Finite time blow-up for a one-dimensional quasilinear parabolic-parabolic chemotaxis system

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Abstract

Finite time blow-up is shown to occur for solutions to a one-dimensional quasilinear parabolic-parabolic chemotaxis system as soon as the mean value of the initial condition exceeds some threshold value. The proof combines a novel identity of virial type with the boundedness from below of the Liapunov functional associated to the system, the latter being peculiar to the one-dimensional setting.

1 Introduction

We study the possible occurrence of blow-up in finite time for solutions to a one-dimensional parabolic system modeling chemotaxis [15]. More precisely, we consider the Keller-Segel chemotaxis model with nonlinear diffusion which describes the space and time evolution of a population of cells moving under the combined effects of diffusion (random motion) and a directed motion in the direction of high gradients of a chemical substance (chemoattractant) secreted by themselves. If \( u \geq 0 \) and \( v \) denote the density of cells and the (rescaled) concentration of chemoattractant, respectively, the Keller-Segel model with nonlinear diffusion reads

\[
\begin{align*}
\partial_t u &= \text{div} \left( a(u) \nabla u - u \nabla v \right) \quad \text{in} \quad (0, \infty) \times \Omega, \\
\varepsilon \partial_t v &= D \Delta v - \gamma v + u - M \quad \text{in} \quad (0, \infty) \times \Omega, \\
a(u) \partial_{\nu} u = \partial_{\nu} v &= 0 \quad \text{on} \quad (0, \infty) \times \partial \Omega, \\
(u, v)(0) &= (u_0, v_0) \quad \text{in} \quad \Omega.
\end{align*}
\]

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In general, $\Omega$ is an open bounded subset of $\mathbb{R}^N$, $N \geq 1$, with smooth boundary $\partial \Omega$, $a$ is a smooth non-negative function, and the parameters $\varepsilon$, $D$, $\gamma$, and $M$ are non-negative real numbers with $D > 0$ and $M > 0$. In addition, the initial data $u_0$ and $v_0$ satisfy

$$u_0 \geq 0, \quad \int_{\Omega} u_0(x) \, dx = |\Omega| \, M, \quad \text{and} \quad \int_{\Omega} v_0(x) \, dx = 0.$$  

The constraints (5) ensure in particular that a solution $(u, v)$ to (1)-(4) satisfies (at least formally) the same properties for positive times, that is,

$$u(t) \geq 0, \quad \int_{\Omega} u(t, x) \, dx = |\Omega| \, M, \quad \text{and} \quad \int_{\Omega} v(t, x) \, dx = 0.$$  

The main feature of (1) is that it involves a competition between the diffusive term $\text{div} \left( a(u) \nabla u \right)$ (spreading the population of cells) and the chemotactic drift term $-\text{div} \left( u \nabla v \right)$ (concentrating the population of cells) that may lead to the blow-up in finite time of the solution to (1)-(4). The possible occurrence of such a singular phenomenon is actually an important mathematical issue in the study of (1)-(4) which is also relevant from a biological point of view: indeed, it corresponds to the experimentally observed concentration of cells in a narrow region of the space which is a preamble to a change of state of the cells. From a mathematical point of view, the blow-up issue has been the subject of several studies in the last twenty years, see the survey [12] and the references therein.

Still, it is far from being fully understood, in particular when $\varepsilon > 0$ (the so-called parabolic-parabolic Keller-Segel model). In that case, the only finite time blow-up result available seems to be that of Herrero & Velázquez who showed in [9, 10] that, when $\Omega$ is a ball in $\mathbb{R}^2$, $D = 1$, and $a \equiv 1$, there are $M > 8\pi$ and radially symmetric solutions $(u, v)$ to (1)-(4) which blow up in finite time. These solutions are constructed as small perturbations of time rescaled stationary solutions to (1)-(4) and a similar result is also true when $\varepsilon = 0$ [8]. The result in [10] actually goes far beyond the mere occurrence of blow-up in finite time as the shape of the blow-up profile is also identified. Recall that the condition $M > 8\pi$ is necessary for the finite blow-up to take place: indeed, it is shown in [10] that, if $\Omega$ is a ball in $\mathbb{R}^2$, $D = 1$, and $a \equiv 1$, radially symmetric solutions to (1)-(4) are global as soon as $M < 8\pi$. We refer to [7, 19] for additional global existence results when $\Omega$ is a bounded domain in $\mathbb{R}^2$, $\varepsilon > 0$, and $a \equiv 1$. In [11] the existence of unbounded solutions is shown for $\varepsilon > 0$ and $a \equiv 1$, but it is not known whether the blow-up takes place in finite or infinite time. The same approach is employed in [13] to obtain unbounded solutions to quasilinear Keller-Segel systems, still without knowing whether the blow-up time is finite or infinite. The finite time blow-up result proved in this paper (Theorem 1) is thus the first one of this kind for quasilinear parabolic-parabolic Keller-Segel systems.

In contrast, for the parabolic-elliptic Keller-Segel system corresponding to $\varepsilon = 0$, several finite time blow-up results are available. There is thus a discrepancy between the two cases $\varepsilon > 0$ and $\varepsilon = 0$ which may be explained as follows. On the one hand, as observed in [14]
when $\varepsilon = 0$, $\Omega$ is a ball of $\mathbb{R}^2$, $a \equiv 1$, and $u_0$ is radially symmetric, it is possible to reduce (1)-(4) to a single parabolic equation for the cumulative distribution function

$$U(t, r) := \int_{B(0, r)} u(t, x) \, dx.$$  

Finite time blow-up is then shown with the comparison principle by constructing appropriate subsolutions. This approach was extended to nonlinear diffusions (non-constant $a$) and arbitrary space dimension $N \geq 1$ in [6]. On the other hand, it has been noticed in [2, 16] that, still for $a \equiv 1$, the moment $M_k$ of $u$ defined by

$$M_k(t) := \int_{\Omega} |x|^k u(t, x) \, dx, \quad k \in (0, \infty),$$

satisfies a differential inequality which cannot hold true for all times for a suitably chosen value of $k > 0$, for it would imply that $u$ reaches negative values in finite time in contradiction with (6). In contrast to the previous approach, this is an obstructive method which provides no information on the blow-up profile and is somehow reminiscent of the celebrated virial identity available for the nonlinear Schrödinger equation (see, e.g., [4, Section 6.5] and the references therein). Nevertheless, it applies to more general sets $\Omega$ [17, 18, 20]. We recently develop further this technique to establish finite time blow-up of radially symmetric solutions to (1)-(4) with $\varepsilon = 0$ in a ball of $\mathbb{R}^N$, $N \geq 2$, when the diffusion is nonlinear, the main idea being to replace the moments by nonlinear functions of the cumulative distribution function $U$. For a related model in $\mathbb{R}^N$ with nonlinear diffusion $a(u) = m u^{m-1}$, $m > 1$, finite time blow-up results were recently established in [3, 21] by looking at the evolution of the second moment $M_2$.

Coming back to the parabolic-parabolic Keller-Segel system (1)-(4) ($\varepsilon > 0$), it seems unlikely that the first approach described above (reduction to a single equation) could work and the purpose of this paper is to show that finite time blow-up results can be established by the second approach in the one-dimensional case ($N = 1$). More precisely, we consider the initial-boundary value problem

(7) \hspace{1cm} \partial_t u = \partial_x (a(u) \partial_x u - u \partial_x v) \quad \text{in} \quad (0, \infty) \times (0, 1),

(8) \hspace{1cm} \varepsilon \partial_t v = D \partial_x^2 v - \gamma v + u - M \quad \text{in} \quad (0, \infty) \times (0, 1),

(9) \hspace{1cm} a(u) \partial_x u = \partial_x v = 0 \quad \text{on} \quad (0, \infty) \times \{0, 1\},

(10) \hspace{1cm} (u, v)(0) = (u_0, v_0) \quad \text{in} \quad (0, 1),

and assume that

(11) \hspace{1cm} \varepsilon > 0, \quad D > 0, \quad \gamma \geq 0, \quad M > 0,

and the initial data $(u_0, v_0) \in W^{1,2}(0, 1; \mathbb{R}^2)$ satisfy

(12) \hspace{1cm} u_0 \geq 0, \quad \int_0^1 u_0(x) \, dx = M, \quad \text{and} \quad \int_0^1 v_0(x) \, dx = 0.
We further assume that $a \in \mathcal{C}^{2}(\mathbb{R})$ and that there are $p \in (1, 2]$, and $c_1 > 0$ such that

\begin{equation}
0 < a(r) \leq c_1 (1 + r)^{-p} \quad \text{for} \quad r \geq 0.
\end{equation}

Our main result then reads as follows.

**Theorem 1** Assume that the parameters $\varepsilon, D, \gamma, M$, the initial data $(u_0, v_0)$, and the function $a$ fulfill the conditions (11), (12), and (13), respectively. Then there is a unique classical maximal solution

\[(u, v) \in \mathcal{C}([0, T_m) \times [0, 1]; \mathbb{R}^2) \cap \mathcal{C}^{1,2}((0, T_m) \times [0, 1]; \mathbb{R}^2)\]

to (7)-(10) with maximal existence time $T_m \in (0, \infty]$. It also satisfies

\begin{equation}
u(t, x) \geq 0, \quad \int_0^1 u(t, x) \, dx = M, \quad \text{and} \quad \int_0^1 v(t, x) \, dx = 0
\end{equation}

for $(t, x) \in [0, T_m) \times [0, 1]$. Introducing

\begin{equation}
F(z_1, z_2) := c_1 (1 + M) + \frac{M^2}{2D} + z_1 + M z_2 + \frac{D + \gamma}{2} z_2^2, \\
\mathcal{P}_q(z_1, z_2, z_3) := \left(1 + \frac{\gamma}{D} + \frac{\gamma}{M} z_2 + \frac{M^{q-2}}{4qD} z_3\right) F(z_1, z_2) + \frac{c_1(q-1)q^{(q-2)/q}D}{(p-1)M^{p-1}} F(z_1, z_2)^{(q-2)/q} - \frac{M^q}{q(q + 1)}
\end{equation}

and

\begin{equation}
m_q(0) := \frac{1}{q} \int_0^1 \left( \int_0^x u_0(y) \, dy \right)^q \, dx,
\end{equation}

for $(z_1, z_2, z_3) \in [0, \infty)^3$ and $q \geq 2$, we have $T_m < \infty$ as soon as $\mathcal{P}_q(m_q(0), \|v_0\|_{H^1}, \varepsilon M) < 0$ for some finite $q \in (2, 2/(2 - p)]$. In particular, if $u_0$ is such that

\begin{equation}
\mathcal{P}_q(m_q(0), 0, 0) < 0 \quad \text{for some finite} \quad q \in (2, 2/(2 - p)],
\end{equation}

there is $\vartheta > 0$ such that $\varepsilon M \in (0, \vartheta)$ and $\|v_0\|_{H^1} < \vartheta$ imply that $\mathcal{P}_q(m_q(0), \|v_0\|_{H^1}, \varepsilon M) < 0$ and thus $T_m < \infty$.

There are functions $u_0$ satisfying (12) and (16) if $M$ is sufficiently large. Indeed, observe that

\begin{equation}
\mathcal{P}_q(0, 0, 0) = \left(1 + \frac{\gamma}{D}\right) \left(c_1 (1 + M) + \frac{M^2}{2D}\right) + \frac{c_1(q-1)q^{(q-2)/q}D}{(p-1)M^{p-1}} \left(c_1 (1 + M) + \frac{M^2}{2D}\right)^{(q-2)/q} - \frac{M^q}{q(q + 1)}
\end{equation}
is negative for sufficiently large $M$ as $q > 2$. Given such an $M > 0$ and choosing the function $u_0(x) = 2M \max \{x + \delta - 1, 0\}/\delta^2$, $x \in (0, 1)$, we have $m_q(0) = (2M)^q \delta/(2q + 1)$ and $P_q(m_q(0), 0, 0) < 0$ for $\delta > 0$ small enough. In fact, if $u_0$ fulfills (16), then the same computation as the one leading to Theorem 1 shows that the corresponding solution to the parabolic-elliptic Keller-Segel system ($\varepsilon = 0$) blows up in a finite time and the last assertion of Theorem 1 states that this property remains true for the parabolic-parabolic Keller-Segel system ($\varepsilon > 0$) provided $\varepsilon$ and $v_0$ are small, that is, in a kind of neighbourhood of the parabolic-elliptic case.

Remark 2 The growth condition required on $a$ in (13) is seemingly optimal: indeed, it is proved in [6] that $T_m = \infty$ if $a(r) \geq c_0 (1 + r)^{-p}$ for some $p < 1$ and $\varepsilon = 0$, and the proof is likely to extend to the case $\varepsilon > 0$.

The proof of Theorem 1 relies on two properties of the Keller-Segel system (7)-(10): first, there is a Liapunov functional [7] which is bounded from below in the one-dimensional case [6] and which provides information on the time derivative of $v$. This will be the content of Section 2 where we also sketch the proof of the local well-posedness of (7)-(10). We next derive an identity of virial type for the $L^q$-norm of the indefinite integral of $u$ in Section 3 which involves in particular the time derivative of $v$. The information obtained on this quantity in the previous section then allow us to derive a differential inequality for the $L^q$-norm of the indefinite integral of $u$ for a suitable value of $q$ which cannot be satisfied for all times if the parameters $\varepsilon$, $D$, $\gamma$, $M$, and the initial data $(u_0, v_0)$ are suitably chosen.

2 Well-posedness and Liapunov functional

In this section, we establish the local well-posedness of (7)-(10) in $W^{1,2}(0, 1; \mathbb{R}^2)$ and recall the availability of a Liapunov functional for this system [7]. To this end, we assume that

(17) \[ 0 < a \in C^2(\mathbb{R}) \]

and define $b \in C^2((0, \infty))$ by

(18) \[ b(1) = b'(1) := 0 \quad \text{and} \quad b''(r) := \frac{a(r)}{r} \quad \text{for} \quad r > 0. \]

Proposition 3 Assume that the parameters $\varepsilon$, $D$, $\gamma$, $M$, and the function $a$ fulfil (11) and (17), respectively. Given initial data $(u_0, v_0) \in W^{1,2}(0, 1; \mathbb{R}^2)$ satisfying (12), there is a unique classical maximal solution

(19) \[ (u, v) \in C([0, T_m] \times [0, 1]; \mathbb{R}^2) \cap C^{1,2}((0, T_m) \times [0, 1]; \mathbb{R}^2) \]

to (7)-(10) with maximal existence time $T_m \in (0, \infty]$ and $(u, v)$ satisfies (14) for $t \in [0, T_m)$. In addition, if $T_m < \infty$, we have

\[ \lim_{t \to T_m} (\|u(t)\|_\infty + \|v(t)\|_\infty) = \infty. \]
Proof. We define $\tilde{a} \in C^2(\mathbb{R}^2; \mathcal{M}_2(\mathbb{R}))$ by

$$\tilde{a}(y) = (\tilde{a}^{m,n}(y))_{1 \leq m, n \leq 2} := \begin{pmatrix} D & 0 \\ -y_2 & a(y_2) \end{pmatrix}$$

for $y = (y_1, y_2) \in \mathbb{R}^2$ and introduce the operators

$$A(y)z := -\partial_x(\tilde{a}(y) \partial_x z),$$

$$(B(y)z(0), B(y)z(1)) := (-\tilde{a}(y) \partial_x z(0), \tilde{a}(y) \partial_x z(1)),$$

and the function

$$f(y) := \begin{pmatrix} -\gamma y_1 + y_2 - M \\ 0 \end{pmatrix}$$

with $z = (z_1, z_2)$. With this notation, an abstract formulation of (7)-(10) reads

$$\partial_t z + A(z)z = f(z),$$

$$B(z)z = 0,$$

$$z(0) = (v_0, u_0),$$

with $z = (v, u)$ and we aim at applying the theory developed in [1]. Owing to (11) and (17), $\tilde{a}(y)$ is a positive definite matrix for all $y \in \mathbb{R}^2$ and we infer from [1, Section 4] that the boundary-value operator $(A, B)$ is normally elliptic. It then follows from [1, Theorems 14.4 & 14.6] that (7)-(10) has a unique classical maximal solution

$$(v, u) := z \in C([0, T_m] \times [0, 1]; \mathbb{R}^2) \cap C^{1,2}((0, T_m) \times [0, 1]; \mathbb{R}^2)$$

for some $T_m \in (0, \infty]$. In addition, $\tilde{a}^{2,1}(y_1, 0) = 0$ for $y_1 \in \mathbb{R}$ and we deduce from [1, Theorem 15.1] that $u(t, x) \geq 0$ for $(t, x) \in [0, T_m] \times [0, 1]$. The property (14) then readily follows from (7)-(10) and (12) by integration. As for the last statement (19), it is a consequence of the lower triangular structure of the matrix $\tilde{a}$ and [1, Theorem 15.5]. □

We next proceed as in [7] to check the availability of a Liapunov functional for (7)-(10).

Lemma 4 Assume that the parameters $\varepsilon, D, \gamma, M,$ and the function $a$ fulfil (11) and (17), respectively. Given initial data $(u_0, v_0) \in W^{1,2}(0, 1; \mathbb{R}^2)$ satisfying (12) and such that $b(u_0) \in L^1(0, 1)$, the corresponding classical solution $(u, v)$ to (7)-(10) satisfies

$$L(u(t), v(t)) + \varepsilon \int_0^t \|\partial_t v(s)\|_2^2 \, ds \leq L(u_0, v_0) \quad \text{for} \quad t \in [0, T_m),$$

where

$$L(u, v) := \int_0^1 \left( b(u) - uv + \frac{D}{2} |\partial_x v|^2 + \frac{\gamma}{2} |v|^2 \right) \, dx.$$
Proof. It follows from (7)-(9) that
\[
\frac{d}{dt} L(u, v) = \int_0^1 (b'(u) - v) \, \partial_t u \, dx + \int_0^1 (D \, \partial_x v \, \partial_x \partial_t v + (\gamma \, v - u) \, \partial_t v) \, dx \\
= - \int_0^1 (b''(u) \, \partial_x v \, \partial_x \partial_t v + (a(u) \, \partial_x u - u \, \partial_x v)) \, dx \\
+ \int_0^1 \partial_t v \left( -D \, \partial_x^2 v + \gamma \, v - u \right) \, dx \\
= - \int_0^1 u \, |\partial_x (b'(u) - v)|^2 \, dx - \int_0^1 (M + \varepsilon \, \partial_t v) \, \partial_v v \, dx \\
\leq -\varepsilon \|\partial_t v\|_2^2,
\] (22)
the last inequality being a consequence of (14). Integrating the previous inequality with respect to time gives (20). □

We next take advantage of the one-dimensional setting to show that $L$ is bounded from below without prescribing growth conditions on $a$. This fact has already been observed in [6] and is peculiar to the one-dimensional case. Indeed, as shown in [7, 11], the occurrence of blow-up is closely related to the unboundeness of the Liapunov functional.

Lemma 5 Assume that the parameters $\varepsilon$, $D$, $\gamma$, $M$, and the function $a$ fulfil (11) and (17), respectively. Given initial data $(u_0, v_0) \in W^{1,2}(0,1; \mathbb{R}^2)$ satisfying (12) and such that $b(u_0) \in L^1(0,1)$, the corresponding classical solution $(u,v)$ to (7)-(10) satisfies
\[
L(u(t), v(t)) \geq -\frac{M^2}{2D} \quad \text{for} \quad t \in [0,T_m).
\] (23)

Proof. Owing to (14), the Poincaré inequality ensures that $\|v(t)\|_{\infty} \leq \|\partial_x v(t)\|_2$ for $t \in [0,T_m)$ so that
\[
\int_0^1 u(t) \, v(t) \, dx \leq \|v(t)\|_{\infty} \|u(t)\|_1 \leq \|\partial_x v(t)\|_2 \|u(t)\|_1.
\]
We use again (14) as well as the non-negativity of $b$ to conclude that
\[
L(u(t), v(t)) \geq \frac{D}{2} \|\partial_x v(t)\|_2^2 - M \|\partial_x v(t)\|_1 \leq \frac{D}{2} \left( \|\partial_x v(t)\|_2 - \frac{M}{D} \right)^2 - \frac{M^2}{2D}
\]
for $t \in [0,T_m)$, from which (23) readily follows. □
3 Finite time blow-up

As already mentioned, the main novelty in this paper is a new identity of virial type which is the cornerstone of the proof that blow-up takes place in finite time under suitable assumptions. Specifically, we assume that the parameters $\varepsilon, D, \gamma, M$, and the function $a$ fulfill the conditions (11) and (13), respectively. Recalling the definition (18) of $b$, we deduce from (13) that

(24) \[ b(r) \leq c_1 (r \ln r - r + 1) 1_{[0,1]}(r) + \frac{c_1(r-1)}{p} 1_{[1,\infty)}(r) \leq c_1 (1 + r), \quad r \geq 0. \]

We also define

(25) \[ A(r) := -\int_r^\infty a(s) \, ds, \quad r \geq 0, \]

and infer from (13) that $A$ is well-defined and satisfies

(26) \[ 0 \leq -A(r) \frac{r}{p-1} r^{2-p}, \quad r \geq 0. \]

Consider next initial data $(u_0, v_0) \in W^{1,2}(0,1; \mathbb{R}^2)$ satisfying (12). If $(u, v)$ denotes the corresponding classical solution to (7)-(10) given by Proposition 3, we define the cumulative distribution functions $U$ and $V$ by

(27) \[ U(t, x) := \int_0^x u(t, y) \, dy \quad \text{and} \quad V(t, x) := \int_0^x v(t, y) \, dy \]

for $(t, x) \in [0, T_m] \times [0, 1]$. It readily follows from (7)-(9) and (14) that $(U, V)$ solves

(28) \[ \varepsilon \partial_t U = \partial_x A(u) - u \partial_x v \quad \text{in} \quad (0, T_m) \times (0, 1), \]

(29) \[ D \partial_t v = D \partial_x v - \gamma V + U - Mx \quad \text{in} \quad (0, T_m) \times (0, 1), \]

the function $A$ being defined in (25), and

(30) \[ U(t, 0) = M - U(t, 1) = 0 \quad \text{and} \quad V(t, 0) = V(t, 1) = 0, \quad t \in [0, T_m). \]

Lemma 6 Introducing $m_q(t) := \|U(t)\|_q^q/q$ for $q \geq 2$, we have

(31) \[ \frac{dm_q}{dt} = \frac{M}{D} m_q - \frac{M^{q+1}}{q(q+1)D} + M^{q-1} A(u(t,1)) - (q-1) \int_0^1 U^{q-2} u A(u) \, dx \]

\[ + \frac{\varepsilon}{qD} \int_0^1 U^{q-1} \partial_x v \, dx - \frac{\gamma}{D} \int_0^1 U^{q-1} v \, dx \]

for $t \in [0, T_m)$. 

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Proof. We infer from (28), (29), and (30) that
\[
\frac{dm_q}{dt} = [U^{q-1} A(u)]_{x=0}^1 - (q-1) \int_0^1 U^{q-2} u A(u) \, dx \\
- \frac{1}{D} \int_0^1 u U^{q-1} \left( \varepsilon \partial_t V + \gamma V - U + M x \right) \, dx \\
= M^{q-1} A(u(t,1)) - (q-1) \int_0^1 U^{q-2} u A(u) \, dx - \frac{\varepsilon}{qD} [U^q \partial_x V]_{x=0}^1 \\
+ \frac{\varepsilon}{qD} \int_0^1 U^q \partial_t v \, dx - \frac{\gamma}{D} \int_0^1 U^{q-1} u V \, dx + \frac{1}{(q+1)D} [U^{q+1}]_{x=0}^1 \\
- \frac{M}{qD} [U^q x]_{x=0}^1 + \frac{M}{D} m_q \\
= M^{q-1} A(u(t,1)) - (q-1) \int_0^1 U^{q-2} u A(u) \, dx + \frac{\varepsilon}{qD} \int_0^1 U^q \partial_t v \, dx \\
- \frac{\gamma}{D} \int_0^1 U^{q-1} u V \, dx - \frac{M^{q+1}}{q(q+1)D} + \frac{M}{D} m_q,
\]
which is the expected identity.

At this point, we notice that the solution to the ordinary differential equation
\[
D \dot{X} = M X - (M^{q+1}/(q(q+1)))
\]
(obtained by neglecting several terms in (31)) is given by
\[
X(t) = \frac{M^q}{q(q+1)} e^{Mt/D} \left( X(0) - \frac{M^q}{q(q+1)} \right),
\]
and thus vanishes at a finite time if \(X(0) < M^q/(q(q+1))\). If a similar argument could be used for (31), we would obtain a positive time \(t_0\) such that \(m_q(t_0) = 0\) which clearly contradicts the properties of \(U(t_0)\): indeed, by (27) and (30), \(x \mapsto U(t_0, x)\) is continuous with \(U(t_0, 1) = M\). Consequently, the solution \((u, v)\) to (7)-(10) no longer exists at this time \(t_0\) and blow-up shall have occurred at an earlier time, thus establishing Theorem 1. For this approach to work, we shall of course control the other terms on the right-hand side of (31) which will in turn give rise to the blow-up criterion stated in Theorem 1. The latter is actually a simple consequence of the following result:

**Theorem 7** Assume that the parameters \(\varepsilon, D, \gamma, M,\) and the initial data \((u_0, v_0)\) are such that
\[
E \left( m_q(0) + L(u_0, v_0) + \frac{M^2}{2D} \right) < 0
\]
for some finite \(q \in (2, 2/(2-p)]\), where
\[
E(z) := \left( 1 + \frac{\gamma}{M} \|v_0\|_{H^1} + \frac{\varepsilon M^{q-1}}{4qD} \right) z + \frac{c_1 (q-1) q^{(q-2)/q}}{(p-1) M^{p-1}} z^{(q-2)/q} - \frac{M^q}{q(q+1)}
\]
for \(z \geq 0\). Then \(T_m < \infty\).
Proof. The starting point of the proof being the identity (31), we first derive upper bounds for the terms on the right-hand side of (31) involving $A$, $\varepsilon$, and $\gamma$. Thanks to (26) and the non-negativity of $U$, it follows from the Hölder inequality that

$$M^{q-1} A(u(t, 1)) - (q - 1) \int_0^1 U^{q-2} u \ A(u) \ dx \leq \frac{c_1(q - 1)}{p - 1} \int_0^1 U^{q-2} u^{2-p} \ dx \leq \frac{c_1(q - 1)q^{(q-2)/q}}{(p - 1)} m_{q-2/q} \left( \int_0^1 u^{((2-p)/2)} \ dx \right)^{2/q}.$$ 

Since $q \in (2,2/(2 - p)]$, we may use the Jensen inequality and (14) to conclude that

$$(33) \quad M^{q-1} A(u(t, 1)) - (q - 1) \int_0^1 U^{q-2} u \ A(u) \ dx \leq \frac{c_1(q - 1)q^{(q-2)/q}}{(p - 1)} M^{2-p} m_{q-2/q}. $$

Next, to estimate the term involving $\gamma$, we adapt an argument from [16] and first claim that

$$(34) \quad V(t, x) \geq V_m(t, x) := \frac{M}{6D} (x^3 - x) + h(t, x), \quad (t, x) \in [0, T_m) \times [0, 1],$$

where $h$ denotes the unique solution to

$$(35) \quad \varepsilon \partial_t h - D \partial_x^2 h + \gamma h = 0, \quad (t, x) \in (0, \infty) \times (0, 1),$$

$$(36) \quad h(t, 0) = h(t, 1) = 0, \quad t \in (0, \infty),$$

$$(37) \quad h(0, x) = \min \left\{ V(0, x) + \frac{M}{6D} (x - x^3), 0 \right\} \leq 0, \quad x \in (0, 1).$$

Indeed, $V_m \leq V$ on $[0, T_m) \times \{0, 1\}$ and $\{0\} \times [0, 1]$, and it follows from the non-negativity of $U$ and the non-negativity of $h$ that

$$\varepsilon \partial_t V_m - D \partial_x^2 V_m + \gamma V_m = \varepsilon \partial_t h - Mx - D \partial_x^2 h + \frac{M\gamma}{6D} (x^3 - x) + \gamma h \leq -Mx \leq U - Mx = \varepsilon \partial_t V - D \partial_x^2 V + \gamma V.$$

The comparison principle then implies (31). We next infer from (34) and the non-negativity of $u$ and $U$ that

$$-\frac{\varepsilon}{D} \int_0^1 U^{q-1} u \ V \ dx \leq -\frac{\varepsilon}{D} \int_0^1 U^{q-1} u \ V_m \ dx = -\frac{\varepsilon}{qD} \left[ U^q V_m \right]_{x=1}^{x=1} + \frac{\varepsilon}{qD} \int_0^1 U^q \partial_x V_m \ dx \leq \frac{\varepsilon}{D} \left( \frac{M}{2D} + \|\partial_x h\|_\infty \right) m_q.$$
We next note that $\partial_x h$ also solves (35) with homogeneous Neumann boundary conditions, the latter property being a consequence of (35) and (36). Since

$$|\partial_x h(0, x)| \leq |v_0(x) + \frac{M}{6D} (1 - 3x^2)| \leq \|v_0\|_\infty + \frac{M}{3D},$$

the comparison principle and the non-negativity of $\gamma$ warrant that $\|\partial_x h(t)\|_\infty \leq \|v_0\|_\infty + (M/3D)$ for $t \geq 0$. Consequently, recalling the Sobolev embedding $\|v_0\|_\infty \leq \|v_0\|_{H^1}$, we end up with

$$(38) - \frac{\gamma}{D} \int_0^1 U^{q-1} u V \, dx \leq \frac{\gamma M}{D^2} \left( 1 + \frac{D}{M} \|v_0\|_{H^1} \right) m_q.$$  

We finally infer from (14), (27), (30), and the Hölder inequality that

$$(39) \frac{\varepsilon}{qD} \int_0^1 U^q \partial_t v \, dx \leq \frac{\varepsilon M^{q/2}}{qD} \int_0^1 U^{q/2} |\partial_t v| \, dx \leq \frac{\varepsilon M^{q/2}}{q^{1/2}D} m_q^{1/2} \|\partial_t v\|_2.$$  

It now follows from (31), (33), (38), and (39) that

$$\frac{dm_q}{dt} \leq \frac{M}{D} \left[ \left( 1 + \frac{\gamma}{D} + \frac{\gamma}{M} \|v_0\|_{H^1} \right) m_q + \frac{c_1(q-1)q^{(q-2)/q}D}{(p-1)M^{p-1}} m_q^{(q-2)/q} - \frac{M^q}{q(q+1)} \right]$$

$$+ \frac{\varepsilon M^{q/2}}{q^{1/2}D} m_q^{1/2} \|\partial_t v\|_2$$

$$\leq \frac{M}{D} E(m_q) - \frac{\varepsilon M^q}{4qD^2} m_q + \frac{\varepsilon M^{q/2}}{q^{1/2}D} m_q^{1/2} \|\partial_t v\|_2.$$  

Owing to (12) and (24), we have $b(u_0) \in L^1(0, 1)$ and it follows from (22), (23), and the above inequality that

$$\frac{d}{dt} \left( m_q + L(u, v) + \frac{M^2}{2D} \right) \leq \frac{M}{D} E(m_q) - \frac{\varepsilon M^q}{4qD^2} m_q + \frac{\varepsilon M^{q/2}}{q^{1/2}D} m_q^{1/2} \|\partial_t v\|_2 - \varepsilon \|\partial_t v\|_2^2$$

$$= \frac{M}{D} E(m_q) + \varepsilon \left( m_q - \frac{2q^{1/2}D}{2q^{1/2}D} m_q^{1/2} \right)^2$$

$$\leq \frac{M}{D} E(m_q).$$  

Using now the monotonicity of $E$ and (23), we end up with

$$\frac{d}{dt} \left( m_q + L(u, v) + \frac{M^2}{2D} \right) \leq \frac{M}{D} E(m_q + L(u, v) + \frac{M^2}{2D}).$$

Assume now for contradiction that $T_m = \infty$. The previous inequality and (32) then warrant that there is a time $t_0 > 0$ such that $m_q(t_0) + L(u(t_0), v(t_0)) + (M^2/2D) = 0$ and hence
m_q(t_0) = 0 by (23). This in turn implies that U(t_0, x) = 0 for all x ∈ [0, 1] and contradicts (30). Consequently, T_m < ∞.

The remaining step towards Theorem 1 is to use the properties of a to simplify the condition (32) derived in Theorem 7.

**Proof of Theorem 1**. It follows from (12), (24), and the Sobolev embedding ∥v_0∥_∞ ≤ ∥v_0∥_{H^1} that

\[
L(u_0, v_0) + \frac{M^2}{2D} \leq \int_0^1 \left( c_1 (1 + u_0) + \frac{D}{2} |\partial_x v_0|^2 + \frac{\gamma}{2} |v_0|^2 + u_0 \|v_0\|_{\infty} \right) \, dx + \frac{M^2}{2D}
\]

\[
\leq c_1 (1 + M) + \frac{M^2}{2D} + \frac{D + \gamma}{2} \|v_0\|_{H^1}^2 + M \|v_0\|_{H^1}
\]

\[
= F(m_q(0), \|v_0\|_{H^1}) - m_q(0),
\]

the function F being defined in Theorem 1. Therefore,

\[
E \left( m_q(0) + L(u_0, v_0) + \frac{M^2}{2D} \right) \leq (E \circ F)(m_q(0), \|v_0\|_{H^1}) = P_q(m_q(0), \|v_0\|_{H^1}, \varepsilon M),
\]

and the condition P_q(m_q(0), \|v_0\|_{H^1}, \varepsilon M) < 0 clearly implies (32) and hence T_m < ∞.

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