Inverse scattering on the line with a transfer condition *

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Abstract

The inverse scattering problem for Sturm-Liouville operators on
the line with a matrix transfer condition at the origin is considered.
We show that the transfer matrix can be reconstructed from the eigen-
values and reflection coefficient. In addition, for potentials with com-
 pact essential support, we show that the potential can be uniquely
reconstructed.

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1 Introduction

In this paper we investigate the inverse scattering problem for the differential equation

\[ \ell y := -\frac{d^2 y}{dx^2} + q(x)y = \zeta^2 y, \quad \text{on } (-\infty, 0) \cup (0, \infty), \quad (1.1) \]

in \( L^2(-\infty, 0) \oplus L^2(0, \infty) = L^2(\mathbb{R}) \) with point transfer condition

\[ \begin{bmatrix} y(0^+) \\ y'(0^+) \end{bmatrix} = M \begin{bmatrix} y(0^-) \\ y'(0^-) \end{bmatrix}. \quad (1.2) \]

Here the entries of \( M \) are taken to be real, \( q \in L^2(\mathbb{R}) \) is assumed to be real valued and obey the growth condition

\[ \int_{-\infty}^{\infty} (1 + |x|)|q(x)|dx < \infty. \quad (1.3) \]

Note that (1.3) gives that \( q \in L^1(\mathbb{R}) \). As usual we denote \( f(0^+) := \lim_{t \downarrow 0} f(t) \) and \( f(0^-) := \lim_{t \uparrow 0} f(t) \). The operator \( L \) in \( L^2(\mathbb{R}) \) is defined by

\[ Ly = \ell y \quad (1.4) \]

on \( \mathbb{R} \setminus \{0\} \) for \( y \) in the domain, \( D(L) \), of \( L \) where

\[ D(L) = \{ y | y, \ell y \in L^2(\mathbb{R}), y|_{(-\infty,0)}^{(j)}, y|_{(0,\infty)}^{(j)} \in AC, j = 0, 1, y \text{ obeys } (1.2) \}. \quad (1.5) \]

As \( q \in L^2(\mathbb{R}) \), \( D(L) \) is independent of \( q \) which ensures that \( D(L) \) is known a priori for the inverse problem.

We will only consider point transfer matrices at the origin and henceforth will refer to them as transfer matrices. In the physical context the transfer matrix represents a change of medium which affects the incident wave as represented by components of the matrix. Our transfer matrices will be real constant transfer matrices i.e. all components will be constants.

Hochstadt and Lieberman, in [8], considered the inverse Sturm-Liouville problem of the unique determination of the potential on a given interval from the spectrum, the boundary conditions and the potential on half of the interval. These results were generalized to the case of eigenfunctions having a discontinuity at the mid-point of the interval in the famous paper by Hald, [7], where, in addition, it was...
shown that one boundary condition can also be uniquely recovered. This in turn was extended by [21] to the case of two interior discontinuities. Similar techniques were then used by [14] to give a uniqueness proof for the inverse Sturm-Liouville problem on a bounded interval with a symmetric potential having two interior jump discontinuities.

Ramm, [17], discusses inverse scattering and spectral one-dimensional problems on the half-line in detail. Some of the main topics included are, invertibility of the steps in the Gel’fand-Levitan and Marchenko inversion procedures, Krein inverse scattering theory and inverse problems.

It should be noted that in [9], Hryniv shows that the potential of a Sturm-Liouville operator depends analytically and Lipschitz continuously on the spectral data i.e. two spectra or one spectrum and the corresponding norming constants. Since he considers $q \in H^{-1}(0,1)$, this means that there could be a discontinuity at an interior point of $(0,1)$. Thus, the inverse problem that Hryniv considers could be thought of as a discontinuous Sturm-Liouville problem on a finite interval where the transfer condition is of a special form which is less general than the transfer condition which we are considering in this paper. In [11, 12, 13] the authors consider the discontinuous Sturm-Liouville operator on a finite interval where the boundary conditions may depend on the eigenparameter. In [11] and [13] a transfer condition equivalent to taking $M = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix}$ in (1.2) is used, whereas in [12] the transfer condition itself is also dependent on the eigenparameter. For each of the various types of problems considered in [11, 12, 13] uniqueness theorems for the solution of inverse problems using the Titchmarsh-Weyl function and spectral data are proven.

In this paper we solve the following inverse problem. Given the scattering data, using the asymptotics developed in [3], we provide a reconstruction of the transfer matrix $M$ and the scattering coefficients. For the case of the potential having compact essential support, given the scattering data, one can determine the Titchmarsh-Weyl $m$-function for (1.1) with separated boundary conditions and transfer condition (1.2), on $[-S,S]$ where $\text{ess supp}(q) \subset [-S,S]$. Consequently the potential can be uniquely reconstructed.

In Section 2 the notation and some basic results are presented. The reflection coefficient is considered in Section 3. Attention is restricted to the compact essential support in Section 4, where the main result is presented. Section 5 is the Appendix in which the details of the
asymptotics used in this paper are presented in detail.

2 Preliminaries

The scattering problem considered in this paper can be treated as two classical half-line problems interacting via the matrix transfer condition (1.2) at the origin.

The operator eigenvalue problem associated with $L$, of (1.4), can be reformulated as a system eigenvalue problem as follows. Let $y_1(t) = y(t)$, $y_2(t) = y(-t)$ and $Y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$ and consider the differential operator in $L^2(0, \infty) \oplus L^2(0, \infty)$ given by

$$TY := -\frac{d^2Y}{dx^2} + QY = \zeta^2 Y,$$

where $Q(t) = \begin{pmatrix} q(t) & 0 \\ 0 & q(-t) \end{pmatrix}$. The domain of $T$ is given by

$$D(T) = \{ Y \mid Y, TY \in (L^2(0, \infty))^2, Y, Y' \in AC, UY(0) = VY'(0) \}$$

(2.2)

where $U = \begin{pmatrix} 1 & -m_{11} \\ 0 & m_{21} \end{pmatrix}$ and $V = \begin{pmatrix} 0 & -m_{12} \\ 1 & m_{22} \end{pmatrix}$. Here $m_{ij}$, for $i, j = 1, 2$, are the entries of the transfer matrix $M$. As the norm on $L^2(0, \infty) \oplus L^2(0, \infty)$ we take

$$\|Y\|^2 = \int_0^\infty Y^T \bar{Y} dx.$$

It should be noted that $Ly = \zeta^2 y, y \in D(L)$, is equivalent to $TY = \zeta^2 Y, Y \in D(T)$. The transfer matrix scattering problem can now be posed as

$$TY = \zeta^2 Y, \quad Y \in D(T).$$

(2.3)

For $F, G \in D(T)$, define the Lagrange form $S(F, G) := \langle TF, G \rangle - \langle F, TG \rangle$, for $F, G \in D(T)$, where

$$\langle F, G \rangle = \int_0^\infty F(x)^T \bar{G}(x) dx.$$

(2.4)

It was shown in \cite{3} Theorem 3.2] that if $\det M \neq 0$ then the operator $T$ is a self-adjoint operator if and only if $\det M = 1$, and hence, after rescaling, for any $M$ with $\det M > 0$, see also \cite{20},
Definition 2.1 [2, p.297] The Jost solutions $f_{+,M}(x,\zeta)$ and $f_{-,M}(x,\zeta)$ are the solutions of (1.1) and (1.2) with
\[
\lim_{x \to \infty} e^{-i\zeta x} f_{+,M}(x,\zeta) = 1 = \lim_{x \to -\infty} e^{i\zeta x} f_{-,M}(x,\zeta).
\] (2.5)

We can now express the Jost solutions $f_{+,M}(x,\zeta)$ and $f_{-,M}(x,\zeta)$ to (2.3) in terms of the classical Jost solutions $f_{+,}(x,\zeta)$ and $f_{-}(x,\zeta)$ (i.e. when $M = I$) by
\[
f_{+,M}(x,\zeta) := \begin{cases} f_{+}(x,\zeta), & x > 0 \\ h_{1}(x,\zeta), & x < 0 \end{cases},
\]
(2.6)
\[
f_{-,M}(x,\zeta) := \begin{cases} f_{-}(x,\zeta), & x < 0 \\ h_{2}(x,\zeta), & x > 0 \end{cases},
\]
(2.7)
where $h_{1}(x,\zeta)$ and $h_{2}(x,\zeta)$ are solutions of (1.1) on $(-\infty, 0)$ and $(0, \infty)$ respectively obeying
\[
\begin{pmatrix} h_{1}(0^{-},\zeta) \\ h_{1}'(0^{-},\zeta) \end{pmatrix} = M^{-1} \begin{pmatrix} f_{+}(0^{+},\zeta) \\ f'_{+}(0^{+},\zeta) \end{pmatrix},
\]
\[
\begin{pmatrix} h_{2}(0^{+},\zeta) \\ h_{2}'(0^{+},\zeta) \end{pmatrix} = M \begin{pmatrix} f_{-}(0^{-},\zeta) \\ f'_{-}(0^{-},\zeta) \end{pmatrix}.
\]
(2.10)

For $M = I$ the existence and asymptotic behaviour of the Jost solutions have been well studied, see for example [4, 16]. In particular
\[
f_{+}(x,\zeta) = e^{i\zeta x} + O \left( \frac{C(x)\rho(x)e^{-\eta x}}{1 + |\zeta|} \right),
\]
(2.8)
and
\[
f_{-}(x,\zeta) = e^{-i\zeta x} + O \left( \frac{C(-x)\tilde{\rho}(x)e^{\eta x}}{1 + |\zeta|} \right),
\]
(2.9)
as $|x| + |\zeta| \to \infty$, where $\eta = \Im(\zeta)$. Here $C(x)$ is a non-negative, non-increasing function of $x$ and
\[
\rho(x) = \int_{x}^{\infty} (1 + |\tau|)|q(\tau)|d\tau, \quad \tilde{\rho}(x) = \int_{-\infty}^{x} (1 + |\tau|)|q(\tau)|d\tau.
\]
(2.10)

For $\xi \in \mathbb{R}$, see [3, Sections 2 and 4], the conjugate Jost solutions take the form
\[
\overline{f}_{+,M}(x,\xi) := \begin{cases} \overline{f}_{+}(x,-\xi), & x > 0 \\ \frac{1}{h_{1}(x,\xi)} = h_{1}(x,-\xi), & x < 0 \end{cases}
\]
(2.11)
which obeys the transfer condition at $x = 0$. Being independent (for $\xi \in \mathbb{R}\setminus\{0\}$), the solutions $f_{+M}(x, \xi)$ and $\overline{f}_{+M}(x, \xi)$ span the solution space of (1.1), with (1.2), so there exist (unique) coefficients $A(\xi)$ and $B(\xi)$ so that

$$f_{-M}(x, \xi) = A(\xi)\overline{f}_{+M}(x, \xi) + B(\xi)f_{+M}(x, \xi). \quad (2.12)$$

Here $A(\xi)$ and $B(\xi)$ are independent of whether $x > 0$ or $x < 0$, and they satisfy the equality $|A(\xi)|^2 - |B(\xi)|^2 = 1$ for $\xi \in \mathbb{R}\setminus\{0\}$. The reflection coefficient is defined as

$$R(\xi) = \frac{B(\xi)}{A(\xi)}, \text{ for } \xi \in \mathbb{R}\setminus\{0\}.$$

From [3] we have that

$$A(\zeta) = \frac{m_{12}}{2i}\zeta + \frac{m_{11} + m_{22}}{2} + \frac{m_{12}}{2} \int_{-\infty}^{\infty} \cos(-\zeta \tau)q(\tau)e^{i|\tau|} d\tau + O\left(\frac{1}{1 + |\zeta|}\right), \quad (2.13)$$

for large $|\zeta|$ and $\Im(\zeta) \geq 0$. For $|\xi|$ large,

$$B(\xi) = -\frac{m_{12}}{2i}\xi + \frac{m_{22} - m_{11}}{2} - \frac{m_{12}}{2} \int_{-\infty}^{\infty} \cos(\xi \tau)q(\tau)e^{-i|\tau|} d\tau + O\left(\frac{1}{1 + |\xi|}\right). \quad (2.14)$$

### 3 The reflection coefficient

In this section, given the reflection coefficient and the eigenvalues we will reconstruct the point transfer matrix as well as the coefficients $A(\zeta)$ and $B(\xi)$. Moreover, for three special cases of the reflection coefficient $R(\xi)$ we will explicitly find the corresponding transfer matrix and $A(\zeta)$.

From [10] p.175 we have the following representation result.

**Lemma 3.1** Let $f$ be a function analytic in the upper half-plane obeying

- $\zeta(f(\zeta) - 1)$ is bounded for $\Im(\zeta) \geq 0$,
- $f(\zeta)$ is continuous for $\zeta \neq 0$ with $\Im(\zeta) \geq 0$,
- $f(\zeta) \neq 0$ for $\zeta \neq 0$ with $\Im(\zeta) \geq 0$,
- $\zeta = 0$ is a first order pole of $f(\zeta)$.  

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If $F(\zeta) = \log f(\zeta)$, then for $\zeta \in \mathbb{C}$ with $\Im(\zeta) > 0$

$$F(\zeta) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{2\Re F(\xi)}{\xi - \zeta} d\xi = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log |f(\xi)|^2}{\xi - \zeta} d\xi.$$

**Theorem 3.2** For $m_{12} = 0$, given the scattering data, $\{R(\xi), \eta_1, \ldots, \eta_N\}$, where $\eta_1, \ldots, \eta_N$ are the eigenvalues of (1.4)-(1.5), the point transfer matrix, $M$, is uniquely determined up to $m_{21}$ and a sign condition. In particular

$$m_{22} = \pm \sqrt{\frac{1 + C_2}{1 - C_2}} \quad \text{and} \quad m_{11} = \pm \sqrt{\frac{1 - C_2}{1 + C_2}}$$

where $C_2 = \lim_{\xi \to \infty} R(\xi)$ (and this limit exists). Moreover,

$$A(\zeta) = \pm \frac{1}{\sqrt{1 - C_2^2}} \prod_{j=1}^{N} \frac{\zeta - i\eta_j}{\zeta + i\eta_j} \exp \left[ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \log |1 - C_2^2| - \log(1 - |R(\xi)|^2) \frac{\xi - \zeta}{\xi - \zeta} d\xi \right].$$

(3.1)

**Proof:** From (2.13) and (2.14), for $m_{12} = 0$, we obtain the following asymptotics

$$A(\zeta) = \frac{m_{11} + m_{22}}{2} + O\left(\frac{1}{1 + |\zeta|}\right), \quad (3.2)$$

$$B(\xi) = \frac{m_{22} - m_{11}}{2} + O\left(\frac{1}{1 + |\xi|}\right). \quad (3.3)$$

As $\det M = 1$ and $m_{12} = 0$ it follows that $m_{11}m_{22} = 1$, that is

$$m_{11} = \frac{1}{m_{22}}, \quad (3.4)$$

making $m_{11} + m_{22} \neq 0$. Thus, given $R(\xi)$ and that $m_{12} = 0$, we have

$$R(\xi) = \frac{B(\xi)}{A(\zeta)} = \frac{m_{22} - m_{11}}{2} + O\left(\frac{1}{1 + |\xi|}\right)$$

$$= \frac{m_{22} - m_{11}}{m_{11} + m_{22}} + O\left(\frac{1}{1 + |\xi|}\right)$$

$$\to \frac{m_{22} - m_{11}}{m_{11} + m_{22}}$$

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as $|\xi| \to \infty$. Denote $C_2 = \frac{m_{22} - m_{11}}{m_{11} + m_{22}}$, then

$$R(\xi) = C_2 + O\left(\frac{1}{1 + |\xi|}\right).$$

Hence $\lim_{\xi \to \infty} R(\xi)$ exists and is $C_2$. Since the reflection coefficient $R(\xi)$ is given $C_2$ is known. By combining (3.4) with the definition of $C_2$ we have

$$m_{22} = \pm \sqrt{\frac{1 + C_2}{1 - C_2}}$$

and

$$m_{11} = \pm \sqrt{\frac{1 - C_2}{1 + C_2}}.$$

Hence, in the case when $m_{12} = 0$, the point transfer matrix can determined up to one parameter ($m_{21}$ being undetermined) from the scattering data.

We now turn our attention to $A(\zeta)$. Let

$$f(\zeta) = \frac{2}{m_{11} + m_{22}} A(\zeta) \prod_{j=1}^{N} \frac{\zeta + i\eta_j}{\zeta - i\eta_j},$$

(3.5)

where $0 < \eta_1 < \cdots < \eta_N$ and $-\eta_j^2, j = 1, \ldots, N$ are the eigenvalues of (1.4), (1.5), see [3, Theorem 3.3]. Using (3.2) we get the asymptotic expression $f(\zeta) = 1 + O(\frac{1}{\zeta})$. Moreover all the properties given in Lemma 3.1 are obeyed. Thus setting $F(\zeta) = \log f(\zeta)$, for $\zeta \in \mathbb{C}$ with $\Im(\zeta) > 0$, we have

$$F(\zeta) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{2\Re F(\xi)}{\xi - \zeta} d\xi = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log |f(\xi)|^2}{\xi - \zeta} d\xi.$$  

(3.6)

Here, as $\xi \in \mathbb{R}$,

$$\log |f(\xi)|^2 = 2 \log \left| \frac{2}{m_{11} + m_{22}} \right| + \log |A(\xi)|^2.$$  

(3.7)

By [3, Lemma 4.2],

$$|A(\xi)|^2 - |B(\xi)|^2 = 1.$$

So, in terms of $R(\xi)$ we have

$$|A(\xi)|^2 (1 - |R(\xi)|^2) = 1,$$

(3.8)

which together with (3.7) gives

$$\log |f(\xi)|^2 = 2 \log \left| \frac{2}{m_{11} + m_{22}} \right| - \log (1 - |R(\xi)|^2).$$  

(3.9)
Combining (3.9) and (3.6) we obtain

\[ F(\zeta) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} 2 \log \left| \frac{2}{m_{11} + m_{22}} \right| \frac{- \log(1 - |R(\xi)|^2)}{\xi - \zeta} d\xi. \]

As \( f(\zeta) = e^{F(\zeta)} \) we conclude

\[
A(\zeta) = \frac{m_{11} + m_{22}}{2} \prod_{j=1}^{N} \frac{\zeta - i\eta_j}{\zeta + i\eta_j} \exp \left[ \frac{1}{2\pi i} \int_{-\infty}^{\infty} 2 \log \left| \frac{2}{m_{11} + m_{22}} \right| \frac{- \log(1 - |R(\xi)|^2)}{\xi - \zeta} d\xi \right]
\]

\[ = \pm \frac{1}{\sqrt{1 - C_2^2}} \prod_{j=1}^{N} \frac{\zeta - i\eta_j}{\zeta + i\eta_j} \exp \left[ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \log \left| 1 - C_2^2 \right| - \log(1 - |R(\xi)|^2) \frac{\xi - \zeta}{\xi - \zeta} d\xi \right]. \]

As a consequence of the above theorem we obtain the following corollary.

**Corollary 3.3** If \( R(\xi) = 0 \) (i.e. the reflectionless case) then the point transfer matrix has the form

\[
M = \begin{pmatrix}
\pm 1 & 0 \\
m_{21} & \pm 1
\end{pmatrix}
\]

and

\[
A(\zeta) = \pm \prod_{j=1}^{N} \frac{\zeta - i\eta_j}{\zeta + i\eta_j}. \tag{3.10}
\]

Furthermore, if \( m_{12} = 0 \) and \( |R(\xi)| = \pm C_2 \) then

\[
m_{22} = \pm \sqrt{\frac{1 + C_2^2}{1 - C_2^2}}, \quad m_{11} = \pm \sqrt{\frac{1 - C_2^2}{1 + C_2}}
\]

and

\[
A(\zeta) = \pm \frac{1}{\sqrt{1 - C_2^2}} \prod_{j=1}^{N} \frac{\zeta - i\eta_j}{\zeta + i\eta_j}.
\]

**Proof:** Since \( R(\xi) = 0 \) we have that \( B(\xi) = 0 \). Thus, by \( \text{(2.14)} \), \( m_{12} = 0 \) and Theorem 3.2 can be applied with \( C_2 = 0 \) to give \( m_{11} = m_{22} = \pm 1 \) and

\[
A(\zeta) = \pm \prod_{j=1}^{N} \frac{\zeta - i\eta_j}{\zeta + i\eta_j}.
\]

If \( m_{12} = 0 \) and \( |R(\xi)| = \pm C_2 \) then Theorem 3.2 can be applied to give

\[
A(\zeta) = \pm \frac{1}{\sqrt{1 - C_2^2}} \prod_{j=1}^{N} \frac{\zeta - i\eta_j}{\zeta + i\eta_j}. \]
We now prove a similar theorem to Theorem 3.2 for the case where \( m_{12} \neq 0 \).

**Theorem 3.4** For \( m_{12} \neq 0 \), given the scattering data \( \{ R(\xi), \eta_1, \ldots, \eta_N \} \), the coefficients of the point transfer matrix, \( M \), obey

\[
m_{12}(C_1m_{11} - m_{21}) = 1
\]

where \( R(\xi) = -1 + \frac{2i}{\xi}C_1 + O \left( \frac{1}{\xi^2} \right) \). In this case

\[
A(\zeta) = \frac{m_{12} \zeta + i(m_{11} + m_{22})}{2i} \prod_{j=1}^{N} \frac{\zeta - i\eta_j}{\zeta + i\eta_j} \times \exp \left[ \frac{1}{2\pi i} \int_{-\infty}^{\infty} 2\log \frac{1}{\sqrt{\xi^2 + m_{12}^2 + (m_{11} + m_{22})^2}} \frac{2}{\xi - \zeta} - \log(1 - |R(\xi)|^2) \, d\xi \right].
\]

**Proof:** As \( m_{12} \neq 0 \) from (2.13), (2.14) and the definition of \( R(\xi) \) we get

\[
R(\xi) = \frac{-\frac{m_{12}}{2i} - \frac{m_{22} - m_{11}}{2} - \frac{m_{12}}{2} \int_{-\infty}^{\infty} \cos(\xi \tau)q(\tau)e^{-i\xi \tau} \, d\tau + O \left( \frac{1}{1 + |\xi|} \right)}{\frac{m_{12}}{2i} + \frac{m_{11} + m_{22}}{2} + \frac{m_{12}}{2} \int_{-\infty}^{\infty} \cos(\xi \tau)q(\tau)e^{i\xi |\tau|} \, d\tau + O \left( \frac{1}{\xi} \right)}
\]

\[
= \left( -1 + \frac{i(m_{22} - m_{11})}{m_{12} \xi} - \frac{i}{\xi} \int_{-\infty}^{\infty} \cos(\xi \tau)q(\tau)e^{-i\xi \tau} \, d\tau + O \left( \frac{1}{\xi^2} \right) \right) \times \left( 1 - \frac{i(m_{11} + m_{22})}{m_{12} \xi} - \frac{i}{\xi} \int_{-\infty}^{\infty} \cos(\xi \tau)q(\tau)e^{i\xi |\tau|} \, d\tau + O \left( \frac{1}{\xi^2} \right) \right)
\]

\[
= -1 + \frac{i(m_{11} + m_{22})}{m_{12} \xi} + \frac{i(m_{22} - m_{11})}{m_{12} \xi} + \frac{i}{\xi} \int_{-\infty}^{\infty} \cos(\xi \tau)q(\tau)e^{i\xi |\tau|} \, d\tau - \frac{i}{\xi} \int_{-\infty}^{\infty} \cos(\xi \tau)q(\tau)e^{-i\xi \tau} \, d\tau + O \left( \frac{1}{\xi^2} \right)
\]

\[
= -1 + \frac{2im_{22}}{m_{12} \xi} - \frac{1}{\xi} \int_{0}^{\infty} q(\tau) \sin(2\xi \tau) \, d\tau + O \left( \frac{1}{\xi^2} \right).
\]

Let

\[
C_1 = \frac{m_{22}}{m_{12}}
\]
then
\[ R(\xi) = -1 + \frac{2i}{\xi} C_1 - \frac{1}{\xi} \int_0^\infty q(\tau) \sin(2\xi \tau) d\tau + O\left(\frac{1}{\xi^2}\right). \] (3.12)

Since the reflection coefficient \( R(\xi) \) is given and \( \int_0^\infty q(\tau) \sin(2\xi \tau) d\tau \) tends to 0 by the Riemann-Lebesgue Lemma, the constant \( C_1 \) is given by
\[ C_1 = \lim_{\xi \to \infty} \frac{\xi (R(\xi) + 1)}{2i}. \]

In addition we have that
\[ \det M = m_{11} m_{22} - m_{12} m_{21} = 1 = m_{12} (C_1 m_{11} - m_{21}). \] (3.13)

Now, let
\[ f(\zeta) = \frac{2i}{m_{12}\zeta + i(m_{11} + m_{22})} A(\zeta) \prod_{j=1}^N \frac{\zeta + i\eta_j}{\zeta - i\eta_j}. \] (3.14)

Substituting (2.13) into (3.14) results in
\[ f(\zeta) = \left(1 + O\left(\frac{1}{\zeta}\right)\right) \prod_{j=1}^N \frac{\zeta + i\eta_j}{\zeta - i\eta_j} \]
\[ = 1 + O\left(\frac{1}{\zeta}\right). \]

Therefore, for \(|\zeta|\) large, \( f(\zeta) - 1 = O\left(\frac{1}{\zeta}\right) \) and all the conditions required in Lemma 3.1 are met by \( f(\zeta) \). Thus setting \( F(\zeta) = \log f(\zeta) \), for \( \zeta \in \mathbb{C} \) with \( \Im(\zeta) > 0 \), we can write
\[ F(\zeta) = \frac{1}{2\pi i} \int_{-\infty}^\infty \frac{2 \log |f(\xi)|^2}{\xi - \zeta} d\xi = \frac{1}{2\pi i} \int_{-\infty}^\infty \frac{\log |f(\xi)|^2}{\xi - \zeta} d\xi. \]

Now by (3.14), together with (3.8),
\[ \log |f(\xi)|^2 = \log \left| \frac{2A(\xi)}{m_{12}\xi + i(m_{11} + m_{22})} \right|^2 \]
\[ = 2 \log \left| \frac{2}{m_{12}\xi + i(m_{11} + m_{22})} \right| + \log |A(\xi)|^2 \]
\[ = 2 \log \frac{2}{\sqrt{\xi^2 m_{12}^2 + (m_{11} + m_{22})^2}} - \log(1 - |R(\xi)|^2), \]
giving
\[
F(\zeta) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{2 \log \sqrt{\xi^2 m_1^2 + (m_{11} + m_{22})^2}}{\xi - \zeta} - \log(1 - |R(\xi)|^2) \, d\xi.
\]

Since \( f(\zeta) = e^{F(\zeta)} \) using (3.14) we obtain
\[
A(\zeta) = \frac{m_{12} \zeta + i(m_{11} + m_{22})}{2i} \prod_{j=1}^{N} \frac{\zeta - i\eta_j}{\zeta + i\eta_j} \times \exp \left[ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{2 \log \sqrt{\xi^2 m_1^2 + (m_{11} + m_{22})^2}}{\xi - \zeta} - \log(1 - |R(\xi)|^2) \, d\xi \right].
\]

The following corollary is a direct consequence of Theorem 3.4 for the case where the exponential term in \( A(\zeta) \) reduces to 1.

**Corollary 3.5** For \( m_{12} \neq 0 \), if
\[
\frac{4}{\xi^2 m_1^2 + (m_{11} + m_{22})^2} = 1 - |R(\xi)|^2, \quad (3.15)
\]
then
\[
A(\zeta) = \frac{m_{12} \zeta + i(m_{11} + m_{22})}{2i} \prod_{j=1}^{N} \frac{\zeta - i\eta_j}{\zeta + i\eta_j}.
\]

Moreover, the coefficients of the transfer matrix are determined by the equations \( K_1 = m_1^2, K_2 = (m_{11} + m_{22})^2, m_{11} m_{22} - m_{21} m_{12} = 1 \) and \( m_{22} = C_1 m_{12} \). Here \( C_1 \) is as in (3.12) and \( K_1, K_2 \) are known and obey \( K_1 > 0 \) and \( K_2 \geq 0 \). This results in four possibilities for the transfer matrix \( M \).

Clearly, in all of the above results in this section, as \( R(\xi) \) is given and we can find \( A(\xi) \) from the relevant equations, it is possible to obtain \( B(\xi) \) since \( B(\xi) = R(\xi)A(\xi) \).

### 4 Compact essential support potentials

For the remainder of the paper we will assume that the potential \( q(x) \) has compact essential support, say ess supp(\( q(\xi) \)).
Lemma 4.1 Let \( \text{ess \, supp}(q) \subset [-S,S] \) for some \( S > 0 \). Given the scattering data \( \{R(\xi), \eta_1, \ldots, \eta_N\} \), the matrix \( W(S,\zeta) \) is uniquely determined. Here

\[
W(x, \zeta) = \begin{bmatrix}
w_1(x, \zeta) & w_2(x, \zeta) \\
w'_1(x, \zeta) & w'_2(x, \zeta)
\end{bmatrix}
\quad \text{with} \quad W(-S, \zeta) = \begin{bmatrix}-1 & 0 \\0 & 1\end{bmatrix} =: H.
\]

and \( w_1, w_2 \) are solutions of (1.1).

Proof: As \( \text{ess \, supp}(q) \subset [-S,S] \), for \( x \leq -S \) and \( \zeta = \xi \in \mathbb{R} \), we have

\[
f_{-,M}(x, \xi) = f_-(x, \xi) = e^{-i\xi x} \quad \text{and} \quad \overline{f}_{-,M}(x, \xi) = \overline{f}_-(x, \xi) = e^{i\xi x}.
\]

By (2.12), for \( x \geq S \),

\[
f_{-,M}(x, \xi) = A(\xi)e^{-i\xi x} + B(\xi)e^{i\xi x},
\]

and

\[
\overline{f}_{-,M}(x, \xi) = \overline{A}(\xi)e^{i\xi x} + \overline{B}(\xi)e^{-i\xi x}.
\]

For \( x \leq -S \), as \( q \) is essentially zero,

\[
w_j(x, \xi) = a_jf_{-,M}(x, \xi) + b_j\overline{f}_{-,M}(x, \xi) = a_je^{-i\xi x} + b_je^{i\xi x}
\]

and for \( x \geq S \),

\[
w_j(x, \xi) = \hat{a}_jf_{-,M}(x, \xi) + \hat{b}_j\overline{f}_{-,M}(x, \xi) = \hat{a}_je^{-i\xi x} + \hat{b}_je^{i\xi x}.
\]

Thus for \( j = 1, 2 \),

\[
\begin{pmatrix}
\hat{a}_j \\
b_j
\end{pmatrix} = \begin{pmatrix}
A(\xi) & \overline{B}(\xi) \\
B(\xi) & \overline{A}(\xi)
\end{pmatrix} \begin{pmatrix}
a_j \\
b_j
\end{pmatrix}.
\]

From the initial value \( W(-S, \zeta) \) it follows that

\[
\begin{pmatrix}
a_1 & a_2 \\
b_1 & b_2
\end{pmatrix} = \begin{pmatrix}
-\frac{e^{-iS\xi}}{2} & -\frac{1}{2\xi}e^{-iS\xi} \\
-\frac{1}{2\xi}e^{iS\xi} & \frac{1}{2\xi}e^{iS\xi}
\end{pmatrix}
\]

so

\[
[w_1(S, \xi) \quad w_2(S, \xi)] = \begin{pmatrix}
e^{-iS\xi} & e^{iS\xi} \\
e^{-iS\xi} & e^{iS\xi}
\end{pmatrix} \begin{pmatrix}
\hat{a}_1 & \hat{a}_2 \\
\hat{b}_1 & \hat{b}_2
\end{pmatrix}
\]

\[
= \begin{pmatrix}
e^{-iS\xi} & e^{iS\xi} \\
e^{-iS\xi} & e^{iS\xi}
\end{pmatrix} \begin{pmatrix}
A(\xi) & \overline{B}(\xi) \\
B(\xi) & \overline{A}(\xi)
\end{pmatrix} \begin{pmatrix}
-\frac{e^{-iS\xi}}{2} & -\frac{1}{2\xi}e^{-iS\xi} \\
-\frac{1}{2\xi}e^{iS\xi} & \frac{1}{2\xi}e^{iS\xi}
\end{pmatrix}
\]

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By Theorems 3.2 and 3.4 given \( R(\xi) \) and \( \eta_1, \ldots, \eta_N \), we can reconstruct \( A(\xi) \) and hence \( B(\xi) \) and thus find \( w_2(S, \xi) \) and \( w_1(S, \xi) \) as above.

We now use the approach given in [1] together with that found in [4, p. 28] in order to prove the unique determination of the potential \( q \) from the scattering data. Let \( v \) be the solution of (1.1) on \([-S, S]\) obeying the transfer condition (1.2) and satisfying the terminal conditions \( v(S) = 0 \) and \( v'(S) = 1 \). The entries of \( W(x, \lambda) \) are entire functions of \( \lambda \) and the determinant is the Wronskian of \( w_1 \) and \( w_2 \) and thus is equal to \(-1\) for all \( x \) and \( \lambda \).

The Titchmarsh-Weyl m-function of (1.1) on \([-S, S]\) for double Dirichlet boundary conditions \( y(-S) = 0 = y(S) \) and the transfer condition (1.2) is that value of \( m \) for which

\[
\psi := w_1 + mw_2
\]

obeys the terminal condition \( \psi(S) = 0 \). Now

\[
\psi(-S, \lambda) = w_1(-S, \lambda) + m(\lambda)w_2(-S, \lambda) = -1.
\]

Let

\[
\Delta(\lambda) := \text{Wron}[w_2, v] = w_2v' - vw_2' = -v(-S, \lambda) = w_2(S, \lambda).
\]

The function \( \Delta(\lambda) \) is entire in \( \lambda \) and the zeros of \( \Delta(\lambda) \) are the eigenvalues of (1.1) with double Dirichlet boundary conditions and the transfer condition (1.2). In addition, \( v \) and \( \psi \) are linearly dependent and as \( \psi(-S, \lambda) = -1 \) we have that \( v(x, \lambda) = -v(-S, \lambda)\psi(x, \lambda) \). Hence

\[
\psi(x, \lambda) = \frac{v(x, \lambda)}{-v(-S, \lambda)} = \frac{v(x, \lambda)}{\Delta(\lambda)}.
\]

If we also define

\[
\Psi(x, \lambda) = \begin{bmatrix} \psi(x, \lambda) & w_2(x, \lambda) \\ \psi'(x, \lambda) & w'_2(x, \lambda) \end{bmatrix}
\]

then from (4.2) it follows that

\[
\Psi(x, \lambda) = W(x, \lambda) \begin{bmatrix} 1 & 0 \\ m(\lambda) & 1 \end{bmatrix},
\]

for all \( x \), and that \( \det \Psi = \det W = -1 \).
Theorem 4.2  Given the Titchmarsh-Weyl m-function, \( m \), to (1.1) on \([-S,S]\) with double Dirichlet boundary conditions and the transfer condition (1.2) and \( \tilde{m} \), the Titchmarsh-Weyl m-function for the same problem but with the potential \( q \) replaced by \( \tilde{q} \). If \( m = \tilde{m} \) then \( q = \tilde{q} \).

Proof: Let tilde (\( \tilde{\cdot} \)) of any quantity, in what follows, denote the same quantity as previously defined but for the problem with \( q \) replaced by \( \tilde{q} \). Since \( m = \tilde{m} \) the eigenvalues for the problem with potential \( q \) coincide with those for the problem with potential \( \tilde{q} \). Note that since we have self-adjointness the algebraic multiplicity of an eigenvalue equals the geometric multiplicity and in addition all the eigenvalues are simple. Thus \( \Delta(\lambda) \) and \( \tilde{\Delta}(\lambda) \) have the same zeros, all of which are simple.

From the asymptotics given in the Appendix it can be seen that \( \Delta \) is of order \( \frac{1}{2} \) and similarly \( \tilde{\Delta} \) is of order \( \frac{1}{2} \). Therefore, as \( \Delta \) and \( \tilde{\Delta} \) are entire functions of order \( \frac{1}{2} \) with the same zeros, we have that

\[ \Delta = c\tilde{\Delta}. \]

So

\[ c = \frac{\Delta}{\tilde{\Delta}} \]

and taking the limit as \( \lambda \) tends to \( -\infty \) gives that \( c = 1 \). Hence \( \Delta = \tilde{\Delta} \).

We now proceed as in [1, 4]. For \( \Delta(\lambda) \neq 0 \), set

\[ P(x, \lambda) = \Psi \tilde{\Psi}^{-1}(x, \lambda) = \begin{bmatrix} \psi(x, \lambda) & w_2(x, \lambda) \\ \psi'(x, \lambda) & w'_2(x, \lambda) \end{bmatrix} \begin{bmatrix} \tilde{w}_2'(x, \lambda) & -\tilde{w}_2(x, \lambda) \\ -\tilde{\psi}'(x, \lambda) & \tilde{\psi}(x, \lambda) \end{bmatrix}. \]

Since \( m = \tilde{m} \) we have that

\[ \Psi = W \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \tilde{\Psi} = \tilde{W} \begin{bmatrix} 1 \\ m \end{bmatrix}, \]

therefore \( P \) has an analytic extension to the entire function

\[ P = W\tilde{W}^{-1} = \begin{bmatrix} w_1(x, \lambda) & w_2(x, \lambda) \\ w'_1(x, \lambda) & w'_2(x, \lambda) \end{bmatrix} \begin{bmatrix} \tilde{w}_2'(x, \lambda) & -\tilde{w}_2(x, \lambda) \\ -\tilde{w}_1'(x, \lambda) & \tilde{w}_1(x, \lambda) \end{bmatrix}, \]

and \( \det P = 1 \). So

\[
\begin{align*}
P_{11} &= -\psi \tilde{w}_2' + w_2 \tilde{\psi}' \\
&= w_2 \psi' - \psi w'_2 + w_2 (\tilde{\psi}' - \psi') - \psi (\tilde{w}_2' - w'_2) \\
&= 1 + \frac{w_2 (\tilde{v}' - v') - v (\tilde{w}_2' - w'_2)}{\Delta(\lambda)}.
\end{align*}
\]
Similarly
\[ P_{12} = \psi \tilde{w}_2 - w_2 \tilde{\psi} = v \tilde{w}_2 - w_2 \tilde{v} \Delta(\lambda). \]

Using Theorem 5.3 and Theorem 5.4 together with the maximum-modulus principle we obtain that \( P_{11} \equiv 1 \) and \( P_{12} \equiv 0 \).

Hence \( \Psi(x, \lambda) = \tilde{\Psi}(x, \lambda) \) and \( w_2(x, \lambda) = \tilde{w}_2(x, \lambda) \) giving that \( q = \tilde{q} \).

**Theorem 4.3** If \( q \) has bounded essential support, then from the scattering data \( \{ R(\xi), \eta_1, \ldots, \eta_N \} \), the potential \( q \) of the scattering problem on the line with transfer condition at the origin can be reconstructed uniquely.

**Proof:** Let \( \text{ess supp}(q) \subset [-S, S] \) for some \( S > 0 \), then from Lemma 1.1 given the scattering data, we can find \( w_1(S, \xi) \) and \( w_2(S, \xi) \). But \( m = -\frac{w_1(S, \xi)}{w_2(S, \xi)} \) so the Titchmarsh-Weyl \( m \)-function for (1.1) on \([-S, S]\) with double Dirichlet boundary conditions and the transfer condition (1.2) is uniquely determined from the scattering data. Now applying Theorem 4.2 gives that the potential \( q \) is uniquely determined by \( m \) on \([-S, S]\).

To show the uniqueness of \( q \) on the whole real line, assume we have two different potentials \( q \) and \( \hat{q} \) with compact essential support. Let \( S \) be so large that \( \text{ess supp}(q) \cup \text{ess supp}(\hat{q}) \subset [-S, S] \), then, as \( q(x) \) is unique on \([-S, S]\) we have \( q = \hat{q} \) on \([-S, S]\) and thus on \( \mathbb{R} \).  

5 Appendix

From [15] we have the following asymptotics for \(-S \leq x < 0\):

\[ w_2(x, \lambda) = \frac{\sin \sqrt{\lambda}(x + S)}{\sqrt{\lambda}} + O \left( \frac{e^{3\sqrt{\lambda}(x + S)}}{\lambda} \right) \]  
(5.3)

\[ w'_2(x, \lambda) = \cos \sqrt{\lambda}(x + S) + O \left( \frac{e^{3\sqrt{\lambda}(x + S)}}{\sqrt{\lambda}} \right) \]  
(5.4)

and for \( S \geq x > 0 \)

\[ v(x, \lambda) = -\frac{\sin \sqrt{\lambda}(S - x)}{\sqrt{\lambda}} + O \left( \frac{e^{3\sqrt{\lambda}(S - x)}}{\lambda} \right) \]  
(5.5)
\[ v'(x, \lambda) = \cos \sqrt{\lambda}(S - x) + O \left( \frac{e^{\sqrt{\lambda}|S-x|}}{\sqrt{\lambda}} \right). \] (5.6)

Since \( w_2 \) and \( v \) obey the transfer condition (1.2) we obtain the following:

For \( m_{12} \neq 0 \) and \( m_{22} \neq 0 \)

\[ w_2(0^+, \lambda) = m_{12} \cos \sqrt{\lambda}S + O \left( \frac{e^{\sqrt{\lambda}|S|}}{\sqrt{\lambda}} \right) \] (5.7)

\[ w'_2(0^+, \lambda) = m_{22} \cos \sqrt{\lambda}S + O \left( \frac{e^{\sqrt{\lambda}|S|}}{\sqrt{\lambda}} \right), \] (5.8)

For \( m_{12} = 0 \) and \( m_{22} \neq 0 \)

\[ w_2(0^+, \lambda) = m_{11} \frac{\sin \sqrt{\lambda}S}{\sqrt{\lambda}} + O \left( \frac{e^{\sqrt{\lambda}|S|}}{\lambda} \right) \] (5.9)

\[ w'_2(0^+, \lambda) = m_{22} \cos \sqrt{\lambda}S + O \left( \frac{e^{\sqrt{\lambda}|S|}}{\sqrt{\lambda}} \right). \] (5.10)

For \( m_{12} \neq 0 \) and \( m_{22} = 0 \)

\[ w_2(0^+, \lambda) = m_{12} \cos \sqrt{\lambda}S + O \left( \frac{e^{\sqrt{\lambda}|S|}}{\sqrt{\lambda}} \right) \] (5.11)

\[ w'_2(0^+, \lambda) = m_{21} \frac{\sin \sqrt{\lambda}S}{\sqrt{\lambda}} + O \left( \frac{e^{\sqrt{\lambda}|S|}}{\lambda} \right). \] (5.12)

For \( m_{12} \neq 0 \) and \( m_{11} \neq 0 \)

\[ v(0^-, \lambda) = -m_{12} \cos \sqrt{\lambda}S + O \left( \frac{e^{\sqrt{\lambda}|S|}}{\sqrt{\lambda}} \right) \] (5.13)

\[ v'(0^-, \lambda) = m_{11} \cos \sqrt{\lambda}S + O \left( \frac{e^{\sqrt{\lambda}|S|}}{\sqrt{\lambda}} \right). \] (5.14)

For \( m_{12} = 0 \) and \( m_{11} \neq 0 \)

\[ v(0^-, \lambda) = -m_{22} \frac{\sin \sqrt{\lambda}S}{\sqrt{\lambda}} + O \left( \frac{e^{\sqrt{\lambda}|S|}}{\lambda} \right) \] (5.15)
\( v'(0^-, \lambda) = m_{11} \cos \sqrt{\lambda} S + O \left( \frac{e^{i|3\sqrt{\lambda}|S}}{\sqrt{\lambda}} \right) \). \quad (5.16)

For \( m_{12} \neq 0 \) and \( m_{11} = 0 \)

\( v(0^-, \lambda) = -m_{12} \cos \sqrt{\lambda} S + O \left( \frac{e^{i|3\sqrt{\lambda}|S}}{\sqrt{\lambda}} \right) \) \quad (5.17)

\( v'(0^-, \lambda) = m_{21} \frac{\sin \sqrt{\lambda} S}{\sqrt{\lambda}} + O \left( \frac{e^{i|3\sqrt{\lambda}|S}}{\sqrt{\lambda}} \right) \). \quad (5.18)

Thus extending \( w_2 \) we obtain for \( S \geq x > 0 \) that if \( m_{12} \neq 0 \) then

\( w_2(x, \lambda) = m_{12} \cos \sqrt{\lambda} S \cos \sqrt{\lambda} x + O \left( \frac{e^{i|3\sqrt{\lambda}|(S+x)}}{\sqrt{\lambda}} \right) \) \quad (5.19)

\( w'_2(x, \lambda) = -m_{12} (\cos \sqrt{\lambda} S) (\sqrt{\lambda} \sin \sqrt{\lambda} x) + O \left( e^{i|3\sqrt{\lambda}|(S+x)} \right) \). \quad (5.20)

If \( m_{12} = 0 \)

\( w_2(x, \lambda) = m_{11} \frac{\sin \sqrt{\lambda} S}{\sqrt{\lambda}} \cos \sqrt{\lambda} x + m_{22} \frac{\sin \sqrt{\lambda} x}{\sqrt{\lambda}} \cos \sqrt{\lambda} S + O \left( \frac{e^{i|3\sqrt{\lambda}|(S+x)}}{\sqrt{\lambda}} \right) \) \quad (5.21)

\( w'_2(x, \lambda) = -m_{11} \sin \sqrt{\lambda} S \sin \sqrt{\lambda} x + m_{22} \cos \sqrt{\lambda} S \cos \sqrt{\lambda} x + O \left( \frac{e^{i|3\sqrt{\lambda}|(S+x)}}{\sqrt{\lambda}} \right) \). \quad (5.22)

Similarly we can extend \( v \) to obtain for \(-S \leq x < 0 \) that if \( m_{12} \neq 0 \) then

\( v(x, \lambda) = -m_{12} \cos \sqrt{\lambda} S \cos \sqrt{\lambda} x + O \left( \frac{e^{i|3\sqrt{\lambda}|(S-x)}}{\sqrt{\lambda}} \right) \) \quad (5.23)

\( v'(x, \lambda) = m_{12} \sqrt{\lambda} \cos \sqrt{\lambda} S \sin \sqrt{\lambda} x + O \left( \frac{e^{i|3\sqrt{\lambda}|(S-x)}}{\sqrt{\lambda}} \right) \). \quad (5.24)

If \( m_{12} = 0 \)

\( v(x, \lambda) = -m_{22} \frac{\sin \sqrt{\lambda} S}{\sqrt{\lambda}} \cos \sqrt{\lambda} x + m_{11} \frac{\sin \sqrt{\lambda} x}{\sqrt{\lambda}} \cos \sqrt{\lambda} S + O \left( \frac{e^{i|3\sqrt{\lambda}|(S-x)}}{\sqrt{\lambda}} \right) \) \quad (5.25)

\( v'(x, \lambda) = m_{22} \sin \sqrt{\lambda} S \sin \sqrt{\lambda} x + m_{11} \cos \sqrt{\lambda} S \cos \sqrt{\lambda} x + O \left( \frac{e^{i|3\sqrt{\lambda}|(S-x)}}{\sqrt{\lambda}} \right) \). \quad (5.26)
Lemma 5.1 For \( m_{12} = 0 \) on the contour \( \Gamma_k \) given in Figure 1

\[
\frac{1}{|\Delta(\lambda)|} = O(\sqrt{\lambda} e^{-2S|3\sqrt{\lambda}|}). \tag{5.27}
\]

Proof: Note that

\[
\Delta(\lambda) = w_2(S) = \left( m_{11} + \frac{1}{m_{11}} \right) \frac{\sin 2\sqrt{\lambda} S}{2\sqrt{\lambda}} + O \left( \frac{e^{2S|3\sqrt{\lambda}|}}{\lambda} \right).
\]

Consider the contour indicated below for \( k \in \mathbb{N} \).

![Figure 1: \( \Gamma_k \) in the \( S\sqrt{\lambda} \)-plane.](image)

Let \( \lambda = \frac{z^2}{S^2} \) then on \( C_1 \) the variable \( z = i\rho + \frac{\pi}{4} + 2k\pi, -k \leq \rho \leq k \), and

\[
|w_2(S)| = \left| \left( m_{11} + \frac{1}{m_{11}} \right) \frac{S \sin (2i\rho + \frac{\pi}{2})}{2i\rho + \frac{\pi}{4} + 2k\pi} + O \left( \frac{e^{2|\rho|}}{k^2} \right) \right|
\]

\[
\geq \left| \left( m_{11} + \frac{1}{m_{11}} \right) \frac{S}{6k\pi} |\cos (2i\rho)| + O \left( \frac{e^{2|\rho|}}{k^2} \right) \right|
\]

\[
\geq \left| \left( m_{11} + \frac{1}{m_{11}} \right) \frac{S}{12k\pi} e^{2|\rho|} + O \left( \frac{e^{2|\rho|}}{k^2} \right) \right|.
\]

Therefore

\[
\frac{1}{|w_2(S)|} \leq \frac{12k\pi}{S} \left| \frac{m_{11}}{1 + m_{11}^2} e^{-2|\rho|} + O(e^{-2|\rho|}) = O(\sqrt{\lambda} e^{-2S|3\sqrt{\lambda}|}) \right|.
\]
Considering $C_2$ and $C_3$ let $z = \pm ik + t$, where $t \in [0, \frac{\pi}{4} + 2k\pi]$ then for $|k|$ large

$$|w_2(S)| = \left| \left( m_{11} + \frac{1}{m_{11}} \right) \frac{S \sin(\pm 2ikt + 2t)}{\pm ik + t} \right| + O \left( \frac{e^{2|k|}}{k^2} \right)$$

$$= \left| \left( m_{11} + \frac{1}{m_{11}} \right) \frac{e^{\pm 2k + 2t} - e^{\pm 2k - 2it}}{2i(\pm ik + t)} \right| \frac{S}{2} + O \left( \frac{e^{2|k|}}{k^2} \right)$$

$$\geq \left| \left( m_{11} + \frac{1}{m_{11}} \right) \frac{S}{2(\pm ik + t)} \right| e^{2|k|} \frac{2}{4} + O \left( \frac{e^{2|k|}}{k^2} \right).$$

Thus

$$\frac{1}{|w_2(S)|} = O(\sqrt{\lambda}e^{-2S|\sqrt{\lambda}|}).$$

Lemma 5.2 For $m_{12} \neq 0$, on the contour $\Upsilon_k$ given in Figure 2,

$$\frac{1}{|\Delta(\lambda)|} = O(e^{-2S|\sqrt{\lambda}|}). \quad (5.28)$$

Proof: In this case

$$\Delta(\lambda) = w_2(S) = m_{12} \cos^2 \sqrt{\lambda} S + O \left( \frac{e^{2S|\sqrt{\lambda}|}}{\sqrt{\lambda}} \right).$$

Hence we consider the contour indicated below for $k \in \mathbb{N}$.

![Figure 2: $\Upsilon_k$ in the $S\sqrt{\lambda}$-plane.](image)
Again, let $\lambda = \frac{z_2^2}{S^2}$ then on $U_1$ the variable $z = i\rho + 2k\pi, -k \leq \rho \leq k$, and

$$|w_2(S)| = m_{12} \cos^2(i\rho + 2k\pi) + O\left(\frac{e^{2|\rho|}}{k}\right)$$

$$= m_{12} \left(\frac{e^{-\rho} + e^\rho}{2}\right)^2 + O\left(\frac{e^{2|\rho|}}{k}\right)$$

$$\geq m_{12} \frac{e^{2|\rho|}}{4} + O\left(\frac{e^{2|\rho|}}{k}\right).$$

Therefore

$$\frac{1}{|w_2(S)|} = O\left(e^{-2S|\sqrt{\lambda}|}\right).$$

Similarly on $U_2$ and $U_3$ set $z = \pm ik + t$ for $t \in [0, 2k\pi]$. Then for large $|k|$ we have that

$$|w_2(S)| = m_{12} \cos^2(\pm ik + t) + O\left(\frac{e^{2|k|}}{k}\right)$$

$$= m_{12} \left(\frac{e^{\pm k + it} + e^{\pm k - it}}{2}\right)^2 + O\left(\frac{e^{2|k|}}{k}\right)$$

$$\geq m_{12} \frac{e^{2|k|}}{4} + O\left(\frac{e^{2|k|}}{k}\right)$$

giving that

$$\frac{1}{|w_2(S)|} = O\left(e^{-2S|\sqrt{\lambda}|}\right).$$

**Theorem 5.3** On the contours $\Gamma_k$ (for $m_{12} = 0$) and $\Upsilon_k$ (for $m_{12} \neq 0$) given in Figure 1 and Figure 2 respectively

$$\frac{\nu \tilde{w}_2 - i\tilde{w}_2}{|\Delta(\lambda)|} = O\left(\frac{1}{\lambda}\right), \quad \text{as} \quad k \to \infty.$$ 

**Proof:** We need to consider four cases. Firstly if $m_{12} = 0$ and $x > 0$
then

\[
\begin{align*}
v \tilde{w}_2 - \tilde{v}w_2 &= \left( \frac{-\sin \sqrt{\lambda}(S - x)}{\sqrt{\lambda}} + O \left( \frac{e^{3\sqrt{\lambda}(S-x)}}{\lambda} \right) \right) \\
&\times \left( m_{11} \frac{\sin \sqrt{\lambda}S}{\sqrt{\lambda}} \cos \sqrt{\lambda}x + m_{22} \frac{\sin \sqrt{\lambda}x}{\sqrt{\lambda}} \cos \sqrt{\lambda}S + O \left( \frac{e^{3\sqrt{\lambda}(S-x)}}{\lambda} \right) \right) \\
&- \left( \frac{-\sin \sqrt{\lambda}(S - x)}{\sqrt{\lambda}} + O \left( \frac{e^{3\sqrt{\lambda}(S-x)}}{\lambda} \right) \right) \\
&\times \left( m_{11} \frac{\sin \sqrt{\lambda}S}{\sqrt{\lambda}} \cos \sqrt{\lambda}x + m_{22} \frac{\sin \sqrt{\lambda}x}{\sqrt{\lambda}} \cos \sqrt{\lambda}S + O \left( \frac{e^{3\sqrt{\lambda}(S-x)}}{\lambda} \right) \right) \\
&= O \left( \frac{e^{2S|3\sqrt{\lambda}|}}{\lambda^{3/2}} \right).
\end{align*}
\]

Hence using Lemma 5.1 we obtain the result for this particular case.

Next consider \( m_{12} = 0 \) and \( x < 0 \) then

\[
\begin{align*}
v \tilde{w}_2 - \tilde{v}w_2 &= \left( \frac{-\sin \sqrt{\lambda}(S - x)}{\sqrt{\lambda}} + O \left( \frac{e^{3\sqrt{\lambda}(S-x)}}{\lambda} \right) \right) \\
&\times \left( m_{22} \frac{\sin \sqrt{\lambda}S}{\sqrt{\lambda}} \cos \sqrt{\lambda}x + m_{11} \frac{\sin \sqrt{\lambda}x}{\sqrt{\lambda}} \cos \sqrt{\lambda}S + O \left( \frac{e^{3\sqrt{\lambda}(S-x)}}{\lambda} \right) \right) \\
&- \left( \frac{-\sin \sqrt{\lambda}(S - x)}{\sqrt{\lambda}} + O \left( \frac{e^{3\sqrt{\lambda}(S-x)}}{\lambda} \right) \right) \\
&\times \left( m_{22} \frac{\sin \sqrt{\lambda}S}{\sqrt{\lambda}} \cos \sqrt{\lambda}x + m_{11} \frac{\sin \sqrt{\lambda}x}{\sqrt{\lambda}} \cos \sqrt{\lambda}S + O \left( \frac{e^{3\sqrt{\lambda}(S-x)}}{\lambda} \right) \right) \\
&= O \left( \frac{e^{2S|3\sqrt{\lambda}|}}{\lambda^{3/2}} \right).
\end{align*}
\]

Again using Lemma 5.1 gives the required result for \( x < 0 \).

So far we have shown that for \( m_{12} = 0 \)

\[
\begin{align*}
\frac{v \tilde{w}_2 - \tilde{v}w_2}{|\Delta(\lambda)|} &= O \left( \frac{1}{\lambda} \right).
\end{align*}
\]

22
It remains to show that the result holds for \( m_{12} \neq 0 \). Again we will consider \( x > 0 \) and \( x < 0 \) separately. Let \( x > 0 \) then

\[
\begin{align*}
    v\tilde{w}_2 - \tilde{v}w_2 &= \left( -\sin \sqrt{\lambda}(S - x) \right) \left( m_{12} \cos \sqrt{\lambda}S \cos \sqrt{\lambda}x + O \left( \frac{e^{\sqrt{\lambda}|S|}}{\sqrt{\lambda}} \right) \right) + O \left( \frac{e^{\sqrt{\lambda}|S - x|}}{\sqrt{\lambda}} \right) \\
    &= \left( -\sin \sqrt{\lambda}(S - x) \right) \left( m_{12} \cos \sqrt{\lambda}S \cos \sqrt{\lambda}x + O \left( \frac{e^{\sqrt{\lambda}|S - x|}}{\sqrt{\lambda}} \right) \right) + O \left( \frac{e^{\sqrt{\lambda}|S|}}{\sqrt{\lambda}} \right) \\
    &= \left( m_{12} \cos \sqrt{\lambda}S \cos \sqrt{\lambda}x + O \left( \frac{e^{\sqrt{\lambda}|S - x|}}{\sqrt{\lambda}} \right) \right) \left( -\sin \sqrt{\lambda}(S - x) \right).
\end{align*}
\]

The result now follows by Lemma 5.2.

Lastly if \( m_{12} \neq 0 \) and \( x < 0 \) then

\[
\begin{align*}
    v\tilde{w}_2 - \tilde{v}w_2 &= \left( -\sin \sqrt{\lambda}(S - x) \right) \left( m_{12} \cos \sqrt{\lambda}S \cos \sqrt{\lambda}x + O \left( \frac{e^{\sqrt{\lambda}|S - x|}}{\sqrt{\lambda}} \right) \right) + O \left( \frac{e^{\sqrt{\lambda}|S|}}{\sqrt{\lambda}} \right) \\
    &= \left( -\sin \sqrt{\lambda}(S - x) \right) \left( m_{12} \cos \sqrt{\lambda}S \cos \sqrt{\lambda}x + O \left( \frac{e^{\sqrt{\lambda}|S - x|}}{\sqrt{\lambda}} \right) \right) + O \left( \frac{e^{\sqrt{\lambda}|S|}}{\sqrt{\lambda}} \right) \\
    &= \left( m_{12} \cos \sqrt{\lambda}S \cos \sqrt{\lambda}x + O \left( \frac{e^{\sqrt{\lambda}|S - x|}}{\sqrt{\lambda}} \right) \right) \left( -\sin \sqrt{\lambda}(S - x) \right).
\end{align*}
\]

Lemma 5.2 again leads to the required result. This completes all four cases.

Similarly we now prove the following theorem.

**Theorem 5.4** On the contours \( \Gamma_k \) (for \( m_{12} = 0 \)) and \( \Upsilon_k \) (for \( m_{12} \neq 0 \)) given in Figure 1 and Figure 2 respectively

\[
\begin{align*}
    \frac{w_2'(\tilde{v}' - v') - v(\tilde{w}_2' - w_2')}{|\Delta(\lambda)|} &= O \left( \frac{1}{\sqrt{\lambda}} \right) \quad \text{as} \quad k \to \infty.
\end{align*}
\]

**Proof:** We again need to consider the same four cases as in Theorem 23.
5.3 So we start with \( m_{12} = 0 \) and \( x > 0 \). In this case

\[
w_2(\tilde{v}' - v') - v(\tilde{w}'_2 - w'_2)
= \left( m_{11} \frac{\sin \sqrt{\lambda} S}{\sqrt{\lambda}} \cos \sqrt{\lambda} x + m_{22} \frac{\sin \sqrt{\lambda} x}{\sqrt{\lambda}} \cos \sqrt{\lambda} S + O \left( \frac{e^{\sqrt{\lambda} |(S+x)|}}{\sqrt{\lambda}} \right) \right) \left( O \left( \frac{e^{\sqrt{\lambda} |(S-x)|}}{\sqrt{\lambda}} \right) \right)
- \left( -\frac{\sin \sqrt{\lambda} (S - x)}{\sqrt{\lambda}} + O \left( \frac{e^{\sqrt{\lambda} |(S-x)|}}{\sqrt{\lambda}} \right) \right) \left( O \left( \frac{e^{\sqrt{\lambda} |(S-x)|}}{\sqrt{\lambda}} \right) \right)
= O \left( \frac{e^{2\sqrt{\lambda} |3\sqrt{\lambda}|}}{\lambda} \right).
\]

Thus by Lemma 5.1 the result holds. Moreover for \( x < 0 \)

\[
w_2(\tilde{v}' - v') - v(\tilde{w}'_2 - w'_2)
= \left( \sin \frac{\sqrt{\lambda} (x + S)}{\sqrt{\lambda}} + O \left( \frac{e^{\sqrt{\lambda} |(x+S)|}}{\sqrt{\lambda}} \right) \right) \left( O \left( \frac{e^{\sqrt{\lambda} |(x-S)|}}{\sqrt{\lambda}} \right) \right)
- \left( -m_{22} \frac{\sin \sqrt{\lambda} S}{\sqrt{\lambda}} \cos \sqrt{\lambda} x + m_{11} \frac{\sin \sqrt{\lambda} x}{\sqrt{\lambda}} \cos \sqrt{\lambda} S + O \left( \frac{e^{\sqrt{\lambda} |(S-x)|}}{\sqrt{\lambda}} \right) \right) \left( O \left( \frac{e^{\sqrt{\lambda} |(S-x)|}}{\sqrt{\lambda}} \right) \right)
= O \left( \frac{e^{2\sqrt{\lambda} |3\sqrt{\lambda}|}}{\lambda} \right),
\]

By Lemma 5.1 the case of \( m_{12} = 0 \) is complete.

Now assume that \( m_{12} \neq 0 \) and \( x > 0 \). Then

\[
w_2(\tilde{v}' - v') - v(\tilde{w}'_2 - w'_2)
= \left( m_{12} \cos \sqrt{\lambda} S \cos \sqrt{\lambda} x + O \left( \frac{e^{\sqrt{\lambda} |(S+x)|}}{\sqrt{\lambda}} \right) \right) \left( O \left( \frac{e^{\sqrt{\lambda} |(S-x)|}}{\sqrt{\lambda}} \right) \right)
- \left( -\frac{\sin \sqrt{\lambda} (S - x)}{\sqrt{\lambda}} + O \left( \frac{e^{\sqrt{\lambda} |(S-x)|}}{\sqrt{\lambda}} \right) \right) \left( O \left( \frac{e^{\sqrt{\lambda} |(S+x)|}}{\sqrt{\lambda}} \right) \right)
= O \left( \frac{e^{2\sqrt{\lambda} |3\sqrt{\lambda}|}}{\lambda} \right),
\]

from which the result follows using Lemma 5.2.
Lastly let \( m_{12} \neq 0 \) and \( x < 0 \). Then
\[
\begin{align*}
\ w_2(v' - v'' - v(\bar{w}_2' - w_2')) & = \left( \frac{\sin \sqrt{\lambda} (x + S)}{\sqrt{\lambda}} + O \left( \frac{e^{\sqrt{\lambda} |x+S|}}{\lambda} \right) \right) \left( O \left( \frac{e^{\sqrt{\lambda} |x+S|}}{\lambda} \right) \right) \\
- & \left( -m_{12} \cos \sqrt{\lambda} S \cos \sqrt{\lambda} x + O \left( \frac{e^{\sqrt{\lambda} |x+S|}}{\sqrt{\lambda}} \right) \right) \left( O \left( \frac{e^{\sqrt{\lambda} |x+S|}}{\sqrt{\lambda}} \right) \right) \\
= & \ O \left( \frac{e^{2S|\sqrt{\lambda}|}}{\sqrt{\lambda}} \right).
\end{align*}
\]
Again using Lemma 5.2 concludes the proof. ■

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