GRAPHS WITH MANY VALENCIES AND FEW EIGENVALUES

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Abstract. Dom de Caen posed the question whether connected graphs with three (distinct) eigenvalues have at most three (distinct) valencies. We do not answer this question, but instead construct connected graphs with four and five eigenvalues and arbitrarily many valencies. The graphs with four eigenvalues come from regular two-graphs. As a side result, we characterize the disconnected graphs and the graphs with three eigenvalues in the switching class of a regular two-graph.

In memory of David Gregory

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1. Introduction

Dom de Caen (see [9, Problem 9] and [4]) posed the question whether connected graphs with three (distinct) eigenvalues have at most three (distinct) valencies. More generally, one may wonder whether the number of valencies in a graph is bounded by a function of the number of eigenvalues. It is important to restrict to connected graphs, because the disjoint union of — for example — the complete bipartite graphs on \(2^m + 2^{n-m}\) vertices have three eigenvalues and \(n + 1\) valencies. Hence the number of valencies is unbounded. Note that by eigenvalues of a graph we mean here the eigenvalues of the adjacency matrix. One may of course pose similar questions for other eigenvalues of graphs. It is for example known that connected graphs with three Laplacian eigenvalues have at most two distinct valencies; see [10].

Mohar [private communication] observed that by adding one vertex and joining it in an arbitrary way to each of the components of the above graph, one obtains a connected graph with many valencies, but interlacing of eigenvalues (see [15]) implies that the number of eigenvalues is at most 7. Thus, it follows that with only 7 eigenvalues, the number of valencies can be arbitrary large.

In this paper, we will further exploit Mohar’s idea to construct connected bipartite graphs with five eigenvalues and (arbitrarily) many valencies. In the same spirit, we will construct connected non-bipartite graphs with five eigenvalues and many valencies. Moreover, we will use regular two-graphs and Seidel switching to construct connected graphs with four eigenvalues and many valencies. As a side result, we characterize the disconnected graphs and the graphs with three eigenvalues in the switching class of a regular two-graph. De Caen’s question remains open though, as is the question whether there are connected bipartite graphs with four eigenvalues and more than four valencies. For background on eigenvalues of graphs we refer to the monographs by Brouwer and Haemers [3] and Cvetković, Doob, and Sachs [6].
2. Bipartite graphs with five eigenvalues

Let us first exploit Mohar’s construction further, in the sense that we want to bring the number of eigenvalues down from 7 to 5. So let us fix a number \( e \) that factors in many different ways (such as \( 2^n \)), and consider the disjoint union of some non-isomorphic complete bipartite graphs with \( e \) edges. Then this graph has spectrum \( \{ \sqrt{e} f, 0^g, -\sqrt{e} f \} \), where \( f \) is the number of components, and \( v = g + 2f \) is the total number of vertices. If we now add a vertex and connect it to each of the components (so that the graph becomes connected), then the eigenvalues of the new graph and the eigenvalues of the original graph interlace [15], which implies that it has spectrum \( \{ \theta_0, \sqrt{e} f - 1, \theta_f, 0^g - 1, \theta_{v-f}, -\sqrt{e} f - 1, \theta_v \} \), where we listed the eigenvalues in non-increasing order. So there are at most 7 distinct eigenvalues, but still there are many (at least about \( 2^f \) valencies). However, it becomes even better if we connect the new vertex to all vertices of one color class of each the bipartite components. In that case the graph remains bipartite, and it is easy to see that the rank of the new graph is the same as the rank of the original graph (indeed, the new incidence matrix has an extra row, but this row is the sum of the other (distinct) rows). This implies that 0 has multiplicity \( g + 1 \), hence \( \theta_f = \theta_{v-f} = 0 \), and the new graph only has five eigenvalues (note that \( \theta_0 = -\theta_v \neq \sqrt{e} \) because the graph is connected, for example). Thus we have the following.

**Proposition 1.** There are connected bipartite graphs with five eigenvalues and arbitrarily many valencies.

Is this the best we can do with bipartite graphs? Bipartite graphs with four eigenvalues are precisely the incidence graphs of so-called uniform multiplicative designs, see [13]. Examples of such graphs are known with up to four distinct valencies. The smallest such example is on 28 vertices, and is constructed from the Fano plane; its spectrum is \( \{ \sqrt{72} \frac{1}{2}, \sqrt{2} \frac{13}{2}, -\sqrt{2} \frac{13}{2}, -\sqrt{72} \frac{1}{2} \} \) and its valencies are 3, 4, 10, and 11 (each occurring 7 times). This graph is actually part of an infinite family of bipartite graphs with four eigenvalues and four valencies that can be obtained from a construction of non-normal uniform multiplicative designs by Ryser [17]. No examples of bipartite graphs with four eigenvalues and more than four valencies are currently known however. Note that there is a strong resemblance between graphs with three eigenvalues and bipartite graphs with four eigenvalues (see [8] and [13]). The following problem resembles De Caen’s problem on graphs with three eigenvalues.

**Problem 2.** Are there connected bipartite graphs with four eigenvalues and more than four valencies?

Note that the above graphs with five eigenvalues can be interpreted as coclique extensions of the spider graph (a disjoint union of edges that is extended by an extra vertex that is adjacent to one vertex of each edge; the spider with \( f \) legs has spectrum \( \{ \sqrt{f + 1}, 1^{f-1}, 0^4, -1^{f-1}, -\sqrt{f + 1} \} \}). Indeed, by replacing vertices with cocliques, and edges by complete bipartite graphs between the corresponding cocliques, we obtain the required graph. In the adjacency matrix this means that we replace a zero by a block of zeros and a one by a block of ones (of appropriate sizes). This clearly does not change the rank of the matrix, so it produces graphs with relatively small rank. It might in fact be fruitful to consider graphs with small
rank and many vertices, as studied by Akbari, Cameron, and Khosrovshahi \cite{1} and Haemers and Peeters \cite{16}.

3. Non-bipartite graphs with five eigenvalues

In the same spirit, we can construct non-bipartite graphs with five eigenvalues and arbitrarily many valencies. Consider again the disjoint union of $f$ complete bipartite graphs with $e$ edges (where $e$ factors in many different ways) and take the complement $\Gamma$ (which is clearly connected). Let $A$ be the adjacency matrix of $\Gamma$, then it is easy to see that $A + I$ has $2f$ distinct rows, and these are linearly independent. Therefore $\Gamma$ has eigenvalue $-1$ with multiplicity $v - 2f$. It is also not so hard to see that $\Gamma$ has eigenvalues $\pm \sqrt{e - 1}$, each with multiplicity at least $f - 1$. Indeed, if a graph has eigenvalue $\theta$ with multiplicity $m$, then its complement has eigenvalue $-1 - \theta$ with multiplicity at least $m - 1$, because the eigenspace of $\theta$ intersects the orthogonal complement of the all-ones vector in a subspace of dimension $m - 1$ or $m$, and the nonzero vectors in this intersection are easily seen to be eigenvectors of the complement of the graph. Thus, the spectrum of $\Gamma$ is $\{\rho_1, -1 + \sqrt{e - 1}, \rho_2, -1 - \sqrt{e - 1}\}$, where $\rho_1$ is the spectral radius, and $\rho_2$ is the remaining eigenvalue. By considering that $\text{tr} A = 0$ and $\text{tr} A^2 = v(v - 1) - 2fe$ (twice the number of edges of $\Gamma$), it follows that $\rho_1 + \rho_2 = v - 2$ and $\rho_1^2 + \rho_2^2 = v^2 - 2v + 2 - 2e(2f - 1)$. Therefore $\rho_{1,2} = -1 + \frac{1}{2}v \pm \frac{1}{2}\sqrt{v^2 - 4e(2f - 1)}$, and it can be shown that these are distinct from the other three eigenvalues of $\Gamma$ (for $f > 1$; we omit the technical details), so that $\Gamma$ has a total of five distinct eigenvalues. Thus this gives a family of connected (non-bipartite) graphs with five eigenvalues and arbitrarily many valencies.

Proposition 3. \textit{There are connected non-bipartite graphs with five eigenvalues and arbitrarily many valencies.}

4. Strong graphs with four eigenvalues

In this section, we will use so-called regular two-graphs to construct (connected, non-bipartite) graphs with four eigenvalues and many valencies. We will now recall some basics on two-graphs; for more details we refer to \cite{3, 14, 18}.

4.1. Regular two-graphs. Let $\Gamma$ be a graph with $(0, 1)$-adjacency matrix $A$. Its Seidel matrix $S = S(\Gamma)$ is defined as $J - I - 2A$. Let $\Pi = \{U, W\}$ be a two-partition of the vertex set $V$ of $\Gamma$. We say a graph — denoted by $\Gamma^\Pi$ — with the same vertex set as $\Gamma$ is obtained by (Seidel) switching $\Gamma$ with respect to $\Pi$ if two distinct vertices $x$ and $y$ are adjacent in $\Gamma^\Pi$ precisely if $x$ and $y$ are adjacent in $\Gamma$ and both are in $U$ or both are in $W$, or if they are not adjacent in $\Gamma$ and one of them is in $U$ and the other one is in $W$. In other words, the edges and non-edges between $U$ and $W$ have been switched. It is well-known that the spectra $S(\Gamma)$ and $S(\Gamma^\Pi)$ are the same. The switching class $[\Gamma]$ of $\Gamma$ is the set $\{\Gamma^\Pi \mid \Pi$ is a two-partition of the vertex set of $\Gamma$, possible with one part empty\} (note that switching induces an equivalence relation on graphs, with switching classes as equivalence classes). There is a one-one correspondence between switching classes of graphs and so-called two-graphs. For the sake of readability however, we will simply call the switching class a two-graph.

We say that the two-graph $[\Gamma]$ is regular if the Seidel matrix $S(\Gamma)$ has exactly two eigenvalues. Note that if the number of vertices of $\Gamma$ is at least two, then the
Seidel matrix $S(\Gamma)$ has at least two eigenvalues. The regular two-graphs containing a complete graph or an empty graph are called trivial.

The graphs in regular two-graphs are examples of so-called strong graphs. We are going to use these strong graphs to show that there are connected graphs with many distinct valencies and exactly four distinct eigenvalues.

In the following, we consider a graph $\Gamma$ in a regular two-graph with $v$ vertices. Let the Seidel matrix of $\Gamma$ have distinct eigenvalues $-\sigma$ and $-\tau$, with respective multiplicities $m_\sigma$ and $m_\tau$. First we will derive some more basic properties of $\Gamma$.

**Lemma 4.** The (adjacency) spectrum of $\Gamma$ is determined by the number of edges $e$ of $\Gamma$. In particular, the spectrum is \{\$\rho_1^1, \rho_2^1, \sigma^{m_\sigma-1}, \tau^{m_\tau-1}\$, where $\rho_1$ and $\rho_2$ are not necessarily distinct from each other or $\sigma$ or $\tau$, and the following equations hold:

$$
\begin{align*}
&m_\sigma + m_\tau = v, \\
&m_\sigma \sigma + m_\tau \tau = -v/2, \\
&m_\sigma \sigma^2 + m_\tau \tau^2 = v^2/4, \\
&\rho_1 + \rho_2 = \sigma + \tau + v/2 = -2\sigma\tau, \\
&\rho_1^2 + \rho_2^2 = \sigma^2 + \tau^2 + 2e - v^2/4.
\end{align*}
$$

**Proof.** The adjacency matrix $A = \frac{1}{2}(J - I - S)$ of $\Gamma$ has eigenvalue $\sigma$ with multiplicity at least $m_\sigma - 1$ and eigenvalue $\tau$ with multiplicity at least $m_\tau - 1$ because, similar as before, the eigenspaces of $S$ intersect the orthogonal complement of the all-ones vector in spaces of dimension at least $m_\tau - 1$ and $m_\sigma - 1$. So we have two unknown eigenvalues, say $\rho_1$ and $\rho_2$. But the sum of the eigenvalues (tr $A$) equals zero and the sum of squares of the eigenvalues (tr $A^2$) equals twice the number of edges. The given equations follow from these sums, and from using that tr $S = 0$ and tr $S^2 = v(v-1)$. We also use the well-known fact that $v-1 = -(2\sigma+1)(2\tau+1)$, which follows from the equation

$$
(S + (2\sigma + 1)I)(S + (2\tau + 1)I) = 0,
$$

and the fact that the diagonal entries of $S^2$ are all $v-1$. \qed

Because $e \leq \binom{v}{2}$ and the number of graphs in $[\Gamma]$ is $2^{v-1}$, there are many graphs in the switching class that have the same spectrum. It is unclear, however, how many of these graphs are non-isomorphic (see also the later remark after Theorem 8).

We remark that for non-trivial regular two-graphs, we have that the eigenvalues $\sigma$ and $\tau$ cannot be 0 or $-1$, and that the multiplicities $m_\sigma$ and $m_\tau$ are larger than 1 (cf. [15, Thm. 6.6]). We also note that if $\Gamma$ is regular, then all eigenvectors of $S$ are also eigenvectors of $A$, and it follows that $\Gamma$ has at most three distinct eigenvalues, so if the regular two-graph is non-trivial, then $\Gamma$ is strongly regular. In the following, we will show that if $\Gamma$ is non-regular, then $\Gamma$ has four distinct eigenvalues. We will also show that $\Gamma$ cannot be bipartite, but first, we will characterize the case that $\Gamma$ is disconnected.

**Proposition 5.** Let $\Gamma$ be a disconnected graph in a non-trivial regular two-graph, as above. Then $\Gamma$ is the disjoint union of an isolated vertex and a connected strongly regular graph with parameters $(-(2\sigma + 1)(2\tau + 1), -2\sigma\tau, \sigma + \tau - \sigma\tau, -\sigma\tau)$. 

Proof. Consider one of the connected components on the set, say, $V_1$ of $v_1$ vertices, and let $V_2$ be the set of remaining $v_2 = v - v_1$ vertices. Then the Seidel matrix $S$ partitions accordingly as

$$S = \begin{bmatrix} S_1 & J \\ J & S_2 \end{bmatrix}.$$ 

Since the Seidel matrix satisfies (2), it follows that

$$S_1 J + J S_2 + (2\sigma + 2\tau + 2)J = 0.$$ 

By considering a column of this equation, it follows that $S_1$ has constant row sum, say $c_1$. Similarly, it follows that $S_2$ has constant column sum, say $c_2$. Clearly, this means that the respective induced graph $\Gamma_i$ is regular with valency $k_i = (v_i - 1 - c_i)/2$, for $i = 1, 2$, respectively. By (11) and (12), it follows that $k_1 + k_2 = (v - 2 - c_1 - c_2)/2 = v/2 + \sigma + \tau = \rho_1 + \rho_2$. Note also that $k_1$ and $k_2$ are eigenvalues of $G$, so they are contained in the spectrum $\{\mu_1, \mu_2, \sigma^{m_\sigma-1}, \tau^{m_\tau-1}\}$. Moreover, we may assume without loss of generality that $\mu_1$ is an eigenvalue of $\Gamma_1$ and $\mu_2$ is an eigenvalue of $\Gamma_2$ (otherwise the ‘component’ not containing either of them has at most two distinct eigenvalues ($\sigma$ and $\tau$), but these are not 0 or $-1$, which is a contradiction), so both $\Gamma_1$ and $\Gamma_2$ are regular graphs with at most three eigenvalues. For the same reason, $\Gamma_2$ must be connected (recall that we already assumed that $\Gamma_1$ is connected), for otherwise it would have a connected component with at most two distinct eigenvalues $\sigma$ and $\tau$. Because $\Gamma_1$ is $k_1$-regular, it follows that it has spectral radius $k_1$, and hence $\mu_1 \leq k_1$, for $i = 1, 2$. From the fact that $k_1 + k_2 = \rho_1 + \rho_2$, it now follows $k_1 = \rho_1$ and $k_2 = \rho_2$. Thus, $\Gamma_1$ is a connected $\rho_1$-regular graph with at most three distinct eigenvalues, for $i = 1, 2$. However, because $-1$ is not an eigenvalue, neither component can be a clique (with at least two vertices). Now two cases remain.

First, if one of the two components, say $\Gamma_2$, is an isolated vertex, then $\rho_2 = k_2 = 0$, and $\Gamma_1$ is connected strongly regular graph with $\rho_1 = k_1 = -2\sigma \tau$ by (11). Now denote the parameters of $\Gamma_1$ by $(v_1, k_1, \lambda_1, \mu_1)$. Then $v_1 = v - 1 = -(2\sigma + 1)(2\tau + 1)$, $\mu_1 = k_1 + \sigma \tau = -\sigma \tau$, and $\lambda_1 = \sigma + \tau + \mu_1 = \sigma + \tau - \sigma \tau$.

Secondly the case remains that both components are connected strongly regular graphs. Assume without loss of generality that $\Gamma_2$ has the smallest valency of the two components, and let it have parameters $(v_2, k_2, \lambda_2, \mu_2)$. Because $k_1 + k_2 = -2\sigma \tau$ by (11), it follows that $k_2 \leq -\sigma \tau$. But then $\mu_2 = k_2 + \sigma \tau \leq 0$, and so $\Gamma_2$ is disconnected, which is a contradiction. Thus, this final case cannot occur. \qed

We note that by switching, one can always isolate every given vertex, and thus obtain the disjoint union of a vertex and a connected strongly regular graph with the given parameters. It is in fact well-known that regular two-graphs are characterized by this property. The contribution of Proposition 5 is that there can be no other disconnected graphs. Another consequence of the correspondence to strongly regular graphs is that $\sigma$ and $\tau$ are integers, except (possibly) if $m_\sigma = m_\tau$, in which case $\sigma$ and $\tau$ are equal to $-\frac{3}{2} \pm \frac{1}{2}\sqrt{v - 1}$.

**Proposition 6.** Let $\Gamma$ be a graph with at most three distinct eigenvalues in a non-trivial regular two-graph, as above. Then $\Gamma$ is strongly regular with parameters $(-(2\sigma + 1)(2\tau + 1) + 1, -\tau(2\sigma + 1), \sigma(1 - \tau), -\tau(\sigma + 1))$ or $(-(2\sigma + 1)(2\tau + 1) + 1, -\sigma(2\tau + 1), \tau(1 - \sigma), -\sigma(\tau + 1))$. 
Proof. By the previous proposition, Γ must be connected, so the spectral radius has multiplicity 1. Consider the spectrum \( \{\rho_1, \rho_2, \sigma^{m-1}, \tau^{m-1}\} \) of Γ. By the assumption that Γ has at most three distinct eigenvalues, we may assume without loss of generality that \( \rho_2 = \sigma \) or \( \rho_2 = \rho_1 \).

Suppose first that \( \rho_2 = \sigma \). Then \( \rho_1 = \tau + v/2 \) by (1). Because \( m_\sigma > 1 \) and \( \rho_1 > \tau \), it follows that \( \rho_1 \) must be the spectral radius. Because \( 2e = \text{tr} A^2 = \rho_1^2 + m_\sigma \sigma^2 + (m_\tau - 1)\tau^2 = \rho_1^2 + v^2/4 - (\rho_1 - v/2)^2 = v\rho_1 \) (where we used (1)), it now follows that Γ is regular. Thus, Γ is strongly regular, and its parameters \((-2\sigma + 1)(2\tau + 1) + 1, -\sigma(2\tau + 1), (1 - \sigma), -\sigma(\tau + 1))\) follow in a straightforward manner. By interchanging the role of \( \tau \) and \( \sigma \) we obtain the other parameter sets in the statement of the proposition.

Finally, suppose that \( \rho_2 = \rho_1 \). Then \( \rho_1 = \rho_2 = -\sigma \tau \) by (1). In this case, assume (without loss of generality) that \( \tau > \sigma \). Then \( \tau \) must be the spectral radius, and \( \sigma < -1 \). But then \( \rho_1 = -\sigma \tau > \tau \), which is a contradiction. \( \square \)

**Proposition 7.** Let Γ be a graph in a non-trivial regular two-graph. Then Γ is not bipartite.

Proof. Suppose that Γ is bipartite. By the previous two propositions it follows that Γ is connected with four distinct eigenvalues. From the bipartiteness, we have that its spectrum \( \{\rho_1, \rho_2, \sigma^{m-1}, \tau^{m-1}\} \) is symmetric about 0. If \( m_\sigma > 2 \) or \( m_\tau > 2 \), then it follows that \( \sigma = -\tau \) and \( \rho_1 = -\rho_2 \), but then the equation \( \rho_1 + \rho_2 = \sigma + \tau + v/2 \) from (1) gives a contradiction. So \( m_\sigma = 2 \) and \( m_\tau = 2 \), so \( v = 4 \). However, the only regular two-graphs on four vertices are the trivial ones. \( \square \)

Besides being of general interest, Propositions 5 and 7 allow us to conclude that if we find graphs in regular two-graphs with many (more than 2) valencies, then they are connected, non-bipartite, and have four distinct eigenvalues. Let us therefore construct such graphs.

### 4.2. Graphs with many valencies from the symplectic two-graph.

Let \( r \) be a positive integer. Let \((\cdot, \cdot)\) denote a non-degenerate symplectic bilinear form on the \( 2r \)-dimensional vector space \( V = \text{GF}(2)^{2r} \) over the binary field \( \text{GF}(2) \). Let Γ be the graph with vertex set \( V \) and \( u \sim v \) if \( (u, v) \neq 0 \). The switching class \([\Gamma]\) is known as the symplectic two-graph; and it is regular with \( \sigma, \tau = \pm 2^{-1} \). It is clear that Γ has \( 0 \) as an isolated vertex. The other component of Γ is known as the symplectic graph \( \text{Sp}(2r) \), which is a strongly regular graph with parameters \( (2^{2r} - 1, 2^{2r-1}, 2^{2r-2}, 2^{2r-2}) \) according to Proposition 5; see also [14] Lemma 10.12.1.

We will use the fact that the graph \( \text{Sp}(2r) \) has every graph on at most \( 2r - 1 \) vertices as an induced subgraph, a result shown by Vu [19] (see also [14] Thm. 8.11.2).

**Theorem 8.** Let \( t \geq 3 \), and Δ be a graph on \( n \) vertices with \( t \) distinct valencies. Then there exists a connected graph on at most \( 2^{n+2} \) vertices with four distinct eigenvalues and at least \( t \) distinct valencies, having Δ as an induced subgraph.

Proof. Let \( r = \lceil \frac{n}{2} \rceil + 1 \) and consider the symplectic two-graph \([\Gamma]\) as described above. Then Δ is an induced subgraph of the component \( \text{Sp}(2r) \) of Γ, say on the vertex set \( U \). Now switch Γ with respect to \( \Pi = (U, V \setminus U) \). Then the resulting graph \( \Gamma^{\Pi} \) has at least \( t \) valencies because the graph \( \text{Sp}(2r) \) is regular, and as the switching class of Γ is a regular two-graph, we obtain that \( \Gamma^{\Pi} \) is connected with exactly four distinct eigenvalues, by Lemma 4 and Propositions 5 and 6. \( \square \)
Note that also the Paley graphs have the property that they contain all graphs on a smaller number of vertices as induced subgraphs (see [2]), so instead of the symplectic two-graphs, one can also use the regular two-graphs that correspond to the Paley graphs. In any case, we conclude the following.

**Proposition 9.** There are connected non-bipartite graphs with four eigenvalues and arbitrarily many valencies.

As a side result of the construction method for Theorem 8, we obtain that it is possible to construct non-isomorphic graphs with the same spectrum in this way, starting from non-isomorphic subgraphs $\Delta$ with the same number of vertices and edges (see also the remark after Lemma 4). This follows in this particular case from the fact that $n$ is small compared to $k = 2^{2r-1}$, so that in $\Gamma^{11}$ there is a unique vertex with valency $n$, and its local graph is $\Delta$. A computational experiment shows that among the $2^{15}$ graphs in the symplectic two-graph on 16 vertices, there are at least four connected non-isomorphic graphs with the same spectrum as $\Gamma$ (which itself is not connected). In total 15 possible spectra occur, and for each of these except for two, we obtain at least two non-isomorphic graphs with that particular spectrum (in one exceptional case we obtain the strongly regular Clebsch graph, and in the other we obtain a graph with spectral radius $4 + \sqrt{27}$ and valencies 7, 9, 11, and 13). The maximum number of distinct valencies for a graph in this regular two-graph is five.

In this context, it is good to mention that Cioabă, Haemers, Vermette, and Wong [5] recently showed that all Friendship graphs except the one on 33 vertices are determined by the spectrum. These graphs also have four distinct eigenvalues of which two are simple, just like the non-regular graphs in a regular two-graph. Also, Van Dam [7, Thm. 4.4, §4.3.5] characterized several regular graphs with four eigenvalues of which two are simple. Moreover, for most of the latter characterizations, Seidel switching plays a key role. In general, however, we expect that almost all graphs with few (say at most 7) eigenvalues are not determined by the spectrum. We expect this even more for graphs with four eigenvalues of which two are simple. For the general question of which graphs are determined by the spectrum, we refer to [11, 12].

5. **Graphs with three eigenvalues**

Let us return in this final section to De Caen’s original question. Currently, we know only of finitely many connected graphs with three eigenvalues and three valencies [1, 8], but let us see whether we can find an example with three eigenvalues and four valencies.

Consider a connected graph with three distinct eigenvalues. As long as the number of distinct valencies is at most three, the partition of the vertices according to their valencies is equitable (see [8]), and this makes a search for (putative parameter sets for) such graphs easier. It is unclear whether the ‘valency partition’ is equitable if the number of valencies is larger than three, or whether such graphs can exist at all. A relatively small putative parameter set with four valencies (in fact, the smallest one according to 15-year old, but unverified, computations) is the following one on 51 vertices and spectrum $\{30^1, 3^{20}, -3^{30}\}$. The computations show that a graph with this spectrum must have valencies 13, 18, 34, and 45 occurring 15, 5, 30, and 1 times, respectively. In fact, using the techniques of [8] it can be shown
that in this particular case, the valency partition is also equitable, with quotient matrix
\[
\begin{bmatrix}
2 & 0 & 10 & 1 \\
0 & 0 & 18 & 0 \\
5 & 3 & 25 & 1 \\
15 & 0 & 30 & 0 \\
\end{bmatrix}.
\]
Quite a bit of this graph is therefore determined. Besides the trivial parts, one can show that the incidence structure between the 5 vertices of valency 18 and 30 vertices of valency 34 is a 2-(5, 3, 9) design, and there is only one such design: three times the full design of all triples on 5 points. We leave it as a problem to the reader to finish the (de-)construction.

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