MAGNETOHYDRODYNAMICS'S TYPE EQUATIONS
OVER CLIFFORD ALGEBRAS

IGOR KONDRASHUK∗†, EDUARDO A. NOTTE-CUELLO‡§ and MARKO A. ROJAS-MEDAR∗∥
∗Grupo de Matemática Aplicada
Dpto. de Ciencias Básicas, Facultad de Ciencias
Universidad del Bío-Bío, Campus Fernando May
Casilla 447, Chillán, Chile
†Dpto. de Matemáticas, Facultad de Ciencias
Universidad de La Serena
Av. Cisternas 1200, La Serena, Chile
‡igor.kondrashuk@ubiobio.cl
§marko@ubiobio.cl
∥enotte@userena.cl

Received 1 October 2009
Accepted 3 March 2010

We study a system of equations modeling the stationary motion of incompressible electrical conducting fluid. Based on methods of Clifford analysis, we rewrite the system of magnetohydrodynamics fluid in the hypercomplex formulation and represent its solution in Clifford operator terms.

Keywords: Clifford algebras; Clifford analysis; Dirac operator; magnetohydrodynamics’s type equations; fluid equations.

1. Introduction
In several situations the motion of incompressible electrical conducting fluid can be modeled by so called equations of magnetohydrodynamics, which correspond to the Navier–Stokes’ equations coupled to the Maxwell’s equations. In the case where there is free motion of heavy ions, not directly due to the electric field (see Schlüter [26] and Pikeln[23]), these equations can be reduced to the following form:

\[-\frac{2}{\rho} \Delta u^* + u^* \cdot \nabla u^* - \frac{1}{\rho} h^* \nabla h^* = f^* - \frac{1}{\rho} \nabla \left( p^* + \frac{\mu}{2} h^* \right),\]

\[-\frac{1}{\mu \sigma} \Delta h^* + u^* \cdot \nabla h^* - h^* \cdot \nabla u^* = -\nabla w^*,\]

\[\text{div} u^* = 0,\]

\[\text{div} h^* = 0,\]

\[u^*|_{\partial \Omega} = 0, \quad u^*|_{\partial \Omega} = 0.\]

337
In these equations we have assumed homogeneous boundary conditions just for simplicity. In standard ways we could treat the non-homogeneous case. Here, $u^*$ and $h^*$ are respectively the unknown velocity and magnetic fields; $\rho^*$ is the unknown hydrostatic pressure; $w^*$ is an unknown function related to the motion of heavy ions (in such way that the density of electric current, $j_0$, generated by this motion satisfies the relation $\text{rot} j_0 = -\sigma \nabla w^*$); $\mu > 0$ is the constant magnetic permeability of the medium; $\sigma > 0$ is the constant electric conductivity; $\eta > 0$ is the constant viscosity of the fluid; $\Phi^*$ is an given external force field.

In this paper, we will consider the problem of existence and uniqueness of strong solutions for the problem (1.1) in a unbounded domain $\Omega$ of $\mathbb{R}^3$. It is appropriate to remind earlier works on the initial-value problems closely related to (1.1) in order to clarify the intended contribution of the present work. The stationary problem corresponding to (1.1) was considered by Chizhonkov [7], while the question of the (local) existence of a solution of the evolution problems was analyzed by Lassner [19], making use of semigroup techniques similar to ones in Fujita and Kato [10]. The more constructive spectral Galerkin method was used by Boldrini and Rojas-Medar [3] to obtain local-time strong solutions. Also, by using this same method in [24] the existence and uniqueness of periodic strong solutions for the magnetohydrodynamics’s type equations have been studied.

By other hand, the nonlinear partial differential equations of mathematical physics have been a little studied by Clifford analysis, in particular, the equations of fluid mechanics. In fact, one of the pioneer work of boundary value problems for elliptic partial differential equations, such as the Stokes and Navier–Stokes equations in bounded domains, was done by Gürlebeck and Sprössing [12], where they made use of certain orthogonal decomposition of the underlying function space in which one of the subspace is the space of null solutions of the corresponding Dirac operator. This approach was later extended to unbounded domains in [11], in particular Cerejeiras and Kähler [4], studied the Stokes operator by means of the Clifford analysis, and after they applied their results to the stationary Navier–Stokes equations. The main argument is the linearization of the nonlinear equations, where they applied in each iteration the results for linear Stokes equations and used the argument of Banach principle of fix point. Later, Kondrashuk, Notte-Cuello and Rojas-Medar [17], using similar ideas, studied the stationary nonlinear equations for asymmetric fluids.

The paper is organize as follows. In Sec. 2 we present some preliminaries about the Clifford algebras of multivectors in a more general context and make transparent the nature of all the objects involved. In Sec. 3 we write the magnetohydrodynamics’s equation over the Clifford formalism, recall some theorems and operators from Clifford analysis and represent the solutions of our equation in terms these operators. In Sec. 4 we present conclusions and comments.

2. Preliminaries over the Clifford Algebra Approach

Let $V$ be a vector space over the real field $\mathbb{R}$ of finite dimension, i.e., $\dim V = n, n \in \mathbb{N}$. By $V^*$ we denote the dual space of $V$. We remind that the space of $k$-tensors (denoted $T_k(V^*)$) are the set of all $k$-linear mappings $\tau_k$ such that

$$\tau_k : V^* \times \cdots \times V^* \rightarrow \mathbb{R}$$
The Clifford algebra $\mathcal{C}$

From here we get the standard relation characterizing the Clifford algebra $\mathcal{C}$ of a metric vector space $(V, g)$ defined as the quotient algebra

$$\mathcal{C}(V, g) = \frac{T(V)}{J_g},$$

where $J_g \subset T(V)$ is the bilateral ideal of $T(V)$ generated by the elements of the form $u \otimes v + v \otimes u = 2g(u, v)$, with $u, v \in V \subset T(V).$ The elements of $\mathcal{C}(V, g)$ are sometime called Clifford numbers.

Let $\rho_2 : T(V) \to \mathcal{Cl}(V, g)$ be the natural projection of $T(V)$ onto the quotient algebra $\mathcal{Cl}(V, g)$; Multiplication in $\mathcal{Cl}(V, g)$ is called Clifford product and defined as

$$AB = \rho_2(A \otimes B),$$

for all $A, B \in \mathcal{Cl}(V, g).$ In particular, for $u, v \in V \subset \mathcal{Cl}(V, g)$, we have

$$u \otimes v = \frac{1}{2}(u \otimes v - v \otimes u) + g(u, v) + \frac{1}{2}(u \otimes v + v \otimes u) - g(u, v)$$

and then

$$\rho_2(u \otimes v) \equiv uv = \frac{1}{2}(u \otimes v - v \otimes u) + g(u, v) = u \wedge v + g(u, v).$$

From here we get the standard relation characterizing the Clifford algebra $\mathcal{Cl}(V, g)$,

$$uv + vu = 2g(u, v).$$

In that follows we take $V = \mathbb{R}^n$, and we denote by $\mathbb{R}^{p,q}$ ($n = p + q$) the real vector space $\mathbb{R}^n$ endowed with a non-degenerated metric $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that, if $\{e_i\}$, $(i = 1, 2, \ldots, n)$ is an orthonormal basis of $\mathbb{R}^{p,q}$, we have

$$g(e_i, e_j) = g_{ij} = \begin{cases} +1, & i = j = 1, \ldots, p \\ -1, & i = j = p + 1, \ldots, p + q = n \\ 0, & i \neq j. \end{cases}$$

The Clifford algebra $\mathcal{Cl}(\mathbb{R}^{p,q}) = \mathbb{R}_{p,q} = \mathcal{Cl}_{p,q}$ is the Clifford algebra over $\mathbb{R}$, generated by 1 and the $\{e_i\}$, $(i = 1, 2, \ldots, n)$ such that $e_i^2 = g(e_i, e_j)$, $e_ie_j = -e_je_i$ ($i \neq j$), and $e_A = e_1e_2 \cdots e_n \neq \pm 1$.

Therefore the universal Clifford algebra $\mathcal{Cl}_{p,q}$ has the dimension $2^n$. Henceforth, each element $a \in \mathcal{Cl}_{p,q}$ shall be written in the form

$$a = \sum a_A e_A$$

where the coefficients $a_A$ are real numbers. For details on this formalism see [25].
3. Magnetohydrodynamics’s Type Equations over Clifford Formalism

Let $\Omega \subset \mathbb{R}^n$ and $\Gamma = \partial \Omega$. Then functions $u$ defined in $\Omega$ with values in $C^{0,0}_{1,0} (p = 0$ and $q = n)$ are considered. These functions may be written as

$$u(x) = \sum_A e_A u_A(x), \quad x \in \Omega,$$

where $u_A(x) \in \mathbb{R}$. Properties such as continuity, differentiability, integrability, and so on, which are ascribed to $u$ have to be possessed by all components $u_A(x)$. In this way, the usual Banach space of these functions are denoted by $C^{0,0} (\Omega, C_{0,0})$, $\mathcal{L}_q (\Omega, C_{0,0})$, and $W^k_0 (\Omega, C_{0,0})$ or in abbreviated form $C^{0} (\Omega)$, $\mathcal{L}_q (\Omega)$ and $W^k_0 (\Omega)$.

Let us now introduce the Dirac operator by

$$D = \sum_{k=1}^n \frac{\partial}{\partial x_k}$$

is easy prove that $D^2 = -\Delta$, where $\Delta$ is the Laplacian operator.

We remind that the subspace of $\mathcal{C}_{1,0}^0$ generated by the basic element $e_A$ with equal length $k$ is denoted by $\mathcal{C}_{k,0}^0$. Its elements are called $k$-vectors. It follows that $\mathcal{C}_{1,0}^0$ is isomorphic to $\mathbb{R}^n$ ($\mathcal{C}_{1,0}^0 \cong \mathbb{R}^n$). In this sense, we can identify each vector $u(x) \in \mathbb{R}^n$ with

$$u(x) = u_1(x)e_1 + \cdots + u_n(x)e_n \in \mathcal{C}_{1,0}^0 \hookrightarrow \mathcal{C}_{0,0}.$$

Then we can calculate $Du(x)$ when $u(x) \in \mathcal{C}_{1,3}^0 \hookrightarrow \mathcal{C}_{0,3}$, as follows

$$Du(x) = \text{Sc}(Du) + \text{biv}(Du),$$

where $\text{Sc}(Du)$ and $\text{biv}(Du)$ are the scalar part and bivector part of $Du$, respectively.

3. Magnetohydrodynamics’s Type Equations over Clifford Formalism

We consider the stationary magnetohydrodynamical systems (1.1) for $u^*, h^* : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ and $p^*, u^* : \mathbb{R}^3 \rightarrow \mathbb{R}$, which is written as

$$-\Delta u^* + \frac{\mu}{\eta} (u^* \cdot \nabla) u^* - \frac{\mu}{\eta} (h^* \cdot \nabla) h^* + \frac{1}{\eta} \nabla p^* = \frac{\beta - \mu}{2\eta} u^*,$$

$$-\Delta h^* + \mu (u^* \cdot \nabla) h^* - \mu (h^* \cdot \nabla) u^* = -\mu \sigma \text{grad} \, u^*, \quad (3.1)$$

$$\text{div} \, u^* = 0; \quad \text{div} \, h^* = 0,$$

where $\pi^* = p^* + \frac{\beta}{\eta} h^2$ and we set $\frac{\mu}{\eta} \pi^* \vec{e}^*$ instead of $\frac{\mu}{\eta} \vec{e}^*$ to facilitate the computations.

For this system we consider the following boundary conditions

$$u^*(x) = 0, \quad h^*(x) = 0 \quad \text{on} \quad \partial \Omega = \Gamma. \quad (3.2)$$

Now, we can write the system (3.1)–(3.2) in the Clifford formalism with

$$u(x), h(x), \vec{f}(x) \in \mathcal{C}_{1,3}^0 \hookrightarrow \mathcal{C}_{0,3}.$$
as

\[
DDu + \frac{\mu}{\eta} M(u) - \frac{\mu}{\eta} M(h) + \frac{1}{\eta} D\pi = 0 \\
DDh + \mu \sigma N(u, h) - \mu \sigma N(h, u) = Dw
\]

(3.3)

where \( \pi = p + \frac{\mu}{2} h^2 \), \(-\Delta = DD = D^2 \) and \( M(u), N(u, h) \) are the operators defined by

\[
M(u) = \left[ Sc(uD) \right] u - \bar{f}/2; \quad N(u, h) = \left[ Sc(uD) \right] h.
\]

3.1. Operators from Clifford analysis

Now, we recall without proof the theorems and operators considered, by example in [4,11,12]. Let a fixed point \( z \) lying in the complement of the closure of \( \Omega \), which contains a non-empty open set. Then we can consider the operator

\[
\tilde{T}f(y) = \int_{\Omega} K_z(x, y) f(x) d\Omega_x,
\]

(3.4)

with \( K_z(x, y) = G(x - y) - G(x - z) \), and where \( G(x) \) is the so-called generalized Cauchy kernel, the Green function of the Dirac operator. Due to the property that \( G(x) \) is a fundamental solution of \( D \), we have that \( K_z(x, y) \) is a monogenic function for \( x \in \Omega \), then \( \tilde{T}f(y) = f(y) \) for \( f \in L^q(\Omega), 1 < q < \infty \). The operator given in (3.4) is a continuous mapping of \( W^k_q(\Omega) \) in \( W^{k+1}_q(\Omega) \), \( 1 < q < \infty \), \( k = 0, 1, ... \) and is bounded operator of \( W^{-1}_q(\Omega) \) in \( L^q(\Omega), 1 < q < \infty \), for details see [4,11,12].

Theorem 1 (Borel-Pompeiu’s formula). If \( f \in W^1_q(\Omega), 1 < q < \infty \), then we have

\[
\tilde{F}_\Gamma f = f - \tilde{T}Df,
\]

with

\[
\tilde{F}_\Gamma f = \int_{\Gamma} K_z(x, y) \alpha(x) f(x) d\Gamma_x
\]

where \( \alpha(x) \) is the outward pointing normal unit vector to \( \Gamma \) at the point \( x \).

Proposition 1. If \( k \in \mathbb{N} \) then the operator

\[
\tilde{F}_k : W^{k-1}_q(\Gamma) \to W^k_q(\Omega) \cap \ker D
\]

is a continuous operator.

Theorem 2 (Plemelj-Sokhotzki’s formula). If \( f \in W^1_q(\Gamma), 1 < q < \infty, l > 0 \), then we have

\[
\text{tr} \tilde{F}_l f = \frac{1}{q} \tilde{F}_l f + \frac{1}{q} \tilde{S}_l f,
\]

whereby

\[
\tilde{S}_l f = 2 \int_{\Gamma} K_z(x, y) \alpha(x) f(x) d\Gamma_x
\]

is the singular integral operator of Cauchy type over the boundary.
Theorem 3. The space \( L^q(\Omega) \), \( 1 < q < \infty \), allows the direct decomposition

\[
L^q(\Omega) = \ker D(\Omega) \cap L^q(\Omega) \oplus D \left( W^{1,q}_0(\Omega) \right).
\]

The above theorem allows to obtain the projections

\[
P : L^q(\Omega) \to \ker D(\Omega) \cap L^q(\Omega)
\]
and

\[
Q : L^q(\Omega) \to D \left( W^{1,q}_0(\Omega) \right),
\]

for \( q = 2 \) these projections are orthoprojections. It was also proven in [4]

\[
Qf = D\Delta^{-1}Df
\]

where \( \Delta^{-1} \), is the solution operator of the Dirichlet problem of the Poisson equation with homogeneous boundary data

\[
-\Delta u = f \quad \text{in} \quad \Omega,
\]
\[
u = 0 \quad \text{on} \quad \Gamma
\]

for \( f \in W^{1,q}_0(\Omega) \), \( 1 < q < \infty \).

Theorem 4. Suppose \( f \in W^{1,q}_0(\Omega) \), \( \pi \in L^q(\Omega, \mathbb{R}) \), \( 1 < q < \infty \); then any solution of the system (3.3) has the representation

\[
\begin{align*}
\hat{u} + \frac{1}{q} Q\hat{T}M(u) - \frac{\mu}{\eta} Q\hat{T}M(h) + \frac{1}{\eta} \hat{T} \pi &= 0 \\
h + \mu \hat{T}Q\hat{T}N(u, h) - \mu \hat{T}Q\hat{T}N(h, u) - \mu \hat{T} Qw &= 0
\end{align*}
\]

(3.5)

Proof. These equations follow the decomposition of the space \( L^q(\Omega) \), see for example [4,16]. In fact, recall that \( Qf = D\Delta^{-1}Df \) and \( \hat{T}f(y) = f(y) \), and the Borel-Pompeiu’s formula imply

\[
\hat{T}D\hat{u} = u - \hat{f} + u, \quad u \in W^{1,q}_0(\Omega)
\]

thus, we can write

\[
\hat{T}\hat{Q}\hat{T}D\hat{u} = \hat{T}(D\Delta^{-1}D)\hat{T}D\hat{u} = \hat{T}D\Delta^{-1}DD\hat{u} = \hat{T}D\hat{u} = u.
\]

(3.6)

Then by applying the \( \hat{T}\hat{Q}\hat{T} \) operator to system (3.3) and using the formula (3.6) we obtain the expected result.
Lemma 1. (1) Let $q/2 < q < \infty$. Then the operator $M: \mathcal{W}_q^1(\Omega) \to \mathcal{W}_{q+1}^1(\Omega)$ is a continuous operator and we have

$$\|\tilde{S}(uD)\|_{\mathcal{W}_{q+1}^1(\Omega)} \leq C_2 \|u\|_{\mathcal{W}_q^1(\Omega)}^2.$$ 

(2) Let $q/2 < q < \infty$. Then the operator $N: \mathcal{W}_q^1(\Omega) \times \mathcal{W}_q^1(\Omega) \to \mathcal{W}_{q+1}^1(\Omega)$ is a continuous operator and we have

$$\|\tilde{S}(uD)|h|\|_{\mathcal{W}_{q+1}^1(\Omega)} \leq C_2 \|u\|_{\mathcal{W}_q^1(\Omega)} \|h\|_{\mathcal{W}_q^1(\Omega)}.$$ 

Proof. The proof of (1) appear in [4] and (2) is similar. Let $L_q(\Omega) \hookrightarrow W_{q+1}^1(\Omega)$, if $r = q/(n+q)$ and suppose $\frac{1}{r} + \frac{1}{s} = 1$, then the Hölder’s inequality results in

$$\int_{\Omega} |u^j_h|^{s+1} \, d\Omega \leq \|u^j_h\|_{L_q(\Omega)} \|u^j_h\|_{L_{r+1}(\Omega)} \leq \|u^j_h\|_{L_{r+1}(\Omega)}.$$ 

From $q = tr$ we have $t = (n+q)/n, s = (n+q)/q$ and $sr = n$. Now, due to the embedding $\mathcal{W}_q^1(\Omega) \hookrightarrow L_n(\Omega)$ for $q > n/2$ we can write

$$\int_{\Omega} |u^j_h|^{s+1} \, d\Omega \leq C_1 \|u^j_h\|_{\mathcal{W}_q^1(\Omega)} \|h\|_{\mathcal{W}_q^1(\Omega)},$$

where $C_1$ is a constant.

On the other hand, we have the following estimates:

$$\|Du\|_{L_q(\Omega)} + \frac{1}{\eta} \|Q\|_{L_{r+1}(\Omega)} \leq C \left\| \frac{\nu}{\eta} \tilde{T} M(u) - \frac{1}{\eta} \tilde{T} M(h) \right\|_{L_{r+1}(\Omega)}$$

and

$$\|Dh\|_{L_q(\Omega)} + \mu \sigma \|Q\|_{L_{r+1}(\Omega)} \leq C \|\mu \sigma \tilde{T} N(u, h) - \mu \sigma \tilde{T} N(h, u)\|_{L_{r+1}(\Omega)}.$$

This norm estimate gives us the possibility to solve our problem by iteration

$$u_i = \frac{\nu}{\eta} \tilde{T} Q \tilde{T} M(h_{i-1}) - \frac{1}{\eta} \tilde{T} Q \tilde{T} M(u_{i-1}) - \frac{1}{\eta} \tilde{T} Q \pi_i,$$

$$h_i = \mu \sigma \tilde{T} Q \tilde{T} N(h_i, u_i) - \mu \sigma \tilde{T} Q \tilde{T} N(u_i, h_i) + \mu \sigma \tilde{T} Q w_i$$

(3.7)

Now, we will prove the convergence of the iterative method. From the first equation of (3.7) we obtain

$$\|u_i - u_{i-1}\|_{W_{q+1}^1(\Omega)} \leq \frac{\nu}{\eta} \|\tilde{T} Q \tilde{T} (M(u_{i-1}) - M(u_{i-2}))\|_{W_{q+1}^1(\Omega)} + \frac{1}{\eta} \|\tilde{T} Q (\pi_{i-1})\|_{W_{q+1}^1(\Omega)} \|$$
and by making use of third equation of \((3.7)\), we obtain
\[
\|u_i - u_{i-1}\|_{W_i^1(\Omega)} \leq 2C_1\|M(u_{i-1}) - M(u_{i-2})\|_{W_i^1(\Omega)}
\]
where
\[
C_1 = \left(\frac{L}{\eta}\right)\|\tilde{T}\|_{L^\infty(Q,T)}\|Q\|_{L^\infty(Q,T)}\|\tilde{T}\|_{W_i^1(\Omega)}.
\]
Now, due to the above lemma, we have
\[
\|\tilde{S}(uD)i\|_{W_i^1(\Omega)} \leq C_2\|u\|_{W_i^1(\Omega)}
\]
\[
\|\tilde{S}(uD)i\|_{W_i^1(\Omega)} \leq C_2\|u\|_{W_i^1(\Omega)}\|h\|_{W_i^1(\Omega)}
\]
then, in a manner similar to \([4]\), this results in
\[
\|M(u_{i-1}) - M(u_{i-2})\|_{W_i^1(\Omega)} \leq C_2\|u_{i-1} - u_{i-2}\|_{W_i^1(\Omega)}(\|u_{i-1}\|_{W_i^1(\Omega)} + \|u_{i-2}\|_{W_i^1(\Omega)}).
\]
With \(L_i = 2C_1C_2(\|u_{i-1}\|_{W_i^1(\Omega)} + \|u_{i-2}\|_{W_i^1(\Omega)})\) we obtain
\[
\|u_i - u_{i-1}\|_{W_i^1(\Omega)} \leq L_i\|u_{i-1} - u_{i-2}\|_{W_i^1(\Omega)}.
\]
Furthermore, we have
\[
\|u_i\|_{W_i^1(\Omega)} \leq \frac{L_i}{\eta}\|\tilde{T}\|_{TQTM(u_{i-1})\|W_i^1(\Omega)} + \frac{L_i}{\eta}\|\tilde{T}\|_{TQTM(u_{i-1})\|W_i^1(\Omega)}
\]
\[
\leq \frac{L_i}{\eta}\|TQTM(u_{i-1})\|_{W_i^1(\Omega)} + \frac{L_i}{\eta}\|\tilde{T}\|_{W_i^1(\Omega)}\|TQTM(u_{i-1})\|_{W_i^1(\Omega)}
\]
\[
\leq 2C_1C_2\|u_{i-1}\|_{W_i^1(\Omega)} + 2C_1\frac{L_i}{\eta}\|W_{i-1}\|_{W_i^1(\Omega)}.
\]
Thus, using arguments similar to \([4, p. 97]\), to ensure that \(\|u_i\|_{W_i^1(\Omega)} \leq \|u_{i-1}\|_{W_i^1(\Omega)}\), we must have that \((p/\eta)/\|\tilde{f}\|_{W_i^1(\Omega)} \leq \frac{1}{16C_1C_2}\) then
\[
\|u_{i-1}\|_{W_i^1(\Omega)} - \frac{1}{16C_1C_2} \leq W
\]
with \(W = [(4C_1C_2)^{-2} - p/\|\tilde{f}\|_{W_i^1(\Omega)}]/(pC_2)]^{1/2}\). As a consequence of this inequality we obtain the estimate
\[
\|u_{i-1}\|_{W_i^1(\Omega)} \leq W + \frac{1}{16C_1C_2} \equiv R
\]
This inequality is valid for any \(i\). Finally, it can be shown that
\[
\|u_i - u_{i-1}\|_{W_i^1(\Omega)} \leq (1 - 4C_1C_2W)\|u_{i-1} - u_{i-2}\|_{W_i^1(\Omega)}
\]
with the condition \(L_i \leq (1 - 4C_1C_2W) \equiv L < 1\).
Thus, from (3.7) we can write the following inequality:

\[ (3.3) \]

Theorem 5. If

by making use this duality between Navier–Stokes equation and quantum processes one

because of many computational difficulties typical for this discipline [1, 2, 18, 27], however

without taking into account any quantum effects. Fluid dynamics is similar to gravitational

limit used in [20] are dual to phenomena of classical fluids dynamics in the AdS space

black holes which is described by

holography.

In this paper we studied the equations of stationary magneto-hydrodynamics in Clifford

4. Conclusion and Discussion

On the other hand, \( h_i \) is calculated by the second equation of (3.7) setting

\[ h'_i = \mu \sigma T Q T N(h_i^{-1}, u_i) - \mu \sigma T Q T N(u_i, h_i^{-1}) + \tilde{T} Q w_i. \]

Thus, from (3.7) we can write the following inequality:

\[
\| h'_i - h_i^{-1} \|_{W^2_1(\Omega)} \leq \mu \sigma \| T Q T (N(h^{-2}, u_i) - N(h^{-1}, u_i)) \|_{W^2_1} \\
+ \mu \sigma \| T Q T (N(u_i, h^{-1}) - N(u_i, h^{-2})) \|_{W^2_1} \\
\leq 2 P_1 \| u_i \|_{W^2_1} \| h_i^{-1} - h_i^{-2} \|_{W^2_1}
\]

where \( P_1 = \tilde{T} C \), with

\[
\tilde{T} = \mu \sigma \| T \|_{\mathcal{L}_c \rightarrow \mathcal{L}_c^{2m, 2m, 2m}} \| Q \|_{\mathcal{L}_C, \mathcal{L}_c^{2m, 2m}} \| (\tilde{T}) \|_{W^2_1(\Omega), \mathcal{L}_c^{2m, 2m}}.
\]

Then, if \( 2 P_1 R \leq 1 \), we have that the sequence \( \{ h'_i \} \) converges in \( W^2_1(\Omega) \).

Consequently, we have proved the following result.

Theorem 5. If \( f \in W^1_4(\Omega) \) satisfies

\[
(\rho/q) \| f \|_{W^1_4(\Omega)} \leq \left( 16 C_1^2 C_2^2 \right)^{-1}
\]

with \( C_1 = (\tilde{T}) \| T \|_{\mathcal{L}_c \rightarrow \mathcal{L}_c^{2m, 2m}} \| Q \|_{\mathcal{L}_C, \mathcal{L}_c^{2m, 2m}} \| (\tilde{T}) \|_{W^2_1(\Omega), \mathcal{L}_c^{2m, 2m}} \) and \( \frac{2}{3} < q < \infty \), then the system

(3.3) has a unique solution \( (u, p, h, w) \in W^1_0(\Omega) \cap \ker \text{div} \times L^q_0(\Omega) \times W^1_0(\Omega) \cap \ker \text{div} \times L^q(\Omega) \).

4. Conclusion and Discussion

In this paper we studied the equations of stationary magneto-hydrodynamics in Clifford

algebra formalism and using the toolkit of Clifford analysis. The instationary case can

be treated in analogous way as in [5]. The classical linear partial differential equations

of nonrelativistic mathematical physics, as is well known are constructed on the basis of

important physical laws. It has recently been found that all these equations can be set in the

context of Clifford analysis. Manipulating with derivatives, it is possible to generate many

important physical laws. It has recently been found that all these equations can be set in the

space with curvature in framework of Clifford formalism we can treat the phenomena of

Thus, from (3.7) we can write the following inequality:

\[ (3.3) \]

Theorem 5. If \( f \in W^1_4(\Omega) \) satisfies

\[
(\rho/q) \| f \|_{W^1_4(\Omega)} \leq \left( 16 C_1^2 C_2^2 \right)^{-1}
\]

with \( C_1 = (\tilde{T}) \| T \|_{\mathcal{L}_c \rightarrow \mathcal{L}_c^{2m, 2m}} \| Q \|_{\mathcal{L}_C, \mathcal{L}_c^{2m, 2m}} \| (\tilde{T}) \|_{W^2_1(\Omega), \mathcal{L}_c^{2m, 2m}} \) and \( \frac{2}{3} < q < \infty \), then the system

(3.3) has a unique solution \( (u, p, h, w) \in W^1_0(\Omega) \cap \ker \text{div} \times L^q_0(\Omega) \times W^1_0(\Omega) \cap \ker \text{div} \times L^q(\Omega) \).

4. Conclusion and Discussion

In this paper we studied the equations of stationary magneto-hydrodynamics in Clifford

algebra formalism and using the toolkit of Clifford analysis. The instationary case can

be treated in analogous way as in [5]. The classical linear partial differential equations

of nonrelativistic mathematical physics, as is well known are constructed on the basis of

important physical laws. It has recently been found that all these equations can be set in the

context of Clifford analysis. Manipulating with derivatives, it is possible to generate many

equations, in particular all the physical laws are among them [25]. It is for this reason that

Clifford analysis represents one of the most remarkable fields of modern mathematics. This

formalism can be applied to the relativistic case without curvature too. Also, we can use

the generalization of this formalism to the spaces with curvature [22]. Knowing how to work

in space with curvature in framework of Clifford formalism we can treat the phenomena of

holography.

In particular, it has been discovered that quantum field theory processes in the horizon of

black holes which is described by \( N = 4 \) supersymmetric quantum field theory in a special

limit used in [20] are dual to phenomena of classical fluids dynamics in the AdS space

without taking into account any quantum effects. Fluid dynamics is similar to gravitational

dynamics [14]. The investigation on the quantum field theory side is complicate procedure

because of many computational difficulties typical for this discipline [1, 2, 18, 27], however

by making use this duality between Navier–Stokes equation and quantum processes one
can study quantum physics phenomena without making detailed calculation of quantum corrections. In such a case, quantum equations written in relativistic formulation for four-dimensional theory can be converted in relativistic Navier–Stokes equation for the fluid [21]. Then, the non-relativistic limit for Navier–Stokes equation can be taken. The main technical issue is to compare the energy-momentum tensor in the four-dimensional conformal field theory lying in the horizon and the energy-momentum tensor of the five-dimensional fluid inside the black hole [21]. The fluid model considered in this paper in analogy with Navier–Stokes equation can have its dual on the gravity side.

Another approach to derive non-relativistic equations from the string theory was proposed in [9,13]. Non-relativistic Schrodinger equation appears due to the conformal group technique in the plane wave limit of AdS space. Schrodinger equation has the same group of symmetry as the Navier–Stokes equation has [15]. This suggests that Navier–Stokes equation can be derived from string theory too, based on conformal group technique. Non-relativistic Schrodinger equation can be treated in the Clifford analysis. For example, instationary equations have been investigated in [5,6], in particular instationary Navier–Stokes and Schrödinger equations. To our knowledge, at present there is no investigation dedicated to the development of Clifford formalism in string theory. Our research suggests that such links can be found, at least in the plane wave limit of AdS space.

Acknowledgments

I. Kondrashuk was supported by Fondecyt (Chile) grants 1040368, 1050512 and by DIUBB grant (UBB, Chile) 082609. E.A. Notte-Cuello was supported by Dirección de Investigación de la Universidad de La Serena, DIULS, grant CD091501. M.A. Rojas-Medar was partially supported by DGI-MEC (Spain) Grant MTM2009-12927 and Fondecyt Grant 1080628.

References

[1] P. Allendes, N. Guerrero, I. Kondrashuk and E. A. Notte Cuello, New four-dimensional integrals by Mellin–Barnes transform, J. Math. Phys. 51 (2010) 05230.
[2] Z. Bern, M. Czakon, L. J. Dixon, D. A. Kosower and V. A. Smirnov, The four-loop planar amplitude and cusp anomalous dimension in maximally supersymmetric Yang–Mills theory, Phys. Rev. D 75 (2007) 085010 [arXiv:hep-th/0610248].
[3] M. A. Rojas-Medar and J. L. Boldrini, Global strong solutions of equations of Magnetohydrodynamic type, J. Austral. Math. Soc. Ser. B 38 (1997) 291–306.
[4] P. Cerejeiras and U. Kähler, Elliptic boundary value problems of fluids dynamics over unbounded domain, Math. Meth. Appli. Sci. 23 (2000) 81–101.
[5] P. Cerejeiras, U. Kähler and F. Sommen, Parabolic Dirac operators and Navier–Stokes equations over time-varying domains, Math. Meth. Appli. Sci. 28 (2005) 1715–1724.
[6] P. Cerejeiras and N. Vieira, Factorization of the non-stationary Schrödinger operator, Adv. Appl. Clifford Alg. 17 (2007) 331–341.
[7] E. V. Chizhonkov, On a system of equations of magnetohydrodynamic type, Soviet Math. Dokl. 30 (1984) 542–545.
[8] C. Constantin and C. Foias, Navier–Stokes Equations (The University of Chicago Press, Chicago and London, 1988).
[9] C. Duval, M. Hassaine and P. A. Horvathy, The geometry of Schrödinger symmetry in gravity background/non-relativistic CFT, Annals Phys. 324 (2009) 1158 [arXiv:0809.3128 [hep-th]].
[10] H. Fujita and T. Kato, On the Navier–Stokes initial value problem, I. Arch. Rational Mech. Anal. 16 (1964) 269–315.
[11] K. Gürlebeck and U. Kähler, J. Ryan and W. Sprössig, Clifford analysis over unbounded domains, *Advances in Applied Mathematics* **19** (1997) 216–239.

[12] K. Gürlebeck and W. Sprössig, Quaternionic analysis and elliptic boundary value problems (Akademieverlag, Berlin, 1989).

[13] M. Hassaine, Conformal symmetry of an extended Schrödinger equation and its relativistic origin, *J. Phys. A* **40** (2007) 5717 [arXiv:hep-th/0701285].

[14] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space Time* (Cambridge Univ. Press, 1973).

[15] M. Hassaine and P. A. Horvathy, Symmetries of fluid dynamics with polytropic exponent, *Phys. Lett. A* **279** (2001) 215 [arXiv:hep-th/0009092].

[16] U. Kähler, On a direct descomposition of the space $L^2(\Omega)$. *J. Analysis Applications* **18**(4) (1999) 839–848.

[17] I. Kondrashuk, E. Notte-Cuello and M. A. Rojas-Medar, Stationary asymmetric fluids and Hodge operator, *Boletín de la Sociedad Española de Matemáticas Aplicadas* **47** (2009) 99–106.

[18] I. Kondrashuk and A. Kotikov, Triangle UD integrals in the position space, *JHEP* **0808** (2008) 106 [arXiv:0803.3420 [hep-th]].

[19] G. Lassueur, Über ein rand-anfangswert-problem der magnetohydrodynamik, *Arch. Rational Mech. Anal.* **25** (1967) 388–405.

[20] J. M. Maldacena, The large N limit of superconformal field theories and supergravity, *Adv. Theor. Math. Phys.* **2** (1998) 231, *Int. J. Theor. Phys.* **38** (1999) 1113 [arXiv:hep-th/9711200].

[21] M. Rangamani, Gravity & Hydrodynamics: Lectures on the fluid-gravity correspondence, arXiv:0905.4352 [hep-th].

[22] E. A. Notte-Cuello, W. A. Rodrigues Jr. and Q. A. G. Souza, The square of the Dirac and spin-Dirac operators on a Riemann-Cartan space, *Rep. Math. Phys.* **60** (2007) 135–157.

[23] S. B. Pilpel, *Grundlagen der Kosmischen Elektrodynamik* (Moscow, 1966).

[24] E. A. Notte-Cuello, M. D. Rojas-Medar and M. A. Rojas-Medar, Periodic strong solutions of the magnetohydrodynamic type equations, *Progressiones* **21**(3) (2002) 199–124.

[25] W. A. Rodrigues Jr. and E. Capelas Oliveira, The many faces of Maxwell, Dirac and Einstein equations. A Clifford bundle approach, *Lecture Notes in Physics* 772 (Springer, New York, 2007).

[26] A. Schlüter, Dynamic des Plasmas, I and II. *Z. Naturforsch.* **5b** (1950), 72–78; **6a** (1951), 73–79.

[27] V. A. Smirnov, *Evaluating Feynman Integrals*, Springer Tracts, *Mod. Phys.* **211** (2004) 1.

[28] R. Temam, *Navier–Stokes Equations*, Rev. edn. (North-Holland, Amsterdam, 1979).