Bound States of Dimensionally Reduced SYM\(_{2+1}\) at Finite N

Francesco Antonuccio, Oleg Lunin, Stephen S. Pinsky

Department of Physics,
The Ohio State University,
Columbus, OH 43210

(March 14, 2021)

Abstract

We consider the dimensional reduction of \(\mathcal{N} = 1\) SYM\(_{2+1}\) to 1+1 dimensions. The gauge groups we consider are U(\(N\)) and SU(\(N\)), where \(N\) is finite. We formulate the continuum bound state problem in the light-cone formalism, and show that any normalizable SU(\(N\)) bound state must be a superposition of an infinite number of Fock states. We also discuss how massless states arise in the DLCQ formulation for certain discretizations.
I. INTRODUCTION.

Solving for the non-perturbative properties of quantum field theories – such as QCD – is typically an intractable problem. In order to gain some insights, however, a number of lower dimensional models have been investigated in the large $N$ (planar) approximation, with a plethora of examples emerging in the last few years (for a review see [1]).

In this work, we will pursue the same theme of studying a lower dimensional field theory, but unlike many previous investigations, we will allow the number of gauge colors, $N$, to be finite. In particular, we will consider the $1 + 1$ dimensional theory that is obtained by dimensionally reducing $2 + 1$ dimensional $\mathcal{N} = 1$ supersymmetric Yang Mills theory. The large $N$ limit of this theory has already been investigated in [2], and was shown to exhibit the phenomena of screening [3,4]. In this work, we will find it advantageous to quantize the theory on the light-cone, and to adopt the light-cone gauge. Since the light-cone Hamiltonian is proportional to the square of the supercharge (from supersymmetry), one may formulate the bound state problem in terms of the supercharge [2,5].

A particular motivation for studying $1 + 1$ dimensional field theories in the light-cone formalism is the simplicity of certain bound states – the t’Hooft pion and Schwinger particle being well known examples of this. Analogs of the t’Hooft pion in a non-supersymmetric theory involving (complex) adjoint fermions have also been discovered [10,13]. All of these bound states are characterized by relatively simple Fock state expansions, and in particular, there is an upper bound on the allowed number of partons appearing in each Fock state. It is therefore of interest to see whether the massless states in the DLCQ [7] formulation of the model studied here also admit simple Fock state expansions in the continuum limit. We will be addressing that question in detail here, while providing a more thorough discussion of our numerical results elsewhere [8].

We should stress that a number of recent string theory developments have sharpened the need to understand supersymmetric Yang-Mills in various dimensions, since they play a crucial role in describing D-brane dynamics (see [11] for a review), and, ultimately, in formulating M(atrix) Theory [1]. An interpretation of the matrix model for M(atrix) Theory at finite $N$ has also been given by Susskind [9], providing additional motivation to study super Yang-Mills at finite $N$. We should stress, however, that in the model we study here, we compactify the null direction $x^-$, rather than in a spatial direction. Furthermore, we drop the zero mode sector [14,15], which is conventional in DLCQ, and therefore we eliminate any possibility of connecting our solutions with an equal time quantization of the same theory with a spatially compactified dimension. However, experience with DLCQ has shown that the massive spectrum is insensitive to how the theory is compactified, and to the zero modes. For the massless spectrum this may not be the case.

The organization of the paper proceeds as follows; in Section II we formulate the bound state problem for a two dimensional matrix model in the light-cone formalism. In particular, we write down explicit expressions for the quantized supercharges of this theory. In Section III, we present an analytical study of the continuum bound state equations for the gauge group SU($N$), and conclude that there can be no normalizable SU($N$) bound state with an upper limit on the number of partons in its Fock state expansion. Remarkably, the proof hinges on the assumption that the eigenstates are normalizable - no further properties concerning the eigenstate wave functions are needed in addition to the fact that they satisfy
the bound state equations. The proof is given in several steps. First we consider the validity of this proposition in the large $N$ approximation, and for massless bound states. We then generalize the proof for finite $N$, and for massive states. In Section IV we discuss in detail the massless solutions that appear in the DLCQ bound state equations. Finally, in Section V, we review some of the implications of our results with regards to the utility of DLCQ as a non-perturbative approach towards solving Yang-Mills field theories at finite and large $N$.

II. FORMULATION OF THE BOUND STATE PROBLEM.

The light-cone formulation of the supersymmetric matrix model obtained by dimensionally reducing $\mathcal{N} = 1$ SYM$_{2+1}$ to 1+1 dimensions has already appeared in [2], to which we refer the reader for explicit derivations. We simply note here that the light-cone Hamiltonian $P^-$ is given in terms of the supercharge $Q^-$ via the supersymmetry relation $\{Q^-, Q^-\} = 2\sqrt{2}P^-$, where

$$Q^- = 2^{3/4}g \int dx^- \text{tr}\left\{(i[\phi, \partial_- \phi] + 2\psi \bar{\psi}) \frac{1}{2\sqrt{2}} \psi\right\}. \quad (2.1)$$

In the above, $\phi_{ij} = \phi_{ij}(x^+, x^-)$ and $\psi_{ij} = \psi_{ij}(x^+, x^-)$ are $N \times N$ Hermitian matrix fields representing the physical boson and fermion degrees of freedom (respectively) of the theory, and are remnants of the physical transverse degrees of freedom of the original 2+1 dimensional theory. This is a special feature of light-cone quantization in light-cone gauge: all unphysical degrees of freedom present in the original Lagrangian may be explicitly eliminated. There are no ghosts.

For completeness, we indicate the additional relation $\{Q^+, Q^+\} = 2\sqrt{2}P^+$ for the light-cone momentum $P^+$, where

$$Q^+ = 2^{1/4} \int dx^- \text{tr}\left[(\partial_- \phi)^2 + i\psi \bar{\partial}_- \psi\right]. \quad (2.2)$$

The (1,1) supersymmetry of the model follows from the fact $\{Q^+, Q^-\} = 0$. In order to quantize $\phi$ and $\psi$ on the light-cone, we first introduce the following expansions at fixed light-cone time $x^+ = 0$:

$$\phi_{ij}(x^-, 0) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{dk^+}{\sqrt{2k^+}} \left(a_{ij}(k^+)e^{-ik^+x^-} + a_{ji}^\dagger(k^+)e^{ik^+x^-}\right); \quad (2.3)$$

$$\psi_{ij}(x^-, 0) = \frac{1}{2\sqrt{\pi}} \int_0^\infty dk^+ \left(b_{ij}(k^+)e^{-ik^+x^-} + b_{ji}^\dagger(k^+)e^{ik^+x^-}\right). \quad (2.4)$$

We then specify the commutation relations

$$[a_{ij}(p^+), a^\dagger_{ik}(q^+)] = \{b_{ij}(p^+), b^\dagger_{ik}(q^+)\} = \delta(p^+ - q^+)\delta_{il}\delta_{jk} \quad (2.5)$$

for the gauge group $U(N)$, or

$$[a_{ij}(p^+), a^\dagger_{ik}(q^+)] = \{b_{ij}(p^+), b^\dagger_{ik}(q^+)\} = \delta(p^+ - q^+)\left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}\right) \quad (2.6)$$
for the gauge group SU(N).

For the bound state eigen-problem \(2P^+P^-|\Psi> = M^2|\Psi>\), we may restrict to the subspace of states with fixed light-cone momentum \(P^+\), on which \(P^+\) is diagonal, and so the bound state problem is reduced to the diagonalization of the light-cone Hamiltonian \(P^-\). Since \(P^-\) is proportional to the square of the supercharge \(Q^-\), any eigenstate \(|\Psi>\) of \(P^-\) with mass squared \(M^2\) gives rise to a natural four-fold degeneracy in the spectrum because of the supersymmetry algebra—all four states below have the same mass:

\[
|\Psi>, \quad Q^+|\Psi>, \quad Q^-|\Psi>, \quad Q^+Q^-|\Psi>.
\]

Although this four-fold degeneracy is realized in the continuum formulation of the theory, this property will not necessarily survive if we choose to discretize the theory in an arbitrary manner. However, a nice feature of DLCQ is that it does preserve the supersymmetry (and hence the exact four-fold degeneracy) for any resolution.

Focusing attention on zero mass eigenstates, we simply note that a massless eigenstate \(|\Psi>\) must also be annihilated by the supercharge \(Q^-\), since \(P^-\) is proportional to \((Q^-)^2\). Thus the relevant eigen-equation is \(Q^-|\Psi> = 0\). We wish to study this equation. However, first we need to state the explicit equation for \(Q^-\), in the momentum representation, which is obtained by substituting the quantized field expressions (2.3) and (2.4) directly into the definition of the supercharge (2.1). The result is:

\[
Q^- = \frac{i2^{-1/4}}{\sqrt{\pi}} \int_0^\infty dk_1dk_2dk_3 \delta(k_1 + k_2 - k_3) \left\{ \frac{1}{2\sqrt{k_1k_2}} \left[ a_{ik}^\dagger(k_1)a_{kj}^\dagger(k_2)b_{ij}(k_3) - b_{ij}^\dagger(k_3)a_{ik}(k_1)a_{kj}(k_2) \right] \right. \\
\left. \frac{1}{2\sqrt{k_1k_3}} \left[ a_{ik}^\dagger(k_3)a_{kj}(k_1)b_{ij}(k_2) - a_{ik}^\dagger(k_1)b_{ij}^\dagger(k_2)a_{kj}(k_3) \right] \right. \\
\left. \frac{1}{2\sqrt{k_2k_3}} \left[ b_{ik}^\dagger(k_1)a_{kj}^\dagger(k_2)b_{ij}(k_3) - a_{ij}^\dagger(k_3)b_{ij}^\dagger(k_3)a_{ik}(k_1)a_{kj}(k_2) \right] \right. \\
\left. \left( \frac{1}{k_1} + \frac{1}{k_2} - \frac{1}{k_3} \right) \left[ b_{ik}^\dagger(k_1)b_{ik}^\dagger(k_2)b_{ij}(k_3) + b_{ij}^\dagger(k_3)b_{ik}(k_1)b_{kj}(k_2) \right] \right\}. \tag{2.8}
\]

In order to implement the DLCQ formulation \(\text{[4]}\) of the theory, we simply restrict the momenta \(k_1, k_2\) and \(k_3\) appearing in the above equation to the following set of allowed momenta: \(\{P_+^+, 2P_+^+, 3P_+^+, \ldots\}\). Here, \(K\) is some arbitrary positive integer, and must be sent to infinity if we wish to recover the continuum formulation of the theory. The integer \(K\) is called the harmonic resolution, and \(1/K\) measures the coarseness of our discretization\(\text{[4]}\). Physically,

\(^1\)We assume the normalization \(\text{tr}[T^aT^b] = \delta^{ab}\), where the \(T^a\)'s are the generators of the Lie algebra of \(\text{SU}(N)\).

\(^2\)Recently, Susskind has proposed a surprising connection between the harmonic resolution arising from the DLCQ of \(M\) theory, and the integer \(N\) appearing in the \(\text{U}(N)\) gauge group for \(\text{M(atrix)}\) Theory (namely, they are the same) \(\text{[4]}\).
1/K represents the smallest unit of longitudinal momentum fraction allowed for each parton. As soon as we implement the DLCQ procedure, which is specified unambiguously by the harmonic resolution $K$, the integrals appearing in the definition of $Q^-$ are replaced by finite sums, and the eigen-equation $Q^-|\Psi\rangle = 0$ is reduced to a finite matrix problem. For sufficiently small values of $K$ (in this case for $K \leq 4$) this eigen-problem may be solved analytically. For values $K > 5$, we may still compute the DLCQ supercharge analytically as a function of $N$, but the diagonalization procedure must be performed numerically. A detailed discussion of the DLCQ analytical and numerical results of this work will appear elsewhere [8].

For now, we concentrate on the structure of the zero mass eigenstates for the continuum theory. Firstly, note that for the U($N$) bound state problem, massless states appear automatically because of the decoupling of the U(1) and SU($N$) degrees of freedom that constitute U($N$). More explicitly, we may introduce the U(1) operators

$$
\alpha(k^+) = \frac{1}{N} \text{tr}[a(k^+)] \quad \text{and} \quad \beta(k^+) = \frac{1}{N} \text{tr}[b(k^+)],
$$

(2.9)

which allow us to decompose any U($N$) operator into a sum of U(1) and SU($N$) operators:

$$
a(k^+) = \alpha(k^+) \cdot 1_{N\times N} + \tilde{a}(k^+) \quad \text{and} \quad b(k^+) = \beta(k^+) \cdot 1_{N\times N} + \tilde{b}(k^+),
$$

(2.10)

where $\tilde{a}(k^+)$ and $\tilde{b}(k^+)$ are traceless $N \times N$ matrices. If we now substitute the operators above into the expression for the supercharge (2.8), we find that all terms involving the U(1) factors $\alpha(k^+), \beta(k^+)$ vanish – only the SU($N$) operators $\tilde{a}(k^+), \tilde{b}(k^+)$ survive. i.e. starting with the definition of the U($N$) supercharge, we end up with the definition of the SU($N$) supercharge. In addition, the (anti)commutation relations $[\tilde{a}_{ij}(k_1), \alpha^\dagger(k_2)] = 0$ and $\{\tilde{b}_{ij}(k_1), \beta^\dagger(k_2)\} = 0$ imply that this supercharge acts only on the SU($N$) creation operators of a fock state - the U(1) creation operators only introduce degeneracies in the SU($N$) spectrum. Clearly, since $Q^-$ has no U(1) contribution, any fock state made up of only U(1) creation operators must have zero mass. The non-trivial problem here is to determine whether there are massless states for the SU($N$) sector. We will address this topic next.

### III. THE PROOF

It was pointed out in the previous section that a zero mass eigenstate is annihilated by the light-cone supercharge (2.8):

$$
Q^-|\Psi\rangle = 0 \quad \text{(3.1)}
$$

We wish to show that if such an SU($N$) eigenstate is normalizable, then it must involve a superposition of an infinite number of Fock states. The basic strategy is quite simple; normalizability will impose certain conditions on the light-cone wave functions as one or several momentum variables vanish. Moreover, if we assume a given eigenstate $|\Psi\rangle$ has at most $n$ partons, then the terms in $Q^-|\Psi\rangle$ consisting of $n+1$ partons must sum to zero, providing relations between the $n$ parton wave functions only. We then show these wave functions must all vanish by studying various zero momentum limits of these relations.
Interestingly, the utility of studying light-cone wave functions at small momenta also appears in the context of light-front QCD$_{3+1}$\[16\].

In order to proceed with a systematic presentation of the proof, we start by considering the large $N$ limit case. This simply means that we consider Fock states that are made from a single trace of a product of boson or fermion creation operators acting on the light-cone Fock vacuum $|0\rangle$. Multiple trace states correspond to $1/N$ corrections to the theory, and are therefore ignored. In this limit, a general state $|\Psi\rangle$ is a superposition of Fock states of any length, and may be written in the form

$$
|\Psi\rangle = \sum_{n=2}^{\infty} \sum_{r=0}^{n} \int_0^{P^+} \frac{dq_1 \ldots dq_n}{\sqrt{q_1 \ldots q_n}} \delta(q_1 + \ldots + q_n - P^+) \times
$$

$$
f_P^{(n,r)}(q_1, \ldots, q_n) \text{tr}[c^{\dagger}(q_1) \ldots c^{\dagger}(q_n)]|0\rangle,
$$

(3.2)

where $c^{\dagger}(q^+)$ represents either a boson or fermion creation operator carrying light-cone momentum $q^+$, and $f_P^{(n,r)}$ denotes the wave function of an $n$ parton Fock state containing $r$ fermions in a particular arrangement $P$. It is implied that we sum over all such arrangements, which may not necessarily be distinct with respect to cyclic symmetry of the trace.

At this point, we simply remark that normalizability of a general state $|\Psi\rangle$ above implies

$$
\int_0^{P^+} \frac{dq_1 \ldots dq_n}{q_1 \ldots q_n} \delta(q_1 + \ldots + q_n - P^+) |f_P^{(n,r)}(q_1, \ldots, q_n)|^2 < \infty
$$

(3.3)

for any particular wave function $f_P^{(n,r)}$. Therefore, any wave function vanishes if one or several of its momenta are made to vanish.

We are now ready to carry out the details of the proof. But first a little notation. We will write $|\Psi_{(n,m)}\rangle$ to denote a superposition of all Fock states – as in (3.2) – with precisely $n$ partons, $m$ of which are fermions. Such a Fock expansion involves only the wave functions $f_P^{(n,m)}$, and the number of them is enumerated by the index $P$. For the special case $|\Psi_{(n,0)}\rangle$ (i.e. no fermions), there is only one wave function, which we denote by $f^{(n,0)}$ for brevity:

$$
|\Psi_{(n,0)}\rangle = \int_0^{P^+} \frac{dq_1 \ldots dq_n}{\sqrt{q_1 \ldots q_n}} \delta(q_1 + \ldots + q_n - P^+) f^{(n,0)}(q_1, \ldots, q_n) \text{tr}[a^{\dagger}(q_1) \ldots a^{\dagger}(q_n)]|0\rangle.
$$

(3.4)

There is another special case we wish to consider; namely, the state $|\Psi_{(n,2)}\rangle$ consisting of $n$ parton Fock states with precisely two fermions. If we place one of the fermions at the beginning of the trace, then there are $n - 1$ ways of positioning the second fermion, yielding $n - 1$ possible wave functions. We will enumerate such wave functions by the subscript index $k$, as in $f_k^{(n,2)}$, where $k = 2, 3, \ldots, n$. The subscript $k$ denotes the location of the second fermion. Explicitly, we have

$$
|\Psi_{(n,2)}\rangle = \sum_{k=2}^{n} \int_0^{P^+} \frac{dq_1 \ldots dq_n}{\sqrt{q_1 \ldots q_n}} \delta(q_1 + \ldots + q_n - P^+) \times
$$

$$
f_k^{(n,2)}(q_1, \ldots, q_k, \ldots, q_n) \text{tr}[b^{\dagger}(q_1)a^{\dagger}(q_2) \ldots b^{\dagger}(q_k) \ldots a^{\dagger}(q_n)]|0\rangle.
$$

(3.5)

Of course, depending upon the symmetry, the $n - 1$ Fock states enumerated in this way need not be distinct with respect to the cyclic properties of the trace. This provides us with additional relations between wave functions – a fact we will make use of later on.
Now let us assume that $|\Psi\rangle$ is a normalizable SU($N$) zero mass eigenstate with at most $n$ partons. Glancing at the form of (2.8), we see that the $n+1$ parton Fock states containing a single fermion in each of the combinations $Q^-|\Psi_{(n,0)}\rangle$ and $Q^-|\Psi_{(n,2)}\rangle$ must cancel each other to guarantee a massless eigenstate. This immediately gives rise to the following wave function relation:

$$\frac{q_1 + 2q_2}{q_1 + q_2} f^{(n,0)}(q_1 + q_2, q_3, \ldots, q_{n+1}) - \frac{q_1 + 2q_{n+1}}{q_1 + q_{n+1}} f^{(n,0)}(q_{n+1} + q_1, q_2, \ldots, q_n) =$$

$$= 2\sqrt{\frac{q_1}{n}} \sum_{k=2}^{n} \frac{q_{k+1} - q_k}{(q_{k+1} + q_k)^{3/2}} f^{(n,2)}_k(q_1, \ldots, q_{k-1}, q_k + q_{k+1}, q_{k+2}, \ldots, q_{n+1}). \quad (3.6)$$

In the limit $q_i \to 0$, for $3 \leq i \leq n$, this last equation is reduced to

$$\frac{1}{\sqrt{q_{i+1}}} f^{(n,2)}_i(q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_{n+1})$$

$$- \frac{1}{\sqrt{q_{i-1}}} f^{(n,2)}_{i-1}(q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_{n+1}) = 0. \quad (3.7)$$

An immediate consequence is that any wave function $f^{(n,2)}_i$ for $i = 3, 4, \ldots, n$, may be expressed in terms of $f^{(n,2)}_2$. Explicitly, we have

$$f^{(n,2)}_i(q_1, q_2, \ldots, q_n) = \frac{q_i}{q_2} f^{(n,2)}_2(q_1, q_2, \ldots, q_n), \quad i = 3, 4, \ldots, n. \quad (3.8)$$

Moreover, the limit $q_2 \to 0$ of equation (3.6) yields the further relation after a suitable change of variables:

$$f^{(n,0)}(q_1, q_2, q_3, \ldots, q_n) = \frac{2}{n} \sqrt{\frac{q_1}{q_2}} f^{(n,2)}_2(q_1, q_2, q_3, \ldots, q_n). \quad (3.9)$$

Finally, because of the cyclic properties of the trace, there is an additional relation between wave functions:

$$f^{(n,2)}_i(q_1, q_2, \ldots, q_i, \ldots, q_n) = -f^{(n,2)}_{n-i+2}(q_i, q_{i+1}, \ldots, q_n, q_1, q_2, \ldots, q_{i-1}). \quad (3.10)$$

Setting $i = 2$ in the above equation, and $i = n$ in equation (3.8), we deduce

$$f^{(n,2)}_2(q_1, q_2, \ldots, q_n) = -\sqrt{\frac{q_1}{q_2}} f^{(n,2)}_2(q_2, q_3, \ldots, q_n, q_1). \quad (3.11)$$

Combining this with equation (3.9), we conclude $(\sqrt{\frac{q_1}{q_2}} + \sqrt{\frac{q_2}{q_1}}) f^{(n,0)}(q_1, \ldots, q_n) = 0$, where we use the fact that the wave functions $f^{(n,0)}$ are cyclically symmetric. Thus $f^{(n,0)}$ must vanish. It immediately follows that $f^{(n,2)}_i$ vanish for all $i$ as well.

To summarize, we have shown that if $|\Psi\rangle$ is a normalizable zero mass eigenstate, where each Fock state in its Fock state expansion has no more than $n$ partons, the contributions $|\Psi_{(n,0)}\rangle$ and $|\Psi_{(n,2)}\rangle$ in this Fock state expansion must vanish. Since we may assume $|\Psi\rangle$ is bosonic, the only other contributions involve Fock states with an even number of fermions:
positive integer \( k \) and our previous analysis demonstrated that \( Q \) and \( a \) fermion is replaced by two bosons. Keeping in mind that we are free to renormalize any wave function by a constant, we end up with the following relation:

\[
\Psi(\mathbf{Q}, n, 4) \equiv 0, \quad \text{and thus the } n+1 \text{ parton states with three fermions in } Q^- | \Psi(n, 4) \rangle \text{ alone must sum to zero.}
\]

We are now ready to perform an induction procedure. Namely, we assume that for some positive integer \( k \) the state \( | \Psi(n,2(k-1)) \rangle \) vanishes. Then the \( n+1 \) parton Fock states in \( Q^- | \Psi \rangle \) which contain \( 2k-1 \) fermions receive contributions only from \( Q^- | \Psi(n,2k) \rangle \) in which a fermion is replaced by two bosons. This has to sum to zero. We therefore obtain a relation among the wave functions \( f_p^{(n,2k)} \) by considering the action of the supercharge (2.3) in which a fermion is replaced by two bosons. Keeping in mind that we are free to renormalize any wave function by a constant, we end up with the following relation:

\[
\sum_P f_p^{(n,2k)}(s_1, \ldots, s_{i-1}, s_i + s_{i+1}, s_{i+2}, \ldots, s_{n+1}) \frac{s_{i+1} - s_i}{(s_{i+1} + s_i)^{3/2}} = 0. \tag{3.12}
\]

It is now an easy task to show that the wave functions \( f_p^{(n,2k)} \) appearing in equation (3.12) must vanish; one simply considers various limits \( s_j \to 0 \) as we did before. This completes our proof by induction. Namely, there can be no non-trivial normalizable massless state with an upper limit on the number of allowed partons. Of course, this proof is valid only in the large \( N \) limit. We now turn our attention to the finite \( N \) case.

Let us define \( Q_{lead}^- \) to be that part of the supercharge \( Q^- \) that replaces a fermion with two bosons, or replaces a boson with a boson and fermion pair. As in the large \( N \) case we begin by assuming that we have a normalizable zero mass eigenstate \( | \Psi \rangle \) which is a sum of Fock states that have at most \( n \) partons. The proof for finite \( N \) consists of two parts. First, we consider bosonic states consisting of only \( n \) parton Fock states that have at most two fermions, and show the wave functions must vanish. We then invoke an induction argument to consider \( n \) parton wave functions involving an even number of fermions, and show they must vanish as well.

The additional complication introduced by the assumption that \( N \) is finite is that a given Fock state may involve more than just a single trace. However, note that \( Q_{lead}^- \) cannot decrease the number of traces; it can either increase the number of traces by one, or leave the number unchanged. Thus we have a natural induction procedure in the number of traces as well. Since the terms in \( Q_{lead}^- \) have only one annihilation operator, it acts on a given product of traces according to the Leibniz rule:

\[
Q_{lead}^- (\text{tr}[A]\text{tr}[B] \ldots ) | 0 \rangle = (Q_{lead}^- \text{tr}[A]) \text{tr}[B] \ldots | 0 \rangle + (-1)^{F(A)} \text{tr}[A]Q_{lead}^- (\text{tr}[B] \ldots ) | 0 \rangle > . \tag{3.13}
\]

Schematically, the general structure of an arbitrary Fock state with \( k \) traces has the form

\[
f_p^{(n,i_1,i_2,\ldots,i_k)} \text{tr}[(b^\dagger)^{i_1}a^\dagger \ldots a^\dagger] \ldots \text{tr}[(b^\dagger)^{i_k}a^\dagger \ldots a^\dagger] | 0 \rangle > , \tag{3.14}
\]

where \( n \) denotes the total number of partons in each Fock state, and the integers \( i_1, i_2, \ldots \) denote the number of fermions in the first trace, second trace, and so on. We will always
order the traces so that the number of fermions in each trace decreases to the right. The index \( P \) labels a particular arrangement of fermions.

We now consider the \( n+1 \) parton Fock states of \( Q_{\text{lead}}^{+} | \Psi\rangle \) that have precisely one fermion. The only possible contributions involve three types of wave functions: \( f_{P}^{(n,0)} \), \( f_{P}^{(n,2)} \) and \( f_{P}^{(n,1,1)} \) (we only include the permutation index \( P \) if there is more than one distinct arrangement). If these three wave functions contribute to the same one fermion Fock state, then the distribution of bosons in the Fock state corresponding to \( f_{P}^{(n,2)} \) determines the distribution of bosons for \( f_{P}^{(n,0)} \) and \( f_{P}^{(n,1,1)} \). We allow \( Q_{\text{lead}}^{+} \) to act only on the first trace in both \( f_{P}^{(n,0)} \) and \( f_{P}^{(n,2)} \), and only on the second one in \( f_{P}^{(n,1,1)} \). If there are more than two traces in these states, they must be identical in all the components, and so don't play a role in the calculation. Thus, it is sufficient to consider states with two traces only. Such a state has the form

\[
| \Phi \rangle = \int_{0}^{P_{\text{max}}} \frac{d^{m+n}q}{\sqrt{q_{1} \cdots q_{m+n}}} \delta(q_{1} + \cdots + q_{n+m} - P^{+}) \\
\times f_{P}^{(n,m,0)}(q_{1}, \ldots, q_{m}| q_{m+1}, \ldots, q_{m+n}) \\
\times \text{tr}[a^\dagger(q_{1}) \cdots a^\dagger(q_{m}) \text{tr}[a^\dagger(q_{m+1}) \cdots a^\dagger(q_{m+n})]| 0] \\
+ \int_{0}^{P_{\text{max}}} \frac{d^{m+n-2}q_{1}dq_{n+m-2}p_{1}p_{2}}{\sqrt{q_{1} q_{n+m-2} p_{1} p_{2}}} \delta(q_{1} + \cdotsconst+ q_{n+m-2} + p_{1} + p_{2} - P^{+}) \ \\
\times f_{P}^{(n+m,1,1)}(p_{1}, q_{1}, \ldots, q_{m}| q_{m+1}, \ldots, q_{m+n}) \\
\times \text{tr}[b^\dagger(p_{1}) a^\dagger(q_{1}) \cdots a^\dagger(q_{m}) \text{tr}[b^\dagger(p_{2}) a^\dagger(q_{m+1}) \cdots a^\dagger(q_{m+n})] + \ \\
\sum_{P} f_{P}^{(n,m,2)}(p_{1}, P | q_{1}, \ldots, q_{m-2}, q_{m+2} | q_{m+1}, \ldots, q_{m+n}) \\
\times \text{tr}\left( b^\dagger(p_{1}) P[ a^\dagger(q_{1}) \cdots a^\dagger(q_{m-2}), b^\dagger(p_{2})] \text{tr}[a^\dagger(q_{m+1}) \cdots a^\dagger(q_{m+n})]\right) | 0],
\]

(3.15)

where we have summed over the index \( P \) representing all possible permutation arrangements of bosons and fermions that contribute. We then find,

\[
F(p, q_{1}, \ldots, q_{m}| q_{m+1}, q_{m+2}, \ldots, q_{m+n}) + \\
+ \left. Q_{\text{lead}}^{+} \left( \frac{q_{m+2} - q_{m+1}}{(q_{m+2} + q_{m+1})^{3/2}} \right) F_{P}^{(n+m,1,1)}(p_{1}, q_{1}, \ldots, q_{m}| q_{m+1}, q_{m+2}, q_{m+3}, \ldots, q_{m+n}) \right) = 0, \quad (3.16)
\]

where \( F \) is the contribution from \( f_{P}^{(n+m,0)} \) and \( f_{P}^{(n+m,2)} \). Now we see that the limit \( q_{m+1} \to 0 \) gives: \( f_{P}^{(n+m,1,1)} \equiv 0 \). Thus if (3.13) represents a contribution to the massless eigenstate \( | \Psi \rangle \), then \( | \Phi \rangle \) takes the form

\[
| \Phi \rangle = \int_{0}^{P_{\text{max}}} \frac{d^{m+n-2}q_{1}dq_{n+m-2}}{\sqrt{q_{1} q_{n+m-2}}} \delta(q_{1} + \cdots + q_{n+m-2} - (P^{+} - K^{+})) \ \\
\times f_{P}^{(n,m,0)}(q_{1}, \ldots, q_{m}| q_{m+1}, \ldots, q_{m+n}) \text{tr}[a^\dagger(q_{1}) \cdots a^\dagger(q_{m})] + \ \\
\int_{0}^{P_{\text{max}}} \frac{dp_{1}dp_{2}}{\sqrt{p_{1} p_{2}}} \delta(p_{1} + p_{2} - K^{+})
\]

9
\[
\sum_P f_P^{(n+m,2)}(p_1, P[q_1, \ldots, q_{m-2}; p_2]|q_{m+1}, \ldots, q_{m+n})
\]
\[
\text{tr}(b^\dagger(p_1)P[a^\dagger(q_1) \ldots a^\dagger(q_{m-2}); b^\dagger(p_2)])\text{tr}(a^\dagger(q_{m+1}) \ldots a^\dagger(q_{m+n})) |0\rangle
\] (3.17)

and \(Q_{\text{lead}}\) acts only on the terms in the square brackets. All these terms have only one trace, which is a scenario we already encountered in the large \(N\) limit case. Using the results of that discussion, we find that the only massless solution of the form (3.17) is the trivial one. This is the starting point of the induction procedure for finite \(N\).

As explained earlier, we look for fermions \((k > 1)\), To finish the proof we need to show that for any \(k\) the only allowed wave function is the trivial one. From the large \(N\) result we know there are no such one trace states. We now consider the state with an arbitrary number of traces, \(a\) having the same form as the first trace. Then without loss of generality, we may assume

\[
\sum_P f_P^{(2k)}(s_1 \ldots s_n)|0\rangle > = \sum_P \int_0^{P^+} \frac{ds_1 \ldots ds_n}{\sqrt{s_1 \ldots s_n}} \delta(s_1 + \ldots + s_n - P^+) f_P^{(2k)}(s_1 \ldots s_n)\text{tr}(c^\dagger(s_1) \ldots c^\dagger(s_1)) \text{tr}(\ldots) \text{tr}(\ldots c^\dagger(s_n)) |0\rangle,
\] (3.18)

then the analog of (3.12) for such states reads:

\[
\sum_i f_P^{(2k)}(s_{j_1} \ldots s_{j_a} | s_{i_1} + s_{i_2} + \ldots s_{i_a + k_a} | s_{i_2} + s_{i_3} + \ldots s_{i_a + k_a} | s_{i_3} + s_{i_4} + \ldots s_{i_a + k_a} | \ldots s_{i_a}) \frac{s_{i_a + 1} - s_i}{(s_{i_a + 1} + s_i)^{3/2}} = 0.
\] (3.19)

Here, \(\sum_i\) means that for each trace we should include one additional term with \(i = j_a + k_a\), \(i + 1 = j_a\) if \(c\) corresponding to both \(j_a + k_a\) and \(j_a\) is \(a\). If the number of traces is \(a\), we introduce

\[j_a = \sum_{b=1}^{a-1} k_b.\]

If any of the blocks \(\text{tr}(\ldots)\) in the state for which (3.19) is written contains two or more fermions, then, as in the large \(N\) case, all the corresponding wave functions \(f_P^{(2k)}\) vanish. So we only need to consider the states of the form:

\[
\sum_P f_P^{(n,k_1+\ldots,\ldots)}(p_1, q_1, \ldots, q_{k_1}|p_2, q_{k_1+1}, \ldots, q_{k_1+k_2}|\ldots) \times
\]
\[
\text{tr}(b^\dagger(p_1)q^\dagger(q_1) \ldots a^\dagger(q_{k_1})) \text{tr}(b^\dagger(p_2)q^\dagger(q_{k_1+1}) \ldots a^\dagger(q_{k_1+k_2})) \ldots |0\rangle.
\] (3.20)

Let \(\tilde{Q}\) denote that part of the supercharge \(Q^-\) which replaces a fermion with two bosons. Let us consider the result of such a change in the first trace. Suppose there are \(a\) traces having the same form as the first trace. Then without loss of generality, we may assume they are the first \(a\) traces. Then using the symmetries of the wave functions we find:

\[
\tilde{Q}|\Psi_{(n,k_1+\ldots,\ldots)}\rangle = -\frac{1}{2\sqrt{2\pi}} \sum_P \int_0^{P^+} dkdp dq f_P^{(n,k_1+\ldots,\ldots)}(p_1, q_1, \ldots, q_{k_1}|p_2, q_{k_1+1}, \ldots, q_{k_1+k_2}|\ldots) \times
\]
\[
\sum_{b=1}^{a} \frac{p_b - 2k}{p_b} \frac{1}{\sqrt{k(p_b - k)}} (-1)^{b+1} \times
\]
\[
\sum_P f_P^{(n,k_1+\ldots,\ldots)}(p_1, q_1, \ldots, q_{k_1}|p_2, q_{k_1+1}, \ldots, q_{k_1+k_2}|\ldots) \times
\]
\[
\text{tr}(b^\dagger(p_1)q^\dagger(q_1) \ldots a^\dagger(q_{k_1})) \text{tr}(b^\dagger(p_2)q^\dagger(q_{k_1+1}) \ldots a^\dagger(q_{k_1+k_2})) \ldots |0\rangle.
\]
\[ \text{tr} \left( b^\dagger(p_1)a^\dagger(q_1) \ldots a^\dagger(q_{k_1}) \right) \ldots \text{tr} \left( a^\dagger(k)a^\dagger(p_h - k)a^\dagger(q_{(h-1)k_1+1}) \ldots a^\dagger(q_{b_1}) \right) \ldots |0\rangle \]
\[ = -\frac{1}{2\sqrt{2}\pi} \sum_P \int_0^{p^+} dk dp dq \frac{p_1 - 2k}{p_1} \frac{1}{\sqrt{k(p_1 - k)}} \text{tr} \left( a^\dagger(k)a^\dagger(p_1 - k)a^\dagger(q_1) \ldots a^\dagger(q_{k_1}) \right) \times \]
\[ \text{tr} \left( b^\dagger(p_2)a^\dagger(q_{k_1+1}) \ldots a^\dagger(q_{k_1+k_2}) \right) \ldots |0\rangle > \sum_{b=1}^{a} (-1)^{b+1} (-1)^{b+1} \times \]
\[ f_P^{(n,k_1+1,\ldots)}(p_1, q_1, \ldots, q_{k_1} | p_2, q_{k_1+1}, \ldots, q_{k_1+k_2} | \ldots). \]

If the above expression vanishes then the only solution is the trivial one in which all wave functions vanish. This finishes the proof of the induction procedure for the finite \( N \) case.

The extension of the proof to massive bound states is straightforward. Firstly, assume \(|\Psi\rangle\) is a normalizable eigenstate of \( 2P^+P^- \) with mass squared \( M^2 \neq 0 \). Then, since \( P^- = \frac{1}{\sqrt{2}}(Q^-)^2 \), the state

\[ |\tilde{\Psi}\rangle \equiv |\Psi\rangle + \alpha Q^- |\Psi\rangle \]

for \( \alpha^2 = \sqrt{2}P^+/M^2 \) is a normalizable eigenstate of the supercharge \( Q^- \), with eigenvalue \( 1/\alpha \). We therefore study the eigen-problem \( Q^- |\tilde{\Psi}\rangle = \frac{1}{\alpha} |\tilde{\Psi}\rangle \). The resulting constraints on the wave functions may be obtained by modifying our original expressions by including a wave function multiplied by a finite constant. However, in our analysis, we always need to let some of the momenta vanish, and therefore this additional contribution vanishes. The analysis (and therefore the conclusions) remains unchanged.

We therefore conclude that any normalizable SU(\( N \)) bound state (massless or massive) that exists in the model must be a superposition of an infinite number of Fock states.

### IV. BOUND STATES IN DLCQ.

In the previous section we proved that the continuum formulation of the theory does not have any normalizable bound states with a finite number of partons. Our proof used the behavior of wave functions at small momenta arising from the normalizability assumption. Neither of these properties can be used in DLCQ, however. Here we consider some simple examples of massless DLCQ solutions with \( n \) bosons to help shed some light on the relation between DLCQ solutions and the solutions of the continuum theory. For simplicity, we work in the large \( N \) limit case.

We write the momentum of a state in DLCQ in terms of the momentum fraction \( q_i \) where \( q_i = \frac{r_i}{r} P^+ \), and the \( r_i \) are positive integers. The wave function of such a state is \( f^{(n,0)}(r_1, \ldots, r_n) \). There are two conditions that must be satisfied to show that it is massless. One is that the coefficient of the term with one additional fermion that is produced by the action of \( Q^- \) is zero. This condition gives the relation,

\[ \frac{2r_n + t}{r_n + t} f^{(n,0)}(r_1, \ldots, r_{n-1}, r_n + t) - \frac{2r_{n-1} + t}{r_{n-1} + t} f^{(n,0)}(r_1, \ldots, r_{n-1} + t, r_n) = 0. \]  

(5.1)

where \( t \) correspond to the momentum fraction of the one fermion. The second is that the coefficient of the state with two fewer bosons and one additional fermion which is also produced by the action of \( Q^- \) is zero. This condition gives the relation,
\[ \sum_{k,t} \frac{t - 2k}{k(t - k)} f^{(n,0)}(r_1, \ldots, r_{n-2}, k, t - k) \delta(r_{n-1} + r_n, t) = 0. \] \hspace{1cm} (5.2)

For the case where all \( r_i = 1 \), and the total harmonic resolution is \( n \), it is trivial that eqn (5.1) is satisfied since there is not enough resolution to increase the number of particles in the state. It is also easy to see from eqn (5.2) since the coefficient of the one term in the sum is zero. Thus the wave function \( f^{(n,0)}(1,1,\ldots,1) \) is a massless state for every resolution.

To discuss additional solutions it is useful to start by considering eqn (5.1). The case \( t = 1 \), gives the equation

\[ f^{(n,0)}(r_1, \ldots, r_{n-2}, r_{n-1}, r_n + 1) = \frac{2r_{n-1} + 1}{2r_n + 1} \frac{r_n + 1}{r_{n-1} + 1} f^{(n,0)}(r_1, \ldots, r_{n-2}, r_{n-1} + 1, r_n). \] \hspace{1cm} (5.3)

This equation is trivial to satisfy if \( r_i = 1 \) for all \( i \). The contributions in eqn (5.2) come from the two terms in the sum, \( k = 1, t = 3 \) and \( k = 2, t = 3 \). Each term has the same coefficient but of opposite sign and cancel. Therefore the state \( f^{(n,0)}(1,1,\ldots,1,2) \) is a massless state for all resolutions.

The next case \( t = 2 \) in eqn (5.1) gives,

\[ f^{(n,0)}(r_1, \ldots, r_{n-2}, r_{n-1}, r_n + 2) = \frac{2r_{n-1} + 2}{2r_n + 2} \frac{r_n + 2}{r_{n-1} + 2} f^{(n,0)}(r_1, \ldots, r_{n-2}, r_{n-1} + 2, r_n). \] \hspace{1cm} (5.4)

Using (5.3) twice we find:

\[ f^{(n,0)}(r_1, \ldots, r_{n-2}, r_{n-1}, r_n + 2) = \frac{2r_{n-1} + 1}{2r_n + 3} \frac{r_n + 2}{r_{n-1} + 1} f^{(n,0)}(r_1, \ldots, r_{n-2}, r_{n-1} + 1, r_n + 1) = \]
\[ = \frac{2r_{n-1} + 1}{2r_n + 3} \frac{r_n + 2}{2r_{n-1} + 3} \frac{r_n + 1}{r_{n-1} + 2} f^{(n,0)}(r_1, \ldots, r_{n-2}, r_{n-1} + 2, r_n). \] \hspace{1cm} (5.5)

Comparing with (5.4) we have:

\[ f^{(n,0)}(r_1, \ldots, r_{n-2}, r_{n-1} + 2, r_n) \left( \frac{(r_n + 1)^2}{(2r_n + 3)(2r_n + 1)} - \frac{(r_{n-1} + 1)^2}{(2r_{n-1} + 3)(2r_{n-1} + 1)} \right) = 0. \] \hspace{1cm} (5.6)

Using relation (5.1) several times we can always express an arbitrary wave function in the following form:

\[ f^{(n,0)}(r_1, \ldots, r_n) = C(r_1, \ldots, r_n) f^{(n,0)}(1, \ldots, 1, L + 1, 1) \] \hspace{1cm} (5.7)

where \( L = r_1 + \ldots + r_n - n \) and \( C(r_1, \ldots, r_n) \) is some nonzero coefficient. The two massless states we found above correspond to \( L = 0 \) and \( L = 1 \). Choosing \( r_1 = \ldots = r_{n-2} = r_n = 1 \) in (5.6) we find,

\[ f^{(n,0)}(1, \ldots, 1, (L - 1) + 2, 1) = 0 \quad for \quad L > 2 \] \hspace{1cm} (5.8)

due to monotonic behavior of the function in the parenthesis. Then using (5.7) we conclude that all the wave functions with \( L > 2 \) vanish. So the only case we need consider is \( L = 2 \). In this case (5.1) has only two nontrivial cases: \( t = 1 \) and \( t = 2 \) which are given by (5.3) and
In the second of these equations we can only have \( r_1 = \ldots = r_n = 1 \) so it is trivially satisfied. Equation (5.3) however gives a nontrivial relation for the wave function:

\[
\begin{align*}
 f^{(n,0)}(1, \ldots, 1, 2, 2) &= f^{(n,0)}(1, \ldots, 2, 1, 2) = \ldots = f^{(n,0)}(2, \ldots, 1, 1, 2) = \frac{10}{9} f^{(n,0)}(1, \ldots, 1, 3).
\end{align*}
\]

(5.9)

finally we must show that eqn(5.2) is satisfied which is straightforward.

These are only a few examples of massless states, and there are in fact many more in DLCQ [8]. In the continuum limit we have proven that there are no massless normalizable states with a finite number of particles. However, there is the possibility that at each finite value of the harmonic resolution, one obtains an exactly massless bound state, but as the harmonic resolution is sent to infinity, the number of Fock states required to keep the bound state massless must also be infinite.

V. CONCLUSIONS

In this work we considered the dimensional reduction of \( \mathcal{N} = 1 \) SYM\( 2+1 \) to \( 1 + 1 \) dimensions, and at finite \( N \). Our main objective was to analyze the structure of bound states both in the continuum and in the DLCQ formulation. We discovered many massless states in the DLCQ formulation, but showed that any massless (or massive) normalizable bound states in the continuum theory must be a superposition of an infinite number of Fock states. Our work therefore shows that any exact analytical treatment of the continuum bound state problem is probably a too ambitious objective for the near future. This scenario is to be contrasted with the simple bound states discovered in a number of \( 1 + 1 \) dimensional theories with complex fermions, such as the Schwinger model, the t’Hooft model, and a dimensionally reduced theory with complex adjoint fermions [11,13]. While these theories offered hope that bound states viewed in the light-cone formalism might be much simpler than in the equal-time quantization approach, we see here that this is not the case. Numerical DLCQ studies of the model are, of course, insensitive to the complexity of the bound state problem, and extensive numerical results on the model studied here will appear elsewhere [8].
REFERENCES

[1] S.J. Brodsky, H.C. Pauli, and S.S. Pinsky, “Quantum Chromodynamics and Other Field Theories on the Light Cone” to appear in Phys.Rept. hep-ph/9705477

[2] Y. Matsumura, N. Sakai, and T. Sakai, Phys.Rev D52:2446-2461,1995 hep-th/9504150

[3] D. J. Gross, I. R. Klebanov, A. V. Matytsin and A. V. Smilga, Nucl.Phys B461 (1996) 109. hep-th/9511104

[4] A. Armoni and J. Sonnenschein “Screening and Confinement in Large $N_f$ QCD and in $N=1$ SYM$_2$” TAUP-2412-97 hep-th/9703114

[5] A. Hashimoto and I. R. Klebanov. Nucl.Phys B434:264-282,1995 hep-th/9409064

[6] T. Banks, W. Fischler, S. Shenker, L. Susskind, Phys. Rev. D55 (1997) 5112, hep-th/9610043.

[7] H.-C. Pauli and S.J.Brodsky, Phys.Lett. D32 (1985) 1993, 2001.

[8] F.Antonuccio, S.S.Pinsky and O.Luin, To appear.

[9] L.Susskind, Another Conjecture About Matrix Theory, hep-th/9704080.

[10] F. Antonuccio, S.S. Pinsky, Phys.Lett B397:42-50,1997, hep-th/9612021.

[11] W. Taylor, “Lectures on D-Branes, Gauge Theory and M(arices)” PUPT-1762, Jun 1997. 80pp. Talk given at 2nd Trieste Conference on Duality in String Theory, Trieste, Italy, 16-20 Jun 1997. hep-th/9801182

[12] D. J. Gross, A. Hashimoto and I. R. Klebanov, The Spectrum of a Large N Gauge Theory Near Transition From Confinement to Screening” NSF-ITP-97-133, Oct 1997. 16pp. hep-th/9710240

[13] S. Pinsky, The Analog of the t’Hooft Pion with Adjoint Fermions” Invited talk at New Nonperturbative Methods and Quantization of the Light Cone, Les Houches, France, 24 Feb - 7 Mar 1997, hep-th/9705242

[14] S. Pinsky, Phys.Rev D56:5040-5049,1997 hep-th/9612073

[15] G. McCartor, D. G. Robertson and S. Pinsky Phys.Rev D56:1035-1049,1997 hep-th/9612083

[16] F.Antonuccio, S.J.Brodsky, S.Dalley, Phys.Lett. B412 (1997) 104-110.