A NOTE ON THE MULTIPlicity OF DETERMINANTAL IDEALS

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Abstract. Herzog, Huneke, and Srinivasan have conjectured that for any homogeneous $k$-algebra, the multiplicity is bounded above by a function of the maximal degrees of the syzygies and below by a function of the minimal degrees of the syzygies. The goal of this paper is to establish the multiplicity conjecture of Herzog, Huneke, and Srinivasan about the multiplicity of graded Cohen-Macaulay algebras over a field $k$ for $k$-algebras $k[x_1, \cdots, x_n]/I$ being $I$ a determinantal ideal of arbitrary codimension.

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1. Introduction

Let $R = k[x_1, \cdots, x_n]$ be a polynomial ring in $n$ variables over a filed $k$, let $deg(x_i) = 1$ and let $I \subset R$ be a graded ideal of arbitrary codimension. Consider the minimal graded free $R$-resolution of $R/I$:

$$0 \longrightarrow \oplus_{j \in \mathbb{Z}} R(-j)^{\beta_{p,j}(R/I)} \longrightarrow \cdots \longrightarrow \oplus_{j \in \mathbb{Z}} R(-j)^{\beta_{1,j}(R/I)} \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

where we denote $\beta_{i,j}(R/I) = Tor_i^R(R/I, k)_j$ the graded Betti number of $R/I$. Many important numerical invariants of $I$ and the associated scheme can be read off from the minimal graded free $R$-resolution of $R/I$. For instance, the Hilbert polynomial, and hence the multiplicity $e(R/I)$ of $I$, can be written down in terms of the shifts $j$ such that $\beta_{i,j}(R/I) \neq 0$ for some $i$, $1 \leq i \leq p$.

Let $c$ denote the codimension of $R/I$. Then $c \leq p$ and equality holds if and only if $R/I$ is Cohen-Macaulay. We define $m_i(I) = \min\{j \in \mathbb{Z} \mid \beta_{i,j}(R/I) \neq 0\}$ the minimum degree shift at the $i$-th step and $M_i(I) = \max\{j \in \mathbb{Z} \mid \beta_{i,j}(R/I) \neq 0\}$ the maximum degree shift at the $i$-th step. We will simply write $m_i$ and $M_i$ when there is no confusion. If $R/I$ is Cohen-Macaulay and has a pure resolution, i.e. $m_i = M_i$ for all $i$, $1 \leq i \leq c$, then Huneke...
and Miller showed in \cite{miller} that
\[ e(R/I) = \prod_{i=1}^{c} \frac{m_i}{c!}. \]

Generalizing their result Herzog, Huneke, and Srinivasan made the following multiplicity conjecture:

**Conjecture 1.1.** If $R/I$ is Cohen-Macaulay then
\[ \frac{\prod_{i=1}^{c} m_i}{c!} \leq e(R/I) \leq \frac{\prod_{i=1}^{c} M_i}{c!}. \]

Conjecture 1.1 has been extensively studied, and partial results have been obtained. It turns out to be true for the following type of ideals:

- Complete intersections \cite{complete}
- Powers of complete intersection ideals \cite{powers}
- Perfect ideals with a pure resolution \cite{perfect}
- Perfect ideals with a quasi-pure resolution (i.e. $m_i \geq M_{i-1}$) \cite{quasi-pure}
- Perfect ideals of codimension 2 \cite{codimension2}
- Gorenstein ideals of codimension 3 \cite{gorenstein}
- Perfect stable monomial ideals \cite{stable}
- Perfect square free strongly stable monomial ideals \cite{square-free}

The goal of this paper is to prove Conjecture 1.1 for determinantal ideals of arbitrary codimension $c$, i.e. ideals generated by the maximal minors of a $t \times (t+c-1)$ homogeneous polynomial matrix. Determinantal ideals have been a central topic in both commutative algebra and algebraic geometry and, due to their important role, their study has attracted many researchers and has received considerable attention in the literature. Some of the most remarkable results about determinantal ideals are due to J.A. Eagon and M. Hochster in \cite{eagon-hochster}, and to J.A. Eagon and D.G. Northcott in \cite{eagon-northcott}. J.A. Eagon and M. Hochster proved that generic determinantal ideals are perfect. J.A. Eagon and D.G. Northcott constructed a finite graded free resolution for any determinantal ideal and, as a corollary, they got that determinantal ideals are perfect. Since then many authors have made important contributions to the study of determinantal ideals and the reader can look at \cite{eagon}, \cite{eagon-northcott} and \cite{eagon-northcott2} for background, history and a list of important papers.

In this short note we verify that determinantal ideals $I$ satisfy Herzog-Huneke-Srinivasan Conjecture which relates the multiplicity $e(R/I)$ to the minimal and maximal shifts in the graded minimal $R$-resolution of $R/I$.

Next we outline the structure of the paper. In section 2, we first recall the basic facts on determinantal ideals $I$ of codimension $c$ defined by the maximal minors of a $t \times (t+c-1)$ homogeneous matrix $A$ and the associated complexes needed later on. We determine the minimal and maximal shifts in the graded minimal free $R$-resolution of $R/I$ in terms of the degree matrix $U$ of $A$ and we state some technical lemmas used in the inductive process of the proof of our main Theorem (cf Theorem 3.1).

Section 3 is completely devoted to proving Conjecture 1.1 for determinantal ideals $I$ of arbitrary codimension. To prove it we use induction on the codimension $c$ of $I$ and for any
c induction on the size t of the homogeneous \( t \times (t + c - 1) \) matrix whose maximal minors generate \( I \) by successively deleting columns and rows of the largest possible degree when we prove the lower bound and columns and rows of the smallest possible degree when we prove the upper bound. We end the paper with an example which illustrates that the upper and lower bounds for the multiplicity \( e(R/I) \) of a determinantal ideal \( I \) given in Theorem 3.1 are sharp.

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2. Determinantal ideals

In the first part of this section, we provide the background and basic results on determinantal ideals needed in the sequel, and we refer to [2] and [5] for more details.

Let \( \mathcal{A} \) be a homogeneous matrix, i.e. a matrix representing a degree 0 morphism \( \phi : F \rightarrow G \) of free graded \( R \)-modules. In this case, we denote by \( I(\mathcal{A}) \) the ideal of \( R \) generated by the maximal minors of \( \mathcal{A} \).

Definition 2.1. An homogeneous ideal \( I \subset R \) of codimension \( c \) is called a determinantal ideal if \( I = I(\mathcal{A}) \) for some \( t \times (t + c - 1) \) homogeneous matrix \( \mathcal{A} \).

Let \( I \subset R \) be a determinantal ideal of codimension \( c \) generated by the maximal minors of a \( t \times (t + c - 1) \) matrix \( \mathcal{A} = (f_{ji})_{i=1,...,t}^{j=1,...,t+c-1} \) where \( f_{ji} \in k[x_1, ..., x_n] \) are homogeneous polynomials of degree \( a_j - b_i \). We assume without loss of generality that \( \mathcal{A} \) is minimal; i.e., \( f_{ji} = 0 \) for all \( i, j \) with \( b_i = a_j \). If we let \( u_{j,i} = a_j - b_i \) for all \( j = 1, \ldots, t + c - 1 \) and \( i = 1, \ldots, t \), the matrix \( \mathcal{U} = (u_{j,i})_{i=1,...,t}^{j=1,...,t+c-1} \) is called the degree matrix associated to \( I \). By re-ordering degrees, if necessary, we may also assume that \( b_1 \geq \ldots \geq b_t \) and \( a_0 \leq a_1 \leq \ldots \leq a_{t+c-2} \). In particular, we have:

\[
(2.1) \quad u_{j,i} \leq u_{j+1,i} \quad \text{and} \quad u_{j,i} \leq u_{j,i+1} \quad \text{for all} \quad i, j.
\]

Note that the degree matrix \( \mathcal{U} \) is completely determined by \( u_{1,1}, u_{2,1}, \ldots, u_{c,1}, u_{2,2}, u_{3,2}, \ldots, u_{c+1,2}, \ldots, u_{t,t}, u_{t+1,t}, \ldots, u_{t+c-1,t} \). Moreover, the graded Betti numbers in the minimal free \( R \)-resolution of \( R/I(\mathcal{A}) \) depend only upon the integers

\[
\{u_{j,i}\}_{1 \leq i \leq t}^{i \leq c} \subset \{u_{j,i}\}_{i=1,...,t}^{j=1,...,t+c-1}
\]

as described below.

Proposition 2.2. Let \( I \subset R \) be a determinantal ideal of codimension \( c \) with degree matrix \( \mathcal{U} = (u_{j,i})_{i=1,...,t}^{j=1,...,t+c-1} \) as above. Then we have:

1. \( m_i = u_{1,1} + u_{2,1} + \cdots + u_{i,1} + u_{i+1,2} + u_{i+2,3} + \cdots + u_{t+i-1,t} \) for \( 1 \leq i \leq c \),
2. \( M_i = u_{c-i+1,1} + u_{c-i+2,2} + \cdots + u_{t+c-i-1,t} + u_{t+c-i+2,t} + \cdots + u_{t+c-1,t} \) for \( 1 \leq i \leq c \).

Proof. We denote by \( \phi : F \rightarrow G \) the morphism of free graded \( R \)-modules of rank \( t \) and \( t + c - 1 \), defined by the homogeneous matrix \( \mathcal{A} \) associated to \( I \). The Eagon-Northcott complex \( \mathcal{D}_0(\phi^*) \) :
\[ 0 \to \land^{t+c-1}G^* \otimes S_{c-1}(F) \otimes \land^t F \to \land^{t+c-2}G^* \otimes S_{c-2}(F) \otimes \land^t F \to \ldots \to \]
\[ \land^t G^* \otimes S_0(F) \otimes \land^t F \to R \to R \to R/I \to 0 \]
gives us a graded minimal free \( R \)-resolution of \( R/I \) (See, for instance \[2\]; Theorem 2.20 and \[3\]; Corollary A2.12 and Corollary A2.13). Now the result follows after an straightforward computation.

We will now fix the notation and prove the technical lemmas needed in the induction process we will use in next section for proving the multiplicity Conjecture for determinantal ideals of arbitrary codimension.

Let \( I \subset R \) be a homogeneous ideal of codimension \( c \). Assume that \( I \) is determinantal and let \( \mathcal{A} \) (resp \( \mathcal{U} \)) be the \( t \times (t+c-1) \) homogeneous matrix (resp. degree matrix) associated to \( I \). Let \( \mathcal{A}' \) (resp \( \mathcal{U}' \)) be the \( (t-1) \times (t+c-2) \) homogeneous matrix (resp. degree matrix) obtained deleting the last column and the last row of \( \mathcal{A} \) and denote by \( I' \) the codimension \( c \) determinantal ideal generated by the maximal minors of \( \mathcal{A}' \). Since the multiplicity of \( R/I \) and \( R/I' \) are completely determined by the corresponding degree matrices, it is enough to consider an example of ideal for any degree matrix. So, from now on, we take

\[
\mathcal{A} := \begin{pmatrix}
u_{1,1} & u_{2,1} & \ldots & u_{c-1,1} & u_{c-1} & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & x_1^{u_{t-1,t-1}} & x_2^{u_{t-1,t-1}} & \ldots & x_{c-1}^{u_{t-1,t-1}} & x_c^{u_{t-1,t-1}} & x_c^{u_{t-1,t-1}} \\
0 & 0 & \ldots & 0 & x_1^{u_{t,t}} & x_2^{u_{t,t}} & \ldots & x_{c-1}^{u_{t,t}} & x_c^{u_{t,t}} & x_c^{u_{t,t}} \\
\end{pmatrix}
\]

and

\[
\mathcal{A}' := \begin{pmatrix}
u_{1,1} & u_{2,1} & \ldots & u_{c-1,1} & u_{c-1} & 0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & x_1^{u_{t-1,t-1}} & x_2^{u_{t-1,t-1}} & \ldots & x_{c-1}^{u_{t-1,t-1}} & x_c^{u_{t-1,t-1}} \\
0 & 0 & \ldots & 0 & x_1^{u_{t,t}} & x_2^{u_{t,t}} & \ldots & x_{c-1}^{u_{t,t}} & x_c^{u_{t,t}} \\
\end{pmatrix}
\]

Let \( J \subset R \) be the codimension \( c - 1 \) determinantal ideal generated by the maximal minors of the \( t \times (t+c-2) \) homogeneous matrix

\[
\mathcal{B} := \begin{pmatrix}
u_{1,1} & u_{2,1} & \ldots & u_{c-1,1} & u_{c-1} & 0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & x_1^{u_{t-1,t-1}} & x_2^{u_{t-1,t-1}} & \ldots & x_{c-1}^{u_{t-1,t-1}} & x_c^{u_{t-1,t-1}} \\
0 & 0 & \ldots & 0 & x_1^{u_{t,t}} & x_2^{u_{t,t}} & \ldots & x_{c-1}^{u_{t,t}} & x_c^{u_{t,t}} \\
\end{pmatrix}
\]
obtained deleting the last column of $\mathcal{A}$. Analogously, we consider $\mathcal{A}''$ (resp. $\mathcal{U}'')$ the $(t-1) \times (t+c-2)$ homogeneous matrix (resp. degree matrix) obtained deleting the first column and the first row of $\mathcal{A}$ and we denote by $I''$ the codimension $c$ determinantal ideal generated by the maximal minors of $\mathcal{A}''$. Let $\mathcal{C}$ be the $t \times (t+c-2)$ homogeneous matrix obtained deleting the first column of $\mathcal{A}$ and we denote by $I'''$ the codimension $c$ determinantal ideal generated by the maximal minors of $\mathcal{C}$.

The ideal $I$ is obtained from $I'$ by a basic double G-link as well as from $K$ by a basic double G-link. Indeed, we have

**Lemma 2.3.** With the above notation, it holds

1. $I = J + x_{c}^{u_{t+c-1,t}} I'$ and $I = K + x_{1}^{u_{1,1}} I''$.
2. The sequences

$$0 \rightarrow J(-u_{t+c-1,t}) \rightarrow I'(-u_{t+c-1,t}) \oplus J \rightarrow J + x_{c}^{u_{t+c-1,t}} I' = I \rightarrow 0$$

and

$$0 \rightarrow K(-u_{1,1}) \rightarrow I''(-u_{1,1}) \oplus K \rightarrow K + x_{1}^{u_{1,1}} I'' = I \rightarrow 0$$

are exact.
3. $e(R/I) = e(R/K) = e(R/I') + u_{t+c-1,t} \cdot e(R/J)$ and $e(R/I) = e(R/I'') + u_{1,1} \cdot e(R/K)$.

**Proof.** (1) The equalities of ideals are immediate.
(2) and (3) follow from [9]; Lemma 4.8.

**Lemma 2.4.** With the above notation, we have

1. $m_{i} = m_{i}(I) = m_{i}(I') + u_{t+i-1,t} = m'_{i} + u_{t+i-1,t}$ for all $1 \leq i \leq c$,
2. $M_{i} = M_{i}(I) = M_{i}(I'') + u_{c-i+1,1} = M''_{i} + u_{c-i+1,1}$ for all $1 \leq i \leq c$,
3. $m_{i}(J) = m_{i}(I) = m_{i}$ for all $1 \leq i \leq c - 1$, and
4. $M_{i}(K) = M_{i}(I) = M_{i}$ for all $1 \leq i \leq c - 1$.

**Proof.** It follows from Proposition 2.2.

### 3. The Multiplicity Conjecture

Using the fact that the ideal $I$ is obtained from the ideal $I'$ (resp. $I''$) by a basic double G-link, we can now show that Conjecture 1.1 is true for determinantal ideals of arbitrary codimension.

**Theorem 3.1.** Let $I \subset R$ be a determinantal ideal of codimension $c$. Then the following lower and upper bounds hold:

1. $e(R/I) \geq \prod_{i=1}^{c} m_{i}$, and
2. $e(R/I) \leq \prod_{i=1}^{c} M_{i}$.

**Proof.** As we explained in section 2, it is enough to prove the result for the ideal $I$ generated by the maximal minors of the $t \times (t+c-1)$ matrix.
We proceed by induction on the codimension $c$ of $I$. If $c = 1$ then $I$ is a principal ideal and the result is trivial. For $c = 2$ the result was proved by Herzog and Srinivasan in [7]. Assume $c \geq 3$. We will now induct on $t$. If $t = 1$ then $I$ is a complete intersection ideal and the result is well known. Assume $t > 1$. Let $\mathcal{A}'$ (resp. $\mathcal{B}$) be the matrix obtained deleting the last column and the last row (resp. the last column) of the matrix $\mathcal{A}$ and let $I'$ (resp. $J$) be the ideal generated by the maximal minors of $\mathcal{A}'$ (resp. $\mathcal{B}$). Let $m_i, m'_i$ and $m_i(J)$ be the minimal shifts in the graded minimal free $R$-resolution of $R/I$, $R/I'$ and $R/J$, respectively (see Proposition 2.2 and Lemma 2.4).

By Lemma 2.3 (3),

$$e(R/I) = e(R/I') + u_{t+c-1,t} \cdot e(R/J),$$

by hypothesis of induction on $c$ and Lemma 2.4 (3), we have

$$e(R/J) \geq \frac{\prod_{i=1}^{c-1} m_i(J) (c-1)!}{(c-1)!} = \frac{\prod_{i=1}^{c-1} m_i}{(c-1)!},$$

and by hypothesis of induction on $t$ we have

$$e(R/I') \geq \frac{\prod_{i=1}^{c} m'_i}{(c)!}.$$

Therefore, since $m_i = m'_i + u_{t+i-1,t}$ (Lemma 2.4 (1)), we have

$$c! e(R/I) \geq \prod_{i=1}^{c} m_i$$

if and only if

$$\prod_{i=1}^{c} m'_i + c u_{t+c-1,t} \prod_{i=1}^{c-1} m_i \geq \prod_{i=1}^{c} m_i = \prod_{i=1}^{c} (m'_i + u_{t+i-1,t}) = u_{t+c-1,t} \prod_{i=1}^{c-1} m_i + \prod_{i=1}^{c} m'_i + c \sum_{r=0}^{c-2} u_{t+r,t} m_1 \cdots m_r m'_{r+2} \cdots m'_{c-1} =$$

$$u_{t+c-1,t} \prod_{i=1}^{c-1} m_i + \prod_{i=1}^{c} m'_i + c \sum_{r=0}^{c-2} u_{t+r,t} m_1 \cdots m_r m'_{r+2} \cdots m'_{c-1}.$$
if and only if
\[(c - 1)u_{t+c-1,t} \prod_{i=1}^{c-1} m_i \geq \sum_{r=0}^{c-2} (u_{t+r,t} m_1 \cdots m_r m'_{r+2} \cdots m'_{c-1} m'_c).\]

Since, for all integer \(i, 1 \leq i \leq c - 1\), and for all integer \(r, 0 \leq r \leq c - 2\), we have the inequalities
\[m_i - m'_{i+1} = (u_{1,1} + u_{2,1} + \cdots + u_{i,1} + u_{i+1,2} + u_{i+2,3} + \cdots + u_{t+i-1,t}) - (u_{1,1} + u_{2,1} + \cdots + u_{i,1} + u_{i+1,1} + u_{i+2,2} + u_{i+3,3} + \cdots + u_{t+i-1,t-1}) = (u_{i+1,2} - u_{i+1,1}) + (u_{i+2,3} - u_{i+2,2}) + \cdots + (u_{t+i-1,t} - u_{t+i-1,t-1}) \geq 0,
\]
and
\[u_{t+c-1,t} \geq u_{t+r,t},\]
we obtain
\[u_{t+c-1,t} \prod_{i=1}^{c-1} m_i \geq (u_{t+r,t} m_1 \cdots m_r m'_{r+2} \cdots m'_{c-1} m'_c)\]
for all \(r, 0 \leq r \leq c - 2\), and the lower bound follows.

(2) The upper bound is proved similarly. We again proceed by induction on the codimension \(c\) of \(I\). If \(1 \leq c \leq 2\) then the result works. So, let us assume \(c \geq 3\). We will now induct on \(t\). If \(t = 1\) then \(I\) is a complete intersection ideal and the result is true. Assume \(t > 1\). Let \(A''\) (resp. \(C\)) be the matrix obtained deleting the first column and the first row (resp. the first column) of the matrix \(A\) and let \(I''\) (resp. \(K\)) be the ideal generated by the maximal minors of \(A''\) (resp. \(C\)). Let \(M_i, M''_i\) and \(M_i(K)\) be the maximal shifts in the graded minimal free \(R\)-resolution of \(R/I, R/I''\) and \(R/K\), respectively.

By Lemma 2.3 (3),
\[e(R/I) = e(R/I'') + u_{1,1} \cdot e(R/K),\]
by hypothesis of induction on \(c\) and Lemma 2.4 (4), we have
\[e(R/K) \leq \prod_{i=1}^{c-1} M_i(K) = \prod_{i=1}^{c-1} M_i\]
and by hypothesis of induction on \(t\) we have
\[e(R/I'') \leq \prod_{i=1}^{c} M''_i.
\]
By Lemma 2.4 (2), \(M_i = M_i(I) = M_i(I'') + u_{c-i+1,1} = M''_i + u_{c-i+1,1}\) for all \(1 \leq i \leq c\). Therefore, we have
\[c! e(R/I) \leq \prod_{i=1}^{c} M_i\]
if and only if
\[\prod_{i=1}^{c} M''_i + cu_{1,1} \prod_{i=1}^{c-1} M_i \leq \prod_{i=1}^{c} M_i =\]
Theorem. □

Example 3.2. Let $I \subset R$ be a codimension $c$ determinantal ideal generated by the maximal minors of a $t \times (t+c-1)$ matrix all whose entries are homogeneous polynomials of fixed degree $1 \leq d \in \mathbb{Z}$. Thus, we have

$$m_i(I) = M_i(I) = td + (i-1)d \quad \text{for all } i, \quad 1 \leq i \leq c.$$  

Therefore, we conclude that

$$e(R/I) = \frac{\prod_{i=1}^{c} m_i(I)}{c!} = \frac{\prod_{i=1}^{c} M_i(I)}{c!} = \frac{\prod_{i=1}^{c} (td + (i-1)d)}{c!} = d^c \binom{t+c-1}{c}.$$
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