Density operators and selective measurements.

Wlodzimierz M. Tulczyjew
Valle San Benedetto, 2
62030 Monte Cavallo, Italy
Associated with
Division of Mathematical Methods in Physics
University of Warsaw
Hoża 74, 00-682 Warszawa
and
Istituto Nazionale di Fisica Nucleare,
Sezione di Napoli
Complesso Universitario di Monte Sant’Angelo
Via Cintia, 80126 Napoli, Italy
tulczy@libero.it

1. Introduction.

It is widely believed that statistical interpretation of quantum mechanics requires that density operators representing quantum states be normalized. We present a description of selective measurements in terms of density operators. The description is inspired by Schwinger’s Algebra of Microscopic Measurements [1], (see also [2]). Density operators used are not normalized. We do not know applications of density operators requiring normalization.

2. Beams of particles.

The physical space is an affine space \( M \) of dimension 3 modelled on a vector space \( V \). There is an Euclidean metric tensor \( g: V \to V^* \).

We consider beams of particles of mass \( m \) and constant energy \( E \) in the direction of a unit vector \( z \in V \). The internal states of the particles are elements of a unitary vector space \( U \) of dimension \( r \) over the field \( \mathbb{C} \) of complex numbers. The elements of \( U \) are \textit{kets} \(|u⟩\) and elements of the dual space \( U^* \) are \textit{bras} \( ⟨a| \). The unitary structure establishes an antilinear isomorphism of \( U \) with \( U^* \) assigning to each ket \(|u⟩\) a unique bra \( ⟨u^†| \). The number

\[
⟨u^†_1|u_2⟩
\]

is the \textit{scalar product} of vectors \(|u_1⟩\) and \(|u_2⟩\).

We introduce a sequence of points

\[
x_0, x_1, x_2, \ldots, x_n
\]

satisfying inequalities

\[
⟨g(z), x_i - x_{i-1}⟩ > 0,
\]

and a corresponding sequence of planes

\[
X_i = \{x \in M; ⟨g(z), x - x_i⟩ = 0\}.
\]

In the immediate neighbourhood of each plane \( X_i \) the beam is not subject to external interaction and is represented by a plane wave

\[
|ψ_i(x)⟩ = |A_i⟩ \exp (ik⟨g(z), x - x_0⟩ + δ_i), \quad |A_i⟩ \in U, \quad k = \frac{\sqrt{2mE}}{\hbar}
\]

with time dependence separated. The probability flux through a unit surface element of the plane \( X_i \) is expressed by

\[
\frac{\hbar k}{m} ⟨A_i^†|A_i⟩.
\]
Between the plane \(X_{i-1}\) and the plane \(X_i\) the beam passes through a selective device. Its action on the wave function is described as the action of a linear transition operator

\[
M_i : U \rightarrow |\psi_{i-1}(x)\rangle \mapsto |\psi_i(x)\rangle = M_i |\psi_{i-1}(x)\rangle. \tag{8}
\]

If \(M_1, M_2, M_3, \ldots, M_n\) is the sequence of transition operators and

\[
|\psi_0(x)\rangle = |A_0\rangle \exp (ik\langle g(z), x - x_0 \rangle) \tag{10}
\]

is the initial state, then

\[
|M_i \cdots M_2 M_1 |A_0\rangle \exp (ik\langle g(z), x - x_0 \rangle) \tag{11}
\]

and

\[
\langle \psi_i^\dagger(x)|\psi_i(x)\rangle = \langle A_0^\dagger|M_1^\dagger M_2^\dagger \cdots M_i^\dagger M_i \cdots M_2 M_1 |A_0\rangle \tag{12}
\]

The flux of particles through unit surface element of \(X_i\) is given by

\[
\frac{\hbar k}{m} \langle A_0^\dagger|M_1^\dagger M_2^\dagger \cdots M_i^\dagger M_i \cdots M_2 M_1 |A_0\rangle. \tag{13}
\]

3. Mixed states and density operators.

The expression (13) can be presented in the form

\[
\frac{\hbar k}{m} \text{tr} (M_i \cdots M_2 M_1 |A_0\rangle \langle A_0^\dagger|M_1^\dagger M_2^\dagger \cdots M_i^\dagger) = 1 \tag{14}
\]

In this new expression the pure initial state is represented by the density operator \(|A_0\rangle \langle A_0^\dagger|\) and can be replaced by a mixed state represented by a positive Hermitian density operator \(T\). The expression

\[
\frac{\hbar k}{m} \text{tr} (M_i \cdots M_2 M_1 T M_1^\dagger M_2^\dagger \cdots M_i^\dagger) \tag{15}
\]

is the result.

4. Selective measurements.

We set the density operator \(T\) in the expression (15) equal to

\[
T = \frac{m}{\hbar kr} I, \tag{16}
\]

where \(I\) is the identity operator. The operator \(T\) is normalized in the sense that

\[
\frac{\hbar k}{m} \text{tr} T = 1 \tag{17}
\]

although this normalization is of no importance. The expression

\[
P_i = \frac{\hbar k}{m} \text{tr} (M_i \cdots M_2 M_1 T M_1^\dagger M_2^\dagger \cdots M_i^\dagger) = \frac{1}{r} \text{tr} (M_i \cdots M_2 M_1 M_1^\dagger M_2^\dagger \cdots M_i^\dagger) \tag{18}
\]

represents the probability of detecting a particle crossing a unit surface element of the plane \(X_i\) in unit time. The state of the initial beam emitted by a source at \(X_0\) is totally mixed.
We want to describe the following experimental arrangement. The initial beam emitted at \( X_0 \) undergoes a preliminary selection by a sequence of devices represented by the sequence of operators
\[
M_1, M_2, \ldots, M_j.
\] (19)
The beam undergoes further selection passing through a sequence of devices represented by operators
\[
N_1 = M_{j+1}, N_2 = M_{j+2}, \ldots, N_{n-j} = M_n.
\] (20)
The particles are detected at \( X_n \) by a non selective detector. After the preliminary selection the state of the beam is represented by the density operator
\[
N = M_j M_{j-1} \cdots M_1 T M_1 \cdots M_{j-1} M_j = \frac{m}{\hbar k r} M_j M_{j-1} \cdots M_1 M_1 \cdots M_{j-1} M_j
\] (21)
with the probability of non selective detection
\[
P_{in} = \frac{\hbar k}{m} \text{tr}(M_j M_{j-1} \cdots M_1 T M_1 \cdots M_{j-1} M_j) = \frac{\hbar k}{m} \text{tr} M
\] (22)
The beam arrives at \( X_n \) in a state represented by the density operator
\[
N_{n-j} N_{j-1} \cdots N_1 M_j M_{j-1} \cdots M_1 T M_1 \cdots M_{j-1} M_j N_1 \cdots N_{n-j-1} N_{n-j}
\] (23)
It is detected with the probability
\[
P_{out} = \frac{\hbar k}{m} \text{tr}(N_{n-j} N_{j-1} \cdots N_1 M_j M_{j-1} \cdots M_1 T M_1 \cdots M_{j-1} M_j N_1 \cdots N_{n-j-1} N_{n-j})
\]
\[
= \frac{\hbar k}{m} \text{tr}(N_1 \cdots N_{n-j-1} N_{n-j} N_{n-j-1} \cdots N_1 M_j M_{j-1} \cdots M_1 T M_1 \cdots M_{j-1} M_j)
\] (24)
\[
= \frac{\hbar k}{m} \text{tr}(NM)
\]
with
\[
N = N_1 \cdots N_{n-j-1} N_{n-j} N_{n-j-1} \cdots N_1.
\] (25)
The density operator \( N \) characterizes the selective detector. The probability \( P_{out} \) is measured at \( X_j \) by the selective detector. This measurement is performed on the mixed state represented by the operator \( M \). The relative probability
\[
P_{out}/P_{in} = \text{tr}(NM)/\text{tr} M
\] (26)
should be considered the result of the selective measurement described. Arbitrary normalization can be imposed on \( M \). Normalization of \( N \) would distort the result of the measurement.

5. An example.
In addition to the metric tensor
\[
g : V \rightarrow V^*
\] (27)
we introduce in the model space \( V \) of the physical space \( M \) an orientation \( o \) defined as an equivalence class of bases.
We analyse the internal states of a beam of particles of spin 1/2. States of particles are represented by wave functions with values in an unitary space of complex dimension 2. The set of hermitian traceless operators in \( U \) is a real vector space \( S \) of dimension 3.

The trace \( \text{tr}(ab) \) of a product is a non negative real number and the mapping
\[
S \times S \rightarrow \mathbb{R} : (a, b) \mapsto \text{tr}(ab)
\] (28)
is bilinear and symmetric. The spectrum of an operator \( a \in S \) is a pair \( \{ \alpha, -\alpha \} \) of real numbers and the spectrum of the operator \( aa \) is the set \( \{ \alpha^2, \alpha^2 \} \). It follows that \( \text{tr}(aa) = 0 \) if and only if \( a = 0 \). In conclusion we have a Euclidean scalar product

\[ (|) : S \times S \to \mathbb{R} : (a, b) \mapsto (a|b) = \frac{1}{2} \text{tr}(ab). \]  

(29)

We introduce the Pauli morphism

\[ \sigma : V \to S. \]  

(30)

This morphism is an isometry such that the operator

\[ \frac{1}{2} \sigma(w) : U \to U \]  

(31)

associated with each unit vector \( w \in V \) is the spin operator in the direction \( w \). Its spectrum is the set \( \{ 1/2, -1/2 \} \) and its eigenvectors represent states of the particle with spin \( 1/2 \) and \( -1/2 \) in the direction of \( w \).

We introduce a number of operators in the space \( U \):

1) The projection operator

\[ K(w) = \frac{1}{2} (I + \sigma(w)) \]  

(32)

associated with a unit vector \( w \in V \). This operator projects onto the space of eigenstates of the spin operator \( 1/2 \sigma(w) \) corresponding to the eigenvalue \( 1/2 \).

2) A phase shift operator

\[ D(\delta) = \exp(i\delta)I. \]  

(33)

3) An attenuation operator

\[ R(\rho) = \exp(-\rho/2)I. \]  

(34)

4) A unitary unimodular operator

\[ G : U \to U. \]  

(35)

This operator represents a rotation

\[ E : V \to V \]  

(36)

in the sense that

\[ G \sigma(w) G^{-1} = \sigma(Ew). \]  

(37)

5) The operator

\[ Q(w) = \frac{1}{2} (I + \sigma(w)) \]  

(38)

associated with a vector \( w \in V \) of norm \( \|w\| \neq 1 \). This operator is not a projection operator.

Consider a beam undergo a preliminary selection by devices represented by \( M_1 = K(w) \) and \( M_2 = D(\delta) \). The vector \( w \) is orthogonal to the direction of the beam and the first of the devices is a Stern-Gerlach filter. It is accompanied by an unavoidable phase shift. The state prepared by these devices is a pure state represented by the density operator

\[ M = M_2 M_1^T M_1^\dagger M_2^\dagger = \frac{m}{2\hbar k} K(w). \]  

(39)

The selective detector is composed of devices represented by operators \( N_1 = R(\rho), N_2 = D(\delta'), \) and \( N_3 = K(w') \). The attenuation \( R(\rho) \) my be due to the beam passing through a potential barrier. The density operator

\[ N = N_1^\dagger N_2^\dagger N_3^\dagger N_3 N_2 N_1 = \exp(-\rho) K(w') \]  

(40)
represents the selective detector. The result of the selective measurement is the relative probability

\[ P_{\text{out}}/P_{\text{in}} = \frac{\text{tr}(NM)}{\text{tr} M} = \frac{1}{2} (1 + (w'|w)) \]  

since

\[ \text{tr} M = 1 \]  

and

\[ \text{tr}(NM) = \text{tr}(\exp(-\rho)K(w')K(w)) \]
\[ = \frac{1}{4} \text{tr}(\exp(-\rho)(I + \sigma(w'))(I + \sigma(w))) \]
\[ = \frac{1}{4} \text{tr}(\exp(-\rho)(I + \sigma(w') + \sigma(w) + \sigma(w')\sigma(w))) \]
\[ = \frac{1}{2} (1 + (w'|w)) \]  

6. References.

[1] J. Schwinger, The Algebra of Microscopic Measurements, Proc. Natl. Acad. Sc. US, 45 (1959)
[2] F. A. Kaempfier, Concepts in Quantum Mechanics, Academic Press, New York and London (1965)