Stochastic models of edge turbulent transport in the thermonuclear reactors

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Abstract. Two-dimensional stochastic model of turbulent transport in the scrape-off layer (SOL) of thermonuclear reactors is considered. Convective instability arisen in the system with respect to perturbations reveals itself in the strong outward bursts of particle density propagating ballistically across the SOL. The criterion of stability for the fluctuations of particle density is formulated. A possibility to stabilize the system depends upon the certain type of plasma waves interactions and the certain scenario of turbulence. A bias of limiter surface would provide a fairly good insulation of chamber walls excepting for the resonant cases. Pdf of the particle flux for the large magnitudes of flux events is modeled with a simple discrete time toy model of 1-dimensional random walks concluding at the boundary. The spectra of wandering times feature the pdf of particle flux in the model and qualitatively reproduce the experimental statistics of transport events.

1. Introduction.
Evidence of high plasma density in the vicinity of all wall components of ITER has been reported in both experiments and numerics. Edge transport leads to severe constraints in the operating window of a next step device since it propagates ballistically with rather high velocities toward the wall provoking the plasma-wall interactions in areas that are not designed for this purpose. Electric currents flowing in the Scrape-Off Layer (SOL) are found to determine the turbulent activity, and their understanding remains an important task.

The large particle flow due to the bursts of density is of strongly intermittent nature and does not appear to fit into the standard view of diffusive transport: the probability distributions function of the particle flux departs from the Gaussian distribution forming a long (apparently exponential) tail which dominates at high positive flux of particles.

In the present paper, we consider a variety of two dimensional fluid models exerted to the Gaussian distributed external random forces. We formulate a simple toy model which describes qualitatively the statistics of transport events in the cross-field system and discuss a possible modification of turbulent transport that would insulate the divertor walls from the rather intensive particle flow.

2. Two-dimensional dissipation free model of cross-field transport.
Let us consider 2D frame of reference comprising of radial x and poloidal y coordinates both normalized to the hybrid Larmor’s radius $\rho$. Time is supposed to be normalized to the inverse ion cyclotron frequency $\Omega_i^{-1}$. Neglecting for the dissipation processes in plasma, under the constant temperatures $T_e >> T_i$, this problem is reduced to the interactions between the normalized particle density field $n(x, y, t)$, the electric potential field $\phi(x, y, t)$, and the vorticity field $\omega(x, y, t)$:
\[
(1) \quad \dot{n} = [n, \phi], \quad \dot{w} = [w, \phi] - g \cdot \nabla \log n, \quad w = \Delta \phi,
\]
in which \([p, q] = \partial_p q - \partial_q p\) is the Poisson brackets. The small parameter \(g = \rho/R\), where \(R\) is the major radius of the torus, in front of the curvature drive term is supposed to be averaged along the magnetic field lines and considered as constant.

In absence of curvature drive \((g=0)\), the equations (1) describe the rotations of density and vorticity gradients around the poloidal cross-field drift \(v_y = -c/B_0 \cdot \partial_x \phi \times \mathbf{e}_z\). Their solutions fit into the generalized Galilean invariance: the electric potential and the poloidal drift are tolerant towards the arbitrary time dependent shifts decaying at \(t \to -\infty\), \(\phi \to \phi + x \cdot \phi(t) + \phi_0(t), v_y \to v_y + \phi_0(t)\). The radial drift is trivial, \(v_r = 0\). At the same time, the density of particles forms the profile-preserving waves traveling in the poloidal direction,

\[
n = \phi_3 \left( x, y - \int_{-\infty}^t v_y \left( x, t' \right) \cdot dt' \right).
\]

For \(g>0\), the particle density is characterized by the Boltzmann’s distributions in the poloidal direction. Among them, those compatible with the Galilean invariance are the solitons (solitary waves)

\[
n \propto \exp \left( -\frac{1}{gT(x, y)} \left| y - \int_{-\infty}^t v_y \left( x, y, t' \right) \cdot dt' \right| \right),
\]

where \(T(x, y)\) is any stationary function twice integrable over its domain. Other potential configurations with the trivial poloidal drifts break the Galilean invariance. For instance, the periodic lattice of high particle density radiating from the center can represent them,

\[
n \propto \exp \left( -\frac{1}{g} \int_{-\infty}^y U \left( y' \right) \cdot dy' \right), \quad \dot{w} = U \left( y \right) \mod 2\pi.
\]

With two concurrent symmetries there can occur either the frustration of one of them or the vanishing of both with the consequent appearance of a complicated dynamic picture that is most likely stochastic. The latter case corresponds to a maximally symmetric motion resulting from the destruction of unperturbed symmetries\(^3\). In particular, instability in the system (1) occurs with respect to any small perturbation either of density or vorticity.

3. **Toward the stochastic model of cross-field transport.**

More realistic models of turbulent transport in the SOL have to include the dissipation processes in plasmas and the sheath boundary conditions\(^7\) setting the particle flux to the ion saturation current and ensuring a vanishing current density into the wall. Instead of accounting for the sheath loss terms that control the parallel transport losses at the boundary, we introduce the Gaussian distributed stochastic forces into the dynamical equations for the fluctuations of density and vorticity,

\[
(2) \quad \nabla \cdot \mathbf{w} = u_0 v \cdot \Delta \mathbf{n} - \sum_{k \geq 1} \frac{(-1)^{k+1}}{k} g_k \nabla \cdot n^k + f_w, \quad \nabla \cdot n = u_0 v \cdot \Delta n + f_n,
\]
in which \(\nabla / \equiv \partial_x + \epsilon y \times \nabla\) is the covariant derivative. The effective curvature drive \(\partial_x \log(1+n/n_0)\) is represented by the series in \(\partial_x n^k\) with the coefficients \(g_k = g/n_0^k\) where \(n_0\) is the mean particle density. The diffusion coefficients \(v\) and \(u_0v\) both are normalized to the Bohm’s value \(T_e/eB\) and govern the damping in the small scales; \(u_0\) is the dimensionless Prandtl number.

Stochastic forces model both the external noise risen in the SOL due to the Langmuir probes\(^5\) and stochastic sources of particles and helicity maintaining the system out of equilibrium, as well as
the internal noise risen due to the microscopic degrees of freedom eliminated from the phenomenological equations (2). The physically important effect of particle escape at the sheath is replaced with a quenched loss of particles at the points for which \( f_n < 0 \). Simultaneously, the particles are supposed to arrive in the SOL in areas where \( f_n > 0 \) modeling the injection of particles from the divertor core. The sheath boundary conditions are taken into account by means of zero mean, \( \langle f_n \rangle = 0 \). Similarly we suppose that \( \langle f_w \rangle = 0 \).

In the stochastic model (2), the covariances of random forces determine the density of particle and helicity flows pumped into the SOL from the divertor core. Under the simplest “white noise” assumption, the particle and helicity random forces are supposed to be uncorrelated in space and time and unrelated to each other,

\[
D_{nn} = J_n \delta(r' - r) \delta(t' - t), \quad D_{ww} = J_w \delta(r' - r) \delta(t' - t)
\]

that corresponds to the uniformly distributed pumps of powers \( J_n \) and \( J_w \).

In the more realistic descriptions of turbulent transport, the covariances of stochastic forces should contain detailed information on the certain type of interactions between plasma waves and the certain scenario of turbulence. The first can be specified in a phenomenological way by the Langevin equation describing the relaxation dynamics in plasmas as a superdiffusion process,

\[
\frac{\partial f_n}{\partial t} = \frac{f_n}{\tau_w} + \frac{f_w}{\sqrt{\beta}},
\]

in which \( \beta \approx \langle f_w^2 \rangle > 0 \), and the non-local covariance operator \( \tau_w^{-1} \) determining the spectrum of reciprocal correlation times between the particle and helicity perturbations is

\[
\tau_w^{-1}(k) = \lambda \nu \kappa^{-2 - 2\gamma}, \quad 0 < 2\gamma < 1,
\]

in the momentum space. In the case of \( 2\gamma \ll 1 \), the Langevin equation (4) with the kernel (5) reproduces the asymptotical dispersion relation typical for the Langmuir waves\(^6\). Spectra with \( 2\gamma \to 1 \) correspond to the ion-acoustic waves traveling in the collisionless plasma. Intermediate values of \( 2\gamma \) correspond to the various types of interactions between these two types of plasma waves described by the Zakharov’s equations\(^7\).

The Langevin equation (4) results in a relation between the covariances of random forces,

\[
D_{nn}(\omega, k) = \frac{1}{\beta} \frac{D_{ww}(\omega, k)}{\omega^2 + \lambda^2 \nu^2 k^{4 - 4\gamma}},
\]

in which the coupling constant \( \lambda > 0 \) establishes the time scale separation between “fast” (helical) and “slow” (density) modes in the model. In the rapid-change limit (\( \lambda \to \infty \)), the spectral density of particle flux \( D_{nn}(\omega k) \) follows up \( D_{ww}(\omega k) \) instantly. In the opposite case of “frozen” configuration (\( \lambda \to 0 \)), the spectral density of particle flux remains static \( \propto \delta(\omega) \) (independent upon the time argument, in \( t \)-representation) that corresponds to a random source of particles operating continuously in time.

The choice of model for the covariance \( D_{ww}(\omega, k) \) would determine the certain scenario of turbulence. In accordance to Kolmogorov’s phenomenological theory of turbulence, the inertial range is fed with energy of large scale motions \( \propto \delta(k) \) (the direct cascade). From one hand, the use of power law model,
\[ \delta(k) = \lim_{\nu \to 0} \int (r/\rho_s)^{-\nu} \exp(i k \cdot r) \cdot dr = k^{-d} \frac{\Gamma(d/2)}{2\pi^{d/2}} \lim_{\nu \to 0} (\nu k \cdot \rho_s), \]

in which \( d \) is the dimensionality of space (\( d=2 \)) leads to a conclusion that it should be \( D_{\omega \omega}(\omega, k) \propto k^{-d} \).

From another hand, the instantaneous spectral balance of particle flux, \( \frac{1}{2} \int D_{\omega \omega}(\omega, k) \cdot d\omega / 2\pi \), should not depend upon neither the reciprocal correlation time \( \tau_c(k) \) at any \( k \) nor the amplitude of helicity flux \( \beta \) that is true provided \( D_{\omega \omega} \propto \lambda \beta \cdot k^{-2\gamma} \). Furthermore, \( D_{\omega \omega} \) has to fit into its proper physical dimension \( T^{-3} \), which is combined from the only dimensional parameters in the theory, \( u_0^3 V^3 k^6 \). Collecting all the above factors together, one obtains that \( D_{\omega \omega} \propto \lambda \beta \cdot u_0^3 V^3 k^{6-2\gamma-2\varepsilon} \).

To make it consistent with the previous conclusion, one has to introduce, in the power exponent, other control parameter \( 2\varepsilon > 0 \) such that \( 6-2\gamma = 2\varepsilon \), for the Kolmogorov scenario of turbulence. Besides the direct cascade, other scenarios of turbulence are possible, for instance, one can consider the inverse cascade when the small scale vortexes are combined into a large scale one. Eventually, the model which we use for the covariance of helicity stochastic force is

\[ D_{\omega \omega}(\omega, k) = \beta \lambda u_0^3 V^3 k^{6-2\gamma-2\varepsilon}. \]

It describes the direct cascade along the line \( 6-2\gamma = 2\varepsilon \) in the space of control parameters (see figure 1) and the inverse cascade as \( 6-2\gamma = 2\varepsilon \) that is \( 4-2\gamma = 2\varepsilon \) in \( d=2 \). All physically appropriate values of control parameters in the stochastic model \((2,4,7)\) form a “band of interest” shown in figure 1.

In the present paper, we discuss the large scale (\( k \to 0 \)) asymptotic behavior of the response functions \( \langle \delta \sigma_n(r, t)/ \delta f(0, 0) \rangle \) and \( \langle \delta \sigma_r(r, t)/ \delta f(0, 0) \rangle \) quantifying the reaction of system onto an external disturbance of both density and vorticity occurring at the origin at time \( t = 0 \). The high order response functions are related to the analogous multipoint distribution functions \( F_n(r_1, t_1; \ldots r_n, t_n) \).

The power-law models for the covariances of random forces have been used extensively in the statistical theory of turbulence\(^8\) (see also the references therein). The models of random walks in random environment with long-range correlations based on the Langevin equation \((4)\) have been discussed in concern with the problem of anomalous scaling of a passive scalar advected by the synthetic compressible turbulent flow\(^9\), then for the purpose of establishing the time scale separation, in the models of self organized criticality\(^10\).

4. Iterative solutions of stochastic problem and their diagram representations.

The linearized homogeneous problem, for the fluctuations of density and vorticity vanishing at \( t \to -\infty \) is satisfied by the retarded Green’s functions

\[ \tilde{\Delta}_n(r, t) = \frac{\theta(t)}{4\pi \nu \cdot t} \exp \left( -\frac{r^2}{4\nu \cdot t} \right), \]

expressing the casualty principle of dynamical problem. Nonlinearities in \((2)\) then can be taken into account by the perturbation theory with respect to the small parameters \( g_k \) that allows for a diagram representation shown on Figure 2. Therein, the external line (a tail) stands for the field \( n \), the double external line denotes the field \( w \), and the bold line represents the magnetic flux \( \nu \). The triangles stay for the random force \( f_n \), and the filled triangles represent \( f_w \). The retarded Green functions are marked by the lines with an arrow which direction marks the time ordering of their arguments. Slashes correspond to the differential operator \( \nabla \). Circles surrounding vertices represent the antisymmetric interaction \( \nu \times \nabla \), and squares are for the poloidal gradient \( \nabla_\nu \).
All correlation functions of fluctuating fields as well as functions expressing the system response for the external perturbations could be found by the multiplication of trees displayed on Figure 2 followed by the averaging over all possible configurations of random forces. In diagrams, this procedure corresponds to all possible contractions of lines ended with the identical triangles (see Figure 3). Diagrams having an odd number of external triangles give zero contributions in average. As a result of contractions, the following new elements (lines) appear in diagrams: (the integration over coinciding arguments is imposed). These are the free propagators of density and vorticity fluctuations, the mixed propagator, and the mixed retarded Green’s function. In diagrams, they are displayed with the lines without an arrow and with the composite directed lines consequently (see Figure 3). The diagram technique introduced in the present section is suitable for the model preserving the continuous symmetry of (1), apart from the sheath.

5. Analytical properties of diagrams. The functional integral formulation of stochastic problem and renormalization.

The iterative solutions of stochastic problem would have a physical meaning provided their diagram series converge in space and time. While calculating the graphs of diagram series, one can see that some of them diverge at the large momenta for $\epsilon < 1$ and therefore require an ultraviolet (UV) cut-off parameter $\Lambda$. Then, the UV divergences reveal themselves as powers and logarithms of $\Lambda$. The power-like divergences arise in the model due to the kinematical effect of dragging of small scale vortexes by the large scale motion and disappear in the proper frame of reference moving along with the large scale motions. To omit them, one can subtract from the divergent diagrams their values at zero external momenta. However, the logarithmic UV-divergences remain in the diagrams for the response function $\langle \delta n(\mathbf{r},t)/\delta f(0,0) \rangle$ and can be handled by the renormalization group method (RG).

To observe the consequent subtraction of logarithmic UV-divergences from the perturbation theory, we have used the quantum field theory formulation of RG method and the conventional dimensional counting arguments. It is important to note that the set of diagrams (Figure 3) arisen in the model (2,4,7) by the iterations with respect to nonlinearities is equivalent to the standard Feynman diagrams of some quantum field theory. This coincidence of diagrams is a particular consequence of the general equivalence between the $t$-local stochastic dynamical problems (in which the interactions contain no time derivatives) and the relevant quantum field theories with the action functional found in accordance to the MSR formalism. In accordance to it, the statistical averages with respect to all admissible configurations of Gaussian distributed random forces in a stochastic dynamical problem can be identified with the functional averages with the weight $\exp S$. In particular, for the problem (2,4,7) the relevant action functional is

$$S = \frac{1}{2} n' D_{nn} n' + \frac{1}{2} w' D_{ww} w' - n'(\partial_n n + [n, \phi] - \nu \cdot \Delta n) + \phi'(w - \Delta \phi)$$

$$- w' \left( \partial_n w + [w, \phi] + \sum_{k \geq 1} \frac{(-1)^k g_k}{k} \cdot \partial_y n^k - u_i \nu \cdot \Delta w \right)$$

(the integrations over space and time are implied). In (8), the auxiliary fields $n'$ and $w'$ are functionally conjugated to the stochastic forces $f_n$ and $f_w$ consequently, and $\phi'$ appears as the Lagrange multiplier for the bounding relation between the electric potential and vorticity fields.

The generating functional of Green’s functions in the quantum field theory is given by the functional integral

$$\Omega(A_\phi) = \int D\Phi \cdot \exp[S + A_\phi \cdot \Phi], \Phi \equiv \{n, n', w, w', \phi, \phi'\}$$
with the arbitrary source functions $A_0$ interpreted as the not random external fields, so that the Green’s function $\langle n'(0,0)n(r,t)\rangle$ for $t > 0$ coincides with the response function $\langle \delta n(x,t) \delta f(x,0) \rangle$. It is important to note that all possible boundary conditions and the damping asymptotic conditions for the fluctuation fields $n$ and $w$ at $t \to \infty$ are supposed to be included into the functional integration domain in (9). The functional integral (9) has the standard Feynman diagram representation which coincides with that one constructed in the previous section. The basic symmetry of (8) is the generalized Galilean invariance with respect to the transformations of fields

$$\partial_a \phi(x,t) \to \partial_a \phi(x,t) - a(t), \quad n(x,t) \to n(x,y-b(t),t), \quad b(t) = \int_{-\infty}^t a(t') dt'$$

with the arbitrary integrable function $a(t)$ decaying as $t \to -\infty$.

Any quantity $Q$ in (8) can be characterized with respect to the independent scale transformations in space and time by its momentum and frequency dimensions, $d^{\rho s}$ and $d^{\nu s}$. These scale transformations are coupled in the free theory that allows for the introduction of total canonical dimension $d_\rho = d^{\rho o} + 2d^{\nu s}$. The scale transformations would be also coupled at a stable fixed point of renormalization group transformation, if it exists that gives the critical dimension $d_{\rho 0} = d^{\rho 0} + d_{\nu 0} + \gamma_0$, in which $\Delta_\nu$ is the critical dimension of frequency, and $\gamma_0$ is the anomalous dimension of $Q$ (both calculated from the appropriate renormalization constants).

Despite the action functional (8) has an infinite number of charges ($\beta$ and $g_s$, $k \geq 1$), its renormalization does not require an infinite number of counterterms since the only correlation function which actually diverges at large momenta is $\langle \bar{n} n \rangle$. Other correlation functions suspicious for being divergent break the Galilean invariance of (8) and therefore do not play a role in renormalization. The only superficially divergent contribution can be subtracted from the perturbation series by the multiplicative renormalization of Prandtl’s number, $u_0 = u Z$, where $u$ is its renormalized value, and the renormalization constant $Z$ contains all poles in $\varepsilon$.

However, the conventional approach of critical phenomena theory is useless for the determining of large scale asymptotes in the problem in question, since the severe instability frustrates the critical behavior in the system preventing its approaching to the formal asymptote predicted by the renormalization group method.

### 6. Instability of iterative solutions in the large scales.

The iterative solutions of stochastic problem would be asymptotically stable in the large scales provided all small perturbations of both density and vorticity damp out with time. The exact response functions found from the Dyson equations (see Figure 3) should have poles located in the lower half-plane of frequency space. In a "proper" perturbation theory, apart from a crossover region, the stability of the response functions is secured by the correct sign of dissipation term $\nu k^2$ which dominates over the dispersion relation in the large scales ($k \to 0$):

$$a(k) = -i\nu k^2 + i\Sigma(a k)$$

where $\Sigma(a k)$ is the sum of infinite series of all 1-irreducible diagrams of perturbation theory.

This is true but with one exception: if the expansion of $\Sigma$ in $k$ near the origin contains a term linear in $k$, for $k \to 0$ its contribution can become more important than the bare one. In particular, it happens in all orders of diagram series (Figure 3) representing the response of particle density to the perturbations of both density and vorticity. Those diagrams contributing as $\Sigma_{\omega n} \approx \nu k^2 A(\varepsilon, \gamma)$ include simultaneously the antisymmetric interaction vertex together with the poloidal gradient and the free propagator of particle density $\Delta_m$. To be specific, let us consider the analytical expression for one of them (the second diagram displayed in Figure 3):

$$\Sigma_{\omega n} \big|_{\text{bare}} \equiv g_s k \int \frac{d p}{(2\pi)^3} \int \frac{d \omega d \omega'}{(2\pi)^3} \frac{k_1 p_z + k + p_z - p_z^2}{(k - p)^2} \Delta_m(\omega', p) \Delta_\omega(\omega - \omega', k - p)$$

under the "white noise" assumption (3), $\Delta_m \approx (\omega^2 + \nu^2 p^2)^{-1}$, and the integral (11) diverges both as $k \to \infty$ for $\varepsilon < 1$ and as $k \to 0$ for $\varepsilon > 1$. As we explained in the previous section, the UV-singularities can be
handled with the RG method, however, for those arisen as $k \to 0$ the RG is useless. Introducing the infrared cut-off parameter, one obtains:

$$\sum_{n=\text{loop}} g_2 m^{2-2\epsilon} = \frac{\nu k_\gamma}{8\pi(u+1)(\epsilon - 1)}$$

where the singularity at $m \to 0$ would compensate the smallness of $g_2$, therefore any fluctuation with $k_i > 0$ appears to be unstable in the large scales.

The accounting of finite reciprocal correlation time between the fluctuations of density and vorticity (4) would regularize the model. It appears that $A_{\text{fin}} = (\omega^2 + \nu^2 \rho^2(1+\delta \rho^2))^{-1}$, and the integral (11) can be computed by its analytic continuation for any momenta excepting for the points, $-1+\epsilon = \gamma \mod 1: A(\epsilon,\gamma)|_{\text{loop}} \approx (u/\lambda)^{(\epsilon-1)/\gamma} g_2/8u \gamma \csc (\pi(\epsilon-1)/\gamma)$.

In accordance to the dispersion relation (10), a density fluctuation arisen in the SOL with some random momenta $(k_x, k_y)$ would be asymptotically stable, in the large scales, if its effective scale $\xi = k_y/k^2$ playing the role of an order parameter does not exceeds its critical value: $\xi < \xi_c = A(\epsilon,\gamma)^{-1}$. In the framework of perturbation theory with respect to the small parameters $g_k$, we have demonstrated that the concurrency of symmetries in the model would cause to instability. However, provided the density and vorticity fluctuations maintaining the system out of equilibrium enter the system in a correlated way, some small scale fluctuations would acquire stability as far as $\xi < \xi_c$. Herewith, the actual dynamical picture would be very complicated since as we have seen above the amplitude of anomalous contribution $A(\epsilon,\gamma)$ could alternate its sign depending upon the values of $\epsilon$ and $\gamma$ (see Figure 4).

7. Stabilization of ballistic transport by the biasing.

The evidence of insulation of ITER wall components from the long range bursts of matter by the biasing of limiter surface has been observed recently in the numerical simulations. The physical ground for such a possibility is lucid: the solutions with $v_i = 0$ which break the Galilean invariance can be frustrated by the biasing $v_i \to v_i - V$. In the present section, we shall be interested in such values of shift $V$ which could provide a negative imaginary part $\text{Im}(\omega) < 0$ in the dispersion relation (10).

For the uncorrelated statistics of random forces (3), the new dispersion relation (in the first order of perturbation theory) in the limit $k \to 0$ reads as following,

$$\omega(k,V) \equiv -i k^2 + i k g_1 g_2 \frac{vm^{2-2\epsilon}}{u+1} \frac{\sqrt{\pi} V^2}{8\nu} \frac{\Gamma(1/2+\epsilon)(u+2)}{\Gamma(2+\epsilon)(1+u)} \log m$$

One can see that for any finite $m > 0$ there exists finite $V < \infty$ such that $\text{Im}(\omega) < 0$, however, $V \to \infty$ as $m \to 0$.

In contrast to it, the accounting for the possible correlations between density and vorticity

$$V = \pi n u^3 \left( \frac{u\lambda}{1+u} \right)^{1/2\gamma} \left( \frac{u(1+u)^{1+2\epsilon/\gamma}}{1+u} \right)^{1/2\gamma} \left[ \frac{1+u}{(1+u)^{1+2\epsilon/\gamma} - 1} \sin \pi(1+2\epsilon)/2\gamma \right] \sin \pi \epsilon / \gamma,$$

fluctuations (4) restores the finite bias $V$ sufficient to suppress the burst of density in the system:

which is singular as $\epsilon/\gamma \in \mathbb{Z}$. The biasing could provide a fairly good insulation of wall components filtering out the most of unstable fluctuations of density excepting the resonant ones. The latter arise when the parameters tuning the certain type of plasma wave interactions and the certain model of energy pump into the system are divisible. In Figure 5, we have displayed the set of points in the space of parameters $\epsilon$ and $\gamma$ where the model (2,4,7) is unstable and cannot be stabilized by the biasing of limiter surface.

8. Statistics of edge transport events.

Large scale instability developed in the model (2,4,7) is related to the appearance and unbounded growth of fluctuations of particle density close to the wall. In accordance to the fluctuation-dissipation
theorem, the fluctuations arisen in the stochastic dynamical system are related to its dissipative properties. In particular, the matrix of exact response functions \( R(k, \omega) \) expressing the perturbations risen due to the random sources \( f_x \) and \( f_y \) determines the matrix of exact dynamical Green’s functions, \( G(k, \omega) \), as following: \( R(k, \omega) - R^T(k, -\omega) = i \omega G(k, \omega) \) where \( R^T \) is the transposed \( R \).

In the limit of large scales \( k \gg 1 \), we take into account for the leading contributions into the self- energy operators in the elements of \( R \):

\[
R = \begin{bmatrix}
-\i \omega - v A_x k_x + v \cdot k^2 & -\i \omega - v A_y k_y + v \cdot u k^2 \\
-\i \omega + v \cdot k^2 & -\i \omega + v \cdot u k^2
\end{bmatrix}^{-1}
\]

large scales. Correspondent advanced Green’s functions appear to be analytic in the lower half-plane of the frequency space:

\[
G_{nm} a(k, t) = \frac{1}{v} \frac{A_k \theta(-t)}{k^4 - A_k^2 \kappa^2}, \quad G_{nm} c(k, t) = -\i \frac{\theta(-t)}{v A_k \kappa - k^2} - \i \frac{\theta(-t)}{u v \kappa^2}
\]

They relate the density of particles in the growing fluctuations arisen inside the chamber at time \( t \) with those achieved at the same moment of time \( t' > t \) at the point \( \mathbf{r} \):

\[
\delta n(\mathbf{r}', t') = \int d\mathbf{r} \int_{t' \leq t} d\mathbf{r} \cdot G_{nm} a(\mathbf{r}' - \mathbf{r}),
\]

in which

\[
G_{nm} a(\mathbf{r}' - \mathbf{r}) = \frac{1}{2v A_k} \left[ \sin A_k (y' - y) \int_0^{A_k \kappa - r} J_0 \left( \frac{A_k \kappa - r}{2} \right) + \cos A_k (y' - y) H_0 \left( \frac{A_k \kappa - r}{2} \right) \right]
\]

and \( J_0 \) and \( H_0 \) are the Bessel and Struve functions respectively. The latter integral is finite if the domain of integration \( \Omega \) is bounded for any value \( |A_k| > 0 \). To be specific, let us consider the circle \( \Omega = B_R \) of radius \( R \) as the relevant domain boundary and suppose that the density of particles incorporated into the growing fluctuations inside the chamber is independent of time and maintained at the stationary rate \( \delta n_0(\mathbf{r}) \). Then the integration with respect to \( \mathbf{r} \) can be calculated at least numerically that gives the fluctuation growth rate \( B(R) \) depending upon the size of chamber such that

\[
\delta n(R, t) = \tau B(R),
\]

where \( \tau \) is the traveling time of a density blob before it achieves the divertor wall being convected by the turbulent motion. It is the distribution of such wandering times that determines the anomalous transport statistics described by the flux pdf in our simplified model, \( P(\delta n) = P(\tau) \cdot B(R) \).

9. Pdf of the wandering times \( \tau \) close to the wall.

The discrete time model we discuss below is similar to the toy model of systems close to a threshold of instability\(^{12}\), and can be considered as a 1-dimensional model of turbulence exhibiting a fair qualitative similarity to the actual flux driven anomalous transport events\(^3\).

We specify the random radial coordinate of a growing blob of high particle density by the real number \( x \in [0, 1] \). Another real number \( R \in [0, 1] \) is for the coordinate of wall. The blob is supposed to be convected by the turbulent flow and grow as long as \( x < R \) and it is destroyed otherwise. We consider \( x \) as a random variable distributed with respect to some given pdf \( P(x < u) = R(u) \). Furthermore, we suppose that with some probability \( 0 < \eta < 1 \) the growing blob can be drawn by the large scale motion either toward the wall or outward it, and therefore, in the frame of reference related to the large scale motion, the coordinate of wall \( R \) can also be considered as a random variable distributed over the unit interval with respect to some pdf \( P(R < u) = Q(u) \). In general, \( F \) and \( Q \) are two arbitrary left-continuous increasing functions satisfying the normalization conditions \( F(0) = Q(0) = 0, F(\infty) = Q(\infty) = 1 \).

The parameter \( \eta \) effectively quantifies the development of turbulence since it introduces a time scale separation between large and small scale motions in the chamber. In the fully developed turbulence ambient, \( \eta = 1 \), the small blobs are dragged by the large scale motions at each time step. In
the contrary, as $\eta=0$, the turbulence is supposed to be undeveloped and the wall appears to be motionless.

The discrete time random process is defined in the following way. At time $t=0$, the variable $x$ is chosen with respect to pdf $F$, and $R$ is chosen with respect to pdf $Q$. If $x < R$, the process continues and goes to time $t=1$. Otherwise, provided $x \geq R$, the process is eliminated. At time $t \geq 1$, the following events happen:

i) with probability $\eta$, the random variable $x$ is chosen with pdf $F$, but the threshold $R$ keeps the value it had at time $t-1$. Otherwise,

ii) with probability $1-\eta$, the random variable $x$ is chosen with pdf $F$, and $R$ is chosen with pdf $Q$.

When $x \geq R$, the process ends, otherwise it continues and goes to time $t + 1$. Eventually, at some time step $t$ the coordinate of blob drops beyond $R$, so that $\tau$ limits the duration of convective phase. The new blob then arises, and the simulation process is repeated.

We shall be interested in the distribution of durations of convection phases $P_\tau(x; F, Q)$ (denoted as $P(\tau)$ in the what following) provided the probability distributions $F$ and $Q$ are known, and the control parameter $\eta$ is fixed.

The excellence of the above toy model is that for the most $\eta$'s $F$ and $Q$ the pdf of convective phases $P(\tau)$ can be computed analytically. Here, we briefly reproduce the main results for a convenience of readers referring them to $^{12}$ for details.

Introducing the auxiliary functions $K(\eta) = \int F^\eta(u) dQ(u)$ and $\delta K(\eta) = K(\eta) - K(\eta+1)$, one can readily calculate $P_\eta(\tau)$ for the marginal cases (i.e., when there is no large scale motions in the vicinity of walls and in the fully developed turbulent ambience): $P_{\eta=0}(\tau) = K(1)^\tau$ and $P_{\eta=1}(\eta) = \delta K(\eta)$. From the first equation, it is obvious that in the case of $\eta = 0$ which corresponds to the undeveloped turbulence, for any choice of $F$ and $Q$, the probability $P(\tau)$ decays exponentially. In the opposite case $\eta = 1$, $P(\tau)$ depends upon the particular choice of pdf $F$ and $Q$. In particular, there exists a class of functions for which $P(\tau)$ forms a power law tail $^{12}$. For instance, one can take $F(u) = (1+\alpha)u^\beta$, $\alpha > -1$, and $G(u) = (1+\beta)(1-u)^\beta$, $\beta > -1$, then $P_{\eta=1}(\tau) \approx \tau^{-(\alpha+\beta)}(1+\beta)\Gamma(2+\beta)/(1+\alpha)^{(1+\beta)}$. It is possible to prove that for any choice of $F$ and $Q$ and for any fixed $0<\eta<1$ the decay of distribution $P(\tau)$ is bounded by exponentials. For the lower bound, one has an estimation $P(\tau) \geq \eta^\tau P_{\eta=1}(\tau) + (1-\eta^\tau) P_{\eta=0}(\tau)$. For the upper bound, it is $P(\tau) \leq \eta^\tau P_{\eta=1}(\tau) + (1-\eta^\tau) P_{\eta=0}(\tau)(1+K(1))^{-\tau} \leq \eta^\tau K(1)(1+K(1))^{-\tau} - \eta^\tau K(1)^{-\tau}$.

The simpler and explicit expressions can be given for $P(\tau)$ provided the probability density functions are uniform $dF(u) = dQ(u) = du$ for all $u \in [0, 1]$. Then we have $P_{\eta=0}(\tau) = 2^{-\tau+1}$, $P_{\eta=1}(\tau) = 1/(\tau+1)(\tau+2)$. For the intermediate values of $\eta$, the decay rate is mixed (see Figure 6).

10. Conclusion.

We have discussed the two-dimensional stochastic model of turbulent transport in the Scrape-Off Layer of ITER. The improvement of understanding of the intermittent ballistic transport in the regions close to the wall components is of crucial importance for the forthcoming nuclear fusion studies and would help to find the optimal engineering solutions for the “next step device”.

The dynamical equations for the fluctuations of particle density and vorticity fields have been supplied with the Gaussian distributed random forces maintaining the system out of equilibrium and modeling the sheath boundary conditions. Solutions of simplified dynamical equations show that there are two concurrent symmetries in the problem. As a result, the system exhibits a convective instability: the density of particles would grow up unboundedly in a response for the small perturbations of either density or vorticity. Dissipation processes in plasmas smear the picture, so that some fluctuations would acquire stability.

In our model, we have formulated a criterion of stability for the fluctuations of particle density. It depends upon the certain plasma wave interactions and the certain local scenario of turbulence. Herewith, the direct turbulent cascade proposed by Kolmogorov and the inverse cascade
are the marginal scenarios of turbulence. These two factors are always on the stage, and their interplay forms a complicated dynamical picture. For instance, it is believed now that the divertor wall components could be isolated from the strong outward bursts of density by biasing part of the limiter surface. We have studied such a possibility in our model and found that the biasing would provide a fairly good insulation from the most of fluctuations of density excepting for some resonant cases that is of a special interest.

Pdf of the particle flux is determined in our model by the spectra of wandering times of high density blobs being convected by the turbulent flow. To estimate these spectra, we have considered a 1-dimesional toy model of turbulence with the control parameter $\eta$ that effectively quantifies the development of turbulence by introducing a time scale separation between large and small scale motions in the chamber. The statistics of particle flux in our model is bounded by exponentials.

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12. Figures

Figure 1: The “band of interest” in the space of parameters of the model (2,4,7).

Figure 2: The first tree – diagrams for the iterative solutions of problem (2,4,7).
Figure 3: The diagram representation of Dyson’s equations for the simplest response functions.

\[
\langle \frac{\delta n}{\delta f_n} \rangle = \frac{\Delta_n(p, \omega)}{\Delta_{mm}(k-p, -\omega)} k + g_2 \frac{\Delta_{nn}(p, \omega)}{\Delta_{nn}(k-p, -\omega)} k + \cdots
\]

\[
\langle \frac{\delta n}{\delta f_n} \rangle = \frac{\Delta_n(p, \omega)}{\Delta_{mm}(k-p, -\omega)} k + g_2 \frac{\Delta_{nn}(p, \omega)}{\Delta_{nn}(k-p, -\omega)} k + \cdots
\]

\[
\Delta = i \varepsilon \varepsilon \kappa \kappa
\]

\[
\square = -i g \gamma
\]

\[
\h = -i \varepsilon \varepsilon \kappa \kappa / \kappa^2
\]

Figure 4: The signature of \( A(\varepsilon, \gamma)^{-1}|_{1\text{-loop}} \) in the space of parameters (grey is for +1, white is for -1). Tilted lines show the values \(-1 + \varepsilon = \gamma \mod 1\) where \( A(\varepsilon, \gamma)^{-1}|_{1\text{-loop}} = 0 \).
Figure 5: The set of points in the space of parameters $\varepsilon$ and $\gamma$ where the model (2,4,7) is unstable and cannot be stabilized by the biasing of limiter surface.

Figure 6: The distributions of wandering times near the wall in the discrete time model: the case of uniform densities, $dF(u) = dG(u) = du$, for all $u \in [0,1)$, at different values of control parameter $\eta$.

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