Unified Theoretical Framework for Unit Root Test and Fractional Unit Root Test

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Abstract

This paper proposes a new testing procedure for the degree of fractional integration of a time series inspired on the unit root test of Dickey-Fuller (1979). The composite null hypothesis is that of \( d \geq d_0 \) against \( d < d_0 \). The test statistics is the same as in Dickey-Fuller test using as output \((1 - L)^{d_0} y_t\) instead of \((1 - L)y_t\) and as input \((1 - L)^{-1 + d_0} y_{t-1}\) instead of \(y_{t-1}\), exploiting the fact that if \( y_t \) is \( I(d) \) then \((1 - L)^{-1 + d_0} y_t\) is \( I(1) \) under the null \( d = d_0 \). If \( d \geq d_0 \), using the generalization of Sowell’s results (1990), we propose a test based on the least favorable case \( d = d_0 \), to control type I error and when \( d < d_0 \) we show that the usual tests statistics diverges to \(-\infty\), providing consistency.

Keywords: Fractional integration, Fractional unit root, Dickey-Fuller test.

1 Introduction and motivation

Since the seminal work by Dickey and Fuller [DF] (1979) formal tests for unit roots (or integration of order one, \( I(1) \)) have been become standard in applied time series analysis and econometrics. Recently, there has been considerable interest in generalizing the Dickey-Fuller type test to take account the fractional integration order. For instance, Dolado, Gonzalo, and Mayoral [DGM] (2002) introduced a test based on an auxiliary regression for the null of unit root against the alternative of fractional integration. Further, the DGM test was refined by Lobato and Velasco [LV] (2006, 2007).

In the basic framework, \( y_t \) denotes a fractionally integrated process whose true order of integration is \( d \), denoted as \( I(d) \),

\[
\Delta^d y_t = \varepsilon_t, \ t = 1, 2, \ldots, n, \tag{1.1}
\]
where $\varepsilon_t$ are independent and identically distributed (i.i.d) random variables with zero mean and finite variance. The fractional difference operator $\Delta^d = (1 - L)^d$ is defined in terms of lag operator $L$ by the formal expansion

$$\Delta^d = \sum_{i=0}^{\infty} \frac{\Gamma(i - d)}{\Gamma(i + 1) \Gamma(-d)},$$

for any $d \neq 1, 2, \cdots$, and $\Gamma(\cdot)$ is the Gamma function.

For the PGD (1), DGM (2002) introduced a test based on an auxiliary regression for the null of the unit root against the alternative of fractional integration, namely

$$H_0 : d = 1 \text{ against } H_1 : d = d_1, \quad (1.2)$$

where $d_1 < 1$. DGM (2002) proposed to test the null hypothesis by means of the $t$ statistic of the coefficient of $\Delta^{d_1} y_{t-1}$ in the ordinary least squares (OLS) regression

$$\Delta y_t = \phi_1 \Delta^{d_1} y_{t-1} + u_t, \quad (t = 1, \cdots, n),$$

where $n$ denotes the sample size. LV (2006, 2007) argue that $\Delta^{d_1} y_{t-1}$ is not the best class of regression one can choose and propose another auxiliary regression model for the hypotheses test (2). They propose to test (1.2) by using the following auxiliary model

$$\Delta y_t = \phi_2 z_{t-1}(d_1) + u_t, \quad (t = 1, \cdots, n),$$

where

$$z_{t-1}(d_1) = \left( \frac{\Delta^{d_2-1} - 1}{1 - d_2} \right) \Delta y_t.$$

The DGM (2002) and LV (2007) tests presents analogy with the original Dickey-Fuller test, but we cannot consider them as generalization of the familiar Dickey-Fuller test in the sense that the conventional $I(1)$ vs $I(0)$ framework is recovered (for the DGM test the conventional framework is recovered only if $d_1 = 0$). Indeed, under the null and $d_1$ known, the $t$ statistic in the regression model of DGM (2002) depend on fractional Brownian motion if $0 \leq d_1 < 0.5$ and $t \to N(0, 1)$ if $0.5 \leq d_1 < 1$. These asymptotic distributions are different from those derived by Dickey and Fuller (1979), which depend only on standard Brownian motion. The implementation of DGM (2002) test, would require tabulations of the percentiles of the functional of fractional Brownian motion, which imply that the inference on the presence of unit root would be conditional on $d_1$, and thus might suffer from misspecification. Note, also, that the assumption of knowing $d_1$ is not clear. Is this assumption means that the true parameter of the integration process under study is known? If this is the case, what is the usefulness of the test (1.2)? When $d_1$ is not taken to be known a priory, a pre-estimation of it is needed to implement the test. In this case, we can perform the test only if the estimator of $d_1$ ($\hat{d}_1$) is sufficiently close to unity (see DGM (2002) for details). Similar criticisms can be formulated for LV (2007) test.
In this article, we provide adequate scope of the more general fractional null hypothesis and fractional alternative (which include the $I(1)$ null and the $I(0)$ null like particular cases). More precisely, this paper is concerned with the hypotheses test for the degree of fractional integration

$$H_0 : d \geq d_0 \text{ against } H_1 : d < d_0,$$

if the true data generating process (DGP) of $y_t$ is fractionally integrated process whose true order of integration is $d$, denotes as $I(d)$

$$\Delta^d y_t = \varepsilon_t,$$

where $d_0 \geq 0$, $\varepsilon_t \sim i.i.d(0, \sigma^2)$ and $y_0 = 0$. This situation arises for instance when testing the null hypothesis for checking the nonstationarity of the process $y_t$.

We propose to test the null hypothesis by means of the estimator and the $t$-statistic of the coefficient of $\Delta^{-1+d_0} y_{t-1}$, in the ordinary least squares (OLS) of the auxiliary autoregression

$$\Delta^{-1+d_0} y_t = \hat{\phi}_n \Delta^{-1+d_0} y_{t-1} + \hat{\varepsilon}_t,$$

or equivalently

$$\Delta^{d_0} y_t = \hat{\rho}_n \Delta^{-1+d_0} y_{t-1} + \hat{\varepsilon}_t,$$

where $\hat{\rho}_n = \hat{\phi}_n - 1$ and $\{\hat{\varepsilon}_t\}$ the residuals. In order to grasp the intuition behind the fractional autoregressive model (1.6), suppose that $y_t \sim FI(d_0)$ and let us consider the relation between $\Delta^{d_0} y_t$ and $\Delta^{-1+d_0} y_{t-1}$. Note that, it is easy to check, that $\Delta^{d_0} y_t = (1 - L) [\Delta^{-1+d_0} y_t]$ and $\Delta^{-1+d_0} y_t \sim I(1)$. Putting $\Delta^{-1+d_0} y_t = x_t$ we can rewrite (1.6) as follows

$$(1 - L)x_t = \hat{\rho}_n x_{t-1} + \hat{\varepsilon}_t, \quad t = 1, 2, \cdots, n. \quad (1.7)$$

The regression model (1.7) is the simple Dickey-Fuller framework to deal with the testing problem

$$H_0 : \rho = 0 \text{ against } H_1 : \rho < 0.$$

The autoregressive model (1.6) can be easily implemented for practical settings and flexible enough to account for broad family of long memory specification of the fractional parameter $d$.

In this article, testing the hypotheses test (1.3) by using the auxiliary regression model (1.5) or equivalently (1.6), provide adequate scope of the more general fractional null hypothesis (which include the $I(1)$ null and $I(0)$ null like particular cases). Our proposal, unlike the proposal of DGM (2002) and LV (2007), not only presents analogy with the original Dickey-Fuller, but we can considered it as generalization of the familiar Dickey-Fuller test in the sense that the conventional $I(1)$ vs $I(0)$ framework is recovered for any value of $d_0 \in \mathbb{R}$ under the null.

Before stating the main result of this note, we give some technical tools that we need for this study. Let $\eta_t = (1 - L)^{-\delta} \varepsilon_t$, with $\delta \in [-0.5, 0.5]$ and $\varepsilon_t$ defined
as above. Let $\sigma_S^2 = var(S_n)$, where $S_t = \sum_{j=1}^m \eta_j$. When $|\delta| < \frac{1}{2}$, we have (see Sowell (1990))

$$
\lim_{n \to \infty} n^{-1-2\delta} \sigma_S^2 = \frac{\sigma_\epsilon^2 \Gamma(1-2\delta)}{(1+2\delta)\Gamma(1+\delta)\Gamma(1-\delta)} \equiv \kappa_\eta^2(\delta),
$$

where $\Gamma(\cdot)$ denote the gamma or generalized factorial function. For the case $\delta = \frac{1}{2}$, we have (see Miu(1998))

$$
\lim_{n \to \infty} (n^{-2} \log^{-1} n) \sigma_S^2 = \frac{2\sigma_\epsilon^2}{\pi} \equiv \kappa_\eta^2(\frac{1}{2}).
$$

(1.10)

If in addition, $E|\epsilon_t|^a < \infty$ for $a \geq \max \left\{4, \frac{-8\delta}{1+2\delta} \right\}$, we have the following useful results applies to this types of processes:

$$
\lim_{n \to \infty} \frac{\kappa_\eta^{-1}(\delta)}{\Gamma(1+\delta)} S_{[nr]} \Rightarrow \frac{1}{\Gamma(1+\delta)} \int_0^r (r-s)\delta dW(s) (\equiv W_\delta(r)),
$$

if $-\frac{1}{2} < \delta < \frac{1}{2}$, and

$$
\lim_{n \to \infty} \frac{\log^{-1} n \kappa_\eta^{-1}(\frac{1}{2})}{\Gamma(1+\delta)} S_{[nr]} \Rightarrow W_{0.5}(r),
$$

if $\delta = 0.5$.

Where $W(r)$ is the standard Brownian motion on $[0,1]$ associated with the $\epsilon_t$ sequence and the symbols $\Rightarrow$, ” and $\Rightarrow_p$ ” denote weak convergence and convergence in probability, respectively.

2  Asymptotic null and alternative distributions

By noting that $d - d_0$ can always be decomposed as $d - d_0 = m + \delta$, where $m \in \mathbb{N}$ and $\delta \in \{-0.5, 0.5\}$, the asymptotic null and alternative of the Dickey-Fuller, normalized bias statistic $n\hat{\rho}_n$ and the Dickey-Fuller $t$-statistic $t\hat{\rho}_n$ in the model (1.6) are provided by the theorem 1.

**Theorem 1.** Let $\{y_t\}$ be generated according DGP (1.4). If regression model (1.6) is fitted to a sample of size $n$ then, as $n \to \infty$,

1. $n\hat{\rho}_n$ verifies that

$$
\hat{\rho}_n = O_p(\log^{-1} n) \quad \text{and} \quad (\log n) \hat{\rho}_n \Rightarrow -\infty, \quad \text{if} \quad d - d_0 = -0.5,
$$

(2.1)

$$
\hat{\rho}_n = O_p(n^{-1-2\delta}) \quad \text{and} \quad n\hat{\rho}_n \Rightarrow -\infty, \quad \text{if} \quad -0.5 < d - d_0 < 0,
$$

(2.2)

$$
\hat{\rho}_n = O_p(n^{-1}) \quad \text{and} \quad n\hat{\rho}_n \Rightarrow \frac{1}{\int_0^1 W_\delta^2(r)dr} \{\text{w}^2(1) - 1\}, \quad \text{if} \quad d - d_0 = 0,
$$

(2.3)

$$
\hat{\rho}_n = O_p(n^{-1}) \quad \text{and} \quad n\hat{\rho}_n \Rightarrow \frac{1}{\int_0^1 W_{\delta,m+1}(1)dr} \{\text{w}^2(1) - 1\}, \quad \text{if} \quad d - d_0 > 0.
$$

(2.4)
2. \( \hat{\phi}_n \) verifies that

\[
t_{\hat{\phi}_n} = O_p(n^{-0.5} \log^{-0.5} n) \quad \text{and} \quad t_{\hat{\phi}_n} \xrightarrow{p} -\infty, \quad \text{if} \quad d - d_0 = -0.5, \quad (2.5)
\]

\[
t_{\hat{\phi}_n} = O_p(n^{-\delta}) \quad \text{and} \quad t_{\hat{\phi}_n} \xrightarrow{p} -\infty, \quad \text{if} \quad -\frac{1}{2} < d - d_0 < 0, \quad (2.6)
\]

\[
t_{\hat{\phi}_n} = O_p(1) \quad \text{and} \quad t_{\hat{\phi}_n} = \frac{\frac{1}{\hat{\phi}_n} \{w^2(1) - 1\}}{\left[\int_{0}^{1} w^2(r)dr\right]^{1/2}}, \quad \text{if} \quad d - d_0 = 0, \quad (2.7)
\]

\[
t_{\hat{\phi}_n} = O_p(n^\delta) \quad \text{and} \quad t_{\hat{\phi}_n} \xrightarrow{P} +\infty, \quad \text{if} \quad 0 < d - d_0 < 0.5, \quad (2.8)
\]

\[
t_{\hat{\phi}_n} = O_p(n^{0.5}) \quad \text{and} \quad t_{\hat{\phi}_n} \xrightarrow{P} +\infty, \quad \text{if} \quad d - d_0 \geq 0.5. \quad (2.9)
\]

where \( w_{\delta, m}(r) \) is \((m - 1)\)-fold integral of \( w_{\delta}(r) \) recursively defined as \\
\[
w_{\delta, m}(r) = \int_{0}^{r} w_{\delta, m-1}(s)ds, \quad \text{with} \quad w_{\delta, 1}(r) = w_{\delta}(r) \quad \text{and} \quad w(r) \quad \text{is the standard Brownian motion.}
\]

**Proof.** See Appendix A.

These properties and distributions are the generalization of those established by Sowell (1990) for the cases \(-\frac{1}{2} < d - 1 < 0, \quad d - 1 = 0\) and \(0 < d - 1 < \frac{1}{2} \).

From (2.1) and (2.2), the rate at which \( \hat{\phi}_n = \hat{\phi}_n - 1 \) converge to zero (i.e. \( \hat{\phi}_n \) converge to 1 ) is slow for nonpositive values of \( d - d_0 \), particularly it is very slow for \(-\frac{1}{2} < d - d_0 < -\frac{1}{4} \). Moreover for \(-\frac{1}{2} < d - d_0 < 0 \), the limiting distribution of \( \hat{\phi}_n \) has nonpositive support and then \( \lim_{n \to \infty} P(\hat{\phi}_n < 1) = 1 \). From (2.3) and (2.4), \( \hat{\phi}_n \) converge to zero at the rate \( n \), when \( d \geq d_0 \). The rate convergence \( n \) is faster than the usual standard rate \( n^{\frac{1}{2}} \), when we deal with stationary \( I(0) \) variables. Then, for \( d - d_0 \geq 0 \), the least squares estimate is super consistent. In the other words, if a first order autoregression model (1.5) is fitted to a sample of size \( n \) generated according an ARFIMA(0,1+d−d_0,0), where \( 1+d-d_0 \) is the order of integration of \( \Delta^{-1} \phi_0 y_t \), then the OLS estimator, \( \hat{\phi}_n \), will converge to 1, when \( d - d_0 \geq 0 \). Figure 1 and figure 2 below illustrates this fact in an obvious way, for the cases where the value of fractional parameter specified under the null are respectively \( d_0 = 1 \) and \( d_0 = 0.5 \).

![Figure 1: Relation between \( \hat{\phi}_n \) and \( d \) when \( d_0 = 1 \).](image1.png)

![Figure 2: Relation between \( \hat{\phi}_n \) and \( d \) when \( d_0 = 0.5 \).](image2.png)

For instance, the figure 2 was made as follows: For a fixed sample \( \{u_{1-n}, \cdots, u_0, \cdots, u_n\} \) generated from \( i.i.d.(0,1) \), with \( n = 1000 \), samples of
$ARFIMA(0,d+0.5,0)$ processes were generated for $d$ varying between 0 and 2.5, with step of 0.01. For each sample $\{x_t, t = 1, \ldots, n\}$ a first order autoregression model (1.5) is fitted and estimate of $\phi$ are calculated. By plotting the parameter $\hat{\phi}_n$ against the fractional parameter $d$, one obtains the figure 2. A general procedure for generating a fractionally integrated series of length $n$ is to apply for $t = 1, \ldots, n$, the formula

$$x_t = \sum_{j=0}^{t-1} \frac{\Gamma(d+j)}{\Gamma(d)\Gamma(j+1)} u_{t-j}.$$  

The relation between $\hat{\phi}_n$ and $d$, highlighted by the results, (2.1), (2.2), (2.3), (2.4) and illustrated by figures 1 and 2, suggests that when we deal with degree of fractional integration test, we have,

$$\phi = 1 \iff d \geq d_0 \quad \text{and} \quad \phi < 1 \iff d < d_0$$

In the other words, the testing problem $H_0 : \phi = 1$ against $H_1 : \phi < 1$ is equivalent to (1.3). Another, important propriety highlighted by Theorem 1 is that the tests are invariant to the original value of $d$, and the asymptotic properties only depend on $d - d_0$. For example, we have used the samples of 10000 observations to estimate the densities (following Sowell (1990)) of $n\hat{\rho}_n$ and $t\hat{\rho}_n$ under $d - d_0 = 0$. The estimated densities are presented in Fig. 3. and Fig.4.

**Figure 3:** Kernel densities estimate of $n\hat{\rho}_n$ statistic under $H_0 : d = d_0$ using 1000 samples of size $n = 250$.

**Figure 4:** Kernel densities estimate of $t\hat{\rho}_n$ statistic under $H_0 : d = d_0$ using 1000 samples of size $n = 250$.

For each statistics $n\hat{\rho}_n$ or $t\hat{\rho}_n$ under $d = d_0$ and for a given size of sample $n = 250$, the estimated densities for different values of $d$ are represented on the same graph. The figures 3 and 4 show that by fitting the regression model (1.6) to the sample generated according (1.4), one obtains the same distribution that those used by Dickey-Fuller (1979, 1981). In other words, as shown below, the proposed test, based on the fractional regression model (1.5) (or equivalently (1.6)) and the hypotheses test (1.3), can be understood and implemented exactly as the simple Dickey and Fuller test for unit root by using the usual tables statistics of the conventional $n\hat{\rho}_n$ statistic (hereafter $Z_1$ ) and $t\hat{\rho}_n$ statistic (hereafter $Z_2$ ).
3 Power and size of the Simple Fractional Dickey-Fuller test

Consider the problem of hypotheses test (1.3) in sample of size \( n \), generating according (1.4). We introduce two nonrandomized test, defined by a function \( \Psi_{i,n} \), \( i = 1, 2 \) on the sample space of the observations \( Z_i, i = 1, 2 \) with critical regions \( C_i, i = 1, 2 \). The \( \Psi_{i,n} \) test for a region \( C_i \) is its indicator function

\[
\Psi_{i,n}(z_i) = \begin{cases} 
1 & \text{if } z_i \in C_i, \\
0 & \text{if } z_i \notin C_i.
\end{cases}
\] (3.1)

Let \( \alpha \) and \( \beta \) respectively the type I error and the type II error of the test \( \Psi_{i,n} \). Since \( H_0 \) and \( H_1 \) are composite, we have

\[
\alpha = \sup_{d \geq 1} \pi_{0,i}(d) = \pi_{0,i}(d_0)
\]
\[
\beta = \pi_{1,i}(d),
\]

where \( \pi_{0,i}(d) = P_{H_0}(Z_i \in C_i) \) and \( \pi_{1,i}(d) = P_{H_1}(Z_i \notin C_i) \). For the alternative hypothesis \( H_1: d < d_0 \), we consider one sided critical regions of the form

\[
C_i = \{ Z_i < c_{n,i}(\alpha) \},
\] (3.2)

where \( \alpha \) is the level of the test and the critical points \( c_{n,i}(\alpha) \), are those used in the standard Dickey-Fuller test for unit root. With these settings, the power function and the size function of the test \( \Psi_{i,n} \), denoted respectively by \( \Pi_{\Psi_{i,n}}(d) \) and \( S_{\Psi_{i,n}}(d) \), are:

\[
\Pi_{\Psi_{i,n}}(d) = 1 - \pi_{1,i}(d), \quad \text{for } d < d_0
\]
\[
S_{\Psi_{i,n}}(d) = 1 - \pi_{0,i}(d), \quad \text{for } d \geq d_0
\]

**Theorem 2.** For a given \( \alpha \), a sequence of tests \( \{ \Psi_{i,n} \} \) defined by (3.1), with critical region (3.2) is consistent i.e.

\[
\lim_{n \to \infty} S_{\Psi_{i,n}}(d) \geq 1 - \alpha, \quad \text{for } d \geq d_0,
\]
\[
\lim_{n \to \infty} \Pi_{\Psi_{i,n}}(d) \to 1, \quad \text{for } -\frac{1}{2} < d - d_0 < 0.
\]

**Proof.** Consider first the statistic \( Z_1 \). For \( -\frac{1}{2} \leq d - d_0 < 0 \), (i.e. \( -\frac{1}{2} < \delta < 0 \)), we have

\[
\lim_{n \to \infty} \Pi_{\Psi_{1,n}}(d) = \lim_{n \to \infty} P_{H_1}(Z_1 < c_{1,n}(\alpha))
\]
\[
= \lim_{n \to \infty} P_{H_1}(n^{2\delta} Z_1 < n^{2\delta} c_{1,n}(\alpha))
\]
\[
= 1,
\]

because, from result (2.2), the limit distribution of \( n^{2\delta} Z_1 \) has nonpositive support and \( \lim_{n \to \infty} n^{2\delta} c_{1,n}(\alpha) = 0 \).
Figure 5: Kernel densities estimates of $Z_1$ using 10000 samples of size $n = 250$, when the true value of $d$ are respectively 0.5, 0.6, 0.7, 0.8, 0.9, 1.5 and the value specified under the null is $d_0 = 0.5$.

With similar reasoning, we show that $\lim_{n \to \infty} S_{\Psi_{1,n}}(d) = 1$ for $d > d_0$ and then $\lim_{n \to \infty} S_{\Psi_{1,n}}(d) \geq 1 - \alpha$. To check this result, let us consider the figure 5 below.

The figure 5, where $c = c_{1,n}(\alpha)$, show that the nominal size $\alpha = \pi_{0,1}(d_0)$ is greater than $\pi_{0,1}(d)$, for $d > d_0$.

Consider, now, the statistic $Z_2$. For $-\frac{1}{2} < d - d_0 < 0$, (i.e. $-\frac{1}{2} < \delta < 0$), we have

$$\lim_{n \to \infty} \Pi_{\Psi_{2,n}}(d) = \lim_{n \to \infty} P_{H_1}(Z_2 < c_{2,n}(\alpha))$$

$$= \lim_{n \to \infty} P_{H_1}(n^\delta Z_2 < n^\delta c_{2,n}(\alpha))$$

$$= 1$$

because, from (2.3), the limit distribution of $n^\delta Z_2$ has nonpositive support and $\lim_{n \to \infty} n^\delta c_{2,n}(\alpha) = 0$.

For $d - d_0 > 0$, with $\gamma = \delta$ if $0 < \delta < 0.5$ and $\gamma = 0.5$ if $d - d_0 \geq 0.5$, we have

$$\lim_{n \to \infty} S_{\Psi_{2,n}}(d) = \lim_{n \to \infty} P_{H_0}(Z_2 > c_{2,n}(\alpha))$$

$$= \lim_{n \to \infty} P_{H_0}(n^{-\gamma} Z_2 > n^{-\gamma} c_{2,n}(\alpha))$$

$$= 1$$

because the limit of $n^{-\gamma} Z_2$ has nonnegative support and $\lim_{n \to \infty} n^{-\gamma} c_{2,n}(\alpha) = 0$. ■

Based on these results, in order to test the nonstationarity of a given sample of the process $y_t$, if we want to test the null hypothesis $H_0 : d \geq 1$ against $H_1 : d < 1$, we use the usual auxiliary regression model

$$\Delta y_t = \hat{\rho}_n y_{t-1} + \hat{\epsilon}_t,$$
and if we want to test the null hypothesis \( H_0 : d \geq 0.5 \) against \( H_1 : d < 0.5 \), we use the following auxiliary regression model

\[
\Delta^{0.5} y_t = \hat{\rho}_n \Delta^{-0.5} y_{t-1} + \hat{\epsilon}_t.
\]

Note that under \( d = 1 \) we have \( y_t \xrightarrow{} I(1) \) and under \( d = 0.5 \), we have \( \Delta^{-0.5} y_{t-1} \xrightarrow{} I(1) \). In the both cases, the conventional framework of \( I(1) \) vs \( I(0) \) is recovered, since the asymptotic distributions of \( Z_1 \) and \( Z_2 \) under \( d = d_0 \) are invariant and they are the same as those derived derived by Dickey and Fuller (1979, 1981).

### 3.1 Size and power of the F-DF test based on \( t \) statistic

In this subsection, in Monte Carlo study, we show that the proposed hypotheses test (1.3) based on the DGP (1.4) and in the auxiliary regression model (1.6) fare very well both in terms of power and size when we use the \( t \) statistic. To investigate the size and power of the hypotheses test (1.3), 10000 samples of \( FI(d) \) Gaussian processes (1.4) are generated and the regression model (1.6) is used to estimate \( t \). The sample size considered is \( n = 50 \) and \( n = 250 \). We will use, as true values of the fractional parameter of integration of the process \( y_t \), three values of \( d \): 0; 0.5; 1 and for each value, we specify various values for \( d_0 \). If the set of the values of \( d_0 \) for a given value of \( d \) is denoted by \( S_d(d_0) \) then the sets which will be used for the three values of \( d \) are respectively

\[
S_0(d_0) = \{-0.4; -0.3; -0.2; -0.1; 0; 0.1; 0.2; 0.3; 0.4\},
S_{0.5}(d_0) = \{0; 0.1; 0.2; 0.3; 0.4; 0.5; 0.6; 0.7; 0.8; 0.9\},
S_1(d_0) = \{0.5; 0.6; 0.7; 0.8; 0.9; 1; 1.1; 1.2; 1.3; 1.4\}.
\]

The tables 1 and 2 contain the simulation results on the size of the F-DF test for the hypotheses test (1.3). The tables 1 and 2 show that the testing problem (1.3) has good performances in terms of size since we have

\[
P(t \geq c_n(\alpha) \mid \delta \geq 0) \geq 1 - \alpha.
\]

The table 3 and Table 4 contain the simulation results on the power of the F-DF test for the hypotheses test (1.3). There are some conclusions to be drawn from it. First, the power of the F-DF test increases with the increase of sample size and \( \delta = d - d_0 \). For example, for \( \alpha = 5\% \), \( d = 1 \) and \( \delta = -0.1 \), power is 12.36\% for \( n = 50 \), 20.76\% for \( n = 250 \) and for \( \alpha = 5\% \), \( d = 1 \) and \( \delta = -0.3 \), power is 48.5\% for \( n = 50 \), 86.05\% for \( n = 250 \). Second, as shown in table 4, for \( n = 250 \), the power of F-DF test is below 50\% for \( (\delta = -0.1) \) and for \( (\alpha = 1\%, \delta = -0.2) \). Third, for a given \( n, \alpha \) and \( \delta \), the power for \( d = 0 \), \( d = 0.5 \) and \( d = 1 \) are approximately similar because the asymptotic under the alternative does not depend on \( d \) but depends on \( \delta = d - d_0 \).
Table 1. Size of the hypotheses test (1.3) when we use $t$ ($n = 50$)

| True value of $d$ | $\alpha \setminus \delta$ | 0.4   | 0.3   | 0.2   | 0.1   | 0     |
|-------------------|----------------------------|-------|-------|-------|-------|-------|
| $d = 0$           | 1%                         | 100   | 99.99 | 99.93 | 99.73 | 98.96 |
|                   | 5%                         | 99.98 | 99.85 | 99.39 | 98.17 | 94.05 |
|                   | 10%                        | 99.85 | 99.48 | 98.26 | 95.78 | 90.34 |
| $d = 0.5$         | 1%                         | 99.98 | 99.97 | 99.90 | 99.74 | 98.91 |
|                   | 5%                         | 99.91 | 99.78 | 99.28 | 98.14 | 95.22 |
|                   | 10%                        | 99.78 | 99.42 | 98.25 | 95.63 | 90.00 |
| $d = 1$           | 1%                         | 100   | 99.99 | 99.97 | 99.79 | 99.12 |
|                   | 5%                         | 99.95 | 99.92 | 99.55 | 98.34 | 95.13 |
|                   | 10%                        | 99.83 | 99.54 | 98.72 | 96.03 | 90.02 |

Table 2. Size of the hypotheses test (1.3) when we use $t$ ($n = 250$)

| True value of $d$ | $\alpha \setminus \delta$ | 0.4   | 0.3   | 0.2   | 0.1   | 0     |
|-------------------|----------------------------|-------|-------|-------|-------|-------|
| $d = 0$           | 1%                         | 100   | 100   | 99.99 | 99.92 | 99.01 |
|                   | 5%                         | 100   | 99.99 | 99.95 | 99.29 | 94.68 |
|                   | 10%                        | 100   | 99.97 | 99.73 | 97.87 | 89.49 |
| $d = 0.5$         | 1%                         | 100   | 99.98 | 99.8 | 99.91 | 99.18 |
|                   | 5%                         | 100   | 100   | 99.97 | 99.27 | 95.44 |
|                   | 10%                        | 100   | 100   | 100   | 97.66 | 90.49 |
| $d = 1$           | 1%                         | 100   | 100   | 100   | 99.92 | 98.89 |
|                   | 5%                         | 100   | 100   | 99.96 | 99.32 | 94.87 |
|                   | 10%                        | 99.98 | 99.97 | 99.8 | 98.60 | 90.15 |

Table 3. Power of the hypotheses test (3) when we use $t$ ($n = 50$)

| True value of $d$ | $\alpha \setminus \delta$ | -0.1 | -0.2 | -0.3 | -0.4 |
|-------------------|----------------------------|------|------|------|------|
| $d = 0$           | 1%                         | 3.61 | 9.53 | 22.23 | 45.82 |
|                   | 5%                         | 12.66 | 26.68 | 47.16 | 73.03 |
|                   | 10%                        | 21.97 | 39.56 | 62.14 | 84.92 |
| $d = 0.5$         | 1%                         | 2.87 | 9.36 | 22.59 | 45.16 |
|                   | 5%                         | 11.9 | 25.17 | 48.34 | 72.74 |
|                   | 10%                        | 21.17 | 39.58 | 63.23 | 84.39 |
| $d = 1$           | 1%                         | 3.2  | 9.39 | 22.91 | 46.23 |
|                   | 5%                         | 12.36 | 26.1 | 48.50 | 73.47 |
|                   | 10%                        | 21.63 | 39.24 | 63.69 | 84.99 |
Table 4. Power of the hypotheses test (1.3) when we use $t$ ($n = 250$)

| True value of $d$ | $\alpha \setminus \delta$ | $-0.1$ | $-0.2$ | $-0.3$ | $-0.4$ |
|------------------|-------------------------|--------|--------|--------|--------|
| $d = 0$          | 1%                      | 7.59   | 30.07  | 68.00  | 95.48  |
|                  | 5%                      | 19.98  | 52.53  | 85.90  | 99.16  |
|                  | 10%                     | 30.62  | 64.77  | 92.95  | 99.87  |
| $d = 0.5$        | 1%                      | 7.3    | 30.06  | 68.54  | 95.60  |
|                  | 5%                      | 20.35  | 52.40  | 85.80  | 99.39  |
|                  | 10%                     | 31.44  | 65.10  | 92.50  | 99.89  |
| $d = 1$          | 1%                      | 7.71   | 29.90  | 68.36  | 95.39  |
|                  | 5%                      | 20.76  | 51.77  | 86.05  | 99.35  |
|                  | 10%                     | 31.71  | 64.29  | 92.41  | 99.87  |

4 Concluding Remarks

In this paper, to distinguish between $FI(d)$ processes, we have proposed a new and appropriate testing procedure in time domain, that extends the familiar Dickey-Fuller (1979) types tests for unit root ($I(1)$ against $I(0)$), by embedding the case $d = 0$ and $d = 1$ in continuum of memory properties. The main idea of our test procedure is the following: in order to test if the process $y_t$ is fractionally integrated of order $d_0$, it suffices to test if the process $x_t = (1 - L)^{-1+d_0}y_t$ is integrated of order 1. We have referred to the test based on this idea as the Fractional Dickey-Fuller (FD-F) test (DGM (2002) gave the same name for their test). The proposed test is based on the OLS estimator ($\hat{\rho}_n$) or its $t$-ratio in the autoregression model

$$\Delta^{d_0}y_t = \hat{\rho}_n\Delta^{-1+d_0}y_{t-1} + \hat{\epsilon}_t, \ t = 1, 2, \cdots, n$$

With this regression model associated with the non explosive feature of $FI(d)$ processes (i.e. convergence speed of $\hat{\rho}_n$ to zero at rate $n$), we have showed that the testing problem $H_0: d \geq d_0$ against $H_1: d < d_0$, is equivalent to $H_0: \rho = 0$ against $H_1: \rho < 0$. We have also showed that the asymptotic distributions for ordinary least squares (OLS) and its $t$-ratio under the null simple hypothesis $H_0: d = d_0$ are identical to those derived by Dickey and Fuller (1979,1981) for the simple case (without drift and trend). This implies that the proposed test can be understood and implemented exactly as the Dickey-Fuller test for unit root by using the usual tables statistics.

This article does not discuss the situation when there is short memory in series, of $AR$ or $MA$ type. This seems a very serious drawback for practical implementation of the tests. Here, we give just an indication when $y_t \sim ARFIMA(p, d, 0)$

$$A(L)\Delta^d y_t = \epsilon_t,$$

where $A(L) = \sum_{j=0}^p \alpha J^L$, with $\alpha_0 = 1$ and the roots of $A(z) = 0$ are outside the unit circle and $\epsilon_t$ is defined as above. Then the fractional augmented Dickey-
Fuller test, for the null hypothesis $d \geq d_0$, is based on the regression model

$$\Delta^d y_t = \hat{\rho}_n \Delta^{-1+d_0} y_{t-1} + \sum_{j=0}^{p} \hat{\alpha}_j \Delta^{d_0} y_{t-j} + \hat{\epsilon}_t$$

The case for the presence of autocorrelation in the error process deserve that one devotes a paper, to take account of the work of Said and Dickey (1984) and Phillips (1987).

**Appendix: Proof of theorem 1**

By denoting $\Delta^{-1+d_0} y_t = x_t$, the OLS estimator of $\rho$ and its $t$-ratio for the auxiliary regression model (6), are given by the usual squares expressions

$$\hat{\rho}_n = \frac{\sum_{t=1}^{n} (\Delta x_t) (x_{t-1})}{\sum_{t=1}^{n} (x_{t-1})^2},$$

$$t_{\hat{\rho}_n} = \frac{\sum_{t=1}^{n} (\Delta x_t) (x_{t-1})}{\left\{s_n^2 \sum_{t=1}^{n} (x_{t-1})^2\right\}^{1/2}},$$

where the variance of the residuals, $s_n^2$ is given by

$$s_n^2 = n^{-1} \sum_{t=1}^{n} (\Delta x_t - \hat{\rho} x_{t-1})^2.$$
For the \( \sum_{t=1}^{n} [\Delta x_t] [x_{t-1}] \) term, we have that

\[
\sum_{t=1}^{n} [\Delta x_t] [x_{t-1}] = \frac{1}{2} \left( \Delta^{-1+d_0} y_n \right)^2 - \frac{1}{2} \sum_{t=1}^{n} (\Delta x_t)^2
\]

For the first term, it follows from (1.9), (1.10), (1.11), (1.12) and the continuous mapping theorem

\[
\frac{1}{2 (\log n) \kappa^2(y_\delta)} \left( \Delta^{-1+d_0} y_n \right)^2 \Rightarrow \frac{1}{2} w^2(1), \text{ if } -0.5 < d - d_0 < 0,
\]

\[
\frac{1}{2n^{1+2\delta} \kappa^2(y_\delta)} \left( \Delta^{-1+d_0} y_n \right)^2 \Rightarrow \frac{1}{2} w^2(1), \text{ if } d - d_0 = 0,
\]

\[
\frac{1}{2n^{1+2\delta} \kappa^2(y_\delta)} \left( \Delta^{-1+d_0} y_n \right)^2 \Rightarrow \frac{1}{2} w^2(1), \text{ if } 0 < d - d_0 < 0.5,
\]

\[
\frac{1}{2n^{2(m+1)} (\log n) \kappa^2(y_\delta)} \left( \Delta^{-1+d_0} y_n \right)^2 \Rightarrow \frac{1}{2} w^2_{0.5,m+1}(1), \text{ if } d - d_0 = m + 0.5,
\]

\[
\frac{1}{2n^{1+2(m+\delta)} \kappa^2(y_\delta)} \left( \Delta^{-1+d_0} y_n \right)^2 \Rightarrow \frac{1}{2} w^2_{\delta,m+1}(1), \text{ if } d - d_0 \neq m + 0.5.
\]

For the second term, we have:

When \( d - d_0 = -0.5 \), by using Lemma 2.1 of Ming Liu (1998) result 2

\[
- \frac{1}{2} \kappa^{-2}(\frac{1}{2}) n^{-1} \sum_{t=1}^{n} (\Delta x_t)^2 \mathcal{P} - \frac{1}{2} \kappa^{-2}(\frac{1}{2}) \text{var}(\Delta x_t) = -1.
\]

When \(-0.5 < d - d_0 < 0\), by using (1.9) and the ergodic theorem (note that here \( d - d_0 = \delta \))

\[
- \frac{1}{2} \kappa^{-2}(\delta) n^{-1} \sum_{t=1}^{n} (\Delta x_t)^2 \mathcal{P} - \frac{1}{2} \kappa^{-2}(\delta) \text{var}(\Delta x_t) = - \frac{1}{2} \frac{1}{\Gamma(1+\delta)} \frac{\Gamma(1+\delta)}{\Gamma(1-\delta)}.
\]

When \( d - d_0 = 0 \), by using (1.9) and the ergodic theorem

\[
- \frac{1}{2} \kappa^{-2}(0) n^{-1} \sum_{t=1}^{n} (\Delta x_t)^2 \mathcal{P} - \frac{1}{2} \kappa^{-2}(0) \text{var}(\Delta x_t) = - \frac{1}{2}.
\]

When \( 0 < d - d_0 < 0.5 \), by using (1.9) and the ergodic theorem (note that here \( d - d_0 = \delta \))

\[
- \frac{1}{2} n^{-1} \sum_{t=1}^{n} (\Delta x_t)^2 \mathcal{P} - \frac{1}{2} \kappa^{-2}(\delta) \text{var}(\Delta x_t) = - \frac{1}{2} \frac{1}{\Gamma(1+\delta)} \frac{\Gamma(1+\delta)}{\Gamma(1-\delta)}.
\]
When \( d - d_0 = m + 0.5 \), by using (1.10), (1.12) and the continuous mapping theorem
\[
\frac{1}{n^{2(m+0.5)}(\log n)^{\kappa_2^2(0.5)}} \sum_{t=1}^{n} (\Delta x_t)^2 \Rightarrow \int_0^1 w^2_{0.5,m}(r)dr.
\] (A17)

When \( d - d_0 \neq m + 0.5 \), by using (1.9), (1.11) and the continuous mapping theorem
\[
\frac{1}{n^{2(m+\delta)}\kappa_2^2(\delta)} \sum_{t=1}^{n} (\Delta x_t)^2 \Rightarrow \int_0^1 w^2_{\delta,m}(r)dr.
\] (A18)

Therefore, when \( d - d_0 = -0.5 \), we have by using (A7) and (A13)
\[
n^{-1}\kappa_2^{-2}(0.5) \sum_{t=1}^{n} [\Delta x_t][x_{t-1}] \xrightarrow{p} -1,
\] (A19)

when \(-0.5 < d - d_0 < 0\), by using (A8) and (A14)
\[
n^{-1}\kappa_2^{-2}(\delta) \sum_{t=1}^{n} |\Delta x_t|[x_{t-1}] \xrightarrow{p} -(1/2 + \delta) \frac{\Gamma(1+\delta)}{\Gamma(1-\delta)}.
\] (A20)

when \( d - d_0 = 0 \), by using (A9) and (A15)
\[
n^{-1}\kappa_2^{-2}(0) \sum_{t=1}^{n} [\Delta x_t][x_{t-1}] \Rightarrow -\frac{1}{2} \{w^2(1) - 1\}
\] (A21)

when \( 0 < d - d_0 < 0.5 \), by using (A10) and (A16)
\[
n^{-1-2\delta}\kappa_2^{-2}(\delta) \sum_{t=1}^{n} [\Delta x_t][x_{t-1}] \Rightarrow \frac{1}{2} w^2_{\delta,1}(1)
\] (A22)

when \( d - d_0 = m + 0.5 \), by using (A11) and (A17)
\[
n^{-2(m+1)}(\log^{-1} n) \kappa_2^{-2}(0.5) \sum_{t=1}^{n} [\Delta x_t][x_{t-1}] \Rightarrow \frac{1}{2} w^2_{0.5,m+1}(1)
\] (A23)

when \( d - d_0 \neq m + 0.5 \)
\[
n^{-1-2(m+\delta)}\kappa_2^{-2}(\delta) \sum_{t=1}^{n} [\Delta x_t][x_{t-1}] \Rightarrow \frac{1}{2} w^2_{\delta,m+1}(1)
\] (24)

Hence, using respectively (A1, A19), (A2, A20), (A3, A21), (A4, A22), (A5, A23), (A6, A24) and the continuous mapping theorem, we obtain that
\[
(\log n) \hat{\rho}_n = \frac{n^{-1}\kappa_2^{-2}(0.5) \sum_{t=1}^{n} [\Delta x_t][x_{t-1}]}{n^{-1}(\log^{-1} n) \kappa_2^{-2}(0.5) \sum_{t=1}^{n} [x_{t-1}]^2} \Rightarrow \frac{-1}{\int_0^1 w^2_{0.5}(r)dr},
\] (A25)
if \(d - d_0 = -0.5\).

\[
\hat{\phi}_n = \frac{n^{-1}\kappa_0^{-2}(\delta) \sum_{t=1}^{n} [\Delta x_t] [x_{t-1}]}{n^{-2-2\delta}\kappa_0^{-2}(\delta) \sum_{t=1}^{n} [x_{t-1}]^2} \Rightarrow \frac{\delta}{\int_0^1 w^2(r) dr}, \tag{A26}
\]

if \(-0.5 < d - d_0 < 0\).

\[
n\hat{\phi}_n = \frac{n^{-1}\kappa_0^{-2}(0) \sum_{t=1}^{n} [\Delta x_t] [x_{t-1}]}{n^{-2}\kappa_0^{-2}(0) \sum_{t=1}^{n} [x_{t-1}]^2} \Rightarrow \frac{1}{2} \left[ \frac{1}{\int_0^1 w^2(r) dr} \right], \tag{A27}
\]

if \(d - d_0 = 0\).

\[
n\hat{\phi}_n = \frac{n^{-1-2\delta}\kappa_0^{-2}(\delta) \sum_{t=1}^{n} [\Delta x_t] [x_{t-1}]}{n^{-2-2\delta}\kappa_0^{-2}(\delta) \sum_{t=1}^{n} [x_{t-1}]^2} \Rightarrow \frac{1}{2} \left[ \frac{1}{\int_0^1 w^2(r) dr} \right], \tag{A28}
\]

if \(0 < d - d_0 < 0.5\).

\[
n\hat{\phi}_n = \frac{n^{-2}\kappa_0^{-2}(0.5) \sum_{t=1}^{n} [\Delta x_t] [x_{t-1}]}{n^{-2(1+m+1)}\kappa_0^{-2}(0.5) \sum_{t=1}^{n} [x_{t-1}]^2} \Rightarrow \frac{1}{2} \left[ \frac{1}{\int_0^1 w^2_{0.5,m+1}(r) dr} \right] \equiv \rho_{1,\infty}, \tag{A29}
\]

if \(d - d_0 = m + 0.5\).

\[
n\hat{\phi}_n = \frac{n^{-2}\kappa_0^{-2}(\delta) \sum_{t=1}^{n} [\Delta x_t] [x_{t-1}]}{n^{-2(1+m+1)}\kappa_0^{-2}(\delta) \sum_{t=1}^{n} [x_{t-1}]^2} \Rightarrow \frac{1}{2} \left[ \frac{1}{\int_0^1 w^2_{0.5,m+1}(r) dr} \right] \equiv \rho_{2,\infty}, \tag{A30}
\]

if \(d - d_0 \neq m + 0.5\).

Now consider the \(t\)-statistic, first notice that

\[
\hat{\sigma}^2 = n^{-1} \sum_{t=1}^{n} (\Delta x_t - \hat{\phi}_n x_{t-1})^2
= n^{-1} \left( \sum_{t=1}^{n} (\Delta x_t)^2 + \hat{\phi}_n^2 \sum_{t=1}^{n} (x_{t-1})^2 - 2\hat{\phi}_n \sum_{t=1}^{n} [\Delta x_t] [x_{t-1}] \right).
\]

Hence, when \(d - d_0 = -0.5\), by using (A1), (A13), (A19) and (A25), it follows

\[
\hat{\sigma}^2 \xrightarrow{p} \text{var}(\Delta x_t) = \frac{4\sigma^2_x}{\pi}. \tag{A31}
\]

When \(-0.5 < d - d_0 < 0\), by using A2, A14, A20 and A26, it follows

\[
\hat{\sigma}^2 \xrightarrow{p} \text{var}(\Delta x_t) = \frac{\sigma^2_x \Gamma(1 - \delta)}{\Gamma^2(1 - \delta)}. \tag{A32}
\]

When \(d - d_0 = 0\), by using A3, A15, A21 and A27, it follows

\[
\hat{\sigma}^2 \xrightarrow{p} \text{var}(\Delta x_t) = \sigma^2_x. \tag{A33}
\]
When $0 < d - d_0 < 0.5$, by using A4, A16, A22 and A28, it follows
\[
\hat{\sigma}^2 \xrightarrow{P} \text{var}(\Delta x_t) = \frac{\sigma_x^2 \Gamma(1 - 2\delta)}{\Gamma^2(1 - \delta)}. \quad (A34)
\]

When $d - d_0 = m + 0.5$, by using A5, A17, A23 and A29, it follows
\[
\frac{n^{-2m}}{\log n} \kappa_\eta^{-2}(0.5) \hat{\sigma}^2 = \int_0^1 w_{0.5,m}^2(r) dr + \rho_{1,\infty}^2 \int_0^1 w_{0.5,m+1}^2(r) dr - \rho_{1,\infty}^{2} w_{0.5,m+1}^2(1). \quad (A35)
\]

When $d - d_0 \neq m + 0.5$, by using A6, A18, A24 and A30, it follows
\[
\frac{n^{-2m-2\delta+1}}{\log n} \kappa_\eta^{-2}(\delta) \hat{\sigma}^2 = \int_0^1 w_{\delta,m}^2(r) dr + \rho_{2,\infty}^2 \int_0^1 w_{\delta,m+1}^2(r) dr - \rho_{2,\infty}^{2} w_{\delta,m+1}^2(1). \quad (A36)
\]

Finally, by using respectively (A1, A19, A31), (A2, A20, A32), (A3, A21, A33), (A4, A22, A34) (A5, A23, A35), (A6, A24, A36) we obtain for the $t$-statistic
\[
t_{\hat{\beta}_n} = \frac{n^{-0.5} \log^{-0.5} n}{} \kappa_\eta^{-1}(0.5) \sum_{t=1}^{n} [\Delta x_t] [x_{t-1}] \xrightarrow{P} \rho_{\rho} \rightarrow -\infty,
\]

and $t_{\hat{\beta}_n} = O_p(n^{0.5} \log^{0.5} n)$ if $d - d_0 = -0.5$.

\[
t_{\hat{\beta}_n} = \frac{n^{-1} \log^{-1} n}{} \kappa_\eta^{-1}(0.5) \sum_{t=1}^{n} [\Delta x_t] [x_{t-1}] \xrightarrow{P} \rho_{\rho} \rightarrow -\infty,
\]

and $t_{\hat{\beta}_n} = O_p(n^{-\delta})$ if $-0.5 < d - d_0 < 0$.

\[
t_{\hat{\beta}_n} = \frac{n^{-1} \log^{-1} n}{} \kappa_\eta^{-1}(0.5) \sum_{t=1}^{n} [\Delta x_t] [x_{t-1}] \xrightarrow{P} \rho_{\rho} \rightarrow -\infty,
\]

and $t_{\hat{\beta}_n} = O_p(1)$ if $d - d_0 = 0$.

\[
t_{\hat{\beta}_n} = \frac{n^{-2(0.5)} \log^{-2(0.5)} n}{} \kappa_\eta^{-2}(0.5) \sum_{t=1}^{n} [\Delta x_t] [x_{t-1}] \xrightarrow{P} \rho_{\rho} \rightarrow -\infty,
\]

and $t_{\hat{\beta}_n} = O_p(n^{0.5})$ if $0 < d - d_0 < 0.5$.

\[
t_{\hat{\beta}_n} = \frac{n^{-1} \log^{-1} n}{} \kappa_\eta^{-1}(0.5) \sum_{t=1}^{n} [\Delta x_t] [x_{t-1}] \xrightarrow{P} \rho_{\rho} \rightarrow -\infty,
\]

and $t_{\hat{\beta}_n} = O_p(n^{0.5})$ if $d - d_0 = m + 0.5$.

\[
t_{\hat{\beta}_n} = \frac{n^{-1} \log^{-1} n}{} \kappa_\eta^{-1}(0.5) \sum_{t=1}^{n} [\Delta x_t] [x_{t-1}] \xrightarrow{P} \rho_{\rho} \rightarrow -\infty,
\]

and $t_{\hat{\beta}_n} = O_p(n^{0.5})$ if $d - d_0 \neq m + 0.5$. 

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