Conformal Dynamics Problem List

Edited by Ben Bielefeld

The following list of unsolved problems was given at the Conformal Dynamics Conference which was held at SUNY Stony Brook in November 1989. Problems were contributed by Ben Bielefeld, Adrien Douady, Curt McMullen, Jack Milnor, Misuhiro Shishikura, Folkert Tangerman, and Peter Veerman.

§1. Local connectivity of Julia sets

Let $f_\lambda(z) = \lambda z + z^2$ where $\lambda = \exp(2\pi i \theta)$. Call such a polynomial parabolic if $\theta$ is rational. If $\theta$ is irrational, and $f_\lambda$ is analytically conjugate to a rotation near 0 we say $f_\lambda$ is a Siegel polynomial. Otherwise we call $f_\lambda$ a Cremer polynomial. Douady and Sullivan [Sul] have shown that the Julia set of a Cremer polynomial is never locally connected. In the generic case, Douady (unpublished) has described specific examples of external rays which do not land, but rather have an entire continuum of limit points in the Julia set.

Question 1. Is there an arc joining 0 to $-\lambda$, in the Julia set of a Cremer polynomial? ($-\lambda$ is the preimage of the fixed point.)

Question 2. Give a plausible topological model for the Julia set of a Cremer polynomial.

Question 3. Make a good computer picture of the Julia set of some Cremer polynomial.

Question 4. Are there any rays landing at 0 for a Cremer polynomial?

Question 5. For which Cremer polynomials is the critical point accessible?

Question 6. If we remove the fixed point from the Julia set of a Cremer polynomial, how many connected components are there in the resulting set $J(f_\lambda) - \{0\}$, i.e., is the number of components countably infinite?

Question 7. Are there any Siegel polynomials whose Siegel disk has a boundary which is not a Jordan curve?

Let $P_c(z) = z^2 + c$ where $c$ ranges over values for which the Julia set of $P_c$ is connected.

Question 8. For which $c$ is the Julia set of $P_c$ locally connected? [Reportedly Yoccoz has recently proved local connectivity except at points on boundaries of hyperbolic components and infinitely renormalizable points.]

Question 9. Is the Julia set of $P_c$ locally connected when $c$ is real?
Question 10. If $c$ is the Feigenbaum point, is the Julia set of $P_c$ locally connected?

Question 11. If one could show that topological conjugacy of $P_{c_1}$ and $P_{c_2}$ implies that $P_{c_1}$ is quasiconformally conjugate to $P_{c_2}$, would this imply local connectivity of the quadratic connectedness locus (i.e., the Mandelbrot set, consisting of all $c$ for which the Julia set of $P_c$ is connected)?

Let $\tau = p/q$, and let $M_\tau$ be the limb of the connectedness locus with interior angle $p/q$.

Question 12. Is the diameter of $M_\tau$ less than $K/q^2$ for some constant $K$ independent of $\tau$? If not, is it at least less than $K \log(q)/q^2$? (Remark: the still unpublished Yoccoz inequality implies that the diameter is bounded by a constant over $q$. Compare [P].)

Question 13. If $P_c$ is nonrecurrent does this imply the existence of a conformal metric (a metric of the form $\rho(z)|dz|$ with integrable singularities) for which $P_c$ is expanding on its Julia set? [Yoccoz has proved local connectivity in the nonrecurrent case.]

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§2. Quasiconformal Surgery (Douady, Bielefeld, Shishikura)

It is possible to investigate rational functions using the technique of quasiconformal surgery as developed in [DH2], [BD] and [S]. There are various methods of gluing together polynomials via quasiconformal surgery to make new polynomials or rational functions. The idea of quasiconformal surgery is to cut and paste the dynamical spaces for two polynomials so as to end up with a branched map whose dynamics combines the dynamics of the two polynomials. One then tries to
find a conformal structure that is preserved under this branched map of the sphere to itself, so that using the Ahlfors-Bers theorem the map is conjugate to a rational function. There are several topological surgeries which experimentally seem to exist, but for which no one has yet been able to find a preserved complex structure.

The first such kind of topological surgery is *mating* of two monic polynomials with the same degree. (Compare [TL].) The first step is to think of each polynomial as a map on a closed disk by thinking of infinity as a circle worth of points, one point for each angular direction. The obvious extension of the polynomial at the circle at infinity is \( \theta \mapsto d\theta \) where \( d \) is the degree of the polynomial. Now glue two such polynomials together at the circles at infinity by mapping the \( \theta \) of the first polynomial to \(-\theta\) in the second. Finally, we must shrink each of the external rays for the two polynomials to a single point. The result should be conjugate to a rational map of degree \( d \). (Surprisingly this construction sometimes seems to make sense even when the filled Julia sets for both polynomials have vacuous interior.)

For instance we can take the rabbit to be the first polynomial, that is \( z^2 + c \) where the critical point is periodic of period 3 (\( c \sim -122561 + 744862i \)). The Julia set appears in the following picture.

![The rabbit](image)

Then for the second polynomial we could take the basilica, that is \( z^2 - 1 \) (it is named after the Basilica San Marco in Venice. One can see the basilica on top and its reflection in the water below). The Julia set for the basilica appears in the following figure.
Next we show the basilica inside-out \((z^3 - 1)/(z^2 - 1)\) which is what we will glue to the rabbit.

And finally we have the Julia set for the mating \((z^3 + c)/(z^2 + 1)\) where \(c = \frac{1 + \sqrt{-3}}{2}\).
**Question 1.** Which matings correspond to rational functions? There are some known obstructions. For example, Tan Lei has shown that matings between quadratic polynomials can exist only if they do not belong to complex conjugate limbs of the Mandelbrot set.

**Question 2.** Can matings be constructed with quasiconformal surgery?

**Question 3.** If one polynomial is held fixed and the other is varied continuously, does the resulting rational function vary continuously? Is mating a continuous function of two variables?

The second type of topological surgery is **tuning**. First take a polynomial $P_1$ with a periodic critical point $\omega$ of period $k$, and assume that no other critical points are in the entire basin of this superattractive cycle. Let $P_2$ be a polynomial with one critical point whose degree is the same as the degree of $\omega$. We also assume that the Julia sets of $P_1$ and $P_2$ are connected. We give two descriptions. For the first description we assume the closure $\overline{B}$ of the immediate basin of $\omega$ is homeomorphic to the closed unit disk $\overline{D}$, and that the Julia set for $P_2$ is locally connected. Now, $P_1^k$ maps $\overline{B}$ to itself by a map which is conjugate to the map $z \mapsto z^d$ of $\overline{D}$, where $d$ is the degree of the critical point. (In fact, if $d > 2$, then there are $d-1$ possible choices for the conjugating homeomorphism, and we must choose one of them.) Intuitively the idea is now the following. Replace the basin $B$ by a copy of the dynamical plane for $P_2$, gluing the “circle at infinity” for this plane onto the boundary of $B$ so that external angles for $P_2$ correspond to internal angles in $\overline{B}$. Now shrink each external ray for $P_2$ to a point. Also, make an analogous modification at each pre-image of $B$. The map from the modified $B$ to its image will be given by $P_2$, and the map on all other inverse images of the modified $B$ will be the identity. The result $P_3$, called $P_1$ tuned with $P_2$ at $\omega$, should be conjugate to a polynomial having the same degree as $P_1$. Conversely $P_2$ is said to be obtained from $P_3$ by renormalization.

In the case of quadratic polynomials, the tunings can be made also in the case when $P_2$ is not locally connected. Also it may be that the case where there is a critical point on the boundary of the basin is different.

As an example we can take $P_1$ to be the rabbit polynomial. Then we can take $P_2(z) = z^2 - 2$ which has the closed segment from -2 to 2 as its Julia set. The following figure shows the resulting quadratic Julia set tuning the rabbit with the segment $(z^2 + c$ where $c \sim -.101096 + .956287i)$.
In the picture we see each ear of the rabbit replaced with a segment.

**Question 4.** Does the tuning construction always give a result which is conjugate to a polynomial? This is true when $P_1$ and $P_2$ are quadratic.

**Question 5.** Can tunings be constructed with quasiconformal surgery?

**Question 6.** Does the resulting polynomial vary continuously with $P_2$? This is true when $P_1$ and $P_2$ are quadratic [DH2].

**Question 7.** Does the resulting tuning vary continuously with $P_1$? (here we consider only $P_1$ with a superstable orbit.)

**Question 8.** Given a sequence $P_{1,k}$ of tending to some limit, do the tunings of $P_{1,k}$ with $P_2$ tend to a limit which is independent of $P_2$?

The third kind of surgery is **intertwining surgery**.

Let $P_1$ be a monic polynomial with connected Julia set having a repelling fixed point $x_0$ which has ray landing on it with rotation number $p/q$. Look at a cycle of $q$ rays which are the forward images of the first. Cut along these rays and we get $q$ disjoint wedges. Now let $P_2$ be a monic polynomial with a ray of the same rotation number landing on a repelling periodic point of some period dividing $q$ (such as 1 or $q$). Slit this dynamical plane along the same rays making holes for the wedges. Fill the holes in by the corresponding wedges above making a new sphere. The new map will given by $P_1$ and $P_2$ except on a neighborhood of the inverse images of the cut rays where it will have to be adjusted to make it continuous. This construction should be possible to do quasiconformally using the methods in [BD] together with Shishikura’s new (unpublished) method of presurgery in the case where the rays in the $P_2$ space land at a repelling orbit. This construction doesn’t seem to work when the rays land at a parabolic orbit.

For instance we can take $P_1(z) = z^2$ and $P_2(z) = z^2 - 2$. The Julia set for $P_1$ is the unit circle with repelling fixed point at 1 and the ray at angle 0 lands on it with rotation number 0. The Julia set for $P_2$ is the
closed segment from -2 to 2 with repelling fixed point 2 and the ray at angle 0 lands on it with rotation number 0. We cut along the 0 ray in both cases. Opening the cut in the first dynamical space gives us one wedge. The space created by opening the cut in the second space is the hole into which we put the wedge. The resulting cubic Julia set is shown in the following picture (the polynomial is $z^3 + az$ where $a \sim 2.55799i$).

![Diagram of a circle intertwined with a segment](image1)

A circle intertwined with a segment

We see in the picture the circle and the segment, and at the inverse image of the fixed point on the segment we see another circle. At the other inverse of the fixed point on the circle we see a segment attached. All the other decorations come from taking various inverses of the main circle and segment.

As a second example we can intertwine the basilica with itself. The ray $1/3$ lands at a fixed point and has rotation number $1/2$. The following is the Julia set for the basilica intertwined with itself (the polynomial here is $z^3 - \frac{3}{4}z + \frac{\sqrt{17}}{4}$).

![Diagram of a basilica intertwined with itself](image2)

A basilica intertwined with itself

**Question 9.** When does an intertwining construction give something which is conjugate to a polynomial?
Question 10. Can intertwinings be constructed with quasiconformal surgery?

Question 11. Does the resulting polynomial vary continuously in $P_2$?

Here is a different kind of continuity question. Consider the space of all monic polynomials

$$z \mapsto z^n + a_{n-1}z^{n-1} + \cdots + a_1z$$

with $|a_1| \geq 1$, so that there is an unattractive fixed point at the origin. Here we do not require that the Julia set be connected. If at least one external ray lands at the origin, then there is a well-defined "rotation number" of these external rays under the map $\theta \mapsto n\theta \pmod{1}$.

Question 12. Does this rotation number extend uniquely to a continuous map from our space of polynomials to $\mathbb{R}/\mathbb{Z}$? (When $a_1 = e^{2\pi i \theta}$, this rotation number map must take the value $\theta$.)

References:

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§3. Thurston’s algorithm for real functions (Bielefeld, Tangerman, Veerman, Milnor)

Work in the space $P = P(n, \mu)$ of piecewise monotone maps $f$ of the interval $I = [0, 1]$ which have $n$ laps, and which map the boundary $\{0, 1\}$ into itself by some specified map $\mu$. Starting with any map $f_0$ in $P$, let
$p_0$ be the unique degree $n$ polynomial in $P$ which has all critical points in $[0,1]$ and which has the same critical values, encountered in the same order, as does $f_0$. Then there is a unique homeomorphism $h_0$ of $[0,1]$ which takes the critical points of $f_0$ to the critical points of $p_0$ and which satisfies $p_0 \circ h_0 = f_0$. It follows that the map $f_1 = h_0 \circ p_0 = h_0 \circ f_0 \circ h_0^{-1}$ is topologically conjugate to $f_0$. Now continue inductively, constructing maps $f_{k+1} = h_k \circ f_k \circ h_k^{-1}$.

\[
\begin{array}{c c c}
I & \overset{f_0}{\longrightarrow} & I \\
\uparrow h_0 & \nabla p_0 & \uparrow h_0 \\
I & \overset{f_1}{\longrightarrow} & I \\
\uparrow h_1 & \nabla f_1 & \uparrow h_1 \\
I & \overset{f_2}{\longrightarrow} & I \\
\end{array}
\]

If $f_0$ is post-critically finite, and if there is no Thurston obstruction, then it follows from Thurston’s argument that the resulting sequence of polynomials $p_k$ converges to a polynomial $p_\infty$ with the same kneading sequence as $f_0$. Computer experiments suggest that in many cases the sequence of maps $f_k$ converges to this same limit. Furthermore, this seems to be true even if $f_0$ is not post-critically finite.

**Question 1.** Formulate and prove some precise statement in this direction.

Now we can consider the same problem except that instead of lifting by polynomials we lift by some other family in $P$. For instance computer experiments suggest that we can choose the $p_k$’s from a family of the form $p(x) = -|x|^{\alpha} + c$ where $\alpha > 1$. (To be more precise, we must first change coordinates with an affine map so that the boundary $0,1$ maps to $0$. This yields the family of maps $x \mapsto k - k|2x - 1|^{\alpha}$ from the unit interval to itself where $0 < k \leq 1$.)

**Question 2.** Given $f_0$ with a preperiodic or periodic kneading sequence does Thurston’s algorithm converge for the specific family above. There is a proof only for $\alpha$ an even integer, which is the polynomial case.

**Question 3.** Give a general property for the lifting family which will guarantee convergence of Thurston’s algorithm for preperiodic or periodic kneading sequences.

**Question 4.** Give a general property for the lifting family which will guarantee convergence of Thurston’s algorithm for arbitrary kneading sequences.
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§4. Stable regions for complex Hénon type maps (Milnor)

Let $f$ be a polynomial diffeomorphism of $\mathbb{C}^2$ with Jacobian determinant $\delta$. Suppose that the set $K^+(f)$ of points with bounded forward orbit, has a non-empty interior. Let $U$ be some connected component of this interior. According to Montel, the set of iterates of $f$ restricted to $U$ possesses a convergent subsequence, which converges say to $g : U \to \mathbb{C}^2$. Evidently the rank $r$ of $g$ is zero or one if $|\delta| < 1$, and is two if $|\delta| = 1$.

**Question 1.** Can $U$ be a wandering component? If so, we must have $|\delta| < 1$. Can the rank of $g$ be either zero or one? Can $U$ be either bounded, or unbounded of finite volume, or of infinite volume?

If $U$ is not a wandering component, then it is strictly periodic under $f$, and, after replacing $f$ by some finite iterate, we may assume that $f(U) = U$. Note then that $f$ commutes with $g$. One useful family of examples is provided by the Hénon maps. Given any two non-zero complex numbers $\lambda$ and $\mu$, there is an essentially unique (quadratic) Hénon map $H_{\lambda,\mu}$ which has a fixed point with eigenvalues $\lambda$ and $\mu$.

**Rank Zero Case.** If $g(U) = x_0 \in U$, then $x_0$ is an attracting fixed point. Examples are provided by the Hénon maps $H_{\lambda,\mu}$ where the eigenvalues $\lambda$ and $\mu$ can be any two numbers in the punctured open disk $D - 0$. If $x_0 \in \partial U$, then it is conjectured that one eigenvalue must be equal to 1. Evidently the other eigenvalue must be in $D - 0$. Here $H_{1,\mu}$ provides an example.

**Rank One Case.** If $g$ has a fixed point in $U$, then $g$ must be a retraction of $U$ onto a Siegel disk. There are examples of the form $H_{\lambda,\mu}$ with $1 = |\lambda| > |\mu|$. (Compare Zehnder.) Can $g$ be a retraction onto a Herman ring (or onto a punctured Siegel disk)?

**Rank Two Case.** If $g$ has a fixed point, then $U$ is a “Siegel bi-disk”. There are examples of the form $H_{\lambda,\mu}$ with $|\lambda| = |\mu| = 1$. (Again see Zehnder.) Can $U$ also be the product of a Herman ring with a Herman ring, or the product of a Herman ring with a Siegel disk?
§5. Geometrically finite maps and Kleinian groups (McMullen)

Geometrically finite rational maps

Let $f(z)$ be a rational map, $C$ its set of critical points, $P = \cup_{i=1}^{\infty} f^n(C)$ its post-critical set and $J$ its Julia set. The map $f$ is \textit{expanding} if $P \cap J = \emptyset$. It is well-known that $f$ is expanding iff some fixed iterate of $f$ uniformly expands the spherical metric on the Julia set; these maps are also called \textit{hyperbolic} or \textit{Axiom A}.

Let the space $\text{Rat}_d$ (respectively $\text{Poly}_d$) of rational (polynomial) maps of degree $d$ be equipped with the topology of uniform convergence. A well-known and fundamental problem is to resolve the following:

\textbf{Conjecture.} The expanding maps form a dense subset of $\text{Rat}_d$ and $\text{Poly}_d$.

Cf. \cite{MSS} where this is related to the problem of invariant measurable line fields supported on the Julia set. (It is known that the set of expanding maps is open). This is not even known in the case of quadratic polynomials. The corresponding problems for real maps are also open.

In many ways an expanding rational map is well-behaved (cf. \cite{Sul}); it is like a Kleinian group with compact convex core in $\mathbb{H}^3$.

More generally, let us say a rational map is \textit{geometrically finite} if $P \cap J$ is a finite set. Equivalently, every critical point in the Julia set is preperiodic. (In this case rationally indifferent cycles are allowed). These maps should be compared to geometrically finite Kleinian groups.

For a geometrically finite rational map $f$:

\textbf{Problem 1.} Show that either the Julia set $J$ is the whole sphere and the action of $f$ on $J$ is ergodic, or the Hausdorff dimension $\delta$ of $J$ is less than 2. In the latter case, what can be said about the $\delta$-dimensional measure of $J$ and the dynamics with respect to this measure class?

\textbf{Problem 2.} Show every component of $J$ is locally connected.

\textbf{Problem 3.} Develop for $f$ an analogue of the Haken decomposition for 3-manifolds. For example, if $J$ is disconnected, can $f$ be constructed by surgery from rational maps with connected Julia sets?
Problem 4. Extend Thurston’s combinatorial theory of critical finite rational maps (those for which \(|P| < \infty\)) to all geometrically finite maps. That is, describe \(f\) up to combinatorial equivalence rel \(\overline{P}\) by a finite amount of topological data, and characterize those combinatorial types which arise as rational maps.

**Convex hyperbolic 3-manifolds**

Let \(N\) be a complete hyperbolic 3-manifold presented as the quotient of \(H^3\) by the action of a Kleinian group \(\Gamma\). The *convex core* of \(N\) is the quotient of the convex hull of the limit set. All closed geodesics in \(N\) are contained in the convex core.

**Question.** Suppose \(\pi_1(N)\) is generated by \(n\) elements. Is there an upper bound \(R_n\) to the radius of an embedded ball entirely contained in the convex core of \(N\)? (Here \(R_n\) should depend only on \(n\)).

The question has a positive answer when \(N\) is a quasifuchsian group. By results of Thurston [Th Ch. 13] there is a pleated surface near every point in the convex core, and this provides an upper bound on the injectivity radius.

The question also has an (easy) positive solution for hyperbolic 2-manifolds, and we know of no counterexample for hyperbolic manifolds of any dimension.

**Critically finite rational maps on \(\mathbb{P}^n\)**

A basic tool aiding the study of critically finite rational maps on the Riemann sphere is the Poincaré metric on the complement of the post-critical set \(P\) (assuming \(|P| > 2\)). This metric is expanded by \(f\). One should be able to apply the same sort of arguments to critically finite rational maps \(f : \mathbb{P}^n \to \mathbb{P}^n, n > 1\), such that the complement of the post-critical set is Kobayashi hyperbolic.

More precisely, say \(f\) is *critical finite* if there exist (possibly reducible) hypersurfaces \(V \subset W \subset \mathbb{P}^n\) such that

\[
f : (\mathbb{P}^n \setminus W) \to (\mathbb{P}^n \setminus V)
\]

is a covering map.

**Problem.** Are there nontrivial examples of critically finite maps with \(\mathbb{P}^n \setminus V\) Kobayashi hyperbolic? How do they behave dynamically?

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