**Witt vectors, semirings, and total positivity**

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**Abstract.** We extend the big and \( p \)-typical Witt vector functors from commutative rings to commutative semirings. In the case of the big Witt vectors, this is a repackaging of some standard facts about monomial and Schur positivity in the combinatorics of symmetric functions. In the \( p \)-typical case, it uses positivity with respect to an apparently new basis of the \( p \)-typical symmetric functions. We also give explicit descriptions of the big Witt vectors of the natural numbers and of the nonnegative reals, the second of which is a restatement of Edrei’s theorem on totally positive power series. Finally we give some negative results on the relationship between truncated Witt vectors and \( k \)-Schur positivity and give nine open questions.

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Introduction

Witt vector functors are certain functors from the category of rings (always commutative) to itself. They come in different flavors, but each of the ones we will consider sends a ring $A$ to a product $A \times A \times \cdots$ with certain addition and multiplication laws of an exotic arithmetic nature. For example, for each prime $p$, there is the $p$-typical Witt vector functor $\mathcal{W}(p,n)$ of length $n \in \mathbb{N} \cup \{\infty\}$. As sets, we have $\mathcal{W}(p,n)(A) = A^{n+1}$. When $n = 1$, the ring operations are defined as follows:

\[
(a_0, a_1) + (b_0, b_1) = (a_0 + b_0, a_1 + b_1 - \sum_{i=1}^{p-1}\frac{1}{p}p^i a_i b_{i-1})
\]

\[
(a_0, a_1)(b_0, b_1) = (a_0 b_0, a_0 b_1 + a_1 b_0 + p a_1 b_1)
\]

$0 = (0, 0)$

$1 = (1, 0)$.

For $n \geq 2$, the formulas in terms of coordinates are too complicated to be worth writing down. Instead we will give some simple examples:

\[
\mathcal{W}(p,n)(\mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}/p^{n+1}\mathbb{Z},
\]

\[
\mathcal{W}(p,n)(\mathbb{Z}) \cong \{ (x_0, \ldots, x_n) \in \mathbb{Z}^{n+1} \mid x_i \equiv x_{i+1} \mod p^{i+1} \}.
\]

In the second example, the ring operations are performed componentwise; in particular, the coordinates are not the same as the coordinates above. For another example, if $A$ is a $\mathbb{Z}[1/p]$-algebra, then $\mathcal{W}(p,n)(A)$ is isomorphic after a change of coordinates to the usual product ring $A^{n+1}$. This phenomenon holds for other kinds of Witt vectors—when all relevant primes are invertible in $A$, then the Witt vector ring of $A$ splits up as a product ring.

The $p$-typical Witt vector functors were discovered by Witt in 1937 [45] and have since become a central construction in $p$-adic number theory, especially in $p$-adic Hodge theory, such as in the construction of Fontaine’s period rings [45] and, via the de Rham–Witt complex of Bloch–Deligne–Illusie, in crystalline cohomology [19]. Also notable, and related, is their role in the algebraic K-theory of $p$-adic rings, following Hesselholt–Madsen [18].

There is also the big Witt vector functor. For most of this chapter, we will think about it from the point of view of symmetric functions, but for the moment, what is important is that it is formed by composing all the $p$-typical functors in an inverse limit:

\[
\mathcal{W}(A) = \lim_n \mathcal{W}(p_1,\infty)(\cdots(\mathcal{W}(p_n,\infty)(A))\cdots),
\]

where $p_1, \ldots, p_n$ are the first $n$ primes, and the transition maps are given by projections $\mathcal{W}(p,\infty)(A) \to A$ onto the first component. (A non-obvious fact is that the $p$-typical functors commute with each other, up to canonical isomorphism: so in fact the ordering of the primes here is unimportant.) This has an adelic flavor, and indeed it is possible to unify the crystalline cohomologies for all primes
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using a (“relative”) de Rham–Witt complex for the big Witt vectors. However the
cohomology of this complex is determined by the crystalline cohomologies and the
comparison maps to algebraic de Rham cohomology; so the big de Rham–Witt
cohomology does not, on its own, provide any new information.

But because of this adelic flavor, it is natural to ask whether the infinite prime
plays a role. The answer is yes, and the purpose of this chapter is to explore how it
does, at least as related to positivity, which we will regard as p-adic integrality for
the infinite prime. We will have nothing to say about other aspects of the infinite
prime, such as archimedean norms or a Frobenius operator at \( p = \infty \).

To explain this in more detail, we need to recall some basics of the theory of
symmetric functions. The big Witt vector functor is represented, as a set-valued
functor, by the free polynomial ring \( \Lambda_Z = \mathbb{Z}[h_1, h_2, \ldots] \). So as sets, we have

\[
W(A) = \text{Hom}_{\text{Alg}}(\Lambda_Z, A) = A \times A \times \cdots
\]

If we think of \( \Lambda_Z \) as the usual ring of symmetric functions in infinitely many formal
variables \( x_1, x_2, \ldots \) by writing

\[
h_n = \sum_{i_1 \leq \cdots \leq i_n} x_{i_1} \cdots x_{i_n},
\]

then the ring operations on \( W(A) \) are determined by two well-known coproducts
in the theory of symmetric functions. The addition law is determined by the
coproduct \( \Delta^+ \) on \( \Lambda_Z \) for which the power-sum symmetric functions \( \psi_n = \sum_i x_i^n \)
are primitive,

\[
\Delta^+(\psi_n) = \psi_n \otimes 1 + 1 \otimes \psi_n,
\]

and the multiplication law is determined by the coproduct for which the power
sums are group-like,

\[
\Delta^x(\psi_n) = (\psi_n \otimes 1)(1 \otimes \psi_n) = \psi_n \otimes \psi_n.
\]

This is also a fruitful point of view for the \( p \)-typical functors: \( W_{(p)} = \mathbb{Z}[\theta_1, \theta_p, \theta_{p^2}, \ldots] \) of \( p \)-typical symmetric functions, where the \( \theta_n \in \Lambda_Z \) are the Witt symmetric functions, which are defined recursively by the relations

\[
\psi_n = \sum_{d|n} d \theta_{n/d}^{n/d}.
\]

The ring operations on the \( p \)-typical Witt functors are equivalent to coproducts on
\( \Lambda_{Z,(p)} \), and these are compatible with the two coproducts on \( \Lambda_Z \). In fact, \( \Lambda_Z \) can
be reconstructed from all the \( \Lambda_{Z,(p)} \) as a kind of Euler product:

\[
\Lambda_Z = \Lambda_{Z,(2)} \odot \Lambda_{Z,(3)} \odot \Lambda_{Z,(5)} \odot \cdots, \tag{0.0.2}
\]

where \( \odot \) is the operation that on representing objects corresponds to composition on functors. This is in fact just another expression of formula (0.0.1), or alternatively of Wilkerson’s theorem [44].
Now, the relation of all this to the infinite prime is that there is a well-known positivity structure on $\Lambda_\mathbb{Z}$. This is the subset $\Lambda_\mathbb{N}$ consisting of symmetric functions that have nonnegative coefficients when viewed as series in the formal variables $x_1, x_2, \ldots$. It is closed under addition and multiplication and contains 0 and 1; so it is a semiring, or more plainly, an $\mathbb{N}$-algebra. It is also well known that the coproducts $\Delta^+$ and $\Delta^x$ above are induced by coproducts on $\Lambda_\mathbb{N}$, and so one might hope to use them to extend the big Witt construction to all $\mathbb{N}$-algebra, and hence to incorporate some positivity information in the usual theory of Witt vectors. This is indeed possible and the primary purpose of this chapter is to write it all down in some detail.

In fact, there is another such model over $\mathbb{N}$, which is also well known. It is the sub-$\mathbb{N}$-algebra $\Lambda^{\text{Sch}}$ consisting of symmetric functions which have nonnegative coordinates in the basis of Schur symmetric functions.

**Theorem A.** The functors $\text{Hom}_{\text{Alg}_{\mathbb{N}}}(\Lambda_\mathbb{N}, -)$ and $\text{Hom}_{\text{Alg}_{\mathbb{N}}}(\Lambda^{\text{Sch}}, -)$ extend the big Witt vector functor $W$ from $\mathbb{Z}$-algebras to $\mathbb{N}$-algebras. Each has a unique comonad structure compatible with that on $W$.

In terms of actual mathematical content, this is just a repackaging of some standard positivity facts in the theory of symmetric functions. Thus a large part of this chapter is expository. Its goal is to convince the reader that there is interesting mathematics where Witt vectors, combinatorial positivity, and semiring theory meet. To this end, I have included a number of open questions, which I hope will serve to focus readers’ minds and stimulate their interest. Most of the questions are precise, of the yes/no variety, and some are no doubt well within reach.

To give an example of something from this territory, I will report one new observation, which is that there is also a positive model for the $p$-typical Witt vector functors.

**Theorem B.** There is a representable comonad on the category of $\mathbb{N}$-algebras which agrees with the $p$-typical Witt vector functor $W_{(p), \infty}$ on $\mathbb{Z}$-algebras.

As with theorem A, the representing object is given by a positivity condition with respect to a $\mathbb{Z}$-basis, in this case of $\Lambda_{\mathbb{Z}, (p)}$. Write $d_p = -\theta_p = (\psi_1^p - \psi_p^p)/p$. Consider the (finite) monomials of the form

$$
\prod_{i,j \geq 0} (\psi_i^p \circ d_p^j)^{m_{ij}},
$$

where $\circ$ denotes the plethysm operation on $\Lambda_{\mathbb{Z}}$, and where $m_{ij} < p$. Then this family of monomials is a $\mathbb{Z}$-basis for $\Lambda_{\mathbb{Z}, (p)}$. Its $\mathbb{N}$-linear span $\Lambda_{\mathbb{N}, (p)}$ is a sub-$\mathbb{N}$-algebra of $\Lambda_{\mathbb{Z}}$, and the functor on $\mathbb{N}$-algebras it represents admits a unique comonad structure compatible with that of $W_{(p), \infty}$. To my knowledge, this basis of $\Lambda_{\mathbb{Z}, (p)}$ has not been considered before.

In one way, the theory around theorem B is more satisfactory than that around theorem A. This is that it also works for the $p$-typical Witt vectors of finite length. I initially hoped that bases of $k$-Schur functions of Lapointe–Lascoux–Morse (see the book [21]) would allow us to define $\mathbb{N}$-algebras of big Witt vectors of finite length.
length, but this turned out to be false. See section \[10\] for some details on this negative result.

There is also a larger purpose to this chapter, which is to show that the formalism of (commutative) semirings—and more broadly, scheme theory over \(\mathbb{N}\)—is a natural and well-behaved formalism, both in general and in its applications to Witt vectors and positivity. It has gotten almost no attention from people working with scheme theory over \(\mathbb{Z}\), but it deserves to be developed seriously—and independently of any applications, which are inevitable in my view. Let me conclude with some words on this.

Arithmetic algebraic geometry can be regarded as the study of systems of polynomials equations over \(\mathbb{Z}\). Such a system is equivalent to a presentation of a \(\mathbb{Z}\)-algebra; so one could say that arithmetic algebraic geometry is the study of the category of \(\mathbb{Z}\)-algebras. Of course, arithmetic algebraic geometers study many other objects, such as nonaffine schemes over \(\mathbb{Z}\), line bundles over them, and so on, but let us consider these as objects of derived interest, as tools for understanding polynomial equations. More to the point, they are formally inevitable once we allow ourselves the category of \(\mathbb{Z}\)-algebras and some general category theory.

Let me recall how this works for algebraic spaces. The category of affine schemes is defined to be the opposite of the category of rings. It has a Grothendieck topology on it, the fppf topology, where covers are given by fppf algebras, those that are faithfully flat and finitely presentable. The category of all algebraic spaces over \(\mathbb{Z}\) (a slight enlargement of the category of schemes) is what one gets by taking the closure of the category of affine schemes under adjoining arbitrary coproducts and quotients by fppf equivalence relations.

This is a completely formal process. (For instance, see Toën–Vaqué \[12\], \[13\].) Given a category \(C\) that looks enough like the opposite of the category of rings and a well-behaved class of equivalence relations, we can produce an algebraic geometry from \(C\) by gluing together objects of \(C\) using the given equivalence relations. In particular, we can do this with the category of \(\mathbb{N}\)-algebras and produce a category that could be called the category of schemes over \(\mathbb{N}\). This brings positivity into algebraic geometry at a very basic level. In arithmetic algebraic geometry today, positivity is typically treated in an ad hoc manner (for example, in global class field theory), much it seems as integrality was before the arrival of scheme theory in the 1950s.

On the other hand, as it appears to me, most people working on semiring theory follow a tradition close to general algebra or even computer science. Scheme theory has had little influence. As someone raised in that tradition, I find this unacceptable. The category of rings is the same as the category of semirings equipped with a map from \(\mathbb{Z}\). In other words, one might say that arithmetic algebraic geometry is nothing more than semiring theory over \(\mathbb{Z}\). One would therefore expect an active interest in finding models over \(\mathbb{N}\) (or the nonnegative reals, or \ldots) of as many of the objects of usual algebraic geometry over \(\mathbb{Z}\) as possible, just as one always tries to find models for objects of arithmetic algebraic geometry (such as moduli spaces) over \(\mathbb{Z}\). Yet such an effort seems to be nearly nonexistent. Perhaps one reason for this is that most existing expositions of scheme theory begin by considering spectra
of prime ideals, and it is less clear how to mimic this approach over \( \mathbb{N} \). Or perhaps people are more interested in designing foundations for specific applications, such as tropical algebraic geometry, rather than developing general tools. Whatever the case, it is important to get beyond this.

So in the first two sections, I give a category-theoretic account of the very basics of semiring theory and algebraic geometry over \( \mathbb{N} \). It is largely expository. I hope it will demonstrate to people who are skeptical that the basic constructions of scheme theory extend to \( \mathbb{N} \), and demonstrate to semiring theorists a point of view on their subject that emphasizes macroscopic ideas, such as flatness, base change, and descent, more than what is common. So at least the general formalism over \( \mathbb{N} \) can be brought up closer to that over \( \mathbb{Z} \).

If arithmetic algebraic geometry provides the motivation here and semiring theory provides the formalism, then algebraic combinatorics provides us with the positivity results. These are needed to define Witt vectors of semirings that do not contain \(-1\); they could be viewed as the analogues at the infinite prime of the slightly subtle \( p \)-adic congruences needed to define the \( p \)-typical Witt vectors for rings that do not contain \( 1/p \). But I also hope that combinatorialists will find something fresh in our emphasis on Witt vectors rather than symmetric functions. While the two are equivalent, they often suggest different questions. For instance, the coproduct \( \Delta^\times \) on \( \Lambda_{\mathbb{Z}} \) has gotten much less attention than \( \Delta^+ \). But these are just the co-operations that induce the multiplication and addition operations on Witt vectors. Although it is not without interest to view Witt vectors only as abelian groups, the real richness of the theory and their role in arithmetic algebraic geometry comes when we remember their full ring structure (or even better, their \( \Lambda \)-ring structure). So to a specialist in Witt vectors, ignoring \( \Delta^\times \) might feel like missing the whole point. Also, aspects of symmetric functions related to the finite primes seem under-studied in the algebraic combinatorics community. For instance, the ring \( \Lambda_{\mathbb{Z}}(p) = \mathbb{Z}[\theta_1, \theta_p, \ldots] \) of \( p \)-typical symmetric functions is, it appears to me, nearly unknown there. (The symmetric function \( \theta_p \) does appear in Macdonald \cite{Mac} as \( \varphi_p \) on p. 120.)

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Conventions

The word \textit{positive} will mean \( > 0 \), and \textit{negative} will mean \( < 0 \). For any subset \( A \) of the field \( \mathbb{R} \) of real numbers, we will write

\[
A_+ := \{ x \in A \mid x \geq 0 \}.
\]

The set \( \mathbb{N} \) of natural numbers is \( \{0, 1, 2, \ldots\} \). The ring \( \mathbb{Z}_p \) of \( p \)-adic integers is \( \lim_n \mathbb{Z}/p^n\mathbb{Z} \), and the field \( \mathbb{Q}_p \) of \( p \)-adic numbers is \( \mathbb{Z}_p[1/p] \).
For any category $C$, we will write $C(X,Y) = \text{Hom}_C(X,Y)$ for short.

1. Commutative algebra over $\mathbb{N}$, the general theory

The primary purpose of this section and the next one is to collect the definitions and formal results of commutative algebra and scheme theory over $\mathbb{N}$ that we will need. The reader is encouraged to skip them at first and refer back only when necessary.

A general reference to the commutative algebra is Golan’s book [16]. While everything here is essentially the same, there are some small differences. For instance, I have preferred to drop the prefix semi wherever possible and do not want to assume $0 \neq 1$. I have also followed the categorical method and used its terminology, because it gives the development a feeling of inevitability that I think is absent from the more element-centric approaches.

1.1. The category of $\mathbb{N}$-modules. The category $\text{Mod}_\mathbb{N}$ of $\mathbb{N}$-modules is by definition the category of commutative monoids, which we typically write additively. Thus an $\mathbb{N}$-module is a set $M$ with a commutative binary operation $+$ and an identity element $0 = 0_M$, and a homomorphism $M \to P$ is function $f: M \to P$ such that $f(0_M) = 0_P$ and $f(x + y) = f(x) + f(y)$ for all $x, y \in M$. As usual, the identity element is unique when it exists; so we will often leave to the reader the task of specifying it.

For example, $\mathbb{N}$ itself is an $\mathbb{N}$-module under usual addition. It represents the identity functor on $\text{Mod}_\mathbb{N}$ in an evident (and unique) way.

1.2. Submodules and monomorphisms. A subset $P$ of an $\mathbb{N}$-module $M$ is said to be a sub-$\mathbb{N}$-module if it admits an $\mathbb{N}$-module structure making the inclusion $P \to M$ a homomorphism. A map of $\mathbb{N}$-modules is injective if and only if it is a monomorphism. So we will often identify monomorphisms and submodules.

The dual statement is false—there are nonsurjective epimorphisms, for instance the usual inclusion $\mathbb{N} \to \mathbb{Z}$.

1.3. Products, coproducts. The category has all products: $\prod_{i \in I} M_i$ is the usual product set with identity $(\ldots, 0, \ldots)$ and componentwise addition

$$(\ldots, m_i, \ldots) + (\ldots, m'_i, \ldots) := (\ldots, m_i + m'_i, \ldots).$$

It also has all coproducts: $\bigoplus_{i \in I} M_i$ is the sub-$\mathbb{N}$-module of $\prod_{i \in I} M_i$ consisting of the vectors $(m_i)_{i \in I}$ such that $m_i = 0$ for all but finitely many $i \in I$.

In particular, we can construct free objects: given any set $S$ and any set map $f: S \to M$, the morphism $\bigoplus_{s \in S} \mathbb{N} \to M$ defined by $(n_s)_{s \in S} \mapsto \sum_{s \in S} n_s f(s)$ is the unique extension of $f$ to an $\mathbb{N}$-module map $\bigoplus_{s \in S} \mathbb{N} \to M$. 
1.4. Internal equivalence relations, quotients, and epimorphisms. A subset $M \subseteq M \times M$ is said to be a $\text{Mod}_N$-equivalence relation if it is both an equivalence relation on $M$ and a sub-$N$-module of $M \times M$.

Given any surjective homomorphism $f : M \to M'$ of $N$-modules, the induced equivalence relation $M \times_M M'$ is clearly a $\text{Mod}_N$-equivalence relation. Conversely, given any $\text{Mod}_N$-equivalence relation on $M$, the set $M/R$ of equivalence classes has a unique $N$-module structure such that the projection $M \to M/R$ sending $x$ to the equivalence class $[x]$ of $x$ is a homomorphism of $N$-modules. In other words, the rules $[x] + [y] = [x + y]$ and $0 = [0]$ give a well-defined $N$-modules structure on $M/R$.

1.5. Generators and relations. Let $R$ be a subset of $M \times M$. The $\text{Mod}_N$-equivalence relation $E$ generated by $R$ is the minimal $\text{Mod}_N$-equivalence relation on $M$ containing $R$. It exists because any intersection of submodules is a submodule and any intersection of equivalence relations is an equivalence relation. It can be constructed explicitly by taking the transitive closure of the (linear) symmetric closure of the reflexive closure of the sub-$N$-module $M \times M$ generated by $R$. Note the contrast with the category of modules over a ring, where taking the transitive closure is unnecessary. This gives the theory a dynamical feel that is absent from usual module theory.

We can construct $N$-modules in terms of generators and relations by combining this with free construction above. Clearly, much of this works in much more general categories, especially categories of algebras, as defined below. We will use such generalizations without comment. In the present case and others, we will often write $M/(\ldots, m_i = n_i, \ldots)$ for $M/E$ if $R = \{\ldots, (m_i, n_i), \ldots\}$.

1.6. Hom and $\otimes$. The set $\text{Hom}(M, P) = \text{Mod}_N(M, P)$ of $N$-module homomorphisms $M \to P$ is itself an $N$-module under pointwise addition:

$$ (f + g)(x) := f(x) + g(x), \quad 0(x) := 0. \quad (1.6.1) $$

We will also use the notation $\text{Hom}_N(M, P)$. It is functorial in $M$ (contravariant) and $P$ (covariant). For any fixed $N$-module $M$, the functor $\text{Hom}(M, -)$ has a left adjoint, which we write $M \otimes -$, or $M \otimes_N -$ for clarity. In other words, $M \otimes M'$ is characterized by the property that a homomorphism $M \otimes M' \to P$ is the same as set map $\langle-,-\rangle : M \times M' \to P$ which is $N$-bilinear in that it is an $N$-module map in each argument if the other argument is fixed. It follows that if we denote the image of an element $(m, m')$ under the universal bilinear map $M \times M' \to M \otimes M'$ by $m \otimes m'$, then $M \otimes M'$ is the commutative monoid generated by symbols $m \otimes m'$, for all $m \in M$ and $m' \in M'$, modulo all relations of the form

$$ (m_1 + m_2) \otimes m' = m_1 \otimes m' + m_2 \otimes m', \quad 0 \otimes m' = 0, $$
$$ m \otimes (m_1' + m_2') = m \otimes m_1' + m \otimes m_2', \quad m \otimes 0 = m \otimes 0 = 0. $$

It is clear that $\otimes$ makes $\text{Mod}_N$ into a symmetric monoidal category with identity $N$. 

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1.7. **N-algebras.** An N-algebra (soon to be understood to be commutative) is defined to be a monoid in Mod\_N with respect to the monoidal structure \( \otimes \). Thus an N-algebra is a set \( A \) with two associative binary operations \(+, \times\) and respective identity elements 0, 1 such that + is commutative and \( \times \) distributes over + and satisfies \( 0 \times x = x \times 0 = 0 \). We usually write \( xy \) for \( x \times y \). We will sometimes use the term semiring as a synonym for N-algebra.

A morphism \( A \to B \) of N-algebras is morphism of monoids in the monoidal category Mod\_N. In other words, it is a function \( f: A \to B \) which satisfies the identities

\[
\begin{align*}
  f(0) &= 0, \\
  f(x + y) &= f(x) + f(y), \\
  f(1) &= 1, \\
  f(xy) &= f(x)f(y).
\end{align*}
\]

The category formed by N-algebras and their morphisms is denoted Alg\_N.

For example, N-module N admits a unique N-algebra structure; multiplication is usual integer multiplication. It is the initial object in Alg\_N. Likewise, 0 with its unique N-algebra structure is the terminal object. For any subring \( A \subseteq \mathbb{R} \), the subset \( A_+ := \{ x \in A \mid x \geq 0 \} \) is a sub-N-algebra of \( A \).

The category of rings is equivalent in an evident way to the full subcategory of Alg\_N spanned by objects in which 1 has an additive inverse.

1.8. **Commutativity assumption.** From now on in this paper, all N-algebras will be understood to be commutative under \( \times \), although for much of the rest of this section, this is just for convenience.

Also for the rest of this section, \( A \) will denote an N-algebra.

1.9. **A-modules and A-algebras.** One defines A-modules and A-algebras in the obvious way. An A-module (also called an A-semimodule in the literature) is an N-module equipped with an action of \( A \) with respect to the monoidal structure \( \otimes \). So it is an N-module \( M \) equipped with an N-module map \( A \otimes M \to M \), written \( a \otimes m \mapsto am \) such that the following identities are satisfied:

\[
1 m = m, \quad (ab)m = a(bm).
\]

A morphism of A-modules \( M \to P \) is an N-linear map \( f: M \to P \) satisfying the identity \( f(am) = af(m) \). The category of A-modules is denoted Mod\_A. We will sometimes write \( \text{Hom}_A(M, P) = \text{Mod}_A(M, P) \) for the set of A-module morphisms. Observe when \( A = \mathbb{N} \), the category just defined agrees with that defined in (1.1).

An A-algebra is an N-algebra \( B \) equipped with a morphism \( i_B: A \to B \). A morphism \( B \to C \) of A-algebras is a morphism \( f: B \to C \) of N-algebras such that \( f \circ i_B = i_C \). The category of A-algebras is denoted Alg\_A. As with modules, when \( A = \mathbb{N} \), the category of N-algebras as defined here agrees with that defined in (1.7).

Also observe that if \( A \) is a ring, then these definitions of A-module and A-algebra agree with the usual ones in commutative algebra. In particular, a Z-module is the same as an abelian group, and a Z-algebra is the same as a ring.
1.10. **Hom** and \( \otimes \). The set \( \text{Hom}_A(M, P) \) has a natural \( A \)-module structure given by pointwise operations. In other words, it is a sub-\( \mathbb{N} \)-module of \( \text{Hom}_\mathbb{N}(M, P) \), and its \( A \)-module structure is given by the identity \((af)(x) = af(x)\). Of course, this uses the commutativity of multiplication on \( A \).

For any fixed \( A \)-module \( M \), the functor \( \text{Hom}_A(M, -) \) has a left adjoint, which we write \( M \otimes_A - \):

\[
\text{Hom}_A(M \otimes_A M', N) = \text{Hom}_A(M', \text{Hom}_A(M, N)).
\]

(Again, when \( A = \mathbb{N} \) this agrees with the functor \( M \otimes - \) defined above.) As above, an \( A \)-linear map \( M \otimes_A M' \to P \) is the same a set map \( \langle -, - \rangle : M \times M' \to P \) which is \( A \)-bi-linear in the sense that it is an \( A \)-module map in each argument when the other argument is held fixed. Thus \( M \otimes_A M' \) equals the quotient of \( M \otimes \mathbb{N} M' \) by all relations of the form

\[
(am) \otimes m' = m \otimes (am'),
\]

with its \( A \)-module structure given by \( a(m \otimes m') = am \otimes m' = m \otimes am' \).

1.11. **Limits and colimits of \( A \)-modules and \( A \)-algebras.** The category \( \text{Mod}_A \) has all limits and colimits. Limits, coproducts, and filtered colimits can be constructed as when \( A \) is a ring. Coequalizers might be less familiar. The coequalizer of a pair of maps \( f, g : M \to P \) in \( \text{Mod}_A \) is the quotient of \( P \) by the \( \text{Mod}_A \)-equivalence relation generated by the subset \( \{(f(x), g(x)) \mid x \in M\} \subseteq P \times P \).

1.12. **Warning: kernels and cokernels.** There are reasonable notions of kernel and cokernel, but we will not need them. The kernel of a map \( f : M \to N \) of \( A \)-modules is defined to be the pull-back \( M \times_N \{0\} \to M \), and the cokernel is the push-out \( N \to N \oplus_M \{0\} \). Many familiar properties of kernels and cokernels from abelian categories fail to hold for modules over semirings. For instance the sum map \( \mathbb{N} \oplus \mathbb{N} \to \mathbb{N} \) has trivial kernel and cokernel, but it is not an isomorphism. So kernels and cokernels play a less prominent role here than they do in abelian categories. Equalizers and coequalizers are more useful.

1.13. **Base change, induced and co-induced modules.** Let \( B \) be an \( A \)-algebra, and let \( M \) be an \( A \)-module. Then \( B \otimes_A M \) and \( \text{Hom}_A(B, M) \) are \( B \)-modules in the evident ways. These constructions give the left and right adjoints of the forgetful functor \( U \) from \( B \)-modules to \( A \)-modules. They are called the \( B \)-modules induced and co-induced by \( A \). It is also clear that the forgetful functor \( U \) is both monadic and comonadic.

1.14. **Limits and colimits of \( A \)-algebras.** Like \( \text{Mod}_A \), the category \( \text{Alg}_A \) also has all limits and colimits. Limits, coproducts, and filtered colimits can again be constructed as when \( A \) is a ring. In particular, coproducts are tensor products

\[
B \sqcup C = B \otimes_A C.
\]

More generally, the coproduct of any family \( (B_i)_{i \in I} \) is the tensor product of all \( B_i \) over \( A \). And as with modules, the coequalizer of two \( A \)-algebra morphisms...
\( f, g: B \to C \) is \( C/R \), where \( R \) be the equivalence relation internal to \( \text{Alg}_A \) on \( C \) generated by the relation \( \{(f(x), g(x)) \in C \times C \mid x \in B\} \).

### 1.15. Base change for algebras

For any \( A \)-algebra \( B \), the forgetful functor \( \text{Alg}_B \to \text{Alg}_A \) has a left adjoint. It sends \( C \) to \( B \otimes_A C \), where the \( B \)-algebra structure is given by the map \( b \mapsto b \otimes 1 \).

### 1.16. Flat modules and algebras

Let \( M \) be an \( A \)-module. Because the functor \( M \otimes_A - : \text{Mod}_A \to \text{Mod}_A \) has a right adjoint, it preserves all colimits. Since finite products and coproducts agree, it also preserves finite products. If it preserves equalizers, we say \( M \) is flat. In this case, \( M \otimes_A - \) preserves all finite limits.

Observe that, while not all monomorphisms are equalizers, it is nevertheless true that tensoring with a flat module preserves monomorphisms. Indeed \( f: N \to P \) is a monomorphism if and only if the diagonal map \( N \to N \times_P N \) is an isomorphism, and this property is preserved by tensoring with a flat module.

Flatness is preserved under base change. An \( A \)-algebra is said to be flat if it is when regarded as an \( A \)-module. If \( A \to B \) and \( B \to C \) are flat, then so is the composition \( A \to C \). More generally, if \( B \) is a flat \( A \)-algebra, and \( M \) is a flat \( B \)-module, then \( M \) is flat as an \( A \)-module.

### 1.17. Examples of flat modules

Any free module is flat. Any filtered colimit of flat modules is flat. We will see in (2.5) below that flatness is a flat-local property. So for example a module is flat if it is flat-locally free.

It is theorem of Govorov and Lazard that any flat module over a ring can be represented as a filtered colimit of free modules. This continues to hold for modules over any \( \mathbb{N} \)-algebra. As over rings, this is tantamount to an equational criterion for flatness for modules over \( \mathbb{N} \)-algebras, but now we must consider all relations of the form \( \sum_i a_i x_i = \sum_i b_i x_i \) instead of just those of the form \( \sum_i a_i x_i = 0 \), as one usually does over rings. See Katsov [21].

If \( S \) is a multiplicative subset of \( A \), let \( A[1/S] \) denote the initial \( A \)-algebra in which every element of \( S \) becomes multiplicatively invertible. Then \( A[1/S] \) is flat because it can be represented as colim_{s \in S} A, where for all \( s, t \in S \) there is a transition map \( A \to A \) from position \( s \) to position \( st \) given by multiplication by \( t \).

But it is completely different if we adjoin additive rather than multiplicative inverses. We will see in (2.12) below that \( \mathbb{Z} \) is not a flat \( \mathbb{N} \)-module. In fact, 0 is the only \( \mathbb{Z} \)-module that is flat over \( \mathbb{N} \). It is also the only \( \mathbb{R} \)-vector space that is flat over \( \mathbb{R}_+ \).

### 2. The flat topology over \( \mathbb{N} \)

The purpose of this section is to give some idea of scheme theory over \( \mathbb{N} \). It is the point of view I prefer for the mathematics of this chapter, but I will not use it in a serious way.

Scheme theory and the flat topology over \( \mathbb{N} \) were first considered in Toën–Vaquié [49]. Lorscheid has considered a different but related approach [30]. In
recent years, positivity structures in algebraic geometry have appeared in some interesting applications, although in an ad hoc way. For example, let us mention
the work of Lusztig [31], Fock–Goncharov [14], and Rietsch [35].

2.1. Flat covers. Let us say that a family \((B_i)_{i \in I}\) of flat \(A\)-algebras is faithful if the family of base change functors

\[
\text{Mod}_A \longrightarrow \prod_{i \in I} \text{Mod}_{B_i}, \quad M \mapsto (B_i \otimes_A M)_{i \in I}
\]  

(2.1.1)

reflects isomorphisms, that is, a map \(M \to N\) of \(A\)-modules is an isomorphism if (and only if), for every \(i \in I\) the induced map \(B_i \otimes_A M \to B_i \otimes_A N\) is. Let us say that a family of flat \(A\)-algebras is an fpqc cover if it has a finite subfamily that is faithful.

2.2. The fpqc topology. For any \(\mathbb{N}\)-algebra \(K\), let \(\text{Aff}_K\) denote the opposite of the category of \(K\)-algebras. For any \(K\)-algebra \(A\), write \(\text{Spec} A\) for the corresponding object in \(\text{Aff}_K\). The fpqc covers form a pretopology on \(\text{Aff}_K\), in the usual way. See Toën–Vaquié, proposition 2.4 [43]. The resulting topology is called the fpqc topology or, less formally, the flat topology.

One might also like to define an fppf topology by requiring that each \(B_i\) be finitely presented as an \(A\)-algebra. The following question is then natural:

**Question 1.** Let \((B_i)_{i \in I}\) be a faithful family of flat \(A\)-algebras. If each \(B_i\) is finitely presented as an \(A\)-algebra, is there a finite faithful subfamily?

When \(K\) is \(\mathbb{Z}\), it is a fundamental fact from scheme theory that the answer is yes. To prove it one combines quasi-compactness in the Zariski topology with the fact that faithfully flat morphisms of finite presentation have open image.

2.3. Faithfully flat descent. Let \((B_i)_{i \in I}\) be a faithful family of flat \(A\)-algebras. Then the family of base change functors (2.1.1) is comonadic. As usual, this is just an application of Beck’s theorem in category theory. (See Borceux [2], theorem 4.4.4, p. 212. See also theorem 2.5 of Toën–Vaquié [43].) Thus the fibered category of modules satisfies effective descent in the comonadic sense. If the family \((B_i)_{i \in I}\) is finite, or more generally an fpqc cover, then the comonadic approach to descent agrees with the Grothendieck’s original one. So in either sense, in the fpqc topology, the fibered category of modules satisfies effective descent, or it is a stack.

Thus descent allows us to recover the category of \(A\)-modules from that of modules over the cover. As usual, and as a consequence, it allows us to recover \(A\) itself:

**2.4. Proposition.** Let \(A\) be an \(\mathbb{N}\)-module, and let \(E\) denote the \(\mathbb{N}\)-algebra (possibly noncommutative, a priori) of natural endomorphisms of the identity functor on \(\text{Mod}_A\). Then the canonical map \(A \to E\) is an isomorphism.

**Proof.** Let \(\varphi\) be such a natural endomorphism. Set \(a = \varphi_A(1)\). Then for any \(M \in \text{Mod}_A\), the map \(\varphi_M : M \to M\) is multiplication by \(a\). Indeed, for any \(m \in M\),
consider the map \( f : A \to M \) determined by \( f(1) = m \). Then we have \( \varphi_M(m) = \varphi_M(f(1)) = f(\varphi_A(1)) = f(a) = am \). 

2.5. Proposition. Let \( B \) be an \( \mathbb{N} \)-algebra, and let \((C_i)_{i \in I}\) be a faithful family of flat \( B \)-algebras.

1. For any finite diagram \( (M_j)_{j \in J} \) of \( B \)-modules, a map \( M \to \varprojlim_j M_j \) is an isomorphism if and only if each map \( C_i \otimes_B M \to \varprojlim_j C_i \otimes_B M_j \) is an isomorphism.

2. A map \( A \to B \) of \( \mathbb{N} \)-algebras is flat if and only if each \( C_i \) is flat as an \( A \)-module.

3. A \( B \)-module \( N \) is flat if and only if each \( C_i \otimes_B N \) is flat as a \( C_i \)-module.

Proof. (1): Since the family \((C_i)_{i \in I}\) is faithful, it is enough to check the map is an isomorphism after base change to each \( C_i \). But since each \( C_i \) is flat, we have \( C_i \otimes_B (\varprojlim_j M_j) = \varprojlim_j (C_i \otimes_B M_j) \).

(2): Consider a finite diagram \( (M_j)_{j \in J} \) of \( A \)-modules. To show that the induced map \( B \otimes_A (\varprojlim_j M_j) \to \varprojlim_j (B \otimes_A M_j) \) is an isomorphism, it is enough by part (1) to show that each map \( C_i \otimes_A (\varprojlim_j M_j) \to \varprojlim_j (C_i \otimes_A M_j) \) is. But this holds because each \( C_i \) is flat over \( A \).

The converse holds because flatness is stable under composition.

(3): Let \( (M_j)_{j \in J} \) be a finite diagram of \( B \)-modules. To show \( N \otimes_B (\varprojlim_j M_j) \to \varprojlim_j (N \otimes_B M_j) \) is an isomorphism, it is enough by part (1) to show that each map \( C_i \otimes_B N \otimes_B (\varprojlim_j M_j) \to \varprojlim_j (C_i \otimes_B N \otimes_B M_j) \) is. This holds because each \( C_i \otimes_B N \) is flat as a \( C_i \)-module.

The converse holds because flatness is stable under base change.

2.6. Algebraic geometry over \( \mathbb{N} \). We can then define the basic objects of algebraic geometry over any \( \mathbb{N} \)-algebra \( K \) in a formal way, as in Toën–Vaquié [43]. A map \( f : \text{Spec} \ B \to \text{Spec} \ A \) is Zariski open if the corresponding map \( A \to B \) is a flat epimorphism of finite presentation. One then defines \( K \)-schemes by gluing together affine \( K \)-schemes along Zariski open maps. In Toën–Vaquié, all this takes place in the category of sheaves of sets on \( \text{Aff}_K \) in the Zariski topology.

Presumably one could define a category of algebraic spaces over \( \mathbb{N} \) by adjoining quotients of flat or étale equivalence relations, under some meaning of these terms, as for example in Toën–Vaquié, section 2.2 [42]. But there are subtleties with both fppf maps, as in the question above, and étale maps, where it is not clear that the first definition that comes to mind is the best one. So some care seems to be needed before we can have complete confidence in the definition. In any case, we will certainly not need this generality.

2.7. Extending fibered categories to nonaffine schemes. Because we have faithfully flat descent for modules, we can define the category \( \text{Mod}_X \) of \( X \)-modules (i.e., quasi-coherent sheaves) over any \( K \)-scheme \( X \). More generally, any fibered category over \( \text{Aff}_K \) for which we have effective descent extends uniquely (up to some
appropriate notion of equivalence) to such a fibered category over the category of $K$-schemes. For example, flat modules have this property by (2.5)(3). So we can make sense of a flat module over any $N$-scheme.

Similarly, using (2.5)(2), there is a unique way of defining flatness for morphisms $X \to Y$ of $K$-schemes that is stable under base change on $Y$ and fpqc-local on $X$.

We will conclude this section with some examples of flat-local constructions and properties.

2.8. Additively idempotent elements. An element $m$ of an $A$-module $M$ is \textit{additively idempotent} if $2m = m$. The set $I(M)$ of such elements is therefore the equalizer

$$I(M) \longrightarrow M \xrightarrow{x \to x} M.$$ 

Thus $I$ has a flat-local nature. Indeed, for any flat $A$-algebra $B$, the induced map $B \otimes_A I(M) \to I(B \otimes_A M)$ is an isomorphism of $B$-modules, and if $(B_i)_{i \in J}$ is an fpqc cover of $A$, then the induced map

$$I(M) \longrightarrow \prod_{i} I(B_i \otimes_A M) \xrightarrow{\prod_{j,j'} I(B_j \otimes_A B_{j'} \otimes_A M)}$$

is an equalizer diagram. It follows that, given an $N$-scheme $X$, this defines an $X$-module $I(M)$ for any $X$-module $M$. Thus we have a morphism of fibered categories $I: \text{Mod}_X \to \text{Mod}_X$.

2.9. Cancellative modules. An $A$-module $M$ is \textit{additively cancellative} if the implication

$$x + y = x + z \Rightarrow y = z$$

holds in $M$. This is equivalent to the following being an equalizer diagram:

$$M^2 \xrightarrow{(x,y) \to (x,y,y)} M^3 \xrightarrow{(x,y,z) \to x+y, (x,y,z) \to x+z} M$$

Therefore, by (2.5), cancellativity is a flat-local property.

Multiplicative cancellation works similarly. For any $a \in A$, let us say that $M$ is \textit{$a$-cancellative} if the implication

$$ax = ay \Rightarrow x = y$$

holds in $M$. This is equivalent to the following being an equalizer diagram:

$$M \xrightarrow{x \to (x,x)} M^2 \xrightarrow{(x,y) \to ax, (x,y) \to ay} M^2 \xrightarrow{(x,y) \to ax, (x,y) \to ay} M$$

Then $a$-cancellativity is also a flat-local property.
2.10. **Strong and subtractive morphisms.** A morphism $M \to N$ of $A$-modules is *strong* if the first of the following induced diagrams

\[
\begin{array}{ccc}
M^2 & \to & M \\
\downarrow & & \downarrow \\
N^2 & \to & N
\end{array}
\quad \text{and}
\begin{array}{ccc}
M \times N & \to & M \\
\downarrow & & \downarrow \\
N & \to & N
\end{array}
\]

is cartesian. It is called *subtractive* if the second is. (The reason for this name is that a submodule $M \subseteq N$ is subtractive if it closed under differences that exist in $N$.) Both properties are flat-local.

2.11. **Additively invertible elements.** For any $A$-module $M$, let

\[
V(M) = \{x \in M \mid \exists y \in M, \ x + y = 0\}
\]

denote the subset of additively invertible elements. Then $V(M)$ is a sub-$A$-module of $M$. The resulting functor is the right adjoint of the forgetful functor $\text{Mod}_{\mathbb{Z} \otimes A} \to \text{Mod}_A$, and so we have

\[
V(M) = \text{Hom}_A(\mathbb{Z} \otimes A, M).
\]

It can also be expressed as an equalizer:

\[
V(M) \xrightarrow{x \mapsto (x, -x)} M^2 \xrightarrow{0} \xrightarrow{+} M
\]

Therefore $V$ also has a local nature, and so we can define an $X$-module $V(M)$ for any module $M$ over any $\mathbb{N}$-scheme $X$. In fact, since $V(M)$ is a group under addition, $V(M)$ is in fact an $(X \times \text{Spec } \mathbb{Z})$-module, and $V$ can be viewed as a morphism $\text{Mod}_X \to f_*(\text{Mod}_{X \times \text{Spec } \mathbb{Z}})$ of fibered categories, where $f$ denotes the canonical projection $X \times \text{Spec } \mathbb{Z} \to X$.

Golan [16] says $M$ is *zerosumfree* if $V(M) = 0$, or equivalently if $(0)$ is a strong submodule. The remarks above imply that being zerosumfree is a flat-local property.

2.12. **Corollary.** Let $A$ be a zerosumfree $\mathbb{N}$-algebra. Then the map $A \to \mathbb{Z} \otimes A$ is flat if and only if $\mathbb{Z} \otimes A = 0$. More generally, zero is the only flat $A$-module which is a group under addition.

*Proof.* Since such a module $M$ is flat, we have $V(M) = M \otimes_A V(A) = 0$. On the other hand, since $M$ is a group, we have $V(M) = M$. \qed

It follows formally that the empty scheme is the only $\mathbb{Z}$-scheme that is flat over a given zerosumfree scheme. Thus we would expect some subtleties involving derived functors when passing from algebraic geometry over $\mathbb{N}$ to that over $\mathbb{Z}$, or even from $\mathbb{R}_+$ to $\mathbb{R}$. 
3. Plethystic algebra for \( \mathbb{N} \)-algebras

Let \( K, K', L \) be \( \mathbb{N} \)-algebras.

3.1. Models of co-\( C \) objects in \( \text{Alg}_K \). Let \( C \) be a category of the kind considered in universal algebra. Thus an object of \( C \) is a set with a family of multinary operations satisfying some universal identities. For example, \( C \) could be the categories of groups, monoids, \( L \)-algebras, \( L \)-modules, Lie algebras over \( L \), loops, heaps, and so on.

Let us say that a covariant functor \( \text{Alg}_K \rightarrow C \) is representable if its underlying set-valued functor is. Let us call the object representing such a functor a co-\( C \) object in \( \text{Alg}_K \). For example, a co-group structure on a \( K \)-algebra \( A \) is the same as a group scheme structure on \( \text{Spec} A \) over Spec \( K \). Likewise, a co-\( L \)-algebra structure on \( A \) is the same as a \( L \)-algebra scheme structure on \( \text{Spec} A \) over Spec \( K \).

3.2. Co-\( L \)-algebra objects in \( \text{Alg}_K \). Unpacking this further, we see a co-\( L \)-algebra object of \( \text{Alg}_K \) is a \( K \)-algebra \( P \) together with \( K \)-algebra maps

\[
\Delta^+: P \rightarrow P \otimes_K P \\
\Delta^\times: P \rightarrow P \otimes_K P
\]

subject to the condition that for all \( A \in \text{Alg}_K \), the set \( \text{Hom}_K(P, A) \) equipped with the binary operations \( + \) and \( \times \) induced by \( \Delta^+ \) and \( \Delta^\times \) is an \( \mathbb{N} \)-algebra, and an \( \mathbb{N} \)-algebra map

\[
\beta: L \rightarrow \text{Alg}_K(P, K).
\]

These properties can of course be expressed in terms of \( P \) itself, without quantifying over any variable algebras such as \( A \) above. For example, \( \Delta^+ \) and \( \Delta^\times \) must be cocommutative and coassociative, and \( \Delta^\times \) should codistribute over \( \Delta^+ \), and so on.

Similarly, the (unique) elements \( 0 \) and \( 1 \) in the \( \mathbb{N} \)-algebras \( \text{Hom}_K(P, A) \) correspond to \( K \)-algebra morphisms

\[
\varepsilon^+: P \rightarrow K, \quad \varepsilon^\times: P \rightarrow K,
\]

which we will need later.

We will often use the term \( K-L \)-bialgebra instead of co-\( L \)-algebra object of \( \text{Alg}_K \). This is not meant to suggest any relation to the usual meaning of the term bialgebra. (Every \( K-L \)-bialgebra has two bialgebra structures in the usual sense—\( \Delta^+ \) and \( \Delta^\times \)—but this is just a coincidence of terminology.)

Let \( \text{Alg}_{K,L} \) denote the category of \( K-L \)-bialgebras. A morphism \( P \rightarrow P' \) of \( \text{Alg}_{K,L} \) is a \( K \)-algebra map compatible with the co-operations \( \Delta^+, \Delta^\times, \) and \( \beta \). In other words, the induced natural transformation of set-valued functors must prolong to a natural transformation of \( L \)-algebra-valued functors.

3.3. Plethystic algebra. In the case where \( K \) and \( L \) are \( \mathbb{Z} \)-algebras, the theory of \( K-L \)-bialgebras was initiated in Tall–Wraith [40] and developed further in Borger–Wieland [5]. It is clear how to extend the general theory developed there.
to \( \mathbb{N} \)-algebras. In almost all cases, the relevant words from \[8\] work as written; at some places, obvious changes are needed. The reader can also refer to Stacey–Whitehouse \[38\], where the general theory is written down for general universal-algebraic categories. (Also, see Bergman–Hausknecht \[1\] for many fascinating case studies taken from different categories, such Lie algebras, monoids, groups, possibly noncommutative rings, and many more.)

Let us list some of the main ideas we will need.

1. The composition product is a functor \(- \odot_L -: \text{Alg}_{K,L} \times \text{Alg}_L \to \text{Alg}_K\). It is characterized by the adjunction

   \[
   \text{Alg}_K(P \odot L A, B) = \text{Alg}_L(A, \text{Alg}_K(P, B)).
   \]

2. It has an extension to a functor \(- \odot_L -: \text{Alg}_{K,L} \times \text{Alg}_{L,K'} \to \text{Alg}_{K,K'}\).

3. This gives a monoidal structure (not generally symmetric) on the category \(\text{Alg}_{K,K}\) of \(K\)-\(K\)-bialgebras. The unit object is \(K[e]\), the one representing the identity functor.

4. A composition \(K\)-algebra is defined to be a monoid with respect to this monoidal structure. The operation is denoted \(\circ\) and the identity is denoted \(e\).

5. An action of a composition \(K\)-algebra \(P\) on a \(K\)-algebra \(A\) is defined to be an action of the monoid object \(P\), or equivalently of the monad \(P \odot K -\). We will write \(f(a)\) for the image of \(f \odot a\) under the action map \(P \odot K A \to A\). A \(P\)-equivariant \(K\)-algebra is a \(K\)-algebra equipped with an action of \(P\). When \(K = \mathbb{N}\), we will also use the term \(P\)-semiring.

6. For any \(K\)-L-bialgebra \(P\), we will call the functor it represents its Witt vector functor: \(W_P = \text{Alg}_K(P, -)\). It takes \(K\)-algebras to \(L\)-algebras. When \(K = L\), a composition structure on \(P\) is then equivalent to a comonad structure on \(W_P\). When \(P\) has a composition structure, then \(W_P(A)\) has a natural action of \(P\), and in this way \(W_P\) can be viewed as the right adjoint of the forgetful functor from \(P\)-equivariant \(K\)-algebras to \(K\)-algebras.

### 3.4. Example: composition algebras and endomorphisms

An element \(\psi\) of a composition \(K\)-algebra \(P\) is \(K\)-algebra-like if for all \(K\)-algebras \(A\) with an action of \(P\), the self map \(x \mapsto \psi(x)\) of \(A\) is a \(K\)-algebra map. This is equivalent to requiring

\[
\Delta^+(\psi) = \psi \otimes 1 + 1 \otimes \psi, \quad \Delta^X(\psi) = \psi \otimes \psi, \quad \beta(c)(\psi) = c,
\]

for all \(c \in K\). (For comparison, one could say \(d \in P\) is \(K\)-derivation-like if it acts as a \(K\)-linear derivation on any \(K\)-algebra. This can also be expressed directly by

\[\text{It is called a biring triple in [40], a plethory in [8], a Tall-Wraith monad object in [1], and a Tall-Wraith monoid in [38]. The term \textit{composition algebra} is both plain and descriptive; so I thought to try it out here. It does however have the drawback in that it already exists in the literature with other meanings.}\]
saying $d$ is primitive under $\Delta^+$ and it satisfies the Leibniz rule $\Delta^+(d) = d \otimes e + e \otimes d$ and the $K$-linearity identity $\beta(c)(d) = 0$.

Now let $G$ be a monoid. Let $P$ be the $K$-algebra freely generated (as an algebra) by the symbols $\psi_g$, for all $g \in G$. Then $P$ has a unique composition such that each $\psi_g$ is $K$-algebra-like and we have

$$\psi_g \circ \psi_h = \psi_{gh}$$

for all $g, h \in G$. Then an action of $P$ on an algebra $A$ is the same as an action of $G$ on $A$.

In this case, the Witt functor is simply $W_P(A) = A^G$, where $A^G$ has the usual product algebra structure.

3.5. Models of co-C objects in $\mathcal{Alg}_K$. Let $C$ be a category of the kind considered above. Let $K \to K'$ be an $\mathbb{N}$-algebra map, and let $P'$ be a co-C object in $\mathcal{Alg}_{K'}$. Then a model for $P'$ over $K$ (or a $K$-model) is a co-C object $P$ in $\mathcal{Alg}_K$ together with an isomorphism $K' \otimes_K P \to P'$ of co-C objects of $\mathcal{Alg}_{K'}$. Then for any $K'$-algebra $A'$, we have

$$W_{P'}(A') = \mathcal{Alg}_{K'}(P', A') = \mathcal{Alg}_K(P, A') = W_P(A').$$

So the Witt vector functor of $P'$ extends $W_P$ from $\mathcal{Alg}_{K'}$ to $\mathcal{Alg}_K$. Conversely, any such extension to a representable functor comes from a unique model of $P'$.

3.6. Flat models of co-C-objects in $\mathcal{Alg}_K$. We will be especially interested in finding $K$-models of $P'$ that are flat as $K$-modules. These can exist only when $P'$ itself is flat, but this will be the case in all our examples. Further we will only consider the case were $K$ is contained in $K'$.

In this context, the composition $P \to K' \otimes_K P \to P'$ is injective. Conversely, if a subset $P \subseteq P'$ admits a flat model structure, then it does so in a unique way. Indeed, since the induced maps $P \otimes_K n \to P' \otimes_K n$ are injective, each co-operation $\Delta$ on $P'$ restricts to at most one on $P$. One might thus say that being a flat model (when $K \subseteq K'$) is a property of a given subset of $P'$, rather than a structure on it.

The case where $C = \mathcal{Alg}_L$ will be of particular interest to us. Then a flat model is just a subset $P \subseteq P'$ such that the following properties hold:

1. $P$ is a flat sub-$K$-algebra of $P'$,
2. the induced map $K' \otimes_K P \to P'$ is a bijection,
3. $\Delta^+(P) \subseteq P \otimes_K P$ and $\varepsilon^+(P) \subseteq K$,
4. $\Delta^-(P) \subseteq P \otimes_K P$ and $\varepsilon^-(P) \subseteq K$,
5. $\beta(L) \subseteq \mathcal{Alg}_K(P, K)$, where $\mathcal{Alg}_K(P, K)$ is regarded as a subset of $\mathcal{Alg}_{K'}(P', K')$.

When $L = \mathbb{N}$, the last condition is always satisfied. (Alternatively, the conditions on the co-units $\varepsilon^+$ and $\varepsilon^-$ are also redundant but not in the absence of (5).)

For other categories $C$, it is usually clear how to modify these conditions. For example, if $C = \mathcal{Mod}_N$, one would drop (4) and (5).
3.7. Models of composition algebras. When $P'$ is a composition $K'$-algebra, we will usually want to descend the composition structure as well. Then a $K$-model of $P'$ (as a composition algebra) is a composition $K$-algebra $P$ together with an action of $P$ on $K'$ and an isomorphism $K' \otimes_K P \to P'$ of composition $K'$-algebras. Giving such a model is equivalent to giving a representable comonad on $\text{Alg}_K$ prolonging $W_{P'}$ in the evident sense.

A flat model when $K \subseteq K'$ is then just a flat sub-$K$-algebra $P \subseteq P'$ satisfying conditions (1)–(5) above plus

(6) $P \circ P \subseteq P$ and $e \in P$

So again, if $P$ admits such a structure, it does so in a unique way.

4. The composition structure on symmetric functions over $\mathbb{N}$

The purpose of this section is to give two different $\mathbb{N}$-models of $\Lambda_{\mathbb{Z}}$, the composition ring of symmetric functions. Since $\Lambda_{\mathbb{Z}}$ represents the usual big Witt vector functor, these give extensions of the big Witt vector functor to $\mathbb{N}$-algebras.

Our treatment is broadly similar to Macdonald’s [32]. He discusses the co-additive structure in example 25 of I.5, the co-multiplicative structure in example 20 of I.7, and plethysm in I.8.

Let $K$ be an $\mathbb{N}$-algebra.

4.1. Conventions on partitions. We will follow those of Macdonald, p. 1 [32].

So, a partition is an element

$$\lambda = (\lambda_1, \lambda_2, \ldots) \in \mathbb{N} \oplus \mathbb{N} \oplus \cdots$$

such that $\lambda_1 \geq \lambda_2 \geq \cdots$. As is customary, we will allow ourselves to omit any number of zeros, brackets, and commas and to use exponents to represent repetition. So for example we have $(3, 2, 2, 1, 0, \ldots) = 3^2 2^2 1$ and $(0, \ldots) = 0$.

The length of $\lambda$ is the smallest $i \geq 0$ such that $\lambda_{i+1} = 0$. The weight of $\lambda$ is $\sum \lambda_i$ and is denoted $|\lambda|$. We also say $\lambda$ is a partition of its weight.

4.2. $\Psi_K$. Let $\Psi_K$ denote the composition $K$-algebra associated to the multiplicative monoid of positive integers. So $\Psi_K = K[\psi_1, \psi_2, \ldots]$, where each $\psi_n$ is $K$-algebra-like and we have $\psi_m \circ \psi_n = \psi_{mn}$.

We will be interested in (flat) models of $\Psi_{\mathbb{Q}}$ over smaller subalgebras, especially $\mathbb{Q}_+$, $\mathbb{Z}$ and $\mathbb{N}$. There are the obvious models $\Psi_{\mathbb{Q}_+}$, $\Psi_{\mathbb{Z}}$, and $\Psi_{\mathbb{N}}$, but we will be more interested in larger ones.

4.3. Symmetric functions. Let $\Lambda_K$ denote the usual $K$-algebra of symmetric functions in infinitely many variables $x_1, x_2, \ldots$ with coefficients in $K$. (See Macdonald, p. 19 [32].) More precisely, $\Lambda_K$ is the set of formal series $f(x_1, x_2, \ldots)$ such that the terms of $f$ have bounded degree and for all $n$, the series $f(x_1, \ldots, x_n, 0, 0, \ldots)$ is a polynomial which is invariant under permuting the variables $x_1, \ldots, x_n$. 

It is clear that \( \Lambda_K \) is freely generated as a \( K \)-module by the \textit{monomial} symmetric functions \( m_\lambda \), where \( \lambda \) ranges over all partitions and where

\[
m_\lambda = \sum_\alpha x_1^{\alpha_1} x_2^{\alpha_2} \cdots
\]

where \( \alpha \) runs over all permutations of \( \lambda \) in \( \mathbb{N}^\infty \). In particular, we have \( K \otimes_{\mathbb{N}} \Lambda_{\mathbb{N}} = \Lambda_K \). When \( K = \mathbb{N} \), this is the unique basis of \( \Lambda_{\mathbb{N}} \), up to unique isomorphism on the index set.

It is well known that when \( K \) is a ring, \( \Lambda_K \) is freely generated as a \( K \)-algebra by the \textit{complete} symmetric functions \( h_1, h_2, \ldots \), where

\[
h_n := \sum_{i_1 \leq \cdots \leq i_n} x_{i_1} \cdots x_{i_n} = \sum_{|\lambda|=n} m_\lambda.
\]

Alternatively, if we write

\[
\psi_n = x_1^n + x_2^n + \cdots,
\]

then the induced map

\[
\Psi_K = K[\psi_1, \psi_2, \ldots] \to \Lambda_K
\]

is an injection when \( K \) is a flat \( \mathbb{Z} \)-algebra and is a bijection when \( K \) is a \( \mathbb{Q} \)-algebra. In particular, \( \Lambda_{\mathbb{N}} \) and \( \Lambda_{\mathbb{Z}} \) are models for \( \Psi_{\mathbb{Q}} = \Lambda_{\mathbb{Q}} \). The \( \psi_n \) have several names: the \textit{Adams}, \textit{Frobenius}, and \textit{power-sum} symmetric functions.

**4.4. Remark: \( \Lambda_{\mathbb{N}} \) is not free as an \( \mathbb{N} \)-algebra.** Indeed, all \( m_\lambda \) are indecomposable additively and, one checks, multiplicatively (except \( m_0 = 1 \)). Therefore any generating set of \( \Lambda_{\mathbb{N}} \) as an \( \mathbb{N} \)-algebra must contain all of them. But they are not algebraically independent, because any monomial in them is a linear combination of the others. For example \( m_1^2 = m_2 + 2m_{1,1} \).

**4.5. Elementary and Witt symmetric functions.** At times, we will use other families of symmetric functions, such as the \textit{elementary} symmetric functions

\[
e_n = \sum_{i_1 < \cdots < i_n} x_{i_1} \cdots x_{i_n},
\]

and the less well-known \textit{Witt} symmetric functions \( \theta_1, \theta_2, \ldots \), which are determined by the relations

\[
\psi_n = \sum_{d|n} d \theta_n^{d/n},
\]

for all \( n \geq 1 \). Probably the most concise way of relating them all is with generating functions in \( 1 + t\Lambda_{\mathbb{Q}}[[t]] \):

\[
\prod_{d \geq 1} (1 - \theta_d t^d)^{-1} = \exp \left( \sum_{j \geq 1} \frac{\psi_j}{j} t^j \right) = \sum_{i \geq 0} b_i t^i = ( \sum_{i \geq 0} e_i(-t)^i )^{-1}.
\]

Indeed, one can check that each expression equals \( \prod_j (1 - x_j t)^{-1} \).
One consequence of note is that the $\theta_d$ generate $\Lambda _\mathbb{Z}$ freely:

$$\Lambda _\mathbb{Z} = \mathbb{Z}[\theta_1, \theta_2, \ldots].$$

Another is that we have

$$\mathbb{Q}_+ [h_1, h_2, \ldots] \subseteq \Psi_{\mathbb{Q}_+}.$$  \hfill (4.5.3)

The Witt symmetric functions are rarely encountered outside the literature on Witt vectors. The following are some decompositions in more common bases:

\[
\begin{align*}
\theta_1 &= \psi_1 = m_1 = e_1 = h_1 = s_1 \\
\theta_2 &= (\psi_1^2 + \psi_2)/2 = -m_2 = -e_2 = -h_1^2 + h_2 = -s_1^2 \\
\theta_3 &= (\psi_1^3 + \psi_3)/3 = -2m_{12} - m_{21} = -e_2 e_1 + e_3 = -h_2 h_1 + h_3 = -s_2 \\
\theta_4 &= (3\psi_1^4 + 2\psi_2 \psi_1^2 - \psi_3^2 + 2\psi_4)/8 = -9m_{14} - 4m_{212} - 2m_{22} - m_{31} \\
&= -e_2 e_1^2 + 3e_3 e_1 - e_4 = -h_1^3 + 2h_2 h_1^2 - h_2^2 - h_3 h_1 + h_4 \\
&= -s_1^4 - s_2^2 - s_3^2 - s_4 \\
\theta_5 &= (\psi_1^5 + \psi_5)/5 = -24m_{15} - 12m_{213} - 6m_{24} - 4m_{312} - 2m_{32} - m_{41} \\
&= -e_2 e_1^4 + e_2 e_1 e_3 e_1^2 - e_3 e_2 e_1 + e_4 e_1 + e_5 \\
&= -h_2 h_1^3 + h_2^2 h_1 + h_3 h_1^2 - h_3 h_2 - h_4 h_1 + h_5 \\
&= -s_{213} - s_{212} - s_{312} - s_{31} - s_{41} \\
\theta_6 &= (13\psi_1^6 - 9\psi_2 \psi_1^4 + 9\psi_2^2 \psi_1^2 - 3\psi_3^2 + 8\psi_3 \psi_1^3 - 4\psi_4^3 + 12\psi_6)/72 \\
&= -130m_{16} - 68m_{214} - 35m_{2212} - 18m_{23} - 24m_{313} - 12m_{321} - 4m_{32} \\
&= -6m_{412} - 3m_{42} - m_{51} \\
&= -e_2 e_1^4 + e_2 e_1^2 + e_3 e_1^3 - e_3 e_2 e_1 - e_4 e_1^2 + e_4 e_2 + e_5 e_1 - e_6 \\
&= -h_2^3 h_1^2 - h_2 h_1^3 + 3h_3 h_2 h_1 - h_2^3 + h_4 h_1^2 - h_4 h_2 - h_5 h_1 + h_6 \\
&= -s_{2212} - s_{221} - 2s_{3212} - 3s_{321} - s_{42} - 2s_{412} - 2s_{42} - s_{51}
\end{align*}
\]

**4.6. Proposition.** Consider the composition algebra structure on $\Lambda _\mathbb{Q}$ induced by the isomorphism $\Psi_\mathbb{Q} \to \Lambda _\mathbb{Q}$ of \hfill [4.3.1]. Then the structure maps on $\Lambda _\mathbb{Q}$ satisfy the following:

\[
\begin{align*}
\Delta ^+ &\colon f(\ldots, x_i, \ldots) \mapsto f(\ldots, x_i \otimes 1, 1 \otimes x_i, \ldots) \\
\Delta ^x &\colon f(\ldots, x_i, \ldots) \mapsto f(\ldots, x_i \otimes x_i, \ldots) \\
\varepsilon ^+ &\colon f(\ldots, x_i, \ldots) \mapsto f(0, 0, \ldots) \\
\varepsilon ^x &\colon f(\ldots, x_i, \ldots) \mapsto f(1, 0, 0, \ldots) \\
f \circ g &= f(y_1, y_2, \ldots),
\end{align*}
\]

for all $f \in \Lambda _\mathbb{Q}$ and $g \in \Lambda _\mathbb{N}$, where $g = y_1 + y_2 + \cdots$ with each $y_j$ a monomial in the variables $x_i$ with coefficient 1.

**Proof.** The maps $\Delta ^+, \Delta ^x, \varepsilon ^+, \varepsilon ^x$ displayed above are manifestly $\mathbb{Q}$-algebra maps. Therefore to show they agree with the corresponding structure maps, it is enough
to consider elements $f$ that run over a set of generators, such as the $\psi_n$. In this case, one can calculate the image of $\psi_n$ under the maps displayed above and observe that it agrees with the image of $\psi_n$ under the structure maps, by (3.4.1).

Similarly, for any fixed $g \in \Lambda_N$, the map $f \mapsto f \circ g$ displayed above is a $\mathbb{Q}$-algebra map. So it is again enough to assume $f = \psi_m$. Then $\psi_m \circ g \mapsto g$ is an $\mathbb{N}$-algebra map $\Lambda_N \to \Lambda_Q$, and to show it agrees with the rule displayed above, it is enough to do so after allowing coefficients in $\mathbb{Q}$. Then the $\psi_n$ are generators, and so it is enough to assume $g = \psi_n$. We are left to check $\psi_m \circ \psi_n = \psi_{mn}$, which is indeed the composition law on $\Psi_Q$.

4.7. Corollary. $\Lambda_N$ and $\Lambda_Z$ are models of the composition algebra $\Lambda_Q$ in a unique way.

Proof. Since both $\Lambda_N$ and $\Lambda_Z$ are free, it is enough to check properties (1)–(6) of (3.6) and (3.7). It is immediate from (4.6) that all the structure maps preserve $\Lambda_N$, which finishes the proof for it.

It is also clear that the structure maps preserve $\Lambda_Z$, except for $\Lambda_Z$ being closed under composition, which we will now show. Because $\Lambda_N$ is a composition algebra and we have $\Lambda_Z = \mathbb{Z} \otimes_N \Lambda_N$, it is enough to show $f(-1) \in \mathbb{Z}$ for all $f \in \Lambda_Z$. To do this, it is enough to restrict to the case where $f$ ranges over a set of algebra generators, such as the $h_n$. But since $\psi_n(-1) = -1$, the identity (4.5.2) implies $h_n(-1) = 0$ for $n \geq 2$ and $h_1(-1) = -1$. So we have $h_n(-1) \in \mathbb{Z}$ for all $n$.

4.8. Explicit description of a $\Lambda_N$-action. For the convenience of the reader, let us say what it means for the composition algebra $\Lambda_N$ to act on an $\mathbb{N}$-algebra $A$ in elementary terms.

For each partition $\lambda$, there is a set map
\[ m_\lambda : A \to A \]
such that the following identities hold
\[
\begin{align*}
m_0(x) &= 1 \\
(m_\lambda m_\mu)(x) &= m_\lambda(x)m_\mu(x) \\
m_\lambda(0) &= \varepsilon^+(m_\lambda) \\
m_\lambda(x + y) &= \Delta^+(m_\lambda)(x, y) := \sum_{\mu, \nu} q^\lambda_{\mu\nu} m_\mu(x)m_\nu(y) \\
m_\lambda(1) &= \varepsilon^+(m_\lambda) \\
m_\lambda(xy) &= \Delta^+(m_\lambda)(x, y) := \sum_{\mu, \nu} r^\lambda_{\mu\nu} m_\mu(x)m_\nu(y) \\
m_1(x) &= x \\
m_\lambda(m_\mu(x)) &= (m_\lambda \circ m_\mu)(x).
\end{align*}
\]

The notation $(m_\lambda m_\mu)(x)$, or more generally $f(x)$, is defined by
\[ f(x) = \sum_{\nu} f_\nu m_\nu(x) \in A, \]
where \( f = \sum_{\nu} f_\nu m_\nu, \ f_\nu \in \mathbb{N}. \) And \( q_{\mu\nu}^\phi \) and \( r_{\mu\nu}^\psi \) are the structure coefficients in \( \mathbb{N} \) for the two coproducts:

\[
\Delta^+(m_\lambda) = \sum_{\mu, \nu} q_{\mu\nu}^\phi m_\mu \otimes m_\nu
\]

\[
\Delta^x(m_\lambda) = \sum_{\mu, \nu} r_{\mu\nu}^\psi m_\mu \otimes m_\nu.
\]

4.9. Example: the toric \( \Lambda_N \)-structure on monoid algebras. Let \( A \) be a commutative monoid, written multiplicatively. The monoid algebra \( \mathbb{N}[A] \) is the set of finite formal sums \( \sum_{i=1}^{n} [a_i] \), where each \( a_i \in A \), and multiplication is the linear map satisfying the law \( [a][b] = [ab] \). Then for any \( f \in \Lambda_N \) define

\[
f(\sum_{i=1}^{n} [a_i]) := f([a_1], \ldots, [a_n], 0, 0, \ldots).
\] (4.9.1)

The right-hand side denotes the substitution \( x_i = [a_i] \) into the symmetric function \( f \). To show this law defines an action of \( \Lambda_N \) on \( \mathbb{N}[A] \), it is enough to observe that (4.9.1) is the restriction of an action of a larger composition algebra on a larger \( \mathbb{N} \)-algebra. By (4.6), it is the restriction of the action of \( \Lambda_Q \) on \( \mathbb{Q}[A] := \mathbb{Q} \otimes \mathbb{N}[A] \) determined by \( \psi_n([a]) = [a^n] \) for all \( a \in A, n \geq 1 \).

We call the \( \Lambda_N \)-structure on \( \mathbb{N}[A] \) the toric \( \Lambda_N \)-structure because Spec \( \mathbb{Z}[A] \) is a toric variety and the \( \Lambda_N \)-structure extends in a canonical way to any nonaffine toric variety, once this concept is defined. (See [3], for example.)

4.10. Example: the Chebyshev line. When \( A = \mathbb{Z} \) above, we have \( \mathbb{N}[A] = \mathbb{N}[x^{\pm 1}] \). Let \( B \) denote the subalgebra of \( \mathbb{N}[A] \) spanned by \( 1, x + x^{-1}, x^2 + x^{-2}, \ldots \). Then \( B \) is the set of invariants under the involution \( \psi_{-1} \) of \( \mathbb{N}[x^{\pm 1}] \) defined by \( x \mapsto x^{-1} \). Since \( \psi_{-1} \) is a \( \Lambda_N \)-morphism and since the category of \( \Lambda_N \)-algebras has all limits, \( B \) is a sub-\( \Lambda_N \)-semiring. It is a model over \( \mathbb{N} \) of the \( \Lambda_2 \)-ring \( \mathbb{Z}[x + x^{-1}] \) called the Chebyshev line in [3], but it is not isomorphic to the \( \mathbb{N} \)-algebra \( \mathbb{N}[y] \). In fact, it is not even finitely generated as an \( \mathbb{N} \)-algebra.

4.11. Flatness for \( \Lambda_N \)-semirings. If a \( \Lambda_N \)-semiring \( A \) is flat over \( \mathbb{N} \), then we have \( \mathbb{Q} \cap A = \mathbb{N} \) as subsets of \( \mathbb{Q} \otimes \mathbb{N} A \). (Recall from (2.9) that \( A \) is additively cancellative and hence maps injectively to \( \mathbb{Q} \otimes \mathbb{N} A \).) Indeed, since \( A \) is \( \mathbb{N} \)-flat, it contains no negative numbers, by (2.12); thus if \( a \in A \) is a rational number, we have \( \binom{n}{0} = m_{1^n}(a) \geq 0 \) for all \( n \), and so \( a \) must in fact be integral.

But as D. Grinberg pointed out to me, non-toric \( \Lambda_N \)-semirings which are flat and finitely presented over \( \mathbb{N} \) are still not hard to create. One example is the sub-\( \mathbb{N} \)-algebra of \( \psi_{-1} \)-invariants of the monoid algebra \( \mathbb{N}[x]/(x^3) \). It is isomorphic to \( \mathbb{N}[y]/(y^2 - y + 2) \), via the map \( y \mapsto x + x^2 \). In [6], De Smit and I proved some classification results about reduced \( \Lambda \)-rings that are finite flat over \( \mathbb{Z} \). It would be interesting to know if there are similar results over \( \mathbb{N} \).
4.12. Example: convergent exponential monoid algebras. Let \( B \) be submonoid of \( \mathbb{C} \) under addition, and let \( \mathbb{N}[e^B]^- \) denote the set of formal series \( \sum_{i=1}^{\infty} e^{b_i} \), where \( b_i \in B \), such that the complex series \( \sum_i e^{b_i} \) converges absolutely. (Of course, \( e \) now denotes the base of the natural logarithm.) We identify series that are the same up to a permutation of the terms. More formally, \( \mathbb{N}[e^B]^- \) is the set of elements \((\ldots, n_b, \ldots) \in \mathbb{N}^B \) such that the sums \( \sum_{b \in S} n_b e^{b} \) are bounded as \( S \) ranges over all finite subsets of \( B \). (Also, note that the notation is slightly abusive in that \( \mathbb{N}[e^B]^- \) depends on \( B \) itself and not just its image under the exponential map.) It is a sub-\( \mathbb{N} \)-module of \( \mathbb{N}^B \) and has a multiplication defined by

\[
\left( \sum_i [e^{b_i}] \right) \left( \sum_j [e^{c_j}] \right) = \left( \sum_{i,j} [e^{b_i+c_j}] \right),
\]

under which it becomes an \( \mathbb{N} \)-algebra. It is the unique multiplication extending that on the monoid algebra \( \mathbb{N}[B] \) which is continuous, in some suitable sense.

For integers \( n \geq 1 \), define

\[
\psi_n \left( \sum_i [e^{b_i}] \right) = \sum_i [e^{nb_i}].
\]

This is easily seen to be an element of \( \mathbb{N}[e^B]^- \), and hence the \( \psi_n \) form a commuting family of \( \mathbb{N} \)-algebra endomorphisms of \( \mathbb{N}[e^B]^- \). The induced endomorphisms of \( \mathbb{Q} \otimes_{\mathbb{N}} \mathbb{N}[e^B]^- \) prolong to a unique action of \( \Lambda_{\mathbb{Q}} \). It follows that for any symmetric function \( f \in \Lambda_{\mathbb{Q}} \), we have

\[
f \left( \sum_i [a_i] \right) = f([a_1], [a_2], \ldots) \in \mathbb{Q} \otimes_{\mathbb{N}} \mathbb{N}^B. \tag{4.12.1}
\]

Indeed, it is true when \( f = \psi_n \); then by the multiplication law above, it is true when \( f \) is a monomial in the \( \psi_n \), and hence for any \( f \in \Psi_{\mathbb{Q}} = \Lambda_{\mathbb{Q}} \). It follows that for any \( f \in \Lambda_{\mathbb{N}} \), the element \( f(\sum_i [a_i]) \) lies in \( \mathbb{N}^B \cap (\mathbb{Q} \otimes_{\mathbb{N}} \mathbb{N}[e^B]^-) = \mathbb{N}[e^B]^- \). So \( \mathbb{N}[e^B]^- \) inherits an action of \( \Lambda_{\mathbb{N}} \) from \( \mathbb{Q} \otimes_{\mathbb{N}} \mathbb{N}[e^B]^- \).

Observe that for all real \( t \geq 1 \), we have operators \( \psi_t \):

\[
\psi_t \left( \sum_i [e^{b_i}] \right) = \sum_i [e^{tb_i}].
\]

Indeed the series \( \sum_i e^{tb_i} \) is absolutely convergent if \( \sum_i e^{b_i} \) is. This is probably important. (See \( \text{[SS1]} \).)

4.13. Example: convergent monoid algebras. Now let \( A \) be a submonoid of the positive real numbers under multiplication, and let \( \log A = \{ b \in \mathbb{R} \mid e^b \in A \} \). Then define

\[
\mathbb{N}[A]^- = \mathbb{N}[e^{\log A}]^-.
\]
4.14. Remark: non-models for $\Lambda_Z$ over $\mathbb{N}$. The $\mathbb{N}$-algebra $\mathbb{N}[e_1, \ldots]$ is a model for $\Lambda_Z$ as a co-$\mathbb{N}$-module object, because $\Delta^+(e_n) = \sum_{i+j=n} e_i \otimes e_j$ has no negative coefficients. But it is not a model as a co-$\mathbb{N}$-algebra object, because

$$\Delta^x(e_2) = e_2 \otimes e_1^2 + e_1^2 \otimes e_2 - 2e_2 \otimes e_2$$

does have a negative coefficient.

The complete symmetric functions

$$h_n = \sum_{i_1 \leq \cdots \leq i_n} x_{i_1} \cdots x_{i_n},$$

behave similarly. The $\mathbb{N}$-algebra $\mathbb{N}[h_1, \ldots]$ is a model for $\Lambda_Z$ over $\mathbb{N}$ as a co-$\mathbb{N}$-module object, because $\Delta^+(h_n) = \sum_{i+j=n} h_i \otimes h_j$, but not as a co-$\mathbb{N}$-algebra object, because $\Delta^x(h_2) = h_2^2 \otimes h_2 - h_2 \otimes h_2 - h_2 \otimes h_2 + 2h_2 \otimes h_2$.

For Witt symmetric functions, we have

$$\Delta^+(\theta_p) = \theta_p \otimes 1 + 1 \otimes \theta_p - \sum_{i=1}^{p-1} \frac{1}{i} \binom{p}{i} \theta_i^p \otimes \theta_i^{p-i}$$
$$\Delta^x(\theta_p) = \theta_p^p \otimes \theta_1 + \theta_1^p \otimes \theta_p + p\theta_p \otimes \theta_p,$$

for any prime $p$. So $\mathbb{N}[\theta_1, \ldots]$ is not even a model as a co-$\mathbb{N}$-module object. Using $-\theta_p$ instead of $\theta_p$, removes the sign from the first formula but adds one in the second. We will see in section 8 that it is possible to circumvent this problem, at least if we care only about a single prime.

5. The Schur model for $\Lambda_Z$ over $\mathbb{N}$

5.1. Schur functions and $\Lambda^{\text{Sch}}$. For any partition $\lambda = (\lambda_1, \ldots, \lambda_r)$, put

$$s_\lambda := \det(h_{\lambda_i-i+j}) \in \Lambda_Z,$$

(5.1.1)

where as usual $h_0 = 1$ and $h_n = 0$ for $n < 0$. For example,

$$s_{321} = \det \begin{pmatrix} h_3 & h_4 & h_5 \\ h_1 & h_2 & h_3 \\ 0 & 1 & h_1 \end{pmatrix}.$$  

(5.1.2)

Such symmetric functions are called Schur functions. They have simple interpretations in terms of representation theory of general linear and symmetric groups, and several of the results we use below are usually proved using such an interpretation. But it is enough for us just to cite the results, and so we will ignore this interpretation. This is discussed more in 5.11 below.

Write

$$\Lambda^{\text{Sch}} := \bigoplus_\lambda \mathbb{N}s_\lambda,$$  

(5.1.3)
where \( \lambda \) runs over all partitions. The Schur functions are well known to form a \( \mathbb{Z} \)-linear basis for \( \Lambda_{\mathbb{Z}} \). (See p. 41 (3.3) of Macdonald’s book \[32\].) Therefore \( \Lambda^{\text{Sch}} \) is an \( \mathbb{N} \)-model for \( \Lambda_{\mathbb{Z}} \) as a module, by way of the evident inclusion \( \Lambda^{\text{Sch}} \to \Lambda_{\mathbb{Z}} \).

5.2. Proposition. \( \Lambda^{\text{Sch}} \) is a \( \mathbb{N} \)-model of \( \Lambda_{\mathbb{Z}} \) as a composition algebra, in a unique way. We also have \( \Lambda^{\text{Sch}} \subseteq \Lambda_{\mathbb{N}} \).

Proof. Since \( \Lambda^{\text{Sch}} \) is free, it is enough to check properties (1)–(6) of (3.6) and (3.7). These all reduce to standard facts about Schur functions, for which we will refer to chapter 1 of Macdonald’s book \[32\].

We have \( \Lambda^{\text{Sch}} \subseteq \Lambda_{\mathbb{N}} \) because of the equality

\[
s_{\lambda} = \sum_{\mu} K_{\lambda\mu} m_{\mu}
\] (5.2.1)

where the \( K_{\lambda\mu} \) are the Kostka numbers, which are in \( \mathbb{N} \), by Macdonald, p. 101 (6.4).

The fact that \( \Lambda^{\text{Sch}} \) is a sub-\( \mathbb{N} \)-algebra follows from the equations

\[
1 = s_0, \quad s_{\lambda}s_{\mu} = \sum_{\nu} \epsilon_{\lambda\mu}^{\nu} s_{\nu},
\]

where the \( \epsilon_{\lambda\mu}^{\nu} \) are the Littlewood–Richardson coefficients, which are in \( \mathbb{N} \), by Macdonald, pp. 113–114, (7.5) and (7.3).

By Macdonald, p. 72 (5.9) and p. 119 (7.9), we have

\[
\Delta^+(s_{\lambda}) = \sum_{\mu\nu} \gamma_{\lambda\mu}^{\nu} s_{\mu} \otimes s_{\nu},
\]

\[
\Delta^-(s_{\lambda}) = \sum_{\mu\nu} c_{\lambda\mu}^{\nu} s_{\mu} \otimes s_{\nu},
\]

where the \( \epsilon_{\lambda\mu}^{\nu} \) are again the Littlewood–Richardson coefficients and the \( \gamma_{\lambda\mu}^{\nu} \) are the Kronecker coefficients, which are in \( \mathbb{N} \) by pp. 114–115 of Macdonald. It follows that \( \Lambda^{\text{Sch}} \) is closed under the co-addition and comultiplication maps.

The containments \( \epsilon^+(\Lambda^{\text{Sch}}), \epsilon^-(\Lambda^{\text{Sch}}) \subseteq \mathbb{N} \) follow from \( \Lambda^{\text{Sch}} \subseteq \Lambda_{\mathbb{N}} \).

Finally, we have

\[
s_{\lambda} \circ s_{\mu} = \sum_{\nu} a_{\lambda\mu}^{\nu} s_{\nu},
\]

where \( a_{\lambda\mu}^{\nu} \in \mathbb{N} \), by Macdonald, p. 136 (8.10). So we have \( s_{\lambda} \circ s_{\mu} \in \Lambda^{\text{Sch}} \). It follows that \( \Lambda^{\text{Sch}} \) is closed under composition, since \( \Lambda^{\text{Sch}} \) is an \( \mathbb{N}-\mathbb{N} \)-bialgebra. Last, it contains the compositional identity element \( e \) because \( e = s_1 \).

5.3. Remark. The three models of \( \Psi_{\mathbb{Q}} \) over \( \mathbb{Q} \) we have seen are \( \Psi_{\mathbb{N}}, \Lambda_{\mathbb{N}}, \) and \( \Lambda^{\text{Sch}} \). The largest of the three is \( \Lambda_{\mathbb{N}} \). The other two are incomparable, even over \( \mathbb{Q}_{+} \). Indeed, we have \( s_{11} = (\psi^2_1 - \psi_2)/2 \) and \( \psi_2 = s^2_1 - 2s_{11} \).
5.4. Remark: $\Lambda^{\text{Sch}}$ is not free as an $\mathbb{N}$-algebra. As in (4.4), any generating set of $\Lambda^{\text{Sch}}$ contains all Schur functions that cannot be written as monomials with coefficient 1 in the other Schur functions. This is true of $s_1 = h_1$, $s_2 = h_2$, and $s_{11} = h_1^2 - h_2$ because they are irreducible in $\Lambda_{\mathbb{Z}} = \mathbb{Z}[h_1, h_2, \ldots]$. But they are not algebraically independent: $s_1^2 = s_2 + s_{11}$.

5.5. Remark: explicit description of a $\Lambda^{\text{Sch}}$-action. This is the same as in (4.8), but with operators $s_\lambda$. Now the identities are the following:

\[
\begin{align*}
s_0(x) &= 1 \\
(s_\lambda s_\mu)(x) &= s_\lambda(x) s_\mu(x) \\
s_\lambda(0) &= \varepsilon^+(s_\lambda) \\
s_\lambda(x + y) &= \Delta^+(s_\lambda)(x, y) := \sum_{\mu, \nu} c_\lambda^{\mu\nu} s_\mu(x) s_\nu(y) \\
s_\lambda(1) &= \varepsilon^x(s_\lambda) \\
s_\lambda(xy) &= \Delta^x(s_\lambda)(x, y) := \sum_{\mu, \nu} \gamma_\lambda^{\mu\nu} s_\mu(x) s_\nu(y) \\
s_1(x) &= x \\
s_\lambda(s_\mu(x)) &= (s_\lambda \circ s_\mu)(x) := \sum_{\nu} a_\lambda^{\nu\mu} s_\mu(x) s_\nu(y),
\end{align*}
\]

where $c_\lambda^{\mu\nu}$, $\gamma_\lambda^{\mu\nu}$, and $a_\lambda^{\nu\mu}$ are the Littlewood–Richardson, Kronecker, and Schur plethysm coefficients, as in the proof of (5.2).

In these terms, a $\Lambda_\mathbb{N}$-action gives rise to a $\Lambda^{\text{Sch}}$-action by the formula in (5.2.1).

5.6. Example. Since we have $\Lambda^{\text{Sch}} \subseteq \Lambda_\mathbb{N}$, any $\Lambda_\mathbb{N}$-semiring is a $\Lambda^{\text{Sch}}$-semiring. For example, in a monoid $\mathbb{N}$-algebra $\mathbb{N}[M]$ with the toric $\Lambda_\mathbb{N}$-structure, we have

\[
s_\lambda([m]) = \begin{cases} [m]^l & \text{if } \lambda = (l) \\ 0 & \text{otherwise} \end{cases}
\]

(5.6.1)

5.7. Example. Consider the monoid $\mathbb{Z}$-algebra $\mathbb{Z}[x]/(x^2 - 1)$ with the $\Lambda_\mathbb{Z}$-structure defined by $\psi_n(x) = x^n$, and let $A$ denote the sub-$\mathbb{N}$-algebra generated by $\eta = -x$. So we have a presentation

\[
A = \mathbb{N}[\eta]/(\eta^2 = 1).
\]

Then $A$ is a sub-$\Lambda^{\text{Sch}}$-semiring. Indeed, for any partition $\lambda$ of $r$, we have

\[
s_\lambda(-x) = (-1)^r s_\lambda(x) = \begin{cases} (-x)^r & \text{if } \lambda = (1^r) \\ 0 & \text{otherwise} \end{cases}
\]

by (5.6.1), where $\lambda'$ is the partition conjugate to $\lambda$. 
5.8. Frobenius lifts and \( p \)-derivations. For each integer \( n \geq 1 \), we have \( \psi_n = m_n \in \Lambda_N \). Therefore every \( \Lambda_N \)-semiring \( A \) has a natural endomorphism \( \psi_n \). It is called the Adams or Frobenius operator. When \( n \) is a prime \( p \), the induced map on \( \mathbb{F}_p \otimes_N A \) is the Frobenius map \( x \mapsto x^p \). To see this, one can reduce to the case of rings, or one can simply observe that we have
\[
d_p := \frac{1}{p}(\psi_1^p - \psi_p) \in \Lambda_N
\]
and hence \( \psi_p(x) + pd_p(x) = x^p \). The operator \(-d_p = \theta_p\) is a \( p \)-derivation in the language of Buium, p. 31 [11].

It is different for \( \Lambda^{\text{Sch}} \)-semirings. Since \( \psi_n \not\in \Lambda^{\text{Sch}} \) for \( n \geq 2 \), we do not generally have Frobenius lifts. However we do have the operators \( d_n := -\theta_n \) for \( n \geq 2 \). Indeed Scharf–Thibon [37] and Doran [12] proved a conjecture of Reutenauer [34] that
\[
-\theta_n \in \Lambda^{\text{Sch}} \quad \text{for} \quad n \geq 2.
\]

Of course, for \( n = 1 \) we have the opposite: \( +\theta_1 \in \Lambda^{\text{Sch}} \). This irksome exception suggests the perspective here might not be the most fundamental one. For example, it is a standard result in the theory of \( p \)-typical Witt vectors (tantamount to the Cartier–Dieudonné–Dwork lemma) that the ring \( \mathbb{Z}[d_{pn} \mid n \geq 0] \) of \( p \)-typical symmetric functions agrees with \( \mathbb{Z}[d_{pn}^n \mid n \geq 0] \), the composition \( \mathbb{Z} \)-algebra generated by the iterated \( p \)-derivations. The families of generators \( d_{pn} \) and \( d_{pn}^n \) agree for \( n \leq 1 \) but not for \( n = 2 \).

**Question 2.** Do the symmetric functions \( d_{pn} - d_{pn}^n \) lie in \( \Lambda^{\text{Sch}} \)? More generally, do \( d_{pn+m} - d_{pn} \circ d_{pn} \)? If not, do they lie in \( \Lambda_N \)?

If the answers are yes, it would suggest that the operators \( d_{pn}^n \) are more fundamental from the point of view of positivity than the \( d_{pn} \). In other words, up to some signs, the iterated \( p \)-derivations generate a larger sub-\( \mathbb{N} \)-algebra than the Witt symmetric functions. Computations have shown that the answer to the stronger question above is yes when \( p = 2 \), \( m + n \leq 3 \); \( p = 3 \), \( m + n = 2 \); and \( p = 5 \), \( m + n = 2 \).

The obvious generalization \( d_{rs} - d_r \circ d_s \in \Lambda^{\text{Sch}} \) is not always true. Computations have shown it is true for \( (r, s) = (2, 3), (2, 5), (3, 5), (5, 3) \) but false for \( (3, 2) \) and \( (5, 2) \).

5.9. Remark: the necessity of nonlinear operators. Observe that, neither \( \Lambda_N \) nor \( \Lambda^{\text{Sch}} \) is generated by linear operators, even after base change to \( \mathbb{R}_+ \). Indeed, because we have \( m_{14} = e_2 = s_1 \), any generating set of \( \Lambda_+ \) or \( \mathbb{R}_+ \otimes_N \Lambda^{\text{Sch}} \) would have to contain a multiple of \( e_2 \), which is nonadditive. So an action of \( \Lambda_+ \) or \( \Lambda^{\text{Sch}} \) on \( \mathbb{N} \)-algebra cannot be expressed entirely in terms of linear operators.

It appears to be the case that over \( \mathbb{Z} \) or \( \mathbb{Z}_p \), any composition algebra that cannot be generated by linear operators can trace its origin to the \( d_n \) operators—in other words, to lifting Frobenius maps to characteristic 0. There are probably theorems to this effect. (For example, Buium [11] classifies ring scheme structures on the plane \( \mathbb{A}^2_{\mathbb{Z}_p} \), which is perhaps the first test case, and he is able to prove such a result.
there. Could it be that the existence of nonlinear composition algebras over \(\mathbb{R}^+\) is due to a similarly identifiable phenomenon? It seems quite optimistic to hope the answer is yes, but if it were, it would be so important that the question should not be dismissed.

5.10. **Remark: composition algebras over number fields.** The composition ring \(\Lambda_Z\) has analogues over rings of integers in general number fields. We will not really use them in this chapter, but they will appear in a several remarks and open questions.

Let \(K\) be a number field, let \(\mathcal{O}_K\) denote its ring of integers, and let \(E\) denote a family of maximal ideals of \(\mathcal{O}_K\). Let \(\Lambda_{\mathcal{O}_K,E}\) denote the composition \(\mathcal{O}_K\)-algebra characterized by the property that an action on any flat \(\mathcal{O}_K\)-algebra \(A\) is the same as a commuting family of \(\mathcal{O}_K\)-algebra endomorphisms \(\psi_p\) such that each \(\psi_p\) reduces to the Frobenius map \(x \mapsto x^{[\mathcal{O}_K : p]}\) on \(A/pA\). This construction and the associated Witt vector functor \(W_{\mathcal{O}_K,E}\) are discussed in much more detail in section 1 of my paper [4]. When \(K = \mathbb{Q}\), it reduces to a special case of the construction of \(\Lambda^{\text{Sch}}\), as is explained there, but otherwise there is no overlap between the two.

5.11. **Remark: representation theory and K-theory.** The standard way of looking at \(\Lambda^{\text{Sch}}\) is from the point of view of the representation theory of the general linear groups \(\text{GL}_n\). Then \(\Lambda^{\text{Sch}}\) is identified with \(\bigoplus_n R^+_{\text{GL}_n}\), where \(R^+_{\text{GL}_n}\) is the representation semiring of \(\text{GL}_n\), and \(\Lambda^{\text{Sch}}\) is identified with \(\bigoplus_n R^+_{\text{GL}_n} T_n^{S_n}\), where \(R^+_{\text{GL}_n} T_n^{S_n}\) is the symmetric part of the representation semiring of the diagonal torus \(T_n\). The positivity properties of \(\Lambda^{\text{Sch}}\) then hold because the (co-)operations on \(\Lambda^{\text{Sch}}\) correspond to (co-)operations on categories of representations, and hence they preserve effectivity.

One has similar behavior in K-theory. Given a partition \(\lambda\) of \(n\), let \(V_\lambda\) be the corresponding irreducible \(\mathbb{Q}\)-linear representation of \(S_n\). Then we can define functors \(S_\lambda\) on amenable linear tensor categories \(\mathcal{C}\) by \(S_\lambda(E) = \text{Hom}_{\mathcal{C}[S_n]}(V_\lambda, E^{\otimes n})\).

Thus the Schur functions \(s_\lambda\) preserve effectivity in the Grothendieck ring \(K_0(\mathcal{C})\), and so the effective sub-semiring \(K_0^+(\mathcal{C})\) is in fact a sub-\(\Lambda^{\text{Sch}}\)-semiring.

While this is a better way of thinking about positivity properties on \(\Lambda^{\text{Sch}}\), the deeper meaning of the connection with Witt vectors is still elusive. It is not clear why one would want to assemble all this representation-theoretic data into an algebraic gadget \(\Lambda_Z\) and then consider the corresponding Witt vector rings. Indeed, the composition algebras \(\Lambda_{\mathcal{O}_K,E}\) of \(\text{GL}_n\) and the corresponding Witt vector functors \(W_{\mathcal{O}_K,E}\) also have arithmetic interest (for instance in the theory of complex multiplication when \(K\) is an imaginary quadratic field), but no representation theoretic interpretation of \(\Lambda_{\mathcal{O}_K,E}\) is known. In fact, no interpretation in terms of something similar to symmetric functions is known.

At the time of this writing, there is still a tension between the following three observations: (1) the connection between Frobenius lifts and the representation theory of \(\text{GL}_n\) appears to be a coincidence, merely an instance of a fundamental algebraic object arising in two unrelated contexts; (2) to define \(\Lambda^{\text{Sch}}\) and \(W^{\text{Sch}}\) one needs the positivity results established with representation theory; (3) \(\Lambda^{\text{Sch}}\) and
$W^{\text{Sch}}$ have some arithmetic interest, as in section 7 below. The situation with $\Lambda_N$ and $W$ is perhaps less mysterious—they seem more important from the arithmetic point of view and also require no nontrivial positivity results from representation theory. Perhaps the resolution will be that $\Lambda^{\text{Sch}}$ and $W^{\text{Sch}}$ are of arithmetic interest only through their relation to $\Lambda_N$ and $W$.

Whatever the case, this is why I have ignored K-theory and used the representation theory as a black box.

6. Witt vectors of $\mathbb{N}$-algebras

For the classical theory of Witt vectors for rings, one can see Bergman’s lecture 26 in Mumford’s book [33], or chapter III of Hazewinkel’s book [17], or §1 of chapter IX of Bourbaki [9] and especially the exercises there. One can also see Witt’s original writings, [45] and pp. 157–163 of [46].

6.1. $W$ and $W^{\text{Sch}}$. For any $\mathbb{N}$-algebra $A$, define the $\mathbb{N}$-algebra of Witt vectors with entries in $A$ by

$$W(A) := \text{Alg}_\mathbb{N}(\Lambda_N, A),$$

and define the $\mathbb{N}$-algebra of Schur-Witt vectors with entries in $A$ by

$$W^{\text{Sch}}(A) := \text{Alg}_\mathbb{N}(\Lambda^{\text{Sch}}, A).$$

They both inherit $\mathbb{N}$-algebra structures from the $\mathbb{N}$-$\mathbb{N}$-bialgebra on $\Lambda_N$ and $\Lambda^{\text{Sch}}$, as explained in section 5.

Since both $\Lambda_N$ and $\Lambda^{\text{Sch}}$ are models of $\Lambda_\mathbb{Z}$ over $\mathbb{N}$, both Witt functors agree with the usual big Witt vector functor on rings. The inclusion $\Lambda^{\text{Sch}} \subseteq \Lambda_N$ induces an $\mathbb{N}$-algebra map

$$W(A) \to W^{\text{Sch}}(A),$$

which is natural in $A$.

Since $\Lambda_N$ is a composition algebra, $W(A)$ naturally has a $\Lambda_N$-action. It is defined, for $a \in W(A)$ and $f \in \Lambda_N$, by

$$f(a) : g \mapsto a(g \circ f) \tag{6.1.1}$$

for all $g \in \Lambda_N$. Equivalently, the comonad structure map $W(A) \to W(W(A))$ is given by $a \mapsto [g \mapsto a(g \circ f)]$. The analogous statements hold for $W^{\text{Sch}}$ and $\Lambda^{\text{Sch}}$.

6.2. Ghost and similar maps. Consider the diagram

$$
\begin{array}{ccc}
\Lambda^{\text{Sch}} & \longrightarrow & \Lambda_N \\
\downarrow & & \downarrow \\
\mathbb{N}[h_1, h_2, \ldots] & \longrightarrow & \Psi_N
\end{array}
$$
Witt vectors, semirings, and total positivity

of sub-$\mathbb{N}$-algebras of $\Lambda_{\mathbb{Z}}$, where the bottom map is dashed to indicate that it is defined only after base change to $\mathbb{Q}_+$. If we apply the functor $\text{Alg}_N(-,A)$ to this diagram, we get the following diagram of Witt vectors:

\[
\begin{array}{ccc}
W^{\text{Sch}}(A) & \xrightarrow{v} & W(A) \\
\sigma \downarrow & & \downarrow w \\
1 + tA[[t]] & \leftarrow \sigma_{gh} & - A^\infty
\end{array}
\]  

(6.2.1)

where in the bottom row we have made use of the following identifications

\[
\text{Alg}_N(\mathbb{N}[h_1,\ldots],A) \xrightarrow{\sim} 1 + tA[[t]], \quad a \mapsto \sum_i a(h_i)t^i
\]

\[
\text{Alg}_N(\Psi_N, A) \xrightarrow{\sim} A^\infty, \quad a \mapsto \langle a(\psi_1), a(\psi_2), \ldots \rangle.
\]

The dashed arrow indicates that $A$ needs to be a $\mathbb{Q}_+$-algebra for $\sigma_{gh}$ to be defined.

Recall from (4.14) that $1 + tA[[t]]$ is in general only a commutative monoid, under multiplication, but the other three are $\mathbb{N}$-algebras.

6.3. Proposition. (1) $v$ is a morphism of $\Lambda^\text{Sch}$-semirings, and $w$ is a morphism of $\Psi_N$-semirings, and $\sigma$ and $\sigma_{gh}$ are morphisms of $\mathbb{N}$-modules: $\sigma_*(x + y) = \sigma_*(x)\sigma_*(y)$, $\sigma_*(0) = 1$.

(2) If $A$ is a $\mathbb{Z}$-algebra, then $v$ and $\sigma$ are bijections. If $A$ is contained in a $\mathbb{Z}$-algebra, then $v$ and $\sigma$ are injections.

(3) If $A$ is a $\mathbb{Q}$-algebra, then $w$ and $\sigma_{gh}$ are bijections. If $A$ is contained in a $\mathbb{Q}$-algebra, then $w$ and $\sigma_{gh}$ are injections.

Proof. (1): The first statement holds because the inclusions $\Lambda^\text{Sch} \rightarrow \Lambda_N$ and $\Psi_N \rightarrow \Lambda_N$ are morphisms of composition algebras. The second follows from the equations $\Delta^*(h_n) = \sum_{i+j=n} h_i \otimes h_j$ and $\varepsilon^*(h_n) = 0$ for $n \geq 1$.

(2): This holds because $\Lambda_N$, $\Lambda^\text{Sch}$, and $\mathbb{N}[h_1,\ldots]$ are models for $\Lambda_{\mathbb{Z}}$ over $\mathbb{N}$.

(3): Similarly, this holds because $\Lambda_N$, $\Psi_N$, and $\mathbb{N}[h_1,\ldots]$ are models for $\Psi_{\mathbb{Q}}$ over $\mathbb{N}$.

6.4. Remark. The map $v: W(A) \rightarrow W^{\text{Sch}}(A)$ can fail to be injective when $A$ is not contained in a ring. For example, it is not injective when $A$ is the Boolean semiring $\mathbb{N}/(1 + 1 = 1)$. See the forthcoming work with Darij Grinberg [7].

6.5. Coordinates for Witt vectors of rings. When $A$ is a ring, the bijection $\sigma$ of (6.2.1) allows us to identify Witt vectors and power series. In other words, the complete symmetric functions give a full set of coordinates on the $\mathbb{Z}$-scheme $W = \text{Spec} \Lambda_{\mathbb{Z}}$. 
Note that there are three other common conventions for identifying Witt vectors and power series. Given any two signs $\varepsilon_1, \varepsilon_2 \in \{+1, -1\}$, the map

$$\sigma_{\varepsilon_2}^\varepsilon: W(A) \sim \to 1 + tA[[t]]$$

$$a \mapsto \left( \sum_i a(e_i)(\varepsilon_1 t)^i \right)^{\varepsilon_2}$$

(6.5.1)

is an isomorphism of $\mathbb{N}$-modules. We have taken $\sigma = \sigma^-_-$ as our convention. It has one advantage over the others, which is that $\sigma_{gh}$ can be defined over $\mathbb{Q}_+$ rather than just $\mathbb{Q}$. Equivalently, the complete symmetric functions can be written as polynomials in the power sums with nonnegative coefficients. This is not true with the elementary symmetric functions, even if we allow ourselves to change the signs of half of them. I am not aware of any other convincing reason to prefer one of these conventions.

The presentation $\Lambda_\mathbb{Z} = \mathbb{Z}[\theta_1, \ldots]$ gives another set of full coordinates:

$$W(A) \sim \to \Lambda_\mathbb{Z}^\infty, \quad a \mapsto (a(\theta_1), a(\theta_2), \ldots)$$

We will call the elements $a(\theta_1), \ldots \in A$ the Witt components or Witt coordinates of $a$. Addition does not have a simple closed form in these coordinates, unlike in the series coordinates above. But the Witt coordinates do have two advantages over the series coordinates. First, they are related to the power sums by a simple, closed-form expression (4.5.1). Second, they behave well with respect to localization. More precisely, let $E$ be a set of prime numbers. Then we have a free presentation

$$\Lambda_{\mathbb{Z}[1/E]} = \mathbb{Z}[1/E][\theta_m \circ \psi_n],$$

where $m$ runs over the positive integers whose prime divisors do not lie in $E$, and $n$ runs over those whose prime divisors do. (See the isomorphisms (1.20.1) and (5.3.1) in [4], for example.)

The components of the image of $a$ under the map

$$w: W(A) \to \Lambda_\mathbb{Z}^\infty, \quad a \mapsto \langle a(\psi_1), a(\psi_2), \ldots \rangle$$

are its ghost components. Their advantage is that all algebraic structure is completely transparent—addition and multiplication are performed componentwise and Frobenius operations are given by scaling the indices. When $A$ is a $\mathbb{Q}$-algebra, $w$ is a bijection, and so the ghost components have no deficiencies over $\mathbb{Q}$. When $A$ is a torsion-free ring (i.e. a flat $\mathbb{Z}$-algebra), $w$ is an injection, and the ghost coordinates are still useful there. We write the ghost components with angle brackets to avoid any confusion with the Witt components.

6.6. Witt vectors for semirings contained in rings. A Witt vector is determined by its series coordinates when $A$ is contained in a ring. Indeed, a map $\Lambda_\mathbb{N} \to A$ is equivalent to a map $a: \Lambda_\mathbb{Z} \to \mathbb{Z} \otimes \mathbb{N} A$ subject to the effectivity condition $a(m\lambda) \in A$ for all partitions $\lambda$. Therefore we have

$$W(A) = \{ \sum_i a_i t^i \in 1 + tA[[t]] \mid \sum_\mu M_\mu a_{\mu_1} a_{\mu_2} \cdots \in A \},$$

(6.6.1)
where $M$ is the transition matrix from the monomial basis $(m_\lambda)_\lambda$ to the basis of monomials in the complete symmetric functions $h_n$:

$$m_\lambda = \sum_\mu M^\mu_\lambda h_{\mu_1} h_{\mu_2} \cdots.$$  

The same formula holds for $W^{\text{Sch}}(A)$ but where $M$ is the transition matrix from the Schur basis to this $h$ basis.

Similar statements hold for the ghost components when $A$ is contained in a $Q$-algebra. For example, we have

$$W^{\text{Sch}}(A) = \{ \langle a_1, a_2, \ldots \rangle \in A^\infty \mid \sum_\mu N^\mu_\lambda a_{\mu_1} a_{\mu_2} \cdots \in A \},$$  

where $N$ is the transition matrix from the Schur basis to the basis of monomials in the power sums:

$$s_\lambda = \sum_\mu N^\mu_\lambda \psi_{\mu_1} \psi_{\mu_2} \cdots.$$  

In this particular case, the matrix $N$ has well-known description: the block where $|\lambda| = |\mu| = n$ is the inverse of the character table of the symmetric group $S_n$. See Macdonald, (7.8) p. 114 [32].

6.7. Example: some explicit effectivity conditions. Let us write out the effectivity conditions on ghost components for $W^{\text{Sch}}(A)$ and $W(A)$ up to weight 4. If a ghost vector $(a_1, \ldots, a_4) \in A^4$ lies in the image of $W^{\text{Sch}}(A)$, then the following eleven elements of $Q \otimes N A$ are contained in $A$:

$$a_1, \, (a_1^2 \pm a_2)/2, \, (a_1^3 \pm 3a_1 a_2 + 2a_3)/6, \, (a_1^3 - a_3)/3,$$

$$\quad (a_1^4 \pm 6a_1^2 a_2 + 3a_1^2 + 8a_1 a_3 \pm 6a_4)/24,$$

$$\quad (3a_1^4 \pm 6a_1^2 a_2 - 3a_1^2 \mp 6a_4)/24, \, \left(2a_1^4 + 6a_2^2 - 8a_1 a_3\right)/24.$$

These are all the conditions corresponding to $s_4$ with $|\lambda| \leq 4$.

In the case of $W(A)$, the following elements are required to be in $A$:

$$a_1, \, a_2, \, a_3, \, a_4,$$

$$\quad (a_1^2 - a_2)/2, \, a_1 a_2 - a_3,$$

$$\quad (a_1^3 - 3a_1 a_2 + 2a_3)/6, \, a_1 a_3 - a_4, \, (a_2^2 - a_4)/2,$$

$$\quad (a_1^2 a_2 - a_2^2 - 2a_1 a_3 + 2a_4)/2, \, \left(a_1^4 - 6a_1^2 a_2 + 3a_1^2 + 8a_1 a_3 - 6a_4\right)/24.$$  

These are all the conditions corresponding to $m_\lambda$ with $|\lambda| \leq 4$. 
6.8. Coordinates for Witt vectors of semirings. For general semirings, there is no purely vector-like description of $W(A)$ or $W^{\mathrm{Sch}}(A)$. Indeed, for any set $T$, the functor $A \mapsto A^T$ is represented by the free $\mathbb{N}$-algebra on $T$, but by (4.4) and (5.4), neither $\Lambda^N$ nor $\Lambda^{\mathrm{Sch}}$ is free as an $\mathbb{N}$-algebra.

Instead Witt vectors are cut out of infinite-dimensional affine space by quadratic relations determined by the structure constants of multiplication in the relevant basis. Let $\mathcal{P}$ denote the set of partitions. For any $\lambda, \mu \in \mathcal{P}$, write $m_{\lambda \mu} = \sum_{\nu} b_{\lambda \mu}^\nu m_{\nu}$, $s_{\lambda \mu} = \sum_{\nu} c_{\lambda \mu}^\nu s_{\nu}$.

Then we have

$$W(A) = \{ a \in A^P \mid a_0 = 1, m_{\lambda \mu} a = \sum_{\nu} b_{\lambda \mu}^\nu a_{\nu} \}$$
$$W^{\mathrm{Sch}}(A) = \{ a \in A^P \mid a_0 = 1, s_{\lambda \mu} a = \sum_{\nu} c_{\lambda \mu}^\nu a_{\nu} \}.$$  

Addition and multiplication are then defined using the structure constants for the coproducts $\Delta^+$ and $\Delta^x$ with respect to the basis in question.

6.9. Topology and pro-structure. It is often better to view $W(A)$ and $W^{\mathrm{Sch}}(A)$ as pro-sets, or pro-discrete topological spaces, as when $A$ is a ring. We do this as follows. Let $(P_i)_{i \in I}$ denote the filtered system of finitely generated sub-$\mathbb{N}$-algebras of $\Lambda^N$. Then we have

$$W(A) = \text{Hom}(\text{colim}_{i \in I} P_i, A) = \lim_{i \in I} \text{Hom}(P_i, A).$$  

So $W(A)$, and similarly $W^{\mathrm{Sch}}(A)$, has the natural structure of a pro-set. When $A$ is a ring, $W(A)$ can be expressed as an inverse limit of rings. I do not know, however, whether the analogous statement holds over $\mathbb{N}$:

**Question 3.** Are $W(A)$ and $W^{\mathrm{Sch}}(A)$ pro-objects in the category of $\mathbb{N}$-algebras? More naturally, is this true as representable functors? Equivalently, can $\Lambda_N$ and $\Lambda^{\mathrm{Sch}}$ be expressed as filtered colimits $\text{colim} P_i$, where each $P_i$ is a co-$\mathbb{N}$-algebra object in $\text{Alg}_{\mathbb{N}}$ which is finitely generated as an $\mathbb{N}$-algebra?

For a stronger form of this question, see (8.1).

6.10. Teichmüller and anti-Teichmüller elements. For any $\mathbb{N}$-algebra $A$, consider the monoid algebra $\mathbb{N}[A]$ with the toric $\Lambda_N$-structure. By the adjunction property of $W$, the $\mathbb{N}$-algebra map $\mathbb{N}[A] \to A$ defined by $[a] \mapsto a$ lifts to a unique $\Lambda_N$-equivariant map

$$\mathbb{N}[A] \to W(A).$$

For $a \in A$, the image of $[a]$ in $W(A)$ is called the **Teichmüller lift** of $a$ and is also denoted $[a]$. Explicitly, $[a]$ is the $\mathbb{N}$-algebra map $\Lambda_N \to A$ determined by

$$[a] : m_\lambda \mapsto \begin{cases} a^r & \text{if } \lambda = (r) \\ 0 & \text{otherwise} \end{cases}$$ (6.10.1)
The Teichmüller map \( a \mapsto [a] \) is a map of monoids \( A \to W(A) \) under multiplication.

For the anti-Teichmüller elements, consider the \( \mathbb{N} \)-algebra \( \mathbb{N}[\eta]/(\eta^2 = 1) \otimes_{\mathbb{N}} \mathbb{N}[A] \), determined by \( \eta \mapsto 1 \) and \( [a] \mapsto a \) for all \( a \in A \) lifts by adjunction to a unique \( \Lambda^\text{Sch} \)-equivariant map \( \mathbb{N}[\eta]/(\eta^2 = 1) \otimes_{\mathbb{N}} \mathbb{N}[A] \to W^\text{Sch}(A) \).

For any \( a \in A \), define the anti-Teichmüller lift \( \{a\} \in W^\text{Sch}(A) \) to be the image of \( \eta \otimes a \). So in \( W(\mathbb{Z} \otimes_{\mathbb{N}} A) \) we have

\[
\{a\} = [-a].
\]

(6.10.2)

We also have

\[
\{a\} : s_\lambda \to \begin{cases} a^r & \text{if } \lambda = (1^r) \\ 0 & \text{otherwise} \end{cases}
\]

(6.10.3)

and

\[
[a]\{b\} = \{ab\}, \quad \{a\}\{b\} = \{ab\}, \quad \{a\} = \{1\}\{a\}.
\]

(6.10.4)

Observe that the anti-Teichmüller lifts exist only in \( W^\text{Sch}(A) \) and not generally in \( W(A) \). For example, the element \( \{1\} \in W^\text{Sch}(\mathbb{N}) \) is not in the sub-\( \mathbb{N} \)-algebra \( W(\mathbb{N}) \). Indeed, its ghost vector \((1, -1, 1, -1, \ldots)\) is not even in \( \mathbb{N}^\infty \).

### 6.11. The involution and the forgotten symmetric functions

Let \( \omega : \Lambda_{\mathbb{Z}} \to \Lambda_{\mathbb{Z}} \) denote the ring map determined by \( \omega(h_n) = e_n \) for all \( n \). Then we have

\[
\omega(s_\lambda) = s_{\lambda'}
\]

for all \( \lambda \), where \( \lambda' \) denotes the conjugate partition. (See Macdonald, p. 23 (2.9).) Therefore we have \( \omega(\Lambda^\text{Sch}) = \Lambda^\text{Sch} \), and so \( \omega \) induces a functorial set map

\[
W^\text{Sch}(A) \to W^\text{Sch}(A), \quad a \mapsto a \circ \omega.
\]

In fact, this map is simply multiplication by \( \{1\} \). Indeed, it follows from the identities of (4.5.2) that

\[
\omega(\psi_n) = (-1)^{n+1} \psi_n
\]
for all \( n \); now combine this with the equality \{1\} = \{1, -1, -1, \ldots\}.

The symmetric functions \( f_\lambda := \omega(m_\lambda) \) are sometimes called the forgotten symmetric functions. Their span \( \Lambda^\text{for} := \omega(\Lambda_N) \) contains \( \Lambda^\text{Sch} \). It represents the functor \( \text{Alg}(\Lambda^\text{for}, A) = \{1\} W(A) \), which is the free \( W(A) \)-module generated by the symbol \{1\}. The induced map \( \{1\} W(A) \to W^\text{Sch}(A) \) is a \( W(A) \)-module map, but not an \( N \)-algebra map. In particular, \( \Lambda^\text{for} \) is not a model for \( \Lambda_\mathbb{Z} \) as co-\( N \)-algebra object.

6.12. Example: the map \( \mathbb{N} \to W^\text{Sch}(A) \) is injective unless \( A = 0 \).

In fact, the map \( \varphi \) from \( \mathbb{N} \) to any nonzero \( \Lambda^\text{Sch} \)-semiring is injective. For if \( m, n \in \mathbb{N} \) have the same image and \( m < n \), then we have

\[
1 = \varphi\left(\binom{n}{m}\right) = s_{1^n}(\varphi(n)) = s_{1^n}(\varphi(m)) = \varphi\left(\binom{m}{n}\right) = 0.
\]

In particular, the map from \( \mathbb{N} \) to any nonzero \( \Lambda_N \)-semiring is injective.

On the other hand, the \( \Lambda^\text{Sch} \)-equivariant map \( \mathbb{N}[\eta]/(\eta^2 = 1) \to W^\text{Sch}(A) \) sending \( \eta \to \{1\} \) is not always injective. Indeed, when \( A = \mathbb{Z}/2\mathbb{Z} \), we have \( [-1] = [1] = 1 \) and hence \( 1 + \eta \mapsto 0 \).

7. Total positivity

In this section, we will give explicit descriptions of \( W(\mathbb{R}_+) \) and \( W^\text{Sch}(\mathbb{R}_+) \) and then use this to describe \( W(\mathbb{N}) \). These are very rich objects, and there is much more to say about them than we can here.

We will find it convenient to use the series normalization \( \sigma_+^+ \) of (6.6), as well as our standard one \( \sigma = \sigma^- \). By (4.5.2), we have

\[
\sigma(x) = \sum_i x(h_i)t^i, \quad \sigma_+^+(x) = \sum_i x(e_i)t^i.
\]

The two are related by the involution \( f(t) \mapsto f(-t)^{-1} \). In other words, we have

\[
\sigma_+^+(x) = \sigma(\{1\}x). \tag{7.0.1}
\]

7.1. Proposition. For any Witt vector \( x \in W(\mathbb{R}_+) \), write \( 1 + a_1 t + a_2 t^2 + \cdots \) for the series \( \sigma_+^+(x) \in 1 + t\mathbb{R}_+[[t]] \). Then for all \( n \), we have

\[
a_n \leq \frac{a_n^n}{n!}. \tag{7.1.1}
\]

In particular, the series \( \sum_n a_n t^n \) converges to an entire function on \( \mathbb{C} \).

Proof. Since \( m_{2,1} = -e_{n-1}e_1 - ne_n \), we have \( a_{n-1}a_1 - na_n \geq 0 \). Then follows by induction.
7.2. Total nonnegativity. A formal series \( \sum_n a_n t^n \in 1 + t \mathbb{R}[[t]] \) is said to be totally nonnegative if all (finite) minors of the infinite matrix \((a_{i-j})_{ij}\) are nonnegative. (To be clear, we understand \(a_0 = 1\) and \(a_n = 0\) for \(n < 0\). Other authors allow more general series.) For example, up to the \(2 \times 2\) minors these inequalities amount to the following:

\[
a_n \geq 0, \quad a_n a_{n+i-j} \geq a_{n+i} a_{n-j}.
\]

Chapter 8 of Karlin’s book [20] has an extensive treatment.

7.3. Proposition. A Witt vector \(x \in W(\mathbb{R})\) lies in \(W^{\text{Sch}}(\mathbb{R}_+)\) if and only if the series corresponding to \(x\) under the bijection \(\sigma: W(\mathbb{R}) \rightarrow 1 + t \mathbb{R}[[t]]\) is totally nonnegative. The same is true for the bijection \(\sigma^+_1\).

Proof. First consider the universal series \(1 + h_1 t + h_2 t^2 + \cdots \in \Lambda_\mathbb{Z}[[t]]\). Then it is a standard fact in algebraic combinatorics that the minors of the matrix \((h_{i-j})_{ij}\) generate \(\Lambda^{\text{Sch}}\) as an \(\mathbb{N}\)-module. (The minors are the so-called skew Schur functions \(s_{\lambda/\mu}\), by Macdonald, p. 70 (5.4) [32]. Their \(\mathbb{N}\)-span contains \(\Lambda^{\text{Sch}}\) because every Schur function is a skew Schur function. For the other containment, see (9.1) and the following text on p. 142 of Macdonald.) Therefore \(x\), viewed as a ring map \(\Lambda_\mathbb{Z} \rightarrow \mathbb{R}\), sends \(\Lambda^{\text{Sch}}\) to \(\mathbb{R}_+\) if and only if the corresponding series is totally nonnegative.

For the second statement, combine the above with (7.0.1) and the fact that \(\{1\}\) is an element of \(W^{\text{Sch}}(A)\) and is invertible.

7.4. Theorem (Edrei [13], Thoma [41]). The totally nonnegative series in \(1 + t \mathbb{R}[[t]]\) are precisely those of the form

\[
e^{\gamma t} \prod_{i=1}^{\infty} \frac{(1 + \alpha_i t)}{1 - \beta_i t},
\]

(7.4.1)

where \(\gamma, \alpha_i, \beta_i \geq 0\) (and both \(\sum \alpha_i\) and \(\sum \beta_i\) converge), and every such representation is unique.

For a proof, see Karlin’s book [20], in which the result is theorem 5.3, p. 412.

7.5. Corollary. Consider a Witt vector \(x \in W(\mathbb{R})\). Then \(x\) lies in \(W^{\text{Sch}}(\mathbb{R}_+)\) if and only if \(\sigma^+_1(x)\) is of the form

\[
e^{\gamma t} \prod_{i=1}^{\infty} \frac{(1 + \alpha_i t)}{1 - \beta_i t},
\]

(7.5.1)

where \(\gamma, \alpha_i, \beta_i \geq 0\). Similarly, \(x\) lies in \(W(\mathbb{R}_+)\) if and only if \(\sigma^+_1(x)\) is of the form

\[
e^{\gamma t} \prod_{i=1}^{\infty} (1 + \alpha_i t),
\]

(7.5.2)

where \(\gamma, \alpha_i \geq 0\).
Proof. The first part follows from (7.3) and (7.4).

Now consider the second part. The first part and (7.1) imply that \( \sigma^+ + (x) \) is of the form (7.5.2) for all \( x \in W(\mathbb{R}_+) \). Conversely because we have \( 1 + \alpha t = \sigma^+_+(\alpha t) \) and \( e^{\gamma t} = \lim_{n \to \infty} \left( 1 + \frac{\gamma t}{n} \right)^n \), any series of the form (7.5.2) is a limit of finite products of series in \( \sigma^+_+(W(\mathbb{R}_+)) \).

Now observe that \( \sigma^+ \) identifies \( W(\mathbb{R}_+) \) with a submonoid of \( 1 + t \mathbb{R}_+[[t]] \). Further it is a closed subset because it is defined by a family of nonnegativity conditions, as in (6.6.1). (Compare Kingman [22][23].)

7.6. Remark: \( W(\mathbb{R}_+) \) as a convergent monoid algebra. Equivalently, the subset \( W^{\text{Sch}}(\mathbb{R}_+) \) consists of the Witt vectors in \( W(\mathbb{R}) \) that can be represented (necessarily uniquely) in the form

\[
\sum_{i=1}^{\infty} [\alpha_i] + \sum_{i=1}^{\infty} \{ \beta_i \} + [\gamma] \xi,
\]

where \( \gamma, \alpha_i, \beta_i \geq 0 \) (and both \( \sum_i \alpha_i \) and \( \sum_i \beta_i \) converge) and where \( \xi \) is the Witt vector with ghost components \( \langle 1, 0, 0, ... \rangle \), or equivalently such that \( \sigma(\xi) = e^t \).

Another interpretation is that the evident map is an isomorphism

\[
\mathbb{N}[\mathbb{R}_+] \otimes \mathbb{N}[\eta]/(\eta^2 = 1) \oplus \mathbb{R}_+ \xi \to W^{\text{Sch}}(\mathbb{R}_+).
\]

Similarly, we have \( W(\mathbb{R}_+) = \mathbb{N}[\mathbb{R}_+] \otimes \mathbb{R}_+ \xi \).

7.7. Remark: \( W(\mathbb{R}_+) \) and entire functions. If we view (7.5.2) as a Hadamard factorization, then we see that yet another interpretation of \( W(\mathbb{R}_+) \) is that it is the set of entire functions \( f \) on \( \mathbb{C} \) of order at most 1 such that \( f(0) = 1 \), the zeros of \( f \) are negative real numbers, and \( p = 0 \) (in the notation of lecture 4 of Levin’s book [29]).

7.8. Corollary. \( W(\mathbb{N}) \), viewed as a subset of \( 1 + t\mathbb{Z}[[t]] \) by the map \( \sigma^+_+ \), agrees with the set of polynomials in \( 1 + t\mathbb{Z}[t] \) with only nonpositive real roots. In particular, \( W(\mathbb{N}) \) is countable.

Proof. We have \( W(\mathbb{N}) = W(\mathbb{Z}) \cap W(\mathbb{R}_+) \). By (7.5), elements on the right-hand side correspond to series of the form (7.5.2) with coefficients in \( \mathbb{Z} \). Certainly this includes all polynomials in \( 1 + t\mathbb{Z}[t] \) with only nonpositive real roots. Conversely, the coefficients of such a series tend to 0, by (7.1); so all such series are polynomials. \( \square \)

7.9. Remark. We can reinterpret this in a way that treats the finite and infinite places of \( \mathbb{Q} \) as similarly as possible. A monic \( p \)-adic polynomial has coefficients in \( \mathbb{Z}_p \) if and only if all its roots are integral over \( \mathbb{Z}_p \). Therefore \( W(\mathbb{N}) \), viewed as a subset of \( 1 + t\mathbb{Q}[[t]] \) via \( \sigma^+_+ \), is the set of polynomials that when written as \( \prod_i (1 + \alpha_i t) \), have the property that every \( \alpha_i \) is integral at each finite place and is real and positive at every infinite place.
Rephrasing again, if $O^\text{IP}_\bar{Q}$ denotes the multiplicative monoid of algebraic numbers which are integral at all finite places and which are real and positive at all infinite places, then we have

$$W(\mathbb{N}) = \mathbb{N}[O^\text{IP}_\bar{Q}]^{\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})}. \quad (7.9.1)$$

**7.10. Counterexample:** $W$ does not preserve surjectivity. Indeed, $\mathbb{N}$ surjects onto a nonzero ring, for instance $\mathbb{Z}/2\mathbb{Z}$. But $W(\mathbb{N})$ is countable while $W$ applied to any nonzero ring is uncountable. Of course, $W$ does preserve surjectivity for maps between rings, by (6.5).

The analogous results are also true for $W^\text{Sch}$. This will be shown in forthcoming work with Darij Grinberg [7].

**7.11. Remark.** There is a multi-dimensional generalization of (7.8). Let $A$ be a discrete subring of $\mathbb{R}^n$, for some $n \geq 0$, and write $A_+ = A \cap \mathbb{R}_+^n$. Then $W(A_+)$, viewed as a subset of $W(A) = 1 + tA[[t]]$, consists of the polynomials which split completely over $\mathbb{R}^n_+$. In particular, this applies to any totally real number field $K$, in which case we have the following generalization of (7.9.1):

$$W(O_K,+) = \mathbb{N}[O^\text{IP}_\bar{Q}]^{\text{Gal}(\bar{\mathbb{Q}}/K)}. \quad (7.11.1)$$

In particular, there are countably many Witt vectors with entries in the algebra of totally nonnegative totally real algebraic integers.

**8. A model for the $p$-typical symmetric functions over $\mathbb{N}$**

**8.1. $p$-typical Witt vectors and general truncation sets.** Following Bergman (lecture 26 of [33]), let us say a set $S$ of positive integers is a truncation set if $d$ is in $S$ whenever one of its multiples is. For any truncation set $S$, write

$$A_{\mathbb{Z},S} := \mathbb{Z}[\theta_d \mid d \in S] \subseteq \mathbb{Z}[\theta_1, \theta_2, \ldots] = A_{\mathbb{Z}},$$

where the $\theta_d$ are the Witt symmetric functions of [4.5]. Then by induction on $S$, we have

$$\mathbb{Q} \otimes_{\mathbb{Z}} A_{\mathbb{Z},S} = \mathbb{Q}[\psi_d \mid d \in S].$$

This and the existence of a retraction $\Lambda_{\mathbb{Z}} \to A_{\mathbb{Z},S}$ imply

$$\Delta^+(A_{\mathbb{Z},S}), \Delta^-(A_{\mathbb{Z},S}) \subseteq A_{\mathbb{Z},S} \otimes_{\mathbb{Z}} A_{\mathbb{Z},S}.$$ 

Therefore $A_{\mathbb{Z},S}$ is a co-$\mathbb{Z}$-algebra object of $A_{\mathbb{Z}}$. For any ring $A$, write

$$W_S(A) := A_{\mathbb{Z}}(A_{\mathbb{Z},S}, A)$$

for the corresponding ring of Witt vectors. Then the induced map $W(A) \to W_S(A)$ is surjective for all $A$. Indeed, any retraction $\Lambda_{\mathbb{Z}} \to A_{\mathbb{Z},S}$ gives a functorial section.
If $S$ is the set of divisors of some integer $m$, then $W_S$ and $\Lambda_{Z,S}$ agree with the objects $W_{Z,E,n}$ and $\Lambda_{Z,E,n}$ defined in [4], where $E$ denotes the set of prime divisors of $m$ and $n \in \mathbb{N}^E$ is the vector with $n_p = \text{ord}_p(m)$.

When $S = \{1, \ldots, p^k\}$ for some prime $p$, the functor $W_S$ is the $p$-typical Witt vector functor of length $k$ (or more traditionally $k+1$) discussed in the introduction. In this case, we will write $W_{(p),k}(A)$ and $\Lambda_{Z,(p),k}$ instead of $W_S$ and $\Lambda_{Z,S}$. When $k = \infty$, we will also write $W_{(p)}$ and $\Lambda_{Z,(p)}$.

**Question 4.** Does $\Lambda_{Z,S}$ have a model $\Lambda_{N,S}$ over $\mathbb{N}$ as a co-$\mathbb{N}$-algebra object? If so, are there models such that $\Lambda_{N,S} \circ \Lambda_{N,S'} \subseteq \Lambda_{N,SS'}$, where $SS' = \{ss' \mid s \in S, s' \in S'\}$?

The purpose of this section is to show the answers are yes in the $p$-typical case, when $S$ and $S'$ contain only powers of the prime $p$.

### 8.2. Positive $p$-typical symmetric functions.

For any prime number $p$, recall the notation

$$d_p := e_p - \psi_p = m_{p-1,1} + \cdots + (p-1)!m_{1p} \in \Lambda_N \cap \Lambda_{Z,(p),k}.$$ 

Let $\Lambda_{N,(p),k}$ denote the sub-$\mathbb{N}$-algebra of $\Lambda_{Z,(p),k}$ generated by the set $\{\psi_p^i \circ d_p^j \mid i + j \leq k\}$, and write

$$A_{k,l} = \mathbb{N}[x_{i,j} \mid k - j \leq i + j \leq k]/(x_{i,j}^p = x_{i+1,j} + px_{i,j+1} \mid i + j \leq k - 1).$$

### 8.3. Lemma.

The $\mathbb{N}$-algebra map $\mathbb{N}[x_{i,j} \mid i + j \leq k] \to \Lambda_{N,(p),k}$ sending $x_{i,j}$ to $\psi_p^i \circ d_p^j$ factors through $A_{k,k}$, and the induced map is an isomorphism

$$A_{k,k} \cong \Lambda_{N,(p),k}.$$ (8.3.1)

In particular, $\mathbb{N}[\psi_p^i \circ d_p^j \mid i + j = k]$ is a free polynomial algebra on $k+1$ generators, and $\Lambda_{N,(p),k}$ is a free module of rank $p^{k(k+1)/2}$ over it.

**Proof.** A morphism $\varphi : A_{k,k} \to \Lambda_{N,(p),k}$ sending $x_{i,j}$ to $\psi_p^i \circ d_p^j$ exists because we have

$$(\psi_p^i \circ d_p^j)^p = e_p \circ (\psi_p^i \circ d_p^j) = (\psi_p + pd_p) \circ (\psi_p^i \circ d_p^j) = \psi_p^{i(1)} \circ d_p^j + p\psi_p \circ d_p^{j+1},$$

since $\psi_p$ commutes under composition with every element. Also, $\varphi$ is clearly surjective. To show injectivity, since $A_{k,k}$ is free over $\mathbb{N}$, it is enough to do so after tensoring with $\mathbb{Q}$. Since $\psi_p^0, \ldots, \psi_p^k$ generate $\mathbb{Q} \otimes_{\mathbb{Z}} \Lambda_{Z,(p),k}$ freely as a $\mathbb{Q}$-algebra, it is enough to show that $x_{0,0}, \ldots, x_{k,0}$ generate $\mathbb{Q} \otimes_{\mathbb{N}} A_{k,k}$ as a $\mathbb{Q}$-algebra. To do this we will show by induction the more general statement that for $l = 0, \ldots, k$, we have $\mathbb{Q}[S_l] = \mathbb{Q} \otimes_{\mathbb{N}} A_{k,l}$, where $\mathbb{Q}[S_l]$ is the sub-$\mathbb{Q}$-algebra generated by the set

$$S_l = \{x_{i,0} \mid k - l \leq i \leq k\} \cup \{x_{k-l,j} \mid 1 \leq j \leq k - l\}.$$
When \( l = 0 \), this is clear from the definition of \( A_{k,0} \).

For \( l \geq 1 \), we may assume by induction that \( \mathbb{Q}[S_{l-1}] = \mathbb{Q} \otimes \mathbb{N} A_{k,l-1} \). Since \( A_{k,l} \) is generated as an \( A_{k,l-1} \)-algebra by \( \{x_{k-l-j,j} \mid j = 0, \ldots, k-l\} \), which is contained in \( S_l \), it is enough to show \( S_{l-1} \subset \mathbb{Q}[S_l] \). Since we have \( x_{k,0}, \ldots, x_{k-l-1,0} \in S_l \), it is enough to show that for \( j = 1, \ldots, k - l + 1 \), we have \( x_{k-l+1-j,j} \in \mathbb{Q}[S_l] \). This follows from the relations

\[
x_{k-l-j+1,j} = x_{k-l-j+2,j-1} + px_{k-l-j+1,j}.
\]

Indeed, the left-hand side is in \( \mathbb{Q}[S_l] \), and so we have

\[
x_{k-l-j+2,j-1} \in \mathbb{Q}[S_l] \implies x_{k-l-j+1,j} \in \mathbb{Q}[S_l].
\]

Now proceed by induction on \( j \), observing that the left-hand side holds when \( j = 1 \).

**8.4. Proposition.** (1) \( \Lambda_{\mathbb{N},(p),k} \) is free as an \( \mathbb{N} \)-module.

(2) It is an \( \mathbb{N} \)-model for \( \Lambda_{\mathbb{Z},(p),k} \) as a co-\( \mathbb{N} \)-algebra object in a unique way.

(3) We have \( \Lambda_{\mathbb{N},(p),k} \circ \Lambda_{\mathbb{N},(p),k'} \subseteq \Lambda_{\mathbb{N},(p),k+k'} \).

**Proof.** (1): This follows from (8.3).

(2): The induced map \( \mathbb{Z} \otimes \mathbb{N} \Lambda_{\mathbb{N}(p),k} \to \Lambda_{\mathbb{Z}(p),k} \) is an injection, and it is a surjection because \( \Lambda_{\mathbb{Z}(p),k} \) is generated as a \( \mathbb{Z} \)-algebra by \( e, d_p, \ldots, p^k e \), all of which are contained in \( \Lambda_{\mathbb{N}(p),k} \). Therefore \( \Lambda_{\mathbb{N}(p),k} \) is a model for \( \Lambda_{\mathbb{Z}(p),k} \) as an algebra.

To show it is a model as a co-\( \mathbb{N} \)-algebra object, it is enough to show

\[
\Delta(x) \in \Lambda_{\mathbb{N}(p),k} \otimes \mathbb{N} \Lambda_{\mathbb{N}(p),k} \tag{8.4.1}
\]

as \( x \) runs over a set of \( \mathbb{N} \)-algebra generators for \( \Lambda_{\mathbb{N}(p),k} \), and where \( \Delta \) runs over the two coproducts \( \Delta^+, \Delta^x \).

First consider the case \( k = 1 \), where we have the generators \( e, \psi_p, d_p \). Since \( e \) and \( \psi_p \) are both \( \mathbb{N} \)-algebra-like elements, (8.4.1) holds for them. For \( d_p \), it follows from the positivity of the coefficients in the following equalities:

\[
\Delta^+(d_p) = d_p \otimes 1 + 1 \otimes d_p + e \otimes e
\]

\[
\Delta^x(d_p) = d_p \otimes \psi_p + \psi_p \otimes d_p + pd_p \otimes d_p.
\]

Now consider the general case. Since \( \Lambda_{\mathbb{N}(p),1} \) is a co-\( \mathbb{N} \)-algebra object in \( \text{Alg}_{\mathbb{N}} \), we can form \( (\Lambda_{\mathbb{N}(p),1})^{\otimes k} \). Consider the map

\[
(\Lambda_{\mathbb{N}(p),1})^{\otimes k} \to (\Lambda_{\mathbb{Z}(p),1})^{\otimes k} \to \Lambda_{\mathbb{Z}(p),k}
\]

of co-\( \mathbb{N} \)-algebra objects defined by \( f_1 \circ \cdots \circ f_k \mapsto f_1 \circ \cdots \circ f_k \). Then its image is equal to \( \Lambda_{\mathbb{N}(p),k} \). Indeed, \( \Lambda_{\mathbb{N}(p),1} \) is generated by \( e, \psi_p, d_p \), all of which commute with each other under composition; so the image is the sub-\( \mathbb{N} \)-algebra generated by all \( \circ \)-words in \( e, \psi_p, d_p \) of length \( k \), which is \( \Lambda_{\mathbb{N}(p),k} \). Thus we have a surjection

\[
(\Lambda_{\mathbb{N}(p),1})^{\otimes k} \to \Lambda_{\mathbb{N}(p),k} \tag{8.4.2}
\]
of co-$\mathbb{N}$-algebra objects, and hence a diagram

$$
\begin{array}{ccc}
(\Lambda_{N,(p),1})^{\otimes k} & \rightarrow & \Lambda_{Z,(p),k} \\
\Delta & \downarrow & \Delta \\
(\Lambda_{N,(p),1})^{\otimes k} \otimes_{\mathbb{N}} (\Lambda_{N,(p),1})^{\otimes k} & \rightarrow & \Lambda_{Z,(p),k} \otimes_{\mathbb{N}} \Lambda_{Z,(p),k}
\end{array}
$$

Therefore $\Delta(\Lambda_{N,(p),k})$ is contained in the image of $(\Lambda_{N,(p),1})^{\otimes k} \otimes_{\mathbb{N}} (\Lambda_{N,(p),1})^{\otimes k}$, which is contained in $\Lambda_{N,(p),k} \otimes_{\mathbb{N}} \Lambda_{N,(p),k}$. Therefore $\Lambda_{N,(p),k}$ is a model as a co-$\mathbb{N}$-algebra object.

(3): Because (8.4.2) is surjective, $\Lambda_{N,(p),k} \circ \Lambda_{N,(p),k'}$ equals the image of

$$(\Lambda_{N,(p),1})^{\otimes k} \circ (\Lambda_{N,(p),1})^{\otimes k'} \rightarrow \Lambda_{Z,(p)}$$

which is $\Lambda_{N,(p),k+k'}$. \qed

8.5. $p$-typical Witt vectors and $\Lambda$-structures for semirings. We can put this together in what should now be a familiar way.

Define

$$
\Lambda_{N,(p)} := \text{colim}_k \Lambda_{N,(p),k}.
$$

It is a flat composition $\mathbb{N}$-algebra. Write

$$
W_{(p)}(A) := \text{Alg}_{\mathbb{N}}(\Lambda_{N,(p)}, A)
$$

for its Witt vector functor. The truncated version is

$$
W_{(p),k}(A) := \text{Alg}_{\mathbb{N}}(\Lambda_{N,(p),k}, A),
$$

and we have

$$
W_{(p)}(A) = \lim_k W_{(p),k}(A).
$$

These Witt functors take values in $\mathbb{N}$-algebras. If $A$ is a ring, then they agree with the usual $p$-typical Witt vector rings.

Giving an action of $\Lambda_{N,(p)}$ on an $\mathbb{N}$-algebra $A$ is equivalent to giving an $\mathbb{N}$-algebra endomorphism $\psi_p : A \rightarrow A$ and a set map $d_p : A \rightarrow A$ satisfying the identities

$$
\begin{align*}
\psi_p(d_p(x)) &= d_p(\psi_p(x)) \\
\psi_p(x) + pd_p(x) &= x^p \\
d_p(x + y) &= d_p(x) + d_p(y) + \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} x^i y^{p-i} \\
d_p(xy) &= d_p(x)\psi_p(y) + \psi_p(x)d_p(y) + pd_p(x)d_p(y) \\
d_p(0) &= 0 \\
d_p(1) &= 0.
\end{align*}
$$

(8.5.1)
Observe that if $A$ is flat over $\mathbb{N}$, then $\psi_p$ determines $d_p$, assuming it exists. This is because additive cancellativity and $p$-cancellativity are flat-local properties, by \cite{ref}. Thus giving a $\Lambda_{N,(p)}$-structure on a flat $\mathbb{N}$-algebra is equivalent to giving an $\mathbb{N}$-algebra endomorphism $\psi_p$ lifting the Frobenius map on $\mathbb{Z}/p\mathbb{Z} \otimes_{\mathbb{N}} A$ and satisfying $\psi_p(x) \leq x^p$ for all $x$. Here we write $a \leq b$ if there exists an element $c \in A$ such that $a + c = b$.

8.6. Remark: a partition-like interpretation of the bases. It is possible to give an interpretation of the bases of $\Lambda_{N,(p),k}$ and $\Lambda_{N,(p)}$ in terms of the language of partitions. For simplicity of notation, let us write $\Lambda_{N,(p),\infty} = \Lambda_{N,(p)}$ and hence allow $k = \infty$.

Over $\mathbb{Z}$, there is a basis of $\Lambda_{\mathbb{Z},(p),k}$ given by monomials in the $d^\circ_i$ or the $\theta_{p^j}$, where $i \geq 0$. They can be indexed by usual partitions $\lambda$ that are $p$-typical in the sense that all parts $\lambda_j$ are powers of $p$. For example, one could use the family $\prod_i \theta_{\lambda_i}$.

Equivalently, we could index them by the $p$-typical multiplicity vectors $m \in \bigoplus_{i \geq 0} \mathbb{N}$, where $m_i$ is the number of $j$ such that $\lambda_j = p^i$.

On the other hand, the $\mathbb{N}$-basis of $\Lambda_{N,(p),k}$ consists of monomials

$$\prod_{i+j \leq k} (\psi_p \circ d^\circ_j)^{m_{ij}},$$

such that when $i+j < k$, we have $m_{ij} < p$. Observe that such a monomial remains a basis element in $\Lambda_{N,(p),k+1}$ if and only if $m_{ij} < p$ when $i+j = k$ as well.

We might think of the vector $m \in \bigoplus_{i,j \geq 0} \mathbb{N}$ as the multiplicity vector of a 2-dimensional $p$-typical partition. Such a partition would be an expression of the form $\sum_{i,j} m_{ij} p^i j$. The basis for $\Lambda_{N,(p),k}$ would then be indexed by all 2-dimensional $p$-typical partitions subject to the conditions that there are less than $p$ parts $(i,j)$ when $i+j < k$, and no parts $(i,j)$ when $i+j > k$.

8.7. Relation to the multiple-prime theory. Since $\psi_p, d_p \in \Lambda_{N}$, we have $\Lambda_{N,(p)} \subseteq \Lambda_{N}$. This induces canonical algebra maps $W(A) \to W_{(p)}(A)$ for all $\mathbb{N}$-algebras $A$. In particular, for each $a \in A$, there is a Teichmüller lift $[a] \in W_{(p)}(A)$. It is the image of the usual Teichmüller lift $[a] \in W(A)$.

On the other hand, we have $\Lambda_{N,(p)} \not\subseteq \Lambda^{\text{Sch}}$, simply because $\psi_p \not\subseteq \Lambda^{\text{Sch}}$. In particular, there is no functorial map $W^{\text{Sch}}(A) \to W_{(p)}(A)$ that agrees with the usual one for rings.

8.8. Some explicit descriptions of $W_{(p),k}(A)$. The presentation \cite{8.3.1} translates directly into finite descriptions of the Witt vectors of finite length:

$$W_{(p),k}(A) = \{(a_{i,j}) \in A^{(i,j) \mid i+j \leq k} \mid a_{i,j}^p = a_{i+1,j} + pa_{i,j+1} \text{ for } i+j < k\}.$$  

For example,

$$W_{(p),1}(A) = \{(a_{00}, a_{10}, a_{01}) \mid a_{00}^p = a_{10} + pa_{01}\}$$
In general, \( W_{(p),k} = \text{Spec} \Lambda_{\mathbb{Z}(p),k} \) is the locus in the \( \mathbb{N} \)-scheme \( \Lambda_{\mathbb{N}}^{(k+2)} \) defined by the \((k+1)\) relations in the algebra \( \Lambda_{k,k} \) of \( \mathbb{S}^2 \).

As usual with Witt vectors, the algebraic structure is not transparent when expressed in coordinates. The simplest nontrivial example is \( W_{(p),1}(A) \), where we have

\[
(a_{00}, a_{10}, a_{01}) + (b_{00}, b_{10}, b_{01}) = (a_{00} + b_{00}, a_{10} + b_{10}, a_{01} + b_{01} + \sum_{i=1}^{p-1} \frac{1}{p^i} a_{00} b_{01}^{p-1})
\]

\[
(a_{00}, a_{10}, a_{01})(b_{00}, b_{10}, b_{01}) = (a_{00}b_{00}, a_{10}b_{10}, a_{01}b_{01} + p a_{00}b_{01})
\]

This is just another expression of the formulas of \( \mathbb{S}^{5.1} \).

**8.9. \( W_{(p),k}(A) \) when \( A \) is contained in a ring.** In this case, we can ignore the relations in the presentation \( \mathbb{S}^{5.1} \) and instead describe \( W_{(p),k}(A) \) in terms of the usual \( p \)-typical Witt vector ring \( W_{(p),k}(\mathbb{Z} \otimes_A A) \) and effectivity conditions corresponding to the generators \( \psi_{p,i} \circ d_{p,j} \). Indeed, a morphism \( \Lambda_{\mathbb{Z}(p),k} \to A \) is equivalent to a morphism \( a: \Lambda_{\mathbb{Z}(p),k} \to \mathbb{Z} \otimes_A A \) such that \( a(\psi_{p,i} \circ d_{p,j}) \in A \) for all \( i, j \). Thus we have

\[
W_{(p),k}(A) = \left\{ a \in W_{(p),k}(\mathbb{Z} \otimes_A A) \mid a(\psi_{p,i} \circ d_{p,j}) \in A \text{ for } i + j \leq k \right\}.
\]

For instance, if \( A \) is contained in a \( \mathbb{Q} \)-algebra, then this permits a recursive description in terms of ghost components: \( W_{(p),k}(A) \) is the set of ghost vectors \( \langle a_0, \ldots, a_k \rangle \in A^{k+1} \) satisfying the following property:

\[
\langle a_1, \ldots, a_k \rangle, \left\langle \frac{a_0^p - a_1}{p}, \ldots, \frac{a_{k-1}^p - a_k}{p} \right\rangle \in W_{(p),k-1}(A).
\]

Thus the conditions are that for all \( i \geq 0 \), the elements

\[
\frac{a_i^p - a_{i+1}}{p}, \quad \frac{1}{p} \left( \frac{a_i^p - a_{i+1}}{p} \right)^p - \frac{1}{p} a_i^p - \frac{a_{i+2}}{p}
\]

\[
\frac{1}{p} \left( \frac{1}{p} \left( \frac{a_i^p - a_{i+1}}{p} \right)^p - \frac{1}{p} a_i^p - \frac{a_{i+2}}{p} \right)^p - \frac{1}{p} \left( \frac{a_{i+1}^p - a_{i+2}}{p} \right)^p - \frac{1}{p} \left( \frac{a_{i+2}^p - a_{i+3}}{p} \right)
\]

and so on lie in \( A \).

For example, \( W_{(p),2}(\mathbb{R}_+) \) is the set of ghost vectors in \( \mathbb{R}_+^2 \) of the form \( \langle a, a^p x, a^p y \rangle \), where \( x \leq 1 \) and

\[
0 \leq x^p - y \leq \frac{(1 - x)^p}{p^p - 1}.
\]

Thus the pre-image of 1 under the projection \( W_{(p),2}(\mathbb{R}_+) \to \mathbb{R}_+ \) onto the first coordinate is the 2-simplex bounded by the curves \( y = x^p \), \( y = x^p - (1 - x)^p / p^p - 1 \) and \( y = 0 \) in the \( xy \)-plane.
8.10. Counterexample: The canonical map $W_{(p),k+1}(A) \to W_{(p),k}(A)$ is not generally surjective. It is surjective when $A$ is a ring or when $k = 0$. But when $k \geq 1$ and $A$ is general, it is not. It is enough to check this in the universal case, when $A = \Lambda_{N,(p),k}$. In other words, it is enough to show the inclusion $\Lambda_{N,(p),k} \to \Lambda_{N,(p),k+1}$ has no retraction in the category $\text{Alg}_N$. So suppose $\varphi$ is such a retraction. We have

$$\varphi(\psi^p_0 \circ dp)^p = \psi^p_0 \circ dp + p\psi^p_0 \circ dp^{(j+1)}.$$ (8.10.1)

If $i + j = k$, the left-hand side is additively indecomposable in $\Lambda_{N,(p),k}$, by (8.3). Therefore we have $\varphi(\psi^p_0 \circ dp^{(j+1)}) = 0$ whenever $i + j = k$, and hence

$$\varphi(\psi^p_0 \circ dp)^p = \varphi(\psi^p_0 \circ dp) + p\varphi(\psi^p_0 \circ dp^{(j+1)}) = 0.$$ But this is impossible because $\varphi$ is a retraction.

8.11. Semirings and the infinite prime. A finite prime $p$ has two roles in $\Lambda_N$: it supplies the property of $p$-adic integrality and the structure of the Frobenius lift $\psi_p$. The fundamental point of this chapter is that it is reasonable for some purposes to think of positivity as $p$-adic integrality at the place $p = \infty$. So the infinite prime assumes the first role here but not the second. This seems meager when compared to the rich role the infinite prime plays in the theory of automorphic forms, for example, but it does have the merit that we can treat the first role purely algebraically, and hence scheme-theoretically.

It is natural to wonder whether there is some analogue of the second role for $p = \infty$ and whether there is an $\infty$-typical theory that can be isolated from the rest of the primes. There is some cause for the hope that flows will appear here.

9. On the possibility of other models

So far, we have not been concerned with whether the $N$-models we have considered are the most natural ones—their mere existence has been interesting enough. The purpose of this short section is to raise some questions in this direction.

As discussed in (5.11), the connection between symmetric functions and arithmetic algebraic geometry is explained by Wilkerson’s theorem, which we interpret as saying that $\Lambda_N$ is the composition ring that controls commuting Frobenius lifts. It is natural to ask whether there are similar, arithmetically satisfying descriptions of the other models we have considered. In the $p$-typical case, the description of (8.20) in terms of bounded Frobenius lifts is perhaps satisfying enough. It would be interesting to find such a description for $\Lambda_N$ or $\Lambda_{\text{Sch}}$. A less satisfying alternative would be to single out $\Lambda_N$ and $\Lambda_{\text{Sch}}$ among all $N$-models by some general properties, and at least this form of the question admits a precise expression:

**Question 5.** Are $\Lambda_N$ and $\Lambda_{\text{Sch}}$ the only flat models for $\Lambda_N$ over $N$? Is $\Lambda_{N,(p)}$ the only flat model for $\Lambda_{N,(p)}$ over $N$? If not are they the maximal or minimal ones?
Whether we ask for models as composition algebras or models as co-$N$-algebra objects, I do not know the answer. I do not even know the answer to analogous questions about integrality at the finite primes. For instance, is $\Lambda_{\mathbb{Z},(p)}$ the maximal integral model for $\mathbb{Z}[1/p] \otimes_{\mathbb{Z}} \Lambda_{\mathbb{Z},(p)}$?

**Question 6.** Over $\mathbb{Q}_+$, there is a third model for $\Lambda_{\mathbb{Q}}$, namely $\Psi_{\mathbb{Q}_+}$. Is there a fourth?

**Question 7.** Let $K$ be a number field, and let $T$ be a set of embeddings $K \to \mathbb{R}$. Do the composition algebras $\Lambda_{\mathcal{O}_K,E}$ of (5.10) have models over the sub-$N$-algebra of $\mathcal{O}_K$ consisting of elements that are nonnegative under all $\sigma \in T$?

We have seen that if $K = \mathbb{Q}$ and $T$ consists of the unique embedding, the answer is yes in two cases: when $E$ consists of all maximal ideals of $\mathbb{Z}$ or when it consists of only one. I do not know the answer in any other case, unless $T$ or $E$ is empty.

10. *$k$-Schur functions and truncated Witt vectors*

Let $\Lambda_{\mathbb{Z},k}$ denote $\mathbb{Z}[h_1, \ldots, h_k]$. Thus, in the notation of of (8.1), we have $\Lambda_{\mathbb{Z},k} = \Lambda_{\mathbb{Z},S}$, where $S$ is the truncation set $\{1, 2, \ldots, k\}$. The purpose of this section is to show how $k$-Schur functions, a recent development in the theory of symmetric functions, allow us to give an $N$-model $\Lambda_{\mathbb{Z},k}^{\text{Sch}}$ for $\Lambda_{\mathbb{Z},k}$ which approaches $\Lambda_{\mathbb{Z},k}^{\text{Sch}}$ as $k$ tends to infinity. Unfortunately, $\Lambda_{\mathbb{Z},k}^{\text{Sch}}$ is only a model as a co-$N$-module object, and not as a co-$N$-algebra object. This would seem to be fatal for any application of $k$-Schur functions to Witt vectors as objects of arithmetic algebraic geometry. But they do have several properties that are good from the point of view of Witt vectors, and there are several parallels with the $p$-typical $N$-models of section 8.

The purpose of this section is to enter the details into the literature, in case they can be of use to anyone else.

10.1. *$k$-Schur functions and $\Lambda_{\mathbb{Z},k}^{\text{Sch}}$.* It is not possible to make a $\mathbb{Z}$-basis for $\Lambda_{\mathbb{Z},k}$ out of usual Schur functions. This is because there are only finitely many Schur functions in any given $\Lambda_{\mathbb{Z},k}$. But Lapointe–Lascoux–Morse [27] discovered certain symmetric functions that form a basis for $\Lambda_{\mathbb{Z},k}$ and are similar to Schur functions in many ways. They call them $k$-$\text{Schur}$ functions and denote them $s^{(k)}_\lambda$, where $\lambda$ runs over all partitions $(\lambda_1, \ldots)$ which are $k$-bounded in the sense that $\lambda_1 \leq k$.

Our reference for $k$-Schur functions will be the book [24] by Lam et al., in particular part 2, which was written by Morse, Schilling, and Zabrocki and is based on lectures by Lapointe and Morse. They consider more than one definition of $k$-Schur function but conjecture that they all agree. For definiteness, we will take

$$s^{(k)}_\lambda := s^{(k)}_\lambda[X; 1]$$

as our definition, where $s^{(k)}_\lambda[X; t]$ is defined there in equation (3.16), p. 81. See pp. 83–84 for a discussion of the relations with the other definitions.
Define
\[ \Lambda^\text{Sch}_k := \bigoplus_{\lambda} \mathbb{N} s^{(k)}_\lambda. \]

As explained in part 2, section 4.5 of [24], the family \( s^{(k)}_\lambda \) forms a \( \mathbb{Z} \)-basis for \( \Lambda^\text{Sch}_k \).

In other words, \( \Lambda^\text{Sch}_k \) is free over \( \mathbb{N} \) and is a model for \( \Lambda^\text{Z},k \) over \( \mathbb{N} \) as a module.

10.2. Proposition. For \( k \geq 0 \), we have

1. \( \Lambda^\text{Sch}_k \) is a model for \( \Lambda^\text{Z},k \) as a co-\( \mathbb{N} \)-module object in \( \text{Alg}_{\mathbb{N}} \).
2. \( s^{(k)}_\lambda = s_\lambda \), if \( k \geq \lambda_1 + l - 1 \), where \( l \) is the length of \( \lambda \).
3. \( \Lambda^\text{Sch}_k \subseteq \Lambda^\text{Sch}_{k+1} \) and \( \Lambda^\text{Sch} = \bigcup_k \Lambda^\text{Sch}_k \).
4. \( \Lambda^\text{Sch}_k \) is finitely presented as an \( \mathbb{N} \)-algebra.

Proof. These are mostly just restatements of results collected in part 2, chapter 4 of the book [24].

1. This follows from corollaries 8.1 and 8.2 of Lam [26]. See sections 4.7 and 4.8 of part 2 of [24].
2. This is property 39 of Lapointe–Morse [28]. The result under the stronger assumption \( k \geq \sum \lambda_i \) is discussed in part 2, section 4.1 of [24].
3. The first statement is theorem 2 of Lam–Lapointe–Morse–Shimozono [25]. See section 4.10 of part 2 of [24]. The second statement follows from the first, together with part (2).
4. This follows from the multiplication rule established in theorem 40 of Lapointe–Morse [28]. See section 4.6 of part 2 of [24]. As a module over the \( \mathbb{N} \)-algebra generated by the \( k \)-rectangular \( k \)-Schur functions, \( \Lambda^\text{Sch}_k \) is freely generated by the \( k! \) \( k \)-irreducible \( k \)-Schur functions. Further, the \( k \)-rectangular \( k \)-Schur functions are algebraically independent. So \( \Lambda^\text{Sch}_k \) can be generated by \( k! + k \) elements with \( \binom{k+1}{2} \) quadratic relations.

10.3. Remark. There are some similarities between \( \Lambda^\text{Sch}_k \) and \( \Lambda^\text{Ni},(p),k \). Compare the preceding proposition with section 8, and especially the presentation of \( \Lambda^\text{Sch}_k \) mentioned in the proof of (4) above with the presentation of (8.3.1).

10.4. Truncated Schur–Witt vectors for semirings. For any \( \mathbb{N} \)-algebra \( A \), define
\[ W^\text{Sch}_k(A) := \text{Alg}_\mathbb{N}(\Lambda^\text{Sch}_k, A). \]

It follows from (10.2) (4) that \( W^\text{Sch}_k(A) \) can be described as the subset of a finite-dimensional affine space \( A^N \) satisfying a finite list of equations.

By (10.2) (2)–(3), we have \( \Lambda^\text{Sch} = \text{colim}_k \Lambda^\text{Sch}_k \) and \( W^\text{Sch}_k(A) = \text{lim}_k W^\text{Sch}_k(A) \).

It follows from (10.2) (1) that \( W^\text{Sch}_k(A) \) inherits an \( \mathbb{N} \)-module structure and that \( W^\text{Sch}_k(A) = W_k(A) \) when \( A \) is a ring. Unlike in when \( A \) is a ring, \( W^\text{Sch}_k(A) \) does not generally inherit an \( \mathbb{N} \)-algebra structure.
10.5. Counterexample: $\Lambda^{\text{Sch}}_k$ is not a co-$\mathbb{N}$-algebra object. It is for $k \leq 2$, but Luc Lapointe found that $\Delta^\wedge(s^{(3)}_{22}) \not\in \Lambda^{\text{Sch}}_3 \otimes \Lambda^{\text{Sch}}_3$, and so it fails for $k = 3$. This can be checked by hand using the following equalities:

$$12s^{(3)}_{22} = \psi^4_1 + 3\psi^2_2 - 4\psi_1\psi_3$$

$$\psi^4_1 = s^{(3)}_{1111} + 2s^{(3)}_{2211} + 2s^{(3)}_{22} + s^{(3)}_{31}$$

$$\psi^2_2 = s^{(3)}_{1111} - 2s^{(3)}_{2111} + 2s^{(3)}_{22} + s^{(3)}_{31}$$

$$\psi_1\psi_3 = s^{(3)}_{1111} - s^{(3)}_{2111} - s^{(3)}_{22} + s^{(3)}_{31}.$$ 

On the other hand, $\Delta^\wedge(s^{(3)}_{22})$ is contained in $\Lambda^{\text{Sch}}_4 \otimes \Lambda^{\text{Sch}}_4$. This is just because it is an $\mathbb{N}$-linear combination of elements of the form $s^\lambda \otimes s^\mu$, where $\lambda$ and $\mu$ are partitions of $2 + 2 = 4$; so we have $s^\lambda = s^{(4)}_{\lambda}$ and $s^\mu = s^{(4)}_{\mu}$ for all $\lambda, \mu$ in question.

It is also not true that $\Lambda^{\text{Sch}}_k \circ \Lambda^{\text{Sch}}_l \subseteq \Lambda^{\text{Sch}}_{kl}$ for all $k, l$. According to my computations, it is true if $k, l \leq 3$ and $(k, l) \neq (3, 3)$. But for $k = l = 3$, it fails: the coefficient of $s^{(9)}_{6311111}$ in $s^{(3)}_{22} \circ s^{(3)}_{22}$ is $-1$. In fact, we have $\Lambda^{\text{Sch}}_3 \circ \Lambda^{\text{Sch}}_3 \not\subseteq \Lambda^{\text{Sch}}_{11}$ and $\Lambda^{\text{Sch}}_3 \circ \Lambda^{\text{Sch}}_3 \subseteq \Lambda^{\text{Sch}}_{12}$.

11. Remarks on absolute algebraic geometry

This volume is a collection of contributions on the theme of the mythical field with one element. One can see this chapter from that point of view, although I have so far avoided making the connection. There are two natural approaches to rigidifying the category of rings—one can look for models over $\mathbb{N}$, or one can add structure, such as a $\Lambda^{\text{Z}}$-ring structure, which we think of as descent data to the absolute point [3]. In this chapter, we have combined the two. I do not have much more to say about the philosophy of the field with one element than I already have said in [3], but this way of thinking does suggest some mathematical questions.

**Question 8.** Is it possible to extend the constructions $W$ and $W^{\text{Sch}}$ to non-affine $\mathbb{N}$-schemes? What about their adjoints $A \mapsto \Lambda^{\text{N}} \circ A$ and $A \mapsto \Lambda^{\text{Sch}} \circ A$?

Over $\mathbb{Z}$, this was done in my paper [5], but there are several complications over $\mathbb{N}$. The most important is that over $\mathbb{Z}$ I used Witt vectors of finite length, because it is better to think of $W(A)$ as a projective system of discrete rings, rather than actually taking the limit. But there is not yet any finite-length version of the big Witt vector functor for $\mathbb{N}$-algebras. But we do have finite-length $p$-typical functors for $\mathbb{N}$-algebras available; so it is probably easier to make immediate progress there.

A similar question is whether the concept of a $\Lambda^{\text{N}}$-structure or $\Lambda^{\text{Sch}}$-structure can be extended to nonaffine $\mathbb{N}$-schemes. Over $\mathbb{Z}$, this is done using the functor $W^* = \text{colim}_n W^*_n$, where $W^*_n$ is the extension of $W_n$ to nonaffine schemes. (One could also use its right adjoint $W_* = \text{lim}_n W^*_n$). So the two questions are indeed closely related. The following question is a natural guide:
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Question 9. Let $X$ be a $\Lambda_N$-scheme that is flat and locally of finite presentation over $\mathbb{N}$. Does there exist a toric variety $Y$ over $\mathbb{N}$ and a surjective $\Lambda_N$-morphism $Y \to X$?

This requires some explanation. By a toric variety over $\mathbb{N}$, I mean an $\mathbb{N}$-scheme that can be formed by gluing together affine $\mathbb{N}$-schemes of the form $\text{Spec} \mathbb{N}[M]$, where $M$ is a commutative monoid, along other schemes of the same form, where all the gluing maps are induced by maps of the monoids. Surjectivity of a morphism of $\mathbb{N}$-schemes can be understood in the sense of the Zariski topos. Finally, while a $\Lambda_N$-structure has at the moment no precise meaning for nonaffine $X$, it is possible to strengthen the question to a precise one that is still open. For instance, we can require only that the base change of $X$ to $\mathbb{Z}$ admit a $\Lambda_\mathbb{Z}$-structure.

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