Explicit bounds on canonical Green functions of modular curves

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Abstract. We prove explicit bounds on canonical Green functions of Riemann surfaces obtained as compactifications of quotients of the hyperbolic plane by Fuchsian groups.

1. Introduction

Let $H$ denote the hyperbolic plane, identified with the complex upper half-plane with holomorphic coordinate $z = x + iy$. Let $\mu_H = y^{-2}dx\,dy$ denote the standard volume form on $H$. The group $\text{SL}_2(\mathbb{R})$ acts isometrically on $H$ by $\gamma z = \frac{az + b}{cz + d}$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R})$ and $z \in H$. The Laplace operator on $H$ is $\Delta = y^2(\partial_x^2 + \partial_y^2)$.

A Fuchsian group is a discrete subgroup of $\text{SL}_2(\mathbb{R})$. A Fuchsian group $\Gamma$ is called cofinite if the volume of $\Gamma \backslash H$ with respect to the measure induced by $\mu_H$ is finite.

Let $\Gamma$ be a cofinite Fuchsian group, and let $X$ be the standard compactification of $\Gamma \backslash H$ obtained by adding the cusps. We assume that $X$ has positive genus. There are two interesting Green functions in this setting. First, we have the hyperbolic Green function $\text{gr}_\Gamma$ outside the diagonal on $\Gamma \backslash H \times \Gamma \backslash H$, given by the structure of $\Gamma \backslash H$ as a quotient of the hyperbolic plane by a Fuchsian group. This appears naturally in the spectral theory of automorphic forms. Second, we have the canonical Green function $\text{gr}_X^{\text{can}}$ outside the diagonal on $X \times X$, given by the structure of $X$ as a compact Riemann surface of positive genus. This is a fundamental object in the intersection theory on arithmetic surfaces developed by Arakelov [1], Faltings [5] and others, where it is used to define local intersection numbers of horizontal divisors at the infinite places.

In this article we derive explicit bounds on $\text{gr}_X^{\text{can}}$ using bounds on $\text{gr}_\Gamma$ established by the author in [3]. Our results are valid for any cofinite Fuchsian group, although we are motivated by the case of arithmetic groups, and in particular congruence subgroups of $\text{SL}_2(\mathbb{Z})$. Bounds on the canonical Green functions of the modular curves $X_1(n)$ are relevant to recent work of Edixhoven, Couveignes et al. [4] and of the author [2], where Arakelov theory is employed to obtain a polynomial-time algorithm for computing Galois representations attached to Hecke eigenforms over finite fields.

The following theorem illustrates our general results. To avoid having to deal with the logarithmic singularity, we only give an upper bound. More precise results are contained in Theorem 2.1.

Theorem 1.1. Let $\Gamma$ be a congruence subgroup of level $n$ of $\text{SL}_2(\mathbb{Z})$ such that the compactification $X$ of $\Gamma \backslash H$ has positive genus. Then the canonical Green function of $X$ satisfies

$$\sup_{X \times X} \text{gr}_X^{\text{can}} \leq 1.6 \cdot 10^4 + 7.7n + 0.088n^2.$$ 

It will make little difference to us whether the quotient $\Gamma \backslash H$ and its compactification $X$ are interpreted as as stacks with generic stabiliser $\Gamma \cap \{\pm 1\}$ or as their coarse moduli spaces, which are Riemann surfaces. To avoid any subtleties, the reader can restrict himself to groups containing neither $-1$ nor any elliptic elements and with all cusps regular, such as $\Gamma_1(n)$ with $n \geq 5$. Two points are useful to keep in mind. First, as in [3], we define integration on $\Gamma \backslash H$ and $X$ in a "stack-like" way, so that in case $-1 \in \Gamma$, integrals over $\Gamma \backslash H$ or $X$ are half of what the naïve definition gives. Second, the space of cusp forms of weight 2 for $\Gamma$, the space of holomorphic differentials on $X$, and the space of holomorphic differentials on the coarse moduli space of $X$ are all isomorphic. We write $g_X$ for the dimension of these spaces, and call it the genus of $X$.

Remark. A different approach to the problem of bounding canonical Green functions was taken by Jorgenson and Kramer [11]. For compact quotients of the upper half-plane, they deduced an interesting expression for the canonical Green function in terms of data associated to the hyperbolic metric. In comparison, our method is less involved and applies more naturally to modular curves.

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2. Notation and statement of results

2.1. Cusps
Let $\Gamma$ be a cofinite Fuchsian group. The cusps of $\Gamma$ correspond to the conjugacy classes of maximal parabolic subgroups of $\Gamma$. For every cusp $\epsilon$ of $\Gamma$, we fix one such subgroup and denote it by $\Gamma_\epsilon$, and we fix $\sigma_\epsilon \in \text{SL}_2(\mathbb{R})$ such that

$$\{\pm 1\} \sigma^{-1}_\epsilon \Gamma_\epsilon \sigma_\epsilon = \left\{ \pm \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \bigg| b \in \mathbb{Z} \right\}.$$ 

For $z \in H$, we write

$$q_\epsilon(z) = \exp(2\pi i \sigma^{-1}_\epsilon z)$$

and

$$y_\epsilon(z) = \Im \sigma^{-1}_\epsilon z = -\log |q_\epsilon(z)| / 2\pi.$$ 

For all $\gamma \in \Gamma$, we write

$$C_\epsilon(\gamma) = |c| \quad \text{if} \quad \sigma^{-1}_\epsilon \gamma \sigma_\epsilon = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$ 

It is known that if $\epsilon$ is a real number satisfying the inequality

$$0 < \epsilon \leq \min_{\gamma \in \Gamma} C_\epsilon(\gamma), \quad (2.1)$$

then for all $z \in H$ and $\gamma \in \Gamma$ one has the implication

$$y_\epsilon(z) > 1/\epsilon \quad \text{and} \quad y_\epsilon(\gamma z) > 1/\epsilon \quad \Rightarrow \quad \gamma \in \Gamma_\epsilon.$$ 

For any $\epsilon$ satisfying (2.1), the image of the strip

$$\{ x + iy \mid 0 \leq x < 1 \text{ and } y > 1/\epsilon \} \subset H$$

under the map

$$H \xrightarrow{\sigma_\epsilon} H \rightarrow \Gamma \setminus H$$

is an open disc $D_\epsilon(\epsilon)$ around $\epsilon$. The map $q_\epsilon$ induces a chart on $\Gamma \setminus H$ identifying $D_\epsilon(\epsilon)$ with the punctured disc $\{ z \in \mathbb{C} \mid 0 < |z| < \exp(-2\pi/\epsilon) \}$. The disc $D_\epsilon(\epsilon)$ has the compactification

$$\bar{D}_\epsilon(\epsilon) = \{ z \in \Gamma \setminus H \mid y_\epsilon(z) \geq 1/\epsilon \} \cup \{ \epsilon \}.$$ 

2.2. The canonical $(1,1)$-form
Let $X$ be a compact connected Riemann surface of genus $g_X \geq 1$. The $\mathbb{C}$-vector space $\Omega^1(X)$ of global holomorphic differentials on $X$ has dimension $g_X$ and is equipped with the inner product

$$\langle \alpha, \beta \rangle = \frac{i}{2} \int_X \alpha \wedge \bar{\beta}.$$ 

The canonical $(1,1)$-form on $X$ is

$$\mu^\text{can}_X = \frac{i}{2g_X} \sum_{\alpha \in B} \alpha \wedge \bar{\alpha},$$

where $B$ is any orthonormal basis of $\Omega^1(X)$ with respect to $\langle \ , \ \rangle$. The form $\mu^\text{can}_X$ is independent of the choice of $B$.

Let us now assume that $X$ is (the coarse moduli space associated to) the compactification of $\Gamma \setminus H$ with $\Gamma$ a cofinite Fuchsian group. We define a smooth and bounded function $F_\Gamma$ on $\Gamma \setminus H$ by

$$F_\Gamma(z) = \sum_{f \in B} (3z)^2 |f(z)|^2,$$

where $B$ is any orthonormal basis for the space of holomorphic cusp forms of weight 2 for $\Gamma$. The $(1,1)$-forms $\mu^\text{can}_X$ and $\mu_H$ are related by

$$\mu^\text{can}_X = \frac{1}{g_X} F_\Gamma \mu_H. \quad (2.2)$$
2.3. Spectral theory for Fuchsian groups

We collect here some facts that we will need. For proofs and further details, we refer to Iwaniec’s book [6] or Hejhal’s two volumes [8] and [9].

Let $\Gamma$ be a cofinite Fuchsian group. The Laplace operator on $\mathbb{H}$ induces an (unbounded, densely defined) self-adjoint operator $\Delta_\Gamma$ on the Hilbert space $L^2(\Gamma \setminus \mathbb{H})$. We will often work with $-\Delta_\Gamma$ instead, since it is non-negative. The spectrum of $-\Delta_\Gamma$ consists of a discrete part and a continuous part. The discrete part consists of eigenvalues, which we denote by $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$, $\lambda_j \to \infty$ as $j \to \infty$.

Let $\{\phi_j\}_{j=0}^\infty$ be a corresponding orthonormal system of eigenfunctions. The continuous part of the spectrum of $-\Delta_\Gamma$ is the interval $[1/4, \infty)$ with multiplicity equal to the number of cusps of $\Gamma$. It corresponds to the non-holomorphic Eisenstein series $E_c(z, s)$.

Every smooth and bounded function on $\Gamma \setminus \mathbb{H}$ has the spectral representation

$$f(z) = \sum_{j=0}^\infty b_j \phi_j(z) + \sum_c \frac{1}{4\pi i} \int_{R_{s=1/2}} b_c(s) E_c(z, s) ds,$$

where $c$ runs over the cusps of $\Gamma$ and the coefficients $b_j$ and $b_c(s)$ are given by

$$b_j = \int_{\Gamma \setminus \mathbb{H}} f \bar{\phi}_j \mu_{\mathbb{H}} \quad \text{and} \quad b_c(s) = \int_{\Gamma \setminus \mathbb{H}} f \bar{E}_c(s, \cdot) \mu_{\mathbb{H}}.$$

For later use, we define, as in [3, §2.5],

$$\Phi_\Gamma(z, \lambda) = \sum_{j: \lambda_j \leq \lambda} |\phi_j(z)|^2 + \sum_c \frac{1}{4\pi i} \int_{R_{s=1/2}} |E_c(z, s)|^2 ds. \quad (2.3)$$

It is known that for $\lambda$ tending to $\infty$ and fixed $z$, this function is bounded linearly in $\lambda$.

2.4. The Green function of a Fuchsian group

Let $\Gamma$ be a cofinite Fuchsian group. For every smooth and bounded function $f$ on $\Gamma \setminus \mathbb{H}$, there exists a unique smooth and bounded function $g_f$ on $\Gamma \setminus \mathbb{H}$ such that

$$\Delta_\Gamma g_f = f - \frac{1}{\text{vol}\Gamma} \int_{\Gamma \setminus \mathbb{H}} f \mu_{\mathbb{H}} \quad \text{and} \quad \int_{\Gamma \setminus \mathbb{H}} g_f \mu_{\mathbb{H}} = 0.$$

There exists a unique function $gr_\Gamma$ on $\Gamma \setminus \mathbb{H} \times \Gamma \setminus \mathbb{H}$ that satisfies $gr_\Gamma(z, w) = gr_\Gamma(w, z)$, is smooth except for a logarithmic singularity along the diagonal, and has the property that if $f$ is a smooth and bounded function on $\Gamma \setminus \mathbb{H}$, then the function $g_f$ is given by

$$g_f(z) = \int_{w \in \Gamma \setminus \mathbb{H}} gr_\Gamma(z, w) f(w) \mu_{\mathbb{H}}(w).$$

The function $gr_\Gamma$ is called the Green function of the Fuchsian group $\Gamma$.

2.5. The canonical Green function of a Riemann surface

Let $X$ be a Riemann surface. Let $*$ denote the star operator on smooth 1-forms, given with respect to any local holomorphic coordinate $z = x + iy$ by

$$*dx = dy, \quad *dy = -dx.$$

If we identify $X$ locally with the hyperbolic plane, an easy calculation shows that the operator $d * d$ sending functions to $(1,1)$-forms is related to the Laplace operator $\Delta$ as follows: if $f$ is any smooth function on $X$, then

$$d * df = \Delta f \cdot \mu_{\mathbb{H}}.$$
Let us now assume that $X$ is compact, connected and of positive genus. Let $\alpha$ be a smooth $(1,1)$-form on $X$. Then there exists a unique smooth function $h_\alpha$ on $X$ such that
\[ d \ast dh_\alpha = \alpha - \left( \int_X \alpha \right) \mu_X^{\text{can}} \quad \text{and} \quad \int_X h_\alpha \mu_X^{\text{can}} = 0. \]

There exists a unique function $\text{gr}_X^{\text{can}}$ on $X \times X$ that satisfies $\text{gr}_X^{\text{can}}(z, w) = \text{gr}_X^{\text{can}}(w, z)$, is smooth except for a logarithmic singularity along the diagonal, and has the property that if $\alpha$ is a smooth $(1,1)$-form on $X$, then the function $h_\alpha$ is given by
\[ h_\alpha(z) = \int_{w \in X} \text{gr}_X^{\text{can}}(z, w) \alpha(w). \]

The function $\text{gr}_X^{\text{can}}$ is called the canonical Green function of $X$.

2.6. The main result
For all $u > 1$, we write
\[ L(u) = \frac{1}{4\pi} \log \frac{u + 1}{u - 1}. \]

If $\Gamma$ is a Fuchsian group and $X$ is the compactification of $\Gamma \backslash H$ obtained by adding the cusps, we write
\[ \zeta_\Gamma = \frac{1}{g_X} \int_{\Gamma \backslash H} \text{F}_\Gamma \mu_X^{\text{can}} = \frac{1}{\text{vol}_\Gamma}. \quad (2.4) \]

We will commit the following abuse of notation: for $z \in H$ and $Z$ a subset of $\Gamma \backslash H$, we write $z \in Z$ if the image of $z$ in $\Gamma \backslash H$ lies in $Z$.

**Theorem 2.1.** Let $\Gamma$ be a cofinite Fuchsian group, and let $X$ be the compactification of $\Gamma \backslash H$ obtained by adding the cusps. Let $\delta$ be a real number with $\delta > 1$. For every cusp $c$ of $\Gamma$, let $\epsilon_c^\prime > \epsilon_c > 0$ be real numbers satisfying the inequalities
\[ \epsilon_c^\prime (\delta + \sqrt{\delta^2 - 1})^{1/2} \leq \min_{\gamma \in \Gamma_c} C_c(\gamma) \quad \text{and} \quad (\delta + \sqrt{\delta^2 - 1}) \epsilon_c \leq \epsilon_c^\prime \]
and small enough such that the discs $D_c(\epsilon_c^\prime)$ are pairwise disjoint. Let $Y$ be the compact subset of $\Gamma \backslash H$ defined by
\[ Y = (\Gamma \backslash H) \setminus \bigcup_c D_c(\epsilon_c). \]

Let $A$ and $B$ be real numbers such that the hyperbolic Green function $\text{gr}_\Gamma$ satisfies
\[ A \leq \text{gr}_\Gamma(z, w) + \sum_{u(z, \gamma w) \leq \delta} (L(u(z, \gamma w) - L(\delta))) \leq B \quad \text{for all } z, w \in Y. \quad (2.5) \]

Let $C > 0$ be such that the function $\Phi_\Gamma(z, \lambda)$ defined by (2.3) satisfies
\[ \Phi_\Gamma(z, \lambda) \leq C \lambda \quad \text{for all } z \in Y \text{ and } \lambda \geq 1/4. \quad (2.6) \]

Let $\eta \in (0, 1/4]$ be such that the spectrum of $-\Delta_\Gamma$ is contained in $\{0\} \cup [\eta, \infty)$. With the notation
\[ S = \sqrt{\left( \frac{1}{4\pi^2} + 4 \right) \zeta_\Gamma}, \]
\[ T(\epsilon) = \frac{\sup_{\Gamma} F_\Gamma}{g_X} \left( \frac{\epsilon}{4\pi} \right)^2 \quad \text{for all } \epsilon > 0, \]
\[ r_s = \frac{1}{24\pi} \left( \sqrt{\frac{2}{\epsilon - 1}} + \arctan \sqrt{\frac{\delta - 1}{2}} \right), \]
\[ A_\epsilon = A + \#(\Gamma \cap \{ \pm 1 \}) \left[ \frac{1}{\epsilon_c^\prime} \left( 1 - \frac{2}{\pi} \arctan \sqrt{\frac{\delta - 1}{2}} \right) - \epsilon_c r_s \right], \]
\[ B_\epsilon = B + \#(\Gamma \cap \{ \pm 1 \}) \left[ \frac{1}{\epsilon_c^\prime} \left( 1 - \frac{2}{\pi} \arctan \sqrt{\frac{\delta - 1}{2}} \right) + \epsilon_c r_s \right]. \]
we have the following bounds on the canonical Green function $\text{gr}^\text{can}_X(z,w)$:

(a) If $z, w \in Y$, we have

$$A - 2S - \zeta_{\Gamma}/\eta \leq \text{gr}^\text{can}_X(z,w) + \sum_{\gamma \in \Gamma \cap \{\pm 1\}} \left( L(u(z,w) - L(\delta)) \right) \leq B + 2S.$$ 

(b) If $\mathfrak{c}$ is a cusp such that $z \in D_\mathfrak{c}(\varepsilon_\mathfrak{c})$, $w \in Y$ and $w \not\in D_\mathfrak{c}(\varepsilon'_\mathfrak{c})$, or such that $w \in D_\mathfrak{c}(\varepsilon_\mathfrak{c})$, $z \in Y$ and $z \not\in D_\mathfrak{c}(\varepsilon'_\mathfrak{c})$, then we have

$$A - 2S - \zeta_{\Gamma}/\eta \leq \text{gr}^\text{can}_X(z,w) \leq B + 2S + T(\varepsilon_\mathfrak{c}).$$

(c) If $\mathfrak{c}, \mathfrak{d}$ are two distinct cusps such that $z \in D_\mathfrak{c}(\varepsilon_\mathfrak{c})$ and $w \in D_\mathfrak{d}(\varepsilon_\mathfrak{d})$, we have

$$A - 2S - \zeta_{\Gamma}/\eta \leq \text{gr}^\text{can}_X(z,w) \leq B + 2S + T(\varepsilon_\mathfrak{c}) + T(\varepsilon_\mathfrak{d}).$$

(d) If $\mathfrak{c}$ is a cusp such that $z, w \in D_\mathfrak{c}(\varepsilon'_\mathfrak{c})$, we have

$$\tilde{A}_\mathfrak{c} - 2S - \zeta_{\Gamma}/\eta \leq \text{gr}^\text{can}_X(z,w) - \#(\Gamma \cap \{\pm 1\}) \cdot \frac{1}{2\pi} \log |q_\mathfrak{c}(z) - q_\mathfrak{c}(w)| \leq \tilde{B}_\mathfrak{c} + 2S + 2T(\varepsilon'_\mathfrak{c}).$$

In earlier work of the author [3], it was described how to compute explicit real numbers $A$ and $B$ as in (2.5) and $C$ as in (2.6) for concrete groups $\Gamma$, such as congruence groups of $\text{SL}_2(\mathbb{Z})$. In Section 4 below, we will show that bounds on the function $F_{\Gamma}$ can likewise be found easily for given groups $\Gamma$.

3. Tools

3.1. Comparison of hyperbolic and canonical Green functions

There is a standard way to relate the hyperbolic and canonical Green functions, which we will use to find explicit bounds on the canonical Green function. Let $\Gamma$ be a cofinite Fuchsian group, and let $X$ be the compactification of $\Gamma \backslash \mathbb{H}$. We define a function $h_{\Gamma}: \Gamma \backslash \mathbb{H} \to \mathbb{R}$ by

$$h_{\Gamma}(z) = \int_{w \in \Gamma \backslash \mathbb{H}} \text{gr}_{\Gamma}(z,w) \mu^\text{can}_X(w)$$

$$= \frac{1}{g_X} \int_{w \in \Gamma \backslash \mathbb{H}} \text{gr}_{\Gamma}(z,w) F_{\Gamma}(w) \mu_{\mathbb{H}}(w).$$

By the defining properties of $\text{gr}_{\Gamma}$, the function $h_{\Gamma}$ satisfies

$$\Delta h_{\Gamma} = \frac{1}{g_X} F_{\Gamma} - \frac{1}{g_X \text{vol}_{\Gamma}} \int_{\Gamma \backslash \mathbb{H}} F_{\Gamma} \mu_{\mathbb{H}}$$

$$= \frac{1}{g_X} F_{\Gamma} - \frac{1}{\text{vol}_{\Gamma}}.$$ 

This implies that the canonical Green function of $X$ can be expressed as

$$\text{gr}^\text{can}_X(z,w) = \text{gr}_{\Gamma}(z,w) - h_{\Gamma}(z) - h_{\Gamma}(w) + \int_{\Gamma \backslash \mathbb{H}} h_{\Gamma} \mu^\text{can}_X.$$

3.2. The Selberg–Harish-Chandra transform

Let $k$ be a real number, and let $\Delta_k = g^2 (\partial_x^2 + \partial_y^2) - iky\partial_x$ denote the Laplace operator of weight $k$ on $\mathbb{H}$. Let $\theta: [1, \infty) \to \mathbb{R}$ be a piecewise smooth function with compact support. We define

$$\theta^{(k)}(z,w) = \left( \frac{w - \overline{z}}{z - \overline{w}} \right)^{k/2} \theta(u(z,w)).$$
Let \( P_{s,k} \) be the generalisation of the Legendre function \( P_{s-1}(u) \) given by Fay [7, §1] (note that our definition of weight is twice that of [7]):

\[
P_{s,k} = \left( \frac{2}{u+1} \right)^s F \left( s - \frac{k}{2}, s + \frac{k}{2}; 1; \frac{u-1}{u+1} \right).
\]  

(3.3)

We define the Selberg–Harish-Chandra transform of weight \( k \) of the function \( \theta \) as

\[
h_k(\theta)(s) = 2\pi \int_1^\infty \theta(u) P_{s,k}(u) du.
\]  

(3.4)

If \( f \) is an eigenfunction of \( -\Delta_k \) with eigenvalue \( s(1-s) \), then we have

\[
\int_{w \in H} \theta(\gamma z, \gamma w) f(w) dw = h_k(\theta)(s) f(z);
\]  

(3.5)

see Fay [7, Theorem 1.5].

3.3. Automorphic forms

For simplicity, we take \( k = 2 \) from now on. We write

\[
\nu(\gamma, z) = \frac{ez+d}{cz+d} = \frac{(ez+d)^2}{|cz+d|^2} \quad \text{for} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R}) \text{ and } z \in \mathbb{H}.
\]

We recall that an automorphic form (of Maaß) of weight 2 for \( \Gamma \) is a smooth function \( f : \mathbb{H} \to \mathbb{C} \) with the following properties:

(1) the function \( f \) satisfies the transformation formula

\[
f(\gamma z) = \nu(\gamma, z) f(z) \quad \text{for all } \gamma \in \Gamma \text{ and } z \in \mathbb{H};
\]

(2) for every cusp \( c \) of \( \Gamma \), there is a real number \( \kappa \) such that \( |f(z)| = O(y_c(z)^\kappa) \) as \( y_c(z) \to \infty \).

A cusp form of weight 2 for \( \Gamma \) is a function \( f \) satisfying (1) and the following condition (which is stronger than (2)):

(2') for every cusp \( c \) of \( \Gamma \) there exists \( \epsilon > 0 \) such that \( |f(z)| = O(\exp(-\epsilon y_c z)) \) as \( y_c(z) \to \infty \).

Let \( L^2(\Gamma \backslash \mathbb{H}, 2) \) denote the Hilbert space of square-integrable automorphic forms of weight 2 for \( \Gamma \), equipped with the Petersson inner product.

Let \( \theta \) be a function as in §3.2. Then we have

\[
\theta^{(2)}(\gamma z, \gamma w) = \frac{\nu(\gamma, z)}{\nu(\gamma, w)} \theta^{(2)}(z, w) \quad \text{for all } \gamma \in \text{SL}_2(\mathbb{R}) \text{ and } z, w \in \mathbb{H}.
\]

Let \( \Gamma \) be a cofinite Fuchsian group. We define

\[
K^{(2)}_{\Gamma,\theta}(z, w) = \sum_{\gamma \in \Gamma} \nu(\gamma, w) \theta^{(2)}(z, \gamma w).
\]  

(3.6)

This function satisfies

\[
K^{(2)}_{\Gamma,\theta}(w, z) = K^{(2)}_{\Gamma,\theta}(z, w)
\]

and, for all \( \gamma \in \Gamma \),

\[
K^{(2)}_{\Gamma,\theta}(\gamma z, w) = \nu(\gamma, z) K^{(2)}_{\Gamma,\theta}(z, w),
K^{(2)}_{\Gamma,\theta}(z, \gamma w) = \nu(\gamma, w)^{-1} K^{(2)}_{\Gamma,\theta}(z, w).
\]

Now (3.5) implies that if \( f \) is an automorphic form of weight 2 for \( \Gamma \) satisfying \( -\Delta_2 f = s(1-s)f \), then

\[
\int_{w \in \Gamma \backslash \mathbb{H}} K^{(2)}_{\Gamma,\theta}(z, w) f(w) \mu_H(w) = h_k^{(2)}(s) f(z).
\]  

(3.7)
4. Explicit bounds on the canonical \((1, 1)\)-form

Let \(\Gamma\) be a cofinite Fuchsian group. In this section we find bounds on the function \(F_\Gamma\) that are easy to evaluate explicitly in concrete cases. We essentially adapt the methods of Iwaniec [6, §7.2] from weight 0 to weight 2. This method is more elementary than that used by Jorgenson and Kramer in [10], and our bounds are easy to make explicit, as the example in Section 6 shows.

For \(z \in \mathbb{H}\) and \(b \geq 1\), we write

\[ N_\Gamma(z, b) = \#\{\gamma \in \Gamma \mid u(z, \gamma z) \leq b\}. \]

**Proposition 4.1.** For every cofinite Fuchsian group \(\Gamma\), all \(z \in \mathbb{H}\) and all \(a > 1\), we have

\[ F_\Gamma(z) \leq \frac{(a - 1)N_\Gamma(z, 2a^2 - 1)}{8\pi (\log \frac{a+1}{2})^2}. \]

**Proof.** Let \((f_1, \ldots, f_g)\) be an orthonormal basis of the space of holomorphic cusp forms of weight 2 for \(\Gamma\). We write

\[ \phi_j(z) = (3z) f_j(z). \]

Then the \(\phi_j\) are annihilated by the operator \(\Delta_2\), and \((\phi_1, \ldots, \phi_g)\) is an orthonormal system in the Hilbert space \(L^2(\Gamma \setminus \mathbb{H}, 2)\) of automorphic forms of weight 2 for \(\Gamma\).

Let \(z \in \mathbb{H}\) and \(a > 1\) be given. We apply \(\S\S 3.2\) and \(3.3\) with

\[ \theta(u) = \begin{cases} 1 & \text{if } 1 \leq u \leq a, \\ 0 & \text{if } u > a. \end{cases} \]

In the Hilbert space \(L^2(\Gamma \setminus \mathbb{H}, 2)\), we consider \(K^{(2)}_{\Gamma, \theta}(z, w)\), as a function of \(w\), and the orthonormal system \((\phi_1, \ldots, \phi_g)\). From Bessel’s inequality and (3.7), we obtain

\[ \sum_{j=1}^g |h^{(2)}_g(0)\phi_j(z)|^2 \leq \int_{w \in \Gamma \setminus \mathbb{H}} |K^{(2)}_{\Gamma, \theta}(z, w)|^2 \mu_\mathbb{H}(w). \]

We note that the left-hand side is equal to \(|h^{(2)}_g(0)|^2 F_\Gamma(z)\). Let us denote the right-hand side by \(\kappa(z)\); this is a \(\Gamma\)-invariant function of \(z\) with values in \([0, \infty)\). The definition (3.6) gives

\[ \kappa(z) = \sum_{\gamma_1, \gamma_2 \in \Gamma} \nu(\gamma_1, w)\theta(\gamma_1 z, \gamma_1 w)\nu(\gamma_2, w)\theta(\gamma_2 z, \gamma_2 w) \mu_\mathbb{H}(w). \]

Putting \(\gamma = \gamma_1 \gamma_2^{-1}\), we obtain after a straightforward computation

\[ \kappa(z) = \sum_{\gamma \in \Gamma} \nu(\gamma, z) \int_{w \in \mathbb{H}} \theta(\gamma z, w) \theta(\gamma z, w) \mu_\mathbb{H}(w). \]

This implies

\[ \kappa(z) \leq \sum_{\gamma \in \Gamma} \int_{w \in \mathbb{H}} \theta(u(z, w)) \theta(u(z, w)) \mu_\mathbb{H}(w). \]

By the definition of \(\theta\), the integral on the right-hand side can be interpreted as the area of the intersection of the discs of area \(2\pi(a - 1)\) around \(z\) and \(\gamma z\), respectively. By the triangle area for the hyperbolic distance, this intersection is empty unless \(u(z, \gamma z) \leq 2a^2 - 1\).

This implies

\[ \kappa(z) \leq 2\pi(a - 1)N_\Gamma(z, 2a^2 - 1), \]

and hence

\[ |h^{(2)}_g(0)|^2 F_\Gamma(z) \leq 2\pi(a - 1)N_\Gamma(z, 2a^2 - 1). \]

We evaluate \(h^{(2)}_g(0)\) using (3.3) and (3.4). The hypergeometric series terminates after two terms and gives

\[ P_{0, 2}(u) = \frac{2}{u + 1}. \]

This implies

\[ h^{(2)}_g(0) = 4\pi \log \frac{u + 1}{2}. \]

This finishes the proof. \(\Box\)
The above proposition does not give the correct asymptotic behaviour of \( F_\Gamma(z) \) for \( z \) close to a cusp of \( \Gamma \). The following result extends our bounds to neighbourhoods of the cusps.

**Lemma 4.2** (cf. Jorgenson and Kramer [10, Theorem 3.1]). Let \( \Gamma \) be a cofinite Fuchsian group, let \( \epsilon \) be a cusp of \( \Gamma \), and let \( \epsilon \) be a real number satisfying (2.1). Then for all \( z \in D_\epsilon(\epsilon) \), we have

\[
F_\Gamma(z) \leq (\epsilon y_\epsilon(z))^2 \exp(4\pi/\epsilon - 4\pi y_\epsilon(z)) \sup_{\partial D_\epsilon(\epsilon)} F_\Gamma
\]

\[
\leq \left\{ \begin{array}{ll}
\sup_{\partial D_\epsilon(\epsilon)} F_\Gamma & \text{if } \epsilon \leq 2\pi, \\
\left( \frac{\epsilon}{2\pi} \exp(2\pi/\epsilon - 1) \right)^2 \sup_{\partial D_\epsilon(\epsilon)} F_\Gamma & \text{if } \epsilon > 2\pi.
\end{array} \right.
\]

**Proof.** Every holomorphic cusp form \( f \) of weight 2 for \( \Gamma \) has a \( q \)-expansion of the form

\[
f(z)dz = \sum_{n=1}^{\infty} a_{\epsilon,n}(f) q_\epsilon(z)^n \cdot d(\sigma_\epsilon^{-1}z) \text{ with } a_{\epsilon,n}(f) \in \mathbb{C}.
\]

This implies

\[
(3z)^2 |f(z)|^2 = y_\epsilon(z)^2 \left| \sum_{n=1}^{\infty} a_{\epsilon,n}(f) q_\epsilon(z)^n \right|^2.
\]

Applying this to an orthonormal basis of the space of holomorphic cusp forms of weight 2, we see that the function

\[
y_\epsilon^{-2} \exp(4\pi y_\epsilon(z)) F_\Gamma(z) = \sum_{f \in B} \left| \frac{f(z)}{q_\epsilon(z)} \right|^2
\]

extends to a subharmonic function on \( \tilde{D}_\epsilon(\epsilon) \). By the maximum principle for subharmonic functions, the function assumes its maximum on the boundary. This implies the first inequality. The second inequality follows from the easily checked fact that the function \((\epsilon y_\epsilon(z))^2 \exp(4\pi/\epsilon - 4\pi y_\epsilon(z))\) for \( y \geq 1/\epsilon \) assumes its maximum at \( y = 1/(2\pi) \) if \( \epsilon > 2\pi \), and at \( y = 1/\epsilon \) if \( \epsilon \leq 2\pi \). \( \square \)

### 5. Proof of the main result

Our proof of Theorem 2.1 is based on the equation (3.2), the bounds on the hyperbolic Green function from [3], and on bounds on the function \( h_F \) defined by (3.1). The proof of the latter bounds occupies most of this section; Theorem 2.1 then follows without difficulties.

**Lemma 5.1.** Let \( \Gamma, Y, \eta \) and \( C \) be as in Theorem 2.1. Then the function

\[
M_\Gamma(z) = \sum_{j \geq 1} \frac{1}{\lambda_j^2} |\phi_j(z)|^2 + \sum_{\epsilon} \frac{1}{4\pi i} \int_{\mathbb{R}_+ = 1/2} \frac{1}{(s(1-s))^2} |E_\epsilon(z,s)|^2 ds
\]

satisfies

\[
M_\Gamma(z) \leq \left( \frac{1}{4\eta^2} + 4 \right) C \quad \text{for all } z \in Y.
\]

**Proof.** Separating the terms with \( \lambda_j \leq 1/4 \), we get (with \( \partial \Phi_\Gamma / \partial \lambda \) taken in a distributional sense)

\[
M_\Gamma(z) = \sum_{j, 0 < \lambda_j \leq 1/4} \frac{1}{\lambda_j^2} |\phi_j(z)|^2 + \int_{1/4}^{\infty} \frac{1}{\lambda^2} \frac{\partial \Phi_\Gamma(z,\lambda)}{\partial \lambda} d\lambda
\]

\[
\leq \frac{1}{\eta^2} \Phi(z, 1/4) + \left[ \frac{1}{\lambda^2} \Phi_\Gamma(z, \lambda) \right]_{\lambda = 1/4}^{\infty} + 2 \int_{1/4}^{\infty} \lambda^{-3} \Phi_\Gamma(z, \lambda) d\lambda
\]

\[
= \left( \frac{1}{\eta^2} - 16 \right) \Phi(z, 1/4) + 2 \int_{1/4}^{\infty} \lambda^{-3} \Phi_\Gamma(z, \lambda) d\lambda
\]

\[
\leq \left( \frac{1}{\eta^2} - 16 \right) C + 2C \int_{1/4}^{\infty} \lambda^{-2} d\lambda
\]

\[
= \left( \frac{1}{\eta^2} - 16 \right) \frac{C}{4} + 8C
\]

\[
= \left( \frac{1}{4\eta^2} + 4 \right) C,
\]

where the second inequality follows from (2.6). \( \square \)
**Lemma 5.2.** Let $\Gamma$, $Y$, $\eta$ and $C$ be as in Theorem 2.1, and let $\zeta_\Gamma$ be as in (2.4). Then we have

$$|h_\Gamma(z)|^2 \leq \left( \frac{1}{4\pi} + 4 \right) C_{\zeta_\Gamma} \text{ for all } z \in Y.$$  

**Proof.** Let $X$ be the compactification of $\Gamma \setminus H$. Since the function $F_\Gamma$ is smooth and bounded, we may consider its spectral representation, say

$$\frac{1}{g_X} F_\Gamma(z) = \sum_{j \geq 0} b_j \phi_j(z) + \sum_{c} \frac{1}{4\pi i} \int_{\Re s=1/2} b_c(s) E_c(z, s) ds. \tag{5.1}$$

Then the definition of $h_\Gamma$ implies that it has the spectral representation

$$h_\Gamma(z) = -\sum_{j \geq 1} \frac{b_j}{\lambda_j} \phi_j(z) - \sum_{c} \frac{1}{4\pi i} \int_{\Re s=1/2} \frac{b_c(s)}{s(1-s)} E_c(z, s) ds. \tag{5.2}$$

(Note the absence of the term corresponding to $j = 0$.) Now the Cauchy–Schwarz inequality implies

$$h_\Gamma(z)^2 \leq M_\Gamma(z) \left( \sum_{j \geq 1} |b_j|^2 + \sum_{c} \frac{1}{4\pi i} \int_{\Re s=1/2} |b_c(s)|^2 \right).$$

Next, it follows from (5.1), the identity $|a_0^2| = 1/\text{vol}_\Gamma$ and (2.2) that

$$\sum_{j \geq 1} |b_j|^2 + \sum_{c} \frac{1}{4\pi i} \int_{\Re s=1/2} |b_c(s)|^2 = \int_{z \in \Gamma \setminus H} \left( \frac{1}{g_X} F_\Gamma(z) - \frac{1}{\text{vol}_\Gamma} \right)^2 \mu_H(z)$$

$$= \frac{1}{g_X} \int_{z \in \Gamma \setminus H} F_\Gamma(z)^2 \mu_H(z) - \frac{1}{\text{vol}_\Gamma} \tag{5.3}$$

$$= \frac{1}{g_X} \int_{\Gamma \setminus H} F_\Gamma \mu_{\text{can}} - \frac{1}{\text{vol}_\Gamma}.$$

Together with Lemma 5.1 and the definition of $\zeta_\Gamma$, this finishes the proof. \qed

We now extend our bounds on $h_\Gamma$ to the discs around the cusps.

**Lemma 5.3.** Let $\Gamma$ be a cofinite Fuchsian group, and let $X$ be the compactification of $\Gamma \setminus H$. Let $c$ be a cusp of $\Gamma$, and let $\epsilon$ be a real number satisfying (2.1). For all $z \in D_\epsilon(c)$, we have

$$h_{\Gamma, \epsilon}^-(z) \leq h_\Gamma(z) \leq h_{\Gamma, \epsilon}^+(z),$$

where

$$h_{\Gamma, \epsilon}^+(z) = \sup_{\partial D_\epsilon(c)} h_\Gamma + \frac{1}{\text{vol}_\Gamma} \log(\epsilon y_\Gamma(z))$$

and

$$h_{\Gamma, \epsilon}^-(z) = \inf_{\partial D_\epsilon(c)} h_\Gamma - \frac{\sup_{\partial D_\epsilon(c)} F_\Gamma}{g_X} \left( \frac{\epsilon}{4\pi} \right)^2 (1 - \exp(4\pi/\epsilon - 4\pi y_\Gamma(z))) + \frac{1}{\text{vol}_\Gamma} \log(\epsilon y_\Gamma(z)).$$

**Proof.** We note that

$$\Delta h_{\Gamma, \epsilon}^+(z) = -\frac{1}{\text{vol}_\Gamma}$$

and

$$\Delta h_{\Gamma, \epsilon}^-(z) = \frac{\sup_{\partial D_\epsilon(c)} F_\Gamma}{g_X} (\epsilon y_\Gamma(z))^2 \exp(4\pi/\epsilon - 4\pi y_\Gamma(z)) - \frac{1}{\text{vol}_\Gamma}.$$  

By the non-negativity of $F_\Gamma$ and Lemma 4.2, this implies

$$\Delta h_{\Gamma, \epsilon}^+(z) \leq h_\Gamma(z) \leq \Delta h_{\Gamma, \epsilon}^-(z).$$

Therefore $h_{\Gamma, \epsilon}^+ - h_\Gamma$ and $h_\Gamma - h_{\Gamma, \epsilon}^-$ are subharmonic functions on $\bar{D}_\epsilon(c)$. By the maximum principle for subharmonic functions, each of these functions assumes its maximum on the boundary. The definitions of $h_{\Gamma, \epsilon}^\pm$ imply that these maxima are non-negative. \qed
Finally, we prove bounds on the integral $\int_{\Gamma \backslash \mathbb{H}} h_{\Gamma} \mu_{X}^{\text{can}}$.

**Lemma 5.4.** Let $\Gamma$ be a cofinite Fuchsian group, let $X$ be the compactification of $\Gamma \backslash \mathbb{H}$, and let $\eta > 0$ be such that the spectrum of $-\Delta_{\Gamma}$ is contained in $\{0\} \cup [\eta, \infty)$. Then we have

$$-\zeta/\eta \leq \int_{\Gamma \backslash \mathbb{H}} h_{\Gamma} \mu_{X}^{\text{can}} \leq 0.$$

**Proof.** We use the spectral representations (5.1) and (5.2). We obtain

$$\int_{\Gamma \backslash \mathbb{H}} h_{\Gamma} \mu_{X}^{\text{can}} = \int_{z \in \Gamma \backslash \mathbb{H}} h_{\Gamma}(z) \cdot \frac{1}{g_{X}} F_{\Gamma}(z) \mu_{ \mathbb{H}}(z)$$

$$= -\sum_{j \geq 1} \frac{|b_{j}|^{2}}{\lambda_{j}} - \sum_{c} \frac{1}{4 \pi i} \int_{\Re(s) = 1/2} \frac{|b_{c}(s)|^{2}}{s(1-s)}.$$

We note that the right-hand side is non-positive. Next, the assumption that the spectrum of $-\Delta_{\Gamma}$ is contained in $\{0\} \cup [\eta, \infty)$ implies

$$\int_{\Gamma \backslash \mathbb{H}} h_{\Gamma} \mu_{X}^{\text{can}} \geq -\frac{1}{\eta} \left( \sum_{j \geq 1} |b_{j}|^{2} + \sum_{c} \frac{1}{4 \pi i} \int_{\Re(s) = 1/2} |b_{c}(s)|^{2} ds \right).$$

Together with (5.3), this proves the claim. □

**Proof of Theorem 2.1.** Part (a) of the theorem follows from the comparison formula (3.2), the bound (2.5) for $g_{\Gamma}$, the bound on $h_{\Gamma}$ given by Lemma 5.2, and the bound on $\int_{\Gamma \backslash \mathbb{H}} h_{\Gamma} \mu_{X}^{\text{can}}$ given by Lemma 5.4.

The proof of parts (b)–(d) is similar. We first note that, by Lemmata 5.2 and 5.3,

$$-S - T(\epsilon) \leq h_{\Gamma}(z) - \frac{1}{\text{vol}_{\Gamma}} \log(\epsilon y_{\Gamma}(z)) \leq S$$

for all $z \in D(\epsilon)$, and similarly with $\epsilon'$ in place of $\epsilon$. Instead of (2.5) we now invoke [3, Proposition 5.5], which gives bounds for the function $g_{\Gamma}$ when one or both variables are near a cusp. As in the proof of (a), it remains to apply the formula (3.2) and Lemma 5.4. □

**6. Example: congruence subgroups of SL$_{2}(\mathbb{Z})$**

Let $\Gamma$ be a congruence subgroup of SL$_{2}(\mathbb{Z})$ such that the corresponding modular curve $X$ has positive genus. Let $n$ be the level of $\Gamma$, i.e. the minimal positive integer with the property that $\Gamma$ contains the kernel of the reduction map SL$_{2}(\mathbb{Z}) \to$ SL$_{2}(\mathbb{Z}/n\mathbb{Z})$.

We start by fixing the various parameters. For the parameter $\delta$ from Theorem 2.1, we take

$$\delta = 2.$$

Selberg’s eigenvalue conjecture predicts that all non-zero eigenvalues of $-\Delta_{\Gamma}$ are at least $1/4$. This is at present not known to be true, but the sharpest known lower bound, due to Kim and Sarnak [12, Appendix 2], allows us to take

$$\eta = 975/4096.$$

We define

$$\epsilon = (\delta + \sqrt{\delta^{2} - 1})^{-3/2} \approx 0.139 \quad \text{and} \quad \epsilon' = (\delta + \sqrt{\delta^{2} - 1})^{-1/2} \approx 0.518.$$

Let $Y_{0}$ denote the compact subset of SL$_{2}(\mathbb{Z}) \backslash \mathbb{H}$ which is the image of the strip

$$\{ x + iy \in \mathbb{H} \mid |x| \leq 1/2 \text{ and } \sqrt{3}/2 \leq y \leq 1/\epsilon \}. $$
For every cusp \( \epsilon \) of \( \Gamma \), we let \( m_\epsilon \) denote the ramification index of \( \epsilon \) over the unique cusp \( \infty \) of \( \text{SL}_2(\mathbb{Z}) \); this equals the index of the corresponding maximal parabolic subgroups considered modulo \( \{ \pm 1 \} \). For the parameters \( \epsilon_\epsilon \) and \( \epsilon'_\epsilon \), we take

\[
\epsilon_\epsilon = m_\epsilon \epsilon \quad \text{and} \quad \epsilon'_\epsilon = m_\epsilon \epsilon'.
\]

Using the definition of \( C_\epsilon(\gamma) \), it is not hard to show that

\[
\min_{\gamma \in \Gamma} C_\epsilon(\gamma) \geq m_\epsilon.
\]

This implies that the parameters \( \epsilon_\epsilon \) and \( \epsilon'_\epsilon \) satisfy the conditions in Theorem 2.1. As in Theorem 2.1, let \( Y \) be the complement of the discs \( D_\epsilon(\epsilon_\epsilon) \). Then \( Y \) is the inverse image of \( Y_0 \) in \( \Gamma \setminus \mathbb{H} \).

We will need an upper bound on the point counting function \( N_\Gamma(z, 17) \) for \( z \in Y_0 \). It is clear from the definition of \( N_\Gamma(z, U) \) that

\[
\sup_{z \in Y} N_\Gamma(z, U) \leq \sup_{z \in Y_0} N_{\text{SL}_2(\mathbb{Z})}(z, U).
\]

Using the methods of [3, §4.3], we have

\[
N_{\text{SL}_2(\mathbb{Z})}(z, 17) \leq 226 \quad \text{for all} \quad z \in Y_0. \quad (6.1)
\]

We next compute suitable \( A \) and \( B \) satisfying (2.5). For this we use (6.1) and the remaining part of [3, §4.3], with the same parameters \( \alpha^\pm \), \( \beta^\pm \) and \( \gamma^\pm \). The result is

\[
A = -3.00 \cdot 10^4 \quad \text{and} \quad B = 1.58 \cdot 10^4.
\]

We next find a suitable value of the parameter \( C \). We use [3, Lemma 2.4], which says

\[
\Phi_\Gamma(z, \lambda) \leq \frac{\pi}{(2\pi - 4)^2} N_{\text{SL}_2(\mathbb{Z})}(z, 17) \lambda \quad \text{for all} \quad z \in \mathbb{H} \quad \text{and all} \quad \lambda \geq 1/4.
\]

The inequality (6.1) implies that we can take

\[
C = 137.
\]

We continue with explicit bounds on the canonical \((1,1)\)-form. For the parameter \( a \) from Proposition 4.1, we take

\[
a = 1.44.
\]

Again using the method from [3, §4.3], we compute an upper bound for \( N_{\text{SL}_2(\mathbb{Z})}(z, 2a^2 - 1) \) for \( z \in Y_0 \). The result is

\[
N_{\text{SL}_2(\mathbb{Z})}(z, 2a^2 - 1) \leq 58 \quad \text{for all} \quad z \in Y_0.
\]

Substituting this in the bound from Proposition 4.1, we see that

\[
\sup_Y F_\Gamma \leq 25.7.
\]

For every cusp \( \epsilon \), Lemma 5.3 implies

\[
\sup_{D_\epsilon(\epsilon_\epsilon)} F_\Gamma \leq \max\left\{1, \left(\frac{\epsilon_\epsilon}{2\pi}\right)^2\right\} \sup_Y F_\Gamma.
\]

From the definition of \( \epsilon_\epsilon \) and the fact that all ramification indices \( m_\epsilon \) are bounded by the level \( n \) of \( \Gamma \), we conclude

\[
\sup_{X} F_\Gamma \leq \max\left\{1, \left(\frac{n\epsilon}{2\pi}\right)^2\right\} \sup_Y F_\Gamma \leq \max\{25.7, 0.0126n^3\}.
\]
Finally, we consider the invariant $\zeta_\Gamma$. Using $\int_X \nu^\text{can}_X = 1$ and $g_X \geq 1$, we make the rather coarse estimate

$$\zeta_\Gamma \leq \sup_X F_\Gamma \leq \max \{ 25.7, 0.0126 n^2 \}.$$ 

Proof of Theorem 1.1. With the above estimates, we obtain the following bounds on the various constants in the theorem:

- $S \leq \max \{ 172, 3.79n \}$,
- $T(\epsilon_\ell) \leq 0.00313 n^2$,
- $T(\epsilon'_\ell) \leq 0.0436 n^2$,
- $\tilde{A}_\ell \geq -3.00 \cdot 10^4 - 0.0279 n$,
- $\tilde{B}_\ell \leq 1.58 \cdot 10^4 + 0.0279 n$.

The theorem follows from Theorem 2.1 and the above bounds.

References

[1] С. Ю. Аракелов, Теория пересечений дивизоров на арифметической поверхности. Известия Академии Наук СССР, серия математическая 38 (1974), № 6, 1179–1192.
S. Yu. ARAKELOV, Intersection theory of divisors on an arithmetic surface. Mathematics of the USSR Izvestiya 8 (1974), 1167–1180. (English translation.)

[2] P. J. Bruin, Modular curves, Arakelov theory, algorithmic applications. Proefschrift (Ph. D. thesis), Universiteit Leiden, 2010.

[3] P. J. Bruin, Explicit bounds on Green functions of Fuchsian groups. Submitted; preprint available at http://arxiv.org/abs/1205.6306.

[4] S. J. Edixhoven and J.-M. Couveignes (with R. S. de Jong, F. Merkl and J. G. Bosman), Computational Aspects of Modular Forms and Galois Representations. Annals of Mathematics Studies 176. Princeton University Press, 2011.

[5] G. Faltings, Calculus on arithmetic surfaces. Annals of Mathematics (2) 119 (1984), 387–424.

[6] H. Iwaniec, Introduction to the Spectral Theory of Automorphic Forms. Revista Matemática Iberoamericana, Madrid, 1995.

[7] J. D. Fay, Fourier coefficients of the resolvent for a Fuchsian group. Journal für die reine und angewandte Mathematik 293/294 (1977), 143–203.

[8] D. A. Hejhal, The Selberg trace formula for $\mathrm{PSL}(2, \mathbb{R})$, Volume 1. Lecture Notes in Mathematics 548. Springer-Verlag, Berlin/Heidelberg, 1976.

[9] D. A. Hejhal, The Selberg trace formula for $\mathrm{PSL}(2, \mathbb{R})$, Volume 2. Lecture Notes in Mathematics 1001. Springer-Verlag, Berlin/Heidelberg, 1983.

[10] J. Jorgenson and J. Kramer, Bounding the sup-norm of automorphic forms. Geometric and Functional Analysis 14 (2005), no. 6, 1267–1277.

[11] J. Jorgenson and J. Kramer, Bounds on canonical Green’s functions. Compositio Mathematica 142 (2006), no. 3, 679–700.

[12] H. H. Kim, Functoriality for the exterior square of $\mathrm{GL}_4$ and the symmetric fourth of $\mathrm{GL}_2$. With appendix 1 by D. Ramakrishnan and appendix 2 by Kim and P. Sarnak. Journal of the A.M.S. 16 (2002), no. 1, 139–183.

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