Energy-Based Control of Nonlinear Infinite-Dimensional Port-Hamiltonian Systems with Dissipation

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Abstract—In this paper, we consider nonlinear PDEs in a port-Hamiltonian setting based on an underlying jet-bundle structure. We restrict ourselves to systems with 1-dimensional spatial domain and 2nd-order Hamiltonian including certain dissipation models that can be incorporated in the port-Hamiltonian framework by means of appropriate differential operators. For this system class, energy-based control by means of Casimir functionals as well as energy balancing is analysed and demonstrated using a nonlinear Euler-Bernoulli beam.

I. INTRODUCTION

The port-Hamiltonian (pH) system formulation in combination with energy-based control schemes has turned out to be an effective tool for the description and control of nonlinear finite-dimensional systems, see [1], [2] for instance. Especially, for the stabilisation of, e.g., non-energy minimal equilibria the control by interconnection based on Casimir functionals as well as the energy balancing methodology are well established.

In the infinite-dimensional scenario, the pH system representation is – in contrast to the finite-dimensional scenario – not unique. With regard to control engineering purposes the Stokes-Dirac approach, see [3], [4], [5], as well as an approach based on jet-bundle structures, see [6], [7], have turned out to be adequate frameworks. These approaches mainly differ in the choice of variables (energy variables versus derivative variables in the Hamiltonian). This has the consequence that for linear partial differential equations (PDEs) the Stokes-Dirac framework is closely connected to well-known functional-analytic methods, in particular to the theory of strongly continuous semigroups [8], whereas the jet-bundle approach is very well suited for systems that allow a variational characterisation including nonlinear systems.

One of the major benefits of infinite-dimensional pH system formulations is that they provide a consistent framework for the development of (finite-dimensional) boundary controllers which are of great practical relevance. In view of this, the two most important schemes that have already been extended to the infinite-dimensional setting are the energy-Casimir method (see, e.g., [9] for the Stokes-Dirac approach and [10], [11] for the jet-bundle framework) as well as the control by energy balancing (see [12]). Note that in [9], [12] as well as in [10], [11] only linear mechanical structures have been considered to show the applicability of the proposed control schemes. It is worth stressing that, similar to the finite-dimensional case, both methods can be used to stabilise, e.g., non-energy minimal rest positions; however, the energy-Casimir method yields dynamic controllers whereas energy balancing usually leads to static control laws.

In this paper, we focus on the description of (nonlinear) mechanical systems on 1-dimensional spatial domains formulated within the jet-bundle approach. To demonstrate the capability of this approach, we refer to [10], [11] for systems with 1st- and 2nd-order Hamiltonian densities. The order of the Hamiltonian density is basically responsible for the number of boundary-port categories that can be introduced. Therefore, in what follows, we confine ourselves to 2nd-order Hamiltonian densities. In general, the considered jet-bundle approach, where the Hamiltonian depends on derivative variables, can be divided in the non-differential case and the more general differential-operator case with respect to the interconnection and damping maps, see [6]. In this setup, for mechanical systems the interconnection map can usually be chosen as the canonical map and nontrivial dissipation maps can be used to include certain damping models.

Therefore, the main contributions of this paper are as follows: i) we study the impact of certain differential operators – used as dissipation mappings – on the boundary ports of a nonlinear infinite-dimensional pH-system in the jet-bundle approach, see Section III; ii) we propose a control scheme based on the energy-Casimir method that exploits a certain boundary-output assignment in the pH-framework including differential operators (e.g. nontrivial dissipation maps) and 2nd-order Hamiltonian densities, see Section IV; iii) the control by energy balancing (EBC) is investigated for nonlinear PDEs within the jet-bundle framework mainly exploiting the geometric properties of the corresponding boundary operators, see Section V. To show the applicability of the proposed theory including the control strategies, the example of a nonlinear Euler-Bernoulli beam subject to structural damping is considered.

II. NOTATION AND PRELIMINARIES

In this paper, we make heavy use of differential-geometric methods, with a notation similar to [13]. To keep the
formulas short and readable, we apply tensor notation and Einstein's convention on sums. The symbols \( \wedge, | \) and \( \mathfrak{d} \) denote the exterior (wedge) product, the natural contraction between tensor fields and the exterior derivative, respectively. It should be noted that the use of pull-back bundles is omitted for ease of presentation. Furthermore, the ranges of the used indices are not indicated when they are clear from the context.

The set of all smooth functions on an arbitrary manifold \( \mathcal{M} \) is denoted by \( C^\infty(\mathcal{M}) \).

In the following, we investigate PDEs with 1-dimensional spatial domain. To be able to distinguish between dependent and independent coordinates, we introduce bundle structures. Let us consider the bundle \( \pi : \mathcal{E} \to \mathcal{B} \), where \( \pi \) is a surjective submersion — called projection — from the total manifold \( \mathcal{E} \) to the base manifold \( \mathcal{B} \). Since we confine ourselves to 1-dimensional spatial domains, \( \mathcal{B} \) only possesses the independent (spatial) coordinate \( z^1 \). Note that \( \partial \mathcal{B} \) represents the boundary of the manifold \( \mathcal{B} \), and the restriction of a mathematical expression to \( \partial \mathcal{B} \) is indicated with \( (\cdot)|_{\partial \mathcal{B}} \).

Furthermore, the total manifold \( \mathcal{E} \) comprises the coordinates \( (z^1, x^\alpha) \) with \( \alpha = 1, \ldots, n \), where \( x^\alpha \) denote the dependent coordinates. Next, let us introduce derivative coordinates (jet variables). To this end, we consider the (higher-order) jet manifold \( J^r(\mathcal{E}) \), possessing the coordinates \( (z^1, x^\alpha, x^\alpha_1, \ldots, x^\alpha_{1^n}) \), where for instance \( x^\alpha_{11} \) denotes the 2nd-order derivative coordinate, i.e. the 3rd derivative of \( x^\alpha \) with respect to the independent coordinate \( z^1 \). Here, \( r \) denotes the highest occurring order of derivatives and, exemplarily, for the 4th jet manifold \( J^4(\mathcal{E}) \), we have the coordinates \( (z^1, x^\alpha, x^\alpha_1, x^\alpha_1, x^\alpha_{11}, x^\alpha_{111}) \).

Furthermore, we introduce the tangent bundle \( \tau_\mathcal{E} : \mathcal{T}(\mathcal{E}) \to \mathcal{E} \) possessing the coordinates \( (z^1, x^\alpha, \dot{z}^1, \dot{x}^\alpha) \), together with the abbreviations \( \partial_1 = \partial/\partial z^1 \) and \( \partial_\alpha = \partial/\partial x^\alpha \) denoting the fibre bases of the bundle. Of special interest is the so-called vertical tangent bundle \( \nu : \mathcal{V}(\mathcal{E}) \to \mathcal{E} \), which is a subbundle of \( \tau_\mathcal{E} \) and equipped with the coordinates \( (z^1, x^\alpha, \dot{x}^\alpha) \). By means of the total derivative \( d_1 = \partial_1 + x_1^\alpha \partial_\alpha + x_{11}^\alpha \partial_{\alpha 1} + x_{111}^\alpha \partial_{\alpha 11} + \ldots \), we are able to introduce the \( n \)th prolongation of a vertical vector field \( \nu = \nu^\alpha \partial_\alpha \), where for instance the 2nd prolongation reads as \( j^2(\nu) = \nu^\alpha \partial_\alpha + d_1(\nu^\alpha) \partial_{\alpha 1} + d_{11}(\nu^\alpha) \partial_{\alpha 11} + \ldots \), with the abbreviations \( \partial_{\alpha 1} = \partial/\partial x^\alpha_1 \), \( \partial_{\alpha 11} = \partial/\partial x^\alpha_{11} \) and \( d_{11} = d_1 \circ d_1 \) denoting the repeated total derivative.

In what follows, we also need some further important structures in order to be able to consider one-forms. The cotangent bundle \( \tau_\mathcal{E}^* = \mathcal{T}^*(\mathcal{E}) \to \mathcal{E} \), where we have the coordinates \( (z^1, x^\alpha, \dot{z}^1, \dot{x}^\alpha) \) and the holonomic bases \( dz^1, dx^\alpha \), allows us to locally define a section \( \omega = \omega_\alpha dz^1 + \omega_\alpha dx^\alpha \), with \( \omega_\alpha, \omega_\alpha \in C^\infty(\mathcal{E}) \), on it. In this paper, we focus on (Hamiltonian) densities \( \mathcal{H} = \int_B \mathcal{H}(\omega) \) and (Hamiltonian) functionals \( \mathcal{E} = \int_B \mathcal{E}(\omega) \), with \( \mathcal{H} \in C^\infty(J^2(\mathcal{E})) \) — i.e. on densities that may depend on derivative coordinates —, where \( \Omega = dz^1 \) denotes the volume element on \( \mathcal{B} \) and \( \Omega_1 = \partial_1 | dz^1 \) the boundary-volume form. The bundle structure \( \pi : \mathcal{E} \to \mathcal{B} \) allows us to construct some further geometric objects like the tensor bundle \( \mathcal{W}_r(\mathcal{E}) = T^*(\mathcal{E}) \wedge T^*(\mathcal{B}) \) with a typical element \( \omega_{\alpha} dx^{\alpha} \wedge dz^1 \) for \( \mathcal{W}_1(\mathcal{E}) \), where \( \omega_{\alpha} \in C^\infty(J^2(\mathcal{E})) \) is met. Moreover, we consider \( k \)th-order linear differential operators \( \mathcal{D} : \mathcal{W}_r(\mathcal{E}) \to \mathcal{V}(\mathcal{E}) \) serving as a map of an element \( \mathcal{W}_r(\mathcal{E}) \) of jet order \( r \) to an element \( \mathcal{V}(\mathcal{E}) \) of jet order \( r + k \).

Further, we introduce the so-called horizontal exterior derivative \( d_h \), meeting \( d_h(\omega) = d_1^1 \wedge d_1(\omega) \) for a form \( \omega : J^r(\mathcal{E}) \to T^*(J^r(\mathcal{E})) \), to be able to make use of Stokes' theorem, see [13] and [6, Appendix] for more details.

III. INFINITE-DIMENSIONAL PH-SYSTEMS

In this section, we extend the pH-framework presented in [6, Definition 4] to 2nd-order Hamiltonian densities with the restriction on 1-dimensional spatial domains by following the findings of [11]. This framework is mainly based on the underlying jet-bundle structures of the PDEs under consideration and makes heavy use of a certain power-balance relation. Moreover, we solely investigate systems with boundary in- and outputs, i.e. in-domain in- and outputs are not addressed within this paper.

Let \( \mathcal{J} \) be a 2nd-order Hamiltonian, i.e. \( \mathcal{J} \in C^\infty(J^2(\mathcal{E})) \), then a pH system formulation using differential operators is

\[
\dot{\mathcal{J}} = (\mathcal{J} - \mathcal{R})(\mathcal{E} \delta \mathcal{J}),
\]

including appropriate boundary conditions. Here, \( \mathcal{J}, \mathcal{R} \) are \( r \)th-order linear vector-valued differential operators and serve as maps \( \mathcal{J}, \mathcal{R} : \mathcal{W}_1(\mathcal{E}) \to \mathcal{V}(\mathcal{E}) \) with the following properties. The interconnection operator \( \mathcal{J} \) characterises the internal power flow, and enjoys the property of formal skew-adjointness, i.e. \( \mathcal{J}(\eta) \mathcal{E} = -\mathcal{J}(\mathcal{E}) \eta + d_h(\eta) \), with \( j = 1 \Omega_1 \) and \( \eta \in \mathcal{V}_1(\mathcal{E}) \). In addition, \( \mathcal{R} \) comprises dissipative effects and is described by a formally self-adjoint, non-negative operator meeting \( \mathcal{R}(\eta) \mathcal{E} = \mathcal{R}(\mathcal{E}) \eta + d_h(\tau) \) with \( \tau = r \Omega_1 \), and the non-negativity relation \( \mathcal{R}(\eta) \eta \geq 0 \).

Furthermore, for 2nd-order Hamiltonian densities, the variational derivative corresponds to \( \delta \mathcal{J} = \delta_h \mathcal{H}(dx^\alpha \wedge \Omega_1) + \delta_\alpha(\cdot) - d_1(\delta_\alpha(\cdot)) + d_{11}(\delta_{\alpha 1}(\cdot)) \). It is of particular interest how the Hamiltonian functional \( \mathcal{H} \) evolves along solutions of the system \( \mathcal{J} \) (well-posedness provided). For \( \mathcal{H} \) the formal change can be deduced to the balance relation

\[
\mathcal{H} = \int_B (\mathcal{J} - \mathcal{R})(\mathcal{E} \delta \mathcal{J}) + (\dot{x} | \delta h^1 \mathcal{J} + \dot{x}_1 | \delta h^{2 \mathcal{J}}) | \partial B, \tag{2}
\]

which states a power-balance relation if \( \mathcal{H} \) represents the total energy of the system. From a control engineering point of view, it is of special interest to introduce power ports by means of \( \mathcal{H} \), where we basically exploit both the boundary operators \( \delta h^1 \mathcal{J} = (\partial_\alpha H - d_1(\partial_\alpha H)) dx^\alpha \wedge \Omega_1 \) and \( \delta h^{2 \mathcal{J}} = \partial_\alpha H dx^\alpha \wedge \Omega_1 \).

Remark 1. It is worth stressing that boundary ports can be generated in two different ways. On the one hand, they can be a direct consequence of the jet variables that may occur in the Hamiltonian, cf. \( \delta h^1 \) and \( \delta h^{2 \mathcal{J}} \); on the other hand, boundary terms can also stem from the differential operators \( \mathcal{J} \) and \( \mathcal{R} \) — depending on their structure — which influence the boundary ports that are due to the jet variables, or even create additional ports.
Remark 2. In the Stokes-Dirac scenario no jet variables occur in $\mathcal{H}$ because energy variables are used; therefore, boundary ports solely stem from the interconnection operator $\mathcal{J}$, see [3] for instance.

Due to the fact that the general introduction of boundary ports is not possible in the operator case, we first consider the so-called non-differential operator case, where the introduction of boundary ports is straightforward. Second, to highlight the fact that $\mathcal{J}$ and $\mathcal{R}$ modify the boundary terms, we additionally investigate the impact of specific dissipation operators in the course of Subsection III-B.

A. Non-Differential Operator Case

In the non-differential operator case the differential operators $\mathcal{J}$ and $\mathcal{R}$ degenerate to bounded linear mappings $\mathcal{J}, \mathcal{R}$. Hence, in this case (1) reads in local coordinates as

$$\dot{x}^\alpha = (\mathcal{J}^{\alpha\beta} - \mathcal{R}^{\alpha\beta}) \delta_\beta \mathcal{H}, \quad \alpha, \beta = 1, \ldots, n,$$

where the skew-symmetric interconnection map $\mathcal{J}$ meets

$$\mathcal{J}^{\alpha\beta} = -\mathcal{J}^{\beta\alpha} \in \mathcal{C}^\infty(\mathcal{J}^1(\mathcal{E})),$$

and the positive semi-definiteness and symmetry of $\mathcal{R}$ implies

$$\mathcal{R}^{\alpha\beta} = \mathcal{R}^{\beta\alpha} \in \mathcal{C}^\infty(\mathcal{J}^1(\mathcal{E})).$$

Consequently, the power-balance relation (2) reduces to

$$\dot{H} = -\int_B \delta_\alpha(\mathcal{H}) \mathcal{R}^{\alpha\beta} \delta_\beta(\mathcal{H}) \, dx^1 + \ldots$$

$$+ (\dot{x}^\alpha \delta_\alpha^{1,1} \mathcal{H} + \dot{x}_i^\alpha \delta_\alpha^{2,2} \mathcal{H}) \mid_{\partial B}.$$ (4)

Here, the first term denotes the distributed dissipation on the domain, while the second part allows us to define power ports with

$$\delta_\alpha^{1,1} \mathcal{H} = \partial_1 \mathcal{H} - d_1(\partial_1 \mathcal{H})$$

$$\delta_\alpha^{2,2} \mathcal{H} = \partial_1 \mathcal{H}.$$ (5)

Remark 3. From (5) it is obvious that in the non-differential operator case the boundary ports solely stem from the derivative variables occurring in the Hamiltonian density.

Henceforth, we suppose that the boundary $\partial B = \{0, L\}$ can be divided into an unactuated part $\partial B_u = 0$ and a fully actuated part $\partial B_a = L$. For $\partial B_a$, we assume that $\dot{x}^\alpha \delta_\alpha^{1,1} \mathcal{H} \mid_{\partial B_a} = 0$ as well as $\dot{x}_i^\alpha \delta_\alpha^{2,2} \mathcal{H} \mid_{\partial B_a} = 0$, i.e., no power exchange takes place through the unactuated boundary part. For $\partial B_u$, we set $\dot{x}^\alpha \delta_\alpha^{1,1} \mathcal{H} \mid_{\partial B_u} = \dot{\xi} \dot{\xi}$ and $\dot{x}_i^\alpha \delta_\alpha^{2,2} \mathcal{H} \mid_{\partial B_u} = \ddot{\xi} \ddot{\xi}$ with the collocated boundary pairs $(\dot{\xi}, \ddot{\xi})$ and $(\dot{\xi}, \ddot{\xi})$ including the index ranges $\xi = 1, \ldots, m$ and $\ddot{\xi} = 1, \ldots, m$. By setting

$$\dot{B}_\alpha \dot{\xi} \ddot{\xi} = \delta_\alpha^{1,1} \mathcal{H} \mid_{\partial B_u},$$

$$\dot{B}_\alpha \ddot{\xi} \dot{\xi} = \delta_\alpha^{2,2} \mathcal{H} \mid_{\partial B_u},$$

we assign the roles of the inputs and outputs in (4), which is of course not unique, see [14]. To highlight the benefits of the pH system representation for the non-differential operator scenario, in the appendix we study the Examples 2 and 3 where the transversal and the longitudinal deflection of a linear and a nonlinear beam structure are investigated.

B. Specific Operators for $\mathcal{R}$

The objective of this section is to study the transversal as well as the longitudinal deflection of a nonlinear beam model subject to structural damping. To this end, we introduce two linear differential operators and investigate their impact on the power-balance equation, more specifically how they affect the geometric boundary ports.

First, we consider the 2nd-order formally self-adjoint operator $\mathcal{R}_A$, locally expressed by

$$\forall \mathcal{R}_A(\eta) = d_1(\mathcal{R}^{\alpha\beta}_A d_1(\eta_\beta)) \partial_\alpha,$$

with $\mathcal{R}_A^{\alpha\beta} = \mathcal{R}_A^{\beta\alpha} \in \mathcal{C}^\infty(\mathcal{R}(\mathcal{H}))$, i.e., the coefficients may in general depend on the spatial coordinate. Furthermore, we are able to deduce the important relation

$$\mathcal{R}_A(\eta) \mid_B = d_1(\mathcal{R}^{\alpha\beta}_A d_1(\eta_\beta) \Omega_1) - d_1(\eta_\beta) \mathcal{R}_A^{\alpha\beta} d_1(\eta_\alpha) \Omega,$$

indicating that $\mathcal{R}_A$ influences the (geometric) boundary ports (first term in (7)) as well as the domain conditions (second term in (7)). The non-negativity of $\mathcal{R}_A$ follows if $d_1(\eta_\beta) \mathcal{R}_A^{\alpha\beta} d_1(\eta_\alpha) \Omega_1 \leq 0$ is met, i.e., the coefficient matrix $[\mathcal{R}_A^{\alpha\beta}]$ must be symmetric as well as negative semi-definite.

Next, to be able to describe the structural damping for the vertical deflection, we exploit the 4th-order operator

$$\forall \mathcal{R}_B(\eta) = d_1(\mathcal{R}^{\alpha\beta}_B d_1(\eta_\beta)) \partial_\alpha,$$

with $\mathcal{R}_A^{\alpha\beta} \in \mathcal{C}^\infty(\mathcal{R}(\mathcal{H}))$, where $\mathcal{R}_A^{\alpha\beta} = \mathcal{R}_A^{\beta\alpha}$ must be met for the self-adjointness of the operator. Similar to (7), we rewrite the expression $\mathcal{R}_B(\eta)$ according to

$$\mathcal{R}_B(\eta) \mid_B = d_1(\mathcal{R}^{\alpha\beta}_B d_1(\eta_\beta) \Omega) + \ldots$$

$$+ d_1(\mathcal{R}^{\alpha\beta}_B d_1(\eta_\beta)) \eta_\beta - \mathcal{R}_A^{\alpha\beta} d_1(\eta_\alpha) d_1(\eta_\alpha) \Omega_1),$$

and find that $\mathcal{R}_B$ also affects the boundary ports as well as the domain conditions. Note that the non-negativity of $\mathcal{R}_B$ follows if $d_1(\eta_\beta) \mathcal{R}_A^{\alpha\beta} d_1(\eta_\alpha) \Omega_1 \geq 0$ is satisfied, i.e., the coefficient matrix $[\mathcal{R}_B^{\alpha\beta}]$ has to be positive semi-definite to meet this requirement.

Now, having the preceding findings at hand, we are able to investigate the following example.

Example 1 (Nonlinear 2nd-order beam structure with structural damping). We consider the following set of nonlinear PDEs, where

$$\rho A \ddot{w}^1 = \frac{3}{2} E A w_{11}^1 \dot{w}_1^1 + E A w_{1}^1 w_{1}^1 + \ldots$$

$$+ E A w_{1}^1 w_{1}^1 - E I \dot{w}_{1111}^1 - \alpha_1 \ddot{w}_{1111}^1$$

(9a)

and

$$\rho A \ddot{w}^2 = E A w_{1}^2 + E A w_{1}^2 + \alpha_2 \ddot{w}_{11}^2$$

(9b)

describes the transversal deflection $w^1$ and

the longitudinal deflection $w^2$ of a nonlinear beam, with $E, I, \rho, A > 0$ as material parameters and $\alpha_1, \alpha_2 > 0$ as damping coefficients; see [15] for instance for a similar model. Note that (9) follows from the calculus of variations [16] where nonlinear strain-deflection relations in the sense of von Kármán [17] and the Euler-Bernoulli hypothesis have
been used; eventually, (linear) structural damping has been included via the damping terms $-\alpha_1 \dot{w}_{11}^1$ and $\alpha_2 \dot{w}_{11}^2$. If we introduce the generalised momenta $p_1 = \rho A \dot{w}^1, p_2 = \rho A \dot{w}^2$ and set the Hamiltonian density to

$$\mathcal{H} = \frac{1}{2\rho A} \left( (p_1)^2 + (p_2)^2 \right) + \ldots + \frac{1}{2} EA((w_1^2)^2 + \frac{1}{4}(w_1^1)^4 + w_1^2(w_1^2)^2) + \frac{1}{2} EI(w_{11}^1)^2,$$

we find that the interconnection tensor

$$\mathcal{J} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

and the linear dissipation operator

$$\mathfrak{R} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ d_{11}(\alpha_1 d_{11}(\cdot)) & 0 \\ 0 & d_{11}(\alpha_2 d_{11}(\cdot)) & -d_{11}(\alpha_2 d_{11}(\cdot)) \end{bmatrix}$$

yield an appropriate pH system representation for (9) according to \( \dot{x} = (\mathcal{J} - \mathfrak{R})(\delta y) \). Note that \( \mathfrak{R} \) comprises the operators \( \mathfrak{R}_A \) and \( \mathfrak{R}_B \), i.e. \( \mathfrak{R} = \mathfrak{R}_A + \mathfrak{R}_B \) with \( \mathfrak{R}^{A}_{ij} = -\alpha_2 \), \( \mathfrak{R}^{B}_{ij} = \alpha_1 \), \( \mathfrak{R}^{ij} = 0 \) for \( i, j \neq 4 \) and \( \mathfrak{R}^{kl}_{B} = 0 \) for \( k, l \neq 3 \). It is worth stressing that the evaluation of the power-balance relation (10) is not as straightforward as in the non-differential operator case. Since this fact is a key point of this paper, we explain the following steps in detail. To this end, we consider the expression

$$\mathcal{H} = \int_{B} (-d_{11}(\alpha_1 \dot{w}_{11}^1)\dot{w}^1 + d_{11}(\alpha_2 \dot{w}_{11}^2)\dot{w}^2)dz_1 + \ldots + (\dot{w}^1 \dot{Q} + \dot{w}^2 \dot{N} + \dot{w}_{11}^1 \dot{M})|_{\partial B_a}.$$

where the shear force, the normal force and the bending moment follow to

$$\dot{Q} = \frac{1}{2} EA(w_1^1)^3 + EA w_1^2 w_1^1 - EI w_{11}^1,$$

$$\dot{N} = EA w_1^1 + \frac{1}{2} EA (w_1^1)^2,$$

$$\dot{M} = EI w_{11}^1,$$

by applying (5). However, if we use (7) and (8), the power-balance relation (11) can be rewritten as

$$\mathcal{H} = - \int_{B} (\alpha_1 (\dot{w}_{11}^1)^2 + \alpha_2 (\dot{w}_{11}^2)^2)dz_1 + \ldots + (\dot{w}^1 \dot{Q} + \dot{w}^2 \dot{N} + \dot{w}_{11}^1 \dot{M})|_{\partial B_a},$$

where it becomes clear that \( \alpha_1, \alpha_2 > 0 \) ensures the non-negativity of \( \mathfrak{R} \) which – provided that no power flow takes place via \( \partial B_a \) – guarantees the energy dissipation according to the structural damping models under consideration. Furthermore, comparing the expressions

$$\dot{Q} = \dot{Q} - \alpha_1 \dot{w}_{11}^1, \quad \dot{N} = \dot{N} + \alpha_2 \dot{w}_{11}^2, \quad \dot{M} = \dot{M} + \alpha_1 \dot{w}_{11}^1$$

with (11) highlights the impact of the dissipation operator \( \mathfrak{R} \) on the (geometric) boundary-port relations. In accordance with Remark [1] we find that \( \dot{Q}, \dot{N} \) and \( \dot{M} \) stem on the one hand, from the application of the boundary operators (5) which are a consequence of the jet variables in \( \mathcal{H} \) and, on the other hand, from the impact of the differential operators. This is important to realise since we use the shear force, the normal force and the bending moment at \( z^1 = L \) as manipulated variables for the controller designs in the following two sections. Therefore, the collocated inputs and outputs are \( \dot{u}^1 = \dot{Q}, \dot{u}^2 = \dot{N}, \dot{u}^1 = \dot{M} \) and \( \dot{y}^1 = \dot{w}^1, \dot{y}^2 = \dot{w}^2, \dot{y}_1 = \dot{w}_{11}^1 \), respectively. Consequently, the boundary maps result in \( \dot{B}_a = \delta_{\alpha, \beta} \) and \( \dot{B}_a = \delta_{\alpha, \beta} \), together with the Kronecker-Delta symbol meeting \( \delta_{\alpha, \beta} = 1 \) for \( \alpha = \beta \) and \( \delta_{\alpha, \beta} = 0 \) for \( \alpha \neq \beta \).

Ex. [1] clearly highlights that differential operators have a strong impact on the power-balance relation and therefore on the introduction of the associated power ports. The aim of the next two sections is to stabilise a certain rest position of the nonlinear beam structure including damping, see Ex. [1]. To this end, we shall use a dynamic pH-controller exploiting Casimir functionals as well as a static control law designed by means of energy balancing.

IV. ENERGY-BASED CONTROL BY MEANS OF CASIMIR FUNCTIONALS

In this section, we extend the results and findings for the energy-Casimir method applied to systems with 2nd-order Hamiltonians in the non-differential operator scenario, see [11], to the differential operator setting with a certain input and output assignment for the boundary ports. For detailed informations concerning the 1st-order and 2nd-order case, we refer to [18] and [11], respectively. Since the introduction of boundary ports in the differential-operator scenario is not straightforward, cf. Ex. [1] we suppose that, independently of the concrete differential operators, the outputs of the considered pH-system can be parameterised via (6). Note that the parameterisation of the corresponding collocated inputs strongly depends on the involved differential operators and cannot be stated in a general form. However, this assumption enables us to extend the energy-Casimir method to the differential-operator scenario as well.

A. Interconnection (Infinite-Finite)

In the following, we are interested in a power-conserving interconnection of the infinite-dimensional plant (1) and a finite-dimensional controller at the actuated boundary \( \partial B_a \) according to

$$\dot{u} + \dot{\tilde{g}} + u + \dot{\tilde{y}} + \dot{\tilde{u}} + \dot{\tilde{y}} = 0.$$

Here, \( \dot{\tilde{u}} \) and \( \dot{\tilde{y}} \) represent the collocated in- and output pairings of the controller, which have been divided into two
parts to take account of both boundary-port categories of the plant. The structure of the dynamic pH-controller can locally be given as

\[
\ddot{y}_{c,j} = \ddot{G}_{c,j} \partial_{\alpha c} H_c \quad \text{and} \quad \ddot{y}_{c,j} = \ddot{G}_{c,j} \partial_{\alpha c} H_c \tag{14}
\]

with \(\alpha_c, \beta_c = 1, \ldots, n_c, \ i = 1, \ldots, \hat{m} \) and \( j = 1, \ldots, \hat{m} \). Note that \(H_c\) can be deduced to \(-\partial_{\alpha c}(H_c)R_{\alpha c, \beta c} \partial_{\beta c}(H_c) + \hat{u}_c \hat{y}_{c,j} + \hat{u}_c \hat{y}_{c,j};\) therefore, based on the power-conserving interconnection, the controller can be used to inject additional damping. As power-conserving feedback structure we set

\[
\ddot{u}_c = \dddot{K} \dddot{y}_c, \quad \ddot{u}_c = \dddot{K} \dddot{y}_c, \quad \ddot{u}_c = -\dddot{K} \dddot{y}_c,
\]

clearly satisfying (13), together with appropriate maps \(\dddot{K}, \dddot{K}\) and their duals \(\dddot{K}^*, \dddot{K}^*\), respectively. It is worth stressing that the closed-loop system still possesses a pH-structure with \(\mathcal{H}_d = \int_B \mathcal{H}(z) + H_c\) as closed-loop Hamiltonian.

Next, we investigate Casimir functionals (structural invariants) of the closed loop, which shall enable us to relate some of the controller states to the plant in order to partially shape \(\mathcal{H}_d\). Note that the controller states that are not related to the plant can be used for the damping injection and thus for the purpose of stabilisation.

**Remark 4.** It should be noted that in this contribution the focus is on a formal approach based on differential-geometric methods. Thus no detailed stability investigations will be carried out as this requires functional-analytic methods in general. However, the relations \(\mathcal{H}_d > 0\) and \(\mathcal{H}_d \leq 0\) serve as necessary conditions for a stability investigation in the sense of Lyapunov. Worth stressing is the fact that for the nonlinear PDE system under investigation, cf. Ex 1 the proof of stability is no trivial task as the verification of the pre-compactness of the closed-loop trajectories is not straightforward.

**B. Determination of the Conditions for Structural Invariants**

Motivated by [10], we introduce the specific functionals

\[
\mathcal{G}^\lambda = x^\lambda + \int_B \mathcal{C}^\lambda dz^1, \quad \mathcal{C}^\lambda \in C^\infty(J^2(\mathcal{E})),
\]

with \(\lambda = 1, \ldots, \tilde{n}_c \leq n_c\), which have to fulfil \(\delta^\lambda = 0\) independently of \(\mathcal{H}\) and \(H_c\) in order to serve as conserved quantities. That is, they have to meet the conditions

\[
\begin{align*}
(J^\lambda_{\alpha c} - R^\lambda_{\alpha c}) &= 0, \quad (\delta^\lambda_{\beta c} - R^\lambda_{\beta c}) = 0 \tag{15a} \\
\mathcal{A}_c \delta^\lambda(\mathcal{C}) &= 0 \tag{15b} \\
(\hat{G}^\lambda_c \hat{K}^\alpha B_{\alpha c} + \delta^\lambda_{\alpha c} \mathcal{C}^\lambda) |_{\partial B_{\alpha c}} &= 0 \tag{15c} \\
(\hat{G}^\lambda_c \hat{K}^\alpha B_{\alpha c} + \delta^\lambda_{\alpha c} \mathcal{C}^\lambda) |_{\partial B_{\alpha c}} &= 0 \tag{15d} \\
(\hat{F}^\alpha \mathcal{C}^\lambda + \hat{F}^\alpha \delta^\lambda \mathcal{C}^\lambda) |_{\partial B_{\alpha c}} &= 0 \tag{15e}
\end{align*}
\]

to qualify as structural invariants. For the derivation of (15) we refer to [11, Eqs. (18) and (19)], where it must be emphasised that only the non-differential operator case is treated there. However, based on the chosen boundary-port parameterisation, the computation can easily be adopted to the specific differential-operator case considered here. Note that (15b) can easily be satisfied by setting \(\mathcal{C}^\lambda = d_1(\mathcal{C}^\lambda)\) with \(\mathcal{C}^\lambda \in C^\infty(J^1(\mathcal{E}))\), as the variational derivative annihilates total derivatives (see [19, Theorem 4.7]), i.e. \(\delta^\lambda \mathcal{C}^\lambda = 0\) is met independently of the concrete function \(\mathcal{C}^\lambda\).

**C. Energy-Casimir Controller for Example 1**

In [11], an energy-based control law was developed for the vertical deflection of a linear Euler-Bernoulli beam without dissipation. The aim of this subsection is to design a Casimir-based controller for the nonlinear beam structure with dissipation of Ex 1 in order to stabilise the desired equilibrium (with arbitrary constants \(a, b \in \mathbb{R}\))

\[
w_1, d = az + b, \quad w_2, d = 0, \quad w_1, d = a. \tag{16}
\]

In the following, we design a nonlinear dynamical controller with \(n_c = 6\), where it must be emphasised that three controller coordinates are related to the plant to properly shape the Hamiltonian such that (16) becomes a part of the minimum. The remaining three controller states are used to inject additional damping in the closed loop. If we consider the total derivatives \(\mathcal{C}^1 = -\frac{1}{2}d_1(\hat{z}^1 w_1^1), \mathcal{C}^2 = -\frac{1}{2}d_1(\hat{z}^2 w_2^1)\) and \(\mathcal{C}^3 = -\frac{1}{2}d_1(\hat{z}^3 w_1^1)\) as Casimir functions, we find that the conditions (15) yield the algebraic restrictions

\[
\begin{align*}
J_c - R_c &= \begin{bmatrix} 0_{3 \times 3} & 0_{3 \times 3} \\ 0_{3 \times 3} & A \end{bmatrix} \\
\hat{G}^5_c &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \hat{G}^5_{c,1} \hat{G}^5_{c,2} \\
\hat{G}^6_c &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \hat{G}^6_{c,1} \hat{G}^6_{c,2}
\end{align*}
\tag{17}
\]

with

\[
A = \begin{bmatrix} -R^4_{c,1} & -J^4_{c,2} & J^4_{c,3} & -R^4_{c,3} \\ -J^4_{c,2} & -R^4_{c,1} & J^4_{c,3} & -R^4_{c,3} \\ -J^4_{c,3} & -R^4_{c,3} & J^4_{c,3} & -R^4_{c,3} \\ J^4_{c,3} & -R^4_{c,3} & -R^4_{c,3} & J^4_{c,3} \end{bmatrix}
\]

for the controller mappings. Then the Casimir functions yield the important relations

\[
\begin{align*}
\mathcal{G}^1 &= x_1 + \int_0^1 C^1 dz^1 = x_1 - w_1^1 |_{L}, \\
\mathcal{G}^2 &= x_2 + \int_0^1 C^2 dz^1 = x_2 - w_2^2 |_{L}, \\
\mathcal{G}^3 &= x_3 + \int_0^1 C^3 dz^1 = x_3 - w_1^1 |_{L}.
\end{align*}
\]

By choosing the initial conditions for the controller states appropriately, the relations between the plant and the controller states result in

\[
x_1 = w_1^1 |_{L}, \quad x_2 = w_2^2 |_{L}, \quad x_3 = w_1^1 |_{L}.
\]

To appropriately shape \(\mathcal{H}_d\), we set \(H_c\) to

\[
H_c = \frac{c_1}{4} x_1 - w_1^1 |_{L} |^{4} + \frac{c_2}{4} x_2 - w_2^2 |_{L} |^{4} + \ldots + \frac{c_4}{4} x_3 - w_1^1 |_{L} |^{4} + \frac{1}{2} M_e x_c^p x_c^p
\]
together with the constants $c_1, c_2, c_3 > 0$ and the positive definite matrix $M_c, M_{c, \mu, \nu} \in \mathbb{R}$ for $\mu, \nu = 4, 5, 6$. Consequently, $\mathcal{H}_d$ evolves along solutions of the closed loop according to

$$\dot{\mathcal{H}}_d = - \int_B \left( \alpha_1 (\dot{w}_{11})^2 + \alpha_2 (\dot{w}_{12})^2 \right) dz^1 + \ldots - x^\mu M_{c, \mu, \nu} R_c^\rho M_{c, \rho} x^\rho,$$

with $\rho, \vartheta = 4, 5, 6$, from which it becomes apparent that the controller injects additional damping.

We want to stress that although the dynamic controller is able to stabilise the desired rest position, there is no systematic approach to determine the remaining degrees of freedom for the controller maps properly. Therefore, in the next section, we propose a static controller based on energy balancing that enables to simplify the controller design compared to the energy-Casimir method.

V. ENERGY-BALANCING CONTROL

The intention of this section is to introduce a further control methodology which is able to shape the plant Hamiltonian (at least partially) and to add damping. In [12], a similar control framework is proposed exploiting a pH system representation based on Stokes-Dirac structures which is suitable for linear PDEs. Therefore, we propose a similar control scheme for pH-systems with 2nd-order Hamiltonian formulated in terms of the jet-bundle framework, which makes heavy use of the geometric properties of the variational derivative as well as both the boundary operators, and is not restricted to linear PDE systems and controllers.

First, we demonstrate the basic principle of the energy-balancing control (EBC) scheme by means of the non-differential operator case, i.e. for systems of the form (3). Then, since the introduction of boundary ports is not straightforward in the differential-operator case, the proposed control methodology is studied on the example of the nonlinear beam structure with structural damping of Ex. 1.

In the following, we consider static control laws that can be divided into an energy-shaping part $\beta$ and a damping-injection part $\eta$ of the form

$$\dot{u}_\xi = \beta \dot{\xi} + \ddot{u}_\xi, \quad \dot{u}_\mu = \beta \dot{\mu} + \ddot{u}_\mu.$$

The aim is to use the energy-shaping input $\beta$ to map the open-loop equations (3) into the target system

$$\dot{\mathcal{H}}_d = \left( J^{\alpha \beta} - R^{\alpha \beta} \right) \delta_\beta \mathcal{H}_d,$$

with the ansatz $\mathcal{H}_d = H + H_a$, where $H_a$ is chosen such that $\mathcal{H}_d$ has a minimum at the desired position of rest. Moreover, we set the (new) input of the target system to

$$\dot{B}_\alpha \ddot{u}_\xi = \delta^{\alpha \vartheta}_\alpha H_d |_{\partial B_a}, \quad \dot{B}_\alpha \ddot{u}_\mu = \delta^{\beta \vartheta}_\alpha H_d |_{\partial B_a},$$

which shall be used to inject some additional damping.

In what follows, we explain in detail how the energy-shaping as well as the damping-injection part have to be determined. It should be noted that since we take no distributed input into account, we are not able to modify the original dynamics of the system (3). Consequently, the matching equations are given as

$$(J^{\alpha \beta} - R^{\alpha \beta}) \delta_\beta H = (J^{\alpha \beta} - R^{\alpha \beta}) \delta_\beta H_d,$$

due to $H_d = H + H_a$, they can be reduced to

$$(J^{\alpha \beta} - R^{\alpha \beta}) \delta_\beta H_a = 0,$$

yielding first conditions for $H_a$. Furthermore, due to the fact that there is no power flow through the unactuated boundary $\partial B_a, H_a$ must satisfy the boundary conditions $\delta^{\alpha \vartheta}_\alpha H_a |_{\partial B_a} = 0$ and $\delta^{\alpha \vartheta}_\alpha H_a |_{\partial B_a} = 0$, cf. (15c). In the following, we make use of the pleasant property that the variational derivative always annihilates total derivatives, which allows us to derive a proper control law in an elegant manner. In accordance with both boundary categories, cf. (6), a suitable choice for the additional Hamiltonian density is given as

$$H_a = \sum_{\vartheta = 1}^{\infty} \hat{h}_\vartheta + \sum_{\mu = 1}^{\infty} \hat{h}_\mu,$$

with $\hat{h}_\vartheta = d_1(\hat{f}_\vartheta)$ and $\hat{h}_\mu = d_1(\hat{f}_\mu)$ meeting the required boundary conditions. Thus, we are able to modify the system’s actuated boundary $\partial B_a$ ensured that (21) is satisfied since $\delta_\vartheta H_a = 0$ holds – due to the special choice for $h_\vartheta$ and $\hat{h}_\mu$, which enables us to develop nonlinear control laws as well. In fact, with regard to control purposes, we are interested how the closed-loop functional $\mathcal{H}_d = \int_B \mathcal{H}_d dz^1$ evolves along solutions of the closed loop. A straightforward evaluation yields

$$\dot{\mathcal{H}}_d = - \int_B \delta_\alpha (H_d) R^{\alpha \beta} \delta_\beta (H_d) dz^1 + \ldots + (x_\alpha \delta^{\alpha \vartheta}_\alpha H_d + x_\alpha \delta^{\alpha \vartheta}_\alpha H_d |_{\partial B_a}),$$

and by recalling that $\delta_\vartheta H_a = 0$ is met, we see that the dissipation effect in the domain remains unchanged. If we consider the boundary expression of (22), with (18) and (20) we find that the energy-shaping control laws result in

$$\dot{B}_\alpha \dot{\xi} = - \delta^{\alpha \vartheta}_\alpha H_a |_{\partial B_a},$$
$$\dot{B}_\alpha \dot{\mu} = - \delta^{\beta \vartheta}_\alpha H_a |_{\partial B_a}.$$

Now, since we mapped the original system into the target system (19), we use its collocated boundary pairs $(\hat{u}_\vartheta, \hat{y}_\vartheta)$ and $(\dot{u}_\mu, \dot{y}_\mu)$ to inject additional damping. If we choose the inputs for the damping-injection part according to

$$\hat{u}_\vartheta = - K^{\vartheta \gamma} \dot{y}_\vartheta, \quad \dot{u}_\mu = - K^{\mu \gamma} \dot{y}_\mu,$$

with $[K^{\vartheta \gamma}] > 0$ and $[K^{\mu \gamma}] > 0$, then (22) follows to

$$\dot{\mathcal{H}}_d = - \int_B (\delta_\alpha (H) R^{\alpha \beta} \delta_\beta (H)) dz^1 - \dot{y}_\vartheta K^{\vartheta \gamma} \dot{y}_\vartheta - \dot{y}_\mu K^{\mu \gamma} \dot{y}_\mu,$$

ensuring that $\dot{\mathcal{H}}_d \leq 0$ is met, i.e. that $\mathcal{H}_d$ is non-increasing along closed-loop solutions; cf. Remark 4 regarding the stability investigation.
A. Energy-Balancing Controller for Example 1

Next, we investigate the impact of linear differential operators to the proposed controller design procedure. In particular, we develop a control law in order to stabilise the desired equilibrium (16) for the nonlinear beam with structural damping of Ex. 1 Due to the fact that the influence of $\mathfrak{R}_A$ and $\mathfrak{R}_B$ cannot be given in general, a parameterisation of the input of (19) according to (20) is not possible. However, for the considered beam structure we set $\hat{w}^1 = \hat{Q} + \delta^{1,1}_x H_A$, $\hat{w}^2 = \hat{N} + \delta^{2,1}_x H_A$ and $\hat{w}^1 = \hat{M} + \delta^{1,2}_x H_A$, therefore, the boundary maps result in $\hat{B}^{\alpha_x}_x = \delta^{\alpha_x}_x$ and $\hat{B}^{\alpha_x}_z = \delta^{\alpha_x}_z$, together with the Kronecker-Delta symbol meeting $\delta_{\alpha\beta} = 1$ for $\alpha = \beta$ and $\delta_{\alpha\beta} = 0$ for $\alpha \neq \beta$.

Following the intention of the proposed control scheme, we choose the additional Hamiltonian according to

$$
\hat{H}_a = \hat{h}_1 + \hat{h}_2 + \hat{h}_1, 
$$

in order to shape the closed-loop Hamiltonian at the actuated boundary $\partial B_A$ and to fulfil the boundary conditions $\delta^{1,1}_x H_A|_{\partial B_A} = 0$ and $\delta^{2,2}_x H_A|_{\partial B_A} = 0$. An evaluation of (23) and (24) yields the EBC laws

$$
\dot{\beta}^1 = -c_1 (w^1_1 - w^1_2)|_L^d L^3, \\
\dot{\beta}^2 = -c_2 (w^2_1 - w^2_2)|_L^d L^3, \\
\dot{\beta}^3 = -c_3 (w^3_1 - w^3_2)|_L^d L^3, 
$$

mapping the beam into the target system $\dot{z}^\alpha = (J_{\alpha \beta} - \mathfrak{R}^{\alpha \beta}_x) \delta_{\beta} H_d$ with the Hamiltonian density $H_d = H + H_a$. If we set the (damping-injection) input for this system to

$$
\hat{w}^1 = -k_1 (\hat{w}^1_1)_L, \\
\hat{w}^2 = -k_2 (\hat{w}^2_1)_L, \\
\hat{w}^1 = -k_3 (\hat{w}^1_1)_L, 
$$

the power-balance relation results in

$$
\dot{\mathcal{H}}_d = \int_B (\alpha_1 (\hat{w}^1_1)^2 + \alpha_2 (\hat{w}^2_1)^2) dz_1 + \ldots \\
- k_1 (\hat{w}^1_1)|_L^d L^2 - k_2 (\hat{w}^2_1)|_L^d L^2 - k_3 (\hat{w}^1_1)|_L^d L^2, 
$$

obviously satisfying $\dot{\mathcal{H}}_d \leq 0$. It should be noted that the EBC scheme suggests a nonlinear PD control law, where the proportional part may be interpreted as a nonlinear spring. From Figure 1 it becomes clearly evident that the proposed controller stabilises the desired equilibrium (16) with $\alpha = 0.01$ and $b = 0.01$.

VI. CONCLUSION

In this contribution, the pH-framework based on the jet-bundle scenario has been used for the modeling and energy-based control for a nonlinear Euler-Bernoulli beam including structural damping. In particular, the control by interconnection method has been adapted for the system class under investigation and for the first time, EBC techniques that have been already considered in the Stokes-Dirac scenario (linear setting), have been analysed for nonlinear PDEs based on the geometric framework exploiting the underlying jet-bundle structure. Future research directions will include the generalisation of EBC techniques to domain inputs in this setting and extensions towards higher-dimensional spatial domains.

APPENDIX

In the appendix, two examples are studied that serve as a motivation for the presented framework in the non-differential operator case. In particular, a linear and a nonlinear Euler-Bernoulli beam are considered where in contrast to Ex. 1 no differential operators appear.

Example 2 (Linear beam structure with viscous damping). First, we investigate a beam with linearised geometric and linear constitutive relations, combined with external viscous damping effects. The corresponding equations of motion in classical notation read as

$$
\rho A \ddot{w}^1 = -EI \frac{\partial^4 w^1}{\partial z^4}, \\
\rho A \ddot{w}^2 = -EI \frac{\partial^4 w^2}{\partial z^4},
$$

with $E, I, \rho, A > 0$ as material parameters, see [16] for more details, and small positive constants $\alpha_1, \alpha_2$ for the damping parameters. Because we are interested in a system description according to (3), we first introduce the generalised momenta $p_1 = \rho A \dot{w}^1$ and $p_2 = \rho A \dot{w}^2$. By means of these new coordinates, the total energy density of the beam can be expressed by

$$
\mathcal{H} = \frac{1}{2 \rho A} ((p_1^2 + (p_2)^2) + \frac{1}{2} EI (w^1_{11})^2 + \frac{1}{2} EA (w^2_{11})^2).
$$

Consequently, we find that (25) can be formulated as

$$
\begin{bmatrix}
\dot{w}^1 \\
\dot{w}^2 \\
\dot{p}_1 \\
\dot{p}_2
\end{bmatrix} = (\mathcal{J} - \mathcal{R})
\begin{bmatrix}
\delta_{w^1, \mathcal{H}} \\
\delta_{w^2, \mathcal{H}} \\
\delta_{p^1, \mathcal{H}} \\
\delta_{p^2, \mathcal{H}}
\end{bmatrix},
$$

Figure 1: Simulation results for $w^1$ for the proposed EBC approach. Beam parameters: $L = 0.54 \text{ m}$, $EI = 14.97 \text{ Nm}^2$, $EA = 50 \text{ N}$, $\rho A = 2.1 \text{ kgm}^{-1}$ and initial rest position $w^1(0,0) = 0$. Controller parameters: $c_1 = 2 \cdot 10^3$, $c_2 = 1000$, $c_3 = 8 \cdot 10^4$, $k_1 = 2200$ and $k_2 = 3 = 1$. 
where the tensors $\mathcal{J}$ and $\mathcal{R}$ correspond to

$$
\mathcal{J} = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{bmatrix}, \quad \mathcal{R} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & \alpha_1 & 0 & 0 \\
0 & 0 & \alpha_2 & 0
\end{bmatrix}.
$$

A straightforward evaluation of (24) yields

$$
\mathcal{H} = -\int_B (\alpha_1 (\dot{w}_1)^2 + \alpha_2 (\dot{w}_2)^2)dz^1 + \ldots
+ (\dot{w}_1 Q + \dot{w}_2 N + \dot{w}_1 M)\big|_{\partial \mathcal{B}_s},
$$

with the shear force $Q = -EIw_{111}$, the normal force $N = EAw_1^2$ and the bending moment $M = EIw_{111}$, where we made extensive use of (5). Because the part of $\mathcal{H}$ which is associated to the transversal vibrations is of 2nd order, we obtain the two boundary ports $(\dot{w}_1, Q)$ and $(\dot{w}_1, M)$. On the other hand, the part of $\mathcal{H}$ which is related to the longitudinal vibrations of the beam is of first order, and thus only one boundary port $(\dot{w}_2, N)$ appears. It is worth stressing that these three ports are real power ports because the Hamiltonian corresponds to the total energy of the beam. Note that if $Q$, $N$ and $M$ serve as controlled variables this fixes the corresponding collocated outputs, and the construction of the maps $\hat{B}_{\alpha \xi}$ and $B_{\nu \xi}$ of (6) becomes straightforward in that case.

Example 3 (Nonlinear 2nd-order beam structure) Next, we consider nonlinear geometric relations within the Euler-Bernoulli beam theory. In particular, we consider nonlinear strain-deflection relations in the sense of von Kármán, see [17] for example. Using this strain-deflection relations leads to the potential energy density

$$
\mathcal{P} = \frac{1}{2} EA((w_1^2)^2 + \frac{1}{4}(w_1^4) + w_1^2(w_2^1)^2) + \frac{1}{2} EI(w_{111})^2,
$$

while for the kinetic energy density we have $\mathcal{K} = \frac{1}{2} \rho A (\dot{w}_1)^2 + (\dot{w}_2^1)^2$. Then, applying Hamilton’s principle (see [16]) on the Lagrangian density $\mathcal{L} = \mathcal{K} - \mathcal{P}$ yields the following equations of motion

$$
w_1 = \frac{E}{\rho} \dot{w}_1 \dot{w}_1 + w_1^2 + w_1 \dot{w}_1 + w_{111} - \frac{EI}{\rho A} w_{111},
$$

$$
w_2 = \frac{E}{\rho} \dot{w}_2 + \frac{E}{\rho} \dot{w}_1 \dot{w}_1.
$$

(27)

Again, by using the generalised momenta $p_1 = \rho Aw_1^2$ and $p_2 = \rho Aw_2^2$, as well as the total energy density $\mathcal{K} + \mathcal{P}$ as Hamiltonian density, we find that the governing equations can also be formulated as

$$
\begin{bmatrix}
\dot{w}_1 \\
\dot{w}_2 \\
p_1 \\
p_2
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\delta_{w_1 \mathcal{H}} \\
\delta_{w_2 \mathcal{H}} \\
\delta_{p_1 \mathcal{H}} \\
\delta_{p_2 \mathcal{H}}
\end{bmatrix}.
$$

(28)

Note that (26) and (28) are structurally equivalent apart from the dissipation effects in (26); however, the nonlinearity of (28) is hidden in $\mathcal{H}$. Moreover, since no dissipation effects have been taken into account, the formal change of the Hamiltonian functional reads as

$$
\mathcal{H} = (\dot{w}_1 Q + \dot{w}_2 N + \dot{w}_1 M)\big|_{\partial \mathcal{B}_s},
$$

with the modified shear force, normal force and bending moment that follow again by using (5) according to

$$
\dot{Q} = \frac{1}{2} EA(w_1^3) + EAw_1 w_1 - EI w_{111},
$$

$$
\dot{N} = EAw_1^2 + \frac{1}{2} EA(w_1^4),
$$

$$
\dot{M} = EI w_{111},
$$

(29)

(30)

respectively.

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