VOXRTX MOTION IN CHARGED FLUIDS

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abstract

A non-relativistic scalar field coupled minimally to electromagnetism supports in the presence of a homogeneous background electric charge density the existence of smooth, finite-energy topologically stable flux vortices. The static properties of such vortices are studied numerically in the context of a two parameter model describing this system as a special case. It is shown that the electrostatic and the mexican hat potential terms of the energy are each enough to ensure the existence of vortex solutions. The interaction potential of two minimal vortices is obtained for various values of the parameters. It is proven analytically that a free isolated vortex with topological charge $N \neq 0$ is spontaneously pinned, while in the presence of an external force it moves at a calculable speed and in a direction $(N/|N|) 90^0$ relative to it. In a homogeneous external current $\mathbf{J}$ the vortex velocity is $\mathbf{V} = -\mathbf{J}$. Other theories with the same vortex behaviour are briefly discussed.

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1. Introduction

We study in detail some aspects of the physics of flux vortices in systems described by a non-relativistic complex scalar field minimally coupled to electromagnetism and in the presence of a background homogeneous electric charge density. The model was introduced and the essential features of its vortex dynamics were discussed in a recent publication [1]. Here we elaborate upon and extend the results of reference [1].

Our motivation to analyze the behaviour of topological solitons in the above model is twofold. First, it is actually a whole class of models potentially relevant to the theoretical treatment of several physical systems. In the absence of any scalar potential one deals with a plasma with infinitely-massive positive charges frozen in a configuration of constant charge density. With a quartic potential for the scalar field the theory can be viewed either as the natural coupling to the electromagnetic field of the dynamical Gross-Pitaevskii equation of superfluidity [2] or as one possible dynamical extension of the static Ginzburg-Landau model of superconductivity, reminiscent in particular of the large friction limit of the system considered in reference [3]. Second, in the context of the above models one can obtain definite theoretical predictions about the gross features of the motion of vortices in two spatial dimensions, without any approximation and insensitive to the fine details of the Hamiltonian. This might prove a reasonable starting point for the understanding of the dynamics of an isolated Abrikosov vortex in thin superconducting films, a subject of considerable interest in the context of ordinary as well as high-$T_c$ superconductors [3] [4] [5] [6].

In section II we introduce the model [1]. Since the bulk of our results refer to two dimensional objects we directly present the model in two spatial dimensions. Section III contains our numerical study of axially symmetric vortex solutions for various values of the topological charge. We explicitly demonstrate the almost identical roles of the mexican hat potential term and the electrostatic term in the energy density of the model. We show that either one of these two terms is enough
to guarantee the existence of non-trivial vortex solutions. The parameter space is divided into two regions according to whether the ratio of the energy of the doubly-charged vortex over the energy of two well-separated single vortices is smaller or larger than one. Finally the two-vortex system is discussed and their interaction potential is obtained numerically. Contrary to the relativistic model\cite{7} \cite{8} the interaction energy is not always a monotonous function of the distance between the two vortices. In a wide region of the parameter space there is a bump in the interaction potential of two vortices, which leads to the possibility of having doubly-charged dumb-bell shaped static solutions. The dynamics of vortices is discussed in section IV. The canonical structure of the model is derived and it is shown that the unambiguous linear momentum is essentially the first moment of the topological density. The momentum conservation law in the absence of any external fields is shown to imply the spontaneous pinning of isolated vortices, while the application of an external force results in the motion of the vortex at a constant calculable speed in a direction $(\hat{N}/|\hat{N}|)\ 90^0$ relative to it\cite{1}, features encountered already in the motion of magnetic bubbles in the ferromagnetic continuum\cite{9} \cite{10}. Finally, in the discussion section V we show that necessary condition for the manifestation of the above Hall-behaviour in the dynamics of topological solitons in a given field theory is that the momentum part of its symmetry algebra admits central extension. Other interesting models sharing this property are commented upon.

2. The model - General features

We will be dealing with a non-relativistic complex scalar field $\Psi$ (the condensate) minimally coupled with coupling $q$ to the electromagnetic potential $(A_0, A_i)$. To make the model physically sensible and mathematically consistent it is necessary to introduce a background (positive-ion) charge density to neutralize the system. For simplicity we take it to be constant and homogeneous throughout. We concentrate on the physics of infinitely long straight vortices i.e. on field configurations uniform along the third spatial direction and define the model directly
in two space dimensions by the lagrangian

\[ \mathcal{L} = \frac{i\gamma}{2} [\Psi^* D_t \Psi - \text{c.c.}] + \gamma q \Psi^2 A_0 - \frac{\gamma^2}{2m} |D_i \Psi|^2 + \frac{1}{8\pi} (E^2 - B^2) - V(|\Psi|) \] (2.1)

with \( B = \epsilon_{ij} \partial_i A_j \), \( E_i = -\frac{1}{c} \partial_t A_i - \partial_i A_0 \), \( D_i \Psi = (\partial_i + iq A_0)\Psi \), \( D_i \Psi = (\partial_i - i\frac{q}{c} A_i)\Psi \). \( \gamma \), \( m \) and \( q \) are parameters, \( c \) is the speed of light and the spatial indices \( i, j \) range from 1 to 2. As it will become clear, although the specific form of the potential \( V \) changes the details of the profile of the vortex solutions, it does not affect their dynamical behaviour. For the discussion of vortex dynamics \( V \) could even be absent but in order to make contact with a variety of models of physical interest we will allow for a mexican hat phenomenological potential

\[ V(|\Psi|) = \frac{1}{8} g (\Psi \Psi^* - \Psi_0^2)^2 \] (2.2)

with quartic self-coupling \( g \). Rescale fields and coordinates according to

\[
x_i \rightarrow \frac{\gamma\kappa}{\Psi_0\sqrt{mg}} x_i \quad t \rightarrow \frac{\gamma\kappa^2}{\Psi_0^2 g} t
\]

\[
\Psi \rightarrow \Psi_0 \Psi \quad A_0 \rightarrow \frac{g \Psi_0^2}{\gamma\kappa^2 q} A_0 \quad A_i \rightarrow \frac{c \Psi_0 \sqrt{mg}}{\gamma\kappa q} A_i
\]

(2.3)

to obtain\(^{[1]}\), in terms of the dimensionless quantities \( t, x_i, \Psi, A_0 \) and \( A_i \) used from now-on, the lagrangian

\[ \mathcal{L} = \frac{1}{2} (\Psi^*(i\partial_t - A_0)\Psi + \text{c.c.}) + A_0 - \frac{1}{2} |D_i \Psi|^2 + \frac{1}{2} (\frac{1}{\lambda} E^2 - B^2) - \frac{1}{8\kappa^2} (\Psi \Psi^* - 1)^2 \] (2.4)

with \( B = \epsilon_{ij} \partial_i A_j \), \( E_i = -\partial_t A_i - \partial_i A_0 \) and \( D_i = \partial_i - iA_i \). We keep the same symbols to simplify our notation.
Classically the model depends on the two free dimensionless parameters $\kappa$ and $\lambda$ defined by

$$
\kappa^2 = \frac{g m^2 c^2}{4\pi q^2 \gamma^4} \quad \lambda = \frac{m^3 c^4}{4\pi q^2 \gamma^4 \Psi_0^2} \tag{2.5}
$$

The conversion to physical units for the lengths and the characteristic times is given for the specific system at hand by the rescaling formulas (2.3). An overall factor $g\Psi_0^4 / \kappa^2$ was dropped from $\mathcal{L}$ since it does not enter the equations of motion. It plays though the role of $1/\hbar$ in the quantum theory and determines the necessary condition for the validity of our semiclassical approximation

$$
\frac{\kappa^2}{g \Psi_0^4} \to 0 \tag{2.6}
$$

In this limit we expect the quantum solitons to resemble closely their classical ascendants studied below.

The equations of motion on the other hand read

$$
i \dot{\Psi} = -\frac{1}{2} D^2 \Psi + A_0 \Psi + \frac{1}{4} \kappa^2 (\Psi^* \Psi - 1) \Psi
$$

$$
\frac{1}{\lambda} \dot{E}_i = \epsilon_{ij} \partial_j B - J_i \tag{2.7}
$$

The Gauss constraint and the remaining Maxwell identity

$$
\frac{1}{\lambda} \partial_i E_i = \rho \quad \dot{B} = -\epsilon_{ij} \partial_i E_j \tag{2.8}
$$

complete the set of field equations of the model. The charge and current densities $\rho$ and $J_i$ respectively, are given by

$$
\rho = \Psi^* \Psi - 1 \quad J_i = \frac{1}{2i} [\Psi^* D_i \Psi - (D_i \Psi)^* \Psi] \tag{2.9}
$$

The corresponding energy functional is the sum of four positive terms $W = W_d +$
$W_b + W_e + W_v$ with

$$W_d = \frac{1}{2} \int d^2 x |D_i \Psi|^2 \quad W_b = \frac{1}{2} \int d^2 x B^2$$

$$W_e = \frac{1}{2\lambda} \int d^2 x E^2 \quad W_v = \frac{1}{8\kappa^2} \int d^2 x (\Psi \Psi^\ast - 1)^2$$

(2.10)

We will discuss exclusively smooth, particle-like, localized configurations with finite energy. Notice that even in the absence of the potential $V$ finiteness of $W_e$ already implies that these configurations must be neutral. A localized configuration with non-zero net electric charge leads to an electric field behaving like $1/r$ at large distances and this makes $W_e$ diverge. This neutrality requirement translates into a boundary condition for $|\Psi|$ at infinity. In fact $|\Psi|$ must tend to one, since otherwise it will lead to infinite charge at large distances. It is possible of course to interpolate to a neutral singular configuration but only at an infinite energy cost. One could in principle contemplate a logarithmically divergent energy due to a non-zero finite charge but it is impossible to allow for an asymptotic value of $|\Psi|$ different from one. We conclude that the electrostatic term exactly like the standard phenomenological potential term, when combined with the requirement of finite energy, leads to the same boundary condition for $|\Psi|$ at infinity. Furthermore, we will explicitly demonstrate in the next section that even without $V$ the field equations do support the existence of non-trivial soliton solutions. We thus have to impose the conditions

$$Q = \int d^2 x \rho = 0$$

(2.11)

and

$$|\Psi(x)| \to 1 \quad and \quad |D_i \Psi| \to 0 \quad as \quad |x| \to \infty$$

(2.12)

Such configurations are known [11] to be classified according to the first homotopy group of $S^1$ into disjoint topological sectors (equivalence classes) characterized by
an integer topological charge $N$, computed from

$$N = \frac{1}{2\pi i} \int d^2 x \, \epsilon_{kl} \partial_k \Psi^* \partial_l \Psi$$  \hspace{1cm} (2.13)

For field configurations satisfying the above boundary conditions one may equivalently use the manifestly gauge-invariant formulas

$$N = \frac{1}{2\pi} \int d^2 x \, B$$

or

$$N = \frac{1}{2\pi i} \int d^2 x \, [\epsilon_{kl}(D_k \Psi)^*(D_l \Psi) - iB(\Psi^*\Psi - 1)]$$  \hspace{1cm} (2.14)

as the integral of the explicitly gauge-invariant topological density

$$\tau = \frac{1}{2\pi i} [\epsilon_{kl}(D_k \Psi)^*(D_l \Psi) - iB(\Psi^*\Psi - 1)]$$ \hspace{1cm} (2.15)

An argument which distinguishes these three definitions and favours the use of the last version will be given in the next section.

A final technical remark is related to the gauge invariance mentioned above. The action (2.4) is invariant under the space dependent gauge transformations $\Psi' = \exp(i\Lambda(x))\Psi$, $A'_i = A_i + \partial_i \Lambda$, $A'_0 = A_0$ for arbitrary function $\Lambda(x)$. For small gauge transformations i.e. for $\Lambda$ such that $\epsilon_{ij}\partial_i \partial_j \Lambda = 0$ the topological charge $N$ is also invariant. Large, and for the $U(1)$ case at hand necessarily singular, transformations for which $\epsilon_{ij}\partial_i \partial_j \Lambda = 2\pi n\delta(x)$, $N$ changes by $n$ units. To solve the field equations one has to eliminate the redundant degrees of freedom by imposing a gauge-fixing condition. We choose to work in the gauge $\nabla \cdot A = 0$.

Equations (2.7) and (2.8) admit the one parameter family of trivial vacuum solutions

$$\Psi = \exp(i\alpha) \quad A_i = 0 \quad A_0 = 0$$  \hspace{1cm} (2.16)

for arbitrary constant value of $\alpha$. Although the small oscillation analysis around a trivial vacuum is of interest by itself and deserves detailed study, the purpose
of this work is to analyse the physics of flux-vortices to which we now turn our attention.

3. Static properties

3.1. Isolated vortices

We will search for static axially symmetric solutions of the field equations in any given non-trivial topological sector. The most general ansatz for such a vortex configuration with topological charge $N$ is in polar coordinates $(r, \phi)$ given by

$$
\Psi(x) = X(r) e^{iN\phi} \quad A_0(x) = f(r) \quad A(x) = \frac{N}{r} (1 - \alpha(r)) \hat{\phi}
$$

At infinity (2.12) implies the boundary conditions

$$
X(r) \to 1 \quad \alpha(r) \to 0 \quad (3.2)
$$

Within the above ansatz the electric and the magnetic fields are

$$
E = -f' \hat{r} \quad B = -\frac{N\alpha'}{r} \quad (3.3)
$$

while the electric charge and current densities take the form

$$
\rho = X^2 - 1 \quad J = \frac{N}{r} \alpha X^2 \hat{\phi} \quad (3.4)
$$

Finally the field equations reduce to the following set of ordinary nonlinear differential equations

$$
X'' + \frac{1}{r} X' - \frac{N^2}{r^2} X \alpha^2 = \frac{1}{2} \kappa^2 (X^2 - 1) X + 2 f X
$$

$$
\frac{1}{\lambda} \left( f'' + \frac{1}{r} f' \right) = -\rho \quad \alpha'' - \frac{1}{r} \alpha' = \alpha X^2 \quad (3.5)
$$

The behaviour of the solutions around the origin $r = 0$ is dictated by the
requirement of smoothness and by the equations themselves to be

\[ X(r) \to c_1 r^{1/|N|} \quad \alpha(r) \to 1 + c_2 r^2 \quad f(r) \to c_3 + \frac{1}{4} \lambda r^2 \]  

(3.6)

where \( c_1, c_2 \) and \( c_3 \) are undetermined constants.

Being unable to solve analytically the equations at hand we proceeded numerically. After some experimentation we decided that among the various methods available the variational approach is the most convenient. The method is based on the fact that the field equations (2.7) and (2.8) for static solutions are identical to the conditions for the minimization of the energy functional (2.10) under the Gauss-law and charge neutrality constraints. This is true even after we insert into the energy functional the spherically symmetric ansatz (3.1)\(^{[11]}\) and leads in that case directly to equations (3.5). We can thus use the variational method and minimize the energy in order to find approximate solutions of our equations.

We approximate the fields of the ansatz by a set of trial functions depending on some number of variational parameters. We then evaluate the energy as a function of these parameters and minimize the resulting expression. The position of the minimum in the parameter space determines the approximate solution, whose energy is the value of the energy functional at the minimum. The variational ansatz we used, compatible with the boundary conditions at large \( r \) is:

\[
X(r) = 1 + e^{-kr} \sum_{l=0}^{m} z_l \frac{r^l}{l!} \quad \alpha(r) = e^{-r} \sum_{l=0}^{m} \alpha_l \frac{r^l}{l!} \]  

(3.7)

\[
f'(r) = e^{-kr} \left[ \frac{c_l}{r} + \sum_{l=0}^{m} c_l \frac{r^l}{l!} \right] + e^{-2kr} \left[ \frac{d_l}{r} + \sum_{l=0}^{2m} d_l \frac{r^l}{l!} \right] \]

(3.8)

The asymptotic analysis of (3.5) requires that \( k = \kappa \) in the above equations. Numerically though it is more accurate to leave \( k \) as an independent variational
parameter. The optimum value of $k$ is not $\kappa$. This is no contradiction. The asymptotic study actually refers to very large distances, where all fields have already reached their vacuum values and do not contribute to the energy. One may in principle put $k = \kappa$ but this frustrates the other variational parameters, which are forced to take unnaturally diverse values in order to lead to the right solution. This induces in general higher numerical errors.

Plugging the above expressions for $X$ and $f'$ into Gauss’ constraint equation (2.8a) we determine the coefficients $c_{-1}$, $d_{-1}$, $c_i$ and $d_i$ in terms of the $z_i$. The boundary conditions at the origin give $\alpha_0 = \alpha_1 = 1$, $z_0 = -1$ for the $N = 1$ vortex. For higher-$N$ vortices one has more conditions due to the faster fall-off of $X$ at $r = 0$. Thus, for the $N = 2$ vortex we obtain in addition to the above that $z_1 = -k$. Finally, the vanishing of the total electric charge eliminates one more of the unknown coefficients of the variational ansatz. In our computations as it will be explained below, we achieve satisfactory accuracy for $m = 10$, i.e. with 16 and 15 independent variational parameters for $N = 1$ and $N = 2$ respectively.

Having solved explicitly Gauss’ constraint as well as the neutrality condition, we insert the variational ansatz into the energy functional, we compute analytically the spatial integrals to end-up with a polynomial of fourth order in the variational parameters. A quasi-Newton minimization procedure in then used to evaluate its minimum. The method converges fairly rapidly within four or five iterations inspite of the large number of parameters. The physical nature of the problem and the proper choice of the variational ansatz which restricts the fields to the right subspace of the configuration space, are the two factors to which it is reasonable to attribute this rapid convergence.

Figure 1 shows the profiles of the $N = 1$ vortex solutions for two different sets of values of the parameters of the model. In figures 1a and 1b we plot the magnitude of the electric current $J$, the magnetic field $B$, the charge density $\rho$ and the magnitude of the electric field $E$ divided by $\lambda$ as functions of the radius $r$ from the center of the vortex for $\lambda = 0.1$ and $\kappa = 2$. Similarly, in figures 1c and 1d
we have the same quantities but for $\lambda = 0.1$ and $\kappa = 0$, the case with no quartic potential.

Before we proceed with further results a few comments about the accuracy of our numerical computations are in order. To obtain an estimate of the error induced by the truncation of the configuration space, we minimized the energy for various values of $m$ in order to check the stability of our results against a broadening of the trial ansatz (3.7), (3.8). The energy at the minimum for $m = 10$ differed from the one for $m = 11$ in the sixth significant digit. Thus we conclude that our energy calculations are correct to within one part in $10^5$ or so. To eliminate the danger that during a minimization process we were accidentally trapped in a local minimum, we repeated each run several times starting with different initial configurations. An additional test is provided by the accuracy with which we verify the virial relation

$$W_b = 2W_e + W_v$$

obtained by the well known scaling argument of Derrick\textsuperscript{[12]}. We define

$$\Delta = \frac{2W_e + W_v - W_b}{W_b}$$

(3.9)

and check that in all our calculations $\Delta$ is smaller than $10^{-4}$. Finally, the spherical symmetry of the ansatz provides another test of the accuracy of our results. Notice that the ansatz is spherically symmetric in the sense that a spatial rotation by some angle $\beta$ can be compensated by a global internal U(1) rotation by $N\beta$. This means that the corresponding generator $Q - NL$ where $L$ is the field angular momentum to be defined in the next section, must vanish for the solution. $Q$ is zero by construction and we check, by explicit evaluation of the integral of the angular momentum density, that for all solutions $L$ is also zero to within one part in $10^4$.

Figure 2 shows the values of the ratio $R = E_{N=2}/2E_{N=1}$ of the energy of the double vortex to twice the energy of the single one, as well as the value of the energy of a single vortex itself divided by $\pi$ for various sets of the parameters
of the model. For $\lambda = 0$ and for static configurations our model reduces to the Ginzburg-Landau equations for a superconductor and $\kappa$ corresponds exactly to the ratio of the penetration depth to the coherence length defined there.

Figure 2d is in perfect agreement with the results obtained previously in the context of the Ginzburg-Landau theory\cite{13,7,8}. We divided all energies by $\pi$ to make easier contact with previous results concerning the Ginzburg-Landau theory, in which for $\kappa = 1$ the $N = n$ vortex solution has energy $E_n = n$. The fact that the energy of a vortex is an increasing function of the parameters $\kappa$ and $\lambda$ is easily understood by the following argument: (we give the argument for $\kappa$). Start with the solution of the static equations for some value of the parameter $\kappa$ and a given $\lambda$ which we keep fixed. By construction it satisfies Gauss’ constraint and is neutral. It can thus be used as a trial initial configuration for the energy minimization for a different value $\kappa' < \kappa$. The energy $E(\kappa')$ of the solution for $\kappa'$ is smaller than the value of the energy functional for the above configuration. The latter is equal to the energy of the solution for the value $\kappa$ minus the positive quantity $(1/8) (\kappa^2 - \kappa'^2) \int d^2x (\Psi\Psi^* - 1)^2$, where $\Psi$ is the solution for $\kappa$. Thus, $E(\kappa') < E(\kappa)$.

Q.E.D. A similar argument with a little extra care due to Gauss’ constraint applies to $\lambda$. The other extreme, the no-potential case $\kappa = 0$ is shown on figure 2c. The critical value of $\lambda$ for the $N = 2$ vortex to have the same energy with two well-separated single vortices is 0.0167.

The critical line separating the parameter space into regions-I and -II, according to the value of $R$ being smaller or larger than one respectively, is shown on figure 3. Regions-I and -II correspond in the Ginzburg-Landau case to ordinary type-I and type-II superconductors. It is interesting to compare the energy of a triple vortex to that of three single vortices far from each other. In contrast to the Ginzburg-Landau case there exists inside region-I another critical line separating the parameter space into two subregions according to whether $E(N = 3)$ is larger or smaller than 3 times $E(N = 1)$. Consistent with the well known facts about the Ginzburg-Landau model this line too passes through the point $(\kappa = 1, \lambda = 0)$. The subregion with $E(N = 3) < 3E(N = 1)$ is further split into two subsubregions.
depending on the value of the ratio $E(N = 4)/4E(N = 1)$ being larger or smaller than one, and so on.

In the context of a relativistic model we would talk about stable and unstable higher-$N$ vortices. For instance, a vortex with $N = 2$ and energy larger than twice the energy of a single one could not be absolutely stable against decay to two single vortices. In the model at hand though as we will see in the next chapter a completely different picture arises\textsuperscript{[1]}.

3.2. The two-vortex system

Our main goal in this section is to compute the energy of two single vortices as a function of their separation $d$. Again we proceed numerically and choose a variational ansatz with the following characteristics: 1) The complex field vanishes at the points $(\pm \frac{d}{2}, 0)$ on the x-axis. 2) For large $d$ the configuration reduces to two well-separated axially symmetric single vortices at a distance $d$ from each other, while 3) for $d \to 0$ it takes the form of an axially symmetric $N = 2$ vortex located at the origin. For the phase $\Theta$ and the magnitude $X$ of the complex field $\Psi$ we write\textsuperscript{[8]}

$$\Theta(x, y) = \tan^{-1}\left[\frac{2xy}{x^2 - y^2 - \frac{d^2}{4}}\right]$$ (3.10)\n
$$X(x) = \omega X^{(1)}(|x - \frac{d}{2}|) X^{(1)}(|x + \frac{d}{2}|) + (1 - \omega) \frac{(|x - \frac{d}{2}|) (|x + \frac{d}{2}|)}{r^2} X^{(2)}(r) + \delta X$$ (3.11)\n
in terms of the previously determined solutions $X^{(1)}$ and $X^{(2)}$ for the $N = 1$ and $N = 2$ vortices respectively. The function $\delta X$ is written in the form

$$\delta X = (|x - \frac{d}{2}|) (|x + \frac{d}{2}|) \cosh^{-1}(kr) \sum_{m=0}^{l} \sum_{n=0}^{m} z_{mn} r^{2m} \cos(2n\phi)$$ (3.12)\n
$\omega$ is a variational parameter, the relative weight of $X^{(1)}$ and $X^{(2)}$ in the configuration. To fix its optimal value we minimized the energy with respect to $\omega$ for $\delta X = 0$. 

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The variational ansatz is completed by a trial set of functions for the gauge potential

\[ A(x) = \omega A^{(1)}(|x - \frac{d}{2}|) + \omega A^{(1)}(|x + \frac{d}{2}|) + (1 - \omega) A^{(2)}(r) + \delta A(x) \]  

(3.13)

where \( \delta A = \delta A_r \hat{r} + \delta A_\phi \hat{\phi} \) with \( \delta A_r \) and \( \delta A_\phi \) defined by

\[ \delta A_r = r \cosh^{-1}(r) \sum_{m=0}^{l} \sum_{n=0}^{m} \alpha_{mn}^r r^{2m} \sin(2n\phi) \]  

(3.14)

\[ \delta A_\phi = r \cosh^{-1}(r) \sum_{m=0}^{l} \sum_{n=0}^{m} \alpha_{mn}^\phi r^{2m} \cos(2n\phi) \]  

(3.15)

As mentioned above \( d = (d, 0) \) to place the two vortices at \((\pm \frac{d}{2}, 0)\) along the \( x \) axis. The angular dependence of the fields was restricted to the form given above by the requirement on the configuration to be invariant under spatial reflection with respect to either one of the coordinate axes.

We used the same numerical procedure as in the isolated vortex case to determine the optimum values of the remaining variational parameters \( \alpha_{ij}^\phi, \alpha_{ij}^r, z_{ij} \) and \( k \). The dominant source of error in our results is due to the truncation of the configuration space. Again an estimate is obtained from the change in the value of the quantity of interest as we vary the number of variational parameters. To achieve an error of the order of 0.1\% in the total energy for all values of \( d \), it was enough to set \( l = 1 \) for \( d > 3 \), while for \( d < 3 \) it was necessary to take \( l = 2 \) i.e. a richer variational ansatz with 18 parameters. The uncertainty in \( W_e \) was a little larger, something like \( 0.2 - 0.3\% \), but since \( W_e \) contributes always a small fraction to the total energy, this error is negligible. Contrary to the isolated vortex case we did not perform the spatial integrations analytically. Instead we carried out these integrations numerically. Appropriate choice of the grid and the boundaries reduced the corresponding error to the order of 0.01\%. 

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In figure 4 we plot the energy divided by $\pi$ of the two-vortex system as a function of their separation $d$ for four sets of the parameters $\kappa$ and $\lambda$ corresponding to the points $A, B, C$ and $D$ shown in figure 3, from well inside region-II, to somewhere in region-I. We have included the point $C$ on the critical line with $(\kappa = 0.5, \lambda = 0.0115)$. Figure 4a corresponds to parameter values far from the critical line in region-II. The force between the vortices is everywhere repulsive. In all other cases one immediately recognizes a region of attraction and a region of repulsion of the two solitons. The local extremum in the energy leads to the exciting possibility of the existence of dumb-bell shaped doubly-charged static solutions of the field equations for a rather wide range of parameters. We will have the opportunity to elaborate on this in a future publication\cite{14}.

Finally, figures 5, 6, 7 and 8 show the total energy density $w_t$, the gauge-invariant topological density $\tau$, the magnetic field $B$ and the electrostatic energy distribution $w_e$ respectively, of the optimum two-vortex configuration corresponding to the parameter values $(\kappa = 0.5, \lambda = 0.0115)$ on the critical line. The pictures in each one of them refer to distances $d = 0, 2, 4, 6$. Notice that of the two definitions $B/2\pi$ and $\tau$ of the topological density the former follows less closely the energy distribution and as such it offers a bad description of the two vortex positions. We have checked, although we do not show it here, that the topological density in terms of the complex field alone is even worse in that respect. Notice also that the distribution of $w_e$ does not follow that of the total energy. It spreads over the whole region between the two vortices and is actually responsible for the bump in their interaction energy. Being so small in magnitude it does not alter the final picture that the two vortices can be safely considered far from each other already for $d = 4$. These features of $w_e$ as well as its negligible magnitude compared to $w_t$ are also shown on figure 9 for the case of an axially symmetric $N = 1$ configuration with $\kappa = 0.5$ and $\lambda = 0.0115$. The virial relation given in the previous section shows that $W_e$ is never dominant. It can at most be $1/3$ of $W_t - W_d$ for any value of the parameters.

We end this section with a remark about the numerical procedure used to
compute $W_e$ at each step of the iterative minimization process. We exploited the linearity of the Poisson equation to compute once and for all the electric field created by the charge density due to the configuration (3.10) to (3.12) in terms of the parameters of the ansatz. Thus the calculation of the electric field after each iteration is immediate. One does not have to solve the Poisson equation again but just plug into the general formula the new values of the parameters. $W_e$ is finally given by a straightforward spatial integration. This way the computation is more accurate and much faster.

4. Vortex Dynamics

4.1. Canonical structure of the model

In order to investigate the dynamics of vortices we start with the canonical structure of the model. We follow the standard procedure$^{[15]}$ applied to gauge theories to determine its fundamental Poisson brackets. As mentioned in a previous section to eliminate the redundant degrees of freedom due to the gauge invariance of the model we impose the condition

$$\nabla \cdot A = 0$$

Since the action does not depend on $\dot{A_0}$, $A_0$ is a dependent variable satisfying the Gauss law constraint which we solve to obtain

$$A_0(x, t) = -\frac{\lambda}{2\pi} \int d^2 x' \ln|x - x'| (\Psi^* \Psi(x', t) - 1)$$

(4.1)

(use was made of $\nabla^2 \ln|\mathbf{x} - \mathbf{x}'| = 2\pi \delta(\mathbf{x} - \mathbf{x'})$). With the above gauge-fixing condition the purely electromagnetic part of the action takes the form

$$S_A = \frac{1}{2} \int d^2 x \left[ \frac{1}{\lambda} (E_T^2 + E_L^2) - B^2 \right]$$

(4.2)

where the transverse and the longitudinal parts of the electric field are $E_T = -\dot{A}$ and $E_L = -\nabla A_0$ respectively.
The canonical momentum $\pi_k$ conjugate to $A_k$ is then $\pi_k = \partial L / \partial \dot{A}_k = -E^T_k / \lambda$, while $\Psi$ and $\pi \equiv \partial L / \partial \dot{\Psi} = i \Psi^*$ are conjugate of each other. Thus, the fundamental Poisson brackets are

$$\{ \Psi(x, t), \Psi^*(x', t) \} = -i \delta(x - x')$$

$$\{ A_i(x, t), \dot{A}_j(x', t) \} = \lambda \delta^{T}_{ij}(x - x')$$

$$\{ \Psi(x, t), A_i(x', t) \} = 0 = \{ \Psi(x, t), \dot{A}_i(x', t) \}$$

where the transverse $\delta$ function is defined as $\delta^{T}_{ij}(x - x') \equiv (\delta_{ij} - \nabla^2 \partial_i \partial_j) \delta(x - x')$.

Finally, the Hamiltonian (energy) of the system is

$$H = \int d^2x \left[ \frac{1}{2\lambda} (E_T^2 + E_L^2) + \frac{1}{2} B^2 + \frac{1}{2} |D\Psi|^2 + \frac{1}{8} \kappa^2 (\Psi^* \Psi - 1)^2 \right]$$

All other Poisson brackets are obtained from the ones given above. It is straightforward to check that

$$\{ A_0(x, t), \Psi(x', t) \} = -\frac{i \lambda}{2\pi} \ln|x - x'| \Psi(x', t)$$

$$\{ A_0(x, t), A_i(x', t) \} = 0 = \{ A_0(x, t), \dot{A}_i(x', t) \}$$

as well as that the equations of motion (2.7) and (2.8) coincide with Hamilton’s equations

$$\dot{\Psi}(x, t) = \{ \Psi(x, t), H \}$$

$$\dot{A}_k(x, t) = \{ A_k(x, t), H \}$$

$$\dot{\pi}_k(x, t) = \{ \pi_k(x, t), H \}$$

We are now ready to proceed with the construction of the field momentum and angular momentum for the model at hand, valid in all topological sectors. We will show that the corresponding conservation laws describe the basic feature of vortex motion in the absence of external forces, namely spontaneous pinning.
4.2. Momentum and angular momentum in all sectors

The Noether expression for the linear momentum in our model is

\[ P^N_k = \int d^2x \left( -\pi \partial_k \Psi - \pi_j \partial_k A_j \right) \] (4.7)

It is derived under the assumption that all fields approach their vacuum values fast enough as \( r \to \infty \) so that all integrals are meaningful and surface terms appearing in intermediate steps are zero. It is straightforward to check that the above expression is ill-defined for any vortex configuration with non-zero topological charge. Under the same assumptions though which led to (4.7) the latter can be brought to the form

\[ P_k = -\epsilon_{ki} \int d^2x x_i \epsilon_{lm} (\partial_l \pi \partial_m \Psi + \partial_l \pi_j \partial_m A_j) \] (4.8)

which is well-defined for any smooth finite-energy field configuration with arbitrary topological charge. The two expressions (4.7) and (4.8) differ by exactly those surface terms which were omitted from the former and make the latter finite and unambiguous in all topological sectors.

All the defining properties of the momentum are verified for \( P_k \). This is no surprise. They are formally valid for \( P^N_k \) and this differs from \( P_k \) only by surface terms. In any case, one can directly and unambiguously show using the equations of motion (2.7), (2.8) and the Poisson brackets derived above first, that \( P_k \) is conserved

\[ \frac{d}{dt} P_k = 0 \] (4.9)

second, that it is the generator of spatial displacements

\[ \{ P_k, F \} = \partial_k F \] (4.10)

and finally, that it is gauge-invariant. The only condition on the configuration \( F = (\Psi, A_0, A_i) \) necessary for the derivation is that \( A_0 \) vanishes faster than \( r^{-1} \).
as \( r \to \infty \). All our vortices have \( A_0 \) approaching zero at infinity exponentially fast and consequently they safely belong to this set of configurations. Furthermore, using Gauss’ constraint we can rewrite the momentum in the manifestly gauge-invariant form\(^1\)

\[
P_k = \epsilon_{ki} \int d^2x \left( 2\pi x_i \tau + \frac{1}{\lambda} E_i B \right) \tag{4.11}
\]

The second term is the Poynting vector, the pure electromagnetic contribution to the field momentum.

Thus, independently of derivation, \( P_k \) is the correct form of the momentum in our theory, valid in any topological sector and reducing to the naive expression in the trivial sector \( N = 0 \) and for configurations allowing for free integrations by parts.

Before we discuss the implications of the above formula of the linear momentum on the motion of vortices, we would like to construct in a similar way the correct form of the angular momentum in our model. Again, the Noether expression

\[
L_N = - \int d^2x \epsilon_{ij} \left[ x_i \left( \pi \partial_j \Psi + \pi_k \partial_j A_k \right) + \pi_i A_j \right] \tag{4.12}
\]

is formally conserved and generates rotations but is divergent in any non-trivial sector. Allow for integrations by parts and ignore surface terms to rewrite it in the form

\[
L = \int d^2x \left[ \frac{1}{2} \mathbf{x}^2 \epsilon_{ij} \left( \partial_i \pi \partial_j \Psi + \partial_i \pi_k \partial_j A_k \right) + \epsilon_{ij} \pi_i A_j \right] \tag{4.13}
\]

or as a manifestly gauge invariant quantity in terms of the topological density \( \tau \) defined in section 2

\[
L = - \int d^2x \left( \pi \mathbf{x}^2 \tau + \frac{1}{\lambda} \mathbf{x} \cdot \mathbf{E} \mathbf{B} \right) \tag{4.14}
\]

Using either one of the last two formulas one can show that for essentially all finite-energy configurations as in the case of the linear momentum, \( L \) is well-
defined, conserved and generates spatial rotations, i.e. it satisfies

\[ \{ L, \Psi(x, t) \} = \epsilon_{ij} x_i \partial_j \Psi \]

\[ \{ L, A_k(x, t) \} = \epsilon_{ij} x_i \partial_j A_k + \epsilon_{kl} A_l \]  \hspace{1cm} (4.15)

\( L \) is then the proper definition of the angular momentum when dealing with topologically non-trivial configurations. As already explained it vanishes for the spherically symmetric neutral solutions studied in section three.

4.3. Vortex motion

In a following paper \([14]\) we will present details of a numerical simulation of the motion of the flux vortices described above under the influence of various external forces. Here we would like for the sake of completeness to predict analytically and without any approximation the essential features of their dynamical behaviour \([1]\). It is already apparent that the momentum \( P_k \) defined in the previous section is not a measure of the translational motion of a vortex but instead it describes its position. In fact the momentum of a static axially symmetric vortex with charge \( N \) centered at \( a \) is \( P_k = 2\pi N \epsilon_{ki} a_i \). It is also clear that a localized free vortex of arbitrary shape moving in formation at constant velocity \( v_i \) would have \( P_k = 2\pi N \epsilon_{ki} (a_i^0 + v_i t) \). For any \( v_i \neq 0 \) this is forbidden by the linear momentum conservation law. A free vortex is spontaneously pinned as a consequence of momentum conservation.

Let us define the "guiding center" \( R \) of a generic configuration with \( N \neq 0 \) by

\[ R_i \equiv -\frac{1}{2\pi N} \epsilon_{ij} P_j \]  \hspace{1cm} (4.16)

Since under a rigid displacement of the whole configuration by \( c \), \( R \) changes to \( R + c \) and for a spherical vortex it coincides with its geometric center, we naturally interpret it as the mean position of the configuration. Note also in support of this interpretation, that \( R \) is related to the first moment of the topological density \( \tau \), so that for nearly spherical configurations \( R \) is close to their "center of topology".
a. In the absence of external forces both $N$ and $P$ are conserved and

$$\frac{d}{dt} \mathbf{R} = 0$$  \hspace{1cm} (4.17)

A generic vortex-like configuration produced in the system will of course fluctuate in its details but it will remain pinned at its initial mean position.

b. Consider next the response of such a vortex to an externally prescribed electric current $\tilde{J}_i(x,t)$. Its effect is studied by adding to the action the term $\delta S = \int d^2 x \, dt \, \tilde{J}_i(x,t) A_i$. For consistency of the model the external current should have zero divergence. The new term in the action modifies the $A_i$ equation of motion (2.7) by the substitution $J_i \rightarrow J_i + \tilde{J}_i$ on the right-hand side. Because of the external current the momentum and the angular momentum are no longer conserved. Instead, their time derivatives are equal to the external force and torque respectively. Indeed one can easily verify Newton’s equation:

$$\frac{d}{dt} P_k = F_k^{\text{Lorentz}} = - \int d^2 x \, \epsilon_{kl} \tilde{J}_l(x,t) B(x,t)$$  \hspace{1cm} (4.18)

as well as

$$\frac{d}{dt} L = \text{Torque} = \int d^2 x \, x_i \tilde{J}_i B$$  \hspace{1cm} (4.19)

Naturally the force on the vortex is opposite to the Lorentz force acting on the external current, while the torque is by definition expressed in terms of the force density $f_k = -\epsilon_{kl} \tilde{J}_l B$ as the integral of $\epsilon_{ij} x_i f_j$. We now use (4.16) and the fact that for arbitrary $\tilde{J}$ $N$ is still conserved to obtain

$$\frac{dR_k}{dt} = - \frac{1}{2\pi N} \int d^2 x \, \tilde{J}_k B$$  \hspace{1cm} (4.20)

In the idealized situation of a homogeneous throughout the vortex external current the above formula simplifies to

$$\frac{dR_k}{dt} = - \tilde{J}_k(t)$$  \hspace{1cm} (4.21)

In general, the naive expectation based on the usual Newtonian reasoning and Galilean invariance would be that the vortex should accelerate in the direction of
the force acting on it. Instead, (4.16) combined with \(dP_k/dt = F_k\) for a generic force \(F_k\), shows that the equation of motion of the mean position of the vortex is

\[
\frac{d}{dt} R_k = -\frac{1}{2\pi N} \epsilon_{kl} F_l
\]  \hspace{1cm} (4.22)

i.e. the vortex moves with speed \(|F|/2\pi|N|\) and at \(\pm 90^0\) relative to the force for positive or negative \(N\) respectively. In the special case of the homogeneous external current considered here, the force is also proportional to \(N\) and all vortices move with the same velocity equal to minus the external current itself. If in particular the latter is due entirely to the condensate charges taken here by convention positive with unit charge density, its value is exactly equal to minus the velocity of the carriers. Thus the vortex will reorganize itself during a transient period following the onset of the external current and it will move with the same speed but opposite to the current carriers. Furthermore it should be pointed-out that in the context of our field theory model no approximation other than the implicit assumption that the vortex remains localized was necessary. The position interpretation (4.16) of the momentum converts Newton’s law into an equation giving directly the velocity of the vortex in terms of the applied force, while the \(\epsilon_{ij}\) of (4.16) makes the vortex move at an angle \((N/|N|) 90^0\) relative to the direction of the external force (Hall-behaviour) \([1][9]\). The above general conclusion, reached without ever solving an equation of motion, does not depend on the details of the Hamiltonian of the system. Any potential \(V\) or any additional higher spatial derivative terms in \(H\) modifies the detailed profile of the vortex solutions but it does not alter their dynamics.

The guiding center interpretation of the momentum (4.16) is not applicable in the case of topologically trivial \((N = 0)\) configurations. Newton’s law (4.18) is of course still true for any external force, but this does not strictly speaking tell us much about the actual motion of the vortex. In contrast to the case of ordinary particle dynamics with Poincaré or even Galilean invariance the relation between momentum and velocity is not a priori known in the present model.
5. Discussion

We presented a general treatment of the motion of flux-vortices in a two parameter field theory model describing as stated in the introduction a rather wide class of idealized physical systems. We argued that independently of the details of the Hamiltonian the vortices exhibit Hall-behaviour, like electrons moving on a plane under the simultaneous action of a perpendicular magnetic and an in-plane electric field. The picture is identical to the one derived previously for the dynamics of magnetic bubbles in the ferromagnetic continuum\(^9\). The fact that the solitons in these two field theories are topologically different, classified by the second homotopy group of \(S^2\) in one case and by the first homotopy group of \(S^1\) in the other, does not seem to play any role in our discussion.

An external current was shown from first principles to pull the vortex in the opposite direction. It is not clear at this point what is the relation of our model with the ones considered previously\(^4\),\(^16\) in connection with superconductivity. It is tempting to think of it as a field-theoretic realization of the hydrodynamic model discussed in reference \([16]\) in the vanishing friction limit. The conclusion though reached there about vortex motion, based on the phenomenological use of the Magnus force\(^{17}\) known from fluid mechanics\(^{18}\), is \(\mathbf{V} = \tilde{\mathbf{J}}\). This is opposite to ours, which is correspondingly reminiscent of the opposite sign Hall effect reported in high-\(T_c\) superconductors\(^{3}\).

The Hall motion of the vortices derived above is attributed to the radical change in the role of the momentum of the theory as a result of the underlying topology. The Hall behaviour of an isolated vortex is exactly due to the fact that the linear momentum (4.11) contains a piece which is equal to the first moment of the topological density. An immediate consequence of this fact combined with the fundamental property (4.10) of the momentum is that

\[
\{P_1, P_2\} = 2\pi N \quad (5.1)
\]

We thus conclude that necessary condition for the manifestation of Hall motion
is that the translation part of the symmetry algebra of the model admits central extension. It is well-known\textsuperscript{[19]} that this is not possible in the case of the Galilean or the Poincaré algebra in any spatial dimension higher than or equal to two. It is not even true for the Euclidean algebra $E(D)$ for $D \geq 3$. Thus, the Hall behaviour encountered above is expected a priori only in two-dimensional systems with spatial $E(2)$ algebra, or in $D$-dimensional systems with translational symmetry alone.
In the system at hand one can immediately check that

$$\{L, P_1\} = P_2 \quad \{L, P_2\} = -P_1$$

and verify that it belongs to the first category as expected.

We would also like to stress that although (4.8) is true even in theories with canonical momentum $\pi$ proportional to the time derivative of the complex field, our previous reasoning does not go through since $P$ is then related to the first moment not of the topological density but of a quantity which actually vanishes for static configurations. Thus, another condition which is necessary is that the equations of motion are first order in the fields which carry the topology in the particular model. In other words, we do not expect Hall-motion of the vortices in the relativistic-like model\textsuperscript{[20]} derived as a dynamical extension of the Ginzburg-Landau theory of a superconductor appropriate for temperatures far below $T_c$.

On the other hand, the above formalism applies to the motion of vortices in a generalization of the present model which allows for dynamical positive-ions forming a lattice, of the $(N_h, N_{ss})$ with $N_{ss} \neq -2N_h$ vortices of the high-$T_c$ superconductor model proposed in reference [21], as well as to the motion of vortices in easy-plane ferromagnetic films\textsuperscript{[22]}, to mention just a few systems of considerable interest.
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FIGURE CAPTIONS

Figure 1: $N = 1$ vortex profiles. (1a) and (1b) correspond to $\kappa = 2$ and $\lambda = 0.1$, while (1c) and (1d) to $\kappa = 0$ and $\lambda = 0.1$

Figure 2: Plots of $R = E_{N=2}/2E_{N=1}$ and of $E_{N=1}/\pi$ as functions of $\kappa$ or $\lambda$ for the values of the second parameter as shown. (2d) corresponds to the Ginzburg-Landau model, while (2c) to the no-potential case.

Figure 3: Regions-I and -II of the parameter space.

Figure 4: The energy of the two-vortex system as a function of their separation for the values of parameters corresponding to the points $A$, $B$, $C$ and $D$ of figure 3. The dashed line is drawn at twice the energy of a single vortex.

Figure 5: The total energy density of the two vortices $d = 0$, 2, 4 and 6. $\kappa = 0.5$ and $\lambda = 0.0115$.

Figure 6: The topological charge density $\tau$ of the two vortices for $d = 0$, 2, 4 and 6. $\kappa = 0.5$ and $\lambda = 0.0115$.

Figure 7: The magnetic field of the two vortices for $d = 0$, 2, 4 and 6. $\kappa = 0.5$ and $\lambda = 0.0115$.

Figure 8: The electrostatic energy density of the two vortices $d = 0$, 2, 4 and 6. $\kappa = 0.5$ and $\lambda = 0.0115$.

Figure 9: Plot of the densities $w_t$ and $150w_e$ for an isolated $N = 1$ vortex. $\kappa = 0.5$ and $\lambda = 0.0115$. 

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