An asymptotic approximation for TCP CUBIC

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Abstract
In this paper, we derive an expression for computing the average window size of a single TCP CUBIC connection under random losses. For this we consider a throughput expression for TCP CUBIC computed earlier under deterministic periodic packet losses. We validate this expression theoretically. We then use insights from the deterministic loss-based model to scale appropriately a sequence of Markov chains with random losses indexed by the probability of loss $p$. We show that this sequence converges to a limiting Markov chain as $p$ tends to 0. The stationary distribution of the limiting Markov chain is then used to derive the average window size for small packet error rates. We then use a simple approximation to extend our current results with negligible queuing to a setup with multiple connections and non-negligible queuing. We validate our model and approximations via simulations.

Keywords TCP CUBIC · High-speed TCP · Asymptotic approximation · Performance analysis

Mathematics Subject Classification 68M12 · 68M11 · 60J10 · 90B18

1 Introduction
The TCP-IP protocol suite forms the backbone of the current Internet, and TCP is a crucial component of it. TCP provides reliable, in-order data transfer and flow and congestion control. In this paper, we focus on TCP congestion control. TCP congestion control has been successful in preventing congestion collapse over the Internet. How-
ever, in [1,2] we see that the traditional TCP congestion control algorithms can be very inefficient over wireless links and over high-speed large-delay networks. A number of high-speed TCP congestion control algorithms have been proposed to address the issue of inefficiency, some notable examples being H-TCP, BIC, CUBIC, Compound and FAST [3]. In this paper, we consider TCP CUBIC congestion control as it is widely used. TCP CUBIC is the default congestion control algorithm on Linux since 2006. In [4], the authors report that of the 30,000 web-servers that they considered, more than 25% used TCP CUBIC.

We first give a brief overview of the literature on traditional additive increase multiplicative decrease (AIMD) TCP, which has been extensively studied using a wide variety of tools. In [5,6], the authors use fluid models to analyze TCP performance. In [5], the author compares the performance of TCP Reno with TCP Vegas using a differential equation-based model for TCP window evolution, whereas in [6], the authors solve for throughput of a large number of TCP Reno, New Reno and SACK flows going through AQM routers. In [7,8], the authors look at optimization-based techniques for performance analysis of TCP. In [7], the authors show that the rate distribution of TCP-like sources in a general network can be expressed as a solution to a global optimization problem. In [8] the authors formulate the rate allocation problem as a congestion control game and show that the Nash equilibrium of the game is a solution to a global optimization problem. In [9], the authors consider providing QoS to TCP and real-time flows through the use of rate control for the real-time flows and RED at the bottleneck queues. In [10], the authors provide expressions for TCP Reno throughput using a simple periodic loss model. In [11], the authors use Markovian models to derive an expression for TCP Reno throughput under random losses.

In [12,13], we see experimental evaluation of high speed TCP variants. The reference [12] compares the performance of CUBIC, HSTCP and TCP SACK in a 10 Gbps optical network. In [13], the authors perform an experimental evaluation of TCP CUBIC in a small buffer regime. The reference [14] is a comprehensive simulation-based analysis of high-speed TCP variants, where they compare the protocols for intra-protocol and inter-protocol fairness. There are many references on simulation/experimental analysis of TCP CUBIC; however, there are fewer analytical results. In [15,16], the authors use Markov chain-based models for TCP CUBIC throughput computations. In [17], the authors analyze performance of TCP CUBIC in a cloud networking environment using mean fields.

More recently, researchers at Google have proposed the BBR algorithm [18,19] for congestion control. The BBR algorithm estimates the bottleneck bandwidth and the round trip time for a connection sequentially. The objective of BBR is to estimate the bandwidth-delay product, i.e., the optimal window size for data transfer. Experimental studies in [18] over Google’s WAN network and Youtube video servers have shown promising improvements over the currently used TCP CUBIC algorithm, especially with respect to the queuing delays and consequentially the round trip latency of the connections.
Main Contributions

In this paper, we derive a mean window size and throughput expression for a single long-lived TCP CUBIC flow with random losses. We also extend our analysis to multiple TCP CUBIC connections over a bottleneck link with non-negligible queuing.

In this paper, we focus on long-lived flows; file transfers and video streaming are common examples. These constitute a significant part of internet traffic. For short-lived flows, the slow start phase is more critical, and has been studied in [20]. The slow start phase does not have a significant impact on the throughput of a long-lived flow and hence is typically ignored in computation of TCP throughput [15,16,21–24]. Long-lived and short-lived connections are studied together in [25].

Throughput expressions for TCP CUBIC have been evaluated under a deterministic loss model in [26]. Also, average window size for TCP CUBIC with random losses has been numerically computed using Markov chains in [15,16]. In [21], we see that the expressions for throughput in [26] are less accurate when compared against the Markov chain-based results in [15]. However, the Markov chain-based results do not yield a closed-form expression and we need to solve for the stationary distribution of a Markov chain for each value of drop rate, \( p \). For small \( p \), this could be computationally expensive as the state space of the Markov chain could be very large. We address this drawback of the Markov chain model in this paper by obtaining an approximation for TCP CUBIC under random losses as a function of \( p \) and round trip time (RTT).

For this, we first theoretically validate the expression for TCP CUBIC throughput (given in [26]) under deterministic periodic losses. We then consider a sequence of TCP CUBIC window evolution processes indexed by the drop rate, \( p \), and show that with a suitable scaling this sequence converges to a limiting Markov chain as \( p \) tends to 0. The appropriate scaling is obtained from the deterministic periodic loss model. The stationary distribution of the limiting Markov chain gives us a closed-form expression for the mean window size and throughput which is more accurate than the expression available in the literature currently. Our approach is based on a similar result used for TCP Reno throughput computation in [22] and TCP Compound throughput computation in [23]. However, our proofs are significantly different.

The organization of the paper is as follows: In Sect. 2, we describe our system model. In Sect. 3, we validate the deterministic loss model expression. In Sect. 4, we show that for \( p > 0 \) and with \( W_{\text{max}} = \infty \), the window size process at RTT epochs, the window size process at loss epochs and the time between the loss epochs have unique stationary distributions and that their means under stationarity are also finite. In Sect. 5, we derive an approximation for mean window size under random losses. In Sect. 6, we briefly describe an approximation to extend our results to a setup with non-negligible queuing. In Sect. 7, we compare our model predictions against ns2 simulations. Section 8 concludes our paper.

2 System model for TCP CUBIC

We consider a single TCP CUBIC connection going through a link with constant RTT (round trip time), as shown in Fig. 1. The packets of the connection may be subject
to channel losses. We assume that a packet can be lost independently of other packets with probability $p$. This is a common assumption, also made in [15,16] and is realistic when the losses are dominated by the losses on the wireless links in a network, an increasingly common phenomenon. Our objective is to compute an expression for TCP CUBIC throughput in this setup, which we develop in Sect. 4.

The window size evolution of TCP CUBIC is based on the time since the last congestion epoch. The window size (say $W_0$) at the last epoch is considered as an equilibrium window size. The TCP CUBIC window update is conservative near $W_0$ and is aggressive otherwise. The aggressive behavior gives TCP CUBIC higher throughput compared to traditional TCP in high-speed networks. The TCP CUBIC window size at time $t$, assuming 0 to be a loss epoch with window size $W_0$ just before loss and no further losses in $(0, t]$, is given by

$$W_{\text{cubic}}(t) = C \left( t - \sqrt[3]{\frac{\beta W_0}{C}} \right)^3 + W_0,$$

where $C$ and $\beta$ are TCP CUBIC parameters. The TCP CUBIC update can be slower as compared to TCP Reno. To ensure a worst case behavior like TCP Reno, the aggregate window update is given by $W(t) = \max\{W_{\text{cubic}}(t), W_{\text{reno}}(t)\}$, where $W_{\text{reno}}(t)$ is given by

$$W_{\text{reno}}(t) = W_0 (1 - \beta) + 3 \beta \frac{t}{2 - \beta \text{RTT}}.$$

In our analysis henceforth, we ignore the Reno mode operation, focusing only on the CUBIC mode. However, we account for the Reno mode operation in the final average window size expression.

In the next section, we discuss a deterministic loss model for TCP CUBIC and use the results developed therein in Sect. 4 to compute the TCP CUBIC average window size.

### 3 Fluid models for TCP CUBIC

We now consider a simple fluid model for TCP CUBIC. For the fluid model, we disregard the discrete nature of the TCP window size and also assume that the window
update is continuous instead of happening at discrete intervals of time. The model that we consider here is a widely used deterministic loss model (see [10,24,26]) used to compute the ‘response function’ of TCP. The TCP response function is an expression for TCP throughput in terms of system parameters such as drop rate, \( p \), and RTT, \( R \).

We consider a single TCP CUBIC flow with constant RTT, \( R \). Each packet can be dropped independently of the others with probability \( p \). Under this assumption, the mean number of packets sent between losses is \( \frac{1}{p} \).

We now consider the TCP window evolution under a deterministic loss model with loss rate \( p \). Let us denote the window size for the deterministic loss model at time \( t \) by \( \hat{W}(t) \). Suppose \( \hat{W}(0) = x \). Let \( \tau_p(x) \) denote the time taken by the process \( \hat{W}(t) \) to send \( \frac{1}{p} \) packets with initial window size \( x \), i.e., \( \tau_p(x) \) satisfies

\[
\frac{1}{R} \int_0^{\tau_p(x)} \hat{W}(t) \, dt = \frac{1}{p}. \tag{3}
\]

The window size \( \hat{W}(t) \) evolves as given by (1) until time \( \tau_p(x) \). At \( t = \tau_p(x) \), \( \hat{W}(t) \) undergoes a window reduction so that \( \hat{W}(\tau_p(x)^+) = (1 - \beta)\hat{W}(\tau_p(x)) \), where \( \beta \) is the multiplicative drop factor. Next, the window size \( \hat{W}(t) \) evolves as given by (1) but now with initial window size, \( \hat{W}(\tau_p(x)) \). Again, at time \( t = \tau_p(x) + \tau_p(\hat{W}(\tau_p(x))) \), \( \hat{W}(t) \) undergoes another loss. This process continues and we take \( \{\hat{W}(t)\} \) to be right continuous.

Suppose there exists a unique \( x_p^* \) such that \( \hat{W}(\tau_p(x_p^*)) = x_p^* \), i.e., the fixed point equation

\[
\hat{W}(\tau_p(x)) = x \tag{4}
\]

has a unique solution. Then, if we start from \( x_p^* \), the process \( \hat{W}(t) \) will have a periodic behavior with period \( \tau_p(x_p^*) \) and \( \hat{W}(t) \in [(1 - \beta)x_p^*, x_p^*] \). The long time average for the process \( \hat{W}(t) \) is then given by

\[
\frac{1}{\tau_p(x_p^*)} \int_0^{\tau_p(x_p^*)} \hat{W}(t) \, dt, \tag{5}
\]

with \( \hat{W}(0) = x_p^* \). Using the above model, the average window size for TCP CUBIC is given by

\[
\mathbb{E}[W(p)] = \sqrt{\frac{C(4 - \beta)}{4\beta}} \left( \frac{R}{p} \right)^{3/2}. \tag{6}
\]

The throughput of the TCP connection is given by \( \frac{\mathbb{E}[W(p)]}{R} \). In Proposition 1, we provide a theoretical justification validating the use of the above expression for mean window size. We prove that (4) has a unique fixed point and, starting from any initial window size, under the deterministic loss model with fluid window sizes, the window evolution for TCP CUBIC is eventually periodic with (6) giving the correct time average window size. In Proposition 1, we ignore the slow start phase and ignore that there may be an upper bound on the maximum window size. These assumptions are also made in [10,24,26].
**Proposition 1** For the deterministic loss model, for any given \( p \in (0, 1) \), there exists a unique \( x \) (denoted by \( x^* \)) such that \( \hat{W}(\hat{x}_p(x)) = x \). For any \( x \geq 1 \) with \( \hat{W}(0) = x \), \( \hat{W}(t) \) converges to \( x^* \) at drop epochs.

**Proof** Existence of \( x^* \) Assuming the initial window size to be \( x \), we have

\[
\hat{W}(\hat{x}_p(x)) = C \left( \hat{x}_p(x) - \sqrt[3]{\frac{\beta_x}{C}} \right) + x.
\]

Solving for the fixed point, \( x^* \) of \( \hat{W}(\hat{x}_p(x)) \) gives us \( \hat{x}_p(x^*) = \sqrt[3]{\frac{\beta x^*}{C}} \). Since \( \frac{1}{p} \) packets are sent in \( \hat{x}_p(x) \), we have

\[
\frac{1}{R} \int_0^{\hat{x}_p(x^*)} \hat{W}(u) \, du = \frac{1}{p}.
\]

The fixed point, \( x^* \), for \( \hat{W}(\hat{x}_p(x)) \) is then given by

\[
x^* = \sqrt[3]{\frac{C}{\beta} \left( \frac{4}{4 - \beta} \frac{R}{p} \right)^3}.
\]

(7)

Since \( \hat{W}(u) \) is strictly increasing and continuous, this point is unique. Thus, for every \( p \in (0, 1) \), there exists a unique \( x^* \) given by (7) such that \( \hat{W}(\hat{x}_p(x^*)) = x^* \).

**Convergence to \( x^* \)**

Let us denote the deterministic process, \( \hat{W}(u) \) at time \( u > 0 \) with \( \hat{W}(0) = x \) by \( \hat{W}(u, x) \) so as to also include the initial window size in the process description explicitly. We define \( J(x) = \sqrt[3]{\frac{\beta x}{C}} \) to be the time taken by \( \hat{W}(t) \) to hit \( x \), given that the initial window size, \( \hat{W}(0) \), before drop was \( x \) and there are no losses in \( (0, u] \), with \( u > J(x) \). We show convergence of the map \( x \rightarrow \hat{W}(\tau_p(x)) \) to the fixed point as time increases, in two steps.

**Step 1**

We first show that, if \( x < x^* \), then \( x < \hat{W}(\tau_p(x), x) \). Since \( x < x^* \), \( J(x) < J(x^*) \).

For \( t < J(x) \), we have \( (t - J(x))^2 < (t - J(x^*))^2 \). Thus \( \hat{W}(t, y) = C(t - J(y))^3 + y \) for \( t < J(y) \) implies \( \frac{d\hat{W}(t, x)}{dt} < \frac{d\hat{W}(t, x^*)}{dt} \) for \( t < J(x) \). Also, \( x = \hat{W}(0, x) < \hat{W}(0, x^*) = x^* \). Therefore, for \( t < J(x) \), \( \hat{W}(t, x) < \hat{W}(t, x^*) \). Hence, we have

\[
\int_0^{J(x)} \hat{W}(u, x) \, du < \int_0^{J(x)} \hat{W}(u, x^*) \, du < \int_0^{J(x^*)} \hat{W}(u, x^*) \, du = \frac{R}{p}.
\]

The second inequality holds because \( \hat{W}(u, x) > 0 \) for all \( u, x \). Therefore we get

\[
\int_0^{\tau_p(x)} \hat{W}(u, x) \, du < \int_0^{\tau_p(x)} \hat{W}(u, x^*) \, du = \frac{R}{p}
\]

and \( x = \hat{W}(J(x), x) < \hat{W}(\tau_p(x), x) \). This shows that, if \( x < x^* \), the window size at loss epochs increases.
Step 2

We now show that if \( x > x_p^* \), then \( x_p^* < \hat{W}(\tau(x), x) < x \). The proof for \( \hat{W}(\tau(x), x) < x \) follows as in the previous proof, and hence we do not show it here.

Now, we prove that if \( x > x_p^* \) then \( x_p^* < \hat{W}(\tau(x), x) \). Suppose \( T_1(x) \) denotes the time when \( \hat{W}(T_1(x), x) = x_p^* \). From (1), we get \( T_1(x) = J(x) + \sqrt{\frac{x_p^* - x}{C}} \). Therefore,

\[
\int_{0}^{T_1(x)} \hat{W}(u, x)du = \frac{C}{4} \left( \left( \frac{x_p^* - x}{C} \right)^{\frac{3}{2}} - J(x)^4 \right) + \left( J(x) + \sqrt{\frac{x_p^* - x}{C}} \right) x.
\]

Substituting \( x = \alpha x_p^* (\alpha > 1) \) and then using (7) for \( x_p^* \) simplifies the above expression to

\[
\int_{0}^{T_1(x)} \hat{W}(u, x)du = \frac{R}{p} \left( \frac{(1 - \alpha)^4}{(4 - \beta)\beta^3} + \alpha^3 + \frac{4\alpha(1 - \alpha)\frac{1}{3}}{(4 - \beta)\beta^3} \right).
\]

Now substitute \( \gamma = (\alpha - 1), \gamma > 0 \), and use \( (4 - \beta)\beta^\frac{1}{3} \leq 3 \) for \( \beta \in (0, 1) \) to get

\[
\int_{0}^{T_1(x)} \hat{W}(u, x)du < \frac{R}{p} \left( (1 + \gamma)^\frac{4}{3} - \gamma^\frac{4}{3} - \frac{4}{3}\gamma^\frac{1}{3} \right).
\]

Using Lemma A.1 in the Appendix, we get, for \( k \in (1, 2), (1 + x)^k - x^k - kx^{k-1} < 1 \). Therefore we have

\[
\int_{0}^{T_1(x)} \hat{W}(u, x)du < \frac{R}{p} = \int_{0}^{\tau_p(x)} \hat{W}(u, x)du
\]

and \( x_p^* = \hat{W}(T_1(x), x) < \hat{W}(\tau_p(x), x) \).

Let \( x_{n+1} = \hat{W}(\tau_p(x_n)) \), for \( n \in \mathbb{N} \). From step 2 above, if \( x_n > x_p^* \), then \( x_p^* < x_{n+1} < x_n \). Thus, if \( x_0 = x > x_p^* \), the sequence \( \{x_n\} \) strictly decreases and as it is bounded below (by \( x_p^* \)), it converges to a limit. Because of continuity of \( \hat{W}(\tau_p(\cdot)) \), the limiting point is a fixed point of \( \hat{W}(\tau_p(\cdot)) \) and, as the fixed point is unique, the limit is \( x_p^* \). Thus we have shown that, for any \( x > x_p^* \), the window size at drop epochs (just before loss) monotonically decreases to \( x_p^* \).

From step 1, if \( x_n < x_p^* \), then \( x_{n+1} > x_n \). Suppose \( x_0 = x < x_p^* \). There are two possibilities for the sequence \( \{x_n\} \): either \( x_n \) exceeds \( x_p^* \) for some \( n \) or \( x_n \leq x_p^* \) for all \( n \). If \( x_n > x_p^* \) for some \( n \), then, from arguments given in the previous paragraph, the sequence \( \{x_n\} \) converges to \( x_p^* \). If the latter case holds, \( \{x_n\} \) strictly increases and is bounded above by \( x_p^* \). Then, from similar arguments as in the preceding paragraph, \( \{x_n\} \) converge to \( x_p^* \).

\[\square\]

In Fig. 2, we illustrate multiple iterations of the equation \( \hat{W}(\tau_p(\cdot)) \) for input \( x \in (0, 100) \). We denote the \( k \)th iteration of \( \hat{W}(\tau_p(\cdot)) \) by \( W_k^p(\cdot) \). We see that as the number
of iterations, \( k \), increases, \( W^k_p(.) \) goes close to the fixed point irrespective of the starting point.

From Eq. (7) in Proposition 1, the time between consecutive losses converges to

\[
\hat{\tau}_p(x^*_p) = 3\sqrt{\frac{\beta x^*_p}{C}} = \left( \frac{4\beta R}{(4-\beta)Cp} \right)^{\frac{1}{4}}.
\]  

Thus, for the TCP CUBIC deterministic loss model, from Eq. (7), the window size at drop epochs converges to \( C_1 p^{-\frac{3}{4}} \) and, from Eq. (8), the time between consecutive losses converges to \( C_2 p^{-\frac{1}{4}} \) for some constants \( C_1 \) and \( C_2 \). These are key insights which we will use in Sect. 4, where we derive an expression for average window size under random losses.

### 4 Model with infinite maximum window size

We consider a single TCP connection with constant RTT, i.e., negligible queuing. We assume that the packets are dropped independently with probability \( p \). We have analyzed this system using Markov chains in [15]. In [15], we derive expressions for average window size numerically when the window size \( W_n \) is bounded by some \( W_{\text{max}} < \infty \). We now derive an approximate expression for average window size for low packet error rates assuming \( W_{\text{max}} = \infty \).

Let \( W_n(p) \) denote the window size at the end of the \( n \)th RTT. Let \( W'_n(p) \) denote the window size at the last drop epoch before the \( n \)th RTT (excluding time epoch \( n \)) and let \( T_n(p) \) be the number of RTTs elapsed between the last drop epoch before the \( n \)th RTT
and the nth RTT. As in the deterministic loss model case, we ignore the Reno mode of operation and consider (1) for window evolution. The process \( \{W'_n(p), T_n(p)\} \) forms a Markov chain. We show that, for \( p \in (0, 1) \), the processes \( \{W_n(p)\} \) and \( \{W'_n(p)\} \) have unique stationary distributions.

**Proposition 2** For any fixed \( p \in (0, 1) \) the Markov chain \( \{W'_n(p), T_n(p)\} \) has a single aperiodic, positive recurrent class with remaining states being transient. Hence, it has a unique stationary distribution.

**Proof** From any state in the state space of the Markov chain \( \{W'_n(p), T_n(p)\} \), a sequence of packet losses will cause the Markov chain to hit the state \((1, 0)\). Therefore, the state \((1, 0)\) in the state space of the Markov chain \( \{W'_n(p), T_n(p)\} \) is reachable from any state in the state space with nonzero probability. The states that can be reached by \((1, 0)\) form a communicating class. The remaining states in the state space are transient.

We now show that the communicating class containing \((1, 0)\) is positive recurrent. For convenience, we drop the \( p \) from our notation. For a state \((z, d)\) in the communicating class, we define the Lyapunov function \( L(z, d) = z + d^4 \). The conditional one-step drift of the Lyapunov function is given by

\[
\mathbb{E}[L(W_{n+1}', T_{n+1}) - L(W'_n, T_n)|(W'_n, T_n) = (z, d)]
\]

\[
= (z + (d + 1)^4)q(z, d) + \left( C \left( Rd - \frac{3\beta z}{C} \right)^3 + z \right) (1 - q(z, d)) - z - d^4
\]

\[
= -d^4 + (d + 1)^4 q(z, d) + C \left( Rd - \frac{3\beta z}{C} \right)^3 (1 - q(z, d)), \quad (9)
\]

where \( q(z, d) = (1 - p)^{C(Rd - \frac{3\beta z}{C})^3 + z} \) is the probability of no loss in the nth RTT.

Let us denote the one-step drift in the Lyapunov function defined in (9) by \( f(z, d) \). The quantity \( C(Rd - \frac{3\beta z}{C})^3 + z \geq (1 - \beta)z \). Therefore we have \( q(z, d) \leq (1 - p)^{(1 - \beta)z} \) and \(-C(Rd - \frac{3\beta z}{C})^3 \leq \beta z \). Also, \( d + 1 \leq 2d, \forall d \in \{1, 2, \ldots\} \). Thus, for the one-step drift, we have

\[
f(z, d) = -d^4 + C \left( Rd - \frac{3\beta z}{C} \right)^3 + \left( (d + 1)^4 - C \left( Rd - \frac{3\beta z}{C} \right)^3 \right) q(z, d)
\]

\[
\leq -d^4 + C(Rd)^3 + (16d^4 + 3\beta z)(1 - p)^{(1 - \beta)z}. \quad (10)
\]

For some \( \epsilon > 0 \), we can choose \( z^* \) such that \( \beta z(1 - p)^{(1 - \beta)z} < \epsilon \) and \( 16(1 - p)^{(1 - \beta)z} \leq \frac{1}{2}, \forall z > z^* \). Thus we have, \( \forall d > 0, z > z^* \),

\[
f(z, d) \leq -\frac{1}{2}d^4 + CR^3d^3 + \epsilon.
\]
We can choose $d^*$ such that $-\frac{1}{2}d^4 + CR^3d^3 < -2\epsilon$, for all $d > d^*$. Therefore, $\forall d > d^*$ and $z > z^*$, $f(z, d) < -\epsilon$.

Consider the first equation in (10). The term $q(z, d) = (1 - p)C(Rd - \sqrt{\frac{Rz^3 + q}{C}})^3$ falls exponentially in $z$ for any fixed $d$, and falls super-exponentially in $d$ for any fixed $z$. Hence, for any fixed $z$, as $d \to \infty$, the term $((d + 1)^4 - C(Rd - \sqrt{\frac{Rz^3}{C}})^3)q(z, d) \to 0$. As $d^4$ is asymptotically larger than any polynomial of degree < 4, for any fixed $z \leq z^*$, we can choose $t(z)$ such that $f(z, d) < -\epsilon$, $\forall d > t(z)$. Similarly, for a fixed $d$, as $z \to \infty$, the term $((d + 1)^4 - C(Rd - \sqrt{\frac{Rz^3}{C}})^3)q(z, d) \to 0$. For a fixed $d$, for $z$ large, $\frac{3\sqrt{Rz}}{C} > Rd$. Therefore, for any fixed $d \leq d^*$, we can choose $w'(d)$ such that $f(z, d) < -\epsilon$, $\forall z > w'(d)$.

Thus the one-step drift $f(z, d) < -\epsilon$ outside of a finite set for some $\epsilon > 0$. Thus, by the mean drift criteria for positive recurrence [27], the communicating class containing $(1, 0)$ is positive recurrent. Also, this class is aperiodic as the state $(1, 0)$ has a nonzero probability of hitting itself in one step (a self-loop with probability $p$). $\Box$

We have shown above that the Markov chain $\{W''_n(p), T_n(p)\}$ has a unique stationary distribution. Let $V_k(p)$ denote the window size at the $k$th loss epoch (just after loss) and let $G'_{V_k(p)}$ denote the time between the $k$th and $(k + 1)$st loss epoch. The following corollary is a consequence of Proposition 2.

Corollary 1 For any $p \in (0, 1)$, the processes $\{W_n(p)\}, \{V_k(p)\}$ and $\{G'_{V_k(p)}\}$ have unique stationary distributions.

Proof For the process $\{W'_n, T_n\}$, consider the inter-visit times to state $(1, 0)$. These epochs are regeneration epochs for the process $\{W'_n, T_n\}$ as well as for the processes $\{W_n(p), \{V_k(p)\} and \{G'_{V_k(p)}\}$. From Proposition 2, $(1, 0)$ is positive recurrent. Therefore, the mean regeneration cycle length,$^1$ $\mathbb{E}[\tau_{1,0}(p)]$, for the $\{W'_n, T_n\}$ process is finite. Since $\mathbb{E}[\tau_{1,0}(p)]$ is also the mean regeneration cycle length for the $\{W_n(p)\}$ process, the $\{W_n(p)\}$ process has a unique stationary distribution. The regeneration cycle length for the processes $\{V_k(p)\}$ and $\{G'_{V_k(p)}\}$ (denoted by $\tau_V(p)$) is given by the number of loss epochs between two consecutive visits to state $(1, 0)$. Since, in each regeneration cycle, $\tau_V(p) \leq \tau_{1,0}(p)$, we get $\mathbb{E}[\tau_V(p)] < \infty$. Hence, the processes $\{V_k(p)\}$ and $\{G'_{V_k(p)}\}$ also have unique stationary distributions. $\Box$

In Proposition 3, we show that, for $p \in (0, 1)$, the TCP window size under stationarity has finite mean.

Proposition 3 For fixed $p \in (0, 1)$ the mean window size is finite, i.e., $\mathbb{E}[W(p)] < \infty$ under stationarity.

Proof Let us denote by $V_k(p)$ the window size at the $k$th congestion epoch, just after loss. For any RTT epoch, $n$, occurring between the $(k - 1)$st and $k$th loss epoch, $^1 \tau_{1,0}(p)$ is the number of RTTs between consecutive visits to state $(1, 0)$.
Consider the process \( \{\hat{W}_n(p)\} \), with \( \hat{W}_n(p) = \frac{V_k(p)}{1-\beta} \) for RTT epoch \( n \), occurring between the \((k-1)st\) and \( kth\) loss epoch. Then,

\[
\mathbb{E} \left[ \sum_{k=1}^{\tau_{1,0}(p)} W_k(p) \middle| W_0 = 1, T_0 = 0 \right] \leq \mathbb{E} \left[ \sum_{k=1}^{\tau_{1,0}(p)} \hat{W}_k(p) \middle| W_0 = 1, T_0 = 0 \right].
\]

Let \( \{T_k(p)\} \) be i.i.d. with distribution \( \mathbb{P}(T_k(p) = m) = (1-p)^{m-1} p, \) for \( m = 1, 2, \ldots \), and independent of the process \( \{W_n(p)\} \). We note that \( T_k(p) \) is stochastically larger than the number of RTTs between any two loss epochs. Hence,

\[
\mathbb{E} \left[ \sum_{k=1}^{\tau_{1,0}(p)} \hat{W}_k(p) \middle| W_0 = 1, T_0 = 0 \right] \leq \frac{1}{1-\beta} \mathbb{E} \left[ \sum_{k=1}^{\tau_V(p)} T_k(p)V_k(p) \middle| W_0 = 1, T_0 = 0 \right] = \frac{1}{1-\beta} \mathbb{E}[\tau_V(p)]\mathbb{E}[T_1(p)V(p)] = \frac{1}{1-\beta} \mathbb{E}[\tau_V(p)]\mathbb{E}[T_1(p)]\mathbb{E}[V(p)].
\]

The mean window size under stationarity is given by

\[
\mathbb{E}[W(p)] = \frac{\mathbb{E}[\sum_{k=1}^{\tau_{1,0}(p)} W_k(p) \middle| W_0 = 1, T_0 = 0]}{\mathbb{E}[\tau_{1,0}(p)]}.
\]

Since the state \((1, 0)\) (for the process \( \{W_n', T_n\} \)) is positive recurrent, \( \mathbb{E}[\tau_{1,0}(p)] < \infty \). Thus, to show \( \mathbb{E}[W(p)] < \infty \), it is sufficient to show that \( \mathbb{E}[V(p)] < \infty \) under stationarity.

Since \( \{V_n\} \) is a countable state space Markov chain, using the result for finiteness of stationary moments in [28], we have that \( \{V_n\} \) has finite mean if

\[
\sup_{i \in A} \mathbb{E}[V_1 | V_0 = i] < \infty,
\]

and there is a \( \delta > 0 \) such that

\[
\mathbb{E}[V_1 | V_0 = i] \leq (1-\delta)i,
\]

for all \( i \in A^c \), where \( A \) is a finite set.

Instead of showing that (11) and (12) holds for the pair \( \{V_0, V_1\} \), we will show that these equations hold for a pair of random variables \( \{\mathcal{V}_0, \mathcal{V}_1\} \) with \( \mathbb{E}[V_1 | V_0 = i] \leq \mathbb{E}[\mathcal{V}_1 | \mathcal{V}_0 = i] \). This will establish finiteness of expectation of the stochastic processes \( \{V_n\} \) and \( \{W_n\} \).

**Construction of \( \{\mathcal{V}_0, \mathcal{V}_1\} \)** Given the initial window size, \( V_0 \), the window size \( V_1 \) at the first congestion epoch (just after loss) depends on the time at which the first congestion happens. The time between two congestion epochs \( \tau(i) \) is a random variable which depends on the initial window size. The probability mass function for \( \tau(i) \) given that the initial window size is \( i \) is
\( P(\tau(i) = m) = (1 - p)^i (1 - p)^{i_1} \cdots (1 - p)^{i_{m-2}} (1 - (1 - p)^{i_{m-1}}), \)

where \( m > 0, i_k = C \left( Rk - \sqrt{\frac{\beta i}{C(1-\beta)}} \right)^3 + \frac{i}{1-\beta}, \) for \( k = 0, 1, \ldots, m - 1, \) and \( i_m \triangleq (1 - \beta)C \left( Rm - \sqrt{\frac{\beta i}{C(1-\beta)}} \right)^3 + i, \) the window size, just after loss, at the end of the \( m \)th RTT.

We now define a pair \( \{V_0, V_1\} \) with a stochastically larger inter-congestion epoch than \( \{V_0, V_1\}. \) Suppose \( V_0 = i, \) then let \( V_1 \) be

\[
V_1 = (1 - \beta)C \left( R\tau(i) - \sqrt{\frac{\beta i}{C(1-\beta)}} \right)^3 + i, 
\]

where the probability distribution function of \( \{\tau(i)\} \) is given by

\[
P(\tau(i) = m) = q_i^{m-1}(1 - q_i),
\]

with \( q_i = (1 - p)^i. \) The random variable \( \tau(i) \) is stochastically larger than \( \tau(i), \) i.e., \( P(\tau(i) > x) \geq P(\tau(i) > x) \) for all \( x \in \{0, 1, 2, 3, \ldots\}. \) As the inter-congestion epoch is stochastically larger, we have \( E[V_1|V_0 = i] \leq E[V_1|V_0 = i]. \) Thus it is sufficient to prove (11) and (12) for \( \{V_0, V_1\}. \)

For the pair \( \{V_0, V_1\}, \) we have

\[
E[V_1|V_0 = i] = (1 - \beta)C E[\tau(i)^3 R^3 - 3\tau(i)^2 R^2 K_i + 3\tau(i)RK_i^2 - K_i^3] + i, 
\]

(13)

where \( K_i = \sqrt[3]{\frac{\beta i}{C(1-\beta)}}. \) The random variable \( \tau(i) \) is geometric with parameter \( q_i \) and its moments are given by the following equations:

\[
E[\tau(i)] = \frac{1}{1 - q_i}, \quad E[\tau(i)^2] = \frac{1 + q_i}{(1 - q_i)^2}, 
\]

and

\[
E[\tau(i)^3] = \frac{1}{1 - q_i} + \frac{6q_i}{(1 - q_i)^3}. 
\]

The above moments are substituted into (13) to obtain

\[
E[V_1|V_0 = i] = (1 - \beta) \left( CR^3 \left( \frac{1}{1 - q_i} + \frac{6q_i}{(1 - q_i)^2} \right) 
- 3CR^2 \left( \frac{\beta i}{C(1-\beta)} \right)^{\frac{1}{3}} \frac{1 + q_i}{(1 - q_i)^2} 
+ 3CR \left( \frac{\beta i}{C(1-\beta)} \right)^{\frac{2}{3}} \frac{1}{1 - q_i} \right) - \beta i + i
\]
\[ \leq (1 - \beta) \left( C R^3 \left( \frac{1}{1 - q_i} + \frac{6q_i}{(1 - q_i)^2} \right) + 3CR \left( \frac{\beta i}{C(1 - \beta)} \right)^{\frac{2}{3}} \frac{1}{1 - q_i} \right) - \beta i + i. \]

As \( i \to \infty, q_i \to 0 \) if \( p < 1 \). Hence, we can choose an \( i^* \) such that, \( \forall i > i^* \),
\[ \frac{\beta i}{2} \geq C R^3 \left( \frac{1}{1 - q_i} + \frac{6q_i}{(1 - q_i)^2} \right) \quad \text{and} \quad \frac{\beta i}{2} \geq 3CR \left( \frac{\beta i}{C(1 - \beta)} \right)^{\frac{2}{3}} \frac{1}{1 - q_i}. \]
Thus we have, for all \( i > i^* \),
\[ \mathbb{E}[V_1 | V_0 = i] \leq (1 - \beta) \left( \frac{\beta i}{2} + \frac{\beta i}{2} \right) - \beta i + i \]
\[ = (1 - \beta^2)i. \]

Thus we have shown that (12) holds for all \( i > i^* \) for the pair \( \{V_0, V_1\} \). Also, since the random variable \( \tau(i) \) has finite moments, from (13) we see that (11) holds for \( A = \{i : i \leq i^*\} \) for \( \{V_0, V_1\} \). This shows that, under stationarity, \( \mathbb{E}[V(p)] < \infty \), which proves the finiteness of \( \mathbb{E}[W(p)] \).

\[ \square \]

5 Asymptotic approximations

We now derive an expression for average window size with random losses. In Sect. 5.1, we derive approximations for \( \{V_k(p)\} \) which will be used in Sect. 5.2 to obtain results for \( \{W_n(p)\} \) and throughput.

5.1 Asymptotic approximations for \( \{V_k(p)\} \)

Consider \( p^{\frac{1}{2}}G^{\frac{p}{x}} \left[ \frac{x}{p^{\frac{3}{4}}} \right] \), which denotes the product of \( p^{\frac{1}{2}} \) with the time for first packet loss when the initial window size is \( \left[ \frac{x}{p^{\frac{3}{4}}} \right] \). The choice of the parameters \( p^{\frac{1}{2}} \) and \( p^{\frac{3}{4}} \) is motivated by the deterministic loss model in Sect. 3 wherein the time between losses is inversely proportional to \( p^{\frac{1}{4}} \) and the time average window size is inversely proportional to \( p^{\frac{3}{4}} \). In Proposition 4, we show that the term \( p^{\frac{1}{2}}G^{\frac{p}{x}} \left[ \frac{x}{p^{\frac{3}{4}}} \right] \) converges to a random variable \( \overline{G}_x \) as \( p \to 0 \) for all \( x \geq 1 \).

\[ ^2 \left[ \cdot \right] \text{ denotes the floor() operation.} \]
Proposition 4 For $x > 0$, as $p \to 0$, $p^{\frac{1}{3}} G^p \left\lfloor \frac{x}{p^{\frac{1}{3}}} \right\rfloor$ converges in distribution to a random variable $\overline{G}_x$, with

$$
\mathbb{P}(\overline{G}_x \geq y) = \exp \left( -xy - \frac{CR^3 y^4}{4} + \sqrt{\frac{\beta x C^2}{(1 - \beta)}} y^3 R^2 - \left( \frac{\beta x C^{1/2}}{1 - \beta} \right)^{\frac{3}{2}} 3 R^2 y^2 \right).
$$

(14)

Also, for any finite $M$, if $x, y \leq M$ the above convergence is uniform in $x$ and $y$, i.e.,

$$
\lim_{p \to 0} \sup_{p^{\frac{1}{3}} \leq x, y \leq M} \left| \mathbb{P} \left( p^{\frac{1}{3}} G^p \left\lfloor \frac{x}{p^{\frac{1}{3}}} \right\rfloor \geq y \right) - \mathbb{P}(\overline{G}_x \geq y) \right| = 0.
$$

(15)

Proof We have

$$
\mathbb{P} \left( G^p \left\lfloor \frac{x}{p^{\frac{1}{3}}} \right\rfloor > \left\lfloor \frac{y}{p^{\frac{1}{3}}} \right\rfloor \right) \leq \mathbb{P} \left( p^{\frac{1}{3}} G^p \left\lfloor \frac{x}{p^{\frac{1}{3}}} \right\rfloor \geq y \right) \leq \mathbb{P} \left( G^p \left\lfloor \frac{x}{p^{\frac{1}{3}}} \right\rfloor \geq \frac{y}{p^{\frac{1}{3}}} \right),
$$

with $\mathbb{P} \left( G^p \left\lfloor \frac{x}{p^{\frac{1}{3}}} \right\rfloor > \left\lfloor \frac{y}{p^{\frac{1}{3}}} \right\rfloor \right) = (1 - p)^{x_0 + x_1 + \cdots + x_i}$, where $x_0 = \left\lfloor \frac{x}{p^{\frac{1}{3}}} \right\rfloor$ is the initial window size (immediately after a loss) and $x_i$ is the window size at the end of the $i$th RTT. Using (1), we get

$$
x_i = C(iR - K)^3 + \frac{x_0}{(1 - \beta)},
$$

with $K = \sqrt{\frac{\beta x_0}{(1 - \beta)C}}$. Let $m = \left\lfloor \frac{y}{p^{\frac{1}{3}}} \right\rfloor$. The term $C(iR - K)^3 + \frac{x_0}{(1 - \beta)} \in [x_i, x_i + 1]$. Therefore,

$$
\mathbb{P} \left( G^p \left\lfloor \frac{x}{p^{\frac{1}{3}}} \right\rfloor > \left\lfloor \frac{y}{p^{\frac{1}{3}}} \right\rfloor \right) \in (1 - p)^{x_0 + C(R - K)^3 + \frac{x_0}{1 - \beta} + \cdots + C(mR - K)^3 + \frac{x_0}{1 - \beta} + (m + 1) - (m + 1) - (mR - K)^3).$$

(16)

The terms on the RHS (right hand side) of Eq. (16) can be simplified as follows:

$$
(R - K)^3 + (2R - K)^3 + \cdots + (mR - K)^3 = R^3 \sum_{i=1}^{m} i^3 - 3R^2 K \sum_{i=1}^{m} i^2 + 3RK^2 \sum_{i=1}^{m} i - mK^3.
$$
Therefore,

\[
x_0 + \left( C(R - K)^3 + \frac{x_0}{1 - \beta} \right) + \cdots + \left( C(mR - K)^3 + \frac{x_0}{1 - \beta} \right) = x_0 + \frac{mx_0}{1 - \beta} + C \left( R^3 \sum_{i=1}^{m} i^3 - 3 R^2 K \sum_{i=1}^{m} i^2 + 3 R K^2 \sum_{i=1}^{m} i - m K^3 \right). \tag{17}
\]

After we expand the series \( \sum_{i=1}^{m} i \), \( \sum_{i=1}^{m} i^2 \) and \( \sum_{i=1}^{m} i^3 \), we see that the RHS of Eq. (17) has terms of the form \( K^n m^j \) with \( n + j \leq 4 \). Now,

\[
\lim_{p \to 0} K^n m^j = \lim_{p \to 0} \left( \frac{\beta x_0}{(1 - \beta) C} \right)^n \left( \frac{y}{p^{\frac{1}{4}}} \right)^j = \lim_{p \to 0} \left( \frac{\beta x}{(1 - \beta) C} \right)^{\frac{y}{4}} p^{-n-j} y^j, \tag{18}
\]

and

\[
\lim_{p \to 0} \left( x_0 + \frac{mx_0}{1 - \beta} \right) = \lim_{p \to 0} \frac{x}{p^{\frac{1}{4}}} + \frac{xy}{p(1 - \beta)}. 
\]

Therefore, for \( n + j < 4 \), \( \lim_{p \to 0} (1 - p)^{K^n m^j} = 1 \). Also, for the term \( (1 - p)^{m+1} \) in Eq. (16), we note that \( \lim_{p \to 0} (1 - p)^{m+1} = 1 \). Therefore, for the limit of Eq. (16) as \( p \to 0 \), we need to only consider terms of the form \( K^n m^j \) with \( n + j = 4 \) and the term \( \frac{xy}{1 - \beta} \). For these terms, we use \( \lim_{p \to 0} (1 - p)^{\frac{1}{2}} = \exp(-1) \) to get

\[
\mathbb{P}\left( G^p \left\lfloor \frac{x}{p^{\frac{3}{4}}} \right\rfloor \geq \left\lfloor \frac{y}{p^{\frac{1}{4}}} \right\rfloor \right) \to \exp\left( -xy - \frac{C R^3 y^4}{4} + \frac{\beta x C^2}{(1 - \beta)} y^2 R^2 - \left( \frac{\beta x C^{1/2}}{1 - \beta} \right)^{\frac{3}{2}} 3 R y^2 \right). \tag{19}
\]

Using similar steps as above, we can show that

\[
\mathbb{P}\left( G^p \left\lfloor \frac{x}{p^{\frac{3}{4}}} \right\rfloor \geq \left\lfloor \frac{y}{p^{\frac{1}{4}}} \right\rfloor \right) \to \exp\left( -xy - \frac{C R^3 y^4}{4} + \frac{\beta x C^2}{(1 - \beta)} y^2 R^2 - \left( \frac{\beta x C^{1/2}}{1 - \beta} \right)^{\frac{3}{2}} 3 R y^2 \right). 
\]

This proves convergence of \( p^{\frac{1}{2}} G^p \left\lfloor \frac{x}{p^{\frac{3}{4}}} \right\rfloor \) in distribution to \( \overline{G}_x \).
We now show uniform convergence of $\mathbb{P}\left(p^{\frac{1}{2}} G_p^{\frac{x}{p^\frac{1}{4}}} \geq y\right)$ to $\mathbb{P}(G_x \geq y)$. Taking logarithms on both sides of (16), we get

\[
\log \mathbb{P}\left(G_p^{\frac{x}{p^\frac{1}{4}}} > \left\lfloor \frac{y}{p^\frac{1}{4}} \right\rfloor \right) \\
\in \left[\left(x_0 + \left(C(R - K)^3 + \frac{x_0}{1 - \beta}\right) + \cdots + \left(C(mR - K)^3 + \frac{x_0}{1 - \beta}\right)\right) + (m + 1) \log(1 - p), \left(x_0 + \left(C(R - K)^3 + \frac{x_0}{1 - \beta}\right) + \cdots + \left(C(mR - K)^3 + \frac{x_0}{1 - \beta}\right)\right) + (m + 1) \log(1 - p)\right].
\]

(20)

Equation (20) has elements of the form $K^n m^j \log(1 - p)$ with $n + j \leq 4$, the term $(x_0 + \frac{x_0m}{1 - \beta}) \log(1 - p)$ and $(m + 1) \log(1 - p)$. The elements with $n + j = 4$ are the only terms that contribute to the limit, i.e., (19). From (18), the remaining elements in (20) are of the form $c(n, j)x^n y^j p^{-\frac{n+j}{4}} \log(1 - p)$ with $n + j < 4$ and $0 \leq n \leq 4$, with $c(n, j)$ being some finite coefficient. If $n + j < 4$, $p^{-\frac{n+j}{4}}$ is of the form $p^{\epsilon(n,j)}$ with $\epsilon(n, j) > -1$. These terms can be grouped together as $f(x, y, p)$, where $f$ has elements of the form $x^n y^j p^{-\frac{n+j}{4}} \log(1 - p)$ with $n + j < 4$ and $0 \leq n \leq 4$, the element $p^{-\frac{3}{4}} x \log(1 - p)$ and the element $\log(1 + p^{-\frac{3}{4}} y) \log(1 - p)$. Letting $T = \max\{1, M\}$, $x^n y^j \leq T^{\frac{n+j}{4}}$, for $x, y \leq M$. Therefore, for $x, y \leq M$, we have

\[
\left|\log \mathbb{P}\left(G_p^{\frac{x}{p^\frac{1}{4}}} > \left\lfloor \frac{y}{p^\frac{1}{4}} \right\rfloor \right) - \log \mathbb{P}(G_x \geq y)\right| \leq |f(x, y, p)|
\]

\[
\leq c_1 T^4 p^\epsilon \log(1 - p),
\]

where the term $T^4$ in the inequality comes from the element with the largest power for $x$ and $y$ in the RHS of (20), which is of the form $c y^4 x^0$, $\epsilon = \min_{n, j, n+j<4} \epsilon(n, j, k) > -1$ and $c_1$ is a constant independent of $p, x, y$. Therefore, we have

\[
\lim_{p \rightarrow 0} \sup_{\frac{3}{4} \leq x \leq M, y \leq M} \left|\log \mathbb{P}\left(G_p^{\frac{x}{p^\frac{1}{4}}} > \left\lfloor \frac{y}{p^\frac{1}{4}} \right\rfloor \right) - \log \mathbb{P}(G_x \geq y)\right| = 0.
\]
We can similarly prove
\[
\lim_{p \to 0} \sup_{p^3 \leq x \leq M, y \leq M} \log P \left( \frac{x}{p^\frac{3}{4}} \geq \left\lfloor \frac{y}{p^\frac{3}{4}} \right\rfloor \right) - \log P(G_x \geq y) = 0.
\]

The result (15) in Proposition 4 follows from the uniform continuity of the \( \exp() \) function on \((-\infty, 0)\).

We now derive a limiting result for the process \( \{V_k(p)\} \) embedded at the loss epochs of the TCP CUBIC window evolution process. The process \( \{V_k(p)\} \) is a Markov chain embedded within the window size process \( \{W_n(p)\} \). Let \( K(x) = \frac{3\beta x}{\sqrt{(1-\beta)}} \). If \( V_0(p) = \left\lfloor \frac{x}{p^\frac{3}{4}} \right\rfloor \), then \( V_1(p) \) is
\[
V_1(p) = (1 - \beta) \left( C \left( G^{\frac{p}{x}} \left\lfloor \frac{x}{p^\frac{3}{4}} \right\rfloor R - K \left( \left\lfloor \frac{x}{p^\frac{3}{4}} \right\rfloor \right) \right)^3 + \frac{1}{1 - \beta} \left\lfloor \frac{x}{p^\frac{3}{4}} \right\rfloor \right),
\]
where \( G^{\frac{p}{x}} \) denotes the time (in multiples of \( R \)) between consecutive losses when the window size immediately after the first of these losses was \( x \).

We now define a Markov chain which serves as the limit for the process \( \{V_n(p)\} \) with appropriate scaling. Define a Markov chain \( \{V_n\} \) as follows: Let \( V_0 \) be a random variable with an arbitrary initial distribution on \( \mathbb{R}^+ \). Define \( V_n \), for \( n \geq 1 \), as
\[
V_n = (1 - \beta) \left( C \left( \overline{G}_{V_{n-1}} R - K (\overline{V}_{n-1}) \right)^3 + \frac{\overline{V}_{n-1}}{1 - \beta} \right),
\]
where \( \overline{G}_{V_{n-1}} \) and \( \overline{V}_{n-1} \) are random variables with distribution given by (14) chosen independently of \( \{\overline{V}_k : k < n - 1\} \). The following proposition shows that the process \( \{V_n\} \) defined by (22) is the appropriate limit for the \( \{V_n(p)\} \) process as \( p \to 0 \).

**Proposition 5** Suppose \( \overline{V}_0 = x \) and \( V_0(p) = \left\lfloor \frac{x}{p^\frac{3}{4}} \right\rfloor \) for some \( x > 0 \) and all \( p > 0 \). Then we have
\[
\lim_{p \to 0} \sup_{x > p^\frac{3}{4}} \left| P_x \left( \left\lfloor \frac{3}{4} V_1(p) \leq a_1, \left\lfloor \frac{3}{4} V_2(p) \leq a_2, \ldots, \left\lfloor \frac{3}{4} V_n(p) \leq a_n \right\rfloor \right) \right| = 0,
\]
where \( a_i \in \mathbb{R}^+ \) for \( i = 1, 2, \ldots, n \) and \( P_x \) denotes the law of the processes when \( \overline{V}_0 = x \) and \( V_0(p) = \left\lfloor \frac{x}{p^\frac{3}{4}} \right\rfloor \).
Proof We prove (23) for \( n = 1, 2 \); the proof for \( n > 2 \) follows by induction.

For \( n = 1 \),

\[
\lim_{p \to 0} \Pr_x(p^{\frac{3}{2}} V_1(p) \leq a_1) = \lim_{p \to 0} \Pr \left( p^{\frac{1}{2}} (1 - \beta) C \left( R G^p \left[ \frac{x}{p^{\frac{1}{2}}} \right] - K \left( \left\lfloor \frac{x}{p^{\frac{1}{2}}} \right\rfloor \right) \right)^3 + x \leq a_1 \right) \\
= \lim_{p \to 0} \Pr \left( p^{\frac{1}{2}} G^p \left[ \frac{x}{p^{\frac{1}{2}}} \right] \leq \frac{(a_1 - x)}{c(1 - \beta)^{\frac{1}{3}}} + K(x) \right) \\
= \Pr \left( \overline{G}_x \leq \frac{(a_1 - x)}{c(1 - \beta)^{\frac{1}{3}}} + K(x) \right) \\
= \Pr_x (\overline{V}_1 \leq a_1).
\]

From Eq. (15) in Proposition 4, the convergence is uniform in \( x \) over any bounded interval. Also, from (21) and (22), for \( x > \frac{a_1}{1 - \beta} \) we have \( \Pr_x(p^{\frac{3}{2}} V_1(p) \leq a_1) = \Pr_x(\overline{V}_1 \leq a_1) = 0 \). Therefore,

\[
\lim_{p \to 0} \sup_{x \geq p^{\frac{3}{2}}} \left| \Pr_x \left( p^{\frac{3}{2}} V_1(p) \leq a_1 \right) - \Pr_x \left( \overline{V}_1 \leq a_1 \right) \right| = 0.
\]

This proves (23) for \( n = 1 \).

We now prove the result for \( n = 2 \). Consider

\[
\Pr_x(p^{\frac{3}{2}} V_1(p) \leq a_1, p^{\frac{3}{2}} V_2(p) \leq a_2) \\
= \int_{0}^{a_1} \Pr \left( \left( 1 - \beta \right) \left( R p^{\frac{1}{2}} G^p \left[ \frac{y}{p^{\frac{1}{2}}} \right] - K(y) \right)^3 + y \leq a_2 \right) \Pr_x(p^{\frac{3}{2}} V_1(p) \in dy).
\]

From Eq. (15) in Proposition 4, the term \( \Pr \left( C (1 - \beta)(R p^{\frac{1}{2}} G^p \left[ \frac{y}{p^{\frac{1}{2}}} \right] - K(y))^3 + y \leq a_2 \right) \) converges to \( \Pr(C(1 - \beta)(R p^{\frac{1}{2}} \overline{G}_x - K(y))^3 + y \leq a_2) \) uniformly in \( y \) for \( y < a_1 \). Therefore, for any given \( \epsilon > 0 \) there exists a \( p^* \) such that, for \( p < p^* \),

\[
\left| \Pr_x(p^{\frac{3}{2}} V_1(p) \leq a_1, p^{\frac{3}{2}} V_2(p) \leq a_2) \\
- \int_{0}^{a_1} \Pr \left( \left( 1 - \beta \right) \left( R p^{\frac{1}{2}} G_y - K(y) \right)^3 + y \leq a_2 \right) \Pr_x(p^{\frac{3}{2}} V_1(p) \in dy) \right| \leq \epsilon.
\]
Now,

\[
\int_0^{a_1} \mathbb{P} \left( C(1 - \beta) \left( Rp^{\frac{1}{3}} G - K(y) \right)^3 + y \leq a_2 \right) \mathbb{P}_x(p^{\frac{3}{4}} V_1(p) \in dy)
\]

\[
= \int_0^\infty \mathbb{P} \left( C(1 - \beta) \left( Rp^{\frac{1}{3}} G - K(y) \right)^3 + y \leq a_2 \right) \mathbb{I}_{\left\{ y \leq a_1 \right\}} \mathbb{P}_x(p^{\frac{3}{4}} V_1(p) \in dy)
\]

\[
= \mathbb{E}_x[g(p^{\frac{3}{4}} V_1(p))],
\]

where the function \( g(y) = \mathbb{P} \left( C(1 - \beta) \left( Rp^{\frac{1}{3}} G - K(y) \right)^3 + y \leq a_2 \right) \mathbb{I}_{\left\{ y \leq a_1 \right\}}. \) For any continuous function \( f \) on \( \mathbb{R}^+ \) with compact support, using Proposition A.1 from the Appendix, we have

\[
\lim_{p \to 0} \sup_{x \geq p^{\frac{3}{4}}} \left| \mathbb{E}_x \left[ f(p^{\frac{3}{4}} V_1(p)) \right] - \mathbb{E}_x \left[ f(V_1) \right] \right| = 0. \tag{24}
\]

The function \( g \) is continuous with compact support. Therefore, using (24), we get

\[
\lim_{p \to 0} \sup_{x \geq p^{\frac{3}{4}}} \left| \mathbb{P}_x \left( p^{\frac{3}{4}} V_1(p) \leq a_1, p^{\frac{3}{4}} V_2(p) \leq a_2 \right) - \mathbb{P}_x(\mathbb{V}_1 \leq a_1, \mathbb{V}_2 \leq a_2) \right| = 0. \]

The proof of (23) for \( n > 2 \) can be done using induction, as follows:

\[
\mathbb{P}_x(p^{\frac{3}{4}} V_1(p) \leq a_1, p^{\frac{3}{4}} V_2(p) \leq a_2, \ldots, p^{\frac{3}{4}} V_{n+1}(p) \leq a_{n+1})
\]

\[
= \int_0^{a_1} \mathbb{P}_y(p^{\frac{3}{4}} V_1(p) \leq a_2, \ldots, p^{\frac{3}{4}} V_n(p) \leq a_{n+1}) \mathbb{P}_x(p^{\frac{1}{3}} V_1(p) \in dy).
\]

Assuming the result holds for \( n \), we have

\[
\left| \mathbb{P}_x(p^{\frac{3}{4}} V_1(p) \leq a_1, p^{\frac{3}{4}} V_2(p) \leq a_2, \ldots, p^{\frac{3}{4}} V_{n+1}(p) \leq a_{n+1}) - \mathbb{E}_x[g_n(p^{\frac{3}{4}} V_1(p))] \right|
\]

\[
\leq \epsilon,
\]

where the function \( g_n(y) = \mathbb{P}_y(\mathbb{V}_1 \leq a_2, \ldots, \mathbb{V}_n \leq a_{n+1}) \mathbb{I}_{\left\{ y \leq a_1 \right\}}. \) The function \( g_n(\cdot) \) is continuous by the induction hypothesis. Using Proposition A.1 from the Appendix gives us the desired result. \( \square \)

Since the finite dimensional distributions of \( \{p^{\frac{3}{4}} V_n(p)\} \) converge to \( \{\mathbb{V}_n\} \), we have:

**Corollary 2** If \( \lim_{p \to 0} p^{\frac{3}{4}} V_0(p) \) converges in distribution to \( \mathbb{V}_0 \), then the Markov chain \( \{p^{\frac{3}{4}} V_n(p)\} \) converges in distribution to the Markov chain \( \{\mathbb{V}_n\} \).
Proof Let \( \hat{\pi}_p \) and \( \hat{\pi} \) be the initial distributions for the processes \( \{ p^{3/2} V_n(p) \} \) and \( \{ \overline{V}_n \} \), respectively. Also, let \( \hat{\pi}_p \) converge weakly to \( \hat{\pi} \). We have

\[
\left| P_{\hat{\pi}_p}(p^{3/2} V_1(p) \leq a_1) - P_{\hat{\pi}}(\overline{V}_1 \leq a_1) \right| \\
= \left| \int_x P_x(p^{3/2} V_1(p) \leq a_1) \hat{\pi}_p(dx) - P_x(\overline{V}_1 \leq a_1) \hat{\pi}(dx) \right| \\
\leq \int_x \left| P_x(p^{3/2} V_1(p) \leq a_1) - P_x(\overline{V}_1 \leq a_1) \right| \hat{\pi}_p(dx) \\
+ \int_x \left( P_x(\overline{V}_1 \leq a_1) \hat{\pi}_p(dx) - P_x(\overline{V}_1 \leq a_1) \hat{\pi}(dx) \right).
\]

From Proposition 5, for any \( \epsilon > 0 \), we can choose a \( p^* \) such that, for all \( p < p^*, \)
\[
\left| P_x(p^{3/2} V_1(p) \leq a_1) - P_x(\overline{V}_1 \leq a_1) \right| \leq \epsilon, \text{ for all } x \geq p^{3/2}. \]
Also, since \( P_x(\overline{V}_1 \leq a_1) \) is a continuous, bounded function in \( x \), we have

\[
\lim_{p \to 0} \left| \int_x P_x(\overline{V}_1 \leq a_1) \hat{\pi}_p(dx) - \int_x P_x(\overline{V}_1 \leq a_1) \hat{\pi}(dx) \right| = 0.
\]

This proves the result in Corollary 2 for \( n = 1 \). The proof for \( n \geq 2 \) follows easily by induction. \( \square \)

We now prove that the limiting Markov chain \( \{ \overline{V}_n \} \) has a unique invariant distribution. For proving that, the given proof requires that (some of) the moments of \( \overline{G}_x \) be uniformly bounded in \( x \), which follows from the following lemma.

**Lemma 1** There exists \( \zeta > 0 \) such that, for all \( t \in (-\zeta, \zeta) \), we have

\[
\sup_{x > 0} \mathbb{E}[e^{t \overline{G}_x}] < \infty.
\]

**Proof** Consider \( H(y) = \sup_{x > 0} P(\overline{G}_x \geq y) \). The function \( H(y) \) upper bounds \( P(\overline{G}_x \geq y) \) for all \( x > 0 \). From (14),

\[
H(y) = \exp \left( -\inf_{x > 0} \left( xy + \frac{CR^3 y^4}{4} - \frac{1}{3} \frac{\beta x C^2}{1 - \beta} y^3 R^2 + \left( \frac{\beta x C^{1/2}}{1 - \beta} \right)^{3/2} \frac{3 R^2 y^2}{2} \right) \right).
\]

Let

\[
f(x, y) = xy + \frac{CR^3 y^4}{4} - \frac{1}{3} \frac{\beta x C^2}{1 - \beta} y^3 R^2 + \left( \frac{\beta x C^{1/2}}{1 - \beta} \right)^{3/2} \frac{3 R^2 y^2}{2}.
\]

Substituting \( x = t^3 \), we get

\[
f(t^3, y) = t^3 y + \frac{CR^3 y^4}{4} - \frac{1}{3} \frac{\beta C^2}{(1 - \beta)} y^3 R^2 t + \left( \frac{\beta C^{1/2}}{1 - \beta} \right)^{3/2} \frac{3 R^2 y^2}{2} t^2.
\]
Now,
\[ \frac{\partial f(t^3, y)}{\partial t} = 3t^2y + 3 \left( \frac{\beta C^{1/2}}{1 - \beta} \right)^{\frac{2}{3}} Ry^2t - \sqrt[3]{\frac{\beta C^2}{(1 - \beta)}} y^3 R^2, \] (26)
and
\[ \frac{\partial^2 f(t^3, y)}{\partial t^2} = 6ty + 3 \left( \frac{\beta C^{1/2}}{1 - \beta} \right)^{\frac{2}{3}} Ry^2. \] (27)

Consider \( f(t^3, y) \) at some fixed \( y > 0 \). From (26) and (27), the function \( f(t^3, y) \) has two stationary points, one of which is a local minimum and the other a local maximum. Let us denote the local minimum by \( t_{\text{min}}(y) \) and the local maximum by \( t_{\text{max}}(y) \). We have \( t_{\text{min}}(y) > 0 \) and \( t_{\text{max}}(y) < 0 \). Also as \( t \to -\infty \), \( f(t^3, y) \to -\infty \), and as \( t \to \infty \), \( f(t^3, y) \to \infty \). Thus, over \( t > 0 \), the function \( f(t^3, y) \) has a unique global minimum. Hence, there exists a unique \( x > 0 \) (corresponding to the local minimum of \( f(t^3, y), t_{\text{min}}(y) \)) which attains the infimum in Eq. (25). Let \( x^*(y) \) denote the \( x \) which attains the infimum in Eq. (25). The minimum \( x^*(y) \) is given by
\[ x^*(y) = \frac{R^3y^3}{8} \left[ - \left( \frac{\beta C^{0.5}}{1 - \beta} \right)^{\frac{2}{3}} + \sqrt{\left( \frac{\beta C^{0.5}}{1 - \beta} \right)^{\frac{4}{3}} + \frac{4}{3} \left( \frac{\beta C^2}{1 - \beta} \right)^{\frac{1}{3}}} \right]^3. \] (28)

Substituting (28) in (25) gives us
\[ H(y) = e^{-\gamma(C, \beta) R^3 y^4}, \]
where \( \gamma(C, \beta) \) is a constant dependent on \( C \) and \( \beta \). In Fig. 3, we illustrate \( H(y) \) and \( \mathbb{P}(\overline{C}_x \geq y) \) for \( x = 0, 0.1, 1 \) (with \( C = 0.4, \beta = 0.3 \) and RTT, \( R = 1 \)). For the version of TCP CUBIC we consider [29], we have \( C = 0.4 \) and \( \beta = 0.3 \). For these values, \( \gamma(C, \beta) = 0.0510 > 0 \). (In fact, numerically evaluating \( \gamma(C, \beta) \) we find that for \( C = 0.4 \), for all \( \beta \in (0, 1) \), \( \gamma(C, \beta) > 0 \).) The function \( H(y) \) is a complementary cumulative distribution function and is decreasing super-exponentially in \( y \). Therefore, the moment generating function (MGF) corresponding to \( H(y) \) is bounded in a neighborhood of 0. Since \( H(y) \) bounds \( \mathbb{P}(\overline{C}_x \geq y) \) for all \( x \), we have
\[ \sup_{x > 0} \mathbb{E}[e^{t\overline{C}_x}] < \infty, \] (29)
for \( t \) in some neighborhood of 0.

**Proposition 6** The Markov chain \( \{\overline{V}_n\} \) is Harris recurrent and has a unique invariant distribution.

**Proof** We first prove that the Markov chain \( \{\overline{V}_n\} \) is Harris irreducible w.r.t. the Lebesgue measure on \( \mathbb{R}^+ \). To prove this, consider a point \( x \) in the state space of \( \{\overline{V}_n\} \). Let \( L(x, A) \) denote the probability of \( \{\overline{V}_n\} \) hitting set \( A \) in a finite time starting
with $V_0 = x$. From Eq. (22), $\mathbb{P}(x, (x(1-\beta), \infty)) = 1$. The distribution of $G_x$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^+$. Therefore, for any set $A$ with nonzero Lebesgue measure such that $A \subseteq ((1-\beta)x, \infty)$, $\mathbb{P}(x, A) > 0$. Hence, $L(x, A) > 0$. Also, $\mathbb{P}(x, (x(1-\beta), x)) > 0$. Therefore, for any set $B \subseteq (0, (1-\beta)x)$ with nonzero Lebesgue measure, there exists $n$ such that $\mathbb{P}^n(x, B) > 0$. Therefore, for any set $C$ with nonzero Lebesgue measure, $L(x, C) > 0$. Thus the Markov chain $\{V_n\}$ is Harris irreducible w.r.t. the Lebesgue measure.

To show the positive recurrence of the Markov chain, we use a result from [30, p. 116]. In our setup, it is sufficient to prove the following results: There exists a $x^*$ such that

1. $\mathbb{E}[\bar{V}_{n+1} - \bar{V}_n | \bar{V}_{n+1} = x] \leq -\epsilon$, for all $x > x^*$ for some $\epsilon > 0$.
2. $\mathbb{E}[\bar{V}_{n+1} | \bar{V}_{n} = x] < \infty$, for all $x \leq x^*$.

From Eq. (22),

$$\mathbb{E}[\bar{V}_{n+1} - \bar{V}_n | \bar{V}_{n} = x] = \mathbb{E} \left[ (1-\beta)C(G_x R - K(x))^3 \right]$$

$$\leq (1-\beta)C(\mathbb{E}[G_x^3] R^3 + \mathbb{E}[G_x] R K(x)^2 - K(x)^3)$$

$$\leq (1-\beta)C(\sup_{x > 0} (\mathbb{E}[G_x^3]) R^3 + \sup_{x > 0} (\mathbb{E}[G_x]) R K(x)^2$$

$$- K(x)^3). \quad (30)$$

Using Taylor’s series expansion, we have

$$\frac{t^4 \tilde{G}_x^4}{4!} \leq e^{t\tilde{G}_x} + e^{-t\tilde{G}_x}. \quad (31)$$
From Lemma 1, \( \sup_{x>0} \mathbb{E}[e^{tG_x}] < \infty \) for \( t \in (-t_0, t_0) \) for some \( t_0 > 0 \). Therefore, using (31), \( \sup_{x>0}\mathbb{E}[G_x^4] \) is finite. We have \( G_x^3 \leq G_x + 1 \) and \( G_x \leq G_x + 1 \). Hence, the terms \( \sup_{x>0}\mathbb{E}[G_x^3] \) and \( \sup_{x>0}\mathbb{E}[G_x^4] \) are finite. Therefore, the RHS of the last inequality in (30) has terms \( x \) and \( x^2 \). Since the \( x \) term dominates \( x^2 \) for large \( x \) and has a negative coefficient, we can find an \( x^* \) and an \( \epsilon > 0 \) such that, for all \( x > x^* \),

\[
\mathbb{E}[\mathbb{V}_{n+1} - \mathbb{V}_n | \mathbb{V}_{n+1} = x] \leq -\epsilon.
\]  

(32)

Also, for \( x < x^* \), we have

\[
\mathbb{E}[\mathbb{V}_{n+1} | \mathbb{V}_n = x] = \mathbb{E}\left[ (1 - \beta) \left( C (G_x R - K(x))^3 \right) \right] + x
\]

\[
\leq (1 - \beta)C \left( \mathbb{E}[G_x^3] R^3 + \mathbb{E}[G_x^4] R K(x)^2 \right) + x^*
\]

\[
\leq (1 - \beta)C \left( \sup_x(\mathbb{E}[G_x^3]) R^3 + \sup_x(\mathbb{E}[G_x^4]) R K(x^*)^2 \right) + x^*
\]

< \infty.

(33)

The last inequality comes from Lemma 1. From Eqs. (32) and (33), we see that the Markov chain \( \{\mathbb{V}_n\} \) is Harris recurrent and has a unique invariant distribution. \( \square \)

5.2 Asymptotic approximations for throughput

Let us denote the average time between losses, under stationarity, by \( \mathbb{E}[G_{V(\infty)}^p] \). The average number of packets sent between two consecutive losses is \( p^{-1} \). Therefore, by Palm calculus [27], the average window size, \( \mathbb{E}[W(p)] \), under stationarity, is

\[
\mathbb{E}[W(p)] = \frac{1}{p\mathbb{E}[G_{V(\infty)}^p]}.
\]

From Proposition 3, \( \mathbb{E}[W(p)] \) is finite.

Let \( \overline{G}_{\overline{V}_\infty} \) denote the random variable with same distribution as the stationary distribution of \( \overline{G}_{\overline{V}_n} \) (existence of which is proved in Proposition 6). Then, from Proposition 4, we expect \( \mathbb{E}[p^{\frac{3}{4}} G_{V(\infty)}^p] \) to be close to \( \mathbb{E}[\overline{G}_{\overline{V}_\infty}] \) for small \( p \), and hence

\[
\mathbb{E}[W(p)] \approx \frac{p^{-\frac{3}{4}}}{\mathbb{E}[\overline{G}_{\overline{V}_\infty}]}.
\]

for small \( p \).

We can evaluate \( \mathbb{E}[\overline{G}_{\overline{V}_\infty}] \) using Monte Carlo simulations. In Fig. 4, we illustrate simulations for evaluating \( \sum_{i=1}^{n} \frac{\overline{G}_{\overline{V}_i}}{n} \) with initial conditions \( \overline{V}_0 = 0.0, 0.1, 2.0 \) for the scaled Markov chain \( \{\overline{V}_n\} \) with parameters, \( C = 0.4, \beta = 0.3 \) (used in [26]) and with RTT \( R = 1 \) s. We see that in these cases, after \( n > 250 \), there is little change in
\[ n^{-1} \sum_{i=1}^{n} G_{V_i}. \] For the TCP CUBIC parameters as given above with RTT \( R = 1 \), and using \( n = 10,000 \), we get \( \mathbb{E}[G_{V_i}] \approx 0.7690. \)

Let \( \overline{C}(R) \) denote our approximation for \( \frac{1}{\mathbb{E}[G_{V_i}]} \) obtained from the Monte Carlo simulation when RTT of flow is \( R \). Thus, the mean window size for a flow with RTT \( R \) is given by

\[ \mathbb{E}[W(p)] \approx p^{-\frac{3}{4}} \overline{C}(R). \quad (34) \]

From (6) in Sect. 3, the mean window size for the fluid model is directly proportional to \( R^{\frac{3}{4}} \). We now check whether this holds for the approximation described in this section.

We have computed \( \overline{C}(1) = 1.3004 \). In Fig. 5, we plot a scatter plot of \( 1.3004R^{\frac{3}{4}} \) against \( \overline{C}(R) \), where \( \overline{C}(R) \) is obtained from a Monte Carlo simulation for a flow with RTT = \( R \). We have chosen twenty equally spaced RTT values in \((0.1, 2)\) s. The difference between \( 1.3004R^{\frac{3}{4}} \) and \( \overline{C}(R) \) is less than 1%. Thus, \( \mathbb{E}[W(p)] \) for our Markov chain approximation is also (approximately) directly proportional to \( R^{\frac{3}{4}} \). Therefore, we can approximate the average window size for a TCP CUBIC flow with RTT \( R \) as \( \mathbb{E}[W(p)] = 1.3004 \left( \frac{R}{p} \right)^{\frac{3}{4}} \). To account for the TCP Reno mode of operation which we ignored in our analysis described before, we make the following approximation:

\[ \mathbb{E}[W(p)] = \max \left\{ 1.3004 \left( \frac{R}{p} \right)^{\frac{3}{4}}, 1.31 \sqrt{\frac{1}{p}} \right\}, \quad (35) \]

where the second term on the RHS approximates the TCP Reno average window size as given in [10]. For the same parameters, the deterministic periodic loss model
Scatter plot data points
\[ y = x \]
R goes from 0.1 sec to 2 sec in steps of 0.1 sec

Fig. 5 Scatter plot of 1.3004\( R^{\frac{3}{4}} \) v/s \( C(R) \)

gives us \( E[W(p)] = \max \left\{ 1.0538 \left( \frac{R}{p} \right)^{\frac{3}{4}}, \frac{1.31}{\sqrt{p}} \right\} \). Thus, in the CUBIC mode, these two approximations differ by more than 18%. In Sect. 7, we see that (35) matches better with ns2 simulations than fluid approximation. For the Reno mode, for both the models we use the same approximation.

In this section, we derived an approximation for throughput for a single TCP CUBIC connection with negligible queuing. In the next section, we describe a simple approximation to extend the results to a more general setup with multiple connections and non-negligible queuing.

6 Extension to multiple TCP connections

The expression (35) was obtained assuming the RTT to be constant, i.e., we assumed that the queuing was negligible. However, when multiple TCP connections go through a link, the queuing may be non-negligible. We can approximate the average window size for TCP CUBIC in this case by replacing \( R \) with \( E[R] \) in (35).

We briefly describe a technique to compute the average RTT of the TCP flows, assuming the bottleneck queue to be a M/G/1 queue. This technique helps us extend our approximation to the case with multiple TCP flows with time-varying RTT (due to queuing).

Here we assume only one bottleneck queue; however, the technique can be easily extended to multiple bottleneck queues [31]. Let \( C \) denote the link speed of the bottleneck link (in bps), let \( \lambda_i \) denote the throughput of flow \( i \) (in packets/s) and let \( \Delta_i \) denote its propagation delay. Let \( E[s_i] \) and \( E[s_i^2] \) denote the mean and second moment of packet lengths of connection \( i \), respectively. Then using the M/G/1 approximation [32],
the RTT, of TCP connection \( i \) is given by

\[
E[R_i] = \frac{\lambda E[s^2]}{2C^2(1 - \rho)} + \frac{E[s]}{C} + \Delta_i,
\]

where

\[
E[s] = \sum_i \frac{\lambda_i}{\lambda} E[s_i], \quad E[s^2] = \sum_i \frac{\lambda_i}{\lambda} E[s_i^2],
\]

\[
\rho = \frac{\lambda E[s]}{C}
\]

and \( \lambda \) is the overall TCP throughput. Also, from Little’s law,

\[
\lambda_i = \frac{(1 - p_i)E[W_i]}{R_i}.
\]

We replace \( R_i \) by \( E[R_i] \) in (35) to compute \( E[W_i] \). The throughput \( \lambda_i \) can then be computed by solving (36) and (37).

For illustration, we consider an example with 15 flows sharing a single bottleneck link of speed 100 Mbps. The packet error rates for five of these flows are 0.01, it is 0.001 for another group of five flows, and for the rest it is set to 0.0001. The propagation delays for the five connections in each group are set to 0.05, 0.1, 0.2, 0.25, and 0.5 s, respectively. The packet sizes were set to 1050 bytes. The average throughput of these flows is computed using the M/G/1 approximation described above.

In Fig. 6, we plot a scatter plot comparing the throughput obtained using the M/G/1 approximation with ns2 simulation. Each point in the plot compares the throughput obtained by a flow as obtained by ns2 simulation with the corresponding approximation obtained from the M/G/1 approximation model. Thus, for our example, we have a total of 15 points corresponding to the 15 flows. We do not show the packet error rate \( p_i \) and propagation delay \( \Delta_i \) corresponding to each flow in the plot to avoid clutter. The model approximations and simulations differ by \(< 5\%\) for most cases, and the maximum difference is 17%.

### 7 Simulation results

We consider a single TCP CUBIC connection with negligible queueing. The link speeds are set to 10 Gbps so that the queueing is negligible. The packet sizes are 1050 bytes, which is the default value in ns2. The maximum window size is set to 40,000. In Tables 1 and 2, we compare the asymptotic approximation of Sect. 5 against the fluid approximate model in [26], our earlier Markov model in [15] and ns2 simulations. Table 1 compares the average window size, whereas Table 2 compares the goodput. In the Tables, we list the results for different values of RTT, \( R \), and PER, \( p \). We see that, unlike TCP Reno, the average window size for TCP CUBIC depends on the RTT of the flow and increases with RTT. This behavior makes TCP CUBIC fairer to flows with larger RTT as compared to TCP Reno and also leads to TCP CUBIC being more efficient over large-delay networks.
Throughput in packets/sec (NS2)

Throughput in pkts/sec (MG1—approximation)

Scatterplot comparing model throughput with simulations

\((\lambda_{\text{ns2}}, \lambda_{\text{model}})\)

Fig. 6 Scatter plot for TCP CUBIC throughput (15 flows, bottleneck link speed: 100 Mbps)

For the deterministic loss model, we use

\[
E[W(p)] = \max\{1.0538 \left(\frac{R}{p}\right)^{\frac{3}{4}}, 1.31\sqrt{p}\}
\]

so as to account for the Reno mode of operation. If the RTT is small (< 0.1 s), TCP CUBIC operates more like Reno. In such cases, the deterministic loss model has accuracy similar to the Markovian models. However, when RTT is large (> 0.1 s) the Markovian models are better (sometimes much better, especially for \(R = 1\) s) than the deterministic periodic loss model. The Markov model in [15] explicitly considers the TCP Reno mode behavior, while here we just use a simple approximation to account for the TCP Reno mode behavior. However, in spite of this we see that the Markov model in [15] performs only marginally better than the current Markovian approximation that we use in this paper. When compared against ns2 simulations, the Markov model in [15] typically has errors < 4%, whereas for the current approximation given in this paper, the errors are < 5% for most cases.

In Tables 1 and 2, we mark an entry in bold when the CUBIC mode approximation is used. In the CUBIC mode of operation, the fluid model performs relatively poorly and has errors > 13.5%. We note that while we assumed \(W_{\text{max}}\) to be infinite for the asymptotic model, our asymptotic model results match well with the Markov chain models of [15] where the maximum window size was finite and only moderately large (2500 packets).

We note that the TCP CUBIC average window size for \(p = 0.01\) is close to the TCP Reno average window size which is 13.1 packets (computed using \(1.31\sqrt{p}\) [10]) when RTT is less than 0.2 s. However, the TCP CUBIC average window size increases with RTT and is close to 40 packets when RTT is 1 s, which is three times better than the average window size for TCP Reno. This behavior makes TCP CUBIC more efficient on large BDP networks.

Using the approach described in Sect. 4, we can compute expressions for average window size for different TCP CUBIC parameters to study the effect of these param-
Table 1  Average window size via different approximations

| Per RTT | Simulations (ns2) | Det. Fluid [26] | Markov chain [15] | Approx. Markov max \( \max \left\{ 1.3004 \left( \frac{R}{p} \right)^{\frac{3}{4}}, \frac{1.31}{\sqrt{p}} \right\} \) |
|---------|------------------|-----------------|-------------------|-------------------------------------------------|
| \( 1 \times 10^{-2} \) | 1 39.97 | 33.33 | 37.44 | 41.19 |
| \( 1 \times 10^{-2} \) | 0.2 14.3 | 13.10 | 13.53 | 13.10 |
| \( 1 \times 10^{-2} \) | 0.1 12.62 | 13.10 | 12.50 | 13.10 |
| \( 1 \times 10^{-2} \) | 0.02 12.08 | 13.10 | 12.41 | 13.10 |
| \( 1 \times 10^{-2} \) | 0.01 11.53 | 13.10 | 12.41 | 13.10 |
| \( 5 \times 10^{-3} \) | 1 69.46 | 56.05 | 63.78 | 69.27 |
| \( 5 \times 10^{-3} \) | 0.2 21.82 | 18.53 | 21.02 | 20.81 |
| \( 5 \times 10^{-3} \) | 0.1 18.29 | 18.53 | 18.09 | 18.53 |
| \( 5 \times 10^{-3} \) | 0.02 17.21 | 18.53 | 17.73 | 18.53 |
| \( 5 \times 10^{-3} \) | 0.01 16.58 | 18.53 | 17.73 | 18.53 |
| \( 1 \times 10^{-3} \) | 1 229.96 | 187.40 | 218.32 | 231.63 |
| \( 1 \times 10^{-3} \) | 0.2 67.83 | 56.05 | 67.92 | 69.58 |
| \( 1 \times 10^{-3} \) | 0.1 44.68 | 41.43 | 44.55 | 41.43 |
| \( 1 \times 10^{-3} \) | 0.02 39.40 | 41.43 | 39.94 | 41.43 |
| \( 1 \times 10^{-3} \) | 0.01 38.71 | 41.43 | 39.94 | 41.43 |
| \( 5 \times 10^{-4} \) | 1 384.43 | 315.17 | 370.12 | 388.56 |
| \( 5 \times 10^{-4} \) | 0.2 113.05 | 94.26 | 114.52 | 117.02 |
| \( 5 \times 10^{-4} \) | 0.1 69.12 | 58.58 | 70.05 | 69.24 |
| \( 5 \times 10^{-4} \) | 0.02 55.89 | 58.58 | 56.66 | 58.58 |
| \( 5 \times 10^{-4} \) | 0.01 55.38 | 58.58 | 56.66 | 58.58 |
| \( 8 \times 10^{-5} \) | 1 1507.19 | 1245.81 | 1487.19 | 1539.87 |
| \( 8 \times 10^{-5} \) | 0.2 430.49 | 372.58 | 454.41 | 462.57 |
| \( 8 \times 10^{-5} \) | 0.1 260.91 | 221.54 | 271.15 | 273.69 |
| \( 8 \times 10^{-5} \) | 0.02 143.99 | 146.46 | 143.42 | 146.46 |
| \( 8 \times 10^{-5} \) | 0.01 140.83 | 146.46 | 142.71 | 146.46 |

eters on TCP performance. In Table 3, we compare the results for \( \beta = 0.2^3 \), which is used by an older version of TCP CUBIC which is also widely used [4]. For this parameter setting, we get

\[
\mathbb{E}[W(p)] = \max \left\{ 1.54 \left( \frac{R}{p} \right)^{\frac{3}{4}}, \frac{1.31}{\sqrt{p}} \right\} . \tag{38}
\]

\(^3\) In the rest of the paper, we have used \( \beta = 0.3 \), which is used by the current version of TCP CUBIC [29].
\begin{table}
\centering
\begin{tabular}{cccccc}
\hline
Per RTT & $\lambda$ & $\lambda$ & $\lambda$ & $\lambda$ & $\lambda$ \\
\hline
\$1 \times 10^{-2}$ & 1 & 39.55 & 32.99 & 37.06 & 40.78 \\
\$1 \times 10^{-2}$ & 0.2 & 70.54 & 64.85 & 66.99 & 64.85 \\
\$1 \times 10^{-2}$ & 0.1 & 124.5 & 129.69 & 123.7 & 129.69 \\
\$1 \times 10^{-2}$ & 0.02 & 595.41 & 648.45 & 614.29 & 648.45 \\
\$1 \times 10^{-2}$ & 0.01 & 1135.12 & 1296.9 & 1228.58 & 1296.9 \\
\$5 \times 10^{-3}$ & 1 & 69.09 & 55.77 & 63.46 & 68.93 \\
\$5 \times 10^{-3}$ & 0.2 & 108.43 & 92.19 & 104.56 & 103.53 \\
\$5 \times 10^{-3}$ & 0.1 & 181.75 & 184.37 & 180.04 & 184.37 \\
\$5 \times 10^{-3}$ & 0.02 & 854.30 & 921.87 & 881.97 & 921.87 \\
\$5 \times 10^{-3}$ & 0.01 & 1645.59 & 1843.74 & 1763.94 & 1843.74 \\
\$1 \times 10^{-3}$ & 1 & 226.72 & 187.21 & 218.10 & 231.40 \\
\$1 \times 10^{-3}$ & 0.2 & 338.76 & 279.95 & 339.25 & 347.46 \\
\$1 \times 10^{-3}$ & 0.1 & 446.28 & 413.89 & 445.09 & 413.89 \\
\$1 \times 10^{-3}$ & 0.02 & 1966.64 & 2069.43 & 1994.96 & 2069.43 \\
\$1 \times 10^{-3}$ & 0.01 & 3862.36 & 4138.87 & 3989.91 & 4138.86 \\
\$5 \times 10^{-4}$ & 1 & 384.16 & 315.01 & 369.94 & 389.37 \\
\$5 \times 10^{-4}$ & 0.2 & 564.90 & 471.05 & 572.30 & 584.81 \\
\$5 \times 10^{-4}$ & 0.1 & 690.72 & 585.51 & 700.16 & 692.04 \\
\$5 \times 10^{-4}$ & 0.02 & 2791.22 & 2927.54 & 2831.66 & 2927.54 \\
\$5 \times 10^{-4}$ & 0.01 & 5528.73 & 5855.07 & 5663.38 & 5855.07 \\
\$8 \times 10^{-5}$ & 1 & 1506.79 & 1245.71 & 1487.07 & 1539.75 \\
\$8 \times 10^{-5}$ & 0.2 & 2151.97 & 1862.77 & 2271.86 & 2312.64 \\
\$8 \times 10^{-5}$ & 0.1 & 2608.56 & 2215.23 & 2711.30 & 2736.66 \\
\$8 \times 10^{-5}$ & 0.02 & 7195.12 & 7322.54 & 7170.32 & 7322.54 \\
\$8 \times 10^{-5}$ & 0.01 & 14,067.18 & 14,645.07 & 14,270.00 & 14,645.07 \\
\hline
\end{tabular}
\caption{Goodput obtained via different approximations}
\end{table}

In this case, the median errors for the deterministic loss model in [26], for the Markov chain model in [15] and the current approximation are 11%, 4.5% and 6.2%, respectively.

\section{Conclusion and future work}

We have derived the throughput expression for a single TCP CUBIC connection with fixed RTT under random packet losses. To this end, we first considered the throughput expression developed for deterministic loss model for TCP CUBIC. We then considered the sequence of TCP window size processes indexed by $p$, the drop rate. We show that, with appropriate scaling, this sequence converges to a limiting Markov chain. The
scaling is obtained using insights from the deterministic loss model. The stationary distribution of the limiting Markov chain is then used to compute the desired throughput expression. While the throughput expression was evaluated for a single TCP CUBIC connection with negligible queuing, we show that this can be extended to a network with multiple TCP connections with non-negligible queuing using simple approximations. We validate all our models and assumptions by comparison with ns2 simulations. The ns2 simulations show a better match with our theoretical model as compared to the deterministic loss model.

While we did not focus on the most recent development in congestion control, viz., Google’s BBR algorithm, in this paper, a theoretical analysis of the BBR algorithm using techniques outlined in this paper would be an interesting direction of future work.

Appendix

Lemma A.1 If \( k \in (1, 2) \), we have

\[
(1 + x)^k - x^k < 1 + kx^{k-1},
\]

for \( x > 0 \).
Proof Consider the function \( f(y) = (1 + y)^k - y^k \). The second derivative of \( f \),

\[
 f^{(2)}(y) = k(k - 1)(1 + y)^{k-2} - k(1 - 1)y^{k-2},
\]
is strictly less than 0 for all \( y > 0 \). Therefore, \( f \) is a strict concave function over \((0, \infty)\).

The tangent to the curve \( f(y) \) at \( y = 0 \) is given by \( g(y) = 1 + ky \). Now, since the function \( f \) is strictly concave in \((0, \infty)\), we have

\[
(1 + y)^k - y^k < 1 + ky,
\]
for \( y > 0 \). Substituting \( x = \frac{1}{y} \), we get

\[
(1 + x)^k - 1 < x^k + kx^{k-1},
\]
for \( y > 0 \). Rearranging terms in the above inequality gives us the desired result. \( \square \)

Proposition A.1 Let \( \{X_p(x), x \in \mathbb{R}^+\} \) be a process, for \( 0 < p < 1 \), which converges to a limiting process \( X(x) \) uniformly, in the sense that

\[
\lim_{p \to 0} \sup_{x, y \leq M} |\mathbb{P}(X_p(x) \leq y) - \mathbb{P}(X(x) \leq y)| = 0,
\]
for any finite \( M \), and for each \( x \). Let the limiting distribution, \( \mathbb{P}(X(x) \leq y) \) be continuous. Then,

\[
\lim_{p \to 0} \sup_{x \leq M} |\mathbb{E}f(X_p(x)) - \mathbb{E}f(X(x))| = 0,
\]
for any \( f : \mathbb{R}^+ \to \mathbb{R} \) continuous with compact support.

Proof Consider a continuous function \( f \) with compact support \([0, K]\). Such a function is uniformly continuous. Therefore, given any \( \epsilon \), there exist \( m \) points \( u_0 = 0 < u_1 < \cdots < u_m = K \) such that

\[
\sup_{u_i < y < u_{i+1}} |f(y) - f(u_i)| < \epsilon,
\]
for all \( i = 1, 2, \ldots, m \). We have

\[
E[f(X_p(x))] = \int_0^K f(u)\mathbb{P}(X_p(x) \in du).
\]

From (41),

\[
\left| E[f(X_p(x))] - \sum_{i=1}^{m-1} f(u_i)\mathbb{P}(X_p(x) \in (u_i, u_{i+1}]) \right| \leq \epsilon.
\]
Similarly,

\[ E[f(X(x))] - \sum_{i=1}^{m-1} f(u_i) \mathbb{P}(X(x) \in (u_i, u_{i+1}^+)) \leq \epsilon. \]

Therefore,

\[
\left| \mathbb{E}f(X_p(x)) - \mathbb{E}f(X(x)) \right| \leq \sum_{i=1}^{m} f(u_i) \left| \mathbb{P}(X_p(x) \in (u_i, u_{i+1}^+)) - \mathbb{P}(X(x) \in (u_i, u_{i+1}^+)) \right| + 2\epsilon \\
\leq \sum_{i=1}^{m} \| f \|_{\infty} \left| \mathbb{P}(X_p(x) \in (u_i, u_{i+1}^+)) - \mathbb{P}(X(x) \in (u_i, u_{i+1}^+)) \right| + 2\epsilon,
\]

where \( \| f \|_{\infty} = \text{sup}\{f(x) : x \in [0, K]\} \). Since \( f \) is continuous over a compact support, it is bounded, and hence \( \| f \|_{\infty} < \infty \). Therefore,

\[
\lim_{p \to 0} \sup_x \left| \mathbb{E}f(X_p(x)) - \mathbb{E}f(X(x)) \right| \\
\leq \lim_{p \to 0} \| f \|_{\infty} \sum_{i=1}^{m} \sup_x \left| \mathbb{P}(X_p(x) \in (u_i, u_{i+1}^+)) - \mathbb{P}(X(x) \in (u_i, u_{i+1}^+)) \right| + 2\epsilon \\
= 2\epsilon.
\]

The second relation follows from the hypothesis (39). Since \( \epsilon \) is arbitrary, we get the desired result. \(\_\_\_\_\_\_\_\)

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