Non-local interactions in a BEC: 
an analogue gravity perspective

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Abstract
We add a minimal correction term to the local Gross–Pitaevskii equation to represent non-locality in the interactions. We show that the effective minimal non-locality can make the healing length decrease more rapidly with the increase of s-wave scattering length leaving the expression of the velocity of sound unaltered. We discuss the implication of this result for a Bose–Einstein condensate being used as an analogue gravity system. The presented result is important in the context of condensed matter physics as well because one can considerably change the size of a quantized vortex at finite s-wave scattering length by tuning the healing length.

Keywords: analogue gravity, BEC, Bogoliubov spectrum, quantized vortex, healing length
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Introduction
A Bose–Einstein condensate (BEC) is considered as an ideal analogue gravity system [1] for various reasons. The nano Kelvin temperature, quantum coherence over large length and time scales, low speed of sound waves, amenability to accurate experimental control and manipulations [2, 3] etc make the BEC one of the most ideal systems to simulate quantum blackhole geometries and Hawking radiation. The small amplitude collective modes (phonons) in a BEC can see a Lorentzian metric depending on a suitable background Newtonian bulk flow of the condensate. Under suitable conditions, such a barotropic, inviscid, irrotational background flow field can generate a sonic event horizon for the phonons which cannot overflow the boundary where the bulk velocity exceeds the velocity of sound. It was Unruh’s 1981 paper [4] which practically started this field of analogue gravity by proposing the
possibility of observing Hawking radiation in analogue systems. Following Unruh’s seminal paper, a lot of attention has been attracted by the field not only in connection with the classic trans-Planckian problem [5], but also for a wide spectrum of issues elegantly compiled in the review by Barceló et al [6].

BEC of interacting bosons is explained on the basis of the Bogoliubov theory. The Bogoliubov theory identifies the ground state of the interacting BEC to be the vacuum of phonon-like collective modes (excitations) which obey the famous Bogoliubov dispersion relation

\[ \epsilon_p = \left[ \frac{g}{m} p^2 + \left( \frac{\rho_p}{m} \right)^2 \right]^{1/2} \]

where \( p \) is the momentum of the excitation, \( g \) is the approximate two-body interaction strength between bosons, \( n \) is the density of the condensate and \( m \) is the mass of a single boson. At very small \( p = \hbar k \), the dispersion relation becomes that of phonon’s i.e.

\[ \epsilon_p = \hbar \omega = cp \]

where \( c = \sqrt{g n / m} \) is the velocity of sound in the condensate. The small amplitude collective oscillations in a BEC can see a curved spacetime over a moving BEC with velocity \( v \) for which the dispersion relation deviates from the exact Bogoliubov dispersion law in the following manner [6]

\[ \epsilon_p = \hbar v' k \pm \left[ \frac{g n}{m} \hbar k^2 + \left( \frac{\hbar^2 k^2}{2m} \right)^2 \right]^{1/2}, \quad (1) \]

where \( i \) stands for \( x, y, z \) and the repeated index implies a sum. In the simplest scenario, the low energy excitations (small \( k \)), having a linear dispersion relation, respond to the curved spacetime geometry and see the sonic horizon (where the bulk velocity becomes supersonic) from which an analogue Hawking radiation is expected.

At small \( k \), one can ignore the under-root quartic term in \( k \) in equation (1) and deal with a linear dispersion relation, but, the approximation breaks down at the horizon where \( v \to c \) and the quartic term becomes relevant. The presence of the quartic term at the horizon can of course give rise to interesting physics. Some of the works done by Parentani and coworkers considering a novel density-density correlation probe [7], studying hydrodynamic flow over several length scales giving Plankian spectrum with a surface gravity independent temperature [8], considering high frequency dispersion over a wider set of conditions [9], are quite important. Another recent work by Fleurov et al taking into account the quantum pressure term, which is normally neglected in the hydrodynamics of the quantum fluids to simplify the scenario, shows a second characteristic length scale which is an important observation [10]. So, the presence of dispersion makes the physics richer, but at the same time, a control over this dispersion can be of immense importance to look into the system more closely and for having the group velocity tunable. There can be small \( k \) problems due to the finite size of the horizon introducing a large wavelength cut off [2] beyond which sound waves would get just diffracted. So, (a) controlled access to small wavelength excitations and (b) the finite (small) velocity of sound are two essential ingredients for the BEC to be used as an analogue system.

In this paper we discuss the scenario where a minimal correction in the local Gross–Pitaevskii (GP) equation arising from the structureless non-locality in the interactions decreases the healing length in such a way, that the quartic term in the dispersion relation can be made very small on tuning the s-wave scattering length keeping the velocity of sound the same as that of the local GP equation. The present analysis indicates a simple way of achieving both the essential ingredients mentioned above by the tuning of s-wave scattering length for repulsive interactions. The variation of scattering length in BEC used as an analogue system has been discussed in [11] in a different context.

Considering the same set of basic conditions, as in the Bogoliubov theory, namely, (a) the macroscopic occupation of the BEC ground state and (b) diluteness of the condensate, Gross and Pitaevskii derived the famous equation for the complex order parameter of the inhomogeneous condensate known as the GP equation [12]. The GP equation not only correctly
The general form of the GP equation is

\[ i\hbar \frac{\partial}{\partial t} \psi(\mathbf{r}, t) = \left( -\frac{\hbar^2 \nabla^2}{2m} + V_{\text{ext}}(\mathbf{r}, t) + g|\psi(\mathbf{r}, t)|^2 \right) \psi(\mathbf{r}, t), \]  

(2)

where \( \psi \) is the order parameter of the BEC and is related to the condensate density by \( n = |\psi(\mathbf{r}, t)|^2 \), \( V_{\text{ext}} \) is the external potential. The GP equation also admits vortex solutions which are particle like (dispersion relation is not linear like that of massless phonons) and are higher energy states stabilized by rotating the BEC [12].

As is already mentioned, the GP equation in its above mentioned form relies on the diluteness condition to keep the nonlinear term local. The diluteness condition reads as \( |a| \ll n^{-1/3} \) i.e. the s-wave scattering length is much smaller than the average range of separation between particles in the BEC. The s-wave scattering length is given by the relation \( g = \frac{4\pi\hbar^2 a}{m} \) following the Born approximation and has to be kept positive, in the simplest scenario, to keep the BEC ground state thermodynamically stable. On the basis of the above mentioned assumptions, the non-local interaction term \( \psi(\mathbf{r}, t) \int d\mathbf{r}' \psi^*(\mathbf{r}', t)V(\mathbf{r}' - \mathbf{r})\psi(\mathbf{r}', t) \) of the general GP equation is written as \( \psi(\mathbf{r}, t) \int d\mathbf{r}' V(\mathbf{r}' - \mathbf{r})|\psi(\mathbf{r}', t)|^2 = g\psi(\mathbf{r}, t) \int d\mathbf{r}' \delta(\mathbf{r}' - \mathbf{r})|\psi(\mathbf{r}', t)|^2 = g|\psi(\mathbf{r}, t)|^2|\psi(\mathbf{r}, t)|^2 \). The local GP equation can be shown to be derived variationally by minimizing a free energy functional. The asymptotic exactness of the GP energy functional at the dilute limit with two-body interactions was rigorously shown by Lieb et al [13, 14]. In this proof, one works at the thermodynamic limit \( N \to \infty, a \to 0 \) to keep \( Na \) finite and this limit corresponds well with \( V(\mathbf{r}' - \mathbf{r}) \) being replaced by a delta function.

Varied non-local forms of the interaction potential have been considered to capture new solutions not admitted by local GP theory [15, 16]. There is a symmetry based recent analysis of non-locality in GP equation in [17]. With the help of non-local nonlinearities, roton minimum in the dispersion relation has been predicted, which is a typical characteristic of superfluids [18–20]. An interesting work [21] considers non-locality in the interactions in the presence of a particular periodic global potential to show that, although the GP free energy and the corresponding solutions show asymptotic correspondence as one moves from the non-local to the local limit, the asymptotic correspondence in the stability of the solutions is not present [22].

In [23], the effect of quantum fluctuations on the mean field model is treated semi-classically and [24] presents an interesting analysis considering non-local interactions preventing the collapse of the condensate. The work of Andreev et al [25] on degenerate boson–fermion system derives a similar dispersion relation based on a third order correction as in this paper but focuses on different issues [26–28]. Rosanov et al have done an analysis on internal oscillations of solitons with non-local nonlinearity on a similar system that considers a correction term similar to ours with an attractive interaction at large distances [29]. There is a nice review article by Yukalov [30] addressing theoretical challenges of BEC theory covering non-local and disordered potentials.

Model

The general form of the GP equation is

\[ i\hbar \frac{\partial}{\partial t} \psi(\mathbf{r}, t) = \left( -\frac{\hbar^2 \nabla^2}{2m} + V_{\text{ext}}(\mathbf{r}, t) \right) \psi(\mathbf{r}, t) + \left( \int d\mathbf{r}' \psi^*(\mathbf{r}', t)V(\mathbf{r}' - \mathbf{r})\psi(\mathbf{r}', t) \right) \psi(\mathbf{r}, t), \]  

(3)
which one gets by replacing quantum mechanical field (creation/annihilation) operators \( \hat{\psi}^\dagger / \hat{\psi} \) by complex numbers (order parameters) \( \psi^\dagger / \psi \) in the Heisenberg equations for the fields, where the order parameter is related to the condensate density as \( |\psi|^2 = n \). One can do this approximation by considering a macroscopic occupation of the BEC where the non-commutativity of the creation and annihilation operators can be ignored at the mean field level as is considered in the Bogoliubov theory.

If one does a Taylor expansion of \( \psi(\mathbf{r}, t) \) about the position \( \mathbf{r} \) keeping in mind the fact that the interaction potential is symmetric, one generates the GP equation for one-dimensional inhomogeneities with an additional nonlinear term to the conventional local GP equation as

\[
i\hbar \frac{\partial}{\partial t} \psi(\mathbf{r}, t) = \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V_{\text{ext}}(\mathbf{r}, t) + g|\psi(\mathbf{r}, t)|^2 \right) \psi(\mathbf{r}, t) + \frac{1}{6} a_0^2 g |\psi(\mathbf{r}, t)| \frac{\partial^2}{\partial x^2} |\psi(\mathbf{r}, t)|^2.
\]

(4)

The above equation effectively considers that, the \( \psi(\mathbf{r}, t) \) is uniform on the \( y-z \) plane (for the sake of simplicity) and also, that there are no small amplitude global modes with spherical or circular symmetry in the system for the unavoidable spatial dependences of the amplitude of such modes. The gradient term in the Taylor expansion does not contribute due to the symmetry of the interaction and we have kept the next higher order term here. The interaction potential \( V(\mathbf{r} - \mathbf{r}) \) is taken here to be the same flat repulsive potential \( V_{\text{eff}} \) considered in the derivation of the conventional local GP equation where \( g = \int V_{\text{eff}} \, d\mathbf{r} \). The \( V_{\text{eff}} \) has a range effectively equal to the s-wave scattering length \( a \) which is the effective range of interaction seen by particles undergoing so-called zero-energy scattering. The spherical symmetry of the interaction is evident from the consideration of s-wave scattering only. It is interesting to note that the first nonlinear term of our expansion is the same as the nonlinear term of the conventional local GP equation. But, not imposing the drastic \( \delta \)-function approximation allows for the appearance of the new term which would have been missing had we applied the \( \delta \)-function approximation in the beginning.

We are not considering the interaction potential to be a \( \delta \)-function. We are considering \( a \) as finite and that legitimizes the inclusion of the correction (nonlinear) term with a second order derivative. Since, there already exists a term with a second order spatial derivative in the local GP equation, the slowness of density variation over space cannot prevent the additional term from appearing in it. The slowness of the spatial variation of the density is considered here to help drop the higher order terms of the expansion. We would like to emphasize the fact that, this is the first minimal correction term that could be added, keeping the GP dynamics local, in order to incorporate the effect of the non-locality of the interactions on top of the \( \delta \)-function approximation. At the limit \( a \to 0 \) (corresponding to the limit taken by Lieb et al [13, 14]) this additional term will obviously disappear and we get back the traditional local GP equation. Apart from relaxing the width of the \( \delta \)-potential, in all other respects, we are following the standard assumptions of the local GP theory. Moreover, our model is also a local one where the extra term brings in the effect of non-locality to some extent.

Let us identify a few properties of the equation (4). Unlike the usual local GP equation, equation (4) cannot be derived variationally from a free energy functional, but it preserves the continuity equation \( \frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{j} \) where the current density \( \mathbf{j} = -\frac{i\hbar}{2m} \left( \psi^\dagger \nabla \psi - \psi \nabla \psi^\dagger \right) \). The total mass of the condensate is conserved even after the perturbation is added. The second most important feature preserved by this equation is that, the spatially uniform but oscillating ground state solution \( \psi_0 = \sqrt{n} e^{i \phi} \) of the local GP equation (equation (2)) is still a solution of the system, although, we cannot call it the ground state solution because the concept of the GP free energy no longer applies. Actually, it is well-known that the GP ground state is, in reality, a metastable state. This conservation of mass is an essential ingredient preserved
despite the addition of the term representing the non-locality. The condition of the dynamical stability of the so-called uniform ground state would remain the same in this modified scenario, however, this is not that important in the analogue gravity context as the dynamical stability of a state with an underlying supersonic velocity.

**Small amplitude collective modes**

Let us consider a condensate without any confining external potential and drop the $V_{\text{ext}}$ term from equation (4). To look at the one-dimensional small amplitude modes, we perturb the uniform oscillatory solution as

$$\psi(x, t) = \psi_0 + \sum_j (u_j(x) e^{-i\omega t} + v_j(x) e^{i\omega t}) e^{-i\mu t}.$$  

Putting this expression in equation (4) and linearizing, one can get the dynamics of the amplitude of the perturbations as

$$\hbar \omega_j u_j = gn u_j + \left(\frac{a^2 gn}{6} - \frac{\hbar^2}{2m}\right) u_j'' + gn v_j + \frac{a^2 gn}{6} v_j''$$

$$-\hbar \omega_j v_j = gn v_j + \left(\frac{a^2 gn}{6} - \frac{\hbar^2}{2m}\right) v_j'' + gn u_j + \frac{a^2 gn}{6} u_j''.$$  

(5)

where we have used $\mu = gn$ which one gets by putting $\psi_0$ in equation (4). Considering the ansatz $u_j = u e^{ikx}$ and $v_j = v e^{ikx}$ we get the dispersion relation as

$$\hbar^2 \omega^2 = \frac{\hbar^2 k^2 gn}{m} + \left(\frac{\hbar^4}{4m^2} - \frac{\hbar^2 a^2 gn}{6m}\right) k^4.$$  

(6)

This dispersion relation immediately tells us, that up to the limit $\frac{4\pi}{3} a^3 = \frac{1}{2\pi}$, the qualitative nature remains the same as that of the Bogoliubov spectrum. The excitation energy, however, reduces at larger $k$ compared to the Bogoliubov spectrum as $a$ approaches its limiting value $(3/8\pi n)^{1/3}$. The possibility of keeping the dispersion relation linear by tuning the s-wave scattering length is the point of interest here from the perspective of BEC as an analogue gravity system. For an $a$ larger than $(3/8\pi n)^{1/3}$ the sign of the coefficient of the quartic term in $k$ would be negative and more than two-body interactions might be important at this stage if superposition of two-body interactions practically fails to accommodate the physics. The bending of the dispersion curve at long-range interactions shows some tendency to the roton minimum, but, in the present context, we are confined to 1D and do not take into account the structures of the interaction potential which stabilizes excitations and brings the dispersion curve back upwards again. However, the indication that a long-range interaction is a generic reason for the creation of another minimum for small wavelength excitations is present here.

**Healing length**

Let us have a look at the change in the healing length $\xi_0$ which is instrumental in demarcating the phonon length scales from the particle like excitations. The Bogoliubov spectrum is given by

$$\hbar^2 \omega^2 = \frac{\hbar^2 k^2 gn}{m} + \frac{\hbar^4 k^4}{4m^2}.$$  

(7)

At small $k$, this dispersion relation takes the form of a phonon dispersion relation $\hbar \omega = \rho c$, where $\rho = \hbar k$. The healing length indicates the point of transition from the phonon spectrum to the particle spectrum where one considers the kinetic and the potential energy balance as $\frac{\hbar^2}{2m} = \frac{\hbar^2}{2m} \simeq gn$. This relation gives a healing length $\xi_0 = \frac{\hbar}{\sqrt{2m}} = \frac{\hbar}{\sqrt{2m}} = \frac{1}{\sqrt{8\pi m}}$. The relationship from which the healing length is derived, being a balance between the kinetic and the potential energy of excitations, indicates that excitations with a smaller length scale than
the healing length are treated as particles (wave packets with mass). Phonon-like excitations typically have wavelengths larger than the healing length. In the modified model, to find the healing length, we have to consider a balance between the quadratic and the quartic terms in the wave number on the right-hand side of equation (6) considering

\[ p^2 c^2 = \left( \frac{1}{4m^2} - \frac{a^2 gn}{6\hbar^2} \right) p^4, \]

we get

\[ \xi = \xi_0 \left( \frac{1}{2} - \frac{a^2}{6\xi_0^2} \right)^{1/2}. \]

So, the healing length decreases with the increase in \( a \) which is a more rapid decrease for a constant \( n \) than the \( a^{-1/2} \) scale of decrease obtained from the conventional local GP model. In fact, whereas the healing length scaling as \( a^{-1/2} \) at a constant \( n \) becomes zero at an infinite \( a \), here it becomes zero at an \( a \) of the order of \( n^{-1/3} \). In figure 1, we compare the variation of healing length with \( a \) for local and non-local cases at various densities of the condensate. An equivalent way of looking at the present scenario is an increase of the effective mass of the particles with an enhancement of the scattering length. It is the Laplacian term appearing as the minimal correction with an opposite sign to the kinetic energy term that renormalizes the effective mass. Interesting to note is that, this increase of the effective mass of the particles is not affecting the expression of the velocity of the sound wave in the BEC which comes from the interactions itself. It is the \( \delta \)-function (local) interaction that fixes the velocity of sound just as in the local GP equation. Thus, the velocity of sound scales as \( \sqrt{a} \) at a constant density and this finite velocity of sound at a vanishing of the healing length is an interesting situation for an analogue system.

**BEC as an analogue system**

Considering a single particle state of the form \( \psi(r, t) = \sqrt{n(r, t)} \exp[i\theta(r, t)/\hbar] \), one can write the local GP equation in the form

\[ \frac{\partial n}{\partial t} + \nabla \cdot (nv) = 0 \]
where \( V = \frac{\hbar^2(\sqrt{n} + \frac{1}{2})^2}{m} \). If we consider the additional term that we have added to the local GP equation and linearize the above two equations for small amplitude density and phase perturbations as \( n \to n + \hat{n}_1 \) and \( \theta \to \theta + \hat{\theta}_1 \), we get the following two equations for \( \hat{n}_1 \) and \( \hat{\theta}_1 \):

\[
\frac{\partial \hat{n}_1}{\partial t} + \frac{1}{m} \nabla \cdot (\hat{n}_1 \nabla \theta + n \nabla \hat{\theta}_1) = 0, \\
\frac{\partial \hat{\theta}_1}{\partial t} + \frac{\nabla \theta \cdot \nabla \hat{\theta}_1}{m} + g\hat{n}_1 - \frac{\hbar^2}{2m} D_2 \hat{n}_1 = 0.
\]

The term \( D_2 \hat{n}_1 \) in 1D will have the form

\[
D_2 \hat{n}_1 = -\frac{\hat{n}_1}{2} n^{-3/2} \frac{\partial^2 n^{1/2}}{\partial x^2} + \frac{n^{-1/2} \partial^2 (n^{-1/2} \hat{n}_1)}{2} - \frac{gma^2 \partial^2 \hat{n}_1}{3\hbar^2 \partial x^2},
\]

where the last term on the rhs is coming from the correction term added to the local GP equation.

The dispersion relation in 1D corresponding to the modified GP equation in the present context would look like

\[
\omega = vk \pm \left[ \frac{gn}{m} k^2 + \left( \frac{\hbar^2}{4m^2} - \frac{a^2 gn}{6m} \right) k^2 \right]^{1/2}.
\]
context of analogue gravity, the present result indicates a range of tunability of the dispersion. This important fact was unnoticed in the context of the BEC so far. Thus, the generic large momentum divergence of the group velocity of wave packets, where undesirable, can be controlled to a good extent and excitations of small wave number can be accessed.

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