Noncommutative gravity in three dimensions coupled to point-like sources

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Abstract
Noncommutative gravity in three dimensions with no cosmological constant is reviewed. We find a solution which describes the presence of a torsional source.

1 What does ‘noncommutative’ mean?

Please see [1] for a review of noncommutative field theory, if you want to study more.

Consider noncommutative coordinates, for example

\[ [x, y] = i \theta , \]

(1)

where \( \theta \) is a real constant. Then the ‘uncertainty’ lies between \( x \) and \( y \), namely,

\[ \Delta x \Delta y \geq \theta , \]

(2)

(where a numerical factor has omitted). This means the existence of the minimal length scale.

If complex combinations of the coordinates, \( z = x + iy \) and \( \bar{z} = x - iy \), are introduced, they satisfy

\[ [z, \bar{z}] = 2 \theta . \]

(3)

There are different representations to describe the noncommutativity; the commutative coordinate formalism with the star product, the Fock space (operator) formalism, etc. (see [1] for details). In this talk, we simply use them identically, unless the identification leads to confusions (thus, we do not use \( \star \) in this talk). For example, we denote the equalities

\[ 1 = \sum_{n=0}^{\infty} |n\rangle \langle n| , \quad z = \sum_{n=0}^{\infty} \sqrt{n+1} |n\rangle \langle n+1| , \quad \bar{z} = \sum_{n=0}^{\infty} \sqrt{n+1} |n+1\rangle \langle n| , \]

(4)

where ket and bra satisfy \( z|0\rangle = 0 \), \( z|n\rangle = \sqrt{2}\theta \sqrt{n+1} |n-1\rangle \), \( \bar{z}|n\rangle = \sqrt{2}\theta \sqrt{n+1} |n+1\rangle \), and so on.

Another example is

\[ 2(-1)^m L_m(2r^2/\theta) e^{-r^2/\theta} = |m\rangle \langle m| , \]

(5)

where \( r^2 = x^2 + y^2 \) and \( L_n(x) \) is the Lagurre polynomial.

For later use, we define \( \frac{1}{z} \) the inverse of \( z, \bar{z} \) as

\[ \frac{1}{z} = \frac{1}{\sqrt{2}\theta} \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} |n+1\rangle \langle n| , \quad \frac{1}{\bar{z}} = \frac{1}{\sqrt{2}\theta} \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} |n\rangle \langle n+1| . \]

(6)

This definition leads to \( z \frac{1}{z} = \frac{1}{z} \bar{z} = 1 \), however,

\[ \frac{1}{z} = \frac{1}{\bar{z}} = 1 - |0\rangle \langle 0| . \]

(7)

Thus the derivative of \( \frac{1}{z} \) and \( \frac{1}{\bar{z}} \) is

\[ \partial_z \frac{1}{z} = \frac{1}{2\theta} \left[ z, \frac{1}{z} \right] = \frac{1}{2\theta} |0\rangle \langle 0| = \frac{1}{\theta} e^{-r^2/\theta} , \quad \partial_{\bar{z}} \frac{1}{\bar{z}} = \frac{1}{2\theta} \left[ \bar{z}, \frac{1}{\bar{z}} \right] = \frac{1}{2\theta} |0\rangle \langle 0| = \frac{1}{\theta} e^{-r^2/\theta} . \]

(8)

Interestingly enough, in the commutative limit, we find

\[ \frac{1}{\theta} e^{-r^2/\theta} \xrightarrow{\theta \rightarrow 0} \pi \delta(x) \delta(y) . \]

(9)

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2Note that here \( \frac{1}{z} \) is defined as operator formalism and this may differ from \( z^{-1} \) in the usual star product formalism.
2 Why ‘noncommutative’?

We expect that noncommutative field theory is worth studying, because:

- the fundamental minimal length $\sqrt{\theta}$ may avoid singularities and infinities which arise in usual field theories.
- the effective field theory of string or brane or M-theory in the presence of the background $B$ field can be expressed as a noncommutative field theory.
- the analysis of quantum Hall systems with a constant magnetic field admits the formulation with noncommuting coordinates.

3 Three dimensional noncommutative gravity

...has a long history.

In this talk, we concentrate our attention on noncommutative gravity in three dimensions. Three dimensional Chern-Simons noncommutative gravity was studied by Balasubramanian et al. and more recently by Cacciatori et al.

We would like to study noncommutative gravity in three dimensions with no cosmological constant. Further, we wish to find exact solutions whose spatial coordinates are mutually noncommutative.

The signature is taken to be Euclidean, and the coordinates are denoted as

$$x^1 = x, \quad x^2 = y, \quad x^3 = \tau, \quad \text{where} \quad [x, y] = i \theta. \quad (10)$$

We define a matrix-valued dreibein one-form and a connection one-form as

$$e = e^a J_a + e^4 i, \quad \omega = \omega^a J_a + \omega^4 i, \quad \text{where} \quad J_1 = \frac{i}{2} \sigma_1, \quad J_2 = -\frac{i}{2} \sigma_2, \quad J_3 = \frac{i}{2} \sigma_3. \quad (11)$$

A matrix-valued torsion two-form and a curvature two-form are given by

$$T = de + \omega \wedge e + e \wedge \omega, \quad R = d\omega + \omega \wedge \omega. \quad (12)$$

The vacuum solution of ‘Einstein equation’ satisfies

$$R = T = 0. \quad (13)$$

Unless the Abelian field $e^4$ and $\omega^4$ vanish, we cannot regard this model as that for noncommutative gravity with an arbitrary value of $\theta$.

4 Difficulties in noncommutative gravity I

If we choose the spin connection as

$$\omega = \frac{\alpha}{2} \left( \frac{1}{z} dz - \frac{1}{\bar{z}} d\bar{z} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right), \quad (14)$$

we obtain the curvature

$$R = \frac{\alpha}{2 \theta} |0\rangle \langle 0| dz \wedge d\bar{z} \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) + \frac{\alpha^2}{4} \left[ \frac{1}{z} \frac{1}{\bar{z}} \right] \left( \begin{array}{cc} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{array} \right) dz \wedge d\bar{z} \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right). \quad (15)$$

In the commutative limit, this can be regarded as the curvature of the spacetime where the point particle with mass $-\alpha/(8G)$ is located at the origin (where $G$ is the Newton constant) \[3\]. For an arbitrary value of $\theta$, however, the Abelian part remains; its interpretation is difficult, as for a theory of gravity.\[3\]

\[3\]See Nair\[4\] for an extension to the other dimensions.
5 Difficulties in noncommutative gravity II

In the matrix form, the local Lorentz transformation can be expressed as
\[ e' = U^{-1} e U, \quad \omega' = U^{-1} \omega U + U^{-1} dU, \] (16)
where \( UU^{-1} = 1 \).

Under these transformation, \( T \) and \( R \) becomes
\[ T' = U^{-1} T U, \quad R' = U^{-1} R U. \] (17)
Then the equations of motion \( T = R = 0 \) is unchanged.

On the other hand, the local translations can be written as
\[ e' = e + d\rho + \omega \rho - \rho \omega, \quad \omega' = \omega, \] (18)
Under these transformation, \( T \) and \( R \) becomes
\[ T' = T + \rho R - R \rho, \quad R' = R. \] (19)
The equations of motion \( T = R = 0 \) is unchanged also by the translation.

Therefore, we can construct vacuum solutions as pure gauge. The matrices which satisfy \( UU^{-1} = 1 \) take the form (modulo rigid rotations)
\[ U = \begin{pmatrix} S & 0 \\ P & S^{-1} \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} S^{-1} & P \\ 0 & S \end{pmatrix}, \] (20)
where \( SS^{-1} = 1, S^{-1}S = 1 - P, P^2 = P \), and \( SP = PS^{-1} = 0 \) are required.

Then the pure-gauge connection is
\[ \omega_g = U^{-1} dU = \begin{pmatrix} S^{-1} & P \\ 0 & S \end{pmatrix} \begin{pmatrix} dS & 0 \\ dP & dS^{-1} \end{pmatrix} = \begin{pmatrix} S^{-1}dS + PdP & PdS^{-1} \\ SdP & SdS^{-1} \end{pmatrix}, \] (21)
Unfortunately, this includes the Abelian part in general \((\text{Tr} \omega)\). Thus the interpretation of this type of solutions is also unclear in a theory of gravity.

6 A solution with a torsional source

Now we choose
\[ e = \frac{i}{2} \left\{ d\tau + \frac{GS}{2i} \left( \frac{1}{z} \frac{dz}{dz} - \frac{1}{2} \frac{d\bar{z}}{d\bar{z}} \right) \right\} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{i}{2} \begin{pmatrix} 0 & dz \\ d\bar{z} & 0 \end{pmatrix}, \] (22)
where \( S \) is a constant and \( \omega = 0 \).

Then we obtain
\[ T = \frac{GS}{4\theta} \left| 0 \right> \left< 0 \right| d\bar{z} \wedge dz \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad R = 0. \] (23)

In the commutative limit, this corresponds to the solution obtained by Deser, Jackiw and ’t Hooft in the case of the mass of the point particle is zero. For a finite \( \theta \), the torsional source has a finite extension.

This one-body solution can be generalized to the \( N \)-body solution,
\[ e = \frac{i}{2} \left\{ d\tau + \sum_{a=1}^{N} \frac{GS_a}{2i} \left( \frac{1}{z - z_a} dz - \frac{1}{2 - z_a} d\bar{z} \right) \right\} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{i}{2} \begin{pmatrix} 0 & dz \\ d\bar{z} & 0 \end{pmatrix}, \] (24)
where \( z_a, 2_a \) and \( S_a \) are constants and \( \frac{1}{z - z_a} \) and \( \frac{1}{2 - z_a} \) are defined by
\[ \frac{1}{z - z_a} = \sum_{m=0}^{\infty} z_a^m \left( \frac{1}{z} \right)^{m+1}, \quad \frac{1}{2 - z_a} = \sum_{m=0}^{\infty} \frac{z_a^m}{2} \left( \frac{1}{2} \right)^{m+1}. \] (25)
7 Wave equation

Now we can write down a wave equation for a massless scalar field around one torsional source,

\[ \left\{ -\partial^2_t + 2(D_z D_{\bar{z}} + D_{\bar{z}} D_z) \right\} \phi = 0, \]

(26)

where

\[ D_z \equiv \partial_z - \frac{G S}{2i} \frac{1}{z} \partial_t, \quad D_{\bar{z}} \equiv \partial_{\bar{z}} + \frac{G S}{2i} \frac{1}{\bar{z}} \partial_t. \]

(27)

Here we changed the signature into the Lorentzian one.

Solving the noncommutative differential equation, we will analyze the wave scattering by the torsional source. The scattering process is nontrivial for a small impact parameter \( \approx \sqrt{\theta} \). The investigation of the wave scattering will be reported elsewhere [6].

8 Open problems

- How can the Abelian fields in this formalism have certain meanings in a theory of gravity?
- How can we obtain ‘conical’ (i.e. massive) solution?
- How can we take global properties of spacetime into account? How and when do we have to use a noncommutative torus and sphere?

References

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