Abstract

A fuzzy version of the ordinary round 2-sphere has been constructed with an invariant curvature. We here consider linear connections on arbitrary fuzzy surfaces of genus zero. We shall find as before that they are more or less rigidly dependent on the differential calculus used but that a large number of the latter can be constructed which are not covariant under the action of the rotation group. For technical reasons we have been forced to limit our considerations to fuzzy surfaces which are small perturbations of the fuzzy sphere.
1 Introduction

A fuzzy version of the ordinary round 2-dimensional sphere $S^2$ has been given (Madore 1992, Madore & Grosse 1992, Grosse & Preˇ snajder 1993, Grosse, Klimˇ c´ ık & Preˇ snajder 1997b, Carow-Watamura & Watamura 1997) which uses a noncommutative geometry based on a filtration of the algebra $M_n$ of $n \times n$ complex matrices and a differential calculus based on a set of derivations which form the irreducible representation of dimension $n$ of the Lie algebra of $SU_2$. The rotation group acts on this fuzzy sphere and the only torsion-free metric connection on it has an invariant ‘Gaussian’ curvature. Recently Dimakis & Madore (1996) have proposed a method of constructing an arbitrary number of differential calculi over a given noncommutative associative algebra. We shall use this method to construct linear connections on an arbitrary fuzzy surface $\Sigma$ of genus zero. We shall find as before that the connections are more or less rigidly dependent on the differential calculus but that a large number of the latter can be constructed associated to metrics which are not covariant under the action of the rotation group. For technical reasons we have been forced to limit our considerations to fuzzy surfaces which are small perturbations of the fuzzy sphere.

There seems to be an intimate connection between the commutation relations of an algebra and the linear curvature which it can support. Evidence of this is to be found in previous articles (Madore 1992, Dimakis & Madore 1996, Madore & Mourad 1996, Madore 1996, 1997). To a certain extent the present results furnish another example.

In Section 2 we recall the general method for constructing differential calculi based on a set of derivations which do not necessarily close to form a Lie algebra and we briefly recall the definition of a linear connection which we shall use. Our calculations are based on the noncommutative equivalent of the moving-frame formalism. In Section 3 we construct the frames and in Section 4 we calculate the associated torsion-free metric connections.

2 General Formalism

Suppose one is given an $M_n$-bimodule $\Omega^1(M_n)$ and an application

$$M_n \xrightarrow{d} \Omega^1(M_n)$$

(2.1)

of $M_n$ into $\Omega^1(M_n)$. Then there is a construction (Connes 1994, Dimakis & Madore 1996, Madore & Mourad 1996) which yields a differential calculus $\Omega^*(M_n)$ over $M_n$ which has $\Omega^1(M_n)$ as module of 1-forms. We construct (2.1) using a set of derivations (Dubois-Violette 1988, Dubois-Violette et al. 1989, Dimakis & Madore 1996). Let $\lambda_a$, $1 \leq a \leq 3$, be a set of linearly independent anti-hermitian elements of $M_n$ such that only matrices proportional to the identity commutate with it. The $*$-algebra generated by the $\lambda_a$ is equal then to $M_n$. Consider the derivations $e_a = \text{ad} \lambda_a$. In order for them to have the correct dimensions one must introduce a mass parameter $\mu$ and replace $\lambda_a$ by $\mu \lambda_a$. We shall set $\mu = 1$. Define the map (2.1) by

$$df(e_a) = e_a f.$$  (2.2)

Under certain conditions, given below, $\Omega^1(M_n)$ has a basis $\theta^a$ dual to the derivations,

$$\theta^a(e_b) = \delta^a_b,$$  (2.3)
a basis which commutes therefore with the elements $f$ of $M_n$:

$$f \theta^a = \theta^a f. \quad (2.4)$$

In such cases $\Omega^1(M_n)$ is free as a left or right module and can be identified with the direct sum of 3 copies of $M_n$:

$$\Omega^1(M_n) = M_n \oplus M_n \oplus M_n. \quad (2.5)$$

The $\theta^a \otimes \theta^b$ form a basis for $\Omega^1(M_n) \otimes_{M_n} \Omega^1(M_n)$. The module $\Omega^2(M_n)$ of 2-forms can be identified as a submodule of $\Omega^1(M_n) \otimes_{M_n} \Omega^1(M_n)$ and the projection onto the 2-forms can be written in the form

$$\pi(\theta^a \otimes \theta^b) = P^{ab}_{\quad cd} \theta^c \otimes \theta^d \quad (2.6)$$

where the $P^{ab}_{\quad cd}$ are complex numbers which satisfy

$$P^{ab\quad ef}_{\quad cd} P^{ef}_{\quad cd} = P^{abcd}. \quad (2.7)$$

The product $\theta^a \theta^b$ satisfies therefore

$$\theta^a \theta^b = P^{ab}_{\quad cd} \theta^c \theta^d. \quad (2.8)$$

If the $\theta^a$ exist then it can be shown (Dimakis & Madore 1996, Madore & Mourad 1996) that the $\lambda_a$ must satisfy the equation

$$2 \lambda_c \lambda_d P^{cd}_{\quad ab} - \lambda_c F^{c}_{\quad ab} - K_{ab} = 0 \quad (2.9)$$

with $F^{c}_{\quad ab}$ and $K_{ab}$ complex numbers.

The structure elements $C^{a}_{\quad bc}$ are defined by the equation

$$d\theta^a = -\frac{1}{2} C^{a}_{\quad bc} \theta^b \theta^c. \quad (2.10)$$

They are related to the coefficients of (2.9) by

$$C^{a}_{\quad bc} = F^{a}_{\quad bc} - 2 \lambda_d P^{(da)}_{\quad bc}. \quad (2.11)$$

The fuzzy sphere has a differential calculus defined by choosing the $\lambda_a$ to be the irreducible $n$-dimensional representation of the Lie algebra of $SU_2$. In this case

$$P^{ab}_{\quad cd} = \frac{1}{2}(\delta^a_c \delta^b_d - \delta^b_c \delta^a_d), \quad (2.12)$$

the $C^{a}_{\quad bc} = F^{a}_{\quad bc}$ are the $SU_2$ structure constants and the $K_{bc}$ vanish.

To discuss the commutative limit it is convenient to change the normalization of the generators $\lambda_a$. We introduce the parameter $\bar{k}$ with the dimensions of (length)$^2$ and define ‘coordinates’ $x_a$ by

$$x_a = i \bar{k} \lambda_a. \quad (2.13)$$

The $x_a$ satisfy therefore the commutation relations

$$[x_a, x_b] = i \bar{k} x_c C^c_{\quad ab}. \quad (2.14)$$

We choose the $\lambda_a$ so that $C_{abc} = r^{-1} \epsilon_{abc}$ where $r$ is a length parameter. These structure constants are in general independent from the structure elements introduced in (2.10) but in the case of the fuzzy sphere they are equal. Introduce the $SU_2$-Casimir metric
The projection of $\Omega_1$ restricted to $V$ if $\theta$ is the element of the calculus $\Omega_1^*$ then we would be tempted to identify $k$ with the inverse of the square of the Planck mass, $k = \mu_P^{-2}$, and consider space-time as fundamentally non-commutative in the presence of gravity.

The differential calculus on the fuzzy sphere has (Madore 1992) a basis

$$\theta^a = -C^a_{\ b c} x^b dx^c - ikr^{-2}\theta x^a.$$  \hspace{1cm} (2.16)

The 1-form $\theta$ can be written

$$\theta = ik^{-1}x_a \theta^a = r^2k^{-2}x_adx^a.$$  \hspace{1cm} (2.17)

In the commutative limit $\theta$ diverges but $k\theta \to r^2A$ where $A$ is the Dirac-monopole potential of unit magnetic charge. The commutative limit of the frame $\theta^a$ is a moving frame on a $U_1$-bundle over $S^2$. In this case it is not the frame bundle. A standard Kaluza-Klein reduction gives rise to the potential $A$ as well as the geometry of the sphere.

The formalism which we shall use resembles that which can be used to describe a manifold $V$ of dimension $d$ which is defined by its metric embedding in a flat euclidean space $\mathbb{R}^n$. If there are no topological complications then generically $n = d(d + 1)/2$ since this is the number of independent components of a local metric on $V$. Let $x^a$ be the coordinates of the embedding space and $y^a$ local coordinates of $V$. Then $V$ is defined locally by equations of the form $x^a = x^a(y^a)$. Let $g_{ab}$ be the components of the flat metric on $\mathbb{R}^n$. The local components $h_{\alpha\beta}$ of the induced metric on $V$ are given by

$$h_{\alpha\beta} = g_{ab} \frac{\partial x^a}{\partial y^\alpha} \frac{\partial x^b}{\partial y^\beta}.$$  

Let $x^a = x^a + h^a(x^b)$ be a variation of the coordinates of the embedding space $\mathbb{R}^n$. The most general variation $h'_{\alpha\beta}$ of $h_{\alpha\beta}$ can be obtained by the induced variation $g'_{ab} = g_{ab} - \partial_a h_b$ of $g_{ab}$:

$$h'_{\alpha\beta} = h_{\alpha\beta} - \partial_a h_b \frac{\partial x^a}{\partial y^\alpha} \frac{\partial x^b}{\partial y^\beta}. \hspace{1cm} (2.18)$$

If $\theta^a$ is a moving frame on $\mathbb{R}^n$ then the new metric $g'$ is given by

$$g'(\theta^a \otimes \theta^b) = g^{ab} + \partial (h^a).$$

A moving frame for $g'$ is given then by $\theta'^a = (\delta^a_b + \Lambda^a_b)\theta^b$ with

$$\Lambda^a_b = -\partial_b \theta^a.$$  \hspace{1cm} (2.19)

The algebra $\mathcal{C}(V)$ of smooth functions on $V$ can be identified with a quotient of the algebra of $\mathcal{C}(\mathbb{R}^n)$ and the differential calculus $\Omega^*(V)$ can be identified with a quotient of the calculus $\Omega^*(\mathbb{R}^n)$. It is convenient to designate by $dx^\alpha$ the element $dx^\alpha$ of $\Omega^1(\mathbb{R}^n)$ restricted to $V$. Then $1 \leq \alpha \leq n$, but the differentials $dx^\alpha$ satisfies $n - d$ relations. The projection of $\Omega^1(\mathbb{R}^n)$ onto $\Omega^1(V)$ can be written in the form

$$dx^\alpha = \pi^\alpha_a dx^a.$$  \hspace{1cm} (2.20)
The moving frame \( \theta^a \) can be chosen such that \( \pi^a_\alpha \theta^a \) is a moving frame on \( V \) for the induced metric with components \( h_{\alpha \beta} \). The transformation to a moving frame for the perturbed metric \( h'_{\alpha \beta} \) is given therefore by

\[
\pi_a^\alpha \delta_b^a + \tilde{\Lambda}_b^a = \pi_a^\alpha - \pi_a^\alpha \partial_b h^a.
\]

There is also a projection of the parallelizable Hopf bundle \( P \) onto \( \Sigma \). Therefore \( \mathcal{C}(\Sigma) \) can be identified as a subalgebra of \( \mathcal{C}(P) \) and the differential calculus \( \Omega^*(\Sigma) \) can be identified with a differential subalgebra of the calculus \( \Omega^*(P) \). We shall use this identification also when considering the differential calculus over \( \Sigma \). In general there is always a projection of the (parallelizable) frame bundle \( L \) of a manifold \( V \) onto \( V \) itself and so \( \Omega^*(V) \) can be considered as a subalgebra of a differential calculus \( \Omega^*(L) \) which has a free module of 1-forms.

### 3 Perturbations of the calculus

We are interested in perturbations of the fuzzy sphere which could be considered as noncommutative versions of a topological sphere which is not invariant under the action of the rotation group. We first introduce a perturbation \( \Omega^*(M_n) \) of the differential calculus \( \Omega^*(M_n) \) of the fuzzy sphere. We shall require that the perturbed differential calculus be based on derivations. We introduce then

\[
e'_a = \frac{1}{ik} \text{ad} x'_a
\]

with

\[
x'_a = x_a + h_a
\]

and where \( h_a \) is a set of 3 elements of \( M_n \) with \( |h_a| << r \). If we write the perturbed version of (2.9) as

\[
2x'_c x'_d P^{cd}_{ab} - ikx'_c F^{tc}_{ab} + k^2 K'_ab = 0
\]

and expand the coefficients as

\[
P^{cd}_{ab} = \frac{1}{2}(\delta^c_a \delta^d_b - \delta^d_a \delta^c_b) + P^{cd}_{(1)ab}, \quad F^{tc}_{ab} = C^{tc}_{ab} + F^{tc}_{(1)ab}, \quad K'_ab = K_{(1)ab}
\]

we find the equation

\[
[x_a, h_b] - [x_b, h_a] + 2x_c x_d P^{cd}_{(1)ab} - ikx_c C^{tc}_{ab} - ikx_c F^{tc}_{(1)ab} + k^2 K_{(1)ab} = 0 \quad (3.2)
\]

for the perturbations. We can suppose that \( P^{cd}_{(1)ab} \) is symmetric in the first two indices since an antisymmetric part can be absorbed in a redefinition of the fifth term. It must be then antisymmetric in the last two if \( P^{cd}_{ab} \) is to be a projector. The \( P^{cd}_{(1)ab} \) could be also chosen to be trace-free in the first two indices since the trace part could be absorbed in a redefinition of the last term.

The most general solution to the equation

\[
[x_a, h_b] - [x_b, h_a] - ikh_c C^{tc}_{ab} = 0 \quad (3.3)
\]

is given by \( h_a = [x_a, f] \) where \( f \) is an arbitrary anti-hermitian matrix. We obtain thereby \( n^2 - 1 \) linearly independent solutions to Equation (3.2) with

\[
P^{cd}_{(1)ab} = 0, \quad F^{tc}_{(1)ab} = 0, \quad K_{(1)ab} = 0. \quad (3.4)
\]
These solutions correspond to the trivial redefinition of the generators \( x_a \) which leave invariant the commutation relations (2.14). The most general such operation is given by the map \( x_a \mapsto u^{-1}x_a u \) with \( u \) a unitary matrix. In the linear approximation \( u \approx 1 + f \).

The commutative limit suggests the Ansatz \( h_a = hx_a \) with \( h \) an arbitrary element of \( M_n \). Equation (3.2) simplifies to the equation

\[
x_{[a} h_{b]} + 2x_c x_d P_{(1)ab}^{cd} - i k x_c F_{(1)ab}^c + k^2 K_{(1)ab} = 0
\]  

with \( P_{(1)ab}^{cd} \) symmetric in the first two indices. We have been unable to find interesting explicit solutions to this equation.

Some solutions to (3.2) for \( h_a \) can be found by inspection. A trivial ‘monopole’ solution, with the \( h_a = f_a \) all proportional to the unit matrix, is given by

\[
P_{(1)ab}^{cd} = 0, \quad F_{(1)ab}^c = 0, \quad k K_{(1)ab} = i f_c C_{(1)ab}^{cd}.
\]  

A ‘dipole’ solution of the form \( h_a = f_{ab} x^b \) is given by

\[
P_{(1)ab}^{cd} = 0, \quad F_{(1)ab}^c = C_{(1)ab}^{cd} f_{cd} - C_{(1)ab}^{de} f_{de}, \quad K_{(1)ab} = 0.
\]  

A ‘quadrupole’ solution of the form \( h_a = f_{abc} x^b x^c \) can also be found with \( f_{abc} \) all proportional to the unit matrix. In the linear approximation this becomes for the solution (3.8)

\[
[x_{[a'} x_{b]}] = i k x_c x_{[a'} e^{(C_{(1)ab})}_{b']c} - 2 x_{[a'} x_{b]} (P_{(1)ab}^{cd} - P_{(1)ab}^{cd}) - k^2 K_{(1)ab} = 0.
\]

In the linear approximation this becomes for the solution (3.8)

\[
[x_{[a'} x_{b]}] = i k x_c x_{[a'} e^{(C_{(1)ab})}_{b']c} - i k x_{[a'} x_{b]} e^{(C_{(1)ab})}_{b']c}.
\]  

The \( x_a' \) are a new set of generators of the matrix algebra which satisfy different commutation relations. The differential calculus is defined in terms of the corresponding derivations \( e'_a \). In the commutative limit the commutation relations (3.10) induce a Poisson structure

\[
\{ x_{[a'}, x_{b]} \} = x_{[a'} e^{(C_{(1)ab})}_{b']c} - x_{[a'} x_{b]} e^{(C_{(1)ab})}_{b']c}.
\]  

on the surface. The \( x_a' \) designate here coordinates in \( \mathbb{R}^3 \). It is interesting to note that although the commutation relations (3.10) do not in principle fix the noncommutative differential calculus, the limiting Poisson structure (3.11) depends on the differential structure of \( \Sigma \). The coordinate transformation \( x_a \mapsto x_{a'} \) is not a symplectomorphism. It was shown some time ago that symplectomorphisms are limits of transformations of the form (3.3); this was one of the results of the authors quoted below in Section 5. We refer to Madore (1995) for a recent discussion.
The condition (2.9) is a necessary condition for the existence of the dual basis $\theta^a$. To show that effectively the $\theta^a$ exist we must construct them explicitly. We write

$$\theta^a = \theta^a + \theta^a_{(1)}.$$ 

Then from the perturbed version of (2.3) we find the relation

$$\theta^a_{(1)}(e_b) + \theta^a(e_{(1)b}) = 0 \quad (3.12)$$

for the perturbation $\theta^a_{(1)}$. We write this as

$$i k \theta^a_{(1)}(e_b) = C^a_{ab} x^b \theta^c - r^{-2} h_b x^a \theta^b + i kr^{-2} C^a_{bc} f^b_{de} x^d x^e \theta^e. \quad (3.13)$$

We shall find below that the $f_{abc}$ must be completely symmetric in all indices. In this case we find that

$$\theta^a_{(1)} = -(2 - \frac{k^2}{r^4}) f^a_{bc} x^b \theta^c - r^{-2} h_b x^a \theta^b + i kr^{-2} C^a_{bc} f^b_{de} x^d x^e \theta^e. \quad (3.14)$$

We can write therefore

$$\theta^a = (\delta^a_b + \Lambda^a_b) \theta^b \quad (3.15)$$

where

$$\Lambda^a_b = -(2 - \frac{k^2}{r^4}) f^a_{bc} x^b - r^{-2} h_b x^a + i kr^{-2} C^a_{bc} f^c_{de} x^d x^e. \quad (3.16)$$

It must be emphasized that there are two distinct differential calculi and two distinct modules of 1-forms. In $\Omega^1(M_n)$ the $\theta^a$ are the preferred basis which commute with the elements of the algebra. The $\theta^a$ satisfy therefore from (3.15) the commutation relations

$$[\theta^a, f] = [\Lambda^a_b, f] \theta^b \neq 0.$$ 

In $\Omega^1(M_n)$ the $\theta^a$ are the preferred basis which commute with the elements of the algebra. The $\theta^a$ satisfy therefore from (3.15) the commutation relations

$$[\theta^a, f] = -[\Lambda^a_b, f] \theta^b \neq 0.$$ 

Since the $\theta^a$ are defined on the derivations $e_a$ and the $\theta^a$ are defined on the derivations $e'_a$, to compare the two we must extend them both to the union of the two sets of derivations. This can be done since the differential $d\theta^a$ can be defined on an arbitrary derivation $X$ by the formula $d\theta^a(X) = X \theta^a$. On the extended set neither basis commutes with the elements of the algebra. We recall that the vector space of all derivations of $M_n$ is of dimension $n^2 - 1$ and the modules of 1-forms we consider here are defined on subspaces thereof of dimension 3.

In the commutative limit both the differential calculi defined above tend to the de Rham differential calculus on the $U_1$-bundle $\mathcal{P}$ over $\Sigma$. In this limit $\Lambda^a_b \rightarrow \tilde{\Lambda}^a_b$ where

$$\tilde{\Lambda}^a_b = -2 f^a_{bc} x^c - r^{-2} h_b x^a = -\partial_b h^a - r^{-2} h_b x^a \quad (3.17)$$

is a change of basis between two moving frames on $\mathcal{P}$. The sphere $S^2$ is defined by the relation $g_{ab} x^a x^b - r^2 = 0$ on the ordinary euclidean coordinates of $\mathbb{R}^3$. We have therefore (2.20) with

$$\pi^a_a = \delta^a_a - r^{-2} x^a x_a$$

and (2.21) follows from (3.17). One verifies that $\pi^a_a \theta^a$ is, by restriction to $\Omega^1(S^2)$, a (redundant) moving frame for the metric on the sphere. The perturbed metric is given therefore by (2.21).
The quadrupole perturbation is rather special. However it can be combined with a redefinition of the generators given by a nonperturbative matrix \( u \). In this way an arbitrarily large number of perturbative solutions to the Equation (2.9) can be found. It is easy to localize the obstruction to higher-order multipole perturbations in the linear approximation. The third term in Equation (3.2) is only quadratic in the variables and cannot therefore combine with higher-order polynomials appearing in the first two terms. Such a polynomial would appear only in the next approximation with terms of the form \( x_{(e)cd} P_{(1)ab}^{cd} \).

4 Linear connections

A linear connection is defined (Dubois-Violette et al. 1995, 1996) as a couple \((D, \sigma)\) where \( D \) is a covariant derivative and \( \sigma \) is a generalized permutation. The unique torsion-free metric connection on the fuzzy sphere, or rather the fuzzy \( U_1 \)-bundle over it, is given by

\[
D \theta^a = -\frac{1}{2} C^a_{\ b} \theta^b \otimes \theta^c, \quad \sigma(\theta^a \otimes \theta^b) = \theta^b \otimes \theta^a. \tag{4.1}
\]

The metric on the fuzzy bundle can be defined by the condition

\[
g(\theta^a \otimes \theta^b) = g^{ab} \tag{4.2}\]

with \( g^{ab} \) the components of the euclidean metric. This defines a metric on the surface in the commutative limit.

The choice of metric is strongly restricted by the choice of differential calculus. If we introduce a new basis \( \tilde{\theta}^a \) of \( \Omega^1(M_n) \) defined by \( \tilde{\theta}^a = (\delta^a_b + \Lambda^a_b) \theta^b \) with the \( \Lambda^a_b \) arbitrary elements of \( M_n \) then of course the coefficients of the metric change accordingly just as in the commutative case. The difference lies in the fact that we cannot introduce a new metric \( \bar{g} \) given by \( \bar{g}(\tilde{\theta}^a \otimes \tilde{\theta}^b) = g^{ab} \). Unless the \( \Lambda^a_b \) lie in the center of the algebra this equation would be inconsistent since \( f \bar{g}^{ab} = g^{ab} f \) but in general

\[
f \bar{g}(\tilde{\theta}^a \otimes \tilde{\theta}^b) = \bar{g}(f \tilde{\theta}^a \otimes \tilde{\theta}^b) \neq \bar{g}(\tilde{\theta}^a \otimes \tilde{\theta}^b f) = \bar{g}(\tilde{\theta}^a \otimes \tilde{\theta}^b)f. \]

A metric on the new calculus can be defined by the condition

\[
g(\theta'^a \otimes \theta'^b) = g^{ab}. \tag{4.3}\]

Since they lie in different differential calculi we must consider \( \theta'^a \neq \tilde{\theta}^a \) if \( \Lambda^a_b \) is given by (3.16).

The perturbed linear connection associated to the perturbed differential calculus is defined by the covariant derivative

\[
D \theta'^a = -\omega'^a_{\ bc} \theta'^b \otimes \theta'^c \tag{4.4}
\]

and a generalized perturbation

\[
\sigma'(\theta'^a \otimes \theta'^b) = S'^{ab}_{\ cd} \theta'^c \otimes \theta'^d.
\]

We write as above

\[
S'^{ab}_{\ cd} = \delta^a_c \delta^b_d + S_{(1)cd}^{ab}
\]

and expand also the coefficients of the maps \( \tau' \) and \( \chi' \) introduced in Madore & Mourad (1996):

\[
T'^{ab}_{\ cd} = 2 \delta^a_c \delta^b_d + T_{(1)cd}^{ab}, \quad \chi'^{ab}_{\ cd} = \chi_{(1)cd}^{ab}.
\]
Then the most general expression for $\omega'^a_{bc}$ is given by
\begin{equation}
\omega'^a_{bc} = \frac{1}{2} C'^a_{bc} + x^a_{(1)bc} - i x_d S'^{ad}_{(1)bc} \tag{4.5}
\end{equation}
and the most general expression for $S'^{ab}_{(1)cd}$ is given by
\begin{equation}
S'^{ab}_{(1)cd} = \frac{1}{2} T'^{ab}_{(1)cd} - 2 P'^{ab}_{(1)cd}. \tag{4.6}
\end{equation}
Comparing (4.5) with (3.9) we see that the condition
\begin{equation}
\omega'^a_{[bc]} = C'^a_{bc}
\end{equation}
that the connection be torsion-free is indeed respected. The most ‘natural’ generalized permutation is given by $\tau = 2$. We have then
\begin{equation}
\sigma(\theta'^a \otimes \theta'^b) = \theta'^b \otimes \theta'^a - 2 P'^{ab}_{(1)cd} \theta'^c \otimes \theta'^d. \tag{4.7}
\end{equation}
A straightforward generalization of the definition of metric compatibility (Dubois-Violette et al. 1995, Madore & Mourad 1996) can be written in the form
\begin{equation}
\omega'^a_{bc} + \omega'^b_{cd} S'^{ad}_{be} = 0 \tag{4.8}
\end{equation}
A short calculations yields that the left-hand side of this equation is given by
\begin{equation}
\omega'^a_{bc} + \omega'^b_{cd} S'^{ad}_{be} = - i x_d (S'^{ad}_{(1)bc} + S'^{ad}_{(1)c} d^a) + \left( \chi'^a_{(1)bc} + \chi'^a_{1}cb + \frac{1}{2} C'^e_{cd} S'^{ad}_{(1)be} \right) \tag{4.9}
\end{equation}
Provided the constraint
\begin{equation}
C'^e_{cd} S'^{ad}_{(1)be} = C'^e_{da} S'^{ad}_{(1)c} d^b \tag{4.10}
\end{equation}
is satisfied the condition that the last term of (4.9) vanish yields the value
\begin{equation}
\chi'^a_{(1)bc} = - \frac{1}{4} \left( C'^e_{cd} S'^{ad}_{(1)be} + C'^e_{da} S'^{ad}_{(1)c} d^e - C'^e_{bd} S'^{ad}_{(1)c} d^e \right) \tag{4.11}
\end{equation}
for the coefficients of $\chi$. The condition of metric compatibility becomes then
\begin{equation}
S'^{ad}_{(1)bc} + S'^{ad}_{(1)c} d^a = 0 \tag{4.12}
\end{equation}
The Equations (4.10) and (4.12) are to be considered as equations for the coefficients of $\tau$. There is a solution to these two equations, given by
\begin{equation}
T'^{ad}_{(1)bc} = - 4 P'^{ad}_{(1)c} d^a \tag{4.13}
\end{equation}
provided that $f_{abc}$ is completely symmetric in all indices. The expression (4.11) for the coefficients of $\chi$ simplifies to
\begin{equation}
\chi'^a_{(1)bc} = 2 f'^a_{bc}. \tag{4.14}
\end{equation}
The corresponding coefficients of $\sigma$ are given by
\begin{equation}
S'^{ad}_{(1)bc} = - 2 (P'^{ad}_{(1)bc} - P'^{ad}_{(1)c} d^a + P'^{ad}_{(1)c} d^a). \tag{4.15}
\end{equation}
There are perhaps other solutions. The final values for the coefficients of the connection are found by using (4.14) and (4.15) in (4.5). The resulting expression is formally the same in the commutative limit with the matrices $x^a$ replaced by coordinates $x^a$.

Using (4.14) and (4.15) we see that the coefficients (4.5) of the perturbed linear connection can be written in the form
\begin{equation}
\omega'^a_{bc} = \frac{1}{2} C'^a_{bc} + 2 f'^a_{bc} + 2 x_d^f (f'^b_{de} C'^a_{ce} + f'^b_{ae} C'^d_{ce} - f'^b_{ec} C'^d_{ae}) \tag{4.16}
\end{equation}
We have constructed a linear connection on a bundle over the surface. It will yield upon reduction à la Kaluza-Klein a linear connection, an electromagnetic potential and a scalar field on the surface itself. From (2.16) we see that the electromagnetic field will have unit magnetic charge. We see here also that the electromagnetic potential can be identified with the component of the frame normal to the surface.
5 Discussion

One can formally write a curvature associated to \((D', \sigma')\) but the definition of curvature is not satisfactory. The main interest of curvature in the case of a smooth manifold \(V\) is the fact that it is local. Riemann curvature can be defined as a map

\[
\Omega^1(C(V)) \xrightarrow{R} \Omega^2(C(V)) \otimes_{C(V)} \Omega^1(C(V)).
\]

If \(\xi \in \Omega^1(C(V))\) then \(R(\xi)\) is local; at a given point it depends only on the value of \(\xi\) at that point. This can be expressed as a bilinearity condition; the above map is a \(C(V)\)-bimodule map. If \(f \in C(V)\) then

\[
fR(\xi) = R(f\xi), \quad R(\xi f) = R(\xi)f.
\]

In the noncommutative case bilinearity is the natural (and only possible) expression of locality. It has not yet been possible to enforce it in a satisfactory manner. We refer to Dubois-Violette et al. (1996) for a recent discussion.

If we compare the expression (3.11) for the induced Poisson structure with the expression (4.16) for the linear connection we see that there is definitely a connection between the two. Further evidence of this is given in another article (Madore 1997). The exact extent to which one is determined by the other is not clear.

Although we are primarily interested in the matrix version of surfaces as an model of an eventual noncommutative theory of gravity they have a certain interest in other, closely related, domains of physics. Without the differential calculus the fuzzy sphere is basically just an approximation to a classical spin \(r\) by a quantum spin \(\tilde{r}\) given by (2.15) with \(\hbar\) in lieu of \(\kappa\). It has been extended in various directions under various names and for various reasons by Berezin (1975), de Wit et al. (1988), Hoppe (1989), Fairlie et al. (1989), Floratos et al. (1989), Cahen et al. (1990) and Bordemann et al. (1991). In order to explain the finite entropy of a black hole it has been conjectured, for example by 't Hooft (1996), that the horizon has a structure of a fuzzy 2-sphere since the latter has a finite number of ‘points’ and yet has an \(SO_3\)-invariant geometry. Matrix descriptions of general \(p\)-dimensional hypersurfaces were first proposed in connection with Dirichlet \(p\)-branes and their relation to \(M\)-theory (Banks et al. 1996). Attempts to endow them with a smooth differential structure has been made (Madore 1996, Grosse et al. 1997a).

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