On stationary solutions to the non-vacuum Einstein field equations

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Abstract

We derive a local curvature estimate for four-dimensional stationary solutions to the inheriting Einstein-Maxwell-Klein-Gordon equations. In particular, it implies that any such stationary geodesically complete solution with vanishing Poynting vector and proper coupling constants (like dark energy) is flat. We also generalize the results to higher dimensions.

1 Introduction

The purpose of this paper is to generalize the results in [2] for vacuum Einstein field equations to non-vacuum Einstein field equations. In [2], we proved that any geodesically complete 4-d spacetime with a timelike Killing field satisfying the vacuum Einstein field equation with nonnegative cosmological constant is actually flat. This result generalized the previous results in [1] and [8] by removing the extra conditions (like chronological condition) imposed on their theorems.

Because the physical spacetime can never be flat, it must violate the assumptions in the theorem in [2], for instance, the geodesic completeness. Also, by the singularity theorem of Penrose and Hawking (see [5] Chapter 8), the geodesic completeness of the physical spacetime is also hardly valid. Moreover, we know that many stationary spacetimes contain black holes, see [3] [10]. In [2], in order to prove the above result, we actually derived a local curvature estimate (see Theorem 1.3 in [2]), which does not require the spacetime to be geodesically complete. So from this point of view, the local curvature estimate in [2] seems to be more important.

In this paper, we are mainly concerned with the Einstein field equations in the presence of two kinds of matters or fields, the electromagnetic fields and scalar fields. For the former case, the interaction of the gravity (a 4-d spacetime $(M,g_M)$) and the electromagnetic field (a 2-form $F$) is described by a system of differential equations called Einstein-Maxwell equation:

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} + \Lambda g_{\alpha\beta} = \kappa (F_{\alpha\gamma} F_{\beta\delta} g^{\gamma\delta} - \frac{1}{4} |F|^2 g_{\alpha\beta})$$

$$dF = d \ast F = 0,$$

(1.1)

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where \( \Lambda, \kappa \) are constants, \( |F|^2 \triangleq F_\zeta F_\eta g^{\zeta \eta} g^{\gamma \delta} \). The second equation in (1.1) satisfied by the 2-form \( F_\alpha \delta \) (electromagnetic field) is called the Maxwell equation.

A scalar field, in physics, associates a value (\( \in \mathbb{R} \) or \( \mathbb{C} \)) to each point of the spacetime, like temperature, pressure etc. In this paper, we consider the Klein-Gordon scalar fields, which can be used to describe the \( \pi^0 \), or \( \pi^+ \), \( \pi^- \) mesons. The Einstein equation coupled with Klein-Gordon equation is the following

\[
\begin{align*}
\text{Ric}(g_M) - \frac{R}{2} g_M + \Lambda g_M &= \kappa'[d\phi \otimes d\phi - \frac{1}{2}(|d\phi|^2 + \frac{m^2}{\hbar^2}|\phi|^2)g_M] \\
\Delta_{g_M} \phi &= \frac{m^2}{\hbar^2} \phi
\end{align*}
\]

(1.2)

where \( \Lambda, \kappa', m \) (mass), \( \hbar \) (reduced planck constant) are constants, \( \phi : M \to \mathbb{R} \) or \( \mathbb{R}^2 \) is a map (scalar field).

The (uncharged) Einstein-Maxwell-Klein-Gordon equation is a combination of (1.1) and (1.2) (see [5] Chapter 3):

\[
\begin{align*}
\text{Ric}(g_M) - \frac{R}{2} g_M + \Lambda g_M &= \kappa (F_\alpha ^{\zeta} F_\beta ^{\delta} g^{\zeta \delta} - \frac{1}{4}|F|^2 g_{\alpha \beta}) \\
&+ \kappa'[d\phi \otimes d\phi - \frac{1}{2}(|d\phi|^2 + \frac{m^2}{\hbar^2}|\phi|^2)g_M] \\
dF = d^* F = 0 \\
\Delta_{g_M} \phi &= \frac{m^2}{\hbar^2} \phi
\end{align*}
\]

(1.3)

It is easy to see that (1.3) reduces to (1.1) or (1.2) when \( \phi = 0 \) or \( F = 0 \) in (1.3). So instead of dealing with (1.1) and (1.2) separately, we can handle only one equation (1.3).

Let \( X \) be a timelike Killing field on \( M \), by which we say the spacetime \( (M, g_M) \) is stationary. Following [12] [13], we say the solution \( (g_M, F, \phi) \) in (1.3) is inheriting if \( F \) and \( \phi \) satisfy

\[
\mathcal{L}_X(F) = 0, \ d\phi(X) = 0,
\]

which means that \( F \) and \( \phi \) inherit the \( X \)-symmetry from the metric tensor \( g_M \).

Denote the electric and magnetic fields (related to \( X \)) by

\[
E = i_X F, \ B = i_X \ast F
\]

(1.5)

respectively, where \( i_X F \) is the tensor contraction of \( F \) with \( X \). Let \( X^* \) be the 1-form on \( M \) obtained from \( X \) by lowering indices. We associate a Riemannian metric tensor \( \hat{g} \) on \( M \) to \( g_M \) (see [2]):

\[
\hat{g} \triangleq -\frac{2}{g_M(X, X)} X^* \otimes X^* + g_M.
\]

(1.6)

The first main result of the paper is the following local curvature estimate:

**Theorem 1.1** Let \( (M, g_M) \) be a 4-d spacetime with a timelike Killing field \( X \) such that \( g_M \) satisfies the inheriting Einstein-Maxwell-Klein-Gordon equation (1.3) (1.4), where \( \Lambda \geq 0 \) and \( \kappa \leq 0, \kappa' \geq 0 \) are constants. Let \( \hat{B}(x_0, a) \) be a \( \hat{g} \)-metric ball centered at \( x_0 \) of radius \( a > 0 \) with compact closure in \( M \) and

\[
\sup_{x \in \hat{B}(x_0, a)} |\kappa|u^{-2} \ |E \wedge B |_{\hat{g}} \leq a^{-2}.
\]

(1.7)
Then for any $0 < \delta < 1$, there is a constant $C_\delta > 0$ depending only on $\delta$ such that

$$\sup_{x \in B(x_0, \frac{a}{2})} |Rm(g_M)|_g \leq \frac{\delta m^2}{\hbar^2} + C_\delta a^{-2}. \tag{1.8}$$

The Hodge dual (on three-space) of the 2-form $E \wedge B$ in (1.7) is a 1-form corresponding to the Poynting vector in physics. A corollary of Theorem 1.1 is the following

**Theorem 1.2** Let $(M, g_M)$ be a geodesically complete 4-d spacetime with a timelike Killing field $X$ such that $g_M$ satisfies the inheriting Einstein-Maxwell-Klein-Gordon equation (1.3) (1.4), where $\Lambda \geq 0$, $\kappa \leq 0$, $\kappa' \geq 0$. We assume the Poynting vector vanishes on $M$, i.e., $E \wedge B \equiv 0$. Then $(M, g_M)$ is flat.

We give a possible physical explanation why Theorem 1.1 or Theorem 1.2 hold for $\Lambda \geq 0$, $\kappa \leq 0$, $\kappa' \geq 0$ (but not the contrary). It is well-known that positive cosmological constant $\Lambda$ and negative energy ($\kappa < 0$, $\kappa' > 0$) will cause the acceleration of the universe. Intuitively, the acceleration will reduce the density of the matter or energy distribution of the universe, hence make the spacetime more "regular".

**Remark 1.1** The spacetime being static implies that the Poynting vector vanishes, i.e. $E \wedge B = 0$.

We say a stationary spacetime is static if the orthogonal complement of the timelike Killing field is an integrable distribution. In general, when $\dim \geq 5$, the "curvature" measuring the difference between stationarity and staticity is difficult to control. In order to generalize Theorems 1.1 and 1.2 to $\dim \geq 5$, we can assume for simplicity that the spacetime is static. In this case, we can obtain analogous results for general dimensions, see Theorems 3.1 3.2.

The paper is organized as follows. In section 2, we prove Theorems 1.1 and 1.2. In section 3, we generalize the results to general dimensions.

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## 2 4-d Einstein-Maxwell-Klein-Gordon equations

### 2.1 Preliminaries

In this section, we recall some useful formulas for stationary spacetimes, their computations could be found in [2], see also [1] [7]. Let $\{x^\alpha\}$ be a local coordinate system on a 4-d stationary spacetime $(M, \tilde{g}, X)$ so that the metric tensor $\tilde{g} = g_{\alpha\beta}dx^\alpha dx^\beta$ takes the following form:

$$\tilde{g} = -u^2(dt + \theta)^2 + g \tag{2.1}$$

where $u = (-\langle X, X \rangle)^{\frac{1}{2}}$, $\theta = \theta dx^i$ and $g = g_{ij}dx^i dx^j$ are $t = x^0$-independent. Here we use $\tilde{g}$ to denote the spacetime metric $g_M$. We use the convention that for tensor indices, the Greek letters $\alpha, \beta, \cdots$ are running from 0 to 3, the Latin letters $i, j, \cdots$ from 1 to 3. Let $e_0 = \frac{\partial}{\partial t}$, $e_i = \frac{\partial}{\partial x^i} - \theta_i \frac{\partial}{\partial t}$ be a local tangent frame satisfying $\langle e_i, e_j \rangle = g_{ij}$ and $\langle e_0, e_j \rangle = 0$. Fix the local orientation of $M$ (on this coordinate
chart) given by the frame \((e_1, e_2, e_3, e_0)\). We use \(*^4\) and \(*\) to denote the Hodge dual operators (with respect to the above orientation and metrics \(\bar{g}\) and \(g\)) on spaces of dimensions 4 and 3 respectively.

We remark that the coordinate system \(\{x^\alpha\}\) and the local tangent frame \(\{e_\alpha\}\) will be used frequently throughout the paper. It is clear that their constructions are irrelevant to the dimensions.

Let \(\omega = u^3 \ast d\theta\) be a spatial 1-form on \(M\). For any Lorentzian metric \(\bar{g}\) of the form (2.1), the Ricci curvature components \(\bar{Ric}(e_\alpha, e_\beta)\) can be expressed as a system of differential equations satisfied by \(u, \omega\) and \(g\) (see [2], (5.1)):

\[
R_{ij} = u^{-1} \nabla_i \nabla_j u + \frac{1}{2} u^{-4} (\omega_i \omega_j - |\omega|^2 g_{ij}) + \bar{Ric}(e_i, e_j)
\]
\[
\triangle u = -\frac{1}{2} u^{-3} |\omega|^2 + u^{-1} \bar{Ric}(X, X)
\]
\[
g^{kl} \nabla_k \omega_l = 3 g^{kl} \omega_k \nabla_l \log u
\]
\[
(*d\omega)_j = \pm 2 u \bar{Ric}(X, e_j).
\]

Let \(\bar{g} \triangleq u^2 g\) be a conformal change of the spatial metric \(g\), the Ricci curvature of \(\bar{g}\) can be computed (see [2], (5.3)):

\[
\bar{R}_{ij} = \frac{1}{2} u^{-4} \omega_i \omega_j + 2 \frac{u_i u_j}{u^2} + \bar{Ric}(e_i, e_j) - u^{-2} \bar{Ric}(X, X) g_{ij}.
\]

The Ricci curvatures of the metrics \(\hat{g}\) (see (1.6)) and \(\bar{g}\) are related (see [2], (5.2)) in the following manner:

\[
\hat{Ric}(X, X) = u^{-2} |\omega|^2 - \bar{Ric}(X, X)
\]
\[
\hat{Ric}(X, e_j) = -\bar{Ric}(X, e_j)
\]
\[
\hat{Ric}(e_i, e_j) = -u^{-4} (|\omega|^2 g_{ij} - \omega_i \omega_j) + \bar{Ric}(e_i, e_j).
\]

For later use, the \(\hat{g}\)-hessian of any time-independent smooth function \(f\) on the space-time can be computed to be (see [2], (2.11)):

\[
\hat{\nabla}^2 f(e_0, e_0) = \frac{1}{2} \langle \nabla u^2, \nabla f \rangle
\]
\[
\hat{\nabla}^2 f(e_0, e_j) = \frac{1}{2} u^{-1} (\ast (\omega \wedge df))(e_j)
\]
\[
\hat{\nabla}^2 f(e_i, e_j) = \nabla_{ij} f.
\]

In particular, we have

\[
\hat{\Delta} f = \tilde{\Delta} f = u^2 \hat{\Delta} f = \Delta f + \langle \nabla \log u, \nabla f \rangle.
\]

### 2.2 Field equations

Denote \(V \circ \phi = \frac{m^2}{\kappa'} |\phi|^2\). Taking trace on (1.3), we find

\[
R = \kappa' |d\phi|^2 + 4\Lambda + 2\kappa' V
\]

and equation (1.3) can be rewritten as

\[
R_{\alpha\beta} = \kappa (F_{\alpha\gamma} F_{\beta\delta} g^{\gamma\delta} - \frac{1}{4} |F|^2 g_{\alpha\beta}) + \kappa' \phi_\alpha \phi_\beta + (\Lambda + \frac{\kappa'}{2} V) g_{\alpha\beta}.
\]
From $\mathcal{L}_XF = 0$, one can derive $\mathcal{L}_X *^4 F = 0$ since $X$ is a Killing field. Denote the electric and magnetic fields related to $X$ by

$$E = i_X F, B = i_X *^4 F$$

respectively. From Cartan’s homotopy formula $\mathcal{L}_X = d i_X + i_X d$ for differential forms, we find $dE = dB = 0$. Using $i_X^2 = 0$, we obtain

$$\mathcal{L}_X E = \mathcal{L}_X B = 0, i_X(E) = i_X(B) = 0,$$

which roughly say that $E$ and $B$ are spatial 1-forms on $M$.

Let $F = \omega_1(x) \wedge u(dt + \theta) + \omega_2(x)$ be a local expression of the electromagnetic field $F$ in the coordinate system $\{x^\alpha\}$ in Section 2.1, where $\omega_1(x)$ and $\omega_2(x)$ are spatial forms of degrees 1 and 2. Since $*^4 F = *^2 \omega_2 \wedge u(dt + \theta) + *^2 \omega_1$, we know $E = -u \omega_1(x)$, $B(x) = -u * \omega_2$. From this, we know $F_{\alpha\beta} F_{\gamma\delta} g^{\alpha\gamma} g^{\beta\delta} = 2u^{-2}(-|E|^2 + |B|^2)$. The Maxwell’s equation $dF = d*^4 F = 0$ is equivalent to a system of differential equations on the electric and magnetic fields $E$ and $B$:

$$\begin{align*}
    dE & = 0 \\
    dB & = 0 \\
    d(u^{-1} * E) & = B \wedge d\theta \\
    d(u^{-1} * B) & = E \wedge d\theta.
\end{align*}$$

Now we compute the components of the Ricci curvature $\bar{Ric}$ in terms of the frame $\{e_\alpha\}$ in Section 2.1. From (2.7), we have

$$\bar{Ric}(X, X) = \kappa \left( \frac{1}{2} |E|^2 + \frac{1}{2} |B|^2 \right) - (\Lambda + \frac{\kappa'}{2} V) u^2.$$  

(2.8)

Since $i_{e_j} F = -E(e_j)(dt + \theta) + i_{e_j} \omega_2$ and

$$\langle i_{e_j} \omega_2, i_{e_j} \omega_2 \rangle = | \star \omega_2 |^2 g_{ij} - (* \omega_2)(e_i)(* \omega_2)(e_j),$$

we have

$$\langle i_{e_j} F, i_{e_j} F \rangle = -u^{-2}(E_i E_j + B_i B_j) + u^{-2} |B|^2 g_{ij}.$$

This implies

$$\bar{Ric}(e_i, e_j) = \kappa [-u^{-2}(E_i E_j + B_i B_j) + \frac{1}{2} u^{-2} (|B|^2 + |E|^2) g_{ij}] + \kappa' \phi_k \phi_j + (\Lambda + \frac{\kappa'}{2} V) g_{ij}.$$  

(2.11)

Hence

$$\bar{Ric}(e_k, e_l) - u^{-2} \bar{Ric}(X, X) g_{kl}$$

$$= -\kappa u^{-2}(E_i E_j + B_i B_j) + \kappa' \phi_k \phi_l + (2\Lambda + \kappa' V) g_{kl}.$$  

(2.12)

To compute the term $\bar{Ric}(X, e_j)$, we need

$$\langle i_X F, i_{e_j} F \rangle = \langle E, -E(e_j)(dt + \theta) + i_{e_j} \omega_2 \rangle$$

$$= \langle E, i_{e_j} (\omega_2) \rangle$$

$$= i_{e_j} (\star (E \wedge \omega_2))$$

$$= -u^{-1} i_{e_j} (\star (E \wedge B)).$$  

(2.13)

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Therefore,
\[ \tilde{Ric}(X, e_j) = -\kappa u^{-1}i_{e_j}((E \wedge B)). \] (2.14)

Combining (2.2), (2.3), (2.11) and (2.12), we have
\[ R_{ij} = u^{-1}\nabla_i\nabla_j u + \frac{1}{2} u^{-4}(\omega_i\omega_j - |\omega|^2 g_{ij}) + \kappa' \phi_i \phi_j + (\Lambda + \frac{\kappa'}{2} V) g_{ij} \]
\[ + \kappa[-u^{-2}(E_i E_j + B_i B_j) + \frac{1}{2} u^{-2}(|E|^2 + |B|^2)g_{ij}], \] (2.15)

and
\[ \tilde{R}_{ij} = 2u^{-2}u_i u_j + \frac{1}{2} u^{-4} \omega_i \omega_j - \kappa u^{-2}(E_i E_j + B_i B_j) \]
\[ + \kappa' \phi_i \phi_j + (2\Lambda + \kappa' V) g_{ij}, \] (2.16)

where \( \tilde{R}_{ij} \) is the Ricci curvature of \( g = u^2 g \).

Denote
\[ E = d\phi_3, \quad B = d\phi_4 \] (2.17)

for some locally defined functions \( \phi_3 \) and \( \phi_4 \). From (2.2), (2.9), (2.10), we have:
\[ \Delta u = -\frac{1}{2} u^{-3}|\omega|^2 + \kappa(\frac{1}{2} u^{-1}|E|^2 + \frac{1}{2} u^{-1}|B|^2) - (\Lambda + \frac{\kappa'}{2} V) u \]
\[ \Delta \phi_3 = \langle d\log u, E \rangle + u^{-2}\langle \omega, B \rangle \] (2.18)
\[ \Delta \phi_4 = \langle d\log u, B \rangle + u^{-2}\langle \omega, E \rangle. \]

Combining (2.6) and (2.18), we have
\[ \hat{\Delta} \log u = u^2 \hat{\Delta} \log u = -\frac{1}{2} u^{-4}|\omega|^2 + \kappa(\frac{1}{2} u^{-2}|E|^2 + \frac{1}{2} u^{-2}|B|^2) - (\Lambda + \frac{\kappa'}{2} V) \]
\[ \hat{\Delta} \phi_3 = u^2 \hat{\Delta} \phi_3 = 2\langle d\log u, E \rangle + u^{-2}\langle \omega, B \rangle \]
\[ \hat{\Delta} \phi_4 = u^2 \hat{\Delta} \phi_4 = 2\langle d\log u, B \rangle + u^{-2}\langle \omega, E \rangle. \] (2.19)

2.3 **A Bochner formula**

In this section, we assume that the condition
\[ \tilde{Ric}(X, e_j) = 0 \] (2.20)
always holds. When condition (2.20) holds, from the last 2 equations of (2.2), we know \( \omega = d\psi \) holds locally for some function \( \psi \) and
\[ \hat{\Delta} \psi = u^2 \hat{\Delta} \psi = 4\langle \omega, d\log u \rangle. \] (2.21)

From (2.14), the condition (2.20) is equivalent to
\[ E \wedge B = 0, \] (2.22)
i.e., the Poynting vector vanishes. Here we assume \( E = B = 0 \) if \( \kappa = 0 \).
Lemma 2.1 Let \( g_H = y_1^{-2}dy_1^2 + c_2y_1^{-2}dy_2^2 + \cdots + c_my_1^{-2}dy_m^2 \), \( c_a > 0 \) be a Riemannian metric on \( m \)-dimensional upper half space \( H = \{(y_1, y_2, \ldots, y_m) : y_i \in R, y_1 > 0 \} \). In terms of the natural frame \( \{ \frac{\partial}{\partial y^a} \} \), the only nonzero components of the Christoffel symbols \( \Gamma^a_{bc} \) of \( g_H \) are:

\[
\Gamma^1_{bb} = \frac{1}{2}c_b b y_1^{1-b}, \quad \Gamma^b_{1b} = -\frac{1}{2}b y_1^{-1}, \quad \Gamma^1_{11} = -y_1^{-1},
\]

where \( b \neq 1 \). Up to the symmetries of curvature tensors, the only nonzero components of the curvature tensor of \( g_H \) are

\[
R_{1a1a} = -\frac{1}{4}c_a c_b b y_1^{-2}, \quad R_{abab} = -\frac{1}{4}c_a c_b a b y_1^{-2},
\]

where \( a, b \neq 1 \) and \( a \neq b \).

In particular, the sectional curvatures of the metric \( g_H \) in Lemma 2.1 are nonpositive.

Let \( g_H = y_1^{-2}(dy_1^2 + dy_2^2 - 2\kappa y_1 dy_2^2 - 2\kappa y_1 dy_2^2) \) be a metric on 4-d upper half space \( H = \{(y_1, y_2, y_3, y_4) : y_i \in R, y_1 > 0 \} \). We define a map \( \Phi \) from the coordinate system \( \{x^a\} \) of Section 2.1 to \( H \) to be \( \Phi = (u^2, \psi, \phi_3, \phi_4) = (y_1, y_2, y_3, y_4) \). Because \( u, \psi, \phi_3, \phi_4 \) are time-independent, \( \Phi \) is also time-independent, hence a map from \( \{x^i\} \) to \( H \).

Lemma 2.2 The map \( \Phi \) satisfies

i) \( \Phi^* g_H = 4d \log u \otimes d \log u + u^{-4}\omega \otimes \omega - 2\kappa u^{-2}E \otimes E - 2\kappa u^{-2}B \otimes B \);

ii) \( \Delta \Phi = u^2 \Delta \Phi = -(2\Lambda + \kappa')V u^2 \frac{\partial^2 \phi}{\partial y^2} + u^{-2}B(\omega, \frac{\partial \phi}{\partial y^2}) + u^{-2}(E, \omega) \frac{\partial \phi}{\partial y^2} \), where \( \Delta \Phi \) (or \( \hat{\Delta} \Phi \)) is the harmonic map Laplacian of \( \Phi \) from Riemannian manifold \( \{x^a, \hat{g}\} \) (or \( \{x^1, \hat{g}\} \)) to \( \{H, g_H\} \).

Proof. Substituting \( m = 4, c_2 = 1, c_3 = -2\kappa, c_4 = -2\kappa \) and \( l_2 = 2, l_3 = 1, l_4 = 1 \) in Lemma 2.1 we find

\[
\Gamma^1_{11} = -y_1^{-1}, \quad \Gamma^1_{22} = \frac{1}{2}y_1^{-1}, \quad \Gamma^1_{33} = \frac{1}{2}y_1^{-1}, \quad \Gamma^1_{44} = -\kappa, \quad \Gamma^2_{12} = -y_1^{-1}, \quad \Gamma^3_{13} = \frac{1}{2}y_1^{-1}.
\]

By direct computations,

\[
\Gamma^1_{ab} \nabla_a \Phi^a \nabla_a \Phi^b = \Gamma^1_{11} \partial_a u^2 \partial_a u^2 + \sum_{b \neq 1} \Gamma^1_{bb} \partial_a y_b \partial_a y_b
\]

\[
= -4u \partial_a u + u^{-2}\omega \partial_a \omega - \kappa(\phi_3)(\phi_3)_\beta - \kappa(\phi_4)(\phi_4)_\beta
\]

\[
\Gamma^2_{ab} \nabla_a \Phi^a \nabla_a \Phi^b = \Gamma^2_{12} (\nabla_a \Phi^a \nabla_a \Phi^2 + \nabla_a \Phi^a \nabla_a \Phi^1)
\]

\[
= -2\omega (\log u)_\alpha - 2\omega (\log u)_\beta
\]

\[
\Gamma^3_{ab} \nabla_a \Phi^a \nabla_a \Phi^b = \Gamma^3_{13} (\nabla_a \Phi^a \nabla_a \Phi^3 + \nabla_a \Phi^a \nabla_a \Phi^1)
\]

\[
= -2(\phi_3)_\beta (\log u)_\alpha - (\phi_4)_\alpha (\log u)_\beta
\]

\[
\Gamma^4_{ab} \nabla_a \Phi^a \nabla_a \Phi^b = -2(\phi_4)_\beta (\log u)_\alpha - (\phi_4)_\alpha (\log u)_\beta.
\]

Using (2.26) and the formula

\[
(\hat{\nabla}_{\alpha \beta} \Phi)^a = \hat{\nabla}_{\alpha \beta} \Phi^a + \Gamma^a_{bc} \nabla_a \Phi^b \nabla_a \Phi^c,
\]

(2.27)
we obtain

\[
\begin{align*}
(\nabla_{\alpha\beta}\Phi)^1 &= 2u^2\nabla_{\alpha\beta}\log u + u^{-2}\omega_\alpha\omega_\beta - \kappa E_\alpha E_\beta - \kappa B_\alpha B_\beta \\
(\nabla_{\alpha\beta}\Phi)^2 &= \nabla_{\alpha\beta}\psi - 2\omega_\alpha\log u_\beta - 2\omega_\beta\log u_\alpha \\
(\nabla_{\alpha\beta}\Phi)^3 &= \nabla_{\alpha\beta}\phi_3 - (\log u)_\alpha E_\beta - (\log u)_\beta E_\alpha \\
(\nabla_{\alpha\beta}\Phi)^4 &= \nabla_{\alpha\beta}\phi_4 - (\log u)_\alpha B_\beta - (\log u)_\beta B_\alpha.
\end{align*}
\]

(2.28)

Taking traces on (2.28) with respect to \( \hat{g} \), we get

\[
\begin{align*}
(\hat{\Delta}\Phi)^1 &= 2u^2\hat{\Delta}\log u + u^{-2}|\omega|^2 - \kappa|E|^2 - \kappa|B|^2 = -(2\Lambda + \kappa'V)u^2 \\
(\hat{\Delta}\Phi)^2 &= \hat{\Delta}\psi - 4\langle\nabla\psi, \nabla\log u\rangle_g = 0 \\
(\hat{\Delta}\Phi)^3 &= \hat{\Delta}\phi_3 - 2\langle d\log u, E\rangle_g = \langle u^{-2}\omega, B\rangle_g \\
(\hat{\Delta}\Phi)^4 &= \hat{\Delta}\phi_4 - 2\langle d\log u, B\rangle_g = \langle u^{-2}\omega, E\rangle_g,
\end{align*}
\]

where we have used (2.19) (2.21).

\[\square\]

**Lemma 2.3** Under the condition (2.21) or (2.22), we have \( \langle \omega, E \rangle = \langle \omega, B \rangle = 0 \). Hence, \( \hat{\Delta}\Phi = -(2\Lambda + \kappa'V)u^2 \frac{\partial^2}{\partial y^2} \).

**Proof.** Assume \( \kappa \neq 0 \), otherwise the result holds trivially because we can take \( E = B = 0 \). In order to show \( \langle \omega, E \rangle = \langle \omega, B \rangle = 0 \), we argue by contradiction. Suppose \( \langle \omega, E \rangle \neq 0 \) at some point \( P \), by (2.22), we have \( B = \lambda E \) holds for some function \( \lambda \) near \( P \). Using i) in Lemma 2.2 and the contracted 2nd Bianchi identity,

\[
\begin{align*}
\frac{1}{2} \tilde{R}_i &= \tilde{\nabla}_j \tilde{R}_{ik}\tilde{g}^{kj} = \tilde{\nabla}_j \left( \frac{1}{2}(\Phi^* h)_{ik} + \kappa'\phi_i\phi_k + (2\Lambda + \kappa'V)g_{ik}\right)\tilde{g}^{jk} \\
&= \frac{1}{4}(tr_g(\Phi^* h))_i + \frac{1}{2}\langle \hat{\Delta}\Phi, \tilde{\nabla}_i\Phi \rangle + [(2\Lambda + \kappa'V)u^{-2}]_i \\
&\quad + \kappa'\langle \hat{\Delta}\phi, \tilde{g}^{\alpha\beta}\rangle + \frac{1}{2}\langle d\phi^2 \tilde{g}^{ij} \rangle,
\end{align*}
\]

(2.30)

since \( tr_g(\Phi^* h) = 2\tilde{R} - 2\kappa'|d\phi|^2 - (12\Lambda + 6\kappa'V)u^{-2}, \) we get

\[
\begin{align*}
\frac{1}{2}\langle \hat{\Delta}\Phi, \tilde{\nabla}_i\Phi \rangle - [(2\Lambda + \kappa'V)u^{-2}]_i + \frac{1}{2}u^{-2}\kappa'(V \circ \phi)_i &= 0.
\end{align*}
\]

Together with ii) in Lemma 2.2 we have

\[
\langle \omega, B \rangle E + \langle \omega, E \rangle B = 0.
\]

This gives \( 2\lambda\langle \omega, E \rangle E = 0 \), which implies \( \lambda = 0 \). Hence \( B \equiv 0 \) near \( P \), from the last equation in (2.19), we know \( \langle \omega, E \rangle \equiv 0 \) near \( P \), which is a contradiction. If \( \langle \omega, B \rangle \neq 0 \), a contradiction can be derived by the same argument. \( \square \)

Now we can apply the standard Bochner formula(for a map not necessarily harmonic, see [1]):

\[
\hat{\Delta}e(\Phi) = 2\langle \nabla\Phi, \hat{\Delta}\Phi \rangle_{\tilde{g}} + 2|\nabla_{\alpha\beta}\Phi|^2 + 2\langle \tilde{R}ic, \Phi^* h \rangle_{\tilde{g}} - 2R_{abcd}\Phi^a\Phi^b\Phi^c\Phi^d\tilde{g}^{\beta\gamma}\tilde{g}^{\alpha\gamma}
\]

(2.31)
Similarly, one can compute
\[ e(\Phi) = \delta^\alpha\beta(\Phi^* g_H)_{\alpha\beta} = 4|\nabla \log u|^2 + u^{-4}|\omega|^2 - 2\kappa u^{-2}|E|^2 - 2\kappa u^{-2}|B|^2. \]  

(2.32)

By using (2.25), Lemma (2.3) and the following formula,
\[ \hat{\nabla}_a \hat{\Delta} \Phi^a = \frac{\partial}{\partial x^a} \hat{\Delta} \Phi^a + \Gamma^a_{bc} \hat{\Delta} \Phi^b \hat{\Phi}^c, \]

one can get
\[ \hat{\nabla}_a \hat{\Delta} \Phi^1 = [- (2\Lambda + \kappa' V) u^2]_{a} + \Gamma^1_{11}(-2\Lambda - \kappa' V) u^2 \Phi^1_{a} = -\kappa' u^2 (V \circ \phi)_{a} \]
\[ \hat{\nabla}_a \hat{\Delta} \Phi^2 = \Gamma^2_{12}(-2\Lambda - \kappa' V) u^2 \Phi^2_{a} = (2\Lambda + \kappa' V) \omega_{a} \]
\[ \hat{\nabla}_a \hat{\Delta} \Phi^3 = \Gamma^3_{13}(-2\Lambda - \kappa' V) u^2 \Phi^3_{a} = (\Lambda + \frac{\kappa'}{2} V) E_{a} \]
\[ \hat{\nabla}_a \hat{\Delta} \Phi^4 = (\Lambda + \frac{\kappa'}{2} V) B_{a}. \]

(2.33)

Hence
\[ \langle \hat{\nabla} \Phi, \hat{\nabla} \Phi \rangle_{\hat{g}} = (2\Lambda + \kappa' V) (|\omega|^2 u^{-4} - \kappa u^{-2}(|E|^2 + |B|^2)) \]
\[ - 2\kappa' \langle \nabla (V \circ \phi), \nabla \log u \rangle. \]

(2.34)

Now we compute the term $|\hat{\nabla}_{\alpha\beta} \Phi|^2$ at a fixed point $P = (x', t')$. By definition,
\[ |\hat{\nabla}_{\alpha\beta} \Phi|^2 = u^{-4}(|\hat{\nabla}_{\alpha\beta} \Phi|^2)_{\hat{g}} + u^{-4}(|\hat{\nabla}_{\alpha\beta} \Phi|^2)_{\hat{g}} - 2\kappa u^{-2}(|\hat{\nabla}_{\alpha\beta} \Phi|^2)_{\hat{g}} - 2\kappa u^{-2}(|\hat{\nabla}_{\alpha\beta} \Phi|^2)_{\hat{g}}. \]

Let $\{x^i\}$ be a spatial normal coordinate system around the fixed point $x'$. Let $F_0 = u^{-1} \frac{\partial}{\partial r}, F_i = \frac{\partial}{\partial \theta^i} - \theta_i \frac{\partial}{\partial r}$, then $\{F_\alpha\}$ is an orthonormal basis of $\hat{g}$ at $(x', t')$, hence
\[ |(\hat{\nabla}_{\alpha\beta} \Phi)|^2 = \sum_{\alpha, \beta} [(\hat{\nabla}_{\alpha\beta} \Phi)^a (F_\alpha, F_\beta)]^2. \]

(2.35)

Combining (2.28) and (2.5), we get
\[ u^{-4}(|\hat{\nabla}_{\alpha\beta} \Phi|^2)_{\hat{g}} = 4|\nabla \log u|^4 + 2u^{-2}\omega \wedge d \log u|^2 \]
\[ + 2 |\nabla \log u| + u^{-4}|\omega|^2 - \kappa u^{-2}(E_i E_j + B_i B_j)|^2 \]
\[ u^{-4}(|\hat{\nabla}_{\alpha\beta} \Phi|^2)_{\hat{g}} = (d \log u, u^{-2}\omega)^2 + u^{-4}|\nabla \omega_i - 2\omega_i (\log u_j) - 2\omega_j (\log u)_i|^2. \]

(2.36)

Similarly, one can compute
\[ u^{-2}(|\hat{\nabla}_{\alpha\beta} \Phi|^3)_{\hat{g}} = (d \log u, u^{-1} E_i)^2 + 2|\frac{1}{2} u^{-2} \omega \wedge u^{-1} E_i|^2 \]
\[ + u^{-2}|\nabla_i E_j - E_i (\log u)_j - E_j (\log u)_i|^2 \]
\[ u^{-2}(|\hat{\nabla}_{\alpha\beta} \Phi|^4)_{\hat{g}} = (d \log u, u^{-1} B_j)^2 + 2|\frac{1}{2} u^{-2} \omega \wedge u^{-1} B_j|^2 \]
\[ + u^{-2}|\nabla_i B_j - B_i (\log u)_j - B_j (\log u)_i|^2. \]

(2.37)

To compute the fourth term on the right hand side of (2.31), we need
\[
R_{abcd} \hat{\Phi}^a \hat{\Phi}^\beta \hat{\Phi}^\gamma \hat{\Phi}^\delta g^{\alpha \gamma} \hat{g}^{\beta \delta} = R_{abcd} \langle \nabla y_a, \nabla y_c \rangle / \langle \nabla y_b, \nabla y_d \rangle \]
\[ = 2 R_{1212} (|\nabla u^2|^2 |\omega|^2 - |\nabla u^2, \omega|^2) + 2 R_{1313} (|\nabla u^2|^2 |E|^2 - |\nabla u^2, E|^2) + 2 R_{1414} (|\nabla u^2|^2 |B|^2 - |\nabla u^2, B|^2) \]
\[ + 2 R_{2424} (|\omega|^2 |B|^2 - \langle \omega, B \rangle^2) + 2 R_{3434} (|E|^2 |B|^2 - \langle E, B \rangle^2), \]
\[ + 2 R_{2323} (|\omega|^2 |E|^2 - \langle \omega, E \rangle^2) \]

(2.38)
where we have used Lemma 2.1.

From Lemma 2.1, we know

$$R_{1212} = -u^{-8}, R_{1313} = R_{1414} = \frac{\kappa}{2}u^{-6}$$
$$R_{2323} = R_{2424} = \kappa u^{-6}, R_{3434} = -\kappa^2 u^{-4}.$$

(2.39)

So

$$- R_{abcd} \phi_a \phi_b \phi_c \phi_d g^{\alpha \beta} g_{\alpha \beta}$$
$$= 8|d \log u \wedge u^{-2}\omega|^2 - 4\kappa|d \log u \wedge u^{-1}E|^2 - 4\kappa|d \log u \wedge u^{-1}B|^2$$
$$- 2\kappa|u^{-2}\omega \wedge u^{-1}E|^2 - 2\kappa|u^{-2}\omega \wedge u^{-1}B|^2 + 2\kappa^2|u^{-1}E \wedge u^{-1}B|^2.$$ 

(2.40)

To compute the third term on the right hand side of (2.31), we rewrite (see (2.11))

$$\langle \hat{\text{Ric}}(e_k, e_l) = -\kappa \frac{2}{u^2}(|E|^2 + |B|^2)g_{kl} + \kappa u^{-2}(|E|^2 g_{kl} - E_k E_l)$$
$$+ \kappa u^{-2}(|B|^2 g_{kl} - B_k B_l) + (\Lambda + \frac{\kappa'}{2} V)g_{kl} + \kappa' \phi_k \phi_l,$$

thus

$$\langle \hat{\text{Ric}}(\Phi^* g_{H}) \rangle_{\hat{g}}$$
$$= [u^{-4}(\omega \omega - |\omega|^2 g^{ij}) + \hat{\text{Ric}}(e_k, e_l)g^{ij}g^{kl}]$$
$$\times [u^{-4}(\omega \omega + 4u^{-2}u_i u_j - 2\kappa u^{-2}E_i E_j - 2\kappa u^{-2}B_i B_j)]$$
$$= [\Lambda + \frac{\kappa'}{2} V - \kappa \frac{2}{u^2}(|E|^2 + |B|^2)e(\Phi) - 4\kappa u^{-2}\omega \wedge d \log u|^2$$
$$+ 3\kappa|u^{-2}\omega \wedge u^{-1}E|^2 + 3\kappa|u^{-2}\omega \wedge u^{-1}B|^2 + 2\kappa^2|u^{-1}E \wedge u^{-1}B|^2 + \kappa' \langle d\phi \otimes d\phi, \Phi^* g_H \rangle.$$ 

(2.41)

Note that $E \wedge B = 0$ holds by condition (2.20). Combining

$$\hat{\Delta}(\frac{\kappa'}{2} V \circ \phi) = \frac{\kappa'}{2} Hess(V)_{ab} \phi_c \phi_d^{bcg} \phi_{\alpha \beta} + \frac{\kappa'}{4} |\nabla V|^2$$

(2.42)

and (2.31) (2.34) (2.36) (2.37) (2.40) (2.41), we have

$$\hat{\Delta}(\frac{\kappa'}{2} e(\Phi) + \frac{\kappa'}{2} V)$$
$$= 4|\nabla \log u|^4 + |2\nabla_{ij} \log u + u^{-4}\omega \omega_{ij} - \kappa u^{-2}(E_i E_j + B_i B_j)|^2$$
$$+ u^{-4}|\nabla \omega_{ij} + 2\omega_i (\log u)_{ij} - 2\omega_j (\log u)_{ij}|^2$$
$$- 2\kappa u^{-2}\nabla_i E_j - E_i (\log u)_{ij} - E_j (\log u)_{ij}|^2$$
$$- 2\kappa u^{-2}\nabla_i B_j - B_i (\log u)_{ij} - B_j (\log u)_{ij}|^2$$
$$+ \langle d \log u, u^{-2}\omega \rangle^2 - 2\kappa \langle d \log u, u^{-1}E \rangle^2 - 2\kappa \langle d \log u, u^{-1}B \rangle^2$$
$$+ 6|u^{-2}\omega \wedge d \log u|^2 + [\Lambda + \frac{\kappa'}{2} V - \kappa \frac{2}{u^2}(|E|^2 + |B|^2)e(\Phi)]$$
$$+ (2\Lambda + \kappa' V)|\omega|^2 u^{-4} - \kappa u^{-2}(|E|^2 + |B|^2) + I$$

(2.43)
where

\[
I = \frac{\kappa'}{2} [\text{Hess}(\phi)_{\alpha\beta} + \frac{1}{2} |\nabla \phi|^2] + \frac{\kappa'}{2} (\phi \otimes \phi, \Phi^* g_H) - 2\kappa' (\nabla (\phi \circ \phi), \nabla \log u) \geq \kappa' \frac{1}{2} |\nabla \phi|^2 - 2d\phi (\nabla \log u) \geq 0
\]

for \(\kappa' \geq 0\).

\[
\hat{\Delta}(\frac{1}{2} e(\phi) + \frac{\kappa'}{2} V) = \Delta(\frac{1}{2} e(\phi) + \frac{\kappa'}{2} V) + (\nabla \log u, \nabla (\frac{1}{2} e(\phi) + \frac{\kappa'}{2} V)) \geq 4|\nabla \log u|^2 + |\nabla_{ij} \log u + u^{-2} \omega_i \omega_j - \kappa u^{-2} (E_i E_j + B_i B_j)|^2
\]

\[+ [\Lambda + \frac{\kappa'}{2} V - \frac{\kappa}{2} u^{-2} (|E|^2 + |B|^2)] e(\phi),
\]

where

\[
\frac{1}{2} e(\phi) + \frac{\kappa'}{2} V = 2|\nabla \log u|^2 + \frac{1}{2} u^{-4} |\omega|^2 - \kappa u^{-2} |E|^2 - \kappa u^{-2} |B|^2 + \frac{\kappa'}{2} V
\]

The most notable feature in formula (2.45) is that each term in the right hand side of (2.45) is nonnegative if \(\Lambda \geq 0, \kappa \leq 0\) and \(\kappa' \geq 0\).

## 2.4 Proof of Theorems 1.1 and 1.2

**Proof.** The idea of the proof is analogous to Theorem 5.3 in [2].

We assume \(\partial \hat{B}(x_0, \alpha) \neq \phi\).

Let \(h(x) = 2|\nabla \log u|^2 + \frac{1}{2} u^{-4} |\omega|^2 - \kappa u^{-2} |E|^2 - \kappa u^{-2} |B|^2 + \frac{\kappa'}{2} V + \Lambda\) be the quantity in (2.46) (up to a constant \(\Lambda\)), \(f(x) = h(x) d_\beta^2(x, \partial \hat{B}(x_0, \alpha))\), and \(x \in \hat{B}(x_0, \alpha)\) such that \(f(x) = \sup_{x \in \hat{B}(x_0, \alpha)} f(x)\).

For any fixed \(0 < \delta < 1\), we have two cases, 1) \(h(\bar{x}) \leq \delta \frac{m^2}{h^2}\), 2) \(h(\bar{x}) > \delta \frac{m^2}{h^2}\).

For case 1), for any \(x \in \hat{B}(x_0, \frac{3\alpha}{4})\), we have \((\frac{3\alpha}{4})^2 h(x) \leq f(\bar{x}) \leq \delta \frac{m^2}{h^2} a^2\), which implies

\[
\sup_{x \in \hat{B}(x_0, \frac{3\alpha}{4})} h(x) \leq 16\delta \frac{m^2}{h^2}.
\]

So we may assume case 2) always holds, and we will show that \(f(x) \leq C \delta\) for some constant \(C\delta\) depending only on \(\delta\). Note that

\[
\kappa' |\nabla (\phi \circ \phi)| \leq 2\kappa' \frac{m^2}{h^2} |d\phi| |\phi| \leq \kappa' \frac{m^2}{h^2} V + \kappa' |d\phi|^2.
\]

We will argue by contradiction. Suppose there is a sequence of 4-Lorentzian manifolds \((M_t, \tilde{g}_t)\) satisfying the equations (1.3) (1.4), and a sequence of \(\hat{g}_t\)-balls
\( \hat{B}(x_l, a_l) \subset M_l \) with compact closure such that \( f(\hat{x}_l) \to \infty \) as \( l \to \infty \), where

\[
f(\hat{x}_l) = \sup_{x \in \hat{B}(x_l, a_l)} h_l(x) d_{\hat{g}_l}(x, \partial \hat{B}(x_l, a_l))
\]

\[
h_l(x) = 2|\nabla \log u_l|^2 + \frac{1}{2} u_l^{-4}|\omega_l|^2 - \kappa u_l^{-2}|E_l|^2 - \kappa_l u_l^{-2}|B_l|^2 + \Lambda_l + \kappa'_l \frac{1}{2} V_l.
\]

(2.48)

Scaling \( u_l \) and \( \hat{g}_l \) by \( u_l(\hat{x}_l)^{-1} \) and \( h_l(\hat{x}_l) \) respectively, one can assume \( u_l(\hat{x}_l) = 1 \) and \( h_l(\hat{x}_l) = 1 \). We still use the same notations \( u_l, \omega_l, E_l, B_l, h_l, \hat{g}_l, \), etc. to denote the corresponding scaled quantities. For any fixed \( 0 < \epsilon < 1 \), any \( x \in \hat{B}(x_0, a_l) \) with

\[
d_{\hat{g}_l}(x, \hat{x}_l) \leq \epsilon f(\hat{x}_l)^\frac{1}{2} h_l^{-\frac{1}{2}}(\hat{x}_l) = \epsilon d_{\hat{g}_l}(\hat{x}_l, \partial \hat{B}(x_0, a_l)),
\]

we have \( d_{\hat{g}_l}(x, \partial \hat{B}(x_0, a_l)) \geq (1 - \epsilon)d_{\hat{g}_l}(\hat{x}, \partial \hat{B}(x_0, a_l)) \), hence

\[
h_l(x) \leq \frac{1}{(1 - \epsilon)^2} h_l(\hat{x}_l).
\]

(2.49)

If there is no ambiguity, we can also omit the subscript \( l \) from \( h_l, u_l, \) etc. It can be shown from (2.49):

\[
h(x) \leq 4 \quad \text{on } \hat{B}(\bar{x}, D).
\]

(2.50)

where \( D = \frac{1}{2} \sqrt{f(\bar{x})} \). In the following, we estimate \( \kappa'|d\phi|^2 \) on \( \hat{B}(\bar{x}, \frac{D}{2}) \). From (2.10) (2.11) (2.14) and (2.4), we know

\[
u^{-2} \hat{R}ic(X, X) \geq \frac{\kappa}{2} u^{-2}(|E|^2 + |B|^2)
\]

\[
u^{-1} \hat{R}ic(X, e_j) = \kappa u^{-2} i_{e_j} (E \wedge B)
\]

\[
\hat{R}ic(e_i, e_j) \geq \frac{\kappa}{2} u^{-2}(|E|^2 + |B|^2) g_{ij} - u^{-4} |\omega|^2 g_{ij}.
\]

(2.51)

Together with (2.50) and (2.48), we get

\[
\hat{R}ic \geq -2\hat{g} \hat{g} \geq -8\hat{g} \quad \text{on } \hat{B}(\bar{x}, D).
\]

(2.52)

Note that we have the Bochner formula:

\[
\hat{\Delta} \kappa'|d\phi|^2 = 2\kappa' \langle \hat{R}ic, d\phi \otimes d\phi \rangle + 2\kappa'|\nabla d\phi|^2 + 2\kappa' \frac{m^2}{h^2} |d\phi|^2
\]

\[
\geq -2\kappa'|d\phi \wedge u^{-2} \omega|^2 + \kappa \kappa' u^{-2} (|E|^2 + |B|^2) |d\phi|^2 + \frac{2}{3} \kappa'^2 |d\phi|^4.
\]

(2.53)

Let \( \xi : [0, \infty) \to [0, 1] \) be a fixed nonnegative smooth non-increasing function such that \( \xi = 1 \) on \( [0, \frac{1}{2}] \) and \( \xi = 0 \) on \( [1, \infty) \). Consider the function \( L(x) = \xi(\frac{d_{\hat{g}}(\hat{x}, x)}{D})(\kappa'|d\phi|^2) \) which is nonnegative and vanishes on \( \partial \hat{B}(\bar{x}, D) \). So \( L(x) \) assumes its maximum at some point \( \hat{x} \in B(\bar{x}, D) \). We temporarily assume \( \hat{x} \) is a smooth point of \( d_{\hat{g}}(\hat{x}, \cdot) \). If \( d_{\hat{g}}(\hat{x}, \hat{x}) \geq \frac{D}{2} \), by (2.53) and Laplacian comparison theorem (see Corollary 1.2 in [11]), one can show

\[
\hat{\Delta} d_{\hat{g}}(\hat{x}, \cdot) \big|_{x=x_{\hat{x}}} \leq \frac{3}{d_{\hat{g}}(\hat{x}, \hat{g})} + 3 \sqrt{\frac{8}{3}} \leq \frac{6}{D} + 6,
\]

(2.54)
hence

\[ 0 \geq \tilde{\Delta} L(x) \mid_{x=\hat{x}} = \xi \tilde{\Delta}(\kappa'|d\phi|^2) - \frac{2}{D^2} \frac{\xi}{\xi} (\kappa'|d\phi|^2) + (\kappa'|d\phi|^2)(\frac{1}{D^2} \xi'' + \frac{1}{D} \xi \tilde{\Delta}d_g) \leq \frac{2}{3} (\kappa'|d\phi|^2) \xi - |\kappa|u^{-2}(|B|^2 + |E|^2) + 2u^{-4}|\omega|^2 + CD^{-1} \kappa'|d\phi|^2 \]

\[ \geq \frac{2}{3} (\kappa'|d\phi|^2) \xi - 16 \kappa'|d\phi|^2 - CD^{-1} \kappa'|d\phi|^2, \]

where we have used (2.48) and (2.50). If \( \hat{x} \) lies in the cut locus of \( \bar{x} \), we can use a standard support function technique as in [14] (or Theorem 3.1 in [11]) to prove that (2.55) still holds.

Multiplying both sides of (2.55) by \( \xi(\hat{x}) \), we find

\[ L(\hat{x})^2 - CL(\hat{x}) \leq 0. \]

which implies \( L(\hat{x}) \leq C \). Therefore,

\[ \sup_{x \in \hat{B}(\bar{x}, \frac{L}{2})} \kappa'|d\phi|^2 \leq C \]  \hspace{1cm} (2.56)

where \( C \) is a universal constant. Combining (2.50) and (2.56), the estimate (2.47) becomes

\[ \kappa'|\nabla(V \circ \phi)| \leq 8\delta^{-1} + C, \quad \text{on} \quad \hat{B}(\bar{x}, \frac{D}{2}), \]  \hspace{1cm} (2.57)

because the quantity \( \frac{m^2}{h^2} \) in (2.47) has been changed to \( \frac{m^2}{h^2} h(\bar{x})^{-1} \leq \delta^{-1} \) after scaling.

From (2.16) and (2.50) (2.57), the Ricci curvature of \( \hat{g} \) is uniformly bounded (independent of \( l \)) on \( \hat{B}(\bar{x}, 1) \). As in the proof of Theorem 5.3 in [2], one can use the horizontal exponential map (w.r.t. metric \( u_l^2 \hat{g}_l \)) to pull back \( \tilde{g}, \omega_l, E_l, B_l, u_l \) to horizontal tangent space, moreover, using [6], one can construct a \( \hat{g} \)-harmonic coordinate system \( \{z^i\} \) (of uniform size \( \{|z| < \delta_1\} \), \( \delta_1 \) independent of \( l \)) on the horizontal tangent space around \( \bar{x}_l \), so that the \( C^{1,\alpha} \)-norm (w.r.t. \( \{z^i\} \) coordinates) of \( \hat{g} \) is uniformly bounded, see Theorem 5.3 (5.31) in [2].

Note that \( u_l, \omega_l, E_l, B_l, \phi_l \) satisfy the following elliptic type equations (see (2.6) (2.19) (2.21)):

\[ u^2 \tilde{\Delta} \log u = -\frac{1}{2} u^{-4} |\omega|^2 + \kappa(\frac{1}{2} u^{-2} |E|^2 + \frac{1}{2} u^{-2} |B|^2) - (\Lambda + \frac{\kappa'}{2} V) \]

\[ u^2 \tilde{\Delta} \phi_3 = 2(d \log u, E) + u^{-2} (\omega, B) \]

\[ u^2 \tilde{\Delta} \phi_4 = 2(d \log u, B) + u^{-2} (\omega, E) \]

\[ u^2 \hat{g}^{kl} \tilde{\nabla}_k \omega_l = 4 \hat{g}^{kl} \omega_k \tilde{\nabla}_l \log u \]

\[ d\omega = \pm 2\kappa \ast (E \wedge B) \]

\[ u^2 \tilde{\Delta} \phi = \frac{m^2}{h^2} \phi \]

where \( 0 \leq \frac{m^2}{h^2} \leq \delta^{-1} \). In the following, we say the \( C^{k,\alpha} \)-norms of quantities \( F_l \) are uniformly bounded, if for any \( 0 < \delta_2 < \delta_1 \), there is a constant \( C_{\delta_2, k, \alpha} \) independent
of \( l \) such that \( |F_l|^{C^k, \alpha} (\{ |l| \leq \delta_2 \}) \leq C_{\delta_2, k, \alpha} \). From the boundedness of \( h_t \), using equations 1-5 in (2.58) and elliptic regularity, one can show that the \( C^{1, \alpha} \)-norms of \( \log u_t \) are uniformly bounded, hence the \( C^{1, \alpha} \)-norms of \( \sqrt{|\mathcal{K}|} E_t, \sqrt{|\mathcal{K}|} B_t, \omega_t \) are also uniformly bounded. Combining (2.57) and the first equation in (2.58), we know the \( C^{2, \alpha} \)-norms of \( \log u_t \) are uniformly bounded. Applying equations 2-5 in (2.58) again, the \( C^{2, \alpha} \)-norms of \( \sqrt{|\mathcal{K}|} E_t, \sqrt{|\mathcal{K}|} B_t, \omega_t \) are also uniformly bounded.

By differentiating the last equation of (2.58), we get

\[
\ddot{\Lambda}(\sqrt{|\mathcal{K}|} \delta) = \frac{m^2}{\hbar^2} \sqrt{\mathcal{K}} (u^{-2} \phi)_i + \ddot{R}_{ik} \sqrt{\mathcal{K}} \phi^i \phi^k.
\]

Since \( \sqrt{|\mathcal{K}|} \delta \), \( \ddot{R}_{ij} \) are uniformly bounded and \( \frac{m^2}{\hbar^2} \leq \delta^{-1}, \frac{\gamma^2}{\hbar^2} \leq 8 \), by \( L^p \)-estimate for elliptic equations, one can show the \( C^{1, \alpha} \)-norms of \( \sqrt{\mathcal{K}} \delta \phi_t \) are uniformly bounded. So the \( C^{3, \alpha} \)-norms of \( \ddot{\mathcal{G}}_t \) are uniformly bounded (using the harmonic coordinates \( \{ z^i \} \) in (2.16)). By repeating the above arguments, one can show that for any \( k \in \mathbb{Z}_+ \), the \( C^{k, \alpha} \)-norms of \( \log u_t, \omega_t, -\sqrt{|\mathcal{K}|} E_t, -\mathcal{K} B_t, \mathcal{K} d\omega_t, \mathcal{K} V_t \) are uniformly bounded. So we can extract a smooth convergent subsequence so that the limit \( u^\infty, \ddot{\mathcal{G}}^\infty, (\sqrt{|\mathcal{K}|} E)^\infty, (\sqrt{|\mathcal{K}|} B)^\infty, (\sqrt{|\mathcal{K}|} d\omega)^\infty, (\mathcal{K} V)^\infty \in C^\infty (|z| < \delta_1) \). Note that the condition (1.8) imply that the limit must satisfy \( (\sqrt{|\mathcal{K}|} E)^\infty \wedge (\sqrt{|\mathcal{K}|} B)^\infty = 0 \). Hence equation (2.45) holds for the limit. The smooth convergence of \( h_t \) and (2.49) imply that \( h^\infty(x) \) will achieve its maximum (1) at the origin \( z^i = 0 \). Since each term in the right hand side of (2.45) is nonnegative, we can apply the strong maximum principle for the limit. This implies that \( h^\infty \equiv \text{const.} = 1 \) and each term on the right hand side of (2.45) vanishes. From \( |\nabla \log u^\infty|^4 = 0 \), we know \( u^\infty \equiv \text{const.} = 1 \). Combining with \( |2\nabla_{ij} \log u + u^{-4}(\omega_j \omega_j - \kappa u^{-2}(E_i E_j + B_i B_j))|^2 \), we find \( (\sqrt{|\mathcal{K}|} E)^\infty = (\sqrt{|\mathcal{K}|} B)^\infty = \omega^\infty = 0 \). From the expression of \( h \), we find \( \mathcal{L} + \frac{1}{2} \mathcal{K} V^\infty \equiv 0 \). The first equation in (2.58) for the limit will give a contradiction: \( 0 = (u^\infty)^2 \ddot{\Lambda} \log u^\infty = -\mathcal{L} - \frac{1}{2} \mathcal{K} V^\infty = -1 \). In conclusion, we have proved

\[
\sup_{\tilde{B}(x_0, \frac{\delta}{8})} h(x) \leq 16 \delta \frac{m^2}{\hbar^2} + C_{\delta} a^{-2}.
\]

From the above proof (on the regularity of \( \log u \)), \( \sup_{\tilde{B}(x_0, \frac{\delta}{8})} |u^{-1} \nabla^2 u| + u^{-2} |\nabla \omega| \) can also be bounded by \( C \delta \frac{m^2}{\hbar^2} + C_{\delta} a^{-2} \). Now combining (2.10) in [2] and formula (2.2), we know \( \sup_{\tilde{B}(x_0, \frac{\delta}{2})} |Rm|_g \leq C \delta \frac{m^2}{\hbar^2} + C_{\delta} a^{-2} \). The estimate (1.8) follows by redefining the constants.

If \( \partial \tilde{B}(x_0, a) \) is empty, by using strong maximum principle on equation (2.45), we conclude that \( Rm \equiv 0 \), which fulfills the estimate (1.8).

\[\square\]

**Proof.** Of Theorem 1.2 By Theorem 3.3 in [2], \((M, \dot{g})\) is geodesically complete. First, let \( a \to \infty \) in (1.8), we find \( |Rm|_g \leq \delta \). Let \( \delta \to 0 \), we find \( Rm \equiv 0 \), i.e., \((M, g_M)\) is flat. \[\square\]

### 3 Higher dimensional static fields

In this section, we generalize our estimate to general dimensions. We may assume the scalar fields can take their values on some manifold. Let \((M, g_M)\) be a spacetime of
dimension $n+1$, $(W, g_W)$ a Riemannian manifold equipped with a Riemannian metric $g_W$, $V : W \to R$ be a fixed function. We consider the (uncharged) Einstein-Maxwell-Klein-Gordon equation, which is the following system of differential equations on the Lorentzian metric $g_M$, a 2-form $F$ on $M$, and a map (scalar field) $\phi : M \to W$:

$$
Ric(g_M) - \frac{R}{2} g_M + \Lambda g_M = \kappa (F_{\alpha\gamma} F_{\beta\delta} g^{\gamma\delta} - \frac{1}{4} |F|^2 g_{\alpha\beta})
+ \kappa' [\phi^* g_W - \frac{1}{2} (|d\phi|^2 + V(\phi)) g_M]
\tag{3.1}
$$

$$
dF = d^* F = 0
$$

$$
\triangle_{g_M, g_W} \phi = \frac{1}{2} (\nabla V) \circ \phi
$$

where $\Lambda, \kappa, \kappa'$ are constants, $\triangle_{g_M, g_W} \phi$ is the harmonic map Laplacian of $\phi$. The third equation in (3.1) may be regarded as a generalized Klein-Gordon equation. Let $X$ be a timelike Killing field on $(M, g_M)$, as before, we say the solution $(g_M, F, \phi)$ to (3.1) is inheriting, if $F$ and $\phi$ satisfies

$$
\mathcal{L}_X (F) = 0, \quad d\phi (X) = 0. \tag{3.2}
$$

**Theorem 3.1** Let $(M, g_M)$ be a static spacetime of dimension $n+1 \geq 4$ with a timelike Killing field $X$, $\phi : M \to W$ a map. We assume

i) $(g_M, F, \phi)$ satisfies the equations (3.1) (3.2) with $\Lambda \geq 0, \kappa \leq 0, \kappa' \geq 0$;

ii) the magnetic field $B = i_X F$ vanishes;

iii) $(W, g_W)$ has nonpositive sectional curvature;

iv) the function $V : W \to R$ is nonnegative and convex, i.e., $\text{Hess}(V) \geq 0$.

Let $\hat{B}(x_0, a)$ be a $\hat{g}$-metric ball centered at $x_0$ of radius $a > 0$ with compact closure in $M$. Then there is a universal constant $C > 0$ such that

$$
\sup_{x \in \hat{B}(x_0, \frac{a}{2})} |\nabla \log u|_{\hat{g}}^2 + |\kappa| u^{-2} |E|^2 + \kappa' n^{-1} |d\phi|^2 \leq C n a^{-2}, \tag{3.3}
$$

where $E = i_X F$ is the electric field.

Before the proof of Theorem 3.1 we have to mention that under the condition that the spacetime is diffeomorphic to $N^n \times R$ so that (2.1) holds globally on $N^n \times R$, analogous result as in Theorem 3.1 on static Einstein-scalar equation has been obtained in [9].

Taking trace on the first equation of (3.1), we get

$$
R = 2 \frac{n+1}{n-1} (\Lambda + \frac{\kappa'}{2} V) + \kappa \frac{n-3}{2(n-1)} |F|^2 + \kappa' |d\phi|^2.
$$

Substituting it into (3.1), we find

$$
R_{\alpha\beta} = \kappa F_{\alpha\gamma} F_{\beta\delta} g^{\gamma\delta} + \kappa' (d\phi \otimes d\phi)_{\alpha\beta} + \frac{1}{n-1} (2\Lambda + \kappa' V - \frac{\kappa}{2} |F|^2) g_{\alpha\beta}, \tag{3.4}
$$
where \( |F|^2 = F^a_c F_{a'b'} g^{c \gamma} g^{d \delta} = 2u^{-2}(|B|^2 - |E|^2) = -2u^{-2}|E|^2 \). Let \( \{x^\alpha\} \) be a local coordinate system so that (2.1) holds. From (3.4) and (2.13), one can show
\[
\bar{Ric}(X, X) = \kappa \frac{n-2}{n-1} |E|^2 - \frac{2\Lambda + \kappa'}{n-1} u^2
\]
\[
\bar{Ric}(X, e_j) = 0
\]
\[
\bar{Ric}(e_i, e_j) = \kappa (-u^{-2}E_i E_j + \frac{1}{n-1} u^{-2} |E|^2 g_{ij}) + \frac{2\Lambda + \kappa'}{n-1} g_{ij}
\]
\[
+ \kappa'(d\phi(e_i), d\phi(e_j)).
\]

The function \( u \) satisfies
\[
\hat{\Delta} \log u = u^{-2} \bar{Ric}(X, X) = \kappa \frac{n-2}{n-1} |E|^2 u^{-2} - \frac{2\Lambda + \kappa'}{n-1}.
\]

The Maxwell equation \( dF = d\ast F = 0 \) is now equivalent to
\[
E = d\phi_3
\]
\[
\hat{\Delta} \phi_3 = 2 \langle d \log u, E \rangle
\]
for some locally defined function \( \phi_3 \). We consider a map \( \Phi = (y_1, y_2) = (u^2, \phi_3) \) from the coordinate system \( \{x^\alpha\} \) to a 2-d upper half space \( H = \{(y_1, y_2) : y_1 > 0\} \) equipped with a metric \( g_H = y_1^{-2} dy_1^2 - 4\kappa \frac{n-2}{n-1} y_1 dy_2^2 \). By similar computations as in Lemma 2.2 (2.28), one can show:
\[
(\nabla_{\alpha\beta} \Phi)^1 = 2u^2 \nabla_{\alpha\beta} \log u - 2 \kappa \frac{n-2}{n-1} E_\alpha E_\beta
\]
\[
(\nabla_{\alpha\beta} \Phi)^2 = \nabla_{\alpha\beta} \phi_3 - (\log u)_\alpha E_\beta - (\log u)_\beta E_\alpha.
\]

It implies that \( \hat{\Delta} \Phi = -\frac{4\Lambda + 2\kappa'}{n-1} u^2 \frac{\partial}{\partial y_{\alpha\beta}} \), where we have used (3.6) and (3.7).

By similar computations as in (2.34) (2.36) (2.37) (2.40) and (2.41), we have
\[
\langle \bar{Ric}(\Phi^* g_H) \rangle = \left[ \bar{Ric}(e_k, e_l) g^{ik} g^{jl} \right] \times \left[ 4 \frac{u_i u_j}{u^2} - 4 \kappa \frac{n-2}{n-1} u^{-2} E_i E_j \right]
\]
\[
= \left[ \frac{2\Lambda + \kappa'}{n-1} - \kappa \frac{n-2}{n-1} u^{-2} |E|^2 \right] e(\Phi) + 4\kappa |u^{-1} E \land d \log u|^2
\]
\[
+ 4\kappa' |d\phi(\nabla \log u)|^2 - 4 \kappa \frac{n-2}{n-1} \kappa' d\phi(\kappa^{-1} E^\#)^2,
\]
\[
|\nabla^2 \Phi|^2 = 4|\nabla \log u|^4 + 4|\nabla_{ij} \log u - \kappa \frac{n-2}{n-1} u^{-2} E_i E_j|^2
\]
\[
- 4 \kappa \frac{n-2}{n-1} \kappa |(d \log u, u^{-1} E)|^2 + u^{-2} |\nabla_i E_j - E_i (\log u)_j - E_j (\log u)_i|^2
\]
\[
- R_{abcd} \Phi^a \Phi^b \Phi^c \Phi^d g^{\gamma \delta} \bar{g}^{\gamma \delta} = -8 \kappa \frac{n-2}{n-1} |d \log u \land u^{-1} E|^2,
\]
\[
\langle \nabla \Phi, \nabla \hat{\Delta} \Phi \rangle = -8 \kappa \frac{n-2}{(n-1)^2} \kappa (\Lambda + \frac{\kappa'}{2} u^{-2} |E|^2 - \frac{4\kappa'}{n-1} (\nabla (V \circ \phi), \nabla \log u), (3.12)
\]
where $E^#$ is the vector field obtained by lifting the indices of $E$. Combining (3.9), (3.10), (3.11), (3.12) and the Bochner formula (2.31), we have

$$\Delta (|\nabla \log u|^2 - \kappa \frac{n-2}{n-1} u^{-2} |E|^2)$$

$$= 2|\nabla \log u|^4 + 2|\nabla_{ij} \log u - \kappa \frac{n-2}{n-1} u^{-2} E_{ij}|^2 - 2\kappa (|d \log u, u^{-1} E|^2$$

$$+ |u^{-1} \nabla_i E_j - u^{-1} E_i (\log u)_j - u^{-1} E_j (\log u)_i|^2) - 2\kappa \frac{n-2}{n-1} \kappa' |d \phi (u^{-1} E^#)|^2$$

$$- 2\kappa \frac{n-3}{n-1} |d \log u \wedge u^{-1} E|^2 - \kappa (4\Lambda + 2\kappa' V) \frac{n-2}{(n-1)^2} u^{-2} |E|^2$$

$$+ (4\Lambda + 2\kappa' V) - 2\kappa \frac{n-2}{n-1} u^{-2} |E|^2 (|\nabla \log u|^2 - \kappa \frac{n-2}{n-1} u^{-2} |E|^2) + I$$

(3.13)

where

$$I = 2\kappa' |d \phi (\nabla \log u)|^2 - \frac{2\kappa'}{n-1} \langle \nabla (V \circ \phi), \nabla \log u \rangle \geq - \frac{\kappa'}{2(n-1)^2} |\nabla V|^2.$$

Each term in the right hand side of (3.13) is nonnegative except $I$. Note that the sectional curvature of $(W, g_W)$ is assumed to be nonpositive and $V$ is convex, we have:

$$\Delta \kappa' |d \phi|^2 = 2\kappa' \langle R\nabla \phi^* g_W - 2\kappa' R_{abcd} \phi^a \phi^b \phi^c \phi^d g^{\alpha \gamma} g'^{\beta \delta}$$

$$+ 2\kappa' |\nabla d \phi|^2 + \kappa' \langle \phi \cdot Hess (V), \hat{g} \rangle$$

$$\geq 2\kappa' \langle \hat{R} \nabla \phi^* g_W + 2\kappa' |\nabla V|^2.$$

Since

$$\langle \hat{R} \nabla \phi^* g_W \rangle = \kappa' \langle d \phi (e_i), d \phi (e_j) \rangle + \frac{(2\Lambda + 2\kappa' V)}{n-1} |d \phi|^2$$

$$+ \kappa [-u^-2 |d \phi (E^#)|^2 + \frac{1}{n-1} u^{-2} |E|^2 |d \phi|^2]$$

(3.15)

$$|\nabla d \phi|^2 \geq \frac{1}{n+1} |\Delta_{g_{g_W}} \phi|^2 = \frac{1}{n+1} |\Delta_{g_{g_W}} \phi|^2 = \frac{1}{4(n+1)} |\nabla V|^2,$$

we obtain

$$\Delta \kappa' |d \phi|^2 \geq 2\kappa' \kappa^2 |d \phi|^4 + \frac{2 \Lambda + 2\kappa' V}{n-1} |d \phi|^2 + \frac{\kappa'}{2(n+1)} |\nabla V|^2$$

$$+ \frac{\kappa' 2}{n-1} u^{-2} |E|^2 |d \phi|^2$$

$$\geq \frac{1}{8n \kappa^2} |d \phi|^4 - \frac{8n}{15(n-1)^2} (\kappa' u^{-2} |E|^2)^2 + \frac{\kappa'}{2(n+1)} |\nabla V|^2.$$  

(3.16)

Combining (3.16) and (3.13), we get

$$\Delta (|\nabla \log u|^2 - \kappa \frac{n-2}{n-1} u^{-2} |E|^2 + \frac{n+1}{(n-1)^2} \kappa' |d \phi|^2)$$

$$\geq 2|\nabla \log u|^4 + \frac{2(n-2)^2}{5(n-1)} \kappa^2 (u^{-2} |E|^2)^2 + \kappa^2 \frac{n+1}{8n(n-1)} |d \phi|^4$$

$$\geq \frac{1}{72} (|\nabla \log u|^2 - \kappa \frac{n-2}{n-1} u^{-2} |E|^2 + \frac{n+1}{(n-1)^2} \kappa' |d \phi|^2)^2,$$

(3.17)
where we have used $2\left(\frac{n-2}{n-1}\right)^2 - \frac{8n}{n^2(n-1)^2} \geq \frac{2}{3}\left(\frac{n-2}{n-1}\right)^2$ and $\frac{1}{8n} \geq \frac{1}{24(n-1)^2}$, for $n \geq 3$.

**Proof.** of Theorem 3.1

Now we are ready to prove the estimate (3.3). We assume $\partial\bar{B}(x_0, a)$ is not empty. Let $h(x) = \left|\nabla \log u\right|^2(x) - \kappa \frac{n-2}{n-1} u^{-2} |E|^2 + \kappa' \frac{n+1}{n-1} |\phi|^2$, $f(x) = h(x)\bar{g}(x, \partial\bar{B}(x_0, a))$, and $f(\bar{x}) = \sup_{x \in \bar{B}(x_0, a)} f(x)$ for some $\bar{x} \in \bar{B}(x_0, a)$.

Note that the function $f(x)$ is invariant under the scaling of the metric. Scaling $u$ and the metric $\hat{g}$ by $u(\bar{x})^{-1}$ and $h(\bar{x})$, we may assume $u(\bar{x}) = 1$ and $h(\bar{x}) = 1$. It is not hard to prove (see (2.49) (2.50)):

$$h(x) \leq 4 \text{ on } \hat{B}(\bar{x}, D),$$

(3.18)

where

$$D = \frac{1}{2} \sqrt{f(\bar{x})}.$$  

(3.19)

From (2.16) in [2], we have

$$\hat{\text{Ric}}(e_0, e_0) = -\bar{\text{Ric}}(e_0, e_0)$$

$$\hat{\text{Ric}}(e_0, e_j) = -\bar{\text{Ric}}(e_0, e_j) = 0$$

(3.20)

$$\hat{\text{Ric}}(e_i, e_j) = \bar{\text{Ric}}(e_i, e_j).$$

Together with (3.5) (3.18), we get

$$\hat{\text{Ric}} \geq -\frac{1}{n-2} h \hat{g}$$

(3.21)

hence

$$\hat{\text{Ric}} \geq -\frac{4}{n-2} \hat{g} \text{ on } \hat{B}(\bar{x}, D).$$

(3.22)

Let $\xi : [0, \infty) \to [0, 1]$ be a fixed nonnegative smooth non-increasing function such that $\xi = 1$ on $[0, \frac{1}{2}]$ and $\xi = 0$ on $[1, \infty)$. Set $L(x) = \xi(\frac{d_\hat{g}(\bar{x}, x)}{D}) h(x)$, and $L(\bar{x}) = \sup_{x \in \hat{B}(\bar{x}, D)} L(x)$ for some $\bar{x} \in \hat{B}(\bar{x}, D)$. From (3.17), we know that $h$ satisfies

$$\Delta h \geq \frac{1}{12} h^2.$$  

(3.23)

Using (3.22) (3.23) and applying maximum principle on $L(x)$ at $\hat{x}$ as in (2.54) (2.55) (2.56), we find at $\hat{x}$:

$$0 \geq \frac{1}{12} \xi h^2 - \frac{2}{D^2} \left(\xi'\right)^2 h$$

$$+ h\left[\frac{1}{D^2} \left(\xi'' + 2n\xi'\right) + \frac{2}{D} \xi' \sqrt{\frac{n}{n-2}}\right].$$

(3.24)

Multiplying both sides of (3.24) by $\xi(\hat{x})$, we find

$$\frac{1}{12} L(\hat{x}) - \frac{Cn}{D^2} - \frac{C}{D} \leq 0,$$

which implies

$$1 \leq C\left(\frac{n}{D^2} + \frac{1}{D}\right),$$

(3.25)
since $L(\hat{x}) \geq L(\bar{x}) = 1$. Hence $D \leq C\sqrt{n}$ for some universal constant $C$. From (3.19), we get the desired estimate $f(\bar{x}) \leq Cn$.

If $\partial B(x_0,a)$ is empty, we can apply the strong maximum principle on equation (3.17) directly to conclude $h \equiv 0$. 

\[ \square \]

A corollary of Theorem 3.1 is the following theorem, which is a generalization of Theorem 1.2 in [2].

**Theorem 3.2** Let $(M, g_M)$ be a static spacetime of dimension $n + 1$ satisfying the assumptions in Theorem 3.1. If we assume $(M, g_M)$ is geodesically complete, then the universal cover of $(M, g_M)$ is isometric to a product $R \times N$ equipped with a product metric $-dt^2 + g_N$, where $g_N$ is a complete Ricci flat Riemannian manifold.

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