A Geometric Approach for Bounding Average Stopping Time *

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Abstract

We propose a geometric approach for bounding average stopping times defined in terms of sums of i.i.d. random variables. We consider stopping times in the hyperspace of sample number and sample sum. Our techniques relies on exploring geometric properties of continuity or stopping regions. Especially, we make use of the concepts of convex hull, convex sets and supporting hyperplane. Explicit formulae and efficiently computable bounds are obtained for average stopping times. Our techniques can be applied to bound average stopping times involving random vectors, nonlinear stopping boundary, and constraints of sample number. Moreover, we establish a stochastic characteristic of convex sets and generalize Jensen’s inequality, Wald’s equations and Lorden’s inequality, which are useful for investigating average stopping times.

1 Introduction

In many areas of engineering and sciences, especially probability and statistics, it is interested to investigate the expectation of stopping times defined in terms of sums of i.i.d. random variables. For example, a frequent topic of random walk \[2, 16, 24, 27] concerns a stopping time which is the smallest positive integer \(n\) such that \(X_1 + \cdots + X_n\) is no less than \(f(n)\), where \(X_1, X_2, \cdots\) are i.i.d. random variables and \(f\) is a function of \(n\). Since many sequential hypothesis testing and estimation procedures can be cast into the context of such stopping time, for analyzing the

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efficiency of statistical inference, it is of practical importance to evaluate the expectation of such stopping time in the area of sequential analysis [6, 8, 10, 11, 15, 18, 28, 29]. Although the literature on such stopping time is abundant, most existing works are focused on the asymptotic analysis of average stopping times (see, e.g., [26, 31] and the references therein). Existing techniques such as Lorden’s inequality [17] for bounding average stopping times are limited to very specific forms of \( f(n) \). In many practical situations, \( f(n) \) can be complicated functions without nice properties such as linearity and monotonicity. The sample number \( n \) may be restricted to a subset of natural numbers, as usually required in group sequential methods [1, 13, 19, 23, 30]. The underlying variables \( X_i \) may be random vectors. However, there lacks of effective technique for obtaining tight bounds for average stopping times, which are general enough to deal with the nonlinearity of the function \( f(n) \), the constraint of the sample number \( n \), and the dimensionality of random variables \( X_1, X_2, \ldots \). Motivated by this situation, we propose a geometric approach to bound average stopping times in a general setting. We consider stopping times in the hyperspace of the tuple \((n, X_1 + \cdots + X_n)\), where \( X_i \) are allowed to be random vectors and \( n \) is a pre-specified subset \( \mathcal{N} \) of natural numbers. A stopping time is represented as the first time \( n \in \mathcal{N} \) that the tuple \((n, X_1 + \cdots + X_n)\) falls into a certain region, referred to as a stopping region (or equivalently, falls outside of a certain region, referred to as a continuity region). Our main idea is to make use of the geometric properties of the continuity region or stopping region. Particularly, we will use concepts such as convexity and supporting hyperplane to develop bounds for average stopping times, which are either explicit or amenable for convex minimization.

The remainder of the paper is organized as follows. In Section 2, we propose to investigate stopping times in a geometric setting, which makes it possible to use geometric concepts such as convex hull, convex set, and supporting hyperplane, etc. Afterward, we establish a probabilistic property of convex sets, which plays a crucial role in bounding average stopping times. In Section 3, we generalize Jensen’s inequality, Wald’s equations and Lorden’s inequality, which are fundamental tools for investigating average stopping times. In Section 4, we establish efficient convex minimization techniques for bounding average stopping times. In Section 5, we develop explicit formulae for bounding average stopping times by virtue of the concept of supporting hyperplane. In Section 6, we propose to bound average stopping times by combining the power of concentration inequalities and the concept of geometric convexity. In Section 7, we extend the techniques to bound average stopping times relevant to continuous-time stochastic processes such as Brownian motion. Section 8 is the conclusion. Most proofs are given in Appendices.

In this paper, we shall use the following notations. The set of positive integers is denoted by \( \mathbb{N} \). The set of non-negative integers is denoted by \( \mathbb{Z}^+ \). The set of real numbers is denoted by \( \mathbb{R} \). The set of non-negative real numbers is denoted by \( \mathbb{R}^+ \). The set of real-valued row matrices of size \( 1 \times d \) is denoted by \( \mathbb{R}^d \). A row matrix in \( \mathbb{R}^d \) is also called a vector. The notation \( \mathbf{0}_d \) denotes
a row matrix of size $1 \times d$ with all elements being 0. The notation $1_d$ denotes a row matrix of size $1 \times d$ with all elements being 1. We use notation $\top$ to denote the transpose of a matrix. We define the following operations of row matrices:

$AB$ denotes the product of $A = [a_1, \cdots, a_d]$ and $B = [b_1, \cdots, b_d]$ in the sense that $AB = [a_1b_1, \cdots, a_db_d]$.

$\frac{A}{B}$ denotes the quotient of $A = [a_1, \cdots, a_d]$ divided by $B = [b_1, \cdots, b_d]$ in the sense that $\frac{A}{B} = [\frac{a_1}{b_1}, \cdots, \frac{a_d}{b_d}]$.

$A^i$ denotes the $i$-th power of $A = [a_1, \cdots, a_d]$ in the sense that $A^i = [a_1^i, \cdots, a_d^i]$.

$|A|$ denotes the absolute value of $A = [a_1, \cdots, a_d]$ in the sense that $|A| = [|a_1|, \cdots, |a_d|]$.

For matrices $A = [a_1, \cdots, a_d]$ and $B = [b_1, \cdots, b_d]$, we write $A \leq B$ if $a_i \leq b_i$ for $i = 1, \cdots, d$.

For matrices $A = [a_1, \cdots, a_d]$ and $B = [b_1, \cdots, b_d]$, we use $< A, B >$ to denote their inner product, that is, $< A, B >$ is equal to $\sum_{i=1}^d a_ib_i$.

For $A = [a_1, \cdots, a_d] \in \mathbb{R}^d$, its $L^p$-norm with $p \geq 1$ is defined as

$$||A||_p = \left( \sum_{i=1}^d |a_i|^p \right)^{\frac{1}{p}}.$$

The $L^p$-norm of the transpose of $A$ is also defined as $||A|\top|_p$, that is, $||A|\top|_p = ||A||_p$.

For a function, $f(v)$, of $v = [v_1, \cdots, v_d] \in \mathbb{R}^d$, we use $\frac{\partial f(v)}{\partial v}$ to denote the gradient of $f(v)$ with respect to $v$, that is,

$$\frac{\partial f(v)}{\partial v} = \left[ \frac{\partial f(v)}{\partial v_1}, \cdots, \frac{\partial f(v)}{\partial v_d} \right].$$

For random vector $X = [x_1, \cdots, x_d]$, we define $X^+ = [\max(0, x_1), \cdots, \max(0, x_d)]$ as the non-negative part of $X$. Similarly, we define $X^- = [\max(0, -x_1), \cdots, \max(0, -x_d)]$ as the non-positive part of $X$.

For a set $\mathcal{S}$, its closure and boundary are denoted by $\overline{\mathcal{S}}$ and $\partial \mathcal{S}$, respectively. The probability of an event $E$ is denoted by $\Pr\{E\}$. The mathematical expectation of a random variable (scalar or vector) $X$ is denoted by $\mathbb{E}[X]$. Let $\mathbb{I}_E$ denote the indicator function such that it assumes value 1 if the event $E$ occurs and it assumes value 0 otherwise. The other notations will be made clear as we proceed.

## 2 Stopping Times and Convex Sets

In this section, we shall propose to investigate stopping times with their geometric representations. We shall also establish a connection between stopping times and convex sets. A stochastic characterization of convex sets is developed.
2.1 Geometric Representations of Stopping Times

Existing methods for bounding the average of a stopping time typically focus on exploring the properties of the function defining the stopping time. Consider, for example, the stopping time mentioned in the introduction of this paper. To bound the expectation of stopping time

\[ N = \inf \{ n \in \mathbb{N} : X_1 + \cdots + X_n \geq f(n) \}, \]  

(1)

conventional wisdom is to explore the function \( f(n) \) for properties such as linearity and monotonicity which could be useful for bounding the average stopping time. We would like to point out that the methods in this direction usually fail to fully exploit the geometric information of the underlying continuity or stopping regions. To clearly address this point, we shall first provide geometric representations of stopping times in the sequel.

Throughout the remainder of this paper, we shall use the following notations and definitions. Let \( 0 \leq N_0 < N_1 < N_2 < \cdots \) be an increasing sequence of integers and define \( \mathcal{N} = \{ N_1, N_2, \cdots \} \). Let \( \mathcal{R} \) be a subset of \( \{(t,s) : t \in \mathbb{R}^+, s \in \mathbb{R}^d\} \). Let \( X = [x_1, \cdots, x_d] \) be a \( d \)-dimensional real-valued random vector with mean \( \mu = \mathbb{E}[X] \). Let \( X_1, X_2, \cdots \) be i.i.d. samples of \( X \). Define \( S_0 = 0 \) and

\[ S_n = \sum_{i=1}^{n} X_i, \quad \bar{X}_n = \frac{S_n}{n} \]

for \( n \in \mathbb{N} \). Our effort will be devoted to stopping times which are defined in terms of sample sum \( S_n \) (or equivalently, sample mean \( \bar{X}_n \)), the region \( \mathcal{R} \) and the set \( \mathcal{N} \). The stopping times defined in this way can be fairly general.

A stopping time can be defined in terms of \( S_n \) as \( N = \inf \{ n \in \mathcal{N} : (n, S_n) \in \mathcal{R} \} \). Such stopping time is associated with the stopping rule: Continue observing \( S_n \) until \( (n, S_n) \in \mathcal{R} \) for some \( n \in \mathcal{N} \). For such stopping rule, the region \( \mathcal{R} \) is referred to as a stopping region.

On the other hand, a stopping time can also be defined as \( N = \inf \{ n \in \mathcal{N} : (n, S_n) \notin \mathcal{R} \} \). Such stopping time is associated with the stopping rule: Continue observing \( S_n \) until \( (n, S_n) \notin \mathcal{R} \) for some \( n \in \mathcal{N} \). For such stopping rule, the region \( \mathcal{R} \) is referred to as a continuity region.

Despite the generality of the above geometric representations, stopping times are usually expressed in algebraic forms. A familiar example is the stopping time defined by (1). In this paper, we propose to investigate stopping times based on their geometric representations. The primary reason is that the bounding of average stopping times can be much more easier by exploiting the geometric properties of the underlying continuity or stopping region \( \mathcal{R} \). As will be seen later, this is especially true when the closure of the region \( \mathcal{R} \) is convex. We discovered that, for a wide variety of stopping times in the context of sequential hypothesis testing and estimation, the corresponding continuity or stopping regions \( \mathcal{R} \) in geometric representations are actually convex. In the worse case that the continuity or stopping regions \( \mathcal{R} \) are not convex, it is
still possible to bound the average stopping time by replacing \( \mathcal{R} \) with its convex hull \( \mathcal{R} \), at the price of extra conservatism.

To illustrate the advantage of geometric representations, consider stopping time

\[
N = \inf\{ n \in \mathbb{N} : f(n, S_n) \geq 0 \},
\]

where \( f(t, s) \) is a bivariate function of \( t \in \mathbb{R}^+ \) and \( s \in \mathbb{R} \). Clearly, the stopping region is

\[
\mathcal{R} = \{(t, s) : t \in \mathbb{R}^+, s \in \mathbb{R}, f(t, s) \geq 0 \}
\]

and the stopping time \( N = \inf\{ n \in \mathbb{N} : (n, S_n) \in \mathcal{R} \} \). Similarly, the continuity region is

\[
\mathcal{R}^c = \{(t, s) : t \in \mathbb{R}^+, s \in \mathbb{R}, f(t, s) < 0 \}
\]

and the stopping time \( N = \inf\{ n \in \mathbb{N} : (n, S_n) \not\in \mathcal{R}^c \} \). It can be shown that if \( f \) is a concave function, then the stopping region \( \mathcal{R} \) is convex. If \( f \) is a convex function, then the continuity region \( \mathcal{R}^c \) is convex. It is important to note that the convexity of the stopping or continuity region may also hold in situations when the function \( f \) is neither convex nor concave. Moreover, even if neither the continuity region nor the stopping region is convex, we can still bound the average stopping time by using their convex hulls. This example demonstrates that, in contrast to using algebraic forms of stopping times, it is possible to exploit the convexity of the continuity or stopping regions in geometric representations under much weaker conditions.

### 2.2 A Stochastic Characteristic of Convex Sets

As discussed in the last subsection, there exists a useful connection between stopping times and convex sets. Since continuity or stopping regions are convex in many situations, it is natural to consider the question of under what conditions the expectation of a random vector will be contained by a convex set. Our investigation indicates that if a set in a finite-dimensional Euclidean space is convex, then the set contains the expectation of any random vector almost surely contained by the set. Conversely, if a set in a finite-dimensional Euclidean space contains the expectation of any random vector almost surely contained by the set, then the set is convex. More formally, we have established the following results.

**Theorem 1** If \( \mathcal{D} \) is a convex set in \( \mathbb{R}^n \), then \( \mathbb{E}[\mathcal{X}] \in \mathcal{D} \) holds for any random vector \( \mathcal{X} \) such that \( \Pr\{\mathcal{X} \in \mathcal{D}\} = 1 \) and that \( \mathbb{E}[\mathcal{X}] \) exists. Conversely, if \( \mathcal{D} \) is a set in \( \mathbb{R}^n \) such that \( \mathbb{E}[\mathcal{X}] \in \mathcal{D} \) holds for any random vector \( \mathcal{X} \) such that \( \Pr\{\mathcal{X} \in \mathcal{D}\} = 1 \) and that \( \mathbb{E}[\mathcal{X}] \) exists, then \( \mathcal{D} \) is convex.

See Appendix A for a proof. This theorem plays a fundamental role in our approach for bounding average stopping times.
We would like to point out that the first assertion of Theorem 1 provides a simple proof of Jensen’s inequality. To see this, note that if a function is convex, then its epigraph, the region above its graph, is a convex set. Hence, if \( f \) is a convex function, then for any random variable \( X \), since \((X, f(X))\) is contained by the epigraph of \( f \), it follows from Theorem 1 that \((E[X], E[f(X)])\) is contained by its epigraph. This implies that \( E[f(X)] \geq f(E[X]) \) by the notion of epigraph.

3 Generalizations of Jensen’s Inequality, Wald’s Equations and Lorden’s Inequality

In this section, we shall generalize Jensen’s inequality, Wald’s equations and Lorden’s inequality, which can be useful for evaluating average stopping times.

3.1 Generalization of Jensen’s Inequality

For investigating the convexity of continuity and stopping regions, we have the following results.

**Theorem 2** Suppose that \( g(z) \) is a multivariate convex function of \( z \in \mathcal{D} \), where \( \mathcal{D} \) is a convex set. Define \( f(t,s) = tg(s) \) for \( t \neq 0 \) and \( s \) such that \( t/s \in \mathcal{D} \). Then, \( f(t,s) \) is a multivariate convex function of \( t > 0 \) and \( s \) such that \( t/s \in \mathcal{D} \). Similarly, \( f(t,s) \) is a multivariate concave function of \( t < 0 \) and \( s \) such that \( t/s \in \mathcal{D} \).

See Appendix B for a proof. As an application of Theorem 2, consider stopping time \( N = \inf\{n \in \mathcal{N} : f(n,S_n) \geq 0\} \), with

\[
f(n,S_n) = \sum_{\ell=1}^{k} \left[ ng_{\ell}(A_{\ell}^\top S_n + \beta_{\ell}) + h_{\ell}(n,S_n) \right],
\]

where \( g_{\ell}, h_{\ell} \) are multivariate convex functions, \( A_{\ell} \in \mathbb{R}^d \), and \( \beta_{\ell} \in \mathbb{R} \). The convexity of the continuity region follows immediately from Theorem 2.

By virtue of Theorem 2 we have derived the following results.

**Theorem 3** Let \( Z \) be a random vector and \( Y \) be a scalar random variable such that \( Z \) and \( \frac{Z}{Y} \) are contained in a convex set \( \mathcal{D} \). Assume that \( g(z) \) is a multivariate convex function of \( z \in \mathcal{D} \). Then,

\[
E \left[ Yg \left( \frac{Z}{Y} \right) \right] \geq E[Y]g \left( \frac{E[Z]}{E[Y]} \right) \quad \text{if } Y \text{ is a positive random variable;}
\]
\[
E \left[ Yg \left( \frac{Z}{Y} \right) \right] \leq E[Y]g \left( \frac{E[Z]}{E[Y]} \right) \quad \text{if } Y \text{ is a negative random variable.}
\]

See Appendix C for a proof. It should be noted that Theorem 3 generalizes Jensen’s inequality.

In the special case that \( Y = 1 \), the first inequality of Theorem 3 reduces to Jensen’s inequality.
3.2 Generalization of Wald’s Equations

Making use of Theorem 3, we have generalized Wald’s equations [29] as follows.

**Theorem 4** Let $X_1, X_2, \cdots$ be i.i.d. samples of random vector $X$ with mean $\mu = \mathbb{E}[X]$ and variance $\nu = \mathbb{E}[(X - \mu)^2]$. Assume that $N$ is an integer-valued random variable such that $\mathbb{E}[N] < \infty$ and that for any possible value $n$ of $N$, the event $\{N = n\}$ depends only on $X_1, \cdots, X_n$. Define

$$S_N = \sum_{i=1}^{N} X_i, \quad \bar{X}_N = \frac{S_N}{N}, \quad V_N = \frac{(S_N - N\mu)^2}{N}.$$

The following assertions hold.

(I): If $g$ is a convex function on a convex set $\mathcal{D}$ in $\mathbb{R}^d$ such that $\mathcal{D}$ contains $\mu$ and the range of $\bar{X}_N$, then

$$\mathbb{E}[N g(\bar{X}_N)] \geq \mathbb{E}[N] g(\mu).$$

(II): If $g$ is a convex function on a convex set $\mathcal{D}$ in $\mathbb{R}^d$ such that $\mathcal{D}$ contains $\nu$ and the range of $V_N$, then

$$\mathbb{E}[N g(V_N)] \geq \mathbb{E}[N] g(\nu).$$

See Appendix D for a proof. To see why the inequality in the first assertion of Theorem 4 is a generalization of Wald’s first equation, consider function $g(x) = x$. By the convexity of $g(x)$, we have $\mathbb{E}[N \bar{X}_N] \geq \mathbb{E}[N] \mu$. On the other hand, by the convexity of $-g(x)$, we have $\mathbb{E}[N(-\bar{X}_N)] \geq \mathbb{E}[N](-\mu)$ or equivalently, $\mathbb{E}[N \bar{X}_N] \leq \mathbb{E}[N] \mu$. Hence, it must be true that $\mathbb{E}[S_N] = \mathbb{E}[N \bar{X}_N] = \mathbb{E}[N] \mu$, which is Wald’s first equation. Similarly, we can demonstrate that the inequality in the second assertion of Theorem 4 is a generalization of Wald’s second equation.

As an illustration of the applications of Theorem 4 consider

$$N = \inf \left\{ n \in \mathcal{N} : n \geq \frac{1}{g(\bar{X}_n)}, \ g(\bar{X}_n) > 0 \right\}.$$

Clearly,

$$N \geq \frac{1}{g(\bar{X}_N)}, \quad g(\bar{X}_N) > 0$$

and thus $N g(\bar{X}_N) \geq 1$ almost surely provided that $\mathbb{E}[N] < \infty$. By using the generalization of Wald’s first equation, we have $\mathbb{E}[N] g(\mu) \geq 1$, which implies the following result.

**Theorem 5** Assume that $g$ is a concave function on a convex set $\mathcal{D}$ in $\mathbb{R}^d$ such that $\mathcal{D}$ contains $\mu$ and the range of $\bar{X}_N$ and that $g(\mu) > 0$. Then, $\mathbb{E}[N] \geq \frac{1}{g(\mu)}$. 

7
As another application example of Theorem 4 consider $L^p$-norm function $g(s) = ||s||_p$, where $s \in \mathbb{R}^d$ and $p \geq 1$. As a consequence of the absolute homogeneity and subadditivity of the $L^p$-norm, we have

$$||\rho \mathbf{x} + (1-\rho)\mathbf{y}||_p \leq ||\rho \mathbf{x}||_p + ||(1-\rho)\mathbf{y}||_p = \rho ||\mathbf{x}||_p + (1-\rho)||\mathbf{y}||_p$$

for arbitrary vectors $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^d$ and $\rho \in [0,1]$. Hence, the $L^p$-norm function $g(s) = ||s||_p$ is a convex function of $s \in \mathbb{R}^d$. Applying Theorem 8 to the $L^p$-norm function $g(s) = ||s||_p$, we have the following results.

**Theorem 6** Let $X_1, X_2, \cdots$ be i.i.d. samples of random vector $X$ with mean $\mu = \mathbb{E}[X]$ and variance $\nu = \mathbb{E}[(X-\mu)^2]$. Assume that $N$ is an integer-valued random variable such that $\mathbb{E}[N] < \infty$ and that for any possible value $n$ of $N$, the event $\{N = n\}$ depends only on $X_1, \cdots, X_n$. Define $S_N = \sum_{i=1}^N X_i$ and $V_N = (S_N - N\mu)^2$. Then,

$$\mathbb{E} ||S_N||_p \geq \mathbb{E}[N] ||\mu||_p,$$

$$\mathbb{E} ||V_N||_p \geq \mathbb{E}[N] ||\nu||_p$$

for all $p \geq 1$.

### 3.3 Generalization of Lorden’s Inequality

In order to obtain tight bounds for stopping times, we need to generalize Lorden’s inequality [17]. In this direction, we have obtained the following result.

**Theorem 7** Let $Z_1, Z_2, \cdots$ be i.i.d. samples of random variable $Z$ such that $\mathbb{E}[Z^2] < \infty$. Assume that $\lambda$ is a random variable independent of $Z_i$ for all $i \in \mathbb{N}$. Define $\mathcal{M}_\lambda = \inf \{n \in \mathbb{N} : \sum_{i=1}^n Z_i \geq \lambda\}$ and $R_\lambda = \sum_{i=1}^{\mathcal{M}_\lambda} Z_i - \lambda$. Then,

$$\mathbb{E}[R_\lambda] \leq \frac{\mathbb{E}[(Z^+)^2]}{\mathbb{E}[Z]} \Pr\{Z < \lambda\} + \mathbb{E}[(Z - \lambda)^+]$$

See Appendix E for a proof.

In the following, we have extended Lorden’s inequality to the case that the increment of sample sizes is not a constant.

**Theorem 8** Let $Z_1, Z_2, \cdots$ be i.i.d. samples of positive random variable $Z$ such that $\mathbb{E}[Z^2] < \infty$. Assume that $\lambda$ is a random variable independent of $Z_i$ for all $i \in \mathbb{N}$. Define $\mathcal{M}_\lambda = \inf \{n \in \mathcal{N} : \sum_{i=1}^n Z_i \geq \lambda\}$ and $R_\lambda = \sum_{i=1}^{\mathcal{M}_\lambda} Z_i - \lambda$. Define $Y = Z_1 + \cdots + Z_{N_1}$ and $K = \max\{N_{i+1} - N_i : \ell \in \mathbb{N}\}$. Then,

$$\mathbb{E}[R_\lambda] \leq \left((K-1)\mathbb{E}[Z] + \frac{\mathbb{E}[Z^2]}{\mathbb{E}[Z]}\right) \Pr\{Y < \lambda\} + \mathbb{E}[(Y - \lambda)^+]$$

See Appendix E for a proof.
4 Bounding Average Stopping Time via Convex Optimization

In this section, we shall demonstrate that the general problem of bounding average stopping times can be converted into problems of convex minimization, which can be readily solved by modern optimization theory and algorithms. In some particular cases, it is possible to obtain explicit bounds for average stopping times.

4.1 Lower Bound on Average Stopping Time

Consider stopping time $N = \inf\{n \in \mathcal{N} : (n, S_n) \in \mathcal{R}\}$, where the set $\mathcal{R}$ is called a stopping region. We have the following results on the average stopping time.

**Theorem 9** Suppose that the stopping region $\mathcal{R}$ is a convex set. Define $\mathcal{A} = \{t \in \mathbb{R}^+ : (t, t\mu) \in \mathcal{R}\}$. Then, $E[N] \geq \min \mathcal{A}$ provided that $\mathcal{A}$ is not empty. Moreover, $E[N] = \infty$ provided that $\mathcal{A}$ is empty.

See Appendix G for a proof.

4.2 Upper Bounds on Average Stopping Time

As mentioned before, a general problem is to bound the stopping time

$$N = \inf\{n \in \mathcal{N} : (n, S_n) \notin \mathcal{R}\},$$

(4)

where $\mathcal{R}$ is called the continuity region. For the boundedness of $E[N]$, consider the following assumptions:

(I) There exist numbers $\lambda > 0$ and $K$ such that

$$N_{\ell+1} \leq \lambda N_\ell + K$$

(5)

for all $\ell \geq 0$.

(II) Either $\limsup_{\ell \to \infty} (N_{\ell+1} - N_\ell) < \infty$ or $\liminf_{\ell \to \infty} \frac{N_{\ell+1}}{N_\ell} > 1$.

(III) $\overline{\mathcal{R}}$ is a convex set containing $(0, 0_d)$.

(IV) $\{(N_0, S_{N_0}) \in \overline{\mathcal{R}}\}$ is a sure event.

(V) There exists a unique positive number $m$ such that $(m, m\mu) \in \partial \mathcal{R}$.

(VI) Each element of $E[|X|^3]$ is finite.
It should be noted sample sizes used in group sequential methods \[1, 13, 19, 23, 30\] typically satisfy the inequality (5) for some numbers \(\lambda > 0\) and \(K\). For the stopping time defined by (4), we have established the following result.

**Theorem 10** If assumptions (I) – (VI) are fulfilled, then \(E[N] < \infty\).

See Appendix H for a proof.

For the purpose of bounding \(E[N]\), define

\[M = \sup \{N_\ell : \ell \in \mathbb{Z}^+, \ N > N_\ell \}. \tag{6}\]

If \(E[N] < \infty\), then it must be true that \(\Pr\{N < \infty\} = 1\). Let \(\ell\) be the index at the termination of the sampling process. This implies that \(\ell\) is a random variable such that \(N = N_\ell\) and \(M = N_{\ell-1}\). It should be noted that \(M\) is not a stopping time and thus \(E[S_M]\) is, in general, not equal to \(E[M]\mu\). In other words, Wald’s first equation \[29\] is not applicable to \(S_M\), although it holds for \(S_N\).

Clearly, as a consequence of the definition of \(M\) and assumption (I), we have \(N \leq \lambda M + K\) and

\[E[N] \leq \lambda E[M] + K. \tag{7}\]

In view of (7), to bound \(E[N]\), it suffices to bound \(E[M]\). For this purpose, we have the following general result.

**Theorem 11** If assumptions (I) – (VI) are fulfilled, then

\[E[M] \leq \max_{(t,s) \in \mathcal{D}} t, \tag{8}\]

where \(\mathcal{D} = \{(t, s) \in \mathbb{R}^2 : t\alpha + \zeta \leq s \leq t\beta + \eta\}\), with

\[\alpha = \mu - \lambda E[(X - \mu)^+], \quad \beta = \mu + \lambda E[(X - \mu)^-], \quad \zeta = (N_0 - K) E[(X - \mu)^+], \quad \eta = (K - N_0) E[(X - \mu)^-].\]

See Appendix I for a proof. It can be checked that \(\mathcal{D}\) is a convex set. Moreover, \(\max_{(t,s) \in \mathcal{D}} t = -\min_{(t,s) \in \mathcal{D}} f(t, s)\), where \(f(t, s) = -t\) is a convex function of \((t, s)\) contained in the convex set \(\mathcal{D}\). Therefore, the upper bound in (8) can be readily evaluated by convex minimization. With recent improvements in computing and in optimization theory, convex minimization is nearly as straightforward as linear programming (see, e.g., [4] for a comprehensive treatment). Convex minimization problems can be solved by contemporary methods such as subgradient projection methods [22], interior-point methods [21], etc.

In the case that \(X\) is a bounded random vector, we have the following result.
Theorem 12 Suppose that $\Pr\{a \leq X \leq b\} = 1$, where $a, b \in \mathbb{R}^d$, and that assumptions (I) – (V) are fulfilled. Define

$$v = \frac{(\mu - a)(b - \mu)}{b - a}, \quad \alpha = \mu - \lambda v, \quad \beta = \mu + \lambda v, \quad \zeta = (N_0 - K) v, \quad \eta = -\zeta$$

and

$$\alpha' = b + \lambda(\mu - b), \quad \beta' = a + \lambda(\mu - a), \quad \zeta' = K(\mu - b), \quad \eta' = K(\mu - a).$$

Then,

$$\mathbb{E}[M] \leq \max_{(t, s) \in D} t, \quad (9)$$

where $D = \{(t, s) \in \mathcal{A} : ta + \zeta \leq s \leq t\beta + \eta, ta' + \zeta' \leq s \leq t\beta' + \eta'\}.$

See Appendix J for a proof.

In many situations, a stopping time is defined in terms of sample mean. Consider stopping time

$$N = \inf\{n \in \mathcal{N} : n \geq g(\overline{X}_n)\}. \quad (10)$$

For such stopping time, we have the following result.

**Theorem 13** Assume that $g$ is a concave function on a convex set $D$ in $\mathbb{R}^d$ such that $\mu$ is an interior point of $D$ and that the range of $\overline{X}_n$ is contained by $D$ for any $n \in \{N_0, N_1, \cdots\}$. Assume that $K$ and $N_0$ are positive integers such that $\{N_0 < g(\overline{X}_{N_0})\}$ is a sure event and that $N_{\ell + 1} - N_\ell \leq K \leq N_0$ for $\ell = 0, 1, 2, \cdots$. Assume that each element of $\mathbb{E}[|X|^3]$ is finite. Then,

$$\mathbb{E}[N] \leq K + \max_{\theta \in \mathcal{D}} g(\theta),$$

where $\mathcal{D} = \{\theta \in D : \alpha \leq \theta \leq \beta\}$ with $\alpha = \mu - \mathbb{E}[(X - \mu)^+]$ and $\beta = \mu + \mathbb{E}[(X - \mu)^-].$

See Appendix K for a proof.

If $\mathcal{N} = \mathbb{N}$, the stopping time defined by (10) becomes $N = \inf\{n \in \mathbb{N} : n \geq g(\overline{X}_n)\}$. For such stopping time, we have the following result.

**Theorem 14** Assume that $g$ is a non-negative concave function on a convex set $D$ in $\mathbb{R}^d$ such that $\mu$ is an interior point of $D$ and that the range of $\overline{X}_n$ is contained by $D$ for any $n \in \mathbb{N}$. Assume that each element of $\mathbb{E}[|X|^3]$ is finite. Then,

$$\mathbb{E}[N] \leq 2 + \max_{\theta \in \mathcal{D}} g(\theta),$$

where $\mathcal{D} = \{\theta \in D : \alpha \leq \theta \leq \beta\}$ with $\alpha = \mu - \mathbb{E}[(X - \mu)^+]$ and $\beta = \mu + \mathbb{E}[(X - \mu)^-].$
See Appendix L for a proof. In the case that $X$ is a bounded random vector, we have the following result for the stopping time defined by (10).

**Theorem 15** Assume that $\Pr\{a \leq X \leq b\} = 1$, where $a, b \in \mathbb{R}^d$. Assume that $g$ is a concave function on a convex set $D$ in $\mathbb{R}^d$ such that $\mu$ is an interior point of $D$ and that the range of $X_n$ is contained by $D$ for any $n \in \{N_0, N_1, \cdots\}$. Assume that $K$ and $N_0$ are positive integers such that \{\begin{align*} N_0 < g(X_{N_0}) \end{align*}\} is a sure event and that $N_{\ell+1} - N_\ell \leq K \leq N_0$ for $\ell = 0, 1, 2, \cdots$. Then,

$$\mathbb{E}[N] \leq K + \max_{\theta \in D} g(\theta),$$

where $D = \{\theta \in D : \mu - v \leq \theta \leq \mu + v\}$ with $v = \frac{(\mu - a)(b - \mu)}{b - a}$.

See Appendix M for a proof.

5 Bounding Average Stopping Time with Supporting Hyperplane

In this section, we shall establish explicit bounds for average stopping times. Consider the stopping time $N$ defined by (4). In view of (7), to bound $\mathbb{E}[N]$, it suffices to bound $\mathbb{E}[M]$. For this purpose, we propose to use the concept of supporting hyperplane to derive explicit bounds for $\mathbb{E}[M]$. In the sequel, we shall use equation $As + Bt = C$, where $A^\top \in \mathbb{R}^d$, $B \in \mathbb{R}$ and $C \in \mathbb{R}$ are constants, to represent a hyperplane which consists of points $(t, s)$ with $t \in \mathbb{R}$ and $s \in \mathbb{R}^d$ satisfying the equation. To bound $\mathbb{E}[M]$ associated with (4), we have the following result.

**Theorem 16** Suppose that assumptions (I) – (VI) are fulfilled. Let $As + Bt = C$, where $C > 0$, be the supporting hyperplane of $R$ passing through $(m, m\mu) \in \partial R$. Let $\mathcal{V} = \{q \in \mathbb{R}^d : each \ element \ of \ q \ assumes \ value \ 0 \ or \ 1\}$. Define

$$\alpha = \mu - \lambda \mathbb{E}[(X - \mu)^+] \quad \beta = \mu + \lambda \mathbb{E}[(X - \mu)^-], \quad \zeta = (N_0 - K) \mathbb{E}[(X - \mu)^+] \quad \eta = (K - N_0) \mathbb{E}[(X - \mu)^-].$$

Then,

$$\mathbb{E}[M] \leq \max \left\{ \frac{C - A[\eta + q(\zeta - \eta)]}{B + A[\beta + q(\alpha - \beta)]} : q \in \mathcal{V} \right\}$$

provided that the minimum of $\{B + A[\beta + q(\alpha - \beta)] : q \in \mathcal{V}\}$ is positive.

See Appendix N for a proof.

From the above theorem, it can be seen that the minimum of $\{B + A[\beta + q(\alpha - \beta)] : q \in \mathcal{V}\}$ is close to the positive quantity $A\mu + B$ if $\alpha$ and $\beta$ are close to $\mu$. Actually, this frequently occurs in practices.

In situations that $X$ is a bounded random vector, we have obtained explicit bounds for $\mathbb{E}[M]$ in connection with the stopping time $N$ defined by (4) as follows.
Theorem 17 Suppose that \( \Pr\{a \leq X \leq b\} = 1 \), where \( a, b \in \mathbb{R}^d \), and that assumptions (I) – (V) are fulfilled. Let \( A \) be the supporting hyperplane of \( \mathcal{R} \) passing through \( (m, m\mu) \). Let \( \mathcal{V} = \{q \in \mathbb{R}^d : \text{each element of } q \text{ assumes value } 0 \text{ or } 1\} \). Define
\[
v = \frac{(\mu - a)(b - \mu)}{b - a}, \quad \alpha = \mu - \lambda v, \quad \beta = \mu + \lambda v, \quad \zeta = (N_0 - K) v, \quad \eta = -\zeta
\]
and
\[
\alpha' = b + \lambda(\mu - b), \quad \beta' = a + \lambda(\mu - a), \quad \zeta' = K(\mu - b), \quad \eta' = K(\mu - a).
\]
Then,
\[
\mathbb{E}[M] \leq \max \left\{ \frac{C - A[\eta + q(\zeta - \eta)]}{B + A[\beta + q(\alpha - \beta)]} : q \in \mathcal{V} \right\}
\]
provided that the minimum of \( \{B + A[\beta + q(\alpha - \beta)] : q \in \mathcal{V}\} \) is positive. Similarly,
\[
\mathbb{E}[M] \leq \max \left\{ \frac{C - A[\eta' + q(\zeta' - \eta')]}{B + A[\beta' + q(\alpha' - \beta')]} : q \in \mathcal{V} \right\}
\]
provided that the minimum of \( \{B + A[\beta' + q(\alpha' - \beta')] : q \in \mathcal{V}\} \) is positive.

See Appendix [O] for a proof.

In the sequel, we shall apply the concept of supporting hyperplane and Lorden’s inequality on overshoot to obtain explicit bounds for average stopping times. Consider stopping time
\[
N = \left\{ n \in \mathbb{N} : n = \frac{m}{K} \in \mathbb{N}, (n, S_n) \notin \mathcal{R} \right\},
\]
where \( K \) is a positive integer. For such stopping time, we have the following results.

Theorem 18 Assume that \( \mathcal{R} \) is a convex set containing \( (0, 0_d) \). Assume that each element of \( \mathbb{E}[(X)^2] \) is finite. Assume that there exists a unique positive number \( m \) such that \( (m, m\mu) \in \partial \mathcal{R} \). Assume that there exists a supporting hyperplane \( A X + B t = C \), where \( C > 0 \), of \( \mathcal{R} \) passing through \( (m, m\mu) \). Define \( Z = BK + A \sum_{i=1}^{K} X_i \). The following assertions hold true.

(I):
\[
\mathbb{E}[N] \leq m + \frac{1}{K} \left( \frac{m}{C} \right)^2 \mathbb{E}[(Z^+)^2]
\]
\[
\leq m + K + \left( \frac{m}{C} \right)^2 \mathbb{E}[(A(X - \mu))^2]
\]
\[
\leq m + K + \left( \frac{m}{C} \right)^2 \|A\|_2^2 \times \mathbb{E}[\|X - \mu\|_2^2].
\]

(II): If the elements of \( X \) are mutually independent, then \( \mathbb{E}[N] \leq m + K + \left( \frac{m}{C} \right)^2 \mathbb{E}[(X - \mu)^2] \).

(III): If \( \Pr\{a \leq X \leq b\} = 1 \), where \( a, b \in \mathbb{R}^d \), then
\[
\mathbb{E}[N] \leq m + \frac{K(mu + v)}{C} - \frac{Km^2uv}{C^2},
\]
where \( u = \frac{1}{2} [A(a + b) + |A(a - b)|] + B \) and \( v = \frac{1}{2} [A(a + b) + |A(b - a)|] + B \). In particular,
\[
\mathbb{E}[N] \leq m + \frac{Ku^2}{v - u} \left( \frac{m}{C} \right)^2 \left( \frac{C}{m} - u \right) \text{ for } u < 0.
\]
See Appendix \[\text{P}\] for a proof.

To illustrate the applications of Theorem \[\text{18}\], consider stopping time

\[N = \inf\{n \in \mathbb{N} : (n, S_n) \notin \mathcal{R}\}.\]  

(11)

We wish to apply assertion (I) of Theorem \[\text{18}\] to derive convenient upper bounds for \(\mathbb{E}[N]\). For this purpose, we define function

\[g(v) = \sup\{t \in \mathbb{R}^+ : (t, tv) \in \mathcal{R}\}\]  

(12)

for \(v \in \mathbb{R}^d\). Note that \(g(v)\) can be \(\infty\). Let \(\nabla(v)\) denote the gradient of \(\ln(g(v))\) with respect to \(v\), that is,

\[\nabla(v) = \frac{\partial \ln(g(v))}{\partial v} = \frac{1}{g(v)} \frac{\partial g(v)}{\partial v}.\]  

(13)

For the stopping time defined by \(11\), we have the following results.

**Theorem 19** Assume that \(\mathcal{R}\) is a convex set containing \((0, 0_d)\). Assume that \(\mathbb{E}[\|X\|_2^2]\) is finite. Assume that \(g(v)\) is differentiable at a neighborhood of \(v = \mu\). Then,

\[\mathbb{E}[N] \leq g(\mu) + 1 + \mathbb{E}\left[\langle \nabla(\mu), X - \mu \rangle^2\right] \leq g(\mu) + 1 + ||\nabla(\mu)||_2^2 \times \mathbb{E}\left[\|X - \mu\|_2^2\right].\]

See Appendix \[\text{Q}\] for a proof. We can apply Theorem \[\text{19}\] to derive a simple bound for the expectation of the first passage time for a random walk with concave boundary. More specifically, consider stopping time

\[N = \inf\{n \in \mathbb{N} : S_n > f(n)\},\]  

(14)

where \(S_n = \sum_{i=1}^n X_i\) is the partial sum of i.i.d scalar random variables \(X_1, X_2, \ldots\), which have the same distribution as \(X\) with mean \(\mu = \mathbb{E}[X]\) and variance \(\sigma^2 = \mathbb{E}[\|X - \mu\|_2^2] < \infty\). Assume that \(f(t)\) is a concave function of \(t \in \mathbb{R}^+\) such that \(f(0) > 0\). Assume that there exists a positive number \(m\) such that

\[m\mu = f(m),\]  

(15)

that is, \(m = g(\mu)\), where the function \(g(.)\) is defined by \(12\) with the continuity region

\[\mathcal{R} = \{(t, s) : t \in \mathbb{R}^+, s \in \mathbb{R}, t \leq f(s)\}.\]

Assume that \(f(t)\) is differentiable in a neighborhood of \(t = m\). Then, \(g(v)\) must be differentiable in a neighborhood of \(v = \mu\). Due to the concavity of the boundary function \(f(.)\), the continuity region \(\mathcal{R}\) is a convex set. To apply Theorem \[\text{19}\] to bound the stopping time in \(14\), we can calculate \(\nabla(\mu)\) by \(13\) as follows.

At a neighborhood of \(v = \mu\), we have

\[g(v)v = f(g(v)).\]
Differentiating both sides of the above equation with the chain rule yields
\[ g'(v)v + g(v) = f'(g(v))g'(v), \]
where \( f'(.) \) and \( g'(.) \) denotes the first derivatives of \( f(.) \) and \( g(.) \), respectively. Solving the equation, we obtain
\[ g'(v) = \frac{g(v)}{f'(g(v)) - v}. \]
It follows that the first derivative, \( g'(\mu) \), of \( g(v) \) at \( v = \mu \) can be obtained as
\[ g'(\mu) = \frac{g(\mu)}{f'(g(\mu)) - \mu} = \frac{m}{f'(m) - \mu}. \]
Therefore, the gradient is
\[ \nabla(\mu) = \frac{g'(\mu)}{g(\mu)} = \frac{m}{[f'(m) - \mu]g(\mu)} = \frac{1}{f'(m) - \mu}. \]
It follows from Theorem 19 that
\[ \mathbb{E}[N] \leq m + 1 + \frac{\sigma^2}{[f'(m) - \mu]^2}. \]
(16)
To use formula (16), we need to obtain \( m \) from equation (15). In many cases, it is possible to derive an explicit expression of \( m \) from equation (15). Even if \( m \) cannot be obtained analytically, it can still be readily computed by numerical methods such as the bisection search method. Due to the concavity of \( f(.) \) and the existence of \( m \) satisfying (15), it must be true that \( t\mu > f(t) \) for large enough \( t > 0 \). For example, we can find such value of \( t \) as \( 2^k \) for some integer \( k > 0 \). Then, the number \( m \) can be obtained by a bisection search from interval \((0, 2^k)\).

6 Bounding Average Stopping Time with Concentration Inequalities

In this section, we shall propose a method for bounding average stopping times by virtue of concentration inequalities. Consider the stopping time defined by (4). Define
\[ \tau = \min\{\ell \in \mathbb{N} : N_\ell > m\}, \]
where \( m \) is the unique positive number such that \((m, m\mu) \in \partial \mathcal{R}\) as defined in assumption (V) of Section 4.2. Define
\[ \mathcal{G}_n = \{z \in \mathbb{R}^d : (n, nz) \in \overline{\mathcal{R}}\} \]
and
\[ \rho(n) = \inf_{z \in \mathcal{G}_n} ||z - \mu||_2 \]
for \( n \in \mathbb{N} \). Let \( \mathcal{L} \) denote the support of the random index \( \ell \) such that \( N_\ell = N \). For \( \ell = 1, 2, \ldots \) and \( k = 1, \ldots, d \), let \( \overline{X}_{N_\ell}^k \) denote the \( k \)-th element of \( \overline{X}_{N_\ell} \). For \( k = 1, \ldots, d \), let \( \mu_k \) denote the \( k \)-th element of \( \mu \). We have the following results.

**Theorem 20** Suppose that assumptions (I)–(VI) are fulfilled. Then,

\[
E[N] \leq N_\tau + \sum_{\ell \geq \tau \in \mathcal{L}} (N_{\ell+1} - N_\ell) \sum_{k=1}^d \Pr \left\{ \left| \overline{X}_{N_\ell}^k - \mu_k \right| \geq \frac{\rho(N_\ell)\sqrt{d}}{\sqrt{d}} \right\}.
\] (17)

Moreover, if \( X \) is a scalar random variable, then,

\[
E[N] \leq N_\tau + \sum_{\ell \geq \tau \in \mathcal{L}} (N_{\ell+1} - N_\ell) \Pr \{ \overline{X}_{N_\ell} \geq \mu + \rho(N_\ell) \} \] (18)

provided that \( \mu \) is less than the infimum of \( \mathcal{G}_{N_\tau} \); and similarly,

\[
E[N] \leq N_\tau + \sum_{\ell \geq \tau \in \mathcal{L}} (N_{\ell+1} - N_\ell) \Pr \{ \overline{X}_{N_\ell} \leq \mu - \rho(N_\ell) \} \] (19)

provided that \( \mu \) is greater than the supremum of \( \mathcal{G}_{N_\tau} \).

See Appendix R for a proof.

It should be noted that for all \( n \in \mathbb{N} \), \( \mathcal{G}_n \) is convex due to the convexity of \( \mathcal{R} \). Consequently, \( \rho(N_\ell) \) can be readily obtained by convex minimization. Making use of the concept of supporting hyperplane, we have the following results.

**Theorem 21** Suppose that assumptions (I)–(VI) are fulfilled. Assume that there exists a supporting hyperplane \( A_\mu + Bt = C \) of \( \mathcal{R} \) passing through \( (m, m\mu) \in \partial \mathcal{R} \). Then,

\[
E[N] \leq N_\tau + \sum_{\ell \geq \tau \in \mathcal{L}} (N_{\ell+1} - N_\ell) \sum_{k=1}^d \Pr \left\{ \left| \overline{X}_{N_\ell}^k - \mu_k \right| \geq \frac{\gamma_\ell}{\sqrt{d}} \right\},
\] (20)

where \( \gamma_\ell = \left( 1 - \frac{m}{N_\ell} \right) \frac{|A_\mu + B|}{\sqrt{AA}} \). Moreover, if \( X \) is a scalar random variable, then,

\[
E[N] \leq N_\tau + \sum_{\ell \geq \tau \in \mathcal{L}} (N_{\ell+1} - N_\ell) \Pr \left\{ \overline{X}_{N_\ell} \geq \mu + \left( 1 - \frac{m}{N_\ell} \right) \frac{B}{A} \right\} \] (21)

provided that \( \mu \) is less than the infimum of \( \mathcal{G}_{N_\tau} \); and similarly,

\[
E[N] \leq N_\tau + \sum_{\ell \geq \tau \in \mathcal{L}} (N_{\ell+1} - N_\ell) \Pr \left\{ \overline{X}_{N_\ell} \leq \mu - \left( 1 - \frac{m}{N_\ell} \right) \frac{B}{A} \right\} \] (22)

provided that \( \mu \) is greater than the supremum of \( \mathcal{G}_{N_\tau} \).

See Appendix S for a proof.

It should be noted that the probabilistic terms in Theorems 20 and 21 can be bounded by concentration inequalities such as Chernoff bounds and Hoeffding inequalities.\[^{16}][7, 9\]
7 Bounds for Average Stopping Times of Brownian Motion

In the last few sections, our techniques for bounding stopping times are devoted to discrete-time stochastic processes. Actually, the principle of such techniques can be extended to continuous-time stochastic processes. To demonstrate this idea, we shall focus on the problem of bounding stopping times pertaining to Brownian motion \[14, 20\].

Let \( W_t \in \mathbb{R}^d \) be a \( d \)-dimensional Brownian motion with mean drift vector \( \mu \) such that \( W_0 = 0_d \) and \( \mathbb{E}[W_t] = t \mu \) for \( t \geq 0 \). Define
\[ \nu = \frac{1}{t} \mathbb{E}[|W_t - t \mu|^2], \quad \overline{W}_t = \frac{W_t}{t}, \quad V_t = (W_t - t \mu)^2, \quad \overline{V}_t = \frac{V_t}{t} \]
for \( t > 0 \). Making use of Theorem 3, we have obtained the following results.

**Theorem 22** Assume that \( T \) is random variable such that \( \mathbb{E}[T] < \infty \) and that for any possible value \( t \) of \( T \), the event \( \{ T = t \} \) depends only on \( \{ W_\tau : 0 \leq \tau \leq t \} \). The following assertions hold.

(I): If \( g \) is a convex function on \( \mathbb{R}^d \), then
\[ \mathbb{E}[Tg(\overline{W}_T)] \geq \mathbb{E}[T]g(\mu). \]

(II): If \( g \) is a convex function of vectors with non-negative elements, then
\[ \mathbb{E}[Tg(V_T)] \geq \mathbb{E}[T]g(\nu). \]

The proof of Theorem 22 is similar to that of Theorem 1 which is given in Appendix D.

Making use of the convexity of the \( L^p \)-norm and Theorem 22 we have the following results.

**Theorem 23** Assume that \( T \) is random variable such that \( \mathbb{E}[T] < \infty \) and that for any possible value \( t \) of \( T \), the event \( \{ T = t \} \) depends only on \( \{ W_\tau : 0 \leq \tau \leq t \} \). Then,
\[ \mathbb{E}[||W_T||_p] \geq \mathbb{E}[T] ||\mu||_p, \]
\[ \mathbb{E}[||V_T||_p] \geq \mathbb{E}[T] ||\nu||_p \]
for all \( p \geq 1 \).

Now, consider stopping time \( T = \inf\{ t > 0 : (t, W_t) \notin \mathcal{R} \} \), where \( \mathcal{R} \) is called the continuity region. We have the following result.

**Theorem 24** Assume that \( \mathcal{R} \) is a convex set containing \((0, 0_d)\) and that there exists a unique positive number \( \tau \) such that \((\tau, \tau \mu) \in \partial \mathcal{R} \). Then, \( \mathbb{E}[T] \leq \tau \).

See Appendix 1 for a proof.

Next, consider stopping time \( T = \inf\{ t > 0 : (t, W_t) \in \mathcal{S} \} \), where \( \mathcal{S} \) is called the stopping region. We have obtained the following results.
Theorem 25  Suppose that the stopping region $\mathcal{R}$ is a convex set. Define $\mathcal{A} = \{ t \in \mathbb{R}^+ : (t, t\mu) \in \mathcal{R} \}$. The following assertions hold.

(I): $\mathbb{E}[T] \geq \inf \mathcal{A}$ provided that the set $\mathcal{A}$ is nonempty.

(II): $\mathbb{E}[T] \leq \sup \mathcal{A}$ provided that $\mathbb{E}[T] < \infty$ and the set $\mathcal{A}$ is nonempty.

(III): $\mathbb{E}[T] = \infty$ provided that the set $\mathcal{A}$ is empty.

See Appendix U for a proof.

Consider stopping times defined in terms of $W_t$. For stopping time $T = \inf \{ t > 0 : t \geq g(W_t) \}$, we have the following result.

Theorem 26  Assume that $g$ is a concave function on $\mathbb{R}^d$ with $g(\mu) > 0$. Then, $\mathbb{E}[T] \leq g(\mu)$.

See Appendix V for a proof.

For stopping time $T = \inf \left\{ t > 0 : t \geq \frac{1}{g(W_t)}, g(W_t) > 0 \right\}$, we have derived the following result.

Theorem 27  Assume that $g$ is a concave function on $\mathbb{R}^d$ with $g(\mu) > 0$. Then, $\mathbb{E}[T] \geq \frac{1}{g(\mu)}$.

See Appendix W for a proof.

8 Conclusion

In this paper, we have established a geometric approach for bounding average stopping times. The central idea of our approach is to explore the geometric convexity of the continuity or stopping regions. Our approach are effective for a wide variety of stopping times which involve random vectors, nonlinear boundary, constraint of sample number, etc. Tight bounds are obtained for stopping times in a general setting, which are explicit or readily computable. A probabilistic characterization is established for convex sets. Extensions are developed for classical results such as Jensen’s inequality, Wald’s equations and Lorden’s inequality.

A Proof of Theorem I

We need some preliminary results. If $X$ is a random variable such that $\Pr\{X < c\} = 1$, then it is clear that $\mathbb{E}[X] \leq c$. However, it is not so obvious that the inequality is strict. Since such strict inequality plays a crucial role in our proof of the theorem, we state it and provide a rigorous proof in the sequel.

Lemma 1  If $X$ is a random variable such that $\Pr\{X < c\} = 1$, then $\mathbb{E}[X] < c$. Similarly, if $X$ is a random variable such that $\Pr\{X > c\} = 1$, then $\mathbb{E}[X] > c$. 

18
Proof. We claim that there exists a positive number $\varepsilon > 0$ such that $\Pr\{X \leq c - \varepsilon\} > 0$. To prove the claim, we use a contradiction method. Suppose that the claim is not true. Then, $\Pr\{X \leq c - \varepsilon\} = 0$ for any $\varepsilon > 0$. It follows that

$$\Pr\{X < c\} = \lim_{\varepsilon \downarrow 0} \Pr\{X \leq c - \varepsilon\} = 0.$$  

This contradicts to the assumption that $\Pr\{X < c\} = 1$. So, we have proved the claim.

Now let $\varepsilon > 0$ be a positive number such that $\Pr\{X \leq c - \varepsilon\} > 0$. Since $\Pr\{X < c\} = 1$, we have

$$\mathbb{E}[X] = \mathbb{E}\left[X \mathbb{1}_{X \leq c - \varepsilon}\right] + \mathbb{E}\left[X \mathbb{1}_{c - \varepsilon < X < c}\right]$$

$$\leq (c - \varepsilon) \Pr\{X \leq c - \varepsilon\} + c \Pr\{c - \varepsilon < X < c\}$$

$$= (c - \varepsilon) \Pr\{X \leq c - \varepsilon\} + c(1 - \Pr\{X \leq c - \varepsilon\})$$

$$= -\varepsilon \Pr\{X \leq c - \varepsilon\} + c < c.$$  

This proves the first assertion. The second assertion can be shown in a similar way. 

\[\square\]

Lemma 2 Assume that $D$ is a closed convex set and $\mathbf{X}$ is a random vector such that $\Pr\{\mathbf{X} \in D\} = 1$, then $\mathbb{E}[\mathbf{X}] \in D$.

Proof. We shall use a contradiction method. Denote $\mu = \mathbb{E}[\mathbf{X}]$. Suppose $\mu \notin D$, i.e., $\mu$ is an exterior point of $D$. By the separating hyperplane theorem [3, Theorem 4.11, page 170], there exists a column vector $\alpha$ such that $\mu \alpha < Z \alpha$ for all $Z \in D$. Since $\Pr\{\mathbf{X} \in D\} = 1$, it must be true that $\Pr\{\mu \alpha < \mathbf{X} \alpha\} = 1$. From Lemma 1 we have

$$\mathbb{E}[\mu \alpha - \mathbf{X} \alpha] < 0,$$

which implies that

$$\mu \alpha < \mathbb{E}[\mathbf{X} \alpha] = \mathbb{E}[\mathbf{X}] \alpha = \mu \alpha.$$  

This is a contradiction. The proof of the lemma is thus completed.  

\[\square\]

Lemma 3 If $X$ is a random variable such that $0 < \Pr\{X < 0\} \leq \Pr\{X \leq 0\} = 1$. Then, $\mathbb{E}[X] < 0$. 

19
**Proof.** We claim that there exists a positive number $\varepsilon > 0$ such that $\Pr\{X \leq -\varepsilon\} > 0$. To prove the claim, we use a contradiction method. Suppose that the claim is not true. Then, $\Pr\{X \leq -\varepsilon\} = 0$ for any $\varepsilon > 0$. It follows that

$$
\Pr\{X < 0\} = \lim_{\varepsilon \to 0} \Pr\{X \leq -\varepsilon\} = 0.
$$

This contradicts to the assumption that $\Pr\{X < 0\} > 0$. So, we have proved the claim.

Now let $\varepsilon > 0$ be a positive number such that $\Pr\{X \leq -\varepsilon\} > 0$. Since $\Pr\{X \leq 0\} = 1$, we have

$$
E[X] = E\left[X I_{\{X \leq -\varepsilon\}}\right] + E\left[X I_{\{-\varepsilon < X \leq 0\}}\right] \\
\leq -\varepsilon \Pr\{X \leq -\varepsilon\} + 0 \times \Pr\{-\varepsilon < X \leq 0\} \\
\leq -\varepsilon \Pr\{X \leq -\varepsilon\} < 0.
$$

This completes the proof of the lemma.

\[\square\]

**Lemma 4** Assume that $D$ is a closed convex set and $\bm{X}$ is a random vector such that $\Pr\{\bm{X} \in D\} = 1$ and $\mu = E[\bm{X}] \in \partial D$, then there exist a column vector $\alpha \neq 0$ and a constant $\beta$ such that $\Pr\{\bm{X}\alpha + \beta = 0\} = 1$.

**Proof.** As a consequence of the convexity of $D$ and the assumption that $\mu = E[\bm{X}] \in \partial D$, it is possible to construct a supporting hyperplane $Z\alpha + \beta = 0$ through $\mu$, where $\alpha \neq 0$ is a column vector and $\beta$ is a constant, such that $Z\alpha + \beta \leq 0$ for all $Z \in D$. By the assumption that $\Pr\{\bm{X} \in D\} = 1$, we have

$$
\Pr\{\bm{X}\alpha + \beta \leq 0\} = 1.
$$

Since $\mu$ is in the supporting hyperplane, we have $E[\bm{X}]\alpha + \beta = 0$. We claim that $\Pr\{\bm{X}\alpha + \beta = 0\} = 1$. To prove this claim, we use a contradiction method. Suppose the claim is not true. Then,

$$
0 < \Pr\{\bm{X}\alpha + \beta < 0\} \leq \Pr\{\bm{X}\alpha + \beta \leq 0\} = 1.
$$

It follows from Lemma 3 that

$$
E[\bm{X}\alpha + \beta] < 0.
$$

This implies that

$$
E[\bm{X}]\alpha + \beta = E[\bm{X}\alpha + \beta] < 0,
$$

which contradicts to the fact that $E[\bm{X}]\alpha + \beta = 0$. The claim is thus established. Hence, it must be true that $\Pr\{\bm{X}\alpha + \beta = 0\} = 1$. This completes the proof of the lemma.
We are now in a position to prove the theorem. The second assertion follows immediately from the notion of convex set and mathematical expectation. So, we only need to show the first assertion. Specifically, we need to show that if $\mathcal{D}$ is a convex set in $\mathbb{R}^n$, then $\mathbb{E}[\mathbf{X}] \in \mathcal{D}$ holds for any random vector $\mathbf{X}$ such that $\Pr\{\mathbf{X} \in \mathcal{D}\} = 1$ and that $\mathbb{E}[\mathbf{X}]$ exists. We shall argue by a mathematical induction on the dimension $n$ of $\mathcal{D}$. For the dimension $n = 1$, the convex set $\mathcal{D}$ must be an interval of the form $\mathcal{D} = [a, b]$, or $\mathcal{D} = (a, b)$, or $\mathcal{D} = [a, b)$. Making use of Lemma 1, it is easy to see $\mathbb{E}[\mathbf{X}] \in \mathcal{D}$ as a consequence of $\Pr\{\mathbf{X} \in \mathcal{D}\} = 1$. Suppose the conclusion $\mathbb{E}[\mathbf{X}] \in \mathcal{D}$ holds for dimension $n - 1$. To complete the induction process, we need to show, based on such hypothesis, that the conclusion $\mathbb{E}[\mathbf{X}] \in \mathcal{D}$ holds for dimension $n$. Let $\mathcal{S}$ denotes the closure of $\mathcal{D}$. By Lemma 2, we have shown $\mathbb{E}[\mathbf{X}] \in \mathcal{S}$. If $\mu = \mathbb{E}[\mathbf{X}]$ is not contained in the boundary of $\mathcal{S}$, then it must be true that $\mu \in \mathcal{D}$. Hence, to show $\mathbb{E}[\mathbf{X}] \in \mathcal{D}$ for dimension $n$, it suffices to show it under the assumption that $\mu = \mathbb{E}[\mathbf{X}]$ is contained in the boundary of $\mathcal{S}$. We proceed as follows. Making use of Lemma 4 and the assumption that $\mu = \mathbb{E}[\mathbf{X}]$ is contained in the boundary of $\mathcal{S}$, we conclude that there exist a column vector $\alpha \neq 0$ and a constant $\beta$ such that $\Pr\{\mathbf{X} \alpha + \beta = 0\} = 1$. Define

$$\mathcal{J} = \mathcal{D} \cap \{Z \in \mathbb{R}^n : Z\alpha + \beta = 0\}.$$ 

Then, $\mathcal{J}$ is convex and

$$\Pr\{\mathbf{X} \in \mathcal{J}\} = 1, \quad \Pr\{\mathbf{X} \alpha + \beta = 0\} = 1.$$ 

Without loss of any generality, assume that the $i$-th element of $\alpha$, denoted by $\alpha_i$, is nonzero. Define a linear transform $\mathcal{T} : \mathcal{J} \mapsto D$ such that for every element $Z = [z_1, \ldots, z_n]$ in $\mathcal{J}$, there exists a corresponding vector $U = [u_1, \ldots, u_n] = \mathcal{T}(Z)$ such that

$$u_i = Z\alpha + \beta, \quad u_\ell = z_\ell, \quad \ell \in \{1, \ldots, n\} \setminus \{i\}$$

or equivalently,

$$U = Z(I + \alpha e_i - e_i^T e_i) + \beta e_i,$$ 

(23)

where $I$ is an identity matrix of size $n \times n$ and $e_i$ is a row matrix with all elements being 0 except the $i$-th element being 1. Note that $D = \{\mathcal{T}(Z) : Z \in \mathcal{J}\}$ must be convex because the transform $\mathcal{T}$ is linear and $\mathcal{J}$ is convex. Define $\mathbf{Y} = [y_1, \ldots, y_n] = \mathcal{T}(\mathbf{X})$. Then,

$$\Pr\{\mathbf{Y} \in D\} = 1, \quad \Pr\{y_i = 0\} = \Pr\{\mathbf{X} \alpha + \beta = 0\} = 1$$

and $\mathbb{E}[y_i] = 0$. Define

$$D^* = \{[u_1, \ldots, u_i, u_{i+1}, \ldots, u_n] : [u_1, \ldots, u_n] \in D\}.$$
Then, \( D^* \) is convex because \( D \) is convex. Define random vector \( V = [v_1, \cdots, v_{n-1}] \) such that \( v_\ell = y_\ell, \ \ell = 1, \cdots, i - 1 \) and \( v_\ell = y_{\ell+1}, \ \ell = i, \cdots, n - 1 \). Then, \( \Pr\{V \in D^*\} = 1 \). Since \( D^* \) is a convex set of \((n - 1)\) dimension and \( \Pr\{V \in D^*\} = 1 \), it follows from the induction hypothesis that \( \mathbb{E}[V] \in D^* \). This implies that \( \mathbb{E}[Y] \in D \).

It can be checked that the determinant of the matrix \( I + \alpha e_i - e_i^\top e_i \) in (23) is equal to \( \alpha_i \), which is nonzero. Hence, \( I + \alpha e_i - e_i^\top e_i \) is invertible, and it follows that
\[
Z = (U - \beta e_i)(I + \alpha e_i - e_i^\top e_i)^{-1}.
\]

This implies that the transform \( \mathcal{T} \) is a one-to-one mapping from \( \mathcal{S} \) to \( D \) and thus the transform is invertible. Note that \( \mathbb{E}[Y] = \mathcal{T}(\mathbb{E}[X]) \) and the transform \( \mathcal{T} \) maps \( \mathcal{S} \) into \( D \). Now, we have \( \mathbb{E}[Y] \in D \). Taking the inverse transform of \( \mathcal{T} \) yields
\[
\mathbb{E}[X] \in \mathcal{S} \subseteq D.
\]

This completes the process of induction and the theorem is thus established.

**B Proof of Theorem [2]**

To show the first assertion, it suffices to show that for any \((t_\ell, s_\ell), \ \ell = 1, \cdots, k \) such that \( t_\ell > 0 \), \( \frac{s_\ell}{t_\ell} \in D \) and positive numbers \( \lambda_\ell, \ \ell = 1, \cdots, k \) such that \( \sum_{\ell=1}^k \lambda_\ell = 1 \),
\[
f \left( \sum_{\ell=1}^k \lambda_\ell t_\ell, \sum_{\ell=1}^k \lambda_\ell s_\ell \right) \leq \sum_{\ell=1}^k \lambda_\ell f(t_\ell, s_\ell).
\]

Define
\[
A = \sum_{\ell=1}^k \lambda_\ell t_\ell, \quad \rho_\ell = \frac{\lambda_\ell t_\ell}{A}, \quad \ell = 1, \cdots, k.
\]

Since \( \rho_\ell, \ \ell = 1, \cdots, k \) are positive numbers satisfying \( \sum_{\ell=1}^k \rho_\ell = 1 \) and the function \( g \) is convex, we have
\[
\sum_{\ell=1}^k \rho_\ell g \left( \frac{s_\ell}{t_\ell} \right) \geq g \left( \sum_{\ell=1}^k \rho_\ell \frac{s_\ell}{t_\ell} \right) = g \left( \sum_{\ell=1}^k \lambda_\ell t_\ell \frac{s_\ell}{A} \right) = g \left( \frac{\sum_{\ell=1}^k \lambda_\ell s_\ell}{A} \right).
\]
It follows that

\[ \sum_{\ell=1}^{k} \lambda_{\ell} f(t_{\ell}, s_{\ell}) = \sum_{\ell=1}^{k} \lambda_{\ell} t_{\ell} g \left( \frac{s_{\ell}}{t_{\ell}} \right) = A \sum_{\ell=1}^{k} \rho_{\ell} g \left( \frac{s_{\ell}}{t_{\ell}} \right) \geq Ag \left( \frac{\sum_{\ell=1}^{k} \lambda_{\ell} s_{\ell}}{A} \right) = f \left( \sum_{\ell=1}^{k} \lambda_{\ell} t_{\ell}, \sum_{\ell=1}^{k} \lambda_{\ell} s_{\ell} \right). \]

This proves the first assertion. The second assertion can be shown in a similar way.

### C Proof of Theorem 3

We shall only show the first assertion, since the second assertion can be shown in a similar way. Define \( f(t, s) = tg \left( \frac{s}{t} \right) \). Since \( g(z) \) is a convex function of \( z \in \mathcal{D} \), it follows from Theorem 2 that \( f(t, s) \) is a convex function of \( t > 0 \) and vector \( s \) such that \( \frac{s}{t} \in \mathcal{D} \). Hence, there exist a column vector \( \alpha \) and number \( \beta \) such that

\[ f(t, s) \geq f(E[Y], E[Z]) + \left( s - E[Z] \right) \alpha + \beta \left( t - E[Y] \right) \]

for \( t > 0 \) and vector \( s \) such that \( \frac{s}{t} \in \mathcal{D} \). As a consequence of this result and the assumption that \( Y > 0 \), \( \frac{Z}{Y} \in \mathcal{D} \), \( \frac{E[Z]}{E[Y]} \in \mathcal{D} \), we have

\[ f(Y, Z) \geq f(E[Y], E[Z]) + (Z - E[Z])\alpha + \beta(Y - E[Y]). \]

Applying the definition of the function \( f \) to the above inequality yields

\[ Yg \left( \frac{Z}{Y} \right) \geq E[Y]g \left( \frac{E[Z]}{E[Y]} \right) + (Z - E[Z])\alpha + \beta(Y - E[Y]). \]

Taking expectations on both sides leads to

\[ E \left[ Yg \left( \frac{Z}{Y} \right) \right] \geq E[Y]g \left( \frac{E[Z]}{E[Y]} \right) + E[Z] - E[Z]\alpha + \beta E[Y - E[Y]] = E[Y]g \left( \frac{E[Z]}{E[Y]} \right). \]
D Proof of Theorem 4

To show the first assertion, we can use the first assertion of Theorem 3 to conclude that

\[
\mathbb{E} \left[ N g(X_N) \right] = \mathbb{E} \left[ N g \left( \frac{1}{N} \sum_{i=1}^{N} X_i \right) \right] \\
\geq \mathbb{E}[N] g \left( \frac{1}{\mathbb{E}[N]} \mathbb{E} \left[ \sum_{i=1}^{N} X_i \right] \right).
\]

By virtue of Wald’s first equation, we have

\[
\mathbb{E} \left[ \sum_{i=1}^{N} X_i \right] = \mathbb{E}[N] \mu.
\]

Hence,

\[
\mathbb{E} \left[ N g(X_N) \right] \geq \mathbb{E}[N] g \left( \frac{1}{\mathbb{E}[N]} \mathbb{E}[N] \mu \right) = \mathbb{E}[N] g(\mu).
\]

To show the second assertion, we can use the first assertion of Theorem 3 to conclude that

\[
\mathbb{E} \left[ N g(V_N) \right] = \mathbb{E} \left[ N g \left( \frac{1}{\mathbb{E}[N]} \mathbb{E} \left[ \sum_{i=1}^{N} X_i - N \mu \right] \right) \right] \\
\geq \mathbb{E}[N] g \left( \frac{1}{\mathbb{E}[N]} \mathbb{E} \left[ \left( \sum_{i=1}^{N} X_i - N \mu \right)^2 \right] \right).\]

By virtue of Wald’s second equation, we have

\[
\mathbb{E} \left[ \left( \sum_{i=1}^{N} X_i - N \mu \right)^2 \right] = \mathbb{E}[N] \nu.
\]

Hence,

\[
\mathbb{E} \left[ N g(V_N) \right] \geq \mathbb{E}[N] g \left( \frac{1}{\mathbb{E}[N]} \mathbb{E}[N] \nu \right) = \mathbb{E}[N] g(\nu).
\]

E Proof of Theorem 7

Define \( \xi = \lambda - Z_1 \). Let \( F_{\xi}(\cdot) \) denotes the cumulative distribution of \( \xi \). Note that

\[
\mathbb{E} \left[ \left( \sum_{i=2}^{N} Z_i - (\lambda - Z_1) \right) \mathbb{I}_{\{Z_1 < \lambda\}} \right] = \mathbb{E} \left[ \left( \sum_{i=2}^{N} Z_i - \xi \right) \mathbb{I}_{\{\xi > 0\}} \right] \\
= \int_{u>0} \mathbb{E} \left[ \left( \sum_{i=2}^{N} Z_i - \xi \right) \mathbb{I}_{\{\xi = u\}} \right] dF_{\xi}(u). \tag{24}
\]
By the definition of $\mathcal{M}_\lambda$, we have

$$
\mathbb{E}\left[ \left( \sum_{i=2}^{\mathcal{M}_\lambda} Z_i - \xi \right) \mid \xi = u \right] = \mathbb{E}\left[ \left( \sum_{i=2}^{\mathcal{M}_u} Z_i - u \right) \mid \xi = u \right],
$$

where

$$
\mathcal{M}_u = \inf \left\{ n \geq 2 : \sum_{i=2}^{n} Z_i \geq u \right\}.
$$

Since the samples $Z_1, Z_2, \cdots$ and $\lambda$ are independent, it follows that $\xi$ and $Z_2, Z_3, \cdots$ are independent. It follows that

$$
\mathbb{E}\left[ \left( \sum_{i=2}^{\mathcal{M}_u} Z_i - u \right) \mid \xi = u \right] = \mathbb{E}\left[ \left( \sum_{i=2}^{\mathcal{M}_u} Z_i - u \right) \right]
$$

for all $u > 0$. Define

$$
\mathcal{M}_u = \inf \left\{ n \in \mathbb{N} : \sum_{i=1}^{n} Z_i \geq u \right\}
$$

for $u > 0$. Since $Z_1, Z_2, \cdots$ are i.i.d. samples of $Z$, it must be true that $\sum_{i=1}^{\mathcal{M}_u} Z_i$ and $\sum_{i=1}^{\mathcal{M}_u} Z_i$ have the same distribution for all $u > 0$. Hence,

$$
\mathbb{E}\left[ \left( \sum_{i=2}^{\mathcal{M}_u} Z_i - u \right) \right] = \mathbb{E}\left[ \left( \sum_{i=1}^{\mathcal{M}_u} Z_i - u \right) \right]
$$

for all $u > 0$. Combining (24) – (27) yields

$$
\mathbb{E}\left[ \left( \sum_{i=2}^{\mathcal{M}_{\lambda}} Z_i - (\lambda - Z_1) \right) 1_{\{Z_1 < \lambda\}} \right] = \int_{u>0} \mathbb{E}\left[ \left( \sum_{i=1}^{\mathcal{M}_u} Z_i - u \right) \right] dF_\xi(u).
$$

By Lorden's inequality [17], we have

$$
\mathbb{E}\left[ \left( \sum_{i=1}^{\mathcal{M}_u} Z_i - u \right) \right] \leq \frac{\mathbb{E}[(Z^+)^2]}{\mathbb{E}[Z]}
$$

for all $u > 0$. Making use of (28) and (29), we have

$$
\mathbb{E}\left[ \left( \sum_{i=2}^{\mathcal{M}_{\lambda}} Z_i - (\lambda - Z_1) \right) 1_{\{Z_1 < \lambda\}} \right] \leq \int_{u>0} \frac{\mathbb{E}[(Z^+)^2]}{\mathbb{E}[Z]} dF_\xi(u)
$$

$$
= \frac{\mathbb{E}[(Z^+)^2]}{\mathbb{E}[Z]} \int_{u>0} dF_\xi(u)
$$

$$
= \frac{\mathbb{E}[(Z^+)^2]}{\mathbb{E}[Z]} \Pr\{\xi > 0\}
$$

$$
= \frac{\mathbb{E}[(Z^+)^2]}{\mathbb{E}[Z]} \Pr\{\lambda - Z_1 > 0\}
$$

$$
= \frac{\mathbb{E}[(Z^+)^2]}{\mathbb{E}[Z]} \Pr\{Z < \lambda\}.
$$

(30)
On the other hand,
\[ E[R_\lambda 1\{Z_1 \geq \lambda\}] = E[(Z_1 - \lambda)^+] = E[(Z - \lambda)^+]. \]  
(31)

Combining (30) and (31) yields
\[ E[R_\lambda] = E[R_\lambda 1\{Z_1 < \lambda\}] + E[R_\lambda 1\{Z_1 \geq \lambda\}] \leq \frac{E[(Z^+)^2]}{E[Z]} \Pr\{Z < \lambda\} + E[(Z - \lambda)^+]. \]

This completes the proof of the theorem.

F Proof of Theorem 8

We need some preliminary result.

Lemma 5 Let \(X_1, X_2, \cdots\) be i.i.d. samples of positive random variable \(X\) such that \(E[X^2] < \infty\). Define \(S_n = \sum_{i=1}^{n} X_i\) for \(n \in \mathbb{N}\). Let \(N_1, N_2, \cdots\) be an increasing sequence of positive integers. Define \(h = \sup_{t \geq 0} (N_{t+1} - N_t)\) with \(N_0 = 0\). Define \(N_t = \inf\{n \in \mathcal{N} : S_n \geq t\}\) for \(t > 0\), where \(\mathcal{N} = \{N_1, N_2, \cdots\}\). Define \(R_t = S_{N_t} - t\). Then, \(E[R_t] \leq (h - 1)E[X] + \frac{E[X^2]}{E[X]}\) for any \(t > 0\).

Proof. Let \(t > 0\). Define \(M_t\) as the largest integer which is less than \(N_t\) and taking value in the set \(\{N_0, N_1, N_2, \cdots\}\). Define \(\mathcal{N}_t = \inf\{n \in \mathbb{N} : S_n \geq t\}\).

We claim that \(S_{h-1+N_t} \geq S_{N_t}\).

To show this claim, note that \(S_k\) is increasing with respect to \(k \in \mathbb{N}\) as a consequence of \(X > 0\). Since \(S_{M_t} < t \leq S_{N_t}\), we have \(M_t \leq \mathcal{N}_t - 1\).

By the definition of \(h\), we have \(S_{N_t} \leq S_{h+M_t} \leq S_{h-1+N_t}\).

The claim is thus true. It follows that \(E[S_{N_t}] \leq E[S_{h-1+N_t}]\).

Since \(h - 1 + \mathcal{N}_t\) is a stopping time, by Wald’s first equation, we have \(E[S_{h-1+N_t}] = (E[\mathcal{N}_t] + h - 1)E[X]\).
Therefore,
\[
E[S_{N_t} - t] \leq E[S_{h + N_t} - t] \\
= E[S_{h + N_t} - S_{N_t}] + E[S_{N_t} - t] \\
= E[S_{h + N_t}] - E[S_{N_t}] + E[S_{N_t} - t] \\
= (E[N_t] + h - 1)E[X] - E[N_t]E[X] + E[S_{N_t} - t] \\
= (h - 1)E[X] + E[S_{N_t} - t].
\]

By Lorden’s inequality
\[
E[S_{N_t} - t] \leq \frac{E[X^2]}{E[X]},
\]
Hence,
\[
E[R_t] = E[S_{N_t} - t] \leq (h - 1)E[X] + \frac{E[X^2]}{E[X]}.
\]
This completes the proof of the lemma.

\[\square\]

We are now in a position to prove the theorem. Define \( \xi = \lambda - Y \). Let \( F_\xi(.) \) denotes the cumulative distribution of \( \xi \). Note that
\[
E \left[ \left( \sum_{i=N_1+1}^{\mathcal{M}_\lambda} Z_i - (\lambda - Y) \right) \mathbb{I}_{(Y<\lambda)} \right] = E \left[ \left( \sum_{i=N_1+1}^{\mathcal{M}_\lambda} Z_i - \xi \right) \mathbb{I}_{(\xi>0)} \right] \\
= \int_{u>0} E \left[ \left( \sum_{i=N_1+1}^{\mathcal{M}_u} Z_i - \xi \right) \mid \xi = u \right] dF_\xi(u). \tag{32}
\]
By the definition of \( \mathcal{M}_\lambda \), we have
\[
E \left[ \left( \sum_{i=N_1+1}^{\mathcal{M}_\lambda} Z_i - \xi \right) \mid \xi = u \right] = E \left[ \left( \sum_{i=N_1+1}^{\mathcal{M}_u} Z_i - u \right) \mid \xi = u \right], \tag{33}
\]
where
\[
\mathcal{M}_u = \inf \left\{ n \in \mathcal{N} \mid n \geq N_2, \sum_{i=N_1+1}^{n} Z_i \geq u \right\}.
\]
Since the samples \( Z_1, Z_2, \cdots \) and \( \lambda \) are independent, it follows that \( \xi \) and \( Z_2, Z_3, \cdots \) are independent. It follows that
\[
E \left[ \left( \sum_{i=N_1+1}^{\mathcal{M}_u} Z_i - u \right) \mid \xi = u \right] = E \left[ \left( \sum_{i=N_1+1}^{\mathcal{M}_u} Z_i - u \right) \right]. \tag{34}
\]
for all $u > 0$. Define
\[ M_u = \inf \left\{ n \in \mathbb{N} : \sum_{i=1}^{n} Z_i \ge u \right\} \]
for $u > 0$, where
\[ \mathbb{N} = \{ N_{\ell} - N_1 : \ell = 2, 3, \ldots \}. \]
Since $Z_1, Z_2, \ldots$ are i.i.d. samples of $Z$, it must be true that $\sum_{i=N_{1}+1}^{M_u} Z_i$ and $\sum_{i=1}^{\mathfrak{M}_u} Z_i$ have the same distribution for all $u > 0$. Hence,
\[ E \left[ \left( \sum_{i=N_{1}+1}^{M_u} Z_i - u \right) \right] = E \left[ \left( \sum_{i=1}^{\mathfrak{M}_u} Z_i - u \right) \right] \quad (35) \]
for all $u > 0$. Combining (32) – (35) yields
\[ E \left[ \left( \sum_{i=N_{1}+1}^{M_u} Z_i - (\lambda - Y) \right) I_{\{Z_1 < \lambda\}} \right] = \int_{u>0} E \left[ \left( \sum_{i=1}^{\mathfrak{M}_u} Z_i - u \right) \right] dF_\xi(u). \quad (36) \]
By Lemma 5, we have
\[ E \left[ \left( \sum_{i=1}^{\mathfrak{M}_u} Z_i - u \right) \right] \le (K - 1) E[Z] + \frac{E[(Z^+)^2]}{E[Z]} \quad (37) \]
for all $u > 0$. Making use of (36) and (37), we have
\begin{align*}
E \left[ \left( \sum_{i=N_{1}+1}^{M_u} Z_i - (\lambda - Y) \right) I_{\{Y < \lambda\}} \right] &\le \int_{u>0} \left( (K - 1) E[Z] + \frac{E[(Z^+)^2]}{E[Z]} \right) dF_\xi(u) \\
&= \left( (K - 1) E[Z] + \frac{E[(Z^+)^2]}{E[Z]} \right) \int_{u>0} dF_\xi(u) \\
&= \left( (K - 1) E[Z] + \frac{E[(Z^+)^2]}{E[Z]} \right) Pr\{\xi > 0\} \\
&= \left( (K - 1) E[Z] + \frac{E[(Z^+)^2]}{E[Z]} \right) Pr\{\lambda - Y > 0\} \\
&= \left( (K - 1) E[Z] + \frac{E[(Z^+)^2]}{E[Z]} \right) Pr\{Y < \lambda\}. \quad (38)
\end{align*}
On the other hand,
\[ E[R_\lambda I_{\{Y \ge \lambda\}}] = E[(Y - \lambda)^+]. \quad (39) \]
Combining (38) and (39) yields
\[ E[R_\lambda] = E[R_\lambda I_{\{Y < \lambda\}}] + E[R_\lambda I_{\{Y \ge \lambda\}}] \le \left( (K - 1) E[Z] + \frac{E[Z^2]}{E[Z]} \right) Pr\{Y < \lambda\} + E[(Y - \lambda)^+]. \]
This completes the proof of the theorem.

28
G Proof of Theorem 9

We shall first show $E[N] \geq \min \mathcal{A}$ under the assumption that $\mathcal{A}$ is not empty. If $E[N] = \infty$, then $E[N] \geq \min \mathcal{A}$ trivially holds. If $E[N] < \infty$, then $\Pr\{N < \infty\} = 1$ and it follows that $S_N$ is well-defined and

$$\Pr\{(N, S_N) \in \mathcal{R}\} = 1.$$  

According to Theorem 1, we have

$$(E[N], E[S_N]) \in \mathcal{R}.$$  

Since $E[N] < \infty$, it follows from Wald’s equation that $E[S_N] = E[N] \mu$. Hence,

$$(E[N], E[N] \mu) \in \mathcal{R},$$  

which immediately implies $E[N] \geq \min \mathcal{A}$.

It remains to show that $E[N] = \infty$ under the assumption that $\mathcal{A}$ is empty. We use a contradiction method. Suppose that $E[N] < \infty$, then $\Pr\{N < \infty\} = 1$ and it follows that

$$\Pr\{(N, S_N) \in \mathcal{R}\} = 1.$$  

According to Theorem 1, we have

$$(E[N], E[S_N]) \in \mathcal{R}.$$  

Since $E[N] < \infty$, it follows from Wald’s equation that $E[S_N] = E[N] \mu$. Hence,

$$(E[N], E[N] \mu) \in \mathcal{R},$$  

which immediately implies that $\mathcal{A}$ is not empty. This is a contradiction. Therefore, it must be true that $E[N] = \infty$ if $\mathcal{A}$ is empty. The proof of the theorem is thus completed.

H Proof of Theorem 10

We need some preliminary results.

Lemma 6 There exist $A, B$ and a positive number $C$ such that $As + Bn = C$ is a supporting hyperplane of $\mathcal{R}$, which passes through $(m, m\mu)$.  


Proof. Since $\mathcal{R}$ is a convex set containing $(0,0_d)$ and $(m,m\mu)$, it must be true that $\mathcal{R}$ contains $(\rho m, \rho m\mu)$ for all $\rho \in (0,1)$. Since $\mathcal{R}$ is a convex set, there exist $A, B$ and $C \geq 0$ such that $As + Bn = C$ is a supporting hyperplane of $\mathcal{R}$ such that $m(A\mu + B) = C$. We claim that $C > 0$. To prove this claim, we use a contradiction method. Suppose $C = 0$. Then, the supporting hyperplane must contain $(0,0_d)$. It follows that the supporting hyperplane contains $(\rho m, \rho m\mu)$ for all $\rho \in (0,1)$. Recall the assumption that there exists a unique positive number $t$ such that $(t,t\mu) \in \partial \mathcal{R}$. Hence, there exists $\rho \in (0,1)$ such that $(\rho m, \rho m\mu) \notin \partial \mathcal{R}$ and $(\rho m, \rho m\mu) \in \mathcal{R}$. This implies that the supporting hyperplane contains some interior point of $\mathcal{R}$. This contradicts to the definition of a supporting hyperplane. Thus, we have shown the claim that $C > 0$. This completes the proof of the lemma.

\[ \blacksquare \]

Lemma 7 There exist $\delta > 0$ and $\Upsilon > 0$ such that $\{N > n\} \subseteq \{|\|X_n - \mu||_2 > \delta\}$ for $n \in \mathcal{N}$ greater than $\Upsilon$.

Proof. According to Lemma 6, there exist $A, B$ and a positive number $C$ such that $As + Bn = C$ is a supporting hyperplane of $\mathcal{R}$, which passes through $(m,m\mu)$. Define

$$T = \inf\{n \in \mathcal{N} : As + Bn \geq C\}.$$ 

Since $C > 0$ and $(0,0_d)$ is contained by the convex set $\mathcal{R}$, it follows that $As + Bt \leq C$ for all $(t,s) \in \mathcal{R}$. This implies that $N \leq T$. Since $C > 0$, we have $A\mu + B = \frac{C}{m} > 0$. Since $A\theta + B$ is a continuous function of $\theta$, there exist $\delta > 0$ and $\Upsilon > 0$ such that

$$A\theta + B \geq \frac{C}{\Upsilon}$$

for all $\theta \in \{\theta \in \mathbb{R}^d : ||\theta - \mu||_2 \leq \delta\}$. Hence,

$$\{N > n\} \subseteq \{T > n\} \subseteq \{n(A\overline{X}_n + B) < C\} \subseteq \left\{A\overline{X}_n + B < \frac{C}{\Upsilon}\right\} \subseteq \{|\|X_n - \mu||_2 > \delta\}$$

for $n \in \mathcal{N}$ greater than $\Upsilon$. This completes the proof of the lemma.

\[ \blacksquare \]

Lemma 8 Let $Y_1, Y_2, \cdots$ be i.i.d. samples of scalar random variable $Y$ which has mean $\alpha = \mathbb{E}[Y]$, variance $\nu = \mathbb{E}[|Y - \alpha|^2]$ and finite $\mathbb{E}[|Y|^3]$. Let $\overline{Y}_n = \frac{1}{n} \sum_{i=1}^{n} Y_i$. Then,

$$\Pr\{|\overline{Y}_n - \alpha| \geq \gamma\} < \exp\left(-\frac{n \gamma^2}{2 \nu}\right) + \frac{2C \mathcal{W}}{n^2 \gamma^3} \text{ for any } \gamma > 0,$$  

(40)

where $\mathcal{W} = \mathbb{E}[|Y - \alpha|^3]$ and $C > 0$ is an absolute constant.
Proof. Note that
\[ \Pr\{Y_n \leq z\} = \Pr\left\{ \frac{\sqrt{n}(Y_n - \alpha)}{\sqrt{\nu}} \leq \frac{\sqrt{n}(z - \alpha)}{\sqrt{\nu}} \right\} \]
and
\[ \Pr\{Y_n \geq z\} = \Pr\left\{ \frac{\sqrt{n}(\alpha - Y_n)}{\sqrt{\nu}} \leq \frac{\sqrt{n}(\alpha - z)}{\sqrt{\nu}} \right\}. \]
Since \( \mathbb{E}[|Y|^3] \) is finite, \( \mathbb{W} \) must be finite. By the non-uniform version of Berry-Essen’s inequality [5, Page 44, Theorem 6.4],
\[ \left| \Pr\{Y_n \leq z\} - \Phi\left( \frac{\sqrt{n}(z - \alpha)}{\sqrt{\nu}} \right) \right| < \frac{C\mathbb{W}}{\sqrt{n\nu^3 + n^2|z - \alpha|^3}} < \frac{C\mathbb{W}}{n^2|z - \alpha|^3} \quad (41) \]
for \( z \in \mathbb{R} \). Making use of (41) and the fact that \( \Phi(x) < \frac{1}{2} \exp\left(-\frac{x^2}{2}\right) \) for any \( x < 0 \), we have
\[ \Pr\{Y_n \leq z\} < \frac{1}{2} \exp\left(-\frac{n}{2} \frac{|z - \alpha|^2}{\nu} \right) + \frac{C\mathbb{W}}{n^2|z - \alpha|^3} \quad \text{for } z \text{ less than } \alpha. \quad (42) \]
In a similar manner, we can show
\[ \Pr\{Y_n \geq z\} < \frac{1}{2} \exp\left(-\frac{n}{2} \frac{|z - \alpha|^2}{\nu} \right) + \frac{C\mathbb{W}}{n^2|z - \alpha|^3} \quad \text{for } z \text{ greater than } \alpha. \quad (43) \]
Finally, combining (42) and (43) yields (40). This completes the proof of the lemma.

We are now in a position to prove the theorem. For \( i = 1, \cdots, d \), let \( \mathbf{x}_i \) be the \( i \)-th component of \( \mathbf{X} \), i.e., \( \mathbf{X} = [\mathbf{x}_1, \cdots, \mathbf{x}_d] \). Let \( \mu_i = \mathbb{E}[\mathbf{x}_i] \) for \( i = 1, \cdots, d \). For \( i = 1, \cdots, d \), let \( \mathbf{x}_{ij}, j = 1, 2, \cdots \) be i.i.d. samples of \( \mathbf{x}_i \) such that \( \mathbf{X}_i = [\mathbf{x}_{i1}, \cdots, \mathbf{x}_{id}] \). Define \( \mathbf{\bar{x}}_n = \frac{1}{n} \sum_{i=1}^{n} \mathbf{x}_{ij} \) for \( i = 1, \cdots, d \). Then, \( \mathbf{\bar{X}}_n = [\mathbf{\bar{x}}_{n1}, \cdots, \mathbf{\bar{x}}_{nd}] \) and
\[ ||\mathbf{\bar{X}}_n - \mu||_2 = \sum_{i=1}^{d} ||\mathbf{\bar{x}}_n - \mu_i||_2. \]
By Lemma 7, we have that there exist \( \delta > 0 \) and \( \Upsilon > 0 \) such that \( \{N > n\} \subseteq \{||\mathbf{\bar{X}}_n - \mu||_2 > \delta\} \) for \( n \in \mathcal{N} \) greater than \( \Upsilon \). Hence,
\[ \Pr\{N > n\} \leq \Pr\{||\mathbf{\bar{X}}_n - \mu||_2 > \delta\} \]
\[ = \Pr\{||\mathbf{\bar{X}}_n - \mu||_2 > \delta^2\} \]
\[ = \Pr\left\{ \sum_{i=1}^{d} ||\mathbf{\bar{x}}_n - \mu_i||_2 > \delta^2 \right\} \]
\[ \leq \sum_{i=1}^{d} \Pr\left\{ ||\mathbf{\bar{x}}_n - \mu_i||_2 > \frac{\delta^2}{d} \right\} \]
\[ = \sum_{i=1}^{d} \Pr\left\{ ||\mathbf{\bar{x}}_n - \mu_i|| > \frac{\delta}{\sqrt{d}} \right\} \]
for $n \in \mathcal{N}$ greater than $\Upsilon$. By virtue of Lemma 8 we have that there exist constants $c_1 > 0$ and $c_2 > 0$ such that

$$\Pr \left\{ | \mathbf{x}_{in} - \mu_i | > \frac{\delta}{\sqrt{d}} \right\} \leq \exp(-nc_1) + \frac{c_2}{n^2}$$

for $i = 1, \cdots, d$. Hence,

$$\Pr \{ N > n \} \leq d \left[ \exp(-nc_1) + \frac{c_2}{n^2} \right]$$

(44)

for $n \in \mathcal{N}$ greater than $\Upsilon$. Let $\tau > 1$ be an integer such that $N_{\tau-1} \leq \Upsilon < N_\tau$. Note that

$$\mathbb{E}[N] = \sum_{n=0}^{N_{\tau}-1} \Pr\{ N > n \} + \sum_{n=N_\tau}^{\infty} \Pr\{ N > n \}$$

$$\leq N_\tau + \sum_{n=N_\tau}^{\infty} \Pr\{ N > n \}$$

$$= N_\tau + \sum_{\ell=\tau}^{\infty} \sum_{n=N_\ell}^{N_{\ell+1}-1} \Pr\{ N > n \}$$

$$\leq N_\tau + \sum_{\ell=\tau}^{\infty} (N_{\ell+1} - N_\ell) \Pr\{ N > N_\ell \}. \quad (45)$$

Combining (44) and (45) yields

$$\mathbb{E}[N] \leq N_\tau + \sum_{\ell=\tau}^{\infty} (N_{\ell+1} - N_\ell) \left[ \exp(-c_1 N_\ell) + \frac{c_2}{N_\ell^2} \right]$$

$$= N_\tau + \sum_{\ell=\tau}^{\infty} (N_{\ell+1} - N_\ell) \exp(-c_1 N_\ell) + c_2 \sum_{\ell=\tau}^{\infty} \frac{N_{\ell+1} - N_\ell}{N_\ell^2}. \quad (46)$$

If $\limsup_{\ell \to \infty} (N_{\ell+1} - N_\ell) < \infty$, then there exists a constant $B > 0$ such that $N_{\ell+1} - N_\ell < B$ for all $\ell \geq \tau$. It follows that

$$\mathbb{E}[N] \leq N_\tau + B \sum_{\ell=\tau}^{\infty} \exp(-c_1 N_\ell) + c_2 B \sum_{\ell=\tau}^{\infty} \frac{1}{N_\ell^2}$$

$$\leq N_\tau + B \sum_{n=1}^{\infty} \exp(-c_1 n) + c_2 B \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$= N_\tau + \frac{B}{\exp(c_1) - 1} + \frac{c_2 B \pi^2}{6} < \infty.$$

It remains to show the boundedness of $\mathbb{E}[N]$ in the case of $\liminf_{\ell \to \infty} \frac{N_{\ell+1}}{N_\ell} > 1$. Making use of (46) and the assumption that there exist numbers $\lambda > 0$ and $K$ such that $N_{\ell+1} \leq \lambda N_\ell + K$ for
all $\ell \geq 0$, we have

$$
\mathbb{E}[N] \leq N_r + \sum_{\ell=\tau}^{\infty} (\lambda N_\ell + K - N_\ell) \exp(-c_1 N_\ell) + c_2 \sum_{\ell=\tau}^{\infty} \frac{\lambda N_\ell + K - N_\ell}{N_\ell^2}
$$

$$
= N_r + K \sum_{\ell=\tau}^{\infty} \exp(-c_1 N_\ell) + c_2K \sum_{\ell=\tau}^{\infty} \frac{1}{N_\ell^2} + (\lambda - 1) \sum_{\ell=\tau}^{\infty} N_\ell \exp(-c_1 N_\ell) + (\lambda - 1) \sum_{\ell=\tau}^{\infty} \frac{1}{N_\ell}
$$

$$
\leq N_r + K \sum_{n=1}^{\infty} \exp(-c_1 n) + c_2K \sum_{n=1}^{\infty} \frac{1}{n^2} + |\lambda - 1| \sum_{n=1}^{\infty} n \exp(-c_1 n) + (\lambda - 1) \sum_{\ell=\tau}^{\infty} \frac{1}{N_\ell}
$$

$$
= N_r + \frac{K}{\exp(c_1) - 1} + \frac{c_2K \pi^2}{6} + |\lambda - 1| \sum_{n=1}^{\infty} n \exp(-c_1 n) + (\lambda - 1) \sum_{\ell=\tau}^{\infty} \frac{1}{N_\ell}.
$$

Note that

$$
\sum_{n=1}^{\infty} n \exp(-c_1 n) < \infty
$$

because

$$
\lim_{n \to \infty} \frac{(n + 1) \exp(-c_1 (n + 1))}{n \exp(-c_1 n)} = \exp(-c_1) < 1.
$$

As a consequence of $\lim \inf_{\ell \to \infty} \frac{N_{\ell+1}}{N_\ell} > 1$, we have

$$
\sum_{\ell=\tau}^{\infty} \frac{1}{N_\ell} < \infty.
$$

Therefore, it must be true that $\mathbb{E}[N] < \infty$ in the case of $\lim \inf_{\ell \to \infty} \frac{N_{\ell+1}}{N_\ell} > 1$. The proof of the theorem is thus completed.

### I Proof of Theorem [11]

Since assumptions (I) – (VI) are fulfilled, it follows from Theorem [10] that $\mathbb{E}[M] < \mathbb{E}[N] < \infty$, which implies $\Pr\{M < \infty\} = 1$ and $\Pr\{N < \infty\} = 1$. Hence, $N$ and $M$ are well-defined random variables. Define

$$
\Delta = S_N - S_M - (N - M)\mu.
$$

Our proof of the theorem relies on some properties of $\Delta$ as stated by the following lemma.

**Lemma 9**

$$
\mathbb{E}[\Delta^+] \leq \mathbb{E}[N - N_0] \mathbb{E}[(X - \mu)^+], \quad (47)
$$

$$
\mathbb{E}[\Delta^-] \leq \mathbb{E}[N - N_0] \mathbb{E}[(X - \mu)^-], \quad (48)
$$

$$
\mathbb{E}[|\Delta|] \leq \mathbb{E}[N - N_0] \mathbb{E}[|X - \mu|]. \quad (49)
$$
Proof. Define
\[ \Delta_\ell = S_{N_\ell} - S_{N_{\ell-1}} - (N_\ell - N_{\ell-1})\mu \]
for \( \ell = 1, 2, \cdots \). Let \( \tau \) denote the stopping index such that \( N_\tau = N \). Note that
\[
\mathbb{E}[\Delta^+] = \sum_{\ell=1}^{\infty} \mathbb{E}[\Delta_\ell^+ \mathbb{I}_{\{\tau = \ell\}}] 
= \sum_{\ell=1}^{\infty} \mathbb{E}[\Delta_\ell^+] 
\leq \mathbb{E}[\Delta_1^+] + \sum_{\ell=2}^{\infty} \mathbb{E}[\Delta_\ell^+] \mathbb{I}_{\{\tau = \ell-1\}} 
\leq \mathbb{E}[\Delta_1^+] \sum_{\ell=2}^{\infty} \mathbb{E}[\Delta_\ell^+] \mathbb{I}_{\{\tau > \ell-1\}}.
\]
Observing that \( \Delta_\ell \) depends only on \( \{X_n : N_{\ell-1} + 1 \leq n \leq N_\ell\} \) and that the event \( \{\tau > \ell-1\} \) depends only on \( \{X_n : 1 \leq n \leq N_{\ell-1}\} \), we have that
\[
\mathbb{E}[\Delta_\ell^+ \mathbb{I}_{\{\tau > \ell-1\}}] = \mathbb{E}[\Delta_\ell^+] \mathbb{I}_{\{\tau > \ell-1\}} = \mathbb{E}[\Delta_\ell^+] \mathbb{P}\{\tau > \ell-1\}
\]
for \( \ell > 1 \). It follows that
\[
\mathbb{E}[\Delta^+] \leq \mathbb{E}[\Delta_1^+] + \sum_{\ell=2}^{\infty} \mathbb{E}[\Delta_\ell^+] \mathbb{I}_{\{\tau > \ell-1\}} = \mathbb{E}[\Delta_1^+] + \sum_{\ell=2}^{\infty} \mathbb{E}[\Delta_\ell^+] \mathbb{P}\{\tau > \ell-1\}.
\]
As a consequence of the identical independence of \( X_1, X_2, \cdots \), we have
\[
\mathbb{E}[\Delta_\ell^+] \leq (N_\ell - N_{\ell-1})\mathbb{E}[(X - \mu)^+], \quad \ell = 1, 2, \cdots.
\]
Hence,
\[
\mathbb{E}[\Delta^+] \leq \left[ N_1 - N_0 + \sum_{\ell=1}^{\infty} (N_{\ell+1} - N_\ell) \mathbb{P}\{\tau > \ell\} \right] \mathbb{E}[(X - \mu)^+] 
= \mathbb{E}[N - N_0] \mathbb{E}[(X - \mu)^+].
\]
This proves (47). By similar arguments we can show the inequalities (48) and (49) regarding \( \mathbb{E}[\Delta^-] \) and \( \mathbb{E}[|\Delta|] \), respectively.

\[\square\]

Lemma 10
\[
\mathbb{E}[M] \alpha + \zeta \leq \mathbb{E}[S_M] \leq \mathbb{E}[M] \beta + \eta,
\]
34
where
\[ \alpha = \mu - \lambda \mathbb{E}[(X - \mu)^+] , \quad \zeta = (N_0 - K) \mathbb{E}[(X - \mu)^+] , \]
\[ \beta = \mu + \lambda \mathbb{E}[(X - \mu)^- ] , \quad \eta = (K - N_0) \mathbb{E}[(X - \mu)^- ] . \]

**Proof.** By the assumption that \( \mathbb{E}[|X|^3] \) is bounded, we have that \( \mathbb{E}[X^+] \) and \( \mathbb{E}[X^-] \) are bounded. By Theorem 10, \( N \) is a stopping time such that \( \mathbb{E}[N] < \infty \). Hence, it follows from Wald’s first equation that
\[
\mathbb{E}[(S_N)^+] \leq \sum_{i=1}^{N} (X_i)^+ = \mathbb{E}[N] \mathbb{E}[X^+] , \quad \mathbb{E}[(S_N)^-] \leq \sum_{i=1}^{N} (X_i)^- = \mathbb{E}[N] \mathbb{E}[X^-].
\]

Thus,
\[
\mathbb{E}[|S_N|] \leq \max \{ \mathbb{E}[(S_N)^+] , \mathbb{E}[(S_N)^-] \}. \tag{50}
\]

By the definition of \( \Delta \), we have
\[ S_M = S_N - \Delta - (N - M) \mu. \tag{51} \]

Hence,
\[
\mathbb{E}[|S_M|] \leq \mathbb{E}[|S_N|] + \mathbb{E}[|\Delta|] + \mathbb{E}[N - M] |\mu|. \tag{52}
\]

From (49) of Lemma 9 we have
\[
\mathbb{E}[|\Delta|] \leq \mathbb{E}[N - N_0] \mathbb{E}[|X - \mu|]. \tag{53}
\]

Since \( \mathbb{E}[M] < \mathbb{E}[N] < \infty \), we have
\[
\mathbb{E}[N - M] < \infty. \tag{54}
\]

Combining (50) – (54) leads to the boundedness of \( \mathbb{E}[|S_M|] \). This establishes the existence of \( \mathbb{E}[S_M] \). Taking expectations on both sides of (51) yields
\[
\mathbb{E}[S_M] = \mathbb{E}[S_N - \Delta - (N - M) \mu]
\]
\[
= \mathbb{E}[S_N] - \mathbb{E}[\Delta] - \mathbb{E}[N] \mu + \mathbb{E}[M] \mu
\]
\[
= \mathbb{E}[N] \mu - \mathbb{E}[\Delta] - \mathbb{E}[N] \mu + \mathbb{E}[M] \mu
\]
\[
= \mathbb{E}[M] \mu - \mathbb{E}[\Delta]
\]
\[
= \mathbb{E}[M] \mu - \mathbb{E}[\Delta^+] + \mathbb{E}[\Delta^-], \tag{55}
\]

where we have used Wald’s equation \( \mathbb{E}[S_N] = \mathbb{E}[N] \mu \) in (55). As a consequence of (56), we have
\[
\mathbb{E}[M] \mu - \mathbb{E}[\Delta^+] \leq \mathbb{E}[S_M] \leq \mathbb{E}[M] \mu + \mathbb{E}[\Delta^-]. \tag{57}
\]

35
In view of \( N \leq \lambda M + K \), we have

\[
E[N - N_0] \leq \lambda E[M] + K - N_0. \tag{58}
\]

Making use of (48), (58) and the second inequality of (57), we have

\[
E[S_M] \leq E[M] \mu + E[N - N_0] E[(X - \mu)^-]
\]
\[
\leq E[M] \mu + (\lambda E[M] + K - N_0) E[(X - \mu)^-]
\]
\[
= E[M] \{\mu + \lambda E[(X - \mu)^-]\} + (K - N_0) E[(X - \mu)^-]
\]
\[
= E[M] \beta + \eta.
\]

Making use of (47), (58) and first inequality of (57), we have

\[
E[S_M] \geq E[M] \mu - E[N - N_0] E[(X - \mu)^+]
\]
\[
\geq E[M] \mu - (\lambda E[M] + K - N_0) E[(X - \mu)^+]
\]
\[
= E[M] \{\mu - \lambda E[(X - \mu)^+]\} - (K - N_0) E[(X - \mu)^+]
\]
\[
= E[M] \alpha + \zeta.
\]

This completes the proof of the lemma. \( \square \)

We are now in a position to prove the theorem. By the definition of \( M \), we have

\[
Pr\{(M, S_M) \in \mathcal{R}\} = 1.
\]

Since \( E[M] < \infty \) and \( E[S_M] \) exists, it follows from Theorem 11 that

\[
(E[M], E[S_M]) \in \mathcal{R}.
\]

The conclusion of the theorem immediately follows from this fact and Lemma (10).

**J Proof of Theorem 12**

We need some preliminary results.

**Lemma 11** Let \( X \) be a random variable with mean \( \mu \) such that \( Pr\{a \leq X \leq b\} = 1 \). If \( g(x) \) is a convex function of \( x \in [a, b] \), then

\[
E[g(X)] \leq \frac{1}{b - a} [(b - \mu)g(a) + (\mu - a)g(b)]. \tag{59}
\]
In particular,

\[
\mathbb{E}[(X - \mu)^+] \leq \frac{(\mu - a)(b - \mu)}{b - a}, \\
\mathbb{E}[(X - \mu)^-] \leq \frac{(\mu - a)(b - \mu)}{b - a}.
\]

(60)

(61)

**Proof.** To show (59), note that, as a consequence of the convexity of the function \(g\),

\[ g(x) \leq \frac{g(b) - g(a)}{b - a} (x - a) + g(a), \quad x \in [a, b]. \]

By the assumption that \(\Pr\{a \leq X \leq b\} = 1\), we have

\[ g(X) \leq \frac{g(b) - g(a)}{b - a} (X - a) + g(a) \]

almost surely. Taking expectation on both sides of the above inequality yields

\[ \mathbb{E}[g(X)] \leq \frac{g(b) - g(a)}{b - a} \mathbb{E}[X - a] + g(a) = \frac{1}{b - a} [(b - \mu)g(a) + (\mu - a)g(b)]. \]

This establishes (59). Applying (59) to convex functions \(g(x) = \max\{x - \mu, 0\}\) and \(g(x) = \max\{\mu - x, 0\}\) yields (60) and (61), respectively.

\[ \square \]

**Lemma 12** Suppose that \(\Pr\{a \leq X \leq b\} = 1\), where \(a, b \in \mathbb{R}^d\), and that the assumptions (I) – (V) are fulfilled. Define

\[ v = \frac{(\mu - a)(b - \mu)}{b - a}, \quad \alpha = \mu - \lambda v, \quad \beta = \mu + \lambda v, \quad \zeta = (N_0 - K)v, \quad \eta = -\zeta \]

and

\[ \alpha' = b + \lambda(\mu - b), \quad \beta' = a + \lambda(\mu - a), \quad \zeta' = K(\mu - b), \quad \eta' = K(\mu - a). \]

Then,

\[
\mathbb{E}[M|\alpha + \zeta \leq \mathbb{E}[SM] \leq \mathbb{E}[M] \beta + \eta], \quad (62)
\]

\[
\mathbb{E}[M|\alpha' + \zeta' \leq \mathbb{E}[SM] \leq \mathbb{E}[M] \beta' + \eta']. \quad (63)
\]

**Proof.** Since \(\Pr\{a \leq X \leq b\} = 1\) and assumptions (I) – (V) are fulfilled, it follows from Theorem 10 that \(\mathbb{E}[M] < \mathbb{E}[N] < \infty\). Since \(\Pr\{a \leq X \leq b\} = 1\), it follows from Lemma 11 that

\[ \mathbb{E}[(X - \mu)^+] \leq v, \quad (64) \]

\[ \mathbb{E}[(X - \mu)^-] \leq v. \quad (65) \]
Making use of (64), (65) and Lemma 9, we have
\[ E[\Delta^+] \leq E[N - N_0] E[(X - \mu)^+] \leq E[N - N_0]v, \]  
(66)
\[ E[\Delta^-] \leq E[N - N_0] E[(X - \mu)^-] \leq E[N - N_0]v. \]  
(67)

Making use of (57), (66), (67) and the fact that \( N \leq \lambda M + K \), we have
\[ E[S_M] \leq E[M]\mu + E[\Delta^-] \leq E[M]\mu + E[N - N_0]v \leq E[M]\mu + E[\lambda M + K - N_0]v \]
\[ = E[M]\beta + \eta, \]
and
\[ E[S_M] \geq E[M]\mu - E[\Delta^-] \geq E[M]\mu - E[N - N_0]v \geq E[M]\mu - E[\lambda M + K - N_0]v \]
\[ = E[M]\alpha + \zeta. \]

This proves (62). It remains to show (63). Recall that
\[ \Delta = S_N - S_M - (N - M)\mu = \sum_{i=M+1}^{N} (X_i - \mu). \]

Since \( \Pr\{a \leq X \leq b\} = 1 \), it follows that
\[ (X - \mu)^+ \leq b - \mu, \quad (X - \mu)^- \leq \mu - a \]
almost surely. Hence,
\[ E[\Delta^+] \leq E \left[ \sum_{i=M+1}^{N} (X_i - \mu)^+ \right] \leq E[N - M](b - \mu), \]
(68)
\[ E[\Delta^-] \leq E \left[ \sum_{i=M+1}^{N} (X_i - \mu)^- \right] \leq E[N - M](\mu - a). \]
(69)

Making use of (57), (68), (69) and the fact that \( N \leq \lambda M + K \), we have
\[ E[S_M] \leq E[M]\mu + E[\Delta^-] \leq E[M]\mu + E[N - M](\mu - a) \leq E[M]\mu + \{(\lambda - 1)E[M] + K\}(\mu - a) \]
\[ = E[M]\lambda(\mu - a) + a] + K(\mu - a) \]
\[ = E[M]\beta' + \eta', \]
and
\[ E[S_M] \geq E[M] \mu - E[\Delta^+] \]
\[ \geq E[M] \mu - E[N - M](b - \mu) \]
\[ \geq E[M] \mu - \{(\lambda - 1)E[M] + K\} (b - \mu) \]
\[ = E[M][b - \lambda(b - \mu)] + K(\mu - b) \]
\[ = E[M] \alpha' + \zeta'. \]

This proves (63). The proof of the lemma is thus completed.

\[ \square \]

We are now in a position to prove the theorem. By the definition of \( M \), we have \( \Pr\{ (M, S_M) \in \mathcal{R} \} = 1 \). Since \( E[M] < \infty \) and \( E[S_M] \) exists, it follows from Theorem 1 that \( (E[M], E[S_M]) \in \mathcal{R} \). The conclusion of the theorem immediately follows from this fact and Lemma 12.

K Proof of Theorem 13

We need some preliminary results.

Lemma 13 \( E[N] < \infty \).

Proof. Since \( \mu \) is an interior point of the convex set \( D \), there exists a number \( \delta > 0 \) such that \( \{ \theta \in \mathbb{R}^d : ||\theta - \mu||_2 \leq \delta \} \subset D \). Since \( g \) is a concave function on \( D \), it must be a continuous function on \( D \). By the bounded-value theorem, we have that there exists a positive number \( \Upsilon \) such that \( g(\theta) \leq \Upsilon \) for any \( \theta \) contained in the set \( \{ \theta \in \mathbb{R}^d : ||\theta - \mu||_2 \leq \delta \} \). This implies that
\[ \{ N > n \} \subseteq \{ ||X_n - \mu||_2 \geq \delta \} \]
for any \( n \in \mathcal{N} \) greater than \( \Upsilon \). Let \( \tau > 1 \) be an integer such that \( N_{\tau - 1} \leq \Upsilon < N_{\tau} \). Using the same technique as that for proving (46), we can show that
\[ E[N] \leq N_{\tau} + \sum_{\ell=1}^{\infty} (N_{\ell+1} - N_\ell) \exp(-c_1 N_\ell) + c_2 \sum_{\ell=\tau}^{\infty} \frac{N_{\ell+1} - N_\ell}{N_\ell^2}, \]
where \( c_1 \) and \( c_2 \) are some positive constants. As a consequence of \( N_{\ell+1} - N_\ell \leq K, \ell = 0, 1, 2, \cdots \), the right hand side of (70) can be readily shown to be bounded.

\[ \square \]

Lemma 14
\[ E[M] < g \left( \frac{E[S_M]}{E[M]} \right). \]
Proof. From Lemma 13 we have $E[M] < E[N] < \infty$, it follows that $S_M$ is well-defined and that $E[S_M]$ exists. By the definition of $M$, we have

$$M < g \left( \frac{S_M}{M} \right)$$

almost surely. Multiplying both sides of the above inequality by $M$ yields

$$M^2 < Mg \left( \frac{S_M}{M} \right)$$

almost surely. Taking expectation on both sides of the above inequality and using Jensen’s inequality yields

$$E^2[M] \leq E[M^2] < E \left[ Mg \left( \frac{S_M}{M} \right) \right].$$

Since $M$ is a positive random variable and $g$ is a concave function, it follows from Theorem 3 that

$$E \left[ Mg \left( \frac{S_M}{M} \right) \right] \leq E[M]g \left( \frac{E[S_M]}{E[M]} \right).$$

Hence,

$$E^2[M] < E[M]g \left( \frac{E[S_M]}{E[M]} \right).$$

Dividing both sides of the above inequality by $E[M]$ yields

$$E[M] < g \left( \frac{E[S_M]}{E[M]} \right).$$

We are now in a position to prove the theorem. By the same argument as that used in the proof of Theorem 11, we can show that

$$E[M] \mu - E[N - N_0]E[(X - \mu)^+] \leq E[S_M] \leq E[M] \mu + E[N - N_0]E[(X - \mu)^-].$$

As a consequence of $N_{\ell+1} - N_\ell \leq K \leq N_0$, $\ell = 0, 1, 2, \cdots$, we have that $N - N_0 \leq M$ and it follows that

$$E[M] \mu - E[M]E[(X - \mu)^+] \leq E[S_M] \leq E[M] \mu + E[M]E[(X - \mu)^-].$$

Hence,

$$\alpha = \mu - E[(X - \mu)^+] \leq \frac{E[S_M]}{E[M]} \leq \mu + E[(X - \mu)^-] = \beta.$$

Finally, the conclusion of the theorem follows from the above inequality and Lemma 14.
L  Proof of Theorem 14

Define
\[ \mathcal{N}(\varepsilon) = \inf \{ n \in \mathbb{N} : n \geq 1 + \varepsilon + g(\mathcal{X}_n) \}, \]
where \( \varepsilon > 0 \). Define \( K = 1 \), \( N_0 = 1 \) and \( N_\ell = \ell + 1 \) for \( \ell \in \mathbb{N} \). Then, \( N_{\ell+1} - N_\ell \leq K \leq N_0 \) for \( \ell = 0, 1, 2, \cdots \). Since \( g \) is non-negative, it must be true that \( N_0 < 1 + \varepsilon + g(\mathcal{X}_{N_0}) \) is a sure event. Therefore, we can apply Theorem 13 to stopping time \( \mathcal{N}(\varepsilon) \) to conclude that
\[ \mathbb{E}[\mathcal{N}(\varepsilon)] \leq 1 + \max_{\theta \in \mathcal{D}} [1 + \varepsilon + g(\theta)] \leq 2 + \varepsilon + \max_{\theta \in \mathcal{D}} g(\theta). \]
Since \( N \leq \mathcal{N}(\varepsilon) \), we have
\[ \mathbb{E}[N] \leq \mathbb{E}[\mathcal{N}(\varepsilon)] \leq 2 + \varepsilon + \max_{\theta \in \mathcal{D}} g(\theta). \]
Since the above inequalities hold for arbitrarily small \( \varepsilon > 0 \), it must be true that \( \mathbb{E}[N] \leq 2 + \max_{\theta \in \mathcal{D}} g(\theta) \). This completes the proof of the theorem.

M  Proof of Theorem 15

As a consequence of the assumption that \( \Pr\{a \leq X \leq b\} = 1 \), where \( a, b \in \mathbb{R}^d \), it must be true that each element of \( \mathbb{E}[|X|^3] \) is finite. It follows from Theorem 13 that
\[ \mathbb{E}[N] \leq K + \max_{\theta \in \mathcal{D}} g(\theta), \]
where \( \mathcal{D} = \{ \theta \in \mathcal{D} : \alpha \leq \theta \leq \beta \} \) with \( \alpha = \mu - \mathbb{E}[(X - \mu)^+] \) and \( \beta = \mu + \mathbb{E}[(X - \mu)^-]. \) By (60) and (61) of Lemma 11 we have
\[ \alpha = \mu - \mathbb{E}[(X - \mu)^+] = \mu - \nu \]
and
\[ \beta = \mu + \mathbb{E}[(X - \mu)^-] = \mu + \nu, \]
respectively. This completes the proof of the lemma.

N  Proof of Theorem 16

To prove Theorem 16 we need some preliminary results.

Lemma 15 Let \( \rho, \varrho \in \mathbb{R} \). Let \( \theta \) and \( \vartheta \) be column vectors such that \( \theta^T, \vartheta^T \in \mathbb{R}^d \). Define
\[ \mathcal{Q} = \{ q \in \mathbb{R}^d : 0_d \leq q \leq 1_d \} \]
and $\mathcal{V} = \{ \mathbf{q} \in \mathcal{D} : \text{each element of } \mathbf{q} \text{ assumes value 0 or 1} \}$. Assume that $\rho + \mathbf{q} \theta > 0$ for all $\mathbf{q} \in \mathcal{V}$. Then,

$$\max \left\{ \frac{\rho + \mathbf{q} \vartheta}{\rho + \mathbf{q} \theta} : \mathbf{q} \in \mathcal{D} \right\} = \max \left\{ \frac{\rho + \mathbf{q} \vartheta}{\rho + \mathbf{q} \theta} : \mathbf{q} \in \mathcal{V} \right\}.$$ 

**Proof.** We shall prove the result of the lemma by a mathematical induction on the dimension $d$. For $d = 1$, we have that $\mathcal{D}$ is the interval $[0, 1]$ and that $\mathcal{V} = \{0, 1\}$. In this case, $\theta$, $\vartheta$ and $\mathbf{q}$ are scalars. If $\theta = 0$, then

$$\frac{\rho + \mathbf{q} \vartheta}{\rho + \mathbf{q} \theta} = \frac{\rho}{\rho},$$

which is a linear function of $\mathbf{q} \in [0, 1]$. If $\theta \neq 0$, then

$$\frac{\rho + \mathbf{q} \vartheta}{\rho + \mathbf{q} \theta} = \frac{\rho - \rho + \theta (\frac{\vartheta}{\rho} + \vartheta^2)}{\rho + \mathbf{q} \theta} = \frac{\rho - \rho}{\rho + \mathbf{q} \theta} + \theta,$$

which is a monotone function of $\mathbf{q} \in [0, 1]$. So, $\frac{\rho + \mathbf{q} \vartheta}{\rho + \mathbf{q} \theta}$ is a monotone function of $\mathbf{q} \in [0, 1]$, regardless the value of $\theta$. As a consequence of such monotonicity, we have

$$\max \left\{ \frac{\rho + \mathbf{q} \vartheta}{\rho + \mathbf{q} \theta} : \mathbf{q} \in [0, 1] \right\} = \max \left\{ \frac{\rho + \mathbf{q} \vartheta}{\rho + \mathbf{q} \theta} : \mathbf{q} \in \{0, 1\} \right\}.$$ 

This proves that the result of the lemma holds when the dimension $d$ is equal to 1. Now we assume that the result of the lemma holds when the dimension $d$ is equal to $k \geq 1$. Based on such induction hypothesis, we need to show that the result of the lemma also holds when the dimension $d$ is equal to $k + 1$. This amounts to prove

$$\max \left\{ \frac{\rho + \mathbf{q} \vartheta}{\rho + \mathbf{q} \theta} : \mathbf{q} \in \mathcal{D}_{k+1} \right\} = \max \left\{ \frac{\rho + \mathbf{q} \vartheta}{\rho + \mathbf{q} \theta} : \mathbf{q} \in \mathcal{V}_{k+1} \right\}.$$ 

Here, $\theta$ and $\vartheta$ are column vectors of size $(k + 1) \times 1$, 

$$\mathcal{D}_k = \{ \mathbf{q} \in \mathbb{R}^k : \mathbf{0}_d \leq \mathbf{q} \leq \mathbf{1}_d \}$$

and $\mathcal{V}_k = \{ \mathbf{q} \in \mathcal{D}_k : \text{each element of } \mathbf{q} \text{ assumes value 0 or 1} \}$ for $k \in \mathbb{N}$. Note that

$$\max \left\{ \frac{\rho + \mathbf{q} \vartheta}{\rho + \mathbf{q} \theta} : \mathbf{q} \in \mathcal{D}_{k+1} \right\} = \max \left\{ \frac{\rho + \mathbf{q} \vartheta}{\rho + \mathbf{q} \theta} : \mathbf{q} \in \mathcal{V}_{k+1} \right\}.$$ 

(71)

Here, $\mathbf{q}'$ is the first element of $\mathbf{q}$, 

$\mathbf{q}''$ is a column vector obtained by eliminating the first element of $\mathbf{q}$, 

$\theta'$ is the first element of $\theta$, 

$\theta''$ is a column vector obtained by eliminating the first element of $\theta$, 

$\vartheta'$ is the first element of $\vartheta$, 

$\vartheta''$ is a column vector obtained by eliminating the first element of $\vartheta$. 

42
Note that
\[
\max \left\{ \frac{q + q'\varphi + q''\varphi''}{\rho + q'\theta' + q''\theta''} : q'' \in D_k, \ q' \in [0, 1] \right\} = \max_{q' \in D_k} \max \left\{ \frac{q + q'\varphi + q''\varphi''}{\rho + q'\theta' + q''\theta''} : q' \in [0, 1] \right\}. \tag{72}
\]
For fixed \(q'' \in D_k\), using previous arguments, it can be shown that \(\frac{q + q'\varphi + q''\varphi''}{\rho + q'\theta' + q''\theta''}\) is a monotone function of \(q' \in [0, 1]\). Hence,
\[
\max \left\{ \frac{q + q'\varphi + q''\varphi''}{\rho + q'\theta' + q''\theta''} : q' \in [0, 1] \right\} = \max \left\{ \frac{q + q'\varphi}{\rho + q'\theta} \right\} \max \left\{ \frac{q + q''\varphi''}{\rho + q''\theta''} : q'' \in D_k, \ q \in [0, 1] \right\}. \tag{73}
\]
It follows from (71), (72) and (73) that
\[
\max \left\{ \frac{q + q'\varphi}{\rho + q'\theta} : q \in D_{k+1} \right\} = \max_{q'' \in D_k} \max \left\{ \frac{q + q''\varphi''}{\rho + q''\theta''} : q'' \in D_k, \ q \in [0, 1] \right\} = \max \left\{ \max_{q'' \in D_k} \frac{q + q''\varphi''}{\rho + q''\theta''}, \max_{q'' \in D_k} \frac{q + q''\varphi''}{\rho + q''\theta''} : q'' \in D_k, \ q \in [0, 1] \right\}.
\]
By the induction hypothesis, we have
\[
\max_{q'' \in D_k} \frac{q + q''\varphi''}{\rho + q''\theta''} = \max_{q'' \in D_k} \frac{q + q''\varphi''}{\rho + q''\theta''}, \quad \max_{q'' \in D_k} \frac{q + q''\varphi''}{\rho + q''\theta''} = \max_{q'' \in D_k} \frac{q + q''\varphi''}{\rho + q''\theta''}.
\]
Therefore,
\[
\max \left\{ \frac{q + q'\varphi}{\rho + q'\theta} : q \in D_{k+1} \right\} = \max \left\{ \max_{q'' \in D_k} \frac{q + q''\varphi''}{\rho + q''\theta''}, \max_{q'' \in D_k} \frac{q + q''\varphi''}{\rho + q''\theta''} : q'' \in D_k, \ q \in [0, 1] \right\} = \max \left\{ \frac{q + q'\varphi}{\rho + q'\theta}, \frac{q + q''\varphi''}{\rho + q''\theta''} : q \in D_{k+1} \right\}.
\]
This completes the proof of the process of the mathematical induction. The proof of the lemma is thus completed.

\[\square\]

**Lemma 16** If \(E[M]\alpha + \zeta \leq E[S_M] \leq E[M]\beta + \eta\), then there exists \(q\) such that \(0_d \leq q \leq 1_d\) and that \(E[S_M] = E[M]\theta + \phi\), where \(\theta = q(\alpha - \beta) + \beta\) and \(\phi = q(\zeta - \eta) + \eta\).

**Proof.** We consider the scalar case. The argument can be readily generalized to the vector case. If \(E[S_M] = E[M]\beta + \eta\), then the lemma holds with \(\theta = \beta\) and \(\phi = \eta\). If \(E[S_M] = E[M]\alpha + \zeta\), then the lemma holds with \(\theta = \alpha\) and \(\phi = \zeta\). Hence, it remains to prove this lemma under the assumption that
\[
E[M]\alpha + \zeta < E[S_M] < E[M]\beta + \eta. \tag{74}
\]
For this purpose, define
\[
\theta_q = q(\alpha - \beta) + \beta, \quad \phi_q = q(\zeta - \eta) + \eta
\]
43
and 
\[ w(q) = \mathbb{E}[S_M] - \mathbb{E}[M]q - \phi_q \]
for \( q \in [0, 1] \). Note that \( w(q) = \mathbb{E}[S_M] - \mathbb{E}[M]q(\alpha - \beta) + \beta - [q(\zeta - \eta) + \eta] \) is a continuous function of \( q \in [0, 1] \). Clearly, as a consequence of (74),
\[ w(0) = \mathbb{E}[S_M] - \mathbb{E}[M]\beta - \eta < 0, \quad w(1) = \mathbb{E}[S_M] - \mathbb{E}[M]\alpha - \zeta > 0. \]
By virtue of the intermediate value theorem, there exists a number \( q^* \in (0, 1) \) such that \( w(q^*) = 0 \). This implies that
\[ \mathbb{E}[S_M] = \mathbb{E}[M]q^* + \phi_{q^*}, \]
where
\[ \theta_{q^*} = q^*(\alpha - \beta) + \beta, \quad \phi_{q^*} = q^*(\zeta - \eta) + \eta \]
with \( q^* \in (0, 1) \). This completes the proof of the lemma.

\[ \Box \]

We are now in a position to prove the theorem. Since the assumptions (I) – (VI) are fulfilled, it follows from Theorem 10 that \( \mathbb{E}[M] \leq \mathbb{E}[N] < \infty \). Hence, \( \mathbb{E}[S_M] \) exists. Since \( C > 0 \) and \( R \) contains \( (0, 0, d) \), it must be true that
\[ As + Bt \leq C \]
for any \((t, s) \in R\). Hence,
\[ \Pr\{AS_M + BM \leq C\} = 1. \]
By Theorem 1, we have
\[ A\mathbb{E}[S_M] + B\mathbb{E}[M] \leq C. \]
By Lemma 10, we have \( \mathbb{E}[M] + \zeta \leq \mathbb{E}[S_M] \leq \mathbb{E}[M] + \eta \). According to Lemma 16 there exist \( \theta^* = q^*(\alpha - \beta) + \beta \) and \( \phi^* = q^*(\zeta - \eta) + \eta \) such that \( \mathbb{E}[S_M] = \theta^*\mathbb{E}[M] + \phi^* \). Hence,
\[ \mathbb{E}[M](A\theta^* + B) + A\phi^* \leq C \]
As a consequence of the assumption that the minimum of \{ \( B + A[\beta + q(\alpha - \beta) \]: \( q \in \mathcal{V} \) \} is positive, we have
\[ \min\{A\theta_q + B : 0_d \leq q \leq 1_d\} > 0 \]
and thus \( A\theta^* + B > 0 \). It follows that
\[ \mathbb{E}[M] \leq \frac{C - A\phi^*}{B + A\theta^*} \leq \max\left\{ \frac{C - A[\eta + q(\zeta - \eta)]}{B + A[\beta + q(\alpha - \beta)]} : 0_d \leq q \leq 1_d \right\}. \]
Invoking Lemma 15 we have
\[ \max\left\{ \frac{C - A[\eta + q(\zeta - \eta)]}{B + A[\beta + q(\alpha - \beta)]} : 0_d \leq q \leq 1_d \right\} = \max\left\{ \frac{C - A[\eta + q(\zeta - \eta)]}{B + A[\beta + q(\alpha - \beta)]} : q \in \mathcal{V} \right\}. \]
This completes the proof of the theorem.
O Proof of Theorem 17

Since \( \Pr\{a \leq X \leq b\} = 1 \) and the assumptions (I) – (V) are fulfilled, it follows from similar arguments to that of Theorem 10 that \( E[M] \leq E[N] < \infty \). Hence, \( E[S_M] \) exists. Since \( C > 0 \) and \( R \) contains \((0, 0, \ldots)\), it must be true that \( As + Bt \leq C \) for any \((t, s) \in R\). Hence, \( \Pr\{AS_M + BM \leq C\} = 1 \). By Theorem 1, we have

\[
A E[S_M] + B E[M] \leq C. 
\]

According to Lemma 12, we have

\[
E[M] \alpha + \zeta \leq E[S_M] \leq E[M] \beta + \eta, 
\]

\[
E[M] \alpha' + \zeta' \leq E[S_M] \leq E[M] \beta' + \eta'. 
\]

The proof of the theorem can be completed by using similar arguments as that of Theorem 16.

P Proof of Theorem 18

We need some preliminary results.

Lemma 17 Assume that each element of \( E[|X|] \) is finite. Then, \( E[N] < \infty \).

Proof. Since \( As + Bt = C \), where \( C > 0 \), is the supporting hyperplane of the continuity region \( R \), passing through the boundary point \((m, m\mu)\), it must be true that \( C = m(A^\top \mu + B) > 0 \). By the definition that \( Z = BK + A \sum_{i=1}^K X_i \), we have \( E[Z] = (A\mu + B)K > 0 \). Note that

\[
E[|Z|] = E\left[|BK + A \sum_{i=1}^K X_i|\right] \leq |A| \sum_{i=1}^K E[|X_i|] + |BK| = K(|A|E[|X|] + |B|), 
\]

where the upper bound is finite as a consequence of the assumption that each element of \( E[|X|] \) is finite. Let \( Z_1, Z_2, \ldots \) be i.i.d. random vectors having the same distribution as that of \( Z \). Define

\[
\tau = \inf \left\{ n \in \mathbb{N} : \sum_{\ell=1}^n Z_\ell > C \right\}. 
\]

(75)

Making use of the fact that \( E[Z] > 0 \), \( E[|Z|] < \infty \) and assertion (i) of Theorem 3.1 in page 83 of Gut’s book [12], we have that

\[
E[\tau] < \infty. 
\]

Now define

\[
\mathcal{N} = \inf\{t \in \mathcal{N} : AS_t + Bt \geq C\}. 
\]

(76)
where $\mathcal{N} = \{ n \in \mathbb{N} : \frac{n}{K} \in \mathbb{N} \}$. Then,

$$E[\mathcal{N}] = K E[\tau] < \infty.$$ 

Since $C > 0$ and the convex set $\mathcal{R}$ contains $(0,0,d)$, it must be true that $As + Bt \leq C$ for all $(t,s) \in \mathcal{R}$. Hence,

$$\{(n,S_n) \in \mathcal{R} \} \subseteq \{ AS_n + Bn \leq C \}$$

for $n \in \mathbb{N}$. This implies that $\mathcal{N} \leq \mathcal{N}$. Hence, $E[\mathcal{N}] \leq E[\mathcal{N}] < \infty$. This completes the proof of the lemma.

\[\square\]

We are now in a position to prove the theorem. By the assumption that each element of $E[|X|^2]$ is finite, it must be true that each element of $E[|X|]$ is finite. Hence, it follows from Lemma 17 that $E[\mathcal{N}] \leq E[\mathcal{N}] < \infty$, where $\mathcal{N}$ is defined by (76). Note that

$$E[Z^2] = E \left[ \left| BK + A \sum_{i=1}^{K} X_i \right|^2 \right]$$

$$= E \left[ A \sum_{i=1}^{K} X_i \right]^2 + 2BK E \left[ A \sum_{i=1}^{K} X_i \right] + (BK)^2$$

$$= E \left[ A \sum_{i=1}^{K} X_i \right]^2 + 2BK^2 A\mu + (BK)^2$$

$$\leq ||A||^2 \sum_{i=1}^{K} E[|X_i|^2] + 2BK^2 A\mu + (BK)^2$$

$$= K||A||^2 \sum_{i=1}^{K} E[|X_i|^2] + 2BK^2 A\mu + (BK)^2.$$

Using the above bound for $E[Z^2]$ and the assumption that $E[|X|^2]$ is finite, we have that $E[Z^2] < \infty$. Let $Z_1, Z_2, \ldots$ be i.i.d. samples of $Z$ and define $\tau$ as (75). In the proof of Lemma 17 we have established that $E[\tau] < \infty$. By Lorden’s inequality

$$E \left[ \sum_{\ell=1}^{\tau} Z_\ell - C \right] \leq \frac{E[(Z^+)^2]}{E[Z]}.$$ 

Using Wald’s equation, we have

$$E \left[ \sum_{\ell=1}^{\tau} Z_\ell \right] = E[\tau] E[Z]$$

and thus

$$E[\tau] E[Z] - C \leq \frac{E[(Z^+)^2]}{E[Z]}.$$
from which we have
\[ \mathbb{E}[\tau] \leq \frac{C}{\mathbb{E}[Z]} + \frac{\mathbb{E}[(Z^+)^2]}{\mathbb{E}^2[Z]}. \]

Hence,
\[
\mathbb{E}[\mathcal{N}] \leq \mathbb{E}[\mathcal{N}] = K\mathbb{E}[\tau] \\
\leq \frac{CK}{\mathbb{E}[Z]} + \frac{K\mathbb{E}[(Z^+)^2]}{\mathbb{E}^2[Z]} \\
= \frac{C}{A\mu + B} + \frac{\mathbb{E}[(Z^+)^2]}{K(A\mu + B)^2} \\
= m + \frac{1}{K} \left( \frac{m}{C} \right)^2 \mathbb{E}[(Z^+)^2], \tag{77}
\]

where we have used the fact that \( \mathbb{E}[Z] = K(A\mu + B) \) and \( C = m(A\mu + B) \). Note that
\[
\mathbb{E}[(Z^+)^2] \leq \mathbb{E}[Z^2] \\
= \mathbb{E} \left[ \left( BK + \sum_{i=1}^{K} AX_i \right)^2 \right] \\
= \mathbb{E} \left[ \left( BK + K A\mu + \sum_{i=1}^{K} A(X_i - \mu) \right)^2 \right] \\
= K^2(A\mu + B)^2 + \mathbb{E} \left[ \left( \sum_{i=1}^{K} A(X_i - \mu) \right)^2 \right] \\
= K^2(A\mu + B)^2 + K \mathbb{E} \left[ (A(X - \mu))^2 \right]. \tag{78}
\]

It follows from (77) and (78) that
\[
\mathbb{E}[\mathcal{N}] \leq K + \frac{C}{A\mu + B} + \frac{\mathbb{E} \left[ (A(X - \mu))^2 \right]}{(A\mu + B)^2} = m + K + \left( \frac{m}{C} \right)^2 \mathbb{E} \left[ (A(X - \mu))^2 \right]. \tag{79}
\]

Note that
\[ A(X - \mu) = \langle A^T, X - \mu \rangle. \]

Since the absolute value of the inner product of two vectors is no greater than the product of their Euclidean norms, we have
\[
\mathbb{E} \left[ (A(X - \mu))^2 \right] \leq \mathbb{E} \left[ ||A^T||_2 \times ||X - \mu||_2 \right] = ||A^T||_2^2 \times \mathbb{E} \left[ ||X - \mu||_2^2 \right] = ||A||_2^2 \times \mathbb{E} \left[ ||X - \mu||_2^2 \right]
\]
Therefore,
\[
\mathbb{E}[\mathcal{N}] \leq m + K + \left( \frac{m}{C} \right)^2 \mathbb{E} \left[ (A(X - \mu))^2 \right] \leq m + K + \left( \frac{m}{C} \right)^2 \ ||A||_2^2 \times \mathbb{E} \left[ ||X - \mu||_2^2 \right].
\]

This establishes assertion (I) of the theorem.
If the elements of $X$ are mutually independent, then $\mathbb{E}\left[ (A(X-\mu))^2 \right] = A^2\mathbb{E}[(X-\mu)^2]$. It follows from this fact and (79) that

$$\mathbb{E}[N] \leq m + K + \left( \frac{m}{C} \right)^2 A^2 \mathbb{E}[(X-\mu)^2].$$

This establishes assertion (II) of the theorem.

It remains to show assertion (III). As a consequence of the definition of $Z$ and the assumption that $\Pr\{a \leq X \leq b\} = 1$, we have $Ku \leq Z \leq Kv$ almost surely. It follows that

$$(Z^+)^2 \leq Z^2 \leq \frac{(Kv)^2 - (Ku)^2}{Kv - Ku} (Z - Ku) + (Ku)^2 = K(u+v)Z - K^2uv$$

almost surely. Hence,

$$\mathbb{E}[(Z^+)^2] \leq K(u+v)\mathbb{E}[Z] - K^2uv = K^2(u+v)(A\mu + B) - K^2uv.$$  \hfill (80)

Making use of (77) and (80), we have

$$\mathbb{E}[N] \leq \frac{C}{A\mu + B} + \frac{\mathbb{E}[(Z^+)^2]}{K(A\mu + B)^2}$$

$$= \frac{C}{A\mu + B} + \frac{K^2(u+v)(A\mu + B) - K^2uv}{K(A\mu + B)^2}$$

$$= \frac{C}{A\mu + B} + \frac{K(u+v)(A\mu + B) - Kuv}{(A\mu + B)^2}$$

$$= \frac{C}{A\mu + B} + \frac{K(u+v)}{A\mu + B} - \frac{Kuv}{(A\mu + B)^2}$$

$$= m + \frac{mK(u+v)}{C} - \frac{m^2Kuv}{C^2},$$

where we have used the assumption that $m(A\mu + B) = C > 0$. This establishes the first inequality of assertion (III).

To show the second inequality of assertion (III), note that

$$(Z^+)^2 \leq \frac{(Kv)^2 - (Ku)^2}{Kv - Ku} (Z - Ku) = \frac{Kv^2}{v - u} (Z - Ku)$$

almost surely for $u < 0$. Hence,

$$\mathbb{E}[(Z^+)^2] \leq \frac{K^2v^2}{v - u} (A\mu + B - u), \quad u < 0.$$  \hfill (81)
Making use of (77) and (81), we have
\[
E[N] \leq \frac{C}{A\mu + B} + \frac{\mathbb{E}[(Z^+)^2]}{K(A\mu + B)^2}
\]
\[
= \frac{C}{A\mu + B} + \frac{K^2v^2}{v-u}(A\mu + B - u)
\]
\[
= \frac{C}{A\mu + B} + \frac{Kv^2}{v-u}(A\mu + B - u)
\]
\[
= m + \frac{Kv^2}{v-u}\left(\frac{m}{C}\right)^2\left(\frac{C}{m} - u\right)
\]
for \( C > 0 > u \).

This completes the proof of the theorem.

Q  Proof of Theorem 19

We need some preliminary results.

**Lemma 18** Let \((m, m\mu)\), where \(m = g(\mu)\), be a boundary point of the continuity region \(\mathcal{R}\). Define
\[
A = -[\nabla(\mu)]^\top, \quad B = 1 - A\mu, \quad C = g(\mu).
\]
Then, \(As + Bt = C\) is the supporting hyperplane for \(\mathcal{R}\) passing through the boundary point \((m, m\mu)\).

**Proof.** Define function
\[
f(t, s) = t - g\left(\frac{s}{t}\right)
\]
for \((t, s) \in \mathcal{R}\) with \(t > 0\). As a consequence of the definition of the function \(g(.)\), it must be true
that \(f(t, s) = 0\) holds for any boundary point \((t, s)\) of \(\mathcal{R}\) with \(t > 0\). In particular, \(f(m, m\mu) = 0\).
Since \(g(v)\) is differentiable at a neighborhood of \(v = \mu\), the function \(f(t, s)\) is differentiable at
a neighborhood of \((t, s) = (m, m\mu)\). Since \(\mathcal{R}\) is convex, it must be true that the tangent plane
to the surface \(f(t, s) = 0\), passing through \((m, m\mu)\), coincides with the supporting hyperplane of
\(\mathcal{R}\). Therefore, to show that \(As + Bt = C\) is the supporting hyperplane of \(\mathcal{R}\) passing through
the boundary point \((m, m\mu)\), it suffices to show that \(C\) is equal to \(m\), and that \(A^\top\) and \(B\) are,
respectively, equal to the partial derivatives of \(f(t, s)\) with respect to \(s\) and \(t\) when \(s = m\mu, \ t = m\).
In other words, it is sufficient to show that
\[
C = m, \quad A^\top = \left.\frac{\partial f(t, s)}{\partial s}\right|_{t=m, s=m\mu}, \quad B = \left.\frac{\partial f(t, s)}{\partial t}\right|_{t=m, s=m\mu}.
\]
Define
\[ h(v) = \frac{\partial g(v)}{\partial v}. \]

Using the chain rule of differentiation, we have
\[ \frac{\partial f(t,s)}{\partial s} = -\frac{1}{t} h \left( \frac{s}{t} \right) \]
and
\[ \frac{\partial f(t,s)}{\partial t} = 1 + h \left( \frac{s}{t} \right) \frac{s^\top}{t^2}. \]

Evaluating such derivatives with \( t = m, s = m\mu \) yields
\[ \frac{\partial f(t,s)}{\partial s} \bigg|_{t=m, s=m\mu} = -\frac{h(\mu)}{g(\mu)} = -\nabla(\mu) = A^\top \]
and
\[ \frac{\partial f(t,s)}{\partial t} \bigg|_{t=m, s=m\mu} = 1 + \frac{h(\mu)}{g(\mu)} \mu^\top = 1 - A\mu = B. \]

Since the boundary point \((m, m\mu)\) is in the supporting hyperplane, it must be true that
\[ C = A(m\mu) + Bm = m(A\mu + B). \]

Observing that \( A\mu + B = 1 \), we have \( C = m \). This completes the proof of the lemma.

\[ \square \]

We are now in a position to prove the theorem. Making use of Lemma\textsuperscript{[18]} and assertion (I) of Theorem\textsuperscript{[18]} we have
\[ \mathbb{E}[N] \leq m + 1 + \left( \frac{m}{C} \right)^2 \mathbb{E} \left[ \langle A^\top, X - \mu \rangle^2 \right], \]
where \( A = -[\nabla(\mu)]^\top \) and \( C = m = g(\mu) \). Hence,
\[ \mathbb{E}[N] \leq g(\mu) + 1 + \mathbb{E} \left[ \langle \nabla(\mu), X - \mu \rangle^2 \right]. \]

Using the fact that the absolute value of the inner product of two vectors is no greater than the product of their norms, we have
\[ \langle \nabla(\mu), X - \mu \rangle^2 \leq ||\nabla(\mu)||_2^2 \times ||X - \mu||_2^2. \]

It follows that
\[ \mathbb{E}[N] \leq g(\mu) + 1 + \mathbb{E} \left[ \langle \nabla(\mu), X - \mu \rangle^2 \right] \leq g(\mu) + 1 + ||\nabla(\mu)||_2^2 \times \mathbb{E}[||X - \mu||_2^2]. \]

This completes the proof of the theorem.
R Proof of Theorem 20

Since the assumptions (I) – (VI) are fulfilled, it follows from Theorem 10 that \( \mathbb{E}[N] < \infty \). Note that

\[
\mathbb{E}[N] = N_\tau + \sum_{\ell \geq \tau} (N_{\ell+1} - N_\ell) \Pr \{ N > N_\ell \}. \tag{82}
\]

By the definitions of \( N, G_n, \rho(n) \) and the convexity of \( \mathcal{R} \), we have

\[
\{ N > N_\ell \} \subseteq \{ (N_\ell, S_{N_\ell}) \in \mathcal{R} \}
\]
\[
\subseteq \{ \overline{X}_{N_\ell} \in \mathcal{G}_{N_\ell} \}
\]
\[
\subseteq \{ || \overline{X}_{N_\ell} - \mu ||_2 \geq \rho(N_\ell) \}
\]

for \( \ell \geq \tau \). Note that

\[
\Pr \{ N > N_\ell \} \leq \Pr \{ || \overline{X}_{N_\ell} - \mu ||_2 \geq \rho(N_\ell) \} \leq \Pr \{ \sum_{i=1}^d \left| X_{N_\ell}^k - \mu_k \right|^2 \geq \frac{[\rho(N_\ell)]^2}{d} \text{ for some } k \text{ among } 1, \cdots, d \} \tag{83}
\]
\[
\leq \sum_{i=k}^d \Pr \left\{ \left| X_{N_\ell}^k - \mu_k \right|^2 \geq \frac{[\rho(N_\ell)]^2}{d} \right\} \tag{84}
\]

for \( \ell \geq \tau \), where we have used the Pigeon-Hole principle in (83). Making use of (82) and (84), we have

\[
\mathbb{E}[N] \leq N_\tau + \sum_{\ell \geq \tau} (N_{\ell+1} - N_\ell) \sum_{k=1}^d \Pr \left\{ \left| X_{N_\ell}^k - \mu_k \right| \geq \frac{\rho(N_\ell)}{\sqrt{d}} \right\}.
\]

If \( X \) is a scalar random variable and \( \mu \) is less than the infimum of \( \mathcal{G}_{N_\ell} \), then it follows from the definitions of \( N, G_n, \rho(n) \) and the convexity of \( \mathcal{R} \) that

\[
\{ N > N_\ell \} \subseteq \{ (N_\ell, S_{N_\ell}) \in \mathcal{R} \}
\]
\[
\subseteq \{ \overline{X}_{N_\ell} \in \mathcal{G}_{N_\ell} \}
\]
\[
\subseteq \{ \overline{X}_{N_\ell} \geq \mu + \rho(N_\ell) \}
\]

for \( \ell \geq \tau \). Hence,

\[
\Pr \{ N > N_\ell \} \leq \Pr \left\{ \overline{X}_{N_\ell} \geq \mu + \rho(N_\ell) \right\} \tag{85}
\]

for \( \ell \geq \tau \). Making use of (82) and (85), we have

\[
\mathbb{E}[N] \leq N_\tau + \sum_{\ell \geq \tau} (N_{\ell+1} - N_\ell) \Pr \left\{ \overline{X}_{N_\ell} \geq \mu + \rho(N_\ell) \right\}.
\]
If $X$ is a scalar random variable and $\mu$ is greater than the supremum of $G_{N,\tau}$, then it follows from the definitions of $N, G_n, \rho(n)$ and the convexity of $R$ that

$$\{N > N_\ell\} \subseteq \{(N_\ell, S_{N_\ell}) \in \mathcal{R}\} \subseteq \{X_{N_\ell} \in G_{N_\ell}\} \subseteq \{X_{N_\ell} \leq \mu - \rho(N_\ell)\}$$

for $\ell \geq \tau$. Hence,

$$\Pr\{N > N_\ell\} \leq \Pr\{X_{N_\ell} \leq \mu - \rho(N_\ell)\}$$

for $\ell \geq \tau$. Making use of (82) and (86), we have

$$\mathbb{E}[N] \leq N_\tau + \sum_{\ell+1 \in \mathcal{L}} (N_{\ell+1} - N_\ell) \Pr\{X_{N_\ell} \leq \mu - \rho(N_\ell)\}.$$ 

This completes the proof of the theorem.

**S Proof of Theorem 21**

We need some preliminary results.

**Lemma 19** For $n \in \mathbb{N}$, $\mu \in \mathbb{R}^d$, the minimum of $||s - n\mu||_2^2$ with respect to $s \in \mathbb{R}^d$ subject to $As + Bn = C$ is equal to $\frac{\frac{a(A\mu + B) - C}{AA}}{2}$.

**Proof.** For $k = 1, \cdots, d$, let $s_k, \mu_k, a_k$ denotes the $k$-th element of $s, \mu$ and $A$, respectively. In other words,

$$s = [s_1, \cdots, s_d], \quad \mu = [\mu_1, \cdots, \mu_d], \quad A = [a_1, \cdots, a_d]^\top.$$ 

For simplicity of notation, define $D = C - Bn$. Then, the problem of minimizing of $||s - n\mu||_2^2$ with respect to $s \in \mathbb{R}^d$ subject to $As + Bn = C$ can be written as the problem of minimizing $\sum_{k=1}^d (s_k - n\mu_k)^2$ with respect to $s_1, \cdots, s_d \in \mathbb{R}$ subject to

$$\sum_{k=1}^d a_k s_k = D.$$ 

We shall solve this problem by the Lagrange-multiplier method. Define

$$f(\xi, s_1, \cdots, s_d) = \sum_{k=1}^d (s_k - n\mu_k)^2 + \xi \left( \sum_{k=1}^d a_k s_k - D \right).$$ 

Note that the partial derivatives

$$\frac{\partial f}{\partial s_k} = 2(s_k - n\mu_k) + \xi a_k, \quad k = 1, \cdots, d$$

$$\frac{\partial f}{\partial \xi} = \sum_{k=1}^d a_k s_k - D.$$ 

(87)
Setting $\frac{\partial f}{\partial s_k} = 0$ yields
\[ s_k = n\mu_k - \frac{\xi a_k}{2}, \quad k = 1, \ldots, d. \]
Substituting the above expression of $s_k$ into (87) and setting $\frac{\partial f}{\partial \xi} = 0$ yields
\[ \sum_{k=1}^{d} a_k \left( n\mu_k - \frac{\xi a_k}{2} \right) = D, \]
i.e.,
\[ n \sum_{k=1}^{d} a_k \mu_k - \frac{\xi}{2} \sum_{k=1}^{d} a_k^2 = D, \]
from which we have
\[ \xi = 2 \frac{n \sum_{k=1}^{d} a_k \mu_k - D}{\sum_{k=1}^{d} a_k^2}. \]
Hence,
\[ (s_k - n\mu_k)^2 = \left( \frac{\xi a_k}{2} \right)^2 = \left( \frac{n \sum_{k=1}^{d} a_k \mu_k - D}{\sum_{k=1}^{d} a_k^2} \right)^2 a_k^2, \quad k = 1, \ldots, d. \]
It follows that
\[ \sum_{k=1}^{d} (s_k - n\mu_k)^2 = \frac{(n \sum_{k=1}^{d} a_k \mu_k - D)^2}{\sum_{k=1}^{d} a_k^2} = \frac{n(A\mu + B - C)^2}{AA^\top}. \]
This completes the proof of the lemma.

We are now in a position to prove the theorem. According to Lemma 19, we have
\[
\rho(n) = \frac{1}{n} \sqrt{\min\{||s - n\mu||_2^2 : As + Bn = C, \ s \in \mathbb{R}^d\}}
\]
\[
= \frac{1}{n} \sqrt{\frac{[n(A\mu + B) - C]^2}{AA^\top}}
\]
\[
= \frac{1}{n} \sqrt{\frac{(\frac{n}{m} - 1)C^2}{AA^\top}}
\]
\[
= \frac{1}{n} \sqrt{\frac{[(\frac{m}{n} - 1)m(A\mu + B)]^2}{AA^\top}}
\]
\[
= \frac{1}{n} \sqrt{\frac{[(n-m)(A\mu + B)]^2}{AA^\top}}
\]
\[
= \frac{1}{n} \sqrt{\frac{[(1 - \frac{m}{n})|A\mu + B|]^2}{\sqrt{AA^\top}}}
\]
\[
= \frac{1}{n} \sqrt{\frac{[(1 - \frac{m}{n})|A\mu + B|]^2}{\sqrt{AA^\top}}}
\]
53
for $n > m$. Hence,
\[ \rho(N_{\ell}) = \frac{(1 - \frac{m}{N_{\ell}})|A\mu + B|}{\sqrt{AA^\top}} = \gamma_{\ell} \quad (88) \]
for $\ell \geq \tau$. Making use of (88) and (17) of Theorem 20 we have (20).

In the case that $X$ is a scalar random variable, we have that
\[ \rho(N_{\ell}) = \gamma_{\ell} = \left(1 - \frac{m}{N_{\ell}}\right)\left|\mu + \frac{B}{A}\right| \quad (89) \]
for $\ell \geq \tau$. As a consequence of (89) and (18), (19) of Theorem 20 we have (21) and (22). This completes the proof of the theorem.

\section{Proof of Theorem 24}

We need some preliminary results. By a similar argument as that for proving Lemma 6, we can show the following result.

\begin{lemma} \label{lemma20}
There exist $A, B$ and a positive number $C$ such that $As + Bt = C$ is a supporting hyperplane of $\mathbb{F}$, which passes through $(\tau, \tau\mu)$.
\end{lemma}

Define
\[ \mathcal{T} = \inf\{t > 0 : AW_t + Bt \geq C\}. \]
By a similar argument as that for proving Lemma 7 we can show the following result.

\begin{lemma} \label{lemma21}
There exist $\delta > 0$ and $\Upsilon > 0$ such that $\{\mathcal{T} > t\} \subseteq \{||W_t - \mu||_2 > \delta\}$ for $t$ greater than $\Upsilon$.
\end{lemma}

The following result is well known (see, e.g., [25, page 55]).

\begin{lemma} \label{lemma22}
Let $B_t$ be a scalar Brownian motion with zero drift and unity diffusion. Then,
\[ \Pr\left\{\sup_{0 \leq s \leq t} |B_s| \geq \lambda\right\} \leq 2 \exp\left(-\frac{\lambda^2}{2t}\right), \quad \lambda > 0, \quad t \geq 0. \]
\end{lemma}

With the above two lemmas, we can show the boundedness of the average stopping time as stated as follows.

\begin{lemma} \label{lemma23}
$\mathbb{E}[\mathcal{T}] < \infty$.
\end{lemma}
Proof. Let $\Sigma$ denote the diffusion matrix of $W_t$, i.e., $\mathbb{E}[(W_t - t\mu)^\top(W_t - t\mu)] = \Sigma$. Then, we can write

$$W_t = t\mu + B_t \Sigma^\top,$$

where $B_t$ is a standard Brownian motion with drift vector $0_d$ and identity diffusion matrix. Note that

$$\Pr\{|W_t - \mu|_2 > \delta\} = \Pr\{|W_t - t\mu|_2 > t\delta\} = \Pr\{|B_t \Sigma^\top|_2 > (t\delta)^2\} = \Pr\{|B_t \Sigma^\top B_t^\top| > (t\delta)^2\}.$$

Since $\Sigma^\top \Sigma$ is a positive semidefinite matrix, there exists an orthogonal matrix $U$ and a diagonal matrix $\Lambda$ with diagonal elements $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d \geq 0$ such that

$$\Sigma^\top \Sigma = U \Lambda U^\top.$$

Hence,

$$B_t \Sigma^\top \Sigma B_t^\top \leq \lambda_1 B_t U U^\top B_t^\top = \lambda_1 B_t B_t^\top,$$

and it follows that

$$\Pr\{|W_t - \mu|_2 > \delta\} \leq \Pr\{|B_t \Sigma^\top B_t^\top| > (t\delta)^2\}.$$

If $\lambda_1 = 0$, then $\Pr\{|W_t - \mu|_2 > \delta\} = 0$ for all $t > 0$. It follows from Lemma 21 that $\Pr\{T > t\} = 0$ for $t$ greater than $\Upsilon$. Consequently, $\mathbb{E}[T] \leq \Upsilon < \infty$. Hence, it remains to show $\mathbb{E}[T] < \infty$ with $\lambda_1 > 0$. In this case, we have

$$\Pr\{|W_t - \mu|_2 > \delta\} \leq \Pr\left\{B_t B_t^\top > \frac{(t\delta)^2}{\lambda_1}\right\}.$$

Let $B_t^k$ denote the $k$-th component of $B_t$, i.e., $B_t = [B_t^1, \ldots, B_t^d]$. Then, $B_t B_t^\top = \sum_{k=1}^d |B_t^k|^2$ and

$$\Pr\{|W_t - \mu|_2 > \delta\} \leq \Pr\left\{\sum_{k=1}^d |B_t^k|^2 > \frac{(t\delta)^2}{\lambda_1}\right\} \leq \Pr\left\{|B_t^k|^2 > \frac{(t\delta)^2}{d\lambda_1} \text{ for some } k \text{ among } 1, \ldots, d\right\} \leq \sum_{k=1}^d \Pr\left\{|B_t^k|^2 > \frac{(t\delta)^2}{d\lambda_1}\right\} = \sum_{k=1}^d \Pr\left\{|B_t^k| > \frac{t\delta}{\sqrt{d\lambda_1}}\right\} = \sum_{k=1}^d \Pr\left\{|B_t^k| > \frac{t\delta}{\sqrt{d\lambda_1}}\right\},$$

55
where $B_t$ is a scalar Brownian motion with zero drift and unity diffusion. Making use of Lemma 22, we have

$$\Pr\left\{ \left| B_t \right| > \frac{t\delta}{\sqrt{d\lambda_1}} \right\} \leq 2 \exp \left( -\frac{\left( \frac{t\delta}{\sqrt{d\lambda_1}} \right)^2}{2t} \right) = 2 \exp \left( -\frac{t\delta^2}{2d\lambda_1} \right).$$

Hence,

$$\Pr\{||W_t - \mu||_2 > \delta\} \leq 2d \exp \left( -\frac{t\delta^2}{2d\lambda_1} \right)$$

for $t > 0$. Invoking Lemma 21, we have that

$$\Pr\{T > t\} \leq \Pr\{||W_t - \mu||_2 > \delta\} \leq 2d \exp \left( -\frac{t\delta^2}{2d\lambda_1} \right)$$

for $t$ greater than $\Upsilon$. If follows that

$$\mathbb{E}[T] = \int_{t=0}^{\infty} \Pr\{T > t\} dt$$

$$\leq \Upsilon + \int_{t=\Upsilon}^{\infty} \Pr\{T > t\} dt$$

$$\leq \Upsilon + \int_{t=\Upsilon}^{\infty} 2d \exp \left( -\frac{t\delta^2}{2d\lambda_1} \right) dt < \infty.$$

This completes the proof of the lemma.

\[\square\]

**Lemma 24** \(\mathbb{E}[T] = \tau.\)

**Proof.** From Lemma 23, we have \(\mathbb{E}[T] < \infty.\) Making use of Wald’s equation, we have

$$\mathbb{E}[W_T] = \mathbb{E}[T] \mu. \quad (90)$$

By the definition of \(T,\) we have

$$AW_T + BT = C$$

almost surely. Taking expectation on both sides of the above equation yields

$$AE[W_T] + B\mathbb{E}[T] = C. \quad (91)$$

Combining (90) and (91) yields

$$(A\mu + B)\mathbb{E}[T] = C. \quad (92)$$

From Lemma 20, we know that there exist $A, B$ and a positive number $C$ such that $As + Bt = C$ is a supporting hyperplane of \(\mathcal{F},\) which passes through $(\tau, \tau \mu)$, where $\tau > 0$. Hence,

$$A(\tau \mu) + B\tau = C > 0$$

56
and it follows that $A\mu + B > 0$. Dividing both sides of (92) by $A\mu + B$ yields

$$E[T] = \frac{C}{A\mu + B} = \frac{A(\tau \mu) + B\tau}{A\mu + B} = \tau.$$  

We are now in a position to prove the theorem. Since $C > 0$ and $(0, 0_d) \in \mathcal{R}$, it follows from Lemma 20 that $As + Bt \leq C$ for all $(t, s) \in \mathcal{R}$. This implies that $T \leq T$ and thus $E[T] \leq E[T]$. Finally, using Lemma 24, we have $E[T] \leq E[T] \leq \tau$. This completes the proof of the theorem.

U Proof of Theorem 25

We shall first show $E[T] \geq \min \mathcal{A}$ under the assumption that $\mathcal{A}$ is not empty. If $E[T] = \infty$, then $E[T] \geq \inf \mathcal{A}$ trivially holds. If $E[T] < \infty$, then $Pr\{T < \infty\} = 1$ and it follows that $W_T$ is well-defined and

$$Pr\{(T, W_T) \in \mathcal{R}\} = 1.$$  

According to Theorem 11 we have

$$(E[T], E[W_T]) \in \mathcal{R}.$$  

Since $E[T] < \infty$, it follows from Wald’s equation that $E[W_T] = E[T]\mu$. Hence,

$$(E[T], E[T]\mu) \in \mathcal{R},$$  

which immediately implies $\sup \mathcal{A} \geq E[T] \geq \inf \mathcal{A}$. This establishes assertions (I) and (II).

It remains to show that $E[T] = \infty$ under the assumption that $\mathcal{A}$ is empty. We use a contradiction method. Suppose that $E[T] < \infty$, then $Pr\{T < \infty\} = 1$ and it follows that

$$Pr\{(T, W_T) \in \mathcal{R}\} = 1.$$  

According to Theorem 11 we have

$$(E[T], E[W_T]) \in \mathcal{R}.$$  

Since $E[T] < \infty$, it follows from Wald’s equation that $E[W_T] = E[T]\mu$. Hence,

$$(E[T], E[T]\mu) \in \mathcal{R},$$  

which immediately implies that $\mathcal{A}$ is not empty. This is a contradiction. Therefore, it must be true that $E[T] = \infty$ if $\mathcal{A}$ is empty. The proof of the theorem is thus completed.
V Proof of Theorem 26

We need some preliminary results.

**Lemma 25** $\mathbb{E}[T] < \infty$.

**Proof.** Note that there exists a number $\delta > 0$ such that $\{ \theta \in \mathbb{R}^d : ||\theta - \mu||_2 \leq \delta \} \subset \mathbb{R}^d$. Since $g$ is a concave function on $\mathbb{R}^d$, it must be a continuous function on $\mathbb{R}^d$. By the bounded-value theorem, we have that there exists a positive number $\Upsilon$ such that $g(\theta) \leq \Upsilon$ for any $\theta$ contained in the set $\{ \theta \in \mathbb{R}^d : ||\theta - \mu||_2 \leq \delta \}$. This implies that

$$\{ T > t \} \subseteq \{ ||W_t - \mu||_2 \geq \delta \}$$

for any $t$ greater than $\Upsilon$. Hence, by the same argument as that of Lemma 23, we can show that $\mathbb{E}[T] < \infty$.

$\square$

We are now in a position to prove the theorem. From Lemma 25, we have $\mathbb{E}[T] < \infty$. Hence, it follows from Wald’s equation that $\mathbb{E}[W_T] = \mathbb{E}[T] \mu$. By the definition of the stopping time $T$, we have

$$T = g \left( \frac{W_T}{T} \right)$$

almost surely. Multiplying both sides of the above inequality by $T$ yields

$$T^2 = Tg \left( \frac{W_T}{T} \right)$$

almost surely. Taking expectation on both sides of the above inequality and using Jensen’s inequality yields

$$\mathbb{E}^2[T] \leq \mathbb{E}[T^2] = \mathbb{E} \left[ Tg \left( \frac{W_T}{T} \right) \right]. \quad (93)$$

Since $T$ is a positive random variable and $g$ is a concave function, it follows from Theorem 3 that

$$\mathbb{E} \left[ Tg \left( \frac{W_T}{T} \right) \right] \leq \mathbb{E}[T]g \left( \frac{\mathbb{E}[W_T]}{\mathbb{E}[T]} \right). \quad (94)$$

Combining (93) and (94) yields

$$\mathbb{E}^2[T] \leq \mathbb{E}[T]g \left( \frac{\mathbb{E}[W_T]}{\mathbb{E}[T]} \right),$$

which implies

$$\mathbb{E}[T] \leq g \left( \frac{\mathbb{E}[W_T]}{\mathbb{E}[T]} \right) = g \left( \frac{\mathbb{E}[T] \mu}{\mathbb{E}[T]} \right) = g(\mu).$$

This completes the proof of the theorem.
Proof of Theorem 27

If $E[T] = \infty$, then the conclusion of the theorem holds trivially. So, it suffices to show the theorem under the assumption that $E[T] < \infty$. Since $E[T]$ is bounded, the Wald’s equation $E[W_T] = E[T] \mu$ holds. As a consequence of the definition of the stopping time $T$ and the fact that $Pr\{T < \infty\} = 1$, we have

$$T = \frac{1}{g(W_T)}, \quad g(W_T) > 0$$

almost surely, which implies that

$$T g(W_T) = 1$$

almost surely. Taking expectation on both sides of the above equation yields

$$E[T g(W_T)] = 1,$$

or equivalently,

$$E\left[T g\left(\frac{W_T}{T}\right)\right] = 1.$$

Since $T$ is a positive random variable and $g$ is a concave function, it follows from Theorem 3 that

$$1 = E\left[T g\left(\frac{W_T}{T}\right)\right] \leq E[T] g\left(\frac{E[W_T]}{E[T]}\right). \tag{95}$$

Applying the Wald’s equation $E[W_T] = E[T] \mu$ to (95) yields

$$1 \leq E[T] g\left(\frac{E[W_T]}{E[T]}\right) = E[T] g\left(\frac{E[T] \mu}{E[T]}\right) = E[T] g(\mu).$$

Since $g(\mu) > 0$, we can conclude from the inequality $1 \leq E[T] g(\mu)$ that $E[T] \geq \frac{1}{g(\mu)}$. This completes the proof of the theorem.

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