Research Article

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Polynomial stability of the wave equation with distributed delay term on the dynamical control

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Abstract: Using the frequency domain approach, we prove the rational stability for a wave equation with distributed delay on the dynamical control, after establishing the strong stability and the lack of uniform stability.

Keywords: distributed delay term, dynamical control, strong stability, lack of uniform stability, polynomial decay.

MSC: 35B35, 35B40, 35L05, 93D15

1 Introduction

In the literature, there are full practical processes that might be modelled by distributed delay systems, which present a wide range of applications in various fields such as micro-organism growth [25], hematopoiesis [1, 2], logistics [4] and traffic flow [21]. In the recent past (last four decades), many researchers have fruitfully investigated on that subject, and successfully applied them in more widespread other areas. They have developed mathematical tools in order to establish polynomial or exponential decays of these systems. We refer readers to [20] for a list of early works, and to [8–10, 12–15, 22, 26, 27] and the references therein, for some other relevant results.

In this paper, we consider the following wave equation with a distributed delay term on the dynamical control:

\[
\begin{align*}
  u_{tt}(x, t) - u_{xx}(x, t) &= 0 \quad \text{in } ]0, 1[ \times (0, +\infty) \\
  u(0, t) &= 0 \\
  u_x(1, t) + \eta(t) &= 0 \quad \forall \, t \in (0, +\infty) \\
  \eta_t(t) - u_t(1, t) + \beta_1 \eta(t) + \int_{\tau_1}^{\tau_2} \beta_2(s) \eta(t-s) ds &= 0 \quad \forall \, t \in (0, +\infty) \\
  u(\cdot, 0) &= u_0, \quad u_t(\cdot, 0) = u_1 \quad \text{in } ]0, 1[ \\
  \eta(0) &= \eta_0 \in \mathbb{R} \\
  \eta(-t) &= f_0(-t) \quad \forall \, t \in (0, \tau_2),
\end{align*}
\] (1)

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where \( \eta \) denotes the dynamical control, while \( \tau_1 \) and \( \tau_2 \) are two real numbers verifying
\[
0 \leq \tau_1 < \tau_2;
\]
moresover \( \beta_1 \) is a positive constant, \( \beta_2 : [\tau_1, \tau_2] \to \mathbb{R} \) is a positive \( L^\infty \) function and the initial data \((u_0, u_1, f_0)\) belong to a suitable space. The damping of the system is made via the indirect damping mechanism. Throughout all paper, we assume that \( \beta_2 : [\tau_1, \tau_2] \to \mathbb{R} \) is a positive \( L^\infty \) function verifying:
\[
\int_{\tau_1}^{\tau_2} \beta_2(s)\,ds < \beta_1. \tag{2}
\]
It is well known that if \( \beta_2 = 0 \) (that is no delay occurs in the system), the energy of problem (1) is polynomially decaying to zero with the rate \( t^{-1} \); see for instance Wehbe [28] for one dimensional case and Toufayli [24] for higher dimension.

But in the presence of a delay, namely for such a below system
\[
\begin{align*}
\frac{\partial u}{\partial t}(x, t) - \Delta u(x, t) &= 0 \quad \text{in } \Omega \times (0, +\infty) \\
u(0, t) &= 0 \\
u(t, 0) &= 0, \quad \frac{\partial u(t, 0)}{\partial \nu} = u_1 \quad \text{in } \Omega, \\
\left. \frac{\partial u}{\partial \nu}(x, t) + \mu_1 u(x, t) + \mu_2 u(x, t - \tau) \right|_{\partial \Omega} &= 0 \quad \text{on } \Gamma_N \times (0, +\infty) \\
u(x, 0) &= u_0, \quad u_t(x, 0) = u_1 \quad \text{in } \Omega \\
\left. u(x, t - \tau) = f_0(x, t - \tau) \right|_{\Gamma_N} &= 0 \quad \text{on } \Gamma_N \times (0, \tau),
\end{align*}
\tag{3}
\]
with constants \( \beta_1 \) and \( \beta_2 \) verifying the following assumption that there exists a positive constant \( \zeta \) verifying
\[
\tau \beta_1 < \zeta < \tau(2\beta_1 - \beta_2), \tag{4}
\]
it has been proved in Gilbert and . al [7] that the energy of problem (3) decays polynomially with the same rate as in [24, 28].

In the case of the wave equations, Nicaise and Pignotti [16] investigated exponential stability results with delay concentrated at \( \tau \) for the system
\[
\begin{align*}
\frac{\partial u}{\partial t}(x, t) - \Delta u(x, t) &= 0 \quad \text{in } \Omega \times (0, +\infty) \\
u(x, t) &= 0 \quad \text{on } \Gamma_D \times (0, +\infty) \\
\frac{\partial u}{\partial \nu}(x, t) + \mu_1 u_t(x, t) + \mu_2 u_t(x, t - \tau) &= 0 \quad \text{on } \Gamma_N \times (0, +\infty) \\
u(x, 0) &= u_0, \quad u_t(x, 0) = u_1 \quad \text{in } \Omega \\
\left. u(x, t - \tau) = f_0(x, t - \tau) \right|_{\Gamma_N} &= 0 \quad \text{on } \Gamma_N \times (0, \tau),
\end{align*}
\tag{5}
\]
under the condition \( \mu_2 < \mu_1 \), by combining inequalities due to Carleman estimates and compactness-uniqueness arguments. Later, they also obtain in [17] the exponential stability with distributed delay of the
Polynomial stability of the wave equation with distributed delay

system

\[
\begin{align*}
\begin{cases}
    u_{tt}(x, t) - \Delta u(x, t) = 0 \text{ in } \Omega \times (0, +\infty) \\
    u(x, t) = 0 \text{ on } \Gamma_D \times (0, +\infty) \\
    \frac{\partial u}{\partial v}(x, t) + \mu_1 u_t(x, t) + \int_{\tau_1}^{\tau_2} \mu_2(s) u_t(x, t - s) \, ds = 0 \text{ on } \Gamma_N \times (0, +\infty) \\
    u(x, 0) = u_0, \quad u_t(x, 0) = u_1 \text{ in } \Omega \\
    u(x, -t) = f_0(x, -t) \text{ on } \Gamma_N \times (0, \tau), \\
    u(x, t) = 0 \text{ on } \partial\Omega \times (0, +\infty) \\
    u_{tt}(x, t) = 0 \text{ in } \Omega \times (0, +\infty).
\end{cases}
\end{align*}
\]  

(6)

under the assumption

\[
\int_{\tau_1}^{\tau_2} \mu_2(s) \, ds < \mu_1.
\]

(7)

In this paper, staying on the one dimensional space, we propose a dynamical boundary moment control \( \eta \) with a distributed delay term, and we look for the possible ways to stabilize the system (1). To our knowledge polynomial stability with distributed delay term has not yet been done, even if the system (3), that is time delay concentrated at \( \tau \), decays polynomially.

The paper is organized as follows: section 2 is devoted to the well posedness of problem (1), while the section 3 deals with the strong stability of problem (1); furthermore, section 4 establishes the lack of uniform stability, and finally in section 5 stands on the polynomial stability of problem (1).

2 Well posedness

In this section, we will establish the well posedness of the problem (1), using the semigroup theory. Let us set

\[
z(\rho, t, s) = \eta(t - s \rho), \quad \rho \in (0, 1), \ s \in (\tau_1, \tau_2), \ t > 0.
\]

(8)

The problem (1) is now equivalent to

\[
\begin{align*}
\begin{cases}
    u_{tt}(x, t) - u_{xx}(x, t) = 0 \text{ in } [0, 1] \times (0, +\infty) \\
    sz_t(\rho, t) + z_\rho(\rho, t) = 0 \text{ in } (0, 1) \times (0, +\infty) \times (\tau_1, \tau_2) \\
    u(0, t) = 0 \quad \forall \ t \in (0, +\infty) \\
    u(1, t) + \eta(t) = 0 \quad \forall \ t \in (0, +\infty) \\
    \eta_t(t) - u_t(1, t) + \beta_1 \eta(t) + \int_{\tau_1}^{\tau_2} \beta_2(s) z(1, t, s) \, ds = 0 \quad \forall \ t \in (0, +\infty) \\
    z(0, t, s) = \eta(t) \quad \forall \ t \in (0, +\infty) \\
    u(\cdot, 0) = u_0, \quad u_t(\cdot, 0) = u_1 \text{ in } [0, 1] \text{ and } \eta(0) = \eta_0 \\
    z(\rho, 0, s) = f_0(-\rho s) \quad \forall \ s \in (0, \tau_2).
\end{cases}
\end{align*}
\]

(9)
Let us set
\[ \mathcal{U} = (u, u_t, \eta, z)^T. \]

Then we have
\[ \mathcal{U}_t = (u_t, u_{tt}, \eta_t, z_t)^T = (u_t, u_{xx}, u_t(1, t) - \beta_1 \eta(t) - \int_0^{\tau_2} \beta_2(s)z(1, t, s)ds, -s^{-1}z_\rho)^T. \]

Therefore problem (9) can be rewritten in an abstract framework:

\[
\begin{cases}
\mathcal{U}_t = \mathcal{A} \mathcal{U} \\
\mathcal{U}(0) = (u_0, u_1, \eta_0, f_0(-s))
\end{cases}
\] (10)

where the operator \( \mathcal{A} \) is defined by
\[
\mathcal{A} (u, v, \eta, z)^T = \left( v, u_{xx}, v(1) - \beta_1 \eta - \int_0^{\tau_2} \beta_2(s)z(1, s)ds, -s^{-1}z_\rho \right)^T,
\]

with domain
\[
\mathcal{D}(\mathcal{A}) = \{(u, v, \eta, z)^T \in (H^2(0, 1) \cap V) \times \mathbb{R} \times L^2((\tau_1, \tau_2); H^1(0, 1)) : z(0) = \eta, u_x(1) + \eta = 0\}.
\]

where
\[
V = \{ u \in H^1(0, 1), u(0) = 0 \}.
\]

Let us now introduce the Hilbert space
\[
\mathcal{H} = V \times L^2(0, 1) \times \mathbb{R} \times L^2((\tau_1, \tau_2); L^2(0, 1))
\]

endowed with the norm
\[
\left\| (u, v, \eta, z)^T \right\|_{\mathcal{H}}^2 = \|u_x\|_{L^2(0, 1)}^2 + \|v\|_{L^2(0, 1)}^2 + |\eta|^2 + \int_0^{\tau_2} \left( \int_0^1 |z(\rho, s)|^2 d\rho \right) ds.
\]

So, the natural associated inner product is
\[
\left\langle \left( \begin{array}{c} u \\ v \\ \eta \\ z \end{array} \right), \left( \begin{array}{c} u^* \\ v^* \\ \eta^* \\ z^* \end{array} \right) \right\rangle_{\mathcal{H}} = \int_0^1 \left( u_x \overline{u}_x + v \overline{v} \right) dx + \eta \overline{\eta} + \int_0^{\tau_2} \left( \int_0^1 \frac{1}{\rho} z(\rho, s) z^*(\rho, s) d\rho \right) ds.
\]

**Proposition 2.1.**

The operator \( \mathcal{A} \) defined above is \( m \)-dissipative.

**Proof.** We see that
\[
\left\langle \mathcal{A} \left( \begin{array}{c} u \\ v \\ \eta \\ z \end{array} \right), \left( \begin{array}{c} u \\ v \\ \eta \\ z \end{array} \right) \right\rangle_{\mathcal{H}} = \left\langle \left( \begin{array}{c} v \\ u_{xx} \\ \eta \\ -s^{-1}z_\rho \end{array} \right), \left( \begin{array}{c} v \\ u_x(1 - \beta_1 \eta - \int_0^{\tau_2} \beta_2(s)z(1, s)ds) \\ \eta \\ z \end{array} \right) \right\rangle_{\mathcal{H}}.
\]
Using Green formula, Cauchy Schwarz's inequality and the definition of $D(\mathcal{A})$ we obtain

$$
\Re\left\langle \mathcal{A}\begin{pmatrix} u \\ v \\ \eta \\ z \end{pmatrix}, \begin{pmatrix} u \\ v \\ \eta \\ z \end{pmatrix} \right\rangle = \Re \left( u(1) - \beta_1 \eta - \int_{t_1}^{t_2} \beta_2(s)z(1, s) ds \right) \eta + \frac{1}{2} \int_{t_1}^{t_2} \beta_2(s) \left( |z(1, s)|^2 - |z(0, s)|^2 \right) ds + \frac{1}{2} \left| \beta_2(s) ds \right|
$$

Now, the relation (2) allows to conclude that $\mathcal{A}$ is dissipative.

Let us prove that the operator $\lambda I - \mathcal{A}$ is surjective for at least one $\lambda > 0$.

For $(f, g, h, k)^T \in \mathcal{H}$, we look for $(u, v, \eta, z)^T \in D(\mathcal{A})$ solution of

$$
\begin{cases}
\lambda u - v = f & \text{in } ]0, 1[ \\
\lambda v - u_{xx} = g & \text{in } ]0, 1[ \\
\lambda \eta - \left( v(1) - \beta_1 \eta - \int_{t_1}^{t_2} \beta_2(s)z(1, s) ds \right) = h & \\
\lambda z + s^{-1}z_\rho = k & \text{in } ]0, 1[.
\end{cases}
$$

(11)
Suppose that we have found \( u \) with the appropriate regularity. It means that we have also found \( \eta \). Then \( v = \lambda u - f \), and we can determine \( z \) by solving the system

\[
\begin{aligned}
\begin{cases}
\quad s^{-1}z_{\rho} + \lambda z = k & \text{in } ]0, 1[ \\
\quad z(0) = \eta.
\end{cases}
\end{aligned}
\]  

(12)

We obtain

\[ z(\rho, s) = \eta e^{-\lambda \rho} + se^{-\lambda \rho} \int_0^\rho k(\sigma, s)e^{\lambda \sigma} \, d\sigma. \]

In particular

\[ z(1, s) = \eta e^{-\lambda s} + se^{-\lambda s} \int_0^1 k(\sigma, s)e^{\lambda \sigma} \, d\sigma. \]

The function \( u \) verifies now

\[
\begin{aligned}
\begin{cases}
-u_{xx} + \lambda^2 u = g + \lambda f & \text{in } ]0, 1[ \\
\quad u(0) = 0 \\
\quad u_s(1) = -z(0)
\end{cases}
\end{aligned}
\]  

(13)

By using Lax-Milgram’s Lemma, the problem (13) admits a unique weak solution. Indeed, multiplying the first equation by \( v \in V \) and by integrating formally by parts, we get

\[
a(u, v) = F(v), \forall v \in V,
\]  

(14)

where the bilinear and continuous form \( a \) is given by

\[
a(u, v) = \int_0^1 \left( u_s v_s + \lambda^2 uv \right) \, dx \quad \forall u, v \in V,
\]

while the linear form \( F \) is

\[
F(v) = \int_0^1 (g + \lambda f)v \, dx + \eta v(1), \quad \forall v \in V.
\]

Since \( a \) is clearly strongly coercive on \( V \) and \( F \) is continuous on \( V \), by Lax-Milgram’s Lemma, problem (13) admits a unique solution \( u \in V \). By taking test functions \( v \in D(0; 1) \), we recover the first identity of (13). This guarantees that \( u \) belongs to \( H^2(0, 1) \). Using now Green’s formula, we see that \( u \) satisfies the third identity of (13).

Finally, we define \( \eta \) and \( v \) by setting

\[
v = \lambda u - f \quad \text{and} \quad \eta = \frac{v(1) - \int_1^{\tau_1} \beta_2(s)z(1, s) \, ds + h}{\beta_1 + \lambda}.
\]

This shows that the operator \( \mathcal{A} \) is m-dissipative on \( \mathcal{H} \) and it generates a \( C_0 \) semigroup of contractions in \( \mathcal{H} \), under Lumer-Phillips theorem. So, we have found \((u, v, \eta, z)^T \in D(\mathcal{A}) \) which verifies (13).

We can now state on the following existence results.

**Theorem 2.2.**

*If \( U_0 = (u_0, u_1, \eta_0, f_0)^T \) belongs to \( \mathcal{H} \), then problem (1) has one and only one weak solution \( U = (u, u_t, \eta, z)^T \) verifying:

\[
\begin{aligned}
\begin{cases}
\quad u \in C([0, \infty), V) \cap C^1([0, \infty), L^2(0, 1)) \\
\quad \eta \in C([0, \infty))
\end{cases}
\end{aligned}
\]  

(15)\*
Furthermore, if \( U_0 = (u_0, u_1, \eta_0, f_0) \) belongs to \( D(A) \), then problem (1) has one and only one strong solution \( U = (u, u_t, \eta, z)^T \) which satisfies

\[
\begin{cases}
    u \in C([0, \infty), H^2(0, 1) \cap V) \cap C^1([0, \infty), V) \cap C^2([0, \infty), L^2(0, 1)) \\
    \eta \in C^1([0, \infty))
\end{cases}
\]  

(16)

Proof. This result is easy to check following the Hille-Yosida theory.

3 Strong stability

In this section, we establish strong stability result. The main result of this subsection is the following.

Theorem 3.1.
The \( C_0 \)-semigroup \( \left( e^{tA} \right)_{t \geq 0} \) is strongly stable on the energy space \( \mathcal{H} \), that is for any \( U_0 \in \mathcal{H} \),

\[
\lim_{t \to 0} \| e^{tA} U_0 \|_{\mathcal{H}} = 0.
\]

Proof. We use the spectral decomposition theory of Sz-Nagy-Foias and Foguel [3, 6, 23]. Following this theory, since the resolvent of \( A \) is compact, it suffices to establish that \( A \) has no eigenvalue on the imaginary axis. For our purpose, it is easy to prove that the resolvent of the operator \( A \) defined in (10) is compact. We are ready now to achieve the proof of theorem 3.1 with the following lemma.

Lemma 3.2.
There is no eigenvalue of \( A \) on the imaginary axis, that is

\( i\mathbb{R} \subset \rho(A) \).

Proof. By contradiction argument, we assume that there exists at least one \( i\lambda \in \sigma(A), \lambda \in \mathbb{R}, \lambda \neq 0 \) on the imaginary axis. Let \( U = (u, v, \eta, z)^T \in D(A) \) be the corresponding normalized eigenvector, that is verifying \( \| U \| = 1 \) and

\[
A(u, v, \eta, z)^T = i\lambda(u, v, \eta, z)^T,
\]

(17)

which is equivalent to

\[
\begin{cases}
    v - i\lambda u = 0 \quad \text{in } (0, 1) \\
    u_{xx} - i\lambda v = 0 \quad \text{in } (0, 1) \\
    v(1) - \beta_1 \eta - \int_{\tau_2}^{\tau_1} \beta_2(s) z(1, s) ds - i\lambda \eta = 0 \\
    s^{-1} z_\rho - i\lambda z = 0 \quad \text{in } (0, 1).
\end{cases}
\]

(18)

Recalling the dissipativity of \( A \), it follows that

\[
0 = \Re \left( \left\langle A(u, v, \eta, z)^T, (u, v, \eta, z)^T \right\rangle_{\mathcal{H}} \right) \leq |\eta|^2 \left( -\beta_1 + \int_{\tau_2}^{\tau_1} \beta_2(s) ds \right) \leq 0;
\]

(19)

that is \( \eta = 0 \).

Owing to the definition of \( z \) in §2, we deduce that \( \eta = z = 0 \).

Now (18) becomes

\[
\begin{cases}
    v - i\lambda u = 0 \quad \text{in } (0, 1) \\
    u_{xx} - i\lambda v = 0 \quad \text{in } (0, 1) \\
    v(1) = 0.
\end{cases}
\]

(20)
From the first equation of (20), we deduce that
\[ u(1) = 0 \]
Setting \( v = i\lambda u \), it remains to find \( u \in V \) which verifies
\[
\begin{cases}
    u_{xx} + \lambda^2 u = 0 & \text{in } (0, 1) \\
    u_x(1) = 0 \\
    u(1) = 0.
\end{cases}
\] (21)

Therefore, from the general theory of ordinary differential equations, we deduce that
\[ u = 0, \text{ on } (0, 1). \] (22)

Now it follows that \( (u, v, \eta, z)^T = (0, 0, 0, 0)^T \) which contradicts the fact that \( \|U\| = 1 \). We conclude that \( \mathcal{A} \) has no eigenvalue on the imaginary axis. \( \square \)

As the conditions of the spectral decomposition theory of Sz-Nagy-Foias and Foguel are full satisfied, the proof of theorem 3.1 is thus completed. \( \square \)

4 Lack of exponential stability

In this section, we will show that the system (1) is lack of exponential decay rate. Our future computations are based on frequency domain approach for exponential stability (see Huang [11] and Prüss [19]), more precisely on the below result.

**Lemma 4.1.**
A \( \mathcal{C}_0 \)-semigroup \( \left( e^{tA} \right) \) of contractions on a Hilbert space \( \mathcal{H} \) is exponentially stable that is
\[
\left\| e^{tA} U_0 \right\|_{\mathcal{H}} \leq C e^{-\omega t} \|U_0\|_{\mathcal{H}} \quad \forall \ U_0 \in \mathcal{H}, \ \forall \ t \geq 0,
\] (23)
for some positive constants \( C \) and \( \omega \), if and only if
\[
\rho(A) \not\supset \{i\beta, \ \beta \in \mathbb{R}\} \equiv i\mathbb{R}
\] (24)
and
\[
\sup_{\beta \in \mathbb{R}} \left\| (i\beta - A)^{-1} \right\|_{L(\mathcal{H})} < \infty,
\] (25)
where \( \rho(A) \) denotes the resolvent set of the operator \( A \).

The main result of the current section is the following.

**Theorem 4.2.**
The system (1) is not exponentially stable in the energy space \( \mathcal{H} \).

**Proof.** Following the lemma 4.1 above, we prove that the condition (25) is not satisfied in the sense that there exists some sequences \( \lambda_n \), \( U_n = (u, v, \eta, z)^T \) and \( F_n = (f_{1n}, f_{2n}, f_{3n}, f_{4n})^T \) such that
\[
(i\lambda_n - \mathcal{A})U_n = F_n;
\] (26)
\[
\|F_n\|_{\mathcal{H}} = O(1);
\] (27)
and
\[ \lim_{n \to +\infty} ||U_n||_{\mathcal{C}} = +\infty. \]  
(28)

The relation (26) is equivalent to
\[
\begin{cases}
  i\lambda_n u - v = f_{1n} \\
  i\lambda_n v - u_{xx} = f_{2n} \\
  i\lambda_n \eta - \left( \nu(1) - \beta_1 \eta - \int_0^{\tau_2} \beta_2(s)z(1, s)ds \right) = f_{3n} \\
  i\lambda_n z + s^{-1}z_\rho = f_{4n}.
\end{cases}
\]  
(29)

We look for a particular solution, defined for \( f_{1n} = f_{3n} = f_{4n} = 0 \), and \( f_{2n} \) will be chosen later. Then (29) becomes
\[
\begin{cases}
  i\lambda_n u - v = 0 \\
  i\lambda_n v - u_{xx} = f_{2n} \\
  i\lambda_n \eta - \left( \nu(1) - \beta_1 \eta - \int_0^{\tau_2} \beta_2(s)z(1, s)ds \right) = 0 \\
  i\lambda_n z + s^{-1}z_\rho = 0.
\end{cases}
\]  
(30)

The fourth equation of (30) combining with the condition \( z(0) = \eta \) gives \( z(\rho, s) = \eta e^{-i\lambda_n \rho} \), that is
\[ z(1, s) = \eta e^{-i\lambda_n s}. \]  
(31)

Combining the first and the second equation of (30), and using the fact that \((u, \nu, \eta, x)^T \in \mathcal{D}(A)\), it follows that
\[
\begin{cases}
  u_{xx} + \lambda_n^2 u = -f_{2n} \\
  u(0) = 0 \\
  u_x(1) = -\eta.
\end{cases}
\]  
(32)

The homogeneous equation associated to (32) can be solved as
\[ u_p(x) = k_1 \cos(\lambda_n x) + k_2 \sin(\lambda_n x), \quad k_1, k_2 \in \mathbb{R}. \]

Notice that \( u_1(x) = \cos(\lambda_n x) \) et \( u_2(x) = \sin(\lambda_n x) \) are both the solutions of the homogeneous equation associated to (32).

Let us denote by \( \mathcal{W}(u_1, u_2) \) the “Wronskien” of the family \((u_1, u_2)\). We have
\[
\mathcal{W}(u_1, u_2) = \begin{vmatrix} \cos(\lambda_n x) & \sin(\lambda_n x) \\ -\lambda_n \sin(\lambda_n x) & \lambda_n \cos(\lambda_n x) \end{vmatrix} = \lambda_n \neq 0.
\]

As \( \mathcal{W}(u_1, u_2) \neq 0 \), the family \((u_1, u_2)\) forms a fundamental system of solutions. Consequently we can search the particular solution of (32) in the form
\[ u_p(x) = k_1(x) \cos(\lambda_n x) + k_2(x) \sin(\lambda_n x) \]  
(33)

where \( k_1 \) and \( k_2 \) are functions which verify
\[
\begin{cases}
  k_1' \cos(\lambda_n x) + k_2' \sin(\lambda_n x) = 0 \\
  -k_1' \lambda_n \sin(\lambda_n x) + k_2' \lambda_n \cos(\lambda_n x) = -f_{2n}.
\end{cases}
\]  
(34)

The equation (34) can be solved as
\[ k_1(x) = \frac{1}{\lambda_n} \int_0^x f_{2n}(s) \sin(\lambda_n s)ds \quad \text{and} \quad k_2(x) = -\frac{1}{\lambda_n} \int_0^x f_{2n}(s) \cos(\lambda_n s)ds. \]  
(35)
Combining (35) and (33), we get
\[ u_0(x) = -\frac{1}{\lambda_n} \int_0^x f_{2n}(s) \sin(\lambda_n(x-s)) \, ds. \]  
(36)

Now the general solution of (32) can be written as
\[ u(x) = k_1 \cos(\lambda_n x) + k_2 \sin(\lambda_n x) - \frac{1}{\lambda_n} \int_0^x f_{2n}(s) \sin(\lambda_n(x-s)) \, ds, \quad k_1, k_2 \in \mathbb{R}. \]  
(37)

On the one hand we have
\[ u(0) = 0 \quad \Rightarrow \quad k_1 = 0. \]

On the other hand we compute
\[ u(1) = k_2 \sin(\lambda_n) - \frac{1}{\lambda_n} \int_0^1 f_{2n}(s) \sin(\lambda_n(1-s)) \, ds; \]
from which follows
\[ k_2 = u(1) \frac{1}{\sin \lambda_n} + \frac{1}{\lambda_n \sin \lambda_n} \int_0^1 f_{2n}(s) \sin(\lambda_n(1-s)) \, ds. \]

Consequently the general solution of (32) can be rewritten as
\[ u(x) = u(1) \frac{\sin(\lambda_n x)}{\sin \lambda_n} + \frac{\sin(\lambda_n x)}{\lambda_n \sin \lambda_n} \int_0^x f_{2n}(s) \sin(\lambda_n(1-s)) \, ds - \frac{1}{\lambda_n} \int_0^x f_{2n}(s) \sin(\lambda_n(x-s)) \, ds. \]  
(38)

Differentiating the above relation, it follows that
\[ u_s(x) = \lambda_n u(1) \frac{\cos(\lambda_n x)}{\sin \lambda_n} + \frac{\cos(\lambda_n x)}{\lambda_n \sin \lambda_n} \int_0^x f_{2n}(s) \sin(\lambda_n(1-s)) \, ds - \frac{1}{\lambda_n} \int_0^x f_{2n}(s) \cos(\lambda_n(x-s)) \, ds \]
that is
\[ u_s(1) = \lambda_n u(1) \cot \lambda_n + \cot \lambda_n \int_0^1 f_{2n}(s) \sin(\lambda_n(1-s)) \, ds - \frac{1}{\lambda_n} \int_0^1 f_{2n}(s) \cos(\lambda_n(1-s)) \, ds. \]  
(39)

Now using (39) and the boundary condition \( u_s(1) = -\eta \), we get
\[ u(1) = -\frac{\eta \tan \lambda_n}{\lambda_n} - \frac{1}{\lambda_n} \int_0^1 f_{2n}(s) \sin(\lambda_n(1-s)) \, ds + \tan \lambda_n \int_0^1 f_{2n}(s) \cos(\lambda_n(1-s)) \, ds. \]  
(40)

From the first and the third equations of (30) we compute
\[ \left( \int_{r_1}^{r_2} \beta_1 + \int_{r_1}^{r_2} \beta_2(s)e^{-i\lambda_n s} \, ds \right) \eta = i\lambda_n u(1) \]
\[ = -i\eta \tan \lambda_n - i \int_0^1 f_{2n}(s) \sin(\lambda_n(1-s)) \, ds \]
\[ + i \tan \lambda_n \int_0^1 f_{2n}(s) \cos(\lambda_n(1-s)) \, ds \]}
that is
\[
\left( i\lambda_n + \beta_1 + \int_{t_1}^{t_2} \beta_2(s)e^{-i\lambda_n s} ds + i \tan \lambda_n \right) \eta = -i \int_0^1 f_{2n}(s) \sin (\lambda_n(1 - s)) \ ds
\]
\[
+ i \tan \lambda_n \int_0^1 f_{2n}(s) \cos (\lambda_n(1 - s)) \ ds.
\]
Let us set
\[
\Pi_n = i\lambda_n + \beta_1 + \int_{t_1}^{t_2} \beta_2(s)e^{-i\lambda_n s} ds + i \tan \lambda_n.
\]
Before computing \( \eta \), let us demonstrate that \( \Pi_n \neq 0 \) with the choice \( \lambda_n = 2n\pi + \frac{1}{\sqrt{n}} \).

We have
\[
\Pi_n = i\lambda_n + \beta_1 + \int_{t_1}^{t_2} \beta_2(s) \frac{\cos (\lambda_n s) + i \sin (\lambda_n s)}{\cos (\lambda_n s) + i \sin (\lambda_n s)} ds + i \tan \left( \frac{1}{\sqrt{n}} \right)
\]
\[
= i\lambda_n + \beta_1 + \int_{t_1}^{t_2} \beta_2(s) \cos (\lambda_n s) ds + i \left( \lambda_n - \int_{t_1}^{t_2} \beta_2(s) \sin (\lambda_n s) ds + \tan \left( \frac{1}{\sqrt{n}} \right) \right).
\]

We have
\[
0 < \beta_1 - \int_{t_1}^{t_2} \beta_2(s) ds < \beta_1 + \int_{t_1}^{t_2} \beta_2(s) \cos (\lambda_n s) ds
\]
So we can deduce that
\( \Pi_n \neq 0 \), and
\[
\eta = -\frac{i}{\Pi_n} \int_0^1 f_{2n}(s) \sin (\lambda_n(1 - s)) \ ds + \frac{i \tan \lambda_n}{\Pi_n} \int_0^1 f_{2n}(s) \cos (\lambda_n(1 - s)) \ ds.
\]
Inserting (41) in (40), it follows that
\[
u(1) = \frac{i \tan \lambda_n}{\lambda_n \Pi_n} \int_0^1 f_{2n}(s) \sin (\lambda_n(1 - s)) \ ds - \frac{i \tan^2 \lambda_n}{\lambda_n \Pi_n} \int_0^1 f_{2n}(s) \cos (\lambda_n(1 - s)) \ ds
\]
\[
- \frac{1}{\lambda_n} \int_0^1 f_{2n}(s) \sin (\lambda_n(1 - s)) \ ds + \frac{\tan \lambda_n}{\lambda_n} \int_0^1 f_{2n}(s) \cos (\lambda_n(1 - s)) \ ds
\]
that is
\[
u(1) = -\frac{i\lambda_n + \beta_1 + \int_{t_1}^{t_2} \beta_2(s)e^{-i\lambda_n s} ds}{\lambda_n \Pi_n} \int_0^1 f_{2n}(s) \sin (\lambda_n(1 - s)) \ ds
\]
\[
+ \frac{\left( i\lambda_n + \beta_1 + \int_{t_1}^{t_2} \beta_2(s)e^{-i\lambda_n s} ds \right) \tan \lambda_n}{\lambda_n \Pi_n} \int_0^1 f_{2n}(s) \cos (\lambda_n(1 - s)) \ ds.
\]
If we take \( \lambda_n \) large enough in (41), we get

\[
u(1) = \frac{C_0}{\lambda_n} \int_0^1 f_{2n}(s) \sin \left( \lambda_n(1-s) \right) ds. \tag{42}\]

Now let us compute \( \lambda_n u(x) \), using (42). We get

\[
\lambda_n u(x) = \lambda_n u(1) \frac{\sin(\lambda_n x)}{\sin \lambda_n} + \frac{\sin(\lambda_n x)}{\sin \lambda_n} \int_0^1 f_{2n}(s) \sin \left( \lambda_n(1-s) \right) ds - \int_0^x f_{2n}(s) \sin \left( \lambda_n(x-s) \right) ds
\]

\[
= C_0 \frac{\sin(\lambda_n x)}{\sin \lambda_n} \int_0^1 f_{2n}(s) \sin \left( \lambda_n(1-s) \right) ds + \frac{\sin(\lambda_n x)}{\sin \lambda_n} \int_0^1 f_{2n}(s) \sin \left( \lambda_n(1-s) \right) ds
\]

\[
- \int_0^x f_{2n}(s) \sin \left( \lambda_n(x-s) \right) ds
\]

\[
= (1 - C_0) \frac{\sin(\lambda_n x)}{\sin \lambda_n} \int_0^1 f_{2n}(s) \sin \left( \lambda_n(1-s) \right) ds - \int_0^x f_{2n}(s) \sin \left( \lambda_n(x-s) \right) ds.
\]

Consequently we have

\[
\lambda_n u(x) = (1 - C_0) \frac{\sin(\lambda_n x)}{\sin \lambda_n} P_n(1) - \frac{P_n(x)}{H_n(x)} \tag{43}
\]

where we set

\[
P_n(x) := \int_0^x f_{2n}(s) \sin \left( \lambda_n(x-s) \right) ds. \tag{44}
\]

Let us set \( f_{2n}(x) := \sin(\lambda_n x) \). Then computing \( P_n(x) \), we obtain

\[
P_n(x) = \int_0^x \sin(\lambda_n s) \sin \left( \lambda_n(x-s) \right) ds
\]

\[
= \int_0^x \sin(\lambda_n s) \left( \sin(\lambda_n x) \cos(\lambda_n s) - \cos(\lambda_n x) \sin(\lambda_n s) \right) ds
\]

\[
= \sin(\lambda_n x) \int_0^x \sin(\lambda_n s) \cos(\lambda_n s) ds - \cos(\lambda_n x) \int_0^x \sin^2(\lambda_n s) ds
\]

\[
= \frac{\sin(\lambda_n x)}{2\lambda_n} \int_0^x \left( \sin^2(\lambda_n s) \right)' ds - \frac{\cos(\lambda_n x)}{2} \int_0^x (1 - \cos(2\lambda_n s)) ds
\]

\[
= \frac{\sin^3(\lambda_n x)}{2\lambda_n} - \frac{x \cos(\lambda_n x)}{2} + \frac{\cos(\lambda_n x) \sin(2\lambda_n x)}{4\lambda_n}
\]

\[
= \frac{\sin^3(\lambda_n x)}{2\lambda_n} - \frac{x \cos(\lambda_n x)}{2} + \frac{\cos^2(\lambda_n x) \sin(\lambda_n x)}{2\lambda_n}
\]

\[
= \frac{\sin(\lambda_n x) - x \cos(\lambda_n x)}{2\lambda_n}.
\]

Recalling the choice of \( \lambda_n \), we have that \( \sin(\lambda_n) = \frac{1}{\sqrt{n}} \), \( \cos(\lambda_n) = 1 \) and \( \lambda_n = 2n\pi \). So we get

\[
P_n(1) = \frac{1}{2\pi n^{3/2}} - \frac{1}{2} = -\frac{1}{2}.
\]
Then it follows that
\[
\int_0^1 |H_n(x)|^2 \, dx \geq \int_0^1 \frac{x^2 \cos^2(\lambda_n x)}{8} \, dx - \frac{C_1}{\lambda_n^2}
\]
(where \(C_1\) is a generic positive constant)
\[
\geq \frac{1}{48} - \frac{C_1}{\lambda_n^2}.
\]
that is
\[
\int_0^1 |H_n(x)|^2 \, dx \geq \frac{1}{48} - \frac{C_1}{\lambda_n}.
\]
(45)

Furthermore we have
\[
\int_0^1 |K_n(x)|^2 \, dx = \int_0^1 \left| (1 - C_0) \frac{\sin(\lambda_n x)}{\sin(\lambda_n)} P_n(1) \right|^2 \, dx
\]
\[
\geq \frac{C_2}{\sin^2(\lambda_n)} \int_0^1 \sin^2(\lambda_n x) \, dx
\]
(where \(C_2\) is a generic positive constant)
\[
\geq C_2 n \left[ \frac{x}{2} - \frac{\sin(2\lambda_n x)}{4\lambda_n} \right]_0^n
\]
that is
\[
\int_0^1 |K_n(x)|^2 \, dx \geq C_3 n + C_4
\]
(46)

with \(C_3\) (positive) and \(C_4\) are generic constants.

Following (43) we have that
\[
\int_0^1 |\lambda_n u(x)|^2 \, dx = \int_0^1 |K_n(x) + H_n(x)|^2
\]
\[
= \int_0^1 |K_n(x)|^2 \, dx + \int_0^1 |H_n(x)|^2 \, dx + 2 \int_0^1 K_n(x) H_n(x) \, dx
\]
(47)

A straightforward calculation using the identity \(2ab \geq -a^2 - b^2\) gives for all \(\varepsilon > 0\):
\[
K_n H_n = \left( \frac{1}{\sqrt{\varepsilon}} K_n \right) \left( \sqrt{\varepsilon} H_n \right)
\]
\[
\geq - K_n^2 / \varepsilon - \varepsilon H_n^2.
\]
(48)

Inserting (48) in (47) it follows that
\[
\int_0^1 |\lambda_n u(x)|^2 \geq \left( 1 - \frac{2}{\varepsilon} \right) \int_0^1 |K_n(x)|^2 \, dx + (1 - 2\varepsilon) \int_0^1 |H_n(x)|^2 \, dx.
\]
(49)

Now combining (49), (46) and (45), we obtain
\[
\int_0^1 |\lambda_n u(x)|^2 \geq C_3 \left( 1 - \frac{2}{\varepsilon} \right) n + C_4 \left( 1 - \frac{2}{\varepsilon} \right) + (1 - \varepsilon) \left( \frac{1}{12} + \frac{C}{\lambda_n} \right).
\]
Consequently there exists a positive constant $\gamma_1$, and another constant $\gamma_2$ such that

$$\int_0^1 \left| \lambda_n u(x) \right|^2 \, dx \geq \gamma_1 n + \gamma_2. \tag{50}$$

We deduce from (50) that

$$\| U_n \|_{\mathcal{C}}^2 \geq \| v \|_{L^2(0,1)}^2 = \int_0^1 \left| \lambda_n u(x) \right|^2 \, dx \geq \gamma_1 n + \gamma_2, \tag{51}$$

which implies

$$\lim_{n \to +\infty} \| U_n \|_{\mathcal{C}} = +\infty.$$ 

On the other hand, according to the choice of $F_n$ we have

$$\| F_n \|_{\mathcal{C}} = \frac{1}{2} \int_0^1 \sin^2(\lambda_n x) \, dx = \frac{1}{2} \frac{\sin(2\lambda_n)}{4\lambda_n}$$

which implies

$$\| F_n \|_{\mathcal{C}} = O(1).$$

Finally we have found some sequences $\lambda_n$, $U_n$ and $F_n$ which verifies (26)-(28). Consequently system (1) is not uniformly stable.

\[\square\]

### 5 Rational stabilization result

In this section, we shall prove that problem (1) is polynomially stable under assumption (2). To obtain this, we use method based on the following result due to Borichev and Tomilov [5]:

**Theorem 5.1.**

Let $A$ be the generator of a $C_0$-semigroup of bounded operators on a Hilbert space $X$ such that $i \mathbb{R} \subset \rho(A)$. Then, we have the polynomial decay

$$\left\| e^{tA} U_0 \right\| \leq \frac{C}{t^{(1/2)_+}} \| U_0 \|, \quad t > 0,$$

if and only if

$$\limsup_{|\lambda| \to +\infty} \frac{1}{|\lambda|^6} \left\| (i\lambda - A)^{-1} \right\| < \infty.$$

The main result of this section is the next theorem.

**Theorem 5.2.**

The semigroup of system (1) decays polynomially as

$$\left\| e^{tA} U_0 \right\| \leq \frac{C}{t} \| U_0 \|, \quad \forall \ U_0 \in \mathcal{D}(A), \quad \forall \ t > 0. \tag{52}$$
Proof. It suffices to show following the results in [18, 28] and the above theorem, that for any \( U = (u, v, \eta, z)^T \in \mathcal{D}(A) \) and \( F = (f, g, h, k)^T \in \mathcal{H} \), the solution of
\[
(iA - A) U = F
\]
verifies
\[
\|U\|_{\mathcal{H}} \leq C\lambda\|F\|_{\mathcal{H}};
\]
where \( \lambda > 0 \) and \( C \) a positive constant.

Problem (1) without delay is the following one
\[
\begin{cases}
  u_{tt}(x, t) - u_{xx}(x, t) = 0 \text{ in } ]0, 1[ \times (0, +\infty) \\
  u(0, t) = 0 \\
  u_x(1, t) + \eta(t) = 0 \ \forall \ t \in (0, +\infty) \\
  u_t(t) - u_x(1, t) + \beta \eta(t) \ \forall \ t \in (0, +\infty) \\
  u(\cdot, 0) = u_0, \ u_t(\cdot, 0) = u_1 \ \text{in } ]0, 1[ , \ \eta(0) = \eta_0 \in \mathbb{R}
\end{cases}
\]
which is well-posed in
\[
\mathcal{H}_0 := V \times L^2(0, 1) \times \mathbb{R}
\]
endowed with the norm
\[
\|(u, v, \eta)^T\|_{\mathcal{H}_0}^2 := \|u_x\|_{L^2(0,1)}^2 + \|v\|_{L^2(0,1)}^2 + \eta^2.
\]
The generator of its semigroup is \( A_0 \) defined by
\[
A_0 (u, v, \eta)^T := (v, u_{xx}, v(1 - \beta_1 \eta))^T
\]
with domain
\[
\mathcal{D}(A_0) := \{(u, v, \eta)^T \in \left(H^2(0, 1) \cap V\right) \times V \times \mathbb{R} : u_x(1) + \eta = 0\}.
\]
Thanks to [28], the operator \( A_0 \) generates a polynomial stable semigroup with optimal decay rate \( t^{-1} \). Therefore the solution \((u^*, v^*, \eta^*)^T\) of
\[
(iA - A_0) \begin{pmatrix} u^* \\ v^* \\ \eta^* \end{pmatrix} = \begin{pmatrix} u \\ v \\ \eta \end{pmatrix}
\]
verifies
\[
\|(u^*, v^*, \eta^*)^T\|_{\mathcal{H}_0} \leq C_0 \lambda \|(u, v, \eta)^T\|_{\mathcal{H}_0}
\]
where \( C_0 \) is a positive constant.

On the other hand the system (59) can be rewritten as
\[
\begin{cases}
  i\lambda u^* - v^* = u \\
  i\lambda v^* - u_{xx} = v \\
  i\lambda \eta^* - v^*(1) + \beta_1 \eta^* = \eta.
\end{cases}
\]
Let \( \varepsilon \) be a positive constant, the choice of which will be made later.

With the help of integrations by parts and using (61), we have
\[
\langle (iA - A) \begin{pmatrix} u \\ v \\ \eta \\ z \end{pmatrix}, \begin{pmatrix} u^* \\ v^* \\ \eta^* \\ -\frac{1}{\varepsilon} z \end{pmatrix} \rangle_{\mathcal{H}} = \langle \begin{pmatrix} i\lambda u - v \\ i\lambda v - u_{xx} \\ i\lambda \eta - v(1) + \beta_1 \eta + \int_{t_2}^{t_3} \beta_2(s)z(1, s)ds \\ i\lambda z + s^{-1} z_{t_3} \end{pmatrix}, \begin{pmatrix} u^* \\ v^* \\ \eta^* \\ -\frac{1}{\varepsilon} z \end{pmatrix} \rangle_{\mathcal{H}}
\]
\[
\frac{1}{\epsilon} \int_{r_1}^{r_2} (s \beta_2(s)) \left( \int_{0}^{1} (i \lambda z + s^{-1} z \rho) \, dp \right) \, ds
\]

\[
= \int_{0}^{1} (i \lambda u - v) \, dx + \int_{0}^{1} (i \lambda v - u_{xx}) \, dx
\]

\[
+ \left( i \lambda \eta - v(1) + \beta_1 \eta + \int_{r_1}^{r_2} \beta_2(s) z(1, s) \, ds \right) \eta
\]

\[
- \frac{1}{\epsilon} \int_{r_1}^{r_2} (s \beta_2(s)) \left( \int_{0}^{1} (i \lambda z + s^{-1} z \rho) \, dp \right) \, ds
\]

\[
= i \lambda \int_{0}^{1} u_x u_t \, dx - \int_{0}^{1} v_x u_x \, dx + i \lambda \int_{0}^{1} v \, dx - \int_{0}^{1} u_{xx} v \, dx
\]

\[
+ \left( i \lambda \eta - v(1) + \beta_1 \eta + \int_{r_1}^{r_2} \beta_2(s) z(1, s) \, ds \right) \eta
\]

\[
- \frac{i \lambda}{\epsilon} \int_{r_1}^{r_2} (s \beta_2(s)) \left( \int_{0}^{1} |z|^2 \, dp \right) \, ds - \frac{1}{\epsilon} \int_{r_1}^{r_2} \beta_2(s) \int_{0}^{1} z \rho \, dp \, ds
\]

\[
\left( (i \lambda A) \begin{pmatrix} u \\ v \\ \eta \\ z \end{pmatrix}, \begin{pmatrix} u^* \\ v^* \\ \eta^* \\ -\frac{1}{\epsilon} z^* \end{pmatrix} \right)_{\mathcal{C}} = i \lambda \int_{0}^{1} u_x u_t \, dx + v(1) \eta + \int_{0}^{1} v_x u_x \, dx + i \lambda \int_{0}^{1} v \, dx
\]

\[
+ \eta v(1) + \int_{0}^{1} u_x v_x \, dx + \left( i \lambda \eta - v(1) + \beta_1 \eta + \int_{r_1}^{r_2} \beta_2(s) z(1, s) \, ds \right) \eta
\]

\[
- \frac{i \lambda}{\epsilon} \int_{r_1}^{r_2} (s \beta_2(s)) \left( \int_{0}^{1} |z|^2 \, dp \right) \, ds - \frac{1}{\epsilon} \int_{r_1}^{r_2} \beta_2(s) \int_{0}^{1} z \rho \, dp \, ds
\]

\[
\left( (i \lambda A) \begin{pmatrix} u \\ v \\ \eta \\ z \end{pmatrix}, \begin{pmatrix} u^* \\ v^* \\ \eta^* \\ -\frac{1}{\epsilon} z^* \end{pmatrix} \right)_{\mathcal{C}} = -\int_{0}^{1} |u_x|^2 \, dx - \int_{0}^{1} |v|^2 \, dx - |\eta|^2 + \left( 2 \beta_1 \eta + \int_{r_1}^{r_2} \beta_2(s) z(1, s) \, ds \right) \eta
\]

\[
- \frac{i \lambda}{\epsilon} \int_{r_1}^{r_2} (s \beta_2(s)) \left( \int_{0}^{1} |z|^2 \, dp \right) \, ds - \frac{1}{\epsilon} \int_{r_1}^{r_2} \beta_2(s) \int_{0}^{1} z \rho \, dp \, ds
\]
Recalling (53) and using (56), we deduce from the above relation that

\[
\begin{align*}
\| (u, v, \eta) \|_{\mathcal{H}_0}^2 &= -\Re \left\langle F, \begin{pmatrix} u^* \\ v^* \\ \eta^* \end{pmatrix} \right\rangle + \Re \left( 2\beta_1 \eta + \int_{r_1}^{r_2} \beta_2(s) z(1, s) ds \right) \eta^* \\
&- \Re \left( \frac{1}{\varepsilon} \int_{r_1}^{r_2} \beta_2(s) \int_0^1 \Re (z_0 \overline{z}) \, dp \, ds \right) \\
&- \frac{1}{\varepsilon} \int_{r_1}^{r_2} \beta_2(s) \left[ |z(s, s)|^2 \right] ds \\
&= -\Re \left\langle F, \begin{pmatrix} u^* \\ v^* \\ \eta^* \end{pmatrix} \right\rangle + \Re \left( 2\beta_1 \eta + \int_{r_1}^{r_2} \beta_2(s) z(1, s) ds \right) \eta^* \\
&- \frac{1}{\varepsilon} \int_{r_1}^{r_2} \beta_2(s) \left( |z(1, s)|^2 - |z(0, s)|^2 \right) \, ds
\end{align*}
\]

Now, using triangular, Cauchy-Schwarz's and Young's inequalities, we get

\[
\| (u, v, \eta) \|_{\mathcal{H}_0}^2 \leq \| F \|_{\mathcal{H}} \left( \frac{1}{\varepsilon} \| (0, 0, 0, z) \|_{\mathcal{H}} + \| (u^*, v^*, \eta^*) \|_{\mathcal{H}_0}^2 \right) + \frac{\beta_1^2}{\varepsilon} \| \eta \|^2 + \frac{\varepsilon}{2} \| \eta^* \|^2 \\
+ \frac{1}{\varepsilon} \int_{r_1}^{r_2} \beta_2(s) |z(1, s)|^2 \, ds + \frac{\varepsilon}{2} \| \eta^* \| \int_{r_1}^{r_2} \beta_2(s) \, ds - \frac{1}{\varepsilon} \int_{r_1}^{r_2} \beta_2(s) |z(1, s)|^2 \, ds
\]
\[ \begin{align*}
&+ \frac{|\eta|^2}{2E} \int_{r_1}^{r_2} \beta_2(s) \, ds \\
&\leq |F|_{2C} \left( \frac{1}{E} \left\| (u, v, \eta, z)^T \right\|_{2C} + \left\| (u^*, v^*, \eta^*)^T \right\|_{2C_0} \right) + \left( \frac{\beta_1^2}{E} + \frac{1}{2E} \int_{r_1}^{r_2} \beta_2(s) \, ds \right) |\eta|^2 \\
&+ \varepsilon |\eta^*|^2 \left( \frac{1}{2} \int_{r_1}^{r_2} \beta_2(s) \, ds + 1 \right) \int_{r_1}^{r_2} \left( s\beta_2(s) \int_0^1 \|z(\rho, s)\|^2 \, d\rho \right) \, ds
\end{align*} \]

that is, using the definition of \( \mathcal{C} \)-norm of \( U = (u, v, \eta, z)^T \) and (60)

\[ \|U\|_{2C}^2 \leq |F|_{2C} \left( \frac{1}{E} \left\| (u, v, \eta, z)^T \right\|_{2C} + \left\| (u^*, v^*, \eta^*)^T \right\|_{2C_0} \right) + \left( \frac{\beta_1^2}{E} + \frac{1}{2E} \int_{r_1}^{r_2} \beta_2(s) \, ds \right) |\eta|^2 \\
+ \int_{r_1}^{r_2} \left( s\beta_2(s) \int_0^1 \|z(\rho, s)\|^2 \, d\rho \right) \, ds \tag{62} \]

From the dissipativity of \( A \), it follows that

\[ \Re \left( (i\lambda I - A) \begin{pmatrix} u \\ v \\ \eta \\ z \end{pmatrix}, \begin{pmatrix} u \\ v \\ \eta \\ z \end{pmatrix} \right) \geq |\eta|^2 \left( \beta_1 - \int_{r_1}^{r_2} \beta_2(s) \, ds \right) ; \]

that is, using (53) and the Cauchy-Schwarz’s inequality

\[ |\eta|^2 \left( \beta_1 - \int_{r_1}^{r_2} \beta_2(s) \, ds \right) \leq \Re \left( (i\lambda I - A) \begin{pmatrix} u \\ v \\ \eta \\ z \end{pmatrix}, \begin{pmatrix} u \\ v \\ \eta \\ z \end{pmatrix} \right) \]

\[ \leq \Re \left( F, U \right)_{2C} \leq |F|_{2C} \|U\|_{2C}. \]

Then we deduce that

\[ |\eta|^2 \leq \frac{1}{\left( \beta_1 - \int_{r_1}^{r_2} \beta_2(s) \, ds \right)} \|F\|_{2C} \|U\|_{2C}. \tag{63} \]

Note further that (59) and (60) combining with the dissipativeness of \( A_0 \) directly yield

\[ \beta_1 |\eta^*|^2 = \Re \left( (i\lambda I - A_0) \begin{pmatrix} u^* \\ v^* \\ \eta^* \end{pmatrix}, \begin{pmatrix} u^* \\ v^* \\ \eta^* \end{pmatrix} \right) \leq \left\| (u, v, \eta)^T \right\|_{2C_0} \left\| (u^*, v^*, \eta^*)^T \right\|_{2C_0} \leq C\lambda \left\| (u, v, \eta)^T \right\|_{2C_0}^2 \]

that is

\[ \beta_1 |\eta^*|^2 \leq C\lambda \|U\|_{2C}^2. \tag{64} \]
So using (64) and (63) in (62), we get

\[ \|U\|^{2}_{\mathcal{H}} \leq \left( \frac{1}{\varepsilon} + C_{0} \lambda \right) \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}} + C_{1} \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}} + C_{2} \lambda \varepsilon \|U\|^{2}_{\mathcal{H}} + \int_{r_{1}}^{r_{2}} \left( s \beta_{2}(s) \int_{0}^{1} z^{2}(\rho, s) \, d\rho \right) \, ds \]

where \( C_{1} \) and \( C_{2} \) defined below are constants not dependent on \( \lambda \)

\[
C_{1} = \frac{\left( \frac{\beta_{1}^{2}}{\varepsilon} + \frac{1}{2\varepsilon} \int_{r_{1}}^{r_{2}} \beta_{2}(s) \, ds \right)}{\beta_{1} - \int_{r_{1}}^{r_{2}} \beta_{2}(s) \, ds},
\]

\[
C_{2} = \frac{C_{0}}{\beta_{1}} \left( \frac{1}{2} \int_{r_{1}}^{r_{2}} \beta_{2}(s) \, ds + 1 \right).
\]

Setting \( \varepsilon = \frac{1}{2C_{2}A} \) in the above relation such that \( C_{2} \lambda \varepsilon = \frac{1}{2} \), we have

\[ \frac{1}{2} \|U\|^{2}_{\mathcal{H}} \leq (C_{3} \lambda + C_{1}) \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}} + \int_{r_{1}}^{r_{2}} \left( s \beta_{2}(s) \int_{0}^{1} z^{2}(\rho, s) \, d\rho \right) \, ds \] (65)

Now we need a best estimation for \( \int_{r_{1}}^{r_{2}} \left( s \beta_{2}(s) \int_{0}^{1} z^{2}(\rho, s) \, d\rho \right) \, ds \).

Following (53) and solving the next Cauchy problem (66)

\[
\begin{cases}
  s^{-1} z_{\rho} + i \lambda z = k \\
  z(0),
\end{cases}
\]

we obtain

\[ z(\rho) = z(0) e^{-i\lambda s_{0}} + s \int_{0}^{\rho} e^{-i\lambda s(\rho-\sigma)} k(\sigma) \, d\sigma, \quad \forall \rho \in (0, 1). \] (67)

Using the triangular inequality, it follows from (67) that

\[ |z(\rho)| \leq |z(0)| + s \int_{0}^{\rho} |k(\sigma)| \, d\sigma, \quad \forall \rho \in (0, 1), \]

that is

\[ |z(\rho)|^{2} \leq |z(0)|^{2} + s^{2} \left( \int_{0}^{\rho} |k(\sigma)| \, d\sigma \right)^{2} + 2 |z(0)| s \left( \int_{0}^{\rho} |k(\sigma)| \, d\sigma \right), \quad \forall \rho \in (0, 1). \] (68)
On the one hand, by Cauchy-Schwarz’s inequality we obtain

\[
\left( \int_0^\rho |k(\sigma)| \, d\sigma \right)^2 \leq \left( \int_0^\rho |k(\sigma)|^2 \, d\sigma \right) \left( \int_0^\rho \, d\sigma \right)
\]

\[
\leq \rho \int_0^\rho |k(\sigma)|^2 \, d\sigma
\]

\[
\leq \int_0^\rho |k(\sigma)|^2 \, d\sigma;
\]

that is

\[
\left( \int_0^\rho |k(\sigma)| \, d\sigma \right)^2 \leq \int_0^\rho |k(\sigma)|^2 \, d\sigma.
\]  \hspace{1cm} (69)

On the other hand, Young’s inequality guarantees that

\[
2 |z(0)| \left( \int_0^\rho |k(\sigma)| \, d\sigma \right) \leq |z(0)|^2 + \rho \int_0^\rho |k(\sigma)|^2 \, d\sigma.
\]  \hspace{1cm} (70)

Combining (68), (69) and (70), it follows that

\[
|z(\rho, s)|^2 \leq 2 |\eta|^2 + 2s^2 \int_0^\rho |k(\sigma)|^2 \, d\sigma.
\]  \hspace{1cm} (71)

Integrating (71) on \((\tau_1, \tau_2) \times (0, 1)\) and making easy computations, we get

\[
\int_{\tau_1}^{\tau_2} \left( s^2 \beta_2(s) \int_0^1 z^2(\rho, s) \, d\rho \right) \, ds \leq 2 |\eta|^2 \int_{\tau_1}^{\tau_2} s \beta_2(s) \, ds + 2 \int_{\tau_1}^{\tau_2} \left( s^3 \beta_2(s) \int_0^1 |k(\sigma)|^2 \, d\sigma \right) \, ds
\]

\[
\leq 2\tau_2 |\eta|^2 \int_{\tau_1}^{\tau_2} \beta_2(s) \, ds + 2\tau_2 \int_{\tau_1}^{\tau_2} \left( s \beta_2(s) \int_0^1 |k(\sigma)|^2 \, d\sigma \right) \, ds
\]

\[
\leq 2\tau_2 \beta_1 |\eta|^2 + 2\tau_2 \int_{\tau_1}^{\tau_2} \left( s \beta_2(s) \int_0^1 |k(\sigma)|^2 \, d\sigma \right) \, ds
\]

We arrive at

\[
\int_{\tau_1}^{\tau_2} \left( s \beta_2(s) \int_0^1 z^2(\rho, s) \, d\rho \right) \, ds \leq C_4 \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}} + 2\tau_2^2 \|F\|_{\mathcal{H}}^2,
\]  \hspace{1cm} (72)

where \(C_4 = \frac{2\tau_2 \beta_1}{\beta_1 - \int_{\tau_1}^{\tau_2} \beta_2(s) \, ds}\) is a constant not dependent on \(\lambda\).

Finally, combining (72) and (66), it follows that

\[
\|U\|_{\mathcal{H}}^2 \leq 2 (C_3 \lambda + C_4) \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}} + 4\tau_2^2 \|F\|_{\mathcal{H}}^2.
\]  \hspace{1cm} (73)

Taking \(\lambda\) sufficiently large in (73), we get \(\|U\|_{\mathcal{H}}^2 \leq C \left( \lambda \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^2 \right)\), from where follows that \(\|U\|_{\mathcal{H}} \leq C \lambda \|F\|_{\mathcal{H}}\). Therefore recalling (53), we conclude that

\[
\limsup_{\lambda \to +\infty} \frac{1}{\lambda} \left\| (i\lambda - A)^{-1} \right\| < \infty.
\]

So from Theorem 5.1, the semigroup decays polynomially with the rate \(t^{-1}\).
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