4.6** Suppose that \( (g_1, g_2) \in S_d \times S_d \) generate an Abelian and transitive subgroup \( G \leq S_d \). Then, for every \( r \in \mathbb{Z} \) such that \( \gcd(r, \operatorname{ord}(g_1)) = \gcd(r, \operatorname{ord}(g_2)) = 1 \), there exists \( h \in S_d \) such that
\[
g_1^r = h g_1 h^{-1} \quad \text{and} \quad g_2^r = h g_2 h^{-1}.
\]

**Proof.** Recall that, due to Proposition 4.2, the group \( G \) has order \( d \). The key fact that is used in the proof of the desired statement is that any \( G \)-set of size \( d \) with a transitive action of \( G \) is isomorphic to the \( G \)-set \( G \) with the standard \( G \)-action by multiplication.

Let us show that \( r \) is coprime to \( \operatorname{ord}(w) \) for every \( w \in G \).
Since \( r \) is coprime to \( \operatorname{ord}(g_1) \) and \( \operatorname{ord}(g_2) \), \( r \) is coprime to \( \operatorname{ord}(g_1^{t_1}) \) and \( \operatorname{ord}(g_2^{t_2}) \) for all integers \( t_1 \) and \( t_2 \).
Since \( G = \langle g_1, g_2 \rangle \) is Abelian, every \( w \in G \) can be written in the form
\[
w = g_1^{t_1} g_2^{t_2}.
\]
Let \( k_1 := \operatorname{ord}(g_1^{t_1}) \) and \( k \) be the least common multiple of \( k_1 \) and \( k_2 \). Since \( r \) is coprime to \( k_1 \) and \( k_2 \), \( r \) is also coprime to \( k \).
Since $w^k = \text{id}$, we conclude that $\text{ord}(w)|k$. Thus we proved that $r$ is coprime to $\text{ord}(w)$ for every $w \in G$.

Since the group $G = \langle g_1, g_2 \rangle$ is Abelian, the formula
\[ \theta(w) := w^r, \quad w \in G \]
defines a group endomorphism $\theta : G \to G$.

Moreover, since $r$ is coprime to $\text{ord}(w)$ for every $w \in G$, $w^r = \text{id}$ if and only if $w = \text{id}$. Hence the homomorphism $\theta$ is injective. Since the group $G$ is finite, the injectivity of $\theta$ implies its surjectivity. Thus $\theta$ is actually an automorphism of $G$.

Precomposing the inclusion homomorphism $G \to S_d$ with $\theta$, we get the new action $a$ of $G$ given by the formula
\[ a(w)(i) := w^r(i). \]

Since $\theta$ is an automorphism, the new action of $G$ on $\{1, 2, \ldots, d\}$ is also transitive. Since $G$ has order $d$ and the new action of $G$ is transitive, we conclude that the new $G$-set is isomorphic to the original one.

The proposition is proved. \qed

Let us use Proposition \ref{prop:automorphism} to prove that the orbit of every Abelian child’s drawing (with respect to the action of GT-shadows) is a singleton:

**Corollary 4.7** Let $[(g_1, g_2)]$ be an Abelian child’s drawing of degree $d$ and $N \in NF_{PB_4}(B_4)$ such that $[(g_1, g_2)]$ is subordinate to $N$. For every $[m, f] \in GT(N)$, we have
\[ [(g_1, g_2)]^{[m, f]} = [(g_1, g_2)]. \]

**Proof.** Let $\psi : F_2 \to \langle g_1, g_2 \rangle$ be the homomorphism that sends $x_{12}$ (resp. $x_{23}$) to $g_1$ (resp. $g_2$). Furthermore, let $h := \psi(f)$.

We know that $[(g_1, g_2)]^{[m, f]}$ is represented by the pair
\[ (g_1^{2m+1}, h^{-1}g_2^{2m+1}h). \]

Since the group $\langle g_1, g_2 \rangle$ is Abelian, $h^{-1}g_2^{2m+1}h = g_2^{2m+1}$. Hence the child’s drawing $[(g_1, g_2)]^{[m, f]}$ is represented by the pair
\[ (g_1^{2m+1}, g_2^{2m+1}). \]

Since $g_1$ (resp. $g_2$) is the image of $x_{12}N_{F_2}$ (resp. $x_{23}N_{F_2}$) and $\text{ord}(x_{12}N_{F_2}) = \text{ord}(x_{12}N_{PB_3})$, $\text{ord}(x_{23}N_{F_2}) = \text{ord}(x_{23}N_{PB_3})$, we have $\text{ord}(g_1)|\text{ord}(x_{12}N_{PB_3})$ and $\text{ord}(g_2)|\text{ord}(x_{23}N_{PB_3})$.

Hence, using the fact that $N_{\text{ord}}$ is a multiple of the orders $\text{ord}(x_{12}N_{PB_3})$, $\text{ord}(x_{23}N_{PB_3})$ and $2m+1$ is coprime with $N_{\text{ord}}$, we conclude that $2m+1$ is coprime with the integers $\text{ord}(g_1)$ and $\text{ord}(g_2)$.

Thus, applying Proposition \ref{prop:automorphism} we conclude that the pairs $(g_1^{2m+1}, g_2^{2m+1})$ and $(g_1, g_2)$ represent the same child’s drawing. \qed

Combining \cite[Proposition 2.5]{[1]} and \cite[Corollary on page 2]{[30]} with Proposition \ref{prop:automorphism} and Corollaries \ref{cor:orbital}, \ref{cor:1}, we deduce the following statement:

**Corollary 4.8** Every Abelian child’s drawing $D$ admits a Belyi pair $(X, \gamma)$ defined over $\mathbb{Q}$.

**Remark 4.9** As far as the author understands, the statement of the above corollary is a consequence of \cite[Corollary 3.4]{[5]} and it was proved directly in paper \cite{[20]} by R.A. Hidalgo (see \cite[Corollary 3.5]{[20]}).
5 Examples of GTSh\(\circ\)-orbits for (non-Abelian) child’s drawings

5.1 An example of degree 6, genus 0 and orbit size 2

Let us denote by \(D_{6,0}\) the degree 6 child’s drawing represented by triple

\[
((1, 4, 5, 2)(3, 6), (1, 6, 3, 2)(4, 5), ((1, 3), (2, 4))).
\] (5.1)

The passport of \(D_{6,0}\) is

\[
(((4, 2), (4, 2), (2, 2, 1, 1))
\]

and its genus is 0. A Belyi map representing \(D_{6,0}\) can be found at [https://beta.lmfdb.org/Belyi/6T10/4.2/](https://beta.lmfdb.org/Belyi/6T10/4.2/).

The \(G_{\mathbb{Q}}\)-orbit of \(D_{6,0}\) has size 2 and the Galois conjugate of \(D_{6,0}\) is represented by the permutation triple

\[
((1, 4, 5, 2)(3, 6), (1, 2, 5, 6)(3, 4), (3, 5)(4, 6)).
\] (5.2)

Using [7], we found an element \(N \in \text{NFI}_{PB_4}(B_4)\) that dominates \(D_{6,0}\). \(N\) is the kernel of a group homomorphism \(PB_4 \to S_{192}\) stored in the file \(E_{de6genus0}\) (see [7, Section 6]). Here are some basic facts about this element of \(\text{NFI}_{PB_4}(B_4)\):

- \(|PB_4 : N| = 289, 207, 845, 356, 544 = 2^{10} \cdot 3^{24};\)
- \(|PB_3 : N_{PB_3}| = 46656 = 2^6 \cdot 3^6;\)
- \(|F_2 : N_{F_2}| = 11664 = 2^4 \cdot 3^6;\)
- the order of the commutator subgroup \([F_2/N_{F_2}, F_2/N_{F_2}]\) is \(729 = 3^6;\)
- \(N_{\text{ord}} := |PB_2 : N_{PB_2}| = 4;\)
- \(N\) is isolated and \(\text{GT}^{\circ}(N)\) is a non-Abelian group of order \(32 = 2^5.\)

The orbit \(\text{GT}^{\circ}(N)(D_{6,0})\) coincides with the \(G_{\mathbb{Q}}\)-orbit of \(D_{6,0}\).

The interesting feature of \(D_{6,0}\) is that the \(\text{GT}\)-shadow \([3, 1_{F_2}] \in \text{GT}^{\circ}(N)\) corresponding to the complex conjugation acts trivially on \(D_{6,0}\). A \(\text{GT}\)-shadow that transforms \(D_{6,0}\) to its Galois conjugate is represented by the pair

\[
(1, yxyx^2y^2x^{-3}y^{-4}) \in \mathbb{Z} \times [F_2, F_2].
\]

In fact, there are \(\text{GT}\)-shadows that transforms \(D_{6,0}\) to its Galois conjugate and belong to the kernel of the virtual cyclotomic character. Here is an example of such a \(\text{GT}\)-shadow

\[
[0, yxy^3xy^2x^{-6}y^{-6}] \in \text{GT}^{\circ}(N).
\]

5.2 An example of degree 5, genus 0 and orbit size 2

Let \(D_{5,0}\) be the child’s drawing of degree 5 represented by the permutation triple

\[
((1, 4, 5, 2), (2, 3, 5, 4), (1, 4)(2, 3)).
\] (5.3)

The passport of \(D_{5,0}\) is

\[
(((4, 1), (4, 1), (2, 2, 1))
\]
and its genus is 0. A Belyi map representing $D_{5,0}$ can be found at https://beta.lmfdb.org/Belyi/5T3/4. The $G_Q$-orbit of $D_{5,0}$ has size 2 and the Galois conjugate of $D_{5,0}$ is represented by the permutation triple

$$(1, 2, 5, 4), (1, 5, 2, 3), (1, 4)(2, 3).$$

(5.4)

Using [7], we found an element $N \in NFI_{PB_4}(B_4)$ that dominates $D_{5,0}$. $N$ is the kernel of a group homomorphism $PB_4 \to S_{160}$ stored in the file $E_{jde}5genus0$ (see [7, Section 6]).

Here are some basic facts about this element of $NFI_{PB_4}(B_4)$:

• $|PB_4 : N| = 250000000000 = 2^{10} \cdot 5^{12};$
• $|PB_3 : N_{PB_3}| = 8000 = 2^6 \cdot 5^3;$
• $|F_2 : N_{F_2}| = 2000 = 2^4 \cdot 5^3;$
• the order of the commutator subgroup $[F_2/N_{F_2}, F_2/N_{F_2}]$ is $125 = 5^3;$
• $N_{ord} := |PB_2 : N_{PB_2}| = 4;$
• $N$ is not settled and the connected component of $N$ in $GTSh^\vartriangleleft$ has exactly two objects; the size of $GT^\vartriangleleft(N)$ is 16.

The orbits $GT^\vartriangleleft(N)(D_{5,0})$ and $G_Q(D_{5,0})$ coincide.

5.3 Examples of child’s drawing subordinate to $N^{(5)}$ and $N^{(29)}$ from list (3.19)

Recall that [8, Section 4] presents basic information about 35 selected elements $N^{(0)}, \ldots, N^{(34)}$ of $NFI_{PB_4}(B_4)$. In this subsection, we describe child’s drawings of degrees 7, 15 and 18 subordinate to $N^{(29)}$ and a child’s drawing of degree 8 subordinate to $N^{(5)}$.

Here is the basic information about $N^{(29)}$:

• $|PB_4 : N^{(29)}| = 2520 = 2^3 \cdot 3^2 \cdot 5 \cdot 7;$
• $|PB_3 : N_{PB_3}^{(29)}| = 136080 = 2^4 \cdot 3^5 \cdot 5 \cdot 7;$
• $|F_2 : N_{F_2}^{(29)}| = 45360 = 2^4 \cdot 3^4 \cdot 5 \cdot 7;$
• $N_{ord}^{(29)} = 6;$
• $N^{(29)}$ is an isolated object of $NFI_{PB_4}(B_4)$ and $GT^\vartriangleleft(N^{(29)})$ is non-Abelian group of order $48 = 2^4 \cdot 3$.

The child’s drawings represented by the following permutation triples

$$(1, 2, 3)(4, 5)(6, 7), (1, 5, 6)(2, 7)(3, 4), (1, 4)(2, 6)(3, 7, 5),$$

(5.5)

$$(1, 2, 3, 4, 5, 6)(7, 8, 9, 10, 11, 12)(13, 14, 15),$$

(1, 2, 6, 12, 9, 15)(3, 7, 13)(4, 11, 14, 5, 8, 10),

(1, 2, 15, 11, 8, 3)(4, 13, 9, 5, 10, 12)(6, 14, 7))

(5.6)
are subordinate to \( N^{(20)} \). We denote by \( D_{7,0} \) (resp. \( D_{15,4}, D_{18,4} \)) the child’s drawing represented by the permutation triple in \((5.5)\) (resp. in \((5.6), (5.7)\)).

The child’s drawings \( D_{7,0}, D_{15,4} \) and \( D_{18,4} \) have degrees 7, 15 and 18, respectively. The passport of \( D_{7,0} \) is \((3, 2, 2), (3, 2, 2), (3, 2, 2)\) and its genus is zero. A Belyi map that represents \( D_{7,0} \) can be found at \url{https://beta.lmfdb.org/Belyi/7T6/3.2.2/3.2.2/3.2.2/a}.

The passports of \( D_{15,4} \) and \( D_{18,4} \) are

\[
(6, 6, 3), (6, 6, 3), (6, 6, 3) \quad \text{and} \quad (6, 6, 6), (6, 6, 6), (3, 3, 3, 3, 3, 3)
\]

respectively. Both child’s drawings \( D_{15,4} \) and \( D_{18,4} \) have genus 4.

The orbit \( GT^\circ (N^{(20)})(D_{7,0}) \) coincides with the \( G_Q \)-orbit of \( D_{7,0} \). In fact, using [7], one can show that there is only one child’s drawing with the passport \((3, 2, 2), (3, 2, 2), (3, 2, 2)\).

Hence the \( G_Q \)-orbit of \( D_{7,0} \) and the \( GT^\circ (N) \)-orbit of \( D_{7,0} \) (for any \( N \in NFI_{PB4}(B_4) \) that dominates \( D_{7,0} \)) must have size 1.

The orbit \( GT^\circ (N^{(20)})(D_{15,4}) \) has two elements\(^{12}\) and the “conjugate” of \( D_{15,4} \) is represented by the permutation triple

\[
\begin{align*}
(1, 6, 5, 4, 3, 2) & (7, 12, 11, 10, 9, 8)(13, 15, 14), \\
(1, 15, 9, 12, 6, 2) & (3, 13, 7)(4, 10, 8, 5, 14, 11), \\
(1, 6, 2, 7, 10, 14) & (3, 11, 9, 4, 8, 15)(5, 12, 13)).
\end{align*}
\]

Currently, database [25] does not contain Belyi maps of degree 15. However, we can prove that

**Claim 5.1** The \( G_Q \)-orbit of \( D_{15,4} \) coincides with the orbit \( GT^\circ (N^{(20)})(D_{15,4}) = \{D_{15,4}, D_{15,4}^*\} \), where \( D_{15,4}^* \) is the child’s drawing represented by the permutation triple in \((5.8)\).

**Proof.** The image of the complex conjugation in \( GT^\circ (N^{(20)}) \) is represented by the pair \((-1, 1_{F_2})\) and

\[
D_{15,4}^* = D_{15,4}^{[-1,1_{F_2}]}.
\]

Hence \( GT^\circ (N^{(20)}) \subset G_Q(D_{15,4}) \). The inclusion \( G_Q(D_{15,4}) \subset GT^\circ (N^{(20)}) \) is a consequence of Corollary 3.9.

We should remark that the \( G_Q \)-orbit of \( D_{15,4} \) is significantly smaller than the number of child’s drawings with the passport \((6, 6, 3), (6, 6, 3), (6, 6, 3)\). Using [7], one can show that the number of child’s drawings with the passport \((6, 6, 3), (6, 6, 3), (6, 6, 3)\) is \( \geq 260 \).

The orbit \( GT^\circ (N^{(20)})(D_{18,4}) \) is a singleton. The child’s drawing \( D_{18,4} \) is special because the corresponding (degree 18) covering of \( \mathbb{CP}^1 - \{0, 1, \infty\} \) is Galois. In fact, the child’s drawing \( D_{18,4} \) can be represented by the subgroup of \( F_2 \) that corresponds to the commutator subgroup \([F_2/N_{F_2}, F_2/N_{F_2}]\). (See Remark 2.8)

\(^{12}\)Please see Appendix B or the description of the command orbit( , ) on page 25 of [7].
Here is the basic information about element $N^{(5)}$ from (3.19):

- $|PB_4 : N^{(5)}| = 24 = 2^3 \cdot 3$;
- $|PB_3 : N^{(5)}_{PB_3}| = 864 = 2^5 \cdot 3^3$;
- $|F_2 : N^{(5)}_{F_2}| = 288 = 2^5 \cdot 3^2$;
- $N^{(5)}_{ord} = 6$;
- $N^{(5)}$ is an isolated object of $NFI_{PB_4}(B_4)$ and the group $GT^\varnothing(N^{(5)})$ is isomorphic to $D_6$ (the dihedral group of order 12).

The child’s drawing $D_{8,0}$ represented by the permutation triple

$$((1, 2, 3)(4, 5, 6), (1, 8, 5)(2, 4, 7), (1, 3, 7, 4, 6, 8)(2, 5)).$$ (5.9)

is subordinate to $N^{(5)}$.

$D_{8,0}$ has genus 0 and its passport is

$$((3, 3, 1, 1), (3, 3, 1, 1), (6, 2)).$$

The number of child’s drawings with this passport is 5.

The orbit $GT^\varnothing(N^{(5)})(D_{8,0})$ is a singleton. Hence the $G_Q$-orbit of $D_{8,0}$ is also a singleton. Since $N^{(19)} \leq N^{(5)}$, $N^{(19)}$ (from the list in (3.19)) also dominates $D_{8,0}$.

The author could not find $D_{8,0}$ in [25]. However, $D_{8,0}$ can be transformed, by the action of $B_3$, to the child’s drawing represented by the Belyi map https://beta.lmfdb.org/Belyi/8T12/6.2/3.3. For the action of $B_3$ on child’s drawings, we refer the reader to [22, Construction 1.1.17].

### A Charming GT-shadows satisfy the simplified hexagon relations

Let us prove the following auxiliary statement:

**Proposition A.1** Let $N \in NFI_{PB_4}(B_4)$. If a pair $(m, f) \in \mathbb{Z} \times F_2$ satisfies the hexagon relations (2.6), (2.7) (modulo $N_{PB_3}$) then

$$f(x, y)f(y, x) \in N_{F_2},$$ (A.1)

where $N_{F_2} := N_{PB_3} \cap F_2$.

**Proof.** Let us consider the element

$$\sigma_1^{2m+1}f^{-1}\sigma_2^{2m+1}f\sigma_1^{2m+1}N_{PB_3} \in B_3/N_{PB_3}.$$ (A.2)

On the one hand, (2.6) implies that

$$(\sigma_1^{2m+1}f^{-1}\sigma_2^{2m+1}f)\sigma_1^{2m+1}N_{PB_3} = f^{-1}\sigma_1\sigma_2\epsilon^m\sigma_1^{-2m}\sigma_1^{2m+1}N_{PB_3} =$$

$$f^{-1}\sigma_1\sigma_2\epsilon^m\sigma_1N_{PB_3} = f^{-1}\theta\epsilon^mN_{PB_3},$$

where $\theta := \sigma_1\sigma_2\sigma_1$. 

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On the other hand, using (2.7) and the relation $\sigma_1 \theta = \theta \sigma_2$, we get

$$\sigma_1^{2m+1} (f^{-1} \sigma_2^{2m+1} N_{PB_3} = \sigma_1^{2m+1} \sigma_2 \sigma_1 c^m \sigma_2^{-2m} f N_{PB_3} =$$

$$\sigma_1^{2m} \theta c^m \sigma_2^{-2m} f N_{PB_3} = \theta \sigma_2^m c^m \sigma_2^{-2m} f N_{PB_3} = \theta f c^m N_{PB_3}.$$ 

Thus $\theta f c^m N_{PB_3} = f^{-1} \theta c^m N_{PB_3}$ and hence

$$\theta f \sigma_1^{2m+1} N_{PB_3} = f^{-1} N_{PB_3}. \quad (A.3)$$

Since $\theta x_{12} \theta^{-1} = x_{23}$ and $\theta x_{23} \theta^{-1} = x_{12}$ and $f \in F_2$, relation (A.3) implies (A.1). \(\square\)

If we assume that $f \in [F_2, F_2]$, then (A.1) implies additional useful properties:

**Proposition A.2** Let $N \in NF_{PB_4}(B_4)$ and $N_{F_2} := N_{PB_3} \cap F_2$. If $f \in [F_2, F_2]$ satisfies (A.1), then

$$f(y,z) f(z,y) \in N_{F_2}, \quad (A.4)$$

and

$$f(z,x) f(x,z) \in N_{F_2}, \quad (A.5)$$

where $z := y^{-1} x^{-1}$.

**Proof.** Relation (A.1) obviously implies that

$$f(x,y) f(y,x) \in N_{PB_3}, \quad (A.6)$$

Since

$$\sigma_1 \sigma_2 x_{12} (\sigma_1 \sigma_2)^{-1} = x_{23} \quad \text{and} \quad \sigma_1 \sigma_2 x_{23} (\sigma_1 \sigma_2)^{-1} = x_{23}^{-1} x_{12}^{-1} c,$$

conjugating (A.6) by $\sigma_1 \sigma_2$ gives us

$$f(y,zc) f(zc,y) \in N_{PB_3}, \quad (A.7)$$

Since $f \in [F_2, F_2]$ and $c \in Z(PB_3)$, relation (A.7) implies (A.4). Conjugating relation (A.7) by $\sigma_1 \sigma_2$, we get

$$f(zc,x) f(x,zc) \in N_{PB_3}, \quad (A.8)$$

Since $f(z,x) \in [F_2, F_2]$ and $c \in Z(PB_3)$, relation (A.8) implies (A.5). \(\square\)

We will now get the versions of relations (4.3) and (4.4) from Section 4 of Drinfeld’s foundational paper [6]:

**Proposition A.3** Let $N \in NF_{PB_4}(B_4)$ and $(m, f) \in \mathbb{Z} \times [F_2, F_2]$. The pair $(m, f)$ satisfies the hexagon relations (2.6), (2.7) (modulo $N_{PB_3}$) if and only if $(m, f)$ satisfies the relations

$$f(x,y) f(y,x) \in N_{F_2}, \quad (A.9)$$

$$x^m f(z,x) f(z,y) f(x,y) \in N_{F_2}, \quad (A.10)$$

where $N_{F_2} := N_{PB_3} \cap F_2$ and $z := y^{-1} x^{-1}$. 33
Proof. Recall that

\[ x := x_{12}, \quad y := x_{23}, \quad z := y^{-1}x^{-1}, \quad w := x^{-1}y^{-1}. \]

Using the relations

\[ \sigma_2^{-1}x_{12}\sigma_2 = x_{23}^{-1}x_{12}c = zc, \quad \sigma_2^{-1}x_{23}\sigma_2 = x_{23} \]

and the properties \( c \in Z(PB_3), \quad f^{-1} \in [F_2, F_2], \) we rewrite the left hand side of (2.6) as follows:

\[ \sigma_1x_{12}^{-1}\sigma_2x_{23}^{-1}fN_{PB_3} = \sigma_1\sigma_2c^mz^mf^{-1}(z, y)y^mfN_{PB_3}. \] (A.11)

Using the relations

\[ \sigma_2^{-1}\sigma_1^{-1}x_{12}\sigma_2x_{23}^{-1}c = zc, \quad \sigma_2^{-1}\sigma_1^{-1}x_{23}\sigma_1\sigma_2 = x_{12} = x \]

and the properties \( c \in Z(PB_3), \quad f^{-1} \in [F_2, F_2], \) we rewrite the right hand side of (2.6) as follows:

\[ f^{-1}\sigma_1\sigma_2x_{12}^{-1}c^mN_{PB_3} = \sigma_1\sigma_2c^mf^{-1}(z, x)x^{-m}N_{PB_3}. \] (A.12)

Combining (A.11) and (A.12), we see that relation (2.6) is equivalent to

\[ x^m(f(z, x))z^mf^{-1}(z, y)y^mf(x, y) \in N_{F_2}. \] (A.13)

Using the relations

\[ \sigma_2\sigma_1x_{12}\sigma_1^{-1}\sigma_2^{-1} = cz_{12}^{-1}x_{23}^{-1} = cw, \quad \sigma_2\sigma_1x_{23}\sigma_1^{-1}\sigma_2^{-1} = x_{12} = x \]

and the properties \( c \in Z(PB_3), \quad f \in [F_2, F_2], \) we rewrite the right hand side of (2.7) as follows:

\[ \sigma_2\sigma_1x_{23}^{-1}c^mN_{PB_3} = x^{-m}f(w, x)\sigma_2\sigma_1c^mN_{PB_3} \] (A.14)

Using the relations

\[ \sigma_2x_{12}\sigma_2^{-1} = cw, \quad \sigma_2x_{23}\sigma_2^{-1} = x_{23} = y \]

and the properties \( c \in Z(PB_3), \quad f \in [F_2, F_2], \) we rewrite the left hand side of (2.7) as follows:

\[ f^{-1}\sigma_2x_{23}^{-1}f\sigma_1x_{12}^{-1}N_{PB_3} = f^{-1}y^mf(w, y)w^m\sigma_2\sigma_1c^mN_{PB_3}. \] (A.15)

Combining (A.15) and (A.14), we see that relation (2.7) is equivalent to

\[ y^mf(w, y)w^mf^{-1}(w, x)x^mf^{-1}(x, y) \in N_{PB_3}. \] (A.16)

Conjugating (A.16) by \( \theta := \sigma_1\sigma_2\sigma_1, \) we see that relation (2.7) is equivalent to

\[ x^m(f(z, x))z^mf^{-1}(z, y)y^mf^{-1}(y, x) \in N_{F_2}. \] (A.17)

We can now prove the desired equivalence. Indeed, if a pair \((m, f) \in \mathbb{Z} \times [F_2, F_2]\) satisfies (2.6) and (2.7) then, due to Proposition A.14 relation A.9 holds.

Moreover, relation (A.1) from Proposition A.2 implies that

\[ f^{-1}(z, y)N_{F_2} = f(y, z)N_{F_2}. \] (A.18)

Hence (A.13) implies (A.10).

We proved that relations (2.6), (2.7) imply relations (A.9), (A.10).

Let us now assume that a pair \((m, f) \in \mathbb{Z} \times [F_2, F_2]\) satisfies relations (A.9) and (A.10).

Due to Proposition A.2 relation (A.18) holds. Using (A.9), (A.10) and (A.18) we deduce relations (A.13) and (A.17). Since we showed above that relation (A.13) (resp. relation (A.17)) is equivalent to (2.6) (resp. (2.7)), we proved that relations (A.9), (A.10) imply hexagon relations (2.6), (2.7).
B A few words about the package GT

The material presented in Section 5 of this paper depends heavily on the software package GT for working with GT-shadows and their action on child’s drawings. See [7] for the detailed documentation of this package.

The package is written in Python and it uses commands and functions from the library SymPy [34]. Child’s drawings are represented by permutation pairs. For a finitely generated group \( G \cong \mathbb{F}_n/K \) (e.g. \( B_4, \, PB_4, \, B_3, \, PB_3, \, F_2 \)), finite index normal subgroups of \( G \) are represented by group homomorphisms \( \psi \) from \( \mathbb{F}_n \) to symmetric groups that satisfy the property \( K \subset \ker(\psi) \). Since \( K \subset \ker(\psi) \), \( \ker(\psi) \) corresponds to exactly one finite index normal subgroup of \( G \cong \mathbb{F}_n/K \) via the correspondence theorem.

Let \( G \) be a finitely generated group, \( \psi \) be a group homomorphism from \( G \) to \( S_q \) and \( N := \ker(\psi) \). Then the quotient group \( G/N \) is identified with the permutation group \( \psi(G) \leq S_q \) and permutations in \( \psi(G) \) represent elements of \( G/N \).

Two mathematical statements used in the package stand out:

- given group homomorphisms \( \psi_1 : \mathbb{F}_n \to S_{d_1}, \psi_2 : \mathbb{F}_n \to S_{d_1} \), we have
  \[ \ker(\psi_1 \times \psi_2) = \ker(\psi_1) \cap \ker(\psi_2) ; \]

- the cyclotomic character \( \chi : G_Q \to \hat{\mathbb{Z}}^\times \) is surjective.

The first statement is often used to check whether two finite index normal subgroups (of a finitely generated group) coincide or not. The second statement was used for testing the package.

A simple trick (related to managing memory) was used to generate words (i.e. elements of \( F_2 \) or elements of \([F_2, F_2]\)) that represent distinct elements of \( F_2/N_{F_2} \) or distinct elements of the commutator subgroup \([F_2/N_{F_2}, F_2/N_{F_2}]\): instead of storing actual instances of the class \texttt{sympy.combinatorics.permutations.Permutation}, we stored only their labels (called “ranks”).

We should mention that some commands of the package may be time consuming and certain commands have limitations due to computer memory. Typically, there is an option of timing the commands that may be time consuming.

When we run the main file \texttt{PaB.py}, a computer creates the list \texttt{listE} of compatible equivalence relations on \texttt{PaB} corresponding to 35 distinct elements

\[
N^{(0)}, \, N^{(1)}, \, \ldots, \, N^{(33)}, \, N^{(34)} \quad \text{(B.1)}
\]

of the poset \texttt{NFI_pb4}(B_4). Table 1, on page 11 in [7], shows basic information about these compatible equivalence relations. For every \( 0 \leq i \leq 34 \), the quotient group \( F_2/N^{(i)}_{F_2} \) is non-Abelian.

To produce the 35 elements in (B.1), we used a generator of all conjugacy classes of group homomorphisms \( \psi : B_4 \to S_d \). It is clear that, for every homomorphisms \( \psi : B_4 \to S_d \), the subgroup \( \ker(\psi) \cap PB_4 \leq PB_4 \) belongs to the poset \texttt{NFI_pb4}(B_4). For more details, we refer the reader to [7, Subsection 4.1.1].

Here are selected things one could do with the groupoid of GT-shadows using the package:

- Given an element \( N \in \texttt{NFI_pb4}(B_4) \), one could generate all GT-shadows with the target \( N \) and all charming GT-shadows with the target \( N \). For example, all GT-shadows with the target \( N^{(i)} \) are found for every \( 0 \leq i \leq 32 \) and all charming GT-shadows with the target \( N^{(i)} \) are found for every \( 0 \leq i \leq 34 \).
• One can test whether an element \( N \in \text{NFI}_{PB4}(B_4) \) is an isolated object of the groupoid \( \text{GTSh}^\triangledown \) (i.e. whether \( \text{GTSh}^\triangledown(N,N) = \text{GT}^\triangledown(N) \)). For example, 27 of the 35 elements in (B.1) are isolated.

• One can compose \( \text{GT} \)-shadows (if they can be composed) and one can compute the inverse\(^{13}\) of a charming \( \text{GT} \)-shadow.

• Given \( K, N \in \text{NFI}_{PB4}(B_4) \) with \( K \leq N \) and a \( \text{GT} \)-shadow \( [m,f] \in \text{GT}(N) \), one can determine whether \( [m,f] \) survives into \( K \), i.e. whether \( [m,f] \) belongs to the image of the natural map \( \text{GT}(K) \to \text{GT}(N) \).

• Given an isolated object \( N \) of the groupoid \( \text{GTSh}^\triangledown \) and the complete list of charming \( \text{GT} \)-shadows with the target \( N \), one can produce a permutation group isomorphic to the group \( \text{GT}^\triangledown(N) = \text{GTSh}^\triangledown(N,N) \).

• Given \( K, N \in \text{NFI}_{PB4}(B_4) \) with \( K \leq N \), one can look for fake charming \( \text{GT} \)-shadows with the target \( N \). See [8, Corollary 3.13].

• Given \( N \in \text{NFI}_{PB4}(B_4) \), one can check whether \( N \) satisfies the strong Furusho property or the weak Furusho property. See [7, Section 5.2] or [8, Section 4.3].

Here are selected things one could do with child’s drawings using the package:

• Given a child’s drawing \( D \), one can compute its passport, its genus and its monodromy group; one can also check whether \( D \) is Galois or not.

• Given child’s drawings \( D \) and \( D' \), one can test whether \( D = D' \) or not.

• Given a triple \( \tau \) of partitions of \( d \in \mathbb{Z}_{>1} \), one can generate all child’s drawings (if any) whose passport is \( \tau \).

• Given an integer \( d > 1 \), one can generate all child’s drawings of degree \( d \).

• Given a child’s drawing \( D \) and \( N \in \text{NFI}_{PB4}(B_4) \), one can check whether \( D \) is subordinate to \( N \).

• Given a child’s drawing \( D \) subordinate to \( N \in \text{NFI}_{PB4}(B_4) \) and a \( \text{GT} \)-shadow \( [m,f] \) with the target \( N \), one can compute the result \( D^{[m,f]} \) of the action of \( [m,f] \) on \( D \).

• Given a child’s drawing \( D \) subordinate to \( N \) and the complete list of \( \text{GT} \)-shadows (resp. charming \( \text{GT} \)-shadows) with the target \( N \), one can compute the orbit \( \text{GT}(N)(D) \) (resp. the orbit \( \text{GT}^\triangledown(N)(D) \)).

• Given an element \( N \in \text{NFI}_{PB4}(B_4) \) and a positive integer \( d \), one can generate all (if any) child’s drawings subordinate to \( N \).

• Given a child’s drawing \( D \), one can find an element \( N \in \text{NFI}_{PB4}(B_4) \) that dominates \( D \).

\(^{13}\)Note that computing the inverse of a charming \( \text{GT} \)-shadow may be time consuming.
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