Divergence of a stationary random vector field can be always positive (a Weiss’ phenomenon)

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Abstract

The divergence of a stationary random vector field at a given point is usually a centered (that is, zero mean) random variable. Strangely enough, it can be equal to 1 almost surely. This fact is another form of a phenomenon disclosed by B. Weiss in 1997.

Introduction

If a random vector field is stationary (that is, shift-invariant in distribution), then the expectation of its divergence must vanish, since it is equal to the divergence of the expectation, thus, the divergence of a constant vector field. This simple argument is conclusive provided that the expectations are well-defined (especially, for Gaussian processes). Waiving existence of first moments we cannot ask about the expected divergence, but we still may ask, whether the divergence can be always (strictly) positive, or not. The answer appears to be negative in dimension one but affirmative in dimension two. The former is evident, while the latter is demonstrated by a non-evident counterexample constructed below following an idea of B. Weiss [1].

The phenomena under consideration manifest themselves equally well in two forms, discrete and continuous. Starting with the discrete setup we treat two lattices \( \mathbb{Z} \) and \( \mathbb{Z}^2 \) as two (infinite) graphs.

\[ \begin{array}{cccc}
-1 & 0 & 1 & 2 \\
\mathbb{Z} & & & \\
\end{array} \]

\[ \begin{array}{cccc}
& & & \\
& & & \\
& & & \\
\mathbb{Z}^2 & & & \\
& & & \\
\end{array} \]

\[ ^1 \text{This research was supported by THE ISRAEL SCIENCE FOUNDATION (grant No. 683/05).} \]
By a vector field we mean a real-valued function on oriented edges such that its sum over the two orientations of an edge vanishes. By the divergence of a vector field we mean the following real-valued function on vertices: given a vertex, we sum over the (two or four) outgoing edges the values of the vector field.

3 5 0 1
a vector field

−8 5 1
its divergence

For a vector field \( v = (v_{x,x+1})_{x \in \mathbb{Z}} \) on \( \mathbb{Z} \), its divergence is given by

\[
(\text{div } v)_x = v_{x,x+1} - v_{x-1,x} \quad \text{for } x \in \mathbb{Z}.
\]

If a random vector field \( v \) on \( \mathbb{Z} \) is stationary then the two random variables \( v_{-1,0} \) and \( v_{0,1} \) are identically distributed. If also \( (\text{div } v)_0 \geq 0 \) a.s., then \( v_{-1,0} \leq v_{0,1} \) a.s., therefore \( \mathbb{P}(v_{-1,0} \leq a < v_{0,1}) = \mathbb{P}(v_{-1,0} \leq a) - \mathbb{P}(v_{0,1} \leq a) = 0 \) for all \( a \in \mathbb{R} \). Letting \( a \) run over a dense countable set we get \( v_{-1,0} = v_{0,1} \) a.s. It means that

a stationary random vector field \( v \) on \( \mathbb{Z} \) cannot satisfy the condition \( \mathbb{P}((\text{div } v)_0 > 0) = 1 \).

In dimension two the situation is different.

**Theorem 1.** There exists a stationary random vector field \( v \) on \( \mathbb{Z}^2 \) whose divergence is equal to 1 everywhere, almost sure.

(For the proof see Sect. 1.) The corresponding result in the continuous setup follows by smoothing, namely, convolution with the indicator function of the square \((-0.5, 0.5) \times (-0.5, 0.5)\). In other words: if, say, \( v = 1 \) on the edge \(((0,0), (0,1))\) and \( v = 0 \) on other edges of \( \mathbb{Z}^2 \), then the first (horizontal) component of the smoothed vector field on \( \mathbb{R}^2 \) at \((x, y)\) is equal to \(1 - |x - 0.5|\) if \(|x - 0.5| < 1, |y| < 0.5\) (and 0 otherwise).
The divergence of the smoothed field is equal to +1 on the square \((-0.5, 0.5)\times(-0.5, 0.5)\) and -1 on \((0.5, 1.5)\times(-0.5, 0.5)\); just the smoothed \(\text{div } v\).

Having \(v\) such that \(\text{div } v = 1\) everywhere on \(\mathbb{Z}^2\) we get the smoothed divergence equal to 1 almost everywhere on \(\mathbb{R}^2\) (and in the distributional sense). Some additional smoothing gives a smooth vector field of divergence 1 everywhere.

\section{The construction and the proof}

In order to keep the matter as discrete as possible, from now on we consider only \(\text{integer-valued}\) vector fields on \(\mathbb{Z}^2\).

\textbf{Lemma 1.} If there exist stationary random vector fields \(v_1, v_2, \ldots\) such that
\begin{enumerate}[(a)]  
  \item \(\mathbb{P}((\text{div } v_n)_x = 1) \to 1\) as \(n \to \infty\), for every vertex \(x\) of the graph \(\mathbb{Z}^2\),
  \item \(\sup_n \mathbb{P}(|(v_n)_y| > C) \to 0\) as \(C \to \infty\), for every edge \(y\) of the graph \(\mathbb{Z}^2\),
\end{enumerate}
then there exists a stationary random vector field \(v\) such that
\begin{enumerate}[(A)]  
  \item \(\mathbb{P}((\text{div } v)_x = 1) = 1\) for every vertex \(x\) of the graph \(\mathbb{Z}^2\),
  \item \(\mathbb{P}(|v_y| > C) \leq \sup_n \mathbb{P}(|(v_n)_y| > C)\) for every \(C\) and every edge \(y\) of the graph \(\mathbb{Z}^2\).
\end{enumerate}

\textbf{Proof.} The distribution \(\mu_n\) of \(v_n\) is a probability measure on the space \(\mathcal{Z}^E\) of all maps \(E \to \mathbb{Z}\); here \(E\) is the (countable) set of all edges of \(\mathbb{Z}^2\). Using the one-point compactification \(\overline{\mathbb{Z}} = \mathbb{Z} \cup \{\infty\}\) of \(\mathbb{Z}\) we may treat \(\mu_n\) as measures on the compact metrizable space \(\overline{\mathbb{Z}}^E\). All probability measures on \(\overline{\mathbb{Z}}^E\) being a compact metrizable space, we take a convergent subsequence: \(\mu_{n_k} \to \mu\).

Let \(v\) be distributed \(\mu\), then \((B)\) is satisfied, and all values of \(v\) are finite a.s. due to \((b)\) and \((B)\). Clearly, \(v\) is stationary. Treating \((\text{div } v)_x\) as a function (of \(v\)) defined and continuous \(\mu\)-almost everywhere (namely, on \(\mathcal{Z}^E\)) we see that \((\text{div } v_{n_k})_x\) converges in distribution to \((\text{div } v)_x\) as \(k \to \infty\). Thus, \((a)\) implies \((A)\). \hfill \Box
Random vector fields $v_n$ will be constructed out of non-random finite fragments. For example, $n = 2$:

The fragment of size $3 \times 3$ (whose construction will be explained later) is repeated, forming a (double-) periodic vector field (non-random). Shifting this periodic field we get $3 \cdot 3 = 9$ periodic fields. These 9 vector fields are the possible values of the random vector field $v_2$; they have equal probabilities ($1/9$), by definition (of $v_2$). Thus, $v_2$ is stationary.

Note that the divergence of $v_2$ at a given point (say, the origin) takes on two values, $+1$ (with probability $8/9$) and $-8$ (with probability $1/9$). Also, the value of $v_2$ on a given horizontal edge takes on three values $-3, 0, 3$ with probabilities $1/9, 7/9, 1/9$ respectively.

The fragment of size $3 \times 3$, used above, is the second element of a sequence, whose $n$-th element is of size $(2^n - 1) \times (2^n - 1)$. The sequence is constructed recursively. Its first element, of size $1 \times 1$, is trivial: just a single vertex, no edges. The $(n + 1)$-th fragment contains four copies of the $n$-th fragment (two of them being turned upside down) connected as follows:
Here are the first three fragments:

For each fragment, the vector field conforms to the given oriented graph, and its divergence is equal to +1 at all vertices except for one vertex. Such vector field exists and is unique, since the graph is a tree; the divergence at the root is equal to \(-((2^n - 1)^2 - 1) = -(2^{2n} - 2^{n+1})\). Note the consistency: the four copies of the \(n\)-th fragment occur in the \((n+1)\)-th fragment as vector fields (not only graphs).

For each \(n\) we construct \(v_n\) out of the \(n\)-th fragment in the same way as we did it for \(n = 2\). Clearly, \(v_n\) is an integer-valued stationary random vector field on \(\mathbb{Z}^2\), and
\[ P\left( (\text{div } v_n)_x = 1 \right) = 1 - \frac{1}{(2^n - 1)^2}, \]

which verifies Condition (a) of Lemma 1. It remains to check Condition (b), which is the point of the next lemma.

**Lemma 2.** \( \sup_n P \left( |(v_n)_y| > C \right) = O(1/\sqrt{C}) \) as \( C \to \infty \), for every edge \( y \) of the graph \(\mathbb{Z}^2\).

**Proof.** First, the maximal possible value of \(|(v_n)_y|\) is equal to \(2^{2n} - 2^{n+1}\) (this is the value on the edge that enters the root of the tree). Second,
\[ P \left( |(v_{n+1})_y| \leq 2^{2n} - 2^{n+1} \right) \geq \frac{4 \cdot (2^n - 1)^2}{(2^{n+1} - 1)^2} \]
since the \((n + 1)\)-th fragment has \((2^{n+1} - 1)^2\) vertices, and out of these, \(4 \cdot (2^n - 1)^2\) vertices belong to the four copies of the \(n\)-th fragment. (For edges the ratio is even larger.) Similarly,
\[ P \left( |(v_{n+i})_y| \leq 2^{2n} - 2^{n+1} \right) \geq \frac{4^i \cdot (2^n - 1)^2}{(2^{n+i} - 1)^2} \]
for \( i = 1, 2, \ldots \). Therefore

\[
\mathbb{P}(|(v_{n+i})_y| \leq 2^{2n}) \geq \left(\frac{2^{n+i} - 2^i}{2^{n+i} - 1}\right)^2 \geq \left(1 - \frac{2^i - 1}{2^{n+i} - 1}\right)^2 \geq (1 - 2^{-n})^2 \geq 1 - 2 \cdot 2^{-n};
\]

\[
\sup_n \mathbb{P}(|(v_n)_y| > 2^{2k}) \leq 2 \cdot 2^{-k};
\]

the lemma follows.

\( \square \)

## 2 Remarks and questions

**Remark 1.** It is easy to see that the constructed sequence \((v_n)_n\) converges in distribution. No need to choose a subsequence (as in the proof of Lemma 1).

**Question 1.** Is the condition \( \mathbb{E} \sqrt{|v_y|} < \infty \) compatible with div \( v = 1 \)?

**Remark 2.** As shown by B. Weiss [1], the sample functions of a stationary complex-valued process on the complex plane can be non-constant entire functions.

### References

[1] B. Weiss (1997): *Measurable entire functions*, Annals of Numerical Mathematics 4, 599–605.

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