Relaxations of AC Minimal Load-Shedding for Severe Contingency Analysis

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Abstract—This work considers the task of finding an AC-Feasible operating point of a severely damaged transmission network while ensuring a minimal amount of active power loads are removed. This AC Minimal Load-Shedding (AC-MLS) task is a non-convex nonlinear optimization problem, which can be incredibly challenging to solve on large-scale transmission system data sets. This work demonstrates that convex relaxations of the AC-MLS problem provide a reliable and scalable method for finding high-quality bounds on the amount of active power load shedding required in the AC-MLS problem. To demonstrate their effectiveness, the solution methods proposed in this work are rigorously evaluated on 1000 $N$-$k$ scenarios on seven power networks ranging in size from 70 to 6000 buses. The most effective relaxations of the AC-MLS problem converges in less than 20 seconds on commodity computing hardware for all 7000 of the scenarios considered.

Index Terms—Nonlinear Optimization, Convex Optimization, AC Minimal Load-Shedding, N-k Contingency, Power System Restoration

NOMENCLATURE

$N$ - The set of nodes in the network
$E$ - The set of from edges in the network
$E^R$ - The set of to edges in the network
$i$ - imaginary number constant
$S = p + iq$ - AC power
$V = v/\theta$ - AC voltage
$Y = g + ib$ - Line admittance
$T = L\theta^t$ - Transformer properties
$Y^s = g^s + ib^s$ - Bus shunt admittance
$W$ - Product of two AC voltages
$b^c$ - Line charging
$s^u$ - Line apparent power thermal limit
$\theta^\Delta$ - Voltage angle difference limit
$S^d = p^d + iq^d$ - AC power demand
$S^g = p^g + iq^g$ - AC power generation
$c_0, c_1, c_2$ - Generation cost coefficients
$\Re\{\cdot\}, \Im\{\cdot\}$ - Real and imag. parts of a complex number
$(\cdot)^*$ - Conjugate of a complex number
$|\cdot|$ - Magnitude of a complex number, $l^2$-norm
$x^l, x^u$ - Lower and upper bounds of $x$, respectively
$x$ - A constant value

I. INTRODUCTION

Restoring a transmission system after significant disruptions, such as hurricanes, floods, or ice storms, is a challenging task with significant consequences on both human and economic welfare. Indeed, a comprehensive approach to transmission system restoration requires considering many factors such as component repair prioritization, cold load pickups, generation dispatch, and standing phase angle requirements, just to name a few [1], [2], [3]. In this work, we focus on a core subproblem of transmission system restoration, namely the AC Minimal Load-Shedding problem (AC-MLS). Given a severely damaged network, for example where as many as 30% of the components are out of service, the AC-MLS problem consists of determining the maximum amount of active power loads that can be served in the damaged network, subject to operating requirements such as bus voltage limits, line thermal limits, and generator capability limits.

The AC-MLS problem is considered to be an incredibly challenging problem to solve. The key challenge arises due to the significant number of damaged components, which makes the normal-operation set point of little assistance in establishing a new AC-feasible operating point for the damaged network. It is well known that this task is “maddeningly difficult” to do by hand [4]. Furthermore, as we demonstrate in this work, AC transmission systems with realistic components, such as bus shunts and line charging, present additional challenges for finding AC-feasible solutions to the AC-MLS problem.

A natural approach to mitigate the computational challenges of AC-MLS is to develop a linearized DC power flow approximation of the problem. However, several studies have demonstrated that this approach has significant limitations in severe contingency cases [2], [5]. To address these shortcomings, various extensions of the DC power flow approximation such as, the LPAC approximation [6], [2], have been considered. However, to the best of our knowledge, no work has attempted to solve the complete non-convex nonlinear AC-MLS problem in its entirety without approximation.

In this work we leverage recent developments in nonlinear programming and convex relaxations of the AC Power Flow equations [7], [8], [9], [10], [11] to investigate the AC-MLS problem. We propose a novel formulation of the AC-MLS problem and consider three relaxations of this formulation. The experimental results demonstrate: (1) the challenges of scaling the AC-MLS problem to realistic networks with thousands of buses; and (2) the success of the proposed convex relaxations in providing strong lower bounds to the AC-MLS problem in...
Model 1 AC-OPF

\[
\begin{align*}
\text{variables: } & S^g_i (\forall i \in N), \ V_i (\forall i \in N) \\
\text{minimize: } & \sum_{i \in N} c_{2i}(R(S^g_i))^2 + c_{1i}R(S^g_i) + c_{0i} \\
\text{subject to: } & v_i^l \leq |V_i| \leq v_i^u \ \forall i \in N \\
& S^g_i \leq S^g_i \leq S^{g\text{u}}_i \ \forall i \in N \\
& S^g_i - S^l_i - Y^*_i |V_i|^2 = \sum_{(i,j) \in E \cup E^R} S_{ij} \ \forall i \in N \\
S_{ij} &= \left( Y^*_{ij} - i \frac{b^c_{ij}}{2} \right) \frac{|V_i|^2}{T_{ij}^2} - Y^*_{ij} \frac{V_i V^*_j}{T_{ij}} \ (i,j) \in E \\
S_{ji} &= \left( Y^*_{ij} - i \frac{b^c_{ij}}{2} \right) \frac{|V_j|^2}{T_{ij}^2} - Y^*_{ij} \frac{V_j V^*_i}{T_{ij}} \ (i,j) \in E \\
|S_{ij}| &\leq s^{\text{u}}_{ij} \ \forall (i,j) \in E \cup E^R \\
- \theta^\Delta_{ij} &\leq \angle(V_i V^*_j) - \theta^\Delta_{ij} \ \forall (i,j) \in E \\
\end{align*}
\]

constraints (1e)-(1f) model Ohm’s Law over lines and transformers. Finally Constraints (1g)-(1h) ensure that line thermal limits and voltage angle difference limits are enforced. A detailed derivation of this formulation and its notations are available in [13].

It is important to emphasize that the AC modeling details, such as bus shunts and line charging (i.e., \( b^c \) in (1e)-(1f)) present unique challenges for developing an AC-MLS formulation. This is in significant contrast to AC-OPF formulations, where incorporating these terms is a trivial extension.

A. AC Feasibility Challenges under Contingencies

At first glance, adapting Model 1 to an AC-MLS formulation seems straightforward. One can make the constant power loads flexible by introducing a decision variable \( z_i^c \in (0, 1) \) and updating the power balance constraint as follows,

\[
S^g_i - z_i^c S^l_i - Y^*_i |V_i|^2 = \sum_{(i,j) \in E \cup E^R} S_{ij} \ \forall i \in N. 
\]

This modification introduces bus-by-bus constant power-factor load shedding into the model, and intuitively, one expects there exists a load shedding that is sufficient to find a new AC operating point. However, after considering this simple extension on several networks under thousands of severe contingencies, it was observed that this model is insufficient to ensure feasibility in all cases.

Figure 1 presents a simple five-bus AC power network featuring bus shunts and line charging to illustrate three unexpected contingency cases that have been identified. To make this example mathematically precise, we assume the following network data, which are representative of realistic network data sets:

\[
\begin{align*}
0.9 &\leq |V_i| \leq 1.1 \ \forall i \in \{1, 2, 3, 4, 5\} \\
10 - i \infty &\leq S^g_i \leq 100 + i \infty \\
Y_{4,5} &= 0 - i25, b^c_{4,5} = 0.08 \\
\end{align*}
\]

Generator Injection Bounds: On the five-bus example network, consider the outage contingency of line 2-3 and line 3-5. This isolates bus 3 and its associated generating unit. The problem in this contingency is that the generator has a non-zero active power injection bound. Hence, when there is no load to serve, the constraints cannot be satisfied. Specifically, in this contingency, Constraints (1c) and (1d) result in,

\[
\begin{align*}
10 &\leq R(S^g_i) \leq 100 \\
R(S^g_i) &= 0 
\end{align*}
\]

only a few seconds. A key advantage of the proposed solution methodology over previous works is that it is guaranteed to converge in polynomial time and provides an optimistic bound on the maximum amount of active load that can be served in the damaged network considering all of the typical operating requirements such as bus voltage limits, line thermal limits, and generator capability limits.

The rest of the paper is organized as follows. First, the scope of the AC power flow model is defined and a number of the feasibility challenges inherent in this model are discussed in Section II. Second, the AC-ML formulation is proposed and various relaxations are derived in Section III-A. A detailed computational evaluation of all proposed formulations is conducted on seven realistic power networks in Section IV. Finally, the given computational evaluation results are provided in Section V.

II. SCOPE OF THE AC NETWORK MODEL

In this work, we solve the AC-MLS problem at a level of network fidelity and scale that is comparable with commercial tools, such as PowerWorld, PSS/E, and PSLF. However, in the interest of leveraging available network datasets, we adapt the AC Optimal Power Flow (AC-OPF) model of MATPOWER [12]. This formulation roughly corresponds to network models from PSS/E v23 and is still an accurate mathematical model of the most common network components, namely generators, constant power loads, fixed bus shunts, \( \pi \)-model lines, and two-winding transformers.

The complete MATPOWER AC-OPF formulation is presented in Model 1. The objective function (1a) minimizes the active power generation costs. Constraints (1b) ensure that the bus voltage magnitudes are maintained within a desired operating range. Constraints (1c) provide generator operation limits for both active and reactive injection. Constraints (1d) capture power balance from Kirchhoff’s Current Law and
which has no feasible assignment. This indicates that model feasibility requires that all generator operating ranges include the zero value, or generators should be uncommitted from the dispatch scenario.

**Bus Shunt Shedding:** On the five-bus example network, consider the outage contingency of line 1-2 and line 2-3. This isolates bus 2 and its associated fixed bus shunt. The key problem in this contingency is that a fixed shunt results in a nonzero active and reactive power injection at a bus. Hence, when no generation is available, Kirchhoff’s Current Law cannot be satisfied. Specifically, in this contingency, Constraints (1b) and (1d) result in,

$$0.9 \leq |V_2| \leq 1.1$$

which has no feasible assignment. This indicates that model feasibility requires the option of shedding fixed bus shunts from the network.

**Bus Voltage Bounds:** On the five-bus example network, consider the outage contingency of line 1-4 and line 3-5. This isolates line 4-5, which has a nonzero line charging value. The key problem in this case is that line charging makes the Ohm’s Law constraints impossible to satisfy when no power flows over the line. Specifically, in this contingency, Constraints (1b), (1e) and (1f) result in,

$$0.9 \leq |V_4| \leq 1.1$$

An analysis of the active power aspect of (6c)-(6d) reveals that one of the following must hold,

$$|V_4| = 0 \lor |V_5| = 0 \lor \angle (V_i V_j^*) = 0$$

while an analysis of the reactive power aspect of (6c)-(6d) reveals that both of the following must hold,

$$|V_4| = |V_5| \land \angle (V_i V_j^*) \neq 0$$

Combining the requirements of (6a), (6b), (6e), (6f) demonstrates that this system of equations has no feasible assignment. This indicates that model feasibility requires the option of either removing lines with charging values or relaxing the lower bounds of bus voltage magnitudes. These observations indicate that the AC-MLS problem requires a disjunctive optimization formulation that is considerably more difficult to solve than traditional AC-OPF formulations.

### III. AC Minimal Load-Shedding Formulations

Motivated by the observations in Section II this work introduces the following decision variables into the AC-MLS problem:

$$z_i^g, z_i^u \in \{0, 1\} \ \forall i \in N$$

$$z_i^d, z_i^t \in \{0, 1\} \ \forall i \in N$$

### Model 2 AC-MLS

**variables:** $S_i^g (\forall i \in N), V_i (\forall i \in N)$

$$z_i^v, z_i^u \in \{0, 1\} \ \forall i \in N$$

$$z_i^d, z_i^t \in \{0, 1\} \ \forall i \in N$$

**maximize:**

$$\sum_{i \in N} z_i^v \sum_{i \in N} z_i^u \sum_{i \in N} z_i^d \sum_{i \in N} |\Re (S_i^d)| z_i^d$$

**subject to:**

$$z_i^v v_i^0 \leq |V_i| \leq z_i^v v_i^u \ \forall i \in N$$

$$z_i^d S_i^d \leq S_i^d \leq z_i^d S_i^u \ \forall i \in N$$

$$S_i^g - z_i^d S_i^d - z_i^t Y_i^a |V_i|^2 = \sum_{(i,j) \in E \cup R} S_{ij} \ \forall i \in N$$

The AC-MLS formulation in Model 2 includes two sources of non-convexity: (1) discrete variables (i.e. $z^v, z^g \in \{0, 1\}$); and (2) products of continuous variables (e.g. $z^*|V|^2, V_i^a V_j$).
Relaxation of the discrete variables is easily achieved by relaxing the integrality requirement and allowing these variables to have a continuous range as follows:

\[ z_i^v, z_i^g \in \{0, 1\} \Rightarrow z_i^v, z_i^g \in (0, 1) \]  

(9)

Throughout this work, C is used to annotate models where the discrete variables have been relaxed to continuous variables (e.g., AC-MSL-C). Relaxation of the continuous variable products is more challenging. To address this we turn to recent developments in convex relaxation of the AC-OPF problem [13], [8], [9], [10], [11].

The Second Order Cone (SOC) Relaxation: In the interest of performance and scalability, we choose to develop a model based on the Second Order Cone (SOC) relaxation of the AC Power Flow equations [9]. Although some relaxations are stronger than SOC [13], [15], [11] and others are faster than SOC [8], the SOC relaxation was selected because it provides an appealing trade-off between bounding strength and runtime performance.

The first insight of the SOC relaxation is that the voltage product terms \( V_i^\ell V_j \), can be lifted into a higher dimensional W-space as follows,

\[ |V_i|^2 \Rightarrow W_{ii} \quad \forall i \in N \]  

(10a)

\[ V_i V_j^* \Rightarrow W_{ij} \quad \forall (i, j) \in E \]  

(10b)

Note that lifting Model [11] into the W-space makes all of the non-convex constraints linear.

The second insight of the SOC relaxation is that this W-space relaxation can be strengthened by adding the valid inequality,

\[ |W_{ij}|^2 \leq W_{ii} W_{jj} \]  

(11a)

which is a convex rotated second-order cone constraint that is supported by a wide variety of industrial-grade optimization tools.

After applying (10a), (10b), and (11a) to Model [2] one non-convex term still remains in the Kirchhoff’s Current Law constraint due to the product of the shunt shedding variable \( z_i^s \) and the lifted voltage magnitude variable \( W_{ii} \) as follows,

\[ S_i^g - z_i^d S_i^d - Y_i^s W_{ii} = \sum_{(i,j) \in E \cup R^E} S_{ij} \quad \forall i \in N \]  

(12)

Observing that the \( z_i^s \) and \( W_{ii} \) variables have tight bounds, McCormick inequalities [16] can be leveraged to provide a convex relaxation of this non-convex variable product. The general form of the McCormick envelope is given by,

\[ \langle xy \rangle^M = \begin{cases} 
xy \geq x^u y + y^u x - x^u y^u \\
xy \geq x^u y + y^u x - x^u y^u \\
xy \leq x^l y + y^l x - x^l y^l \\
xy \leq x^l y + y^l x - x^l y^l 
\end{cases} \]  

(M-CONV)

In this specific case, a new variable \( W_{ii}^a \) is introduced to represent the product of \( z_i^s \) and \( W_{ii} \) as follows:

\[ W_{ii}^a = \langle z_i^s W_{ii} \rangle^M \quad \forall i \in N \]  

(13)

Model 3 SOC-MLS

variables: \( S_i^g(\forall i \in N), W_{ij}(\forall (i, j) \in E), W_{ii}(\forall i \in N) \) : real

\[ W_{ii}^a \quad \forall i \in N \]

\[ z_i^v, z_i^g \in \{0, 1\} \quad \forall i \in N \]

\[ z_i^v, z_i^g \in (0, 1) \quad \forall i \in N \]

maximize: (8a)

subject to:

\[ \langle z_i^v W_{ii} \rangle^M \quad \forall i \in N \]

\[ (14a) \]

\[ W_{ii}^a \quad \forall i \in N \]

\[ z_i^v, z_i^g \in \{0, 1\} \quad \forall i \in N \]

\[ z_i^v, z_i^g \in (0, 1) \quad \forall i \in N \]

With this final relaxation, a convex SOC relaxation of the AC-MLs problem is developed.

The complete SOC-MLS formulation is presented in Model 3. The objective function remains the same as the AC-MLS formulation, i.e., (8a). Constraints (14a) enforce the bus voltage magnitudes’ operating range in the lifted W-space variables. Constraints (14b) provide the convex relaxation of the shunt shedding and voltage magnitude variable products, and Constraints (14c) incorporate the relaxation variable \( W_{ii}^a \) into Kirchhoff’s Current Law. Constraints (14d)–(14g) implement the established W-space convex relaxation of \( (1e), (1f), \) and \( (1h) \). A detailed derivation of these constraints is available in [13].

C. Formulations Summary

In summary, this work considers the following four variants of the proposed AC-MLs formulation from Model 2

1) AC-MLs - a nonconvex Mixed Integer NonLinear Program (MINLP), which is a challenging model to solve reliably at scale.

2) AC-MLs-C - a nonconvex NonLinear Program (NLP), which is more likely to converge at scale, but may not satisfy the integrality requirements of AC-MLs.

3) SOC-MLs - a Mixed Integer Second-Order Cone Program (MISOCP), which is guaranteed to converge but only provides a lower bound on the optimal value of AC-MLS.

4) SOC-MLs-C - a Second-Order Cone Program (SOCP), which is a weaker but more scalable variant of SOC-MLS.
A core motivation for considering all of these variants is that they provide a range of trade-offs in model accuracy and performance. This allows us to clearly quantify how each relaxation impacts the AC-MLS solutions.

D. Practical Implementation Considerations

There are a number of practical implementation details for solving the proposed models that are worth mentioning.

Single-Objective Form: Although one would like solve the multi-objective function in AC-MLS directly, there is limited support for multi-objective functions in industrial optimization software. To work around this limitation, the lexicographic multi-objective function \( f(S) \) can be modeled as a single objective function as follows,

\[
\sum_{i \in N} M^v z^v_i + M^q z^q_i + M^s z^s_i + |\mathbb{R}(S_i^d)| z^d_i \quad (15)
\]

The key challenge in this representation is to find values of the scaling parameters \( M^v, M^q, M^s \) such that the optimal solution does not change. For the datasets considered in this work, we found the following method of setting these parameters to be sufficient: \( M^* = 10 \times \max_{i \in N} (|\mathbb{R}(S_i^d)|) \), \( M^q = M^* \), \( M^s = 10 \times M^* \), for the datasets considered in this work.

Data Processing: Typical AC-OPF studies are conducted on test cases that have been curated by subject-matter experts to ensure that the network data is meaningful and of high quality \( [17] \). However, in the context of severe contingency analysis, the AC-MLS is subject to network data where hundreds of the components are out of service, which can lead to unexpected network structures. A notable example of this is the occurrence of dangling buses. These buses are connected to the network by a single line and have no load or generation and hence have limited utility.

In this work, we observed that conducting the following data processing steps on damaged networks greatly aided the convergence of all formulations considered herein. First, each connected component in the network should be solved independently. For simplicity, in this work, we only solved the largest connected component. Second, all dangling buses should be put out of service.

IV. Computational Evaluation

To evaluate the effectiveness of the Minimal Load-Shedding formulations, we consider a severe contingency analysis use case and conduct a detailed computational study on a variety of realistic network data sets. In the interest of making the evaluation exacting, we consider one thousand randomly selected \( N-k \) damage scenarios where 30\% of the network’s lines are removed. This type of scenario was observed to be particularly challenging in \( [22] \). The evaluation is presented in four parts: (1) first, the details of the test cases and computational tools are introduced; (2) then, a detailed study of the reliability and scalability of each formulation is conducted; (3) this is followed by an analysis of the optimality gap of the relaxations; (4) finally, the evaluation is concluded with proof-of-concept load shedding analysis to demonstrate the value of the proposed minimal load-shedding formulations.

A. Test Cases and Computational Setting

In this study, seven cases are selected from the IEEE PES PGLib AC-OPF v17.08 benchmark library \( [18] \). The details of these cases are presented in Table I. The selection of these cases was designed to: (1) provide a representative sample of some of the most realistic data sets available; and (2) span a wide range of problem sizes, from small (e.g., 73 buses) to realistic (e.g., >2000 buses), to highlight the scalability properties of the proposed formulations.

TABLE I  TEST CASE SIZE AND N-K DAMAGE SCENARIO OVERVIEW.

| Test Case | | | k | Scenarios |
|---|---|---|---|---|
| IEEE RTS 96 \( [12] \) | 73 | 120 | 36 | 1000 |
| PSERC 240 \( [19] \) | 240 | 448 | 134 | 1000 |
| PEGASE 1394 \( [20] \) | 1354 | 1991 | 397 | 1000 |
| RTE 1888 \( [21] \) | 1888 | 2531 | 759 | 1000 |
| Polish 238 kwp \( [18] \) | 2383 | 2896 | 869 | 1000 |
| Polish 3120kW \( [18] \) | 3120 | 3593 | 1108 | 1000 |
| RTE 6468 \( [21] \) | 6468 | 9000 | 2700 | 1000 |

All of the Minimal Load-Shedding formulations were implemented in Julia v0.5 using the optimization modeling layer JuMP.jl v0.17 \( [22] \) and PowerModels.jl v0.4 \( [23] \). The NLP and SOCP formulations were solved with Ipopt \( [24] \) using the HSL MA27 linear algebra solver. The MISOCP formulation was solved using Pajarito.jl v0.5 \( [25] \) using Ipopt and Gurobi v7.0. The MINLP formulation was solved using the default version of Bonmin \( [26] \), which uses open source solvers. All of the solvers were configured to stop when the optimality gap was less than 1e–6. The continuous formulations (i.e. AC-MLS-C, SOC-MLS-C) were given a time limit of 150 seconds, while the discrete formulations (i.e. AC-MLS, SOC-MLS) were given a time limit of 1500 seconds. All of the formulations were evaluated on HPE ProLiant XL170r servers with two Intel 2.10 GHz CPUs and 128 GB of memory.

B. Algorithm Reliability and Scalability

Table II presents the algorithm status and runtime results of all four formulations across all seven test cases. The status categories indicate the following properties: converged, the solver algorithm completed normally; time limit, the solver algorithm reached the prescribed time limit; error, the solver algorithm had an unexpected failure, usually due to numerical accuracy issues. Note that the runtimes are an average value for all of the scenarios for each combination of formulation and status value.

The results are summarized as follows: (1) For the smallest test case, IEEE RTS 96, all of the formulations worked effectively. (2) As the test cases increased in size to the 2000-3000 bus range, the reliability and runtime of the AC-MLS formulation became a significant issue. (3) When the test case was very large (>

6000 buses), the AC-MLS-C relaxation’s
runtime became a significant limiting factor. (4) Both the SOC-MLS and SOC-MLS-C were reliable and scalable on all of the considered networks; however, SOC-MLS-C was at least ten times faster than SOC-MLS. Given the algorithmic success of SOC-MLS/SOC-MLS-C over AC-MLS/AC-MLS-C, the next important topic to investigate is the bounding quality of these convex relaxations.

C. Relaxation Strength

Table [III] presents a comparison of optimality gaps across all four formulations. Given that the AC-MLS formulation is the only one that provides feasible solutions to this problem, the (percentage) optimality gap is given by $100 \times (\text{AC-MLS - Relaxation})/\text{AC-MLS}$. The values in this table are averages over all scenarios; it is important to note that these averages only consider scenarios where all four algorithms converged. Hence the number of scenarios considered in the average decreases with the network size.

The results are summarized as follows: (1) All of the optimally gaps were surprisingly small, well below 1%. (2) Although there were a few differences in the gap between SOC-MLS and SOC-MLS-C, the gaps were extremely close. Overall, these results suggest that the SOC-MLS-C relaxation is sufficient to provide a high quality lower bound to AC-MLS. Furthermore, these quality results indicate that the additional runtime needed to solve SOC-MLS is likely not warranted by the small improvement in optimality gap.

D. Proof-of-Concept Load-Shedding Analysis

Given that the previous results have demonstrated the efficacy of the SOC-MLS-C relaxation, this section provides a brief and preliminary analysis of the load-shedding results provided by that formulation.

Figure 2 presents the distribution of active power load shed in the one thousand $N$-k scenarios for each of the networks considered. Two observations become clear: (1) in many of these severe contingencies the amount of load shed can be significant, e.g. $>20\%$ of the total active power; and (2) there do seem to be significant variations in the effects of $N$-30$\%$ contingencies across these networks, both in the mean and variance of total load-shed. It is also important to recall that these distributions are for the lower bound on active power load shedding provided by the SOC-MLS-C relaxation. The true distributions of AC-MLS may increase the amount of active power load shed. However, the results from Table [III] suggest that these changes would be minor.

V. CONCLUSION

This work has proposed a formulation of the AC-MLS problem for use in severe contingency analysis of AC power networks. The work has shown that even if the complete AC-MLS problem is challenging to compute on large-scale networks (e.g., $>2000$ buses), the convex relaxation SOC-MLS-C is a fast and reliable alternative, which provides high-quality bounds for the AC-MLS problem. The value of these high-quality bounds are demonstrated by a proof-of-concept load-shedding analysis study.

Despite the success of the SOC-MLS-C formulation in this work, its critical shortcoming is that it rarely provides feasible solutions to the AC-MLS problem. Hence, developing a scalable algorithm for finding AC-MLS feasible solutions remains a valuable open question for future work.

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**TABLE II**

| Status | AC-MLS | AC-MLS-C | SOC-MLS | SOC-MLS-C | AC-MLS | AC-MLS-C | SOC-MLS | SOC-MLS-C |
|--------|--------|---------|--------|---------|--------|---------|--------|---------|
| IEEE RTS 96 (n=1000) | converged | 98.60% | 94.40% | 100.00% | 100.00% | 16.18 | 0.14 | 0.50 | 0.07 |
| time limit | 1.40% | | | | | | | |
| error | | | | | | | | |
| PSERC 240 (n=1000) | converged | 71.30% | 98.70% | 100.00% | 100.00% | 18.46 | 1.54 | 4.10 | 0.32 |
| time limit | 3.70% | 0.70% | | | 1614.58 | 69.82 | | |
| error | 25.00% | 0.60% | | | 1500.00 | 20.63 | | |
| PEGASE 1354 (n=1000) | converged | 94.10% | 100.00% | 100.00% | 100.00% | 221.36 | 5.46 | 32.97 | 2.26 |
| time limit | 1.70% | | | | | | | |
| error | 4.20% | | | | | | | |
| RTE 1888 (n=1000) | converged | 84.30% | 88.40% | 100.00% | 100.00% | 99.38 | 10.53 | 67.21 | 2.53 |
| time limit | 0.10% | 11.50% | | | 1655.67 | 151.00 | | |
| error | 15.60% | 0.10% | | | 1500.00 | 130.41 | | |
| Polish 2383wp (n=1000) | converged | 86.00% | 100.00% | 99.80% | 100.00% | 121.58 | 7.64 | 38.43 | 3.04 |
| time limit | 1.30% | | | | 1639.76 | | | |
| error | 12.70% | 0.20% | | | 1500.00 | | | |
| Polish 3120wp (n=1000) | converged | 15.70% | 99.90% | 99.80% | 100.00% | 659.79 | 9.49 | 67.67 | 4.03 |
| time limit | 74.00% | 0.10% | | | 1531.44 | 150.75 | | |
| error | 10.30% | 0.20% | | | 1500.00 | | | 633.83 |
| RTE 6468 (n=1000) | converged | 14.50% | 35.60% | 99.60% | 100.00% | 865.42 | 57.17 | 648.47 | 12.34 |
| time limit | 3.10% | 58.70% | | | 1608.88 | 152.38 | | |
| error | 82.40% | 5.70% | 0.40% | | 1500.00 | 92.89 | 1500.00 | |

**TABLE III**

| Case | Obj. Val. AC-MLS | Optimality Gap (%) AC-MLS-C | Optimality Gap (%) SOC-MLS | Optimality Gap (%) SOC-MLS-C |
|------|------------------|-----------------------------|---------------------------|-----------------------------|
| IEEE RTS 96 (n=914) | 2.463e+04 | 0.0000% | 0.0044% | 0.0044% |
| PSERC 240 (n=907) | 1.935e+06 | -0.0049% | 0.0010% | 0.0010% |
| PEGASE 1354 (n=927) | 2.001e+06 | 0.0022% | 0.0024% | 0.0037% |
| RTE 1888 (n=1000) | 1.101e+06 | 0.0027% | 0.0004% | 0.0006% |
| Polish 2383wp (n=859) | 5.759e+05 | 0.0084% | 0.0062% | 0.0077% |
| Polish 3120wp (n=155) | 1.107e+06 | 0.0079% | 0.0138% | 0.0186% |
| RTE 6468 (n=554) | 9.496e+06 | -0.0151% | 0.0003% | 0.0003% |

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