Differentially rotating disks of dust: Arbitrary rotation law

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Abstract

In this paper, solutions to the Ernst equation are investigated that depend on two real analytic functions defined on the interval $[0, 1]$. These solutions are introduced by a suitable limiting process of Bäcklund transformations applied to seed solutions of the Weyl class. It turns out that this class of solutions contains the general relativistic gravitational field of an arbitrary differentially rotating disk of dust, for which a continuous transition to some Newtonian disk exists. It will be shown how for given boundary conditions (i.e. proper surface mass density or angular velocity of the disk) the gravitational field can be approximated in terms of the above solutions. Furthermore, particular examples will be discussed, including disks with a realistic profile for the angular velocity and more exotic disks possessing two spatially separated ergoregions.

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1 Introduction

Differentially rotating disks of dust have already been studied by Ansorg and Meinel [1]. They considered the class of hyperelliptic solutions to the Ernst equation introduced by Meinel and Neugebauer [2], see also [3]-[6]. These hyperelliptic solutions depend on a number of complex parameters and a real potential function. Ansorg and Meinel concentrated on the case in which one complex parameter can be prescribed. They determined the real potential function in order to satisfy a particular boundary condition valid for all disks of dust. To generate their solutions, they used Neugebauer’s and Meinel’s rigorous solution [7, 8, 9] to the boundary value problem of a rigidly rotating disk of dust which also belongs to the hyperelliptic class.

A subclass of Ansorg’s and Meinel’s solutions is made up of Bäcklund transforms of seed solutions of the Weyl class. Solutions of this type are of particular interest since their mathematical structure is much simpler than that of the more general hyperelliptic solutions.

With this in mind, the following questions arise:

• Is it possible to find solutions corresponding to more general differentially rotating disks of dust by increasing the number of prescribed complex parameters?

• If so, is there a rapidly converging method for approximating arbitrary differentially rotating disks of dust with given boundary conditions (i.e. proper mass density or angular velocity)?

• Is it perhaps possible to construct such a method by restriction to the much simpler solutions of the Bäcklund type?

To answer these questions, the paper is organized as follows. In the first section the metric tensor, Ernst equation, and boundary conditions are introduced and the class of solutions of the Bäcklund type is represented. As will be discussed in the second section, the properties of these solutions can be used to obtain more general solutions by a suitable limiting process. Since these more general solutions depend on two real analytic functions defined on the interval [0,1], a rapidly converging numerical scheme to satisfy arbitrary boundary conditions for disks of dust can be created. This is depicted in the third section. Finally, the fourth section contains particular examples of differentially rotating disks of dust, including disks with a realistic profile for the angular velocity and more exotic disks possessing two spatially separated ergoregions.

In what follows, units are used in which the velocity of light as well as Newton’s constant of gravitation are equal to 1.

1.1 Metric Tensor, Ernst equation, and boundary conditions

The metric tensor for axisymmetric stationary and asymptotically flat space-times reads as follows in Weyl-Papapetrou-coordinates ($\rho, \zeta, \varphi, t$):

$$ds^2 = e^{-2U}[e^{2k}(d\rho^2 + d\zeta^2) + \rho^2 d\varphi^2] - e^{2U}(dt + a d\varphi)^2.$$ 

For this line element, the vacuum field equations are equivalent to a single complex equation – the so-called Ernst equation [22, 23]

$$\left(\Re f\right) \Delta f = (\nabla f)^2,$$ \hspace{1cm} (1)

\footnote{The construction of solutions to the Ernst equation by means of Bäcklund transformations belongs to the powerful analytic methods developed by several authors [10]-[20]. For a detailed introduction see [11].}
\[
\Delta = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial \zeta^2}, \quad \nabla = \left( \frac{\partial}{\partial \rho}, \frac{\partial}{\partial \zeta} \right),
\]

where the Ernst potential \( f \) is given by

\[ f = e^{2U} + \frac{ib}{\rho} \quad \text{with} \quad b, \zeta = \frac{e^{4U}}{\rho} a, \quad b, \rho = -\frac{e^{4U}}{\rho} a. \quad (2) \]

The remaining function \( k \) can be calculated from the Ernst potential \( f \) by a line integration:

\[ k, \rho = (U, \rho)^2 - (U, \zeta)^2 + \frac{1}{4} e^{-4U} [(b, \rho)^2 - (b, \zeta)^2] \]
\[ k, \zeta = 2U, \rho U, \zeta + \frac{1}{2} e^{-4U} b, \rho b, \zeta. \]

To obtain the boundary conditions for differentially rotating disks of dust, one has to consider the field equations for an energy-momentum-tensor

\[ T^{ik} = \epsilon u^i u^k = \sigma_p(\rho)e^{U-k}\delta(\zeta)u^i u^k, \]

where \( \epsilon \) and \( \sigma_p \) stand for the energy-density and the invariant (proper) surface mass-density, respectively, \( \delta \) is the usual Dirac delta-distribution, and \( u^i \) denotes the four-velocity of the dust material.\(^3\)

Integration of the corresponding field equations from the lower to the upper side of the disk (with coordinate radius \( \rho_0 \)) yields the conditions (see [24], pp. 81-83)

\[ 2\pi \sigma_p = e^{U-k}(U, \zeta + \frac{1}{2} Q) \]
\[ e^{4U} Q^2 + Q(e^{4U})_\zeta + (b, \rho)^2 = 0 \]

for \( \zeta = 0^+, 0 \leq \rho \leq \rho_0 \) and

\[ Q = -\rho e^{-4U} [b, \rho b, \zeta + (e^{2U})_\rho (e^{2U})_\zeta]. \]

Note that boundary condition (4) for the Ernst potential \( f \) does not involve the surface mass-density \( \sigma_p \). This condition comes from the nature of the material the disk is made of. Therefore, equation (4) will be referred to as the dust-condition. Instead of prescribing the proper surface mass-density \( \sigma_p \) [which leads to the boundary condition (3)] one can alternatively assume a given angular velocity \( \Omega = \Omega(\rho) = u^\phi/u^t \) of the disk which results in the boundary condition (\( \zeta = 0^+, 0 \leq \rho \leq \rho_0 \)):

\[ \Omega = \frac{Q}{a, \zeta - a Q}. \quad (6) \]

The following requirements due to symmetry conditions and asymptotical flatness complete the set of boundary conditions:

- Regularity at the rotation axis is guaranteed by
  \[ \frac{\partial f}{\partial \rho}(0, \zeta) = 0. \]

- At infinity asymptotical flatness is realized by \( U \to 0 \) and \( a \to 0 \). For the potential \( b \) this has the consequence \( b \to b_\infty = \text{const} \). Without loss of generality, this constant can be set to 0, i.e. \( f \to 1 \) at infinity.

- Finally, reflectional symmetry with respect to the plane \( \zeta = 0 \) is assumed, i.e. \( f(\rho, -\zeta) = \overline{f(\rho, \zeta)} \) (with a bar denoting complex conjugation).

\(^3u^i\) has only \( \phi \)- and \( t \)- components.
1.2 Solutions of the Bäcklund type

For a given integer \( q \geq 1 \), a set \( \{Y_1, \ldots, Y_q\} = \{Y_\nu\}_{\nu=1}^q \) of complex parameters, and a real analytic function \( g \) defined on the interval \([0, 1]\), the following expression

\[
f = f_0 = \exp \left( - \frac{1}{Z_D} \int_{1}^{0} \frac{(-1)^q g(x^2)}{Z_D} \, dx \right), \quad Z_D = \sqrt{1 + \frac{\bar{\gamma} \gamma}{\rho^2}}\sqrt{(1 + \bar{\gamma} \gamma) 1 + \frac{\bar{\gamma} \gamma}{\rho^2}}, \quad \Re(Z_D) < 0
\]

\[
\lambda_\nu = \sqrt{\frac{\gamma_\nu \gamma_\nu}{\rho^2}}, \quad \gamma_\nu = \frac{1}{\rho^2} \left(2^q \lambda_\nu \lambda_\nu - 1\right)
\]

\[
\alpha_\nu = \frac{1 - \gamma_\nu}{1 + \gamma_\nu}, \quad \gamma_\nu = \exp \left( \lambda_\nu (Y_\nu + i\zeta) \int_{1}^{0} \frac{(-1)^q g(x^2)}{Z_D} \, dx \right), \quad \alpha_\nu \bar{\alpha}_\nu = 1
\]

satisfies the Ernst equation. With the additional requirement that for each parameter \( Y_\nu \) there is also a parameter \( Y_\mu \) with \( Y_\nu = -Y_\mu \), reflection symmetry, \( f(\rho, \zeta) = \bar{f}(\rho, \zeta) \), is ensured. Moreover, the parameters \( Y_\nu \) are assumed to lie outside the imaginary interval \([-i, i]\).

The above Ernst potential \( f = f(\rho/\rho_0, \zeta/\rho_0; \{Y_\nu\}_{\nu=1}^q; g) \) is obtained by a Bäcklund transformation applied to the real seed solution \( f_0 \), see (13). On the other hand, as demonstrated in appendix A, it can be constructed from the hyperelliptic solutions by a suitable limiting process (see also (13)). The particular ansatz chosen for the seed solution \( f_0 \) guarantees a resulting Ernst potential which corresponds to a disk-like source of the gravitational field (see also section 1.2 of (13)). Furthermore, \( f \) does not possess singularities at \( (\rho, \zeta) = \rho_0(|Y_\nu|, -\Re(Y_\nu)) \). This is due to the fact that \( \alpha_\nu \lambda_\nu \) is a function of \( \lambda_\nu^2 \), and this means that \( f \) does not behave like a square root function near the critical points \( (\rho, \zeta) = \rho_0(|Y_\nu|, -\Re(Y_\nu)) \), but rather like a rational function. Now, in the whole area of physically interesting

\[4\text{In the following, the notation } \{Y_1, \ldots, Y_q\} \text{ will be abbreviated by } \{Y_\nu\}_{\nu=1}^q.\]

\[5\text{Hence, the set } \{Y_\nu\}_{\nu=1}^q \text{ consists of real parameters and/or pairs of complex conjugate parameters.}\]
solutions that will be treated in the subsequent sections, each zero of the denominator is cancelled by a corresponding zero of the numerator in (7) such that the resulting gravitational field is regular outside the disk.

The real function \( g \) that enters the Ernst potential is assumed to be analytic on \([0, 1]\) in order to guarantee an analytic behaviour of the angular velocity \( \Omega \) for all \( \rho \in [0, \rho_0] \). Moreover, the additional requirement

\[
g(1) = 0
\]

leads to a surface mass density \( \sigma_p \) of the form

\[
\sigma_p(\rho) = \sigma_0 \psi_p[(\rho/\rho_0)^2 \sqrt{1 - (\rho/\rho_0)^2}] \quad \text{[with \( \psi_p \) analytic in \([0, 1]\), \( \psi_p(0) = 1 \]} \quad (8)
\]

and therefore ensures that \( \sigma_p \) vanishes at the rim of the disk.

In this article the question as to whether the above expression for the Ernst potential is sufficiently general to approximate arbitrary differentially rotating disks of dust is investigated. Of particular interest is a rapidly converging method to perform this approximation. To this end, the set \( \{Y_\nu\}_{q} \) of complex parameters will be translated into an analytic function \( \xi : [0, 1] \rightarrow \mathbb{R} \).

Thus the Ernst potential will depend on two real analytic functions defined on \([0, 1]\):

\[
f = f(\rho/\rho_0, \zeta/\rho_0; \xi; g),
\]

which eventually proves to be sufficient to satisfy both the dust condition \((4)\) and the boundary condition \((3)\) [or alternatively \((6)\)]. The rapid and accurate approximation can be realized since both \( g \) and \( \xi \) are analytic on \([0, 1]\) and thus permit elegant expansions in terms of Chebyshev polynomials.

2 Generalization of the Bäcklund type solutions by a limiting process

As demonstrated in \(4\) for the Bäcklund type solutions with \( q = 1 \), the dust condition \((4)\) can be satisfied by an appropriate choice of the function \( g \) if the complex parameters \( Y_\nu \) are prescribed. To fulfil a second boundary condition, \((3)\) or \((6)\), the set \( \{Y_\nu\}_{q} \) of these parameters has to be translated into a real analytic function \( \xi \).

To this end, consider the following equalities for the above solutions \( f = f(\{Y_\nu\}_{q}; g) \) which are proved in appendix B:

\[
f[\{Y_1, \ldots, Y_{q-2}, Y_{q-1}, Y_q\}; g] = f[\{Y_1, \ldots, Y_{q-2}\}; g]
\]

if \( Y_{q-1} = -Y_q \in \mathbb{R} \quad (9)\)

\[
f[\{Y_1, \ldots, Y_{q-2}, Y_{q-1}, Y_q\}; g] = f[\{Y_1, \ldots, Y_{q-2}\}; g]
\]

if \( Y_{q-1} = Y_q \quad (10)\)

\[
\lim_{t \to \infty} f[\{Y_1, \ldots, Y_{q-1}, it\}; g] = f[\{Y_1, \ldots, Y_{q-1}\}; g]
\]

if \( t \in \mathbb{R} \quad (11)\)

\[
\lim_{Y_q \to \infty} f[\{Y_1, \ldots, Y_{q-2}, Y_{q-1}, Y_q\}; g] = f[\{Y_1, \ldots, Y_{q-2}\}; g]
\]

if \( Y_{q-1} = -Y_q \quad (12)\)

\footnote{In the following the Ernst potentials \( f \) given by \((9)\) are considered as complex functions depending on the set \( \{Y_\nu\}_{q} \) of complex parameters and on \( g \).}
In order to find an approximation scheme, the desired function \( \xi = \xi(\{Y_\nu\}_q) \) is supposed to be invariant under the modifications \((\text{13})\) of the set \( \{Y_\nu\}_q \) that do not effect the Ernst potential. This property will be necessary to solve the boundary conditions uniquely.

It is realized by the real analytic function

\[
\xi(x^2; \{Y_\nu\}_q) = \frac{1}{x} \ln \left[ \prod_{\nu=1}^{q} \frac{iY_\nu - x}{iY_\nu + x} \right], \quad x \in [-1, 1],
\]

which can be proved by considering that for each parameter \( Y_\nu \) there is also a parameter \( Y_\mu \) with \( Y_\nu = -\overline{Y_\mu} \), and that, moreover, the parameters \( Y_\nu \) do not lie on the imaginary interval \([-i, i]\).

The set \( \mathcal{X} \) of all functions \( \xi = \xi(x^2; \{Y_\nu\}_q), q \in \mathbb{N} \), which are defined by \((\text{13})\) forms a dense subset of the set \( \mathcal{A} \) of all real analytic functions on \([0, 1] \). Now, for a given function \( g \), each \( \xi \in \mathcal{X} \) is mapped by \((\text{13})\) onto a uniquely defined Ernst potential \( f \in \mathcal{E} \):

\[
\Phi_g : \mathcal{X} \rightarrow \mathcal{E}, \quad \Phi_g(\xi) = f(\{Y_\nu\}_q; g),
\]

where the set \( \{Y_\nu\}_q \) results from \( \xi \) by \((\text{13})\). In the following, it is assumed that this mapping \( \Phi_g \) can be extended to form a continuous function defined on \( \mathcal{A} \). Then, given the two real functions \( g \) and \( \xi \), defined and analytic on the interval \([0, 1] \), the Ernst potential

\[
f(\xi; g) = \lim_{q \to \infty} f(\{Y_\nu^{(q)}\}_q; g)
\]

exists and is independent of the particular choice of the sequence \( \{\{Y_\nu^{(q)}\}_q\}_{q=q_0}^{\infty} \) which serves to represent \( \xi \) by

\[
\xi(x^2) = \frac{1}{x} \lim_{q \to \infty} \ln \left[ \prod_{\nu=1}^{q} \frac{iY^{(q)}_\nu - x}{iY^{(q)}_\nu + x} \right] \quad \text{for} \quad x \in [-1, 1].
\]

This provides the groundwork for the approximation scheme that will be developed in the next section. The treatment additionally assumes that the boundary conditions \((\text{3})\) and \((\text{4})\) [or \((\text{4})\) and \((\text{3})\)] interpreted as functions of \( g \) and \( \xi \) are invertible. The accurate and rapid convergence of the numerical methods justifies this assumption although a rigorous proof cannot be given.

### 3 An approximation scheme for arbitrary differentially rotating disks of dust

It is now possible to attack general boundary value problems for differentially rotating disks of dust. With the above generalized solutions \( f = f(\xi; g) \) the boundary conditions [see formulas \((\text{4})\) \((\text{3})\) \((\text{6})\)] become a problem of inversion to determine \( g \) and \( \xi \) from \( \sigma_\rho \) or \( \Omega \):

\[
(A) \quad S(g; \xi) = \{ e^{U-k[\kappa, \zeta]} + 4Q/\sigma_\nu \sqrt{1 - (\rho/\rho_0)^2} \}(\xi; g) \leq 2\pi\psi \quad \text{or}

\[
(A') \quad O(g; \xi) = \{ Q/\Omega(0)(a, \zeta - a, Q) \}(\xi; g) \leq \Omega(0) = \Omega^0
\]

\[
(B) \quad D(g; \xi) = \{ \rho^2 \left[ Q^2 (e^{4U} + Q(\rho^2)) \zeta + (b, \rho)^2 \right] \}(\xi; g) \leq 0, \quad g(1) \leq 0
\]

\[
\text{Here,} \quad \mathcal{E} \text{ denotes the set of all Ernst potentials corresponding to disk-like sources.}
\]

\[
\text{The mathematical aspects of this assumption will be discussed in section 5.}
\]
This inversion problem is tackled in the following manner:

1. The only way to treat the complicated system (13) numerically seems to be by restricting it to a finite, discretized version and solving this by means of a Newton-Raphson method.

2. For this method, a good initial guess for the solution is needed. As shown in appendix C.1, there exists a representation of the functions \( g \) and \( \xi \) in terms of Chebyshev-polynomials in terms of Chebyshev-polynomials \( T_j(\tau) = \cos[j \arccos(\tau)] \):

\[
F_j(v_k) \doteq 0 \quad (1 \leq j, k \leq N_1 + N_2 - 1): \]

\[
\cdot F_j = D_j \quad (1 \leq j \leq N_1 - 1), \quad F_{N_1} = \varepsilon(g_m; \xi_n) - \varepsilon, \]

\[
F_{N_1+j-1} = S_j - 2\pi\psi_j \quad \text{or} \quad F_{N_1+j-1} = O_j - \Omega^*_j \quad (2 \leq j \leq N_2), \]

\[
v_k = g_{k+1} \quad (1 \leq k \leq N_1 - 1), \quad v_{N_1+k-1} = \xi_k \quad (1 \leq k \leq N_2) \]

\[
\cdot g(x^2) \approx \sum_{j=1}^{N_1} g_j T_{j-1}(2x^2 - 1) - \frac{1}{2}g_1, \quad g(1) \doteq 0 \Rightarrow g_1 = -2 \sum_{j=2}^{N_1} g_j \]

\[
\cdot \xi(x^2) \approx \sum_{j=1}^{N_2} \xi_j T_{j-1}(2x^2 - 1) - \frac{1}{2}\xi_1 \]

\[
\cdot \psi_p(x^2) \approx \sum_{j=1}^{N_2} \psi_j T_{j-1}(2x^2 - 1) - \frac{1}{2}\psi_1, \]

\[
\psi_p(0) \doteq 1 \Rightarrow \psi_1 = 2 \sum_{j=2}^{N_2} (-1)^j \psi_j + 2 \]

\[
\cdot \Omega^*[(\rho/\rho_0)^2] = \Omega(\rho)/\Omega(0): \]

\[
\Omega^*(x^2) \approx \sum_{j=1}^{N_2} \Omega^*_j T_{j-1}(2x^2 - 1) - \frac{1}{2}\Omega^*_1, \]

\[
\Omega^*(0) \doteq 1 \Rightarrow \Omega^*_1 = 2 \sum_{j=2}^{N_2} (-1)^j \Omega^*_j + 2 \]

\[
\cdot S(x^2 = \rho^2/\rho_0^2; g; \xi) \approx \sum_{j=1}^{N_2} S_j(g_m; \xi_n) T_{j-1}(2x^2 - 1) - \frac{1}{2}S_1(g_m; \xi_n) \]

\[
\cdot O(x^2 = \rho^2/\rho_0^2; g; \xi) \approx \sum_{j=1}^{N_2} O_j(g_m; \xi_n) T_{j-1}(2x^2 - 1) - \frac{1}{2}O_1(g_m; \xi_n) \]

\[
\cdot D(x^2 = \rho^2/\rho_0^2; g; \xi) \approx \sum_{j=1}^{N_1-N_2} D_j(g_m; \xi_n) T_{j-1}(2x^2 - 1) - \frac{1}{2}D_1(g_m; \xi_n) \]
The function \( \varepsilon(g_m; \xi_n) = M^2/J \) is determined using (16) for the above functions \( g \) and \( \xi \).

4. For the above system, the boundary values are assumed to be given in the form of the \( \psi_k \)'s or \( \Omega^* \)'s \((k = 2, \ldots, N_2)\). Moreover, some \( \varepsilon \ll 1 \) has to be prescribed. Then, good initial \( v_k \)'s come from the Newtonian expansion. The Newton-Raphson method improves the \( v_k \)'s and yields a very accurate solution to (15) for the chosen small \( \varepsilon \). Now, this solution serves as the initial estimate for the \( v_k \)'s belonging to a marginally increased value for \( \varepsilon \). Again, the Newton-Raphson method improves the solution, and one continues in this manner until this procedure ceases to converge. This occurs for some finite value \( \varepsilon_0 \), at the latest for \( \varepsilon = 1 \). A further discussion of this limit is given below.

5. A rather technical detail is the retranslation of the \( \xi_j \) into a set \( \{Y_\nu \}_q \) which then gives a satisfactory approximation of \( \xi \) in terms of (13). There are many ways to do this. Here, the following one has been chosen. One rewrites equation (13) in the equivalent form

\[
\exp \left[ x \xi(x^2; \{Y_\nu \}_q) \right] = \prod_{\nu=1}^{q} \frac{iY_\nu - x}{iY_\nu + x} = \frac{P_q(-x)}{P_q(x)} \quad \text{with} \quad P_q(x) = \sum_{\nu=1}^{q} b_\nu x^{\nu}.
\]

The coefficients \( b_\nu \) of the polynomial \( P_q \) can be determined by evaluating the left hand side at \( q \) arbitrary different points \( x_\mu \in [0, 1] \) and solving the following linear system:

\[
\exp \left[ x \xi(x^2; \{Y_\nu \}_q) \right] \sum_{\nu=1}^{q} b_\nu x^{\nu} = \sum_{\nu=1}^{q} b_\nu (-x_\mu)^{\nu}.
\]

The zeros of \( P_q \) determine the \( Y_\nu \).

The above scheme has been performed for many different prescribed surface mass densities and angular velocities. This provides strong evidence for the conjecture that, in this manner, all Newtonian disks can be extended into the relativistic regime. It has been found that the value for \( \varepsilon_0 \), the limiting parameter for the convergence of this scheme, depends on the chosen profile for \( \psi_p \) (or equivalently for \( \Omega^* \)). It is illustrated in appendix C.2, how the Ernst potential always tends to the extreme Kerr solution as \( \varepsilon \to 1 \). This supports a conjecture by Bardeen and Wagoner. But \( \varepsilon_0 = 1 \) does not hold for all given surface mass densities. Even in the Newtonian regime there are surface mass densities for which a realistic physical disk cannot be found since the corresponding angular velocity would become imaginary. If one chooses a profile for \( \sigma_p \) not very different from these, then the Newtonian limit still might exist, but some \( \varepsilon_0 < 1 \) turns up, beyond which the method does not converge. In the case of prescribed angular velocity, the situation is similar. Here, for any sequence \( f = f(g_\varepsilon) \) the angular velocity \( \Omega^* \) tends for all \( x^2 \in [0, 1] \) to 1 as \( \varepsilon \to 1 \). So, each nonuniform rotation law will lead to some \( \varepsilon_0 < 1 \) (see section 4 for examples).

The above expansions in terms of Chebyshev-polynomials allow a very accurate representation with only a small number of coefficients. However, the retranslation of \( \xi \) (see the above point 5) leads to functions that are not especially well suited for an approximation. In particular, if the boundary condition \( \psi_p \) is chosen to be close to those for which there is no Newtonian disk, then the accuracy cannot be driven.

\[9\text{Here, zeros of Chebyshev-polynomials have been used.}\]
particularly high by the computer program used, although the method in principle allows arbitrary approximation (see section 4.2). For $\psi_p$'s sufficiently far away from those critical ones, the accuracy obtained was very high. By choosing appropriate values for $N_1$ and $N_2$ one can always achieve extremely good agreement with the dust condition (12 digits and beyond) which ensures a realistic physical interpretation of the solution. The accuracy to which the second boundary condition, (3) or (6), can be satisfied, depends on the parameter $\varepsilon$. It is usually around 8 digits in the weak relativistic regime, and falls as $\varepsilon$ increases, but is still around 4 digits as $\varepsilon$ tends to $\varepsilon_0$. These values arose for $N_1 = 30$, $N_2 = 12$, and typical $\psi_p$'s (like $\psi_p$'s depending linearly on $x^2$) and $\Omega^*$'s (e. g. the realistic one considered in section 4.1). The number $q$ of the parameters $Y_\nu$ by which $\xi$ is represented, was chosen to be between 20 and 30 (independently of $N_2$).

What remains to be discussed is the regularity of the Ernst potentials that were obtained. For a few of the solutions, the functions $e^{2U}$ and $b$ were plotted over the coordinates $\rho$ and $\zeta$. Moreover, the agreement of the alternative representations of $M$ and $J$, as given by the behaviour of the Ernst potential at infinity

\[ U = -\frac{M}{r} + \mathcal{O}(r^{-2}), \quad b = -2J \frac{\cos \theta}{r^2} + \mathcal{O}(r^{-3}), \quad (r = \sqrt{\rho^2 + \zeta^2}, \ \zeta = r \cos \theta) \]

with the results from formulas (16) yields good confirmation of the regularity. This agreement was checked for all solutions that were calculated.

4 Representative examples

From the numerous solutions obtained, three particular sets of differentially rotating disks are discussed in more detail. The first one is an example of disks revolving with a realistic rotation law. The second set illustrates the break down of the numerical method for a specially prescribed surface mass density $\sigma_p$ at some $\varepsilon_0 < 1$. On the other hand it is demonstrated that, for the same $\sigma_p$, regular solutions can be found in the highly relativistic regime. Finally, the third example concerns the occurrence of a second ergoregion for a particular series of disks and, moreover, the gradual merging of the two spatially separated ergoregions as $\varepsilon$ increases.

The deviations between the boundary values obtained for particular numerical solutions and the given boundary conditions are listed in tables. The quantities $\Delta_D$, $\Delta_\Omega$, and $\Delta_\sigma$ therein are defined by

\[ \Delta_D = \max_{x^2 \in [0,1]} |D_{\text{obt}}(x^2; g; \xi)| \]

\[ \Delta_\Omega = \max_{x^2 \in [0,1]} \left| \Omega_{\text{obt}}^*(x^2) - \Omega_{\text{giv}}^*(x^2) \right| \]

\[ \Delta_\sigma = \max_{x^2 \in [0,1]} \left| \psi_{p\text{obt}}^*(x^2) - \psi_{p\text{giv}}^*(x^2) \right| , \]

where the indices 'obt' and 'giv' refer to obtained and given quantities, respectively. Moreover, by letters (a), . . . , (e), special examples are marked, for which illustrative graphs have been made. Here, curves drawn in the same line style belong to the same solution. The graphs show the dimensionless quantities $\rho_0 \sigma_p$ and $\rho_0 \Omega$ as well as $g$ and $\xi$ plotted against the normalized radial coordinate $\rho/\rho_0$ and $x$, respectively.
4.1 Disks possessing a realistic rotation law

As motivated by observations in astrophysics the rotation law of a galaxy is often modelled by an equation of the form (see [27])

\[
\Omega(\rho) = \frac{\Omega(0)}{\sqrt{1 + \rho^2 / \rho_1^2}}.
\]

Here, the parameter \(\rho_1\) varies for different galaxies. In the following series of solutions illustrated in figure 1, \(\rho_1 = 0.7 \rho_0\) has been chosen. As described in section 3, there is a limiting parameter \(\varepsilon_0 \approx 0.935\), for which the numerical method ceases to converge.

| \(\varepsilon = M^2/J\) | [a] | [b] | [c] | [d] | [e] | [f] | [g] | [h] | [i] |
|------------------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| \(\Delta_D \cdot 10^{12}\) | 0.5 | 0.65 | 0.72 | 0.79 | 0.86 | 0.9 | 0.92 | 0.93 | 0.9336 | 0.9344 | 0.9345 |
| \(\Delta_\Omega \cdot 10^6\) | 0.02 | 0.04 | 0.09 | 0.7 | 6 | 30 | 60 | 120 | 170 | 180 | 200 |

Figure 1: Disks possessing the rotation law (17) with \(\rho_1 = 0.7 \rho_0\)
\((N_1 = 30, N_2 = 12)\).

4.2 Disks with a critical surface mass density

For the following sequence of solutions, a surface mass density of the form

\[
\sigma_p(\rho) = \sigma_0 \left( 1 - 3 \frac{\rho^2}{\rho_0^2} + \beta \frac{\rho^4}{\rho_0^4} \right) \sqrt{1 - \frac{\rho^2}{\rho_0^2}}
\]

(18)
It turns out that for $\beta > \beta_N \approx 7$ no Newtonian disks with a real angular velocity can be found. On the other hand, for $\beta = 5.5$, all relativistic solutions for $0 \leq \varepsilon \leq 1$ exist. The table and graphs of figure 2 refer to the case $\beta = 6$. Starting here from the Newtonian solution, one soon recognizes a first limiting parameter $\varepsilon_0 \approx 0.60$ for which the method breaks down. However, by coming from solutions with $\beta = 5.5$ and $\varepsilon$ close to 1, it is possible to create highly relativistic solutions with $\beta = 6$. In fact, there is another limiting parameter, $\varepsilon_1 \approx 0.97$, above which the solutions with $\beta = 6$ exist once again. Due to the nearness to the critical surface mass density (for $\beta = \beta_N$), the accuracy obtained for the boundary condition (3) is not very high.

4.3 Disks possessing spatially separated ergoregions

The particular set of disks depicted in figure 3 demonstrates the occurrence of a second ergoregion\textsuperscript{10}. These solutions do not satisfy a specially prescribed boundary condition (4) or (6), but have been constructed in the following manner as intermediate solutions.

\textsuperscript{10}An ergoregion is a portion of the $(\rho, \zeta)$-space within which the function $e^{2U}$ is negative.
Figure 3: Example for a series of disks possessing spatially separated ergoregions. In the uppermost picture, the rims of the ergoregions in the $(\rho/\rho_0, \zeta/\rho_0)$-space are to be seen ($N_1 = 40, N_2 = 9$).
If one investigates solutions with surface mass densities similar to those of (18), one recognizes two minima for \(e^{2U}\) (taken as a function of \(\rho\), \(0 \leq \rho \leq \rho_0\), \(\zeta = 0\)), say at \(\rho_a\) and \(\rho_b > \rho_a\). Now, for a particular choice of \(\sigma_p\) it is possible to get \(e^{2U}(\rho_a) > 0\) and \(e^{2U}(\rho_b) < 0\), whilst by another choice one can achieve \(e^{2U}(\rho_a) < 0\) and \(e^{2U}(\rho_b) > 0\). This makes clear, that disks with spatially separated ergoregions can be constructed by interpolating between these solutions.

For the chosen example, there is only a narrow interval \((\varepsilon_a, \varepsilon_b)\) for which the two separated ergoregions occur. As can be seen from figure 3, after creation of the second ergoregion at \(\varepsilon_a \approx 0.8403\), both ergoregions grow as \(\varepsilon\) increases. Eventually, at \(\varepsilon_b \approx 0.8415\), the ergoregions merge into one ergoregion.

5 Discussion of mathematical aspects

As already mentioned in section 2, the assumption that the function \(\Phi_g\) introduced in (14) can be extended to form a continuous mapping defined on \(A\), lies at the heart of the above numerical methods. Although this assumption seems to be intuitive, it is not trivial. Consider the following example:

For any analytic function \(\psi : [0, 1] \to \mathbb{R}\) one finds the equality

\[
\lim_{q \to \infty} \sum_{\nu=1}^{q} \ln \left[ 1 + \frac{1}{q} \psi \left( \frac{\nu}{q} \right) \right] = \int_{0}^{1} \psi(t) \, dt.
\]

From this it follows that

\[
2 \int_{0}^{1} \phi(t) \, dt = \lim_{q \to \infty} \sum_{\nu=1}^{q} \ln \frac{q + x\phi(\nu/q)}{q - x\phi(\nu/q)} \quad \text{with} \quad \psi(t) = \pm x \phi(t).
\]

Hence, the function \(\xi(x^2) \equiv 2\) can be represented by any sequence of the form

\[
Y^{(q)}_{\nu} = \frac{1}{x \phi(\nu/q)} \quad \text{with} \quad \int_{0}^{1} \phi(t) \, dt = 1.
\]

Since these sequences might be quite different from each other, it is rather surprising that all of them approximate the same Ernst potential given by (10). But this follows from the above assumption.

This already indicates the difficulties which are connected with a rigorous proof of this assumption because the Ernst potential is only given in terms of the set \(\{Y_{\nu}\}_q\) and not directly in terms of \(\xi\).

A further conjecture is strongly confirmed by extensive numerical investigations:

For the hyperelliptic class of solutions represented by (14) in appendix A, the functions \(\xi\) and \(g\) are given by

\[
\xi(x^2) = \frac{1}{2x} \ln \left[ \prod_{\nu=1}^{p} \frac{iX_{\nu} - x}{iX_{\nu} + x} \right],
\]

\[
g(x^2) = \text{sign} \left( \prod_{\nu=1}^{p} X_{\nu} \right) A_g(x^2) h(x^2),
\]

\[
A_g(x^2) = \sqrt{\prod_{\nu=1}^{p} (ix - X_{\nu})(ix - X_{\nu}^*)}, \quad A_g(x^2) > 0.
\]

11To verify this formula one simply expands the logarithms in the form \(\ln(1 + \varepsilon) = \varepsilon + \mathcal{O}(\varepsilon^2)\) and notes that the resulting sum tends to the Riemann integral of the right hand side.
In particular, in this formulation, the solution for the Neugebauer-Meinel-disk \[7, 8, 9\] assumes the form \( f = f(\xi; g) \) where

\[
\xi(x^2) = \frac{1}{2x} \ln \frac{x^2 - C_1(\mu)x + C_2(\mu)}{x^2 + C_1(\mu)x + C_2(\mu)},
\]

\[
C_1(\mu) = \sqrt{2[1 + C_2(\mu)]}, \quad C_2(\mu) = \frac{1}{\mu} \sqrt{1 + \mu^2},
\]

\[
g(x^2) = -\frac{1}{\pi} \text{arsinh}[\mu(1 - x^2)],
\]

and the parameter \( \mu, 0 < \mu < \mu_0 = 4.62966184..., \) is related to the angular velocity by

\[
\mu = 2\Omega^2 \rho_0^2 e^{-2V_0}, \quad V_0 = U(\rho = 0, \zeta = 0).
\]

As already mentioned, a direct proof of the above assumptions promises to be very complicated. But there might be an alternative proof which relies on relating a general solution of the Ernst equation to the solution of a so-called Riemann-Hilbert problem, see \[18, 21, 28, 29\]. In this treatment, an appropriately introduced matrix function, from which the Ernst potential can be extracted, is supposed to be regular on a two-sheeted Riemann surface of genus zero except for some given curve, where it possesses a well-defined jump behaviour. The freedom of two jump functions defined on this curve corresponds to the freedom to choose \( \xi \) and \( g \). Now, if one succeeds in finding a particular formulation of a Riemann-Hilbert problem in which \( \xi \) and \( g \) are involved, then the final solution for \( f \) proves to depend only on \( \xi \) (and \( g \)) and not on a particular global representation in terms of \( \{Y_\nu\}_q \). This deserves further investigation.

There is very strong numerical evidence for the validity of both assumptions. For various functions \( \xi \) (and functions \( g \)), different representations \( \{Y_\nu\}_q \) have been seen to approximate the same Ernst potential. In particular, the approximation of the Neugebauer-Meinel-solution in terms of Bäcklund solutions was carried out to give an agreement up to the 12th digit with the hyperelliptic solution, which confirms both assumptions.

### A The transition from the hyperelliptic solutions to the Bäcklund type solutions

In this section the Bäcklund type solutions are derived from the hyperelliptic class. The latter is assumed to be given in the form represented in \[12\] for an even integer \( p \geq 2 \):

\[
f = \exp \left( \sum_{\nu=1}^{p} \int_{X_\nu}^{X^{(\nu)}} \frac{X^p dX}{V(X)} - u_p \right)
\]

\[
V(X) = \sqrt{(X + iz)(X - iz)} \prod_{\nu=1}^{p} (X - X_\nu)(X - \bar{X}_\nu), \quad z = \frac{1}{\rho_0}(\rho + i\zeta)
\]

\[\text{The parameters } K^{(\nu)}, \text{ the upper integration limits } K^{(\nu)}, \text{ and the integration variable } K \text{ have to be replaced by their 'normalized' values } X_\nu = K^{(\nu)}/\rho_0, X^{(\nu)} = K^{(\nu)}/\rho_0, \text{ and } X = K/\rho_0, \text{ respectively.}\]
by the following assumptions:

To this end, the above expression for $f$ is rewritten in the equivalent form:

$$ f = \exp \left[ \sum_{\nu=1}^{p} \left( \int_{X_{2\nu-1}}^{X_{2\nu}} \frac{A(X) dX}{V(X)} + \int_{X_{2\nu}}^{X_{2\nu+1}} \frac{A(X) dX}{V(X)} \right) - \int_{-1}^{1} \frac{(-1)^{q} g(x^2) dx}{Z_D} \right] $$

The Jacobian inversion problem (20) reads as follows in a similarly rewritten form (1 ≤ $\mu$ ≤ $q$):

$$ \sum_{\nu=1}^{q} \left( \int_{X_{2\nu-1}}^{X_{2\nu}} \frac{A(X) dX}{V(X)(X - Y_{\mu})} + \int_{X_{2\nu}}^{X_{2\nu+1}} \frac{A(X) dX}{V(X)(X - Y_{\mu})} \right) = \int_{-1}^{1} \frac{(-1)^{q} g(x^2) dx}{(ix - Y_{\mu})Z_D} $$

Furthermore

$$ \int_{X_{2\nu-1}}^{X_{2\nu}} \frac{A(X) dX}{V(X)(X - Y_{\mu})} + \int_{X_{2\nu}}^{X_{2\nu+1}} \frac{A(X) dX}{V(X)(X - Y_{\mu})} = $$

$$ - \int_{X_{2\nu}}^{X_{2\nu+1}} \frac{A(X) dX}{V(X)(X - Y_{\mu})} + \int_{X_{2\nu-1}}^{X_{2\nu}} \frac{A(X) dX}{V(X)(X - Y_{\mu})} $$

with $X^{(2\nu)}$ now lying in the other sheet of the Riemann surface.

In the limit $\varepsilon \to 0$, one obtains

$$ \lim_{\varepsilon \to 0} \int_{X_{2\nu}}^{X_{2\nu+1}} \frac{A(X) dX}{V(X)(X - Y)} = \left\{ \begin{array}{ll} \pm \pi \delta_{\nu \mu}/|\lambda_{\mu}(Y_{\mu} + iz)| & \text{for } Y = Y_{\mu} \\ 0 & \text{for } Y = Y_{\mu} \end{array} \right. $$

The set $\{iX_{\nu}\}_p$ consists of arbitrary real parameters and/or pairs of complex conjugate parameters (in order to guarantee reflectional symmetry). The $z$-dependent values for the $X^{(\nu)}$ as well as the integration paths on a two-sheeted Riemann surface result from the Jacobian inversion problem (20).

The transition to the Bäcklund type solutions (7) can be obtained in the limit $\varepsilon \to 0$ by the following assumptions:

- $p = 2q$
- $X_{2\nu-1} = Y_{\nu} + \varepsilon \beta_{\nu}$, $X_{2\nu} = Y_{\nu}$ (1 ≤ $\nu$ ≤ $q$), $\{\beta_{\nu}\}_q$ arbitrary
- $g(x^2) = (-1)^{q} h(x^2) A(ix)$, $A(X) = \prod_{\nu=1}^{q} (X - Y_{\nu})(X - Y_{\mu})$.

To this end, the above result from the Jacobian inversion problem (20) reads as follows in a similarly rewritten form (1 ≤ $\mu$ ≤ $q$):

$$ \sum_{\nu=1}^{q} \left( \int_{X_{2\nu-1}}^{X_{2\nu}} \frac{A(X) dX}{V(X)(X - Y_{\mu})} + \int_{X_{2\nu}}^{X_{2\nu+1}} \frac{A(X) dX}{V(X)(X - Y_{\mu})} \right) = \int_{-1}^{1} \frac{(-1)^{q} g(x^2) dx}{(ix - Y_{\mu})Z_D} $$

Furthermore

$$ \int_{X_{2\nu-1}}^{X_{2\nu}} \frac{A(X) dX}{V(X)(X - Y_{\mu})} + \int_{X_{2\nu}}^{X_{2\nu+1}} \frac{A(X) dX}{V(X)(X - Y_{\mu})} = $$

$$ - \int_{X_{2\nu}}^{X_{2\nu+1}} \frac{A(X) dX}{V(X)(X - Y_{\mu})} + \int_{X_{2\nu-1}}^{X_{2\nu}} \frac{A(X) dX}{V(X)(X - Y_{\mu})} $$

with $X^{(2\nu)}$ now lying in the other sheet of the Riemann surface.

In the limit $\varepsilon \to 0$, one obtains

$$ \lim_{\varepsilon \to 0} \int_{X_{2\nu}}^{X_{2\nu+1}} \frac{A(X) dX}{V(X)(X - Y)} = \left\{ \begin{array}{ll} \pm \pi \delta_{\nu \mu}/|\lambda_{\mu}(Y_{\mu} + iz)| & \text{for } Y = Y_{\mu} \\ 0 & \text{for } Y = Y_{\mu} \end{array} \right. $$
with $\delta_{\mu \nu}$ being the usual Kronecker symbol and $\lambda_\mu$ as defined in (7).

The second term amounts to
\[
\lim_{\varepsilon \to 0} \int_{X^{(2\nu)}}^{X^{(2\nu-1)}} \frac{A(X) dX}{V(X)(X - Y)} = \int_{X^{(2\nu)}} \frac{dX}{(X - Y)(X - i\varepsilon)(X + i\varepsilon)} = \\
\frac{1}{\lambda(Y)(Y + i\varepsilon)} \ln \left( \frac{[\lambda(X^{(2\nu-1)}) - \lambda(Y)] [\lambda(X^{(2\nu)}) + \lambda(Y)]}{[\lambda(X^{(2\nu-1)}) + \lambda(Y)] [\lambda(X^{(2\nu)}) - \lambda(Y)]} \right),
\]
where for evaluation of the second integral the substitution
\[
\lambda = \sqrt{\frac{X - i\varepsilon}{X + i\varepsilon}}
\]
has been used.

Hence, the Jacobian inversion problem reads as follows in the limit $\varepsilon \to 0$:
\[
\prod_{\nu=1}^{q} \frac{[\lambda(X^{(2\nu-1)}) - \lambda_{\mu}] [\lambda(X^{(2\nu)}) + \lambda_{\mu}]}{[\lambda(X^{(2\nu-1)}) + \lambda_{\mu}] [\lambda(X^{(2\nu)}) - \lambda_{\mu}]} = -\gamma_{\mu} \tag{21}
\]
\[
\prod_{\nu=1}^{q} \frac{[\lambda(X^{(2\nu-1)}) - \lambda^*_{\mu}] [\lambda(X^{(2\nu)}) + \lambda^*_{\mu}]}{[\lambda(X^{(2\nu-1)}) + \lambda^*_{\mu}] [\lambda(X^{(2\nu)}) - \lambda^*_{\mu}]} = \overline{\gamma}_{\mu} \tag{22}
\]
and in an analogous manner
\[
f = f_0 \prod_{\nu=1}^{q} \frac{[\lambda(X^{(2\nu-1)}) + 1] [\lambda(X^{(2\nu)}) - 1]}{[\lambda(X^{(2\nu-1)}) - 1] [\lambda(X^{(2\nu)}) + 1]} \tag{23}
\]
[with $\gamma_{\mu}, \lambda^*_{\mu}$ and $f_0$ as defined in (7).]

Instead of evaluating the quantities $\lambda(X^{(\nu)})$, ($1 \leq \nu \leq 2q$), the coefficients $b_\nu$ and $c_\nu$ ($1 \leq \nu \leq q$) of the polynomial
\[
P(\lambda) = \prod_{\nu=1}^{q} [\lambda - \lambda(X^{(2\nu-1)})] [\lambda + \lambda(X^{(2\nu)})]
\]
\[
\quad = \lambda^{2q} + \lambda \sum_{\nu=1}^{q} b_\nu \lambda^{2\nu - 2} + \sum_{\nu=1}^{q} c_\nu \lambda^{2\nu - 2} \tag{24}
\]
are determined. Since
\[
\frac{P(\lambda_{\mu})}{P(-\lambda_{\mu})} = -\gamma_{\mu}, \quad \frac{P(\lambda^*_{\mu})}{P(-\lambda^*_{\mu})} = \overline{\gamma}_{\mu}, \quad f = f_0 \frac{P(-1)}{P(1)}, \tag{25}
\]
the following system of linear equations for the quantities $b_\nu, c_\nu, P(1)$, and $P(-1)$ emerges:
\[ \sum_{\nu=1}^{q} \left[ b_{\nu} \alpha_{\mu} \lambda_{\mu}^{2\nu-1} + c_{\nu} \lambda_{\mu}^{2\nu-2} \right] = -\lambda_{\mu}^{2q}, \]

\[ \sum_{\nu=1}^{q} \left[ b_{\nu} \alpha_{\mu}^* (\lambda_{\mu}^*)^{2\nu-1} + c_{\nu} (\lambda_{\mu}^*)^{2\nu-2} \right] = -(\lambda_{\mu}^*)^{2q} \]  \hspace{1cm} (26)

\[ \sum_{\nu=1}^{q} (b_{\nu} - c_{\nu}) + P(-1) = 1 \]

\[ \sum_{\nu=1}^{q} (b_{\nu} + c_{\nu}) - P(1) = -1, \]

with \( \alpha_{\mu} \) and \( \alpha_{\mu}^* \) as defined in (\ref{eq:alpha}).

Finally, if the solution of this linear system for \( P(\pm 1) \) is expressed by means of Cramer’s rule, the desired form (\ref{eq:backlund}) of the Bäcklund type is obtained.

\section*{B \hspace{1cm} Invariance properties of the Ernst potential}

For the proof of the properties (\ref{eq:invariance}) of the Ernst (\ref{eq:ernst}) potential, the Ernst potential (\ref{eq:ernst}) is reformulated by

\[ f(\{Y_{\nu}\}_q; g) = f_0 \frac{D(-1; \{Y_{\nu}\}_q; g)}{D(1; \{Y_{\nu}\}_q; g)} \]  \hspace{1cm} (27)

with

\[ \cdot D(\lambda; \{Y_{\nu}\}_q; g) \]

\[ = \begin{vmatrix} a_1 & (a_1 x_1) & \cdots & (a_1 x_1^{q-1}) & 1 & x_1 & \cdots & x_1^q \\ a_2 & (a_2 x_2) & \cdots & (a_2 x_2^{q-1}) & 1 & x_2 & \cdots & x_2^q \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{2q+1} & (a_{2q+1} x_{2q+1}) & \cdots & (a_{2q+1} x_{2q+1}^{q-1}) & 1 & x_{2q+1} & \cdots & x_{2q+1}^q \end{vmatrix} \]

\[ \cdot a_1 = \lambda, \quad a_{2\nu} = \alpha_{\nu} \lambda_{\nu}, \quad a_{2\nu+1} = \alpha_{\nu}^* \lambda_{\nu}^*, \]

\[ \cdot x_1 = \lambda^2, \quad x_{2\nu} = \lambda_{\nu}^2, \quad x_{2\nu+1} = (\lambda_{\nu}^*)^2. \]

The above expression for \( D(\lambda; \{Y_{\nu}\}_q; g) \) is a Vandermonde-like determinant. These determinants have been studied in detail by Steudel, Meinel and Neugebauer (\cite{30}).

By their \textit{reduction formula} (\textit{8}) of (\cite{30}), \( D \) assumes the form:

\[ D(\lambda; \{Y_{\nu}\}_q; g) \]

\[ = V_{q,q+1}(a_r; b_r | x_r) \quad \text{[with } b_r = 1 \text{ for } r = 1 \ldots (2q + 1)\text{]} \]

\[ = \sum_{P} \varepsilon_{P} \left( \prod_{j=1}^{q} a_{r(j)} \right) V_q[x_{r(1)}, \ldots, x_{r(q)}] V_{q+1}[x_{r(q+1)}, \ldots, x_{r(2q+1)}] \]

where

\[ \cdot \text{the sum runs over all permutations } P = \{r(1), \ldots, r(2q+1)\} \text{ of } (1, 2, \ldots, 2q+1) \]

\[ \text{with } r(k) < r(j) \text{ for } k < j < q \text{ as well as for } q \leq k < j \]
\[
\varepsilon_P = \begin{cases} 
+1 & \text{for } P \text{ even} \\
-1 & \text{for } P \text{ odd} 
\end{cases}
\]

- the Vandermonde determinants are given by
\[
\mathcal{V}_N[x_1, \ldots, x_N] = \prod_{k>j} (x_k - x_j).
\]

In this formulation the following properties can be proved:

(A) If \( x_{2q+1} = x_{2q} \) then
\[
D(\lambda; \{Y_{\nu}\}_{q}; g) = (-1)^q (a_{2q} - a_{2q+1}) \left[ \prod_{j=1}^{2q-1} (x_{2q} - x_j) \right] D(\lambda; \{Y_{\nu}\}_{q-1}; -g)
\]

(B) If \( x_{2q} = 1 + \kappa \epsilon + O(\epsilon^2) \), \( x_{2q+1} = 1 - \kappa \epsilon + O(\epsilon^2) \), and \( (a_{2q}a_{2q+1}) = 1 + O(\epsilon) \), then
\[
D(\mp 1; \{Y_{\nu}\}_{q}; g) = \kappa \epsilon \left[ \prod_{j=2}^{2q-1} (1 - x_j) \right] (a_{2q} + a_{2q+1} \pm 2) D(\mp 1; \{Y_{\nu}\}_{q-1}; g) + O(\epsilon^2).
\]

With (A) the equalities \[3\] and \[10\] can be derived whilst (B) serves to confirm \[1\] and \[12\]. In order to prove (A) consider the following groups of permutations separately:

\begin{align*}
P_1 & : r(q - 1) = 2q, \ r(q) = 2q + 1 \\
P_2 & : r(2q) = 2q, \ r(2q + 1) = 2q + 1 \\
P_3 & : r(q) = 2q, \ r(2q + 1) = 2q + 1 \\
P_4 & : r(q) = 2q + 1, \ r(2q + 1) = 2q
\end{align*}

For \( x_{2q+1} = x_{2q} \), all terms belonging to \( P_1 \) and \( P_2 \) vanish while all terms belonging to \( P_3 \) and \( P_4 \) possess a common factor, \([a_{2q} \prod_{j=1}^{2q-1} (x_{2q} - x_j)] \) and \([a_{2q+1} \prod_{j=1}^{2q-1} (x_{2q} - x_j)] \), respectively. After reordering (from which the factor \((-1)^q\) results), (A) is easily obtained.

The proof for (B) works similarly. Now, eight groups of permutations have to be considered separately:

\begin{align*}
P_{1a} & : r(1) = 1, \ r(q - 1) = 2q, \ r(q) = 2q + 1 \\
P_{1b} & : r(q + 1) = 1, \ r(2q) = 2q, \ r(2q + 1) = 2q + 1 \\
P_{2a} & : r(q) = 2q, \ r(q + 1) = 1, \ r(2q + 1) = 2q + 1 \\
P_{2b} & : r(1) = 1, \ r(q) = 2q, \ r(2q + 1) = 2q + 1 \\
P_{3a} & : r(q) = 2q + 1, \ r(q + 1) = 1, \ r(2q + 1) = 2q \\
P_{3b} & : r(1) = 1, \ r(q) = 2q + 1, \ r(2q + 1) = 2q \\
P_{4a} & : r(q - 1) = 2q, \ r(q) = 2q + 1, \ r(q + 1) = 1 \\
P_{4b} & : r(1) = 1, \ r(2q) = 2q, \ r(2q + 1) = 2q + 1
\end{align*}
All terms of permutations with a coinciding first index can be combined to give\([9]\):

\[
\{P_{1a}, P_{1b}\} \implies O(\varepsilon^3)
\]

\[
\{P_{2a}, P_{2b}\} \implies a_2 q F + O(\varepsilon^2)
\]

\[
\{P_{3a}, P_{3b}\} \implies a_{2q+1} F + O(\varepsilon^2)
\]

\[
\{P_{4a}, P_{4b}\} \implies \pm 2 F + O(\varepsilon^2)
\]

with

\[
F = (-1)^{q+1} \kappa \epsilon \prod_{j=2}^{2q-1} (1 - x_j) D(\pm 1; \{Y_{\nu}\}_{q-1}; -g).
\]

C Newtonian and ultrarelativistic limits

C.1 The Newtonian limit

In the limit of small functions \(g\) and \(\xi\), i.e.,

\[
g(x^2) = \varepsilon g_0(x^2) + O(\varepsilon_g^2), \quad \xi(x^2) = \varepsilon \xi_0(x^2) + O(\varepsilon^2),
\]

the Ernst potential \(f = f(\xi; g)\) as introduced in section 2 is given by

\[
f(\xi; g) = 1 - \varepsilon_g \int_{-1}^{1} \frac{g(x^2)dx}{Z_D} - i \varepsilon_g \xi_0 \int_{-1}^{1} \frac{(ix)g(x^2)\xi_0(x^2)dx}{Z_D} + O(\varepsilon_g^2) + O(\varepsilon_\xi^2).
\]

(28)

In this section, the above property will be proved and the functions \(g_0\) and \(\xi_0\) will be derived as they result from the Newtonian expansion of the boundary conditions.

C.1.1 The Ernst potential for small functions \(g\) and \(\xi\)

Due to the assumption that the function \(\Phi_g\) introduced in (14) can be extended to form a continuous mapping defined on \(\mathcal{A}\) (see sections 2 and 5), the representation of \(\xi\) in terms of \(\{Y_{\nu}\}_q\) can be chosen arbitrarily. Here, the following set \(\{Y_{\nu}\}_q\) is used:

\[
\cdot \quad q = 4r
\]

\[
\begin{cases}
Y_{4\nu-3} = Z_{\nu}(1 + \varepsilon_\xi z_{\nu}), & Y_{4\nu-2} = -\overline{Y}_{4\nu-3} \\
Y_{4\nu-1} = \overline{Z}_{\nu}(1 - \varepsilon_\xi z_{\nu}), & Y_{4\nu} = -\overline{Y}_{4\nu-1}
\end{cases}, \quad \Re(Z_{\nu}) \neq 0, \quad z_{\nu} \in \mathbb{R} \\
(\nu = 1 \ldots r)
\]

Then, it follows from (13) that \(\xi(x^2) = \varepsilon \xi_0(x^2) + O(\varepsilon_\xi^2)\) with

\[
\xi_0(x^2) = -4i \sum_{\nu=1}^{r} \frac{z_{\nu}(Z_{\nu} - \overline{Z}_{\nu})(x^2 - Z_{\nu}

To evaluate the Ernst potential in this limit, the formulation [21][26] in appendix A is used and the following steps are performed:

\footnote{Here the requirements \(a_1 = \mp 1, x_1 = 1\) are necessary. Additionally, for \(P_{4a}\) and \(P_{4b}\), the constraint \(a_{2q} a_{2q+1} = 1 + O(\epsilon)\) is needed.}
C.1.2 The functions $g$ for any family of Ernst potentials $t$ rotationally asymmetric disks of dust with the parameter $\varepsilon = 1 + f^2$. Thus, with any zero $\tilde{\gamma}^2$. Finally, if $\varepsilon \to 0$, the coefficients $b_\nu$ of the polynomial \((24)\) vanish. This can be seen by considering the solution to linear system (20).

$$b_\nu = \frac{D_\nu}{D} :$$

$$\begin{align*}
D &= \begin{pmatrix}
a_2 & \cdots & (a_2x_2^{q-1}) & 1 & x_2 & \cdots & x_2^{q-1} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
a_{2q+1} & \cdots & (a_{2q+1}x_{2q+1}^{q-1}) & 1 & x_{2q+1} & \cdots & x_{2q+1}^{q-1}
\end{pmatrix} \\
\cdot a_{2q} = \alpha_\eta \lambda_\eta, \quad a_{2q+1} = \alpha_\eta^* \lambda_\eta, \quad x_{2q} = \lambda_\eta^2, \quad x_{2q+1} = (\lambda_\eta^*)^2
\end{align*}$$

$D_\nu$ is derived from $D$ by replacing the $\nu$-th column by the vector $\{-x_2^\nu, \ldots, -x_{2q+1}^\nu\}$.

For $1 \leq \nu \leq q$, $D_\nu$ can be expanded in terms of Vandermonde determinants

$$V_{q+1}(x_{r(1)}, \ldots, x_{r(q+1)}), \quad r(\eta) \in \{2, \ldots, 2q + 1\}, \quad r(\eta) < r(\mu) \text{ for } \eta < \mu.$$  

In the limit $\varepsilon \to 0$, any set $\{x_{r(\eta)}\}_{q+1}$ contains at most $q$ different values, and therefore all $D_\nu$ vanish. On the other hand, $D$ remains finite (here only Vandermonde determinants $V_q$ are involved), and hence all $b_\nu$ tend to zero.

2. Thus, with any zero $\tilde{\lambda}_\nu$ of the Polynomial \((24)\), $(-\tilde{\lambda}_\nu)$ also becomes a zero as $\varepsilon \to 0$. This set of zeros is ordered in the following way:

$$\{\lambda(X^{(1)}), -\lambda(X^{(2)}), \ldots, -\lambda(X^{(2q-1)}), -\lambda(X^{(2q)})\} = \{\tilde{\lambda}_1, -\tilde{\lambda}_1, \ldots, \tilde{\lambda}_q, -\tilde{\lambda}_q\},$$

Suppose there is a $\lambda_\mu$ different from all zeros:

$$\lambda_\mu \neq \lambda(X^{(2\nu-1)}) = \lambda(X^{(2\nu)}) \quad \text{and} \quad \lambda_\mu \neq -\lambda(X^{(2\nu-1)}) \quad \text{for all } \nu = 1 \ldots q.$$  

Then, since $\gamma_\mu \neq -1$ for small $g$, \((21)\) cannot be satisfied.

3. This gives rise to the following ansatz ($\nu = 1 \ldots q$):

$$\lambda^2(X^{(2\nu-1)}) = \lambda_\nu^2 + \varepsilon_\xi \kappa_{2\nu-1} + O(\varepsilon_\xi^2), \quad \lambda^2(X^{(2\nu)}) = \lambda_\nu^2 + \varepsilon_\xi \kappa_{2\nu} + O(\varepsilon_\xi^2),$$

by which the system \((21) (22)\) can easily be solved to get the set $\{\kappa_{2\nu}\}_{2q}$.

4. Finally, if $g(x^2) = \varepsilon_\eta \eta_0(x^2) + O(\varepsilon_\eta^2)$ is considered, then \((28)\) follows from \((23)\) by inserting the values obtained for $\{\lambda(X^{(\nu)})\}_{2q}$.

C.1.2 The functions $g_0$ and $\xi_0$ as resulting from the boundary conditions

For any family of Ernst potentials $f = f(g_0; \xi_0)$ describing a sequence of differentially rotating disks of dust with the parameter $\varepsilon = M^2/J$ \([M \text{ and } J \text{ as defined in } (10)]\), the following expansion is valid (see \([24, \text{ pp. 83-89}])$:

$$f = 1 + \varepsilon_2(\rho, \zeta)\varepsilon^2 + i b_3(\rho, \zeta)\varepsilon^3 + O(\varepsilon^4).$$
By comparison with (28) one gets
\[ \cdot \varepsilon_\theta = \varepsilon^2, \quad \varepsilon_\phi = \varepsilon, \]
\[ \cdot e_2(\rho, \xi) = - \int_{-1}^{1} \frac{g_0(x^2)dx}{Z_D}, \quad b_3(\rho, \xi) = - \int_{-1}^{1} \frac{(ix)g_0(x^2)\xi_0(x^2)dx}{Z_D}. \]

If the boundary conditions,
\[ \cdot \sigma_\rho(\rho) = \sigma_0 \psi_2[(\rho/\rho_0)^2] \sqrt{1 - (\rho/\rho_0)^2} \varepsilon^2 + \mathcal{O}(\varepsilon^4) \quad \text{with } \psi_2(0) = 1 \quad \text{or} \]
\[ \cdot \Omega(\rho) = \Omega_0 \Omega_1 [(\rho/\rho_0)^2] \varepsilon + \mathcal{O}(\varepsilon^3) \quad \text{with } \Omega_1(0) = 1, \]
are given, then it follows from equations (3-4) that
\[ \cdot (e_2)_\xi = 4\pi \sigma_0 \psi_2 \sqrt{1 - (\rho/\rho_0)^2} \quad \text{or} \quad (e_2)_\rho = 2\Omega_0^2 \Omega_1^2 \rho \quad \text{and} \]
\[ \cdot (b_3)_\rho = 2\rho \Omega_0 \Omega_1 (e_2)_\xi. \]

By expressing \( e_2 \) and \( b_3 \) in terms of \( g_0 \) and \( \xi_0 \) in these equations, one gets Abelian integral equations for \( \xi_0 \) and \( g_0 \). Their solutions read as follows:
\[ g_0(x^2) = -4\sigma_0 (1 - x^2) \int_0^{\pi/2} (\sin^2 \phi) \psi_2 (\cos^2 \phi + x^2 \sin^2 \phi) d\phi \]
\[ g_0(x^2)\xi_0(x^2) = 8\sigma_0 \Omega_0 (1 - x^2) \int_0^{\pi/2} (\sin^2 \phi) \psi_1 (\cos^2 \phi + x^2 \sin^2 \phi) d\phi \]

(with \( \psi_1(x^2) = \Omega_1(x^2) \psi_2(x^2) \)).

Note that only one of the functions \( \psi_2 \) and \( \Omega_1 \) can be prescribed since both represent different boundary conditions of the same Newtonian potential \( e_2 \). Likewise, the constants \( \sigma_0 \) and \( \Omega_0^2 \) depend on each other. Moreover, these constants in terms of \( \psi_2 \) and \( \Omega_1 \) are prescribed by the equation \( \varepsilon = M^2/J \).

C.2 The ultrarelativistic limit

It is difficult to relate the functions \( g \) and \( \xi \) of an Ernst potential \( f = f(g; \xi) \) to its physical properties like \( M \) and \( J \). Nevertheless, if a sequence \( f(g_\varepsilon; \xi_\varepsilon) \) can be extended to arbitrary values \( \varepsilon < 1 \), then, in the limit \( \varepsilon \to 1 \), the universal solution of an extreme Kerr black hole is reached. It is illustrated how this limit results from the form (5) of the Ernst potential.

If the limit \( \rho_0 \to 0 \) is considered for finite values of \( r = \sqrt{\rho^2 + \xi^2} \), then by using the formulation (27) one gets (with \( \xi = r \cos \theta \)):
\[ f = \left( 1 - \frac{\rho_0}{r} \right) \int_{-1}^{1} (-1)^q g(x^2)dx + \mathcal{O}(\rho_0^2) \]
\[ \left[ \frac{E_1 r + \rho_0 |E_1 \cos \theta - (\rho^2 - 1)^q E_2|}{E_1 r + \rho_0 |E_3 \cos \theta + (\rho^2 - 1)^q E_2|} + \mathcal{O}(\rho_0^3) \right]. \]

The \( E_j \) do not depend on \( \rho \) and \( \xi \) but on \( g \) and \( \xi \). In particular:
\[ \cdot E_1 = \begin{bmatrix} b_1 & (b_1 Z_1) & \cdots & (b_1 Z_1^{q-1}) & 1 & \cdots & Z_1^{q-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ b_{2q} & (b_{2q} Z_{2q}) & \cdots & (b_{2q} Z_{2q}^{q-1}) & 1 & \cdots & Z_{2q}^{q-1} \end{bmatrix} \]
\[ E_2 = \begin{bmatrix} b_1 & (b_1 Z_1) & \cdots & (b_1 Z_1^{q-2}) & 1 & Z_1 & \cdots & Z_1^q \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{2q} & (b_{2q} Z_{2q}) & \cdots & (b_{2q} Z_{2q}^{q-2}) & 1 & Z_{2q} & \cdots & Z_{2q}^q \end{bmatrix} \]

\[ b_{2\nu - 1} = -\tanh \left[ \frac{1}{2} \int_{-1}^{1} \frac{(-1)^q g(x^2)dx}{ix - Y_\nu} \right], \quad b_{2\nu} \overline{v}_{2\nu - 1} = 1 \]

Clearly, if \( E_1 \neq 0 \) then \( \lim_{\rho_0 \to 0} f = 1 \). The Ernst potential passes to an ultrarelativistic limit if \( E_1 \) and \( \rho_0 \) tend simultaneously to zero such that

\[ \Omega_U = \lim_{\rho_0 \to 0} \frac{(-1)^q E_1}{2E_2 \rho_0} \]

exists. Then one gets

\[ f = \frac{2\Omega_U r + E_4 \cos \theta - 1}{2\Omega_U r + E_4 \cos \theta + 1}. \]

The only Ernst potential of this form which is asymptotically flat and regular for \( r > 0 \) is the extreme Kerr solution. The constant \( \Omega_U \) is then real and describes the ‘angular velocity of the horizon’. Moreover, \( J = 1/(4\Omega_U^2) = M^2 \), and hence \( \varepsilon = 1 \).

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\[ ^{14} \text{It can be shown that } E_1^2 \in \mathbb{R}. \text{ Hence, the ultrarelativistic limit for the family } f(g_1; \xi) \text{ is performed when some function } E_a = E_a(g_1; \xi) = E_a(\xi)E_1^2(\xi), \text{ which is independent of the representation } \{ Y_\nu \}_q, \text{ vanishes.} \]
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