Expanding graphs, Ramanujan graphs, and 1-factor perturbations

Pierre de la Harpe and Antoine Musitelli
adress email: Pierre.delaHarpe@math.unige.ch and musitel0@math.unige.ch

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Abstract

We construct \((k \pm 1)\)-regular graphs which provide sequences of expanders by adding or subtracting appropriate 1-factors from given sequences of \(k\)-regular graphs. We compute numerical examples in a few cases for which the given sequences are from the work of Lubotzky, Phillips, and Sarnak (with \((\text{Fried–91}, \text{and later work})\) based on random techniques: for all \(k\) of the spectral gap.

\(1\) Introduction

Let \(X = (V, E)\) be a simple finite graph with \(n\) vertices, where \(V\) denotes the vertex set and \(E\) the set of geometrical edges of \(X\). The adjacency matrix \(A\) of \(X\), with rows and columns indexed by \(V\), is defined by \(A_{v, w} = 1\) if there exists an edge connecting \(v\) and \(w\), and \(A_{v, w} = 0\) otherwise (in particular \(A_{v, v} = 0\)). The eigenvalues of \(X\), which are those of \(A\), constitute a decreasing sequence \(\lambda_0(X) \geq \lambda_1(X) \geq \ldots \geq \lambda_{n-1}(X)\); the spectral gap \(\lambda_0(X) - \lambda_1(X)\) of \(X\) is positive if and only if \(X\) is connected. Let us assume from now on that \(X\) is \(k\)-regular for some \(k \geq 3\), namely that \(\sum_w A_{v, w} = k\) for all \(v \in V\), so that \(\lambda_0(X) = k\).

Recall that, for any infinite sequence \((X_i)_{i \in I}\) of connected \(k\)-regular simple finite graphs with increasing vertex sizes, we have \(\lim \inf_{i \to \infty} \lambda_1(X_i) \geq 2\sqrt{k-1}\). A graph \(X\) is said to be a Ramanujan graph if it is connected and if \(|\mu| \leq 2\sqrt{k-1}\) for any eigenvalue \(\mu \neq \pm k\) of \(X\). From elaborate arithmetic constructions, we know explicit infinite sequences of Ramanujan graphs for degree \(k\) when \(k - 1\) is the order of a finite field; but the existence of such sequences is an open problem for other degrees, for example when \(k = 7\). It is thus interesting to find sequences of expanders of degree \(k\), namely infinite sequences \((X_i)_{i \in I}\) of \(k\)-regular connected simple finite graphs with increasing vertex sizes such that \(\inf_{i \in I}(k - \lambda_1(X_i))\) is strictly positive, and indeed as large as possible (short of being equal to \(2\sqrt{k-1}\)).

For all this, see for example [Lubot–94], [Valet–97], [Colin–98], and [DuSaV–03].

The object of the present Note is to examine a procedure of construction of sequences of expanders \((X_i)_{i \in I}\) of degree \(k\) by perturbation of sequences of Ramanujan graphs. When \(k - l\) is the order of a finite field, we obtain estimates \(\lambda_l(X_i) \leq l - 1 + 2\sqrt{(k - l)}\); for example, for \(k = 7\) and \(l = 1\), this corresponds to a spectral gap

\[
7 - \lambda_1(X_i) \geq 6 - 2\sqrt{6} \approx 1.52 \quad \text{for all } i \in I,
\]

to be compared with the Ramanujan minoration by

\[
7 - \lim \inf_{i \in I} \lambda_1(X_i) \leq 7 - 2\sqrt{6} \approx 2.10
\]
of the spectral gap.

We insist on finding explicit constructions, but we record however the following results of J. Friedman (see [Fried–91], and later work) based on random techniques: for all \(k \geq 3\) and all \(\epsilon > 0\), there exists sequences \((X_i)_{i \in I}\) of connected \(k\)-regular simple finite graphs with increasing vertex sizes and with \(\lambda_1(X_i) \leq 2\sqrt{k-1} + \epsilon\) for all \(i \in I\).

Let \(X = (V, E)\) be a graph. If \(X\) is not bipartite, we denote by \(\overline{X} = (V, \overline{E})\) the complement of \(X\); two distinct vertices are adjacent in \(\overline{X}\) if and only if they are not so in \(X\). If \(X\) is bipartite, given with a bipartition \(V = V_0 \sqcup V_1\), we denote by \(\overline{X} = (V, \overline{E})\) the bipartite complement of \(X\); two vertices \(v \in V_0\), \(w \in V_1\) are adjacent in \(\overline{X}\) if and only if they are not in \(X\). A matching of a graph \(X\) is a subset \(M\) of \(E\) such that any vertex \(x \in V\) is incident with at most one edge of \(M\), and a perfect matching (also called 1-factor) is a subset \(F\) of \(E\) such that any vertex \(x \in V\) is incident with exactly one edge of \(F\).
Let \( X = (V,E) \) be a graph. If \( F \) is a perfect matching of \( X \), we denote by \( X - F \) the graph \((V,E \setminus F)\); if \( X \) is \( k \)-regular, then \( X - F \) is \((k-1)\)-regular. If \( F \) is a perfect matching of \( \overline{X} \), we denote by \( X + F \) the graph \( \overline{X} - F \); if \( X \) is \( k \)-regular, then \( X + F \) is \((k+1)\)-regular.

The basic observation for the present Note is the set of inequalities

\[ |\lambda_j(X + F) - \lambda_j(X)| \leq 1 \]

for any perfect matching \( F \) of \( X \) (for \( X - F \)) or of \( \overline{X} \) (for \( X + F \)), and for all \( j \in \{0,\ldots,n-1\} \), where \( n = |V| \) (Proposition 2). We can apply this to the Ramanujan graphs \( X^{p,q} \) and their complements (notation of [DaSaV-03], see below). In Section 3, we describe an algorithm for finding perfect matchings in regular bipartite graphs (thus concentrating on pairs \((p,q)\) for which the graph \( X^{p,q} \) is bipartite). In conclusion, we report some numerical computations.

## 2 Graphs of the form \( X^{p,q} \pm F \)

Let us recall the definition of the graphs \( X^{p,q} \).

If \( R \) is a commutative ring with unit, the Hamilton quaternion algebra \( \mathbb{H}(R) \) over \( R \) is the free module \( R^4 \) with basis \( \{1,i,j,k\} \), where multiplication is defined by \( i^2 = j^2 = k^2 = -1 \), and \( ij = -ji = k \), plus circular permutations of \( i,j,k \). A quaternion \( q = a_0 + a_1 i + a_2 j + a_3 k \) has a conjugate \( \overline{q} = a_0 - a_1 i - a_2 j - a_3 k \) and a norm \( N(q) = q \overline{q} = a_0^2 + a_1^2 + a_2^2 + a_3^2 \).

Let \( p \in \mathbb{N} \) be an odd prime. If \( p \equiv 1 \pmod{4} \), a theorem of Jacobi shows that there are exactly \( p + 1 \) quaternions in \( \mathbb{H}(\mathbb{Z}) \) of norm \( p \) of the form \( a_0 + a_1 i + a_2 j + a_3 k \) with \( a_0 \equiv 0 \pmod{2} \), and \( a_0 \geq 0 \). From those with \( a_0 \geq 2 \), say \( 2s \) of them, we obtain \( \alpha_1,\ldots,\alpha_s \) as above. Those of the form \( a_1 i + a_2 j + a_3 k \), say \( 2t \) of them, \(^1\) occur in pairs \((\alpha,-\alpha)\); we select arbitrarily one, say \( \alpha \), from each pair, and we set

\[ S_p = \{\alpha, \overline{\alpha}, \ldots, \alpha, \overline{\alpha}\} \quad \text{with} \quad 2s = p + 1. \]

If \( p \equiv 3 \pmod{4} \), there are quaternions in \( \mathbb{H}(\mathbb{Z}) \) of norm \( p \) of the form \( a_0 + a_1 i + a_2 j + a_3 k \) with \( a_0 \equiv 0 \pmod{2} \), and \( a_0 \geq 0 \). From those with \( a_0 \geq 2 \), say \( 2s \) of them, we obtain \( \alpha_1,\ldots,\alpha_s \) as above. Those of the form \( a_1 i + a_2 j + a_3 k \), say \( 2t \) of them, occur in pairs \((\alpha,-\alpha)\); we select arbitrarily one, say \( \alpha \), from each pair, and we set

\[ S_p = \{\alpha, \overline{\alpha}, \ldots, \alpha, \overline{\alpha}\} \quad \text{with} \quad 2s = p + 1. \]

Observe that \( t/4 \) is the number of solutions in \( \mathbb{N} \) of the equation \( a_1^2 + a_2^2 + a_3^2 = p \), and that we have again \( |S_p| = 2s = p + 1 \) by Jacobi’s theorem. Observe also that we can have \( s = 0 \) (case of \( p = 3 \)), as well as \( t = 0 \) (case of \( p = 7 \pmod{8} \)), or both \( s \) and \( t \) positive (case of \( p = 19 \), with \( s = 4 \) and \( t = 12 \)).

Let \( q \) be another odd prime, \( q \neq p \), and let \( \tau_q : \mathbb{H}(\mathbb{Z}) \to \mathbb{H}(\mathbb{F}_q) \) denote reduction modulo \( q \). The equation \( x^2 + y^2 + 1 = 0 \) has solutions in \( \mathbb{F}_q \). We choose one solution; then the mapping \( \psi_q : \mathbb{H}(\mathbb{F}_q) \to M_2(\mathbb{F}_q) \) defined by

\[ \psi_q(a_0 + a_1 i + a_2 j + a_3 k) = \begin{pmatrix} a_0 + a_1 x + a_3 y & -a_2 y + a_1 + a_3 x \\ -a_1 y - a_2 + a_3 x & a_0 - a_1 x - a_2 y \end{pmatrix} \]

is an algebra isomorphism and \( \psi_q(\tau_q(S_p)) \) is in the group \( GL_2(\mathbb{F}_q) \) of invertible elements of \( M_2(\mathbb{F}_q) \). We denote by \( \phi : GL_2(q) \to PGL_2(q) \) the reduction modulo the centre, and we set

\[ S_{p,q} = \{ \phi(\psi_q(\tau_q(S_p))) \} \subset PGL_2(q). \]

It follows from the definitions that \( S_{p,q} \) is symmetric: if \( s \in S_{p,q} \) is the image of \( \alpha \in S_p \) (notation as above), then \( s^{-1} \) is the image of \( \overline{\alpha} \); if \( s \) is the image of \( \beta \in S_p \), then \( s^{-1} = 1 \). Moreover, it is known that \( |S_{p,q}| = p + 1 \). There are now two cases to consider.

Either \( p \) is a square modulo \( q \). Then \( S_{p,q} \subset PSL_2(q) \) and indeed \( S_{p,q} \) generates \( PSL_2(q) \). By definition, \( X^{p,q} \) is the Cayley graph of \( PSL_2(q) \) with respect to \( S_{p,q} \); more precisely, \( X^{p,q} = (V,E) \) with \( V = PSL_2(q) \) and \( \{v,w\} \in E \) if \( v^{-1} w \in S_{p,q} \). It is a \((p+1)\)-regular graph with \( q(q^2 - 1) \) vertices which is connected, non-bipartite, and which is a Ramanujan graph.

Or \( p \) is not a square modulo \( q \). Then \( S_{p,q} \cap PSL_2(q) = \emptyset \) and \( S_{p,q} \) generates \( PGL_2(q) \). By definition, \( X^{p,q} \) is the Cayley graph of \( PGL_2(q) \) with respect to \( S_{p,q} \). It is a \((p+1)\)-regular bipartite graph with \( q(q^2 - 1) \) vertices which is connected and which is a Ramanujan graph.

See [DaSaV-03] for proofs of a large part of the facts stated above, including the connectedness of the graphs \( X^{p,q} \) when \( p > 5 \) and \( q > p^3 \), and the expanding property of this family. For the proof that \( (X^{p,q})_q \) is actually...
a family\(^2\) of Ramanujan graphs, see the original papers ([LuPhS–88], with a large part obtained independently in [Margu–88]), as well as [Sarna–90].

Table I shows the spectrum of \(X^{3,q}\) for \(q \in \{5, 7, 11\}\) and Table II that of \(X^{5,q}\) for \(q \in \{7, 11\}\). Numerical computations of eigenvalues reported in this paper have been computed with Mathlab.

**Proposition 1**  If the graph \(X^{p,q}\) is bipartite, \(X^{p,q}\) and its bipartite complement \(\overline{X^{p,q}}\) have perfect matchings.

**Proof**  This is a case of the “Marriage Theorem”; see for example Corollary 1.1.4 in [LovPl–86]. Here is another reason for \(X^{p,q}\) (bipartite or not): any connected vertex-transitive graph of even order has a perfect matching (Section 3.5 in [GodRo–01]); this applies in particular to Cayley graphs of finite groups of even order, such as \(PGL_2(q)\) and \(PSL_2(q)\).

**Proposition 2**  Let \(X = (V,E)\) be a finite graph with \(n\) vertices and with eigenvalues \(\lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_{n-1}\). Let \(F\) be a matching of \(X\) [respectively of the complement \(\overline{X}\)] and let \(\mu_0 \geq \mu_1 \geq \ldots \geq \mu_{n-1}\) be the eigenvalues of \(X - F\) [respectively \(X + F\)]. Then \(\left|\mu_j - \lambda_j\right| \leq 1\) for \(j \in \{0,1,\ldots,n-1\}\).

**Proof**  Outside diagonal entries, the adjacency matrix \(A_F\) of \((V,F)\) coincides with a matrix of permutation (the permutation being a non-empty product of transpositions with disjoint supports, one transposition for each edge in \(F\)). Thus \(\|A_F\| \leq 1\). Here, the norm of a matrix acting on the Euclidean space \(\mathbb{R}^V\) is the operator norm \(\|A_F\| = \sup \{\|Af\|_2 \mid f \in \mathbb{R}^V, \|f\|_2 \leq 1\}\), where \(\|f\|_2^2 = \sum_{v \in V} |f(v)|^2\).

Thus Proposition 1 follows from the classical Courant-Fischer-Weyl minimax principle, according to which eigenvalues of symmetric operators are norms of appropriate restrictions of these operators. See e.g. Chapter III in [Bhati–97].

3 Tables

There are several standard efficient algorithms to find a perfect matching \(F\) in a graph \(X\); see [LovPl–86] and [West–01], among others. We will not describe here the details of the algorithm we have used. Eigenvalues of \(X - F\) can then be computed with Mathlab.

The eigenvalues of a graph of the form \(X^{p,q} - F\) depend on the choice of \(F\). Table III gives for each of three pairs \((p,q)\) the values of the spectral gaps \(p - \lambda_1(X^{p,q} - F)\) corresponding to four different \(F\). Table III shows that there are situations \((p = 5, q = 7)\) with \(\lambda_0(X - F) = k - 1 < \lambda_0(X) = k\) and \(\lambda_1(X - F) > \lambda_1(X)\).

Table IV shows the full spectrum of \(X^{3,5} - F\) for one specific \(F\). Tables V to VII show the ten largest eigenvalues of three graphs of the form \(X^{p,q} + F\). Observe that the multiplicities in Tables IV to VII are much less than those of the unperturbed graphs.

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\(^2\)The family is indexed by the set of all odd primes \(q\), and \(p\) is a fixed arbitrary odd prime.
| Table I: spectra of $X^{3,q}$ | q=5 | q=7 | q=11 |
|-----------------------------|-----|-----|-----|
|                            | eigenvalues | multiplicities | eigenvalues | multiplicities | eigenvalues | multiplicities |
| -4.0000                    | 1             | -4.0000     | 1             | -3.2361     | 30             |
| -3.0000                    | 12           | -3.0000     | 24           | -3.0000     | 33             |
| -2.0000                    | 28           | -2.8284     | 30           | -2.7321     | 10             |
| -1.0000                    | 4            | -2.0000     | 28           | -2.6180     | 24             |
| 0.0000                     | 30           | -1.4142     | 24           | -2.3723     | 10             |
| 1.0000                     | 4            | -1.0000     | 40           | -2.0468     | 36             |
| 2.0000                     | 28           | 0.0000      | 42           | -2.0000     | 10             |
| 3.0000                     | 12           | 1.0000      | 40           | -1.6180     | 36             |
| 4.0000                     | 1            | 1.4142      | 24           | -1.5616     | 33             |
|                            |              | 2.0000      | 28           | -0.9191     | 36             |
|                            |              | 2.8284      | 30           | -0.7321     | 30             |
|                            |              | 3.0000      | 24           | -0.3820     | 24             |
|                            |              | 4.0000      | 1            | 0.0000      | 30             |
|                            |              |              |              | 0.3820      | 12             |
|                            |              |              |              | 0.6180      | 36             |
|                            |              |              |              | 0.7321      | 10             |
|                            |              |              |              | 1.0000      | 52             |
|                            |              |              |              | 1.2361      | 30             |
|                            |              |              |              | 1.9191      | 36             |
|                            |              |              |              | 2.0000      | 20             |
|                            |              |              |              | 2.5616      | 33             |
|                            |              |              |              | 2.6180      | 12             |
|                            |              |              |              | 2.7321      | 30             |
|                            |              |              |              | 3.0468      | 36             |
|                            |              |              |              | 3.3723      | 10             |
|                            |              |              |              | 4.0000      | 1              |

| Table II: spectra of $X^{5,q}$ | q=7 | q=11 |
|-----------------------------|-----|-----|
|                            | eigenvalues | multiplicities | eigenvalues | multiplicities |
| -6.0000                    | 1            | -4.0243     | 36             |
| -4.0000                    | 21           | -3.7321     | 30             |
| -3.0000                    | 16           | -3.0000     | 65             |
| -2.8284                    | 42           | -2.2361     | 30             |
| -2.0000                    | 21           | -1.7321     | 10             |
| -1.4142                    | 12           | -1.6180     | 60             |
| -1.0000                    | 48           | -1.3723     | 10             |
| 0.0000                     | 14           | -1.2361     | 12             |
| 1.0000                     | 48           | -0.5616     | 33             |
| 1.4142                     | 12           | -0.2679     | 30             |
| 2.0000                     | 21           | -0.1638     | 36             |
| 2.8284                     | 42           | 0.6180      | 60             |
| 3.0000                     | 16           | 1.0000      | 30             |
| 4.0000                     | 21           | 1.7321      | 10             |
| 6.0000                     | 1            | 1.7818      | 36             |
|                            |              | 2.2361      | 30             |
|                            |              | 3.0000      | 50             |
|                            |              | 3.2361      | 12             |
|                            |              | 3.4063      | 36             |
|                            |              | 3.5616      | 33             |
|                            |              | 4.3723      | 10             |
|                            |              | 6.0000      | 1              |

| Table III: spectral gaps for $X^{p,q} - F$ | p=3,q=5 | p=3,q=7 | p=5,q=7 |
|---------------------------------------------|---------|---------|---------|
| 0.4457                                      | 0.2499  | 0.7910  |
| 0.3025                                      | 0.1862  | 0.7732  |
| 0.2993                                      | 0.1785  | 0.7367  |
| 0.2702                                      | 0.0272  | 0.7152  |
| eigenvalues | multiplicities |
|-------------|----------------|
| -3.0000     | 1              |
| -2.5543     | 8              |
| -2.5450     | 4              |
| -2.1542     | 4              |
| -2.0000     | 6              |
| -1.8829     | 8              |
| -1.2929     | 8              |
| -1.0000     | 3              |
| -0.8302     | 4              |
| -0.5086     | 4              |
| -0.4394     | 4              |
| 0.0000      | 4              |
| 0.4394      | 4              |
| 0.5086      | 8              |
| 0.8302      | 4              |
| 1.0000      | 3              |
| 1.2929      | 8              |
| 1.8829      | 8              |
| 2.0000      | 6              |
| 2.1542      | 4              |
| 2.5450      | 4              |
| 2.5543      | 8              |
| 3.0000      | 1              |

Table V: largest eigenvalues for $X^{3.5} + F$

| eigenvalues | multiplicities |
|-------------|----------------|
| 3.2578      | 1              |
| 3.2163      | 1              |
| 3.1707      | 1              |
| 3.1707      | 1              |
| 3.1998      | 1              |
| 3.3208      | 1              |
| 3.3431      | 1              |
| 3.2214      | 1              |
| 3.4417      | 1              |
| 3.2418      | 1              |
| 3.4902      | 1              |
| 3.3046      | 1              |
| 3.5358      | 1              |
| 3.5525      | 1              |
| 3.6211      | 1              |
| 3.5653      | 1              |
| 3.6822      | 1              |
| 3.5935      | 1              |
| 3.8466      | 1              |
| 3.6547      | 1              |
| 5.0000      | 1              |
| 5.0000      | 1              |
Table VI: largest eigenvalues for $X^+$ + $F$

| eigenvalues | multiplicities | eigenvalues | multiplicities | eigenvalues | multiplicities |
|-------------|----------------|-------------|----------------|-------------|----------------|
| 3.6042      | 1              | 3.6199      | 1              | 3.6138      | 1              |
| 3.6130      | 1              | 3.6478      | 1              | 3.6431      | 1              |
| 3.6349      | 1              | 3.6594      | 1              | 3.6524      | 1              |
| 3.6728      | 1              | 3.6826      | 1              | 3.6726      | 1              |
| 3.6892      | 1              | 3.6996      | 1              | 3.6922      | 1              |
| 3.6971      | 1              | 3.7203      | 1              | 3.7131      | 1              |
| 3.7073      | 1              | 3.7468      | 1              | 3.7275      | 1              |
| 3.7505      | 1              | 3.7548      | 1              | 3.7461      | 1              |
| 3.7697      | 1              | 3.7752      | 1              | 3.7985      | 1              |
| 5.0000      | 1              | 5.0000      | 1              | 5.0000      | 1              |

Table VII: largest eigenvalues for $X^+$ + $F$

| eigenvalues | multiplicities | eigenvalues | multiplicities | eigenvalues | multiplicities |
|-------------|----------------|-------------|----------------|-------------|----------------|
| 4.3702      | 1              | 4.3388      | 1              | 4.3229      | 1              |
| 4.4015      | 1              | 4.4378      | 1              | 4.4340      | 1              |
| 4.4271      | 1              | 4.4326      | 1              | 4.3882      | 1              |
| 4.4625      | 1              | 4.4790      | 1              | 4.4117      | 1              |
| 4.4888      | 1              | 4.5124      | 1              | 4.4671      | 1              |
| 4.4971      | 1              | 4.5618      | 1              | 4.5585      | 1              |
| 4.5519      | 1              | 4.5925      | 1              | 4.5875      | 1              |
| 4.5976      | 1              | 4.6417      | 1              | 4.6341      | 1              |
| 4.6512      | 1              | 4.6892      | 1              | 4.7260      | 1              |
| 7.0000      | 1              | 7.0000      | 1              | 7.0000      | 1              |

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