Monotonicity in first-passage percolation

Jean-Baptiste Gouéré *

Abstract

We consider standard first-passage percolation on \( \mathbb{Z}^d \). Let \( e_1 \) be the first coordinate vector. Let \( a(n) \) be the expected passage time from the origin to \( n e_1 \). In this short paper, we note that \( a(n) \) is increasing under some strong condition on the support of the distribution of the passage times on the edges.

1 Introduction and results

First passage percolation. We consider the graph \( \mathbb{Z}^d, \ d \geq 2 \), obtained by taking \( \mathbb{Z}^d \) as vertex set and by putting an edge between two vertices if the Euclidean distance between them is 1. We consider a family of non-negative i.i.d.r.v. \( \tau = (\tau(e))_{e \in \mathcal{E}} \) indexed by the set of edges \( \mathcal{E} \) of the graph. We interpret \( \tau(e) \) as the time needed to travel along the edge \( e \) (the graph is unoriented).

If \( a \) and \( b \) are two vertices of \( \mathbb{Z}^d \), we call path from \( a \) to \( b \) any finite sequence of vertices \( r = (a = x_0, ..., x_k = b) \) such that, for all \( i \in \{0, ..., k - 1\} \), the vertices \( x_i \) et \( x_{i+1} \) are linked by an edge. We denote by \( \mathcal{C}(a, b) \) the set of such paths. The time needed to travel along a path \( r = (x_0, ..., x_k) \) is defined by:

\[
\tau(r) = \sum_{i=0}^{k-1} \tau(x_i, x_{i+1}).
\]

Then, the time needed to go from \( a \) to \( b \) is defined by:

\[
T(a, b) = \inf \{ \tau(r) : r \in \mathcal{C}(a, b) \}.
\]

Let \( e_1, \ldots, e_d \) denote the canonical basis vectors of \( \mathbb{R}^d \). We are interested in the sequence \( (a(n)) \) defined by:

\[
a(n) = \mathbb{E}(T(0, ne_1)).
\]

We write \( T'(0, ne_1) \) and \( a'(n) \) for the passage times and expected passage times obtained when the paths are restricted to \( \{(x_1, \ldots, x_d) : 0 \leq x_1 \leq n\} \).

*Postal address: Université d’Orléans MAPMO B.P. 6759 45067 Orléans Cedex 2 France E-mail: jbgouere@univ-orleans.fr
Main result and related results. We denote by $S_-$ the infimum of the support of the distribution of the $\tau(e)$. We denote by $S_+$ the supremum of the support.

Theorem 1 Assume $0 < S_- \text{ and } S_+ \leq 2S_-$. Then the sequence $(a_n)$ is non-decreasing. More precisely, we have:

$$a(n) \geq a(n-1) + S_- \left[1 - \frac{(S_+ - S_-)^2}{S_-^2}\right].$$

As soon as the distribution of the $\tau(e)$ is not a Dirac distribution there exists, with probability one, infinitely many random $N$ such that $T(0, (N - 1)e_1) > T(0, Ne_1)$. However, monotonicity of expected passage times seems quite natural and was already conjectured by Hammersley and Welsh in [3]. In [2], Alm and Wierman proved the monotonicity for an Euclidean first-passage percolation model. We are not aware of any other positive results.

On the other hand, van den Berg proved in [7] that, when $d = 2$, one has $a'(2) < a'(1)$ when $\tau(e) = 1$ with small probability and $\tau(e) = 0$ otherwise. Note that we still have $a'(2) < a'(1)$ if, instead of setting $\tau(e) = 0$ we set $\tau(e) = \varepsilon$ for a small enough $\varepsilon$. A related result was given by Joshi in [6].

We refer to the review by Howard [5] for a more detailed account.

Further remarks.
- The same result holds for the $a'(n)$.
- The proof gives that $T(0, ne_1)$ stochastically dominates the mean of $n$ dependent copies of $T(0, (n - 1)e_1)$ (see [3] and [2]).

With the same strategy one can prove for example the following result:

$$a(n) \geq a(n-1) \text{ as soon as } S_- > 0 \text{ and } \left(\frac{a(n)n^{-1}}{S_-} - 1\right) \left(\frac{E(\tau(e))}{S_-} - 1\right) \leq \frac{1}{2} \quad (1)$$

where $e$ is a fixed edge. We show how to adapt the proof of Theorem 1 to prove this result below the proof of Theorem 1. In particular, using the inequality $a(n) \leq nE(\tau(e))$, we get that $a$ is non-decreasing as soon as:

$$S_- > 0 \text{ and } E(\tau(e)) \leq (1 + 2^{-1/2})S_-.$$

This gives a sufficient condition with no assumption on $S_+$ which can be infinite. However, this sufficient condition is still strong and we do not see how to give any significantly weaker condition.

1. Let us sketch a proof. Fix $a$ and $b$ such that $S_- < a < b < S_+$. For each $n$, consider a box $\{n-C, \ldots, n\} \times \{-D, \ldots, D\}^{d-1}$. Let $A_n$ be the following event: $\tau(e) \leq a$ for edges inside the boundary of the box and $\tau(e) \geq b$ for edges inside the box. For suitably chosen large $C$ and $D$ and for $n > C$, we have $T(0, (n - 1)e_1) > T(0, ne_1)$ as soon as $A_n$ occurs. As the $A_n$ are local event of fixed positive probability, the result follows.

2. Indeed, $a'(1)$ can only increase while $a'(2)$ increases by at most $\varepsilon E(N)$ where $N$ is the length of a geodesic for the initial passage times. Using $T'(0, 2e_1) \leq 2$ one can check that any geodesic must remain in a random box of subgeometrical height. Therefore $E(N)$ is finite and the result follows.
Fix the distribution of $\tau(e)$. Assume $S_+ > 0$ and $E\tau(e) < \infty$. Then the conditions in (1) are true for large enough $n$ and $d$. This is due to the fact that $a(n)n^{-1}$ can be made arbitrarily close to $S_-$.

2 Proofs

Proof of Theorem 1. For all $i$ we consider the following sets of edges:

- $H^i$: the set of edges $(x, x + e_1)$ where $x = (x_1, \ldots, x_d)$ is such that $x_1 = i$.
- $V^i$: the set of edges $(x, x + e_k)$ where $x_1 = i$ and $k$ belongs to $\{2, \ldots, d\}$.

We define new passage times $\tau^i(e)$ as follows:

- If $e$ belongs to $H^i$ then $\tau^i(e) = 0$.
- If $e$ belongs to $V^i$ then $\tau^i(e) = +\infty$.
- Otherwise, $\tau^i(e) = \tau(e)$.

We denote by $T^i(a, b)$ the time needed to travel from $a$ to $b$ with the passage times $\tau^i(e)$. Note, for all $n \geq 1$ and all $i \in \{0, n - 1\}$, the following:

$$T^i(0, ne_1) \text{ and } T(0, (n - 1)e_1) \text{ have the same distribution.} \quad (2)$$

We now compare $T^i(0, ne_1)$ and $T(0, ne_1)$. Let $\pi$ be a path from 0 to $ne_1$ such that $\tau(\pi) = T(0, ne_1)$. We modify this path as follows. Each time the path goes, in this order, through an edge $(x, y) \in V^i$, we replace this part of the path by $(x, x + e_1, y + e_1, y)$. We denote by $\pi^i$ the modified path. We have

$$\tau^i(\pi^i) \leq \tau(\pi) - S_- \text{card}(\pi \cap H_i) + (S_+ - S_-)\text{card}(\pi \cap V_i)$$

where, for example, $\text{card}(\pi \cap H_i)$ denotes the number of edges of $H_i$ used by $\pi$. The term involving $H_i$ is due to the time saved by the modification of the passage times. The term involving $V_i$ is partly due to the time left by the modification of the path. We thus get

$$T^i(0, ne_1) \leq T(0, ne_1) - S_- \text{card}(\pi \cap H_i) + (S_+ - S_-)\text{card}(\pi \cap V_i) \quad (3)$$

and then

$$\sum_{i=0}^{n-1} T^i(0, ne_1) \leq nT(0, ne_1) - S_- \sum_{i=0}^{n-1} \text{card}(\pi \cap H_i) + (S_+ - S_-)\sum_{i=0}^{n-1} \text{card}(\pi \cap V_i). \quad (4)$$

Note

$$\sum_{i=0}^{n-1} \text{card}(\pi \cap H_i) \geq n \quad (5)$$

as $\pi$ is a path from 0 to $ne_1$. But

$$T(0, ne_1) = \tau(\pi) \geq S_- \sum_{i=0}^{n-1} \text{card}(\pi \cap V_i) + S_- \sum_{i=0}^{n-1} \text{card}(\pi \cap H_i) \geq S_- \sum_{i=0}^{n-1} \text{card}(\pi \cap V_i) + S_- n \quad (6)$$
and, moreover,
\[ T(0, ne_1) \leq \tau(0, e_1, 2e_2, \ldots, ne_1) \leq nS_+ . \]

Therefore:
\[
\sum_{i=0}^{n-1} \text{card}(\pi \cap V_i) \leq \frac{T(0, ne_1) - nS_-}{S_-} \leq \frac{nS_+ - nS_-}{S_-} .
\] (7)

From (4) and (7) we get:
\[
\sum_{i=0}^{n-1} T^i(0, ne_1) \leq nT(0, ne_1) - nS_- + \frac{n(S_+ - S_-)^2}{S_-} .
\] (8)

Taking expectations and using (2) we get:
\[
na(n - 1) \leq na(n) - nS_- \left[ 1 - \frac{(S_+ - S_-)^2}{S_-^2} \right] .
\]

The proof follows. □

Proof of (1). The proof is essentially the same. The main difference lies in the definition of the new passage times \( \tau^i(e) \). We let \( \tilde{\tau} \) be an independent copy of \( \tau \). We then set:
- If \( e \) belongs to \( H^i \) then \( \tau^i(e) = 0 \).
- If \( e \) belongs to \( V^i \) then \( \tau^i(e) = +\infty \).
- If \( e \) belongs to \( V^{i+1} \) then \( \tau^i(e) = \tilde{\tau^i(e)} \).
- Otherwise, \( \tau^i(e) = \tau(e) \).

Instead of (3) we can write, after taking conditional expectation w.r.t. \( \tau \) :
\[
T^i(0, ne_1) \leq T(0, ne_1) - S_- \text{card}(\pi \cap H_i) + (E(\tau(e)) - S_-) \text{card}(\pi \cap V_i) + (E(\tau(e)) - S_-) \text{card}(\pi \cap V_{i+1}) .
\] (9)

Instead of (4) we can get :
\[
\sum_{i=0}^{n-1} T^i(0, ne_1) \leq nT(0, ne_1) - S_- \sum_{i=0}^{n-1} \text{card}(\pi \cap H_i) + 2(E(\tau(e)) - S_-) \sum_{i=0}^{n} \text{card}(\pi \cap V_i) .
\] (10)

Using (5), an equality similar to (6) and taking expectation, we get:
\[
na(n - 1) \leq na(n) - S_- n + 2(E(\tau(e)) - S_-) \frac{a(n) - nS_-}{S_-}
\]
and thus:
\[
a(n - 1) \leq a(n) - S_- \left( 1 - 2 \left( \frac{E(\tau(e))}{S_-} - 1 \right) \left( \frac{a(n)n^{-1}}{S_-} - 1 \right) \right) .
\]

The proof follows. □
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