On the existence of an energy gap in one-dimensional Lesanovsky’s model

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We study the quantum lattice gas model in one dimension introduced by Lesanovsky [1], who showed that the exact ground state and a couple of excited states can be obtained analytically. The Hamiltonian of the model depends solely on the parameter $z$, the meaning of which is a fugacity in the corresponding classical lattice gas model. For small $z$ (0 < $z$ < 1), we prove that there is an energy gap between the ground state and the excited states by applying Knabe’s method [2].

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I. INTRODUCTION

Recently there has been a growing interest in exploring the physics of strongly interacting systems using Rydberg atoms, atoms in states of high principal quantum number $n$. It was argued that a variety of exotic systems can be simulated with Rydberg atoms in optical lattices [1, 3–9]. These atoms are interacting via the van der Waals-type interaction, which is strongly enhanced when $n$ is very large. This interaction naturally leads to the Rydberg blockade, i.e., a simultaneous excitation of two nearby atoms to Rydberg states is forbidden. The first experiments on Rydberg atoms in quasi-one-dimensional optical lattices were carried out [10].

In Ref. [1] Lesanovsky has introduced a solvable model of Rydberg lattice gas in one dimension. The model can be thought of as a quantum Ising chain with long-range interaction in a transverse and longitudinal field. The model depends solely on the parameter $z$, the meaning of which is a fugacity in the corresponding classical lattice gas with hard-core constraint. Lesanovsky showed that the exact zero-energy ground state is a weighted superposition of states, each of which is labeled by a configuration of Rydberg states with the Rydberg blockade. He also obtained explicit expressions for a couple of excited states. The results of exact diagonalization suggest that it is likely that one of excited states he obtained corresponds to the first excited state. If this is the case, the energy gap is nonvanishing since the analytical expression for the energy of the first excited state is nonzero for finite $z$. To date, however, there is no rigorous proof of the existence of an energy gap.

In this brief report, we show that the existence of an energy gap for small $z$ can be proved without knowing the explicit expression for the energy of the first excited state. The idea is to use the method proposed by Knabe in [2], which was applied to show the existence of the gap in one-dimensional (1D) Affleck-Kennedy-Lieb-Tasaki models with various spins [11]. The method enables one to get lower bounds for energy gaps of infinite chains by diagonalizing finite-size chains with open boundaries. Other systems where this method is applicable include a sawtooth chain and spin ladders [12, 13].

II. LESANOVSKY’S MODEL

We consider a system of hard-core bosons on a lattice. The Hilbert space at each lattice site is spanned by $|n_i\rangle$, where $n_i = 1$ (0) indicates that the site $i$ is occupied (empty). With the identification $|\uparrow\rangle \leftrightarrow |1\rangle$ and $|\downarrow\rangle \leftrightarrow |0\rangle$, the operator that creates/annihilates the hard-core boson at site $i$ can be expressed as $\sigma_i^+ = (\sigma_i^x + i\sigma_i^y)/2$, where $\sigma_i^\alpha$ ($\alpha = x, y, z$) are the standard Pauli matrices. The model can be defined on any lattice in any dimension [14]. However, for simplicity, we restrict our attention to the case of 1D lattice with periodic boundary conditions [1]. An extension of a proof of the gap and exact results for the excited states in higher dimensional cases will be presented in a separate publication [13]. The Hamiltonian for the 1D chain of length $N$ is given by

$$H = \sum_{i=1}^{N} P_{i-1}[\sigma_i^+ + zP_i + z^{-1}n_i]P_{i+1},$$

where the parameter $z$ is real and positive. Here $P_i = (1 - \sigma_i^z)/2$ and $n_i = (1 + \sigma_i^z)/2$ are projectors on the occupied and empty states on the $i$-th site, respectively. The periodic boundary conditions imply $P_{N+1} = P_1$, $n_{N+1} = n_1$, and so on. In the following, we are interested in the restricted Hilbert space in which $n_i, n_{i+1} = 0$ for all $i$, i.e., an occupied state $|1\rangle$ is always accompanied by an empty state $|0\rangle$ on either side. This exclusion rule follows naturally from the Rydberg blockade. In the subspace, as shown in [14], the ground state of $H$ is unique [10].

III. KNABE’S METHOD

In this section, we provide a summary of Knabe’s method [2]. Let us consider a Hamiltonian of the form

$$\mathcal{H} = \sum_{i=1}^{N} Q_i,$$

with periodic boundary conditions ($Q_{N+1} = Q_1$). Here, $Q_i$ ($i = 1, 2, ..., N$) are projection operators, i.e., $Q_i^2 = Q_i$, and their commutators satisfy $[Q_i, Q_j] = 0$ if $|i-j| > 0$.
Note that these $Q_i$'s take the same form as $Q_1$, but act on different sites. The Hamiltonian is positive semi-definite by construction. If it is known that the ground-state energy is zero, then the inequality

$$\mathcal{H}^2 \geq \epsilon \mathcal{H}, \quad \epsilon > 0$$

implies that the energy gap (the lowest non-vanishing eigenvalue of $\mathcal{H}$) is larger than $\epsilon$.

Knabe has shown that one can derive such an inequality if the same model on a finite chain with open boundaries satisfies

$$h_{n,i}^2 \geq \epsilon_n h_{n,i}, \quad \epsilon_n > \frac{1}{n},$$

where $n \geq 2$ and $h_{n,i} := \sum_{j=i}^{i+n-1} Q_j$. Thus, the existence of the energy gap is established if one can show that the inequality is satisfied for some integer $n$. Note that for such $n$, a lower bound for the energy gap is given by

$$\epsilon = \frac{n}{n-1} \left( \epsilon_n - \frac{1}{n} \right).$$

IV. PROOF OF THE GAP

Let us apply Knabe’s method to the Hamiltonian $\mathcal{H}$. For convenience we introduce a scaled Hamiltonian

$$\mathcal{H} = H/(z + z^{-1}) = \sum_{i=1}^{N} Q_i,$$

$$Q_i = \frac{1}{z + z^{-1}} P_{i-1} \sigma^x P_i + z^{-1} n_i | P_{i+1}. (7)$$

The condition that the ground-state energy is zero is satisfied since the scaling of the Hamiltonian does not change the zero energy. The local Hamiltonians $Q_i$ ($i = 1, 2, ..., N$) are the projection operators that satisfy $Q_i^2 = Q_i$, which can be verified using $\sigma^x \sigma^x = \sigma^x$ and $P_i \sigma^x n_i = \sigma^x$. It is also easy to see that $[Q_i, Q_j] = 0$ if $|i - j| > 1$.

We first examine $h_{2,i} = Q_i + Q_{i+1}$. In the restricted Hilbert space, $h_{2,i}$ is expressed as the following matrix:

$$h_{2,i} = \frac{1}{z + z^{-1}} \begin{pmatrix} 2z & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & z & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & z^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & z^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & z^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & z & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & z \\ \end{pmatrix} \ (8)$$

The order of the basis states is $|0000\rangle$, $|0001\rangle$, $|0010\rangle$, $|0100\rangle$, $|1000\rangle$, $|0101\rangle$, $|1010\rangle$, $|1001\rangle$. One can diagonalize $h_{2,i}$ analytically and find that the energy gap (the lowest non-vanishing eigenvalue of $h_{2,i}$) is given by $\epsilon_2 = 1/(1+z^2)$ which is greater than $1/2$ when $0 < z < 1$. Therefore, from (4) and (5), the Hamiltonian $H$ has an energy gap if $0 < z < 1$ and a lower bound for the gap is $(1-z^2)/z$. It should be noted that, to our knowledge, this is the first example in which Knabe’s method is successfully applied to a proof of the gap without the aid of numerical diagonalization.

The condition $0 < z < 1$ is sufficient for the existence of the energy gap, but of course not optimal. An improved condition can be obtained by considering $h_{n,i}$ with $n > 2$. For example, if we take $n = 4$, we find that the energy gap of $h_{4,i}$ is $\epsilon_4 = x/(1+z^2)$ where $x$ is the smallest root of the cubic equation: $x^3 - (4+3z^2)x^2 + (5+7z^2+2z^4)x - (2+4z^2+z^4) = 0$. The condition $\epsilon_4 > 1/4$ yields $0 < z < 1.3263$. A sufficient condition for the existence of the gap can be extended by diagonalizing the Hamiltonians for longer chains with open boundaries.

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on an arbitrary lattice was shown in [14]. Note that the

signs of the off-diagonal elements of $H$ can be changed

by means of the unitary transformation $H \rightarrow \Sigma^z H \Sigma^z$, 

where $\Sigma^z := \prod_{i=1}^{N} \sigma_i^z$.

[17] A generalization of Knabe’s method including the case where $Q_i^2 \leq Q_i$ was discussed in D. Schuricht and S. Rachel, Phys. Rev. B 78, 014430 (2008).

[18] Here we write $A \geq B$ to denote that $A - B$ is positive semi-definite.