UNIFORM BOUNDS FOR THE NUMBER OF RATIONAL POINTS ON
SYMMETRIC SQUARES OF CURVES WITH LOW MORDELL–WEIL RANK

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ABSTRACT. A central problem in Diophantine geometry is to uniformly bound the number
of \( K \)-rational points on a smooth curve \( X/K \) in terms of \( K \) and its genus \( g \). A recent paper by
Stoll proved uniform bounds for the number of \( K \)-rational points on a hyperelliptic curve
\( X \) provided that the rank of the Jacobian of \( X \) is at most \( g - 3 \). Katz, Rabinoff and Zureick-
Brown generalized his result to arbitrary curves satisfying the same rank condition.

In this paper, we prove conditional uniform bounds on the number of rational points on
the symmetric square of \( X \) outside its algebraic special set, provided that the rank of the
Jacobian is at most \( g - 4 \). We also find rank-favorable uniform bounds in the hyperelliptic
case.

1. INTRODUCTION

Let \( X \) be a curve of genus \( g \geq 2 \) and Mordell–Weil rank \( r \), defined over a number field
\( K \). In 1983, Faltings famously proved the Mordell conjecture asserting the finiteness of the
number of \( K \)-rational points of \( X \). This result leads to the natural question of the existence
of uniform bounds. A central conjecture in Diophantine geometry is

**Conjecture 1.1.** There exists a constant \( B(g, K) \) such that any smooth curve \( X \) over \( K \) of genus
\( g \geq 2 \) has at most \( B(g, K) \) rational points.

Recent years witnessed progress on this uniformity conjecture for curves with low
Mordell–Weil rank. Specifically, we have the following two results.

**Theorem 1.2** ([Sto13, Theorem 9.1]). Let \( X \) be a hyperelliptic curve over \( \mathbb{Q} \) of genus \( g \geq 3 \) and
suppose \( r \leq g - 3 \). If \( r = 0 \), then

\[
\#X(\mathbb{Q}) \leq 33(g - 1) + 1.
\]

If \( r \geq 1 \), then

\[
\#X(\mathbb{Q}) \leq 8rg + 33(g - 1) - 1.
\]

**Theorem 1.3** ([KRZB16, Theorem 5.1]). Let \( X \) be any smooth curve over \( \mathbb{Q} \) of genus \( g \geq 3 \), and
suppose that \( r \leq g - 3 \). Then

\[
\#X(\mathbb{Q}) \leq 84g^2 - 98g + 28.
\]

Another question that Faltings’s theorem naturally lends itself to is whether a similar
finiteness statement holds for higher degree points, i.e. for some \( d \in \mathbb{Z}_{>0} \), points \( P \in \)
\( X(\overline{\mathbb{Q}}) \) such that \( [Q(P) : Q] \leq d \). Due to another result of Faltings [Fal94]
concerning points on subvarieties of abelian varieties, the answer is yes once we exclude points coming from
the algebraic special set \( S(\text{Sym}^d X) \) (cf. Definition 3.2) of the \( d \)th symmetric power \( \text{Sym}^d X \)
of the curve \( X \).

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The proofs of the above uniformity results involve variations of the $p$-adic method known as the Chabauty–Coleman method utilizing different theories of $p$-adic integration and Berkovich curves. This $p$-adic technique has been generalized from curves to symmetric powers of curves through the work of Siksek [Sik09] and Park [Par16]. Specifically, the work of Siksek established the general setup for symmetric power Chabauty–Coleman, and Park’s work used tropical intersection theory to produce a conditional, effective bound on the number of points on the symmetric powers of curves with good reduction at $p$ lying outside of its algebraic special set.

1.4. Statement of results. In this paper, we prove conditional uniform bounds on the number of points on the symmetric square of a curve $X$, with relatively low Mordell–Weil rank, lying outside of its algebraic special set.

**Theorem 1.5.** Let $X$ be a smooth, projective, geometrically integral curve over $\mathbb{Q}$ with genus $g \geq 4$ satisfying both $r \leq g - 4$ and Assumption 3.3. Then the number of points in $(\text{Sym}^2 X)(\mathbb{Q})$ lying outside its algebraic special set is at most

$$288g^4 + \frac{1616}{3}g^3 - \frac{2900}{9}g^2 + \frac{11654}{9}g - \frac{4012}{9}.$$ 

We also obtain a rank-favorable uniform bound for hyperelliptic curves.

**Theorem 1.6.** Let $H$ be a hyperelliptic curve over $\mathbb{Q}$ of genus $g \geq 4$ satisfying both $r \leq g - 4$ and Assumption 3.3. Then the number of points in $(\text{Sym}^2 H)(\mathbb{Q})$ lying outside its algebraic special set is at most

$$96g^3r + \frac{2192}{9}g^2r + \frac{27037}{18}g^2 - \frac{1184}{3}gr - \frac{4043}{9}g + \frac{736}{3}r + \frac{1429}{9}.$$ 

From this we obtain an immediate corollary concerning rational torsion packets of $\text{Sym}^2 H$. Recall that a rational torsion packet for $\text{Sym}^2 H$ is the inverse image of the group of rational torsion points of the Jacobian $J$ of $H$ under an Abel–Jacobi map $\text{Sym}^2 H \hookrightarrow J$.

**Corollary 1.7.** Let $H$ be a hyperelliptic curve over $\mathbb{Q}$ of genus $g \geq 4$ satisfying Assumption 3.3. Then, the size of a rational torsion packet of $\text{Sym}^2 H$ lying outside its algebraic special set is at most

$$\frac{27037}{18}g^2 - \frac{4043}{9}g + \frac{1429}{9}.$$ 

1.8. Overview of Proof. Our proof proceeds in the spirit of [Sto13] and [KRZB16]. Specifically, we first cover $\text{Sym}^2 X(\mathbb{Q}_p)$ with residue polydisks and polyannuli. To generalize symmetric power Chabauty to annuli, we extend Park’s results concerning tropical intersection theory for power series to Laurent series, which is made possible due to the robustness of Rabinoff’s [Rab12] theory of polyhedral subdomains. Finally, we perform a case-by-case analysis of the common zeros of the $p$-adic integrals coming from Chabauty’s method using our above results.

1.9. Outline of paper. We begin in § 2 with a discussion of Chabauty and Coleman’s method and state some relevant facts concerning $p$-adic integration. In § 3, we introduce the setup for symmetric square Chabauty needed to obtain uniform bounds on the number of points. In § 4, we partition $\text{Sym}^2 X(\mathbb{Q}_p)$ into residue disks and annuli making use
of Stoll’s results and a combinatorial argument. In § 5, we briefly review tropical analytic geometry and generalize Park’s work on tropical intersection theory to the setting of Laurent series. In § 6, we perform the actual calculations of the number of common zeros of the \( p \)-adic integrals coming from Chabauty’s method using the Newton polygons and mixed volumes. We conclude in § 7 by summing up all of our cases to achieve uniform bounds.

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2. Background

2.0.1. Notation. Let \( X \) be a smooth, projective, geometrically integral curve over \( \mathbb{Q} \). Let \( J \) be the Jacobian of \( X \). Let \( X_\mathbb{L} \) or \( J_\mathbb{L} \) represent the base change of the curve or the Jacobian respectively to any field extension \( \mathbb{L} \) of \( \mathbb{Q} \). Let \( g \) and \( r \) denote the genus and Mordell-Weil rank of \( X \) respectively. Let \( X(\mathbb{L}) \) denote the \( \mathbb{L} \)-rational points of \( X \). Let \( \mathbb{Q}_p \) denote the \( p \)-adics, and let \( \mathbb{C}_p \) denote the completion of the algebraic closure of \( \mathbb{Q}_p \). Let \( H^0(J_\mathbb{L}, \Omega^1) \) and \( H^0(X_\mathbb{L}, \Omega^1) \) denote the vector spaces of regular differentials on \( J_\mathbb{L} \) and \( X_\mathbb{L} \), respectively.

We use \( \mathcal{X} \) to refer to a model of a curve \( X \) over \( \mathcal{O}_{\mathbb{K}_\mathbb{P}} \) for some valued field \( \mathbb{K}_\mathbb{P} \) with residue field \( \kappa \). We say that the model \( \mathcal{X} \) is proper if the Zariski closure of any point \( P \in X(\mathbb{K}_\mathbb{P}) \) contains exactly one point \( P \) in \( \mathcal{X}_\kappa \). We also say it is regular. In particular, regularity implies that \( P \) is a smooth point of \( \mathcal{X}_\kappa \) for all \( P \).

Properness gives us a reduction map

\[ \text{red}_p : X(\mathbb{K}_\mathbb{P}) \to \mathcal{X}_\kappa(\kappa). \]

We call the preimage of a point under the reduction map a residue tube. Stoll showed that by contracting the \( \mathbb{P}^1 \) components of a proper regular model, we can obtain a proper model \( \mathcal{X} \) where all points reduce to either smooth points or ordinary double points (nodes). Then each residue tube is either analytically isomorphic to a \( p \)-adic disk or a \( p \)-adic annulus considered as a rigid analytic space, by the following theorem.

**Theorem 2.1** ([BL85, Proposition 2.3]). With notation as above, let \( \mathcal{X} \) be a proper model of \( X \). Let \( \mathbb{P} \) be a uniformizer of \( \mathbb{K}_\mathbb{P} \), and let \( P \in \mathcal{X}(\kappa) \) be a point on the special fiber \( \mathcal{X}_\kappa \). If \( P \) is a smooth point, then the preimage of \( P \) under the reduction map is analytically isomorphic to \( \mathbb{P} \mathcal{O}_{\mathcal{X}_\kappa} \). If \( P \) is a node, its preimage is analytically isomorphic to a \( p \)-adic annulus.

2.2. The Chabauty–Coleman method. For a detailed exposition of Chabauty and Coleman’s method, we refer the reader to [MP07].

In 1941, Chabauty [Cha41] showed that the number of rational points on \( X \) is finite when \( r \leq g - 1 \). His main idea was to consider \( X(\mathbb{Q}) \) in the more tractable spaces.
Let \( \overline{J(Q)} \) represent the \( p \)-adic closure of \( J(Q) \) inside \( J(Q_p) \). This topological space carries the structure of a \( p \)-adic Lie group of dimension \( r' \), which sits inside the compact \( p \)-adic Lie group \( J(Q_p) \cong (\mathbb{Z}_p)^g \oplus \overline{J(Q_p)}_{\text{tors}} \). Note that the bottom arrow factors as \( J(Q) \hookrightarrow \overline{J(Q)} \hookrightarrow J(Q_p) \).

The idea behind Chabauty’s method is to bound \( \#(\overline{J(Q)} \cap X(Q_p)) \) instead of \( \#X(Q) \). Under the condition \( r \leq g - 1 \), [MP07, Lemma 4.2] tells us that \( r' \leq r \leq g - 1 \). Using this fact, Chabauty constructed locally analytic functions \( f \) on \( X(Q_p) \) satisfying \( f(P) = f(Q) \) for \( P, Q \in \overline{J(Q)} \cap X(Q_p) \) reducing to the same point on \( X_{\mathbb{F}_p}(\mathbb{F}_p) \). Then he used the fact that locally analytic functions, which are not locally constant, cannot achieve the same value infinitely often to deduce that \( \#(\overline{J(Q)} \cap X(Q_p)) \) is finite.

Later, Coleman used Newton polygons to give an effective bound on the number of rational points.

**Theorem 2.3** ([Col85, Theorem 4]). If \( p \) is a prime of good reduction such that \( p > 2g \) and \( r \leq g - 1 \), then

\[
\#X(Q) \leq \#X_{\mathbb{F}_p}(\mathbb{F}_p) + (2g - 2).
\]

There have been several refinements of Coleman’s bound, most of which either seek to remove the assumption of \( p \) being of good reduction or to gain some dependence on the rank of the Jacobian. Specifically [LT02, Corollary 1.2] and [Sto13, Corollary 6.7] accomplished this by utilizing the theory of proper regular models and using Clifford’s theorem rather than Riemann-Roch as the geometric input.

In the past five years, there have been major leaps proving uniform bounds for the number of rational points on curves which satisfy a Chabauty-like rank condition. Note that the previous results did not produce uniform bounds for the following reasons. First, the smallest prime of good reduction could be arbitrarily large, and second, a regular proper model of \( X \) can have arbitrarily long chains of \( \mathbb{P}^1 \)'s on its special fiber. To ameliorate these issues, we forgo regularity of the model \( \mathcal{X} \) of \( X \), with the drawback that we must also analyze the integrals on \( p \)-adic annuli.

The first uniformity result comes from Stoll [Sto13, Theorem 9.1], who proves rank-favorable (depending on the rank) uniform bounds for hyperelliptic curves of Mordell–Weil rank \( r \leq g - 3 \). The work of Katz, Rabinoff, and Zureick-Brown [KRZB16, Theorem 5.1] extends Stoll’s uniform bound to arbitrary curves satisfying this rank condition using non-Archimedean potential theory on Berkovich curves and the theory of linear systems and divisors on metric graphs. However, the bound is not rank-favorable. For our purposes, the main input is \( p \)-adic integration, which we recall below.

### 2.4. \( p \)-adic integration

In this section, we define and discuss the difference between the abelian and the Berkovich–Coleman integral. We also state conditions under which the two integrals agree, and why such a fact is necessary to prove uniform bounds.
**Definition 2.5.** Since $J$ is an abelian variety over $\mathbb{C}_p$, we can consider the abelian logarithm, which is defined to be the unique $\mathbb{C}_p$-Lie group homomorphism $\log: J(\mathbb{C}_p) \to \mathbb{C}_p^g$ such that its derivative $d\log: \text{Lie}(J) \to \text{Lie}(\text{Lie}(J)) = \text{Lie}(J)$ is the identity map. Thus, $\log$ is also a local diffeomorphism. It is also a well-known fact that $\text{Lie}(J)$ is the dual of $H^0(J_{\mathbb{C}_p}, \Omega^1)$.

Thus, we have a bilinear map

$$\text{Lie}(J) \times H^0(J_{\mathbb{C}_p}, \Omega^1) \to \mathbb{C}_p$$

and we denote the evaluation pairing by $\langle \cdot, \cdot \rangle$.

**Definition 2.6 (Abelian integral).** For $P, Q \in J(\mathbb{C}_p)$ and $\omega \in H^0(J_{\mathbb{C}_p}, \Omega^1)$, define

$$\text{Ab} \int_{P}^Q \omega := \langle \log(P), \omega \rangle$$

and

$$\text{Ab} \int_{Q}^P \omega := \text{Ab} \int_{P}^0 \omega - \text{Ab} \int_{Q}^0 \omega.$$

**Definition 2.7.** We say that a differential $\omega$ is **good** if the corresponding functional on the Lie algebra vanishes on $\log(J(Q))$.

Recall from above that to obtain uniform bounds, we must work with a model of the curve at a prime $p$ of possibly bad reduction. The problem we run into is that some of the residue tubes are annuli, and as the abelian integral cannot be expressed as a power series on these annuli, Coleman’s argument fails to carry over.

To overcome this problem, we consider both the abelian integral and the Berkovich–Coleman integral. We only need the Berkovich–Coleman integral on residue tubes, and therefore only define it on these spaces.

**Definition 2.8 (Berkovich–Coleman integral).** Let $\omega \in H^0(J_{\mathbb{C}_p}, \Omega^1)$. Let $P, Q \in J(\mathbb{C}_p)$ be in the same residue tube. Suppose $\omega$ on the residue tube has local parameter $t$. Then we can write

$$\omega = \sum_{n=-\infty}^{\infty} a_n t^n \frac{d}{t}. $$

Let $f$ denote the function

$$f(t) := \sum_{n \neq 0} a_n t^n,$$

and choose a branch of the logarithm by defining

$$\text{Log}(t) := \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(t-1)^n}{n}.$$

Finally, define

$$\text{BC} \int_{Q}^P \omega := (f(P) + a_0 \text{Log}(P)) - (f(Q) - a_0 \text{Log}(Q)).$$
The abelian integral is the integral that is used to power the classical Chabauty-Coleman method, but the Berkovich–Coleman integral has the advantage that it can be evaluated as a Laurent series on an annulus, whereas the abelian integral may not have such a simple representation. In order to reap the benefits of both integrals, we look for situations in which they are equal.

The crucial theorem is that for a codimension 2 space of differentials, these two integrals evaluate to the same number.

**Theorem 2.9** ([Sto13, Proposition 7.3]). Let $A_P$ be an annulus in $X(\mathbb{C}_p)$. Let $V$ be the subspace of $H^0(J_{\mathbb{C}_p}, \Omega^1)$ consisting of all $\omega$ such that

$$\int_P^Q \omega = \int_P^Q \omega$$

for all $P, Q \in A_P$, and such that the Laurent series expansion of $\omega$ on $A_P$ has no $dt/t$ term. Then $V$ has codimension at most 2.

### 3. Symmetric power Chabauty

#### 3.0.1. Notation.
Define the symmetric power $\text{Sym}^d X$ to be the quotient of the $d$-fold Cartesian product of $X$ by the action of the symmetric group $S_d$. A point of $\text{Sym}^d X(\mathbb{Q})$ can be represented as a $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable multiset of $d$ elements of $X(\mathbb{Q})$.

#### 3.1. Excluding the algebraic special set.
Siksek [Sik09] and Park [Par16] generalized Chabauty’s method to the symmetric power $\text{Sym}^d X$ of a curve $X$ satisfying $r \leq g - d$. An immediate problem of applying Chabauty’s method to higher dimensional varieties is that the Albanese could be trivial. This problem is easily resolved when considering $\text{Sym}^d X$ since its Albanese is the Jacobian of $X$. Another immediate problem is that $\# \text{Sym}^d X(\mathbb{Q})$ may be infinite. Park eliminates this problem by excluding the points in the algebraic special set of $\text{Sym}^d X$.

**Definition 3.2.** Let $Y$ be a projective variety over $\overline{\mathbb{Q}}$. The **algebraic special set** of $Y$ is the Zariski closure of the union of the images of all nonconstant rational maps $f: G \to Y$ of group varieties $G$ defined over $\overline{\mathbb{Q}}$ into $Y$. We denote the special set of $Y$ by $S(Y)$. For a projective variety $Y$ over $\mathbb{Q}$, let $S(Y)$ be the closed subscheme of $Y$ whose base extension $S(Y)_{\overline{\mathbb{Q}}}$ is $S(Y_{\overline{\mathbb{Q}}})$.

By a result of Faltings, we have that $\#(\text{Sym}^d X \setminus S(\text{Sym}^d X))(\mathbb{Q})$ is finite, and we can bound it using a Chabauty-like technique. The rank condition allows one to find not one good differential but $d$ linearly independent good differentials $\omega_1, \ldots, \omega_d$ from which one can construct $d$ locally analytic functions $\eta_1, \ldots, \eta_d$ on $(\text{Sym}^d X)(\mathbb{Q}_p)$ whose common zero set contains the $\mathbb{Q}$-rational points of $\text{Sym}^d X$.

In order to bound the number of common zeros, we make the same assumption made by Park. Let $\Lambda_X$ denote the $d$-dimensional vector space spanned by the $d$ linearly independent forms $\omega_1, \ldots, \omega_d$. Let $(\text{Sym}^d X)^{an}$ denote the analytification of $(\text{Sym}^d X)$, and let

$$(\text{Sym}^d X)^{\eta=0} := \left\{ P \in (\text{Sym}^d X)(\mathbb{C}_p) : \eta\omega(P) = 0, \forall \omega \in \Lambda_X \right\} \subseteq (\text{Sym}^d X)^{an}(\mathbb{C}_p).$$
A priori, this zero locus inherits the structure of a $p$-adic manifold, as in [Ser64, Section 3] of dimension $r' \leq r$, and hence is locally an open $p$-adic disk of dimension $r'$. By convergence properties of $\eta$, we can shrink this open disk to a closed $p$-adic disk which still contains all of the zero of $\eta$ we wish to bound. The closed $p$-adic disk inherits an affinoid rigid analytic structure, and as we are in the affinoid setting $\text{Sp} A$, the irreducible components of $\text{Sp} A$ are of the form $\text{Sp} A/f$ where $f$ is a minimal prime of the Noetherian, Jacobson ring $A$ by [BGR84, Theorems 5.2.6/1,3]. To this end, we define rigid analytic components of $(\text{Sym}^d X)^{\eta=0}$ as the components of these closed $p$-adic disks.

**Assumption 3.3 ([Par16, Assumption 1.0.7]).** There exist good differentials $\omega_1, \ldots, \omega_d$ such that every positive dimensional rigid analytic component of $(\text{Sym}^d X)^{\eta=0}$ is contained within $\mathcal{S}(\text{Sym}^d X)^{an}$, where $\mathcal{S}(\text{Sym}^d X)^{an}$ denotes the rigid analytification of the algebraic special set of $\text{Sym}^d X$.

**Corollary 3.4 ([Par16, Corollary 4.2.4]).** Given our assumption, we have that the rational points of $\text{Sym}^d(X) \setminus \mathcal{S}(\text{Sym}^d X)$ are contained within the finitely many zero-dimensional components of $(\text{Sym}^d X)^{\eta=0}$.

### 3.5. Generalization of Park’s setup.

In the remainder of this section, we extend Park’s work to the case of arbitrary reduction modulo $p$. Let $K'$ be the compositum of all extensions of $Q_p$ of degree at most $d$. By the proof of [Sto13, Proposition 5.3] (which we state in § 4), there exists a proper model $\mathcal{X}/\mathcal{O}_{K'}$ of $X_{K'}$, whose residue tubes are the disks and annuli in said proposition. Then for any $d \geq 1$, this gives us a reduction map

$$\text{red}_p : (\text{Sym}^d X)(Q_p) \rightarrow (\text{Sym}^d \mathcal{X})(\mathbb{F}_p)$$

$$(Q_1, \ldots, Q_d) \mapsto \{\text{red}_p(Q_1), \ldots, \text{red}_p(Q_d)\},$$

where $\text{red}_p'$ denotes the reduction map $X(K') \rightarrow \mathcal{X}(\mathbb{F}_p)$ given by the model. This is well-defined because $Q_1, \ldots, Q_d \in X(K')$ by definition of $K'$.

We call the preimage of a point $[\mathcal{F}_1, \ldots, \mathcal{F}_d]$ under this reduction map a **residue polytube**. If $\mathcal{F}_1, \ldots, \mathcal{F}_d$ are all smooth, we call the preimage a **polydisk**.

Consider any $\omega \in H^0(J_{Q_p}, \Omega^1)$. We define the function on $\text{Sym}^d X(Q_p)$

$$\eta: \text{Sym}^d X(Q_p) \rightarrow Q_p$$

$$(Q_1, \ldots, Q_d) \mapsto \int_{0}^{[Q_1 + \ldots + Q_d - \text{O}]} \omega,$$

where $\text{O}$ is some fixed degree $d$ divisor of $X_Q$, and the integral is as in § 2.4. Siksek noted that if the residue polytube is a polydisk, we can expand $\omega$ on the residue polytube in local coordinates as

$$\omega = \omega(u_1, \ldots, u_d) = \sum_{i=1}^{d} \omega_i(u_1, \ldots, u_d) du_i \in \mathbb{Z}_p[[u_1, \ldots, u_d]],$$

and it can be shown that $\eta$ is given by the formal antiderivative of $\omega$ on residue polydisks.

The main obstruction we face when $p$ is of bad reduction is that preimages under the reduction map may now be annuli. We use the Berkovich–Coleman integral because it has a Laurent series representation on a residue annulus. However, we want $d$ integrals.
which actually equal the abelian integral, so we work under the rank condition \( r \leq g - d - 2 \) instead of Park’s rank condition \( r \leq g - d \).

**Corollary 3.6.** Let \( \mathbb{A}_P \) be an annulus coming from a node \( \overline{P} \). If \( r \leq g - d - 2 \), there exist \( d \) linearly independent differentials \( \omega_1, \ldots, \omega_d \) on \( J_{\mathbb{A}_P} \) which vanish on \( \log \overline{\mathbb{Q}} \), such that for all \( \omega \) in the span of \( \omega_1, \ldots, \omega_d \)

\[
\int_{\mathbb{A}_P} \omega = \int_{\mathbb{A}_P} \omega,
\]

and the corresponding Laurent series on \( \mathbb{A}_P \) have no \( dt/t \) term.

**Proof.** Let \( V \) be the subspace of \( H^0(J_{\mathbb{A}_P}, \Omega^1) \) consisting of all \( \omega \) such that \( \int_{\mathbb{A}_P} \omega = \int_{\mathbb{A}_P} \omega \), and such that the Laurent series expansion of \( \omega \) has no \( dt/t \) term. By Theorem 2.9, \( V \) has codimension at most 2. Note that one codimension comes from the condition of having no \( dt/t \) term. As \( \dim H^0(J_{\mathbb{A}_P}, \Omega^1) = g \), there is a \( g - 2 \) dimensional subspace on which the integrals are equal. Since \( r \leq g - d - 2 \), we can choose \( d \) linearly independent differentials in this subspace that vanish on \( \log \overline{\mathbb{Q}} \). \( \square \)

In particular, as all the residue expansions of our \( \omega_i \) do not contain a \( dt/t \) term, we note that the residue expansions of the \( \eta_i \) can be obtained by formally anti-differentiating, and do not depend on a \( p \)-adic logarithm. If we suppose Assumption 3.3 the rational points of \( \text{Sym}^2 X \) outside of its special set correspond to zero-dimensional components of the common vanishing locus of the \( \eta_i \), so it suffices to bound these.

### 3.7. Expressing integrals as pure power or Laurent series

The first step is to rewrite the multivariate Laurent series without any mixed monomial terms.

**Definition 3.8.** We define a pure power (resp. Laurent) series to be a power (resp. Laurent) series with no mixed monomial terms.

Siksek showed that on residue polydisks, we can write the integrals \( \eta_i \) as pure power series. The same argument shows that on any residue polytube, we express the integrals \( \eta_i \) as pure Laurent series. Consider the residue polytube \( U \subseteq (\text{Sym}^d X)(\mathbb{C}_p) \) over \( \{P_1, \ldots, P_d\} \in (\text{Sym}^d X)(\overline{\mathbb{F}_p}) \). Let \( U_i \) be the individual residue tube over \( P_i \), and let \( t_i \) be a uniformizer which induces an analytic isomorphism from \( U_i \) to a disk or annulus. Let \( \{Q_1, \ldots, Q_d\}, \{Q'_1, \ldots, Q'_d\} \in U \) such that \( Q_i, Q'_i \in U_i \). Then, by translation invariance of \( \omega \), we can write:

\[
\eta([Q_1, \ldots, Q_d]) = \int_{\emptyset}^{[Q_1, \ldots, Q_d]} \omega
\]

\[
= \int_{\emptyset}^{[Q_1, \ldots, Q_d - 0]} \omega
\]

\[
= \int_{\emptyset}^{[Q'_1, \ldots, Q'_d]} \omega + \int_{Q'_1}^{Q_1} \omega + \cdots + \int_{Q'_d}^{Q_d} \omega
\]

\[
= C + \int_{t_1(Q'_1)}^{t_d(Q_1)} (t_1^{-1})^* \omega + \cdots + \int_{t_d(Q'_d)}^{t_d(Q_d)} (t_d^{-1})^* \omega,
\]
where $C = \int_{Q_1^0}^{Q_1^d} \omega$ is a constant. We have shown that if we choose $\omega$ as in Corollary 3.6 then $\int_{Q_i^0}^{t_i(Q_i)} (t_i - 1) \omega = f_i(t_i)$ for some one variable power or Laurent series $f_i$. Then

$$\eta(t_1, \ldots, t_d) = C + \sum_{i=1}^{d} f_i(t_i).$$

Thus we have obtained a pure representation of our Laurent or power series. Note that this pure series is a function on $U_1 \times \cdots \times U_d \subseteq X^d(C_p)$ not $(\text{Sym}^d X)(C_p)$.

**Remark 3.9.** Note that by [Sik09, Lemma 2.3], if $U_i$ is a $p$-adic disk, we can choose $t_i$ such that the power series obtained by formally anti-differentiating $\omega$ on $U_i$ has no constant term, i.e. they vanish at 0. We call this “centering the disk.”

Combining this section with Corollary 3.6 gives the following generalization of [Par16, Proposition 3.2.2]. The second part relates the number of zeros of these functions to the number of rational points in the residue polytube.

**Proposition 3.10.** For each residue polytube $U \subseteq (\text{Sym}^d X)(C_p)$, let $K$ be the compositum of the fields of definition of all of the points in $U$. Then there exist good $\omega_1, \ldots, \omega_d \in H^0(J_{Q_p}, \Omega^1)$, such that for each $1 \leq i \leq d$ there exists a pure Laurent series or power series $\eta_i$ with coefficients in $K$ satisfying

$$\int_{Q_1}^{Q_2} \omega_i = \eta_i(t(Q_2)),$$

for some $Q_1 \in U$ and all $Q_2 \in U$, where $t$ is an analytic isomorphism from $U$ to a disk or annulus. If

$$\text{red}_p(Q_1) = \text{red}_p(Q_2) = \{P_1, \ldots, P_d\} = \bigcup_{i=1}^{r} S_i$$

where $S_i$ consists of $s_i$ copies of the same point in $X_s(F_p)$, then each point in $U$ corresponds to $N = \prod_{i=1}^{r} (s_i)!$ common zeros of $\eta_1, \ldots, \eta_d$.

### 4. Partitioning $\text{Sym}^2 X(Q_p)$ into disks and annuli

In this section, we compute how many residue disks and annuli cover $(\text{Sym}^2 X)(Q_p)$. First, we recall Stoll’s results about partitioning into disks and annuli, stated below.

**Proposition 4.1** ([Sto13, Proposition 5.3]). Let $X$ be any smooth projective geometrically integral curve over a $p$-adic field $k/Q_p$ of genus $g$ and let $q$ be the size of the residue field. Then there is a number $0 \leq t \leq g$ such that $X(k)$ can be written as a disjoint union of the set of $k$-points of at most $(5q + 2)(g - 1) - 3q(t - 1)$ open disks and at most $2(g - 1) + (t - 1)$ open annuli in $X$. 

4.1.1. Notation. Note that \( \mathbb{Q}_p \) only has three quadratic extensions, one of which is unramified and two of which are ramified. Let \( \mathbb{Q}_p^{p,2} \) denote the unramified extension of \( \mathbb{Q}_p \), and let \( K_1 \) and \( K_2 \) denote the other two ramified extensions. Let

\[
D_1 = (5p + 2)(g - 1) - 3p(t - 1),
\]
\[
D_2 = (5p^2 + 2)(g - 1) - 3p^2(t - 1),
\]
\[
\alpha = 2(g - 1) + (t - 1).
\]

Finally, let \( P^\sigma \) denote the Galois conjugate of a quadratic point \( P \).

4.2. Disks and annuli in the symmetric square case. Let \( X \) be a curve with rank \( r \leq g - 4 \). For each point \( \{P, Q\} \in (\text{Sym}^2 \mathcal{X})(\mathbb{F}_p) \), we fix an ordering and take the preimage in \( X \times X \), which is a product of disks or annuli.

For our purposes, we need to understand the possible reduction types in the \( d = 2 \) case. A point in \( \text{Sym}^2 X(\mathbb{Q}_p) \setminus \mathcal{S}(\text{Sym}^2 X(\mathbb{Q}_p)) \) could reduce to one of the following:

1. (a) \( P_1 = P_2^\sigma \in \mathcal{X}_s(\mathbb{F}_p^2) \setminus \mathcal{X}_s(\mathbb{F}_p) \) and both are smooth points on the special fiber;
   (b) \( P_1 = P_2^\sigma \in \mathcal{X}_s(\mathbb{F}_p^2) \setminus \mathcal{X}_s(\mathbb{F}_p) \) and both are nodes on the special fiber;
2. (a) \( P_1 = P_2 \in \mathcal{X}_s(\mathbb{F}_p) \) and both are smooth points on the special fiber;
   (b) \( P_1 = P_2 \in \mathcal{X}_s(\mathbb{F}_p) \) and both are nodes on the special fiber;
3. (a) \( P_1 \neq P_2, P_1, P_2 \in \mathcal{X}_s(\mathbb{F}_p) \) and both are smooth points on the special fiber;
   (b) \( P_1 \neq P_2, P_1, P_2 \in \mathcal{X}_s(\mathbb{F}_p) \) both are nodes on the special fiber;
   (c) \( P_1 \neq P_2, P_1, P_2 \in \mathcal{X}_s(\mathbb{F}_p) \) one is a node and the other is a smooth point on the special fiber.

Using Stoll’s bounds on the number of residue disks and annuli, we get the following results. We elaborate on this table below.

| Case | Number of Disks/Annuli | Type of preimage |
|------|-----------------------|------------------|
| 1(a) | \( \frac{1}{2}D_2 \) | \( p\mathcal{O}_{\mathbb{Q}_p^2} \times p\mathcal{O}_{\mathbb{Q}_p^2} \) |
| 1(b) | \( \frac{1}{2}\alpha \) | \( A(\mathbb{Q}_p^2) \times A(\mathbb{Q}_p^2) \) |
| 2(a) | \( D_1 \) | \( p\mathcal{O}_{K_1} \times p\mathcal{O}_{K_1} \) |
| 2(b) | \( \alpha \) | \( A(K_1) \times A(K_1) \) |
| 3(a) | \( \begin{pmatrix} D_1 \\ 2 \end{pmatrix} \) | \( p\mathbb{Z}_p \times p\mathbb{Z}_p \) |
| 3(b) | \( \begin{pmatrix} \alpha \\ 2 \end{pmatrix} \) | \( A(\mathbb{Q}_p) \times A(\mathbb{Q}_p) \) |
| 3(c) | \( D_1\alpha \) | \( p\mathbb{Z}_p \times A(\mathbb{Q}_p) \) |

Table 1. Partition data for \( \text{Sym}^2(\mathbb{Q}_p) \)
Case 1. In the first case, \( q = p^2 \). Since \( P \) determines \( \mathbb{P}_\sigma \), we are counting pairs of conjugate residue disks / annuli. Because each point has a distinct conjugate that is not itself, and is in fact in a different residue tube, we simply multiply Stoll’s bound by \( \frac{1}{2} \).

Case 2. In the second case, \( q = p \) because points which reduce to \( \mathbb{F}_p \) must lie in a ramified extension of \( \mathbb{Q}_p \). Thus, from each of the two ramified extensions of \( \mathbb{Q}_p \), we get \( D_1 \) disk \( \times \) disk, and \( \alpha \) annulus \( \times \) annulus. However, note that in the second case, there is a two-to-one correspondence between common zeros of \( \eta_1 \) and \( \eta_2 \) and points in the residue polydisk/polyannulus.

Case 3. In the third case, \( q = p \), and disk \( \times \) annulus is possible. We get \( (D_1) \) disk \( \times \) disk, \( (\frac{\alpha}{2}) \) annulus \( \times \) annulus, and \( D_1 \alpha \) disk \( \times \) annulus.

We obtain bounds on each case by bounding the number of zeros in each case and multiplying it by the number of disks and annuli for that case. We then sum up these bounds to achieve a global bound.

5. Bounding zero-dimensional components

In this section, we briefly recall background on tropical analytic geometry, in particular Rabinoff’s [Rab12] theory of polyhedral subdomains. We conclude by discussing relevant lemmas from [Sto13] and [KRZB16] concerning the Laurent series representations of the \( p \)-adic integrals.

5.0.1. Notation. For the rest of this section, let the field \( K \) denote \( \mathbb{C}_p \). Let \( B_1^K = Sp K\langle x_1 \rangle \), where \( Sp \) is the functor that takes a quotient of a Tate algebra to a rigid analytic space whose points consist of the maximal spectrum of the Tate algebra.

5.1. Tropical analytic geometry. In [Rab12], Rabinoff expounded on the relationship between tropical and rigid analytic geometry. Roughly speaking, for a “nice” polyhedron \( P \subseteq \mathbb{R}^n \), Rabinoff defined an affinoid open sub-domain \( \mathcal{U}(P) \) of the analytification of an affine toric variety associated to combinatorial data attached to \( P \).

Definition 5.2. We refer the reader to [Rab12, Notation 2.2, Definition 2.3 parts (i) and (iv)] for the definition of integral \( \Gamma \)-affine polyhedron, and to Definition 3.12 for the cone of unbounded directions.

Throughout this section, let \( P \) be an integral \( \Gamma \)-affine polyhedron in \( \mathbb{R}^d \).

Definition 5.3. Let \( \sigma \) be the cone of unbounded directions of \( P \), and let \( S_P = \sigma^\vee \cap \mathbb{Z}^d \). Then define

\[
K\langle U(P) \rangle := \left\{ \sum_{u \in S_P} a_u x^u \divides a_u \in K, v(a_u) + \langle u, v \rangle \to \infty \text{ for all } v \in P \right\}.
\]

By [Rab12, Proposition 6.9], \( K\langle U(P) \rangle \) is a \( K \)-affinoid algebra. One can think of \( K\langle U(P) \rangle \) as the set of power series that converge on the points of \( K^n \) whose valuation belongs to the polyhedron \( P \).

Example 5.4. For \( P = \{ (x_1, \ldots, x_d) \in \mathbb{R}^d \mid x_i \geq m_i \text{ for } 1 \leq i \leq d, m_i \in \mathbb{Q}_{>0} \} \), \( K\langle U(P) \rangle \) is isomorphic to the usual Tate algebra over \( K \).
Example 5.5. If \( P = \prod_{i=1}^{n}[r_i, s_i] \), then \( K\langle U(P) \rangle \) is the algebra of Laurent series which converge on the points with valuation in \( P \).  

Definition 5.6. Given a polynomial \( f \in K\langle U(P) \rangle \), define 
\[
H(f) := \{(u,v(a_u)) \mid u \in S_P, a_u \neq 0 \}.
\]
We call this the height graph of \( f \) with respect to \( 0 \). For \( w \in P \), define 
\[
m_w(f) := \min((w,1) \cdot (u,v(a_u))) \mid (u,v(a_u)) \in H(f) \).
\]
Let \( \text{vert}_w(f) := \{(u,v(a_u)) \in S_P \mid (w,1) \cdot (u,v(a_u)) = m_w(f) \} \).

Geometrically, \( \text{vert}_w(f) \) can be thought of as the lower faces of the regular subdivision given by the valuation.

Definition 5.7. For any \( f \in K\langle U(P) \rangle \), define 
\[
\text{Trop}(f) = \{w \in P \mid \# \text{vert}_w(f) > 1 \}.
\]
Note that the above is not the classical definition of \( \text{Trop}(f) \), however this equivalence follows from the fundamental theorem of tropical geometry, which is [MS15, Theorem 3.2.5].

Definition 5.8. For \( f_1, \ldots, f_n \in K\langle U(P) \rangle \), define 
\[
V(f_1, \ldots, f_n) := \text{Sp}(K\langle U(P) \rangle/(f_1, \ldots, f_n)).
\]

Definition 5.9. Let \( f_1, \ldots, f_d \in K\langle U(P) \rangle \), \( Y_i = V(f_i) \), and \( Y = \bigcap_{i=1}^{d} Y_i \). Assume that \( Y \) is zero-dimensional. Then the intersection multiplicity at \( w \) is defined as 
\[
i(w, Y_1, \ldots, Y_d) = \dim_K H^0(Y \cap U_{(w)}, \mathcal{O}_{Y \cap U_{(w)}})
\]
where we view \( \{w\} \) as a zero-dimensional polytope. In simpler terms, this intersection multiplicity at \( w \) is the number of common zeros of the \( f_i \) that have the same coordinate-wise valuation as \( w \), counted with multiplicity.

Also define \( \gamma_w(f_i) = \pi(\text{conv}(\text{vert}_w(f_i))) \), where \( \pi: \mathbb{Z}^d \times \mathbb{R} \to \mathbb{Z}^d \) is the projection onto the first coordinate, and \( \text{conv} \) denotes the convex hull of a set of points. Let \( \gamma_i = \gamma_w(f_i) \) when it’s not ambiguous.

Definition 5.10 (Mixed volume). Let \( P_1, \ldots, P_d \) be bounded polyhedrons in \( \mathbb{R}^d \). Define the function 
\[
f(\lambda_1, \ldots, \lambda_d) := \text{vol}(\lambda_1 P_1 + \cdots + \lambda_d P_d),
\]
where \( + \) is the Minkowski sum. The mixed volume of the \( P_i \), denoted \( \text{MV}(P_1, \ldots, P_d) \) is defined as the coefficient of the \( \lambda_1 \cdots \lambda_d \) term of \( f(\lambda_1, \ldots, \lambda_d) \).

Theorem 5.11 ([Rab12, Theorem 11.7]). Suppose \( f_1, \ldots, f_d \in K\langle U(P) \rangle \) have finitely many common zeros, and let \( w \in \bigcap_{i=1}^{d} \text{Trop}(f_i) \) be an isolated point. Let \( \gamma_i \) be as above. Let \( Y_i = V(f_i) \). Then 
\[
i(w, Y_1, \ldots, Y_d) = \text{MV}(\gamma_1, \ldots, \gamma_d).
\]

1There is an error in [Rab12, Example 6.8]. The indexing of the summation should be over \( \mathbb{Z}^n \).

2This is different from Rabinoff’s definition since we use different sign conventions on the exponents of the Laurent series.
It can be shown that \( \text{vert}_P(f_i) \) is finite, so Theorem 5.11 implies that the \( \gamma_i \), and hence the intersection multiplicity information depend only on a finite number of terms of \( f_i \). Thus, we can approximate each series by a polynomial. The following definition and theorem allow us to bound the number of common zeros of a set of (Laurent) polynomials.

**Definition 5.12 (Newton polygon).** Given a polynomial \( f = \sum_{u \in S} a_u x^u, \ S \subseteq \mathbb{Z}^d \) finite, define

\[
\text{New}(f) = \text{conv}\{u : a_u \neq 0\} \subseteq \mathbb{R}^n.
\]

If \( f \in K \langle \mathcal{U}(P) \rangle \) is a Laurent series, define

\[
\text{New}(f) = \text{New}(f,P) = \pi(\text{conv}(\text{vert}_P(f))).
\]

Since \( \text{vert}_P(f) \) is finite, this is well-defined.

**Theorem 5.13 ([CLO06]).** Let \( f_1, \ldots, f_d \in K[x_1^{\pm 1}, \ldots, x_d^{\pm 1}] \) be Laurent polynomials with finitely many common zeros. Then the number of common zeros with multiplicity of \( f_i \) in \( (K^\times)^d \) is given by

\[
\text{MV}(\text{New}(f_1), \ldots, \text{New}(f_d)).
\]

The following theorem generalizes [Par16, Theorem 5.3.13] to Laurent series.

**Theorem 5.14.** Let \( f_1, \ldots, f_d \in K \langle \mathcal{U}(P) \rangle \) have finitely many common zeros. Let \( S_1, \ldots, S_d \subseteq \mathbb{Z}^d \) be finite sets such that \( S_i \) contains \( \pi(\text{vert}_P(f_i)) \) for each \( i \). Define the **auxiliary polynomials**

\[
g_i = \sum_{u \in S_i} a_u x^u.
\]

Then if \( g_1, \ldots, g_d \) have finitely many common zeros,

\[
\# \left( (K^\times)^d \cap \bigcap_{i=1}^d V(f_i) \right) \leq \text{MV}(\text{New}(g_1), \ldots, \text{New}(g_d)).
\]

**Proof.** By construction \( \gamma_w(f_i) = \gamma_w(g_i) \) for all \( i \) and \( w \in P \). By Theorem 5.11, the number of common zeros of the \( f_i \) with valuation \( w \in P \) is determined only by \( \gamma_w(f_i) \). Thus, the number of common zeros of the \( f_i \) with valuation in \( P \) is equal to the number of common zeros of the \( g_i \), with valuation in \( P \). This is obviously less than or equal to the number of common zeros with any valuation. But by Bernstein’s theorem the **total** number of common zeros of the \( g_i \) in \( (K^\times)^d \) is

\[
\text{MV}(\text{New}(g_1), \ldots, \text{New}(g_d)).
\]

This proves the theorem. Note that \( \text{MV}(\text{New}(g_1), \ldots, \text{New}(g_d)) \) may give us zeros with valuations outside of \( P \), and thus our theorem only states an inequality. \( \square \)

**5.15. Deformation of Laurent series.** In order to count points outside the algebraic special set, we need to count zero-dimensional components of the vanishing locus of the \( \eta_i \), even if there are (infinite) positive-dimensional components. In this subsection, we show that we can deform a set of Laurent series to have finite intersection, without changing their tropicalizations and \( \gamma_i \).
**Definition 5.16.** Let \( P \subseteq \mathbb{R}^d \) be an integral \( \Gamma \)-affine polyhedron. For \( f_1, \ldots, f_d \in K\langle U(P) \rangle \), define \( N_0(f_1, \ldots, f_d) \) to be the number of zero-dimensional components of \( \bigcap_{i=1}^d V(f_i) \), counted with multiplicity. If \( Y = V(f_1, \ldots, f_d) \) is finite, then define

\[
N(f_1, \ldots, f_d) = \dim H^0(Y, \mathcal{O}_Y).
\]

Note that for the rest of this paper, we mean non-degenerate series to mean that for every variable \( t \), some power of \( t \) appears with non-zero coefficient.

**Theorem 5.17 ([Par16, Theorem 6.1.7]).** Let \( f_1, \ldots, f_d \in K\langle U(P) \rangle \) be non-degenerate Laurent series. There exist \( g_1, \ldots, g_d \) with \( \text{Trop}(f_i) = \text{Trop}(g_i) \) and \( \gamma_w(f_i) = \gamma_w(g_i) \) for all \( w \in P \) such that the \( g_i \) have finitely many common zeros, and

\[
N_0(f_1, \ldots, f_d) \leq N(g_1, \ldots, g_d).
\]

Furthermore, if the \( f_i \) are Laurent polynomials, then the \( g_i \) can be chosen to be Laurent polynomials as well, with \( \text{New}(g_i) = \text{New}(f_i) \).

**Proof.** The proof of [Par16, Theorem 6.1.7] does not use the condition that the expansion is a power series and not a Laurent series. The conditions \( \text{Trop}(f_i) = \text{Trop}(g_i) \) and \( \gamma_w(f_i) = \gamma_w(g_i) \) immediately imply that \( \text{New}(g_i) = \text{New}(f_i) \). Thus, the proof immediately generalizes to the Laurent series case, but we will explain the main idea.

We deform the \( f_i \) one at a time, by inductively finding \( g_1, \ldots, g_r \) satisfying

1. \( \text{Trop}(f_i) = \text{Trop}(g_i) \) and \( \gamma_w(f_i) = \gamma_w(g_i) \) for \( i \in \{1, \ldots, r\} \)
2. \( \operatorname{codim} \bigcap_{i=1}^r V(g_i) = r \)
3. \( N_0(f_1, \ldots, f_d) \leq N_0(g_1, \ldots, g_r, f_{r+1}, \ldots, f_d) \)

for \( 1 \leq r \leq d \). The statement is clear for \( r = 1 \), by taking \( g_1 = f_1 \). Given \( g_1, \ldots, g_r \), let \( C_1, \ldots, C_t \) be the codimension \( r \) irreducible components of \( \bigcap_{i=1}^r V(g_i) \), and let \( P_i \in C_i \) for \( i = 1, \ldots, t \), such that \( P_i \neq 0 \). By [Par16, Lemma 5.6] there exists a polynomial \( h \) such that \( h \) does not vanish at any of the \( P_i \), and \( M(h) \subseteq M(f_{i+1}) \) (see [Par16, Definition 5.4]), so that \( g_{r+1} = f_{r+1} + \varepsilon h \) has the same tropicalization and \( \gamma_w \) as \( f_{r+1} \) for small enough \( \varepsilon \).

We will also elaborate slightly on why these deformations do not decrease the number of zero-dimensional components. Let \( G_i(t_1, \ldots, t_d, \varepsilon) = g_i(t_1, \ldots, t_d) \) for \( 1 \leq i \leq r \), let \( G_{r+1} = f_{r+1}(t_1, \ldots, t_d) + \varepsilon h(t_1, \ldots, t_d) \), and let \( f_i = f_i(t_1, \ldots, t_d) \) for \( i > r + 1 \). Define \( I \) to be the product of all non-maximal minimal prime ideals containing the ideal \( (g_1, \ldots, g_r, f_{r+1}, \ldots, f_d) \), and let \( f \) be an element of \( I \) which does not vanish on any of the zero-dimensional components, which exists by the prime avoidance theorem. Consider the map

\[
\alpha : \text{Sp} K\langle U(P) \rangle_f \times B_K^1 \to B_K^1.
\]

Let \( Y = V(G_1, \ldots, G_r, G_{r+1}, F_{r+2}, \ldots, F_d) \). Then \( Y \cap \alpha^{-1}(0) \) consists of all points \( p \) which do not vanish at \( f \) but vanish at \( g_1, \ldots, g_r \) and \( f_{r+1}, \ldots, f_d \). This implies that \( p \) is not on a positive dimensional component, because if it were then \( p \) vanishes at \( I \ni f \).

Thus the size of \( Y \cap \alpha^{-1}(0) \) is equal to the number of zero-dimensional components of \( V(g_1, \ldots, g_r, f_{r+1}, \ldots, f_d) \). Then [Rab12, Theorem 10.2], implies that for small \( |\varepsilon| \), this is also equal to the number of common zeros of \( g_1, \ldots, g_r, f_{r+1} + \varepsilon h, f_{r+1}, \ldots, f_d \), away from the positive-dimensional components of \( V(g_1, \ldots, g_r, f_{r+1}, \ldots, f_d) \). \( \square \)
Remark 5.18. In Theorem 5.14, if the auxiliary polynomials \( g_i \) have infinitely many common zeros, we can choose Laurent polynomials \( g'_1, \ldots, g'_d \) with finitely many common zeros by Theorem 5.17.

Remark 5.19. In order to bound \( N(g_1, \ldots, g_d) \), we have to also count “degenerate” zeros, where some of the coordinates are zero (cf. [Par16, Section 6.2]).

5.20. Relevant results concerning Laurent series expansions.

Remark 5.21. Since we want to bound the number of common zeros of the \( \eta_i \) with valuations in \([1/e, a] \times [1/e, b]\) where \( a \) or \( b \) could be infinite, it suffices to bound the number of common zeros of the \( \eta_i \) with valuations in \( P = [1/e, M] \times [1/e, N] \) as \( N \to a, M \to b, \) and \( M, N < \infty \). Note that then \( \eta_i \in \mathbb{K}(\mathcal{U}(P)) \), and the theorems in this section apply. In particular, the local expansions coming from symmetric power Chabauty lie in some affinoid algebra \( \mathbb{K}(\mathcal{U}(P)) \).

Definition 5.22 ([Sto06, Section 6]). Let \( e \) be the ramification index of \( \mathbb{K}/\mathbb{Q}_p \), and define
\[
\delta(e, k) = \max\{N \geq 0 \mid ev(k + 1) + N \leq ev(k + N + 1)\}.
\]

Lemma 5.23 ([Sto06, Lemma 6.1]). If \( p > e + 1 \), then \( \delta(e, k) \leq e \lfloor k/(p - e - 1) \rfloor \). In particular
\[
k + \delta(e, k) \leq \mu_e k
\]
where
\[
\mu_e = \frac{p - 1}{p - e - 1}.
\]

Lemma 5.24 ([Par16, Lemma 6.2.4]). Let \( \mathbb{K}/\mathbb{Q}_p \) be an extension with ramification index \( e \). Let \( f \in \mathbb{K}[t] \) be such that \( f' \in \mathcal{O}_\mathbb{K}[t] \) and such that \( f \) has no constant term. Let \( k - 1 = \text{ord}_{t=0}(f' \mod p) \). Then if \( w \in [1/e, \infty) \) and \( u > k + \delta(e, k - 1) \) or \( u = 0 \), then \( (u, v(a_u)) \notin \text{vert}_w(f) \).

Proof. By definition \( u > k + \delta(e, k - 1) \) means
\[
ev(k) + u - k > ev(u),
\]
so
\[
v(u) - v(k) < \frac{1}{e}(u - k) < w(u - k),
\]
or
\[
v(a_u) + uw > v(a_k) + kw,
\]
since \( v(ua_u) \geq 0 \), and \( v(ka_k) = 0 \). This shows that \( (u, v(a_u)) \notin \text{vert}_w(f) \). Since \( a_0 = 0 \), indubitably \( (0, v(a_0)) \notin \text{vert}_w(f) \).

Thus any Laurent series can be approximated by pure Laurent polynomials whose degree is less than \( k + \delta(e, k - 1) \).

Corollary 5.25 ([KRZB16, Corollary 4.18]). Let \( \mathbb{A}_\mathbb{P} \) be an annulus coming from a node. Let \( \omega \) be a good differential 1-form contained in \( V \), where \( V \) is as in Theorem 2.9. Then the number of zeros of \( \eta \) on \( \mathbb{A}_\mathbb{P} \) is at most \( 4(2g - 2) \).
Remark 5.26. By Theorem 5.11 this is equivalent to the statement that there there exists an interval \([c_1, c_2]\) of length at most \(8g - 8\) such that \(u \notin [c_1, c_2]\) implies \(u \notin \text{vert}_w(\eta)\) for \(w \in (0, a)\), where \(A_P\) is the annulus given by
\[
\{z \mid 0 < v(z) < a\}
\]
for any \(a\) because Theorem 5.11 counts with multiplicity.

Lemma 5.27. Let \(P\) be a polyhedron, and let \(F = f_1(t_1) + \cdots + f_d(t_d)\) be a pure Laurent series in \(K\langle U(P)\rangle\). Let \(w \in P\). Then
\[
\text{vert}_w(F) \subseteq \text{vert}_w(f_1) \cup \cdots \cup \text{vert}_w(f_d).
\]

Proof. Note that we are considering the \(f_i\) as a function of \(d\) variables implicitly and thus \(\text{vert}_w(F)\) lies in the same ambient space as \(\text{vert}_w(f_i)\). Using the same notation as in Proposition 3.10, let \(u \in \text{vert}_w(F)\). Since \(F\) is a pure Laurent series, we can assume without loss of generality that \(u = (u_1, 0, \ldots, 0)\). Then \(u \in \text{vert}_w(f_1)\). \(\square\)

Because of the above two lemmas, we can bound our Newton polygons by the convex hull of all the vertices of all the pure Laurent series.

Remark 5.28. In the situation of Proposition 3.10, if \(U = U_1 \times \cdots \times U_d\), and \(U_j\) is a disk, then \(\eta_i = f_{i,1}(t_1) + \cdots + f_{i,d}(t_d)\) can be chosen so that \(f_{i,j} \in K[t_j]\) has no constant term and satisfies Lemma 5.25. If \(U_j\) is an annulus, then \(f_{i,j}\) satisfies Corollary 5.24.

6. Calculation of the number of zero-dimensional components

In this section we bound the number of zero-dimensional components of the vanishing locus of the \(\eta_i\) on each tube, the same way as Park did, by taking the mixed volume of the convex hulls of the vertices of the individual components of the pure Laurent series or power series. Throughout this section, assume \(p \geq 5\) is a prime.

6.0.1. Notation. For a fixed residue polytube \(U\), consider the pullback \(U_1 \times U_2\), given by fixing an order and pulling back from the symmetric square to the Cartesian product of \(X\). Let \(\omega_1\) and \(\omega_2\) be the differentials given by Proposition 3.10, and let the expansions of the corresponding integrals as pure Laurent or power series on \(U_1 \times U_2\) be
\[
\eta_1 = f_{1,1}(t_1) + f_{1,2}(t_2), \quad \eta_2 = f_{2,1}(t_1) + f_{2,2}(t_2),
\]
where the first index corresponds to the differential and the second index corresponds to the disk/annulus. Let
\[
k_{i,j} - 1 = \text{ord}_{t_j = 0}(f'_{i,j} \mod p).
\]

Let \(D_1, D_2\), and \(\alpha\) be as in § 4. Let \(\mu_\alpha\) be as in Lemma 5.23. If \(K\) is a finite extension of \(Q_p\), let \(\mu_\alpha\) denote the smooth points and nodes respectively in the image of the reduction map
\[
X(K) \to X_{\alpha}(\mathbb{F}_q),
\]
where \(\mathbb{F}_q\) is the residue field of \(K\).

Before we proceed with our computations, we recall some useful facts.
**Definition 6.1.** For an $n \times n$ matrix $A = (a_{ij})$, define the **permanent** of $A$ to be

$$
\text{Per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)}.
$$

**Lemma 6.2 ([MP07, Lemma A.4]).** Let $V$ be as in Theorem 2.9. Then

$$
\sum_{P \in \text{sm}(K)} \text{ord}_P \omega \leq 2g - 2
$$

for all $\omega \in V$.

We split into 7 cases as described in § 4. Recall that we have three types of points in $(\text{Sym}^2 \mathcal{H}_s)(\mathbb{F}_p)$ that quadratic points reduce to. For a point $\{Q_1, Q_2\}$ reducing to $\{P_1, P_2\}$, let $e$ be the ramification index of the field of definition of $Q_1$ and $Q_2$. We can check that given the type of reduction, the ramification index is uniquely determined. In particular, $e = 1, 2, 1$ in Cases 1, 2, and 3, respectively.

**6.3. Case 1(a).** In this case $e = 1$. Consider a residue polydisk $U_1 \times U_2$ which is the preimage of $\{P, P^o\}$ for some smooth points $P, P^o \in \mathcal{H}_s(\mathbb{F}_{p^2}) \setminus \mathcal{H}_s(\mathbb{F}_p)$. By Lemma 5.24 and Lemma 5.27, $\eta_1$ and $\eta_2$ have auxiliary polynomials with Newton polygons contained in

$$
X_i = \text{conv}(e_1, e_2, a_{i,1}e_1, a_{i,2}e_2),
$$

where $a_{i,j} = k_{i,j} + \delta(e, k_{i,j} - 1)$, and $e_i$ is the $i$-th standard vector. (We can take $X_i$ to be these quadrilaterals instead of the triangles in [Par16], because $\eta_1$ and $\eta_2$ can be chosen, by centering the disks, so that they have zero constant terms.)

![Figure 1](image-url)

**FIGURE 1.** The Newton polygon is contained in $X_i$.

Note that in this case, if $\{Q_1, Q_2\}$ reduces to $\{P, P^o\}$ with $P \in \mathcal{H}_s(\mathbb{F}_{p^2}) \setminus \mathcal{H}_s(\mathbb{F}_p)$, then we must have $Q_1 = Q_2^o$ as $(Q_1, Q_2)$ is a rational point on the symmetric square, so they are either both zero on both nonzero — hence the common zeros of the $\eta_i$ which correspond to points in the residue polydisk are either in $(K^\times)^2$ or equal to $(0, 0)$.

Thus, $N_0(\eta_1, \eta_2)$ is at most the mixed volume of the $X_i$, plus one for the possible solution at $(0, 0)$, which we can directly compute to be

$$
N_0(\eta_1, \eta_2) \leq \frac{1}{2}(a_{1,2}a_{2,1} + a_{2,2}a_{1,1} - 2) + 1 = \frac{1}{2}(\text{Per}(a_{i,j}) - 2) + 1
$$
with \(a_{i,j} \leq \mu_1 k_{i,j}\) as above. By definition,

\[ k_{i,j} = \text{ord}_P \omega_i + 1, \]

and

\[ a_{1,1}a_{2,2} + a_{1,2}a_{2,1} \leq \mu_1^2 (k_{1,1}k_{2,2} + k_{1,2}k_{2,1}). \]

So if we sum up across all the disks involved in Stoll’s bound, noting that the choice of one disk determines the other since they must be centered at conjugate points, so only one summation is needed, we get

\[
\sum_{U_1 \times U_2} N_0(\eta_1, \eta_2) \leq \frac{1}{2} \sum_{P \in \text{sm}(Q_{p^2})} \frac{1}{2}(\mu_1^2 (k_{1,1}k_{2,2} + k_{1,2}k_{2,1}) - 2) + 1
\]

\[
\leq \frac{\mu_1^2}{2} \sum_{P \in \text{sm}(Q_{p^2})} (\text{ord}_P \omega_1 + 1)(\text{ord}_P \sigma \omega_2 + 1)
\]

\[
\leq \frac{\mu_1^2}{2}(D_2 + (2g - 1)^2),
\]

where the inequality comes from Lemma 6.2. By the second half of Proposition 3.10, \(N_0(\eta_1, \eta_2)\) is an upper bound for the number of rational points on the symmetric square that pulls back to \(U\) in this case, so the total number of rational points which reduce to a pair of conjugate smooth points is at most

\[
\frac{\mu_1^2}{2}(D_2 + (2g - 1)^2). \tag{6.3.1}
\]

6.4. Case 2(a). In this case \(e = 2\). Consider residue tube \(U\) which is the preimage of \(\{P, P\}\) for some smooth point \(P \in \mathcal{B}_s(\mathbb{F}_p)\). We fix one of the ramified extensions \(K_i\) and pull back to the Cartesian product to get a polydisk \(U_1 \times U_2 \subseteq X^2(K_i)\).

In this case, we must count degenerate zeros. We bound the number of degenerate zeros in the following way. Since our power series are of the form \(f_{i,1}(t_1) + f_{i,2}(t_2)\), where the \(f_{i,j}\) all have no constant term, if without loss of generality we set \(t_1 = 0\), we are now looking for the number of zeros of the \(f_{i,2}\), which we crudely bound by the sum of the number of common zeros of each \(f_{1,2}\) and \(f_{2,2}\). This is equal to the sum of the lengths of the sides of the quadrilateral on the axes, by Newton polygons.
Thus, by a mixed volume computation, we get

\[ N_0(\eta_1, \eta_2) \leq \frac{1}{2} (\text{Per}(a_{i,j}) - 2) + (a_{1,1} - 1) + (a_{1,2} - 1) + (a_{2,1} - 1) + (a_{2,2} - 1) + 1, \]

where the extra terms come from accounting for zeros where one or both coordinates are zero, and the \( \frac{1}{2}(\text{Per}(a_{i,j}) - 2) \) is the bound on the number of non-degenerate zeros. Using the inequality \( a_{i,j} \leq u_2 k_{i,j} \), we get that this is at most

\[ \frac{\mu_2^2}{2} (k_{1,1} k_{2,2} + k_{1,2} k_{2,1}) + \mu_2 (k_{1,1} + k_{1,2} + k_{2,1} + k_{2,2}) - 4. \]

Now, sum up the first term in the above over all the disks, and use the definition of \( k_{i,j} \) to get

\[ \frac{\mu_2^2}{2} \sum_{P \in \text{sm}(K_i)} (\text{ord}_P \omega_1 + 1) (\text{ord}_P \omega_2 + 1) + (\text{ord}_P \omega_1 + 1) (\text{ord}_P \omega_2 + 1) \]

\[ \leq \mu_2^2 (D_1 + (2g-1)^2). \]

Also, if we sum up the other terms, we get

\[ \sum_{P \in \text{sm}(K_i)} \mu_2 (k_{1,1} + k_{1,2} + k_{2,1} + k_{2,2}) = \mu_2 \left( \sum_{P \in \text{sm}(K_i)} 2 (\text{ord}_P \omega_1 + \text{ord}_P \omega_2) + 4 \right) \]

\[ \leq 4 \mu_2 (2g-2) + 4 \mu_2 D_1. \]

Thus in total, there are at most

\[ \mu_2^2 (D_1 + (2g-1)^2) + 4 \mu_2 ((2g-2) + D_1) - 4D_1. \] (6.4.1)

rational points which reduce to \( \{P, P\} \) for some smooth point \( P \).

**Remark 6.5.** Because the above bound is for one ramified extension, we should multiply the bound for \( N_0(\eta_1, \eta_2) \) by 2 to account for both ramified extensions. However, by the second half of Proposition 3.10, there is a two-to-one correspondence between zeros and points, so we must also multiply by \( \frac{1}{2} \).

### 6.6. Case 3(a).

In this case \( e = 1 \). Consider a residue polydisk \( U_1 \times U_2 \) which is the preimage of \( \{P_1, P_2\} \) for some *distinct* smooth points \( P_1, P_2 \in \mathcal{X}_k(F_p) \). By the same reasoning as before,

\[ N_0(\eta_1, \eta_2) \leq \frac{1}{2} (\text{Per}(a_{i,j}) - 2) + (a_{1,1} - 1) + (a_{1,2} - 1) + (a_{2,1} - 1) + (a_{2,2} - 1) + 1. \]

By Lemma 5.23, \( a_{i,j} \leq \mu_1 k_{i,j} \), so the above is bounded by

\[ \frac{\mu_1^2}{2} (k_{1,1} k_{2,2} + k_{1,2} k_{2,1}) + \mu_1 (k_{1,1} + k_{1,2} + k_{2,1} + k_{2,2}) - 4. \]

Note that \( k_{i,j} \) depends on \( P = \{P_1, P_2\} \), and

\[ k_{i,j} = \text{ord}_{P_j} \omega_i + 1. \]
We sum up the linear terms over the possible disks. In this case, the points reduce to arbitrary different disks so we must take a double sum, so we have
\[
\sum_{U_1 \times U_2} N_0(\eta_1, \eta_2) \leq \sum_{P_1, P_2 \in \text{sm}(Q_p)} \mu_1(k_{1,1} + k_{1,2} + k_{2,1} + k_{2,2})
\]
\[
= \mu_1 \sum_{P_1, P_2 \in \text{sm}(Q_p)} \text{ord}_{P_1} \omega_1 + \text{ord}_{P_2} \omega_1 + \text{ord}_{P_1} \omega_2 + \text{ord}_{P_2} \omega_2 + 4
\]
\[
= 2\mu_1 \left( \sum_{P_1, P_2 \in \text{sm}(Q_p)} \text{ord}_{P_1} \omega_1 + \text{ord}_{P_2} \omega_1 \right) + 4\mu_1 \left( \frac{D_1}{2} \right).
\]

Now notice that we have \(D_1\) disks in \(\text{sm}(Q_p)\) under consideration, and thus each \(P_1 \in \text{sm}(Q_p)\), has \(D_1 - 1\) possible paired \(P_2 \in \text{sm}(Q_p)\). Thus, by Lemma 6.2 we can bound the above by
\[
\mu_1 \left( (4g - 4)(D_1 - 1) + 4 \left( \frac{D_1}{2} \right) \right).
\]

We bound the summation of the first term as followss.
\[
\frac{\mu_1^2}{2} \sum_{P_1, P_2 \in \text{sm}(Q_p)} (\text{ord}_{P_1} \omega_1 + 1)(\text{ord}_{P_2} \omega_2 + 1) + (\text{ord}_{P_1} \omega_2 + 1)(\text{ord}_{P_2} \omega_1 + 1)
\]
\[
\leq \frac{\mu_1^2}{2} \left( \sum_{P_1 \in \text{sm}(Q_p)} \text{ord}_{P_1} \omega_1 + 1 \right) \cdot \left( \sum_{P_2 \in \text{sm}(Q_p)} \text{ord}_{P_2} \omega_2 + 1 \right)
\]
\[
\leq \frac{\mu_1^2}{2}(2g - 2 + D_1)^2.
\]

Thus, the final bound in this case is
\[
\frac{\mu_1^2}{2}(2g - 2 + D_1)^2 + \mu_1 \left( (4g - 4)(D_1 - 1) + 4 \left( \frac{D_1}{2} \right) \right) - 4 \left( \frac{D_1}{2} \right).
\]

6.7. Case 1(b), 2(b), and 3(b). Consider a residue polyannulus \(U_1 \times U_2\) which is the preimage of \(\{P_1, P_2\}\) for some \(P_1, P_2\) which are both nodes on \(\mathcal{S}_s\). In Case 1(b), \(P_1, P_2 \in \mathcal{S}_s(\mathbb{F}_p^2) \setminus \mathcal{S}_s(\mathbb{F}_p)\) and \(P_1 = P_2^\ast\). In Case 2(b), \(P_1, P_2 \in \mathcal{S}_s(\mathbb{F}_p)\) and \(P_1 = P_2\). In Case 3(b), \(P_1, P_2 \in \mathcal{S}_s(\mathbb{F}_p)\) and \(P_1 \neq P_2\). Also, \(e = 1, 2, 1\) in Cases 1(b), 2(b), and 3(b) respectively.

By Lemma 5.27 and Corollary 5.25, if \(P_1\) and \(P_2\) are both nodes on \(\mathcal{S}_s\), then both \(\eta_1\) and \(\eta_2\) can be approximated by Laurent polynomials whose Newton polygons are contained in \(\text{conv}(c_{i,1}e_1, c_{2,1}e_1, c_{1,2}e_2, c_{2,2}e_2)\), for some \(c_{i,j}\) with \(c_{2,j} - c_{1,j} \leq 8g - 8\), according to [KRZB16, Corollary 4.18]. It can be checked that the mixed volume of two of these is at most \(16(2g - 2)^2\). In the annulus case, there are no degenerate zeros because 0 is not in the annulus. Thus, for Cases 1, 2, 3 there are a total of
\[
16(2g - 2)^2 \left( \frac{1}{2} \alpha + \alpha + \left( \frac{\alpha}{2} \right) \right) = 16(2g - 2)^2 \left( \frac{3}{2} \alpha + \left( \frac{\alpha}{2} \right) \right)
\]
zero-dimensional components. So our final bound is

\[ 16(2g - 2)^2 \left( \frac{3}{2} \alpha + \left( \frac{\alpha}{2} \right) \right). \quad (6.7.1) \]

**6.8. Case 3(c).** In this case \( e = 1 \). Consider a residue polytube \( U_1 \times U_2 \) which is the preimage of \( \{ P_1, P_2 \} \) where \( P_1, P_2 \subseteq \mathcal{X}_s(F_p) \), and \( P_1 \) is a smooth point on \( \mathcal{X}_s \) and \( P_2 \) is a node. In this case the Newton polygons can be bounded by

\[
\text{conv}(0, a_{1,1}e_1, a_{2,1}e_1, c_{1,2}e_2, c_{2,2}e_2)
\]

with \( c_{1,2} - c_{2,2} \leq 8g - 8 \), again by [KRZB16, Corollary 4.18]. It can be shown that the mixed volume of any two of these is at most

\[
(4g - 4)(a_{1,1} + a_{2,1}).
\]

As \( U_2 \) is an annulus, the only degenerate zeros must be those for which the first component is 0. Here each \( \eta_i \) is the sum of a power series and a Laurent series. By centering the disk \( U_1 \), we can ensure that the power series have no constant term, so degenerate zeros are common zeros of the two Laurent series. Therefore, again by [KRZB16, Corollary 4.18], we have at most \( 4(2g - 2) \) degenerate zeros. Thus, in total, we have a bound of

\[
N_0(\eta_1, \eta_2) \leq (4g - 4)(a_{1,1} + a_{2,1}) + 4(2g - 2).
\]

The sum over all such pairs \( \{ P_1, P_2 \} \) is bounded by

\[
2(2g - 2) \sum_{P_1 \in \text{sm}(Q_p)} \sum_{P_2 \in \text{ns}(Q_p)} \mu_1(2 + \text{ord}_p \omega_1 + \text{ord}_p \omega_2) + 2 \\
\leq (4g - 4)(2\alpha \mu_1(2g - 2) + (2\mu_1 + 2)D_1 \alpha).
\]

Thus, the bound in this case is

\[
(8g - 8)(\alpha \mu_1(2g - 2) + (\mu_1 + 1)D_1 \alpha). \quad (6.8.1)
\]

To summarize our computations, we have the following table.

| Case | Number of zero-dimensional components |
|------|--------------------------------------|
| 1(a) | \( \frac{\mu_2}{2}(D_2 + (2g - 1)^2) \) |
| 2(a) | \( \mu_2^2(D_1 + (2g - 1)^2) + 4\mu_2((2g - 2) + D_1) - 4D_1 \) |
| 3(a) | \( \frac{\mu_2}{2}(2g - 2 + D_1)^2 + \mu_1 \left( (4g - 4)(D_1 - 1) + 4 \left( \frac{D_1}{2} \right) \right) - 4 \left( \frac{D_1}{2} \right) \) |
| 1(b), 2(b), 3(b) | \( 16(2g - 2)^2 \left( \frac{3}{2} \alpha + \left( \frac{\alpha}{2} \right) \right) \) |
| 3(c) | \( (8g - 8)(\alpha \mu_1(2g - 2) + (\mu_1 + 1)D_1 \alpha) \) |

**Table 2.** Data for zero-dimensional components
7. Uniform Bounds

The above computations give us the following theorem by summing up over the various cases.

**Theorem 1.5.** Let \( X \) be a smooth, projective, geometrically integral curve over \( \mathbb{Q} \) with genus \( g \geq 4 \) satisfying both \( r \leq g - 4 \) and Assumption 3.3. Then the number of points in \((\text{Sym}^2 X)(\mathbb{Q})\) lying outside its algebraic special set is at most

\[
288g^4 + \frac{1616}{3}g^3 - \frac{2900}{9}g^2 + \frac{11654}{9}g - \frac{4012}{9}.
\]

**Proof.** Let \( p = 5 \). By adding up Equations 6.3.1, 6.4.1, 6.6.1, 6.7.1 and 6.8.1, we get that there are at most

\[
128g^4 + 128g^3t + 32g^2t^2 + \frac{1616}{3}g^3 - \frac{1256}{3}g^2t - 344gt^2 - \frac{8858}{9}g^2 - 380gt + 662t^2 + \frac{11654}{9}g - 206t - \frac{4012}{9}
\]

rational points. Since \( 0 \leq t \leq g \), we plug in \( t = 0 \) for all the negative terms and \( t = g \) for all of the positive terms in the above expression to get our statement. \( \Box \)

7.1. Rank-favorable bounds for hyperelliptic curves. Using Proposition 7.2, we can compute rank-favorable bounds if \( X \) is a hyperelliptic curve.

**Proposition 7.2 ([Sto13, Proposition 7.7]).** Let \( \Delta_\mathfrak{P} \) be an annulus in \( X \) (the preimage of a node \( \mathfrak{P} \)). Assume \( p > e + 1 \), where \( e \) is the ramification index of \( k \). Then there exists a dimension \( g - r - 2 \) space of good differential forms \( \omega \) such that the number of zeros of \( \eta \) in \( \Delta(\mathfrak{P}^{\text{unr}}) \) is at most \( 2\mu_e r \), where \( \mu_e \) is as in Lemma 5.23.

**Remark 7.3.** By Theorem 5.11, this is equivalent to the statement that there exists an interval \([c_1, c_2]\) of length at most \( 2\mu_e r \) such that \( u \not\in [c_1, c_2] \) implies \( u \vert_{\text{vert},w}(\eta) \), where \( \Delta_\mathfrak{P} \) is the annulus given by

\[
\{ z \mid 0 < v(z) < a \}.
\]

Thus, for each point \( \{P_1, P_2\} \in (\text{Sym}^2 \mathcal{X})_s(\mathbb{F}_p) \), where \( P_2 \) is a node, we can choose \( \omega_1 \) and \( \omega_2 \) such that the number of zeros of \( \eta_i \) on \( \Delta_\mathfrak{P}_i \) is at most \( 2\mu_e r \). Then by adjusting the bounds computed in Equations 6.3.1, 6.4.1, 6.6.1, 6.7.1, 6.8.1 using Proposition 7.2, we can generate rank-favorable uniform bounds on the number of quadratic points of hyperelliptic curves.

**Theorem 1.6.** Let \( H \) be a hyperelliptic curve over \( \mathbb{Q} \) of genus \( g \geq 4 \) satisfying both \( r \leq g - 4 \) and Assumption 3.3. Then the number of points in \((\text{Sym}^2 H)(\mathbb{Q})\) lying outside its algebraic special set is at most

\[
96g^3r + \frac{2192}{9}g^2r + \frac{27037}{18}g^2 - \frac{1184}{3}gr - \frac{4043}{9}g + \frac{736}{3}r + \frac{1429}{9}.
\]

**Proof.** Our proof follows the same lines as the proof of Theorem 1.5. Specifically, the cases for which the computations change are 1(b), 2(b), 3(b), and 3(c). For cases 1(b), 2(b), and 3(b), by Proposition 7.2, we can now say that our Newton polygon is contained within

\[
\text{conv}(c_{1,1}e_1, c_{2,1}e_1, c_{1,2}e_2, c_{2,2}e_2)
\]
for some $c_{i,j}$ with $c_{2,1} - c_{1,1} \leq 8g - 8$ and $c_{2,2} - c_{1,2} \leq 2\mu_\epsilon r$. The mixed volume of two of these is at most $8(4g - 4)(\mu_\epsilon r) \leq 8\mu_\epsilon r(2g - 2)$. Thus our bound for case 1(b), 2(b), 3(b) is
\[
8\mu_1 r(2g - 2) \left( \frac{1}{2} \alpha \right) + 8\mu_2 r(2g - 2)(\alpha) + 8\mu_1 r(2g - 2) \left( \frac{\alpha}{2} \right).
\]
Now we adapt our bound of case 3(c). We can choose differentials so that the Newton polygons are contained in
\[
\text{conv}(0, a_{1,1}e_1, a_{2,1}e_1, c_{1,2}e_2, c_{2,2}e_2)
\]
where $c_{2,2} - c_{1,2} \leq 2\mu_1 r$ by Proposition 7.2. Then by a similar computation as before, we have at most
\[
2\mu_1 r(\alpha \mu_1 (2g - 2) + (\mu_1 + 1)D_1 \alpha)
\]
zero dimensional components. Then, as above we can plug in explicit values of $p, \mu_1$ and $\mu_2$ to get a bound
\[
\frac{128}{3} g^3 r + \frac{128}{3} g^2 tr + \frac{32}{3} g t^2 r + \frac{2192}{9} g^2 r - \frac{776}{9} g tr - \frac{104}{36} t^2 r + \frac{43049}{36} g^2 - \frac{7255}{6} gt
\]
\[
+ \frac{1225}{4} t^2 - \frac{5624}{9} gr + \frac{2072}{9} tr - \frac{32087}{36} g - \frac{5305}{12} t + \frac{736}{3} r + \frac{1429}{9}.
\]
Using $0 \leq t \leq g$ as before gives an upper bound of
\[
\frac{96 g^3 r + 2192}{9} g^2 r + \frac{27037}{18} g^2 - \frac{1184}{3} gr - \frac{4043}{9} g + \frac{736}{3} r + \frac{1429}{9}.
\]
\]

\textbf{Corollary 1.7.} Let $H$ be a hyperelliptic curve over $\mathbb{Q}$ of genus $g \geq 4$ satisfying Assumption 3.3. Then, the size of a rational torsion packet of $\text{Sym}^2 H$ lying outside its algebraic special set is at most
\[
\frac{27037}{18} g^2 - \frac{4043}{9} g + \frac{1429}{9}.
\]

\textit{Proof.} Take the rank-favorable bound given in Theorem 1.6. If we let $r = 0$, we immediately get the size of torsion packets. Thus our statement is proved. \hfill \square

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