Leading soft theorem for multiple gravitinos

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ABSTRACT: We compute leading soft theorem for multiple gravitinos (and graviton) in an arbitrary theory of supergravity with an arbitrary number of finite energy particles with arbitrary mass and arbitrary spin by extending Sen’s approach [1] to fermionic symmetry. Our result is true for any compactification of type II and Heterotic superstring theory. Our result is valid at all orders in perturbation for four and higher spacetime dimensions.
1 Introduction

In a scattering event, a particle whose momentum in the centre of mass frame is much lower than other particles is called a soft particle and relation between the $S$-matrix of hard particles with and without the soft particles is known as the soft theorem. The soft theorems capture certain universal features of the theory. Study of soft theorems is an old subject [2–12]. However, in the last few years the interest on soft theorems has been renewed because of its connection to asymptotic symmetry [13–18]. It was argued that the flat space $S$-matrix in four dimensions should possess the symmetry of a asymptotically flat space. This symmetry is spontaneously broken and the graviton is the Goldstone boson of this spontaneously broken symmetry. Similarly, soft photon theorem can also be understood as a consequence of the large gauge transformations. These studies established the relation between seemingly different phenomena - Asymptotic symmetry, soft theorem and memory effect. In subsequent papers the study of asymptotic symmetry was extended to higher than four dimensions [17, 19, 20] but our understanding of asymptotic symmetries in an arbitrary dimension is far from being complete. The spacetime dimension independent treatment mostly relies on Feynman diagrammatic techniques. In this approach, one starts from a specific Lagrangian and then computes only a subclass of the Feynman diagrams which contribute to the (sub-)leading soft theorem(s). Soft photon and soft graviton theorem were computed in [21–60]. The new impetus in this direction is Sen’s work [1, 67]. This method relies on covariantization of 1PI effective action with respect to the soft field. So the result doesn’t depend on any particular Lagrangian or on asymptotic symmetry. This powerful method was used to compute sub-sub-leading soft graviton theorem [68] and also to compute multiple (sub-)leading soft graviton theorem [64]. It has been noted that the soft-photon theorem is universal at leading order [5, 6]. It was also found that soft-graviton theorem is universal not only in the leading order but also in the sub-leading order [69]. In a recent paper [70], the soft theorems has been investigated when two different type of massless particles are present.

In four and higher dimensions the theories of massless particles are severely constrained by Poincare invariance and Unitarity. Massless particles with spin $> 2$ cannot minimally couple; they can couple only through the field strength. So the only particles which possess gauge invariance and can have minimal coupling are spin $1, 3/2, 2$. We already have a complete understanding of soft photon and soft graviton theorem. However, we still don’t have many results about soft gluon and soft gravitino theorem. These computations involve subtlety in the sense that the leading soft factors don’t commute and their commutator is also leading order in soft momenta. At the level complexity, the soft gravitino theorem is more subtle than photon or graviton but significantly less subtle than that of the gluon. This is because even though the commutator of two soft factors is non-vanishing, the commutator of three soft factors vanishes in the case of gravitino but it doesn’t vanish for gluon. However, for a specific type of theories, soft gluon theorem can be conveniently computed using CHY formalism [72–75]. This advantage is not currently available for soft gravitino/photino. In this paper, we wish to derive the leading order soft theorem for gravitino in a general quantum field theory with local Supersymmetry in an arbitrary number of dimensions. Soft gravitino operator is a fermionic soft operator. Though a lot is known about bosonic soft theorems, the available literature for fermionic soft theorems is significantly little. Single soft photino theorem was computed in [76].
Amplitudes with one and two soft gravitinos for four-dimensional supergravity theories were computed for a particular model in [77–79]. The result for single soft gravitino in $D = 4$ can also be obtained from asymptotic symmetry [80, 81]. We generalize the result to the case with an arbitrary number of soft gravitinos. In our work, we follow Sen’s covariantization approach [1, 67, 68]. The advantage of this method is that it is valid for arbitrary theories, to all orders in perturbation theory and in arbitrary dimensions, as long as there is no infrared divergence. In this paper, we mostly follow the notation and conventions of [68]. We have summarized our notation and convention in section A. We find that for multiple gravitinos the leading order result is universal.

An important aspect of quantum theories with massless particles is IR divergence. In $D = 4$ loop diagrams suffer from IR divergences. In QED, there is a procedure to write IR finite $S$ matrix element [82]. This procedure has also been understood from the perspective of asymptotic symmetries [83–85]. There has been some recent progress for quantum gravity [86, 87]. In our work, we don’t have any new result regarding the IR divergence of supergravity theories. Our result is valid for any theory of supergravity in $D \geq 4$.

Background independence of String field theory implies that String field theory in presence of a soft field is obtained simply by deforming the world-sheet CFT by a marginal super-conformal operator which corresponds to that field. Recently Sen has proved background independence in superstring field theory [88]. So our analysis is also valid for any supersymmetric compactification of superstring theory.

### 1.1 Main result

Our main result is equation (5.1) where we have written the soft factor for arbitrary number of external soft gravitino. Consider an amplitude $\Gamma_{M+N}(\{p_i\}, \{k_u\})$ for $M$ soft gravitinos and $N$ any other hard particles. It is related to the amplitude of $N$ hard particles $\Gamma_N(\{p_i\})$ in the following way

$$
\Gamma_{M+N}(\{p_i\}, \{k_u\}) = \left[ \prod_{i=1}^{M} S_{u_i} + \sum_{A=1}^{\lfloor M/2 \rfloor} \prod_{i=1}^{A} M_{u_i,v_i} \prod_{j=1}^{M-2A} S_{r_j} \right] \Gamma_N(\{p_i\}) + \mathcal{O}(1/k^{M-1}) \quad (1.1)
$$

Various terms in this expression are explained below

1. $p_i$ are the momenta of the hard particles, $k_u$ are momenta of the soft particles

2. $S_u$ is the soft factor for single soft gravitino. It is given by

$$
S_u = \kappa \sum_{i=1}^{N} \left( \epsilon_{(u)}^\alpha \frac{p_i^\mu}{p_i \cdot k_u} \bar{Q}_\alpha \right) \quad (1.2)
$$

Here $\kappa$ is the gravitational coupling constant. $\epsilon_{(u)}^\alpha$ is the polarization of the gravitino; it has a Lorentz vector index & a majorana spinor index and it is grassmann odd. The

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[1] In $D \leq 3$ there is no graviton and gravitino.
gravitino polarization (in the harmonic gauge) satisfies the transversality condition and gamma traceless condition.

\[(k_u)\mu \epsilon^{(u)}_\mu = 0 \text{ , } \quad \gamma^{\alpha\beta}_\mu \epsilon^{(u)}_\mu = 0\] (1.3)

\(Q_\alpha\) are the supersymmetry charges/generators. Since \(S_u\) is a product of two grassmann odd quantities, it is grassmann even. Two single soft factors do not commute with each other.

\[S_u S_v \neq S_v S_u\] (1.4)

3. Whenever there is more than one gravitino, they can combine pairwise to give a soft graviton which in-turn couples to the hard particles. \(M_{uv}\) encodes these type of contributions.

The explicit expression for \(M_{uv}\) is given by

\[M_{uv} = \kappa^2 \sum_{i=1}^{N} \frac{1}{2} \frac{\epsilon^{(u)}_\mu \theta \epsilon^{(v)}_\nu}{p_i \cdot (k_u + k_v)} \left( \frac{p_i^\mu p_i^\nu}{p_i \cdot k_v} + \frac{1}{2} \eta^{\mu\nu} p_i \cdot (k_u - k_v) + \frac{(k_v p_i^\mu - k_u p_i^\mu)}{k_u \cdot k_v} \right)\] (1.5)

\(M_{uv}\) is neither symmetric nor anti-symmetric in its (particle-)indices

\[M_{uv} \neq \pm M_{vu}\] (1.6)

4. Since the single soft factors for gravitino do not commute, the final expression for arbitrary soft gravitino depends on the choice of ordering. In section 5.1, we demonstrate that any order can be obtained from any other ordering. However, our expression is not manifestly symmetric on the various soft gravitinos.

5. The first term is the product of single-soft gravitino factors. The single-soft factors appear in a particular order and the explicit form of second piece changes depending on the ordering of soft factors because two soft factors don’t commute.

6. In the second term, \([M/2]\) denotes the greatest integer which is less than or equal to \(M/2\). \(A\) counts the number of pairs of gravitinos giving a soft graviton. The subscripts \(\{r_j, u_i, v_i\}\) can take values from \(1, ..., M\) and \(v_i > u_i\) and \(r_j\)’s are also ordered with the largest \(r_j\) appearing on the right.

7. The supersymmetry algebra may contain central charges. In this case, the gravitino supermultiplet contains graviphoton. In presence of central charge, there are additional contributions to \(M_{uv}\) due to graviphoton couplings. In presence of central charge the expression of \(M_{uv}\) is modified as follows

\[M_{uv} \rightarrow \tilde{M}_{uv} = M_{uv} + \kappa^2 \sum_{i=1}^{N} \frac{e_i^{(u)} \theta \epsilon^{(v)}_i}{p_i \cdot (k_u + k_v)} \left( \frac{p_i^\mu p_i^\nu}{p_i \cdot k_v} + \frac{1}{2} \eta^{\mu\nu} p_i \cdot (k_u - k_v) + \frac{(k_v p_i^\mu - k_u p_i^\mu)}{k_u \cdot k_v} \right)\] (1.7)

e_i is the charge of the \(i^{th}\) external state under symmetry generated by graviphoton. \(Z\) is an element of the Clifford algebra such that \(Z_{\alpha\beta}\) commutes with all other element of the Clifford algebra and

\[Z_{\alpha\beta} = Z_{\beta\alpha}\] (1.8)
We checked the gauge invariance of (1.1).

In presence of soft graviton, we have to multiply the above expression by soft factors of the graviton. For \( M_1 \) soft gravitino and \( M_2 \) soft graviton equation (1.1) is modified as given below

\[
\Gamma_{N+M_1+M_2}(\{p_i\}, \{k_r\}) = \left[ \prod_{j=1}^{M_2} \tilde{S}_{u_j} \right] \left[ \prod_{i=1}^{M_1} \mathcal{S}_{u_i} \right] + \sum_{A=1}^{\lfloor M_1/2 \rfloor} \prod_{i=1}^{A} \mathcal{M}_{u_i v_i} \prod_{j=1}^{M_1-2A} \mathcal{S}_{r_j} \left\{ \right\} \Gamma_{N}(\{p_i\})
\] (1.9)

\( \tilde{S}_u \) is the leading soft factor for graviton. It given by

\[
\tilde{S}_u = \kappa \sum_{i=1}^{N} \left( \frac{\zeta^{(u)}_{\mu\nu} p^\mu_i p^\nu_i}{p_i \cdot k_u} \right)
\] (1.10)

here \( \zeta_{\mu\nu} \) is the polarization of soft graviton.

The rest of the paper is as follows. In section 2, we derive the vertices from the 1PI effective action. Then we start with the simplest case of single soft gravitino in section 3. We show the gauge invariance of the expression. Then we compute the expression for the two soft gravitinos in section 4. The coupling of the gravitinos to the graviton is essential to show the gauge invariance of the expression for two soft gravitinos. Then we write down the expression for multiple soft gravitinos in section 5. We don’t present any derivation of this result. We check the gauge invariance of this expression. Our conjectured result is based on the computation in section 3, section 4 and appendix B. In section 6, we derive the contribution to the soft theorem, when the supersymmetry algebra contains central charges. Then we present our brief conclusion and potential future directions.

2 Set-up

We are interested in deriving the leading order soft theorem for gravitino in an arbitrary theory of supergravity. Our starting point is a globally super-symmetric Lagrangian which is invariant under some number of Majorana supersymmetry\(^2\). So the usual (dimension-dependent) restriction for the existence of a globally super-symmetric Lagrangian applies. We promote the global symmetry to a local one by replacing all the derivatives with covariant derivatives. At the leading order, only the minimal coupling of gravitino with matter fields contribute. We don’t assume anything about the multiplet in which matter fields are sitting. Our analysis is valid for the matter in any supersymmetry multiplet.

Let \( \Phi_m \) be any quantum field which transforms under some reducible representation of the Poincare group, supersymmetry and the internal symmetry group(s). The transform of the fields under the global supersymmetry is given by

\[
\Phi_m \longrightarrow \Phi_m + (\theta^a Q_\alpha)_m^a \Phi_n
\] (2.1)

\( Q_\alpha \) are supersymmetry generators. They satisfy the following algebra

\[
\left\{ Q_\alpha, Q_\beta \right\} = -\frac{1}{2} \gamma^\mu_{\alpha\beta} P_\mu
\] (2.2)

\(^2\)From Coleman-Mandula theorem and HLS theorem, the maximum number of super-charge is 32.
Here $P_\mu$ is the momentum generator. The indices $\alpha, \beta$ are the collection of all possible spinor indices, not the indices for the minimal spinor (of that dimension). So, in a theory of more than one supersymmetry, $Q_{\alpha}$ are the collection of all the super-charges. Gamma matrices are in Majorana representation and symmetric in the spinor indices.

Now we will evaluate the vertex that describes the coupling of a soft gravitino to any hard particle. We start from the quadratic term of the 1PI effective action

$$S = \frac{1}{2} \int \frac{d^D p_1}{(2\pi)^D} \frac{d^D p_2}{(2\pi)^D} \Phi_m(p_1) K^{mn}(p_2) \Phi_n(p_2) (2\pi)^D \delta^{(D)}(p_1 + p_2)$$

The kinetic term is invariant under global supersymmetry transformation. This implies

$$K^{n_1 m_3} (Q_{\alpha})_{m_3} m_2 + K^{m_3 m_2} (Q_{\alpha})_{m_3} n_1 = 0$$

Later we will need the propagator. Let’s assume it has the following form:

$$\Xi(q) \left( q^2 + M^2 \right)^{-1}$$

The local-supersymmetry transformation of the vierbein $e_{\alpha}^a \mu$ and the gravitino $\Psi_{\mu \alpha}$ are given by

$$\delta e_{\alpha}^a \mu = \frac{1}{2} \theta \gamma^a \Psi_{\mu}$$

$$\delta \Psi_{\mu \alpha} = D_{\mu} \theta_{\alpha} + \frac{1}{4} \omega_{\mu ab} \gamma^{ab} \theta_{\alpha}$$

Here $\theta$ is the local supersymmetry parameter. Now consider a small fluctuation

$$E_{\alpha}^a \mu = \delta e_{\alpha}^a \mu - \kappa \zeta_a^\mu e^{ik \cdot x}$$

$$\Psi_{\mu \alpha} = \kappa \epsilon_{\mu \alpha} e^{ik \cdot x}$$

Here $\kappa$ is the gravitational coupling constant. In linear order of fluctuation we get the following expression for the super-covariant derivative

$$D_{\alpha} = \partial_{\alpha} - \kappa \zeta_a^\mu \partial_{\mu} - i \kappa \epsilon_{\alpha}^a Q_{\alpha} - i \kappa \frac{1}{2} \omega_a^{cd} J_{cd}$$
2.2 Vertex of one soft gravitino to matter

The coupling of one soft gravitino to matter fields at linear order can be found by covariantizing the derivative in (2.3). So due the interaction with gravitino, the momenta of hard particle changes by $\delta q = -\kappa \Psi^\mu Q_\alpha$.

The coupling of gravitino with the matter field can then be found just from the quadratic part of the 1PI effective action by making the following changes in (2.3) [68]:

- $\delta (D)(p_1 + p_2)$ gets replaced by $\delta (D)(p_1 + p_2 + k)$ where $k$ is the momenta of soft gravitino.
- The change in kinetic operator $K^{mn}$ due to shift in momenta has to be substituted.

So we get

$$S^{(L)} = \frac{1}{2} \int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \Phi_m(p_1) \left[ -\frac{\partial K(p_2)}{\partial p_{2\mu}} \kappa \epsilon^\mu Q_\alpha \Phi_n(p_2) (2\pi)^D \delta(D)(p_1 + p_2 + k) \right]$$

(2.13)

So the vertex is given by

$$-i \kappa \frac{\partial K(p_1)}{\partial p_{1\mu}} \epsilon^\mu Q_\alpha$$

(2.14)

2.3 External particles

Since we compute only $S$-matrix elements, all the external particles satisfy on-shell and transversality condition. The external particle of polarization $\varepsilon_{i,m}$ and momenta $p_i$ satisfies the conditions:

$$\varepsilon_{i,m} K^{mn}(q) = 0$$

(2.15a)

$$p_i^2 + M_i^2 = 0$$

(2.15b)

2.4 Coupling of two soft gravitinos to a soft graviton

When we have more than one soft gravitino, we would need to consider the minimal coupling of gravitino with graviton. To derive this vertex, we followed [68]. The graviton coupling to any matter field can be written as:

$$S = \frac{1}{2} \int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} (2\pi)^D \delta(D)(k_1 + k_2 + p)$$

$$\Phi_m(k_1) \left[ -\zeta_{\mu\nu} k_2^\nu \frac{\partial}{\partial k_2\mu} K^{mn}(k_2) + \frac{1}{2} (p_\delta \delta_\sigma - p_\sigma \delta_\delta) \frac{\partial}{\partial k_2\mu} K^{mp}(k_2) (J^{ab})_p n \right] \Phi_n(k_2)$$

(2.16)

where $\zeta_{\mu\nu}$ is the graviton polarization.

The kinetic term for the gravitino, in the harmonic gauge, is given by

$$K^{\mu\alpha\nu\beta}(p) = (p_\mu \gamma^\alpha)_{\alpha\beta} \eta^{\mu\nu}$$

(2.17)



The angular momentum generator is

$$(J^{ab})_{\mu,\alpha}^{\nu,\beta} = (J^a)_\mu^{\nu} \delta_\alpha^\beta + (J^b)_\alpha^{\beta} \delta_\mu^{\nu}$$

(2.18)
where $J_{V}^{ab}$ and $J_{S}^{ab}$ are angular momentum generator in vector and spinor representations respectively.

\begin{align}
(J_{V}^{ab})_{\mu}^{\nu} &= \delta^{a}_{\mu} \eta^{\nu b} - \delta^{b}_{\mu} \eta^{a\nu} \\
(J_{S}^{ab})_{\alpha}^{\beta} &= -\frac{1}{2} (\gamma^{ab})_{\alpha}^{\beta} \\
\gamma^{ab} &\equiv \frac{1}{2} (\gamma^{a} \gamma^{b} - \gamma^{b} \gamma^{a})
\end{align}

Our gamma matrix convention is given in (A.2). Our convention is that all the particles are incoming; the gravitino has momentum $k_1$ and $k_2$ and the graviton has momenta $p$. The momentum conservation implies

\begin{equation}
p + k_1 + k_2 = 0
\end{equation}

So the vertex \((V^{\mu\nu;\mu_2})_{\alpha\beta}\) is given by

\begin{equation}
-i \kappa \left[ k_2^2 (\gamma_{\mu_1})^{\alpha\beta} \eta^{\mu\nu} + \frac{1}{4} (p_d \delta_{\mu_2}^{\nu} - p_c \delta_{d}^{\mu_2}) (\gamma_{\nu_1} \gamma_{\nu_2})^{\alpha\beta} \eta^{\mu\nu} + (p^{\mu} \eta^{\mu_2} - p^\nu \eta^{\mu_2}) (\gamma_{\mu_1}^{\alpha\beta}) \right]
\end{equation}

### 2.5 Note on Feynman diagrams

We use a red double-arrowed line for soft-gravitino, a blue wavy line to denote soft gravitons, a violet wavy line for graviphoton, Cyan double arrowed line for hard fermionic particles (including hard gravitinos) and black line to denote hard bosonic particles.

![Figure 1: Conventions for Feynman diagrams](image)

### 3 Single soft gravitino

In this section, we compute the leading order contribution to soft gravitino theorem due to one soft gravitino. The only diagram that contributes to this process is depicted in figure [2].

The expression for the propagator is given in equation (2.5) In this diagram, the propagator carries momenta $p_i + k$ and $M_i$ is the mass of the $i$-th particle. Let’s denote the corresponding propagator by $\Xi_{m_1 n_1}(p_i + k)$. The contribution to figure 2 is given by:

\begin{equation}
\Gamma_{N+1}^{m_1...m_N}(\{p_i\}, k) = \left[ i \kappa \sum_{i=1}^{N} \left( \frac{\partial K}{\partial p_{\mu i}} \right) \epsilon_{\mu}^{a} Q_{a} \right] \Xi_{m_1 n_1} \left( \frac{2 M_{i}}{p_i \cdot k} \right) \Gamma_{N}^{m_{i+1}...m_{i+1}}(\{p_i\})(3.1)
\end{equation}

4We use a double arrowed line for Majorana particles because they are their own anti-particle; they only have $Z_2$ charge.
where in the second step, we have used the on-shell condition (2.15b) for external hard particle and the fact that gravitino is soft. Now we will use (2.7) and (2.8) to simplify the expression

\[
\frac{\partial \mathcal{K}(-p_i) \epsilon_{\mu}^{\alpha} Q_{\alpha}}{\partial p_{i\mu}} z_{m_i \tilde{n}_i} = \epsilon_{\mu}^{\alpha} \left( \frac{\partial \mathcal{K}(-p_i) Q_{\alpha}}{\partial p_{i\mu}} z \right)^{m_i} \tilde{n}_i
\]

\[
= -\epsilon_{\mu}^{\alpha} \left( \frac{\partial \mathcal{K}(-p_i) Q_{\alpha}}{\partial p_{i\mu}} \right)^{m_i} \tilde{n}_i
\]

\[
= -\epsilon_{\mu}^{\alpha} \left( \mathcal{K}(-p_i) \frac{\partial z}{\partial p_{i\mu}} Q_{\alpha} + 2i p_{i\mu} Q_{\alpha} \right)^{m_i} \tilde{n}_i
\]

(3.2)

From first step to second step we have used (2.7) and from second step to third step we have used (2.8). Now the first term drops out because of the on-shell condition (2.15a). Hence we obtain

\[
\Gamma_{N+1}^{m_1...m_N}(\{p_i\}, k) = \left[ \kappa \sum_{i=1}^{N} \left( \frac{p_{i\mu} \epsilon_{\mu}^{\alpha}}{p_i \cdot k} Q_{\alpha} \right)^{m_i} \tilde{n}_i \right] \Gamma_{N}^{m_1...m_{i-1}n_im_{i+1}...m_N}(\{p_i\})
\]

(3.3)

**Soft operator** We define the soft operator \( S_u \) as

\[
S_u = \kappa \sum_{i=1}^{N} \left( \frac{p_{i\mu} \epsilon_{\mu}^{[u] \alpha}}{p_i \cdot k_u} Q_{\alpha} \right)
\]

(3.4)

where \( u \) labels the soft gravitino. So the above result can be re-written as:

\[
\Gamma_{N+1}^{m_1...m_N}(\{p_i\}, k) = \left[ S^{m_i \tilde{n}_i} \right] \Gamma_{N}^{m_1...m_{i-1}n_im_{i+1}...m_N}(\{p_i\})
\]

(3.5)

### 3.1 Gauge invariance

As a consistency check, we check the gauge invariance of equation (3.3). We put pure gauge polarization for the gravitino

\[
\epsilon_{\alpha\mu} = k_{\mu} \theta_{\alpha}
\]

(3.6)

Here \( \theta_{\alpha} \) is a Majorana spinor. For pure gauge gravitino the amplitude should vanish. From (3.3), we obtain

\[
\theta_{\alpha} \sum_{i=1}^{N} (Q_{\alpha})^{m_i \tilde{n}_i} \Gamma_{N}^{m_1...m_{i-1}n_im_{i+1}...m_N}(p_i) = 0
\]

(3.7)

This is the ward-identity for the global super-symmetry.
3.2 Effect of Infrared divergence

Now we will briefly discuss infrared divergences in supergravity. Our discussion is based on section 6 of [68] and on section 2.3 of [64]. We mention only the key points. We are using 1PI effective action and we assumed that the 1PI vertices do not give any inverse soft momenta. Here we will check that assumption. We wish to check the relevance of IR divergences due to loop momenta for our computation.

Consider the Feynman diagram in figure 3, if the external momenta are finite, then we can see that the amplitude does not have IR divergence for \( D \geq 4 \). This can be concluded by naive power-counting. We have three powers of \( \ell \) in the denominator, one from each of the propagators with momenta \( p_i + \ell \), \( p_j - \ell \) and \( \ell \). The key difference with respect to graviton case is that the propagator with momenta \( \ell \) just gives one power of \( \ell \) because it is a fermionic particle. So in this case, we will have \( D \) powers of \( \ell \) in numerator due to loop integral and hence the amplitude goes like \( \ell^{D-3} \) for small loop momentum \( \ell \). So the diagram is free of IR divergence in \( D \geq 4 \).

But when the momenta \( k \to 0 \), then the propagator carrying momentum \( p_i + k + \ell \) gives another power of \( \ell \) and makes the result logarithmically divergent in \( D = 4 \). But the leading soft gravitino goes like \( k^{-1} \). So in \( D \geq 4 \), IR divergences due to the loop doesn’t change the leading order result.

However, in supergravity, there are other diagrams which diverge more severely in the infrared. Consider the Feynman diagram in figure 4. In this case, the internal massless particle is graviton (it can also be photon/graviphoton). This diagram is logarithmic divergent in \( D = 5 \). So the leading order answer still holds for \( D \geq 5 \).

In \( D = 4 \), the diagram in figure 4 can potentially contribute to the same order as the leading order answer and hence it needs more careful analysis. For leading soft graviton and soft photon theorem, the amplitude with and without the soft particle has exactly same divergence and hence the leading soft factor is not affected by IR divergence [6, 24, 39]. In this case, an analogous statement holds.

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We are thankful to Ashoke Sen for discussion on this section and correcting one mistake in an earlier version of the draft.
4 Two soft gravitinos

Now we will consider the amplitude with $N$ hard particles and 2 soft gravitinos. There are essentially four different types of Feynman diagrams which contribute this process

1. The class of diagrams where the two soft gravitinos are attached to different legs (for example, figure 5). These diagrams are easy to evaluate. The computation for these type of diagrams is essentially the same as single soft gravitino. The contribution from figure 5 is given by

$$
\kappa^2 \sum_{i=1}^{N} \frac{\epsilon^{(1);\alpha}_{\mu} P_{\mu}^{\alpha}}{p_i \cdot k_1} Q_{\alpha} \sum_{j=1; j \neq i}^{N} \frac{\epsilon^{(2);\beta}_{\nu} P_{\nu}^{\beta}}{p_j \cdot k_2} Q_{\beta} \Gamma(\{p_i\})
$$

2. The class of diagrams where the soft gravitinos are attached to the same leg. There are mainly three types of such diagrams - (figure 6, figure 7, figure 8). Figure 6, Figure 7 denote the diagrams where the soft gravitino directly attaches the same hard-particles. These two diagrams differ only in order of attaching to the hard particle. Figure 8 captures the process when the soft gravitinos combine to give a soft graviton and then the soft graviton attaches to the hard particles.

Now will evaluate these diagrams

**Evaluation of figure 6**  

The contribution from the Feynman diagram in figure 6 is given by

$$
\Gamma^{(1)}_{N+2} = \kappa^2 \sum_{i=1}^{N} \frac{\partial K^{\mu \rho}(-p_i) \left[ \epsilon^{(1);\alpha}_{\mu} Q_{\alpha} \Xi(-p_i - k_1) \right]_{pq} \partial K^{\nu \rho}(-p_i - k_1) \left[ \epsilon^{(2);\beta}_{\nu} Q_{\beta} \Xi(-p_i - k_1 - k_2) \right]_{rs} \Gamma_N(\{p_i\})}{(2p_i \cdot k_1)(2p_i \cdot (k_1 + k_2))}
$$

Using (2.7) and (2.8) we can simplify this expression and we get

$$
\kappa^2 \sum_{i=1}^{N} \frac{\epsilon^{(1);\alpha}_{\mu} P_{\mu}^{\alpha}}{p_i \cdot k_1} \frac{\epsilon^{(2);\beta}_{\nu} P_{\nu}^{\beta}}{p_i \cdot (k_1 + k_2)} Q_{\alpha} Q_{\beta} \Gamma_N(\{p_i\})
$$

(4.3)
Figure 5: Feynman diagram for double soft gravitino - I

Figure 6: Feynman diagram for double soft gravitino - II

Figure 7: Feynman diagram for double soft gravitino - III

Evaluation of figure 7 The contribution due to figure 7 can obtained from equation (4.2) by interchanging $1 \leftrightarrow 2$

\[
\Gamma^{(2)}_{N+2}(\{p_i\},k_1,k_2) = \kappa^2 \sum_{i=1}^{N} \frac{\epsilon^{(2)\beta}_{\mu_2} p_i^\mu}{p_i \cdot (k_1 + k_2)} \frac{\epsilon^{(1)\alpha}_{\mu_1} p_i^\mu}{p_i \cdot k_2} Q_\alpha Q_\beta \Gamma_N(\{p_i\})
\]

\[
= \kappa^2 \sum_{i=1}^{N} \left[ \eta_{\mu_1\nu_1} \eta_{\mu_2\nu_2} + \frac{\eta_{\mu_1\nu_2} \eta_{\mu_2\nu_1}}{2(k_1 \cdot k_2)} \right] \frac{\epsilon^{(1)\alpha}_{\mu_1} p_i^\mu}{p_i \cdot (k_1 + k_2)} \frac{\epsilon^{(2)\beta}_{\mu_2} p_i^\mu}{p_i \cdot k_2} Q_\alpha Q_\beta + \frac{1}{2}(p_i^\mu)_{\alpha\beta} \Gamma_N(\{p_i\})
\]

Evaluation of figure 8 Now we would like to evaluate the figure 8

\[
- \left[ \epsilon^{(1)\alpha}_{\mu_1} (\mathcal{Y}_{\mu_2\mu_3\mu_4})_{\alpha\beta} \epsilon^{(2)\beta}_{\mu_2} \right] \left[ \eta_{\mu_1\nu_1} \eta_{\mu_2\nu_2} + \frac{\eta_{\mu_1\nu_2} \eta_{\mu_2\nu_1}}{2(k_1 \cdot k_2)} \right] \frac{1}{2} \left[ \frac{-\kappa p_i^\mu}{\partial p_{\nu_2} \cdot 2 p_i \cdot (k_1 + k_2)} \right]
\]
here we have defined soft gravitinos. The total contribution can be written as
\[
\Gamma = \left[\epsilon_\mu^{(1)\alpha}(\gamma^\mu\gamma_1\mu_2)\eta_{\mu\nu}\eta_{\nu2}\eta_{\nu1}\eta_{\nu2} - \frac{2}{D-2}\eta_{\mu1\nu2}\eta_{\nu1\nu2}\right]\left[2p^\mu_{1}p^\nu_{2}\right]
\]
where we have not written the factor $[4(k_1 \cdot k_2)(p_i \cdot (k_1 + k_2))^{-1}$ in the second step. The last piece of the graviton propagator doesn’t contribute due to gamma-traceless condition of gravitino and other two pieces are equal
\[
\left[\epsilon_\mu^{(1)\alpha}(\gamma^\mu\gamma_1\mu_2)\eta_{\mu\nu}\eta_{\nu2}\right]\left[2p^\mu_{1}p^\nu_{2}\right] = 2\left[\epsilon_\mu^{(1)\alpha}(\gamma^\mu\gamma_1\mu_2)\eta_{\mu\nu}\eta_{\nu2}\right]p_i p_i p_{12}
\]
Using the explicit expression for $(\gamma^\mu\gamma_1\mu_2)_{\alpha\beta}$ from (2.21) and putting back the propagator factor, we get
\[
\Gamma^{(3)}_{N+2}(\{p_i\}, k_1, k_2) = \kappa^2 \sum_{i=1}^{N} \epsilon_\mu^{(1)\alpha}(\gamma^\mu\gamma_1\mu_2)_{\alpha\beta} \left[-\eta^\mu_{\nu}p_i \cdot k_2 - \frac{1}{2}\eta^\mu_{\nu}(k_1 + k_2)d_{\nu\epsilon\delta} + (k_2^\mu p_{\nu}^\epsilon - k_{\nu}^\mu p_i^\epsilon)\right] \epsilon_\nu^{(2)\beta}
\]
where we have not written the factor $[4(k_1 \cdot k_2)(p_i \cdot (k_1 + k_2))^{-1}$ in the second step. The last piece of the graviton propagator doesn’t contribute due to gamma-traceless condition of gravitino and other two pieces are equal
\[
\left[\epsilon_\mu^{(1)\alpha}(\gamma^\mu\gamma_1\mu_2)\eta_{\mu\nu}\eta_{\nu2}\right]\left[2p^\mu_{1}p^\nu_{2}\right] = 2\left[\epsilon_\mu^{(1)\alpha}(\gamma^\mu\gamma_1\mu_2)\eta_{\mu\nu}\eta_{\nu2}\right]p_i p_i p_{12}
\]
Using the explicit expression for $(\gamma^\mu\gamma_1\mu_2)_{\alpha\beta}$ from (2.21) and putting back the propagator factor, we get
\[
\Gamma^{(3)}_{N+2} = \kappa^2 \sum_{i=1}^{N} \frac{C_{12}(p_i)}{(p_i \cdot (k_2 + k_1))} \Gamma_N(\{p_i\})
\]
After simplifying the second term and using gamma-traceless condition for gravitino, we get
\[
\Gamma^{(3)}_{N+2} = \kappa^2 \sum_{i=1}^{N} \frac{C_{12}(p_i)}{(p_i \cdot (k_2 + k_1))} \Gamma_N(\{p_i\})
\]
here we have defined
\[
C_{uv}(p_i) = \frac{1}{2} \epsilon_\mu^{(u)} \eta_\nu \epsilon_\nu^{(v)} \left[\frac{1}{2}\eta^\mu_{\nu}p_i \cdot (k_u - k_v) + (k_u^\mu p_i^\nu - k_v^\nu p_i^\mu)\right]
\]
Note that
\[
\epsilon_\mu^{(u)} \eta_\nu \epsilon_\nu^{(v)} = -\epsilon_\nu^{(u)} \eta_\mu \epsilon_\mu^{(v)} \quad \Rightarrow \quad C_{uv}(p_i) = C_{vu}(p_i)
\]
Now we add the contributions from (4.1), (4.3), (4.4) and (4.8) to get the full answer for two soft gravitinos. The total contribution can be written as
\[
\Gamma_{N+2}(\{p_i\}, k_1, k_2) = \left[S_i S_2 + M_{12}\right] \Gamma_N(\{p_i\})
\]
we have already defined $S_u$ in (3.4). $M_{uv}$ is defined as follows
\[
M_{uv} = \kappa^2 \sum_{i=1}^{N} \frac{1}{2} \frac{C_{uv}(p_i)}{(p_i \cdot (k_u + k_v))} \left[\frac{1}{2}\eta^\mu_{\nu}p_i \cdot (k_u - k_v) + (k_u^\mu p_i^\nu - k_v^\nu p_i^\mu)\right]
\]
Some properties of $S_u$ and $M_{uv}$
• Two soft operators don’t commute

\[ [S_u, S_v] = -\frac{\kappa^2}{2} \sum_{i=1}^{N} \left[ \epsilon^{(u);\alpha}_{\mu} \epsilon^{(v);\beta}_{\nu} \frac{p_i^\mu}{p_i \cdot k_u} \frac{p_i^\nu}{p_i \cdot k_v} \right] \]  

(4.13)

• While writing the result for two soft gravitinos, we could have chosen the other ordering of soft factors but both results should match i.e.

\[ S_u S_v + \mathcal{M}_{uv} = S_v S_u + \mathcal{M}_{vu} \]  

(4.14)

Above equation can be explicitly verified by noting that:

\[ \mathcal{M}_{vu} - \mathcal{M}_{uv} = \kappa^2 \sum_{i=1}^{N} \epsilon^{(u)}_{\mu} \cdot p_i \epsilon^{(v)}_{\nu} \cdot p_i \left( -\frac{1}{2} \eta_{\mu\nu} \right) \]  

(4.15)

We already computed \( S_u S_v - S_v S_u \) in (4.13). Hence (4.14) is satisfied.

• Three soft operators satisfy Jacobi identity.

\[ [S_u, [S_v, S_w]] + [S_v, [S_w, S_u]] + [S_w, [S_u, S_v]] = 0 \]  

(4.16)

In this particular case, each term in the above equation are individually zero.

\[ [S_u, [S_v, S_w]] = 0 \]  

(4.17)

This is not true for soft gluon operator(s). Though (4.16) is true for soft gluon operator, (4.17) doesn’t hold for soft gluon operator. This fact makes the computation of the soft factors for multiple soft gluon even more cumbersome.

• Some more properties of \( \mathcal{M}_{uv} \) are listed below

\[ \mathcal{M}_{uv} \neq \pm \mathcal{M}_{vu} \]  

(4.18a)

\[ \mathcal{M}_{u_1 v_1} \cdot \mathcal{M}_{u_2 v_2} = \mathcal{M}_{u_2 v_2} \cdot \mathcal{M}_{u_1 v_1} \]  

(4.18b)

\[ S_w \cdot \mathcal{M}_{uv} = \mathcal{M}_{uv} \cdot S_w \]  

(4.18c)

4.1 Gauge invariance

As a consistency check, we check the gauge invariance of the result obtained in (4.11). The right-hand side should vanish when one puts any of the gravitinos as a pure gauge. Here we will put \( \epsilon^{(2)} \) as a pure gauge and check if RHS vanishes or not.

\[ \epsilon^{(2)}_{\mu \alpha} = k_{2\mu} \theta_2^\alpha \]  

(4.19)

So for pure gauge, the first term in (4.11) vanishes because \( Q^\beta \) directly hits \( \Gamma_N(\{ p_i \}) \) and gives zero due to supersymmetry ward-identity (3.7). The second piece gives:

\[ \mathcal{M}_{12}(\epsilon^{(1)}_{1 \alpha}, k_{2}^\beta \theta_2^\alpha) = \kappa^2 \sum_{i=1}^{N} \left( 1 - \frac{\epsilon^{(1)}_{\mu} \eta_{\mu \nu}}{2 p_i \cdot (k_1 + k_2)} \right) \left[ \frac{k_{2\mu}}{2 k_1 \cdot k_2} p_i \cdot k_2 + \frac{k_{2\mu}}{2 k_1 \cdot k_2} p_i \cdot (k_1 - k_2) + \frac{1}{2 k_1 \cdot k_2} (k_{2\mu} p_i \cdot k_2 - k_{2\mu} p_i \cdot k_1) \right] \]

\[ = \kappa^2 \sum_{i=1}^{N} \left( 1 - \frac{\epsilon^{(1)}_{\mu} \eta_{\mu \nu}}{2 p_i \cdot (k_1 + k_2)} \right) \left[ \frac{k_{2\mu}}{2 k_1 \cdot k_2} \right] = 0 \]  

(4.20)
where in the last step we have used momentum conservation \( \sum_{i=1}^{N} p_i = 0 \).

One should be able to show the gauge invariance when \( \epsilon^{(1)} \) is pure gauge. But in this case, first term in (4.11) does not give ward-identity directly and also \( M_{12} \) term does not vanish. But one can check that the sum is gauge invariant. Alternative we can use (4.14) to express the amplitude in the other ordering of soft factors

\[
\Gamma_{N+2}(\{p_i\}, k_1, k_2) = \left[ S_2 S_1 + M_{21} \right] \Gamma_N(\{p_i\})
\]

(4.21)

In this representation, it is obvious that the RHS vanishes for pure-gauge \( \epsilon^{(1)} \). In general,

\[
M_{uv}(\epsilon^\mu_\alpha, k^\mu, \theta_\alpha) = 0
\]

(4.22a)

\[
M_{uv}(k^\mu, \theta_\alpha, \epsilon^\mu_\alpha) \neq 0
\]

(4.22b)

In order to express the result for an arbitrary number of soft gravitinos, we always choose an ordering amongst the external gravitinos. The gauge invariance would be manifest if when one puts the last gravitino as pure gauge. Using relations of the form (4.14), one can check the gauge invariance for pure gauge configuration of any soft particle.

**Symmetrized form the amplitude**  The expression for the soft factor in (4.11) is not manifestly symmetric on the gravitinos. That form was useful to prove gauge invariance. Now we use (4.16) and (4.17) to write the answer in a form which is manifestly symmetric on the gravitinos

\[
\Gamma_{N+2}(\{p_i\}, k_1, k_2)
\]

\[
= \frac{1}{2} \left[ S_1 S_2 + S_2 S_1 + M_{12} + M_{21} \right] \Gamma_N(\{p_i\})
\]

(4.23)

\[
= \left[ \frac{1}{2} (S_1 S_2 + S_2 S_1) + \kappa^2 \sum_{i=1}^{N} \frac{1}{p_i \cdot (k_1 + k_2)} \left[ C_{12}(p_i) + \frac{1}{4} (p_i \cdot \epsilon^{(1)})(\epsilon^{(2)} \cdot p_i) \frac{p_i \cdot (k_1 - k_2)}{(p_i \cdot k_2)(p_i \cdot k_1)} \right] \right] \Gamma_N(\{p_i\})
\]

Apart from the last term, other terms are clearly symmetric under the exchange \( 1 \leftrightarrow 2 \).

## 5 Arbitrary number of soft gravitinos

Now we consider the amplitude with an arbitrary number of soft gravitations. In this case, the following type of diagrams can contribute:

- When some soft gravitinos attach on one leg and some on separate legs, but none of them form pairs to give soft graviton as shown in figure 9.
- When some soft gravitinos attach on one leg and some on separate legs and some form pairs to give soft graviton as shown in figure 10.
- All gravitino attach on the same leg, but none of them forming pairs to give soft graviton as shown in figure 11.
- Some gravitinos form pairs and give a soft graviton while some attach directly to external leg as shown in figure 12.
We performed explicit computation for three soft gravitino, which is shown in appendix B. By looking at the pattern followed in two and three soft gravitino case, we propose the following
expression for $M$-soft gravitinos.

$$\Gamma_{N+M}(\{p_i\}, \{k_u\}) = \left[ \prod_{i=1}^{M} S_{u_i} + \sum_{A=1}^{[M/2]} \prod_{i=1}^{A} M_{u_i v_i} \prod_{j=1}^{M-2A} S_{r_j} \right] \Gamma_N(\{p_i\})$$  \hspace{1cm} (5.1)

where $\lfloor M/2 \rfloor$ denotes the greatest integer which is less than or equal to $M/2$. Now we will explain various terms.

1. The first term is very similar to the leading soft factor for multiple soft photon or multiple soft graviton. The other terms are there because of the fact that soft gravitino factors don’t commute. We always write the first factor in a particular order, for example, $S_{u_1}, \ldots, S_{u_M}$ $u_1 < u_2 \ldots < u_M$ and then the particular form of the second term depends on this choice of ordering for the first term. This way to write in particular ordering also turns out to be convenient to check gauge invariance.

2. In the second term, $A$ counts the number of pairs of gravitinos giving a soft graviton. For each pair, we have a factor of $C_{uv}$ coming from gravitino-graviton-gravitino vertex which combines with a factor due to the use of anti-commutation relation to bring the first term in particular order, to give $M_{uv}$. The subscripts $\{r_j, u_i, v_i\}$ can take values from $1, \ldots, M$ and $v_i > u_i$ and $r_j$’s are also ordered with the largest $r_j$ appearing on the right.

The disadvantage of the expression (5.1) is that it relies on the choice of an ordering. The expression is not manifestly invariant under alternation of the ordering. Now, we will show that the expression is actually invariant under rearrangement; We can go to any particular ordering starting from any other ordering. Our strategy is as follows:

1. We first show that any two consecutive entries can be interchanged.

2. By repeating this operation (of interchanging any two consecutive entries) many times, we can obtain any ordering starting from any other ordering.\footnote{Theorem 2.1 in this note gives a proof of the above statement.}

### 5.1 Re-arrangement

Here we show that any two consecutive terms of equation (5.1) can be interchanged. Consider the $i^{th}$ and $(i + 1)^{th}$ particle. We write the expression (5.1)

$$\Gamma_{N+M}(\{p_i\}, \{k_u\}) = \left[ S_{u_1} \ldots S_{u_i} S_{u_{i+1}} \ldots S_{u_M} + M_{u_1 u_2} S_{u_3} \ldots S_{u_i} S_{u_{i+1}} \ldots S_{u_M} \\
M_{u_2 u_3} S_{u_1} \ldots S_{u_i} S_{u_{i+1}} \ldots S_{u_{M-1}} S_{u_M} + \ldots + M_{u_1 u_i} S_{u_2} \ldots S_{u_{i+1}} \ldots S_{u_{M-1}} S_{u_M} \\
M_{u_2 u_i} S_{u_1} \ldots S_{u_{i+1}} \ldots S_{u_{M-1}} S_{u_M} + \ldots + M_{u_i u_{i+1}} S_{u_1} \ldots S_{u_{M-1}} S_{u_M} + \\
M_{u_1 u_2} M_{u_3 u_4} \ldots S_{u_i} S_{u_{i+1}} \ldots S_{u_{M-1}} S_{u_M} + \ldots + M_{u_1 u_2} \ldots M_{u_i u_{i+1}} \ldots S_{u_{M-1}} S_{u_M} \right] \Gamma_N(\{p_i\})$$  \hspace{1cm} (5.2)

Here the $i^{th}$ and $(i + 1)^{th}$ particle can appear only in three different ways.
• **Possibility I**: Both the $i^{\text{th}}$ and $(i+1)^{\text{th}}$ gravitino appear as $S$

\[ \left[ A S_u S_{u+1} B \right] \Gamma_N(\{p_i\}) \tag{5.3} \]

where $A$ and $B$ involves all the other $M-2$ gravitinos. The other gravitinos appear as ordered multiplications of $S_u$ and $\mathcal{M}_{uv}$’s in all possible ways.

• **Possibility II**: Both the $i^{\text{th}}$ and $(i+1)^{\text{th}}$ gravitino appear in $\mathcal{M}_{uv}$ together

\[ \left[ \tilde{A} \mathcal{M}_{ui+1 u+1} \tilde{B} \right] \Gamma_N(\{p_i\}) \tag{5.4} \]

Here $\tilde{A}$ and $\tilde{B}$ involves all the other $M-2$ gravitinos. Again the other gravitinos appear as ordered multiplications of $S_u$ and $\mathcal{M}_{uv}$’s in all possible ways. This would imply

\[ A = \tilde{A} , \quad B = \tilde{B} \tag{5.5} \]

So same $A$ and $B$ appear in (5.3) and in (5.4). Adding (5.3) and (5.4) we get

\[ \left[ A(S_u S_{u+1} + \mathcal{M}_{ui+1 u+1})B \right] \Gamma_N(\{p_i\}) \tag{5.6} \]

• **Possibility III**: At least one of them appears as $\mathcal{M}$ and if both of them appear in $\mathcal{M}_{uv}$, they don’t appear together. The possibility of both of them to appear together in $\mathcal{M}_{uv}$ has already been taken into account in possibility II.

\[ \sum_{j=1, j \neq i, i+1}^{N} \left[ \mathcal{M}_{ui+1} C_{i+1}(\epsilon_{ui+1}) + \mathcal{M}_{ujui+1} C_i(\epsilon_u) \right] \Gamma_N(\{p_i\}) \tag{5.7} \]

Here $C_{i+1}(\epsilon_{ui+1})$ is the all possible arrangements of all the gravitinos except $u_j$ and $u_i$ and similarly $C_i(\epsilon_u)$ is the all possible arrangements of all the gravitinos except $u_j$ and $u_{i+1}$.

Now if started with an ordering whether $u_{i+1}$ appeared before $u_i$ then we can repeat the same analysis. Equation (5.7) is same in both cases, but (5.2) and (5.3) will be replaced by $i \leftrightarrow i+1$. Hence instead of (5.6) we would have got

\[ \left[ A(S_{ui+1} S_u + \mathcal{M}_{ui+1 u}) B \right] \Gamma_N(\{p_i\}) \tag{5.8} \]

But now we can use (4.14) to see that (5.6) and (5.8) a essentially same. Hence the final answer is same irrespective of which order we choose for the soft factors.

### 5.2 Gauge invariance

We have proved the expression for multiple soft gravitinos can be rearranged to any particular ordering. Using this, we can bring any gravitino to be the rightmost. So we will show the gauge invariance of the expression only for the rightmost gravitino.

The right-most gravitino can appear only in two ways

1. It can appear in $S_u$. Since it is the right-most gravitino, it will directly hit the amplitude of the hard-particle and hence zero by (3.7).

2. Or it can appear in $\mathcal{M}_{uv}$. Again it will always appear as the 2nd index. But this vanishes because of (4.22a).
6 Two soft gravitinos in presence of central charge

In case of extended supersymmetries\(^7\), one can have central charges in the supersymmetry algebra. The supersymmetry algebra in (2.2) modifies to

\[
\{ Q_\alpha, Q_\beta \} = -\frac{1}{2} \gamma^\mu_{\alpha\beta} P_\mu - \frac{1}{2} Z_{\alpha\beta} U
\]

(6.1)

\(U\) is (are) the generator(s) of \(U(1)\) symmetry(ies) generated by the central charge(s). As explained below equation (2.2), \(\alpha, \beta\) are some (ir-)reducible spinor indices. In this language the existence of central charge is equivalent to the condition that there exists element(s) \(Z_{\alpha\beta}\) in the Clifford algebra such that, \(Z_{\alpha\beta}\) satisfies

\[
Z_{\alpha\beta} = Z_{\beta\alpha}
\]

(6.2)

In general, there can be higher form central charges. For example, in \(D = 11\), the supersymmetry algebra is of the form

\[
\{ Q_\alpha, Q_\beta \} = -\frac{1}{2} \gamma^\mu_{\alpha\beta} P_\mu + \gamma^\mu_{\alpha\beta\gamma\delta} A_{\mu\gamma\delta\nu}
\]

(6.3)

But for our purpose, we ignored any higher form central charges. This is because the higher form central charges can only minimally couple to extended objects (of appropriate dimensions), whereas we consider the scattering of only point-like states.

In this case the commutator of two soft operators in (4.13) is modified as follows

\[
[S_u, S_v] = -\frac{\kappa^2}{2} \sum_{\mu=1}^{N} \left[ \epsilon^{(u)}_{\mu} (p_1 \cdot k)^{\alpha_1}_{\alpha} + \epsilon^{(v)}_{\mu} (Z_{\alpha\beta}) \eta^{\mu\nu}_1 \right] \left( p_1 \cdot k_u p_1 \cdot k_v \right)
\]

(6.4)

In presence of the central term the computation in section 4 will be modified.

Graviphoton and new interaction [90] When we gauge the global supersymmetry with central charge to get supergravity, we get a \(U(1)^N\) gauge symmetry generated by spin 1 bosons (graviphoton) present in the graviton multiplet. These graviphotons couple to the gravitino and to any matter which carries the central charge. The coupling of the graviphoton to gravitino is completely fixed by supersymmetry and is related to that of graviton. The gravitino-gravitino-graviphoton three point function \((\tilde{V}_{\mu\nu};\gamma_1)^{\alpha\beta}\) is given by

\[
-\kappa \left[ k^\mu_1 (Z)^{\alpha\beta}_1 \eta^{\mu\nu} \right] - \frac{\kappa}{2} \left[ (k_1 + k_2) c \delta_d^\mu_1 (Z) \gamma^\gamma_{\beta} \eta^{\alpha\gamma} \right] + i \kappa \left[ (k^\mu_2 \eta^{\mu_1\nu} - k^\nu \eta^{\mu_1\mu})(Z)^{\alpha\beta} \right]
\]

(6.5)

Evaluation of figure 13 Now we would like to evaluate the figure 13. We don’t write the factor \([4 (k_1 \cdot k_2) (p_1 \cdot (k_1 + k_2))^{-1}\] coming from graviphoton propagator and the internal propagator.

\[
\epsilon_i \frac{-i \kappa}{2} \left[ \epsilon^{(1)}_{\mu} (\tilde{V}_{\mu\nu};\gamma_1)^{\alpha\beta} \epsilon^{(2)}_{\nu} \right] \eta_{\mu_1\mu_2} p_1^\mu_2
\]

(6.6)

\(^7\)We are thankful to Matteo Bertolini, Atish Dabholkar, Kyriakos Papadodimas, Cumrun Vafa for discussion on this point.
Using the explicit expression for $(\tilde{\mathcal{V}}^{\lambda k\mu})_{\alpha\beta}$ we get

$$
\Gamma^{(4)}_{N+2}(\{p_i\},k_1,k_2) = \frac{(-i\kappa)^2}{2} \sum_{i=1}^{N} \epsilon_i \epsilon_{\mu\alpha}(Z)^{\alpha\beta}\left[\eta^{\mu\nu} p_i \cdot k_2 + \frac{1}{2} \eta^{\mu\nu}(k_1 + k_2)\epsilon^d e^e - (k_2^\mu p_i^\nu - k_1^\nu p_i^\mu)\right] \epsilon^{(2)}_{\nu\beta} \\
\left[\frac{1}{(p_i \cdot (k_2 + k_1))(k_1 \cdot k_2)}\right] \Gamma_N(\{p_i\})
$$

\hfill (6.7)

Simplifying the above expression we get,

$$
\Gamma^{(4)}_{N+2}(\{p_i\},k_1,k_2) = \frac{\kappa^2}{2} \sum_{i=1}^{N} \epsilon_i \epsilon_{\mu\alpha}(Z)^{\alpha\beta}\left[-\eta^{\mu\nu} p_i \cdot k_2 + \frac{1}{2} \eta^{\mu\nu} p_i \cdot (k_1 + k_2) + (k_2^\mu p_i^\nu - k_1^\nu p_i^\mu)\right] \epsilon^{(2)}_{\nu\beta} \\
\left[\frac{1}{(p_i \cdot (k_2 + k_1))(k_1 \cdot k_2)}\right] \Gamma_N(\{p_i\})
$$

\hfill (6.8)

In this case the definition of $C_{uv}(p_i)$ in (4.9) will be modified as follows

$$
\tilde{C}_{uv}(p_i) = C_{uv}(p_i) + \frac{\kappa^2}{2} \sum_{i=1}^{N} \epsilon_i (\epsilon^{(u)}_{\mu\nu} Z \epsilon^{(v)}_{\mu\nu}) \left[\frac{1}{2} \eta^{\mu\nu} p_i \cdot (k_u - k_v) + \frac{(k_2^\mu p_i^\nu - k_1^\nu p_i^\mu)}{k_u \cdot k_v}\right]
$$

\hfill (6.9)

Note that

$$
\tilde{C}_{uv}(p_i) = \tilde{C}_{vu}(p_i)
$$

\hfill (6.10)

So the final answer is given by

$$
\left[S_1 S_2 + \tilde{M}_{12}\right] \Gamma(\{p_i\})
$$

\hfill (6.11)

where

$$
\tilde{M}_{uv} = M_{uv} + \frac{\kappa^2}{2} \sum_{i=1}^{N} \epsilon_i (\epsilon^{(u)}_{\mu\nu} Z \epsilon^{(v)}_{\mu\nu}) \left[\frac{p_i^\mu p_i^\nu}{p_i \cdot k_u} + \frac{1}{2} \eta^{\mu\nu} p_i \cdot (k_u - k_v) + \frac{(k_2^\mu p_i^\nu - k_1^\nu p_i^\mu)}{k_u \cdot k_v}\right]
$$

\hfill (6.12)

Note that the relations in equations (4.18a), (4.18b), (4.18c) remain the same if we replace $M_{uv}$ with $\tilde{M}_{uv}$. In this particular case

$$
S_u S_v - S_v S_u = -\tilde{M}_{uv} + \tilde{M}_{vu}
$$

\hfill (6.13)
equation (5.1) is modified as follows

\[ \mathcal{M}_{uv} - \mathcal{M}_{uv} = \mathcal{M}_{vu} - \mathcal{M}_{uv} + \frac{\kappa^2}{2} \sum_{i=1}^{N} e_i \left[ \frac{\epsilon^{(u)}_{\mu} Z \epsilon^{(v)}_{\nu}}{p_i \cdot (k_u + k_v)} \right] \left[ \frac{p_i^{\mu} p_i^{\nu} - (k_u \cdot k_v)}{k_u \cdot k_v} \right] \]

Now,

\[ \mathcal{M}_{uv} = \mathcal{M}_{vu} - \mathcal{M}_{uv} + \frac{\kappa^2}{2} \sum_{i=1}^{N} e_i \left[ \frac{\epsilon^{(u)}_{\mu} Z \epsilon^{(v)}_{\nu}}{p_i \cdot (k_u + k_v)} \right] \left[ \frac{p_i^{\mu} p_i^{\nu} - (k_u \cdot k_v)}{k_u \cdot k_v} \right] \]

We already computed \( S_u S_v - S_v S_u \) in (6.4). Hence (6.13) is satisfied.

### 6.1 Gauge invariance

As explained in section 4.1, it is easier to prove gauge invariance if put pure gauge polarization for the gravitino adjacent to \( \Gamma_N \). So we consider pure gauge polarization for the second gravitino

\[ \epsilon_{2a}^{(u)} = k_2^a \theta_2^a \] (6.15)

Now as a consistency check we check the gauge invariance. So for pure gauge

\[ \mathcal{M}_{uv} = \mathcal{M}_{vu} - \mathcal{M}_{uv} + \frac{\kappa^2}{2} \sum_{i=1}^{N} e_i \left[ \frac{\epsilon^{(u)}_{\mu} Z \epsilon^{(v)}_{\nu}}{p_i \cdot (k_u + k_v)} \right] \left[ \frac{p_i^{\mu} p_i^{\nu} - (k_u \cdot k_v)}{k_u \cdot k_v} \right] \]

where in the last step we have used momentum conservation and (central-)charge conservation

\[ \sum_{i=1}^{N} p_i = 0 \quad , \quad \sum_{i=1}^{N} e_i = 0 \] (6.17)

### 6.2 Presence of soft graviton

Following [1, 67, 68] it is easy to include soft graviton into this calculation. The vertex for the leading soft graviton \((\zeta_{1\mu} \rho_{1} \rho_{1}^{\nu})\) commutes with the vertex for soft graviton and also commutes with the vertex for any other soft graviton. So, in presence \( M_1 \) soft gravitino and \( M_2 \) soft graviton equation (5.1) is modified as follows

\[ \Gamma_{N+M_1+M_2}(\{p_i\}, \{k_r\}) = \left[ \prod_{j=1}^{M_2} \tilde{S}_{u_j} \right] \left[ \prod_{i=1}^{M_1} S_{u_i} \right] + \sum_{A=1}^{[M_1/2]} A \prod_{i=1}^{A} M_{u,v_i} \prod_{j=1}^{M_1-2A} S_{r_j} \] (6.18)

\( \tilde{S}_u \) is the leading soft factor for graviton, given in equation (1.10).

We know that leading and sub-leading soft factors for multiple gravitons are universal. In this paper, we derived the leading order expression for multiple soft gravitinos and we found that it is also universal. These three soft theorems are inter-related by supersymmetry. One way to argue this is to observe that all these three soft theorems follow from covariantizing the action with respect to the soft field. In supergravity, the structure of the covariant derivative is uniquely fixed by supersymmetry.
7 Conclusion

In this paper, we have computed leading multiple soft gravitino theorems for an arbitrary theory of supergravity. One natural question to ask is that what is the structure of sub-leading soft gravitino theorem and how the structure of the subleading soft gravitino theorem is related to that of sub-leading and sub-subleading soft graviton theorem. One can use this approach to compute soft photino theorem and correction to soft photino theorem in presence of gravitino, photon and graviton. Another interesting question is to derive the result for multiple soft gravitinos from the analysis of asymptotic symmetries and from CFT living on $\mathcal{F}^\pm$ following [91–94]. Following [95] one could also try to verify this result from world-sheet methods. We leave these questions for future work.

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A Notation and convention

Our notation is as follows

Curved space index $\mu, \nu, \rho, \sigma$ (A.1a)

Tangent space index $a, b$ (A.1b)

$SO(d, 1)$ spinor index $\alpha, \beta$ (A.1c)

Soft-particle index $u, v$ (A.1d)

Hard-particle index $i, j$ (A.1e)

Number of Soft-particles $M$ (A.1f)

Number of Hard-particles $N$ (A.1g)

Polarization of the graviton $\zeta_{\mu\nu}$ (A.1h)

Polarization of the gravitino $\epsilon_{\mu a}$ (A.1i)

A.1 Gamma matrix and spinor convention

We use the following the gamma matrix convention

$$\{\gamma^a, \gamma^b\} = -2\eta^{ab}$$ (A.2)

and we get

$$[\gamma^a, \gamma^b] = -2\eta^{ab}\gamma^c + 2\eta^{ac}\gamma^b$$ (A.3)
The basic spinors are defined as

$$\psi_{\alpha}$$ (A.4)

gamma matrix is

$$(\gamma^{\mu})_{\alpha}^{\beta}$$ (A.5)

We raise and lower the indices as follows (NW-SE convention)

$$\psi^{\alpha} = C^{\alpha\beta} \psi_{\beta}$$, \quad $$\psi_{\alpha} = \psi^{\beta} C_{\beta\alpha}$$ (A.6)

Here $$C_{\alpha\beta}$$ satisfies

$$C^{\alpha\beta} C_{\gamma\beta} = \delta_{\alpha}^{\gamma}$$, \quad $$C_{\beta\alpha} C^{\beta\gamma} = \delta_{\gamma}^{\alpha}$$ (A.7)

$$(\gamma^{\mu})_{\alpha\beta}$$ is given by

$$\gamma^{\mu}_{\alpha\beta} = (\gamma^{\mu})_{\alpha}^{\gamma} C_{\gamma\beta}$$ (A.8)

### A.2 Majorana spinor

For two Majorana spinors $$\psi_{1}$$ and $$\psi_{2}$$

$$(\psi_{1})^{\alpha}(\psi_{2})_{\alpha} = (\psi_{2})^{\alpha}(\psi_{1})_{\alpha}$$ (A.9)

### B Three soft gravitinos

In this appendix, we present the explicit computation for three soft gravitinos. This computation is instructive to understand the soft factor for multiple gravitinos in section 5. In this section, we will write only $$\Gamma_{N+3}$$ instead of $$\Gamma_{N+3}(\{p_i\}, k_1, k_2, k_3)$$ to denote the amplitude with the soft gravitinos and similarly we will write $$\Gamma_{N}$$ instead of $$\Gamma_{N}^{(1)}(\{p_i\})$$ to denote the amplitudes involving only the hard-particles. For three soft gravitons, the different contributions are as follows:

- First consider Feynman diagrams where all three gravitinos attach to separate external legs (figure 14). In this case the contribution will be just the multiplication of individual factors. So we get

$$\Gamma_{N+3}^{(1)} = \kappa^{3} \sum_{i=1}^{N} \frac{p_{i}^{\mu} \epsilon_{\mu}}{p_{i} \cdot k_{i}} \mathcal{Q}_{\alpha_{1}} \sum_{j=1,j \neq i}^{N} \frac{p_{j}^{\mu} \epsilon_{\mu}}{p_{j} \cdot k_{j}} \mathcal{Q}_{\alpha_{2}} \sum_{k=1,k \neq i,j}^{N} \frac{p_{k}^{\mu} \epsilon_{\mu}^{(3)}}{p_{k} \cdot k_{3}} \mathcal{Q}_{\alpha_{3}} \Gamma_{N}(\{p_{i}\})$$ (B.1)

- Now we consider the case when two gravitino attach same leg and the third one on different leg as shown in figure 15. The contribution from such configurations are given by

$$\Gamma_{N+3}^{uv|w;1} = \kappa^{3} \sum_{i=1}^{N} \frac{p_{i}^{\mu} \epsilon_{\mu}^{(u)}}{p_{i} \cdot k_{u}} \mathcal{Q}_{\alpha_{u}} \frac{p_{i}^{\mu} \epsilon_{\mu}^{(v)}}{p_{i} \cdot (k_{u} + k_{v})} \mathcal{Q}_{\alpha_{v}} \sum_{j=1,j \neq i}^{N} \frac{p_{j}^{\mu} \epsilon_{\mu}^{(w)}}{p_{j} \cdot k_{w}} \mathcal{Q}_{\alpha_{w}} \Gamma_{N}(\{p_{i}\})$$ (B.2)

where $$u, v, w$$ can take values 1, 2, 3. We can have different contributions depending on the order in which gravitinos attach.
The third possibility consists of the diagrams when all gravitinos attaching to the same external leg (figure 16). The contribution is given by

\[
\Gamma_{uvw}^{N+3} = \kappa^3 \sum_{i=1}^{N} \epsilon_i^{(u)\alpha_u} p_i^\mu \epsilon_i^{(v)\alpha_v} p_i^\nu \epsilon_i^{(w)\alpha_w} p_i^\rho \frac{Q_{\alpha_u} Q_{\alpha_v} Q_{\alpha_w}}{\Gamma_N(\{p_i\})}
\]

We have 6 diagrams which can be obtained by interchanging the external soft gravitinos. In (B.3), this obtained from various permutations of \(u, v\) and \(w\).

Now we consider the diagrams in which any two soft gravitinos combine to give a soft graviton and then the soft graviton attaches to the external leg; the left-over (lonely !) third one directly attaches. This can also give rise to two scenarios i.e. the internal soft
graviton and the leftover lonely gravitino can attach to same hard particles or to different hard particles.

Figure 17: Feynman diagram for three soft gravitinos - IV

In the case when they attach on separate legs as shown in figure 17, we just have the multiplication of two factors:

$$\Gamma_{uv|w;2}^{N+3} = \kappa^3 \sum_{i=1}^{N} \left[ \frac{C_{uv}(p_i)}{p_i \cdot k_u + k_v} \right] \sum_{j=1, j \neq i}^{N} \frac{\epsilon_{\alpha\mu}^{(u)\alpha\mu} p_j^\mu}{p_j \cdot k_w} \frac{C_{uv}(p_i)}{(p_i \cdot (k_u + k_v))} \Gamma_N(\{p_i\})$$

(B.4)

Since any two gravitinos can combine to give the internal soft graviton (and the third one will attach to the separate leg), there are three possibilities.

Now we can have the case when both the internal soft graviton and the left-over soft gravitino attach to same external leg as shown in figure 18

Figure 18: Feynman diagram for three soft gravitinos - V

$$\Gamma_{uv|w;3}^{N+3} = \kappa^3 \sum_{i=1}^{N} \left[ \frac{\epsilon_{\alpha\mu}^{(u)\alpha\mu} p_j^\mu}{p_j \cdot k_w} Q_{\alpha\omega} \frac{C_{uv}(p_i)}{(p_i \cdot (k_u + k_v))} \right] \Gamma_N(\{p_i\})$$

(B.5)

We will have another diagram in which the graviton attaches first and gravitino later i.e.

$$\Gamma_{uv|w;4}^{N+3} = \kappa^3 \sum_{i=1}^{N} \left[ \frac{C_{uv}(p_i)}{p_i \cdot (k_u + k_v)} \frac{\epsilon_{\alpha\mu}^{(u)\alpha\mu} p_j^\mu}{p_j \cdot (k_1 + k_2 + k_3)} Q_{\alpha\omega} \right] \Gamma_N(\{p_i\})$$

(B.6)

Adding the contributions from (B.4), (B.5) and (B.6), we get

$$\Gamma_{uv|w;4}^{N+3} = \Gamma_{uv|w;2}^{N+3} + \Gamma_{uv|w;3}^{N+3} + \Gamma_{uv|w;4}^{N+3} = \kappa^2 \left[ \sum_{i=1}^{N} \frac{C_{uv}(p_i)}{p_i \cdot (k_u + k_v)} \right] S_w \Gamma_N(\{p_i\})$$

(B.7)
Figure 19: Feynman diagram for three soft gravitinos - VI

Now we write the contributions due to six diagrams of the type shown in figure 16. We choose a particular ordering. We choose $Q_\gamma$ to be the right-most. The $\Gamma_{N+3}^{123}$ remains the same.

$$\Gamma_{N+3}^{123} = \kappa^3 \sum_{i=1}^{N} \frac{\epsilon^{(1)}_{\mu} p_i^\mu}{p_i \cdot k_1} \frac{\epsilon^{(2)}_{\nu} p_i^\nu}{p_i \cdot (k_1 + k_2)} \frac{\epsilon^{(3)}_{\rho} p_i^\rho}{p_i \cdot (k_1 + k_2 + k_3)} Q_{\alpha} Q_{\beta} Q_{\gamma} \Gamma_N(\{p_i\})$$ (B.8)

Now we bring any other expression into this particular ordering by using (2.2). For example,

$$\Gamma_{N+3}^{132} = \kappa^3 \sum_{i=1}^{N} \frac{\epsilon^{(1)}_{\mu} p_i^\mu}{p_i \cdot k_1} \frac{\epsilon^{(2)}_{\nu} p_i^\nu}{p_i \cdot (k_1 + k_3)} \frac{\epsilon^{(3)}_{\rho} p_i^\rho}{p_i \cdot (k_1 + k_2 + k_3)} Q_{\alpha} Q_{\gamma} Q_{\beta} \Gamma_N(\{p_i\})$$

$$= \kappa^3 \sum_{i=1}^{N} \frac{\epsilon^{(1)}_{\mu} p_i^\mu}{p_i \cdot k_1} \frac{\epsilon^{(2)}_{\nu} p_i^\nu}{p_i \cdot (k_1 + k_3)} \frac{\epsilon^{(3)}_{\rho} p_i^\rho}{p_i \cdot (k_1 + k_2 + k_3)} \left[ Q_{\alpha} Q_{\beta} Q_{\gamma} + \frac{1}{2} (p_i)^{\beta\gamma} Q_{\alpha} \right] \Gamma_N(\{p_i\})$$ (B.9)

Following the same philosophy, we obtain

$$\Gamma_{N+3}^{213} = \kappa^3 \sum_{i=1}^{N} \frac{\epsilon^{(1)}_{\mu} p_i^\mu}{p_i \cdot k_1} \frac{\epsilon^{(2)}_{\nu} p_i^\nu}{p_i \cdot (k_1 + k_2)} \frac{\epsilon^{(3)}_{\rho} p_i^\rho}{p_i \cdot (k_1 + k_2 + k_3)} \left[ Q_{\alpha} Q_{\beta} Q_{\gamma} + \frac{1}{2} (p_i)^{\alpha\beta} Q_{\gamma} \right] \Gamma_N(\{p_i\})$$ (B.10a)

$$\Gamma_{N+3}^{231} = \kappa^3 \sum_{i=1}^{N} \frac{\epsilon^{(1)}_{\mu} p_i^\mu}{p_i \cdot k_2} \frac{\epsilon^{(2)}_{\nu} p_i^\nu}{p_i \cdot (k_2 + k_3)} \frac{\epsilon^{(3)}_{\rho} p_i^\rho}{p_i \cdot (k_1 + k_2 + k_3)} \left[ Q_{\alpha} Q_{\gamma} Q_{\beta} - \frac{1}{2} (p_i)^{\alpha\gamma} Q_{\beta} + \frac{1}{2} (p_i)^{\alpha\beta} Q_{\gamma} \right] \Gamma_N(\{p_i\})$$ (B.10b)

$$\Gamma_{N+3}^{321} = \kappa^3 \sum_{i=1}^{N} \frac{\epsilon^{(1)}_{\mu} p_i^\mu}{p_i \cdot k_2} \frac{\epsilon^{(2)}_{\nu} p_i^\nu}{p_i \cdot (k_2 + k_3)} \frac{\epsilon^{(3)}_{\rho} p_i^\rho}{p_i \cdot (k_1 + k_2 + k_3)} \left[ Q_{\alpha} Q_{\beta} Q_{\gamma} - \frac{1}{2} (p_i)^{\alpha\gamma} Q_{\beta} \right.

$$+ \frac{1}{2} (p_i)^{\alpha\beta} Q_{\gamma} + \frac{1}{2} (p_i)^{\beta\gamma} Q_{\alpha} \right] \Gamma_N(\{p_i\})$$ (B.10c)

$$\Gamma_{N+3}^{312} = \kappa^3 \sum_{i=1}^{N} \frac{\epsilon^{(1)}_{\mu} p_i^\mu}{p_i \cdot k_3} \frac{\epsilon^{(2)}_{\nu} p_i^\nu}{p_i \cdot (k_3 + k_1)} \frac{\epsilon^{(3)}_{\rho} p_i^\rho}{p_i \cdot (k_1 + k_2 + k_3)} \left[ Q_{\alpha} Q_{\beta} Q_{\gamma} - \frac{1}{2} (p_i)^{\alpha\gamma} Q_{\beta} + \frac{1}{2} (p_i)^{\alpha\beta} Q_{\gamma} \right] \Gamma_N(\{p_i\})$$ (B.10d)
Adding all the contributions from equation (B.8), (B.9) (B.10a)-(B.10d) we get:

\[
\Gamma_N^{ex} = \kappa^2 \sum_{i=1}^{N} \sum_{r \neq s \neq u \neq v} \left[ \frac{1}{p_i \cdot (k_r + k_u)} \right] \left[ \frac{1}{p_i \cdot (k_r + k_u)} \right] \frac{\epsilon_\mu^{(1)} p_\mu^i}{p_i \cdot k_1} \frac{\epsilon_\nu^{(2)} p_\nu^i}{p_i \cdot k_2} \frac{\epsilon_\rho^{(3)} p_\rho^i}{p_i \cdot k_3} Q_\alpha Q_\beta Q_\gamma \Gamma_N(p_i) + \frac{1}{2} \kappa^2 \sum_{i=1}^{N} \left[ \frac{\epsilon_\mu^{(1)} p_\mu^i}{p_i \cdot k_1} \right] \frac{\epsilon_\nu^{(2)} p_\nu^i}{p_i \cdot (k_2 + k_3)} \frac{\epsilon_\rho^{(3)} p_\rho^i}{p_i \cdot k_3} \frac{\epsilon_\sigma^{(4)} p_\sigma^i}{p_i \cdot (k_2 + k_3)} \Gamma_N(p_i) + \frac{1}{2} \kappa^2 \sum_{i=1}^{N} \left[ \frac{\epsilon_\mu^{(1)} p_\mu^i}{p_i \cdot k_1} \right] \frac{\epsilon_\nu^{(2)} p_\nu^i}{p_i \cdot k_2} \frac{\epsilon_\rho^{(3)} p_\rho^i}{p_i \cdot k_3} \frac{\epsilon_\sigma^{(4)} p_\sigma^i}{p_i \cdot (k_2 + k_3)} \Gamma_N(p_i) + \frac{1}{2} \kappa^2 \sum_{i=1}^{N} \left[ \frac{\epsilon_\mu^{(1)} p_\mu^i}{p_i \cdot k_1} \right] \frac{\epsilon_\nu^{(2)} p_\nu^i}{p_i \cdot k_2} \frac{\epsilon_\rho^{(3)} p_\rho^i}{p_i \cdot k_3} \frac{\epsilon_\sigma^{(4)} p_\sigma^i}{p_i \cdot (k_2 + k_3)} \Gamma_N(p_i) + \frac{1}{2} \kappa^2 \sum_{i=1}^{N} \left[ \frac{\epsilon_\mu^{(1)} p_\mu^i}{p_i \cdot k_1} \right] \frac{\epsilon_\nu^{(2)} p_\nu^i}{p_i \cdot (k_2 + k_3)} \frac{\epsilon_\rho^{(3)} p_\rho^i}{p_i \cdot k_3} \frac{\epsilon_\sigma^{(4)} p_\sigma^i}{p_i \cdot k_2} \Gamma_N(p_i)
\]

(B.11)

We add the contributions from (B.1), (B.2) and (B.7) to get the full answer. The full result can be written as

\[
\Gamma_N^{ex}(\{p_i\}, \{k_u\}) = \left[ S_1 S_2 S_3 + M_{12} S_1 + M_{13} S_2 \right] \Gamma_N(\{p_i\})
\]

where $S_u$ and $M_{uv}$ is defined in (3.4) and (4.12) respectively. The above answer matches with the proposed answer (5.1) with $M = 3$.

### B.1 Rearrangement

We have written the answer for a particular ordering (1 \(-\) 2 \(-\) 3). In this case, we explicitly demonstrate the rearrangement. Let’s say we want to write in the order 1 \(-\) 3 \(-\) 2. We apply the identity (4.14) for $u = 2, v = 3$

\[
\Gamma_N^{ex}(\{p_i\}, \{k_u\}) = \left[ S_1 (S_3 S_2 - M_{23} + M_{32}) + M_{12} S_3 + M_{13} S_2 \right] \Gamma_N(\{p_i\})
\]

(B.13)

### B.2 Gauge invariance

The gauge invariance of (B.12) is easiest to show if we put pure gauge polarization for the last one, for example, 3rd gravitino in (B.12), 2nd gravitino in (B.13). Because the answer can always be rearranged to any particular ordering, we can always bring any particular gravitino to be the last entry. So, it’s sufficient to show gauge invariance for the pure gauge polarization for the last one.

Let’s consider (B.12) and pure gauge polarization for the 3rd gravitino. The first and the second term vanish as in equation (3.7) and third & fourth term vanish because of (4.22a).

### Symmetric form

Now we will write the answer (B.13) in the form which is manifestly symmetric in all the gravitinos

\[
\Gamma_N^{ex}(\{p_i\}, \{k_u\}) = \left[ \frac{1}{3!} S_1 S_2 S_3 \right] + \kappa^2 \sum_{r \neq s \neq u \neq v} \sum_{i=1}^{N} \left[ \frac{1}{p_i \cdot (k_r + k_u)} \right] \left[ \frac{1}{p_i \cdot (k_r + k_u)} \right] \frac{\epsilon_\mu^{(1)} p_\mu^i}{p_i \cdot k_1} \frac{\epsilon_\nu^{(2)} p_\nu^i}{p_i \cdot k_2} \frac{\epsilon_\rho^{(3)} p_\rho^i}{p_i \cdot k_3} \frac{\epsilon_\sigma^{(4)} p_\sigma^i}{p_i \cdot (k_2 + k_3)} \Gamma_N(p_i)
\]

(B.14)
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