INTEGRAL SEQUENCES COUNTING PERIODIC POINTS

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1. Introduction

An existing dialogue between number theory and dynamical systems is advanced. A combinatorial device gives necessary and sufficient conditions for a sequence of non-negative integers to count the periodic points in a dynamical system. This is applied to study linear recurrence sequences which count periodic points. Instances where the $p$-parts of an integer sequence themselves count periodic points are studied. The Mersenne sequence provides one example, and the denominators of the Bernoulli numbers provide another. The methods give a dynamical interpretation of many classical congruences such as Euler-Fermat for matrices, and suggest the same for the classical Kummer congruences satisfied by the Bernoulli numbers.

Let $M_n = 2^n - 1, n \geq 1$ denote the $n$-th term of the Mersenne sequence $(M_n)$. This sequence is of interest in number theory because it is expected to contain infinitely many prime terms, and in dynamics because it counts the periodic points in the simplest expanding dynamical system. Let $T: S^1 \to S^1$ be the squaring map $T(z) = z^2$, and let $\text{Per}_n(T)$ denote the set of points of period $n$ under $T$, that is the set of solutions of the equation $T^n(z) = z$. Then it is easy to check that $|\text{Per}_n(T)| = M_n$.

Other classical sequences arise in a similar way. Let $L_n$ denote the $n$-th term of the Lucas sequence $1, 3, 4, 7, \ldots$, and let $X$ denote the set of all doubly-infinite strings of 0’s and 1’s in which every zero is followed by a 1, and $T: X \to X$ the left shift defined by $(Tx)_n = x_{n+1}$. Then $|\text{Per}_n(T)| = L_n$.

The Lehmer-Pierce sequences (generalising the Mersenne sequence; see [4]) also arise in counting periodic points. Let $f(x)$ denote a monic,
integral polynomial with degree \( d \geq 1 \) and roots \( \alpha_1, \ldots, \alpha_d \). Define
\[
\Delta_n(f) = \prod_{i} |\alpha_i^n - 1|,
\]
which is non-zero for \( n \geq 1 \) if no \( \alpha_i \) is a root of unity. When \( f(x) = x - 2 \), we obtain \( \Delta_n(f) = M_n \). Sequences of the form \( (\Delta_n(f)) \) were studied by Pierce and Lehmer with a view to understanding the special form of their factors, in the hope of using them to produce large primes. In dynamics they arise as sequences of periodic points for toral endomorphisms. Let \( X = \mathbb{T}^d \) denote the \( d \)-dimensional additive torus. Then the companion matrix \( A_f \) of \( f \) acts on \( X \) by multiplication \( \mod 1 \), \( T(x) = A_f x \mod 1 \). It requires a little thought to check that \( |\text{Per}_n(T)| = \Delta_n(f) \) under the same ergodicity condition that no \( \alpha_i \) is a root of unity (see [4]). Notice that the Lehmer-Pierce sequences are the absolute values of integer sequences which could have mixed signs.

Our final examples illuminate the same issue of signed sequences whose absolute value counts periodic points. The Jacobsthal-Lucas sequence \( R_n = |(-2)^n - 1| \) counts points of period \( n \) for the map \( z \mapsto z^{-2} \) on \( S^1 \). The sequence \( S_n = |2^n + (-3)^n| \) counts periodic points in a certain continuous automorphism of a 1-dimensional solenoid, see [3] or [10].

Following [12], call a sequence \( u_n \) of non-negative integers realisable if there is a set \( X \) and a map \( T : X \rightarrow X \) such that \( u_n = |\text{Per}_n(T)| \). The examples above were of sequences that are realisable by continuous maps of compact spaces; it turns out that any realisable sequence is in fact realisable by such a map.

It is natural to ask what is required of a sequence in order that it be realisable. For example, could the Fibonacci sequence, the more illustrious cousin of the Lucas sequence, be realised in this way? The answer is no, and a simple proof will follow in the next section. In fact a sequence of non-negative integers satisfying the Fibonacci recurrence is realisable if and only if it is a non-negative integer multiple of the Lucas sequence (see [12] and [13]).

The statements of the main theorems now follow. For the first, note that if \( (u_n) \) is any sequence of integers, then it is reasonable to ask if the sequence \( (|u_n|) \) of absolute values is realisable. For example, the sequence \( 1, -3, 4, -7, \ldots \) is a signed linear recurrence sequence whose absolute values are realisable.

The first theorem gives a generalisation of the observation about realisable sequences which satisfy a linear recurrence relation. The definitions are standard but they will be recalled in the next section. Recall that the \( \mathbb{C} \)-space of all solutions of a binary recurrence relation
has dimension 2. The realisable subspace is the subspace generated by the realisable solutions. For the Fibonacci recurrence, the realisable subspace has dimension 1 and is spanned by the Lucas sequence.

**Theorem 1.1.** Let \( u_n \) denote the \( n \)-th term of an integer sequence which satisfies a non-degenerate binary recurrence relation. Let \( \Delta \) denote the discriminant of the characteristic polynomial associated to the recurrence relation. Then the realisable subspace has

1. dimension 0 if \( \Delta < 0 \),
2. dimension 1 if \( \Delta = 0 \) or \( \Delta > 0 \) and non-square, and
3. dimension 2 if \( \Delta > 0 \) is a square.

Theorem 1.1 surely has a generalisation to higher degree which characterises the realisable subspace in terms of the factorisation of the characteristic polynomial \( f \). The second theorem is a partial result in that direction, giving a restriction on the dimension of the realisable subspace under the assumption that the characteristic polynomial has a dominant root.

**Theorem 1.2.** Let \( f \) denote the characteristic polynomial of a non-degenerate linear recurrence sequence with integer coefficients. If \( f \) is separable and has \( l \) irreducible factors and a dominant root then the dimension of the realisable subspace is \( \leq l \).

It is not clear if there is an exact result, but the deep result of Kim, Ormes and Roush [8] on the Spectral Conjecture of Boyle and Handelman [1] gives a checkable criterion for a given linear recurrence sequence to be realised by an irreducible subshift of finite type.

**Example 1.3.** Consider the sequences which satisfy the Tribonacci relation

\[
\tag{1}
\begin{align*}
\quad u_{n+3} &= u_{n+2} + u_{n+1} + u_n.
\end{align*}
\]

The sequence 1, 3, 7, 11, 21, . . . satisfies (1) and is realisable. This is the sequence of traces \( T(A^n) \), where \( A \) is the companion matrix to \( f(x) = x^3 - x^2 - x - 1 \),

\[
A = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 1 & 1
\end{pmatrix}.
\]

For an explanation of this remark, turn to the proof of Corollary 2.4. Theorem 1.2 says that any realisable sequence which satisfies (1) is a multiple of this one.

The third theorem consists of a pair of examples. Given a sequence \( u_n \) and a prime \( p \), write \( \lbrack u_n \rbrack_p \) for the \( p \)-part of \( u_n \). We say a sequence
is locally realisable at \( p \) if the sequence \( [u_p]_p \) is itself realisable. We say the sequence is everywhere locally realisable if it is locally realisable at \( p \) for all primes \( p \). If a sequence is everywhere locally realisable, then for each \( n \geq 1 \), \( [u_p]_p = 1 \) for all but finitely many \( p \), and it is realisable by Corollary 2.2. We will sometimes use the term globally realisable for a sequence when we wish to emphasize the distinction with local realisability. Consider the Bernoulli numbers, which are defined by the formula

\[
\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.
\]

Then \( B_n \in \mathbb{Q} \) for all \( n \), and \( B_n = 0 \) for all odd \( n > 1 \).

**Theorem 1.4.**

1. Any Lehmer–Pierce sequence is everywhere locally, and hence globally, realisable.
2. Let \( b_n \) denote the denominator of \( B_{2n} \) for \( n \geq 1 \). Then \( (b_n) \) is everywhere locally, and hence globally, realisable.

The maps in Theorem 1.4 are endomorphisms of groups. Theorem 1.4 and Lemma 2.1 suggest a dynamical interpretation of composite versions of the classical Kummer congruences; see section 4 below.

2. **Combinatorial dynamics**

We begin with a simple remark that shows the Fibonacci sequence is not realisable. No map can have 1 fixed point and exactly 2 points of period 3, as any point of least period 3 must have an orbit of length 3 comprising points all of period 3. More generally, for any prime \( p \), the number of non-fixed points of period \( p \) must be divisible by \( p \) because their orbits occur in cycles of length \( p \). Using a generalisation of this kind of reasoning, the following characterisation emerges.

**Lemma 2.1.** Let \( u = (u_n) \) be a sequence of non-negative integers, and let \( u \ast \mu \) denote the Dirichlet convolution of \( u \) with the Möbius function \( \mu \). Then \( u \) is realisable if and only if \((u \ast \mu)(n) \equiv 0 \mod n \) and \((u \ast \mu)(n) \geq 0 \) for all \( n \geq 1 \).

To see why this holds, notice that the set of points of period \( n \) is the disjoint union of the set of points of least period \( d \) for \( d \) running through the divisors of \( n \), and the number of points with least period \( d \) is a multiple of \( d \). The Dirichlet convolution is the usual definition from analytic number theory: \( \mu(1) = 1, \mu(n) = 0 \) unless \( n \) is square-free and \( \mu(n) = (-1)^k \) if \( n \) is the product of \( k \) distinct primes, and the Dirichlet
convolution of an arithmetical function $g$ with $\mu$ is given by

$$(g * \mu)(n) = \sum_{d \mid n} \mu(d) g(n/d).$$

Finally, the result is obtained using the Möbius inversion formula. For brevity, write $u_n^* = (u * \mu)(n)$ for $n \geq 1$.

**Corollary 2.2.** The sum and product of two realisable sequences are both realisable.

This may be seen either using elementary properties of the Dirichlet convolution or using the realising maps: if $u$ and $v$ are realisable, then the Cartesian product of the realizing maps realises $(u_n v_n)$, while the disjoint union realises $(u_n + v_n)$.

Notice that if $n = p^r$, for a prime $p$ and $r > 0$ an integer, Lemma 2.1 requires that

$$u_{p^r} \equiv u_{p^{r-1}} \mod p^r$$

for any realisable sequence $u$.

**Corollary 2.3.** Let $a$ denote a positive integer and let $p$ and $r$ be as above. Then

$$a^{p^r} \equiv a^{p^{r-1}} \mod p^r.$$  

**Proof.** This is the statement of the Euler-Fermat Theorem, which may be seen because the sequence $u_n = a^n$ is realisable. For example, the left shift $T$ on $\{0, 1, \ldots, a - 1\} \mathbb{Z}$ has $|\text{Per}_n(T)| = a^n$.  

This kind of observation — that periodic points in full shifts give simple proofs of many elementary congruences — is folklore; indeed the paper [2] gives a rather complicated proof of Euler–Fermat using a dynamical system.

Lemma 2.1 does more with no additional effort. The following is a generalisation of the Euler-Fermat Theorem for integral matrices which will be used in the proof of Theorem 1.1.

**Corollary 2.4.** Let $A$ denote a square matrix with integer entries and let $p$ and $r$ be as above. Then

$$\text{trace}(A^{p^r}) \equiv \text{trace}(A^{p^{r-1}}) \mod p^r.$$  

**Proof.** It is sufficient to assume $A$ has non-negative entries, since any matrix has such a representative mod $p^r$. For non-negative entries, $(\text{trace}(A^n))$ is realisable: Let $G_A$ be the labeled graph with adjacency matrix $A$ and $T_A$ the edge-shift on the set of labels of infinite paths on $G_A$. Then the number of points of period $n$ for this system is $\text{trace}(A^n)$ (see [11] for the details).  

We now state the consequences of Lemma 2.1 in their most general form for matrix traces.

**Corollary 2.5.** Let $A$ denote a square matrix with integer entries and let $A_n$ denote the sequence $\text{trace}(A^n)$. Then for all $n \geq 1$

$$A_n^* \equiv 0 \mod n.$$ 

Before the proof of Theorem 1.1, we begin with some notation (for a lively account of the general properties of linear recurrence sequences, see [14]). Suppose we are given a binary recurrence sequence $u = (u_n)$. This means that $u_1$ and $u_2$ are given as initial values with subsequent terms defined by a recurrence relation

$$u_{n+2} = Bu_{n+1} - Cu_n. \tag{2}$$

The polynomial $f(x) = x^2 - Bx + C$ is the characteristic polynomial of the recurrence relation. We will write $A_f = \begin{pmatrix} 0 & 1 \\ -C & B \end{pmatrix}$ for the companion matrix of $f$. The zeros $\alpha_1$ and $\alpha_2$ of $f$, are the characteristic roots of the recurrence relation. The assumption on non-degeneracy means that $\alpha_1/\alpha_2$ is not a root of unity. The discriminant of the recurrence relation is $\Delta = B^2 - 4C$. Of course, if $\Delta = 0$ then the roots of $f$ coincide, if $\Delta < 0$ the roots are non-real and distinct, if $\Delta > 0$ is a square then the roots are rational and in the other case, the roots are real and distinct but irrational.

The general solution of the recurrence relation in these cases is as follows:

- $\Delta = 0$: $u_n = (\gamma_1 + \gamma_2 n)\alpha_1^n$ (here $\alpha_1 = \alpha_2$).
- $\Delta \neq 0$: $u_n = \gamma_1 \alpha_1^n + \gamma_2 \alpha_2^n$.

**Proof.** (of Theorem 1.1)

Assume first that $\Delta = 0$, and let $p$ denote any prime which does not divide $\gamma_2$. Then the congruence in Corollary 2.4 is plainly violated at $n = p$ unless $\gamma_2 = 0$. In that case, $|\gamma_1 \alpha_1^n|$ is realisable and the space this generates is 1-dimensional.

If $\Delta > 0$ is a square, then the roots are rationals and plainly, must be integers. We claim that for any integers $\gamma_1$ and $\gamma_2$, the sequence $|\gamma_1 \alpha_1^n + \gamma_2 \alpha_2^n|$ is realisable. In fact (up to multiplying and adding full shifts) this sequence counts the periodic points for an automorphism on a one-dimensional solenoid, see [4] or [10].

The two cases where $\Delta \neq 0$ is not a square are similar. Write $\alpha = e + f \sqrt{\Delta}$ for one of the roots of $f$ and let $K = \mathbb{Q}(\alpha)$ denote the quadratic number field generated by $\alpha$. Write $T_{K|\mathbb{Q}}: K \to \mathbb{Q}$ for
the usual field trace. The general integral solution to the recurrence is $u_n = T_K|\mathbb{Q}((a + b\sqrt{\Delta})\alpha^n)$, where $a$ and $b$ are both integers or both half-odd integers. Write $v_n = T_K|\mathbb{Q}(a\alpha^n)$ and $w_n = T_K|\mathbb{Q}(b\sqrt{\Delta}\alpha^n)$. Now $v_n = a\text{trace}(A_f^n)$, where $A_f$ denotes the companion matrix of $f$. Hence it satisfies $v_p \equiv v_1 \mod p$ for all primes $p$ by Corollary 2.4.

Let $p$ denote any inert prime for $K$. The residue field is isomorphic to the field $\mathbb{F}_{p^2}$. Moreover, the non-trivial field isomorphism restricts to the Frobenius at the finite field level. Reducing mod $p$ gives the congruence

$$w_p = T\left(\sqrt{\Delta}\alpha\right) \equiv \sqrt{\Delta}\alpha^p - \sqrt{\Delta}\alpha \mod p.$$  

Thus, $w_p \equiv -w_1 \mod p$ for all inert primes $p$. On the other hand, $v_p \equiv v_1 \mod p$ for all inert primes $p$.

If $|u_n|$ is realisable then $|u_p| \equiv |u_1| \mod p$ by Corollary 2.3. If $u_p \equiv -u_1 \mod p$ for infinitely many primes $p$ then $v_p + w_p \equiv v_1 - w_1 \equiv -v_1 - w_1 \mod p$. We deduce that $p|v_1$ for infinitely primes and hence $v_1 = 2ae = 0$. We cannot have $e = 0$ by the non-degeneracy, so $a = 0$.

If $u_p \equiv u_1 \mod p$ then, by a similar argument, we deduce that $bf = 0$. We cannot have $f = 0$ again, by the non-degeneracy so $b = 0$. This proves that when $\Delta \neq 0$ is not a square, the realisable subspace must have rank less than 2.

Suppose firstly that $\Delta > 0$. We will prove that the rank is precisely 1. In this case, there is a dominant root. If this root is positive then all the terms of $u_n$ are positive. If the dominant term is negative then the sequence of absolute values agrees with the sequence obtained by replacing $\alpha$ by $-\alpha$ and the dominant root is now positive. In the recurrence relation (2) $C = N_K|\mathbb{Q}(\alpha)$, the field norm, and $B = T_K|\mathbb{Q}(\alpha)$. We are assuming $B > 0$. If $C < 0$ then the sequence $u_n = T(A_f^n)$ is realisable because the matrix $A_f$ has non-negative entries. If $C > 0$ then we may conjugate $A_f$ to such a matrix (this leaves the sequence of traces invariant). To see this, let $E$ denote the matrix

$$E = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}.$$  

Then

$$E^{-1}A_fE = \begin{pmatrix} k & 1 \\ Bk - k^2 - C & B - k \end{pmatrix}.$$  

If $B$ is even, take $k = B/2$. Then the lower entries in $E^{-1}A_fE$ are $(B^2 - 4C)/4 = \Delta/4 > 0$ and $B/2 > 0$. If $B$ is odd, take $k = (B+1)/2$. Then the lower entries are $(B^2 - 1 - 4C)/4 = (\Delta - 1)/4 \geq 0$ and $(B - 1)/2 \geq 0$. In both cases we have conjugated $A_f$ to a matrix with non-negative entries. Since we know that the sequence of traces of a
matrix with non-negative entries is realisable, we have completed this part of the proof.

Finally, we must show that when $\Delta < 0$, both sequences $v_n$ and $w_n$ are not realisable in absolute value. Assume $a \neq 0$, and then note that $v_1 = 2ae \neq 0$ by the non-degeneracy assumption. For all primes $p$ we have $v_p \equiv v_1$ by the remark above. Since the roots $\alpha_1$ and $\alpha_2$ are complex conjugates, $|\alpha_1| = |\alpha_2|$. Let $\beta = \frac{1}{2\pi} \arg(\alpha_1/\alpha_2)$; $\beta$ is irrational by the non-degeneracy assumption. The sequence of fractional parts of $p\beta$, with $p$ running through the primes, is dense in $(0,1)$ (this was proved by Vinogradov [16]; see [15] for a modern treatment). It follows that there are infinitely many primes $p$ for which $v_p v_1 < 0$. Therefore, if $|v_n|$ is realisable then it satisfies $v_p \equiv v_1 \mod p$ and $-v_p \equiv v_1 \mod p$ for infinitely many primes. We deduce that $v_1 = 0$ which is a contradiction. With $w_n$ we may argue in a similar way to obtain a contradiction to $w_1 \neq 0$. If $|w_n|$ is realisable then Lemma 2.1 says $|w_p^2| \equiv |w_p| \equiv |w_1|$ for all primes $p$. Arguing as before, $w_p^2 \equiv w_1$ for both split and inert primes. However, the sequence $\{p^2\beta\}$, $p$ running over the primes, is dense in $(0,1)$. (Again, this is due to Vinogradov in [16] or see [5] for a modern treatment. The general case of $\{F(p)\}$, where $F$ is a polynomial can be found in [2].) We deduce that $w_p^2 w_1 < 0$ for infinitely many primes. This means $w_p^2 \equiv w_1 \mod p$ and $w_p^2 \equiv -w_1 \mod p$ infinitely often. This forces $w_1 = 0$ - a contradiction.

3. Proof of Theorem 1.2

The proof of Theorem 1.2 uses the methods introduced in the proof of Theorem 1.1.

Proof. Let $d$ denote the degree of $f$. In the first place we assume $l = 1$, thus $f$ is irreducible. The irreducibility of $f$ implies that the rational solutions of the recurrence are given by $u_n = T_{K/Q}(\gamma \alpha^n)$, where $K = \mathbb{Q}(\alpha)$, and $\gamma \in K$. We write $\gamma_i, \alpha_i, i = 1, \ldots, d$ for the algebraic conjugates of $\gamma$ and $\alpha$. The dominant root hypothesis says, after relabelling, $|\alpha_1| > |\alpha_i|$ for $i = 2, \ldots, d$. We will show that if $u_n$ is realisable then $\gamma \in \mathbb{Q}$.

Let $p$ denote any inert prime. If $p$ is sufficiently large, the dominant root hypothesis guarantees that $u_p, \ldots, u_{p^d}$ will all have the same sign. Using Lemma 2.1 several times, we deduce that

$$u_p \equiv u_{p^2} \equiv \cdots \equiv u_{p^d} \equiv \pm u_1 \mod p.$$
Therefore \( u_p + \cdots + u_{pd} \equiv \pm du_1 \mod p \), the sign depending upon the sign of \( u_1 \). However,
\[
u_p + \cdots + u_{pd} \equiv T_{K|\mathbb{Q}}(\gamma) T_{K|\mathbb{Q}}(\alpha) \mod p.
\]
We deduce a fundamental congruence
\[
T_{K|\mathbb{Q}}(\gamma) T_{K|\mathbb{Q}}(\alpha) \equiv \pm d T_{K|\mathbb{Q}}(\gamma \alpha) \mod p.
\]
Since this holds for infinitely many primes \( p \), the congruence is actually an equality,
\[
(3) \quad T_{K|\mathbb{Q}}(\gamma) T_{K|\mathbb{Q}}(\alpha) = \pm d T_{K|\mathbb{Q}}(\gamma \alpha).
\]
The next step comes with the observation that if \( u_n \) is realisable then \( u_{rn} \) is realisable for every \( r \geq 1 \). Thus equation (3) now reads
\[
(4) \quad T_{K|\mathbb{Q}}(\gamma) T_{K|\mathbb{Q}}(\alpha^r) = \pm d T_{K|\mathbb{Q}}(\gamma \alpha^r).
\]
Dividing equation (4) by \( \alpha^r \) and letting \( r \to \infty \) we obtain the equation
\[
T_{K|\mathbb{Q}}(\gamma) = \pm d \gamma_1.
\]
This means that one conjugate of \( \gamma \) is rational and hence \( \gamma \) is rational.

The end of the proof in the case \( l = 1 \) can be re-worked in a way that makes it more amenable to generalisation. The trace is a \( \mathbb{Q} \)-linear map on \( K \) so its kernel has rank \( d - 1 \). Thus every element \( \gamma \) of \( K \) can be written \( q + \gamma_0 \) where \( q \in \mathbb{Q} \) and \( T_{K|\mathbb{Q}}(\gamma_0) = 0 \). Noting that \( T_{K|\mathbb{Q}}(q) = dq \) and cancelling \( d \), this simply means equation (4) can be written
\[
u_r = \pm q T_{K|\mathbb{Q}}(\alpha^r),
\]
for all \( r \geq 1 \) confirming that the realisable subspace has rank \( \leq 1 \).

The general case is similar. Each of the irreducible factors of \( f \) generates a number field \( K_j, j = 1, \ldots, l \) of degree \( d_j = [K_j : \mathbb{Q}] \). The solutions of the recurrence look like
\[
u_n = \sum_{j=1}^{l} T_{K_j|\mathbb{Q}}(\gamma_j \alpha_j^n),
\]
where each \( \gamma_j \in K_j \). Let \( L \) denote the compositum of the \( K_j \). Using the inert primes of \( L \) and noting that each is inert in each \( K_j \), we deduce an equation
\[
(5) \quad \sum_{j=1}^{l} \frac{d}{d_j} T_{K_j|\mathbb{Q}}(\gamma_j) T_{K_j|\mathbb{Q}}(\alpha_j) = \pm d \sum_{j=1}^{l} T_{K_j|\mathbb{Q}}(\gamma_j \alpha_j).\]
As before, replace $\alpha_j$ by $\alpha^r_j$, and cancel $d$ so that

$$u_r = \pm \sum_{j=1}^{l} \frac{1}{d_j} T_{K_j|\mathbb{Q}}(\gamma_j) T_{K_j|\mathbb{Q}}(\alpha^r_j)$$

Each $\gamma_j$ can be written $\gamma_j = q_j + \gamma_{0j}$, where $T_{K_j|\mathbb{Q}}(\gamma_{0j}) = 0$. Noting that $T_{K_j|\mathbb{Q}}(q_j) = d_j q_j$ we deduce that

$$u_r = \pm \sum_{j=1}^{l} q_j T_{K_j|\mathbb{Q}}(\alpha^r_j)$$

which proves that the realisable subspace has rank $\leq l$. \hfill \Box

4. Proof of Theorem 1.4

It is sufficient to construct local maps $T_p : X_p \to X_p$ for each prime $p$. Then Corollary 2.2 guarantees a global realisation by defining

$$T = \prod_p T_p$$
on $X = \prod_p X_p$.

If the maps $T_p$ are group endomorphisms then the map $T$ is a group endomorphism.

Proof. As motivation, consider the Mersenne sequence. For each prime $p$, let $U_p \subset S^1$ denote the group of all $p$th power roots of unity. Define the local endomorphism $S_p : x \mapsto x^2$ on $U_p$. Then $|\text{Per}_n(S_p)| = [2^n - 1]_p$ so $S_p$ gives a local realisation of the Mersenne sequence.

An alternative approach is to use the $S$-integer dynamical systems from [3]: for each prime $p$, define $T_p$ to be the automorphism dual to $x \mapsto 2x$ on $\mathbb{Z}(p)$ (the localisation at $p$). Then by [3],

$$|\text{Per}_n(T_p)| = \prod_{q \leq \infty; q \neq p} |2^n - 1|_q = [2^n - 1]_p$$

by the product formula. This approach gives a convenient proof of the general case. We may assume that the polynomial $f$ is irreducible; let $K = \mathbb{Q}(\xi)$ for some zero of $f$. Then for each prime $p$, let $S$ comprise all places of $K$ except those lying above $p$, and let $T_p$ be the $S$-integer map dual to $x \mapsto \xi x$ on the ring of $S$-integers in $K$. Then by the product formula

$$|\text{Per}_n(T_p)| = \left( \prod_{v|p} [\xi^n - 1]_v \right)^{-1} = [\Delta_n(f)]_p$$

as required.

For the Bernoulli denominators, define $X_p = \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. For $p = 2$ define $T_p$ to be the identity. For $p > 2$, let $g_p$ denote an element
of (multiplicative) order \((p - 1)/2\). Define \(T_p : X_p \to X_p\) to be the endomorphism \(T_p(x) = g_p x \mod p\). Plainly \(|\text{Per}_n(T_p)| = p\) if and only if \(p - 1|2n\); for all other \(n\), \(|\text{Per}_n(T_p)| = 1\). The von Staudt–Clausen Theorem ([6], [9]) states that

\[ B_{2n} + \sum \frac{1}{p} \in \mathbb{Z}, \]

where the sum ranges over primes \(p\) for which \(p - 1|2n\). Thus \(|\text{Per}_n(T_p)| = \max\{1, |B_{2n}|_p\}\) and this shows the local realisability of the Bernoulli denominators.

The following statements seem plausible upon numerical investigation.

1. the denominators

\[ 12, 120, 252, 240, 132, 32760, \ldots \]

of \(B_{2n}/2n\) form a sequence that is everywhere locally realisable;

2. the numerators

\[ 1, 1, 1, 1, 1, 691, 1, 3617, 43867, \ldots \]

of \(B_{2n}/2n\) form a realisable sequence that is not locally realisable at the irregular primes 37, 59, 67, 101, 103, 131, 149, 157, \ldots .

3. the denominators

\[ 24, 240, 504, 480, 264, 65520, 24, 16320, \ldots \]

of \(B_{2n}/4n\) form a realisable sequence that is not locally realisable at the primes 2, 3, 5, 7, 11, 13 but seems to be locally realisable for large primes.

Taking these remarks together with \(n = p^r\) in Lemma 2.1, suggests a dynamical interpretation of the Kummer congruences. These are stated now, for a proof see [9].

**Theorem 4.1.** If \(p\) denotes a prime and \(p - 1\) does not divide \(n\) then \(n \equiv n' \mod (p - 1)p^r\) implies

\[ (1 - p^{n-1})B_{n} \equiv (1 - p^{n'-1})B_{n'} \mod p^{r+1}. \]

**References**

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