SPECTRAL PROPERTIES OF THE BCS GAP EQUATION
OF SUPERFLUIDITY

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ABSTRACT. We present a review of recent work on the mathematical aspects of the BCS gap equation, covering our results of [9] as well as our recent joint work with Hamza and Solovej [8] and with Frank and Naboko [6], respectively. In addition, we mention some related new results.

1. Introduction

In this paper we shall describe our recent mathematical study [8, 6, 9] of one of the current hot topics in condensed matter physics, namely ultra cold fermionic gases consisting of neutral spin-$\frac{1}{2}$ atoms. The kinetic energy of these atoms is described by the non-relativistic Schrödinger operator, and their interaction by a pair potential $\lambda V$ with $\lambda$ being a coupling parameter. As experimentalists are nowadays able to vary the inter-atomic potentials, the form of $\lambda V$ in actual physical systems can be quite general; see the recent reviews in [5] and [4]. Our primary goal concerns the study of the superfluid phases of such systems. According to Bardeen, Cooper and Schrieffer [2] (BCS) the superfluid state is characterized by the existence of a non-trivial solution of the gap equation

$$\Delta(p) = -\frac{\lambda}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \hat{V}(p-q) \frac{\Delta(q)}{E(q)} \tanh \frac{E(q)}{2T} dq$$

at some temperature $T \geq 0$, with $E(p) = \sqrt{(p^2 - \mu)^2 + |\Delta(p)|^2}$. Here, $\mu > 0$ is the chemical potential and $\hat{V}(p) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} V(x)e^{-ipx} dx$ denotes the Fourier transform of $V$. The function $\Delta(p)$ is the order parameter and represents the wavefunction of the Cooper pairs. Despite the fact that the BCS equation (1) is highly non-linear, we shall show in Theorem 1 (see also [8, Thm 1]) that the existence of a non-trivial solution to (1) at some temperature $T$ is equivalent to the fact that a certain linear operator, given in (6) below, has a negative eigenvalue. For $T = 0$ this operator is given by $| - \Delta - \mu | + \lambda V$. This rather astonishing possibility of reducing a non-linear to a linear problem allows for a more thorough mathematical study. Using
spectral-theoretic methods, we are able to give a precise characterization of the class of potentials leading to a non-trivial solution for (1). In particular, in Theorem 2 (see also [6, Thm 1]) we prove that for all interaction potentials that create a negative eigenvalue of the effective potential on the Fermi sphere (see (9) below; a sufficient condition for this property is that \( \int_{\mathbb{R}^3} V(x) dx < 0 \)), there exists a critical temperature \( T_c(\lambda V) > 0 \) such that (1) has a non-trivial (i.e., not identically vanishing) solution for all \( T < T_c(\lambda V) \), whereas there is no such solution for \( T \geq T_c(\lambda V) \). Additionally, we shall determine in Theorem 2 the precise asymptotic behavior of \( T_c(\lambda V) \) in the small coupling limit. We extend this result in Theorem 3 (see also [9, Thm 1]) and give a derivation of the critical temperature \( T_c \) valid to second order Born approximation. More precisely, we shall show that

\[
T_c = \mu \frac{8e^{-2/\gamma}}{\pi} e^{\pi/(2\sqrt{|b_\mu|})} \tag{2}
\]

where \( \gamma \approx 0.577 \) denotes Euler’s constant, and where \( b_\mu < 0 \) is an effective scattering length. To first order in the Born approximation, \( b_\mu \) is related to the scattering amplitude of particles with momenta on the Fermi sphere, but to second order the expression is more complicated. The precise formula is given in Eq. (14) below. For interaction potentials that decay fast enough at large distances, we shall show that \( b_\mu \) reduces to the usual scattering length \( a_0 \) of the interaction potential in the low density limit, i.e., for small \( \mu \). Our formula thus represents a generalization of a well-known formula in the physics literature [7, 13].

In the case of zero temperature, the function \( E(p) \) in (1) describes an effective energy-momentum relation for quasi particles, and

\[
\Xi := \inf_p E(p) = \inf_p \sqrt{(p^2 - \mu)^2 + |\Delta(p)|^2}
\]

is called the energy gap of the system. It is of major importance for applications, such as the classification of different types of superfluids. In fact, \( \Xi \) is the spectral gap of the corresponding second quantized BCS Hamiltonian. (See [2] and [12] or the appendix in [8].)

An important problem is the classification of potentials \( V \) for which \( \Xi > 0 \). This questions turns out to be intimately related to the continuity of the momentum distribution \( \gamma(p) \), which will be introduced in the next section. In the normal (i.e., not superfluid) state, \( \Delta = 0 \) and \( \gamma \) is a step function at \( T = 0 \), namely \( \gamma(p) = \theta(|p| - \sqrt{\mu}) \). According to the picture presented in standard textbooks the appearance of a superfluid phase softens this step function and \( \gamma(p) \) becomes continuous. We are going to prove in this paper that if \( V(x)|x| \in L^{6/5} \) and \( \int V < 0 \) then indeed both strict positivity of \( \Xi > 0 \) and continuity \( \gamma \) hold. It remains an open problem to find examples of potentials such that the gap vanishes in cases where a superfluid phase occurs.
One of the difficulties involved in evaluating $\Xi$ is the potential non-uniqueness of the solution of the BCS gap equation. For interaction potentials that have nonpositive Fourier transform, however, we shall show that the BCS pair wavefunction is unique, and has zero angular momentum. In this case, we shall prove in Theorem 5 (see also [9, Thm. 2]) similar results for $\Xi$ as for the critical temperature. It turns out that, at least up to second order Born approximation,

$$\Xi = T_c \frac{\pi}{e^\gamma}$$

in this case. This equality is valid for any density, i.e., for any value of the chemical potential $\mu$. In particular, $\Xi$ has exactly the same exponential dependence on the interaction potential, described by $b_\mu$, as the critical temperature $T_c$.

2. Preliminaries and main results

We consider a gas of spin $1/2$ fermions at temperature $T \geq 0$ and chemical potential $\mu > 0$, interacting via a local two-body interaction potential of the form $2\lambda V(x)$. Here, $\lambda > 0$ is a coupling parameter, and the factor 2 is introduced for convenience. We assume that $V$ is real-valued and has some mild regularity properties, namely $V \in L^1(\mathbb{R}^3) \cap L^{3/2}(\mathbb{R}^3)$. In the BCS approximation, the system is described by the BCS functional $\mathcal{F}_T$, derived by Leggett in his seminal paper [11], based on the original work of BCS [2].

The BCS functional $\mathcal{F}_T$ is related to the pressure of the system and is given by

$$\mathcal{F}_T(\gamma, \alpha) = \int (p^2 - \mu) \gamma(p) dp + \int |\alpha(x)|^2 V(x) dx - TS(\gamma, \alpha),$$

where the entropy $S$ is

$$S(\gamma, \alpha) = - \int \text{Tr}_{C^2} [\Gamma(p) \log \Gamma(p)] dp, \quad \Gamma(p) = \begin{pmatrix} \gamma(p) & \alpha(p) \\ \bar{\alpha}(p) & 1 - \gamma(p) \end{pmatrix}.$$ 

The functions $\gamma(p)$ and $\bar{\alpha}(p)$ are interpreted as the momentum distribution and the Cooper pair wave function, respectively. They satisfy the matrix constraint $0 \leq \Gamma(p) \leq 1$ for all $p \in \mathbb{R}^3$. In terms of the BCS functional the occurrence of superfluidity is described by minimizers with $\alpha \neq 0$. We remark that in the case of the Hubbard-model this functional was studied in [3].

For an arbitrary temperature $0 \leq T < \infty$ the BCS gap equation, which is the Euler-Lagrange equation associated with the functional $\mathcal{F}_T$, reads

$$\Delta(p) = -\frac{\lambda}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} V(p-q) \frac{\Delta(q)}{E(q)} \tanh \frac{E(q)}{2T} dq,$$ 

where $E(p) = \sqrt{(p^2 - \mu)^2 + |\Delta(p)|^2}$. The order parameter $\Delta$ is related to the expectation value of the Cooper pairs $\alpha$ via $2\alpha(p) = \Delta(p)/E(p)$. We present in the following a thorough mathematical study of this equation. In
order to do so, we shall not attack the equation (5) directly, but exploit the fact that $\alpha$ is a critical point of the semi-bounded functional $\mathcal{F}_T$.

The key to our studies is the observation in [8] that the existence of a non-trivial solution to the non-linear equation (5) can be reduced to a linear criterion, which can be formulated as follows.

**THEOREM 1** ([8, Theorem 1]). Let $V \in L^{3/2}, \mu \in \mathbb{R},$ and $\infty > T \geq 0$. Define

$$K_{T,\mu} = (p^2 - \mu)\frac{e^{(p^2 - \mu)/T} + 1}{e^{(p^2 - \mu)/T} - 1}.$$

Then the non-linear BCS equation (5) has a non-trivial solution if and only if the linear operator

$$K_{T,\mu} + \lambda V,$$

acting on $L^2(\mathbb{R}^3)$, has at least one negative eigenvalue.

Hence we are able to relate a non-linear problem to a linear problem which is much easier to handle. The operator $K_{T,\mu}$ is understood as a multiplication operator in momentum space. In the limit $T \to 0$ this operator reduces to $|−\Delta - \mu|^1/2 K_{T,\mu} - 1$.

### 2.1. The critical temperature

Theorem 1 enables a precise definition of the critical temperature, by

$$T_c(\lambda V) := \inf\{T | K_{T,\mu} + \lambda V \geq 0\}. \quad (7)$$

The symbol $K_{T,\mu}(p)$ is point-wise monotone in $T$. This implies that for any potential $V$, there is a critical temperature $0 \leq T_c(\lambda V) < \infty$ that separates two phases, a superfluid phase for $0 \leq T < T_c(\lambda V)$ from a normal phase for $T_c(\lambda V) \leq T < \infty$. Note that $T_c(\lambda V) = 0$ means that there is no superfluid phase for $\lambda V$. Using the linear criterion (7) we can classify the potentials for which $T_c(\lambda V) > 0$, and simultaneously we can evaluate the asymptotic behavior of $T_c(\lambda V)$ in the limit of small $\lambda$. This can be done by spectral theoretical methods. Applying the Birman-Schwinger principle one observes that the critical temperature $T_c$ can be characterized by the fact that the compact operator

$$\lambda(\text{sgn} V)|V|^{1/2}K_{T_c,\mu}^{-1}|V|^{1/2} \quad (8)$$

has $-1$ as its lowest eigenvalue. This operator is singular for $T_c \to 0$, and the key observation is that its singular part is represented by the operator

$$\lambda \ln(1/T_c) \mathcal{V}_\mu,$$

where $\mathcal{V}_\mu : L^2(\Omega_\mu) \mapsto L^2(\Omega_\mu)$ is given by

$$(\mathcal{V}_\mu u)(p) = \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{\mu}} \int_{\Omega_\mu} \hat{V}(p-q)u(q) \, d\omega(q). \quad (9)$$

Here, $\Omega_\mu$ denotes the 2-sphere with radius $\sqrt{\mu}$, and $d\omega$ denotes Lebesgue measure on $\Omega_\mu$. We note that the operator $\mathcal{V}_\mu$ has appeared already earlier in the literature [3, 10].

Our analysis here is somewhat similar in spirit to the one concerning the lowest eigenvalue of the Schrödinger operator $p^2 + \lambda V$ in two space
dimensions \[14\]. This latter case is considerably simpler, however, as \(p^2\) has a unique minimum at \(p = 0\), whereas \(K_{T,\mu}(p)\) takes its minimal value on the Fermi sphere \(p^2 = \mu\), meaning that its minimum is highly degenerate. Hence, in our case, the problem is reduced to analyzing a map from the \(L^2\) functions on the Fermi sphere \(\Omega_\mu\) (of radius \(\sqrt{\mu}\)) to itself. Let us denote the lowest eigenvalue of \(\mathcal{V}_\mu\) as
\[e_\mu(V) := \inf \text{spec} \mathcal{V}_\mu.
\]
Whenever this eigenvalue is negative then the critical temperature is non-zero for all \(\lambda > 0\), and we can evaluate its asymptotics. Moreover, the converse is “almost” true:

**THEOREM 2** ([6] Theorem 1]). Let \(V \in L^{3/2}(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)\) be real-valued, and let \(\lambda > 0\).

(i) Assume that \(e_\mu(V) < 0\). Then \(T_c(\lambda V)\) is non-zero for all \(\lambda > 0\), and
\[
\lim_{\lambda \to 0} \lambda \ln \frac{\mu}{T_c(\lambda V)} = -\frac{1}{e_\mu(V)}. \tag{10}
\]
(ii) Assume that \(e_\mu(V) = 0\). If \(T_c(\lambda V)\) is non-zero, then \(\ln(\mu/T_c(\lambda V)) \geq c\lambda^{-2}\) for some \(c > 0\) and small \(\lambda\).
(iii) If there exists an \(\epsilon > 0\) such that \(e_\mu(V - \epsilon|V|) = 0\), then \(T_c(\lambda V) = 0\) for small enough \(\lambda\).

As we see, the occurrence of superfluidity as well as the asymptotic behavior of \(T_c(\lambda V)\) is governed by \(e_\mu(V)\). A sufficient condition for \(e_\mu(V)\) to be negative is \(\int V < 0\). But one can easily find other examples. Eq. \((10)\) shows that the critical temperature behaves like \(T_c(\lambda V) \sim \mu e^{1/(\lambda e_\mu(V))}\). In other words it is exponentially small in the coupling.

In the following, we shall derive the second order correction, i.e., we will compute the constant in front of the exponentially small term in \(T_c\). For this purpose, we define an operator \(\mathcal{W}_\mu\) on \(L^2(\Omega_\mu)\) via its quadratic form
\[
\langle u | \mathcal{W}_\mu | u \rangle = \int_0^\infty dp \left( \frac{|p|^2}{||p|^2 - \mu} \left[ \int_{S^2} d\Omega (|\hat{\varphi}(p)|^2 - |\hat{\varphi}(\sqrt{\mu}p/|p|)|^2) \right] + \frac{1}{|p|^2} \int_{S^2} d\Omega |\hat{\varphi}(\sqrt{\mu}p/|p|)|^2 \right). \tag{11}
\]
Here, \(\hat{\varphi}(p) = (2\pi)^{-3/2} \int_{\Omega_\mu} \hat{V}(p - q)u(q)d\omega(q)\), and \((|p|, \Omega) \in \mathbb{R}_+ \times S^2\) denote spherical coordinates for \(p \in \mathbb{R}^3\). We note that since \(V \in L^1(\mathbb{R}^3)\), \(\int_{S^2} d\Omega |\hat{\varphi}(p)|^2\) is Lipschitz continuous in \(|p|\) for any \(u \in L^2(\mathbb{R}^3)\), and hence the radial integration is well-defined, even in the vicinity of \(p^2 = \mu\). In fact the operator \(\mathcal{W}_\mu\) can be shown to be Hilbert-Schmidt class, see [9] Section 3.

For \(\lambda > 0\), let
\[
\mathcal{E}_\mu = \lambda^2 \frac{\pi}{2\sqrt{\mu}} \mathcal{V}_\mu - \lambda^2 \frac{\pi}{2\mu} \mathcal{W}_\mu, \tag{12}
\]
and let $b_\mu(\lambda)$ denote its ground state energy,

$$b_\mu(\lambda) = \inf \text{spec } \mathcal{B}_\mu.$$  \hspace{1cm} (13)

We note that if $e_\mu < 0$, then also $b_\mu(\lambda) < 0$ for small $\lambda$. In fact, if the eigenfunction corresponding to the lowest eigenvalue $e_\mu$ of $\mathcal{V}_\mu$ is unique and equals $u \in L^2(\Omega_\mu)$, then

$$b_\mu(\lambda) = \langle u|\mathcal{B}_\mu|u \rangle + O(\lambda^3) = \lambda \frac{\pi e_\mu}{2\sqrt{\mu}} - \lambda^2 \frac{\pi}{2\mu} \langle u|\mathcal{V}_\mu|u \rangle + O(\lambda^3).$$ \hspace{1cm} (14)

In the degenerate case, this formula holds if one chooses $u$ to be the eigenfunction of $\mathcal{V}_\mu$ that yields the largest value $\langle u|\mathcal{V}_\mu|u \rangle$ among all such (normalized) eigenfunctions.

With the aid of $b_\mu(\lambda)$, we can now recover the next order of the critical temperature for small $\lambda$.

**THEOREM 3 ([9, Theorem 1]).** Let $V \in L^1(\mathbb{R}^3) \cap L^{3/2}(\mathbb{R}^3)$ and let $\mu > 0$. Assume that $e_\mu = \inf \text{spec } \mathcal{V}_\mu < 0$, and let $b_\mu(\lambda)$ be defined in (13). Then the critical temperature $T_c$ for the BCS equation is strictly positive and satisfies

$$\lim_{\lambda \to 0} \left( \ln \left( \frac{\mu}{T_c} \right) + \frac{\pi}{2\sqrt{\mu b_\mu(\lambda)}} \right) = 2 - \gamma - \ln(8/\pi).$$ \hspace{1cm} (15)

Here, $\gamma \approx 0.577$ denotes Euler’s constant.

The Theorem says that, for small $\lambda$,

$$T_c \sim \mu \frac{8e^{\gamma-2}}{\pi} e^{\pi/(2\sqrt{\mu b_\mu(\lambda)})}.$$ \hspace{1cm} (16)

Note that $b_\mu(\lambda)$ can be interpreted as a (renormalized) effective scattering length of $2\lambda V(x)$ (in second order Born approximation) for particles with momenta on the Fermi sphere. In fact, if $V$ is radial and $\int_{\mathbb{R}^3} V(x)dx < 0$, it is not difficult to see that for small enough $\mu$ the (unique) eigenfunction corresponding to the lowest eigenvalue $e_\mu$ of $\mathcal{V}_\mu$ is the constant function

$u(p) = (4\pi\mu)^{-1/2}$. (See [6, Section 2.1].) For this $u$, we have

$$\lim_{\mu \to 0} \langle u|\mathcal{B}_\mu|u \rangle = (\lambda/4\pi) \int_{\mathbb{R}^3} V(x)dx - (\lambda/4\pi)^2 \int_{\mathbb{R}^6} \frac{V(x)V(y)}{|x-y|}dxdy \equiv a_0(\lambda).$$

Here, $a_0(\lambda)$ equals the scattering length of $2\lambda V$ in second order Born approximation. Assuming additionally that $V(x)|x| \in L^1$ and bearing in mind that $b_\mu(\lambda) = \langle u|\mathcal{B}_\mu|u \rangle + O(\lambda^3)$ for small enough $\mu$, we can, in fact, estimate the difference between $b_\mu(\lambda)$ and $a_0(\lambda)$. Namely we prove in [9, Proposition 1] that

$$\lim_{\mu \to 0} \frac{1}{\sqrt{\mu}} \left( \frac{1}{\langle u|\mathcal{B}_\mu|u \rangle} - \frac{1}{a_0(\lambda)} \right) = 0.$$ \hspace{1cm} (17)

This yields the approximation

$$T_c \approx \mu \frac{8e^{\gamma-2}}{\pi} e^{\pi/(2\sqrt{a_0(\lambda)})}.$$
in the limit of small $\lambda$ and small $\mu$. This expression is well-known in the physics literature \[7, 13\]. We point out, however, that our formula (16) is much more general since it holds for any value of $\mu > 0$.

2.2. Energy Gap at Zero Temperature. Consider now the zero temperature case $T = 0$. In this case, it is natural to formulate a functional depending only on $\alpha$ instead of $\gamma$ and $\alpha$. In fact, for $T = 0$ the optimal choice of $\gamma(p)$ in $F_T$ for given $\hat{\alpha}(p)$ is clearly

$$\gamma(p) = \begin{cases} \frac{1}{2} (1 + \sqrt{1 - 4|\hat{\alpha}(p)|^2}) & \text{for } p^2 < \mu \\ \frac{1}{2} (1 - \sqrt{1 - 4|\hat{\alpha}(p)|^2}) & \text{for } p^2 > \mu \end{cases}.$$ (17)

Subtracting an unimportant constant, this leads to the zero temperature BCS functional

$$F_0(\alpha) = \frac{1}{2} \int_{\mathbb{R}^3} |p^2 - \mu| \left(1 - \sqrt{1 - 4|\hat{\alpha}(p)|^2}\right) dp + \lambda \int_{\mathbb{R}^3} V(x)|\alpha(x)|^2 dx. \quad (18)$$

The variational equation satisfied by a minimizer of (18) is then

$$\Delta(p) = -\frac{\lambda}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \hat{V}(p - q) \frac{\Delta(q)}{E(q)} dq,$$ (19)

with $\Delta(p) = 2E(p)\hat{\alpha}(p)$. This is simply the BCS equation (5) at $T = 0$. For a solution $\Delta$, the energy gap $\Xi$ is defined as

$$\Xi = \inf_{p} E(p) = \inf_{p} \sqrt{(p^2 - \mu)^2 + |\Delta(p)|^2}.$$ (20)

It has the interpretation of an energy gap in the corresponding second-quantized BCS Hamiltonian (see, e.g., [12] or the appendix in [8].)

A priori, the fact that the order parameter $\Delta$ is non vanishing does not imply that $\Xi > 0$. Strict positivity of $\Xi$ turns out to be related to the continuity of the corresponding $\gamma$ in (17). In fact, we are going to prove in Lemma 1 that if $V$ decays fast enough, i.e., $V(x)|x| \in L^{6/5}(\mathbb{R}^3)$, the two properties, $\Xi > 0$ and $\gamma(p)$ continuous, are equivalent. Both properties hold true under the assumption that $\int V < 0$:

**THEOREM 4.** Let $V \in L^{3/2} \cap L^1$, with $V(x)|x| \in L^{6/5}(\mathbb{R}^3)$ and $\int V = (2\pi)^{3/2}\hat{V}(0) < 0$. Let $\alpha$ be a minimizer of the BCS functional. Then $\Xi$ defined in (20) is strictly positive, and the corresponding momentum distribution $\gamma$ in (17) is continuous.

One of the difficulties involved in evaluating $\Xi$ is the potential non-uniqueness of minimizers of (18), and hence non-uniqueness of solutions of the BCS gap equation (19). The gap $\Xi$ may depend on the choice of $\Delta$ in this case. For potentials $V$ with non-positive Fourier transform, however, we can prove the uniqueness of $\Delta$ and, in addition, we are able to derive the precise asymptotic of $\Xi$ as $\lambda \to 0$.

In the following we will restrict our attention to radial potentials $V$ with non-positive Fourier transform. We also assume that $\hat{V}(0) = (2\pi)^{-3/2} \int V(x)dx < 0$. It is easy to see that $e_\mu = \inf \text{spec } \nu_\mu < 0$ in this case, and that the
(unique) eigenfunction corresponding to this lowest eigenvalue of $V_{\mu}$ is the constant function.

In particular we have the following asymptotic behavior of the energy gap $\Xi$ as $\lambda \to 0$.

**THEOREM 5** ([9, Theorem 2]). Assume that $V \in L^1(\mathbb{R}^3) \cap L^{3/2}(\mathbb{R}^3)$ is radial, with $\hat{V}(p) \leq 0$ and $\hat{V}(0) < 0$. Then there is a unique minimizer (up to a constant phase) of the BCS functional (18) at $T = 0$. The corresponding energy gap, $\Xi = \inf_p \sqrt{(p^2 - \mu)^2 + |\Delta(p)|^2}$, is strictly positive, and satisfies

$$\lim_{\lambda \to 0} \left( \ln \left( \frac{\mu}{\Xi} \right) + \frac{\pi}{2\sqrt{p} b_{\mu}(\lambda)} \right) = 2 - \ln(8).$$

(21)

Here, $b_{\mu}(\lambda)$ be defined in (13).

The Theorem says that, for small $\lambda$,

$$\Xi \sim \mu^8 e^{\pi/(2\sqrt{\mu} b_{\mu}(\lambda))}.$$

In particular, in combination with Theorem [3], we obtain the universal ratio

$$\lim_{\lambda \to 0} \frac{\Xi}{T_c} = \frac{\pi}{e^\gamma} \approx 1.7639.$$

That is, the ratio of the energy gap $\Xi$ and the critical temperature $T_c$ tends to a universal constant as $\lambda \to 0$, independently of $V$ and $\mu$. This property has been observed before for the original BCS model with rank one interaction [2, 12], and in the low density limit for more general interactions [7] under additional assumptions. Our analysis shows that it is valid in full generality at small coupling $\lambda \ll 1$.

3. Sketch of the proof of Theorem [1]

The backbone of our analysis is the linear criterion in Theorem [1]. As a first step towards its proof, one has to prove that the functional $\mathcal{F}_T(\gamma, \alpha)$ in [1] attains a minimum on the set

$$\mathcal{D} = \{(\gamma, \alpha) \mid \gamma \in L^1(\mathbb{R}^3, (1+p^2)dp), \alpha \in H^1(\mathbb{R}^3), 0 \leq \gamma \leq 1, |\hat{\alpha}|^2 \leq \gamma(1-\gamma) \}.$$

This can be done by proving lower semi-continuity of $\mathcal{F}_T$ on $\mathcal{D}$. See [8, Prop. 1] for details. Theorem [1] is then a direct consequence of the equivalence of the following three statements [8, Theorem 1]:

(i) The normal state $(\gamma_0, 0)$, with $\gamma_0 = [e^{(p^2-\mu)/T} + 1]^{-1}$ being the Fermi-Dirac distribution, is unstable under pair formation, i.e.,

$$\inf_{(\gamma, \alpha) \in \mathcal{D}} \mathcal{F}_T(\gamma, \alpha) < \mathcal{F}_T(\gamma_0, 0).$$

(ii) There exists a pair $(\gamma, \alpha) \in \mathcal{D}$, with $\alpha \neq 0$, such that

$$\Delta(p) = \frac{p^2 - \mu}{2 - \gamma(p) \hat{\alpha}(p)}$$

(22)

satisfies the BCS gap equation (5).
(iii) The linear operator $K_{T, \mu} + V$ has at least one negative eigenvalue.

The proof of the equivalence of these three statements consists of the following steps. First, it is straightforward to show that (i) $\Rightarrow$ (ii). By evaluating the stationary equations in both variables, $\gamma$ and $\alpha$, one shows that the combination (22) satisfies the BCS equation (5).

To show that (iii) $\Rightarrow$ (i), first note that $(\gamma_0, 0)$ is the minimizer of $F_T$ in the case $V = 0$. Consequently $\frac{d}{dt} F_T(\gamma_0, tg)|_{t=0} = 0$ for general $g$. Moreover, a simple calculation shows that

$$\frac{d^2}{dt^2} F(\gamma_0, tg)|_{t=0} = 2 \langle g | K_{T, \mu} + \lambda V | g \rangle.$$ 

If $K_{T, \mu} + \lambda V$ has a negative eigenvalue, we thus see that $F(\gamma_0, tg) < F_T(\gamma_0, 0)$ for small $t$ and an appropriate choice of $g$.

The hardest part in showing the equivalence of the three statements is to show that (ii) $\Rightarrow$ (iii). Given a pair $(\tilde{\gamma}, \tilde{\alpha})$ such that the corresponding $\Delta$ in (22) satisfies the BCS equation (5), we note that if $\hat{\alpha} = m(p) \tilde{\alpha}(p)$ and $\gamma(p) = 1/2 + m(p)(\gamma(p) - 1/2)$, the pair $(\gamma, \alpha)$ yields the same $\Delta$ and hence also satisfies (5). Moreover, with the choice

$$m(p) = \frac{p^2 - \mu}{\frac{1}{2} - \gamma(p)} \tanh \frac{E(p)}{2T}$$

(where $E(p) = \sqrt{(p^2 - \mu)^2 + |\Delta(p)|^2}$), the new pair $(\gamma, \alpha)$ satisfies additionally

$$\frac{2E(p)}{\tanh \frac{E(p)}{2T}} = \frac{p^2 - \mu}{\frac{1}{2} - \gamma(p)}$$

(23)

$$\frac{\lambda}{(2\pi)^3} \int \hat{V}(p - q) \hat{\alpha}(q) dq = - \frac{p^2 - \mu}{\frac{1}{2} - \gamma(p)}.$$ 

(24)

Note that in the case $V = 0$, i.e., $\Delta = 0$, the equation (23) reduces to

$$2K_{T, \mu}(p) = \frac{p^2 - \mu}{\frac{1}{2} - \gamma_0}.$$ 

Using this fact, together with (24), we thus obtain

$$\langle \alpha | K_{T, \mu} + \lambda V | \alpha \rangle = \frac{1}{2} \left( \alpha \left| \frac{p^2 - \mu}{\frac{1}{2} - \gamma_0} - \frac{p^2 - \mu}{\frac{1}{2} - \gamma} \right| \alpha \right).$$ 

(25)

Using the definition of $E(p)$ and the strict monotonicity of the function $x \mapsto x/\tanh \frac{x}{2T}$ for $x \geq 0$, we infer from (23) that

$$\frac{p^2 - \mu}{\frac{1}{2} - \gamma_0} \leq \frac{p^2 - \mu}{\frac{1}{2} - \gamma},$$

with strict the inequality on the set where $\Delta \neq 0$. Consequently, the expression (25) is strictly negative. Hence $K_{T, \mu} + \lambda V$ has a negative eigenvalue. This shows that (ii) implies (iii).
4. Proof of Theorems 2 and 3

For a (not necessarily sign-definite) potential \( V(x) \) let us use the notation

\[
V(x)^{1/2} = (\text{sgn} V(x)) |V(x)|^{1/2}.
\]

From our definition of the critical temperature \( T_c \) it follows immediately
that for \( T = T_c \) the operator \( K_{T,\mu} + \lambda V \) has and eigenvalue 0 and no negative eigenvalue. If \( \psi \) is the corresponding eigenvector, one can rewrite the
eigenvalue equation in the form

\[
-\psi = \lambda K_{T,\mu}^{-1} V \psi.
\]

Multiplying this equation by \( V^{1/2}(x) \), one obtains an eigenvalue equation
for \( \varphi = V^{1/2} \psi \). This argument works in both directions and is called the
Birman-Schwinger principle (see [6, Lemma 1]). In particular it tells us that
the critical temperature \( T_c \) is determined by the fact that for this value of
\( T \) the smallest eigenvalue of

\[
B_T = \lambda V^{1/2} K_{T,\mu}^{-1} |V|^{1/2}
\]
equals \(-1\). Note that although \( B_T \) is not self-adjoint, it has real spectrum.

Let \( \mathfrak{F} : L^1(\mathbb{R}^3) \to L^2(\Omega_\mu) \) denote the (bounded) operator which maps
\( \psi \in L^1(\mathbb{R}^3) \) to the Fourier transform of \( \psi \), restricted to the sphere \( \Omega_\mu \). Since
\( V \in L^1(\mathbb{R}^3) \), multiplication by \( |V|^{1/2} \) is a bounded operator from \( L^2(\mathbb{R}^3) \) to
\( L^1(\mathbb{R}^3) \), and hence \( \mathfrak{F} |V|^{1/2} \) is a bounded operator from \( L^2(\mathbb{R}^3) \) to \( L^2(\Omega_\mu) \).

Let

\[
m_\mu(T) = \max \left\{ \frac{1}{4\pi \mu} \int_{\mathbb{R}^3} \left( \frac{1}{K_{T,\mu}(p)} - \frac{1}{p^2} \right) dp, 0 \right\},
\]

and let

\[
M_T = K_{T,\mu}^{-1} - m_\mu(T) \mathfrak{F}^* \mathfrak{F}.
\]

As in [6, Lemma 2] one can show that \( V^{1/2} M_T |V|^{1/2} \) is a Hilbert-Schmidt
operator on \( L^2(\mathbb{R}^3) \), and its Hilbert Schmidt norm is bounded uniformly in
\( T \). In particular, the singular part of \( B_T \) as \( T \to 0 \) is entirely determined
by \( V^{1/2} \mathfrak{F}^* \mathfrak{F} |V|^{1/2} \).

Since \( V^{1/2} M_T |V|^{1/2} \) is uniformly bounded, we can choose \( \lambda \) small enough
such that \( 1 + \lambda V^{1/2} M_T |V|^{1/2} \) is invertible, and we can then write \( 1 + B_T \) as

\[
1 + B_T = 1 + \lambda V^{1/2} (m_\mu(T) \mathfrak{F}^* \mathfrak{F} + M_T) |V|^{1/2}
\]

\[
= \left( 1 + \lambda V^{1/2} M_T |V|^{1/2} \right) \left( 1 + \frac{\lambda m_\mu(T)}{1 + \lambda V^{1/2} M_T |V|^{1/2}} V^{1/2} \mathfrak{F}^* \mathfrak{F} |V|^{1/2} \right).
\]

Then \( B_T \) having an eigenvalue \(-1\) is equivalent to

\[
\frac{\lambda m_\mu(T)}{1 + \lambda V^{1/2} M_T |V|^{1/2}} V^{1/2} \mathfrak{F}^* \mathfrak{F} |V|^{1/2}
\]
having an eigenvalue $-1$. The operator in (29) is isospectral to the selfadjoint operator
\[
\mathfrak{F}|V|^{1/2} \frac{\lambda m_\mu(T)}{1 + \lambda V^{1/2} M T_c |V|^{1/2}} V^{1/2} \mathfrak{F}^*,
\]
acting on $L^2(\Omega_\mu)$.

At $T = T_c$, $-1$ is the smallest eigenvalue of $B_T$, hence (29) and (30) have an eigenvalue $-1$ for this value of $T$. Moreover, we can conclude that $-1$ is actually the smallest eigenvalue of (29) and (30) in this case. For, if there were an eigenvalue less than $-1$, we could increase $T$ and, by continuity, find some $T > T_c$ for which there is an eigenvalue $-1$. Using (28), this would contradict the fact that $B_T$ has no eigenvalue $-1$ for $T > T_c$.

Consequently, the equation for the critical temperature can be written as
\[
\lambda m_\mu(T_c) \inf \text{spec} \mathfrak{F}|V|^{1/2} \frac{1}{1 + \lambda V^{1/2} M T_c |V|^{1/2}} V^{1/2} \mathfrak{F}^* = -1.
\]
This equation is the starting point for the proof of Theorems 2 and 3.

**Proof of Theorem 2.** Up to first order in $\lambda$ the equation (31) reads
\[
\lambda m_\mu(T_c) \inf \text{spec} \mathfrak{F}[V - \lambda V M T_c V + O(\lambda^2)] \mathfrak{F}^* = -1,
\]
where the error term $O(\lambda^2)$ is uniformly bounded in $T_c$. Note that $\mathfrak{F} V \mathfrak{F}^* = \sqrt{\mu} V_\mu$ defined in (9). Assume now that $e_\mu = \inf \text{spec} V_\mu$ is strictly negative. Since $V^{1/2} M T_c V^{1/2}$ is uniformly bounded, it follows immediately that
\[
\lim_{\lambda \to 0} \lambda m_\mu(T_c) = -\frac{1}{\inf \text{spec} \mathfrak{F} V \mathfrak{F}^*} = -\frac{1}{\sqrt{\mu} e_\mu}.
\]
Together with the asymptotic behavior $m_\mu(T) \sim \mu^{-1/2} \ln(\mu/T)$ as $T \to 0$, this implies the leading order behavior of $\ln(\mu/T_c)$ as $\lambda \to 0$ and proves the statement in $(i)$.

In order to see $(ii)$ it suffices to realize that, in the case $\mathfrak{F} V \mathfrak{F}^* \geq 0$, Eq. (32) yields $m_{T_c} \geq \text{const} / \lambda^2$.

The statement $(iii)$ is a consequence of the fact that
\[
\mathfrak{F}|V|^{1/2} \frac{1}{1 + \lambda V^{1/2} M T_c |V|^{1/2}} V^{1/2} \mathfrak{F}^* \geq \mathfrak{F}[V - \lambda |V|] \mathfrak{F}^* \geq 0,
\]
for $\lambda$ small enough. We refer to [6] for details.

**Proof of Theorem 3.** To obtain the next order, we use Eq. (32) and employ first order perturbation theory. Since $\mathfrak{F} V \mathfrak{F}^*$ is compact and $\inf \text{spec} \mathfrak{F} V \mathfrak{F}^* < 0$ by assumption, first order perturbation theory implies that
\[
m_\mu(T_c) = \frac{-1}{\lambda \langle u|\mathfrak{F} V \mathfrak{F}^*|u \rangle - \lambda^2 \langle u|\mathfrak{F} V M T_c V \mathfrak{F}^*|u \rangle + O(\lambda^3)},
\]
where $u$ is the (normalized) eigenfunction corresponding to the lowest eigenvalue of $\mathfrak{F} V \mathfrak{F}^*$. (In case of degeneracy, one has to choose the $u$ that minimizes the $\lambda^2$ term in the denominator of (33) among all such eigenfunctions.)
Eq. (33) is an implicit equation for $T_c$. Since $\mathcal{F}_V M_T V \mathcal{F}^*$ is uniformly bounded and $T_c \to 0$ as $\lambda \to 0$, we have to evaluate the limit of $\langle u | \mathcal{F}_V M_T V \mathcal{F}^* | u \rangle$ as $T \to 0$. To this aim, let $\varphi = V \mathcal{F}^* u$. Then

$$\langle u | \mathcal{F}_V M_T V \mathcal{F}^* | u \rangle = \int_{\mathbb{R}^3} \frac{1}{K_{T,\mu}(p)} |\hat{\varphi}(p)|^2 \, dp - m_\mu(T) \int_{\Omega_\mu} |\hat{\varphi}(p)|^2 \, d\omega(p)$$

(34)

Recall that $K_{T,\mu}(p)$ converges to $|p^2 - \mu|$ as $T \to 0$. Using the Lipschitz continuity of the spherical average of $|\hat{\varphi}(p)|^2$ (see [9, Eq. (29)]) it is easy to see that

$$\lim_{T \to 0} \langle u | \mathcal{F}_V M_T V \mathcal{F}^* | u \rangle = \langle u | W_\mu | u \rangle,$$

(35)

with $W_\mu$ defined in (11). In particular, combining (33) and (35), we have thus shown that

$$\lim_{\lambda \to 0} \left( m_\mu(T_c) + \frac{1}{\inf \text{spec} \left( \lambda \sqrt{\mu} V - \lambda^2 W_\mu \right)} \right) = 0.$$

(36)

The statement follows by using the asymptotic behavior ([9, Lemma 1])

$$m_\mu(T) = \frac{1}{\sqrt{\mu}} \left( \ln \frac{\mu}{T} + \gamma - 2 + \ln \frac{8}{\pi} + o(1) \right)$$

(37)

in the limit of small $T$, where $\gamma \approx 0.5772$ is Euler’s constant. $\Box$

5. PROOF OF THEOREMS 4 AND 5

5.1. Sufficient condition for $\Xi > 0$. If $\epsilon_\mu(V) < 0$ we know that the BCS equation (19) has a solution, meaning the system shows a superfluid phase for $T = 0$. This is not sufficient, however, to guarantee the existence of a positive gap $\Xi > 0$ nor the continuity of the momentum distribution $\gamma$. Unlike the case of the critical temperature, we lack a linear criterion which allows a precise characterization of potentials $V$ giving rise to a strictly positive gap. We are, however, able to derive sufficient conditions, namely a fast enough decay of $V$. Under such assumptions one can show the equivalence of the positivity of $\Xi$ and the continuity of $\gamma$. Both hold true if additionally $\int V < 0$. It remains an open problem to find examples for $V$ such that $\epsilon_\mu < 0$ but $\Xi = 0$.

Lemma 1. Assume that $V \in L^{3/2}$ and that $V(x)|x| \in L^{6/5}(\mathbb{R}^3)$. Then $\Xi > 0$ if and only if $\gamma$ is continuous.

Proof. It is easy to deduce [8] from the BCS equation (19) that $\hat{\alpha}$ is in $C^0(\mathbb{R}^3)$. Because of (17) the continuity of $\gamma$ is equivalent to the fact that
$|\hat{\alpha}| \equiv 1/4$ on the Fermi $\Omega_\mu$. From the relation $\Delta(p) = 2E(p)\hat{\alpha}(p)$ one obtains

$$|\hat{\alpha}(p)|^2 = \frac{\frac{1}{4}}{\sqrt{\frac{(p^2 - \mu)^2}{|\Delta(p)|^2} + 1}}, \quad (38)$$

and we can conclude that $|\hat{\alpha}|^2 = 1/4$ on the Fermi surface if and only if $\Delta(p)$ does not vanish on $\Omega_\mu$. Namely, suppose that $\Delta$ vanishes at some $p'$ on the Fermi surface. Since $\alpha \in H^1(\mathbb{R}^3)$ we see that $\alpha \in L^2(\mathbb{R}^3) \cap L^6(\mathbb{R}^3)$ and hence, together with $V(x)|x| \in L^{6/5}$, Hölder’s inequality implies that $\Delta(x)|x| = V(x)\alpha(x)|x| \in L^1(\mathbb{R}^3)$. We thus infer that $\Delta(p)$ is Lipschitz continuous, meaning that $\Delta(p)$ cannot decay slower to 0 than linear. Hence there is a $\delta$ such that $\lim_{p \to p'} \frac{(p^2 - \mu)^2}{|\Delta(p)|^2} \geq \delta$ and $|\alpha(p')|^2 \leq \frac{1}{4} \frac{1}{\sqrt{3\delta + 1}} < \frac{1}{4} \quad \Box$

**Proof of Theorem 4.** Let $\alpha$ be a global minimizer of the BCS functional $F_0$. Then for any $\hat{g} \in C_0^\infty(\mathbb{R}^3)$ such that $|\hat{\alpha} + \epsilon \hat{g}| \leq 1/2$ for $\epsilon$ small enough,

$$\frac{d^2}{d\epsilon^2}F(\alpha + \epsilon \hat{g})\bigg|_{\epsilon=0} \geq 0. \quad (39)$$

A straightforward calculation yields

$$\frac{d^2}{d\epsilon^2}F(\alpha + \epsilon \hat{g})\bigg|_{\epsilon=0} = 2\langle g|E(-i\nabla) + \lambda V|g\rangle + 8 \int \frac{|p^2 - \mu|\Re(\hat{\alpha}\hat{g})|^2}{[1 - 4|\hat{\alpha}|^2]^{3/2}}. \quad (40)$$

Assume now that $\Xi = 0$. This means that $\Delta$ has to vanish at some point $p' \in \Omega_\mu$. Then there has to be an open neighborhood on $\Omega_\mu$ on which $\Delta$ vanishes. In fact, according to the argument in the proof of Lemma 1 (Eq. (38) and Lipschitz continuity of $\Delta$) there is a neighborhood $N_\delta(p') \subset \mathbb{R}^3$ in the vicinity of $p'$ where $|\hat{\alpha}|^2 < 1/4 - \delta$ for some $\delta > 0$, and hence $\Delta$ vanishes on $N_\delta(p') \cap \Omega_\mu$. Note that $\Delta$ cannot vanish at one point on the Fermi surface since otherwise $|\hat{\alpha}| = 1/2$ except on one point, which contradicts the continuity of $\hat{\alpha}$.

We shall now construct an appropriate trial sequence $\hat{g}_n$, essentially supported in $N_\delta$, such that

$$\lim_{n \to \infty} \left[ \langle g_n|E(-i\nabla)|g_n\rangle + 8 \int \frac{|p^2 - \mu|\Re(\hat{\alpha}\hat{g}_n)|^2}{[1 - 4|\hat{\alpha}|^2]^{3/2}} \right] = 0 \quad (41)$$

and

$$\lim_{n \to \infty} \langle g_n|V|g_n\rangle = \int_{\mathbb{R}^3} V(x)dx < 0. \quad (42)$$

This gives a contradiction to (39).

For the construction of $g_n$ let $\psi_n \in L^2(\Omega_\mu)$ be supported in $N_\delta(p') \cap \Omega_\mu$ such that $\psi_n(s) \to \delta(s - p')$ as $n \to \infty$. Choose also $f_n \in L^2(\mathbb{R}_+; t^2dt)$ such that $f_n(t) \to \delta(\sqrt{t} - t)$, and let $\hat{g}_n(p) = \psi_n(s)f_n(|p|)$. Observe that on $N(p')$, $E(p) = \frac{|p^2 - \mu|}{\sqrt{1 - 4|\hat{\alpha}(p)|^2}} \leq c|p^2 - \mu|$ for some constant $c$, and thus grows linearly in $|p|$ close to $\sqrt{\mu}$. Hence one easily sees that the problem here is equivalent to the existence of a negative eigenvalue of the relativistic operator $|p| + V$ in one dimension. Using the Birman-Schwinger
principle, it is easy to see that the latter always has a negative eigenvalue if ∫V < 0.

5.2. Proof of Theorem 5. The energy gap of the system at zero temperature, Ξ = inf_p E(p), with
\[ E(p) = |p^2 - \mu|/\sqrt{1 - 4|\hat{\alpha}(p)|^2} = \sqrt{|p^2 - \mu|^2 + |\Delta(p)|^2}, \]
depends on the behavior of |\Delta(p)| on the Fermi sphere. The function Δ is not unique, in general and need not be radial even in case V is radial.

Under the assumption that \( \hat{V} \) is non-positive and \( \hat{V}(0) < 0 \), we shall argue in the following that the minimizer of the BCS functional (18) at \( T = 0 \) is unique [9, Lemma 3]. If, in addition, V is radial, this necessarily implies that also the minimizer has to be radial. Since \( \hat{V} \leq 0 \),
\[ \int_{\mathbb{R}^d} \hat{\alpha}(p) \hat{V}(p - q) \hat{\alpha}(q) \, dp dq \geq \int_{\mathbb{R}^d} |\hat{\alpha}(p)| |\hat{V}(p - q)| |\hat{\alpha}(q)| \, dp dq. \] (43)

Hence, if \( \hat{\alpha}(p) \) is a minimizer of \( F_0 \), (18), so is |\( \hat{\alpha}(p) \)|.

Assume now there are two different minimizers \( f \neq g \), both with non-negative Fourier transform. Since \( t \to 1 - \sqrt{1 - 4t} \) is strictly convex for \( 0 \leq t \leq 1/2 \) we see that \( \psi = \frac{1}{\sqrt{2}} f + i \frac{1}{\sqrt{2}} g \), satisfies
\[ F_0(\psi) < \frac{1}{2} F_0(f) + \frac{1}{2} F_0(g). \]
This is a contradiction to \( f, g \) being distinct minimizers, and hence \( f = g \).

In particular, the absolute value of a minimizer has to be unique. If \( \hat{\alpha} \) is the unique non-negative minimizer, then one easily sees from the BCS equation (using \( \int V < 0 \)) that \( \hat{\alpha} \) is, in fact, strictly positive. Hence any minimizer is non-vanishing. But [13] is strict for non-vanishing functions, unless \( \hat{\alpha}(p) = |\hat{\alpha}(p)| e^{i\kappa} \) for some constant \( \kappa \in \mathbb{R} \).

To summarize, we have just argued that for \( \hat{V} \leq 0 \), \( \hat{V}(0) < 0 \) and V radial, the solution of the BCS equation is unique, up to a constant phase, and it is radially symmetric. This will enable us to apply the same methods as we used for the critical temperature \( T_c \) in order to derive the asymptotic behavior of Ξ.

The variational equation (19) for the minimizer of \( F_0 \) can be rewritten in terms of \( \alpha \) as
\[ (E(-i\nabla) + \lambda V(x)) \alpha(x) = 0. \] (44)
That is, \( \alpha \) is an eigenfunction of the pseudodifferential operator \( E(-i\nabla) + \lambda V(x) \), with zero eigenvalue. Since \( \hat{V} \leq 0 \) and \( \hat{\alpha}(p) \) is non-negative we can even conclude that \( \alpha \) has to be the ground state.

Similarly to the proof of Theorem 3, we can now employ the Birman-Schwinger principle to conclude from (44) that \( \phi_\lambda = V^{1/2} \alpha \) satisfies the eigenvalue equation
\[ \lambda V^{1/2} \frac{1}{\sqrt{(p^2 - \mu)^2 + |\Delta(p)|^2}} |V|^{1/2} \phi_\lambda = -\phi_\lambda. \] (45)
Moreover, there are no eigenvalues smaller than $-1$ of the operator on the left side of (45).

Let

$$\tilde{m}_\mu(\Delta) = \max \left\{ \frac{1}{4\pi \mu} \int_{\mathbb{R}^3} \left( \frac{1}{\sqrt{(p^2 - \mu)^2 + |\Delta(p)|^2}} - \frac{1}{p^2} \right) \, dp, \, 0 \right\}. \quad (46)$$

Similarly to (27), we split the operator in (45) as

$$V^{1/2} \frac{1}{E(-i\nabla)} |V|^{1/2} = \tilde{m}_\mu(\Delta) V^{1/2} \tilde{\gamma}^* \tilde{\gamma} |V|^{1/2} + V^{1/2} M_\Delta |V|^{1/2}. \quad (47)$$

Again one shows that $V^{1/2} M_\Delta |V|^{1/2}$ is bounded in Hilbert-Schmidt norm, independently of $\Delta$. Moreover, as in the proof of Theorem 3 (cf. Eqs. (28)–(30)), the fact that the lowest eigenvalue of $V^{1/2} E(-i\nabla)^{-1} |V|^{1/2}$ is $-1$ is, for small enough $\lambda$, equivalent to the fact that the selfadjoint operator on $L^2(\Omega_\mu)$

$$\tilde{\gamma} |V|^{1/2} \frac{\lambda \tilde{m}_\mu(\Delta)}{1 + \lambda V^{1/2} M_\Delta |V|^{1/2}} V^{1/2} \tilde{\gamma}^*$$

has $-1$ as its smallest eigenvalue. This implies that $\lim_{\lambda \to 0} \lambda \tilde{m}_\mu(\Delta) = -1/(\sqrt{\mu} e_\mu)$ and hence, in particular, $\tilde{m}_\mu(\Delta) \sim \lambda^{-1}$ as $\lambda \to 0$. The unique eigenfunction corresponding to the lowest eigenvalue $e_\mu < 0$ of $V_\mu$ is, in fact, a positive function, and because of radial symmetry of $V$ it is actually the constant function $u(p) = (4\pi \mu)^{-1/2}$.

We now give a precise characterization of $\Delta(p)$ for small $\lambda$.

**Lemma 2.** Let $V \in L^1 \cap L^3/2$ be radial, with $\hat{V} \leq 0$ and $\hat{V}(0) < 0$, and let $\Delta$ be given in (18), with $\alpha$ the unique minimizer of the BCS functional (18). Then

$$\Delta(p) = -f(\lambda) \left( \int_{\Omega_\mu} \hat{V}(p - q) \, d\omega(q) + \lambda \eta_\lambda(p) \right) \quad (48)$$

for some positive function $f(\lambda)$, with $\|\eta_\lambda\|_{L^\infty(\mathbb{R}^3)}$ bounded independently of $\lambda$.

**Proof.** Because of (45), $\tilde{\gamma} |V|^{1/2} \phi_\lambda$ is the eigenfunction of (47) corresponding to the lowest eigenvalue $-1$. Note that because of radial symmetry, the constant function $u(p) = (4\pi \mu)^{-1/2}$ is an eigenfunction of (47). For small enough $\lambda$ it has to be eigenfunction corresponding to the lowest eigenvalue (since it is the unique ground state of the compact operator $\tilde{\gamma} V \tilde{\gamma}^*$). We conclude that

$$\phi_\lambda = f(\lambda) \frac{1}{1 + \lambda V^{1/2} M_\Delta |V|^{1/2}} V^{1/2} \tilde{\gamma}^* u = f(\lambda) \left( V^{1/2} \tilde{\gamma}^* u + \lambda \xi_\lambda \right) \quad (49)$$

for some normalization constant $f(\lambda)$. Note that $\|\xi_\lambda\|_2$ uniformly bounded for small $\lambda$, since both $V^{1/2} M_\Delta |V|^{1/2}$ and $V^{1/2} \tilde{\gamma}^*$ are bounded operators.
From (44) and the definition $\phi_{\lambda} = V^{1/2} \alpha$ we know that

$$\Delta(p) = 2E(p)\hat{\alpha}(p) = -2\lambda \widetilde{V}\alpha(p) = -2\lambda|V|^{1/2}\phi_{\lambda}(p).$$

In combination with (49) this implies that

$$\Delta(p) = -2\lambda f(\lambda) \left( \widetilde{V}\phi_{\lambda}(p) + \lambda \eta_{\lambda}(p) \right),$$

with $\eta_{\lambda} = |V|^{1/2} \xi_{\lambda}$. With $\|\eta_{\lambda}\|_{\infty} \leq (2\pi)^{-3/2} \|\eta_{\lambda}\|_{1} \leq (2\pi)^{-3/2} \|V\|_{1} \|\xi_{\lambda}\|_{2}$ by Schwarz’s inequality, we arrive at the statement of the Lemma.

With the aid of Lemma 2 and Lipschitz continuity of $\int_{\Omega_{\mu}} \hat{V}(p-q) \, d\omega(q)$ (which follows from $V \in L^{1}(\mathbb{R}^{3})$) it is not difficult to see that

$$\tilde{m}_{\mu}(\Delta) = \frac{1}{\sqrt{\mu}} \left( \ln \frac{\mu}{\Delta(\sqrt{\mu})} - 2 + \ln 8 + o(1) \right)$$

as $\lambda \to 0$. From Eq. (47) we now conclude that

$$\tilde{m}_{\mu}(\Delta) = \frac{1}{\lambda(u|\hat{\psi}V\hat{\psi}^{*}|u) - \lambda^{2} \langle u|\hat{\psi}V\Delta V\hat{\psi}^{*}|u \rangle + O(\lambda^{3})},$$

(51)

where $u(p) = (4\pi\mu)^{-1/2}$ is the normalized constant function on the sphere $\Omega_{\mu}$. Moreover, with $\varphi = V\hat{\psi}^{*}u$,

$$\langle u|\hat{\psi}V\Delta V\hat{\psi}^{*}|u \rangle = \int_{\mathbb{R}^{3}} \frac{1}{E(p)} |\varphi(p)|^{2} \, dp - \tilde{m}_{\mu}(\Delta) \int_{\Omega_{\mu}} |\varphi(\sqrt{\mu}p/|p|)|^{2} \, d\omega(p)$$

$$= \int_{\mathbb{R}^{3}} \left( \frac{1}{E(p)} |\varphi(p)|^{2} - |\varphi(\sqrt{\mu}p/|p|)|^{2} \right) + \frac{1}{p^{2}} |\varphi(\sqrt{\mu}p/|p|)|^{2} \, dp.$$ 

Using Lemma 2 and the fact that $\lim_{\lambda \to 0} f(\lambda) = 0$, we conclude that

$$\lim_{\lambda \to 0} \langle u|\hat{\psi}V\Delta V\hat{\psi}^{*}|u \rangle = \langle u|\mathcal{W}_{\mu}|u \rangle,$$

(52)

with $\mathcal{W}_{\mu}$ defined in (11). (Compare with Eqs. (34) and (35).) In combination with (50) and (51) and the definition of $B_{\mu}$ in (12), this proves that

$$\lim_{\lambda \to 0} \left( \ln \frac{\mu}{\Delta(\sqrt{\mu})} + \frac{\pi}{2\sqrt{\mu} \langle u|B_{\mu}|u \rangle} \right) = 2 - \ln(8).$$

The same holds true with $\langle u|B_{\mu}|u \rangle$ replaced by $b_{\mu}(\lambda) = \inf \text{spec} B_{\mu}$, since under our assumptions on $V$ the two quantities differ only by terms of order $\lambda^{3}$.

Now, by the definition of the energy gap $\Xi$ in (20), $\Xi \leq \Delta(\sqrt{\mu})$. Moreover,

$$\Xi \geq \min_{|p^{2} - \mu| \leq \Xi} |\Delta(p)|,$$

from which it easily follows that $\Xi \geq \Delta(\sqrt{\mu})(1 - o(1))$, using Lemma 2. This proves Theorem 5.
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