Calculation of multi-loop superstring amplitudes

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Abstract
The multi-loop interaction amplitudes in the closed, oriented superstring theory are obtained by the integration of local amplitudes. The local amplitude is represented by a sum over the spinning string local amplitudes. The spinning string local amplitudes are given explicitly through super-Schottky group parameters and through interaction vertex coordinates on the \(1\!\!1\) complex, non-split supermanifold. The obtained amplitudes are free from divergences. They are consistent with the world-sheet spinning string symmetries. The vacuum amplitude vanishes along with 1-, 2- and 3-point amplitudes of massless states. The vanishing of the above-mentioned amplitude occurs after the integration of the corresponding local amplitude has been performed over the super-Schottky group limiting points and over interaction vertex coordinate, except for those \(3\!\!2\) variables which are fixed due to \(SL(2)\)-symmetry.

Keywords: superstrings, multi-loop amplitudes, finiteness, vacuum amplitude, nullification

1. Introduction

In the Ramond–Neveu–Schwarz theory \([1]\) the superstring interaction amplitudes are obtained by a summation of the spinning (fermion) string interaction amplitudes. The string world-sheet carries the zweibein \(e^m_n\) and the 2D-gravitino field \(\phi^m_n\). Owing to the local 2D-symmetries, the amplitudes are independent of the above-mentioned gauge fields \(e^m_n\) and \(\phi^m_n\). Due to a hidden space–time symmetry of the superstring, it is expected that the vacuum superstring amplitude is nullified along with 1-, 2- and 3-point amplitudes of massless states. Every \(n\)-loop amplitude is represented by an integral of a local amplitude that is calculated over \((3n - 3|2n - 2)\) complex moduli (if \(n > 1\)) and over interaction vertex coordinates on the complex \((1|1)\) supermanifold.
In [2] the supermanifold is specified as the Riemann surface with spin structure [3]. Grassmann moduli are assigned to the 2D-gravitino field which, therefore, is not conformally flat. Multi-loop amplitudes in [2] depend on the location of the 2D-gravitino field [2, 4, 5] that means a loss of the 2D supersymmetry in these calculations. True two-loop amplitudes have been given in [6]. They have been reproduced [7] in the pure spinor formulation [8]. Additionally, the low-energy limit of the four-point, 3-loop amplitude has been obtained [9] in this formulation.

Arbitrary-loop RNS-amplitudes have been obtained in [10], the gauge [2] being employed. The period matrix on the (1|1) complex supermanifold is, generally, dependent on Grassmann moduli. In this case the integration of the local amplitude over the fundamental region of the modular group leads to the loss of 2D supersymmetry. As the result, the spinning string amplitudes depend on $\phi_p$. To restore the supersymmetry, the integration over the fundamental region of the modular group must be supplemented [11] by the integral around the boundary of the region. The loss of the supersymmetry in [2] occurs due to the above-mentioned boundary integral has been ignored and, also, because of an incomplete calculation of the ghost zero mode contribution to the integration measure [10].

If the Riemann surface genus $n \leq 3$, then the period matrix entries can be taken [6] as the moduli set. In this case the boundary integral does not arise. If $n = 3$ and the moduli setting [2] is used, then the boundary integral is removed [10] by a re-definition of the local amplitude. The integration measures are given by modular forms for both $n = 2$ and $n = 3$. Unlike the two-loop case, the GSO projection of the four-point, three-loop amplitudes ceases to depend on $\phi_p$ due to the integration over vertex coordinates, just as it arises in each of the spin structures. If $n > 3$, then periods of superscalar functions depend on Grassmann moduli for any choice of moduli variables. The boundary integral is present in the expression for the amplitude, and the integration measures are not modular forms.

A study of the $n > 2$-loop amplitudes is hampered because these amplitudes cease to depend on $\phi_p$ only due to the integration over moduli and the interaction vertex coordinates. In particular, the finiteness, the nullification of the vacuum amplitude and the vanishing of the $m < 4$-point, massless state amplitudes have not been verified in [10] for superstring $n > 3$-loop amplitudes.

In the present paper these basic properties of the superstring amplitudes are verified employing the supercovariant gauge [12–16]. In this case zweibein and 2D-gravitino field are conformally flat. Local spinning string amplitudes have a manifest $SL(2)$-symmetry. The string world-sheet is specified as the $(1|1)$ non-split supermanifold. The supermanifold carries a ‘superspin’ structure [15–18] which is a supersymmetric extension of the relevant spin structure [3]. Local spinning amplitudes with any number of loops have been explicitly calculated [16] for all the superspin structures. In doing so the supermanifold has been described by ‘super-Schottky’ groups that are superconformal extensions [18] of the Schottky groups. Partition functions have been computed from equations [16, 18] that realize the requirement that the spinning amplitudes are independent of infinitesimal local variations of both the vierbein and the gravitino field. So they are consistent with the gauge invariance of the fermion string (the same method has been also employed in [10]).

The fundamental domain of the group of the local symmetries of the superstring is ambiguous since its ‘boundary’ depends [5, 11] on Grassmann parameters. Further, the integration over degenerated surfaces is ambiguous with respect to those replacements of the integration variables which admix Grassmann variables to the boson ones [11]. Then the calculation is guided by a preservation of local symmetries of the amplitude.
The integration of the local amplitudes is considered in the present paper. In doing so, the supermanifold is described by super-Schottky groups \[16, 17\] and the local amplitudes \[16\] are employed. The super-Schottky groups and the mentioned amplitudes are reviewed in sections 2, 4–6 of this paper. A number of the results \[16\] are modified to be more adapted for goals of the paper. To avoid unnecessary complication, the deriving of these modified formula is planned to give elsewhere.

A supercovariant integration of the local amplitudes is proposed in section 3 of the paper. In sections 7 and 8 the finiteness of the superstring amplitudes is established. The nullification of the vacuum amplitude and a vanishing the \(m \leq 4\)-point amplitudes of the massless states is verified. The preservation of the local symmetries of the amplitudes is argued.

2. Superspin structures

The genus-\(n\) super-spin structure presents a superconformal extension of the relevant genus-\(n\) spin given by the set of transformations \(\Gamma^{(0)}_{a,s}\) and \(\Gamma^{(0)}_{b,s}\) (where \(s = 1, \ldots n\)) corresponding to the round of \(A_s\) and \(B_s\)-cycles on the Riemann surface. Then

\[
\Gamma^{(0)}_{b,s} : \{z \to g_s(z), \; \theta \to \vartheta \}
\]

where \(l_{b,s}\) and \(l_{a,s}\) are the theta function characteristics assigned to the given handle \(s\). They can be restricted by 0 and 1/2. The Schottky transformation \(g_s(z)\) (the \(s\) index is omitted) is

\[
g(z) = \frac{az + b}{cz + d} \quad \text{where} \quad ad - bc = 1.
\]

The parameters in (2) are expressed through \(u\) and \(v\) limiting points and the complex multiplier \(k (|k| \leq 1)\) as follows

\[
a = \frac{u - kv}{\sqrt{k (u - v)}}, \quad d = \frac{ku - v}{\sqrt{k (u - v)}} \quad \text{and} \quad c = \frac{1 - k}{\sqrt{k (u - v)}}.
\]

The Schottky transformation (2) turns the boundary of the Schottky circle \(C_v\) into the boundary of the \(C_u\) circle. In this case

\[
C_v = \{z : |cz + d|^2 = 1\}, \quad C_u = \{z : |cz + a|^2 = 1\}.
\]

Using (3), one can see that \(v\) lies inside \(C_v\) and outside \(C_u\). Correspondingly, \(u\) is inside \(C_u\) and outside \(C_v\). In the Ramond case \((l_{b,s} = 1/2)\) a square root cut appears on \(z\)-plane between \(u_r\) and \(v_r\). The set of \(n\) forming transformations (1) and their group products forms the Schottky group of the genus-\(n\). If all the Schottky circles of the forming transformations are separated from each other, then the exterior of them can be taken as the fundamental region \[19\]. Circles (4) can be replaced by other relevant curves \[19\]. The exterior of all these curves can be taken as the fundamental region.

In the superstring theory the transformations (1) are replaced \[15–17\] by \(SL(2)\) transformations \(\Gamma_{a,s}\) and \(\Gamma_{b,s}\) as follows

\[
\Gamma_{a,s} = \Gamma^{(0)}_{a,s} \Gamma_s \quad \Gamma_{b,s} = \Gamma^{(0)}_{b,s} \Gamma_s,
\]

where \(\Gamma^{(0)}_{a,s}\) and \(\Gamma^{(0)}_{b,s}\) are given by (1) while \(\Gamma_s\) transforms \((z \vartheta)\) as follows

\[
z = z^{(i)} + \vartheta^{(i)} z^{(i)} + \vartheta \left(1 + \frac{\vartheta z^{(i)}}{2}\right) + z^{(i)},
\]

\[
\vartheta = \vartheta^{(i)} \left(1 + \frac{\vartheta z^{(i)}}{2}\right) + z^{(i)},
\]
\[ \varepsilon_i(z) = \frac{\mu_i(z-v_i) - \nu_i(z-u_i)}{u_i - v_i}, \quad \varepsilon'_i = \partial_z \varepsilon_i(z), \]  

and \((u_i, v_i)\) and \((v_i, u_i)\) are limiting points of transformations (5). The set of the transformations (5) for \(s = 1, \ldots, n\) together with their group products forms the genus-\(n\) super-Schottky group. The superconformal \(p\)-tensor \(T_p(t)\) is changed under the \(SL(2)\) transformation \(\Gamma(t) = \{ t \to t' = (z(t)/|\vartheta(t)|) \}\) as follows

\[ T_p(t) = T_p(t)Q(p)G(t), \quad Q(p)^{-1}(t) = D(t)\vartheta(t); \quad D(t) = \partial \partial_z + \partial \partial_{v_i}. \]  

Further, \(\Gamma_{b,s}\) turns the ‘boundary’ of the \(\hat{C}_u\) ‘circle’ to the ‘boundary’ of \(\hat{C}_v\) where

\[ \hat{C}_u = \{ t : (-1)^{2s+1} Q_{\Gamma_{b,s}(t, \vartheta(t))}(\hat{C}_{b,s}(t)) = 1 \}, \quad \hat{C}_v = \{ t : (-1)^{2s+1} Q_{\Gamma_{b,s}(t, -\vartheta(t))}(\hat{C}_{b,s}(t)) = 1 \}. \]

In this case \(l_{2s}\) is assigned to the right movers and \(l'_{2s}\) is associated with the left ones. Another pair of congruent ‘circles’ is

\[ \hat{C}_v = \{ t : (c_z \varepsilon^{(s)} + d_z)^2 = 1 \} \quad \text{and} \quad \hat{C}_u = \{ t : (-c_z \varepsilon^{(s)} + a_z)^2 = 1 \}, \]

\(\varepsilon^{(s)}\) being defined by (6). ‘Circles’ (9) and (10) are different from each other and from (4) by terms that are proportional to Grassmann quantities. Therefore, the integration of the conformal \((1/2, 1/2)\)-tensor over the Schottky-group fundamental region destroys \(SL(2)\)-symmetry. To preserve the symmetry, the fundamental region can be bounded by a ‘step function’ factor

\[ B_{LL}(t, \tilde{t}; \{ q, \tilde{q} \}) = \prod_{G} \theta(\ell_{G}^L(t)\ell_{G}^{\tilde{L}}(\tilde{t}) - 1) \theta(1 - \ell_{G}^L(\Gamma_{b,G}; -(L, t))\ell_{G}^{\tilde{L}}(\Gamma_{b,G}; -(L', \tilde{t}))), \]

where the step function \(\theta(x)\) is understood [10, 11] as the Taylor series in the ‘soul’ part of its argument (that is in the part proportional to the Grassmann parameters). In doing so, the known relation \(\delta t/d x = \delta (x)\) is used where \(\delta (x)\) is the Dirac delta-function. Further, \(\ell_{G}^L(t)\) is a relevant ‘curve’ and \(\ell_{G}^{\tilde{L}}(\tilde{t})\) is assigned to the super-Schottky group transformation \(\Gamma_{b,G}(L; t)\) (for instance, \(\hat{C}_v\) or \(\hat{C}_u\) can be taken as \(\ell_{G}^L(t)\)). When integration variables are replaced, the arguments of the step functions are correspondingly replaced, too. As the result, the integral of \((1/2, 1/2)\)-supertensor is independent of the choice of the integration variables. The expansion of \(\theta(x)\) over the Grassmann quantities originates boundary terms in the integral.

3. Integration of the local amplitude

The superstring amplitude is calculated by the integration of the local amplitude over the interaction vertex coordinates \(t_j = (z_j, \vartheta_j)\) and over super-Schottky group parameters excepting those \((3|2)\) parameters, which are fixed due to \(SL(2)\) symmetry. If \(U_a, V_a\) and \(u_b, v_b\) are fixed, then the local amplitude receives a factor \(|\mathcal{H}(U_a, V_a, u_b)|^2\) where [16]

\[ \mathcal{H}(U_a, V_a, u_b) = (u_a - u_b)(v_a - u_b) \left[ 1 - \frac{\mu_a \mu_b}{2(u_a - u_b)} - \frac{v_a \mu_b}{2(v_a - u_b)} \right]. \]
Due to this factor, the amplitude does not depend on a choice of the set of fixed parameters. The integration region over \( t_j = (z_j|\tilde{v}_j) \) is restricted by the step function product

\[
\tilde{B}_{L,L}^{(n)}(\{t_j, \tilde{t}_j\}; \{q, \tilde{q}\}) = \prod_{j=1}^{m} B_{L,L}^{(n)}(\{t_j, \tilde{t}_j\}; \{q, \tilde{q}\}). \tag{13}
\]

The integration over the super-Schottky group parameters is performed in such a way that the period matrix is kept in the fundamental region of the modular group. The modular transformation on the supermanifold is a globally defined, holomorphic superconformal transition \( t \to \tilde{t} = \tilde{t}(t, \{q\}; L), q \to \tilde{q}(\{q\}; L) \) and \( L \to \tilde{L}(L) \). In this case the period matrix \( \omega(\{q\}, L) \) is changed just as the period matrix \( \omega(\{q\}_0) \) on the Riemann surface \([20] \), that is

\[
\omega(\{q\}, L) = [A\omega(\{\tilde{q}\}, \tilde{L}) + B][C\omega(\{\tilde{q}\}, \tilde{L}) + D]^{-1}, \tag{14}
\]

where integer \( A, B, C \) and \( D \) matrices satisfy certain conditions [20]. All the Grassmann parameters being equal to zero, the period matrix on the supermanifold is reduced to the period matrix \( \omega(\{q\}_0) \) on the Riemann surface. In this case [13, 14, 16, 21]

\[
2\pi i \omega_{2p}(\{q\}_0) \equiv \ln \frac{(u_s - u_p)(v_s - v_p)}{(u_s - v_p)(v_s - u_p)} + \sum_{i\neq j} \ln \frac{[u_s - g_i(u_p)] [v_s - g_i(v_p)]}{[u_s - g_i(v_p)] [v_s - g_i(u_p)]}, \quad (s \neq p); \tag{15}
\]

\[
2\pi i \omega_{2s}(\{q\}_0) \equiv \ln k_s + \sum_{i\neq j} \ln \frac{[u_s - g_i(u_p)] [v_s - g_i(v_p)]}{[u_s - g_i(v_p)] [v_s - g_i(u_p)]}, \tag{16}
\]

where \( I \) is the identical transformation. The summation in (15) is performed over all the Schottky group transformations \( I \) whose leftmosts are not group powers of \( g_s \) and rightmosts are not group powers of \( g_p \). In (16) the summation is performed over all those \( g_i \), whose leftmosts and rightmosts are not group powers of \( g_p \). A jumping from the given branch of the logarithmic function to an other branch corresponds to a certain transformation (14) with \( C = 0 \) and \( A = D = I \). Hence spin structures can be re-defined by the fixing of the branch of the logarithmic function instead of the usual \( |\text{Re } \omega_{2p}(\{q\}_0)| \leq 1/2 \) condition.

Transformations (14) with \( B = C = 0 \) contain the interchange between handles and the replacement of the given transformation of the super-Schottky group by its inverse. If (32) limiting points are fixed, then the interchange touches only \((n - 2) \) of \( n \) handles. So, integrating over the moduli, one can either order the handles in any manner, or multiplying the integrand by \( 2^{-\sigma(n-2)}(n - 2)! \). Remaining transformations with \( B = C = 0 \) correspond to re-definitions of the set of the basic transformations.

The basic transformation set at zeroth \((\mu, \nu)\) is usually [20] fixed by the condition that entries \( \text{Im} \omega_{2s}(\{q\}_0) = y_s(\{q, \tilde{q}\}_0) \) of the matrix \( \text{Im} \omega(\{q\}_0) = y(\{q, \tilde{q}\}_0) \) obey relations as follows

\[
[F y(\{q, \tilde{q}\}_0) F^T]_{ij} \geq y_{ij}(\{q, \tilde{q}\}_0), \tag{17}
\]

where \( F \) is the \( n \)-dimensional integer, unimodular matrix whose last \( n - j + 1 \) entries of every column \( f_j \) are relatively prime. In addition, the fundamental region is bounded [20] as follows

\[
|\text{det}(C\omega(\{q\}_0) + D)|^2 \geq 1, \quad |\text{Re } \omega_{2p}(\{q\}_0)| \leq 1/2, \tag{18}
\]

where \( C \) and \( D \) are any of the integer matrices in (14). Restrictions (17) and (18) of the integration region over the Schottky-group parameters can be realized multiplying the
integrands by relevant steps function factor. At non-zeroth ($\mu$, $\nu$)-parameters $\omega(\{q\}_0)$-matrix is replaced by $\omega(\{q\}, L)$ and $\omega(\{q\}_a)$ is replaced by $\omega(\{q\}, L')$. The questioned step function factor $O(\omega_L, \omega_{L'})$ is as follows

$$O(\omega_L, \omega_{L'}) = \prod_j \theta(G_j(\omega_L, \omega_{L'})),$$

where $\theta(x)$ is understood as the Taylor series in the ‘sole’ part of its argument. The $G_j(\omega, \mathcal{W}) = 0$ relations determine the boundary of the integration region.

The period matrix does not changed under isomorphic re-definitions

$$\Gamma_{x,t}(l_{2a}) \rightarrow \Gamma_{G_j} \Gamma_{x,t}(l_{2a}) \Gamma_{G_j}^{-1}, \quad \Gamma_{b,t}(l_{2a}) \rightarrow \Gamma_{G_j} \Gamma_{b,t}(l_{2a}) \Gamma_{G_j}^{-1}$$

(20)

of the forming transformations (5) where $\Gamma_{G_j}$ is any super-Schottky group transformation (generically, $G_j$ depends on $x$). So the integration region over the Schottky-group limiting points must be more bounded. For instance, this can be achieved by the requirement that every $u_s$ is exterior of all the Schottky circles with the exception of $C_{u_s}$. The Schottky circles can be replaced by relevant curves $t_{L'}^L(t)$. In this case the integrand is multiplied by the step-function product $B_{L,L'}(\{q, \bar{q}\})$ which is built from the step functions products $B_{L,L'}^{(n)}(\{q, \bar{q}\})$ given as

$$B_{L,L'}^{(n)}(\{q, \bar{q}\}) = \prod_{G=G_j} \theta(t_{L'}^L(U_l) t_{L'}^{L'}(U_t) - 1) \theta(1 - t_{L'}^L(\Gamma_{h,G^{-1}}(L, U_l)) t_{L'}^{L'}(\Gamma_{h,G^{-1}}(L', U_t))),$$

(21)

where $G_j$ is assigned to the $\Gamma_{h,t}(l_{2a})$ transformation. If $(U_l, V_i, U_s)$ are fixed due to $SL(2)$-symmetry, then $B_{L,L'}^{(n)}(\{q, \bar{q}\})$ is as follows

$$B_{L,L'}^{(n)}(\{q, \bar{q}\}) = \prod_{s=1}^m B_{L,L'}^{(n)}(\{q, \bar{q}\}).$$

(22)

Perhaps, equation (21) needs in some refinement unessential for goals of this paper.

Henceforth, the $n$-loop, $m$-point amplitude $A_{n,m}(\{p_j, \xi^{(j)}\})$ for the interaction states carrying 10-dimensional momenta $\{p_j\}$ and the polarization tensors $\xi^{(j)}$, is represented as follows

$$A_{n,m}(\{p_j, \xi^{(j)}\}) = \frac{g^2 \pi^2 (m - 2)! \int L_{L'}^L} {2^n 2^{n-2}(n - 2)!} \sum_{L,L'} |H(U_l, V_i, U_s)|^2 Z_{L,L'}^{(n)}(\{q, \bar{q}\}) O(\omega_L, \omega_{L'}) \times B_{L,L'}^{(n)}(\{q, \bar{q}\}) B_{L,L'}^{(m)}(\{t_r, \bar{t}_r\}; \{q, \bar{q}\}) \times \langle \prod_{r=1}^m V(t_r, I_r; p_r; \xi^{(r)}) (dq d\bar{q})^r (dr d\bar{r}) \rangle,$$

(23)

where $Z_{L,L'}^{(n)}(\{q, \bar{q}\})$ is the integration measure (the partition function), $\langle \ldots \rangle$ denotes the vacuum expectation of the interaction vertex product, $g$ is the coupling constant and $L (L')$ labels the super-spin structures of right (left) movers. The $H(U_l, V_i, U_s)$ factor (12) arises because (3|2) super-Schottky group parameters are fixed due to $SL(2)$-symmetry. The integration is performed over vertex coordinates $\{t_r, \bar{t}_r\}$ on the (1|1) complex supermanifold and over $(3n - 3|2n - 2)$ complex moduli parameters of the super-Schottky group. For any Grassmann variable $\eta$ we define $\int d\eta = 1$. For any boson variable $x$ we define $dxd\bar{x} = d(Re x) d(Im x)/(4\pi)$. Step function factors are given by (13), (19) and by (22).

It follows from (18) that $|\omega(\{q\}_0, G)| \geq 1$ where $G$ is any one among the Schottky group transformations. If the multiplier $k_G$ of the $G$ transformation is small, then
\[ |\omega((q_0)_{G,G})|^2 \approx |(2\pi)^{-1} \ln |k_G|^2 \] and, therefore, \(|k_G| \leq e^{-3s/2}\). Seemingly, conditions (17) and (18) admit only small \(k_G\). In any case, it is accepted in the paper that all \(k_G\) are small enough so that all the Schottky group transformations are loxodromic and the \(|k_G| \approx 1\) multipliers do not contained in the fundamental region (17) and (18).

The integrand in (23) are calculated through vacuum correlators

\[ (X^N(t_1, t_2)) = -\delta^{NM} \hat{K}_{L,L'}(t_1, t_2; \{ q \}) \]

of the superfields \(X^N(t, \bar{t})\) of the matter \((N = 0, \ldots, 9)\) and through the correlator

\[ (C(t_1, t_2)B(t_2, t_3)) = -G_{gh}(t_1, t_2; \{ q \}) \]

of \((C, B)\)-ghosts where \(C\) is the vector supermultiplet and \(B\) is the \(3/2\)-tensor one. In (24) the ‘mostly plus’ metric is implied. The fields are normalized in correspondence with the action \(S\) given by

\[ S = \int \frac{d^2 x}{\pi} d\theta d\bar{\theta} [B\overline{\partial C} - \overline{B}\partial C - 2\overline{D}X^N DX^N]. \]

We consider the massless boson interaction amplitudes. Then the interaction \(V(t, \bar{t}; p; \zeta)\) vertex [13] is as follows

\[ V(t, \bar{t}; p; \zeta) = 4\zeta_{MN} [D(t)X^M(t, \bar{t})]\overline{[D(\bar{t})X^N(t, \bar{t})]} \exp[i(p_X^M(t, \bar{t})], \]

where \(p = \{p_M\}\) is 10-momentum of the interacting boson while \(\zeta_{MN}\) is its polarization tensor, \(p^M\zeta_{MN} = \rho^N\zeta_{MN} = 0\), and \(p_2 = 0\). The spinor derivative \(D(t)\) is defined in (8). The summation over twice repeated indexes is implied. We use the ‘mostly plus’ metric. The dilaton \(\zeta_{MN}\) tensor is given by the transverse Kronecker symbol \(\delta^M_N\).

### 4. Vacuum correlator of the matter superfields

As in [16], the correlator (24) is calculated through a super-holomorphic \(R_L(t, t'\{ q \})\) Green function which is changed under (5) as follows

\[ R_L(t^b, t'^b; \{ q \}) = R_L(t, t'; \{ q \}) + J_L(t'^b; \{ q \}; L), \quad R_L(t^a, t'; \{ q \}) = R_L(t, t'; \{ q \}), \]

where \(t^b = \Gamma_{a,r}(t), t'^a = \Gamma_{a,s}(t)\). The \(J_L(t; \{ q \}; L)\) functions have periods as follows

\[ J_L(t^b; \{ q \}; L) = J_L(t; \{ q \}; L) + 2\pi i\omega_{tr}(\{ q \}, L), \]

\[ J_L(t'^b; \{ q \}; L) = J_L^{(\nu)}(t; \{ q \}; L) + 2\pi i\delta_{rs}. \]

The vacuum correlator (24) is given by

\[ 4\hat{K}_{L,L'}(t, t'; t'^b, t'^b; \{ q, \bar{q} \}) = R_L(t,t'; \{ q \}) + \overline{R_L}(t,t'; \{ \bar{q} \}) + I_{LL'}(t, t'; t', t'^b; \{ q, \bar{q} \}), \]

\[ I_{LL'}(t, t'; t', t'^b; \{ q, \bar{q} \}) = \mathcal{I}_L(t, t'; \{ q, \bar{q} \}; L, L')\Omega_{LL'}((\{ q, \bar{q} \})L_L')^{-1}\mathcal{I}_L(t, t'; \{ q, \bar{q} \}; L, L'), \]

where

\[ \mathcal{I}_L(t, t'; \{ q, \bar{q} \}; L, L') = J_L(t, \{ q \}; L) + \overline{J_L}(t, \{ \bar{q} \}; \overline{L'}), \]

\[ \Omega_{LL'}((\{ q, \bar{q} \})L_L') = 2\pi i(\omega((\{ q, \bar{q} \}), L') - \omega(\{ q, \}, L)). \]
The dilaton-vacuum transition constant (the one-point amplitude) is given by (23) for \( m = 1 \). The \( \bar{I}_{L,L'}(t, \tilde{t}; \{ q, \tilde{q} \}) \) vacuum expectation of the single interaction vertex is as follows

\[
\bar{I}_{L,L'}(t, \tilde{t}; \{ q, \tilde{q} \}) = 2D(t)D(\tilde{t})\bar{I}_{L,L'}(t, \tilde{t}; t', \tilde{t}'; \{ q, \tilde{q} \})_{\lambda = \nu}.
\]  

(33)

Integrating it by parts and using relations (29), one obtains that the dilaton-vacuum transition constant is proportional to the vacuum amplitude.

At zeroth \((\mu, \nu_1)\)-parameters, \( R_L(t, t'; \{ q \}) \) is reduced to \( R_L(t, t'; \{ q \}) \) where

\[
R_L(t, t'; \{ q \}) = R_b(z, z'; \{ q \}) - \vartheta \vartheta R_{\alpha}(z, z'; \{ q \}).
\]

(34)

The boson Green function \( R_b(z, z'; \{ q \}) \) is as follows [13, 14, 16, 21]

\[
R_b(z, z'; \{ q \}) = \ln(z - z') + \sum_\Gamma \ln \left( \frac{|z - g_\Gamma(z')|}{|z - g_\Gamma(\infty)|} \right).
\]

(35)

It differs only in scalar zero mode from its usual expression [2]. The \( J_r(z; \{ q \}) \) scalar function at zeroth \((\mu, \nu)\) is given by [13, 14, 16, 21]

\[
J_r(z; \{ q \}) = \sum_\Gamma \ln \frac{z - g_\Gamma(u_r)}{z - g_\Gamma(v_r)},
\]

(36)

where the summation is performed over all the group products \( \Gamma \) except those whose rightmost is a power of \( g_r \). The fermion Green function \( R_{\alpha}(z, z'; \{ q \}) \) for the even spin structure \((l_1, l_2) \) can be written down as follows [16]

\[
R_{\alpha}(z, z'; \{ q \}) = \exp \left\{ \frac{1}{2} [R_b(z, z; \{ q \}) + R_b(z', z'; \{ q \})] - R_b(z, z'; \{ q \}) \right\} \frac{\Theta[L](z - z'\omega)}{\Theta[L](0\omega)},
\]

(37)

where \( \Theta[L](z - z'\omega) \) is the theta function with characteristics \( L = (l_1, l_2) \) and \( z \) is related to \( z' \) by the Jacobi mapping. For odd spin structures it is given in [16]. The function (37) is equal to the fermion Green function in [2].

If \((\mu_2, \nu_2)\)-parameters are presented, then vacuum correlators in the Neveu–Schwarz sector are obtained by a ‘naive’ supersymmetrization [14, 22] of relevant expressions at zeroth \((\mu_1, \nu_1)\). In the Ramond sector the genus-1 functions \( R_L^{(1)}(t_1, t_2) \) and the having periods scalar function \( J^{(1)}(t; \{ q \}) \) are expressed through the genus-1 functions \( R_b(z_1^{(s)}, z_2^{(s)}; \{ q \}) \), \( R_{\alpha}(z_1^{(s)}, z_2^{(s)}; \{ q \}) \) and \( J^{(1)}(z; \{ q \}) \) at \( \mu_1 = \nu_1 = 0 \) as follows

\[
R_L^{(1)}(t_1, t_2) = R_b(z_1^{(s)}, z_2^{(s)}; \{ q \}) - \vartheta \vartheta R_{\alpha}(z_1^{(s)}, z_2^{(s)}; \{ q \})
- \varepsilon'_1 \varepsilon'_2 \Xi(z_1; \{ q \}, L_0) + \vartheta \vartheta \varepsilon'_1 \Xi(z_1; \{ q \}, L_0) + \varepsilon'_1 \varepsilon'_2,
\]

(38)

\[
J^{(1)}(t; \{ q \}) = J^{(1)}(z^{(s)}; \{ q \}) = \ln \left( \frac{z^{(s)} - u_s}{z^{(s)} - v_s} \right).
\]

(39)

where \( z^{(s)} \) and \( \vartheta \) are related with \( z \) and \( \vartheta \) by (6), and

\[
\Xi(z_1; \{ q \}, L_0) = (z - z') R_{\alpha}(z, z'; \{ q \})|_{z_1 \to \infty}.
\]

(40)

The third and fourth terms on the right side of (38) hold the decreasing of \( K_L(t_1, t_2; \{ q \}) \) at \( z_1 \to \infty \) and at \( z_2 \to \infty \). The \( \varepsilon \varepsilon' \) term is added to have, in addition, the decreasing of \( [R_L^{(1)}(t_1, t_2) - \ln(z_1 - z_2)] \) at \( z_1 \to \infty \) and at \( z_2 \to \infty \).
The vacuum correlators on the higher genus supermanifolds has been given in [16] where the proportional to \((\mu_s, \nu_s)\) terms of the correlators are calculated through genus-1 functions. Now we express these terms through genus-\(n\) functions at zeroth \((\mu_s, \nu_s)\). In this case \(R_L(t, t'; \{q\})\) are represented as follows

\[
R_L(t, t'; \{q\}) = R_{bl}(z, t'; \{q\}) - \partial R_{fl}(z, t'; \{q\}),
\]

where \(R_{bl}(t, t'; \{q\})\) and \(R_{fl}(t, t'; \{q\})\) are found from equations as follows (the proof is planned elsewhere)

\[
\begin{align*}
R_{bl}(z, t'; \{q\}) &= R_0(z, z'; \{q\}_0) - \sum_i \int_{C_i} \partial_{\zeta_i} R_0(z, z; \{q\}_0) \varepsilon_i(z) \frac{dz_i}{2\pi i} R_{bl}(z, t'; \{q\}), \\
R_{fl}(z, t'; \{q\}) &= R_0(z, z'(q)_0) \vartheta + \sum_i \int_{C_i} \partial_{\zeta_i} R_0(z, z; \{q\}_0) \varepsilon_i(z) \frac{dz_i}{2\pi i} \partial_{\zeta_i} R_{fl}(z, t'; \{q\}).
\end{align*}
\]

The integration is performed in the positive direction along the \(C_s\) contour, which surrounds the \(C_u\) and \(C_v\) circles (4) and the cut (if \(h_s = 0\) between \(u_s\) and \(v_s\). Further,

\[
J_{i}(t; \{q\}; L) = J_{br}(z; \{q\}; L) + \partial J_{fr}(z; \{q\}; L).
\]

The equations for \(J_{br}(z; \{q\}; L)\) and \(J_{fr}(z; \{q\}; L)\) are as follows

\[
\begin{align*}
J_{br}(z; \{q\}; L) &= J_i(z; \{q\}_0) + \sum_i \int_{C_i} \partial_{\zeta_i} R_0(z, z; \{q\}_0) \varepsilon_i(z) \frac{dz_i}{2\pi i} J_{fr}(z; \{q\}; L), \\
J_{fr}(z; \{q\}; L) &= - \sum_i \int_{C_i} R_{bl}(z, z; \{q\}_0) \varepsilon_i(z) \frac{dz_i}{2\pi i} \partial_{\zeta_i} J_{br}(z; \{q\}; L).
\end{align*}
\]

The \(J_i(\{q\}_0)\) scalar function at zeroth \((\mu, \nu)\) is given by (36).

The period matrix is found to be

\[
2\pi i \omega_{mr}(\{q\}; L) = 2\pi i \omega_{mr}(\{q\}_0) + \sum_i \int_{C_i} \partial_{\zeta_i} J_{mr}(z; \{q\}_0) \frac{dz_i}{2\pi i} \varepsilon_i(z) J_{fr}(z; \{q\}; L).
\]

5. Ghost superfield correlator

The superfield \(C\) in (25) has discontinuity [16] under twists about non-contractible cycles. It is a peculiarity of the scheme [16] which, among other things, calculates the \(B\)-superfield zero-mode contribution to the integration measure. The Green function \(G_{gh}(t, t'; \{q\})\) in (25) is represented through the Green function \(G(t, t'; \{q\})\) that is changed under \(t \rightarrow t^\rho = (z^\rho_m | \vartheta^\rho)^{\rho}\) transformation (5) as follows \((p = a, b)\)

\[
G(t^\rho, t'; \{q\}) = Q_{t^\rho}^{-2}(t) G(t, t'; \{q\}) + \sum_{\chi} Y_{p,\chi}(t) \chi (t'; \{q\}),
\]

where summation is performed over \(\chi = (k, u, v, \mu, \nu)\). The \([G_{gh}(t, t'; \{q\}) - G(t, t'; \{q\})]\) difference is a linear combination [16] of the \(Y_{p,\chi}(t) \chi (t'; \{q\})\) terms where \(N_0\) and \(N_0'\) are assigned to those (32) parameters that are fixed by \(SL(2)\)-symmetry. A superconformal \(3/2\)-tensor \(\chi (t'; \{q\})\) has a singularity at \(z' \rightarrow \infty\). So it is not a zero mode. Further, \(Y_{p,\chi}\) is a polynomial of degree 2 in \((z, \vartheta)\) as follows
are expressed through polynomials \( Y_{p,N}^{(r)}(t) \). The fermion Green function \( G_{\text{f}}(x, y, t; \{ q \}) \) is given by [16]

\[
G_{\text{f}}(x, y, t; \{ q \}) = \frac{1}{[\pi]^9} \sum_{\Gamma} \exp\left[ \frac{i}{2} \left( \frac{1}{2} \int_{\Gamma} dT \right) \right] G_{\text{f}}(x, y, \Gamma; \{ q \}),
\]

where \( G_{\text{f}}(x, y, \Gamma; \{ q \}) \) is defined as

\[
G_{\text{f}}(x, y, \Gamma; \{ q \}) = \sum_{\alpha} \phi_{\alpha}(x) \phi_{\alpha}^*(y),
\]

and the rest polynomials can be found in [16].

All the \((\mu_s, \nu_s)\) parameters being equal to zero, \( G(t, t'; \{ q \}) \) is reduced to \( G(t, t'; \{ q \}) \) where

\[
G(t, t'; \{ q \}) = G_{\text{b}}(z, z'; \{ q \}) \phi(t) + \phi(t) G_{\text{b}}(z, z'; \{ q \}).
\]

The boson Green function \( G_{\text{b}}(z, z'; \{ q \}) \) is given by [16]

\[
G_{\text{b}}(z, z'; \{ q \}) = \sum_{\alpha} \phi_{\alpha}(z) \phi_{\alpha}^*(z'), \quad \{ q \} = \{ \sigma \}
\]

where \( \sigma \) is defined as

\[
\sigma = \sum_{\alpha} \phi_{\alpha}(z) \phi_{\alpha}^*(z'), \quad \{ q \} = \{ \sigma \}
\]

and the rest polynomials can be found in [16].
where

\[ \Phi^{(1)}_{\sigma, \gamma}(z; \{q\}_0) = \sum_g e^{\Omega_g(\sigma)} (e_g z + d_g)^2 (e_g z + d_g)^2. \]

\[ \Phi^{(2)}_{\sigma, \gamma}(z; \{q\}_0) = \sum_g e^{\Omega_g(\sigma)} (e_g z + d_g)^2 (e_g z + d_g)^2. \] (57)

The summation is performed over all the Schottky-group elements. It can be proved \[\text{[16]}\] that

\[ \Phi_{\gamma, \alpha}(z; \{q\}_0) = \sum_{\alpha, \mu, \nu} \widetilde{M}_{\alpha, \gamma}(\{\sigma\}) \chi_{\alpha}(z; \{q\}_0), \] (58)

\[ \widetilde{M}_{\alpha, \gamma}(\{\sigma\}) = \int_{C_{\gamma}} \Phi_{\gamma, \alpha}(z; \{q\}_0) Y_{\alpha}(z) \frac{dz}{2\pi i}. \] (59)

where \(C_{\gamma}\)-contour is defined as in (42).

In the calculation \[\text{[16]}\] of the integration measure an axillary Green function \(S_\sigma(t, t'; \{q\})\) is also used. If all \((\mu', \nu')\) are nullified, then

\[ S_\sigma(t, t'; \{q\}) = G_\sigma(z, z'; \{q\}_0) \theta' + i S^{(f)}_\sigma(z, z'; \{q\}_0) \] (60)

and

\[ S^{(f)}_\sigma(z, z'; \{q\}_0) = G_\sigma(z, z'; \{q\}_0) \]

\[ - \sum_{r=1}^n \sum_{\alpha_r} \int_{C_{\alpha_r}} G_\sigma(z, z''; \{q\}_0) \frac{dz''}{2\pi i} Y_{\alpha}(z) \psi_{\gamma, \alpha}(z'; \{q\}_0). \] (61)

where \(\alpha_r = (\mu_r, \nu_r)\) and \(C_{\gamma}\) is defined by (4). Furthermore,

\[ \Phi_{\gamma, \gamma}(z; \{q\}_0) = \sum_{N=\mu, \nu} M_{\gamma, N}(\{\sigma\}) \psi_{\gamma, N}(z; \{q\}_0), \] (62)

where

\[ M_{\gamma, N}(\{\sigma\}) = \int_{C_{\gamma}} \Phi_{\gamma, N}(z; \{q\}_0) \frac{dz}{2\pi i}. \] (63)

In [16] the proportional to \((\mu, \nu)\) terms of \(S_\sigma(t, t'; \{q\})\) are calculated through genus-1 functions. Now we present equations that calculate these terms through function \[\text{([60])}\] at zeroth \((\mu, \nu)\). In this case \(S_\sigma(t, t'; \{q\})\) is represented as

\[ S_\sigma(t, t'; \{q\}) = S^{(b)}_\sigma(z, t'; \{q\}_0) + i S^{(f)}_\sigma(z, t'; \{q\}_0). \] (64)

where the right side functions are calculated from equations as follows

\[ S^{(b)}_\sigma(z, t'; \{q\}_0) = G_\sigma(z, z'; \{q\}_0) \theta' + \sum_j \int_{C_j} \frac{dz'}{2\pi i} G_\sigma(z, z'; \{q\}_0) \varepsilon_j(z') S^{(f)}_{\sigma}(z', t'; \{q\}_0), \]

\[ S^{(f)}_{\sigma}(z, t'; \{q\}_0) = S^{(f)}_\sigma(z, t'; \{q\}_0) - \sum_j \int_{C_j} \frac{dz'}{2\pi i} S^{(f)}_{\sigma}(z, z'; \{q\}_0) \]

\[ \times [-2\varepsilon_j + \varepsilon_j(z') \partial_z] S^{(b)}_{\sigma}(z', t'; \{q\}_0), \] (65)

\(\varepsilon_j(z)\) being given by (7).
6. Integration measure

The integration measures (partition functions) $Z_{L,L'}({\{q, \bar{q}\}})$ in (23) are obtained from equations [16, 18] which provide the independence of the superstring amplitudes under local variations of the vierbein and of the gravitino field. Due to a separation in the right and left movers, the integration measure in (23) is as follows

$$Z_{L,L'}^{(n)}({\{q, \bar{q}\}}) = (4\pi)^5 \det \Omega_{L,L'}^{(n)}({\{q, \bar{q}\}}) Z_{L,L'}^{(n)}({\{q, \bar{q}\}}),$$

where $Z_{L,L'}^{(n)}({\{q\}})$ is a holomorphic function of $q$ and $\Omega_{L,L'}^{(n)}({\{q, \bar{q}\}})$ is given by (32). Furthermore,

$$Z_L^{(n)}({\{q\}}) = Z_{m,L}^{(n)}({\{q\}_0}) \bar{Z}_{gh,L}^{(n)}({\{q\}_0}) \gamma_L^{(n)}({\{q\}}) \prod_s (u_s - \nu_s - \mu_s)_{s}^{-1},$$

where $Z_{m,L}^{(n)}({\{q\}_0})$ is due to the matter fields at zeroth $(\mu, \nu)$-parameters and $\ln \gamma_L^{(n)}({\{q\}})$ is proportional to $(\mu, \nu)$-parameters. In this paper the entries of (67) are given in a more convenient form than in [16]. In particular, $Z_{m,L}^{(n)}({\{q\}_0})$ is represented as follows

$$Z_{m,L}^{(n)}({\{q\}_0}) = \Theta^5[\Gamma(0 \omega^{(0)}) \prod_{m=1}^{\infty} (1 - k_m)^{15},$$

where $\omega^{(0)} \equiv \omega({\{q\}_0})$ and $\Theta[\Gamma(0 \omega^{(0)})]$ is the theta-function whose characteristics are $L = (l_1, l_2)$. The product in (68) is calculated over those Schottky group multipliers $k$ which are not powers of the other ones. Furthermore,

$$\bar{Z}_{gh,L}^{(n)}({\{q\}_0}) = e^{-\pi \sum_{\sigma, \tau} \lambda_{\sigma, \tau} \omega^{(m)}_{\sigma, \tau} \det \bar{M}_L^{(n)}({\sigma, \tau}) [-\det \bar{M}_L^{(n)}(-{\sigma, \tau})]^{-1/2}$$

$$\times \left( \prod_s \left( \frac{(1 - k_s)^{2(-1)^{2s}2s - 1 + 2l_s}}{(1 + (-1)^{2s}2s \sqrt{k_s})^2 k_s^{3/2}} \right) \prod_{m=2}^{\infty} m \mathcal{P}_m^{(n)}(\sigma, k) \mathcal{P}_m^{(n)}(-{\sigma, k}) \right)^{-1},$$

where $\bar{M}_L^{(n)}({\sigma, \tau})$ is given by (59), and

$$\mathcal{P}_m^{(n)}(\sigma, k) = \frac{1 - k^{-1/2} \omega(\sigma, k)}{1 - k},$$

where $\Omega(\sigma, k) \equiv \Omega^{(n)}_{(kl)}({\{\sigma\}_l})$ and $\Omega^{(n)}_{(kl)}({\{\sigma\}_l})$ is given by (53) for the Schottky group product $\Gamma_{kl}$ whose multiplier is $k$. The right side of (69) is independent of the choice the $\sigma$-set [16]. In particular, it is unchanged under the $\sigma_i \rightarrow -\sigma_i$ replacement of each $\sigma_i$ from the questioned $\sigma$-set.

Nominally (68) can be re-written down as (the proof is planned elsewhere)

$$\bar{Z}_{gh,L}^{(n)}({\{q\}_0}) = e^{\pi \sum_{\sigma, \tau} \lambda_{\sigma, \tau} \omega^{(m)}_{\sigma, \tau} \prod_{m=1}^{\infty} m \mathcal{P}_m^{5(n)}(\sigma, k) \mathcal{P}_m^{5(n)}(-{\sigma, k})$$

but the product over $k$ for $m = 1$ might be divergent for $n > 1$ when the Ramond handles are presented. Furthermore,

$$\gamma_L^{(n)}({\{q\}}) = \exp[\text{trace ln}(1 + \bar{K}) + \text{trace ln}(1 + \bar{S}) + \text{trace ln}(1 + \bar{M})],$$

where $\bar{K}$ and $\bar{S}$ are formed by the set of $\bar{K}_i$ and, respectively, of $\bar{S}_r$ operators. The operators perform the integration over $z'$ along the $C_z$ contour, their kernels are $\bar{K}_i(z, z')dz'/\left(2\pi i\right)$ and, respectively, $\bar{S}_r(z, z')dz'/\left(2\pi i\right)$ where
\[\tilde{S}_s(z, z') = \sum_r \int_{C_r} \frac{dz}{2\pi i} G_0(z, z'; \{q\}_0) \tilde{S}_s(z, z'; \{q\}_0) [-2\varepsilon_s' + \varepsilon_s(z') \partial_1]. \tag{73}\]

\[\tilde{K}_s(z, z') = \sum_r \int_{C_r} \frac{dz}{2\pi i} \partial_1 \tilde{R}_s(z, z'; \{q\}_0) \tilde{S}_s(z, z'; \{q\}_0) \varepsilon_s'(z'). \tag{74}\]

The \(M_{am}\) entry of the \(M\) matrix is given by

\[M_{am} = \sum_j \int_{C_a} \frac{dz'}{2\pi i} \left( \alpha_a \right)(z'; \{q\}_0) [-2\varepsilon_s' + \varepsilon_s(z') \partial_1] \int_{C_a} \tilde{S}_a^{(a)}(z', t'; \{q\}_0) \varepsilon_s'(z') \alpha_m \varepsilon_s(\alpha_m) \wedge \varepsilon_s(\alpha_s) \delta_{\alpha_m} \delta_{\varepsilon_m}(t), \tag{75}\]

where \(\alpha_a\) runs the \((\mu_a, \nu_a)\) set, \(\beta_m\) runs the \((\mu_m, \nu_m)\) set and \(\chi_{\alpha_a}(z'; \{q\}_0)\) is defined by (58). If \(j = m\), then the integration contour over \(z'\) is situated inside the integration contour over \(z_1\).

7. Uncertainties of the amplitude

The calculation of the amplitude (23) includes Grassmann integrations together with integrations over local variables. If the integrand has singularities the integrations are ambiguous with respect to a non-split replacement of the integration variables as it is seen for an easy integral

\[I(\alpha) = \int \frac{dx dy dz d\beta}{|z - \alpha\beta|^p} \theta(1 - |z|^2), \tag{76}\]

where \(z = x + iy\), \((\alpha, \beta)\) are Grassmann variables and \(p\) characterizes the strength of the singularity. For the sake of simplicity we bound the integration region by \(|z|^2 \leq 1\). Equation (76) is re-written as follows

\[I(\alpha) = \int \frac{dx dy}{|z|^p} \theta(1 - |z|^2) d\alpha d\beta + p^2 \int \frac{dx dy}{4|z|^{p+2}} \theta(1 - |z|^2) d\alpha d\beta. \tag{77}\]

The first integral is equal to zero since its integrand does not contain Grassmann variables. The second integral is divergent at \(z = 0\), if \(\text{Re} p > 0\). On the other side, in (76) one can introduce the \(\tilde{z} = z - \alpha\beta\) variable instead of \(z\). Then the Grassmann variables will be present only in the step function \(\theta(|\tilde{z} + \alpha\beta|^2)\). Grassmann integrations being performed, the integral is reduced to the integral along the circle \(|z|^2 = 1\). Thus for any \(p\) the result is finite as follows

\[I(\alpha) = - \int \frac{dx dy}{|z|^p} \alpha \beta \delta(|z|^2 - 1) \left[ \frac{d\beta}{dz^2} (|z|^2 - 1) \right] = - \frac{\pi p}{2}. \tag{78}\]

So (76) depends on the integration variables. This ambiguity arises because the integrand is expanded in a series over the Grassmann variables in the singular point \(z = 0\). If the integral is convergent, then it does not depend on the choice of the integration variables provided that a transition to new integration variables remains it to be convergent. If the amplitude is divergent, a cutoff parameter appears. In this case the amplitude becomes to depend on \(\{N_0\}\) set (12) that falls the theory. So the ambiguity is resolved due to \(SL(2)\)-symmetry.

The integration measure (69) has a singularity at \(k_s \to 0\), (as it has been discussed, the \(k_s \to 1\) multipliers are not contained in the integration region). The leading singularity is \(1/(k_s^3 \ln^3[k_s])\) when \(l_2 = 0\), but it is canceled in the sum over \(l_2 = 1/2\) and \(l_2 = 0\) spin structures. The resulted singularity is \(1/(k_s^3 \ln^3[k_s])\) that is the integrable singularity. The same singularity arises at \(l_2 = 1/2\). So, the superstring amplitude is integrable at \(k_s \to 0\),
Due to the singularity at \( z = z' \) in the Green function, the vacuum expectation of the interaction vertices in (23) is singular in nodal regions where some number \( m_1 > 1 \) of the interaction vertices go to the same point \( z_0 \). If a number of \((u_s, v_s)\)-pairs also move to \( z_0 \) then in addition the Riemann surface degenerates. If \( 1 < m_1 < (m - 1) \) and \( m > 3 \), then the strength of the singularity depends on the 10-energy invariant corresponding to the given reaction channel. In this case the integral is calculated [23] for those energies where it is convergent (below the reaction threshold). Being analytically continued to the over-threshold energies, the amplitude receives singularities that are required by the unitarity equations. So only configurations where \( m_i = 0 \), \( m_i = 1 \), \( m_i = (m - 1) \) and \( m_i = m \) might lead to divergences in the amplitude.

From (68), (69) and (71), the term in (66) leading at \( u_s \to v_s \), is proportional to the genus-1 integration measure \( Z_{L_1\ell', L_2\ell} (k_s, \tilde{k}_s) \)

\[
Z_{L_1\ell', L_2\ell} (k_s, \tilde{k}_s) = \left( -\frac{2\pi}{\ln|k_s|} \right)^3 \begin{cases} (-1)^{2l_1+2l_2+2l_3+2l_4} \Theta^4[l_{1s}, l_{2s}] \left( \frac{\ln k_s}{2\pi i} \right) \\ \times \Theta^4[l'_{1s}, l'_{2s}] \left( \frac{\ln \tilde{k}_s}{2\pi i} \right) \times |k_s^{-1} \left( \prod_{m} (1 - k_s^m) \right)|^{24}. \end{cases}
\]

(79)

If no interaction vertices are nearby \( v_s \), then the integrand of (23) is approximated by the \( A_{n, m}^{1,(\alpha)} (\{q, \tilde{q}\}; \{t, \tilde{t}\}) \) function as follows

\[
A_{n, m}^{1,(\alpha)} (\{q, \tilde{q}\}; \{t, \tilde{t}\}) = \frac{1}{|u_s - v_s - \mu_l|^{2\ell_1}} \sum_{L_1\ell_1, L_2\ell_2} \Theta^{1,\ell_1} \left( \frac{\ln k_s}{2\pi i}, -\frac{\ln \tilde{k}_s}{2\pi i} \right) Z_{L_1\ell_1, L_2\ell_2} (k_s, \tilde{k}_s) \\
\times \left[ A_{n-1, m}^{1,(\alpha)} (\{q, \tilde{q}\}; \{t, \tilde{t}\}) + \ldots \right],
\]

(80)

where \( A_{n-1, m}^{1,(\alpha)} (\{q, \tilde{q}\}; \{t, \tilde{t}\}) \) is the integrand for the \((n - 1)\)-loop, \( m \)-point amplitude. The step function \( \Theta^{1,\ell_1} \left( \frac{\ln k_s}{2\pi i}, -\frac{\ln \tilde{k}_s}{2\pi i} \right) \) bounds the fundamental region of the genus-1 modular group [20]. The ‘dots’ in (80) codify the terms which depend on \((u_s|k_s)\), on \((v_s|\tilde{k}_s)\) and on their complex conjugate. Being summed either over \( L_1 \) or over \( L_2' \), the \( Z_{L_1\ell_1, L_2\ell_2} (k_s, \tilde{k}_s) \) function (79) is nullified due to the Riemann identities. It is the well known result [23] for the vacuum function on the torus. If either \( L_1 \) or \( L_2' \) is odd, then (79) is equal to zero by itself. Therefore, the singularity in (80) at \( u_s \to v_s \) might appear only because of those ‘dots’ terms in (80) which depend on both \( L_1 \) and \( L_2' \) genus-1 spin structures.

The discussed terms in (68) and in (37) are of the order of \((u_s - v_s)^2\). So they are too small to degenerate the divergence in the amplitude. If \( l_{1s} = 1/2 \) and \( l_{2s} = 0 \), then linear in \((u_s - v_s)\) terms appear in (69), but they are nullified in the amplitude due to the symmetry of (69) with respect to the \( \sigma \to -\sigma \) replacement. The factor (72) in (67) has no singularities at \( u_s = v_s \) due to the small \((|u_s - v_s|)\) size of the \( C_1 \)- contour. Therefore, the discussed configuration does not originates the amplitude divergence.

If, in addition, a single interaction vertex coordinate \( z \) moves to \( v_s \), then due to the \( \hat{X}_{L'\ell'} (t, \tilde{t}; t', \tilde{t}'; \{q\}) \) correlator (30), additional terms might contribute to the coefficient at the discussed singularity. Under the discussed conditions the correlator (30) is given by equation (A.7) of appendix. In particular, in this case the genus-1 superholomorphic function \( R_k (t, t'; \{q\}) \) in (30) is as follows
\[ R_L(t, t'; \{ q \}) = R_{\text{bd}}(v_s, t'; \{ q \}_{n-1}) - \text{ln}(z^{(n)} - z^2) \big| z \to \infty \]
\[ \times \left[ \partial_{t_s} R_{\text{bd}}(v_s, t'; \{ q \}_{n-1}) + \left( \frac{\mu_s - \nu_s}{u_s - v_s} \right) R_{\text{bd}^{(n)}}(v_s, t'; \{ q \}_{n-1}) \right] \]
\[ - \varepsilon_t(z^{(n)}) R_{\text{bd}^{(n)}}(v_s, t'; \{ q \}_{n-1}) = \frac{\partial}{\partial z} \Xi(z^{(n)}; \{ q \}_0, L_s), \quad (81) \]

\[ \Xi(z^{(n)}; \{ q \}_0, L_s) = \left[ 1 - \frac{\mu_s \nu_s}{2(u_s - v_s)} \right] \left( 1 + z' \Xi(z^{(n)}; \{ q \}_0, L_s) R_{\text{bd}^{(n)}}(v_s, t'; \{ q \}_{n-1}) \right) \]
\[ + \left( \Xi(z^{(n)}; \{ q \}_0, L_s) - 1 \right) (R_{\text{bd}^{(n)}}(v_s, t'; \{ q \}_{n-1}) \Xi(z^{(n)}; \{ q \}_0, L_s) \right) \]
\[ + \varepsilon_t(z^{(n)}) \partial_{t_s} R_{\text{bd}^{(n)}}(v_s, t'; \{ q \}_{n-1})), \quad (82) \]

where \( \Xi(z^{(n)}; \{ q \}_0, L_s) \) is defined by (40), \( R_{\text{bd}}(z^{(n)}; z^2; \{ q \}_0) \) is the same as in (38), \( \varepsilon_t(z^{(n)}) \) is defined by (7), and

\[ \Xi(z; \{ q \}_0, L_s) = (z - z') \left[ R_{\text{bd}}(z, z'; \{ q \}_0) \right] \Xi(z; \{ q \}_0, L_s). \quad (83) \]

The functions \( R_{\text{bd}^{(n)}}(v_s, t'; \{ q \}_{n-1}) \) and \( R_{\text{bd}^{(n)}}(v_s, t'; \{ q \}_{n-1}) \) are defined by (28) and (41) on the genus-(n-1) supermanifold obtained by removing of the handle \( 's \) from the questioned genus-n supermanifold. The first term on the right-hand side of (81) is independent of \( t \) and of \( L_s \). This term is inessential for the matter under consideration.

From (35), the ‘body’ of the second term on the right-hand side of (81) is proportional to \((u_s - v_s)\). So this term has no pole at \((u_s - v_s)\). In addition to the function (81), the vacuum correlator (A.7) contains the term expressed through the function (A.9). Like the second term on the right-hand side of (81), its ‘body’ is proportional to \((u_s - v_s)\) and has no pole at \((u_s = v_s)\). The function (A.9) in the genus-1 case is independent of \( L_s \) because the scalar function (39) does not depend on \( L_s \). So the \( L_s \)-dependent term appear only due to the last term on the right-hand side of (81). To integrate over \( d^2 t \), it is convenient to introduce \( t'^{(l)} = (z^{(l)})^{(l)^{(l)}} \) as the integration variable instead of \( t = \Xi(z) \) and to bound the integration region by the ‘circles’ (10). The integration over \( \partial^{(l)} \) being performed, the \( L_s \)-dependent terms in the vacuum expectation of the vertices (27) are presented by the expression bilinear in the product of functions (82) and its complex conjugate. The expression contains the factor \(|u_s - v_s - \mu_s \nu_s|^2/|u_s - v_s|^2\) that reduces the \( 1/|u_s - v_s - \mu_s \nu_s|^2 \) singularity in the amplitude to the \( 1/|u_s - v_s|^2 \) one. The \( L_s \)-dependent contributions to the questioned expression are linear and bilinear in \( \Xi(z^{(n)}; \{ q \}_0, L_s) = 1 \), in \( \Xi(z; \{ q \}_0, L_s) \) and in their complex conjugate. Due to the Riemann identities, these terms disappear in the sum over spin structures that agrees with the nullification of the one-point, two-point and vacuum functions on the torus [23].

Moreover, the \( d^2 z^{(o)} \) integration reduces the considered \( 1/|u_s - v_s|^2 \) singularity to the singularity \( \ln|u_s - v_s|^2 \) that is integrable in the every given spin structure. Indeed, if \(|z^{(o)} - v_s| >> |u_s - v_s|\); then \( \Xi(z^{(o)}; \{ q \}_0, L_s) \sim (u_s - v_s)^2/(z^{(o)} - v_s)^2 \). This case the \( |z^{(o)} - v_s| \sim |u_s - v_s| \) region mostly contributes to the integral over \( d^2 z^{(o)} \). So the \( 1/|u_s - v_s|^2 \) singularity is compensated by the \( \sim |u_s - v_s|^2 \) smallness of the integration region. As the result, the discussed singularity disappears. The region \(|z^{(o)} - v_s| >> |u_s - v_s| \) is important only in the integration of the term including functions (83) and \( \varepsilon_t(z^{(n)}) \Xi(z^{(n)}; \{ q \}_0, L_s) \) times their complex conjugate. The above-mentioned terms are \( \sim |u_s - v_s|^2/|z^{(o)} - v_s|^2 \) at \(|z^{(o)} - v_s| >> |u_s - v_s| \). The integration of this expression over the \(|z^{(o)} - v_s| >> |u_s - v_s| \)
region gives $\ln|u_1 - v_1|^2$, as was stated in the beginning of this paragraph. By aforesaid, this singularity disappears in the sum over spin structures.

If a number $m_1$ of the vertices moving to $v_s$ is $m_1 = m$, then the amplitude is proportional to the genus-$(n-1)$ vacuum amplitude. As is verified in the next section, the vacuum amplitude of the arbitrary genus is nullified and therefore in the $m_1 = m$ configuration the singularity in the integrand (23) does not appears. If $m_1 = (m - 1)$, and the rest vertex corresponds to the dilaton emission, then the amplitude is again proportional to the genus-$(n-1)$ vacuum amplitude. Otherwise the divergence is absent in the every spin structure just as it appears in the $m_1 = 1$ case. Moreover, the divergence does not appear when $|u_1 - v_1| \ll \rho \to \infty$ where $\rho$ is the minimal distance between $u_1$ or $v_1$ and any other limiting point of the Schottky group. Indeed, such a configuration is reduced by a relevant $L(2)$-transformation to the above-considered case $u_1 \to v_1$.

So, integrand in (23) has no non-integrable singularities. In this case divergences in the superstring amplitudes might appear due to a degeneration of the Riemann surface into the sum of the genus > 1 surfaces as it demonstrated on figure 1.

Such degenerate configurations do not happen in two- and three-loop amplitudes because three limiting points of the Schottky group are fixed due to $L(2)$-symmetry. So an absence of non-integrable singularities in the integrand ensures the finiteness of two- and three-loop amplitudes, but not the finiteness of amplitudes with more than three loops. It is verified in the next section that the integration over degenerated configurations of higher genuses does not generate divergences and, therefore, divergences in amplitude (23) do not appear.

8. Finiteness of the superstring amplitudes

As was said in the end of the previous section, we now discussed the moduli configuration where the genus-$n$ supermanifold becomes degenerate into a sum of the lower genus supermanifolds. In this case the genus-$n$ super-Schottky group is degenerate into a group product of the lower genus sub-groups. This happens when all the distances between the local
limiting points of the given sub-group go to zero. Another configuration is where distances between the local limiting points of the given sub-group all are not so large as the minimal distance $\rho \rightarrow \infty$ between the above-mentioned points and the rest limiting points of the genus-$n$ group. Both the configurations correspond to the same degeneration of the genus-$n$ supermanifold into the lower genus supermanifolds. Indeed, they are reduced to each other by means of a relevant $L(2)$ transformation. The spin structure of every one from the lower genus supermanifolds is supposed to be even. Otherwise its contribution to the amplitude disappears due to the theta-constant factor in (68).

If the genus-$n$ supermanifold degenerates into a sum of the genus-$n_1$ supermanifold and of the genus-$(n - n_1)$ one, and no interaction vertices are nearby $v_p$, then the integrand $A_{n,m}(\{q_i, \bar{q}_j\}, \{t_{ij}, \bar{t}_{ij}\})$ of (23) is factorized as follows

$$A_{n,m}(\{q, \bar{q}\}, \{t, \bar{t}\}) = Z_n(\{q, \bar{q}\})[A_{n-n_1,m}(\{q, \bar{q}\}_{n-n_1}, \{t, \bar{t}\}) + \ldots],$$

where $Z_n(\{q, \bar{q}\})$ depends only on moduli variables of the genus-$n_1$ supermanifold and $A_{n-n_1,m}(\{q, \bar{q}\}_{n-n_1}, \{t, \bar{t}\})$ is the integrand in (23) for the $(n - n_1)$-loop, $m$-point amplitude. The ‘dots’ encode the correction terms Generally, they are functions of both the genus-2 supermanifold variables and the genus-$(n - 2)$ supermanifold ones. In the explicit form $Z_n(\{q, \bar{q}\})$ is

$$Z_n(\{q, \bar{q}\}) = \sum_{L_1, L_2} Z_n^{(n)}(\{q, \bar{q}\}) \mathcal{O}^{(n)}(\omega_{L_1}, \omega_{L_2}) \mathcal{B}^{(n)}_{L_1, L_2}(\{q, \bar{q}\})$$

where $(L_1, L_2)$ is the spin structure of the genus-$n_1$ supermanifold. Other definitions are the same as in (23) for $n = n_1$. Unlike (23), the function (85) does not contain the factor (12).

We begin with the case when the genus-$n$ supermanifold degenerates into a sum of the genus-2 supermanifold and of the genus-$(n - 2)$ one. Then the integrand of the amplitude is given by (84) for $n_1 = 2$. We suppose that the genus-2 supermanifold is formed by $s$- and $p$- handles (figure 1). For definiteness, $u_s, v_s$ and $u_p$ limiting points are supposed to move to $v_p$.

We show that this configuration does not generate divergences in the amplitude (23) and that the genus-2 vacuum amplitude vanishes.

In doing so, we consider the $\mathcal{A}(U_p, \overline{U}_p, V_p, V_p, u_s, \pi_s)$ function$^2$ that is obtained by integrating (84) for $n_1 = 2$ over $V = (s, \tau_s), \mu_s$ and their complex conjugate. So

$$\mathcal{A}(U_p, \overline{U}_p, V_p, V_p, u_s, \pi_s) = \int Z_2(\{q, \bar{q}\}) d\bar{\tau}_s d\mu_s d\bar{\mu}_s d\bar{\bar{\tau}}_s d\bar{\bar{\mu}}_s$$

As far as the genus-1 function is nullified, the integral is convergent at $|v_s - v_p| \gg |u_p - v_p|$. So there is no necessity to introduce a cutoff restricting the size of the integrated configuration.

Accordingly to the previous section, the integral is convergent at the $v_s = u_s$ point. We verify that the integral is really equal to zero. For this purpose we consider the integral

$$\mathcal{A}(U_p, V_p, \overline{U}_p, \overline{V}_p, U_s, \overline{U}_s) = \int Z_2(U_s, U_p, \overline{U}_s, \overline{U}_p, V_s, V_p, \overline{V}_s, \overline{V}_p) d\bar{\tau}_s d\mu_s d\bar{\bar{\tau}}_s d\bar{\bar{\mu}}_s$$

without the integration with respect to $d\mu_s d\bar{\mu}_s$. It is evident that

$$\mathcal{A}(U_p, V_p, \overline{U}_p, \overline{V}_p, U_s, \overline{U}_s) = \int \mathcal{A}(U_p, V_p, \overline{U}_p, \overline{V}_p, U_s, \overline{U}_s) d\mu_s d\bar{\mu}_s$$

Like (86), the integral (87) is convergent and does not require any cut-off. In this case (87) has an explicit symmetry under those $SL(2)$ transformations which are independent of the spin structure. In particular, it is invariant under the super-boost transformation of its arguments.

$^2$ For the sake of brevity, an explicit dependence on the Schottky group multipliers is omitted.
The super-boost transformation $(z|\bar{z}) \to (\xi|\bar{\xi})$ is defined as follows
\[ z = \bar{z} + z_0 + \bar{v}_0, \quad \bar{v} = \bar{\bar{v}} + \bar{v}_0, \quad (89) \]
where $z_0$ and $\bar{v}_0$ are the transformation parameters. Therefore, $(87)$ depends on differences between $u_s$, $u_p$ and $v_p$. Then $\mathcal{A}(U_p, V_p, U_\bar{p}, V_\bar{p}, U_s, U_\bar{s})$ can be written down as follows
\[ \mathcal{A}(U_p, V_p, U_\bar{p}, V_\bar{p}, U_s, U_\bar{s}) = \sum_{m=-1}^{3} \sum_{n=-1}^{3} V_{mn} \psi_{mn}(r_s, w_p, \bar{r}, \bar{w}_p), \quad (90) \]
where $\psi_{mn}(r_s, w_p, \bar{r}, \bar{w}_p)$ are certain functions of $r_s = (u_p - u_s)$ and of $w_p = (u_p - v_p = \mu_p v_p)$, and $V_1 = \mu_s\mu_p v_p$, $V_2 = (\mu_p - \mu_s)$, $V_3 = (\nu_p - \mu_s)$. In this case
\[ \mathcal{A}(U_p, V_p, U_\bar{p}, V_\bar{p}, U_s, U_\bar{s}) = -\sum_{m=-1}^{3} \sum_{n=-1}^{3} U_{mn} \psi_{mn}(r_s, w_p, \bar{r}, \bar{w}_p), \quad (91) \]
$\psi_{mn}(r_s, w_p, \bar{r}, \bar{w}_p)$ being identical to that in $(90)$, and
\[ U_1 = \mu_p v_p, \quad U_2 = \mu_3 = -1. \quad (92) \]
The right-hand part of $(90)$ to be invariant under transformation $(89)$ with $\bar{v}_0 = 0$ requires that
\[ \psi_{1n}(r_s, w_p, \bar{r}, \bar{w}_p) = \partial_{u_s} \psi_{3n}(r_s, w_p, \bar{r}, \bar{w}_p), \quad \psi_{mn}(r_s, w_p, \bar{r}, \bar{w}_p) = \partial_{w_p} \psi_{mn}(r_s, w_p, \bar{r}, \bar{w}_p). \quad (93) \]
Since the genus-1 function is nullified, it follows from the equations of section 4 that at $\mu_s = \nu_s = 0$ only the region $|v_i - u_p|^2 \sim |v_p - u_p|^2$ could contribute to $(87)$. Since the considered moduli configuration $(u_p$ and $v_p$ both move to $v_p$ while $u_s$ is fixed, figure 2) does not correspond a degenerate surface, it is reasonable to expect that the integration over this region does not generate divergences in the amplitude that is explicitly verified as follows.

Indeed, due to the small size of the integration region, the function
\[ \mathcal{A}_0(u_p, v_p, \bar{u}_p, \bar{v}_p, u_s, \bar{u}_s) \equiv \mathcal{A}(U_p, V_p, U_\bar{p}, V_\bar{p}, U_s, U_\bar{s})|_{v_p = 0} \quad (94) \]
has no a singularity at \( u_p = v_p \). In this case \( \psi_{m\ell}(r_s, w_p, \bar{r}_s, \bar{w}_p) \) at both \( m = 1 \) and \( n = 1 \) has no a singularity at \( u_p = v_p \) as well as at \( w_p = 0 \). Then, from (93), the \( \psi_{1l}(r_p, w_p, \bar{r}_p, \bar{w}_p) \) and \( \psi_{1m}(r_s, w_p, \bar{r}_s, \bar{w}_p) \) functions also have no singularity as at \( u_p = v_p \), so at \( w_p = 0 \). Therefore, function (94) has no singularity at \( u_p = v_p \) and at \( w_p = 0 \).

Also the singularity does not appear when \( \tilde{\mathcal{A}}(U_p, V_p, \nabla_p, \bar{V}_p, u_s, \pi_s) \) is integrated with respect to the \( u_s \) positions over the region where \( u_p, v_s \), and \( u_s \) all move to \( v_p \). Indeed, due to the smallness of this region, the singularity does not appear in the integral of \( \psi_{m\ell}(r_s, w_p, \bar{r}_s, \bar{w}_p) \) with both \( m = 1 \) and \( n = 1 \). If \( \psi_{1l}(r_p, w_p, \bar{r}_p, \bar{w}_p) \) and \( \psi_{1m}(r_s, w_p, \bar{r}_s, \bar{w}_p) \) are integrated, this region does not contribute to the integral the \( u_s \) positions because \( \psi_{1l}(r_p, w_p, \bar{r}_p, \bar{w}_p) \) and \( \psi_{1m}(r_s, w_p, \bar{r}_s, \bar{w}_p) \) each is the derivative (93) with respect to \( u_s \). The ‘dots’ terms in (84) have no singularity at \( u_p = v_p \) for the same reasons as in the genus-1 case which has been considered in the previous section. Therefore, the considered configuration generates no singularity at \( u_p = v_p \) in the amplitude.

Further, \( \tilde{\mathcal{A}}(U_p, V_p, \nabla_p, \bar{V}_p, u_s, \pi_s) \) has the symmetry under the \( SL(2) \) transformation

\[
\tilde{f}(z) = f(z - \hat{f}(z))d\varepsilon_p(z), \quad \hat{f} = \sqrt{f'(z)}\left(1 + \frac{1}{2}\varepsilon_p(\zeta)\right)\tilde{f}'(z) - \varepsilon_p(z)
\]

that reduces \( \mu_p \) and \( \nu_p \) to zero, \( u_p, v_p \), and \( u_s \) being unchanged. In this case

\[
\tilde{f}(z) = z + \frac{(z - u_p)(z - v_p)}{(u_s - u_p)(u_s - v_p)}\mu_s\varepsilon_p(u_s).
\]

Being invariant under this transformation, \( \tilde{\mathcal{A}}(U_p, V_p, \nabla_p, \bar{V}_p, u_s, \pi_s) \) is expressed through the same function at \( \mu_p = \nu_p = \bar{\mu}_p = \bar{\nu}_p = 0 \) as follows

\[
\tilde{\mathcal{A}}(U_p, V_p, \nabla_p, \bar{V}_p, u_s, \pi_s) = \left|1 - \frac{\mu_p\nu_p}{v_p - u_p}\right|^2\tilde{\mathcal{A}}_0(u_p, v_p, \bar{\pi}_p, \bar{\bar{\pi}}_p, u_s, \bar{\pi}_s),
\]

where \( \tilde{\mathcal{A}}_0(u_p, v_p, \bar{\pi}_p, \bar{\bar{\pi}}_p, u_s, \bar{\pi}_s) \) is defined by (94). To obtain (97), the integrand in (88) is represented as follows

\[
\tilde{\mathcal{A}}(U_p, V_p, \nabla_p, \bar{V}_p, U_s, \bar{U}_s)\,d\mu_p\,d\bar{\mu}_p = \left|d\mu_p\,d\bar{\mu}_p\tilde{\mathcal{A}}(U_p, V_p, \nabla_p, \bar{V}_p, U_s, \bar{U}_s)[\tilde{H}(U_p, V_p, U_s)]^2\right| \times [\tilde{H}(U_p, V_p, U_s)]^{-2},
\]

where \( \tilde{H}(U_p, V_p, U_s) \) is given by (12). The transformation (95) reduces the expression inside the square brackets reduces to the expression of the same form with \( \mu_p = \nu_p = 0 \) and where \( \mu_s \) is replaced by \( \bar{\mu}_s \). By dimension reasons, this expression is proportional to \( \bar{\mu}_s\bar{\nu}_s \). So \( \tilde{H}(U_p, V_p, U_s) \) in the expression behind the square brackets can be replaced by \( (u_p - u_s)(u_s - v_p)\left|1 - \mu_p\nu_p/(u_p - v_p)\right|\}. \) Thereupon the integration over \( \mu_s \) and \( \bar{\pi}_s \) being performed, the relation (97) appears.

By aforesaid, the left-hand side of (97) is finite at \( u_p \to v_p \). Therefore, \( \tilde{\mathcal{A}}_0(u_p, v_p, \bar{\pi}_p, \bar{\bar{\pi}}_p, u_s, \bar{\pi}_s) \) is equal to zero at \( u_p = v_p \).

Due to \( L(2) \) symmetry, \( |(u_p - u_s)(u_s - v_p)|^2\tilde{\mathcal{A}}_0(u_p, v_p, \bar{\pi}_p, \bar{\bar{\pi}}_p, u_s, \bar{\pi}_s) \) does not depend on \( u_{ps}, v_{ps} \), and \( u_{ts} \). Then \( \tilde{\mathcal{A}}_0(u_p, v_p, \bar{\pi}_p, \bar{\bar{\pi}}_p, u_s, \bar{\pi}_s) \) is equal to zero identically in its arguments and the contribution to the amplitude of the discussed configuration disappears. At the same time, it follows from (23), (85) and (86) that the genus-2 vacuum amplitude \( A_{2,0} \) is given as
\begin{equation}
A_{2,0} = \int [(u_p - v_p)(u_s - v_p)]^2 \tilde{A}_0(u_p, v_p, p_r, p_s, u_s, \pi_s)^2 d^2 k_p d^2 k_s.
\end{equation}

Therefore, the vacuum genus-2 amplitude vanishes. The nullification of the vacuum amplitude arises once the vacuum function was integrated over those limiting points of the genus-2 super-Schottky group which are not fixed by \textit{SL}(2)-symmetry. The local in the moduli space vacuum function is not nullified.

If a single interaction vertex coordinate \( z_a \) also moves to \( v_p \), then the amplitude integrand is obtained from (84) by a modification of the \( \tilde{A}_{n-m,m}(\{q, \bar{q}\}_{(a-n)}, \{t_j, \bar{t}_j\}) \) function as follows
\begin{equation}
\tilde{A}_{n-m,m}(\{q, \bar{q}\}_{(a-n)}, \{t_j, \bar{t}_j\}) \rightarrow \langle V_a(\{t_j\}, \{q, \bar{q}\}) \rangle \tilde{A}_{n-m,m}(\{q, \bar{q}\}_{(a-n)}, \{t_j, \bar{t}_j\})_{j=\alpha},
\end{equation}
where \( \langle V_a \rangle \) is that part of \( \langle \prod_{a=1}^m V(t, \bar{t}; p_a; \zeta^{(a)}) \rangle \) in (23) which depends on \( t_a = (z_a)(\bar{z}_a) \). Correspondingly, \( \tilde{A}(U_p, V_p, U_p, V_p, U_p, U_p) \) in (87) is replaced by \( \tilde{A}(U_p, V_p, U_p, V_p, U_p, U_p) \) where
\begin{equation}
\tilde{A}(U_p, V_p, U_p, V_p, U_p, U_p) = \int \tilde{Z}_2(U_p, U_p, U_p, V_p, V_p, V_p) \times \langle V_a \rangle d z_a d \bar{z}_a d \bar{z}_a d^2 \bar{v}_i d^2 \bar{v}_j d^2 \bar{v}_k.
\end{equation}

Then function (86) is replaced by \( \tilde{A}(U_p, U_p, V_p, V_p, U_p, \bar{U}_p) \) which is obtained by the integration over \( \mu_s \) and \( \bar{\mu}_s \) of (101). Like (87), function (101) keeps the symmetry under the super-boost transformation (89). Therefore, function \( \tilde{A}(U_p, U_p, V_p, V_p, U_p, \bar{U}_p) \) has no pole at \( u_p = v_p \) just as function (86) in the vacuum configuration. At the same time, the vacuum correlator (A.7) is not invariant under the change of \( t \) by transformation (95). Therefore, unlike (87), function (101) is not symmetrical under this transformation. As the result, \( \tilde{A}(U_p, U_p, V_p, V_p, U_p, \bar{U}_p) \) is not nullified. There is nullified its sole term which contains only the dilaton-type pairings (33) that are invariant under the discussed transformation. If a number of the vertices moving to \( v_p \) is either \( (m - 1) \), or \( m \), then the absence of divergences in the amplitude is argued just as in the genus-1 case.

The nullification of the vacuum amplitude and the finiteness of amplitudes (23) on the genus \( \geq 2 \) supermanifolds is established by the induction method. In doing so, we consider the configuration where the genus-\( n \) supermanifold degenerates into the sum of the genus-\( n_1 \) > 2 supermanifold and of the genus-(\( n - n_1 \)) one. Then the integrand of the amplitude (23) is given by (84) and (85). We select certain handles ‘\( s \)’ and ‘\( p \)’ among other handles of the genus-\( n_1 \) supermanifold. Then we integrated over the super-Schottky group points on the genus-\( n_1 \) supermanifold excepting those points which are assigned to the ‘\( s \)’ and ‘\( p \)’ handles. For definiteness we again suppose that all the limiting points on the genus-\( n_1 \) supermanifold move to \( v_p \). In line with the genus-2 case, we consider the integral
\begin{equation}
\tilde{A}_{n_1}(U_p, V_p, U_p, V_p, U_p, U_p) = \int \tilde{Z}_2^{(n_1)}(U_p, U_p, U_p, V_p, V_p, V_p) d^2 w_i d^2 \bar{w}_j d^2 \bar{w}_k
\end{equation}
where
\begin{equation}
\tilde{Z}_2^{(n_1)}(U_p, U_p, U_p, V_p, V_p, V_p) = \int Z_{n_1}(\{q, \bar{q}\}_{n_1}) \prod_{i=1}^{n_1-2} d^2 U_i d^2 \bar{U}_i d^2 V_i d^2 \bar{V}_i.
\end{equation}
The integration in (103) is performed over the limiting points on genus-\( n_1 \) supermanifold except those limiting points which are assigned to the ‘\( s \)’ and ‘\( p \)’ handles (we omit an explicit dependence on the Schottky group multipliers). Besides, we consider the \( \tilde{A}_{n_1}(U_p, U_p, U_p, V_p, V_p, U_p, \bar{U}_p) \) function that is obtained by the integration over \( \mu_s \) and \( \bar{\mu}_s \) of
expression (102) as follows

$$\tilde{A}_n(U_p, \bar{U}_p, V_p, \bar{V}_p, u_\mu, \pi_\nu) = \int \tilde{A}_n(U_p, \bar{U}_p, V_p, \bar{V}_p, U_\mu, \bar{U}_\nu) d\mu d\nu.$$  

(104)

Following the induction method, we agree that any genus-$g < n_1$ vacuum amplitude vanishes and that integral (102) is convergent. Then function (103) is symmetrical with respect to transformations (89) and (95). So function (104) vanishes identically in its arguments. Then the vacuum amplitude is nullified just as in the genus-2 case. The configurations where the vertices in number $m_1 = 1$, $m_1 = (m - 1)$, or $m_1 = m$ move to $v_p$ do not generate divergences in (23) just as it appears in the genus-2 case.

To calculate two- and three-point amplitudes, we consider configurations where two or, respectively, three vertices move to $v_p$ as well as the limiting points of the Schottky group on the genus-$n_1$ supermanifold. In the leading approximation the dependence on the vertex coordinates appears through the vacuum expectation $\langle \ldots \rangle$ of the product of the questioned vertices. Generally, correlator (30) is not invariant under $SL(2)$-transformations, but receives two additional terms, every term being dependent on only one of the points of the correlator. In particular, the discussed configuration has no symmetry under transformation (95) except only the case when the total 10-momentum of the states in $\langle \ldots \rangle$ is preserved. Then the nullification of two- and three-point amplitudes is verified using transformations (89) and (95) just as the nullification of the vacuum amplitude.

The consideration of the more than three-point configurations on degenerated supermanifolds is beyond the present paper. Such a consideration is hampered by singularities in the interaction vertex coordinate space that appear due to the singularity of the vacuum correlator (30), when the interaction vertex coordinates move. In addition, due to massless states are present, there are no regions in the complex 10-invariant space where integral over interaction vertex coordinates is simultaneously convergent in all the nodal domains.

Local spinning string amplitudes are covariant under the supermodular transformations [17]. Such transformations, generally, depend on the superspin structure [17] because of terms proportional to Grassmann parameters. At the same time, the integration of a single superspin structure is ill-defined due to the singularities of the integration measure (66). In this case the modular invariance of the whole amplitude (23) implies a relevant regularization of the ill-defined integrations. The same is true for those $SL(2)$-transformations which depend on the superspin structure (in particular, for super-Schottky group transformations).

The desired regularization is seemingly achieved by a spin-structure independent cut-off of the period matrix entries. Indeed, it follows from section 4 that the period matrix entries cease to depend on the spin structure of the degenerate handle. Correspondingly, the cut-off of the Schottky group parameters does not depend on the spin structure of the degenerate handle. The regularization is realized by introducing relevant step-function multipliers in (23) which, in accordance previous section, are treated as Taylor series over Grassmann variables.

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Appendix. Correlators on degenerated surfaces

It is supposed that the genus-$n$ supermanifold is degenerated into a sum of the genus-$n_1$ supermanifold and of the genus-$n_2$ one, $n_2 = n - n_1$. In doing so, it is assumed that the local limiting points of the genus-$n_1$ supermanifold are moved to $v_p$ which are among them. Further, if $t = (z|\vartheta)$ and $z$ is moved to $v_p$, then the super-holomorphic Green function $R_L(t, t'; \{ q \})$ in (24) is approximated as follows

$$R_L(t, t'; \{ q \}) \approx \int d\vartheta_2 R_{L_1}(t; \vartheta_2; \{ q \}_1)D(\tilde{t}_2)R_{L_2}(\tilde{t}_2, t'; \{ q \}_2) + R_{dL_2}(\tilde{t}_2^{(0)}, t'; \{ q \}_2),$$

(A.1)

where $R_{dL_2}(\tilde{t}_2^{(0)}, t'; \{ q \}_2)$ is defined by (41) on the genus-$n_2$ supermanifold, $\tilde{t}_2 = (v_p|\vartheta_2)$, $\tilde{t}_2^{(0)} = (v_p|0)$ and

$$R_{L_1}(t; \vartheta_2; \{ q \}_1) = [R_{L_1}(t; \tilde{t}_2; \{ q \}_1) - \ln(z - z_2)|z - z_2|_{z_2 \to -\infty},$$

(A.2)

where $t_2 = (z_2|\vartheta_2)$. Equation (A.1) follows directly from (35), (37) and (42).

If the $m_1$th handle is assigned to the genus-$n_1$ supermanifold and a certain point $z'$ is not nearby $v_p$, then in accordance with (A.1), the superscalar function $J_{m_1}(t'; \{ q \}; L)$ is approximated as

$$J_{m_1}(t'; \{ q \}; L) \approx \int d\vartheta_2 J_{m_1}(\vartheta_2; \{ q \}_1; L_1)D(\tilde{t}_2)R_{L_2}(\tilde{t}_2, t'; \{ q \}_2),$$

(A.3)

where

$$J_{m_1}(\vartheta_2; \{ q \}_1; L_1) = J_{m_1}(\tilde{t}_2; \{ q \}_1; L_1)(v_p - z_2)|z_2 \to -\infty.$$

(A.4)

If $z$ is nearby $v_p$, then $J_{m_1}(t; \{ q \}; L) \approx J_{m_1}(t; \{ q \}_1; L_1)$. If, in addition, the $m_1$th handle is assigned to the genus-$n_1$ supermanifold, then $\omega_{m_1 m_1}(\{ q \}; L) \approx \omega_{m_1 m_1}(\{ q \}_1; L_1)$. If the $m_2$th handle is assigned to the genus-$n_2$ supermanifold and $z$ is nearby $v_p$, then the superscalar function $J_{m_2}(t; \{ q \}; L)$ is approximated as follows

$$J_{m_2}(t; \{ q \}; L) \approx \int d\vartheta_2 J_{m_2}(\vartheta_2; \{ q \}_1; L_2)D(\tilde{t}_2)J_{m_2}(\tilde{t}_2; \{ q \}_2; L_2) + J_{m_2}(v_p; \{ q \}_2; L_2),$$

(A.5)

where $J_{m_2}(v_p; \{ q \}_2; L_2) = J_{m_2}(\tilde{t}_2; \{ q \}_2; L_2)$ at $\vartheta_2 = 0$. In accordance with both (A.3) and (A.5), the $\omega_{m_2 m_2}(\{ q \}; L)$ entry of the period matrix is approximated by

$$\omega_{m_2 m_2}(\{ q \}; L) \approx \int d\vartheta_2 J_{m_2}(\vartheta_2; \{ q \}_1; L_2)D(\tilde{t}_2)J_{m_2}(\tilde{t}_2; \{ q \}_2; L_2).$$

(A.6)

If $m_2$ and $m_1$ are both associated with the genus-$n_2$ supermanifold and $z$ is not nearby $v_p$, then $J_{m_2}(t; \{ q \}; L) \approx J_{m_2}(t; \{ q \}_2; L_2)$ and $\omega_{m_2 m_2}(\{ q \}; L) \approx \omega_{m_2 m_2}(\{ q \}_2; L_2)$.

As the result, under questioned conditions the correlator (30) is given by

$$\hat{X}_{L,L'}(t, t', t'; \{ q \}) = \left[ \int d\vartheta_2 X_{L,L'}(t, t; \vartheta_2; \{ q \}_1)D(\tilde{t}_2) + \int d\vartheta_2 X_{L,L'}(t, t'; \vartheta_2; \{ q \}_1)D(\tilde{t}_2) \right]
\times \hat{X}_{L,L'}(\tilde{t}_2, \tilde{t}_2'; t', t'; \{ q \}_2) + \hat{X}_{L,L'}(v_p, v'_p; t', t'; \{ q \}_2).$$

(A.7)
where \( \tilde{\tau}_2 = (v_2|\partial_2) \) and

\[
\chi_{L_r,L_\ell}(t, \tau; \vartheta_2; \{ q \}_1) = \mathcal{R}_{L_r}(t, \vartheta_2; \{ q \}_1) + \sum_{m_1} \tilde{I}_{m_1}(t, \tau; \{ q, \bar{q} \}_1; L_1, L_1') \mathcal{J}_{m_1}(\vartheta_2; \{ q \}_1; L_1).
\]

(A.8)

In turn,

\[
\tilde{I}_{m_1}(t, \tau; \{ q, \bar{q} \}_1; L_1, L_1') = \sum_{m_1} \mathcal{J}_{m_1}(t; \{ q \}_1; L_1)
+ \mathcal{J}_{m_1}(t; \{ q \}_1; L_1') \Omega_{L_r,L_\ell}(\{ q, \bar{q} \}_1; \{ q \}_1) \mathcal{I}_{m_1,m_1}^{-1},
\]

(A.9)

\( \Omega_{L_r,L_\ell}(\{ q, \bar{q} \}_1; \{ q \}_1) \) being given by (32) and \( \tilde{X}_{L_r,L_\ell}(t_2, \vartheta_2); \{ q \}_2) \) at \( t_2 = 0 \).

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