Quantum groupoids and deformation quantization

PING XU *
Research Institute for Math. Science
Kyoto University, Kyoto, JAPAN
AND
Department of Mathematics
The Pennsylvania State University
University Park, PA 16802, USA
e-mail: ping@math.psu.edu

Abstract

The purpose of this Note is to unify quantum groups and star-products under a general umbrella: quantum groupoids. It is shown that a quantum groupoid naturally gives rise to a Lie bialgebroid as a classical limit. The converse question, i.e., the quantization problem, is posed. In particular, any regular triangular Lie bialgebroid is shown quantizable. For the Lie bialgebroid of a Poisson manifold, its quantization is equivalent to a star-product.

Version française abrégée

Groupoïdes quantiques et quantification par déformation

Résumé Cette note a pour but d’unifier groupes quantiques et star-produits sous une même enseigne: les groupoïdes quantiques. Nous montrons que tout groupoïde quantique admet de manière naturelle une bigébroïde de Lie comme limite classique. Le problème réciproque de quantification est posé, et nous le résolvons dans le cas des bigébroïdes de Lie triangulaires régulières. Enfin, quantifier la bigébroïde associée à une variété de Poisson revient à y construire un star-produit.

Soit $A$ une algèbre de Lie, son algèbre enveloppante universelle $UA$ possède une structure d’algèbroïde de Hopf cocommutative. En particulier, lorsque $A$ est l’algèbroïde de Lie d’un fibré tangent $TP$, cette structure d’algèbroïde de Hopf est celle de l’algèbre $D(P)$ des opérateurs différentiels sur $P$.

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Les principaux résultats présentés dans cette note sont:

**Theorem A** A tout star-produit sur une variété $P$ correspond un groupoïde quantique $D_\hbar(P)$, déformation de l’algébroïde de Hopf de $D(P)$.

**Theorem B** Un groupoïde quantique $U_\hbar A$ admet de manière naturelle une bigébroïde de Lie $(A, A^*)$ comme limite classique. La structure de Poisson induite par cette bigébroïde de Lie coïncide avec celle qui est associée à la $*$-algèbre $C_\infty(P)[[\hbar]]$.

Réciproquement, une quantification d’une bigébroïde de Lie $(A, A^*)$ est un groupoïde quantique $U_\hbar A$ dont la limite classique est $(A, A^*)$.

**Theorem C** Toute bigébroïde de Lie triangulaire régulière admet une quantification.

Nous croyons que, dans le cadre général décrit ci-dessus, les techniques utilisées dans la théorie des groupes quantiques peuvent donner une meilleure compréhension des star-produits sur une variété de Poisson.

1 Introduction

Poisson tensors in many aspects resemble classical triangular r-matrices in quantum group theory. A notion unifying both Poisson structures and Lie bialgebras was introduced in [13], called Lie bialgebroids. Integration theorem for Lie bialgebroids encomposes both Drinfeld theorem of integration of Lie bialgebras on the one hand, and Karasev-Weinstein theorem of existence of local symplectic groupoids for Poisson manifolds on the other hand. Quantization of Lie bialgebras leads to quantum groups, while quantization of Poisson manifolds is the so called star-products. It is therefore natural to expect that there should exist some intrinsic connection between these two quantum objects. The purpose of the Note is to connect these two concepts under the general framework of quantum groupoids (or QUE algebroids), i.e., Hopf algebroid deformation of the universal enveloping algebras of Lie algebroids.

Given a Lie algebroid $A$, its universal enveloping algebra $UA$ carries a natural cocommutative Hopf algebroid structure. In particular, when $A$ is the tangent bundle Lie algebroid $TP$, this is the Hopf algebroid structure on $D(P)$, the algebra of differential operators on $P$.

We will see that a star-product on $P$ corresponds to a quantum groupoid $D_\hbar(P)$, which is a Hopf algebroid deformation of $D(P)$. In general, we show that a quantum groupoid $U_\hbar A$ naturally induces a Lie bialgebroid $(A, A^*)$ as a classical limit. Then, we pose the general question of quantization of Lie bialgebroids, which, as special cases, encomposes quantization of Lie bialgebras and deformation quantization of Poisson manifolds. In particular, we prove that any regular triangular Lie bialgebroid is quantizable.

Our main motivation is that this general framework may provide some new insights in understanding star products of Poisson manifolds via the methods in quantum group theory.

It is worth noting that the notion of Hopf algebroids was introduced by Lu [12] essentially by translating the axioms of Poisson groupoids to their quantum counterparts, while the case that base algebras are commutative was already studied by Maltsev [14] in 1992.

2 Hopf algebroids
Definition 2.1 A Hopf algebroid consists of the following data:

1) a total algebra $H$ with product $m$, a base algebra $R$, a source map: an algebra homomorphism $\alpha: R \to H$, and a target map: an algebra anti-homomorphism $\beta: R \to H$ such that the images of $\alpha$ and $\beta$ commute in $H$, i.e., $\forall a, b \in R$, $\alpha(a)\beta(b) = \beta(b)\alpha(a)$. There is then a natural $(R, R)$-bimodule structure on $H$ given by $a \cdot h = \alpha(a)h$ and $h \cdot a = \beta(a)h$. Thus, we can form the $(R, R)$-bimodule product $H \otimes_R H$. It is easy to see that $H \otimes_R H$ again admits an $(R, R)$-bimodule structure. This will allow us to form the triple product $H \otimes_R H \otimes_R H$ and so on.

2) a co-product: an $(R, R)$-bimodule map $\Delta: H \to H \otimes_R H$ with $\Delta(1) = 1 \otimes 1$ satisfying the co-associativity:

$$(\Delta \otimes \text{id}_H)\Delta = (\text{id}_H \otimes \Delta)\Delta: H \to H \otimes_R H \otimes_R H;$$

(1)

3) the product and the co-product are compatible in the following sense:

$$\Delta(h)(\beta(a) \otimes 1 - 1 \otimes \alpha(a)) = 0, \text{ in } H \otimes_R H, \forall a \in R \text{ and } h \in H, \text{ and}$$

(2)

$$\Delta(h_1 h_2) = \Delta(h_1)\Delta(h_2), \forall h_1, h_2 \in H, \text{ (see the remark below)};$$

(3)

4) a co-unit map: an $(R, R)$-bimodule map $\epsilon: H \to R$ satisfying $\epsilon(1_H) = 1_R$ (it follows then that $\epsilon\beta = \epsilon\alpha = \text{id}_R$) and

$$(\epsilon \otimes \text{id}_R)\Delta = (\text{id}_H \otimes \epsilon)\Delta = \text{id}_R: H \to H.$$  

(4)

Here we have used the identification: $R \otimes_R H \cong H \otimes_R R \cong H$ (note that both maps on the left hand sides of Equation (4) are well-defined).

We will denote this Hopf algebroid by $(H, R, \alpha, \beta, m, \Delta, \epsilon)$.

Remark (1) It is clear that any left $H$-module is automatically an $(R, R)$-bimodule. Now given any left $H$-modules $M_1$ and $M_2$, for $m_1 \in M_1$, $m_2 \in M_2$ and $h \in H$, define,

$$h \cdot (m_1 \otimes_R m_2) = \Delta(h)(m_1 \otimes m_2).$$

(5)

The RHS is a well-defined element in $M_1 \otimes_R M_2$ due to Equation (2). In particular, when taking $M_1 = M_2 = H$, we see that the RHS of Equation (3) makes sense. In fact, Equation (3) implies that $M_1 \otimes_R M_2$ is again a left $H$-module.

(2) The compatibility condition (Equations (2) and (3) ) is equivalent to the following one in Lu’s original definition [12]: the kernel of the map

$$\Psi: H \otimes H \otimes H \to H \otimes_R H: \sum h_1 \otimes h_2 \otimes h_3 \mapsto \sum (\Delta h_1)(h_2 \otimes h_3)$$

(6)

is a left ideal of $H \otimes H^{op} \otimes H^{op}$, where $H^{op}$ denotes $H$ with the opposite product.

(3) In our definition above, we choose not to require the existence of antipodes because many interesting examples, as shown below, often do not admit antipodes.

The following proposition follows immediately from definition.

Proposition 2.2 Let $(H, R, \alpha, \beta, m, \Delta, \epsilon)$ be a Hopf algebroid. For any left $H$-modules $M_1$ and $M_2$, $M_1 \otimes_R M_2$ is again a left $H$-module. Moreover, the tensor product is associative: $(M_1 \otimes_R M_2) \otimes_R M_3 \cong M_1 \otimes_R (M_2 \otimes_R M_3)$. The category of $H$-modules in fact becomes a monoidal category.
Example 2.1 Let $D$ denote the algebra of all differential operators on a smooth manifold $M$, and $R$ the algebra of smooth functions on $M$. Then $D$ is a Hopf algebroid over $R$. Here, $\alpha = \beta$ is the inclusion $R \hookrightarrow D$, while the coproduct $\Delta : D \rightarrow D \otimes_R D$ is defined as

$$\Delta(D)(f, g) = D(fg), \quad \forall D \in D, f, g \in R. \quad (7)$$

Note that $D \otimes_R D$ is simply the space of bidifferential operators. Clearly, $\Delta$ is co-commutative. As for the co-unit, we take $\epsilon : D \rightarrow R$, the natural projection from a differential operator to its $0$th-order part. In this case, left $D$-modules are $D$-modules in the usual sense, and the tensor product is the usual tensor product of $D$-modules over $R$.

We note, however, that this Hopf algebroid does not admit an antipode in any natural sense. Given a differential operator $D$, its antipode, if it exists, would be the dual operator $D^*$. However, the latter is a differential operator on $1$-densities, which does not possess any canonical identification with a differential operator on $R$.

Example 2.2 The construction above can be generalized to show that the universal enveloping algebra $UA$ of a Lie algebroid $A$ admits a co-commutative Hopf algebroid structure.

Again we take $R = C^\infty(P)$, and let $\alpha = \beta : C^\infty(M) \rightarrow UA$ be the inclusion. For the co-product, we set

$$\Delta(f) = f \otimes_R 1, \quad \forall f \in C^\infty(P);$$

$$\Delta(X) = X \otimes_R 1 + 1 \otimes_R X, \quad \forall X \in \Gamma(A).$$

This extends to a co-product $\Delta : UA \rightarrow UA \otimes_R UA$ by the compatibility condition: Equations (2) and (3). Alternatively, we may identify $UA$ as the subalgebra of $D(G)$ consisting of right invariant differential operators on a (local) Lie groupoid $G$ integrating $A$, and then restrict the co-product $\Delta_G$ on $D(G)$ to this subalgebra. This is well defined since $\Delta_G$ maps right invariant differential operators to right invariant bidifferential operators. Finally, the co-unit map is defined as the projection $\epsilon : UA \rightarrow C^\infty(P)$.

Example 2.3 Let $P$ be a smooth manifold and $D$ the ring of differential operators on $P$. Let $D[[h]]$ be the space of formal power series in $h$ with coefficients in $D$. The Hopf algebroid structure on $D$ induces a Hopf algebroid structure on $D[[h]]$, whose structure maps will be denoted by the same symbols.

Let $\varphi = 1 \otimes_R 1 + hB_1 + \cdots \in D \otimes_R D[[h]]$ be a formal power series in $h$ with coefficients being bidifferential operators. For any $f, g \in C^\infty(P)[[h]]$, set $f \ast_h g = \varphi(f, g)$. This product is associative iff the following identity holds:

$$(\Delta \otimes_R id)(\varphi)^{12} = (id \otimes_R \Delta)(\varphi)^{23}, \quad (8)$$

where $\varphi^{12} = \varphi \otimes 1 \in (D \otimes_R D) \otimes D[[h]]$ and $\varphi^{23} = 1 \otimes \varphi \in D \otimes (D \otimes_R D)[[h]]$. Note that both sides of the above equation are well defined elements in $D \otimes_R D \otimes_R D[[h]]$.

Equation (8) reminds us the equation of a twistor defining a triangular Hopf algebra (see Section 10 in [4]). Thus, it is not surprising that our $\varphi$ here can be used to produce a new Hopf algebroid structure on $D[[h]]$.

Now assume that $\varphi = 1 \otimes_R 1 + hB_1 + \cdots \in D \otimes_R D[[h]]$ satisfies Equation (3). Then, $\{f, g\} = B_1(f, g) - B_1(g, f), \quad \forall f, g \in C^\infty(P)$, is a Poisson bracket, and $f \ast_h g = \varphi(f, g)$ defines a star product on $P$, which is a deformation quantization of this Poisson structure (1).

Let $D_h = D[[h]]$ be equipped with the usual multiplication, and $R_h = C^\infty(P)[[h]]$ with the $\ast$-product above. Define $\alpha : R_h \rightarrow D_h$ and $\beta : R_h \rightarrow D_h$, respectively, by $\alpha(f)g = f \ast_h g$ and $\beta(f)g = g \ast_h f, \quad \forall f, g \in C^\infty(P)$. From the associativity of $\ast_h$, it follows that $\alpha$ is an algebra homomorphism while $\beta$ is an anti-homomorphism, and $\alpha(R_h)$ commutes with $\beta(R_h)$. Moreover the associativity of $\ast_h$ implies that
Lemma 2.3 Let $M_1$ and $M_2$ be any $D$-modules. Then the map

$$
\Phi : M_1[[h]] \otimes_R M_2[[h]] \rightarrow M_1 \otimes_R M_2[[h]]
$$

$$(m_1 \otimes m_2) \mapsto \varphi \cdot (m_1 \otimes m_2)$$

is well defined and establishes an isomorphism between these two vector spaces.

In particular, when $M_1 = M_2 = D$, we obtain an isomorphism $\Phi : D_2 \otimes_R D_2 \rightarrow D_1 \otimes_R D_1[[h]]$. Now define $\Delta_h : D_2 \rightarrow D_2 \otimes_R D_2$ by

$$\Delta_h = \varphi^{-1} \Delta \varphi,$$

and let $\epsilon : D_2 \rightarrow R$ be the projection. Here, Equation (10) means that $\Delta_h(x) = \Phi^{-1}(\Delta(x)\varphi)$, $\forall x \in D_2$.

The following result can be easily verified.

**Theorem A** $(D_2, R_2, \alpha, \beta, m, \Delta_2, \epsilon)$ is a Hopf algebroid.

Given any $D$-modules $M_1$ and $M_2$, the Hopf algebroid structure on $D_2$ induces a $D$-module structure on $M_1[[h]] \otimes_R M_2[[h]]$. It is easy to see that the map $\Phi$ as in Lemma 2.3 is in fact an isomorphism of $D$-modules. Hence $M_1[[h]] \otimes_R M_2[[h]]$ and $M_2[[h]] \otimes_R M_1[[h]]$ are isomorphic $D$-modules, where the isomorphism is given by $\Phi^{-1} \circ \sigma \circ \Phi$, with $\sigma$ being the flipping.

3 Deformation of Hopf algebroids and quantum groupoids

**Definition 3.1** A deformation of a Hopf algebroid $(H, R, \alpha, \beta, m, \Delta, \epsilon)$ over a field $k$ is a topological Hopf algebroid $(H_2, R_2, \alpha_2, \beta_2, m_2, \Delta_2, \epsilon_2)$ over the ring $k[[h]]$ of formal power series in $h$ such that

(i). $H_2$ is isomorphic to $H[[h]]$ as $k[[h]]$ module with unit $1_H$, and $R_2$ is isomorphic to $R[[h]]$ as $k[[h]]$ module with unit $1_R$;

(ii). $\alpha_2 = \alpha(\mod h), \beta_2 = \beta(\mod h), \ m_2 = m(\mod h), \epsilon_2 = \epsilon(\mod h)$;

(iii). $\Delta_2 = \Delta(\mod h)$

**Remark** The meaning of (i) and (ii) is clear. However, for Condition (iii), we need the following simple fact:

**Lemma 3.2** Under the hypothesis (i) and (ii) as in Definition 3.1, set $V_2 = H_2 \otimes_R H_2$. Then $V_2/hV_2$ is isomorphic to $H \otimes_R H$ as a vector space.

Let $\tau : H_2 \otimes_R H_2 \rightarrow H \otimes_R H$ denote the composition of the projection $V_2 \rightarrow V_2/hV_2$ with the isomorphism $V_2/hV_2 \cong H \otimes_R H$. We shall also use the notation $h \mapsto 0$ to denote this map when the meaning is clear from the context. Then, Condition (iii) means that $\lim_{h \rightarrow 0} \Delta_2(x) = \Delta(x)$ for any $x \in D$. 

\[
\varphi(\beta(f) \otimes 1 - 1 \otimes \alpha(f)) = 0 \quad \text{in} \quad D \otimes_R D[[h]], \quad \forall f \in C^{\infty}(P).
\]
Definition 3.3 A quantum groupoid (or a QUE algebroid) is a deformation of the Hopf algebroid $UA$ (see Example 2.2) of a Lie algebroid $A$.

Let $(U_hA \cong UA[[h]])$, $R_h(\cong C^\infty(P)[[h]])$, $\alpha_h, \beta_h, m_h, \Delta_h, \epsilon_h$ be a quantum groupoid. It is well known that
\[
\{f, g\} = \lim_{h \to 0} \frac{1}{h}(f \ast_h g - g \ast_h f), \quad \forall f, g \in C^\infty(P)
\]
defines a Poisson structure on the base manifold $P$.

Define
\[
\delta f = \lim_{h \to 0} \frac{1}{h}(\alpha_h f - \beta_h f) \in UA, \quad \forall f \in C^\infty(P),
\]
\[
\Delta^1 X = \lim_{h \to 0} \frac{1}{h}(\Delta_h X - (1 \otimes_h X + X \otimes_h 1)) \in UA \otimes_R UA, \quad \forall X \in \Gamma(A), \quad \text{and}
\]
\[
\delta X = \Delta^1 X - \Delta^{1,op}_h X \in UA \otimes_R UA.
\]
A routine calculation using the axioms of Hopf algebroids leads to

Proposition 3.4

(i). $\delta f \in \Gamma(A)$ and $\delta X \in \Gamma(\wedge^2 A)$ for any $f \in C^\infty(P)$ and $X \in \Gamma(A)$;

(ii). $\delta(fg) = f\delta g + g\delta f$ for any $f, g \in C^\infty(P)$;

(iii). $\delta(fX) = f\delta X + \delta f \wedge X$ for any $f \in C^\infty(P)$ and $X \in \Gamma(A)$;

(iv). $\rho(\delta f)g = \{f, g\}$ for any $f, g \in C^\infty(P)$, where $\rho : A \to TP$ is the anchor of the Lie algebroid $A$.

Properties (i)-(iii) allow us to extend $\delta$ to a well defined degree 1 derivation $\delta : \Gamma(\wedge^A) \to \Gamma(\wedge^{A+1})$. It is not difficult to prove:

Proposition 3.5

\[
\delta^2 = 0; \quad \text{and} \quad \delta[X, Y] = [\delta X, Y] + [X, \delta Y], \quad \forall X, Y \in \Gamma(A).
\]

4 Quantization of Lie bialgebroids

Recall that a Lie bialgebroid \([3, 10]\) is a pair of Lie algebroids $(A, A^*)$ satisfying the compatibility condition: Equation \([12]\), where $\delta : \Gamma(\wedge^A) \to \Gamma(\wedge^{A+1})$ is the differential induced from the Lie algebroid $A^*$. Lie bialgebroids include usual Lie bialgebras. Besides, for any Poisson manifold $P$, the pair $(TP, T^*P)$ carries a natural Lie bialgebroid structure. More generally, given a Lie algebroid $A$ and $\Lambda \in \Gamma(\wedge^2 A)$ satisfying $[\Lambda, \Lambda] = 0$, $(A, A^*)$ is a Lie bialgebroid, called a triangular Lie bialgebroid \([13]\). It is called regular if $A$ is of constant rank. Another interesting example arises from the classical dynamical Yang-Baxter equation \([3]\):

Example 4.1 Let $\mathfrak{g}$ be a simple Lie algebra with Cartan subalgebra $\mathfrak{h}$. A classical dynamical $r$-matrix \([3]\) is a $\mathfrak{g} \otimes \mathfrak{g}$ valued equivariant function $r : \mathfrak{h}^* \to \mathfrak{g} \otimes \mathfrak{g}$ satisfying the classical dynamical Yang-Baxter equation:
\[
\text{Alt}(dr) + [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0
\]
such that \( r^{12} + r^{21} \) is a constant function valued in \((S^2 g)^g\). Given a classical dynamical r-matrix \( r \), let \( A = T\hbar^* \times g \), equipped with the product Lie algebroid. Define \( \Lambda \in \Gamma(\wedge^2 A) \) by \( \Lambda = \sum \xi_\alpha \wedge h_\alpha + r \). Here \( \{ h_\alpha \} \) is a basis of \( \mathfrak{h} \), \( \xi_\alpha \) its dual basis considered as a constant vector field on \( \mathfrak{h}^* \), and \( \xi_\alpha \wedge h_\alpha \) is considered as a section of \( \wedge^2 A \) in an evident sense. It is simple to check that \( \Lambda \) satisfies the hypothesis as in Theorem 2.1 in [11], and therefore induces a coboundary (or exact as called in [11]) Lie bialgebroid \((A, A^*)\).

The base space of a Lie bialgebroid \((A, A^*)\) is naturally equipped with a Poisson structure, which is defined by the bundle map \( \rho \circ \rho^*: T^*P \rightarrow TP \). Here \( \rho \) and \( \rho^* \) are the anchors of \( A \) and \( A^* \) respectively.

A combination of Propositions 3.4 and 3.5 leads to

**Theorem B** A quantum groupoid \( U_\hbar A \) naturally induces a Lie bialgebroid \((A, A^*)\) as a classical limit. The induced Poisson structure of this Lie bialgebroid on the base \( P \) coincides with the one induced from the base \(*\)-algebra \( R_\hbar(\cong C^\infty(P)[[\hbar]]) \).

Such a Lie bialgebroid \((A, A^*)\) is called the classical limit of the quantum groupoid \( U_\hbar A \). Conversely,

**Definition 4.1** A quantization of a Lie bialgebroid \((A, A^*)\) is a quantum groupoid \( U_\hbar A \) whose classical limit is \((A, A^*)\).

It is a deep theorem of Etingof and Kazhdan [3] that every Lie bialgebra is quantizable. On the other hand, the existence of \(*\)-products for arbitrary Poisson manifolds was recently proved by Kontsevich [9]. In terms of Hopf algebroids, this amounts to saying that the Lie bialgebroid \((TP, T^* P)\) of a Poisson manifold \( P \) is always quantizable. It is therefore natural to expect:

**Conjecture** Every Lie bialgebroid is quantizable.

In fact, by modifying Fedosov’s method (see [3] [4]), one can prove the following:

**Theorem C** Any regular triangular Lie bialgebroid is quantizable.

**Remark.** (1). In [6], Etingof and Varchenko developed a theory of \( \eta \)-Hopf algebroids, which was aimed to provide a general language for elliptic quantum groups and the quantum dynamical Yang-Baxter equation. The relation of these studies to quantization of the Lie bialgebroids as in Example 4.1 is the subject of work in progress.

(2). According to the general spirit of deformation theory, any deformation corresponds to a certain cohomology. In particular, the deformation of Hopf algebras is controlled by the cohomology of a certain double complex arising from the Hopf algebra structure [8]. It is natural to ask what is the proper cohomology theory controlling the deformation of a Hopf algebroid, and in particular what is the premier obstruction to the quantization problem [1].

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