Analytical Investigation of Step-Square Wave in Time and Frequency Domain Using Transform Equations

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Abstract. A unit step function can be used for generating a pulse of given width and height. The same pulse can be used for generating a stepped waveform of known levels and a given width in time-domain which when translated to frequency or complex S-domain provides the fundamental constituent frequencies. This paper is about studying the frequency of the above-generated functions in time as well in the frequency domain using transform equations. The result of this simulations are indicative of the correlation between step square waving and reductions in total harmonic distortion (THD) during DC to AC conversion. A decrease in the pulse width difference between each step shows better conversion with reduced harmonics when subjected to simple passive filter of low transfer function.

1. Introduction

In today’s broad field of wave applications for theoretical and physical interest, transforms are used in understanding the fundamental component and simplification of the complex equations at different domain [1]. Such commonly used transform include the Fourier and Laplace transform which when carried out on a function f(t) in time domain produces the signal in the frequency domain F(ω) and complex frequency s-domain F(s) respectively. On a general term, the Fourier Transform decomposes any function into a sum of sinusoidal basis functions where each of these basis functions is a complex exponential of a different frequency [1-2]. The Fourier transform therefore offers a unique way of viewing any function - as the sum of simple sinusoids. Through this, noise reduction and harmonics cancelation system could be achieved for any signal. For a square like waveform, described as a pulse train, or pulse wave, is a periodic waveform consisting of instantaneous transitions between two distinct levels easily generated by the switching High and low state of a DC voltage source has found much application in today worlds such as timings circuits, PWM, H-bridge, DC-DC & DC-AC conversion, and generation etc. [1-13]. In most application where independent source are required to combine in steps or linearly every individual source offers additional input to either the stability of the output or extra source of noise hence, This work is about studying step square wave forms analytically using MATLAB R2016a, deriving their expressions and behavior at different domain by analyzing their transform. The effect of amplitude and pulse width change would be considered at this domain, and also the effect of combining the signals before and after subjection to filter circuit of known transfer function \( H(s||j\omega) \) would be explored since in the real world circuits behavior with different sources are mostly linear—adding or subtracting based on the combination [2, 7, 8, 10].
2. Theory and transform equations

The Fourier transform $\mathcal{F}(t)$ is intimately related with the Laplace transform $\mathcal{L}(s)$ which is also used for the solution of differential equations and the analysis of filters systems. The Laplace and the Fourier transforms for a causal function are the same, provided that the real part of $s$ is zero [3, 14] therefore more focus is on the Fourier transform. For any given function $f(t)$ the Fourier transform and the inverse transform can be written respectively as

$$\mathcal{F}\{f(t)\} = F(f) = \int_{-\infty}^{\infty} f(t)e^{-2\pi ift} dt \quad (1)$$

$$\mathcal{F}^{-1}\{F(f)\} = \int_{-\infty}^{\infty} F(f)e^{2\pi ift} df = f(t) \quad (2)$$

The Laplace transform on the other hand for a function $f(t)$ takes it to a function of a complex variable $F(s)$ in the frequency domain. Where $s = \sigma + \omega$. The transform and inverse operator is given as

$$\mathcal{L}\{f(t)\} = F(s) = \int_{0^-}^{\infty} f(t)e^{-st} dt \quad (3)$$

$$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i} \lim_{T \to \infty} \int_{y-iT}^{y+iT} F(s)e^{st} ds = f(t) \quad (4)$$

If we consider a unit square wave in time domain $f(t)$ generated by a sequence of close-open-close of a light switch. The resulting wave can be described as

$$f(t) = \begin{cases} A & \text{if } -\tau \leq t \leq \tau \\ 0 & \text{otherwise} \end{cases} \quad (5)$$

Where $\tau = \text{On Time}$ & $A = \text{DC Amplitude}$. Since a square wave is the sum of sinusoid with different fundamental frequency it can also be expressed as

$$f(t) = u(t + \tau) - u(t - \tau) \quad (6)$$

Where $u(t)$ is called the step function defined as

$$u(t) = \begin{cases} A & t > 0 \\ 0 & t < 0 \end{cases} \quad (7)$$

Where $f(t)$ is a linear sum of two unit step function which has been time delayed and inversed and the other time advanced. Hence The Fourier transform on this square pulse should give the $\text{sinc}(t)$ function described mathematically as

$$\mathcal{F}\{f(t)\} = F(f) = \int_{-\infty}^{\infty} f(t)e^{-2\pi ift} dt$$
\[
\int_{-\tau}^{\tau} Ae^{-2\pi jft} dt = \frac{A}{-2\pi jf} \left[ e^{-2\pi jft} \right]_{-\tau}^{\tau} = \frac{A}{-2\pi jf} [e^{-2\pi jft} - e^{2\pi jft}] = \frac{A\tau}{\pi f} \left[ \frac{e^{2\pi jft} - e^{-2\pi jft}}{2i} \right] = \frac{A\tau}{\pi f} \sin(\pi ft) = A\tau [\text{sinc}(ft)]
\]

The transfer function \( H(s) = \frac{1}{s+1} = \frac{1}{j\omega + 1} \) of a known filter can then be linearly applied to condition the wave. Where

\[
H(s) = \frac{1}{s+1} = \frac{1}{j\omega + 1}
\]

where \( R \) and \( C \) are unity otherwise

\[
= \frac{1}{sc + 1} = \frac{1}{\omega + 1}
\]

3. Analytical plots from transform equations

Considering the theoretical equations in section II, the equivalent model is created using MATLAB R2016a to obtain various analytical plots of a square pulse derived from constituent unit step functions in the time domain. The acquired pulse is then further combined to obtain waves of different width, height and steps. The resulting pulse train is then subjected to various transform from time domain to frequency domain and complex frequency S-domain shown in figure 1-12.

4. Discussion and application of results

In this work we have modeled a DC to AC conversion using analytical approach in MATLAB and studied the effect of different changes along the conversion process. The design is based on Unit step function for creating square pulse and step square wave. In Figure 1. An alternating (sinusoidal) wave obtained analytically from a unit step function — square pulse — step square wave + transform + filter. It was observed that the higher the step number the lower the transfer function of the filter required for the conversion and hence an overall reduction in the total harmonic distortion (THD) for the system. In figure 2, an impulse function was created in time domain and transformed, the result in the frequency and complex frequency domain are a DC with zero frequency component likewise the unit step function in time domain produces an impulse function in the frequency domain and other fundamental frequency peak attenuated. From figure 3, the unit step function have been used to derive a unit square pulse. Its transform into the frequency domain at each step is shown in figure 4 and the unit step square wave function in the frequency domain is described as a sinc (\( \omega \)) function shown at the bottom right corner of Figure 3 and in the bottom row of figure 4 the single side double sided band of the Fourier transform is show with their imaginary components. In figure 5 the effect of change the width and height respectively is illustrated in both time and frequency domain using transform. For change in width the fundamental frequency is observed to start converging and as the pulse width tends to infinity in the time domain the frequency converges to an impulse while a change in amplitude or pulse height only increase the power spectrum energy and no clear difference in frequency shift was observed at different heights. In figure 6 the unit step square wave of varying width are added and transformed, the effect of adding this unit step is linear. The result of addition is increase in the number of step and in the frequency domain the fundamental frequency is found to start converging and as the step increases the lesser the harmonics present in the transformed wave. In figure 7, the step...
wave have been inverted and low pass filtered at different transfer function. In figure 8, the delayed and inverted step square wave have been linearly added to its original wave and the resulting wave is full cycle step wave. Figure 9 and 10 shows the effect of applying filter to the resulting wave at different cut off frequency in the transfer function of the low pass filter. It is seen that a very nice AC signal was produced and is shown in figure 11 to figure 12. In figure 12 (a) and 12 (b) respectively, a single cycle and multiple cycle filtered step square wave is then transformed into frequency domain.

**Figure 1.** A step square wave is $F(\omega)$ transform and low pass Filtered at various $f_c$ then inverse $F(\omega)$ producing a sinusoid $f(t)$

**Figure 2.** $\delta(t)$ and $u(t)$ transformed to $\delta(\omega)$ and $u(\omega)$

**Figure 3.** Converting Unit pulse into square pulse then fourier transform
Figure 4. Fourier transform of all function in figure 3.

Figure 5. Effect of width and height change in $t$ and $\omega$ domain.

Figure 6. Effect of adding signal (step pulse) in $t$ and $\omega$ domain.

Figure 7. Effect of filtering on step pulse in $t$ and $\omega$ domain.

Figure 8. Effect of adding step signal in time domain.

Figure 9. Filtering at different $H(s || j\omega)$.
5. Conclusion
In this research, we have generated a step square wave function for analytical investigation of the square pulse from a unit step function in the time domain using MATLAB R2016a. The square wave has been subjected to analytical transform in frequency and time domain where the effect of width and height change, time delay and advance, time reversal and inversion and filtering where considered. By increasing the step number the transfer function required to generate a perfect sinusoid also reduce making it possible for DC to AC conversion with simple low pass filter—preferable single pole low passive filter. Also by increasing the width of the pulse in the time domain, the fundamental frequency present tend to converge reducing the number of harmonics present in the system. When the height was change and the power spectral density was plotted, the power intensity of the spectrum also increased. Combining the width change and the height change a step square wave was generated which when subjected to the transform produced a much lower harmonics distortion. This is very much important in real world application where reduction of total harmonic distortion (THD) in the system is necessary. In conclusion, we could generalize that when the width of a pulse in the time domain tend towards a DC produces an Impulse in the frequency domain where all the fundamental frequency has converged and center at zero. Subsequently, an impulse in the time domain would also produce DC in the frequency domain.
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