Simulation of wave propagation in a periodic layered elastomer using spectral element method

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Abstract. The problem of wave propagation in a periodic elastomer is investigated. Dispersion equation for a periodic layered composite made of elastomers is formulated and solved employing the spectral element method. The problem is formulated in a variational form taking into account that stiffness tensor has no symmetry when an elastomer is considered. The results are compared with the same calculations made using the transfer matrix method. The method can be naturally extended to dielectric elastomers and to the three-dimensional case.

1. Introduction
In recent years, acoustic metamaterials and phononic crystals have received special attention since they provide unique properties such as negative refraction [1], band-gaps [2, 3], tunable properties [4, 5] etc. Employment of periodic composite structures has a great potential in development of sensors/actuators along with the systems of active noise reduction, adaptive optics, energy harvesting etc. Introduction of dielectric elastomers, as components of these structures, allows more efficient manipulating of the wave energy flow. However, wave propagation and diffraction in acoustic metamaterials with elastomers have been studied for a relatively low number of problems [6, 7]. The standard finite element software, e.g. COMSOL Multiphysics, does not include tools for the simulation of arbitrary stiffness, piezoelectric and dielectric constants tensors. The main objective of this study is to extend the spectral element method [8] to the simulation of periodic elastomer composite, also called phononic crystal, and provide investigation results of the propagating waves properties.

2. Statement of the problem
2.1. Finite layered phononic crystal between two half-planes
Let us consider a layered phononic crystal (PnC) situated between two isotropic elastic half-planes, see Fig. 1. In the context of this work we consider a 2D problem. The PnC is made of \( M \) elastic unit-cells of thickness \( H \) composed of \( L \) sublayers. Therefore, Cartesian coordinate system \( x = (x_1, x_2) \) is introduced so that the lower boundary of the PnC is parallel to \( Ox_1 \) axis. Index \( n = 1, \ldots, ML \) is used to number sub-layers \( V^{(n)} = \{ |x_1| \leq \infty, z_{n-1} \leq x_2 \leq z_n \} \) of finite thickness \( h_n = z_n - z_{n-1} \). The lower \( \{ x_2 \leq z_0 \} \) and upper \( \{ x_2 \geq z_{ML} \} \) half-planes are denoted as \( V^{(0)} \) and \( V^{(ML +1)} \) respectively. Material properties of the \( n \)-th layer \( (n = 1, \ldots, ML) \) and
Figure 1. The geometry of the problem for a finite phononic crystal between two half-planes.

half-planes \((n = 0, ML + 1)\) are given by tensor \(C^{(n)}_{ijkl}\), which in general case for the elastomer has no symmetry, and the mass density \(\rho^{(n)}\).

The governing equations for the time-harmonic wave motion with the angular frequency \(\omega\) in \(n\)-th layer \(V^{(n)}\) are written in terms of the components of stress tensor \(\sigma_{ij}\) and the displacement vector with two non-zero components \(u = \{u_1, u_2\}\) as follows:

\[
\sum_{j=1}^{2} \frac{\partial \sigma^{(n)}_{ij}(x)}{\partial x_j} + \rho^{(n)} \omega^2 u^{(n)}_i(x) = 0. \tag{1}
\]

Constitutive equations

\[
\sigma^{(n)}_{ij} = C^{(n)}_{ijkl} \frac{\partial u^{(n)}_k(x)}{\partial x_l} \tag{2}
\]

relate components of the stress tensor \(\sigma_{ij}\) and the displacement vector \(u_k\). For in-plane motion, the substitution of constitutive equations (2) into (1) gives governing equations in terms of the displacement vector \(u\) for \(n\)-th layer as follows:

\[
\sum_{j=1}^{2} \sum_{k=1}^{2} \sum_{l=1}^{2} C^{(n)}_{ijkl} \frac{\partial^2 u^{(n)}_k(x)}{\partial x_l \partial x_j} + \rho^{(n)} \omega^2 u^{(n)}_i(x) = 0. \tag{3}
\]

Hereafter, upper index \(n\) is omitted where possible for easier perception of the equations.

2.2. Infinite layered phononic crystal

An infinite phononic crystal with a two-layered unit-cell is considered as shown in figure 2, where governing equations (1) are valid. To describe wave propagating at angle \(\theta\) with respect to \(Ox_2\), wavenumber \(\zeta\) is introduced following the Floquet-Bloch theory for periodic structures. Therefore, the following periodic boundary conditions are assumed for an infinite periodic structure:

\[
u_k(x_1, H) = u_k(x_1, 0)e^{-i\zeta \cos \theta H}, \quad 0 \leq x_1 \leq d, \tag{4}\]
3. The spectral element method

The problem of a finite phononic crystal situated between two half-spaces can be solved using the extended transfer matrix method [9]. At the same time, infinite PnC can be simulated with the mesh-based numerical methods, for example finite element method. However, the standard software, e.g. COMSOL Multiphysics, usually does not include modules, which allow to consider arbitrary stiffness and piezoelectric tensor. Therefore, the spectral element method is applied to solve the boundary value problem (3)–(7). The variational form

\[
\sum_{i=1}^{2} \sum_{j=1}^{2} \oint_{\partial V} \sigma_{ij}(x) v_k^l(x) n_j \, d\ell - \sum_{i=1}^{2} \sum_{j=1}^{2} \int_V \sigma_{ij}(x) \frac{\partial v_k^l(x)}{\partial x_j} \, dV + \rho \omega^2 \sum_{i=1}^{2} \int_V u_i(x) v_k^l(x) \, dV = 0, \tag{8}
\]

of the governing equations for a given domain \( V \) with boundary \( \partial V \) is used in the spectral element method.

Interpolation polynomials based on Gauss-Legendre-Lobatto grid \( C^{li}_k(\xi^l_i) \), where \( k = 1, 2 \), are employed as basis functions

\[
u_k(x) = \sum_{l_1=1}^{M_1} \sum_{l_2=1}^{M_2} \sum_{i_1=1}^{N} \sum_{i_2=1}^{N} u_{l_1l_2i_1i_2}^k C^{l_1i_1}_1 C^{l_2i_2}_2 = \sum_{l} u_l^k C^{li}_1 C^{l_2}_2. \tag{9}\]

The global coordinates \( x_i \) are transformed into local coordinates \( \xi^l_i \) at the each element \( l \):

\[
\xi^l_i = \frac{2x_i - x_i^{l+1} - x_i^l}{x_i^{l+1} - x_i^l}, \quad x_i = \frac{x_i^{l+1} - x_i^l}{2} \xi^l_i + \frac{x_i^{l+1} + x_i^l}{2}, \quad i = 1, 2. \tag{10}\]
Employing periodic Floquet boundary conditions (6) and (7), contour integrals can be rewritten as

$$\frac{d}{dx_i}v_i = r_i \frac{d}{d\xi_i}, \quad \nu_i = \frac{2}{x_{i+1}^l - x_i^l}, \quad i = 1, 2.$$ 

Here index \(I(l_1, l_2, i_1, i_2)\) is introduced for simplicity, and \(I\) is in the range 1, \ldots, \(G = M_1 M_2 N^2\), more details can be found in [10]. According to the Bubnov-Galerkin method, projection functions coincide with basis functions

$$v_i^k(x) = \delta_{k' l} C_{l_1}^i \left(\xi_1^l\right) C_{l_2}^i \left(\xi_2^l\right). \quad (11)$$

Here index \(I'(l'_1, l'_2, i'_1, i'_2)\) is used to address projection functions in the same way as index \(I\).

Then, the relations for stresses can be also written via (9) and (11):

$$\sigma_{ij}(x) = \sum_{k=1}^{2} \sum_{l=1}^{G} C_{ijkl} \frac{\partial}{\partial x_l} \left( \sum_{i} u_k^i C_{i1}^i \left(\xi_1^l\right) C_{i2}^i \left(\xi_2^l\right) \right) =$$

$$= \sum_{k=1}^{2} \sum_{l=1}^{G} u_k^i \left( C_{ijkl1} \frac{dC_{i1}^i \left(\xi_1^l\right)}{dx_1} C_{i2}^i \left(\xi_2^l\right) + C_{ijkl2} C_{i1}^i \left(\xi_1^l\right) \frac{dC_{i2}^i \left(\xi_2^l\right)}{dx_2} \right).$$

Employing periodic Floquet boundary conditions (6) and (7), contour integrals can be rewritten as follows:

$$\oint \sum_{i=1}^{2} \sum_{j=1}^{G} \sigma_{ij}(x) v_i^k(x) n_j(x) d\ell =$$

$$= \sum_{i=1}^{2} \int_{0}^{d} \left( -\sigma_{i2} (x_1, 0) v_i^k (x_1, 0) + \sigma_{i2} (x_1, H) v_i^k (x_1, H) \right) dx_1 +$$

$$\sum_{i=1}^{2} \int_{0}^{H} \left( -\sigma_{i1} (0, x_2) v_i^k (0, x_2) + \sigma_{i1} (d, x_2) v_i^k (d, x_2) \right) dx_2.$$  

These integrals can be evaluated using quadrature formula based on \(N\) Gauss-Legendre-Lobatto grid nodal points \(\chi_k\):

$$\int_{-1}^{1} f(x) dx = \sum_{k=1}^{N} w_k f(\chi_k). \quad (12)$$

Let us consider contour integrals

$$P_1 = -\sum_{i=1}^{2} \int_{0}^{d} \sigma_{i2} (x_1, 0) v_i^k (x_1, 0) dx_1.$$  

Using (2), (9) and (11), one can derive

$$P_1 = -\sum_{i=1}^{2} \sum_{k=1}^{2} \sum_{l=1}^{G} u_k^i \left( \int_{0}^{d} C_{ijkl1} \frac{dC_{i1}^i \left(\xi_1^l\right)}{dx_1} C_{i2}^i \left(\xi_2^l\right) \delta_{k' l} C_{i1}^j \left(\xi_1^l\right) C_{i2}^j \left(\xi_2^l\right) dx_1 + \right.$$  

$$+ \int_{0}^{d} C_{ijkl2} C_{i1}^i \left(\xi_1^l\right) \frac{dC_{i2}^i \left(\xi_2^l\right)}{dx_2} \delta_{k' l} C_{i1}^j \left(\xi_1^l\right) C_{i2}^j \left(\xi_2^l\right) dx_1 \right).$$
Gauss-Legendre-Lobatto polynomials $C^i(\chi_j)$ are non-zero only if $i = j$: $C^i(\chi_j) = \delta_{ij}$. The values of derivative at the nodal points $\chi_k$ are designated as

$$\frac{dC^i(\chi_k)}{d\xi} = D_k^i.$$ 

Therefore, the following expressions are valid taking into account (12):

$$C^{i2}(\xi_2^i) C^{i2}(\xi_2^i) = \delta_{i21} \delta_{i21} \delta_{i21}, \quad x_2 = 0,$$

$$\frac{dC^{i2}(\xi_2^i)}{dx_2} C^{i2}(\xi_2^i) = \delta_{i21} \delta_{i21} D_i^1 \delta_{i21}, \quad x_2 = 0,$$

$$\int_0^d \frac{dC^{i1}(\xi_1^i)}{dx_1} C^{i1}(\xi_1^i) \, dx_1 = \delta_{i11} \omega_i \delta_{i11},$$

$$\int_0^d C^{i1}(\xi_1^i) C^{i1}(\xi_1^i) \, dx_1 = \frac{1}{\omega_i} \delta_{i11} \delta_{i11}.$$ 

Then, integral $P_1$ is calculated as:

$$P_1 = - \sum_{k=1}^{2} \sum_{l} u_k^l \delta_{i1l} \delta_{i1l} \delta_{i1l} \omega_i \sum_{i=1}^{2} \delta_{k'l} \left( C_{i2k1} \delta_{i21} D_i^{1l} + C_{i2k2} D_i^{1l} \delta_{i11} \frac{r_2^{l2}}{r_1^{l1}} \right).$$

The remaining are calculated in the same manner:

$$P_2 = e^{-i\cos\theta H} \sum_{i=1}^{2} \int_0^d \sigma_{i2}(x_1, 0) u_k^{l'}(x_1, H) \, dx_1 =$$

$$e^{-i\cos\theta H} \sum_{k=1}^{2} \sum_{l} u_k^l \delta_{i1l} \delta_{i1l} \delta_{i1l} \omega_i \sum_{i=1}^{2} \delta_{k'l} \left( C_{i2k1} \delta_{i21} D_i^{1l} + C_{i2k2} D_i^{1l} \delta_{i11} \frac{r_2^{l2}}{r_1^{l1}} \right).$$

$$P_3 = - \sum_{i=0}^{2} \int_0^H \sigma_{i1}(0, x_2) u_k^{l'}(0, x_2) \, dx_2 =$$

$$- \sum_{k=1}^{2} \sum_{l} u_k^l \delta_{i1l} \delta_{i1l} \delta_{i1l} \omega_i \sum_{i=1}^{2} \delta_{k'l} \left( C_{i1k1} D_i^{1l} \delta_{i22} \frac{r_1^{l1}}{r_2^{l2}} + C_{i1k2} \delta_{i11} D_i^{2l2} \right).$$

$$P_4 = \sum_{i=1}^{2} \int_0^H \sigma_{i1}(0, x_2) e^{-i\sin\theta} u_k^{l'}(d, x_2) \, dx_2 =$$

$$e^{-i\sin\theta d} \sum_{k=1}^{2} \sum_{l} u_k^l \delta_{i1l} \delta_{i1l} \delta_{i1l} \omega_i \sum_{i=1}^{2} \delta_{k'l} \left( C_{i1k1} D_i^{1l} \delta_{i22} \frac{r_1^{l1}}{r_2^{l2}} + C_{i1k2} \delta_{i11} D_i^{2l2} \right).$$

To satisfy periodic Floquet boundary conditions (4) and (5), the following equalities must be satisfied:

$$u_k^{l1}_M = u_k^{l1}_1 e^{-i\cos\theta H},$$

(13)
\[ u_k^{M_1 l_2 N_2} = u_k^{l_2 l_2} e^{-i\zeta \sin \theta d}. \] (14)

Therefore, the following system can be composed if a given surface load is applied (the components \( f' \) in the right-hand side describe this load):

\[
\sum_{l} \sum_{k=1}^{2} \left( \tilde{A}_{l,l}^{l'k}(k) + \omega^2 \tilde{b}_{l,l}^{l'k}(k) \right) u_{l}^{l' l_2} = f',
\]

Three summands in the variational form (8) are evaluated separately. Thus, the contour integral gives

\[
\tilde{A}_{l,l}^{l'k} = \sum_{i=1}^{2} \delta_{l,1} \delta_{l_2,i} w_{l_1} \sum_{i} \delta_{k,i} \left( C_{i2k1} \delta_{i1} D_{i1}^{l_1} + C_{i2k2} D_{i2}^{l_2} \delta_{i1} \frac{r_{l_1}}{r_1} \right) \left( \delta_{l_2,2} \delta_{i2} e^{-i\zeta \cos \theta H} - \delta_{l_2,1} \delta_{i2} \right) + \delta_{l_2,1} \delta_{l_1} w_{l_2} \sum_{i} \delta_{k,i} \left( C_{i1k1} \delta_{i1} D_{i1}^{l_1} + C_{i1k2} \delta_{i1} \frac{r_{l_2}}{r_2} \right) \left( \delta_{l_1,1} \delta_{i2} e^{-i\zeta \sin \theta d} - \delta_{l_1,1} \delta_{i2} \right).
\]

The volume integrals can be separated into diagonal matrix

\[
\tilde{A}_{l,l}^{l'k} = -\frac{\rho}{r_1 r_2} \delta_{l_1,1} \delta_{l_2,2} \delta_{kk'} \delta_{i_1,1} w_{i_1} \delta_{i_2,2} w_{i_2} \delta_{j_2,1},
\]

which is frequency dependent, and matrix

\[
A_{l,l}^{l'k} = \delta_{l_1,1} \delta_{l_2,2} \left( -\frac{r_{l_1}}{r_2} \delta_{i_1,1} w_{i_1} \right) C_{k'1k1} \sum_{n=1}^{N} w_n D_{n}^{l_1} D_{n}^{l_1} - \frac{r_{l_2}}{r_2} \delta_{i_2,2} w_{i_2} C_{k'2k2} \sum_{n=1}^{N} w_n D_{n}^{l_2} D_{n}^{l_2} - C_{k'1k2} D_{n}^{l_1} D_{n}^{l_2} w_{i_1} w_{i_2} - C_{k'2k1} D_{i_1}^{l_1} D_{i_2}^{l_2} w_{i_1} w_{i_2}.
\]

Boundary conditions (13) and (14) are also taken into account so that

\[
\tilde{A}_{l,l}^{l'k}(l_1', M_2, i_1', N) I(l_1', M_2, i_1', N) I = A_{l,l}^{l'k}(l_1', M_2, i_1', N) I = 0,
\]

\[
\tilde{A}_{l,l}^{l'k}(l_1, M_1, i_2', N, i_2') I = A_{l,l}^{l'k}(l_1, M_1, i_2', N, i_2') I = 0,
\]

except for

\[
\tilde{A}_{l,l}^{l'k}(l_1, M_1, i_1', N) I(l_1', M_2, i_1', N) I(l_1', M_2, i_1', 1) = -e^{-i\zeta \cos \theta H},
\]

\[
\tilde{A}_{l,l}^{l'k}(1, M_1, i_1', 1) I(1, M_2, i_1', 1) = -e^{-i\zeta \sin \theta d},
\]

where \( I = \Gamma G, l_1 = \Gamma M_1, l_2 = \Gamma M_2, i_1 = \Gamma N, i_2 = \Gamma N. \)

Finally, the dispersion equation

\[
\det \left( A_{l,l}^{l'k} + \tilde{A}_{l,l}^{l'k}(k) - \omega^2 \tilde{A}_{l,l}^{l'k}(k) \right) = 0
\]

can be written with respect to the wavenumber \( \zeta \). The latter can be reduced to the eigenvalue problem:

\[
\mathbf{B} - \lambda_n \mathbf{E} = 0, \quad \mathbf{B} = \mathbf{A}^{-1} \left( \mathbf{A} + \tilde{A}(\zeta) \right), \quad \omega_n = \sqrt{\lambda_n},
\]

here \( \mathbf{E} \) is the identity matrix.
4. Conclusions
In this study, the spectral element method [8] was extended to the simulation of periodic elastomer composite. The obtained mathematical model is convenient for the investigation of the properties of the waves propagating in phononic crystals and acoustic metamaterials including elastomers. The method can be naturally extended to dielectric elastomers and to the three-dimensional case.

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