The bright $N$-soliton solution of a multi-component modified nonlinear Schrödinger equation

Yoshimasa Matsuno

Division of Applied Mathematical Science, Graduate School of Science and Engineering, Yamaguchi University, Ube, Yamaguchi 755-8611, Japan

E-mail: matsuno@yamaguchi-u.ac.jp

Received 5 September 2011, in final form 21 October 2011
Published 18 November 2011
Online at stacks.iop.org/JPhysA/44/495202

Abstract

A direct method is developed for constructing the bright $N$-soliton solution of a multi-component modified nonlinear Schrödinger equation. Specifically, the two different expressions of the solution are obtained both of which are expressed as a ratio of determinants. A simple relation is found between them by employing the properties of the Cauchy matrix. The proof of the solution reduces to the bilinear equations among the bordered determinants in which Jacobi’s identity and related formulas play a central role. Finally, the bright $N$-soliton solution is presented for a (2+1)-dimensional nonlocal model equation arising from the multi-component system as the number of dependent variables tends to infinity.

PACS numbers: 05.45.Yv, 42.81.Dp, 02.30.Jr

1. Introduction

The study of a multi-component system of nonlinear partial differential equations (PDEs) is of current interest in the theory of nonlinear waves. Of particular concern are the multi-component generalizations of the nonlinear Schrödinger (NLS) equation because of their wide applicability in real physical systems such as nonlinear optics, nonlinear water waves and plasma physics [1–3]. The integrable two-component NLS system has been introduced for the first time by Manakov to describe the propagation of the polarized electric field in an optical fiber and explored in detail by means of the inverse scattering transform (IST) method [4]. After this remarkable work, various variants of integrable multi-component NLS systems have been proposed and analyzed by the exact method of solution such as the IST and Hirota’s direct method. In this paper, we consider the following multi-component system of nonlinear PDEs which is a hybrid of the coupled NLS equation and coupled derivative NLS equation:

$$i q_{j,t} + q_{j,xx} + \mu \left( \sum_{k=1}^{n} |q_k|^2 \right) q_j + i \gamma \left( \sum_{k=1}^{n} |q_k|^2 \right) q_j = 0, \quad j = 1, 2, \ldots, n,$$  \quad (1.1)
where \( q_j = q_j(x,t), \ j = 1, 2, \ldots, n, \) are complex-valued functions of \( x \) and \( t, \mu \) and \( \gamma \) are the real constants, \( n \) is an arbitrary positive integer and the subscripts \( x \) and \( t \) appended to \( q_j \) denote partial differentiations. The integrability of the above system has been established by constructing the Lax pair and an infinite number of conservation laws [5]. For the two special cases \( \gamma = 0 \) and \( \mu = 0 \) reduced from the system (1.1) which correspond to the multi-component NLS and multi-component derivative NLS equations, respectively, their integrability has already been verified in [6, 7]. In the context of plasma physics, the two-component system with \( \mu = 0 \) is a model equation for the propagation of polarized Alfvén waves. The single bright soliton solution to this system has been obtained by means of the IST [8]. The two-component system with \( \mu \neq 0 \) and \( \gamma \neq 0 \) has been derived as a model for describing the propagation of ultra-short pulses in birefringent optical fibers, together with its soliton solutions [9]. Quite recently, we obtained the general bright multi-soliton solution (i.e. the N-soliton solution with \( N \) being an arbitrary positive integer which vanishes at infinity) of the two-component system by using a direct method [10]. It is important to remark that the constant \( \mu \) must be non-negative to support smooth bright solitons. The negative case is worth studying as well which will be treated in a separate issue.

The purpose of this paper is to extend the results obtained in [10] to the general \( n \)-component system. Specifically, we present the bright \( N \)-soliton solution of the \( n \)-component system (1.1) in the form of compact determinantal expressions. Although the construction of the solution can be done following a similar procedure to that developed in [10] for the two-component system, we provide a novel proof of the solution using the expansion formulas for the bordered determinant.

This paper is organized as follows. In section 2, we first transform the system (1.1) to a gauge equivalent system and then recast it to a system of bilinear equations by introducing appropriate dependent variable transformations. Subsequently, the bright \( N \)-soliton solution to the bilinear equations is presented. It has a simple structure expressed in terms of certain determinants. In section 3, we introduce some notations associated with the bright \( N \)-soliton solution and then prove several key formulas for determinants. In section 4, we perform the proof of the bright \( N \)-soliton solution using an elementary theory of determinants in which Jacobi’s identity and related formulas will play a central role. In section 5, we provide an alternative expression of the bright \( N \)-soliton solution and compare it with the corresponding solution presented in section 2. We then find a simple relation between the two types of solutions by employing the properties of the Cauchy matrix. This result leads to an alternative proof of the solution in a very simple way. In section 6, we discuss a (2+1)-dimensional nonlocal modified NLS equation arising from the continuum limit \( n \to \infty \) of the system (1.1). Specifically, we demonstrate that its bright \( N \)-soliton solution can be generated simply from that of the \( n \)-component system. Section 7 is devoted to concluding remarks where we will comment on existing literature about the bright \( N \)-soliton solutions of the multi-component NLS equation (equation (1.1) with \( \gamma = 0 \)) and show that the solutions obtained in this paper include these known solutions as special cases.

2. Bilinearization and the bright \( N \)-soliton solution

2.1. Bilinearization

We first apply the gauge transformations

\[
q_j = u_j \exp \left[ -i \gamma \frac{1}{2} \int_{-\infty}^{x} \sum_{k=1}^{n} |u_k|^2 \, dx \right], \quad j = 1, 2, \ldots, n, \quad (2.1)
\]
to the system (1.1) subjected to the boundary conditions $q_j \to 0$, $u_j \to 0$, $j = 1, 2, \ldots, n$, as $|x| \to \infty$, where $u_j = u_j(x, t)$, $j = 1, 2, \ldots, n$, are complex-valued functions of $x$ and $t$.

Then, we obtain the system of nonlinear PDEs for $u_j$:

$$
i u_{j,x} + u_{j,t} + \mu \left( \sum_{k=1}^{n} |u_k|^2 \right) u_j + i\gamma \left( \sum_{k=1}^{n} u_k^* u_{j,k}^* \right) u_j = 0, \quad j = 1, 2, \ldots, n, 
$$

(2.2)

where the asterisk appended to $u_k$ denotes complex conjugate. This notation will be used frequently hereafter. The second step in our analysis is given by the following proposition.

**Proposition 2.1.** By means of the dependent variable transformations

$$
u_j = \frac{g_j}{f}, \quad j = 1, 2, \ldots, n, 
$$

(2.3)

the system of nonlinear PDEs (2.2) can be decoupled into the following system of bilinear equations for $f$ and $g_j$:

$$(i D_x + D_t^2) g_j \cdot f = 0, \quad j = 1, 2, \ldots, n, 
$$

(2.4)

$$D_x f \cdot f^* = \frac{i\gamma}{2} \sum_{k=1}^{n} |g_k|^2, 
$$

(2.5)

$$D_t^2 f \cdot f^* = \mu \sum_{k=1}^{n} |g_k|^2 + \frac{i\gamma}{2} \sum_{k=1}^{n} D_x g_k \cdot g_k^*. 
$$

(2.6)

Here, $f = f(x, t)$ and $g_j = g_j(x, t)$, $j = 1, 2, \ldots, n$, are the complex-valued functions of $x$ and $t$ and the bilinear operators $D_x$ and $D_t$ are defined by

$$D^n_x D^m_t f \cdot g = \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^m \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^n f(x, t) g(x', t') \bigg|_{x' = x, t' = t}, 
$$

(2.7)

where $m$ and $n$ are the non-negative integers.

**Proof.** Substituting (2.3) into (2.2) and rewriting the resultant equation in terms of the bilinear operators, equation (2.2) can be rewritten as

$$\frac{1}{f^2} (iD_x g_j \cdot f + D_t^2 g_j \cdot f) + \frac{g_j}{f^2} \left( -f^* D_x^2 f \cdot f + \mu \sum_{k=1}^{n} |g_k|^2 + i\gamma \sum_{k=1}^{n} g_k^* D_x g_k \cdot f \right) = 0, 
$$

(2.8)

for $j = 1, 2, \ldots, n$.

Insert the identity

$$f^* D_x^2 f \cdot f = f D_x^2 f \cdot f^* - 2 f_x D_x f \cdot f^* + f(D_x f \cdot f^*)_x
$$

(2.9)

into the second term on the left-hand side of (2.8). Then, equation (2.8) becomes

$$\frac{1}{f^2} (iD_x g_j \cdot f + D_t^2 g_j \cdot f) + \frac{g_j}{f^2} \left[ f \left( -D_x^2 f \cdot f^* + \mu \sum_{k=1}^{n} |g_k|^2 + D_x f \cdot f^*_x + i\gamma \sum_{k=1}^{n} g_k^* g_k \right) 
+ f_x \left( 2D_x f \cdot f^* - i\gamma \sum_{k=1}^{n} |g_k|^2 \right) \right] = 0, \quad j = 1, 2, \ldots, n. 
$$

(2.10)

As easily confirmed by a direct calculation, the left-hand side of (2.10) becomes zero by virtue of equations (2.4)–(2.6).
It now follows from (2.3) and (2.5) that
\[-\frac{i\gamma}{2} \sum_{k=1}^{n} |u_k|^2 = \frac{\partial}{\partial x} \ln f^*,\]
which, substituted into (2.1), yields the solution of the system (1.1) in the form
\[q_j = \frac{g_j f^*}{f^2}, \quad j = 1, 2, \ldots, n.\]  
(2.12)

Note that for the \(n\)-component NLS equation (the system (1.1) with \(\gamma = 0\)), the solution (2.12) simplifies to \(q_j = g_j / f\). Indeed, if \(\gamma = 0\), then the bilinear equation (2.5) reduces to \(D_x f \cdot f^* = 0\). Thus, the ratio \(f^*/f\) turns out to be an arbitrary function of \(t\) which can be set to 1 under an appropriate boundary condition.

2.2. The bright N-soliton solution

We now state the main result in this paper as follows.

**Theorem 2.1.** The bright N-soliton solution of the system of bilinear equations (2.4)–(2.6) is given by the determinants \(f\) and \(g_j\) (\(j = 1, 2, \ldots, n\)), where
\[f = \begin{vmatrix} A & I \\ -I & B \end{vmatrix}, \quad g_j = \begin{vmatrix} A & I \\ -I & B \end{vmatrix} \begin{pmatrix} z^T \\ 0 \end{pmatrix}, \quad j = 1, 2, \ldots, n.\]  
(2.13)

Here, \(A, B\) and \(I\) are \(N \times N\) matrices and \(z, a_j\) and \(0\) are \(N\)-component row vectors defined below and the symbol \(T\) denotes the transpose:

\[A = (a_{jk})_{1 \leq j, k \leq N}, \quad a_{jk} = \frac{1}{2} \frac{z_j z_k^*}{p_j + p_k^*}, \quad z_j = \exp(p_j x + i p_j^2 t),\]  
(2.14a)

\[B = (b_{jk})_{1 \leq j, k \leq N}, \quad b_{jk} = \frac{(\mu + i\gamma p_k)c_{jk}}{p_j^* + p_k}, \quad c_{jk} = \sum_{s=1}^{n} a_{js} a_{sk}^*,\]  
(2.14b)

\[I = (\delta_{jk})_{1 \leq j, k \leq N}, : N \times N\;\text{unit matrix},\]  
(2.14c)

\[z = (z_1, z_2, \ldots, z_N), \quad a_j = (\alpha_{j1}, \alpha_{j2}, \ldots, \alpha_{jN}), \quad 0 = (0, 0, \ldots, 0).\]  
(2.14d)

The above bright \(N\)-soliton solution is characterized by \((n + 1)N\) complex parameters \(p_j\) (\(j = 1, 2, \ldots, n\)) and \(\alpha_{sj}\) (\(s = 1, 2, \ldots, n; \; j = 1, 2, \ldots, N\)). The former parameters determine the amplitude and velocity of the solitons, whereas the latter determine the polarizations and the envelope phases of the solitons. The conditions \(p_j + p_k^* \neq 0\) for all \(j\) and \(k\) and \(p_j \neq p_k\) for \(j \neq k\) may be imposed to ensure the regularity of the solution. In the special case of \(n = 2\), (2.13) and (2.14) reproduce the bright \(N\)-soliton solution presented in [10]. The proof of theorem 2.1 will be given in section 4.

To simplify the proof of theorem 2.1, the following observation is useful.

**Proposition 2.2.** If we introduce the gauge transformations

\[f = \tilde{f}, \quad g_j = \exp \left[i \left(\frac{\mu}{\gamma} \tilde{x} + \left(\frac{\mu}{\gamma^2} \tilde{t} \right) \right) \right] \tilde{g}_j, \quad j = 1, 2, \ldots, n.\]  
(2.15a)
then the bilinear equations (2.4)–(2.6) recast to

\[(\mathbf{i} \partial_{\tilde{t}} + \mathbf{D}_{\tilde{x}}^{2}) \tilde{g}_j \cdot \tilde{f} = 0, \quad j = 1, 2, \ldots, n, \quad (2.16)\]

\[D_{\tilde{x}} \tilde{f} \cdot \tilde{f}^* = \frac{iY}{2} \sum_{k=1}^{n} |\tilde{g}_k|^2 . \quad (2.17)\]

\[D_{\tilde{x}}^{2} \tilde{f} \cdot \tilde{f}^* = \frac{iY}{2} \sum_{k=1}^{n} \mathbf{D}_{\tilde{x}} \tilde{g}_k \cdot \tilde{g}_k^*.\quad (2.18)\]

respectively.

**Proof.** The proof can be performed by a straightforward calculation. \(\square\)

Thus, the form of equations (2.4) and (2.5) is unchanged, whereas equation (2.6) becomes a simplified equation with \(\mu = 0\). Consequently, the proof of the \(N\)-soliton solution may be performed for the corresponding solution with \(\mu = 0\). Hence, in the analysis developed in the following sections, we put \(\mu = 0\) without loss of generality.

### 3. Notation and some basic formulas for determinants

In this section, we first introduce the notation for matrices and then provide some basic formulas for determinants.

#### 3.1. Notation

We define the following matrices associated with the \(N\)-soliton solution (2.13) with (2.14):

\[D = \begin{pmatrix} A & I \\ -I & B \end{pmatrix}, \quad (3.1)\]

\[D(a^*; b) = \begin{pmatrix} A & I & 0^T \\ -I & B & b^T \\ 0 & a^* & 0 \end{pmatrix}, \quad D(a^*; z) = \begin{pmatrix} A & I & z^T \\ -I & B & 0^T \\ 0 & a^* & 0 \end{pmatrix}, \quad D(z^*; z) = \begin{pmatrix} A & I & z^T \\ -I & B & 0^T \\ 0 & a^* & 0 \end{pmatrix}. \quad (3.2)\]

Note the positions of the vectors \(a^*, b, z\) and \(z^*\) in the above expressions. The matrices which include more than two vectors will be introduced as well. For example,

\[D(a^*, z^*; b, z) = \begin{pmatrix} A & I & 0 & z^T \\ -I & B & b^T & 0^T \\ 0 & a^* & 0 & 0 \\ z^* & 0 & 0 & 0 \end{pmatrix}, \quad D(a^*, z^*; z, z') = \begin{pmatrix} A & I & z^T & z'^T \\ -I & B & 0^T & 0^T \\ 0 & a^* & 0 & 0 \\ z^* & 0 & 0 & 0 \end{pmatrix}. \quad (3.3)\]
3.2. Formulas for determinants

Let \( A = (a_{jk}) \) be an \( M \times M \) matrix with \( M \) being an arbitrary positive integer, \( A_{jk} \) be the cofactor of the element \( a_{jk} \) and \( a, b, a_j \) and \( b_j \) \((j = 1, 2, \ldots, n)\) be \( M \)-component row vectors. The following well-known formulas are used frequently in our analysis [11]:

\[
\frac{\partial}{\partial x} |A| = \sum_{j,k=1}^{M} \frac{\partial a_{jk}}{\partial x} A_{jk}, \tag{3.4}
\]

\[
|A a^T |z = |A| z - \sum_{j,k=1}^{M} A_{jk} a_j b_k, \tag{3.5}
\]

\[
|A(a_1, a_2; b_1, b_2)|/|A| = |A(a_1; b_1)||A(a_2; b_2)| - |A(a_1; b_2)||A(a_2; b_1)|. \tag{3.6}
\]

Formula (3.4) is the differentiation rule of the determinant and (3.5) is the expansion formula for a bordered determinant with respect to the last row and last column. Formula (3.6) is Jacobi’s identity.

The following two formulas may not be well known but are very important in our analysis. In particular, formula (3.7) gives rise to the expansion formulas for the bordered determinant (see (3.9) and (3.10) below):

\[
|A(a_1, \ldots, a_n; b_1, \ldots, b_n)|/|A|^{n-1} = \begin{vmatrix}
|A(a_1; b_1)| & \ldots & |A(a_1; b_n)| \\
\vdots & \ddots & \vdots \\
|A(a_n; b_1)| & \ldots & |A(a_n; b_n)|
\end{vmatrix} (n \geq 2), \tag{3.7}
\]

\[
|A + \epsilon \sum_{s=1}^{n} b_s^T a_s| = |A| + \sum_{m=1}^{n'} (-\epsilon)^m \sum_{1 \leq s_1 < \cdots < s_m \leq n} |A(a_{s_1}, \ldots, a_{s_m}; b_{s_1}, \ldots, b_{s_m})| \\
= |A| + \sum_{m=1}^{n'} (-\epsilon)^m m! \sum_{s_1, \ldots, s_m=1}^{n} |A(a_{s_1}, \ldots, a_{s_m}; b_{s_1}, \ldots, b_{s_m})|. \tag{3.8}
\]

Here, \( \epsilon \) is an arbitrary parameter, the notation \( b_s^T a_s \) on the left-hand side of (3.8) represents an \( M \times M \) matrix whose \((j, k)\) element is given by \( \beta_{jk} a_{jk} \) and \( n' = \min(n, M) \). Formula (3.7) is a variant of the Sylvester theorem in the theory of determinants.

**Proof of (3.7).** The proof proceeds by a mathematical induction. For \( n = 2 \), (3.7) reduces to Jacobi’s identity (3.6). Assume that formula (3.7) is true for \( n - 1 \) \((n \geq 3)\). Let \( L \) be the left-hand side of (3.7). Recall that the determinant changes its sign if any two rows (or columns) are interchanged. Applying this rule repeatedly to \( L \) when \( a_n \) and \( b_n \) are shifted in front of \( a_1 \) and \( b_1 \), respectively,

\[
L = |\hat{A}(a_1, \ldots, a_{n-1}; b_1, \ldots, b_{n-1})|/|A|^{n-1},
\]

where \( \hat{A} = A(a_1, b_2) \) is an \((M + 1) \times (M + 1)\) matrix which is assumed to be nonsingular, i.e. \( |\hat{A}| \neq 0 \). In view of the inductive hypothesis, \( L \) can be written as

\[
L = |\hat{A}|^{n-1} \begin{vmatrix}
|\hat{A}(a_1; b_1)| & \ldots & |\hat{A}(a_1; b_{n-1})| \\
\vdots & \ddots & \vdots \\
|\hat{A}(a_{n-1}; b_1)| & \ldots & |\hat{A}(a_{n-1}; b_{n-1})|
\end{vmatrix}. \]

It follows from Jacobi’s identity (3.6) that

\[
|\hat{A}(a_1; b_1)|/|A| = |A(a_1, a_2; b_2)|/|A| = |A(a_1; b_2)|/|A| = |A(a_1; b_1)|/|A|.
\]

6
which, substituted into $L$, recasts $L$ into the form

$$L = \frac{1}{|A(a_e; b_e)|^{n-2}} |\{ |A(a_e; b_e)| |A(a_j; b_e)| - |A(a_j; b_e)| |A(a_e; b_j)| \}_{1 \leq j, k \leq n-1}|.$$  

Referring to the property of the bordered determinant, the above expression simplifies to

$$L = \frac{1}{|A(a_e; b_e)|^{n-2}} \begin{vmatrix} |A(a_e; b_e)| |A(a_1; b_1)| & \cdots & |A(a_e; b_e)| |A(a_1; b_{n-1})| & |A(a_1; b_e)| \\ \vdots & \ddots & \vdots & \vdots \\ |A(a_e; b_e)| |A(a_1; b_1)| & \cdots & |A(a_e; b_e)| |A(a_{n-1}; b_{n-1})| & |A(a_{n-1}; b_e)| \\ |A(a_e; b_1)| & \cdots & |A(a_e; b_{n-1})| & 1 \end{vmatrix}.$$  

Extract the factor $|A(a_e; b_e)|$ from the $j$th row ($j = 1, 2, \ldots, n-1$) and then multiply the last column by the same factor. Then, the resultant expression is seen to be equal to the right-hand side of (3.7).

**Proof of (3.8).** For $n = 1$, it follows by using the property of the bordered determinant that

$$|A + \epsilon b_f^T a_1| = \begin{vmatrix} a_{11} + \epsilon \beta_{11} \alpha_{11} & \cdots & a_{1M} + \epsilon \beta_{11} \alpha_{1M} \\ \vdots & \ddots & \vdots \\ a_{M1} + \epsilon \beta_{1M} \alpha_{11} & \cdots & a_{MM} + \epsilon \beta_{1M} \alpha_{1M} \end{vmatrix} = \epsilon \begin{vmatrix} A & b_f^T \\ -a_{11} & \epsilon^{-1} \end{vmatrix}.$$  

Repeated use of the above modification yields

$$|A + \epsilon \sum_{j=1}^{n} b_f^T a_j| = \epsilon^n \begin{vmatrix} A & b_f^T & \cdots & b_f^T \\ -a_{11} & \epsilon^{-1} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ -a_{n} & 0 & \cdots & \epsilon^{-1} \end{vmatrix}.$$  

Expanding the determinant in powers of $\epsilon^{-1}$, it is found that

$$|A + \epsilon \sum_{j=1}^{n} b_f^T a_j| = |A| + \epsilon^n \sum_{m=1}^{\min(n, M)} \sum_{1 \leq s_1 < \cdots < s_m \leq n} |A(a_{s_1}, \ldots, a_{s_m}; b_{s_1}, \ldots, b_{s_m})| \epsilon^{-(n-m)}.$$  

The above expression coincides with (3.8) for $n \leq M$ since $n' = \min(n, M) = n$. For $M + 1 \leq n$, on the other hand, the determinant $|A(a_{s_1}, \ldots, a_{s_m}; b_{s_1}, \ldots, b_{s_m})|$ becomes zero identically for $M + 1 \leq m \leq n$, as confirmed by the Laplace expansion of the determinant with respect to the last $m$ rows, for example. This implies that the summation with respect to $m$ is truncated at $m = M$ which is in accordance with (3.8) since $n' = \min(n, M) = M$. The second line of (3.8) follows from the facts that any permutation of the indices $\{s_1, s_2, \ldots, s_m\}$ does not alter the value of the determinant $|A(a_{s_1}, \ldots, a_{s_m}; b_{s_1}, \ldots, b_{s_m})|$ and the total number of the permutation is $m!$, and if the determinant includes at least two same rows (or columns), then it becomes zero identically.

Suppose that $|A| \neq 0$. Expanding the determinant on the right-hand side of (3.7) with respect to the first column and using (3.7) with $n$ replaced by $n-1$, we then obtain an expansion.
formula

\[ |A(a_1, \ldots, a_n; b_1, \ldots, b_n)| = \frac{1}{|A|} \sum_{j=1}^{n} (-1)^{j-1} |A(a_j, b_1)||A(a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n; b_2, \ldots, b_n)|. \]  

(3.9)

Similarly, the expansion with respect to the first row gives

\[ |A(a_1, \ldots, a_n; b_1, \ldots, b_n)| = \frac{1}{|A|} \sum_{j=1}^{n} (-1)^{j-1} |A(a_1, b_j)||A(a_2, \ldots, a_n; b_1, \ldots, b_{j-1}, b_{j+1}, \ldots, b_n)|. \]  

(3.10)

The above two formulas will be used effectively to prove the bilinear equations (2.5) and (2.6).

4. Proof of the bright \( N \)-soliton solution

In this section, we show that the bright \( N \)-soliton solution (2.13) with (2.14) satisfies the system of bilinear equations (2.4)–(2.6). The proof can be performed for \( \mu = 0 \), as noted at the end of section 2. We first prove some formulas associated with the determinants \( f \) and \( g_j \) \( \text{for} \ j = 1, 2, \ldots, n \) and then proceed to the proof.

4.1. Formulas

In terms of the notation introduced in section 3.1 (see (3.1) and (3.2)), \( f \) and \( g_j \) are written in the form

\[ f = |D|, \quad g_j = -|D(a_j^*; z)|, \quad j = 1, 2, \ldots, n. \]  

(4.1)

The differentiation rules of \( f \) and \( g_j \) with respect to \( t \) and \( x \) are given by the following formulas:

Lemma 4.1.

\[ f_t = -\frac{i}{2} \left[ |D(z^*; z_t)| - |D(z^*; z)| \right], \]  

(4.2)

\[ f_x = -\frac{i}{2} |D(z^*; z)|, \]  

(4.3)

\[ f_{xx} = -\frac{1}{2} \left[ |D(z^*; z_x)| + |D(z^*; z)| \right], \]  

(4.4)

\[ g_{j,t} = -|D(a_j^*; z_t)| + \frac{i}{2} |D(a_j^*, z^*; z, z)|, \]  

(4.5)

\[ g_{j,x} = -|D(a_j^*; z_x)|, \]  

(4.6)

\[ g_{j,xx} = -|D(a_j^*; z_{xx})| + \frac{i}{2} |D(a_j^*, z^*; z, z)|. \]  

(4.7)

Here, \( z_t, z_x \) and \( z_{xx} \) are \( N \)-component row vectors given by \( z_t = (ip_1^2 z_1, ip_2^2 z_2, \ldots, ip_N^2 z_N) \), \( z_x = (p_1 z_1, p_2 z_2, \ldots, p_N z_N) \) and \( z_{xx} = (p_1^2 z_1, p_2^2 z_2, \ldots, p_N^2 z_N) \), respectively.
Proof. We prove (4.2). Let \( D = (d_{jk})_{1 \leq j,k \leq 2N} \) be a \( 2N \times 2N \) matrix and \( D_{jk} \) be the cofactor of the element \( d_{jk} \). It follows by applying formula (3.4) to the determinant \( f \) given by (2.13) that

\[
f_t = \frac{i}{2} \sum_{j,k=1}^{N} D_{jk} (p_j^* - p_k^*) z_j z_k^*\]

\[
= \frac{i}{2} \sum_{j,k=1}^{N} D_{jk} (z_j z_k^* - z_j^* z_k),
\]

where in passing to the second line, use has been made of the relations \( p_j z_j = z_j, \quad p_k^* z_k^* = z_k, \). Referring to formula (3.5) with \( z = 0 \) and taking into account the notation (3.2), the above expression reduces to the right-hand side of (4.2). A key feature in the proof is that the factor \( (p_j + p_k^*)^{-1} \) in the element \( a_{jk} \) has been canceled after differentiation with respect to \( t \). Using formulas (3.4) and (3.5) as well as some basic properties of determinants, formulas (4.3)–(4.7) can be proved in the same way.

The complex conjugate expressions of \( f, f_x \) and \( g_j \) can be expressed as follows.

Lemma 4.2.

\[
f^* = |\bar{D}|, \quad \bar{D} = \begin{pmatrix}{A} & I \\ -I & B - iyC \end{pmatrix}, \quad (4.8)
\]

\[
f_x^* = -\frac{1}{2} [\bar{D}(z^*; z)], \quad (4.9)
\]

\[
g_j^* = |\bar{D}(z^*; a_j)|. \quad (4.10)
\]

Proof. We prove (4.8). It follows from (2.14a) and (2.14b) that \( A^* = A^T \) and \( B^* = B^T - iyC^T \), where \( C \) is an \( N \times N \) matrix with elements \( c_{jk} \) defined by (2.14b). These relations lead to the expression of \( f^* \):

\[
f^* = \begin{vmatrix} A^* & I \\ -I & B^* \end{vmatrix} = \begin{vmatrix} A^T & I \\ -I & (B - iyC)^T \end{vmatrix}.
\]

Since \( |A^T| = |A| \) for any square matrix \( A \), the above expression reduces to the right-hand side of (4.8) after multiplying the \( j \)th row and \( k \)th column \((j,k = 1, 2, \ldots, N)\) by a factor \(-1\). Differentiating (4.8) with respect to \( x \) and applying formula (3.5) with \( z = 0 \) to the resulting expression, formula (4.9) follows immediately. The proof of formula (4.10) can be performed in the same way.

The following formulas will be used in the proof of (2.5) and (2.6).

Lemma 4.3.

\[
|\bar{D}| = |D| + \frac{1}{2} |D(z^*; \tilde{z})|. \quad (4.11)
\]

\[
|D(b_k^*; \tilde{z})| = |\bar{D}(a_k^*; z)|. \quad (4.12)
\]

\[
|\bar{D}(a_k^*; b_k)| = -|D(b_k^*; a_k)| - \frac{1}{2} |D(b_k^*; z^*; a_k, \tilde{z})|. \quad (4.13)
\]
\[ |\tilde{D}(\mathbf{a}_z; \mathbf{z})| = |D(\mathbf{b}_z^*; \mathbf{z}) + \frac{1}{2} |D(\mathbf{b}_z^*; \mathbf{z}; \tilde{\mathbf{z}})|. \]  
\[ (4.14) \]

\[ |D(\mathbf{z}; \mathbf{z})| = 2i\gamma \sum_{k=1}^{n} |D(\mathbf{z}; \mathbf{a}_k)|. \]  
\[ (4.15) \]

\[ |\tilde{D}(\mathbf{a}_z; \mathbf{a}_k)| = |D(\mathbf{b}_z^*; \mathbf{a}_k)|. \]  
\[ (4.16) \]

where \( \tilde{\mathbf{z}} \) and \( \mathbf{b}_k \) are \( N \)-component row vectors given respectively by
\[
\tilde{\mathbf{z}} = (z_1/p_1, z_2/p_2, \ldots, z_N/p_N)
\]
and
\[
\mathbf{b}_k = (\alpha k_1 p_*^1, \alpha k_2 p_*^2, \ldots, \alpha k_N p_*^N).
\]

**Proof.** First, we prove (4.11). A direct calculation using the elements of \( B \) and \( C \) given by
\[
(2.14)
\]
gives
\[
b_{jk} - i\gamma c_{jk} = -\frac{1}{p_j^*} p_k a_{jk}.
\]
The determinant \( |\tilde{D}| \) from (4.8) is now modified to the form
\[
|\tilde{D}| = \begin{vmatrix}
A & I \\
-I & (-\frac{1}{p_j^*} p_k) a_{jk}
\end{vmatrix}
\]
\[
= \begin{vmatrix}
(-\frac{1}{p_j^*} p_k) a_{jk} & I \\
-I & B
\end{vmatrix},
\]
where the last line of the above expression follows immediately from the property of the determinant. Definition (2.14a) of \( a_{jk} \) now gives
\[
-\frac{1}{p_j^*} p_k a_{jk} = a_{jk} - \frac{z_j z_k^*}{2p_j}.
\]
In view of the property of the bordered determinant, \( |\tilde{D}| \) is modified in the form
\[
|\tilde{D}| = \begin{vmatrix}
A & I & \tilde{\mathbf{z}}^T \\
-I & B & 0^T \\
\frac{\tilde{\mathbf{z}}}{2} & 0 & 1
\end{vmatrix},
\]
which is seen to coincide with (4.11) by applying formula (3.5). The proof of (4.12)–(4.14) can be performed in the same way. Hence, we omit the proof.

Let us now proceed to the proof of (4.15). To this end, it is to be noted that the determinant \( f \) can be rewritten in the form
\[
f = \begin{vmatrix}
A & I \\
-I & B
\end{vmatrix},
\]
where \( \tilde{A} \) and \( \tilde{B} \) are \( N \times N \) matrices defined by
\[
\tilde{A} = (\tilde{a}_{jk})_{1 \leq j, k \leq N}, \quad \tilde{a}_{jk} = \frac{1}{2} p_j^* + p_k^* z_j^* z_k.
\]
\[
\tilde{B} = (\tilde{b}_{jk})_{1 \leq j, k \leq N}, \quad \tilde{b}_{jk} = i\gamma c_{jk} p_k^* + \frac{1}{2} p_j^* + p_k^* z_j^* z_k.
\]
Indeed, the above expression of \( f \) is derived from \( f \) from (4.1) by extracting the factors \( z_j \) and \( z_k^* \) from the \( j \)th row and \( k \)th column, respectively for \( j, k = 1, 2, \ldots, N \) and then multiplying the \( (N + j) \)th row and \( (N + k) \)th column by the factors \( z_j^* \) and \( z_k \), respectively, for \( j, k = 1, 2, \ldots, N \). Using formulas (3.4) and (3.5) gives an alternative expression of \( f \)
\[
f_x = -i\gamma \sum_{k=1}^{n} |D(\mathbf{b}_z^*; \mathbf{a}_k)|.
\]
Formula (4.15) follows by comparing this expression with (4.3). Formula (4.16) comes from the complex conjugate expression of (4.15). \( \square \)
4.2. Proof of (2.4)

The proof of (2.4) proceeds following the same procedure as that of the same equation for \( n = 2 \) [10]. Let \( P_1 \) be

\[
P_1 = (iD_i + D_i^2)g_{ij} \cdot f.
\]

(4.17)

Substituting (4.1)–(4.7) into (4.17), \( P_1 \) becomes

\[
P_1 = -|D(a_r^*; z; z, z_1)|D| + |D(a_r^*; z)|D(x^*; z_1)| - |D(a_r^*; z)|D(z^*; z)|
- |[D(a_r^*; z)] + |D(a_r^*; z_1)]|\).
\]

(4.18)

Referring to Jacobi’s identity (3.6) and the fundamental formula \( \alpha[D(a; b_1)] + \beta[D(a; b_2)] = |D(a; \alpha b_1 + \beta b_2)| (\alpha, \beta \in \mathbb{C}) \), \( P_1 \) simplifies to \( P_1 = -|D(a_r^*; iz + z_{xx})| \). Since \( iz + z_{xx} = 0 \) by (2.14a), the last column of the determinant consists only of zero elements, implying that \( P_1 = 0 \).

\[\square\]

4.3. Proof of (2.5)

The equation to be proved is \( P_2 = 0 \), where

\[
P_2 = D_s f \cdot f^* - \frac{i\gamma^2}{2} \sum_{k=1}^n |g_{kk}|^2.
\]

(4.19)

Substituting (4.1), (4.3) and (4.8)–(4.10) into (4.19), \( P_2 \) becomes

\[
P_2 = -\frac{1}{2} |\hat{D}|D(z^*; z)| + \frac{1}{2} |D|D(z^*; z)\| + \frac{i\gamma^2}{2} \sum_{k=1}^n |D(a_k^*; z)||\hat{D}(z^*; a_k)|.
\]

(4.20)

Further simplification is possible with the use of (4.11), (4.15) and (4.16) with (4.13), giving rise to

\[
P_2 = \frac{i\gamma^2}{2} \sum_{k=1}^n (-|\hat{D}(a_k^*; z)||D(z^*; a_k)| + |D(a_k^*; z)||\hat{D}(z^*; a_k)|).
\]

(4.21)

Applying Jacobi’s identity (3.6) to the middle term and replacing \( |D(b_k^*; \bar{z})| \) by the right-hand side of (4.12) in the resultant expression, \( P_2 \) reduces to

\[
P_2 = \frac{i\gamma^2}{2} \sum_{k=1}^n (-|\hat{D}(a_k^*; z)||D(z^*; a_k)| + |D(a_k^*; z)||\hat{D}(z^*; a_k)|).
\]

(4.22)

It now follows from (3.8) that

\[
|\hat{D}(a_k^*; z)| = |D(a_k^*; z)| + \sum_{m=1}^{n''} \frac{(i\gamma)^m}{m!} \sum_{k_1, \ldots, k_m=1}^n |D(a_k^*; a_{k_1}, \ldots, a_{k_m}; z, a_{k_1}, \ldots, a_{k_m})|,
\]

(4.23a)

\[
|\hat{D}(z^*; a_k)| = |D(z^*; a_k)| + \sum_{m=1}^{n''} \frac{(i\gamma)^m}{m!} \sum_{k_1, \ldots, k_m=1}^n |D(z^*; a_{k_1}, \ldots, a_{k_m}; a_k, a_{k_1}, \ldots, a_{k_m})|,
\]

(4.23b)

where \( n'' = \min(n - 1, N - 1) \). Referring to the expansion formulas (3.9) and (3.10), one has

\[
|D(a_k^*; a_{k_1}, \ldots, a_{k_m}; z, a_{k_1}, \ldots, a_{k_m})| = |D|^{-1} |D(a_k^*; z)||D(a_{k_1}^*; \ldots, a_{k_m}^*; a_{k_1}, \ldots, a_{k_m})|
+ |D|^{-1} \sum_{l=1}^m (-1)^l |D(a_{k_l}^*; z)||D(a_{k_1}^*; a_{k_{l+1}}, \ldots, a_{k_m}^*; a_{k_1}, \ldots, a_{k_{l-1}}, a_{k_l}, \ldots, a_{k_m})|.
\]

(4.24a)
$|D(z^*; a^*_1, \ldots, a^*_m; a_k, \ldots, a_{k_m})| = |D|^{-1}|D(z^*; a_k)| |D(a^*_1, \ldots, a^*_m; a_k, \ldots, a_{k_m})|$

$$+ |D|^{-1} \sum_{\ell=1}^{m} (-1)^\ell |D(z^*; a_k)| |D(a^*_1, \ldots, a^*_\ell; a_k, \ldots, a_{k_m})|. \quad (4.24b)$$

By introducing (4.23) into (4.22) and then using (4.24), $P_2$ takes the form

$$P_2 = \frac{i\gamma}{2|D|} \sum_{n=1}^{n'} \frac{(iy)^m}{m!} \prod_{l=1}^{m} (-1)^l$$

$$\times \sum_{k,k_1,\ldots,k_n=1} \left[ |D(a_k^*; z)| |D(z^*; a_k)| |D(a^*_1, a^*_2, \ldots, a^*_m; a_k, \ldots, a_{k_m})| \right]$$

$$+ |D(a^*_1; z)| |D(z^*; a_k)| |D(a^*_1, a^*_2, \ldots, a^*_m; a_k, \ldots, a_{k_m})|. \quad (4.25)$$

Interchange the indices $k$ and $k_j$ in the first term and then shift the row vector $a^*_k$ in front of $a_{k+1}$ and the column vector $a_k$ in front of $a_{k}$, respectively. This leads to the following relation:

$$|D(a^*_1, a^*_2, \ldots, a^*_m; a_k, \ldots, a_{k_m})|$$

$$\rightarrow |D(a^*_{k_+1}, a^*_1, \ldots, a^*_m; a_k, \ldots, a_{k_m})|$$

$$= |D(a^*_1, a^*_2, \ldots, a^*_m; a_k, a_k, \ldots, a_{k_m})|. \quad (4.24a)$$

Note that the value of the determinant is not altered since the total signature caused by the above manipulation is $(-1)^{2(n'-1)} = 1$. Thus, the first term on the right-hand side of (4.25) coincides with the second term and consequently, $P_2 = 0$.

4.4. Proof of (2.6)

Instead of proving (2.6) directly, we differentiate (2.5) by $x$ and add the resultant expression to (2.6) and then prove the equation $P_3 = 0$, where

$$P_3 = f_x f^*_x - f_x f^*_{x^*} - \frac{iy}{2} \sum_{k=1}^{n} g_{kx}^* g_{k}^*.$$

(4.26)

This reduces the total amount of calculations considerably and the proof becomes transparent. It now follows from (4.1), (4.3), (4.4), (4.6) and (4.8)–(4.10) that

$$P_3 = -\frac{1}{2} \left[ |D(z^*; z_x)| + |D(z^*; z)| \right] |D| - \frac{1}{4} |D(z^*; z)| |\tilde{D}(z^*; z)|$$

$$+ \frac{1}{2} \sum_{k=1}^{n} |D(a^*_k; z_x)| |\tilde{D}(z^*; a_k)|. \quad (4.27)$$

Differentiation of (4.15) with respect to $x$ gives

$$|D(z^*; z_x)| + |D(z^*; z)| = -i\gamma x \sum_{k=1}^{n} |D(b^*_k; z_x; a_k, z)|. \quad (4.28)$$

Inserting (4.15) and (4.28) into (4.27), $P_3$ can be put into the form

$$P_3 = \frac{iy}{2} \sum_{k=1}^{n} \left[ |\tilde{D}| |D(b^*_k; z^*; a_k, z)| + |D(z^*; z)| |\tilde{D}(a^*_k; b^*_k)| + |D(a^*_k; z_x)| |\tilde{D}(z^*; a_k)| \right]. \quad (4.29)$$
Note from (4.11), (4.13), (4.14) and Jacobi’s identity (3.6) that
\[
|\tilde{D}| |D(b_i^*, z; \mathbf{a}_k, z)| + |D(z^*; z)||\tilde{D}(\mathbf{a}_k^*, \mathbf{b}_j)| = - |D(z^*; \mathbf{a}_k)| |\tilde{D}(\mathbf{a}_k^*; z)|
\]
\[
+ \frac{1}{2} |D(b_i^*, z^*; z, \tilde{z})| = - |D(z^*; \mathbf{a}_k)| |\tilde{D}(\mathbf{a}_k^*; z)|.
\]
After substituting (4.30) into (4.29), \(P_3\) becomes
\[
P_3 = \frac{iy}{2} \sum_{k=1}^{n} [-|\tilde{D}(\mathbf{a}_k^*; z)| |D(z^*; \mathbf{a}_k)| + |D(\mathbf{a}_k^*; z)| |\tilde{D}(\mathbf{a}_k^*; z)|].
\]
This expression reduces to (4.22) if one replaces \(z_i\) by \(z\). Hence, the proof of the relation \(P_3 = 0\) completely parallels that of \(P_2 = 0\) with \(P_2\) from (4.22).

5. An alternative expression of the bright \(N\)-soliton solution

Here, we present an alternative expression of the bright \(N\)-soliton solution in terms of the determinants with smaller sizes when compared with those given by (2.13). Explicitly, we write it as a theorem.

**Theorem 5.1.** The determinants \(f'\) and \(g_j'\) \((j = 1, 2, \ldots, n)\) given below satisfy the system of bilinear equations (2.4)–(2.6):
\[
f' = |A' + B'|, \quad g_j' = \begin{vmatrix} A' + B' & \mathbf{y}^T \\ -\mathbf{a}_j' & 0 \end{vmatrix}, \quad j = 1, 2, \ldots, n,
\]
where \(A'\) and \(B'\) are \(N \times N\) matrices and \(\mathbf{y}\) and \(\mathbf{a}_j'\) are \(N\)-component row vectors defined below:
\[
A' = (a'_{jk})_{1 \leq j, k \leq N}, \quad a'_{jk} = \frac{1}{2} y_j y_k^* q_j + q_k^* c_j^*, \quad y_j = \exp(q_j x + i q_j^2 t), \quad j = 1, 2, \ldots, n
\]
\[
B' = (b'_{jk})_{1 \leq j, k \leq N}, \quad b'_{jk} = \frac{(\mu - iy q_j^2)}{q_j + q_k^*} c_j^*, \quad c_j^* = \sum_{s=1}^{n} \alpha'_j \alpha'_s^*, \quad j = 1, 2, \ldots, n\]
\[
\mathbf{y} = (y_1, y_2, \ldots, y_n), \quad \mathbf{a}_j' = (\alpha'_{j1}, \alpha'_{j2}, \ldots, \alpha'_{jn}).
\]
Here, \(q_j\) \((j = 1, 2, \ldots, N)\) and \(\alpha'_{s j}\) \((s = 1, 2, \ldots, n; j = 1, 2, \ldots, N)\) are complex parameters characterizing the solution.

**Proof.** The proof of the solution can be performed in the same way as that of (2.13) with (2.14). Indeed, the proof of (2.4), (2.5) and (2.6) reduces respectively to relations (4.18), (4.22) and (4.31) in which the matrix \(D\) may be replaced simply by the matrix \(A' + B'\).

Let us show that the determinants \(f\) and \(g_j\) from (2.13) are closely related to the determinants \(f'\) and \(g_j'\) given by (5.1). The following lemma is useful for this purpose.

**Lemma 5.1.** The determinants \(f\) and \(g_j\) given by (2.13) can be rewritten in the form
\[
f = |I + AB|, \quad g_j = \begin{vmatrix} I + AB & \mathbf{z}^T \\ -\mathbf{a}_j^* & 0 \end{vmatrix}, \quad j = 1, 2, \ldots, n.
\]
Proof. Multiplying \( f \) from (2.13) by a factor \( \begin{vmatrix} b & -l \\ l & o \end{vmatrix} \) and performing the operation of matrix multiplication, the first expression of (5.3) follows immediately. Similarly, the second expression is obtained if one multiplies \( g_j \) by a factor \( \begin{vmatrix} 0 & w \\ 0 & w \end{vmatrix} \). Indeed,

\[
\begin{vmatrix} A & I \\ -I & B \end{vmatrix} \begin{vmatrix} B & -l \\ l & O \end{vmatrix} = \begin{vmatrix} I + AB & -A \\ O & I \end{vmatrix} = |I + AB|,
\]

\[
\begin{vmatrix} A & I \\ -I & B \end{vmatrix} \begin{vmatrix} B & -I \\ I & O \end{vmatrix} = \begin{vmatrix} I + AB & -A \\ 0 & I \end{vmatrix} = \begin{vmatrix} I + AB & z_j^T \\ 0 & 0 \end{vmatrix} = -a_j^* 0.
\]

Since the value of each factor multiplied from the right is 1, (5.3) follows. \( \square \)

We now establish the following theorem.

Theorem 5.2. Under the parameterization \( q_j = -p_j^* (j = 1, 2, \ldots, N) \) and \( \alpha_{ij}' = -\alpha_{ij}/(2c_i^j) \) \( (s = 1, 2, \ldots, n; j = 1, 2, \ldots, N) \), the determinants \( f, f', g_j \) and \( g_j' \) satisfy the relations

\[
f = c|A|f',
\]

\[
g_j = c|A|g_j', \quad j = 1, 2, \ldots, n,
\]

where

\[
c = (-1)^N \prod_{l=1}^N (4c_l^j c_l), \quad c_l = \prod_{m=1}^N \frac{p_l + p_m^*}{(p_l - p_m)}, \quad l = 1, 2, \ldots, N.
\]

The parameters \( p_j (j = 1, 2, \ldots, N) \) are assumed to satisfy the conditions \( p_l + p_m^* \neq 0 \) for all \( l \) and \( m \) and \( p_l \neq p_m \) for \( l \neq m \).

Proof. Let \( \tilde{A} \) be a Cauchy matrix of the form \( \tilde{A} = (\frac{1}{2} \frac{1}{p_l + p_m}) \). Then, \( A = (\alpha_{ij}) \tilde{A}(\alpha_{ij}') \).

Since \( |\tilde{A}| \neq 0 \) by virtue of the well-known formula for \( |\tilde{A}| \) and the conditions imposed on the parameters \( p_j (j = 1, 2, \ldots, N) \), the inverse of \( \tilde{A} \) exists, implying that \( A^{-1} \) exists as well.

Actually, it reads \( A^{-1} = (\alpha_{ij}')^{-1} \tilde{A}^{-1}(\alpha_{ij})^{-1} \). Using the explicit expression of \( \tilde{A}^{-1} \), i.e. \( \frac{2c_l^j c_l}{p_j^* + p_k^* c_l^j c_l} \) [11], the inverse matrix \( A^{-1} \) can be written in the form

\[
A^{-1} = \begin{pmatrix} 2c_l^j c_k^j & 1 \\ p_k^* + p_l^* c_l^j c_l & 0 \end{pmatrix}.
\]

Applying the basic properties of determinants to \( f \) and \( g_j \) from (5.3) gives

\[
f = |A||A^{-1} + B|,
\]

\[
g_j = |A| \begin{vmatrix} A^{-1} + B & A^{-1}z_j^T \\ -a_j^* & 0 \end{vmatrix}, \quad j = 1, 2, \ldots, n.
\]
The $j$th element of the column vector $A^{-1}z^T$ is

$$
(A^{-1}z^T)_j = \sum_{l=1}^{N} (A^{-1})_{jl}z_l
$$

$$
= \frac{2\epsilon^+}{z_j^*} \sum_{l=1}^{N} \frac{1}{p_j^* + p_l} \prod_{m=1}^{N} (p_l - p_m)
$$

$$
= \frac{2\epsilon^+}{z_j^*} \sum_{l=1}^{N} \prod_{m=1}^{N} (p_l - p_m).
$$

(5.10)

By Euler’s formula, the sum in the last line turns out to be 1 and hence $(A^{-1}z^T)_j = 2\epsilon^+ / z_j^*$. Introducing this relation to $g_j$,

$$
g_j = |A| \begin{vmatrix} A^{-1} + B & \mathbf{z}^T / A_j \ \
-\mathbf{a}_j^* & 0 \end{vmatrix}, \quad j = 1, 2, \ldots, n.
$$

(5.11)

where $\mathbf{z} = (2\epsilon_i^+ / z_1^*, 2\epsilon_i^+ / z_2^*, \ldots, 2\epsilon_i^+ / z_N^*)$ is an $N$-component row vector.

The next step is to modify the determinants $f'$ and $g_j'$. By means of the parameterization $q_j = -p_j^*$, $y_j$ from (5.2a) is related to $z_j$ from (2.1a) by the relation $y_j = z_j^{*\gamma - 1}$. Similarly, the relation $c_{jk}' = c_{jk}/(4\epsilon_i c_{ki})$ follows from (2.1b) and (5.2b) and the parameterization $\alpha_{ij}' = -\alpha_{ij}/(2\epsilon_i)$. Substitution of these relations into $f'$ gives

$$
f' = \begin{vmatrix} 1 & (\mu - iyq_j) c_{jk}' \ 
2q_j + q_k^* & (\mu + iyq_j) c_{jk}' \end{vmatrix}
$$

$$
= \frac{(-1)^N}{\prod_{l=1}^{N} (4\epsilon_i c_{kl})} \begin{vmatrix} 2\epsilon^+ c_{jk} & 1 \ 
p_j^* + p_k^* z_j^* z_k & p_j^* + p_k^* \end{vmatrix}
$$

$$
= c^{-1}|A^{-1} + B|.
$$

(5.12)

where in passing to the second line, the factor $1/(2\epsilon^+_i)$ has been extracted from the $j$th row ($j = 1, 2, \ldots, N$) and the factor $-1/(2\epsilon_i)$ from the $k$th column ($k = 1, 2, \ldots, N$), respectively. The similar procedure applied to $g_j'$ leads to the expression

$$
g_j' = c^{-1}|A^{-1} + B| \begin{vmatrix} A^{-1} + B & \mathbf{z}^T / A_j \ 
-\mathbf{a}_j^* & 0 \end{vmatrix}, \quad j = 1, 2, \ldots, n.
$$

(5.13)

Relation (5.4) follows from (5.8) and (5.12), whereas relation (5.5) follows from (5.11) and (5.13).

Thus, we have obtained two different expressions for the bright $N$-soliton solution of the system of nonlinear PDEs (2.2). Explicitly, they read $u_j = g_j / f = g_j' / f'$ ($j = 1, 2, \ldots, n$).

The following proposition provides an alternative proof of theorem 5.1.

**Proposition 5.1.** If $f$ and $g_j$ given respectively by (5.4) and (5.5) satisfy the system of bilinear equations (2.4)–(2.6), then $f'$ and $g_j'$ satisfy the same system of equations, and vice versa.

**Proof.** Substituting (5.4) and (5.5) into (2.4) and using the definition of the bilinear operators,

$$
c^2|A|^2 (D_t g_j' \cdot f' + D_x^2 g_j' \cdot f') + c^2 (D_t^2 |A| \cdot |A|) g_j' f' = 0.
$$

(5.14)
The Cauchy type determinant $|A|$ can be modified, after extracting the factor $z_j$ from the $j$th row $(j = 1, 2, \ldots, N)$ and the factor $z_k^*$ from the $k$th column $(k = 1, 2, \ldots, N)$, respectively, in the form
\[ |A| = \prod_{j=1}^{N} (z_j z_j^*) \left| \left( \frac{1}{2} p_j + p_j^* \right) \right| \]
\[ = \exp \left[ \sum_{j=1}^{N} (p_j + p_j^*) x + i \sum_{j=1}^{N} (p_j^2 - p_j^*) \right] \left| \left( \frac{1}{2} p_j + p_j^* \right) \right|. \tag{5.15} \]

Differentiation of $|A|$ with respect to $x$ gives
\[ |A|_x = \sum_{j=1}^{N} (p_j + p_j^*) |A|, \quad |A|_{xx} = \left( \sum_{j=1}^{N} (p_j + p_j^*) \right)^2 |A|. \tag{5.16} \]

It immediately follows from (5.16) that
\[ D_x^2 |A| \cdot |A| = 2(|A||A|_{xx} - |A|_x^2) = 0. \tag{5.17} \]

It is seen from (5.14) and (5.17) and the relation $c|A| \neq 0$ that $f'$ and $g'$ satisfy the bilinear equation (2.4). The remaining part of the proposition can be proved in the same way if one uses (5.17) and the reality of $|A|$, i.e. $|A|^* = |A|$ which is a consequence of the Hermitian nature of the matrix $A$. The proof of the converse proposition proceeds in the same way if one uses the relation $D_x^2 |A|^{-1} \cdot |A|^{-1} = 0$ in place of (5.17).

\[ \square \]

6. A continuum model

The $n$-component system (1.1) yields a continuum model when one takes a limit $n \to \infty$. It represents a $(2+1)$-dimensional nonlinear modified NLS equation of the form
\[ i q_t + q_{xx} + \mu \left( \int_{-\infty}^{\infty} |q|^2 \, dy \right) q + i \gamma \left( \int_{-\infty}^{\infty} |q|^2 \, dy \right)_x = 0, \quad q = q(x, y, t). \tag{6.1} \]

Recall that when $\gamma = 0$, this equation reduces to a $(2+1)$-dimensional nonlinear NLS equation proposed by Zakharov [12]. The exact method of solution for equation (6.1) can be developed following the same procedure as that for the system of nonlinear PDEs (1.1). Hence, we summarize the main results.

First, the application of the gauge transformation
\[ q = u \exp \left[ -\frac{i \gamma}{2} \int_{-\infty}^{x} \int_{-\infty}^{\infty} |u(x, y, t)|^2 \, dx \, dy \right], \quad u = u(x, y, t), \tag{6.2} \]

to the system (6.1) subjected to the boundary conditions $q \to 0$, $u \to 0$ $|x| \to \infty$ transforms (6.1) to a nonlinear nonlinear PDE for $u$:
\[ i u_t + u_{xx} + \mu \left( \int_{-\infty}^{\infty} |u|^2 \, dy \right) u + i \gamma \left( \int_{-\infty}^{\infty} u^* u_x \, dy \right) u = 0. \tag{6.3} \]

The proposition below is an analog of proposition 2.1.

**Proposition 6.1.** By means of the dependent variable transformation
\[ u = \frac{g}{f}, \tag{6.4} \]
equation (6.3) can be decoupled into the following system of bilinear equations for $f = f(x, t)$ and $g = g(x, y, t)$:
\[ (i D_t + D_x^2) g \cdot f = 0. \tag{6.5} \]
\[
D_x f \cdot f^* = i\gamma \int_{-\infty}^{\infty} \left| g \right|^2 \, dy, \quad (6.6)
\]
\[
D_x^2 f \cdot f^* = \mu \int_{-\infty}^{\infty} \left| g \right|^2 \, dy + \frac{i\gamma}{2} \int_{-\infty}^{\infty} D_x g \cdot g^* \, dy. \quad (6.7)
\]

**Proof.** The proof proceeds exactly as that of proposition 2.1. Formally, one may simply replace the sum \( \sum_{k=1}^{n} \) by the integral \( \int_{-\infty}^{\infty} \).

It follows from (6.2), (6.4) and (6.6) that

\[
q = \frac{g f^*}{f^2}, \quad (6.8)
\]

which is just a continuum limit of (2.13).

The following theorem can be derived from a continuum limit of the bright \( N \)-soliton solution given by theorems 2.1 and 5.1.

**Theorem 6.1.** The system of bilinear equations (6.5)–(6.7) admits the following two different expressions \( f, g \) and \( f', g' \) for the bright \( N \)-soliton solution:

\[
\begin{align*}
 f &= \begin{vmatrix} A & I \\ -I & B \end{vmatrix}, \\
 g &= \begin{vmatrix} A & I \\ -I & B \end{vmatrix} \begin{bmatrix} x^T \\ 0 \end{bmatrix}, \\
 f' &= \begin{vmatrix} A' + B' \end{vmatrix}, \\
 g' &= \begin{vmatrix} A' + B' \end{vmatrix} \begin{bmatrix} y^T \\ -a'^* \end{bmatrix}.
\end{align*} 
\quad (6.9)
\]

Here, \( A \) and \( B \) are \( N \times N \) matrices given respectively by (2.14a) and (2.14b), with \( c_{jk} \) being replaced by \( \int_{-\infty}^{\infty} \alpha_j(y)\alpha_k^*(y) \, dy \), \( A' \) and \( B' \) are \( N \times N \) matrices given respectively by (5.2a) and (5.2b), with \( c'_{jk} \) being replaced by \( \int_{-\infty}^{\infty} \alpha'_j(y)\alpha'^*_k(y) \, dy \) and \( \mathbf{a} = \mathbf{a}(y) = (\alpha_1, \alpha_2, \ldots, \alpha_N) \) and \( \mathbf{a}' = \mathbf{a}'(y) = (\alpha'_1, \alpha'_2, \ldots, \alpha'_N) \) are \( N \)-component row vectors where \( \alpha_j \) and \( \alpha'_j \) \( (j = 1, 2, \ldots, N) \) are continuous functions of \( y \).

**Proof.** The proof can be performed in the same way as that of theorems 2.1 and 5.1.

**Theorem 6.2.** Under the parameterization \( q_j = -p_j^* \) and \( \alpha'_j = -\alpha_j / (2c_j^*) \) \( (j = 1, 2, \ldots, N) \), the determinants \( f, f', g \) and \( g' \) satisfy the relations

\[
\begin{align*}
 f &= c|A|f', \\
 g &= c|A|g',
\end{align*} 
\quad (6.11)
\]

where \( c \) is defined by (5.6) and the parameters \( p_j \) \( (j = 1, 2, \ldots, N) \) are specified such that \( p_l + p_m^* \neq 0 \) for all \( l \) and \( m \) and \( p_l \neq p_m \) for \( l \neq m \).

**Proof.** The proof parallels theorem 5.2.

**Proposition 6.2.** If \( f \) and \( g \) given by (6.9) satisfy the system of bilinear equations (6.5)–(6.7), then \( f' \) and \( g' \) given by (6.11) and (6.12) satisfy the same system of equations, and vice versa.

**Proof.** The proof is completely parallel to that of proposition 5.1.
7. Concluding remarks

In this paper, we have obtained two different expressions for the bright $N$-soliton solution of an $n$-component modified NLS equation and found a simple relationship between them. We have also presented the bright $N$-soliton solution of a continuum model arising from the system as the number $n$ of the dependent variables tends to infinity. These solutions include, as special cases, existing solutions for a multi-component NLS equation. Actually, when $\gamma = 0$, the system of nonlinear PDEs (1.1) reduces to an $n$-component NLS equation. It admits the bright $N$-soliton solution of the form $q_j = g_j/f = g_j'/f'$ $(j = 1, 2, \ldots, n)$. This fact follows from (2.12) and the relations $f'' = f$ and $f''' = f'$. See (2.13) and (5.1). Solution (2.13) with $\gamma = 0$ has been obtained in [13] using a direct method, whereas solution (5.3) with $\gamma = 0$ has been constructed by means of the IST [14]. An alternative expression (5.1) of the solution with $\gamma = 0$ has been derived by employing a method of algebraic geometry [15]. For a continuum model (6.1) with $\gamma = 0$, the solution takes the form $q = g/f$. The bright $N$-soliton solution (6.9) with $\gamma = 0$ has been found in [16] by a direct method.

In a future work, we will investigate the various features of the multi-component bright solitons. In particular, we will be concerned with the effect of the parameters $\mu$ and $\gamma$ which characterize the different types of nonlinearities on the interaction process of solitons.

Acknowledgments

This work was partially supported by the grant-in-aid for Scientific Research (C) no 22540228 from Japan Society for the Promotion of Science.

References

[1] Hasegawa A and Kodama Y 1995 Solitons in Optical Communications (New York: Oxford)
[2] Kivshar Y S and Agrawal G P 2003 Optical Solitons from Fibers to Photonic Crystals (New York: Academic)
[3] Maimistov A I 2010 Solitons in nonlinear optics Quantum Electron. 40 756–81
[4] Manakov S V 1974 On the theory of two-dimensional stationary self-focusing of electromagnetic waves Sov. Phys.—JETP 38 248–53
[5] Hisakado M and Wadati M 1995 Integrable multi-component hybrid nonlinear Schrödinger equations J. Phys. Soc. Japan 64 408–13
[6] Makhan’kov V G and Pashaev O K 1982 Nonlinear Schrödinger equation with noncompact isogroup Theor. Math. Phys. 53 979–87
[7] Fordy A P 1984 Derivative nonlinear Schrödinger equations and Hermitian symmetric spaces J. Phys. A: Math. Gen. 17 1235–45
[8] Morris H C and Dodd R K 1979 The two component derivative nonlinear Schrödinger equation Phys. Scr 20 505–8
[9] Hisakado M, Iizuka T and Wadati M 1994 Coupled hybrid nonlinear Schrödinger equation and optical solitons J. Phys. Soc. Japan 63 2887–94
[10] Matsuno Y 2011 The $N$-soliton solution of a two-component modified nonlinear Schrödinger equation Phys. Lett. A 375 2090–4
[11] Vein R and Dale P 1999 Determinants and Their Applications in Mathematical Physics (New York: Springer)
[12] Zakharov V E 1980 The inverse scattering method Solitons (Topics in Current Physics) ed R K Bullough and P J Caudrey (Berlin: Springer) pp 243–85
[13] Ablowitz M J, Ohta Y and Trubatch A D 1999 On discretizations of a vector nonlinear Schrödinger equation Phys. Lett. A 253 287–304
[14] Tsuchida T 2004 $N$-soliton collision in the Manakov model Prog. Theor. Phys. 111 151–82
[15] Dubrovin B A, Malanyuk T M, Krichever I M and Makhan’kov V G 1988 Exact solutions of the time-dependent Schrödinger equation with self-consistent potentials Sov. J. Part. Nucl. 19 252–69
[16] Maruno K and Ohta Y 2008 Localized solutions of a (2+1)-dimensional nonlocal nonlinear Schrödinger equation Phys. Lett. A 372 4446–50