Quantum to classical crossover under dephasing effects in a two-dimensional percolation model

Junjie Qi\textsuperscript{1,2}, Haiwen Liu\textsuperscript{3}, Chui-Zhen Chen\textsuperscript{4}, Hua Jiang\textsuperscript{4}, and X. C. Xie\textsuperscript{1,2*}

\textsuperscript{1}International Center for Quantum Materials and School of Physics, Peking University, Beijing 100871, China; \textsuperscript{2}Collaborative Innovation Center of Quantum Matter, Beijing 100871, China; \textsuperscript{3}Center for Advanced Quantum Studies, Department of Physics, Beijing Normal University, Beijing 100875, China; \textsuperscript{4}Institute for Advanced Study and School of Physical Science and Technology, Soochow University, Suzhou 215006, China

Received April 16, 2019; accepted June 11, 2019; published online September 2, 2019

Scaling theory predicts complete localization in $d = 2$ in quantum systems belonging to the orthogonal class (i.e., with time-reversal symmetry and spin-rotation symmetry). The conductance $g$ behaves as $g \sim \exp(-L/l)$ with system size $L$ and localization length $l$ in the strong disorder limit. However, classical systems can always have metallic states in which Ohm’s law shows a constant $g$ in $d = 2$. We study a two-dimensional quantum percolation model by controlling dephasing effects. The numerical investigation of $g$ aims at simulating a quantum-to-classical percolation evolution. An unexpected metallic phase, where $g$ increases with $L$, generates immense interest before the system becomes completely classical. Furthermore, the analysis of the scaling plot of $g$ indicates a metal-insulator crossover.

quantum percolation model, dephasing effects, metal-insulator crossover

PACS number(s): 64.60.ah, 71.30.+h, 73.23-b

Citation: J. Qi, H. Liu, C.-Z. Chen, H. Jiang, and X. C. Xie, Quantum to classical crossover under dephasing effects in a two-dimensional percolation model, Sci. China-Phys. Mech. Astron. 63, 227811 (2020), \url{https://doi.org/10.1007/s11433-019-9449-9}

1 Introduction

It has been understood that the scaling properties of the conductance $g$ are determined by one-parameter scaling theory [1, 2]. The conductance $g$ behaves as $g \sim \exp(-L/l)$ with system size $L$ and localization length $l$. However, classical systems can always have metallic states in which Ohm’s law shows a constant $g$ in $d = 2$. We study a two-dimensional quantum percolation model by controlling dephasing effects. The numerical investigation of $g$ aims at simulating a quantum-to-classical percolation evolution. An unexpected metallic phase, where $g$ increases with $L$, generates immense interest before the system becomes completely classical. Furthermore, the analysis of the scaling plot of $g$ indicates a metal-insulator crossover.

Phase coherence is one of the key features that determines whether a system is quantum or classical. However, phase coherence can be easily lost in real systems, and the system tends to be a classical one under dephasing effects. Thus, a metallic state may appear instead of complete localization. The mechanism of how the phase coherence affects the MIT...
in a quantum-to-classical evolution is not clear until now.

In the present paper, we study a two-dimensional (2D) quantum percolation model [3-6] describing the dynamics of a quantum particle moving in a random system. According to one-parameter scaling theory, all the states in such a 2D quantum system are localized under the Anderson disorder [7, 8]. On the other side, classical percolation theory always has a threshold \( P_c \) of the percolation transition. Therefore, we investigate the evolution process in details. We introduce the dephasing mechanism to destroy the quantum coherence for the purpose of switching from a quantum percolation model to a classical one. Here, we consider a two-terminal device with a central region of a 2D quantum percolation model and two ideal leads. We control in such a way that the dephasing process only takes place in the central region. Specifically, the dephasing process is introduced by using Böttiker’s virtual probes [9, 10]. These virtual probes are coupled to the lattice sites with the current-conserving condition. We calculate the conductance \( g \) for a finite-size system numerically by using the Landauer-Büttiker formula combined with the non-equilibrium Green function method [10-14]. We find that an unexpected metallic phase appears before the system is entirely switched into a classical one. The conductance \( g \) in the novel metallic phase increases with the system size \( L \). This is not consistent with the classical Ohm’s law, which shows the conductance \( g \) is constant in 2D. Thus, it is more appropriate to say that the novel metallic phase contains semi-quantum and semi-classical contribution. Furthermore, the scaling function of \( g \) under different coherent lengths \( L_c \) is inspected to gain deep insights. We show that the novel metallic phase maybe a consequence of metal-insulator crossover.

The rest of the paper is organized as follows. In sect. 2, we introduce the quantum percolation model. The dephasing mechanism is brought in to simulate a quantum-to-classical evolution. In sect. 3, we calculate the conductance numerically with the Landauer-Büttiker formula accompanied by the non-equilibrium Green function method. We show the numerical results of the conductance with the disorder and dephasing. An unexpected metallic phase appears near the region \( P = 1 \). The novel phase may contain semi-quantum and semi-classical contribution. We then investigate the scaling behaviour of the conductance \( g \). The evidence shows a metal-insulator crossover. Finally, we give a brief summary in sect. 4.

2 Model and method

We start from considering a two-terminal device as shown in Figure 1. The Hamiltonian of the central region is the 2D quantum percolation model [3-6, 15, 16] which can be written as:

\[
H = \sum_i \epsilon_i c_i^\dagger c_i + \sum_{\langle i,j \rangle} t_{ij} (c_i^\dagger c_j + c_j^\dagger c_i),
\]

where the sum \((i, j)\) goes over the nearest neighbor sites. The on-site energy \( \epsilon_i \) obeys the uniform distribution over the interval \([-W/2, W/2]\) with the disorder strength \( W \). The bond between the nearest sites is either present \( t_{ij} = 1 \) with probability \( P \) or absent \( t_{ij} = 0 \) with probability \( 1 - P \). As shown in Figure 1, the central region is sandwiched between the left (L) and right (R) leads with size \( N = L \times L \). Unlike the central region, we assume that the L(R) leads are both ideal conductors. When the probability \( P < P_c \) in such a percolation model, the electrons injected from the left lead cannot flow through the central region to the right lead. Thus, there is no current in the device. Once the probability exceeds the threshold value \( P_c \), current can flow into the right lead. The conductance is calculated by applying the Landauer-Büttiker formula combined with the non-equilibrium Green function method [10-14]. We only add the dephasing effects in the central region. The dephasing mechanism is introduced by using Böttiker’s virtual probes [9, 10] to simulate the quantum to classical percolation evolution. We assume that the lattice sites are randomly chosen to be connected to the virtual leads with the dephasing probability \( p_v \). \( t_v \) is the hopping element between a virtual lead and a lattice site. The black dots in Figure 1 show the lattice sites \( i \) which are coupled by the virtual leads. There are totally \( N_v = p_v \times N \) virtual leads in the central region.

We add a small bias \( V = V_L - V_R \) between the left lead and right lead, which can drive a current flowing along the longitudinal direction. Either real or virtual lead current \( I_p \) \((p = L, R, 1, 2, \ldots, N_v)\) is given by multiprobe Landauer-Büttiker formula [9, 10]

\[
I_p = \frac{2e}{h} \sum_{q \neq p} T_{p \rightarrow q} (V_p - V_q),
\]

where \( V_p \) is the bias in the lead \( p \). The transmission function from lead \( q \) to lead \( p \) is expressed as \( T_{p \rightarrow q} = \text{Tr}[\Gamma_p G^\dagger G^q] \).

Figure 1 (Color online) A percolation lattice model with size \( N = L \times L \) is sandwiched between the left (L) and right (R) leads. The sites with black dots \( \bullet \) are connected to the Büttiker’s virtual leads randomly.
where the line width function $\Gamma_p = i(\Sigma_r^p - \Sigma_r^{\dagger}p)$, with the retarded self-energy $\Sigma_r^p$. The retarded Green function can be calculated by $G^r = [G^a]^\dagger = [EI - H - \Sigma_r^p\Sigma_r^{\dagger}]^{-1}$, where $E$ is the Fermi energy. After we get the current $I_L$, the conductance can be directly obtained as $g = (V_L - V_R)/I_L$. We note that the percolation probability $P$, the dephasing probability $p_v$ and the dephasing strength $t_v$ can affect the coherent length $L_\phi$ remarkably [17].

3 Unexpected metallic phase

We come to the main results of our work. Firstly, let us investigate the conductance $g$ of the quantum percolation model versus the probability $P$ by varying the parameters: the width $L$, the dephasing effects $p_v$ and $t_v$ at a given Fermi energy $E = -1$ eV and the disorder strength $W = 3$ or 4, as shown in Figure 2. The average numerical results are calculated repeatedly more than 5000 times. In the quantum percolation model, the conductance $g$ is zero at small probability $P$. When $P$ increases, electron clusters span from one side of the lattice to the opposite side, then the conductance $g$ becomes nonzero.

It is clearly found that $g$ decreases with $L$ increasing in Figure 2(a), which indicates that the states far from $P = 1$ are localized due to the disorder strength $W = 3$. The states near the probability $P = 1$ are not localized due to the finite size effects of numerical computation, which can be overcome by increasing the disorder strength $W$. All the states are localized due to the disorder strength $W = 4$ in Figure 2(c). This is consistent with the one-parameter scaling theory. Then, we bring in the dephasing mechanism in the system. For example, we add 20% virtual leads of the lattice sites to the system with dephasing strength $t_v = 0.5$ along with the disorder. We find that all the curves of $g$ with different $L$ cross at a single point about $P_c \approx 0.87$ in Figure 2(b) and $P_c \approx 0.92$ in Figure 2(d). More remarkably, the conductance $g$ increases monotonously with the width $L$ in the region for $P_c < P < 1$. That means an unexpected metallic phase occurs beyond the cross point $P_c$. The insets in Figure 2(b) and (d) are the zoom-in of the unexpected metallic phase. The most fascinating of the metallic phase is that its nature cannot be attributed to either the quantum class or the classical class alone. Next let us see the details. Based on the one parameter scaling theory, all the states are localized in quantum systems. Noticing that the dephasing mechanism is introduced here, our model under-

![Figure 2](Color online) The conductance $g$ vs. the probability $P$ (a) when the disorder strength is $W = 3$ and (c) $W = 4$ by increasing the width $L = 20, 32, 48, 54, 66, 72$. (b), (d) Add dephasing effects by Büttiker’s virtual probes to (a) and (c) with $p_v = 0.2, t_v = 0.5$, respectively. The insets are the zoom-in of the metallic region. (e) The conductance $g$ vs. the probability $P$ by the width expansion of $L = 80, 96, 108$ for $W = 4$. (f) Larger dephasing values are act on the model for $W = 4$. In all subplots, we take the Fermi energy $E = -1$ eV.
goes a quantum-to-classical evolution. When the dephasing strength is large enough, our system will be totally classical (see Figure 2(f)). Although metal is common in a 2D classical model, Ohm’s law gives a fixed conductance \( g \) regardless of the system size \( L \). Thus, the novel metallic phase is not a classical one. At a moderate dephasing strength of Figure 2(d), the system can be in a transitional situation interplay between the quantum and classical percolations. Hence, the metallic percolation phase should contain the semi-quantum and semi-classical contributions. Furthermore, we find in Figure 2(e) that the curves do not cross but merge when increasing the system size up to \( L = 108 \) with the dephasing strength and disorder unchanged. This is the hallmark of Ohm’s law in the classical category. As above, when we increase the dephasing strength to a larger value in Figure 2(f), there is also an overlap trend of these curves. Consequently, we can explain the results in Figure 2(e) and (f) from a classical perspective.

One of the hallmarks of quantum systems, compared to classical ones, is the existence of phase coherence. Because the phase coherent length is an important length scale in quantum transport, we will study it in detail below. The current under dephasing effects contains the phase-coherent part and phase-incoherent part. If the current flows directly from the left lead to the right lead without going through the virtual leads, the phase coherence will sustain. Thus, the current is denoted as phase-coherent part. In comparison, if the current goes through the virtual leads, its phase coherence will be destroyed. This is the so-called phase-incoherent part in our manuscript. When increasing the dephasing effects, the ratio of the phase-incoherent part also increases. At a certain value \( p_v \) and \( t_v \), both parts could have equal percentage. Thus, the system size \( L \) is recognized to be equal to the phase coherent length \( L_{\phi} \) [17]. When the system size \( L \) is shorter than the phase coherent length \( L_{\phi} \), the system is totally in the phase-coherent region. In Figure 3, we show the phase coherent length \( L_{\phi} \) versus the probability \( P \) at the fixed disorder and dephasing strength. It should be noted that \( L_{\phi} \) is a continuous value in realistic systems. The dashdotted lines may be the possible tracks of \( L_{\phi} \). The step-like performance of \( L_{\phi} \) shown in Figure 3 is due to the constraints of numerical algorithm in which \( L_{\phi} \) is an integer equal to the system size \( L \). If we add more virtual probes coupling with the lattice sites by increasing \( p_v \), we find that the coherent length \( L_{\phi} \) gets conceivably smaller by comparing the two figures in Figure 3(a) and (c). This agrees with the common belief that dephasing always destroys the quantum coherence. In Figure 3(d), we keep increasing the dephasing strength \( t_v \) on the basis of Figure 3(c). As a result, the coherent length \( L_{\phi} \) continues to decrease. In addition, the disorder strength increases, the coherent length \( L_{\phi} \) also decreases (see Figure 3(b)). Now we can use the phase coherent length \( L_{\phi} \) obtained in Figure 3 to uncover the physical origin of Figure 3. In the quantum percolation limit, \( L_{\phi} ≫ l \) (the localization length), the system is strongly localized with \( g \sim \exp(-L/l) \). Thus, the system is an insulator (see Figure 2(c)). When the dephasing mechanism is brought in, the phase coherent length becomes finite. During the quantum-to-classical evolution, there would be a transition at a suitable \( L_{\phi} \). At a certain dephasing strength, \( L_{\phi} \) is comparable with the system size \( L \) before becoming completely classical. Meanwhile, when \( L_{\phi} < l \) (the localization length), the system is in the ballistic-like transport, and hence a metallic behaviour appears. For a percolation system, the larger probability \( P \) is, the longer the localization length \( l \) is. Thus, \( l \) can exceed \( L_{\phi} \) near \( P = 1 \) and is shorter than \( L_{\phi} \) in the region of small \( P \). In particular, the region near \( P = 1 \) presents better ballistic-like properties. When \( L_{\phi} \) keeps on getting smaller, a real sample can be divided into several phase-coherent blocks with length \( L_{\phi} \). In each phase-coherent block, quantum principle is valid. The whole sample can be viewed as an ensemble of small phase-coherent blocks in the classical regime. More virtual leads \( (p_v) \) and larger dephasing strengths \( (t_v) \) contribute to a smaller coherent length \( L_{\phi} \). A similar situation occurs when increasing the system size \( L \) in Figure 2(e). In such a case, the system should be classical and the conductance \( g \) obeys an ohmic scaling law. When \( L_{\phi} ≪ L \), the curves of the conductance \( g \) should merge.

At last, we follow the standard one-parameter scaling analysis [7] of the data in Figure 2 and show the results in Figure 4. The characteristic length \( \xi \) is obtained by collapsing data of the conductance \( g \) into a single curve. The curve represents

![Figure 3](https://example.com/figure3.png)

Figure 3 (Color online) Plot the coherent length \( L_{\phi} \) vs. the probability \( P \) for (a) the dephasing probability \( p_v = 0.2 \) and strength \( t_v = 0.5 \); (c) the dephasing probability \( p_v = 0.3 \) and strength \( t_v = 0.5 \); (d) the dephasing probability \( p_v = 0.3 \) and strength \( t_v = 0.7 \). In (a), (c) and (d), the disorder strength is \( W = 4 \). (b) We take the disorder strength \( W = 6 \), the dephasing probability \( p_v = 0.2 \) and strength \( t_v = 0.5 \). The Fermi energy is at \( E = -1 \) eV in all cases. The dashed orange lines may be the possible continuous tracks of \( L_{\phi} \). The broken blue lines show the numerical results of \( L_{\phi} \).
the scaling function. We inspect the scaling behaviour of the conductance $g$ on two branches: the metallic side $P > P_c$ (see Figure 4(a)) and the insulating side $P < P_c$ (see Figure 4(b)). The figure shows that all the data of conductance can merge into a single curve for different system sizes $L$. The inset is the plot of $\ln \xi$ versus the probability $P$, where $\xi(P)$ diverges at $P_c$. By requiring $\xi(P) \propto |P - P_c|^{-\nu}$ in the vicinity of $P_c$, we can extract the critical exponent $\nu$.

We analyze the divergence of $\xi$ in terms of a power law $\xi(P) \propto |P - P_c|^{-\nu}$ on the metallic side shown in Figure 5. However, we find a linear fit with slope $\nu = 8.07$, much higher than any known 2D disordered systems. Based on percolation theory, the critical exponent with classical systems gives $\nu = 4/3$ in $d=2$ [18]. Beyond this, we have inspected more scaling curves by changing the dephasing parameters. By employing a power law, the critical exponent $\nu$ is still very large with the dephasing altered. The reason is probably that a power law is not suitable in our case. It is important to stress that $\nu$ in our case is not a true critical exponent, since its value changes with different system sizes and dephasing parameters. Finally, we argue that this is due to the nature of a metal-insulator crossover. Hence, the Harris criterion does not apply here.

4 Conclusion

In conclusion, we investigate the whole evolution process from 2D quantum to classical percolation. Without dephasing, the system is a quantum percolation model with localized states. When increasing the dephasing effects in the quantum percolation model, the system switches towards the classical one gradually. Remarkably, an unexpected metallic phase exists at a moderate dephasing strength, and the behaviour of the conductance deviates from the classical Ohm’s law. The scaling behaviour of the conductance in the presence of dephasing effects suggests a metal-insulator crossover.

This work was supported by the National Basic Research Program of China (Grant Nos. 2015CB921102, 2017YFA0303301, and 2017YFA0304600), and National Natural Science Foundation of China (Grant Nos. 11504008, 11574245, 11674028, and 11822407). We gratefully acknowledge the inspiration of early work and helpful discussions with Junren Shi. They find the tendency of the metal-insulator transition [6] which inspires our following research presented in this work. We elaborate the cause of the metallic phase by the systematic investigation.

1 E. Abrahams, P. W. Anderson, D. C. Licciardello, and T. V. Ramakrishnan, Phys. Rev. Lett. 42, 673 (1979).
2 A. Altland and M. R. Zirnbauer, Phys. Rev. B 55, 1142 (1997).
3 Y. Shapir, A. Aharony, and A. B. Harris, Phys. Rev. Lett. 49, 486 (1982).
4 Y. Meir, A. Aharony, and A. B. Harris, Europhys. Lett. 10, 275 (1995).
5 I. Chang, Z. Lev, A. B. Harris, J. Adler, and A. Aharony, Phys. Rev. Lett. 74, 2094 (1995).
6 J.-R. Shi, S. He, and X. C. Xie, arXiv: cond-mat/9904393 (1999).
7 A. MacKinnon and B. Kramer, Z. Physik B - Condensed Matter 53, 1 (1983).
8 F. Evers and A. D. Mirlin, Rev. Mod. Phys. 80, 1355 (2008), arXiv: 0707.4378.
9 M. Büttiker, Phys. Rev. B 33, 3020 (1986).
10 S. Datta, Electronic Transport in Mesoscopic Systems (Cambridge University Press, Cambridge, 1995).
11 R. Landauer, Philos. Mag. 21, 863 (1970).
12 D. S. Fisher and P. A. Lee, Phys. Rev. B 23, 6851 (1981).
13 Y. Meir and N. S. Wingreen, Phys. Rev. Lett. 68, 2512 (1992).
14 A. P. Jauho, N. S. Wingreen, and Y. Meir, Phys. Rev. B 50, 5528 (1994).
15 B. S. Dillon and H. Nakanishi, Eur. Phys. J. B 87, 286 (2014), arXiv: 1310.4224.
16 B. S. Dillon and H. Nakanishi, arXiv: 1708.07472.
17 Y. Xing, Q. Sun, and J. Wang, Phys. Rev. B 77, 115346 (2008), arXiv: 0806.2173.
18 D. Stauffer and A. Aharoni, Introduction to Percolation Theory (2nd ed.) (Taylor Francis, Abingdon, 1994).