On rational solutions of multicomponent and matrix KP hierarchies

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Abstract

We derive some rational solutions for the multicomponent and matrix KP hierarchies generalising an approach by Wilson. Connections with the multicomponent version of the KP/CM correspondence are discussed.

1 Introduction

The KP hierarchy is an integrable hierarchy of partial differential equations generated by a pseudo-differential operator of the form

\[ L = D + u_1 D^{-1} + u_2 D^{-2} + \ldots \] (1)

Here the \((u_i)_{i \geq 1}\) are elements of a differential algebra \(A\) of smooth functions of a variable \(x\) (with \(D = \partial/\partial x\)) and a further infinite family of variables \(t = (t_i)_{i \geq 1}\). The evolution of \(L\) is determined by the following system of Lax-type equations:

\[ \partial_k L = [B_k, L] \quad (k \geq 1) \] (2)

where \(\partial_k = \partial/\partial t_k\) and \(B_k = (L^k)_+\). All these equations commute, and the variable \(x\) may be identified with \(t_1\). This hierarchy of equations is naturally viewed as a dynamical system defined on an infinite-dimensional Grassmann manifold, as discovered by Sato [9]. In [10] Segal and Wilson developed a very general framework for building solutions to the KP hierarchy out of the points of a certain Grassmannian \(Gr(H)\) of closed subspaces in a Hilbert space \(H\).

The multicomponent KP hierarchy (mcKP) is a generalisation of the KP hierarchy obtained by replacing \(A\) with the differential algebra of \(r \times r\) matrices whose entries belong to an algebra of smooth functions of a variable \(x\) and \(r\) further families of variables

\[ t^{(1)} = (t^{(1)}_i)_{i \geq 1} \ldots \ t^{(r)} = (t^{(r)}_i)_{i \geq 1} \]

which we will collectively denote by \(\tilde{t}\). The operator \(L\) now becomes

\[ L = D + U_1 D^{-1} + U_2 D^{-2} + \ldots \] (3)
where \((U_i)_{i \geq 1}\) are \(r \times r\) matrices. To define the evolution of \(L\) we introduce a set of \(r\) matrix pseudo-differential operators \(R_\alpha\) of the form\(^3\)

\[
R_\alpha = E_\alpha + R_{1\alpha} D^{-1} + R_{2\alpha} D^{-2} + \ldots
\]  

where \(E_\alpha\) is the matrix with 1 in the entry \((\alpha, \alpha)\) and zero elsewhere. These operators are required to satisfy the equations \([L, R_\alpha] = 0, [R_\alpha, R_\beta] = 0\) and \(\sum_\alpha R_\alpha = I_r\) (it can be shown that such operators do exist). The evolution equations for the mcKP hierarchy are then

\[
\partial_{k\alpha} L = [B_{k\alpha}, L] \quad (k \geq 1, \alpha = 1 \ldots r)
\]

where \(\partial_{k\alpha} = \partial / \partial t_1^{(\alpha)}\) and \(B_{k\alpha} = (L^k R_\alpha)_+\). The variable \(x\) may be identified with \(\sum_\gamma t_1^{(\gamma)}\); if we define also for each \(k \geq 2\) the new variables \(t_k = \sum_\alpha t_k^{(\gamma)}\) then the corresponding flows determine a sub-hierarchy of equations that we call the matrix KP hierarchy.

In [12] George Wilson obtained a complete classification of the rational solutions to the KP hierarchy. More precisely, he proved that the coefficients of an operator \(L\) satisfying (1) are proper (i.e., vanishing at infinity) rational functions of \(x\) exactly when \(L\) comes from a point of a certain sub-Grassmannian of \(Gr(H)\), which he called the adelic Grassmannian. This space turns out to be in one-to-one correspondence with the phase space of the rational Calogero-Moser system [13]; this provides a geometric explanation for the phenomenon, first noticed by Krichever in [6], that the poles of a rational solution to the KP equation evolve as a system of point particles described by the Calogero-Moser Hamiltonian. On the other hand, the adelic Grassmannian may also be seen as the moduli space of isomorphism classes of right ideals in the Weyl algebra [2], thereby linking the subject to the emerging field of noncommutative algebraic geometry. This web of connections is sometimes referred to as the “KP/CM correspondence” (see also [1] for a wider perspective on that matter).

The purpose of this paper is to establish some rationality results, obtained in the author’s PhD Thesis [11], for the solutions of multicomponent and matrix KP hierarchies. Our main motivation is to understand how the above-mentioned results generalise to the multicomponent setting. The paper is organised as follows. In section 2 we briefly recall the mapping between points in the \(r\)-component Segal-Wilson Grassmannian \(Gr(r)\) and solutions of multicomponent KP. In section 3 we analyse the multicomponent rational Grassmannian \(Gr^{rat}(r)\) and display an explicit formula for the Baker and tau functions associated to these points. In section 4 we prove two rationality results: in Theorem 2 we consider mcKP solutions coming from certain (very special) points in \(Gr^{rat}(r)\), whereas in Theorem 3 we restrict to the matrix KP hierarchy but consider a much larger subset of \(Gr^{rat}(r)\). Finally in section 5 we briefly comment on the relevance of these results for the multicomponent version of the KP/CM correspondence.

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2 Preliminaries

We start by briefly recalling the definition of the \(r\)-component Segal-Wilson Grassmannian [10, 8]. Consider the Hilbert space \(H^{(r)} = L^2(S^1, \mathbb{C}^r)\); its elements can be thought of as functions \(\mathbb{C} \rightarrow \mathbb{C}^r\)

\(^3\)Greek indices will henceforth run from 1 to \(r\).
by embedding $S^1$ in the complex plane as the circle $\gamma_R$ with centre 0 and radius $R \in \mathbb{R}^+$. We have the splitting $H^{(r)} = H^{(r)}_+ \oplus H^{(r)}_-$ in the two subspaces consisting of functions with only positive (resp. negative) Fourier coefficients, with associated orthogonal projections $\pi_{\pm}$. The Segal-Wilson Grassmannian of $H^{(r)}$, denoted by $\text{Gr}(r)$, is the set of all closed linear subspaces $W \subseteq H^{(r)}$ such that $\pi_+|_W$ is a Fredholm operator of index zero and $\pi_-|_W$ is a compact operator.

If we take the elements of $H^{(r)}$ to be row vectors for definiteness, the loop group $\text{LGL}(r, \mathbb{C})$ naturally acts on $H^{(r)}$ (hence on $\text{Gr}(r)$) by matrix multiplication from the right. We define $\Gamma_+(r)$ as the subgroup consisting of diagonal matrices of the form

$$\text{diag}(g_1, \ldots, g_r) \quad \text{with } g_\alpha \in \Gamma_+ \quad (6)$$

where $\Gamma_+$ is the group of analytic functions $g \colon S^1 \to \mathbb{C}^*$ that extend to holomorphic functions on the disc $\{ z \in \mathbb{C}P^1 \mid |z| \leq R \}$ and such that $g(0) = 1$ (cfr. [10]). It follows that for each $g_\alpha$ there exists a holomorphic function $f_\alpha$ such that $g_\alpha = e^{f_\alpha}$ (with $f_\alpha(0) = 0$), and by letting $f_\alpha = \sum_{i \geq 1} t_i^{(\alpha)} z^i$ we have

$$g_\alpha(z) = \exp \sum_{i \geq 1} t_i^{(\alpha)} z^i \quad (7)$$

Thus a generic matrix $g \in \Gamma_+(r)$ may be written in the form

$$g = \exp \text{diag} \left( \sum_{i \geq 1} t_i^{(1)} z^i, \ldots, \sum_{i \geq 1} t_i^{(r)} z^i \right) \quad (8)$$

and is totally described by the family of coefficients $\bar{t} = (t^{(1)}, \ldots, t^{(r)})$.

We now recall the mapping between points of $\text{Gr}(r)$ and solutions to the multicomponent KP hierarchy. In what follows we will say that a matrix-valued function $\psi(z)$ belongs to a subspace $W \in \text{Gr}(r)$ if and only if each row of $\psi$, seen as an element of $H^{(r)}$, belongs to $W$.

For every $W \in \text{Gr}(r)$ we define

$$\Gamma_+(r)^W := \{ g \in \Gamma_+(r) \mid Wg^{-1} \text{ is transverse} \} \quad (9)$$

where “transverse” means that the orthogonal projection $Wg^{-1} \to H_+^{(r)}$ is an isomorphism. For any $g \in \Gamma_+(r)^W$ the reduced Baker function associated to $W$ and $g$ is the matrix-valued function $\tilde{\psi}$ whose row $\tilde{\psi}_\alpha$ is the inverse image of $e_\alpha \in H_+^{(r)}$ (the $\alpha$-th element of the canonical basis of $\mathbb{C}^r$) by $\pi_+|_{Wg^{-1}}$. It follows straightforwardly that

$$\tilde{\psi}_W(g, z) = I_r + \sum_{i \geq 1} W_i(g) z^{-i} \quad (10)$$

for some matrices $(W_i)_{i \geq 1}$. Now, since each row of the matrix $\tilde{\psi}_W$ belongs to the subspace $Wg^{-1}$, each row of the product matrix $\tilde{\psi}_Wg$ will belong to $W$; the Baker function associated to $W$ is the map $\psi_W$ which sends $g \in \Gamma_+(r)^W$ to this matrix:

$$\psi_W(g, z) = \left( I_r + \sum_{i \geq 1} W_i(g) z^{-i} \right) g(z) \quad (11)$$

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2 More generally, we could consider the subgroup determined by a maximal torus of type $\mathfrak{t}$ (where \( \mathfrak{t} \) is any partition of $r$) in $\text{GL}(r, \mathbb{C})$; this gives rise to the so-called Heisenberg flow of type $\mathfrak{t}$. However, as shown e.g. in [11], these flows are simply the pullback on $\text{Gr}(r)$ of multiple lower-dimensional mKP flows.
Notice that for every $\eta \in \text{LGL}(r, \mathbb{C})$ one has $\tilde{\psi}_W(g, z) = \tilde{\psi}_W(g \eta^{-1}, z)$, so that

$$\psi_W(g, z) = \psi_W(g \eta^{-1}, z) \cdot \eta$$

(12)

We now recall that expressions such as (11) are in one-to-one correspondence with zeroth-order pseudo-differential operators of the form

$$K_W = I_r + \sum_{i \geq 1} W_i(t) D^{-i}$$

(13)

where we consider $g$ as a function of the coefficients $t$ defined by (8). From $K_W$ we can define a first order pseudo-differential operator $L_W$ via the following prescription (“dressing”):

$$L_W = K_W D K_W^{-1}$$

(14)

The following result was proved in [3]:

**Theorem 1.** For any point $W \in \text{Gr}(r)$ the operator $L_W$ defined by (14) satisfies the multicomponent KP equation.

We remark that the correspondence (14) is not one-to-one: if $K' = KC$ where $C = I_r + \sum_{j \geq 1} C_j D^{-j}$ for some family $(C_j)_{j \geq 1}$ of constant diagonal matrices then $L = L'$. At the level of Gr(r) this “gauge freedom” is expressed by the action of the group $\Gamma_-(r)$ consisting of diagonal $r \times r$ matrices of the form $\text{diag}(h_1, \ldots, h_r)$, where each $h_\alpha$ belongs to the group (denoted $\Gamma_-$ in [10]) of analytic functions $h: S^1 \to \mathbb{C}^*$ that extend to holomorphic functions on the disc \{ $z \in \mathbb{C}P^1 \mid |z| \geq R$ \} and such that $h(\infty) = 1$.

Another way to describe the Baker function associated to a point of Gr(r) relies on the so-called tau function. Here we do not need to enter into the details of its definition (see again [3]); the key fact is that to each subspace $W \in \text{Gr}(r)$ we can associate certain holomorphic functions on $\Gamma_+(r)$, denoted $\tau_W$ and $\tau_{W, \alpha \beta}$ for each pair of indices $\alpha \neq \beta$, determined up to constant factors, such that the following equality (“Sato’s formula”) holds:

$$\tilde{\psi}_W(g, z)_{\alpha \beta} = \begin{cases} 
\tau_W(g q_{z, \alpha}) & \text{if } \alpha = \beta \\
\tau_W(g) z^{-1} \frac{\tau_{W, \alpha \beta}(g q_{z, \beta})}{\tau_W(g)} & \text{if } \alpha \neq \beta
\end{cases}$$

(15)

where for each $z \in \mathbb{C}$ and $\alpha = 1, \ldots, r$ we define $q_{\alpha}$ to be the element of $\Gamma_+(r)$ that has $q_{\alpha}(\zeta) := 1 - \frac{\zeta}{z}$ at the $(\alpha, \alpha)$ entry and 1 elsewhere on the diagonal.

### 3 The multicomponent rational Grassmannian

#### 3.1 Definition

For the sake of brevity we will denote by $\mathcal{R}$ the space of rational functions on the complex projective line $\mathbb{C}P^1$, by $\mathcal{P}$ the subspace of polynomials and by $\mathcal{R}_-$ the subspace of proper (i.e., vanishing at infinity) rational functions. We consider the space $\mathcal{R}'$ of $r$-tuples of rational functions with the direct sum decomposition

$$\mathcal{R}' = \mathcal{P}^r \oplus \mathcal{R}_-^r$$

(16)
and associated canonical projection maps $\pi_+: \mathcal{R}^r \to \mathcal{P}^r$ and $\pi_-: \mathcal{R}^r \to \mathcal{R}_r^c$.

We define the Grassmannian $\mathbb{G}_{\text{rat}}(r)$ as the set of closed linear subspaces $W \subseteq \mathcal{R}^r$ for which there exist polynomials $p, q \in \mathcal{P}$ such that

$$p\mathcal{P}^r \subseteq W \subseteq q^{-1}\mathcal{P}^r$$

The virtual dimension of $W$, denoted $\text{vdim } W$, is the index of the (Fredholm) operator $p_+ := \pi_+|_W$; one has

$$\text{vdim } W = \dim W' - r \deg p$$

where $W' := W/p\mathcal{P}^r$ is finite-dimensional. We denote by $\mathbb{G}_{\text{rat}}(r)$ the subset of $\mathbb{G}_{\text{rat}}(r)$ consisting of subspaces of virtual dimension zero.

This space can be embedded in the $r$-component Segal-Wilson Grassmannian $\text{Gr}(r)$ by the following procedure: given $W \in \mathbb{G}_{\text{rat}}(r)$, we choose the radius $R \in \mathbb{R}^+$ involved in the definition of $\text{Gr}(r)$ such that every root of the polynomial $q$ appearing in (17) is contained in the open disc $|z| < R$; then the restrictions $f|_{|z|R}$ for all $f \in W$ determine a linear subspace whose $L^2$-closure belongs to $\text{Gr}(r)$. This embedding automatically defines a topology on $\mathbb{G}_{\text{rat}}(r)$ and its subspaces by restriction.

**Lemma 1.** A subspace $W \in \mathbb{G}_{\text{rat}}(r)$ has virtual dimension zero if and only if the codimension of the inclusion $W \subseteq q^{-1}\mathcal{P}^r$ coincides with $r \deg q$.

**Proof.** Condition (17) may be rewritten as $q_p\mathcal{P}^r \subseteq qW \subseteq \mathcal{P}^r$, so that the codimension of $W$ in $q^{-1}\mathcal{P}^r$ is the same as the codimension of $qW$ in $\mathcal{P}^r$, namely $\dim \mathcal{P}^r/qW$. Taking the quotient of both those spaces by the common subspace $qp\mathcal{P}^r$ we get an isomorphic linear space which is the quotient of two finite-dimensional spaces:

$$\frac{\mathcal{P}^r}{qW} \cong \frac{\mathcal{P}^r/qp\mathcal{P}^r}{qW/qp\mathcal{P}^r}$$

Moreover, $qW/qp\mathcal{P}^r \cong W/p\mathcal{P}^r = W'$, so that

$$\text{codim}_{q^{-1}\mathcal{P}^r} W = r(\deg q + \deg p) - \dim W'$$

By using (18) we finally get

$$\text{codim}_{q^{-1}\mathcal{P}^r} W = r \deg q - \text{vdim } W$$

from which the lemma follows. \qed

Let’s introduce a more algebraic description for $\mathbb{G}_{\text{rat}}(r)$ in the same vein as [12]. For every $k = 1, \ldots, r$, $s \in \mathbb{N}$ and $\lambda \in \mathbb{C}$ we define the linear functional $\text{ev}_{k,s,\lambda}$ on $\mathcal{P}^r$ by

$$\langle \text{ev}_{k,s,\lambda}, (p_1, \ldots, p_r) \rangle = p^{(s)}_k(\lambda)$$

These functionals are easily seen to be linearly independent; we denote by $\mathcal{C}^{(r)}$ the linear space they generate, and think of it as a space of “differential conditions” we can impose on $r$-tuples of polynomials. We also set

$$\mathcal{C}^{(r)}_\lambda := \text{span} \{ \text{ev}_{k,s,\lambda} \}_{1 \leq k \leq r, s \in \mathbb{N}}$$
and
\[ \mathcal{C}_{r,\lambda}^{(r)} := \text{span}\{\text{ev}_{k,s,\lambda}\}_{1 \leq k \leq r, 0 \leq s < t} \]
with the convention that \( \mathcal{C}_{0,\lambda}^{(r)} = \{0\} \).

Given \( c \in \mathcal{C}^{(r)} \), the finite set of points \( \lambda \in \mathbb{C} \) such that the projection of \( c \) on \( \mathcal{C}_\lambda^{(r)} \) is nonzero will be called the support of \( c \). For every linear subspace \( C \subseteq \mathcal{C}^{(r)} \), its annihilator

\[ V_C := \{ (p_1, \ldots, p_r) \in \mathcal{P}^r \mid \langle c, (p_1, \ldots, p_r) \rangle = 0 \text{ for all } c \in C \} \]
is a linear subspace in \( \mathcal{P}^r \).

**Lemma 2.** A subspace \( W \subseteq \mathcal{R}^r \) belongs to \( \text{Gr}^{\text{rat}}(r) \) if and only if there exist a finite-dimensional subspace \( C \subseteq \mathcal{C}^{(r)} \) and a polynomial \( q \) such that \( W = q^{-1}V_C \); moreover \( W \in \text{Gr}^{\text{rat}}(r) \) if and only if \( r \deg q = \dim C \).

**Proof.** Let’s suppose that \( W = q^{-1}V_C \); for some finite-dimensional subspace \( C \) in \( \mathcal{C}^{(r)} \) and let \( \{\lambda_1, \ldots, \lambda_m\} \) be the support of \( C \). For every \( i = 1, \ldots, m \) let \( t_i \) be the maximum value of \( s \) for the functionals \( \text{ev}_{k,s,\lambda_i} \) involved in the elements of \( C \). By letting \( p := \prod_{i=1}^m (z - \lambda_i)^{t_i+1} \) it follows that \( p \mathcal{P}^r \subseteq V_C \), hence a fortiori \( q(p)\mathcal{P}^r \subseteq V_C \subseteq \mathcal{P}^r \) and dividing by \( q \) we see that \( W \in \text{Gr}^{\text{rat}}(r) \). Now, if \( r \deg q = \dim C \) then \( \text{codim}_{\mathcal{P}^r} V_C = \dim C = r \deg q \) and lemma \ref{lemma:codimension} implies that \( W \in \text{Gr}^{\text{rat}}(r) \).

For the converse, let \( W \in \text{Gr}^{\text{rat}}(r) \). Then there exist \( p, q \in \mathcal{P} \) such that \( q(p)\mathcal{P}^r \subseteq qW \subseteq \mathcal{P}^r \); this means in particular that the linear space \( qW \) is obtained by imposing a certain (finite) number of linearly independent conditions in the dual space of \( \mathcal{P}^r / q(p)\mathcal{P}^r \). The latter can be identified with \( \mathcal{C}^r \otimes U \), where \( U \) is the linear space of polynomials with degree less than \( \deg p + \deg q \), so that \( qW \) is determined by a finite-dimensional subspace in \( (\mathcal{C}^r \otimes U)^* \). On the other hand, this space is generated by the elements of \( \mathcal{C}^{(r)} \) (e.g. using the functionals \( \frac{1}{q} \text{ev}_{k,s,0} \) that extract the \( s \)-th coefficient of the \( k \)-th polynomial). Thus, there exists a linear subspace \( C \subseteq \mathcal{C}^{(r)} \) of finite dimension such that \( V_C = qW \). If moreover \( \text{vdim} W = 0 \), then by \( \text{(13)} \) the linear space \( W' \cong qW / q(p)\mathcal{P} \) has dimension \( r \deg p \), so that it must be defined by \( r \deg p \) linearly independent conditions. It follows that the subspace \( C \) has dimension \( r \deg q \).

In the sequel, the subspace \( q^{-1}V_C \) singled out by this lemma will be denoted simply by \( (C, q)^* \).

**Lemma 3.** Two subspaces \( W_1 = (C, q_1)^* \) and \( W_2 = (C, q_2)^* \) determined by the same conditions space \( C \) in \( \text{Gr}^{\text{rat}}(r) \) lie in the same \( \Gamma_-(r) \)-orbit.

Actually, the matrix \( \eta := \frac{q_2}{q_1} I_r \) belongs to \( \Gamma_-(r) \) (notice that \( q_1 \) and \( q_2 \) are both of degree \( r \dim C \)) and is such that \( W_1 \eta = W_2 \).

We say that a finite-dimensional subspace \( C \subseteq \mathcal{C}^{(r)} \) is homogeneous if it admits a basis consisting of “1-point conditions”, i.e. differential conditions each one involving a single point:

\[ C = \bigoplus_{\lambda \in \mathbb{C}} C_\lambda \quad \text{where} \quad C_\lambda := C \cap \mathcal{C}_\lambda^{(r)} \quad \text{(20)} \]

We denote by \( \text{Gr}^{\text{hom}}(r) \) the set of subspaces \( (C, q)^* \in \text{Gr}^{\text{rat}}(r) \) such that \( C \) is homogeneous, \( \dim C_\lambda = r n_\lambda \) for some natural numbers \( n_\lambda \in \mathbb{N} \) and \( q = q_C \), where

\[ q_C := \prod_{\lambda \in \mathbb{C}} (z - \lambda)^{n_\lambda} \quad \text{(21)} \]
3.2 The Baker and tau functions

We would now like to determine the Baker function of a point \((C, q)^* \in \text{Gr}^{\text{rat}}(r)\). By Lemma 8 it is enough to consider the case \(q = z^d\). Thus, we suppose that \(\dim C = rd\) and take the subspace \(W = (C, z^d)^*\); we claim that \(\psi_W\) must have the form

\[
\psi_W(g, z) = \left( I_r + \sum_{j=1}^{d} W_j(g) z^{-j} \right) g(z)
\]

Indeed, by standard arguments (cfr. [12, Sect. 4]), on the one hand we have that each row of the matrix-valued function \(z^d \psi_W(g, z)\) (for every fixed \(g\)) belongs to the \(L^2\)-closure of \(V_C\) in \(H^r_+(r)\) (for some value of the radius \(R\)), and on the other hand that each functional \(\text{ev}_{k,s,\lambda}\) extends uniquely to a continuous functional on \(H^r_+(r)\); hence \((z^d \psi_W)_\alpha \in V_C\) for every \(\alpha\). Now,

\[
(z^d \psi_W)_{\alpha \beta} = \left( z^d \delta_{\alpha \beta} + \sum_{j=1}^{d} W_{j\alpha \beta}(g) z^{-d-j} \right) g_{\beta}(z)
\]

and if we require every matrix element to be a polynomial, we see that every \(W_j\) with \(j > d\) must be the zero matrix.

To determine the matrices \(W_1, \ldots, W_d\), let \(\{c_1, \ldots, c_{rd}\}\) be a basis for \(C\); for each \(\alpha\) we take the \(\alpha\)-th row of \(z^d \psi_W\) and impose the equalities \(\langle c_i, (z^d \psi_W)_\alpha \rangle = 0\) (with \(i = 1, \ldots, rd\)). This yields the following linear system of equations:

\[
\langle c_1, \left( z^d \delta_{\alpha \beta} + \sum_{j=1}^{d} W_{j\alpha \beta}(g) z^{-d-j} \right) g_{\beta}(z) \rangle = 0
\]

In other words, we have a family of \(r\) linear systems, each of which involves \(rd\) equations, for a total of \(r^2d\) scalar equations. The unknowns are of course the \(ms\) entries of the \(d\) matrices \(\{W_1, \ldots, W_d\}\); the coefficients of these unknowns involve, as in the scalar case, the \(g_{\beta}\)'s and their derivatives evaluated at the points in the support of \(C\).

The tau functions associated to \(W = (C, z^d)^* \in \text{Gr}^{\text{rat}}(r)\) are readily obtained by imitating the calculations in [3]; they are built out of the following family of \((rd + 1) \times (rd + 1)\) matrices (indexed by \(\alpha, \beta = 1, \ldots, r\)):

\[
M_{\alpha \beta} :=
\begin{pmatrix}
\langle c_1, g_1 \rangle & \cdots & \langle c_{rd}, g_1 \rangle & 0 \\
\vdots & \vdots & \vdots & \vdots \\
\langle c_1, g_r \rangle & \cdots & \langle c_{rd}, g_r \rangle & 0 \\
\langle c_1, z g_1 \rangle & \cdots & \langle c_{rd}, z g_1 \rangle & 0 \\
\vdots & \vdots & \vdots & \vdots \\
\langle c_1, z g_r \rangle & \cdots & \langle c_{rd}, z g_r \rangle & 0 \\
\vdots & \vdots & \vdots & \vdots \\
\langle c_1, z^{d-1} g_1 \rangle & \cdots & \langle c_{rd}, z^{d-1} g_1 \rangle & 0 \\
\vdots & \vdots & \vdots & \vdots \\
\langle c_1, z^{d-1} g_r \rangle & \cdots & \langle c_{rd}, z^{d-1} g_r \rangle & 0 \\
\langle c_1, z^{d} g_1 \rangle & \cdots & \langle c_{rd}, z^{d} g_1 \rangle & \delta_{\alpha \beta}
\end{pmatrix}
\]
where in the last column the only nonzero element is on the $\beta$-th row of each block. Notice that in an expression like $(c_i, g_\gamma)$, $g_\gamma$ must be interpreted as the row vector having $g_\gamma$ in its $\gamma$-th entry and zero elsewhere.

In terms of these matrices, the “diagonal” tau function $\tau_W$ is simply the cofactor of the element $\delta_{\alpha\beta} = 1$ in the lower right corner, or equivalently the determinant of the $rd \times rd$ minor obtained by deleting the last column and the last row:

$$\tau_W(g) = \det((c_i, z^{j-1}g_\gamma))_{i=1,\ldots,rd \atop j=1,\ldots,\gamma=1,\ldots,r}$$  \hspace{1cm} (25)

This is just the matrix of coefficients of the linear system (23), so that the system has a solution exactly when $g \in \Gamma_+(r)^W$, as expected. The “off-diagonal” tau function $\tau_{W\alpha\beta}$ (with $\alpha \neq \beta$) is the cofactor of the element $z^{-1}$ in the last column:

$$\tau_{W\alpha\beta}(g) = (-1)^{r-\beta} \det((c_i, z^{j-1}(g_\gamma + \delta_{j\beta} \delta_{\gamma\beta}(zg_\alpha - g_\gamma))))_{i=1,\ldots,rd \atop j=1,\ldots,\gamma=1,\ldots,r}$$  \hspace{1cm} (26)

Observe that an expression such as $z^{j-1}g_\gamma$ may equivalently be read as $\partial^{j-1}_{\gamma\gamma} g_\gamma$; this will be useful in what follows.

4 Rational solutions

We can now prove a rationality result for the solutions of the mcKP hierarchy coming from subspaces in $\text{Gr}^{\text{hom}}(r)$ defined by a set of conditions whose support is a single point.

Theorem 2. Let $W \in \text{Gr}^{\text{hom}}(r)$ and suppose that $W = (C, q_C)^*$ with $C$ supported on a single point $\lambda \in \mathbb{C}$. Then:

1. Each diagonal entry of the reduced Baker function $\tilde{\psi}_W$ is a rational function of the times $t_1^{(1)}, \ldots, t_1^{(r)}$ that tends to 1 as $t_1^{(\alpha)} \to \infty$ for any $\alpha$;

2. The tau function $\tau_W$ is a polynomial in $t_1^{(1)}, \ldots, t_1^{(r)}$ with constant leading coefficient.

Proof. The two statements are equivalent by virtue of the diagonal part of Sato’s formula (15), so it suffices to prove one of them; we choose the first. By hypothesis we have $W = (C, (z - \lambda)^d)^*$ with $C \subseteq \mathcal{C}_\lambda^{(r)}$ of dimension $rd$; let $(c_1, \ldots, c_{rd})$ be a basis for it. Consider the subspace $U := (C, z^d)^* \in \text{Gr}^{\text{rat}}(r)$; its tau function is given by (25). To compute the diagonal elements of the corresponding Baker function we use Sato's formula for $U$:

$$\tilde{\psi}_{U\alpha\alpha} = \frac{\tau_U(gq_{\zeta\alpha})}{\tau_U(g)}$$  \hspace{1cm} (27)

(here we consider $\zeta$ as a parameter and $z$ as a variable). Since we are only interested in the times with subscript 1 we will work in the stationary setting, i.e. we put $t_k^{(\alpha)} = 0$ for every $k \geq 2$, $\alpha = 1 \ldots m$. Each condition $c_i$ is supported at $\lambda$, hence we can define a family of polynomials $\{\phi_{i\gamma}\}_{\gamma=1,\ldots,r}$ (with $\phi_{i\gamma}$ only depending on $t_1^{(\gamma)}$) by the equation

$$\langle c_i, g_\gamma \rangle = g_\gamma(\lambda)\phi_{i\gamma}$$  \hspace{1cm} (28)
To apply (27) we need to know the determinant of the matrices $\partial_{t\gamma}^{j-1}(c_1,g_\gamma)$ and $\partial_{t\gamma}^{j-1}(c_1,g_\gamma(1-\delta_{\alpha\gamma}\zeta))$. As for the first, by using (28) its generic element can be written in the form

$$\partial_{t\gamma}^{j-1}(c_1,g_\gamma) = \partial_{t\gamma}^{j-1}(g_\gamma(\lambda)\phi_{i\gamma}) = g_\gamma(\lambda)(\partial_{t\gamma} + \lambda)^{j-1}\phi_{i\gamma} \quad (29)$$

For the second matrix we have

$$\partial_{t\gamma}^{j-1}(c_1,g_\gamma(1-\delta_{\alpha\gamma}\zeta)) = \partial_{t\gamma}^{j-1}(c_1,g_\gamma) - \partial_{t\gamma}^{j-1}(c_1,\delta_{\alpha\gamma}g_\gamma\zeta) \quad (30)$$

The first term is exactly (29), whereas the second is

$$\delta_{\alpha\gamma}\partial_{t\gamma}^j(c_1,g_\gamma)\zeta^{-1} = \delta_{\alpha\gamma}\partial_{t\gamma}^j(g_\gamma(\lambda)\phi_{i\gamma})\zeta^{-1} = \delta_{\alpha\gamma}g_\gamma(\lambda)(\partial_{t\gamma} + \lambda)^j\phi_{i\gamma}\zeta^{-1}$$

By putting all together, equation (30) becomes

$$g_\gamma(\lambda)(\partial_{t\gamma} + \lambda)^{j-1}(\phi_{i\gamma} - \delta_{\alpha\gamma}(\partial_{t\gamma}\phi_{i\gamma} + \lambda\phi_{i\gamma})\zeta^{-1})$$

that we can rewrite as

$$g_\gamma(\lambda)(1 - \delta_{\alpha\gamma}\frac{\lambda}{\zeta})(\partial_{t\gamma} + \lambda)^{j-1}(\phi_{i\gamma} - \delta_{\alpha\gamma}\frac{1}{\zeta - \lambda}\partial_{t\gamma}\phi_{i\gamma}) \quad (31)$$

But $(1 - \delta_{\alpha\gamma}\frac{1}{\zeta}) = q_{c\alpha}(\lambda)$, so plugging (29) and (31) into (27) we obtain

$$\tilde{\psi}_U^{\alpha\alpha}(g,\zeta) = (q_{c}(\lambda))^d \frac{\det((\partial_{t\gamma} + \lambda)^{j-1}(\phi_{i\gamma} - \delta_{\alpha\gamma}\frac{1}{\zeta - \lambda}\partial_{t\gamma}\phi_{i\gamma}))}{\det((\partial_{t\gamma} + \lambda)^{j-1}\phi_{i\gamma})} \quad (32)$$

The factor $(q_{c}(\lambda))^d$ disappears when we go back from $\psi_U$ to $\psi_W$; we are left with the ratio of two determinants of matrices with polynomial entries in the times $t_1^{(\gamma)}$, which is clearly a rational function. Moreover, if we expand the numerator of (32) by linearity over the sum, we see that the term obtained by always choosing $\phi_{i\gamma}$ exactly reproduces the polynomial at the denominator, and all the other terms involve a polynomial which has degree strictly lower than $\tau$ in some $t_1^{(\gamma)}$ (since we replace $\phi_{i\gamma}$ with one of its derivatives); this proves that $\tilde{\psi}_W^{\alpha\alpha} \rightarrow 1$ as all the $t_1^{(\alpha)}$ tend to infinity.

We now consider the evolution on $\text{Gr}(r)$ described by the flows of the matrix KP hierarchy. Recall from the Introduction that these are given by the vector fields $\partial/\partial t_k$, where $t_k = \sum_{\gamma}t_1^{(\gamma)}$. Equivalently, we can take the matrix $g \in \Gamma_+(r)$ to be of the form

$$g = \text{diag}(e^{\xi(r^{-1},t,z)}, \ldots, e^{\xi(r^{-1},t,z)}) \quad (33)$$

where $t = (t_k)_{k\geq 1}$ (and $t_1 = x$). Let’s define $\tilde{g} := e^{\xi(r^{-1},t,z)}$ and $h := e^{\xi(t,z)}$ (so that $g = \tilde{g}I_r$ and $\tilde{g}r = h$); then the Baker and tau functions for the matrix KP hierarchy are naturally expressed in terms of $h$ only, since that single function completely controls the flows of the hierarchy.

**Theorem 3.** Let $W \in \text{Gr}^{\text{hom}}(r)$, then:

1. $\tilde{\psi}_W(h,z)$ is a matrix-valued rational function of $x$ that tends to $I_r$ as $x \rightarrow \infty$;
2. $\tau_W(h)$ and $\tau_W^{\alpha \beta}(h)$ are polynomial functions of $x$ with constant leading coefficients.

Proof. Let $W = (C, q_C)^*$ with $C$ homogeneous and let $(\lambda_1, \ldots, \lambda_d)$ be its support with each point counted according to its multiplicity (so that the $\lambda_i$ are not necessarily distinct); finally for each $i = 1 \ldots d$ let $(c_{ij})_{j=1 \ldots r}$ be a set of $r$ linearly independent conditions at $\lambda_i$. In this way we get a basis $(c_{11}, \ldots, c_{dr})$ for $C$ made of 1-point conditions. Now consider the subspace $U := (C, z^d)^* \in \text{Gr}_{\text{rat}}(r)$; it is related to $W$ by the following element of $\Gamma_-(r)$:

$$\eta = \prod_{i=1}^d g_z(\lambda_i)^{-1} I_r = \exp \left( \sum_{k \geq 0} \sum_{i=1}^d \frac{\lambda_i^k}{k} z^{-k} \right) I_r \quad (34)$$

This corresponds to multiplying the tau function by

$$\hat{\eta} = \prod_{\alpha=1}^r \exp \left( - \sum_{k \geq 0} \sum_{i=1}^d \lambda_i^{k(\alpha)} \right) = \prod_{\alpha=1}^r \prod_{i=1}^d g_\alpha(\lambda_i)^{-1} = \prod_{i=1}^d h(\lambda_i)^{-1} \quad (35)$$

since $g_\alpha = \tilde{g}$ for every $\alpha$ and $\tilde{g}^r = h$.

Let’s define the family of polynomials $\{\phi_{ij\gamma}\}$ (for $i = 1 \ldots d$, $j = 1 \ldots r$, $\gamma = 1 \ldots r$) by the equation

$$\langle c_{ij}, g_\gamma \rangle = \tilde{g}(\lambda_i) \phi_{ij\gamma} \quad (36)$$

Again, this works precisely because each $c_{ij}$ is a 1-point condition; notice that, although $g_\gamma = \tilde{g}$ for every $\gamma$, the polynomials $\phi$ still depend on $\gamma$ since it is the index of the only nonzero entry of the row vector on which $c_{ij}$ acts.

We can now easily compute the tau functions associated to $U$ by retracing the same steps as in the proof of theorem 22 but now using the polynomials defined by (33). Then $\tau_U(g)$ is the determinant of the matrix whose generic element is

$$\tilde{g}(\lambda_i)(\partial_{1\gamma} + \lambda_i)^{k-1} \phi_{ij\gamma}$$

The term $\tilde{g}(\lambda_i)$ does not depend on the row indices $(k, \gamma)$ so that we can factor it out from the determinant and get

$$\tau_U = \prod_{i=1}^d (\tilde{g}(\lambda_i))^r \det \left((\partial_{1\gamma} + \lambda_i)^{k-1} \phi_{ij\gamma} \right) \quad (37)$$

But $\prod_i (\tilde{g}(\lambda_i))^r = \prod_i h(\lambda_i)$ is exactly the inverse of (35), so that

$$\tau_W = \det \left((\partial_{1\gamma} + \lambda_i)^{k-1} \phi_{ij\gamma} \right) \quad (38)$$

is the determinant of a matrix with polynomial entries in $s_i^{(\alpha)} = \tilde{x}$, hence a polynomial in $x$ and the coefficient of the top degree term involves only the constants $\lambda_i$. By Sato’s formula, this implies that $\tilde{\psi}_W^{\alpha\alpha} \to 1$ as $x \to \infty$.

Now take $\alpha, \beta \in \{1, \ldots, m\}$, $\alpha \neq \beta$ and consider the off-diagonal tau function $\tau_{U/\alpha \beta}(g)$; it is given by the determinant of a matrix $M_{\alpha \beta}(g)$ which coincides with the one involved in the definition of $\tau_U(g)$ except for the row corresponding to $k = d - 1$, $\gamma = \beta$ which is replaced by the row $\langle c_{ij}, \partial_{1\alpha}^d g_\alpha \rangle$. But since $g_\alpha = g_\beta = \tilde{g}$ we can again collect out of the determinant the same factor as before, so that $\tau_{W/\alpha \beta}(g) = \det \Phi_{ij,k\gamma}$ with

$$\Phi_{ij,k\gamma} := \begin{cases} (\partial_{1\gamma} + \lambda_i)^{k-1} \phi_{ij\gamma} & \text{if } k \neq d \text{ or } \gamma \neq \beta \\ (\partial_{1\alpha} + \lambda_i)^d \phi_{ij\alpha} & \text{if } k = d \text{ and } \gamma = \beta \end{cases} \quad (39)$$
This means that \( \tau \) and we can subtract this row multiplied by \( \lambda_1 \) (say) to the row \( [40] \) without altering the determinant, so that
\[
\Phi_{ij,k} := \begin{cases} 
(\partial_1 + \lambda_i)^{k-1} \phi_{ij} & \text{if } k \neq d \text{ or } \gamma \neq \beta \\
(\partial_1 + \lambda_i)^{d-1}((\lambda_i - \lambda_1)\phi_{ij} + \partial_1 \phi_{ij}) & \text{if } k = d \text{ and } \gamma = \beta
\end{cases}
\]
This means that \( \tau_{W_{\alpha \beta}} \) is the determinant of a matrix whose generic entry is equal or of degree strictly lower than the corresponding one on \( \tau_W \); it follows that the degree of \( \tau_{W_{\alpha \beta}} \) is strictly lower than \( \tau_W \), and this (again by Sato’s formula) implies that the off-diagonal components of \( \tilde{\psi}_W \) tend to zero as \( x \to \infty \).

To see how some of the new solutions look like, take for example the homogeneous subspace with 1-point support \( W = (C, \frac{1}{z-\lambda})^* \) with \( C \) generated by the \( r \) conditions
\[
c_k = ev_{k,1,\lambda} + a_{k1} ev_{0,0,\lambda} + \cdots + a_{kr} ev_{r,0,\lambda} \quad (k = 1 \ldots r)
\]
determined by the \( r \times r \) matrix \( A = (a_{ij}) \). These are, in a sense, the most general conditions involving only the functionals \( ev_{k,s,\lambda} \) with \( s \leq 1 \). For the sake of brevity, let’s set \( t_{\lambda} := r^{-1}(x + 2\lambda + 3\lambda^2 + \ldots) \); then
\[
\tau_W(t) = \det(t_{\lambda} I_r + A^t) \quad \tau_{W_{\alpha \beta}}(t) = -\text{cof}_{\beta,\alpha}(t_{\lambda} I_r + A^t)
\]
\[
\tilde{\psi}(t, z) = I_r - (t_{\lambda} I_r + A^t)^{-1} \frac{1}{z - \lambda}
\]
where \( \text{cof}_{\beta,\alpha} \) stands for the \((\beta, \alpha)\)-cofactor of a matrix and \( A^t \) for the transpose of \( A \).

More generally, we could take the element of \( \text{Gr}^{\text{hom}}(r) \) determined by the support \( \{\lambda_1, \ldots, \lambda_n\} \) and, for each \( i = 1 \ldots n \), a matrix \( A_i \) that specifies a set of conditions of the form \( [41] \) to be imposed at the point \( \lambda_i \). The tau function of such a subspace is the determinant of the following block matrix:
\[
\begin{pmatrix}
Y_1 & \ldots & Y_n \\
\lambda_1 Y_1 + I_r & \ldots & \lambda_n Y_n + I_r \\
\vdots & \ddots & \vdots \\
\lambda_1^{n-1} Y_1 + (n-1)\lambda_1^{n-2} I_r & \ldots & \lambda_n^{n-1} Y_n + (n-1)\lambda_n^{n-2} I_r
\end{pmatrix}
\]
where \( Y_i := t_{\lambda_i} I_r + A_i^t \). The off-diagonal tau functions \( \tau_{W_{\alpha \beta}} \) are obtained in the usual way (i.e., replacing the \( \beta \)-th line of the bottom blocks with the \( \alpha \)-th line of the blocks \( \lambda_i^n Y_i + n\lambda_i^{n-1} I_r \)), and the matrix Baker function is then given by Sato’s formula. When \( r = 1 \), these subspaces are exactly the ones described in \([13] \) Sect. 3).

5 Relationship with the multicomponent KP/CM correspondence

Recall from \([13] \) that a crucial ingredient of the scalar KP/CM correspondence is the bijective map \( \beta : C \to \text{Gr}^\text{ad} \) between the phase space of the Calogero-Moser system \( C \) and the adelic Grassmannian.
$Gr^{ad}$. In the multicomponent case, an analogous mapping should be defined between the phase space of the Gibbons-Hermsen system [5] (also known as “spin Calogero-Moser”) and some space of solutions for the matrix KP hierarchy, as suggested by a well-known calculation [7].

In Wilson’s unpublished notes [14], a map that fulfils this rôle is conjectured. In order to describe (part of) the definition of this map let’s recall (see e.g. [13, Sect. 8]) that the completed phase space of the $n$-particle, $r$-component Gibbons-Hermsen system is the (smooth, irreducible) affine algebraic variety defined by the following symplectic reduction:

$$C_{n,r} = \{ (X,Y,v,w) \mid [X,Y] + vw = -I \} / GL(n, \mathbb{C})$$

where $(X,Y,v,w) \in \text{End}(\mathbb{C}^n) \oplus \text{End}(\mathbb{C}^n) \oplus \text{Hom}(\mathbb{C}^n, \mathbb{C}^r) \oplus \text{Hom}(\mathbb{C}^r, \mathbb{C}^n)$ and $GL(n, \mathbb{C})$ acts as a change of basis in $\mathbb{C}^n$. Now denote by $C''_{n,r}$ the subspace of $C_{n,r}$ consisting of equivalence classes of quadruples for which $Y$ is diagonalisable with distinct eigenvalues; from each of these classes we can select a representative such that $Y = \text{diag}(\lambda_1, \ldots, \lambda_n)$. $X$ is a Moser matrix associated to $Y$ (i.e. $X_{ij} = 1/(\lambda_i - \lambda_j)$ and $X_{ii} = \alpha_i$ is any complex number) and each pair $(v_i, w_i)$ (where we denote by $v_i$ the $i$-th row of $v$ and by $w_i$ the $i$-th column of $w$) belongs to the algebraic variety $\{ (\xi, \eta) \in \mathbb{C}^r \times \mathbb{C}^r \mid \xi \cdot \eta = -1 \} / \mathbb{C}^*$, where $\lambda \in \mathbb{C}^*$ acts as

$$\lambda.(\xi, \eta) = (\lambda \xi, \lambda^{-1} \eta) \quad (42)$$

Notice in particular that none of the $v_i$’s and $w_i$’s can be the zero vector, by virtue of the normalisation condition $v_iw_i = -1$.

According to [14], a point $[X,Y,v,w] \in C''_{n,r}$ corresponds to the subspace $W \in Gr^{rat}(r)$ defined by the following prescriptions:

1. functions in $W$ are regular except for (at most) a simple pole in each $\lambda_i$ and a pole of any order at infinity;
2. if $f = \sum_{k \geq -1} f^{(i)}_k (z - \lambda_i)^k$ is the Laurent expansion of $f \in W$ in $\lambda_i$ then:
   (a) $f^{(i)}_{-1}$ is a scalar multiple of $v_i$, and
   (b) $(f^{(i)}_{0} + \alpha_i f^{(i)}_{-1}) \cdot w_i = 0$.

Our purpose is now to show how these prescriptions translate in the language of the previous sections.

So suppose that $W \in Gr^{rat}(r)$ satisfies conditions (1) (2); we must find a finite-dimensional, homogeneous space of conditions $C$ such that $W = (C, q_C)^*$ where $q_C$ is given by (21). Condition (1) clearly implies that $C$ has support $\{ \lambda_1, \ldots, \lambda_n \}$ and $q_C = \prod_{i=1}^n (z - \lambda_i)$. Thus, we only need to find, for each $i = 1 \ldots n$, a subspace $C'_i \subseteq \mathbb{C}_{\lambda_i}^r$ of dimension $r$ whose elements satisfy the conditions (2a) (2b). Let’s define the $n$ polynomials

$$q_i := \frac{q_C}{z - \lambda_i} = \prod_{j \neq i} (z - \lambda_j)$$

for $i = 1 \ldots n$; then by a direct computation we have that

$$ev_{k,0,\lambda_i}(q_C f) = q_i(\lambda_i) f^{(i)}_{-1,k}$$
and similarly
\[ \text{ev}_{k,1,\lambda_i}(q_{C}f) = \sum_{\ell \neq i} \prod_{j \neq i, \ell} (\lambda_i - \lambda_j) f_{j-1,k}^{(i)} + q_i(\lambda_i) f_{0,k}^{(i)} \]
whence (putting \( \delta_i := \sum_{\ell \neq i} (\lambda_i - \lambda_{\ell})^{-1} \))
\[ q_i(\lambda_i) f_{0,k}^{(i)} = (\text{ev}_{k,1,\lambda_i} - \delta_i \text{ev}_{k,0,\lambda_i})(q_{C}f) \]
Using these expressions, (2a) reads
\[ \text{ev}_{k,0,\lambda_i}(q_{C}f) = q_i(\lambda_i)a_i v_{ik} \quad \text{for all } k = 1 \ldots r \]
for some \( a_i \in \mathbb{C} \), but only \( r - 1 \) of these conditions are independent because of the \( C^* \)-action on \( v_i \). As we already noticed, at least one entry of \( v_i \) is nonzero, say \( v_{i1} \neq 0 \). Then we can use the \( C^* \)-action to normalise \( v_{i1} = 1 \), so that \( \text{ev}_{1,0,\lambda_i}(q_{C}f) = q_i(\lambda_i)a_i \) and the remaining \( r - 1 \) conditions can be written as
\[ \text{ev}_{k,0,\lambda_i}(q_{C}f) - v_{ik} \text{ev}_{1,0,\lambda_i}(q_{C}f) = 0 \quad \text{for all } k = 2 \ldots r \]
The further condition (2b) directly translates as
\[ \sum_{k=1}^{r} w_{ki} \left( \text{ev}_{k,1,\lambda_i}(q_{C}f) + (\alpha_i - \delta_i) \text{ev}_{k,0,\lambda_i}(q_{C}f) \right) = 0 \]
where \( w_{i1} \) is fixed by the equation \( v_i w_i = -1 \). We conclude that, in the \( (r - 1) \)-dimensional affine cell where \( v_{i1} \neq 0 \), we can take as \( C_i \) the subspace
\[ C_i = \left( \text{ev}_{2,0,\lambda_i} - v_{i2} \text{ev}_{1,0,\lambda_i}, \ldots, \text{ev}_{r,0,\lambda_i} - v_{ir} \text{ev}_{1,0,\lambda_i}, \sum_{k=1}^{r} w_{ki}(\text{ev}_{k,1,\lambda_i} + (\alpha_i - \delta_i) \text{ev}_{k,0,\lambda_i}) \right) \]
Analogous descriptions are available in the other cells, where \( v_{ik} = 0 \) for every \( k < k^* \) and \( v_{ik}^* \neq 0 \). Repeating this argument for every \( i = 1 \ldots n \), we obtain a homogeneous subspace \( C = C_1 \oplus \cdots \oplus C_n \) in \( \mathcal{E}^{(r)} \) such that \( W = (C, q_{C})^* \), as we wanted.
To see a concrete example in the simplest possible case \( (n = r = 2) \), consider the space associated to a point
\[ \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}, \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}, \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}, \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} \in C_{2,2}'' \]
Suppose further that \( v_{i1} \neq 0 \) and \( v_{21} \neq 0 \); the corresponding subspace \( W \in \text{Gr}^{\text{hom}}(2) \) is then given by \( (z - \lambda_1)^{-1}(z - \lambda_2)^{-1}V_C \), where \( C \subseteq \mathcal{E}^{(2)} \) is generated by the four conditions
\[
\begin{align*}
c_{11} &= \text{ev}_{2,0,\lambda_1} - v_{12} \text{ev}_{1,0,\lambda_1} \\
c_{12} &= w_{11} \text{ev}_{1,1,\lambda_1} + w_{21} \text{ev}_{2,1,\lambda_1} + w_{11}(\alpha_1 + \delta) \text{ev}_{1,0,\lambda_1} + w_{21}(\alpha_1 + \delta) \text{ev}_{2,0,\lambda_1} \\
c_{21} &= \text{ev}_{2,0,\lambda_2} - v_{22} \text{ev}_{1,0,\lambda_2} \\
c_{22} &= w_{12} \text{ev}_{1,1,\lambda_2} + w_{22} \text{ev}_{2,1,\lambda_2} + w_{12}(\alpha_2 - \delta) \text{ev}_{1,0,\lambda_2} + w_{22}(\alpha_2 - \delta) \text{ev}_{2,0,\lambda_2}
\end{align*}
\]
where \( \delta := \delta_2 = -\delta_1 = (\lambda_2 - \lambda_1)^{-1} \), \( w_{11} = -1 - v_{12}w_{21} \) and \( w_{12} = -1 - v_{22}w_{22} \). After some computation we find
\[ \tau_W(t) = (t_{\lambda_1} + \alpha_1)(t_{\lambda_2} + \alpha_2) + \delta^2(v_{22}w_{21} + w_{11})(v_{12}w_{22} + w_{12}) \]
\[ \tau_{W12}(t) = -w_{12}v_{22}(t_{\lambda_1} + \alpha_1) - w_{11}v_{12}(t_{\lambda_2} + \alpha_2) + \delta(2v_{12}v_{22}(w_{12}w_{21} - w_{11}w_{22}) + v_{22}w_{12} - v_{12}w_{11}) \]

\[ \tau_{W21}(t) = -w_{22}(t_{\lambda_1} + \alpha_1) - w_{21}(t_{\lambda_2} + \alpha_2) + \delta(2w_{21}w_{22}(v_{22} - v_{12}) + w_{21} - w_{22}) \]

We remark that \( \tau_W \) coincides with the determinant of the matrix \( X \) under the evolution described by the Gibbons-Hermsen flows \( H_k = \text{tr} Y^k \) (taking \( t_k \) as the time associated to \( H_k \)):

\[ \tau_W(t) = \det \left( \begin{array}{cc} \alpha_1 + t_{\lambda_1} & \frac{\delta(\lambda_2)}{\delta(\lambda_1)} w_{12}w_{21} \\ \frac{\delta(\lambda_1)}{\delta(\lambda_2)} w_{22}w_{12} & \alpha_2 + t_{\lambda_2} \end{array} \right) \]

This result fits nicely with the general formulae derived in [14].

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