On a $q$-Deformation of the Discrete Painlevé I equation and $q$-orthogonal Polynomials

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Abstract

I present a $q$-analog of the discrete Painlevé I equation, and a special realization of it in terms of $q$-orthogonal polynomials.

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1 Introduction

Recently, there is growing interest in difference versions of Painlevé equations \([1]\). A first example, the discrete Painlevé I equation (dPI), which is a nonlinear non-autonomous ordinary difference equation of the form

$$\frac{n + \alpha}{R_n} = \beta (R_{n+1} + R_n + R_{n-1}) + \gamma,$$

appeared in the theory of matrix models for 2-dimensional quantum gravity, cf. \([2]-[4]\). There it occurs as lowest non-trivial term in the so-called ‘string equation’.

Other discrete Painlevé equations were subsequently found. A discrete version of Painlevé II was obtained in \([6]\), starting from unitary rather than hermitean one-matrix models, and independently in \([7]\) in connection with similarity reductions of integrable lattice equations. Next, using the new method of singularity confinement, \([8]\), discrete analogues of Painlevé III, IV and V were found in \([9]\). As the singularity confinement approach is based on what is still a conjecture (namely, the non-propagation of spontaneous singularities in integrable mappings), proofs of integrability of these equations had still to be provided. For dPI and dPII, the situation was clear from the beginning, because their very construction (either by the approach of orthogonal polynomials or by the similarity approach) led to the existence of isomonodromic deformation problems for these equations, cf. \([5]\) respectively \([7]\). However, for the new discrete equations, dPIII-dPV, a priori no Lax pair was known. For dPIV and dPV the situation is still open, but in \([10]\) a Lax pair for dPIII was presented (among new isomonodromy problems for dPI and dPII, and their variants). Interestingly enough, this Lax pair is a discrete isomonodromic deformation problem of a linear \(q\)-difference equation, rather than of a differential equation. It is to my knowledge the first time that \(q\)-difference equations arise in connection with Painlevé equations. To study the asymptotics of such type of systems, it is necessary to go back to the historic works of the Birkhoff school on this subject, cf. e.g. \([11]-[15]\), that unfortunately seem to have fallen into oblivion since the 1940’s. Although it is relatively easy to find special solutions for the dPIII equation, cf. \([16]\), the analysis of the corresponding \(q\)-difference system seems quite complicated. As the theory of isomonodromic deformations for \(q\)-difference systems still needs to be developed, it would be useful to have at ones disposal some slightly less complicated systems than the one for dPIII, in order to study the features of this kind of analysis.

In this note, I would like to introduce for that purpose a new discrete Painlevé type of system, namely a \(q\)-deformed version of the dPI equation \((1)\). This is probably the simplest \(q\)-deformed transcendent, although it is naturally written as a third order difference equation rather than a second-order one like \((1)\). Nevertheless, the \(q\)-deformed discrete Painlevé I equation, carries –like dPIII– an underlying monodromy problem of \(q\)-difference type. This system is a natural candidate to be investigated from an analytical point of view, namely by constructing asymptotic solutions and the connection coefficients between them and formulating an inverse (Riemann- Hilbert type) of problem along the lines laid out in \([12]\). The isomonodromic deformation can then be used to ‘solve’ the corresponding Painlevé transcendent. I will leave this program for a future publication, and in this note I will confine myself to establishing a connection between the \(q\)-deformed Painlevé I and \(q\)-orthogonal polynomials, very much along the lines in which this connection was made in the one-matrix models. However, in order to do do this, we will notice that one needs to modify slightly the equation, namely by introducing one term higher in the string equation. Furthermore,
by making this connection we will be naturally led to some discrete-time analogue of the Volterra equation, the similarity reduction of which is the \( q \)-deformed Painlevé I equation.

### 2 \( q \)-Hermite Polynomials

\( q \)-Analysis, \([17, 18]\) is very fashionable nowadays, mainly because of its use in quantum groups and the investigations on \( q \)-special functions in connection with their representation theory, cf. e.g. \([19]-[22]\). In this note I would like to draw attention to a new class of \( q \)-special functions, namely \( q \)-difference deformed versions of Painlevé transcendents, in particular a \( q \)-deformed version of the dPI equation (1). It is well-known that the latter equation arises as the lowest non-trivial term in the string equation for one-matrix models via orthogonal polynomials. A special subcase, namely if \( \alpha = \beta = 0 \) in (1), gives rise to the Hermite polynomials. In seeking a \( q \)-deformed version it is natural, therefore, to start from the basic \( q \)-Hermite polynomials, defined e.g. in \([17]\). Let us first review these, mainly for the purpose of fixing the notations.

First, let me recall the definition of the \( q \)-exponential function

\[
\begin{aligned}
e^x_q &= \sum_{n=0}^{\infty} \frac{x^n}{(n)_q}, \\
(e^x_q)^{-1} &= e^{-x}_q, \quad (2)
\end{aligned}
\]

introducing also the \( q \)-numbers \( (n)_q \equiv \frac{q^n - 1}{q - 1} \). Sometimes it is useful to work with Andrew’s notation, \([18]\),

\[
(a; q)_\infty \equiv \prod_{j=0}^{\infty} (1 - q^j a) , \quad (a; q)_n \equiv \frac{(a; q)_\infty}{(aq^n; q)_\infty},
\]

in terms of which we have e.g.

\[
e^{-1}_{q-1} = ((q - 1)x; q)_\infty , \quad (q < 1) . \quad (3)
\]

Now we can introduce the basic \( q \)-Hermite polynomials by means of the generating function

\[
S_q(x, t) = e^{xt} e^{ \hat{a} t^2} = \sum_{n=0}^{\infty} \frac{t^n}{(n)_q} H_n(x) , \quad (4)
\]

Furthermore, introducing the \( q \)-differentiation and the \( q \)-dilatation operator

\[
qD_x f(x) \equiv \frac{f(qx) - f(x)}{(q-1)x} , \quad qT_x f(x) \equiv f(qx) ,
\]

and recalling that the \( q \)-exponential obeys the difference rules

\[
qD_x e^{ax}_q = a e^{ax}_q , \quad qD_x e^{ax}_{q^{-1}} = a e^{qax}_{q^{-1}} ,
\]

one can derive for \( S_q \) the following relations

\[
qD_x S_q = t S_q , \quad (5a)
\]

\[
qD_t S_q = (x + (2) q \hat{a} t) qT_x^{-1} qT_t S_q , \quad (5b)
\]

\[\footnote{In contrast with the original exponential function, the \( q \)-exponential function \( e^x_q \) has a finite radius of convergence \( R = |1 - q|^{-1} \) as \(|q| < 1 \), and simple poles at \( x = (1 - q)^{-1} q^{-j} \) for \( j = 0,1,2,\ldots \).} \]
leading to the relations

\[ qD_x H_n = (n)_q H_{n-1} , \]
\[ H_{n+1} = x q^n qT_x^{-1} H_n + \bar{a}(2)_q(n)_q q^{n-1} qT_x^{-1} H_{n-1} . \] (6a)

The last equation can be converted into

\[ H_{n+1} = x H_n + \bar{a}(2)_q^{-1}(n)_q q^n H_{n-1} , \] (6c)

whereas, from (6a) and (6b) one can also derive the second order \( q \)-difference equation

\[ q^{-n}(n)_q H_n = x qD_x qT_x^{-1} H_n + \bar{a}(2)_q^{-1} qD_x qT_x^{-1} qD_x H_n , \] (6d)

i.e. the \( q \)-analogue of the Hermite equation. Using the fact that \( qT_x^{-1} qD_x = qD_x qT_x^{-1} \), and that \( qD_x = (2)_q x qD_x q \), we can convert the last equation into

\[ qD_x \left[ e_{-a}^{-2} x^2 qT_x^{-1} qD_x H_n \right] + a(2)_q^{-1} q^{-n}(n)_q e_{-2}^{-a x^2} H_n = 0 , \] (7)

identifying \( a = (-\bar{a}(q^{-1} + 1)^2)^{-1} \), which we take to be real positive. From (7) it is clear that the basic \( q \)-Hermite polynomials are orthogonal with respect to the Jackson integral

\[ \int_{-c}^c f(x) d_q x \equiv (1 - q)c \sum_{n=0}^\infty q^n [f(q^n c) + f(-q^n c)] , \quad c = \left( (1 - q^2) a \right)^{-1/2} , \quad (0 < q < 1) , \]

with weight function \( w_q(x) = e_{-2}^{-a x^2} \). Introducing the following representation for the \( q \)-Gamma function, [17, 18],

\[ \Gamma_q(x) = q^{-x} \int_0^{1/(1-q)} t^{x-1} e_{-t}^{-1} d_q t = \frac{(q; q)_\infty}{(q^2; q)_\infty} (1 - q)^{1-x} , \] (8)

in terms of which we can express the normalization constant, we have the orthogonality condition

\[ \int_{-c}^c d_q x e_{-2}^{-a x^2} H_n(x) H_m(x) = 2 q^{2(n+1)}(n)_{q^2} \sqrt{a} \Gamma_q \left( \frac{1}{2} \right)^2 \delta_{n,m} . \] (9)

Finally, the dependence on the variable \( a \) (or, equivalently, on \( \bar{a} \)) is significant. In fact, from

\[ q^2 D_a S_q = \frac{t^2}{(2)_q^2 a^2} S_q , \]

we derive

\[ q^2 D_a H_n = \frac{(n)_q(n-1)_q}{(2)_q^2 a^2} H_{n-2} . \] (10)

3 \ A \ q-deformed Painlevé I Equation

Comparing the Lax pair for dPI, cf. e.g. [6], with the one for the \( q \)-Hermite polynomials, namely eqs. (6a) and (6b), it is natural to pose the following \( q \)-linear system

\[ x P_n(x) = P_{n+1}(x) + R_n P_{n-1}(x) , \] (11a)
\[ qD_x P_n(x) = A_n P_{n-1}(x) + B_n P_{n-3}(x) , \] (11b)
the compatibility of which leads to the set relations
\begin{align*}
A_{n+1} &= qA_n + 1, \\ B_{n+1} + R_nA_{n-1} &= q(A_nR_{n-1} + B_n), \\ R_nB_{n-1} &= qB_nR_{n-3}.
\end{align*}

Eqs. (12a) and (12c) are readily solved as
\begin{align*}
A_n &= (n)_q + \alpha q^n, \\ B_n &= \beta q^{-n}R_nR_{n-1}R_{n-2}.
\end{align*}

Inserting these expressions into (12b) we obtain the following third-order nonlinear non-autonomous ordinary difference equation
\begin{equation}
\beta q^{-n} \left( q^{-1}R_{n+1} - qR_{n-2} \right) = q\left( \frac{(n)_q + \alpha q^n}{R_n} - \frac{(n-1)_q + \alpha q^{n-1}}{R_{n-1}} \right),
\end{equation}

which we will refer to as qPI. The connection is made clear by rewriting (14) in the form
\begin{equation}
q^{-n} \left[ \gamma + \beta q^{-2} \left( R_{n+1} + R_n + R_{n-1} \right) + \beta(q^{-2} - 1) \sum_{j=-\infty}^{n-2} R_j \right] = \left( \frac{(n)_q + \alpha q^n}{R_n} \right),
\end{equation}

namely by performing one (formal) ‘integration’. The infinite sum term in (15) will clearly disappear in the limit \( q \to 1 \), in which case we immediately recover dPI , (1).

In [23] a \( q \)-deformed Painlevé equation related to a special subcase of the discrete PII has been presented which shows some similarity to eq. (14). It is not clear, however, whether their equation can be reduced to (14) by coalescence. Furthermore, the approach in [23] does not focuss on isomonodromic deformation problems, but rather starts from factorizing Schrödinger operators. What is interesting about (14), apart from being probably the simplest \( q \)-deformed transcendent that is now at our disposal, is that there is a connection with \( q \)-orthogonal polynomials, along the same lines as for the original dPI. However, in order to establish this connection one has to modify slightly eq. (14) as we shall see.

### 4 q-Orthogonal Polynomials

We look for an explicit realization of the qPI equation in terms of \( q \)-orthogonal polynomials, in much the same way as the original discrete Painlevé I equation (1) is solved by means of orthogonal polynomials. It is this solution that gives rise to a connection with discrete hermitian one-matrix models.

For this purpose let us generalize the orthogonality conditions (9) to

\begin{equation}
\int d_qx e_{-ax^2} e_{-bx^4} P_n(x) P_m(x) = h_n \delta_{n,m}.
\end{equation}

For the integration limits in the Jackson integral (16) we take the smallest zero of the weight function \( w_q(x) = e_{-ax^2} e_{-bx^4} \), i.e. \( x = c \), where \( c = \min \left\{ \left(1 - q^2a\right)^{-1/2}, \left(1 - q^4b\right)^{-1/4} \right\} \), (for fixed positive \( a, b \), taking \( 0 < q < 1 \)) to ensure positivity of the measure. The polynomials are supposed to be normalized such that
\begin{equation}
P_n(x) = x^n + \cdots \quad \Rightarrow \quad D_x P_n = (n)_q P_{n-1} + \cdots.
\end{equation}
Of course one may go further and introduce in \([16]\) a weight factor depending on an arbitrary number of \(q\)-exponential factors, but I will refrain from doing so in the present note.

From the orthogonality condition \([16]\) we can derive the isomonodromy problem \([11a]\), with \(R_n = h_n/h_{n-1}\) (as in the usual case). However, \([11b]\) needs to be modified as follows from the following relation for the weight function

\[
q D_x w_q(x) = -\left[(2)_q ax + (4)_q bx^3 + (q-1)(2)_q(ab x)^5\right] w_q(q x) ,
\]

which indicates that one has to push already to fifth order terms in order to take into account second and fourth order coupling constants \(a, b\). This means that eq. \([11b]\) is not applicable here, and performing the calculation

\[
\int d_q x w_q(x) [(q D_x P_n)P_m + P_n (q D_x P_m)] + (q-1) \int d_q x w_q(x)(q D_x P_n)(q D_x P_m)
= \int d_q x w_q(x)(q T_x P_n)(q T_x P_m) \left[(2)_q ax + (4)_q bx^3 + (q-1)(2)_q(ab x)^5\right] ,
\]

we are led to a linear problem consisting of \([11a]\) together with

\[
q D_x P_n(x) = A_n P_{n-1}(x) + B_n P_{n-3}(x) + C_n P_{n-5} .
\]

Compatibility will now lead to

\[
\begin{align*}
A_{n+1} &= q A_n + 1 , \quad \text{(19a)} \\
B_{n+1} + R_n A_{n-1} &= q (A_n R_{n-1} + B_n) , \quad \text{(19b)} \\
C_{n+1} + R_n B_{n-1} &= q (B_n R_{n-3} + C_n) , \quad \text{(19c)} \\
R_n C_{n-1} &= q C_n R_{n-5} . \quad \text{(19d)}
\end{align*}
\]

Eqs. \([19a]\) and \([19d]\) again are readily solved as

\[
A_n = (n)_q + \alpha q^n , \quad C_n = \gamma q^{-n} R_n R_{n-1} R_{n-2} R_{n-3} R_{n-4} ,
\]

and from \([19b]\) and \([19c]\) we get the system

\[
q \frac{(n)_q + \alpha q^n}{R_n} - \frac{(n-1)_q + \alpha q^{n-1}}{R_{n-1}} = \gamma q^{-n} \left( q^{-1} R_{n+1} \tilde{B}_{n+1} - q R_{n-2} \tilde{B}_n \right) , \quad \text{(21a)}
\]

\[
\tilde{B}_n - \tilde{B}_{n-1} = q^{-2} R_{n+1} - R_{n-4} , \quad \text{(21b)}
\]

in which \(\tilde{B}_n = q^{-n} \gamma R_n R_{n-1} R_{n-2} \tilde{B}_{n-1}\). Of course, one can derive from \([21]\) a closed sixth order difference equation by eliminating the \(\tilde{B}_n\). This is not so illuminating, and I will not give the formula. However, by formally solving \([21b]\) as

\[
\tilde{B}_n = \beta \gamma^{-1} + q^{-2} (R_{n+1} + R_n + R_{n-1} + R_{n-2} + R_{n-3}) + (q^{-2} - 1) \sum_{j \leq n-4} R_j ,
\]

we see that the case of \([15]\) is included for \(\gamma \to 0\).

On the other hand, from the orthogonality condition, performing the calculation \([17]\) one derives relations between the coefficients \(A_n, B_n, C_n\). For \(A_n\) and \(C_n\) one finds again \([20]\) for the special choice \(\alpha = 0\) respectively \(\gamma = (1 - q^{-1})q^4(2)_q^{-1}(4)_q^{-1} ab\). Furthermore, for \(B_n\) we obtain

\[
\tilde{B}_n + (q-1)q^{-2n+4} \gamma R_{n-2} R_{n-3} R_{n-4} \tilde{B}_{n-2}
= q^2(4)_q^{-1} b \gamma^{-1} + q^{-2} (R_{n+1} + R_n + R_{n-1} + R_{n-2}) + q^{2-n} R_{n-3} ,
\]

(22)
which turns out to be consistent with (21), and in addition we have the following equation

\[ q^n \frac{(n)_q}{R_n} = (2)_{q^{-1}}a + (4)_{q^{-1}}b (R_{n+1} + R_n) - \gamma q^{-n} R_{n-1} R_{n-3} \]

\[ + \gamma q^{-4} [R_{n+1} (R_{n+2} + R_{n+1} + R_n) + R_n (R_{n+1} + R_n + R_{n-1})] \]

\[ + \gamma \tilde{B}_n [\tilde{B}_{n+2} - q^2 (4)_{q^{-1}} b \gamma^{-1} - q^{-2} (R_{n+3} + R_{n+2} + R_{n+1} + R_n)] \]

\[ + (1-q) \gamma q^{-2n} R_n R_{n-1} R_{n-2} R_{n-3} R_{n-4} , \] (23)

which is the actual $q$-deformed string equation (at least for the lower order terms), identifying $\beta = q^2 (4)_{q^{-1}} b$. Also eq. (23) is consistent with (21), which can be checked explicitly by tedious algebra. Thus, the orthogonality is consistent with the compatibility condition, which in turn implies that the $q$-orthogonal polynomials are indeed solutions of the system (18).

5 Discrete-Time Flows

The dependence on the ‘coupling constants’ $a, b, \ldots$ as time-flow parameters in (16) is significant, and will lead to a discrete-time version of the Volterra system. In fact for the polynomials $P_n$, we can derive from (16) the $q$-difference time-evolutions of the form

\[ q^2 D_a P_n = Q_n P_{n-2} , \] (24a)

\[ q^2 D_b P_n = U_n P_{n-2} + V_n P_{n-4} , \] (24b)

where the coefficients $Q_n$, $U_n$ and $V_n$ are determined by the orthogonality condition. If we would have included additional higher-order exponents in the weight function of (16) this would lead to a discrete-time Volterra hierarchy. Let us illustrate this by working out explicitly the lowest order time-flow, namely in terms of the variable $a$. From (24a) together with (11a) we obtain the relations

\[ q^2 D_a R_n = Q_n - Q_{n+1} , \] (25a)

\[ (q T_a R_n) Q_{n-1} = Q_n R_{n-2} , \] (25b)

from which one can derive that

\[ Q_n = c \tilde{R}_n \tilde{R}_{n-1} , \quad \tilde{R}_n = R_n + (1-q^{-2}) a Q_{n+1} , \quad c = \text{constant} . \] (26)

This then leads immediately to the following exact discrete-time $q$-deformation of the Volterra equation

\[ (q^2 D_a \tilde{R}_n) = c \left[ \tilde{R}_n \tilde{R}_{n-1} - q^2 q^2 T_a (\tilde{R}_{n+1} \tilde{R}_n) \right] . \] (27)

It is slightly surprising that the exact integrable discretization of the Volterra systems turns out to take a form which resembles closely the original continuous-time equation, provided one considers the proper variables $\tilde{R}_n$. Furthermore, solving (25a) by taking $Q_n = (q T_a h_n) / h_{n-2}$, recalling that $R_n = h_n / h_{n-1}$, we are led to the following equation for $h_n$

\[ h_{n-1} (q T_a h_n) = \left[ h_n + (1-q^2) a (q T_a h_{n+1}) \right] \left[ h_{n-1} + (1-q^2) a (q T_a h_n) \right] , \] (28)
where taking $c = 1$ is consistent with the orthogonality. On the other hand, from the orthogonality condition (14) one can derive by a calculation similar to (17)

$$- q^2 D_a h_n = q^2 T_a \left( h_{n+1} + \frac{h_n^2}{h_{n-1}} \right) + (1 - q^2) a \frac{(q^2 T_a h_n)^2}{h_{n-2}},$$

which can be shown to be consistent with (28). Let me mention that eq. (28) is a special subcase of a universal lattice equation derived some years ago in [24].

Of course, the above derivation can be extended to the higher-order time-variables, e.g. starting from (24b). In general this will lead to more complicated systems and we will abstain in this note from presenting their derivation. What is important to note, however, is that for all the coefficients $a, b, \ldots$ in the weight function (which in certain contexts are interpreted as coupling constants), the $q$-deformation in terms of the spectral parameter $x$ leads necessarily also to a $q$-deformation in terms of these higher-order ‘time’-variables which in turn leads to integrable discrete-time systems like the discrete Volterra system of (27).

6 Discussion

I have shown that it is fairly straightforward to obtain a $q$-analogue of the discrete Painlevé I equation by straightforwardly $q$-deforming the continuous isomonodromy problem for discrete PI. In this way one obtains eq. (14), which is associated with probably the simplest isomonodromic deformation problem of a linear $q$-difference equation. Having this system at our disposal, we can now seriously embark on the more serious problem of investigating the full asymptotics for the corresponding transcendentals.

Furthermore, I have put forward a connection with $q$-orthogonal polynomials. For this one needs to consider a slightly more complicated system including 5th order terms in the “string equation”. The connection with orthogonal polynomials is interesting, because on the one hand it establishes what is, in fact, a special similarity solution of the discrete-time Volterra hierarchy. On the other hand, it suggests a discretization of the Hermitian one-matrix model, which –on the continuum level– is well known to be related to orthogonal polynomials of Painlevé type. It is suggestive, therefore, to try to trace back the $q$-orthogonal polynomials to the corresponding matrix models. If one were naive, one would be tempted to write down an expression of the form

$$Z_N(a, b) = \int_{N \times N} [d_q H] e^{-a tr H^2} e^{-b tr H^4},$$

for the $q$-deformation of the partition function of the Hermitian one-matrix model, in which $[d_q H]$ denotes a proper $q$-analogue of the Haar measure. However, eq. (30) cannot be correct, because the $q$-exponents do not decompose in a natural way for linear combinations in their arguments, nor do the Jackson integrals allow for arbitrary changes of variables. What one needs is an interpretation of an expression of the form of (30) which should lead –after integration over the “angle variables” associated with an Hermitian matrix $H$– to a multiple Jackson integral over the eigenvalues of $H$ of the form

$$Z_N = \int \prod_{i=1}^N d_q \lambda_i e^{-a \lambda_i^2} e^{-b \lambda_i^4} \prod_{i<j=1}^N (\lambda_i - \lambda_j)^2.$$

(31)
It is this expression for a ‘partition function’ that has a direct connection with the qPI equation of section 4. However, how to arrive at this expression starting from a (discrete) matrix integral needs some further study which is beyond the scope of this paper.

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References

[1] B. Grammaticos and A. Ramani, Discrete Painlevé Equations: Derivation and Properties, in Proc. of the NATO ASI on Differential Geometric Methods in Nonlinear Partial Differential Equations. ed. P. Clarkson (Kluwer, to be published).

[2] E. Brézin and V.A. Kazakov, Phys. Lett. 236B (1990) 144.

[3] D. J. Gross and A.A. Migdal, Phys. Rev. Lett. 64 (1990) 127; Nucl. Phys. B340 (1990) 333.

[4] M. Douglas, Phys. Lett. 238B (1990) 176; M.R. Douglas and S.H. Shenker, Nucl. Phys. B335 (1990) 635.

[5] A.S. Fokas, A.R. Its and A. Kitaev, Commun. Math. Phys. 142 (1991) 313.

[6] V. Periwal and D. Shevitz, Phys. Rev. Lett. 64 (1990) 1326.

[7] F.W. Nijhoff and V.G. Papageorgiou, Phys. Lett. 153A (1991) 337.

[8] B. Grammaticos, A. Ramani and V.G. Papageorgiou, Phys. Rev. Lett. 67 (1991) 1825.

[9] A. Ramani, B. Grammaticos and J. Hietarinta, Phys. Rev. Lett. 67 (1991) 1829.

[10] V.G. Papageorgiou, F.W. Nijhoff, B. Grammaticos and A. Ramani, Phys. Lett. 164A (1992) 57.

[11] R.D. Carmicheal, Am. J. Math. 34 (1912) 147.

[12] G.D. Birkhoff, Proc. Natl. Acad. Arts and Sci. 49 (1913) 521.

[13] C.R. Adams, Ann. Math. Ser. II 30 (1929) 195.

[14] W.J. Trjitzinski, Acta Math. 61 (1933) 1.

[15] J. Le Caine, Am. J. Math. 43 (1942) 585.

[16] B. Grammaticos, F.W. Nijhoff, V. Papageorgiou, A. Ramani and J. Satsuma, Linearization and Solutions of the Discrete Painlevé III Equation, Preprint LPN-Paris VII (May 1993).
[17] H. Exton, *q-Hypergeometric Functions and Applications*, (Ellis, Horwood Ltd., 1983).

[18] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, (Cambridge Univ. Press, 1990).

[19] T.H. Koornwinder, Proc. Koninkl. Acad. v. Wetenschappen **92A** (1989) 97.

[20] D.B. Fairlie, *q-Analysis and Quantum Groups*, Preprint Univ. of Durham DTP-90-45.

[21] R. Floreanini and L. Vinet, Lett. Math. Phys. **22** (1991) 45, ibid. **25** (1991) 151; J. Math. Phys. **33** (1992) 1385; Ann. Phys. **221** (1993) 53.

[22] I.B. Frenkel and N. Yu. Reshetikhin, Commun. Math. Phys. **146** (1992) 1.

[23] V. Spiridonow, L. Vinet and A. Zhedanov, Lett. Math. Phys. **29** (1993) 63.

[24] F.W. Nijhoff, G.R.W. Quispel and H.W. Capel, Phys. Lett. **97A** (1983) 125.