A Monotonous Intuitionistic Fuzzy TOPSIS Method with Linear Orders under Two Novel Admissible Distance Measures

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Abstract—All intuitionistic fuzzy TOPSIS methods contain two key elements: (1) the order structure, which can affect the choices of positive ideal-points and negative ideal-points, and construction of admissible distance/similarity measures; (2) the distance/similarity measure, which is closely related to the values of the relative closeness degrees and determines the accuracy and rationality of decision-making. For the order structure, many efforts are devoted to constructing some score functions, which can strictly distinguish different intuitionistic fuzzy values (IFVs) and preserve the natural partial order for IFVs. This paper proves that such a score function does not exist, namely the application of a single monotonous and continuous function does not distinguish all IFVs. For the distance or similarity measure, some examples are given to show that classical similarity measures based on the normalized Euclidean distance and normalized Minkowski distance do not meet the axiomatic definition of intuitionistic fuzzy similarity measures. Moreover, some illustrative examples are given to show that classical intuitionistic fuzzy TOPSIS methods do not ensure the monotonicity with the natural partial order or linear orders, because they may yield counter-intuitive results. To overcome this limitation, we propose a novel intuitionistic fuzzy TOPSIS method, using two new admissible distances with the linear orders measured by a score degree/similarity function and accuracy degree, and prove that they are monotonous under these two linear orders. This is the first result with a strict mathematical proof on the monotonicity with the linear orders for the intuitionistic fuzzy TOPSIS method. Finally, we show two practical examples and comparative analysis with other decision-making methods to illustrate the efficiency of the developed TOPSIS method.

Index Terms—Intuitionistic fuzzy set, Distance measure, Similarity measure, TOPSIS, Multi-attribute decision making.

I. INTRODUCTION

Zadeh (1965) [1] presented the fuzzy set (FS) theory by applying membership degree to measure the importance of a fuzzy element, which generalized the concept of crisp set, characterized by a characteristic function taking value 0 or 1, by taking any value in the unit interval [0, 1]. However, due to the limitation of a membership function that only indicates two (supporting and opposing) opposite states of fuzziness, the fuzzy set cannot express the neutral state of “this and also that”. According to this, Atanassov (1986) [2] generalized Zadeh’s fuzzy set by proposing the concept of an intuitionistic fuzzy value (IFV) by Xu [3]. Thereafter, various multi-attribute decision making (MADM) methods were developed and widely applied. This paper focuses on the intuitionistic fuzzy TOPSIS method.

Being one of the most well-known MADM methods, TOPSIS was first proposed by Hwang and Yoon (1981) [4]. The main idea of this TOPSIS is that the most desirable alternative should be nearest from the positive ideal-point and meanwhile furthest from the negative ideal-point. Noting that the total order structure ‘≤’ and the absolute distance ‘|·|’ on the real line $\mathbb{R}$ are admissible (the bigger the real number, the nearer from the maximum, and the further from the minimum). This can naturally guarantee the establishment of the TOPSIS and further ensure its monotonicity. Due to the complex two-dimensional structure of the space of all IFVs, all existing IF distance measurements for IFVs are not admissible under linear orders on IFVs. Therefore, we [5] introduced the concept of admissible distance measure with the linear order ‘$\leq_{xV}$‘ of Xu and Yager [6] and constructed an admissible distance measure $\varrho$ under the linear order ‘$\leq_{xV}$‘.

Through careful analysis of the TOPSIS method, it has been found that the TOPSIS method contains two key elements: (1) the order structure, which can affect the choices of positive ideal-points and negative ideal-points; (2) the distance/similarity measure, which is closely related to the relative closeness degrees and determines the accuracy and rationality of decision-making. Therefore, all improvements on IF TOPSIS method provide certain improvements on the order structure and the distance/similarity measures.

To rank IFVs, score function is a useful tool. Xu and Yager [6] introduced the linear order ‘$\leq_{xV}$‘ for IFVs by applying a score function and an accuracy function. According to the TOPSIS idea [4], Zhang and Xu [7] proposed another
linear order \( \leq_{\mathbb{Z}_A} \) for IFVs by applying a similarity function and an accuracy function. Recently, Xing et al. [8] defined a linear order for IFVs by using a score function expressed by the Euclidean distance from the maximum point \((1,0)\). Wang et al. [9] classified the existing score functions of IFVs into two types, one type consists of score functions without abstention group influence [10], [11], [12], [13], [14], [15], and the other consists of score functions with abstention group influence [16]. Zeng et al. [17] illustrated that these existing score functions, the existing accuracy functions in [18], [19], and the measure methods in [20], [21], [7] for ranking IFVs have a drawback that they may cause some unreasonable ranking results or they cannot distinguish some different IFVs. To overcome this drawback, Zeng et al. [17] proposed a new score function \( S_{c,k} \) for IFVs, which was a monotonically increasing injective with Atanassov’s partial order \( \subset \) (see [17, Theorems 3.1 and 3.2]). However, we constructed an example to show that this inference [17] does not hold and proved that such a score function does not exist, i.e., there is no any continuous injective from the space of all IFVs to \( \mathbb{R} \) that is increasing with Atanassov’s partial order \( \subset \). This means that the score function \( S_{c,k} \) has the same drawback. This trouble mainly arises from the fact that the space of all IFVs has a two-dimensional structure, which is not homeomorphic to any closed interval on the real line \( \mathbb{R} \). In fact, we proved that the application of a single monotonous and continuous function does not distinguish all IFVs.

Being a pair of dual concepts, the normalized distance measures and similarity measures have been widely studied. Similarly to the axiomatic definition of similarity measure for fuzzy sets [22], [23], Li and Cheng [24] gave an axiomatic definition of similarity measures for IFSs by using normalization (S1), symmetry (S3), and compatibility with Atanassov’s partial order (S4), i.e., the condition \( I_1 \subset I_2 \subset I_3 \) implies that the similarity measure between \( I_1 \) and \( I_3 \) is smaller than that between \( I_1 \) and \( I_2 \) and between \( I_2 \) and \( I_3 \). Xu [25] introduced some new IF similarity measures and applied them to some practical MADM problems. Xu and Chen [26] presented a comprehensive overview of distance and similarity measures of IFSs and proposed some continuous distance and similarity measures for IFSs using the weighted Hamming distance, weighted Euclidean distance, and weighted Hausdorff distance. Iancu [27] defined some IF similarity measures using Frank \( t \)-norms \( T_F \). Szmidt and Kacprzyk [28] pointed out that the third parameter (indeterminacy degree) should be considered when calculating distances for IFSs. Because of the duality between distance and similarity measures, various three-dimensional IF distance and similarity measures including indeterminacy degrees were introduced [29], [30], [31], [32]. However, Atanassov’s partial order only reveals the magnitudes of membership degrees and non-membership degrees between two IFSs, and thus many three-dimensional IF similarity measures considering indeterminacy degrees might not meet the axiomatic condition (S4). In fact, we constructed three examples to show that Euclidean similarity measure [30], [33], [31], Minkowski similarity measure in [25], [31], [30], [33], and a modified similarity measure [26] based on the idea of the above TOPSIS does not satisfy the axiomatic condition (S4). In particular, it should be pointed out that some existing IF distance and similarity measures are unreasonable for dealing with some practical decision-making problems. For example, Mitchell [34] showed that Li and Cheng’s similarity measure [24] may lead to counter-intuitive situations in some cases. Chen et al. [35] showed some counterexamples to illustrate that the similarity measures in [36], [37], [38], [34], [19], [39] may produce unreasonable results in some cases. As noted above, the application of a single monotonous and continuous function does not distinguish all IFVs. This means that all continuous IF similarity measures are unable to distinguish every pair of different IFSs. For example, when we apply a continuous IF similarity measure for pattern recognition, we always encounter the case that we cannot determine the classification result. Therefore, the comparative analysis on the indistinguishability is meaningless ([40, Tables 2–5], [36, Tables 1–2], [35, Tables 1, 2, 5, 6], [29, Table 2]), because all continuous IF similarity measures will also encounter the same indistinguishability problem. On the other hand, all existing distance and similarity measures are only admissible with Atanassov’s partial order, and thus decision-making results obtained by these distance or similarity measures can only guarantee the monotonicity with the partial order. Therefore, to obtain monotonous decision-making methods, we need to develop new admissible distance and similarity measures with linear orders.

During the past decade, various generalized IF TOPSIS methods have been developed. For example, Boran et al. [41] first extended the TOPSIS method to IF group decision making with IFV weights. Then, some similar IF TOPSIS methods with IFV/linguistic weights were developed [42], [33], [43] and widely applied to practical decision-making problems [42], [44], [45], [46], [47]. Our Examples 6 below in this paper show that the TOPSIS methods in [41], [44], [33], [43], [46] may yield unreasonable results even when dealing with the simplest decision-making problems. Chen et al. [48] proposed a MADM method with crisp numerical weights based on the TOPSIS method and a new similarity measure for the IF situation. Zeng et al. [17] pointed out that the MADM method [48] cannot distinguish the alternatives in some special situations. Furthermore, our Example 8 below in this paper demonstrates that the MADM method [48] is not monotonous with the linear order \( \leq_{XY} \) of Xu and Yager [6].

Based on a new distance measure, Shen et al. [29] developed an extended IF TOPSIS method and applied it to credit risk evaluation.

Inspired by the above discussions, this paper establishes a monotonous IF TOPSIS under two popular linear orders, \( \leq_{XY} \) in [6] and \( \leq_{\mathbb{Z}_A} \) in [7]. More precisely, the main contributions of this paper are as follows:

1. We construct some counterexamples to illustrate that Euclidean similarity measure ([33], [30]), Minkowski similarity measure ([25]), and modified Euclidean similarity measure ([26]) do not satisfy the axiomatic definition of IF similarity measures (see Examples 1–3).

2. We prove that there is no any continuous and injective function from the space of all IFVs to the real line \( \mathbb{R} \) that is increasing with Atanassov’s partial order \( \subset \). This indicates
the nonexistence of continuous similarity measure distinguishing between every pair of IFVs. Therefore, the comparative analysis on the indistinguishability for IF similarity measures is meaningless, but unfortunately this is a common problem for all continuous IF similarity measures.

(3) We construct three simple examples to show that some classical IF TOPSIS methods [33, 48, 44, 41, 43, 46] are not monotonous with Atanassov’s partial order ‘⊂’ or the linear order ‘≤’ (see Examples 6–8), which may yield counter-intuitive results. To overcome this limitation, by proposing two new admissible distances with the linear order ‘≤’ or ‘<xy’, we develop a novel IF TOPSIS method and prove that it is monotonically increasing with these two linear orders.

(4) We provide two practical examples and a comparative analysis with other MADM methods to illustrate the efficiency of our proposed TOPSIS method.

II. Preliminaries

A. Intuitionistic fuzzy set (IFS)

Definition 2.1 ([49, Definition 1.1]): Let X be the universe of discourse. An intuitionistic fuzzy set (IFS) I in X is defined as an object in the following form

$$I = \{ (x, \mu_I(x), \nu_I(x)) \mid x \in X \},$$

where the functions

$$\mu_I : X \rightarrow [0, 1],$$

and

$$\nu_I : X \rightarrow [0, 1],$$

define the degree of membership and the degree of non-membership of an element x ∈ X to the set I, respectively, and for every x ∈ X,

$$\mu_I(x) + \nu_I(x) \leq 1.$$  (2)

Let IFS(X) denote the set of all IFSs in X. For I ∈ IFS(X), the indeterminacy degree πI(x) of an element x belonging to I is defined by πI(x) = 1 − μI(x) − νI(x). In [3, 31], the pair ⟨μI(x), νI(x)⟩ is called an intuitionistic fuzzy value (IFV) or an intuitionistic fuzzy number (IFN). For convenience, use α = ⟨μα, να⟩ to represent any IFV α, which satisfies μα ∈ [0, 1], να ∈ [0, 1], and 0 ≤ μα + να ≤ 1. Additionally, s(α) = μα − να and h(α) = μα + να are called the score degree and the accuracy degree of α, respectively. Let I denote the set of all IFVs, i.e., I = {⟨μ, ν⟩ ∈ [0, 1]2 | μ + ν ≤ 1}.

Motivated by the basic operations on IFVs, Xu et al. [31, 6] introduced the following basic operational laws for IFVs.

Definition 2.2 ([31, Definition 1.2.2]): Let α = ⟨μα, να⟩, β = ⟨μβ, νβ⟩ ∈ I. Define

(i) α ∩ β = ⟨μα ∨ μβ, να ∨ νβ⟩,

(ii) α ∪ β = ⟨μα ∧ μβ, να ∧ νβ⟩,

(iii) α ⊕ β = ⟨μα + μβ, να + νβ⟩,

(iv) α ⊙ β = ⟨μαμβ, νανβ⟩,

(v) λα = ⟨(λμα), 1 − (1 − λ)να⟩, λ > 0,

(vi) λ(μα) = ⟨(1 − (1 − λ)μα), (1 − λ)να⟩, λ > 0.

B. Orders for IFSs

Atanassov’s order ‘⊂’ [49], defined by that α ⊂ β if and only if α ∩ β = α, is a partial order on I. To compare any two IFVs, Xu and Yager [6] introduced the following linear order ‘≤xy’ (see also [3, Definition 3.1] and [31, Definition 1.1.3]):

Definition 2.3 ([6, Definition 1]): Let α1 and α2 be two IFVs.

- If s(α1) < s(α2), then α1 is smaller than α2, denoted by α1 ≤xy α2.
- If s(α1) = s(α2), then
  - if h(α1) = h(α2), then α1 = α2;
  - if h(α1) < h(α2), then α1 <xy α2.

If α1 ≤xy α2 or α1 = α2, then denote it by α1 ≤xy α2.

Alongside Xu and Yager’s order ‘≤xy’ in Definition 2.3, Szmidt and Kacprzyk [21] proposed another comparison function, ρ(α) = \frac{1}{2}(1 + \mu(\alpha))(1 − \mu(\alpha)) for IFVs, which is a partial order. However, it sometimes cannot distinguish between two IFVs. Although Xu’s method [3] constructs a linear order for ranking any two IFVs, its procedure has the following disadvantages: (1) It may result in that the less we know, the better the IFV, which is not reasonable. (2) It is sensitive to a slight change of the parameters. (3) It is not preserved under multiplication by a scalar, namely, α ≤xy β might not imply λα ≤xy λβ, where λ is a scalar (see [50, Example 1]). To overcome such shortcomings of the above two ranking methods, Zhang and Xu [7] improved Szmidt and Kacprzyk’s method [21], according to Hwang and Yoon’s idea [4] and technique for order preference by similarity to an ideal solution. They also introduced a similarity function L(α), called the L-value in [7], for any IFV α = ⟨μα, να⟩, as follows:

$$L(\alpha) = \frac{1 - \nu_\alpha}{(1 - \mu_\alpha) + (1 - \nu_\alpha)} = \frac{1 - \nu_\alpha}{1 + \pi_\alpha}. \quad (3)$$

In particular, if να < 1, then

$$L(\alpha) = \frac{1}{1 - \nu_\alpha} + 1. \quad (4)$$

Furthermore, Zhang and Xu [7] introduced the following order ‘≤xy’ for IFVs by applying the similarity function L(α).

Definition 2.4 ([7]): Let α1 and α2 be two IFVs.

- If L(α1) < L(α2), then α1 ≤xy α2;
- If L(α1) = L(α2), then
  - if h(α1) = h(α2), then α1 = α2;
  - if h(α1) < h(α2), then α1 <xy α2.

If α1 ≤xy α2 or α1 = α2, then denote it by α1 ≤xy α2.

C. IF distance and similarity measure

Li and Cheng [24] introduced a similarity measure for IFSs, which was then improved by Mitchell [51] as follows. More results on the similarity measure can be found in [30].

Definition 2.5 ([51]): Let X be the universe of discourse and S : IFS(X) × IFS(X) → [0, 1] be a mapping. S(α) is called an admissible similarity measure with the order ⊂ on IFS(X) if it satisfies the following conditions: for any I1, I2, I3 ∈ IFS(X),
(S1) \( 0 \leq S(I_1, I_2) \leq 1 \).

(S2) \( S(I_1, I_2) = 1 \) if and only if \( I_1 = I_2 \).

(S3) \( S(I_1, I_2) = S(I_2, I_1) \).

(S4) If \( I_1 \subset I_2 \subset I_3 \), then \( S(I_1, I_3) \leq S(I_1, I_2) \) and \( S(I_1, I_3) \leq S(I_2, I_3) \).

Remark 1: The admissible similarity measure with the order \( \subset \) was also called similarity measure by Hung and Yang [52] and Szmidt [30]. When no ambiguity is possible, we simply call it similarity measure.

Definition 2.6: Let \( X \) be the universe of discourse and \( I_1, I_2 \in \text{IFS}(X) \). If \((\mu_{I_1}(x), \nu_{I_1}(x)) \leq S_{\text{xy}}(\mu_{I_2}(x), \nu_{I_2}(x))\) holds for all \( x \in X \), then we say that \( I_1 \) is smaller than or equal to \( I_2 \) with the linear order \( \leq_{S_{\text{xy}}} \), denoted by \( I_1 \leq_{S_{\text{xy}}} I_2 \).

Based on Definition 2.6, we introduce the improved similarity measure definition for IFSs below:

Definition 2.7: Let \( X \) be the universe of discourse and \( S : \text{IFS}(X) \times \text{IFS}(X) \rightarrow [0, 1] \) be a mapping. \( S(\_\_\_) \) is called an admissible similarity measure with the order \( \leq_{S_{\text{xy}}} \) on \( \text{IFS}(X) \) if it satisfies the conditions (S1)–(S3) in Definition 2.5, and the following one (S4'):

(S4') For any \( I_1, I_2, I_3 \in \text{IFS}(X) \), if \( I_1 \leq_{S_{\text{xy}}} I_2 \leq_{S_{\text{xy}}} I_3 \), then \( S(I_1, I_3) \leq S(I_1, I_2) \) and \( S(I_1, I_3) \leq S(I_2, I_3) \).

The normalized Hamming distance in [28] is:

\[
d_{\text{Ha}}(I_1, I_2) = \frac{1}{2n} \sum_{j=1}^{n} H_j,
\]
where \( H_j = |\mu_{I_1}(x_j) - \mu_{I_2}(x_j)| + |\nu_{I_1}(x_j) - \nu_{I_2}(x_j)| + |\pi_{I_1}(x_j) - \pi_{I_2}(x_j)| \).

The normalized Euclidean distance in [28] is:

\[
d_{\text{Eu}}(I_1, I_2) = \sqrt{\frac{1}{2n} \sum_{j=1}^{n} E_j},
\]
where \( E_j = \left(\mu_{I_1}(x_j) - \mu_{I_2}(x_j)\right)^2 + \left(\nu_{I_1}(x_j) - \nu_{I_2}(x_j)\right)^2 + \left(\pi_{I_1}(x_j) - \pi_{I_2}(x_j)\right)^2 \).

The normalized Minkowski distance in [28], [30] is:

\[
d_m^{(\alpha)}(I_1, I_2) = \frac{1}{2n} \sum_{j=1}^{n} M_j,
\]
where \( M_j = \left(\mu_{I_1}(x_j) - \mu_{I_2}(x_j)\right)\alpha + \left(\nu_{I_1}(x_j) - \nu_{I_2}(x_j)\right)\alpha + \left(\pi_{I_1}(x_j) - \pi_{I_2}(x_j)\right)\alpha \) and \( \alpha \geq 1 \).

By using the normalized Hamming distance, Szmidt and Kacprzyk [53] introduced the following similarity measure \( S_{\text{SK}}^{1} \) for two IFSs \( I_1 \) and \( I_2 \):

\[
S_{\text{SK}}^{1}(I_1, I_2) = \frac{d_{\text{Ha}}(I_1, I_2)}{d_{\text{Ha}}(I_1, I_2^0)},
\]
where \( I_2^0 \) is the complement of \( I_2 \). If we replace the normalized Hamming distance with the normalized Euclidean distance, we obtain the following "similarity measure" \( S_{\text{SK}}^{2} \) for two IFSs \( I_1 \) and \( I_2 \):

\[
S_{\text{SK}}^{2}(I_1, I_2) = \frac{d_{\text{Eu}}(I_1, I_2)}{d_{\text{Eu}}(I_1, I_2^0)}.
\]

Clearly, both \( S_{\text{SK}}^{1} \) and \( S_{\text{SK}}^{2} \) are not similarity measures, because their values may exceed 1. Then, the following similarity measures were introduced by Szmidt [30] using the normalized Hamming and Euclidean distances:

\[
S_{\text{SK}}^{1}(I_1, I_2) = 1 - d_{\text{Ha}}(I_1, I_2),
\]
and

\[
S_{\text{SK}}^{2}(I_1, I_2) = 1 - d_{\text{Eu}}(I_1, I_2).
\]

Based on the normalized Minkowski distance \( d_m^{(\alpha)} \), Xu [25] introduced the following Minkowski similarity measure (see also [33], [26]): for \( \alpha > 0 \),

\[
S_m^{(\alpha)}(I_1, I_2) = 1 - d_m^{(\alpha)}(I_1, I_2).
\]
Clearly, \( S_{\text{SK}}^{1}(I_1, I_2) = S_m^{1} \) and \( S_{\text{SK}}^{2}(I_1, I_2) = S_m^{2} \).

Motivated by the idea of the TOPSIS of Hwang and Yoon [4], Xu and Chen [26] modified Eqs. (8) and (9) as follows:

\[
S_{\text{SC}}^{1}(I_1, I_2) = \frac{d_{\text{Ha}}(I_1, I_2^0)}{d_{\text{Ha}}(I_1, I_2) + d_{\text{Ha}}(I_1, I_2^0)},
\]
and

\[
S_{\text{SC}}^{2}(I_1, I_2) = \frac{d_{\text{Eu}}(I_1, I_2^0)}{d_{\text{Eu}}(I_1, I_2) + d_{\text{Eu}}(I_1, I_2^0)}.
\]

III. THE DRAWBACKS OF SOME EXISTING SIMILARITY MEASURES

This section illustrates that the similarity measures defined by Eqs. (11), (12), and (14) do not meet the property (S4) in the axiomatic definition of intuitionistic fuzzy similarity measures.

Example 1: Let the universe of discourse \( X = \{x_1\} \), and \( I_1 = \left\{ \frac{(0.1)}{x_1} \right\} \), \( I_2 = \left\{ \frac{(0.1)}{x_1} \right\} \), \( I_3 = \left\{ \frac{(0.4)}{x_1} \right\} \). Clearly, \( I_1 \subset I_2 \subset I_3 \). By direct calculation, one has

\[
S_{\text{SC}}^{2}(I_1, I_2) = 1 - \sqrt{\frac{0 - 0.1^2 + 1 - 0^2 + 0 - 0.9^2}{2}}
\]

\[= 1 - \sqrt{0.91},
\]

and

\[
S_{\text{SC}}^{2}(I_1, I_3) = 1 - \sqrt{\frac{0 - 0.4^2 + 1 - 0^2 + 0 - 0.6^2}{2}}
\]

\[= 1 - \sqrt{0.76},
\]

and thus \( S_{\text{SC}}^{2}(I_1, I_2) < S_{\text{SC}}^{2}(I_1, I_3) \). This, together with \( I_1 \subset I_2 \subset I_3 \), implies that the formula \( S_m^{1} \) defined by Eq. (11) is not a similarity measures on IFSs.

Example 2: Let the universe of discourse \( X = \{x_1\} \) and \( I_1 = \left\{ \frac{(0.1)}{x_1} \right\} \), \( I_2 = \left\{ \frac{(0.2)}{x_1} \right\} \), \( I_3 = \left\{ \frac{(0.3)}{x_1} \right\} \) be three IFSs on \( X \) such that \( 0 < \mu_2 < \mu_3 < 0.5 \). Clearly, \( I_1 \subset I_2 \subset I_3 \). Fix any \( \alpha > 1 \). By direct calculation, one has

\[
S_{m}^{(\alpha)}(I_1, I_2) = 1 - \sqrt{\left(\mu_2\right)^{\alpha} + 1 + (1 - \mu_2)^{\alpha}},
\]
and

\[
S_{m}^{(\alpha)}(I_1, I_3) = 1 - \sqrt{\left(\mu_3\right)^{\alpha} + 1 + (1 - \mu_3)^{\alpha}},
\]

Let \( \Gamma(x) = 1 - \frac{x^{\alpha+1} + (1-x)^{\alpha}}{2} \) \((x \in (0,0.5))\). Noting that \( \alpha > 1 \) and \( x \in (0,0.5) \), by direct calculation, we get \( \Gamma(x) = \ldots \ldots \)
and the continuity is under the topology of subset of \( I \). Therefore, the formula \( S^\alpha_m \) defined by Eq. (12) is not a similarity measures on IFSs for any \( \alpha > 1 \).

Example 3: Let the universe of discourse \( X = \{ x_1 \} \), and \( I_1 = \{ (0, 1) \} \), \( I_2 = \{ (0.9, 0.01) \} \), and \( I_3 = \{ (0.001, 0.007) \} \). Clearly, \( I_1 \subset I_2 \subset I_3 \). By direct calculation, one has

\[
S^2_m(I_1, I_2) = \sqrt{\frac{\max(0, -0.92^2 + |1 - 0.92|^2 + 0 - 0.09)^2}{2}} + \sqrt{\frac{\max(0, 0.072^2 + |1 - 0.072|^2 + 0 - 0.09)^2}{2}}
\]

\approx 0.09141,

and

\[
S^2_m(I_1, I_3) = \sqrt{\frac{\max(0, 0.0072^2 + |1 - 0.0072|^2 + 0 - 0.09)^2}{2}} + \sqrt{\frac{\max(0, 0.00072^2 + |1 - 0.00072|^2 + 0 - 0.09)^2}{2}}
\]

\approx 0.09148,

and thus \( S^2_m(I_1, I_2) < S^2_m(I_1, I_3) \). This, together with \( I_1 \subset I_2 \subset I_3 \), implies that the formula \( S^\alpha_m \) defined by Eq. (14) is not a similarity measures on IFSs.

IV. A REMARK ON SCORE FUNCTIONS FOR IFVs

Zeng et al. [17] introduced the following score value \( S_{CK}(\cdot) \) for IFV \( \alpha = (\mu_\alpha, \nu_\alpha) \):

\[
S_{CK}(\alpha) = (\mu_\alpha - \nu_\alpha) - (1 - \mu_\alpha - \nu_\alpha) \times \frac{\log_2(2 - \mu_\alpha - \nu_\alpha)}{100}. \tag{15}
\]

Then, they proved the following basic properties for the score function \( S_{CK}(\cdot) \).

**Theorem 4.1** ([17, Theorem 3.1]): Assume that \( \alpha \) and \( \beta \) are two IFVs. If \( \alpha \neq \beta \), then \( S_{CK}(\alpha) \neq S_{CK}(\beta) \).

**Theorem 4.2** ([17, Theorem 3.2]): Assume that \( \alpha \) and \( \beta \) are two IFVs. If \( \alpha \supset \beta \), then \( S_{CK}(\alpha) > S_{CK}(\beta) \).

However, the following example shows that Theorem 4.1 does not hold.

**Example 4:** Choose \( \alpha = (0, 0) \) and \( \beta = \left( \frac{99}{200}, \frac{101}{200} \right) \). By direct calculation, one has

\[
S_{CK}(\alpha) = (0 - 0) - (1 - 0 - 0) \times \frac{\log_2 2}{100} = - \frac{1}{100},
\]

and

\[
S_{CK}(\beta) = \left( \frac{99}{200} - \frac{101}{200} \right) - \left( 1 - \frac{99}{200} - \frac{101}{200} \right) \times \frac{\log_2(2 - 1)}{100} = - \frac{1}{100}.
\]

This implies that Theorem 4.1 does not hold since \( \alpha \neq \beta \).

In fact, we can prove that there is no any continuous function from \( \mathbb{I} \) to \( \mathbb{R} \) simultaneously meeting the conditions in Theorems 4.1 and 4.2, which indicates that the two-dimensional structure of IFVs is too complex to distinguish all IFVs with only a single monotonous and continuous function, where the monotonicity is under the Atanassov’s order “\( \subset^* \)”, and the continuity is under the topology of subset of \( \mathbb{R}^2 \).

**Theorem 4.3:** There is no any continuous function \( f : \mathbb{I} \to \mathbb{R} \) satisfying the following two conditions:

1. \( f \) is injective, i.e., for any \( \alpha, \beta \in \mathbb{I} \) with \( \alpha \neq \beta \), \( f(\alpha) \neq f(\beta) \);
2. \( f \) is increasing under the partial order \( \subset \), i.e., for any \( \alpha, \beta \in \mathbb{I} \) with \( \alpha \subset \beta \), \( f(\alpha) \leq f(\beta) \).

**Proof:** Suppose on the contrary that there exists a continuous function \( f : \mathbb{I} \to \mathbb{R} \) simultaneously satisfying the conditions (1) and (2).

(i) Let \( \varphi(\nu) = f((0.25, \nu)) \) (\( \nu \in [0.25, 0.75] \)). Clearly, \( \varphi \) is continuous since \( f \) is continuous on \( \mathbb{I} \). For any \( 0.25 \leq \nu_1 \leq \nu_2 \leq 0.75 \), by \( (0.25, \nu_1) \supset (0.25, \nu_2) \) and condition (2), one has \( \varphi(\nu_1) = f((0.25, \nu_1)) \leq f((0.25, \nu_2)) = \varphi(\nu_2) \). This, together with condition (1), implies that \( \varphi(\nu) \) is strictly decreasing on \( [0.25, 0.75] \). Thus, \( \varphi(((0.25, 0.75)) = f((0.25, 0.75)), f((0.25, 0.25))) \) by the intermediate value theorem.

(ii) Let \( \psi(\nu) = f((\nu, 0.25)) \) (\( \nu \in [0, 0.25] \)). Clearly, \( \psi \) is continuous since \( f \) is continuous on \( \mathbb{I} \). For any \( 0 \leq \mu_1 \leq \mu_2 \leq 0.25 \), by \( (\mu_1, 0.25) \subset (\mu_2, 0.25) \) and condition (2), one has \( \psi(\mu_1) = f((\mu_1, 0.25)) \leq f((\mu_2, 0.25)) = \psi(\mu_2) \). This, together with condition (1), implies that \( \psi(\nu) \) is strictly increasing on \( [0, 0.25] \). Thus, \( \psi(0, 0.25)) = \psi(0), \psi(0.25)) = f((0, 0.25)), f((0.25, 0.25))) \) by the intermediate value theorem.

Summing (i) and (ii), one can easily verify that \( \Lambda = \varphi((0.25, 0.75)) \cap \psi((0, 0.25)) = \max\{f((0.25, 0.75)), f((0.25, 0.25))\} \) is a non-degenerate interval, i.e.,

\[
\max\{f((0.25, 0.75)), f((0.25, 0.25))\) \leq f((0.25, 0.25)),
\]

implying that, for any \( \xi \in \Lambda \), there exist \( \nu \in (0.25, 0.75) \) and \( \mu \in [0, 0.25) \) such that \( \varphi(\nu) = \xi \) and \( \psi(\mu) = \xi \), and thus there exist \( 0 < \mu < 0.25 < \nu \leq 0.75 \) such that \( \varphi(\nu) = f((0.25, \nu)) = \xi = f((\mu, 0.25)) = \psi(\mu) \). This contradicts with condition (1).

**Fig. 1.** Geometrical interpretation of the proof of Theorem 4.3

Shen et al. [29] pointed out that many existing distance measures cannot determine the classification results for some pattern recognition problems (see [29, Table 2]), i.e., their dual similarity measures cannot distinguish between some
pair of IFVs. To overcome this drawback, they proposed a new distance measure $d_{ab}$ as follows: for $\alpha = (\mu_\alpha, \nu_\alpha)$, $\beta = (\mu_\beta, \nu_\beta) \in \tilde{I}$,

$$d_{ab}(\alpha, \beta) = \sqrt{\frac{(\bar{\mu}_\alpha - \bar{\mu}_\beta)^2 + (\bar{\nu}_\alpha - \bar{\nu}_\beta)^2}{2}},$$

where $\bar{\mu}_\alpha = \mu_\alpha(1 + \frac{2}{3} \pi_\alpha(1 + \pi_\alpha))$, $\bar{\nu}_\alpha = \nu_\alpha(1 + \frac{2}{3} \pi_\alpha(1 + \pi_\alpha))$, $\bar{\mu}_\beta = \mu_\beta(1 + \frac{2}{3} \pi_\beta(1 + \pi_\beta))$, and $\bar{\nu}_\beta = \nu_\beta(1 + \frac{2}{3} \pi_\beta(1 + \pi_\beta))$.

Fix $\beta \in \tilde{I}$ and define $G(\alpha) = 1 - d_{ab}(\alpha, \beta)$ for $\alpha = (\mu_\alpha, \nu_\alpha) \in \tilde{I}$. From [29, Theorem 1], it follows that, for $\alpha_1, \alpha_2 \in \tilde{I}$ with $\alpha_1 < \alpha_2$, one has $G(\alpha_1) < G(\alpha_2)$, i.e., the function $G$ satisfies the condition (2) of Theorem 4.3. This, together with Theorem 4.3 and the continuity of $G$, implies that $G$ is not injective, and thus there exist two different IFVs $\alpha_1$ and $\alpha_2 \in \tilde{I}$ such that $G(\alpha_1) = G(\alpha_2)$, implying that $d_{ab}(\alpha_1, \beta) = d_{ab}(\alpha_2, \beta)$. Therefore, the distance measure $d_{ab}$ has the same drawback (see Example 5). In fact, by Theorem 4.3, we conclude that there is no any continuous distance measure that can overcome the above drawback. Therefore, the comparative analysis in [40, Tables 2–5], [36, Tables 1–2], [35, Tables 1, 2, 5, 6], and [29, Table 2] on the indistinguishability is meaningless.

**Example 5:** Let $\beta = (0, 0)$ and $\alpha = (x, y) \in \tilde{I}$. Then,

$$d_{sb}(\alpha, \beta) = \frac{\sqrt{1 + \frac{2}{3}(1 - x - y)(2 - x - y)^2 \times (x^2 + y^2)}}{2} = 0.5,$$

i.e.,

$$\left[1 + \frac{2}{3}(1 - x - y)(2 - x - y)^2 \times (x^2 + y^2)\right] = 0.5. \quad (16)$$

Clearly, Eq. (16) has infinitely many solutions. This means that there exist infinitely many IFVs, whose distances from $\beta$ are all equal to 0.5.

### V. A MONOTONOUS IF TOPSIS METHOD WITH THE LINEAR ORDERS $\leq_{xy}$ AND $\leq_{yx}$

Suppose that there are $n$ alternatives $A_i (i = 1, 2, \ldots, n)$ evaluated with respect to $m$ attributes $O_j (j = 1, 2, \ldots, m)$. The sets of the alternatives and attributes are denoted by $A = \{A_1, A_2, \ldots, A_n\}$ and $O = \{O_1, O_2, \ldots, O_m\}$, respectively. The rating (or evaluation) of each alternative $A_i \in A (i = 1, 2, \ldots, n)$ on each attribute $O_j (j = 1, 2, \ldots, m)$ is expressed with an IFS $r_{ij} = (\mu_{ij}, \nu_{ij})$, denoted by $r_{ij} = (\mu_{ij}, \nu_{ij})$ for short, where $\mu_{ij} \in [0, 1]$ and $\nu_{ij} \in [0, 1]$ are respectively the satisfaction (or membership) degree and dissatisfaction (or non-membership) degree of the alternative $A_i$ on the attribute $O_j$ satisfying the condition $0 \leq \mu_{ij} + \nu_{ij} \leq 1$. A multi-attribute decision-making (MADM) problem with IFSs is expressed in matrix for shown in Table I.

### TABLE I

| $O_1$ | $O_2$ | \ldots | $O_m$ |
|------|------|-------|-------|
| $A_1$ | $(\mu_{11}, \nu_{11})$ | $(\mu_{12}, \nu_{12})$ | \ldots | $(\mu_{1m}, \nu_{1m})$ |
| $A_2$ | $(\mu_{21}, \nu_{21})$ | $(\mu_{22}, \nu_{22})$ | \ldots | $(\mu_{2m}, \nu_{2m})$ |
| \vdots | \vdots | \vdots | \vdots | \vdots |
| $A_n$ | $(\mu_{n1}, \nu_{n1})$ | $(\mu_{n2}, \nu_{n2})$ | \ldots | $(\mu_{nm}, \nu_{nm})$ |

To follow the common sense, a good method for MADM must guarantee the monotonicity, i.e., the higher the score of each attribute of the alternative is, the higher the ranking is. Meanwhile, decision-making results obtained by this good method must be consistent with our intuitive judgment, when dealing with the simplest problems, for which the decision-making results can be obtained by direct observation and comparison (see Examples 6 and 7). In analyzing the TOPSIS method with crisp score values in $[0, 1]$, we can find that the unit interval $[0, 1]$ has excellent algebraic and topological structures as follows: (1) The unit interval $[0, 1]$ has a natural linear order structure $\leq$ and a natural metric structure $|\cdot|$. (2) This natural metric structure $|\cdot|$ can ensure that the smaller the value in $[0, 1]$ is, the farther away from 1, and the closer away from 0; (3) The order topology induced by the linear order $\leq$ is consistent with the topology induced by the metric $|\cdot|$. These good structures of $[0, 1]$ can ensure that the TOPSIS method of Hwang and Yoon [4] is monotonous. Recently, we [5] proved that the space $\tilde{I}$ of all IFVs with the order topology induced by the linear order $\leq_{xy}$, defined in Definition 2.3, is not metrizable, i.e., there is no such good distance for $\tilde{I}$ with the linear order $\leq_{xy}$. Nevertheless, we still construct an admissible distance with the order $\leq_{xy}$ in [5]. In the following, we will show that this distance is very important for our proposed monotonous IF TOPSIS method.

#### A. Limitation in TOPSIS method of Li [33]

First, we recall a fundamental IF TOPSIS method from [33] and use two examples show that the TOPSIS method in [33] does not have the basic monotonicity with the partial order $\subset$, which may lead the decision-making result to be unreasonable and inconsistent with the actual situation, when dealing with some even simplest decision-making problems.

The main process of IF TOPSIS method in [33, Section 3.3] is summarized as follows:

**Step 1:** Determine the alternatives $A = \{A_1, A_2, \ldots, A_n\}$ and attributes $O = \{O_1, O_2, \ldots, O_m\}$, respectively;

**Step 2:** Construct the IF decision matrix $R = (r_{ij})_{m \times n}$, as shown in Table I;

**Step 3:** Determine IF weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_m)^T$ of the attributes expressed, where $\omega_j = (\rho_j, \vartheta_j) \in \tilde{I}$;

**Step 4:** Compute the weighted IF decision matrix $T = (T_{ij} = \omega_j \otimes r_{ij})$ by Definition 2.2 (v), i.e., $T_{ij} = (\rho_j \mu_{ij} + \vartheta_j \nu_{ij})$;

**Step 5:** Determine the IF positive ideal-point $A^+ = ((\mu^+_1, \nu^+_1), (\mu^+_2, \nu^+_2), \ldots, (\mu^+_m, \nu^+_m))^T$ and the IF negative
ideal-point $A^- = (\mu_1, \nu_1^r), (\mu_2, \nu_2^r), \ldots, (\mu_m, \nu_m^r)^T$ as follows:

$$
\mu_j^+ = \max_{1 \leq i \leq n} \{ \bar{\mu}_{ij} \}, \quad \nu_j^+ = \min_{1 \leq i \leq n} \{ \bar{\nu}_{ij} \}, \quad \mu_j^- = \min_{1 \leq i \leq n} \{ \bar{\mu}_{ij} \}, \quad \nu_j^- = \max_{1 \leq i \leq n} \{ \bar{\nu}_{ij} \};
$$

Step 6: Compute the Euclidean distances $d_{\text{Eu}}(A_i, A^+)$ and $d_{\text{Eu}}(A_i, A^-)$ of the alternatives $A_i$ ($i = 1, 2, \ldots, n$) from $A^+$ and $A^-$ by using Eq. (6);

Step 7: Calculate the relative closeness degrees $C_i$ of the alternatives $A_i$ ($i = 1, 2, \ldots, n$) to the IF positive ideal-point $A^+$ by the following formula:

$$
C_i = \frac{d_{\text{Eu}}(A_i, A^-)}{d_{\text{Eu}}(A_i, A^+) + d_{\text{Eu}}(A_i, A^-)};
$$

Step 8: Rank the alternatives $A_i$ ($i = 1, 2, \ldots, n$) according to the nonincreasing order of the relative closeness degrees $C_i$ and select the most desirable alternative.

Example 6: Suppose that there exist 4 alternatives $A_1$, $A_2$, $A_3$, $A_4$ evaluated with respect to 2 benefit attributes $\theta_1$, $\theta_2$. The sets of the alternatives and attributes are denoted by $\{A_1, A_2, A_3, A_4\}$ and $\{\theta_1, \theta_2\}$, respectively. Assume that the IF weight vector of $\theta_1$ and $\theta_2$ is $\omega = (\langle 1, 0 \rangle, \langle 1, 0 \rangle)^T = (\omega_1, \omega_2)^T$. The IF decision-making matrix is expressed as shown in Table II.

| $\theta_1$ | $\theta_2$ |
|------------|------------|
| $A_1$      | (0, 1)     |
| $A_2$      | (0.9, 0.01)|
| $A_3$      | (0.901, 0.007)|
| $A_4$      | (1, 0)     |

If we use the above TOPSIS method [33, Section 3.3], by direct calculation, it can be verified that the weighted IF decision matrix is given as shown in Table III.

| $\omega_j \times \theta_1$ | $\omega_j \times \theta_2$ |
|-------------------------|-------------------------|
| $A_1$                   | (0, 1)                  |
| $A_2$                   | (0.9, 0.01)             |
| $A_3$                   | (0.901, 0.007)          |
| $A_4$                   | (1, 0)                  |

The IF positive ideal-point $A^+$ and the IF negative ideal-point $A^-$ are obtained as follows:

$$
A^+ = (\langle 1, 0 \rangle, \langle 1, 0 \rangle) \quad \text{and} \quad A^- = (\langle 0, 1 \rangle, \langle 0, 1 \rangle),
$$

respectively. According to the Euclidean distance of the alternatives $A_1$, $A_2$, $A_3$, and $A_4$ from $A^+$ and $A^-$ obtained in [33, Eqs. (3.27) and (3.28)], the relative closeness degrees $C_i$ of the alternatives $A_1$, $A_2$, $A_3$, and $A_4$ to the intuitionistic fuzzy positive ideal-point can be calculated as follows:

$$
C_i = \frac{d_{\text{Eu}}(A_i, A^-)}{d_{\text{Eu}}(A_i, A^+) + d_{\text{Eu}}(A_i, A^-)} = 0,
$$

$$
C_2 = \frac{d_{\text{Eu}}(A_2, A^-)}{d_{\text{Eu}}(A_2, A^+) + d_{\text{Eu}}(A_2, A^-)} = 0.9085917,
$$

$$
C_3 = \frac{d_{\text{Eu}}(A_3, A^-)}{d_{\text{Eu}}(A_3, A^+) + d_{\text{Eu}}(A_3, A^-)} = 0.9085194,
$$

and

$$
C_4 = \frac{d_{\text{Eu}}(A_4, A^-)}{d_{\text{Eu}}(A_4, A^+) + d_{\text{Eu}}(A_4, A^-)} = 1,
$$

respectively. Therefore, the ranking order of $A_1$, $A_2$, $A_3$, and $A_4$ is $A_4 \succ A_2 \succ A_3 \succ A_1$. However, $A_4 \succ A_3 \succ A_2 \succ A_1$ by a direct observation since $\langle 1, 0 \rangle \succ (0.901, 0.007) \succ (0.9, 0.01) \succ (0, 1)$. This means that the ranking order obtained by the IF TOPSIS method in [33] is not consistent with the real situation.

The following example demonstrates that the above IF TOPSIS may yield some unreasonable decision-making results, even if we restrict the normalized weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_m)^T$ in Step 3 to be positive real numbers, i.e., $\omega_j \in [0, 1]$ and $\sum_{j=1}^m \omega_j = 1$.

Example 7: Suppose that there exist 4 alternatives $A_1$, $A_2$, $A_3$, $A_4$ evaluated with respect to 2 benefit attributes $\theta_1$, $\theta_2$. The sets of the alternatives and attributes are denoted by $\{A_1, A_2, A_3, A_4\}$ and $\{\theta_1, \theta_2\}$, respectively. Assume that the weight vector of $\theta_1$ and $\theta_2$ is $\omega = (0.5, 0.5)^T = (\omega_1, \omega_2)^T$. The IF decision-making matrix is expressed as shown in Table IV.

| $\omega_j \times \theta_1$ | $\omega_j \times \theta_2$ |
|-------------------------|-------------------------|
| $A_1$                   | (0.1)                   |
| $A_2$                   | (0.99, 0.0001)          |
| $A_3$                   | (0.990199, 4.9 \times 10^{-4}) |
| $A_4$                   | (1, 0)                  |

By direct calculation, it can be verified that the weighted intuitionistic fuzzy decision matrix is given as shown in Table V.

| $\omega_j \cdot \theta_1$ | $\omega_j \cdot \theta_2$ |
|--------------------------|--------------------------|
| $A_1$                    | (0, 1)                   |
| $A_2$                    | (0.9, 0.01)              |
| $A_3$                    | (0.901, 0.007)           |
| $A_4$                    | (1, 0)                   |

If we use the TOPSIS method in [33], by Example 6, we know that the ranking order of $A_1$, $A_2$, $A_3$, and $A_4$ is $A_4 \succ A_2 \succ A_3 \succ A_1$. This is also an unreasonable decision-making result.

Remark 2: (1) Examples 6 and 7 shows that (2) Careful readers can verify that by applying the TOPSIS methods in [41], [44], [43], [46] to Examples 6, the same result can be obtained. This means that the TOPSIS methods in [41], [44], [43], [46] may produce unreasonable results when dealing with the simplest decision-making problems.
B. Limitation in TOPSIS method of Chen et al. [48]

The above two examples show that the TOPSIS method in [33] is not monotonic with Atanassov’s partial order \( \preceq \). Recently, Chen et al. [48] developed a monotonic TOPSIS method with the partial order \( \preceq \) based on a new similarity measure. However, the following example shows that the TOPSIS methods in [48] is not monotonic with the linear order \( \leq_{xy} \).

The main process of IF TOPSIS method in [48] is summarized as follows:

Step 1: Determine the alternatives \( A = \{A_1, A_2, \ldots, A_n\} \) and attributes \( \Theta = \{\Theta_1, \Theta_2, \ldots, \Theta_m\} \), respectively, and construct the IF decision matrix \( R = (r_{ij} = \langle \mu_{ij}, \nu_{ij} \rangle)_{m \times n} \), as shown in Table I;

Step 2: Determine the IF positive ideal-point \( \mathbf{A}^+ = ((\mu^+_1, \nu^+_1), (\mu^+_2, \nu^+_2), \ldots, (\mu^+_m, \nu^+_m))^T \) and the IF negative ideal-point \( \mathbf{A}^- = ((\mu^-_1, \nu^-_1), (\mu^-_2, \nu^-_2), \ldots, (\mu^-_m, \nu^-_m))^T \) as follows:

\[
\{ \mu^+_j, \nu^+_j \} = \left\{ \begin{array}{ll}
\left\{ \max_{1 \leq i \leq n} \{ \mu_{ij} \}, \min_{1 \leq i \leq n} \{ \nu_{ij} \} \right\}, \quad & \Theta_j \in \Theta^+, \\
\left\{ \min_{1 \leq i \leq n} \{ \mu_{ij} \}, \max_{1 \leq i \leq n} \{ \nu_{ij} \} \right\}, \quad & \Theta_j \in \Theta^-, 
\end{array} \right.
\]

and

\[
\{ \mu^-_j, \nu^-_j \} = \left\{ \begin{array}{ll}
\left\{ \min_{1 \leq i \leq n} \{ \mu_{ij} \}, \max_{1 \leq i \leq n} \{ \nu_{ij} \} \right\}, \quad & \Theta_j \in \Theta^+, \\
\left\{ \max_{1 \leq i \leq n} \{ \mu_{ij} \}, \min_{1 \leq i \leq n} \{ \nu_{ij} \} \right\}, \quad & \Theta_j \in \Theta^-, 
\end{array} \right.
\]

where \( \Theta^+ \) is the set of benefit attributes and \( \Theta^- \) is the set of cost attributes;

Step 3: Compute the degree of indeterminacy \( \pi_j = 1 - \mu^+_j - \nu^+_j \) of the positive ideal-point \( \langle \mu^+_j, \nu^+_j \rangle \) for each attribute \( \Theta_j \) (\( j = 1, 2, \ldots, n \));

Step 4: Compute the degree of indeterminacy \( \pi_j = 1 - \mu^-_j - \nu^-_j \) of the negative ideal-point \( \langle \mu^-_j, \nu^-_j \rangle \) for each attribute \( \Theta_j \) (\( j = 1, 2, \ldots, n \));

Step 5: Compute the degree of similarity \( g_{ij}^+ \) between the evaluating IF \( r_{ij} \) of the alternative \( A_i \) with respect to the attribute \( \Theta_j \) and the positive ideal-point \( \langle \mu^+_j, \nu^+_j \rangle \) of the attribute \( \Theta_j \) to construct the positive similarity matrix

\[
G^+ = (g^+_{ij})_{m \times n}, \quad \text{where} \quad g^+_{ij} = 1 - \frac{2(\mu^+_j - \mu_{ij}) - (\nu^+_j - \nu_{ij})}{3} \times (1 - \pi^+_j + \pi_{ij}) + \frac{2(\mu^+_j - \mu_{ij}) - (\nu^+_j - \nu_{ij})}{3} \times \pi^+_j + \pi_{ij};
\]

Step 6: Compute the degree of similarity \( g_{ij}^- \) between the evaluating IF \( r_{ij} \) of the alternative \( A_i \) with respect to the attribute \( \Theta_j \) and the negative ideal-point \( \langle \mu^-_j, \nu^-_j \rangle \) of the attribute \( \Theta_j \) to construct the positive similarity matrix

\[
G^- = (g^-_{ij})_{m \times n}, \quad \text{where} \quad g^-_{ij} = 1 - \frac{2(\mu^-_j - \mu_{ij}) - (\nu^-_j - \nu_{ij})}{3} \times (1 - \pi^-_j + \pi_{ij}) + \frac{2(\mu^-_j - \mu_{ij}) - (\nu^-_j - \nu_{ij})}{3} \times \pi^-_j + \pi_{ij};
\]

Step 7: Compute the weighted positive score \( S_i^+ = \sum_{j=1}^{m} \omega_j g_{ij}^+ \) and the weighted negative score \( S_i^- = \sum_{j=1}^{m} \omega_j g_{ij}^- \) of each alternative \( A_i \) (\( i = 1, 2, \ldots, n \)), where \( \omega_j \) is the weight of criterion \( \Theta_j \) such that \( \omega_j \in (0, 1] \) and \( \sum_{j=1}^{m} \omega_j = 1 \); and

Step 8: Compute the relative degree of closeness \( T(A_i) = \frac{S_i^+}{S_i^+ + S_i^-} \) of each alternative \( A_i \). The larger the value of \( T(A_i) \), the better the preference order of alternative \( A_i \). Then, rank the alternatives \( A_i \) (\( i = 1, 2, \ldots, n \)) according to the nonincreasing order of the relative closeness degrees \( T(A_1), T(A_2), \ldots, T(A_n) \).

**Example 8:** Suppose that there exist 4 alternatives \( A_1, A_2, A_3, A_4 \) evaluated with respect to 2 benefit attributes \( \Theta_1, \Theta_2 \). The sets of the alternatives and attributes are denoted by \( \{A_1, A_2, A_3, A_4\} \) and \( \{\Theta_1, \Theta_2\} \), respectively. Assume that the weight vector of \( \Theta_1, \Theta_2 \) is \( \omega = (0.5, 0.5)^T \). The intuitionistic fuzzy decision-making matrix is expressed as shown in Table VI.

| \( \Theta_1 \) | \( \Theta_2 \) |
|----------------|----------------|
| \[0, 1\]     | \[0, 1\]    |
| \(A_1\) | \(A_2\) | \(A_3\) | \(A_4\) |
| \(0.64, 0.36\) | \(0.64, 0.36\) | \(0.64, 0.36\) | \(0.64, 0.36\) |

The IF positive ideal-point \( \mathbf{A}^+ \) and the IF negative ideal-point \( \mathbf{A}^- \) are obtained as follows:

\[
\mathbf{A}^+ = \langle (1, 0), (1, 0) \rangle \quad \text{and} \quad \mathbf{A}^- = \langle (0, 1), (0, 1) \rangle,
\]

respectively. By using the TOPSIS method in [48], by direct calculation, we obtain the positive similarity matrix \( G^+ \) and the negative similarity matrix \( G^- \) as follows:

\[
G^+ = (g^+_{ij})_{4 \times 2} = \begin{bmatrix} 0 & 0 \\ 0.615 & 0.615 \\ 0.64 & 0.64 \\ 1 & 1 \end{bmatrix},
\]

and

\[
G^- = (g^-_{ij})_{4 \times 2} = \begin{bmatrix} 1 & 1 \\ 0.385 & 0.385 \\ 0.36 & 0.36 \\ 0 & 0 \end{bmatrix}.
\]

Then, the weighted positive scores \( S_i^+ = \omega_1 g^+_{i1} + \omega_2 g^+_{i2} \) (\( i = 1, 2, 3, 4 \)) and the weighted negative scores \( S_i^- = \omega_1 g^-_{i1} + \omega_2 g^-_{i2} \) (\( i = 1, 2, 3, 4 \)) of the alternatives \( A_1, A_2, A_3, A_4 \) can be calculated as follows:

\[
S^+_1 = 0, \quad S^+_2 = 0.615, \quad S^+_3 = 0.64, \quad S^+_4 = 1,
\]

and

\[
S^-_1 = 1, \quad S^-_2 = 0.385, \quad S^-_3 = 0.36, \quad S^-_4 = 0.
\]

Therefore, the relative degree of closeness \( T(A_i) = \frac{S_i^+}{S_i^+ + S_i^-} \) of each alternative \( A_i \). The larger the value of \( T(A_i) \), the better the preference order of alternative \( A_i \). Then, rank the alternatives \( A_i \) (\( i = 1, 2, \ldots, n \)) according to the nonincreasing order of the relative closeness degrees \( T(A_1), T(A_2), \ldots, T(A_n) \).

**Summing up Examples 6–8:** an interesting question is whether there exists an IF TOPSIS method that is monotonic with the linear order \( \leq_{xy} \) or \( \leq_{zx} \)? In the following section, we will construct an IF TOPSIS method that is monotonic with the linear order \( \leq_{xy} \) or \( \leq_{zx} \).
VI. A MONOTONOUS IF TOPSIS METHOD

The fundamental cause for counterintuitive decision-making results in Examples 6–8 lies in the structure of metrics for IFVs. In [5], we defined a metric $\varrho$ in $\mathbb{I}$ as follows: for $\alpha, \beta \in \mathbb{I}$,

$$\varrho(\alpha, \beta) = \begin{cases} \frac{1}{2} \left(1 + |s(\alpha) - s(\beta)|\right), & s(\alpha) \neq s(\beta), \\ \frac{1}{4} (|h(\alpha) - h(\beta)|), & s(\alpha) = s(\beta), \end{cases}$$  \hspace{1cm} (17)

where $s(\alpha)$ and $h(\alpha)$ are the score degree and the accuracy degree of $\alpha$, respectively. Furthermore, we [5] proved the following basic properties of $\varrho$.

Theorem 6.1 ([5]):

(1) $\varrho(\alpha, \beta) \in [0, 1]$ and $\varrho(\alpha, \beta) = 0$ if and only if $\alpha = \beta$.
(2) $\varrho(\alpha, \beta) = 1$ if and only if $\alpha = 0, 1$ or $\beta = 0, 1$.
(3) $\varrho(\alpha, \beta) = \varrho(\beta, \alpha)$.
(4) For any $\alpha, \beta, \gamma \in \mathbb{I}$, $\varrho(\alpha, \beta) + \varrho(\beta, \gamma) \geq \varrho(\alpha, \gamma)$.
(5) For any $\alpha, \beta, \gamma \in \mathbb{I}$, if $\alpha \leq_{xy} \beta \leq_{xy} \gamma$, then $\varrho(\alpha, \beta) \leq \varrho(\alpha, \gamma)$ and $\varrho(\beta, \gamma) \leq \varrho(\beta, \gamma)$.

Based on the similarity function $L(\alpha)$, similarly to the metric $\varrho$ defined by Eq. (17), we defined another metric $\tilde{\varrho}$ in $\mathbb{I}$ as follows: for $\alpha, \beta \in \mathbb{I}$,

$$\tilde{\varrho}(\alpha, \beta) = \begin{cases} \frac{1}{2} \left(1 + |L(\alpha) - L(\beta)|\right), & L(\alpha) \neq L(\beta), \\ \frac{1}{4} (|h(\alpha) - h(\beta)|), & L(\alpha) = L(\beta), \end{cases}$$  \hspace{1cm} (18)

where $L(\alpha)$ and $h(\alpha)$ are the similarity function and the accuracy degree of $\alpha$, respectively. Now we can prove that the metric $\tilde{\varrho}$ defined by Eq. (18) has the following basic properties.

Theorem 6.2:

(1) $\tilde{\varrho}(\alpha, \beta) \in [0, 1]$ and $\tilde{\varrho}(\alpha, \beta) = 0$ if and only if $\alpha = \beta$.
(2) $\tilde{\varrho}(\alpha, \beta) = 1$ if and only if $(\alpha = 0, 1)$ or $\beta = 0, 1$.
(3) $\tilde{\varrho}(\alpha, \beta) = \tilde{\varrho}(\beta, \alpha)$.
(4) For any $\alpha, \beta, \gamma \in \mathbb{I}$, $\tilde{\varrho}(\alpha, \beta) + \tilde{\varrho}(\beta, \gamma) \geq \tilde{\varrho}(\alpha, \gamma)$.
(5) For any $\alpha, \beta, \gamma \in \mathbb{I}$, if $\alpha \leq_{xy} \beta \leq_{xy} \gamma$, then $\tilde{\varrho}(\alpha, \beta) \leq \tilde{\varrho}(\alpha, \gamma)$ and $\tilde{\varrho}(\beta, \gamma) \leq \tilde{\varrho}(\beta, \gamma)$.

For the MADM problem with IFPS, by using the two metrics defined by Eqs. (17) and (18), we propose a new IF TOPSIS method as follows:

Step 1: (Construct the decision matrix) Supposing that the decision-maker gave the rating (or evaluation) of each alternative $A_i \in A (i = 1, 2, \ldots, n)$ on each attribute $O_j$ ($j = 1, 2, \ldots, m$) in the form of IFNs $r_{ij} = (\mu_{ij}, \nu_{ij})$, construct an IF decision matrix $R = (r_{ij})_{m \times n}$ as shown in Table 1.

Step 2: (Normalize the decision matrix) Transform the IF decision matrix $R = (r_{ij})_{m \times n}$ to the normalized IF decision matrix $\overline{R} = (\overline{r}_{ij})_{m \times n} = (\langle \mu_{ij}, \nu_{ij} \rangle)_{m \times n}$ as follows:

$$\overline{r}_{ij} = \begin{cases} r_{ij}^C, & \text{for benefit attribute } O_j, \\ r_{ij}^C, & \text{for cost attribute } O_j, \end{cases}$$

where $r_{ij}^C$ is the complement of $r_{ij}$.

Step 3: (Determine the positive and negative ideal-points) Determine the IF positive ideal-point $A^+ = (\langle \mu^+, \nu^+ \rangle, \langle \mu^+, \nu^+ \rangle, \ldots, \langle \mu^+, \nu^+ \rangle)^\top$ and IF negative ideal-point $A^- = (\langle \mu^-, \nu^- \rangle, \langle \mu^-, \nu^- \rangle, \ldots, \langle \mu^-, \nu^- \rangle)^\top$ as follows:

$$\mu_j^+ = \max_{1 \leq i \leq n} \{\overline{r}_{ij}\}, \nu_j^+ = \min_{1 \leq i \leq n} \{\overline{r}_{ij}\}, \mu_j^- = \min_{1 \leq i \leq n} \{\overline{r}_{ij}\}, \nu_j^- = \max_{1 \leq i \leq n} \{\overline{r}_{ij}\}.$$

Step 4: (Compute the weighted similarity measures) Compute the weighted similarity measures between the alternatives $A_i (i = 1, 2, \ldots, n)$ and the IF positive ideal-point $A^+$ and between the alternatives $A_i (i = 1, 2, \ldots, n)$ and the IF negative ideal-point $A^-$ by using the following formulas:

$$S(A_i, A^+) = 1 - \sum_{j=1}^{m} \omega_j \cdot \tilde{\varrho}(\langle \mu_{ij}, \nu_{ij} \rangle, \langle \mu_j^+, \nu_j^+ \rangle),$$  \hspace{1cm} (19)

and

$$S(A_i, A^-) = 1 - \sum_{j=1}^{m} \omega_j \cdot \tilde{\varrho}(\langle \mu_{ij}, \nu_{ij} \rangle, \langle \mu_j^-, \nu_j^- \rangle).$$  \hspace{1cm} (20)

By Theorems 6.1 and 6.2, it is easy to see that the similarity measures obtained by Eqs. (19) and (20) are admissible similarity measures with the orders $\leq_{xy}$ and $\leq_{yx}$, respectively.

Step 5: (Compute the relative closeness degrees) Calculate the relative closeness degrees $C_i$ of the alternatives $A_i (i = 1, 2, \ldots, n)$ to the IF positive ideal-point $A^+$ by using the following formula:

$$C_i = \frac{S(A_i, A^+)}{S(A, A^+) + S(A_i, A^-)}. $$  \hspace{1cm} (23)

Step 6: (Rank the alternative) Rank the alternatives $A_i (i = 1, 2, \ldots, n)$ according to the nonincreasing order of the relative closeness degrees $C_i$ and select the most desirable alternative.

Remark 3: By Theorems 6.1 and 6.2, it is easy to see that $S(A_i, A^+) + S(A_i, A^-)$ in Eq. (23) is always nonzero. This overcomes the limitation that many TOPSIS method may lead to the situation that the denominator is equal to 0 when computing the relative closeness degree.

Theorem 6.3 (Monotonicity): Using Eqs. (19) and (21), the above proposed method is increasing with the linear order $\leq_{xy}$, i.e., for the MADM problem expressed in Table 1, if there exist 1 \leq i_1, i_2 \leq n such that $\overline{r}_{i_1j} \leq_{xy} \overline{r}_{i_2j}$ holds for all 1 \leq j \leq m, then $C_{i_1} \leq C_{i_2}$, i.e., $A_{i_1}$ is better than $A_{i_2}$ ranked by the proposed method. In particular, the proposed method is increasing with Atanassov's order ‘\$’.

Proof: Let $A^+ = (\langle \mu^+, \nu^+ \rangle, \langle \mu^+, \nu^+ \rangle, \ldots, \langle \mu^+, \nu^+ \rangle)^\top$ and $A^- = (\langle \mu^-, \nu^- \rangle, \langle \mu^-, \nu^- \rangle, \ldots, \langle \mu^-, \nu^- \rangle)^\top$ be the IF positive ideal-point and the IF negative ideal-point obtained by Step 3, respectively. Clearly, $\langle \mu^+, \nu^+ \rangle \leq_{xy} \overline{r}_{i_1j} \leq_{xy} \overline{r}_{i_2j} \leq_{xy} \langle \mu^+, \nu^+ \rangle$. By Theorem 6.1, one has

$$\tilde{\varrho}(\mu^+, \nu^+), r_{i_1j} \leq \tilde{\varrho}(\mu^+, \nu^+), r_{i_2j},$$

and

$$\tilde{\varrho}(\mu^+, \nu^+), r_{i_2j} \leq \tilde{\varrho}(\mu^+, \nu^+), r_{i_1j}.$$
and thus,
\[ S(A_{i1}, A^-) \geq S(A_{i2}, A^-), \]
and
\[ S(A_{i1}, A^+) \leq S(A_{i2}, A^+) \]  by Eqs. (19) and (21).

This, together with Eq. (23), implies that

(1) if \( S(A_{i1}, A^+) = 0 \), then
\[ \varepsilon_{i1} = \frac{S(A_{i1}, A^+)}{S(A_{i1}, A^+) + S(A_{i1}, A^-)} = 0 \leq \varepsilon_{i2}; \]

(2) if \( S(A_{i1}, A^+) > 0 \), then \( S(A_{i2}, A^+) \geq S(A_{i1}, A^+) > 0 \), and thus
\[ \varepsilon_{i1} = \frac{S(A_{i1}, A^+)}{S(A_{i1}, A^+) + S(A_{i1}, A^-)} = \frac{1}{1 + \frac{S(A_{i1}, A^-)}{S(A_{i1}, A^+)}} \]
\[ \leq \frac{1}{1 + S(A_{i2}, A^+)/S(A_{i1}, A^+)} = S(A_{i2}, A^+) \]
\[ \leq \frac{S(A_{i1}, A^+)}{S(A_{i1}, A^+) + S(A_{i1}, A^-)} = \varepsilon_{i2}. \]

Therefore, \( \varepsilon_{i1} \leq \varepsilon_{i2} \).

By Theorem 6.2, similarly to the proof of Theorem 6.3, it is not difficult to check that the following result holds.

**Theorem 6.4 (Monotonicity):** Using Eqs. (20) and (22), the above proposed method is increasing with the linear order \( \leq_{x} \), i.e., for the MADM problem shown in Table I, if there exist \( 1 \leq i_1, i_2 \leq n \) such that \( r_{i_1j} \leq_{x} r_{i_2j} \) holds for all \( 1 \leq j \leq m \), then \( \varepsilon_{i_1} \leq \varepsilon_{i_2} \), i.e., \( A_{i_2} \) is better than \( A_{i_1} \) ranked by the proposed method. In particular, the proposed method is increasing with Atanassov’s order ‘\(<\’’.

### VII. ILLUSTRATIVE EXAMPLES

This section provides two practical examples to illustrate the efficiency of the above proposed TOPSIS method. One is an IF MADM problem on the choice of suppliers in the supply chain management (see Example 9). The preference order obtained by the proposed TOPSIS method is slightly different from the results obtained by those TOPSIS methods in [48], [54], [17], although their most desirable alternatives are consistent. By analyzing the weights and comparative analysis with weighted averaging aggregation operators induced by four classes of common triangular norms, we show that the proposed preference order is more reasonable. The other is an IF MADM problem on the choice of project managers. The preference order obtained by the proposed TOPSIS method is consistent with the results obtained by the TOPSIS methods in [48], [54], [17].

**Example 9 ([48, Example 5.1]):** Assume that there are five alternatives \( A_1, A_2, A_3, A_4, \) and \( A_5 \) of suppliers and four attributes \( \theta_1, \theta_2, \theta_3, \) and \( \theta_4 \) to assess these five alternatives, so as to choose the best supplier among these five alternatives in the supply chain management, where \( \theta_1 \) is the “Product Quality”, \( \theta_2 \) is the “Service”, \( \theta_3 \) is the “DelIVERY”, \( \theta_4 \) is the “Price” and \( \theta_1, \theta_2, \theta_3, \theta_4 \) are benefit attributes, with weight vector \( \omega = (0.25, 0.4, 0.2, 0.15)^\top \).

Step 1: (Construct the decision matrix) The decision matrix \( R = (r_{ij})_{5 \times 4} \) given by the decision maker is listed in Table VII.

|   | \( \theta_1 \) | \( \theta_2 \) | \( \theta_3 \) | \( \theta_4 \) |
|---|---|---|---|---|
| \( A_1 \) | (0.6, 0.3) | (0.5, 0.2) | (0.2, 0.5) | (0.1, 0.6) |
| \( A_2 \) | (0.8, 0.2) | (0.8, 0.1) | (0.6, 0.1) | (0.3, 0.4) |
| \( A_3 \) | (0.6, 0.3) | (0.4, 0.3) | (0.4, 0.2) | (0.5, 0.2) |
| \( A_4 \) | (0.9, 0.1) | (0.5, 0.2) | (0.2, 0.3) | (0.1, 0.5) |
| \( A_5 \) | (0.7, 0.1) | (0.3, 0.2) | (0.6, 0.2) | (0.4, 0.2) |

Step 2: (Normalize the decision matrix) Since \( \theta_1, \theta_2, \theta_3, \) and \( \theta_4 \) are all benefit attributes, one has \( R = (r_{ij})_{5 \times 4} = R \).

Step 3: (Determine the positive and negative ideal-points) The IF positive ideal-point is
\[ A^+ = (0.9, 0.1), (0.8, 0.1), (0.6, 0.1), (0.5, 0.2) \]^\top, \]
and IF negative ideal-point is
\[ A^- = (0.6, 0.3), (0.3, 0.3), (0.2, 0.5), (0.1, 0.6) \]^\top.

Steps 4 and 5: (Compute the relative closeness degrees) Calculate the relative closeness degrees \( \varepsilon_i \) of the alternatives \( A_i \) \((i = 1, 2, 3, 4, 5)\) to the IF positive ideal-point \( A^+ \) by Eqs. (19), (21), and (23):
\[ \varepsilon_1 = 0.3665, \varepsilon_2 = 0.6360, \varepsilon_3 = 0.4643, \varepsilon_4 = 0.5231, \varepsilon_5 = 0.5053. \]

Step 6: (Rank the alternative) Because \( \varepsilon_2 > \varepsilon_4 > \varepsilon_5 > \varepsilon_3 > \varepsilon_1 \), the preference order of the alternatives \( A_i \) \((i = 1, 2, 3, 4, 5)\) is: \( A_2 > A_4 > A_5 > A_3 > A_1 \).

Repeating Steps 1–3, by applying Eqs. (20) and (22), we obtain the following result:

Step 4 and 5: (Compute the relative closeness degrees) Calculate the relative closeness degrees \( \varepsilon_i \) of the alternatives \( A_i \) \((i = 1, 2, 3, 4, 5)\) to the IF positive ideal-point \( A^+ \) by Eqs. (20), (22), and (23):
\[ \varepsilon_1 = 1.050, \varepsilon_2 = 0.8535, \varepsilon_3 = 0.4925, \varepsilon_4 = 0.5528, \varepsilon_5 = 0.5138. \]

Step 6: (Rank the alternative) Because \( \varepsilon_2 > \varepsilon_4 > \varepsilon_5 > \varepsilon_3 > \varepsilon_1 \), the preference order of the alternatives \( A_i \) \((i = 1, 2, 3, 4, 5)\) is: \( A_2 > A_4 > A_5 > A_3 > A_1 \).

**Comparative analysis**

From Table VIII, which shows a comparison of the preference orders of the alternatives in Example 9 for different MADM methods, we observe that our two results are exactly the same, which are different from the result \( A_2 > A_5 > A_4 > A_3 > A_1 \) in [48], [54], [17]. Because the weights of the attributes \( \theta_1 \) and \( \theta_2 \) are larger than that of \( \theta_3 \) and \( \theta_4 \), i.e., \( \theta_1 \) and \( \theta_2 \) are more important than \( \theta_3 \) and \( \theta_4 \), we know that \( \theta_1 \) and \( \theta_2 \) are more affected than \( \theta_3 \) and \( \theta_4 \) for decision making. Thus, from \( r_{i1} \supset r_{i2} \supset r_{i3} \supset r_{i4} \) in Table VII, we know that \( A_4 \) may be better than \( A_5 \). To illustrate this, we use the Archimedean t-conorm and t-norm based intuitionistic fuzzy weighted averaging (ATFS-IFWA) operator introduced by Xia et al. ([55, Definition 6]) to aggregate the decision matrix \( R \). The scores and L-values for alternatives \( A_1, A_2, A_3, A_4, A_5 \) obtained by these t-norms are listed in Figures 2–5, respectively. From Figures 2–5, we know that (1) \( A_4 \) is always better than \( A_5 \); (2) the preference order of the alternatives \( A_i \) \((i = 1, 2, 3, 4, 5)\) is \( A_2 > A_4 > A_5 > A_3 > A_1 \), which is consistent with our results, when the parameter \( \gamma \) is changed to be near 0. Therefore, our method is more effective.
A COMPARISON OF THE PREFERENCE ORDERS OF THE ALTERNATIVES IN EXAMPLE 9 FOR DIFFERENT MADM METHODS

| Methods                          | Preference orders |
|----------------------------------|-------------------|
| Chen et al.’s TOPSIS method in [48] | $A_2 \succ A_3 \succ A_4$ |
| Wang and Wei’s TOPSIS method in [54]  | $A_2 \succ A_3 \succ A_4$ |
| Zeng et al.’s VIKOR method in [17]   | $A_2 \succ A_3 \succ A_4$ |
| Shen et al.’s TOPSIS method in [29]  | $A_2 \succ A_3 \succ A_4$ |
| Xu’s IFWA operator method in [3]     | $A_2 \succ A_3 \succ A_4$ |
| Our TOPSIS method by Eqs. (19) and (21) | $A_2 \succ A_3 \succ A_4$ |
| Our TOPSIS method by Eqs. (20) and (22) | $A_2 \succ A_3 \succ A_4$ |

1) Using Hamacher t-norms $T_{\gamma}^H$ ($\gamma \in (0, +\infty)$) with the additive generators $\tau(x) = \log \left[\frac{1+(1-\gamma)x}{\gamma} \right]$ in [56], the scores and $L$-values for alternatives $A_1, A_2, A_3, A_4, A_5$ obtained by $T_{\gamma}^H$ are obtained as shown in Fig. 2.

(2) Using Frank t-norms $T_{\gamma}^F$ ($\gamma \in (0, 1) \cup (1, +\infty)$) with the additive generators $\tau(x) = \log \left(\frac{\gamma-1}{\gamma-1-x} \right)$ in [56], the scores and $L$-values for alternatives $A_1, A_2, A_3, A_4, A_5$ obtained by $T_{\gamma}^F$ are obtained as shown in Fig. 3.

(3) Using Dombi t-norms $T_{\gamma}^D$ ($\gamma \in (0, +\infty)$) with the additive generators $\tau(x) = \left(\frac{1}{1+\gamma} \right)^\gamma$ in [56], the scores and $L$-values for alternatives $A_1, A_2, A_3, A_4, A_5$ obtained by $T_{\gamma}^D$ are obtained as shown in Fig. 4.

Example 10 ([17, Example 5.2]): Assume that there is a committee of a company, which decides to choose a project manager from five alternatives $A_1, A_2, A_3, A_4, A_5$ with four attributes $O_1, O_2, O_3, O_4$, where $O_1$ is “Self-Confidence”, $O_2$ is “Personality”, $O_3$ is “Past Experience”, $O_4$ is the “Profitability in Project Management” and $O_1, O_2, O_3, O_4$ are all benefit attributes, with weight vector $\omega = (0.1, 0.2, 0.3, 0.4)$.

Assume the decision matrix $R = (r_{ij})_{5 \times 4}$ given by the committee is as listed in Table IX.

| Table IX - The Decision Matrix $R$ |
|-----------------------------------|
| $O_4$ | $O_3$ | $O_2$ | $O_1$ |
| $A_1$ (0.4, 0.5) | (0.3, 0.6) | (0.4, 0.4) | (0.5, 0.3) |
| $A_2$ (0.4, 0.4) | (0.5, 0.4) | (0.4, 0.5) | (0.3, 0.4) |
| $A_3$ (0.4, 0.6) | (0.5, 0.5) | (0.4, 0.6) | (0.4, 0.6) |
| $A_4$ (0.3, 0.4) | (0.2, 0.6) | (0.1, 0.9) | (0.4, 0.4) |
| $A_5$ (0.5, 0.4) | (0.3, 0.6) | (0.3, 0.5) | (0.47, 0.5) |

Step 1: (Normalize the decision matrix) Since $O_1, O_2, O_3, O_4$ and $O_2$ are all benefit attributes, one has $\overline{R} = (\overline{r})_{5 \times 4} = R$.

Step 2: (Determine the positive and negative ideal-points) The IF positive ideal-point is

$A^+ = (0.5, 0.4, 0.5, 0.4, 0.4, 0.5, 0.3)^\top$.

and IF negative ideal-point is

$A^- = (0.3, 0.6, 0.2, 0.6, 0.1, 0.9, 0.3, 0.6)^\top$. 

Fig. 4. Scores and $L$-values for alternatives obtained by Hamacher t-norms $T_{\gamma}^H$

Fig. 5. Scores and $L$-values for alternatives obtained by Aczél-Alsina t-norms $T_{\gamma}^{AA}$
Step 3: (Compute the relative closeness degrees) Calculate the relative closeness degrees $c_i$ of the alternatives $A_i$ ($i = 1, 2, 3, 4, 5$) to the IF positive ideal-point $A^+$ by Eqs. (19), (21), and (23): $c_1 = 0.6311$, $c_2 = 0.5552$, $c_3 = 0.5058$, $c_4 = 0.3980$, $c_5 = 0.5307$.

Step 4: (Rank the alternative) Because $c_1 > c_2 > c_3 > c_4$, the preference order of the alternatives $A_i$ ($i = 1, 2, 3, 4, 5$) is: $A_1 \succ A_2 \succ A_3 \succ A_5 \succ A_4$.

Repeating Steps 1–2, by applying Eqs. (20) and (22), we obtain the following result:

Step 3: (Compute the relative closeness degrees) Calculate the relative closeness degrees $c_i$ of the alternatives $A_i$ ($i = 1, 2, 3, 4, 5$) to the IF positive ideal-point $A^+$ by Eqs. (20), (22), and (23): $c_1 = 0.6805$, $c_2 = 0.5808$, $c_3 = 0.5042$, $c_4 = 0.3566$, $c_5 = 0.5457$.

Step 4: (Rank the alternative) (Rank the alternative) Because $c_1 > c_2 > c_3 > c_4$, the preference order of the alternatives $A_i$ ($i = 1, 2, 3, 4, 5$) is: $A_1 \succ A_2 \succ A_3 \succ A_5 \succ A_4$.

From Table X, which shows a comparison of the preference orders of the alternatives in Example 10 for different MADM methods, we observe that our two results are consistent with the preference orders obtained by the MADM methods in [48], [54], [17].

| Methods | Preference orders |
|---------|-------------------|
| Chen et al.’s TOPSIS method in [48] | $A_1 \succ A_2 \succ A_3 \succ A_5 \succ A_4$ |
| Wang and Wei’s TOPSIS method in [54] | $A_1 \succ A_2 \succ A_3 \succ A_4 \succ A_5$ |
| Zeng et al.’s VIKOR method in [17] | $A_1 \succ A_2 \succ A_3 \succ A_4 \succ A_5$ |
| Our TOPSIS method by Eqs. (19) and (21) | $A_1 \succ A_2 \succ A_3 \succ A_4 \succ A_5$ |
| Our TOPSIS method by Eqs. (20) and (22) | $A_1 \succ A_2 \succ A_3 \succ A_4 \succ A_5$ |

VIII. Conclusions

This paper is devoted to establishing a monotonous IF TOPSIS method with two typical linear orders, ‘$\leq_{xy}$’ in [6] and ‘$\leq_{xZ}$’ in [7]. Noting that the TOPSIS method is closely related to the order structure and the metric/similarity measure, we first discuss some examples to show that some classical similarity measures in [33], [30], [25], [26], including Euclidean similarity measure, Minkowski similarity measure, and modified Euclidean similarity measure, do not satisfy the axiomatic definition of IF similarity measures. Then, we prove the nonexistence of a continuous function that can distinguish IFV by a real number and is increasing with Atanassov’s order ‘c’. As a direct corollary, we prove that there is no any continuous similarity measure that can distinguish between each pair of IFVs. Moreover, we show some illustrative examples to demonstrate that some classical IF TOPSIS methods in [44], [41], [48], [33], [43], [46] are not monotonous with Atanassov’s partial order ‘c’ or the linear order ‘$\leq_{xZ}$’, which may yield counter-intuitive results. To overcome this limitation, by using two new admissible distances with the linear order ‘$\leq_{xy}$’ or ‘$\leq_{xZ}$’, we develop a novel IF TOPSIS method and prove that it is monotonically increasing with these two linear orders. Finally, we show two practical examples with comparative analysis to other MADM methods to illustrate the efficiency of our TOPSIS method. In the future, we will further study the general construction of linear orders and admissible distance/similarity measures for IFVs, which is very important for building more effective IF TOPSIS methods.

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