ON THE STRUCTURAL THEOREM OF PERSISTENT HOMOLOGY

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Abstract. We study the categorical framework for the computation of persistent homology, without reliance on a particular computational algorithm. The computation of persistent homology is commonly summarized as a matrix theorem, which we call the Matrix Structural Theorem. Any of the various algorithms for computing persistent homology constitutes a constructive proof of the Matrix Structural Theorem. We show that the Matrix Structural Theorem is equivalent to the Krull-Schmidt property of the category of filtered chain complexes. We separately establish the Krull-Schmidt property by abstract categorical methods, yielding a novel nonconstructive proof of the Matrix Structural Theorem.

These results provide the foundation for an alternate categorical framework for decomposition in persistent homology, bypassing the usual persistence vector spaces and quiver representations.

But the power of homology is seldom of much efficacy, except in those happy dispositions where it is almost superfluous.

with apologies to Edward Gibbon

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1. Manifestations of the Structural Theorem

1.1. Introduction. During the last decade, persistent homology \cite{16,6} has achieved great success as a powerful and versatile tool, particularly for Topological Data Analysis (TDA) of point clouds. The term point cloud usually means a finite subset of points in a Euclidean space $\mathbb{R}^k$, where the dimension $k$ can be large, and the number of points is often very large. Many excellent surveys and introductions are available in the literature \cite{7,28,15,34,19,33}. Decomposition plays a central role both in the theory and in the applications of persistent homology. The ubiquitous “barcode diagrams” encode a decomposition in terms of the types and multiplicities of indecomposable summands. This data is an invariant, independent of the choice of decomposition. The summands represented by long barcodes contain important characteristic information, while the summands represented by short barcodes only contain random “noise” and may be disregarded. A number of “stability theorems” \cite{10,28} provide a firm foundation for this intuitively appealing interpretation of the long and short barcode invariants. In this paper we consider the interplay between the algorithmic and the categorical underpinnings for decomposition in persistent homology.

It is helpful to first review analogous decomposition issues for the much more familiar context of finite-dimensional vector spaces (over a fixed field $\mathbb{F}$). The ordinary Gaussian elimination algorithm can construct a basis for a vector space. Any choice of basis then constitutes a decomposition of the vector space, wherein the linear span of each basis element is a one-dimensional vector space. The direct sum of these one-dimensional summands is canonically identified (naturally isomorphic) to the original vector space. A one-dimensional vector space cannot be further decomposed as a sum of nonzero (dimensional) summands. This means that one-dimensional vector spaces are the indecomposable objects, in the category of vector spaces. Since all one-dimensional vector spaces are mutually isomorphic, there is just one type of indecomposable (object) in the category of vector spaces. The familiar dimension of a vector space is just the multiplicity of the indecomposable (one-dimensional) summands in a decomposition, and this multiplicity is an invariant independent of the choice of decomposition.

Now setting aside what we know about Gaussian elimination, we ask more abstractly why is it that any vector space is actually decomposable? We first observe that decomposability is a categorical property, since it involves both objects (vector spaces) and morphisms (linear maps). The theory of Krull-Schmidt categories \cite{21} provides an appropriate, albeit abstract, categorical setting for questions of decomposability. The axioms of a Krull-Schmidt category guarantee that every object admits an essentially unique decomposition as a finite sum of indecomposable objects. For the category of vector spaces, this essential uniqueness encodes the familiar fact that the dimension is an invariant independent of the choice of decomposition. The
goal then becomes to verify (and understand) the Krull-Schmidt property for the category of vector spaces. A concrete constructive verification of the Krull-Schmidt axioms for the category of vector spaces follows easily from basic properties of Gaussian elimination and linearity, but this is more in line with describing how to perform a decomposition rather than why vector spaces are decomposable. Fortunately there is a complementary abstract tool available. A theorem of Atiyah [3] dating back to the early years of category theory provides a very useful criterion for verifying the Krull-Schmidt property of a category. For the category of vector spaces, Atiyah’s criterion reduces to checking certain elementary properties of linear maps. So Atiyah’s theorem nonconstructively answers the abstract question of why any vector space admits a decomposition, complementing our understanding of how to constructively decompose a given vector space via Gaussian elimination.

In this paper we consider analogous questions of how and why decomposition works in persistent homology. The following picture summarizes one common description of the transformation from point cloud data to barcodes invariants:

The initial stages, going from a point cloud to a filtered chain complex, will be briefly reviewed in Section 1.2 below. The primary focus of this paper will be the final stages, going from filtered chain complexes to barcodes. At the homology step, the homology functor $H_n$ of the chosen dimension/degree $n$ takes a filtered chain complex (which is a diagram of chain complexes) to a persistence vector space (which is a diagram of vector spaces). The key final step is to compute barcode invariants by decomposition of a persistence vector space. An important insight [6] is that persistence vector spaces are quiver representations. A concrete consequence is the applicability of decomposition algorithms from quiver representation theory, showing how to decompose a persistence vector space and compute the barcodes. An abstract consequence is that the appropriate category of quiver representations is Krull-Schmidt by Atiyah’s theorem, showing why all of this works. So the Krull-Schmidt property of persistence vector spaces nicely ties together the theoretical and computational aspects. But there is one problem with this picture.
The standard computational algorithms for persistent homology [16, 35, 36] do not work by decomposing a persistence vector space. The following picture summarizes how barcodes are normally computed:

The initial stages of the picture, going from a point cloud to a filtered chain complex, are unchanged. The key reduction step [16, 35, 36] is the construction of a special type of basis. Each basis element is interpreted as either a creator or as a destroyer of a homology class. The selection step consists of keeping those creators and destroyers that correspond to nonzero barcodes of the desired homology dimension/degree $n$, and discarding the remaining basis elements. The question remains of why there should exist such algorithms operating on filtered complexes, rather than on persistence vector spaces.

In this paper we provide a categorical framework for the standard persistent homology algorithms, using an equivalence of categories to unify the two pictures above:

Our Categorical Structural Theorem (Theorem 1.6) is the foundation of the framework. The theorem asserts that the category of filtered chain complexes is Krull-Schmidt, and provides an intuitive classification of indecomposables. This leads to an alternate framework for persistent homology, where the barcodes describe the Krull-Schmidt decomposition of an object in a quotient of the category of filtered chain complexes. The barcodes are exactly the same as in the standard framework, because the quotient category is equivalent to the category of persistence vector spaces. This framework gives a unified answer for why and how decomposition actually works in persistent homology. We no longer need to rely on the Krull-Schmidt property of the category of persistence vector spaces as an indirect
1.2. **Topological Data Analysis by Example.** This paper focuses on the final stages of Topological Data Analysis (TDA), going from a filtered chain complex to barcode invariants. In this section we present a simple example to illustrate the stages leading up to the Structural Theorem, namely going from a point cloud to a filtered simplicial complex. A reader familiar with TDA may skip this section, which is similar to material in introductory papers such as [6, 10, 19] and textbooks such as [15, 34]. In our example, we use the $\alpha$-complex construction [15, 14], which is suitable for low dimensions. We note that for large point clouds in high dimensions, the Vietoris-Rips construction [7, 28] is often preferable.

**Example 1.1.** The first step is to construct a Delaunay complex, the second step is to construct a filtration of the Delaunay complex. We illustrate the construction of the Delaunay simplicial complex associated to a point cloud. Figure 1 shows a point cloud consisting of four points in in $\mathbb{R}^2$ labeled by $n \in \{1, 2, 3, 4\}$, together with the Voronoi cell $V(n)$ of each labeled point. We recall [15] that a Voronoi cell $V(n)$ contains all the points $x \in \mathbb{R}^2$ such that $n$ is the closest labeled point to $x$ (or one of the closest if several are equidistant). Figure 2 shows the Delaunay...
simplicial complex encoding the intersections of the Voronoi cells. We recall that the simplex \([n_0, \ldots, n_k]\), where \(n_i \in \{1, 2, 3, 4\}\) and \(n_0 < \cdots < n_k\), is included in the Delaunay complex iff \(V(n_0) \cap \cdots \cap V(n_k) \neq \emptyset\). For example, the simplex \([1, 2]\) is included because \(V(1) \cap V(2) \neq \emptyset\), but the simplex \([3, 4]\) is not included because \(V(3) \cap V(4) = \emptyset\).

The \(\alpha\) construction assigns to each Delaunay simplex \([n_0, \ldots, n_k]\) a real non-negative “birth parameter” \(b([n_0, \ldots, n_k])\). Let \(B_r(n)\) denote the closed ball of radius \(r\) centered at the labeled point \(n\), and consider the subset \(A_r(n) = B_r(n) \cap V(n)\) of the Voronoi cell \(V(n)\). The birth parameter of the Delaunay simplex \([n_0, \ldots, n_k]\) is defined to be the smallest value of \(r\) such that \(A_r(n_0) \cap \cdots \cap A_r(n_k) \neq \emptyset\). A value of \(r\) is called a “threshold” if it is the birth parameter for some Delaunay simplex. The integer “level” \(p\) indexes the thresholds in increasing order, as illustrated in Figures 3 through 6.

The \(\alpha\) construction produces a filtration of the Delaunay complex, and the simplicial homology \([20]\) of this filtered complex is described in terms of the barcode.
Figure 4. Level $p = 2$ is the threshold $r = 1 = b([1, 2])$.

Figure 5. Level $p = 3$ is the threshold $r = 1.12 = b([1, 3]) = b([1, 4]) = b([2, 3]) = b([2, 4])$.

Figure 6. Level $p = 4$ is the threshold $r = 1.25 = b([1, 2, 3]) = b([1, 2, 4])$. 
invariants \[28, 15, 34\]. Conventionally the filtration and the corresponding barcodes are indexed by the real-valued threshold parameter \(r\), which for our example yields the \(H_1\) barcode diagram of Figure 7. The diagram indicates that the first homology \(H_1\) detects two one-dimensional “holes” that appear at \(r = 1.12\) and are filled in at \(r = 1.25\). In this paper we will index filtrations and the corresponding barcodes by the integer-valued level \(p\), which for our example yields the \(H_1\) barcode diagram of Figure 8. This diagram indicates the same information, namely that the first homology \(H_1\) detects two one-dimensional “holes” that appear at \(p = 3\) (which corresponds to \(r = 1.12\)) and are filled in at \(p = 4\) (which corresponds to \(r = 1.25\)).

1.3. Matrix Structural Theorem. For simplicity, we start with the ungraded version of the structural theorem. A differential matrix is a square matrix \(D\) satisfying \(D^2 = 0\). We’ll say a differential matrix is Jordan if it is in Jordan normal form, meaning it decomposes as a block-diagonal matrix built from copies of the two differential Jordan block matrices

\[
J = \begin{bmatrix} 0 & \end{bmatrix}, \quad K = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.
\]

We’ll say a differential matrix \(D\) is almost-Jordan if there exists a permutation matrix \(P\) such that the differential matrix \(P^{-1}DP\) is Jordan. Given an almost-Jordan differential matrix \(D\), it is trivial to construct such a permutation matrix \(P\). We will say a square matrix \(B\) is triangular if it is upper-triangular and invertible.

The standard algorithm for computing persistent homology is based on the papers \([16, 35, 36]\). The result of a persistent homology computation, not depending on a choice of algorithm, is conveniently summarized \([13, 28]\) as a matrix factorization:

**Theorem 1.2.** (Ungraded Matrix Structural Theorem) Any differential matrix \(D\) factors as \(D = BDB^{-1}\) where \(D\) is an almost-Jordan differential matrix and \(B\) is a triangular matrix.
It is the triangular condition that makes this interesting: without the triangular condition, this would follow immediately from the ordinary Jordan normal form. Furthermore, the matrix $D$ is unique, as we show in Appendix A. We’ll call $D$ the persistence canonical form of the differential matrix $D$. A column of the triangular matrix $B$ is in $\ker D$ iff the corresponding column of $D$ is zero. We will say that $B$ is normalized if each such column has diagonal entry equal to 1. It is always possible to normalize $B$ by scalar multiplication of columns, but even with normalization $B$ is not unique in general. A constructive proof of Theorem 1.2 follows from any of the algorithms for computing persistent homology. In Appendices B.1 and B.2 we discuss the matrix reduction approach to computing persistent homology.

Example 1.3. Consider the filtered simplicial complex shown in Figure 9. With the usual convention for an adapted basis, the ordering of basis elements prioritizes the level of the filtration over the degree/dimension of the simplex. The initial basis of simplices is then ordered so the level (denoted by prescript) is nondecreasing, and within each level the degree (denoted by postscript) is nondecreasing. Using lexicographic order to break any remaining ties, the initial adapted basis is $a_1, b_1, c_1, ac_1, abc_1, abc_2$, and the boundary operator over the field $\mathbb{R} = \mathbb{Q}$ of rationals is represented by the differential matrix

$$
D = \begin{pmatrix}
1a_0 & 1b_0 & 2ab_1 & 3c_0 & 4bc_1 & 5ac_1 & 6abc_2 \\
1a_0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\
1b_0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\
2ab_1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
3c_0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
4bc_1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
5ac_1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
6abc_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$
The persistence canonical form is

\[
D = \begin{bmatrix}
\begin{array}{cccccccc}
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0
\end{array}
\end{bmatrix}
\]

as verified by checking that \( D = B^{-1}DB \) for the triangular (and normalized) matrix

\[
B = \begin{bmatrix}
\begin{array}{cccccccc}
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0
\end{array}
\end{bmatrix}
\]

The persistence canonical form \( D \) is almost-Jordan in general, and in this example it happens to be actually Jordan. The matrix \( B \) represents the basis change to the new adapted basis \( 1a_0, 1b_0, 2ab_{1}, 3c_0, b_{c1}, 5c_1, 6abc_2 \). The level remains nondecreasing because \( B \) is triangular. Each basis element retains pure degree, although Theorem 1.2 does not explicitly address issues of degree. The matrix \( D \) represents the boundary operator relative to the new adapted basis.

We prefer to prioritize degree over level in ordering the elements of an adapted basis. This has the advantage of encoding the degree in the block structure of the matrix. The following version of the structural theorem is then manifestly compatible with the grading by degree:

**Theorem 1.4.** (Matrix Structural Theorem) Any block-superdiagonal differential matrix \( D \) factors as \( D = BDB^{-1} \) where \( D \) is a block-superdiagonal almost-Jordan differential matrix and \( B \) is a block-diagonal triangular matrix.

The block-diagonal structure of \( B \) ensures that the transformed basis elements retain pure degree. The persistence canonical form \( D \) inherits the block-superdiagonal structure of the differential \( D \). It is always possible to normalize \( B \) by scalar multiplication of columns as in the ungraded case. Any of the algorithmic proofs of Theorem 1.2 \([16, 35, 36]\) can be used to prove Theorem 1.4 by keeping track of degrees. We discuss this point for the standard algorithm in Appendix B.2.

**Example 1.5.** We again consider the filtered chain complex of Example 1.3, but with basis order prioritizing degree over level. Now the degree of basis elements (denoted by postscript) is nondecreasing, and within a degree the level (denoted by prescript) of basis elements is nondecreasing. Using lexicographic order to break any
remaining ties, the initial adapted basis is now $\{a_0, b_0, c_0, ab_1, bc_1, ac_1, abc_2\}$, and the boundary operator over the field $\mathbb{F} = \mathbb{Q}$ of rationals is now represented by the block-superdiagonal differential matrix

$$D = \begin{bmatrix}
1a_0 & 1b_0 & 1c_0 & 2ab_1 & 1bc_1 & 5ac_1 & 6abc_2 \\
1a_0 & 0 & 0 & -1 & 0 & -1 & 0 \\
1b_0 & 0 & 0 & 1 & -1 & 0 & 0 \\
3c_0 & 0 & 0 & 0 & 1 & 1 & 0 \\
2ab_1 & 0 & 0 & 0 & 0 & 0 & 1 \\
4bc_1 & 0 & 0 & 0 & 0 & 0 & 1 \\
5ac_1 & 0 & 0 & 0 & 0 & 0 & -1 \\
6abc_2 & 0 & 0 & 0 & 0 & 0 & 0 
\end{bmatrix}.$$

The persistence canonical form inherits the block-superdiagonal structure

$$D = \begin{bmatrix}
1a_0 & 1b_0 & 1c_0 & 2ab_1 & 1bc_1 & 5ac_1 & 6abc_2 \\
1a_0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1b_0 & 0 & 0 & 1 & 0 & 0 & 0 \\
3c_0 & 0 & 0 & 0 & 1 & 0 & 0 \\
2ab_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
4bc_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
5ac_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
6abc_2 & 0 & 0 & 0 & 0 & 0 & 0 
\end{bmatrix}.$$

as verified by checking that $D = B^{-1}DB$ for the block-diagonal triangular (and normalized) matrix

$$B = \begin{bmatrix}
1a_0 & 1b_0 & 1c_0 & 2ab_1 & 1bc_1 & 5ac_1 & 6abc_2 \\
1a_0 & 1 & -1 & -1 & 0 & 0 & 0 \\
1b_0 & 0 & 1 & 0 & 0 & 0 & 0 \\
3c_0 & 0 & 0 & 1 & 0 & 0 & 0 \\
2ab_1 & 0 & 0 & 0 & 1 & -1 & 0 \\
4bc_1 & 0 & 0 & 0 & 0 & 1 & 0 \\
5ac_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
6abc_2 & 0 & 0 & 0 & 0 & 0 & -1 
\end{bmatrix}.$$

The persistence canonical form $D$ is almost-Jordan, but not actually Jordan in this example. The matrix $B$ represents the basis change to the new adapted basis $\{a_0, b_0, c_0, ab_1, bc_1, ac_1, abc_2\}$. Since $B$ is block-diagonal, each basis element remains pure degree, and the degree remains nondecreasing. Since $B$ is triangular, the level remains nondecreasing within each degree. The computation of this particular matrix $B$ via the standard matrix reduction algorithm is worked out in Appendix B.2.
1.4. Categorical Structural Theorem and Structural Equivalence. A Krull-Schmidt category is an additive category where objects decompose nicely as direct sums of indecomposable objects. In Chapter 2, we study the additive category of filtered chain complexes in the setting of Krull-Schmidt categories, starting with a review of Krull-Schmidt categories in section 2.1. A filtered complex will be called basic if its boundary operator can be represented by differential matrix consisting of a single Jordan block. We will use nonconstructive categorical methods to prove the following structural theorem for the category of filtered complexes:

**Theorem 1.6.** *(Categorical Structural Theorem)* The category of filtered complexes is Krull-Schmidt. A filtered complex is indecomposable iff it is basic.

In chapter 3 we will prove the equivalence of the matrix and the categorical versions of the structural theorem. One direction is proved in section 3.1:

**Proposition 1.7.** *(Forward Structural Equivalence)* The Matrix Structural Theorem implies the Categorical Structural Theorem.

This is followed by a detailed example of a Krull-Schmidt decomposition computation. The other direction is proved in section 3.2:

**Proposition 1.8.** *(Reverse Structural Equivalence)* The Categorical Structural Theorem implies the Matrix Structural Theorem.

Combining the Categorical Structural Theorem [1.6] and the Reverse Structural Equivalence Proposition [1.8] yields a nonconstructive categorical proof of the Matrix Structural Theorem [1.4]. This contrasts with the various constructive algorithmic proofs of Theorem [1.4], which are discussed in Appendix B.2. The constructive algorithmic proofs explain how persistent homology works, the nonconstructive proof explains why persistent homology works.

2. Proving the Categorical Structural Theorem

2.1. Additive and Krull-Schmidt Categories. This section reviews the relevant background from category theory. General references for category theory include [24, 4, 1]. Additive categories are discussed in [24, 32]. Krull-Schmidt categories are discussed in [21, 27, 3].

**Definition 2.1.** A category is additive if:

1. Each $\text{Hom}(X,Y)$ is an abelian group, and the morphism composition map $\text{Hom}(Y,Z) \times \text{Hom}(X,Y) \rightarrow \text{Hom}(X,Z)$ is biadditive/bilinear.
2. There exists a zero object 0.
3. Any finite collection of objects $X_1, X_2, \ldots, X_n$ has a direct sum $X_1 \oplus X_2 \oplus \cdots \oplus X_n$.

An additive category is linear over the field $\mathbb{F}$ if each $\text{Hom}(X,Y)$ is a finite-dimensional $\mathbb{F}$-vector space, and each map $\text{Hom}(Y,Z) \times \text{Hom}(X,Y) \rightarrow \text{Hom}(X,Z)$ describing composition of morphisms $(g,f) \mapsto g \circ f$ is $\mathbb{F}$-bilinear. All of the categories we will be studying are linear.

The endomorphism ring of an object $X$ in an additive category is the Abelian group $\text{Hom}(X,X)$ of endomorphisms, with multiplicative structure given by composition of endomorphisms. In a linear category, the endomorphism ring is an $\mathbb{F}$-algebra.
Definition 2.2. A ring is local if:

1. \( 1 \neq 0 \).
2. If an element \( f \) does not have a multiplicative inverse, then the element \( 1 - f \) has a multiplicative inverse.

The local property is important, because a finite direct sum decomposition into summands with a local endomorphism rings is essentially unique:

Theorem 2.3. (e.g. [21] Theorem 4.2) Let \( X \) be an object in an additive category, and suppose there are two finite decompositions

\[ X_1 \oplus \cdots \oplus X_m = X = Y_1 \oplus \cdots \oplus Y_n \]

into (nonzero) objects with local endomorphism rings. Then \( m = n \) and there exists a permutation \( \pi \) such that \( X_i \simeq Y_{\pi(i)} \) for each \( 1 \leq i \leq m \).

An object \( X \) in an additive category is decomposable if it is the direct sum \( X = Y \oplus Z \) of two nonzero objects \( Y \) and \( Z \). An indecomposable object, often abbreviated as an indecomposable, is a nonzero object that is not decomposable.

Lemma 2.4. An object is indecomposable if it has a local endomorphism ring.

Proof. We will show that the endomorphism ring of a decomposable object \( X \) is not local. We may assume that \( X = Y \oplus Z \) with \( Y \) and \( Z \) nonzero. Then neither \( f = 1_Y \oplus 0_Z \) nor \( 1_X - f = 1_Y \oplus 1_Z - f = 0_Y \oplus 1_Z \) has a multiplicative inverse in the ring \( \text{Hom}(X, X) = \text{Hom}(Y \oplus Z, Y \oplus Z) \). □ □

A Krull-Schmidt category has properties that guarantee both the existence and essential uniqueness of finite direct sum decompositions of any object, see e.g. [21, 27] for more details:

Definition 2.5. An additive category is Krull-Schmidt if:

1. Every object admits a finite decomposition as a sum of indecomposables.
2. Every indecomposable has a local endomorphism ring.

Recall that an additive category is Abelian if every morphism has a kernel and a cokernel, every monic morphism is normal (is the kernel of some morphism), and every epic morphism is conormal (is the cokernel of some morphism). Additional information about Abelian categories is outlined in Appendix 5.2. Note that Definition 2.5 of Krull-Schmidt category does not assume that the additive category is Abelian, or even the existence of kernels and cokernels. We are primarily interested in the linear category of filtered chain complexes, which is not Abelian. But we will use Abelian categories and their subcategories to show that this category is nonetheless Krull-Schmidt. Atiyah’s Criterion [3, 21] provides a very general sufficient condition for an Abelian category to be Krull-Schmidt. Since all of our categories are linear, we will only need the following special case:

Theorem 2.6. (Atiyah’s Criterion) A linear Abelian category is Krull-Schmidt.

The proof of Atiyah’s Criterion is nonconstructive. It neither provides an algorithm to decompose a given object as a direct sum of indecomposables, nor a classification of indecomposables.
2.2. Persistence Objects and Filtered Objects. Persistence objects [6] and filtered objects [32] are described by categorical diagrams. Suppose that $\mathcal{X}$ is a linear Abelian category (and therefore Krull-Schmidt by Theorem 2.6). We will study persistence indexed by an integer $p \in \mathbb{Z}$, with $\leq$ denoting the standard partial order. A persistence object in $\mathcal{X}$ is a diagram $\cdots \rightarrow p_{-1}X \rightarrow p_{-1}X \rightarrow p_{-1}X \rightarrow \cdots$. A morphism of persistence objects $\bullet f : \bullet X \rightarrow \bullet X'$ is a commutative diagram of “ladder” type

$$
\begin{array}{ccccccc}
\cdots & \rightarrow & p_{-1}X & \rightarrow & p_{-1}X & \rightarrow & p_{-1}X \\
& & \downarrow \scriptstyle{p_{-1}f} & & \downarrow \scriptstyle{p_{-1}f} & & \\
\cdots & \rightarrow & p_{-1}X' & \rightarrow & p_{-1}X' & \rightarrow & p_{-1}X'.
\end{array}
$$

The category of persistence objects in $\mathcal{X}$ is Abelian, with pointwise kernels, cokernels, and direct sums. The set of morphisms $\bullet X \rightarrow \bullet X'$ between two persistence objects is a vector space, but not finite-dimensional in general. We will say a categorical diagram is tempered if all but finitely many of its arrows are iso(morphisms). A tempered diagram has a global finiteness property, distinct from the relative finiteness property normally conferred by the term “tame”. The set of morphisms $\bullet X \rightarrow \bullet X'$ between two tempered persistence objects is a finite-dimensional vector space. This is because $p_{\pm 1}f \circ \alpha = \beta \circ p_{\pm 1}f$ in a commutative square with parallel isomorphisms $\alpha$ and $\beta$, determining $p_{\pm 1}f$ in terms of $p_{\pm 1}f$. The tempered persistence objects comprise a strictly full Abelian subcategory of the persistence objects. Theorem 2.6 now yields:

**Proposition 2.7.** Let $\mathcal{X}$ be a linear Abelian category. The category of tempered persistence objects in $\mathcal{X}$ is Krull-Schmidt.

We next discuss subobjects in a linear Abelian category $\mathcal{X}$. We make the additional assumption that the category $\mathcal{X}$ is concrete, meaning that an object in $\mathcal{X}$ is a set with some additional features, and a morphism in $\mathcal{X}$ is a map of sets compatible with the additional features. For example, the linear Abelian category $\mathcal{V}$ of (finite-dimensional) vector spaces is a concrete linear Abelian category. An inclusion $X \hookrightarrow X'$ in $\mathcal{X}$ is an arrow that is an inclusion of the underlying sets. We say $X$ is a subobject of $X'$ iff such an inclusion arrow exists. An inclusion arrow is monic [24, 4], and the composition of inclusion arrows is an inclusion arrow. Any object $X'$ in $\mathcal{X}$ has a zero subobject $0 \hookrightarrow X'$, and is its own subobject $X' \hookrightarrow X'$. A subobject $X \hookrightarrow X'$ is proper if $X \neq X'$. A nonzero object is said to be simple if it does not have a proper nonzero subobject. A simple object is obviously indecomposable, but an indecomposable object need not be simple.

A filtered object in a concrete linear Abelian category $\mathcal{X}$ is a special type of tempered persistence object in $\mathcal{X}$. We say a tempered persistence object $\bullet X$ in $\mathcal{X}$ is bounded below if there exists an integer $j$ such that $\bullet X = 0$ whenever $p \leq j$. We say $\bullet X$ is a filtered object if it is bounded below and if every arrow is an inclusion
The filtered objects in $X$ comprise a strictly full subcategory of the tempered persistence objects. The properties of monics have several consequences. A filtered object diagram has a categorical limit and a colimit \([21, 4]\). The limit is 0 since the diagram is bounded below. The colimit $X$ is $kX$ for $k$ sufficiently large (satisfying $pX = X$ whenever $k \leq p$). Finally, any summand of a filtered object is isomorphic to a filtered object. Combining these facts with Proposition \([2, 7]\) yields:

**Lemma 2.8.** Let $X$ be a linear Abelian category. The category of filtered objects in $X$ is Krull-Schmidt. A filtered object $\bullet X$ in $X$ is indecomposable iff its colimit $X$ is an indecomposable object in $X$.

Here a filtered object $pZ$ with a decomposable colimit $X \oplus Y$ decomposes as the direct sum of the filtered objects $pZ \cap X$ and $pZ \cap Y$.

We note that the filtered objects comprise a subcategory of the tempered persistence objects, but this subcategory is not Abelian because a morphism of filtered objects may have a kernel and/or cokernel that is not a filtered object. So Lemma \([2, 8]\) is not merely a corollary of Theorem \([2, 6]\). Finally we observe that the category of persistence objects in a (concrete) linear Abelian category is itself a (concrete) linear Abelian category, to which Lemma \([2, 8]\) applies.

### 2.3. Chain Complexes and Filtered Chain Complexes.

A **persistence vector space** is a persistence object in the (concrete) linear Abelian category $X = \mathcal{V}$ of (finite-dimensional) $F$-vector spaces. Tempered persistence vector spaces are well-understood via the theory of quiver representations. A nonempty subset $I \subseteq \mathbb{Z}$ will be called an interval if $c \in I$ whenever $a \leq c \leq b$ with $a \in I$ and $b \in I$. We associate to an interval $I \subseteq \mathbb{Z}$ the **interval persistence vector space** $\bullet I$ constructed as follows: $pI = F$ whenever $p \in I$, $pI = 0$ whenever $p \notin I$, and every arrow $F \to F$ is the identity morphism 1. We will often omit the bullet prescript when context allows. For example, the interval persistence vector space $[1, 4) = \bullet [1, 4)$ is the diagram of vector spaces

$$
\cdots \longrightarrow 0 \longrightarrow F \overset{1}{\longrightarrow} F \overset{1}{\longrightarrow} F \longrightarrow 0 \longrightarrow \cdots
$$

associated to the interval $[1, 4) = \{1, 2, 3\} \subseteq \mathbb{Z}$. Proposition \([2, 7]\) applies to the linear Abelian category of tempered persistence vector spaces. Furthermore, the well-studied representation theory of $A_n$ quivers (see e.g. \([30]\)) carries over by a limiting argument to prove the following structural theorem for the category of tempered persistence vector spaces:

**Theorem 2.9.** The category of tempered persistence vector spaces is Krull-Schmidt. A tempered persistence vector space is indecomposable iff it is isomorphic to an interval.
Theorem 2.9 can be applied to cochain complexes. A cochain complex, or co-
complex for short, is a tempered persistence vector space \( \cdots V_{p-1} \xrightarrow{\partial_{p-1}} V_p \xrightarrow{\partial_{p+1}} V_{p+1} \xrightarrow{\partial_{p+2}} \cdots \)
with the property that the composition of successive arrow \( \partial_{p+1} \circ \partial_{p} \) is zero. The kernel of a morphism between cocomplexes is a cocomplex, as is the cokernel, so the cocomplexes comprise a strictly full Abelian subcategory of the tempered persistence vector spaces. Theorem 2.6 and Theorem 2.9 now yield the structural result:

**Proposition 2.10.** The category \( \mathcal{C}^{\text{op}} \) of cocomplexes is linear and Abelian, and therefore Krull-Schmidt. A cocomplex is indecomposable iff it is isomorphic to an interval cocomplex.

Chain complexes are dual to cochain complexes. A complex (short for chain complex) is a tempered diagram \( V_{\bullet} \) in \( \mathcal{V} \) of type

\[
\cdots \leftarrow V_{n-2} \xleftarrow{\partial_{n-1}} V_{n-1} \xleftarrow{\partial_n} V_n \xleftarrow{\partial_{n+1}} \cdots
\]

with the property that the composition of successive arrows \( \partial_{n-1} \circ \partial_n \) is zero. A morphism of complexes \( f_{\bullet} : V_{\bullet} \rightarrow V'_{\bullet} \) is a commutative ladder diagram. The category \( \mathcal{V} \) of vector spaces is isomorphic to its opposite category \( \mathcal{V}^{\text{op}} \) via the duality functor that takes a vector space to its dual and a linear map to its transpose/adjoint [24, 4]. Duality takes the category \( \mathcal{C}^{\text{op}} \) of cocomplexes to the category of complexes \( \mathcal{C} \). A complex is called an interval complex if its dual is an interval cocomplex, and Proposition 2.10 becomes:

**Proposition 2.11.** The category \( \mathcal{C} \) of complexes is linear and Abelian, and therefore Krull-Schmidt. A complex is indecomposable iff it is isomorphic to an interval complex.

The interval complexes are easily classified. An interval complex \( I_{\bullet} \) is associated to an interval \( I \subseteq \mathbb{Z} \) as follows: \( n \in I \) whenever \( I_n = F \), and \( n \notin I \) whenever \( I_n = 0 \). Since adjacent nonzero arrows in a complex cannot be iso(morphisms), the interval complexes are in bijective correspondence with the intervals \( I \subseteq \mathbb{Z} \) of cardinality at most two. We will often omit the bullet postscript when the context allows. We denote by \( J[n] = \{n\} \subseteq \mathbb{Z} \) the intervals of cardinality one. For example, the complex \( J[1] = J[1]_{\bullet} \) is the diagram of vector spaces

\[
\cdots \leftarrow 0 \leftarrow F \leftarrow 0 \leftarrow 0 \leftarrow \cdots
\]

\[
n = 0 \quad n = 1 \quad n = 2 \quad n = 3
\]

The indecomposable complex \( J[n] \) is simple. We denote by \( K[n] = [n, n+1] \subseteq \mathbb{Z} \) the intervals of cardinality two. For example, the complex \( K[1] = K[1]_{\bullet} \) is the
diagram of vector spaces

\[ \cdots \leftarrow 0 \leftarrow \mathbb{F} \leftarrow 1 \leftarrow \mathbb{F} \leftarrow 0 \leftarrow \cdots \]
\[ n = 0 \quad n = 1 \quad n = 2 \quad n = 3 \]

The indecomposable complex \( K[n] \) has exactly one nonzero proper subobject \( J[n] \hookrightarrow K[n] \). For example, the inclusion of complexes \( J[1] \hookrightarrow K[1] \) is the commutative ladder diagram

\[ \cdots \leftarrow 0 \leftarrow \mathbb{F} \leftarrow 0 \leftarrow 0 \leftarrow \cdots \]
\[ \cdots \leftarrow 0 \leftarrow \mathbb{F} \leftarrow \mathbb{F} \leftarrow 0 \leftarrow \cdots \]
\[ n = 0 \quad n = 1 \quad n = 3 \quad n = 4 \]

We now return to the the Categorical Structural Theorem 1.6. A filtered complex is a diagram in the category \( C \) of complexes

\[ \cdots \rightarrow p-1 V_* \rightarrow p V_* \rightarrow p+1 V_* \rightarrow \cdots . \]

We will say a filtered complex is basic if its colimit \( V_* \) is isomorphic to an interval complex. The first statement of Proposition 2.11 tells us that the category \( C \) of complexes is linear and Abelian. Then Lemma 2.8 tells us that the category of filtered complexes is Krull-Schmidt. The second statement of Proposition 2.11 classifies the indecomposable filtered complexes, completing the proof of:

**Theorem 1.6.** *(Categorical Structural Theorem)* The category of filtered complexes is Krull-Schmidt. A filtered complex is indecomposable iff it is basic.

The basic filtered complexes are easily classified since we know all proper subobjects of interval complexes, namely \( 0 \hookrightarrow J[n] \), \( 0 \hookrightarrow K[n] \), and \( J[n] \hookrightarrow K[n] \). Details and examples of basic filtered complexes appear in Chapter 4.

### 3. Categorical Frameworks for Persistent Homology

#### 3.1. Standard Framework using Persistence Vector Spaces.

The structural theorem for the category of tempered persistence vector spaces, Theorem 2.9, is the foundation for the standard framework for persistent homology.

For each integer \( n \), the homology of degree \( n \) is a functor \( H_n : C \rightarrow V \) from the category \( C \) of complexes to the category \( V \) of vector spaces. An object \( C \) in \( C \) is a diagram of vector spaces

\[ \cdots \leftarrow V_{n-1} \leftarrow \partial_n V_n \leftarrow \partial_{n+1} V_{n+1} \leftarrow \cdots \]
where $\partial_n \circ \partial_{n+1} = 0$. Then $\text{im} \partial_{n+1} \hookrightarrow \ker \partial_n$ is a subobject inclusion of vector spaces, and the homology is the quotient vector space (cokernel)

$$H_n(C) = \ker \partial_n / \text{im} \partial_{n+1}.$$ 

More generally, the homology functor $H_n$ takes a diagram in $\mathcal{C}$ to a diagram in $\mathcal{V}$.

Denote by $\mathcal{F}$ the category of filtered complexes. An object $F$ in $\mathcal{F}$ is a diagram of complexes

$$\cdots \longrightarrow V_{p-1} \longrightarrow V_p \longrightarrow V_{p+1} \longrightarrow \cdots,$$

which is tempered and bounded below, and which has monic arrows. Denote by $\mathcal{P}$ the category of tempered persistence vector spaces. The homology functor $H_n$ takes the diagram $F$ to the diagram of vector spaces

$$\cdots \longrightarrow H_n(V_{p-1}) \longrightarrow H_n(V_p) \longrightarrow H_n(V_{p+1}) \longrightarrow \cdots,$$

which is tempered and bounded below, but which need not have monic arrows in general. So an object $F$ in $\mathcal{F}$ goes to an object $P_n(F)$ in $\mathcal{P}$. Similarly a morphism in $\mathcal{F}$, which is a commutative ladder diagram of complexes, goes to a morphism in $\mathcal{P}$, which is a commutative ladder diagram of vector spaces. The resulting functor $P_n : \mathcal{F} \to \mathcal{P}$ is the persistent homology of degree $n$.

The standard framework for studying the persistent homology functors $P_n : \mathcal{F} \to \mathcal{P}$ is based on the structural theorem for the category $\mathcal{P}$, Theorem 2.9. It suffices to work with an appropriate Krull-Schmidt subcategory of the Krull-Schmidt category $\mathcal{P}$. A filtered complex $F$ is studied by decomposing the persistence vector space $P_n(F)$ as a sum of indecomposables. Since the diagram $P_n(F)$ is bounded below, all of its indecomposables are bounded below. The persistence vector spaces that are bounded below comprise a full Abelian subcategory of $\mathcal{P}$, which we will denote by $\text{im} P_n$. Despite the notation, the category $\text{im} P_n$ does not depend on $n$; it is always the same subcategory of $\mathcal{P}$. The isomorphism class of an indecomposable in the Krull-Schmidt category $\text{im} P_n$ is described by the familiar barcode. An interval $I \subseteq \mathbb{Z}$ will be called a barcode if it is bounded below. A barcode persistence vector space is a persistence vector space $\cdot I$ corresponding to a barcode $I \subseteq \mathbb{Z}$.

**Theorem 3.1.** The persistent homology functor $P_n : \mathcal{F} \to \mathcal{P}$ factors as

$$\mathcal{F} \to \text{im} P_n \to \mathcal{P}.$$ 

The category $\text{im} P_n$ is Krull-Schmidt. An object in $\text{im} P_n$ is indecomposable iff it is isomorphic to a barcode persistence vector space.

We can now express the standard framework for persistent homology in terms of the functor $\mathcal{F} \to \text{im} P_n$ which takes a filtered complex to a persistence vector space in $\text{im} P_n$. The Krull-Schmidt property of $\text{im} P_n$ then allows decomposition as a sum of indecomposables. Each indecomposable in $\text{im} P_n$ is a barcode persistence vector space, which is specified up to isomorphism by its barcode $I \subseteq \mathbb{Z}$. An object in $\text{im} P_n$ is determined up to isomorphism by its set of barcodes.
3.2. Alternate Framework using Quotient Categories. The structural theorem for the category of filtered complexes, Theorem 1.6, is the foundation for our alternate framework for persistent homology.

We will work with an appropriate Krull-Schmidt quotient category of the Krull-Schmidt category $\mathcal{F}$. Recall in general [14] that an object of a quotient category of $\mathcal{F}$ is an object of $\mathcal{F}$, and a morphism is an equivalence class of morphisms of $\mathcal{F}$. Our quotient category $\text{coim} P_n$ is defined via the following equivalence relation (congruence) on morphisms: two morphisms $f$ and $f'$ in $\mathcal{F}$ are equivalent iff the morphisms $P_n(f)$ and $P_n(f')$ in $\mathcal{P}$ are equal. Note that the category $\text{coim} P_n$ now depends on the integer $n$; each $\text{coim} P_n$ is a different quotient category of $\mathcal{F}$.

Example 3.2. We return to the filtered simplicial complex of Example 1.3 as shown in Figure 10.

We first consider the subobject shown in Figure 11. In $\mathcal{F}$, this is a proper nonzero subobject. In the quotient category $\text{coim} P_0$, the subobject inclusion becomes an isomorphism between nonzero objects. In the quotient category $\text{coim} P_1$, this becomes a proper zero subobject.

Now consider another subobject as shown in Figure 12. In $\mathcal{F}$, this is a proper nonzero subobject. In the quotient category $\text{coim} P_0$, this remains a proper nonzero subobject. In the quotient category $\text{coim} P_1$, the inclusion morphism becomes an isomorphism between nonzero objects.
We recall that a quotient of a Krull-Schmidt category is Krull-Schmidt in general. This is because an indecomposable in \( \mathcal{F} \) becomes either a zero object or an indecomposable with a local endomorphism ring in the quotient category (see e.g. [22] p. 431). The classification of indecomposables in the quotient category \( \text{coim} P_n \) is now easily obtained from Theorem 1.6. This is independent of the well-known classification of indecomposables in the category of persistence vector spaces (Theorem 2.9). Using the classification of indecomposables in each of the Krull-Schmidt categories \( \text{coim} P_n \) and \( \text{im} P_n \), it is now easy to verify that the functor \( \text{coim} P_n \to \text{im} P_n \) is full, faithful, and essentially surjective. Recalling [24, 4] that a functor satisfying these conditions is an equivalence of categories, we have:

**Theorem 3.3.** The persistent homology functor \( P_n : \mathcal{F} \to \mathcal{P} \) factors as

\[ \mathcal{F} \to \text{coim} P_n \to \text{im} P_n \to \mathcal{P}, \]

where the functor \( \text{coim} P_n \to \text{im} P_n \) is an equivalence of categories.

The isomorphism class of an indecomposable object \( \bullet X \) in the Krull-Schmidt category \( \text{coim} P_n \) can be specified as \( I_n \). Here the integer \( n \) is the degree/dimension label of the category \( \text{coim} P_n \). The interval subset \( I \subseteq \mathbb{Z} \) is defined by the rule: \( p \in I \) iff the complex \( p X \) at level \( p \) is isomorphic to \( J[n] \). We note that the classification of indecomposables in \( \text{coim} P_n \) does not reference homology. This point will be illustrated in detail in Example 4.4 below. We remark that our naming choices for \( \text{coim} P_n \) and \( \text{im} P_n \) are intended to emphasize the parallel between Theorem 3.3 and the factorization of a morphism in an Abelian category, see Section 5.2. Section 5.1 reviews an analogous functor factorization in the simpler setting of plain (not persistent) homology.

We can now express the alternate framework for persistent homology in terms of the functor \( \mathcal{F} \to \text{coim} P_n \) which takes a filtered complex in \( \mathcal{F} \) to the same filtered complex viewed as an object in \( \text{coim} P_n \). The Krull-Schmidt property of \( \text{coim} P_n \) then allows decomposition as a sum of indecomposables in \( \text{coim} P_n \). Each indecomposable is specified up to isomorphism by \( I_n \). An object in \( \text{coim} P_n \) is determined up to isomorphism by the collection of intervals \( I_n \subseteq \mathbb{Z} \) indexing its decomposition. This framework obviates the need for auxiliary objects such as persistence vector spaces, while providing exactly the same information about filtered complexes as the standard framework. These two frameworks are further compared in Section 5.4.

4. Proving Structural Equivalence

4.1. **Forward Structural Equivalence.** We now consider in more detail matrix representations of a filtered complex and its automorphisms. The first step is to associate to a filtered complex a finite-dimensional vector space with an appropriately adapted basis. A filtered complex is a diagram of complexes indexed by the integer level \( p \), displayed below together with its colimit:

\[
\cdots \hookrightarrow_{-1} V_\bullet \hookrightarrow_0 V_\bullet \hookrightarrow_1 V_\bullet \hookrightarrow_2 V_\bullet \hookrightarrow_3 V_\bullet \hookrightarrow \cdots \quad V_\bullet
\]

\[
\text{colim}
\]
A filtered complex becomes a “lattice” diagram of finite-dimensional vector spaces:

\[
\begin{array}{ccccccc}
& & & & & & \vdots \\
V_{-1} & \rightarrow & V_0 & \rightarrow & V_1 & \rightarrow & V_2 & \rightarrow & \cdots & V_2 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \vdots \\
\cdots & \rightarrow & V_1 & \rightarrow & V_0 & \rightarrow & V_1 & \rightarrow & V_1 & \cdots & V_1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \vdots \\
\cdots & \rightarrow & V_0 & \rightarrow & V_0 & \rightarrow & V_0 & \rightarrow & V_0 & \cdots & V_0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \vdots \\
\cdots & \rightarrow & V_{-1} & \rightarrow & V_{-1} & \rightarrow & V_{-1} & \rightarrow & V_{-1} & \cdots & V_{-1} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \vdots \\
& & & & & & \vdots
\end{array}
\]

colim

In the colimit complex, the composition \( \partial_{n-1} \circ \partial_n : V_n \rightarrow V_{n-2} \) is zero for all \( n \). Since the diagram is tempered, \( \partial_{n-1} \circ \partial_n \) is an isomorphism for all but finitely many \( n \). It follows that the complex is bounded, meaning that the vector space \( V_n \) is zero-dimensional for all but finitely many \( n \). The direct sum \( V = \bigoplus_{n \in \mathbb{Z}} V_n \) is then a finite-dimensional vector space associated to the filtered complex. A vector \( v \in V \) is said to have pure degree iff \( v \in V_n \subseteq V \) for some integer \( n \). The integer \( n \) is then called the degree of the pure degree vector \( v \), and is encoded by a postscript \( v_n \). The (filtration) level of a degree \( n \) vector \( v_n \in V \) is the smallest integer \( p \) such that \( v_n \in V_p \subseteq V_n \). The level of the degree \( n \) vector \( v_n \) is encoded by a prescript \( v_p n \).

Gaussian elimination constructs an adapted basis for a filtered vector space. Summing over degrees, we obtain an adapted basis of a filtered complex, meaning a basis of the vector space \( V \) satisfying the three conditions:

- Every basis element has pure degree.
- For each \( n \) and \( p \), the vector space \( _p V_n \) is spanned by the basis vectors with degree equal to \( n \) and level less than or equal to \( p \).
- The basis elements are ordered so that degree is nondecreasing, and within each degree the level is nondecreasing.

A block-diagonal triangular matrix \( B \) transforms an adapted basis to a new adapted basis, representing an automorphism of the filtered complex. Here we assume that the block structure of the matrix is compatible with degrees of the basis elements.

The colimit boundary \( \partial = \bigoplus_{n \in \mathbb{Z}} \partial_n \) of a filtered complex is a linear endomorphism \( \partial : V \rightarrow V \). The colimit boundary \( \partial \) is represented by a matrix \( D \) relative to an adapted basis. The matrix representative \( D \) is block-superdiagonal because \( \partial \) is homogeneous of degree \(-1\), and \( D^2 = 0 \) because \( \partial^2 = 0 \). If additionally the matrix representative is almost-Jordan, we will say the adapted basis is special. The Matrix Structural Theorem 1.4 yields:

**Proposition 4.1.** A filtered complex admits a special adapted basis.
Proof. Choose an adapted basis. Let \( D \) be the block-superdiagonal differential matrix representing \( \partial \) relative to the adapted basis. Theorem 1.4 provides a block-diagonal triangular matrix \( B \) such that \( D = B^{-1}DB \) is almost-Jordan. So the matrix \( B \) transforms the original adapted basis to a special adapted basis. \( \square \)

Corollary 4.2. A filtered complex admits a finite decomposition as a sum of basic filtered complexes.

Proof. Choose a special adapted basis, and denote by \( D \) the corresponding almost-Jordan block-superdiagonal differential matrix representative. Let \( P \) be a permutation matrix such that the matrix \( P^{-1}DP \) is Jordan. Each Jordan block of this matrix represents a basic subobject of the filtered complex. The decomposition into Jordan blocks represents the decomposition of the filtered complex as a direct sum of basic filtered complexes. \( \square \)

To verify the Krull-Schmidt property, we will also need:

Lemma 4.3. A basic filtered complex has local endomorphism ring.

Proof. We first show that the colimit complex of a basic filtered complex has local endomorphism ring. The colimit complex is isomorphic to an interval complex. An interval complex is an indecomposable in the linear Abelian category of complexes, so it has local endomorphism ring by Atiyah’s Criterion 2.6. (Or less abstractly, it is easy to check that the endomorphism ring of an interval complex is isomorphic to the field \( \mathbb{F} \).)

The proof is completed by checking that the endomorphism ring of a basic filtered complex maps isomorphically to the endomorphism ring of its colimit interval complex. In general, the endomorphism ring of a filtered object maps \textit{injectively} to the endomorphism ring of its colimit. We need to show that the endomorphism ring of a basic filtered complex maps \textit{surjectively} to the endomorphism ring of its colimit. It suffices to show that an endomorphism of an interval complex restricts to an endomorphism of any subobject. There are two types of interval complexes to consider. If the interval complex is isomorphic to \( J[n] \), then the subobjects are 0 and \( J[n] \), and any endomorphism restricts. If the interval complex is isomorphic to \( K[n] \), then the subobjects are 0, \( J[n] \), and \( K[n] \), and any endomorphism restricts. \( \square \)

Assembling the pieces proves the main result of this section:

Proposition 1.7. (Forward Structural Equivalence) The Matrix Structural Theorem implies the Categorical Structural Theorem.

Proof. We first prove that a filtered complex is indecomposable iff it is basic. A basic filtered complex has a local endomorphism ring by Lemma 4.3, so it is indecomposable by Lemma 2.4. An indecomposable filtered complex is a finite direct sum of basic filtered complexes by Corollary 4.2. The direct sum cannot have more than one summand, because that would contradict the indecomposability. So an indecomposable filtered complex is basic.

Now it remains to check the two conditions of Definition 2.5. Since a basic filtered complex is indecomposable, Corollary 4.2 asserts that every filtered complex admits a finite decomposition as a sum of indecomposables. Since an indecomposable filtered complex is basic, Lemma 4.3 asserts that every indecomposable has a local endomorphism ring. \( \square \)
Example 4.4. Let $F$ be the filtered complex of Example 1.5. The initial adapted basis consists of appropriately ordered simplices: $1a_0, 1b_0, 3c_0, 2ab_1, 4bc_1, 5ac_1, 6abc_2$. The block-superdiagonal differential matrix $D$ represents the colimit boundary operator relative to the initial adapted basis.

The triangular block-diagonal matrix $B$ represents an automorphism of the filtered complex. This automorphism takes the initial adapted basis to the transformed adapted basis $1a_0, 1b_0, 3c_0, 2ab_1, 4bc_1, 5ac_1, 6abc_2$. This transformed adapted basis is special, because the block-superdiagonal differential matrix representative $\tilde{D} = B^{-1}DB$ is almost-Jordan:

$$D = \begin{bmatrix}
1a_0 & 1b_0 & 3c_0 & 2ab_1 & 4bc_1 & 5ac_1 & 6abc_2 \\
1a_0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1b_0 & 0 & 0 & 1 & 0 & 0 & 0 \\
3c_0 & 0 & 0 & 0 & 1 & 0 & 0 \\
2ab_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
4bc_1 & 0 & 0 & 0 & 0 & 1 & 0 \\
5ac_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
6abc_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}.$$ 

We have retained the shading denoting the super-diagonal blocks, and we have also boldfaced the nonzero entries and the diagonal entries of zero columns. An almost-Jordan differential matrix $P^{-1}DP$ is Jordan iff the matrix $DP$, which is related to $D$ by a permutation of columns, has each boldfaced $1$ immediately following the boldfaced $0$ in the same row. Permuting columns 3 and 4 suffices for this example, and

$$P = \begin{bmatrix}
1a_0 & 1b_0 & 3c_0 & 2ab_1 & 4bc_1 & 5ac_1 & 6abc_2 \\
1a_0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1b_0 & 0 & 1 & 0 & 0 & 0 & 0 \\
3c_0 & 0 & 0 & 1 & 0 & 0 & 0 \\
2ab_1 & 0 & 0 & 0 & 1 & 0 & 0 \\
4bc_1 & 0 & 0 & 0 & 0 & 1 & 0 \\
5ac_1 & 0 & 0 & 0 & 0 & 0 & 1 \\
6abc_2 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}.$$
produces the Jordan matrix

\[
P^{-1} D P = \begin{bmatrix}
    \frac{1}{a} & 0 & 0 & 0 & 0 & 0 \\
    \frac{1}{b} & 0 & 0 & 0 & 0 & 1 \\
    \frac{1}{c} & 0 & 0 & 0 & 0 & 0 \\
    \frac{1}{d} & 0 & 0 & 0 & 0 & 0 \\
    \frac{1}{e} & 0 & 0 & 0 & 0 & 0 \\
    \frac{1}{f} & 0 & 0 & 0 & 0 & 0 
\end{bmatrix}
\]

The decomposition of the Jordan matrix into its Jordan blocks represents the decomposition of the filtered complex into indecomposable/basic summands. We now list the indecomposable summands, denoting by \( \langle v \rangle \) the linear span of a vector \( v \in V \):

- **The Jordan block matrix** \( \frac{1}{a} \begin{bmatrix} 0 \end{bmatrix} \) represents the filtered complex

\[
\begin{array}{cccccccc}
    : & : & : & : & : & : & : & : \\
    \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
    \cdots \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow \cdots \rightarrow 0 \\
    \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
    \cdots \rightarrow 0 \rightarrow \langle \frac{1}{a} \rangle \rightarrow \langle \frac{1}{a} \rangle \rightarrow \langle \frac{1}{a} \rangle \rightarrow \langle \frac{1}{a} \rangle \rightarrow \langle \frac{1}{a} \rangle \rightarrow \cdots \rightarrow \langle \frac{1}{a} \rangle \\
    \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
    \cdots \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow \cdots \rightarrow 0 \\
    \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
    : & : & : & : & : & : & : & : \\
\end{array}
\]

\( p = 0 \quad p = 1 \quad p = 2 \quad p = 3 \quad p = 4 \quad p = 5 \)

This filtered complex is basic because in \( F \) it is isomorphic to the filtered complex

\[
\cdots \rightarrow 0 \rightarrow J[0] \rightarrow J[0] \rightarrow J[0] \rightarrow J[0] \rightarrow J[0] \rightarrow \cdots \rightarrow J[0], \\
\]

\( p = 0 \quad p = 1 \quad p = 2 \quad p = 3 \quad p = 4 \quad p = 5 \)

which has the interval complex \( J[0] \) as colimit. So the filtered complex is an indecomposable object in the category \( F \), and also an indecomposable object in the quotient category \( \text{coim} P_0 \) where it is isomorphic to \([1, \infty)_0\). Here the subscript 0 labels the degree/dimension of the quotient category \( \text{coim} P_0 \), and the interval subset \([1, \infty) \subseteq \mathbb{R}\) encodes the levels \( p \) that are isomorphic to \( J[0] \). The equivalence \( \text{coim} P_0 \rightarrow \text{im} P_0 \) corresponds to the homology functor \( H_0 \) acting on a diagram of complexes, producing the indecomposable barcode persistence vector space \([1, \infty)\):

\[
\cdots \rightarrow 0 \rightarrow Q \rightarrow Q \rightarrow Q \rightarrow Q \rightarrow Q \rightarrow \cdots . \\
\]

\( p = 0 \quad p = 1 \quad p = 2 \quad p = 3 \quad p = 4 \quad p = 5 \)
For any $n \neq 0$, the filtered complex is a zero object in the quotient category $\text{coim} P_n$.

- The Jordan block matrix
  \[
  \begin{pmatrix}
  1 & 0 \\
  a & 1
  \end{pmatrix}
  \]
  represents the filtered complex.

This filtered complex is basic because in $\mathcal{F}$ it is isomorphic to the filtered complex

\[
\cdots \xrightarrow{0} J[0] \xrightarrow{1} K[0] \xrightarrow{1} K[0] \xrightarrow{1} K[0] \xrightarrow{1} K[0] \xrightarrow{1} K[0] \xrightarrow{1} K[0] \xrightarrow{1} \cdots K[0],
\]

which has the interval complex $K[0]$ as colimit. So the filtered complex is an indecomposable object in the category $\mathcal{F}$, and also an indecomposable object in the quotient category $\text{coim} P_0$ where it is isomorphic to $[1, 2)_0$. Here the subscript 0 labels the degree/dimension of the quotient category $\text{coim} P_0$, and the interval subset $[1, 2) \subseteq \mathbb{Z}$ encodes the levels $p$ that are isomorphic to $J[0]$. The equivalence $\text{coim} P_0 \rightarrow \text{im} P_0$ corresponds to the homology functor $H_0$ acting on a diagram of complexes, producing the indecomposable barcode persistence vector space $[1, 2)$:

\[
\cdots \xrightarrow{0} \mathbb{Q} \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} \cdots .
\]

For any $n \neq 0$, the filtered complex is a zero object in the quotient category $\text{coim} P_n$. 

The Jordan block matrix \[
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
\]
represents the filtered complex...
The Jordan block matrix \[
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
\]
represents the filtered complex...

This filtered complex is basic because in \(\mathcal{F}\) it is isomorphic to the filtered complex

\[
\cdots \hookrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots \hookrightarrow 0
\]

which has the interval complex \(K[1]\) as colimit. So the filtered complex is an indecomposable object in the category \(\mathcal{F}\), and also an indecomposable object in the quotient category \(\text{coim} P_1\) where it is isomorphic to \([5,6)\). Here the subscript 1 labels the degree/dimension of the quotient category \(\text{coim} P_1\), and the interval subset \([5,6) \subseteq \mathbb{Z}\) encodes the levels \(p\) that are isomorphic to \(J[1]\). The equivalence \(\text{coim} P_1 \rightarrow \text{im} P_1\) corresponds to the homology functor \(H_1\) acting on a diagram of complexes, producing the indecomposable barcode persistence vector space \([5,6):\)

\[
\cdots \rightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots
\]

For any \(n \neq 1\), the filtered complex is a zero object in the quotient category \(\text{coim} P_n\).

This completes the decomposition of the filtered complex \(F\) in the category \(\mathcal{F}\). As an object in the quotient category \(\text{coim} P_5\), the filtered complex \(F\) is isomorphic to \([0, \infty) \oplus [1,2) \oplus [3,4)\). As an object in the quotient category \(\text{coim} P_1\), the filtered complex \(F\) is isomorphic to \([5,6)\). For any other value of \(n\), the filtered complex \(F\) is a zero object in the quotient category \(\text{coim} P_n\).

4.2. Reverse Structural Equivalence. Special adapted bases help to intermediumiate between the Matrix Structural Theorem and Categorical Structural Theorem. In Proposition 4.1 we established the existence of a special adapted basis using...
the Matrix Structural Theorem \[1.4 \]. Now in the reverse direction, we establish the existence of a special adapted basis using the Categorical Structural Theorem \[1.6 \].

**Proposition 4.5.** A filtered complex admits a special adapted basis.

**Proof.** The Categorical Structural Theorem decomposes the filtered complex as a finite direct sum of indecomposables. Each indecomposable summand is a basic filtered complex, so it admits a special adapted basis. With appropriate ordering, the union over the summands of these basis elements is a special adapted basis for the direct sum filtered complex. □ □

An automorphism of a filtered complex transforms an adapted basis to another adapted basis. The change of basis is represented by a matrix $B$, which is block-diagonal because an automorphism preserves the degree of basis elements. But the matrix $B$ need not be triangular in general. We call a filtered complex nondegenerate if \( \dim(V_{p+1}^n) \leq \dim(V_p^n) + 1 \) for any $p$ and any $n$.

**Lemma 4.6.** If a filtered complex is nondegenerate, then any change of adapted basis is represented by a triangular matrix $B$.

**Proof.** An automorphism takes a basis element of degree $n$ and level $p$ to a linear combination of basis elements of degree $n$ and level at most $p$. A filtered complex is nondegenerate iff an adapted basis contains no pair of elements with the same degree and same level. In this case the linear combination does not contain any basis elements that appear later in the ordering of the basis. The matrix $B$ is then triangular, since it has no nonzero entries below the diagonal. □ □

We will construct nondegenerate filtered complexes by using the upper-left submatrices of a differential matrix. We illustrate submatrices with an example:

**Example 4.7.** The upper-left submatrices are indicated below for the block-superdiagonal differential matrix \( D : \mathbb{Q}^7 \rightarrow \mathbb{Q}^7 \) given by:

\[
D = \begin{bmatrix}
0 & 0 & 0 & -1 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

Note that for each integer $0 < p < 7$, the upper-left submatrix \( D_p : \mathbb{Q}^p \rightarrow \mathbb{Q}^p \) is itself a block-superdiagonal differential matrix. We remark that the matrix $D$ had appeared previously in Example \[1.5 \], representing the degenerate (not nondegenerate) filtered complex of Example \[1.8 \].

**Lemma 4.8.** Any block-superdiagonal differential matrix $D$ represents the colimit boundary of some nondegenerate filtered complex.
Proof. Let \( D : \mathbb{F}^m \to \mathbb{F}^m \) be a block-superdiagonal differential matrix. We construct a filtered complex

\[
\cdots \hookrightarrow V_{-1} \hookrightarrow V_0 \hookrightarrow V_1 \hookrightarrow V_2 \hookrightarrow V_3 \hookrightarrow \cdots \quad \mathbb{V}
\]

by specifying for each integer \( p \) the complex \( pV_\bullet \) at level \( p \):

- For \( p \leq 0 \), the complex is the zero complex.
- For \( 1 < p < m \), the complex is specified by the block-superdiagonal differential submatrix \( pD : \mathbb{F}^p \to \mathbb{F}^p \).
- For \( m \leq p \), the complex is specified by the initial block-superdiagonal differential matrix \( D : \mathbb{F}^m \to \mathbb{F}^m \).

The arrows are the subobject inclusions \( pV_\bullet \hookrightarrow p+1 V_\bullet \). Then the diagram is a filtered complex since the zero complex is a limit and the complex \( D : \mathbb{F}^m \to \mathbb{F}^m \) is a colimit. It only remains to observe that the filtered complex is nondegenerate, and that the matrix \( D \) represents its colimit boundary. □ □

Note that the block structure of the differential matrix \( D \) is important in the preceding proof. If a differential matrix does not have block-superdiagonal structure, then an upper-left submatrix need not be a differential matrix in general.

Now we have assembled the ingredients to prove:

**Proposition 1.8.** (Reverse Structural Equivalence) The Categorical Structural Theorem implies the Matrix Structural Theorem.

Proof. Let \( D \) be a block-superdiagonal differential matrix. Lemma \( 4.8 \) lets us choose a nondegenerate filtered complex that is represented by \( D \). Proposition \( 4.5 \) lets us make a change of basis to a special adapted basis. The block-diagonal matrix \( B \) representing the basis change is triangular by Lemma \( 4.6 \). Finally, the block-superdiagonal differential \( D = B^{-1} DB \) is almost-Jordan because the adapted basis is special. □ □

5. Concluding Remarks and Directions for Further Development

5.1. Encoding Homology. Section \( 3.2 \) presents our alternate framework for persistent homology, based on the Krull-Schmidt decomposition of a filtered complex afforded by the Categorical Structural Theorem \( 1.6 \). In this section we outline the analogous alternate framework for the homology of “plain” (i.e. not filtered) complexes (see Section \( 2.3 \)). The encoding of homology within a decomposition is easier to explain in this simpler setting, and the explanation carries over mutatis mutandis to the more complicated persistent homology framework. The basic idea is to “compute” homology (or persistent homology) by discarding from a decomposition those summands that are a priori known to have zero homology (respectively persistent homology). Note that this idea cannot be implemented in all situations. Even for plain homology, it works with coefficients in a field \( \mathbb{F} \), but fails in the fundamental case of integer coefficients. For persistent homology, it works for the “ordinary” case of filtered complexes as discussed previously, but fails for zigzag persistent homology as discussed in Section 5.3 below.
The category $\mathcal{C}$ of chain complexes is Krull-Schmidt (by Proposition 2.11), and any indecomposable complex is isomorphic to $J[m]$ or to $K[m]$ for some integer $m$. Denote by $H_n : \mathcal{C} \to \mathcal{V}$ the degree-$n$ homology functor from $\mathcal{C}$ to the category $\mathcal{V}$ of vector spaces (see Section 3.1). The following result is the analogue of Theorem 3.3 for this simpler setting:

**Proposition 5.1.** The (plain) homology functor $H_n : \mathcal{C} \to \mathcal{V}$ factors as

$$\mathcal{C} \to \text{coim } H_n \to \text{im } H_n \to \mathcal{V},$$

where the functor $\text{coim } H_n \to \text{im } H_n$ is an equivalence of categories.

In this setting $\text{im } H_n$ is just another name for $\mathcal{V}$, and $\text{im } H_n \to \mathcal{V}$ is the identity. The interesting part is the quotient functor $\mathcal{C} \to \text{coim } H_n$, where the quotient category $\text{coim } H_n$ is defined via the following equivalence relation (congruence) on morphisms: two morphisms $f$ and $f'$ in $\mathcal{C}$ are equivalent iff the morphisms $H_n(f)$ and $H_n(f')$ in $\mathcal{V}$ are equal. By the Krull-Schmidt property, any complex $C$ in $\mathcal{C}$ is isomorphic to a direct sum with appropriate multiplicities of the indecomposable complexes $J[m]$ and $K[m]$ for various $m$. The key property required to encode homology in this framework is: an indecomposable complex goes to zero under the quotient functor iff it goes to zero under the homology functor $H_n$. Namely, $J[m]$ goes to zero unless $m = n$, and $K[m]$ goes to zero for all $m$. So working with a complex $C$ in the quotient category $\text{coim } H_n$ amounts to discarding from a decomposition of $C$ those indecomposable summands that are a priori known to go to zero under the homology functor $H_n$. Each indecomposable summand that remains is canonically isomorphic to $J[n]$, and the set of these isomorphisms contains the data for the usual “basis of homology cycles” of the homology vector space $H_n(C)$.

5.2. **Kernels and Cokernels.** It is well-known that the representations of a quiver constitute an Abelian category, see for example [30]. This means that Abelian categories are relevant to persistent homology, and this has been studied in the paper [11]. We now rapidly review the fundamental constructs in an Abelian category, referring to Freyd’s classic [17] or the more modern approach of [32] for details. Recall that an additive category is pre-Abelian if any morphism $f : X \to Y$ admits a kernel, $\text{ker } f \to X$, and a cokernel, $Y \to \text{coker } f$, each characterized by standard universal properties. Then the image, $\text{im } f \to Y$, is defined as the kernel of the cokernel, and the coimage, $X \to \text{coim } f$ as the cokernel of the kernel. Any morphism $f : X \to Y$ in a pre-Abelian category factors uniquely as ([32] Lemma 3.12): 

$$X \to \text{coim } f \to \text{im } f \to Y.$$

Finally, a pre-Abelian category is Abelian iff $\text{coim } f \to \text{im } f$ is always an isomorphism. ([32] Definition 5.1; this is widely known as the “rank theorem” for the Abelian category $\mathcal{V}$ of finite-dimensional vector spaces.)

The standard framework for persistent homology (Section 3.1) focuses on the category $\text{im } P_n$. The category $\text{im } P_n$ is Abelian, so each morphism $f : X \to Y$ has a kernel, cokernel, image, and coimage. Furthermore the category $\text{im } P_n$ is Krull-Schmidt, so the objects $\text{ker } f$, $\text{coker } f$, $\text{im } f$, and $\text{coim } f$ can be decomposed in terms of barcodes. The paper [11] presents algorithms for computing the barcode
invariants of these objects for the case when \( f : X \hookrightarrow Y \) is the inclusion of a subobject, and discusses the case of general \( f \) in terms of mapping cylinders.

Our alternate framework for persistent homology (Section 3.2) focuses on the quotient category \( \text{coim} P_n \). Theorem 3.3 asserts that the persistent homology functor \( P_n : F \to \mathcal{P} \) factors as

\[
F \to \text{coim} P_n \to \text{im} P_n \to \mathcal{P},
\]

where the functor \( \text{coim} P_n \to \text{im} P_n \) is an equivalence of categories. Since \( \text{im} P_n \) is Abelian, the equivalence immediately implies that the quotient category \( \text{coim} P_n \) is also Abelian. In a forthcoming paper \[29\], we study algorithms for constructing \( \ker f \to X \) and \( Y \to \text{coker} f \) for a general morphism \( f : X \to Y \) in the quotient category \( \text{coim} P_n \). (We note that the category \( F \) of filtered complexes is pre-Abelian but not Abelian \[32\], but the quotient functor \( F \to \text{coim} P_n \) does not preserve kernels and cokernels.)

5.3. Zigzag Persistent Homology. Zigzag persistent homology was introduced in \[8, 9\] and further studied in \[31, 25, 26\]. In this section we apply the categorical techniques of Chapters 2 and 3 to the general zigzag case. A reader who is not interested in the zigzag case may skip this section, and continue to the concluding discussion in Section 5.4. Our main result Theorem 3.3 applies to “ordinary persistent homology” (i.e. not the zigzag generalization). This result is also informally outlined in the flowchart diagrams of Section 1.1. We will show below that this result only partially generalizes to the zigzag case. Theorem 3.3 asserts that the ordinary persistent homology functor factors as

\[
F \to \text{coim} P_n \to \text{im} P_n \to \mathcal{P},
\]

where each category is Krull-Schmidt. This assertion generalizes to the zigzag case. Theorem 3.3 further asserts that the functor \( \text{coim} P_n \to \text{im} P_n \) is an equivalence of categories. This assertion does not generalize to the zigzag case. For the general zigzag case, the indecomposables of the Krull-Schmidt category \( \text{im} P_n \) are still classified by intervals as in \[8\]. But now the classification of indecomposables is more complicated for the Krull-Schmidt category \( \text{coim} P_n \), as illustrated by the example at the end of the section.

We proceed to an outline of the categorical framework for the general zigzag case. Let \( X \) be a linear Abelian category. We describe the sources of the leftward-directed arrows of a zigzag diagram as a subset \( L \subseteq \mathbb{Z} \). For any \( L \subseteq \mathbb{Z} \), we define an \( L \)-persistence object in \( X \) to be a diagram \( \bullet X \) in the category \( X \) of type

\[
\cdots \xleftarrow{p-1} p^{-1}X \xleftrightarrow{p} pX \xleftrightarrow{p+1} p+1X \xrightarrow{p} \cdots ,
\]

where the arrow directions are specified by the rule: \( p^{-1}X \xleftarrow{p} pX \) if \( p \in L \) and \( p^{-1}X \xrightarrow{p} pX \) if \( p \notin L \). For example, an \( L \)-persistence object with \( L = \{0, 2, 3\} \subseteq \mathbb{Z} \) is:

\[
\cdots \xrightarrow{-1} -1X \xleftarrow{0} 0X \xrightarrow{1} 1X \xleftarrow{2} 2X \xrightarrow{3} 3X \xrightarrow{\cdots} \]

\[
p = -1 \quad p = 0 \quad p = 1 \quad p = 2 \quad p = 3
\]
We recover the previous definition of “ordinary” persistence object from Section 2.2 by choosing $L = \emptyset$. A morphism of $L$-persistence objects $f : X \to X'$ is a commutative diagram of “ladder” type:

\[ \cdots \leftrightarrow p_{-1}X \leftrightarrow p_0X \leftrightarrow p_1X \leftrightarrow \cdots \\
\downarrow_{p_{-1}f} \downarrow_{pf} \downarrow_{p_1f} \\
\cdots \leftrightarrow p_{-1}X' \leftrightarrow p_0X' \leftrightarrow p_1X' \leftrightarrow \cdots. \]

Recall that a categorical diagram is tempered if all but finitely many of its arrows are iso(morphisms). The proof of Proposition 2.7 readily generalizes to:

**Proposition 5.2.** Let $\mathcal{X}$ be a linear Abelian category. Then for any $L \subseteq \mathbb{Z}$, the category of tempered $L$-persistence objects in $\mathcal{X}$ is Krull-Schmidt.

An “$L$-filtered object” is the generalization of a filtered object to the zigzag case. For any $L \subseteq \mathbb{Z}$, an $L$-filtered object in a concrete linear Abelian category $\mathcal{X}$ is a special type of tempered $L$-persistence object in $\mathcal{X}$. Recall that in a concrete linear Abelian category we denote by $X \hookrightarrow X'$ the inclusion of a subobject. We say a tempered $L$-persistence object $X$ is an $L$-filtered object if it is bounded below and if every arrow is an inclusion arrow:

\[ \cdots \leftrightarrow p_{-1}X \leftrightarrow p_0X \leftrightarrow p_1X \leftrightarrow \cdots. \]

For example, an $L$-filtered object with $L = \{0, 2, 3\} \subseteq \mathbb{Z}$ is:

\[ \cdots \leftrightarrow _{-1}X \leftrightarrow _0X \leftrightarrow _1X \leftrightarrow _2X \leftrightarrow _3X \leftrightarrow \cdots. \]

\[ p = -1 \quad p = 0 \quad p = 1 \quad p = 2 \quad p = 3 \]

We recover the previous definition of “ordinary” filtered object from Section 2.2 by choosing $L = \emptyset$. It is important to note that an $L$-filtered object need not admit a categorical limit and colimit in the general case $L \neq \emptyset$. Consequently only the first assertion of Lemma 2.8 generalizes:

**Lemma 5.3.** Let $\mathcal{X}$ be a linear Abelian category. Then for any $L \subseteq \mathbb{Z}$, the category of $L$-filtered objects in $\mathcal{X}$ is Krull-Schmidt.

But the classification of filtered objects in terms of colimits is not available for the general zigzag case $L \neq \emptyset$.

For any $L \subseteq \mathbb{Z}$ we can now introduce $L$-persistent homology, commonly known as “zigzag persistent homology.” $\mathcal{F}$ now denotes the Krull-Schmidt (by Lemma 5.3) category of $L$-filtered complexes, where an object is a diagram of complexes. $\mathcal{P}$ now denotes the Krull-Schmidt (by Proposition 5.2) category of $L$-persistence vector spaces, where an object is a diagram of vector spaces. The homology functor $H_n$ takes a diagram of complexes to a diagram of vector spaces, resulting in a functor $P_n : \mathcal{F} \to \mathcal{P}$ which we call the $L$-persistent homology of degree $n$. This is the functor that is commonly known as zigzag persistent homology. We recover the “ordinary” (i.e. not “zigzag”) persistent homology functor by choosing $L = \emptyset$. 
We now consider the factorization properties of the $L$-persistent homology functors for an arbitrary subset $L \subseteq \mathbb{Z}$, obtaining a partial generalization of Theorem 3.3 for the ordinary case where $L = \emptyset$. The category $\text{im} P_n$, comprised of the $L$-persistence vector spaces that are bounded below, is a full Abelian subcategory of $\mathcal{P}$ and therefore Krull-Schmidt. (The category $\text{im} P_n$ does not depend on $n$; it is always the same subcategory of $\mathcal{P}$.) The well-studied representation theory of $A_\infty$ quivers (see e.g. [8, 30]) still carries over by a limiting argument to classify the indecomposable $L$-persistence vector spaces in terms of intervals $I \subseteq \mathbb{Z}$ that are bounded below. The category $\text{coim} P_n$ is a categorical quotient of $\mathcal{F}$, defined via the following equivalence relation (congruence) on morphisms: two morphisms $f$ and $f'$ in $\mathcal{F}$ are equivalent iff the morphisms $P_n(f)$ and $P_n(f')$ in $\mathcal{P}$ are equal. For any $L \subseteq \mathbb{Z}$, the category $\mathcal{F}$ and its quotient $\text{coim} P_n$ are both Krull-Schmidt. But the classification of their indecomposable summands in terms of colimits is not available for the general zigzag case $L \neq \emptyset$. Consequently for arbitrary $L \subseteq \mathbb{Z}$, we only have the following partial generalization of Theorem 3.3:

**Theorem 5.4.** For any $L \subseteq \mathbb{Z}$, the $L$-persistent homology functor $P_n : \mathcal{F} \to \mathcal{P}$ factors as

$$\mathcal{F} \to \text{coim} P_n \to \text{im} P_n \to \mathcal{P}.$$  

Recall that for the “ordinary” case $L = \emptyset$, we compared the classification of indecomposables in the two categories to prove that $\text{coim} P_n \to \text{im} P_n$ is an equivalence of categories. But for the general zigzag case where $L \neq \emptyset$, the following example shows that the functor $\text{coim} P_n \to \text{im} P_n$ is not an equivalence in general:

**Example 5.5.** We consider $L$-persistent homology with $L = \{0, 2, 3\} \subseteq \mathbb{Z}$. Start with the $L$-filtered simplicial complex shown in Figure 13. In the quotient category $\text{coim} P_1$ this becomes the (indecomposable) object

$$0 \longrightarrow J[1] \longrightarrow K[1] \leftarrow K[1] \leftarrow K[1] \leftarrow J[1] \leftarrow 0.$$  

$p \leq -3 \quad p = -2 \quad p = -1 \quad p = 0 \quad p = 1 \quad p = 2 \quad p \geq 3$

Compare with the $L$-filtered simplicial complex shown in Figure 14. In the quotient category $\text{coim} P_1$ this becomes the (decomposable) object

$$0 \longrightarrow J[1] \longrightarrow K[1] \leftarrow 0 \longrightarrow K[1] \leftarrow J[1] \leftarrow 0.$$  

$p \leq -3 \quad p = -2 \quad p = -1 \quad p = 0 \quad p = 1 \quad p = 2 \quad p \geq 3$
This pair of objects is not isomorphic in the quotient category $\text{coim} P_1$. But these non-isomorphic objects become isomorphic in $\text{im} P_n$, since both go to the same (decomposable) object

$$
0 \longrightarrow F \longrightarrow 0 \leftarrow 0 \longrightarrow 0 \leftarrow 0 \longrightarrow F \leftarrow 0.
$$

$$p \leq -3 \quad p = -2 \quad p = -1 \quad p = 0 \quad p = 1 \quad p = 2 \quad p \geq 3$$

It follows that the functor $\text{coim} P_n \rightarrow \text{im} P_n$ cannot be an equivalence of categories; an equivalence would not take a non-isomorphic pair to an isomorphic pair. (Furthermore, an equivalence would not take an indecomposable object to a decomposable object.)

5.4. What is the Best Framework for Persistent Homology? To conclude, we make a few general remarks about comparing categorical frameworks. A functor $F : \mathcal{A} \rightarrow \mathcal{B}$ transforms objects and morphisms in a category $\mathcal{A}$ to objects and morphisms in another category $\mathcal{B}$. The usefulness of a functor for studying objects and morphisms in the category $\mathcal{A}$ depends on various criteria for the category $\mathcal{B}$. Such criteria are discussed in many Algebraic Topology textbooks, for example [20], in the context of the “plain” homology functors $H_n$. Here we briefly consider some key criteria in the context of the persistent homology functors $P_n$, including the zigzag case of Appendix 5.3.

A functor $F : \mathcal{A} \rightarrow \mathcal{B}$ may be useful if the category $\mathcal{B}$ has additional structure. We will call this the **structural criterion** for the category $\mathcal{B}$. Krull-Schmidt categories and Abelian categories are relevant examples of categories with additional structure. A categorical structure tends to be useful in applications if it is amenable to algorithmic computation. The most important applications of persistent homology are based on algorithmic decompositions of objects in various Krull-Schmidt categories. As we have shown, the algorithms for persistent homology actually compute a Krull-Schmidt decomposition of a filtered complex in $\mathcal{F}$. In the standard framework (Section 3.1) the functor $\mathcal{F} \rightarrow \text{im} P_n$ takes this decomposition of a filtered complex in $\mathcal{F}$ to a decomposition of a persistence vector space in the category $\text{im} P_n$ of persistence vector spaces. In our alternate framework (Section 3.2) the functor $\mathcal{F} \rightarrow \text{coim} P_n$ takes this decomposition of a filtered complex in $\mathcal{F}$ to a decomposition of a filtered complex in the quotient category $\text{coim} P_n$. For “ordinary” persistent homology we have the equivalence of categories $\text{coim} P_n \rightarrow \text{im} P_n$ (Theorem 3.3), so the alternate framework and the standard framework perform equally well on the structural criterion.

But in the general zigzag case the categories $\text{coim} P_n$ and $\text{im} P_n$ are not equivalent (Appendix 5.3). Decomposition algorithms in the Krull-Schmidt category $\text{im} P_n$
are known, and furthermore this category is known to be Abelian. So the standard framework $\mathcal{F} \to \text{im } P_n$ performs well on the structural criterion. But decomposition algorithms in the more complicated Krull-Schmidt category $\text{coim } P_n$ do not appear to be known at present, and furthermore it is not clear whether this category is Abelian. So at present a putative alternate framework $\mathcal{F} \to \text{coim } P_n$ for the general zigzag case does rather badly on the structural criterion.

An important countervailing consideration is whether the functor $F : \mathcal{A} \to \mathcal{B}$ loses, or forgets, too much of the information present in the original category $\mathcal{A}$. For example, the functor may be losing too much information if it takes non-isomorphic objects in $\mathcal{A}$ to isomorphic objects in $\mathcal{B}$ (as we saw in Example 5.5 for a zigzag case of the functor $\text{coim } P_n \to \text{im } P_n$). Typically we work with concrete categories, where objects are represented as sets with additional features, such as algebraic or topological features. Then we would like the sets representing objects of $\mathcal{B}$ to retain features of the sets representing objects of $\mathcal{A}$. We will call this the \textit{representational criterion} for the category $\mathcal{B}$. Our alternate framework for persistent homology (Section 3.2) is based on the quotient functor $\mathcal{F} \to \text{coim } P_n$. Our alternate framework performs very well on the representational criterion, because a filtered complex in $\mathcal{F}$ goes to the very same filtered complex in the quotient category $\text{coim } P_n$. The standard framework (Section 3.1) is based on the functor $\mathcal{F} \to \text{im } P_n$. The standard framework does not perform well on the representational criterion, because a persistence vector space in $\text{im } P_n$ does not retain the algebraic features of a filtered complex in $\mathcal{F}$. These arguments carry over to the general zigzag case, where a putative alternate framework would also perform better on the representational criterion.

In conclusion, we argue that our alternate framework for persistent homology is better than the standard framework based on the representational criterion and structural criterion described here, in the setting of “ordinary” persistent homology.

\textbf{Appendix A. Bruhat Uniqueness Lemma}

Here we establish the uniqueness of the persistence canonical form $D$ appearing in the Matrix Structural Theorem 1.4, as well as in the ungraded version Theorem 1.2. Our result generalizes the uniqueness statement for the usual Bruhat factorization of an invertible matrix [2,18].

It is convenient to make the following definitions. We call an (upper) triangular matrix $U$ \textit{unitriangular} if it is unipotent, meaning that each diagonal entry is 1. We call a matrix $M$ \textit{quasi-monomial} if each row has at most one nonzero entry and each column has at most one nonzero entry. We remark that a unitriangular matrix is always square, but a quasi-monomial matrix need not be square. The key to proving uniqueness is:

\textbf{Lemma A.1.} \textit{Suppose }$M_1 U = V M_2$, \textit{where }$M_1$ \textit{and }$M_2$ \textit{are quasi-monomial and }$U$ \textit{and }$V$ \textit{are unitriangular. Then }$M_2 = M_1$.

In the following proof, the term \textit{row-pivot} denotes a matrix entry that is the leftmost nonzero entry in its row, and \textit{column-pivot} denotes a matrix entry that is the bottommost nonzero entry in its column.

\textit{Proof.} The first half of the proof consists of showing that every nonzero entry of $M_2$ is also an entry of $M_1$. A nonzero entry of the quasi-monomial matrix $M_2$ is a column-pivot. Similarly a nonzero entry of the quasi-monomial matrix $M_1$ is a
row-pivot. It now suffices to show that a column-pivot of \( M_2 \) is a row-pivot of \( M_1 \). Since \( V \) is unitriangular, \( VM_2 \) has the same column-pivots as \( M_2 \). Similarly since \( U \) is unitriangular, \( M_1U \) has the same row-pivots as \( M_1 \). It now suffices to prove that a column-pivot of \( S = VM_2 \) is a row-pivot of \( S = M_1U \). Suppose to the contrary that some column-pivot of \( S \) is not a row-pivot of \( S \). Let \( x \) be the leftmost such column-pivot. Since \( x \) is not a row-pivot, there exists a row-pivot \( y \) to the left of \( x \) in the same row. If \( y \) were a column-pivot of \( S = VM_2 \), then it would be a column-pivot of \( M_2 \). But the quasi-monomial matrix \( M_2 \) cannot have two nonzero entries \( y \) and \( x \) in the same row. So \( y \) is not a column-pivot of \( S \), and there exists a column-pivot \( z \) below \( y \) in the same column. If \( z \) were a row-pivot of \( S = U M_1 \), then it would be a row-pivot of \( M_1 \). But the quasi-monomial matrix \( M_1 \) cannot have two nonzero entries \( z \) and \( y \) in the same column. So \( z \) is a column-pivot of \( S \) that is not a row-pivot of \( S \), and \( z \) is to the left of (and below) \( x \). This is a contradiction, because \( x \) is the leftmost such column-pivot.

The second half of the proof consists of showing that every nonzero entry of \( M_1 \) is also an entry of \( M_2 \). This is analogous to the first half, and we omit the details. The two matrices then have the same nonzero entries, so they must also have the same zero entries. Since all the entries of the two matrices are the same, we have proved \( M_2 = M_1 \).

Recall that a matrix \( M \) is Boolean if every non-zero entry is 1. An almost-Jordan differential matrix \( D \) is Boolean and quasi-monomial.

**Proposition A.2.** Suppose \( P_1A = BP_2 \) where \( P_1 \) and \( P_2 \) are Boolean quasi-monomial and \( A \) and \( B \) are invertible triangular. Then \( P_2 = P_1 \).

**Proof.** Factor \( A = T_1U \) as the product of an invertible diagonal matrix \( T_1 \) and a unitriangular matrix \( U \). Factor \( B = VT_2 \) as the product of a unitriangular matrix \( V \) and an invertible diagonal matrix \( T_2 \). Then \((P_1T_1)U = V(T_2P_2)\), with \((P_1T_1)\) and \((T_2P_2)\) quasi-monomial. Lemma A.1 then gives the \( P_1T_1 = T_2P_2 \). Since the quasi-monomial matrices \( P_1 \) and \( P_2 \) are Boolean, the conclusion follows. \( \square \)

We remark that any permutation matrix \( P \) is Boolean and quasi-monomial, so Proposition A.2 generalizes the standard uniqueness result for Bruhat factorization of an invertible matrix [2][18].

The uniqueness of the persistence canonical form \( D \) appearing in Theorem 1.2 and in the Matrix Structural Theorem 1.4 now follows easily:

**Corollary A.3.** Suppose \( D \) is a differential matrix and \( B_1 \) and \( B_2 \) are invertible triangular matrices. If both differential matrices \( D_1 = B_1^{-1}DB_1 \) and \( D_2 = B_2^{-1}DB_2 \) are almost-Jordan, then \( D_2 = D_1 \).

**Proof.** \( D_1(B_1^{-1}B_2) = (B_1^{-1}B_2)D_2 \), and the result follows from Proposition A.2 \( \square \)

**Appendix B. Constructively Proving the Matrix Structural Theorem**

**B.1. Linear Algebra of Reduction.** In this section we discuss column-reduction of a matrix \( M : \mathbb{F}^m \to \mathbb{F}^n \), including its application to describing the kernel and image of the matrix. Column-reduction of a differential matrix \( D \) is a standard tool in the computation of persistent homology, where it is usually just called reduction.
We prefer the more precise terminology in order to maintain the distinction with row-reduction, since both are used for Bruhat factorization.

As in Appendix A, the term column-pivot denotes a matrix entry that is the bottommost nonzero entry in its column. A matrix \( R \) is said to be column-reduced if each row has at most one column-pivot.

**Definition B.1.** A column-reduction of a matrix \( M \) is an invertible triangular matrix \( V \) such that \( R = MV \) is column-reduced.

A column-reduction \( V \) exists for any matrix \( M \), but is not unique in general. Column-reduction algorithms used for persistent homology usually prioritize computational efficiency. For our computational examples, we will use a column-reduction algorithm that is popular for Bruhat factorization. This algorithm is easy to implement, but is not very efficient computationally. The algorithm starts at the leftmost column of \( M \) and proceeds rightward by successive columns as follows:

- If the current column is zero, do nothing.
- If the current column is nonzero, add an appropriate multiple of the current column to each column to the right in order to zero the entries to the right of the column-pivot (in the same row).

Stop if the current column is the rightmost column, otherwise proceed to the column immediately to the right and repeat. By design, the resulting matrix \( R \) has the property that any column-pivot has only zeros to the right of it (in the same row). So a row of \( R \) cannot contain more than one column-pivot, implying that \( R \) is column-reduced. The invertible triangular column-reduction matrix \( V \) is constructed by performing the same column operations on the identity matrix \( I \), where \( I \) has same number of columns as \( M \).

We briefly discuss some linear-algebraic properties of column-reduction. A column-reduction easily yields a basis for the kernel of a matrix as well as a basis for the image. By contrast, Gaussian elimination easily yields a basis for the image a matrix, but requires additional back-substitution to produce a basis for the kernel. Column-reduction algorithms are therefore a convenient alternative to Gaussian elimination for matrix computations in general, and this fact seems to be underappreciated. We use a variant of the usual adapted basis for a filtered vector space, disregarding the ordering of basis elements. We’ll say that a basis of a finite-dimensional vector space \( X \) is *almost-adapted* to a subspace \( Y \subseteq X \) if \( Y \) is spanned by the set of basis elements that are contained in \( Y \). Proposition B.1 yields:

**Corollary B.2.** Let \( V : \mathbb{F}^m \to \mathbb{F}^m \) be a column-reduction of a matrix \( M : \mathbb{F}^m \to \mathbb{F}^n \). Then:

1. The nonzero columns of the column-reduced matrix \( R = MV \) are a basis of \( \text{im} \, M \).
2. The columns of the invertible triangular matrix \( V \) are a basis of \( \mathbb{F}^m \), and this basis is almost-adapted to \( \ker M \).

**Proof.**

1. The nonzero columns of \( R \) span \( \text{im} \, M \). The nonzero columns of \( R \) are linearly independent because \( R \) is column-reduced.
(2) The columns of $V$ are a basis of $\mathbb{F}^m$ because $V$ is invertible. This basis is almost-adapted to $\ker M$ because the nonzero columns of $R = MV$ are linearly independent. □

Example B.3. We compute in detail a column-reduction of the matrix $M : \mathbb{Q}^4 \to \mathbb{Q}^3$, which is presented below with a column augmentation by the identity matrix $I$.

\[
\begin{array}{ccc}
M & = & \begin{bmatrix}
1 & -2 & 0 & -8 \\
2 & -4 & 6 & 2 \\
1 & -2 & 2 & -2 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \\
\frac{I}{I} & \mapsto & \begin{bmatrix}
1 & 0 & -2 & -6 \\
2 & 0 & 2 & 6 \\
1 & 0 & 0 & 0 \\
1 & 2 & -2 & 2 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \\
\mapsto & \begin{bmatrix}
1 & 0 & -2 & 0 \\
2 & 0 & 2 & 0 \\
1 & 0 & 0 & 0 \\
1 & 2 & -2 & 8 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & -3 \\
0 & 0 & 0 & 1
\end{bmatrix}
\end{array}
\]

The result of the computation is a factorization $R = MV$, where $R$ is column-reduced and $V$ is invertible triangular (and unipotent). We describe each step of the computation:

(1) The first column of $M$ is nonzero, so it has a column-pivot. At the next processing step, boldface the column-pivot for clarity, and add an appropriate multiple of the first column to each column to the right in order to zero the entries to the right of the column-pivot (in the same row).

(2) At this point the second column is zero, so requires no processing step.

(3) At this point the third column is nonzero, so it has a column-pivot. At the next processing step, boldface the column pivot, and add an appropriate multiple of the third column to each column to the right in order to zero the entries to the right of the column-pivot (in the same row).

(4) At this point the fourth column is zero, so requires no processing step.

Columns 1 and 3 of $R$ are the nonzero columns, so they are a basis of $\text{im} M \subseteq \mathbb{Q}^3$. The four columns of $V$ are a basis of $\mathbb{Q}^4$ that is almost-adapted to $\ker M$. Columns 2 and 4 of $V$ correspond to the zero columns of $R$, so they are a basis of $\ker M \subseteq \mathbb{Q}^4$.

B.2. Matrix Structural Theorem via Reduction. The standard algorithm of persistent homology [16, 35, 36] starts with a differential matrix $D$ and constructs a matrix $B$ satisfying the conditions of:

Theorem 1.2. (Ungraded Matrix Structural Theorem) Any differential matrix $D$ factors as $D = BDB^{-1}$ where $D$ is an almost-Jordan differential matrix and $B$ is a triangular matrix.

The matrix formulation of the standard algorithm constructs a matrix $B = \hat{V}$ from a column-reduction $V$ of a differential matrix $D$, as discussed in [13] for $F = \mathbb{Z}/2\mathbb{Z}$. Since $R = DV$ is column-reduced, there exists at most one nonzero column of $R$ that has its column-pivot in row $k$. Here $1 \leq k \leq m$ where $m$ is the number of rows of the square matrix $D$. $V$ is constructed one column at a time using the following rule:

- If there exists a nonzero column of $R$ that has its column-pivot in row $k$, then column $k$ of $\hat{V}$ is equal to this column of $R$. 

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- If there exists a nonzero column of $R$ that has its column-pivot in row $k$, then column $k$ of $\hat{V}$ is equal to this column of $R$. 

 If there does not exist a nonzero column of $R$ that has its column-pivot in row $k$, then column $k$ of $\hat{V}$ is equal to column $k$ of $V$.

The matrix $\hat{V}$ is invertible triangular, because each column is nonzero and has its column-pivot on the diagonal.

Unlike its progenitor $V$, the matrix $\hat{V}$ contains all of the nonzero columns of $R$. We introduce the “pivot matrix” of $R$ to encode the combinatorial data needed to recover $R$. For any column-reduced matrix $M$, the pivot matrix of $M$ is constructed by replacing every column-pivot of $M$ with 1, and every other nonzero entry of $M$ with 0. It follows that the pivot matrix is is Boolean and quasi-monomial (Appendix A).

**Lemma B.4.** Let $D$ be the pivot matrix of $R$. Then $\hat{V}D = R$.

**Proof.** Supposing column $k$ of $R$ is nonzero, let $j$ be the row number of the unique nonzero entry in column $k$ of $D$. Then by construction, column $j$ of $\hat{V}$ is equal to column $k$ of $R$. □ □

But like its progenitor $V$, the matrix $\hat{V}$ is a column-reduction of the differential $D$:

**Lemma B.5.** $D\hat{V} = R$.

**Proof.** The triangular matrix $V$ is a column-reduction of $D$, as per Definition B.1. We now show that the triangular matrix $\hat{V}$ is also a column-reduction of $D$. Recall that $\hat{V}$ is constructed as a modification of $V$, by replacing a (possibly empty) subset of the columns of $V$ with columns of $R$. Every column of $R$ is in ker $D$, because $DR = D^2V = 0$. So $\hat{R} := D\hat{V}$ is constructed as a modification of $R = DV$, by replacing a (possibly empty) subset of the nonzero columns of $R$ by zero columns. This particular modification preserves the column-reduced property, so $\hat{R} = D\hat{V}$ is column-reduced. It follows that $\hat{V}$ is a column-reduction of $D$, as per Definition B.1.

Since $\hat{V}$ and $V$ are column-reductions of the same matrix $D$, we see from Corollary B.2 that $\hat{R}$ and $R$ have the same number (namely rank $D$) of nonzero columns. It follows that in the construction of $\hat{R}$ as a modification of $R$, *none* of the nonzero columns of $R$ can be replaced by zero columns. This establishes the equality $\hat{R} = R$, and the conclusion $D\hat{V} = R$ follows. □ □

We can now complete the constructive proof of Theorem 1.2:

**Proof.** Letting $B = \hat{V}$, Lemmas B.4 and B.5 give $BD = R = DB$. The pivot matrix $D = B^{-1}DB$ is a differential matrix, since it is conjugate to the differential matrix $\hat{D}$. It only remains to check that the differential matrix $D$ is almost-Jordan. This requires constructing a permutation matrix $P$ such that $P^{-1}DP$ is Jordan. Since the differential matrix $D$ is furthermore Boolean and quasi-monomial, $P$ can be constructed by the procedure previously explained in Example 4.4. □ □

An immediate corollary of the proof is the important and generally known fact that the almost-Jordan differential $D$ can be easily constructed as the pivot matrix of the column-reduced matrix $R = DV$. Furthermore, while $R$ may depend on the choice of column-reduction $V$, Corollary A.3 guarantees that the pivot matrix $D$ is an invariant of $D$ (which we call the *persistence canonical form of $D$* in Section 1.3) independent of the choice of column-reduction $V$. The matrix $D$ contains all
of the data for the multiplicities of the summands in a decomposition, and this is independent of the choice of decomposition because of the Krull-Schmidt property. The data required to compute a particular decomposition is conveniently encoded by in the matrix $\hat{V}$, which is a column-reduction with additional special properties. These points are illustrated in the example at the end of the section. In the language of the standard framework, one says that the “barcodes” are contained in $D$, and the “creators and destroyers” of persistent homology are contained in $\hat{V}$.

We also note that the invertible triangular matrix $B = \hat{V}$ produced by the standard algorithm is not in general normalized (see the discussion in Section 1.3 following Theorem 1.2). But it is easy to construct a diagonal matrix $T$ such that the invertible diagonal matrix $B = \hat{V}T$ is normalized. This will also be illustrated in the example at the end of the section.

The graded case is an easy modification:

**Theorem 1.4.** (Matrix Structural Theorem) Any block-superdiagonal differential matrix $D$ factors as $D = BDB^{-1}$ where $D$ is a block-superdiagonal almost-Jordan differential matrix and $B$ is a block-diagonal triangular matrix.

**Proof.** Let $D$ be a block-superdiagonal differential matrix $D$. Then the invertible triangular column-reduction matrix $V$ produced by a reduction algorithm, such as [16, 35, 36] or our Appendix B.1, is block-diagonal. If $V$ is block-diagonal, then so is the invertible triangular matrix $B = \hat{V}$ constructed by the standard algorithm from $V$ and $R = DV$. □ □

The following example of a standard algorithm computation illustrates both block-structure and normalization.

**Example B.6.** We work with block-superdiagonal differential $D : Q^7 \to Q^7$ of Example 1.5, which is presented below with a column augmentation by the identity matrix $I$. The identity matrix is block-diagonal with respect to the grading structure inherited from $D$. We first compute a column-reduction of $D$:
The result of the computation is a factorization $R = DV$, where $R$ is column-reduced and $V$ is invertible triangular (and unipotent). The intervening steps are omitted for brevity. The block-superdiagonal almost-Jordan differential $D$ is now easily computed as the pivot matrix of $R$, by setting every column-pivot to 1 and every other nonzero entry to 0:

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The barcode invariants can be computed from the matrix $D$ and the filtration levels of the basis elements.

Proceeding to compute a particular decomposition as in Example 4.4, we use the standard algorithm to construct $\hat{V}$ as a modification of $V$. Each nonzero column of $R$ replaces the column of $V$ that has its column-pivot in the same row. Then $\hat{V}$ inherits the block-diagonal structure of $V$:

$$V = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \hat{V}$$

We list the columns of $\hat{V}$ that are equal to columns of $R$; this data is also encoded by the nonzero entries of the pivot matrix $D$:

- Column 2 of $\hat{V}$ is equal to column 4 of $R$; row 2 column 4 of $D$ has entry 1.
- Column 3 of $\hat{V}$ is equal to column 5 of $R$; row 3 column 5 of $D$ has entry 1.
- Column 6 of $\hat{V}$ is equal to column 7 of $R$; row 6 column 7 of $D$ has entry 1.

Each of the remaining columns of $\hat{V}$ is equal to the corresponding column of $V$. One may now check by matrix multiplication that $\hat{V}^{-1}DV = D$, where $D$ is the pivot matrix of $R$ as above.

The invertible triangular matrix $\hat{V}$ is not normalized: column 6 of $\hat{V}$ corresponds to a zero column of $D$, but its diagonal entry is not equal to 1. We can normalize by scalar multiplication of the appropriate columns. Let $T$ be the diagonal matrix with 1 in the first five diagonal entries and $-1$ in the last two. Then the invertible triangular matrix $B = \hat{V}T$ is normalized, and this is the matrix that appears in Example 1.5. Note that $B^{-1}DB = D = \hat{V}^{-1}DV$ by Corollary A.3.
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