An embedding problem for finite local torsors over twisted curves

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Abstract
In his previous paper, the author proposed as a problem a purely inseparable analogue of the Abhyankar conjecture for affine curves in positive characteristic and gave a partial answer to it, which includes a complete answer for finite local nilpotent group schemes. In the present paper, motivated by the Abhyankar conjectures with restricted ramifications due to Harbater and Pop, we study a refined version of the analogous problem, based on a recent work on tamely ramified torsors due to Biswas–Borne, which is formulated in terms of root stacks. We study an embedding problem to conclude that the refined analogue is true in the solvable case.

KEYWORDS
embedding problems, fundamental group schemes, root stacks, tamely ramified torsors

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1 INTRODUCTION

1.1 The Abhyankar conjectures with restricted ramifications

In [17], Grothendieck defined the étale fundamental group \( \pi_1^{\text{ét}}(X) \) for a scheme \( X \) as a profinite group which controls the Galois theory for finite étale coverings of \( X \). If \( X/\mathbb{C} \) is a complex algebraic variety, then one can calculate the étale fundamental group \( \pi_1^{\text{ét}}(X) \) of \( X \) as the profinite completion of the topological fundamental group \( \pi_1^{\text{top}}(X(\mathbb{C})) \) of the associated analytic space \( X(\mathbb{C}) \),

\[
\pi_1^{\text{ét}}(X) \simeq \pi_1^{\text{top}}(X(\mathbb{C})).
\]

In fact, by the Lefschetz principle, this description can be adopted to any algebraically closed field of characteristic 0. For example, we can find that the étale fundamental group \( \pi_1^{\text{ét}}(\mathbb{A}^1_k) \) of the affine line \( \mathbb{A}^1_k \) over an algebraically closed field \( k \) of characteristic 0 is trivial, i.e. \( \pi_1^{\text{ét}}(\mathbb{A}^1_k) = 0 \) because the associated Riemann surface \( \mathbb{A}^1(\mathbb{C}) \) is simply connected.

However, if the base field \( k \) is of positive characteristic \( p > 0 \), then the situation is more complicated. For example, it is known that the étale fundamental group \( \pi_1^{\text{ét}}(\mathbb{A}^1_k) \) of the affine line \( \mathbb{A}^1_k \) over an algebraically closed field of characteristic \( p > 0 \) is highly nontrivial. Indeed, the Artin–Schreier coverings of \( \mathbb{A}^1_k \) contribute to make the fundamental group very big,

\[
\dim_{\overline{F}_p} \text{Hom}(\pi_1^{\text{ét}}(\mathbb{A}^1_k), \overline{F}_p) = \infty.
\]

In particular, \( \pi_1^{\text{ét}}(\mathbb{A}^1_k) \) is far from topologically finitely generated.
The Abhyankar conjecture for affine curves partially describes the étale fundamental group $\pi^\text{ét}_1(U)$ of an affine smooth curve $U$ in positive characteristic $p > 0$ from the viewpoint of the inverse Galois problem. In [1], Abhyankar asked which finite groups occur as a quotient of $\pi^\text{ét}_1(U)$ for an affine smooth curve $U$ defined over an algebraically closed field $k$ of characteristic $p > 0$ and proposed a conjectural answer to the question. The conjecture was solved affirmatively by Raynaud and Harbater [31], [19]. The Abhyankar conjecture says that contrary to the étale fundamental group $\pi^\text{ét}_1(U)$ itself, the inverse Galois problem over $U$ has a concise answer and the finite quotients of $\pi^\text{ét}_1(U)$ can be completely determined by the topological one $\pi^\text{ét}_{10p}(U(\mathbb{C}))$ of a Riemann surface $U(\mathbb{C})$ of the same type $(g, n)$, where $g = \dim_k H^1(X, \mathcal{O}_X)$ denotes the genus of the smooth compactification $X$ of $U$ and $n$ denotes the cardinality of the complement $X \setminus U$ of $U$.

**Theorem 1.1.** (The Abhyankar conjecture, cf. [1], [31], [19]) With the above notation, let $G$ be a finite group. Then $G$ occurs as a quotient of $\pi^\text{ét}_1(U)$ if and only if the quotient $G(p') = G/p(G)$ can be generated by at most $2g + n - 1$ elements, where $p(G)$ is the normal subgroup of $G$ generated by all the $p$-Sylow subgroups.

Note that the “only if” part is proved by Abhyankar himself [1]. In fact, it can be also deduced from Grothendieck’s result, which gives a description of the maximal pro-prime-to-$p$ quotient of the étale fundamental group, i.e.

$$\pi^\text{ét}_1(U)^{(p')} \simeq \hat{F}_{2g+n-1},$$

(1.1)

where $\hat{F}_{2g+n-1}$ denotes the free profinite group of rank $2g + n - 1$. For example, in the case where $U = \mathbb{A}^1_k$ the affine line, the conjecture claims that the set of finite quotients of $\pi^\text{ét}_1(\mathbb{A}^1_k)$ coincides with the set of quasi-$p$-groups, where a finite group $G$ is said to be quasi-$p$ if $G = p(G)$. The conjecture had been widely open even in the case where $U = \mathbb{A}^1_k$ until Serre gave a partial answer to the problem.

**Theorem 1.2.** (Serre, cf. [34].) Suppose given a short exact sequence of finite groups

$$1 \longrightarrow G' \longrightarrow G \longrightarrow G'' \longrightarrow 1,$$

which satisfies the following conditions.

(i) $G$ is a quasi-$p$-group.
(ii) $G''$ appears as a quotient of $\pi^\text{ét}_1(\mathbb{A}^1_k)$.
(iii) $G'$ is solvable.

Then $G$ also appears as a quotient of $\pi^\text{ét}_1(\mathbb{A}^1_k)$. In particular, any solvable quasi-$p$-group appears as a quotient of $\pi^\text{ét}_1(\mathbb{A}^1_k)$.

Serre solved an embedding problem to prove the theorem. More precisely, let $\bar{\phi} : \pi^\text{ét}_1(\mathbb{A}^1_k) \to G''$ be a chosen surjective homomorphism. He proved that the corresponding embedding problem

$$\begin{array}{ccc}
\pi^\text{ét}_1(\mathbb{A}^1_k) & \longrightarrow & G'' \\
\downarrow \bar{\phi} & & \\
1 & \longrightarrow & G'
\end{array}$$

always has a solution, i.e. there exists a surjective lifting $\phi : \pi^\text{ét}_1(\mathbb{A}^1_k) \to G$ of $\bar{\phi}$ (after modifying $\bar{\phi}$).

Serre’s theorem was used in Raynaud’s proof. After Serre, in [31], Raynaud proved the conjecture for the affine line $U = \mathbb{A}^1_k$ and, soon after, in [19], Harbater solved the conjecture for general $U$ by applying the method of formal patching and by making use of Raynaud’s result. Actually, Harbater proved the conjecture in a stronger form, which we call the strong Abhyankar conjecture.
Theorem 1.3. (The strong Abhyankar conjecture due to Harbater, cf. [19].) With the above notation, let \( U \) be an affine smooth curve over \( k \) of type \((g, n)\) and let \( G \) be a finite group such that \( G^{(p')} = G/p(G) \) can be generated by \( 2g + n - 1 \) elements. Then there exists a finite étale connected Galois covering \( V \rightarrow U \) with Galois group \( G \) which is tamely ramified along \( X \setminus U \) except for one point \( x_0 \in X \setminus U \).

Independently of Harbater, in [30], Pop gave a further refinement of Theorem 1.1 in terms of embedding problems. Let us recall his approach. With the notation in Theorem 1.3, let \( G \) be a finite group with \( G(p') \) generated by \( 2g + n - 1 \) elements. Then we get an embedding problem

\[
\begin{array}{cccc}
\pi_1^{\text{ét}}(U) \\
\downarrow \phi \\
1 & \rightarrow & p(G) & \rightarrow & G & \rightarrow & G^{(p')} & \rightarrow & 1,
\end{array}
\]

(1.2)

where \( \phi \) is a fixed surjection, which always exists due to (1.1). Let us fix an étale Galois \( G(p') \)-cover

\[
V_0 \rightarrow U
\]

(1.3)

which realizes the surjective homomorphism \( \phi \). Note that \( V_0 \rightarrow U \) is tamely ramified along \( X \setminus U \). Furthermore, let \( Y_0 \) be the normalization of \( X \) in \( V_0 \). Under this situation, Pop solved the embedding problem (1.2) in the following form (see [30] for the full statements).

Theorem 1.4. (Pop, cf. [30, Theorem B and Corollary].) With the above notation, if the quasi-\( p \)-group \( p(G) \) appears as a quotient of \( \pi_1^{\text{ét}}(A^1_k) \), then there exists a surjective lifting \( \phi : \pi_1^{\text{ét}}(U) \rightarrow G \) of \( \phi \) in (1.2) such that a corresponding étale Galois \( G \)-cover \( V \rightarrow U \) is ramified only at one point in \( Y_0 \). In particular, Theorem 1.3 is a consequence of Raynaud’s solution of the Abhyankar conjecture for the affine line, i.e. Theorem 1.1 for the affine line \( A^1_k \).

Note that the theorem holds for an arbitrary affine curve \( U \) over \( k \), a priori which has no relation with the affine line \( A^1_k \).

1.2 | Main theorem

In [26] and [27, Chapter II], Nori introduced a new invariant \( \pi^N(X) \), which he called the fundamental group scheme, as a generalization of Grothendieck’s étale fundamental group \( \pi_1^{\text{ét}}(X) \) for a scheme \( X \) defined over a field \( k \). The fundamental group scheme \( \pi^N(X) \) (if it exists) is a profinite \( k \)-group scheme which classifies \( G \)-torsors over \( X \), where we can take as \( G \) an arbitrary (not necessarily étale) finite \( k \)-group scheme. In the case where \( k \) is of characteristic 0, then, by a theorem of Cartier, any finite \( k \)-group scheme \( G \) is étale. Hence, if \( k \) is an algebraically closed field of characteristic 0, then we can compute \( \pi^N(X) \) as a pro-constant \( k \)-group scheme associated with \( \pi_1^{\text{ét}}(X) \). On the other hand, if \( k \) has positive characteristic \( p > 0 \), then the former group cannot be recovered from the latter one in general. For example, finite local torsors, which are sometimes called purely inseparable coverings, over \( X \) make the group scheme \( \pi^N(X) \) larger. In [29], the author formulated a purely inseparable analogue of the Abhyankar conjecture to estimate the difference between these two fundamental groups \( \pi_1^{\text{ét}}(X) \) and \( \pi^N(X) \) from the viewpoint of the inverse Galois problem.

Question 1.5. (Purely inseparable analogue of the Abhyankar conjecture [29, Question 3.3].) Let \( U \) be an affine smooth curve over an algebraically closed field \( k \) of positive characteristic \( p > 0 \) and let \( G \) be a finite local \( k \)-group scheme. If there exists an injective homomorphism \( \chi(G) \hookrightarrow (\mathbb{Q}_p/\mathbb{Z}_p)^{\oplus p + n - 1} \), then does \( G \) appear as a finite quotient of \( \pi^N(U) \)? Here, \( \chi(G) \) stands for the group of characters of \( G \), i.e. \( \chi(G) \defeq \text{Hom}(G, \mathbb{G}_m,k) \), and \( p \) is the \( p \)-rank of the compactification \( X \) of \( U \) and \( n = \#(X \setminus U) \).

Here recall that if a finite local \( k \)-group scheme \( G \) appears as a quotient of \( \pi^N(U) \), then the character group \( \chi(G) \) must be embedded into the group \( (\mathbb{Q}_p/\mathbb{Z}_p)^{\oplus p + n - 1} \) (cf. [29, Proposition 3.1]).
In [7], Biswas–Borne introduced the notion of tamely ramified $G$-torsors for an arbitrary finite group scheme $G$ as a generalization of tamely ramified Galois covers in the usual sense (cf. Definition 4.14). Therefore, we can formulate a purely inseparable analogue of the strong Abhyankar conjecture due to Harbater (cf. Theorem 1.3).

**Question 1.6.** (Purely inseparable analogue of the strong Abhyankar conjecture.) Let $U$ be an affine smooth curve over an algebraically closed field $k$ of positive characteristic $p > 0$ and let $G$ be a finite local $k$-group scheme. If there exists an injective homomorphism $\mathcal{X}(G) \hookrightarrow (\mathbb{Q}_p / \mathbb{Z}_p)^{\oplus \gamma + n - 1}$, then does there exist a $G$-torsor over $U$ which is tamely ramified above $X \setminus U$ except for one point such that it realizes a surjective homomorphism $\pi^N(U) \twoheadrightarrow G$?

In this paper, toward the strong analogue, we will consider the inverse Galois problem over root stacks. Let $X$ be a projective smooth curve over $k$ of genus $g \geq 0$ and of $p$-rank $\gamma \geq 0$. Let $\emptyset \neq U \subset X$ be a nonempty open subscheme with $#X \setminus U = n \geq 1$. We shall denote by $X_0$ the one-punctured smooth affine curve $X \setminus \{x_0\}$ and consider $D = (x_i)_{i=1}^{n-1}$ as a family of reduced distinct Cartier divisors on $X$. For each family $r = (r_i)_{i=1}^{n-1} \in \prod_{i=1}^{n-1} \mathbb{Z}_{\geq 0}$ of integers, we denote by $\sqrt[r]{D/X_0}$ the root stack associated with $X_0$ and the data $(D, p^r)$. Biswas–Borne showed that finite flat torsors which are representable by a $k$-scheme over root stacks give candidates of tamely ramified torsors (cf. [7, §3.3]; see also Theorem 4.15(1)). Hence, Question 1.6 has the following interpretation, but the author is not sure if these are equivalent to each other (cf. Remark 1.11(1)).

**Question 1.7.** (Stacky counterpart of Question 1.6.) With the above notation, let $G$ be a finite local $k$-group scheme. Suppose that there exists an injective homomorphism $\mathcal{X}(G) \hookrightarrow (\mathbb{Q}_p / \mathbb{Z}_p)^{\oplus \gamma + n - 1}$ of abelian groups. Then, do there exist an $(n-1)$-tuple $r = (r_i)_{i=1}^{n-1}$ of integers $r_i \geq 0$ and a Nori-reduced $G$-torsor $Y \longrightarrow \sqrt[r]{D/X_0}$ such that $Y$ is representable by a $k$-scheme?

Note that if Question 1.6 has an affirmative answer, then so does Question 1.7 (cf. [7, §3.3]; see also Theorem 4.15(1)). Toward the strong analogue, we shall study an embedding problem over the root stacks and conclude that Question 1.7 has an affirmative answer at least in the solvable case. Namely, as the main result of this paper, we will prove the following result.

**Theorem 1.8.** (Cf. Theorem 4.30.) Suppose given an exact sequence of finite local $k$-group schemes

$$1 \longrightarrow G' \longrightarrow G \longrightarrow G'' \longrightarrow 1.$$ 

(1.4)

Suppose that the following conditions are satisfied.

(i) There exists an injective homomorphism $\mathcal{X}(G) \hookrightarrow (\mathbb{Q}_p / \mathbb{Z}_p)^{\oplus \gamma + n - 1}$.

(ii) There exists a surjective $k$-homomorphism $\pi^N(\sqrt[r]{D/X_0}) \overset{\text{def}}{=} \varprojlim \pi^N(\sqrt[r]{D/X_0}) \twoheadrightarrow G''$.

(iii) $G'$ is solvable.

Then there exist an $(n-1)$-tuple $r$ of non negative integers and a Nori-reduced $G$-torsor $Y \longrightarrow \sqrt[r]{D/X_0}$ which is representable by a $k$-scheme. In particular, Question 1.7 has an affirmative answer for any finite local solvable $k$-group scheme $G$.

Here, let us recall the definition of solvable group schemes $G$. For an affine group scheme $G$ over a field $k$, we denote by $D(G)$ the derived subgroup of $G$ (cf. [40, §10.1], [7, Definition A.1]). Then it turns out that the quotient $G^{ab} \overset{\text{def}}{=} G/D(G)$ is abelian and the natural surjection $G \longrightarrow G^{ab}$ is universal among morphisms $G \longrightarrow H$ into abelian $k$-group schemes (cf. [7, Appendix §A.1]). In particular, $G$ is abelian if and only if $D(G) = 1$. Moreover, by the universality of the morphism $G \longrightarrow G^{ab}$, it can be seen that any automorphism of an affine $k$-group scheme $G$ induces an automorphism of the derived subgroup $D(G)$. In particular, if $G'$ is a normal closed subgroup scheme of $G$, then the derived subgroup $D(G')$ of $G'$ is also normal in $G$. For any integer $m > 0$, we define the subgroup scheme $D^m(G)$ of $G$ inductively by

$$D^0(G) \overset{\text{def}}{=} G, \quad D^m(G) \overset{\text{def}}{=} D(D^{m-1}(G)).$$
Then an affine $k$-group scheme $G$ is said to be solvable if $D^n(G) = 0$ for sufficiently large $m > 0$ (cf. [40, §10.1]). It turns out that an affine $k$-group scheme $G$ is solvable if and only if it admits a sequence of closed subgroup schemes

$$1 = G_1 < G_{i-1} < \cdots < G_0 = G$$

such that each $G_i$ is normal in $G_{i-1}$ with $G_{i-1}/G_i$ abelian (cf. [40, §16, Exercise 2]).

By restricting to the affine curve $U$, as a consequence of Theorem 1.8, we get the following result, which is motivated by Serre’s Theorem 1.2. This is a generalization of one of the previous results due to the author (cf. [29, Proposition 3.4]).

**Corollary 1.9.** (Cf. Corollary 4.33.) For any finite local solvable $k$-group scheme $G$, the group scheme $G$ appears as a quotient of $\pi^N(U)$ if and only if the character group $\chi(G)$ can be embedded into the abelian group $\left(\mathbb{Q}_p/\mathbb{Z}_p\right)^{\oplus y+n-1}$. Namely, the purely inseparable analogue of the Abhyankar conjecture (Question 1.5) has an affirmative answer for any finite local solvable $k$-group scheme $G$.

On the other hand, in [7], Biswas–Borne also proved that if $G$ is abelian, then there exists an equivalence of categories between the category of tamely ramified $G$-torsors and the category of $G$-torsors of root stacks which are representable by a scheme (cf. [7, §3.4]; see also Theorem 4.15(2)). Therefore, as a consequence of Theorem 1.8, we have the following result.

**Corollary 1.10.** (Cf. Corollary 4.34.) With the above notation, let $G$ be a finite local abelian $k$-group scheme. Suppose that there exists an injective homomorphism $\chi(G) \hookrightarrow \left(\mathbb{Q}_p/\mathbb{Z}_p\right)^{\oplus y+n-1}$. Then there exists an $(n-1)$-tuple $r = (r_i)_{i=1}^{n-1}$ of integers $r_i \geq 0$ and a tamely ramified $G$-torsor $Y \to X_0$ with ramification data $(D, p^r)$ such that the restriction $Y \times_{X_0} U \to U$ gives a Nori-reduced $G$-torsor. Namely, the purely inseparable analogue of the strong Abhyankar conjecture (Question 1.6) has an affirmative answer for any finite local abelian $k$-group scheme $G$.

The following are further remarks.

**Remark 1.11.**

1. As we state just before Corollary 1.10, by [7, §3.4], for a finite abelian $k$-group scheme $G$, a $G$-torsor $Y \to \sqrt[n]{D}/X$ which is representable by a $k$-scheme gives rise to a tamely ramified $G$-torsor $Y \to X$ with ramification data $(D, p^r)$. However, the argument in [7, §3.4] does not work for non-abelian $k$-group schemes, which is noticed by David Rydh (cf. [7, Appendix §B], see also Remark 4.17). In this point, the author is not sure whether or not Questions 1.6 and 1.7 are equivalent to each other.

2. Although Questions 1.6 and 1.7 are motivated by Harbater’s Theorem 1.3, our proof of the main theorem is done by solving embedding problems and it is more like Pop’s approach (cf. Theorem 1.4). In fact, one can ask whether an analogous result holds in our situation as follows.

Let $G$ be a finite local $k$-group scheme. Then we have an exact sequence

\[1 \to G' \to G \to \text{Diag}(\chi(G)) \to 0,\]

where $\text{Diag}(\chi(G))$ is the diagonalizable group scheme associated with the group $\chi(G)$ of characters of $G$, and $G'$ is the kernel of the natural surjective homomorphism $G \to \text{Diag}(\chi(G))$. Then one can check that $\chi(G') = 1$. Now we have reached the following question motivated by Pop’s Theorem 1.4.

**Question 1.12.** (Purely inseparable analogue of Theorem 1.4.) With the same notation as in Questions 1.6 and 1.7, suppose given a finite local $k$-group scheme $G$ whose character group $\chi(G)$ can be embedded into the group $\left(\mathbb{Q}_p/\mathbb{Z}_p\right)^{\oplus y+n-1}$. Suppose further that $G' \overset{\text{def}}{=} \text{Ker}(G \to \text{Diag}(\chi(G)))$ is a quotient of the Nori fundamental group scheme $\pi^N(\mathbb{A}_k^1)$ of the affine line $\mathbb{A}_k^1$, i.e. Question 1.5 has an affirmative answer for $U = \mathbb{A}_k^1$ and for $G'$. Then do there exist an $(n-1)$-tuple
r and a Nori-reduced $G$-torsor over $\sqrt[r]{D/X_0}$ which is representable by a $k$-scheme? Moreover, does there exist a Nori-reduced $G$-torsor over $U$ which is tamely ramified above $X \setminus U$ except for one point?

One of our missing points for this direction is that we have no answer to the embedding problem

$$
\begin{array}{cccccc}
1 & \longrightarrow & G' & \longrightarrow & G & \longrightarrow & \text{Diag}(\mathbb{X}(G)) & \longrightarrow & 1
\end{array}
$$

unless $G'$ is solvable (cf. Lemma 4.31). On the other hand, under the assumption that $G'$ is solvable, the former question in Question 1.12 has an affirmative answer without the further assumption on $G'$, which is a part of the main theorem (Theorem 1.8).

Finally we explain the organization of the present paper.

In §2, we recall the notion of fundamental group schemes and fundamental gerbes, and their tannakian interpretations, following Borne–Vistoli [9] and Tonini–Zhang [37]. In [6], Biswas–Borne studied the fundamental gerbes of proper tame stacks, e.g. root stacks associated with proper $k$-schemes. On the other hand, we have to deal with root stacks associated with non-proper curves, so more general setups, e.g. a tannakian interpretation of the Nori fundamental gerbes for non-proper tame stacks might be required. However, our main interest is the fundamental group schemes of smooth root stacks, for which a more elementary setup is sufficient, hence one can access the proof of the main theorem in a more direct way (cf. §4.2), especially without relying on the tannakian interpretation in terms of Frobenius divided sheaves (cf. §2.5), which will be used for dropping the smoothness assumption (cf. Appendix B).

In §3, we recall the definition and basic properties of root stacks, for which the main references are [11] and [36, Chapter 1]. In §2 and §3, there are no original results and, all the results are well-understood by the experts.

In §4, after reviewing a recent work due to Biswas–Borne on tamely ramified torsors [7, §3], we shall give a proof of Theorem 1.8. The idea is that a quite similar argument as in the proof of [29, Proposition 3.4] still works even if one replaces an affine curve $U$ by the pro-system of root stacks $\{\sqrt[r]{D/X_0}\}_r$. The main ingredients of the proof are Lieblich’s work on twisted sheaves [24] (cf. §4.4) and a theorem of Alper on vector bundles over algebraic stacks having good moduli spaces [4]. The former one ensures that our embedding problem is always unobstructed (cf. Lemma 4.31), and the latter one ensures the existence of representable torsors (cf. Lemma 4.18).

In Appendix §A, as a related topic, we consider the possibility of a generalization of the Katz–Gabber correspondence (cf. [21]) for finite local torsors, and prove that a naïve generalization is far from true (cf. Proposition A.6).

### 1.3 | Notation

In the present paper, $k$ always means a field. We denote by $\text{Vec}_k$ the category of finite dimensional vector spaces over a field $k$. For an affine $k$-group scheme $G$, we denote by $\text{Rep}(G)$ the category of finite dimensional left $k$-linear representations of $G$. For each $(V, \varphi) \in \text{Rep}(G)$, we denote by $V^G$ the $G$-invariant subspace of $V$, i.e. $V^G = \{v \in V \mid \varphi(v) = v \otimes 1\}$.

In the present paper, for the basic definitions and notions about algebraic stacks, we follow Stacks Project [35]. In particular, as a definition of algebraic spaces and algebraic stacks, we adopt [35, Definition 025X] and [35, Definition 026O] respectively. For any algebraic stack $\mathcal{X}$, we denote by $|\mathcal{X}|$ the underlying topological space (cf. [35, 04XE]) and by $|\mathcal{X}|_0$ the set of closed points of $\mathcal{X}$.

For a ring $A$, we denote by $(\text{Aff}/A)$ the category of affine $A$-schemes. We consider the fibered categories

$$
\text{Vect} \subseteq \text{QCoh}_{\text{fp}} \subseteq \text{QCoh}
$$

over $(\text{Aff}/A)$ of locally free sheaves of finite rank, quasi-coherent sheaves of finite presentation and quasi-coherent sheaves respectively. For any fibered category $\mathcal{X}$ over $(\text{Aff}/A)$, we define the categories

$$
\text{Vect}(\mathcal{X}) \subseteq \text{QCoh}_{\text{fp}}(\mathcal{X}) \subseteq \text{QCoh}(\mathcal{X})
$$
to be

\[ \text{Vect}(\mathcal{X}) \overset{\text{def}}{=} \text{Hom}_A(\mathcal{X}, \text{Vect}) \]

and similarly \( \text{Qcoh}_p(\mathcal{X}) \overset{\text{def}}{=} \text{Hom}_A(\mathcal{X}, \text{Qcoh}_p) \) and \( \text{Qcoh}(\mathcal{X}) \overset{\text{def}}{=} \text{Hom}_A(\mathcal{X}, \text{Qcoh}) \). For a field \( k \), we have \( \text{Vect}(\text{Spec } k) = \text{Vec}_k \). If \( \mathcal{X} \) is a Noetherian algebraic stack over a scheme \( S \), we can also consider the category \( \text{Coh}(\mathcal{X}) \) of coherent sheaves on \( \mathcal{X} \). If \( \mathcal{X} = B_S G \) is the classifying stack of an affine flat and finitely presented \( S \)-group scheme \( G \), then \( \text{Qcoh}(B_S G) \) is nothing but the category of \( G \)-equivariant sheaves over \( S \), i.e. quasi-coherent sheaves \( F \) on \( S \) endowed with an action of \( G \) (cf. [3, §2.1]). Therefore, we get a natural functor \( \text{Qcoh}(B_S G) \to \text{Qcoh}(S) \) which maps each \( G \)-equivariant sheaf \( F \) to the \( G \)-invariant subsheaf \( F^G \). In the case where \( S = \text{Spec } k \) is the spectrum of a field \( k \), all the three categories \( \text{Coh}(BG), \text{Vect}(BG) \) and \( \text{Rep}(G) \) are canonically equivalent to each other,

\[ \text{Coh}(BG) = \text{Vect}(BG) = \text{Rep}(G). \]

For a scheme \( X \) over a field \( k \) of characteristic \( p > 0 \) and a positive integer \( n > 0 \), we define the \( n \)th Frobenius twist \( X^{(n)} \) by the Cartesian diagram

\[
\begin{array}{ccc}
X^{(n)} & \longrightarrow & X \\
\downarrow & & \downarrow \\
\text{Spec } k & \longrightarrow & \text{Spec } k,
\end{array}
\]

where \( F \) is the absolute Frobenius of \( \text{Spec } k \). Then the \( n \)th power of the absolute Frobenius morphism \( F^n : X \to X \) factors uniquely through \( X^{(n)} \) and we get a \( k \)-morphism \( X \to X^{(n)} \), which is denoted by \( F^{(n)} \) and is called the \( n \)th relative Frobenius morphism of \( X \). If \( k \) is perfect, the projection \( X^{(n)} \to X \) is an isomorphism of schemes, but not of \( k \)-schemes. Moreover, in this case, we can also consider the \((-n)\)th Frobenius twist \( X^{(-n)} \) of \( X \) by using the inverse morphism \( F^{-n} \) of \( F^n : \text{Spec } k \to \text{Spec } k \). If \( \mathcal{X} \to \text{Spec } k \) is a fibered category of groupoids over a field \( k \) of characteristic \( p > 0 \), we define the absolute Frobenius morphism \( F : \mathcal{X} \to \mathcal{X} \) as the collection of maps

\[ \mathcal{X}(S) \to \mathcal{X}(S); \quad \xi \mapsto F_S^*(\xi) = \xi F_S \]

for \( S \in (\text{Aff }/k) \) (cf. [37, Notations and conventions]). If \( \mathcal{X} = X \) is a scheme over \( k \), then this coincides with the one in the previous sense. Moreover, we can also define the relative Frobenius morphisms \( F^{(n)} : \mathcal{X} \to \mathcal{X}^{(n)} \) for \( n > 0 \) in the same manner as before.

## 2 | FUNDAMENTAL GERBES AND THEIR TANNAKIAN INTERPRETATIONS

### 2.1 | Fundamental gerbes

Let \( k \) be a field. A \textit{finite stack over } \( k \) is an algebraic stack \( \Gamma \) over \( k \) which has finite flat diagonal and admits a flat surjective morphism \( U \to \Gamma \) for some finite \( k \)-scheme \( U \) (cf. [9, Definition 4.1]). A \textit{finite gerbe over } \( k \) is a finite stack over \( k \) which is a gerbe in the fppf topology. A finite stack \( \Gamma \) is a finite gerbe if and only if it is geometrically connected and geometrically reduced (cf. [9, Proposition 4.3]). A finite gerbe \( \Gamma \) over \( k \) is a \textit{tannakian gerbe over } \( k \) in the sense of [33, III §2]. A \textit{profinite gerbe over } \( k \) is a tannakian gerbe over \( k \) which is equivalent to a projective limit of finite gerbes over \( k \) (cf. [9, Definition 4.6]).

Let \( \mathcal{X} \) be a fibered category in groupoids over \( k \). Suppose that \( \mathcal{X} \) is \textit{inflexible over } \( k \) in the sense of [9, Definition 5.3], namely, it admits a profinite gerbe \( \Pi \) over \( k \) together with a morphism \( \mathcal{X} \to \Pi \) such that, for any finite stack \( \Gamma \) over \( k \), the induced functor

\[ \text{Hom}_k(\Pi, \Gamma) \to \text{Hom}_k(\mathcal{X}, \Gamma) \]
is an equivalence of categories (cf. [9, Theorem 5.7]). If such a gerbe \( \Pi \) exists for \( \mathcal{X}/k \), it is unique up to unique isomorphism, so we denote it by \( \Pi^{N}_{\mathcal{X}/k} \) or \( \Pi^{N}_{\mathcal{X}} \) for simplicity, and call it the \textit{Nori fundamental gerbe} for \( \mathcal{X} \) over \( k \) (cf. [9]). If \( \mathcal{X} \) is an algebraic stack of finite type over \( k \), from the definition of \( \Pi^{N}_{\mathcal{X}/k} \), for any finite group scheme \( G \) over \( k \), there exists a natural bijection

\[
\text{Hom}_{k}\left( \Pi^{N}_{\mathcal{X}/k}, B_{k}G \right) \overset{\cong}{\longrightarrow} \text{Hom}_{k}\left( \mathcal{X}, B_{k}G \right) = H^{1}_{\text{fppf}}(\mathcal{X}, G).
\]

In this sense, the Nori fundamental gerbe \( \Pi^{N}_{\mathcal{X}/k} \) classifies \( G \)-torsors over \( \mathcal{X} \) for any finite group scheme \( G \) over \( k \). If we suppose that an inflexible algebraic stack \( \mathcal{X} \) over \( k \) has a \( k \)-rational point \( x \in \mathcal{X}(k) \), then the composition \( \text{Spec} k \overset{x}{\longrightarrow} \mathcal{X} \longrightarrow \Pi^{N}_{\mathcal{X}/k} \) defines a section \( \xi \in \Pi^{N}_{\mathcal{X}/k}(k) \), whence \( \Pi^{N}_{\mathcal{X}/k} \cong B_{k}\text{Aut}_{k}(\xi) \). We denote by \( \pi^{N}(\mathcal{X}, x) \) the group scheme \( B_{k}\text{Aut}_{k}(\xi) \) over \( k \), which is nothing other than the \textit{fundamental group scheme} of \( (\mathcal{X}, x) \) in the sense of Nori [27, Chapter II]. Namely, for any finite group scheme \( G \) over \( k \), the set of homomorphisms \( \text{Hom}_{k}(\pi^{N}(\mathcal{X}, x), G) \) is naturally bijective onto the set of isomorphism classes of pointed \( G \)-torsors over \( (\mathcal{X}, x) \). We call the profinite group scheme \( \pi^{N}(\mathcal{X}, x) \) the \textit{Nori fundamental group scheme} of \( (\mathcal{X}, x) \).

Let \( \mathcal{X} \) be an inflexible algebraic stack of finite type over \( k \). A morphism \( \mathcal{X} \longrightarrow \Gamma \) into a finite gerbe \( \Gamma \) is said to be \textit{Nori-reduced} (cf. [9, Definition 5.10]) if for any factorization \( \mathcal{X} \longrightarrow \Gamma' \longrightarrow \Gamma \) where \( \Gamma' \) is a finite gerbe and \( \Gamma' \longrightarrow \Gamma \) is faithful, then \( \Gamma' \longrightarrow \Gamma \) is an isomorphism. According to [9, Lemma 5.12], for any morphism \( \mathcal{X} \longrightarrow \Gamma \) into a finite gerbe, there exists a unique factorization \( \mathcal{X} \longrightarrow \Delta \longrightarrow \Gamma \), where \( \Delta \) is a finite gerbe, \( \mathcal{X} \longrightarrow \Delta \) is Nori-reduced and \( \Delta \longrightarrow \Gamma \) is representable. A \( G \)-torsor \( P \longrightarrow \mathcal{X} \) is said to be \textit{Nori-reduced} if the morphism \( \mathcal{X} \longrightarrow B_{G} \Gamma \) is Nori-reduced.

Next we introduce two variants of the Nori fundamental gerbe, namely, the \textit{étale} and \textit{local fundamental gerbes} (cf. [9, §8], [37, §4, Definition 4.1]).

**Definition 2.1.** (Cf. [37, §3, Definition 3.1].) A finite stack \( \Gamma \) over \( k \) is said to be \textit{étale} if it admits a finite étale surjective morphism \( U \longrightarrow \Gamma \) from a finite étale \( k \)-scheme \( U \).

If \( k \) is of characteristic 0, then any finite gerbe is étale. Hence, suppose that \( k \) is of positive characteristic \( p > 0 \). To define a well-defined notion of \textit{finite local stack}, we need more preliminaries. For each \( k \)-algebra \( A \), we denote by \( A_{\text{ét}} \) as the union of \( k \)-subalgebras of \( A \) which are finite étale over \( k \) (cf. [37, §2, Definition 2.1]). Note that \( A_{\text{ét}} \) depends on the base field \( k \). If \( A \rightarrow B \) is a surjective \( k \)-algebra homomorphism with nilpotent kernel, then the induced homomorphism \( A_{\text{ét}} \rightarrow B_{\text{ét}} \) is an isomorphism (cf. [37, §2, Remark 2.2]). Moreover, for the 1\textsuperscript{st} relative Frobenius homomorphism

\[
A^{(1)} = A \otimes_{k} k \longrightarrow A ; \ a \otimes \lambda \longmapsto a^{p} \lambda,
\]

the induced morphism \( (A^{(1)})_{\text{ét}} \rightarrow A_{\text{ét}} \) is an isomorphism (cf. [37, §2, Remark 2.3]). If \( A \) is a finite \( k \)-algebra, then there exists an integer \( n \geq 0 \) such that the image of the relative Frobenius homomorphism \( A^{(n)} \rightarrow A \) is an étale \( k \)-algebra and the residue fields of \( A^{(n)} \) are separable over \( k \) (cf. [37, Lemma 2.4]). In particular, the surjective homomorphism \( A^{(n)} \rightarrow (A^{(n)})_{\text{red}} \) induces an isomorphism \( (A^{(n)})_{\text{ét}} \cong (A^{(n)})_{\text{red}} \).

For each affine scheme \( U = \text{Spec} A \) over \( k \), we define \( U^{\text{ét}} \overset{\text{def}}{=} \text{Spec} A_{\text{ét}} \). Note that the canonical morphism \( U \rightarrow U^{\text{ét}} \) is faithfully flat. Let \( R \Rightarrow U \) be a flat groupoid where \( R \) and \( U \) finite over \( k \). Then the induced morphisms \( R_{\text{red}} \Rightarrow U_{\text{red}} \) define a groupoid. Moreover, if the residue fields of \( R \) and \( U \) are separable over \( k \), then the induced morphisms \( R_{\text{red}} \Rightarrow U_{\text{red}} \) also define a groupoid which is canonically isomorphic to the first one \( U^{\text{ét}} \Rightarrow U^{\text{ét}} \) (cf. [37, §3, Lemma 3.4]).

**Definition 2.2.** (Cf. [37, §3 Definition 3.5].) Let \( \Gamma \) be a finite stack over a field \( k \) and let \( U \longrightarrow \Gamma \) be a finite atlas from an affine \( k \)-scheme \( U \) with \( R = U \times_{\Gamma} U \). We define an \textit{étale quotient} \( \Gamma^{\text{ét}} \) as the stack associated with the groupoid \( R^{\text{ét}} \Rightarrow U^{\text{ét}} \).

**Lemma 2.3.** (Cf. [37, §3, Lemma 3.6].) \textit{With the same notation as in Definition 2.2, for any finite étale stack \( E \) over \( k \), the induced morphism}

\[
\text{Hom}_{k}(\Gamma^{\text{ét}}, E) \longrightarrow \text{Hom}_{k}(\Gamma, E)
\]
is an equivalence. Moreover, for any $i \geq 0$, the morphism $\Gamma^{\text{ét}} \to \left(\Gamma^{(i)}\right)^{\text{ét}}$ is an equivalence and for any sufficiently large $i \gg 0$, the functor $\Gamma^{(i)} \to \left(\Gamma^{(i)}\right)^{\text{ét}}$ has a section, whence the relative Frobenius morphism $\Gamma \to \Gamma^{(i)}$ factors through $\Gamma^{\text{ét}}$.

In particular, the étale quotient $\Gamma^{\text{ét}}$ for a finite stack $\Gamma$ is independent of the choice of atlas $U \to \Gamma$ and is unique up to unique isomorphism.

**Definition 2.4.** (Cf. [37, §3, Definition 3.9].) A finite stack $\Gamma$ over $k$ is said to be local if $\Gamma^{\text{ét}} = \text{Spec } k$. Moreover, we define a pro-étale gerbe, or pro-local gerbe in the same manner as [9, Definition 4.6].

**Definition 2.5.** Let $\mathcal{X}$ be a fibered category in groupoids over $k$. An étale fundamental gerbe (respectively local fundamental gerbe) for $\mathcal{X}$ is a pro-étale gerbe (respectively a pro-local gerbe) over $k$ to which a morphism $\mathcal{X} \to \Pi$ such that for any finite étale stack (respectively finite local stack) $\Gamma$ over $k$, the induced functor

$$\text{Hom}_k(\Pi, \Gamma) \to \text{Hom}_k(\mathcal{X}, \Gamma)$$

is an equivalence of categories. If such a one $\Pi$ exists, it is unique up to unique isomorphism and we denote by $\Pi^{\text{ét}}_{\mathcal{X}/k}$ (respectively $\Pi^{\text{loc}}_{\mathcal{X}/k}$) the étale fundamental gerbe (respectively the local fundamental gerbe) for $\mathcal{X}$.

**Proposition 2.6.** Let $\mathcal{X}$ be an algebraic stack of finite type over $k$. Then we have the following.

1. The étale fundamental gerbe $\Pi^{\text{ét}}_{\mathcal{X}/k}$ exists if and only if $\mathcal{X}$ is geometrically connected, or equivalently, $H^0\left(\mathcal{O}_\mathcal{X}\right)^{\text{ét}} = k$.
2. Suppose that $\mathcal{X}$ is reduced. Then the local fundamental gerbe $\Pi^{\text{loc}}_{\mathcal{X}/k}$ exists if and only if $H^0\left(\mathcal{O}_\mathcal{X}\right)$ does not contain nontrivial purely inseparable extensions of $k$.
3. If $\mathcal{X}$ is inflexible, then the étale fundamental gerbe $\Pi^{\text{ét}}_{\mathcal{X}/k}$ and the local fundamental gerbe $\Pi^{\text{loc}}_{\mathcal{X}/k}$ exist.

**Proof.**

1. See [37, §2, Lemma 2.7; §4, Proposition 4.3].
2. See [37, §7, Theorem 7.1].
3. See [37, §4, Remark 4.2].

## 2.2 Tannakian reconstruction and recognition

A pseudo-abelian category is an additive category $C$ together with a family $J_C$ of sequences of the form $c' \to c \to c''$ in $C$ (cf. [37, Definition]). A $\mathbb{Z}$-linear functor $\Phi : C \to D$ of pseudo-abelian categories is said to be exact if it maps any sequence of $J_C$ to a sequence isomorphic to a one of $J_D$.

Let $R$ be a ring. If $\mathcal{X}$ is a fibered category over $(\text{Aff}/R)$, then we consider the category $\text{Vect}(\mathcal{X})$ of vector bundles on $\mathcal{X}$ as a pseudo-abelian category with the family $J_{\text{Vect}(\mathcal{X})}$ of pointwise exact sequences

$$\mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$$

i.e. for any affine $R$-scheme $T$ and any object $\xi \in \mathcal{X}(T)$, the sequence

$$\mathcal{F}'(\xi) \to \mathcal{F}(\xi) \to \mathcal{F}''(\xi)$$

of vector bundles on $T$ is exact. If $C$ and $D$ are $R$-linear, monoidal and pseudo-abelian categories, we denote by $\text{Hom}_{R,\otimes}(C, D)$ the category whose objects are $R$-linear exact monoidal functors and whose morphisms are natural monoidal isomorphisms. If $f : \mathcal{X} \to \mathcal{Y}$ is a base preserving functor of categories $\mathcal{X}$ and $\mathcal{Y}$ over $(\text{Aff}/R)$, $f^* \in \text{Hom}_{R,\otimes}(\text{Vect}(\mathcal{Y}), \text{Vect}(\mathcal{X}))$. 
Let $C$ be a pseudo-abelian monoidal $R$-linear category. We define an fpqc stack $\Pi_C$ in groupoids over $R$ by attaching to each affine $R$-scheme $T$ the category

$$\Pi_C(T) \overset{\text{def}}{=} \text{Hom}_{R,\mathcal{S}}(C, \text{Vect}(T)).$$

Note that there exists a natural monoidal $R$-linear functor over $(\text{Aff} / R)$

$$\Phi : C \to \text{Vec}(\Pi_C) ; \ c \mapsto (\Pi_C(T) \ni \xi \mapsto \xi(c) \in \text{Vec}(T)). \quad (2.1)$$

A pseudo-abelian monoidal $R$-linear category $C$ over $(\text{Aff} / R)$ is said to satisfy \textit{tannakian recognition} (cf. [37, §1]) if the functor (2.1) is an equivalence and for any sequence $\chi : c' \to c \to c''$, $\Phi(\chi)$ is pointwise exact if and only if $\chi \in J_C$.

Let $\mathcal{X}$ be a fibered category over $(\text{Aff} / R)$. Then there exists a base preserving functor

$$\mathcal{X} \to \Pi_{\text{Vec}(\mathcal{X})}. \quad (2.2)$$

A fibered category $\mathcal{X}$ in groupoids over $(\text{Aff} / R)$ is said to satisfy \textit{tannakian reconstruction} (cf. [37, §1]) if the functor (2.2) is an equivalence.

Then a classical Tannaka duality (cf. [12, Théorème 1.12]) can be restated in the following way.

\textbf{Theorem 2.7.} (Cf. [37, §1, Example 1.5] Let $k$ be a field. If $T$ is a $k$-tannakian category, then it satisfies tannakian recognition and $\Pi_T$ is a tannakian gerbe over $k$. Conversely, if $\Pi$ is a tannakian gerbe over $k$, then it satisfies tannakian reconstruction and $\text{Vec}(\Pi)$ is a $k$-tannakian category.

Moreover, Nori’s reconstruction theorem (cf. [27, Chapter I, §2.2, Proposition 2.9]) can be restated in the following way.

\textbf{Proposition 2.8.} Let $G$ be an affine group scheme over $k$ and let $\mathcal{X}$ be a fibered category in groupoids over $k$. Then there exists a natural equivalence of categories,

$$\text{Hom}_k(\mathcal{X}, BG) \overset{\sim}{\longrightarrow} \text{Hom}_{k,\mathcal{S}}(\text{Vec}(BG), \text{Vec}(\mathcal{X})).$$

This is valid because $BG = \Pi_{\text{Vec}(BG)}$ and the natural functor

$$\text{Hom}_k(\mathcal{X}, \Pi_{\text{Vec}(BG)}) \to \text{Hom}_{k,\mathcal{S}}(\text{Vec}(BG), \text{Vec}(\mathcal{X})).$$

is an equivalence of categories (cf. [37, §1]).

\section*{2.3 Tannakian interpretation in the pseudo-proper case: Essentially finite bundles}

In this subsection, we recall a tannakian interpretation of the Nori fundamental gerbe under a properness assumption, which was originally given by Nori [26] for the fundamental group scheme. We shall follow a simplified argument due to Borne–Vistoli [9, §7].

\textbf{Definition 2.9.} (Cf. [9, §7, Definition 7.1]) A fibered category $\mathcal{X}$ over $k$ is said to be \textit{pseudo-proper} if it satisfies the following conditions.

(i) There exists a quasi-compact scheme $U$ and a morphism $U \to \mathcal{X}$ which is representable, faithfully flat, quasi-compact and quasi-separated.
(ii) For any locally free sheaf $E$ of $\mathcal{O}_X$-module on $\mathcal{X}$, the $k$-vector space $H^0(\mathcal{X}, E)$ is finite-dimensional.

\textbf{Example 2.10.} (Cf. [9, §7, Examples 7.2].)
(1) A finite stack $\Gamma$ over $k$ is pseudo-proper.
(2) A tannakian gerbe $\Phi$ over $k$ is pseudo-proper.

Now let $\mathcal{X}$ be a pseudo-proper fibered category in groupoids over a field $k$. Then, the category $\text{Vect}(\mathcal{X})$ of vector bundles on $\mathcal{X}$ is a $k$-linear rigid tensor category with finite-dimensional Hom vector spaces, in which the idempotents split, and the Krull–Schmidt theorem holds in $\text{Vect}(\mathcal{X})$. Namely, every object $E$ of $\text{Vect}(\mathcal{X})$ can be described as a direct product of indecomposable objects $E_i$ $(1 \leq i \leq n)$, $E \cong \bigoplus_{i=1}^{n} E_i$, and moreover such a description of $E$ is unique up to isomorphism.

**Definition 2.11.** (Cf. [9, §7, Definition 7.5].) A vector bundle $E \in \text{Vect}(\mathcal{X})$ is said to be finite if there exist $f$ and $g$ in $\mathbb{N}[t]$ with $f \neq g$ such that $f(E) \cong g(E)$, or equivalently, the set of isomorphism classes of indecomposable components of all the powers of $E$ is finite.

In particular, if $E$ is a finite bundle over $\mathcal{X}$, then all the indecomposable components are also finite. Moreover, if $E$ and $E'$ are finite bundles on $\mathcal{X}$, then the direct sum $E \oplus E'$, the tensor product $E \otimes_{\mathcal{O}_X} E'$ and the dual $E^\vee$ are also finite (cf. [9, §7, Proposition 7.6]).

**Definition 2.12.** (Cf. [9, §7, Definition 7.7].) A vector bundle $E$ over a pseudo-proper fibered category in groupoids over $k$ is said to be essentially finite if it is the kernel of a homomorphism between two finite bundles. We denote by $\text{EFin}(\mathcal{X})$ the category of essentially finite bundles over $\mathcal{X}$.

**Example 2.13.** (Cf. [9, §7, Proposition 7.8].) Let $\Phi$ be a profinite gerbe over a field $k$. Then all the vector bundles on $\Phi$ are essentially finite, whence

$$\text{Vect}(\Phi) = \text{EFin}(\Phi).$$

Indeed, if we write $\Phi = \lim_{\leftarrow i \in I} \Gamma_i$ with $\Gamma_i$ finite, then $\text{Vect}(\Phi) = \lim_{\leftarrow i \in I} \text{Vect}(\Gamma_i)$. Therefore, it suffices to prove the claim in the case where $\Phi = \Gamma$ is finite. Since $\Gamma$ is a finite gerbe over $k$, there exists a faithfully flat representable morphism $\pi : T \cong \text{Spec } K \longrightarrow \Gamma$ from the spectrum of a field $K$ which is finite over $k$. First note that the vector bundle $\pi_* \mathcal{O}_T$ is finite on $\Gamma$ because

$$\pi_* \mathcal{O}_T \otimes \pi_* \mathcal{O}_T \cong \pi_* (\mathcal{O}_T \otimes \pi^* \pi_* \mathcal{O}_T) \cong \pi_* (\mathcal{O}_T^{\oplus d}),$$

for some $d > 0$. Now fix an arbitrary vector bundle $E \in \text{Vect}(\Gamma)$. Let $E$ be of rank $r > 0$. Since $T$ is the spectrum of a field, we have $\pi^* E \cong \mathcal{O}_T^{\oplus r}$, whence $E \hookrightarrow \pi_* \pi^* E \cong \pi_* \mathcal{O}_T^{\oplus r}$. Again by using the fact that $T$ is the spectrum of a field together with flat descent, we can easily see that the cokernel of this inclusion is also a vector bundle over $\Gamma$ and can be embedded into $\pi_* \mathcal{O}_T^{\oplus m}$ for some $m > 0$. As $\pi_* \mathcal{O}_T$ is a finite bundle over $\Gamma$, we can conclude that $E$ is essentially finite.

**Theorem 2.14.** (Nori, Borne–Vistoli, cf. [26], [9, §7, Theorem 7.9].) Let $\mathcal{X}$ be an inflexible and pseudo-proper fibered category over a field $k$ and let $\mathcal{X} \longrightarrow \Pi_{\mathcal{X}/k}^N$ be the Nori fundamental gerbe for $\mathcal{X}/k$. Then the pullback functor

$$\text{Vect} \left( \Pi_{\mathcal{X}/k}^N \right) \longrightarrow \text{Vect}(\mathcal{X})$$

is fully faithful and gives an equivalence of tensor categories between $\text{Vect} \left( \Pi_{\mathcal{X}/k}^N \right)$ and $\text{EFin}(\mathcal{X})$. In particular, the category $\text{EFin}(\mathcal{X})$ of essentially finite bundles over $\mathcal{X}$ is a tannakian category over $k$.

**Corollary 2.15.** (Cf. [9, §7, Corollary 7.10].) Let $\Phi$ be a tannakian gerbe over a field $k$. Then, the full subcategory $\text{EFin}(\Phi)$ consisting of essentially finite bundles is a tannakian category over $k$.

Let us introduce the following notation.
**Definition 2.16.** Let $k$ be a field and let $\mathcal{T}$ be a tannakian category over $k$. We define the full tannakian subcategory $\text{EFin}(\mathcal{T})$ of $\mathcal{T}$ to be the essential image of the fully faithful functor $\text{EFin}(\Pi_\mathcal{T}) \hookrightarrow \mathcal{T}$ (cf. Theorem 2.7 and Corollary 2.15). We denote by $\hat{\Pi}_\mathcal{T}$ the tannakian gerbe $\Pi_{\text{EFin}(\mathcal{T})}$ associated with the tannakian category $\text{EFin}(\mathcal{T})$ over $k$. By definition, $\hat{\Pi}_\mathcal{T}$ is a profinite gerbe over $k$.

### 2.4 Formalism for tannakian interpretations

Let $k$ be a field and consider two fibered categories $\mathcal{X}$ and $\mathcal{X}_\mathcal{T}$ over $(\text{Aff}/k)$ together with a base preserving functor $\pi : \mathcal{X} \to \mathcal{X}_\mathcal{T}$. Define $\mathcal{T}(\mathcal{X}) \overset{\text{def}}{=} \text{Vect}(\mathcal{X}_\mathcal{T})$, which is a pseudo-abelian rigid monoidal $k$-linear category and which admits a $k$-linear monoidal exact functor $\pi^* : \mathcal{T}(\mathcal{X}) \to \text{Vect}(\mathcal{X})$. We will apply the following formalism to the fibered categories $\mathcal{X}_\mathcal{T} = \mathcal{X}^\infty (\text{cf.} §2.5)$ and $\mathcal{X}_\mathcal{T} = \text{Spec} k$ (cf. §4.1)

**Axiom 2.17.** (Cf. [37, §5, Axioms 5.2].) Let $L \overset{\text{def}}{=} \text{End}_{\mathcal{T}(\mathcal{X})}(1_{\mathcal{T}(\mathcal{X})})$ and consider the following conditions.

(A) $\mathcal{T}(\mathcal{X}) = \text{QCoh}_L(\mathcal{X}_\mathcal{T})$.

(B) The functor $\pi^* : \mathcal{T}(\mathcal{X}) \to \text{Vect}(\mathcal{X})$ is faithful.

(C) For all finite étale stacks $\Gamma$ over $L$, the following functor is an equivalence of categories,

$$\text{Hom}_L(\mathcal{X}_\mathcal{T}, \Gamma) \to \text{Hom}_L(\mathcal{X}, \Gamma).$$

The following provides us a formalism to get a tannakian interpretation of the étale fundamental gerbe.

**Theorem 2.18.** (Tonini–Zhang, cf. [37, §5, Theorem 5.8].)

1. Suppose that Axiom 2.17(A) is satisfied and that $L$ is a field. Then Axiom 2.17(B) is satisfied and $\mathcal{T}(\mathcal{X})$ is an $L$-tannakian category. Moreover, for any tannakian gerbe $\Gamma$ over $L$, the functor

$$\text{Hom}_L(\mathcal{X}_\mathcal{T}, \Gamma) \to \text{Hom}_L(\mathcal{X}_\Gamma, \Gamma)$$

is an equivalence of categories.

2. (Cf. [37, §5, Proposition 5.7].) Suppose that Axiom 2.17(A) is satisfied. If $\mathcal{X}$ is connected and Axiom 2.17(B) is satisfied, then $L$ is a field.

3. Suppose that Axioms 2.17(A), (C) are satisfied and that $L$ is a field. Then the morphism $\mathcal{X} \to \Pi_{\mathcal{T}(\mathcal{X})} \overset{\text{def}}{=} (\Pi_{\mathcal{T}(\mathcal{X})})^\text{ét}$ gives the étale fundamental gerbe for $\mathcal{X}$ over $L$.

$$\text{Vect}(\Pi_{\mathcal{T}(\mathcal{X})}) \simeq \text{EFin}(\mathcal{T}(\mathcal{X})).$$

From now on, we shall assume that $k$ is a field of positive characteristic $p > 0$. If $\mathcal{X}$ be a fibered category over $(\text{Aff}/k)$, then the Frobenius pullback

$$F^* : \text{Vect}(\mathcal{X}) \to \text{Vect}(\mathcal{X})$$

is defined by applying the absolute Frobenius pointwise. The functor $F^*$ is an $F^p$-linear exact monoidal functor.

**Definition 2.19.** (Cf. [37, §5, Definition 5.11].) For each integer $i \geq 0$, we define the category $\mathcal{T}_i(\mathcal{X})$ to be the category of tuples $(F, G, \lambda)$ where

1. $F \in \text{Vect}(\mathcal{X})$,
2. $G \in \mathcal{T}(\mathcal{X})$, and
3. $\lambda : F^\text{ét}_i \mathcal{T} \to G_{|\mathcal{X}} \overset{\text{def}}{=} \pi_i^* G$ is an isomorphism.
A morphism \((F; G, \lambda) \rightarrow (F', G', \lambda')\) is a pair of morphisms \(F \rightarrow F'\) and \(G \rightarrow G'\) which are compatible with the isomorphisms \(\lambda\) and \(\lambda'\). The category \(T(\mathcal{X})\) is \(F_p\)-linear monoidal and rigid with the unit object \(1_{T(\mathcal{X})} = (\mathcal{O}_X, \mathcal{O}_X, \text{id})\). We endow \(T(\mathcal{X})\) with a \(k\)-structure via
\[
k \rightarrow \text{End}_{T(\mathcal{X})}(1_{T(\mathcal{X})}) : a \mapsto (a, a^{\rho^i}).
\]

We consider \(T(\mathcal{X})\) as a pseudo-abelian category together the distinguished set \(J_{T(\mathcal{X})}\) of sequences which are pointwise exact. The forgetful functor \(T(\mathcal{X}) \rightarrow \text{Vect}(\mathcal{X})\) is \(k\)-linear monoidal and exact.

There exists a \(k\)-linear monoidal and exact functor
\[
T(\mathcal{X}) \rightarrow T_{+1}(\mathcal{X}) ; (F; G, \lambda) \mapsto (F, F^* G, F^* \lambda).
\]

We define \(T_\infty(\mathcal{X})\) as the direct limit of the categories \(T_i(\mathcal{X})\). The category \(T_\infty(\mathcal{X})\) is a \(k\)-linear monoidal rigid category. The following provides us a formalism to get a tannakian interpretation of the Nori fundamental gerbe.

**Theorem 2.20.** (Tonini–Zhang, cf. [37, §5, Theorem 5.14].) Suppose that Axiom 2.17(A) is satisfied for \(\pi_T : \mathcal{X} \rightarrow \mathcal{X}_T\), that \(L = L_0 = \text{End}_{T(\mathcal{X})}(1_{T(\mathcal{X})})\) is a field and the following condition holds for \(\mathcal{X}\).

For any \(F \in \text{QCoh}_{fp}(\mathcal{X})\), if \(F^* F \in \text{Vect}(\mathcal{X})\), then \(F \in \text{Vect}(\mathcal{X})\). \hspace{1cm} (2.4)

Then we have the following.

1. For any \(i \in \mathbb{N} \cup \{\infty\}\), the ring \(L_i \overset{\text{def}}{=} \text{End}_{T_i(\mathcal{X})}(1_{T_i(\mathcal{X})})\) is a field, \(T_i(\mathcal{X})\) is an \(L_i\)-tannakian category and \(\Pi_{T_i(\mathcal{X})}\) is a tannakian gerbe over \(L_i\), and the functor \(T_i(\mathcal{X}) \rightarrow \text{Vect}(\mathcal{X})\) is faithful monoidal and exact.

2. The functors \(T_i(\mathcal{X}) \rightarrow T_{i+1}(\mathcal{X})\) and \(T_i(\mathcal{X}) \rightarrow T_\infty(\mathcal{X})\) are faithful monoidal, exact and compatible with the forgetful functors \(T_i(\mathcal{X}) \rightarrow \text{Vect}(\mathcal{X})\). The functor \(T_\infty(\mathcal{X}) \rightarrow \text{Vect}(\mathcal{X})\) induces a morphism \(\mathcal{X} \rightarrow \Pi_{T_\infty(\mathcal{X})}\), whence \(\mathcal{X}\) is a fibred category over \(L_\infty\). Moreover,
\[
L_\infty = \{x \in H^0(\mathcal{O}_X) \mid x^{\rho^i} \in L_0 \text{ for some } i \geq 0\}
\]
is purely inseparable over \(L_0\).

3. For any \(i \in \mathbb{N} \cup \{\infty\}\), we have
\[
\text{EFin}(T_i(\mathcal{X})) = \{(F, G, \lambda) \mid G \in \text{EFin}(T_i(\mathcal{X}))\}
\]
in \(T_i(\mathcal{X})\), and \(\text{EFin}(T_\infty(\mathcal{X})) \approx \lim_{i \rightarrow 1} \text{EFin}(T_i(\mathcal{X}))\). Moreover, the morphism \(\mathcal{X} \rightarrow \Pi_{T_\infty(\mathcal{X})} \overset{\text{def}}{=} (\Pi_{T_\infty(\mathcal{X})})^{\text{loc}}\) is the local fundamental gerbe for \(\mathcal{X}\) over \(L_\infty\).

4. Suppose also that Axiom 2.17(C) holds. Then \(\mathcal{X} \rightarrow \tilde{\Pi}_{T_\infty(\mathcal{X})}\) (cf. Definition 2.16) is the Nori fundamental gerbe for \(\mathcal{X}\) over \(L_\infty\) and we have an equivalence of \(L_\infty\)-tannakian categories,
\[
\text{Vect}(\Pi_{\mathcal{X}/L_\infty}) \simeq \text{EFin}(T_\infty(\mathcal{X})),
\]

**Example 2.21.** (Cf. [37, §5, Remark 5.15].) If \(\mathcal{X}\) is a reduced fibred category in groupoids over a field \(k\). Then the condition (2.4) is fulfilled.

### 2.5 Example: Frobenius divided sheaves

The results of this subsection will be used only in Appendix §B. In §B, the Nori fundamental gerbes of non-smooth tame stacks will be studied, which is not necessary for the proof of the main theorem. Thus, the reader can access the main theorem without following the results in this subsection.

Let \(k\) be a perfect field of characteristic \(p > 0\).
**Definition 2.22.** (Cf. [37, §6, Definition 6.20].) Let \( \mathcal{X} \) be a fibered category in groupoids over \( k \). We define the fibered category \( \mathcal{X}^{(\infty)} \) in groupoids over \( k \) together with a morphism \( \mathcal{X} \to \mathcal{X}^{(\infty)} \) as follows. Consider the sequence of the relative Frobenius morphisms

\[
\mathcal{X} \to \mathcal{X}^{(1)} \to \mathcal{X}^{(2)} \to \ldots \to \mathcal{X}^{(i)} \to \ldots
\]

and define \( \mathcal{X}^{(\infty)} \) to be the limit

\[
\mathcal{X}^{(\infty)} \overset{\text{def}}{=} \lim_{i \in \mathbb{N}} \mathcal{X}^{(i)}
\]

and the morphism \( \mathcal{X} \to \mathcal{X}^{(\infty)} \) to be the natural morphism. We define the category \( \text{Fdiv}(\mathcal{X}) \) of **Frobenius divided sheaves** on \( \mathcal{X} \), or shortly **\( F \)-divided sheaves** on \( \mathcal{X} \), by

\[
\text{Fdiv}(\mathcal{X}) \overset{\text{def}}{=} \text{Vect}(\mathcal{X}^{(\infty)}).
\]

We apply the formalism discussed in the previous section to the morphism of fibered categories

\[
\pi = \pi_{\text{Fdiv}} : \mathcal{X} \to \mathcal{X}_{\text{Fdiv}} = \mathcal{X}^{(\infty)}.
\]

**Theorem 2.23.** (dos Santos, Tonini–Zhang, cf. [13], [37, §6, Theorem 6.23]) Let \( \mathcal{X} \) be a geometrically connected algebraic stack of finite type over \( k \). Then all the Axioms 2.17(A), (B) and (C) are satisfied for \( \mathcal{X} \to \mathcal{X}^{(\infty)} \) and \( L = \text{End}_{\text{Fdiv}(\mathcal{X})}(1_{\text{Fdiv}(\mathcal{X})}) = k \). Therefore, \( \text{Fdiv}(\mathcal{X}) \) is a \( k \)-tannakian category.

**Remark 2.24.** Moreover, the associated tannakian gerbe \( \Pi_{\text{Fdiv}(\mathcal{X})} \) is banded by a pro-smooth affine group scheme (cf. [13], [37]).

As a consequence, we get tannakian interpretations of the fundamental gerbes in positive characteristic.

**Corollary 2.25.** (Gieseker, dos Santos, Esnault–Hogadi, Tonini–Zhang, cf. [15], [13], [14], [37].) Let \( \mathcal{X} \) be a geometrically connected algebraic stack of finite type over \( k \). Then we have the following.

1. There exists a natural equivalence of tannakian categories over \( k \),

\[
\text{Vect}
\left(
\Pi_{\mathcal{X}/k}^F
\right) \simeq \text{EFin}(\text{Fdiv}(\mathcal{X})).
\]

2. If moreover \( \mathcal{X} \) is geometrically reduced, then there exists a natural equivalence of tannakian categories over \( k \), which extends the one given in (1),

\[
\text{Vect}
\left(
\Pi_{\mathcal{X}/k}^N
\right) \simeq \text{EFin}(\text{Fdiv}_\infty(\mathcal{X})).
\]

### 3 | ROOT STACKS AND QUASI-COHERENT SHEAVES OF THEM

#### 3.1 | Finite linearly reductive group schemes

In this subsection, we recall the definition of **linearly reductive** group schemes. First, a finite flat group scheme \( G \) over a scheme \( S \) is called diagonalizable if it is isomorphic to the Cartier dual of a constant abelian group scheme. To a finite abelian group \( \Gamma \), one can associate the diagonalizable group scheme, which we denote by \( \text{Diag}_S(\Gamma) \), over \( S \) (cf. [40, Section 2.2]). Note that the Cartier dual of \( \text{Diag}_S(\Gamma) \) is canonically isomorphic to the constant group scheme associated with \( \Gamma \). The formulation of \( \text{Diag}_S(\Gamma) \) is compatible with any base change, i.e. for any morphism \( S' \to S \) of schemes, there exists
a canonical isomorphism

\[ \text{Diag}_{S'}(\Gamma) \xrightarrow{\simeq} \text{Diag}_S(\Gamma) \times_S S' \]

(cf. [18, §1, 1.1.2]).

Let \( S \) be the spectrum of a strictly henselian local ring or a separably closed field. For an affine flat \( S \)-group scheme \( G \), we will denote by \( \%G \) the group of characters of \( G \), namely \( \%G \overset{\text{def}}{=} \text{Hom}_{S-G}(G, \mathbb{G}_m) \). If \( G \) is diagonalizable, then the character group \( \%G \) recovers the original group scheme \( G \), i.e. there exists a canonical isomorphism

\[ G \xrightarrow{\simeq} \text{Diag}_S(\%G) . \]

In fact, the correspondence \( \Gamma \mapsto \text{Diag}_S(\Gamma) \) gives an anti-equivalence between the category of finite abelian groups and the category of finite diagonalizable \( S \)-group schemes and a quasi-inverse functor is given by \( G \mapsto \%G \).

A finite flat group scheme \( G \) over a scheme \( S \) is said to be \textit{linearly reductive} (cf. [3, §2]) if the functor

\[ \text{Qcoh}(B_S G) \rightarrow \text{Qcoh}(S); \ P \mapsto P^G \]

is exact. If \( S = \text{Spec} \ k \) is the spectrum of a field \( k \), then the latter condition can be replaced by the condition that the functor

\[ \text{Rep}(G) \rightarrow \text{Vec}_k; \ V \mapsto V^G \]

is exact, or equivalently, \( \text{Rep}(G) \) is a semisimple category. In [3], a classification of finite flat linearly reductive group schemes (cf. [3, Theorem 2.16]) is given for a general base scheme \( S \). Here, we recall the classification theorem only in a special case.

**Proposition 3.1.** (Cf. [3, Proposition 2.10].) \( S \) is the spectrum of a separably closed field \( k \), then a finite flat group scheme \( G \) over \( S \) is linearly reductive if and only if it admits an exact sequence

\[ 1 \rightarrow \Delta \rightarrow G \rightarrow H \rightarrow 1 \]

such that \( H \) is constant and tame and \( \Delta \) is a diagonalizable group scheme. Moreover, the extension admits a splitting after an fpqc cover of \( S \). Here, a finite étale group scheme over \( S \) is said to be tame if the degree is prime to the characteristic of \( k \).

### 3.2 Tame stacks

Let \( S \) be a scheme and \( \mathcal{X} \) be an algebraic stack locally of finite presentation over \( S \) with finite inertia \( I_{\mathcal{X}/S} \). In this case, according to [22], [28, Definition 11.1.1], there exists a coarse moduli space \( \pi : \mathcal{X} \rightarrow X \), i.e. a morphism into an algebraic space over \( S \) satisfying the following two conditions.

(i) The induced map \( \mathcal{X}(\xi) \rightarrow X(\xi) \) is bijective for any geometric point \( \xi \).

(ii) The morphism \( \pi \) is universal for maps to algebraic spaces.

The morphism \( \pi : \mathcal{X} \rightarrow X \) is proper, and the map \( \theta_\mathcal{X} \rightarrow \pi_* \theta_\mathcal{X} \) is an isomorphism (cf. [28, Theorem 11.1.2(ii)]).

**Definition 3.2.** Under the above situation, \( \mathcal{X} \) is said to be \textit{tame} if the functor \( \pi_* : \text{Qcoh}(\mathcal{X}) \rightarrow \text{Qcoh}(X) \) is exact.

**Example 3.3.** Let \( G \) be a finite flat group scheme of finite presentation over \( S \). Then the structure morphism \( \pi : B_S G \rightarrow S \) is a coarse moduli of the classifying stack \( B_S G \) and if we identify \( \text{Qcoh}(B_S G) \) with the category of \( G \)-equivariant sheaves over \( S \), the functor \( \pi_* : \text{Qcoh}(B_S G) \rightarrow \text{Qcoh}(S) \) is nothing but the one taking the \( G \)-invariant part, i.e. \( \pi_* P = P^G \). Therefore, \( B_S G \) is tame if and only if \( G \) is a linearly reductive \( S \)-group scheme.
Theorem 3.4. (Cf. [3, §3, Theorem 3.2].) With the same notation as in Definition 3.2, the following are equivalent.

(a) \( \mathcal{X} \) is tame.
(b) There exists an fppf cover \( \mathcal{X}' \to X \), a linearly reductive group scheme \( G \) over \( X' \) acting on a finite \( X' \)-scheme \( U \) of finite presentation such that

\[ \mathcal{X} \times_X X' \cong [U/G]. \]

Corollary 3.5. (Cf. [3, §3, Corollaries 3.3, 3.4 and 3.5]; [4, §7, Proposition 7.4].) Let \( \mathcal{X} \to S \) be as above and let \( \pi : \mathcal{X} \to X \) be a coarse moduli space. Then we have the following.

(1) Suppose that \( \mathcal{X} \) is tame. For any morphism of \( S \)-algebraic spaces \( \mathcal{X}' \to X \), the coarse moduli space of \( \mathcal{X} \times_X X' \) is the projection \( \mathcal{X} \times_X X' \to X' \).
(2) Suppose that \( \mathcal{X} \) is tame. If \( \mathcal{X} \) is flat over \( S \), then \( X \) is also flat over \( S \).
(3) If \( \mathcal{X} \) is tame, then for any morphism of schemes \( S' \to S \), \( \mathcal{X} \times_S S' \) is a tame stack over \( S' \).
(4) The algebraic stack \( \mathcal{X} \) over \( S \) is tame if and only if for any morphism \( \text{Spec} k \to S \) with \( k \) algebraically closed, the fiber \( \mathcal{X} \times_S \text{Spec} k \) is tame.
(5) The morphism \( \pi : \mathcal{X} \to X \) is a universal homeomorphism.

3.3 Root stacks and quasi-coherent sheaves of them

Let \( S = \text{Spec} k \) be the spectrum of a field \( k \) and \( \mathcal{X} \) be an algebraic stack over \( k \). Recall that there exists a natural equivalence of groupoids

\[ \text{Pic}(\mathcal{X}) \cong \text{Hom}_k(\mathcal{X}, B_k \mathbb{G}_m), \]

where \( \text{Pic}(\mathcal{X}) \) is the groupoid of invertible sheaves over \( \mathcal{X} \). Under this equivalence, each invertible sheaf \( \mathcal{L} \) on \( \mathcal{X} \) corresponds to a \( \mathbb{G}_m \)-torsor \( P(\mathcal{L}) \to \mathcal{X} \) defined by \( P(\mathcal{L}) \overset{\text{def}}{=} \text{Spec}_k \left( \bigoplus_{i \in \mathbb{Z}} \mathcal{L}^\otimes i \right) \). Moreover, under the above equivalence, each global section \( s \in H^0(\mathcal{X}, \mathcal{L}) \) of an invertible sheaf \( \mathcal{L} \) on \( \mathcal{X} \) corresponds to a \( \mathbb{G}_m \)-equivariant function \( \theta(s) : \mathbb{A}^1_k \to \mathbb{A}^1_k \). This amounts to saying that there exists a natural equivalence between the groupoid \( \text{Hom}_k(\mathcal{X}, [\mathbb{A}^1_k \to \mathbb{G}_m]) \) and the groupoid of all the pairs \( (\mathcal{L}, s) \) of an invertible sheaf \( \mathcal{L} \) on \( \mathcal{X} \) together with a global section \( s \in H^0(\mathcal{X}, \mathcal{L}) \), where the action of \( \mathbb{G}_m \) on \( \mathbb{A}^1_k \) is defined by the natural multiplication. Since

\[ [\mathbb{A}^n_k \to \mathbb{G}^n_m] = [\mathbb{A}^1_k \to \mathbb{G}_m] \times_k \cdots \times_k [\mathbb{A}^1_k \to \mathbb{G}_m] \]

for each positive integer \( n > 0 \), the groupoid \( \text{Hom}_k(\mathcal{X}, [\mathbb{A}^n_k \to \mathbb{G}^n_m]) \) is equivalent to the groupoid of families \( (\mathcal{L}, s)_i \) of such pairs \( (\mathcal{L}_i, s_i) \).

Definition 3.6. Suppose given a morphism \( \mathcal{X} \to [\mathbb{A}^n_k \to \mathbb{G}^n_m] \), which corresponds to a family \( (\mathcal{L}, s) \), and an \( n \)-tuple \( r = (r_i)_{i=1}^n \) of positive integers \( r_i > 0 \). We define the algebraic stack \( \sqrt{\mathcal{X}(\mathcal{L}, s)/\mathcal{X}} \) as the 2-fiber product

\[ \sqrt{\mathcal{X}(\mathcal{L}, s)/\mathcal{X}} \longrightarrow [\mathbb{A}^n_k \to \mathbb{G}^n_m] \]

\[ \mathcal{X} \longrightarrow [\mathbb{A}^n_k \to \mathbb{G}^n_m], \]

where \( \vartheta_r \) is the \( r \)th power map \( a = (a_i)_{i=1}^n \mapsto a^r = (a_i^r)_{i=1}^n \). We denote by \( \pi_r \), or simply \( \pi \) if no confusion occurs, the natural map \( \sqrt{\mathcal{X}(\mathcal{L}, s)/\mathcal{X}} \to \mathcal{X} \).
From the definition, for any two $n$-tuples $\mathbf{r}$ and $\mathbf{r'}$ with $\mathbf{r} \mid \mathbf{r'}$, which means $r_i \mid r'_i$ for any $i$, there exists a natural $\mathcal{X}$-morphism

$$\sqrt[\psi]{(L,s)/\mathcal{X}} \longrightarrow \sqrt[\psi]{(L,s)/\mathcal{X}}$$

which stems from the endomorphism $\theta_{r'/r}$ of $[\mathbb{A}^n_k/\mathbb{G}_m^n]$, where $\mathbf{r'} / \mathbf{r} = (r'_1 / r_1, \ldots, r'_n / r_n)$. In fact, if we put $\mathcal{Y} = \sqrt[\psi]{(L,s)/\mathcal{X}}$, then we have $\sqrt[\psi]{(L',s')/\mathcal{Y}} = \sqrt[\psi]{(L,s)/\mathcal{X}}$, where $(L', s')$ is the family of pairs which defines the morphism $\mathcal{Y} \longrightarrow [\mathbb{A}^n_k/\mathbb{G}_m^n]$. Moreover, if $f : \mathcal{Y} \longrightarrow \mathcal{X}$ is a morphism of algebraic stacks over $k$, then we have

$$\mathcal{Y} \times_{\mathcal{X}} \sqrt[\psi]{(L,s)/\mathcal{X}} = \sqrt[\psi]{(f^*L, f^*s)/\mathcal{Y}}.$$ 

**Example 3.7.**

1. Let $X$ be a scheme over $k$. If all the $s_i$ are nowhere vanishing sections, or equivalently, if the associated $\mathbb{G}_m$-equivalent morphisms $P(L_0) \longrightarrow \mathbb{A}^1_k$ factor through $\mathbb{G}_m$, then the morphism $X \longrightarrow [\mathbb{A}^n_k/\mathbb{G}_m^n]$ factors through $[\mathbb{G}_m^n/\mathbb{G}_m^n] = S$, whence $\pi : \sqrt[\psi]{(L,s)/X} \longrightarrow X$.

2. Let $X = \text{Spec } A$ be an affine scheme. If $L_i = \mathcal{O}_X$ for any $i$, then we have

$$\sqrt[\psi]{(L,s)/X} \simeq \left[ (\text{Spec } A[t]/(t^r - s) )/\mu_r \right].$$

**Proposition 3.8.** With the above notation, put $\mathcal{Y} = \sqrt[\psi]{(L,s)/\mathcal{X}}$. Then we have the following.

1. (Cf. [11, Theorem 2.3.2].) The diagonal $\Delta : \mathcal{Y} \longrightarrow \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}$ is finite.

2. (Cf. [11, Corollary 2.3.6].) The morphism $\pi : \mathcal{Y} \longrightarrow \mathcal{X}$ is faithfully flat and quasi-compact.

3. (Cf. [36, Theorem 1.2.31 and Proposition 1.2.32].) If $\mathcal{X} = X$ is an algebraic space over $k$, then $\mathcal{Y}$ is a tame stack over $k$ with $\pi : \mathcal{Y} \longrightarrow X$ the coarse moduli space of $\mathcal{Y}$.

**Proof.**

1. It suffices to show the map $\theta_r : [\mathbb{A}^n_k/\mathbb{G}_m^n] \longrightarrow [\mathbb{A}^n_k/\mathbb{G}_m^n]$ has finite diagonal. Since the natural projection $\mathbb{A}^n_k \longrightarrow [\mathbb{A}^n_k/\mathbb{G}_m^n]$ is a smooth surjective morphism, we are reduced to the case where $\mathcal{X} = \mathcal{X} = \text{Spec } k[x_1, \ldots, x_n]$ and $\mathcal{Y} = \sqrt[\psi]{(\mathcal{O}_\mathcal{X}, x_1^n, \ldots, x_n^n)} / \mathbb{G}_m^n = \left[ \mathbb{A}^n_k/\mu_r \right]$. Namely, it suffices to show that the morphism $\left[ \mathbb{A}^n_k/\mu_r \right] \longrightarrow \mathbb{A}^n_k$ has finite diagonal. However, since the natural morphism $\mathbb{A}^n_k \longrightarrow \left[ \mathbb{A}^n_k/\mu_r \right]$ is a finite flat surjective morphism, we get a Cartesian diagram with two vertical arrows finite flat and surjective,

$$\begin{array}{ccc}
\text{Spec } \frac{k[x,y,t]}{(s-ty^r-t^r)} & \longrightarrow & \text{Spec } \frac{k[x,y]}{(x^r-y^r)} \\
\downarrow & & \downarrow \\
[\mathbb{A}^n_k/\mu_r] & \longrightarrow & [\mathbb{A}^n_k/\mu_r] \times_{[\mathbb{A}^n_k/\mu_r]} \left[ \mathbb{A}^n_k/\mu_r \right].
\end{array}$$

Since the top horizontal arrow is finite, this implies that the bottom one is also finite. This completes the proof.

2. It suffices to show the claims for $\theta_r : [\mathbb{A}^n_k/\mathbb{G}_m^n] \longrightarrow [\mathbb{A}^n_k/\mathbb{G}_m^n]$. However, since the diagram

$$\begin{array}{ccc}
\mathbb{A}^n_k & \longrightarrow & \mathbb{A}^n_k/\mathbb{G}_m^n \\
\downarrow \quad \quad \downarrow \\
\mathbb{A}^n_k & \longrightarrow & \left[ \mathbb{A}^n_k/\mathbb{G}_m^n \right]
\end{array}$$


is commutative, where the two horizontal arrows are the universal $G_m^n$-torsors and the left vertical arrow is the $r$th power map, it suffices to show the claims for the $r$th power map $\mathbb{A}^n_k \to \mathbb{A}^n_k$ (cf. [35, Lemma 06FM]), which are standard facts.

(3) Since the problem is Zariski local for $X$, we may assume that $X = \text{Spec } A$ is an affine scheme over $k$ and the invertible sheaves $\mathcal{L}_i$ are trivial. Then, thanks to Theorem 3.4, the claim follows from the description given in Example 3.7(2).

\[ \square \]

**Proposition 3.9.** (Cf. [36, Proposition 1.2.35].) With the same notation as in Proposition 3.8, we have the following.

1. $\mathcal{O}_X \xrightarrow{\pi^*} \pi_*(\mathcal{F} \otimes \mathcal{E}).$
2. For any quasi-coherent sheaves $\mathcal{E}$ on $X$ and $\mathcal{F}$ on $Y$, $\pi_*(\mathcal{F} \otimes \mathcal{E}) \xrightarrow{\pi^*} \pi_*(\mathcal{F} \otimes \pi^* \mathcal{E}).$
3. For any quasi-coherent sheaf $\mathcal{E}$ on $X$, $\mathcal{E} \xrightarrow{\pi^*} \pi_*(\mathcal{E}).$

Therefore, the functor $\pi^* : \text{Qcoh}(X) \to \text{Qcoh}(Y)$ is fully faithful.

**Proof.** (3) follows from (1) and (2). Therefore, it suffices to show (1) and (2). As explained in the proof of [36, Proposition 1.2.35], by considering an fppf cover $U \to X$ from a $k$-scheme $U$, the problems can be reduced to the case where $X = X$ is a scheme, whence, by Proposition 3.8, $Y$ is a tame stack with $\pi : Y \to X$ the coarse moduli space. Then (1) holds as we recall at the beginning of the previous Subsection §3.2. Moreover, for the second claim (2), see [4, Proposition 4.5]. \[ \square \]

We will apply the above arguments to the following specific situation.

**Definition 3.10.** Let $X$ be a locally Noetherian scheme over $k$. Let $D = (D_i)_{i \in I}$ be a finite family of reduced irreducible distinct effective Cartier divisors $D_i$ on $X$. We set $D^\circ = \bigcup_{i \in I} D_i \subset X$. Furthermore, suppose given a family $r = (r_i)_{i \in I}$ of integers $r_i > 0$. For each $i \in I$, we denote by $s_{D_i}$ the canonical section of $\mathcal{O}_X(D_i)$, i.e. $s_{D_i} : \mathcal{O}_X \to \mathcal{O}_X(D_i)$ is the natural inclusion. Let $\theta_X(D) = (\theta_X(D_i))_{i \in I}$ and $s_D = (s_{D_i})_{i \in I}$. For each $r = (r_i)_{i \in I}$ with $r_i > 0$, we define

$$X^r \overset{\text{def}}{=} \sqrt[\theta_X(D)]{X} \overset{\text{def}}{=} \sqrt[\theta_X(D), s_D]{X}$$

and call it the root stack associated with $X$ and the data $(D, r)$.

**Proposition 3.11.** (Cf. [8, §2.4.1], [7, Lemmas 3.2 and 3.4].) With the above notation, we have the following.

1. For any open affine neighbourhood $U = \text{Spec } A$ together with local equations $s_i = 0$ for $D_i$ on $U$ so that $s_i : \mathcal{O}_X(D_i) \mid_U \xrightarrow{\pi^*} \mathcal{O}_U$, we have an isomorphism

$$U \times_X \sqrt[D/X]{} \simeq \left[ [\text{Spec } A[t]/(t^r - s)] \right].$$

2. For any closed point $\xi = [x] \in |\sqrt[D/X]|_0 = |X|_0$, there exists a unique gerbe $G_\xi \to \text{Spec } k(x)$, which we call the residual gerbe at $\xi$, such that $G_\xi$ is a Noetherian algebraic substack of $\sqrt[D/X]$ and the image of $|G_\xi|_0 \to |\sqrt[D/X]|_0$ is $\xi$. Moreover, for any $\xi = [x] \in |\sqrt[D/X]|_0$, the residual gerbe $G_\xi$ at $\xi$ is neutral and non-canonically isomorphic to $B_{k(x)}^r\mu_{x, r}$, where the
index $r_x = (r_{x,i})_{i \in I}$ is defined to be

$$r_{x,i} = \begin{cases} 1 & \text{if } x \notin D_i, \\ r_i & \text{if } x \in D_i. \end{cases}$$

(3.1)

Proof.

(1) follows from the description given in Example 3.7(2) together with the fact that

$$\big( O(U(D_i), s_{D_i}) \big) \overset{s_i}{\cong} \big( O_U, s_i \big)$$

for any $i \in I$.

(2) The uniqueness of such a gerbe follows from [35, Lemma 06MT]. Let us prove the existence. The problem is Zariski local, by (1), we can replace $\sqrt[\text{Zar}}]{D/X}$ by the quotient stack $\mathcal{X} \equiv \left[ \left( \text{Spec} A[t]/(t^r - s) \right)/\mu_r \right]$. For any $x \in \text{Spec } A$, we have

$$\mathcal{X} \times_X X = \left[ \left( \text{Spec} k(x)[t]/(t^r - s(x)) \right)/\mu_r \right] \cong \prod_{s_i(x) = 0} \left[ \left( \text{Spec} k(x)[t_i]/(t_i^{r_i}) \right)/\mu_{r_i} \right],$$

hence

$$\left( \mathcal{X} \times_X X \right)_{\text{red}} = \prod_{s_i(x) = 0} \left[ \text{Spec } k(x)/\mu_{r_i} \right] \simeq B_{k(x)} \mu_{x}$$

gives a desired gerbe $G_{\xi}$ above $x$. □

The following is a variant of a theorem due to Alper [4, §10, Theorem 10.3].

**Proposition 3.12.** With the same notation as in Definition 3.10, let $r'$ be another index with $r \mid r'$. Then we have the following.

1. For any closed point $\xi$ of $\mathcal{X}^{r'}$, the induced morphism $\pi_{\xi} : G_{\xi} \to G_{\xi}$ between the residual gerbes is a gerbe. More precisely, each closed point $\xi = [x] \in |\mathcal{X}^{r'}|_0 = |X|_0$ gives rise to the 2-Cartesian diagram

$$\begin{array}{ccc}
B_{k(x)}(\mu_{r_x/\mu_x}) & \to & G_{\xi} \\
\downarrow & & \downarrow \\
\text{Spec } k(x) & \to & G_{\xi}.
\end{array}$$

2. Let $E$ be a vector bundle on $\mathcal{X}^{r'}$. Suppose that for any closed point $\xi$ of $\mathcal{X}^{r'}$, we have $\pi_{\xi}^* \pi_{\xi'}^* (E|_{G_{\xi}}) \overset{\simeq}{\to} E|_{G_{\xi'}}$. Then $\pi_{\xi}^* \pi_{\xi'}^* E \overset{\simeq}{\to} E$ and $\pi_{\xi}^* E$ is a vector bundle on $\mathcal{X}^{r'}$.

3. The functor $\pi^* : \text{Vect}(\mathcal{X}) \to \text{Vect}(\mathcal{X}^{r'})$ is fully faithful and the essential image consists of all the vector bundles $E$ on $\mathcal{X}^{r'}$ such that for any closed point $\xi$ of $\mathcal{X}$, we have $\pi_{\xi}^* \pi_{\xi'}^* (E|_{G_{\xi}}) \overset{\simeq}{\to} E|_{G_{\xi'}}$. Moreover, the essential image of the functor $\pi^* : \text{Vect}(\mathcal{X}) \to \text{Vect}(\mathcal{X}^{r'})$ is closed under taking subquotients.
Proof.

(1) By taking an fppf covering $T \to \mathcal{X}$ with $T$ a scheme, the assertion can be deduced from Proposition 3.11(2) (cf. [35, Lemma 06QF]).

(2) Take fppf coverings $f : T \to \mathcal{X}$ and $f' : T' \to \mathcal{X}'$ with $T$ and $T'$ schemes and consider the commutative diagram

$$
\begin{array}{ccc}
T'' & \to & T' \\
\downarrow f'' & & \downarrow f' \\
\mathcal{X}' \times_{\mathcal{X}} T & \to & \mathcal{X}' \\
\downarrow \pi' & & \downarrow \pi \\
T & \to & \mathcal{X}
\end{array}
$$

where all the squares are 2-Cartesian. Note that $f'' : T'' \to \mathcal{X}' \times_{\mathcal{X}} T$ is an fppf covering with $T''$ a scheme. Then, for any morphism $\phi : E_1 \to E_2$ of vector bundles on $\mathcal{X}'$, $\phi$ is an isomorphism if and only if $g^* f'^* \phi = f''^* g^* \phi$ is an isomorphism, and the latter condition is equivalent to the condition that $g^* \phi$ is an isomorphism.

Then the claims are equivalent to saying that $g^* \pi^*_s E \simeq g^* E$ and $f^* \pi^*_s E$ is a vector bundle on $T$. However, since $f : T \to \mathcal{X}$ is flat, by flat base changing, we get

$$
g^* \pi^*_s E \simeq \pi'^*_s f^* \pi^*_s E \simeq \pi'^*_s g^* E
$$

and similarly, $f^* \pi^*_s E \simeq \pi'_* g^* E$. Hence, the problem is reduced to the absolute case $\pi : \mathcal{X}' \to \mathcal{X}$, which follows from [4, §10 Theorem 10.3].

(3) The description of the essential image is immediate from (2). The full faithfulness follows from Proposition 3.9(3). The last assertion is a consequence of (1).

\begin{proposition}
(Cf. [5, Proposition 3.9(c)].) With the same notation as in Definition 3.10, suppose further that $X$ is smooth over a perfect field $k$, that each $D_i$ is smooth over $k$, and that $D$ is a simple normal crossings divisor on $X$. Then the root stack $\sqrt{D}/X$ is a smooth algebraic stack over $k$.

Proof. Since the problem is Zariski local, we may assume that

$$\mathcal{X} \overset{\text{def}}{=} \sqrt{D}/X \simeq [Z/\mu_r]$$

for some $Z = \text{Spec } A[t]/(t^r - s)$ with $X = \text{Spec } A$ (cf. Proposition 3.11(1)), where $s_1, \ldots, s_n$ is a regular sequence in $A$. Let us begin with showing that $Z$ is smooth over $k$. Let $p : Z \to X$ be the composition of the quotient map with the coarse moduli space map $\pi$. Let $D' \overset{\text{def}}{=} p^{-1}(D)$. Then note that $D' = \{ V(t_i) \}_{i=1}^n$, where $V(t_i) = \{ t_i = 0 \}$, and $t_1, \ldots, t_n$ is a regular sequence. By [35, Lemma 0BIA], the scheme theoretic intersection $\bigcap_{i=1}^n D_i$ is a regular scheme. However, since $A[t]/(t^r - s, t_1, \ldots, t_n) = A/(s_1, \ldots, s_n) = 0_{\mu_r^n}$,

by [35, Lemma 00NU], we can find that $Z$ is regular over $k$. However, as $k$ is perfect, by [35, Lemma 0CBP], the same is true after base change any algebraic extension of $k$. Namely $Z$ is a geometrically regular and hence by [35, Lemma 038X], $Z$ is a smooth $k$-scheme.

Let $W \to \mathcal{X}$ be a smooth atlas and put

$$W' \overset{\text{def}}{=} W \times_{\mathcal{X}} Z.$$
Since the projection \( W' \rightarrow Z \) is smooth and \( Z \) is a smooth \( k \)-scheme, \( W' \) is smooth over \( k \). Then we get the commutative diagram

\[
\begin{array}{ccc}
W' & \longrightarrow & W \\
\downarrow & & \downarrow \\
\text{Spec } k. & & 
\end{array}
\]

Since \( W' \rightarrow W \) is an fppf covering, by [35, Lemma 05B5], \( W \) must be a smooth \( k \)-scheme. This completes the proof. \( \square \)

4  EMBEDDING PROBLEM OVER ROOT STACKS

4.1 Local fundamental group schemes

Let \( k \) be a field of characteristic \( p > 0 \) and let \( \mathcal{X} \) be a reduced algebraic stack over \( k \) such that \( H^0(\mathcal{X}) \) contains no nontrivial purely inseparable extension of \( k \), which admits the local fundamental gerbe \( \mathcal{X} \rightarrow \Pi_{\mathcal{X}/k}^{\text{loc}} \) (cf. Definition 2.5 and Proposition 2.6(2)). By applying the formalism in \( \S 2.4 \), to the structure morphism \( \mathcal{X} \rightarrow \text{Spec } k = \mathcal{X}_{/k} \), we get a tannakian interpretation of the local fundamental gerbe \( \Pi_{\mathcal{X}/k}^{\text{loc}} \). With the same notation as in \( \S 2.4 \), we consider the categories

\[
D_i(\mathcal{X}) \overset{\text{def}}{=} T_i(\mathcal{X})
\]

for \( i \geq 0 \) with \( \mathcal{X}_{/k} = \text{Spec } k \) and the functors

\[
D_i(\mathcal{X}) \rightarrow D_{i+1}(\mathcal{X}) ; (F, V, \lambda) \mapsto (F^i F_k V, F^i \lambda).
\]

Recall that \( D_\infty(\mathcal{X}) = \lim_i D_i(\mathcal{X}) \).

**Theorem 4.1.** (Tonini–Zhang, cf. [37, \S 7, Theorem 7.1.]) Let \( k \) be a field of characteristic \( p > 0 \) and let \( \mathcal{X} \) be a reduced algebraic stack over \( k \) such that \( H^0(\mathcal{X}) \) contains no nontrivial purely inseparable extension of \( k \). Then, all the categories \( D_i(\mathcal{X}) (i \geq 0) \) and \( D_\infty(\mathcal{X}) \) are tannakian categories over \( k \) and the above functors \( D_i(\mathcal{X}) \rightarrow D_{i+1}(\mathcal{X}) \) are \( k \)-linear monoidal exact functors. Moreover, there exists a canonical equivalence of tannakian categories

\[
D_\infty(\mathcal{X}) \overset{\sim}{\rightarrow} \text{Vect}(\Pi_{\mathcal{X}/k}^{\text{loc}}).
\]

**Proof.** All the statements except for the last equivalence are immediate consequences of Theorem 2.20(1) and (2). However, by Theorem 2.20(3), we have \( D_\infty(\mathcal{X}) = \text{EF} in(D_\infty(\mathcal{X})) \) and the morphism \( \mathcal{X} \rightarrow \Pi_{D_\infty(\mathcal{X})}^{\text{loc}} = \Pi_{D_\infty(\mathcal{X})}^{\text{loc}} \) is the local fundamental gerbe over \( k \). This completes the proof.

Furthermore, if \( k \) is perfect, then the local fundamental gerbe \( \Pi_{\mathcal{X}/k}^{\text{loc}} \) is neutral.

**Proposition 4.2.** (Romagny–Tonini–Zhang, cf. [32, \S 2.]) Let \( \Gamma \) be a pro-local gerbe over a perfect field of characteristic \( p > 0 \). Then \( \Gamma(k) \neq \emptyset \) and, for any two objects \( \xi, \xi' \in \Gamma(k) \), there exists exactly one isomorphism \( \xi \overset{\sim}{\rightarrow} \xi' \). In other words, the tannakian category \( \text{Vect}(\Gamma) \) has a neutral fiber functor which is unique up to unique isomorphism.

**Proof.** Since \( \text{Vect}(\Gamma) \overset{\sim}{\rightarrow} \text{Vect}(\Pi_{\mathcal{X}/k}^{\text{loc}}) \), we may assume that \( \Gamma = \Pi_{\mathcal{X}/k}^{\text{loc}} \) for some reduced algebraic stack \( \mathcal{X} \) over \( k \). Then by Theorem 4.1, it suffices to show the claim for \( D_\infty(\mathcal{X}) \). Indeed, the functors \( D_i(\mathcal{X}) \rightarrow \text{Vec}_k ; (F, V, \lambda) \mapsto F^{-i} V \) define neutral fiber functors compatible with transition functors \( D_i(\mathcal{X}) \rightarrow D_{i+1}(\mathcal{X}) \), whence we get a neutral fiber functor of \( D_\infty(\mathcal{X}) \). Suppose given two neutral fiber functors \( \omega \) and \( \omega' \) of \( D_\infty(\mathcal{X}) \). Then the sheaf of isomorphisms \( P \overset{\text{def}}{=} \text{Isom}_k^\otimes(\omega, \omega') \) is
a torsor over $k$ under a pro-local $k$-group scheme, which admits a unique $k$-rational point Spec $k = P_{\text{red}} \to P$. Therefore, there exists a unique isomorphism $\omega \sim \omega'$. This completes the proof.

**Corollary 4.3.** (Ünver, Zhang, Romagny–Tonini–Zhang, cf. [39], [41], [32, §2].) Let $X$ be a reduced algebraic stack over a perfect field $k$ with $X(k) \neq \emptyset$. Then the local fundamental gerbe $\Pi loc(X, x)$ is independent of the choice of the $k$-rational point $x$. More precisely, for any two $k$-rational points $x, x' \in X(k)$, there exists exactly one isomorphism of pro-local $k$-group schemes $\Pi loc(X, x) \sim \Pi loc(X, x')$. Moreover, for any $k$-rational point $x \in X(k)$, there exist equivalences of categories

$$\hom_k (\Pi loc(X, x), G) \sim \hom (\Pi loc_{X/k}, BG) \sim H^1_{\text{fppf}} (X, \mu_{p^m}).$$

for any finite local $k$-group scheme $G$.

Hence, from now on, we write $\Pi loc(X)$ instead of $\Pi loc(X, x)$ for the local fundamental group scheme of $(X, x)$. The local fundamental group scheme $\Pi loc(X)$ has a more explicit description as follows. Let $L(X)$ be the category of pairs $(P, G)$ where $G$ is a finite local $k$-group scheme and $P \to X$ is a $G$-torsor. Then the category $L(X)$ is cofiltered and the projective limit

$$\lim_{(P, G) \in L(X)} (P, G)$$

exists and the underlying group scheme of the limit is canonically isomorphic to $\Pi loc(X)$.

Let $X$ be a reduced algebraic stack of finite type over an algebraically closed field $k$ of characteristic $p > 0$. Let $\Pi loc(X)$ be the local fundamental group scheme of $X$. As $k$ is algebraically closed of characteristic $p > 0$, a finite local group scheme $G$ over $k$ is linearly reductive if and only if it is diagonalizable, i.e. $G \simeq \text{Diag}(A)$ for some abelian $p$-group $A$ (cf. Proposition 3.1). Therefore, the isomorphism class of the maximal linearly reductive quotient of $\Pi loc(X)$ is determined completely by the $p$-primary torsion subgroup of the group $\chi (\Pi loc(X)) = \hom (\Pi loc(X), \mathbb{G}_m)$ of characters of the local fundamental group scheme $\Pi loc(X)$, which can be calculated as follows

$$\chi (\Pi loc(X))[p^\infty] = \lim_{m>0} \hom_k (\Pi loc(X), \mu_{p^m}) \sim \lim_{m>0} H^1_{\text{fppf}} (X, \mu_{p^m}).$$

Hence, we have seen the following.

**Proposition 4.4.** If $k$ is an algebraically closed field of characteristic $p > 0$ and $X$ is a reduced algebraic stack of finite type over $k$, then there exists a canonical isomorphism:

$$\Pi loc(X)_{\text{lin.red}} \sim \lim_{m>0} \text{Diag} (H^1_{\text{fppf}} (X, \mu_{p^m})).$$

**Corollary 4.5.** Let $X$ be a proper smooth connected curve over an algebraically closed field of characteristic $p > 0$ with $p$-rank $\gamma$ and let $\emptyset \neq U \subseteq X$ be a nonempty open subscheme of $X$ with $n = \# (X \setminus U) \geq 1$. Then we have the following.

$$\Pi loc(U)_{\text{lin.red}} \sim \begin{cases} \text{Diag} \left( \left( \mathbb{Q}_p / \mathbb{Z}_p \right)^{\oplus n} \right) & \text{if } n = 0, \\ \text{Diag} \left( \left( \mathbb{Q}_p / \mathbb{Z}_p \right)^{\oplus n+1} \right) & \text{if } n > 0. \end{cases}$$

**Proof.** If $n = 0$, this is standard. In the case where $n > 0$, see [29, Proposition 3.2].

**Remark 4.6.** Let $X$ be a reduced algebraic stack over an algebraically closed field $k$ of characteristic $p > 0$ and $\Pi loc(X)$ the local fundamental group scheme of $X/k$. We can identify the local fundamental gerbe $\Pi loc_{X/k}$ with the classifying stack $\mathcal{B}_k\Pi loc(X)$ (cf. Corollary 4.3). Suppose given a $k$-homomorphism $\Pi loc(X) \to G$ to a finite local $k$-group scheme $G$ and
the corresponding $G$-torsor $P \rightarrow \mathcal{X}$. Then, under the assumption that $k$ is an algebraically closed field, the following conditions are equivalent.

1. The homomorphism $\pi^{\text{bc}}(\mathcal{X}) \rightarrow G$ is surjective.
2. The corresponding morphism $\Pi_{\mathcal{X}/k}^{\text{loc}} = B_k \pi^{\text{loc}}(\mathcal{X}) \rightarrow B_k G$ is Nori-reduced.
3. There exists no strict closed subgroup scheme $H \subset G$ such that the $G$-torsor $P \rightarrow \mathcal{X}$ is reduced to an $H$-torsor $\mathcal{P} \rightarrow \mathcal{Y}$, i.e., $\mathcal{P} \simeq \text{Ind}_{G}^{H}(\mathcal{Q})$.
4. If we denote by $\mathcal{E}_{\infty}(P)$ the image of the regular representation $(k[G], \rho_{\text{reg}})$ of $G$ in $D_{\infty}(\mathcal{X})$ under the restriction functor $\text{Rep}(G) \rightarrow \text{Rep}(\pi^{\text{loc}}(\mathcal{X})) \simeq D_{\infty}(\mathcal{X})$ (cf. Theorem 4.1), then
   \[
   \dim_k \text{Hom}_{D_{\infty}(\mathcal{X})}(1, \mathcal{E}_{\infty}(P)) = 1.
   \]

Indeed, the equivalence between (a) and (b) is a consequence of Proposition 4.2. The equivalence between (a) and (c) is immediate from Corollary 4.3. The implication (a) $\Rightarrow$ (d) follows from the fact that $\text{Hom}(\mathcal{Y}, (k[G], \rho_{\text{reg}})) = k[G] G = k$. Conversely, let us suppose that the condition (a) is not satisfied. If $H$ denotes the image of the given homomorphism $\pi^{\text{loc}}(\mathcal{X}) \rightarrow G$, then $H$ is a closed subgroup scheme of $G$ with $H \neq G$ and the restriction map $\text{Rep}(G) \rightarrow \text{Rep}(\pi^{\text{loc}}(\mathcal{X}))$ factors through the category $\text{Rep}(H)$, which implies that
   \[
   k[G]^H = \text{Hom}_{\text{Rep}(H)}(1, (k[G], \rho_{\text{reg}})) \subseteq \text{Hom}_{D_{\infty}(\mathcal{X})}(1, \mathcal{E}_{\infty}(P)).
   \]

As $H \neq G$, we have $\dim k[G]^H \neq 1$ and hence the condition (d) does not hold. This proves the implication (d) $\Rightarrow$ (a).

4.2 Nori fundamental gerbes of root stacks in positive characteristic

We shall use the same notation as in §3.3. Let $X$ be a geometrically connected and geometrically reduced scheme of finite type over the spectrum $S = \text{Spec} \, k$ of a perfect field $k$ of characteristic $p > 0$. Let $D = (D_i)_{i \in I}$ be a finite family of reduced irreducible effective Cartier divisors on $X$ and put $D = \bigcup_{i \in I} D_i \subset X$. For each $r = (r_i)_{i \in I}$ with $r_i > 0$, as in §3.3, we put
   \[
   \mathfrak{x}^r = \sqrt{D/X}.
   \]

In this subsection, motivated by [6], the Nori fundamental gerbes associated with root stacks are studied, dropping the properness assumption on $X$. However, under the smoothness assumption as put in Proposition 3.13, a more direct approach can be applied, which is enough to prove the main theorem.

**Proposition 4.7.** Under the same assumption as in Proposition 3.13, if we put $U = X \setminus D$, then the induced morphisms
   \[
   \Pi^N_U \rightarrow \Pi^N_X
   \]
are gerbes.

**Proof.** First note that, for any finite gerbe $\Gamma$ over $k$, a morphism $\Pi^N_{\mathfrak{x}^r} \rightarrow \Gamma$ into $\Gamma$ is a gerbe if and only if the composition $\mathfrak{x}^r \rightarrow \Pi^N_{\mathfrak{x}^r} \rightarrow \Gamma$ is Nori-reduced (cf. Remark 4.8). Therefore, it suffices to show that, for any Nori-reduced morphism $\mathfrak{x}^r \rightarrow \Gamma$ into a finite gerbe $\Gamma$, the composition $U \rightarrow \mathfrak{x}^r \rightarrow \Gamma$ is still Nori-reduced. As $k$ is assumed to be perfect, without loss of generality, we may assume that $k$ is an algebraically closed field (cf. [9, §6]) and hence may assume that $\Gamma = B G$ for some finite $k$-group scheme $G$. If the morphism $U \rightarrow \mathfrak{x}^r \rightarrow \Gamma$ is not Nori-reduced, then it factors through $B H$ where $H$ is a strict subgroup scheme of $G$. One can show that the resulting morphism $U \rightarrow BH$ is extended to a morphism $\mathfrak{x}^r \rightarrow BH$. Indeed, the problem is Zariski local (cf. Proposition 2.8), we may assume that $\mathfrak{x}^r = [Z/\mu_r]$ with $Z = \text{Spec} \, O_X[t]/(t^r - s)$. As we saw in the proof of Proposition 3.13, $Z$ is smooth over $k$. Therefore, by applying [27, Chapter II Proposition 6] to the open immersion $Z \times_{\mathfrak{x}^r} U \hookrightarrow Z$, we get an extension $Z \rightarrow BH$ of the
morphism $Z \times_{\mathcal{X}^r} U \longrightarrow U \longrightarrow BH$.

By dividing by the action of $\mu_r$, we get a desired extension $\mathcal{X}^r = [Z/\mu_r] \longrightarrow BH$. However, as $H \neq G$, this contradicts with the Nori-reducedness of the morphism $\mathcal{X}^r \longrightarrow BG$. Therefore, the morphism $U \longrightarrow \mathcal{X}^r \longrightarrow BG$ is Nori-reduced. □

Remark 4.8. Let $\mathcal{X}$ be an inflexible fibered category over a field $k$ with $\mathcal{X} \longrightarrow \Pi^N_{\mathcal{X}}$ its Nori fundamental gerbe. Let $\phi : \Pi^N_{\mathcal{X}} \longrightarrow \Gamma$ be a morphism into a finite gerbe $\Gamma$ over $k$. Then $\phi$ is a gerbe if and only if the composition $\mathcal{X} \longrightarrow \Pi^N_{\mathcal{X}} \xrightarrow{\phi} \Gamma$ is Nori-reduced. Indeed, as $\Pi^N_{\mathcal{X}}$ itself is inflexible (cf. [9, Proposition 5.4]), by the universal property of the Nori fundamental gerbe $\mathcal{X} \longrightarrow \Pi^N_{\mathcal{X}}$, we may assume that $\mathcal{X} = \Pi^N_{\mathcal{X}}$, hence may assume that $\mathcal{X}$ is pseudo-proper (cf. [9, Example 7.2(b)]), in which case the equivalence is already remarked in [2, Remark 1.14].

Proposition 4.9. Let $r$ and $r'$ be two indices with $r \mid r'$. The natural morphism $\Pi^N_{\mathcal{X}^r/k} \longrightarrow \Pi^N_{\mathcal{X}^{r'}/k}$ is a gerbe. Moreover, for any finite $k$-group scheme $G$ and any $G$-torsor $\mathcal{Y} \longrightarrow \mathcal{X}^r$, there exists a $G$-torsor $\mathcal{Y} \longrightarrow \mathcal{X}^r$ such that $\mathcal{Y} \simeq \mathcal{X} \times_{\mathcal{X}^r} \mathcal{X}^{r'}$ if and only if for any closed point $\xi$ of $\mathcal{X}^r$, the composition

$$G'_\xi \longrightarrow \mathcal{Y}^r \longrightarrow B_k G$$

factors through the gerbe $G'_\xi \longrightarrow G_\xi$ (cf. Proposition 3.12(1)), where $G_\xi$ (respectively $G'_\xi$) denotes the residual gerbe of $\mathcal{X}^r$ (respectively of $\mathcal{X}^{r'}$) at $\xi$.

Proof. First let us show the second assertion. Thanks to Proposition 2.8, we have a commutative diagram where horizontal arrows are equivalence of categories,

$$
\begin{align*}
\text{Hom}_k(\mathcal{X}^r, BG) \xrightarrow{\sim} &\text{Hom}_k(\text{Vect}(BG), \text{Vect}(\mathcal{X}^r)) \\
\text{Hom}_k(\mathcal{X}^r, BG) \xrightarrow{\sim} &\text{Hom}_k(\text{Vect}(BG), \text{Vect}(\mathcal{X}^r)).
\end{align*}
$$

Therefore, by applying Proposition 3.12(3), we get the second assertion.

We prove the first assertion of the proposition under the same assumption as in Proposition 3.13. In the general case, see Appendix § B. Under the assumption in Proposition 3.13, for any indices $r \mid r'$, the commutative diagram

$$
\begin{align*}
\Pi^N_{\mathcal{X}^r} \longrightarrow &\Pi^N_{\mathcal{X}^{r'}} \\
\downarrow \quad &\downarrow \\
\Pi^N_{\mathcal{X}^{r'}}
\end{align*}
$$

induces the restriction functors

$$
\text{Vect}(\Pi^N_{\mathcal{X}^r}) \xrightarrow{u} \text{Vect}(\Pi^N_{\mathcal{X}^{r'}}) \xrightarrow{v} \text{Vect}(\Pi^N_{\mathcal{X}^{r'}}),
$$

where $v$ and $vou$ are fully faithful by Proposition 4.7. Thus, it follows that the restriction functor $u : \text{Vect}(\Pi^N_{\mathcal{X}^r}) \longrightarrow \text{Vect}(\Pi^N_{\mathcal{X}^{r'}})$ is fully faithful as well. Now let us show that $\Pi^N_{\mathcal{X}^r} \longrightarrow \Pi^N_{\mathcal{X}^{r'}}$ is a gerbe. If we write $\Pi^N_{\mathcal{X}^r/k} = \lim \longrightarrow \Gamma_i$, where
\( X' \longrightarrow \Gamma_i \) are Nori-reduced with \( \Gamma_i \) finite gerbes (cf. [9, Proof of Theorem 5.7]), then all the projections \( \Pi^N_{X'/k} \longrightarrow \Gamma_i \) are gerbes (cf. Remark 4.8). We have to show that the composition \( \Pi^N_{X'/k} \longrightarrow \Pi^N_{X/k} \longrightarrow \Gamma_i \) is still a gerbe for each \( i \).

However, as \( \Gamma_i \) is finite, the map \( \Pi^N_{X'} \longrightarrow \Gamma_i \) is a gerbe if and only if the restriction functor \( \text{Vect}(\Gamma_i) \longrightarrow \text{Vect}\left(\Pi^N_{X'}\right) \) is fully faithful (cf. [37, Remark B.7]). The latter condition is fulfilled because the restriction functor is the composition of fully faithful functors \( \text{Vect}(\Gamma_i) \longrightarrow \text{Vect}\left(\Pi^N_{X'}\right) \). This completes the proof. \( \square \)

Remark 4.10. Proposition 4.9 particularly implies that if \( r \mid r' \), then the natural map between the local fundamental group schemes is surjective, i.e. \( \pi \text{loc}(X') \longrightarrow \pi \text{loc}(X) \). One can prove this fact without the smoothness assumption. Indeed, for any surjective homomorphism \( \pi \text{loc}(X) \longrightarrow G \) onto a finite local \( k \)-group scheme \( G \), let \( H \subseteq G \) be the image of the composition of the homomorphisms \( \pi \text{loc}(X') \longrightarrow \pi \text{loc}(X) \). Then we obtain a commutative diagram

\[
\begin{array}{ccc}
H^1_{\text{fppf}}(X', H) & \longrightarrow & H^1_{\text{fppf}}(X', G) \\
\uparrow & & \uparrow \\
H^1_{\text{fppf}}(X, H) & \longrightarrow & H^1_{\text{fppf}}(X, G),
\end{array}
\]

where the injectivity of the horizontal maps is valid because the root stacks \( X' \) and \( X' \) are reduced and the reduced subscheme of the quotient space \( G/H \) is trivial, i.e. \( (G/H)_\text{red} = \text{Spec } k \). Then one can deduce from Propositions 2.8 and 3.12 together with the identification \( H^1_{\text{fppf}}(-, G) = \text{Hom}_k(-, BG) \) that the above diagram is Cartesian. This implies that \( H = G \).

4.3 | Torsors over root stacks and tamely ramified torsors

We will continue to use the same notation as in the previous subsection.

Definition 4.11. Let \( X \) be a scheme and let \( D \) be an effective Cartier divisor of \( X \). Put \( U = X \setminus D \). A finite flat cover \( Y \longrightarrow X \) which is étale over \( U \) is said to be tamely ramified along \( D \) if for any point \( x \in D \), all the connected components of \( Y \times_X \text{Spec } \theta_{X,x} \) are of the form \( \text{Spec } \theta_{X,x}[T]/(T^m - a) \) where \( m \) is a positive integer which is prime to the characteristic of \( k(x) \) and \( a \) is a local equation of \( D \) in \( \text{Spec } \theta_{X,x} \), where \( \theta_{X,x} \) is the strict henselization of the local ring \( \theta_{X,x} \) at \( x \).

Lemma 4.12. (Abhyankar, cf. [18, Theorem 2.3.2 and Corollary 2.3.4].) Let \( X \) be a normal scheme and let \( D \subseteq X \) be a simple normal crossings divisor. Let \( f : Y \longrightarrow X \) be a finite flat cover which is étale over \( U = X \setminus D \). Then the following are equivalent.

(a) \( Y \) is normal and \( f \) is tamely ramified above the generic points of \( D \).
(b) \( f : Y \longrightarrow X \) is tamely ramified along \( D \).

Definition 4.13. Suppose given a locally Noetherian \( k \)-scheme \( X \) and an \( n \)-tuple \( s = (s_i)_{i=1}^n \) of regular functions \( s_i \in H^0(X, \theta_X) \). A Kummer morphism associated with the data \( (s, r) \) is a finite flat \( k \)-morphism \( Y \longrightarrow X \) defined by

\[
\theta_Y = \theta_X[t]/(t^r - s) \overset{\text{def}}{=} \bigotimes_{i=1}^n \theta_X[t_i]/(t_i^r_i - s_i),
\]

which has a natural action of \( \mu_r = \prod_{i=1}^n \mu_{r_i} \).

Definition 4.14. (Cf. [7, Definition 2.2].) Let \( G \) be a finite \( k \)-group scheme. A tamely ramified \( G \)-torsor over \( X \) with ramification data \( (D, r) \) is a scheme \( Y \) endowed with an action of \( G \) and a finite flat \( G \)-invariant morphism \( Y \longrightarrow X \) such that for any closed point \( x \) of \( X \), there exists a monomorphism \( \mu_{e_x} \longrightarrow G \) (cf. (3.1)) defined over an extension \( k'/k \) such
that in a fppf neighbourhood of \( x \) in \( X \), the morphism \( Y \to X \) is isomorphic to
\[
(Z \times_k k') \times^{\mu_{r_x}} G,
\]
where \( Z \) is the Kummer morphism associated with the data \( \left( s = (s_i)_{i=1}^n, r_x \right) \), where each \( s_i \) is a local equation of \( D_i \).

**Theorem 4.15.** (Biswas–Borne, cf. [7, §3].) Let \( Y \) be a scheme endowed with an action of a finite \( k \)-group scheme \( G \) and a finite flat \( G \)-invariant morphism \( Y \to X \). Then we have the following.

1. If \( Y \to X \) is a tamely ramified \( G \)-torsor with ramification data \( (D, r) \), then the morphism \( Y \to X \) uniquely factors through a \( G \)-torsor \( Y \to \sqrt[\mu_r]{D/X} \).
2. Suppose that \( G \) is abelian. Then the converse of (1) is true. Namely, the morphism \( Y \to X \) factors through a \( G \)-torsor \( Y \to \sqrt[\mu_r]{D/X} \) if and only if \( Y \) is a tamely ramified \( G \)-torsor over \( X \) with ramification data \( (D, r) \).

Particularly, the first result (1) indicates that a \( G \)-torsor \( Y \to \sqrt[\mu_r]{D/X} \) over the root stack \( \sqrt[\mu_r]{D/X} \) which is representable by a \( k \)-scheme gives a candidate of a tamely ramified \( G \)-torsor over \( X \) with ramification data \( (D, r) \) in the sense of Definition 4.14.

**Remark 4.16.** With the above notation, suppose that \( X \) is a regular connected \( k \)-scheme, \( D \) is a simple normal crossings divisor and \( G \) is a finite constant \( k \)-group scheme. Then, for a finite flat \( G \)-invariant morphism \( Y \to X \) whose restriction to \( X \setminus D \) is an étale Galois \( G \)-cover, the following are equivalent.

(a) The morphism \( Y \to X \) is tamely ramified along \( D \) in the sense of Definition 4.11.
(b) The morphism \( Y \to X \) is a tamely ramified \( G \)-torsor with ramification data \( (D, r) \) for some \( n \)-tuple \( r \) in the sense of Definition 4.14.
(c) The morphism \( Y \to X \) (uniquely) factors through a \( G \)-torsor \( Y \to \sqrt[\mu_r]{D/X} \) for some \( n \)-tuple \( r \).

Indeed, the implication (a) \( \implies \) (b) is immediate from Definitions 4.11 and 4.14. The implication (b) \( \implies \) (c) is nothing other than the assertion of Theorem 4.15(1). Finally, the implication (c) \( \implies \) (a) can be deduced from Abhyankar’s lemma (cf. Lemma 4.12). Indeed, since the problem is Zariski local, without loss of generality, we may assume that \( [Z/\mu_r] \cong \sqrt[\mu_r]{D/X} \), where \( Z \) is a Kummer morphism. Then, we get a commutative diagram

\[
\begin{array}{ccc}
P & \to & Y \\
\downarrow & & \downarrow \\
Z & \to & \sqrt[\mu_r]{D/X} \\
\downarrow & & \downarrow \\
X & & X
\end{array}
\]

with the 2-Cartesian square, where \( P \to Z \) is a \( G \)-torsor and \( Z \to \sqrt[\mu_r]{D/X} \) is the quotient map \( Z \to [Z/\mu_r] \cong \sqrt[\mu_r]{D/X} \). Let \( x_i \) denote the generic point of the divisor \( D_i \) for any \( i \). Then, the above commutative diagram implies that the ramification index \( e_{x_i} \) at \( x_i \) divides \( r_i \). As \( Y \) is representable, the composition map
\[
B_{\mu_r} = G_{x_i} \to \sqrt[\mu_r]{D/X} \to BG
\]
is faithful, hence \( \mu_{r_i} \) is étale group scheme, i.e. \( r_i \) is prime-to-\( p \). As \( e_{x_i} \) divides \( r_i \), this implies that \( e_{x_i} \) is prime-to-\( p \). Therefore, \( Y \to X \) is tamely ramified above \( x_i \). Moreover, since \( \sqrt[\mu_r]{D/X} \) is regular (cf. Proposition 3.13), so is \( Y \). Now, Lemma 4.12 implies that the condition (a) holds.

**Remark 4.17.** To figure out the reason why the abelianness assumption is required in Theorem 4.15(2), let us recall the argument due to Biswas–Borne. The problem is Zariski local, so we may assume that \( \sqrt[\mu_r]{D/X} = [Z/\mu_r] \). Let
$P \overset{\text{def}}{=} \text{Isom}_{[Z/\mu_r]}(Z \wedge \mu_r G, Y)$ be the fppf sheaf of isomorphisms of $G$-torsors over $[Z/\mu_r]$. Then $P$ is a right $N = Z \wedge \mu_r G$-torsor over $[Z/\mu_r]$, where the action of $\mu_r$ on $G$ is defined by conjugation. For example, if $G$ is abelian, then the conjugacy action is trivial, hence $N = [Z/\mu_r] \times_k G$ and Proposition 2.8 can be applied to show that $P$ descents to a $G$-torsor over $[Z/\mu_r]$. Then $P$ is a right $N = Z \wedge \mu_r G$-torsor over $[Z/\mu_r]$, where the action of $\mu_r$ on $G$ is defined by conjugation. For example, if $G$ is abelian, then the conjugacy action is trivial, hence $N = [Z/\mu_r] \times_k G$ and Proposition 2.8 can be applied to show that $P$ descents to a $G$-torsor over $X$. In the general case, it does not seem that $\mathcal{N}$ is a fibered category over schemes which satisfies a tannakian reconstruction (§ 2.2), and the same argument cannot be applied. In fact, as explained in [7, Appendix §B], David Rydh constructs examples of $G = \mu_p \ltimes \alpha_p$-torsors over smooth root stacks $\sqrt{D/X}$ for which the associated sheaf $P$ of isomorphisms of $G$-torsors cannot be trivialized by any fppf cover $X' \to X$.

For later use, we shall show the following lemma.

**Lemma 4.18.** Suppose that $X$ is a connected smooth curve over an algebraically closed field $k$ of characteristic $p > 0$ with $D = (x_i)_{i \in I}$, where $x_i$ are finite distinct closed points of $X$. Let $G$ be a finite $k$-group scheme. Let $\mathcal{Y} \to \mathfrak{X}^r$ be a Nori-reduced $G$-torsor. Then there exists a family $r' = (r'_i)_{i \in I}$ of integers $r'_i > 0$ with $r'|r$ and a $G$-torsor $Y' \to \mathfrak{X}^{r'}$ with $Y'$ representable by a $k$-scheme such that

$$\mathcal{Y} \approx \mathcal{Y} \times_{\mathfrak{X}^{r'}} \mathfrak{X}^{r'}$$

as a $G$-torsor over $\mathfrak{X}^{r'}$.

**Proof.** First note that if a $G$-torsor $\mathcal{Y} \to \mathfrak{X}^{r'}$ is representable, then as remarked in [7, Remark 3.6(2)], $\mathcal{Y}$ can be represented by a $k$-scheme, and by [6, §3.2 Proposition 11], the latter condition is equivalent to the condition that the composition of morphisms

$$G_{x_1} \to \mathfrak{X}^{r'} \to BG$$

is representable for any $i \in I$. Since the problem is Zariski local, we may assume that $\#I = 1$, so let us put $x = x_1$, $r = r_1$. Moreover, we may assume that $\mathfrak{X}^{r'} = [Z/\mu_r]$ where $Z \to X$ is a Kummer morphism. Then we have $G_x = B\mu_r$ (cf. Proposition 3.11(2)). Let $\mu_r \to G$ be the homomorphism of $k$-group schemes associated with the composition $B\mu_r = G_x \to [Z/\mu_r] = \mathfrak{X}^{r'} \to BG$ and set $\mu_{r''} \overset{\text{def}}{=} \text{Ker}(\mu_r \to G)$. Then, by the definition, the restriction

$$[Z/\mu_{r''}] \to [Z/\mu_r] = \mathfrak{X}^{r'} \to BG \quad (4.1)$$

is trivial on inertia groups. Therefore, by [6, §2.3 Corollary 5], which can be applied to non-proper tame stacks as discussed in the first paragraph of the proof of Proposition 4.9(1), the morphism (4.1) factors through the coarse moduli space $Z' \overset{\text{def}}{=} Z/\mu_{r''}$. By dividing by the natural action of $\mu_{r''}$, we get a morphism

$$\mathfrak{X}' = [Z'/\mu_{r'}] \to BG,$$

which is representable by the construction. Therefore, we can conclude that the $G$-torsor $\mathcal{Y} \to \mathfrak{X}'$ descends to a $G$-torsor $Y \to \mathfrak{X}'$ which is representable by a $k$-scheme. This completes the proof. \hfill \square

### 4.4 Brauer groups of root stacks

In this subsection, we always work with the lisse-étale topology (cf. [23]). Let $\mathcal{X}$ be an algebraic stack over a field $k$. We denote by $\text{Lis-ét}(\mathcal{X})$ the lisse-étale site of $\mathcal{X}$. A gerbe over $\mathcal{X}$ always means a gerbe over the site $\text{Lis-ét}(\mathcal{X})$ (cf. [35, Definition 06NZJ]). To each gerbe $G \to \mathcal{X}$, we associate a sheaf $I(G)$ on $G$, which we call the inertia sheaf, as follows. For any object $(U, u) \in \text{Lis-ét}(\mathcal{X})$ and any section $x : U \to G$, $I(G)(x) \overset{\text{def}}{=} \text{Aut}_{G(U)}(x)$ is the group of automorphisms of the section $x$. 
Definition 4.19. (Cf. [24, Definition 2.2.1.6].) Let \( F \) be a sheaf on \( \mathcal{G} \). Then \( F \) admits a right action \( F \times I(\mathcal{G}) \to F \) of the inertia sheaf \( I(\mathcal{G}) \), which is called the \textit{inertial action} on \( F \), as follows. For any object \((U, u) \in \text{Lis-\text{\(\acute{e}t\)}}(\mathcal{X})\) and any section \( x : U \to \mathcal{G} \), there exists a natural action
\[
F(x) \times I(\mathcal{G})(x) \to F(x); (a, \sigma) \mapsto \sigma^* a.
\]
Here, note that each element \( \sigma \in I(\mathcal{G})(x) \) induces an isomorphism \( \sigma^* : F(x) \to F(x) \).

A gerbe \( f : \mathcal{G} \to \mathcal{X} \) is said to be \textit{abelian} if for any \((U, u) \in \text{Lis-\text{\(\acute{e}t\)}}(\mathcal{X})\) and any section \( x : U \to \mathcal{G} \), the sheaf \( \text{Aut}_{U}(x) \) on \( \text{Lis-\text{\(\acute{e}t\)}}(U) \) is abelian. If a gerbe \( f : \mathcal{G} \to \mathcal{X} \) is abelian, for any \((U, u) \in \text{Lis-\text{\(\acute{e}t\)}}(\mathcal{X})\) and any two sections \( x, y \in \mathcal{G}_{(U, u)} \), there exists a canonical isomorphism \( \text{Aut}_{U}(x) \cong \text{Aut}_{U}(y) \) of sheaves. Moreover, there exists a sheaf \( \mathcal{A} \) of abelian groups on \( \text{Lis-\text{\(\acute{e}t\)}}(\mathcal{X}) \) such that for any \((U, u) \in \text{Lis-\text{\(\acute{e}t\)}}(\mathcal{X})\) and any section \( x : U \to \mathcal{G} \), there exists an isomorphism \( u^* \mathcal{A} \cong \text{Aut}_{U}(x) \) of sheaves such that for any \((U, u) \in \text{Lis-\text{\(\acute{e}t\)}}(\mathcal{X})\) and any two sections \( x, y : U \to \mathcal{G} \), the diagram
\[
\begin{array}{ccc}
u^* \mathcal{A} & \longrightarrow & \text{Aut}_{U}(x) \\
\downarrow & & \downarrow \\
\text{Aut}_{U}(y) & \longrightarrow & 
\end{array}
\]
is commutative (cf. [35, Lemma 0CJY]). In other words, there exists an isomorphism of sheaves on \( \mathcal{G} \),
\[
\mathcal{A}_{\mathcal{G}} \overset{\text{def}}{=} f^* \mathcal{A} \cong I(\mathcal{G}).
\]
In this case, such a gerbe \( \mathcal{G} \to \mathcal{X} \) is called an \( \mathcal{A} \)-gerbe. More precisely, for a given abelian sheaf \( \mathcal{A} \) on \( \text{Lis-\text{\(\acute{e}t\)}}(\mathcal{X}) \), an \( \mathcal{A} \)-gerbe is a gerbe \( \mathcal{G} \to \mathcal{X} \) together with an isomorphism \( \mathcal{A}_{\mathcal{G}} \cong I(\mathcal{G}) \) of sheaves on \( \mathcal{X} \). Let \( \mathcal{A} \) be an abelian sheaf on \( \text{Lis-\text{\(\acute{e}t\)}}(\mathcal{X}) \). Let \( \mathcal{G}_1, \mathcal{G}_2 \) be \( \mathcal{A} \)-gerbes over \( \mathcal{X} \). Then \( \mathcal{G}_1 \) is said to be isomorphic to \( \mathcal{G}_2 \) if there exists an equivalence of fibered categories \( \phi : \mathcal{G}_1 \to \mathcal{G}_2 \) over \( \mathcal{X} \) such that the composition
\[
\mathcal{A}_{\mathcal{G}_2} \overset{\text{def}}{=} I(\mathcal{G}_2) \to \phi^* I(\mathcal{G}_1) \overset{\text{def}}{=} \phi^* \mathcal{A}_{\mathcal{G}_1} = \mathcal{A}_{\mathcal{G}_2}
\]
is the identity (cf. [24, Definition 2.2.1.3]). Then the set of isomorphism classes of \( \mathcal{A} \)-gerbes can be described as the second cohomology group \( H^2_{\text{Lis-\text{\(\acute{e}t\)}}}(\mathcal{X}, \mathcal{A}) \). The unit of the abelian group \( H^2_{\text{Lis-\text{\(\acute{e}t\)}}}(\mathcal{X}, \mathcal{A}) \) is represented by the fibered category
\[
\text{TORS}_{\mathcal{X}}(\mathcal{A}) \longrightarrow \mathcal{X}
\]
which classifies \( \mathcal{A} \)-torsors on the site \( \text{Lis-\text{\(\acute{e}t\)}}(\mathcal{X}) \).

From now on, we shall consider the multiplicative group \( \mathcal{A} = \mathbb{G}_m \). Let \( f : \mathcal{G} \to \mathcal{X} \) be a \( \mathbb{G}_m \)-gerbe. Then \( \mathcal{G} \) is an algebraic stack over \( k \) (cf. [20, Proposition 1.1]). In this case, any quasi-coherent sheaf \( F \) on \( \mathcal{G} \) admits a left action of \( \mathbb{G}_m \), which comes from the left \( \mathcal{O}_C \)-module structure.

Definition 4.20. (Cf. [24, Definition 3.1.1.1].) With the above notation, a quasi-coherent sheaf \( F \) on \( \mathcal{G} \) is said to be a \textit{twisted sheaf} if the right action associated with the left \( \mathbb{G}_m \)-action on \( F \) coincides with the inertial action \( F \times I(\mathcal{G}) \to F \) (cf. Definition 4.19). Here, recall that \( \mathbb{G}_m, \mathcal{G} = I(\mathcal{G}) \).

Let us recall the following result.

Proposition 4.21. (Cf. [24, Lemma 3.1.1.8].) Let \( \mathcal{X} \) be an algebraic stack over a field \( k \). A \( \mathbb{G}_m \)-gerbe \( \mathcal{G} \) is isomorphic to \( \mathbb{B}\mathbb{G}_m, \mathcal{X} \) as a \( \mathbb{G}_m \)-gerbe if and only if there exists a twisted invertible sheaf \( \mathcal{L} \) on \( \mathcal{G} \).
Proof. First note that there exists an equivalence of groupoids

$$\text{Hom}_\mathcal{X}(\mathcal{G}, B\mathbb{G}_m, \mathcal{X}) \cong \text{Pic}(\mathcal{G}).$$

Let $\phi : \mathcal{G} \to B\mathbb{G}_m, \mathcal{X}$ be a morphism of fibered categories over $\mathcal{X}$ and $\mathcal{L}$ the invertible sheaf on $\mathcal{G}$ corresponding to $\phi$. We have to show that $\phi$ is an isomorphism of $\mathbb{G}_m$-gerbes if and only if $\mathcal{L}$ is a twisted sheaf on $\mathcal{G}$. Indeed, $\mathcal{L}$ is a twisted sheaf on $\mathcal{G}$ if and only if, for any object $(U, u) \in \text{Lis-ét}(\mathcal{X})$ and any section $x : U \to \mathcal{G}$, the diagram

$$x^* \mathcal{L} \times \mathbb{G}_m, U \to x^* \mathcal{L} \times \text{Aut}_U(x) \to x^* \mathcal{L}$$

is commutative, where the action $x^* \mathcal{L} \times \mathbb{G}_m, U \to x^* \mathcal{L}$ is given by $(f, a) \mapsto a^{-1} \cdot f$ and the one $x^* \mathcal{L} \times \text{Aut}_U(x) \to x^* \mathcal{L}$ is the inertial action (cf. Definition 4.19). However, the commutativity of the diagram is equivalent to the commutativity of the following diagram,

$$\text{Aut}_U(x) \to \text{Aut}_U(x^* \mathcal{L}) \to \mathbb{G}_m, U \to \text{Aut}_U(\phi \circ x),$$

or equivalently, to saying that the composition

$$\mathbb{G}_m, U \to \text{Aut}_U(x^* \mathcal{L}) \to \text{Aut}_U(\phi \circ x) \to \mathbb{G}_m, U$$

is the identity. This completes the proof. \qed

Now we apply the above arguments to root stacks. Let $X$ be a geometrically connected quasi-compact smooth scheme over the spectrum $S = \text{Spec } k$ of a field $k$ with $\eta$ the generic point. Let $\mathcal{D} = (D_i)_{i \in I}$ be a finite family of reduced irreducible effective Cartier divisors on $X$ and put $D = \bigcup_{i \in I} D_i \subset X$. For each $r = (r_i)_{i \in I}$ with $r_i > 0$, as in §3.3, we put

$$\mathfrak{X}^r = \sqrt[\eta]{\mathcal{D}/X}.$$

Recall that there exists a natural quasi-compact morphism $\eta \to \mathfrak{X}^r$.

**Proposition 4.22.** (Cf. [24, Proposition 3.1.3.3].) With the above notation, suppose further that $D_i$ are smooth and that $D$ is a simple normal crossings divisor on $X$. Then the restriction map

$$H^2_{\text{lis-ét}}(\mathfrak{X}^r, \mathbb{G}_m) \to H^2_{\text{lis-ét}}(\eta, \mathbb{G}_m)$$

is injective.

As a consequence, we have the following.

**Corollary 4.23.** With the same notation as in Proposition 4.22, if $X$ is a connected smooth curve defined over an algebraically closed field, then we have $H^2_{\text{lis-ét}}(\mathfrak{X}^r, \mathbb{G}_m) = 0$.

This is an immediate consequence of Proposition 4.22 because if $X$ is a connected smooth curve defined over an algebraically closed field, then the Brauer group of the generic point $\eta$ vanishes (cf. [35, Lemma 03RF]). Let us prove Proposition 4.22.
Lemma 4.24. With the same notation as in Proposition 4.22, the root stack $\mathcal{X}^r$ is a smooth quasi-compact algebraic stack over $S$.

Proof. The smoothness follows from Proposition 3.13. Recall that we have assumed that $X$ is quasi-compact. As the morphism $\mathcal{X}^r \to X$ is of finite type (cf. §3.3), $\mathcal{X}^r$ is quasi-compact. □

Lemma 4.25. With the same notation as in Proposition 4.22, let $\mathcal{G} \to \mathcal{X}^r$ be a $\mathbb{G}_m$-gerbe. Then $\mathcal{G}$ is a smooth quasi-compact algebraic stack over $k$.

Proof. The classifying stack $B\mathbb{G}_m$ is a smooth quasi-compact algebraic stack. As any $\mathbb{G}_m$-gerbe $\mathcal{G} \to \mathcal{X}^r$ is locally isomorphic to $B\mathbb{G}_m$, the gerbe $\mathcal{G}$ must be smooth and quasi-compact. □

Proof of Proposition 4.22. Let $\mathcal{G} \to \mathcal{X}^r$ be a $\mathbb{G}_m$-gerbe and $[\mathcal{G}] \in H^2_{\text{lis-ét}}(\mathcal{X}^r, \mathbb{G}_m)$ the corresponding class. Suppose that $[\mathcal{G}] = 0$ in $H^2_{\text{lis-ét}}(\mathcal{X}^r, \mathbb{G}_m)$. We have to show that $[\mathcal{G}] = 0$. Since $[\mathcal{G}] = 0$, by Proposition 4.21, there exists an invertible $\mathbb{G}_m$-twisted sheaf $\mathcal{L}_\eta$. Since $\mathcal{X}^r \to X$ is quasi-compact, so is $\mathcal{L}_\eta \to X$. Therefore, the sheaf $i_* \mathcal{L}_\eta$ is a quasi-coherent $\mathcal{G}$-twisted sheaf (cf. [23, Proposition (13.2.6)(i)]). Since $\mathcal{G}$ is Noetherian (Lemma 4.25), according to [24, Proposition 3.1.1.9], we can write $\mathcal{M} \overset{\text{def}}{=} i_* \mathcal{L}_\eta$ as the colimit of coherent $\mathcal{G}$-twisted subsheaves of it as follows

$$\mathcal{M} = \lim_{\lambda} \mathcal{M}_\lambda$$

However, since $\mathcal{M}_\lambda$ is coherent, we have $\mathcal{M}_\lambda = i^* \mathcal{M} = \lim_{\lambda} i^* \mathcal{M}_\lambda = i^* \mathcal{M}_\lambda$ for some $\lambda$. By replacing $\mathcal{M}_\lambda$ with $\mathcal{M}_\lambda \overset{\text{def}}{=} \mathcal{M}_\lambda^\vee$, if necessary, we obtain a reflexible $\mathcal{G}$-twisted sheaf $\mathcal{L}$ of rank one such that $i^* \mathcal{L} \simeq \mathcal{L}_\eta$. However, since $\mathcal{G}$ is smooth over $k$ (cf. Lemma 4.25), any reflexible sheaf of rank one must be an invertible sheaf. Therefore, again by using Proposition 4.21, we can conclude that $[\mathcal{G}] = 0$. This completes the proof. □

4.5 | Embedding problems over root stacks

Let $k$ be an algebraically closed field of characteristic $p > 0$ and let $S = \text{Spec} k$. Let $X$ be a projective smooth curve over $k$ of genus $g \geq 0$ and of $p$-rank $\gamma \geq 0$. Let $\emptyset \neq U \subset X$ be a nonempty open subscheme with $\# X \setminus U = n \geq 1$. Let $X \setminus U = \{ x_0, x_1, \ldots, x_{n-1} \}$. Fix an integer $m$ with $0 \leq m \leq n - 1$. We shall denote by $X_m$ the smooth affine curve $X \setminus \{ x_0, x_1, \ldots, x_m \}$ and consider $D = (x_i)_{i=m+1}^{n-1}$ as a family of reduced distinct Cartier divisors on $X_m$. For each family $r = (r_i)_{i=m+1}^{n-1} \in \prod_{i=m+1}^{n-1} \mathbb{Z}_{\geq 0}$ of integers, we denote by $\mathcal{X}^r_m = \sqrt[p^r]{D/X_m}$ the root stack associated with $X_m$ and the data $(D, p^r)$.

Then there exists an exact sequence

$$0 \to \text{Pic}(X_m) \to \text{Pic}(\mathcal{X}^r_m) \to \prod_{i=m+1}^{n-1} H^0(x_i, \mathbb{Z}/p^r\mathbb{Z}) \to 0, \quad (4.2)$$

where each group $H^0(x_i, \mathbb{Z}/p^r\mathbb{Z})$ is nothing but the Picard group of the residual gerbe $G_{x_i}$ of $\mathcal{X}^r_m$ at the closed point $x_i$ (cf. [8]). From the definition of root stacks, for any two families $r, r'$ of positive integers with $r \leq r'$, there exists a natural morphism of algebraic stacks

$$\mathcal{X}^{r'}_m \to \mathcal{X}^r_m.$$ 

Then the exact sequence (4.2) implies that there exists an exact sequence

$$0 \to \text{Pic}(X_m) \to \varprojlim_{r} \text{Pic}(\mathcal{X}^r_m) \to (\mathbb{Q}_p/\mathbb{Z}_p) \otimes_{m-m-1} \to 0. \quad (4.3)$$
Since $X_m$ is affine, there exists a surjective homomorphisms $\text{Pic}^0(X) \to \text{Pic}(X_m)$, hence $\text{Pic}(X_m)$ is divisible. Thus, the exact sequence (4.3) implies that the abelian group $\varprojlim_r \text{Pic}(\mathcal{X}_m^{p^r})$ is $p$-divisible.

**Lemma 4.26.** Let $r = (r_i)_{i=m+1}^{n-1}$ be a family of positive integers. Then we have the following.

(1) For any quasi-coherent sheaf $E$ on $\mathcal{X}_m^{p^r}$, we define the fppf sheaf $W(E)$ of abelian groups on $\mathcal{X}_m^{p^r}$ to be

\[ W(E)(Y) \overset{\text{def}}{=} \Gamma(Y, E \otimes \mathcal{O}_Y) \]

for any morphism $Y \to \mathcal{X}_m^{p^r}$ from a scheme $Y$. Then $H^q_{\text{fppf}}(\mathcal{X}_m^{p^r}, W(E)) = 0$ for any $q > 0$.

(2) $H^0_{\text{fppf}}(\mathcal{X}_m^{p^r}, \mathbb{G}_m) = H^0(X_m, \mathcal{O}_{X_m})^\times$.

(3) $H^1_{\text{fppf}}(\mathcal{X}_m^{p^r}, \mathbb{G}_m) = \text{Pic}(\mathcal{X}_m^{p^r})$.

(4) $H^2_{\text{fppf}}(\mathcal{X}_m^{p^r}, \mathbb{G}_m) = 0$.

**Proof.**

(1) First note that there exists a natural isomorphism

\[ H^q_{\text{fppf}}(\mathcal{X}_m^{p^r}, W(E)) \overset{\cong}{\to} H^q_{\text{lis-ét}}(\mathcal{X}_m^{p^r}, W(E)) = H^q(\mathcal{X}_m^{p^r}, E) \]

for any $q \geq 0$ (cf. [10, Théorème B.2.5]). Next since $X_m$ is affine, we have

\[ H^q(\mathcal{X}_m^{p^r}, E) \cong H^0(X_m, R^q\pi_{p^r}^*E) \]

for any $q \geq 0$. On the other hand, since $\pi_{p^r} : \mathcal{X}_m^{p^r} \to X_m$ is cohomologically affine and $\mathcal{X}_m^{p^r}$ has affine diagonal (cf. Proposition 3.8(1)(3)), $R^q\pi_{p^r}^*E = 0$ for any $q > 0$ (cf. [4, Remark 3.5]). Therefore we can conclude that

\[ H^q_{\text{fppf}}(\mathcal{X}_m^{p^r}, W(E)) = 0 \]

for any $q > 0$.

(2) This follows from

\[ H^0_{\text{fppf}}(\mathcal{X}_m^{p^r}, \mathbb{G}_m) \cong \varprojlim_{(U, U) \in \text{Lis-ét}(\mathcal{X}^{p^r})} \mathbb{G}_m(U) \]

\[ \cong \left( \left\{ \varprojlim_{(U, U) \in \text{Lis-ét}(\mathcal{X}^{p^r})} \Gamma(U, \mathcal{O}_U) \right\}^\times \right) \cong H^0(X_m, \mathcal{O}_{X_m})^\times \]

\[ \cong H^0(X_m, \mathcal{O}_{X_m})^\times = H^0(X_m, \mathcal{O}_{X_m})^\times. \]

Here, recall that $\pi_{p^r}^*\mathcal{O}_{\mathcal{X}_m^{p^r}} = \mathcal{O}_{X_m}$.

(3) See [10].

(4) This follows from Corollary 4.23 together with [10, Théorème B.2.5].

□
Lemma 4.27. We have the following.

(1) For any integer \( s \geq 1 \) and any two families \( r, r' \) of positive integers with \( (s)_{i=m+1}^{n-1} \leq r \leq r' \), we have \( \text{Pic}(\mathfrak{X}_m^p)^{[p^s]} = \text{Pic}(\mathfrak{X}_m^{p^{r'}})^{[p^s]} \).

(2) For any family \( r \) of positive integers, we have \( \text{Pic}(\mathfrak{X}_m^p)/p\text{Pic}(\mathfrak{X}_m^{p^s}) \cong (\mathbb{Z}/p^s\mathbb{Z})^{\oplus n-m-1} \).

Proof. This can be verified by using the exact sequence (4.2) and the fact that \( \text{Pic}(X_m) \) is divisible. \( \square \)

Lemma 4.28. For any integer \( s > 0 \), we have \( \varinjlim_{T} H_1^{fppf}(\mathfrak{X}_m^p, \mathbb{Z}/p^s\mathbb{Z}) \cong \mathbb{Z}/p^s\mathbb{Z}^{\oplus s+n-1} \).

Proof. By Proposition 4.7, the restriction maps

\[ \varinjlim_{r'} H_1^{fppf}(\mathfrak{X}_{m-1}^p, \mu_{p^s}) \longrightarrow \varinjlim_{r} H_1^{fppf}(\mathfrak{X}_m^p, \mu_{p^s}) \]

are injective for all \( 0 < m \leq n - 1 \). Since \( \varinjlim_r H_1^{fppf}(\mathfrak{X}_{n-1}^p, \mu_{p^s}) = H_1^{fppf}(U, \mu_{p^s}) \cong (\mathbb{Z}/p^s\mathbb{Z})^{\oplus n-1} \) (cf. [29, Proposition 3.2]), this reduces us to the case when \( m = 0 \). In this case, as \( H_0^{fppf}(X, \mathbb{Z}/p^s\mathbb{Z}) \cong k \times \mathbb{Z}/p^s\mathbb{Z} \) (cf. [29, Proposition 3.2]), the assertion follows from the exact sequence (4.2) together with the fact that \( \text{Pic}(X_0) \) is a \( p \)-divisible group. \( \square \)

Lemma 4.29. We have the following.

(1) Let \( N \) be a finite flat abelian \( \mathfrak{X}_m^p \)-group scheme. If \( N \) and its Cartier dual \( N^\vee \) are of height one, then we have \( H_2^{fppf}(\mathfrak{X}_m^p, N) = 0 \).

(2) If \( G \) is a finite local diagonalizable \( k \)-group scheme, we have \( \varinjlim_r H_2^{fppf}(\mathfrak{X}_m^p, G) = 0 \).

Proof.

(1) See the last assertion of Proposition C.2.

(2) As \( G \) is a successive extension of \( \mu_p \), without loss of generality, we may assume that \( G = \mu_p \). We will prove that the transition map

\[ H_2^{fppf}(\mathfrak{X}_m^p, \mu_p) \longrightarrow H_2^{fppf}(\mathfrak{X}_{m+1}^p, \mu_p) \]

is the zero map for all \( r \). By Lemma 4.26(4), for any family \( r \) of positive integers, there exists an isomorphism of abelian groups \( \text{Pic}(\mathfrak{X}_m^p)/p\text{Pic}(\mathfrak{X}_m^{p^s}) \cong H_2^{fppf}(\mathfrak{X}_m^p, \mu_p) \). Therefore it suffices to show that the composition
is the zero map. To prove this, it suffices to notice that there exists a commutative diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \text{Pic}(X_m) & \rightarrow & \text{Pic}(X'_m) & \rightarrow & \prod_{i=m+1}^{n-1} \mathbb{Z}/p^i \mathbb{Z} & \rightarrow & 0 \\
0 & \rightarrow & \text{Pic}(X_m) & \rightarrow & \text{Pic}(X'_{m+1}) & \rightarrow & \prod_{i=m+1}^{n-1} \mathbb{Z}/p^{r+1} \mathbb{Z} & \rightarrow & 0 \\
\end{array}
\]

where the right vertical sequence is exact, and the isomorphism \( \text{Pic}(X'_m)/p\text{Pic}(X'_{m+1}) \cong \prod_{i=m+1}^{n-1} \mathbb{Z}/p^i \mathbb{Z} \) is Lemma 4.27(2). This completes the proof.

Now we prove the main theorem. As a notation, we define \( \pi_{\text{loc}}(X_p^\infty) \) to be the projective limit of the pro-system \( \{ \pi_{\text{loc}}(X_p^r) \}_{r} \), i.e.

\[
\pi_{\text{loc}}(X_p^\infty) \overset{\text{def}}{=} \lim_{\rightarrow} \pi_{\text{loc}}(X_p^r).
\]  \hspace{1cm} (4.4)

**Theorem 4.30.** Suppose given an exact sequence of finite local \( k \)-group schemes

\[
1 \rightarrow G' \rightarrow G \overset{\pi}{\rightarrow} G'' \rightarrow 1.
\]  \hspace{1cm} (4.5)

Suppose that the following conditions are satisfied.

(i) There exists an injective homomorphism \( \chi(G) \hookrightarrow \left( \mathbb{Q}_p / \mathbb{Z}_p \right)^{\oplus r + n - 1} \).

(ii) There exists a surjective \( k \)-homomorphism \( \overline{\phi} : \pi_{\text{loc}}(X_p^\infty) \twoheadrightarrow G'' \).

(iii) \( G' \) is solvable.

Then there exist an \((n - m - 1)\)-tuple \( r \) of nonnegative integers and a Nori-reduced \( G \)-torsor \( Y \rightarrow X_m^p \) which is representable by a \( k \)-scheme.

To prove the theorem, we need the following lemma.

**Lemma 4.31.** Suppose given an exact sequence of finite local \( k \)-group schemes,

\[
1 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 1
\]  \hspace{1cm} (4.6)

together with a surjective homomorphism \( \overline{\phi} : \pi_{\text{loc}}(X_p^\infty) \twoheadrightarrow G'' \). If \( G' \) is abelian, then \( \overline{\phi} \) can be lifted to a homomorphism \( \pi_{\text{loc}}(X_p^\infty) \rightarrow G \).
Proof. As $G'$ is a finite abelian $k$-group scheme with $k$ perfect, it decomposes into a direct product $G' = G'_1 \times G'_2$ of a finite local unipotent group scheme $G'_1$ and a finite local diagonalizable group scheme $G'_2$. As the decomposition is preserved under any automorphism of $G'$, we get a commutative diagram of group schemes with exact rows and columns

\[
\begin{array}{ccccccc}
\downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & G'_1 & \rightarrow & G'_2 & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \\
G'_1 & \rightarrow & G & \rightarrow & G'' & \rightarrow & 1, \\
\downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & G/\ker & \rightarrow & G'' & \rightarrow & 1 \\
\end{array}
\]

which allows us to assume that $G' = G'_1$, or that $G' = G'_2$. In the latter case, as the automorphism group scheme $\text{Aut}(G')$ is étale, the extension (4.6) is central, i.e. the conjugacy action of $G''$ on $G'$ is trivial, hence Giraud’s theory of non-abelian cohomology (cf. [16, p. 284, Remarque 4.2.10]) can be applied, which implies that the assertion follows from the vanishing of the cohomology group $\varprojlim_{\mathbb{F}_p} H^2_{\text{fppf}}(\mathfrak{X}^p_m, G')$ (cf. Lemma 4.29(2)).

Therefore, we have only to deal with the case when $G'$ is a finite local abelian unipotent group scheme. As $G'$ is obtained by successive extensions of finite local group schemes of height one, without loss of generality, we may assume that $G'$ is of height one. Similarly, we may further assume that the Cartier dual $G'^D$ is also of height one. Take an $m$-tuple $\mathbf{r}$ so that $\overline{\phi}$ factors through $\pi^\text{loc}(\mathfrak{X}^p_m)$. The exact sequence (4.6) gives rise to a gerbe $BG \rightarrow BG''$. By pulling back the gerbe along the morphism

\[
\mathfrak{X}^p_m \rightarrow Br^\text{loc}(\mathfrak{X}^p_m) \rightarrow BG'',
\]

we get a gerbe $\mathcal{G} \rightarrow \mathfrak{X}^p_m$. If $P'' \rightarrow \mathfrak{X}^p_m \rightarrow BG''$ denotes the $G''$-torsor associated with the above morphism, then the composition $P'' \rightarrow \mathfrak{X}^p_m \rightarrow BG''$ factors through the neutral section $\text{Spec} \ k \rightarrow BG''$ and the restriction $\mathcal{G} \times_{\mathfrak{X}^p_m} P''$ is isomorphic to the trivial gerbe $BG' \times_k P''$ over $P''$.

\[
\begin{array}{ccc}
BG' \times_k P'' & \rightarrow & \mathcal{G} \\
\downarrow & & \downarrow \\
P'' & \rightarrow & \mathfrak{X}^p_m
\end{array}
\]

Therefore, it turns out that $\mathcal{G} \rightarrow \mathfrak{X}^p_m$ is an $N$-gerbe over $\mathfrak{X}^p_m$, where

\[
N \overset{\text{def}}{=} P'' \wedge^G G' \text{ is a finite flat abelian } \mathfrak{X}^p_m\text{-group scheme. Here the action of } G'' \text{ on } G' \text{ is given by the conjugation, which is well-defined because } G' \text{ is abelian. As } G' \text{ and its Cartier dual } G'^D \text{ are of height one, the same is true for the twist } N, \text{i.e. both } N \text{ and } N^D \text{ are of height one as well. Therefore, by Lemma 4.29(1), we have } H^2_{\text{fppf}}(\mathfrak{X}^p_m, N) = 0, \text{ hence } \mathcal{G} \simeq BN. \text{ This completes the proof.}
\]

Proof of Theorem 4.30. Thanks to Lemma 4.18, we have only to solve the embedding problem

\[
\begin{array}{ccccccc}
1 & \rightarrow & G' & \rightarrow & G & \rightarrow & G'' \\
\downarrow & & \downarrow & & \downarrow & & \\
\overset{\sim}{\varphi} & & \pi & & \pi & & 1
\end{array}
\]
By virtue of Lemmas 4.28 and 4.31, a similar argument in the proof of [29, Proposition 3.4] can work even after replacing $U$ by the pro-system of root stacks $\{\mathfrak{X}_m^{p^r}\}_r$. Recall that, thanks to Corollary 4.3, for any finite local $k$-group scheme $G$, there exists a canonical bijective map

$$\text{Hom}\left(\pi^{\text{loc}}(\mathfrak{X}_m^{p^r}, G)\right) \xrightarrow{\sim} H^1_{\text{fppf}}(\mathfrak{X}_m^{p^r}, G)$$

of pointed sets.

First we assume that $G'$ is abelian, in which case, thanks to Lemma 4.31, without loss of generality, we may assume that the extension (4.5) is split (cf. [30, Appendix, 1], a) and b)). Again by considering the decomposition $G' = G'_1 \times G'_2$ into the direct product of a unipotent group scheme and a diagonalizable group scheme, we may assume that $G'$ is either unipotent or diagonalizable. In the latter case, as $\text{Aut}(G')$ is étale, we get $G' = G'' \times G'''$. Moreover, as $G'$ is a successive extension of $\mu_p$, we are reduced to the case when $G' = \mu_p$, in which case we have the inequalities

$$\dim_{\mathbb{F}_p} \text{Hom}_k(G'', \mu_p) < \dim_{\mathbb{F}_p} \text{Hom}_k(G, \mu_p) \leq \gamma + n - 1,$$

where the second one is due to the condition (i). Therefore, by Lemma 4.28, one can see that the inclusion

$$H^1_{\text{fppf}}(\mathfrak{X}_m^{p^r}, \mu_p) \supset \text{Ker}\left(\text{Hom}_k(\pi^{\text{loc}}(\mathfrak{X}_m^{p^r}, \mu_p) \to \text{Hom}_k(K_r, \mu_p)\right) \xrightarrow{\sim} \text{Hom}_k(G'', \mu_p),$$

is strict for sufficiently large $r$, where we set $K_r = \text{Ker}(\pi^{\text{loc}}(\mathfrak{X}_m^{p^r} \to G'')$ for any index $r$ for which $\tilde{\phi}$ factors through $\pi^{\text{loc}}(\mathfrak{X}_m^{p^r})$ (cf. (4.4)). Thus, one can take an element

$$g \in \text{Hom}(\pi^{\text{loc}}(\mathfrak{X}_m^{p^r}, \mu_p) \setminus \text{Ker}(\text{Hom}_k(\pi^{\text{loc}}(\mathfrak{X}_m^{p^r}, \mu_p) \to \text{Hom}_k(K_r, \mu_p))$$

for some $r$ so that the homomorphism $(g, \tilde{\phi}) : \pi^{\text{loc}}(\mathfrak{X}_m^{p^r}, G'') = \mu_p \times G'' = G$ gives rise to a surjective lifting $\phi : \pi^{\text{loc}}(\mathfrak{X}_m^{p^r}) \to G$ of $\tilde{\phi}$. This completes the proof in the case where $G'$ is diagonalizable.

Let us assume that $G'$ is abelian and unipotent. As the Frobenius kernels $\text{Ker}(F^{(i)} : G' \to G''(i \geq 1)$ are stable under any automorphism of $G'$, they are normal in $G$. Thus, by taking the quotients by them, we are reduced to the case where $G''$ is of height one. Similarly for the Frobenius kernels of the Cartier dual $G^{\text{cd}}$ of $G'$, and thus we may further assume that the Cartier dual $G^{\text{cd}}$ is of height one as well. Recall that such a $G'$ must be isomorphic to a direct sum of $\alpha_p$. Namely, we are reduced to the case where $G' = \alpha_p^t$ for some integer $t > 0$. Moreover, without loss of generality, we may assume that the conjugacy action $G'' \to \text{Aut}(G'')$ is irreducible, i.e. $G'$ does not contain a nontrivial strict subgroup scheme $1 \neq H \subseteq G'$ which is stable under the conjugacy action by $G''$. Now we use the same notation as in the proof of Lemma 4.31. Suppose $\tilde{\phi}$ factors through $\pi^{\text{loc}}(\mathfrak{X}_m^{p^r})$ for some $r$ and let $P'' \to \mathfrak{X}_m^{p^r}$ be the $G''$-torsor associated with the resulting homomorphism $\tilde{\psi} : \pi^{\text{loc}}(\mathfrak{X}_m^{p^r}) \to G''$. Fix a section $\sigma_0 : G'' \to G$ of $\pi$. Put $N = P'' \rtimes G''. G'$. Then, the liftings of the given surjective homomorphism $\tilde{\psi}$ are completely parametrized by the abelian group $H^1_{\text{fppf}}(\mathfrak{X}_m^{p^r}, N)$ (cf. Remark 4.32).

Let $\text{Hom}_{G''}(G'', G)$ be the set of sections of the surjective homomorphism $\pi : G \to G''$, i.e.

$$\text{Hom}_{G''}(G'', G) \overset{\text{def}}{=} \{\sigma \in \text{Hom}(G'', G) | \pi \circ \sigma = \text{id}_{G''}\}.$$
Moreover, set
\[ P'' \wedge \text{Hom}_{G''}(G'', G) \overset{\text{def}}{=} \left\{ P'' \wedge G'' \sigma G \mid \sigma \in \text{Hom}_{G''}(G'', G) \right\} \subset H^1_{fppf}(\mathfrak{X}_m^{p\sigma}, G). \]

Note that
\[ P'' \wedge \text{Hom}_{G''}(G'', G) \subseteq H^1_{fppf}(\mathfrak{X}_m^{p\sigma}, N) \subseteq H^1_{fppf}(\mathfrak{X}_m^{p\sigma}, G). \]

Since \( G' \) is irreducible under the conjugacy action of \( G'' \), for a lifting \( \psi \in H^1_{fppf}(\mathfrak{X}_m^{p\sigma}, N) \) of \( \overline{\psi} \), if \( \text{Im}(\psi) \cap G' \neq 1 \), then the \( G'' \)-orbit of \( \text{Im}(\psi) \cap G' \) generates \( G' \). However, as \( G' \) is abelian, the \( G'' = \text{Im}(\overline{\psi}) \)-orbit coincides with the \( \text{Im}(\psi) \)-orbit of \( \text{Im}(\psi) \cap G' \) by the conjugacy action. Therefore, the complement
\[ H^1_{fppf}(\mathfrak{X}_m^{p\sigma}, N) \setminus (P'' \wedge \text{Hom}_{G''}(G'', G)) \]
exactly consists of surjective liftings of \( \overline{\psi} \). Therefore, we have to show that the inclusion is strict, i.e.
\[ P'' \wedge \text{Hom}_{G''}(G'', G) \not\subset H^1_{fppf}(\mathfrak{X}_m^{p\sigma}, N). \]

We denote by \( Z(G'', G') \) the set of crossed homomorphisms \( G'' \rightarrow G' \) with respect to the conjugacy action \( G'' \rightarrow \text{Aut}_k(G') \). Namely, each element \( z \in Z(G'', G') \) is a morphism \( z : G'' \rightarrow G' \) of \( k \)-schemes satisfying the condition that
\[ z(g_1g_2) = z(g_1) + g_1^{-1}z(g_2)g_1 \]
for any sections \( g_1, g_2 \in G'' \), where the multiplication of \( G' \) is written additively. Note that \( Z(G'', G') \) is a subset of the group \( G'(k[G'']) \) of \( k[G''] \)-valued points of \( G' \). As \( G' = \alpha_p^{G'} \), the group \( G'(k[G'']) = \alpha_p(k[G''])^{G'} \) has a natural structure of \( k \)-vector space and \( Z(G'', G') \) becomes a \( k \)-subspace of \( G'(k[G'']) \). As both the group schemes \( G' \) and \( G'' \) are finite, the \( k \)-vector space \( Z(G'', G') \) is finite dimensional. Moreover, the map
\[ \text{Hom}_{G''}(G'', G') \rightarrow Z(G'', G') \ ; \sigma \mapsto \sigma \cdot \sigma_0^{-1} \]
is bijective with the inverse map \( z \mapsto z \cdot \sigma_0 \), and the composition
\[ Z(G'', G') \xrightarrow{z \cdot \sigma_0 \sigma_0^{-1}} P'' \wedge \text{Hom}_{G''}(G'', G) \hookrightarrow H^1_{fppf}(\mathfrak{X}_m^{p\sigma}, N) \]
is a \( k \)-linear map. By Lemma C.2, we have
\[ \dim_k Z(G'', G') < \infty = \dim_k H^1_{fppf}(\mathfrak{X}_m^{p\sigma}, N), \]
hence \( P'' \wedge \text{Hom}_{G''}(G'', G) \neq H^1_{fppf}(\mathfrak{X}_m^{p\sigma}, N) \). Therefore, \( \overline{\psi} \) has a surjective lifting onto \( G \). This implies the theorem is true if \( G' \) is abelian.

Let us prove the theorem in the general case by induction on the order of \( G' \). As \( G' \) is solvable, the derived subgroup scheme \( D(G') \) is a strict subgroup scheme of \( G' \), i.e. \( D(G') \not\subset G' \). Since \( D(G') \) is also normal in \( G \), we get a commutative
As we have already seen that the theorem is true for $G^{\text{ab}}$, we can take a surjective homomorphism $\pi^\text{loc} \left( \mathfrak{X}_m^\infty \right) \to G / D(G')$.

As the order of $D(G')$ is strictly less than the order of $G'$, by the induction hypothesis, the theorem is true for the exact sequence

$$1 \longrightarrow D(G') \longrightarrow G \longrightarrow G / D(G') \longrightarrow 1.$$

Therefore, there exist an $m$-tuple $r$ and a Nori-reduced $G$-torsor over $\mathfrak{X}_m^r$ which is representable by a $k$-scheme. This completes the proof of the theorem.

Remark 4.32. Let

$$1 \longrightarrow G' \longrightarrow G \longrightarrow G'' \longrightarrow 1$$

be an exact sequence of finite local $k$-group schemes. Let $\mathcal{X}$ be a reduced algebraic stack over $k$ which admits the local fundamental group scheme $\pi^\text{loc}(\mathcal{X})$. Let $\overline{\phi} : \pi^\text{loc}(\mathcal{X}) \to G''$ be a homomorphism of $k$-group schemes. Let $P''$ denote a $G''$-torsor over $\mathcal{X}$ which corresponds to the homomorphism $\overline{\phi}$ via the bijection in Corollary 4.3. Then, the set of liftings $\phi : \pi^\text{bc}(\mathcal{X}) \to G$ of $\overline{\phi}$ is naturally bijective into the set

$$S(\overline{\phi}) \overset{\text{def}}{=} \left\{ [P] \in H^1_{\text{fppf}}(\mathcal{X}, G) ; \ [P \wedge^G G''] = [P''] \text{ in } H^1_{\text{fppf}}(\mathcal{X}, G'') \right\}.$$

Recall that for any finite local $k$-group scheme $H$, the category $\text{Hom}_k(\mathcal{X}, BH)$ is a setoid (cf. Proposition 4.2), and it is actually equivalent to the set $H^1_{\text{fppf}}(\mathcal{X}, H)$. In particular, the isomorphism class $[P''']$ in $H^1_{\text{fppf}}(\mathcal{X}, G'')$ gives rise to a unique morphism $\xi'' : \mathcal{X} \to BG''$ up to unique isomorphism. We set

$$G \overset{\text{def}}{=} BG \times_{BG'', \xi''} \mathcal{X},$$

which is a gerbe over $\mathcal{X}$. By forgetting isomorphisms into $\xi''$ in $\text{Hom}(\mathcal{X}, BG'')$, we obtain a natural map $\text{Hom}_{\mathcal{X}}(\mathcal{X}, G) / \simeq \to S(\overline{\phi})$. However, as $\text{Hom}(\mathcal{X}, BG'')$ is a setoid, this map has to be bijective (cf. [35, Lemma 04SD]).

Corollary 4.33. Suppose given an exact sequence of finite local $k$-group schemes

$$1 \longrightarrow G' \longrightarrow G \longrightarrow G'' \longrightarrow 1.$$

Suppose that the following conditions are satisfied.
(i) There exists an injective homomorphism \( \chi(G) \hookrightarrow (\mathbb{Q}_p / \mathbb{Z}_p)^{\oplus r + n - 1} \).

(ii) There exists a surjective \( k \)-homomorphism \( \pi_{\text{loc}}(U) \twoheadrightarrow G'' \).

(iii) \( G' \) is solvable.

Then there exists a Nori-reduced \( G \)-torsor over \( U \). In particular, for any finite local solvable \( k \)-group scheme \( G \), the group scheme \( G \) appears as a quotient of \( \pi_{\text{loc}}(U) \) if and only if the character group \( \chi(G) \) can be embedded into the abelian group \( (\mathbb{Q}_p / \mathbb{Z}_p)^{\oplus r + n - 1} \).

**Proof.** This is nothing other than Theorem 4.30 specialized in the case when \( m = n - 1 \).

**Corollary 4.34.** Let \( G \) be a finite local abelian \( k \)-group scheme. Suppose that there exists an injective homomorphism \( \chi(G) \hookrightarrow (\mathbb{Q}_p / \mathbb{Z}_p)^{\oplus r + n - 1} \). Then there exists an \((n - 1)\)-tuple \( r \) of integers \( r_i \geq 0 \) and a tamely ramified \( G \)-torsor \( Y \rightarrow X_0 \) with ramification data \( (D, p^r) \) such that the restriction \( Y \times_{X_0} U \rightarrow U \) gives a Nori-reduced \( G \)-torsor.

**Proof.** This is immediate from Theorems 4.15(2) and 4.30.

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Let $k$ be an algebraically closed field of characteristic $p > 0$. We denote by $t$ the coordinate of the affine line $\mathbb{A}^1_k$, i.e. $\mathbb{A}^1_k = \text{Spec } k[t]$. Let $\mathcal{D} \overset{\text{def}}{=} \text{Spec } k[[t^{-1}]]$ be the formal disk and $\eta \overset{\text{def}}{=} \text{Spec } k((t^{-1}))$ the formal punctured disk. Note that there exist natural morphisms $\eta \longrightarrow \mathbb{A}^1_k$ and $\eta \longrightarrow \mathcal{D}$.

**Definition A.1.** Let $G$ be a finite group. An étale Galois $G$-cover $f : Y \longrightarrow \mathbb{G}_m,k = \text{Spec } k[t, t^{-1}]$ is said to be special if the following conditions are satisfied.

(i) $f$ is tamely ramified above $t = 0$.

(ii) The monodromy group of $f$, i.e. the image of the homomorphism $\pi^{	ext{et}}_1(\mathbb{G}_m,1) \longrightarrow G$ which corresponds to $f$ (up to conjugacy) has a unique $p$-Sylow subgroup.

Then the Katz–Gabber correspondence for finite étale coverings can be stated as follows.
**Theorem A.2.** (Katz–Gabber, cf. [21].) Let \( G \) be a finite group. The morphism \( \eta : \mathbb{G}_{m,k} \to G_{m,k} \) induces an equivalence of categories between the category of special \( G \)-covers of \( G_{m,k} \) and the category of \( G \)-covers of \( \eta \).

In particular, the theorem includes the following fact.

**Proposition A.3.** The morphism \( \eta : \mathbb{A}_1^k \to \mathbb{A}_1^k \) induces an isomorphism of pro-\( p \)-groups

\[
\text{Gal} \left( \overline{k((t^{-1})/k((t^{-1}))} \right)^{(p)} \cong \pi_1^{et}(\mathbb{A}_1^k,1)^{(p)},
\]

where \( \overline{k((t^{-1})} \) is a separable closure of \( k((t^{-1}) \) and \((-)^{(p)} \) means the maximal pro-\( p \)-quotient.

**Example A.4.** Let us consider \( G = \mathbb{F}_p \). The natural inclusion \( k[t] \to k((t^{-1})) \) induces an isomorphism

\[
H^1_{et}(\mathbb{A}_1^k, \mathbb{F}_p) = k[t]/\mathcal{P}(k[t]) \cong k((t^{-1}))/\mathcal{P}(k((t^{-1}))) \cong H^1_{et}(\eta, \mathbb{F}_p),
\]

where \( \mathcal{P} \) is the \( \mathbb{F}_p \)-linear map \( f \mapsto f - f^p \). This is valid because

\[
k((t^{-1})) = k[t] \oplus t^{-1}k[[t^{-1}]] \quad \text{and} \quad \mathcal{P}(t^{-1}k[[t^{-1}]]) = t^{-1}k[[t^{-1}]],
\]

where the second equation can be seen as follows. For an arbitrary element \( f \in t^{-1}k[[t^{-1}]] \), we have

\[
f = \sum_{i=0}^{\infty} \mathcal{P}(f^i) = \mathcal{P} \left( \sum_{i=0}^{\infty} f^i \right)
\]

with \( \sum_{i=0}^{\infty} f^i \in t^{-1}k[[t^{-1}]] \), hence \( f \in \mathcal{P}(t^{-1}k[[t^{-1}]]). \)

Let us consider what happens for finite local torsors. To ease of notation, for a reduced inflexible algebraic stack \( \mathcal{X} \) over a field \( k \), we write \( \tau(\mathcal{X}) \) for the local fundamental group scheme \( \pi_{\text{loc}}(\mathcal{X}) \) of \( \mathcal{X} \). Since the natural inclusion \( \eta : \mathcal{D} \to \mathcal{D} \) is an open immersion and \( \mathcal{D} \) is normal, according to [27, Chapter II, §2], the natural homomorphism \( \tau(\eta) \to \tau(\mathcal{D}) \) is surjective.

**Lemma A.5.** We have the following.

(1) For any integer \( m > 0 \), the composition of natural maps

\[
H^1_{fpf}(\mathbb{G}_m, \mu_{p^m}) \to H^1_{fpf}(\eta, \mu_{p^m}) \to H^1_{fpf}(\eta, \mu_{p^m}) / H^1_{fpf}(\mathcal{D}, \mu_{p^m})
\]

is an isomorphism of abelian groups.

(2) The composition of natural maps

\[
H^1_{fpf}(\mathbb{A}_1^k, \alpha_p) \to H^1_{fpf}(\eta, \alpha_p) \to H^1_{fpf}(\eta, \alpha_p) / H^1_{fpf}(\mathcal{D}, \alpha_p)
\]

is an isomorphism of abelian groups.
Proof.

(1) This follows from the descriptions of cohomology groups

\[ H^1_{\text{fppf}}(\eta, \mu_p^m) \cong k((t^{-1}))^x / k((t^{-1}))^{x_p^m} \cong t^{-Z} / t^{-p^mZ} \times k[t^{-1}]^x / k[t^{-1}]^{x_p^m}, \]

\[ H^1_{\text{fppf}}(\mathcal{O}, \mu_p^m) \cong k[[t^{-1}]]^x / k[[t^{-1}]]^{x_p^m}, \]

\[ H^1_{\text{fppf}}(\mathbb{G}_m, \mu_p^m) \cong t^{-Z} / t^{-p^mZ}. \]

(2) This follows from the descriptions of cohomology groups

\[ H^1_{\text{fppf}}(\eta, \alpha_p) \cong k((t^{-1}))/ k((t^{-1}))^p \cong k[t]/ k[t]^p \oplus k[[t^{-1}]]/ k[[t^{-1}]]^p, \]

\[ H^1_{\text{fppf}}(\mathcal{O}, \alpha_p) \cong k[[t^{-1}]]/ k[[t^{-1}]]^p, \]

\[ H^1_{\text{fppf}}(\mathbb{A}^1_k, \alpha_p) \cong k[t]/ k[t]^p. \]

Proposition A.6. There exists a canonical isomorphism of affine $k$-group schemes

\[ \varpi^{ab}(\eta) \cong \varpi^{ab}(\mathcal{O}) \times \varpi^{ab}(\sqrt[p]{0/\mathbb{A}^1_k}), \]

where $\varpi\left(\sqrt[p]{0/\mathbb{A}^1_k}\right)$ is defined to be

\[ \varpi\left(\sqrt[p]{0/\mathbb{A}^1_k}\right) \overset{\text{def}}{=} \lim_{r>0} \varpi\left(\sqrt[p]{0/\mathbb{A}^1_k}\right). \]

Proof. Since we have natural $k$-homomorphisms $\varpi(\eta) \rightarrow \varpi(\mathcal{O})$ and $\varpi(\eta) \rightarrow \varpi(\sqrt[p]{0/\mathbb{A}^1_k})$, there exists a canonical homomorphism

\[ \varpi(\eta) \rightarrow \varpi(\mathcal{O}) \times \varpi(\sqrt[p]{0/\mathbb{A}^1_k}). \]

We shall show that this homomorphism induces the desired isomorphism. Since we have

\[ \varpi^{ab}(\mathcal{O} \bigsqcup \sqrt[p]{0/\mathbb{A}^1_k}) = \varpi^{ab}(\mathcal{O}) \times \varpi^{ab}(\sqrt[p]{0/\mathbb{A}^1_k}), \]

by the universal property of the local fundamental group scheme, it suffices to show that for any abelian local finite $k$-group scheme $G$, the induced map

\[ H^1_{\text{fppf}}(\mathcal{O} \bigsqcup \sqrt[p]{0/\mathbb{A}^1_k}, G) = H^1_{\text{fppf}}(\mathcal{O}, G) \oplus H^1_{\text{fppf}}(\sqrt[p]{0/\mathbb{A}^1_k}, G) \rightarrow H^1_{\text{fppf}}(\eta, G) \]

is an isomorphism. By virtue of Lemma 4.29, we may assume that $G$ is isomorphic to $\mu_p$ or $\alpha_p$ (cf. [16, p. 284, Remarque 4.2.10]). Since there exist canonical isomorphisms

\[ H^1_{\text{fppf}}\left(\sqrt[p]{0/\mathbb{A}^1_k}, \mu_p\right) \overset{\sim}{\rightarrow} H^1_{\text{fppf}}(\mathbb{G}_m, \mu_p), \]

\[ H^1_{\text{fppf}}(\mathbb{A}^1_k, \alpha_p) \overset{\sim}{\rightarrow} H^1_{\text{fppf}}(\sqrt[p]{0/\mathbb{A}^1_k}, \alpha_p), \]
(cf. Lemma 4.28 and Proposition 4.9 respectively), the claim is a consequence of Lemma A.5.

**Corollary A.7.** There exists a natural isomorphism

$$\text{Ker}(\varpi^{ab}(\eta) \to \varpi^{ab}(\mathfrak{D})) \cong \varpi^{ab}\left(\sqrt[p^n]{0/A_1^1_k}\right).$$

**APPENDIX B: FROBENIUS DIVIDED SHEAVES ON ROOT STACKS**

The aim of this appendix is to complete the proof of Proposition 4.9, namely to prove the proposition without the smoothness assumption. We will use the same notation as in §4.2. Let $X$ be a geometrically connected and geometrically reduced scheme of finite type over the spectrum $S = \text{Spec } k$ of a perfect field $k$ of characteristic $p > 0$. Let $\mathbf{D} = \{D_i\}_{i \in I}$ be a finite family of reduced irreducible effective Cartier divisors on $X$ and put $D = \bigcup_{i \in I} D_i \subset X$. For each $r = (r_i)_{i \in I}$ with $r_i > 0$, as in §3.3, we put

$$\mathfrak{F} = \sqrt[r]{D/X}.$$

By definition of root stacks, the $n$th Frobenius twist commutes with the root construction,

$$\left(\sqrt[r]{D/X}\right)^{(n)} = \sqrt[r]{D^{(n)}/X^{(n)}}.$$

**Proposition B.1.** Let $r$ and $r'$ be two indices with $r \mid r'$. With the same notation as in §2.5, we have the following.

1. The functor $\pi^* : \text{Fdiv}(\mathfrak{F}) \to \text{Fdiv}(\mathfrak{F}')$ is fully faithful and the essential image consists of all the Frobenius divided sheaves $\mathcal{E} = (E_i, \sigma_i)_{i=0}^\infty$ with $\pi^* E_i|_{q^r_{\xi}} \cong E_i|_{q^r_{\xi}}$ for any $i$ and for any closed point $\xi$ of $\mathfrak{F}'$, where $q^r_{\xi}$ denotes the residual gerbe of $\mathfrak{F}'$ at $\xi$ (cf. Proposition 3.11(2)).

2. The functor $\pi^* : \text{Fdiv}_\infty(\mathfrak{F}) \to \text{Fdiv}_\infty(\mathfrak{F}')$ is fully faithful and the essential image consists of all the objects $(F, E, \lambda)$ of $\text{Fdiv}_\infty(\mathfrak{F}')$ with $\pi^* E|_{q^r_{\xi}} \cong F|_{q^r_{\xi}}$ and $\pi^* \pi^* E_i|_{q^r_{\xi}} \cong E_i|_{q^r_{\xi}}$ for any $i$ and for any closed point $\xi$ of $\mathfrak{F}'$.

3. The induced morphism of tannakian gerbes $\Pi_{\text{Fdiv}_\infty(\mathfrak{F}')} \to \Pi_{\text{Fdiv}_\infty(\mathfrak{F})}$ is a gerbe.

**Proof.**

1. For the full faithfulness, let us begin with an observation. Let $E_i (i = 1, 2)$ be two vector bundles on $\mathfrak{F}$ which admit Frobenius descents, i.e. there exist vector bundles $E_i^{(1)}$ on $\mathfrak{F}^{(1)}$ together with isomorphisms $\sigma_i : F^{(1)} E_i^{(1)} \cong E_i$. Suppose given an $O_{\mathfrak{F}}$-linear map $\alpha : E_1 \to E_2$ and an $O_{\mathfrak{F}^{(1)}}$-linear map $\beta : E_1^{(1)} \to E_2^{(1)}$. Then, by Proposition 3.9(3), the diagram

$$\begin{array}{ccc}
F^{(1)*} E_1^{(1)} & \xrightarrow{\sigma_1} & E_1 \\
\downarrow \quad F^{(1)*} \alpha \downarrow & & \downarrow \alpha \\
F^{(1)*} E_2^{(1)} & \xrightarrow{\sigma_2} & E_2
\end{array}$$

is commutative if and only if it is commutative after applying the pullback functor $\pi^*$. Therefore, the full faithfulness of the functor $\pi^* : \text{Fdiv}(\mathfrak{F}) \to \text{Fdiv}(\mathfrak{F}')$ follows from this observation together with Proposition 3.9(3).

For the description of the essential image, by virtue of Proposition 3.12, it suffices to show that for any vector bundle $E$ on $\mathfrak{F}$, if $\pi^* E$ admits a Frobenius descent which is of the form $F^{(1)*} E^{(1)}$ for some vector bundle $E^{(1)}$ on $\mathfrak{F}^{(1)}$, the isomorphism $\sigma : F^{(1)*} \pi^* E \cong \pi^* E$ defines a canonical isomorphism $F^{(1)*} E^{(1)} \cong E$. However, since $F^{(1)*} \pi^* E^{(1)} = \pi^* F^{(1)*} E^{(1)}$, the pushforward $\pi_* \sigma$ defines a desired isomorphism $F^{(1)*} E^{(1)} \cong E$. 

Similarly, the assertion is a consequence of Propositions 3.9(3) and 3.12. Let us begin with showing the full faithfulness.

Fix an arbitrary integer $j > 0$. Let $F_i (i = 1, 2)$ be two vector bundles on $\mathcal{X}$ such that there exist $F$-divided sheaves $G_i = \{ E^{(n)}_i, \sigma^{(n)}_i \}_{n=0}^{\infty}$ on $\mathcal{X}$ together with isomorphisms $\lambda_i : F_j^* F_i \xrightarrow{\simeq} G_i|_{\mathcal{X}} = E_i^{(0)}$. Suppose given an $\mathcal{O}_\mathcal{X}$-linear map $\alpha : F_1 \rightarrow F_2$ and a morphism $\beta = \{ \beta(n) \}_{n=0}^{\infty} : G_1 \rightarrow G_2$ of $F$-divided sheaves. Then, by Proposition 3.9(3), the diagram

$$
\begin{array}{c}
F_j^* F_1 \\
\downarrow \alpha_j \\
F_j^* F_2
\end{array}
\xrightarrow{h_j}
\begin{array}{c}
E_1^{(0)} \\
\downarrow \lambda_1 \\
E_2^{(0)}
\end{array}
$$

is commutative if and only if it is commutative after applying the pullback functor $\pi^*$. From the definition of $\text{Fdiv}_{\infty}(-)$ together with the full faithfulness of the functor $\pi^* : \text{Fdiv}(\mathcal{X}) \rightarrow \text{Fdiv}(\mathcal{X}')$ (cf. (1)), the observation implies that the functor $\text{Fdiv}_{\infty}(\mathcal{X}) \rightarrow \text{Fdiv}_{\infty}(\mathcal{X}')$ is fully faithful.

Let us discuss on the description of the essential image. Fix an arbitrary integer $j > 0$. Let $(F_i, G_i, \lambda)$ be an object of $\text{Fdiv}_j(\mathcal{X}')$ satisfying $\pi_* \pi_*(F|_{\mathcal{X}}) \xrightarrow{\simeq} F|_{\mathcal{X}}$ and $\pi_* \pi_*(E|_{\mathcal{X}}) \xrightarrow{\simeq} E|_{\mathcal{X}}$ for any $i$ and for any closed point $\xi$ of $\mathcal{X}'$.

From Proposition 3.12, we have $\pi_* \pi_*(F) \xrightarrow{\simeq} F$ in $\text{Vect}(\mathcal{X}')$. On the other hand, by (1), we also have $\pi_* \pi_*(G) \xrightarrow{\simeq} G$ in $\text{Fdiv}(\mathcal{X}')$. The isomorphism $\lambda : F_j^* F \xrightarrow{\simeq} G|_{\mathcal{X}}$, then descends to a one $\pi_* \lambda : F_j^* F \xrightarrow{\simeq} (\pi_* G)|_{\mathcal{X}}$, which implies that $(F, G, \lambda)$ descends to an object of $\text{Fdiv}_j(\mathcal{X}')$. This completes the proof.

(3) This follows from (2) together with the last assertion in Proposition 3.12(3).
\[ \Gamma(U, \mathcal{F}(E(1))) / \Gamma(U, \mathcal{F}) p \] is of infinite dimension. Since \( f \) is generically étale, the commutative diagram

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{F} & \mathcal{X} \\
\downarrow f & & \downarrow f \\
U & \xrightarrow{F} & U
\end{array}
\]

is generically Cartesian. Since \( F \) is a flat morphism on the smooth curve \( U \), it follows that the canonical base change map

\[ (f_*\mathcal{F}(1)) \longrightarrow f_* (\mathcal{F}(1)) \]

is a generically isomorphism of torsion-free coherent sheaves, hence its kernel is trivial and its cokernel is a torsion coherent sheaf over \( U \). Therefore, \( \Gamma(U, f_* (\mathcal{F}(1))) / \Gamma(U, f_*\mathcal{F}) p \) is of infinite dimension if and only if \( \Gamma(U, (f_*\mathcal{F})(1)) / \Gamma(U, f_*\mathcal{F}) p \) is of infinite dimension. However, as \( U \) is an affine dense open subscheme of \( \mathbb{P}^1_k \), any torsion-free coherent sheaf is a free \( \mathcal{O}_U \)-module, and \( \Gamma(U, \mathcal{O}_U) / \Gamma(U, \mathcal{O}_U) p \) is of infinite dimension. Therefore, \( \Gamma(U, (f_*\mathcal{F})(1)) / \Gamma(U, f_*\mathcal{F}) p \) is of infinite dimension. This completes the proof.

As an application, one can see the following result.

**Proposition C.2.** Let \( U \) be an affine smooth curve over a perfect field \( k \) of characteristic \( p > 0 \) with function field \( K \). Let \( D = (x_i)_{i=1}^m \) be a family of distinct closed points of \( U \). Let \( r = (r_i)_{i=1}^m \in \prod_{i=1}^m \mathbb{Z}_{\geq 1} \) be a family of integers, we denote by \( \mathcal{X} = \sqrt[\pi]{D} / U \) the associated root stack. Let \( N \) be a finite flat abelian \( \mathcal{X} \)-group scheme. If both \( N \) and its Cartier dual \( N^\vee \) are of height one, then the first cohomology group \( H^1_{\text{fppf}}(\mathcal{X}, N) \) is infinite dimensional over \( K \). Furthermore, we have \( H^q_{\text{fppf}}(\mathcal{X}, N) = 0 \) for \( q > 1 \).

**Proof.** As \( N^\vee \) is of height one, there exists a locally free \( \mathcal{O}_{\mathcal{X}} \)-module \( \omega_{N^\vee} \) of finite rank together with the exact sequence of fppf abelian sheaves

\[ 0 \longrightarrow N \longrightarrow V(\omega_{N^\vee}) \xrightarrow{\phi} V(\omega_{N^\vee}(1)) \longrightarrow 0 \]

(cf. [25, Chapter III, Theorem 5.1]), where \( V(\omega_{N^\vee}) \) is the vector group over \( \mathcal{X} \) associated with \( \omega_{N^\vee} \), which is defined to be

\[ V(\omega_{N^\vee})(Y) \overset{\text{def}}{=} \text{Hom}_{\mathcal{O}_Y}(\omega_{N^\vee} \otimes \mathcal{O}_Y, \mathcal{O}_Y) \]

for any morphism \( Y \longrightarrow \mathcal{X} \) from a scheme \( Y \). Moreover, as \( N \) is of height one, the homomorphism \( \phi : V(\omega_{N^\vee}) \longrightarrow V(\omega_{N^\vee}(1)) \) is the \( p \)th power Frobenius map \( \phi = F(1) \) of the vector group \( V(\omega_{N^\vee}) \) (cf. [25, Chapter III, §5]). As \( V(\omega_{N^\vee}) = W(\omega_{N^\vee}(1)) \), by the same argument as in the proof of Lemma 4.26(1), we have \( H^q_{\text{fppf}}(\mathcal{X}, V(\omega_{N^\vee})) = 0 \) for \( q > 0 \) and the same is true for \( V(\omega_{N^\vee}(1)) = V(\omega_{N^\vee}(1)) \). Therefore, by considering the long exact sequence associated with the above short exact sequence, we have

\[ H^1_{\text{fppf}}(\mathcal{X}, N) \cong \Gamma(\mathcal{X}, \omega_{N^\vee}(1)) / \Gamma(\mathcal{X}, \omega_{N^\vee})^p, \]

which is an infinite dimensional \( k \)-vector space by Lemma C.1, and \( H^q_{\text{fppf}}(\mathcal{X}, N) = 0 \) for \( q > 1 \). This completes the proof.

**Remark C.3.** In the proof of [29, Proposition 3.4], the infiniteness of the cohomology group \( H^1_{\text{fppf}}(U, \mathcal{A}_p) \) for an affine smooth geometrically connected curve over \( k \) is mentioned, but is wrongly explained. As in Lemma C.1, one should have taken a dominant morphism \( U \longrightarrow \mathbb{A}^1_k \) which is generically étale.