cPCWE - PERTURBED CONVECTIVE WAVE EQUATION BASED ON COMPRESSIBLE FLOWS

Stefan Schoder
Institute of Fundamentals and Theory in Electrical Engineering (IGTE)
Graz University of Technology
8010 Graz, Austria
stefan.schoder@tugraz.at

ABSTRACT
This work derives a variant of the perturbed convective wave equation based on the acoustic perturbation equations for compressible flows. In particular, the derivation reformulates the relation of Helmholtz’s decomposition to the acoustic and source potential definition. The detailed roadmap of a possible implementation is presented algorithmically. Finally, initial results on the sound prediction capabilities concerning a mixing layer example are presented.

Keywords Aeroacoustics · Fluid dynamics · Acoustics · Helmholtz’s decomposition · Flow solver · Acoustic Energy Conversion

1 Introduction
Aeroacoustic analogies (e.g., Lighthill’s) compute noise radiation efficiently. However, the obtained fluctuating field only converges to the acoustic field in steady flow regions. As first recognized by Phillips [1] and Lilley [2], the source terms responsible for mean flow-acoustics interactions should be part of the wave operator. This can also be achieved for Lighthill’s analogy, which can be adapted for a uniform background flow to account for convection effects inside the wave operator [3].

Another approach for computing aeroacoustics is based on a systematic decomposition of the field properties assumed to be related to acoustics and the ‘pure’ fluid motion. This approach circumvents that sources depend on the acoustic solution and provides a rigorous definition of acoustics inside flow regions (at least for nearly incompressible flows, as shown in [4]). Ribner [5] formulated the dilatation equation such that the fluctuating pressure is decomposed in a pseudo pressure and an acoustic pressure part \( p' = p^0 + p^a \). Hardin and Pope [6] updated this idea and formulated their viscous/acoustic splitting technique expansion about the incompressible flow (EIF), where they introduced a density correction \( \rho_1 \). The EIF formalism was modified over the years substantially in [7–9].

Being more general, applicable for a wider Mach number range, and starting from the linearized Euler equations (LEE), the field variables \( (\rho, u, p) \) are Reynolds decomposed in a temporal mean component \( \langle \cdot \rangle \) and a fluctuating component \( \cdot' \). Bailly et al. [10, 11] indicate important aeroacoustic source terms on the momentum equation of the LEE. Over the years, the LEE were modified to guarantee that only acoustic waves are propagated [12]. Significant contributions based on the LEE were derived by [13, 14, 15, 16, 17]. Ewert and Schröder [18] proposed a different decomposition technique, leading to the acoustic perturbation equations (APE). Instead of decomposing the flow field, the source terms of the wave equations are projected onto the acoustic modes obtained from LEE. Hüppe [19] derived a computationally efficient reformulation of the APE-2 system and named it perturbed convective wave equation (PCWE). Several aeroacoustic low Mach number flow applications have been addressed using the PCWE model successfully [20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36]. The PCWE is valid for obtaining near-field acoustics of incompressible flows [4]. Focusing on a two-way coupling of the flow and acoustic variables, Ewert and Kreuzinger developed a computational workflow for low Mach numbers recently [37]. A first attempt to establish a scalar wave equation for intermediate subsonic Mach number flows was made by the AWE-PO [32, 38, 39] recently. In this sense, seeking a rigorous definition and an accurate formulation for a large variety of Mach number flows is still...
cPCWE

a challenge worth investigating. Kempf and Munz [41] use a high-order DG method for a compressible flow solver with the APE-4 for accurate sound predictions. With the idea of APE and Helmholtz’s decomposition, this work aims to extend the PCWE equation for subsonic Mach number flows (the so-called cPCWE).

Regarding the state-of-the-art, we will derive a wave equation based on the fundamental ideas of [18]. First, the ideas for incompressible flows are collected and joined to a valid concept (methodological generalization). Second, we extend the concept to compressible flows (physical generalization). The derived aeroacoustic model leads to a stable scalar wave equation that solves an (approximated) compressible potential for subsonic flows. The wave operator only excites longitudinal wave modes (acoustic modes) and includes mean convection effects. The goal of this new wave equation is that it holds for subsonic Mach number flows and recovers the PCWE in the incompressible limit. The investigation on the cPCWE is beneficial to increase further the theoretical understanding of flow-induced sound of subsonic Mach number flows.

2 Wave equation

In this section, a convective wave equation based on compressible flow data (cPCWE) is derived. The cPCWE can be derived efficiently from the APE-1 system [18] and the use of Helmholtz’s decomposition [42]. Firstly, the variables are defined by linearization to distinguish between steady and fluctuating flow components. Secondly, vortical and acoustical perturbations are separated such that the following definitions are obtained

\[
\begin{align*}
  p &= p_0 + p', \quad \rho = \rho_0 + \rho', \\
  u &= u_0 + u', \quad v = v + v_a = u_0 + \nabla \times A - \nabla \psi_a,
\end{align*}
\]

with the pressure \( p \), the mean pressure \( p_0 \), the perturbation pressure \( p' \), the fluid dynamic perturbation pressure \( p_v \), the acoustic perturbation pressure \( p_a \), the density \( \rho \), the mean density \( \rho_0 \), the perturbation density \( \rho' \), the velocity \( u \), the mean velocity \( u_0 \), the perturbation velocity \( u' \), the fluid dynamic perturbation velocity \( u_v = \nabla \times A \), the acoustic perturbation velocity \( u_a = -\nabla \psi_a \) and the vector potential \( A \). To eliminate the compressible part of the flow velocity, the Poisson equation \( \Delta s = \nabla \cdot \mathbf{u}' \) has to be solved and the vortical fluctuating velocity can be obtained by \( u_v = u' - \nabla s \). We define the acoustic field as irrotational by the acoustic scalar potential \( \psi_a \). Considering a general compressible flow, we arrive at the following perturbation equations

\[
\begin{align*}
  \frac{\partial p'}{\partial t} + u_0 \cdot \nabla p' + \rho_0 c_0^2 \nabla \cdot u_a &= 0, \\
  \rho_0 \frac{\partial u_a}{\partial t} + \rho_0 \nabla (u_0 \cdot u_a) + \nabla p' &= \rho_0 \nabla \Phi_p
\end{align*}
\]

with the isentropic speed of sound \( c_0 \) and the source potential \( \Phi_p \). In this preliminary derivation, we neglect viscous effects and discard the vorticity mode according to [18]. Rewriting equation (5) yields the definition of the fluctuating pressure

\[ p' = \rho_0 \frac{\partial \psi_a}{\partial t} + \rho_0 u_0 \cdot \nabla \psi_a + \rho_0 \Phi_p = \rho_0 \frac{D \psi_a}{Dt} + \rho_0 \Phi_p. \tag{6} \]

The first part accounts for the acoustic pressure \( p_a \) and the second part is a result of the following Poisson equation [18]

\[ \Delta \Phi_p = -\nabla \cdot [(u_v \cdot \nabla)u_v'] + (u_0 \cdot \nabla)u_v + (u_v \cdot \nabla)u_0 + T' \nabla s_0 - s' \nabla T_0. \tag{7} \]

The first part of the source term includes the self-noise of vortical structures, the second and third terms the shear-noise interactions, the fourth and fifth term account for thermal effects. Substituting (6) into (4) yields the cPCWE

\[ \frac{1}{c_0^2} \frac{D^2 \psi_a}{Dt^2} - \Delta \psi_a = -\frac{1}{\rho_0 c_0^2} \frac{D \Phi_p}{Dt}. \tag{8} \]

This convective wave equation describes acoustic sources generated by compressible flow structures and their wave propagation through flowing media. In addition, instead of the original unknowns \( p_a \) and \( v_a \), just one scalar \( \psi_a \) unknown. As shown in [33] and consistent with the pressure correction equation, the fluctuating vortical pressure in the overall domain can be recovered by

\[ p_v = \rho_0 \Phi_p. \tag{9} \]

Finally, we have derived a scalar wave equation that separates the source generation processes of compressible flows and the linear acoustic propagation.
3 Computational workflow

The algorithm can be used to compute the acoustic results. From an algorithmic view, we have to do four tasks. Firstly, we are solving the compressible flow equations. Secondly, we solve Poisson’s equation to obtain the filtered velocity in every time step

\[ \Delta \phi = \nabla \cdot u' \quad \text{with} \quad u_v = u' - \nabla \phi. \]  \hspace{1cm} (10)

Every time step can be processed independently. The filtered velocity is used to assemble the cPCWE source. Thirdly, we filter this source with the same operator matrix . Finally, we are solving the wave propagation simulation for the time series. To sum up, depending on the number of processors and processes one can submit, the algorithm is of similar complexity than the original PCWE or any other aeroacoustic wave equation. The solution processes for the Laplace equations can be highly parallelized. In the presence of internal wall boundaries near the sources, proper wall boundary conditions must be found for Poisson’s equation.

4 Alternative formulation of the cPCWE

By inserting the definition of the fluctuating pressure again into the wave equation, a Poisson equation similar to Doak’s idea for time-stationary momentum fluctuations can be derived

\[ \rho_0 \Delta \psi_a = \frac{1}{c_0^2} \frac{Dp'}{Dt}. \]  \hspace{1cm} (11)

5 Preliminary results

As in the previous study, this two-dimensional isothermal mixing layer application is considered to assess the validity of acoustic wave equations. The acoustic intensity

\[ L_I = 10 \log \frac{I}{I_0} \]  \hspace{1cm} (12)

is used, with \( I = \langle p'^2 \rangle / (\rho_0 c_0) \) and \( I_0 = 10^{-12} \text{W.m}^{-2} \) to compare the cPCWE results to the acoustic predictions of Lighthill’s equation (LH) and the conservation equations (DNS) as reference. In figure the intensity predicted by cPCWE shows good agreement with the one predicted by LH and the DNS. The deviations of the cPCWE and LH results are less than 1.5 dB in the rapid flow region and less than 2 dB in the slow flow region.

6 Future Work

This short working paper presents the derivation of the cPCWE, and we are happy to receive feedback. First results are presented for a two-dimensional isothermal mixing layer. The convergence of the numerical algorithm is assessed next. Furthermore, we are looking forward to collaborations on the topic to advance the theoretical understanding of flow-induced sound for subsonic flows.

References

[1] O. M. Phillips, On the generation of sound by supersonic turbulent shear layers, Journal of Fluid Mechanics 9 (1) (1960) 1–28.
Figure 1: Preliminary computations of the acoustic intensity $L_I$ depending on the angle $\theta$ a) in the rapid flow below the mixing layer and b) in the slow flow region above the mixing layer. – – DNS [40], - - - Lighthill’s equation [40], - - - cPCWE.

[2] G. M. Lilley, On the noise from jets, AGARD-CP-131, 1974.
[3] X. Gloerfelt, C. Bailly, D. Juvé, Direct computation of the noise radiated by a subsonic cavity flow and application of integral methods, Journal of sound and vibration 266 (1) (2003) 119–146.
[4] P. Maurerlehner, S. Schoder, J. Tieber, C. Freidhager, H. Steiner, G. Brenn, K.-H. Schafer, A. Ennemoser, M. Kaltenbacher, Aeroacoustic formulations for confined flows based on incompressible flow data, Acta Acust. 6 (2022) 45, doi:10.1051/aacus/2022041
[5] H. S. RIBNER, AERODYNAMIC SOUND FROM FLUID DILATATIONS - A Theory of the Sound from Jets and Other Flows, Tech. rep., Institute for Aerospace Studies, University of Toronto (1962).
[6] J. C. Hardin, D. S. Pope, An acoustic/viscous splitting technique for computational aeroacoustics, Theoretical and Computational Fluid Dynamics 6 (5-6) (1994) 323–340.
[7] W. Z. Shen, J. N. Sørensen, Aeroacoustic modelling of low-speed flows, Theoretical and Computational Fluid Dynamics 13 (4) (1999) 271–289.
[8] W. Z. Shen, J. N-ocute, R. S-ocute, R. S-ocute, Comment on the aeroacoustic formulation of Hardin and Pope, AIAA journal 37 (1) (1999) 141–143.
[9] S. A. Slimon, M. C. Soteriou, D. W. Davis, Computational aeroacoustics simulations using the expansion about incompressible flow approach, AIAA journal 37 (4) (1999) 409–416.
[10] C. Bailly, D. Juve, Numerical solution of acoustic propagation problems using linearized Euler equations, AIAA journal 38 (1) (2000) 22–29.
[11] C. Bogey, C. Bailly, D. Juvé, Computation of flow noise using source terms in linearized Euler’s equations, AIAA journal 40 (2) (2002) 235–243.
[12] W. De Roeck, G. Rubio, M. Baelmans, W. Desmet, Toward accurate hybrid prediction techniques for cavity flow noise applications, International journal for numerical methods in fluids 61 (12) (2009) 1363–1387.
[13] C.-D. Munz, M. Dumbser, M. Zucchini, The multiple pressure variables method for fluid dynamics and aeroacoustics at low Mach numbers, Numerical methods for hyperbolic and kinetic problems 7 (2003) 335–359.
[14] M. Munz, C.-D.and Dumbser, S. Roller, Linearized acoustic perturbation equations for low Mach number flow with variable density and temperature, Journal of Computational Physics 224 (1) (2007) 352–364.
[15] S. Roller, T. Schwartzkopff, R. Fortenbach, M. Dumbser, C.-D. Munz, Calculation of low Mach number acoustics: a comparison of MPV, EIF and linearized Euler equations, ESAIM: Mathematical Modelling and Numerical Analysis 39 (3) (2005) 561–576.
[16] J. Seo, Y. J. Moon, Perturbed compressible equations for aeroacoustic noise prediction at low mach numbers, AIAA journal 43 (8) (2005) 1716–1724.
5

[17] J. H. Seo, Y. J. Moon, Linearized perturbed compressible equations for low Mach number aeroacoustics, Journal of Computational Physics 218 (2) (2006) 702–719.

[18] R. Ewert, W. Schröder, Acoustic perturbation equations based on flow decomposition via source filtering, Journal of Computational Physics 188 (2) (2003) 365–398.

[19] A. Hüppe, J. Grabinger, M. Kaltenbacher, A. Reppenhagen, G. Dutzler, W. Kühnel, A Non-Conforming Finite Element Method for Computational Aeroacoustics in Rotating Systems, in: 20th AIAA/CEAS Aeroacoustics Conference, 2014, p. 2739.

[20] S. Schoder, M. Kaltenbacher, Hybrid aeroacoustic computations: State of art and new achievements, Journal of Theoretical and Computational Acoustics 27 (04) (2019) 1950020.

[21] S. Schoder, M. Weitz, P. Maurerlehner, A. Hauser, S. Falk, S. Kniesburges, M. Döllinger, M. Kaltenbacher, Hybrid aeroacoustic approach for the efficient numerical simulation of human phonation, The Journal of the Acoustical Society of America 147 (2) (2020) 1179–1194.

[22] S. Schoder, C. Junger, M. Kaltenbacher, Computational aeroacoustics of the eaa benchmark case of an axial fan, Acta Acustica 4 (5) (2020) 22.

[23] S. Schoder, A. Wurzinger, C. Junger, M. Weitz, C. Freidhager, K. Roppert, M. Kaltenbacher, Application limits of conservative source interpolation methods using a low mach number hybrid aeroacoustic workflow, Journal of Theoretical and Computational Acoustics 29 (01) (2021) 2050032.

[24] S. Schoder, P. Maurerlehner, A. Wurzinger, A. Hauser, S. Falk, S. Kniesburges, M. Döllinger, M. Kaltenbacher, Aeroacoustic source term characterization of the human voice production–perturbed convective wave equation, Applied Sciences 11 (6) (2021) 2614.

[25] S. Schoder, C. Junger, M. Weitz, M. Kaltenbacher, Conservative source term interpolation for hybrid aeroacoustic computations, in: 25th AIAA/CEAS aeroacoustics conference, 2019, p. 2538.

[26] S. Falk, S. Kniesburges, S. Schoder, B. Jakubaß, P. Maurerlehner, M. Echtnerach, M. Kaltenbacher, M. Döllinger, 3d-fv-fe aeroacoustic larynx model for investigation of functional based voice disorders, Frontiers in physiology 12 (2021) 616985.

[27] M. Lasota, P. Šidlof, M. Kaltenbacher, S. Schoder, Impact of the sub-grid scale turbulence model in aeroacoustic simulation of human voice, Applied Sciences 11 (4) (2021) 1970.

[28] P. Maurerlehner, S. Schoder, C. Freidhager, A. Wurzinger, A. Hauser, F. Kraxberger, S. Falk, S. Kniesburges, M. Echtnerach, M. Döllinger, et al., Efficient numerical simulation of the human voice, e & i Elektrotechnik und Informationstechnik 138 (3) (2021) 219–228.

[29] S. Schoder, K. Roppert, opencfs: Open source finite element software for coupled field simulation–part acoustics, arXiv preprint arXiv:2207.04443 (2022).

[30] S. Schoder, F. Kraxberger, S. Falk, A. Wurzinger, K. Roppert, S. Kniesburges, M. Döllinger, M. Kaltenbacher, Error detection and filtering of incompressible flow simulations for aeroacoustic predictions of human voice, The Journal of the Acoustical Society of America 152 (3) (2022) 1425–1436.

[31] L. Tieghi, S. Becker, A. Corsini, G. Delibra, S. Schoder, F. Czwielong, Machine-learning clustering methods applied to detection of noise sources in low-speed axial fan, Journal of Engineering for Gas Turbines and Power (2022).

[32] S. Schoder, M. Kaltenbacher, É. Spieser, H. Vincent, C. Bogey, C. Bailly, Aeroacoustic wave equation based on pierce’s operator applied to the sound generated by a mixing layer, in: 28th AIAA/CEAS Aeroacoustics 2022 Conference, 2022, p. 2896.

[33] M. Tautz, Aeroacoustic Noise Prediction of Automotive HVAC Systems, FAU University Press, 2019.

[34] J. Valášek, M. Kaltenbacher, P. Sváček, On the application of acoustic analogies in the numerical simulation of human phonation process, Flow, Turbulence and Combustion 102 (1) (2019) 129–143, doi:10.1007/s10494-018-9900-z.

[35] M. Lasota, P. Šidlof, P. Maurerlehner, M. Kaltenbacher, S. Schoder, Anisotropic minimum dissipation subgrid-scale model in hybrid aeroacoustic simulations of human phonation, The Journal of the Acoustical Society of America 153 (2) (2023) 1052–1063.

[36] S. Schoder, A. Wurzinger, Dataset cylincf-01 creation pipeline: Circular cylinder in a cross flow, mach number 0.03 and reynolds number 200, arXiv preprint arXiv:2303.05265 (2023).

[37] R. Ewert, J. Kreuzinger, Hydrodynamic/acoustic splitting approach with flow-acoustic feedback for universal subsonic noise computation, Journal of Computational Physics 444 (2021) 110548, doi:10.1016/j.jcp.2021.110548.
[38] E. Spieser, Modélisation de la propagation du bruit de jet par une méthode adjointe formulée pour l’acoustique potentielle, Ph.D. thesis, University Lyon (2020).

[39] S. Schoder, É. Spieser, H. Vincent, C. Bogey, C. Bailly, Noise prediction using the aeroacoustic wave equation based on pierce’s operator, AIAA Journal (2022 (in review)).

[40] S. Schoder, É. Spieser, H. Vincent, C. Bogey, C. Bailly, Acoustic modeling using the aeroacoustic wave equation based on pierce’s operator, AIAA Journal (2023) 1–10.

[41] D. Kempf, C.-D. Munz, Zonal direct-hybrid aeroacoustic simulation of trailing edge noise using a high-order discontinuous galerkin spectral element method, Acta Acustica 6 (2022) 39. doi:10.1051/aacus/2022030

[42] S. Schoder, M. Kaltenbacher, K. Roppert, Helmholtz’s decomposition applied to aeroacoustics, in: 25th AIAA/CEAS Aeroacoustics Conference, 2019, p. 2561.

[43] S. Schoder, K. Roppert, opencfs-data: Data pre-post-processing tool for opencfs–aeroacoustics source filters, arXiv preprint arXiv:2302.03637 (2023).

[44] P. Doak, Momentum potential theory of energy flux carried by momentum fluctuations, Journal of sound and vibration 131 (1) (1989) 67–90.

[45] S. Schoder, Helmholtz’s decomposition for aeroacoustics using a standard flow solver, arXiv preprint arXiv:2207.08144 (2022).