From asymptotic freedom to $\theta$ vacua: Qubit embeddings of the O(3) nonlinear $\sigma$ model

Stephan Caspar $^*$ and Hersh Singh $^†$

InQubator for Quantum Simulation (IQuS), Department of Physics, University of Washington, Seattle, Washington 98195-1550, USA

Conventional lattice formulations of $\theta$ vacua in the 1 + 1-dimensional O(3) nonlinear sigma model suffer from a sign problem. Here, we construct the first sign-problem-free regularization for arbitrary $\theta$. Using efficient lattice Monte Carlo algorithms, we demonstrate how a Hamiltonian model of spin-$\frac{1}{2}$ degrees of freedom on a 2-dimensional spatial lattice reproduces both the infrared sector for arbitrary $\theta$, as well as the ultraviolet physics of asymptotic freedom. Furthermore, as a model of qubits on a two-dimensional square lattice with only nearest-neighbor interactions, it is naturally suited for studying the physics of $\theta$ vacua and asymptotic freedom on near-term quantum devices. Our construction generalizes to $\theta$ vacua in all CP($N - 1$) models, solving a long standing sign problem.

INTRODUCTION

The strong interactions of the standard model described by quantum chromodynamics (QCD) pose a challenging problem for classical computation. While nonperturbative lattice Monte Carlo (MC) methods are a powerful tool for studying static properties of strongly coupled quantum field theories (QFTs) like QCD [1–4], questions involving real time dynamics, finite density or nontrivial $\theta$ vacua are still out of reach for lattice MC methods due to severe sign problems [5, 6].

Emerging quantum platforms provide an exciting possibility for investigating QFTs in previously inaccessible regimes. They are not directly affected by the sign problems arising in classical lattice MC methods. However, bosonic lattice field theories such as QCD have infinite-dimensional local Hilbert spaces, while hardware degrees of freedom (DOF) are usually finite-dimensional, mostly qubits. A significant effort is underway to explore different embeddings of QFTs in qubits, with a multitude of ideas emerging from bosonic field theory [7–10], nonlinear sigma models (NL$\sigma$Ms) [11–20] and gauge theories [21–37].

The 1 + 1-dimensional O(3) NL$\sigma$M has a long history as a prototype for QCD, due to similarities such as asymptotic freedom, dynamical transmutation and the generation of a nonperturbative mass gap, as well as a topological $\theta$ term. The O(3) NL$\sigma$M with a $\theta$-term is formally defined by the continuum action

$$S_\theta[\vec{\phi}] = \frac{1}{g^2} \int d^2 x (\partial_\mu \vec{\phi})^2 + i \theta Q[\vec{\phi}],$$

(1)

where $\vec{\phi} \in \mathbb{R}^3$ with $|\vec{\phi}|^2 = 1$, and

$$Q[\vec{\phi}] = \frac{1}{8\pi} \int d^2 x \varepsilon_{\mu\nu} \frac{\vec{\phi}}{2} \cdot (\partial^\mu \vec{\phi}) \times (\partial^\nu \vec{\phi})$$

(2)

is the integer topological charge, making the theory 2$\pi$-periodic in $\theta$. Both $\theta = 0, \pi$ points are well-understood, analytically as well as on the lattice. Exact S-matrices have been conjectured for both $\theta = 0$ and $\theta = \pi$ [38–41] and their integrability has been confirmed using non-perturbative lattice MC methods [42–44].

However, general, non-integrable $\theta$ remain challenging. As a topological effect, it cannot be studied directly in perturbation theory about the free ultraviolet (UV) fixed point, although some analytic progress has been made by perturbing about the $\theta = \pi$ integrable point [45]. Nonperturbatively, the inclusion of a $\theta$ term causes a sign problem when discretizing the action in Eq. (1) on a 2-dimensional spacetime lattice. Even though improved actions combined with cluster algorithms have been shown to tame both cutoff effects and the sign problem to allow a reliable extrapolation from modest volumes around $\theta \approx 0$ [46, 47], and even fully solve the sign problem at $\theta = \pi$ [43], so far there are no known lattice MC methods which allow a fully controlled study of arbitrary $\theta$ vacua.

Motivated by the prospect of quantum simulation to address these challenges, we develop an embedding of the O(3) NL$\sigma$M at arbitrary $\theta$ into a 2-dimensional Heisenberg antiferromagnet, such that a controlled continuum limit can be taken. Remarkably, not only does this model allow the systematic study of $\theta$ vacua on quantum hardware, it also enables the first sign problem free algorithm for classical computations at arbitrary $\theta$. This extends a similar proposals put forward in Refs. [11, 12, 14, 21] for classical and quantum simulation of

![RG flow diagram of O(3) NL$\sigma$Ms $S_\theta$ defined in Eq. (1). $S_\theta$ is a family of asymptotically-free QFTs which all flow into the trivial IR fixed point, except at $\theta = \pi$ where it reaches the SU(2)$_1$ WZW fixed point. At small $|\theta - \pi|$, the RG flow of $S_\theta$ passes arbitrarily close to the WZW fixed point, on its way to the trivial fixed point.](https://example.com/RG_diagram.png)
Alternating

$$J' > 0, \quad J_{x,y} = J(1 + (-1)^{x+y} \gamma),$$

Columnar

$$J' < 0, \quad J_{x,y} = J(1 + (-1)^x \gamma),$$

where $J > 0$ is always antiferromagnetic, and $\gamma$ is the staggering parameter, as shown in Fig. 2. In both these cases, the continuum limit of the O(3) NLSM with a $\theta$ term can be obtained from odd or even $L_Y$, by taking the limit $L_Y \to \infty$ at fixed $\gamma L_Y$ such that $L_X \gg L_Y \gg 1$ is maintained.

To demonstrate the continuum limit, we need to recover the physics of the theory described by Eq. (1) at all scales, from the UV to the IR. For all $\theta$, the continuum action $S_\theta$ defined in Eq. (1), describes an asymptotically-free theory, controlled in the UV by the fixed point of two free bosons. The coupling $g$ is a relevant coupling and thus drives the theory away from the free UV fixed point into a strongly coupled theory in the IR. While all $S_\theta$ theories flow out of the same UV fixed point, non-perturbative effects lead to different RG trajectories for different $\theta$. Figure 1 shows a conjectured RG flow diagram for the O(3) NLSM at arbitrary $0 \leq \theta \leq \pi$. For all $\theta \neq \pi$ the theory flows to the trivial massive fixed point in the IR. However, at $\theta = \pi$, the theory undergoes a second order phase transition and the low-energy physics changes completely. The mass-gap vanishes and the IR physics is described by a nontrivial conformal field theory ($CFT$) called the SU(2)$_1$ WZW theory [53]. Interestingly, the two ideas of staggering [54, 55] and D-theory [12] can be combined with the qubit Hamiltonian of Eq. (3) to reproduce the physics of both IR and UV.

**IR physics of the $\theta$ vacua.** For $\gamma = 0$, the Hamiltonian of Eq. (3) reduces to the ordinary Heisenberg antiferromagnet. For a fixed $L_Y$ and $\gamma = 0$, this model has been studied in condensed matter literature as spin ladders, and is known to be described by the O(3) NLSM at low-energies with $\theta = 2\pi S L_Y$ [56–58]. Under this identification, the translation-by-one symmetry (\(\phi \mapsto -\phi\)) on the lattice scale becomes the charge conjugation symmetry (\(\phi \mapsto -\phi\)) in the continuum. Therefore, a $\theta$-term can be induced in the IR by introducing a staggered coupling which breaks this symmetry [54, 55, 59]. Ref. [55] showed that for spin-S ladders with alternating staggering $\theta = 2\pi S L_Y(1 + \gamma f(L_Y))$, where $f(L_Y)$ is a non-universal function. Therefore the low-energy physics of the $\theta$ vacua can be studied by varying the staggering parameter $\gamma$ [60]. However, to obtain the continuum limit of the O(3) NLSM, we must also obtain the physics of asymptotic freedom in the UV, which we now turn to.

**Regulating the UV: Asymptotic Freedom.** The continuum

\[ H = \sum_{(x,y)} J_{x,y} \vec{S}_{x,y} \cdot \vec{S}_{x+1,y} + J' \sum_{(x,y)} \vec{S}_{x,y} \cdot \vec{S}_{x,y+1}, \quad (3) \]

where $\vec{S}_i$ are the spin operators acting on two-dimensional Hilbert space at the site $(x,y)$, $J_{x,y}$ are the couplings along the $x$ direction, $J'$ is the coupling along the $y$ direction, and the 2d lattice has dimensions $L_X \times L_Y$. We consider the following two configurations for staggering the couplings:

Alternating: $J' > 0, \quad J_{x,y} = J(1 + (-1)^{x+y} \gamma),$ (4)

Columnar: $J' < 0, \quad J_{x,y} = J(1 + (-1)^x \gamma),$

FIG. 2. Two configurations for the staggered interactions, described in Eq. (4), considered as a regularization of the 1+1-dimensional O(3) NLSM with a $\theta$ term. For the alternating staggering, all couplings are antiferromagnetic, while for the columnar case, the transverse coupling $J'$ is ferromagnetic and $J_{z} = J(1 \pm \gamma)$ is antiferromagnetic. All interactions are of the Heisenberg $\vec{S}_i \cdot \vec{S}_j$ type.

FIG. 3. Proposed embedding of the O(3) NLSM with a $\theta$ term into a 2-dimensional array of ultracold atoms. The alternating staggering described in Eq. (4) and Fig. 2 arises naturally from distance-dependent antiferromagnetic interactions by deforming a rectangular lattice.
The physics is therefore frozen along the \( L_Y \) direction, and the system effectively becomes one-dimensional, described by the \( 1 + 1 \) O(3) NLσM with an effective coupling \( g^2 \sim 1/L_Y \). Since the correlation length diverges exponentially in \( L_Y \), a continuum QFT can be defined in the limit of \( L_Y \) large. Therefore, in this limit, the spin-\( \frac{1}{2} \) Hamiltonian of Eq. (3) is a lattice regularization of the O(3) NLσM with an arbitrary \( \theta \) at all scales, including asymptotic freedom in the UV.

**Extension to CP\((N-1)\) models.** All methods in this paper are straightforward to extend from O(3) = CP(1) to the entire family of CP\((N-1)\) models, which also allow for a \( \theta \) term. Both \( \theta = 0, \pi \) have been considered before in the D-theory formulation \([13–15, 51]\) using a Heisenberg model of SU\((N)\) spins, where the SU\((N)\) representations are chosen such that spontaneous symmetry breaking (SSB) of the type SU\((N) \to U(N-1)\) occurs \([61, 62]\). This ensures that the continuum CP\((N-1)\) fields arise as Goldstone modes as the continuum limit \((L_Y \to \infty)\) is taken. Since the discussion of charge conjugation symmetry is identical to that for the O(3) model, the staggering patterns of Eq. (4) will induce \( \theta \neq 0, \pi \) in these constructions as well.

**METHODS**

In this work, we study the Hamiltonian defined in Eq. (3) by performing MC sampling of the partition function \( Z = \text{Tr} e^{-\beta H} \) using a worm algorithm \([63–65]\) on a spacetime lattice. The dimensions \( L_X \times L_Y \) of the 2-dimensional spatial lattices were varied in the range \( 32 \leq L_X \leq 1024 \) and \( L_Y = 3, 5, 7 \), with periodic boundary conditions in \( L_X \) and open boundary conditions in \( L_Y \). The couplings \( J = |J'| = 1 \) were held fixed and only the staggering \( \gamma \) was varied in the range \( 0 \leq \gamma L_Y \leq 2 \). The Heisenberg model with this level of staggering is not frustrated and no sign problem occurs in the MC sampling. In our computations, the imaginary-time extent \( \beta \) was also discretized into \( L_T \) timesteps of size \( \epsilon \), such that \( \beta = \epsilon L_T \). Strictly speaking, the Hamiltonian model of Eq. (3) is recovered only by extrapolating to the \( \epsilon \to 0 \) limit. Alternatively, one can develop a cluster algorithm directly in the continuous time limit \([66]\). However, since we are interested in studying the continuum limit of a relativistic field theory, we perform MC computations at a fixed \( \epsilon = 1.0 \), which gives a transfer matrix model with the same continuum limit. The physical extent \( \beta \) is chosen such that \( \beta = L_X/c \), where \( c \) is the speed of light of this system.

For all combinations of the parameters we calculate the second-moment correlation length \( \xi_2(L_X, g) \) from the spin-spin correlation function \( \langle \tilde{S}_{x,y} \cdot \tilde{S}_{x',y'} \rangle \). This long distance length scale has been extensively studied at \( \theta = 0 \) \([44, 52]\) and is easy to extract from lattice results. The calculation is then repeated with doubled volume \( 2L_X \) but fixed bare couplings \( g_{\text{bare}} = (L_Y, J, J', \gamma) \). This macroscopic change in scale \( L_X \to 2L_X \) defines a discrete variant of the \( \beta \)-function, known as the step-scaling function \([42]\)

\[
F_\xi(z) = \frac{\xi_2(2L_X, g_{\text{bare}})}{\xi_2(L_X, g_{\text{bare}})}, \quad z = \frac{\xi_2(L_X, g_{\text{bare}})}{L_X}, \tag{5}
\]

where \( z \) defines a renormalized coupling. In the continuum limit \( \xi_2 \to \infty \), at constant \( z \), the step-scaling function \( F_\xi(z) \) becomes a universal function, which uniquely characterizes the corresponding QFT.

**RESULTS**

Figure 4 shows numerical results for the step-scaling function for the O(3) NLσM at various \( \theta \), computed using the qubit Hamiltonian of Eq. (3). Results from both alternating (left panel) and columnar (right panel) staggering configurations are shown. To guide the reader, we show three continuous curves: the perturbative prediction (dotted line) and nonperturbative MC results for \( \theta = 0 \) (black dashed line), and \( \theta = \pi \) (black solid line). The perturbative curve is a two-loop computation \([44]\) valid in the UV \((z \gg 1)\) and shows asymptotic freedom near the UV fixed point \( F(z) = 2 \) at \( z \to \infty \). The \( \theta = 0 \) curve was obtained with the standard lattice action in Ref. \([44]\), and shows the flow from the UV to the trivial IR fixed point \((F_\xi(0) = 1)\) at \( z = 0 \).

The \( \theta = \pi \) curve (black solid line) is a polynomial fit in \( z^{-2n} \) up to order \( n = 5 \) to our MC results with \( \gamma L_Y = 0 \). This shows the RG flow from the asymptotically-free UV fixed point at \( z = \infty \) to the SU(2)\(_1\) WZW fixed point in the IR at \( z = z^* \). We estimate the location of the nontrivial IR fixed point to be \( z^* \approx 0.28 \) where \( F(z^*) = 2 \), which is the discrete equivalent of a vanishing \( \beta \)-function. We emphasize that the physics of all scales, from asymptotic freedom in the UV to the SU(2)\(_1\) WZW theory in the IR is reproduced by this model.

The remaining curves show new results for non-zero \( \gamma L_Y \sim |\theta - \pi|/\pi \). The \( \theta = 0, \pi \) curves form a lower and upper bound on all step scaling curves \( 0 \leq \theta \leq \pi \). All curves closely follow the perturbative two-loop calculation (dotted line) at large \( z \) down to \( z \approx 0.75 \). At lower values of the renormalized coupling, non-perturbative effects start to dominate, leading to divergent trajectories.

For small staggering \( \gamma L_Y \) the curves closely track the \( \theta = \pi \) curve. But since \( \theta \) is a relevant perturbation about the WZW fixed point, the RG trajectories cannot reach the nontrivial fixed point at \( z^* \) and ultimately have to flow away to the trivial fixed point at \( z = 0 \), consistent with the fact that these theories are massive. These theories can be made to pass arbitrarily close to the SU(2)\(_1\) WZW fixed point by choosing smaller and smaller \( \gamma L_Y \), without the need for any fine tuning, exemplified
by the curve $\gamma L_Y = 0.01$ in Fig. 4. This is the phenomenon
of conformal walking, which is also exhibited by QCD-like
4-dimensional non-Abelian gauge theories near the conformal
window [67], or technicolor extensions of the Standard model
[47].

As the staggering $\gamma$ is increased further, the step scaling
curves trace out the entire area bounded by the two curves
$\theta = 0, \pi$, demonstrating that all $\theta$ vacua are contained in this
model. However, this only yields a qualitative relationship
between $\gamma L_Y$ and $\theta$. Semi-classical results from large-$L_Y$
expansions [55, 68] suggest that the relationship should be
linear, $\theta = 2\pi S L_Y (1 + \gamma f(L_Y))$ with $f(L_Y) \to f(\infty)$
approaching a finite constant in the large-$L_Y$ limit. Numerically,
we observe that a value of $\gamma L_Y = 1.0$ ($\gamma L_Y = 0.25$) ap-
proximates the $\theta = 0$ curve with the alternating (columnar)
staggering. Additional data also show a periodic reappearance
of $\theta = \pi$ around values of $\gamma L_Y = 2.0$ ($\gamma L_Y = 0.5$). From
this we estimate the asymptotic values $f(\infty) \approx 1.0$ for alter-
nating and $f(\infty) \approx 0.25$ for columnar staggering. The small
discontinuities of the step-scaling curves between different
values of $L_Y = 3, 5, 7$ suggest that the corrections to $f(L_Y)$
at finite $L_Y$ are mild, especially considering that similar values
of the renormalized coupling $z = \xi_2(L_X, \theta)$ were obtained
with drastically different lattice spacings (usually $L_X = 64$ for
the smaller $L_Y$ compared to $L_X = 1024$ for the larger $L_Y$).

Similar results are also observed with even ladders, where
the staggering $\gamma$ is a perturbation about the $\theta = 0$ theory.
Preliminary results from $L_Y = 2, 4, 6$ with alternating and
columnar staggers suggest that $\theta = \pi$ can also be obtained in
this way. These results strongly motivate a conjecture: the
continuum limit ($L_Y \to \infty$, with $L_Y$ either odd or even) for
each fixed $\gamma L_Y$ is in fact a unique QFT corresponding to the
$1 + 1$-dimensional O(3) NL$\sigma$M with a fixed $\theta$,

$$\theta \equiv \pi L_Y + \gamma f \pmod{2\pi}, \quad (6)$$

where $f$ is a non-universal constant which depends on the
details of the model such as choice of couplings, staggering
configuration, and whether $L_Y$ is odd or even. While we
have provided strong evidence in favor of this identification,
there are many paths forward to establish this more rigorously.
For example, odd and even $L_Y$ could be used to self-validate
this conjecture, by showing that their step-scaling functions
agree by appropriately tuning $\gamma L_Y$. Further, a comparison
with the approach of Ref. [46] using topological lattice actions
would be very illuminating. In that approach, $\theta$ appears as a
manifestly topological parameter and thus does not require an
empirical identification like Eq. (6). It would also be interesting
to connect with analytical results on the $\theta$ vacua based on
semiclassical instanton methods.

CONCLUSIONS

In this work, we have shown how to implement the $1 + 1$
dimensional O(3) NL$\sigma$M at arbitrary $\theta$ using qubit degrees of
freedom. While the motivation behind this work is the quan-
tum simulation of $\theta$ vacua on near-term quantum hardware,
interestingly, this result also advances lattice computations of
QFTs using classical MC methods. On the classical side, it
provides the first sign-problem-free MC algorithm for arbitrary
$\theta$. Our numerical results, obtained with an efficient worm algo-

FIG. 4. Step-scaling function of the O(3) NL$\sigma$M with various $\theta$. We show step-scaling curves for different values of $\gamma L_Y \sim |\theta - \pi|/\pi$ with odd
$L_Y$, obtained from alternating (left) and columnar staggering (right), as defined in Eq. (4). For a fixed $\gamma L_Y$, we show MC results for $L_Y = 3$
(solid line), $L_Y = 5$ (dashed lines) and $L_Y = 7$ (dotted lines). The dotted black curve is a two-loop perturbative prediction [44]. The dashed
black line is the step-scaling function for $\theta = 0$ obtained in Ref. [44]. The solid black line is an $O(z^{-10})$ fit to the $\gamma L_Y = 0$ data, which
corresponds to the step-scaling function of the O(3) NL$\sigma$M at $\theta = \pi$. These curves mimic the RG flow diagram shown in Fig. 1, and arrows on
the $\theta = 0, \pi$ curves indicate RG flow from UV to IR. All curves agree in the perturbative UV regime, while nonperturbative effects from the $\theta$
term lead to divergent trajectories in the IR.
this model, and we conjecture a simple prescription of how the continuum limit can be reached and the physics at all scales can be studied.

This construction enables real-time simulation of $\theta$ vacua in the $O(3)_{NL\sigma M}$ on near-term quantum hardware. These theories can be regularized at any lattice spacing through an embedding into a 2-dimensional square lattice of qubits with nearest-neighbor Heisenberg-type interactions. The alternating staggering is a prime candidate for an analog quantum simulation platform like ultracold atoms, with uniform pairwise interactions, and couplings that can be arranged through the trapping pattern shown in Fig. 3. On digital quantum hardware like superconducting qubits or trapped ions, either staggering can be implemented using standard Suzuki-Trotter decompositions. Interestingly, the limit $L_N \gg L_V$ is also amenable to DMRG-type algorithms on tensor networks, which would be powerful complementary approach to lattice MC and quantum simulation going forward.

In lattice field theory, the $1+1$-dimensional $O(3)_{NL\sigma M}$ has been long considered an ideal testbed for static properties of QCD, exhibiting many of its features, including asymptotic-freedom and $\theta$ vacua. Even more possibilities open up once we have access to realtime dynamics using quantum platforms. For instance, accessing nontrivial $\theta$ would allow the study of inelastic scattering processes in an asymptotically-free theory, which would have been impossible in the integrable $\theta = 0, \pi$ theories.

Formulating QFTs using qubits can yield unexpected advantages. For the $O(3)_{NL\sigma M}$, this approach has the rather remarkable feature that it completely circumvents a sign problem present in conventional lattice formulations of the $\theta$ term, and is amenable to efficient cluster algorithms. Extension to the entire family of $CP(N - 1)$ models is straightforward. This is encouraging on the path forward towards studying QCD with novel classical and quantum algorithms. Our results demonstrate that there is no fundamental obstruction to studying $\theta$ vacua with discrete degrees of freedom, but whether such ideas might one day even help with the sign problems in QCD remains to be seen.

ACKNOWLEDGMENTS

We learned about D-theory from Shailesh Chandrasekharan and Uwe-Jens Wiese and are grateful to them for many enlightening conversations over the years. We thank Martin Savage for inspiring discussions and important feedback on the manuscript. We would also like to acknowledge stimulating conversations with Tanmoy Bhattacharya, Anthony Ciavarella, Mendel Nguyen and Mithat Unsal on related matters.

The material presented here was funded in part by the DOE QuantiSED program through the theory consortium “Intersections of QIS and Theoretical Particle Physics” at Fermilab with Fermilab Subcontract No. 666484, in part by Institute for Nuclear Theory with US Department of Energy Grant DE-FG02-00ER41132, and in part by U.S. Department of Energy, Office of Science, Office of Nuclear Physics, Inqubator for Quantum Simulation (IQuS) under Award Number DOE (NP) Award DE-SC0020970.

[1] C. Ratti, Lattice QCD and heavy ion collisions: A review of recent progress, Reports on Progress in Physics 81, 084301 (2018).
[2] Z. Davoudi, W. Detmold, P. Shanahan, K. Orginos, A. Parreño, M. J. Savage, and M. L. Wagman, Nuclear matrix elements from lattice QCD for electroweak and beyond-Standard-Model processes, Physics Reports Nuclear Matrix Elements from Lattice QCD for Electroweak and beyond–Standard-Model Processes, 900, 1 (2021).
[3] M. T. Hansen and S. R. Sharpe, Lattice QCD and Three-Particle Decays of Resonances, Annual Review of Nuclear and Particle Science 69, 65 (2019).
[4] M. Constantinou, A. Courtoy, M. A. Ebert, M. Engelhardt, T. Giani, T. Hobbs, T.-J. Hou, A. Kusina, K. Kutak, J. Liang, H.-W. Lin, K.-F. Liu, S. Liuti, C. Mezrag, P. Nadolsky, E. R. Nocera, F. Olnes, J.-W. Qiu, M. Radici, A. Radyushkin, A. Rajan, T. Rogers, J. Rojo, G. Schierholz, C. P. Yuan, J.-H. Zhang, and R. Zhang, Parton distributions and lattice-QCD calculations: Toward 3D structure, Progress in Particle and Nuclear Physics 121, 103908 (2021).
[5] S. D. Hsu and D. Reeb, On the sign problem in dense qcd, International Journal of Modern Physics A 25, 53 (2010).
[6] V. Goy, V. Bornyakov, D. Boyd, A. Molochkov, A. Nakamura, A. Nikolaev, and V. Zakharov, Sign problem in finite density lattice qcd, Progress of Theoretical and Experimental Physics 2017, 031D01 (2017).
[7] S. P. Jordan, K. S. Lee, and J. Preskill, Quantum computation of scattering in scalar quantum field theories, arXiv preprint arXiv:1112.4833 (2011).
[8] S. P. Jordan, K. S. Lee, and J. Preskill, Quantum algorithms for quantum field theories, Science 336, 1130 (2012).
[9] K. Yeter-Aydeniz, E. F. Dumitrescu, A. J. McCaskey, R. S. Bennink, R. C. Pooser, and G. Siopsis, Scalar quantum field theories as a benchmark for near-term quantum computers, Physical Review A 99, 032306 (2019).
[10] N. Klco and M. J. Savage, Digitization of scalar fields for quantum computing, Physical Review A 99, 052335 (2019).
[11] S. Chandrasekharan, B. Scarlet, and U. J. Wiese, From spin ladders to the 2D O(3) model at non-zero density, Computer Physics Communications Proceedings of the Europhysics Conference on Computational Physics Computational Modeling and Simulation of Complex Systems, 147, 388 (2002).
[12] R. Brower, S. Chandrasekharan, S. Riederer, and U. J. Wiese, D-theory: Field quantization by dimensional reduction of discrete variables, Nuclear Physics B 693, 149 (2004).
[13] B. B. Beard, M. Pepe, S. Riederer, and U.-J. Wiese, Efficient Cluster Algorithm for CP(N-1) Models, Computer Physics Communications 175, 629 (2006).
[14] C. Lafflamme, W. Evans, M. Dalmonte, U. Gerber, H. Mejía-Díaz, W. Bietenholz, U. J. Wiese, and P. Zoller, CP(N-1) quantum field theories with alkaline-earth atoms in optical lattices, Annals of Physics 370, 117 (2016).
[15] W. Evans, U. Gerber, M. Hornung, and U. J. Wiese, SU(3) quantum spin ladders as a regularization of the CP(2) model at
non-zero density: From classical to quantum simulation, Annals of Physics 398, 94 (2018).

[16] F. Bruckmann, K. Jansen, and S. Kühn, O(3) nonlinear sigma model in S1+1 dimensions with matrix product states, Physical Review D 99, 074501 (2019).

[17] H. Singh and S. Chandrasekharan, A qubit regularization of the $O(3)$ sigma model, Phys. Rev. D100, 054505 (2019).

[18] T. Bhattacharya, A. J. Buser, S. Chandrasekharan, R. Gupta, and H. Singh, Qubit regularization of asymptotic freedom, Physical Review Letters 126, 172001 (2021).

[19] H. Singh, Qubit $O(n)$ nonlinear sigma models, arXiv:1911.12353 [hep-lat, physics:quant-ph] (2019).

[20] H. Singh, Large-charge conformal dimensions at the SO(N)$/$Wilson-Fisher fixed point, arXiv:2203.00059 [hep-lat] (2022).

[21] S. Chandrasekharan and U. J. Wiese, Quantum link models: A discrete approach to gauge theories, Nuclear Physics B 492, 455 (1997).

[22] R. Brower, S. Chandrasekharan, and U. J. Wiese, QCD as a quantum link model, Phys. Rev. D60, 094502 (1999).

[23] I. Raychowdhury and J. R. Strikyer, Solving gauss’s law on digital quantum computers with loop-string-hadron digitization, Physical Review Research 2, 033039 (2020).

[24] R. Anishetty, M. Mathur, and I. Raychowdhury, Prepotential formulation of su(3) lattice gauge theory, Journal of Physics A: Mathematical and Theoretical 43, 035403 (2009).

[25] D. Banerjee, M. Bögli, M. Dalmonte, E. Rico, P. Stebler, U.-J. Wiese, and P. Zoller, Atomic quantum simulation of $u(n)$ and $su(n)$ non-abelian lattice gauge theories, Physical review letters 110, 125303 (2013).

[26] E. Zohar, J. I. Cirac, and B. Reznik, Quantum simulations of lattice gauge theories using ultracold atoms in optical lattices, Reports on Progress in Physics 79, 014401 (2015).

[27] M. C. Banuls, K. Cichy, J. I. Cirac, K. Jansen, and S. Kühn, Efficient basis formulation for $\{(+1)\times1\}$-dimensional su(2) lattice gauge theory: spectral calculations with matrix product states, Physical Review X 7, 041046 (2017).

[28] C. Muschik, M. Heyl, E. Martinez, T. Monz, P. Schindler, B. Vogell, M. Dalmonte, P. Hauke, R. Blatt, and P. Zoller, U(1) Wilson lattice gauge theories in digital quantum simulators, New Journal of Physics 19, 103020 (2017).

[29] T. V. Zache, F. Hendriksen, F. Jendrzejewski, M. Oberthaler, J. Berges, and P. Hauke, Quantum simulation of lattice gauge theories using wilson fermions, Quantum science and technology 3, 034010 (2018).

[30] A. Alexandru, P. F. Bedaque, S. Harmalkar, H. Lamm, S. Lawrence, N. C. Warrington, N. Collaboration, et al., Gluon field digitization for quantum computers, Physical Review D 100, 115301 (2019).

[31] J. Bender and E. Zohar, Gauge redundancy-free formulation of compact qed with dynamical matter for quantum and classical computations, Physical Review D 102, 114517 (2020).

[32] Z. Davoudi, M. Hafezi, C. Monroe, G. Pagano, A. Seif, and A. Shaw, Towards analog quantum simulations of lattice gauge theories with trapped ions, Physical Review Research 2, 023015 (2020).

[33] N. Klco, M. J. Savage, and J. R. Strikyer, Su (2) non-abelian gauge field theory in one dimension on digital quantum computers, Physical Review D 101, 074512 (2020).

[34] A. F. Shaw, P. Lougovski, J. R. Strikyer, and N. Wiebe, Quantum algorithms for simulating the lattice schwinger model, Quantum 4, 306 (2020).

[35] V. Kasper, T. V. Zache, F. Jendrzejewski, M. Lewenstein, and E. Zohar, Non-abelian gauge invariance from dynamical decoupling, arXiv preprint arXiv:2012.08620 (2020).

[36] A. J. Buser, H. Gharibyan, M. Hanada, M. Honda, and J. Liu, Quantum simulation of gauge theory via orbifold lattice, Journal of High Energy Physics 2021, 1 (2021).

[37] J. F. Haase, L. Dellantonio, A. Celi, D. Paulson, A. Kan, K. Jansen, and C. A. Muschik, A resource efficient approach for quantum and classical simulations of gauge theories in particle physics, Quantum 5, 393 (2021).

[38] A. B. Zamolodchikov and A. B. Zamolodchikov, Factorized S-matrices in two dimensions as the exact solutions of certain relativistic quantum field theory models, Annals of Physics 120, 253 (1979).

[39] J. Balog and Á. Hegedüs, TBA equations for excited states in the O(3) and O(4) nonlinear -model, Journal of Physics A: Mathematical and General 37, 1881 (2004).

[40] J. Balog and Á. Hegedüs, The finite size spectrum of the 2-dimensional O(3) nonlinear $\sigma$-model, Nuclear Physics B 829, 425 (2010).

[41] M. Lüscher, Quantum non-local charges and absence of particle production in the two-dimensional non-linear $\sigma$-model, Nuclear Physics B 135, 1 (1978).

[42] M. Lüscher, P. Weisz, and U. Wolff, A numerical method to compute the running coupling in asymptotically free theories, Nuclear Physics B 359, 221 (1991).

[43] W. Bietenholz, A. Pochinsky, and U.-J. Wiese, Testing Haldane’s Conjecture in the O(3) Model by a Meron Cluster Simulation, Nuclear Physics B – Proceedings Supplements 47, 727 (1996).

[44] S. Caracciolo, R. G. Edwards, A. Pelissetto, and A. D. Sokal, Asymptotic scaling in the two-dimensional O(3) $\sigma$ model at correlation length 10$^3$, Physical Review Letters 75, 1891 (1995).

[45] J. Balog, P. Forgacs, and L. Pallá, A two-dimensional integrable axionic sigma-model and T-duality, arXiv:hep-th/0004180 10.1016/S0370-2693(00)00645-6 (2000).

[46] M. Bogli, F. Niedermayer, M. Pepe, and U. J. Wiese, Non-trivial $\theta$-vacuum effects in the 2-d o(3) model, JHEP 04, 117.

[47] P. de Forcrand, M. Pepe, and U. J. Wiese, Walking near a conformal fixed point: The 2-d o(3) model at $\theta \approx \pi$ as a test case, Physical Review D 86, 075006 (2012).

[48] K. G. Wilson and J. B. Kogut, The Renormalization group and the epsilon expansion, Phys.Rept. 12, 75 (1974).

[49] K. G. Wilson, The renormalization group and critical phenomena, Reviews of Modern Physics 55, 583 (1983).

[50] J. B. Kogut, An introduction to lattice gauge theory and spin systems, Reviews of Modern Physics 51, 659 (1979).

[51] B. B. Beard, M. Pepe, S. Riederer, and U.-J. Wiese, Study of CP$\left(\pi - 1\right)$ $\theta$-vacua by cluster simulation of SU($n$) quantum spin ladders, Physical Review Letters 94, 010603 (2005).

[52] S. Caracciolo, R. G. Edwards, A. Pelissetto, and A. D. Sokal, Wolff type embedding algorithms for general nonlinear sigma models, Nucl.Phys. B403, 475 (1993).

[53] R. Shankar and N. Read, The $\theta = \pi$ nonlinear sigma model is massless, Nuclear Physics B 336, 457 (1990).

[54] I. Affleck, Field Theory Methods and Quantum Critical Phenomena, in Les Houches Summer School in Theoretical Physics: Fields, Strings, Critical Phenomena (1988).

[55] M. A. Martin-Delgado, R. Shankar, and G. Sierra, Phase Transitions in Staggered Spin Ladders, Physical Review Letters 77, 3443 (1996).

[56] G. Sierra, The nonlinear sigma model and spin ladders, Journal of Physics A: Mathematical and General 29, 3299 (1996).

[57] F. D. M. Haldane, Continuum dynamics of the 1-D Heisenberg antiferromagnetic identification with the O(3) nonlinear sigma model, Phys.Lett. A93, 464 (1983).

[58] F. D. M. Haldane, Nonlinear field theory of large spin Heisenberg antiferromagnets. Semiclassically quantized solitons of the
one-dimensional easy Axis Neel state, Phys.Rev.Lett. **50**, 1153 (1983).

[59] I. Affleck and F. D. M. Haldane, Critical theory of quantum spin chains, Physical Review B **36**, 5291 (1987).

[60] L. C. Venuti, C. D. E. Boschi, E. Ercolessi, F. Ortolani, G. Morandi, S. Pasini, and M. Roncaglia, Particle content of the nonlinear sigma model with a $\theta$-term: A lattice model investigation, Journal of Statistical Mechanics: Theory and Experiment **2005**, L02004 (2005).

[61] N. Read and S. Sachdev, Valence-bond and spin-Peierls ground states of low-dimensional quantum antiferromagnets, Physical Review Letters **62**, 1694 (1989).

[62] N. Read and S. Sachdev, Some features of the phase diagram of the square lattice SU(N) antiferromagnet, Nuclear Physics B **316**, 609 (1989).

[63] N. Prokof’ev and B. Svistunov, Worm Algorithm for Problems of Quantum and Classical Statistics, arXiv:0910.1393 [cond-mat, physics:hep-lat] (2010).

[64] N. Prokof’ev and B. Svistunov, Worm Algorithms for Classical Statistical Models, Phys.Rev.Lett. **87**, 160601 (2001).

[65] U. J. Wiese and H. P. Ying, A determination of the low energy parameters of the 2-d Heisenberg antiferromagnet, Zeitschrift für Physik B Condensed Matter **93**, 147 (1994).

[66] B. B. Beard and U.-J. Wiese, Simulations of Discrete Quantum Systems in Continuous Euclidean Time, Physical Review Letters **77**, 5130 (1996).

[67] T. Banks and A. Zaks, On the phase structure of vector-like gauge theories with massless fermions, Nuclear Physics B **196**, 189 (1982).

[68] G. Sierra, On the application of the nonlinear sigma model to spin chains and spin ladders, Lect.Notes Phys. **478**, 137 (1997).