Aspects of Muchnik’s paradox in restricted betting *

George Barmpalias  Lu Liu

Wednesday 19th January, 2022 at 02:54

Abstract. Muchnik’s paradox says that enumerable betting strategies are not always reducible to enumerable strategies whose bets are restricted to either even rounds or odd rounds. In other words, there are outcome sequences \( x \) where an effectively enumerable strategy succeeds, but no such parity-restricted effectively enumerable strategy does. We characterize the effective Hausdorff dimension of such \( x \), showing that it can be as low as \( 1/2 \) but not less. We also show that such reals that are random with respect to parity-restricted effectively enumerable strategies with packing dimension as low as \( \log \sqrt{3} \). Finally we exhibit Muchnik’s paradox in the case of computable integer-valued strategies.

George Barmpalias
State Key Lab of Computer Science, Institute of Software, Chinese Academy of Sciences, Beijing, China. E-mail: barmpalias@gmail.com. Web: http://barmpalias.net

Lu Liu
Department of Mathematics, Central South University, ChangSha 410083, China
E-mail: g.jiayi.liu@gmail.com.

*Supported by NSFC grant No. 11971501.
1 Introduction

A casino may be determined by the set of binary outcome sequences $x$ that it produces, where sequential bets can be placed by players using wagers they can afford (namely at most their current capital). A betting strategy may be qualified by the amount of information that it uses, for example its computational or arithmetical complexity. Suppose that a player can amass unbounded fortune while betting on a casino sequence $x$, starting from a finite initial capital.

Can this player also win when she is restricted to bet on a specific subset of stages/bits, say the even or the odd bits of $x$?

It turns out that the answer depends on the computational complexity or effectivity of the strategies considered. To be more precise, we introduce some standard formalism. Following Ville (1939), a betting strategy can be modeled by a non-negative function $\sigma \mapsto M(\sigma)$ on the finite binary strings $2^{<\omega}$ representing the capital of the player at state $\sigma$, such that

$$M(\sigma) = \frac{M(\sigma * 0) + M(\sigma * 1)}{2}. \hspace{1cm} (2)$$

We call such functions martingales and note that they have a probabilistic interpretation: the expected capital gain after a bet is zero. In this formalization of strategies the wagers and favorable outcomes are implicit: if $M(\sigma * 0) < M(\sigma * 1)$ then the player bets on 1 the amount $M(\sigma * 1) - M(\sigma)$, which he doubles or loses depending on the outcome. The case $M(\sigma * 0) > M(\sigma * 1)$ is analogous while $M(\sigma * 0) = M(\sigma * 1)$ means that the player does not bet. Betting strategies that account for inflation or saving (permanently taking capital out of the casino) are modeled by supermartingales which are functions $M$ such that (2) holds with $=$ replaced by $\geq$, hence the expected gain is no more than 0. We say that $M$ succeeds on $x$ if $\limsup_n M(x | n) = \infty$ and say that $M$ is

- even-parity if $M(\sigma * i) = M(\sigma)$ for all $\sigma$ of even length and $i < 2$;
- odd-parity if $M(\sigma * i) = M(\sigma)$ for all $\sigma$ of odd length and $i < 2$;

and refer to either case as a single-parity (super)martingale.

Muchnik (2009) showed for the class of computable strategies, question (1) has a positive answer:

every computable martingale is the product of an even-parity and an odd-parity martingale, and the same is true of supermartingales. \hspace{1cm} (3)
Betting on $x$ with an even-parity strategy is quite stronger to betting on a casino sequence consisting of the even bits of $x$: an even betting strategy has access to all the previous (including the odd) bits of $x$, which may inform the current bet. For example, if $x$ is algorithmically random then a computable even-parity strategy succeeds on $x \oplus x$ but no computable strategy succeeds on $x$.\(^1\) Computable martingales were proposed as a foundation of algorithmic information by Schnorr (1971a) but do not capture the standard notion of randomness by Martin-Löf (1966). For the latter we need to consider the class of left-c.e. (super)martingales $M$, where a function is left-c.e. if its values are uniformly computably approximable from below: $x$ is Martin-Löf random iff no left-c.e. (super)martingale succeeds on $x$. A left-c.e. (super)martingale whose wager-function is also left-c.e. is called strongly left-c.e. (super)martingale; note that a left-c.e. (super)martingale is not necessarily strongly left-c.e.

Muchnik (2009) showed that (1) has a negative answer for the class of left-c.e. (super)martingales:

\[ \text{Muchnik’s paradox: there exists } x \text{ such that some left-c.e. martingale succeeds on } x \text{ but no even-parity or odd-parity left-c.e. (super)martingale succeeds on } x. \] (4)

This means that, in the context of enumerable bets (left-c.e. strategies) a player can win on $x$, but can no longer win if restricted to single-parity strategies. This has been called Muchnik’s paradox and was discussed extensively by Chernov et al. (2008) in the context of online complexity and prediction, as well as by Bauwens (2014) who produced Kolmogorov complexity versions of it and related it to real-life scenarios and the theory of causality of timeseries.

Muchnik (2009) does not give any information about the descriptive complexity of $x$ in (4), other than it is not Martin-Löf random. The present work deals with two aspects of (4):

(a) How predictable can $x$ be, while still immune to single-parity guessing?

(b) Does this paradox occur in the context of other strategy restrictions such as minimum bets?

Our results. In §4 we show that there exists $x$ which is strongly predictable, in the sense that betting on successive pairs of bits we need at most three guesses to guess correctly

\[ \text{(out of the possible four possibilities) but no single-parity left-c.e. supermartingale succeeds on } x. \] (5)

This strong version of (4) implies the following.

**Theorem 1.1** (Packing dimension and single-parity betting). There exists $x$ with effective packing dimension $\text{Dim}(x) \leq \log \sqrt{3}$ and all single-parity left-c.e. supermartingales fail on $x$.

that the effective Hausdorff dimension $\text{dim}(x)$ of $x$ is at most $\log \sqrt{3} \approx 0.79$. As we explain in §2, this means that some left-c.e. martingale succeeds on $x$ exponentially fast.

Our main result is the following characterization of the effective Hausdorff dimension of the reals that can exhibit Muchnik’s paradox.

**Theorem 1.2** (Hausdorff dimension and single-parity betting).

(i) if $\text{dim}(x) < 1/2$ then there exist even-parity and odd-parity (strongly) left-c.e. martingales $N, T$ which succeed on $x$: $\lim_n N(z \upharpoonright n) = \lim_n T(z \upharpoonright n) = \infty$.

\[^1\] $x \oplus y := x_0y_0x_1y_1 \ldots$, where $x := x_0x_1 \ldots$ and $y := y_0y_1 \ldots$
(ii) \( \exists x \) with \( \dim(x) = 1/2 \) and all single-parity left-c.e. supermartingales fail on \( x \).

Moreover \( x \) in (ii) can be chosen a left-c.e. real.

The construction of \( x \) in clause (ii) is dynamic, and \( \dim(x) \) approaches 1/2 through bets on successively longer blocks of outcomes. In this sense the betting strategy for (5) in §4 is simpler and more compact, at the expense of achieving a larger dimension bound \( \log \sqrt{3} > 1/2 \).

In §3 we examine the above problem, namely the reducibility of a strategy to single-parity strategies, in the context of computable integer-valued (super)martingales. Such restricted strategies have been studied by Bienvenu et al. (2010), Teutsch (2014), Barmpalias et al. (2015) and are a special case of the the restricted wager strategies studied by Chalcraft et al. (2012), Peretz and Bavly (2015), Peretz (2015) and Barmpalias and Fang (2020). This aspect is interesting for the following reason: on the one hand computable strategies are reducible to single-parity strategies by (3), on the other hand this decomposition by Muchnik (2009) involves scaling wagers which is not possible if the strategies are integer-valued.

**Theorem 1.3** (Integer-valued). There exists \( z \) such that some computable integer-valued martingale succeeds on \( z \) but each computable integer-valued single-parity supermartingale fails on \( z \). Moreover we can ensure that \( \dim(z) = 0 \).

Single-parity strategies have an apparent similarity to the monotonous or single-sided strategies studied by Barmpalias et al. (2020): these are strategies that bet on a single outcome, 0 or 1. Indeed the analogue of the decomposition (3) as well as (i) of Theorem 1.2 were shown for single-sided strategies, and in §3 we prove an analogue of Theorem 1.3 for single-sided integer-valued strategies. However left-c.e. single-sided strategies can be much more complex than single-parity strategies as we explain in §2.1, and this is the reason that (ii) of Theorem 1.2 was only shown for the special case of strongly left-c.e. single-sided strategies in (Barmpalias et al., 2020). Recently, using different methods, Barmpalias and Liu (2021) showed that there exists a non-random \( x \) such that all single-sided left-c.e. supermartingales fail on \( x \). It is not known whether such a real can have effective Hausdorff dimension < 1. This and related problems are discussed in §6 along with a summary of our results.

**A word on notation.** We maintain some regularity in our notation, using variables \( i, j, s \) for non-negative integers, \( \sigma, \tau \) for strings, \( x, y, z \) for reals, \( M, N, T, D \) for (super)martingales.

## 2 Background on enumerable and restricted strategies

We show some facts about (super)martingales that are used in the next sections and background that is directly relevant to our results and is not readily available in the literature in this form. General background can be found in the textbooks (Shen et al., 2017) and (Downey and Hirschfeldt, 2010).

### 2.1 Enumerable strategies and restrictions

Intuitively, the complexity of a strategy is an expression of the information that the casino has about the player: if a casino plays against a class of highly complex strategies, the outcome sequences that they need to produce in order to ensure that the strategies do not win is also complex. We typically gauge complexity in terms of arithmetical definability or computability. We say that \( x \) is random with respect to a class \( M \) of (super)martingales if no member of \( M \) succeeds on \( x \).
Enumerable strategies represent a step of complexity beyond the computable strategies. From the point of view of a constructive observer, bets are placed on the various outcomes at various times (stages), and the final capital at a particular state (namely prefix of binary outcomes) of the game is only a limit of a computable sequence of compound bets. This is made precise by:

A martingale is left-c.e. if and only if it can be written as the sum of a uniformly computable sequence of martingales. \cite[Lemma 2.5]{DowneyHirschfeldt2010}.

Every computable supermartingale is bounded above by a computable martingale, but it is well-known that the optimal left-c.e. supermartingale is not bounded above by any left-c.e. martingale. In fact, there is no uniform list of all left-c.e. martingales, and this is the reason it is often easier to work with supermartingales; this can sometimes be done without loss of generality as whenever some left-c.e. supermartingale succeeds on a real $x$, there exists a left-c.e. martingale that succeeds on $x$. Another fact concerns the standard winning condition $\limsup_n M(x \upharpoonright n) = \infty$ to the stronger $\lim_n M(x \upharpoonright n) = \infty$. It is known that if $\limsup_n M(x \upharpoonright n) = \infty$ for a left-c.e. supermartingale $M$ then there exists a left-c.e. martingale $N$ such that $\lim_n N(x \upharpoonright n) = \infty$; see \cite[§6.3]{DowneyHirschfeldt2010}.

In the case of restricted strategies one needs to be careful as many of the above facts may fail. For example, for some wager-restricted martingales, including the integer-valued martingales, the equivalence between the two winning conditions fails; see \cite[Barmpalias and Fang, 2020]{Teutsch2014}.

Single-parity strategies preserve more of the standard properties above due to the fact that, by Chernov et al. \cite{Chernovetal2008}, they can be expressed as unrestricted strategy with online access to an oracle (the previous outcomes that were off-limits for betting). We may say that $(\sigma, \tau) \mapsto N(\tau \mid \sigma)$ is an online martingale if for each $|\sigma| = |\tau|:

$$N(\hat{\tau} \ 0 \mid \sigma) + N(\hat{\tau} \ 1 \mid \sigma) = 2N(\hat{\tau} \mid \hat{\sigma})$$

where $\hat{\sigma}, \hat{\tau}$ are the predecessors of $\sigma, \tau$ (similar for supermartingales). This means that the bet placed at $\hat{\tau}$ is conditional on the last bit of $\sigma$. Then it is clear that if $M$ is a left-c.e. even-parity martingale, there exists an online left-c.e. martingale $N$ such that $M(\sigma \oplus \tau) = N(\tau \mid \sigma)$.

\textbf{Lemma 2.1.} If $M$ is an even-parity left-c.e. martingale then

(a) there exists a uniformly computable family $(N_i)$ of even-parity martingales such that $M = \sum_i N_i$.

(b) if $\limsup_n M(x \upharpoonright n) = \infty$ there exists a left-c.e. even-parity martingale $N$ with $\lim_n N(x \upharpoonright n) = \infty$.

Both clauses also hold for odd-parity martingales, and (b) also holds when $M$ is a supermartingale.

\textbf{Proof.} A direct relativization of \cite[Lemma 2.5]{DowneyHirschfeldt2010} shows that (6) holds for online martingales. Then for (a) it suffices to apply this result to an online left-c.e. martingale $N$ such that $M(\sigma \oplus \tau) = N(\tau \mid \sigma)$. In particular, let $(N_i)$ be a computable family of online left-c.e. martingales such that $N(\tau \mid \sigma) = \sum_i N_i(\tau \mid \sigma)$. Then $M(\sigma \oplus \tau) = \sum_i N'_i(\sigma \oplus \tau)$ where $N'_i(\sigma \oplus \tau) := N_i(\tau \mid \sigma)$ are uniformly computable and even-parity.

A direct relativization of the argument in \cite[§6.3.1]{DowneyHirschfeldt2010} for left-c.e. supermartingales shows that it also holds for online supermartingales. Hence for (b) it suffices to apply this fact to an online left-c.e. supermartingale $N$ such that $M(\sigma \oplus \tau) = N(\tau \mid \sigma)$. The case of odd-parity martingales is symmetric. \hfill $\square$

5
Surprisingly, the closely related single-sided strategies can be more complex and do not necessarily satisfy the decomposition of (i) of Lemma 2.1. The reason for this is that the wager $|M(\sigma) - M(\sigma \ast 0)|$ of a left-c.e. martingale is not necessarily left-c.e. while the wagers of an effective mixture of computable single-sided strategies are. In this case the closest analogue is that given $i < 2$:

for every strictly $i$-sided left-c.e. martingale $M$, there exists a uniformly computable sequence $(N_i)$ of $i$-sided martingales such that the partial sums $S_n = \sum_{i<n} N_i$ are strictly $i$-sided and converge to $M$. (Barmpalias et al., 2020, Lemma 2.7)

where strictly $i$-sided means that $M(\sigma_i) > M(\sigma \ast (1 - i))$ for all $\sigma$. The point here is that if we view $M$ as an enumeration of bets, we only know that the aggregate bet at any initial segment of outcomes is single-sided, while individual sub-bets in the enumeration could favor either outcome 0,1. This issue was discussed by Barmpalias et al. (2020, §2.2) where these sub-bets were called intermediate. For a decomposition into single-sided strategies we need the stronger hypothesis that $M$ is strongly l.c.e., which means that the wagers of $M$ are also l.c.e.: 

for every strongly left-c.e. strictly $i$-sided martingale $M$, there exists a uniformly computable sequence $(N_i)$ of $i$-sided martingales such that the partial sums $S_n = \sum_{i<n} N_i$ converge to $M$. (Barmpalias et al., 2020, Lemma 2.8)

This is the reason that (Barmpalias et al., 2020, Theorem 3.3) was restricted to strongly left-c.e. martingales: there exists $x$ with $\dim(x) = 1/2$ and each single-sided strongly left-c.e. martingale fails on $x$. Constructing a non-random $x$ such that all single-sided left-c.e. (super)martingales fail on $x$ is considerably more challenging and requires different methods, see (Barmpalias and Liu, 2021). Moreover we do not know how low the effective Hausdorff dimension of such a real can be (other than it must be at least 1/2), and indeed whether it can be less than 1.

An even-parity left-c.e. supermartingale $M$ is optimal if for any even-parity left-c.e. supermartingale we have $M' = O(M)$; a similar definition applies for odd-parity left-c.e. supermartingales. Let $\lambda$ denote the empty string.

**Lemma 2.2.** There exist optimal even-parity and odd-parity left-c.e. supermartingales $M,N$ respectively such that $M(\lambda) < 1/2, N(\lambda) < 1/2$.

**Proof.** The weighted sum of two even-parity left-c.e. supermartingales is an even-parity left-c.e. supermartingale. Moreover there exists a universal enumeration $(M_i)$ of all left-c.e. even-parity supermartingales $M$ with $M(\lambda) \leq 1$. Hence $\sum_{i} 2^{-i} \cdot M_i$ is an optimal left-c.e. even-parity supermartingale. A similar argument applies for the case of odd-parity left-c.e. supermartingales. 

**2.2 Speed of capital-gain and fractal dimensions**

The martingale approach to algorithmic information theory was introduced by Schnorr (1971b,c) who also showed some interest in the rate of success of (super)martingales $M$, and in particular the classes

$$S_h(M) = \left\{ x \mid \limsup_n \frac{M(x \upharpoonright n)}{h(n)} = \infty \right\}$$

and $$\hat{S}_h(M) = \left\{ x \mid \liminf_n \frac{M(x \upharpoonright n)}{h(n)} = \infty \right\}$$

where $h : \mathbb{N} \to \mathbb{N}$ is a computable non-decreasing function. Lutz (2000, 2003) showed that the Hausdorff dimension of a class of reals can be characterized by the exponential ‘success rates’ of
left-c.e. supermartingales, and in that light defined the effective Hausdorff dimension \( \dim(x) \) of a real \( x \) as the infimum of the \( s \in (0, 1) \) such that \( x \in S_{h}(M) \) for some left-c.e. supermartingale \( M \), where \( h(n) = 2^{(1-s)n} \). Mayordomo (2002) showed that

\[
\dim(x) = \liminf_{n} \frac{K(x \upharpoonright n)}{n}
\]

where \( K \) denotes the prefix-free Kolmogorov complexity. Martin-Löf random reals have effective dimension 1, but the converse does not hold. Moreover there are computably random reals (no computable martingale succeeds on them) of effective dimension 0.

Athreya et al. (2007) showed that the packing dimension of C. Tricot (1982); Sullivan (1984) can be effectivized and characterized in a similar way:

\[
\text{Dim}(x) := \inf \{ s \in (0, 1) : x \in \hat{S}_{h}(M) \} = \limsup_{n} \frac{K(x \upharpoonright n)}{n}
\]

where \( h(n) = 2^{(1-s)n} \) and the infimum is taken over all left-c.e. supermartingales \( M \). We call \( \text{Dim}(x) \) the effective packing dimension of \( x \). Note that \( \dim(x) \leq \text{Dim}(x) \) for all \( x \).

Lemma 2.3. Suppose that \( (V_{i}) \) is uniformly c.e., \( V_{i} \subseteq 2^{2i} \) and for each \( \sigma \in V_{i} \) at most three of the four 2-bit extensions of \( \sigma \) belong to \( V_{i+1} \). Then each \( x \) which has a prefix in each \( V_{i} \) has \( \text{Dim}(x) \leq \log \sqrt{3} \).

Proof. By the hypothesis, for each \( \sigma \in V_{i} \) we have \( \mu_{\sigma}(V_{i+1}) \leq 3/4 \).

Define \( M \) by \( M(\lambda) = 1 \) and for each \( i, \sigma \in V_{i} \), \( \sigma' > \sigma \) with \( |\sigma'| = |\sigma| + 2 \) let

\[
M(\sigma') = \begin{cases} 
(4/3) \cdot M(\sigma) & \text{if } \sigma' \in V_{i+1} \\
0 & \text{if } \sigma' \notin V_{i+1}
\end{cases}
\]

Clearly \( M \) is left-c.e. on the strings of even length where it is defined, and

\[
4M(\sigma) \geq M(\sigma \ast 00) + M(\sigma \ast 01) + M(\sigma \ast 10) + M(\sigma \ast 11)
\]

so it is uniquely extendible to a supermartingale on \( 2^{\omega} \). By the definition of \( M \) it follows that for each \( n \) and \( \sigma \in 2^{2n} \) such that \( \forall i \leq n \sigma \upharpoonright 2i \in V_{i} \) we have \( M(\sigma) \geq (4/3)^{n} \) so

\[
\liminf_{n} \frac{M(x \upharpoonright n)}{(4/3)^{n/2}} > 0.
\]

For each \( s > \log \sqrt{3} \) we have \( 2^{(1-s)n} < (4/3)^{n/2} \) hence

\[
s > \log \sqrt{3} \quad \Rightarrow \quad \liminf_{n} \frac{M(x \upharpoonright n)}{2^{(1-s)n}} = \infty
\]

which shows that \( \text{Dim}(x) \geq \log \sqrt{3} \), by the definition of effective packing dimension. □

For further background on algorithmic dimension see (Downey and Hirschfeldt, 2010, Chapter 13).
2.3 The van Lambalgen theorem

The case of single-parity betting is superficially relevant to the van Lambalgen theorem, so we briefly review the interesting ways that the latter fails under computability restrictions.

Recall that $z$ is random with respect to the class $\mathcal{M}$ of supermartingales if $\mathcal{M}(z \upharpoonright n) = O(1)$ for each $M \in \mathcal{M}$. If $\mathcal{M}$ contains somewhat effective supermartingales that can be relativized to an oracle $y$, we may consider the class $\mathcal{M}^y$ of the members of $\mathcal{M}$ relativized to oracle $y$ and say that $z$ is $y$-random with respect to $\mathcal{M}$ if it is random with respect to $\mathcal{M}^y$. We say that $x, y$ are

- **mutually random** with respect to $\mathcal{M}$ if $x$ is $y$-random with respect to $\mathcal{M}$ and $y$ is $x$-random with respect to $\mathcal{M}$.

- **weakly mutually random** with respect to $\mathcal{M}$ if $x$ is random with respect to $\mathcal{M}$ and $y$ is $x$-random with respect to $\mathcal{M}$; or $x$ is $y$-random with respect to $\mathcal{M}$ and $y$ is random with respect to $\mathcal{M}$.

The van Lambalgen theorem says that the following are equivalent:

(i) $x \oplus y$ is random with respect to the class $\mathcal{M}$ of left-c.e. supermartingales;

(ii) $x, y$ are mutually random with respect to the class $\mathcal{M}$ of left-c.e. supermartingales;

(iii) $x, y$ are weakly mutually random with respect to the class $\mathcal{M}$ of left-c.e. supermartingales.

Restriction to the computable strategies has the following effects on this equivalence. We refer to randomness with respect to the class of computable supermartingales as computable randomness; mutual and weakly mutual computable randomness is defined analogously.

- there are $x, y$ such that $x \oplus y$ is computably random but $x, y$ are not mutually computably random; hence (i) → (ii) fails for computable strategies; (Yu, 2007)

- if $x, y$ are mutually is computably random then $x \oplus y$ is computably random; hence (ii) → (i) holds for computable strategies; (Bauwens, 2020)

- there are $x, y$ such that $x \oplus y$ is not computably random but $x, y$ are weakly mutually computably random; hence (iii) → (i) fails for computable strategies; (Bauwens, 2020).

Miyabe and Rute (2013) studied van Lambalgen’s theorem in the context of computable strategies and a notion of uniform relativization, and Chakraborty et al. (2017) considered the case of time-bounded computable strategies.

Both the single-parity strategies we considered earlier and the relativized strategies in the van Lambalgen theorem use the information in the even bits of the outcome in order to bet on the odd bits (or vice-versa). However the access to the even bits in the van Lambalgen theorem is unrestricted, while in the case of single-parity strategies it is online in the sense that when placing a bet on round $t$ the player only has access on the even bits $i < t$ (as well as the odd bits $< t$ where he was allowed to bet). This type of online advice was explored by Chernov et al. (2008) and Bauwens (2014), and can be equivalently formulated in terms of online program-size complexity or online semimeasures.
3 Integer-valued strategies and single-parity irreducibility

When the wagers of a strategy are restricted to a specific set of values, their success may be limited. Such restrictions were studied by Chalcraft et al. (2012), Peretz and Bavly (2015), Peretz (2015) while Bienvenu et al. (2010), Teutsch (2014), Barmpalias et al. (2015) focused on the specific case of integer wagers. Barmpalias and Fang (2020) studied the case where the permissible granularity for the wagers decreases at given rates.

We show that in the case of integer-valued supermartingales the reducibility to single-sided or single-parity strategies no longer holds, even in the case of computable strategies.

Definition 3.1. We define two computable integer-valued martingales $N, D$. Martingale $N$ only bets on outcome 1 and is given by:

- $N(\lambda) = 5$ and if $N(\sigma) = 0$ then $N(\sigma 0) = N(\sigma 1) = 0$
- otherwise, $N(\sigma 0) = N(\sigma) - 1$ and $N(\sigma * 1) = N(\sigma) + 1$.

Martingale $D$ bets the minimal wager 1 on the two outcomes alternately and is given by:

- $D(\lambda) = 5$ and if $D(\sigma) = 0$ then $D(\sigma 0) = D(\sigma 1) = 0$
- if $D(\sigma) > 0$ and $|\sigma|$ is even then $D(\sigma 0) = D(\sigma) + 1$ and $D(\sigma 1) = D(\sigma) - 1$
- if $D(\sigma) > 0$ and $|\sigma|$ is odd then $D(\sigma 0) = D(\sigma) - 1$ and $D(\sigma 1) = D(\sigma) + 1$.

Lemma 3.2. If $N(\rho) > 2, D(\rho) > 2$ then for each integer-valued supermartingale $M$

(i) if $M$ is single-parity, there exists $\tau > \rho$ such that $N(\tau) > c$ and $\forall \tau' > \tau, M(\tau') = M(\tau)$.

(ii) if $M$ is single-sided there exists $\tau > \rho$ such that $D(\tau) > c$ and $\forall \tau' > \tau, M(\tau') = M(\tau)$.

Proof. For (i), without loss of generality, assume that $M$ is an even-parity supermartingale and for each odd position $i$ after $|\rho|$ let $\tau(i) = 1$. Then for any choice of values in the even positions of $\tau$ after $|\rho|$ we have $N(\tau) \geq 1$. Since $M(\tau * j) = M(\tau)$ whenever $|\tau|$ is even and since $M$ is integer-valued, we may fix a finite segment of the even positions of $\tau$ after $|\rho|$ so that $\forall \tau' > \tau, M(\tau') = M(\tau)$. The conclusion then follows if we replace the constructed $\tau$ with $\tau * 1^c$.

For (ii), without loss of generality, assume that $M$ is 0-sided. Recall that $D$ bets on 0 at even positions and on 1 at odd positions. If on each odd position $i$ after $|\rho|$ we let $\tau(i) = 1$, then for any choice of values in the even positions of $\tau$ after $|\rho|$ we have $D(\tau) \geq 1$. Since $M(\tau * 1) \leq M(\tau)$ for all $\tau$ and since $M$ is integer-valued, we may fix a finite segment of the even positions of $\tau$ after $|\rho|$ so that $\forall \tau' > \tau, M(\tau') = M(\tau)$ and $|\tau|$ is even. The conclusion then follows if we replace the constructed $\tau$ with $\tau * (01)^c$. □

Theorem 3.3 (Integer-valued irreducibility).

(a) There exists $z$ such that some computable integer-valued martingale succeeds on $z$ but each computable integer-valued single-parity supermartingale fails on $z$.

(b) There exists $z$ such that some computable integer-valued martingale succeeds on $z$ but each computable integer-valued single-sided supermartingale fails on $z$. 

9
Proof. Recall the martingales $N, D$ of Definition 3.1. For (a), let $(M_i)$ be a list of all computable integer-valued single-parity supermartingales. By (i) of Lemma 3.2 we can define $z$ by initial segments so that $M_i(z \upharpoonright_n) = O(1)$ for all $i$ and $\lim_n N(z \upharpoonright_n) = \infty$. This proves (a). For (b) let $(M'_i)$ be a list of all computable integer-valued single-sided supermartingales. By (ii) of Lemma 3.2 we can define $z$ by initial segments so that $M'_i(z \upharpoonright_n) = O(1)$ for all $i$ and $\lim_n D(z \upharpoonright_n) = \infty$. This proves (b). \hfill \Box

Since martingale $N$ only bets on 1, the definition of $z$ in Theorem 3.3 by initial segments can be modified by interpolating arbitrarily long blocks ‘010101010...’ without losing the accumulated capital of $N$ along $z$. By choosing these segments sufficiently long we can ensure that the Kolmogorov complexity of $z$ drops sufficiently at the end of those segments, so that

$$\lim \inf \frac{K(z \upharpoonright_n)}{n} = 0$$

so, by the characterizations of §2.2, $\dim(z) = 0$.

4 Packing dimension and parity-betting: proof of Theorem 1.1

We show that there are highly predictable reals, where we can predict each consecutive pair of bits in 3-out-of-4 guesses, yet no single-parity left-c.e. supermartingale succeeds on them.

Theorem 4.1. There exists $x$ and a Martin-Löf test $(V_i)$ such that $V_i \subseteq 2^\omega$, $x \upharpoonright 2^\omega \in V_i$, $\mu_\sigma(V_i) \leq 3/4$ for each $\sigma \in V_i$, and no single-parity left-c.e. supermartingale succeeds on $x$.

By Lemma 2.3 we have $\text{Dim}(x) \leq \log \sqrt{3}$, hence Theorem 1.1.

Lemma 4.2. Given non-negative $m_0, m_1$ there exists an even-parity martingale $M_0$ on $2^{\omega^2}$ such that

$$M_0(00) = m_0 \land M_0(10) = m_1 \land M_0(\lambda) = \max\{m_0, m_1\}/2. \tag{8}$$

Moreover for every even-parity martingale $M$ on $2^{\omega^2}$ such that $M(00) \geq m_0 \land M(10) \geq m_1$ we have $M(\tau) \geq M_0(\tau), \tau \in 2^{\omega^2}$.

Proof. If $M$ is even-parity we have

$$M(\lambda) = M(0) = M(1) = \frac{M(00) + M(10)}{2} = \frac{M(10) + M(11)}{2} \tag{9}$$

which, along with (8), uniquely determines the values of $M_0(01), M_0(11)$:

- if $m_0 \geq m_1$ then $M_0(11) = m_0 - m_1$ and $M_0(01) = 0$
- if $m_0 < m_1$ then $M_0(01) = m_1 - m_0$ and $M_0(11) = 0$

which shows the first clause of the claim. On the other hand, for any martingale $M$ with $M(00) \geq m_0, M(10) \geq m_1$, condition (9) implies $M(\lambda) \geq \max\{m_0, m_1\}$, which in turn implies $M(\tau) \geq M_0(\tau), \tau \in 2^{\omega^2}$ as required. \hfill \Box

Lemma 4.3. Given $m_0, m_1, n_0, n_1 \geq 0$ there exist unique even and odd betting martingales $M_0, N_0$ on $2^{\omega^2}$ such that $M_0(\lambda) = \max\{m_0, m_1\}/2$, $M_0(00) = m_0, M_0(10) = m_10, N_0(0) = n_0, N_0(1) = n_1$ and
Lemma 4.4. Given \( m \)

\[ \begin{align*}
\text{Proof.} & \quad \text{Let } N_0 \text{ be the unique odd-parity martingale on } 2^{\leq 2} \text{ determined by } N_0(0) = n_0, N_0(1) = n_1 \text{ so } D_N := N - N_0 \text{ is a non-negative odd-parity martingale.} \\
\text{Lemma 4.4.} & \quad \text{Given } m_{00}, m_{10}, n_0, n_1, c \geq 0 \text{ suppose that } M, N \text{ are even and odd betting supermartingales such that } M(\eta 00) \geq m_{00}, N(\eta 0) \geq n_0, M(\eta 10) \geq m_{10}, N(\eta 1) \geq n_1, M(\eta) + N(\eta) \leq c \text{ and }
\end{align*} \]

\[ m_{00} + n_0 \geq c \wedge m_{10} + n_1 \geq c. \tag{10} \]

where \( \eta \) is a string of even length. Then

\[ (n_0 \leq n_1 \Rightarrow M(\eta 01) + N(\eta 01) \leq c) \wedge (n_0 > n_1 \Rightarrow M(\eta 11) + N(\eta 11) \leq c). \]

\textbf{Proof.} Without loss of generality we may assume \( \eta = \lambda \), and that \( M, N \) are martingales: in the case that they are mere supermartingales we may consider the unique even and odd betting martingales \( M', N' \) that agree with \( M, N \) on the strings \( \eta = 2^{\leq 2} \); then \( M'(\lambda) \leq M(\lambda), N'(\lambda) \leq N(\lambda), N(i) = N'(i j) = N'(i), i, j < 2 \); so the hypothesis on \( M, N \) applies on \( M', N' \) and the application of the statement on \( M', N' \) gives

- \( n_0 \leq n_1 \Rightarrow M(01) + N(0) = M'(\eta 01) + N'(\eta 0) \leq c \)
- \( n_0 > n_1 \Rightarrow M(11) + N(1) = M'(11) + N'(1) \leq c. \)

Given \( m_{00}, m_{10}, n_0, n_1, M, N \) consider the martingales \( M_0, N_0, D_M, D_N \) given by Lemma 4.3 so

\[ M + N = (M_0 + N_0) + D_M + D_N \quad \text{and} \quad c_0 := M_0(\lambda) + N_0(\lambda) \leq c. \tag{11} \]

There are two rounds of bets corresponding to the two bits:

- \( M_0 \) bets only on the second round (his bets may be dependent on the outcome of the first round)
- \( N_0 \) bets only on the first round, so \( N_0(i j) = N_0(i), i, j < 2. \)

If \( n_0 \leq n_1 \) then \( N_0(0) \) is the loser so by the first of (10), \( M_0(00) = m_{00}, N_0(0) = n_0 \) and the last of (11), \( M(\eta 00) \) has to win (at least) the amount \( N_0(\lambda) - N_0(0) \) that \( N \) loses under outcome 0, plus the difference \( c - c_0 \) from the initial capital \( c_0 \) of \( M_0 + N_0 \). As a consequence, \( M_0(01) \) has to lose the same amount so, since \( M(0) = M(\lambda) \):

\[ M_0(01) + N_0(01) = M_0(01) + N_0(0) \leq M(0) - (c - c_0) + N(\lambda) = c_0 - (c - c_0). \tag{12} \]

By (11) we have \( D_M(\lambda) + D_N(\lambda) \leq c - c_0. \) Since \( D_M, D_N \) are single-parity martingales, at the end of the two rounds and at any outcome they can at most double their initial capital, which is \( \leq c - c_0, \) so

\[ D_M(01) + D_N(01) \leq 2 \cdot (c - c_0). \]
Combining the above with (12) and (11) we get that \( M(01) + N(01) \) is bounded above by:
\[
M_0(01) + N_0(01) + D_M(01) + D_N(01) \leq c_0 - (c - c_0) + 2 \cdot (c - c_0) = c_0 + (c - c_0) = c
\]
as required. The case \( n_0 > n_1 \) is symmetric. \( \square \)

**Lemma 4.5.** Given \( c \geq 0 \), string \( \eta \) of even length and left-c.e. even and odd betting supermartingales \( M, N \) we can effectively enumerate \( V \subseteq 2^{\leq 2} \) such that \( |V| \leq 3 \) and if \( M(\eta) + N(\eta) \leq c \) then \( \exists \tau \in V : M(\tau) + N(\tau) \leq c \).

**Proof.** We first enumerate \( \eta 00, \eta 10 \) into \( V \) and wait until a stage \( s \) such that
\[
M_s(\eta 00) + N_s(\eta 00) > c \land M_s(\eta 10) + N_s(\eta 10) > c
\]
where \( (M_s), (N_s) \) are computable non-decreasing even and odd betting supermartingales converging to \( M, N \). These approximations can be effectively obtained from any left-c.e. approximations to \( M, N \), so the above is a \( \Sigma_1^0 \) condition. If and when this occurs at some stage \( s \) we let \( m_{00} := M_s(\eta 00), m_{10} := M_s(\eta 10), n_0 := N_s(\eta 00), n_1 := N_s(\eta 10) \) and
- if \( n_0 \leq n_1 \) we enumerate \( \eta 01 \) into \( V \)
- if \( n_0 > n_1 \) we enumerate \( \eta 11 \) into \( V \).

If the third string never gets enumerated into \( V \), we have \( |V| = 2 \) and \( M(\tau) + N(\tau) \leq c \) for at least one of \( 00, 10 \); otherwise \( |V| = 3 \) and by Lemma 4.4 \( M(\tau) + N(\tau) \leq c \) holds for the last string \( \tau \) that was enumerated into \( V \). \( \square \)

We may now complete the proof of Theorem 4.1. Let \( M, N \) be as in Lemma 2.2 and let \( D(\sigma) := M(\sigma) + N(\sigma), \) so \( D(\iota) \leq 1 \). Then by nested application of Lemma 4.5 we can define an array \( (V_i) \) satisfying the conditions of Lemma 2.3 and such that for each \( \sigma \in V_i \) that \( D(\sigma) \leq 1 \) there exists \( \sigma' \in V_{i+1} \) such that \( D(\sigma') \leq 1 \). This condition shows that there are infinite paths \( x \) through the tree defined by \( (V_i) \) such that \( \forall n \ (x \upharpoonright 2n \in V_n \land D(x \upharpoonright 2n) \leq 1) \). By Lemma 2.3 it follows that \( D(x \upharpoonright n) = O(1) \) and by the optimality of \( D \) by Lemma 2.2, we have \( D'(x \upharpoonright n) = O(1) \) for any single-parity left-c.e. supermartingale \( D' \).

5 Hausdorff dimension and parity-betting: proof of Theorem 1.2

Effective Hausdorff dimension dimension, as discussed in §2.2, can be characterized of in terms of effective statistical tests. Given \( s \in (0, 1) \), an \( s \)-test is a uniformly c.e. sequence \( (V_i) \) of sets of strings such that \( \sum_{\sigma \in V_k} 2^{-|\sigma|} < 2^{-k} \) for each \( k \). As reported by Downey and Hirschfeldt (2010, §13.6):
\[
given s \in (0, 1) \text{ one can effectively obtain an effective list of all } s \text{-tests.} \tag{13}
\]
Since \( s < 1 \), the condition \( \sum_{\sigma \in V_k} 2^{-|\sigma|} < 2^{-k} \) means that the length of each string in \( V_k \) is more than \( k \). These observations will be used in the proof (i) of Theorem 1.2, which is Lemma 5.1 below. We say that \( x \) is weakly \( s \)-random if it avoids all \( s \)-tests \( (V_i) \), in the sense that there are only finitely many \( i \) such that \( x \) has a prefix in \( V_i \). Tadaki (2002) showed that the weak \( s \)-randomness of \( x \) is equivalent to \( \exists c \ \forall n \ K(x \upharpoonright n) > s \cdot n - c \), so by (7):
\[
dim(x) = \sup\{s : x \text{ is weakly } s \text{-random}. \tag{14}\}
Clause (i) of Theorem 1.2 is the following lemma.

**Lemma 5.1.** If $\dim z < 1/2$ then there are strongly left-c.e. martingales $N, T$, such that $N$ is even-parity, $T$ is odd-parity and $\lim_n N(\uparrow z_n) = \lim_n T(\uparrow z_n) = \infty$.

**Proof.** Let $(V_i)$ be a universal $1/2$-test, so that every $z$ which is not weakly $1/2$-random has prefixes in infinitely many $V_i$. We define computable families $(N_\sigma), (T_\sigma)$ of even-parity and odd-parity strategies respectively, indexed by strings, and let:

$$N := \sum_i \sigma_{\infty} \sum N_\sigma \quad \text{and} \quad T := \sum_i \sigma_{\infty} \sum T_\sigma.$$  

Since each $N_\sigma$ is even-parity, the same is true of $N$, and in the same way $T$ is odd-parity. For each $i, \sigma$ strategy $N_\sigma$ starts with $N_\sigma(\lambda) = 2^{-1/2}$ and at each $\sigma < \sigma$ of even length it bets all capital on $\sigma(|\rho|)$, while placing no bets at odd positions. Formally:

$$N_\sigma(\rho) = \begin{cases} 
0 & \text{if } |\hat{\rho}| \text{ is even and } \rho \notin \sigma \\
2 \cdot N_\sigma(\hat{\rho}) & \text{if } |\hat{\rho}| \text{ is even and } \rho \leq \sigma \\
N_\sigma(\hat{\rho}) & \text{if } |\hat{\rho}| \text{ is odd or } |\rho| > |\sigma| 
\end{cases}$$

where $\hat{\rho}$ denotes the predecessor of $\rho$.

Then for each $\rho > \sigma$ we have $N_\sigma(\rho) = 1$. If $\dim z < 1/2$ then $z$ has prefixes in infinitely many $V_i$ so $\lim_n N(\uparrow z_n) = \infty$. The definition of the odd-parity $T_\sigma$ is analogous, as well as the proof that $\lim_n T(\uparrow z_n) = \infty$, provided that $\dim z < 1/2$. \hfill $\square$

**Lemma 5.2.** Given a left-c.e. supermartingale $(M_\lambda) \to M$ and $n$ we can effectively define a left-c.e. martingale $(N_\lambda) \to N$ on $2^{\leq n}$ such that $\forall \sigma \in 2^{\leq n} N(\sigma) \leq M(\sigma)$ and $\forall \sigma \in 2^n M(\sigma) = N(\sigma)$. Moreover if $M$ is even-parity, so can $N$, and the same holds for odd-parity.

We call the martingale $N$ of Lemma 5.2 the martingale-floor of $M$ on $2^{\leq n}$ and denote it by $M^n$.

Let $N, T$ be optimal even and odd parity left-c.e. supermartingales so by (b) of Lemma 2.1 it suffices to define $(\sigma_n), \sigma_1 < \sigma_1+1$ such that

$$M(\sigma_n) := N(\sigma_n) + T(\sigma_n) = O(1) \land \lim_n \frac{K(\sigma_n)}{|\sigma_n|} = 1/2 \quad (15)$$

and let $x := \lim_n \sigma_n$. For the second of (15) it suffices to define a prefix-free machine $V$ such that

$$K_V(\sigma_n) \leq |\sigma_n| \cdot q_n \quad \text{where} \quad q_n := 1/2 + 3/(n + 2). \quad (16)$$

and $K_V$ is the Kolmogorov complexity with respect to $V$. Without loss of generality we can assume that $M(\lambda) < 2^{-1}$. For the first of (15) it suffices that

$$M(\sigma_n) \leq 2^{-1} + \sum_{i < n} 2^{-i-2}. \quad (17)$$

One way to think about this requirement is to try to ensure that $M(\sigma_n) - M(\sigma_{n-1}) \leq 2^{-n-1}$ for all $n$. Supposing inductively that $\sigma_{n-1}$ has been determined, the task of keeping $M(\sigma_n) - M(\sigma_{n-1})$ small potentially involves changing the approximation to $\sigma_n$ a number of times, since $M$ is a left-c.e. supermartingale. This instability of the final value of $\sigma_n$ is in conflict with (16). The idea for handling this
in order for the growth of $\sigma$ must have seen at many times, and in order to satisfy (16) this works if $M$ is a martingale, so any increase in the initial capital has an immediate effect on a given of outcome trials, but we may overcome this issue as follows.

**Lemma 5.3** (Growth along special extensions). Let $N, T$ be even and odd-parity left-c.e. supermartingales with canonical approximations $(N_s)$, $(T_s)$, and let $M_s := N_s + T_s$. Given even $k, \tau < \tau$ of even length and $s$:

$$\forall t > s \left( M_t^k(\sigma) - M_s^k(\sigma) < 2^{-p} \Rightarrow M_t(\tau) < M_s(\tau) + 2^{[|r|−|\sigma|]/2−p} \right)$$

where $M^k := N^k + T^n$ and $N^k, T^k$ are the martingale-floors of $N, T$ and $(N^k_s), (T^k_s)$ are their canonical martingale approximations.

**Proof.** Since $M^k$ agrees with $M$ on $2^k$ it suffices to show that for all $s$ and even $k$

$$\forall \tau \in 2^k, \sigma < \tau \forall t > s \left( M_t(\sigma) - M_s(\sigma) < 2^{-p} \Rightarrow M_t^k(\sigma) < M_s^k(\sigma) + 2^{[|r|−|\sigma|]/2−p} \right)$$

where $|\sigma|$ is assumed to be even. Since $N^k_s \leq N^k, T^k_s \leq T^k$ are martingales, the increase $M^k_\tau - M^k_s(\tau) - M^k_s(\sigma) = (N^k_\tau - N^k_s(\tau) + T^k_\tau - T^k_s(\tau))$ must have come due to an increase $M^k_\tau - M^k_s(\sigma) = (N^k_\tau - N^k_s(\sigma) + T^k_\tau - T^k_s(\sigma))$ of the two initial capitals, and the capital gain from betting the latter on the $|\tau|−|\sigma|$ bits from $\sigma$ to $\tau$. Since each $N^k_\tau - N^k_s, T^k_\tau - T^k_s$ actually bets on $(|\tau|−|\sigma|)/2$ of these bits it follows that the capital gain will be at most $2^{[|r|−|\sigma|]/2}$ times the initial capital increase. □

Let $(M_s)$ be a canonical approximation to $M$ and let $M^k_s, M^k_s$ be as in Lemma 5.3. For each $n$ the segment $\sigma_n$ as well as its approximations will have a fixed length $s_n$ which we define later. For the approximations to $\sigma_n$ with $n > 0$, we will apply Lemma 5.3 certain values $p_n$ of $p$. By the bound given in Lemma 5.3 in order for the growth of $M^s(\sigma_n)$ at each length $s_n$ to be bounded above by $2^{−n−2}$, we need to set:

$$p_n = s_n/2 + n + 2. \quad (18)$$

We now motivate and define the value of $s_n$: Suppose that $n > 0$ and our choice of $\sigma_{n−1}$ has settled, but that now we are forced to choose a new value of $\sigma_n$ because the capital on some initial segment has increased by too much. What does Lemma 5.3 tell us about the increase in capital, $2^{p_n}$ say, that must have seen at $\sigma_{n−1}$ in order for this to occur? A bound for $p_n$ gives a corresponding bound on the number of times that $\sigma_n$ will have to be chosen: after $\sigma_{n−1}$ has settled the approximation to the next initial segment $\sigma_n$ can change at most $2^{p_n}$ many times. Overall, $\sigma_n$ can then change at most $2^{\Sigma_{i<n}p_i} \cdot 2^{p_n}$ many times, and in order to satisfy (16), at each of these changes we need to enumerate to the machine $V$ a description of length $q_n \cdot s_n$. In order to keep the weight of these requests bounded, we will aim at keeping the total weight of the requests corresponding to $\sigma_n$ bounded above by $2^{−n}$, for which it is sufficient that:

$$2^{−s_nq_n} \cdot 2^{p_n} \cdot 2^{\Sigma_{i<n}p_i} < 2^{−n} \iff 2^{p_n−s_nq_n} < 2^{−n−\Sigma_{i<n}p_i} \iff s_nq_n - p_n > n + \sum_{i<n} p_i. \quad (19)$$

By (18) we get $p_n = q_n^2 = n + 2 + s_n \cdot (q_n(1/2) - q_n)$ so (19) reduces to:

$$s_n \cdot (q_n - 1/2) > 2n + 2 + \sum_{i<n} p_i \iff s_n \geq 2n + 2 + \sum_{i<n} p_i / q_n − 1/2.$$
Hence it suffices to set:

\[ s_n = 3 \cdot \frac{2n + 2 + \sum_{i<n} p_i}{q_n - 1/2} = (n + 2) \cdot \left( 2 + \sum_{i<n} p_i \right). \tag{20} \]

We are ready to inductively define the approximations \( \sigma_n[s] \) of \( \sigma_n \) for all \( n \), in stages \( s \). At stage \( s + 1 \) the segment \( \sigma_n \) requires attention if \( n > 0 \) and

\[ \left( \sigma_n[s] \downarrow \land M_{s+1}(\sigma_n[s]) > 2^{-1} + \sum_{i<n} 2^{-i-2} \right) \lor \sigma_n[s] \uparrow. \]

In the argument below the suffix \( [s] \) on an expression, as in \( M(\sigma_n)[s] \), indicates that all parameters in the expression (here \( M, \sigma_n \)) are evaluated at the end of stage \( s \). Let \( \sigma_0[s] = \lambda \) for all \( s \) and \( s_0 = 0 \).

**Construction of \( (\sigma_n) \).** At stage \( s + 1 \) pick the least \( n \leq s \) such that \( \sigma_n \) requires attention, if such exists.

(a) If \( \sigma_n[s] \uparrow \), let \( \sigma_n[s + 1] \) be the leftmost extension \( \tau \) of \( \sigma_{n-1}[s] \) with \( |\tau| = s_n, M_s(\tau) \leq M(\sigma_{n-1})[s] \).

(b) If \( \sigma_n[s] \downarrow \), set \( \sigma_i[s + 1] \uparrow \) for all \( i \geq n \).

In any case, let \( k \leq s \) least (if such exists) such that \( \sigma_k[s + 1] \downarrow \) and \( K_{V_i}(\sigma_k[s + 1]) > q_k \cdot s_k \), and issue a \( V \)-description of \( \sigma_k[s + 1] \) of length \( q_k \cdot s_k \).

**Verification.** Since \( M_s \) is a supermartingale, the string \( \tau \) of clause (a) exists, so the construction of \( (\sigma_n[s]) \) is well-defined. In any interval of stages where \( \sigma_{n-1} \) remains defined, successive values of \( \sigma_n \) are lexicographically increasing. It follows that each \( \sigma_n[r] \) converges to a final value \( \sigma_n \) such that \( \sigma_n < \sigma_{n+1} \). The real \( x \) determined by the initial segments \( \sigma_n \) is thus left-c.e. and by the construction:

\[ M(\sigma_n)[s + 1] \leq 2^{-1} + \sum_{i<n} 2^{-i-2} \]

at all stages \( s \) where \( \sigma_n \) is defined, so \( M(x \uparrow_n) < 1 \) for all \( n \).

It remains to show that the weight of \( V \) is bounded above by 1. Suppose that \( \sigma_n \) gets newly defined at stage \( s + 1 \) and at stage \( t > s + 1 \) it becomes undefined, while \( \sigma_{n-1}[j] \downarrow \) for all \( j \in [s, t] \). Then

\[ M(\sigma_n)[s + 1] \leq M(\sigma_{n-1})[s] \quad \text{and} \quad M_t(\sigma_n[s + 1]) > 2^{-1} + \sum_{i<n} 2^{-i-2} \]

where the latter is due to the fact that \( \sigma_n \) becomes undefined at stage \( t \). By Lemma 5.3 and (18),

\[ M_t^\text{fin}(\sigma_{n-1}[s + 1]) - M_t^\text{fin}(\sigma_{n-1})[s + 1] > 2^{-p_n}, \]

which means that the weight of the \( V \)-descriptions that we enumerate for strings of length \( s_n \) is at most \( 2^{s_nq_n} \cdot 2\sum_{i<n} p_i \). By the definition of \( s_n \) in (20) and (19) this weight is bounded above by 2\(^{-n} \). So the total weight of the descriptions that are enumerated into \( V \) is at most 1.

### 6 Conclusion

Muchnik’s paradox says that some reals are predictable with respect to left-c.e. supermartingales but unpredictable with respect to single-parity left-c.e. supermartingales. Informally, *some left-c.e.*
strategies are irreducible to single-parity strategies. We have characterized the power of single-parity left-c.e. supermartingales and martingales in terms of effective Hausdorff dimension: reals with \( \dim(x) < 1/2 \) are predictable with respect to both odd and even parity martingales, while there exists \( x \) with \( \dim(x) = 1/2 \) on which all single-parity supermartingales fail. Moreover, using a different argument, we showed that there are reals \( x \) with effective packing dimension as low as \( \log \sqrt{3} \approx 0.79 \), yet no single-parity left-c.e. supermartingale succeeds on \( x \). The following question arises:

how low can the effective packing dimension of reals exhibiting Muchnik’s paradox be?

Since \( \dim(x) \leq \Dim(x) \), our results say that it cannot be less than 1/2; however \( [1/2, \log \sqrt{3}) \) is a gray area. Questions regarding the predictability of reals exhibiting Muchnik’s paradox are questions about the power of single-parity betting.

We also exhibited Muchnik’s paradox in the case of computable integer-valued (super)martingales. This is interesting since this phenomenon does not occur in the case of computable (super)martingales. Integer-valued strategies represent a specific example of a wager-restriction, so a related question is which wager restrictions permit the irreducibility of a computable strategy to computable single-parity strategies?

Finally we discussed the case of left-c.e. single-sided strategies and why they appear to be more powerful than the single-parity left-c.e. strategies. Using different methods, Barmpalias and Liu (2021) showed that there exists non-random \( x \) such that no single-sided left-c.e. supermartingale succeeds on \( x \). However the analogue of our Theorem 1.2 remains open:

How small can the effective Hausdorff and packing dimensions of \( x \) be if no single-sided left-c.e. supermartingale succeeds on \( x \)?

Only the lower bound 1/2 is known; for example, we don’t know if such \( x \) can have \( \dim(x) < 1 \).

References

K. B. Athreya, J. M. Hitchcock, J. H. Lutz, and E. Mayordomo. Effective strong dimension in algorithmic information and computational complexity. *SIAM J. Comput.*, 37:671–705, 2007.

G. Barmpalias and N. Fang. Granularity of wagers in games and the possibility of savings. *Information and Computation*, 275, 2020.

G. Barmpalias and L. Liu. Irreducibility and enumerability in betting strategies. Preprint; [https://faculty.csu.edu.cn/liujiayi/zh_CN/lwcg/77976/content/35629.htm#lwcg](https://faculty.csu.edu.cn/liujiayi/zh_CN/lwcg/77976/content/35629.htm#lwcg), 2021.

G. Barmpalias, R. Downey, and M. McInerney. Integer-valued betting strategies and Turing degrees. *Journal of Computer and System Sciences*, 81:1387–1412, 2015.

G. Barmpalias, N. Fang, and A. Lewis-Pye. Monotonous betting strategies in warped casinos. *Information and Computation*, 271:104480, 2020.

B. Bauwens. Asymmetry of the Kolmogorov complexity of online predicting odd and even bits. In E. W. Mayr and N. Portier, editors, *31st International Symposium on Theoretical Aspects of Computer Science (STACS 2014)*, *STACS 2014, March 5-8, 2014, Lyon, France*, volume 25 of *LIPIcs*, pages 125–136. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2014.
B. Bauwens. Uniform van Lambalgen’s theorem fails for computable randomness. *Inf. Comput.*, 271: 104486, 2020.

L. Bienvenu, F. Stephan, and J. Teutsch. How powerful are integer-valued martingales? In *Programs, Proofs, Processes, 6th Conference on Computability in Europe, CiE 2010, Ponta Delgada, Azores, Portugal, June 30 - July 4, 2010. Proceedings*, pages 59–68, 2010.

J. C. Tricot. Two definitions of fractional dimension. *Math. Proc. Cambridge Philos. Soc.*, 91:57–74, 1982.

D. Chakraborty, S. Nandakumar, and H. Shukla. On resource-bounded versions of the van Lambalgen theorem. In T. Gopal, G. Jäger, and S. Steila, editors, *Theory and Applications of Models of Computation*, pages 129–143, Cham, 2017. Springer International Publishing.

A. Chalcraft, R. Dougherty, C. Freiling, and J. Teutsch. How to build a probability-free casino. *Information and Computation*, 211:160–164, 2012.

A. V. Chernov, A. Shen, N. K. Vereshchagin, and V. Vovk. On-line probability, complexity and randomness. In Y. Freund, L. Györfi, G. Turán, and T. Zeugmann, editors, *Algorithmic Learning Theory, 19th International Conference, ALT 2008, Budapest, Hungary, October 13-16, 2008. Proceedings*, volume 5254 of *Lecture Notes in Computer Science*, pages 138–153. Springer, 2008.

R. G. Downey and D. Hirschfeldt. *Algorithmic Randomness and Complexity*. Springer, 2010.

J. H. Lutz. Gales and the constructive dimension of individual sequences. In *Automata, languages and programming (Geneva, 2000)*, volume 1853 of *Lecture Notes in Comput. Sci.*, pages 902–913. Springer, Berlin, 2000.

J. H. Lutz. The dimensions of individual strings and sequences. *Information and Computation*, 187, 2003.

P. Martin-Löf. The definition of random sequences. *Information and Control*, 9:602–619, 1966.

E. Mayordomo. A Kolmogorov complexity characterization of constructive Hausdorff dimension. *Inform. Process. Lett.*, 84(1):1–3, 2002.

K. Miyabe and J. Rute. Van lambalgen’s theorem for uniformly relative schnorr and computable randomness. In *Proceedings of the 12th Asian Logic Conference*, pages 251–270. World Scientific, 2013.

A. A. Muchnik. Algorithmic randomness and splitting of supermartingales. *Problems of Information Transmission*, 45(1):54–64, 2009.

R. Peretz. Effective martingales with restricted wagers. *Information and Computation*, 245:152–164, 2015.

R. Peretz and G. Bavly. How to gamble against all odds. *Games and Economic behavior*, 94:157–168, 2015.

C. Schnorr. *Zufälligkeit und Wahrscheinlichkeit. Eine algorithmische Begründung der Wahrscheinlichkeitsrechnung*. Springer-Verlag, Berlin, 1971a. Lecture Notes in Mathematics, Vol. 218.
C. Schnorr. A unified approach to the definition of random sequences. *Mathematical Systems Theory*, 5(3):246–258, 1971b.

C. Schnorr. *Zufälligkeit und Wahrscheinlichkeit. Eine algorithmische Begründung der Wahrscheinlichkeitsrechnung*. Springer-Verlag, Berlin, 1971c. Lecture Notes in Mathematics, Vol. 218.

A. Shen, V. A. Uspensky, and N. Vereshchagin. *Algorithmic Randomness and Complexity*. Mathematical Surveys and Monographs. American Mathematical Society, 2017.

D. Sullivan. Entropy, Hausdorff measures old and new, and limit sets of geometrically finite Kleinian groups. *Acta Math.*, 153:259–277, 1984.

K. Tadaki. A generalization of Chaitin’s halting probability $\Omega$ and halting self-similar sets. *Hokkaido Math. J.*, 31(1):219–253, 2002.

J. Teutsch. A savings paradox for integer-valued gambling strategies. *International Journal of Game Theory*, 43(1):145–151, 2014.

J. Ville. *Etude critique de la notion de collectif*. Monographies des Probabilités. Calcul des Probabilités et ses Applications. Gauthier-Villars, Paris, Paris, 1939.

L. Yu. When van Lambalgen’s theorem fails. *Proc. Amer. Math. Soc.*, 135(3):861–864 (electronic), 2007. ISSN 0002-9939.