Deterministic transformations of three-qubit entangled pure states

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The states of three-qubit systems split into two inequivalent types of genuine tripartite entanglement, namely the Greenberger-Horne-Zeilinger (GHZ) type and the \( W \)-type. A state belongs to one of these classes can be stochastically transformed only into a state within the same class by local operations and classical communications. We provide local quantum operations, consisting of the most general two-outcome measurement operators, for the deterministic transformations of three-qubit pure states in which the initial and the target states are in the same class. We explore these transformations, originally having the standard GHZ and the standard \( W \) states, under the local measurement operators carried out by a single party and \( p \) (\( p = 2, 3 \)) parties (successively). We find a notable result that the standard GHZ state cannot be deterministically transformed to a GHZ-type state in which its all bipartite entanglements are nonzero, i.e., a transformation can be achieved with unit probability when the target state has at least one vanishing bipartite concurrence.

I. INTRODUCTION

Quantum entanglement, as a bizarre nonclassical correlation, took its first steps into quantum theory in the middle of the second quarter of the last century \cite{1, 2}. Since then entanglement theory has flourished in company with quantum information theory \cite{3, 4}, and it has its roots in a wide range of exceptional discoveries such as quantum teleportation \cite{5}, dense coding \cite{6}, quantum cryptography \cite{7}, and remote state preparation \cite{8}. Furthermore, during the last two decades, various studies have been presented to characterize different types of nonclassical correlations, and entanglement is still the most remarkable among all these. This assessment is due to the high performance that entanglement shows when used as a resource.

The use of entanglement as a resource \cite{3, 9} in quantum information and quantum computation requires its quantification, characterization, and manipulation. Bipartite entanglement is well understood from all these aspects indeed, however, many problems are waiting to be solved for multipartite entanglement. In this respect, entanglement manipulation of multipartite pure states with local operations and classical communications (LOCC)–the free operations in the context of resource theory of entanglement—is often seen as one of the fundamental tasks that has been widely studied in the theory of quantum information. With this motivation, in this work we are concerned with the case of three-qubit pure states: deterministic transformations of Greenberger-Horne-Zeilinger (GHZ) type states and \( W \)-type states.

In general, for \( n \geq 3 \)–the number of particles–the problem becomes much more complicated. A necessary and sufficient condition of the possibility of a deterministic LOCC transformation of three-qubit pure states was given in Ref. \cite{10}. The researchers \cite{11} presented all of the necessary and sufficient conditions for the possibility of converting truly multipartite rank-2 states into each other. Spedalieri \cite{12} studied the properties of deterministic LOCC transformations of three-qubit pure states with tripartite entanglement. A systematic treatment of the transformations of \( W \)-type states was given in Ref. \cite{13}. Distilling maximally entangled tripartite GHZ states via LOCC \cite{14, 15} and selective information manipulation \cite{16} was investigated. A protocol for the optimal distillation of the asymmetric \( W \) states, as well as the symmetric \( W \) state, from an arbitrary \( W \) class state was introduced in Ref. \cite{17}. Cui et al. \cite{18} derived upper bounds and lower bounds for the optimal probability of transformation from a GHZ state to a GHZ-class state. The optimal local transformations of flip and exchange symmetric multi-qubit states were obtained in Ref. \cite{19}. It was shown \cite{20} that if the initial state \( |\psi\rangle \) and final state \( |\phi\rangle \) are genuine tripartite pure states in the GHZ class then \( |\psi\rangle \) can be transformed to \( |\phi\rangle \) by separable operations if and only if \( |\psi\rangle \) can be transformed to \( |\phi\rangle \) by deterministic LOCC. Roughly speaking, most studies have tended to focus on the conditions for the probabilistic and the deterministic transformations rather than the measurement operators (i.e., protocols for optimal transformations). Providing a simple and practical protocol for the deterministic transformations of three-qubit entangled pure states is the subject of this paper.

In this paper, we first introduce an explicit and comprehensive protocol for the deterministic transformations of a GHZ-type state into another GHZ-type state. To assess whether and how the standard GHZ state is transformed into a GHZ-type state deterministically, we use the most general local quantum operations–canonical operators for three-qubit systems. We reveal that all GHZ class states, except the ones with all three bipartite concurrences are nonzero, can be obtained by deterministic transformations of the standard GHZ state. After that, we present local quantum operations which allow three parties to transform a \( W \)-type state into another \( W \)-type state in three steps with unit probability. These operations consist of the most general two-outcome measurement operators. We also apply the same protocol to the standard \( W \) state to show how it is transformed into a general \( W \)-type state.

The rest of the paper proceeds as follows. In Sec. \ref{sec:2} we recall some definitions for thee-qubit pure states and their entanglement parameters. We provide the local measurement operators for the deterministic transformations of a GHZ-type state into another GHZ-type state in Sec. \ref{sec:3}. We then present...
in Sec. [IV] the local measurement operators for the deterministic transformations of a $W$-type state into another $W$-type state. In Sec. [V] we conclude our work with a summary.

### II. THREE-QUBIT PURE STATES

This section contains some definitions for key terms of three-qubit pure states and their entanglement parameters that are needed for a clear understanding of the presented work. Let us commence with the canonical form of three-qubit pure states. Following the approach presented in Refs. [21, 22], one can express the canonical form of three-qubit pure states such that

$$
|\psi\rangle = \lambda_0 |000\rangle + \lambda_1 e^{i\phi} |100\rangle + \lambda_2 |101\rangle + \lambda_3 |110\rangle + \lambda_4 |111\rangle, \quad (\lambda_i \geq 0),
$$

where the coefficients $\lambda_i$ satisfy $\sum_{i=0}^{4} \lambda_i^2 = 1$. It is well known that if two arbitrary states $|\psi\rangle$ and $|\phi\rangle$ are related by local unitaries, i.e., if these two states are local unitary equivalent (LUeq), then they are equal (mostly written $|\psi\rangle \sim |\phi\rangle$) from the information theoretic point of view. When deterministic transformations of entangled pure states are investigated, the LUeq forms of entangled pure states constitute one of the most crucial points of the whole process. In two-qubits case, for instance, the maximally entangled pure state, $(|00\rangle + |11\rangle)/\sqrt{2}$, can be transformed into the state $a|00\rangle + b|11\rangle$ with unit probability. A two-outcome measurement, carried out by one of the parties, with the measurement operators $a|0\rangle\langle 0| + b|1\rangle\langle 1|$ and $b|0\rangle\langle 0| + a|1\rangle\langle 1|$ yields one of the states $a|00\rangle + b|11\rangle$ and $b|00\rangle + a|11\rangle$, respectively. These two states are LUeq under the unitary transformation $|0\rangle \leftrightarrow |1\rangle$ on both qubits. Thus, having a LUeq form of the target state in a deterministic transformation makes the problem easier to examine. In this sense, a LUeq form of the state $(1)$ was presented in Ref. [23]:

$$
|\psi\rangle = \lambda_0' |000\rangle + \lambda_1' e^{i\phi'} |100\rangle + \lambda_2' |101\rangle + \lambda_3' |110\rangle + \lambda_4' |111\rangle, \quad (\lambda_i' \geq 0),
$$

where the coefficients $\lambda_i'$ satisfy $\sum_{i=0}^{4} \lambda_i'^2 = 1$, as usual. As introduced in Ref. [23], local unitary equivalence of the states $(1)$ and $(2)$ implies

$$
\frac{\lambda_0' \lambda_2 - \lambda_1' \lambda_3}{\lambda_0' \lambda_2 + \lambda_1' \lambda_3} = \frac{\lambda_0 \lambda_2 - \lambda_1 \lambda_3}{\lambda_0 \lambda_2 + \lambda_1 \lambda_3},
$$

As one knows, two arbitrary three-qubit pure states are LUeq if and only if their entanglement parameters are the same. In this way, the unitary invariants–entanglement parameters–can be found to be

$$
C_{AC} = 2\lambda_0' \lambda_2' - 2\lambda_0 \lambda_2, \quad C_{AB} = 2\lambda_1' \lambda_3' - 2\lambda_0 \lambda_3,
$$

$$
C_{BC} = 2(\lambda_2' \lambda_3' - e^{i\phi'} \lambda_1' \lambda_4') = 2(\lambda_2 \lambda_3 - e^{i\phi} \lambda_1 \lambda_4),
$$

$$
\tau = 4\lambda_0'^2 \lambda_2^2 = 4\lambda_0^2 \lambda_2^2.
$$

where $\tau$ is three-tangle and $C_{AB}$ is the concurrence–bipartite entanglement–between the qubits $A$ and $B$ [24]. Apart from these, a phase of the entanglement was introduced in Ref. [10] such that

$$
\cos \varphi_3 = \frac{\lambda_0' C_{BC} + \lambda_2' C_{AB} - \lambda^2 \tau}{C_{AB} C_{AC} C_{BC}}, \quad \varphi_3 \in [0, \pi].
$$

Here, the phase $\varphi_3$ is read as the entanglement phase (EP), and it becomes indefinite when $C_{AB} C_{AC} C_{BC} = 0$. Then, a state whose entanglement phase $\varphi_3$ is definite has been referred as an EP-definite state and a state whose entanglement phase $\varphi_3$ is indefinite has been referred as an EP-indefinite state [10].

Essentially, two arbitrary states $|\psi\rangle$ and $|\phi\rangle$ are considered to be in the same class if there is a nonzero probability of success for the both transformations $|\psi\rangle \rightarrow |\phi\rangle$ and $|\phi\rangle \rightarrow |\psi\rangle$ through stochastic local operations and classical communications (SLOCC). In the three-qubit case, there are two classes of genuine tripartite entangled states which cannot be converted into each other by SLOCC, namely, the Greenberger-Horne-Zeilinger (GHZ) and $W$ class states [25, 26]. If the three-tangle is nonzero, then the three-qubit state is of GHZ class. However, if three-tangle is zero and the reduced density matrices $\rho_A \equiv Tr_{BC} |\psi\rangle \langle \psi|, \rho_B$, and $\rho_C$ have rank two, then the state $|\psi\rangle$ is a $W$-type state. A general three-qubit $W$-type state is given by

$$
|\psi_W\rangle = \lambda_0 |000\rangle + \lambda_1 |100\rangle + \lambda_2 |101\rangle + \lambda_3 |110\rangle,
$$

where all the bipartite entanglements are nonzero and $\tau = 0$, i.e., $\lambda_i > 0$ for $i = 0, 2, 3$ and $\lambda_1 \geq 0$. Additionally, the standard GHZ state is given by

$$
|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle),
$$

and the standard $W$ state is given by

$$
|\text{W}\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle).
$$

These two states being completely inaccessible from each other by means of SLOCC.

To establish an effective protocol, it is crucial to start with suitable measurement operators. The complexity of the optimal transformation of $n$-qubit ($n \geq 3$) systems is due to local
quantum operators having vast numbers of parameters. The most general local measurement operators acting on qubits are two-by-two complex matrices:
\[ M = y_1 e^{i\delta_1} |0\rangle \langle 0| + y_2 e^{i\delta_2} |0\rangle \langle 1| + y_3 e^{i\delta_3} |1\rangle \langle 0| + y_4 e^{i\delta_4} |1\rangle \langle 1|, \]
where \( y_k \geq 0 \) and \( \delta_k \in [0, 2\pi) \) for \( k = 1, 2, 3, 4 \). Two operators \( M \) and \( M' \) are in the same equivalence class \( (M \equiv M') \) if they both transform states in one equivalence class to states in some other equivalence class with the same probability of success. In this context, the equivalence classes of local measurements, which allows one to write the operators with the minimal number of parameters, were defined in Ref. [23]. Throughout this paper, while the use of the canonical operators (i.e., the most general local measurement operators) simplifies the state transformations, the canonical forms of local measurement operators [23] will be used. These are given by
\[ M_{A_k} = a_{00k} |0\rangle \langle 0| + a_{10k} e^{i\theta_{A_k}} |1\rangle \langle 0| + a_{11k} |1\rangle \langle 1|, \]
\[ M_{B_k} = b_{00k} |0\rangle \langle 0| + b_{01k} e^{i\theta_{B_k}} |0\rangle \langle 1| + b_{11k} |1\rangle \langle 1|, \]
\[ M_{C_k} = c_{00k} |0\rangle \langle 0| + c_{01k} e^{i\theta_{C_k}} |0\rangle \langle 1| + c_{11k} |1\rangle \langle 1|, \]
for the parties \( A, B, \) and \( C \), respectively. Here \( \theta_{A_k} \in [0, 2\pi) \) \( (x = a, b, c) \) and all the coefficients are real. It is important to stress that to be able to apply a deterministic LOCC transformation to a given state, all the outputs are supposed to be LUeq. We have two LUeq states given in Eqs. \([1] \) and \([2] \), therefore, it is required to focus on a general two-outcome local operations for the desired deterministic transformations. To recap, the key ingredient for the protocol described in this paper is determining a right threshold; we will consider a general two-outcome local measurement of the form \([10], [11], [12]\), and these operations yield two states \( |\xi_1\rangle \) and \( |\xi_2\rangle \) which are LUeq (\( |\xi_1\rangle \sim |\xi_2\rangle \)).

III. TRANSFORMATIONS OF GHZ-TYPE STATES

We now proceed to examine the deterministic LOCC transformations of three-qubit GHZ-type states. We will discuss this problem under subsections IIIA, IIIB, and IIIC of this section in which each of them concerned with a certain final state. More specifically, in section IIIA the target state has only one nonzero bipartite entanglement, and in section IIIB the target state has two nonzero bipartite entanglements. Section IIIC is addressed to the final state where all bipartite entanglements are nonzero.

A. States with only one nonzero bipartite entanglement–local measurements by a single party

The transformation under scrutiny is the following. We initially have the standard GHZ state given in Eq. \([7]\) and aim to obtain a GHZ-type state, which has only one nonzero bipartite entanglement, via local quantum operations. There are three GHZ-type states with only one nonzero bipartite entanglement:
\[ \lambda_0 |000\rangle + \lambda_1 |100\rangle + \lambda_4 |111\rangle, \quad (C_{BC} = 2\lambda_1 \lambda_4), \]
\[ \lambda_0 |000\rangle + \lambda_2 |101\rangle + \lambda_4 |111\rangle, \quad (C_{AC} = 2\lambda_0 \lambda_2), \]
\[ \lambda_0 |000\rangle + \lambda_3 |110\rangle + \lambda_4 |111\rangle, \quad (C_{AB} = 2\lambda_0 \lambda_3). \]

The local operations carried out by a single party suffice to achieve the desired transformations. Suppose that the party \( q \) performs a local operation–consisting of a set of the measurement operators \( \{M_{ik}\} \)–to the GHZ state given in Eq. \([7]\). Then, the output states are obtained such that
\[ |\psi_k\rangle = \frac{M_{ik} |\text{GHZ}\rangle}{\sqrt{p_k}}, \quad (q = A, B, C), \]
where \( p_k = \langle \text{GHZ} | M_{ik}^\dagger M_{ik} |\text{GHZ}\rangle \). The measurement operators satisfy the normalization relation \( \sum_k M_{ik}^\dagger M_{ik} = I \) where \( I \) denotes the identity operator. It should be noted that in Eq. \([16]\), while the party \( q \) carries out a local measurement, the other two parties do not perform any measurement on their respective systems, e.g., for \( q = B \) the Eq. \([16]\) should be read as \( \langle I \otimes B_k \otimes I | \langle \text{GHZ} | / \sqrt{p_k} \rangle \). In the following we will present the set of the measurement operators \( \{M_{ik}\} \) for parties \( A, B, \) and \( C \) successively.

First, consider a general two-outcome local operation on the first qubit of the state given in Eq. \([7]\) with the measurement operators given by
\[ M_{A_k} = \lambda_0 |0\rangle \langle 0| + \lambda_1 |1\rangle \langle 0| + \lambda_4 |1\rangle \langle 1|, \]
\[ M_{B_k} = \lambda'_0 |0\rangle \langle 0| - \lambda'_1 |1\rangle \langle 0| + \lambda'_4 |1\rangle \langle 1|, \]
where \( \lambda'_0 = \lambda_0 / \kappa, \lambda'_1 = \lambda_1 / \kappa, \lambda'_4 = \kappa \lambda_4, \kappa = \sqrt{\lambda_0^2 + \lambda_1^2}/\lambda_4, \) and \( \sum_{k=0}^4 \lambda'_0^2 M_{A_k}^\dagger M_{A_k} = I \). The state after the measurements performed by party \( A \), i.e., \( |\psi_k\rangle = \langle M_{A_k} \otimes I \otimes I | \langle \text{GHZ} | / \sqrt{p_k} \rangle \), will be one of the states
\[ |\psi_1\rangle = \lambda_0 |000\rangle + \lambda_1 |100\rangle + \lambda_4 |111\rangle, \]
\[ |\psi_2\rangle = \lambda'_0 |000\rangle - \lambda'_1 |100\rangle + \lambda'_4 |111\rangle, \]
with probabilities \( p_k = \langle \text{GHZ} | M_{A_k}^\dagger M_{A_k} |\text{GHZ}\rangle = 1/2 \) for \( k = 1, 2 \). The states given in Eqs. \([19]\) and \([20]\) are LUeq. The local unitary transformations
\[ U_A = -\frac{\lambda_1 - i\lambda_0 \sigma_z}{\sqrt{\lambda_0^2 + \lambda_1^2}}, \quad U_B = i \sigma_y, \quad U_C = -\sigma_z, \]
on the qubits \( A, B, \) and \( C \), respectively, will transform the state \( |\psi_2\rangle \) into the state \( |\psi_1\rangle \): \( \langle U_A \otimes U_B \otimes U_C | \psi_2 \rangle = | \psi_1 \rangle \). Here \( \sigma_z = |0\rangle \langle 0| + |1\rangle \langle 1|, \sigma_x = -i |0\rangle \langle 1| + |1\rangle \langle 0|, \) and \( \sigma_z = |0\rangle \langle 0| - |1\rangle \langle 1| \) are the Pauli matrices. While all bipartite entanglements are initially zero, by the measurement on the first
qubit, the bipartite entanglement between the second and the third qubits becomes nonzero \((C_{BC} = 2\lambda_1\lambda_4)\). However, the bipartite entanglements between the first qubit and the other two qubits is still zero \((C_{AB} = C_{AC} = 0)\). In other words, the measurement carried out by the party \(A\) has no effect on the bipartite entanglements between the particles \(A - B\) and \(A - C\).

Second, consider a general two-outcome local operation on the second qubit of the state given in Eq. (7) with the measurement operators \(\{M_{bk}\}\) on the standard GHZ state. Then, for the final state, we have \(C_{q\tilde{q}} = C_{q\tilde{q}} = 0\) and \(C_{q\tilde{q}} \neq 0\).

To sum up, while all bipartite entanglements are zero for the initial state (7), when the party \(q\) performs a local operation on the measurement operators \(\{M_{bk}\}\), the bipartite entanglement between the first qubit and the other two qubits becomes nonzero \((C_{BC} = 2\lambda_1\lambda_4)\). However, the bipartite entanglement between the third qubit and the other two qubits is zero \((C_{AC} = C_{BC} = 0)\).

B. States with only one vanishing bipartite entanglement–local measurements by two parties

We now aim to obtain a GHZ-type state which has only one vanishing bipartite entanglement, starting with the standard GHZ state given in Eq. (7), via local quantum operations. We will carry out the desired transformations in two steps, and for the first steps we will exploit the results obtained in the subsection IIIA.

Let us consider the case that in the first step of the entire transformation the party \(A\) performs a local operation to the state (7). In that case, from the results obtained in the subsection IIIA, the state

\[
|\psi\rangle = |\lambda_0 \rangle \langle 000| + |\lambda_1 \rangle \langle 100| + |\lambda_4 \rangle \langle 111|
\]

on the qubits \(A, B,\) and \(C\), respectively, will transform the state (25) into the state (24). By the measurement on the second qubit, the bipartite entanglement between the first and the third qubits becomes nonzero \((C_{AC} = 2\lambda_0\lambda_2)\). However, the bipartite entanglement between the second qubit and the other two qubits is zero \((C_{AB} = C_{BC} = 0)\).

Third, consider a general two-outcome local operation on the third qubit of the state given in Eq. (7) with the measurement operators given by

\[
M_C = \lambda_0 |0 \rangle \langle 0| + \lambda_3 |0 \rangle \langle 1| + \lambda_4 |1 \rangle \langle 1|,
\]
to the state given in Eq. (32), one of the states
\[ |\phi_1\rangle = \mu_0 |000\rangle + \mu_1 |100\rangle + \mu_2 |101\rangle + \mu_4 |111\rangle, \] (34)
\[ |\phi_2\rangle = \mu'_0 |000\rangle + \mu'_1 |100\rangle - \mu'_2 |101\rangle + \mu'_4 |111\rangle, \] (35)
is obtained with probabilities \( p_1 = p_2 = 1/2 \), respectively, where \( \lambda_0 \mu_1 = \lambda_1 \mu_0, \mu'_0 = \mu_0/\kappa, \mu'_1 = \mu_1/\kappa, \mu'_2 = \kappa \mu_2, \mu'_4 = \kappa \mu_4 \), and \( \kappa = \sqrt{\mu_0^2 + \mu_1^2 / \mu_2^2 + \mu_4^2} \). Also, the condition for the deterministic transformation, \( \Sigma_{k=1}^2 M_{C_k} M_{B_k} = I \), gives \( \lambda_4 = 1/\sqrt{2} \). The states given in Eqs. (39) and (40) are LUEq. The local unitary transformations
\[ U_A = \frac{\mu_0 \sigma_x - \mu_1 \sigma_z}{\sqrt{\mu_0^2 + \mu_1^2}}, \quad U_B = \frac{\mu_2 - i \mu_3 \sigma_y}{\sqrt{\mu_2^2 + \mu_3^2}}, \quad U_C = -i \sigma_y, \] (36)
on the qubits A, B and C, respectively, will transform the state (35) into the state (34). As a result, deterministic transformations of the GHZ state (31) into a GHZ-type state via local operations performed by the party A first and the party B second can be expressed as follows:
\[ \langle \text{GHZ} \rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \]
\[ \downarrow \{ M_{B_k} \}_{k=1,2} \]
\[ |\psi\rangle = \lambda_0 |000\rangle + \lambda_1 |100\rangle + \frac{1}{\sqrt{2}} |111\rangle \] (37)
\[ \downarrow \{ M_{B_k} \}_{k=1,2}, \quad (\lambda_0 \mu_1 = \lambda_1 \mu_0) \]
\[ |\phi\rangle = \mu_0 |000\rangle + \mu_1 |100\rangle + \mu_2 |101\rangle + \mu_4 |111\rangle, \] where the sets of the measurement operators \( \{ M_{A_k} \}_{k=1,2} \) and \( \{ M_{B_k} \}_{k=1,2} \) are given in Eqs. (17) and (18) and Eq. (33), respectively. We note that for the final state |\phi\rangle given in Eq. (37), we have \( C_{AB} = 0, C_{AC} = 2 \mu_0 \mu_2 \) and \( C_{BC} = 2 \mu_1 \mu_4 \), i.e., there is no bipartite entanglement between the first and the second qubits—the parties performing sequential measurements.

One can also consider the party C as the particle which performs a local operation in the second step instead of the party B. Then, in the second step, if the party C performs the measurement operators
\[ M_{C_1} = \frac{\mu_0}{\sqrt{2} \lambda_0} |0\rangle \langle 0| + \mu_3 |0\rangle \langle 1| + \mu_4 |1\rangle \langle 1|, \] (38)
\[ M_{C_2} = \frac{\mu'_0}{\sqrt{2} \lambda_0} |0\rangle \langle 0| - \mu'_3 |0\rangle \langle 1| + \mu'_4 |1\rangle \langle 1|, \]
to the state given in Eq. (32), one of the states
\[ |\phi_1\rangle = \mu_0 |000\rangle + \mu_1 |100\rangle + \mu_3 |110\rangle + \mu_4 |111\rangle, \] (39)
\[ |\phi_2\rangle = \mu'_0 |000\rangle + \mu'_1 |100\rangle - \mu'_3 |110\rangle + \mu'_4 |111\rangle, \] (40)
is obtained with probabilities \( p_1 = p_2 = 1/2 \), respectively, where \( \lambda_0 \mu_1 = \lambda_1 \mu_0, \mu'_0 = \mu_0/\kappa, \mu'_1 = \mu_1/\kappa, \mu'_3 = \kappa \mu_3, \mu'_4 = \kappa \mu_4 \), and \( \kappa = \sqrt{\mu_0^2 + \mu_1^2 / \mu_3^2 + \mu_4^2} \). Also, the condition for the deterministic transformation, \( \Sigma_{k=1}^2 M_{C_k} M_{C_k} = I \), gives \( \lambda_4 = 1/\sqrt{2} \). The states given in Eqs. (39) and (40) are LUEq. The local unitary transformations
\[ U_A = \frac{\mu_0 \sigma_x - \mu_1 \sigma_z}{\sqrt{\mu_0^2 + \mu_1^2}}, \quad U_B = -i \sigma_y, \quad U_C = \frac{\mu_3 - i \mu_4 \sigma_y}{\sqrt{\mu_3^2 + \mu_4^2}}, \] (41)
on the qubits A, B and C, respectively, will transform the state (40) into the state (39). As a result, deterministic transformations of the GHZ state (31) into a GHZ-type state via local operations performed by the party A first and the party C second can be expressed as follows:
\[ \langle \text{GHZ} \rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \]
\[ \downarrow \{ M_{A_k} \}_{k=1,2} \]
\[ |\psi\rangle = \lambda_0 |000\rangle + \lambda_1 |100\rangle + \frac{1}{\sqrt{2}} |111\rangle \] (42)
\[ \downarrow \{ M_{C_k} \}_{k=1,2}, \quad (\lambda_0 \mu_1 = \lambda_1 \mu_0) \]
\[ |\phi\rangle = \mu_0 |000\rangle + \mu_1 |100\rangle + \mu_3 |110\rangle + \mu_4 |111\rangle, \]
where the sets of the measurement operators \( \{ M_{A_k} \}_{k=1,2} \) and \( \{ M_{C_k} \}_{k=1,2} \) are given in Eqs. (17) and (18) and Eq. (33), respectively. We note that for the final state |\phi\rangle given in Eq. (42), we have \( C_{AC} = 0, C_{AB} \neq 0 \) and \( C_{BC} \neq 0 \), i.e., the local measurements on the first and the third qubits create bipartite entanglements between the first and the second qubits, between the second and the third qubits, but do not create an entanglement between the first and the third qubits.

Now, finally, consider the case that in the first step of the entire transformation the party B performs a local operation to the state (31). In that case, from the results obtained in the subsection A, the state
\[ |\psi\rangle = \lambda_4 |000\rangle + \lambda_2 |101\rangle + \lambda_4 |111\rangle, \] (43)
can be obtained deterministically by performing the local measurement operators given in Eqs. (22) and (23). Then, in the second step, if the party C performs the measurement operators
\[ M_{C_1} = \mu_0 |0\rangle \langle 0| + \frac{\mu_1}{\sqrt{2} \lambda_2} |0\rangle \langle 1| + \frac{\mu_4}{\sqrt{2} \lambda_4} |1\rangle \langle 1|, \] (44)
\[ M_{C_2} = \mu'_0 |0\rangle \langle 0| - \frac{\mu'_1}{\sqrt{2} \lambda_2} |0\rangle \langle 1| + \frac{\mu'_4}{\sqrt{2} \lambda_4} |1\rangle \langle 1|, \]
to the state given in Eq. (43), one of the states
\[ |\phi_1\rangle = \mu_0 |000\rangle + \mu_1 |100\rangle + \mu_2 |101\rangle + \mu_4 |111\rangle, \] (45)
\[ |\phi_2\rangle = \mu'_0 |000\rangle - \mu'_1 |100\rangle + \mu'_2 |101\rangle - \mu'_4 |111\rangle, \] (46)
is obtained with probabilities \( p_1 = p_2 = 1/2 \), respectively, where \( \lambda_4 \mu_2 = \lambda_2 \mu_4, \lambda_4 \mu_1 = \lambda_3 \mu_3, \mu_1 \mu_4 = \mu_3 \mu_5, \mu'_0 = \mu_0/\kappa, \mu'_1 = \kappa \mu_2, \mu'_2 = \kappa \mu_3, \mu'_3 = \kappa \mu_4, \mu'_4 = \kappa \mu_4 / \sqrt{(\mu_2^2 + \mu_3^2)(\mu_2^2 + \mu_4^2)} \). Also, the condition for the
deterministic transformation, \( \sum_{k=1}^{2} M_{Ck}^\dagger M_{Ck} = I \), gives \( \lambda_1 = 1/\sqrt{2} \). The states given in Eq. (45) and Eq. (46) are LUeq.

The local unitary transformations

\[
U_A = -i\sigma_y, \quad U_B = \frac{\mu_2 \sigma_z + \mu_4 \sigma_x}{\sqrt{\mu_2^2 + \mu_4^2}}, \quad U_C = \frac{\mu_3 i - i\mu_4 \sigma_z}{\sqrt{\mu_3^2 + \mu_4^2}},
\]

(47)
on the qubits A, B and C, respectively, will transform the state (40) into the state (39). As a result, deterministic transformations of the pure GHZ state into a GHZ-type state via local operations performed by the party B first and the party C second can be expressed as follows:

\[
|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)
\]

\[
\downarrow \{M_{Bk}\}_{k=1,2}
\]

\[
|\psi\rangle = \frac{1}{\sqrt{2}}(|000\rangle + \lambda_2 |101\rangle + \lambda_4 |111\rangle)
\]

\[
\downarrow \{M_{Ck}\}_{k=1,2}
\]

\[
|\phi\rangle = \mu_0 |000\rangle + \mu_1 |100\rangle + \mu_2 |101\rangle + \mu_3 |110\rangle + \mu_4 |111\rangle,
\]

(48)

where the sets of the measurement operators \( \{M_{Bk}\}_{k=1,2} \) and \( \{M_{Ck}\}_{k=1,2} \) are given in Eqs. (22) and (23) and Eq. (44), respectively. We note that for the final state \( |\phi\rangle \) given in Eq. (48), we have \( C_{BC} = 0 \) (\( \mu_1 \mu_4 = \mu_2 \mu_3 \)), \( C_{AB} \neq 0 \) and \( C_{AC} \neq 0 \), i.e., there is no bipartite entanglement between the second and the third qubits—the qubits performing sequential measurements.

To sum up, when the party \( q \) first performs a local operation with the measurement operators \( \{M_{Bk}\} \) to the state (7), and the party \( \tilde{q} \) second performs a local operation with the measurement operators \( \{M_{\tilde{B}k}\} \) to the state obtained from the local operation performed by party \( q \) in the first step, then the bipartite entanglements between the parties \( q \) and \( \tilde{q} \) and between the parties \( \tilde{q} \) and \( \hat{q} \) become nonzero, \( C_{q\tilde{q}} \neq 0 \) and \( C_{\tilde{q}\hat{q}} \neq 0 \). On the other hand, the bipartite entanglement between the parties \( q \) and \( \hat{q} \) is zero, \( C_{q\hat{q}} = 0 \) (see Fig. 2).

C. State with nonzero bipartite entanglements—local measurements by three parties

In subsections III A and III B we have discussed the deterministic transformations of the GHZ state (7) (an EP-indefinite state) into an EP-indefinite GHZ-type state by LOCC. This means the final states have at least one vanishing bipartite entanglement. We now aim to obtain the most general GHZ-type state (see Fig. 3)—all bipartite entanglements are nonzero (an EP-definite state)—starting with the standard GHZ state given in Eq. (7), via local quantum operations. We will try to carry out the desired transformation in three steps, and for the first two steps we will exploit the results obtained in the subsection III B. As discussed in the previous subsection, the state

\[
|\phi\rangle = \mu_0 |000\rangle + \mu_1 |100\rangle + \mu_2 |101\rangle + \mu_3 |110\rangle + \mu_4 |111\rangle,
\]

(49)

can be deterministically obtained by local operations respectively performed by the first and second parties. We have \( \tau = 4\mu_0^2 \mu_4^2 \), \( C_{BC} = 2\mu_1 \mu_4 \), \( C_{AB} = 0 \), and \( C_{AC} = 2\mu_0 \mu_2 \) for the initial state (49). Then, if the party \( C \) performs a two-outcome local operation by the measurement operators

\[
M_{Ck} = c_{00k} |0\rangle \langle 0| + c_{01k} e^{i\theta_k} |0\rangle \langle 1| + c_{11k} |1\rangle \langle 1|,
\]

(50)

for \( k = 1, 2 \), on the third qubit of the state \( |\phi\rangle \) given in Eq. (49), one can obtain

\[
(I \otimes I \otimes M_{Ck}) |\phi\rangle = c_{00k} \mu_0 |000\rangle + [c_{00k} \mu_1 + c_{01k} e^{i\theta_k} \mu_2] |100\rangle + c_{01k} e^{i\theta_k} \mu_4 |110\rangle + c_{11k} \mu_2 |101\rangle + c_{11k} \mu_4 |111\rangle
\]

\[
= \sqrt{\beta_k} |\zeta_k\rangle,
\]

(51)

where

\[
|\zeta_1\rangle = \alpha_0 |000\rangle + \alpha_1 e^{i\delta_1} |100\rangle + \alpha_2 |101\rangle + \alpha_3 |110\rangle + \alpha_4 |111\rangle,
\]

(52)

\[
|\zeta_2\rangle = \alpha_0' |000\rangle + \alpha_1' e^{i\delta_1'} |100\rangle + \alpha_2' |101\rangle + \alpha_3' |110\rangle + \alpha_4' |111\rangle,
\]

(53)
\[ p_k = \langle \phi | I \otimes I \otimes (M_G^\dagger M_C) | \phi \rangle \] for \( k = 1, 2 \), \( \theta_{i3} = 0 \), \( \theta_{i3} = \pi \), \( \alpha = 0 \), and \( \alpha' = \pi \). The states given in Eqs. (52) and (53) are LUeq (for the relations between the coefficients \( \alpha_i \) and \( \alpha'_i \) replace \( \lambda \) with \( \alpha \) in Eq. (4)). Also, for the case of deterministic transformation, it is required that \( \sum_{k=1}^{2} M_{ik}^\dagger M_{ik} = I \). The constraints on the deterministic transformation—LU equivalence of the output states \( |z_i \rangle \) and \( |z'_i \rangle \) and \( p_1 + p_2 = 1 \)—yields
\[
\mu_0^2 + \mu_1^2 = \mu_2^2 + \mu_3^2 = \frac{1}{2},
\]
for the initial state (49), and
\[
\alpha_2 \alpha_3 (\alpha_2 \alpha_3 - \alpha_1 \alpha_4) = 0
\]
for the final states. The Eq. (55) can also be written such that
\[
C_{AB} C_{AC} C_{BC} = 0,
\]
where \( C_{AB} = 2 \alpha_0 \alpha_3 \), \( C_{AC} = 2 \alpha_0 \alpha_2 \), \( C_{BC} = 2 (\alpha_2 \alpha_3 - \alpha_1 \alpha_4) \), and \( \alpha_0 \neq 0 \). The result (56) suggests that the final state must have at least one vanishing bipartite concurrence. However, the final states (52) and (53) have nonzero bipartite entanglements (this is the case we study). Hence, deterministic transformation of the GHZ state \( |7\rangle \) into a GHZ-type state is possible if the target state satisfies the condition given by the Eq. (56). In other words, the standard GHZ state, which is an EP-indefinite state, can not be deterministically transformed to a state with all bipartite entanglements are nonzero (an EP-definite state).

IV. TRANSFORMATIONS OF W-TYPE STATES

We now provide the local measurement operators for the deterministic transformations of a W-type state into another W-type state. A general three-qubit W-type state is given by
\[
|\psi_W \rangle = \lambda_0 |000 \rangle + \lambda_1 |100 \rangle + \lambda_2 |101 \rangle + \lambda_3 |110 \rangle,
\]
where all the bipartite entanglements are nonzero and \( \tau = 0 \), i.e., \( \lambda_i > 0 \) for \( i = 0, 2, 3 \) and \( \lambda_i \geq 0 \) (see Fig. 4). As given in Ref. [13], a deterministic LOCC transformation from a W-type state
\[
|\chi \rangle = x_0 |000 \rangle + x_1 |100 \rangle + x_2 |101 \rangle + x_3 |110 \rangle
\]
to another W-type state
\[
|\chi' \rangle = x'_0 |000 \rangle + x'_1 |100 \rangle + x'_2 |101 \rangle + x'_3 |110 \rangle,
\]
is possible if and only if \( x_i \geq x'_i \) for \( i = 1, 2, 3 \). Needless to say, the state (58) can be transformed into the canonical form
\[
|\chi \rangle = x_1 |000 \rangle + x_0 |100 \rangle + x_3 |101 \rangle + x_2 |110 \rangle,
\]
by the unitary transformation \( \sigma_2 \) on the first qubit, that is \( |0 \rangle \leftrightarrow |1 \rangle \). We consider the case where the initial state is \( |\chi \rangle \) given in Eq. (60) with \( x_i > 0 \) for \( i = 1, 2, 3 \) and \( x_0 \geq 0 \). Then, the party A performs a general two-outcome local operation to the state (60) with the canonical measurement operators
\[
M_{A_k} = \sqrt{p_k} \alpha_0 \beta_k \langle 0 | \langle 0 | + (-1)^{k-1} \sqrt{P_{k-1}} \sqrt{1 - \alpha_0^2 \beta_k^2} \langle 1 | \langle 1 |,
\]
\[
\sqrt{p_k} \beta_k \langle 1 | \langle 1 |,
\]

where \( \alpha_0 \geq 0 \) and \( \sum_{k=1}^{3} M_{A_k}^\dagger M_{A_k} = I \). We then have the states
\[
|\psi_k \rangle = \langle M_{A_k} \otimes I \otimes I | |\chi \rangle = \sqrt{p_k} \alpha_0 |000 \rangle + (-1)^{k-1} \alpha_1 |100 \rangle + x_3 |101 \rangle + x_2 |110 \rangle,
\]
for \( k = 1, 2 \), where we also have \( x_0^2 + x_1^2 = \alpha_0^2 + \alpha_2^2 \). Here, the probabilities are found to be \( p_1 = (\alpha_1 + x_0)/2 \alpha_1 \) and \( p_2 = (\alpha_1 - x_0)/2 \alpha_1 \). The local unitary transformation \( \sigma_2 \) on the three particles apiece allows us to transform the state (62) for \( k = 2 \) into the state (63) for \( k = 1 \), i.e., \( |\chi \rangle = |\psi_1 \rangle \). To recap, the party A can transform the state (60) into the state (62) (for \( k = 1 \)) with unit probability via the local measurement operators (61):
\[
|\chi \rangle = x_1 |000 \rangle + x_0 |100 \rangle + x_3 |101 \rangle + x_2 |110 \rangle
\]
\[
\downarrow \left\{ \begin{array}{c} (M_{A_k}, p_k = \frac{1}{2} + \frac{(-1)^{k-1} x_0}{2 \alpha_1}) \end{array} \right\}_{k=1,2}
\]
\[
|\psi \rangle = \alpha_0 |000 \rangle + \alpha_1 |100 \rangle + x_3 |101 \rangle + x_2 |110 \rangle,
\]
where \( x_1 \geq \alpha_0 \) and \( x_0 \leq \alpha_0 \). The state \( |\psi \rangle \) given in Eq. (63) is the most general state that can be deterministically obtained by the measurements on the first qubit of the source state \( |\chi \rangle \) given in Eq. (60). Next, the party B performs a two-outcome measurement to the second qubit of the state \( |\psi \rangle \) given in Eq. (63) with the canonical measurement operators
\[
M_{B_k} = \sqrt{p_k} \beta_k \langle 0 | \langle 0 | + (-1)^{k-1} \sqrt{P_{k-1}} \sqrt{1 - \alpha_0^2 \beta_k^2} \langle 1 | \langle 1 |,
\]
\[
\sqrt{p_k} \beta_k \langle 1 | \langle 1 |,
\]
\[
|\phi_k \rangle = \langle I \otimes M_{B_k} \otimes I | |\psi \rangle = \sqrt{p_k} \beta_k \langle 0 | \langle 0 | + (-1)^{k-1} \beta_k |100 \rangle + x_3 |101 \rangle + \beta_3 |110 \rangle,
\]
where \( \beta_1 \geq 0 \) and \( \sum_{k=1}^{3} M_{B_k}^\dagger M_{B_k} = I \). The resulting state will then be one of the states
\[
|\phi_1 \rangle = \langle I \otimes M_{B_1} \otimes I | |\psi \rangle = \sqrt{p_1} |000 \rangle + (-1)^{k-1} \beta_k |100 \rangle + x_3 |101 \rangle + \beta_3 |110 \rangle,
\]
where the probabilities are found to be \( p_1 = (\beta_1 + \alpha_1)/2 \beta_1 \) and \( p_2 = (\beta_1 - \alpha_1)/2 \beta_1 \), respectively, and \( x_0^2 + x_1^2 = \beta_1^2 + \beta_3^2 \). The states \( |\phi_1 \rangle \) and \( |\phi_2 \rangle \) given in Eq. (65) are LUeq. The unitary transformation \( -\sigma_2 \) carried out by the party A, and the unitary transformation \( \sigma_2 \) carried out by the party B will transform the state \( |\phi_2 \rangle \) into the state \( |\phi_1 \rangle \), i.e., \( -\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \).\]
As a result, we have

$$\ket{\psi} = \alpha_0 \ket{000} + \alpha_1 \ket{100} + x_3 \ket{101} + x_2 \ket{110}$$

$$\downarrow \left\{ \left( M_{\phi_k}, p_k = \frac{1}{2} + \frac{(-1)^{k-1} \alpha_1}{2 \beta_1} \right) \right\}_{k=1,2}$$

$$\ket{\phi} = \alpha_0 \ket{000} + \beta_1 \ket{100} + x_3 \ket{101} + \beta_3 \ket{110},$$

where $x_2 \geq \beta_3$ and $\alpha_1 \leq \beta_1$. The first two qubits together can transform the state $\ket{\chi}$ given in Eq. (60) to the state $\ket{\phi}$ given in Eq. (66) with unit probability (by combining the Eq. (63) and Eq. (66)). Lastly, a two-outcome measurement performed by the party $C$ to the third qubit of the state $\ket{\phi}$ given in Eq. (66) with the canonical measurement operators

$$M_{\chi_k} = \sqrt{p_k} \ket{0} \bra{0} + \frac{(-1)^{k-1}}{\sqrt{p_3 - k}} \sqrt{1 - \frac{p_2}{x_3^2}} \ket{1} \bra{1}$$

$$+ \frac{p_2}{x_3} \ket{1} \bra{1},$$

where $\gamma_2 \geq 0$ and $\sum_{k=1}^{2} M_{\chi_k} M_{\chi_k} = I,$ gives one of the states

$$\ket{\phi_k} = \left( I \otimes I \otimes M_{\chi_k} \right) \ket{\phi}$$

$$= \alpha_0 \ket{000} + \frac{(-1)^{k-1} \gamma_1}{2 \gamma_1} \ket{100} + \gamma_2 \ket{101} + \beta_3 \ket{110},$$

with probabilities $p_1 = (\gamma_1 + \beta_1)/2 \gamma_1$ and $p_2 = (\gamma_1 - \beta_1)/2 \gamma_1$, respectively. Here we have that $x_2^2 + \beta_3^2 = \gamma_1^2 + \gamma_2^2$. The states $\ket{\phi_1}$ and $\ket{\phi_2}$ given in Eq. (65) are LUEq. The unitary transformation $-\sigma_z$ carried out by the first qubit, and unitary transformation $\sigma_z$ carried out by the second qubit will transform the state $\ket{\phi_2}$ into the state $\ket{\phi_1}$, i.e., $-\sigma_z \otimes \sigma_z \otimes I \ket{\phi_2} = \ket{\phi_1}$. We finally have

$$\ket{\phi} = \alpha_0 \ket{000} + \beta_1 \ket{100} + x_3 \ket{101} + \beta_3 \ket{110}$$

$$\downarrow \left\{ \left( M_{\chi_k}, p_k = \frac{1}{2} + \frac{(-1)^{k-1} \beta_1}{2 \gamma_1} \right) \right\}_{k=1,2}$$

$$\ket{\phi} = \alpha_0 \ket{000} + \gamma_1 \ket{100} + \gamma_2 \ket{101} + \beta_3 \ket{110},$$

where $x_3 \geq \gamma_2$ and $\beta_1 \leq \gamma_1$. In conclusion, we obtained the entire transformation such that

$$\ket{\chi} \rightarrow M_{\chi_k} \rightarrow \ket{\psi} \rightarrow M_{\chi_k} \rightarrow \ket{\phi}.$$

Here, the initial state $\ket{\chi}$ given in Eq. (60) and the final state $\ket{\phi}$ given in Eq. (69) attest the if and only if condition [13] for the deterministic transformation $\ket{\chi} \rightarrow \ket{\phi}$: $x_1 \geq \alpha_0$, $x_2 \geq \beta_1$, $x_3 \geq \gamma_2$, and $x_3 \leq \gamma_1$.

The canonical form of the standard $W$ state given in Eq. (8) is, by taking $x_0 = 0$ and $x_1 = x_2 = x_3 = 1/\sqrt{3}$ in (60), $\ket{000} + \ket{101} + \ket{110})/\sqrt{3}$. Then, deterministic transformation of the standard $W$ state into a $W$-type state can be written such that

$$\ket{W} = \frac{1}{\sqrt{3}} (\ket{000} + \ket{101} + \ket{110})$$

$$\downarrow \left\{ \left( M_{\chi_k}, p_k = \frac{1}{2} \right) \right\}_{k=1,2}$$

$$\ket{\psi} = \alpha_0 \ket{000} + \alpha_1 \ket{100} + \frac{1}{\sqrt{3}} \ket{101} + \frac{1}{\sqrt{3}} \ket{110}$$

$$\downarrow \left\{ \left( M_{\chi_k}, p_k = \frac{1}{2} + \frac{(-1)^{k-1} \alpha_1}{2 \beta_1} \right) \right\}_{k=1,2}$$

$$\ket{\phi} = \alpha_0 \ket{000} + \beta_1 \ket{100} + \frac{1}{\sqrt{3}} \ket{101} + \beta_3 \ket{110},$$

where the local measurement operators are given in Eqs. (61), (64), and (67). As one can easily notice, when the party $q$ carries out a local operation then the bipartite entanglement between the other two parties remains unchanged while the bipartite entanglements between the party $q$ and the other two parties decrease.

V. CONCLUSION

In summary, the present paper sets out to examine the deterministic LOCC transformations of three-qubit entangled pure states. While an arbitrary three-qubit pure state can exist in one of the two inequivalent SLOCC classes of tripartite entanglement, we discussed the deterministic transformations under two separate sections.

We first presented local quantum operations for the deterministic transformations of a GHZ-type state into another GHZ-type state. By using both two LUEq forms of three-qubit entangled pure states and the canonical forms of local measurement operators [23], we introduced a simple and practical protocol, offering an alternative point of view. We originally had the standard GHZ state and applied our protocol to obtain a GHZ-type state with only one nonzero bipartite entanglement and only one vanishing bipartite entanglement. The former was achieved by a single party and the later was achieved by the cooperation of two parties in two steps.

Next, we aimed to obtain the most general GHZ-type state—the state with all bipartite entanglements nonzero. The most significant finding to emerge from this study is that the GHZ state (and a GHZ-type state with at least one vanishing bipartite entanglement) cannot be deterministically transformed to a GHZ-type state with all bipartite entanglements are nonzero. In other words, for the target state, if the bipartite entanglements satisfy the relation $C_{\Lambda_0}C_{\Lambda_1}C_{\Lambda_2} = 0$ then the deterministic transformation is possible. This result contributes to our understanding of GHZ-type states transformation.

Finally, we presented local quantum operations for the deterministic transformations of a $W$-type state into another $W$-type state. Here we again used the canonical form of local
measurement operators and achieved the transformations in three steps (i.e., with the cooperation of three parties). Each step of the entire transformation is also a deterministic transformation. Furthermore, the entire transformation gives the if and only if condition \[13\] for the deterministic transformations of \(W\)-type states.

\[1\] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. 47, 777 (1935).
\[2\] E. Schrödinger, Naturwiss. 23, 807 (1935).
\[3\] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. 81, 865 (2009).
\[4\] L. Amico, R. Fazio, A. Osterloh, and V. Vedral, Rev. Mod. Phys. 80, 517 (2008).
\[5\] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. 70, 1895 (1993).
\[6\] C. H. Bennett and S. J. Wiesner, Phys. Rev. Lett. 69, 2881 (1992).
\[7\] A. K. Ekert, Phys. Rev. Lett. 67, 661 (1991).
\[8\] A. K. Pati, Phys. Rev. A 63, 014302 (2000).
\[9\] E. Chitambar and G. Gour, Rev. Mod. Phys. 91, 025001 (2019).
\[10\] H. Tajima, Annals of Physics 329, 1 (2013).
\[11\] S. Turgut, Y. Güll, and N. K. Pak, Phys. Rev. A 81, 012317 (2010).
\[12\] F. M. Spedalieri (2001), arXiv:quant-ph/0110179.
\[13\] S. Kını Ş and S. Turgut, Journal of Mathematical Physics 51, 092202 (2010).
\[14\] O. Cohen and T. A. Brun, Phys. Rev. Lett. 84, 5908 (2000).
\[15\] A. Acín, E. Jané, W. Dür, and G. Vidal, Phys. Rev. Lett. 85, 4811 (2000).
\[16\] O. Cohen and T. A. Brun, Phys. Rev. Lett. 84, 5908 (2000).
\[17\] A. Yildiz, Phys. Rev. A 82, 012317 (2010).
\[18\] W. Cui, W. Helwig, and H.-K. Lo, Phys. Rev. A 81, 012111 (2010).
\[19\] G. Karpat and Z. Gedik, Physics Letters A 376, 75 (2011).
\[20\] Y. Chen and H. Kan, Phys. Rev. A 90, 062340 (2014).
\[21\] A. Acín, A. Andrianov, L. Costa, E. Jané, J. I. Latorre, and R. Tarrach, Phys. Rev. Lett. 85, 1560 (2000).
\[22\] A. Acín, A. Andrianov, E. Jané, and R. Tarrach, Journal of Physics A: Mathematical and General 34, 6725 (2001).
\[23\] G. Torun and A. Yildiz, Phys. Rev. A 89, 032320 (2014).
\[24\] V. Coffman, J. Kundu, and W. K. Wootters, Phys. Rev. A 61, 052306 (2000).
\[25\] W. Dür, G. Vidal, and J. I. Cirac, Phys. Rev. A 62, 062314 (2000).
\[26\] A. Acín, D. Bruß, M. Lewenstein, and A. Sanpera, Phys. Rev. Lett. 87, 040401 (2001).
\[27\] C. Sabín and G. García-Alcaine, Eur. Phys. J. D 48, 435 (2008).