LINEAR RESOLUTIONS OVER KOSZUL COMPLEXES AND KOSZUL HOMOLOGY ALGEBRAS

JOHN MYERS

ABSTRACT. Let $R$ be a standard graded commutative algebra over a field $k$, let $K$ be its Koszul complex viewed as a differential graded $k$-algebra, and let $H$ be the homology algebra of $K$. This paper studies the interplay between homological properties of the three algebras $R$, $K$, and $H$. In particular, we introduce two definitions of Koszulness that extend the familiar property originally introduced by Priddy; one which applies to $K$ (and, more generally, to any connected differential graded $k$-algebra) and the other, called strand-Koszulness, which applies to $H$. The main theoretical result is a complete description of how these Koszul properties of $R$, $K$, and $H$ are related to each other. This result shows that strand-Koszulness of $H$ is stronger than Koszulness of $R$, and we include examples of classes of algebras which have Koszul homology algebras that are strand Koszul.

INTRODUCTION

Koszul complexes are classical objects of study in commutative algebra. Their structure reflects many important properties of the rings over which they are defined, and these reflections are often encoded in the product structure of their homology. Indeed, Koszul complexes are more than just complexes — they are, in fact, the prototypical examples of differential graded (= DG) algebras in commutative ring theory, and thus the homology of a Koszul complex carries an algebra structure. These homology algebras encode (among other things) the Gorenstein condition [6], the Golod condition [17], and whether or not the ring is a complete intersection [22]. The present paper follows in the spirit of these results, by studying how certain properties of Koszul complexes and their homology algebras are interrelated with properties of the rings from which they originate.

Let $R$ be a standard graded commutative algebra over a field $k$, let $K$ be its Koszul complex, and let $H$ be the Koszul homology algebra. The ground field $k$ can be resolved by free modules over the algebras $R$ and $H$, and by semifree modules over the algebra $K$ (the definition of semifree module belongs to DG homological algebra, and it will be recalled in the body of the paper). We compare the three resolutions of $k$ over the three different algebras, with the goal being to identify when linearity of one of the resolutions implies linearity of one of the others. Linearity of the free resolution of $k$ over $R$ means that the algebra belongs to the class of Koszul algebras, a familiar class whose theory was first laid down by Priddy in [27] and which has become a class of central importance in the homology theory of both

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noncommutative and commutative graded algebras. See, for example, the survey [15] and the monograph [26] for expositions on the noncommutative side, and the survey [12] directed at applications in commutative algebra.

On the other hand, what we mean by “linearity” of the resolutions over $K$ and $H$ requires elaboration. First of all, both $K$ and $H$ have more structure than $R$, and the resolutions need to take these extra structures into account if they are to be of any use. Indeed, $K$ of course has its differential, and the grading on $R$ induces an extra (internal) degree on both $K$ and $H$. Choosing to resolve $k$ over $K$ by bigraded semifree modules instead of just bigraded free modules over the underlying algebra of $K$ (which is just an exterior algebra) effectively takes the differential of $K$ and its extra grading into account. We then say that $k$ has a linear resolution over $K$, or that $K$ is a Koszul DG algebra, when certain bigraded differential Tor’s vanish, in complete analogy with the non-DG definition of Koszul. This definition applies to all connected differential bigraded algebras, not just Koszul complexes.

Our definition follows in the spirit of (but is not exactly identical to) the definition offered by He and Wu in [20] who deal with connected cochain DG algebras instead of chain algebras, which is where our focus lies. The differential accompanying the former type of algebra raises homological degree, while the one accompanying the latter lowers homological degree. Cochain DG algebras tend to occur more naturally in geometric and topological settings whereas chain DG algebras occur in algebraic ones; for example, He and Wu’s first example of a Koszul cochain DG algebra is the minimal model of the de Rham complex of a smooth manifold satisfying certain extra hypotheses. See also the dictionary [7], where some distinctions (and similarities) between the topological and algebraic sides of DG homological algebra are elucidated.

Shifting attention to the bigraded algebra $H$, we recall that $H_{ij} \neq 0$ implies $i = j = 0$ or $j - i > 0$, a property that we describe concisely by saying $H$ lives in positive strands. This shape makes it possible to perform a totalization process, that we call strand totalization, to produce a connected graded $k$-algebra $H'$ with $H'_n = \bigoplus_{j-i=n} H_{ij}$ for each $n$. Then, when we speak of $k$ having a linear free resolution over $H$, we mean that $H'$ is a Koszul algebra in the usual sense, and we say $H$ is strand Koszul.

The complete relationship between the three notions of Koszulness is expressed in

**Theorem A.** Let $R$ be a standard graded commutative algebra over a field $k$ and let $K$ be its Koszul complex. The algebras $R$ and $K$ are Koszul simultaneously.

With all of the definitions securely in place, the proof comes by repackaging a construction due to Tate. In [3], Avramov wrote that “...for most purposes one can replace the ring by its Koszul complex"; our Theorem A lends more evidence to the veracity of this claim.

Shifting attention to the bigraded algebra $H$, we recall that $H_{ij} \neq 0$ implies $i = j = 0$ or $j - i > 0$, a property that we describe concisely by saying $H$ lives in positive strands. This shape makes it possible to perform a totalization process, that we call strand totalization, to produce a connected graded $k$-algebra $H'$ with $H'_n = \bigoplus_{j-i=n} H_{ij}$ for each $n$. Then, when we speak of $k$ having a linear free resolution over $H$, we mean that $H'$ is a Koszul algebra in the usual sense, and we say $H$ is strand Koszul.

The complete relationship between the three notions of Koszulness is expressed in

**Theorem B.** Let $R$ be a standard graded commutative algebra over a field $k$, let $K$ be its Koszul complex, and let $H$ be the homology algebra of $K$. The following are equivalent.

1. The algebra $H$ is strand Koszul.
The algebra $K$ is Koszul and the trigraded Eilenberg-Moore spectral sequence
\[ E^2_{pqj} = \text{Tor}^H_{p} (k, k)_{qj} \Rightarrow \text{Tor}^K_{p+q} (k, k)_j \]
degenerates on the second page.

The algebra $R$ is Koszul and the trigraded Avramov spectral sequence
\[ E^2_{pqj} = \text{Tor}^H_{p} (k, \text{Tor}^Q_{q} (k, k))_{qj} \Rightarrow \text{Tor}^R_{p+q} (k, k)_j \]
degenerates on the second page, where $Q$ is a commutative polynomial ring over which $R$ is an algebra.

Furthermore, if one of the above statements is true (and hence all are), then we have the following equations involving Poincaré-Betti series:
\[ P^R_k(s, t) = P^H_k(s, s, t) \quad \text{and} \quad P^R_k(s, t) = (1 + st)^n P^H_k(s, s, t). \]

Eilenberg-Moore spectral sequences were first introduced as an aid to solve problems in topology; their relevance to problems in commutative ring theory was first noted by Avramov in [1] who then derived the spectral sequence in (3) in [2]. The relations involving the Poincaré-Betti series are simple consequences of the degeneracy of the spectral sequences.

Thus strand-Koszulness of $H$ is stronger than Koszulness of $R$ and $K$. For example, the Koszul algebra
\[ R = \frac{k[x, y, z, u]}{(x^2, xy, xz + u^2, xu, y^2 + z^2, zu)}, \]
which is listed as “isotope 63ne” in Roos’ catalog [28, Appendix A], has a Koszul homology algebra which is not strand Koszul (verification by Macaulay2 [18] using the DGAlgebras package written by Frank Moore). We note that out of the 104 algebras of embedding dimension 4 that Roos catalogs, there are only two Koszul algebras which have Koszul homology algebras that are not strand Koszul. Thus, in combination with our next result, we have that strand-Koszulness of $H$ is “nearly” equivalent to Koszulness of $R$, as long as $R$ has embedding dimension $\leq 4$.

**Theorem C.** Let $R$ be a standard graded commutative algebra over a field $k$, let $H$ be the Koszul homology algebra of $R$, and suppose that one of the following statements is true.

1. The algebra $R$ is Koszul with embedding dimension $\leq 3$.
2. The algebra $R$ is Koszul and Golod.
3. The algebra $R$ is a quadratic complete intersection.
4. The algebra $R$ is artinian Gorenstein of socle degree 2, and either $k$ does not have characteristic 2 or the embedding dimension of $R$ is odd.
5. The algebra $R$ is Koszul and the defining ideal of $R$ is minimally generated by three elements.
6. The defining ideal of $R$ is the edge ideal of a path on $\geq 3$ vertices.

Then $H$ is strand Koszul.

The proof uses a mixture of techniques. Previously established theory makes quick work of the implication “(1)-(3) $\Rightarrow$ $H$ is strand Koszul,” while our proof of the implication “(4)-(6) $\Rightarrow$ $H$ is strand Koszul” is based on (noncommutative) Gröbner basis techniques and computations of presentations of $H$ as a quotient of a free algebra.
We point out that the strand totalizations $H'$ are not new; for example, a question posed by Boocher, D’Alì, Grifo, Montaño, and Sammartano [9] asks whether or not the algebra $H'$ is generated by degree-1 elements when $R$ is a Koszul integral domain (this replaces the same question posed earlier by Avramov without the hypothesis that $R$ is an integral domain — the answer to that question turned out to be negative, see [9, Remark 3.2]). In [16], Fröberg and Lőfwal study $H'$ and uncover a connection between this algebra and (part of) the homotopy Lie algebra of $R$.

The relation between strand totalizations and Koszulness of $R$ was also considered by Croll et al. [13] who showed that if there is an element $ζ ∈ H^1 \subseteq H'_{1,2}$ such that $H' · ζ = H' > 1$, then $R$ is Koszul.

Section 1 of this paper sets up definitions and notations and proves some preliminary results, while section 2 is dedicated to the proofs of Theorems A and B and also contains some results on relationships between Poincaré-Betti series, Hilbert series, and low-degree Betti numbers. The third and final section contains the proof of Theorem C.

1. Definitions and preliminaries

Throughout this paper $k$ denotes a field. Elements of graded objects will always be assumed homogeneous and thus all elements of graded objects have degrees. Our main reference for commutative algebra is [11]; for graded ring theory we use [20] and [24]; for DG homological algebra we cite [14] and [5].

1.1. Connected $\mathbb{Z}$-graded algebras. A graded (=$\mathbb{Z}$-graded) $k$-algebra $A = \bigoplus_{j ∈ \mathbb{Z}} A_j$ is said to be connected when $A_0 = k$, $A_j \neq 0$ implies $j ≥ 0$, and each homogeneous component $A_j$ is finite dimensional.

All graded $k$-algebras in this paper will be assumed connected.

Let $A$ be a graded $k$-algebra.

The field $k$ can be considered a graded $k$-algebra concentrated in degree 0. The natural map $A → k$ (called the augmentation) is a morphism of graded algebras and we define the augmentation ideal of $A$ to be $A_+ = \ker (A → k)$.

Let $M$ be a graded left $A$-module. The $q$-th shift of $M$ is defined to be the graded module $$M(q) = \bigoplus_{j ∈ \mathbb{Z}} M(q)_j, \quad \text{with} \quad M(q)_j = M_{q+j},$$

The module $M$ is said to be finite when $M_j = 0$ for all $j << 0$ and each $M_j$ is finite dimensional.

If $M$ is finite, then it has a graded minimal free resolution of the form $$0 \leftarrow M \leftarrow \bigoplus_{j ∈ \mathbb{Z}} A(-j)^{β_{0,j}} \leftarrow \bigoplus_{j ∈ \mathbb{Z}} A(-j)^{β_{1,j}} \leftarrow \cdots,$$

where the integers $β^A_{p,j}(M) = β_{p,j}$ are uniquely determined by $M$ and are called the bigraded Betti numbers of $M$. They are all finite, and they can be measured via the dimensions of certain Tor-spaces:

$$β^A_{p,j}(M) = \dim_k \text{Tor}^A_p(k, M)_j.$$

We define the bigraded Poincaré series of $M$ to be the formal series

$$P^A_M(s, t) = \sum_{p,j} β^A_{p,j}(M)s^pt^j ∈ \mathbb{Z}[[s]][(t)].$$
We will say the algebra $A$ is Koszul if $\beta^A_{p,ij}(k) \neq 0$ implies $p = j$ for all $p$ and $j$.

1.2. Connected $\mathbb{Z}^2$-graded algebras. A bigraded (= $\mathbb{Z}^2$-graded) $k$-algebra $A = \bigoplus_{i,j \in \mathbb{Z}} A_{ij}$ is said to be connected when $A_{0,0} = k$, $A_{ij} \neq 0$ implies $i, j \geq 0$, and each homogeneous component $A_{ij}$ is finite dimensional. Given an element $a \in A_{ij}$, the integers $i$ and $j$, the pair $(i, j)$, and the sum $i + j$ will be called, respectively, the homological degree, internal degree, bidegree, and total degree of $a$. The homological degree of $a$ will be denoted $|a|$, its internal degree denoted $\text{deg} a$, and its bidegree denoted $\text{bideg} a$. We will write $A_i$ for the $\mathbb{Z}$-graded $k$-space $A_{i,*}$.

All bigraded $k$-algebras in this paper will be assumed connected.

Let $A$ be a bigraded $k$-algebra.

As in the connected $\mathbb{Z}$-graded setting, the field $k$ can be considered a bigraded $k$-algebra concentrated in bidegree $(0,0)$. The natural map $A \to k$ (called the augmentation) is a morphism of bigraded algebras and we define the augmentation ideal of $A$ to be $A_+ = \ker (A \to k)$.

Let $M$ be a bigraded left $A$-module. The p-th homological shift of $M$ is defined to be the bigraded module

$$\Sigma^p M = \bigoplus_{i,j \in \mathbb{Z}} (\Sigma^p M)_{ij}, \quad (\Sigma^p M)_{ij} = M_{i-p,j}.$$  

The q-th internal shift of $M$ is defined to be the bigraded module

$$M(q) = \bigoplus_{i,j \in \mathbb{Z}} M(q)_{ij}, \quad M(q)_{ij} = M_{i,q+j}.$$  

The module $M$ is said to be finite if $M_i = 0$ for all $i < 0$ (as above, we write $M_i$ for $M_{i,*}$) and each $M_i$ is finite as an $A_0$-module.

If we restrict our attention to finite bigraded $A$-modules, then a bigraded version of Nakayama's lemma holds true (see [26, Chapter 1, Lemma 4.1] for the $\mathbb{Z}$-graded version), and thus a theory of bigraded minimal free resolutions can be developed. Indeed, if $M$ is finite, then it has a bigraded minimal free resolution of the form

$$(1.2.1) \quad 0 \leftarrow M \leftarrow \bigoplus_{i,j \in \mathbb{Z}} \Sigma^i A(-j)^{\beta^A_{i,j}}, \quad \bigoplus_{i,j \in \mathbb{Z}} \Sigma^i A(-j)^{\beta^A_{i,j}} \leftarrow \cdots ,$$  

where the integers $\beta^A_{p,ij}(M) = \beta^A_{p,ij}$ are uniquely determined by $M$ and are called the trigraded Betti numbers of $M$. They are all finite, and they can be measured via the dimensions of certain Tor-spaces:

$$\beta^A_{p,ij}(M) = \text{dim}_k \text{Tor}^A_{p,ij} (k, M)_{ij}.$$  

We define the trigraded Poincaré series of $M$ to be the formal series

$$P^A_M(r, s, t) = \sum_{p,i,j} \beta^A_{p,ij}(M) r^p s^i t^j \in \mathbb{Z}[[r]][(s,t)].$$  

For later applications, we note the following proposition which shows that the “shape” of the augmentation ideal $A_+$ is reflected in the “shape” of the bigraded minimal free resolution of $k$.

**Proposition 1.3.** Let $A$ be a connected bigraded $k$-algebra. If

$$(A_+)_{ij} \neq 0 \Rightarrow i, j - i > 0,$$

then

$$\beta^A_{p,ij}(k) \neq 0 \Rightarrow i, j - i \geq p.$$
1.4. Strand totalizations. Let $V = \bigoplus_{i,j \in \mathbb{Z}} V_{ij}$ be a bigraded $k$-vector space. We define an exact functor $V \mapsto V'$ from the category of $\mathbb{Z}^2$- to $\mathbb{Z}$-graded vector spaces via the formulas

$$V' = \bigoplus_{n \in \mathbb{Z}} V'_n, \quad V'_n = \bigoplus_{j-i=n} V_{ij}.$$ 

The $\mathbb{Z}$-graded vector space $V'$ is called the strand totalization of $V$, and we define the strand degree of an element $v \in V_{ij}$ to be the difference $j - i$. The functor $(-)'$ carries a bigraded $k$-algebra $\mathcal{A}$ to the $\mathbb{Z}$-graded $k$-algebra $\mathcal{A}'$, and it sends a bigraded $\mathcal{A}$-module $M$ to the $\mathbb{Z}$-graded $\mathcal{A}'$-module $M'$. Its action on shifts is explained in the formula

$$(\Omega^p M(q))' = M'(p + q).$$

We will say that a bigraded $k$-algebra $\mathcal{A}$ lives in positive strands if its strand totalization $\mathcal{A}'$ is connected (as a $\mathbb{Z}$-graded algebra), while a finite bigraded $\mathcal{A}$-module $M$ is said to live in positive strands if $M'$ is a finite $\mathcal{A}'$-module. If $M$ is such a module, then applying $(-)'$ to the bigraded minimal free resolution (1.2.1) produces the exact sequence

$$0 \leftarrow M' \leftarrow \bigoplus_{i,j \in \mathbb{Z}} \mathcal{A}'(i - j)_{\beta_{i,j}^0} \leftarrow \bigoplus_{i,j \in \mathbb{Z}} \mathcal{A}'(i - j)_{\beta_{i,j}^1} \leftarrow \cdots,$$

which is the graded minimal free resolution of $M'$ over $\mathcal{A}'$. This yields the following formula relating the bigraded Betti numbers of $M'$ and the trigraded Betti numbers of $M$:

$$\beta^A_{pq}(M') = \sum_{j-i=q} \beta^A_{pji}(M).$$

We will say a bigraded algebra $\mathcal{A}$ which lives in positive strands is strand Koszul when $\mathcal{A}'$ is Koszul (as a $\mathbb{Z}$-graded algebra). The last displayed equality then yields

**Proposition 1.5.** Let $\mathcal{A}$ be a connected bigraded $k$-algebra which lives in positive strands. The algebra $\mathcal{A}$ is strand Koszul if and only if $\beta^A_{pji}(k) \neq 0$ implies $p = j - i$. 

1.6. Exterior and divided-power algebras. Let $R$ be a commutative connected $\mathbb{Z}$-graded $k$-algebra viewed as a connected bigraded $k$-algebra concentrated in homological degree 0. If $F$ is a free bigraded $R$-module with a fixed basis $T = \{t_1, \ldots, t_n\}$ such that each $|t_i|$ is odd, we write $\Lambda_R(T)$ or $\Lambda_R(t_1, \ldots, t_n)$ to stand for the exterior algebra of the module $F$. We shall refer to $\Lambda_R(T)$ as the exterior algebra generated over $R$ by the exterior variables $t_1, \ldots, t_n$. Note that $\Lambda_R(T)$ is a connected bigraded $k$-algebra.

If each $|t_i|$ is even instead of odd, then we write $\Gamma_R(T)$ or $\Gamma_R(t_1, \ldots, t_n)$ for the divided-power algebra generated over $R$ by the divided-power variables $t_1, \ldots, t_n$. This is the free bigraded $R$-module generated by monomials of the form

$$t_1^{(a_1)} \cdots t_n^{(a_n)}, \quad a_1, \ldots, a_n \in \mathbb{N},$$

where $t_i^{(0)} = 1_R$ and $t_i^{(1)} = t_i$ for each $i = 1, \ldots, n$. The module $\Gamma_R(T)$ is given the structure of an $R$-algebra via the multiplication rules:

$$t_i^{(a)} t_j^{(b)} = t_j^{(b)} t_i^{(a)} \quad \text{for all } i, j = 1, \ldots, n \text{ and } a, b \in \mathbb{N},$$

$$t_i^{(a)} t_i^{(b)} = \frac{(a + b)!}{a!b!} t_i^{(a+b)} \quad \text{for all } i = 1, \ldots, n \text{ and } a, b \in \mathbb{N}.$$ 

Note that $\Gamma_R(T)$ is a connected bigraded $k$-algebra.
1.7. DG algebras and modules. A DG $k$-algebra is a bigraded $k$-algebra $A$ equipped with a $k$-linear endomorphism $\partial^A : A \to A$ of bidegree $(-1, 0)$, called the differential, such that $\partial^A \circ \partial^A = 0$ and which satisfies the Leibniz rule:

$$\partial(ab) = \partial(a)b + (-1)^{|a|}a\partial(b) \quad \text{for all } a, b \in A.$$ 

We write $A^\bullet$ for the underlying bigraded $k$-algebra obtained by forgetting the differential of $A$. If no confusion will arise, we will omit the superscript from $\partial^A$ and write $\partial$ in its place.

We will say that a DG $k$-algebra $A$ is connected if the bigraded algebra $A^\bullet$ is connected and there is an inclusion $\partial A_1 \subseteq A_{0,+}$. For such an algebra, we define the augmentation ideal of $A$ to be

$$A_+ = A_{0,+} \oplus A_1 \oplus A_2 \oplus \cdots.$$ 

Since $A_i = 0$ for all $i < 0$, there is an inclusion $A_0 \subseteq Z(A)$ into the subspace of cycles and hence $Z(A)$ is a connected bigraded $k$-algebra (and the inclusion $A_0 \subseteq Z(A)$ is one of subalgebras). The space of boundaries $B(A)$ is an ideal of the algebra $Z(A)$, and so the homology $H(A)$ is a bigraded $k$-algebra; but even more, the homology is a connected bigraded $k$-algebra since $\partial A_1 \subseteq A_{0,+}$. The inclusion morphism $A_0 \to Z(A)$ composed with the canonical projection $Z(A) \to H(A)$ yields a morphism $A_0 \to H(A)$ of bigraded $k$-algebras; hence all bigraded $H(A)$-modules are also bigraded $A_0$-modules.

All DG $k$-algebras in this paper will be assumed to be connected.

Let $A$ be a DG $k$-algebra. A (left) DG $A$-module is a bigraded left $A^\bullet$-module $M$ equipped with a $k$-linear endomorphism $\partial^M : M \to M$ of bidegree $(-1, 0)$, called a differential, such that $\partial^M \circ \partial^M = 0$ and which satisfies the Leibniz rule:

$$\partial^M(am) = \partial^A(a)m + (-1)^{|a|}a\partial^M(m) \quad \text{for all } a \in A, m \in M.$$ 

Right DG $A$-modules are defined analogously. As with DG algebras, we will write $M^\bullet$ for the underlying bigraded $A^\bullet$-module obtained by forgetting the differential of $M$.

Letting $M$ and $N$ be two left DG $A$-modules, a function $\alpha : M \to N$ is called a morphism (of DG modules) if it induces an $A^\bullet$-linear map $M^\bullet \to N^\bullet$ of bidegree $(0, 0)$ and if $\partial^N \circ \alpha = \alpha \circ \partial^M$. If the induced $H(A)$-linear map $H(\alpha) : H(M) \to H(N)$ is an isomorphism, then $\alpha$ is called a quasi-isomorphism.

A DG $A$-module $M$ will be called homologically finite if the homology $H(M)$ is finite as a bigraded $H(A)$-module. If $M^\bullet$ is a finite free bigraded $A^\bullet$-module, then $M$ is called a semifree DG $A$-module.

Remark 1.8. Our definition of semifree applies only to DG modules whose underlying bigraded module is finite, since these are the only DG modules that will interest us. For the definition of semifree in the absence of boundedness hypotheses, see, for example, [14].

1.9. Minimal semifree resolutions and Koszul DG algebras. Let $A$ be a DG $k$-algebra (recall that $A$ is assumed to be connected). In this section all DG $A$-modules are assumed to be homologically finite.

Let $M$ be a left DG $A$-module.

There exists a semifree DG $A$-module $F$ and a quasi-isomorphism $\varepsilon : F \to M$; such a morphism is called a semifree resolution of $M$. If $N$ is a right DG $A$-module,
we then define
\[ \text{Tor}^A(N, M) = H(N \otimes_A F). \]
This definition is independent (up to isomorphism) of the choice of resolution; this follows essentially from (a DG version of) the Comparison Theorem, just as it does in non-DG homological algebra.

Existence of semifree resolutions is established in, for example, [14] (see also [5]). These references deal with the more general case of DG modules without any boundedness hypotheses. The reason that we restrict ourselves to homologically finite DG modules is because in this setting one can prove the existence of minimal semifree resolutions; by definition, these are semifree resolutions \( \varepsilon : F \to M \) such that \( \partial F \subseteq A_+ F \). For such resolutions it follows that
\[ \text{Tor}^A(k, M) = k \otimes_A F. \]
For each \( p \) and \( j \), we define
\[ \beta^A_{pj}(M) = \dim_k \text{Tor}^A_p(k, M)_j. \]
These numbers are called the \textit{bigraded Betti numbers} of \( M \). We define the \textit{bigraded Poincaré series} of \( M \) to be the formal series
\[ P^A_M(s, t) = \sum_{p, j} \beta^A_{pj}(M)s^pt^j \in \mathbb{Z}((s, t)). \]
We say that \( A \) is a \textit{Koszul} DG \( k \)-algebra if \( \beta^A_{pj}(k) \neq 0 \) implies \( p = j \).

1.10. Example. Consider the graded \( k \)-algebra \( R = k[X]/(X^2) \) with \( X \) in internal degree 1, and let \( K \) be the exterior algebra generated over \( R \) by a single exterior variable \( t \) of bidegree \((1, 1)\). Define a differential on \( K \) by setting \( \partial t = x \) where \( x \) denotes the image of \( X \) in \( R \). Equipped with this differential, the algebra \( K \) is a connected DG \( k \)-algebra; in fact, \( K \) is just the Koszul complex of \( R \) (see section 1.11 immediately below this example).

Let \( V = \bigoplus_{i=0}^\infty kv_{2i} \) be a bigraded \( k \)-space generated by elements \( v_{2i} \) of bidegree \((2i, 2i)\), and consider the semifree DG \( K \)-module \( F = K \otimes_k V \) with differential
\[ \partial(a \otimes v_{2i}) = \begin{cases} \partial a \otimes v_{2i} + (-1)^{|a|}axt \otimes v_{2i-2} & : i \geq 1, \\ \partial a \otimes v_0 & : i = 0. \end{cases} \]
Then the natural map \( \varepsilon : F \to k \) which restricts to the augmentation \( R \to k \) in homological degree 0 is the minimal semifree resolution of \( k \) over \( K \). Thus \( \text{Tor}^K(k, k) \cong V \) as a bigraded \( k \)-space, from which it follows that \( K \) is a Koszul DG algebra.

The resolution \( F \) can be unraveled into the sequence
\[ 0 \leftarrow R^{(-i)} \leftarrow R^{(-i+1)} \leftarrow R^{(-i+2)} \leftarrow \cdots \]
of free graded \( R \)-modules, which the reader will note is simply the graded minimal free resolution of \( k \) over \( R \). Also, note that both \( K \) and \( R \) are Koszul algebras. Neither of these observations are coincidences; see Proposition 2.2 and Corollary 2.3.
1.11. **Koszul complexes.** Let $R$ be a commutative connected $\mathbb{Z}$-graded $k$-algebra. Let $\{x_1, \ldots, x_n\}$ be a $k$-basis of $R_1$ and let $t_1, \ldots, t_n$ be exterior variables each of bidegree $(1, 1)$. The **Koszul complex** of $R$ is the exterior algebra $K^R = \Lambda_R(t_1, \ldots, t_n)$ (see section 1.6). We define a differential on $K^R$ by setting $\partial t_i = x_i$ for each $i = 1, \ldots, n$, and extending to all of $K^R$ via the Leibniz rule. Equipped with this differential, the algebra $K^R$ is a connected DG $k$-algebra; it is independent (up to isomorphism of DG algebras) of the choice of basis of $R_1$.

We define the **Koszul homology algebra** of $R$ to be the homology algebra $H^R = H(K^R)$; note that it is a connected bigraded $k$-algebra which lives in positive strands.

If the algebra $R$ under consideration is understood and no confusion is likely to arise, we will write $K$ and $H$ in place of $K^R$ and $H^R$.

2. **Koszulness of $R$, $K^R$, and $H^R$**

Throughout the rest of this paper we let $R$ be a standard graded $k$-algebra ($k$ = field) of embedding dimension $n$ (written edim $R = n$). By definition, this means that there is a commutative polynomial ring $Q = k[x_1, \ldots, x_n]$, generated by variables of degree 1, a homogeneous ideal $J$ of $Q$ contained in $(x_1, \ldots, x_n)^2$, and an isomorphism $R \cong Q/J$ of graded algebras. We write $x_i$ for the image of $X_i$ in $R$.

We set as our first task to show that the graded minimal free resolution of $k$ over $R$ and the minimal semifree resolution of $k$ over $K$ ($= K^R$) are the same. We begin by recalling a construction due to Tate [31].

2.1. **Tate resolutions.** The construction begins with the Koszul complex $K$. If $T_1 = \{t_1, \ldots, t_n\}$ is the set of exterior variables generating $K$, we set the notation

$$R \langle T_1 \rangle = K.$$

Now choose cycles $\zeta_1, \ldots, \zeta_m \in Z_1(R \langle T_1 \rangle)$ whose homology classes minimally generate the module $H_1(R \langle T_1 \rangle)$. Letting $T_2 = \{t_{n+1}, \ldots, t_{n+m}\}$ be a set of divided-power variables with bideg$(t_{n+i}) = (2, \deg \zeta_i)$ for each $i = 1, \ldots, m$, we set

$$R \langle T_1, T_2 \rangle = R \langle T_1 \rangle \otimes_R \Gamma_R(T_2), \quad \partial t_{n+i}^{(j)} = \zeta_i t^{(j-1)},$$

and extend the differential from $R \langle T_1 \rangle$ to all of $R \langle T_1, T_2 \rangle$ via the Leibniz rule. By construction we have

$$H_1(R \langle T_1, T_2 \rangle) = 0.$$

One then adjoins to $R \langle T_1, T_2 \rangle$ a set $T_3$ of exterior variables in homological degree 3 in order to “kill” cycles whose homology classes minimally generate $H_2(R \langle T_1, T_2 \rangle)$. The result is the DG algebra

$$R \langle T_1, T_2, T_3 \rangle = R \langle T_1, T_2 \rangle \otimes_R \Lambda_R(T_3)$$

with $H_2(R \langle T_1, T_2, T_3 \rangle) = 0$. This process continues, alternating between adjoining sets $T_{2i}$ of divided-power variables (in even homological degree) and sets $T_{2i+1}$ of exterior variables (in odd homological degree) to produce a DG algebra

$$R \langle T \rangle, \quad T = \bigcup_{i \geq 1} T_i$$

which is a graded free resolution of $k$ over $R$. But even more, Gulliksen [19] and Schoeller [29] proved that this construction yields the minimal graded free resolution.
Proposition 2.2. Let $F$ be the minimal graded free resolution of $k$ over $R$.

(1) The resolution $F$ has a DG $K$-module structure making it the minimal semifree resolution of $k$ over $K$.

(2) There are equalities

$$P_k^R(s, t) = (1 + st)^n P_k^K(s, t) \quad \text{and} \quad H_R(t)(1 - t)^n P_k^K(-1, t) = 1.$$

Proof. We carry over the notation introduced above in 2.1, supposing that $F = R \langle T \rangle$ has been constructed according to Tate’s method.

(1): The inclusion $K \subseteq R \langle T \rangle$ expresses $R \langle T \rangle$ as a DG $K$-module. However, we have $R \langle T \rangle = K \otimes_R R \langle T \geq 2 \rangle$, and since $R \langle T \geq 2 \rangle$ is a free bigraded $R$-module, we conclude that $R \langle T \rangle$ is a semifree DG $K$-module. Minimality of $R \langle T \rangle$ over $K$ then follows immediately from minimality of $R \langle T \rangle$ over $R$.

(2): We have that $\text{Tor}_R^R(k, k) \cong k \otimes_R R \langle T \rangle \cong k \langle T \rangle$ and $\text{Tor}_K^K(k, k) \cong k \otimes_K R \langle T \rangle \cong k \otimes_R (R \langle T \geq 2 \rangle) \cong k \langle T \geq 2 \rangle$.

Thus we have

$$\text{Tor}_R^R(k, k) \cong \Lambda_k(T_1) \otimes_k \text{Tor}_K^K(k, k)$$

as bigraded vector spaces, from which the first equality follows. The second then follows from the first in view of the well-known equality $H_R(t)(1 - t)^n P_k^K(-1, t) = 1$ (see [26, Chapter 2, Proposition 2.1]).

□

Theorem A from the introduction immediately follows, in the form of

Corollary 2.3. The algebra $R$ is Koszul if and only if $K$ is Koszul.

Remark 2.4. Let $F$ be the minimal graded free resolution of $k$ over $R$. Viewing $F$ as the minimal semifree resolution of $k$ over $K$ as in the proposition, recall that minimality means $\partial F \subseteq K_+ F$ where

$$K_+ = R_+ \oplus K_1 \oplus K_2 \oplus \cdots.$$

However, since minimality of $F$ as a semifree resolution over $K$ arises from minimality of $F$ as a free resolution over $R$, it satisfies the stronger form of minimality expressed by $\partial F \subseteq R_+ F$.

Remark 2.5. Recall that the algebra $\text{Ext}_R^k(k, k)$ is the bigraded universal enveloping algebra of a connected bigraded Lie algebra $\pi^*(R)$ called the homotopy Lie algebra of $R$; see, for example, [4]. The subspace $\pi^{>2}(R)$ is a Lie subalgebra of $\pi^*(R)$ and its universal enveloping algebra is the (bigraded) vector space dual of $\text{Tor}_K^K(k, k)$; see [3]. Thus the first equality in part (2) of Proposition 2.2 (and therefore the corollary as well) follows from the Poincaré-Birkhoff-Witt Theorem ([30]).

By using the construction described above in 2.1, Tate proved that

$$P_k^R(s, t) = \frac{(1 + st)^n}{\prod_{i=1}^c (1 - s^2 t d_i)}$$

if $R$ is a complete intersection cut out by homogeneous forms of degrees $d_1, \ldots, d_c \geq 2$. Conversely, Assmus [22] proved that if $P_k^R(s, t)$ is given by the above formula, then $R$ is a complete intersection. Combining these results with Proposition 2.2 yields
Proposition 2.6. The algebra $R$ is a complete intersection if and only if

$$P^R_k(s, t) = \frac{1}{\prod_{i=1}^c (1 - s^{2t}d_i)}$$

for some integers $d_1, \ldots, d_c \geq 2$.

Having established that the algebras $R$ and $K$ are Koszul simultaneously, we now bring the homology algebra $H (= H^R)$ into the picture. Our main tools to study the relationship between $H$ and the algebras $R$ and $K$ will come from the machinery of Eilenberg-Moore spectral sequences introduced below in 2.9; we begin preparing the way for their introduction by recalling certain algebra and module structures which can be defined on Tor’s. Recall from the beginning of this section that $R$ is a graded $Q$-algebra, where $Q$ is a commutative polynomial ring.

2.7. Tor-algebras and Tor-modules. Let $M$ be a graded $R$-module with scaling map

$$\mu^M : R \otimes Q M \to M$$

and let

$$\mu^{KQ} : K^Q \otimes Q K^Q \to K^Q$$

be the product map on the Koszul complex $K^Q$. We then have a sequence of $Q$-linear maps

$$H(R \otimes Q K^Q) \otimes Q H(M \otimes Q K^Q) \xrightarrow{\alpha} H((R \otimes Q K^Q) \otimes Q (M \otimes Q K^Q))$$

$$\xrightarrow{\gamma} H((R \otimes Q M) \otimes Q (K^Q \otimes Q K^Q))$$

$$\xrightarrow{\beta} H(M \otimes Q K^Q)$$

where $\alpha$ is the “external homology product” of [23] and $\beta = \mu^M \otimes Q \mu^{KQ}$. Since $K^Q$ resolves $k$ over $Q$, we therefore have a pair of maps

$$\text{Tor}^Q (R, k) \otimes Q \text{Tor}^Q (M, k) \to \text{Tor}^Q (M, k),$$

$$\text{Tor}^Q (R, k) \otimes Q \text{Tor}^Q (R, k) \to \text{Tor}^Q (R, k),$$

where the second is obtained from the first by specializing to $M = R$. The second map endows $\text{Tor}^Q (R, k)$ with a bigraded $Q$-algebra structure and the first endows $\text{Tor}^Q (M, k)$ with a (left) bigraded $\text{Tor}^Q (R, k)$-module structure.

The change-of-rings isomorphism

$$K^R \cong R \otimes Q K^Q$$

yields an isomorphism $H \cong \text{Tor}^Q (R, k)$ which one can check is an isomorphism of bigraded $k$-algebras. This implies, in particular, that $\dim_k H_{ij} = \beta^{ij}_k(R)$ for all $i$ and $j$, and thus the Betti table of $R$ over $Q$ determines the dimensions of $H$.

We thus have a way to manufacture a bigraded $H$-module from a graded $R$-module $M$: First pass from $M$ to the $\text{Tor}^Q (R, k)$-module $\text{Tor}^Q (M, k)$, and then invoke the isomorphism $H \cong \text{Tor}^Q (R, k)$. In the sequel we will only be interested in the case $M = k$, where one easily proves

Proposition 2.8. There is an isomorphism

$$\text{Tor}^Q (k, k) \cong \bigoplus_{i=0}^n \Sigma^i k(-i)^{(i)}$$
of left bigraded $H$-modules. In particular, there are $k$-linear isomorphisms
\[
\text{Tor}_p^H (k, \text{Tor}_Q^Q (k, k))_{qj} \cong \bigoplus_{i=0}^{n} \text{Tor}_p^H (k, k)_{q-i, j-i}
\]
for all $p, q, j$ and an equality
\[
P_{\text{Tor}_H^Q (k, k)}^H (s, s, t) = (1 + st)^n P_{k}^H (s, s, t).
\]

2.9. Eilenberg-Moore spectral sequences. We will use two spectral sequences.

The first is the Eilenberg-Moore spectral sequence, reading as
\[
E_2^{pqj} = \text{Tor}_p^H (k, k)_{qj} \Rightarrow \text{Tor}_{p+q}^K (k, k)_j
\]
and with differentials acting as
\[
d^r_{pqj} : E_r^{pqj} \rightarrow E_r^{p-r, q+r-1, j}.
\]

For a derivation of the sequence (in the bigraded setting) see, for example, [14].

The second spectral sequence of interest was first described in [2]; it reads as
\[
E_2^{pqj} = \text{Tor}_p^H (k, \text{Tor}_Q^Q (k, k))_{qj} \Rightarrow \text{Tor}_{p+q}^K (k, k)_j
\]
and has differentials acting as in the first spectral sequence. Note that this spectral sequence uses the $H$-module structure on $\text{Tor}_Q^Q (k, k)$ described in 2.7. This spectral sequence will be called the Avramov spectral sequence.

Remark 2.10. Avramov derived his spectral sequence from the machinery of Eilenberg-Moore spectral sequences and thus both spectral sequence can accurately be called Eilenberg-Moore spectral sequences. We note that our decision to name the second spectral sequence after Avramov conflicts with Roos’ usage in [28], who refers to the first spectral sequence above as the Avramov spectral sequence.

A convergent spectral sequence $E_2^{pqj} \Rightarrow E$ of trigraded vector spaces generates inequalities of sums of dimensions. Indeed, for all $p, q, j$ we have
\[
\dim_k E_2^{pqj} \geq \dim_k E_\infty^{pqj} = \dim_k (F_p E_{p+q, j}/F_{p-1} E_{p+q, j}),
\]
and thus
\[
\sum_{p+q=m} \dim_k E_2^{pqj} \geq \sum_{p+q=m} \dim_k E_\infty^{pqj} = \dim_k E_{mj}
\]
for all $m, j$. The following proposition then follows.

Proposition 2.11.

(1) The Eilenberg-Moore spectral sequence yields a coefficient-wise inequality
\[
P_{k}^K (s, t) \leq P_{k}^H (s, s, t),
\]
and equality holds if and only if the spectral sequence degenerates on the second page (i.e., $E_2 = E_\infty$).

(2) The Avramov spectral sequence yields a coefficient-wise inequality
\[
P_{k}^R (s, t) \leq P_{\text{Tor}_H^Q (k, k)}^H (s, s, t),
\]
and equality holds if and only if the spectral sequence degenerates on the second page.

We now arrive at the central theoretical result of the paper, Theorem B from the introduction, renamed here as
Theorem 2.12. The following statements are equivalent.

1. The algebra $H$ is strand Koszul.
2. The algebra $K$ is Koszul and the Eilenberg-Moore spectral sequence degenerates on the second page.
3. The algebra $R$ is Koszul and the Avramov spectral sequence degenerates on the second page.

Furthermore, if one of the above statements is true (and hence all are), then there are equalities

$$P^K_k(s, t) = P^H_k(s, s, t) \quad \text{and} \quad P^R_k(s, t) = (1 + st)^n P^H_k(s, s, t).$$

Proof. (1) $\Rightarrow$ (2): We shall first prove that $E^2 = E^\infty$. To do so, we observe that if the differential $d_{pqj}^2: E^2_{pqj} \to E^2_{p-2, q+1, j}$ were nonzero, then necessarily $\text{Tor}^H_{p}(k, k)_{qj} = E^2_{pqj} \neq 0$ and $\text{Tor}^H_{p-2}(k, k)_{q+1, j} = E^2_{p-2, q+1, j} \neq 0$.

But $H$ is strand Koszul, and thus Proposition 1.5 implies both $p = j - q$ and $p - 2 = j - (q + 1)$. This is absurd, which means that we must have $d_{pqj}^2 = 0$ and therefore $E^2 = E^\infty$.

Now, the coefficient-wise inequality in Proposition 2.11(1) turns into an equality, and it yields equalities

$$(*) \quad \beta^K_{ij}(k) = \sum_{p+q=i} \beta^H_{pqj}(k)$$

for each $i$ and $j$. Thus if $\hat{\beta}^H_{ij}(k) \neq 0$, then there is a pair $p, q$ with $p + q = i$ and $\beta^K_{pqj}(k) \neq 0$. Proposition 1.5 then gives $p = j - q$, and hence $i = j$. This proves $K$ is Koszul.

(2) $\Rightarrow$ (1): From Proposition 2.11 and degeneracy of the spectral sequence, we still have the equalities $(*)$. Thus if $\beta^H_{pqj}(k) \neq 0$ for some $p, q, j$, then $\beta^K_{p+q,j}(k) \neq 0$. Since $K$ is Koszul, this means that $p = j - q$, and by Proposition 1.5, this means $H$ is strand Koszul.

(2) $\Leftrightarrow$ (3): The algebras $R$ and $K$ are Koszul simultaneously by Corollary 2.3. Propositions 2.2, 2.11, and 2.8 give

$$(1 + st)^n P^K_k(s, t) = P^R_k(s, s, t) \leq P^{H}_{\text{Tor}^H_Q(k, k)}(s, s, t) = (1 + st)^n P^H_k(s, s, t).$$

Therefore, by Proposition 2.11, the two spectral sequences degenerate on the second page simultaneously. \qed

We finish this section by noting that even when the Avramov spectral sequence does not degenerate predictably, it is still possible to obtain relations from it between the low-degree Betti numbers of $k$ over $R$ and $k$ over $H$. These relations are presented in
Theorem 2.13. The following equalities hold:

\[
\beta^R_{2,2}(k) = \binom{n}{2} + \beta^H_{1,1,2}(k), \\
\beta^R_{2,j}(k) = \beta^H_{1,1,j}(k) \text{ for } j > 2, \\
\beta^R_{3,j}(k) = \beta^H_{1,2,j}(k) + n\beta^H_{1,1,j-1}(k) \text{ for } j > 3, \\
\beta^R_{4,j}(k) = \beta^H_{1,3,j}(k) + n\beta^H_{1,2,j-1}(k) + \binom{n}{2} \beta^H_{1,1,j-2}(k) + 2\beta^H_{2,2,j}(k) \text{ for } j > 4.
\]

Proof. Combining the Avramov spectral sequence with Proposition 2.8 produces a spectral sequence

\[
E^2_{pqj} = \bigoplus_{i=0}^{n} \text{Tor}_p^H(k, k)_{q-i,j-i} \Rightarrow \text{Tor}_p^R(k, k)_{j}
\]

with differentials on the second page acting on tridegrees as

\[
d^2_{pqj} : E^2_{pqj} \rightarrow E^2_{p-2,q+1,j}.
\]

The algebra \( H \) satisfies the hypotheses of Proposition 1.3; thus if \( E^2_{pqj} \neq 0 \), then there is an \( i \), \( 0 \leq i \leq n \), such that \( \beta^H_{p,q-2-i,j-1}(k) \neq 0 \). Hence \( q-i \geq p \), so that necessarily \( q \geq p \). Thus from (2.10.1) we have

\[
\beta^R_{m,j}(k) = \sum_{p+q-m \geq p, q \geq p} \dim_k E^\infty_{pqj}
\]

for all \( m \) and \( j \).

In particular, we have

\[
\beta^R_{2,j}(k) = \dim_k E^\infty_{0,2,j} + \dim_k E^\infty_{1,1,j}.
\]

However, according to the description of the differentials \( d^2_{pqj} \) given above, we must have \( E^\infty_{0,2,j} = E^2_{0,2,j} \) and \( E^\infty_{1,1,j} = E^2_{1,1,j} \). Thus

\[
\beta^R_{2,j}(k) = \sum_{i=0}^{n} \binom{n}{i} \beta^H_{0,2-i,j-1}(k) + \sum_{i=0}^{n} \binom{n}{i} \beta^H_{1,1-i,j-1}(k)
\]

\[
= \binom{n}{2} \beta^H_{0,0,j-2}(k) + \beta^H_{1,1,j}(k),
\]

from which the equations for \( \beta^R_{2,j}(k) \) follow.

Similarly, we have

\[
\beta^R_{3,j}(k) = \dim_k E^\infty_{0,3,j} + \dim_k E^\infty_{1,2,j}
\]

\[
= \dim_k E^\infty_{0,3,j} + \dim_k E^2_{1,2,j}
\]

\[
= \dim_k E^\infty_{0,3,j} + \beta^H_{1,2,j}(k) + n\beta^H_{1,1,j-1}(k).
\]

But

\[
E^2_{0,3,j} = \bigoplus_{i=0}^{n} \text{Tor}_i^H(k, k)_{3-i,j-i} = \text{Tor}_0^H(k, k)_{0,j-3},
\]

so that \( E^\infty_{0,3,j} = 0 \) if \( j > 3 \). The equations for \( \beta^R_{3,j}(k) \) follow.
Finally, we have
\[ \beta_{4,j}^{R}(k) = \dim_{k} E_{0,4,j}^{\infty} + \dim_{k} E_{1,3,j}^{\infty} + \dim_{k} E_{2,2,j}^{\infty} \]
\[ = \dim_{k} E_{0,4,j}^{\infty} + \dim_{k} E_{1,3,j}^{2} + \dim_{k} \ker d_{2,2,j}^{2}. \]

One checks just as above that \( E_{0,4,j}^{\infty} = 0 \) and \( \ker d_{2,2,j}^{2} = E_{2,2,j}^{2} \) when \( j > 4 \); the equations for \( \beta_{4,j}^{R}(k) \) then follow. \( \square \)

Remark 2.14. The first two equations for \( \beta_{2,j}^{R}(k) \) refine well-known existing relations; see, for example, [11, Theorem 2.3.2].

3. Classes of Koszul homology algebras which are strand Koszul

In this section we describe several classes of standard graded algebras that possess Koszul homology algebras which are strand Koszul. Our goal is to systematically work through the proof of Theorem C from the introduction.

We begin with a result of Avramov. Recall that \( n \) denotes the embedding dimension of our fixed standard graded algebra \( R \). We define depth \( R \) to be the maximal length of a (homogeneous) regular sequence in the augmentation ideal \( R_{+} \).

**Proposition 3.1.** Let \( R \) be a standard graded algebra and let \( H \) be its Koszul homology algebra. If \( n - \text{depth } R \leq 3 \), then the algebra \( R \) is Koszul if and only if \( H \) is strand Koszul.

**Proof.** Avramov proved in [1, Theorem 5.9] that the Eilenberg-Moore spectral sequence degenerates on the second page when \( n - \text{depth } R \leq 3 \). Apply Theorem 2.12. \( \square \)

By using bar constructions, Iyengar [21] gave an alternate construction of the Avramov spectral sequence and described its first page as
\[ E_{pqj}^{1} = \left( \overline{H}^{p} \otimes_{k} \text{Tor}_{q}^{Q}(k,k) \right)_{qj}, \]
where \( \overline{H} \) is the cokernel of the unit map \( k \to H \). We can thus add a series to the inequality in Proposition 2.11(2) to produce
\[ P_{k}^{R}(s,t) \leq P_{H}^{Q}(k,k)(s,s,t) \leq \frac{(1 + st)^{n}}{1 - t(P_{R}^{Q}(s,t) - 1)}. \]
If the series at the two ends are equal, then \( R \) is called a *Golod algebra*. These rings have been intensively studied in the homology theory of commutative local rings (see [4] for an overview).

**Proposition 3.2.** Let \( R \) be a standard graded algebra and let \( H \) be its Koszul homology algebra. If \( R \) is Koszul and Golod, or if \( R \) is a quadratic complete intersection, then \( H \) is strand Koszul.

**Proof.** If \( R \) is a Golod algebra, then the inequalities in (3.1.1) are equalities. If \( R \) is Koszul as well, we can then apply Proposition 2.11 and Theorem 2.12.

If \( R \) is a quadratic complete intersection, then \( H \) is an exterior algebra generated by \( H_{1,2} \); this is a result of Assmus [22]. Hence the strand totalization \( H' \) is an exterior algebra generated in degree 1, which is well-known to be Koszul. \( \square \)
In the rest of this section, the technique we use for establishing strand-Koszulness comes from the theory of noncommutative Gröbner bases. Our main reference is [8], but we will collect here the main definitions and results for the reader’s convenience.

3.3. **Noncommutative Gröbner bases.** Let \( k \) be a field and write

\[
F = k \langle \zeta_1, \ldots, \zeta_n \rangle
\]

for the polynomial algebra generated by noncommuting variables \( \zeta_1, \ldots, \zeta_n \) of degree 1. We consider the degree-lexicographic ordering on the monomials of \( F \) with the variables ordered \( \zeta_1 < \cdots < \zeta_n \).

Let \( G = \{ \gamma_1, \ldots, \gamma_t \} \) be a collection of elements of \( F \) which generate an ideal \( I = (G) \). Multiplying through by scalars if necessary, for each \( i = 1, \ldots, t \) we can write \( \gamma_i = \mu_i - \alpha_i \) where \( \mu_i \) is the leading monomial of \( \gamma_i \) (with coefficient 1) and \( \alpha_i \) is a \( k \)-linear combination of monomials each \( < \mu_i \). A nonzero monomial \( \mu \in F \) is said to be reduced (with respect to \( G \)) if it does not contain any of the \( \mu_i \)'s as a sub-monomial; we then say \( G \) forms a Gröbner basis of \( I \) if the set of images in \( F/I \) of the reduced monomials form a \( k \)-basis for the quotient algebra. In any case these images linearly span the quotient algebra, so to verify that a set of generators of \( I \) is a Gröbner basis we need only check linear independence.

The relevance of Gröbner bases is explained by the following fact: If a Gröbner basis of \( I \) exists which is quadratic, then the quotient \( F/I \) is necessarily Koszul (in commutative algebra such algebras are called \( G \)-quadratic). Indeed, this follows from a filtration argument; see, for example, [26, Chapter 4, Theorem 7.1].

Suppose \( R \) is artinian Gorenstein of socle degree 2. Since \( H_{n,*} \) coincides with the socle of \( R \), we have that \( H_{n,n+2} \cong k \) and \( H_{n,j} = 0 \) for all \( j \neq n+2 \). In fact, more is true: Avramov and Golod proved in [6] that \( H \) is a Poincaré algebra (this result is independent of socle degree), and hence for all \( i, j \) there are \( k \)-linear isomorphisms

\[
H_{ij} \rightarrow \text{Hom}_k (H_{n-i,n+2-j}, H_{n,n+2})
\]

given by \( h \mapsto \lambda_h \) where \( \lambda_h \) is (left) multiplication by \( h \). In particular, the Betti table of \( R \) over \( Q \) (see section 2.7) looks like

|   | 0 | 1 | 2 | \cdots | n-1 | n |
|---|---|---|---|-------|-----|---|
| 0 | 1 | - | - | \cdots | -   | -  |
| 1 | - | 0 | b_2 | \cdots | b_{n-1} | -  |
| 2 | - | - | - | \cdots | -   | 1  |

where \( b_i = b_{n-i} \) for each \( i = 1, 2, \ldots, \lfloor n/2 \rfloor \).

**Theorem 3.4.** Let \( R \) be an artinian Gorenstein standard graded \( k \)-algebra of socle degree 2, and set \( n = \text{edim} R \). If \( k \) does not have characteristic 2, or if \( n \) is odd, then the Koszul homology algebra of \( R \) is strand Koszul.

**Proof.** Suppose first that \( n \) is odd, set \( c = \lfloor n/2 \rfloor \), and consider the multiplication maps

\[
\mu_i : H_{i,i+1} \otimes_k H_{n-i,n-i+1} \rightarrow H_{n,n+2}
\]

for \( i = 1, 2, \ldots, c \). As can be seen by the Betti table above, these are the only nonzero multiplication maps besides the ones that involve \( H_{0,0} \). Since \( H_{n,n+2} \) is 1-dimensional, choosing a nonzero element allows us to identify it with \( k \) and thereby to consider the multiplication maps \( \mu_i \) as bilinear forms. By the Poincaré
condition on $H$, for each $i = 1, 2, \ldots, c$ we can choose bases $\zeta_{i,1}, \ldots, \zeta_{i,b_i}$ of $H_{i,i+1}$ and $\eta_{n-i,1}, \ldots, \eta_{n-i,b_i}$ of $H_{n-i,n-i+1}$ such that the matrix representing the bilinear form $\mu_i$ is equal to the $b_i \times b_i$ identity matrix.

Setting $T = \bigcup_{i=1}^c \{ \zeta_{i,1}, \ldots, \zeta_{i,b_i}, \eta_{n-i,1}, \ldots, \eta_{n-i,b_i} \}$, we then have

$$H' \cong k \langle T \rangle / I$$

where each $\zeta_{i,j}$ and $\eta_{n-i,j}$ has (strand) degree 1, and the ideal $I$ is generated by the following four types of elements:

1. all degree-2 monomials, except for monomials of the form $\zeta_{i,j} \eta_{n-i,j}$ and $\eta_{n-i,j} \zeta_{i,j}$ for $i = 1, \ldots, c$ and $j = 1, \ldots, b_i$,
2. commutators $\eta_{n-i,j} \zeta_{i,j} - (-1)^{(n-1)} \eta_{n-i,j} \zeta_{i,j}$ for $i = 1, \ldots, c$ and $j = 1, \ldots, b_i$,
3. elements of the form $\zeta_{i,j} \eta_{n-i,j} - \zeta_{i,1} \eta_{n-1,1}$ for $i = 2, \ldots, c$ and $j = 1, \ldots, b_i$,

and

4. elements of the form $\zeta_{1,j} \eta_{n-1,j} - \zeta_{1,1} \eta_{n-1,1}$ for $j = 2, \ldots, b_1$.

With respect to the ordering

$$\zeta_{1,1} < \cdots < \zeta_{1,b_1} < \cdots < \zeta_{c,1} < \cdots < \zeta_{c,b_c} < \eta_{n-c,1} < \cdots$$

$$\cdots < \eta_{n-c,b_c} < \cdots < \eta_{n-1,1} < \cdots < \eta_{n-1,b_1},$$

the only reduced monomial of degree $\geq 2$ is $\zeta_{1,1} \eta_{n-1,1}$; since this monomial is nonzero in $H'$, we conclude that the above generators of $I$ form a quadratic Gröbner basis. Hence $H'$ is Koszul.

If $n$ is even, the same argument as above goes through as long as $k$ does not have characteristic 2; indeed, the “middle” multiplication map

$$H_{n/2,n/2+1} \otimes H_{n/2,n/2+1} \to H_{n,n+2}$$

can then be diagonalized. \(\square\)

Now suppose $R$ is Koszul and that $J$ (the defining ideal of $R$; see the beginning of section 2) is minimally generated by three elements. In [10] it is shown that $H$ is generated by its linear strand and that the Betti table of $R$ over $Q$ must be one of the following four:

|   | 0  | 1  | 2  | 3  |
|---|----|----|----|----|
| 0 | 1  | –  | –  | –  |
| 1 | –  | 3  | 2  | –  |

|   | 0  | 1  | 2  | 3  |
|---|----|----|----|----|
| 0 | 1  | –  | –  | –  |
| 1 | –  | 3  | –  | –  |

|   | 0  | 1  | 2  | 3  |
|---|----|----|----|----|
| 0 | 1  | –  | –  | –  |
| 1 | –  | 3  | 1  | –  |

|   | 0  | 1  | 2  | 3  |
|---|----|----|----|----|
| 0 | 1  | –  | –  | –  |
| 1 | –  | –  | 2  | 1  |

If the Betti table of $R$ is one of the two in the top row, then $H$ is obviously strand Koszul. The bottom-left Betti table corresponds to an $H$ which is an exterior algebra generated by $H_{1,2}$, and hence $H$ is again strand Koszul. Thus we may assume the Betti table of $R$ over $Q$ is in the bottom-right position.

**Theorem 3.5.** Let $R$ be a Koszul standard graded algebra. If the defining ideal of $R$ is minimally generated by three elements, then the Koszul homology algebra of $R$ is strand Koszul.
Proof. From the Betti table of $R$ over $Q$, it follows that there is a set of (algebra) generators $\{\zeta_1, \zeta_2, \zeta_3, \eta\}$ of $H$ such that each $\zeta_i$ has bidegree $(1, 2)$, $\eta$ has bidegree $(2, 3)$, $\zeta_1 \eta \neq 0$, and $\zeta_2 \eta = \zeta_3 \eta = 0$. We then have

$$H' \cong \frac{k \langle \zeta_1, \zeta_2, \zeta_3, \eta \rangle}{I}$$

where $I$ is generated by the following relations:

1. all commutators $\eta \zeta_i - \zeta_i \eta$ for $1 \leq i \leq 3$,
2. all skew commutators $\zeta_i \zeta_j + \zeta_j \zeta_i$ for $1 \leq i < j \leq 3$,
3. all squares $\zeta_1^2, \zeta_2^2, \zeta_3^2, \eta^2$ and $\zeta_2 \eta, \zeta_3 \eta$,
4. a linear combination $a \zeta_1 \zeta_2 + b \zeta_1 \zeta_3 + c \zeta_2 \zeta_3$ where not all $a, b, c, \in k$ are zero.

If $c \neq 0$, then with the ordering $\zeta_1 < \zeta_2 < \zeta_3 < \eta$ the reduced monomials of degree $\geq 2$ are $\zeta_1 \eta$, $\zeta_1 \zeta_2$, and $\zeta_1 \zeta_3$. These form a basis of $H'_2$, and hence the list (1)-(4) of relations forms a quadratic Gröbner basis of $I$. The algebra $H'$ is therefore Koszul.

If $c = 0$ and $a, b \neq 0$, then the reduced monomials of degree $\geq 2$ are $\zeta_1 \eta$, $\zeta_1 \zeta_2$, and $\zeta_2 \zeta_3$. These again form a basis of $H'_2$, and hence the relations in the list (1)-(4) form a quadratic Gröbner basis of $I$. Hence $H'$ is Koszul.

If $c = 0$ and one of $a$ or $b$ is also 0, then $H'$ is the quotient of a skew-polynomial algebra by an ideal generated by quadratic monomials; such algebras are Koszul (see [26, Chapter 4, Theorem 8.1]).

Finally, we turn our attention to algebras of the form

$$R = \frac{k[X_1, \ldots, X_n]}{(X_1X_2, X_2X_3, \ldots, X_{n-1}X_n)}, \quad n \geq 3.$$
Every squarefree multidegree \( v \in \mathbb{Z}^n \) (not equal to \((0, \ldots, 0)\)) can be decomposed uniquely into a sum of \( p_{i,r} \)'s with maximal support; for example,

\[ v = (1, 1, 1, 0, 1, 1, 0, 1, 1, 1, 1) \in \mathbb{Z}^{12} \]

decomposes as \( v = p_{1,3} + p_{5,2} + p_{9,4} \). Such a decomposition will be called a complete decomposition of \( v \).

The polynomial ring \( Q = k[X_1, \ldots, X_n] \) is \( \mathbb{Z}^n \)-graded by assigning the variable \( X_i \) the multidegree \( e_i \), where \( e_i \) denotes the \( i \)-th standard basis element of \( \mathbb{Z}^n \). The defining ideal \( (X_1X_2, X_2X_3, \ldots, X_{n-1}X_n) \) of \( R \) is homogeneous with respect to this \( \mathbb{Z}^n \)-grading, and hence \( R \) inherits the \( \mathbb{Z}^n \)-grading from \( Q \).

Given multidegrees \( v, w \in \mathbb{Z}^n \), we define monomials

\[ x^v = x_1^{v_1} \cdots x_n^{v_n} \in R \text{ and } t^w = t_1^{w_1} \cdots t_n^{w_n} \in K. \]

The monomial \( x^v t^w \in K \) is then assigned the multigraded bidegree \((|w|, v + w) \in \mathbb{Z}^{n+1} \), where \(|w|\) is the usual homological degree of the monomial and \( v + w \) is called the multigraded internal degree. The differential on \( K \) preserves this latter degree and hence the homology \( H \) inherits the \( \mathbb{Z}^{n+1} \)-grading from \( K \). For each \( i \in \mathbb{Z} \) and \( u \in \mathbb{Z}^n \), we let \( K_{i,u} \) and \( H_{i,u} \) be the \( k \)-subspaces of \( K \) and \( H \), respectively, which are spanned by all homogeneous elements of multigraded bidegree \((i,u)\); for each \( j \in \mathbb{N} \) we then have decompositions

\[ K_{ij} = \bigoplus_{u \in \mathbb{N}^n, |u| = j} K_{i,u} \text{ and } H_{ij} = \bigoplus_{u \in \mathbb{N}^n, |u| = j} H_{i,u}. \]

Hence the multigraded bidegrees on \( K \) and \( H \) refine the usual bidegrees. The strand degree of a homogeneous element of multigraded bidegree \((i,u)\) is \(|u| - i\).

We now have the terminology and notation needed to state the result which serves as the basis for our study of \( H \). In particular, we note that \( H_{*\cdot u} = 0 \) if \( u \) is not squarefree (see, for example, [25, Corollary 1.40]), so this result describes the dimensions of all nonzero \( H_{i,u} \).

**Proposition 3.7** (Boocher, D’Alì, Grifo, Montaño, Sammartano [9]).

1. The algebra \( H \) is generated by the subspaces \( H_{1,2} \) and \( H_{2,3} \).
2. Let \( u \in \mathbb{Z}^n \) be a squarefree multidegree with complete decomposition \( u = \sum_{p=1}^{m} p_{r_p} \). Then

\[ \dim_k H_{*,u} = \begin{cases} 1 & : r_p \neq 1 \text{ (mod 3)} \text{ for each } p, \\ 0 & : \text{otherwise}. \end{cases} \]

In the former case, \( \dim_k H_{*,u} = 1 \) exactly when \( i = \sum_{p=1}^{m} \lfloor 2r_p/3 \rfloor \).

For all \( i = 1, \ldots, n-1 \) and \( j = 1, \ldots, n-2 \), the monomials \( x_{i+1} t_i \) and \( x_{j+1} t_j t_{j+2} \) are cycles in \( K \); we may thus define homology classes

\[ \zeta_{i,i+1} = \text{cls} (x_{i+1} t_i) \in H_{1,2} \text{ and } \eta_{j,j+1,j+2} = \text{cls} (x_{j+1} t_j t_{j+1}) \in H_{2,3}. \]

**Proposition 3.8.** The sets \( \{ \zeta_{1,2}, \ldots, \zeta_{n-1,n} \} \) and \( \{ \eta_{1,2,3}, \ldots, \eta_{n-2,n-1,n} \} \) form \( k \)-bases for the spaces \( H_{1,2} \) and \( H_{2,3} \), respectively.

**Proof.** That the first set is a basis of \( H_{1,2} \) follows from the relations of \( R \); see, for example [11, Theorem 2.3.2].
To prove the second set is a basis of $H_{2,3}$, we will consider spaces of the form $H_{2,u}$ where $u$ is a squarefree multidegree with $|u| = 3$. If $H_{2,u} \neq 0$, then by Proposition 3.7 we must have $\text{supp}(u) = \{j, j + 1, j + 2\}$ for some $j$.

The space $K_{2,u}$ is three-dimensional, with basis

$$\{x_{j+1}t_jt_{j+2}, x_jt_{j+1}t_{j+2}, x_{j+2}t_{j+1}\}.$$ 

Hence a basis element of the one-dimensional space $H_{2,u}$ is given by the homology class of a linear combination of these three elements which is also a cycle. However, the only such linear combinations are of the form

$$ax_{j+1}t_jt_{j+2} + bx_jt_{j+1}t_{j+2} + x_{j+2}t_{j+1}, \quad a, b \in k,$$

and since

$$\partial(t_jt_{j+1}t_{j+2}) = (x_jt_{j+1}t_{j+2} + x_{j+2}t_{j+1}) - x_{j+1}t_jt_{j+2},$$

it follows that $\{\eta_{j,j+1,j+2}\}$ is a basis of $H_{2,u}$. \hfill$\square$

The homology classes $\zeta_{i,i+1}$ and $\eta_{j,j+1,j+2}$ have multigraded bidegrees $(1, e_{i} + e_{i+1})$ and $(2, e_{j} + e_{j+1} + e_{j+2})$, respectively, and thus they all have strand degree 1. Setting

$$F = k \langle \zeta_{1,2}, \ldots, \zeta_{n-1,n}, \eta_{1,2,3}, \ldots, \eta_{n-2,n-1,n} \rangle,$$

there is a natural morphism $\varphi : F \rightarrow H$ of $\mathbb{Z}^{n+1}$-graded $k$-algebras; by Propositions 3.7 and 3.8, it is surjective. Let $I$ be the ideal of $F$ generated by the following ten types of elements. First, certain commutator and skew-commutator generators:

1. $\eta_{j,j+1,j+2}\eta_{i,i+1,i+2} - \eta_{i,i+1,i+2}\eta_{j,j+1,j+2}$ for all $1 \leq i < j \leq n - 2$, 
2. $\eta_{j,j+1,j+2}\eta_{i,i+1} - \eta_{i,i+1}\eta_{j,j+1,j+2}$ for all $1 \leq i \leq j \leq n - 2$, 
3. $\zeta_{i,i+1}\eta_{j,j+1,j+2} - \eta_{j,j+1,j+2}\zeta_{i,i+1}$ for all $1 \leq j < i \leq n - 1$, 
4. $\zeta_{j,j+1}\zeta_{i,i+1} + \zeta_{i,i+1}\zeta_{j,j+1}$ for all $1 \leq i < j \leq n - 1$.

Then we have the “string” generators (so-called because the subscripts form strings of consecutive numbers):

5. $\eta_{i,i+1}2\zeta_{i,i+1,i+2} - \zeta_{i,i+1}\eta_{i,i+1,i+2}$ for all $1 \leq i \leq n - 4$ (if $n \geq 5$), 
6. $\zeta_{i,i+1}\zeta_{i,i+1,i+2}$ for all $1 \leq i \leq n - 3$ (if $n \geq 4$).

Finally, the “overlap” generators:

7. $\zeta_{i,i+1}\zeta_{j,j+1}$ for all $j$ and all $i = j - 1, j$, 
8. $\zeta_{i,i+1}\eta_{j,j+1,j+2}$ for all $j$ and all $i = j - 1, j, j + 1, j + 2$, 
9. $\eta_{j,j+1,j+2}\zeta_{i,i+1}$ for all $j$ and all $i = j - 1, j, j + 1, j + 2$, and
10. $\eta_{j,j+1,j+2}\zeta_{i,i+1,i+2}$ for all $j$ and all $i = j - 2, j - 1, j, j + 1, j + 2$.

It is easy to show that $J \subseteq \ker \varphi$.

We consider the strand-degree-lexicographic ordering on the monomials of $F$ with respect to the ordering

$$\zeta_{1,2} < \eta_{1,2,3} < \zeta_{2,3} < \eta_{2,3,4} < \cdots < \zeta_{n-2,n-1} < \eta_{n-2,n-1,n} < \zeta_{n-1,n}.$$ 

We then have

**Lemma 3.9.**

1. The monomial $\zeta_{i,i+1}\zeta_{j,j+1}$ is reduced if and only if $i + 2 < j$.
2. The monomial $\zeta_{i,i+1}\eta_{j,j+1,j+2}$ is reduced if and only if $i + 1 < j$.
3. The monomial $\eta_{j,j+1,j+2}\zeta_{i,i+1}$ is reduced if and only if $j + 3 < i$.
4. The monomial $\eta_{j,j+1,j+2}\zeta_{i,i+1,i+2}$ is reduced if and only if $j + 2 < i$. 


Suppose $1 \leq i \leq n$, $1 \leq r \leq n - i + 1$, and $r \not\equiv 1 \pmod{3}$. Define

\[
\mu_{i,r} = \begin{cases} 
\zeta_{i,i+1} & : r = 2, \\
\eta_{i,i+1} & : r \equiv 2 \pmod{3}, r > 2, \\
\eta_{i,i+1} + i + 2 & : r \equiv 0 \pmod{3}.
\end{cases}
\]

By the lemma, each $\mu_{i,r}$ is reduced, and the following proposition shows that all reduced monomials in $T$ are products of these monomials.

**Proposition 3.10.** Let $\mu \in F$ be a reduced monomial (not equal to 1) of multigraded internal degree $u \in \mathbb{Z}^{n}$. Then

1. The multidegree $u$ is squarefree.
2. If the complete decomposition of $u$ is

   \[
   u = p_{i_1,r_1} + \cdots + p_{i_m,r_m} \quad (i_1 < \cdots < i_m),
   \]

then $r_p \not\equiv 1 \pmod{3}$ for each $p = 1, \ldots, m$ and $\mu = \mu_{i_1,r_1} \cdots \mu_{i_m,r_m}$. In particular, there is exactly one reduced monomial in $F_{i,u}$.

**Proof.** We shall prove both statements simultaneously by inducing on the internal degree $|u|$. If $|u| = 2$, then necessarily $\mu = \zeta_{i,i+1} = \mu_{i,2}$ for some $i$ with $1 \leq i \leq n - 1$.

Assume $|u| > 2$ and that statements (1) and (2) both hold for reduced monomials of internal degree $< |u|$. If

\[
\mu = \nu \zeta_{i,i+1}
\]

for some $i$ and some submonomial $\nu$, then $\nu$ is reduced and our inductive hypotheses imply that $u - e_i - e_{i+1}$ is squarefree, and that if this multidegree has complete decomposition $p_{j_1,s_1} + \cdots + p_{j_\ell,s_\ell}$ ($j_1 < \cdots < j_\ell$), then $s_q \not\equiv 1 \pmod{3}$ for each $q = 1, \ldots, \ell$ and

\[
\nu = \mu_{j_1,s_1} \cdots \mu_{j_\ell,s_\ell}.
\]

Since $\mu$ is reduced, by Lemma 3.9 we must have $i > j_\ell + s_\ell$. Hence

\[
\mu = \mu_{j_1,s_1} \cdots \mu_{j_\ell,s_\ell} \mu_{i,2}
\]

and the complete decomposition of $u$ is

\[
p_{j_1,s_1} + \cdots + p_{j_\ell,s_\ell} + p_{i,2}.
\]

Thus statements (1) and (2) both hold for $u$ and $\mu$.

If instead of the decomposition $\mu = \nu \zeta_{i,i+1}$ we have $\mu = \nu \eta_{i,i+1} + i+2$, then Lemma 3.9 implies $i > j_\ell + s_{\ell-1}$. If $i = j_\ell + s_\ell$, then the complete decomposition of $u$ is

\[
p_{j_1,s_1} + \cdots + p_{j_\ell,s_\ell + 2}
\]

and $\mu = \mu_{j_1,s_1} \cdots \mu_{j_\ell,s_{\ell-1} + 2}$. If $i > j_\ell + s_\ell$, then the complete decomposition of $u$ is

\[
p_{j_1,s_1} + \cdots + p_{j_\ell,s_\ell} + p_{i,3}
\]

and $\mu = \mu_{j_1,s_1} \cdots \mu_{j_\ell,s_\ell + 2} \mu_{i,3}$. In either case, statements (1) and (2) both hold for $u$ and $\mu$. \qed

**Theorem 3.11.** Let $n$ be an integer $\geq 3$ and set

\[
R = \frac{k[X_1,X_2,\ldots,X_n]}{(X_1X_2, X_2X_3, \ldots, X_{n-1}X_n)}.
\]

The Koszul homology algebra of $R$ is strand Koszul.
Proof. We will prove that the surjection \( \varphi : F \rightarrow H \) induces an isomorphism \( F/I \cong H \) (having already shown \( I \subseteq \ker \varphi \)). For a fixed multidegree \( u \in \mathbb{Z}^n \), the map \( \varphi \) induces a surjection

\[
(F/I)_{*,u} \rightarrow H_{*,u}
\]

of \( k \)-spaces. To show that the surjection is an isomorphism we will prove that

\[
\dim_k (F/I)_{*,u} = \dim_k H_{*,u}.
\]

Let \( u \) have complete decomposition \( \sum_{p=1}^m p_i p_r \).

We recall that the images of the reduced monomials linearly span the quotient algebra \( F/I \). Thus if \( (F/I)_{*,u} \neq 0 \), then by Proposition 3.10 we have that \( \dim_k (F/I)_{*,u} = 1 \), the multidegree \( u \) is squarefree, and \( r_p \neq 1 \) (mod 3) for each \( p = 1, \ldots, m \). By Proposition 3.7, we then have \( \dim_k H_{*,u} = 1 \). This establishes (*)

Now, the isomorphism \( F/I \cong H \) and Propositions 3.7 and 3.10 show that the images of the reduced monomials in \( F/I \) form a basis, and hence the generators (1)-(10) of \( I \) form a Gröbner basis. Each of these generators has strand degree 2, and hence \( H' \) is Koszul.

Remark 3.12. In addition to algebras cut out by edge ideals of paths, the paper [9] also contains results on algebras whose defining ideals are edge ideals of cycles. If \( R \) denotes one of these latter types of algebras, then its Koszul homology algebra \( H \) has a more complicated structure than what we’ve just seen above; indeed, Theorem 3.15 of [9] states that \( H \) is no longer generated in its linear strand if \( n \equiv 1 \) (mod 3) and \( n > 4 \) (where \( n \) is the number of vertices in the cycle). However, if \( n \not\equiv 1 \) (mod 3), then just as we saw above, \( H \) is generated by the subspaces \( H_{1,2} \) and \( H_{2,3} \); so, could it be that \( H \) is strand Koszul?

Not always. Indeed, if \( n = 9 \), then by running the following Macaulay2 [18] code,

```macaulay2
needsPackage "DGAlgebras"
Q = QQ[x_1..x_9]
I = ideal(x_1*x_2, x_2*x_3, x_3*x_4, x_4*x_5, x_5*x_6, x_6*x_7, x_7*x_8, x_8*x_9, x_9*x_1)
R = Q/I
H = HH koszulComplexDGA R
k = coker vars H
F = res(k,LengthLimit=>2)
peek betti vars H
one sees that \( \beta_{2,6,9}^H(k) = 1 \), and so \( H \) is not even strand quadratic.
```

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Department of Mathematics, SUNY Oswego, Oswego, New York, USA

E-mail address: john.myers@oswego.edu