NON-COMMUTATIVE CREPANT RESOLUTION OF MINIMAL NILPOTENT ORBIT CLOSURES OF TYPE A AND MUKAI FLOPS

WAHEI HARA

Abstract. In this article, we construct a non-commutative crepant resolution (=NCCR) of a minimal nilpotent orbit closure $\overline{B(1)}$ of type A, and study relations between an NCCR and crepant resolutions $Y$ and $Y^+$ of $\overline{B(1)}$. More precisely, we show that the NCCR is isomorphic to the path algebra of the double Beilinson quiver with certain relations and we reconstruct the crepant resolutions $Y$ and $Y^+$ of $\overline{B(1)}$ as moduli spaces of representations of the quiver. We also study the Kawamata-Namikawa’s derived equivalence between crepant resolutions $Y$ and $Y^+$ of $\overline{B(1)}$ in terms of an NCCR. We also show that the P-twist on the derived category of $Y$ corresponds to a certain operation of the NCCR, which we call multi-mutation, and that a multi-mutation is a composition of Iyama-Wemyss’s mutations.

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1. Introduction

The aim of this article is to study non-commutative crepant resolutions (=NCCR) of a minimal nilpotent orbit closure $\overline{B(1)}$ of type A. The notion of NCCR was first introduced by Van den Bergh [VdB04b] in relation to the study of the derived categories of algebraic varieties. We can regard the concept of NCCR as a generalization of the notion of crepant resolution. Van den Bergh introduced it with an expectation that all crepant resolutions, whether commutative or not, have equivalent derived categories. This expectation is a special (and non-commutative) version of a more general conjecture that K-equivalence implies derived equivalence. We note that the study of NCCR is also motivated by theoretical physics (see Introduction of [Le12]).

2010 Mathematics Subject Classification. 14B05, 14E16, 14F05.
Keywords and phrases. Derived category; Mukai flop; Nilpotent orbit closure; Non-commutative crepant resolution; P-twist; Quiver representation; Moduli space.
An NCCR of a Gorenstein algebra $R$ is defined as an endomorphism ring $\text{End}_R(M)$ of a (maximal) Cohen-Macaulay $R$-module $M$ such that $\text{End}_R(M)$ is a (maximal) Cohen-Macaulay $R$-module and has finite global dimension (see Definition 2.1 and Lemma 2.2). In relation to NCCR, it is natural to ask the following questions.

1. Construct an NCCR of $R$ and characterize a module $M$ that gives the NCCR.

2. Construct a derived equivalence between the NCCR and a (commutative) crepant resolution.

3. Construct a (commutative) crepant resolution as a moduli space of modules over the NCCR.

For example, in [BLV10], Buchweitz, Leuschke, and Van den Bergh studied about these problems for a determinantal variety. In this article, we deal with the above problems for a minimal nilpotent orbit closure $B(1)$ of type A. We also study about the derived equivalences for Mukai flops from the point of view of NCCR.

1.1. NCCR of minimal nilpotent orbit closures of type A. Let $V = \mathbb{C}^N$ be a complex vector space of dimension $N \geq 2$. Let us consider a subset $B(1)$ of $\text{End}_\mathbb{C}(V)$ that is given by

$$B(1) := \{ X \in \text{End}_\mathbb{C}(V) \mid X^2 = 0, \dim \text{Ker} X = N - 1 \}.$$  

This is a minimal nilpotent orbit of type A. It is well known that the closure $\overline{B(1)}$ of the orbit $B(1)$ is normal and has only symplectic singularities, and thus the affine coordinate ring $R$ of $\overline{B(1)}$ is normal and Gorenstein. Since $\text{codim}_{\overline{B(1)}}(\partial B(1)) \geq 2$, we have a $\mathbb{C}$-algebra isomorphism $R \simeq H^0(\overline{B(1)}, \mathcal{O}_{B(1)})$. Let $H$ be a subgroup of $\text{SL}_N$ such that $\text{SL}_N / H \simeq B(1)$. It is easy to see that $H$ is isomorphic to a subgroup of $\text{SL}_N$

$$H \simeq \left\{ A = \begin{pmatrix} c & 0 & \cdots & 0 & 0 \\ * & A' & \vdots \\ * & * & c \end{pmatrix} \mid A' \in \text{GL}_{N-2}, \ c \in \mathbb{C}^\times, \ c^2 \cdot \text{det}(A') = 1 \right\}.$$  

Let $\mathcal{M}_a$ be a homogeneous line bundle on $B(1)$ that corresponds to the character $H \ni A \mapsto c^{-a} \in \mathbb{C}^\times$ and we set $M_a := H^0(B(1), \mathcal{M}_a)$. We prove that a direct sum of $R$-modules $(M_a)_a$ gives an NCCR of $R$.

**Theorem 1.1** (see 3.4 and 2.18).

(a) $M_a$ is a Cohen-Macaulay $R$-module for $-N + 1 \leq a \leq N - 1$.

(b) For $0 \leq k \leq N - 1$, the $R$ module $\bigoplus_{a=-N+k+1}^k M_a$ gives an NCCR $\text{End}_R \left( \bigoplus_{a=-N+k+1}^k M_a \right)$ of $R$.

The proof of Theorem 1.1 is based on the theory of tilting bundles on the crepant resolutions $Y$ and $Y^+$. We note that the two crepant resolutions $Y$ and $Y^+$ of $\overline{B(1)}$ are the total spaces of the cotangent bundles on $\mathbb{P}(V)$ and $\mathbb{P}(V^*)$, respectively. Let $\pi : Y \to \mathbb{P}(V)$ and $\pi' : Y^+ \to \mathbb{P}(V^*)$ be the projections. We show that, for all $k \in \mathbb{Z}$, the bundles

$$\mathcal{T}_k := \bigoplus_{a=-N+k+1}^k \pi^* \mathcal{O}_{\mathbb{P}(V)}(a) \text{ and } \mathcal{T}_k^+ := \bigoplus_{a=-N+k+1}^k \pi'^* \mathcal{O}_{\mathbb{P}(V^*)}(a)$$  

are the desired tilting bundles.
are tilting bundles on $Y$ and $Y^+$, respectively (Theorem 3.4). We also show that there is a canonical isomorphism of $R$-algebras

$$\Lambda_k := \text{End}_Y(T_k) \cong \text{End}_{Y^+}(T^+_{N-k-1})$$

and that this algebra is isomorphic to the one that appears in Theorem 1.1 (b). Moreover, by the theory of tilting bundles, we have an equivalence of categories $D_b(Y) \cong D_b(\text{mod}(\Lambda_k))$ between the derived category of a crepant resolution and of an NCCR. In Section 3.3, we provide another NCCR $\Lambda'$ of $R$ that is not isomorphic to $\Lambda_k$ but is derived equivalent to $\Lambda_k$.

1.2. NCCR as the path algebra of a quiver. Next, we describe an NCCR $\Lambda_k$ of $R$ as the path algebra of the double Beilinson quiver with some relations. We note that similar results for non-commutative resolutions of determinantal varieties are obtained by Buchweitz, Leuschke, and Van den Bergh [BLV10], and Weyman and Zhao [WZ12].

Let $S = \text{Sym}(V \otimes V^*)$ be the symmetric algebra of a vector space $V \otimes \mathbb{C} V^*$. Let $v_1, \ldots, v_N$ be the standard basis of $V = \mathbb{C}^N$ and $f_1, \ldots, f_N$ the dual basis of $V^*$. We regard $x_{ij} = v_j \otimes f_i \in S$ as the variables of the affine coordinate ring of an affine variety $\text{End}_{\mathbb{C}}(V) \cong V^* \otimes \mathbb{C} V$. Since $B(1)$ is a closed subvariety of $\text{End}_{\mathbb{C}}(V)$, $R$ is a quotient of $S$.

**Theorem 1.2** (= Thm. 3.8). As an $S$-algebra, the non-commutative algebra $\Lambda_k$ is isomorphic to the path algebra $S\tilde{\Gamma}$ of the double Beilinson quiver $\tilde{\Gamma}$ with $N$ vertices

![Diagram](image)

with relations

$$v_i v_j = v_j v_i, \quad f_i f_j = f_j f_i, \quad v_j f_i = f_i v_j = x_{ij} \text{ for all } 1 \leq i, j \leq N,$$

and $$\sum_{i=1}^{N} f_i v_i = 0 = \sum_{i=1}^{N} v_i f_i.$$

Building on this theorem, we can also show that the two crepant resolutions $Y$ and $Y^+$ are recovered from the quiver $\tilde{\Gamma}$ as moduli spaces of representations (Theorem 4.1). The idea of the proof is based on the fact that crepant resolutions $Y$ and $Y^+$ are moduli spaces that parametrize representations of Nakajima’s quiver of type $A_1$. We show that there is a natural correspondence between stable representations of Nakajima’s quiver of type $A_1$ and representations of $\tilde{\Gamma}$. At the end of Section 4, we also characterize simple representations of the quiver, namely we show that a simple representation corresponds to a point of a crepant resolution that lies over a non-singular point of $\overline{B(1)}$ (see Theorem 4.13).

We note that these relations between a crepant resolution $Y$ (or $Y^+$) and an NCCR $\Lambda_k$ can be considered as a generalization of McKay correspondence. Classical McKay correspondence states that, for a finite subgroup $G \subset \mathbb{C} \mathbb{L}_2$, there are many relations between the geometry of a quotient variety $\mathbb{C}^2 / G$ and representations of the group $G$. In the modern context, McKay correspondence is understood as
relationships (e.g. a derived equivalence) between the crepant resolution \( \mathbb{C}^2/G \) of \( \mathbb{C}^2/G \) and a quotient stack \([\mathbb{C}^2/G]\). We often say that the crepant resolution \( \mathbb{C}^2/G \) is a “geometric resolution” of \( \mathbb{C}^2/G \). On the other hand, since a coherent sheaf on a quotient stack \([\mathbb{C}^2/G]\) is canonically identified with a module over the skew group algebra \( \mathbb{C}[x,y][G] \), we say that a smooth stack \([\mathbb{C}^2/G]\) is an “algebraic resolution” of \( \mathbb{C}^2/G \). Thus, we can interpret McKay correspondence as a correspondence between geometric and algebraic resolutions. In our case, a geometric resolution of \( \mathbb{B}(1) \) is \( Y \) (or \( Y^+ \)) and an algebraic resolution is the NCCR \( \Lambda_k \).

1.3. Mukai flops, P-twists and mutations. It is well-known that the diagram of two crepant resolutions

\[
Y \xrightarrow{\phi} \mathbb{B}(1) \xleftarrow{\phi^+} Y^+
\]

is a local model of a class of flops that are called Mukai flop. Let \( \widetilde{Y} \) be a blowing-up of \( Y \) along the zero-section \( j(\mathbb{P}(V)) \subset Y \). Then, the exceptional divisor \( E \subset \widetilde{Y} \) is naturally identified with the universal hyperplane in \( \mathbb{P}(V) \times \mathbb{P}(V^*) \). Let \( \widetilde{Y} := \widetilde{Y} \cup E \mathbb{P}(V) \times \mathbb{P}(V^*) \) and \( \mathcal{L}_k \) a line bundle on \( \widetilde{Y} \) such that \( \mathcal{L}_k|_{\widetilde{Y}} = \mathcal{O}_k(kE) \) and \( \mathcal{L}_k|_{\mathbb{P}(V) \times \mathbb{P}(V^*)} = \mathcal{O}(-k,-k) \). By using a correspondence \( Y \xleftarrow{\hat{\phi}} \hat{\widetilde{Y}} \xrightarrow{\hat{\phi}^+} Y^+ \), we define functors

\[
\text{KN}_k := R\hat{\phi}_*(L\hat{q}^*(-) \otimes \mathcal{L}_k) : D^b(Y) \to D^b(Y^+)
\]

and

\[
\text{KN}^+_k := R\hat{\phi}_*(L\hat{q}^*(\mathcal{L}_k)) : D^b(Y^+) \to D^b(Y).
\]

According to the result of Kawamata and Namikawa [Kaw02, Na03], the functors \( \text{KN}_k \) and \( \text{KN}^+_k \) give equivalences between \( D^b(Y) \) and \( D^b(Y^+) \). On the other hand, by using tilting bundles \( T_k \) and \( T^+_{N-k-1} \) above, we get equivalences

\[
\Psi_k : D^b(Y) \xrightarrow{\sim} D^b(\text{mod}(\Lambda_k)) \quad \text{and} \quad \Psi^+_{N-k-1} : D^b(Y^+) \xrightarrow{\sim} D^b(\text{mod}(\Lambda_k)).
\]

By composing \( \Psi_k \) and the inverse of \( \Psi^+_{N-k-1} \), we have an equivalence \( D^b(Y) \to D^b(Y^+) \). Although this functor seems to be different from the functor \( \text{KN}_k \) of Kawamata and Namikawa at a glance, we prove the following.

**Theorem 1.3** (= Thm 5.3). *Our functor \( (\Psi^+_{N-k-1})^{-1} \circ \Psi_k \) (resp. \( (\Psi^+_{N-k-1})^{-1} \circ \Psi^+_k \)) coincides with the Kawamata-Namikawa’s functor \( \text{KN}_k \) (resp. \( \text{KN}^+_k \)).*

We note that our proof of Theorem 5.3 gives an alternative proof for the result of Kawamata and Namikawa that states the functors \( \text{KN}_k \) and \( \text{KN}^+_k \) give equivalences of categories.

The \( R \)-algebras \( \Lambda_k \) and \( \Lambda_{k-1} \) are related by the operation that we call multi-mutation. We introduce a multi-mutation functor

\[
\nu_k^\pm : D^b(\text{mod}(\Lambda_k)) \to D^b(\text{mod}(\Lambda_{k-1}))
\]

(see Definition 5.6) as an analog of Iyama-Wemyss’s mutation functor [IW14] (we call it IW mutation, for short) that Wemyss applied to his framework of “Homological MMP” for 3-folds (see [We14]). We show that a multi-mutation functor \( \nu_k^\pm \) gives an equivalence of categories. Moreover, we prove that our multi-mutation functor
is obtained by composing IW mutation functors \(N - 1\) times (Theorem 5.9). Dually, we introduce a multi-mutation functor \(\nu^+_k : D^b(\text{mod}(\Lambda_k)) \to D^b(\text{mod}(\Lambda_{k+1}))\) and show that a multi-mutation \(\nu^+_k\) is also a composition of \(N - 1\) IW mutation functors. Whereas, it is well-known that the derived category \(D^b(Y)\) of a crepant resolution \(Y\) has a non-trivial auto-equivalence called \(P\)-twist (see Definition 2.24). We show that a composition of multi-mutations corresponds to a \(P\)-twist on \(D^b(Y)\) in the following sense:

**Theorem 1.4** (= Thm. 5.18). Let

\[
\begin{align*}
\nu^-_{N+k} : D^b(\text{mod}(\Lambda_{N+k})) & \to D^b(\text{mod}(\Lambda_{N+k-1})) \\
\nu^+_N : D^b(\text{mod}(\Lambda_{N+k-1})) & \to D^b(\text{mod}(\Lambda_{N+k}))
\end{align*}
\]

be multi-mutation functors. Then we have the following diagram of equivalence functors commutes

\[
\begin{array}{ccc}
D^b(Y) & \xrightarrow{\Psi_{N+k}} & D^b(\text{mod}(\Lambda_{N+k})) \\
\downarrow P_k & & \downarrow \nu^-_{N+k} \\
D^b(Y) & \xrightarrow{\Psi_{N+k-1}} & D^b(\text{mod}(\Lambda_{N+k-1})) \\
\downarrow & & \downarrow \nu^+_N \\
D^b(Y) & \xrightarrow{\Psi_{N+k}} & D^b(\text{mod}(\Lambda_{N+k}))
\end{array}
\]

where \(P_k : D^b(Y) \to D^b(Y)\) is the \(P\)-twist defined by a \(\mathbb{P}^{N-1}\)-object \(j_* O_{\mathbb{P}(V)}(k)\).

This theorem means, under the identification \(\Psi_{N+k} : D^b(Y) \xrightarrow{\sim} D^b(\text{mod}(\Lambda_{N+k}))\), a composition of two multi-mutation functors

\[
\nu^+_{N+k-1} \circ \nu^-_{N+k} \in \text{Auteq}(D^b(\text{mod}(\Lambda_{N+k})))
\]

corresponds to a \(P\)-twist \(P_k \in \text{Auteq}(D^b(Y))\). Donovan and Wemyss proved that, in the case of three dimensional flops, a composition of two IW mutation functors corresponds to a spherical-like twist [DW16]. Our theorem says, in the case of Mukai flops, a composition of many IW mutations corresponds to a \(P\)-twist.

As a corollary of the theorem above, we can prove the following functor isomorphism that was first proved by Cautis [Ca12] and later by Addington-Donovan-Meachan [ADM15]. This result gives an example of “flop-flop=twist” results that are widely observed [To07, DW16, DW15].

**Corollary 1.5** (= 5.20, cf. [ADM15, Ca12]). We have a functor isomorphism

\[
\text{KN}_{N+k} \circ \text{KN}'_{-k} \simeq P_k
\]

for all \(k \in \mathbb{Z}\).

1.4. **Plan of the article.** In Section 2, we provide some basic definitions and recall some fundamental results that we need in later sections. In Section 3, we construct an NCCR of a minimal nilpotent orbit closure of type \(A\), and interpret it as the path algebra of a quiver. In Section 4, we reconstruct the crepant resolutions from the quiver that gives the NCCR as moduli spaces of representations of the quiver. Furthermore, we study simple representations of the quiver. In Section 5, we study derived equivalences of the Mukai flop and \(P\)-twists on a crepant resolution via an NCCR.

\[1\]This statement is suggested by Michael Wemyss in our private communication.
1.5. Notations. In this paper, we always work over the complex number field \( \mathbb{C} \). Moreover, we adopt the following notations.

- \( V = \mathbb{C}^N \): \( N \)-dimensional vector space over \( \mathbb{C} \) \((N \geq 2)\).
- \( \mathbb{P}(V) := V \setminus \{0\}/\mathbb{C}^{	imes} \): projectivization of a vector space \( V \).
- \( |E| \): the total space of a vector bundle \( E \).
- \( \text{mod}(A) \): the category of finitely generated right \( A \)-modules.
- \( D^b(A) \): the (bounded) derived category of an abelian category \( A \).
- \( D^b(X) := D^b(\text{coh}(X)) \): the derived category of coherent sheaves on a variety \( X \).
- \( \Phi^X_P, \Phi^X_{P} \rightarrow Y \) : A Fourier-Mukai functor from \( D^b(X) \) to \( D^b(Y) \) whose kernel is \( P \in D^b(X \times Y) \).
- \( \text{Sym}_R^k M \): \( k \)-th symmetric product of a \( R \)-module \( M \).

Acknowledgments. The author would like to express his gratitude to his supervisor Professor Yasunari Nagai for beneficial conversations and helpful advices. He encouraged me to tackle the problems studied in this paper. The author would like to thank Professor Michael Wemyss for reading the previous version of this paper and suggesting Theorem 5.9. The author also thanks Hiromi Ishii for many advice on drawing TikZ pictures.

2. Preliminaries

2.1. Non-commutative crepant resolutions.

Definition 2.1. Let \( R \) be a Cohen-Macaulay (commutative) algebra and \( M \) a non-zero reflexive \( R \)-module. We set \( \Lambda := \text{End}_R(M) \). We say that the \( R \)-algebra \( \Lambda \) is a non-commutative crepant resolution (= NCCR) of \( R \) or \( M \) gives an NCCR of \( R \) if

\[ \text{gldim } \Lambda_p = \dim R_p \]

for all \( p \in \text{Spec } R \) and \( \Lambda \) is a (maximal) Cohen-Macaulay \( R \)-module.

If we assume that \( R \) is Gorenstein, we can relax the definition of NCCR.

Lemma 2.2 (\cite{IW14}). Let us assume that \( R \) is Gorenstein and \( M \) is a non-zero reflexive \( R \)-module. In this case, an \( R \)-algebra \( \Lambda := \text{End}_R(M) \) is an NCCR of \( R \) if and only if \( \text{gldim } \Lambda < \infty \) and \( \Lambda \) is a (maximal) Cohen-Macaulay \( R \)-module.

The theory of NCCR has strong relationship to the theory of tilting bundle.

Definition 2.3. Let \( X \) be a variety. A vector bundle \( T \) (of finite rank) on \( X \) is called a tilting bundle if

1. \( \text{Ext}_X^i(T, T) = 0 \) for \( i \neq 0 \).
2. \( T \) classically generates the category \( D(\text{Qcoh}(X)) \), i.e. for \( E \in D(\text{Qcoh}(X)) \), \( \text{RHom}_X(T, E) = 0 \) implies \( E = 0 \).

Example 2.4. In \cite{Bei79}, Beilinson showed that the following vector bundles on a projective space \( \mathbb{P}^n \)

\[ T = \bigoplus_{k=0}^{n} \mathcal{O}_{\mathbb{P}^n}(k), \quad T' = \bigoplus_{k=0}^{n} \Omega_{\mathbb{P}^n}^{k}(k+1) \]

are tilting bundles. Note that these tilting bundles come from full strong exceptional collections of the derived category \( D^b(\mathbb{P}^n) \) of \( \mathbb{P}^n \) that are called the Beilinson collections.
Once we find a tilting bundle on a variety, we can construct an equivalence between the derived category of the variety and the derived category of a non-commutative algebra that is given as the endomorphism ring of the tilting bundle. This is a generalization of classical Morita theory.

**Theorem 2.5.** Let $T \in \mathbb{D}^b(X)$ be a tilting bundle on a smooth quasi-projective variety $X$. If we set $\Lambda := \text{End}_X(T)$, we have an equivalence of categories

$$\text{RHom}_X(T, -) : \mathbb{D}^b(X) \simto \mathbb{D}^b(\text{mod}(\Lambda)),$$

and the quasi-inverse of this functor is given by

$$- \otimes_\Lambda T : \mathbb{D}^b(\text{mod}(\Lambda)) \simto \mathbb{D}^b(X).$$

For the proof of Theorem 2.5, see [HV07, Theorem 7.6] or [TU10, Lemma 3.3].

The following conjecture is due to Bondal, Orlov, and Van den Bergh.

**Conjecture 2.6** ([VdB04b], Conjecture 4.6). Let $R$ be a Gorenstein $\mathbb{C}$-algebra. Then, all crepant resolutions of $R$ and all NCCRs of $R$ are derived equivalent.

Van den Bergh showed that Conjecture 2.6 holds if $R$ is of dimension 3 and has only terminal singularities [VdB04a, VdB04b]. The existence of an NCCR and a derived equivalence between crepant resolutions and NCCRs are studied in many literatures [Boc12, BLV10, Da10, HNT7, Ka08, SV17a, SV15, SV17b, TU10].

In the rest of this subsection, we recall the basic property of reflexive modules.

**Lemma 2.7** ([BH93], Proposition 1.4.1). Let $R$ be a noetherian ring and $M$ a finitely generated $R$-module. Then the following are equivalent.

1. The module $M$ is reflexive,
2. For each $p \in \text{Spec } R$, one of the following happens
   a. $\text{depth}(R_p) \leq 1$ and $M_p$ is a reflexive $R_p$-module, or
   b. $\text{depth}(R_p) \geq 2$ and $\text{depth}(M_p) \geq 2$.

By using this lemma, we have the following.

**Proposition 2.8.** Let $R$ be a normal Cohen-Macaulay domain and $M$ a (maximal) Cohen-Macaulay $R$-module. Then, $M$ is reflexive.

**Proof.** Let $p$ be a prime ideal of $R$. If $\dim R_p \leq 1$, then the ring $R_p$ is regular and hence $M_p$ has finite projective dimension. Therefore, by the Auslander-Buchsbaum formula ([BH93, Theorem 1.3.3])

$$\text{proj.dim}(M_p) + \text{depth } M_p = \dim R_p,$$

$M_p$ is projective and hence free. If $\dim R_p \geq 2$, we have $\text{depth}(M_p) \geq 2$ by the assumption. \hfill \square

**Proposition 2.9.** Let $R$ be a normal Cohen-Macaulay domain and $M, N$ (maximal) Cohen-Macaulay $R$-modules. Then, the $R$-module $\text{Hom}_R(N, M)$ is reflexive.

**Proof.** If $\dim R_p \leq 1$, then $M_p$ and $N_p$ are free and hence $\text{Hom}_R(N, M)_p$ is also free. Next we assume that $R$ is local and $\dim R \geq 2$. Then, it is enough to show that the depth of $\text{Hom}_R(N, M)$ is greater than or equal to 2. Let us consider the resolution of $N$

$$R^\oplus N_i \xrightarrow{\varphi_i} R^\oplus N_0 \rightarrow N \rightarrow 0.$$
By applying the functor $\text{Hom}_R(-, M)$, we have an exact sequence

$$0 \to \text{Hom}_R(N, M) \to M^\oplus N_0 \xrightarrow{\varphi^*} M^\oplus N_1 \to \text{coker}(\varphi^*) \to 0.$$ 

Then, by using Depth Lemma twice, we have the result. □

2.2. Nilpotent orbit closures. In this subsection, we recall some basic properties of nilpotent orbit closures. The singularity of nilpotent orbit closures gives an important class of symplectic singularities (see [Bea00]). First, we recall the notion of symplectic singularity.

**Definition 2.10 ([Bea00]).** Let $X$ be an algebraic variety. We say that $X$ is a **symplectic variety** if

(i) $X$ is normal.

(ii) The smooth part $X_{\text{sm}}$ of $X$ admits a symplectic 2-form $\omega$.

(iii) For every resolution $f : Y \to X$, the pull back of $\omega$ to $f^{-1}(X_{\text{sm}})$ extends to a global holomorphic 2-form on $Y$.

Let $X$ be an algebraic variety. We say that a point $x \in X$ is a **symplectic singularity** if there is an open neighborhood $U$ of $x$ such that $U$ is a symplectic variety.

Symplectic singularities belong to a good class of singularities that appears in minimal model theory.

**Proposition 2.11 ([Bea00]).** A symplectic singularity is Gorenstein canonical.

For symplectic singularities, we can consider the following reasonable class of resolutions.

**Definition 2.12.** Let $X$ be a symplectic variety. A resolution $\phi : Y \to X$ of $X$ is called **symplectic** if the extended 2-form $\omega$ on $Y$ is non-degenerate. In other words, the 2-form $\omega$ defines a symplectic structure on $Y$.

**Proposition 2.13.** Let $X$ be a symplectic variety and $\phi : Y \to X$ a resolution. Then, the following statements are equivalent

(1) $\phi$ is a crepant resolution,

(2) $\phi$ is a symplectic resolution,

(3) the canonical divisor $K_Y$ of $Y$ is trivial.

Next, we recall the definition of nilpotent orbit closures and some basic properties of them. Let $\mathfrak{g}$ be a complex Lie algebra. For $u \in \mathfrak{g}$, we define a linear map $\text{ad}_u : \mathfrak{g} \to \mathfrak{g}$ by $x \mapsto [u, x]$. In the following, we assume that the Lie algebra $\mathfrak{g}$ is semi-simple, i.e. the bilinear form $\kappa(u, v) := \text{trace}(\text{ad}_u \circ \text{ad}_v)$ is non-degenerate. An element $v \in \mathfrak{g}$ is nilpotent if the corresponding linear map $\text{ad}_v$ is nilpotent. Let $G$ be the adjoint algebraic group of $\mathfrak{g}$. Then, $G$ acts on $\mathfrak{g}$ via the adjoint representation. An orbit $O = G \cdot v \subset \mathfrak{g}$ of $v$ under this action is called a nilpotent orbit if the element $v$ is nilpotent.

**Proposition 2.14 ([Pa91]).** The normalization $\tilde{O}$ of a nilpotent orbit closure $\overline{O}$ in a complex semi-simple Lie algebra $\mathfrak{g}$ has only symplectic singularities. Hence, the singularity of $\tilde{O}$ is Gorenstein canonical.

Let $V = \mathbb{C}^N$ be a $N$-dimensional vector space and

$$B(r) := \{ X \in \text{End}_\mathbb{C}(V) \mid X^2 = 0, \text{rank}(X) = r \} \subset \mathfrak{sl}(V) \simeq \mathfrak{sl}_N.$$
This is a nilpotent orbit of type A. We note that we have
\[ \overline{B(r)} = \{ X \in \text{End}_C(V) \mid X^2 = 0, \text{rank}(X) \leq r \} = \bigcup_{k=1}^{r} B(r). \]

If we consider nilpotent orbit closures of type A, we need not to take the normalization.

**Proposition 2.15 ([KF79]).** Let \( r \geq 1 \). Then, the variety \( \overline{B(r)} \) is normal, and hence has only symplectic singularities. In particular, the variety \( \overline{B(r)} \) is Gorenstein, and has only canonical (equivalently, rational) singularities.

Moreover, we can show that the variety \( \overline{B(r)} \) has symplectic (equivalently, crepant) resolutions.

In the later sections, we study the case \( r = 1 \).

### 2.3. The variety \( \overline{B(1)} \) and its crepant resolutions \( Y \) and \( Y^+ \).

Let \( V = \mathbb{C}^N \) be an \( N \)-dimensional vector space and \( \text{End}_C(V) \) an endomorphism ring of \( V \). Then, the \( \text{SL}_N := \text{SL}(N, \mathbb{C}) \) acts on \( \text{End}_C(V) \) via the adjoint representation
\[ \text{Adj} : \text{SL}_N \rightarrow \text{GL}(\text{End}_C(V)), \ A \mapsto (X \mapsto AXA^{-1}). \]

Let \( X_0 \) be an matrix in \( B(1) \) such that
\[ X_0 : = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{pmatrix} \in \text{End}_C(V). \]

Then, we have
\[ \text{SL}_N \cdot X_0 = B(1). \]

In the following, we consider homogeneous vector bundles on the orbit \( B(1) \). They correspond to linear representations of the stabilizer subgroup \( \text{Stab}_{\text{SL}_N}(X_0) \) of \( \text{SL}_N \).

**Lemma 2.16.** The stabilizer subgroup \( \text{Stab}_{\text{SL}_N}(X_0) \) is given by
\[
\text{Stab}_{\text{SL}_N}(X_0) = \left\{ \begin{pmatrix} c & 0 & \cdots & 0 \\ * & A & \vdots \\ * & * & c \end{pmatrix} : \begin{array}{c} \text{A } \in \text{GL}_{N-2}, \ c \in \mathbb{C} \setminus \{0\}, \ c^2 \cdot \det(A) = 1 \end{array} \right\}.
\]

**Proof.** Let \( A = (a_{ij}) \in \text{SL}_N \). Then, we have
\[
AX_0 = \begin{pmatrix} a_{1N} & 0 & \cdots & 0 \\ a_{2N} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{NN} & 0 & \cdots & 0 \end{pmatrix}, \ X_0A = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ a_{11} & a_{12} & \cdots & a_{1N} \end{pmatrix}.
\]

Thus, if \( AX_0 = X_0A \), we have \( a_{11} = a_{NN}, a_{12} = \cdots = a_{1N} = 0, \) and \( a_{2N} = \cdots = a_{N-1,N} = 0. \) \( \square \)
Definition 2.17. (1) For \( a \in \mathbb{Z} \), we define a character \( m_a : \text{Stab}_{SL_N}(X_0) \to \mathbb{C}^\times \) as

\[
\text{Stab}_{SL_N}(X_0) \ni \begin{pmatrix} c & 0 & \cdots & 0 & 0 \\ * & A & \vdots & \vdots & 0 \\ * & * & \ddots & \ddots & 0 \\ * & * & \cdots & * & c \end{pmatrix} \mapsto c^{-a} \in \mathbb{C}^\times.
\]

(2) Let \( M_a \) be a line bundle on \( B(1) \) that corresponds to the character \( m_a \).

(3) We set \( M_a := H^0(B(1), M_a) \). Then \( M_a \) is a reflexive \( R \)-module.

Next, let us consider a resolution \( Y \) of \( B(1) \). The resolution \( Y \) is given by

\[
Y := \{(X, L) \in \text{End}_C(V) \times \mathbb{P}(V) \mid X(V) \subset L, X^2 = 0\}
\]

and a left \( SL_N \)-action on \( Y \) is given by

\[
A \cdot (X, L) := (AXA^{-1}, AL)
\]

for \( A \in SL_N \) and \( (X, L) \in Y \). Via the second projection \( \pi : Y \to \mathbb{P}(V) \), one can see that \( Y \) is isomorphic to the total space of the cotangent bundle \( \Omega_{\mathbb{P}(V)} \) on \( \mathbb{P}(V) \). Note that the embedding \( Y \subset \text{End}_C(V) \times \mathbb{P}(V) \) is determined by a composition of injective bundle maps

\[
\Omega_{\mathbb{P}(V)} \subset V^* \otimes_C \mathcal{O}_{\mathbb{P}(V)} \subset V^* \otimes_C V \otimes_C \mathcal{O}_{\mathbb{P}(V)}.
\]

Let \( j : \mathbb{P}(V) \to Y \) be the zero-section, and then \( j(\mathbb{P}(V)) \) is given by

\[
j(\mathbb{P}(V)) = \{(0, L) \in \text{End}_C(V) \times \mathbb{P}(V)\}.
\]

On the other hand, the image of the first projection \( \phi : Y \to \text{End}_C(V) \) is just \( B(1) \), and if we set \( U := Y \setminus j(\mathbb{P}(V)) \), then, \( \phi \) contracts \( j(\mathbb{P}(V)) \) to a point \( 0 \in B(1) \), and \( U \) is isomorphic to \( B(1) \) via the morphism \( \phi : Y \to B(1) \). Thus, the first projection \( \phi \) gives a resolution of \( B(1) \). Since the affine variety \( B(1) \) is a symplectic variety, the canonical divisor of \( B(1) \) is trivial. On the other hand, since \( Y \) is isomorphic to the total space of the cotangent bundle on a projective space, the canonical divisor of \( Y \) is also trivial. Thus, the resolution of singularities \( \phi : Y \to B(1) \) is a crepant resolution, and in this case, is symplectic resolution of \( B(1) \).

Let us set \( \mathcal{O}_Y(a) := \pi^* \mathcal{O}_{\mathbb{P}(V)}(a) \).

Lemma 2.18. Under the identification \( U \simeq B(1) \), the homogeneous vector bundle \( \mathcal{M}_a \) is isomorphic to \( \mathcal{O}_Y(a)|_U \).

Proof. We first note that \( \mathcal{O}_Y(a)|_U \) is a homogeneous line bundle on \( U \). Let \( L_0 := X_0(V) \), then \( L_0 \) is a line in \( V \). Let \( y_0 := (X_0, L_0) \in U \) be a point. The fiber of the line bundle \( \mathcal{O}_Y(a)|_U \) at \( y_0 \in U \) is canonically isomorphic to \( L_0^{\otimes -a} \). Note that the action of \( \text{Stab}_{SL_N}(X_0) \) on \( L_0 \) is given by

\[
\begin{pmatrix} c & 0 & \cdots & 0 & 0 \\ * & A & \vdots & \vdots & 0 \\ * & * & \ddots & \ddots & 0 \\ * & * & \cdots & * & c \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}
\]

...
Therefore, the character \( \text{Stab}_{\text{SL}_N}(X_0) \to \text{GL}(L_0^{\otimes -a}) \) that determines \( \mathcal{O}_Y(a)|_U \) coincides with the one that defines \( \mathcal{M}_a \). \( \square \)

Next, we study the other crepant resolution \( Y^+ \) of \( \overline{B(1)} \). Let \( \mathbb{P}(V^*) \) be a dual projective space, that is

\[ \mathbb{P}(V^*) = \{ H \subset V \mid H \text{ is a hyperplane in } V \}. \]

The variety \( Y^+ \) is defined by

\[ Y^+ := \{(X,H) \in \text{End}_C(V) \times \mathbb{P}(V^*) \mid X(V) \subset H, X(H) = 0 \}. \]

An \( \text{SL}_N \)-action on \( Y^+ \) is given by \( A \cdot (X,H) = (AXA^{-1}, AH) \). Let \( \phi^+: Y \to \overline{B(1)} \) be the first projection and \( \pi': Y^+ \to \mathbb{P}(V^*) \) the second projection. As in the case of \( Y, Y^+ \) is isomorphic to the total space of the cotangent bundle \( \Omega_{\mathbb{P}(V^*)} \) on \( \mathbb{P}(V^*) \) via the second projection \( \pi': Y^+ \to \mathbb{P}(V^*) \), and the first projection \( \phi^+: Y^+ \to \overline{B(1)} \) gives a crepant resolution of \( \overline{B(1)} \). The morphism \( \phi^+: Y \to \overline{B(1)} \) contracts the zero section \( j': \mathbb{P}(V^*) \to Y^+ \). Let \( U^+ := Y^+ \setminus j'(\mathbb{P}(V^*)) \) and \( \mathcal{O}_{Y^+}(a) := (\pi')^* \mathcal{O}_{\mathbb{P}(V^*)}(a) \).

As in the above, we can show the following.

**Lemma 2.19.** Under the identification \( U^+ \simeq B(1) \), the homogeneous vector bundle \( \mathcal{M}_a \) is isomorphic to \( \mathcal{O}_{Y^+}(-a)|_{U^+} \).

2.4. **Iyama-Wemyss’s mutation.** In the present subsection, we recall some basic definitions and properties about Iyama-Wemyss’s mutation.

**Definition 2.20.** Let \( R \) be a \( d \)-singular Calabi-Yau ring \( (d\text{-SCY}, \text{for short}) \). A reflexive \( R \)-module \( M \) is say to be a **modifying module** if \( \text{End}_R(M) \) is a (maximal) Cohen-Macaulay \( R \)-module.

**Definition 2.21.** Let \( A \) be a ring, \( M,N \) \( A \)-modules, and \( N_0 \in \text{add } N \). A morphism \( f: N_0 \to M \) is called a **right (add } N \)-approximation** if the map

\[ \text{Hom}_A(N,N_0) \xrightarrow{f^0} \text{Hom}_A(N,M) \]

is surjective.

Let \( R \) be a normal \( d\text{-SCY} \) ring and \( M \) a modifying \( R \)-module. For \( 0 \neq N \in \text{add } M \), we consider

1. a right (add } N \)-approximation of \( M, a : N_0 \to M \).
2. a right (add } N^* \)-approximation of \( M^*, b : N_1^* \to M^* \).

Let \( K_0 := \text{Ker}(a) \) and \( K_1 := \text{Ker}(b) \).

**Definition 2.22.** With notations as above, we define the **right mutation** of \( M \) at \( N \) to be \( \mu^R_N(M) := N \oplus K_0 \) and the **left mutation** of \( M \) at \( N \) to be \( \mu^L_N(M) := N \oplus K_1^* \).

In [IW13], Iyama and Wemyss proved the following theorem.

**Theorem 2.23 ([IW14]).** Let \( R \) be a normal \( d\text{-SCY} \) ring and \( M \) a modifying module. Assume that \( 0 \neq N \in \text{add } M \). Then

1. \( R \)-algebras \( \text{End}_R(M), \text{End}_R(\mu^R_N(M)), \text{End}_R(\mu^L_N(M)) \) are derived equivalent.
2. If \( M \) gives an NCCR of \( R \), so do its mutations \( \mu^R_N(M) \) and \( \mu^L_N(M) \).

\(^2\text{We do not give the definition here but note that this is equivalent to say that } R \text{ is Gorenstein and } \dim R_m = d \text{ for all maximal ideal } m \subset R \text{[HT08].} \)
The equivalence between $\text{End}_R(M)$ and $\text{End}_R(\mu_L^N(M))$ is given as follows. Let $Q := \text{Hom}_R(M, N)$ and

$$V := \text{Image}(\text{Hom}_R(M, N_1) \to \text{Hom}_R(M, K_1^*)) .$$

Then, one can show that $V \oplus Q$ is a tilting $\Lambda := \text{End}_R(M)$-module and there is an isomorphism of $R$-algebras

$$\text{End}_R(\mu_L^N(M)) \simeq \text{End}_\Lambda(V \oplus Q).$$

Thus, we have an equivalence

$$T_N := \text{RHom}(V \oplus Q, \cdot : D^b(\text{mod}(\text{End}_R(M)))) \to D^b(\text{mod}(\text{End}_R(\mu_L^N(M)))) .$$

In this paper, we only use left IW mutations and hence we call them simply the IW mutation functor.

In the later section, we introduce a concept of multi-mutations and prove that a multi-mutation can be written as a composition of IW mutation functors.

2.5. P-twists. In this subsection, we recall the definition of P-twists and their basic properties.

**Definition 2.24.** An object $E$ in the derived category $D^b(X)$ of a variety $X$ of dimension $2n$ is called a $\mathbb{P}$-object if we have $E \otimes \omega_X \simeq E$ and

$$\text{Hom}(E, E[i]) \simeq H^i(\mathbb{P}^n; \mathbb{C})$$

for all $i \in \mathbb{Z}$. For a $\mathbb{P}$-object $E$, the P-twist $P_E : D^b(X) \to D^b(X)$ by $E$ is defined as follow

$$P_E(F) := \text{Cone} \left( \text{Cone}(E \otimes \text{RHom}(E, F)[-2] \to E \otimes \text{RHom}(E, F)) \xrightarrow{ev} F \right) .$$

See Lemma 5.15 for a basic example of $\mathbb{P}$-object.

**Proposition 2.25 ([HT06]).** A P-twist gives an auto-equivalence of $D^b(X)$.

The notion of P-twist was first introduced by Huybrechts and Thomas in their paper [HT06] as an analogue of the notion of spherical twist. Spherical twists give an important class of auto-equivalences on the derived category of a Calabi-Yau variety. In contrast, P-twists give a significant class of auto-equivalences on the derived category of a (holomorphic) symplectic variety. In Section 5.2, we study P-twists on the symplectic variety $Y = |\Omega_{\mathbb{P}(V)}|$ that is explained in the above section, from the viewpoint of NCCR.

3. Non-commutative crepant resolutions of $\overline{B(1)}$

3.1. The existence of NCCRs of $\overline{B(1)}$ and relations between CRs. In this section, we study non-commutative crepant resolutions of a minimal nilpotent closure $\overline{B(1)} \subset \text{End}(V)$ where $V = \mathbb{C}^N$. We always assume $N \geq 2$. Let $R$ be the affine coordinate ring of $\overline{B(1)}$. By Proposition 2.15, the $\mathbb{C}$-algebra $R$ is Gorenstein and normal. Note that $B(1) = B(1) \cup \{0\}$ as set and hence we have

$$\text{codim}_{\overline{B(1)}}(\overline{B(1)} \setminus B(1)) = N \geq 2.$$ 

Thus, we have a $\mathbb{C}$-algebra isomorphism

$$R \simeq H^0(B(1), \mathcal{O}_{B(1)}).$$

**Lemma 3.1.** Let $\mathcal{F}$ be a coherent sheaf on $Y$ that satisfies
(a) \( \text{Ext}^i_Y(F, O_Y) = 0 \) for \( i > 0 \), and
(b) \( H^i(Y, F) = 0 \) for \( i > 0 \).

Then, the push-forward \( \phi_* F =: M \) is a Cohen-Macaulay \( R \)-module.

**Proof.** Since the resolution \( \phi : Y \to \overline{B(1)} = \text{Spec } R \) is crepant, we have \( \phi^* O_{\overline{B(1)}} \cong O_Y \). Thus, we have

\[
\text{Ext}^i_Y(F, O_Y) \cong \text{Ext}^i_Y(F, \phi^* O_{\overline{B(1)}})
\]

\[
\cong \text{Ext}^i_R(\phi_* F, R)
\]

\[
\cong \text{Ext}^i_R(M, R)
\]

and hence we have

\[
\text{Ext}^i_R(M, R) = 0
\]

for \( i > 0 \). Let \( m \subset R \) be a maximal ideal that corresponds to the origin \( 0 \in \overline{B(1)} \), \( (R, \hat{m}) \) the \( m \)-adic completion of \( (R_m, m) \), and \( \hat{M} \) the \( m \)-adic completion of \( M_m \).

Since the local algebra \( \hat{R} \) is Gorenstein, the canonical module \( \omega_{\hat{R}} \) is isomorphic to \( \hat{R} \) as a \( \hat{R} \)-module. Thus, by Grothendieck’s local duality theorem (see [BH93, Theorem 3.5.8]), we have

\[
H^i(\hat{M}) = \text{Hom}_{\hat{R}}\left( \text{Ext}^{2N-i-2}_{\hat{R}}(\hat{M}, \hat{R}), E(\hat{R}/\hat{m}) \right)
\]

where \( E(\hat{R}/\hat{m}) \) is the injective hull of the residue field \( \hat{R}/\hat{m} \). Therefore, we have

\[
H^i_m(\hat{M}) = 0
\]

for \( i < 2N - 2 = \dim R \) and hence \( M \) is a Cohen-Macaulay \( R \)-module (see [BH93, Theorem 3.5.7]).

**Lemma 3.2.** Let \( E \) be a vector bundle on \( \mathbb{P}(V) \) such that

\[
H^i(\mathbb{P}(V), E(k)) = 0
\]

for all \( i > 0 \) and \( k \geq 0 \). Then, we have

\[
H^i(Y, \pi^* E) = 0
\]

for all \( i > 0 \).

**Proof.** Let \( Z \) be a total space of a vector bundle \( V^* \otimes_C O_{\mathbb{P}(V)}(-1) \). Then, \( Y \) is embedded in \( Z \) via the Euler sequence

\[
0 \to \Omega_{\mathbb{P}(V)} \to V^* \otimes_C O_{\mathbb{P}(V)}(-1) \to O_{\mathbb{P}(V)} \to 0.
\]

Since \( (V \otimes_C O_{\mathbb{P}(V)}(-1))/\Omega_{\mathbb{P}(V)} \cong O_{\mathbb{P}(V)} \), the ideal sheaf \( I_{Y/Z} \) is isomorphic to \( O_Z \). Thus, we have an exact sequence on \( Z \)

\[
0 \to O_Z \to O_Z \to O_Y \to 0.
\]

Let \( \pi_Z : Z \to \mathbb{P}(V) \) be the projection. Then, we have

\[
H^i(Z, \pi_Z^* E) \cong H^i(\mathbb{P}(V), E \otimes R\pi_Z^* O_Z)
\]

\[
\cong H^i(\mathbb{P}(V), E \otimes \pi_Z^* O_Z) \quad (\text{since } \pi_Z \text{ is affine})
\]

\[
\cong \bigoplus_{k \geq 0} \text{Sym}^k V \otimes_C H^i(\mathbb{P}(V), E(k))
\]

and this is zero for \( i > 0 \) by the assumption. Thus we have

\[
H^i(Z, \pi_Z^* E \otimes O_Y) = H^i(Y, \pi^* E) = 0
\]
Definition 3.3. For an integer $k \in \mathbb{Z}$, let $T_k := \bigoplus_{a=-N+k+1}^k \mathcal{O}_Y(a)$ be a vector bundle on $Y$ and $\Lambda_k := \text{End}_Y(T_k)$ the endomorphism ring of $T_k$.

Note that the $R$-algebra structure of $\Lambda_k$ does not depend on the choice of the integer $k$. Nevertheless, we adopt this notation to emphasize that the algebra $\Lambda_k$ is given as the endomorphism ring of a bundle $T_k$.

Theorem 3.4. The following hold.

1. For all $k \in \mathbb{Z}$, the vector bundle $T_k$ is a tilting bundle on $Y$.
2. For all $-N + 1 \leq a \leq N - 1$, we have
   \[ \phi_* \mathcal{O}_Y(a) = M_a, \]
   and $M_a$ is a (maximal) Cohen-Macaulay $R$-module.
3. If $0 \leq k \leq N - 1$, then we have an isomorphism
   \[ \text{End}_Y(T_k) \cong \text{End}_R \left( \bigoplus_{a=-N+k+1}^k M_a \right) \]
4. The $R$-module
   \[ \bigoplus_{a=-N+k+1}^k M_a \]
   gives an NCCR $\Lambda_k$ of $R$ for $0 \leq k \leq N - 1$.
5. There is an equivalence of categories
   \[ \text{RHom}_Y(T_k, -) : \text{D}^b(Y) \sim \text{D}^b(\text{mod}(\Lambda_k)). \]

We note that (1) and (5) of Theorem 3.4 are also obtained by Toda and Uehara in [TU10]. They also study the perverse heart of $\text{D}^b(Y)$ that corresponds to $\text{mod}(\Lambda_0)$ via the derived equivalence.

Proof. Let $T = \bigoplus_{a=0}^{N-1} \mathcal{O}_{\mathbb{P}(V)}(a)$ is a tilting bundle on $\mathbb{P}(V)$. Then, we have
\[ H^i(\mathbb{P}(V), T^* \otimes T \otimes \mathcal{O}_{\mathbb{P}(V)}(k)) = 0 \]
for all $i > 0$ and $k \geq 0$. Thus, by Lemma 3.2 we have
\[ H^i(Y, T_0^* \otimes T_0) = 0 \]
and hence $T_0$ is a tilting bundle on $Y$. Since other bundles $T_k (k \in \mathbb{Z})$ are obtained from $T_0$ by twisting $\mathcal{O}_Y(k)$, $T_k (k \in \mathbb{Z})$ are also tilting bundles on $Y$. This shows (1).

On the other hand, by Lemma 3.2 we have
\[ H^i(Y, \mathcal{O}_Y(a)) = 0 \text{ for } i > 0 \]
if $a \geq -N + 1$. Therefore, if $-N + 1 \leq a \leq N - 1$, we have
\[ H^i(Y, \mathcal{O}_Y(a)) = 0, \]
\[ \text{Ext}^i_Y(\mathcal{O}_Y(a), \mathcal{O}_Y) = 0 \]
for all $i > 0$. Thus, by Lemma 3.1 we have the $R$-module $\phi_* \mathcal{O}_Y(a) = H^0(Y, \mathcal{O}_Y(a))$ is Cohen-Macaulay if $-N + 1 \leq a \leq N - 1$. In particular, if $-N + 1 \leq a \leq N - 1$, $\phi_* \mathcal{O}_Y(a)$ is a reflexive $R$-module by Proposition 2.8. By Lemma 2.18, $\phi_* \mathcal{O}_Y(a)$ and $M_a$ are isomorphic outside the unique singular point $0 \in B(1)$. Thus, we have
\[ \phi_* \mathcal{O}_Y(a) \cong M_a \] for \(-N + 1 \leq a \leq N - 1\) and hence \(M_a\) is (maximal) Cohen-Macaulay as an \(R\)-module if \(-N + 1 \leq a \leq N - 1\). This shows (2).

Next, we prove (3). By Lemma 3.1 and (1), we have \(\text{End}_Y(T_k) \cong \phi_*(T_k^* \otimes T_k)\) is a (maximal) Cohen-Macaulay \(R\)-module and hence is reflexive by Proposition 2.8. On the other hand, the \(R\)-module \(\text{End}_R(k \bigoplus_{a=-N+k+1}^k M_a)\) is also reflexive for \(0 \leq k \leq N - 1\) by Proposition 2.9. These two reflexive \(R\)-modules are isomorphic to each other outside the unique singular point 0 \(\in B(1)\). Thus, we have

\[ \text{End}_Y(T_k) \cong \text{End}_R \left( \bigoplus_{a=-N+k+1}^k M_a \right). \]

Finally, (4) follows from (1), (2), and (3). (5) follows from (1).

It is easy to see that the dual statements hold for \(Y^+\).

**Theorem 3.5.** Let \(T_k^+ := \bigoplus_{a=-N+k+1}^k \mathcal{O}_Y(a)\). Then, the following hold.

1. For all \(k \in \mathbb{Z}\), the vector bundle \(T_k^+\) is a tilting bundle on \(Y^+\).
2. For all \(-N + 1 \leq a \leq N - 1\), we have \(\phi^+_a \mathcal{O}_Y(a) = M_{-a}\).
3. If \(0 \leq k \leq N - 1\), then we have an isomorphism
   \[ \text{End}_{Y^+}(T_k^+) \cong \text{End}_R \left( \bigoplus_{a=-N+k+1}^k M_{-a} \right). \]
4. For all \(k \in \mathbb{Z}\), there is a canonical isomorphism
   \[ \text{End}_{Y^+}(T_k^+) \cong \Lambda_{N-k-1}. \]
5. There is an equivalence of categories
   \[ \text{RHom}_{Y^+}(T_k^+, -) : \text{D}^b(Y^+) \xrightarrow{\sim} \text{D}^b(\text{mod}(\Lambda_{N-k-1})). \]

**Proof.** We only show (4). By Lemma 2.18 and Lemma 2.19 we have \(\Lambda_k = \text{End}_Y(T_k)\) and \(\text{End}_{Y^+}(T_{N-k-1}^+)\) are isomorphic to each other on the smooth locus \(B(1)\). Since both algebras are Cohen-Macaulay as \(R\)-modules and hence are reflexive, we have an isomorphism

\[ \Lambda_k = \text{End}_Y(T_k) \cong \text{End}_{Y^+}(T_{N-k-1}^+). \]

This is what we want. \(\square\)

### 3.2. NCCRs as the path algebra of a quiver.

The aim of this subsection is to describe the NCCR \(\Lambda_k\) of \(B(1)\) as the path algebra of a quiver with relations.

As in the above subsection, let \(Z\) be the total space of a vector bundle \(V^* \otimes \mathcal{O}_{\mathbb{P}(V)}(-1)\). Let \(\pi_Z : Z \to \mathbb{P}(V)\) the projection, and we set \(\mathcal{O}_Z(a) := \pi_Z^* \mathcal{O}_{\mathbb{P}(V)}(a)\), \(T_Z := \bigoplus_{a=-N+1}^0 \mathcal{O}_Z(a)\), and \(\Lambda(Z) := \text{End}_Z(T_Z)\). Then, the algebra \(\Lambda_k\) is a quotient algebra of \(\Lambda(Z)\). First, we describe the non-commutative algebra \(\Lambda(Z)\) as the path algebra of a quiver with certain relations.
Note that $Z$ is a crepant resolution of an affine variety $\text{Spec} \, H^0(Z, O_Z)$. We set $\tilde{R} := H^0(Z, O_Z)$. Then, the algebra $\tilde{R}$ is described as follows.

\[ \tilde{R} := H^0(\text{Spec}(\mathbb{P}(V)), \bigoplus_{k \geq 0} \text{Sym}^k V \otimes_{\mathbb{C}} O_{\mathbb{P}(V)}(k)) \]

\[ \simeq \bigoplus_{k \geq 0} \text{Sym}^k V \otimes_{\mathbb{C}} \text{Sym}^k V^* \]

Let $S$ be the affine coordinate ring of $\text{End}_{\mathbb{C}}(V)$, i.e.

\[ S := \bigoplus_{k \geq 0} \text{Sym}^k (V \otimes_{\mathbb{C}} V^*). \]

Let $v_1, \ldots, v_N$ be the standard basis of $V = \mathbb{C}^N$ and $f_1, \ldots, f_N \in V^*$ the dual basis. If we set $x_{ij} := v_j \otimes f_i$, the algebra $S$ is isomorphic to the polynomial ring with $N^2$ variables

\[ S \simeq \mathbb{C}[(x_{ij})_{i,j=1,\ldots,N}] . \]

The affine variety $\text{Spec} \, \tilde{R}$ is embedded in $\text{End}_{\mathbb{C}}(V) = \text{Spec} \, S$ via the canonical surjective homomorphism of algebras

\[ S := \bigoplus_{k \geq 0} \text{Sym}^k (V \otimes_{\mathbb{C}} V^*) \rightarrow \bigoplus_{k \geq 0} \text{Sym}^k V \otimes_{\mathbb{C}} \text{Sym}^k V^* . \]

Next, we define quivers that we use later.

**Definition 3.6.** Let $\Gamma$ be the Beilinson quiver

\[
\begin{align*}
- f_1 & \rightarrow f_1 \rightarrow - f_1 \rightarrow - f_1 \rightarrow \\
0 & : 1 : \cdots : N - 2 : N - 1
\end{align*}
\]

and $\tilde{\Gamma}$ the double Beilinson quiver

\[
\begin{align*}
0 & \quad 1 \quad \cdots \quad N - 2 \quad N - 1 \\
\downarrow & \quad \downarrow \quad \cdots \quad \downarrow \quad \downarrow
\end{align*}
\]

Here, $v_i, f_j$ serve as the label for $N$ different arrows.

Next, we show that the non-commutative algebra $\Lambda_Z$ has a description as the path algebra of the double Beilinson quiver with certain relations.

**Theorem 3.7.** The non-commutative algebra $\Lambda_Z$ is isomorphic to the path algebra $S\tilde{\Gamma}$ of the double Beilinson quiver $\tilde{\Gamma}$ over $S$ with relations

\[
\begin{align*}
v_i v_j &= v_j v_i \text{ for all } 1 \leq i, j \leq N \\
f_i f_j &= f_j f_i \text{ for all } 1 \leq i, j \leq N \\
v_j f_i &= f_i v_j = x_{ij} \text{ for all } 1 \leq i, j \leq N.
\end{align*}
\]
Proof. First, for \( a, b \in \mathbb{Z}_{>0} \) we have

\[
\text{Hom}_Z(\mathcal{O}_Z(a), \mathcal{O}_Z(b)) \simeq \text{Hom}_{\mathbb{P}(V)}(\mathcal{O}_{\mathbb{P}(V)}(a), (\pi_Z)_* \mathcal{O}_Z \otimes \mathcal{O}_{\mathbb{P}(V)}(b))
\]

\[
\simeq \text{Hom}_{\mathbb{P}(V)} \left( \mathcal{O}_{\mathbb{P}(V)}(a), \left( \bigoplus_{k \geq 0} \text{Sym}^k V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}(V)}(1) \right) \otimes \mathcal{O}_{\mathbb{P}(V)}(b) \right).
\]

Moreover, if \( b \geq a \), we have

\[
\text{Hom}_Z(\mathcal{O}_Z(a), \mathcal{O}_Z(b)) \simeq \bigoplus_{k \geq 0} \text{Sym}^k V \otimes_{\mathbb{C}} \text{Sym}^{k+b-a} V^*.
\]

and if \( b \leq a \), we have

\[
\text{Hom}_Z(\mathcal{O}_Z(a), \mathcal{O}_Z(b)) \simeq \bigoplus_{k \geq 0} \text{Sym}^{k+a-b} V \otimes_{\mathbb{C}} \text{Sym}^k V^*.
\]

We define the action \( v : \mathcal{O}_Z(a) \to \mathcal{O}_Z(a-1) \) of \( v \in V \) on \( \mathcal{T}_Z \) as a morphism that correspond to a morphism

\[
\mathcal{O}_{\mathbb{P}(V)}(a) \to v \otimes \mathcal{O}_{\mathbb{P}(V)}(a) \subset V \otimes \mathcal{O}_{\mathbb{P}(V)}(a) \subset \bigoplus_{k \geq 0} \text{Sym}^k V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}(V)}(a+k-1)
\]

via the adjunction. This morphism \( v : \mathcal{O}_Z(a) \to \mathcal{O}_Z(a-1) \) corresponds to an element

\[
v \otimes 1 \in V \otimes_{\mathbb{C}} \mathbb{C} \subset \bigoplus_{k \geq 0} \text{Sym}^{k+1} V \otimes_{\mathbb{C}} \text{Sym}^k V^*
\]

via the isomorphism

\[
\text{Hom}_Z(\mathcal{O}_Z(a), \mathcal{O}_Z(a-1)) \simeq \bigoplus_{k \geq 0} \text{Sym}^{k+1} V \otimes_{\mathbb{C}} \text{Sym}^k V^*.
\]

We also define the action \( f : \mathcal{O}_Z(a) \to \mathcal{O}_Z(a+1) \) of \( f \in V^* \) on \( \mathcal{T}_Z \) as the morphism that is the pull-back of the morphism

\[
f : \mathcal{O}_{\mathbb{P}(V)}(a) \to \mathcal{O}_{\mathbb{P}(V)}(a+1)
\]

by \( \pi_Z : \mathbb{Z} \to \mathbb{P}(V) \). Note that this morphism corresponds to a morphism

\[
\mathcal{O}_{\mathbb{P}(V)}(a) \xrightarrow{f} \mathcal{O}_{\mathbb{P}(V)}(a+1) \subset \bigoplus_{k \geq 0} \text{Sym}^k V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}(V)}(a+k+1)
\]

via the adjunction, and also corresponds to an element

\[
1 \otimes f \in \mathbb{C} \otimes_{\mathbb{C}} V^* \subset \bigoplus_{k \geq 0} \text{Sym}^k V \otimes_{\mathbb{C}} \text{Sym}^{k+1} V^*
\]

via the isomorphism

\[
\text{Hom}_Z(\mathcal{O}_Z(a), \mathcal{O}_Z(a+1)) \simeq \bigoplus_{k \geq 0} \text{Sym}^k V \otimes_{\mathbb{C}} \text{Sym}^{k+1} V^*.
\]
Now, it is clear that \( v_1, \ldots, v_N \) and \( f_1, \ldots, f_N \) generate \( \Lambda_z \) as a \( S \)-algebra and satisfy the commutative relation
\[
\begin{align*}
v_i v_j &= v_j v_i \\
f_i f_j &= f_j f_i
\end{align*}
\]
for any \( i, j = 1, \ldots, N \).

Next, we check that the relation
\[
f_i v_j = v_j f_i = x_{ij}
\]
is satisfied. By adjunction, the map
\[
f_i v_j : \mathcal{O}_Z(a) \to \mathcal{O}_Z(a)
\]
corresponds to the composition
\[
\begin{align*}
\mathcal{O}_{\mathbb{P}(V)}(a) &\to v_j \otimes \mathcal{O}_{\mathbb{P}(V)}(a) \subset \bigoplus_{k \geq 0} \text{Sym}^k V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}(V)}(a + k - 1) \\
&\xrightarrow{(\pi_z) \cdot f_i} \bigoplus_{k \geq 0} \text{Sym}^k V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}(V)}(a + k),
\end{align*}
\]
where the map \((\pi_z) \cdot f_i\) is the direct sum of the maps
\[
\text{Sym}^k V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}(V)}(a + k - 1) \xrightarrow{id \otimes f_i} \text{Sym}^k V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}(V)}(a + k).
\]
Thus, this map factors through as
\[
\mathcal{O}_{\mathbb{P}(V)}(a) \to v_j \otimes \mathcal{O}_{\mathbb{P}(V)}(a) \xrightarrow{id \otimes f_i} v_j \otimes \mathcal{O}_{\mathbb{P}(V)}(a + 1) \subset \bigoplus_{k \geq 0} \text{Sym}^k V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}(V)}(a + k).
\]
Similarly, the map
\[
v_j f_i : \mathcal{O}_Z(a) \to \mathcal{O}_Z(a)
\]
corresponds to the composition
\[
\begin{align*}
\mathcal{O}_{\mathbb{P}(V)}(a) &\xrightarrow{f_i} \mathcal{O}_{\mathbb{P}(V)}(a + 1) \subset \bigoplus_{k \geq 0} \text{Sym}^k V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}(V)}(a + k + 1) \\
&\xrightarrow{(\pi_z) \cdot v_j} \bigoplus_{k \geq 0} \text{Sym}^k V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}(V)}(a + k),
\end{align*}
\]
by adjunction, where the map \((\pi_z) \cdot v_j\) is the direct sum of maps
\[
\text{Sym}^k V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}(V)}(a + k) \xrightarrow{v_j \otimes id} \text{Sym}^{k+1} V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}(V)}(a + k).
\]
Thus, this map factors through as
\[
\mathcal{O}_{\mathbb{P}(V)}(a) \xrightarrow{f_i} \mathcal{O}_{\mathbb{P}(V)}(a + 1) \xrightarrow{v_j \otimes id} v_j \otimes \mathcal{O}_{\mathbb{P}(V)}(a + 1) \subset \bigoplus_{k \geq 0} \text{Sym}^k V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}(V)}(a + k).
\]
Thus, \( f_i v_j \) and \( v_j f_i \) defines the same element in \( \text{Hom}_Z(\mathcal{O}_Z(a), \mathcal{O}_Z(a)) \), and they correspond to an element
\[
x_{ij} = v_j \otimes f_i \in V \otimes V^* \subset \bigoplus_{k \geq 0} \text{Sym}^k V \otimes \text{Sym}^k V^* (= \tilde{R})
\]
via the isomorphism
\[
\text{Hom}_Z(\mathcal{O}_Z(a), \mathcal{O}_Z(a)) \simeq \bigoplus_{k \geq 0} \text{Sym}^k V \otimes \text{Sym}^k V^*.
\]
Thus, we have the relation
\[ f_i v_j = v_j f_i = x_{ij}. \]

It is clear that \( v_1, \ldots, v_N \) and \( f_1, \ldots, f_N \) do not have other relations. Therefore, we have the result. □

The following is one of the main theorems in this paper.

**Theorem 3.8.** The non-commutative algebra \( \Lambda_k \) is isomorphic to the path algebra \( \tilde{S}\Gamma \) of the double Beilinson quiver \( \tilde{\Gamma} \) with relations
\[
\begin{align*}
v_i v_j &= v_j v_i \text{ for all } 1 \leq i, j \leq N, \\
f_i f_j &= f_j f_i \text{ for all } 1 \leq i, j \leq N, \\
v_j f_i &= f_i v_j = x_{ij} \text{ for all } 1 \leq i, j \leq N, \\
\text{and } \sum_{i=1}^N f_i v_i &= 0 = \sum_{i=1}^N v_i f_i
\end{align*}
\]

**Proof.** By the exact sequence
\[
0 \to O_Z \to O_Z \to O_Y \to 0,
\]
we have an exact sequence
\[
0 \to \Lambda_Z \xrightarrow{i} \Lambda_Z \to \Lambda_k \to 0.
\]
Note that the map \( i : \Lambda_Z \to \Lambda_Z \) is given by the multiplication of \( \sum_{i=1}^N x_{ii} = \sum_{i=1}^N v_i \otimes f_i \in S \). Thus, the result follows from Theorem 3.7. □

**Remark 3.9.** If we work over the base field \( \mathbb{C} \) instead of \( S \), we have
\[ \Lambda_k \simeq \mathbb{C}\tilde{\Gamma}/J' \]
and \( J' \) is an ideal that is generated by
\[
\begin{align*}
v_i v_j &= v_j v_i, \quad f_i f_j = f_j f_i, \quad v_j f_i = f_i v_j, \\
f_k v_j f_i &= f_i v_j f_k, \quad v_j f_i v_l = v_l f_i v_j \\
\sum_{i=1}^N f_i v_i &= 0 = \sum_{i=1}^N v_i f_i.
\end{align*}
\]
The isomorphism \( \mathbb{C}\tilde{\Gamma}/J \to \mathbb{C}\tilde{\Gamma}/J' \) is given by \( v_i \mapsto v_i, \quad f_i \mapsto f_i, \quad x_{ij} \mapsto v_j f_i \).

**Example 3.10.** Let us consider the case \( N = 2 \). In this case, the affine surface \( \overline{B(1)} \) is given by
\[
\overline{B(1)} = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mid a^2 + bc = 0 \right\},
\]
and hence has a Du Val singularity of type \( A_1 \) at the origin. The resolution \( Y = |\Omega_{\mathbb{P}^1}| \to \overline{B(1)} \) is the minimal resolution, and the NCCR \( \Lambda_k \) is isomorphic to the smash product \( \mathbb{C}[x, y] \sharp G \), where \( G \) is a subgroup of \( SL_2 \)
\[ G = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \subset SL_2. \]
The quiver that gives the NCCR $\Lambda_k$ is given by

$$
\begin{array}{ccc}
0 & \xrightarrow{f_1} & 1 \\
\wedge & & \wedge \\
\wedge & & \wedge \\
\end{array}
$$

and the relations (over $\mathbb{C}$) are given by $f_1v_1 + f_2v_2 = 0$, $v_1f_1 + v_2f_2 = 0$.

This quiver (with relations) coincides with the one that is described in Weyman and Zhao’s paper [WZ12 Example 6.15]. In [WZ12 Section 6], Weyman and Zhao studied a description of an NCCR of a (maximal) determinantal variety of symmetric matrices as the path algebra of a quiver. Since the surface $\overline{B(1)}$ is isomorphic to a (maximal) determinantal variety of symmetric matrices

$$\left\{ \begin{pmatrix} a & b \\ b & c \end{pmatrix} \mid ac - b^2 = 0 \right\},$$

they obtained the above description of $\Lambda_k$ as a special case.

3.3. **Remark: Alternative NCCRs of $\overline{B(1)}$.** The NCCR $\Lambda_k$ of $\overline{B(1)}$ that is constructed in the above subsection came from the Beilinson collection of $\mathbb{P}(V)$

$$\mathcal{D}^b(\mathbb{P}(V)) = \langle \mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(N-1) \rangle.$$  

In this subsection, we construct an NCCR of $R$ of another type from the different Beilinson collection

$$\mathcal{D}^b(\mathbb{P}(V)) = \langle \Omega^{N-1}(N), \Omega^{N-2}(N-1), \ldots, \Omega^1(2), \mathcal{O}(1) \rangle.$$

**Definition 3.11.**

(1) We define a representation $n_1 : \text{Stab}_{\text{SL}_N}(X_0) \to \text{SL}_{N-1}$ as

$$\text{Stab}_{\text{SL}_N}(X_0) \ni \begin{pmatrix} c & 0 & \cdots & 0 & 0 \\ * & \star & A & \vdots & \star \\ * & * & & A & \star \end{pmatrix} \mapsto \begin{pmatrix} c & 0 & \cdots & 0 \\ * & \star & \vdots & \star \end{pmatrix} \in \text{SL}_{N-1}.$$  

For $0 \leq a \leq N - 1$, we define a representation $n_a$ by

$$n_a := \bigwedge^a n_1.$$  

(2) Let $N_a$ be a vector bundle on $B(1)$ that corresponds to the representation $n_a$.  

(3) We set $N_a := H^0(B(1), N_a)$. Then $N_a$ is a reflexive $R$-module.  

(4) We define an $R$-algebra $\Lambda'$ by

$$\Lambda' := \text{End}_R \left( \bigoplus_{a=0}^{N-1} N_a \right).$$
As in Lemma 2.18, we can relate the homogeneous vector bundle $N_a$ with a (co)tangent bundle on a projective space. We note that we have an isomorphism between vector bundles on $\mathbb{P}(V)$$\begin{align} \bigwedge^a(T_{\mathbb{P}(V)}(-1)) & \simeq (\Omega_{\mathbb{P}(V)}^a)^*(-a) \\ & \simeq \Omega_{\mathbb{P}(V)}^{N-a-1}(N) \otimes O(-a) \\ & \simeq \Omega_{\mathbb{P}(V)}^{N-a-1}(N-a). \end{align}$

Here, $T_{\mathbb{P}(V)}$ is the tangent bundle on $\mathbb{P}(V)$ and $\Omega_{\mathbb{P}(V)}$ is the cotangent bundle on $\mathbb{P}(V)$.

**Lemma 3.12.** We have $\pi^*\Omega_{\mathbb{P}(V)}^{a-1}(a)|_U \simeq N_{N-a}$.

The proof is completely same as in Lemma 2.18.

We want to show that the algebra $N'$ is an NCCR of $R$. In order to show this, we need the following lemma.

**Lemma 3.13 ([BLV10], Corollary 3.24).** Let $\mathcal{M}_a^{b}(-c) := \text{Hom}_{\mathbb{P}(V)}(\Omega_{\mathbb{P}(V)}^{b-1}(b), \Omega_{\mathbb{P}(V)}^{a-1}(a))(-c)$. Then, the cohomology $H^d(\mathbb{P}(V), \mathcal{M}_a^{b}(-c))$ is not zero only in the following cases:

1. If $d - c > 0$, then $d = 0$ and, necessarily, $c < 0$.
2. If $d - c = 0$, then $c + b \in [\max\{a, b\}, \min\{N, a + b - 1\}]$.
3. If $d - c = -1$, then $c - a \in [\max\{0, N - a - b - 1\}, \min\{N - b, N - a\}]$.
4. If $d - c < -1$, then $d = N - 1$, and necessarily, $c > N$.

In particular, if $c \leq 0$, we have $H^d(\mathbb{P}(V), \mathcal{M}_a^{b}(-c)) = 0$ for all $d > 0$.

From this lemma, we can obtain the following corollaries.

**Corollary 3.14.** For $0 \leq a \leq N - 1$, we have $N_a \simeq \phi_*\pi^*\Omega_{\mathbb{P}(V)}^{N-a-1}(N-a)$ and $N_a$ is Cohen-Macaulay.

**Proof.** Let $k \geq 0$ be a non-negative integer. Note that, $\Omega_{\mathbb{P}(V)}^{N-a-1}(N-a) \otimes O(k) \simeq \mathcal{M}_{a+1}^N(k)$ and

\[
(\Omega_{\mathbb{P}(V)}^{N-a-1}(N-a))^* \otimes O(k) \simeq \Omega_{\mathbb{P}(V)}^a(N) \otimes O(N+a) \otimes O(k) \\
\simeq \Omega_{\mathbb{P}(V)}^a(a+1) \otimes O(k-1) \\
\simeq \mathcal{M}_a^{N+1}(k).
\]

Thus, by Lemma 3.13 and Lemma 3.2, we have

\[ H^i(Y, \pi^*\Omega_{\mathbb{P}(V)}^{N-a-1}(N-a)) = 0 = H^i(Y, \pi^*(\Omega_{\mathbb{P}(V)}^{N-a-1}(N-a))^*) \]

for $i > 0$, and hence by Lemma 3.1 we have the $R$-module $\phi_*\pi^*\Omega_{\mathbb{P}(V)}^{N-a-1}(N-a)$ is (maximal) Cohen-Macaulay. In particular, $\phi_*\pi^*\Omega_{\mathbb{P}(V)}^{N-a-1}(N-a)$ is reflexive and hence we have the desired isomorphism.

**Corollary 3.15.** The bundle

\[ T' := \bigoplus_{a=1}^N \pi^*\Omega_{\mathbb{P}(V)}^{a-1}(a) \]

is a tilting bundle on $Y$ and there is an isomorphism as $R$-algebras

\[ N' \simeq \text{End}_Y(T'). \]
In particular, the $R$-module $\bigoplus_{a=0}^{N-1} N_a$ gives an NCCR $\Lambda'$ of $R$.

**Proof.** The bundle $(T')^* \otimes T'$ is the direct sum of $\pi^* \mathcal{M}_a^b(0)$. By Lemma 3.13 and Lemma 3.2 we have

$$H^i(Y, \pi^* \mathcal{M}_a^b(0)) = 0$$

for $i > 0$ and hence we have

$$\text{Ext}_Y^i(T', T') = H^i(Y, (T')^* \otimes T') = 0$$

for $i > 0$. It is clear that the bundle generates the category $\text{D}^b(\text{Qcoh}(Y))$. Therefore, the bundle $T'$ is tilting. \hfill $\square$

**Corollary 3.16.** Let us assume $N \geq 3$. In this case, although the two NCCRs $\Lambda_k, \Lambda'$ of $R$ are not isomorphic to each other, there is an equivalence of categories

$$\text{D}^b(Y) \simeq \text{D}^b(\Lambda_k) \simeq \text{D}^b(\Lambda').$$

**Proof.** The $R$-rank of the first NCCR $\Lambda_k$ is just $2N$ and the $R$-rank of the second NCCR $\Lambda'$ is

$$2 \sum_{a=1}^{N} \text{rank} \Omega_{\pi(Y)}^{a-1} = 2N.$$

Thus, if $N \geq 3$, $\Lambda_k$ and $\Lambda'$ are not isomorphic to each other but have the equivalent derived categories, where the equivalence is given by the composition

$$\text{D}^b(\Lambda') \xrightarrow{- \otimes \Lambda'} \text{D}^b(Y) \xrightarrow{\text{RHom}_Y(T_k, -)} \text{D}^b(\Lambda_k).$$

This shows the result. \hfill $\square$

At the end of this subsection, we give another type of tilting bundles that we use in the later section (Section 5.2.2).

**Proposition 3.17.** The vector bundle

$$S_k = \bigoplus_{a=-N+2}^{0} \mathcal{O}_Y(a) \oplus \left( \pi^* \Omega_{\pi(Y)}^k \otimes \mathcal{O}_Y(1) \right)$$

and its dual vector bundle $S_k^*$ are tilting bundle on $Y$ for all $0 \leq k \leq N - 1$.

**Proof.** As in Lemma 3.15, the claim follows from direct computations using Lemma 3.13 \hfill $\square$

4. From an NCCR to crepant resolutions

4.1. **Main theorem.** In this section, we recover the crepant resolutions $Y$ and $Y^+$ of $\overline{B(1)}$ from the NCCR $\Lambda_k$. Again, let $\Gamma$ be the double Beilinson quiver

```
\begin{tikzcd}
0 & 1 & \cdots & N-2 & N-1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
f_N & f_N & \cdots & f_N & f_N \\
\end{tikzcd}
```
with relations
\[ v_iv_j = v_jv_i \text{ for all } 1 \leq i, j \leq N, \]
\[ f_if_j = f_jf_i \text{ for all } 1 \leq i, j \leq N, \]
\[ v_jf_i = f_iv_j = x_{ij} \text{ for all } 1 \leq i, j \leq N, \]
and \( \sum_{i=1}^{N} f_iv_i = 0 = \sum_{i=1}^{N} v_if_i. \)

For a commutative \( \mathbb{C} \)-algebra \( A \), let \( \overline{\mathcal{R}}(A) \) the set of representations \( W \) of the quiver \( \overline{\Gamma} \) (with the above relations)

\[
\begin{array}{cccc}
W_0 & W_1 & \cdots & W_{N-1} \\
\vdots & \vdots & \ddots & \vdots \\
v_N & v_1 & \cdots & v_N \\
\end{array}
\]

such that, for each \( i \), \( W_i \) is a (constant) rank 1 projective \( A \)-module and \( W \) is generated by \( W_0 = A \).

The goal of this section is to show the following theorem.

**Theorem 4.1** (cf. [VdB04b], Section 6). \( Y \) is the fine moduli space of the functor \( \overline{\mathcal{R}}. \) The universal bundle is \( \mathcal{T}_{N+1} \).

Recall that the NCCR \( \Lambda_k \) is isomorphic to the path algebra \( S\overline{\Gamma}/J \) where \( J \) is the ideal generated by the above relations. Therefore, Theorem 4.1 means that we can recover a crepant resolution \( Y \) of \( \overline{B(\overline{\Gamma})} \) (and a tilting bundle on \( Y \)) from the NCCR \( \Lambda_k \) as a moduli space of \( \Lambda_k \)-modules (and its universal bundle). The other crepant resolution \( Y^+ \) is also recovered as the fine moduli space of another functor \( \overline{\mathcal{R}}^+ \) (see Remark 4.10).

4.2. **Projective module of rank 1.** Let \( A \) be a (commutative, noetherian) \( \mathbb{C} \)-algebra. In this subsection, we recall some basic properties of projective \( A \)-modules of (constant) rank 1. First, we recall the following fundamental result for projective modules. One can find the following proposition in Chapter II, §5, 2, Theorem 1 of [Bourbaki].

**Proposition 4.2.** Let \( M \) be a finitely generated \( A \)-module. Then, the following are equivalent.

(i) \( M \) is projective.

(ii) For all \( p \in \text{Spec} A \), there exists a non-negative integer \( r(p) \in \mathbb{Z}_{\geq 0} \) such that \( M_p \cong A_p^{r(p)} \).

(iii) There exist \( f_1, \ldots, f_r \in A \) such that they generate the unite ideal of \( A \) and \( M_{f_i} \) is a free \( A_{f_i} \)-module for each \( i \).

From this proposition, we have the following.

**Corollary 4.3.** Let \( M \) be a finitely generated \( A \)-module. Then, the following are equivalent.

(1) The sheaf on \( \text{Spec} A \) that associates to \( M \) is an invertible sheaf.
(2) $M$ is a projective $A$-module of constant rank 1.

Thus, if we consider projective modules of constant rank 1, the symmetric product of them coincides with the tensor product.

Lemma 4.4. Let $P$ be a (finitely generated) projective $A$-module of (constant) rank 1. Let $\mathfrak{S}_k$ be a group of permutations of the set $\{1, 2, \ldots, k\}$. Then, for any $m_1, m_2, \ldots, m_k \in P$ and any $\sigma \in \mathfrak{S}_k$, we have

$$m_1 \otimes m_2 \otimes \cdots \otimes m_k = m_{\sigma(1)} \otimes m_{\sigma(2)} \otimes \cdots \otimes m_{\sigma(k)}$$

in $P^\otimes k$. In particular, we have

$$P^\otimes k \simeq \text{Sym}_A^k P$$

as an $A$-module.

Proof. This is the direct consequence of Proposition 4.2 (iii) and the gluing property of sheaves. □

Corollary 4.5. Let $P$ be a (finitely generated) projective $A$-module of (constant) rank 1. For any $u \in P^\vee = \text{Hom}_A(P, A)$ and $m_1 \ldots m_k \in \text{Sym}_A^k P$, we have

$$u(m_1) \cdot m_1 \cdots \widehat{m_i} \cdots m_j \cdots m_k = u(m_j) \cdot m_1 \cdots \widehat{m_i} \cdots m_j \cdots m_k$$

in $\text{Sym}_A^{k-1} P$ for all $1 \leq i < j \leq k$. In particular, the map

$$\text{Sym}_A^k P \to \text{Sym}_A^{k-1} P, \ m_1 \ldots m_k \mapsto u(m_i) \cdot m_1 \cdots \widehat{m_i} \cdots m_k$$

is well-defined and does not depend on the choice of $i$.

Corollary 4.5 will be used in Section 4.4 to construct a representation of $\tilde{\Gamma}$ from a projective module $P$ of constant rank 1.

4.3. An easy case. In order to prove Theorem 4.1, we first study an easier functor $R$. For commutative $\mathbb{C}$-algebra $A$, let $\mathcal{R}(A)$ be the set of representation $W$ of Beilinson quiver $\Gamma$

$$W_0 : \cdots : W_1 : \cdots : W_{N-2} : W_{N-1} :$$

$$-f_N \leftrightarrow -f_1 \leftrightarrow -f_2 \leftrightarrow -f_3 \leftrightarrow \cdots$$

with usual relations

$$f_if_j = f_jf_i \ (i, j = 1, \ldots, N)$$

such that each $W_i$ is rank 1 projective $A$-module and $W$ is generated by $W_0 = A$.

Let us consider a rank 1 projective $A$-module $P$ and split injective morphism $\alpha : P \to V \otimes_A C$. For the pair $(P, \alpha)$, we define a representation $W_\alpha$ of $\Gamma$ as follows. Let $(W_\alpha)_k := \text{Sym}_A^k P^\vee$ where $P^\vee := \text{Hom}_A(P, A)$ is the dual of $P$. The action of $f \in V$ is defined by

$$f : \text{Sym}_A^k P^\vee \to \text{Sym}_A^{k+1} P^\vee, \ u^1 \cdots u^k \mapsto \alpha^\vee(f)u^1 \cdots u^k.$$

By construction, we have $W_\alpha \in \mathcal{R}(A)$.

Proposition 4.6. For any $W \in \mathcal{R}(A)$, there exists a unique pair $(P, \alpha)$ as above such that $W \simeq W_\alpha$. 

Proof. Let $W \in \mathcal{R}(A)$. Since $W$ is generated by the first component $W_0 = A$, we have a surjective morphism

$$\pi : V^* \otimes_{\mathbb{C}} A \to W_1.$$ 

Since $W_1$ is a projective $A$-module, the morphism $\pi$ is split surjection. If $W = W_\alpha$ for some $(P, \alpha)$, then we have $P = W_1^\vee = \text{Hom}_A(W_1, A)$ and $\alpha = \pi^\vee = \text{Hom}_A(\alpha, -)$. This shows the uniqueness of $(P, \alpha)$.

For arbitrary $W$, since $W$ is generated by $W_0$, $W$ is a quotient of a $A$-module

$$\bigoplus_{i=0}^{N-1} \text{Sym}_A^i(V^* \otimes_{\mathbb{C}} A/\text{Ker} \pi) \simeq \bigoplus_{i=0}^{N-1} \text{Sym}_A^i P^\vee.$$ 

However, $W$ and $\bigoplus_{i=0}^{N-1} \text{Sym}_A^i P^\vee$ have the same $A$-rank $N - 1$, we have

$$W \simeq \bigoplus_{i=0}^{N-1} \text{Sym}_A^i P^\vee.$$ 

This shows the lemma. □

Thus, we have the next result.

**Corollary 4.7.** The functor $\mathcal{R}$ is represented by the projective space $\mathbb{P}(V)$ and the universal sheaf is $\bigoplus_{a=0}^{N-1} \mathcal{O}(a)$. 

In the next subsection, we prove Theorem 4.1 by using Proposition 4.6.

### 4.4. Proof of Theorem 4.1

Let us consider a projective $A$-module $P$ of rank 1 and a pair of morphisms $(\alpha, \beta)$, where

$$\alpha : P \hookrightarrow V \otimes_{\mathbb{C}} A$$

$$\beta : P^\vee \to V^* \otimes_{\mathbb{C}} A$$

that satisfies $\beta^\vee \circ \alpha = 0$ (equivalently, $\alpha^\vee \circ \beta = 0$) and $\alpha$ is injective and split. We note that the triple $(P, \alpha, \beta^\vee)$ is a (stable) representation of Nakajima’s quiver of type $A$ over the commutative algebra $A$. Via the basis $v_1, \ldots, v_N$ of $V$, we set the matrix

$$(a_{ij}) := \alpha \circ \beta^\vee : V \otimes_{\mathbb{C}} A \to V \otimes_{\mathbb{C}} A.$$ 

For a triple $(P, \alpha, \beta)$ as above, we define a representation $W_{\alpha\beta}$ as follows. We set $(W_{\alpha\beta})_a := \text{Sym}_A^a P^\vee$. The action of $f \in V^*$ is given by

$$f : \text{Sym}_A^a P^\vee \to \text{Sym}_A^{a+1} P^\vee, \ u^1 \cdots u^a \mapsto \alpha^\vee(f)u^1 \cdots u^a.$$ 

The action of $v \in V$ is given by

$$v : \text{Sym}_A^a P^\vee \to \text{Sym}_A^{a-1} P^\vee, \ u^1 \cdots u^a \mapsto u^j(\beta^\vee(v)) \cdot u^1 \cdots \hat{u}^j \cdots u^a.$$ 

This map is well-defined and does not depend on the choice of $j$ by Corollary 4.5.

First, we need to check the following

**Lemma 4.8.** For a triple $(P, \alpha, \beta)$ as above, we have $W_{\alpha\beta} \in \tilde{\mathcal{R}}(A)$.

**Proof.** We need to check the following.

(1) $v_i v_j = v_j v_i$ and $f_i f_j = f_j f_i$.
(2) $v_i f_i = f_i v_i = a_{ij}$.
(3) $\sum_{i=1}^N f_i v_i = 0 = \sum_{i=1}^N v_i f_i$.
(4) $W_{\alpha\beta}$ is generated by $(W_{\alpha\beta})_0$. 

This shows the lemma.
(1) and (4) trivially follows from the construction of $W_{\alpha \beta}$. We need to check (2) and (3). First, we check (2). The action on $f_i v_j$ on $(W_{\alpha \beta})_k = \text{Sym}_A^k P^\vee$ is given by

$$f_i v_j (u^1 \cdots u^k) = u^1 (\beta^\vee(v_j)) \cdot \alpha^\vee(f_i) u^1 \cdots \hat{u}^i \cdots u^k,$$

for some $l$. On the other hand, $v_j f_i$ acts on $(W_{\alpha \beta})_k$ by

$$v_j f_i (u^1 \cdots u^k) = v_j (\alpha^\vee(f_i) u^1 \cdots u^k) = u^1 (\beta^\vee(v_j)) \cdot \alpha^\vee(f_i) u^1 \cdots \hat{u}^i \cdots u^k = f_i v_j (u^1 \cdots u^k).$$

We note that we also have

$$v_j f_i (u^1 \cdots u^k) = \alpha^\vee(f_i) (\beta^\vee(v_j)) \cdot u^1 \cdots u^k$$

and $\alpha^\vee(f_i) (\beta^\vee(v_j)) = f_i ((\alpha \circ \beta^\vee)(v_j)) = a_{ij} \in A$. Hence we have

$$(v_j f_i) (u^1 \cdots u^k) = (f_i v_j) (u^1 \cdots u^k) = a_{ij} \cdot u^1 \cdots u^k.$$

This shows (2). Next, we check (3). From the above computation, we have

$$\left( \sum_{i=1}^N f_i v_i \right) (u^1 \cdots u^k) = \left( \sum_{i=1}^N u^1 (\beta^\vee(v_i)) \cdot \alpha^\vee(f_i) \right) \cdot u^1 \cdots \hat{u}^i \cdots u^k.$$

Thus, we have to show that

$$\sum_{i=1}^N u^1 (\beta^\vee(v_i)) \cdot \alpha^\vee(f_i) = 0.$$

Let us consider the composition

$$P^\vee \xrightarrow{\beta} V^* \otimes_C A \xrightarrow{\alpha^\vee} P^\vee.$$

Note that $\beta(u^i) = \sum_{i=1}^N (\beta(u^i)) (v_i) \cdot f_i = \sum_{i=1}^N u^1 (\beta^\vee(v_i)) \cdot f_i$. Hence we have

$$\sum_{i=1}^N u^1 (\beta^\vee(v_i)) \cdot \alpha^\vee(f_i) = (\alpha^\vee \circ \beta)(u^i) = 0.$$

The same argument shows that we have

$$\sum_{i=1}^N v_i f_i = 0.$$

This shows (3). \qed

Next, we show the next proposition.

**Proposition 4.9.** For any $W \in \hat{R}(A)$, there exists a unique $(P, \alpha, \beta)$ as above such that $W \simeq W_{\alpha \beta}$.

**Proof.** By forgetting the action of $V$, we can regard $W$ as an object in $R$. Thus, by Proposition 4.6, there exist a projective $A$-module $P$ and a split injective morphism $\alpha : P \to V^* \otimes_C A$ such that $W \simeq W_\alpha$. We want to construct the morphism $\beta : P^\vee \to V^* \otimes_C A$.

The action of $v_i \in V$ on $W_1 = P^\vee$

$$v_i : P^\vee \to W_0 = A$$
is an element in $\text{Hom}_A(P^\vee, A) \simeq P$. Let $p_i \in P$ an element in $P$ that corresponds to $v_i \in V$ via the above isomorphism. By using this, we set a morphism

$$\gamma : V \otimes \mathbb{C} A \rightarrow P$$

by

$$v_i \otimes 1 \mapsto p_i,$$

and we set $\beta := \gamma^\vee$. In order to complete the proof, we need to check the next two properties.

1. $\alpha^\vee \circ \beta = 0$,
2. The given action of $V$ on $W$ coincides with the one that is determined by $\beta$.

First, we check (1). For $u \in P^\vee$, we have

$$\beta(u) = \sum_{i=1}^{N} (\beta(u))(v_i) \cdot f_i.$$

Therefore, we have

$$(\alpha^\vee \circ \beta)(u) = \sum_{i=1}^{N} (\beta(u))(v_i) \cdot \alpha^\vee(f_i) = \sum_{i=1}^{N} f_i((\beta(u))(v_i)) = \sum_{i=1}^{N} (f_i v_i)(u) = 0.$$

The last equality follows from the relation $\sum_{i=1}^{N} f_i v_i = 0$. This shows (1). Next, we check (2). We show that the action

$$v_i : W_k \rightarrow W_{k-1}$$

coincides with the desired one by induction on $k$. For $k = 1$, this is true by the construction of $\beta$. Let us assume $k > 1$. By definition, we have

$$(v_i f_j)(u_1 \cdots u_k) = v_j(\alpha^\vee(f_j)u_1 \cdots u_k).$$

On the other hand, by the relation and the induction hypothesis, we have

$$(v_i f_j)(u_1 \cdots u_k) = a_{ji} \cdot u_1 \cdots u_k = \alpha^\vee(f_j)(\beta^\vee(v_i)) \cdot u_1 \cdots u_k$$

Since $\alpha^\vee : V^* \otimes \mathbb{C} A \rightarrow P^\vee$ is surjective, we can replace $\alpha^\vee(f_j)$ in the above equation by arbitrary $u \in P^\vee$, and hence we have

$$v_j(wu_1 \cdots u_k) = u(\beta^\vee(v_i)) \cdot u_1 \cdots u_k.$$ 

This shows (2) and the proof is completed.

The triple $(P, \alpha, \beta^\vee)$ gives a representation of the Nakajima’s quiver $\overline{Q^\beta}$ over $A$ of dimension vector $(1, N)$, where $Q$ is the $A_1$ quiver (i.e. a point). As it was explained above, the variety $Y$ is given by

$$Y = \{(L, X) \in \mathbb{P}(V) \times \text{End}_\mathbb{C}(V) \mid X(V) \subset L, X(L) = 0\}.$$ 

This is a description of $Y$ as the Nakajima’s quiver variety of type $A_1$ with dimension vector $(1, N)$. From this presentation of $Y$, we find that $Y$ represents the functor $\overline{R}$. Moreover, since Nakajima’s quiver varieties admit a natural symplectic structure,
we can say that a symplectic structure of $Y$ can be recovered from the NCCR as well.

For the details of Nakajima’s quiver variety and the notation that we used above, see [Gi09].

**Remark 4.10.** Let $\tilde{R}^+(A)$ be a set consists of the representations of $\tilde{\Gamma}$ with the relations in Theorem 3.8 of dimension vector $(1, 1, \ldots, 1)$ and generated by the last component $W_{N-1}$. Then, the dual argument shows that the functor $\tilde{R}^+$ represented by the variety $Y^+$.

### 4.5. Simple representations

In the rest of this section, we determine simple representations that are contained in $\tilde{R}(\mathbb{C})$.

**Lemma 4.11.** A representation $W = (W_k)_k \in \tilde{R}(\mathbb{C})$ is simple if and only if it is generated by the last component $W_{N-1}$.

*Proof.* If $W$ is not generated by $W_{N-1}$, the subrepresentation $W'$ that is generated by $W_{N-1}$ defines a non-trivial subrepresentation of $W$, and hence $W$ is not simple.

On the other hand, let $W' = (W'_k)_k$ be a non-zero subrepresentation of $W$. Then, the last part of subrepresentation $W'_{N-1}$ coincides with the one $W_{N-1}$ of $W$. Indeed, since $W'$ is non-zero, there exists $k$ such that $W'_k = \text{Sym}^k P$, where $P$ is a one-dimensional vector space over $\mathbb{C}$. As the map $\alpha^*: V^* \to P$ is surjective, there exists $f \in V$ such that the image of the map

$$\alpha^*(f)^{N-k-1}: \text{Sym}^k P \to \text{Sym}^{N-1} P$$

is non-zero. Therefore, we have $W'_{N-1} \neq 0$ and hence we have $W'_{N-1} = W_{N-1}$. Thus, if $W$ is generated by the last component $W_{N-1}$, the subrepresentation $W'$ should be $W$ itself. \hfill \square

**Corollary 4.12.** A representation $W = (W_k)_k \in \tilde{R}(\mathbb{C})$ is simple if and only if the map $\beta: P^\vee \to V$ is injective.

*Proof.* Let $W$ be a simple representation. Then, by Lemma 4.11, $W$ is generated by the last part $W_{N-1}$. Thus, for at least one $i$, the map $v_i: V_i = P^\vee \to W_0 = \mathbb{C}$ is non-zero. Therefore, if we set an element $p_i \in P$ that corresponds $v_i$ via the identification $P \simeq \text{Hom}_{\mathbb{C}}(P^\vee, \mathbb{C})$, the map $\gamma: V \to P$, $v_i \mapsto p_i$ is non-zero and hence surjective. Recall that the morphism $\beta: P \to V$ is defined as the dual map of $\gamma$. Thus, we have that the map $\beta$ is injective.

On the other hand, if $\beta$ is injective, we have that the representation $W$ is generated by $W_{N-1}$ from the construction. \hfill \square

Let $W \in \tilde{R}(\mathbb{C})$ and $(P, \alpha, \beta)$ a triple that defines $W$. Then, $\alpha(P) \subset V$ defines a line in $V$ and the composition $\alpha \circ \beta^\vee$ defines an element in $\text{End}_C(V)$. Moreover, a pair $(\alpha(P), \alpha \circ \beta^\vee) \in \mathbb{P}(V) \times \text{End}_C(V)$ defines a point of $Y$ that corresponds to $W$ via the identification $\tilde{R}(\mathbb{C}) \simeq Y(\mathbb{C})$.

If $\beta$ is not injective, $\beta$ must be zero and hence the corresponding point of $Y$ belongs to the zero section

$$j(\mathbb{P}(V)) = \{(L, 0) \in \mathbb{P}(V) \times \text{End}_C(V)\}.$$

Conversely, if the point $(\alpha(P), \alpha \circ \beta^\vee) \in Y$ lies on the zero section, the map $\beta$ must be zero and hence not injective.

By summarizing the above discussion, we have the following theorem.
Theorem 4.13. Let $W$ be a representation that belongs to the set $\tilde{\mathcal{K}}(\mathbb{C})$. Then, the following are equivalent.

1. $W$ is simple.
2. $W$ is generated by the last component $W_{N-1}$.
3. $W$ is corresponds to a point of $Y$ that lies over the non-singular part of $B(1)$ via the identification $\tilde{\mathcal{K}}(\mathbb{C}) \simeq Y(\mathbb{C})$.

Of course, the corresponding argument holds for $Y^+$ and $\tilde{\mathcal{K}}^+$.

5. Kawamata-Namikawa’s equivalence for Mukai flops and P-twists

In this section, we always assume $N \geq 3$.

5.1. Kawamata-Namikawa’s equivalence. Recall that the map $\phi : Y \to B(1)$ contracts the zero section $j : \mathbb{P}(V) \to Y$ to $0 \in B(1)$. This is a flopping contraction and the flop is $Y^+ = |\Omega_{\mathbb{P}(V^*)}| \xrightarrow{\sigma} B(1)$, where $\mathbb{P}(V^*)$ is the dual projective space of $\mathbb{P}(V)$. In the following, we write $\mathbb{P} := \mathbb{P}(V)$ and $\mathbb{P}^\vee := \mathbb{P}(V^*)$ for short.

\[
\begin{array}{c}
\mathbb{P} \\
\downarrow^j \downarrow^\phi \\
Y \\
\downarrow^\phi \\
B(1) \\
\downarrow^{\phi^+} \\
Y^+ \\
\downarrow^{\phi^+} \\
\mathbb{P}^\vee
\end{array}
\]

As in the above sections, let $\pi : Y \to \mathbb{P}$ and $\pi' : Y^+ \to \mathbb{P}^\vee$ be the projections, and we set $\mathcal{O}_Y(1) := \pi^*\mathcal{O}_\mathbb{P}(1)$ and $\mathcal{O}_{Y^+}(1) := (\pi')^*\mathcal{O}_{\mathbb{P}^\vee}(1)$. Then, the vector bundles

\[
\mathcal{T}_k := \bigoplus_{a=-N+k+1}^k \mathcal{O}_Y(a),
\]

\[
\mathcal{T}^+_k := \bigoplus_{a=-N+k+1}^k \mathcal{O}_{Y^+}(a)
\]
on $Y$, $Y^+$, respectively, are tilting bundles. Moreover, we have an $R$-algebra isomorphism

\[
\Lambda_k := \text{End}_Y(\mathcal{T}_k) \simeq \text{End}_{Y^+}(\mathcal{T}^+_k),
\]

by Theorem 3.5 (4).

By using the above tilting bundles, we have equivalences of categories

\[
\Psi_k := \text{RHom}_Y(\mathcal{T}_k, -) : \text{D}^b(Y) \xrightarrow{\sim} \text{D}^b(\text{mod}(\Lambda_k)),
\]

\[
(\Psi^\dagger_{N-k-1})^{-1} := - \otimes^{L}_{\Lambda_k} \mathcal{T}^+_{N-k-1} : \text{D}^b(\text{mod}(\Lambda_k)) \xrightarrow{\sim} \text{D}^b(Y^+),
\]

and by compositing these equivalences, we have an equivalence

\[
n\text{KN}_k := \text{RHom}_Y(\mathcal{T}_k, -) \otimes^{L}_{\Lambda_k} \mathcal{T}^+_{N-k-1} : \text{D}^b(Y) \xrightarrow{\sim} \text{D}^b(Y^+).
\]

By construction, the inverse of the equivalence $n\text{KN}_k$ is given by

\[
(n\text{KN}_k)^{-1} \simeq n\text{KN}^\dagger_{N-k-1} := \text{RHom}_{Y^+}(\mathcal{T}^+_{N-k-1}, -) \otimes_{\Lambda_k}^{L} \mathcal{T}^k.
\]

On the other hand, the equivalence between $\text{D}^b(Y)$ and $\text{D}^b(Y^+)$ is first given by Kawamata and Namikawa in terms of the Fourier-Makai transform. We recall their construction of Fourier-Makai type equivalences. Let $\tilde{Y}$ be a blowing-up of $Y$ at the zero section $\mathbb{P}$. Then, $\tilde{Y}$ is also a blowing-up of $Y^+$ at $\mathbb{P}^\vee$. Since the normal
bundle of \( j : \mathbb{P} \hookrightarrow Y \) is isomorphic to \( \Omega_E^1 = \mathcal{O}_\mathbb{P}(\Omega_\mathbb{P}^1) \subset \tilde{Y} \) can be embedded in the fiber product \( \mathbb{P} \times \mathbb{P}^V \) by the Euler sequence. We set \( \hat{Y} := \tilde{Y} \cup_E (\mathbb{P} \times \mathbb{P}^V) \), and let \( \hat{q} : \hat{Y} \to Y \) and \( \hat{p} : \hat{Y} \to Y^+ \) be projections.

Let \( \mathcal{L}_k \) be a line bundle on \( \hat{Y} \) such that \( \mathcal{L}_k|_{\tilde{Y}} = \mathcal{O}_{\tilde{Y}}(kE) \) and \( \mathcal{L}_k|_{\mathbb{P} \times \mathbb{P}^V} = \mathcal{O}(-k, -k) \).

The Kawamata-Namikawa’s functors are given by

\[
\begin{align*}
\text{KN}_k & \colon R\hat{p}^\ast(L\hat{q}_\ast(-) \otimes \mathcal{L}_k) : \text{D}b(Y) \to \text{D}b(Y^+), \\
\text{KN}'_k & \colon R\hat{q}^\ast(L\hat{p}_\ast(-) \otimes \mathcal{L}_k) : \text{D}b(Y^+) \to \text{D}b(Y).
\end{align*}
\]

The following result is due to Kawamata and Namikawa.

**Theorem 5.1** ([Kaw02, Na03]). The functors \( \text{KN}_k \) and \( \text{KN}'_k \) are equivalences.

**Remark 5.2.** By the definition of the functor \( \text{KN}_k \), the following diagram commutes

\[
\begin{array}{ccc}
\text{D}b(Y) & \xrightarrow{\text{KN}_k} & \text{D}b(Y^+) \\
\downarrow{\sim \mathcal{O}_Y(1)} & & \downarrow{\sim \mathcal{O}_{Y^+}(-1)} \\
\text{D}b(Y) & \xrightarrow{\text{KN}'_{k+1}} & \text{D}b(Y^+).
\end{array}
\]

The same holds for our equivalence \( \text{nKN}_k \):

\[
\begin{array}{ccc}
\text{D}b(Y) & \xrightarrow{\text{nKN}_k} & \text{D}b(Y^+) \\
\downarrow{\sim \mathcal{O}_Y(1)} & & \downarrow{\sim \mathcal{O}_{Y^+}(-1)} \\
\text{D}b(Y) & \xrightarrow{\text{nKN}_{k+1}} & \text{D}b(Y^+).
\end{array}
\]

**Theorem 5.3.** Our functor \( \text{nKN}_k \) (resp. \( \text{nKN}'_k \)) coincides with the Kawamata-Namikawa’s functor \( \text{KN}_k \) (resp. \( \text{KN}'_k \)).

Note that in the proof of Theorem 5.3, we do not use the fact that the functors \( \text{KN}_k \) and \( \text{KN}'_k \) are equivalences. Thus, our proof of Theorem 5.3 gives an alternative proof for Theorem 5.1 in this local model of the Mukai flop.

**Proof.** It is easy to see that \( \text{KN}'_{N-k-1} \) is the left and right adjoint of \( \text{KN}_k \). Thus, it is enough to show that the following diagram commutes.

\[
\begin{array}{ccc}
\text{D}b(Y) & \xleftarrow{\text{KN}'_{N-k-1}} & \text{D}b(Y^+) \\
\downarrow{\Psi_k} & & \downarrow{\Psi'_{N-k-1}} \\
\text{D}b(\Lambda_k) & \xleftarrow{\text{KNN}_{N-k-1}} & \text{D}b(\Lambda_k)
\end{array}
\]

We note that the composition \( \Psi_k \circ \text{KNN}_{N-k-1} \) is given by

\[
\text{RHom}_{Y^+}(\text{KN}_k(T_k), -) : \text{D}b(Y^+) \to \text{D}b(\Lambda_k).
\]

Now, Theorem 5.3 follows from Lemma 5.4. \( \square \)
Lemma 5.4. Let \( k \in \mathbb{Z} \) a fixed integer. Then we have

\[
\text{KN}_k(\mathcal{O}_Y(a)) \simeq \mathcal{O}_{Y^+}(-a)
\]

for all \(-N + k + 1 \leq a \leq k\) and hence we have an isomorphism

\[
\text{KN}_k(\mathcal{T}_k) \simeq \mathcal{T}^+_N - k - 1.
\]

Proof. By Remark 5.2, it is enough to show the isomorphism of functors for \( k = 0 \). Recall that the correspondence \( \tilde{Y} \) is given by \( \tilde{Y} = Y \cup E \times \mathbb{P} \). Hence, we have an exact sequence on \( Y \times Y^+ \)

\[
0 \to \mathcal{O}_Y \to \mathcal{O}_Y \oplus \mathcal{O}_{E \times \mathbb{P}^\vee} \to \mathcal{O}_E \to 0.
\]

We use this sequence to compute the Fourier-Mukai functor \( \text{KN}_0 := \Phi_{\mathcal{O}_Y} \). First, we have

\[
\Phi_{\mathcal{O}_{E \times \mathbb{P}^\vee}}(\mathcal{O}_Y(a)) = R\Gamma(\mathbb{P}, \mathcal{O}_P(a)) \otimes j'_\ast \mathcal{O}_{\mathbb{P}^\vee}
\]

\[
= \begin{cases} 
0 & \text{if } a = 0 \\
 j'_\ast \mathcal{O}_{\mathbb{P}^\vee}(-1)[-N + 1] & \text{if } a = -N + 1,
\end{cases}
\]

The exceptional divisor \( E \subset \tilde{Y} \) is a universal hyperplane section over \( P \) and hence a divisor on \( P \times P^\vee \) of bi-degree \((1, 1)\). Thus, we have an exact sequence

\[
0 \to \mathcal{O}_{P \times \mathbb{P}^\vee}(-1, -1) \to \mathcal{O}_{E \times \mathbb{P}^\vee} \to \mathcal{O}_E \to 0.
\]

From the same computation as above, we have

\[
\Phi_{\mathcal{O}_{P \times \mathbb{P}^\vee}(-1, -1)}(\mathcal{O}_Y(a)) = R\Gamma(\mathbb{P}, \mathcal{O}_P(a - 1)) \otimes j'_\ast \mathcal{O}_{\mathbb{P}^\vee}(-1)
\]

\[
= \begin{cases} 
0 & \text{if } -N + 1 < a \leq 0 \\
 j'_\ast \mathcal{O}_{\mathbb{P}^\vee}(-1)[-N + 1] & \text{if } a = -N + 1,
\end{cases}
\]

and hence we have

\[
\Phi_{\mathcal{O}_E}(\mathcal{O}_Y(a)) = \begin{cases} 
j'_\ast \mathcal{O}_{\mathbb{P}^\vee} & \text{if } a = 0 \\
0 & \text{if } -N + 1 < a < 0 \\
j'_\ast \mathcal{O}_{\mathbb{P}^\vee}(-1)[-N + 2] & \text{if } a = -N + 1.
\end{cases}
\]

Furthermore, since we have

\[
\mathcal{O}_{\tilde{Y}}(E) \simeq \tilde{q}^\ast \mathcal{O}_{Y}(-1) \otimes \tilde{p}^\ast \mathcal{O}_{Y^+}(-1),
\]

and

\[
R\tilde{p}_\ast \mathcal{O}_E(kE) = \begin{cases} 
0 & \text{for all } k = 1, \ldots, N - 2, \\
j'_\ast \mathcal{O}_{\mathbb{P}^\vee}(-N)[-N + 2] & \text{for } k = N - 1,
\end{cases}
\]

we have

\[
\Phi_{\mathcal{O}_Y}(\mathcal{O}_Y(a)) = R\tilde{p}_\ast(\mathcal{O}_Y(-aE)) \otimes \mathcal{O}_{Y^+}(-a)
\]

\[
= \mathcal{O}_{Y^+}(-a)
\]

for \(-N + 1 < a \leq 0\), and \( \Phi_{\mathcal{O}_Y}(\mathcal{O}_Y(-N + 1)) \) lies on the exact triangle

\[
\mathcal{O}_{Y^+}(-N - 1) \to \Phi_{\mathcal{O}_E}(\mathcal{O}_Y(-N + 1)) \to \mathcal{O}_{P^\vee}(-1)[-N + 2].
\]

From the above, we can compute \( \text{KN}_0(\mathcal{O}_Y(a)) \) for \(-N + 1 \leq a \leq 0\). If \( a = 0 \), \( \text{KN}_0(\mathcal{O}_Y) \) lies on the exact triangle

\[
\text{KN}_0(\mathcal{O}_Y) \to \mathcal{O}_{Y^+} \oplus j'_\ast \mathcal{O}_{\mathbb{P}^\vee} \to j'_\ast \mathcal{O}_{\mathbb{P}^\vee},
\]
and hence we have

\[ KN_0(\mathcal{O}_Y) \cong \mathcal{O}_Y. \]

If \(-N + 1 < a < 0\), we have

\[ KN_0(\mathcal{O}_Y(a)) \cong \mathcal{O}_Y(-a). \]

Finally, if \(a = -N + 1\), \(KN_0(\mathcal{O}_Y(-N + 1))\) lies on the exact triangle

\[ KN_0(\mathcal{O}_Y(-N + 1)) \to \Phi_{\mathcal{O}_Y}^\vee(\mathcal{O}_Y(-N + 1)) \to j'_*\mathcal{O}_{\mathcal{Y}}^\vee(-1)[-N + 2]. \]

This triangle coincides with the above one that gives the object \(\Phi_{\mathcal{O}_Y}^\vee(\mathcal{O}_Y(-N + 1))\) and hence we have

\[ KN_0(\mathcal{O}_Y(-N + 1)) \cong \mathcal{O}_Y(N - 1). \]

Thus, we have the isomorphism \(KN_0(T_0) \cong T_0^+\) that we want.

5.2. \textbf{P-twists and Mutations.} In this section, we introduce equivalences \(\nu_{N+k}^+\) and \(\nu_{N+k-1}^-\) between the derived categories of non-commutative algebras \(\Lambda_{N+k}\) and \(\Lambda_{N+k-1}\). We show that a composition of multi-mutation functors \(\nu_{N+k-1}^- \circ \nu_{N+k}^+\) corresponds to an autoequivalence \(P_k\) of \(D^b(Y)\) that is a \(P\)-twist defined by a \(\mathbb{P}^{N-1}\)-object \(j_*\mathcal{O}_P(k)\).

5.2.1. \textit{Definition of multi-mutation.} First, we define a multi-mutation functor \(\nu_{N+1}^-: D^b(\text{mod}(\Lambda_{N-1})) \to D^b(\text{mod}(\Lambda_{N-2}))\). Recall that the algebras \(\Lambda_{N-1}\) is given by

\[ \Lambda_{N-1} = \text{End}_R \left( \bigoplus_{a=0}^{N-1} M_a \right). \]

Let us consider the canonical surjective morphism \(R^{\oplus N} \twoheadrightarrow M_{-1}\). Note that this morphism is given by the push-forward of the canonical surjection \(V \otimes_C \mathcal{O}_Y^+ \twoheadrightarrow \mathcal{O}_Y^+(1)\) by \(\phi^+\). Then, we define a \(\Lambda_{N-1}\)-module \(C\) as

\[ C := \text{Image} \left( \text{Hom}_R \left( \bigoplus_{a=0}^{N-1} M_a, R^{\oplus N} \right) \to \text{Hom}_R \left( \bigoplus_{a=0}^{N-1} M_a, M_{-1} \right) \right), \]

and set a \(\Lambda_{N-1}\)-module \(S\) as

\[ S := \text{Hom}_{\Lambda_{N-1}} \left( \bigoplus_{a=0}^{N-1} M_a, \bigoplus_{a=0}^{N-1} M_a \right) \oplus C. \]

**Lemma 5.5.** The following hold.

(i) There exists an isomorphism of \(\Lambda_{N-1}\)-modules

\[ S \cong \text{RHom}_{\mathcal{Y}^+} (T_0^+, T_1^+). \]

(ii) The \(\Lambda_{N-1}\)-module \(S\) defined above is a tilting generator of the category \(D^b(\text{mod}(\Lambda_{N-1}))\).

(iii) We have an isomorphism between \(R\)-algebras

\[ \text{End}_{\Lambda_{N-1}}(S) \cong \Lambda_{N-2}. \]

**Proof.** The (ii) and (iii) follow from (i). First, we have

\[ \text{RHom}_{\mathcal{Y}^+}(T_0^+, T_1^+) = \text{RHom}_{\mathcal{Y}^+}(T_0^+, \bigoplus_{a=-N+2} \mathcal{O}_Y^+ (a)) \oplus \text{RHom}_{\mathcal{Y}^+}(T_0^+, \mathcal{O}_Y^+ (1)). \]
As explained above, we have
\[ M_{-a} = \phi_a^+ \mathcal{O}_{Y^+}(a) \]
for all \(-N + 1 \leq a \leq N - 1\), and we have
\[
\operatorname{RHom}_Y^+(T_0^+, \bigoplus_{a=-N+2}^{\infty} \mathcal{O}_Y(a)) = \operatorname{Hom}_Y^+(T_0^+, \bigoplus_{a=-N+2}^{\infty} \mathcal{O}_Y(a))
\]
\[
= \operatorname{Hom}_R(\bigoplus_{a=0}^{N-1} M_a, \bigoplus_{a=0}^{N-2} M_a).
\]

Next, since the sheaf \( \mathcal{H}om_Y^+(T_0^+, \mathcal{O}_{Y^+}(1)) \) on \( Y^+ \) is a vector bundle and hence is torsion free, the \( R \)-module \( \mathcal{H}om_Y^+(T_0^+, \mathcal{O}_{Y^+}(1)) = \phi_a^+ \mathcal{H}om_Y^+(T_0^+, \mathcal{O}_{Y^+}(1)) \) is also torsion free. Since two \( R \)-modules \( \operatorname{Hom}_R(\bigoplus_{a=0}^{N-1} M_a, M_{-1}) \) and \( \operatorname{Hom}_Y^+(T_0^+, \mathcal{O}_{Y^+}(1)) \) are isomorphic in codimension one, the natural map
\[
\operatorname{Hom}_Y^+(T_0^+, \mathcal{O}_{Y^+}(1)) \rightarrow \operatorname{Hom}_R(\bigoplus_{a=0}^{N-1} M_a, M_{-1})
\]
is injective. Let us consider the surjective morphism \( V \otimes_\mathbb{C} \mathcal{O}_{Y^+} \rightarrow \mathcal{O}_{Y^+}(1) \). We note that the map
\[
\operatorname{Hom}_Y^+(T_0^+, V \otimes_\mathbb{C} \mathcal{O}_{Y^+}) \rightarrow \operatorname{Hom}_Y^+(T_0^+, \mathcal{O}_{Y^+}(1))
\]
is surjective because we have a vanishing of an extension
\[
\operatorname{Ext}^1_{Y^+}(T_0^+, \pi^a \Omega_{\mathbb{P}^v}(1)) = H^1(Y^+, \bigoplus_{a=0}^{N-0} \pi^a \Omega_{\mathbb{P}^v}(a + 1)) = 0
\]
from the same argument as in the proof of Corollary 3.14. Thus, we have the following commutative diagram
\[
\begin{array}{ccc}
\operatorname{Hom}_Y^+(T_0^+, V \otimes_\mathbb{C} \mathcal{O}_{Y^+}) & \xrightarrow{\sim} & \operatorname{Hom}_R(\bigoplus_{a=0}^{N-1} M_a, V \otimes_\mathbb{C} M_0) \\
\downarrow & & \downarrow \\
\operatorname{Hom}_Y^+(T_0^+, \mathcal{O}_{Y^+}(1)) & \xrightarrow{\sim} & \operatorname{Hom}_R(\bigoplus_{a=0}^{N-1} M_a, M_{-1})
\end{array}
\]
and hence we have \( \operatorname{RHom}_Y^+(T_0^+, \mathcal{O}_{Y^+}(1)) = \operatorname{Hom}_Y^+(T_0^+, \mathcal{O}_{Y^+}(1)) = C. \)

From the above lemma, we can define the equivalence:

**Definition 5.6.** We set
\[
\nu_{N-1} := \operatorname{RHom}_{A_{N-1}}(S, -) : \mathbf{D}^b(\mathcal{M}(A_{N-1})) \xrightarrow{\sim} \mathbf{D}^b(\mathcal{M}(A_{N-2})).
\]
We call this functor \( \nu_{N-1} \) the *multi-mutation functor*. By Lemma 5.5, multi-mutation \( \nu_{N-1} \) coincides with the functor
\[
\operatorname{RHom}_{A_{N-1}}(\operatorname{RHom}_Y^+(T_0^+, T_1^+), -) : \mathbf{D}^b(\mathcal{M}(A_{N-1})) \xrightarrow{\sim} \mathbf{D}^b(\mathcal{M}(A_{N-2})),
\]
and hence the following diagram commutes
\[
\begin{array}{ccc}
\mathbf{D}^b(Y^+) & \xrightarrow{\Psi_{N-1}^+} & \mathbf{D}^b(\mathcal{M}(A_k)) \\
\downarrow \Psi_{N-2} & & \downarrow \nu_{N-1} \\
\mathbf{D}^b(\mathcal{M}(A_{N-2}))
\end{array}
\]
We also define a multi-mutation functor $\nu_k^{-}: D^b(\text{mod}(\Lambda_k)) \to D^b(\text{mod}(\Lambda_{k-1}))$ by using the following commutative diagram.

$$
\begin{array}{cccc}
D^b(Y^+) & \xrightarrow{\Psi_{N-k-1}} & D^b(Y^+) & \xrightarrow{\Psi_{N-k-1}} \\
\downarrow{\Psi_n^{-}} & & \downarrow{\Psi_n^{-}} & \\
D^b(\text{mod}(\Lambda_k)) & \xrightarrow{F_{\nu}^{-}} & D^b(\text{mod}(\Lambda_{N-1})) & \xrightarrow{\nu_{N-1}^{-}} \\
\end{array}
$$

where the functor $F_{\nu}^{i}: D^b(\text{mod}(\Lambda_i)) \to D^b(\text{mod}(\Lambda_j))$ is given by the composition

$$
F_{\nu}^{i}: D^b(\text{mod}(\Lambda_i)) \xrightarrow{- \otimes \Lambda_i^{T_{N-1}^{-1}} \otimes Y} D^b(Y^+) \xrightarrow{- \otimes \text{Hom}_Y(T_{N-1}^{-1}, \cdot)} D^b(Y^+) \xrightarrow{\text{RHom}_Y(T_{N-1}^{-1}, \cdot)} D^b(\text{mod}(\Lambda_j)).
$$

5.2.2. Connection between multi-mutations and IW mutations. In the following, we explain the multi-mutation functor $\nu_{N-1}^{-}$ is given by a composition of IW mutations. For definitions and basic properties of IW mutations, see Section 2.4. Let us consider the long Euler sequence on $P^\nu = \mathbb{P}(V^*)$

$$
0 \to \mathcal{O}_{P^\nu}(-N + 1) \to V^* \otimes \mathcal{O}_{P^\nu}(-N + 2) \to \bigwedge^{N-2} V \otimes \mathcal{O}_{P^\nu}(-N + 3) \to \cdots \\
\to \bigwedge^2 V \otimes \mathcal{O}_{P^\nu}(-1) \to V \otimes \mathcal{O}_{P^\nu} \to \mathcal{O}_{P^\nu}(1) \to 0.
$$

By applying the functor $(\phi^+)^* \circ (\pi^j)^*$ to the above sequence, we have a resolution of the module $M_{-1}$ by other modules $M_0, \ldots, M_{N-1}$:

$$
0 \to M_{-1} \to V^* \otimes \mathbb{C} M_{N-2} \to \bigwedge^2 V \otimes \mathbb{C} M_{N-3} \to \cdots \to \bigwedge^2 V \otimes \mathbb{C} M_1 \to V \otimes \mathbb{C} M_0 \to M_{-1} \to 0.
$$

We splice this sequence into short exact sequences

$$
\begin{align*}
0 \to M_{-1} & \to V^* \otimes \mathbb{C} M_{N-2} \to L_{N-2} \to 0 \\
0 \to L_{N-2} & \to \bigwedge^2 V \otimes \mathbb{C} M_{N-3} \to L_{N-3} \to 0 \\
& \vdots \\
0 \to L_k & \to \bigwedge^k V \otimes \mathbb{C} M_{k-1} \to L_{k-1} \to 0 \\
& \vdots \\
0 \to L_1 & \to V \otimes \mathbb{C} M_0 \to M_{-1} \to 0
\end{align*}
$$
and set \( L_{N-1} := M_{N-1}, \ L_0 := M_{-1}, \ W := \bigoplus_{a=0}^{N-2} M_a, \) and \( E_k := W \oplus L_k. \) By dualizing above morphisms, we have a map \( \bigwedge^k V^* \otimes \mathcal{M}_{k-1}^* \rightarrow \mathcal{L}_k^*. \) Since the module \( \mathcal{M}_a \) is reflexive, the above map is surjective. Then, applying the functor \( - \oplus W^* \), we have a surjective map

\[
\left( \bigwedge^k V^* \otimes \mathcal{M}_{k-1}^* \right) \oplus W^* \rightarrow E_k^*.
\]

First, we prove the following

**Lemma 5.7.** The map \( \left( \bigwedge^k V^* \otimes \mathcal{M}_{k-1}^* \right) \oplus W^* \rightarrow E_k^* \) is a right \((\text{add} \ W^*)\)-approximation.

**Proof.** Let us consider the exact sequence

\[
0 \rightarrow \mathcal{L}_{k-1}^* \rightarrow \bigwedge^k V^* \otimes \mathcal{M}_{k-1}^* \rightarrow \mathcal{L}_k^* \rightarrow 0.
\]

We have to show that the map

\[
\text{Hom}_R(W^*, \bigwedge^k V^* \otimes \mathcal{M}_{k-1}^*) \rightarrow \text{Hom}_R(W^*, \mathcal{L}_k^*)
\]

is surjective. First, by definition, we have \( \mathcal{M}_{k-1}^* \simeq \mathcal{M}_{-k+1} \approx (\phi^\dagger)_* \mathcal{O}_{Y^+}(k-1). \) On \( Y^+ \), there is a canonical short exact sequence

\[
0 \rightarrow (\pi^\dagger)^* \bigwedge^{k-1} \mathcal{T}_{\mathcal{P}^\vee} \otimes \mathcal{O}_{Y^+}(-1) \rightarrow \bigwedge^k V^* \otimes \mathcal{O}_{Y^+}(k-1) \rightarrow (\pi^\dagger)^* \bigwedge^k \mathcal{T}_{\mathcal{P}^\vee} \otimes \mathcal{O}_{Y^+}(-1) \rightarrow 0,
\]

where \( T_{\mathcal{P}^\vee} \) is the tangent bundle on \( Y^+ \). Put \( \mathcal{H}_k^*_k := (\pi^\dagger)^* \bigwedge^k \mathcal{T}_{\mathcal{P}^\vee} \otimes \mathcal{O}_{Y^+}(-1) \) and \( \mathcal{H}_k := \mathcal{H}_k^*. \) Since the first non-trivial term of the above exact sequence does not have higher cohomology, we have an isomorphism

\[
\mathcal{L}_k^* \simeq (\phi^\dagger)_* (\mathcal{H}_k^*)
\]

by induction on \( k \). Furthermore, since the third non-trivial term of the above exact sequence and its dual have no higher cohomology, it follows from Lemma 3.1 that the module \( \mathcal{L}_k^* \) is (maximal) Cohen-Macaulay, and hence, the module \( \text{Hom}_R(W^*, \mathcal{L}_k^*) \) is reflexive by Proposition 2.9. In addition, by Proposition 2.8, Lemma 3.1, and Proposition 3.17, the module

\[
\text{Hom}_{Y^+} \left( \bigoplus_{a=0}^{N-2} \mathcal{O}_{Y^+}(a), \mathcal{H}_k^* \right)
\]

is also reflexive. Therefore, we have an isomorphism

\[
\text{Hom}_{Y^+} \left( \bigoplus_{a=0}^{N-2} \mathcal{O}_{Y^+}(a), \mathcal{H}_k^* \right) \simeq \text{Hom}_R(W^*, \mathcal{L}_k^*).
\]

On the other hand, again by Proposition 3.17, we have the vanishing of an extension group

\[
\text{Ext}_{Y^+}^1 \left( \bigoplus_{a=0}^{N-2} \mathcal{O}_{Y^+}(a), \mathcal{H}_{k-1}^* \right) = 0.
\]
This vanishing says that the map
\[
\text{Hom}_Y \left( \bigoplus_{a=0}^{N-2} \mathcal{O}_{Y+}(a) \wedge \bigwedge^k \mathcal{O}_{Y+}(a), \mathcal{O}_{Y+}(k-1) \right) \to \text{Hom}_Y \left( \bigoplus_{a=0}^{N-2} \mathcal{O}_{Y+}(a), \mathcal{H}_k^+ \right)
\]
is surjective. Thus, we have the morphism
\[
\text{Hom}_R \left( W^*, \bigwedge^k V^* \otimes C \mathcal{O}_{Y+}(k-1) \right) \to \text{Hom}_R \left( W^*, L^* \right)
\]
is also surjective. \(\square\)

Since the kernel of the approximation \((\bigwedge^k V^* \otimes C \mathcal{O}_{Y+}(k-1)) \oplus W^* \to E_k^+\) is isomorphic to \(L^*_{k-1}\), the \(R\)-module \(E_{k-1}\) is isomorphic to a (left) IW mutation \(\mu^k_{W}(E_k)\) of \(E_k\) at \(W\). Thus, by Theorem 2.23, we have a derived equivalence
\[
T_W : D^b(\text{mod(End}_R(E_k))) \sim D^b(\text{mod(End}_R(E_{k-1}))).
\]
However, in this case, we can show directly that the functor \(T_W\) actually gives an equivalence of categories. As in the proof of Lemma 5.7, put \(\mathcal{H}_k := (\pi')^* \Omega^k_{E_Y} \otimes \mathcal{O}_{Y+}(1).

Lemma 5.8. (1) We have an isomorphism of \(R\)-algebras \(\text{End}_R(E_k) \simeq \text{End}_{Y^+}(S^+_k)\), where \(S^+_k := \bigoplus_{a=0}^{N-2} \mathcal{O}_{Y+}(a) \oplus \mathcal{H}_k\) is a tilting bundle on \(Y^+\) that is given in Proposition 3.17. (2) We have an isomorphism of functors
\[
T_W \simeq R\text{Hom}_{\text{End}_R(E_k)}(R\text{Hom}_{Y^+}(S^+_k, S^+_{k-1}), -).
\]
(3) In particular, IW mutation functor \(T_W\) gives an equivalence of categories, and the following diagram of functors commutes
\[
\begin{array}{ccc}
D^b(Y^+) & \xrightarrow{S_k} & D^b(\text{mod(End}_R(E_k))) \\
\downarrow S_{k-1} & & \downarrow T_W \\
D^b(\text{mod(End}_R(E_{k-1}))) & & \\
\end{array}
\]
where \(S_k := R\text{Hom}_{Y^+}(S^+_k, -) : D^b(Y^+) \to D^b(\text{mod(End}_R(E_k)))\).

Proof. We can prove this lemma by using almost same arguments as in Lemma 5.5. The different point from Lemma 5.5 is that the vanishing of \(\text{Ext}^i_{Y^+}(S^+_k, S^+_{k-1})\) for \(i > 0\) is non-trivial. However, this vanishing follows from direct computations using Proposition 3.13. \(\square\)

Now we ready to prove the following result that gives a correspondence between multi-mutations and IW mutations.

Theorem 5.9. An equivalence obtained by composing \(N - 1\) IW mutation functors
\[
T_W \circ T_W \circ \cdots \circ T_W : D^b(\text{mod}(\Lambda_{N-1})) \to D^b(\text{mod}(\Lambda_{N-2}))
\]
is isomorphic to a multi-mutation functor \(\nu^-_{N-1}\). Here, we note that \(\text{End}_R(E_{N-1}) = \Lambda_{N-1}\) and \(\text{End}_R(E_0) = \Lambda_{N-2} \).
Proof. By Lemma 5.5 (3), we have a commutative diagram

\[
\begin{array}{cccc}
S_{N-1} & \xrightarrow{S_{N-2}} & S_0 \\
\downarrow & & \\
D^b(Y^+)_\text{mod(End}_R(EN_{-1})) & \xrightarrow{T_W} & D^b(\text{End}_R(EN_{-2})) & \xrightarrow{T_W} \cdots \xrightarrow{T_W} D^b(\text{End}_R(E_0))
\end{array}
\]

Hence, we have \(T_W \circ T_W \circ \cdots \circ T_W \simeq S_0 \circ S_{N-1} = \Psi_1^+ \circ (\Psi_0^+)^{-1} \simeq \nu_{N-1}^-. \quad \Box

Remark 5.10. Applying the same argument, we can prove that a multi-mutation functor

\[\nu_{k}^- : D^b(\text{mod}(\Lambda_k)) \rightarrow D^b(\text{mod}(\Lambda_{k-1}))\]

is written as a composition of IW mutation functors if \(1 \leq k \leq N-1\). In other cases, the above argument cannot be applied because we only know that the module \(\bigoplus_{a=-N+k+1} M_a\) gives an NCCR if \(0 \leq k \leq N - 1\) (see Theorem 3.4).

Next, we discuss the case of multi-mutations \(\nu_{k}^+\).

Theorem 5.11. A multi-mutation functor

\[\nu_{N-2}^+ : D^b(\text{mod}(\Lambda_{N-2})) \rightarrow D^b(\text{mod}(\Lambda_{N-1}))\]

can be written as a composition of \(N-1\) IW mutation functors.

Proof. Let us consider the long Euler sequence on \(P\)

\[
0 \rightarrow \mathcal{O}_P(-1) \rightarrow V \otimes \mathcal{O}_P \rightarrow \bigwedge^{N-2} V^* \otimes \mathcal{O}_P(1) \rightarrow \cdots
\]

\[
\rightarrow \bigwedge^2 V^* \otimes \mathcal{O}_P(N-3) \rightarrow V^* \otimes \mathcal{O}_P(N-2) \rightarrow \mathcal{O}_P(N-1) \rightarrow 0.
\]

Applying a functor \(\phi_* \circ \pi^*\), we have a long exact sequence

\[
0 \rightarrow M_{-1} \rightarrow V \otimes R \rightarrow \bigwedge^{N-2} V^* \otimes M_1 \rightarrow \cdots
\]

\[
\rightarrow \bigwedge^2 V^* \otimes M_{N-3} \rightarrow V^* \otimes M_{N-2} \rightarrow M_{N-1} \rightarrow 0.
\]

Using completely same argument as in the proof of Theorem 5.9, we have an equivalence of categories

\[T_W \circ T_W \circ \cdots \circ T_W : D^b(\text{mod}(\Lambda_{N-2})) \rightarrow D^b(\text{mod}(\Lambda_{N-1}))\]

and this functor is isomorphic to the functor \(\Psi_{N-1} \circ \Psi_{N-2}^{-1} \simeq \nu_{N-2}^+\) under the above identification of algebras. \(\Box\)

Remark 5.12. As in Remark 5.10, we can show that a multi-mutation functor

\[\nu_{k}^+ : D^b(\text{mod}(\Lambda_k)) \rightarrow D^b(\text{mod}(\Lambda_{k+1}))\]

can be described as a composition of IW mutation functors if \(0 \leq k \leq N - 2\).

Remark 5.13. From the proof of theorems, we notice that the object

\[
\mu_W^L(\cdots (\mu_W^L(\bigoplus_{a=0}^N M_a)) \cdots)),
\]

\[\mu_W^L(\cdots (\mu_W^L(\bigoplus_{a=0}^N M_a)) \cdots)),
\]
which obtained from $\bigoplus_{a=0}^{N-1} M_a$ after taking IW mutations at $W$ $(2N - 2)$-times, coincides with the original module $\bigoplus_{a=0}^{N-1} M_a$:

$$\mu_W^2(\cdots(\mu_W^1(\bigoplus_{a=0}^{N-1} M_a))\cdots) = \bigoplus_{a=0}^{N-1} M_a.$$  

If the ring $R$ is complete normal 3-sCY and $M$ is a maximal modifying module, Iyama and Wemyss proved that two times mutation $\mu_N^2 \mu_N^1(M)$ of $M$ at an indecomposable summand $N$ coincides with $M$ [IW14 Summary 6.25]:

$$\mu_N^2 \mu_N^1(M) = M.$$  

Although the module $W$ that we used for mutations is not indecomposable, I think we can regard our equality of modules as a generalization of Iyama-Wemyss’s one. The number of mutations we need seems to be related to the dimension of a fiber of a crepant resolution (or $\mathbb{Q}$-factorial terminalization).

**Corollary 5.14.** The equivalence from $\mathbb{D}^b(Y)$ to $\mathbb{D}^b(Y^+)$ obtained by the composition

$$D^b(Y) \xrightarrow{\Psi_0} D^b(\text{mod } \Lambda_0) \xrightarrow{\nu_{N-2} \cdots \nu_0} D^b(\text{mod } \Lambda_{N-1}) \xrightarrow{(\Psi_0^*)^{-1}} D^b(Y^+)$$

is the inverse of the (original) Kawamata-Namikawa’s functor $KN'_0$.

By the above remark, the functor $\nu_{N-2}^+ \cdots \nu_0^+$ can be written as the composition of $(N-1)^{N-1}$ IW mutation functors. On the other hand, two tilting bundles $T_0$ and $T_0^+$ provide projective generators of the perverse hearts $^0\text{Per}(Y/A_{N-2})$ and $^0\text{Per}(Y^+/A_{N-2}^t)$ respectively (see [TU10 Example 5.3]). Please compare this corollary with [We14 Theorem 4.2].

5.2.3. Multi-mutations and P-twists. Next, we explain that a composition of two multi-mutation functors corresponds to a P-twist on $\mathbb{D}^b(Y)$. First, we recall that the object $j_* \mathcal{O}_P(k)$ is a $\mathbb{P}^{N-1}$-object in $\mathbb{D}^b(Y)$. This fact is well-known but I give the proof here for reader’s convenience.

**Lemma 5.15.** $j_* \mathcal{O}_P(k)$ is a $\mathbb{P}^{N-1}$-object in $\mathbb{D}^b(Y)$.

**Proof.** It is enough to show the case if $k = 0$. Let us consider the spectral sequence

$$E_2^{p,q} = H^p(Y, \text{Ext}_Y^q(j_*, \mathcal{O}_P, j_* \mathcal{O}_P)) \Rightarrow \text{Ext}_Y^{p+q}(j_*, \mathcal{O}_P, j_* \mathcal{O}_P).$$

Since there is an isomorphism

$$\text{Ext}_Y^q(j_*, \mathcal{O}_P, j_* \mathcal{O}_P) \simeq j_* \bigwedge^q \mathcal{N}_P/Y \simeq j_* \Omega_P^q,$$

we have

$$E_2^{p,q} = H^p(\mathbb{P}, \Omega_P^q)$$

$$= \begin{cases} 
\mathbb{C} & \text{if } 0 \leq p = q \leq N - 1, \\
0 & \text{otherwise.}
\end{cases}$$

For the definition of maximal modifying $R$-modules, see [IW14 Definition 4.1]. We note that a module that gives an NCCR is a maximal modifying module if $R$ is a normal $d$-sCY ring [IW14 Proposition 4.5].
Therefore, we have
\[ \text{Ext}^i_Y(j_*\mathcal{O}_T, j_*\mathcal{O}_P) = \begin{cases} C & \text{if } i = 2k \text{ and } k = 0, \ldots, N - 1, \\ 0 & \text{otherwise}. \end{cases} \]
This shows the lemma. \(\square\)

**Definition 5.16.** Let \( P_k \) be a P-twist that is defied by the \( \mathbb{P}^{N-1} \)-object \( j_*\mathcal{O}_P(k) \). More explicitly, the functor \( P_k \) is given by
\[ P_k(E) = \text{Cone} \left( \text{Cone} (j_*\mathcal{O}_P(k)[-2] \to j_*\mathcal{O}_P(k)) \otimes \mathbb{C} \text{RHom}_Y(j_*\mathcal{O}_P(k), E) \overset{ev}{\to} E \right). \]

**Remark 5.17.** By the definition of the functor \( P_k \), the following diagram commutes.
\[
\begin{array}{ccc}
D^b(Y) & \xrightarrow{P_k} & D^b(Y) \\
\downarrow \otimes \mathcal{O}_Y(-1) & & \downarrow \otimes \mathcal{O}_Y(-1) \\
D^b(Y) & \xrightarrow{P_{k-1}} & D^b(Y)
\end{array}
\]
The following is one of main results in this paper.

**Theorem 5.18.** The following diagram of equivalence functors commutes
\[
\begin{array}{ccc}
D^b(Y) & \xrightarrow{\Psi_{N+k}} & D^b(\text{mod}(\Lambda_{N+k})) \\
\downarrow P_k & & \downarrow \nu_{N+k}^- \\
D^b(Y) & \xrightarrow{\Psi_{N+k-1}} & D^b(\text{mod}(\Lambda_{N+k-1})) \\
\downarrow & & \downarrow \nu_{N+k-1}^- \\
D^b(Y) & \xrightarrow{\Psi_{N+k}} & D^b(\text{mod}(\Lambda_{N+k})).
\end{array}
\]
In particular, if we fix the identification \( \Psi_{N+k} : D^b(Y) \to D^b(\text{mod}(\Lambda_{N+k})) \), a composition of two multi-mutation functors
\[ \nu_{N+k-1}^+ \circ \nu_{N+k}^- \in \text{Auteq}(D^b(\text{mod}(\Lambda_{N+k}))) \]
corresponds to a P-twist \( P_k \in \text{Auteq}(D^b(Y)) \).

**Remark 5.19.** If \( 1 \leq N + k \leq N - 1 \) (i.e. if \( -N + 1 \leq k \leq -1 \)), multi-mutation functors \( \nu_{N+k}^- \) and \( \nu_{N+k-1}^+ \) are can be written as compositions of IW mutation functors. Thus, in the case of Mukai flops, we can interpret a P-twist on \( Y \) as a composition of many IW mutation functors. This is a higher dimensional generalization of the result of Donovan and Wemyss [DW16].

**Proof of Theorem 5.18**. It is enough to show the theorem for one \( k \). Here, we prove the case if \( k = -1 \). Recall that the composition
\[
D^b(Y) \xrightarrow{\Psi_{N-1}} D^b(\text{mod}(\Lambda_{N-1})) \xrightarrow{\nu_{N-1}^-} D^b(\text{mod}(\Lambda_{N-2}))
\]
coincides with the functor
\[ \text{RHom}_Y(S \otimes_{\Lambda_{N-1}} \mathcal{T}_{N-1}, -) : D^b(Y) \to D^b(\text{mod}(\Lambda_{N-2})). \]
By Theorem 5.3 and Lemma 5.5, we have \( S \otimes_{\Lambda_{N-1}} \mathcal{T}_{N-1} \simeq \text{KN}_0(T_1^+). \) On the other hand, the equivalence that is given by the composition of functors
\[
D^b(Y) \xrightarrow{P_{-1}} D^b(Y) \xrightarrow{\Psi_{N-2}} D^b(\text{mod}(\Lambda_{N-2}))
\]
Hence, the object $\text{KN}'_{0}(\mathcal{T}_1^+) \simeq \mathcal{T}_{N-2}$.

Recall that the tilting bundles are given by

$$\mathcal{T}_{N-2} = \bigoplus_{a=-2}^{N-2} \mathcal{O}_Y(a), \quad \mathcal{T}_1^+ = \bigoplus_{a=-N+2}^{1} \mathcal{O}_Y(a).$$

By Lemma 5.4, we have

$$\text{KN}'_{0}(\mathcal{O}_Y(a)) \simeq \mathcal{O}_Y(-a)$$

for $-N + 2 \leq a \leq 0$. Therefore, we have to compute the object $\text{KN}'_{0}(\mathcal{O}_Y(1))$. As in Lemma 5.4, we use the exact sequence

$$0 \to \mathcal{O}_Y \to \mathcal{O}_Y \oplus \mathcal{O}_P \times \mathcal{P} \to \mathcal{O}_E \to 0.$$

An easy computation shows that we have

$$\Phi^{Y \to Y}_{\mathcal{O}_P \times \mathcal{P}}(\mathcal{O}_Y(1)) \simeq V \otimes C \cdot j_* \mathcal{O}_P,$$

$$\Phi^{Y \to Y}_{\mathcal{O}_P}(\mathcal{O}_Y(1)) \simeq j_* \mathcal{T}_P(-1),$$

$$\Phi^{Y \to Y}_{\mathcal{O}_P}(\mathcal{O}_Y(1)) \simeq I_{P/Y}(-1),$$

where $\mathcal{T}_P$ is the tangent bundle on $P = \mathbb{P}(V)$ and $I_{P/Y}$ the ideal sheaf of $j : P \subset Y$.

Thus, we have the following exact triangle

$$\text{KN}'_{0}(\mathcal{O}_Y(1)) \to I_{P/Y}(-1) \oplus (V \otimes C \cdot j_* \mathcal{O}_P) \to j_* \mathcal{T}_P(-1).$$

By combining this triangle with the split triangle

$$V \otimes C \cdot j_* \mathcal{O}_P \to I_{P/Y}(-1) \oplus (V \otimes C \cdot j_* \mathcal{O}_P) \to I_{P/Y}(-1),$$

we have the following diagram

$$\begin{array}{c}
\text{KN}'_{0}(\mathcal{O}_Y(1)) \quad \downarrow \quad I_{P/Y}(-1) \\
\uparrow \quad \uparrow \\
\text{KN}'_{0}(\mathcal{O}_Y(1)) \quad \downarrow \quad I_{P/Y}(-1) \\
I_{P/Y}(-1) \quad \downarrow \quad I_{P/Y}(-1)
\end{array}$$

$$\begin{array}{c}
\text{KN}'_{0}(\mathcal{O}_Y(1)) \quad \downarrow \quad I_{P/Y}(-1) \\
\uparrow \quad \uparrow \\
\text{KN}'_{0}(\mathcal{O}_Y(1)) \quad \downarrow \quad I_{P/Y}(-1) \\
I_{P/Y}(-1) \quad \downarrow \quad I_{P/Y}(-1)
\end{array}$$

Hence, the object $\text{KN}'_{0}(\mathcal{O}_Y(1)) \in \text{D}^b(Y)$ is a sheaf, and if we set $\mathcal{F} = \text{KN}'_{0}(\mathcal{O}_Y(1))$, the sheaf $\mathcal{F}$ lies on the exact sequence

$$0 \to j_* \mathcal{O}_P(-1) \to \mathcal{F} \to \mathcal{O}_Y(-1) \to j_* \mathcal{O}_P(-1) \to 0.$$

Recall that $j_* \mathcal{O}_P(-1)$ is a $\mathbb{P}^{N-1}$ object that defines the $P$-twist $P_{-1}$. In particular, $\text{Ext}^2_\mathcal{F}(j_* \mathcal{O}_P(-1), j_* \mathcal{O}_P(-1)) = C \cdot h$. Let $C(h)$ be an object in $\text{D}^b(Y)$ that lies on the exact triangle

$$j_* \mathcal{O}_P(-1)[-2] \xrightarrow{h} j_* \mathcal{O}_P(-1) \to C(h).$$

Then, we have an exact triangle

$$\mathcal{O}_Y(-1)[-1] \to C(h) \to \mathcal{F}.$$
Let \( e : C(h) \to \mathcal{F} \) be the morphism that appears in the above triangle.

Next, we compute the objects \( P_{-1}(\mathcal{K}N'_0(\mathcal{O}_{Y^+}(a))) \) for \(-N + 2 \leq a \leq 1\). Recall that the P-twist \( P_{-1} \) is given by

\[
P_{-1}(E) := \text{Cone}(C(h) \otimes \mathcal{R} \text{Hom}_Y(j_! \mathcal{O}_\mathbb{P}(-1), E) \to E).
\]

Since we have

\[
\mathcal{R} \text{Hom}_Y(j_! \mathcal{O}_\mathbb{P}(-1), \mathcal{O}_Y(b)) \simeq \mathcal{R} \Gamma(\mathbb{P}, \mathcal{O}_\mathbb{P}(-N + b + 1))[-N + 1]
\]

by adjunction, we have

\[
\mathcal{R} \text{Hom}_Y(j_! \mathcal{O}_\mathbb{P}(-1), \mathcal{O}_Y(b)) = 0
\]

for \(0 \leq b \leq N - 2\), and hence we have

\[
P_{-1}(\mathcal{K}N'_0(\mathcal{O}_{Y^+}(a))) \simeq \mathcal{P}(\mathcal{O}_Y(-a)) = \mathcal{O}_Y(-a)
\]

for \(-N + 2 \leq a \leq 0\). It is remaining to compute the object \( P_{-1}(\mathcal{K}N'_0(\mathcal{O}_{Y^+}(1))) = P_{-1}(\mathcal{F})\). From the above computation, we have

\[
\mathcal{R} \text{Hom}_Y(j_! \mathcal{O}_\mathbb{P}(-1), \mathcal{O}_Y(-1)) \simeq \mathcal{R} \Gamma(\mathbb{P}, \mathcal{O}_\mathbb{P}(-N))[-N + 1] \simeq \mathbb{C}[-2N + 2].
\]

On the other hand, by the exact triangle

\[
j_! \mathcal{O}_\mathbb{P}(-1)[-2] \xrightarrow{\mathcal{L}} j_! \mathcal{O}_\mathbb{P}(-1) \to C(h)
\]

that defines \( C(h) \) and the computation

\[
\mathcal{R} \text{Hom}_Y(j_! \mathcal{O}_\mathbb{P}(-1), j_! \mathcal{O}_\mathbb{P}(-1)) = \bigoplus_{i=0}^{N-1} \mathbb{C}[-2i],
\]

we have

\[
\mathcal{R} \text{Hom}_Y(j_! \mathcal{O}_\mathbb{P}(-1), C(h)) = \mathbb{C} \oplus \mathbb{C}[-2N + 1].
\]

Hence, by the exact triangle

\[
\mathcal{O}_Y(-1)[-1] \to C(h) \xrightarrow{\mathcal{L}} \mathcal{F}
\]

that we obtained above, we have

\[
\mathcal{R} \text{Hom}_Y(j_! \mathcal{O}_\mathbb{P}(-1), \mathcal{F}) = \mathbb{C},
\]

and thus, the object \( P_{-1}(\mathcal{F}) \) lies on the exact triangle

\[
C(h) \xrightarrow{\mathcal{L}} \mathcal{F} \to P_{-1}(\mathcal{F}).
\]

Since we have

\[
\text{Hom}_Y(C(h), \mathcal{F}) \simeq \mathbb{C}
\]

from the above, we have the map \( \mathcal{L} : C(h) \to \mathcal{F} \) coincides with the map \( e : C(h) \to \mathcal{F} \) up to non-zero scaler. Therefore, we have

\[
P_{-1}(\mathcal{F}) \simeq \mathcal{O}_Y(-1)
\]

and hence \( P_{-1}(\mathcal{K}N'_0(T^+_1)) \simeq \mathcal{T}_{N-2}. \) This is what we want. \( \square \)

Theorem 5.18 recovers the following result that was first proved by Cautis, and later Addington-Donovan-Meachan in different ways. Our approach that uses non-commutative crepant resolutions and their mutations gives a new alternative proof for their result.
Corollary 5.20 ([Ca12 ADM15]). We have a functor isomorphism

\[ \text{KN}'_{-k} \circ \text{KN}_{N+k} \simeq P_k \]

for all \( k \in \mathbb{Z} \).

Proof. Let us consider the next diagram

\[
\begin{array}{ccc}
\text{D}^b(Y) & \xrightarrow{\Psi_{N+k}} & \text{D}^b(\text{mod}(\Lambda_{N+k})) \\
\downarrow P_k & & \downarrow \Psi_{N+k}^{-1} \\
\text{D}^b(Y) & \xleftarrow{\Psi_{k}} & \text{D}^b(\text{mod}(\Lambda_{N+k-1})).
\end{array}
\]

Since \((\Psi_{N+k}^{-1})^{-1} \circ \Psi_{N+k} \simeq \text{KN}_{N+k}\) and \((\Psi_{N+k-1}^{-1})^{-1} \circ \Psi_{k}^{-1} \simeq \text{KN}'_{-k}\) by Theorem 5.3, we have \( \text{KN}_{-k} \circ \text{KN}_{N+k} \simeq P_k \).

We note that, in order to prove this corollary, Cautis used an elaborate framework “categorical \( \mathfrak{sl}_2 \)-action” that is established by Cautis, Kamnitzer, and Licata [CKL10 CKL13]. Addington, Donovan, and Meachan provided two different proofs. The first one uses a technique of semi-orthogonal decomposition, and the second one uses the variation of GIT quotients and “window shifts”.

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Department of Mathematics, School of Science and Engineering, Waseda University, 3-4-1 Ohkubo, Shinjuku, Tokyo 169-8555, Japan
E-mail address: waheyhey@ruri.waseda.jp