On completions of symmetric and antisymmetric block diagonal partial matrices

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Abstract
A partial matrix is a matrix where only some of the entries are given. We determine the maximum rank of the symmetric completions of a symmetric partial matrix where only the diagonal blocks are given and the minimum rank and the maximum rank of the antisymmetric completions of an antisymmetric partial matrix where only the diagonal blocks are given.

1 Introduction
Let $K$ be a field. A partial matrix over $K$ is a matrix where only some of the entries are given and they are elements of $K$. A completion of a partial matrix is a specification of the unspecified entries. We say that a submatrix of a partial matrix is specified if all its entries are given. The problem of determining whether, given a partial matrix, a completion with some prescribed property exists and related problems have been widely studied: we quote, for instance, the papers [1], [2], [3], [4], [5], [6], [7], [8], [9], [10].

In [5], Cohen, Johnson, Rodman and Woerdeman determined the maximum rank of the completions of a partial matrix in terms of the ranks and the sizes of its maximal specified submatrices. From the results in [5], we get easily the minimum rank and the maximum rank of the completions of a partial matrix where only the diagonal blocks are given:

**Theorem 1.** (See [5].) Let $n_1, \ldots, n_k$ be nonzero natural numbers with $n_1 \leq n_2 \leq \ldots \leq n_k$. Let $A_i \in M(n_i \times n_i, K)$ for $i = 1, \ldots, k$ and $r_i = \text{rank}(A_i)$. Let $A$ be the partial matrix where only the diagonal blocks are given and whose diagonal blocks are $A_1, \ldots, A_k$. Then we have:

(i) the minimum of $\{\text{rk}(\tilde{A})| \tilde{A} \text{ completion of } A\}$ is

$$\max\{r_i | i = 1, \ldots, k\}$$

(ii) the maximum of $\{\text{rk}(\tilde{A})| \tilde{A} \text{ completion of } A\}$ is

$$\min\left\{\sum_{i=1}^{k} n_i, 2\left(\sum_{i=1}^{k-1} n_i\right) + r_k\right\}.$$

Here we determine the maximum rank of the symmetric completions of a symmetric partial matrix where only the diagonal blocks are given (see Theorem 5) and the minimum rank and the maximum rank of the antisymmetric completions of an antisymmetric partial matrix where only the diagonal blocks are given (see Theorem 9). In [3], [4], [9], the analogous problem has been solved for hermitian matrices.

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2 Notation

Notation 2. • We say that a diagonal matrix $A \in M(n \times n, K)$ is b-diagonal if all the nonzero elements of the diagonal are at the beginning, that is either it is the zero matrix or there exists $r \in \{1, \ldots, n\}$ such that $a_{i,i} \neq 0$ if and only if $i \in \{1, \ldots, r\}$.

• We say that a sequence of elementary operations on the rows and on the columns of a square matrix is symmetric if it is given by an elementary operation on the rows, the same elementary operation on the columns, another elementary operation on the rows, the same elementary operation on the columns and so on.

• For any $r, m, n \in \mathbb{N} - \{0\}$ with $r \leq \min\{m, n\}$, we define $T_{m,n}^r$ to be the matrix $m \times n$ with entries in $K$ such that

$$(T_{m,n}^r)_{i,j} = \begin{cases} 1 & \text{if } (i, j) = (m, n), (m - 1, n - 1), \ldots, (m - r + 1, n - r + 1), \\ 0 & \text{otherwise}, \end{cases}$$

for any $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$.

• For any $r, m, n \in \mathbb{N} - \{0\}$ with $r \leq \min\{m, n\}$, we define $E_{m,n}^r$ to be the matrix $m \times n$ with entries in $K$ such that

$$(E_{m,n}^r)_{i,j} = \begin{cases} 1 & \text{if } (i, j) = (1,1), \ldots, (r,r), \\ 0 & \text{otherwise}, \end{cases}$$

for any $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$.

• For any $r, m, n \in \mathbb{N} - \{0\}$ with $r \leq \min\{m, n\}$ and $r$ even, we define $R_{m,n}^r$ to be the matrix $m \times n$ with entries in $K$ such that

$$(R_{m,n}^r)_{i,j} = \begin{cases} 1 & \text{if } (i, j) = (1,2), (3,4), \ldots, (r-1,r), \\ -1 & \text{if } (i, j) = (2,1), (4,3), \ldots, (r,r-1), \\ 0 & \text{otherwise}, \end{cases}$$

for any $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$.

We define $T_{m,n}^0$, $E_{m,n}^0$ and $R_{m,n}^0$ to be the zero matrix $m \times n$.

We write $E_{n,n}^r$ instead of $E_{n,n}^r$ and $R_{n,n}^r$ instead of $R_{n,n}^r$ for simplicity. We omit the subscript in $T_{m,n}^r$, $E_{m,n}^r$ and $R_{m,n}^r$ when their size is clear from the context.

Examples.

$$T_{4,5}^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad E_4^2 = E_{4,4}^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad R_{5,6}^4 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Notation 3. Let $n_1, \ldots, n_k$ be nonzero natural numbers. Let $A_i \in M(n_i \times n_i, K)$ for $i = 1, \ldots, k$.

We denote by $\text{Diag}(A_1, \ldots, A_k)$ the block diagonal matrix whose diagonal blocks are $A_1, \ldots, A_k$ (thus the entries out of the diagonal blocks are 0).

We denote by $\text{diag}(A_1, \ldots, A_k)$ the partial matrix where only the diagonal blocks are given and whose diagonal blocks are $A_1, \ldots, A_k$. We call such a matrix a block diagonal partial matrix.
3 Completions of symmetric block diagonal partial matrices

Remark 4. Let $K$ be a field of characteristic different from 2. Let $n_1, \ldots, n_k$ be nonzero natural numbers. Let $A_i \in M(n_i \times n_i, K)$ be symmetric for $i = 1, \ldots, k$ and $r_i = \text{rank}(A_i)$. It is well known that, by a symmetric sequence of elementary operations $h_i$, we can change the matrix $A_i$ into a b-diagonal matrix $D_i$.

Then there exists a completion of rank $l$ of the partial matrix $\text{diag}(A_1, \ldots, A_k)$ if and only if there exists a completion of rank $l$ of the partial matrix $\text{diag}(D_1, \ldots, D_k)$.

Proof. Let

$I_1 = \{1, \ldots, n_1\}, \quad I_2 = \{n_1 + 1, \ldots, n_1 + n_2\}, \quad \ldots, \quad I_k = \left\{ \left( \sum_{i=1}^{k-1} n_i \right) + 1, \ldots, \sum_{i=1}^{k} n_i \right\}$.

Suppose there exists a symmetric completion of the partial matrix $\text{diag}(A_1, \ldots, A_k)$ of rank $l$. We apply the symmetric sequence of elementary operations $h_1$ to the rows and the columns with index in $I_1$, then we apply the symmetric sequence of elementary operations $h_2$ to the rows and the columns with index in $I_2$ and so on. In this way we get a symmetric completion of $\text{diag}(D_1, \ldots, D_k)$ of rank $l$.

Conversely, suppose there exists a symmetric completion of $\text{diag}(D_1, \ldots, D_k)$ of rank $l$. Apply the symmetric sequence of elementary operations $h_1^{-1}$ to the rows and to the columns with index in $I_1$, then we apply the symmetric sequence of elementary operations $h_2^{-1}$ to the rows and the columns with index in $I_2$ and so on. In this way we get a symmetric completion of $\text{diag}(A_1, \ldots, A_k)$ of rank $l$.

Theorem 5. Let $K$ be a field of characteristic different from 2 and let $n_1, \ldots, n_k$ be nonzero natural numbers with $n_1 \leq n_2 \leq \ldots \leq n_k$. Let $A_i \in M(n_i \times n_i, K)$ for $i = 1, \ldots, k$ be symmetric matrices and let $r_i = \text{rank}(A_i)$. Let $A$ be the partial matrix $\text{diag}(A_1, \ldots, A_k)$. Then the maximum of $\{ r_k(A) \mid \text{A symmetric completion of } A \}$ is

$$
\min \left\{ \sum_{i=1}^{k} n_i, \quad 2 \left( \sum_{i=1}^{k-1} n_i \right) + r_k \right\}. \quad (1)
$$

Proof. By Theorem 4 any completion of $A$ has rank less or equal than the number in (1). Note that this can be proved directly in an easy way: clearly any completion of $A$ has rank less or equal than $\sum_{i=1}^{k} n_i$; besides any completion of $A$ has rank less or equal than $2 \left( \sum_{i=1}^{k-1} n_i \right) + r_k$, since the submatrix of the completion given by the first $\sum_{i=1}^{k-1} n_i$ rows has rank less or equal than $\sum_{i=1}^{k-1} n_i$ and the submatrix given by the last $n_k$ rows has rank less or equal than $\sum_{i=1}^{k-1} n_i + r_k$ (in fact its submatrix given by the first $\sum_{i=1}^{k-1} n_i$ columns has rank less or equal than $\sum_{i=1}^{k-1} n_i$ and the remaining part, that is $A_k$, has rank $r_k$).

Now we prove, by induction on $k$, that we can complete $A$ to a symmetric matrix whose rank is the number in (1). By Remark 4 we can suppose that $A_i$ is b-diagonal for $i = 1, \ldots, k$.

Case $k = 2$.

Let $t = \max\{ n_1 - r_1, n_2 - r_2 \}$. Observe that $t \leq n_1$ if and only if $n_2 - r_2 \leq n_1$.

- If $t \leq n_1$, we consider the following symmetric completion of $A$:

$$
\begin{pmatrix}
A_1 & T_{n_1,n_2}^t \\
T_{n_2,n_1}^t & A_2
\end{pmatrix}.
$$

- If $t > n_1$, we consider the following symmetric completion of $A$:

$$
\begin{pmatrix}
A_1 & T_{n_1,n_2}^t \\
T_{n_2,n_1}^t & A_2
\end{pmatrix}.
$$


By swapping the last $t$ rows of the upper blocks of the matrix with the last $t$ rows of the lower blocks of the matrix, we can see that the rank is $n_1 + n_2$.

- If $t > n_1$, then $t = n_2 - r_2$ and $n_2 - r_2 > n_1$. In this case we consider the following symmetric completion of $A$:

$$
\begin{pmatrix}
A_1 & T_{n_1}^{n_1} & T_{n_1}^{n_2} \\
T_{n_2}^{n_1} & A_2 
\end{pmatrix}
$$

By swapping the first $n_1$ rows of with the last $n_1$ rows, we can see that the rank is $n_1 + r_2 + n_1$.

So, in the case $k = 2$ we have constructed a symmetric completion of rank equal to

$$
\begin{cases}
    n_1 + n_2 & \text{if } n_2 - r \leq n_1, \\
    2n_1 + r_2 & \text{if } n_2 - r > n_1,
\end{cases}
$$

that is $\min\{n_1 + n_2, 2n_1 + r_2\}$.

**Induction step.** Let $k \geq 3$.

- If $n_k - r_k \geq \sum_{i=1}^{k-1} n_i$, we can consider the following symmetric completion:

$$
\begin{pmatrix}
A_1 & 0 & \cdots & 0 \\
0 & \ddots & \cdots & \cdots \\
\cdots & \cdots & \ddots & \cdots \\
\cdots & \cdots & \cdots & 0 \\
0 & \cdots & 0 & A_{k-1}
\end{pmatrix}
$$

where $s = \sum_{i=1}^{k-1} n_i$. By swapping the first $s$ rows and the last $s$ rows, we can see that its rank is $2s + r_k = 2(\sum_{i=1}^{k-1} n_i) + r_k$.

- Suppose $n_k - r_k < \sum_{i=1}^{k-1} n_i$. Let $P$ the submatrix of $A$ given by the last $\sum_{i=2}^{k} n_i$ rows and the last $\sum_{i=2}^{k} n_i$ columns. By induction assumption, we can complete $P$ to a symmetric matrix with rank

$$
\min \left\{ \sum_{i=2}^{k} n_i, 2 \left( \sum_{i=2}^{k-1} n_i \right) + r_k \right\}.
$$

We consider two cases: the case where we complete $P$ to a symmetric matrix of rank $\sum_{i=2}^{k} n_i$ and the case where we complete $P$ to a symmetric matrix of rank $2 \left( \sum_{i=2}^{k-1} n_i \right) + r_k$. We state that, in both cases, we can complete $A$ to a symmetric matrix of rank $\sum_{i=1}^{k} n_i$.

- Case we complete $P$ to a symmetric matrix of rank $\sum_{i=2}^{k} n_i$.
By a symmetric sequence of elementary operation $h$, we can change the completion of $P$ into a diagonal matrix $D$ where all the elements of the diagonal are nonzero; then we can complete $\text{diag}(A_1, D)$ to the symmetric matrix

$$
\begin{pmatrix}
A_1 & T^{n_1-r_1}_{n_1, \sum_{i=2,...,k} n_i, n_1} \\
T^{n_1-r_1}_{\sum_{i=2,...,k} n_i, n_1} & D
\end{pmatrix},
$$

which has rank $ \sum_{i=1,...,k} n_i$. Then, by applying the symmetric sequence of elementary operations $h^{-1}$, we get a symmetric completion of $A$ of rank $ \sum_{i=1,...,k} n_i$.

- Case we complete $P$ to a symmetric matrix of rank $2 \left( \sum_{i=2,...,k-1} n_i \right) + r_k$.

By a symmetric sequence of elementary operation $h$, we can change the completion of $P$ into a diagonal matrix $L$ such that $L_{i,j} \neq 0$ if and only if $i \leq 2 \left( \sum_{i=2,...,k-1} n_i \right) + r_k$; define

$$
t = \max \left\{ n_1 - r_1, \sum_{i=2,...,k} n_i - r_k(L) \right\} = \max \left\{ n_1 - r_1, \sum_{i=2,...,k} n_i - 2 \left( \sum_{i=2,...,k-1} n_i \right) - r_k \right\};
$$

observe that $t \leq n_1$; then we can complete $\text{diag}(A_1, L)$ to the symmetric matrix

$$
\begin{pmatrix}
A_1 & T^t_{n_1, \sum_{i=2,...,k} n_i, n_1} \\
T^t_{\sum_{i=2,...,k} n_i, n_1} & L
\end{pmatrix},
$$

which has rank $ \sum_{i=1,...,k} n_i$. Then, by applying the symmetric sequence of elementary operations $h^{-1}$, we get a symmetric completion of $A$ of rank $ \sum_{i=1,...,k} n_i$. □

## 4 Completions of antisymmetric block diagonal matrices

**Remark 6.** (a) If $K$ is a field of characteristic 2, the set of the antisymmetric matrices $n \times n$ over $K$ is equal to the set of the matrices $n \times n$ over $K$.

(b) If $K$ is a field of characteristic different from 2, we can change, by a symmetric sequence of elementary operations, an antisymmetric matrix into a diagonal block matrix whose diagonal blocks are all equal to

$$
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}.
$$

In fact, if in the $i$-th row (and thus in the $i$-th column) there is a nonzero element, by applying a symmetric sequence of elementary operations, we can suppose that the entry $(i, i+1)$ (and thus the entry $(i+1, i)$) is nonzero and all the other entries of the $i$-th row and the $i$-th column are zero. By another symmetric sequence of elementary operations, we can get easily the form described above.

**Notation 7.** For every $n \in \mathbb{N}$, we denote by $\langle n \rangle$ the “even part” of $n$, that is

$$
\langle n \rangle = \begin{cases} 
n & \text{if } n \text{ is even}, \\
n - 1 & \text{if } n \text{ is odd}.
\end{cases}
$$
**Remark 8.** Let $K$ be a field of characteristic different from 2. Let $n_1, \ldots, n_k$ be nonzero natural numbers. Let $A_i \in M(n_i \times n_i, K)$ be antisymmetric matrices for $i = 1, \ldots, k$ and let $r_i = \text{rank}(A_i)$. Then there exists an antisymmetric completion of rank $l$ of the partial matrix $\text{diag}(A_1, \ldots, A_k)$ if and only if there exists an antisymmetric completion of rank $l$ of the partial matrix $\text{diag}(R_{n_1}^r, \ldots, R_{n_k}^r)$.

(It can be proved as Remark[3].)

**Theorem 9.** Let $K$ be a field of characteristic different from 2. Let $n_1, \ldots, n_k$ be nonzero natural numbers with $n_1 \leq n_2 \leq \ldots \leq n_k$. Let $A_i \in M(n_i \times n_i, K)$ be antisymmetric matrices for $i = 1, \ldots, k$ and let $r_i = \text{rank}(A_i)$ (obviously the $r_i$ are even numbers). Let $A$ be the partial matrix $\text{diag}(A_1, \ldots, A_k)$. Then we have:

(i) the minimum of $\{rk(\tilde{A}) | \tilde{A} \text{ antisymmetric completion of } A\}$ is

$$\max \{r_i | i = 1, \ldots, k\}$$

(ii) the maximum of $\{rk(\tilde{A}) | \tilde{A} \text{ antisymmetric completion of } A\}$ is

$$\min \left\{ \left\langle \sum_{i=1}^{s_i} n_i \right\rangle, 2 \left( \sum_{i=1}^{s_i} n_i \right) + r_k \right\}.$$  

**Proof.** (i) Let $A'_1, \ldots, A'_k$ be the matrices $A_1, \ldots, A_k$ ordered according to the rank, i.e. let $A'_1, \ldots, A'_k$ be such that $\{A'_1, \ldots, A'_k\} = \{A_1, \ldots, A_k\}$ and $s_1 \leq \ldots \leq s_k$, where $s_i = \text{rank}(A'_i)$. Obviously we can complete $\text{diag}(A'_1, \ldots, A'_k)$ to an antisymmetric matrix of rank $l$ if and only if we can complete $\text{diag}(A_1, \ldots, A_k)$ to an antisymmetric matrix of rank $l$. Let $A' = \text{diag}(A'_1, \ldots, A'_k)$. By Remark[3] we can suppose that $A'_i = R_{m_i}^r$ for $i = 1, \ldots, k$, where $m_i$ is the number of the rows and of the columns of $A'_i$.

Observe that, since $s_i \leq s_{i+1}$, then $s_i \leq m_{i+1}$. Thus we can complete the matrix $A'$ to the following antisymmetric matrix:

$$
\begin{pmatrix}
R_{m_1}^{s_1} & R_{m_2}^{s_1} & R_{m_3}^{s_1} & \cdots & R_{m_k}^{s_1} \\
R_{m_1}^{s_2} & R_{m_2}^{s_2} & R_{m_3}^{s_2} & \cdots & R_{m_k}^{s_2} \\
R_{m_1}^{s_3} & R_{m_2}^{s_3} & R_{m_3}^{s_3} & \cdots & R_{m_k}^{s_3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
R_{m_1}^{s_k} & R_{m_2}^{s_k} & R_{m_3}^{s_k} & \cdots & R_{m_k}^{s_k}
\end{pmatrix},
$$

where we omitted the sizes of the off-diagonal matrices for simplicity (the sizes are obliged). The rank of this completion is clearly $\max \{s_i | i = 1, \ldots, k\}$, which is equal to $\max \{r_i | i = 1, \ldots, k\}$.

Finally, observe that, obviously, the rank of any completion of $A$ is greater or equal than $\max \{r_i | i = 1, \ldots, k\}$.

(ii) The same argument as in Theorem[3] proves that any antisymmetric completion of $A$ has rank less or equal than

$$\min \left\{ \left\langle \sum_{i=1}^{m_i} n_i \right\rangle, 2 \left( \sum_{i=1}^{m_i-1} n_i \right) + r_k \right\}.$$
Now we prove, by induction on \( k \), that we can complete \( A \) to an antisymmetric matrix whose rank is the number above. By Remark \( \mathbb{R} \) we can suppose that \( A_i = R_{n_i}^r \) for \( i = 1, \ldots, k \).

**Case \( k = 2 \).**

Let \( t = \max\{n_1 - r_1, n_2 - r_2\} \). Observe that \( t \leq n_1 \) if and only if \( n_2 - r_2 \leq n_1 \).

- If \( t \leq n_1 \), we consider the following antisymmetric completion of \( A \):
  \[
  \begin{pmatrix}
  R_{n_1}^r & -T_{n_1,n_2}^t \\
  T_{n_2,n_1}^t & R_{n_2}^r 
  \end{pmatrix},
  \]

  By swapping the last \( t \) rows of the upper blocks of the matrix with the last \( t \) rows of the lower blocks of the matrix, we can see that the rank is \( \langle n_1 + n_2 \rangle \).

- If \( t > n_1 \), then \( t = n_2 - r_2 \) and \( n_2 - r_2 > n_1 \). In this case we consider the following antisymmetric completion of \( A \):
  \[
  \begin{pmatrix}
  R_{n_1}^r & -T_{n_1,n_2}^n \\
  T_{n_2,n_1}^n & R_{n_2}^r 
  \end{pmatrix},
  \]

  By swapping the first \( n_1 \) rows of with the last \( n_1 \) rows, we can see that its rank is \( n_1 + r_2 + n_1 \).

So, in the case \( k = 2 \) we have constructed a completion of rank equal to

\[
\begin{cases}
\langle n_1 + n_2 \rangle & \text{if } n_2 - r_2 \leq n_1, \\
2n_1 + r_2 & \text{if } n_2 - r_2 > n_1,
\end{cases}
\]

that is \( \min\{\langle n_1 + n_2 \rangle, 2n_1 + r_2\} \).

**Induction step.** Let \( k \geq 3 \).

- If \( n_k - r_k \geq \sum_{i=1}^{k-1} n_i \), we can consider the following antisymmetric completion:
  \[
  \begin{pmatrix}
  R_{n_1}^r & 0 & \cdots & 0 \\
  0 & \ddots & \ddots & \ddots \\
  \vdots & \ddots & \ddots & \ddots \\
  \vdots & \ddots & \ddots & 0 \\
  0 & \cdots & 0 & R_{n_k-1}^r \\
\end{pmatrix}
  \begin{pmatrix}
  -T_{s,n_k}^s \\
  T_{n_k,s}^s \\
  R_{n_k}^r
  \end{pmatrix},
  \]

  where \( s = \sum_{i=1}^{k-1} n_i \). By swapping the first \( s \) rows with the last \( s \) rows, we can see that its rank is \( 2s + r_k = 2 \left( \sum_{i=1}^{k-1} n_i \right) + r_k \).
• Suppose \( n_k - r_k < \sum_{i=1}^{k-1} n_i \). By induction assumption, we can complete the submatrix \( P \) of \( A \) given by the last \( \sum_{i=2}^{k} n_i \) rows and the last \( \sum_{i=2}^{k} n_i \) columns to an antisymmetric matrix with rank

\[
\min \left\{ \left( \sum_{i=2}^{k} n_i \right), \ 2 \left( \sum_{i=2}^{k-1} n_i \right) + r_k \right\}.
\]

We consider two cases: the case where we complete \( P \) to a matrix of rank \( \sum_{i=2}^{k} n_i \) and the case we complete \( P \) to a matrix of rank \( 2 \left( \sum_{i=2}^{k-1} n_i \right) + r_k \). We will show that, in both cases, we can complete \( A \) to an antisymmetric matrix of rank \( \sum_{i=1}^{k} n_i \).

- Case we complete \( P \) to a matrix of rank \( \sum_{i=2}^{k} n_i \).

By a symmetric sequence of elementary operations \( h \), we can change the completion of \( P \) into the matrix \( R_{\sum_{i=2}^{k} n_i}^{\sum_{i=2}^{k} n_i} \). Then we can complete

\[
\text{diag} \left( R_{\sum_{i=1}^{k} n_i}^{\sum_{i=1}^{k} n_i}, R_{\sum_{i=2}^{k} n_i}^{\sum_{i=2}^{k} n_i} \right)
\]

to the antisymmetric matrix

\[
\begin{pmatrix}
R_{\sum_{i=1}^{k} n_i}^{\sum_{i=1}^{k} n_i} & -T_{\sum_{i=1}^{k} n_i}^{\sum_{i=1}^{k} n_i}^{\sum_{i=1}^{k} n_i} \\
T_{\sum_{i=1}^{k} n_i}^{\sum_{i=1}^{k} n_i} & R_{\sum_{i=2}^{k} n_i}^{\sum_{i=2}^{k} n_i}
\end{pmatrix},
\]

which has rank \( \sum_{i=1}^{k} n_i \), in fact:

if \( \sum_{i=2}^{k} n_i \) is even, the rank is

\[
\begin{cases}
\sum_{i=1}^{k} n_i & \text{if } n_1 - r_1 \text{ is even}, \\
\sum_{i=1}^{k} n_i - 1 & \text{if } n_1 - r_1 \text{ is odd},
\end{cases}
\]

if \( \sum_{i=2}^{k} n_i \) is odd, the rank is

\[
\begin{cases}
\sum_{i=1}^{k} n_i - 1 & \text{if } n_1 - r_1 \text{ is even}, \\
\sum_{i=1}^{k} n_i & \text{if } n_1 - r_1 \text{ is odd}.
\end{cases}
\]

Then, by applying the symmetric sequence of elementary operations \( h^{-1} \), we get an antisymmetric completion of \( A \) of rank \( \sum_{i=1}^{k} n_i \).

- Case we complete \( P \) to a matrix of rank \( 2 \left( \sum_{i=2}^{k-1} n_i \right) + r_k \).

By a symmetric sequence of elementary operations \( h \), we can change the completion of \( P \) into \( R_{\sum_{i=2}^{k} n_i}^{2\left(\sum_{i=2}^{k-1} n_i\right)+r_k} \); define

\[
t = \max \left\{ n_1 - r_1, \sum_{i=2}^{k} n_i - 2 \left( \sum_{i=2}^{k-1} n_i \right) - r_k \right\};
\]

observe that \( t \leq n_1 \); then we can complete \( \text{diag} \left( R_{\sum_{i=1}^{k} n_i}^{\sum_{i=1}^{k} n_i}, R_{\sum_{i=2}^{k} n_i}^{2\sum_{i=2}^{k-1} n_i+r_k} \right) \) to the antisymmetric matrix

\[
\begin{pmatrix}
R_{\sum_{i=1}^{k} n_i}^{\sum_{i=1}^{k} n_i} & -T_{\sum_{i=1}^{k} n_i}^{\sum_{i=1}^{k} n_i}^{2\sum_{i=2}^{k-1} n_i+r_k} \\
T_{\sum_{i=1}^{k} n_i}^{\sum_{i=1}^{k} n_i} & R_{\sum_{i=2}^{k} n_i}^{2\sum_{i=2}^{k-1} n_i+r_k}
\end{pmatrix},
\]

which has rank \( \sum_{i=1}^{k} n_i \); then, by applying the symmetric sequence of elementary operations \( h^{-1} \), we get a completion of \( A \) of rank \( \sum_{i=1}^{k} n_i \).
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