Existence results for second-order monotone differential inclusions on the positive half-line

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Abstract

Consider in a real Hilbert space $H$ the differential equation (inclusion) $(E)$:

$$p(t)u''(t) + q(t)u'(t) \in Au(t) + f(t) \text{ for a.a. } t > 0,$$

with the condition $(B)$:

$$u(0) = x \in D(A),$$

where $A: D(A) \subset H \to H$ is a (possibly set-valued) maximal monotone operator whose range contains 0; $p, q \in L^\infty(0, \infty)$, with $\text{ess inf } p > 0$ and $q^+ \in L^1(0, \infty)$. More than four decades ago, V. Barbu established the existence of a unique bounded (on $[0, \infty)$) solution to $(E), (B)$, in the particular case $p \equiv 1$, $q \equiv 0$ and $f \equiv 0$. Subsequently the existence of bounded solutions in the homogeneous case ($f \equiv 0$) has been further investigated by H. Brezis (1972), N. Pavel (1976), L. Véron (1974-76), and by E.I. Poffald and S. Reich (1984) when $A$ is an $m$-accretive operator in a Banach space. The non-homogeneous case has received less attention from this point of view. R.E. Bruck solved affirmatively this problem (in 1980), but under the restrictive condition that $A$ is coercive (and $p \equiv 1$, $q \equiv 0$, $f \in L^\infty(0, \infty; H)$). On the other hand, much attention has been paid by several authors to the asymptotic behavior of bounded solutions (if they exist) as $t \to \infty$, both in the homogeneous and nonhomogeneous case. Recently, I established jointly with H. Khatibzadeh [Set-Valued Var. Anal. DOI 10.1007/s11228-013-0270-3] the existence of (weak and strong) bounded solutions to $(E), (B)$, in the case $p \equiv 1$, $q \equiv 0$, under the optimal condition $tf(t) \in L^1(0, \infty; H)$. In this paper, this result is extended to the general case of non-constant functions $p, q$ satisfying the mild conditions above, thus compensating for the lack of existence theory for such kind of second order problems. Note that our results open up the possibility to apply Lions’ method of artificial viscosity towards approximating the solutions of some nonlinear parabolic and hyperbolic problems, as shown in the last section of the paper.

Keywords. Strong solution, weak solution, bounded solution, smoothing effect, minimization problem, the method of artificial viscosity

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1 Introduction

Let $H$ be a real Hilbert space with the inner product $(\cdot, \cdot)$ and the induced norm $\|x\| = (x, x)^{1/2}$. Consider the second-order differential equation (inclusion)

$$p(t)u''(t) + q(t)u'(t) \in Au(t) + f(t) \quad \text{for a.a. } t \in \mathbb{R}_+ := [0, \infty),$$

with the condition

$$u(0) = x \in D(A),$$

where

(H1) $A: D(A) \subset H \to H$ is a (possibly set-valued) maximal monotone operator whose graph contains $[0, 0]$;

(H2) $p, q \in L^\infty(\mathbb{R}_+) := L^\infty(\mathbb{R}_+; \mathbb{R})$, with $\text{ess inf } p > 0$ and $q^+ \in L^1(\mathbb{R}_+)$, where $q^+(t) := \max\{q(t), 0\}$;

and $f$ is a given $H$-valued function whose (required) properties will be specified later.

It is worth pointing out that in (H1) one can assume without any loss of generality that the range $\mathcal{R}(A)$ of $A$ contains the null vector, since this case reduces to the previous one for a maximal monotone operator obtained from $A$ by shifting its domain.

Information on monotone operators can be found in [6], [10], [24].

We continue with some historical comments:

It was V. Barbu who established for the first time the existence of a unique bounded solution to equation (E) subjected to (B), in the special case $p \equiv 1$, $q \equiv 0$ and $f \equiv 0$, in [4, 5] (see also Chapter V in [6]), followed by the nice paper by H. Brezis [9], who considered a more general condition at $t = 0$; see also N. Pavel [27], as well as E.I. Poffald and S. Reich [29] in the case when $A$ is an $m$-accretive operator in a Banach space. L. Véron [30, 31] paid attention to the same problem (existence of bounded solutions) in the case of (sufficiently smooth) variable coefficients $p(t)$, $q(t)$ and $f \equiv 0$. The existence of bounded solutions in the non-homogeneous case (i.e., when $f$ is not the null function) has received less attention. Recall that Bruck [11] established the existence of a bounded solution on $\mathbb{R}$ of equation (E) (implying that all solutions of (E) are bounded on $\mathbb{R}_+$), in the case $p \equiv 1$, $q \equiv 0$, $f \in L^\infty(\mathbb{R})$, under the restrictive condition that $A$ is coercive. We also mention the relatively recent article by Apreutesei [2], addressing the case of sufficiently smooth coefficients $p$, $q$, with $p(t) \geq p_0 > 0$, $q(t) \geq q_0 > 0$, and $x \in D(A)$.

On the other hand, there has been a great deal of work pertaining to the asymptotic behavior of bounded solutions (if they exist) as $t \to \infty$ of (E), (B), including the case of periodic or almost periodic forcing. See [23, 32, 21, 22] for the case $p \equiv 1$, $q \equiv 0$, $f \equiv 0$. The case $p \equiv 1$, $q \equiv 0$ and $f$ periodic or almost periodic was thoroughly analyzed by Biroli [7, 8], Bruck [11, 12], and by Poffald and Reich [28, 29] in the case when $A$ is an $m$-accretive operator in a Banach space. In recent years Djafari Rouhani
and Khatibzadeh have established various results on the asymptotic behavior of bounded solutions (if they exist), as $t \to \infty$, for both constant and variable coefficients $p, q$, and for both the homogeneous and non-homogeneous case of $(E)$ (see [13]-[18]).

In order to compensate for the lack of existence theory for such kind of second order problems, I have recently started working on this subject. Recall that (as in [1]) equation $(E)$ can be written as

\[(a(t)u'(t))' \in b(t)Au(t) + b(t)f(t),\tag{1.1}\]

where

\[a(t) = \exp \left( \int_0^t \frac{q(s)}{p(s)} \, ds \right), \quad b(t) = \frac{a(t)}{p(t)}.\]

Denote by $X$ the weighted space $L^2_b(\mathbb{R}_+; H) = L^2(\mathbb{R}_+; H; b(t)dt)$, which is a real Hilbert space with the scalar product

\[(f, g)_X = \int_0^\infty b(t)(f(t), g(t))dt,
\]

and the induced norm

\[\|f\|_X^2 = (f, f)_X.\]

Recently [25, 26], we proved the existence of a unique strong solution $u \in X$ to equation $(E)$ subjected to $u(0) = x \in \overline{D(A)}$, under the above conditions $(H1), (H2)$, where instead of $q^+ \in L^1(\mathbb{R}_+)$ we had a different condition on $q$: either $\text{ess inf } q > 0$ or $\text{ess sup } q < 0$. Note that there we did replace the usual boundedness (on $\mathbb{R}_+$) condition by a different one, namely $u \in X$, which may or may not imply boundedness of $u$. More precisely, we proved that if $x \in D(A)$ and $f \in X$, then (cf. Theorem 3.1 in [25]), there exists a unique $u \in X$, with $u', u'' \in X$, such that $u(0) = x$ and $u$ satisfies equation $(E)$ for a.a. $t > 0$. Since

\[a(t)\|u(t)\|^2 = \|x\|^2 + \int_0^t \frac{d}{ds} (a(s)\|u(s)\|^2) \, ds
\]

\[= \|x\|^2 + \int_0^t qb\|u(s)\|^2 \, ds + 2 \int_0^t a(u, u') \, ds
\]

\[\leq \|x\|^2 + M(\|u\|_X^2 + \|u\|_X\|u'\|_X) < \infty,
\]

it follows that

\[\|u(t)\| = O\left(\exp \left( -\frac{1}{2} \int_0^t \frac{q}{p} \, ds \right) \right).\tag{1.2}\]

If $x \in \overline{D(A)}$ and $f \in X$, then (cf. Theorem 1.1 in [26]) there exists a unique $u \in C(\mathbb{R}_+; H) \cap X$, such that $u', u'' \in L^2_b([\varepsilon, \infty); H)$ for all $\varepsilon > 0$, such that $u(0) = x$ and $u$ satisfies equation $(E)$ for a.a. $t > 0$. Therefore (1.2) is again valid for $t \geq \varepsilon$. So, if $\text{ess inf } q > 0$, then $\|u(t)\|$ decays exponentially to zero. In the case $\text{ess sup } q < 0$ and
$f \in X$, $\|u(t)\|$ could be unbounded. This fact is illustrated by the simple scalar equation 
\[ u'' - u' = 1, \quad t > 0, \quad u(0) = x, \] 
that has a unique solution in $X$ (here $X = L^2(\mathbb{R}_+; e^{-t}dt)$, $u(t) = x - t$, which is unbounded (and so are all the other solutions).

It is worth pointing out that the sign condition on $q$ (i.e., either $\text{ess inf } q > 0$ or $\text{ess sup } q < 0$) was essential in our previous treatment \cite{25,26}. However, by an inspection of the proof of Theorem 1.1 in \cite{26} we can see that if in addition $A$ is strongly monotone, then the existence of $u \in X$ follows in absence of the sign condition on $q$. Of course, strong monotonicity is a very restrictive condition.

Our aim in this paper is to derive existence of bounded (on $\mathbb{R}_+$) solutions $u$ to equation (E) subjected to $u(0) = x \in \overline{D(A)}$, under $(H1)$ and $(H2)$ above, including the alternative assumption $q^+ \in L^1(\mathbb{R}_+)$ (which allows $q(t)$ to be ”close” or equal to zero), plus appropriate conditions on the nonhomogeneous term $f$. Of course, replacing the condition $u \in X$ by a boundedness one leads to a different problem that requires separate analysis.

Note that our assumptions on $p$ and $q$ are weaker than those previously used by other authors.

Concerning the methodology we use in this paper, note that, while in \cite{25,26} we performed a global analysis within the space $X$ defined above, here we derive existence on $\mathbb{R}_+$ (of bounded solutions, or bounded solutions in a generalized sense, as specified below) by a limiting process applied to a sequence of two-point boundary value problems on $[0, n], n = 0, 1, ...$ This approach has some common features with that used in \cite{19} for the particular equation
\[ u''(t) \in Au(t) + f(t), \quad t > 0, \] (1.3)
subjected to $u(0) = x \in \overline{D(A)}$. Existence of bounded solutions on $\mathbb{R}_+$ for equation (1.3) in the case of a general maximal monotone $A$ was first established in \cite{19}. More precisely, in \cite{19} a concept of a weak solution was defined for equation (1.3), and the existence of a unique, bounded, weak solution $u = u(t)$, $t \geq 0$, was established under the optimal condition $tf(t) \in L^1(\mathbb{R}_+; H)$ (simple examples involving $A = 0$ show that this class of the $f$’s cannot be enlarged if we want to have bounded solutions); if, in addition, $f \in L^2_{\text{loc}}([0, \infty); H)$, the solution $u$ of equation (1.3) is strong (i.e., $u$ is twice differentiable and satisfies (1.3) for a.a. $t > 0$). In this paper we extend this existence result to the case of variable coefficients $p$ and $q$ satisfying $(H2)$. In Theorems 3.3, 3.4 we establish the existence of weak and strong solutions $u$ to (E), (B) satisfying
\[ \sup_{t \geq 0} a_-(t)\|u(t)\|^2 < \infty, \] (C)
where $a_-(t) = \exp \left(-\int_0^t q^-/p\right)$, $q^-(t) = -\min \{q(t), 0\}$. If, in addition to $(H2)$, $q^- \in L^1(\mathbb{R}_+)$ (i.e., $q \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$), then (C) becomes a real boundedness condition,
and (cf. Corollary 3.6) for each pair \((x, f) \in D(A) \times L^1(\mathbb{R}_+; H; tdt)\) there exists a unique, weak, bounded solution \(u\) of \((E), (B)\). If in addition \(f \in L^2_{loc}(\mathbb{R}; H)\) then \(u\) is even strong.

Note that we use a constructive method, suitable for the numerical approximation and for the variational approach when \(A\) is a subdifferential operator. The smoothing effect on the starting values \(x \in D(A)\) is pointed out. In the last section of the paper we show how our results can be used to approximate the solutions of some parabolic and hyperbolic problems by the method of artificial viscosity introduced by J.L. Lions [20].

2 Some auxiliary results

For a given \(T \in (0, \infty)\) denote by \(X_T\) the weighted space \(L^2(0, T; H; b(t)dt)\), where \(b = b(t)\) is the function defined in Section 1, restricted to the interval \([0, T]\). \(X_T\) is a Hilbert space equipped with the scalar product

\[
(f_1, f_2)_{X_T} = \int_0^T b(t) (f_1(t), f_2(t)) \, dt,
\]

and the induced norm. In fact, under our conditions below, \(X_T\) coincides with the usual \(L^2(0, T; H)\) algebraically and topologically. The need for the weight \(b(t)\) will become obvious in what follows.

**Proposition 2.1.** Assume that \(p, q \in L^\infty(0, T), \) with \(\text{ess inf } p > 0\). Define \(B : D(B) \subset X_T \to X_T\) by

\[
D(B) := \{v \in X_T; v', v'' \in X_T, v(0) = x, v(T) = y\},
\]

\[
Bv = -pv'' - qv',
\]

where \(x, y \in H\) are given vectors. Then, \(B\) is a maximal monotone operator in \(X_T\). More precisely, \(B\) is the subdifferential of the proper, convex, lower semicontinuous function \(\Psi : X_T \to [0, \infty)\) defined by

\[
\Psi(v) = \frac{1}{2} \int_0^T a(t) \|v'(t)\|^2 dt + j(v(0) - x) + j(v(T) - y),
\]

where \(j\) is the indicator function of the set \(\{0\} \subset H\).

**Proof.** Note that \(Bv = -\frac{p}{a}(aw')'\) for all \(v \in D(B)\), so the monotonicity of \(B\) in \(X_T\) (equipped with the scalar product \((\cdot, \cdot)_{X_T}\) defined above) follows easily. The rest of the proof is similar to the proof of Lemma 2.2 in [25], so we omit it. 

\[\Box\]
Lemma 2.2. Let $A$ satisfy (H1), $p, q \in L^\infty(0, T)$, with $\operatorname{ess} \inf \ p > 0$, and let $f \in L^2(0, T; H)$. Then, for all $x, y \in D(A)$, there exists a unique $u = u(t) \in W^{2,2}(0, T; H)$ satisfying
\[
P(t)u''(t) + q(t)u'(t) \in Au(t) + f(t) \text{ for a.a. } t \in (0, T),
\]
\[u(0) = x, \ u(T) = y.\]

Proof. We can assume without any loss of generality that $y = 0$ (otherwise, one can use the substitution $\tilde{u}(t) = u(t) - y$. Let $A_\lambda$ be the Yosida approximation of $A$ for $\lambda > 0$, i.e., $A_\lambda z = \lambda^{-1}(z - J_\lambda z)$, $z \in H$, where $J_\lambda = (I + \lambda A)^{-1}$ (the resolvent of $A$). Denote by $\bar{A}$ the realization of $A$ in $X_T$, i.e., $\bar{A} = \{[v, w] \in X_T \times X_T : [v(t), w(t)] \in A \text{ for a.a. } t \in (0, T)\}$. Note that $\bar{A}_\lambda + B$ is maximal monotone in $X_T$ for all $\lambda > 0$, where $\bar{A}_\lambda$ is the Yosida approximation of $\bar{A}$ and $B$ is the operator defined above (see Proposition 2.1), where $y = 0$. Therefore, for each $\lambda > 0$ there exists a $u_\lambda \in W^{2,2}(0, T; H)$ that satisfies
\[\begin{align*}
-pu''_\lambda - qu'_\lambda + A_\lambda u_\lambda + \lambda u_\lambda &= -f \text{ a.e. in } (0, T), \\
u_\lambda(0) &= x, \ u_\lambda(T) = 0.
\end{align*}\]
Equation (2.3) can be equivalently written as
\[(au'_\lambda)' = b(A_\lambda u_\lambda + \lambda u_\lambda + f) \text{ a.e. in } (0, T).\]
If we multiply equation (2.5) by $u_\lambda(t)$, integrate the resulting equation over $[\tau, T]$, and use the fact that $A_\lambda 0 = 0$, we obtain
\[
\int_{\tau}^{T} ((au'_\lambda)', u_\lambda) \, dt \geq \lambda \int_{\tau}^{T} b\|u_\lambda\|^2 \, dt + \int_{\tau}^{T} b(f, u_\lambda) \, dt,
\]
which implies (see (2.4))
\[
-a(\tau)(u'_\lambda(\tau), u_\lambda(\tau)) - \int_{\tau}^{T} a\|u'_\lambda\|^2 \, dt \geq \lambda \int_{\tau}^{T} b\|u_\lambda\|^2 \, dt + \int_{\tau}^{T} b(f, u_\lambda) \, dt.
\]
Therefore
\[
\frac{1}{2}a(\tau) \frac{d}{d\tau} \|u_\lambda(\tau)\|^2 \leq \int_{\tau}^{T} b\|f\| \cdot \|u_\lambda\| \, ds.
\]
Integrating this inequality over $[0, t]$ yields
\[
\frac{1}{2}a(t)\|u_\lambda(t)\|^2 - \frac{1}{2}\|x\|^2 - \frac{1}{2}\int_{0}^{t} b_s\|u_\lambda\|^2 \, ds \leq \int_{0}^{T} d\tau \int_{\tau}^{T} b\|f\| \cdot \|u_\lambda\| \, ds = \int_{0}^{T} \tau b\|f\| \cdot \|u_\lambda\| \, d\tau,
\]
which implies
\[
a(t)\|u_\lambda(t)\|^2 + \int_{0}^{t} \frac{q_\tau}{p} a\|u_\lambda\|^2 \, d\tau \leq
\]
\[
\left(\|x\|^2 + 2 \int_0^T \tau b \|f\| \cdot \|u_\lambda\| \, d\tau\right) + \int_0^t \frac{q^+}{p} a \|u_\lambda\|^2 \, d\tau, \quad 0 \leq t \leq T.
\]

(2.9)

Recall that \( q^+(t) := \max \{q(t), 0\} \) and \( q^-(t) := -\min \{q(t), 0\} \). It follows by the Gronwall-Bellman lemma that

\[
a(t) \|u_\lambda(t)\|^2 \leq \left(\|x\|^2 + 2 \int_0^T \tau b \|f\| \cdot \|u_\lambda\| \, d\tau\right) \exp \left(\int_0^t \frac{q^+}{p} \, d\tau\right), \quad 0 \leq t \leq T,
\]

(2.10)

and so

\[
\|u_\lambda(t)\|^2 \leq M \left(\|x\|^2 + 2 \int_0^T \tau b \|f\| \cdot \|u_\lambda\| \, d\tau\right), \quad 0 \leq t \leq T,
\]

(2.11)

where \( M = \exp \left(\int_0^T \frac{q^+}{p} \, d\tau\right) \). Denoting \( C_\lambda = \sup\{\|u_\lambda(t)\| : 0 \leq t \leq T\} \), we obtain from (2.11)

\[
C_\lambda^2 \leq M \left(\|x\|^2 + 2C_\lambda \int_0^T \tau b \|f\| \, d\tau\right), \quad 0 \leq t \leq T,
\]

(2.12)

which shows that \( \sup_{\lambda > 0} C_\lambda < \infty \), i.e.,

\[
\sup\{\|u_\lambda(t)\| : 0 \leq t \leq T, \ \lambda > 0\} < \infty.
\]

(2.13)

On the other hand, if we take \( \tau = 0 \) in (2.7), we get

\[
\int_0^T a \|u'_\lambda\|^2 \, dt \leq -(u'_\lambda(0), x) + \int_0^T b \|f\| \cdot \|u_\lambda\| \, dt,
\]

(2.14)

which implies (see also (2.13))

\[
\|u'_\lambda\|^2 \leq C_1 \|u'_\lambda(0)\| + C_2.
\]

(2.15)

where \( C_1, \ C_2 \) are some positive constants. Now, multiplying (2.5) by \( A_\lambda u_\lambda \) and then integrating over \([0, T]\), we obtain

\[
\int_0^T \left((au'_\lambda)' , A_\lambda u_\lambda\right) \, dt \geq \int_0^T b \|A_\lambda u_\lambda\|^2 \, dt + \int_0^T b(f, A_\lambda u_\lambda) \, dt.
\]

(2.16)

Integrating by parts in (2.16) leads to

\[
\|A_\lambda u_\lambda\|^2 \leq -(u'_\lambda(0), A_\lambda x) + \|f\|_{X_T} \|A_\lambda u_\lambda\|_{X_T},
\]

(2.17)

since \( (u'_\lambda , (A_\lambda u_\lambda)' ) \geq 0 \). Recall that \( \|A_\lambda x\| \leq \|A^0 x\| \) for all \( \lambda > 0 \), where \( A^0 \) is the minimal section of \( A \). Therefore, (2.17) implies

\[
\|A_\lambda u_\lambda\|^2 \leq C_3 \|u'_\lambda(0)\| + C_4.
\]

(2.18)

By (2.3), (2.13), (2.15), and (2.18) we can see that

\[
\|u'_\lambda\|^2 \leq C_5 (\|u'_\lambda\|^2 + \lambda \|u_\lambda\|^2 + \|A_\lambda u_\lambda\|^2 + \|f\|^2) \leq C_6 \|u'_\lambda(0)\| + C_7,
\]

(2.19)
for all \( \lambda \in (0, \lambda_0] \), where \( \lambda_0 \) is an arbitrarily fixed positive number. On the other hand, using the obvious relation
\[
\int_0^T (T - t) u_\lambda''(t) \, dt = -T u_\lambda'(0) - x,
\]
and (2.19), we derive
\[
\|u_\lambda'(0)\| \leq C_8\|u_\lambda''\|_{X_T} + C_9
\leq C_8 \sqrt{C_6\|u_\lambda'(0)\| + C_7 + C_9}
\leq C_{10} \sqrt{\|u_\lambda'(0)\| + C_{11}},
\]
which shows that \( \sup_{0 < \lambda \leq \lambda_0} \|u_\lambda'(0)\| < \infty \). So, according to (2.15), (2.18), and (2.19), the sequences \( u_\lambda', u_\lambda'', A_\lambda u_\lambda \) \( (0 < \lambda \leq \lambda_0) \) are all bounded in \( X_T \). Now, for \( \lambda, \mu \in (0, \lambda_0] \), we derive from (2.5)
\[
\int_0^T \left( (a(u_\lambda' - u_\mu'))', u_\lambda - u_\mu \right) \, dt = \int_0^T b(A_\lambda u_\lambda - A_\mu u_\mu + \lambda u_\lambda - \mu u_\mu, u_\lambda - u_\mu) \, dt,
\]
which implies
\[
- \int_0^T a\|u_\lambda' - u_\mu'\|^2 \, dt = \int_0^T b(A_\lambda u_\lambda - A_\mu u_\mu, J_\lambda u_\lambda - J_\mu u_\mu) \, dt + \int_0^T b(A_\lambda u_\lambda - A_\mu u_\mu, \lambda A_\lambda u_\lambda - \mu A_\mu u_\mu) \, dt + \int_0^T b(\lambda u_\lambda - \mu u_\mu, u_\lambda - u_\mu) \, dt.
\]
(2.21)
The first term of the right hand side of (2.21) is nonegative since \( A_\lambda u_\lambda(t) \in AJ_\lambda u_\lambda(t) \), so by the information above we easily obtain
\[
\|u_\lambda' - u_\mu'\|_{X_T} \leq C_{12}(\lambda + \mu).
\]
(2.22)
We also have
\[
\|u_\lambda(t) - u_\mu(t)\| = \|\int_0^t (u_\lambda' - u_\mu') \, dt\|
\leq C_{13}\|u_\lambda' - u_\mu'\|_{X_T}, \quad 0 \leq t \leq T.
\]
(2.23)
Therefore, there exists \( u \in W^{2,2}(0, T; H) \), such that \( u_\lambda \to u \) in \( C([0, T]; H) \), \( u_\lambda' \to u' \) strongly in \( X_T \), and \( u_\lambda'' \to u'' \) weakly in \( X_T \), as \( \lambda \to 0^+ \). Obviously, \( u(0) = x \) and \( u(T) = 0 \). This \( u \) is also a solution to equation (2.1). Indeed, we can pass to the limit as \( \lambda \to 0^+ \) in (2.3) regarded as an equation in \( X_T \). Note that \( A_\lambda u_\lambda(t) \in AJ_\lambda u_\lambda(t) \), for all \( t \in [0, T] \), and
\[
\|J_\lambda u_\lambda - u\|_{X_T} \leq \lambda\|A_\lambda u_\lambda\|_{X_T} + \|u_\lambda - u\|_{X_T} \to 0, \quad \text{as} \ \lambda \to 0^+,
\]
so by the demiclosedness of \( \bar{A} \), the weak limit in \( X_T \) of \( A_\lambda u_\lambda \) belongs to \( Au \), i.e.,

\[-f + pu'' + qu' \in Au \quad \text{for a.a.} \ t \in (0, T).\]

It remains to prove that \( u \) is unique. Let \( v \in W^{2,2}(0, T; H) \) be another solution of problem (2.1), (2.2). We have

\[
\left( a(u' - v') \right)' \in b(Au - Av) \quad \text{for a.a.} \ t \in (0, T).
\]

Multiplying (2.24) by \( u(t) - v(t) \) and integrating over \([0, T]\) we obtain

\[-\int_0^T a\|u' - v'\|^2 \, dt \geq 0,
\]

which shows that \( u' - v' \equiv 0 \), i.e., \( u - v \) is a constant function. Since \( u(0) - v(0) = 0 \), it follows that \( u \equiv v \). \( \square \)

**Remark 2.3.** For similar results we refer to [1]. Note that here \( p \) and \( q \) satisfy weaker conditions. Note further that, not only the conclusion of Lemma 2.2, but also some steps of its proof will be used later.

**Lemma 2.4.** Assume that \( A \) satisfies (H1), \( p, q \in L^\infty(0, T) \), with \( \text{ess inf} \ p > 0 \), \( f \in L^2(0, T; H) \), and \( x, y \in D(A) \). For \( \lambda > 0 \) denote by \( u_\lambda \) the unique solution of

\[
p(t)u_\lambda''(t) + q(t)u'_\lambda(t) = A_\lambda u_\lambda(t) + f(t) \quad \text{for a.a.} \ t \in (0, T),
\]

\[
u_\lambda(0) = x, \quad u_\lambda(T) = y
\]

(which exists by Lemma 2.2). Then, \( u_\lambda \to u \) in \( C([0, T]; H) \) as \( \lambda \to 0^+ \), where \( u \) is the solution of problem (2.1), (2.2). Moreover, \( u'_\lambda \to u' \) in \( C([0, T]; H) \) and \( u''_\lambda \to u'' \) weakly in \( X_T \), as \( \lambda \to 0^+ \).

**Proof.** Following a procedure similar to that used in the proof of Lemma 2.2, we obtain \( u_\lambda \to u \) in \( C([0, T]; H) \), \( u'_\lambda \to u' \) strongly in \( X_T \), and \( u''_\lambda \to u'' \) weakly in \( X_T \), as \( \lambda \to 0^+ \). So actually \( u'_\lambda \to u' \) in \( C([0, T]; H) \) (by Arzelà’s criterion). \( \square \)

### 3 Main Results

We start this section by defining the concepts of strong and weak solution for equation (E) (respectively, equation (E) plus condition (B)) we shall use in what follows.

Note that in general we shall work under our assumptions (H1) and (H2) introduced in Section 1. For an interval \( J \subset \mathbb{R} \), open or not, denote by \( L^p_{\text{loc}}(J; H) \) (resp. \( W^{k,p}_{\text{loc}}(J; H) \)) the space of all \( H \)-valued functions defined on \( J \), whose restrictions to compact intervals \([a, b] \subset J\) belong to \( L^p(a, b; H) \) (respectively, to \( W^{k,p}(a, b; H) \)).
**Definition 3.1.** Let $f \in L^2_{\text{loc}}([0, \infty); H)$ and let $x \in \overline{D(A)}$. A $H$-valued function $u = u(t)$ is said to be a **strong solution** of equation (E) (respectively, of equation (E) plus condition (B)) if $u \in C([0, \infty); H) \cap W^{2,2}_{\text{loc}}((0, \infty); H)$ and $u(t)$ satisfies equation (E) for a.a. $t > 0$ (and, in addition, $u(0) = x$, respectively).

Denote $Y = L^1(0, \infty; H; t \sqrt{a_-(t)} dt)$, where $a_-(t) = \exp\left(-\int_0^t \frac{a^{-\prime}(\tau)}{p(\tau)} d\tau\right)$. Obviously, $Y$ is real Banach space with respect to the norm

$$\|f\|_Y = \int_0^\infty \|f(t)\| t \sqrt{a_-(t)} dt.$$ 

If $f \in Y$ we cannot expect in general existence of strong solutions for (E), so we need the following definition.

**Definition 3.2.** Let $f \in Y$ and let $x \in \overline{D(A)}$. A $H$-valued function $u = u(t)$ is said to be a **weak solution** of equation (E) (respectively, of equation (E) plus condition (B)) if there exist sequences $u_n \in C([0, \infty); H) \cap W^{2,2}_{\text{loc}}((0, \infty); H)$ and $f_n \in Y \cap L^2_{\text{loc}}([0, \infty); H)$, such that: (i) $f_n$ converges to $f$ in $Y$; (ii) $u_n(t)$ satisfies equation (E) with $f = f_n$ for a.a. $t > 0$ and all $n \in \mathbb{N}$; and (iii) $u_n$ converges uniformly to $u$ on any compact interval $[0, T]$ (and, in addition, $u(0) = x$, respectively).

The concept of a weak solution for such second order differential inclusions was previously introduced in [19] in the case $p \equiv 1$ and $q \equiv 0$.

Note that the couple (E), (B) is an incomplete problem. While in [25], [26] we added the condition $u \in L^2(\mathbb{R}^+; H; b(t) dt)$, in this paper we consider a boundedness condition on $\mathbb{R}^+: \sqrt{a_-} \|u\| \in L^\infty(\mathbb{R}^+)$.  

**Theorem 3.3.** Assume (H1) and (H2) hold. If $x \in D(A)$, and $f \in Y \cap L^2_{\text{loc}}([0, \infty); H)$, then there exists a unique strong solution $u$ of (E), (B) which satisfies

$$\sup_{t \geq 0} a_-(t) \|u(t)\|^2 < \infty. \quad (C)$$

Moreover, $\sqrt{a_-} u' \in L^2(\mathbb{R}^+; H)$ and $t^{3/2} u'' \in L^2_{\text{loc}}([0, \infty); H)$. If, in addition, $x \in D(A)$, then $u \in W^{2,2}_{\text{loc}}([0, \infty); H)$.

**Proof.** Let us assume in a first stage that $x \in D(A)$ (and $f \in Y \cap L^2_{\text{loc}}([0, \infty); H)$, as specified in the statement of the theorem). For each $\lambda > 0$ and $n \in \mathbb{N}$, denote by $u_{n\lambda}$, $u_n$ the solutions of the following problems

$$pu_{n\lambda}'' + qu_{n\lambda}' = A_\lambda u_{n\lambda} + f \quad \text{a.e. in } (0, n), \quad (3.1)$$

$$u_{n\lambda}(0) = x, \quad u_{n\lambda}(n) = 0, \quad (3.2)$$
and
\[
pu_n'' + qu_n' \in Au_n + f \quad \text{a.e. in } (0, n), \tag{3.3}
\]
\[
u_n(0) = x, \quad u_n(n) = 0. \tag{3.4}
\]
Lemma 2.2 ensures the existence and uniqueness of \(u_{n\lambda}, u_n \in W^{2,2}(0, n; H)\). By a computation similar to that performed in Lemma 2.2 (see (2.10)), we get
\[
a_-(t)\|u_{n\lambda}(t)\|^2 \leq \|x\|^2 + 2\int_0^n \tau b\|f\| \cdot \|u_{n\lambda}\| d\tau, \quad 0 \leq t \leq n. \tag{3.5}
\]
Denoting \(M_{n\lambda} = \sup_{0 \leq t \leq n} \sqrt{a_-(t)}\|u_{n\lambda}(t)\|\), we can derive from (3.5) the following quadratic inequality
\[
M_{n\lambda}^2 \leq \|x\|^2 + 2\int_0^{\infty} \frac{\tau}{p_0} \|a_+ \sqrt{a_-} \|f\|Y_{M_{n\lambda}}, \tag{3.6}
\]
where \(a_+(t) = \exp \left( \int_0^t \sqrt{a_-} \frac{d\tau}{p} \right)\) and \(p_0 = \text{ess inf } p > 0\), which shows that \(M_{n\lambda} \leq E = E(x, f) := D + \sqrt{D^2 + \|x\|^2}, D := a_+(\infty) \|f\|Y\). Thus,
\[
\sup_{0 \leq t \leq n} a_-(t)\|u_{n\lambda}(t)\|^2 \leq E^2. \tag{3.7}
\]
Similarly, it follows from (3.3), (3.4)
\[
\sup_{0 \leq t \leq n} a_-(t)\|u_n(t)\|^2 \leq E^2. \tag{3.8}
\]
Now, let \(0 < R < m < n\), with \(m, n \in \mathbb{N}\). For a.a. \(t \in (0, m)\) we have
\[
\frac{1}{2} \frac{d}{dt} \left[ a_+ d \frac{d}{dt} \|u_n - u_m\|^2 \right] = \frac{d}{dt} \left[ a_+ d \left( u_n' - u_m', u_n - u_m \right) \right] = \left( (a_+ u_n' - u_m')' \right)' = 4a \|u_n' - u_m'\|^2. \tag{3.9}
\]
By (3.9) we have
\[
\int_0^m (m-t)a(t)\|u_n' - u_m'\|^2 dt \leq \frac{1}{2} \int_0^m (m-t) \frac{d}{dt} \left[ a_+ d \frac{d}{dt} \|u_n - u_m\|^2 \right] dt = 0 + \frac{1}{2} \int_0^m a\left[ d \frac{d}{dt} \|u_n - u_m\|^2 \right] dt = \frac{1}{2} a\left[ d \frac{d}{dt} \|u_n - u_m\|^2 \right] \bigg|_0^m - \frac{1}{2} \int_0^m \frac{aq}{p} \|u_n - u_m\|^2 dt \\
\leq \frac{1}{2} a(m)\|u_n(m)\|^2 + \frac{1}{2} \int_0^m \frac{aq}{p} \|u_n - u_m\|^2 dt. \tag{3.10}
\]
As in the proof of Lemma 2.2, we can derive an inequality for \( u_k \) similar to (2.9) and therefore (see (3.8)) we have for all \( t \in [0, k] \)

\[
\int_0^t \frac{aq^-}{p} \|u_k\|^2 d\tau \leq (\|x\|^2 + 2 \int_0^k \tau b\|f\| \cdot \|u_k\| d\tau) + \int_0^t \frac{aq^+}{p} \|u_k\|^2 d\tau \\
\leq (\|x\|^2 + 2DE) + E^2a_+(\infty)\left(\int_0^\infty \frac{q^+}{p} d\tau\right) := F < \infty. \tag{3.11}
\]

Combining (3.8), (3.10) and (3.11) leads to

\[
(m - R) \int_0^R a(t)\|u'_n - u'_m\|^2 dt \leq \int_0^m (m - t)a(t)\|u'_n - u'_m\|^2 dt \\
\leq \frac{1}{2} E^2a_+(\infty) + 2F. \tag{3.12}
\]

This shows that \( (u'_n) \) is a Cauchy sequence in \( X_R = L^2(0, R; H) \). Therefore \( (u_n) \) is Cauchy in \( C([0, R]; H) \), since

\[
\|u_n(t) - u_m(t)\| = \| \int_0^t [u'_n(\tau) - u'_m(\tau)] d\tau \| \leq \text{Const.} \|u'_n - u'_m\|_{X_R}, \quad 0 \leq t \leq R.
\]

Since \( R \) was arbitrarily chosen, it follows that there exists a function \( u \in C([0, \infty); H) \) \( \cap W^{1,2}_{loc}([0, \infty); H) \), such that \( u_n \to u \) in \( C([0, R]; H) \) (so \( u(0) = x \)) and \( u'_n \to u' \) in \( X_R \), for all \( R > 0 \). In addition, \( a_-\|u\|^2 \in L^\infty(\mathbb{R}_+) \) (cf. (3.8)).

We intend to show that \( u \) is a strong solution to equation \( (E) \) by passing to the limit in equation (3.3) as \( n \to \infty \). To this purpose we establish next a local \( L^2 \)-estimate for \( u''_n \).

By a reasoning from the proof of Lemma 2.2 (see (2.14)), we have

\[
\int_0^n a\|u'_{n\lambda}\|^2 dt \leq -(u'_\lambda(0), x) + \int_0^n b\|f\| \cdot \|u_{n\lambda}\| dt. \tag{3.13}
\]

For a given \( R > 0 \), arbitrary but fixed, we derive from (3.7) and (3.13)

\[
\int_0^R a\|u'_{n\lambda}\|^2 dt \leq \|x\| \cdot \|u'_{n\lambda}(0)\| + \left(\int_0^R + \int_R^n\right) b\|f\| \cdot \|u_{n\lambda}\| dt \\
\leq \|x\| \cdot \|u'_{n\lambda}(0)\| + Ea_+(\infty)\left(\int_0^R \sqrt{a_-} \|f\| dt + \frac{1}{R} \int_R^\infty \frac{\sqrt{a_-}}{p} \|f\| dt\right) \\
\leq \|x\| \cdot \|u'_{n\lambda}(0)\| + K_1, \tag{3.14}
\]

where \( K_1 \) is a positive constant (depending on \( R, x, \) and \( f \)). Now, multiplying the equation
\((au_{n\lambda}')' = b(A_{\lambda}u_{n\lambda} + f)\) by \((R - t)^3A_{\lambda}u_{n\lambda}\) and integrating over \([0, R]\), we obtain
\[
\int_0^R (R - t)^3b\|A_{\lambda}u_{n\lambda}\|^2 dt
= \int_0^R \left((au_{n\lambda}')', (R - t)^3A_{\lambda}u_{n\lambda}\right) dt - \int_0^R (R - t)^3b(f, A_{\lambda}u_{n\lambda}) dt
= - R^3(u_{n\lambda}'(0), A_{\lambda}x) - \int_0^R (R - t)^3a(u_{n\lambda}', (A_{\lambda}u_{n\lambda})') dt
+ 3 \int_0^R (R - t)^2a(u_{n\lambda}', A_{\lambda}u_{n\lambda}) dt - \int_0^R (R - t)^3b(f, A_{\lambda}u_{n\lambda}) dt
\leq R^3\|A^0x\| \cdot \|u_{n\lambda}'(0)\| - \{\text{nonnegative term}\}
+ 3 \int_0^R a((R - t)^2u_{n\lambda}', (R - t)^2A_{\lambda}u_{n\lambda}) dt + \int_0^R (R - t)^2 + \frac{2}{3}b\|f\| \cdot \|A_{\lambda}u_{n\lambda}\| dt
\leq R^3\|A^0x\| \cdot \|u_{n\lambda}'(0)\| + 3 \left( \int_0^R (R - t)a\|u_{n\lambda}'\|^2 dt \right)^{\frac{1}{2}} \left( \int_0^R (R - t)^3a\|A_{\lambda}u_{n\lambda}\|^2 dt \right)^{\frac{1}{2}}
+ \left( \int_0^R (R - t)^3b\|A_{\lambda}u_{n\lambda}\|^2 dt \right)^{\frac{1}{2}} \left( \int_0^R (R - t)^3b\|f\|^2 dt \right)^{\frac{1}{2}}.
\tag{3.15}
\]
By (3.14) and (3.15) it follows that
\[
\int_0^R (R - t)^3b\|A_{\lambda}u_{n\lambda}\|^2 dt \leq K_2\|u_{n\lambda}'(0)\| + K_3.
\tag{3.16}
\]
Denoting \(Z_R = L^2(0, R/2; H)\), we derive from (3.14) and (3.16)
\[
\|u_{n\lambda}'\|^2_{Z_R} \leq K_4\|u_{n\lambda}'(0)\| + K_5, \quad \|A_{\lambda}u_{n\lambda}\|^2_{Z_R} \leq K_6\|u_{n\lambda}'(0)\| + K_7,
\tag{3.17}
\]
so that, according to (3.1), we also have
\[
\|u_{n\lambda}\|^2_{Z_R} \leq K_8\|u_{n\lambda}'(0)\| + K_9.
\tag{3.18}
\]
On the other hand,
\[
\frac{R}{2}u_{n\lambda}'(0) = u_{n\lambda}(R/2) - x - \int_0^{R/2} \left( \frac{R}{2} - t \right)u_{n\lambda}''(t) dt,
\]
hence (see (3.7) and (3.18))
\[
\|u_{n\lambda}'(0)\| \leq K_{10}\|u_{n\lambda}''\|_{Z_R} + K_{11}
\leq K_{12}\sqrt{\|u_{n\lambda}'(0)\| + K_{13}},
\tag{3.19}
\]
which shows that \(\|u_{n\lambda}'(0)\| \leq K_{14}\), where \(K_{14}\) is a constant depending on \(R, x, f\).
Therefore, according to (3.18), we have
\[
\|u_{n\lambda}''\|^2_{Z_R} \leq K_{15} := K_8K_{14} + K_9.
\tag{3.20}
\]
By Lemma 2.4 we know that, for each \( n \in \mathbb{N} \), \( u''_{n\lambda} \) converges weakly in \( L^2(0, n; H) \) (hence in particular in \( Z_R \)) to \( u''_n \) as \( \lambda \to 0^+ \). This piece of information combined with (3.20) shows that \( (u''_n) \) is bounded in \( Z_R \). Now, we are in a position to take to the limit in (3.3), regarded as an equation in \( Z_R \), to deduce that \( u \) (the limit of \( u_n \) in \( C([0, R/2]; H) \), hence in \( Z_R \)) belongs to \( W^{2,2}(0, R/2; H) \) and satisfies equation (E) for a.a. \( t \in (0, R/2) \).

We have used the demiclosedness of the realization of \( A \) in \( Z_R \) (see also the proof of Theorem 3.1 in [19]). Since \( R \) was arbitrarily chosen, this completes the proof of the theorem in the case \( x \in D(A) \).

Now we assume that \( x \in \overline{D(A)} \) (and \( f \in Y \cap L^2_{\text{loc}}([0, \infty); H) \), as specified in the statement of the theorem). Let \( x_k \in D(A) \) such that \( \|x_k - x\| \to 0 \). For each \( k \) denote by \( u_k \) the strong solution of (E) satisfying \( u_k(0) = x_k \) and \( \sqrt{a_-}\|u_k\| \in L^\infty(\mathbb{R}_+) \) (whose existence is ensured by the previous part of the proof). For each \( k \) let \( u_{kn} \), \( u_{kn\lambda} \) be the corresponding approximations (see problems (3.3), (3.4) and (3.1), (3.2) above).

First, we have for a.a. \( t \in (0, n) \)

\[
\frac{1}{2} \frac{d}{dt} \left[ a(\frac{d}{dt}) \|u_{kn} - u_{jn}\|^2 \right] \geq a\|u'_{kn} - u'_{jn}\|^2,
\]

which is similar to (3.9) above. Hence the function \( t \to a(t) \frac{d}{dt} \|u_{kn}(t) - u_{jn}(t)\|^2 \) is non-decreasing on \([0, n]\). Since it equals zero at \( t = n \), it follows that it is \( \leq 0 \) for all \( t \in [0, n] \), hence the function \( t \to \|u_{kn}(t) - u_{jn}(t)\| \) is nonincreasing on \([0, n]\). In particular,

\[
\|u_{kn}(t) - u_{jn}(t)\| \leq \|x_k - x_j\| \ \forall t \in [0, n].
\]

Therefore,

\[
\|u_k(t) - u_j(t)\| \leq \|x_k - x_j\| \ \forall t \geq 0,
\]

which shows that there exists a function \( u \in C([0, \infty); H) \) such that \( u_k \) converges to \( u \) in \( C([0, R]; H) \) for all \( R \in (0, \infty) \), so in particular \( u(0) = x \). According to (3.3) (where \( E = E(x_k, f) \) is bounded), we also have \( \sqrt{a_-}\|u\| \in L^\infty(\mathbb{R}_+) \).

On the other hand, we have for a.a. \( t \in (0, n) \)

\[
\frac{1}{2} \frac{d}{dt} \left[ a(t) \frac{d}{dt} \|u_{kn}(t)\|^2 \right] = \frac{d}{dt} \left( au'_{kn}, u_{kn} \right) = \left( (au'_{kn})', u_{kn} \right) + a\|u'_{kn}\|^2 \\
\geq b(f, u_{kn}) + a\|u'_{kn}\|^2 \\
\geq -b\|f\| \cdot \|u_{kn}\| + a\|u'_{kn}\|^2 \\
\geq -\frac{1}{p_0} \left( \sqrt{a_-}\|f\| \right) \cdot \left( \sqrt{a_-}\|u_{kn}\| \right) + a\|u'_{kn}\|^2 \\
\geq -\frac{E(x_k, f)}{p_0} \left( \sqrt{a_-}\|f\| \right) + a\|u'_{kn}\|^2,
\]

(3.24)
where \( p_0 = \text{ess inf} \ p > 0 \). To obtain the last inequality we have used (3.8). Now we multiply (3.24) by \( t \) and then integrate over \([0, n]\) to derive

\[
\int_0^n t a \|u_{kn}'\|^2 dt \leq M - \frac{1}{2} \int_0^n \frac{d}{dt} \|u_{kn}\|^2 dt \\
\leq M + \frac{1}{2} \|x_k\|^2 + \int_0^n \frac{aq}{p} \|u_{kn}\|^2 dt \\
\leq M + \frac{1}{2} \|x_k\|^2 + \frac{1}{p_0} \int_0^n a_+ q^+ (a_- \|u_{kn}\|^2) dt \\
\leq M + \frac{1}{2} \|x_k\|^2 + \frac{1}{p_0} E(x_k, f)^2 a_+ (\infty) \|q^+\|_{L^1(\mathbb{R}_+)} \\
\leq M_1,
\]

where \( M \) and \( M_1 \) are finite constants, since \( \|x_k\| \) and \( E(x_k, f) \) are bounded sequences.

Therefore,

\[
\int_0^\infty t a \|u_k'\|^2 dt \leq M_1,
\]

i.e., the sequence \( (\sqrt{t a_-} u_k') \) is bounded in \( L^2(\mathbb{R}_+; H) \). In fact this sequence is convergent in \( L^2(\mathbb{R}_+; H) \) (and so its limit \( \sqrt{t a_-} u' \in L^2(\mathbb{R}_+; H) \)). Indeed, multiplying (3.21) by \( t \) and then integrating the resulting inequality over \([0, n]\), we get

\[
\int_0^n t a(t) \|u_{kn}' - u_{jn}'\|^2 dt \leq - \frac{1}{2} \int_0^n a(t) \frac{d}{dt} \|u_{kn}' - u_{jn}'\|^2 dt \\
= \frac{1}{2} \|x_k - x_j\|^2 + \frac{1}{2} \int_0^n \frac{aq}{p} \|u_{kn} - u_{jn}\|^2 dt \\
\leq \frac{1}{2} \|x_k - x_j\|^2 + \frac{1}{2} \int_0^n \frac{a_+ q^+}{p} \|u_{kn} - u_{jn}\|^2 dt \\
\leq \frac{1}{2} \left[ 1 + \frac{1}{p_0} a_+ (\infty) \|q^+\|_{L^1(\mathbb{R}_+)} \right] \|x_k - x_j\|^2.
\]

To derive the last inequality we have used inequality (3.22). From (3.27) it follows that

\[
\int_0^\infty t a(t) \|u_k' - u_j'\|^2 dt \leq c \|x_k - x_j\|^2,
\]

where \( c = \frac{1}{2} \left[ 1 + \frac{1}{p_0} a_+ (\infty) \|q^+\|_{L^1(\mathbb{R}_+)} \right] \), which shows that \( (\sqrt{t a_-} u_k') \) is Cauchy, hence convergent, in \( L^2(\mathbb{R}_+; H) \), as asserted.

In the following we shall prove that \( t^{3/2} u'' \in L^2_{loc}([0, \infty); H) \) and \( u \) is a strong solution of equation (E). To this purpose, it is enough to establish a local \( L^2 \)-estimate for \( u_k' \). We need to use \( u_{kn\lambda} \), the solution of

\[
(au_{kn\lambda}')' = b(A_\lambda u_{kn\lambda} + f) \quad \text{a.e. in } (0, n); \quad u_{kn\lambda}(0) = x_k, \quad u_{kn\lambda}(n) = 0.
\]
For a given \( R > 0, R < n, \) define \( h_R(t) = \min\{t, R-t\}, 0 \leq t \leq R. \) Multiplying equation (3.29) by \( h_R^3(t)A_\lambda u_{kn\lambda}(t) \) and integrating over \([0, R]\), we get the following inequality (which is similar to (3.15) above)

\[
\int_0^R h_R^3 b\|A_\lambda u_{kn\lambda}\|^2 dt \leq -3 \int_0^R h_R^2 h_R'(u_{kn\lambda}', A_\lambda u_{kn\lambda}) dt - \int_0^R h_R^3 b(f, A_\lambda u_{kn\lambda}) dt
\]

\[
\leq 3 \int_0^R a h_R^{3/2} \|A_\lambda u_{kn\lambda}\| \cdot h_R^{1/2} \|u_{kn\lambda}'\| dt
\]

\[
+ \int_0^R h_R^3 b(f) \cdot \|A_\lambda u_{kn\lambda}\| dt.
\]

(3.30)

Here, following an idea from [12], we have used the function \( h_R \) in order to get rid of the term involving \( A_\lambda x_k \) which is no longer bounded. Of course, the information we get is a bit weaker, but enough to ensure that \( u \) is a strong solution. Arguing as before (see (3.25)), we find that

\[
\int_0^R t a\|u_{kn\lambda}'\|^2 dt \leq M_1,
\]

where \( M_1 \) is the same constant as in (3.25). It follows that

\[
\int_0^R h_R a\|u_{kn\lambda}'\|^2 dt \leq M_1.
\]

(3.31)

By (3.30) and (3.31) we see that \( \{t^{3/2}A_\lambda u_{kn\lambda}; \lambda > 0\} \) is uniformly bounded in \( L^2(0, R/2; H) \) and so is \( \{t^{3/2}u_{kn\lambda}''; \lambda > 0\} \) (by using the equation \( pu_{kn\lambda}'' + qu_{kn\lambda}' = A_\lambda u_{kn\lambda} + f \)). Consequently, the sequence \( \{t^{3/2}u_k''\} \) is also bounded in \( L^2(0, R/2; H) \). For a small \( \delta > 0, \) denote \( Z_{\delta,R} = L^2(\delta, R/2; H) \). In this space, \( u_k \) converges strongly to \( u, u_k' \) converges strongly to \( u' \) (cf. (3.28)), while \( u_k'' \) converges weakly to \( u'' \) (in fact, \( t^{3/2}u_k'' \in L^2(0, R/2; H) \)). Passing to the limit in the equation

\[
pu_k'' + qu_k' \in Au_k + f,
\]

regarded as an equation in \( Z_{\delta,R}, \) we see that \( u \) satisfies equation (E) for a.a. \( t \in (\delta, R/2), \) hence for a.a. \( t > 0, \) since \( \delta \) and \( R \) were arbitrarily chosen.

To complete the proof, let us show that \( u \) is unique. Assume that \( v \) is another strong solution of (E), (E) satisfying \( \sqrt{a_+} \|v\| \in L^\infty(\mathbb{R}_+). \) Then, for a.a. \( t > 0, \)

\[
\frac{1}{2} \frac{d}{dt} \left[ a(t) \frac{d}{dt} \|u(t) - v(t)\|^2 \right] = \frac{d}{dt} \left( a(u' - v'), u - v \right)
\]

\[
= \left( (a(u' - v'))', u - v \right) + a\|u' - v'\|^2
\]

\[
\geq a\|u' - v'\|^2
\]

\[
\geq 0.
\]

(3.32)
This shows that the function \( t \to a(t) \frac{d}{dt}\|u(t) - v(t)\|^2 \) is nondecreasing on \([0, \infty)\). It is also nonnegative, since it vanishes at \( t = 0 \). Hence, for \( 0 < t < s \) we have
\[
0 \leq a(t) \frac{d}{dt}\|u(t) - v(t)\|^2 \leq a(s) \frac{d}{ds}\|u(s) - v(s)\|^2.
\] (3.33)

Integrating (3.33) with respect to \( s \) over \([t, T]\) yields
\[
(T - t) a(t) \frac{d}{dt}\|u(t) - v(t)\|^2 \leq a(T)\|u(T) - v(T)\|^2 - \int_t^T \frac{q(s)}{p(s)} a(s)\|u(s) - v(s)\|^2 ds
\]
\[
\leq K + K \int_t^T \frac{q^-(s)}{p(s)} ds
\]
\[
\leq K + M(T - t),
\] (3.34)

where \( K \) is a positive constant (depending on \( u \) and \( v \)), and \( M = K \cdot \text{ess sup} \frac{q^-}{p} < \infty \).

Dividing (3.34) by \( T - t \) and letting \( T \to \infty \), we get
\[
0 \leq a(t) \frac{d}{dt}\|u(t) - v(t)\|^2 \leq M \ \forall t \geq 0.
\] (3.35)

Combining (3.32) and (3.35) leads to
\[
\int_0^\infty a(t)\|u'(t) - v'(t)\|^2 dt \leq \frac{M}{2},
\]
which implies
\[
\liminf_{t \to \infty} \sqrt{a(t)} \|u'(t) - v'(t)\| = 0.
\] (3.36)

Since
\[
0 \leq a(t) \frac{d}{dt}\|u(t) - v(t)\|^2 = 2a(t)(u'(t) - v'(t), u(t) - v(t))
\]
\[
\leq 2\sqrt{K} \sqrt{a(t)}\|u'(t) - v'(t)\|,
\]
it follows by (3.36) that
\[
\lim_{t \to \infty} a(t) \frac{d}{dt}\|u(t) - v(t)\|^2 = 0,
\]
which clearly implies that \( a(t) \frac{d}{dt}\|u(t) - v(t)\| = 0 \ \forall t \geq 0 \). Hence \( \|u(t) - v(t)\| = \|u(0) - v(0)\| = 0 \ \forall t \geq 0 \). □

**Theorem 3.4.** Assume (H1) and (H2) hold. Then, for each \( x \in \overline{D(A)} \) and \( f \in Y \), there exists a unique weak solution \( u \) of (E), (F), (C), and \( \sqrt{ta_-}u' \in L^2(\mathbb{R}_+; H) \).

**Proof.** Let \( x \in \overline{D(A)} \) and let \( f_1, f_2 \in Y \cap L^2_{loc}([0, \infty); H) \). Denote by \( u(t, x, f_i) \), \( i = 1, 2 \), the corresponding strong solutions given by Theorem 3.3 and by \( u_n(t, x, f_i) \) their
approximations \((i = 1, 2, \ n \in \mathbb{N})\), as defined above (see (3.3) and (3.4)). It is easily seen that for a.a. \(t \in (0, n)\)

\[
\frac{1}{2} \frac{d}{dt} \left[ a(t) \frac{d}{dt} \|u_n(t; x, f_1) - u_n(t; x, f_2)\|^2 \right] \geq -b(t) \|f_1(t) - f_2(t)\| \cdot \|u_n(t; x, f_1) - u_n(t; x, f_2)\|. \tag{3.37}
\]

Integrating (3.37) over \([s, n]\) yields

\[
a(s) \frac{d}{ds} \|u_n(s; x, f_1) - u_n(s; x, f_2)\|^2 \leq 2 \int_s^n b \|f_1 - f_2\| \cdot \|u_n(\tau; x, f_1) - u_n(\tau; x, f_2)\| d\tau.
\]

A new integration, this time over \([0, t]\), leads to

\[
\int_0^t a(s) \frac{d}{ds} \|u_n(s; x, f_1) - u_n(s; x, f_2)\|^2 ds \leq 2 \int_0^n \int_s^n b \|f_1 - f_2\| \cdot \|u_n(\tau; x, f_1) - u_n(\tau; x, f_2)\| d\tau ds.
\]

Therefore,

\[
a(t) \|u_n(t; x, f_1) - u_n(t; x, f_2)\|^2 \leq C_n + \int_0^t \frac{q^+(s)}{p(s)} a(s) \|u_n(s; x, f_1) - u_n(s; x, f_2)\|^2 ds, \tag{3.38}
\]

where

\[
C_n = 2 \int_0^n sb \|f_1 - f_2\| \cdot \|u_n(s; x, f_1) - u_n(s; x, f_2)\| ds.
\]

Using the Gronwall-Bellman lemma, we derive from (3.38)

\[
a_-(t) \|u_n(t; x, f_1) - u_n(t; x, f_2)\|^2 \leq C_n \leq \frac{2 \ a_+ (\infty)}{p_0} \int_0^n sa_-(s) \|f_1 - f_2\| \cdot \|u_n(s; x, f_1) - u_n(s; x, f_2)\| ds.
\]

Recall that \(p_0 = \text{ess inf } p\). This implies

\[
\sqrt{a_-}(t) \|u_n(t; x, f_1) - u_n(t; x, f_2)\| \leq C \int_0^n s \sqrt{a_-}(s) \|f_1 - f_2\| ds, \tag{3.39}
\]

where \(C = 2a_+ (\infty)/2\). This leads to

\[
\sqrt{a_-(t)}\|u(t; x, f_1) - u(t; x, f_2)\| \leq C \int_0^\infty s \sqrt{a_-}(s) \|f_1 - f_2\| ds
\]

\[= C\|f_1 - f_2\|_Y, \quad \forall t \geq 0. \tag{3.40}
\]
From inequality (3.40), we can easily derive the existence of a unique weak solution \( u(t;x,f) \) for each \( x \in D(A) \) and \( f \in X \). Indeed, it is sufficient to observe that \( f \) can be approximated (with respect to the norm of \( Y \)) by a sequence \( (f_k) \) of smooth functions with compact support \( \subset (0, \infty) \) and use (3.40) with \( f_1 := f_k \) and \( f_2 := f_j \). Note that (3.25) holds for \( u'_n(t;x,f_k) \) with \( E(x,f_k) \) (which is also bounded), so (3.26) also holds true for \( u'(t;x,f_k) \). Therefore, \( \sqrt{ta_u'} \in L^2(\mathbb{R}_+;H) \) (as the weak limit in \( L^2(\mathbb{R}_+;H) \) of the sequence \( (\sqrt{ta_u'}) \)). This completes the proof of the theorem.

\[ \square \]

**Remark 3.5.** If we assume the stronger condition \( q \in L^1(\mathbb{R}_+) \), then condition (C) becomes

\[
\sup_{t \geq 0} \|u(t)\| < \infty. \tag{C1}
\]

In this case we can state the following result

**Corollary 3.6.** Assume that \((H1)\) holds, \( p \in L^\infty(\mathbb{R}_+) \), with \( \text{ess inf } p > 0 \), and \( q \in L^\infty(\mathbb{R}_+) \cap L^1(\mathbb{R}_+) \). Then, for each \( x \in D(A) \) and \( f \in Y := L^1(0, \infty;H;tdt) \), there exists a unique weak solution \( u \) of problem (E1), (E2), (C1), satisfying \( t^{1/2}u' \in L^2(\mathbb{R}_+;H) \). If in addition \( f \in L^2_{loc}(0, \infty; H) \), then \( u \) is a strong solution satisfying \( t^{3/2}u'' \in L^2_{loc}(0, \infty; H) \). If further \( x \in D(A) \), then \( u \in W^{2,2}_{loc}(0, \infty; H) \).

**Proof.** By the uniqueness property it follows that every strong solution of problem (E1), (E2), (C1), associated with \((x,f) \in D(A) \times [X \cap L^2_{loc}(0, \infty; H)]\), denoted \( u(t;x,f) \), can be obtained by the limiting procedure developed in the proof of Theorem 3.3. By (3.23) and (3.40), we have

\[
\|u(t;x_1,f_1) - u(t;x_2,f_2)\| \leq \|u(t;x_1,f_1) - u(t;x_2,f_1)\| + \|u(t;x_2,f_1) - u(t;x_2,f_2)\| \\
\leq \|x_1 - x_2\| + \hat{C}\|f_1 - f_2\|_Y \quad \forall t \geq 0, \tag{3.41}
\]

which is valid for all \((x_i,f_i) \in D(A) \times [Y \cap L^2_{loc}(0, \infty; H)], i = 1, 2 \) (i.e., for strong solutions), and can be extended to all \((x_i,f_i) \in D(A) \times Y\). Inequality (3.41) shows that for each pair \((x,f) \in D(A) \times Y\) there exists a unique weak solution \( u(t;x,f) \) of problem (E1), (E2), (C1). The rest of the proof follows easily from Theorems 3.3 and 3.4. \( \square \)

**4 Some Comments**

a. **Regularity of weak solutions.** If \( u = u(t;x,f) \) is the weak solution given by Theorem 3.4 corresponding to some \((x,f) \in D(A) \times Y\) and \( f \in L^2_{loc}([t_0, \infty); H) \) for some \( t_0 > 0 \), then \( u \) is a strong solution on \([t_0, \infty)\) (i.e., a weak solution becomes strong once \( f \) becomes locally square integrable). The proof of this assertion follows by the uniqueness property of weak solutions.
b. The condition $f \in Y$ is optimal in the results above. As in [19], we consider the simple example $H = \mathbb{R}$, $A = 0$, $p \equiv 1$, $q \equiv 0$, $f(t) = (t + 1)^{-2-\delta}$, $\delta > 0$. For all $x \in \mathbb{R}$, problem (E), (B), (C) has a unique solution. Obviously, $f \in Y$ and the result is in line with Corollary 3.6. If $\delta = 0$ then $f$ is no longer a member of $Y$ and all solutions of equation (E) are unbounded.

c. The contraction semigroup generated by $u(t; x, 0)$, $x \in D(A)$. For all $x \in D(A)$, define $S(t)x := u(t; x, 0)$, $t \geq 0$, where $u$ is the solution given by Theorem 3.3. Then, according to (3.41), the family $\{S(t) : D(A) \to D(A)\}$ is a semigroup of contractions. In the special case $p \equiv 1$ and $q \equiv 0$, the infinitesimal generator of this semigroup is precisely the square root $A^{1/2}$ of $A$, as defined by V. Barbu (see Chapter V of [6] and [9] for details on this semigroup and its generator).

d. Smoothing effect on the starting values. Let $f \in Y \cap L^2_{\text{loc}}([0, \infty); H)$. Then, the solution $u(t; x, f)$ starting from $x \in \overline{D(A)}$ is a strong one, so in particular $u(t; x, f) \in D(A)$ for a.a. $t > 0$. This is a smoothing effect: if, for example, $A$ is a partial differential operator, then $D(A)$ contains functions which are more regular than those in $\overline{D(A)}$. In the case when $p$, $q$, $f$ are smooth functions, it is expected that for any $x \in \overline{D(A)}$, $u(t; x, f) \in D(A)$ for all $t > 0$ and that $u(t; x, f)$ satisfy equation (E) for all $t > 0$ (not just for a.a. $t > 0$). In the special case $p \equiv 1$, $q \equiv 0$, $f \equiv 0$, this does happen (see [6], p. 315). In this case $u(t; x, 0) = S(t)x$, where $\{S(t); t \geq 0\}$ is the semigroup generated by $A^{1/2}$, which is a nice operator, and $u(t; x, 0)$ is the solution of $u' + A^{1/2}u \equiv 0$.

e. Variational approach. Assume that $A$ is the subdifferential of a proper, convex, lower semicontinuous function $\phi : H \to (-\infty, +\infty]$. Since the graph of $A$ contains $[0, 0]$, one can assume that $0 = \phi(0) = \min\{\phi(z) : z \in H\}$. Recall that for all $(x, f) \in D(A) \times [Y \cap L^2_{\text{loc}}([0, \infty); H)]$ the solution $u = u(t; x, f)$ given by Theorem 5.3 on a given interval $[0, R]$ is the limit of $(u_n)$ in $C([0, R]; H)$. Since $u_n$ is the solution of the two-point boundary value problem (3.3), (3.4), it is the minimizer of the functional $\Psi_n : L^2(0, n; H) \to (-\infty, +\infty]$ defined by $\Psi_n(v) = \frac{1}{2} \int_0^n \left( a\|v\|^2 dt + b\phi(v) + b(f, v) \right) dt$, if $v \in W^{1,2}(0, n; H)$, $\phi(v) \in L^1(0, n)$, $v(0) = x$, $v(n) = 0$, and $\Psi_n(v) = +\infty$, otherwise.

In fact, any (weak) solution $u(t; x, f), (x, f) \in \overline{D(A)} \times Y$, can be approximated on compact intervals by minimizers $u_n$ associated with $(\bar{x}, \bar{f}) \in D(A) \times [Y \cap L^2_{\text{loc}}([0, \infty); H)]$ close to $(x, f)$ in $H \times Y$. 

\[ \text{Gheorghe Moroşanu} \]
5 Approximation by the method of artificial viscosity

Let \( \varepsilon > 0 \) be a small number and let \( p(t) = \varepsilon, \forall t \geq 0 \). In this case, equation \( (E) \) can be regarded as an approximate one for the following reduced equation

\[
q(t)u'(t) \in Au(t) + f(t) \quad \text{for a.a. } t > 0.
\]

Equation \( (E_0) \) where \( q^+ \equiv 0 \) (i.e., \( q(t) \leq 0 \) for a.a. \( t > 0 \)) is particularly significant for applications to parabolic and hyperbolic PDE problems, as explained below. It is expected that any solution \( u_\varepsilon(t; x, f) \) of \( (E) \) (with \( p \equiv \varepsilon \)), \( (B) \), satisfying \( \sqrt{a^-\|u_\varepsilon(\cdot; x, f)\|} \in L^\infty(\mathbb{R}^+) \), approximate in some sense the solution \( u(\cdot; x, f) \) of \( (E_0), (B) \), for \( \varepsilon \) small enough. The advantage is that \( u_\varepsilon \) is more regular (with respect to \( t \)) than \( u \). This method of approximation (called the method of artificial viscosity, due to the term involving \( \varepsilon \) in \( (E) \)) was introduced and studied by J.L. Lions [20] mainly in the case of linear PDE problems. See also [3]. Here we have more general problems on the whole positive half line that require separate analysis. Hopefully some results on this subject will be obtained later. In the following we just present some examples which seem suitable for the artificial viscosity method.

Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^k \) with a smooth boundary \( \Gamma \). Let \( \beta : D(\beta) \subset \mathbb{R} \rightarrow \mathbb{R} \) be a (possibly set-valued) maximal monotone mapping, with \( 0 \in D(\beta) \) and \( 0 \in \beta(0) \). Consider the nonlinear diffusion-reaction equation

\[
u_t - \text{div} \left( r(x) \text{grad} u \right) + \beta(u) \ni f(t, x), \ (t, x) \in (0, \infty) \times \Omega, \tag{E_1}
\]

with the Dirichlet boundary condition

\[
u = 0 \quad \text{on } (0, \infty) \times \Gamma, \tag{DBC}
\]

and the initial condition

\[
u(0, x) = u_0(x), \quad x \in \Omega.
\]

The function \( r = r(x) \) in \( (E_1) \) is assumed to be a nonnegative smooth function. Obviously, \( (E_1), (DBC) \) can be expressed as an equation of the form \( (E_0) \) in \( H = L^2(\Omega) \) with \( q \equiv -1 \), and \( A \) a maximal monotone operator in \( H \). The corresponding approximate equation (i.e., Eq. \( (E) \) with \( p \equiv \varepsilon \)) is

\[
\varepsilon u_{tt} - u_t \in -\text{div} \left( r(x) \text{grad} u \right) + \beta(u) + f(t, x), \ (t, x) \in (0, \infty) \times \Omega,
\]

with the same boundary condition \( (DBC) \). Note that this is an elliptic type equation with respect to \( (t, x) = (t, x_1, ..., x_k) \).

The nonlinear wave equation

\[
u_{tt} - \Delta u + \beta(u_t) \ni f(t, x), \quad (t, x) \in (0, \infty) \times \Omega,
\]

are also considered. For these we shall present some examples which seem suitable for the artificial viscosity method.
with \([DBC]\), could be also examined. It is well known that this equation can be represented (by using the substitution \(v = u_t\)) as an equation of the form \([L_0]\) with \(q \equiv -1\) in the product (phase) space \(H = H^1_0(\Omega) \times L^2(\Omega)\) (see, e.g., [24], p. 205). It is easily seen that the approximate equation (i.e., \([L]\) with \(p \equiv \varepsilon\)), associated with the wave equation above, is equivalent to

\[
\varepsilon^2 u_{tttt} - 2\varepsilon u_{ttt} + u_{tt} - \Delta u + \beta(u_t - \varepsilon u_{tt}) \ni f(t, x),
\]

which obviously provides solutions which are more regular (with respect to \(t\)) than those of the wave equation.

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