Fermion Mass Matrices in terms of the Cabibbo–Kobayashi–Maskawa Matrix and Mass Eigenvalues

D. Falcone∗†, O. Pisanti∗, and L. Rosa∗†

∗ Dipartimento di Scienze Fisiche, Università di Napoli, Mostra d’Oltremare, Pad.19, I-80125, Napoli, Italy;
† INFN, Sezione di Napoli, Napoli, Italy.

e-mail: falcone@na.infn.it
e-mail: pisanti@na.infn.it
e-mail: rosa@na.infn.it

Abstract

A parameter free, model independent analysis of quark mass matrices is carried out. We find a representation in terms of a diagonal mass matrix for the down (up) quarks and a suitable matrix for the up (down) quarks, such that the mass parameters only depend on the six quark masses and the three angles and phase appearing in the Cabibbo-Kobayashi-Maskawa matrix. The results found may also be applied to the Dirac mass matrices of the leptons.

PACS numbers: 12.15.Ff, 12.15.Hh, 14.60.P
Many attempts have been made to connect the quark mixing matrix with the quark mass matrices introducing extra symmetries (or Ansätze) to cast the mass matrices in some particular form \[1\]. Branco, Lavoura and Mota \[2\] have been able to show that for three families the Nearest-Neighbor Interactions (NNI) form of mass matrices corresponds to a choice of basis. Indeed, within the Standard Model \[3\], the NNI form can be obtained by applying a particular transformation to the fermionic fields without observable consequences. Relying on their result, in ref. \[4, 5, 6\] the problem of finding mass matrices for the fermions, as a function of physical parameters only, has been addressed. Due to the NNI form they get very complicated relationships between mass matrices parameters and the Cabibbo-Kobayashi-Maskawa matrix, \(K\). In particular, Harayama and Okamura \[4\] obtained formulae in which two arbitrary phases (that is to say not determined by physical parameters) still remain. Koide \[5\] showed that the two phases can be eliminated by a change of phases of matrix elements. Finally Takasugi \[6\] investigated the connections between NNI basis and the USY (Universal Strength for Yukawa couplings) form of Yukawa coupling \[8\], leaving for future works the problem of expressing quark mass matrices in terms of physical parameters.

In this paper we concentrate on this last problem. Using a particular basis for quark (lepton) fields we find a representation of mass matrices in which there are exactly ten free parameters, nine moduli and one phase. In this basis it is possible to obtain relatively easy expressions for the Cabibbo-Kobayashi-Maskawa matrix elements and, more interestingly, it is possible to invert these relations linking the mass matrices with the physical parameters. We quote exact and approximate formulae. We analyze the quark-phase conventions and determine the expression for the observable phase appearing in the mass matrices. We conclude with brief final remarks.

In what follows we concentrate on the mass and the weak-charged-current terms of the Standard Model Lagrangian \[3\]. We write them as follows:

\[
L = \bar{u}_L^0 \tilde{M}_u^0 u_R^0 + \bar{d}_L^0 \tilde{M}_d^0 d_R^0 + g \bar{u}_L^0 W^+ d_L^0 + h.c.
\]  

(1)

(summation over family indices is intended).

It is possible to perform, with no physical consequences \[2\], the following transformations on the quark fields (a similar argument applies to leptons):

\[
\begin{align*}
& \begin{cases} 
  u_L^0 = U u_L' \\
  u_R^0 = V_u u_R' 
\end{cases} \\
& \begin{cases} 
  d_L^0 = U d_L \\
  d_R^0 = V_d d_R 
\end{cases}
\end{align*}
\]  

(2)

where the only constraint on the matrices \(U, V_u, V_d\) is that they must be unitary. We
choose $U$, $V_d$ and $V_u$ so to have

$$L = \bar{u}L\bar{M}_u u' R + \bar{d}L M_d d_R + g\bar{u}' L W^+ d_L + h.c.,$$

(3)

with

$$M_d \equiv \text{diag}(m_d, m_s, m_b) = U\tilde{M}_d V_d$$

(4)

and

$$\tilde{M}_u \equiv \begin{pmatrix}
0 & m_{12} & m_{13} \\
 m_{21} & 0 & m_{23} \\
0 & m_{32} & m_{33}
\end{pmatrix} = U\tilde{M}_u V_u.$$  

(5)

The $m_{ij}$ are complex numbers, $m_{ij} = N\rho_{ij} \exp (ir_{ij})$, with $N = m_t + m_c + m_u$ a suitable normalization constant.

The two matrices $U$ and $V_d$ are determined by solving the two eigenvalue problems

$$U\tilde{M}_d \tilde{M}_d^\dagger U = \text{diag}(m_{d_1}^2, m_{s_1}^2, m_{b_1}^2), \quad V_d \tilde{M}_d \tilde{M}_d^\dagger V_d = \text{diag}(m_{d_2}^2, m_{s_2}^2, m_{b_2}^2),$$

(6)

while $V_u$ is chosen in such a way to get the three zeroes in eq. (4) (we use the following notation: $A_i$ is the i-th row, $A_i^*$ is the i-th column of a matrix $A$ and $\times$ represents cross product):

$$(V_u)_1 \propto (U\tilde{M}_u)_1 \times (U\tilde{M}_u)_3.$$

$$(V_u)_2 \propto (V_u)_1^* \times (U\tilde{M}_u)_2.$$

$$(V_u)_3 \propto (V_u)_1^* \times (V_u)_2^*,$$

(7)

where the multiplicative constants are determined requiring the $(V_u)_i$ to be norm-one vectors.

In this way $\tilde{M}_u$ contains twelve real parameters, six moduli and six phases. Depending on the arbitrariness of the differences between quark phases we have the possibility to remove five of them remaining with one phase and six moduli, which must be compared with the seven physical parameters given by the up quark masses and Cabibbo-Kobayashi-Maskawa angles and phase.

Obviously, different choices of phases correspond to different representations of $K$ so, to compare our result with the various parametrizations of $K$, the phases must be chosen in an appropriate way. It is possible to reproduce the usual representations of $K$ [9, 10],

$$K = \begin{pmatrix}
c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{13}} \\
-s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{13}} & s_{23}e^{i\delta_{13}} \\
s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{13}} & -s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{13}} & c_{23}e^{i\delta_{13}}
\end{pmatrix},$$

(8)
two last terms refer to the representations of $K$ we can rotate out all the phases except for the combination $\Phi = r_{12} - r_{13}$. Starting from these ones, by redefining the left and right-handed up quark phases as

$$(u^r_L, c^r_L, t^r_L) \rightarrow (u^r_L e^{i r_{12}}, c^r_L e^{i r_{13}}, t^r_L)$$

(9)

we can rotate out all the phases except for the combination $\Phi = r_{12} - r_{13}$. After these transformations, all the $m_{ij}$ become real but $m_{12} = N \rho_{12} e^{i \Phi}$. Thus $\Phi$ is the only combination of phases that has physical relevance. This can be seen calculating, for example, the imaginary part of the fourth order invariant of $K$ defined by

$$J \equiv Im \left( \Delta^{(4)}_{\alpha \rho} \right) \equiv Im \left( K_{\beta \sigma} K_{\gamma \tau} K^{*}_{\beta \sigma} K^{*}_{\gamma \tau} \right) = s_2^2 s_3^2 c_1 c_2 c_3 \sin \delta \simeq \lambda^2 A^2 \eta$$

(10)

(no summation on repeated indices is intended and $\alpha$, $\beta$, $\gamma$, $(\rho, \sigma, \tau)$ cyclic), where the two last terms refer to the representations of $K$ given in eq. (8). The expression of $J$ found within our representation is given (after having calculated $K$) in eq. (16).

With another change of basis,

$$\begin{align*}
\left\{ u^r_L &= S_L u_L \\
   u^r_R &= S_R u_R \right.
\end{align*}$$

(11)

we obtain

$$L = \bar{u}_L M_u u_R + \bar{d}_L M_d d_R + g \bar{u}_L W^+ S^r_L d_L + h.c.,$$

(12)

where $S_L$ and $S_R$ are chosen to diagonalize $\hat{M}_u$,

$$M_u \equiv diag(m_u, m_c, m_t) = S^r_L \hat{M}_u S_R.$$  

(13)

Consequently $K$ is given by $K_{ij} = (S^r_L)_{ij}$ where $S_L$ is the unitary matrix which diagonalizes the product $\hat{M}_u \hat{M}_u^t$. We find for the eigenvectors of $\hat{M}_u \hat{M}_u^t$ the following expression:

$$u_i = (\alpha_i, \beta_i, \gamma_i)/\sqrt{|\alpha_i|^2 + |\beta_i|^2 + |\gamma_i|^2},$$

(14)

with $(l_i = m_i/N, \ i = u, c, t)$

$$\begin{align*}
\alpha_i &= \left( \rho_{21}^2 + \rho_{23}^2 - l_i^2 \right) \left( \rho_{32}^2 + \rho_{33}^2 - l_i^2 \right) - \rho_{23}^2 \rho_{33}^2 \\
\beta_i &= \left( l_i^2 - \rho_{32}^2 \right) m_{23} m_{12}^{*} + m_{23} m_{32} m_{12}^{*} m_{33}^{*} \\
\gamma_i &= \left( l_i^2 - \rho_{21}^2 \right) m_{32} m_{12}^{*} + m_{33} m_{13}^{*} - \rho_{23}^2 m_{32} m_{12}^{*} \\
\end{align*}$$

(15)
so that \( K_{ij} = (u^*_i)_j \). \( J \) is given by

\[
J = \frac{\alpha^*_u \alpha_c (l^2_u - l^2_c)}{(|\alpha_u|^2 + |\beta_u|^2 + |\gamma_u|^2)(|\alpha_c|^2 + |\beta_c|^2 + |\gamma_c|^2)} \rho_{12}\rho_{13}\rho_{23}\rho_{32}\rho_{33} \sin \Phi. \tag{16}
\]

In the basis given by eq. (2) it is very easy to obtain the mass matrix \( \tilde{M}_u \) as a function of \( K \) and of the quark masses; indeed we have

\[
\tilde{M}_u \tilde{M}_u^\dagger = K^\dagger \text{diag}(m^2_u, m^2_c, m^2_t)K \equiv N^2(a_{ij} + ib_{ij}) \tag{17}
\]

\((i, j = 1, 2, 3 \text{ and } b_{ii} = 0)\), that is

\[
\begin{pmatrix}
\rho^2_{12} + \rho^2_{13} & \rho_{12}\rho_{13}e^{i(r_{12} - r_{23})} & \rho_{12}\rho_{23}e^{i(r_{12} - r_{23})} & \rho_{12}\rho_{32}e^{i(r_{12} - r_{23})} + \rho_{13}\rho_{33}e^{i(r_{12} - r_{23})} \\
\rho_{13}\rho_{23}e^{-i(r_{12} - r_{23})} & \rho^2_{21} + \rho^2_{23} & \rho_{23}\rho_{32}e^{-i(r_{21} - r_{23})} & \rho_{23}\rho_{33}e^{-i(r_{21} - r_{23})} \\
\rho_{12}\rho_{23}e^{-i(r_{13} - r_{23})} + \rho_{13}\rho_{32}e^{-i(r_{13} - r_{23})} & \rho_{23}\rho_{32}e^{i(r_{13} - r_{23})} & \rho^2_{31} + \rho^2_{32} & \rho_{32}\rho_{33}e^{i(r_{13} - r_{23})} \\
\rho_{13}\rho_{23}e^{i(r_{13} - r_{23})} & \rho_{23}\rho_{32}e^{-i(r_{13} - r_{23})} & \rho_{32}\rho_{33}e^{-i(r_{13} - r_{23})} & \rho^2_{33}
\end{pmatrix} = \begin{pmatrix}
a_{11} & a_{12} + ib_{12} & a_{13} + ib_{13} \\
a_{12} - ib_{12} & a_{22} & a_{23} + ib_{23} \\
a_{13} - ib_{13} & a_{23} - ib_{23} & a_{33}
\end{pmatrix}. \tag{18}
\]

Given a particular representation of \( K \) these are the most general equations relating mass matrix and physical parameters. Depending on the phase choice one can reduce the three imaginary equations in (18) to just one, but we keep all of them to allow any arbitrary phase convention in \( K \). Solving eq. (18) we find:

\[
\tan r_{12} = \frac{b_{13}\rho^2_{23} - a_{12}b_{23} + a_{13}b_{12}}{a_{13}\rho^2_{23} - a_{12}a_{23} + b_{12}b_{23}}, \quad \tan r_{13} = \frac{a_{23}b_{12} + a_{12}b_{23}}{a_{12}a_{23} - b_{12}b_{23}}, \quad \tan r_{23} = \frac{b_{23}}{a_{23}}.
\]

\[
\rho_{12} = \pm \sqrt{a_{11} - \frac{a_{13}^2 + b_{13}^2}{\rho^2_{23}}}, \quad \rho_{13} = -\sqrt{a_{12}^2 + b_{12}^2}, \quad \rho_{21} = \pm \sqrt{\frac{l^2_u l^2_c}{a_{11}a_{33} - a_{13}^2 - b_{13}^2}},
\]

\[
\rho_{23} = \pm \sqrt{a_{22} - \frac{l^2_u l^2_c}{a_{11}a_{33} - a_{13}^2 - b_{13}^2}}, \quad \rho_{32} = \text{sign}(\Delta) \sqrt{a_{33} - \frac{a_{23}^2 + b_{23}^2}{\rho^2_{23}}},
\]

\[
\Delta = \rho_{12}(\rho_{13}^2 - a_{12}a_{23} + b_{12}b_{23}).
\]

Similar formulae hold for the lepton Dirac masses if the exchanges \((d, s, b) \rightarrow (e, \mu, \tau)\) and \((u, c, t) \rightarrow (\nu_e, \nu_\mu, \nu_\tau)\) are performed.
TABLE 1

|       | $\rho_{12}^2$ | $\rho_{13}^2$ | $\rho_{21}^2$ | $\rho_{23}^2$ | $\rho_{32}^2$ | $\rho_{33}^2$ | J    |
|-------|--------------|--------------|--------------|--------------|--------------|--------------|------|
| exact | 9.54 $\cdot$ 10^{-8} | 8.10 $\cdot$ 10^{-5} | 3.07 $\cdot$ 10^{-9} | 1.68 $\cdot$ 10^{-3} | 7.54 $\cdot$ 10^{-3} | 0.985 | 2.9 $\cdot$ 10^{-5} |
| approx.| 9.54 $\cdot$ 10^{-8} | 8.04 $\cdot$ 10^{-5} | 3.07 $\cdot$ 10^{-9} | 1.67 $\cdot$ 10^{-3} | 7.60 $\cdot$ 10^{-3} | 0.993 | 3.3 $\cdot$ 10^{-5} |

For the sake of utility we quote here the expressions for the $\rho_{ij}^2$, obtained when the Wolfenstein parametrization for $K$ is used. These expressions are approximated up to the fifth order in $\lambda$ and neglecting $l_i^2 (l_i^2)$ with respect to $l_e^2 (l_e^2)$:

$$\rho_{12}^2 \simeq l_2^2 \lambda^2 \left[ (\eta^2 + \rho^2) (1 - \lambda^2) + \rho \lambda^2 \right], \quad \rho_{13}^2 \simeq l_2^2 A^2 \lambda^6 \left[ 1 + \eta^2 - \rho (2 - \rho) \right],$$

$$\rho_{21}^2 \simeq \frac{l_2^2}{A^2} \left[ 1 - 2 A^2 \lambda^4 (1 - \rho + \eta^2 \lambda^2) \right], \quad \rho_{23}^2 \simeq l_3^2 A^2 \lambda^4,$$

$$\rho_{32}^2 \simeq \frac{l_2^2}{A^2} \left\{ 1 - \lambda^2 + \lambda^4 \left[ \frac{1}{4} + A^2 (2 - \lambda^2 + A^2 \lambda^4 (1 + \eta^2)) \right] \right\}, \quad \rho_{33}^2 \simeq l_3^2,$$

$$\tan r_{12} \simeq \frac{\eta}{4 \rho^2} \left[ \lambda^4 (\rho - 1 - 4 A^2 \rho^2) + 2 \rho (\lambda^2 - 2 \rho) \right], \quad \tan r_{13} \simeq \frac{\eta}{1 - \rho},$$

$$\tan r_{23} \simeq -\frac{l_2^2}{2 l_3^2} \eta \lambda^2 \left[ 2 - \lambda^2 (1 - 2 A^2) \right].$$

In table 1 both the exact and approximate values are reported. They are obtained using the central values of the measured ranges of $K$ and the following quark masses evaluated at $m_{Z'} = 91.187$ GeV: $m_t = 180$ GeV, $m_c = 0.661$ GeV, $m_u = 0.00222$ GeV. We quote also the values of the elements of the down quark matrix when $M_u = \text{diag}(m_u, m_c, m_t)$ and $(\tilde{M}_d)_{ij} = N' \sigma_{ij} e^{i \psi_{ij}}$ ($N' = m_d + m_s + m_b$, $m_d = 0.00442$ GeV, $m_s = 0.0847$ GeV, $m_b = 2.996$ GeV). We did not report the results for the $\sigma_{ij}$’s when the approximate formulae are used since the weaker hierarchy between down quark masses makes it necessary a higher order approximation.

In conclusion, we exhibit a representation for the fermion mass matrices where the
expression of the Cabibbo-Kobayashi-Maskawa matrix is relatively easy. We solve the problem of inverting the relationship between the mass matrices and physical parameters. The manageable formulae we find can be useful in investigating the various hypotheses formulated on this sector of the Standard Model.

We are very much indebted with F. Buccella for his suggestions and for the many discussions we had. We also thank G. Mangano for discussions and Zhi-zhong Xing for useful comments. We are very grateful to P. Vitale for her support.

REFERENCES

[1] H. Fritzsch, Phys. Lett. B 73, 317 (1978); Nucl. Phys. B 155, 182 (1979); F. Wilczek and A. Zee, Phys. Rev. Lett. 42, 421 (1979); H. Georgi and C. Jarlskog, Phys. Lett. B 86, 297 (1979); P. Ramond, R. G. Roberts, and G. G. Ross, Nucl. Phys. B 406, 19 (1993); K. Harayama, N. Okamura, A. I. Sanda, and Zhi-Zhong Xing, Preprint DPNU 96/36, hep-ph/9607461.

[2] G. C. Branco, L. Lavoura, and F. Mota, Phys. Rev. D 11, 3443 (1989).

[3] P. Langacker, Phys. Rep. 72, 185 (1981).

[4] K. Harayama and N. Okamura, Phys. Lett. B 387, 614 (1996).

[5] Y. Koide, Preprint US 97/01, hep-ph/9701345.

[6] E. Takasugi, Preprint OU-HET 264, hep-ph/9705263.

[7] N. Cabibbo, Phys. Rev. Lett. 10, 531 (1963); M. Kobayashi and T. Maskawa, Prog. Theor. Phys. 49 n.2, 652 (1973).

[8] G. C. Branco and J. I. Silva-Marcos, Phys. Lett. B 359, 166 (1995).

[9] Particle Data Group, Phys. Rev. D 54, 1 (1996);

[10] L. Wolfenstein, Phys. Rev. Lett. 51, 1945 (1983).

[11] C. Jarlskog, Phys. Rev. Lett. 55, 1039 (1985); O. W. Greenberg, Phys. Rev. D 32, 1841 (1985); I. Dunietz, O. W. Greenberg, and D. -d. Wu, Phys. Rev. Lett. 55, 2935 (1985); C. Hamazaoui and A. Barroso, Phys. Lett. B 154, 202 (1985); D. -d. Wu, Phys. Rev. D 33, 860 (1986).

[12] H. Fusaoka and Y. Koide, Preprint AMU-97-01, US-97-03, hep-ph/9706211.