THE MONOID OF MONOTONE INJECTIVE PARTIAL SELFMAPS OF THE
POSET \((\mathbb{N}_0^3, \leq)\) WITH COFINITE DOMAINS AND IMAGES

OLEG GUTIK AND OLHA KROKHMALNA

ABSTRACT. Let \( n \) be a positive integer \( \geq 2 \) and \( \mathbb{N}_0^n \) is the \( n \)-th power of positive integers with the product order of the usual order on \( \mathbb{N} \). In the paper we study the semigroup of injective partial monotone selfmaps of \( \mathbb{N}_0^n \) with cofinite domains and images. We show that the group of units \( H(\mathbb{N}) \) of the semigroup \( \mathcal{P}O_{\infty}(\mathbb{N}_0^n) \) is isomorphic to the group \( \mathcal{S}_n \) of permutations of an \( n \)-element set, and describe the subsemigroup of idempotents of \( \mathcal{P}O_{\infty}(\mathbb{N}_0^n) \). Also in the case \( n = 3 \) we describe the subsemigroup of partial bijections of the poset \( \mathbb{N}_0^3 \) and Green’s relations on the semigroup \( \mathcal{P}O_{\infty}(\mathbb{N}_0^3) \).

1. Introduction and preliminaries

We shall follow the terminology of [19] and [44].

In this paper we shall denote the cardinality of the set \( A \) by \( |A| \). We shall identify all sets \( X \) with their cardinality \( |X| \). For an arbitrary positive integer \( b \) by \( \mathcal{S}_n \) we denote the group of permutations of an \( n \)-elements set. Also, for infinite subsets \( A \) and \( B \) of an infinite set \( X \) we shall write \( A \subseteq^* B \) if and only if there exists a finite subset \( A_0 \) of \( A \) such that \( A \setminus A_0 \subseteq B \).

An algebraic semigroup \( S \) is called inverse if for any element \( x \in S \) there exists a unique \( x^{-1} \in S \) such that \( xx^{-1}x = x \) and \( x^{-1}xx^{-1} = x^{-1} \). The element \( x^{-1} \) is called the inverse of \( x \in S \).

If \( S \) is a semigroup, then we shall denote the subset of idempotents in \( S \) by \( E(S) \). If \( S \) is an inverse semigroup then \( E(S) \) is closed under multiplication and we shall refer to \( E(S) \) as a band (or the band of \( S \)). If the band \( E(S) \) is a non-empty subset of \( S \) then the semigroup operation on \( S \) determines the following partial order \( \leq \) on \( E(S) \): \( e \leq f \) if and only if \( ef = fe = e \). This order is called the natural partial order on \( E(S) \). A semilattice \( E \) is a commutative semigroup of idempotents. A semilattice \( E \) is called linearly ordered or a chain if its natural order is a linear order.

If \( S \) is a semigroup, then we shall denote the Green relations on \( S \) by \( \mathcal{R}, \mathcal{L}, \mathcal{J}, \mathcal{D} \) and \( \mathcal{H} \) (see [22] or [19] Section 2.1]):

\[
\begin{align*}
a\mathcal{R}b & \text{ if and only if } aS^1 = bS^1; \\
a\mathcal{L}b & \text{ if and only if } S^1a = S^1b; \\
a\mathcal{J}b & \text{ if and only if } S^1aS^1 = S^1bS^1; \\
\mathcal{D} & = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}; \\
\mathcal{H} & = \mathcal{L} \cap \mathcal{R}.
\end{align*}
\]

The \( \mathcal{R} \)-class (resp., \( \mathcal{L} \)-, \( \mathcal{H} \)-, \( \mathcal{D} \)- or \( \mathcal{J} \)-class) of the semigroup \( S \) which contains an element \( a \) of \( S \) will be denoted by \( R_a \) (resp., \( L_a, H_a, D_a \) or \( J_a \)).

If \( \alpha : X \to Y \) is a partial map, then by \( \text{dom} \alpha \) and \( \text{ran} \alpha \) we denote the domain and the range of \( \alpha \), respectively.

Let \( \mathcal{J}_\lambda \) denote the set of all partial one-to-one transformations of an infinite set \( X \) of cardinality \( \lambda \) together with the following semigroup operation: \( x(\alpha \beta) = (x\alpha)\beta \) if \( x \in \text{dom}(\alpha \beta) = \{ y \in \text{dom} \alpha : y\alpha \in \text{dom} \beta \} \).
The semigroup $\mathcal{S}_\alpha$ is called the symmetric inverse semigroup over the set $X$ (see [19, Section 1.9]). The symmetric inverse semigroup was introduced by Wagner [48] and it plays a major role in the semigroup theory. An element $\alpha \in \mathcal{S}_\alpha$ is called cofinite, if the sets $\lambda \setminus \text{dom} \alpha$ and $\lambda \setminus \text{ran} \alpha$ are finite.

For an arbitrary partial order called a partial map $\lambda$ and the symmetric inverse semigroup was introduced by Wagner [48] and it plays a major role in the semigroup theory. An element $\alpha \in \mathcal{S}_\alpha$ is called cofinite, if the sets $\lambda \setminus \text{dom} \alpha$ and $\lambda \setminus \text{ran} \alpha$ are finite.

If $X$ is a non-empty set and $\leq$ is reflexive, antisymmetric, transitive binary relation on $X$ then $\leq$ is called a partial order on $X$ and $(X, \leq)$ is said to be a partially ordered set or shortly a poset.

Let $(X, \leq)$ be a partially ordered set. A non-empty subset $A$ of $(X, \leq)$ is called:

- a chain if the induced partial order from $(X, \leq)$ onto $A$ is linear, i.e., any two elements from $A$ are comparable in $(X, \leq)$;
- an $\omega$-chain if $A$ is order isomorphic to the set of negative integers with the usual order $\leq$;
- an anti-chain if any two distinct elements from $A$ are incomparable in $(X, \leq)$.

For an arbitrary $x \in X$ and non-empty $A \subseteq X$ we denote

$$\uparrow x = \{y \in X : x \leq y\}, \quad \downarrow x = \{y \in X : y \leq x\}, \quad \uparrow A = \bigcup_{x \in A} \uparrow x \quad \text{and} \quad \downarrow A = \bigcup_{x \in A} \downarrow x.$$  

We shall say that a partial map $\alpha : X \rightarrow X$ is monotone if $x \leq y$ implies $(x)\alpha \leq (y)\alpha$ for $x, y \in \text{dom} \alpha$.

Let $\mathbb{N}$ be the set of positive integers with the usual linear order $\leq$ and $n$ be an arbitrary positive integer greater then or equal 2. On the Cartesian power $\mathbb{N}^n = \mathbb{N} \times \cdots \times \mathbb{N}$ we define the product partial order, i.e.,

$$(i_1, \ldots, i_n) \leq (j_1, \ldots, j_n) \quad \text{if and only if} \quad i_k \leq j_k \quad \text{for all} \quad k = 1, \ldots, n.$$  

Later the set $\mathbb{N}^n$ with this partial order will be denoted by $\mathbb{N}^n$.

For an arbitrary positive integer $n \geq 2$ by $\mathcal{P}\mathcal{O}_\times(\mathbb{N}^n_\leq)$ we denote the semigroup of injective partial monotone selfmaps of $\mathbb{N}^n_\leq$ with cofinite domains and images. Obviously, $\mathcal{P}\mathcal{O}_\times(\mathbb{N}^n_\leq)$ is a submonoid of the semigroup $\mathcal{S}_\omega$ and $\mathcal{P}\mathcal{O}_\times(\mathbb{N}^n_\leq)$ is a countable semigroup.

Furthermore, we shall denote the identity of the semigroup $\mathcal{P}\mathcal{O}_\times(\mathbb{N}^n_\leq)$ by $\mathbb{I}$ and the group of units of $\mathcal{P}\mathcal{O}_\times(\mathbb{N}^n_\leq)$ by $H(\mathbb{I})$.

The bicyclic semigroup (or the bicyclic monoid) $\mathcal{C}(p, q)$ is the semigroup with the identity 1 generated by two elements $p$ and $q$, subject only to the condition $pq = 1$. The bicyclic semigroup is bisimple and every one of its congruences is either trivial or a group congruence. Moreover, every homomorphism $h$ of the bicyclic semigroup is either an isomorphism or the image of $\mathcal{C}(p, q)$ under $h$ is a cyclic group (see [19, Corollary 1.32]). The bicyclic semigroup plays an important role in algebraic theory of semigroups and in the theory of topological semigroups. For example a well-known Andersen’s result [11] states that a (0–)simple semigroup with an idempotent is completely (0–)simple if and only if it does not contain an isomorphic copy of the bicyclic semigroup. Semigroup and shift-continuous topologizations of the bicyclic monoid and its generalizations, they embedding into compact-like topological semigroups was studied in [5] [6] [7] [8] [9] [11] [14] [18] [20] [21] [24] [25] [26] [28] [31] [33] [35] [43] [46] and [2] [3] [4] [10] [12] [33] [42], respectively.

The bicyclic monoid is isomorphic to the semigroup of all bijections between upper-sets of the poset $(\mathbb{N}, \leq)$ (see: see Exercise IV.1.11(ii) in [47]). So, the semigroup of injective isotone partial selfmaps with cofinite domains and images of positive integers is a generalization of the bicyclic semigroup. Hence, it is a natural problem to describe semigroups of injective isotone partial selfmaps with cofinite domains and images of posets with $\omega$-chain.

The semigroups $\mathcal{I}_\times(\mathbb{N})$ and $\mathcal{I}_\times(\mathbb{Z})$ of injective isotone partial selfmaps with cofinite domains and images of positive integers and integers, respectively, are studied in [34] and [35]. It was proved that the semigroups $\mathcal{I}_\times(\mathbb{N})$ and $\mathcal{I}_\times(\mathbb{Z})$ have similar properties to the bicyclic semigroup: they are bisimple and every non-trivial homomorphic image $\mathcal{I}_\times(\mathbb{N})$ and $\mathcal{I}_\times(\mathbb{Z})$ is a group, and moreover the semigroup $\mathcal{I}_\times(\mathbb{N})$ has $\mathbb{Z}(+) = \mathbb{Z}(+) \times \mathbb{Z}(+)$ as a maximal group image and $\mathcal{I}_\times(\mathbb{Z})$ has $\mathbb{Z}(+) \times \mathbb{Z}(+)$, respectively.
In the paper \[36\] algebraic properties of the semigroup \(I^\lambda_\text{cf}\) of cofinite partial bijections of an infinite cardinal \(\lambda\) are studied. It is shown that \(I^\lambda_\text{cf}\) is a bisimple inverse semigroup and that for every non-empty chain \(L\) in \(E(I^\lambda_\text{cf})\) there exists an inverse subsemigroup \(S\) of \(I^\lambda_\text{cf}\) such that \(S\) is isomorphic to the bicyclic semigroup and \(L \subseteq E(S)\). described the Green relations on \(I^\lambda_\text{cf}\) and proved that every non-trivial congruence on \(I^\lambda_\text{cf}\) is a group congruence. Also, the structure of the quotient semigroup \(I^\lambda_\text{cf}/\sigma\), where \(\sigma\) is the least group congruence on \(I^\lambda_\text{cf}\), is described.

In the paper \[32\] the semigroup \(\mathcal{I}_\infty(Z^n_{\text{lex}})\) of monotone injective partial selfmaps of the set of \(L_n \times_{\text{lex}} \mathbb{Z}\) having cofinite domain and image, where \(L_n \times_{\text{lex}} \mathbb{Z}\) is the lexicographic product of \(n\)-elements chain and the set of integers with the usual linear order was studied. There it is described the Green relations on \(\mathcal{I}_\infty(Z^n_{\text{lex}})\), showed that the semigroup \(\mathcal{I}_\infty(Z^n_{\text{lex}})\) is bisimple and established its projective congruences. Also, in \[32\] there it is proved that \(\mathcal{I}_\infty(Z^n_{\text{lex}})\) is finitely generated, every automorphism of \(\mathcal{I}_\infty(Z)\) is inner, and it is showed that in the case \(n \geq 2\) the semigroup \(\mathcal{I}_\infty(Z^n_{\text{lex}})\) has non-inner automorphisms. In \[32\] we proved that for every positive integer \(n\) the quotient semigroup \(\mathcal{I}_\infty(Z_n^{\text{lex}})/\sigma\), where \(\sigma\) is a least group congruence on \(\mathcal{I}_\infty(Z_n^{\text{lex}})\), is isomorphic to the direct power \((Z(+))^{2n}\). The structure of the sublattice of congruences on \(\mathcal{I}_\infty(Z_n^{\text{lex}})\) which are contained in the least group congruence is described in \[29\].

In the paper \[30\] algebraic properties of the semigroup \(\mathcal{P}_\infty(N^2_\leq)\) are studied. The properties of elements of the semigroup \(\mathcal{P}_\infty(N^2_\leq)\) as monotone partial bijection of \(N^2_\leq\) are described and showed that the group of units of \(\mathcal{P}_\infty(N^2_\leq)\) is isomorphic to the cyclic group of order two. Also in \[30\] the subsemigroup of idempotents of \(\mathcal{P}_\infty(N^2_\leq)\) and the Green relations on \(\mathcal{P}_\infty(N^2_\leq)\) are described. In particular, there was proved that \(D = \mathcal{I}\) in \(\mathcal{P}_\infty(N^2_\leq)\). In \[31\] the natural partial order \(\leq\) on the semigroup \(\mathcal{P}_\infty(N^2_\leq)\) is described and shown that it coincides with the natural partial order which is induced from symmetric inverse monoid \(S_{N \times N}\) over the set \(N \times N\) onto the semigroup \(\mathcal{P}_\infty(N^2_\leq)\). Also, there it is proved that the semigroup \(\mathcal{P}_\infty(N^2_\leq)\) is isomorphic to the semidirect product \(\mathcal{P}_\infty(N^2_\leq) \rtimes Z_2\) of the monoid \(\mathcal{P}_\infty^+(N^2_\leq)\) of orientation-preserving monotone injective partial selfmaps of \(N^2_\leq\) with cofinite domains and images by the cyclic group \(Z_2\) of the order two. The congruence \(\sigma\) on the semigroup \(\mathcal{P}_\infty(N^2_\leq)\), which is generated by the natural order \(\leq\) on the semigroup \(\mathcal{P}_\infty(N^2_\leq): \sigma = \alpha \sigma \beta\) if and only if \(\alpha\) and \(\beta\) are comparable in \((\mathcal{P}_\infty(N^2_\leq), \leq)\) was described. It is proved that the quotient semigroup \(\mathcal{P}_\infty^+(N^2_\leq)/\sigma\) is isomorphic to the free commutative monoid \(\mathcal{AM}_\omega\) over an infinite countable set and show that quotient semigroup \(\mathcal{P}_\infty(N^2_\leq)/\sigma\) is isomorphic to the semidirect product of the free commutative monoid \(\mathcal{AM}_\omega\) by the group \(Z_2\).

In the paper \[38\] the semigroup \(\mathbb{I}_{\infty}\) of all partial co-finite isometries of positive integers is studied. The semigroup \(\mathbb{I}_{\infty}\) is a some generalization of the bicyclic monoid and it is a submonoid of \(I'_\infty(N)\). Green's relations on the semigroup \(\mathbb{I}_{\infty}\), its band is described there and it is proved that \(\mathbb{I}_{\infty}\) is a simple \(E\)-unitary \(F\)-inverse semigroup. Also there is described the least group congruence \(c_{mg}\) on \(\mathbb{I}_{\infty}\) and proved that the quotient-semigroup \(\mathbb{I}_{\infty}/c_{mg}\) is isomorphic to the additive group of integers. An example of a non-group congruence on the semigroup \(\mathbb{I}_{\infty}\) was presented. Also there it was proved that a congruence on the semigroup \(\mathbb{I}_{\infty}\) is a group congruence if and only if its restriction onto an isomorphic copy of the bicyclic semigroup in \(\mathbb{I}_{\infty}\) is a group congruence.

In the paper \[39\] submonoids of the monoid \(I_\infty'(N)\) of almost monotone injective co-finite partial selfmaps of positive integers \(N\). Let \(C^\leq N\) be a subsemigroup \(I_\infty'(N)\) which is generated by the partial shift \(n \mapsto n + 1\) and its inverse partial map. In \[39\] it was shown that every automorphism of a full inverse subsemigroup of \(I_\infty'(N)\) which contains the semigroup \(C^\leq N\) is the identity map. Also there is constructed a submonoid \(\mathbb{I}_\infty^{[3]}\) of \(I_\infty'(N)\) with the following property: if \(S\) is an inverse submonoid of \(I_\infty'(N)\) such that \(S\) contains \(\mathbb{I}_\infty^{[3]}\) as a submonoid, then every non-identity congruence \(C\) on \(S\) is a group congruence. Also, there it was proved that that if \(S\) is an inverse submonoid of \(I_\infty'(N)\) such that \(S\) contains \(C^\leq N\) as a submonoid then \(S\) is simple and the quotient semigroup \(S/C_{mg}\), where \(C_{mg}\) is minimum group congruence on \(S\), is isomorphic to the additive group of integers.

We observe that the semigroups of all partial co-finite isometries of integers established in \[15, 16, 37\].
The monoid $\mathbf{I}^n_{\infty}$ of cofinite partial isometries of the $n$-th power of the set of positive integers $\mathbb{N}$ with the usual metric for a positive integer $n \geq 2$ was studied in [10]. The semigroup $\mathbf{I}^n_{\infty}$ is a submonoid of $\mathcal{P}\mathcal{O}_\infty(N^n_{\leq})$ for any positive integer $n \geq 2$. In [10] it was proved that for any integer $n \geq 2$ the semigroup $\mathbf{I}^n_{\infty}$ is isomorphic to the semidirect product $\mathcal{I}_n \ltimes (\mathcal{P}_\infty(N^n), \cup)$ of the free semilattice with the unit $(\mathcal{P}_\infty(N^n), \cup)$ by the symmetric group $\mathcal{J}_n$.

Later in this paper we shall assume that $n$ is an arbitrary positive integer $\geq 2$.

In this paper we study the semigroup of injective partial monotone selfmaps of $\mathcal{P}_\infty(N^n_{\leq})$. We show that the group of units $H(\mathcal{I})$ of the semigroup $\mathcal{P}\mathcal{O}_\infty(N^n_{\leq})$ is isomorphic to the group $\mathcal{J}_n$ and describe the subgroup of idempotents of $\mathcal{P}\mathcal{O}_\infty(N^n_{\leq})$. Also in the case $n = 3$ we describe the property of elements of the semigroup $\mathcal{P}\mathcal{O}_\infty(N^n_{\leq})$ as partial bijections of the poset $N^n_{\leq}$ and Green’s relations on the semigroup $\mathcal{P}\mathcal{O}_\infty(N^n_{\leq})$.

2. Properties of elements of the semigroup $\mathcal{P}\mathcal{O}_\infty(N^n_{\leq})$ as monotone partial permutations

In this short section we describe properties of elements of the semigroup $\mathcal{P}\mathcal{O}_\infty(N^n_{\leq})$ as monotone partial transformations of the poset $N^n_{\leq}$.

It is obvious that the group of units $H(\mathcal{I})$ of the semigroup $\mathcal{P}\mathcal{O}_\infty(N^n_{\leq})$ consists of exactly all order isomorphisms of the poset $N^n_{\leq}$ and hence Theorem 2.8 of [28] implies that the following theorem.

**Theorem 1.** For any positive integer $n$ the group of units $H(\mathcal{I})$ of the semigroup $\mathcal{P}\mathcal{O}_\infty(N^n_{\leq})$ is isomorphic to the group $\mathcal{J}_n$ of permutations of an $n$-elements set. Moreover, every element of $H(\mathcal{I})$ permutates coordinates of elements of $\mathbb{N}^n$, and only so permutations are elements of $H(\mathcal{I})$.

Since every $\alpha \in \mathcal{P}\mathcal{O}_\infty(N^n_{\leq})$ is a cofinite monotone partial transformations of the poset $N^n_{\leq}$ the following statement holds.

**Lemma 1.** If $(1, \ldots, 1) \in \text{dom} \alpha$ for some $\alpha \in \mathcal{P}\mathcal{O}_\infty(N^n_{\leq})$ then $(1, \ldots, 1)\alpha = (1, \ldots, 1)$.

For an arbitrary $i = 1, \ldots, n$ define

$$\mathcal{K}_i = \{(1, \ldots, \underbrace{m, \ldots, 1}_{\text{ith}}, \ldots, 1) \in \mathbb{N}^n : m \in \mathbb{N}\}$$

and by $\text{pr}_i : \mathbb{N}^n \to \mathbb{N}^n$ denote the projection on the $i$-th coordinate, i.e., for every $(m_1, \ldots, m_i, \ldots, m_n) \in \mathbb{N}^n$ put $(m_1, \ldots, \underbrace{m_i, \ldots, m_i}_{\text{ith}}, \ldots, m_n)\text{pr}_i = (1, \ldots, \underbrace{m_i, \ldots, m_i}_{\text{ith}}, \ldots, 1)$.

**Lemma 2.** Let $\{\mathcal{X}_1, \ldots, \mathcal{X}_k\}$ be a set of points in $\mathbb{N}^n \setminus \{(1, \ldots, 1)\}$, $k \in \mathbb{N}$. Then the set $\mathbb{N}^n \setminus (\uparrow \mathcal{X}_1 \cup \ldots \cup \uparrow \mathcal{X}_k)$ is finite if and only if $k \geq n$ and for every $\mathcal{K}_i$, $i = 1, \ldots, n$, there exists $\mathcal{X}_j \in \{\mathcal{X}_1, \ldots, \mathcal{X}_k\}$ such that $\mathcal{X}_j \in \mathcal{K}_i$.

**Proof.** ($\Leftarrow$) Without loss of generality we may assume that $\mathcal{X}_j \in \mathcal{K}_i$ for every positive integer $j \leq n$. Then simple verifications imply that the set $\mathbb{N}^n \setminus (\uparrow \mathcal{X}_1 \cup \ldots \cup \uparrow \mathcal{X}_k)$ is finite, and hence so is the set $\mathbb{N}^n \setminus (\uparrow \mathcal{X}_1 \cup \ldots \cup \uparrow \mathcal{X}_k)$.

($\Rightarrow$) Suppose to the contrary that there exist a subset $\{\mathcal{X}_{1,1}, \ldots, \mathcal{X}_{1,n}\}$ and an integer $i \in \{1, \ldots, n\}$ such that $\mathbb{N}^n \setminus (\uparrow \mathcal{X}_1 \cup \ldots \cup \uparrow \mathcal{X}_k)$ is finite and $\mathcal{X}_j \notin \mathcal{K}_i$ for any $j \in \{1, \ldots, k\}$.

The definition of $\mathcal{K}_i$ ($i = 1, \ldots, n$) implies that $\mathcal{K}_i$ with the induced partial order from $\mathbb{N}^n_{\leq}$ is an $\omega$-chain such that $\downarrow \mathcal{K}_i = \mathcal{K}_i$. Hence, for any $\mathcal{X} \in \mathbb{N}^n$ we have that either $\mathcal{K}_i \cup \uparrow \mathcal{X}$ is finite or $\mathcal{K}_i \cap \uparrow \mathcal{X} = \emptyset$. Then by our assumption we get that the set $\mathbb{N}^n \setminus (\uparrow \mathcal{X}_1 \cup \ldots \cup \uparrow \mathcal{X}_n)$ is infinite, a contradiction. The inequality $k \geq n$ follows from the above arguments. 

Later for an arbitrary non-empty subset $A$ of $\mathbb{N}^n$ by $\varepsilon_A$ we shall denote the identity map of the set $\mathbb{N}^n \setminus A$. It is obvious that the following lemma holds.
**Lemma 3.** For an arbitrary non-empty subset $A$ of $\mathbb{N}^n$, $\varepsilon_A$ is an element of the semigroup $\mathcal{P} \mathcal{O}_\infty(\mathbb{N}^n)$, and hence so are $\varepsilon_A \alpha$, $\alpha \varepsilon_A$ and $\varepsilon_A \alpha \varepsilon_A$ for any $\alpha \in \mathcal{P} \mathcal{O}_\infty(\mathbb{N}^n)$.

**Proposition 1.** For arbitrary element $\alpha$ of the semigroup $\mathcal{P} \mathcal{O}_\infty(\mathbb{N}^n)$ there exists a unique permutation $\sigma : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$ such that $(\mathcal{K}_i \cap \text{dom } \alpha) \alpha \subseteq \mathcal{K}_i$ for any $i = 1, \ldots, n$.

**Proof.** Lemma 3 implies that without loss of generality we may assume that $(1, \ldots, 1) \notin \text{dom } \alpha$ and $(1, \ldots, 1) \notin \text{ran } \alpha$.

Since for any $i = 1, \ldots, n$ the set $\mathcal{K}_i$ with the induced order from the poset $\mathbb{N}^n$ is an $\omega$-chain, the set $\mathcal{K}_i \cap \text{dom } \alpha$ contains the least element $x_i$. By Lemma 2 the set $\mathbb{N}^n \setminus (\uparrow p_i \cup \cdots \cup \uparrow p_n)$ is finite and hence so is $\text{dom } \alpha \setminus (\uparrow p_i \cup \cdots \cup \uparrow p_n)$. Since $\alpha$ is a cofinite partial bijection of $\mathbb{N}^n$ we have that $(\uparrow p_i \cup \cdots \cup \uparrow p_n) \alpha = (\uparrow p_i) \alpha \cup \cdots \cup (\uparrow p_n) \alpha$ and the set $\mathbb{N}^n \setminus ((\uparrow p_i) \alpha \cup \cdots \cup (\uparrow p_n) \alpha)$ is finite. Also, since $\alpha$ is a monotone partial bijection of the poset $\mathbb{N}^n$ we obtain that $(\uparrow p_i) \alpha \subseteq \uparrow (\ell_i) \alpha$ for all $i = 1, \ldots, n$. Then by Lemma 2 there exists a permutation $\sigma : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$ such that $(\uparrow p_i) \alpha \in \mathcal{K}_i$ for any $i = 1, \ldots, n$, because $\mathbb{N}^n \setminus (\uparrow p_i) \alpha \cup \cdots \cup (\uparrow p_n) \alpha \subseteq \mathbb{N}^n \setminus (\uparrow p_i) \alpha \cup \cdots \cup (\uparrow p_n) \alpha$ and the set $\mathbb{N}^n \setminus (\uparrow p_i) \alpha \cup \cdots \cup (\uparrow p_n) \alpha$ is finite. This implies that $(\sigma) \alpha \in \mathcal{K}_i$ for all $\sigma \in \mathcal{K}_i \cap \text{dom } \alpha$ and any $i = 1, \ldots, n$.

The proof of uniqueness of the permutation $\sigma$ for $\alpha \in \mathcal{P} \mathcal{O}_\infty(\mathbb{N}^n)$ is trivial. This completes the proof of the proposition. $\square$

Theorem 2 and Proposition 1 imply the following corollary.

**Corollary 1.** For every element $\alpha$ of the semigroup $\mathcal{P} \mathcal{O}_\infty(\mathbb{N}^n)$ there exists a unique element $\sigma$ of the group of units $H(\mathbb{I})$ of $\mathcal{P} \mathcal{O}_\infty(\mathbb{N}^n)$ such that $(\mathcal{K}_i \cap \text{dom } \alpha) \sigma \alpha \subseteq \mathcal{K}_i$ and $(\mathcal{K}_i \cap \text{dom } \alpha) \sigma^{-1} \alpha \subseteq \mathcal{K}_i$ for all $i = 1, \ldots, n$.

**Lemma 4.** There does not exist a finite family $\{L_1, \ldots, L_k\}$ of chains in the poset $\mathbb{N}^n$ such that $\mathbb{N}^2 = L_1 \cup \cdots \cup L_k$. Moreover, every co-finite subset in $\mathbb{N}^n$ has no property.

**Proof.** Suppose to the contrary that there exists a positive integer $k$ such that $\mathbb{N}^2 = L_1 \cup \cdots \cup L_k$ and $L_i$ is a chain for each $i = 1, \ldots, k$. Then $\{(1, k+1), (2, k), \ldots, (k, 2), (k+1, 1)\}$ is an anti-chain in the poset $\mathbb{N}^2$ which contains exactly $k + 1$ elements. Without loss of generality we may assume that $L_i \cap L_j = \emptyset$ for $i \neq j$. Since $\mathbb{N}^2 = L_1 \cup \cdots \cup L_k$, by the pigeonhole principle (or by the Dirichlet drawer principle, see [13] Section 7.3) there exists a chain $L_i$, $i = 1, \ldots, k$, which contains at least two distinct elements of the set $\{(1, k+1), (2, k), \ldots, (k, 2), (k+1, 1)\}$, a contradiction.

Assume that $A$ is a co-finite subset of $\mathbb{N}^n$ such that $A = \mathbb{N}^n \setminus \{x_1, \ldots, x_p\}$ for some positive integer $p$. For every $i = 1, \ldots, p$ we put $L_{k+i} = \{x_i\}$. Then for every finite partition $\{L_1, \ldots, L_k\}$ of $A$ such that $L_i$ is a chain for each $i = 1, \ldots, k$ the following family $\{L_1, \ldots, L_k, L_{k+1}, \ldots, L_{k+p}\}$ is a finite partition of the poset $\mathbb{N}^n$ such that $L_i$ is a chain for each $i = 1, \ldots, k + p$. This contradicts the above part of the proof, and hence the second statement of the lemma holds. $\square$

For any distinct $i, j \in \{1, \ldots, n\}$ we denote

$$\mathcal{K}_{i,j} = \{(x_1, \ldots, x_n) \in \mathbb{N}^n : x_k = 1 \text{ for all } k \in \{1, \ldots, n\} \setminus \{i, j\}\}$$

and

$$\mathcal{K}_{i,j}^c = \mathcal{K}_{i,j} \setminus (\mathcal{K}_i \cup \mathcal{K}_j)$$

**Lemma 5.** Let $n$ be a positive integer $\geq 3$. Let $\sigma_i$ be an arbitrary element of $\mathcal{K}_i \setminus \{1, \ldots, 1\}$ for $i = 3, \ldots, n$ and $\overline{y}_{1,2}$ be an arbitrary element of $\mathcal{K}_{1,2}$. Then there exists a finite family $\{L_1, \ldots, L_k\}$ of chains in the poset $\mathbb{N}^n$ such that $L_1 \cup \cdots \cup L_k = \mathbb{N}^n \setminus (\uparrow \overline{y}_{1,2} \cup \uparrow \sigma_3 \cup \cdots \cup \uparrow \sigma_n)$. 


Proof. Let \( \overline{x} = (x_1, \ldots, x_i, \ldots, 1) \) for \( i = 3, \ldots, n \) and \( \overline{x}_{1,2} = (y_1, y_2, 1, \ldots, 1) \). Then for any element \( \overline{a} = (a_1, \ldots, a_n) \) of the set \( \mathbb{N}^n \setminus \left( \uparrow \overline{y}_{1,2} \cup \uparrow \overline{x}_3 \cup \cdots \cup \uparrow \overline{x}_n \right) \) the following conditions hold:

(i) \( a_i < x_i \) for any \( i = 3, \ldots, n \);
(ii) if \( a_1 \geq y_1 \) then \( a_2 < y_2 \);
(iii) if \( a_2 \geq y_1 \) then \( a_1 < y_1 \).

These conditions imply that
\[
\mathbb{N}^n \setminus \left( \uparrow \overline{y}_{1,2} \cup \uparrow \overline{x}_3 \cup \cdots \cup \uparrow \overline{x}_n \right) = \bigcup \{ S(k_3, \ldots, k_n) : k_3 < x_3, \ldots, k_n < x_n \},
\]
where
\[
S(k_3, \ldots, k_n) = \bigcup \{ L_i(k_3, \ldots, k_n) : i = 1, \ldots, y_1 - 1 \} \cup \bigcup \{ R_j(k_3, \ldots, k_n) : j = 1, \ldots, y_2 - 1 \},
\]
with
\[
L_i(k_3, \ldots, k_n) = \{ (i, p, k_3, \ldots, k_n) \in \mathbb{N}^n : p \in \mathbb{N} \}
\]
and
\[
R_j(k_3, \ldots, k_n) = \{ (p, j, k_3, \ldots, k_n) \in \mathbb{N}^n : p \in \mathbb{N} \}.
\]
We observe that for arbitrary positive integers \( i, j, k_3, \ldots, k_n \), the sets \( L_i(k_3, \ldots, k_n) \) and \( R_j(k_3, \ldots, k_n) \) are chains in the poset \( \mathbb{N}^n \). Since the set \( \mathbb{N}^n \setminus \left( \uparrow \overline{y}_{1,2} \cup \uparrow \overline{x}_3 \cup \cdots \cup \uparrow \overline{x}_n \right) \) is the union of finitely many sets of the form \( S(k_3, \ldots, k_n) \), the above arguments imply the required statement of the lemma. \( \Box \)

**Proposition 2.** Let \( \alpha \) be an element of the semigroup \( \mathcal{P} \mathcal{O}_\infty(\mathbb{N}^n) \) such that \( (X_i \cap \text{dom } \alpha) \alpha \subseteq X_i \) for all \( i = 1, \ldots, n \). Then \( (X_{i_1, i_2} \cap \text{dom } \alpha) \alpha \subseteq X_{i_1, i_2} \) for all distinct \( i_1, i_2 = 1, \ldots, n \).

**Proof.** Suppose to the contrary that there exists \( \overline{x} \in X_{i_1, i_2} \cap \text{dom } \alpha \) such that \( (\overline{x}) \alpha \not\in X_{i_1, i_2} \). By Theorem 1 without loss of generality we may assume that \( i_1 = 1 \) and \( i_2 = 2 \), i.e., \( \overline{x} \in X_{1,2} \) and \( (\overline{x}) \alpha \not\in X_{1,2} \). By Lemma 1 \( \overline{x} 
ot\in (1, \ldots, 1) \).

For every \( i = 3, \ldots, n \) we put \( \overline{x}_i = (1, 1, \ldots, x_i, \ldots, 1) \in \text{dom } \alpha \) is the smallest element of \( X_i \) such that \( (\overline{x}_i) \alpha \not\in (1, \ldots, 1) \). There exists \( x_{1,2} = (x_1, x_2, 1, \ldots, 1) \in \text{dom } \alpha \cap X_{1,2} \) such that \( \overline{x} \leq \overline{x}_{1,2} \). Since \( \alpha \in \mathcal{P} \mathcal{O}_\infty(\mathbb{N}^n) \), \( (\overline{x}) \alpha \leq (\overline{x}_{1,2}) \alpha \not\in X_{1,2} \).

Now, the monotonicity of \( \alpha \) implies that \( (\uparrow \overline{x}_{1,2}) \alpha \subseteq (\uparrow \overline{x}_{1,2}) \alpha \) and \( (\uparrow \overline{x}_i) \alpha \subseteq (\uparrow \overline{x}_i) \alpha \) for any \( i = 3, \ldots, n \). By our assumption we have that
\[
X_{1,2} \cap \text{ran } \alpha \subseteq (\mathbb{N}_0^n \setminus \{ \uparrow \overline{x}_{1,2} \cup \uparrow \overline{x}_3 \cup \cdots \cup \uparrow \overline{x}_n \}) \alpha.
\]
Since the partial transformation \( \alpha \) preserves chains in the poset \( \mathbb{N}_0^n \), Lemma 5 implies that the set \( X_{1,2} \cap \text{ran } \alpha \) is a union of finitely many chains, which contradicts Lemma 3. The obtained contradiction implies the assertion of the proposition. \( \Box \)

**Theorem 2.** Let \( \alpha \) be an element of the semigroup \( \mathcal{P} \mathcal{O}_\infty(\mathbb{N}^n) \) such that \( (X_i \cap \text{dom } \alpha) \alpha \subseteq X_i \) for all \( i = 1, 2, 3 \). Then the following assertions hold:

(i) if \( (x_1, x_2, x_3) \in \text{dom } \alpha \) and \( (x_1, x_2, x_3) = (x_1, x_2, x_3) \) then \( x_1 \leq x_1, x_2 \leq x_2 \) and \( x_3 \leq x_3 \) and hence \( (\overline{x}) \alpha \leq (\overline{x}) \alpha \) for any \( \overline{x} \in \text{dom } \alpha \);

(ii) there exists a smallest positive integer \( n_\alpha \) such that \( (x_1, x_2, x_3) \alpha = (x_1, x_2, x_3) \) for all \( (x_1, x_2, x_3) \in \text{dom } \alpha \cap \uparrow (n_\alpha, n_\alpha, n_\alpha) \).

**Proof.** (i) We shall prove the inequality \( x_1 \leq x_1 \) by induction. The proofs of the inequalities \( x_2 \leq x_2 \) and \( x_3 \leq x_3 \) are similar.

By Proposition 2 we have that if \( x_1 = 1 \) then \( x_1 = 1 \), as well.

Next we shall show the following statement holds:

if for some positive integer \( p > 1 \) the inequality \( x_1 < 1 \) implies \( x_1 \leq x_1 \) then the equality \( x_1 = p \) implies \( x_1 \leq x_1 \), too.
Suppose to the contrary that there exists \((x_1, x_2, x_3) \in \text{dom } \alpha\) such that \(x_1 = p = (x_1, x_2, x_3)\) and \((x_1, x_2, x_3)\alpha = (x_1^\alpha, x_2^\alpha, x_3^\alpha)\) and \(x_1 + 1 \leq x_1^\alpha\). We define a partial map \(\varpi: \mathbb{N}^3 \rightarrow \mathbb{N}^3\) with \(\text{dom } \varpi = \mathbb{N}^3 \setminus \{1\} \times L(x_2) \times L(x_2)\) and \(\text{ran } \varpi = \mathbb{N}^3\) by the formula
\[
(i_1, i_2, i_3)\varpi = \begin{cases} (i_1 - 1, i_2, i_3), & \text{if } i_2 \in L(x_2) \text{ and } i_3 \in L(x_2); \\ (i_1, i_2, i_3), & \text{otherwise}, \end{cases}
\]
where \(L(x_2) = \{1, \ldots, x_2\}\) and \(L(x_3) = \{1, \ldots, x_3\}\). It is obvious that \(\varpi \in \mathcal{PO}_\infty(\mathbb{N}_\leq^3)\), and hence \(\gamma\varpi^k \in \mathcal{PO}_\infty(\mathbb{N}_\leq^3)\) for any positive integer \(k\) and any \(\gamma \in \mathcal{PO}_\infty(\mathbb{N}_\leq^3)\). This observation implies that without loss of generality we may assume that \(x_1^\alpha = x_1 + 1\). Then the assumption of the theorem implies that there exists the smallest element \((i_m, 1, 1)\) of \(\mathcal{K}_1\) such that \(i_m^\alpha > x_1^\alpha + 1\), where \((i_m, 1, 1) = (i_m, 1, 1)\). Since \((^\uparrow(i_m, 1, 1))\alpha \subseteq \uparrow(i_m^\alpha, 1, 1), (^\uparrow(x_1, x_2, x_3))\alpha \subseteq \uparrow(x_1^\alpha, x_2^\alpha, x_3^\alpha)\) and the set \(\mathbb{N}^3 \setminus \text{ran } \alpha\) is finite, our assumption implies that the set
\[
\mathcal{I}_{x_1}(\alpha) = \{(x_1, p_2, p_3) \in \text{dom } \alpha: p_2, p_3 \in \mathbb{N}\}
\]
is a union of finitely many subchains of the poset \((\mathbb{N}^3, \leq)\). This contradicts Lemma \(\text{(iii)}\) because the set \(\mathcal{I}_{x_1}(\alpha)\) with the induced partial order from \(\mathbb{N}^3\) is order isomorphic to a some cofinite subset of the poset \(\mathbb{N}^3\). The obtained contradiction implies the requested inequality \(x_1^\alpha \leq x_1\) and hence we have that statement \((i)\) holds.

The last assertion of \((i)\) follows from the definition of the poset \(\mathbb{N}^3_\leq\):

\((ii)\) Fix an arbitrary element \(\alpha\) of the semigroup \(\mathcal{PO}_\infty(\mathbb{N}^3_\leq)\) such that \((\mathcal{X}_i \cap \text{dom } \alpha)\alpha \subseteq \mathcal{X}_i\) for all \(i = 1, 2, 3\). Suppose to the contrary that for any positive integer \(n\) there exists \((x_1, x_2, x_3) \in \text{dom } \alpha \cap \uparrow(n, n, n)\) such that \((x_1, x_2, x_3)\alpha \neq (x_1, x_2, x_3)\). We put \(N_{\text{dom } \alpha} = |\mathbb{N}^3 \setminus \text{dom } \alpha| + 1\) and \(M_{\text{dom } \alpha} = \max \{\{x_1: (x_1, x_2, x_3) \notin \text{dom } \alpha\}, \{x_2: (x_1, x_2, x_3) \notin \text{dom } \alpha\}, \{x_3: (x_1, x_2, x_3) \notin \text{dom } \alpha\}\} + 1\).

The definition of the semigroup \(\mathcal{PO}_\infty(\mathbb{N}^3_\leq)\) implies that the positive integers \(N_{\text{dom } \alpha}\) and \(M_{\text{dom } \alpha}\) are well defined. Put \(n_0 = \max \{N_{\text{dom } \alpha}, M_{\text{dom } \alpha}\}\). Then our assumption implies that there exists \((x_1, x_2, x_3) \in \text{dom } \alpha \cap \uparrow(n_0, n_0, n_0)\) such that \((x_1, x_2, x_3)\alpha = (x_1^\alpha, x_2^\alpha, x_3^\alpha) \neq (x_1, x_2, x_3)\). By statement \((i)\) we have that \((x_1^\alpha, x_2^\alpha, x_3^\alpha) < (x_1, x_2, x_3)\). We consider the case when \(x_1^\alpha < x_1\). In the cases when \(x_2^\alpha < x_2\) or \(x_3^\alpha < x_3\) the proofs are similar. We assume that \(x_1 \leq x_2\) and \(x_1 \leq x_3\). By statement \((i)\) the partial bijection \(\alpha\) maps the set \(S = \{(x, y, z) \in \mathbb{N}^3: x, y, z \leq x_1 - 1\}\) into itself. Also, by the definition of the semigroup \(\mathcal{PO}_\infty(\mathbb{N}^3_\leq)\) the partial bijection \(\alpha\) maps the set
\[
\{(x_1, 1, 1), \ldots, (x_1, 1, x_1), (x_1, 2, 1), \ldots, (x_1, 2, x_1), \ldots, (x_1, x_1, 1), \ldots, (x_1, x_1, 1)\}
\]
into \(S\), too. Then our construction implies that
\[
|S \setminus \text{dom } \alpha| = |\mathbb{N}^3 \setminus \text{dom } \alpha| = N_{\text{dom } \alpha} - 1
\]
and
\[
|\{(x_1, 1, 1), \ldots, (x_1, 1, x_1), (x_1, 2, 1), \ldots, (x_1, 2, x_1), \ldots, (x_1, x_1, 1), \ldots, (x_1, x_1, 1)\}| \geq N_{\text{dom } \alpha},
\]
a contradiction. In the case when \(x_2 \leq x_1\) and \(x_2 \leq x_3\) or \(x_3 \leq x_1\) and \(x_3 \leq x_2\) we get contradictions in similar ways. This completes the proof of existence of a such positive integer \(n_\alpha\) for any \(\alpha \in \mathcal{PO}_\infty(\mathbb{N}^3_\leq)\). The existence of such minimal positive integer \(n_\alpha\) follows from the fact that the set of all positive integers with the usual order \(\leq\) is well-ordered.

Theorem \(\text{(iii)}\) and Proposition \(\text{(i)}\) imply the following corollary.

Corollary 2. For arbitrary element \(\alpha\) of the semigroup \(\mathcal{PO}_\infty(\mathbb{N}^3_\leq)\) there exist elements \(\sigma_1, \sigma_2\) of the group of units \(H(\mathbb{I})\) of \(\mathcal{PO}_\infty(\mathbb{N}^3_\leq)\) and a smallest positive integer \(n_\alpha\) such that
\[
(x_1, x_2, x_3)\sigma_1\alpha = (x_1, x_2, x_3)\alpha\sigma_2 = (x_1, x_2, x_3)
\]
for each \((x_1, x_2, x_3) \in \text{dom } \alpha \cap \uparrow(n_\alpha, n_\alpha, n_\alpha)\).

Corollary \(\text{(ii)}\) implies
Corollary 3. \(|\mathbb{N}^3 \setminus \text{ran } \alpha| \leq |\mathbb{N}^3 \setminus \text{dom } \alpha|\) for an arbitrary \(\alpha \in \mathcal{PO}_\infty(\mathbb{N}_\leq^3)\).

3. Algebraic properties of the semigroup \(\mathcal{PO}_\infty(\mathbb{N}_\leq^3)\)

Proposition 3. Let \(X\) be a non-empty set and let \(\mathcal{PB}(X)\) be a semigroup of partial bijections of \(X\) with the usual composition of partial self-maps. Then an element \(\alpha\) of \(\mathcal{PB}(X)\) is an idempotent if and only if \(\alpha\) is an identity partial self-map of \(X\).

Proof. The implication \((\Leftarrow)\) is trivial.

\((\Rightarrow)\) Let an element \(\alpha\) be an idempotent of the semigroup \(\mathcal{PB}(X)\). Then for every \(x \in \text{dom } \alpha\) we have that \((x)\alpha = (x)\alpha\) and hence we get that \(\text{dom } \alpha^2 = \text{dom } \alpha\) and \(\text{ran } \alpha^2 = \text{ran } \alpha\). Also since \(\alpha\) is a partial bijective self-map of \(X\) we conclude that the previous equalities imply that \(\text{dom } \alpha = \text{ran } \alpha\).

Fix an arbitrary \(x \in \text{dom } \alpha\) and suppose that \((x)\alpha = y\). Then \((x)\alpha = (x)\alpha\alpha = (y)\alpha = y\). Since \(\alpha\) is a partial bijective self-map of the set \(X\) we have that the equality \((y)\alpha = y\) implies that the full preimage of \(y\) under the partial map \(\alpha\) is equal to \(y\). Similarly the equality \((x)\alpha = y\) implies that the full preimage of \(y\) under the partial map \(\alpha\) is equal to \(x\). Thus we get that \(x = y\) and our implication holds.

Proposition 3 implies the following corollary.

Corollary 4. An element \(\alpha\) of \(\mathcal{PO}_\infty(\mathbb{N}_\leq^n)\) is an idempotent if and only if \(\alpha\) is an identity partial self-map of \(\mathbb{N}_\leq^n\) with the cofinite domain.

Corollary 4 implies the following proposition.

Proposition 4. Let \(n\) be a positive integer \(\geq 2\). The subset of idempotents \(E(\mathcal{PO}_\infty(\mathbb{N}_\leq^n))\) of the semigroup \(\mathcal{PO}_\infty(\mathbb{N}_\leq^n)\) is a commutative submonoid of \(\mathcal{PO}_\infty(\mathbb{N}_\leq^n)\) and moreover \(E(\mathcal{PO}_\infty(\mathbb{N}_\leq^n))\) is isomorphic to the free semilattice with unit \((\mathcal{P}^*(\mathbb{N}^n), \cup)\) over the set \(\mathbb{N}^n\) under the mapping \((\varepsilon)\mathfrak{h} = \mathbb{N}^n \setminus \text{dom } \varepsilon\).

Later we shall need the following technical lemma.

Lemma 6. Let \(X\) be a non-empty set, \(\mathcal{PB}(X)\) be a semigroup of partial bijections of \(X\) with the usual composition of partial self-maps and \(\alpha \in \mathcal{PB}(X)\). Then the following assertions hold:

\(\text{(i)}\) \(\alpha = \gamma \alpha\) for some \(\gamma \in \mathcal{PB}(X)\) if and only if the restriction \(\gamma|_{\text{dom } \alpha}: \text{dom } \alpha \to X\) is an identity partial map;

\(\text{(ii)}\) \(\alpha = \alpha \gamma\) for some \(\gamma \in \mathcal{PB}(X)\) if and only if the restriction \(\gamma|_{\text{ran } \alpha}: \text{ran } \alpha \to X\) is an identity partial map.

Proof. \(\text{(i)}\) The implication \((\Leftarrow)\) is trivial.

\(\text{(\Rightarrow)}\) Suppose that \(\alpha = \gamma \alpha\) for some \(\gamma \in \mathcal{PB}(X)\). Then \(\text{dom } \alpha \subseteq \text{dom } \gamma\) and \(\text{dom } \alpha \subseteq \text{ran } \gamma\). Since \(\gamma: X \to X\) is a partial bijection, the above arguments imply that \((x)\gamma = x\) for each \(x \in \text{dom } \alpha\). Indeed, if \((x)\gamma = y \neq x\) for some \(y \in \text{dom } \alpha\) then since \(\alpha: X \to X\) is a partial bijection we have that either

\[(x)\alpha = (x)\gamma \alpha = (y)\alpha \neq (x)\alpha, \quad \text{if } y \in \text{dom } \alpha,\]

or \((y)\alpha\) is undefined. This completes the proof of the implication.

The proof of \(\text{(ii)}\) is similar to that of \(\text{(i)}\).

Lemma 6 implies the following corollary.

Corollary 5. Let \(n\) be a positive integer \(\geq 2\) and \(\alpha\) be an element of the semigroup \(\mathcal{PO}_\infty(\mathbb{N}_\leq^n)\). Then the following assertions hold:

\(\text{(i)}\) \(\alpha = \gamma \alpha\) for some \(\gamma \in \mathcal{PO}_\infty(\mathbb{N}_\leq^n)\) if and only if the restriction \(\gamma|_{\text{dom } \alpha}: \text{dom } \alpha \to \mathbb{N}^n\) is an identity partial map;

\(\text{(ii)}\) \(\alpha = \alpha \gamma\) for some \(\gamma \in \mathcal{PO}_\infty(\mathbb{N}_\leq^n)\) if and only if the restriction \(\gamma|_{\text{ran } \alpha}: \text{ran } \alpha \to \mathbb{N}^n\) is an identity partial map.
The following theorem describes the Green relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$ and $\mathcal{D}$ on the semigroup $\mathcal{P}\mathcal{O}_{\infty}(\mathbb{N}_3^3)$.

**Theorem 3.** Let $\alpha$ and $\beta$ be elements of the semigroup $\mathcal{P}\mathcal{O}_{\infty}(\mathbb{N}_3^3)$. Then the following assertions hold:

(i) $\alpha\mathcal{L}\beta$ if and only if $\alpha = \mu\beta$ for some $\mu \in H(\mathbb{I})$;

(ii) $\alpha\mathcal{R}\beta$ if and only if $\alpha = \beta\nu$ for some $\nu \in H(\mathbb{I})$;

(iii) $\alpha\mathcal{H}\beta$ if and only if $\alpha = \beta\nu$ for some $\mu, \nu \in H(\mathbb{I})$;

(iv) $\alpha\mathcal{D}\beta$ if and only if $\alpha = \mu\beta\nu$ for some $\mu, \nu \in H(\mathbb{I})$.

**Proof.** (i) The implication ($\Rightarrow$) is trivial.

($\Leftarrow$) Suppose that $\alpha\mathcal{L}\beta$ in the semigroup $\mathcal{P}\mathcal{O}_{\infty}(\mathbb{N}_3^3)$. Then there exist $\gamma, \delta \in \mathcal{P}\mathcal{O}_{\infty}(\mathbb{N}_3^3)$ such that $\alpha = \gamma\beta$ and $\beta = \delta\alpha$. The last equalities imply that $\text{ran} \alpha = \text{ran} \beta$.

Next we consider the following cases:

(i) $(\mathcal{X}_i \cap \text{dom } \gamma) \alpha \subseteq \mathcal{X}_i$ and $(\mathcal{X}_j \cap \text{dom } \beta) \beta \subseteq \mathcal{X}_j$ for all $i, j = 1, 2, 3$;

(ii) $(\mathcal{X}_i \cap \text{dom } \alpha) \alpha \subseteq \mathcal{X}_i$ for all $i = 1, 2, 3$ and $(\mathcal{X}_j \cap \text{dom } \beta) \beta \nsubseteq \mathcal{X}_j$ for some $j = 1, 2, 3$;

(iii) $(\mathcal{X}_j \cap \text{dom } \alpha) \subseteq \mathcal{X}_j$ for some $i = 1, 2, 3$ and $(\mathcal{X}_j \cap \text{dom } \beta) \beta \subseteq \mathcal{X}_j$ for all $j = 1, 2, 3$;

(iv) $(\mathcal{X}_i \cap \text{dom } \alpha) \alpha \nsubseteq \mathcal{X}_i$ and $(\mathcal{X}_j \cap \text{dom } \beta) \beta \nsubseteq \mathcal{X}_j$ for some $i, j = 1, 2, 3$.

Suppose that case (i) holds. Then Proposition 1 and the equalities $\alpha = \gamma\beta$ and $\beta = \delta\alpha$ imply that

(1) $(\mathcal{X}_i \cap \text{dom } \gamma) \gamma \subseteq \mathcal{X}_i$ and $(\mathcal{X}_j \cap \text{dom } \delta) \delta \subseteq \mathcal{X}_j$, for all $i, j = 1, 2, 3$,

and moreover we have that $\alpha = \gamma\delta\alpha$ and $\beta = \delta\gamma\beta$. Hence by Lemma 3 we have that the restrictions $(\gamma\delta)_{\text{dom } \alpha} : \text{dom } \alpha \rightarrow \mathbb{N}^3$ and $(\delta\gamma)_{\text{dom } \beta} : \text{dom } \beta \rightarrow \mathbb{N}^3$ are identity partial maps. Then by condition 1 we obtain that the restrictions $\gamma_{\text{dom } \alpha} : \text{dom } \alpha \rightarrow \mathbb{N}^3$ and $\delta_{\text{dom } \beta} : \text{dom } \beta \rightarrow \mathbb{N}^3$ are identity partial maps, as well. Indeed, otherwise there exists $\pi \in \text{dom } \alpha$ such that either $(\pi)\gamma \nsubseteq \pi$ or $(\pi)\delta \nsubseteq \pi$, which contradicts Theorem 2(ii). Thus, the above arguments imply that in case (i) we have the following equality $\alpha = \beta$.

Suppose that case (i) holds. By Corollary 4 there exists an element $\mu$ of the group of units $H(\mathbb{I})$ of the semigroup $\mathcal{P}\mathcal{O}_{\infty}(\mathbb{N}_3^3)$ such that $(\mathcal{X}_j \cap \text{dom } \beta) \mu \beta \subseteq \mathcal{X}_j$ for all $j = 1, 2, 3$, and since $\alpha\mathcal{L}\beta$ we have that $\alpha = \gamma\beta = \gamma\mu\beta = \gamma(\mu^{-1}\mu)\beta = (\gamma\mu^{-1})(\mu\beta)$ and $\beta = (\mu\beta)\alpha$. Hence we get that $\alpha\mathcal{L}(\mu\beta)$,

$(\mathcal{X}_i \cap \text{dom } \alpha) \alpha \subseteq \mathcal{X}_i$ and $(\mathcal{X}_j \cap \text{dom } \beta) \mu\beta \subseteq \mathcal{X}_j$ for all $i, j = 1, 2, 3$. Then we apply case (i) for elements $\alpha$ and $\mu\beta$ and obtain the equality $\alpha = \mu\beta$, where $\mu$ is the above determined element of the group of units $H(\mathbb{I})$.

In case (i) the proof of the equality $\alpha = \mu\beta$ is similar to case (i).

Suppose that case (i) holds. By Corollary 4 there exist elements $\mu_\alpha$ and $\mu_\beta$ of the group of units $H(\mathbb{I})$ of the semigroup $\mathcal{P}\mathcal{O}_{\infty}(\mathbb{N}_3^3)$ such that $(\mathcal{X}_j \cap \text{dom } \alpha) \mu_\alpha \alpha \subseteq \mathcal{X}_j$ and $(\mathcal{X}_j \cap \text{dom } \beta) \mu_\beta \beta \subseteq \mathcal{X}_j$ for all $i, j = 1, 2, 3$, and since $\alpha\mathcal{L}\beta$ we have that $\alpha = \gamma\beta = \gamma\mu_\beta \beta = \gamma(\mu^{-1}_\beta \mu_\beta) \beta = (\gamma\mu^{-1}_\beta)(\mu_\beta)$. Hence we get that $\mu_\alpha \alpha = (\gamma\mu^{-1}_\beta)(\mu_\beta)$ and $\beta = \delta\alpha = \delta\mu_\alpha \alpha = \delta(\mu^{-1}_\alpha \mu_\alpha) \alpha = (\mu^{-1}_\alpha)(\mu_\alpha \alpha)$. The last two equality imply that $\mu_\beta \beta \subseteq \mathcal{X}_j$ for all $i, j = 1, 2, 3$. Then we apply case (i) for elements $\mu_\alpha \alpha$ and $\mu_\beta \beta$ and obtain the equality $\mu_\alpha \alpha = \mu_\beta \beta$. Hence $\alpha = \mu^{-1}_\alpha \mu_\alpha \alpha = \mu^{-1}_\alpha \mu_\beta \beta$. Since $\mu_\alpha, \mu_\beta \in H(\mathbb{I})$, $\mu = \mu^{-1}_\alpha \mu_\beta \beta \in H(\mathbb{I})$ as well.

The proof of assertion (ii) is dual to that of (i).

Assertion (iii) follows from (i) and (ii).

(iv) Suppose that $\alpha\mathcal{D}\beta$ in $\mathcal{P}\mathcal{O}_{\infty}(\mathbb{N}_3^3)$. Then there exists $\gamma \in \mathcal{P}\mathcal{O}_{\infty}(\mathbb{N}_3^3)$ such that $\alpha\mathcal{L}\gamma$ and $\gamma\mathcal{R}\beta$. By statements (i) and (ii) there exist $\mu, \nu \in H(\mathbb{I})$ such that $\alpha = \mu\gamma$ and $\gamma = \beta\nu$ and hence $\alpha = \mu\beta\nu$. Converse, suppose that $\alpha = \mu\beta\nu$ for some $\mu, \nu \in H(\mathbb{I})$. Then by (i), (ii), we have that $\alpha\mathcal{L}(\beta\nu)$ and $(\beta\nu)\mathcal{R}\beta$, and hence $\alpha\mathcal{D}\beta$ in $\mathcal{P}\mathcal{O}_{\infty}(\mathbb{N}_3^3)$.

Theorem 3 implies Corollary 6 which gives the inner characterization of the Green relations $\mathcal{L}$, $\mathcal{R}$, and $\mathcal{H}$ on the semigroup $\mathcal{P}\mathcal{O}_{\infty}(\mathbb{N}_3^3)$ as partial permutations of the poset $\mathbb{N}_3^3$.

**Corollary 6.**

(i) Every $\mathcal{L}$-class of $\mathcal{P}\mathcal{O}_{\infty}(\mathbb{N}_3^3)$ contains exactly 6 distinct elements.
(ii) Every $R$-class of $PO_{\infty}(N_3^2)$ contains exactly 6 distinct elements.
(iii) Every $H$-class of $PO_{\infty}(N_3^3)$ contains at most 6 distinct elements.

Proof. Statements (i), (ii) and (iii) are trivial and they follow from the corresponding statements of Theorem 3.

Lemma 7. Let $\alpha, \beta$ and $\gamma$ be elements of the semigroup $PO_{\infty}(N_3^3)$ such that $\alpha = \beta\alpha\gamma$. Then the following statements hold:

(i) if $(\mathcal{K}_i \cap \text{dom } \beta)\beta \subseteq \mathcal{K}_i$ for any $i = 1, 2, 3$, then the restrictions $\beta|_{\text{dom } \alpha}: \text{dom } \alpha \to N^3$ and $\gamma|_{\text{ran } \alpha}: \text{ran } \alpha \to N^3$ are identity partial maps;
(ii) if $(\mathcal{K}_i \cap \text{dom } \gamma)\gamma \subseteq \mathcal{K}_i$ for any $i = 1, 2, 3$, then the restrictions $\beta|_{\text{dom } \alpha}: \text{dom } \alpha \to N^3$ and $\gamma|_{\text{ran } \alpha}: \text{ran } \alpha \to N^3$ are identity partial maps;
(iii) there exist elements $\sigma_\beta$ and $\sigma_\gamma$ of the group of units $H(\mathbb{I})$ of $PO_{\infty}(N_3^3)$ such that $\alpha = \sigma_\beta\alpha\sigma_\gamma$.

Proof. (i) Assume that the inclusion $(\mathcal{K}_i \cap \text{dom } \beta)\beta \subseteq \mathcal{K}_i$ holds for any $i = 1, 2, 3$. Then one of the following cases holds:

(1) $(\mathcal{K}_i \cap \text{dom } \alpha)\alpha \subseteq \mathcal{K}_i$ for any $i = 1, 2, 3$;
(2) there exists $i \in \{1, 2, 3\}$ such that $(\mathcal{K}_i \cap \text{dom } \alpha)\alpha \notin \mathcal{K}_i$.

If case (1) holds then the equality $\alpha = \beta\alpha\gamma$ and Proposition 1 imply that $(\mathcal{K}_i \cap \text{dom } \gamma)\gamma \subseteq \mathcal{K}_i$ for any $i = 1, 2, 3$. Suppose that $\text{(x)}\beta < \text{x}$ for some $\text{x} \in \text{dom } \alpha$. Then by Theorem 2(i) we have that

$$(\text{x})\alpha = (\text{x})\beta\alpha\gamma \leq (\text{x})\alpha\gamma \leq (\text{x})\alpha,$$

which contradicts the equality $\alpha = \beta\alpha\gamma$. The obtained contradiction implies that the restriction $\beta|_{\text{dom } \alpha}: \text{dom } \alpha \to N^3$ is an identity partial map. This and the equality $\alpha = \beta\alpha\gamma$ imply that the restriction $\gamma|_{\text{ran } \alpha}: \text{ran } \alpha \to N^3$ is an identity partial map too.

Suppose that case (2) holds. Then by Corollary 1 there exists an element $\sigma$ of the group of units $H(\mathbb{I})$ of the semigroup $PO_{\infty}(N_3^3)$ such that $(\mathcal{K}_i \cap \text{dom } \alpha)\sigma\alpha \subseteq \mathcal{K}_i$ for any $i = 1, 2, 3$. Now, the equality $\alpha = \beta\alpha\gamma$ implies that

$$\alpha\sigma = \beta\alpha\gamma\sigma = \beta\alpha\gamma\sigma = \beta\sigma\gamma\sigma = \beta(\sigma\gamma\sigma).$$

By case (1) we have that the restrictions $\beta|_{\text{dom } \alpha}: \text{dom } \alpha \to N^3$ is an identity partial map, which implies that $\beta\alpha = \alpha$. Then we have that $\alpha = \beta\alpha\gamma = \alpha\gamma$ and hence by Corollary 5 the restriction $\gamma|_{\text{ran } \alpha}: \text{ran } \alpha \to N^3$ is an identity partial map, which completes the proof of statement (i).

(ii) The proof of this statement is dual to (i). Indeed, assume that the inclusion $(\mathcal{K}_i \cap \text{dom } \gamma)\gamma \subseteq \mathcal{K}_i$ holds for any $i = 1, 2, 3$. Then one of the following cases holds:

(1) $(\mathcal{K}_i \cap \text{dom } \alpha)\alpha \subseteq \mathcal{K}_i$ for any $i = 1, 2, 3$;
(2) there exists $i \in \{1, 2, 3\}$ such that $(\mathcal{K}_i \cap \text{dom } \alpha)\alpha \notin \mathcal{K}_i$.

If case (1) holds then the equality $\alpha = \beta\alpha\gamma$ and Proposition 1 imply that $(\mathcal{K}_i \cap \text{dom } \beta)\beta \subseteq \mathcal{K}_i$ for any $i = 1, 2, 3$. Similar as in the proof of statement (i) Theorem 2(i) implies that the restrictions $\beta|_{\text{dom } \alpha}: \text{dom } \alpha \to N^3$ and $\gamma|_{\text{ran } \alpha}: \text{ran } \alpha \to N^3$ are identity partial maps.

Suppose that case (2) holds. Then by Corollary 1 there exists an element $\sigma$ of the group of units $H(\mathbb{I})$ of the semigroup $PO_{\infty}(N_3^3)$ such that $(\mathcal{K}_i \cap \text{dom } \alpha)\sigma\alpha \subseteq \mathcal{K}_i$ for any $i = 1, 2, 3$. Now, the equality $\alpha = \beta\alpha\gamma$ implies that

$$\sigma\alpha = \sigma\beta\alpha\gamma = \sigma\beta\alpha\gamma = \sigma(\beta\alpha\gamma) = (\sigma\beta\alpha\gamma)\gamma.$$

By case (1) we have that the restrictions $\gamma|_{\text{ran } \alpha}: \text{ran } \alpha \to N^3$ is an identity partial map, which implies that $\alpha = \sigma\gamma$. Then we have that $\alpha = \beta\alpha\gamma = \beta\alpha$ and hence by Corollary 5 the restriction $\beta|_{\text{dom } \alpha}: \text{dom } \alpha \to N^3$ is an identity partial map as well, which completes the proof of statement (ii).

(iii) Assume that $\alpha = \beta\alpha\gamma$. By the Lagrange Theorem (see: [11] Section 1.5) for every element $\sigma$ of the group of permutations $\mathcal{S}_n$ the order of $\sigma$ divides the order of $\mathcal{S}_n$. This, Proposition 1 and the
equality $\alpha = \beta \alpha \gamma$ imply that
\[(2) \quad (\mathcal{H}_i \cap \text{dom } \beta^6) \beta^6 \subseteq \mathcal{H}_i \quad \text{and} \quad (\mathcal{H}_i \cap \text{dom } \gamma^6) \gamma^6 \subseteq \mathcal{H}_i, \quad \text{for any } i = 1, 2, 3.\]

Also, the equality $\alpha = \beta \alpha \gamma$ implies that
\[\alpha = \beta \alpha \gamma = \beta (\beta \alpha \gamma) \gamma = \beta^2 \alpha \gamma^2 = \ldots = \beta^6 \alpha \gamma^6.\]

Then statements (i), (ii) and conditions (2) imply that the restrictions $\beta^6|_{\text{dom } \alpha} : \text{dom } \alpha \rightarrow \mathbb{N}^3$ and $\gamma^6|_{\text{ran } \alpha} : \text{ran } \alpha \rightarrow \mathbb{N}^3$ are identity partial maps. By Corollary 11 there exist unique elements $\sigma_\beta, \sigma_\gamma \in H(\mathbb{I})$ such that $(\mathcal{H}_i \cap \text{dom } \beta) \beta \sigma_{\beta}^{-1} \subseteq \mathcal{H}_i, (\mathcal{H}_i \cap \text{dom } \beta) \sigma \beta \subseteq \mathcal{H}_i, (\mathcal{H}_i \cap \text{dom } \alpha) \gamma \sigma_{\gamma}^{-1} \subseteq \mathcal{H}_i$ and $(\mathcal{H}_i \cap \text{dom } \gamma) \sigma \gamma \subseteq \mathcal{H}_i$ for all $i = 1, 2, 3$. Then we have that
\[(3) \quad \beta^6 = (\mathbb{I} \mathbb{I}) (\mathbb{I} \mathbb{I}) (\mathbb{I} \mathbb{I}) =
= (\beta \sigma_{\beta}^{-1} \sigma \beta) (\beta \sigma_{\beta}^{-1} \sigma \beta) (\beta \sigma_{\beta}^{-1} \sigma \beta) =
= (\beta \sigma_{\beta}^{-1} \sigma \beta) (\beta \sigma_{\beta}^{-1} \sigma \beta) (\beta \sigma_{\beta}^{-1} \sigma \beta) (\sigma \beta)\]
and
\[(4) \quad \gamma^6 = (\mathbb{I} \mathbb{I}) (\mathbb{I} \mathbb{I}) (\mathbb{I} \mathbb{I}) =
= (\gamma \sigma_{\gamma}^{-1} \sigma \gamma) (\gamma \sigma_{\gamma}^{-1} \sigma \gamma) (\gamma \sigma_{\gamma}^{-1} \sigma \gamma) =
= (\gamma \sigma_{\gamma}^{-1} \sigma \gamma) (\gamma \sigma_{\gamma}^{-1} \sigma \gamma) (\gamma \sigma_{\gamma}^{-1} \sigma \gamma).\]

We claim that $(\mathbb{I})(\beta \sigma_{\beta}^{-1}) = \mathbb{I}$ for any $\mathbb{I} \in \text{dom } \alpha$. Assume that $(\mathbb{I})(\beta \sigma_{\beta}^{-1}) \neq \mathbb{I}$ for some $\mathbb{I} \in \text{dom } \alpha$. Then the choice of the element $\sigma \beta \in H(\mathbb{I})$, Theorem 2 (i) and (3) imply that
\[(\mathbb{I})\beta^6 = (\mathbb{I})(\beta \sigma_{\beta}^{-1} \sigma \beta) (\beta \sigma_{\beta}^{-1} \sigma \beta) (\beta \sigma_{\beta}^{-1} \sigma \beta) (\sigma \beta) <
< (\mathbb{I})(\beta \sigma_{\beta}^{-1} \sigma \beta) (\beta \sigma_{\beta}^{-1} \sigma \beta) (\beta \sigma_{\beta}^{-1} \sigma \beta) (\sigma \beta) \leq
\leq (\mathbb{I})(\beta \sigma_{\beta}^{-1} \sigma \beta) (\beta \sigma_{\beta}^{-1} \sigma \beta) (\beta \sigma_{\beta}^{-1} \sigma \beta) (\sigma \beta) <
< (\mathbb{I})(\beta \sigma_{\beta}^{-1} \sigma \beta) (\beta \sigma_{\beta}^{-1} \sigma \beta) (\sigma \beta) \leq
\leq (\mathbb{I})(\beta \sigma_{\beta}^{-1} \sigma \beta) (\sigma \beta) <
< (\mathbb{I})(\sigma \beta) \leq
\leq \mathbb{I},\]
which contradicts that the restriction $\beta^6|_{\text{dom } \alpha} : \text{dom } \alpha \rightarrow \mathbb{N}^3$ is an identity partial map. Hence we have that $(\mathbb{I})(\beta \sigma_{\beta}^{-1}) = \mathbb{I}$ for any $\mathbb{I} \in \text{dom } \alpha$, which implies that the equality $(\mathbb{I}) \beta = (\mathbb{I}) \sigma \beta$ holds for any $\mathbb{I} \in \text{dom } \alpha$.

Using 11 as in the above we prove the equality $(\mathbb{I}) \gamma = (\mathbb{I}) \sigma \gamma$ holds for any $\mathbb{I} \in \text{ran } \alpha$.

The obtained equalities and the definition of the composition of partial maps imply statement (iii).

**Lemma 8.** Let $\alpha$ and $\beta$ be elements of the semigroup $\mathcal{P}(\mathbb{N}^3)$ and $A$ be a cofinite subset of $\mathbb{N}^3$. If the restriction $(\alpha \beta)|_A : A \rightarrow \mathbb{N}^3$ is an identity partial map then there exists an element $\sigma$ of the group of units $H(\mathbb{I})$ of $\mathcal{P}(\mathbb{N}^3)$ such that $(\mathbb{I}) \alpha = (\mathbb{I}) \sigma$ and $(\mathbb{I}) \beta = (\mathbb{I}) \sigma^{-1}$ for all $\mathbb{I} \in A$ and $\mathbb{I} \in (A) \alpha$.

**Proof.** We observe that one of the following cases holds:

(1) $(\mathcal{H}_i \cap A) \alpha \subseteq \mathcal{H}_i$ for any $i = 1, 2, 3$;

(2) there exists $i \in \{1, 2, 3\}$ such that $(\mathcal{H}_i \cap A) \alpha \not\subseteq \mathcal{H}_i$.

If case (1) holds then the assumption of the lemma and Proposition 11 imply that $(\mathcal{H}_i \cap (A) \alpha) \beta \subset \mathcal{H}_i$ for any $i = 1, 2, 3$. Suppose that $(\mathbb{I}) \alpha < \mathbb{I}$ for some $\mathbb{I} \in A$. Then by Theorem 2 (i) we have that
\[(\mathbb{I}) \alpha \beta < (\mathbb{I}) \beta \leq \mathbb{I}\]
which contradicts the assumption of the lemma. Similar we show that the case $(7) \beta < \gamma$ for some $\gamma \in (A)\alpha$ does not hold. The obtained contradictions implies that $(x)\alpha = x$ and $(x)\beta = x$ for all $x \in A$.

Suppose that case (2) holds. Then by Corollary 11 there exists an element $\sigma$ of the group of units $H(\mathbb{I})$ of the semigroup $\mathcal{P}O_{\infty}(\mathbb{N}_3)$ such that $(\mathcal{K}_i \cap \text{dom } \alpha)\sigma_{\alpha} \subseteq \mathcal{K}_i$ for any $i = 1, 2, 3$. Now, the assumption of the lemma implies that

$$(x)\alpha \beta = (x)\alpha \mathbb{I} \beta = (x)\alpha \sigma^{-1} \beta = x,$$

and hence by the above part of the proof we get that $(x)\alpha \sigma = x$ and $(y)\sigma^{-1} \beta = x$ for all $y \in (A)\alpha$. The obtained equalities and the definition of the composition of partial maps imply the statement of the lemma.

**Lemma 9.** Let $\alpha, \beta, \gamma$ and $\delta$ be elements of the semigroup $\mathcal{P}O_{\infty}(\mathbb{N}_3)$ such that $\alpha = \gamma \beta \delta$. Then there exist $\gamma^*, \delta^* \in \mathcal{P}O_{\infty}(\mathbb{N}_3)$ such that $\alpha = \gamma^* \beta \delta^*$, dom $\gamma^* = \text{dom } \alpha$, ran $\gamma^* = \text{ran } \beta$, dom $\delta^* = \text{ran } \beta$ and ran $\delta^* = \text{ran } \alpha$.

**Proof.** For a cofinite subset $A$ of $\mathbb{N}_3$ by $\iota_A$ we denote the identity map of $A$. It is obvious that $\iota_A \in \mathcal{P}O_{\infty}(\mathbb{N}_3)$ for any cofinite subset $A$ of $\mathbb{N}_3$. This implies that $\alpha = \iota_{\text{dom } \alpha} \alpha \iota_{\text{ran } \alpha}$ and $\beta = \iota_{\text{dom } \beta} \beta \iota_{\text{ran } \beta}$, and hence we have that

$$\alpha = \iota_{\text{dom } \alpha} \alpha \iota_{\text{ran } \alpha} = \iota_{\text{dom } \alpha} \gamma \beta \delta \iota_{\text{ran } \alpha} = \iota_{\text{dom } \alpha} \gamma \iota_{\text{dom } \beta} \beta \iota_{\text{ran } \beta} \delta \iota_{\text{ran } \alpha}.$$

We put $\gamma^* = \iota_{\text{dom } \alpha} \gamma \iota_{\text{dom } \beta}$ and $\delta^* = \iota_{\text{ran } \beta} \delta \iota_{\text{ran } \alpha}$. The above two equalities and the definition of the semigroup operation of $\mathcal{P}O_{\infty}(\mathbb{N}_3)$ imply that $\text{dom } \gamma^* \subseteq \text{dom } \alpha$, $\text{ran } \gamma^* \subseteq \text{dom } \beta$, $\text{dom } \delta^* \subseteq \text{ran } \beta$ and $\text{ran } \delta^* \subseteq \text{ran } \alpha$. Similar arguments and the equality $\alpha = \gamma^* \beta \delta^*$ imply the converse inclusions which implies the statement of the lemma.

**Theorem 4.** $\mathcal{D} = \mathcal{J}$ in $\mathcal{P}O_{\infty}(\mathbb{N}_3)$.

**Proof.** The inclusion $\mathcal{D} \subseteq \mathcal{J}$ is trivial.

Fix any $\alpha, \beta \in \mathcal{P}O_{\infty}(\mathbb{N}_3)$ such that $\alpha \mathcal{J} \beta$. Then there exist $\gamma, \delta, \alpha, \beta \in \mathcal{P}O_{\infty}(\mathbb{N}_3)$ such that $\alpha = \gamma \delta \alpha$ and $\beta = \gamma \beta \delta \beta$. Then only one of the following cases holds:

1. $(\mathcal{K}_i \cap \text{dom } (\gamma \alpha \beta)) \gamma \alpha \beta \subseteq \mathcal{K}_i$ for any $i = 1, 2, 3$;
2. there exists $i \in \{1, 2, 3\}$ such that $(\mathcal{K}_i \cap \text{dom } (\gamma \alpha \beta)) \gamma \alpha \beta \not\subseteq \mathcal{K}_i$.

If case (1) holds then Lemma 1(i) implies that $(\gamma \alpha \beta) : \text{dom } \alpha \rightarrow \mathbb{N}_3$ and $(\delta \beta \alpha) : \text{ran } \alpha \rightarrow \mathbb{N}_3$ are identity partial maps. Now by Lemma 5 there exist elements $\sigma_{\alpha}$ and $\sigma_{\beta}$ of the group of units $H(\mathbb{I})$ of the semigroup $\mathcal{P}O_{\infty}(\mathbb{N}_3)$ such that $(x)\gamma = (x)\sigma_{\alpha}$, $(y)\beta = (y)\sigma_{\beta}^{-1}$, $(u)\beta = (u)\sigma_{\beta}$ and $(v)\delta = (v)\sigma_{\delta}^{-1}$, for all $x \in \text{dom } \alpha$, $y \in \text{dom } \alpha \gamma$, $u \in \text{ran } \alpha$ and $v \in \text{ran } (\alpha)\delta = \text{ran } \beta$. Then the above arguments imply that $\alpha = \sigma_{\alpha} \beta \sigma_{\delta}^{-1}$ and hence by Theorem 3(iv) we get that $\alpha \mathcal{D} \beta$ in $\mathcal{P}O_{\infty}(\mathbb{N}_3)$.

If case (2) holds then we have that

$$\alpha = \gamma \alpha \beta \delta \alpha = (\gamma \alpha \beta)^2 \alpha (\delta \beta \alpha)^2 = \ldots = (\gamma \alpha \beta)^6 \alpha (\delta \beta \alpha)^6$$

and

$$\beta = \gamma \beta \alpha \beta \delta \beta = (\beta \gamma \alpha)^2 (\delta \alpha \beta)^2 = \ldots = (\gamma \beta \alpha)^6 (\delta \alpha \beta)^6.$$

We put

$$\gamma^* = \gamma \beta (\gamma \beta)^5 \quad \text{and} \quad \delta^* = \delta \beta (\delta \beta)^5.$$
Lemma 7(i) implies that $(\gamma_\alpha \gamma_\beta^2): \text{dom } \alpha \to \mathbb{N}^3$ and $(\delta_2^2 \delta_3): \text{ran } \alpha \to \mathbb{N}^3$ are identity partial maps. Now by Lemma 8 there exist elements $\sigma_\alpha$ and $\sigma_\beta$ of the group of units $H(I)$ of the semigroup $\mathcal{P}E_\infty(\mathbb{N}^3_\alpha)$ such that $(\pi)\gamma_\alpha = (\pi)\sigma_\alpha$, $(\pi)\gamma_\beta^2 = (\pi)\sigma_\beta^{-1}$, $(\pi)\delta_3^2 = (\pi)\sigma_\beta$ and $(\pi)\delta_3 = (\pi)\sigma_\beta^{-1}$, for all $\pi \in \text{dom } \alpha$, $\bar{y} \in (\text{dom } \alpha)\gamma_\alpha = \text{ran } \alpha = \text{dom } \beta$, $\bar{u} \in \text{ran } \alpha$ and $\bar{v} \in (\text{ran } \alpha)\delta_3^2 = (\text{ran } \alpha)\delta_3^2 = \text{ran } \beta$. Then the above arguments imply that $\alpha = \sigma_\alpha \beta \sigma_\beta^{-1}$ and hence by Theorem 8(iv) we get that $\alpha \mathcal{D} \beta$ in $\mathcal{P}E_\infty(\mathbb{N}^3_\alpha)$. □

Acknowledgements

We thank the referee for many comments and suggestions.

References

[1] O. Andersen, Ein Bericht über die Struktur abstrakter Halbgruppen, PhD Thesis, Hamburg, 1952.
[2] L. W. Anderson, R. P. Hunter, and R. J. Koch, Some results on stability in semigroups. Trans. Amer. Math. Soc. 117 (1965), 521–529.
[3] T. Banakh, S. Dimitrova, and O. Gutik, The Rees-Sushchikewitsch Theorem for simple topological semigroups, Mat. Stud. 31 (2009), no. 2, 211–218.
[4] T. Banakh, S. Dimitrova, and O. Gutik, Embedding the bicyclic semigroup into countably compact topological semigroups, Topology Appl. 157 (2010), no. 18, 2803–2814.
[5] S. Bardyla, Classifying locally compact semitopological polycyclic monoids, Mat. Visn. Tov. Im. Shevchenka 13 (2016), 21–28.
[6] S. Bardyla, On a semitopological $\alpha$-bicyclic monoid, Visn. L’viv. Univ., Ser. Mekh.-Mat. 81 (2016), 9–22.
[7] S. Bardyla, On locally compact shift-continuous topologies on the $\alpha$-bicyclic monoid, Topol. Algebra Appl. 6 (2018), 34–42.
[8] S. Bardyla, On locally compact semitopological graph inverse semigroups, Mat. Stud. 49 (2018), no. 1, 19–28.
[9] S. Bardyla, On locally compact topological graph inverse semigroups, Topology Appl. 267 (2019), 106873. DOI: 10.1016/j.topol.2019.106873
[10] S. Bardyla, Embedding of graph inverse semigroups into CLP-compact topological semigroups, Topology Appl. 272 (2020), 107058. DOI: 10.1016/j.topol.2020.107058
[11] S. Bardyla and O. Gutik, On a semitopological polycyclic monoid, Algebra Discrete Math. 21 (2016), no. 2, 163–183.
[12] S. Bardyla and A. Ravsky, Closed subsets of compact-like topological spaces, Preprint arXiv:1907.12129.
[13] G. Berman and K. D. Fryer, Introduction to Combinatorics, New-York, Academic Press, 1972.
[14] M. O. Bertman and T. T. West, Conditionally compact bicyclic semitopological semigroups, Proc. Roy. Irish Acad. A76 (1976), no. 21–23, 219–226.
[15] O. Bezushchak, On growth of the inverse semigroup of partially defined co-finite automorphisms of integers, Algebra Discrete Math. (2004), no. 2, 45–55.
[16] O. Bezushchak, Green’s relations of the inverse semigroup of partially defined co-finite isometries of discrete line, Visn., Ser. Fiz.-Mat. Nauky, Kyiv. Univ. Im. Tarasa Shevchenka (2008), no. 1, 12–16.
[17] P. J. Cameron, Permutation Groups, Cambridge Univ. Press, London, 1999.
[18] I. Chuchman and O. Gutik, Topological monoids of almost monotone injective co-finite partial selfmaps of the set of positive integers, Carpathian Math. Publ. 2 (2010), no. 1, 119–132.
[19] A. H. Clifford and G. B. Preston, The Algebraic Theory of Semigroups, Vol. I., Amer. Math. Soc. Surveys 7, Providence, R.I., 1961; Vol. II., Amer. Math. Soc. Surveys 7, Providence, R.I., 1967.
[20] C. Eberhart and J. Selden, On the closure of the bicyclic semigroup, Trans. Amer. Math. Soc. 144 (1969), 115–126.
[21] I. R. Filhel and O. V. Gutik, On the closure of the extended bicyclic semigroup, Carpathian Math. Publ. 3 (2011), no. 2, 131–157.
[22] J. A. Green, On the structure of semigroups, Ann. Math. (2) 54 (1951), 163–172.
[23] P. A. Grillet, Semigroups. An Introduction to the Structure Theory, Marcel Dekker, New York, 1995.
[24] O. Gutik, On the dichotomy of a locally compact semitopological bicyclic monoid with adjoined zero, Visn. L’viv. Univ., Ser. Mekh.-Mat. 80 (2015), 33–41.
[25] O. Gutik, On locally compact semitopological 0-bisimple inverse $\omega$-semigroups, Topol. Algebra Appl. 6 (2018), 77–101.
[26] O. Gutik and K. Maksymyk, On semitopological interassociates of the bicyclic monoid, Visn. L’viv. Univ., Ser. Mekh.-Mat. 82 (2016), 98–108.
[27] O. V. Gutik and K. M. Maksymyk, On semitopological bicyclic extensions of linearly ordered groups, Mat. Metody Fiz.-Mekh. Polya 59 (2016), no. 4, 31–43.

THE MONOID OF MONOTONE INJECTIVE PARTIAL SELFMAPS OF THE POSET $(\mathbb{N}^3, \leq)$ ...
