Gauss Decomposition, Wakimoto Realisation and Gauged WZNW Models

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Abstract

The implications of gauging the Wess-Zumino-Novikov-Witten (WZNW) model using the Gauss decomposition of the group elements are explored. We show that, contrary to standard gauging of WZNW models, this gauging is carried out by minimally coupling the gauge fields. We find that this gauging, in the case of gauging an abelian vector subgroup, differs from the standard one by terms proportional to the field strength of the gauge fields. We prove that gauging an abelian vector subgroup does not have a nonlinear sigma model interpretation. This is because the target-space metric resulting from the integration over the gauge fields is degenerate. We demonstrate, however, that this kind of gauging has a natural interpretation in terms of Wakimoto variables.

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1 Introduction

Wess-Zumino-Novikov-Witten (WZNW) theories provide a unifying framework for a large class of conformally invariant models in two dimensions. They furnish a Lagrangian realisation for the Kac-Moody algebra which is the building ground for many other theories [1]. The WZNW models are each associated with a Lie algebra. An important tool in establishing the connection between the WZNW and other theories resides in what is known as the Gauss decomposition.

The essence of this decomposition consists in expressing locally a generic element in the Lie group as a product of three matrices: A lower triangular matrix corresponding to the step operators of the negative roots, a diagonal matrix corresponding to the Cartan subalgebra and an upper triangular matrix corresponding to the step operators of the positive roots.

The Gauss decomposition has also been used to find a free field representation for the WZNW models with arbitrary Kac-Moody algebra [2]. The generators of the Kac-Moody current algebra are then realised as functions of a set of free scalar fields together with a $\beta\gamma$ system, where $\beta$ is a one-form while $\gamma$ is a zero-form [2,3]. This realisation is known as the Wakimoto representation [3] and has the consequence that the correlation functions of the WZNW model are expressed as some linear combinations of the correlation functions corresponding to these free fields.

The Gauss decomposition is also at the heart of connecting Toda theories (including Liouville theory) to gauged WZNW models [4-6]. In this decomposition, Toda theories arise as gauge-fixed versions of gauged WZNW actions upon imposing some constraints on parts of the Kac-Moody currents [5]. On the other hand, gauged WZNW models provide a way of describing the motion of strings on geometrically nontrivial backgrounds [7]. These backgrounds are sometimes singular. It is therefore crucial to understand the gauged WZNW model in the Gauss decomposition. This is the aim of this paper.

In section two, we find all the global symmetries that respect the Gauss decomposition and leave the WZNW action invariant. We find that gauging these symmetries is carried out by minimally coupling the gauge fields. In other words, all ordinary derivatives are replaced by covariant derivatives. This is in contrast to standard gauging of the WZNW model where minimal coupling does not lead to a gauge invariant action. We show that, when gauging abelian subgroups, the axial gauging in the Gauss decomposition is identical to the standard gauged WZNW model. On the other hand, the abelian vector gauging
in the Gauss decomposition differs from the standard gauged WZNW model by a term proportional to the field strength of the gauge fields.

We show also that the vector gauging in the Gauss decomposition does not lead to a nonlinear sigma model. This is because the metric is degenerate. We explicitly prove this in the case of the group $SL(2, R)$ and $SL(3, R)$. The general case of the group $SL(N, R)$ is finally studied in details.

Although this vector gauging does not have a nonlinear sigma model model interpretation we show, in section four, that it does have a natural interpretation in terms of Wakimoto variables. We prove that the gauging of $r$ abelian $U(1)$ symmetries in the Wakimoto representation reduces the number of degrees of freedom by $2r$. In section three we give a brief review of the Wakimoto variables and trace back their origin to the WZNW model.

2 Gauging in the Gauss Decomposition

In this paper, we specify a Lie algebra $G$ by its Cartan subalgebra $H$ in a basis $\{H_i, \, i = 1, \ldots, r_G\}$, where $r_G$ is the rank of $G$, together with a set of step operators $\{E_{\alpha}, \, E_{-\alpha}\}$ corresponding to the set of positive roots $\alpha \in \Phi^+$. We denote by $M_G$ the Lie group whose Lie algebra is $G$. The action for the WZNW model defined on the group manifold $M_G$ is given by

$$I(g) = \frac{k}{8\pi} \int_{\partial B} d^2x \sqrt{-\gamma} \gamma^{\mu\nu} Tr \left( g^{-1} \partial_{\mu} g \right) \left( g^{-1} \partial_{\nu} g \right) + \frac{ik}{12\pi} \int_{B} d^3y \epsilon^{\mu\nu\rho} Tr \left( g^{-1} \partial_{\mu} g \right) \left( g^{-1} \partial_{\nu} g \right) \left( g^{-1} \partial_{\rho} g \right),$$

where $g \in M_G$ and $B$ is a three-dimensional manifold whose boundary is the the two-dimensional surface $\partial B$.

Using the Polyakov-Wiegmann formula [8]

$$I(g_1g_2) = I(g_1) + I(g_2) + \frac{k}{4\pi} \int d^2x P_+^{\mu\nu} Tr \left( g_1^{-1} \partial_{\mu} g_1 \right) \left( \partial_{\nu} g_2 g_2^{-1} \right),$$

where it is convenient to define the quantities

$$P_+^{\mu\nu} = \sqrt{-\gamma} \gamma^{\mu\nu} + i\epsilon^{\mu\nu}, \quad P_-^{\mu\nu} = \sqrt{-\gamma} \gamma^{\mu\nu} - i\epsilon^{\mu\nu}$$

one can show that the WZNW action is invariant under

$$g(z, \bar{z}) \rightarrow \bar{\Omega}(\bar{z}) g(z, \bar{z}) \Omega(z).$$
Here \( z \) and \( \bar{z} \) are the complex coordinates on the world-sheet and our conventions are such that \( \gamma^{\bar{z}z} = 1 \) and the antisymmetric tensor is given by \( \epsilon^{\bar{z}z} = i \). The above transformations generate two commuting copies of a Kac-Moody algebra at level \( k \).

Let us assume that the group element \( g \) can be written according to the Gauss decomposition

\[
g = g_- g_0 g_+ ,
\]

where

\[
g_+ = \exp \left( \sum_{\alpha \in \Phi^+} \phi^\alpha_{-} E_{-\alpha} \right) , \quad g_+ = \exp \left( \sum_{\alpha \in \Phi^+} \phi^\alpha_{+} E_{\alpha} \right) , \quad g_0 = \exp \left( \sum_{i=1}^{r} \lambda_i H_i \right) .
\]

The fields \( \phi^\alpha_{-} \), \( \phi^\alpha_{+} \) and \( \lambda^i \) are to be interpreted as the coordinates of the target-space non-linear sigma model associated to the WZNW model. It is also convenient to think of \( g_- \), \( g_+ \) and \( g_0 \) as matrices. The matrices \( g_- \) and \( g_+ \) are, respectively, lower triangular and upper triangular matrices with units along the diagonals, while \( g_0 \) is a diagonal matrix.

A multiple use of the Polyakov-Wiegmann formula yields

\[
I(g) = I(g_0) + \frac{k}{4\pi} \int d^2 x \gamma^{\mu\nu} \text{Tr} \left( g_-^{-1} \partial_\mu g_- g_0 \partial_\nu g_+ g_+^{-1} g_0^{-1} \right) ,
\]

\[
I(g_0) = \frac{k}{8\pi} \int d^2 x \sqrt{-\gamma} \gamma^{\mu\nu} \text{Tr} \left( g_0^{-1} \partial_\mu g_0 \right) \left( g_0^{-1} \partial_\nu g_0 \right) .
\]

In deriving this expression we have made use of the fact that

\[
I(g_-) = I(g_+) = 0 ,
\]

\[
\text{Tr} \left( g_-^{-1} \partial_\mu g_- \right) \left( \partial_\nu g_0 g_0^{-1} \right) = \text{Tr} \left( g_0^{-1} \partial_\mu g_0 \right) \left( \partial_\nu g_+ g_+^{-1} \right) = 0 .
\]

This is because \( g_-^{-1} \partial_\mu g_- \) is a lower triangular matrix with zeros along the diagonal and when multiplied by a lower triangular matrix or a diagonal matrix gives a lower triangular matrix with zeros along the diagonal. Therefore the resulting matrix of this multiplication is always traceless. A similar argument applies to \( \partial_\mu g_+ g_+^{-1} \).

We would like now to look for transformations that are symmetries of the action (2.7) and which, at the same time, preserve the Gauss decomposition. Namely, transformations that take \( g_- \) and \( g_+ \) to matrices that are still, respectively, lower and upper triangular with ones along the diagonals and changes \( g_0 \) to another diagonal matrix. There are, indeed, two sets of such transformations. The first set consists of the transformations

\[
g_- (z, \bar{z}) \rightarrow U(\bar{z}) g_- (z, \bar{z}) U^{-1}(\bar{z})
\]

\[
g_0 (z, \bar{z}) \rightarrow U(\bar{z}) g_0 (z, \bar{z}) V^{-1}(z)
\]

\[
g_+ (z, \bar{z}) \rightarrow V(z) g_+ (z, \bar{z}) V^{-1}(z)
\]

(2.9)
where $U(\bar{z})$ and $V(z)$ are two diagonal matrices having the form

$$U(\bar{z}) = \exp \left( \sum_{i=1}^{r_1} u^i(\bar{z}) H_i \right), \quad V(z) = \exp \left( \sum_{i=1}^{r_2} v^i(z) H_i \right)$$

(2.10)

and $r_1$ and $r_2$ are smaller or equal to the rank of the Lie algebra $r^G$.

The second set of of transformations is given by

$$
\begin{align*}
g_-(z, \bar{z}) & \to \Omega_-(\bar{z}) g_-(z, \bar{z}) \\
g_0(z, \bar{z}) & \to g_0(z, \bar{z}) \\
g_+(z, \bar{z}) & \to g_+(z, \bar{z}) \Omega_+(z)
\end{align*}
\tag{2.11}
$$

The matrices $\Omega_-(\bar{z})$ and $\Omega_+(z)$ are, respectively, of the same nature as $g_-$ and $g_+$ and they are written as

$$
\begin{align*}
\Omega_-(\bar{z}) &= \exp \left( \sum_{\alpha \in \Phi_1^+} \omega^\alpha_-(\bar{z}) E_{-\alpha} \right), \\
\Omega_+(z) &= \exp \left( \sum_{\alpha \in \Phi_2^+} \omega^\alpha_+(z) E_{\alpha} \right)
\end{align*}
\tag{2.12}
$$

Here $\Phi_1^+$ and $\Phi_2^+$ are two subsets of $\Phi^+$, the set of positive roots.

Our aim is to make the matrices $U(\bar{z}), V(z), \Omega_-(\bar{z})$ and $\Omega_+(z)$ depend on both $z$ and $\bar{z}$. This necessitates the introduction of four gauge fields corresponding to the four different transformations. These we denote by $A_\mu, B_\mu, M_\mu, N_\mu (\mu = z, \bar{z})$ and they correspond, respectively, to the transformations generated by $U, V, \Omega_-, \Omega_+$. They are written as

$$
\begin{align*}
A_\mu &= \sum_{i=1}^{r_1} A^i_\mu H_i, \quad B_\mu = \sum_{i=1}^{r_2} B^i_\mu H_i \\
M_\mu &= \sum_{\alpha \in \Phi_1^+} M^\alpha_\mu E_{-\alpha}, \quad N_\mu = \sum_{\alpha \in \Phi_2^+} N^\alpha_\mu E_{\alpha}
\end{align*}
\tag{2.13}
$$

and they transform as

$$
\begin{align*}
A_\mu &\to A_\mu - U^{-1} \partial_\mu U, \quad B_\mu \to B_\mu - V^{-1} \partial_\mu V \\
M_\mu &\to \Omega_- M_\mu \Omega_-^{-1} - \Omega_-^{-1} \partial_\mu \Omega_-, \quad N_\mu \to \Omega_+^{-1} N_\mu \Omega_+ - \Omega_+^{-1} \partial_\mu \Omega_+
\end{align*}
\tag{2.14}
$$

where $U, V, \Omega_-$ and $\Omega_+$ are now functions of both $z$ and $\bar{z}$. The gauged WZNW action in the Gauss decomposition, for both sets of transformations, is found by minimally coupling the gauge fields (replacing ordinary derivatives by gauge covariant derivatives). The gauged WZNW action corresponding to the first set of transformations in (2.9) is found to be

$$I(g, A, B) = \frac{k}{8\pi} \int d^2x \sqrt{-\gamma} \gamma^\mu \gamma^\nu Tr \left( g_{\mu}^{-1} D_\mu g_0 \right) \left( g_{\nu}^{-1} D_\nu g_0 \right) + \frac{k}{4\pi} \int d^2x P^\mu \gamma^\nu Tr \left( g_{\mu}^{-1} D_\mu g_0 D_\nu g_0 + g_0^{-1} g_+ g_0^{-1} \right), \quad (2.15)$$
where

\begin{align*}
D_\mu g_0 &= \partial_\mu g_0 + A_\mu g_0 - g_0 B_\mu \\
D_\mu g_- &= \partial_\mu g_- + A_\mu g_- - g_- A_\mu \\
D_\mu g_+ &= \partial_\mu g_+ + B_\mu g_+ - g_+ B_\mu .
\end{align*}

(2.16)

On the other hand gauging the second set of transformations in (2.11), yields

\begin{align*}
I(g, M, N) &= \frac{k}{8\pi} \int d^2 x \sqrt{-\gamma} \gamma^{\mu\nu} Tr \left( g_0^{-1} \partial_\mu g_0 \right) \left( g_0^{-1} \partial_\nu g_0 \right) \\
&+ \frac{k}{4\pi} \int d^2 x P^{\mu\nu}_- Tr \left( g_-^{-1} D_\mu g_- g_0 D_\nu g_+ g_+^{-1} g_0^{-1} \right) ,
\end{align*}

(2.17)

where the covariant derivatives are now given by

\begin{align*}
D_\mu g_- &= \partial_\mu g_- + M_\mu g_- \\
D_\mu g_+ &= \partial_\mu g_+ + g_+ N_\mu .
\end{align*}

(2.18)

Let us now compare this kind of gauging to what is usually written down for gauged WZNW models. Let us start by exploring the abelian vector gauging. This gauging reflects the invariance of the gauged WZNW action under the transformations

\[ g \rightarrow hgh^{-1}, \quad A_\mu \rightarrow A_\mu - h^{-1} \partial_\mu h , \]

(2.19)

where \( h \) is a group element of the Cartan subalgebra. The usual gauged WZNW action is then [9]

\begin{align*}
I(g, A) = I(g) + \frac{k}{4\pi} \int d^2 x Tr \left( P^{-\mu\nu}_- A_\mu \partial_\nu g g^{-1} - P^{\mu\nu}_+ A_\mu g^{-1} \partial_\nu g \\
+ \sqrt{-\gamma} \gamma^{\mu\nu} A_\mu A_\nu - P^{\mu\nu} A_\mu g A_\nu g^{-1} \right) .
\end{align*}

(2.20)

In the Gauss decomposition, the transformations (2.19) would correspond to setting \( V = U = h \) and \( B_\mu = A_\mu = A_\mu \).

Using the Gauss decomposition for \( g \) in the action (2.20), we get

\begin{align*}
I(g, A) = I(g_0) + \frac{k}{4\pi} \int d^2 x P^{\mu\nu}_- Tr \left( g_-^{-1} D_\mu g_- g_0 D_\nu g_+ g_+^{-1} g_0^{-1} \right) \\
- i \frac{k}{2\pi} \int d^2 x \epsilon^{\mu\nu} Tr \left( A_\mu \partial_\nu g_0 g_0^{-1} \right) ,
\end{align*}

(2.21)

where the covariant derivatives are as written in (2.16) upon setting \( A_\mu = B_\mu = A_\mu \).

Notice that this last action differs from the gauged WZNW action (2.15) previously written for the Gauss decomposition. This difference is due to the presence of the last term in (2.21) and which can be written as

\[ i \frac{k}{2\pi} \int d^2 x \epsilon^{\mu\nu} Tr \left( H_i H_j \right) \lambda^i F^j_{\mu\nu} , \]

(2.22)
where we have used the expression for $g_0$ in (2.6) and for $A_\mu$ the expression (2.13). Here $F^{ij}_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu$ and the indices $i, j$ span the Cartan subalgebra that we are gauging. The appearance of terms proportional to the field strength of the gauge field in gauged nonlinear sigma models have already been advocated in refs.[10,11]. The inclusion of such terms in gauged WZNW models has many consequences on the geometrical interpretation of these models [12].

The other interesting abelian gauging is the axial gauging. This is characterised by the transformations

$$g \rightarrow hgh, \quad A_\mu \rightarrow A_\mu - h^{-1} \partial_\mu h.$$  \hspace{1cm} (2.23)

This is obtained in the Gauss decomposition by setting $V^{-1} = U = h$ and $A_\mu = -B_\mu = A_\mu$. The standard gauged WZNW action corresponding to this axial gauging is given by [9]

$$I(g, A) = I(g) + \frac{k}{4\pi} \int d^2 x Tr \left[ \sqrt{-\gamma} \gamma^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + P^{\mu\nu} A_\mu g^{-1} \partial_\nu g + \sqrt{-\gamma} \gamma^{\mu\nu} A_\mu A_\nu + P^{\mu\nu} A_\mu g A_\nu g^{-1} \right].$$  \hspace{1cm} (2.24)

In the Gauss decomposition, this action reduces exactly to the one that is written in (2.15) upon setting $A_\mu = -B_\mu = A_\mu$. Finally, the kind of gauging corresponding to our second set of transformations in (2.11) cannot be compared to the standard gauged WZNW model. This is because one cannot, in general, gauge a transformation of the form $g \rightarrow \Omega_1 g \Omega_2$ if $\Omega_1$ and $\Omega_2$ are non-abelian and different.

Let us now apply the Gauss decomposition to some simple examples. Since the axial gauging, in the Gauss decomposition, leads to the usual gauged WZNW action, we will concentrate here on the vector gauging which does differ from the usual gauged WZNW model.

**The $SL(2, R)$ Case**

A generic group element $g$ is parametrised through the Gauss decomposition

$$g = g_0 g_+ g_- = \begin{pmatrix} 1 & 0 \\ \chi & 1 \end{pmatrix} \begin{pmatrix} e^{\phi} & 0 \\ 0 & e^{-\phi} \end{pmatrix} \begin{pmatrix} 1 & \psi \\ 0 & 1 \end{pmatrix}.$$  \hspace{1cm} (2.25)

The $U(1)$ vector transformation (obtained by setting $U = V$ in (2.9)) is chosen to be generated by the $U(1)$ matrix

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  \hspace{1cm} (2.26)

The gauged WZNW (obtained by setting $B_\mu = A_\mu$ in (2.15)) is given by

$$I_{sl(2)} = \frac{k}{4\pi} \int d^2 x \left[ \sqrt{-\gamma} \gamma^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + P^{\mu\nu} \psi \left( \partial_\mu - 2A_\mu \right) \chi \left( \partial_\nu + 2A_\nu \right) \psi \right].$$  \hspace{1cm} (2.27)
In terms of the different fields, the infinitesimal gauge transformations are
\[ \delta \chi = -2\varepsilon \chi , \quad \delta \psi = 2\varepsilon \psi , \quad \delta \phi = 0 , \quad \delta A_\mu = -\partial_\mu \varepsilon . \]  
(2.28)

The integration over the gauge fields \( A_\mu \) is a simple Gaussian integration and its contribution inside the partition function is given by
\[ 2\pi \det \left( \left( \frac{2k}{\pi} e^{2\phi} \psi \chi \right) \right)^{-1} \]  
(2.29)

The logarithm of this determinant can be calculated using Zeta-function regularisation and the heat kernel results (see refs.[13]). The finite contribution comes from the coefficients \( a_1 \) in the expansion of the heat kernel and is given by
\[ 2\pi \det \left( \left( \frac{2k}{\pi} e^{2\phi} \psi \chi \right) \right)^{-1} = \frac{1}{8\pi} (2\phi + \log(\psi \chi)) R^{(2)} + \text{const.} , \]  
(2.30)

where \( R^{(2)} \) is the two-dimensional scalar curvature.

Therefore, the effective action is
\[ I_{sl(2)} = k \frac{1}{4\pi} \int d^2x \sqrt{-\gamma} \left[ \gamma^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} (2\phi + \log(\psi \chi)) R^{(2)} \right] . \]  
(2.31)

This action is still invariant under the local transformations for \( \chi, \psi \) and \( \phi \) given in (2.28). We could, therefore, choose a gauge such that \( \chi = 1 \). This choice leads to a non-propagating Faddeev-Popov ghosts. Therefore the only remaining fields are \( \phi \) and \( \psi \). However, \( \psi \) has no kinetic term. Thus, the target-space metric of the corresponding non-linear sigma model is degenerate (has zero as eigenvalue). Consequently, this kind of gauging has no non-linear sigma model interpretation and therefore cannot describe the motion of strings in a curved background. It has, though, a natural interpretation in terms of Wakimoto variable as seen in the next section.

The \( SL(3, R) \) Case

In complete analogy with the \( SL(2, R) \) case, we specify a generic \( SL(3, R) \) group element by
\[ g = g_0 g_+ = \begin{pmatrix} 1 & 0 & 0 \\ \chi_1 & 1 & 0 \\ \chi_2 & \chi_3 & 1 \end{pmatrix} \begin{pmatrix} e^{\phi_1} & 0 & 0 \\ 0 & e^{\phi_2} & 0 \\ 0 & 0 & e^{-(\phi_1 + \phi_2)} \end{pmatrix} \begin{pmatrix} 1 & \psi_1 & \psi_2 \\ 0 & 1 & \psi_3 \\ 0 & 0 & 1 \end{pmatrix} . \]  
(2.32)

We choose to gauge the vector transformations generated by the two \( U(1) \) matrices
\[ H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} , \quad H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} . \]  
(2.33)
We need, therefore, to introduce two gauge fields \( A^1_\mu \) and \( A^2_\mu \) according to (2.13). The different fields transformations are

\[
\delta \chi_1 = - (\varepsilon_1 - \varepsilon_2) \chi_1, \quad \delta \chi_2 = - (2\varepsilon_1 - \varepsilon_2) \chi_2, \quad \delta \chi_3 = - (\varepsilon_1 - 2\varepsilon_2) \chi_3
\]

\[
\delta \psi_1 = (\varepsilon_1 - \varepsilon_2) \psi_1, \quad \delta \psi_2 = (2\varepsilon_1 - \varepsilon_2) \psi_2, \quad \delta \psi_3 = (\varepsilon_1 + 2\varepsilon_2) \psi_3
\]

\[
\delta \phi_1 = 0, \quad \delta \phi_2 = 0, \quad \delta A^1_\mu = - \partial_\mu \varepsilon_1, \quad \delta A^2_\mu = - \partial_\mu \varepsilon_2, \quad (2.34)
\]

where \( \varepsilon_1 \) and \( \varepsilon_2 \) are two infinitesimal gauge parameters. The gauge invariant action is given by

\[
I_{sl(3)} = \frac{k}{4\pi} \int d^2 x \sqrt{-\gamma} \gamma^{\mu \nu} \left[ \partial_\mu \phi_1 \partial_\nu \phi_1 + \partial_\mu \phi_2 \partial_\nu \phi_2 + \partial_\mu \phi_1 \partial_\nu \phi_2 \right]
\]

\[
+ \frac{k}{4\pi} \int d^2 x P_{\mu \nu} \left\{ e^{(\phi_1 - \phi_2)} \left( \partial_\mu \chi_1 - (A^1_\mu - A^2_\mu) \chi_1 \right) \left( \partial_\nu \psi_1 + (A^1_\nu - A^2_\nu) \psi_1 \right) \right.
\]

\[
+ e^{(\phi_1 + 2\phi_2)} \left( \partial_\mu \chi_3 - (A^1_\mu + 2A^2_\mu) \chi_3 \right) \left( \partial_\nu \psi_3 + (A^1_\nu + 2A^2_\nu) \psi_3 \right) \right.
\]

\[
+ e^{(2\phi_1 + \phi_2)} \left[ \chi_3 \psi_3 \left( \partial_\mu \chi_1 - (A^1_\mu - A^2_\mu) \chi_1 \right) \left( \partial_\nu \psi_1 + (A^1_\nu - A^2_\nu) \psi_1 \right) \right.
\]

\[
+ \left( \partial_\mu \chi_2 - (2A^1_\mu + A^2_\mu) \chi_2 \right) \left( \partial_\nu \psi_2 + (2A^1_\nu + A^2_\nu) \psi_2 \right) \right.
\]

\[
- \chi_3 \left( \partial_\mu \chi_1 - (A^1_\mu - A^2_\mu) \chi_1 \right) \left( \partial_\nu \psi_2 + (2A^1_\nu + A^2_\nu) \psi_2 \right) \right.
\]

\[
- \psi_3 \left( \partial_\mu \chi_2 - (2A^1_\mu + A^2_\mu) \chi_2 \right) \left( \partial_\nu \psi_1 + (A^1_\nu - A^2_\nu) \psi_1 \right) \left\} \right. \right.
\]

\[
(2.35)
\]

The integration over the gauge fields yields the following determinant

\[
F \equiv 9\pi e^{3(\phi_1 + \phi_2)} \left[ suv e^{2\phi_2} (u - 1) (v - 1) + r^2 e^{-\phi_2} + ru e^{-\phi_1} \right]. \quad (2.36)
\]

The gauge invariant variables \( u, v, r \) and \( s \) are defined by

\[
u = \frac{\chi_1 \chi_3}{\chi_2}, \quad v = \frac{\psi_1 \psi_3}{\psi_2}, \quad r = \chi_1 \psi_1, \quad s = \chi_2 \psi_2. \quad (2.37)
\]

This determinant is regularised in the same manner as in the \( SL(2, R) \) case. In terms of these new variables, the final effective action is

\[
I_{sl(3)} = \frac{k}{4\pi} \int d^2 x \left( \sqrt{-\gamma} \gamma^{\mu \nu} \left( \partial_\mu \phi_1 \partial_\nu \phi_1 + \partial_\mu \phi_2 \partial_\nu \phi_2 + \partial_\mu \phi_1 \partial_\nu \phi_2 \right) \right.
\]

\[
+ 9s^2 F^{-1} e^{2(\phi_1 + \phi_2)} P_{\mu \nu} \partial_\mu u \partial_\nu v \right. \left. + \frac{k}{3\pi} \int d^2 x \sqrt{-\gamma} \log(F) R^{(2)}. \quad (2.38)
\]

Again, the fields \( r \) and \( s \) have no kinetic terms and the non-linear sigma model interpretation breaks down. This kind of gauging has also a representation in terms of Wakimoto variables.
The $SL(N,R)$ Case

To end this series of examples, we would like to consider the general case of the group $SL(N,R)$. We will present a simple argument which shows that the abelian vector gauging in the Gauss decomposition leads always to a degenerate metric. Let us denote the entries of $g_-, g_+$ and $g_0$ by

$$
(g^-)_{\alpha\beta} = \begin{cases} 
0 & \alpha < \beta \\
1 & \alpha = \beta \\
\chi(\alpha-1)(\alpha-2)+\beta & \alpha > \beta 
\end{cases}
$$

$$
(g^+)_{\alpha\beta} = \begin{cases} 
\psi(\alpha-1)(N-\frac{\alpha}{2})+\beta-\alpha & \alpha < \beta \\
1 & \alpha = \beta \\
0 & \alpha > \beta 
\end{cases}
$$

$$
(g_0)_{ij} = \begin{cases} 
e & i = j , \sum_{l=1}^N \phi_l = 0 \\
0 & i \ne j (2.39)
\end{cases}
$$

The WZNW action has then the general form

$$
I_{sl(n)} = \frac{k}{4\pi} \int d^2x \left[ \sqrt{-\gamma} \gamma^{\mu\nu} M^{ij} \partial_\mu \phi_i \partial_\nu \phi_j + \tilde{P}_{\mu\nu} M^{\alpha\beta} \partial_\mu \chi_\alpha \partial_\nu \psi_\beta \right] . \quad (2.40)
$$

Here $M^{ij}$ ($i, j = 1, \ldots, r^G$) is a constant matrix while $M^{\alpha\beta}$ ($\alpha, \beta \in \Phi^+$) is a function of the fields $\phi_i$, $\chi_\alpha$ and $\psi_\alpha$.

Suppose that we are gauging $r$, $r \leq r^G$, abelian $U(1)$ symmetries then we need to introduce $r$ gauge fields $A^{\tilde{i}}_{\mu}$, $\tilde{i} = 1, \ldots, r$. The gauged WZNW action in the Gauss decomposition takes then the general form

$$
I^g_{sl(n)} = \frac{k}{4\pi} \int d^2x \left[ \sqrt{-\gamma} \gamma^{\mu\nu} M^{ij} \partial_\mu \phi_i \partial_\nu \phi_j + \tilde{P}_{\mu\nu} M^{\alpha\beta} \partial_\mu \chi_\alpha \partial_\nu \psi_\beta - \tilde{V}_{i\alpha} \partial_\mu \chi_\alpha \partial_\nu \psi_\beta \right] . \quad (2.41)
$$

The quantity $V_{i\alpha}^\rho$ is a constant matrix and is diagonal in the indices $\rho$, $\alpha$. The gauge transformations are

$$
\delta \chi_\alpha = -\varepsilon^{\tilde{i}} V_{i\alpha}^\beta \chi_\beta , \quad \delta \psi_\alpha = \varepsilon^{\tilde{i}} V_{i\alpha}^\beta \psi_\beta , \quad \delta \phi_i = 0 , \quad \delta A^{\tilde{i}}_{\mu} = -\partial_\mu \varepsilon^{\tilde{i}} . \quad (2.42)
$$

and the matrix $M^{\alpha\beta}$ satisfies

$$
- M^{\rho\beta} V_{i\rho}^\alpha + M^{\alpha\rho} V_{i\rho}^\beta - \frac{\partial M^{\alpha\beta}}{\partial \chi_\rho} V_{i\rho}^\sigma \chi_\sigma + \frac{\partial M^{\alpha\beta}}{\partial \psi_\rho} V_{i\rho}^\sigma \psi_\sigma = 0 . \quad (2.43)
$$

The integration over the gauge fields leads to the nonlinear sigma model

$$
I^g_{sl(n)} = \frac{k}{4\pi} \int d^2x \left[ \sqrt{-\gamma} \gamma^{\mu\nu} M^{ij} \partial_\mu \phi_i \partial_\nu \phi_j + \tilde{P}_{\mu\nu} M^{\alpha\beta} \partial_\mu \chi_\alpha \partial_\nu \psi_\beta \right] , \quad (2.44)
$$
where $\tilde{M}^{\alpha\beta}$ is given by

$$\tilde{M}^{\alpha\beta} = M^{\alpha\beta} - (L^{-1})^{ij} M^{\alpha\rho} V^{\rho}_{\rho j} V^{\gamma}_{i\tau} \chi_{\chi \psi_{\sigma}}$$

Therefore, the integration over the gauge fields yields a nonlinear sigma model having a target-space metric which is block diagonal and takes the form

$$G^{ab} = \begin{pmatrix} M^{ij} & 0 \\ 0 & \tilde{M}^{\alpha\beta} \end{pmatrix},$$

where $a, b = 1, \ldots, N^2 - 1$. Notice, however, that

$$\tilde{M}^{\alpha\beta} V^{\sigma}_{i\alpha} \chi_{\sigma} = \tilde{M}^{\alpha\beta} V^{\sigma}_{i\beta} \psi_{\sigma} = 0 \quad (2.47)$$

Hence the metric $G^{ab}$ has $2r$ zero eigenvalues and is not invertible. This is expected since we still have not fixed our gauge. The effect of gauge fixing will, however, remove (or project out) $r$ null eigenvalues. Therefore by exhausting all the gauge freedom we are still left with $r$ vanishing eigenvalues for the metric $G^{ab}$. Hence the metric is still degenerate even after gauge fixing. In the next section we will show that the gauging of the abelian vector subgroup in the Gauss decomposition has a better interpretation in terms of Wakimoto variables.

## 3 The Wakimoto Realisation and the WZNW Model

We will summarise here the so-called Wakimoto representation of the WZNW model [3,14]. This is the representation of the Kac-Moody current algebra in terms of a set of free scalar fields $\phi_i, i = 1, \ldots, r^G$, and a set of bosonic $\beta\gamma$ system, where $\beta^\alpha$ and $\gamma_\alpha$ are respectively of spin one and zero. The operator product expansions of these fields are

$$\phi_i(z)\phi_j(w) = \kappa^{-1}\delta_{ij} \log(z - w), \quad \beta^\alpha(z)\gamma_\sigma(w) = \delta^\alpha_\sigma \frac{1}{z - w}.$$  \quad (3.1)$$

The currents of the Kac-Moody current algebra are then realised as functions of these free fields.

The Sugawara construction leads to the energy-momentum tensor

$$T(z)_{sug} = T_{\beta\gamma}(z) + T_{\phi}(z)$$

$$T_{\beta\gamma}(z) \equiv \partial \gamma_\alpha(z) \beta^\alpha(z) : \alpha \in \Phi^+$$

$$T_{\phi}(z) \equiv \frac{1}{2} \kappa \delta^{ij} \partial \phi_i(z) \partial \phi_j(z) : - \rho_i \partial^2 \phi^i,$$  \quad (3.2)
where \( \rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha \) is half the sum of the positive roots and \( \kappa = h^G + k \), where \( h^G \) is the dual Coxeter number of \( G \). The central charges coming from the different energy momentum tensors are given by

\[
\frac{c_{\text{sug}}}{k + h^G} = c_{\gamma} + c_{\phi} = \left( \dim G - r^G \right) + \left( r^G - \frac{12 \rho^2}{k + h^G} \right),
\]

(3.3)

where the "strange formula" of Freudenthal-De Vries, \( 12 \rho^2 = h^G \dim G \), has been used.

The Sugawara energy-momentum tensor is the Noether current of the action (in the conformal gauge)

\[
S_{\text{waki}} = \frac{k}{2\pi} \int d^2 z \left[ \partial_z \gamma_\alpha \beta^\alpha + \kappa \delta^{ij} \partial_z \phi_i \partial_z \phi_j + \rho_i \phi^j R(2) \right].
\]

(3.4)

Since the WZNW action is a realisation of the Kac-Moody algebra, this action should derive from the WZWN model. This is indeed the case as explained in great details in ref.[2]. Let us sketch this for the \( SL(N,R) \) group. In the Gauss decomposition the WZWN action takes the general form in (2.40). The spin one fields, \( \beta^\alpha \), are obtained from the WZNW action through the change of variables

\[
\beta^\alpha = M^{\alpha\beta} \partial_z \psi_\beta
\]

(3.5)

while the spin zero fields \( \gamma_\alpha \) are identified with \( \chi_\alpha \). It is clear then that the last term in the WZNW action (2.40) reproduces the first term of the Wakimoto action in (3.4). On the other hand, the terms involving \( \phi_i \) in the Wakimoto action are obtained as a combination of the contribution of \( I(g_0) \) (the first term in in the WZNW action (2.40)) and the contribution arising from the Jacobian of the change of variables made in (3.5). We refer the reader to ref.[2] for a complete account.

4 Gauging the Wakimoto Realisation

The Wakimoto action is invariant under the global transformations

\[
\delta \gamma_\alpha = -\epsilon^\iota Q^\beta_{\iota\alpha} \gamma_\beta, \quad \delta \beta^\alpha = \epsilon^\iota Q^\beta_{\iota\alpha} \beta^\beta, \quad \delta \phi_i = 0,
\]

(4.1)

where \( \iota = 1, \ldots, r \) and the constant matrix \( Q^\beta_{\iota\alpha} \) is diagonal in the indices \( \alpha, \beta \). Gauging these transformations requires the introduction of \( r \) gauge fields \( A^\iota_z \) transforming as \( \delta A^\iota_z = -\partial_z \epsilon^\iota \). The gauged action acquires the form

\[
S_{waki}^g = \frac{k}{2\pi} \int d^2 z \left[ \left( \partial_z \gamma_\alpha - A^\iota_z Q^\beta_{\iota\alpha} \gamma_\beta \right) \beta^\alpha + \kappa \delta^{ij} \partial_z \phi_i \partial_z \phi_j + \rho_i \phi^j R(2) \right].
\]

(4.2)
Integrating out the gauge fields is equivalent to imposing the constraints

\[ Q_{i\alpha}^{\beta} \gamma_\beta \beta^\alpha = 0 \ . \] (4.3)

These constraints would allow us to eliminate \( r \) fields of the type \( \beta^\alpha \). Let us, without loss of generality, eliminate the first \( r \) fields of the type \( \beta^\alpha \). We have then

\[ \beta_{\mu}^{\tilde{\alpha}} = -\gamma_\tilde{\beta}^{-1} (Q^{-1})^{\tilde{\beta} \tilde{\alpha}} Q_{i\alpha}^{\beta} \gamma_\beta \beta^\alpha , \] (4.4)

where the indices \( \tilde{\alpha}, \tilde{\beta} = 1, \ldots, r \) and \( \bar{\alpha}, \bar{\beta} = r + 1, \ldots, \frac{1}{2} \left( \text{dim} \mathcal{G} - r^G \right) \) and we have used the fact that \( Q_{i\alpha}^{\beta} \) is diagonal in the indices \( \alpha, \beta \).

Substituting for \( \beta^\alpha \) in the action (4.2), we obtain

\[ S_{waki}^\phi = \frac{k}{2\pi} \int d^2 z \left[ \partial_z \tilde{\gamma}_{\tilde{\alpha}}^{\tilde{\beta}} \tilde{\beta}^\alpha + \kappa \delta_{ij} \partial_z \phi_i \partial_z \phi_j + \rho_i \phi^i R^{(2)} \right] , \] (4.5)

where the new \( \tilde{\beta} \tilde{\gamma} \) system is defined by

\[ \tilde{\gamma}_{\tilde{\alpha}} = -\sum_{\tilde{\alpha}, \tilde{i} = 1}^{r} (Q^{-1})^{\tilde{\alpha} \tilde{i}} Q_{i\alpha}^{\beta} \log \gamma_{\tilde{\alpha}} \log \gamma_{\tilde{\alpha}} , \quad \tilde{\beta}^\alpha = \gamma_{\tilde{\alpha}} \beta^\alpha . \] (4.6)

In these last equation there is no summation over the index \( \tilde{\alpha} \).

It is crucial to notice that both \( \tilde{\gamma}_{\tilde{\alpha}} \) and \( \tilde{\beta}^\alpha \) are gauge invariant variables. Therefore, any gauge fixing that we could have imposed on the original \( \beta \gamma \) system would always result in \( \left[ \frac{1}{2} \left( \text{dim} \mathcal{G} - r^G \right) - r \right] \) pairs of \( \tilde{\beta} \tilde{\gamma} \). Therefore, the central charge corresponding to the energy-momentum tensor of the action (4.5) is given by

\[ c_{\beta \tilde{\gamma}} + c_\phi = \left[ \left( \text{dim} \mathcal{G} - r^G \right) - 2r \right] + \left( r^G - \frac{12 \rho^2 \mathcal{G}}{k + h^G} \right) = \frac{k \text{dim} \mathcal{G}}{k + h^G} - 2r . \] (4.7)

Hence we see that there are \( 2r \) (instead of \( r \)) degrees of freedom removed by gauging \( r \) abelian symmetries in the Wakimoto action.

Let us now take into account that the Wakimoto action descends from the WZNW model. We will here consider again the case of the \( SL(N, R) \) group.

The gauged WZNW action for the \( SL(N, R) \) group is written in (2.41). Let us choose a gauge such that

\[ A_\tilde{i}^i = 0 , \ \tilde{i} = 1, \ldots, r \ . \] (4.8)

The Faddeev-Popov action for this gauge fixing is written as

\[ S_{gh} = \frac{k}{2\pi} \int d^2 z \sum_{\tilde{i} = 1}^{r} b_{\tilde{i}} \partial_z c_{\tilde{i}} \] (4.9)
The action (2.41) reduces then to

\[ I_{g_{\text{sl}}(n)} = \frac{k}{2\pi} \int d^2z \left[ M^{ij} \partial_z \phi_i \partial_{\bar{z}} \phi_j + M^{\alpha \beta} \left( \partial_z \chi_{\alpha} - V^\rho_{\alpha} A^\rho_i \partial_{\bar{z}} \chi_{\rho} \right) \partial_{\bar{z}} \psi_{\beta} \right] + S_{gh}. \] (4.10)

If we change now to the variables \( \gamma_\alpha \) and \( \beta^\alpha \) as defined in (3.5) and take into account the contribution due to the Jacobian (see ref.[2]) then we find

\[ I_{g_{\text{sl}}(n)} = S_{g_{\text{waki}}} + S_{gh}. \] (4.11)

Therefore the gauged Wakimoto action is a gauge-fixed version of the gauged WZNW model in the Gauss decomposition.

There is, however, a difference between gauging the Wakimoto action and the gauged WZNW model. This difference resides in the following remark. In gauging the Wakimoto action, the one-form fields \( \beta^\alpha \) were allocated the transformation

\[ \beta^\alpha \rightarrow \beta^\alpha + \epsilon_i(z, \bar{z}) Q^\alpha_{i\beta} \beta^\beta \] (4.12)

However, if \( \beta^\alpha \) is defined in terms of the field \( \phi_i, \chi_\alpha \) and \( \psi_\alpha \) as in (3.5), then this transformation is certainly a non-local one. This is because \( \beta^\alpha \) depends on \( \partial_z \psi_\alpha \) and any local transformation on \( \psi_\alpha \) would appear through its derivative in \( \beta^\alpha \), which is not what is written above.

The most important point of this section is that we are able to give an interpretation in terms of Wakimoto variables to the gauged WZNW model in the Gauss decomposition. As seen before, this kind of gauging does not have a nonlinear sigma model interpretation.

## 5 Conclusions

In the Wess-Zumino-Novikov-Witten models, there are only two possible symmetries that can be gauged: The vector gauging \( g \rightarrow hgh^{-1} \), where \( h \) can be abelian or non-abelian group element, and the axial gauging \( g \rightarrow hgh \), where in this case \( h \) must be abelian.

If, however, \( g \) admits a Gauss decomposition, \( g = g_0 g_+ \), then there are two different symmetries which respect the Gauss decomposition and which can be gauged. The first such symmetry is an abelian one \( g \rightarrow UgV \) and contains the vector symmetry \((V = U^{-1})\) and the axial symmetry \((V = U)\). The second symmetry is a non-abelian one \( g \rightarrow \Omega_- g \Omega_+ \), where \( \Omega_- \) and \( \Omega_+ \) are two group elements corresponding respectively to the lowering and raising step operators.

Our results show that the gauging of these two kinds of symmetries in the Gauss decomposition is carried out by minimally coupling the gauge fields. In the case of the
abelian gauging we found that the axial gauging in the Gauss decomposition is identical to the standard axial gauging of the WZNW model. The vector gauging in the Gauss decomposition, on the other hand, differs from the standard vector gauging of the WZNW model by by a term proportional to the field strength of the gauge fields. The theory resulting from this vector gauging is still conformally invariant and has a nice interpretation in terms of Wakimoto variables. However, the resulting nonlinear sigma model has a degenerate metric.

We conclude therefore that the gauging of the WZNW models depend very much on the local parametrisation of the group elements. In other words, once the nonlinear sigma model corresponding to the WZWN model is written down, the gauging of the isometries of this sigma model may not lead to a theory which describes the motion of strings on nontrivial backgrounds.

There are still interesting issues to be explored in this context. First of all, the supersymmetric case is straightforward. It seems to us that the duality between the vector gauging and the abelian gauging found in [15] is broken in the Gauss decomposition. This needs further investigations. Furthermore, Sfetsos and Tseytlin [16] have also considered a chiral gauging of the WZNW model (which differs from the standard gauging) and it would be of interest to explore the connection between our gauging in the Gauss decomposition and this chiral gauging. Finally, since the Gauss decomposition is used to connect gauged WZNW models and Toda theories, we feel that it is necessary to re-examine the non-abelian gauging in the Gauss decomposition. This would then shed some light on the origin of the exactly conformally action for non-abelian Toda theories found in ref.[17]. We hope to return to these points in the near future.

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