Semidiscrete Toda lattices

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Abstract

Integrable cut-off constraints for semidiscrete Toda lattice are studied in this paper. Lax presentation for semidiscrete analog of the C-series Toda lattice is obtained. Nonlocal variables that allow to express symmetries of the infinite semidiscrete lattice are introduced and cut-off constraints or a certain type compatible with symmetries of the infinite lattice are classified.

1 Introduction

Various Toda chains and lattices are being examined already for a considerable period of time. One-dimensional Toda chain that describes a system of particles on a line with exponential interaction between each pair of neighboring particles was introduced by M. Toda in 1967 in his article [1]. After that Bogoyavlensky [2] introduced generalized one-dimensional Toda chains corresponding to simple Lie algebras in 1976. In the end of 1970-ties and in the beginning of 1980-ties a series of papers (see [3]–[7]) where generalized two-dimensional Toda lattices were considered appeared almost at the same time. In 1991 Suris [8] studied discrete generalized one-dimensional Toda chains. This paper is focused on semidiscrete two-dimensional Toda lattices.

It is well-known (e.g. see [9] for the details) that in the continuous case the Laplace invariants $h(j) = c(j) - a(j)b(j) - b_y(j)$ of a sequence of hyperbolic second order differential operators

$$L_j = \partial_x \partial_y + a(j)\partial_x + b(j)\partial_y + c(j),$$

linked by Darboux–Laplace transformations

$$L_{j+1}D_j = D_{j+1}L_j,$$ (2)

where $D_j = \partial_x + b(j)$, satisfy the two-dimensional Toda lattice:

$$(\ln h(j))_{xy} = h(j - 1) - 2h(j) + h(j + 1).$$ (3)

Using the substitution $h(j) = \exp(q(j + 1) - q(j))$ one can rewrite the lattice (3) in the form

$$q_{xy}(j) = \exp(q(j + 1) - q(j)) - \exp(q(j) - q(j - 1)),$$ (4)

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1The reference given here is far from being complete, but the format of this article does not allow us to give full review on the subject.
and using the substitution $h(j) = u_{xy}(j)$ one obtains the following lattice:

$$u_{xy}(j) = \exp(u(j - 1) - 2u(j) + u(j + 1)).$$

(5)

The latter system (or, more exactly, its reduction defined by the constraints $u(-1) = u(r) = -\infty$ for some natural $r$) is a particular case of the so-called *systems of exponential type*, i.e. the equations of the form $u_{xy} = \exp(Ku)$, where $K$ is a constant matrix, and $\exp(Ku)$ denotes the vector such that its $j$-th coordinate is equal to the exponent of the $j$-th coordinate of the vector $Ku$. A system of exponential type is proven to be Darboux integrable (i.e. a full set of $y$-integrals of this system is obtained) if and only if matrix $K$ is the Cartan matrix of a simple Lie algebra (see papers [6, 7]). Moreover, systems of exponential type corresponding to simple Lie algebras were explicitly integrated in [4]. Such complete description of integrable systems of exponential type is based on the notion of *characteristic algebra* of a system of equations. It is proved that the existence of a full set of $y$-integrals of a system is equivalent to finite dimensionality of its characteristic algebra; the latter condition holds for a system of exponential type if and only if matrix $K$ is the Cartan matrix of a Lie algebra of series $A - D$ or the Cartan matrix of an exceptional Lie algebra. These systems are called *generalized* or *finite* two-dimensional Toda lattices. Lax presentations for all generalized two-dimensional Toda lattices were obtained in [5] using methods of theory of Lie algebras. In the most simple case of the boundary condition $u(-1) = u(1) = -\infty$ the system (5) is reduced to the famous Liouville equation. This hyperbolic PDE had been explicitly integrated by Liouville [10] in 1853.

Despite the fact that complete description of all explicitly integrable finite two-dimensional Toda lattices had been provided, some problems still remained unsolved. In particular, it was not clear whether $D$-series two-dimensional Toda lattice is a reduction of an $A$-series lattice until this problem was affirmatively resolved by Habibullin [11] in 2005 ($B$-series and $C$-series lattices are reductions of an $A$-series Toda lattice defined by involute constraints $h(−j) = h(j − 1)$ and $h(−j) = h(j)$ respectively). Besides this, explicit integrability and existence of a full set of $y$-integrals are not the only attributes of a concept of integrable equation in our days. A new idea based on the notion of integrability as existence of a Lax presentation was proposed in [11]. Although this approach to generalized Toda lattices is systematic since it provides an integrable cut-off constraint together with a Lax pair (it was an additional problem to find a Lax pair for a finite Toda lattice before), it appears to be not very effective.

One more approach to the problem of classification of integrable cut-off constraints for two-dimensional Toda lattice developed in [12] is based on the study of boundary conditions compatible with symmetries of the infinite Toda lattice.

The structure of this article is as follows. Habibullin’s direct approach that allows to find integrable cut-off constraints together with corresponding Lax presentation is described in the Section 2. Section 3 contains Shabat’s method [13] of obtaining symmetries for two-dimensional infinite Toda and the symmetry approach to the problem of classification of integrable cut-off constraints for two-dimensional Toda lattice developed by Gürel and Habibullin [12] in the continuous case. Semidiscrete Toda lattice is introduced in the Section 4 (here we follow the article [14] by Adler and Startsev). Lax presentation for the semidiscrete analog of a $C$-series Toda lattice is obtained in the Section 5. The symmetry approach to the problem of classification of integrable cut-off constraints for the Toda lattice in the semidiscrete case is developed in the Section 6.

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When this article was almost completed the author discovered the preprint on an article [15] where, in particular, Lax presentation for semidiscrete $C$-series Toda lattice is obtained.
2 Lax pair approach in the continuous case

Before going over to Habibullin’s straightforward method of obtaining Lax presentations for the finite two-dimensional Toda lattices we’ll display explicitly the cut-off constraints corresponding to $A-D$-series Toda lattices in terms of the variables $q(j)$. $A$-series Toda lattice is defined by the trivial boundary conditions

$$q(-1) = \infty, \quad q(r+1) = -\infty$$

for some $r \in \mathbb{N}$. $B$-series lattice is defined the following cut-off constraint:

$$q(0) = 0, \quad q(r+1) = -\infty$$

for some $r \in \mathbb{N}$. $C$-series lattices are generated by cut-off constraints of the form

$$q(0) = q(1), \quad q(r+1) = -\infty,$$

and $D$-series lattices are generated by a more complicate cut-off constraint:

$$q(0) = -\ln \left( e^{q(2)} - \frac{q_x(1)q_y(1)}{2\sinh q(1)} \right), \quad q(r+1) = -\infty.$$  

It is well-known that Darboux-Laplace transformations provide a Lax presentations for the infinite two-dimensional Toda lattice: equations (4) are equivalent to the compatibility conditions of the following linear system of equations:

$$\begin{align*}
\psi_x(j) &= \psi(j+1) + q_x(j)\psi(j) \\
\psi_y(j) &= -h(j-1)\psi(j-1),
\end{align*}$$

where $h(j) = \exp(q(j+1) - q(j))$. We are interested in finite Toda lattices. Therefore we need to obtain Lax presentations for Toda equations reduced by the above cut-off constraints. Consider an arbitrary cut-off constraint of the form

$$q(-1) = F(q(0), q(1), \ldots, q(k))$$

for some $k \in \mathbb{N}$. Since the equation for $\psi_y(j)$ contains $\psi(-1)$, it is rather natural to make an attempt to express $\psi(-1)$ via the dynamic variables $\psi(0), \psi(1), \ldots, \psi(k)$ in a way consistent with the constraint (11). This would have allowed one to obtain a Lax presentation for the reduced lattice. However, such procedure works only for the trivial boundary conditions and this approach produces a Lax pair only for an $A$-series Toda lattice (see [11]). This obstacle led Habibullin to the other idea: since the infinite Toda lattice (4) is symmetric about the interchange of variables $x \leftrightarrow y$, it also admits another Lax presentation:

$$\begin{align*}
\varphi_x(j) &= -h(j-1)\varphi(j-1) \\
\varphi_y(j) &= \varphi(j+1) + q_y(j)\varphi(j),
\end{align*}$$

This means that one may seek for a closure of a Lax pair (10) using not only this Lax pair but to find a common closure for both Lax pairs, that is, to eliminate the variables $\psi(-1)$ and $\varphi(-1)$ using the whole set of variables

$$\psi(0), \psi(1), \ldots, \psi(r), \varphi(0), \varphi(1), \ldots, \varphi(r)$$

in a way consistent with the constraint (11). This approach provides Lax pairs for all generalized Toda lattices of the series $A - D$; moreover, in some cases these Lax pairs do not coincide with the ones that were obtained using Lie algebraic methods in [5]. The following Proposition (see [11]) holds.
Proposition 1. Let \( k < k_1 \) and \( m < m_1 \) be integers. Consider vectors

\[
\Psi = (\psi(k+1), \psi(k+2), \ldots, \psi(k_1 - 1))^t, \quad \Phi = (\varphi(m+1), \varphi(m+2), \ldots, \varphi(m_1 - 1))^t;
\]

then all generalized two-dimensional Toda lattices of the series \( A - D \) admit Lax presentation of the following form:

\[
\begin{pmatrix} \Psi \\ \Phi \end{pmatrix}_x = \begin{pmatrix} A & K \\ M & C \end{pmatrix} \begin{pmatrix} \Psi \\ \Phi \end{pmatrix}, \quad \begin{pmatrix} \Psi \\ \Phi \end{pmatrix}_y = \begin{pmatrix} B & L \\ N & D \end{pmatrix} \begin{pmatrix} \Psi \\ \Phi \end{pmatrix},
\]

(13)

where matrices \( A, B, \ldots, N \) look as follows:

\[
A = \begin{pmatrix}
q_x(k+1) & 1 & 0 & \ldots & 0 \\
0 & q_x(k+2) & 1 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & 1 \\
A_{k+1} & A_{k+2} & A_{k+3} & \ldots & A_{k-1}
\end{pmatrix}, \quad K = \begin{pmatrix}
0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & 0 \\
K_{m+1} & K_{m+2} & \ldots & K_{m-1}
\end{pmatrix},
\]

\[
B = \begin{pmatrix}
B_{k+1} & B_{k+2} & \ldots & 0 & B_{k-1} \\
-h(k+1) & 0 & \ldots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & -h(k_1 - 2) & 0 \\
0 & 0 & \ldots & 0 & 0
\end{pmatrix}, \quad L = \begin{pmatrix}
L_{m+1} & L_{m+2} & \ldots & L_{m-1} \\
0 & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0
\end{pmatrix},
\]

\[
C = \begin{pmatrix}
C_{m+1} & C_{m+2} & \ldots & 0 & C_{m-1} \\
-h(m+1) & 0 & \ldots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & -h(m_1 - 2) & 0 \\
0 & 0 & \ldots & 0 & 0
\end{pmatrix}, \quad M = \begin{pmatrix}
M_{k+1} & M_{k+2} & \ldots & M_{k-1} \\
0 & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0
\end{pmatrix},
\]

\[
D = \begin{pmatrix}
q_y(m+1) & 1 & 0 & \ldots & 0 \\
0 & q_y(m+2) & 1 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & 1 \\
D_{m+1} & D_{m+2} & D_{m+3} & \ldots & D_{m-1}
\end{pmatrix}, \quad N = \begin{pmatrix}
0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & 0 \\
N_{k+1} & N_{k+2} & \ldots & N_{k-1}
\end{pmatrix}.
\]

Remark 1. One of the advantages of this approach is that cut-off constraints on the left edge and on the right edge are independent from each other. Therefore not only generalized Toda lattices of the series \( A - D \) admit Lax presentation of the form (13), but any finite Toda lattice defined by any of the constraints (6)–(9) on the left and on the right edges also admits such Lax presentation. Moreover, there are examples of boundary conditions (11) such that corresponding system is not a system of exponential type although it admits Lax presentation of the form (13) (see [11]).
Remark 2. Probably the lattice (4) admits infinitely many cut-off constraints that are integrable in this sense. However, complete classification of all such cut-off constraints seems to be absolutely impossible due to the fact that one has to solve a huge system of differential-difference equations in order to determine the free entries of the matrices $A, B, \ldots, N$. Besides this, Habibullin’s approach is not very effective in the sense that even if one is interested in a Lax pair for a certain cut-off constraint, it is quite unclear how to choose the integers $k, m, k_1$ and $m_1$ and there is absolutely no indication on how many entries in each of the above matrices should be left undetermined.

Functions $\psi(j)$ and $\varphi(j)$, satisfying the equations (10) and (12), are also solutions of the hyperbolic equations $L^{\psi}(j)\psi(j) = 0$ and $L^{\varphi}(j)\varphi(j) = 0$ respectively, where the operators $L^{\psi}(j)$ and $L^{\varphi}(j)$ are defined by the equations:

$$L^{\psi}(j) = \partial_x\partial_y - q_x(j)\partial_y + h(j - 1), \quad L^{\varphi}(j) = \partial_x\partial_y - q_y(j)\partial_x + h(j - 1).$$

It is easy to verify that in terms of the variable $h$ the cut-off constraints corresponding to the series $B$ and $C$ are nothing but involutions $h(-j) = h(j + 1)$ and $h(-j) = h(j)$ respectively. Combining this with (14), one obtains the following identities for a $B$-series lattice:

$$h^{\psi}(0) = h(-1) = h(2) = h^{\psi}(2), \quad h^{\varphi}(0) = h(0) = h(1) = k^{\psi}(2).$$

It is well known that two linear hyperbolic operators have the same Laplace invariants if and only if they are linked by a gauge transformation. In terms of wave functions this means that there exists a multiplier $R = R(x, y)$ such that $\varphi(0) = R\psi(2)$ (i.e for any solution $\psi(2)$ of one of this equations the function $\varphi(0) = R\psi(2)$ is the solution to the other equation). Similarly, one establishes that

$$h^{\psi}(1) = h(1) = h(0) = h^{\varphi}(1), \quad k^{\psi}(1) = h(0) = h(1) = k^{\varphi}(1);$$

this means that there exists another multiplier $S = S(x, y)$ such that $\psi(1) = S\varphi(1)$. The identities obtained are nothing but a particular case of Habibullin’s general idea to consider common reductions of two Lax pairs for the infinite Toda lattice at the left edge. In the case of a $C$-series Toda lattice one immediately arrives at the following identities:

$$h^{\varphi}(0) = h(-1) = h(1) = h^{\psi}(1), \quad k^{\varphi}(0) = h(0) = h^{\psi}(1),$$
$$h^{\psi}(0) = h(0) = h^{\varphi}(1), \quad k^{\psi}(0) = h(-1) = h(1) = k^{\varphi}(1).$$

Hence there exist the multipliers $R$ and $S$ such that $\varphi(0) = R\psi(1)$, $\psi(0) = S\varphi(1)$. It is also a particular case of the above Habibullin’s approach. The cut-off constraint corresponding to $D$-series lattices is more complicate; in this case the two Lax pairs are not gauge equivalent in any point. The examples of boundary conditions for the series $B$ and $C$ demonstrate that the idea to consider two Lax pairs together and to look for their common reduction is rather natural.
3 Symmetry approach in the continuous case

Another approach to the problem of integrable cut-off constraints for the two-dimensional Toda lattice proposed by Gürel and Habibullin [12] is based on the study of boundary conditions compatible with symmetries of the infinite lattice. However, symmetries of the infinite two-dimensional Toda lattice cannot be expressed in terms of the dynamic variables (on the contrary to the onedimensional case). In the two-dimensional case one has to introduce nonlocal variables in order to define symmetries of the infinite Toda lattice.

Denote \( q_x(j) \) by \( b(j) \) (this variable has nothing in common with coefficients of hyperbolic operators from the previous Section). Hence the Toda lattice can be rewritten as follows:

\[
b_y(j) = h(j) - h(j - 1).
\]

Define the nonlocal variables \( b^{(1)}(j) \) by the identities

\[
\partial_x b(j) = b^{(1)}(j) - b^{(1)}(j - 1), \quad \partial_y b^{(1)}(j) = \partial_x h(j).
\]

Compatibility of these equations is provided by the Toda lattice (15):

\[
\partial_y \partial_x b(j) = \partial_y \left( b^{(1)}(j) - b^{(1)}(j - 1) \right) = \partial_x \left( h(j) - h(j - 1) \right) = \partial_x \partial_y b(j).
\]

Further on, it is necessary to determine the derivatives \( \partial_x b^{(1)}(j) \), but this, however, requires to introduce nonlocalities \( b^{(2)}(j) \) of higher order. Set

\[
\partial_x b^{(1)}(j) = b^{(1)}(j) (b(j + 1) - b(j)) + b^{(2)}(j) - b^{(2)}(j - 1);
\]

hence the compatibility condition \( \partial_y \partial_x b^{(1)}(j) = \partial_x \partial_y b^{(1)}(j) \) obviously leads to the following identity:

\[
\partial_y b^{(2)}(j) = h(j)b^{(1)}(j + 1) - h(j + 1)b^{(1)}(j).
\]

The following Proposition is established straightforwardly by induction.

**Proposition 2.** Nonlocal variables \( b^{(k)}(j) \), where \( k = 2, 3, \ldots \), satisfy the following identities:

\[
\begin{align*}
\partial_y b^{(k)}(j) &= h(j)b^{(k-1)}(j+1) - h(j+k-1)b^{(k-1)}(j), \\
\partial_x b^{(k)}(j) &= b^{(k)}(j) (b(j + k) - b(j)) + b^{(k+1)}(j) - b^{(k+1)}(j-1).
\end{align*}
\]

Compatibility of these equations for each \( k \) is provided by the first of them, but for \( k + 1 \).

The formulas (17) were obtained by A.B. Shabat [13] in order to derive the hierarchy of symmetries of the infinite two-dimensional Toda lattice. The following Proposition holds.

**Proposition 3.** The flows

\[
\begin{align*}
q_x(j) &= b^2(j) + b^{(1)}(j) + b^{(1)}(j - 1), \\
q_y(j) &= b^3(j) + b^{(2)}(j) + b^{(2)}(j - 1) + b^{(2)}(j - 2) + \\
&\quad + b^{(1)}(j)(2b(j) + b(j + 1)) + b^{(1)}(j - 1)(2b(j) + b(j - 1))
\end{align*}
\]

define symmetries of the lattice (13).
The nonlocal variables $b^{(k)}(j)$ also provide an opportunity to obtain first integrals for the two-dimensional Toda lattice (4). Indeed, it is easy to verify that the functions

$$
I_2 = \sum_{j \in \mathbb{Z}} \left( b^2(j) + 2b^{(1)}(j) \right) \quad I_3 = \sum_{j \in \mathbb{Z}} \left( b^3(j) + 3b^{(1)}(j)(b(j) + b(j + 1)) + 3b^{(2)}(j) \right)
$$

are $y$-integrals for the lattice (4). Symmetries and integrals of higher order are expressed in terms of the nonlocal variables $b^{(k)}(j)$ of higher grading, but we are not going to discuss them in this article.

The main idea of the article [12] is to examine what finite Toda lattices admit symmetries and, in particular, to classify all cut-off constraints, compatible with the symmetries (18), (19). Straightforward calculations show that the symmetry (18) is compatible only with trivial boundary condition (i.e. only $A$-series Toda lattices admit symmetry (18)) and that the symmetry (19) is compatible, for example, with the cut-off constraints corresponding to the series $A$, $B$, and $C$. It appears that all boundary conditions for the Toda lattice that are known to be integrable in some sense are compatible with the symmetry (19). In order to formulate the corresponding theorem one has to introduce the new set of dynamic variables (as well as the new set of nonlocal variables) such that the symmetries (18) are (19) are expressed in a more compact way in their terms.

Denote $u = e^{q(-1)}$ and $v = e^{q(0)}$; then the symmetries (18) and (19) can be rewritten respectively as follows:

$$
\begin{align*}
&u_t = -u_{xx} - 2ru, \quad v_t = v_{xx} + 2rv, \quad \text{(20)} \\
u_t = u_{xxx} + 3ru - 3su + 3r_x u, \quad v_t = v_{xxx} + 3rv_x + 3sv, \quad \text{(21)}
\end{align*}
$$

where $r = b^{(1)}(0)$ and $s = b^{(2)}(0) + r(\ln v)_x$ are the new nonlocal variables that satisfy the relations

$$
r_y = (uv)_x, \quad s_y = (uv_x)_x.
$$

The following theorem is the main result of the article [12].

**Theorem 1.** If a cut-off constraint of the form $u = F(v, v_x, v_y, v_{xy})$ is compatible with the symmetry (21), then it is trivial: $u = 0$. A cut-off constraint $u = F(v, v_x, v_y, v_{xy})$ compatible with the symmetry (21) has one of the following forms:

i) $u = a$, where $a = \text{const}$;

ii) $u = av$, where $a = \text{const}$;

iii) $u = \frac{vy_x}{a-v^2} + \frac{vuv_x}{(a-v^2)^2}$, where $a = \text{const}$.

These cut-off constraints correspond with the well-known boundary conditions for the two-dimensional Toda lattice, corresponding the the Lie algebras of the series $A$–$D$. Cut-off constraints of more general form provide all boundary conditions for the Toda lattice that were considered in literature (see [12]).

4 Infinite lattice in the semidiscrete case

In the semidiscrete case as well as in the continuous case the Laplace invariants of a sequence of hyperbolic second order operators linked by Darboux-Laplace transformations satisfy the system of differential-difference equations called the semidiscrete Toda lattice. We’ll follow the article [14] and use this idea to obtain the semidiscrete Toda equations.
Consider a sequence of hyperbolic differential-difference operators

\[ \mathcal{L}_j = \partial_x T + a_n(j)\partial_x + b_n(j)T + c_n(j), \]

where \(a_n(j), b_n(j)\) and \(c_n(j)\) are functions depending on discrete variable \(n \in \mathbb{Z}\) and on continuous variable \(x \in \mathbb{R}\) and where \(T\) is a shift operator: \(T\psi_n(x) = \psi_{n+1}(x)\). Obviously, the operator \(\mathcal{L}_j\) can be factorized in two different ways:

\[ \mathcal{L}_j = (\partial_x + b_n(j))(T + a_n(j)) + a_n(j)k_n(j) = (T + a_n(j))(\partial_x + b_{n-1}(j)) + a_n(j)h_n(j), \]

where \(k_n(j) = \frac{c_n(j)}{a_n(j)} - (\ln a_n(j))' - b_n(j)\) and \(h_n(j) = \frac{c_n(j)}{a_n(j)} - b_{n-1}(j)\) are the Laplace invariants of differential-difference operator \(\mathcal{L}_j\).

Suppose any two neighboring operators \(\mathcal{L}_j\) and \(\mathcal{L}_{j+1}\) are linked by a Darboux-Laplace transformation, that is, satisfy the following relation:

\[ \mathcal{L}_{j+1}\mathcal{D}_j = \mathcal{D}_{j+1}\mathcal{L}_j, \]

where \(\mathcal{D}_j = \partial_x + b_{n-1}(j)\). This operator relation can be rewritten in terms of coefficients as follows:

\[
\begin{cases}
  k_n(j+1) = h_n(j) \\
  (\ln \frac{h_n(j)}{h_{n+1}(j)})'_{x} = h_{n+1}(j+1) - h_{n+1}(j) - h_n(j) + h_n(j-1)
\end{cases}
\]

Using the new variables \(q_n(j)\) that are introduced by the equations \(h_n(j) = \exp(q_{n+1}(j+1) - q_n(j))\), one obtains the semidiscrete Toda lattice:

\[ q_{n,x}(j) - q_{n+1,x}(j) = \exp(q_{n+1}(j+1) - q_n(j)) - \exp(q_{n+1}(j) - q_n(j-1)). \tag{22} \]

Similarly to the continuous case it is natural to examine integrable reductions of the semidiscrete Toda lattice. Trivial boundary conditions \(h_n(-1) = h_n(r) = 0\) lead to the system that should be called the semidiscrete lattice corresponding to the \(A\)-series Lie algebra. In terms of the variable \(q\) this reduction is defined by the cut-off constraint \(q_n(-1) = \infty\), \(q_n(r+1) = -\infty\). Now we’ll find out what involutions does the semidiscrete lattice in terms of the variables \(h\), i.e. the system

\[ (\ln \frac{h_n(j)}{h_{n+1}(j)})'_{x} = h_{n+1}(j+1) - h_{n+1}(j) - h_n(j) + h_n(j-1), \tag{23} \]

admit. It is easy to verify that the reflection \(h_n(-j) = h_{n+j+c}(j - d)\) defines a reduction of the lattice \[23\] if and only if \(d = -2c\). This means that the situation in the semidiscrete case differs from the one in the continuous case since in the continuous case reflections about both semi-integer and integer points define reductions of the Toda lattice (these reductions correspond to the series \(B\) and \(C\) respectively). If \(c = -1\), then one obtains the cut-off constraint \(h_n(-j) = h_{n+j-1}(j - 2)\) (this equation should be satisfied for all \(n \in \mathbb{Z}\)). For \(j = 2\) one arrives at the following cut-off constraint in terms of the variable \(q\):

\[ q_n(-1) - q_{n-1}(-2) = q_{n+1}(1) - q_n(0). \tag{24} \]

The lattice \[22\] satisfying the cut-off constraint \[24\] on the left edge and the trivial boundary condition \(q_n(r+1) = -\infty\) for a certain \(r \in \mathbb{N}\) on the right edge is called semidiscrete Toda lattice corresponding to the \(C\)-series. It is easy to verify that this system converges to the generalized two-dimensional \(C\)-series Toda lattice in the continuum limit.
Straightforward calculation shows that the boundary condition \((24)\) defines an \(n\)-integral of the semidiscrete lattice. Indeed, the function
\[
\mu(x) = q_{n,x}(-1) + q_{n,x}(0) - h_n(0),
\]  
where \(q_n(j)\) is a solution of a \(C\)-series semidiscrete Toda lattice, does not depend on \(n\).

**Remark 3.** The existence of the discrete-time conservation law \((25)\) is equivalent to the cut-off constraint \((21)\). Indeed, one can easily verify that if the value \((25)\) does not depend on \(n\) for a certain solution \(q_n(j)\) of the infinite lattice \((22)\), then the relation \((21)\) holds. However the function \(\mu\) may be different for various solutions of the Toda lattice.

## 5 Lax pair for the semidiscrete \(C\)-series lattice

Exactly as in the continuous case, Darboux-Laplace transformations allow to obtain Lax presentations for the semidiscrete Toda lattice \((22)\). It is easy to verify that the equations \((22)\) are equivalent to the compatibility conditions for the following linear system of equations:
\[
\begin{cases}
\psi_{n,x}(j) = q_{n,x}(j)\psi_n(j) + \psi_n(j + 1) \\
\psi_{n+1}(j) = \psi_n(j) + h_n(j - 1)\psi_n(j - 1)
\end{cases}
\]  
(26)

Darboux-Laplace transformations on discrete variable (i.e. transformations with \(D\)-operator being discrete) also lead to Lax presentation for the semidiscrete lattice and this Lax presentation is not equivalent to the first one:
\[
\begin{cases}
\varphi_{n,x}(j) = \exp(q_{n+1}(j) - q_{n+1}(j - 1))\varphi_n(j - 1) - \varphi_n(j) \\
\varphi_{n-1}(j) = -\exp(q_n(j) - q_{n+1}(j))\varphi_n(j) + \varphi_n(j + 1)
\end{cases}
\]  
(27)

Wave functions \(\psi_n(j)\) and \(\varphi_n(j)\) that satisfy the equations \((26)\) and \((27)\) are solutions of the hyperbolic differential-difference equations \(L^\psi(j)\psi_n(j) = 0\) and \(L^\varphi(j)\varphi_n(j) = 0\) respectively, where
\[
L^\psi(j) = \partial_x T - \partial_x - q_{n+1,x}(j)T + q_{n,x}(j) - h_n(j),
\]
\[
L^\varphi(j) = \partial_x T^{-1} + \exp(q_n(j) - q_{n+1}(j))\partial_x + T^{-1} + \exp(q_n(j) - q_{n+1}(j)) - \exp(q_n(j) - q_{n+1}(j - 1)).
\]

Laplace invariants of these operators are as follows:
\[
\begin{align*}
h_n^\psi(j) &= h_n(j), & k_n^\psi(j) &= h_n(j - 1), & h_n^\varphi(j) &= h_n(j - 1), & k_n^\varphi(j) &= h_{n+1}(j). 
\end{align*}
\]  
(28)

One can easily verify that the cut-off constraint \(h_n(-j) = h_{n+j-1}(-j-2)\) corresponding to the \(C\)-series together with the condition \((28)\) lead to the following relations:
\[
h_n^\psi(-1) = h_{n+1}(0) = k_n^\psi(0),
\]
\[
h_n^\varphi(-1) = h_{n+1}(-1) = k_n^\varphi(0).
\]

Laplace invariants of two linear hyperbolic differential-difference operators are equal if and only if these operators are gauge-equivalent (exactly as in the continuous case). In terms of wave functions this means that there exists a function \(R_n = R_n(x)\) such that \(\varphi_n(-1) = R_n\psi_{n+1}(0)\). Similarly, the following relations hold:
\[
h_n^\psi(-1) = h_n(-1) = h_n^\varphi(0),
\]
\[
k_n^\psi(-1) = h_n(-2) = h_{n+1}(0) = k_n^\varphi(0).
\]
Hence there exists a function $S_n = S_n(x)$ such that $\psi_n(-1) = \varphi_n(0)$. Therefore, the $C$-series cut-off constraint makes the two Lax pairs \([26], [27]\) gauge-equivalent at one point. According to the general Habibullin’s approach it allows to obtain a Lax presentation for the finite lattice by considering common closure of two Lax pairs for the infinite lattice. More precisely, the following Proposition holds.

**Proposition 4.** Toda lattice \([22]\) with the boundary condition \([24]\) on the left edge and with the trivial boundary condition $q_n(r + 1) = -\infty$ for a certain $r \geq 1$ on the right edge is equivalent to the compatibility condition for the following linear system:

$$
\begin{align*}
\partial_x(\Psi) &= A\Psi, \\
\partial_x(\Phi) &= M\Psi + C\Phi, \\
T(\Psi) &= B\Psi + L\Phi, \\
T^{-1}(\Phi) &= D\Phi,
\end{align*}
$$

where

$$
\Psi = \begin{pmatrix} \psi(0) \\
\psi(1) \\
\vdots \\
\psi(r) \end{pmatrix}, \quad \Phi = \begin{pmatrix} \varphi(0) \\
\varphi(1) \\
\vdots \\
\varphi(r) \end{pmatrix},
$$

and matrices are as follows:

$$
A = \begin{pmatrix}
p_n & 0 & 0 & \ldots & 0 & 0 \\
\nu \cdot a_{n+1}(1) & -1 & 0 & \ldots & 0 & 0 \\
0 & a_{n+1}(2) & -1 & \ldots & 0 & 0 \\
0 & 0 & \ddots & \ddots & 0 & 0 \\
0 & 0 & 0 & \ddots & -1 & 0 \\
0 & 0 & 0 & \ldots & a_{n+1}(r) & -1
\end{pmatrix}, \quad B = \begin{pmatrix}
b_n(0) & \nu^{-1} & 0 & 0 & \ldots & 0 \\
0 & b_n(1) & -1 & 0 & \ldots & 0 \\
0 & 0 & b_n(2) & -1 & \ldots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \ldots & -1 \\
0 & 0 & 0 & \ldots & b_n(r)
\end{pmatrix},
$$

$$
C = \begin{pmatrix}
q_{n,x}(0) & 1 & 0 & \ldots & 0 \\
0 & q_{n,x}(1) & 1 & \ldots & 0 \\
0 & \ddots & \ddots & \ddots & 0 \\
0 & 0 & \ldots & 1 & 0 \\
0 & 0 & \ldots & q_{n,x}(r) & 0
\end{pmatrix}, \quad D = \begin{pmatrix}
1 & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 1 & 0 \\
0 & 0 & \ldots & h_n(r-1) & 1
\end{pmatrix},
$$

$$
L = \begin{pmatrix}
f_n & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & \ldots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & \ldots & 0
\end{pmatrix}, \quad M = \begin{pmatrix}
g_n & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & \ldots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & 0 & 0
\end{pmatrix},
$$

Matrix elements are defined as follows:

$$
a_n(j) = \exp(q_n(j) - q_n(j - 1)), \\
b_n(j) = -\exp(q_n(j) - q_{n+1}(j)), \\
f_n = (-1)^n \exp(-q_n(-1)), \\
g_n = (-1)^n \exp(q_{n+1}(0)), \\
p_n = u_{n,x}(-1) + u_{n+1,x}(0), \\
\nu = \nu(x) = \exp \left( -\int (1 + \mu(x)) dx \right).
$$
Remark 4. It is easy to verify that in the semidiscrete case the change of variables $h_n(j) \rightarrow q_n(j)$ could be performed modulo addition of an arbitrary function to all dynamical variables $q_n(j)$. Besides this, if $q_n(j)$ is a solution to the system (22), then $q_n(j) + \varepsilon$ is also a solution to this system for an arbitrary function $\varepsilon = \varepsilon(x)$. This means that having all solutions of the half-infinite Toda lattice with the boundary condition

$$q_{n,x}(-1) + q_{n,x}(0) - h_n(0) = 0, \quad q_{n,x}(r + 1) + q_{n,x}(r) - h_n(r) = 0,$$

one can find all solutions of this lattice satisfying the constraint (25) for an arbitrary function $\mu$ by a proper choice of a function $\varepsilon$. Therefore we’ll assume that $\mu = 0$ for a $C$-series semidiscrete Toda lattice further on. This means that we’ll consider a bit more rigid constraint (29) instead of the constraint (24). This ambiguity in the change of variables $h_n(j) \rightarrow q_n(j)$ also leads to the necessity to introduce an awkward multiplier $\nu$ in the entries of Lax matrices as well.

Remark 5. Lax presentation for semidiscrete Toda lattice with boundary conditions

$$q_{n,x}(-1) + q_{n,x}(0) - h_n(0) = 0, \quad q_{n,x}(r + 1) + q_{n,x}(r) - h_n(r) = 0$$

for a certain $r \geq 1$ is obtained similarly.

6 Symmetry approach in the semidiscrete case

Symmetries of two-dimensional Toda lattice in the semidiscrete case as well as in the continuous case are expressed in terms of nonlocal variables. We’ll state the basic propositions and formulas for the semidiscrete case that are similar to the ones from the Section 3. Introduce the following notation: $\partial_n = I - T$, where $I$ is the identity operator and $b_n(j) = q_{n,x}(j)$. Hence the lattice (22) can be rewritten as follows:

$$\partial_n b_n(j) = h_n(j) - h_n(j - 1). \quad (30)$$

Define the nonlocal variables $b_n^{(1)}(j)$ by the following formulas:

$$\partial_x b_n(j) = b_n^{(1)}(j) - b_n^{(1)}(j - 1), \quad \partial_n b_n^{(1)}(j) = \partial_x h_n(j).$$

Compatibility of these equations is provided by the Toda lattice (30):

$$\partial_n \partial_x b_n(j) = \partial_n \left( b_n^{(1)}(j) - b_n^{(1)}(j - 1) \right) = \partial_x \left( h_n(j) - h_n(j - 1) \right) = \partial_x \partial_n b_n(j).$$

The next step is to determine the derivatives $\partial_x b_n^{(1)}(j)$ but this, however, requires to introduce nonlocalities $b_n^{(2)}(j)$ of the second order. Let

$$\partial_x b_n^{(1)}(j) = b_n^{(1)}(j) \left( b_{n+1}(j + 1) - b_n(j) \right) + h_n(j) \left( b_n^{(1)}(j - 1) - b_n^{(1)}(j) \right) + b_n^{(2)}(j) - b_n^{(2)}(j - 1);$$

then the compatibility condition $\partial_n \partial_x b_n^{(1)}(j) = \partial_x \partial_n b_n^{(1)}(j)$ obviously leads to the following relation:

$$\partial_n b_n^{(2)}(j) = h_n(j) b_n^{(1)}(j + 1) - h_{n+1}(j + 1) b_n^{(1)}(j).$$

The following Proposition is proved by standard inductive reasoning.
**Proposition 5.** Nonlocal variables $b_n^{(k)}(j)$, where $k = 2, 3, \ldots$, satisfy the following equations:

\[
\begin{cases}
\partial_n b_n^{(k)}(j) = h_n(j) b_{n+1}^{(k-1)}(j + 1) - h_{n+k-1}(j + k - 1) b_{n+1}^{(k-1)}(j) \\
\partial_x b_n^{(k)}(j) = b_n^{(k)}(j)(b_{n+k}(j + k) - b_n(j)) + h_{n+k-1}(j + k - 1)(b_n^{(k)}(j - 1) - b_n^{(k)}(j)) + b_{n+1}^{(k+1)}(j - b_{n+1}^{(k+1)}(j - 1)
\end{cases}
\]  

(31)

The compatibility of these equations for each $k$ is provided by the first of them, but for $k + 1$.

In the semidiscrete case symmetries of the Toda lattice are defined as follows.

**Proposition 6.** The flows

\[
\begin{align*}
q_{n,t}(j) &= b_n^{(2)}(j) + b_n^{(1)}(j) + b_n^{(1)}(j - 1), \\
q_{n,t}(j) &= b_n^{(2)}(j) + b_n^{(2)}(j - 1) + b_n^{(2)}(j - 2) + b_n^{(1)}(j)(2b_n(j) + b_n(j + 1)) + b_n^{(1)}(j - 1)(2b_n(j) + b_n(j - 1)) - b_n^{(1)}(j)h_n(j + 1) - b_n^{(1)}(j - 1)h_n(j) - b_n^{(1)}(j - 2)h_n(j - 1)
\end{align*}
\]

(32)

(33)

define symmetries of the lattice \((22)\).

New set of dynamic variables that are more convenient for our approach is introduced in the same way as in the continuous case:

\[
\begin{align*}
u_n &= e^{-q_n(-2)}, & v_n &= e^{q_{n+1}(-1)}, & w_n &= q^{-q_n(0)}, & z_n &= e^{q_{n+1}(1)}.
\end{align*}
\]

(34)

Replace the nonlocal variables $b_n^{(2)}(-2)$, $b_n^{(1)}(-2)$, $b_n^{(2)}(0)$ and $b_n^{(1)}(0)$ as follows:

\[
\begin{align*}
r_n &= b_n^{(1)}(-2), & s_n &= b_n^{(2)}(-2) + r_n(b_n(-1) - h_n(-1)), & \\
\rho_n &= b_n^{(1)}(0), & \sigma_n &= b_n^{(2)}(0) + \rho_n(b_n(1) - h_n(1)).
\end{align*}
\]

The following Proposition is proved by straightforward calculation.

**Proposition 7.** In terms of the new variables the symmetry \((22)\) is expressed as follows:

\[
\begin{align*}
u_{n,t} &= u_{n,xx} + 2r_n u_n, & v_{n,t} &= -v_{n,xx} - 2r_n v_n, & w_{n,t} &= w_{n,xx} + 2\rho_n w_n, & z_{n,t} &= -z_{n,xx} - 2\rho_n z_n,
\end{align*}
\]

(35)

and the symmetry \((33)\) can be rewritten in the following way:

\[
\begin{align*}
u_{n,t} &= u_{n,xxx} + 3\sigma_n u_{n,x} - 3s_n u_n + 3\rho_n u_{n,x}, & v_{n,t} &= v_{n,xxx} + 3r_n v_{n,x} + 3s_n v_n, & w_{n,t} &= w_{n,xxx} + 3\sigma_n w_n + 3\rho_n w_{n,x}, & z_{n,t} &= z_{n,xxx} + 3\rho_n z_{n,x} + 3\sigma_n z_n.
\end{align*}
\]

(36)

(37)

These formulas are more convenient then the formulas \((32, 33)\) because of their compactness. Besides this, the flow defining the symmetry for each of the dynamic variables is expressed only in terms of this variable and two nonlocalities. And, finally, the first of the identities \((31)\) has a very compact form in terms of these variables. The following Proposition holds.

**Proposition 8.** Difference derivatives of the functions $r_n$, $s_n$, $\rho_n$, $\sigma_n$ are total derivatives with respect to $x$:

\[
\begin{align*}
\partial_n r_n &= \partial_x(u_n v_n), & \partial_n s_n &= \partial_x \left( u_{n,x} u_n - \frac{1}{2}(u_n v_n)^2 \right), \\
\partial_n \rho_n &= \partial_x(w_n z_n), & \partial_n \sigma_n &= \partial_x \left( z_{n,x} w_n - \frac{1}{2}(w_n z_n)^2 \right).
\end{align*}
\]

(38)

(39)
We are going to find the cut-off constraints for the infinite semidiscrete Toda lattice that are compatible with the symmetries (32) and (33). For convenience, this should be done in terms of the new set of variables (34) rather than in terms of the old one. In theory, it is possible to consider as complicate cut-off constraints as one wants; but from the practical point of view of obtaining a complete description of all integrable cut-off constraints of a certain form, we’ll restrict ourselves by the following case. We’ll examine the relations between four dynamic variables consecutive in the variable \( j \) and their derivatives with respect to \( n \) and with respect to \( x \). Without loss of generality one may assume that \( j = -2, -1, 0, 1 \), i.e. that we are looking for a relation between the variables \( u_n, v_n, w_n, z_n \) and their derivatives. However, cut-off constraints may impose additional relations between nonlocal variables (as it happens in the continuous case) and these relations may be as complicate as one wants as well. Therefore here we also have to restrict ourselves by some reasonable limitations on the relations considered. Indeed, we’ll assume that the nonlocal variables \( r_n, s_n, \rho_n \) and \( \sigma_n \) are independent from all dynamic variables except for

\[

u_n, \ v_n, \ w_n, \ z_n, \ u_{n+1}, \ v_{n+1}, \ w_{n+1}, \ z_{n+1}, \ldots
\]

Under the above assumptions the following Propositions hold.

**Proposition 9.** The only cut-off constraint of the form \( u_n = F(v_{n-1}, v_n, w_n, w_{n+1}, z_n, z_{n+1}) \) compatible with the symmetry (35) is the trivial one: \( u_n = 0 \).

**Proposition 10.** If the dynamic variables

\[
v_{n-1}, \ v_n, \ w_n, \ w_{n+1}, \ z_n, \ z_{n+1}, \ v_{n-1,x}, \ w_{n,x}, \ z_{n,x}
\]

are independent, then among all cut-off constraints of the form \( u_n = F(v_{n-1}, v_n, w_n, w_{n+1}, z_n, z_{n+1}) \) the only constraint compatible with the symmetry (36,37) is the trivial one: \( u_n = 0 \).

Therefore in order to find non-trivial boundary conditions compatible with the symmetry (36,37) one has to impose a relation between the variables (40). The direct but rather cumbersome calculation leads to the following Theorem.

**Theorem 2.** Let the dynamic variables (40) be linked by a relation of the form

\[
v_{n-1,x} = H(v_{n-1}, v_n, w_n, w_{n+1}, z_n, z_{n+1}, w_{n,x}, z_{n,x})
\]

and suppose a non-trivial cut-off constraint of the form \( u_n = F(v_{n-1}, v_n, w_n, w_{n+1}, z_n, z_{n+1}) \) is compatible with the symmetry (36,37). Then

\[
H = \frac{v_{n-1}w_{n,x}}{w_n} + v_{n-1}z_nw_n,
\]

and cut-off constraint is as follows:

\[
u_n = \frac{z_{n+1}w_{n+1}}{v_n}.
\]

Obviously, the cut-off constraint (42) is exactly the same as the boundary condition (24) and the relation (41) is equivalent to the relation (29) that has already appeared before. This means that in the semidiscrete case as well as in the continuous one the natural class of cut-off constraints of the form \( u_n = F(v_{n-1}, v_n, w_n, w_{n+1}, z_n, z_{n+1}) \) leads to finite Toda lattices corresponding to classical simple Lie algebras (more precisely, to their semidiscrete analogs).

**Remark 6.** The cut-off constraints on the right edge compatible with symmetries (35) or (36,37) are examined similarly.

---

3Nonlocal variables \( r_{n\pm 1}, \rho_{n\pm 1}, s_{n\pm 1}, \sigma_{n\pm 1}, \ldots \) are expressed in term of the dynamic variables and nonlocalities \( r_n, \rho_n, s_n, \sigma_n \) using the equations (38,39).
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