Generalized down-up algebras revisited from a viewpoint of Gröbner basis theory

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ABSTRACT

The so-called generalized down-up algebras are revisited from a viewpoint of Gröbner basis theory. Particularly, it is shown explicitly that generalized down-up algebras are solvable polynomial algebras (provided \( \lambda \omega \neq 0 \)), and by means of homogeneous Gröbner defining relations, the associated graded structures of generalized down-up algebras, namely the associated graded algebras, Rees algebras, and the homogenized algebras of generalized down-up algebras, are explored comprehensively.

ARTICLE HISTORY

Received 3 October 2021
Revised 8 January 2022
Communicated by Ellen Kirkman

KEYWORDS

Filtered algebra; generalized down-up algebra; graded algebra; Gröbner basis

2020 MATHEMATICS SUBJECT CLASSIFICATION
16W70; 16Z05

1. Introduction

Let \( K \) be a field, and \( U(\mathfrak{sl}(2,K)) \) the enveloping algebra of the 3-dimensional Lie algebra \( \mathfrak{sl}(2,K) = Kx \oplus Ky \oplus Kz \) defined by the bracket product: \( [x,y] = z, [z,x] = 2x, [z,y] = -2y \). Due to their importance in physics, particularly in quantum group theory, deformations of \( U(\mathfrak{sl}(2,K)) \) over \( K \) have been studied in great generality in various contexts (e.g., \cite{12, 13, 24}), among which, the down-up algebras introduced in \cite{1, 2} and the generalized down-up algebras introduced in \cite{5} have covered a large class of deformations of \( U(\mathfrak{sl}(2,K)) \), and many basic structural properties of down-up algebras and generalized down-up algebras have been explicitly established respectively in pure ring theoretic methods, methods of representation theory, and geometric methods (e.g., \cite{3, 4, 8, 9, 11}). However, it seems that there has been no an unified approach to the investigation of generalized down-up algebras by using the Gröbner basis methods, though in \cite{16, 17} the Gröbner defining relations of more general algebras similar to \( U(\mathfrak{sl}(2,K)) \) (including generalized down-up algebras) have been established. In this note, we revisit generalized down-up algebras via their Gröbner defining relations, recapture most of the known structural properties of such algebras, show explicitly that if \( \lambda \omega \neq 0 \) then generalized down-up algebras are all solvable polynomial algebras in the sense of \cite{10} (thereby they are equipped with an effective Gröbner basis theory), and by means of the homogeneous Gröbner defining relations, the associated graded structures of generalized down-up algebras, namely the associated graded algebras, Rees algebras, and the homogenized algebras of generalized down-up algebras, are explored comprehensively.
Throughout this note, $K$ denotes an algebraically closed field, $K^* = K - \{0\}$, and all $K$-algebras considered are associative with multiplicative identity 1. If $S$ is a nonempty subset of an algebra $A$, then we write $\langle S \rangle$ for the two-sided ideal of $A$ generated by $S$.

Moreover, for the reader’s convenience of understanding the Auslander regularity and the Cohen–Macaulay property of an algebra in Sections 4 and 5, let us recall that a finitely generated algebra $A$ of finite Gelfand–Krillov dimension $n$ is said to

(a) be Auslander regular if $A$ has finite global homological dimension and, for every finitely generated left $A$-module $M$, every integer $j \geq 0$ and every (right) $A$-submodule $N$ of $\text{Ext}^j_A(M, A)$ we have that $j(N) \geq j$, where $j(N)$ is the grade number of $N$ which is the least integer $i$ such that $\text{Ext}^i_A(M, A) \neq 0$;
(b) satisfy the Cohen–Macaulay property if for every finitely generated left $A$-module $M$ we have the equality: $\text{GK.dim}M + j(M) = n$, where $\text{GK.dim}$ denotes the Gelfand–Krillov dimension.

Concerning the Auslander regularity and the Cohen–Macaulay property of filtered rings and graded rings, in particular, of the generalized down-up algebras and their associated graded structures, one is referred to [13, 18, 20].

2. The Gröbner defining relations of generalized down-up algebras

In this section, we review, in a little more detail, the introduction of generalized down-up algebras in the sense of [5], so as to strengthen the connection of such algebras with other important algebras. Moreover, by referring to [5, 16, 17], we conclude that the set of defining relations of a generalized down-up algebra forms a Gröbner basis in the sense of [23].

Let $K\langle X \rangle = K\langle X_1, X_2, X_3 \rangle$ be the free $K$-algebra generated by $X = \{X_1, X_2, X_3\}$. Consider the algebra $A = K\langle X \rangle / \langle G \rangle$ with $G$ consisting of

\begin{align*}
&g_{31} = X_3X_1 - \lambda X_1X_3 + \gamma X_3, \\
&g_{12} = X_1X_2 - \lambda X_2X_1 + \gamma X_2, \\
&g_{32} = X_3X_2 - \omega X_2X_3 + f(X_1),
\end{align*}

where $\lambda, \gamma, \omega \in K$, and $f(X_1)$ is a polynomial in the variable $X_1$. In [5], this algebra is referred to as a generalized down-up algebra, and is denoted by $L(f, \lambda, \omega, \gamma)$.

Convention. For the purpose of this note and for saving notation, throughout this note we will use $A$, with the presentation $A = K\langle X \rangle / \langle G \rangle$, to denote a generalized down-up algebra with the set of defining relations $G$ as described above.

As one may see from the literature, or as illustrated below, the algebras $A$ defined above are mainly stemming from two topics:

1. The investigation of deformations of the enveloping algebra $U(\mathfrak{sl}(2, K))$ of the 3-dimensional Lie algebra $\mathfrak{sl}(2, K)$.
   (a) With $\lambda = \omega = 1$, $\gamma = 2$ and $f(X_1) = -X_1$ in (*) , it is clear that $A = U(\mathfrak{sl}(2, K))$.
   (b) In [24], Smith introduced a family of algebras $A$ similar to $U(\mathfrak{sl}(2, K))$, that is, $G$ consists of
      \begin{align*}
      g_{31} &= X_1X_3 - X_3X_1 = X_3, \\
      g_{12} &= X_1X_2 - X_2X_1 = -X_2, \\
      g_{32} &= X_3X_2 - X_2X_3 = f(X_1).
      \end{align*}
   (c) Let $\zeta \in K^*$. With $\lambda = \zeta^4$, $\omega = \zeta^2$, $\gamma = -(1 + \zeta^2)$, $a = 0 = c$, and $b = -\zeta$, the algebra $A$ coincides with Woronowicz’s deformation of $U(\mathfrak{sl}(2, K))$ which was introduced in the noncommutative differential calculus [26].
(d) If \( f(X_1) = bX_1^2 + X_1 \) and \( \lambda \gamma \alpha b \neq 0 \), then \( A \) coincides with Le Bruyn’s conformal \( \mathfrak{sl}_2 \) enveloping algebra [13, Lemma 2] which provides a special family of Witten’s deformation of \( U(\mathfrak{sl}(2, K)) \) in quantum group theory [28].

2. The investigation of down-up algebras in the sense of \([1, 2]\).

(e) Let \( K(X_1, X_2) \) be the free \( K \)-algebra generated by \( \{X_1, X_2\} \). The down-up algebra \( A(\alpha, \beta, \gamma) \), in the sense of \([1, 2]\), was introduced in the study of algebras generated by the down and up operators on a differential or uniform partially ordered set (poset), that is \( A(\alpha, \beta, \gamma) = K(X_1, X_2)/(S) \) with \( S \) consisting of

\[
\begin{align*}
  f_1 &= X_1^2 X_2 - \alpha X_1 X_2 X_1 - \beta X_2 X_1^2 - \gamma X_1,  \\
  f_2 &= X_1 X_2^2 - \alpha X_2 X_1 X_2 - \beta X_1^2 X_1 - \gamma X_2,
\end{align*}
\]

where \( \alpha, \beta, \gamma \in K \). If, in the foregoing definition of a generalized down-up algebra \( A \), the polynomial \( f(X_1) \) has degree one, then all down-up algebras \( A(\alpha, \beta, \gamma) \) are retrieved for suitable choices of the parameters of \( A \) \([3–5, 9]\), that is, each down-up algebra is isomorphic to some generalized down-up algebra \( A \). For more detailed argumentation on this result, see \([3, Lemma 1.1]\).

At this stage, also let us point out that the down-up algebras have been connected with many more important algebras, for instance, the generalized Weyl algebra (see \([5]\)), the hyperbolic rings (see \([11, Proposition 3.0.1]\)), and the parafermionic (parabosonic) algebra which is closely related to the cubic Artin–Schelter regular algebras (see \([7]\)), namely the natural action of \( GL(2) \) on the parafermionic algebra for \( D = 2 \) extends as an action of the quantum group \( GL_{p, q}(2) \) on the generic cubic Artin–Schelter regular algebra of type \( S_1 \) with the defining relations

\[
\begin{align*}
  g_1 &= X_2 X_1^2 + q r X_1 X_2 - (q + r) X_1 X_2 X_1,  \\
  g_2 &= X_2^2 X_1 + q r X_1 X_2^2 - (q + r) X_2 X_1 X_2,
\end{align*}
\]

where \( q, r \in \mathbb{C} \).

Now, let \( A \) be a generalized down-up algebra. Then, in either of the following two cases:

(a) \( \deg f(X_1) = n \leq 2 \), \( X_1, X_2 \), and \( X_3 \) are all assigned the degree 1;

(b) \( \deg f(X_1) = n \geq 1 \), \( X_1 \) is assigned the degree 1, but \( X_2 \) and \( X_3 \) are all assigned the degree \( n \),

the set \( \mathcal{G} = \{g_{31}, g_{12}, g_{32}\} \) of defining relations of \( A \) forms a Gröbner basis for the ideal \( \langle \mathcal{G} \rangle \) in the sense of \([23]\), where in both cases the monomial ordering used on \( K(X) \) is the graded lexicographic ordering

\[
X_2 \prec_{\text{grlex}} X_1 \prec_{\text{grlex}} X_3.
\]

Thereby \( A \) has the PBW \( K \)-basis \( B = \{a_k^i a_j^l \in \mathbb{N}^3 \} \), where \( a_k \) is the coset represented by \( X_k \) in \( A, 1 \leq k \leq 3 \). One may refer to \(([17, Ch. 4, Section 3], [5, Theorem 2.1])\) for detailed argumentations, though the notion of a Gröbner basis is not obviously used by \([5]\). For convenience of later usage, we especially record this result here.

**Proposition 2.1.** With notation as above, the following statements hold.

(i) In both the cases (a) and (b) above, \( \mathcal{G} \) is a Gröbner basis of the ideal \( \langle \mathcal{G} \rangle \) with respect to the monomial ordering \( X_2 \prec_{\text{grlex}} X_1 \prec_{\text{grlex}} X_3 \) on \( K(X) \).

(ii) The generalized down-up algebra \( A = K(X)/(\mathcal{G}) \) has the PBW \( K \)-basis

\[
B = \{a_k^i a_j^l \in \mathbb{N}^3 \},
\]

where \( a_k \) is the coset represented by \( X_k \) in \( A, 1 \leq k \leq 3 \). \( \square \)
Remark. The reason that in the above argumentation we distinguish the cases (a) and (b), is to obtain Theorem 4.2(i), Theorem 4.3(ii), and Theorem 5.3(iv) in Sections 4 and 5, respectively.

### 3. Generalized down-up algebras are solvable polynomial algebras provided $\lambda \omega \neq 0$

In an example of [17, p. 154], without proof in detail it is concluded that if $\lambda \omega \neq 0$, then the corresponding generalized down-up algebras are all solvable polynomial algebras in the sense of [10]. It means that the Gröbner basis theory and computational methods for solvable polynomial algebras [10, 16, 17] can be completely applied to generalized down-up algebras. Instead of going to make a long story about this topic, in this section we just give a detailed argumentation of the conclusion mentioned above, and from this fact we re-derive two basic structural properties of generalized down-up algebras given in [5].

First recall from [10, 16, 17, 21] the following definitions. Suppose that a finitely generated $K$-algebra $A = K[a_1, \ldots, a_n]$ has the PBW $K$-basis $B = \{a^\alpha = a_1^{\alpha_1} \cdots a_n^{\alpha_n} \mid \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n\}$, and that $\prec$ is a total ordering on $B$. Then every nonzero element $f \in A$ has a unique expression

$$f = \lambda_1 a^{\alpha(1)} + \lambda_2 a^{\alpha(2)} + \cdots + \lambda_m a^{\alpha(m)},$$

such that $a^{\alpha(1)} \prec a^{\alpha(2)} \prec \cdots \prec a^{\alpha(m)},$

where $\lambda_j \in K^*$, $a^{\alpha(j)} = a_1^{\alpha_{1j}} a_2^{\alpha_{2j}} \cdots a_n^{\alpha_{nj}} \in B$, $1 \leq j \leq m$.

Noticing that elements of $B$ are conventionally called monomials, the leading monomial of $f$ is defined as $\text{LM}(f) = a^{\alpha(m)}$, the leading coefficient of $f$ is defined as $\text{LC}(f) = \lambda_m$, and the leading term of $f$ is defined as $\text{LT}(f) = \lambda_m a^{\alpha(m)}$.

**Definition 3.1** Suppose that the $K$-algebra $A = K[a_1, \ldots, a_n]$ has the PBW basis $B$. If $\prec$ is a total ordering on $B$ that satisfies the following three conditions:

1. $\prec$ is a well-ordering (i.e., every nonempty subset of $B$ has a minimal element);
2. For $a^\beta, a^\gamma, a^\delta, a^\eta \in B$, if $a^\beta \neq 1$, $a^\delta \neq a^\eta$, and $a^\beta = \text{LM}(a^\gamma a^\delta a^\eta)$, then $a^\delta \prec a^\gamma$ (thereby $1 \prec a^\gamma$ for all $a^\gamma \neq 1$);
3. For $a^\beta, a^\gamma, a^\delta, a^\eta \in B$, if $a^\gamma \prec a^\delta$, $\text{LM}(a^\gamma a^\delta a^\eta) \neq 0$, and $\text{LM}(a^\gamma a^\delta a^\eta) \notin \{0, 1\}$, then $\text{LM}(a^\gamma a^\delta a^\eta) \prec \text{LM}(a^\gamma a^\delta a^\eta)$,

then $\prec$ is called a monomial ordering on $B$ (or a monomial ordering on $A$).

**Definition 3.2.** A finitely generated $K$-algebra $A = K[a_1, \ldots, a_n]$ is called a solvable polynomial algebra if $A$ has the PBW $K$-basis $B = \{a^\alpha = a_1^{\alpha_1} \cdots a_n^{\alpha_n} \mid \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n\}$ and a monomial ordering $\prec$ on $B$, such that for $\lambda_{ji} \in K^*$ and $f_{ji} \in A$,

$$a_i a_j = \lambda_{ji} a_j a_i + f_{ji}, \ 1 \leq i < j \leq n,$$

$$\text{LM}(f_{ji}) \prec a_i a_j \quad \text{whenever} \quad f_{ji} \neq 0.$$

By [10], every (two-sided, respectively one-sided) ideal in a solvable polynomial algebra $A$ has a finite Gröbner basis with respect to a given monomial ordering; in particular, for one-sided ideals there is a noncommutative Buchberger algorithm which has been successfully implemented in the computer algebra system Plural.

Let $K\langle X \rangle = K\langle X_1, X_2, X_3 \rangle$ be the free $K$-algebra generated by $X = \{X_1, X_2, X_3\}$, and let $A = K\langle X \rangle/\langle G \rangle$ be a generalized down-up algebra as defined in Section 2. Writing $A = K[a_1, a_2, a_3]$, where $a_i$ is the coset represented by $X_i$ in $A$, $1 \leq i \leq 3$, we now start on showing the following

**Theorem 3.3.** If $\lambda \omega \neq 0$ and $\deg f(X_1) \geq 1$ in the defining relations of $A$, then $A = K[a_1, a_2, a_3]$ is a solvable polynomial algebra in the sense of Definition 3.2.
Proof. First, it follows from Proposition 2.1 that \( A \) has the PBW \( K \)-basis
\[
B = \{ a_2^i a_1^j a_3^k \mid (i, j, k) \in \mathbb{N}^3 \}.
\]

Second, assigning \( a_1 \) the degree 1, \( a_2 \) and \( a_3 \) the degree \( n = \deg(f(a_1)) \), if we take the graded lexicographic ordering \( a_2 \prec_{\text{grlex}} a_1 \prec_{\text{grlex}} a_3 \) on \( B \), then the following statements hold:

(a) \( \prec_{\text{grlex}} \) is clearly a well-ordering on \( B \), i.e., \( \prec_{\text{grlex}} \) satisfies the condition (1) of Definition 3.1.

(b) \( \prec_{\text{grlex}} \) satisfies the condition (2) and (3) of Definition 3.1. To see this, note that \( \prec_{\text{grlex}} \) compares degree first, namely if \( u, v \in B \), then \( u \prec_{\text{grlex}} v \) implies \( \deg u < \deg v \). Also note that \( a_1, a_2, a_3, a_1 a_3, a_2 a_1, a_2 a_3 \in B \), and \( \lambda \omega \neq 0 \). It follows that the generators of \( A \) satisfy
\[
\begin{align*}
    a_3 a_1 &= \lambda a_1 a_3 - \gamma a_3 & &\text{with } \LM(\gamma a_3) = a_3 \prec_{\text{grlex}} a_1 a_3 & &\text{if } \gamma \neq 0, \\
    a_1 a_2 &= \lambda a_2 a_1 - \gamma a_2 & &\text{with } \LM(\gamma a_2) = a_2 \prec_{\text{grlex}} a_2 a_1 & &\text{if } \gamma \neq 0, \\
    a_3 a_2 &= \omega a_2 a_3 - f(a_1) & &\text{with } \LM(f(a_1)) \prec_{\text{grlex}} a_2 a_3.
\end{align*}
\]

Thus, for any \( u = a_2^i a_1^j a_3^k, v = a_2^l a_1^m a_3^n \in B \), the above properties of generators give rise to
\[
uv = (a_2^i a_1^j a_3^k)(a_2^l a_1^m a_3^n) = \rho_{u,v} a_2^{i+l} a_1^{j+m} a_3^{k+n} + h \quad \text{with } \rho_{u,v} \in K^*, \quad h \in K\text{-span}B,
\]
\[
\deg h < \deg(a_2^{i+l} a_1^{j+m} a_3^{k+n}) = (i_1 + i_2 + \ell_1 + \ell_2)n + j_1 + j_2,
\]
thereby \( \LM(h) \prec_{\text{grlex}} a_2^{i+l} a_1^{j+m} a_3^{k+n} = \LM(uv) \).

This enables us to verify easily that \( \prec_{\text{grlex}} \) satisfies the condition (2) and (3) of Definition 3.1.

Summing up, \( \prec_{\text{grlex}} \) is a monomial ordering on \( B \) and therefore \( A \) is a solvable polynomial algebra in the sense of Definition 3.2.

Concerning down-up algebras (see example (e) of Section 2), it follows from \([4, 5, 9]\) and Theorem 3.3 that we have the following corollary.

Corollary 3.4. If \( \lambda \omega \neq 0 \) and \( f(X_1) = X_1 \), then all down-up algebras \( A(\alpha, \beta, \gamma) \) with \( \alpha = \lambda + \omega \) and \( \beta = -\lambda \omega \) are solvable polynomial algebras.

Now, combining Proposition 2.1(i) and Theorem 3.3 with \([10, 16, 17]\) we are able to recapture the following two basic properties of generalized down-up algebras. One may compare all proofs we presented below with those given in \([5]\).

Corollary 3.5. Let \( A = K[X]/(G) \) be a generalized down-up algebra in the sense of Section 2. The following statements hold.

(i) \( A \) has Gelfand–Kirillov dimension three.

(ii) If \( \lambda \omega \neq 0 \), then \( A \) is a Noetherian domain.

Proof.

(i) Based on Proposition 2.1(i), this is just a special case of a more general result given in \([17, \text{p. 167, Example 3}]\), where the Ufnarovski graph \([27]\) associated to a Gröbner basis is employed.

(ii) Since \( A \) is now a solvable polynomial algebra (Theorem 3.3), this follows from the classical result that a solvable polynomial algebra is a domain and every (left, right) ideal of \( A \) has a finite Gröbner basis \([10]\).
4. The associated graded algebras of generalized down-up algebras

In the literature concerning algebras similar to $U(\mathfrak{sl}(2,K))$, particularly in [5, 9, 13], filtered-graded techniques have been employed for establishing some important structural properties, such as global dimension, Auslander regularity, Artin–Schelter regularity, Koszulity, etc. By using Proposition 2.1(i) and certain related results established in [16, 17, 22], in this section we first present clearly the Gröbner defining relations of the associated graded algebra of a generalized down-up algebra, and then we derive (or recapture) two homological properties of generalized down-up algebras.

For our purpose, we start with a little generality on the naturally filtered structure and the associated graded structure for a $K$-algebra $A = K\langle X \rangle / \langle G \rangle$, where $K\langle X \rangle = K\langle X_1, X_2, X_3 \rangle$ is the free $K$-algebra generated by $X = \{X_1, X_2, X_3\}$, and $\langle G \rangle$ is the (two-sided) ideal of $K\langle X \rangle$ generated by a subset $G \subset K\langle X \rangle$. Let $\mathbb{B}$ be the standard $K$-basis of $K\langle X \rangle$. If $X_1, X_2,$ and $X_3$ are assigned the positive degrees $n_1, n_2,$ and $n_3$ respectively, then $K\langle X \rangle$ is turned into an $\mathbb{N}$-graded algebra, i.e., $K\langle X \rangle = \bigoplus_{t \in \mathbb{N}} K\langle X \rangle_t$ with $K\langle X \rangle_t = K\text{-span}\{u \in \mathbb{B} \mid \deg u = t\}$, such that $K\langle X \rangle_{t_1} K\langle X \rangle_{t_2} \subseteq K\langle X \rangle_{t_1 + t_2}$ for all $t_1, t_2 \in \mathbb{N}$. If we further consider the grading filtration $F K \langle X \rangle$ of $K\langle X \rangle$, that is $F K \langle X \rangle = \{F_q K \langle X \rangle\}_{q \in \mathbb{N}}$ with $F_q K \langle X \rangle = \bigoplus_{t \leq q} K\langle X \rangle_t$, then $K\langle X \rangle$ is turned into an $\mathbb{N}$-filtered algebra, i.e., $K\langle X \rangle = \bigcup_{q \in \mathbb{N}} F_q K \langle X \rangle$ and $F_q K \langle X \rangle F_{q'} K \langle X \rangle \subseteq F_{q + q'} K \langle X \rangle$ for all $q, q' \in \mathbb{N}$. Taking this grading filtration $F K \langle X \rangle$ of $K\langle X \rangle$ into account, the algebra $A$ has the induced filtration $F A = \{F_q A\}_{q \in \mathbb{N}}$ with $F_q A = (F_q K \langle X \rangle + \langle G \rangle) / \langle G \rangle$, such that $A = \bigcup_{q \in \mathbb{N}} F_q A$ and $F_q A F_{q'} A \subseteq F_{q + q'} A$ for all $q, q' \in \mathbb{N}$, thereby $A$ is turned into a filtered $K$-algebra. Thus, with respect to $FA$, $A$ has its associated $\mathbb{N}$-graded $K$-algebra $G(A) = \bigoplus_{q \in \mathbb{N}} G(A)_q$ with $G(A)_q = F_q A / F_{q-1} A$.

For a nonzero $f = h_0 + h_1 + \cdots + h_q \in K\langle X \rangle$ with $h_i \in K\langle X \rangle_i$ and $h_q \neq 0$, we write

$$\text{LH}(f) = h_q$$

for the highest degree homogeneous element of $f$ and call $\text{LH}(f)$ the leading homogeneous element of $f$.

**Proposition 4.1.** Let the algebra $A = K\langle X \rangle / \langle G \rangle$ and its associated graded algebra $G(A)$ be as fixed above, and let $\prec_{\text{grlex}}$ be the graded lexicographic monomial ordering on $K\langle X \rangle = K\langle X_1, X_2, X_3 \rangle$ with respect to some positive degrees $n_1, n_2,$ and $n_3$ assigned to $X_1, X_2,$ and $X_3$ respectively. Then,

(i) $G(A) \cong K\langle X \rangle / \langle \text{LH}(\langle G \rangle) \rangle$, where $\text{LH}(\langle G \rangle) = \{\text{LH}(f) \mid f \in \langle G \rangle\}$.

(ii) With respect to $\prec_{\text{grlex}}$ $G$ is a Gröbner basis of $\langle G \rangle$ if and only if $\text{LH}(\langle G \rangle)$ is a homogeneous Gröbner basis of the graded ideal $\langle \text{LH}(\langle G \rangle) \rangle$.

**Proof.** The assertions (i) and (ii) are just special cases of [22] and [16, Ch. III, Ch. IV], or more precisely, the special cases of [17, Ch. 2, Theorem 3.2; Ch. 4, Proposition 2.2].

Now, turning to generalized down-up algebras, it follows from Propositions 2.1 and 4.1 that we have the following

**Theorem 4.2.** With notation as in Section 2, let $A = K\langle X \rangle / \langle G \rangle$ be a generalized down-up algebra and suppose that for $f(X_1)$ in $g_{32} \in G$, the degree $\text{degf}(X_1) \geq 1$. The following statements hold.

(i) If $\text{degf}(X_1) = n \leq 2$, say $f(X_1) = aX_1^2 + bX_1 + c$, $X_1$, $X_2$, and $X_3$ are all assigned the degree 1, then $\text{LH}(\langle G \rangle) = \{X_3 X_1 - \lambda X_1 X_3, X_1 X_2 - \lambda X_2 X_1, X_3 X_2 - \omega X_2 X_3 + aX_3^2\}$ is a homogeneous Gröbner basis of the ideal $\langle \text{LH}(\langle G \rangle) \rangle$ with respect to the monomial ordering $X_3 \prec_{\text{grlex}} X_1 \prec_{\text{grlex}} X_3$ on $K\langle X \rangle$, such that $\text{LM}(\langle \text{LH}(\langle G \rangle) \rangle) = \{X_3 X_1, X_1 X_2, X_3 X_2\}$.

If $\text{degf}(X_1) = n \geq 1$, $X_1$ is assigned the degree 1, but $X_2$ and $X_3$ are all assigned the degree $n$, then $\text{LH}(\langle G \rangle) = \{X_3 X_1 - \lambda X_1 X_3, X_1 X_2 - \lambda X_2 X_1, X_3 X_2 - \omega X_2 X_3\}$ is a homogeneous Gröbner
Let $G(A)$ be the associated graded algebra $G(A)$ of $A$ as described above, then $G(A) \cong K\langle X \rangle/(\mathbf{LH}(G))$. \hfill \Box$

**Theorem 4.3.** With notation and assumption as in Theorem 4.2, the following statements hold.

(i) $GK.dimG(A) = 3 = GK.dimA$, and $G(A)$ has global homological dimension $gl.dimG(A) = 3$, $A$ has $gl.dimA \leq 3$.

(ii) If $\omega \neq 0$ and $f(X_1) = aX_1^2 + bX_1 + c$ with $ab \neq 0$, then $G(A)$ is the associated graded algebra of the conformal $\mathfrak{sl}(2, K)$ enveloping algebra, thereby all results obtained in [13] hold true for $G(A)$, particularly $G(A)$ is an Auslander regular algebra satisfying Cohen–Macaulay property (see the definitions given in Section 1), and so too is $A$.

(iii) If $\lambda \omega \neq 0$ and $\deg f(X_1) = n \geq 1$, then $G(A)$ is a solvable polynomial algebra in the sense of Definition 3.2.

(iv) If $\deg f(X_1) = n \geq 1$, then, with $\lambda \omega \neq 0$, $G(A)$ is an Auslander regular algebra satisfying Cohen–Macaulay property, and so too is $A$.

**Proof.**

(i) This follows from Theorem 4.2 and [17, Ch. 5, Corollary 7.6].

(ii) By Theorem 4.2, if $\deg f(X_1) = n \leq 2$, $X_1$, $X_2$, and $X_3$ are all assigned the degree 1, then $G(A) \cong K\langle X \rangle/(\mathbf{LH}(G))$ where $\mathbf{LH}(G) = \{X_3X_1 - \lambda X_1X_3, X_1X_2 - \lambda X_2X_1, X_3X_2 - \omega X_2X_3 + aX_1^2\}$. Note that if $\lambda \omega a \neq 0$, then $G(A)$ is a conformal $\mathfrak{sl}(2, K)$ enveloping algebra, thereby all results obtained in [13] hold true for $A$, particularly $G(A)$ is an Auslander regular algebra satisfying Cohen–Macaulay property, and so too is $A$.

(iii) By Theorem 3.3, $A = K[a_1, a_2, a_3]$ is a solvable polynomial algebra with respect to the graded lexicographic monomial ordering $a_2 \prec_{\text{grlex}} a_1 \prec_{\text{grlex}} a_3$, where $a_i$ is the coset represented by $X_i$ in $A$ ($= K\langle X \rangle/(G)$), and $a_1$ is assigned the degree 1 and $a_2, a_3$ are all assigned the degree $n = \deg f(X_1)$. So the proof of this assertion is just an analogue of the proof of a similar result established in [21, Section 3] and [16, Ch. IV, Section 4] (where general quadric solvable polynomial algebras are considered).

(iv) By Theorem 4.2, if $\deg f(X_1) = n \geq 1$, $X_1$ is assigned the degree 1, but $X_2$ and $X_3$ are all assigned the degree $n$, then $G(A) \cong K\langle X \rangle/(\mathbf{LH}(G))$ where $\mathbf{LH}(G) = \{X_3X_1 - \lambda X_1X_3, X_1X_2 - \lambda X_2X_1, X_3X_2 - \omega X_2X_3\}$ with $\mathbf{LM}(\mathbf{LH}(G)) = \{X_3X_1, X_1X_2, X_3X_2\}$. Note that if $\lambda \omega \neq 0$, then $G(A)$ is clearly a skew polynomial algebra over $K$. It follows from [20] (or by a similar argumentation as given in [6, 13, 18]) that $G(A)$ is an Auslander regular algebra satisfying Cohen–Macaulay property, and so too is $A$.

5. The homogenized algebras of generalized down-up algebras

Let $K\langle X \rangle = K\langle X_1, X_2, X_3 \rangle$ be the free $K$-algebra generated by $X = \{X_1, X_2, X_3\}$ and consider a generalized down-up algebra $A = K\langle X \rangle/(G)$ in the sense of Section 2. In this section, by using the Gröbner basis techniques we establish some basic structural properties of the homogenized algebras of generalized down-up algebras (see the definition below).

We start by recalling the general notion of a homogenized algebra and some relevant results from the literature (for instance, [13–17, 22, 25]).
Let the free \( K \)-algebra \( K(X) = K(X_1, ..., X_n) \) be equipped with the graded structure \( K(X) = \bigoplus_{q \in \mathbb{N}} K(X)_q \) by assigning each generator \( X_i \) a positive degree \( \deg X_i = m_i > 0, 1 \leq i \leq n \).

Considering the free \( K \)-algebra \( K(X, T) = K(X_1, ..., X_n, T) \) generated by \( \{X_1, ..., X_n, T\} \), let \( K(X, T) \) be equipped with the graded structure \( K(X, T) = \bigoplus_{q \in \mathbb{N}} K(X, T)_q \) by assigning \( \deg X_i = m_i \) for \( 1 \leq i \leq n \), and \( \deg T = 1 \).

Then,

(a) for a nonzero element \( f = \sum_{i=1}^{m} h_i \in K(X) \) with \( h_i \in K(X)_q \), \( q_1 < q_2 < \cdots < q_m \), and \( h_m \neq 0 \), the degree-\( q_m \) homogeneous element \( f = \sum_{i=1}^{m} T^{d_m-q_i}h_i \) of \( K(X, T) \) is referred to as the homogenization of \( f \) in \( K(X, T) \) with respect to \( T \);

(b) for a nonempty subset \( S \subset K(X) \), writing \( I = \langle S \rangle \) for the ideal generated by \( S \), \( A = K(X)/I \), and writing \( \tilde{S} = \{f \mid f \in S\} \cup \{X_iT - TX_i \mid 1 \leq i \leq n\} \), the quotient algebra \( H(A) = K(X, T)/\langle \tilde{S} \rangle \) is referred to as the homogenized algebra of \( A \) with respect to \( T \).

Remark. Related to the theory of quantum groups, the study of homogenized enveloping algebra was proposed by S.P. Smith in [25], and the study of homogenized down-up algebra was proposed by G. Benkart and T. Roby in [2]. In [13], homogenized conformal \( \mathfrak{sl}_2 \) enveloping algebras were used to study modules over conformal \( \mathfrak{sl}_2 \) enveloping algebras. Concerning the representation theory and noncommutative geometry of homogenized enveloping algebras, the reader is referred to, for instance, [13–15]. A study of homogenized algebras, via Gröbner defining relations of algebras, was initiated in [22, part 2] and further extended in [16, 17].

Furthermore, consider the \( \mathbb{N} \)-grading filtration \( FK(X) = \{F_qK(X)\}_{q \in \mathbb{N}} \) of \( K(X) \) with each \( F_qK(X) = \bigoplus_{\ell \leq q} K(X)_\ell \), which gives rise to the \( \mathbb{N} \)-filtration \( FA = \{F_qA\}_{q \in \mathbb{N}} \) of the algebra \( A = K(X)/I \) with each \( F_qA = F_qK(X) + I/I \). Since \( F_{q_1}AF_{q_2}A \subseteq F_{q_1+q_2}A \) for all \( q_1, q_2 \in \mathbb{N} \), this naturally gives rise to an \( \mathbb{N} \)-graded algebra \( \tilde{A} = \bigoplus_{q \in \mathbb{N}} F_qA \) with \( \tilde{A}_q = F_qA \).

This graded algebra is usually referred to as the Rees algebra of \( A \) determined by \( FA \) (see [20], or [16, 17] for more detailed discussion on \( \tilde{A} \)). Concerning the algebras \( H(A) \) and \( \tilde{A} \), the following proposition stem from [22] and [17, Section 7.2].

**Proposition 5.1.** With notation and the degrees assigned to the \( X_i \) and \( T \) as fixed above, let \( \tilde{I} \) be the ideal of \( K(X, T) \) generated by \( \tilde{I} = \{f \mid f \in I\} \cup \{X_iT - TX_i \mid 1 \leq i \leq n\} \). Taking a graded lexicographic ordering \( \prec_{\text{grlex}} \) on the standard \( K \)-basis \( B \) of \( K(X) \) such that

\[
X_{i_1} \prec_{\text{grlex}} X_{i_2} \prec_{\text{grlex}} \cdots \prec_{\text{grlex}} X_{i_n}, \quad 1 \leq i \leq n,
\]

and extending \( \prec_{\text{grlex}} \) to the graded lexicographic ordering \( \prec_{T-\text{grlex}} \) on the standard \( K \)-basis \( B(T) \) of \( K(X, T) \) such that

\[
T \prec_{T-\text{grlex}} X_{i_1} \prec_{T-\text{grlex}} X_{i_2} \prec_{T-\text{grlex}} \cdots \prec_{T-\text{grlex}} X_{i_n}, \quad 1 \leq i \leq n,
\]

if \( G \) is a Gröbner basis of \( I = \langle S \rangle \) with respect to \( \prec_{\text{grlex}} \), then

\[
\tilde{G} = \{g \mid g \in G\} \cup \{X_iT - TX_i \mid 1 \leq i \leq n\}
\]

is a homogeneous Gröbner basis of the graded ideal \( \langle \tilde{I} \rangle \) in \( K(X, T) \) with respect to \( \prec_{T-\text{grlex}} \).

\( \langle \tilde{S} \rangle = \langle \tilde{I} \rangle \), and there is a graded algebra isomorphism

\[
H(A) = K(X, T)/\langle \tilde{G} \rangle \cong \tilde{A}.
\]

Applying Proposition 5.1 to generalized down-up algebras, we may derive the following

**Theorem 5.2.** Let \( A = K(X)/\langle \tilde{G} \rangle \) be a generalized down-up algebra with \( \deg f(X_i) = n \geq 1 \) in \( \tilde{G} \). With notation as in Section 2 and those fixed above, the following statements hold.
(i) $\tilde{G} = \{ \tilde{g} \mid g \in G \} \cup \{ X_i T - TX_i \mid 1 \leq i \leq 3 \}$ is a homogeneous Gröbner basis of the ideal $\langle \tilde{G} \rangle$ in the free $K$-algebra $K\langle X, T \rangle = K\langle X_1, X_2, X_3, T \rangle$ with respect to the graded lexicographic monomial ordering

$$T <_{T-\text{grlex}} X_2 <_{T-\text{grlex}} X_1 <_{T-\text{grlex}} X_3$$
on K\langle X, T \rangle$, where

$$\deg T = 1 = \deg X_1, \quad \deg X_2 = \deg X_3 = n = \deg f(X_1).$$

(ii) The homogenized algebra $H(A)$ of $A$ is isomorphic to the Rees algebra of $A$, i.e., $H(A) = K\langle X, T \rangle / \langle \tilde{G} \rangle \cong \hat{A}$, where the filtration $F_A$ of $A$ is the one induced by the grading filtration of $K\langle X, T \rangle$ (degrees of $T, X_1, X_2,$ and $X_3$ are assigned as in (i)).

Proof. (i) By Proposition 2.1, the set $G$ of defining relations of $A$ forms a Gröbner basis of the ideal $\langle G \rangle$ with respect to the graded lexicographic monomial ordering

$$X_2 <_{\text{grlex}} X_1 <_{\text{grlex}} X_3,$$

where $\deg X_1 = 1$, $\deg X_2 = \deg X_3 = n = \deg f(X_1) \geq 1$. It follows from Proposition 5.1 that the assertions (i) and (ii) hold true.

Theorem 5.3. Let $A = K\langle X \rangle / \langle G \rangle$ be a generalized down-up algebra with $\deg f(X_1) = n \geq 1$ in $G$. With notation as made above, the following statements hold.

(i) The Gelfand–Kirillov dimension of $\hat{A}$ and $H(A)$ are all equal to 4.

(ii) $\text{gldim} \hat{A} = \text{gl.dim} H(A) = 4$.

(iii) The Hilbert series of $\hat{A}$ and $H(A)$ is $\frac{1}{1-t}$. In particular, the assertions (i), (ii), and (iii) hold true for all down-up algebras.

(iv) If $f(X_1) = aX_1^2 + bX_1 + c$, then $\hat{A}$ is a classical quadratic Koszul algebra, and so too is $H(A)$.

Moreover, if $\lambda \omega \neq 0$ in $G$, then the following statements hold.

(v) $\hat{A}$ is an $\mathbb{N}$-graded solvable polynomial algebra in the sense of Definition 3.2, and so too is $H(A)$.

(vi) $\hat{A}$ is a Noetherian domain, and so too is $H(A)$.

(vii) $\hat{A}$ is an Auslander regular algebra satisfying the Cohen–Macaulay property, and so too is $H(A)$.

Proof. Due to the graded algebra isomorphism $\hat{A} \cong H(A)$ (Proposition 5.1), it is clear that we need only to prove all assertions for $\hat{A}$ below.

By Theorem 5.2, taking the monomial ordering

$$T <_{T-\text{grlex}} X_2 <_{T-\text{grlex}} X_1 <_{T-\text{grlex}} X_3$$
on $K\langle X, T \rangle$, where $\deg T = 1 = \deg X_1$, $\deg X_2 = \deg X_3 = n = \deg f(X_1)$, the Gröbner basis $\tilde{G}$ of $\langle \tilde{G} \rangle$ is now consisting of

$$\tilde{g}_{31} = X_3 X_1 - \lambda X_1 X_3 + \gamma TX_3,$$

$$\tilde{g}_{12} = X_1 X_2 - \lambda X_2 X_1 + \gamma TX_3,$$

$$\tilde{g}_{32} = X_3 X_2 - \omega X_2 X_3 + \sum_{i=0}^{n} a_i T^{2n-i} X_1^i \quad \text{(provided } f(X_1) = \sum_{i=0}^{n} a_i X_1^i),$$

such that $LM(\tilde{G}) = \{ X_3 X_1, X_1 X_2, X_3 X_2, X_1 T, X_2 T, X_3 T \}$. So, the assertions (i), (ii), and (iii) follow from [17, Ch. 7, Corollary 3.6].
(iv) If \( f(X_1) = aX_1^2 + bX_1 + c, \) then by Proposition 2.1 (or the argumentation made before it) we know that in this case the Gröbner basis \( \mathcal{G} \) is obtained with respect to the monomial ordering \( X_2 \prec_{\text{grlex}} X_1 \prec_{\text{grlex}} X_3, \) but \( X_1, X_2, X_3 \) are all assigned the natural degree 1. Thus, \( \mathcal{G} \) consists of quadratic relations. Hence \( \mathcal{A} \) is a classical quadratic Koszul algebra.

We next proceed to deal with the case that \( \lambda \omega \neq 0 \) in the Gröbner basis \( \mathcal{G} \) of \( A. \)

(v) By Theorem 3.3, \( A = K[a_1, a_2, a_3] \) is a solvable polynomial algebra with respect to the graded lexicographic monomial ordering \( a_2 \prec_{\text{grlex}} a_1 \prec_{\text{grlex}} a_3, \) where \( a_i \) is the coset represented by \( X_i \) in \( A, a_1 \) is assigned the degree 1, and \( a_2, a_3 \) are all assigned the degree \( n = \deg f(X_1). \) So the proof of this assertion is just an analogue of the proof of a result established in [21, Section 3] and [16, Ch. IV, Section 4] (where general quadric solvable polynomial algebras are considered).

(vi) As in the proof of Corollary 3.5, since \( \mathcal{A} \) is now a solvable polynomial algebra by the assertion of (v), this follows from the classical result that a solvable polynomial algebra is a domain and every (left, right) ideal of \( A \) has a finite Gröbner basis [10].

(vii) By Theorem 4.3(iii), \( A \) and its associated graded algebra \( G(A) \) are Auslander regular algebra satisfying Cohen–Macaulay property. It follows from [19], [20, Ch. III], and [16, Ch. III]) that \( \tilde{A} \) is an Auslander regular algebra satisfying Cohen–Macaulay property, and so too is \( H(A). \) 

Remark. In [5], it was already shown, in a quite different way, that if \( \lambda \omega \neq 0, \) then \( H(A) \) is a Noetherian domain and an Artin Achelter regular algebra of global dimension 4.

Acknowledgment

The authors are grateful to the anonymous referee for his/her comments and suggestions that are quite helpful for improving the manuscript.

Funding

This project was supported by the National Natural Science Foundation of China (11861061 to Rabigul Tuniyaz and 12061068 to Gulshadam Yunus).

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