THE UNIVERSAL COVER OF AN AFFINE THREE-MANIFOLD WITH HOLONYM
OF SHRINKABLE DIMENSION \leq TWO.

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Abstract. An affine manifold is a manifold with an affine structure, i.e.
a torsion-free flat affine connection. We show that the universal cover of a
closed affine 3-manifold $M$ with holonomy group of shrinkable dimension
(or discompacit´e in French) less than or equal to two is diffeomorphic to
$\mathbb{R}^3$. Hence, $M$ is irreducible. This follows from two results: (i) a simply
connected affine 3-manifold which is 2-convex is diffeomorphic to $\mathbb{R}^3$, whose
proof using the Morse theory takes most of this paper; and (ii) a closed
affine manifold of holonomy of shrinkable dimension less or equal to $d$ is
d-$\text{convex}$. To prove (i), we show that 2-convexity is a geometric form of
topological incompressibility of level sets. As a consequence, we show that
the universal cover of a closed affine three-manifold with parallel volume
form is diffeomorphic to $\mathbb{R}^3$, a part of the weak Markus conjecture. As
applications, we show that the universal cover of a hyperbolic 3-manifold
with cone-type singularity of arbitrarily assigned cone-angles along a link
removed with the singular locus is diffeomorphic to $\mathbb{R}^3$. A fake cell has an
affine structure as shown by Gromov. Such a cell must have a concave point
at the boundary.

Our research is to understand the geometrical and topological properties
of manifolds with flat real projective or affine structures, particularly in low-
dimensions. A hyperbolic structure on a manifold can be naturally regarded
as a real projective structure using the Klein model of hyperbolic space. Our
interest stems from the fact that the eight 3-dimensional homogeneous Riemannian structures can be considered naturally as real projective or product
real projective structures, as observed by Thurston, Kapovich, and many oth-
ers even earlier (see [16] for a proof and also Molnár [37] and Thiel [50]).
(A product real projective structure is a geometric structure modelled on a
product of real projective spaces and a product of projective transformation
groups.) The research on these homogeneous Riemannian structures led to
the explosion of discoveries on 3-manifold topology as initiated by Thurston in

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the 1980’s, from which we gained a very practical detailed and comprehensive understanding of 3-manifolds even unparalleled in the presently ever expanding study of 4-manifolds for example. However, we feel that we may need to look at more general geometric structures to gain deeper insights. The study of contact structures and foliations form a line of such research.

Classical affine and projective geometries form a very rich field giving us much insight into Euclidean, spherical, and hyperbolic geometries. A comprehensive treatment of classical affine and projective geometry by Berger [4] for example shows the depth, beauty, and fertility of this subject quite clearly. We feel that a study of real projective and affine geometric structures has many interesting global properties and are attempting to develop tools of study (see Benoist [2], Goldman [24, 25], Choi [16], and Choi-Goldman [17] for references). Most of local properties of affine or projective differential geometry were discovered much earlier while there seems to be only a small number of global results. More importantly the study of real projective and affine structures are important in understanding representations of discrete subgroups of Lie groups. However, it is beyond the subject of this paper.

Topological and geometrical properties of real projective and affine structures on 3-manifolds are completely unknown at the moment. For example, we do not have an example of a 3-manifold not admitting a real projective structure. Sullivan and Thurston [49] made a conjecture that all 3-manifolds admit some analytic geometric structure. This conjecture implies the Poincaré conjecture for example. (It is shown by Smillie [47] that some 3-manifolds that are homeomorphic to connected sums of Lens spaces do not admit flat affine structures.)

In this paper, we study affine structure on 3-manifolds. Affine geometry is the study of properties of the vector space $\mathbb{R}^n, n \geq 1$ invariant under the action of the group $\text{Aff}(\mathbb{R}^n)$ of affine transformations according to Felix Klein’s Erlangen program. An affine structure on a smooth $n$-manifold is given by a maximal atlas of charts to $\mathbb{R}^n$ where the transition functions are affine. That is, an affine structure gives a way to lift affine geometry on a manifold locally and consistently. An affine structure can also be given by a torsion-free flat affine connection. An affine manifold is a manifold with an affine structure. A local diffeomorphism between affine manifolds is affine if it preserves the affine structures locally.

A so-called equi-affine manifold is an affine manifold with a volume form parallel under its connection. Equivalently, an equi-affine manifold is an affine manifold which admits a compatible atlas of charts where the transition functions are volume-preserving affine maps. We will prove that a closed equi-affine 3-manifold must have a universal cover diffeomorphic to a 3-cell using Morse theory adapted to our geometric situation. In particular such an affine
3-manifold is irreducible and does not contain a fake cell. This answers a question of Carrière [13] and sharpens Theorem 2 of Smillie [47]. (A much shorter version of this paper was written as a proceedings paper [15].)

A quotient of the complete affine space $\mathbb{R}^n$ by the properly discontinuous and free action of a discrete subgroup of $\text{Aff}(\mathbb{R}^n)$ is an example of an affine manifold. A complete affine manifold is an affine manifold affinely diffeomorphic to such a quotient. For example, a closed manifold with a flat Riemannian metric is affine since it is isometric with the quotient of the complete Euclidean space by a group of Euclidean isometries (which are affine also). A closed flat Lorentz manifold is also affine since it is modeled locally on $\mathbb{R}^n$ and the group of Lorentzian affine transformations (which are affine) on $\mathbb{R}^n$ equipped with a Lorentzian inner product.

Assigning an affine structure to a smooth manifold $M$ is equivalent to assigning a pair $(\text{dev}, h)$ where $\text{dev}$ is an immersion from $\tilde{M}$ equivariant with respect to a homomorphism $h$ from the group $\pi_1(M)$ of deck transformations of $\tilde{M}$ to $\text{Aff}(\mathbb{R}^n)$. $\text{dev}$ is said to be a developing map, and $h$ a holonomy homomorphism, whose image is said to be the holonomy group. $\text{dev}$ is obtained by analytically continuing local charts and $h$ by piecing together the transition functions as we continue (see Ratcliffe [44, Chapter 8]).

A complete affine manifold is characterized by the fact that the developing map is a diffeomorphism onto $\mathbb{R}^n$ since its universal cover is affinely diffeomorphic to $\mathbb{R}^n$. Numerous incomplete examples were constructed by Sullivan and Thurston [49]. They include some Seifert spaces (see Section 2), and bundles over surfaces with fibers homeomorphic to tori.

Affine 2-manifolds were classified by Nagano and Yagi [11]. They are always homeomorphic to tori or Klein bottles as was shown by Benzécri-Milnor [3] and [38]. Complete affine 3-manifolds are rather limited and classified (see Fried-Goldman [21]).

A form on an affine manifold is parallel if it is parallel with respect to the affine coordinate charts, i.e., its covariant derivative with respect to the affine connection is zero. An affine manifold admits a parallel volume form if and only if $h(\pi_1(M))$ lies in the group of equi-affine transformations, i.e., affine transformations with Jacobian identically 1.

One of the conjectures in the study of affine manifolds is that of Markus [36]: A closed affine manifold is complete if and only if it admits a parallel volume form.

The above conjecture is verified in cases when the holonomy group $\Gamma$ is abelian, nilpotent, and solvable of rank $\leq n$ by Smillie [46], Fried, Goldman, Hirsch [22], and [23] and when $\Gamma$ is distal by Fried [20].
Carrière introduced a set of notions about holonomy groups \[4\]. In this paper, we will use the word “shrinkable dimension” for the French “discom-pacité” formerly translated as discompactedness \[11\]. Given a group of affine transformations, the shrinkable dimension measures the maximal codimension of the degenerating limiting sequences of ellipsoids under the affine action of the group (see Section \[9\]). That is, one measures the maximal “shrinking” dimension of ellipsoids under the group action (see Figure 1).

**Figure 1.** The shrinkable dimension: Linear maps \(\vartheta_i\) mapping 2-disks of radius \(\varepsilon\) in the standard \(\varepsilon\)-ball to ellipsoids. We are provided with a fixed Euclidean metric on \(\mathbb{R}^3\).

Let \(L : \text{Aff}(\mathbb{R}^n) \to \text{GL}(n, \mathbb{R})\) denote the homomorphism taking the linear part of affine transformations. Carrière \[3\] proved that if the shrinkable dimension of \(\Gamma\), or \(\text{sd}(L(\Gamma))\), for the holonomy group \(\Gamma\) is \(\leq 1\), then \(\tilde{M}\) is affinely diffeomorphic to \(\mathbb{R}^n\) or a half space \(\mathbb{R}^n_+\), given by \(x_1 > 0\) where \(x_1\) is an affine coordinate. If in addition \(L(\Gamma)\) is a subgroup of \(\text{SL}(n, \mathbb{R})\), then \(\tilde{M}\) is affinely diffeomorphic to \(\mathbb{R}^n\). In particular, if \(L(\Gamma)\) lies in \(\text{SO}(n - 1, 1)\), then \(M\) is a complete affine manifold, which means that any closed flat Lorentzian manifold is complete.

Since the conjecture of Markus is very difficult to attack, Carrière proposed a new set of conjectures that if the shrinkable dimension of \(\Gamma\) is \(\leq i\) for some integer \(0 < i < n - 1\), then \(\pi_j(\tilde{M})\) is trivial for \(j \geq i\) \[7\]. (See also \[6, Section 3.3\].)

In this paper, we will prove a partial answer to the above conjecture as a corollary to our main Theorem \[1\]:

**Corollary 1.** The universal cover of a closed affine three-manifold with holonomy group of shrinkable dimension less than or equal to 2 is diffeomorphic to \(\mathbb{R}^3\).

We need to introduce the powerful notion of the (projective) completion that is defined for incomplete geometric structures due to Kuiper \[33\] (see Kamishima-Tan \[32\]). The affine space \(\mathbb{R}^n\) has a Euclidean metric \(\mu\) and
the distance metric $d$ induced from it; we choose such a metric standard with respect to an affine coordinate system. The developing map $\text{dev}$ of $M$ induces a Riemannian metric $\mu$ on $\tilde{M}$ from the Euclidean metric of $\mathbb{R}^n$. (An arc is straight if it is a geodesic for $\mu$.) $\mu$ induces a distance metric $d$ on $\tilde{M}$. The completion $\tilde{\tilde{M}}$ of $\tilde{M}$ is obtained by Cauchy completing $\tilde{M}$ with respect to $d$. The set $\tilde{\tilde{M}} = \tilde{M} - \tilde{M}$ is said to be the ideal set. $M$ and $\tilde{\tilde{M}}$ are topologically independent of the choice of $\text{dev}$. The developing map $\text{dev}$ and every deck transformation $\vartheta$ extends uniquely on $\tilde{\tilde{M}}$ to a distance-nonincreasing map and a homeomorphism respectively. For convenience, the extended maps are denoted by the same symbols $\text{dev}$ and $\vartheta$ respectively (see [11]).

An $m$-simplex in $\tilde{M}$ is a compact $m$-ball $A$ imbedded in $\tilde{M}$ such that its manifold interior $A^o$ is included in $\tilde{M}$ and $\text{dev}|A$ is a (topological) imbedding onto a closed nondegenerate affine $m$-simplex in $\mathbb{R}^n$. We introduce the “key” definition given by Carri`ere [6] adopted in the language of completions. (For a definition of $i$-convexity for real projective manifolds, see [13] or [14].)

**Definition 1.** We say that $M$ is $m$-convex, $0 < m < n$, if the following condition holds: Whenever $T \subset \tilde{M}$ is an $(m + 1)$-simplex with sides $F_1, \ldots, F_{m+2}$ such that $T^o \cup F_2 \cup \cdots \cup F_{m+2}$ does not meet $\tilde{\tilde{M}}$, then $T$ does not meet $\tilde{\tilde{M}}$ and is a subset of $\tilde{\tilde{M}}$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{tetrahedron.png}
\caption{An example of a tetrahedron $T$ detecting not being 2-convex in $\tilde{M}$: The gray area denotes $\tilde{\tilde{M}}$ and the dark area $\tilde{\tilde{M}} \cap F_1^o = \tilde{\tilde{M}} \cap T$ in Figures 2 and 3.}
\end{figure}

**Proposition 1.** Let $T$ be an affine $(m+1)$-simplex in $\mathbb{R}^n$ with sides $F_1, \ldots, F_{m+2}$. Every affine immersion of $T^o \cup F_2 \cup \cdots \cup F_{m+2}$ into $M$ extends to one of $T$ if and only if $M$ is $m$-convex.

A heuristic idea of 2-convexity is as follows: Consider a closed room and remove from it all solid objects. Then the completion $\tilde{A}$ of the remainder $A$ is the union of $A$ and the boundary points. Then $A$ will be 2-convex if all solid objects do not have “corners” or convex points or points which we can touch by an interior of a side of a tetrahedron, which we move around.
Any convex domain in $\mathbb{R}^n$ is $m$-convex for every $m$, and the complement of disjoint lines in $\mathbb{R}^n$ is 2-convex. However, the complement of a compact convex set in $\mathbb{R}^n$ is not $m$-convex for any $m$. (See Section 2 for explanations and more examples.) A last class of examples are given in Section 9 (see Theorem 3).

The boundary $\delta M$ of an affine manifold $M^3$ has a concave point $p$ if a convex open totally geodesic 2-disk meets $\delta M$ in a compact set containing $p$. A sufficient condition for $\delta M$ to have no concave point is that each point of $\delta M$ has a neighborhood affinely diffeomorphic to a convex domain in $\mathbb{R}^3$ or to a 3-dimensional submanifold $\Omega$ in $\mathbb{R}^3$ with $\delta \Omega$ with principal curvatures of both signs when $\delta \Omega$ is considered as an isometrically imbedded two-dimensional submanifold of the Euclidean space $\mathbb{R}^3$.

One can find $(m+1)$-simplices in $\tilde{M}$ inside an affine coordinate neighborhood in $\tilde{M}$, and there are many more $(m+1)$-simplices in $\tilde{M}$ (see Remark 4). The following lemma gives us our principal criterion of $m$-convexity.

**Proposition 2.** $M$ is not $m$-convex if and only if there exists an $(m+1)$-simplex $T$ with a side $F_1$ such that $T \cap \tilde{M}_\infty = F_1^0 \cap \tilde{M}_\infty \neq \emptyset$.

**Proof.** Suppose that every $(m+1)$-simplex $T$ has the property that $T$ do not meet $\tilde{M}_\infty$; $T \cap \tilde{M}_\infty$ is a subset of the union of two or more sides but not less than two sides; or if $T \cap \tilde{M}_\infty$ is a subset of a side $F$, then $F \cap \tilde{M}_\infty$ is not a subset of $F^o$. Then one see easily that the definition for $m$-convexity is satisfied by $M$.

Conversely, if $M$ is $m$-convex, and $T$ is an $(m+1)$-simplex with sides $F_1, \ldots, F_{m+2}$ such that $T^o \cup F_2 \cup \cdots \cup F_{m+2} \subset \tilde{M}$, then $T \subset \tilde{M}$. Consequently, there is no $(m+1)$-simplex $T$ with $T \cap \tilde{M}_\infty = F_1^o \cap \tilde{M}_\infty \neq \emptyset$. $\square$
Clearly, $M$ is $m$-convex if and only if any of the covers of $M$ is $m$-convex. It is easy to see that 1-convexity is equivalent to convexity, and $i$-convexity implies $j$-convexity whenever $i \leq j \leq n$ (see Section 2).

Our main theorem is:

**Theorem 1.** Let $M$ be a two-convex affine three-manifold. Then $\tilde{M}$ is diffeomorphic to $\mathbb{R}^3$.

Note above that $M$ does not need to be a closed 3-manifold in the premise of the theorem: $M$ could be an open, noncompact 3-manifold.

**Theorem 2.** Let $M$ be a closed $n$-dimensional affine manifold. If the holonomy group of $M$ has shrinkable dimension less than or equal to $d$, then $M$ is $d$-convex.

The above theorem almost answers another conjecture of Carrière [3, Section 3.3]. The proof is an essentially straightforward generalization of Carrière’s argument [3] with some modification involving the definition of shrinkable dimension. (However, we feel that our definition is just as useful in application.)

The above theorem combined with Theorem 1 implies Corollary 1.

**Corollary 1.** Let $M$ be a closed affine three-manifold. Then $\tilde{M}$ is diffeomorphic to $\mathbb{R}^3$.

**Corollary 2.** Let $M$ be a closed affine three-manifold with parallel volume form. Then the universal cover of $M$ is diffeomorphic to $\mathbb{R}^3$.

**Corollary 3.** Let $M$ be a closed $n$-dimensional affine manifold with holonomy of shrinkable dimension less than or equal to two ($n \geq 4$). Then the universal cover $\tilde{M}$ is foliated by totally geodesic leaves which are diffeomorphic to $\mathbb{R}^3/\Gamma$ for $\Gamma$ depending on the leaves.

**Proof.** Obviously $\tilde{M}$ is foliated by 3-dimensional leaves which are defined by setting the first $n-3$ coordinates of $x_i \circ \text{dev}$, $i = 1, \ldots, n$, equal to constants. Let $l$ be a leaf. Then since $M$ is 2-convex, $l$ is a 2-convex affine 3-manifold by Proposition 1. Hence, the universal cover of $l$ is diffeomorphic to $\mathbb{R}^3$ and $l$ is diffeomorphic to $\mathbb{R}^3/\Gamma$. \qed

We are curious how much the above tells you about the topology of $\tilde{M}$. In fact, we may choose any $n-3$ coordinate directions to obtain the above result. Our conjecture is that the universal cover of $M$ is diffeomorphic to $\mathbb{R}^n$. In particular, our conjecture implies that the universal cover of a closed affine 4-manifold with parallel symplectic form is diffeomorphic to $\mathbb{R}^4$.

As an application of Theorem 2, we see that the universal covers of the submanifolds listed in Proposition 8 are diffeomorphic to $\mathbb{R}^3$ respectively. By
the result of Gromov [26], any compact 3-manifold $M$ with nonempty boundary admits an affine structure with trivial holonomy. We see that if $M$ is reducible, then the boundary of $M$ always has a concave point, and given any affine structure on a fake cell, the boundary of it always has a concave point. (Here is an example of geometry and topology interacting.)

Lastly, we have the applications to singular hyperbolic manifolds. Singular hyperbolic manifolds with cone-type singularity will be defined in Section 10 as was done by Thurston (see Hodgson [31].)

**Theorem 3.** Let $M$ be a singular hyperbolic 3-manifold with cone-type singular locus along a link $L$. Then the universal cover of $M - L$ is diffeomorphic to $\mathbb{R}^3$.

This is also implied by Theorem 1.2.1 of Kojima [34], which is a stronger result (see also Kojima [33]) since he shows $M - L$ is atoridal.

We also make the following conjecture with a proof to be supplied shortly: Let $M$ be a singular hyperbolic 3-manifold with cone-type singular locus along a graph $K$ with cone-angle $\leq 2\pi$. Then the universal cover of $M - K$ is diffeomorphic to $\mathbb{R}^3$.

We think that this conjecture has some relation to a recent conjecture of Casson announced in the summer of 1998 at Tokyo Institute of Technology that the cone-angles of such manifolds can always be infinitesimally deformed.

In this paper, we will prove Theorems 1, 2, and 3, as the other propositions were proved already in this introduction.

In Section 1, we review materials on completions needed in this paper. In Section 2, we list examples on $d$-convexity and discuss $d$-convexity. We prove Propositions 1 and 8.

The purpose of Sections 3 to 7 is to prove Theorem 1. Section 8 to prove Theorem 2 and Section 9 to prove Theorem 3. For simplicity, we assume that $M = \tilde{M}$ or $M$ is simply connected from Sections 3 to 7, and we assume otherwise in Sections 1, 2, 8, and 9.

Since $M$ is simply connected in Sections 3 to 7, the developing map $\text{dev}$ is defined on $M$ itself. Let $M$ have a coordinate function $x_1$ given by composing the developing map $\text{dev} : M \to \mathbb{R}^3$ to $\mathbb{R}^3$ with an affine coordinate function $x_1$ of $\mathbb{R}^3$. This determines level sets, which are open surfaces, in $M$ with $x_1^{-1}(t)$ denoted by $M_t$.

Our plan to prove Theorem 1 is to show geometric incompressibility, i.e. $2$-convexity, implies topological incompressibility of level sets. That is, $2$-convexity implies that each component of the level set in $M$ is incompressible. Since $M$ is simply connected, each component is homeomorphic to a disk. A result of Palmeira [43] says that a simply connected 3-manifold foliated by leaves diffeomorphic to $\mathbb{R}^2$ is diffeomorphic to $\mathbb{R}^3$. Thus, we obtain our result that $M$ is diffeomorphic to $\mathbb{R}^3$. To prove that each component surface $F$ of a
level set is incompressible, we show that there are no compressing disks of \( F \). An amenable disk is an embedded disk where the height function \( x_1 \) is Morse. We deform any disk with boundary in \( F \) to an amenable disk and we cancel critical points to homotope it to an imbedded disk in \( F \). This is accomplished by showing that 2-convexity implies the following properties:

- A property associated with a 3-simplex in \( \hat{M} \) (Section 3).
- Filling of a drum with the bottom side removed (see Theorem 3 in Section 4).
- Filling of an amenable disk with only one critical point and annuli with two critical points (Section 5).
- A homotopy of an amenable disk with only three critical points into a disk in \( F \) (Section 6).
- A homotopy of an amenable disk with arbitrarily many critical points into a disk in \( F \) (Section 7).
- The incompressibility of each component of \( M_t \).

We note that this method of Morse cancellation was already employed by Floyd-Hatcher [19] in their work on punctured-torus bundles over circles. (Hatcher-Thurston [27] shortly generalized this result to two-bridge knots.) Our method only differs from theirs slightly and we provide some detailed analysis steps missing from their work but needed in this paper (mainly in the Appendix).

In Section 3, as a preliminary result, we show that given a 2-convex affine 3-manifold \( M \), if \( T \subset \hat{M} \) is a 3-simplex with a side \( F_1 \) meeting \( \hat{M}_\infty \), then there is no exposed point of \( \hat{M}_\infty \cap F_1 \) in \( F_0 \). (Note that even when \( M \) is 2-convex a 3-simplex may still meet \( \hat{M}_\infty \) in a side, but not in the interior of a side.) This will be used to prove that a solid-cylinder map \( f : \Omega \times [a, b] \to \hat{M} \) can only be a map into \( \hat{M} \).

![Figure 4. A figure of a drum](image)

Section 4 is on the filling of a drum: An embedding from a smooth manifold (perhaps with corners) to another smooth manifold is a topological imbedding, which is a restriction of a smooth immersion in some ambient manifolds. (The word “imbedding” will be reserved for topological imbeddings.) A drum with the bottom side removed is the union of an embedded image of a surface \( \Omega \) into
a level $b$ and the image of embedding from $\delta \Omega \times [a, b]$ to $M$ mapping $(x, t)$ into level $t$ for each $t \in [a, b]$ which is transversal to the level sets. The following process describes the process of the filling of a drum with the bottom side removed. Let $\Omega$ be the surface with smooth boundary in the plane $\mathbb{R}^2$. We show that if there is an embedding from $\delta \Omega \times [a, b]$ to $M$ preserving levels, that is, mapping $(x, t)$ to a point of $M_t$ for each $t \in [a, b]$, and if the embedding restricted to $\delta \Omega \times \{b\}$ extends to one from $\Omega \times \{b\}$ to $M_b$, then the embedding extends to one from $\Omega \times [a, b]$ into $M$ preserving levels.

The proof that the filling of a drum is always possible uses the flow argument based on noticing the flow argument can be applied to ideal boundary in some limited sense.

In Sections 5, 6, and 7, the result of Section 4 will be modified further to show that if an embedded disk $D$ in $M$ has the boundary in a level set $M_t$, then the boundary bounds a disk in $M_t$. Any embedding $F$ of a surface $\Omega$ into $M$ where $F(\delta \Omega) \subset M_t$ for some $t$ can be modified to a so-called amenable embedding $F' : \Omega \to M$ of level $t$ such that $x_1 \circ F'$ is a Morse function and the boundary $\delta \Omega$ maps into $M_t$. Our strategy is to reduce the number of critical points by cancellation.

By a filling of an amenable imbedding $F : \Omega \to M$ of level $t$, we mean the process of finding a subsurface $\Omega'$ in $M_t$ and a 3-dimensional submanifold $N$ of $M$ such that $\Omega$ and $\Omega'$ meet in their common boundary and $\Omega \cup \Omega'$ forms the boundary of $N$. Here $\Omega'$ is unique due to Lemma 1.

In Section 5, we show that given an amenable imbedding of a disk with one critical point, then the boundary bounds a disk in $M_t$. Then we study amenable imbeddings of annuli with two critical points. We show that the amenable disk and annuli are fillable.

In Section 6, we will show that given an amenable disk $D$ of level $t$ with three critical points, $\delta D$ bounds a disk in $M_t$. This is the most technical part of the paper and the reason is that we want to cancel singularities without introducing new ones, and hence, we need to construct deformations carefully. However, the manner in which we do is entirely similar to what one would do in p.l. 3-manifold theory consisting of various disk moves and isotopies.

In Section 7, we do an inductive argument by reducing the number of critical points of amenable disks, showing that given an amenable disk $D$ of level $t$, $\delta D$ bounds a disk in $M_t$. This result, the incompressible surface theory, and Corollary 3 of Palmeira [43] shows that $M$ is diffeomorphic to $\mathbb{R}^3$. This completes the proof of Theorem 1.

In Section 8, we discuss shrinkable dimension. We will give examples of shrinkable dimensions for some Lie groups. The basic idea behind this definition is that along the direction of real maximal tori the affine action of the given Lie group can shrink and stretch indefinitely and in the other directions,
the Lie group action can do so only finitely. We now assume that $M$ is a closed affine $n$-manifold. We give the proof of Theorem 2. The main idea is to follow Carrière’s argument [3] based on infinite rays. However, we need to perturb the $(d + 1)$-simplices sometimes to control the size of the ideal set with rather involved argument. (We point out that Carrière’s argument [3] does not carry out this step and hence we believe it contains a negligible gap, which we fill here.)

In Section 9, we prove Theorem 3. Since the universal cover $(M - L)^\sim$ has a hyperbolic structure, $(M - L)^\sim$ develops into the standard disk in $\mathbb{R}P^3$. Since the standard disk is in the affine patch of $\mathbb{R}P^3$, it follows that $(M - L)^\sim$ has an affine structure. The set of ideal points $(M - L)_\infty^\sim$ consists of lines corresponding to $L$ and points of the sphere of infinity $S_\infty^2$. A large part of Section 9 is devoted to proving that $(M - L)^\sim$ is 2-convex. By Theorem 1, we obtain Theorem 3.

In the appendix, we prove technical Lemma 8 needed in Section 7. That is, we show that there exists a suitable coordinate system on a connected neighborhood including a saddle and a relative maximum point so that our Morse function takes an integral form.

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1. Properties of Completions

In this section, we will capture the necessary facts about the completions due to Kuiper [35] needed in this paper. Let $M$ be an affine $n$-manifold, which is not necessarily simply connected. Proposition 3 shows that we can tell precisely about how two balls meet in $\tilde{M}$ by looking at the way that their developed images meet. Remark 1 is used in many occasions in this paper. The two are basic tools of this paper and other papers (see [11], [12], [14], and [16]).

Associated with an affine structure on an $n$-manifold $M$, there is a development pair $(\text{dev}, h)$ where $\text{dev}: \tilde{M} \to \mathbb{R}^n$ is an immersion and $h: \pi_1(M) \to \text{Aff}(\mathbb{R}^n)$ is a homomorphism such that $h(\vartheta) \circ \text{dev} = \text{dev} \circ \vartheta$ for $\vartheta \in \pi_1(M)$. 
Proof. The map \( \text{dev}| \text{Cl}(A) \) is a \( d \)-isometry. The lemma follows.

A \emph{tame} subset of \( \tilde{M} \) is a compact ball \( A \) imbedded in \( \tilde{M} \) such that \( \text{dev}|A \) is an imbedding onto a convex subset of \( \mathbb{R}^n \) and \( A^o \subset \tilde{M} \). For example, a simplex is tame.

**Proposition 4.** The closure of a \( d \)-bounded convex subset of \( \tilde{M} \) is a tame subset of \( \tilde{M} \). Conversely, the interior of a tame subset is a convex subset of \( \tilde{M} \).

**Definition 2.** A \( k \)-ball in \( \tilde{M} \) is a subspace homeomorphic to a \( k \)-ball such that its interior is a subset of \( \tilde{M} \) and \( \text{dev}|A \) is an imbedding onto a \( k \)-ball in a \( k \)-dimensional affine subspace \( P^k \) of \( \mathbb{R}^n \).

**Proposition 5.** Let \( A \) be a \( k \)-ball in \( \tilde{M} \), and \( B \) and \( l \)-ball. Suppose that \( A^o \cap B^o \neq \emptyset \), \( \text{dev}(A) \cap \text{dev}(B) \) is a compact manifold in \( \mathbb{R}^n \) with interior equal to \( \text{dev}(A)^o \cap \text{dev}(B)^o \), and \( \text{dev}(A)^o \cap \text{dev}(B)^o \) is arcwise-connected. Then \( \text{dev}|A \cup B \) is a homeomorphism onto \( \text{dev}(A) \cup \text{dev}(B) \).

Proof. First, we prove injectivity: Let \( x \in A, y \in B, \) and \( z \in A^o \cap B^o \). Suppose that \( \text{dev}(x) = \text{dev}(y) \). There is a path \( \gamma \) in \( \text{dev}(A) \cap \text{dev}(B) \) from \( \text{dev}(z) \) to \( \text{dev}(x) \) such that \( \gamma|[0,1] \) maps into \( \text{dev}(A)^o \cap \text{dev}(B)^o \). Since \( \text{dev}|A \) is an imbedding onto \( \text{dev}(A) \), there is a lift \( \gamma_A \) of \( \gamma \) into \( A^o \) from \( z \) to \( x \). Similarly, there is a lift \( \gamma_B \) of \( \gamma \) into \( B^o \) from \( z \) to \( y \). Note that \( \gamma_A|[0,1] \) agrees with \( \gamma_B|[0,1] \) by the uniqueness of lifts of arcs for immersions. Thus, \( x = y \) since \( M \) is a complete metric space.

By injectivity, there is a well-defined inverse function \( f \) to \( \text{dev}|A \cup B \). \( f \) restricted to \( \text{dev}(A) \) equals the inverse map of \( \text{dev}|A \), and \( f \) restricted to \( \text{dev}(B) \) the inverse map of \( \text{dev}|B \). Since both inverse maps are continuous,
and \( \text{dev}(A) \) and \( \text{dev}(B) \) are closed, \( f \) is continuous. (See also Proposition 1.3.1 in \[6\].)

**Proposition 6.** Let \( \{K_i\} \) be a sequence of uniformly \( d \)-bounded tame subsets of \( \tilde{\mathcal{M}} \) where \( K_i \subset K_{i+1} \) for each \( i, i = 1, 2, \ldots \). Then the closure \( K' \) of \( \bigcup_{i=1}^{\infty} K_i \) is a tame subset of \( \tilde{\mathcal{M}} \), and \( \text{dev}(K') \) equals the closure of \( \bigcup_{i=1}^{\infty} \text{dev}(K_i) \).

**Proof.** Straightforward.

**Remark 1.** We use the above proposition in the following form: (See Carrière \[6, Section 3\] also.) Let \( T' \) be an affine \( m \)-simplex in \( \mathbb{R}^n \) and \( x \) a point of \( \tilde{\mathcal{M}} \) such that \( \text{dev}(x) \) is a point of \( \delta T' \). Let \( T'_t \) denote the image of \( T' \) under the action of dilatation with center \( \text{dev}(x) \) by a factor \( t, 0 < t \leq 1 \). Let \( A \) be the subset of \( (0, 1] \) such that for \( t \in A \) there exists a tame set \( T_t \) in \( \tilde{\mathcal{M}} \) such that \( x \in \delta T_t \) and \( \text{dev}(T_t) = T'_t \). Then we can show

(a) \( A \) includes \( (0, \epsilon) \) for a small positive number \( \epsilon \).

(b) \( T_t \subset T_{t'} \) if \( t < t', t, t' \in A \), and for each \( t \), \( T_t \) is unique.

(c) \( A \) is an open subset of \( (0, 1] \).

(d) \( A \) is of form \( (0, \tau) \) for a positive number \( \tau \) or equals \( (0, 1] \).

(e) The closure \( T_A \) of \( \bigcup_{t \in A} T_t \) is a tame set such that \( \text{dev}|T_A \) is an imbedding onto the closure of \( \bigcup_{t \in A} T'_t \). Thus \( T_A \) is an \( m \)-simplex in \( \tilde{\mathcal{M}} \) (however, not in \( \mathcal{M} \) in general).

**Proof.** (a) For a coordinate neighborhood \( U \) of \( x \), \( \text{dev}|U \) is a coordinate chart of \( x \). Assume for convenience, that \( \text{dev}(U) \) is a convex open ball. Hence for \( \epsilon \) small enough, there exists \( T_t \) in \( U \) for \( t < \epsilon \).

(b) Since each \( T_t \) contains \( x \) in \( \delta T_t \), \( T_t \) includes \( T_k \) for small enough \( k \) so that \( T_k \subset U \). This follows since the maximal totally geodesic connected submanifold including \( T_t \) or \( T_k \) is determined by its tangent space at \( x \). Hence, \( T_t \) and \( T_{t'} \) includes \( T'_k \) for \( k < \epsilon, t, t' \). By Proposition 3, this means \( T_t \subset T_{t'} \) since \( \text{dev}(T_t) \subset \text{dev}(T_{t'}) \). The uniqueness follows from this statement.

(c) Since \( T_t \) is a compact subset of \( \tilde{\mathcal{M}} \), and \( \text{dev}|T_t \) is an imbedding, there exists a compact neighborhood \( U \) of \( T_t \) such that \( \text{dev}|U \) is an imbedding to its image. Hence, for \( t' \) sufficiently near \( t \), \( U \) includes \( T_{t'} \), and \( t' \in A \).

(d) \( A \) is connected since \( T_t \subset T_{t'} \) if \( t < t' \) for \( t, t' \in A \). Thus (d) follows from (a) and (c).

(e) follows from Proposition 3.

2. \( d \)-convexity: Examples

We start this section by showing that 1-convexity is equivalent to convexity. Next, we give some examples of \( m \)-convex manifolds and prove Propositions 7.
We also prove Proposition 1 from the introduction. Lastly, we show that $i$-convexity implies $j$-convexity if $j \geq i$.

The affine manifold $M$ is said to be convex if its universal cover $\tilde{M}$ is a convex set. The following proposition was proved by Carrièere (see [3]). We give a brief proof based on our definition of 1-convexity.

**Proposition 7.** The following statements are equivalent.

1. $M$ is 1-convex.
2. $M$ is convex.
3. $M$ is affinely diffeomorphic to $\Omega/\Gamma$ where $\Omega$ is a convex domain in $\mathbb{R}^n$ and $\Gamma$ is a group of affine transformations acting on $\Omega$ freely and properly discontinuously.

**Proof.** We show that (1) implies (2). Suppose we are given two segments $s_1$ and $s_2$ in $\tilde{M}$ with endpoints $x$ and $y$ of $s_1$ and $z$ and $t$ of $s_2$. Suppose $t = y$ and $s_1$ and $s_2$ is in general position. We will show that there exists a segment in $\tilde{M}$ with endpoints $x$ and $z$. Let us give affine parameters $f_1$ and $f_2 : [0, 1] \to \tilde{M}$ to $s_1$ and $s_2$ respectively so that $f_1(0) = y$, $f_2(0) = y$, $f_1(1) = x$, and $f_2(1) = z$. For $t \in (0, 1]$, define $T_t$ be the triangle in $\tilde{M}$ with vertices $f_1(t)$, $f_2(t)$, and $y$ if it exists. By Remark 1, we see that $A$ is of form $(0, \tau)$ for $\tau > 0$ or equal $(0, 1]$.

Suppose $A$ of form $(0, \tau)$. The closure $T'$ of $\bigcup_{t \in A} T_t$ is a triangle in $\tilde{M}$. Since the two edges of $T'$ are subset of $s_1$ and $s_2$, they do not meet $\tilde{M}_\infty$. Since $M$ is 1-convex, this means $T' \subset \tilde{M}$ and $\tau \in A$. This is a contradiction, and $A = (0, 1]$.

This means that $M$ includes a triangle $T$ with vertices $x$, $y$, $z$, and $x$ and $z$ is connected by a segment.

Let $x$ and $y$ be arbitrary points of $\tilde{M}$. Then $x$ and $y$ are connected by a broken geodesic. The above result tells you that we can find a broken geodesic with less number of segments than our initial choice. An induction shows that $x$ and $y$ are connected by a segment.

(2) Since $\tilde{M}$ is a tame set, $\text{dev}|\tilde{M} \to \text{dev}(\tilde{M})$ is an imbedding, and hence (3) follows.

That (3) implies (1) follows from the fact that $\tilde{M}$ is affinely diffeomorphic to a convex domain in $\mathbb{R}^n$. $\blacksquare$

We will give some examples illustrating $m$-convexity. Initial examples are affine manifolds whose developing maps are imbeddings.

A convex domain is easily seen to be $i$-convex for each $i$ since the convex hull of any finite set of points of the domain is still in the domain. By the above proposition, every convex affine manifold $M$ is $i$-convex for every $i$ since $\tilde{M}$ is affinely diffeomorphic to a convex domain in $\mathbb{R}^n$.

Let $x_1, x_2$, and $x_3$ denote the standard coordinates of $\mathbb{R}^3$. Remove from $\mathbb{R}^3$ the set given by $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$. Then the remainder is an affine
manifold $M$ which is simply connected. Clearly, $\tilde{M} = M$ and $\hat{M}$ can be identified with $\mathbb{R}^3$ removed with the set given by $x_1 > 0, x_2 > 0, x_3 > 0$. Then $M$ is not 2-convex, since the 3-simplex given by $x_1, x_2, x_3 \geq -1$ and $x_1 + x_2 + x_3 \leq 0$ in $\hat{M}$ has unique side whose interior intersects $M_\infty = \hat{M} - \tilde{M}$ (see Proposition 2 and Figure 4).

Remove from $\mathbb{R}^3$ the set given by $x_1 \geq 0, x_2 \geq 0$. Then the remaining affine manifold $M$ is simply connected and $M = \tilde{M}$ and $\tilde{M}$ can be identified with $\mathbb{R}^3$ removed with the set given by $x_1 > 0, x_2 > 0$. Here $\tilde{M}_\infty$ equals the union of the set given by $x_1 = 0$ and $x_2 \geq 0$ and the set given by $x_2 = 0$ and $x_1 \geq 0$. Then $M$ is 2-convex since given any 3-simplex $T$ in $\mathbb{R}^3$ meeting $\tilde{M}_\infty$ in the interior of a side $F_1, T^o$ or the boundary of $F_1$ meets $\tilde{M}_\infty$. $M$ is not 1-convex since there exists a 2-simplex $F$ in $\tilde{M}$ with an edge $e$ intersecting the line given by $x_1 = x_2 = 0$ in a point of $e^o$ and $F$ meets $\tilde{M}_\infty$ exactly at this point only.

A wedge in $\mathbb{R}^3$ is a convex set given by inequalities $l_1(x) \geq c_1$ and $l_2(x) \geq c_2$ for affine functions $l_1$ and $l_2$ and constants $c_1$ and $c_2$, where $l_1(x) = c_1$ and $l_2 = c_2$ define distinct planes respectively. If we remove from $\mathbb{R}^3$ a collection of disjoint wedges, it follows easily that the remaining affine manifold is 2-convex.

A convex cone in the affine space $\mathbb{R}^3$ is a convex domain $\Omega$ with a distinguished point $p$ such that every point of $\Omega$ other than $p$ has an infinite ray in $\Omega$ from $p$ passing through it. A proper convex cone is a cone that has no complete affine line in it. If we remove from $\mathbb{R}^3$, a collection of disjoint properly convex cones, the remaining affine manifold is not 2-convex.

Another example is an affine manifold in $\mathbb{R}^3$ given by $x_1^2 - x_2^2 - x_3^2 < -1$. This is 2-convex but not 1-convex. The affine manifold given by $x_1^2 - x_2^2 - x_3^2 > 1$ is 1-convex and 2-convex.

Lastly, we mention that given a closed real projective surface $\Sigma$ of negative Euler characteristic, a trivial circle bundle $\Sigma$ over $\Sigma$ carries an affine structure by Benzécri suspension (see [1], [2], and [10]). If $\Sigma$ is not convex, then $M$ is not convex but 2-convex. We omit the simple proof of this fact.

We prove Proposition [1]: Suppose that $M$ is not $m$-convex, and let $p : \tilde{M} \to M$ denote the universal covering map. Then there exists an $(m + 1)$-simplex $T$ in $\tilde{M}$ such that $T \cap \tilde{M}_\infty = F_1^o \cap \tilde{M}_\infty \neq \emptyset$ for a side $F_1$ of $T$. Since $T$ is an $(m + 1)$-simplex, $\text{dev}|T$ is an imbedding onto $\text{dev}(T)$ and $\text{dev}(T)$ is an $(m + 1)$-simplex in $\mathbb{R}^n$. Let $f$ be the map $(\text{dev}\{\text{dev}(T)\})^{-1}$ restricted to $\text{dev}(T^o) \cup \text{dev}(F_2) \cup \text{dev}(F_3) \cdots \cup \text{dev}(F_{m+2})$. $f$ is an affine immersion to $\tilde{M}$. It is easy to see that $f$ does not extend to all of $\text{dev}(T)$. Hence, $p \circ f$ is an affine immersion which does not extend to $\text{dev}(T)$.

Suppose that $M$ is $m$-convex. Let $\text{dev} : \tilde{M} \to \mathbb{R}^n$ be the developing map. Let $f : T^o \cup F_2 \cup \cdots \cup F_{m+2}$ be an affine immersion into $M$ where $T$ is an $m$-simplex with sides $F_1, \ldots, F_{m+2}$ in $\mathbb{R}^n$. Let $\tilde{f}$ be the lift of $f$ to $\tilde{M}$. Then $\text{dev} \circ \tilde{f}$ is also an affine map from $T^o \cup F_2 \cup \cdots \cup F_{m+2}$ into $\mathbb{R}^n$. Since the rank of
the map $\text{dev} \circ \hat{f}$ is maximal, the map extends to a global affine transformation on $\mathbb{R}^n$, and $\text{dev} \circ \hat{f}$ is an imbedding. Hence, $\text{dev}|\hat{f}(T')$ is an imbedding onto an open $m$-simplex $\text{dev} \circ \hat{f}(T')$ in $\mathbb{R}^n$. Since $\hat{f}(T')$ is a convex subset of $\tilde{M}$, the closure $T''$ of $\hat{f}(T')$ is a tame subset of $\tilde{M}$ such that $\text{dev}|T''$ is an imbedding onto $\text{Cl}(\hat{f}(T'))$ by Proposition 4. Since $T''$ is an $m$-simplex with sides $\hat{f}(F_2), \ldots, \hat{f}(F_{m+2})$ and a remaining side $F'_1$, $\hat{f}(T')$ and $\hat{f}(F_2), \ldots, \hat{f}(F_{m+2})$ are subsets of $\tilde{M}$; and $M$ is $m$-convex, we have $T'' \subset M$. Clearly, $\hat{f}$ extends to an affine embedding $f' : T \to T'$, and $p \circ f'$ extends $f$.

**Proposition 8.** (a) Let $\Omega$ be a convex open domain in $\mathbb{R}^3$. Then $\Omega$ removed with the closed set that is the union of complete lines (not necessarily finitely many) is 2-convex.

(b) The interior of a closed 3-dimensional submanifold $M$ of $\mathbb{R}^3$ with smooth boundary without a concave point is 2-convex.

(c) The interior of a compact affine manifold $M$ with trivial holonomy and smooth boundary without a concave point is 2-convex.

**Proof.** (a) Let $\Omega'$ be the convex domain $\Omega$ in $\mathbb{R}^3$ removed with the closed set that is the union of a collection of complete lines. Let $T$ be an affine 3-simplex with sides $F_1, F_2, F_3$, and $F_4$ and an affine immersion $f : T^o \cup F_1 \cup F_3 \cup F_4$ into $\Omega'$. Since the rank of the affine map $f$ is maximal, $f$ is an imbedding and the image equals $T^o \cup F_1 \cup F_3 \cup F_4$ for an affine 3-simplex $T'$ in $\mathbb{R}^3$ with sides $F'_1, F'_2, F'_3, F'_4$. Since the vertices of $T'$ are in $\Omega$, $T'$ is a subset of $\Omega$.

Suppose that $T'$ is not a subset of $\Omega'$. Since $\text{bd}\Omega'$ is the union of a sphere $\text{bd}\Omega$ and the union of removed lines, if $T'$ meets $\text{bd}\Omega'$, then $T'$ meets $\text{bd}\Omega$ or a line $l$. Since $\Omega$ is convex and vertices of $T'$ are in $\Omega$, $T'$ do not meet $\text{bd}\Omega$. Thus $T'$ meets $l$, one of the removed lines; more precisely, $F^o_1$ meets $l$. $l$ meets $F^o_1$ tangentially since otherwise $l$ meets $T^o$, which is a contradiction. Since $l$ has endpoints in $\text{bd}\Omega$, $l$ must meet the boundary of the 2-simplex $F_1$, which is compact in the open set $\Omega$. But since $\delta F_1$ is a subset of $F_2 \cup F_3 \cup F_4$, this is absurd. Therefore, $T' \subset \Omega'$ and $f$ extends to an affine embedding $f' : T \to T'$.

By Proposition 1, $\Omega'$ is 2-convex.

To prove (b), it is sufficient to prove (c). Let $M$ be a compact affine manifold with smooth boundary and trivial holonomy. We assume that $\delta M$ has no concave point. Since the holonomy of $M$ is trivial, the developing map $\text{dev} : \tilde{M} \to \mathbb{R}^3$ factors into the composition of the covering map $p : \tilde{M} \to M$ and $\text{dev}' : M \to \mathbb{R}^3$. Thus the induced metric $d$ on $\tilde{M}$ agrees with that induced from the metric $d'$ on $M$ where $d'$ is induced from $\mathbb{R}^3$ by $\text{dev}'$. Since $M$ is compact, $d'$ is complete, and so is $d$ on $\tilde{M}$. Thus, the $d$-completion of the universal cover $\tilde{M}^o$ of $M^o$ can be identified with $\tilde{M}$, and $\tilde{M}^\infty$ with $\delta\tilde{M}$.

If $M^o$ is not 2-convex, then by Proposition 2 there exists a 3-simplex $T$ in $\tilde{M}^o$ with sides $F_1, F_2, F_3, F_4$ such that $T \cap \delta\tilde{M} = F^o_1 \cap \delta\tilde{M} \neq \emptyset$. Let $x$ be a
point of $F^1_i \cap \delta \tilde{M}$. Let $N$ be an open ball-neighborhood of $x$ in $\tilde{M}$ such that $p|N$ is a diffeomorphism onto $p(N)$ a neighborhood of $p$ in $N$. Then the open surface $p(F^o_1 \cap N)$ shows that $p(x)$ is a concave point of $\delta M$. Since this is a contradiction, $M$ is 2-convex.

\(\square\)

Remark 2. If an affine manifold $M$ is $i$-convex, then it is $j$-convex for each $j$, $j \geq i$.

Proof. Let $M$ be $i$-convex with $i \leq n-2$. We will now show $M$ is $(i+1)$-convex. The rest follows by induction.

Let $T$ be an $(i+2)$-simplex in $\tilde{M}$ with sides $F_1, \ldots, F_{i+3}$, and assume $F_2, \ldots, F_{i+3}$ and $T^o$ do not meet $M_\infty$. $T$ can be considered the union of $(i+1)$-simplices $S_t$ for $t \in [0, 1]$ with common side $G_2$ equal to $F_2 \cap F_3$. In fact, we let $S_t$ to be the convex hull of $tv_2 + (1-t)v_3$ and $G_2$. Each $S_t$ has sides $G_1^t, \ldots, G_{i+2}^t$ such that $G_2^t = G_2$ and $G_i^t$ is a subset of $F_1$ and for $0 < t < 1$, the interior of $G_i^t$ is a subset of $F_i^o$. Since $G_2^t, \ldots, G_{i+2}^t$ are respectively subsets of $F_2 \cup \cdots \cup F_{i+3}$, it follows that $G_2^t, \ldots, G_{i+2}^t$ does not meet $M_\infty$. Since $M$ is $i$-convex, it follows that $S_t$ are all subsets of $\tilde{M}$ and so is $T$. Hence $\tilde{M}$ is $(i+1)$-convex. By induction the rest follows. \(\square\)

3. Tetrahedral convexity and solid-cylinder convexity

As we said above, from Sections 3 to 7 we assume that our affine 3-manifold is simply connected. We have $M = \tilde{M}$ in this case. The purpose of this section is to show that two-convexity implies “tetrahedral two-convexity”, Proposition \ref{tetrahedral} and further “solid-cylinder two-convexity”, Theorem \ref{solid-cylinder}. This theorem will be used to prove the filling of a drum with the bottom side removed in the next section.

A point $x$ of a convex subset $A$ of $\mathbb{R}^n$ is said to be exposed if there exists a supporting affine hyperplane $H$ at $x$ such that $H \cap A = \{x\}$ (see Berger \cite[p. 361]{Berger}). Compare the following proposition to Proposition \ref{exposed}. Although there is no three-simplex with the interior of a side meeting the ideal set $M_\infty$ and meeting $M_\infty$ only there, there could still be a three-simplex meeting $M_\infty$ at a side.

Proposition 9. Let $M$ be a two-convex closed affine three-manifold which is simply connected. Let $T$ be a convex compact three-simplex in $\tilde{M}$, $F_1$ a side of $T$ such that $T \cap M_\infty = F_1 \cap M_\infty \neq \emptyset$, and $P$ the affine two-space including $\text{dev}(F_1)$. Then the convex hull $K$ of $\text{dev}(F_1 \cap M_\infty)$ in $P$ has no exposed point belonging to $\text{dev}(F_1^o)$.

Proof. Suppose not. Let $x'$ be an exposed point of $K$ belonging to $\text{dev}(F_1^o)$, and $H$ the supporting line in $P$ such that $H \cap K = \{x'\}$. It is clear that $x'$ is an element of $\text{dev}(F_1 \cap M_\infty)$. Let $s'$ be the segment $\delta H \cap \text{dev}(F_1)$, and $s$ the
inverse image in \( F_1 \) of \( s' \) under \( \text{dev}|F_1 \); let \( x \) be the point of \( s \) corresponding to \( x' \). Then \( s - \{ x \} \subset M \), and \( x \in M_\infty \). A component of \( \text{dev}(F_1) - s' \) is disjoint from \( K \). Let \( \alpha \) be the closure of the corresponding component of \( \delta F_1 - s \) disjoint from \( M_\infty \cap F_1 \). Since \( \alpha \) is a compact arc disjoint from \( M_\infty \), there exists a compact three-ball neighborhood \( U \) of \( \alpha \) in \( M \). By Proposition 3, we may assume without loss of generality that \( \text{dev}|T \cup U \) is an imbedding onto a compact subset of \( \mathbb{R}^3 \). We may choose \( U \) so that \( \text{dev}(U) \) is a compact neighborhood of \( \text{dev}(\alpha) \) consisting of points of \( d \)-distance from \( \text{dev}(\alpha) \) less than or equal to \( \epsilon \) for some small positive constant \( \epsilon \).

Let \( v_1' \) be the vertex of \( \text{dev}(T) \) opposite to \( \text{dev}(F_1) \), and \( T' \) a convex three-simplex in \( \mathbb{R}^3 \) with vertex \( v_1' \) satisfying the following condition:

- The triangle in the unit tangent bundle at \( v_1' \) determined by \( T' \) matches with that determined by \( \text{dev}(T) \).
- \( F_1' \cap \text{dev}(F_1) = \text{dev}(s) \) where \( F_1' \) is the side of \( T' \) opposite to \( v_1' \).
- \( \delta T' - F_1' \subset \text{dev}(T \cup U) \).

Let \( T_t \) for \( t \in (0, 1) \) be the convex 3-simplex that is the image of \( T' \) under the dilatation with center \( v_1' \) by the magnification factor \( t \). Let \( v_1 \) be the vertex of \( T \) corresponding to \( v_1' \), and consider the subset \( A \) of \( (0, 1) \) whose element \( t \) is such that \( M \) includes a convex three-simplex \( T_t \) such that \( \delta T_t \) contains \( v_1 \) and \( \text{dev}|T_t \) is an imbedding onto \( T_t' \). By Remark 3, \( A \) is of form \( (0, \tau) \) or \( (0, 1) \).

Suppose that \( A \) is of form \( (0, \tau) \). By Remark 3, there exists a compact convex three-ball \( T_\tau \) in \( M \) such that \( \text{dev}|T_\tau \) is an imbedding onto \( T_\tau' \) and \( T_\tau' \subset M \).

The third condition above states that there exists a compact subset \( D \) of \( T \cup U \) such that \( \text{dev}(D) \) equals \( \delta T' - F_1' \). There exists a compact subset \( D_\tau \) of \( T_\tau \), such that \( \text{dev}|D_\tau \) is an imbedding onto \( \delta T_\tau' - F_1' \) where \( F_1' \) is a side of \( T_\tau' \) opposite to \( v_1' \). We have \( \text{dev}|D_\tau \subset \text{dev}(D) \). Since \( \text{dev}|T \cup U \cup T_\tau \) is an imbedding by Proposition 3 (up to choosing \( U \) carefully), we obtain \( D_\tau \subset D \subset M \). Hence, \( T_\tau \cap M_\infty \) is included in the interior of the side of \( T_\tau \) corresponding to \( F_1' \). By definition of two-convexity, it follows that \( T_\tau \subset M \) and \( \tau \in A \), a contradiction.

Suppose now \( A = (0, 1) \). Then \( M \) includes a three-simplex \( T_1 \) such that \( \text{dev}|T_1 \) is an imbedding onto \( T' \). By Proposition 4, \( \text{dev}|T \cup T_1 \) is an imbedding onto \( \text{dev}(T) \cup \text{dev}(T_1) \). As \( \text{dev}(s) \) is a subset of \( T' \), we obtain \( s \subset T_1 \subset M \); however, this contradicts \( x \in M_\infty \).

Remark 3. We will meet this type of changing the direction or tilting of a side again in Section 8.

Let \( (\text{dev}, h) \) be a development pair of \( M \). We have affine coordinate functions \( x_1, x_2, x_3 : \mathbb{R}^3 \to \mathbb{R} \). For convenience, we let \( x_i \) denote the map \( x_i \circ \text{dev} \) defined on \( M \) for each \( i \), \( i = 1, 2, 3 \). Let \( M_t \) denote \( x_t^{-1}(t) \cap M \), which is a closed subset of \( M \).
Lemma 1. For each $t, t \in \mathbb{R}$, each component of $M_t$ is a properly imbedded open surface or an empty set. In particular, the union of a collection of disjoint simple closed curves in $M_t$ forms the boundary of at most one subsurface of $M_t$.

Proof. Since $M$ is without boundary, $M_t$ is also without boundary. Let $U$ be a component of $M_t$. Since $\text{dev}|U$ is an immersion onto an open subset of an affine hyperplane of $\mathbb{R}^3$, $U$ is not compact. Hence, $U$ is an open surface. The conclusion follows. □

We now define solid-cylinder maps: Given an embedded compact surface $\Omega$ in the plane $\mathbb{R}^2$ with smooth boundary and the interval $[a, b]$ of $\mathbb{R}$ for $a, b \in \mathbb{R}$, $a < b$, think of $\Omega \times [a, b]$ as sitting in the Euclidean space $\mathbb{R}^2 \times \mathbb{R}$. A continuous map $f : \Omega \times [a, b] \to \tilde{M}$ is called a solid-cylinder map if $f$ satisfies the following conditions:

(i) $f|\Omega \times [a, b]$ is an injective map into $M$.
(ii) $\text{dev} \circ f|\Omega \times [a, b]$ is a $C^1$-immersion.
(iii) $x_1 \circ \text{dev} \circ f|\Omega \times \{t\}$ is a constant map with value $t$ for each $t, t \in [a, b]$.
(iv) $f|\delta \Omega \times [a, b]$ is an embedding into $M$.

In particular, $f|\Omega \times \{t\}$ is an embedding into $M_t$ for each $t, t \in [a, b]$. We will often call $M_t$ and $\Omega \times \{t\}$ $t$-level sets. Thus, $f$ preserves levels.

A priori, $f(\Omega \times \{b\})$ may intersect $M_\infty$. We will show that this does not happen while $M$ is 2-convex.

Lemma 2. Let $f : \Omega \times [a, b] \to \tilde{M}$ be a solid-cylinder map. For every point $x$ of $\Omega^\circ \times \{b\}$, there exists a compact neighborhood $V$ such that $f|V$ is an imbedding onto a three-simplex $T$ in $\tilde{M}$ with a side $F_1$ such that $T - F_1 \subset M$, $f(x) \in F_1^\circ$, and $x_1 \circ \text{dev}(F_1) = \{b\}$.

Proof. The conditions (ii) and (iii) show that $\text{dev} \circ f$ is an immersion of $\Omega \times [a, b]$ into the half space $H$ of $\mathbb{R}^3$ given by $x_1 \leq b$ while $\text{dev} \circ f(\Omega \times \{b\})$ is the subset of the boundary $\delta H$, which is an affine hyperplane of $\mathbb{R}^3$. Thus, $\Omega \times [a, b]$ includes a compact neighborhood $V$ of $x$ so that $\text{dev} \circ f|V$ is an imbedding onto a three-simplex $T'$ with a side $F'_1 \subset \delta H$. Since $T'$ is a neighborhood of $\text{dev} \circ f(x)$ in $H$, we have $\text{dev} \circ f(x) \in F'_1^\circ$. Let $T = f(V)$. Then $\text{dev}|T$ is an imbedding onto $T'$, and the lemma follows. □

Theorem 4. Suppose that $M$ is a 2-convex affine 3-manifold which is simply connected. If $f$ is a solid-cylinder map, then $f$ is a map into $M$.

Proof. By condition (i), $f|\Omega \times \{b\}$ is a map into $\tilde{M}$. Define $K = (f|\Omega \times \{b\})^{-1}(M_\infty)$. By condition (iv), $K$ is a compact subset of $\Omega^\circ \times \{b\}$. Suppose that $K$ is not empty. Let $r$ be the function on $\mathbb{R}^3$ defined by $r = x_2^2 + x_3^2$. Let $r' = r \circ \text{dev} \circ f$, and choose a maximum point $x$ of $r'$ restricted to $K$. By
above lemma, $\tilde{M}$ includes a convex three-simplex $T$ with a side $F_1$ such that $f(x) \in F_1$ and $T - F_1 \subset M$.

Since $F_1 \subset f(\Omega \times \{b\})$, we have $M_\infty \cap F_1 \subset f(K)$; hence, $M_\infty \cap F_1 = f(K) \cap F_1$. Then $\text{dev} \circ f(x)$ is an exposed point of the convex hull of $\text{dev}(F_1 \cap f(K)) \supset \text{dev}(F_1 \cap M_\infty)$ since $x$ is a maximum point of $r'$ in $K$. This contradicts Proposition $\Box$. Hence, $K$ is empty.

4. Filling in a drum with the bottom side removed

The purpose of this section is to prove the strongest form of two convexity that we can always fill in the drum with a bottom side removed. Let $\delta \Omega$ be the boundary of a compact surface $\Omega$, which is a union of disjoint simple closed curves.

**Theorem 5.** Let $M$ be a 2-convex affine 3-manifold which is simply connected. Let $f : \delta \Omega \times [a, b] \to M$ be a differentiable map satisfying the following conditions:

(i) $f|\delta \Omega \times \{t\}$ is an embedding into a union of disjoint simple closed curves in $M_t$ for each $t$.

(ii) $f|\delta \Omega \times [a, b]$ is transversal to each level set.

(iii) $f|\delta \Omega \times \{a\}$ extends to an embedding $f' : \Omega \times \{a\} \to M_b$.

Then $f$ and $f'$ extend to a common solid-cylinder embedding $F$ from $\Omega \times [a, b]$ to $M$.

The pair of maps $f$ and $f'$ satisfying (i), (ii), and (iii) or the union of their images are said to be a drum with the top side removed.

Actually the way we will use this theorem is that $a$ in (iii) will be replaced by $b$. In this case the pair $f$ and $f'$ is said to be a drum with the bottom side removed. Hence, we use this theorem with the direction of “filling” reversed.

By the above theorem, there exists a surface $\Omega' \subset M_a$ such that $\Omega'$ is diffeomorphic to $\Omega$, and $\Omega' \cup \text{Im} f \cup \text{Im} f'$ forms the boundary of $F(\Omega \times [a, b])$.

It is sufficient to prove the above theorem when $a = 0$ and $b > 0$.

The plan to prove this theorem is as follows:

- We define a flow, defined on $M$ only partially and not complete, which induces the movement of $f(\delta \Omega, t)$ and sends level sets to level sets as $t$ increases from 0 to $b$.
- This flow will be used to define a level-preserving embedding from $\Omega \times [0, \tau)$ to $M$ extending $f'$ and $f$ where $\tau$, $0 < \tau < b$, denotes the level at which we cannot extend the flow.
- Using flow estimates with respect to $d$ on $M$, we show that the above map extends to a continuous map $F : \Omega \times [0, \tau] \to \tilde{M}$.
- By showing $\text{dev} \circ F$ is $C^1$, we prove that $F$ is a solid-cylinder map.
- Using Theorem $\Box$ we show $F$ is a map into $M$ and conclude that $\tau = b$. 
Recall that $\mu$ is the Euclidean metric on $M$ induced by $\text{dev}$, and $M$ has standard coordinates $x_1, x_2,$ and $x_3$. During the proof of the above theorem, we will state everything with respect to this Riemannian metric and the coordinates.

We first need a vector field for our flow. The image of $f$ is the union of disjoint embedded annuli in $M$. Let us call it $\Sigma$. For each point $(x, s)$ of $\delta \Omega \times \{s\}$, the derivative $\dot{f}(x, s)$ of $f(x, s)$ with respect to $s$ yields a vector $w$ at $f(x, s)$ tangent to $\Sigma$. This defines a smooth tangent vector field $w$ to $\Sigma$. Let $w^P$ be the component vector field of $w$ orthogonal to the $x_1$-direction. Since $w^P$ is a smooth vector field, we can find a smooth vector field $v^P$ extending $w^P$ defined on a neighborhood $U$ of $\Sigma$ and orthogonal to the $x_1$-direction.

Let $\phi$ be a smooth function which has compact support in $U$, has its range of values lying between 0 and 1, and is equal to 1 on $\Sigma$. Then $\phi v^P$ is regarded as a smooth vector field defined on $M$ orthogonal to the $x_1$-direction. Since $\phi$ is not zero only in a compact subset, the derivatives of $\phi$ of each order are uniformly bounded in $M$ by a constant depending only on the order. Similarly, those of $v^P$ of each order are uniformly bounded in the support of $\phi$ by a constant depending only on the order. Hence, those of $\phi v^P$ of each order are uniformly bounded above in $M$ by constants depending only on the order.

Let $e_1$ denote the vector field on $M$ in the $x_1$-direction with norm 1, and $W$ the vector field $\phi v^P + e_1$. Then $W$ extends $w$ and is a smooth vector field such that the norms of itself and its derivatives of order less than or equal to two are bounded above by a uniform constant $c_w$.

Let $O_0$ denote $f'(\Omega \times \{0\})$, and $\Phi_t$ the flow generated by $\tilde{W}$ where $\Phi_0$ equals the identity map of $M$. Let us denote by $A$ the set of elements $t$ of $[0, b]$ such that $\Phi_t(x)$ is defined for every $x \in O_0$. Since $O_0$ is compact, $A$ is open.

If $\Phi_t(x)$ is defined, then $\Phi_{t'}(x)$ is defined for $0 \leq t' \leq t$. Thus, $A$ is a connected subset.

Now, we prove that $A$ is closed. Suppose not. Then $A = [0, \tau)$ for some $\tau \in [0, b]$. Let $t \in [0, \tau)$. Since the $x_1$-component of $\tilde{W}$ equals 1 identically, and $\tilde{W}$ extends $w$, it follows that $\Phi_t(O_0)$ is a compact surface in $M_t$ with boundary equal to the union of disjoint smoothly imbedded closed curves $\Phi_t(\delta O_0)$, which equals $f(\delta \Omega \times \{t\})$. Let $O_t$ denote this surface and $O_{[0,\epsilon]}$ the union of $O_t$ for $t \in [0, \epsilon]$, where $\epsilon$ is a small positive number so that $\epsilon < \tau$.

We define an injective immersion $F_{\tau} : \Omega \times [0, \tau) \to M$ by $F_{\tau}(x, s) = \Phi_s(f'(x, 0))$ for $x \in \Omega$ and $s \in [0, \tau)$. Then $F_{\tau}[\Omega \times [0, t)]$ for each $t$, $0 < t < \tau$, is a solid-cylinder embedding extending $f'$ and $f[\delta \Omega \times [0, t])$ simultaneously.

From the definition of $F_{\tau}$, we obtain for $t, s, 0 < t + s < \tau$, $t, s > 0$, and every $x \in \Omega$,

$$F_{\tau} \circ T_s(x, t) = \Phi_s \circ F_{\tau}(x, t)$$
where $T_s$ is a translation by a vector $(0, s)$ in $\mathbb{R}^3$. While $t \in [0, \epsilon]$ for any positive number $\epsilon$ such that $\epsilon < \tau$, $s$ may take any value in $[0, \tau - \epsilon)$ for the above equation to make sense. Thus, for $s \in [0, \tau - \epsilon)$, the following diagram of diffeomorphisms is commutative:

\[
\begin{array}{ccc}
\Omega \times [0, \epsilon] & \xrightarrow{F_{\tau}} & \mathcal{O}_{[0,\epsilon]} \\
\downarrow T_s & & \downarrow \Phi_s \\
\Omega \times [s, s + \epsilon] & \xrightarrow{F_{\tau}} & \mathcal{O}_{[s,s+\epsilon]}.
\end{array}
\] (1)

Since the differential of $\tilde{W}$ is bounded with respect to $\mu$, Lemma 3 [1, p. 63] implies that

\[ d(\Phi_s(x), \Phi_s(y)) \leq e^{K|s|}d(x, y) \]

for every $s \in [0, \tau)$, $x, y \in \mathcal{O}_0$, and a constant $K$ independent of $s$. Then we have

\[ d(F_{\tau}(x, s), F_{\tau}(y, s)) \leq e^{K|\tau|}d(f'(x, 0), f'(y, 0)) \]

for every $x, y \in \Omega$ and $s, 0 < s < \tau$. If the second $d$ below denotes the standard Euclidean distance metric on $\Omega \times [0, b]$, we have for every $x, y \in \Omega$ and an independent positive constant $R'$

\[ d(F_{\tau}(x, s), F_{\tau}(y, s)) \leq R'd((x, s), (y, s)). \] (2)

It is easy to show that the following inequality holds if $(y, s)$ is replaced by $(y, s')$ for $s' \in [0, \tau)$ and $R'$ is replaced by another independent positive constant $R$:

\[ d(F_{\tau}(x, s), F_{\tau}(y, s')) \leq Rd((x, s), (y, s')). \] (3)

Since $\tilde{M}$ is complete, $F_{\tau}$ extends to a continuous map $F : \Omega \times [0, \tau] \to \tilde{M}$. Therefore,

\[ \Phi_s(x) = F \circ T_s \circ F_{\tau}^{-1}(x) \] (4)

is well-defined with range in $\tilde{M}$ for $x \in \mathcal{O}_{[0,\epsilon]}, s \in [0, \tau - \epsilon]$. Moreover, since $T_s \to T_{\tau-\epsilon}$ pointwise as $s \to \tau - \epsilon$, it follows that $\Phi_s | \mathcal{O}_{[0,\epsilon]}$ converges to $\Phi_{\tau-\epsilon}$ pointwise as $s \to \tau - \epsilon$.

We claim that $\text{dev} \circ F$ is a $C^1$-immersion. Let $D\Phi_s(x)$ for $x$ and $\Phi_s(x)$ in the image of $F_{\tau}$ denote the $3 \times 3$-matrix of the differential $T_x(M) \to T_{\Phi_s(x)}(M)$ of $\Phi_s$ at $x$, and $D\tilde{W}(x)$ that of the differential $T_x(M) \to T_x(M)$ of $\tilde{W}$ at $x$. (Recall the global coordinates $x_1, x_2, x_3$ introduced earlier in this section.) Then

\[ \frac{d}{ds} D\Phi_s(x) = D\tilde{W}(\Phi_s(x)) D\Phi_s(x) \] (5)

for $x \in \mathcal{O}_{[0,\epsilon]}$ and $0 \leq s < \tau - \epsilon$ (see Lemma 4 of [1, p. 64]). Since the norms of $D\tilde{W}$ and $D^2 \tilde{W}$ are uniformly bounded above by $C_{\tilde{W}}$ and $\Phi_0$ is the identity map,
then the proof of Lemma 4 of [1, p. 65] shows that the family of $3 \times 3$-matrix-valued functions $D\Phi_s|O_{[0,\epsilon]}$ for $s \in [0, \tau - \epsilon)$ and that of $9 \times 3$-matrix-valued functions $D^2\Phi_s|O_{[0,\epsilon]}$ for $s \in [0, \tau - \epsilon)$ are respectively uniformly bounded. Hence, the family of $3 \times 3$-matrix-valued functions $D\Phi_s|O_{[0,\epsilon]}$ for $s \in [0, \tau - \epsilon)$ forms a bounded, equicontinuous family of functions. (We benefited from a conversation with D.-H. Chae who works in Navier-Stokes equations).

Since $\text{dev}$ preserves coordinates $x_1, x_2, \text{ and } x_3$, $D\text{dev} \circ \Phi_s|O_{[0,\epsilon]}$ equals $D\Phi_s|O_{[0,\epsilon]}$ as matrix-valued functions, and $D^2\text{dev} \circ \Phi_s|O_{[0,\epsilon]}$ equals $D^2\Phi_s|O_{[0,\epsilon]}$. By the Ascoli-Arzelà theorem (see [51, p. 85]), choose a subsequence $\Phi_{s_j}$ such that $\{s_j\} \to \tau - \epsilon$ and $D\text{dev} \circ \Phi_{s_j}|O_{[0,\epsilon]}$ converges uniformly to a continuous $3 \times 3$-matrix-valued function $\Upsilon$ defined on $O_{[0,\epsilon]}$.

Since $\text{dev} \circ \Phi_{s_j}|O_{[0,\epsilon]}$ converges pointwise to $\text{dev} \circ \Phi_{\tau - \epsilon}|O_{[0,\epsilon]}$, Theorem 7.17 [45, p. 152] (a standard fact) implies that $\text{dev} \circ \Phi_{\tau - \epsilon}|O_{[0,\epsilon]}$ is continuously differentiable with $D\text{dev} \circ \Phi_{\tau - \epsilon}|O_{[0,\epsilon]} = \Upsilon$.

We obtain the following equation from equation 5 by taking determinant of the both sides

$$\frac{d}{ds} \det D\Phi_s(x) = \det D\tilde{W}(\Phi_s(x)) \det D\Phi_s(x)$$

If the initial value $f(0)$ of an ordinary differential equation $\frac{d}{ds} f = gf$ for real valued solution $f$ is positive, then the solution is positive always, as it can be seen from the solution

$$f(s) = \exp[\int_0^s gds] f(0)$$

Since $\det D\Phi_0 = 1$, $\Upsilon$ has values in nonsingular $3 \times 3$-matrices.
By equation 4, the following diagram is commutative:

$$
\begin{array}{ccc}
\Omega \times [0, \varepsilon] & \xrightarrow{F_\tau} & O_{[0,\varepsilon]} \\
T_{\tau-\varepsilon} & \downarrow \Phi_{\tau-\varepsilon} & \\
\Omega \times [\tau - \varepsilon, \tau] & \xrightarrow{F} & \mathcal{M}.
\end{array}
$$

Thus,

$$
\text{dev} \circ F|\Omega \times [\tau - \varepsilon, \tau] = (\text{dev} \circ \Phi_{\tau-\varepsilon}) \circ F_\tau \circ T_{\tau-\varepsilon}^{-1}|\Omega \times [\tau - \varepsilon, \tau].
$$

Since $D\text{dev} \circ \Phi_{\tau-\varepsilon} = \Upsilon$, it follows that $\text{dev} \circ F|\Omega \times [\tau - \varepsilon, \tau]$ is of class $C^1$, and so is $\text{dev} \circ F$ on $\Omega \times [0, \tau]$. Moreover, since $\Upsilon$ has values in nonsingular matrices, $\text{dev} \circ F|\Omega \times [\tau - \varepsilon, \tau]$ is an immersion. Since $F$ extends $F_\tau$, and $F_\tau$ is an immersion into $\mathcal{M}$, $\text{dev} \circ F$ is an immersion, and $F : \Omega \times [0, b] \to \mathcal{M}$ is a solid-cylinder map.

By Theorem 4, $F$ is a map into $\mathcal{M}$. Hence, our vector field $\tilde{W}$ is defined on a compact neighborhood of the image of $F$ in $\mathcal{M}$. Recall that $F_\tau : \Omega \times [0, \tau) \to M$ is given by $F_\tau(x, s) = \Phi_s(f'(x, 0))$. One defines $\Phi_\tau(f'(x, 0))$ by letting it equal to $F(x, \tau)$. Then $\Phi_\tau$ is clearly a flow generated by $\tilde{W}$, and $\tau$ belongs to our set $A$. This is a contradiction. Therefore, $A$ is closed and $A = [0, b]$.

Define $F : \Omega \times [0, b] \to M$ by $F(x, s) = \Phi_s(f'(x, 0))$ for $s \in [0, b]$. Then $F$ satisfies the conclusion of Theorem 4.

5. Filling a Disk with One Critical Point and an Annulus with Two Critical Points

Let $\Omega$ be a compact surface. Suppose that $F : \Omega \to M$ is a smooth embedding with the property that $F(\delta\Omega)$ is a subset of $M_t$ for some $t \in \mathbb{R}$, $x_1 \circ F$ is a Morse function with $k$ critical values, and $t$ not a critical value. Then $F$ is said to be an amenable embedding of level $t$ with $k$ critical values. We say that the image of $F$ is an amenable surface, and if $x_1 \circ F$ increases in the inner direction on $\delta\Omega$, then $F$ is a positive amenable embedding; otherwise, $F$ is a negative amenable embedding.

The purpose of this section is to fill in an amenable disk with one critical point and an amenable annulus with two critical points. From the following Proposition 10, these have the smallest number of critical points in their respective homeomorphism classes. The filling of the disk follows from the drum filling that we obtained above. The proof of the filling of annulus is as follows: The idea is to cut off the surface into two parts at the level $k$ just above the level of the saddle point, where the higher one has the unique maximum point and the lower one has the saddle point. The higher surface is homeomorphic to a disk and we can use the filling of the disk. The lower surface is homeomorphic to a pair of pants with two boundary components at a lower level.
and one at the level just above the level of the saddle. The lower surface is cut and pasted near the saddle into the union of two annuli by finding two disks in a small neighborhood of the saddle in $M$. Since the higher surface is filled, $M_k$ include a disk $D$ with which the higher surface bound a 3-ball. $D$ is then altered according to the surgery so that $D$ together with the two annuli form a drum with the bottom side removed. The top side may consist of the union of two disjoint disks or an annulus. We then fill in the drum and take the union with the ball filling the higher disk from which we add or subtract to fit in the original surface. Depending on whether the top side is the union of two disjoint disks or an annulus, the result is a 3-ball or a solid torus. (The both processes are described in Figure 12.)

The level sets of $x_1 \circ F$ form a singular foliation on $\Omega$ with components of $\partial \Omega$ as leaves. Let us call the foliation the level-set foliation on $\Omega$ for $F$. For such foliations indices of singularities are well-defined. (See Casson-Bleiler [9, p. 71].)

**Proposition 10.** The sum of indices of all singularities of the level-set foliation on $\Omega$ for $F$ equals the Euler characteristic of $\Omega$.

**Proof.** This follows since the boundary components are leaves of the foliation. See [7, p. 72].

Each point of $M$ has a neighborhood $U$ such that $\text{dev}|U$ is a chart. Hence, $U$ includes a convex open ball neighborhood of $x$.

**Proposition 11.** Let $D$ be a compact disk with smooth boundary, and $F : D \to M$ an amenable embedding of level $t$ with one critical point. Then there exists an embedding $G : D \to M$ homotopic to $F$ relative to $\partial D$ with $F(D') \cap G(D') = \emptyset$. Moreover, the union of the imbedded disks $F(D)$ and $G(D)$ is the boundary of a compact three-ball in $M$.

**Proof.** Assume without loss of generality that $F$ is positive. Let $\Sigma = F(D)$, and let $f_1$ denote $x_1|\Sigma$. Let $k$ be the unique critical value of $f_1$ and $z$ the critical point. Then $z$ is the maximum point of $f_1$. Choose a convex open ball neighborhood $B$ of $z$. Then there exists a real number $k'$ near $k$, $k' < k$, so that the level set $f^{-1}(k')$ at $k'$ is a simple closed curve in $\Sigma \cap B$, say $\alpha$, which equals the boundary of a disk $\Sigma_\alpha$ in $\Sigma'$. Since $M_{k'} \cap B$ is a convex open disk, $\alpha$ is the boundary of a disk $D_{k'}^2$ in $M_{k'}$. Now, the union of $D_{k'}^2$ and $\Sigma_\alpha$ is the boundary of a three-ball in $B$. Let us call the ball $B_1$.

By Theorem 3, it follows that there exists a disk $D_t^2$ in the level $t$ so that the union of $D_{k'}^2$, $D_t^2$, and the embedded annulus $f_1^{-1}([k', t])$ is the boundary of a three-ball. Let us call it $B_2$. Then $B_1 \cup B_2$ is the three-ball with boundary $\Sigma \cup D_t^2$. The proposition follows easily now.
Proposition 12. Let $A$ be a compact annulus, and $F : A \rightarrow M$ a positive amenable embedding of level $t$ with two critical points. Then one of the following holds:

(α) $F(\delta A)$ is the boundary of the union of two disjoint disks $D_1$ and $D_2$ in $M_t$ so that $D_1 \cup D_2 \cup F(A)$ is the boundary of a compact subset of $M$ homeomorphic to a three-ball.

(β) There exists an embedding $G : A \rightarrow M$ homotopic to $F$ relative to $\delta A$, so that $G(A) \cup F(A)$ equals the boundary of a compact subset in $M$, homeomorphic to a solid torus.

Proof. The critical points of $A$ consist of one index 1 critical point and one index $-1$ critical point since $\chi(A) = 0$ by Proposition 10.

Let $\Sigma$ be the image $F(A)$, and $f_1 = x_1|\Sigma$. Let $l$ and $k$, $l > k$, be critical values, and $w$ and $z$ the corresponding critical points respectively. Then $w$ is a local maximum point and $z$ a saddle point. Let $\epsilon$ be a small positive number so that $k + \epsilon < l$. ($\epsilon$ will be chosen more precisely below.)

Let $\Sigma_0 = f_1^{-1}([k + \epsilon, l])$. Then the higher surface $\Sigma_0$ is a positive amenable disk with one critical point. By Proposition 11, $M_{k+\epsilon}$ includes a disk $D_{k+\epsilon}$, the union of which with $\Sigma_0$ is the boundary of a compact three-ball in $M$. Let $B_1$ denote the three-ball.

Obviously, $t < k < l$. Let $\Sigma_1 = f_1^{-1}([t, k + \epsilon])$, which is homeomorphic to a sphere removed with three disjoint open disks, i.e., a pair of pants. Let $\delta_1 \Sigma_1$ denote the unique boundary components of $\Sigma_1$ at the level $k + \epsilon$, and $\delta_2 \Sigma_1$ the union of the two boundary components in the level $t$. Then $D_{k+\epsilon}$ satisfies $\delta D_{k+\epsilon} = \delta \Sigma_0 = \delta_1 \Sigma_1$.

Near the saddle point of $\Sigma$, we will do the smooth surgery for the lower surface $\Sigma_1$ as indicated by Figure 7.

We will make a chart near $z$ so that $\Sigma$ is mapped to a canonically shaped saddle. (i) Let $B$ be an open ball neighborhood of $z$ in $M$. Choose a coordinate system $u^1, u^2$ in a neighborhood $V$ of $z$ in $\Sigma \cap B$ so that $u^1(z) = u^2(z) = 0$.
and \( f_1 = k + (u^1)^2 - (u^2)^2 \) hold in \( V \) (see [39]). Extend \( u^1 \) and \( u^2 \) to smooth functions on \( B \), and let \( F : B \to \mathbb{R}^3 \) be given by \( F(x) = (x_1(x), u^1(x), u^2(x)) \). Then \( D_z F \) is nonsingular. Hence, there exists a neighborhood \( U \) of \( z \) in \( B \) so that \( F|U \) is a diffeomorphism onto a neighborhood of \((k, 0, 0)\) in \( \mathbb{R}^3 \). We may assume without loss of generality that \( \Sigma \cap U \) is a connected subsurface.

For a positive number \( \delta \), we let 
\[
B_{\delta}(k, 0, 0) = \{(x_1, u^1, u^2) \in \mathbb{R}^3 | |x_1 - k| < \delta, (u^1)^2 < \delta, (u^2)^2 < \delta\}.
\]
Let \( \Sigma' = F(\Sigma \cap U) \). We choose \( \delta', \delta' > 0 \), so that \( B_{\delta'}(k, 0, 0) \subset F(U) \) and \( B_{\delta'}(k, 0, 0) \cap \Sigma' \) equals 
\[
\{(x_1, u^1, u^2) \in B_{\delta'}(k, 0, 0) | x_1 = k + (u^1)^2 - (u^2)^2\}.
\]
That is, \( \Sigma'' = B_{\delta'}(k, 0, 0) \cap \Sigma' \) is realized as a graph of a function.

(ii) Let \( \mathcal{F} \) denote the foliation on \( M \) whose leaves are components of level sets of \( x_1 \). We choose imbeddings of two \( \mathcal{F} \)-transverse disks in \( B_{\delta'}(k, 0, 0) \) that does a surgery on \( \Sigma_1 \), and obtain a union of disjoint \( \mathcal{F} \)-transverse annuli with smooth boundary. We may choose our cutoff number \( \epsilon \) to be less than \( 3\delta' \). Let \( \delta = 3\epsilon \) and define domains
\[
D = \{(x_1, u^1) | k + (u^1)^2 - \delta < x_1 < k + \delta, (u^1)^2 < \delta\},
\]
\[
D^u = \{(x_1, u^1) \in D | x_1 \geq k + (u^1)^2 - \delta/2\},
\]
\[
D^d = \{(x_1, u^1) \in D | x_1 \leq k + (u^1)^2 - \delta/2\}.
\]

We define a continuous map \( g : D \to \mathbb{R} \) by
\[
g(x_1, u^1) = \begin{cases} 
\sqrt{\delta/2}, & (x_1, u^1) \in D^u \\
\sqrt{k + (u^1)^2 - x_1}, & (x_1, u^1) \in D^d.
\end{cases}
\]
We smooth this function. Let \( N(D^d) \) be the set of points \((x_1, u^1)\) of \( D \) with \( x_1 < k + (u^1)^2 - \delta/4 \) and \( \Phi^d : D \to [0, 1] \) a function with bounded derivatives such that
\[
\Phi^d(x_1, u^1) = \begin{cases} 
1, & (x_1, u^1) \in D^d \\
0, & (x_1, u^1) \in D - N(D^d).
\end{cases}
\]
Since \( \sqrt{k + (u^1)^2 - x_1} \) is defined on \( N(D^d) \), we define
\[
g^s(x_1, u^1) = (1 - \Phi^d(x_1, u^1)) \sqrt{\delta/2} + \Phi^d(x_1, u^1) \sqrt{k + (u^1)^2 - x_1}.
\]
Then \( g^s \) is a positive-valued smooth function such that
\[
\begin{align*}
g^s(x_1, u^1) &= \sqrt{k + (u^1)^2 - x_1} \quad \text{for} \quad (x_1, u^1) \in D^d, \\
g^s(x_1, u^1) &= \sqrt{\delta/2} \quad \text{for} \quad (x_1, u^1) \in D - N(D^d), \\
g^s(x_1, u^1) &\geq \sqrt{k + (u^1)^2 - x_1} \quad \text{for} \quad (x_1, u^1) \in N(D^d) - D^d.
\end{align*}
\]

Since \( \delta = 3\epsilon \), the line in \( D \) given by \( x_1 = k + \epsilon \) intersects the arc \( x_1 = k + (u^1)^2 - \delta/2 \) in \( D \) at two points. Let \( R^\epsilon \) be the region in \( D \) given by \( x_1 \leq k + \epsilon \) and \( x_1 \geq k + (u^1)^2 - \delta/2 \). Let \( G^+ : D \to B_3(k, 0, 0) \) be given by \( G^+(x_1, u^1) = (x_1, u^1, g^s(x_1, u^1)) \) and \( G^- : D \to B_3(k, 0, 0) \) by \( G^-(x_1, u^1) = (x_1, u^1, -g^s(x_1, u^1)) \) for \( (x_1, u^1) \in D \). Then \( G^+ \) and \( G^- \) are embeddings into \( B_3(k, 0, 0) \). Let \( \mathcal{E} \) be the foliation on \( B_3(k, 0, 0) \) with leaves that are level sets under \( x_1 \). Let \( G^+(R^\epsilon) \) be denoted by \( D^+ \), and \( G^-(R^\epsilon) \) by \( D^- \). Then \( D^+ \cup D^- \) are disjoint imbedded disks transverse to \( \mathcal{E} \).

Let us recall
\[
\Sigma'' = \{(x_1, u^1, u^2) \in B_3(k, 0, 0) | x_1 = k + (u^1)^2 + (u^2)^2\};
\]
let
\[ \Sigma_1'' = \{(x_1, u^1, u^2) \in \Sigma'' | x_1 \leq k + \epsilon\}, \]
\[ \Sigma_\delta'' = \{(x_1, u^1, u^2) \in \Sigma'' | \sqrt{\delta/2} \leq u^2 \leq \sqrt{\delta/2}\}, \]
so that \((\Sigma_1'' - \Sigma_\delta'') \cup D^+ \cup D^-\) is a smooth manifold properly imbedded in \(B_\delta(k, 0, 0)\) diffeomorphic to the union of two disjoint disks transverse to \(E\), a result of a surgery, since it is the union of
\[ G^+((\{x_1, u^1 \in D | x_1 \leq k + \epsilon\}) \text{ and } G^-((\{x_1, u^1 \in D | x_1 \leq k + \epsilon\}), \]
two manifolds diffeomorphic to disks.

Let \(P\) be the plane in \(B_\delta(k, 0, 0)\) given by \(x_1 = k + \epsilon\). Then \(P \cap \Sigma_1''\) is the union of two arcs on \(P\) given by \(k + \epsilon = k + (u_1)^2 - (u_2)^2\). Recall that \(R\) is the region in \(D\) given by \(x_1 \leq k + \epsilon\) and \(x_1 \geq k + (u_1)^2 - \delta/2\). Let \(I\) be the segment consisting of points of \(R\) whose \(x_1\)-values equal to \(k + \epsilon\). Then the union of \(P \cap \Sigma_\delta'', G^+ (I), \) and \(G^- (I)\) is the boundary of a closed disk \(E'\) in \(P\), and the union of \(E', \Sigma_\delta'', D^+, \) and \(D^-\) is the boundary of a compact three-ball \(B\) in \(B_\delta(k, 0, 0)\).
(iii) We do the smooth surgery now. Let $\mathcal{D}^+$, $\mathcal{D}^-$, $E$, and $\Sigma_\delta$ be $(F\mid U)^{-1}(\mathcal{D}^+)$, $(F\mid U)^{-1}(\mathcal{D}^-)$, $(F\mid U)^{-1}(E')$, and $(F\mid U)^{-1}(\Sigma_\delta')$ respectively. Recall that $\Sigma_1 = f_1^{-1}([t, k + \epsilon])$. Let

$$\Sigma_1^s = (\Sigma_1 - \Sigma_\delta) \cup \mathcal{D}^+ \cup \mathcal{D}^-,$$

which is the result of a surgery and is diffeomorphic to the union of two disjoint annuli. Since $(F\mid U)^{-1}$ maps leaves of $E$ into leaves of $\mathcal{F}$, $\Sigma_1^s$ is transverse to $\mathcal{F}$.

(iv) We now try to find the top face for the above annuli. Recall that $M_{k+\epsilon}$ includes the disk $D_{k+\epsilon}$ with $\delta D_{k+\epsilon} = \delta\Sigma_0 = \delta_1\Sigma_1$ from the beginning of the proof. $\delta D_{k+\epsilon} \cap E$ is a union of two compact arcs $\alpha_1$ and $\alpha_2$ in $\delta_1\Sigma_1$ where $\alpha_1 \cup \alpha_2 = \Sigma_\delta \cap \delta_1\Sigma_1$. Since $E - \alpha_1 - \alpha_2$ is included in one of two components of $M_{k+\epsilon} - \delta_1\Sigma_1$, we have the following two cases:

(a) $E - \alpha_1 - \alpha_2 \subset D_{k+\epsilon}^o$,

(b) $(E - \alpha_1 - \alpha_2) \cap D_{k+\epsilon} = \emptyset$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure11}
\caption{$D_{k+\epsilon}$ and $E$.}
\end{figure}

(a) In this case $D_{k+\epsilon} - (E^o \cup \alpha_1^o \cup \alpha_2^o)$ is the union of two disjoint compact disks, which is denoted by $\mathcal{O}$. Then the boundary of $\mathcal{O}$ equals

$$\delta D_{k+\epsilon} - \Sigma_\delta^o \cup (F\mid U)^{-1}(G^+(I_\epsilon)) \cup (F\mid U)^{-1}(G^-(I_\epsilon)),$$

which is the union of boundary components of $\Sigma_1^s$ at the level $k + \epsilon$.

We see that $\mathcal{O}$ forms the top of the drum $\mathcal{O} \cup \Sigma_1^s$ with the bottom side removed. Since $\Sigma_1^s$ is transverse to $\mathcal{F}$, Theorem 5 shows that there exists a surface in $M_t$ which is the union of two disjoint compact disks and whose boundary equals $\delta_2\Sigma_1$. Call the disks $D_1$ and $D_2$. Then $\mathcal{O} \cup \Sigma_1^s \cup D_1 \cup D_2$ is the boundary of a compact subset $N_2$ of $M$, which is the union of two disjoint subsets homeomorphic to three-balls. Let

$$B_2 = N_2 \cup (F\mid U)^{-1}(\mathcal{B}).$$

Then $B_2$ is homeomorphic to a three-ball.

Recall that the union of $\Sigma_0 = f_1^{-1}([k + \epsilon, l])$, where $l$ is the maximal value of $f_1$ on $\Sigma$, and the disk $D_{k+\epsilon}$ in $M_{k+\epsilon}$ equals the boundary of a compact three-ball $B_1$ in $M$. Since $\delta B_2$ equals $D_{k+\epsilon} \cup \Sigma_1 \cup D_1 \cup D_2$, the balls $B_1$ and $B_2$
meet exactly at $D_{k+\epsilon}$. Consequently, $B_1 \cup B_2$ is homeomorphic to a three-ball. Moreover, the boundary of $B_1 \cup B_2$ equals $\Sigma \cup D_1 \cup D_2$. Hence, we obtain $(\alpha)$.

(b) In this case, $D_{k+\epsilon} \cup E$ is a compact annulus in the level $k+\epsilon$ with smooth boundary, as described in equation 9. As in (a), Theorem 5 shows that there exists a compact annulus $\Omega_2$ with boundary $\delta_2 \Sigma_1$ in the level $t$ such that the union of $D_{k+\epsilon} \cup E$, $\Omega_2$, and $\Sigma_1$ is the boundary of a compact three-manifold $N_2$ homeomorphic to $\Omega_2 \times [t, k + \epsilon]$, a solid torus.

Let

$$B_2 = (N_2 - (F|U)^{-1}(B)) \cup \Sigma_\delta.$$  

While $B_2$ is a compact three-manifold still homeomorphic to a solid torus, the boundary of $B_2$ equals the union of $D_{k+\epsilon}$, $\Omega_2$, and $\Sigma_1$. We can show as in (a) that $B_1 \cup B_2$ is homeomorphic to a solid torus, and the boundary of $B_1 \cup B_2$ equals $F(A) \cup \Omega_2$. Hence, there exists an embedding $G : A \to \Omega_2$ homotopic to $F$ relative to $\delta A$. We obtained $(\beta)$. 

\[ \square \]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig12.png}
\caption{Our filling processes}
\end{figure}
6. Replacing disks and controlling the relative maximum points

In this section, we show that if a portion of an amenable embedding $F$ is an amenable embedding homotopic to an embedding of a disk into a level set then we can replace this portion by an amenable disk with one critical point and still obtain an amenable embedding (see Proposition 13) provided the interior of the imbedded disk is disjoint from the image of $F$. We modify this result further a bit in order to control the location and the size of the second relative maximum point using Lemma 3 (see (iv) and (v) of Proposition 13 below). We even produce gradient-like vector fields for these functions.

Let $F$ be an amenable embedding of a surface $\Omega$. Suppose that $\Omega$ is a subsurface of $\Omega^s$ diffeomorphic to a disk so that $F|\Omega$ is a positive amenable embedding, of level $t$, homotopic relative to $\delta\Omega$ to an embedding $G: \Omega \to W$ for a subsurface $W$ in $M$. Let us fix orientations of $\Omega$ and $R^2$ so that $(x_2 \circ G, x_3 \circ G)$ is an orientation preserving map into $R^2$. We let $f = x_1 \circ F$.

**Proposition 13.** Suppose that $W$ includes no components of $F(f^{-1}(t))$ other than $F(\delta\Omega_1)$. Let $\delta$ be an arbitrary positive number. Then there exists a smooth amenable embedding $F': \Omega \to M$ with following properties:

(i) $F'$ is homotopic to $F$ relative to $\Omega - \Omega_1$.

(ii) The maximum point $y$ of $x_1 \circ F'|\Omega_1$ is unique with maximum value less than $t + \delta$.  

(iii) The number of critical points of $x_1 \circ F'$ is less than or equal to that of $x_1 \circ F$.

**Proof.** Assume without loss of generality that $t > 1$. Since $W$ is a compact subset of $M$, a compact neighborhood $N$ in $M$ includes $W$. Let $N_t(W)$ be a compact-disk neighborhood of $W$ in the interior of $M \cap N$. Hence, there exists a $\delta'$, $0 < \delta' < 1/2, \delta$, so that $p \in M$ obtained by starting from a point $x \in N_t(W)$ and going in the positive or negative $x_1$-direction by a distance $s$ less than or equal to $\delta'$ still lies in $N$. Let this point $p$ be denoted by $x \pm se_1$ where the sign depends on the direction. The set of all such points is denoted by $N(W)$, which is a neighborhood of $W$ homeomorphic to $N_t(W) \times [t - \delta', t + \delta']$.

Let $N'_t(W)$ denote $M_t \cap N(W)$ for $t' \in [t - \delta', t + \delta']$. We also require that the interval $[t - \delta', t + \delta']$ contains no critical value of $f$. Since we assumed that $W$ includes no components of $F(f^{-1}(t))$ other than $F(\delta\Omega_1)$, we can choose $\delta'$ sufficiently small so that $F^{-1}(N_t(W))$ is a simple closed curve $\alpha_{t'}$, which is a component of the $t'$-level set of $x_1 \circ F$. Thus, $F^{-1}(N(W))$ is a compact annulus that is a neighborhood of $\delta\Omega_1$ and is foliated by simple closed curves $\alpha_{t'}$. In particular, whenever $t', t'' \in [t - \delta', t + \delta']$, the union of two level curves $\alpha_{t'}$ and $\alpha_{t''}$, $t' \neq t''$, is the boundary of a compact annulus $A_{t', t''}$ included in $F^{-1}(N(W))$.  


We will now construct an embedding from \( \Omega \) into \( N(W) \). Assume without loss of generality that \( t > 0 \). There is a diffeomorphism
\[
\Phi : A_{t,t+\delta'/2} \rightarrow E
\]
where \( E \) is the annulus in \( \mathbb{R}^2 \) whose boundary is the union of two circles of respective radii \( t \) and \( t - \delta'/2 \) with centers at the origin \( O \) mapping \( \alpha_{\epsilon'} \) to a circle of radius \( t - (t' - t) \) with center \( O \). Clearly \( \Phi \) extends to a diffeomorphism of \( \Omega \) with the disk \( D \) in \( \mathbb{R}^2 \) of radius \( t \) with center \( O \).

We choose a smooth function \( f' : D \rightarrow [0, 3\delta'/4] \) as follows:
\[
f'(x, y) = t - \sqrt{x^2 + y^2} \text{ for } t - \delta'/2 \leq \sqrt{x^2 + y^2} \leq t
\]
and constant along circles with center \( O \) with unique critical point \( O \) taking the maximum value \( 3\delta'/4 \). Then \( f' \circ \Phi \) has constant value \( t' - t \) on \( \alpha' \) for \( t' \in [t, t + \delta'/2] \), and \( \Phi^{-1}(O) \) is the maximum point of \( f' \circ \Phi \). Consequently, \( \Phi^{-1}(O) \) does not belong to \( A_{t,t+\epsilon} \) for \( \epsilon < \delta'/2 \).

Let us define a map \( G' : \Omega \rightarrow N(W) \) by \( G'(x) = G(x) + (f' \circ \Phi(x))e_1 \). Since this can be seen as a graph map, \( G' \) is an embedding. Moreover,
\[
x_1 \circ G' = x_1 \circ F \text{ on } A_{t,t+\delta'/2}
\]
since \( x_1 \circ G' \) has constant value \( t' \) on \( \alpha' \). Therefore, \( G'|_{\alpha'} \) is an embedding onto a simple closed curve in \( M' \) for \( t' \in [t, t + \delta'/2] \), while so is \( F|_{\alpha'} \).

Let \( \epsilon \) be a positive number so that \( 2\epsilon \) is less than \( d(\Omega, \delta N(W)) \). Since \( G'|_{\delta \Omega} = F|_{\delta \Omega} \), it follows that for every \( \epsilon, \epsilon > 0 \), there exists \( \gamma, 0 < \gamma < \delta'/2 \), such that
\[
d(F(x), F(\delta \Omega_1)) < \epsilon, \\
d(G'(x), F(\delta \Omega_1)) < \epsilon, \\
d(G'(x), F(x)) < \epsilon \text{ if } x \in A_{t,t+\gamma}.
\]
Each point of \( F(A_{t,t+\gamma}) \cup G'(A_{t,t+\gamma}) \) is contained in an open \( \epsilon \)-ball included in \( N(W) \). Let \( \epsilon_1 < \gamma \). Since the balls in the cover are convex, we obtain
\[
hF(x) + (1 - h)G'(x) \in N(W) \text{ if } x \in A_{t,t+\epsilon_1}.
\]
for every real number $h$, $0 \leq h \leq 1$.

We smooth $G'$ and $F$ near $\delta \Omega_1$. Let $\epsilon$ be a real number with $0 < \epsilon < \epsilon_1, \delta'$, and $\phi_\epsilon : \Omega \to [0,1]$ a smooth function such that $\phi_\epsilon$ is 1 on $\Omega - \Omega_1 \cup A_{t,t+\epsilon/4}$ and 0 on the compact disk $\Omega_1^0 - A_{t,t+\epsilon}$ and is constant along $\alpha_{t'}$ for $t' \in [t,t+\epsilon]$. We define $F_\epsilon : \Omega \to N(W)$ by

$$F_\epsilon(x) = \phi_\epsilon(x)F(x) + (1-\phi_\epsilon(x))G''(x).$$

Then $F_\epsilon$ equals $F$ on $\Omega - \Omega_1^0$ and equals $G'$ on $\Omega_1^0 - A_{t,t+\epsilon}$. In particular, equation (12) shows that

$$x_1 \circ G' = x_1 \circ F_\epsilon \quad \text{on} \quad \Omega_1.$$  

(14)

Moreover, $F_\epsilon$ is homotopic with $F$ relative to $\Omega - \Omega_1$ since $F_\epsilon$ depends continuously on $\epsilon$ for $\epsilon > 0$ and, as $\epsilon \to 0$, $F_\epsilon|\Omega_1$ converges to $G'|\Omega_1$ which is homotopic to $F|\Omega_1$ relative to $\delta \Omega_1$.

We now show that $F_\epsilon$ is an embedding. Let us orient the curves $\alpha_{t'}$ continuously for $t \leq t' \leq t + \delta$. Since the derivative of $F$ along $\alpha_t$ is identical with that of $G'$, a continuity argument shows that there exists $\epsilon_2$ so that for $\epsilon \leq \epsilon_2$, the derivative of $F$ on $A_{t,t+\epsilon}$ along $\alpha_{t'}$ has positive $\mu$-inner product with that of $G'$. Thus, the derivative of $F_\epsilon$ along $\alpha_t'$ is a nonzero vector field. Since the derivatives of $F$ and $G'$ along $\alpha_{t'}$ have zero $x_1$-components, that of $F_\epsilon$ has zero $x_1$-component.

Since for $0 < \epsilon < \delta'$, at each point of $A_{t,t+\epsilon}$, the $x_1$-component of the derivative of $G'$ in the inner direction orthogonal to $\alpha_{t'}$ is positive, equation (14) shows the positivity of that of $F_\epsilon$. Therefore, $F_\epsilon$ is an immersion at points of $A_{t,t+\epsilon}$ for $\epsilon < \epsilon_1, \epsilon_2, \delta'$. Since $F$ and $G'$ are immersions on $\Omega - \Omega_1$ and on $\Omega_1 - A_{t,t+\epsilon}$ respectively, and $F_\epsilon$ equals $F$ or $G'$ on the respective sets, $F_\epsilon$ is an immersion on all of $\Omega$ for $\epsilon < \epsilon_1, \epsilon_2, \delta'$.

Since we have $F_\epsilon|\alpha_t = F|\alpha_t = G|\alpha_t$, Theorem 2.1.4 [8], p. 37 shows that there exists a positive constant $\epsilon_3$ so that $F_\epsilon|\alpha_{t'}$ is an embedding into $M_{t'}$ for every $\epsilon$ and $t'$ with $0 < \epsilon < \epsilon_3$ and $t' \in [t,t+\epsilon]$. This implies that $F_\epsilon$ is the desired embedding for $\epsilon < \epsilon_i, \delta', i = 1, 2, 3$.

Lemma 3. Let $\mathcal{D}$ be a compact disk in the plane $\mathbb{R}^2$, $z$ a point of $\delta \mathcal{D}$, and $W$ a compact disk with smooth boundary in an open surface. Let $U$ be a compact-disk neighborhood of $z$ in $\mathcal{D}$ where $U \cap \delta \mathcal{D}$ is diffeomorphic to a closed interval. Suppose that $f : U \cup \delta \mathcal{D}$ is an imbedding into $W$ so that $f(\delta \mathcal{D}) \subset \delta W$ and $f$ extends to a smooth map $f'$ in a neighborhood of $U \cup \delta \mathcal{D}$. Then there exists a diffeomorphism $g : \mathcal{D} \to W$ extending $f$.

Proof. We claim that $f$ extends to an embedding $f''$ of an open neighborhood $U'$ of $U \cup \delta \mathcal{D}$ in $\mathcal{D}$. We may assume without loss of generality that $\mathcal{D}$ and $W$ equal the standard disk in $\mathbb{R}^2$. One can easily extend $f|\delta \mathcal{D}$ to an immersion $f_1$ from a neighborhood $N_1$ of $\delta \mathcal{D}$ in $\mathcal{D}$ to a neighborhood of $\delta W$ in $W$. Let
Let $N$ be an open neighborhood of $U$ in $\mathcal{D}$ so that $f'|N : N \to W$ is a well-defined immersion extending $f|U \cup (N \cap \delta \mathcal{D})$. We can do this since the ranks of the derivatives of $f$ at points of $U$ equal 2 and hence those of $f$ at points of $\delta \mathcal{D}$ sufficiently near points of $U$ are also two. Let $\psi : \mathcal{D} \to [0,1]$ be a smooth function such that $\psi$ equals 0 on $U$ and 1 on $\mathcal{D} - N$. Define a map $f'' : N_1 \cup N \to W$ by $f''(x) = \psi(x)f_1(x) + (1 - \psi(x))f'(x)$, so that $f''$ equals $f$ on $U \cup \delta \mathcal{D}$. Since $(f'|N)|N \cap \delta \mathcal{D}$ equals $f|N \cap \delta \mathcal{D}$, the derivative of $f_1$ in the angular direction equals that of $f''|N$ on $N \cap \delta \mathcal{D}$.

This and the fact that $f_1$ and $f'|N$ are immersions show that $\mathcal{D}$ includes a sufficiently thin neighborhood $N_2$ of $\delta \mathcal{D}$ such that $f''$ restricted to $N_2$ is an immersion as an explicit matrix calculation in polar coordinates will verify: Give the polar coordinates on $\mathcal{D}$ and $W$ so that $\theta$ increases along the boundary orientation of $\mathcal{D}$ and $r$ decreases in the radial inward direction of the standard disk $\mathcal{D}$. Assume that $f_1$ and $f'$ preserves orientation. Then under the coordinate system $(r, \theta)$,

$$Df_1 = \left( \begin{array}{cc} \alpha & 0 \\ \beta & b \end{array} \right), \quad Df' = \left( \begin{array}{cc} \alpha' & 0 \\ \beta' & b' \end{array} \right)$$

on $\delta \mathcal{D}$ for $\alpha, b, \alpha', b' > 0$ since $\delta \mathcal{D} \to \delta W$ and the inward vector of $\mathcal{D}$ is mapped to inward vector of $W$ for both of $f_1$ and $f'$. We have

$$Df'' = \left( \begin{array}{cc} \psi_r(f_{11} - f_1') + \psi \alpha + (1 - \psi)\alpha' & 0 \\ * & \psi_\theta(f_{12} - f_2') + \psi b + (1 - \psi)b' \end{array} \right)$$

where $f_{11}$ and $f_1'$ denote the $r$-component of $f_1$ and $f'$ respectively and $f_{12}$ and $f_2'$ denote the $\theta$-components of $f_1$ and $f'$ respectively. Since $f_1 = f'$ on $\delta \mathcal{D} \cap N$, it follows that $Df''$ is not zero on $\delta \mathcal{D}$. Therefore, $f''$ is an immersion on a neighborhood of $\delta \mathcal{D}$.

Since $f''|U \cup \delta \mathcal{D}$ is injective, $N \cup N_2$ includes a neighborhood $U'$ of $U \cup \delta \mathcal{D}$ in $\mathcal{D}$ so that $f''|U'$ is an embedding.

In $U'$, there is a neighborhood $A$ of $U \cup \delta \mathcal{D}$, diffeomorphic to a compact annulus, with one boundary component equal to $\delta \mathcal{D}$ and the other in $\mathcal{D}^\circ$. Then $f''|A$ is a diffeomorphism of $A$ into an annulus $A'$ in $W$ whose one boundary component is $\delta W$. Since $\mathcal{D}^\circ - A^\circ$ is a compact disk with smooth boundary and so is $W^\circ - f(A')^\circ$, there is a diffeomorphism $f'''$ between these disks extending the map $f''$ restricted to the component of $\delta A$ in $\mathcal{D}^\circ$. One can now smooth $f''$ and $f'''$ to a diffeomorphism $\mathcal{D} \to \mathcal{D}$ (see Theorem 8.1.9 [30, p. 182]).

We will need two definitions: A smooth vector field $\zeta$ on $\Omega$ is an $f$-gradient-like vector field for a real-valued Morse function $f$ if

- $\zeta(f) > 0$ in the complement of critical points of $f$, and
- given each critical point $p$ of $f$, there are coordinates $(u^1, u^2)$ in a neighborhood $U$ of $p$ so that $f = f(p) + (-1)^p(u^1)^2 + (-1)^q(u^2)^2$, and $\zeta$ has
coordinates \((-1)^p u^1, (-1)^q u^2\) throughout \(U\), where \(p\) and \(q\) are integers equal to 1 or 2.

(See Milnor \[39, p. 20]\.)

A cubical neighborhood of a point \(x\) in \(M\) is an open convex ball \(B\) with compact closure \(\text{Cl}(B)\) in \(M\) so that \(\text{dev}|B\) is an embedding onto the set

\[
\{(x_1, x_2, x_3) \in \mathbb{R}^3 ||x_i - a_i| < b \text{ for every } i = 1, 2, 3\}
\]

for some real numbers \(a_1, a_2, a_3, b, b > 0\). A bottom side of \(B\) is the side of \(B\) corresponding to \(x_1 = a_1 - b\). \(B\) has natural coordinates \(x_1 \circ \text{dev}, x_2 \circ \text{dev}, \) and \(x_3 \circ \text{dev}\), which were denoted by \(x_1, x_2, \) and \(x_3\).

Proposition 13 can be extended as follows:

Let \(B\) be a cubical neighborhood of \(z \in \delta \Omega_1\). Suppose that there exists an \(x_1 \circ F\)-gradient-like vector field \(\zeta\) on \(\Omega - \Omega_1\) so that the map \((x_2 \circ F, x_3 \circ F)\) restricted on a compact-disk neighborhood \(V\) of \(z\) in \(\Omega_1\), \(V \subset F^{-1}(B)\), is an orientation-preserving diffeomorphism onto an open subset of \(\mathbb{R}^2\) with the given orientation. (Recall that we fix an orientation of \(\Omega\) and \(\mathbb{R}^2\) so that \((x_2 \circ G, x_3 \circ G) : \Omega_1 \rightarrow \mathbb{R}^2\) is orientation-preserving.) Then we can choose \(F'\) so that the following statements hold in addition to (i), (ii), and (iii) in Proposition 13:

(iv) There exists an \(x_1 \circ F'\)-gradient-like vector field \(\zeta'\) on \(\Omega\) extending \(\zeta\) smoothly so that the closure \(\Gamma\) of the \(\zeta'\)-trajectory from \(z\) to the relative maximum point \(y\) of \(x_1 \circ F'\) is a smooth arc and lies in the interior of \(V\) in \(\Omega_1\). (We have \(F'(y) \in B\).

(v) \((x_2 \circ F', x_3 \circ F')|V = (x_2 \circ F, x_3 \circ F)|V\).

Proof. To prove (iv) and (v), we will change \(F_1\) and \(G\) in the proof of Proposition 13. Since \((x_2 \circ F, x_3 \circ F)|V\) maps into \(W\) by the orientation condition, Lemma 3 shows that there is a diffeomorphism \(G_1 : \Omega_1 \rightarrow W\) so that

\[
(x_2 \circ G_1, x_3 \circ G_1)|V = (x_2 \circ F, x_3 \circ F)|V
\]

and

\[
G_1|\delta \Omega_1 = F|\delta \Omega_1.
\]

Recall \(F'\) and \(\Phi\) from equations 13 and 10 respectively. Since \(\Phi\) extends to a neighborhood of \(\Omega_1\), it follows that there exists an \(f'\)-gradient-like vector field \(\zeta_1\) on \(D\) so that \(\zeta_1\) pulls back to an \(f' \circ \Phi\)-gradient-like vector field on \(\Omega_1\) extending \(\zeta\).

Let \(V' = \Phi(V)\), and \(\Gamma_1\) the closure of the \(\zeta_1\)-trajectory from \(\Phi(z)\) to \(O\). There exists \(\epsilon_4, \epsilon_4 > 0\), so that \(\Gamma_1\) intersected with a circle of radius \(t - \epsilon\) with center \(O\) for \(0 \leq \epsilon \leq \epsilon_4\) is a unique point of the interior of \(V'\) in \(D\).

Let \(\Psi : D \rightarrow D\) be a diffeomorphism fixing all points of distance \(t'\) from \(O\), \(t - \epsilon_4 \leq t' \leq t\), and so that \(\Psi(\Gamma_1)\) is a subset of the interior of \(V'\) in \(D\). We define \(G'_1 : \Omega_1 \rightarrow N(W)\) by \(G'_1(x) = G_1(x) + f' \circ \Psi^{-1} \circ \Phi(x)\epsilon_1\).
Let \( \zeta_2 \) denote the vector field defined on \( \Omega_1 \) pulled-back from \( \zeta_1 \) by \( \Psi^{-1} \circ \Phi \). Then \( \zeta_2 \) extends \( \zeta_1 \); \( \zeta_2 \) is \( f' \circ \Psi^{-1} \circ \Phi \)-gradient-like vector field. We let \( \zeta' \) denote the extended vector field. Let \( \Gamma \) be \( \Phi^{-1} \circ \Psi(\Gamma_1) \), which is the closure of the \( \zeta' \)-trajectory from \( z \) to \( \Phi^{-1} \circ \Psi(0) \), the maximum point of \( f' \circ \Psi^{-1} \circ \Phi \) on \( \Omega_1 \), and \( \Gamma \) is a subset of the interior of \( V \) in \( \Omega_1' \).

![Figure 14.](image)

We will need the following corollary later.

**Corollary 4.** Let \( N(\Omega_1) \) be the neighborhood \( \Omega_1 \cup A_{t-\delta,t} \) of \( \Omega_1 \) in \( \Omega^o \). Assume that \( W \) includes no component of \( F(f^{-1}(t)) \), other than \( F(\delta \Omega_1) \). Then there exists an amenable embedding \( F'' : \Omega \rightarrow M \) with the following properties:

(i) \( F'' \) is homotopic to \( F \) relative to \( \Omega - N(\Omega_1)^o \).

(ii) The maximum point \( y \) of \( x_1 \circ F''|_{N(\Omega_1)} \) is unique with maximum value less than \( t \).
The number of critical points of $x_1 \circ F'$ is less than or equal to that of $x_1 \circ F$.

The number of components of $(x_1 \circ F')^{-1}(t)$ is less than that of $(x_1 \circ F)^{-1}(t)$ by one or more.

**Proof.** By the above choice of $\delta'$, $F|N(\Omega_1)$ is a positive amenable embedding of level $t - \delta'$ and homotopic to an imbedding into $N(W)$ relative to $N(\Omega_1) - \Omega_1'$. Since $N(W)$ is diffeomorphic to a three-ball $N_t(W) \times [t - \delta', t + \delta']$, $F|N(\Omega_1)$ is homotopic relative to $\delta N(\Omega_1)$ to an embedding into $N_{t-\delta'}(W)$. Now, Proposition 13 implies the conclusions of the corollary. \qed

7. Homotopy of a Disk with Three Critical Points

We prove that an amenable embedding of a disk with three critical points is homotopic to an embedding into a level set of $M$. The proof involves Morse cancellation theory in this geometric setting. Such a disk can either have two relative maximum points and a saddle point or have one maximum and a relative minimum point and a saddle point. In the first case, by using Proposition 13, we try to push the one relative maximum very near the saddle and then we cancel them by constructing the explicit homotopy in a small convex open set called a cubical neighborhood. To do this, we use a flow line of a gradient-like vector field in the neighborhood connecting the relative maximum to the saddle point that we constructed at the end of Section 6 (see Milnor [39]). In the neighborhood of the union of arcs in the flow lines containing the saddle point and relative maximum and a point above the relative maximum, we find an expression of $f$ as an integral and modify the integral expression. In the second case, we use the 3-ball and the solid torus obtained in Proposition 12 to obtain homotopies. The both processes are captured in Figures 17 and 19 respectively.

**Proposition 14.** Let $\mathcal{D}$ be a disk; let $F : \mathcal{D} \rightarrow M$ be a positive amenable embedding of level $t$ with three critical values. Then there exists an embedding $G : \mathcal{D} \rightarrow M_t$ homotopic to $F$ relative to $\delta \mathcal{D}$

**Proof.** Let $f = x_1 \circ F$. Since the Euler characteristic of $\mathcal{D}$ is 1, $f$ has two relative extreme points and a saddle point by Proposition 10. Let $v_1, v_2,$ and $v_3$ denote critical values, and $z_1, z_2,$ and $z_3$ the corresponding critical points in $\mathcal{D}$ respectively. By relabeling if necessary, we can assume without loss of generality that there are only following cases:

(a) $z_1$ is a maximum point, $z_2$ a local maximum point, and $z_3$ a saddle point where $v_1 > v_2 > v_3 > t$.

(b) $z_1$ is a maximum point, $z_2$ a local minimum point, and $z_3$ a saddle point where $v_1 > v_3 > v_2 \ (v_1, v_3 > t)$. 

Figure 15. Examples of (a) and (b).

(a) Let $B$ be a small cubical neighborhood of $F(z_3)$ in $M$ so that $F^{-1}(B)$ is an open disk neighborhood of $z_3$. We can write $F(x) = (f(x), x_2 \circ F(x), x_3 \circ F(x))$ for $x \in F^{-1}(B)$ under the natural coordinates of $B$. Let $\mathcal{V} = F^{-1}(B)$ whose closure $\text{Cl}(\mathcal{V})$ is a compact disk in $\mathcal{D}$. By choosing $B$ sufficiently small, we may assume without loss of generality that the map defined by

$$x \mapsto (x_2 \circ F(x), x_3 \circ F(x))$$

is a diffeomorphism of $\text{Cl}(\mathcal{V})$ onto a closed rectangular disk in $\mathbb{R}^2$, corresponding to the bottom side of $\text{Cl}(B)$. (See the proof of Proposition [12].)

Choose $\epsilon, \epsilon > 0$, so that $v_3 + \epsilon < v_2$ and for the components $\alpha_1$ and $\alpha_2$ of $f^{-1}(v_3 + \epsilon)$, which are simple closed curves, both $F(\alpha_1)$ and $F(\alpha_2)$ pass $B$. Since $v_1, v_2 > v_3$, it follows from the flow argument that the curves $\alpha_1$ and $\alpha_2$ are the respective boundaries of disjoint disks $D_1$ and $D_2$ in $\mathcal{D}$. We may assume without loss of generality that $D_1^\circ$ contains $z_1$, and $D_2^\circ$ contains $z_2$.

There exists a coordinate system $u^1, u^2$ in a neighborhood $U$ of $z_3$ so that

$$f = v_3 - (u^1)^2 + (u^2)^2$$

holds in $U$ (see the diagram 5 of Milnor [10, p. 15]). For a gradient-like vector field $\zeta$ for $f$, we are able to choose $\epsilon$ sufficiently small and points $r$ and $z$ so that the following statements hold:

- Let $r$ be the point on $D_1^\circ \cap \mathcal{V}$ on the $\zeta$-trajectory from $z_3$ to $z_1$ such that $f(r) > v_3 + \epsilon + \delta$ for some number $\delta, 0 < \delta < \epsilon$. The closure $\Gamma_1$ of the $\zeta$-trajectory from $z_3$ to $r$ is a subset of $\mathcal{V}$. ($\Gamma_1$ is a smooth simple arc with endpoints $r$ and $z_3$.)
- Let $z$ be the point on $\delta D_2 \cap \mathcal{V}$ on the $\zeta$-trajectory from $z_3$ to $z_2$. The closure $\Gamma_2$ of the $\zeta$-trajectory from $z_3$ to $z$ is a simple arc in $\mathcal{V}$. 
We will now modify $F$. By Proposition 11, $F(\alpha_1)$ and $F(\alpha_2)$ are respective boundaries of disks $E_1$ and $E_2$ in $M_{v_3+\varepsilon}$. Since the union of $E_2$ and $F(D_2)$ is the boundary of a three-ball, if $E_1$ is included in $E_2$, then $F(D_1)$ has to lie in the three-ball as $F$ is an imbedding. This contradicts the assumption that $f(z_1)$ is greater than $f(z_2)$. Therefore, $E_2$ does not include $E_1$, and $E_2$ is disjoint from $F(\alpha_1)$.

Recall that $(x_2 \circ F, x_3 \circ F)$ is an imbedding from $V$ to the subset of $\mathbb{R}^2$ corresponding to the bottom side of the cubical neighborhood. (See Figure 14.) We naturally choose an orientation on $D$ so that $(x_2 \circ F, x_3 \circ F)$ is orientation-preserving. Let $\alpha_2$ be given a boundary orientation from $D_2$, and so the image of $V - D_2$ lies to the right of $(x_1 \circ F(\alpha_2 \cap V), x_3 \circ F(\alpha_2 \cap V))$ in $\mathbb{R}^2$.

Let $G : D_2 \to E_2$ be the imbedding homotopic to $F|D_2$. We claim that $(x_1 \circ G, x_2 \circ G)$ on $V \cap D_2$ is also orientation preserving.

Suppose not. Then $E_2$ lies to the right of $F(\alpha_2)$ in $M_{v_3+\varepsilon}$ since $(x_2 \circ G, x_3 \circ G)$ is orientation-reversing immersion. Hence, $E_2$ is a disk in $M_{v_3+\varepsilon}$ that is the closure of the right component of $M_{v_3+\varepsilon} - F(\alpha_2)$. Since $B \cap M_{v_3+\varepsilon} - F(\mathcal{D})$ is the union of two components $F(\alpha_1 \cap \mathcal{V})$ and $F(\alpha_2 \cap \mathcal{V})$ and nothing else, it follows that $E_2 \cap B$ is a component of $B \cap M_{v_3+\varepsilon}$ removed with these two arcs, to the right of $F(\alpha_2 \cap \mathcal{V})$. Since $F(\alpha_1 \cap \mathcal{V})$ lies to the right of $F(\alpha_2 \cap \mathcal{V})$, the closure of $E_2$ contains some points of $F(\alpha_1)$. This is absurd since the boundary of $E_2$ equals $F(\alpha_2)$ and $E_2^o$ does not meet $F(\alpha_1)$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure16.png}
\caption{Figure 16.}
\end{figure}

By the above claim, Proposition 13, and the subsequent result, there exists an amenable embedding $F' : \mathcal{D} \to M$ with the following properties:

(i) $F'$ is homotopic to $F$ relative to $\mathcal{D} - D_2^o$.

(ii) The maximum point $y$ of $x_1 \circ F'|\mathcal{D}_2$ is unique with maximum value less than $t + \varepsilon + \delta$.

(iii) The number of critical points of $x_1 \circ F'$ equals that of $x_1 \circ F$.

(iv) There exists a gradient-like vector field $\zeta'$ on $\mathcal{D}$ extending $\zeta|\mathcal{D} - D_2^o$ smoothly so that the closure of the $\zeta'$-trajectory $\Gamma_3$ from $z$ to $y$ is a smooth arc and lies in $\mathcal{V} \cap D_2$.

(v) $(x_2 \circ F', x_3 \circ F') = (x_2 \circ F, x_3 \circ F)$ holds on $\text{Cl}(\mathcal{V}) \cap \mathcal{D}_2$. 

To obtain (v), we may have to take a slightly larger cubical neighborhood \( B \). In fact, since \( F' = F \) on \( \mathcal{D} - \mathcal{D}_2 \), we have \((x_2 \circ F', x_3 \circ F') = (x_2 \circ F, x_3 \circ F)\) on \( \mathcal{V} \). We let \( F' = F \) on \( \mathcal{D} - \mathcal{D}_2 \). Then \( F'(x) = (f'(x), x_2 \circ F(x), x_3 \circ F(x)) \) for every \( x \in \mathcal{V} \) in the natural coordinates of \( B \).

Since \( \Gamma_1 \cap \Gamma_2 = \{z\} \), \( \Gamma_2 \cap \Gamma_3 = \{z\} \), and \( \Gamma_1 \cap \Gamma_3 = \emptyset \) hold, the union \( \Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \) is a smoothly imbedded compact arc containing \( z \) in the interior and endpoints \( r \) and \( y \). Clearly, \( F' (\Gamma) \) is a subset of \( B \). There exists a positive constant \( \delta_1 \) so that the minimum \( d \)-distance from \( \delta \text{Cl}(B) \) and \( F'(\Gamma) \) is greater than \( 2\delta_1 \). Let \( B' \) be a cubical neighborhood of \( F'(\Gamma) \) in \( B \) so that the \( d \)-distance from each point of \( \delta \text{Cl}(B') \) to \( \delta \text{Cl}(B) \) equals \( 2\delta_1 \). For \( \delta_1 \), the equation \[ 13 \] shows that there exists \( \epsilon_5, \epsilon > 0 \), such that

\[
d(F'(x), F(x)) \leq \delta_1 \quad \text{if} \quad x \in A_{t,t+\epsilon_5}.
\]

We may assume that \( F' = F \) for \( \epsilon < \epsilon_5 \). Therefore, we have obtained:

**Lemma 4.** \( F'(x) \) does not belong to \( B' \) if \( x \in \mathcal{D} - \mathcal{V} \).

**Figure 17.** The cancellation steps in the first case

By a slight extension of the arguments in Section 5 of Milnor [39], we prove Lemma 8 in the appendix that there exists an open neighborhood \( U \) of \( \Gamma \) in \( F^{-1}(B') \) with a coordinate chart \( u = (u^1, u^2) \) so that the \( \zeta' \)-trajectory extending \( \Gamma \) is mapped into the \( u^1 \)-axis, \( u(r) = (-1, 0) \), \( u(z_3) = (0, 0) \), \( u(y) = (1, 0) \), and for every \( x, x \in U \),

\[
f'(x) = f'(r) + 2 \int_{-1}^{u^1(x)} v(t) dt - (u^2(x))^2
\]

for a real-valued function \( v \) defined on the real line \( \mathbb{R} \) that is positive for \( 0 < t < 1 \), and zero at 0 and 1 and negative elsewhere.
The property of $v$ also can be obtained from the equation:

$$2\int_0^1 v(t)dt = f'(y) - f'(z_3) > 0 \quad \text{and} \quad 2\int_{-1}^0 v(t)dt = f'(r) - f'(z_3) < 0$$

since $f'$ is decreasing on the interval in $\Gamma$ from $r$ to $z$ and increasing on that in $\Gamma$ from $z$ to $y$. Thus, $v$ is negative on the interval $(-1,0)$ and positive on $(0,1)$ and negative on $(1,1+\mu]$ for some small $\mu, \mu > 0$.

Now we choose a negative-valued function $w$ defined on $\mathbb{R}$ agreeing with $v$ in $\mathbb{R} - (-1,1+\mu)$ satisfying the following conditions:

(i) $(x,0)$ belongs to $u(U)$ for $x \in [-1-2\mu, 1+2\mu]$,
(ii) $2\int_{-1}^{1+\mu} w(t)dt = f'(y') - f'(r) < 0$ where $y' = u^{-1}(1+\mu,0)$, $f'(y') > f'(z)$, and
(iii) $\int_{-1}^x w(t)dt \geq \int_{-1}^x v(t)dt$ for $x \in \mathbb{R}$.

![Figure 18. Graphs of functions $v$ and $w$ and their primitive functions $f'$.](image)

One can find a nonnegative smooth function $s : \mathbb{R}^2 \to [0,1]$ that equals 1 on the segment in the $u_1$-axis with endpoints $(-1-2\mu, 0)$ and $(1+2\mu, 0)$, has a support in a compact subset of $u(U)$, and whenever $s(u_1,u_2) \neq 0$, we have $\partial s/\partial u^2(u_1,u_2) < 0$ for $u_2 > 0$, $\partial s/\partial u^2(u_1,u_2) > 0$ for $u_2 < 0$. Define a real-valued function $g$ on $U$ by

$$g(x) = f(r) - (u_2^2(x))^2$$

$$+ 2(1 - s(u_1(x),u_2(x))) \int_{-1}^{u_1(x)} v(t)dt + 2s(u_1(x),u_2(x)) \int_{-1}^{u_1(x)} w(t)dt.$$ 

Then $g(x) = f(x)$ for $x \in U - K$, where $K$ is the support of $s \circ u$. By (iii) and the condition on $\partial s/\partial u^2$, $\partial g/\partial u^2$ is nonzero whenever $u_2 \neq 0$. If $u_2 = 0$, $\partial g/\partial u^1$ is nonzero since $w$ is nonzero everywhere. Thus, $g$ has no critical points in $U$. Since the map given by $x \to (x_2 \circ F(x), x_3 \circ F(x))$ for $x \in V$
is a homeomorphism of $V$ onto an open rectangle on $\mathbb{R}^2$, the map given by
$$x \to (g(x), x_2 \circ F(x), x_3 \circ F(x))$$
for $x \in U$ and $x \to (f'(x), x_2 \circ F(x), x_3 \circ F(x))$ for $x \in V - U$ is an embedding into a subset of $B$ with the natural coordinates. Denote this map by $\hat{g} : V \to B$. Define a map $F'' : D \to M$ by $F''(x) = \hat{g}(x)$ if $x \in V$ and $F''(x) = F'(x)$ if $x \in D - V$. Since $\hat{g}(U) \subset B'$, Lemma 4 and the fact that $F'|D - U$ and $\hat{g} : V \to B$ are injective imply that $F''$ is an embedding. Clearly, $F''$ is an amenable embedding homotopic to $F$, and $x_1 \circ F''$ has a unique critical point, which is a maximum point. Hence, the conclusion follows from Proposition 11.

(b) Let $\epsilon$ be a small positive number so that $v_3 - \epsilon > v_2$. The level set $f^{-1}(v_3 - \epsilon)$ is the union of two disjoint simple closed curves $\alpha_1$ and $\alpha_2$. Since $z_1$ is a local maximum point and $z_3$ a saddle point, Proposition 10 implies that the union of $\alpha_1$ and $\alpha_2$ is the boundary of an annulus $A$ in $D^o$ with $z_1, z_3 \in A^o$. We may assume without loss of generality that $\alpha_1$ is the boundary of a disk $D_1$ in $D^o$ such that $\alpha_2 \subset D_1$.

Let $D_2$ be the closed disk in $D^o$ with boundary $\alpha_2$. Then $D_2 \subset D_1$ is a compact subset of $M$ homeomorphic to a solid torus.

(a) The map $F|D_2$ is a negative amenable embedding of level $v_3 - \epsilon$ with one critical point. By reversing the $x_1$-direction and using Proposition 4, we obtain a disk $D'$ in $M_{v_3 - \epsilon}$ with boundary $F(\alpha_2)$ such that $D' \cup F(D_2)$ is a boundary of a three-ball $B_2$. By Lemma 3, $D' = D_2$. Hence, $B_1 \cap B_2 = D_2$ and $B_1 \cup B_2$ is a compact subset of $M$ homeomorphic to a three-ball. The component $A_1$ of $f^{-1}([t, v_3 - \epsilon])$ including $\delta D$ is homeomorphic to an annulus, and the boundary component of $A_1$ in the level $v_3 - \epsilon$ equals $\alpha_1$, where $F(\alpha_1) = \delta D_1$. By Theorem 3, there exists a disk $D_t$ in $M_t$ so that the union of $F(A_1)$, $D_t$, and $D_1$ is the boundary of a three-ball $B_3$. Then $B_1 \cup B_2 \cup B_3$ is homeomorphic to a three-ball and the boundary of $B_1 \cup B_2 \cup B_3$ equals $D_t \cup F(D)$. Hence, $F$ is homotopic relative to $\delta D$ to an embedding $G : D \to D_t$.

(β) As in (a), $F|D_2$ is homotopic relative to $\delta D_2$ to an embedding $j' : D_2 \to M_{v_3 - \epsilon}$. Clearly, $F(\alpha_1)$ either is disjoint from $j'(D_2)$ or a subset of $j'(D_2)^o$.

(i) Suppose that $F(\alpha_1)$ is disjoint from $j'(D_2)$. Then the union of $F(\alpha_1)$ and $F(\alpha_2)$ is the boundary of the annulus $J(A)$, such that $J(A) \cup j'(D_2)$ is an imbedded disk in $M_{v_3 - \epsilon}$. Let $H_1 : D \to M$ be defined by $H_1(x) = F(x)$ if $x \in D - D_2$, $H_1(x) = J(x)$ if $x \in A$, and $H_1(x) = j'(x)$ if $x \in D_2$. Then...
Figure 19. The cancellation steps in the second case (α) and (β).

$H_1|\mathcal{D}_1$ is an imbedding onto the disk $J(A) \cup j'(\mathcal{D}_2)$. Theorem 8.1.8 in [30, p. 181] shows that $H_1|\mathcal{D}_1$ is homotopic relative to $\delta\mathcal{D}_1$ to an embedding $H_2$ from $\mathcal{D}_1$ onto $J(A) \cup j'(\mathcal{D}_2)$ in $M_{v_3-\epsilon}$.

Applying Proposition 13 to $F|\mathcal{D}_1$ and $H_2$, we obtain a smooth amenable embedding with a unique critical point, which is a maximum point. Proposition 11 completes the proof.

(ii) Suppose that $F(\alpha_1) \subset j'(\mathcal{D}_2)^\circ$. Then by Lemma 1, $J(A)$ is an annulus in $j'(\mathcal{D}_2)$ with boundary $F(\alpha_1) \cup F(\alpha_2)$. Define a map $H_3 : \mathcal{D} \to M$ by $H_3(x) = F(x)$ if $x \in \mathcal{D} - \mathcal{D}_1^\circ$, and $H_3(x) = J(x)$ if $x \in A$, and $H_3(x) = j'(x)$ if $x \in \mathcal{D}_2$. Then $H_3$ is homotopic to $F$ relative to $\mathcal{D} - \mathcal{D}_1^\circ$. Let us choose a smooth closed curve $\beta$ in the open disk $j'(\mathcal{D}_2) - J(A)$, so that the union of $\beta$ and $F(\alpha_1)$ is the boundary of an annulus, which we call $A_1$. Then $H_3$ is homotopic relative to $\mathcal{D} - \mathcal{D}_1^\circ$ to an imbedding $H_4 : \mathcal{D} \to M$ defined by $H_4(x) = F(x)$ if $x \in \mathcal{D} - \mathcal{D}_1^\circ$, $H_4|A$ is an embedding onto $A_1$ so that $H_4(\alpha_1) = F(\alpha_1)$, and $H_4(\alpha_2) = \beta$, and $H_4|\mathcal{D}_2$ is an embedding onto the disk that is the closure of $j'(\mathcal{D}_2) - A_1 - J(A)$. Now, $H_4|\mathcal{D}_1$ is an imbedding onto the closure of $j'(\mathcal{D}_2) - J(A)$, a disk. Again by Theorem 8.1.8 in [30, p. 181], $H_4|\mathcal{D}_1$ is homotopic relative to $\delta\mathcal{D}_1$ to an embedding $H_5$ from $\mathcal{D}_1$ onto the closure of $j'(\mathcal{D}_2) - J(A)$ in $M_{v_3-\epsilon}$. 


Similarly to (i), the proof is completed by Proposition 13 and Proposition 11.

8. THE INCOMPRESSIBLE LEVEL SURFACES

We prove that every amenable disk with boundary in a level set is homotopic to an embedding into the level set. Our basic idea is to find a highest saddle point and cut the embedding just below the level of the saddle point. Then what is above is the union of components that are either disks with three critical points, annuli with two critical points, and a disk with one critical point. There is exactly one component that is not a disk with one critical point. We reduce the number of the critical points on it by methods we described in Sections 5 and 6.

Since every embedded disk with boundary in a level set can be deformed to an amenable disk, this shows that each component of a level set is incompressible in $M$. Finally, we show that each component of a level set is diffeomorphic to $\mathbb{R}^2$ using the incompressible surface theory. By a result of Palmeira [13], $M$ is diffeomorphic to $\mathbb{R}^3$. This completes the proof of Theorem 1.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{induction_moves.png}
\caption{Induction Moves}
\end{figure}

By Proposition 11, the following \textit{induction hypothesis} is satisfied for $k = 1$:

\textbf{Hypothesis 1.} Let $D$ be a standard disk and $F: D \to M$ a positive amenable embedding of level $t$ with $k$ critical values. Then there exists an embedding $G$ of $D$ into $M_t$ homotopic to $F$ relative to $\partial D$.

Now assume that the induction hypothesis holds for $k \leq i - 1$ for an integer $i$, $i > 1$. We will show that the induction hypothesis holds for $k = i$. Let $F: D \to M$ be a positive amenable embedding of level $t$ with $i$ critical values. Let $f = x_1 \circ F$, $v$ the maximum of the critical values of index 1, and $z$ the saddle point corresponding to $v$. We choose $\epsilon$, $\epsilon > 0$, so that $[v - \epsilon, v]$ contains no other critical value. Letting $v' = v - \epsilon$, we see that the components of $f^{-1}(v')$ are simple closed curves $\alpha_1, \ldots, \alpha_n$ and the components of $f^{-1}([v', \infty))$ smooth subsurfaces $A_1, \ldots, A_m$ of $D^e$ with boundaries each of whose components equals $\alpha_i$ for some $i$. We may assume without loss of generality that exactly one component $A_i$ contains the unique saddle point $z$ in the interior. Since $F|A_i$ is positive amenable, $A_i$ has at least one local maximum point.
Proposition 11 show that \( \chi(A_1) \geq 0 \). Hence, \( A_1 \) is diffeomorphic to (i) a disk, or (ii) an annulus. Since there is no saddle point in \( A_j \) for \( j > 1 \), we have \( \chi(A_j) > 0 \). Thus, each \( A_j \) for \( j = 2, \ldots, m \) is diffeomorphic to a compact disk and contains a unique critical point, which is a local maximum point.

(i) In this case, every \( A_i \) is diffeomorphic to a disk. Since each \( A_i \) has a unique boundary component, we have \( m = n \) and may assume without loss of generality that \( \alpha_i \) is the boundary of \( A_i \) in \( D \) for each \( i, i = 1, 2, \ldots, n \). Since \( \chi(A_1) = 1 \) and \( A_1 \) has one saddle point, \( A_1 \) has three critical points, one of which is a saddle point and the other two are local extreme points. One of them is a local maximum point, and if the other one is a local minimum point, then the level of the highest saddle \( v \) is higher than that of the local minimum point, and since \( [v - \epsilon, v) \) contains no critical value, \( v - \epsilon \) is still higher than the level of the local minimum point. This is a contradiction since \( A_1 \) has no points below \( v - \epsilon \). Hence \( A_1 \) has two local maximum points and a saddle point.

Since by Propositions 11 and 14 each \( F(\alpha_i) \) for \( i = 1, \ldots, n \), is the boundary of a compact disk in \( M_{i'} \), the closure of precisely one component of \( M_{i'} - F(\alpha_i) \) is homeomorphic to a disk by Lemma 1. Therefore, we can define an innermost closed curve among \( F(\alpha_1), \ldots, F(\alpha_n) \) as follows: \( F(\alpha_j) \) is an innermost closed curve (with respect to \( M_{i'} \)) if the disk with boundary \( F(\alpha_j) \) in \( M_{i'} \) includes no \( F(\alpha_i) \) for \( i \neq j \). We consider an innermost \( F(\alpha_j), j \neq 1, \) included in the disk in \( M_{i'} \) with boundary \( F(\alpha_1) \). By Corollary 1, we obtain an amenable embedding \( F' : D \to M \) homotopic to \( F \) relative to \( D - N(A_j) \) for a compact-disk neighborhood \( N(A_j) \) of \( A_j \) with equal number of singularities, and the number of components of \( f^{-1}(v') \) where \( f' = x_1 \circ F' \) is less than that of \( f^{-1}(v) \) by one. Hence, by an induction, we obtain a positive amenable embedding \( F'' : D \to M \) homotopic to \( F \) relative to

\[
D - \bigcup_{j=2}^{n} N(A_j)
\]

so that \( F''(\alpha_1) \) is now an innermost component and the number of critical points of \( F'' \) equals that of \( F \). (By our choice in Corollary 1 we may assume that \( N(A_j) \) is disjoint from \( A_1 \) and \( N(A_k) \) for \( k \neq j, k = 2, \ldots, n \).) Then by Proposition 14, \( F''|A_1 \) is homotopic to an embedding onto a disk in \( M_{i'} \) relative to \( \delta A_1 \). By Proposition 13 we obtain an amenable embedding \( F''' : D \to M \) homotopic to \( F'' \) relative to \( D - A_1^0 \) and \( x_1 \circ F''' \) has two less number of critical points than \( x_1 \circ F'' \). Then by the induction hypothesis, \( F \) is homotopic to an embedding \( G : D \to M \) relative to \( \delta D \).

(ii) We have \( m = n - 1 \), and may assume without loss of generality that \( \alpha_1 \) and \( \alpha_n \) are the boundary components of the annulus \( A_1 \) and that \( \alpha_i \) is the boundary component of a compact disk \( A_i \) for each \( i, i = 2, \ldots, n - 1 \). We may
further assume without loss of generality that \( \alpha_1 \) is the boundary of a disk \( D \) in \( \mathbb{D} \) such that \( \alpha_\circ \subset \mathbb{D} \). Let \( D \) be the disk with boundary \( \alpha_\circ \) in \( \mathbb{D} \); we have \( D \subset \mathbb{D} \). Then \( F|\mathbb{D} \) is a positive amenable embedding of level \( \nu \), and the map \( F|\mathbb{D} \) is a negative amenable embedding of level \( \nu \) with the number of critical points less than that of \( F \) by at least two. Reversing the direction of \( x_1 \) and applying the induction hypothesis show that \( F|\mathbb{D} \) is homotopic to an embedding \( H_2 : \mathbb{D} \to \mathbb{M} \nu \) relative to \( \delta \mathbb{D} \). Applying Proposition \[ \mathbb{H} \] to \( H_2 \), we obtain a positive amenable embedding \( H_1 : \mathbb{D} \to \mathbb{M} \) homotopic to \( F|\mathbb{D} \) relative to \( \mathbb{D} - \mathbb{D} \), so that \( x_1 \circ H_1 \) has only three critical points, which are a local maximum point, a saddle point, and a local minimum point respectively.

By Proposition \[ \mathbb{H} \], we obtain an embedding \( H : \mathbb{D} \to \mathbb{M} \nu \) homotopic to \( F|\mathbb{D} \) relative to \( \delta \mathbb{D} \). Let \( W \) be the image disk \( H(\mathbb{D}) \), where \( \delta W = F(\alpha_1) \).

Let us define a map \( F' : \mathbb{D} \to \mathbb{M} \) defined by \( F'(x) = F(x) \) if \( x \in D - \mathbb{D} \) and \( F'(x) = H(x) \) if \( x \in \mathbb{D} \). Now \( F' \) may not be an embedding but is homotopic to \( F \) relative to \( \mathbb{D} - \mathbb{D} \). For each \( i, i = 2, \ldots, n - 1, A_i \) either is a subset of \( \mathbb{D} \) or is disjoint from \( \mathbb{D} \), and \( F(\alpha_i) \) either is included in \( W \) or is disjoint from \( W \) since \( F \) is an imbedding. Let

\[
J = \{ j | F(\alpha_j) \subset W, A_j \not\subset \mathbb{D}, 2 \leq j \leq n - 1 \}.
\]

Since \( \alpha_j, j \in J \), is the boundary of a disk \( A \) with one critical point, Proposition \[ \mathbb{H} \] shows that \( F'|A \) is homotopic relative to \( \delta A \) to an embedding of \( A \) into \( W \). Suppose that \( F(\alpha_j), j \in J \), is an innermost one among \( F(\alpha_k) \) for \( k \in J \) in \( \mathbb{M}_\nu \). Corollary \[ \mathbb{H} \] implies that \( F' \) is homotopic relative to \( \mathbb{D} - N(\alpha_j) \) to a map \( F'' : \mathbb{D} \to \mathbb{M} \) so that \( x_1 \circ F''(N(\alpha_j)) < \nu \). By an induction eliminating such innermost \( \alpha_j, j \in J \), we obtain an imbedding \( L : \mathbb{D} \to \mathbb{M} \) so that \( L(x) \in W \) if and only if \( x \in \mathbb{D}, x_1 \circ L(N(\alpha_j)) < \nu \) for \( j \in J \), and \( L \) is homotopic to \( F \) relative to

\[
\mathbb{D} - \mathbb{D} = \bigcup_{j \in J} N(\alpha_j),
\]

where \( N(\alpha_j) \) is obtained as above for each \( j \), and \( L \) is smooth except on \( \delta \mathbb{D} \).
(The neighborhoods \( N(\alpha_j), j = 2, \ldots, n - 1 \), are assumed to be mutually disjoint.)

By Proposition \[ \mathbb{H} \], we obtain an amenable embedding \( L' : \mathbb{D} \to \mathbb{M} \) homotopic to \( L \) relative to \( \mathbb{D} - \mathbb{D} \), and have a unique critical point in \( \mathbb{D} \), which is a local maximum point. Thus we reduced the number of critical points by at least two. The induction hypothesis implies that there exists an embedding \( G : \mathbb{D} \to \mathbb{M} \) homotopic to \( L' \) relative to \( \delta \mathbb{D} \) and, hence, to \( F \). This completes the induction argument.

We proved:

**Theorem 6.** Let \( F : \mathbb{D} \to \mathbb{M} \) be a positive amenable embedding of level \( t \). Then there exists an embedding \( G : \mathbb{D} \to \mathbb{M} \) homotopic to \( F \) relative to \( \delta \mathbb{D} \).
We will follow Chapter 6 of [29]. (Peter Scott and a number of topologists told me that the following lemma is true.) Let $M$ be an orientable smooth open three-manifold. A surface is a properly imbedded connected two-manifold—possibly noncompact. We say that a surface $F$ is incompressible in $M$ if none of the following conditions is satisfied.

- $F$ is a two-sphere that is the boundary of a homotopy three-cell in $M$, or
- there is a two-cell $D$ in $M$ such that $D \cap F = \delta D$, and $\delta D$ not contractible in $F$.

**Lemma 5.** Let $S$ be a two-sided surface in $M$, and let $i_* : \pi_1(S) \to \pi_1(M)$ be the homomorphism induced by the inclusion map. Then if the kernel of $i_*$ is not trivial, then there is an embedded two-cell $D$ in $M$, transversal to $S$, such that $D \cap S = \delta D$, and $\delta D$ is not contractible in $S$.

**Corollary 5.** If $S$ is a two-sided incompressible surface in $M$, then $i_*$ is injective.

Now, let $M$ be a simply connected open two-convex affine three-manifold. Recall that $x_1$ is a function on $M$ defined by $x_1 = x_1 \circ \text{dev}$ for a coordinate function $x_1$ on $\mathbb{R}^3$. We defined $M_t$ to be the inverse image $x_1^{-1}(t)$ in $M$ for a real number $t$. Let $F$ be a component of $M_t$, which is a properly imbedded submanifold of $M$. By Lemma 1, $F$ is not homeomorphic to a sphere. We claim that $F$ is incompressible. Suppose that there is an embedded two-cell $D$ in $M$, transversal to $F$, such that $D \cap F = \delta D$ holds. It is easy to see that $D$ can be perturbed to be an imbedded disk so that $x_1|D$ is a Morse function. Theorem 6 shows that $\delta D$ bounds a disk in $F$.

Since $M$ is simply connected, $F$ is simply connected by Corollary 3. Hence $F$ is diffeomorphic to $\mathbb{R}^2$. We conclude that $M$ is foliated by leaves that are diffeomorphic to $\mathbb{R}^2$. The leaves are closed in $M$ since $M_t$ is closed in $M$. By Corollary 3 of Palmeira [13], $M$ is diffeomorphic to $\mathbb{R}^3$. This completes the proof of Theorem 1.

9. **The shrinkable dimension and $d$-convexity**

We first give our definition of shrinkable dimension and list Lie groups where this dimension is calculated. The shrinkable dimension of an affine group $\Gamma$ is the number $d$ such that for any element of the linear part of group $L(\Gamma)$ there is a direction of at least $(n - d)$-dimension of the standard $n$-ball which does not shrink to the origin. (See Figure 1.) The dimension is calculated for Lie groups using the representation restricted to the maximal tori.

The purpose of this section is to prove Theorem 3. $M$ is a closed affine $n$-manifold with a development map $\text{dev} : \tilde{M} \to \mathbb{R}^n$ and the holonomy homomorphism $h : \pi_1(M) \to \text{Aff}(\mathbb{R}^n)$. We assume that $h(\pi_1(M))$ having shrinkable
dimension less than equal to \( d \). If \( M \) is not \( d \)-convex, then \( \tilde{M} \) includes a \((d+1)\)-simplex such that \( T \cap \tilde{M}_\infty = \tilde{F}_1 \cap \tilde{M}_\infty \neq \emptyset \) for a side \( F_1 \) of \( T \) by Proposition \([2]\).

We prove that if the diameter of \( \tilde{F}_1 \cap \tilde{M}_\infty \) is small, then we use the method of Carrière in \([3]\). That is, we look at a ray starting from the vertex \( v_1 \) opposite \( F_1 \) and ending at a point of \( \tilde{F}_1 \cap \tilde{M}_\infty \). We look at a sequence of balls that this ray projected to \( M \) passes through and using the fact that the shrinkable dimension is less than or equal to \( d \), we show that there exists a sequence of ellipsoids on the ray that does not degenerate in \( n-d \) dimensions and hence meet a compact subset of \( T \cap \tilde{M} \) infinitely often. Since these ellipsoids are subsets of a fixed compact subset of \( \tilde{M} \), we obtain a contradiction to proper-discontinuity of the deck transformation group action of \( \tilde{M} \). (See Figure 21.)

If the diameter of \( \tilde{F}_1 \cap \tilde{M}_\infty \) is not so small, then we change \( T \) by changing the normal direction of \( F_1 \) in the confine of \( \tilde{M} \). This can be done, and for a generic choice of the normal direction, we can show that the diameter can become arbitrarily small. Our major effort is directed toward controlling the size of \( \tilde{F}_1 \cap \tilde{M}_\infty \) (see Figure 24). This will complete the proof of Theorem 2.

Recall the standard Euclidean metric \( d \) on \( \mathbb{R}^n \). We give a definition of shrinkable dimension, which is stronger than that of Carrière \([3]\) but probably just as useful. Let \( d' \) be the pull-back metric \( f^*d \) for a linear automorphism \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \); let \( \Gamma \) be a subgroup of \( \text{GL}(n, \mathbb{R}) \) acting on \( \mathbb{R}^n \) in a standard manner. For \( \epsilon > 0 \), let \( E_\epsilon \) be the standard \( n-d' \)-ball of radius \( \epsilon \) given by \( \{ v \in \mathbb{R}^n | d'(O,v) \leq \epsilon \} \) where \( O \) is the origin. For \( i \geq 1 \), an \( i \)-\( d' \)-ball \( E_i \) of radius \( \epsilon \) is the intersection of \( E_\epsilon \) with a vector subspace of dimension \( i \) of \( \mathbb{R}^n \). We say that \( \Gamma \) has **shrinkable dimension** less than or equal to \( d \) for \( 0 \leq d \leq n-1 \), or say \( \text{sd} \Gamma \leq d \), if given \( \epsilon, \epsilon > 0 \), there exists a positive constant \( \eta \) depending only on \( \epsilon \) such that for every \( \vartheta \in \Gamma \) there exists an \((n-d)\)-\( d' \)-ball \( E_{\epsilon,\vartheta} \) of radius \( \epsilon \) such that

\[
d'(\delta \vartheta(E_{\epsilon,\vartheta}),O) \geq \eta
\]

where \( \delta \vartheta(E_{\epsilon,\vartheta}) \) is the boundary of \( \vartheta(E_{\epsilon,\vartheta}) \). (See Figure 1.) Since \( \vartheta \) is linear, the condition is independent of \( \epsilon \). That is, if the condition is satisfied for a certain \( \epsilon \), then the condition is satisfied for every positive \( \epsilon \). Moreover, the condition is independent of which Euclidean metric we use; that is, if \( d' \) is given by the pull-back metric \( g^*d \) for another linear automorphism \( g : \mathbb{R}^n \rightarrow \mathbb{R}^n \), then the above condition is true for a different choice of \( \delta \). This follows since \( d' \) and \( d \) are quasi-isometric metrics on \( \mathbb{R}^n \). Thus, we need to verify equation \([13]\) only for a certain metric.

We give some examples: Let \( G \) be a connected reductive subgroup of \( \text{GL}(n, \mathbb{R}) \). Then \( G \) admits an internal Cartan decomposition

\[
G = KTK
\]
where \( K \subset G \) is a maximal compact subgroup and \( T \) a real maximal torus (see [28, p. 249]). Suppose that we can conjugate simultaneously so that \( K \subset O(n) \), and \( T \subset D(n) \), where \( O(n) \) is the orthogonal subgroup and \( D(n) \) the group of diagonal matrices. Thus, with respect to a fixed basis \( f \) of \( \mathbb{R}^n \) the matrix \( M(g) \) of every element \( g \) of \( G \) can be written \( M(g) = R_1(g)D(g)R_2(g) \) where \( R_1(g), R_2(g) \in O(n) \) and \( D(g) \) is a diagonal matrix with positive diagonal element. We may further assume that the diagonal entries of \( D(g) \) can be written in the decreasing order, i.e., \( a_{11} \geq a_{22} \geq \cdots \geq a_{nn} \). For each element \( g \) of \( G \), let \( n(g) \) be the number of diagonal elements of \( D(g) \) less than 1. (We remark that \( n(g) \) or \( n(g^{-1}) < n/2 \) if \( a_{ii} = 1 \) for a certain number of \( i \).

If \( n(g) \leq d \) for every \( g \in G \), then we claim that the shrinkable dimension of \( G \) is less than or equal to \( d \). Let \( d' \) be the metric pulled-back from \( d \) by the linear map sending the basis \( f \) to the standard basis of \( \mathbb{R}^n \). We consider the \((n - d) \cdot d'\)-ball \( E_{\epsilon}^{n-d} \) of radius \( \epsilon \) that is the intersection of \( E_{\epsilon} \) with the vector subspace given by \( x_{n-d+1} = 0, \ldots, x_{n} = 0 \) where \( x_1, \ldots, x_n \) are coordinates associated with \( f \). For each \( g \in G \), let \( E_{\epsilon,g} \) be \( R_2(g)^{-1}(E_{\epsilon}^{n-d}) \), an \((n - d) \cdot d'\)-ball of radius \( \epsilon \). Then \( g(E_{\epsilon,g}) = R_1(g)D(g)(E_{\epsilon}^{n-d}) \). Since \( a_{11}, \ldots, a_{n-d,n-d} \geq 1 \), it follows that

\[
d'(\delta M(g)(E_{\epsilon,g}), O) = d'(\delta R_1(g)D(g)(E_{\epsilon}^{n-d}), O) \geq \epsilon.
\]

Let \( \text{SL}(n, \mathbb{R}) \) be the group of \( n \times n \)-matrices of determinant 1, and \( \text{Sp}(2n, \mathbb{R}) \) the group of symplectic linear maps \( \mathbb{R}^{2n} \to \mathbb{R}^{2n} \) with respect to the standard symplectic form. Let \( O(p, q) \) denote the subgroup of \( \text{GL}(n, \mathbb{R}) \) of linear maps preserving the quadratic form given by

\[
x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2, p + q = n.
\]

From the dimension of the real maximal torus and their standard representation (see [3] and [12, p. 310]), we can deduce

\[
\text{sdSL}(n, \mathbb{R}) \leq n - 1, \quad \text{sdO}(2, q) \leq 2 \quad (2 \leq q), \quad \text{sdSp}(2n, \mathbb{R}) \leq n.
\]

(20)

There is a homomorphism \( L \) from the group \( \text{Aff}(\mathbb{R}^n) \) of affine transformations of \( \mathbb{R}^n \) to \( \text{GL}(n, \mathbb{R}) \) given by sending \( g \) to its linear part \( L(g) \). A subgroup \( G \) of \( \text{Aff}(\mathbb{R}^n) \) of \( \mathbb{R}^n \) has shrinkable dimension less than or equal to \( d \) if its linear part \( L(G) \) has shrinkable dimension less than or equal to \( d \).

We will now give the proof of Theorem 3. Let \( M \) be a closed affine \( n \)-manifold and \( (\text{dev}, h) \) a development pair. Then by assumption the linear part of the holonomy group \( h(\pi_1(M)) \) has shrinkable dimension less than or equal to \( d \). Suppose that \( M \) is not \( d \)-convex. We will derive a contradiction.

Let \( p : \tilde{M} \to M \) be the covering map, and \( F \) a fundamental domain of \( M \). Since \( \tilde{M} \) is an open manifold, each point \( x \) of \( F \) is contained in the interior of a convex compact ball \( D \) in \( \tilde{M} \) such that \( p|D \) is an imbedding onto a closed
ball in $M$. (We can do this by choosing $D$ within a chart.) Since $\text{Cl}(F)$ is compact, using the Lebesgue number, we may assume that for each point $x$ of $F$, $\tilde{M}$ includes a compact $d$-ball $B$ of radius $2\epsilon$ with center $x$ so that $p|B$ is an imbedding onto a ball in $M$, and $\text{dev}|B$ is an imbedding onto a $d$-ball of radius $2\epsilon$ with center $\text{dev}(x)$ (see Proposition 3). We say that $B$ is a tiny ball of radius $2\epsilon$ and center $x$.

Let $\delta$ be a positive number for this $\epsilon$ satisfying equation (19) for the linear part of $h(\pi_1(M))$. Since $M$ is not $d$-convex, there exists a $(d+1)$-simplex $T$ in $\tilde{M}$ and $F_1$ the side of $T$ so that

$$T \cap \tilde{M}_\infty = F_1^0 \cap \tilde{M}_\infty \neq \emptyset$$

by Proposition 2. By definition of simplices, $\text{dev}|T$ is an imbedding onto a $(d+1)$-simplex $\text{dev}(T)$ in $\mathbb{R}^\alpha$. Let $v_1$ be the vertex of $\text{dev}(T)$ opposite to $\text{dev}(F_1)$.

We first study the case where the $d$-diameter of $\tilde{M}_\infty$ in $T$ is less than or equal to $\delta/2$ following Carrière’s argument [3]. Let $x \in F_1 \cap \tilde{M}_\infty$ and $\alpha$ the line starting from $v_1$ and ending at $x$. We have $\text{Cl}(\alpha) - \alpha = \{x\}$, where $\text{Cl}(\alpha)$ is a segment in $T$ with endpoints $x$ and $v_1$. Let $M$ have a complete metric denoted by $d$. Then $\tilde{M}$ has an induced complete metric, which is also denoted by $d$. Under the covering map $p : \tilde{M} \to M$, $p|\alpha$ is a $d$-semi-infinite arc in $M$. Then since $M$ is compact, $p|\alpha$ passes arbitrarily close to a point $z$ in $M$.

Let $\tilde{z}$ be a point of $\tilde{M}$ corresponding to $z$ in the fundamental domain $F$. $\tilde{M}$ includes a tiny ball $E$ with $d$-radius $2\epsilon$ and center $\tilde{z}$.

Since $p|\alpha$ enters $p(K)$ for the tiny ball $K$ in $E$ of $d$-radius $\epsilon/2$ with center $\tilde{z}$ infinitely often, we may choose a monotone sequence of points $\{z_i\}$, $z_i \in \alpha$, converging to $x$, so that $p(z_i) \in p(K)$ for each $i$. There exists a sequence $\{z'_i\}$ of points of $E$ such that $p(z'_i) = p(z_i)$ and, hence, $z_i = \psi_i(z'_i)$ for a deck transformation $\psi_i$ for each $i$. For each $i$, $p(E)$ includes a compact ball $E_i$ with $p(z_i) \in E_i$ so that $E_i = p(E'_i)$ for a tiny ball $E'_i$ in $E$ of $d$-radius $\epsilon$ with center $z'_i$. (See Figure 21 from now on.) Since the holonomy group $h(\pi_1(M))$ has shrinkable dimension less than or equal to $d$, it follows that for each $\psi_i$, $E'_i$ includes a compact set $K'_i$ such that $\text{dev}(K'_i)$ is a compact $(n-d)$-$d$-ball of radius $\epsilon$ with center $\text{dev}(z'_i)$ and

$$d(h(\psi_i)(\text{dev}(z'_i)), h(\psi_i)(\delta\text{dev}(K'_i))) \geq \delta$$

for every $i$; that is, we have for each $i$

$$d(\text{dev}(z_i), \delta\text{dev}(\psi_i(K'_i))) \geq \delta.$$

Since $\text{dev}|\psi_i(K'_i)$ is a $d$-isometry, $K'_i$ is an $(n-d)$-$d$-ball satisfying

$$d(z_i, \delta\psi_i(K'_i)) \geq \delta.$$
The set $\text{dev}(T)$ is a subset of a unique $(d + 1)$-dimensional affine subspace $P^{d+1}$ in $\mathbb{R}^n$. Let $S$ be the component of $\text{dev}^{-1}(P^{d+1}) \cap \tilde{M}$ including $T^o$. Let $l : \mathbb{R}^n \to \mathbb{R}^{n-d-1}$ be the affine function whose zero set equals $P^{d+1}$. Since we have $\psi_i(K'_i) \cap S = (l \circ \text{dev} |_{\psi_i(K'_i)})^{-1}(O)$, the set $\psi_i(K'_i) \cap S$ equals a convex ball of dimension $\geq 1 = (n - d) - (n - d - 1)$. Therefore, $\psi_i(K'_i) \cap S$ includes a segment $s_i$ so that $z_i \in s_i$ and $d(z_i, \delta s_i) \geq \delta$ for each $i$ by equation 21.

Let $T_{3\delta/4}(x)$ be the subset of points of $T$ of $d$-distance $3\delta/4$ from $x$. Since the $d$-diameter of $F_1 \cap \tilde{M}_\infty$ is less than or equal to $\delta/2$, $T_{3\delta/4}(x)$ is a compact subset of $\tilde{M}$ not meeting $\tilde{M}_\infty$; hence, $T_{3\delta/4}(x)$ is a compact subset of $\tilde{M}$. Let $l_i = s_i \cap T$. Then $l_i$ is a segment in $T$ so that $z_i \in l_i^o$. An elementary geometric argument shows that $l_i$ meets $T_{3\delta/4}(x)$ if $z_i$ is in the $\delta/8$-neighborhood of $x$ in $T$ (compare with Carrière [3, Fig. 4]). Since $l_i \subset \psi_i(E)$ for the above compact subset $E$ of $\tilde{M}$, $\psi_i(E) \cap T_{3\delta/4}(x)$ is not empty for infinitely many $i$. Since $T_{3\delta/4}(x)$ is a compact subset of $\tilde{M}$, this contradicts the fact that the deck transformation group acts on $\tilde{M}$ properly discontinuously.

Suppose now that the $d$-diameter of $F_1 \cap \tilde{M}_\infty$ is greater than $\delta/2$. By tilting the face $F_1$ of $T$, we will now produce a new $(d + 1)$-simplex $T'$ with $T' \cap \tilde{M}_\infty \subset F_1 \cap \tilde{M}_\infty \neq \emptyset$. 

\textbf{Figure 21.}
where $F'_1$ is a side of $T'$ and the $d$-diameter of $F'_1 \cap \tilde{M}_\infty$ less than or equal to $\delta/4$. This will complete the proof of Theorem 2.

Let us consider the convex hull $K'$ of $\text{dev}(F_1 \cap \tilde{M}_\infty)$ in $\mathbb{R}^n$. Since $\text{dev}(F'_0)$ is a subset of $P^d$ for a $d$-dimensional affine subspace $P^d$ of $P^{d+1}$, $K'$ is a subset of $P^d$. For the purpose of choosing $T'$ and $F'_1$, we will now choose the following objects (see Figure 23):

$L^d_{i-1}$: Since $K'$ is convex, there exists an exposed point $x'$ of $K'$ with respect to $P^d$; that is, $x'$ is the unique intersection point of $K'$ with an affine hyperplane $L^d_{i-1}$ in $P^d$. Then we have $x' \in \text{dev}(F_1 \cap \tilde{M}_\infty)$ and $L^d_{i-1} \cap K' = \{x'\}$.

$L^d_0$: We choose a $(d-1)$-simplex $L^d_0$ in $L^d_0 \cap \text{dev}(F'_0)$ containing $x'$ in the interior and included in the $d$-ball of radius $\leq \delta/8$ with center $x'$.

$l^d_{i-2}$: Let $l^d_{i-2}$ for $i = 1, \ldots, d$ denote the sides of $L^d_{i-1}$.

Let $P^d_0$ be the component of $P^d - L^d_0$ disjoint from $K'$. Choose a segment $s$ of $d$-length $\leq \delta/8$ in the intersection of the closure of the component of $P^d_0$ and $\text{dev}(F'_0)$, transverse to $L^d_0$, and starting from $x'$.

$L^d_i$: We choose a point $x_0$ in $s$ sufficiently close to $x'$ so that the affine hyperplane $L^d_i$ in $P^d$ including $x_0$ and $l^d_{i-2}$ is disjoint from $K'$ for every $i$, $i = 1, \ldots, d$.

$L^d_i$: Let $L^d_i \cap \text{dev}(F_1)$ be denoted by $L^d_i$ for each $i$, $i = 1, \ldots, d$.

$\mathcal{H}^d_i$: For each $i$, $i = 0, \ldots, d$, let $\mathcal{H}^d_i$ denote the intersection of $\text{dev}(F_1)$ with the closure of the component of $P^d - L^d_i$ disjoint from $K'$. Then $\mathcal{H}^d_i \cap K' = \emptyset$ for $i = 1, \ldots, d$, and $\mathcal{H}^d_0 \cap K' = \{x'\}$.

If we let $\tilde{M}_P = \text{dev}^{-1}(P^{d+1}) \cap \tilde{M}$, then $\tilde{M}_P$ includes a compact $(p+1)$-ball neighborhood $U_i$ of the inverse image of $\mathcal{H}^d_i$ under $\text{dev}|T$ for each $i$, $i = 1, \ldots, d$. 

**Figure 22.**
By Proposition 5 and choosing $U_i$ appropriately, we may assume without loss of generality that $\text{dev}(T \cup U_i)$ is an imbedding onto a compact subset of $P^{d+1}$.

Let $P^2$ be a two-dimensional affine subspace including the segment $s$ and $v_1$. Then $P^2 \cap \text{dev}(F_i)$ is a segment $t$ passing through the interior of $\bigcap_{i=1}^d H_i$ and $x_0$. Choose a point $x'_1$ in $t \cap \bigcap_{i=1}^d H_i$, and a point $x_1$ in the exterior of $\text{dev}(T)$ on the semi-infinite line starting from $v_1$ passing through $x'_1$. We choose $x'_1$ and $x_1$ so that their $d$-distances from $x'_0$ are less than $\delta/4$. Then $x'_0 = \lambda v_1 + \mu x_1 + \nu x'$ where $\lambda, \mu, \nu$ are the barycentric coordinates and $\lambda + \mu + \nu = 1$. We choose $x_1$ sufficiently close to $x'_0$ so that $\mu > 0$. 

$$
\mu > 0.
$$

(22)

For each $i$, $i = 1, 2, \ldots, d$, $P^{d+1}$ includes a $(d+1)$-simplex $T^{d+1}_i$ with the following properties:

- The $d$-simplex in the unit tangent sphere at $v_1$ determined by $T^{d+1}_i$ matches with that determined by $\text{dev}(T)$.
- $F^{d}_1 \cap \text{dev}(F_i) = L^{d-1}_i$ where $F^{d}_1$ is the side of $T^{d+1}_i$ opposite to $v_1$.
- $x_0, x_1 \in F^{d}_1$.

Then it is easy to see that $T^{d+1}_i \supset H^{d}_i$, and $T^{d+1}_i \cap K' = \emptyset$. Hence, $x'$ is not an element of $T^{d+1}_i$. Since $F^{d}_1 \cap L^{d-1}_0 \supset L^{d-2}_i$ and we tilted $F^{d}_1$ away from $x'$, we have $T^{d+1}_i \cap (L^{d-1}_0)^o = \emptyset$.

Since $L^{d-1}_1, \ldots, L^{d-1}_d$ are in general position in $P^{d+1}$, $F^{d}_1, \ldots, F^{d}_d$ are in general position in $P^{d+1}$. Hence, $\bigcap_{i=1}^d F^{d}_1$ is a segment containing $x_0$ and $x_1$.

Recall $\tilde{M}_P = \text{dev}^{-1}(P^{d+1}) \cap \tilde{M}$. We may choose $x_1$ close enough to $x'_1$ so that $T^{d+1}_1 \subset \text{dev}(T \cup U_i)$ for every $i$, $i = 1, \ldots, d$. Let $\tilde{T}^{d+1}_1$ be the $(d + 1)$-simplex that is the inverse image of $T^{d+1}_1$ under the imbedding $\text{dev}|T \cup U_i$, where $\text{dev}|\tilde{T}^{d+1}$ is an imbedding onto $T^{d+1}_1$. Since $T \cup U_i$ is a subset of $\tilde{M}$, so is

Figure 23.
\[ \hat{T}^{d+1} \]. Letting \( T^* = \bigcup_{i=1}^{d} T_i^{d+1} \) and \( \hat{T}^* = \bigcup_{i=1}^{d} \hat{T}_i^{d+1} \), we obtain by Proposition 5 and an induction that \( \text{dev}(T \cup \hat{T}^*) \) is an imbedding onto \( \text{dev}(T) \cup T^* \) in \( P^{d+1} \).

For a simplex \( R \) in \( \mathbb{R}^n \) and points \( w_1, \ldots, w_m \), let \( [R, w_1, \ldots, w_m] \) denote the affine simplex spanned by the vertices of \( R \) and \( w_1, \ldots, w_m \) if it is nondegenerate. Since \( l_i^{d-2} \subset F_i^{d,i} \) and \( x_0, x_1 \in F_i^{d,i} \), the \( d \)-simplex \( [l_i^{d-2}, x_0, x_1] \) is a subset of \( F_i^{d,i} \). The \( (d+1) \)-simplex \( [\mathcal{L}_0^{d-1}, x_0, x_1] \) is included in the \( d \)-ball of radius \( \delta/4 \) with center \( x' \) by our choice of \( \mathcal{L}_0^{d-1} \), \( x_0 \), and \( x_1 \), and its sides are
\[ [l_i^{d-2}, x_0, x_1], \ldots, [l_d^{d-2}, x_0, x_1], [\mathcal{L}_0^{d-1}, x_0], \text{ and } [\mathcal{L}_0^{d-1}, x_1]. \]

The affine subspace \( P^{d+1} \) includes a \( (d+1) \)-simplex \( N \) with vertex \( v_1 \) such that the \( d \)-simplex in the unit tangent bundle at \( v_1 \) determined by \( N \) matches with that determined by \( \text{dev}(T) \), and the side \( F_1^N \) opposite to \( v_1 \) includes \( [\mathcal{L}_0^{d-1}, x_1] \).

**Lemma 6.** The simplex \( [\mathcal{L}_0^{d-1}, x_0, x_1] \) is included in \( N \) and includes
\[ N - (\text{dev}(T) - \text{dev}(F_1)) - \bigcup_{i=1}^{d} (T_i^{d+1} - F_i^{d,i}). \]

**Proof.** We assume that the affine functions in this proof are never constant functions. The set \( \text{dev}(T) \) can be described as a set of points of \( P^{d+1} \) where the real-valued affine functions \( f_1, \ldots, f_{d+1} \) are simultaneously nonnegative.
We assume without loss of generality that the zero set of \( f_i \) correspond to the side \( \text{dev}(F_i) \) of \( \text{dev}(T) \) respectively. Thus, \( f_1(x_0) = f_1(x') = 0 \), and \( f_1(v_1) > 0 \), although \( f_1(x_1) < 0 \) since \( x_1 \) does not belong to \( \text{dev}(T) \).

Let \( D \) be the set of points of \( P^{d+1} \) where \( f_2, \ldots, f_{d+1} \) are simultaneously nonnegative. \( N \) can be described as the set of points of \( D \) whose values under an affine function \( g_1 \) are nonnegative. Since the boundary of \( N \) in \( D \) is a zero set of \( g_1 \) restricted to \( D \), \( g_1 \) is zero on the simplex \( [\mathcal{L}^{d-1}_0, x_1] \). Thus, \( g_1(x_1) = 0 \) and \( g_1(v_1) > 0 \) while \( g_1(x') = 0 \) since \( \mathcal{L}^{d-1}_0 \) is a subset of the side \( F^N_1 \) of \( N \) opposite to \( v \) and \( x' \in \mathcal{L}^{d-1}_0 \).

Applying \( f_1 \) to the equation \( x_0 = \lambda v_1 + \mu x_1 + \nu x' \) where \( \lambda + \mu + \nu = 1 \), we obtain \( 0 = \lambda f_1(v_1) + \mu f_1(x_1) \). By equation 22, we obtain \( \lambda > 0 \). Applying \( g_1 \) to the above equation again, we obtain \( g_1(x_0) > 0 \), and hence \( x_0 \in N^o \). Therefore, \( [\mathcal{L}^{d-1}_0, x_0, x_1] \) is a subset of \( N \) while \( [\mathcal{L}^{d-1}_0, x_1] \) is a subset of the side \( F^N_1 \).

Since \( x_1 \) does not belong to \( \text{dev}(T) \), and \( [\mathcal{L}^{d-1}_0, x_0] \) is included in \( F_1 \), \( f_1 \) is negative in the interior of \( [\mathcal{L}^{d-1}_0, x_0, x_1] \).

Let \( h_i \) for each \( i, i = 1, \ldots, d \), be an affine function nonnegative on \( T^{d+1}_i \) and have \( F^d_{1,i} \) included in the zero set. Since \( T^{d+1}_i \cap (\mathcal{L}^{d-1}_0) = \emptyset \) by our choice earlier, \( h_i \) is negative on \( (\mathcal{L}^{d-1}_0)^c \). Since \( x_0 \) and \( x_1 \) are on \( F^d_{1,i} \) for every \( i \), \( h_i \) is negative in the interior of \( [\mathcal{L}^{d-1}_0, x_0, x_1] \) for every \( i, i = 1, \ldots, d \).

If \( x \) is a point of

\[
N - (\text{dev}(T) - \text{dev}(F_1)) - \bigcup_{i=1}^{d}(T^{d+1}_i - F^d_{1,i})
\]

Then \( x \in D \) and \( g_1(x) \geq 0 \), \( f_1(x) \leq 0 \), and \( h_i(x) \leq 0 \) for every \( i, i = 1, \ldots, d \). Suppose that \( x \) is not in \( [\mathcal{L}^{d-1}_0, x_0, x_1] \). Then choose a point \( y \) in the interior of \( [\mathcal{L}^{d-1}_0, x_0, x_1] \) Since \( \overline{xy'} \) does not meet \( F^N_1 \), the open line \( xy' \) meets the interior of a side of \([\mathcal{L}^{d-1}_0, x_0, x_1]\) of the form \([l^{d-2}_1, x_0, x_1], \ldots, [l^{d-2}_d, x_0, x_1], \) or \([\mathcal{L}^{d-1}_0, x_0]\). Since \( f_1, h_1, \ldots, h_d \) are affine functions, \( f_1, h_1, \ldots, h_d \) are simultaneously negative on \( \overline{xy'} \). But one of \( f_1, h_1, \ldots, h_d \) is zero at the point of \( \overline{xy'} \) intersecting the side. This is a contradiction. \( \square \)
The universal cover of an affine three-manifold

Remark 4. In fact, \([\mathcal{L}^{d-1}, x_0, x_1]\) is the closure of

\[ N - (\text{dev}(T) - \text{dev}(F_1)) - \bigcup_{i=1}^{d} (T_i^{d+1} - F_1^{d+1}) \]

in \(N\).

Let \(N_t\) for \(t \in (0,1]\) be the convex \((d+1)\)-simplex that is the image of \(N\) under the dilatation with center \(v_1\) by the magnification factor \(t\). Let \(v'_1\) denote the vertex of \(T\) corresponding to \(v_1\). Let us consider the subset \(A\) of \((0,1]\) whose element \(t\) is such that \(\tilde{M}\) includes a \((d+1)\)-simplex \(N_t\) such that \(\delta N_t \ni v'_1\) and \(\text{dev}|N_t\) is an imbedding onto \(N_t\). By Remark 4, \(A\) is of form \((0,t)\) or \((0,1]\).

If \(A\) is of form \((0,1]\), then \(N_1 \subset \tilde{M}\) and \(\text{dev}|N_1 \cup T\) is an imbedding onto \(N \cup \text{dev}(T)\) by Proposition 3. This means that \(x' \in N_1\), which is a contradiction since \(x'\) is a point of \(\tilde{M}_\infty \cap F_1^*\). Therefore, \(A\) is of form \((0,t)\) for \(0 < t \leq 1\). By Remark 4, there exists a \((d+1)\)-simplex \(N_t\) in \(\tilde{M}\) such that \(\text{dev}|N_t\) is an imbedding onto \(N_t\). Since \(t\) does not belong to \(A\), we have that \(N_t \cap \tilde{M}_\infty \neq \emptyset\).

Again by Proposition 3 and an induction, \(\text{dev}|N_t \cup T \cup T^*\) is an imbedding onto \(N_t \cup \text{dev}(T) \cup T^*\). Hence, \(\text{dev}(N_t \cap \tilde{M}_\infty)\) is disjoint from \(\text{dev}(T) - \text{dev}(F_1)\) and \(T_i^{d+1} - F_1^{d+1}\) for every \(i\) since they are images of subsets of \(\tilde{M}\). By Lemma 3, \(\text{dev}(N_t \cap \tilde{M}_\infty)\) is a subset of \([\mathcal{L}^{d-1}, x_0, x_1]\) since \(N_t \subset N\). Since \([\mathcal{L}^{d-1}, x_0, x_1]\) is included in a \(d\)-ball of radius \(\delta/4\), the \(d\)-diameter of \(N_t \cap \tilde{M}_\infty\) is less than or equal to \(\delta/2\).

Let \(T' = N_t\). Then it follows that

\[ T' \cap \tilde{M}_\infty \subset F_1^{\prime \prime} \cap \tilde{M}_\infty \neq \emptyset \]

for a side \(F_1^{\prime \prime}\) and the \(d\)-diameter of \(F_1^{\prime \prime} \cap \tilde{M}_\infty\) is less than or equal to \(\delta/2\). This completes the proof of Theorem 2.

10. The singular hyperbolic manifolds

We prove in this section Theorem 3. We will first give definition of hyperbolic manifolds with cone-type singular locus, and prove Claim 4, which implies the Theorem 3.

In this section, we identify the hyperbolic space \(H^3\) with the interior of the standard sphere, i.e., the \(d\)-radius 1 with center \(O\), in \(R^3\) in the real projective space \(RP^3\) and the group of hyperbolic isometries with the projectivized group \(PSO(1,3)\) of Lorentz isometric linear transformations of \(R^4 - \{O\}\) in \(GL(4,R)\).

Let \(l\) be the line in the hyperbolic space that is the intersection of the z-axis with the ball \(H^3\), for convenience, and \(N_c(l)\) is the closed hypercyclic neighborhood, i.e., given by points of \(H^3\) of hyperbolic distance less than or equal to \(\delta\) for a fixed \(c > 0\). Let \((N_c(l) - l)\) denote the universal cover of
Lemma 7. The completion \((N_c(l) - l)\) of \((N_c(l) - l)\) can be identified with the union of \(\text{Cl}(l)\) and \((N_c(l) - l)\) with the metric topology such that a Cauchy sequence of points \(p_i\) in \((N_c(l) - l)\) converges to a point \((0,0,t)\) in \(\text{Cl}(l)\) for \(t \in [-1,1]\) if and only if \(\text{dev}_l(p_i)\) converges to \((0,0,t)\) in \(\text{Cl}(l)\).

Proof. Straightforward.

Our manifold

\[((N_c(l) - l) - \{(0,0,1),(0,0,1)\})/ <s_{r,\theta}, s_{1,\theta} >\]

is homeomorphic to a solid torus \(T\) if \(r \neq 1\). The solid torus has a locus of singularity \(l'\) corresponding to \(l\) which is said to be the core singular locus. We say that \(T\) is a solid-torus with cone-type core-singularity \(l'\) of angle \(\theta\) and radius \(c\).

A pair of a three-manifold \(M\) and a link \(L\) in \(M\) is said to have a singular hyperbolic structure with cone-type singular locus \(L\) if \(M - L\) has a hyperbolic structure and for each component \(K\) of \(L\), there exists a closed neighborhood \(N_c(K)\) which is isometric to a solid-torus with cone-type singularity \(\theta_K\) for \(\theta_K > 0\) of radius \(c, c > 0\).

Let \(p : (M - L) \rightarrow M - L\) denote the universal covering map. Since \(M - L\) has a hyperbolic structure, \((M - L)\) has a developing map \(\text{dev} : (M - L) \rightarrow H^3\) and a holonomy homomorphism \(h : \pi_1(M - L) \rightarrow \text{PSO}(1,2)\). Since the standard ball \(H^3\) is a subset of an affine patch \(\mathbb{R}^3\) in \(\mathbb{R}P^3\), it follows that \(\text{dev}\) induces an affine structure on \((M - L)\). (However, the affine structure doesn’t descend to that of \(M - L\).) Theorem \(\Box\) and the following claim prove that \((M - L)\) is diffeomorphic to \(\mathbb{R}^3\) and prove Theorem \(\Box\).

Claim 1. \((M - L)\) is 2-convex.
We begin the proof of the claim. Let $N(L)$ denote the neighborhood of the link $L$ that is the union of $N_{c_i}(K_i)$, $c_i > 0$, for each component $K_i$ of $L$ where $N_{c_i}(K_i)$’s are mutually disjoint. Then $(M - L)$ equals the union of the connected submanifold $(M - N(L)^o)$ the inverse image of $M - N(L)^o$ under $p$ covering $M - N(L)^o$ and $(N(L) - L)$, the inverse image under $p$ of $N(L) - L$. Each component of $(N(L) - L)$ is a cover of $N_{c_i}(K_i) - K_i$, which we denote by $C^j_i$ for some $j$ in the countable index set $J_i$. Hence $(N(L) - L)$ consists of infinitely many components each isometric to $(N_c(l) - l)$ for some $c$ with respect to hyperbolic metrics. Hence, there exists a projective map $f^j_i : C^j_i \rightarrow (N_c(l) - l)$. Finally, the intersection of $(M - N(L)^o)$ and $(N(L) - L)$ is a union of infinitely many disjoint 2-cells which cover the tori $\delta N(K_i)$.

Suppose that $(M - L)$ is not 2-convex. Then there exists a 3-simplex $T$ in $(M - L)$ with $T \cap (M - L)_{\infty} = F^i_1 \cap (M - L)_{\infty} \neq \emptyset$ by Proposition 3. Hence, there exists an affine geodesic $k$ in $T$ starting from the vertex $v_1$ of $T$ opposite $F_1$ and ending at a point $y$ of $F^\infty_1 \cap (M - L)_{\infty}$. $p|k$ is a geodesic with respect to the hyperbolic metric. Suppose that $p|l$ is infinitely long under the hyperbolic metric. Then $\text{dev}|k$ is an infinitely long geodesic in $H^3$ and hence $\text{dev}(y)$ is a point of the sphere at infinity $\delta H^3$. However, note that $\text{dev}(y)$ is in the affine convex hull of four points which are images under $\text{dev}$ of the vertices of $T$. Since these points are in $H^3$ and $H^3$ is a strictly convex subset of $\mathbb{R}^3$, $\text{dev}(y)$ does not belong to $\delta H^3$. Therefore, $p|k$ is finitely long.

By looking at the geometry of $\text{dev}|k$ and $\text{dev}(C^j_i)$ in $H^3$, we see that $k$ cannot leave $C^j_i$ and re-enter the same $C^j_i$. Since $k$ is finitely long, $k$ cannot enter infinitely many $C^j_i$’s. Thus $p|k$ either stops at a point of $M - L$ or it enters $N_{c_i}(K_i)$ for a fixed $K_i$ and every $\epsilon$, $0 < \epsilon < c_i$. The first possibility is impossible since $x$ belongs to $(M - L)_{\infty}$. The second possibility implies that $k$ enters a component $C^j_i$ covering $N_{c_i}(K_i)$ and never leaves it. Letting $C^j_{i,\epsilon}$ denote the cover of $N_{c_i}(K_i)$ in $C^j_i$, $k$ enters $C^j_{i,\epsilon}$ for each $\epsilon > 0$ and never leaves it. We assume without loss of generality that $k$ is in $C^j_i$ by making $k$ smaller if necessary. The indices $i, j$ are fixed from now on.

Since the map $f^j_i : C^j_i \rightarrow (N_{c_i}(l) - l)$ is hyperbolic and hence projective, $\text{dev} \circ f^j_i = \vartheta \circ \text{dev}$ for some $\vartheta$ in $\text{PSO}(3, 1)$. This map $\vartheta$ is quasi-isometric with respect to $d$ on the standard ball $H^3$ in $\mathbb{R}P^3$. Since we will be looking at one $C^j_i$, we may assume without loss of generality that $\vartheta = I$ by choosing another $\text{dev}$ if necessary. $C^j_i$ has a path-metric $d'$ induced from the Riemannian metric from $\mathbb{R}^3$ induced by $\text{dev}$. The restriction to $C^j_i$ of the induced path metric $d$ on $(M - L)$ agrees with $d'$ on $C^j_i$ as the length of path in $(M - L)$ with boundary in $C^j_i$ can be shortened by a path in $C^j_i$ as we can easily show. Hence, the closure $\text{Cl}(C^j_i)$ in $(M - L)$ can be identified with the completion.
of $C^j_i$. Since the completion of $C^j_i$ is quasi-isometric to $(N_c(l) - l)$, we easily see that $\text{Cl}(C^j_i)$ is quasi-isometric to $(N_c(l) - l)$ by an extension map $f$ of $f^j_i$. Actually $f$ is an isometry.

Note that $(M - L)$ is the union of the closure of $(M - N(L))$ and the closure of $C^j_i$ in $(M - L)$ for each $j$. Let $l^j_i$ be the subset of $\text{Cl}(C^j_i)$ corresponding to $\text{Cl}(l)$ in $(N_c(l) - l)$ by $f$ (see Lemma [8]). Hence, $\text{Cl}(C^j_i)$ is the union of $C^j_i$ and the segment $l^j_i$ corresponding to the singular locus, and $x$ belongs to $l^j_i$ since $k$ is a subset of $C^j_i$.

Since $f(k)$ is a subset of $(N_c(l) - l)$, $\text{dev}(f(k))$ is a subset of unique open totally geodesic half-plane $P$ in $H^3$ with boundary $l$. $P \cap N_c(l)$ lifts to a unique subspace $P'$ of $(N_c(l) - l)$ including $k$. Then let $P'' = f^{-1}(P')$. It follows that $\text{dev} \circ f|P'' = \text{dev}|P''$ is an imbedding onto the convex set $P \cap N_c(l)$. Hence, the closure $\text{Cl}(P'')$ of $P''$ in $\text{Cl}(C^j_i)$ is a tame set, a 2-ball, and obviously $\text{Cl}(P'')$ includes the segment $l^j_i$ in its boundary.

Since $\text{dev}(T)$ and $\text{dev}(\text{Cl}(P''))$ are convex compact subset of $\mathbb{R}^3$, $\text{dev}(T) \cap \text{dev}(\text{Cl}(P''))$ is a convex set including $\text{dev}(k)$. Since $k \subset T^{o}$ and $k \subset P''$, $\text{dev}(T^{o}) \cap \text{dev}(P'')$ is not empty and $\text{dev}(T) \cap \text{dev}(\text{Cl}(P''))$ is its closure. Proposition [8] implies that $\text{dev}|T \cup \text{Cl}(P'')$ is an imbedding onto $\text{dev}(T) \cup \text{dev}(\text{Cl}(P''))$. Since $\text{Cl}(l) = \text{dev}(l^j_i)$, this means that

$$T \cap l^j_i = (\text{dev}|T \cup \text{Cl}(P''))^{-1}(\text{dev}(T) \cap \text{Cl}(l)).$$

Since $\text{dev}(T)$ is a subset of $H^3$, $\text{Cl}(l)$ is a segment with endpoints in $\delta H^3$, both sets contain $\text{dev}(x)$, it follows that $l$ is tangent to $\text{dev}(F^o_i)$ at $x$. Otherwise, $l$ intersects $\text{dev}(T^{o})$, which is impossible since $l$ is included in the ideal set of $(M - L)$. Furthermore, $\text{dev}(T) \cap \text{Cl}(l)$ is not a subset of $\text{dev}(F^o_i)$ by a geometric consideration that $\text{dev}(T)$ is a subset of $H^3$ and $\text{Cl}(l)$ has endpoint in the boundary of $H^3$. Hence, $T \cap l^j_i$ is not a subset of $F^o_i$. This contradicts our assumption. Hence, $(M - L)$ is 2-convex, and the proof of Theorem [8] is complete by Theorem [1].

**Appendix**

The purpose of this section is to prove Lemma [8]. The essential step to verify this lemma is Lemma 4.7 of Milnor [32, p. 43].

**Lemma 8.** Let $\Omega$ be a 2-disk. Let $f : \Omega \to \mathbb{R}$ be a smooth function with a local maximum $y$ and a saddle point $z$ with a $f$-gradient-like vector field $\zeta$. Let $\Gamma_1$ be a flow line from $z$ to a point $r$ with $f(r) > f(y)$ and $\Gamma_2$ a flow line from $z$ to $y$. Then there exists a small neighborhood $U$ of $\Gamma_1 \cup \Gamma_2$ with coordinates $(u_1, u_2)$ so that $r$ has coordinates $(-1, 0)$, $z (0, 0)$, and $y (1, 0)$; $f$ can be written
as
\[ f(x) = \int_{-1}^{u_1(x)} v_1(t)dt + f(r) - u_2^2, \]
and \((u_1, u_2) : U \to \mathbb{R}^2\) mapping \(\Gamma_1 \cup \Gamma_2\) to the segment \((-1, 0)(1, 0)\).

The plan to prove this lemma is we do this first for the flow line \(\Gamma_2\) only. For neighborhoods \(V_y\) and \(V_z\) of the endpoints \(y\) and \(z\) of \(\Gamma_2\) respectively, we have the canonical forms of \(f\). We modify our gradient-like vector field \(\zeta\) a little. Then we find a square \(S\) on \(\Gamma_2\), and choose \(\zeta\) as above. We assume further that \(\text{Cl}(\Gamma_2)\) is a subset of a component \(\alpha\) homeomorphic to disks, that the image of \(\text{Cl}(\Gamma_2)\) extends the functions of \(\text{Cl}(V_y)\) and \(\text{Cl}(V_z)\) by comparing functions near arcs in the boundaries of \(V_y\) and \(V_z\) respectively. The extension of the integral form to a neighborhood of \(\Gamma_1 \cup \Gamma_2\) is completely similar to the extension to \(S\) from \(V_y \cup V_z\).

We will show first that there exists a neighborhood \(U_2\) of \(\Gamma_2\) with coordinates \((u_1, u_2)\) so that \(z\) has coordinates \((0, 0)\), and \(y(1, 0)\); \(\tilde{f}\) can be written
\[ f(x) = \int_0^{u_1(x)} v_1(t)dt + f(z) - u_2(x)^2 \]
on \(U_2\); and \((u_1, u_2) : U_2 \to \mathbb{R}^2\) sending \(\Gamma_2\) to \((0, 0)(1, 0)\).

Choose a neighborhood \(V_z\) for \(z\) so that there exists a coordinate system \((v_1, v_2)\) where \(z\) has coordinates \((0, 0)\), \(v_2\) on \(\Gamma_2\) equals 0, \(f\) is written as \(f(z) + v_1^2 - v_2^2\), and the gradient-like vector field \(\zeta\) takes the canonical form \((v_1, -v_2)\), and choose \(V_y\) for \(y\) so that there exists a coordinate system \((v'_1, v'_2)\) where \(y\) has coordinates \((0, 0)\), \(v'_2\) on \(\Gamma_2\) equals 0, \(f\) is written as \(f(y) - v'_1^2 - v'_2^2\), and \(\zeta\) takes the canonical form \((-v'_1, -v'_2)\). Actually, we choose \(V_z\) and \(V_y\) a little bit smaller so that the respective neighborhoods \(V'_z\) and \(V'_y\) of their closures can be coordinatized as above. We assume further that \(\text{Cl}(V_z)\) and \(\text{Cl}(V_y)\) are homeomorphic to disks, that the image of \(\text{Cl}(V'_z)\) under the coordinate chart \((v_1, v_2)\) equals a region given by \(|v_1| < \ell\) and \(|v_2| < \ell\) and \(|v_1^2 - v_2^2| < \ell'\) for some positive numbers \(\ell\) and \(\ell'\) — thus the boundary of the image of \(\text{Cl}(V'_z)\) consists of four arcs in hyperbolas and eight segments — and that \(\text{Cl}(V'_y)\) under the coordinate chart \((v'_1, v'_2)\) equals a disk of radius \(\ell''\) with center \(O\) (see Figure 21). The boundary of \(V_z\) includes an arc segment \(\alpha\) where \(f\) is constant and the boundary of \(V_y\) includes an arc segment \(\beta\) where \(f\) is constant such that \(\alpha^o\) and \(\beta^o\) meet \(\Gamma_2\) at unique points. We assume without loss of generality that \(\alpha\) is a subset of a component \(\alpha'\) of a level set \(f^{-1}(f(z) + \kappa)\), and \(\beta\) is a subset of a component \(\beta'\) of \(f^{-1}(f(y) - \kappa)\) for a small positive number \(\kappa\). We choose \(\alpha\) and \(\beta\) so that \(v_2\) takes values \((-\kappa', \kappa')\) on \(\alpha\) and \(v'_2\) values \((-\kappa', \kappa')\) on \(\beta\) for
another small positive number $\kappa'$. We also assume that the $(v_1, v_2)$-coordinates of $\Gamma_2 \cap \alpha^o$ is $(\sqrt{\kappa}, 0)$ and the $(v_1', v_2')$-coordinates of $\Gamma_2 \cap \beta^o$ is $(-\sqrt{\kappa}, 0)$.

By altering $\zeta$ near $\alpha \cap \Gamma_2$, taking $\alpha$ sufficiently small, and a partition of unity argument, we assume that $\zeta$ is parallel to the vector field $(1, 0)$ in a neighborhood $U_z$ of $\alpha$ in $V'_z$ and on the $v_1$-axis. For simplicity, we let $V'_z = U_z \cup V_z$ from now on. We also alter $\zeta$ near $\beta \cap \Gamma_2$ using a partition of unity and taking $\beta$ sufficiently small so that $\zeta$ is parallel to the vector field $(1, 0)$ in a neighborhood $U_y$ of $\beta$ in $V'_y$ and on the $v_1'$-axis. We let $V'_y = U_y \cup V_z$ from now on.

Morse theory tells us that there exists a homeomorphism $\psi$ from $\alpha'$ to $\beta'$ induced by the flow generated by the $f$-gradient-like vector field $\zeta$. Let $A$ be the component of $D - \alpha' - \beta'$ not containing $y$ and $z$ homeomorphic to an open annulus. By Lemma 4.7 of Milnor [39, p. 43], we may alter our gradient-like vector field $\zeta$ within a compact subset of $A$, and $\psi$ accordingly, so that $\psi(\alpha) = \beta$ and $v_2' \circ \psi = \pm v_2$ holds on $\alpha$. We change the sign of $v_2'$ if necessary so that $v_2' \circ \psi = v_2$ holds on $\alpha$. We assume that $\Gamma_2$ is still the flow line from $z$ to $y$, which we can achieve by a self-diffeomorphism of $A$ supported in a compact subset of $A$ after we obtain a new flow line $\Gamma'_2$ from $z$ to $y$. We let $S$ denote the disk that was obtained by sweeping $\alpha$ to $\beta$ using the flow. (See Figure 27)

Introduce a coordinate $u_1$ on $\Gamma_2$ so that it agrees with $v_1$ in $V_z \cap \Gamma_2$ and with $v'_1$ in $V_y \cap \Gamma_2$; we may have to change the coordinate function $v'_1$ in $V_y$ by a constant if necessary so that $f(x) = f(y) - (v'_1 - c)^2 - (v'_2)^2$ on $V_y$
for a constant $c$, $c > 0$. We chose $u_1$ so that $u_1(z) = 0$, and then we see that $u_1(y) = c$. Restricted on the arc in $\Gamma_2$ between $z$ and $y$, $f$ is strictly increasing, and $f$ is decreasing elsewhere on $\Gamma_2$. We may write

$$f(x) = 2 \int_0^{u_1(x)} v(t) dt + f(z)$$

(25)

on $\Gamma_2$ where $v$ is a real-valued function with $v(t) > 0$ on $(0, c)$, $v(0) = v(c) = 0$, negative elsewhere, and equals a negative constant $C$ outside a bounded interval of $\mathbb{R}$. Let $\epsilon$ be the supremum of $u_1$ on $\Gamma_2 \cap V'_2$ and $\epsilon'$ that of $c - u_1$ on $\Gamma_2 - V'_y$. For $0 < t < \epsilon$, $v(t) = t$, and for $c - \epsilon' < t < c$, $v(t) = -t + c$ so that our equation 25 extends those of the local forms of $f$ in $V_z \cap \Gamma_2$ and $V_y \cap \Gamma_2$ respectively.

Let $\Gamma'_k$ for $k, |k| < \kappa$ denote the flow line from the point of $\alpha$ with $v_2$-coordinate $k$ to that of $\beta$ with $v'_2$-coordinate $k$. We introduce a coordinate function $u_2$ on the disk $S$ by letting $v_2(x)$ for $x \in S$ equal to $k$ if $\Gamma'_k$ passes through $x$. Then $u_2$ extends the coordinates $v_2$ on $V'_2$ and $v'_2$ on $V'_y$ smoothly by the third paragraph above. We let $u_2$ denote the extended function. We extend $u_1$ on $S$ by letting $u_1(x)$ equal to the value of $u_1$ on the point of $\Gamma_2$ intersected with the level curve passing through $x$.

We claim that for $\kappa$, $\kappa > 0$, sufficiently small, if $|k| < \kappa$, we can write for $x \in \Gamma'_k$

$$f(x) = 2 \int_0^{u_1(x)+\eta(x)} v(t) dt - u_2(x)^2 + f(z)$$

(26)

for a small $C^\infty$-function $\eta$ defined on $S$.

Let $C^0[\epsilon, c - \epsilon']$ be the Banach space of continuous real-valued functions defined on $[\epsilon, c - \epsilon']$ with the sup norm $|| \cdot ||_{\infty}$. Let $\mathcal{O}$ be the open subset of $C^0[\epsilon, c - \epsilon']$ of functions $g$ on $[\epsilon, c - \epsilon']$ such that $|g(r)|$ is less than $\min\{\epsilon/2, \epsilon'/2\}$ for each $r$.

We define a continuous functional $\mathcal{F} : \mathcal{O} \to C^0[\epsilon, c - \epsilon']$ for a real valued function $g$ defined on $[\epsilon, c - \epsilon']$ by letting the function $\mathcal{F}(g) : [\epsilon, c - \epsilon'] \to \mathbb{R}$ of variable $r$ equal to

$$k(r) = 2 \int_0^{r+g(r)} v(t) dt - f(z).$$

We claim that $\mathcal{F}$ is a $\sigma$-proper map (see Elworthy-Tromba [18, Section 3]). That is, the domain $\mathcal{O}$ can be written as a countable union of closed sets on each of which $\mathcal{F}$ is a proper map. $\mathcal{O}$ equals $\bigcup_{n=1}^{\infty} \mathcal{O}_n$ where $\mathcal{O}_n$ equals the subset consisting of elements $g$ of $C^0[\epsilon, c - \epsilon']$ with $||g||_{\infty} \leq \min\{\epsilon/2, \epsilon'/2\}(1 - 1/2^n)$.

The following calculations shows us that $\mathcal{F}|\mathcal{O}_n$ is a proper map for each $n$: Let
\( k_1 \) and \( k_2 \) correspond to \( g_1 \) and \( g_2 \) in \( \mathcal{O}_n \) under \( \mathcal{F} \) respectively and observe that
\[
|k_1(r) - k_2(r)| = 2|\int_0^{r+g_1(r)} v(t)dt - \int_0^{r+g_2(r)} v(t)dt| \\
= 2|\int_{r+g_1(r)}^{r+g_2(r)} v(t)dt| \\
\geq |g_1(r) - g_2(r)| \inf_{[\alpha,\epsilon]} |v| 
\] (27)

This implies that if \( k_i \to k \) in \( \mathcal{F}(\mathcal{O}_n) \), then \( g_i \) is a Cauchy sequence for any \( g_i \) in \( \mathcal{O}_n \) satisfying \( k_i = \mathcal{F}(g_i) \). Hence, \( (\mathcal{F}(\mathcal{O}_n))^{-1}(K) \) for a sequentially compact set \( K \) is compact.

The differential of this functional \( \mathcal{F} \) at 0 is given by \( g \mapsto vg \). Since \( v \) is never zero in \( (0, c) \), this map is a linear isomorphism of the space of real-valued smooth functions over \( [\epsilon, c - \epsilon'] \). We define \( f_k : [\epsilon, c - \epsilon'] \to \mathbb{R} \) by letting \( f_k(r) \) equal to \( f(x) \) for \( x \) satisfying \( u_1(x) = r \) and \( u_2(x) = k \). Since \( \mathcal{F} \) is a \( \sigma \)-proper-map, by the infinite dimensional Sard theorem of Smale [48], we can always solve the equation
\[
f_k(r) = 2 \int_0^{r+\eta_k(r)} v(t)dt - k^2 + f(z)
\]
for a continuous function \( \eta_k \) defined on \( [\epsilon, c - \epsilon'] \) if \( |k| \) is sufficiently small.

Let \( S_\kappa \) denote the union of \( \Gamma_l \) for \( |l| < \kappa \). Assume \( \kappa \) is sufficiently small. Our equation 26 would follow if we define \( \eta \) by letting \( \eta(x) = \eta_{u_2(x)}(u_1(x)) \) for \( x \in S_\kappa \) and show \( \eta \) to be \( C^\infty \). The fact that \( \eta \) is smooth will follow if the two-variable function \( \eta' : [\epsilon, c - \epsilon'] \times [-\kappa, \kappa] \) defined by letting \( \eta'(r, k) \) to be \( \eta_k(r) \) is smooth. Since the function \( f' \) defined by letting \( f'(r, k) \) equal to \( f_k(r) \) is smooth on \( S_\kappa \), \( k \mapsto f_k \) is continuous. Hence, the inequality 27 shows that \( k \mapsto \eta_k \) is also a continuous function, and the continuity of \( \eta' \) follows. The smoothness of \( \eta' \) follows by partial differentiating with respect to variables \( r \) and \( k \) the function
\[
f'(r, k) = 2 \int_0^{r+\eta'(r,k)} v(t)dt - k^2 + f(z)
\]
and a bootstrap argument.

For \( \kappa \) sufficiently small, we let \( u_1' = u_1 + \eta \) for each \( x \) in \( S_\kappa \). Then \( u_1' \) is a coordinate function on \( S_\kappa \). Near \( U_z \), \( f \) is of form \( f(z) + u_1^2 - u_2^2 \) and of form \( f(x) = 2 \int_0^{u_1'(x)} v(t)dt - u_2(x)^2 + f(z) \) in \( U_z \cap S_\kappa \) where \( v(t) = t \) for \( t \in (0, \epsilon'') \) for some \( \epsilon'' > \epsilon \). Since the value of \( u_1'(x) \) is near \( \epsilon \) on \( \alpha \) if \( \kappa \) is sufficiently small, an integration shows that \( f(x) = (u_1'(x))^2 - u_2(x)^2 + f(z) \) for \( x \) in \( S_\kappa \) sufficiently near \( \alpha \) since \( \epsilon < \epsilon'' \). Hence, for \( x \) in \( S_\kappa \cap V_z \) sufficiently near \( \alpha \), we have \((u_1'(x))^2 = (v_1(x))^2\). Since \( u_1' \) and \( v_1 \) are positive sufficiently near \( \alpha \), \( u_1' \) extends \( v_1 \) smoothly.
Near $\beta$, $f$ is of form $f(y) - (v'_1 - c)^2 - (u_2)^2$ and of form $f(x) = 2 \int_0^{u'_1(x)} v(t) dt - u_2(x)^2 + f(z)$ in $S_\kappa$. Since $v(t) = -t + c$ for $t \in (c - \epsilon'', c)$ for some $\epsilon''$, $\epsilon'' > \epsilon'$, and $f(y) - f(z) = 2 \int_0^{c} v(t) dt$, it follows that

$$f(y) - f(z) = 2 \int_0^{c-\epsilon''} v(t) dt + (\epsilon'')^2. \tag{28}$$

If a point $x$ in $S_\kappa$ is sufficiently near $\beta$, then $u'_1(x) > c - \epsilon''$. For such $x$, since $v(t) = c - t$ for $c - \epsilon'' < t < c$, we can write

$$f(x) = 2 \int_{c-\epsilon''}^{u'_1(x)} v(t) dt + 2 \int_0^{c-\epsilon''} v(t) dt - (u_2(x))^2 + f(z)$$

$$= -(u'_1(x) - c)^2 + (\epsilon'')^2 + 2 \int_0^{c-\epsilon''} v(t) dt - (u_2(x))^2 + f(z). \tag{29}$$

In $V'_y$, $f$ is of form

$$f(x) = -(v'_1(x) - c)^2 - (v'_2(x))^2 + f(z) + f(y) - f(z).$$

By equations 28 and 29 and this equation, we obtain $(v'_1(x) - c)^2 = (u'_1(x) - c)^2$. Since $v'_1(x) < c$ near $\beta$ and so is $u'_1(x)$, we obtain $v'_1(x) = u'_1(x)$ near $\beta$ in $S_\kappa \cap V'_y$ assuming that $\kappa$ is chosen sufficiently small. Thus $u'_1|S_\kappa$ extends $v'_1|V'_y$ smoothly. Let us denote by $u'_1$ the coordinate function extending $v_1$ and $v'_1$ to $V_z \cup V_y \cup S_\kappa$. Then $(u'_1, u_2)$ forms a global coordinate system of $V_z \cup V_y \cup S_\kappa$.

Since $f$ takes canonical form in $V_z$ and $V_y$, it is easy to see that $f$ can be written of the form

$$f(x) = \int_0^{u'_1(x)} v(t) dt + f(z) - (v_2)^2$$

in $V_z \cup V_y \cup S_\kappa$. We let $U = V_z \cup V_y \cup S_\kappa$ and $u_1 = u'_1/c$ and we let our new $v$ be $c$ multiplied by $v$. Then $z$ has coordinates $(0, 0)$ and $y (1, 0)$, $\Gamma_2$ is on the $u_1$-axis, and $f$ has the form of equation 24.

To prove Lemma 8 fully, we need to extend our integral expression 24 to a neighborhood of $\Gamma_1$ to obtain the expression 23. However, this can be done by the identical method by patching coordinates together.

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