An Introduction to Quantum Computing.

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1 Introduction.

Physics has often progressed very rapidly as the precision of measurements has increased. For instance, it was the precise measurements of Tycho Brahe which were instrumental in Kepler’s deduction of the elliptic orbit, a result which later formed a cornerstone of Newton’s universal law of gravitation. There is a minimum level of precision in the measurement of the planetary orbits, below which it becomes impossible to distinguish between elliptical orbits with minuscule eccentricity, and circular orbits. Once data of the requisite precision to notice a difference was available, all that was left for someone to do is to put the pieces together, and voila you get Newton!

Quantum Optics, plays an essential role in quantum metrology, a field in which the level of precision has increased exponentially over the past two decades. New techniques of increasing precision in quantum optics have increased the significant digits of some of the experimentally measured fundamental constants by orders of magnitude. Quantum optics is so robust that it
also of immediate use in testing theories of gravity and quantum field theory, for example, the L.I.G.O. collaboration are using a 2 km baseline Michelson Interferometer to search for gravitational radiation.

These unheralded successes are currently pushing into new domains of experimental precision, and we now have more direct access to the deeper layers of nature. Every week new quantum computing components are brought into being, by the sheer effort of those working in the field. With each new switch, isolation mechanism, algorithm, etc., the goal of scalable robust quantum computing becomes more eminent. If we are successful in constructing quantum computers, the effect will be more revolutionary than anything before, including the classical computer and the internet. The vast expanse of Hilbert Space will then be in the throes of man.

The fields of quantum optics and quantum computing are closely related to one another. Very often breakthroughs in quantum optics are implemented in quantum information processing, storage and quantum communication devices. For example, two ways in which cavity QED techniques may be used to perform quantum computations are (from [8])

1. Quantum information can be represented by photon states, with atoms trapped in cavities providing the non-linear interactions between photons, necessary for entanglement.

2. Quantum information can be represented by atoms in different states, where photons are used to communicate between the different atoms/states.
Any realization of these schemes would at some point have to address the problem of precision control of population transfer, as a means to generate single photons. Such precision is a per-requisite for realizing any completely quantum technology, that is, any technology based on computational components whose functionality depends \textit{a priori} on quantum non-linearities, an example of which is entanglement.

2 Field Quantization.

This treatment of the field quantization will closely (but not exactly) follow chapter 2 of Gerry et.al., given in [1]. In order that we understand the interaction of quantized modes of the electromagnetic field with “atoms”, (whose definition will, for the moment remain general; we will define an atom to be any bound state of electrons in a potential $V(r)$.) we must first understand the properties of the quantized fields themselves. In the following we begin with the simple case of a single mode field confined to a 1-d cavity. This clearly represents an idealized situation, but we will later generalize to the case of a multimode field in some three dimensional cavity.

2.1 Single mode field.

We begin as always, with the one-dimensional square well, but in the context of quantized modes of the electromagnetic field, which will be relevant for
our later analysis of quantized modes of optical cavities, etc. One fruitful and interesting scenario to investigate for our purposes, is the case of a radiation field confined to a one dimensional cavity free of sources (i.e. there are no currents, charges, or any dielectric media in the cavity), oriented along what we choose to be the $z$ axis, with perfectly conducting walls at $z = 0$ and $z = l$, therefore the transverse electric field must vanish at the boundary.

Recall that in SI units, the source-free Maxwell equations, which our single mode field must satisfy, are

\[
\nabla \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t} \tag{2.1}
\]

\[
\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{B}}{\partial t} \tag{2.2}
\]

\[
\nabla \cdot \mathbf{E} = 0 \tag{2.3}
\]

\[
\nabla \cdot \mathbf{B} = 0 \tag{2.4}
\]

We will assume that the field is polarized in the x-direction i.e. $\mathbf{E}(\mathbf{r}, t) = e_x E_x(z, t)$ and hence $\mathbf{B}(\mathbf{r}, t) = e_y B_y(z, t)$. If we identify $q(t)$ as the canonical position as defined in the Hamiltonian formalism and similarly identify $\dot{q}(t)$ as the canonical momentum, then the solution for the components is

\[
E_x(z, t) = \left(\frac{2\omega^2}{V \varepsilon_0}\right)^{1/2} q(t) \sin(kz) \tag{2.5}
\]

and

\[
B_y(z, t) = \left(\frac{\mu_0 \varepsilon_0}{k}\right) \left(\frac{2\omega^2}{V \varepsilon_0}\right)^{1/2} \dot{q}(t) \cos(kz) \tag{2.6}
\]
where the wave-number $k$ is related to the frequency $\omega$ by $k = \omega/c$. Moreover, the boundary conditions on the electric field at the interface of the perfect conductor at $z = 0$ and $z = L$, constrain the values of $k$ to be

$$k = \left(\frac{n\pi}{L}\right), \ n = 1, 2, .. \quad (2.7)$$

and therefore the allowed frequencies are

$$\omega = c \left(\frac{n\pi}{L}\right), \ n = 1, 2, .. \quad (2.8)$$

We can invert the 2 equations giving $E_x$ and $B_y$ in terms of the canonical position and momentum $q(t)$ and $\dot{q}(t)$, and obtain the expressions for $q(t)$ and $\dot{q}(t)$ in terms of $E_x$ and $B_y$, namely

$$q(t) = E_x(z,t) \left(\frac{V\varepsilon_0}{2\omega^2}\right)^{1/2} \csc(kz) \quad (2.9)$$

$$\dot{q}(t) = B_y(z,t) \left(\frac{k}{\mu_0\varepsilon_0}\right) \left(\frac{V\varepsilon_0}{2\omega^2}\right)^{1/2} \sec(kz) \quad (2.10)$$

From these expressions, it is apparent that the Hamiltonian for the field is

$$H = \frac{1}{2} \int dV \left[ \varepsilon_0 E^2(r,t) + \frac{1}{\mu_0} B^2(r,t) \right] \quad (2.11)$$

Now,

$$\varepsilon_0 E^2(r,t) = \varepsilon_0 E(r,t) \cdot E(r,t) = \varepsilon_0 (e_x \cdot e_x) E_{x}^2(z,t)$$
\[ \varepsilon_0 E_x^2(z, t) \quad (2.12) \]

and similarly,
\[ \frac{1}{\mu_0} B^2(r, t) = \frac{1}{\mu_0} B_y^2(z, t) \quad (2.13) \]

Therefore
\[ H = \frac{1}{2} \int dV \left[ \varepsilon_0 E_x^2(z, t) + \frac{1}{\mu_0} B_y^2(z, t) \right] \quad (2.14) \]

From (1) we have
\[ \varepsilon_0 E_x^2(z, t) = \frac{2\omega^2}{V} q^2(t) \sin^2(kz) \quad (2.15) \]

and
\[ \frac{1}{\mu_0} B_y^2(z, t) = \frac{2}{V} p^2(t) \cos^2(kz) \quad (2.16) \]

Therefore (6) becomes
\[ H = \frac{1}{2} \int \frac{dV}{V} \left[ \omega^2 q^2(t) \sin^2(kz) + p^2(t) \cos^2(kz) \right] \quad (2.17) \]

Since,
\[ \cos^2 x = \frac{1 + \cos 2x}{2} \quad (2.18) \]

and,
\[ \sin^2 x = \frac{1 - \cos 2x}{2} \quad (2.19) \]

we may write the Hamiltonian as
\[ H = \frac{1}{2} \int \frac{dV}{V} \left[ \omega^2 q^2(t) (1 + \cos 2kz) + p^2(t) (1 - \cos 2kz) \right] \quad (2.20) \]
Now the cosine terms drop out of because of the periodic boundary conditions and therefore,

\[ H = \frac{1}{2} \left( p^2 + \omega^2 q^2 \right) \]  

(2.21)

and so the system is equivalent to harmonic oscillator with unit mass. \( \dot{q}(t) = p(t) \).

Now that we have the canonical momentum and canonical position, is is relatively easy to quantize the field by replacing the variables \( H \) with \( \hat{H} \) and \( q(t) \) and \( \dot{q}(t) \) with the hermitean (observable) operators \( \hat{q} \) and \( \hat{p} \), respectively. Moreover, we must require that the observables obey the canonical commutation relation

\[ [\hat{q}, \hat{p}] = i\hbar \hat{I}_{n \times n} \]  

(2.22)

which will write from here on out simply as

\[ [\hat{q}, \hat{p}] = i\hbar \]  

(2.23)

with the \( n \times n \) matrix identity operator \( \hat{I}_{n \times n} \) implied. Having promoted \( \hat{q} \) and \( \hat{p} \) to operators, we are thereby led to the operators for the electric and magnetic fields

\[ \hat{E}_x = \left( \frac{2\omega^2}{V\varepsilon_0} \right)^{1/2} \hat{q} \sin(kz) \]  

(2.24)

\[ \hat{B}_y = \left( \frac{\mu_0\varepsilon_0}{k} \right) \left( \frac{2\omega^2}{V\varepsilon_0} \right)^{1/2} \hat{p} \cos(kz) \]  

(2.25)
and naturally, the Hamiltonian operator becomes
\[
\hat{H} = \frac{1}{2} (\hat{p}^2 + \omega^2 \hat{q}^2)
\] (2.26)

Now we define the non-hermitean creation, \( \hat{a}^\dagger \), and annihilation, \( \hat{a} \), operators as follows[1]:
\[
\sqrt{2\hbar \omega} \hat{a}^\dagger = (\omega \hat{q} - i\hat{p})
\] (2.27)
\[
\sqrt{2\hbar \omega} \hat{a} = (\omega \hat{q} + i\hat{p})
\] (2.28)

Defining
\[
\mathcal{E}_0 = (\hbar \omega / V \varepsilon_0)^{1/2}
\] (2.29)
and
\[
\mathcal{B}_0 = (\mu_0 / k) (\varepsilon_0 \hbar \omega^3 / V)^{1/2}
\] (2.30)

it follows that we can write the operators for the electric and magnetic fields as [1]:
\[
\hat{E}_x (z, t) = \mathcal{E}_0 \left( \hat{a}^\dagger (t) + \hat{a} (t) \right) \sin (kz)
\] (2.31)
\[
\hat{B}_y (z, t) = i\mathcal{B}_0 \left( \hat{a}^\dagger (t) - \hat{a} (t) \right) \cos (kz)
\] (2.32)

From now on we will suppress hats, \( ^\wedge \), on operators and just write \( \hat{a}^\dagger = a^\dagger \), \( \hat{a} = a \), \( \hat{E}_x = E_x \), etc. The benefit of working with creation and annihilation
operators is that we are allowed to utilize the simplicity of their algebra.

\[
[a, a^\dagger] = aa^\dagger - a^\dagger a = 1
\]  

(2.33)

These commutation relations allow us to write the Hamiltonian operator as

\[
H = \hbar \omega \left( a^\dagger a + \frac{1}{2} \right)
\]  

(2.34)

In the Heisenberg representation, a general operator \( \hat{O} \) will obey Heisenberg’s equation of motion [2]

\[
\frac{d\hat{O}}{dt} = \frac{\partial \hat{O}}{\partial t} + \frac{i}{\hbar} \left[ H, \hat{O} \right]
\]  

(2.35)

Which in the case that \( \hat{O} \) does not depend explicitly on the time coordinate, becomes

\[
\frac{d\hat{O}}{dt} = \frac{i}{\hbar} \left[ H, \hat{O} \right]
\]  

(2.36)

Therefore for the creation and annihilation operators we have the following time evolution equations[1]

\[
\frac{da^\dagger}{dt} = i\omega a^\dagger
\]  

(2.37)

\[
\frac{da}{dt} = -i\omega a
\]  

(2.38)

which implies that

\[
a^\dagger (t) = a^\dagger (0) e^{i\omega t}
\]  

(2.39)
and

\[ a(t) = a(0) e^{-i\omega t} \]  

(2.40)

We may expand \( e^{-i\omega t} \) as

\[ e^{-i\omega t} = 1 - i\omega t - \frac{\omega^2 t^2}{2!} + i \frac{\omega^3 t^3}{3!} + ... \]  

(2.41)

which allows us to write \( a(t) \) as

\[ a(t) = a(0) \left( 1 - i\omega t - \frac{\omega^2 t^2}{2!} + i \frac{\omega^3 t^3}{3!} + ... \right) \]  

(2.42)

A useful combination of operators will be \( a^{\dagger}a = n \), a combination known as the number operator. If applied to the \( n^{th} \) eigenstate of the Hamiltonian \( |n\rangle \) (we will later come to identify \( |n\rangle \) as the \( n \) photon state),

\[ a^{\dagger}a |n\rangle = n |n\rangle \]  

(2.43)

this operator gives the value \( n \) of the eigenstate occupied. The energy eigenvalue problem can then be written as

\[ H |n\rangle = \hbar \omega \left( a^{\dagger}a + \frac{1}{2} \right) |n\rangle \]  

(2.44)

\[ = \hbar \omega \left( n + \frac{1}{2} \right) |n\rangle = E_n |n\rangle \]  

(2.45)
Therefore

\[ E_n = \hbar \omega \left( n + \frac{1}{2} \right) . \]  

(2.46)

Where \( E_0 \) is the ground state energy, since 0 is the lowest value which may be taken by \( n \) as can be seen from acting on the state \( |0\rangle \) with the annihilation operator \( a \).

\[ a |0\rangle = 0 \]  

(2.47)

Since

\[ a |n\rangle = \sqrt{n} |n - 1\rangle . \]  

(2.48)

and

\[ a^\dagger |n\rangle = \sqrt{n + 1} |n + 1\rangle . \]  

(2.49)

It follows that any arbitrary eigenstate \( |n\rangle \) can be written in terms of the vacuum state as (e.g. [1])

\[ |n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle . \]  

(2.50)

The states \( |n\rangle \) form a complete basis for the Hamiltonian \( H \), and are orthonormal

\[ \langle m | n \rangle = \delta_{mn} \]  

(2.51)

The non vanishing matrix elements of the creation and annihilation operators are

\[ \langle n - 1 | a | n \rangle = \sqrt{n} \]  

(2.52)
\[ \langle n + 1 | a^\dagger | n \rangle = \sqrt{n + 1} \] (2.53)

### 2.2 Quantum fluctuations of the single mode field.

Recall the operator for the electric field given in (8),

\[ \hat{E}_x (z, t) = \mathcal{E}_0 \left( \hat{a}^\dagger (t) + \hat{a} (t) \right) \sin (kz). \] (2.54)

The eigenstates of the Hamiltonian \( |n\rangle \), do not form a basis for the operator \( \hat{E}_x (z, t) \). This is implied by the fact that the number operator \( n = a^\dagger a \), which does commute with the Hamiltonian, does not commute with the electric field operator \( E_x \).

To see this, let us first calculate the average field \( \langle E_x \rangle \)

\[ \langle E_x \rangle = \langle n | \hat{E}_x (z, t) | n \rangle = \mathcal{E}_0 \left[ \langle n | a | n \rangle + \langle n | a^\dagger | n \rangle \right] \sin (kz) \] (2.55)

which, by (10) and (11), become

\[ = \mathcal{E}_0 [0 + 0] \sin (kz) = \left\langle \hat{E}_x \right\rangle = 0 \] (2.56)

that is, the average field is zero.

The energy density of the field \( E = e_x E_x \) is proportional to the mean square of \( E_x \), [1]

\[ \langle E_x^2 \rangle = 2\mathcal{E}_0^2 \sin^2 (kz) \left( n + \frac{1}{2} \right) \] (2.57)
The variance is defined as \[1\]

\[
\langle (\Delta E_x)^2 \rangle = \langle E_x^2 \rangle - \langle \hat{E}_x \rangle^2
\]  

(2.58)

i.e., its the mean square of the standard deviation, which for the eigenstate \(|\varphi_n\rangle\) becomes

\[
\Delta E_x = \sqrt{\langle E_x^2 \rangle - \langle \hat{E}_x \rangle^2} = \sqrt{2\mathcal{E}_0^2 \sin^2(kz) \left(n + \frac{1}{2}\right)}
\]  

(2.59)

that is,

\[
\Delta E_x = \sqrt{2\mathcal{E}_0 \sin(kz)} \left(n + \frac{1}{2}\right)^{1/2}
\]  

(2.60)

It is interesting to note that even for \(n = 0\) we have

\[
\Delta E_x = \sqrt{2\mathcal{E}_0 \sin(kz)} \frac{1}{\sqrt{2}} = \mathcal{E}_0 \sin(kz)
\]  

(2.61)

these are called the vacuum fluctuations of the field, since they correspond to the eigenstate of the vacuum \(|0\rangle\), the state with zero photons. \[1\]

In the case of the electromagnetic field confined to a 1D cavity, the eigenstates of the Hamiltonian, namely \(|n\rangle\), correspond to states of photon number \(n\). One important fact is that the number operator \(n = a^\dagger a\) and the electric field operator \(E_x\) do not commute,

\[
[n, E_x] = \mathcal{E}_0 \sin(kz) (a^\dagger - a)
\]  

(2.62)
The generalized uncertainty relations state that for any two operators $A$ and $B$ satisfying $[A, B] = C$, it follows that the product of the uncertainties of $A$ with that of $B$ obey the inequality

$$\Delta A \Delta B \geq \frac{1}{2} |\langle C \rangle|$$  \hspace{1cm} (2.63)

It follows therefore, that the number operator and the electric field obey the following uncertainty relations

$$\Delta n \Delta E_x \geq \frac{1}{2} \mathcal{E}_0 |\sin (kz)| \left| \langle a^t - a \rangle \right|$$  \hspace{1cm} (2.64)

This implies a number-phase uncertainty relation [1]

$$\Delta n \Delta \phi \geq 1$$  \hspace{1cm} (2.65)

where $0 < \phi < 2\pi$ is the phase angle associated with the creation and annihilation operators. In fact, it will turn out, that the situation is not actually quite that simple. It turns out to be a very slippery task to define a unique phase operator, and in fact is not possible in general [5],[6],[7]. It can be shown, however, that for proper definitions of the phase,namely those given in [4], that the photon number states $|\varphi_n\rangle$ have a uniform phase distribution $(\Delta \phi/\Delta n) \approx \text{constant}$ for $0 \leq \phi \leq 2\pi$. For more on the number phase uncertainty relations see [3],[4] and the references given in [1].
2.3 Multimode fields.

In free space in the absence of any sources, the source free Maxwell equations are still valid, (joking but obviously true). We write the electric and magnetic fields in terms of the vector potential $A(r, t)$ which satisfies the wave equation [1]

$$\nabla^2 A - \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} = 0$$  \hspace{1cm} (2.66)

and we choose the Coulomb gauge condition (which will become useful later on)

$$\nabla \cdot A = 0$$  \hspace{1cm} (2.67)

The electric field is then given by [1]

$$E = -\frac{\partial A}{\partial t}$$  \hspace{1cm} (2.68)

and the magnetic field is

$$B = \nabla \times A$$  \hspace{1cm} (2.69)

As long as $L \gg \frac{1}{k}$, we can model free space as cubic cavity, with sides of length $L$, therefore we may impose periodic boundary conditions on the faces of the cube [1]. This allows us to deal with the mathematically simpler case of having a denumerably infinite set of normal modes, rather than a non-denumerably infinite set of modes [1]. We require plane waves in the $x_i$ di-
rection, where \((i = 1, 2, 3)\) and \((x_1 = x, x_2 = y, x_3 = z)\), to satisfy the condition

\[ e^{ik_{x_i}x_i} = e^{ik_{x_i}(x_i+L)} \]  \hspace{1cm} (2.70)

which leads the following conditions for the direction numbers \(k_{x_i}\)

\[ k_x = \left(\frac{2\pi}{L}\right) m_x \] \hspace{1cm} (2.71)

\[ k_y = \left(\frac{2\pi}{L}\right) m_y \] \hspace{1cm} (2.72)

\[ k_z = \left(\frac{2\pi}{L}\right) m_z \] \hspace{1cm} (2.73)

where

\[ m_x = m_y = m_z = 0, \pm 1, \pm 2, ... \] \hspace{1cm} (2.74)

Now, the wave vector

\[ \mathbf{k} = (k_x, k_y, k_z) = \frac{2\pi}{L} (m_x, m_y, m_z) \] \hspace{1cm} (2.75)

Moreover, \(k = ||\mathbf{k}|| = \sqrt{k \cdot \mathbf{k}} = \omega_k/c\). Distinct normal modes of the fields are specified by distinct sets of integers \((m_x, m_y, m_z)\). Therefore, the total numbers of modes in the interval \((\Delta m_x, \Delta m_y, \Delta m_z)\) is [1]

\[ \Delta m = \Delta m_x \Delta m_y \Delta m_z = 2 \left(\frac{L}{2\pi}\right)^3 \Delta k_x \Delta k_y \Delta k_z \] \hspace{1cm} (2.76)

taking into account a factor of 2 for the two independent polarizations. In the
limit that \( L \to \infty, \triangle m \to dm \) and we have \((V = L^3)\)

\[
dm = dm_x dm_y dm_z = \left(\frac{V}{4\pi^3}\right) dk_x dk_y dk_z
\]

(2.77)

going to spherical coordinates this is

\[
k = (k_x, k_y, k_z) = k (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)
\]

(2.78)

and therefore

\[
dm = \left(\frac{V}{4\pi^3}\right) k^2 dk d\Omega
\]

(2.79)

or using \( k = \omega_k / c \) we can write this as

\[
dm = \left(\frac{V}{4\pi^3}\right) \frac{\omega_k^2 k}{c^3} d\omega_k d\Omega
\]

(2.80)

Integrating over the solid angle \( \Omega \), we obtain

\[
dm = \frac{V}{\pi^2} k^2 dk = V \rho_k dk
\]

(2.81)

Where \( \rho_k = k^2 / \pi^2 \). We may also write this for \( d\omega_k \) as

\[
dm = V \frac{\omega_k^2}{\pi^2 c^3} d\omega_k = V \rho_k (\omega_k) d\omega_k
\]

(2.82)

for which

\[
\rho_k (\omega_k) = \frac{\omega_k^2}{\pi^2 c^3}
\]

(2.83)
Having “pasted” the cubic grid on our space, we may proceed to expand the vector potential as

\[ A(\mathbf{r}, t) = \sum_{k,s} e_{ks} [A_{ks}(t) e^{i\mathbf{k} \cdot \mathbf{r}} + A_{ks}^*(t) e^{-i\mathbf{k} \cdot \mathbf{r}}] \]  \hspace{1cm} (2.84)

where \( A_{ks} \in \mathbb{C} \) is the amplitude of the field and \( e_{ks} \in \mathbb{R} \) is a polarization vector [1]. Moreover the sum over \( k \) is the sum over the distinct sets of integers \((m_x, m_y, m_z)\), and the sum over \( s \) is the sum over the two polarization directions [1], which must obey the orthonormality relations

\[ e_{ks} \cdot e_{ks'} = \delta_{kk'} \delta_{ss'} \]  \hspace{1cm} (2.85)

The Coulomb gauge condition requires that \( \mathbf{k} \cdot e_{ks} = 0 \), which is known as the \textit{transversality} condition [1].

The wave equation and the Coulomb gauge lead to the following relations for the amplitudes \( A_{ks} \):

\[ \frac{d^2 A_{ks}}{dt^2} + \omega_k^2 A_{ks} = 0 \]  \hspace{1cm} (2.86)

The solution to this differential equation is

\[ A_{ks}(t) = A_{ks} e^{-i\omega_k t} \]  \hspace{1cm} (2.87)
Thus, the electric and magnetic fields become [1]

\[
E(r, t) = i \sum_{k,s} \omega_k e_{ks} \left[ A_{ks} e^{i(k \cdot r - \omega_k t)} + A_{ks}^* (t) e^{-i(k \cdot r - \omega_k t)} \right]
\]

(2.88)

\[
B(r, t) = \frac{i}{c} \sum_{k,s} \omega_k \left( \frac{k}{|k|} \times e_{ks} \right) \left[ A_{ks} e^{i(k \cdot r - \omega_k t)} + A_{ks}^* (t) e^{-i(k \cdot r - \omega_k t)} \right]
\]

(2.89)

The energy of the field is

\[
H = \frac{1}{2} \int dV \left[ \varepsilon_0 E \cdot E + \frac{1}{\mu_0} B \cdot B \right]
\]

(2.90)

Now [1],

\[
\left( \frac{k}{|k|} \times e_{ks} \right) \cdot \left( \frac{k}{|k|} \times e_{ks'} \right) = \delta_{ss'}
\]

(2.91)

and

\[
\left( \frac{k}{|k|} \times e_{ks} \right) \cdot \left( \frac{k}{|k|} \times e_{ks'} \right) = -e_{ks} \cdot e_{-ks'}
\]

(2.92)

Taking our periodic boundary conditions into account, we have

\[
\int e^{\pm i(k-k') \cdot r} dV = \delta_{kk'} V
\]

(2.93)

Therefore, the contribution to \( H \) from the electric field is

\[
\frac{1}{2} \int dV \left[ \varepsilon_0 E \cdot E \right] = \varepsilon_0 V \sum_{k,s} \omega_k^2 A_{ks} A_{ks}^* - R
\]

(2.94)
The contribution from the magnetic field is

\[ \frac{1}{2} \int dV \frac{1}{\mu_0} \mathbf{B} \cdot \mathbf{B} = \varepsilon_0 V \sum_{k,s} \omega_k^2 A_{ks} A^*_{ks} + R \]  

(2.95)

where,

\[ R = \frac{1}{2} \varepsilon_0 V \sum_{k,s} \omega_k^2 e_{ks} \cdot e_{-ks'} \left[ A_{ks} (t) A_{-ks'} (t) + A^*_{ks} (t) A^*_{-ks'} (t) \right] \]  

(2.96)

Therefore, the total energy in the field is

\[ H = 2 \varepsilon_0 V \sum_{k,s} \omega_k^2 A_{ks} (t) A^*_{ks} (t) \]  

(2.97)

but since \( A_{ks} (t) = A_{ks} e^{-i\omega_k t} \), this may be written as

\[ H = 2 \varepsilon_0 V \sum_{k,s} \omega_k^2 A_{ks} A^*_{ks} \]  

(2.98)

### 2.4 Quantization of the multimode field.

We may quantize the field by introducing the canonical position and momentum operators \( q_{ks} \) and \( p_{ks} \), respectively through the definitions

\[ A_{ks} = \frac{1}{2\omega_k \sqrt{\varepsilon_0 V}} [\omega_k q_{ks} + ip_{ks}] \]  

(2.99)

\[ A^*_{ks} = \frac{1}{2\omega_k \sqrt{\varepsilon_0 V}} [\omega_k q_{ks} - ip_{ks}] \]  

(2.100)
in which case the Hamiltonian becomes

$$H = \frac{1}{2} \sum_{k,s} \left( p_{ks}^2 + \omega_k^2 q_{ks}^2 \right)$$  \hspace{1cm} (2.101)$$
as it should.

The canonical variables obey the canonical commutation relations

$$[q_{ks} , q_{k's}] = 0 = [p_{ks} , p_{k's}]$$  \hspace{1cm} (2.102)$$

$$[q_{ks} , p_{k's}] = i\hbar \delta_{kk'} \delta_{ss'}$$  \hspace{1cm} (2.103)$$

Just as we did for the single mode field, we may define the creation and annihilation operators for the multimode fields

$$\sqrt{2\hbar \omega_k} a_{ks} = \omega_k q_{ks} + ip_{ks}$$  \hspace{1cm} (2.104)$$

$$\sqrt{2\hbar \omega_k} a_{ks}^\dagger = \omega_k q_{ks} - ip_{ks}$$  \hspace{1cm} (2.105)$$

The creation and annihilation operators obey the following commutation relations

$$[a_{ks} , a_{k's}^\dagger] = \left[ a_{ks}^\dagger , a_{k's}^\dagger \right] = 0$$  \hspace{1cm} (2.106)$$

$$[a_{ks} , a_{k's}^\dagger] = \delta_{kk'} \delta_{ss'} \delta \left( k' - k \right)$$  \hspace{1cm} (2.107)$$

Just as was the case for the single mode field, the number operator for the
mode $ks$ is $n_{ks} = a_{ks}^\dagger a_{ks}$, and the Hamiltonian is

$$H = \sum_{k,s} \hbar \omega_k \left( a_{ks}^\dagger a_{ks} + \frac{1}{2} \right)$$  \hspace{1cm} (2.108)$$

$$= \sum_{k,s} \hbar \omega_k \left( n_{ks} + \frac{1}{2} \right)$$  \hspace{1cm} (2.109)$$

Each mode is independent of all the rest and has the eigenstates $|n_{ks}\rangle$. If we let $j$ denote the $j^{th}$ mode $k_j s_j$, then we may write the $n^{th}$ photon number state of the $j^{th}$ mode as

$$|\{n_j\}\rangle = \prod_j \frac{\left( a_j^\dagger \right)^{n_j}}{\sqrt{n_j!}} |\varphi_0^{(j)}\rangle$$  \hspace{1cm} (2.110)$$

The energy eigenvalue equation is then

$$H |\{n_j\}\rangle = E |\{n_j\}\rangle$$  \hspace{1cm} (2.111)$$

where [1]

$$E = \sum_j \hbar \omega_j \left( n_j + \frac{1}{2} \right)$$  \hspace{1cm} (2.112)$$

A multimode photon state is the tensor product of all of the individual mode number states, that is

$$|n_1, n_2, n_3...\rangle = |n_1\rangle \otimes |n_2\rangle \otimes |n_3\rangle \otimes ...$$  \hspace{1cm} (2.113)$$

$$= |\{n_j\}\rangle$$  \hspace{1cm} (2.114)$$
The number states are orthogonal, that is

\[ \langle \{ n_{j'} \}| \{ m_j \} \rangle = \prod_{j,j'} \delta_{n_j m_j} \delta_{j j'} \quad \quad (2.115) \]

The multimode vacuum state is

\[ |\{ 0 \}_j \rangle = |0 \rangle \otimes |0 \rangle \otimes |0 \rangle \otimes \ldots \quad \quad (2.116) \]

The action of the creation and annihilation operators on \( j^{th} \) mode of the multimode photon number state are given by

\[ a_j |\{ n_j \} \rangle = \sqrt{n_j} |\{ (n-1) \}_j \rangle \quad \quad (2.117) \]

\[ a_{j}^\dagger |\{ n_j \} \rangle = \sqrt{n_j + 1} |\{ (n+1) \}_j \rangle \quad \quad (2.118) \]

Quantization requires that the amplitudes \( A_{ks} \) become the operators:

\[ \hat{A}_{ks} = \left( \frac{\hbar}{2 \omega_k \varepsilon_0 V} \right)^{1/2} a_{ks} \quad \quad (2.119) \]

Which therefore allows us to define a vector potential operator as well as electric and magnetic field operators, which are, respectively,

\[ \hat{\mathbf{A}}(r, t) = \sum_{k,s} \left( \frac{\hbar}{2 \omega_k \varepsilon_0 V} \right)^{1/2} e_{ks} \left[ a_{ks} e^{i(\mathbf{k} \cdot \mathbf{r} - \omega_k t)} + a_{ks}^\dagger e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega_k t)} \right] \quad \quad (2.120) \]
\[ \hat{E}(\mathbf{r}, t) = i \sum_{k,s} \left( \frac{\hbar \omega_k}{2 \varepsilon_0 V} \right)^{1/2} e_{ks} \left[ a_{ks} e^{i(k \cdot \mathbf{r} - \omega_k t)} - a^\dagger_{ks} e^{-i(k \cdot \mathbf{r} - \omega_k t)} \right] \] (2.121)

\[ \hat{B}(\mathbf{r}, t) = \frac{i}{c} \sum_{k,s} \omega_k \left( \frac{k}{|k|} \times e_{ks} \right) \left( \frac{\hbar \omega_k}{2 \varepsilon_0 V} \right)^{1/2} \left[ a_{ks} e^{i(k \cdot \mathbf{r} - \omega_k t)} - a^\dagger_{ks} e^{-i(k \cdot \mathbf{r} - \omega_k t)} \right] \] (2.122)

where the operators \( a_{ks} = a_{ks}(0) \) form a basis of the Heisenberg representation.

The time dependent creation and annihilation operators are, respectively

\[ a^\dagger_{ks}(t) = a^\dagger_{ks}(0) e^{i \omega_k t} \] (2.123)

\[ a_{ks}(t) = a_{ks}(0) e^{-i \omega_k t} \] (2.124)

It can be seen that the magnetic field is weaker than the electric field by a factor of \( 1/c \), which is what we should expect and is a reassuring sign we haven’t gone off track. The magnetic field couples to the spin magnetic moment of the electrons which is negligible for the aspects of quantum optics which we will investigate.

Some interesting things to note are:

- A single mode plane wave has electric field components given by

\[ \hat{E}(\mathbf{r}, t) = i \left( \frac{\hbar \omega}{2 \varepsilon_0 V} \right)^{1/2} e_x \left[ a e^{i(k \cdot \mathbf{r} - \omega t)} - a^\dagger e^{-i(k \cdot \mathbf{r} - \omega t)} \right] \] (2.125)

- Quantum optics often works in the domain of optical radiation, whose
wavelength \( \lambda \) is on the order \( 10^3 \text{Å} \). Therefore in these situations we may approximate

\[
e^{\pm ik \cdot r} \approx 1 \pm i k \cdot r \quad (2.126)
\]
since, in these situations it is true that

\[
\frac{\lambda}{2\pi} = \frac{1}{|k|} \gg |r_{\text{atom}}| \quad (2.127)
\]

and hence the electric field can be expanded as

\[
\hat{E}(r, t) \approx \hat{E}(t) = i \left( \frac{\hbar \omega}{2\varepsilon_0 V} \right)^{1/2} e_x [ae^{-i\omega t} - a^\dagger e^{i\omega t}] \quad (2.128)
\]

which is known as the dipole approximation.

3 Thermal Modes.

Consider a single mode field in thermodynamic equilibrium with the walls of a cavity of absolute temperature \( T \). The density operator \( \rho \) for the system is [2]

\[
\rho = Z^{-1} e^{-H/kT} \quad (3.1)
\]

where \( H \) is the Hamiltonian, \( k \) is Boltzmann’s constant and \( Z \) is called the partition function, and it is introduced as a normalization factor, in order that
trace of $\rho$ be one\cite{2}. That is

\[ Z = \text{Tr} \left\{ e^{-H/kT} \right\}. \]  

(3.2)

The density matrix for the system is diagonal in the Hamiltonian eigenbasis $|\varphi_n\rangle$ since it is in thermodynamic equilibrium. The diagonal matrix components, gives the population of the stationary state $|\varphi_n\rangle$, which in this case are all the same. They are:

\[ \rho_{nn} = \langle n | Z^{-1} e^{-H/kT} | n \rangle \]  

(3.3)

\[ = Z^{-1} e^{-E_n/kT}. \]  

(3.4)

Since $E_n = \hbar \omega \left( n + \frac{1}{2} \right)$, the partition function becomes\cite{1}

\[ Z = \exp \left( -\frac{\hbar \omega}{2kT} \right) \sum_n \exp \left( -\frac{\hbar \omega_n}{2kT} \right) \]  

(3.5)

Since $\exp (-\hbar \omega/kT) < 1$ we may sum the series

\[ \sum_n \exp (-\hbar \omega/kT) = \frac{1}{1 - \exp (-\hbar \omega/kT)} \]  

(3.6)

Therefore

\[ Z = \frac{\exp (-\hbar \omega/kT)}{1 - \exp (-\hbar \omega/kT)} \]  

(3.7)

The off-diagonal terms vanish, hence there are no coherences between sta-
tionary states, and
\[
\rho_{nm} = \langle n | Z^{-1} e^{-H/kT} | m \rangle \tag{3.8}
\]
\[
= Z^{-1} e^{-E_m/kT} \langle n | m \rangle = 0. \tag{3.9}
\]

We observe that in thermodynamic equilibrium, the populations of the stationary states decrease exponentially with the energy. Since there are no coherences, the system in this case may be considered to be a statistical mixture of the states \(|n\rangle\).

The probability that the thermal mode is in the \(n^{th}\) thermally excited state is
\[
P_n = \langle n | \rho_{th} | n \rangle \tag{3.10}
\]
\[
= \frac{\exp \left( -E_n/kT \right)}{\sum_n \exp \left( -E_n/kT \right)} \tag{3.11}
\]
The density operator may be written as[1]
\[
\rho_{th} = \sum_{n'=0}^{\infty} \sum_{n=0}^{\infty} |n'\rangle \langle n | \rho_{th} | n \rangle \langle n | \tag{3.12}
\]
\[
= \frac{1}{Z} \sum_{n=0}^{\infty} \exp \left( -E_n/kT \right) |n\rangle \langle n | \tag{3.13}
\]
\[
= \sum_{n=0}^{\infty} P_n |n\rangle \langle n | \tag{3.14}
\]
The average photon number of the thermal field is [1]

\[
n = \text{Tr} (n \rho_{th}) = \frac{1}{\exp (\hbar \omega / kT) - 1} \tag{3.15}
\]

from which it follows that for \( kT \gg \hbar \omega \rightarrow \)

\[
n \approx kT / \hbar \omega \tag{3.16}
\]

While for \( \hbar \omega \gg kT \rightarrow \)

\[
n \approx \hbar \omega / kT \tag{3.17}
\]

4 The interaction of atoms and electromagnetic waves

The Hamiltonian for a system consisting an electron bound to an atom in the presence of external fields is

\[
H (r, t) = \frac{1}{2m} [\mathbf{P} + e \mathbf{A} (r, t)]^2 - e \Phi (r, t) + V (r) \tag{4.1}
\]

The gauge invariant electric and magnetic fields are given by

\[
\mathbf{E} (r, t) = -\nabla \Phi (r, t) - \frac{\partial \mathbf{A}}{\partial t} \tag{4.2}
\]

\[
\mathbf{B} (r, t) = \nabla \times \mathbf{A} (r, t) \tag{4.3}
\]
Gauge invariance means that these fields are invariant under the gauge transformations
\[ \Phi' (r, t) = \Phi (r, t) - \frac{\partial \chi (r, t)}{\partial t} \]  \hspace{1cm} (4.4)
\[ A' (r, t) = A (r, t) + \nabla \chi (r, t) \] \hspace{1cm} (4.5)

Therefore the equation governing the time evolution of the system in the Schrodinger representation, is the Schrodinger equation
\[ H (r, t) \Psi (r, t) = i\hbar \frac{\partial \Psi (r, t)}{\partial t}. \] \hspace{1cm} (4.6)

In quantum mechanics all operators are invariant under obey a global $U(1)$ similarity transformation; for some operator $A, A' = UAU^\dagger$, where $U$ is some unitary operator. Moreover, all state vectors are invariant under multiplication by a common $U$. This essentially means that given some quantum mechanical representation of system with all of its operators, and states, etc., we may obtain an equivalent description of that same system if we simultaneously transform all of the states and operators of the theory in the manner prescribed above. The resulting transformed theory will lead to all of the same results as the original theory. The usefulness of this fact is that a particular operator may take on a more tractable form in the transformed theory. Therefore, we may exploit this fact to simplify the Hamiltonian. It will prove useful for us to define the unitary operator $R$ which takes us to another representa-
tion $\Psi'(r,t)$ of the eigenstate $\Psi(r,t)$, by the action of $R$ on $\Psi(r,t)$, namely

$$R\Psi(r,t) = \Psi'(r,t).$$  \hfill (4.7)

The transformed Hamiltonian obeys its own Schrödinger equation

$$H'(r,t)\Psi'(r,t) = i\hbar \frac{\partial \Psi'(r,t)}{\partial t}$$  \hfill (4.8)

where \cite{1}

$$H'(r,t) = RH R^\dagger + i\hbar \frac{\partial R}{\partial t} R^\dagger$$  \hfill (4.9)

Choosing

$$R = \exp \left( -ie\chi(r,t)/\hbar \right)$$  \hfill (4.10)

which amounts to choosing the Coulomb gauge, we have

$$H' = \frac{1}{2m} \left[ P + eA'(r,t) \right]^2 - e\Phi'(r,t) + V(r)$$  \hfill (4.11)

It is important to note that we will be working in the Coulomb gauge, which is not relativistically covariant, but for which $\Phi(r,t) = 0$ and $\nabla \cdot A = 0$ (the transversality condition), therefore the radiation field is completely determined by the vector potential. If there are no sources near the atom, then $A$ satisfies the homogeneous wave equation

$$\nabla^2 A - 1/c^2 \frac{\partial^2 A}{\partial t^2} = 0.$$  \hfill (4.12)
whose solution has the form

\[ \mathbf{A} = A_0 e^{i(k \cdot r - \omega_k t)} + A^\dagger_0 e^{-i(k \cdot r - \omega_k t)} \]  

(4.13)

In the Coulomb gauge the radiation field is completely determined by the vector potential as can be seen from the Hamiltonian

\[ H(r, t) = \frac{\mathbf{P}^2}{2m} + \frac{e}{m} \mathbf{A} \cdot \mathbf{P} + \frac{e^2}{m} \mathbf{A}^2 + V(r) \]  

(4.14)

The transformed Hamiltonian becomes[1]

\[ H' = \frac{1}{2m} [\mathbf{P} + e(\mathbf{A} + \nabla \chi)]^2 - e \frac{\partial \chi}{\partial t} + V(r) \]  

(4.15)

Since \(|k| = 2\pi/\lambda\), it follows that for \(|r| \sim 5a_0\) (Bohr radius) and \(\lambda \sim 500\text{nm}\) (optical radiation), \(k \cdot r \ll 1\), thus we may invoke the dipole approximation which gives the first order interactions and which also implies that on length scales \(\sim |r|_{\text{atom}}\), the vector potential is locally uniform, and we may make use of the fact that \(\mathbf{A}(r, t) \simeq \mathbf{A}(t)\). If we choose our gauge function to be

\[ \chi(r, t) = -\mathbf{A} \cdot \mathbf{r} \]  

(4.16)

it follows that

\[ \nabla \chi(r, t) = -\mathbf{A}(t) \]  

(4.17)

\[ \frac{\partial \chi}{\partial t}(r, t) = -\mathbf{r} \cdot \frac{\partial \mathbf{A}}{\partial t} = -\mathbf{r} \cdot \mathbf{E}(t) \]  

(4.18)
which means that
\[ H'(r,t) = \frac{p^2}{2m} + V(r) + \epsilon \mathbf{r} \cdot \mathbf{E}(t) \]  
(4.19)

we recognize the quantity \( d = -\epsilon \mathbf{r} \) as the electric dipole moment. and we may write
\[ H'(r,t) = H_0 - d \cdot \mathbf{E}(t) \]  
(4.20)

### 4.1 Interaction of an atom with a classical dipole field

Let us begin with the case of a classical field of frequency \( \omega \), given by [1]
\[ \mathbf{E}(t) = E_0 \cos(\omega t) \Theta(t) \]  
(4.21)

where
\[ \Theta(t) = \begin{cases} 
1 & t > 0 \\
0 & t < 0 
\end{cases} \]  
(4.22)

just means that the field is turned on at a time \( t = 0 \). We can study the interaction of an atom with this field by using perturbation theory. Expanding to first order just amounts to using the dipole approximation, \( k \cdot r \ll 1 \), which we have seen previously is satisfied in the case of atoms interacting with a classical electromagnetic field.

Given an atom, in some initial state \( |i\rangle \), we can expand of the atomic state of the atom for all \( t > 0 \), in a basis of uncoupled atomic states \( |k\rangle \), which span
the space of $H^{int}$,

$$|\psi(t)\rangle = \sum_k C_k(t) e^{-iE_k t/\hbar} |k\rangle \quad (4.23)$$

where the amplitudes $C_k(t)$ are normalized such that

$$\sum_k |C_k(t)|^2 = 1. \quad (4.24)$$

Now, working in the Schrodinger picture, the atomic state at time $t$, $|\psi(t)\rangle$, must obey the time dependent Schrodinger equation, which is:

$$i\hbar \frac{\partial |\psi(t)\rangle}{\partial t} = (H_0 + H^{int}) |\psi(t)\rangle \quad (4.25)$$

where, in the dipole approximation, as we know from the last section, $H^{int} = -d \cdot E(t)$. Substituting our expression for $|\psi(t)\rangle$ into the Schrodinger equation, and then multiplying from the left by $\langle l | e^{-iE_k t/\hbar}$, we (denoting time derivatives with dots $\dot{}$) obtain a set of coupled first order differential equations for the amplitudes

$$\dot{C}_k(t) = -i\frac{\hbar}{\hbar} \sum_k C_k(t) \langle l | H^{int} |k\rangle e^{i\omega_{lk} t}$$

Where,

$$\omega_{lk} = (E_l - E_k)/\hbar, \quad (4.26)$$

are the transition frequencies between atomic states $|l\rangle$ and $|k\rangle$. In order that we may solve these equations we must also subject them to the condition, that the initial atomic state is $|i\rangle$, which implies that $C_k(0) = 1$. As the state evolves
in time, the initial state will transfer to other other $|f\rangle$ with a probability given by

$$P_{i\rightarrow f}(t) = |C_f(t)|^2. \quad (4.27)$$

To further simplify the task of solving this set of coupled equations analytically, we expand the amplitudes as a power series in some coupling parameter $0 < \lambda < 1$ (which measures the strength of the interaction relative to scale at which our theory breaks down, $\lambda = 1$).

$$C_i(t) = C_i^{(0)}(t) + \lambda C_i^{(1)}(t) + \lambda^2 C_i^{(2)}(t) + ... \quad (4.28)$$

Inserting the expression for $C_i(t)$ into (IV.1) we obtain a recursion for the $n^{th}$ amplitude

$$\dot{C}_i^{(n)}(t) = -\frac{i}{\hbar} \sum_k C_k^{(n-1)}(t) \langle l| H_{lk}^{\text{int}} |k\rangle e^{i\omega_{lk}t} \quad (4.29)$$

which leads to a coupled set of equations for all of the $C_i^n(t)$, which up to second order are given by

$$\dot{C}_i^{(0)}(t) = 0 \quad (4.30)$$

$$\dot{C}_i^{(1)}(t) = -\frac{i}{\hbar} \sum_k C_k^{(0)}(t) \langle l| H_{lk}^{\text{int}} |k\rangle e^{i\omega_{lk}t} \quad (4.31)$$

$$\dot{C}_i^{(2)}(t) = -\frac{i}{\hbar} \sum_k C_k^{(1)}(t) \langle l| H_{lk}^{\text{int}} |k\rangle e^{i\omega_{lk}t} \quad (4.32)$$

The only surviving terms in the sum are those for $k = i$. Therefore the first
order amplitude becomes, upon integrating on time

\[ C_f^{(1)}(t) = -\frac{i}{\hbar} \int_0^t dt' H^{\text{int}}_{fi} e^{i\omega_f t'} C_i^{(0)}(t') \]  

(4.33)

using the recursion relation, inserting this value for \( C_i^{(1)}(t) \) into the equation for \( C_i^{(2)}(t) \) and integrating on time enables us to find \( C_i^{(2)}(t) \), (see [1])

\[ C_f^{(2)}(t) = -\frac{i}{\hbar} \sum_l \int_0^t dt' H^{\text{int}}_{fl}(t') e^{i\omega_f t'} C_i^{(1)}(t') \]  

(4.34)

\[ = \left(-\frac{i}{\hbar}\right)^2 \sum_l \int_0^t dt' \int_0^{t'} dt'' H^{\text{int}}_{fl}(t') e^{i\omega_f t'} H^{\text{int}}_{li}(t'') e^{i\omega_f t''} C_i^{(0)}(t'') \]  

(4.35)

The total transition probability as a function of time, for a transition from state \( |i\rangle \) to a state \( |f\rangle \) is:

\[ P_{i\rightarrow f}(t) = |C_f^{(0)}(t) + C_f^{(1)}(t) + C_f^{(2)}(t) + ...|^2 \]  

(4.36)

We have up to now, neglected taking account of the fact that the dipole moment operator \( d \) only has non-vanishing matrix elements for states of opposite parity. Taking this into account we see that up to first order \( C_i^{(0)}(t') = 1 \), so that

\[ C_f^{(1)}(t) = -\frac{i}{\hbar} \int_0^t dt' H^{\text{int}}_{fi} e^{i\omega_f t'} \]  

(4.37)

\[ = \frac{1}{2\hbar} (d \cdot E_0)_{fi} \left[ \frac{e^{i(\omega+\omega_f)t} - 1}{(\omega + \omega_f)} - \frac{e^{-i(\omega+\omega_f)t} - 1}{(\omega - \omega_f)} \right] \]  

(4.38)

When the radiation frequency \( \omega \) is near the atomic transition frequency \( \omega_{fi} \), we
will have resonance, and in this case we may neglect the first “anti-resonant” term since the second term will clearly dominate. This is the so called \textit{rotating wave approximation}. With this in mind, the first order transition probability becomes [1]

\[
P_{i \to f}(t) = \left| \frac{(\mathbf{d} \cdot \mathbf{E}_0)_{fi}}{\hbar^2} \right| \sin^2 \left( \frac{\Delta t}{2} \right) \frac{\Delta^2}{\Delta^2} \tag{4.39}
\]

where we have introduced the notation \(\Delta = \omega - \omega_{fi}\), which is known as the \textit{detuning parameter}.

\section{Interaction of an atom with a quantized dipole field}

Earlier, while working in Heisenberg representation, we found that the quanta of a single mode electric field, in the absence of sources of any kind, were given by

\[
\hat{E}(t) = i \left( \frac{\hbar \omega}{2 \varepsilon_0 V} \right)^{1/2} e \left[ a e^{-i\omega t} - a^\dagger e^{i\omega t} \right] \tag{4.40}
\]

switching to the Schrödinger representation, this becomes

\[
\hat{E} = i \left( \frac{\hbar \omega}{2 \varepsilon_0 V} \right)^{1/2} e \left[ a - a^\dagger \right] \tag{4.41}
\]

The free Hamiltonian is

\[
H_0 = H^{atom} + H^{field} \tag{4.42}
\]
where

\[ H^{\text{atom}} = \frac{P^2}{2m} + V(r) \quad (4.43) \]

and

\[ H^{\text{field}} = \hbar \omega a^\dagger a \quad (4.44) \]

are the source free Hamiltonians of the atomic system and the field, respectively. We have suppressed the vacuum energy in our expression for \( H^{\text{field}} \) because it does not contribute to the dynamics. The interaction Hamiltonian is

\[ H^{\text{int}} = -d \cdot E(t) = -i \left( \frac{\hbar \omega}{2 \varepsilon_0 V} \right)^{1/2} (d \cdot e) (a - a^\dagger) \quad (4.45) \]

\[ = d \cdot \mathcal{E}_0 (a^\dagger - a) \quad (4.46) \]

where,

\[ \mathcal{E}_0 = i \left( \frac{\hbar \omega}{2 \varepsilon_0 V} \right)^{1/2} e \quad (4.47) \]

We have thus quantized the atomic system as well as the field system. If we wish to combine the distinct atom and field systems into one, atom-field system, we must remember that the state space of the atom-field system will in general be a linear superposition of the eigenstates of \( H^{\text{atom}} \) and \( H^{\text{field}} \).

Consider an atomic system in the initial state \( |a\rangle \). If we combine this atomic system with the field system which initially contains \( n \) photons, then we will have the atom-fields system which is initially in the state

\[ |i\rangle = |a\rangle |n\rangle. \quad (4.48) \]
Since the interaction Hamiltonian \( H^{\text{int}} \) is proportional to \((a^\dagger - a)\), it follows that for the \( n^{\text{th}} \) eigenstate \(|n\rangle\), the only non-vanishing matrix elements of \( H^{\text{int}} \) (in the atom-field eigenbasis) are the following

\[
\langle H^{\text{int}} \rangle = \sum_{i=1,2} \langle f_i | H^{\text{int}} | i \rangle = (d \cdot \mathcal{E}_0)_{ba} \langle b, m | (a^\dagger - a) | a, n \rangle = \quad (4.49)
\]

\[
= (d \cdot \mathcal{E}_0)_{ba} \left( \sqrt{n+1} \delta_{n,n+1} - \sqrt{n} \delta_{n,n-1} \right). \quad (4.50)
\]

where

\[
(d \cdot \mathcal{E}_0)_{ba} = \langle b | d | a \rangle \cdot \mathcal{E}_0 \quad (4.51)
\]

The quantity \( \langle b | d | a \rangle = d_{ba} \) gives the transition dipole moments between the states \(|b\rangle\) and \(|a\rangle\). Therefore the interaction Hamiltonian couples the \( n^{\text{th}} \) state to either the \( n+1 \) or \( n-1 \) state. In fact \( H^{\text{int}} \) induces a transition from the initial state of the atom-field system \(|i\rangle\) to the state \(|f_1\rangle = |b\rangle |n-1\rangle\) by absorption of a photon or to the state \(|f_2\rangle = |b\rangle |n+1\rangle\), by the emission of a photon. The energies of these states are [1]

\[
|i\rangle = |a\rangle |n\rangle \quad \leftrightarrow \quad E_i = E_a + n\hbar \omega \quad (4.52)
\]

\[
|f_1\rangle = |b\rangle |n-1\rangle \quad \leftrightarrow \quad E_{f_1} = E_b + (n - 1) \hbar \omega \quad (4.53)
\]

\[
|f_2\rangle = |b\rangle |n+1\rangle \quad \leftrightarrow \quad E_{f_2} = E_b + (n + 1) \hbar \omega \quad (4.54)
\]

where \( E_a \) and \( E_b \) are the energy eigenvalues of the respective atomic states \(|a\rangle\) and \(|b\rangle\).
Let us compare the results of the semi-classical versus the quantum treatment of this problem. In both cases, absorption is forbidden in any state for which \( n = 0 \) (zero photons in the system), for obvious reasons. However, for the quantum case of emission, even if \( n = 0 \) transitions may occur known as spontaneous emission a phenomenon with no semi-classical analog. In cases where \( n > 0 \), we then speak of the stimulated emission of an additional photon. In the classical case, if you start with no field, i.e no photon, then you will never have a photon later, but a photon later is almost a certainty in the quantum case. The matrix elements of the interaction are in the case of adsorption [1]

\[
\langle f_1 | H^{\text{int}} | i \rangle = \langle b, n - 1 | H^{\text{int}} | a, n \rangle = -(d \cdot E_0)_{ba} \sqrt{n} \tag{4.55}
\]

\[
= -(d \cdot E_0)_{ba} \sqrt{n} \tag{4.56}
\]

and for the case of emission, are

\[
\langle f_2 | H^{\text{int}} | i \rangle = \langle b, n + 1 | H^{\text{int}} | a, n \rangle = (d \cdot E_0)_{ba} \sqrt{n + 1} \tag{4.57}
\]

\[
= (d \cdot E_0)_{ba} \sqrt{n + 1} \tag{4.58}
\]

where, just as before,

\[
(d \cdot E_0)_{ba} = \langle b | d | a \rangle \cdot E_0 \tag{4.59}
\]

Fermi’s golden rule tells us that the rates of emission and absorption are proportional to square modulus of the matrix element coupling initial \( |i\rangle \) and final states \( |f_1\rangle, |f_2\rangle \), which in the case of a single mode (monochromatic) field
coupled to an atom, whose final state space is spanned by $|a\rangle$, $|b\rangle$. The transition matrix elements are given by: (see [1])

$$W_{i\rightarrow|f\rangle} = \frac{\pi}{2} \sum_{|f\rangle} \left| \frac{(d \cdot \mathcal{E}_0)_{fi}}{\hbar^2} \right|^2 \frac{\delta \left( \omega - \omega_{fi} \right)}{}.$$  \hspace{1cm} (4.60)

Moreover, since,

$$(d \cdot \mathcal{E}_0)_{ba} = \langle b | d | a \rangle \cdot \mathcal{E}_0$$  \hspace{1cm} (4.61)

this becomes,

$$W_{i\rightarrow|f\rangle} = \frac{\pi}{2} \sum_{|f\rangle} \left| \frac{\langle b | d | a \rangle \cdot \mathcal{E}_0}{\hbar^2} \right|^2 \frac{\delta \left( \omega - \omega_{fi} \right)}{}.$$  \hspace{1cm} (4.62)

Therefore the ratio of these rates is given by

$$\frac{\left| \langle f_2 | H^{\text{int}} | i \rangle \right|^2}{\left| \langle f_1 | H^{\text{int}} | i \rangle \right|^2} = \frac{n+1}{n}.$$  \hspace{1cm} (4.63)

For the time being, let’s just focus on two atomic states $|a\rangle$ and $|b\rangle$, and ignore the rest. This allows us to write the state vector as

$$|\psi(t)\rangle = C_i(t) |a\rangle |n\rangle e^{-iE_a t/\hbar} e^{-i\omega t} + C_{f_1}(t) |b\rangle |n-1\rangle e^{-iE_{b_1} t/\hbar} e^{-i(n-1)\omega t}$$

$$+ C_{f_2}(t) |b\rangle |n+1\rangle e^{-iE_{b_2} t/\hbar} e^{-i(n+1)\omega t}$$

since we already said that the initial state was $|\psi(t)\rangle = |a\rangle |n\rangle$, so therefore we have:

$$C_i(0) = 1$$  \hspace{1cm} (4.65)
\[ C_{f1}(0) = C_{f2}(0) = 0 \] (4.66)

We can use perturbation theory just before to obtain the probability amplitudes for all times, \( t > 0 \).

\[ C_{f1}^{(1)}(t) = -\frac{i}{\hbar} \int_0^t dt' \langle f_1 | H^{int} | i \rangle e^{i\omega_{f1}t'} C_{i}^{(0)}(t') \quad \text{(absorption)} \] (4.67)

\[ C_{f2}^{(1)}(t) = -\frac{i}{\hbar} \int_0^t dt' \langle f_2 | H^{int} | i \rangle e^{i\omega_{f2}t'} C_{i}^{(0)}(t') \quad \text{(emission)} \] (4.68)

where \( \omega_{f1i} = (E_{f1} - E_i) / \hbar \) and \( \omega_{f2i} = (E_{f2} - E_i) / \hbar \). Therefore the probability that the atom under goes a transition to the final state \( |b\rangle \) is the sum \( C_f^{(1)} = C_{f1}^{(1)} + C_{f2}^{(1)} \), i.e.

\[ C_f^{(1)} = \frac{i}{\hbar} (d \cdot E_0)_{ab} \left[ \sqrt{n+1} \left( \frac{e^{i(\omega + \omega_{ba})t} - 1}{\omega + \omega_{ba}} \right) - \sqrt{n} \left( \frac{e^{-i(\omega + \omega_{ba})t} - 1}{\omega - \omega_{ba}} \right) \right] \] (4.69)

where \( \omega_{ba} = (E_b - E_a) / \hbar \). If the initial state \( |a\rangle \) happens to be the excited state then \( \omega_{ba} < 0 \). In this case, for radiation frequencies \( \omega \sim (-\omega_{ba}) \), and we get spontaneous emission.

### 4.3 The Jaynes-Cummings Model

Perturbation theory breaks down in situation where we have a driving field of near resonance frequency. This is because the resonance causes large population transfers and the assumption made in perturbation theory that \( C_i(t) \approx 1 \), no longer holds. Therefore we must take another approach. One
such approach can be understood by noticing that for the case of near resonance, most of the population is transferred to the near resonant state, so that the other states may be neglected. The two most dominant states remain, and the resulting system of differential equations are much more tractable.

The *Jaynes-Cummings* Hamiltonian is (see[1],[10])

\[
H = \frac{1}{2} \hbar \omega_0 \sigma_3 + \hbar \omega a^\dagger a + \hbar \lambda (\sigma_+ a + \sigma_+ a^\dagger).
\]  
(4.70)

The *interaction term* is

\[
H^{\text{int}} = \hbar \lambda (\sigma_+ a + \sigma_+ a^\dagger)
\]  
(4.71)

and it induces the transitions,

\[
|e\rangle |n\rangle \leftrightarrow |g\rangle |n+1\rangle
\]  
(4.72)

or

\[
|e\rangle |n-1\rangle \leftrightarrow |g\rangle |n\rangle.
\]  
(4.73)

The product states $|e\rangle |n\rangle$, $|g\rangle |n+1\rangle$, etc., span what is known the *bare* basis, they are *bare* states of the *Jaynes-Cummings* model. For any given $n$, the dynamics of the system is confined to a $2D$ space of product states spanned by

\[
\{|e\rangle |n-1\rangle, |g\rangle |n-1\rangle, |e\rangle |n\rangle, |g\rangle |n\rangle\}.
\]  
(4.74)
We can therefore define general product states for any \( n \)

\[
|\psi_{1n}\rangle = |e\rangle |n\rangle \quad (4.75)
\]

\[
|\psi_{2n}\rangle = |g\rangle |n + 1\rangle \quad (4.76)
\]

It follows that

\[
\langle \psi_{1n} | \psi_{2n} \rangle = 0 \quad (4.77)
\]

The matrix representation of \( H \)

\[
H_{ij} = \langle \psi_{in} | H | \psi_{jn} \rangle \quad (4.78)
\]

In this basis becomes

\[
H_{ij} = \begin{bmatrix}
    n\omega + \frac{1}{2} \hbar \omega_0 & \hbar \lambda \sqrt{n + 1} \\
    \hbar \lambda \sqrt{n + 1} & (n + 1) \omega - \frac{1}{2} \hbar \omega_0
\end{bmatrix} \quad (4.79)
\]

For any given \( n \), the energy eigenvalues are

\[
E_{\pm} (n) = \left( n + \frac{1}{2} \right) \hbar \omega \pm \hbar \Omega_n (\Delta) \quad (4.80)
\]

where \( \Delta = (\omega_0 - \omega) \), is the detuning parameter of the atomic transition frequency and the monochromatic field and

\[
\Omega_n (\Delta) = [\Delta^2 + 4 \lambda^2 (n + 1)]^{1/2} \quad (4.81)
\]
is the damped \textit{Rabi oscillation frequency}, which in the case of resonance, i.e., $\Delta = 0$, becomes

$$\Omega_n (0) = 2\lambda \sqrt{n + 1} \quad (4.82)$$

The set of energy eigenstates form what are known as the \textit{dressed states}, and these are a linear combination of the \textit{bare states} which are

$$|n, +\rangle = \cos (\Phi_n/2) |\psi_{1n}\rangle + \sin (\Phi_n/2) |\psi_{2n}\rangle \quad (4.83)$$

$$|n, -\rangle = -\sin (\Phi_n/2) |\psi_{1n}\rangle + \cos (\Phi_n/2) |\psi_{2n}\rangle \quad (4.84)$$

with $\Phi_n$ given by

$$\Phi_n = \tan^{-1} \left( \frac{2\lambda \sqrt{n + 1}}{\Delta} \right) = \tan^{-1} \left( \frac{\Omega_n (0)}{\Delta} \right) \quad (4.85)$$

Moreover,

$$\sin (\Phi_n/2) = \frac{1}{\sqrt{2}} \left[ \frac{\Omega_n (\Delta) - \Delta}{\Omega_n (\Delta)} \right]^{1/2} \quad (4.86)$$

$$\cos (\Phi_n/2) = \frac{1}{\sqrt{2}} \left[ \frac{\Omega_n (\Delta) + \Delta}{\Omega_n (\Delta)} \right]^{1/2} \quad (4.87)$$

The dressed states $|n, \pm\rangle$ comprise the Jaynes-Cummings doublet. The $\hbar \Omega_n (\Delta)$ term splits the energies of the bare states $|\psi_{1n}\rangle$, $|\psi_{2n}\rangle$, an effect known as the \textit{dynamic stark shift}. In the case of exact resonance $\Delta = 0$, the bare states become degenerate, but the dynamic stark shift splitting of the dressed states endures. In the exact resonance limit the dressed states can be represented in
the basis of bare states as

\[ |n, +\rangle = \frac{1}{\sqrt{2}} (|e\rangle |n\rangle + |g\rangle |n + 1\rangle) \]  \hspace{1cm} (4.88)

\[ |n, -\rangle = \frac{1}{\sqrt{2}} (-|e\rangle |n\rangle + |g\rangle |n + 1\rangle) \]  \hspace{1cm} (4.89)

To obtain the dynamics in the dressed state basis, let us consider the case of an atom field system, for which the field is prepared in some superposition of initial states as

\[ |\psi_f (0)\rangle = \sum_n C_n |n\rangle \]  \hspace{1cm} (4.90)

and for which an atom, initially in the state \(|e\rangle\) gets injected into the field. Therefore the initial state of the atom-field system is

\[ |\psi_{af} (0)\rangle = |\psi_f (0)\rangle |e\rangle \]

\[ = \sum_n C_n |n\rangle |e\rangle = \sum_n C_n |\psi_{1n}\rangle. \]  \hspace{1cm} (4.92)

Now, from (IV.2) and (IV.3), it follows that

\[ |\psi_{1n}\rangle = \cos (\Phi_n /2) |n, +\rangle - \sin (\Phi_n /2) |n, -\rangle \]  \hspace{1cm} (4.93)

therefore the initial state of the atom-field system is

\[ |\psi_{af} (0)\rangle = \sum_n C_n [\cos (\Phi_n /2) |n, +\rangle - \sin (\Phi_n /2) |n, -\rangle] \]  \hspace{1cm} (4.94)
One nice feature of the dressed states, is that they are stationary states of the Hamiltonian, and a consequence, the time evolution of $H$ is

$$|\psi_{af}(t)\rangle = \exp\left(-\frac{i}{\hbar}Ht\right)|\psi_{af}(0)\rangle$$

$$= \sum_n C_n \left[ \cos\left(\Phi_n/2\right)|n,+\rangle e^{-iE_+(n)t/\hbar} - \sin\left(\Phi_n/2\right)|n,-\rangle e^{-iE_-(n)t/\hbar} \right]$$

### 4.4 Jaynes-Cummings with large de-tuning

In the forgoing we have been working with an “on resonance” approximation, in which the detuning parameter vanishes i.e., when $\Delta = 0$. A more interesting case is the one in which detuning exists to the extent that direct atomic transitions do not occur, but where dispersive interactions between a single and a cavity field do occur.

The effective Hamiltonian in the case of large detuning is given by

$$H_{eff} = \hbar\chi \left[ \sigma_+\sigma_- + a^\dagger a\sigma_3 \right]$$

Where,

$$\chi = \lambda^2/\Delta$$

The transition operators are the projections

$$\sigma_+ = |e\rangle \langle g| \quad \sigma_- = |g\rangle \langle e|$$
Note that

\[ \sigma_+ \sigma_- = |e\rangle \langle e| \]  

(4.100)

is the emission projector, and the inversion operator \( \sigma_3 \) is

\[ \sigma_3 = |e\rangle \langle e| - |g\rangle \langle g| \]  

(4.101)

The transition and inversion operators obey the Pauli algebra

\[ [\sigma_+, \sigma_-] = \sigma_3 \]  

(4.102)

\[ [\sigma_3, \sigma_\pm] = 2\sigma_\pm \]  

(4.103)

Suppose that the initial state of the atom-field system has the configuration of an atom in the ground state and the field in a number state, i.e.

\[ |\psi(0)\rangle = |g\rangle |n\rangle \]  

(4.104)

The time evolved state becomes

\[ |\psi(t)\rangle = e^{-iH_{\text{eff}}t/\hbar} |\psi(0)\rangle = e^{i\chi nt} |g\rangle |n\rangle. \]  

(4.105)

Similarly, for the initial conditions

\[ |\psi(0)\rangle = |e\rangle |n\rangle \]  

(4.106)
We get
\[ |\psi(t)\rangle = e^{-iH_{\text{eff}}t/\hbar} |\psi(0)\rangle = e^{i\chi(n+1)t} |e\rangle |n\rangle. \]  
(4.107)

and nothing happens except the production of unmeasurable phase factors.

However, if the initial state is a coherent state of the field, that is in the case where
\[ |\psi(0)\rangle = |g\rangle |\alpha\rangle \quad \text{Coherent initial state} \]  
(4.108)

We obtain
\[ |\psi(t)\rangle = e^{-iH_{\text{eff}}t/\hbar} |\psi(0)\rangle = |g\rangle |\alpha e^{i\chi t}\rangle \]  
(4.109)

Similarly, for the initial state
\[ |\psi(0)\rangle = |e\rangle |\alpha\rangle \]  
(4.110)

we have
\[ |\psi(t)\rangle = e^{-iH_{\text{eff}}t/\hbar} |\psi(0)\rangle = e^{-i\chi t} |e\rangle |\alpha e^{-i\chi t}\rangle \]  
(4.111)

For either case of the initial coherent field state, the coherent state amplitude gets rotated in phase space by the angle \( \theta = \chi t \). The direction of rotation depends on which initial states the atom is in.

Let us now consider the case of an atom in an initial superposition of ground and excited states, which for simplicity, we assume takes the form of a balanced state with the form:
\[ |\psi_{\text{atom}}\rangle = \frac{1}{\sqrt{2}} \left( |g\rangle + e^{i\phi} |e\rangle \right) \quad \phi \leftrightarrow \text{phase} \]  
(4.112)
For an initial state
\[
|\psi(0)\rangle = |\psi_{\text{atom}}\rangle |\alpha\rangle
\]  
(4.113)

We obtain
\[
|\psi(t)\rangle = e^{-iH_{\text{eff}}t/\hbar} |\psi(0)\rangle = \frac{1}{\sqrt{2}} \left( |g\rangle |\alpha e^{i\chi t}\rangle + e^{-i(\chi t-\phi)} |e\rangle |\alpha e^{-i\chi t}\rangle \right)
\]  
(4.114)

which is a much more interesting state, since now the atom and field are entangled.

Taking \(\chi t = \pi/2\), for which \(e^{i\chi t} = i\), and \(e^{-i\chi t} = -i\), we have the entangled state
\[
|\psi\left(\frac{\pi}{2\chi}\right)\rangle = \frac{1}{\sqrt{2}} \left( |g\rangle |i\alpha\rangle - ie^{i\phi} |e\rangle |-i\alpha\rangle \right)
\]  
(4.115)

which can be understood by analogy to the Schrodinger’s cat paradox. With this analogy, our atomic states correspond to the radioactive atom in the paradox, and the two phase-separated coherent field states play the role of Schrodinger’s cat. Moreover, the above entangled state corresponds the entangled state
\[
|\psi_{\text{atom-cat}}\rangle = \frac{1}{\sqrt{2}} \left( |\text{atom not decayed}\rangle |\text{cat alive}\rangle + |\text{atom decayed}\rangle |\text{cat dead}\rangle \right)
\]  
(4.116)

Coherent states differing in phase by \(\pi\) are maximally distinguishable, and there is effectively no overlap between the states, that is for \(|\alpha|\) sufficiently large. Very large values of \(|\alpha|\) are macroscopically distinguishable, while moderate values of \(|\alpha|\), mesoscopically distinguishable.
5 Application of CQED to Quantum Information Processing

5.1 The Fabry-Perot Cavity

We start this section with a brief overview the Fabry-Perot cavity. One of the most essential components of a Fabry-Perot cavity is a partially silvered mirror, which partially reflects and transmits incidents light $E_a$ and $E_b$, which has the effect of producing output fields $E_{a'}$ and $E_{b'}$, which are related by the unitary transformation:

$$
\begin{bmatrix}
E_{a'} \\
E_{b'}
\end{bmatrix} =
\begin{bmatrix}
\sqrt{R} & \sqrt{1-R} \\
\sqrt{1-R} & -\sqrt{R}
\end{bmatrix}
\begin{bmatrix}
E_a \\
E_b
\end{bmatrix}
$$

where $R$ is the reflectivity of the mirror.

A Fabry-Perot (FP) cavity is made from two plane parallel mirrors of reactivities $R_1$ and $R_2$, incident upon which is light from outside the cavity $E_{int}$. Inside the cavity, light bounces back and forth between the two mirrors acquiring a phase shift $e^{i\phi}$ on each trip. The internal cavity field is

$$
E_{cav} = \sum_k E_k = \frac{\sqrt{1-R}E_{int}}{1+e^{i\phi}\sqrt{R_1R_2}}
$$

One of the most important things about the Fabry Perot cavity for purposes of CQED is the power in the internal cavity field mode as a function of of the
power and frequency of the input field,
\[
\frac{P_{\text{cav}}}{P_{\text{in}}} = \left| \frac{E_{\text{cav}}}{E_{\text{in}}} \right|^2 = \frac{1 - R_1}{\left| 1 + e^{i\phi} \sqrt{R_1R_2} \right|^2} \tag{5.3}
\]

Frequency selectivity arises because of constructive and destructive interference between the cavity mode and the reflected light front. Another indispensable feature is that on resonance, the cavity field achieves a maximum which is approximately \((1 - R)^{-1}\) times the incident field.

### 5.2 Quantum Computation

Quantum information can be encoded with single photons in the *dual rail representation*
\[
c_0 |01\rangle + c_1 |10\rangle \tag{5.4}
\]
Arbitrary unitary transformations can be applied to such quantum information using phase shifters, beam splitters, and nonlinear optical Kerr media [1].

The *single photon representation* of a qubit is attractive because it represents the information saturation limit of the electromagnetic field and single photons, by today’s standards can be generated relatively easily and moreover and most importantly, *arbitrary qubit operations become possible, in general in the dual-rail representation*. The difficult part in this approach is making the photon - photon scattering amplitudes large enough for entanglement to occur. In an optical cavity this is commonly implemented with
optical nonlinear Kerr media. However, in reality even the best non-linear Kerr media are weak, and are unable to provide a cross phase modulation of $180^\circ$ between single photon states. It is estimated, that even in the best cases, approximately 50 photons would have to be absorbed for each $180^\circ$ cross phase modulated photon.

Despite its drawbacks, the optical quantum computer does provide us with some insight into the architecture and design of a quantum computer. Assuming we had sufficiently good components available, we could construct an optical quantum computer, which will be almost entirely comprised of optical interferometers. The quantum information is encoded in both the photon number states and the photon phase. Interferometers perform the function of switching between the two representations. Stability however, becomes a major issue, and if a massive representation of a qubit is chosen, then stable interferometers would be a challenge to construct because of the relatively short scale of the de Broglie wavelengths of the qubits.

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