Infinite rate mutually catalytic branching in infinitely many colonies.
Construction, characterization and convergence

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Abstract

We construct a mutually catalytic branching process on a countable site space with infinite “branching rate”. The finite rate mutually catalytic model, in which the rate of branching of one population at a site is proportional to the mass of the other population at that site, was introduced by Dawson and Perkins in [DP98]. We show that our model is the limit for a class of models and in particular for the Dawson-Perkins model as the rate of branching goes to infinity. Our process is characterized as the unique solution to a martingale problem. We also give a characterization of the process as a weak solution of an infinite system of stochastic integral equations driven by a Poisson noise.

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1 Introduction and main results

1.1 Background and Motivation

In [DP98], Dawson and Perkins considered the following mutually catalytic model:

\[ Y_{t,i}(k) = Y_{0,i}(k) + \int_0^t \sum_{l \in S} A(k,l)Y_{s,i}(l) \, ds \]

\[ + \int_0^t (\gamma Y_{s,1}(k)Y_{s,2}(k))^{1/2} \, dW_{s,i}(k), \quad t \geq 0, k \in S, i = 1, 2. \]

Here $S$ is a countable set that is thought of as the site space. (In fact, Dawson and Perkins made the explicit choice $S = \mathbb{Z}^d$.) The matrix $A$ is defined by

\[ A(k,l) = a(k,l) - 1_{\{k = l\}}, \]

where $a$ is a symmetric transition matrix of a Markov chain on $S$. Finally, $(W_i(k), k \in S, i = 1, 2)$ is an independent family of one-dimensional Brownian motions. Dawson and Perkins studied the long-time behavior of this model and also constructed the analogous model in the continuous setting on $\mathbb{R}$ instead of $S$. One can think of $\gamma$ as being the branching rate for this model.

In this paper we study (under weaker assumptions on the matrix $A$) a model that formally corresponds to the case $\gamma = \infty$. This infinite rate mutually catalytic branching process can be characterized by a certain martingale problem. In this paper we show that this martingale problem is well posed and its solution $X$ is the unique solution of a system of stochastic differential equations driven by a certain Poisson noise. In fact, we construct the solution via approximate solutions of this system of SDEs. Furthermore, we show that $X$ is the limit of the Dawson-Perkins processes as $\gamma \to \infty$.

This is the second part in a series of three papers. In the first paper [KM08], we studied the infinite rate mutually catalytic branching process in the case where $S$ is a singleton. In [KM09], we investigated the long-time behaviour for the case where $S$ is countable. There we establish a dichotomy between segregation and coexistence of types depending on the potential properties of the migration mechanism $A$.

An alternative construction of the infinite rate mutually catalytic branching process via a Trotter type approximation scheme can be found [Oel08] and [KO09].

1.2 The main results

We have to introduce some notation. Let $A = (A(k,l))_{k,l \in S}$ be a matrix on $S$ satisfying the following assumptions:

\[ A(k,l) \geq 0 \quad \text{for} \quad k \neq l \]

and

\[ ||A|| := \sup_{k,l \in S} |A(k,l)| + |A(l,k)| < \infty. \]

Let

\[ E = [0, \infty)^2 \setminus (0, \infty)^2. \]

For $u, v \in [0, \infty)^S$ define

\[ \langle u, v \rangle = \sum_{k \in S} u(k)v(k) \in [0, \infty]. \]

Similarly, for $x \in ([0, \infty)^2)^S$ and $\zeta \in [0, \infty)^S$ define

\[ \langle x, \zeta \rangle = \sum_{k \in S} x(k)\zeta(k) \in [0, \infty]^2. \]
By Lemma IX.1.6 of [Lig85], there exists a $\beta \in (0, \infty)^S$ and an $M \geq 1$ such that

$$\sum_{k \in S} \beta(k) < \infty,$$

$$\sum_{l \in S} \beta(l)(|A(k, l)| + |A(l, k)|) \leq M \beta(k) \quad \text{for all } k \in S. \quad (1.5)$$

We fix this $\beta$ for the rest of this paper. Note that for the transpose matrix $A^*$ of $A$, we have $\|A^*\| = \|A\| < \infty$ and (1.6) holds with the same $\beta$. Hence, in what follows, $A$ could be replaced by $A^*$. We will make use of this fact in Section 4 when we construct a dual process.

Let us define the Liggett-Spitzer spaces follows:

$$L^\beta = \{ u \in [0, \infty)^S : \langle u, \beta \rangle < \infty \},$$

$$L^\beta,2 = \{ x \in ([0, \infty)^2)^S : \langle x, \beta \rangle \in [0, \infty)^2 \},$$

$$L^\beta,E = L^\beta,2 \cap E^S.$$

For $u \in \mathbb{R}^S$ define

$$\|u\|_\beta = \sum_{k \in S} |u(k)|\beta(k). \quad (1.7)$$

Let $A^f(k) = \sum_{l \in S} A(k, l)f(l)$ if the sum is well-defined. Let $A^n$ denote the $n$th matrix power of $A$ (note that this is well-defined and finite by (1.3)) and define

$$p_t(k, l) := e^{tA}(k, l) := \sum_{n=0}^{\infty} \frac{t^n A^n(k, l)}{n!}.$$

Let $S$ denote the (not necessarily Markov) semigroup generated by $A$, that is

$$S_t f(k) = \sum_{l \in S} p_t(k, l)f(l) \quad \text{for } t \geq 0. \quad (1.8)$$

We will use the notation $Af$, $S_t f$ and so on also for $[0, \infty)^2$ valued functions $f$ with the obvious meaning.

Note that for $f \in L^\beta$, the expressions $Af$ and $S_t f$ are well-defined and that (recall $M$ from (1.6))

$$\|Af\|_\beta \leq M \|f\|_\beta \quad \text{and} \quad \|S_t f\|_\beta \leq e^{Mt} \|f\|_\beta. \quad (1.10)$$

We define the matrix $A = (|A(k, l)|)_{k,l \in S}$ and denote the corresponding semigroup by $(S_t)_{t \geq 0}$. Clearly, for any $f \in L^\beta$ and $k \in S$, we have

$$Af(k) \leq Af(k), \quad S_t f(k) \leq S_t f(k) \quad \text{and} \quad f(k) \leq S_t f(k). \quad (1.9)$$

As above, it is easy to check that

$$\|Af\|_\beta \leq M \|f\|_\beta, \quad (1.10)$$

$$\|S_t f\|_\beta \leq e^{Mt} \|f\|_\beta \quad \text{for all } t \geq 0. \quad (1.11)$$

Therefore, we trivially have

$$Af(k) \leq \frac{M \|f\|_\beta}{\beta(k)} \quad \text{for all } f \in L^\beta, \quad (1.12)$$

$$S_t f(k) \leq \frac{e^{Mt} \|f\|_\beta}{\beta(k)} \quad \text{for all } f \in L^\beta, \quad t \geq 0. \quad (1.13)$$
Let $D_{L^β, E} = D_{L^β, E}[0, ∞)$ be the Skorohod space of càdlàg $L^β, E$-valued functions.

We will employ a martingale problem in order to characterize the (bivariate) process $X ∈ D_{L^β, E}$ that will be the limit of the Dawson-Perkins models as $γ → ∞$. In order to formulate this martingale problem for $X$, we need some more notation. For $x = (x_1, x_2)$ and $y = (y_1, y_2) ∈ \mathbb{R}^2$ we introduce the lozenge product

$$x ∘ y := -(x_1 + x_2)(y_1 + y_2) + i(x_1 - x_2)(y_1 - y_2) \quad (1.14)$$

(with $i = √{-1}$) and define

$$F(x, y) = \exp(x ∘ y). \quad (1.15)$$

Note that $x ∘ y = y ∘ x$. For $x, y ∈ (\mathbb{R}^2)^S$ we write

$$\langle \langle x, y \rangle \rangle = \sum_{k ∈ S} x(k) ∘ y(k) \quad (1.16)$$

whenever the infinite sum is well-defined and let

$$H(x, y) = \exp(\langle \langle x, y \rangle \rangle). \quad (1.17)$$

Define

$$L^f, β = \{ y ∈ (0, ∞)^2 : y(k) ≠ 0 \text{ for only finitely many } k ∈ S \} \quad (1.18)$$

and

$$L^f, E = L^f, β ∩ E^S. \quad (1.19)$$

Finally, define the spaces

$$L^β, ∞ = \{ f ∈ [0, ∞)^S : \langle f, g \rangle < ∞ \text{ for all } g ∈ L^β \}$$

$$= \{ f ∈ L^β : \sup_{k ∈ S} f(k)/β(k) < ∞ \} \quad (1.20)$$

and

$$L^β, E = \{ η = (η_1, η_2) ∈ E^S : η_1, η_2 ∈ L^β, ∞ \}. \quad (1.21)$$

As a subspace, $L^β, ∞$ inherits the norm of $L^β$.

Note that the function $H(x, y)$ is well-defined if either $x ∈ (\mathbb{R}^2)^S$ and $y ∈ L^f, E$ or $x ∈ L^β, E$ and $y ∈ L^β, E$. Our main theorem is the following.

**Theorem 1.1 (a)** For all $x ∈ L^β, E$, there exists a unique solution $X ∈ D_{L^β, E}$ of the following martingale problem: For each $y ∈ L^f, E$, the process $M^{x, y}$ defined by

$$M^{x, y}_t := H(X_t, y) - H(x, y) - \int_0^t \langle \langle AX_s, y \rangle \rangle H(X_s, y) ds \quad (MP_1)$$

is a martingale with $M^{x, y}_0 = 0$.

(b) For any $x ∈ L^β, E$ and $y ∈ L^β, E$, the process $M^{x, y}$ is well-defined and is a martingale.

(c) Denote by $P_x$ the distribution of $X$ with $X_0 = x$. Then $(P_x)_{x ∈ L^β, E}$ is a strong Markov family.

Unfortunately, the characterization of $X$ as the solution of the martingale problem (MP1) does not shed much light on properties of the process $X$ such as: Is $X$ continuous or discontinuous? If it is discontinuous what is the structure of jump formation? These questions will be answered by a different representation of $X$ as a solution to a system of stochastic differential equations of jump type.

The stochastic parts of the single coordinates in the Dawson-Perkins process defined in (1.1) are two-dimensional isotropic diffusions and are hence time-transformed planar Brownian motions. When we speed up these motions, at any positive time, they will be close to their absorbing points at $E$. Hence, a crucial role in the subsequent considerations will be played by the harmonic measure $Q$ of planar Brownian motion $B$ on $(0, ∞)^2$. That is, if
If \( x \in [0, \infty)^2 \) and \( \tau = \inf \{ t > 0 : B_t \notin (0, \infty)^2 \} \), then we define
\[
Q_x = P_x[B_\tau \in \cdot].
\] (1.22)

If \( x = (u, v) \in (0, \infty)^2 \), then the harmonic measure \( Q_x \) has a one-dimensional Lebesgue density on \( E \) that can be computed explicitly
\[
Q_{(u,v)}(d(\bar{u}, \bar{v})) = \begin{cases} 
\frac{4}{\pi} \frac{uv \bar{u}}{4u^2v^2 + (\bar{u}^2 + v^2 - u^2)^2} \, d\bar{u}, & \text{if } \bar{v} = 0, \\
\frac{4}{\pi} \frac{uv \bar{v}}{4u^2v^2 + (\bar{v}^2 + u^2 - v^2)^2} \, d\bar{v}, & \text{if } \bar{u} = 0.
\end{cases}
\] (1.23)

Furthermore, trivially we have \( Q_x = \delta_x \) if \( x \in E \).

As the next goal is to define a measure for the jumps that drive the process \( X \), we need to describe the infinitesimal dynamics of \( X \). These will be defined in terms the \( \sigma \)-finite measure \( \nu \) on \( E \) that arises as the vague limit (on \( E \setminus \{(1,0)\} \)) of \( \epsilon^{-1}Q_{(1,\epsilon)} \) as \( \epsilon \to 0 \). Using (1.23), it is easy to see that \( \nu \) has a one-dimensional Lebesgue density given by
\[
\nu(d(u,v)) = \begin{cases} 
\frac{4}{\pi} \frac{u}{(1-u)^2(1+u)^2} \, du, & \text{if } v = 0, \\
\frac{4}{\pi} \frac{v}{(1+v)^2} \, dv, & \text{if } u = 0.
\end{cases}
\] (1.24)

We use \( \nu \) to define the Poisson point process (PPP) that will be the driving force of the equations. Let \( N_0 \) be the PPP on \( S \times \mathbb{R}_+ \times \mathbb{R}_+ \times E \) with intensity
\[
N_0 = \ell_S \otimes \lambda \otimes \lambda \otimes \nu,
\] (1.25)
where \( \lambda \) is the Lebesgue measure on \( \mathbb{R}_+ \) and \( \ell_S \) is the counting measure on \( S \). Assume that the process \( X = ((X_{t,1}(k), X_{t,2}(k)) \in E, k \in S, t \geq 0) \) is given. The measure \( \nu \) is the limit of the \( Q \) only at the point \((1,0) \in E \). The limits of \( \epsilon^{-1}Q_{(u,\epsilon)} \) and \( \epsilon^{-1}Q_{(\epsilon,v)} \) can be obtained by simple transformations of \( \nu \) (see [KM08 discussion before (5.5)]). To this end, we define the functions
\[
J_i(y, z) = y_2z_3-i + (y_1 - 1)z_i \quad \text{for } y, z \in E, \ i = 1, 2
\] (1.26)
and
\[
J = (J_1, J_2).
\]

Define the functions \( I_1, I_2 \) and \( J := I_1 + I_2 \) that will serve as intensities for the driving noise by
\[
I_{t,i}(k) := 1_{\{X_{t-i}(k) > \cdot\}} \frac{AX_{t-3-i}(k)}{X_{t-i}(k)} \quad \text{for } k \in S, \ t \geq 0, \ i = 1, 2.
\] (1.27)

Now, given \( X \), we define a new point process \( \mathcal{N} \) by
\[
\mathcal{N} = \mathbb{N}_0 \{ \{k\}, dt, dy \} := N_0 \{ \{k\}, dt, [0, I_t(k)], dy \}
\] (1.28)

Let \( \mathcal{N} \) be the corresponding compensator measure, that is, the measure on \( S \times \mathbb{R}_+ \times E \) given by
\[
\mathcal{N}(\{k\}, dt, dy) = I_t(k) dt \nu(dy).
\]

Finally, define the martingale measure
\[
\mathcal{M} := \mathcal{N} - \mathcal{N}'.
\] (1.29)
Let $x \in \mathbb{L}^{\beta,E}$. By a weak solution of the following system of stochastic differential equations

$$X_t(k) = x(k) + \int_0^t AX_s(k) \, ds + \int_0^t \int_E J(y, X_s(k)) \mathcal{M}([k], ds, dy)$$

(1.30)

we mean a pair $(\mathcal{N}_0, X)$ such that $\mathcal{N}_0$ is a PPP described in (1.25), $X$ is a $D_{L,\beta,E}$ valued process, $\mathcal{M}$ is derived from $\mathcal{N}_0$ and $X$ as described above and (1.30) holds for all $t \geq 0$ and $k \in S$. We say that the solution is unique if the distribution of $X$ is the same for all weak solutions.

**Theorem 1.2** For any $x \in \mathbb{L}^{\beta,E}$ there exists a unique weak solution $(\mathcal{N}_0, X)$ of (1.30) and $X$ solves $\text{(MP}_{\gamma\gamma}^1)$. Now let us go back to the Dawson-Perkins model. We would like to clarify our initial motivation that the process described in Theorems 1.1 1.2 is indeed the limit of Dawson-Perkins process as $\gamma \to \infty$. Let $Y^\gamma = (Y_1^\gamma, Y_2^\gamma)$ be a solution of (1.1) with $Y_0 \in \mathbb{L}^{\beta,E}$. This process with our slightly relaxed assumptions on $\mathcal{A}$ can be constructed in a way similar to the construction of Dawson and Perkins (see also [CDG04]). Furthermore, let $X$ be a solution of $\text{(MP}_{\gamma\gamma}^1)$ with $X_0 = Y_0$.

Clearly, the continuous processes $Y^\gamma$ cannot converge to the discontinuous process $X$ in the Skorohod topology on $D_{L,\beta,E}$. Hence, in order to get a limit theorem, we use the weaker Meyer-Zheng topology (see [MZ84]). Roughly speaking, convergence in the Meyer-Zheng topology means convergence for Lebesgue almost all time points. More precisely, for any $f \in D_{L,\beta,E}$ let $\psi(f)$ denote the image measure on $[0, \infty) \times D_{L,\beta,E}$ of $e^{-t} dt$ under the map $t \mapsto (t, f(t))$. Note that $\psi$ is injective and hence weak convergence in the space of probability measures on $[0, \infty) \times D_{L,\beta,E}$ defines a topology on $D_{L,\beta,E}$ that is called the Meyer-Zheng topology (Meyer and Zheng [MZ84] call it the pseudo-path topology).

For the convergence of $Y^\gamma$ to $X$, it is not crucial that in (1.1) the noise term has the special form of a product. In fact, it is necessary only that the noise is isotropic, strictly positive in $(0, \infty)^2$ and vanishing at the boundary in such a way that it admits a solution with each coordinate nonnegative. Hence, consider the equation

$$Y_{t,i}(k) = Y_{0,i}(k) + \int_0^t \sum_{l \in S} A(l,k) Y_{s,i}(l) \, ds + \int_0^t \gamma^{1/2} \sigma(Y_s(k)) \, dW_{s,i}(k), \quad t \geq 0, k \in S, i = 1, 2.$$  

(1.31)

Here $(W_t(k), k \in S, i = 1, 2)$ is an independent family of one-dimensional Brownian motions and $\sigma : [0, \infty)^2 \to [0, \infty)$ is measurable and fulfills the following assumptions:

**Assumption 1.3**

(i) $\sigma(x) = 0$ for all $x \in E$.

(ii) $\inf \sigma(C) > 0$ for any compact $C \subset (0, \infty)^2$.

(iii) For each $y \in \mathbb{L}^{\beta,2}$ and $\gamma > 0$, (1.31) admits a (weak) $\mathbb{L}^{\beta,2}$-valued solution.

Of course, $\sigma(x) = \sqrt{x_1 x_2}$ is the case considered in (1.1) and it satisfies the above assumptions.

**Theorem 1.4** Assume that (i) and (ii) hold and that for each $\gamma > 0$, we have chosen an $\mathbb{L}^{\beta,2}$-valued solution $Y^\gamma$ of (1.31). Assume that $X_0 := Y_0^\gamma \in \mathbb{L}^{\beta,E}$ does not depend on $\gamma$. Then, for each sequence $\gamma_n \to \infty$, in $D_{L,\beta,E}$ equipped with the Meyer-Zheng topology, we have

$$Y^{\gamma_n} \Rightarrow X \quad \text{as} \quad n \to \infty.$$ 

(1.32)

### 1.3 Organization of the paper

We prove Theorems 1.1 1.2 via an approximation procedure. In Section 2 we will construct a sequence of processes with only finitely many jumps and which have a non-trivial dynamics only on a finite subset $S_m$ of $S$. Then we will show that this sequence converges to the process $X^m$ that solves a system of equations similar to (1.30) with the difference that we suppress jumps greater than $1/m$. In Section 3 we will show that the sequence $(X^m)_{m \in \mathbb{N}}$ converges to a process that solves $\text{(MP}_{\gamma\gamma}^1)$ and (1.30). In Section 4 we will show uniqueness for $\text{(MP}_{\gamma\gamma}^1)$. Section 5 is devoted to the proof of Theorem 1.2. Theorem 1.4 is proved in Section 6.
2 Approximating sequence of processes with a finite number of jumps

The aim of this section is to construct a sequence of approximating processes that should converge to the process solving the martingale problem (MP). Let \((S_m)_{m\in\mathbb{N}}\) be a sequence of finite subsets of \(S\) such that \(S_m \uparrow S\) as \(m \to \infty\). We will define a sequence of approximating processes

\[
\bar{X} = ((\bar{X}_{t,1}(k), \bar{X}_{t,2}(k)) \in E, k \in S, t \geq 0)
\]

in a way that they may change values only for \(k \in S_m\) and stay constant for \(k \in S \setminus S_m\). To this end let us define the matrix \(A_m\) by

\[
A_m(k, l) = \begin{cases} A(k, l), & \text{if } k, l \in S_m, \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}
\]

Let \((S^m_t)_{t \geq 0}\) be the semigroup generated by \(A_m\) and let \(p^m_t = e^{tA_m}\) denote its kernel, that is, for \(f \in L^\beta\)

\[
S^m_t f(k) = \sum_{l \in S} p^m_t(k, l)f(l). \tag{2.2}
\]

Clearly, for any \(f \in L^\beta\),

\[
A_m f(k) \leq Af(k) \quad \text{for all } k \in S, \tag{2.3}
\]

\[
S^m_t f(k) \leq S_t f(k) \quad \text{for all } k \in S. \tag{2.4}
\]

Fix \(\varepsilon > 0\) and \(m, n \in \mathbb{N}\). Let \(\mathcal{N}_0\) be the PPP on \(S \times \mathbb{R}_+ \times \mathbb{R}_+ \times E\) with the intensity \(\bar{N}_0\) given by (1.25). Now given a process \(\bar{X}\) which is adapted to filtration generated by \(\mathcal{N}_0(\cdot, dt, \cdot, \cdot)\), define \(\bar{I}_i\) and \(\bar{I}\) similarly as for \(X\) in (1.27). We can now define a point process by

\[
\bar{N}((k), dt, dy) := \mathcal{N}_0((k), dt, [0, \bar{I}_i(k)], dy). \tag{2.5}
\]

Let \(\tilde{\mathcal{N}}\) be the corresponding compensator measures and define the martingale measures

\[
\tilde{M} := \bar{N} - \tilde{\mathcal{N}}. \tag{2.6}
\]

Denote by \(\Delta \bar{X}_s = \bar{X}_s - \bar{X}_{s-}\) the jump of \(\bar{X}\) at time \(s\) and define the stopping time

\[
\bar{\tau} := \inf \left\{ t : \sum_{s \leq t} \mathbf{1}_{\{\Delta \bar{X}_s \neq 0\}} \geq n \right\}. \tag{2.7}
\]

We will show later that for our process \(\bar{X}\) we have in fact \(\bar{\tau} > 0\) and the jumps of \(\bar{X}\) do not accumulate. Define the following indicator functions for \(y = (y_1, y_2), z = (z_1, z_2) \in E:\)

\[
\bar{h}_i(y, z) := \mathbf{1}_{\{n \geq |y_1 - 1|z_1 \geq \varepsilon\}} \mathbf{1}_{\{y_2 = 0\}} + \mathbf{1}_{\{n \geq y_2, z_1 \geq \varepsilon\}},
\]

\[
\bar{g}_i(y, z) := \mathbf{1}_{\{y_2, z_1 > n\}} + \mathbf{1}_{\{y_2, z_1 < \varepsilon\}} = \mathbf{1}_{\{y_2 > 0\}} (1 - \bar{h}_i(y, z)).
\]

Finally, for \(y, z \in E\) and \(i = 1, 2\) (recall (1.26)), define

\[
\bar{J}_{1,i}(y, z) := \bar{h}_i(y, z)(y_1 - 1)z_i,
\]

\[
\bar{J}_{2,i}(y, z) := \bar{h}_{3-i}(y, z)y_2z_{3-i},
\]

\[
\bar{J}_i(y, z) := \bar{J}_{1,i}(y, z) + \bar{J}_{2,i}(y, z). \tag{2.8}
\]
and
\[ \bar{K}_i(y, z) := \tilde{y}_{3-i}(y, z) y_{2z3-i}. \] (2.9)

To simplify the notation, introduce the processes
\[ \bar{M}_{t,i}(k) \equiv \int_0^{t \wedge \bar{T}} \int_E \bar{J}_i(y, \bar{X}_{s-}(k)) \bar{M}({k}, ds, dy) \] (2.10)
and
\[ \bar{N}_{t,i}(k) \equiv \int_0^{t \wedge \bar{T}} \int E \bar{K}_i(y, \bar{X}_{s-}(k)) \bar{N}'({k}, ds, dy) \] (2.11)
\[ + \int_{t \wedge \bar{T}} \bar{A}_m \bar{X}_{s-}(k) 1_{\{\bar{X}_{s-}(k) > 0\}} ds. \]

Now we are ready to define the process \( \bar{X} \) starting at \( x \in \mathbb{L}^{\beta, E} \) as a solution of some stochastic equations. To this end we need the following proposition.

**Proposition 2.1** For any \( x \in \mathbb{L}^{\beta, E} \), there exists a solution \( \bar{X} \in D_{\mathbb{L}^{\beta, E}} \) to the following system of equations
\[ \bar{X}_{t,i}(k) = x_i(k) + \int_0^t A_m \bar{X}_{s,i}(k) ds + \bar{M}_{t,i}(k) - \bar{N}_{t,i}(k), \quad k \in S_m, i = 1, 2, \] (2.12)
and
\[ \bar{X}_i(k) = x(k) \text{ for } t \geq 0 \text{ and } k \in S \setminus S_m. \] (2.13)

We will prove the proposition via a series of lemmas. First we will need the following.

**Lemma 2.2** Recall the measure \( \nu \) from \ref{assumptions}. Let \( \epsilon > 0 \).

(i) We have
\[ \nu\{(0) \times (\epsilon, \infty)\} = \frac{1}{\pi} \frac{1}{1 + \epsilon^2} \leq \frac{2}{\pi} (1 \wedge \epsilon^{-2}), \]
and
\[ \nu\{([0, \infty) \setminus (1 - \epsilon, 1 + \epsilon)) \times \{0\}\} = \begin{cases} \frac{8}{\pi} \frac{1}{\epsilon(4 - \epsilon^2)} - \frac{2}{\pi}, & \text{if } \epsilon \leq 1, \\ \frac{2}{\pi} \frac{1}{\epsilon(2 + \epsilon)}, & \text{if } \epsilon > 1, \end{cases} \leq \frac{2}{\pi} (\epsilon^{-1} \wedge \epsilon^{-2}). \]

(ii) We have
\[ \int y_2 \nu(dy) = 1. \]
and
\[ \int_{\{|y_1 - 1| \geq \epsilon\}} (y_1 - 1) \nu(dy) = \begin{cases} \frac{1}{\pi} \log \left( \frac{2 + \epsilon}{2 - \epsilon} \right) - \frac{4}{\pi} \frac{1}{2 + \epsilon}, & \text{if } \epsilon \leq 1, \\ \frac{1}{\pi} \log \left( \frac{2 + \epsilon}{2 - \epsilon} \right) + \frac{2}{\pi} \frac{1}{2 + \epsilon}, & \text{if } \epsilon > 1, \end{cases} \]

hence
\[ \left| \int_{\{|y_1 - 1| \geq \epsilon\}} (y_1 - 1) \nu(dy) \right| \leq \frac{2}{\pi}. \]

Furthermore, for all \( n \geq 0 \) we get as a consequence
\[ \left| \int_{\{n \geq |y_1 - 1| \geq \epsilon, y_2 = 0\} \cup \{n \geq y_2 \geq \epsilon\}} (y_1 - 1) \nu(dy) \right| \leq \frac{4}{\pi}. \]
(iii) We have
\[ \int_{\{y_1 - 1 \geq \epsilon\}} (y_1 - 1) \nu(dy) = \frac{1}{\pi} \log(1 + 2/\epsilon) + \frac{2}{\pi} \frac{1}{2 + \epsilon} \leq \frac{4}{\pi} \epsilon^{-1}. \]

(iv) For \( p \in (1, 2) \), we have
\[ m_p := \int_E |y_1 - 1|^p \nu(dy) \leq \frac{4}{\pi} \frac{p^2 - 2p + 2}{p(p - 1)(2 - p)} < \infty. \] (2.14)

Proof. The statements (i)-(iii) are derived by simple calculus. We omit the details. For (iv) note that
\[ m_p \leq \frac{4}{\pi} \int_0^\infty \frac{v}{(1 + v^2)^2} dv + \frac{4}{\pi} \int_0^1 u(1 - u)^{2-p} du + \frac{4}{\pi} \int_0^\infty u(1 - u)^{2-p} (1 + u)^2 du \]
and the right hand side equals the right hand side of (2.14).

Note that Lemma 2.2 implies \( \int (y_1 - 1) \nu(dy) = 0 \) in the sense of a Cauchy principal value.

Lemma 2.3 For any \( x \in \mathbb{R}^d \), \( \tilde{X} \in D_{L^{\beta,E}} \) solves (2.12)-(2.13) if and only if \( \tilde{X} \) solves
\[ \tilde{X}_{t,i}(k) = x_i(k) + \int_0^t \left( \int_0^{t \wedge \tau} \int_E \tilde{J}_{1,i}(y, \tilde{X}_{s} - (k)) \tilde{M}(\{k\}, ds, dy) + \int_0^{t \wedge \tau} \int_E \tilde{J}_{2,i}(y, \tilde{X}_{s} - (k)) \tilde{N}(\{k\}, ds, dy) \right) ds, k \in S_m, i = 1, 2, \] (2.15)
and (2.13).

Proof. Suppose that \( \tilde{X} \in D_{L^{\beta,E}} \) solves (2.12)-(2.13). Fix an arbitrary \( k \in S_m \). Note that since \( \tilde{X}_{s,i}(k) \tilde{X}_{s,3-i}(k) = 0 \), we have (compare Lemma 2.2 (ii))
\[ \int_0^t \int_E y_2 \tilde{X}_{s,3-i}(k) \tilde{N}(\{k\}, ds, dy) \]
\[ = \int_0^t A_m \tilde{X}_{s,i}(k) 1_{\tilde{X}_{s,3-i}(k) > 0} ds \int_E y_2 \nu(dy) \]
\[ = \int_0^t A_m \tilde{X}_{s,i}(k) 1_{\tilde{X}_{s,3-i}(k) > 0} ds, \quad i = 1, 2. \] (2.16)
We use this in the last equality to obtain
\[
\int_0^{t \wedge \tau} \int_E \bar{J}_{2,i}(y, \bar{X}_{s-}(k)) \bar{M}({k}, ds, dy) - \bar{N}_{t,i}(k)
\]
\[
= \int_0^{t \wedge \tau} \int_E \bar{J}_{2,i}(y, \bar{X}_{s-}(k)) \bar{N}({k}, ds, dy)
\]
\[
- \int_0^{t \wedge \tau} \int E (\bar{J}_{2,i} + \bar{K}_i)(y, \bar{X}_{s-}(k)) \bar{N}'({k}, ds, dy)
\]
\[
- \int_{t \wedge \tau}^t A_m \bar{X}_{s,i}(k) 1_{\{\bar{X}_{s-3}(k) > 0\}} ds
\]
\[
= \int_0^{t \wedge \tau} \int E \bar{J}_{2,i}(y, \bar{X}_{s-}(k)) \bar{N}({k}, ds, dy)
\]
\[
- \int_0^{t \wedge \tau} \int E \bar{Y}_{s-3}(k) \bar{N}'({k}, ds, dy)
\]
\[
- \int_{t \wedge \tau}^t A_m \bar{X}_{s,i}(k) 1_{\{\bar{X}_{s-3}(k) > 0\}} ds
\]
\[
= \int_0^{t \wedge \tau} \int E \bar{J}_{2,i}(y, \bar{X}_{s-}(k)) \bar{N}({k}, ds, dy)
\]
\[
- \int_{t \wedge \tau}^t A_m \bar{X}_{s,i}(k) 1_{\{\bar{X}_{s-3}(k) > 0\}} ds.
\]

Now recalling (2.16) we get that \( \bar{X}(k) \) satisfies (2.15) for \( k \in S_m \).

To get the converse implication, that is, if \( \bar{X}(k) \) satisfies (2.15) for \( k \in S_m \) then it also satisfies (2.12), one should just reverse the above argument. \( \square \)

By the above lemma to prove Proposition 2.1 it is enough to show that there exists a solution to (2.15) and (2.13) in \( D_{X,E} \). Let us start the construction of a solution. For this we will need some auxiliary process \( Y \in D_{X,E} \). Also for \( Y \in D_{X,E} \) denote

\[
I_{t,i}^Y(k) = 1_{\{s \geq 0, i = 1, 2\}} A_m Y_{s,i}(k)
\]

We next construct a process \( Y \) that describes the evolution of the process \( \bar{X} \) until the first jump occurs.

**Lemma 2.4** Let \( x \in L^{X,E} \). Then there exists a \( Y \in D_{X,E} \) that solves the following system of equations:

\[
Y_{t,i}(k) = x_i(k) + \int_0^t 1_{\{Y_{s-3-i}(k) = 0\}} A_m Y_{s,i}(k) ds
\]

\[
- \int_0^t \int_E \bar{J}_{i,1}(y, Y_{s-}(k)) I_{t,i}^Y(k) \nu(dy) ds, \quad k \in S_m, i = 1, 2,
\]

\[
Y_{t,i}(k) = x_i(k), \quad k \in S \setminus S_m, i = 1, 2.
\]

Moreover, if \( x_i(k) > 0, then

\[
Y_{t,i}(k) > 0 \quad \text{for all } t > 0.
\]

**Proof.** For \( k \in S \setminus S_m \) the result is trivial. Now we will construct the solution on the set \( S_m \). First we will handle the case where \( x_1(k) + x_2(k) > 0 \) for all \( k \in S_m \).
Define
\[ S_{i,m} = \{ k \in S_m : x_i(k) > 0 \} \quad \text{for } i = 1, 2. \]

Define
\[ Y_{t,3-i}(k) = 0 \quad \text{for all } t \geq 0, k \in S_{i,m}, i = 1, 2. \] (2.19)

Now we will construct a solution to the following system of equations
\[
Y_{t,i}(k) = x_i(k) + \int_0^t 1_{\{Y_{s,3-i}(k) = 0\}} A_m Y_{s,i}(k) \, ds
- \int_0^t \int_E \tilde{J}_{1,i}(y,Y_{s,-}(k)) P_s^Y(k) \nu(dy) \, ds, \quad t \geq 0, k \in S_{i,m}, i = 1, 2.
\] (2.20)

It is easy to check that the above system of equations has a non-explosive solution due to the following estimate on the drift term (see Lemma 2.2(ii))
\[
\left| \int_E \tilde{J}_{1,i}(y,Y_{s,-}(k)) P_s^Y(k) \nu(dy) \right|
= \left| \int_E \tilde{J}_{1,i}(y,Y_{s,-}(k)) 1_{\{Y_{s,i}(k) > 0\}} \frac{A_m Y_{s,3-i}(k)}{Y_{s,i}(k)} \nu(dy) \right|
\leq \frac{4}{\epsilon} 1_{\{Y_{s,i}(k) > 0\}} A_m Y_{s,3-i}(k) ds, \quad k \in S_{i,m}, i = 1, 2,
\]

where we also used that \( Y_{s,3-i}(k) = 0 \) and hence \( A_m Y_{s,3-i}(k) \geq 0 \) for \( k \in S_{i,m} \).

Next we are going to show that \( Y_{t,i}(k) > 0 \) for \( k \in S_{i,m} \). Let \( k \in S_{i,m} \). Then by the calculations similar to the above we get (see Lemma 2.2(iii))
\[
\int_E \tilde{J}_{1,i}(y,Y_{s,-}(k)) P_s^Y(k) \nu(dy) \leq \frac{4}{\epsilon} 1_{\{Y_{s,i}(k) > 0\}} Y_{s,i}(k) A_m Y_{s,3-i}(k).
\]

This implies that
\[
Y_{t,i}(k) \geq x_2(k) + \int_0^t 1_{\{Y_{s,3-i}(k) = 0\}} \sum_{l \neq k} A(k,l) Y_{s,i}(l) \, ds - V_t
\]
where
\[
V_t = \int_0^t 1_{\{Y_{s,3-i}(k) = 0\}} (-A(k,k)) Y_{s,i}(k) \, ds
+ \frac{4}{\epsilon} \int_0^t 1_{\{Y_{s,i}(k) > 0\}} Y_{s,i}(k) A_m Y_{s,3-i}(k) ds.
\]

Since \( \frac{\partial V_t}{\partial t} \) is linear in \( Y_{t,i}(k) \) we immediately get that \( Y_{t,i}(k) \) satisfying (2.20) does not hit 0 in finite time, which means that
\[
Y_{t,i}(k) > 0 \quad \text{for all } t > 0, k \in S_{i,m}.
\] (2.21)

Using (2.21), it is easy to check that \( Y \) defined by (2.19), (2.20) indeed solves the system of equations (2.17) and satisfies (2.18). So the lemma is proved for \( x \in L^3,E \) such that \( x_1(k) + x_2(k) > 0 \) for all \( k \in S_m \).

Now denote
\[
S_{3,m} = \{ k \in S_m : x_1(k) = x_2(k) = 0 \},
\]
and suppose that \( S_{3,m} \neq \emptyset \). We can define the approximating sequence of processes \( Y^l \) with
\[
(Y^l_{0,1}(k),Y^l_{0,2}(k)) = (0,1/l), \quad k \in S_{3,m},
Y^l_{0}(k) = x(k), \quad k \in S \setminus S_{3,m},
\]

\[
Y^l_{t}(k) = x(k), \quad k \in S \setminus S_{3,m},
\]

\[
Y^l_{t}(k) = x(k), \quad k \in S \setminus S_{3,m},
\]
that solves (2.17) and satisfies (2.18) (each $Y^t$ is constructed as above). By letting $l \to \infty$ it is easy to show that any limit point of $\{Y^t\}_{t \geq 1}$ also solves (2.17) and satisfies (2.18). We leave the details to the reader. \hfill \Box

The process $Y$ constructed in Lemma 2.4 will describe the evolution of the process $\tilde{X}$ until the first jump. Now we will have to define the time of jump and what happens at the time of the first jump. Define the point process $Y$. The process $\tilde{X}(k)$. For $i = 1, 2$ let us define

$$\tilde{X}_i(k) = \begin{cases} 1 & t < \tau, \\ Y_t(k), & k \in S \setminus \{k^*\}, \end{cases}$$

$$\Delta \tilde{X}_{\tau,i}(k^*) = \left( \int_0^{\tau - \tau^*} - \int_0^{\tau} \right) \int_E \left( \tilde{J}_{1,i} \left( y, Y_{s,-}(k^*) \right) + \tilde{J}_{2,i} \left( y, Y_{s,-}(k^*) \right) \right) N^Y(\{k^*\}, ds, dy), \quad i = 1, 2,$$

$$\tilde{X}_{\tau,i}(k^*) = \tilde{X}_{\tau,-i}(k^*) + \Delta \tilde{X}_{\tau,i}(k^*), \quad i = 1, 2.$$
Lemma 2.6 \{\tilde{X}_{t\wedge \tau}\}_{t \geq 0} \in D_{L, \beta, \epsilon}.

**Proof.** By our construction and of \(\tilde{X}\) and by Lemma 2.4 we get that
\[
\{\tilde{X}_{t\wedge \tau^-}\}_{t \geq 0} = \{Y_{t\wedge \tau^-}\}_{t \geq 0} \in D_{L, \beta, \epsilon}.
\]
So we just have to check what happens at the moment of jump to show that
\[
\tilde{X}_\tau(k^*) \in E.
\quad (2.24)
\]
As it follows by our construction of \(\tilde{X}\) and by Lemma 2.4 only one type is present at time \(\tau^-\) at site \(k^*\) and let us assume without loss of generality that \(\tilde{X}_{\tau^-}(k^*) = 0\) and \(\tilde{X}_{\tau^-}(k^*) > 0\). The jump that can push \(\tilde{X}_1(k^*)\) from zero comes from the jump of \(N^\gamma(\{k^*\}, ds, dy)\) at time \(\tau\). However, if such a jump at time \(\tau\) is of height \(y = (0, v)\), and \(\tilde{h}^2(y, \tilde{X}_{\tau^-}(k^*)) = 1\) then \(\tilde{X}_{\tau,1}(k^*)\) gets the value
\[
v \tilde{X}_{\tau^-}(k^*),
\]
but at the same time the jump of the process \(\tilde{X}_2\) at time \(\tau\) equals to
\[
\left(\int_0^{\tau^+} - \int_0^{\tau^-}\right) \int_E \tilde{J}_{1,2}(y, \tilde{X}_{\tau^-}(k^*)) N^\gamma(\{k^*\}, ds, dy) = -\tilde{X}_{\tau^-}(k^*)
\]
and this brings \(\tilde{X}_{\tau,2}(k^*)\) to zero, which means that there is still just one type present at the site \(k^*\) at time \(\tau\). If the jumps of \(N^\gamma(\{k^*\}, ds, dy)\) at time \(\tau\) is of height \(y = (u, 0)\), then this may change the positive value of \(\tilde{X}_{\tau^-2}(k^*)\) to another value (although it still stays positive) but the value of \(\tilde{X}_{\tau,1}(k^*)\) still stays at zero.

The same argument says that one cannot get two types present at the same site \(k^*\) from the situation when \(\tilde{X}_{\tau^-1}(k^*) > 0\) and \(\tilde{X}_{\tau^-2}(k^*) = 0\). Hence (2.24) follows and we are done. \(\square\)

**Lemma 2.7** \(\tilde{X}_i\) constructed above solves (2.13), (2.13) for \(t \leq \tau\).

**Proof.** Immediate by construction. \(\square\)

**Proof of Proposition 2.8** Lemma 2.7 shows that we have constructed solution to (2.13), (2.13) until the first jump of \(\tilde{X}\). Now we repeat the construction iteratively and thus construct the solution of (2.13), (2.13) until \(\tilde{\tau}\) – the time of \(n\)th jump of \(\tilde{X}\) – and then continue the solution satisfying (2.13), (2.13) after time \(\tilde{\tau}\) (no jumps after that). This finishes the construction of solution to (2.13), (2.13) and by Lemma 2.3 we are done. \(\square\)

The next aim is to show that \(\tilde{X}\) converges almost surely as \(n \to \infty\) to some process \(\bar{X}\) and to identify \(\bar{X}\) as the solution of a system of stochastic integral equations. In order to describe these equations, we will need some notation. Similarly as \(\tilde{M}\), we denote by \(\bar{M}\) the corresponding point process that is defined in terms of \(M_0\) and \(\tilde{X}\) instead of \(\tilde{X}\). Define \(\bar{h}_i = \lim_{n \to \infty} \tilde{h}_i\) and \(\bar{y}_i = \lim_{n \to \infty} \tilde{y}_i\), that is
\[
\bar{h}_i(y, x) = 1_{\{y_1 \geq x \geq 0\}} + 1_{\{y_2 \geq 0\}} + 1_{\{y_2 \geq \epsilon\}},
\bar{y}_i(y, x) = 1_{\{y_2 < \epsilon\}}.
\]
Define \(\bar{J}_i, \bar{K}_i\) and the processes \(\bar{M}_i\) and \(\bar{N}_i\) as above but with \(\bar{h}_i\) and \(\bar{y}_i\) replaced by \(\bar{h}_i\) and \(\bar{y}_i\), respectively, and with \(\bar{\tau}\) replaced by \(\infty\).

**Proposition 2.8** As \(n \to \infty\), we have the convergence
\[
\tilde{X} \xrightarrow{n \to \infty} \bar{X} \quad \text{P-a.s.},
\]
Lemma 2.9

Let \( \bar{X} \) solves the following system of equations:

\[
\bar{X}_{t,i}(k) = x_i(k) + \int_0^t A_m \bar{X}_{s,i}(k) \, ds + \bar{M}_{t,i}(k) - \bar{N}_{t,i}(k)
\]  

(2.25)

if \( k \in S_m \) and \( \bar{X}_{t,i}(k) = x_i(k) \) otherwise. Moreover, the processes \( \bar{M}_i(k), i = 1, 2, k \in S \), are martingales.

The proposition will be proved via a series of lemmas. Our first task is to show that the integrals with respect to the martingale measures that appear in the definition of \( \bar{X} \) are in fact martingales. Since the integrand functions are bounded by \( n \) and the total number of jumps is bounded by \( n \) we get by Proposition II.1.28 of [JS87] that \( \bar{M}_i(k) \) are local martingales. Moreover, since \( \bar{M}_i(k), i = 1, 2, k \in S_m \), are jump local martingales with bounded jumps we get that they in fact are martingales (see Corollary II.3 and Theorem II.28 in [Pr04]).

Now we are ready to give a bound on the expected values of \( \bar{X}_{t,i}(k), i = 1, 2 \). Since all the stochastic integrals are martingales, we get for all \( k \in S_m \) that

\[
E \left[ \bar{X}_{t,i}(k) \right] = x_i(k) + E \left[ \int_0^t (A_m \bar{X}_{s,i})(k) \, ds \right] - E \left[ \bar{N}_{t,i}(k) \right]
\]

\[
\leq x_i(k) + \int_0^t A_m E \left[ \bar{X}_{s,i} \right](k) \, ds
\]

\[
\leq S^m x_i(k).
\]

Note that the last inequality follows easily by the semigroup theory.

Now recall (2.4) and the definition of \( \bar{X}_i(k) \) for \( k \notin S_m \) to infer that

\[
E \left[ \bar{X}_{t,i}(k) \right] \leq S_t \bar{X}_{0,i}(k) \quad \text{for all } k \in S, \quad \forall > 0, \quad m, n \geq 1, \quad i = 1, 2.
\]

(2.26)

Now we will get the so-called mild form of equation (2.12). Recall that \( p_{l}^m \) is the kernel for \( S^m_t \). Then one can easily check that if \( \bar{X} \) solves (2.12), then it also satisfies the following mild form the equation:

\[
\bar{X}_{t,i}(k) = S^m_t \bar{X}_{0,i}(k) + \sum_{l \in S} \int_0^{t \wedge \tau} p_{l-s}^m(k,l) \, d\bar{M}_{s,i}(l)
\]

\[
- \sum_{l \in S} \int_0^{t \wedge \tau} p_{l-s}^m(k,l) \, d\bar{N}_{s,i}(k) \quad \text{for } k \in S_m.
\]

(2.27)

Now we are ready to derive the martingale decomposition for the product \( \bar{X}_{t,1}(k) \bar{X}_{t,2}(k) \).

Lemma 2.9 Let \( k_1, k_2 \in S, k_1 \neq k_2 \), and let \( t > 0 \). Then

\[
\bar{X}_{t,1}(k_1) \bar{X}_{t,2}(k_2) = S^m_t \bar{X}_{0,1}(k_1) S^m_t \bar{X}_{0,2}(k_2)
\]

\[
- \sum_{l_1, l_2 \in S} \int_0^{t \wedge \tau} p_{l_1-s}^m(k_1,l_1) p_{l_2-s}^m(k_2,l_2) \left( \bar{X}_{s,1}(l_1) A_m \bar{X}_{s,2}(l_2) + \bar{X}_{s,2}(l_2) A_m \bar{X}_{s,1}(l_1) \right) \, ds
\]

\[
+ M_t - N_t,
\]

where \( M_t \) and \( N_t \) are given by

\[
M_t = \sum_{l_1, l_2 \in S} \int_0^{t \wedge \tau} p_{l_1-s}^m(k_1,l_1) p_{l_2-s}^m(k_2,l_2) \left( \bar{X}_{s,1}(l_1) d\bar{M}_{s,2}(l_2) + \bar{X}_{s,2}(l_2) d\bar{M}_{s,1}(l_1) \right),
\]

\[
N_t = \sum_{l_1, l_2 \in S} \int_0^{t \wedge \tau} p_{l_1-s}^m(k_1,l_1) p_{l_2-s}^m(k_2,l_2) \left( \bar{X}_{s,1}(l_1) d\bar{N}_{s,2}(l_2) + \bar{X}_{s,2}(l_2) d\bar{N}_{s,1}(l_1) \right).
\]
Proof. This is an easy application of integration by parts formula and the fact that
\[ \bar{X}_{t, 1}(k) \bar{X}_{t, 2}(k) = 0 \quad \text{for all} \ t \geq 0, \ k \in S. \]

Corollary 2.10 Let \( k_1, k_2 \in S \) and \( t > 0 \). Then we have
\[ \mathbb{E} \left[ \bar{X}_{t, 1}(k_1) \bar{X}_{t, 2}(k_2) \right] \leq S_t x_1(k_1) S_t x_2(k_2). \]

Proof. For \( k_1, k_2 \in S_m \), the result is an immediate consequence of the previous lemma and \((2.4)\). If \( k \not\in S_m \), then \( \bar{X}_{t, 1}(k) = x_1(k) \). Hence, by \((1.9)\), we get
\[ \mathbb{E} \left[ \bar{X}_{t, 1}(k_1) \bar{X}_{t, 2}(k_2) \right] = x_1(k_1) \mathbb{E} \left[ \bar{X}_{t, 2}(k_2) \right] \leq S_t x_1(k_1) S_t x_2(k_2). \]

For the case \( k_2 \not\in S_m \), the estimate is similar. \( \square \)

Now we will give the \( L^p \) bounds for the martingales \( \tilde{M}, i = 1, 2 \).

Lemma 2.11 For any \( p \in (1, 2) \), there exists a constant \( c_p < \infty \) such that for all \( k \in S \), \( T > 0 \) and \( i = 1, 2 \), we have
\[ \mathbb{E} \left[ \sup_{t \leq T} |\tilde{M}_{t,i}(k)|^p \right] \leq c_p \int_0^T ((A + 1)S_t x_1(k) + 1)((A + 1)S_t x_2(k) + 1) \, ds. \]

Proof. First note that for \( k \not\in S_m \), we have \( \tilde{M}_{t,i}(k) \equiv 0 \) and hence the estimate is trivial.

Now let \( k \in S_m \). Note that
\[ \mathbb{E} \left[ \sup_{t \leq T} |\tilde{M}_{t,i}(k)|^p \right] \leq 2^{p-1} \mathbb{E} \left[ \sup_{t \leq T} |C_t|^p \right] + 2^{p-1} \mathbb{E} \left[ \sup_{t \leq T} |D_t|^p \right] \]
\[ =: L_1 + L_2, \]

where
\[ C_t := \int_0^{t \land \bar{\tau}} \int_E \bar{J}_{2,i}(y, \bar{X}_{s-}(k)) \bar{\mathcal{M}}(\{k\}, ds, dy) \]
and
\[ D_t := \int_0^{t \land \bar{\tau}} \int_E \bar{J}_{1,i}(y, \bar{X}_{s-}(k)) \bar{\mathcal{M}}(\{k\}, ds, dy) \]
are martingales of finite variation. As the point process \( \tilde{N} \) has no double points, the square variation process of \( C \) is
\[ [C, C]_t = \int_0^{t \land \bar{\tau}} \int_E \bar{J}_{2,i}(y, \bar{X}_{s-}(k))^2 \bar{\mathcal{N}}(\{k\}, ds, dy). \]
Hence, by the Burkholder-Gundy-Davis inequality (see, e.g., [DM83, Theorem VII.92]) we get with \( c'_p = \frac{p}{2} \):
\(2^{p-1}(4p)^p\)

\[
L_1 \leq c'_p E \left[ \left( \int_0^T \int_{R^+} \tilde{J}_{2,i}(y, \tilde{X}_s-(k))^p \tilde{N}({\{k\}, ds, dy}) \right)^{p/2} \right]
\]

\[
\leq c'_p E \left[ \int_0^T \int_{R^+} \tilde{J}_{2,i}(y, \tilde{X}_s-(k))^p \tilde{N}({\{k\}, ds, dy}) \right]
\]

\[
= c'_p E \left[ \int_0^T \int_{R^+} \tilde{J}_{2,i}(y, \tilde{X}_s-(k))^p \tilde{N}''({\{k\}, ds, dy}) \right]
\]

\[
\leq c'_p E \left[ \int_0^T \int_{R^+} y_2^p \tilde{X}_{s,3-i}(k)^p A_m \tilde{X}_{s,i}(k) \mathbf{1}_{\{x_2 \tilde{X}_{s,3-i}(k) \geq \varepsilon\}} \nu(dy) \right]
\]

\[
\leq c'_p E \left[ \int_0^T \tilde{X}_{s,3-i}(k)^p A_m \tilde{X}_{s,i}(k) ds \right]
\]

\[
\leq c'_p E \left[ \int_0^T (\tilde{X}_{s,3-i}(k)+1) A_m \tilde{X}_{s,i}(k) ds \right]
\]

\[
\leq c'_p \int_0^T (A+1)S_s x_1(k)+1) (A+1)S_s x_2(k)+1) ds.
\]

(2.29)

where the last inequality follows by Corollary 2.10 and 2.3. For the right hand side of (2.30) we use the trivial bound

\[
c'_p \int_0^T ((A+1)S_s x_1(k)+1) (A+1)S_s x_2(k)+1) ds.
\]

Hence we are done for \(L_1\).

Now we treat the \(L_2\) term. Recall \(m_p\) from Lemma 2.2(iv). Again by the Burkholder-Gundy-Davis inequality we get (using the trivial bound \(\tilde{h} \leq 1\) and letting \(c''_p = m_p c'_p\))

\[
L_2 \leq c''_p E \left[ \left( \int_0^T \int_{R^-} \tilde{J}_{1,i}(y, \tilde{X}_s-(k))^2 \tilde{N}({\{k\}, ds, dy}) \right)^{p/2} \right]
\]

\[
\leq c''_p E \left[ \int_0^T \int_{R^-} |y_1-1|^p \tilde{X}_{s,1}(k)^p \tilde{N}({\{k\}, ds, dy}) \right]
\]

\[
= c''_p E \left[ \int_0^T \int_{R^-} |y_1-1|^p \tilde{X}_{s,1}(k)^p \tilde{N}''({\{k\}, ds, dy}) \right]
\]

\[
\leq c''_p E \left[ \int_0^T \tilde{X}_{s,i}(k)^p A_m \tilde{X}_{s,i}(k) ds \right]
\]

\[
\leq c''_p E \left[ \int_0^T (\tilde{X}_{s,i}(k)+1) A_m \tilde{X}_{s,i}(k) ds \right]
\]

\[
\leq c''_p \int_0^T (S_s x_1(k)+1) A S_s x_2(k)+1) ds,
\]

where the last inequality follows by Corollary 2.10 and 2.3. Again, the right hand side of (2.31) is trivially bounded by (2.30). Now the claim holds with \(c_p = c'_p + c''_p\). \(\square\)
Remark 2.12 Note that the bound in the above lemma is uniform in \( m, n, \epsilon \).

Now we will show that for any \( t > 0 \), we can bound the number of jumps of \( \tilde{X}_t \) (that are greater than a certain size) on \([0, t]\) uniformly in \( n \). Let \( \bar{h}_i, \delta \) be defined as \( ar{h}_i \) with \( \delta \) instead of \( \epsilon \).

Lemma 2.13 For any \( k \in S_m, t > 0, i = 1, 2 \) and \( \delta > 0 \) we have

\[
\mathbb{E} \left[ \int_0^t \int_E \bar{h}_i(y, \tilde{X}_{s-}(k)) \tilde{N}(\{k\}, ds, dy) \right] \leq \frac{4}{\pi} \delta^{-2} \int_0^t S_{s\tau - i}(k) AS_s x_i(k) ds.
\]

Proof. By Lemma 2.2(i), Corollary 2.10 and (2.3), the left hand side is bounded by

\[
\frac{4}{\pi} \delta^{-2} \mathbb{E} \left[ \int_0^t \tilde{X}_{s,i}(k) A_m \tilde{X}_{s,i-1}(k) ds \right] \leq \frac{4}{\pi} \delta^{-2} \int_0^t S_{s,i}(k) AS_s x_i(k) ds.
\]

Proof of Proposition 2.8. Fix arbitrary \( t > 0 \). Define the following events

\[
B^n = \left\{ \omega: \int_0^t \int_E \bar{h}_i(y, \tilde{X}_{s-}(k)) \tilde{N}(\{k\}, ds, dy) < n/2 \right\} \cap \left\{ \int_0^t \int_E \bar{h}_3-i(y, \tilde{X}_{s-}(k)) \tilde{N}(\{k\}, ds, dy) < n/2 \right\}
\]

\[
C^n = \left\{ \omega: \int_0^t \int_E 1_{\{y_2 \tilde{X}_{s-}(k) \geq n\}} + 1_{\{y_2=0\}} 1_{\{|y_1-1| \tilde{X}_{s-}(k) \geq n\}} \tilde{N}(\{k\}, ds, dy) \right. \]
\[
+ \int_0^t \int_E 1_{\{y_2 \tilde{X}_{s-}(k) \geq n\}} + 1_{\{y_1>0\}} 1_{\{|y_1-1| \tilde{X}_{s-}(k) \geq n\}} \tilde{N}(\{k\}, ds, dy) = 0 \right\}
\]

\[
D^n = B^n \cap C^n.
\]

Clearly \( B^n \subset \{ \bar{t} > t \} \) and it is easy to see that on \( B^n \), \( \tilde{X} \) solves the system of equations (2.25). Moreover, by definition, \( B^n \subset B^{n+1} \). Hence in order to derive the claim it is enough to get that

\[
P[D^n] \longrightarrow 1 \quad \text{as } n \to \infty.
\]

(2.34)

However, by Lemma 2.13 and the Markov inequality, we get that there exist constants \( c_1 \) and \( c_2 \) depending only on the initial state \( x \) and on \( m \) such that

\[
P[(B^n)^c] \leq \frac{c_1(x,m)}{\epsilon^2 n},
\]

\[
P[(C^n)^c] \leq \frac{c_2(x,m)}{n^2},
\]

and (2.34) follows immediately. Now use Lemma 2.11 to see that \( \tilde{M}_i(k) \) is a martingale, since the \( L^1 \)-limit of martingales is a martingale.

3 Existence of a solution to \( (\text{MP}_1) \)

In what follows let \( (\epsilon_m)_{m \in \mathbb{N}} \) be a sequence such that

\[
\epsilon_m \downarrow 0.
\]

Let \( \tilde{X}^m \) be the process \( \tilde{X} \) defined in Proposition 2.8 with \( \epsilon \) replaced by \( \epsilon_m \). If no ambiguities occur, we will simply write \( \tilde{X} = \tilde{X}^m \). This and the next section will be devoted to the proof of the following theorem.
Theorem 3.1 Let $X_0^m = x \in L^{\beta,E}$ for all $m \in \mathbb{N}$. As $m \to \infty$, the processes $X^m$ converge weakly in $D_{L^{\beta,E}}$ to $X$ which is the unique solution to the martingale problem $\text{MP}_1$ with $X_0 = x$.

The strategy of the proof is pretty much standard. First we prove tightness of the sequence of the processes and show that every convergent subsequence satisfies the above martingale problem. Then we will show the uniqueness of the solution to the martingale problem $\text{MP}_1$.

This section is devoted to the proof of the following proposition which is the first step in the proof of the above theorem.

Proposition 3.2 Let $X_0^m = x \in L^{\beta,E}$ for all $m \in \mathbb{N}$.

(i) The sequence $(X^m)_{m \in \mathbb{N}}$ is tight in $D_{L^{\beta,E}}$.

(ii) Any limit point of $(X^m)_{m \in \mathbb{N}}$ in $D_{L^{\beta,E}}$ solves the martingale problem $\text{MP}_1$.

3.1 Proof of Proposition 3.2(i): Tightness

The strategy for showing tightness in Proposition 3.2 is to do two things:

(1) We show that the so-called compact containment condition holds for $X$ (see Lemma 3.5).

(2) Let $\text{Lip}_f(L^{\beta,E}; \mathbb{C})$ denote the space of bounded Lipschitz functions on $L^{\beta,E}$ that depend on only finitely many coordinates. We use moment estimates for the coordinate processes $X(k)$ and Aldous’s criterion to show that for $f \in \text{Lip}_f(L^{\beta,E}; \mathbb{C})$, the sequence $(f(X^m^t))_{t \geq 0}$ is tight in $D_{R}$ (Lemma 3.9).

By the Stone-Weierstraß theorem, $\text{Lip}_f(L^{\beta,E}; \mathbb{C}) \subset C_b(L^{\beta,E}; \mathbb{C})$ is dense in the topology of uniform convergence on compacts. Hence (1) and (2) imply tightness of $X^m$ by Theorem 3.9.1 of [EK86].

3.1.1 Compact containment

First by (2.26), Proposition 2.8 and Fatou’s lemma it is easy to get
\[
E \left[ X_{t,i}(k) \right] \leq S_{t} x_i(k) \quad \text{for} \quad k \in S, \ i = 1, 2, \tag{3.1}
\]
and hence by (1.13),
\[
E \left[ \langle X_{t,i}, \beta \rangle \right] \leq e^{MT} \langle x_i, \beta \rangle \quad \text{for} \quad i = 1, 2.
\]

By Corollary 2.10 and Fatou’s lemma, again it is easy to get for $k_1, k_2 \in S$
\[
E[\bar{X}_{t,1}(k_1)\bar{X}_{t,2}(k_2)] \leq S_{t} x_1(k_1) S_{t} x_2(k_2). \tag{3.2}
\]

Now we derive bounds on $\sup_{t \leq T}(\bar{X}_{t,i}, \beta)$, $i = 1, 2$. Define the negative parts of the diagonal entries of $A_m$ and $A$
\[
D_m(k) = \max\{-A_m(k,k), 0\}, \quad D(k) = \max\{-A(k,k), 0\}.
\]

Recall $\|A\|$ from (1.3) and note that
\[
\sup_{k \in S} D_m(k) \leq \sup_{k \in S} D(k) \leq \|A\| < \infty. \tag{3.3}
\]

Lemma 3.3 Let $\phi$ be a non-negative function on $S$. For any $T, K > 0$,
\[
P \left[ \sup_{t \leq T}(\bar{X}_{t,i}, \phi) > K \right] \leq K^{-1} \left\langle S_{T} x_i + \int_0^T (D S_s x_i + A S_s x_i) ds, \phi \right\rangle.
\]
Proof. First assume that \( \phi \) has finite support. Define

\[
M_{t,i}(k) \equiv \bar{X}_{t,i}(k) + \int_0^t D_m(k) \bar{X}_{s,i}(k) \, ds + \bar{N}_{t,i}(k),
\]

where (compare (2.11))

\[
\bar{N}_{t,i}(k) = \int_0^t \int_{\mathbb{R}^+} \bar{K}_i(y, \bar{X}_s(k)) \mathcal{N}'(\{k\}, ds, dy).
\]

It is easy to see by Proposition 2.8 that \( M_i(k) \) is a non-negative submartingale and hence so is \( \langle M_t, \phi \rangle \). Since both integrals are nonnegative, we have \( \bar{X}_{t,i}(k) \leq M_{t,i}(k) \) and thus Doob’s inequality yields

\[
P\left[ \sup_{t \leq T} \langle \bar{X}_{t,i}, \phi \rangle > K \right] \leq P\left[ \sup_{t \leq T} \langle M_{t,i}, \phi \rangle > K \right] \leq E[\langle M_{t,i}, \phi \rangle] / K.
\]

(3.5)

Recall (3.1) and note that hence

\[
E\left[ \int_0^T D_m(k) \bar{X}_{s,i}(k) \, ds \right] \leq D_m(k) \int_0^T S_{x_i}(k) \, ds.
\]

Now one can easily check that

\[
E[\bar{N}_{t,i}(k)] \leq \int y_2 \nu(dy) E\left[ \int_0^T A \bar{X}_m(k) \, ds \right] \leq \int_0^T AS_{x_i}(k) \, ds,
\]

where the last inequality follows by (3.1) and Lemma 2.2(ii). Hence the result for \( \phi \) with finite support follows by (3.5). For general \( \phi \in [0, \infty)^S \), the claim follows by monotone convergence. \[ \square \]

Lemma 3.4 A subset \( C \subset \mathbb{L}^{2,E} \) is relatively compact if and only if

(i) \( \sup_{y \in C} \|y_i\|_\beta < \infty \) for \( i = 1, 2 \), and

(ii) for every \( \delta > 0 \) there exists a finite \( F \subset S \) such that

\[
\sup_{y \in C} \|y_i 1_{S \setminus F}\|_\beta < \delta \quad \text{for} \quad i = 1, 2.
\]

Proof. Simple. \[ \square \]

Lemma 3.5 (Compact containment condition) For every \( T > 0 \) and every \( \delta > 0 \) there exists a compact set \( \Gamma_\delta \subset \mathbb{L}^{\beta,E} \) such that for every \( m \in \mathbb{N} \)

\[
P\left[ \bar{X}_m(t) \in \Gamma_\delta \text{ for all } t \in [0,T] \right] \geq 1 - \delta.
\]

Proof. Fix \( T > 0 \). By Lemma 3.4 it is enough to show the following:

(i) For any \( \delta > 0 \), there exists \( K > 0 \) such that for all \( m \in \mathbb{N} \)

\[
P\left[ \sup_{t \leq T} \langle \bar{X}_m(t), \beta \rangle > K \right] \leq \delta, \quad i = 1, 2.
\]

(3.7)
(ii) For any $\delta > 0$, there exists a finite set $F \subset S$ such that for all $m \in \mathbb{N}$,

$$
P \left[ \sup_{t \leq T} |\bar{X}_{t,i}^m 1_{F^c}^m(\cdot, \beta)| > \delta \right] \leq \delta, \ i = 1, 2. \quad (3.8)
$$

For $i = 1, 2$, $x_i \in \mathbb{L}^\beta$, hence

$$
\|S_T x_i\|_\beta \leq e^{\sup_k |A(k,k)| T} \|S_T x_i\|_\beta \leq e^{MT} e^{\sup_k |A(k,k)| T} \|x_i\|_\beta < \infty.
$$

Similarly, we get $\int_0^T DS_s x_i + AS_s x_i \, ds \in \mathbb{L}^\beta$. Hence the claim follows from Lemma 3.3 \hfill \square

### 3.1.2 Tightness of coordinate processes

The next step is to show for $f \in \text{Lip}_p(\mathbb{L}^{2, \beta}, \mathbb{C})$, that $(f(\bar{X}_t^m)_{t \geq 0})_{m \in \mathbb{N}}$ is tight in $D_2$. To this end, for $k \in S$ and $i = 1, 2$, we estimate the $p$-th moments ($p \in (1, 2)$) of the differences $\bar{X}_{t,i}(k) - \bar{X}_{s,i}(k)$ for $t - s$ small. This will be the cornerstone for applying Aldous’s criterion for tightness.

Recall that by Proposition 2.8, $\bar{X}$ solves the following system of equations:

$$
\bar{X}_{t,i}(k) = \left\{ 
\begin{array}{ll}
   x(k) + \int_0^t A_m \bar{X}_s(k) \, ds + \bar{M}_t(k) - \bar{N}_t(k), & \text{if } k \in S_m, \\
   x(k), & \text{otherwise,}
\end{array}
\right.
$$

where $\bar{M}_t(k), i = 1, 2, k \in S$ are martingales.

#### Lemma 3.6

For any $p \in (1, 2)$, there exists a constant $c_p$, such that for all $k \in S$, $T > 0$ and $i = 1, 2$, we have

$$
E \left[ \sup_{t \leq T} |\bar{M}_{t,i}(k)|^p \right] \leq c_p \int_0^T ((A + 1)S_s x_1(k) + 1)((A + 1)S_s x_2(k) + 1) \, ds,
$$

**Proof.** By Proposition 2.8, $\bar{M}$ almost surely (recall the implicit dependency of $\bar{M}$ on $n$). Hence by Lemma 2.11 and Fatou’s lemma, the claim follows. \hfill \square

From the above lemma it is easy to derive the following result that gives the bound (uniform in $m$) on the moments of the increments of $\bar{X}_i(k)$.

#### Lemma 3.7

For any $r_1, r_2 \in (0, 1]$ such that $1 < r_1/r_2 < 2$, there exists a constant $c = c(r_1, r_2)$ such that for all $T > 0$, $k \in S$ and $i = 1, 2$, we have

$$
E \left[ \sup_{t \leq T} |\bar{X}_{t,i}(k) - x_i(k)|^{r_1} \right] 
\leq c \max_{j=1,2} \left( \int_0^T \frac{2}{i'=1} ((A + 1)S_s x_{i'}(k) + 1) \, ds \right)^{r_j}.
$$

**Proof.** For $k \in S \setminus S_m$, the result is trivial. Hence now let $k \in S_m$. By Proposition 2.8 equation (2.3) and the triangle inequality we get

$$
E \left[ \sup_{t \leq T} |\bar{X}_{t,i}(k) - x_i(k)|^{r_1} \right] \leq R_1 + R_2 + R_3, \quad (3.10)
$$

where

$$
R_1 = E \left[ \sup_{t \leq T} \left( \int_0^t A\bar{X}_{s,i}(k) \, ds \right)^{r_1} \right],
$$

$$
R_2 = E \left[ \sup_{t \leq T} |\bar{M}_{t,i}(k)|^{r_1} \right],
$$

$$
R_3 = E \left[ \sup_{t \leq T} |\bar{N}_{t,i}(k)|^{r_1} \right].
$$
As in $R_1$ the integrand is nonnegative, and using Jensen’s inequality and (3.11), we get
\[
R_1 \leq \left( \mathbf{E} \left[ \int_0^T A X_{s,i}(k) \, ds \right] \right)^{r_1} \leq \left( \int_0^T A S x_i(k) \, ds \right)^{r_1}.
\] (3.11)

For $R_2$, use Jensen’s inequality and Lemma 3.6 (with $p = r_1/r_2$) to get that for some constant $c_{r_1/r_2} < \infty$,
\[
R_2 \leq \left( \mathbf{E} \left[ \sup_{t \leq T} |\mathcal{M}_{t,i}(k)|^{r_1/r_2} \right] \right)^{r_2} \leq c_{r_1/r_2} \left( \int_0^T ((A + 1) S x_1(k) + 1) \, ds \right)^{r_2}.\] (3.12)

For $R_3$ we get by Jensen’s inequality and by 3.6 that
\[
R_3 \leq \left( \mathbf{E} \left[ \int_0^T \int_E K_i(y, X_s(k)) N_i(\{k\}, ds, dy) \right] \right)^{r_1} \leq \left( \int_0^T A S x_i(k) \, ds \right)^{r_1}.
\] (3.13)

Combining (3.11), (3.12), (3.13) gives the claim of this lemma. \(\square\)

**Lemma 3.8** Fix arbitrary $T > 0$. Let $(\tau_m)_{m \in \mathbb{N}}$ be a sequence of stopping times bounded by $T$. Then for any $r_1 \in (0, 1)$ and $k \in S$, we have
\[
\lim_{\delta \downarrow 0} \lim_{m \to \infty} \mathbf{E} \left[ |\bar{X}_{\tau_m + \delta, i}(k) - \bar{X}_{\tau_m, i}(k)|^{r_1} \right] = 0.\] (3.14)

**Proof.** Without loss of generality, we may assume $\delta \leq 1$. We define the stopping time
\[
\sigma_{m,K} = \inf \{ t : \langle \bar{X}_{t,1} + \bar{X}_{t,2}, \beta \rangle \geq K \}
\]
and let
\[
R_1 := \mathbf{E} \left[ |\bar{X}_{\tau_m + \delta, i}(k) - \bar{X}_{\tau_m, i}(k)|^{r_1} 1_{\{\sigma_{m,K} \leq T + 1\}} \right],
\]
\[
R_2 := \mathbf{E} \left[ |\bar{X}_{\tau_m + \delta, i}(k) - \bar{X}_{\tau_m, i}(k)|^{r_1} 1_{\{\sigma_{m,K} > T + 1\}} \right].
\]

Let $p > 1$ be such that $pr_1 \leq 1$ and define $q > 1$ by $1/p + 1/q = 1$. Then by Hölder’s inequality, we have
\[
R_1 \leq \left( \mathbf{E} \left[ |\bar{X}_{\tau_m + \delta, i}(k) - \bar{X}_{\tau_m, i}(k)|^{pr_1} \right] \right)^{1/p} \mathbf{P} [\sigma_{m,K} \leq T + 1]^{1/q} \] (3.15)
\[
\leq 2 \left( \mathbf{E} \left[ \sup_{t \leq T + 1} \bar{X}_{t,i}(k)^{pr_1} \right] \right)^{1/p} \mathbf{P} \left[ \sup_{t \leq T + 1} \langle \bar{X}_{t,1} + \bar{X}_{t,2}, \beta \rangle \geq K \right]^{1/q}.
\]

By Lemma 3.7 we have
\[
h(T) := \sup_{m \in \mathbb{N}} \left( \mathbf{E} \left[ \sup_{t \leq T + 1} \bar{X}_{t,i}(k)^{pr_1} \right] \right)^{1/p} < \infty.
\]

Let $\delta_1 > 0$. By Lemma 3.3 we can choose $K$ sufficiently large such that
\[
\sup_{m \in \mathbb{N}} \mathbf{P} \left[ \sup_{t \leq T + 1} \langle \bar{X}_{t,1} + \bar{X}_{t,2}, \beta \rangle \geq K \right]^{1/q} \leq \frac{\delta_1}{h(T)}.
\]

This implies that
\[
\sup_{m \in \mathbb{N}} R_1 \leq \delta_1.
\] (3.16)
Now we turn to $R_2$. Let $r_2 \in (r_1/2, r_1)$. By the strong Markov property of $\bar{X}^m$ and Lemma 3.7, we obtain

\[
\mathbb{E} \left[ |X_{\tau_{m, \bar{X}^m} + \delta, i}(k) - \bar{X}_{\tau_{m, \bar{X}^m}}^m(k)|^{r_1} \mathbf{1}_{\{\sigma_{m, K} > T + 1\}} \right]
\leq \mathbb{E} \left[ |X_{\tau_{m, \bar{X}^m} + \delta, i}(k) - \bar{X}_{\tau_{m, \bar{X}^m}}^m(k)|^{r_1} \mathbf{1}_{\{\bar{X}_{\tau_{m, \bar{X}^m} + \delta, i}^m(\bar{X}) \leq K\}} \right]
= \mathbb{E} \left[ |X_{\tau_{m, \bar{X}^m} + \delta, i}(k) - \bar{X}_{\tau_{m, \bar{X}^m}}^m(k)|^{r_1} \mathbf{1}_{\{\bar{X}_{\tau_{m, \bar{X}^m} + \delta, i}^m(\bar{X}) \leq K\}} \right]
\leq c(r_1, r_2) \max_{j=1,2} \mathbb{E} \left[ \left( \int_0^\delta \prod_{l'=1}^2 (A + 1)S_sX_{\tau_{m, \bar{X}^m}, l'}(k) + 1 \right) ds \right]^{r_j}
\times \mathbf{1}_{\{\bar{X}_{\tau_{m, \bar{X}^m} + \delta, i}^m(\bar{X}) \leq K\}}.
\]

(3.17)

Note that on the event \{$(\bar{X}_{\tau_{m, 1}^m} + \bar{X}_{\tau_{m, 2}^m}, \beta) \leq K$\}, by (1.10), (1.11) and (1.12), we have

\[
((A + 1)S_s\bar{X}_{\tau_{m, \bar{X}^m}}^m(k) \leq (M + 1)e^{M_kK/\beta(k)}.
\]

Hence for some constant $c(r_1, r_2, M, T)$, the right hand side of (3.17) is bounded by

\[
c(r_1, r_2, M, T) \max_{j=1,2} \left( \int_0^\delta \left( \frac{K}{\beta_k} + 1 \right)^2 ds \right)^{r_j} \longrightarrow 0 \text{ as } \delta \downarrow 0,
\]

uniformly in $m$. Together with (3.16), this implies

\[
\limsup_{\delta \downarrow 0} \sup_{m \geq 1} (R_1 + R_2) \leq \delta_1.
\]

(3.19)

Since $\delta_1 > 0$ was arbitrary, the limit is in fact 0. This finishes the proof.

\[\Box\]

**Lemma 3.9** Let $f \in \text{Lip}_f(L^{1, E}; \mathbb{C})$. Then $f(\bar{X}_t^m)_{t \geq 0}$, $m \in \mathbb{N}$, is tight in $D_{\mathbb{C}}$.

**Proof.** Let $T > 0$, $(\tau_m)_{m \in \mathbb{N}}$ and $r_1 \in (0,1)$ be as in Lemma 3.8. Since $f$ is Lipschitz and depends on only finitely many coordinates, by Lemma 3.8 we have

\[
\lim_{\delta \downarrow 0} \sup_{m \to \infty} \mathbb{E} |f(\bar{X}_{\tau_{m, \bar{X}^m} + \delta}) - f(\bar{X}_{\tau_{m, \bar{X}^m}})|^{r_1} = 0.
\]

Hence by Aldous’s tightness criterion (see [Ald78]), the claim follows.

\[\Box\]

### 3.2 Proof of Proposition 3.2(ii): Martingale problem for limit points

In the previous subsection, we proved that $(\bar{X}^m)_{m \in \mathbb{N}}$ is tight in $D_{\mathbb{R}^{1, E}}$ and is hence relatively compact by Prohorov’s theorem. Let $X$ be an arbitrary limit point of that sequence. Then there exists a subsequence $(\bar{X}^{m_k})_{k \in \mathbb{N}}$ such that

\[
\bar{X}^{m_k} \rightharpoonup X \text{ as } k \to \infty
\]

weakly in $D_{\mathbb{R}^{1, E}}$. In order to ease the notation, in this section we will assume that the sequences $(\epsilon_m)_{m \in \mathbb{N}}$ and $(S_m)_{m \in \mathbb{N}}$ were chosen such that

\[
\bar{X}^m \rightharpoonup X \text{ as } m \to \infty
\]

First, we derive estimates on the first and second moment of a limit point $X$.

**Lemma 3.10** For all $t > 0$, $k, l \in S$ with $k \neq l$ and $i = 1, 2$ we have

\[
\mathbb{E} [X_{t,i}(k)] \leq S_{t,0,i}(k),
\]

\[
\mathbb{E} [X_{t,1}(k)X_{t,2}(l)] \leq S_{t,0,1}(k)S_{t,0,2}(l).
\]

(3.20)
For every \( p \in (0, 1] \), there exists a constant \( c(p) \) such that
\[
E \left[ \sup_{t \leq T} X_{t,i}(k)^p \right] \leq c(p) \left( x_i(k)^p + \int_0^T ((A + 1)S_x x_1(k) + 1) \times ((A + 1)S_x x_2(k) + 1) \, ds + 1 \right). 
\] (3.21)

Moreover for any non-negative function \( \phi \) on \( S, T, K > 0 \), and \( i = 1, 2 \),
\[
P \left[ \sup_{t \leq T} (X_{t,i}, \phi) > K \right] \leq \frac{\langle S_T x_i + \int_0^T DS_x x_i + A S_x x_i \, ds, \phi \rangle}{K}. 
\] (3.22)

**Proof.** The inequalities in (3.20) follow from (3.21) and (3.22), respectively, with the help of Fatou’s lemma by switching to the Skorohod space with the a.s. convergence instead of weak convergence of the processes. For the same reasons, (3.21) follows from Lemma 3.7. Here we also used the trivial inequality \( a^p \leq a + 1 \) for \( p \leq 1 \) and \( a \geq 0 \).

Equation (3.22) follows from Lemma 3.8 again by the properties of the weak convergence. \( \square \)

Now we have to identify the equation for the limiting point \( X \). For this goal, it will be enough to identify the compensator measures of the limits of the martingales \( \bar{M}_i(k) \). At this stage it will be more convenient for us to use another representation of those processes. Let \( \bar{N}_\Delta^m(\{k\}, \cdot) = \bar{N}_\Delta^m(\{k\}, \cdot), k \in S \), be the family of point process on \( \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \) induced by the processes \( \bar{X}^m \), that is
\[
\bar{N}_\Delta^m(\{k\}, dt, dz) = \sum_s 1_{\{\Delta X^m_s(k) \neq 0\}} \delta_{(s, \Delta X^m_s(k))}(dt, dz).
\]

Let \( \bar{N}_\Delta^m = \bar{N}_\Delta^m \) denote the corresponding compensator measure and let \( \bar{M}_\Delta := \bar{N}_\Delta - \bar{N}_\Delta^m \). Furthermore, define \( \bar{N}_\Delta, \bar{N}_\Delta^m \) and \( \bar{M}_\Delta \) similarly, but with \( \bar{X} \) replaced by \( X \).

Recall from Proposition 2.3 that
\[
\bar{X}_t(k) = \bar{X}_0(k) + \int_0^t A_m \bar{X}_s(k) ds + \int_0^t \int_E \bar{J}(y, \bar{X}_s(k)) \bar{M}(\{k\}, ds, dy) \tag{3.23}
\]

where for \( i = 1, 2 \),
\[
\bar{J}_i(y, z) = (1_{\{|y_1| \leq \epsilon_m\}} 1_{y_2 = 0} + 1_{\{|y_2, z_i \geq \epsilon_m\}} (y_1 - 1)z_i + 1_{\{|y_2, 3 - z_i \leq \epsilon_m\}} y_2 z_{3 - i}
\]

and
\[
\bar{K}_i(y, z) = 1_{\{|y_2, 3 - i \leq \epsilon_m\}} y_2 z_{3 - i}. \]

As \( \bar{M} \) is a compensated jump measure and \( \bar{N}' \) is absolutely continuous, we get
\[
\bar{M}_\Delta(\{k\}, dt, A) = \int_E 1_{A \setminus \{0\}} (J(y, \bar{X}_{t -}(k))) \bar{M}(\{k\}, dt, dy)
\]

and
\[
\bar{X}_t(k) = x(k) + \int_0^t A_m \bar{X}_s(k) ds + \int_0^t \int_{\mathbb{R}^2} z \bar{M}_\Delta(\{k\}, ds, dz) \tag{3.24}
\]

\[
- \int_0^t \int_E \bar{K}(y, \bar{X}_s(k)) \bar{N}(\{k\}, ds, dy). \tag{3.25}
\]
Lemma 3.11  The weak limit point $X$ is a solution of
\[ X_t(k) = x(k) + \int_0^t A X_s(k) \, ds + \int_0^t \int_{\mathbb{R}^2} z \mathcal{M}_{\Delta} \{ \{ k \}, ds, dz \}, \tag{3.24} \]
where
\[ \mathcal{M}_{\Delta} \{ \{ k \}, dt, dz \} = \mathcal{N}_{\Delta} - \mathcal{N}'_{\Delta}. \]
The compensator measure of the point process $\mathcal{N}_{\Delta}$ is given by
\[ \mathcal{N}'_{\Delta} \{ \{ k \}, dt, A \} = \int_E 1_{A \setminus \{ 0 \}} \left( J(y, X_{t-}(k)) \right) \mathcal{N} \{ \{ k \}, dt, dy \}, \quad A \subset \mathbb{R}^2, \]
where
\[ J_i(y, z) = (y_1 - 1)z_i + y_2z_{3-i}. \]

**Proof.**  Note that $\bar{K} \to 0$ as $m \to \infty$. Hence the third integral in (3.23) vanishes as $m \to \infty$. By Theorem IX.2.4 of [JS87], it is thus enough to check for all $k \in S$ that
\[ \left( \bar{X}^m \{ k \}, \int_{\mathbb{R}^2} \bar{N}'_{\Delta} \{ \{ k \}, dt, dz \} G(z) \right) \overset{m \to \infty}{\to} \left( X(k), \int_{\mathbb{R}^2} \mathcal{N}'_{\Delta} \{ \{ k \}, dt, dz \} G(z) \right), \tag{3.25} \]
for any continuous $G \in C^+_0(\mathbb{R}^2)$ which is 0 in some neighborhood of 0. Note that
\[ \int_{\mathbb{R}^2} \mathcal{N}'_{\Delta} \{ \{ k \}, dt, dz \} G(z) = \int_E G(\bar{J}(y, \bar{X}_{t-}(k))) \mathcal{N}' \{ \{ k \}, dt, dy \}
= \left( \int_E G(\bar{J}(x, \bar{X}_{t-}(k))) \nu(dx) \right) \bar{I}(k) \, dt \tag{3.26} \]
and similarly
\[ \int_{\mathbb{R}^2} \mathcal{N}'_{\Delta} \{ \{ k \}, dt, dz \} G(z) = \left( \int_E G(J(x, X_{t-}(k))) \nu(dx) \right) I(k) \, dt. \tag{3.27} \]
By Skorohod’s theorem, we may assume that $\bar{X}$ and $X$ are defined on one probability space such that $\bar{X}$ converges almost surely to $X$ (and not only weakly). Furthermore, note that $\bar{J} \uparrow J$ as $m \to \infty$. Moreover by the estimates similar to (3.23) it is easy to check that for our choice of function $G$,
\[ \int_E G(\bar{J}(x, z(k))) \nu(dx) \bar{I}(z, k) \to \int_E G(J(x, z(k))) \nu(dx) I(z, k), \quad m \to \infty, \]
uniformly on $z$ in compacts of $L^{\beta, E}$. Here
\[ \bar{I}(z, k) := \sum_{i=1}^2 1_{\{ z_i(k) > 0 \}} \frac{A_{z_3-i}(k)}{z_i(k)}. \]
Hence the right hand side of (3.26) converges to that of (3.27) and we get (3.25). □

For $y \in \mathbb{R}^2$, define $h_y : E \to \mathbb{C}$ by
\[ h_y(z) := e^{(z-(1,0)) \circ y} - 1 - (z-(1,0)) \circ y, \]
Furthermore, for $x \in E$, let
\[ h_{x,y}(z) := \begin{cases} h_{x_1,y}(z), & \text{if } x_1 > 0, \\ h_{x_2(y_2,y_1)}(z), & \text{if } x_2 > 0, \\ 0, & \text{otherwise}. \end{cases} \tag{3.28} \]
Note that
\[ h_{x,y}(z) = e^{J(z,x) \circ y} - 1 - J(z,x) \circ y. \tag{3.29} \]
Lemma 3.12 For any \( x, y \in E \), we have
\[
\int_E h_{x,y} \, d\nu = 0.
\]

**Proof.** By symmetry, it is enough to consider the case \( x = (1,0) \). Note that \( h_{(1,0),y} = h_y \).

A simple application of Itô’s formula shows that \( h_y \) is harmonic. Recall that \( Q_z \) is the harmonic measure for planar Brownian on \( E \). Since \( h_y \) grows at most linearly, and since the first hitting time of \( E \) of planar Brownian motion has any \( p \)-th moment for \( p \in [1/2,1) \) (see \[KM08\] Lemma 3.6 or \[Bar77\] Equation (3.8) with \( \alpha = \pi/2 \)), we get \( \int h_y dQ_z = h_y(z) \). Recall that \( \nu \) is the vague limit of \( Q_{(1,\epsilon)}/\epsilon \) as \( \epsilon \to 0 \). Hence we can hope that \( \int h_y d\nu \) can be written as the limit of \( \epsilon^{-1} \int h_y dQ_{(1,\epsilon)} \). In fact, since \( h_y \) grows at most linearly and since \( h_y(1,0) = 0 \), by \[KM08\] Lemma 5.5, we get
\[
\int h_y \, d\nu = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int h_y \, dQ_{(1,\epsilon)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} h_y(1,\epsilon) = 0. \tag*{\square}
\]

Now we are ready to write the martingale problem for any limiting point \( X \).

**Lemma 3.13** Let \( y \in L^{f,E} \) and let \( X \) be any limit point of \( \bar{X} \). Then
\[
M_t := e^{\langle X_t, y \rangle} - e^{\langle X_0, y \rangle} - \int_0^t e^{\langle A X_s, y \rangle} - e^{\langle X_s - y \rangle} \, ds \tag{3.30}
\]
is a martingale.

**Proof.** By Itô’s formula applied to \( X \) solving (3.24) we get
\[
M_t := e^{\langle X_t, y \rangle} - e^{\langle X_0, y \rangle} - \int_0^t e^{\langle A X_s, y \rangle} - e^{\langle X_s - y \rangle} \, ds - \sum_{k \in S} \int_0^t \int_E N'(\{k\}, \beta, \nu) e^{\langle X_s - y \rangle} \times \left[ e^{J(z, X_{s-}(k)) \circ y(k)} - 1 - J(z, X_{s-}(k)) \circ y(k) \right] \, ds \tag{3.31}
\]
is a local martingale. By the definition of \( N' \) and \( h_{x,y} \) in (3.25), using Lemma 3.12 we get
\[
\int_0^t \int_E N'(\{k\}, \beta, \nu) e^{\langle X_s - y \rangle} [e^{J(z, X_{s-}(k)) \circ y(k)} - 1 - J(z, X_{s-}(k)) \circ y(k)] \, ds = \int_0^t e^{\langle X_s, y \rangle} \int_E h_{X_s, y(k)} \, d\nu = 0.
\]
Hence
\[
M_t = e^{\langle X_t, y \rangle} - e^{\langle X_0, y \rangle} - \int_0^t e^{\langle A X_s, y \rangle} e^{\langle X_s - y \rangle} \, ds, \tag{3.32}
\]
and now it remains to show that \( M \) is in fact a martingale. Applying (3.21), (1.12), (1.11), for all \( T > 0 \), we get
\[
\mathbf{E} \left[ \sup_{t \leq T} \left| \int_0^t e^{\langle A X_s, y \rangle} e^{\langle X_s - y \rangle} \, ds \right| \right] \leq \mathbf{E} \left[ \int_0^T |A X_{s,1}| + |A X_{s,2}| \, ds \right] \leq \int_0^T |A S_{x_1} + A S_{x_2}, y| \, ds \leq \sum_{k \in S} \sum_{i=1}^2 \int_0^T M e^{M_s} \frac{\| x_i \|_\beta}{\beta(k)} |y(k)| \, ds < \infty. \tag{3.33}
\]
Note that the last inequality follows since \( y \) has finite support. Since the exponents in (3.32) have nonpositive real part, they are bounded. Hence we conclude that
\[
\mathbb{E} \left[ \sup_{t \leq T} |M_t| \right] < \infty.
\]
But this implies that \( M \) is indeed a martingale. \( \square \)

By Lemma 3.13, any limit point of \( \bar{X} \) solves the martingale problem \((MP_1)\). Hence the proof of Proposition 3.2(ii) is now complete. \( \square \)

4 Uniqueness of solutions to the martingale problem \((MP_1)\)

This section is devoted to the proof of the following proposition.

**Proposition 4.1** There is a unique solution to the martingale problem \((MP_1)\) and the map \( x \mapsto P_x \) is measurable.

The proposition will be proved via a series of lemmas.

Recall from (1.19) that \( L^f,E \) is the space of \( y \in E^S \) with only finitely many nonzero coordinates. Let \( A^* \) and \( A \) be the transpose matrices of \( A \) and \( A \), respectively, and let \( S^* \) and \( S \) be the corresponding semigroups. Recall the definition of \( L^\beta,E_\infty \) from (1.20) and let \( D_{L^\beta,E_\infty} = D_{L^\beta,E_\infty}[0,\infty) \) be the Skorohod space of \( L^\beta,E_\infty \)-valued càdlàg paths.

We will define a \( D_{L^\beta,E_\infty} \) valued process \( Y = (Y_1, Y_2) \) that solves the martingale problem which is dual to \((MP_1)\).

Recall the function \( H \) from (1.17).

**Proposition 4.2** Let \( Y_0 = y \in L^f,E \). Then there exists the process \( Y \in D_{L^\beta,E_\infty} \) which satisfies the following martingale problem: For all \( x \in L^\beta,E \),
\[
M^t,x,y := H(x, Y_t) - H(x, Y_0) - \int_0^t \langle \langle x, A^* Y_s \rangle \rangle H(x, Y_s) \, ds \quad \text{(MP}^*_1\text{)}
\]
is martingale.

**Proof.** The existence of the process \( Y \in D_{L^\beta,E} \) that solves the martingale problem \((MP_1)\) for all \( x \in L^f,E \) follows immediately from the Proposition 3.2 since the assumptions on \( A \) are satisfied by \( A^* \) as well. To finish the proof we have to show that \( Y \) takes in fact values in the subspace \( L^\beta,E_\infty \) and that \( Y \) satisfies \((MP^*_1)\) for all \( x \in L^\beta,E \) (not only for \( x \in L^f,E \)).

**Step 1.** First we show that \( Y \) takes values in \( L^\beta,E_\infty \). It is enough to show that for all \( \phi \in L^\beta \) and \( i = 1, 2 \), we have
\[
P \left[ \sup_{t \leq T} \langle Y_{t,i}, \phi \rangle > K \right] \to 0 \quad \text{as} \quad K \to \infty. \quad (4.1)
\]
By (3.22) in Lemma 3.10 for any \( \phi \in L^\beta \) and \( K > 0 \), we get
\[
P \left[ \sup_{t \leq T} \langle Y_{t,i}, \phi \rangle > K \right] \leq K^{-1} \left\langle \frac{S^*_T Y_{0,i} + \int_0^T D S^*_s Y_{0,i} + A S^*_s Y_{0,i}}{\phi} \right\rangle. \quad (4.2)
\]
It is enough to show that the right hand side in (4.2) is finite. To this end, we estimate (recall \((3.3)\) and \((1.13)\))
\[
\left\langle \int_0^T D S^*_s Y_{0,i}, \phi \right\rangle \leq \|A\| \int_0^T \langle Y_{0,i}, S_s \phi \rangle \, ds
\]
\[
\leq \|A\| \|\phi\|_{\beta} \left( \int_0^T e^{M_s} \, ds \right) \sum_{k \in S} \frac{Y_{0,i}(k)}{\beta(k)} \quad (4.3)
\]
\[
< \infty.
\]
Note that the last inequality follows from the assumption that $Y_{0,i}$ has finite support. Similarly we obtain finiteness of the other expressions in (4.2). Hence (4.1) follows.

**Step 2.** Now we show that $Y$ satisfies (MP$_t$) for all $x \in \mathbb{L}^{\beta,E}$. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{L}^{f,E}$ such that

$$x_n \uparrow x \quad \text{as} \quad n \to \infty.$$

Then $M^{*,x_n,y}$ is a martingale for any $n \in \mathbb{N}$. By (4.2), for any $T > 0$,

$$\sup_{s \leq T} |H(x_n, Y_s) - H(x, Y_s)| \xrightarrow{n \to \infty} 0 \quad \text{a.s. (and hence in } L^1). \quad (4.4)$$

Note that

$$|\langle x_n, A^*Y_s \rangle| \leq \langle x_1 + x_2, A^*(Y_{s,1} + Y_{s,2}) \rangle = \langle A(x_1 + x_2), Y_{s,1} + Y_{s,2} \rangle.$$

Consequently, for all $T > 0$ and $t \in [0, T]$, we get

$$\left| \int_0^t \langle x_n, A^*Y_s \rangle H(x_n, Y_s) \, ds \right| \leq \int_0^t |\langle x_n, A^*Y_s \rangle| \, ds \leq \int_0^T |\langle x_n, A^*Y_s \rangle| \, ds \leq \int_0^T \langle A(x_1 + x_2), Y_{s,1} + Y_{s,2} \rangle \, ds. \quad (4.5)$$

By Lemma 3.10 the expectation of the right hand side is bounded by

$$\int_0^T \langle A(x_1 + x_2), \mathbf{S}_s'(Y_{0,1} + Y_{0,2}) \rangle \, ds \leq \int_0^T \langle \mathbf{S}_s A(x_1 + x_2), Y_{0,1} + Y_{0,2} \rangle \, ds \leq M \left( \int_0^T e^{Ms} \, ds \right) \left( \|x_1\|_\beta + \|x_2\|_\beta \right) \sum_{k \in \mathbb{S}} \frac{Y_{0,1}(k) + Y_{0,2}(k)}{\beta(k)} < \infty. \quad (4.6)$$

By dominated convergence, the integral term in the definition of $M^{*,x_n,y}$ converges in $L^1$ to the corresponding integral term for $M^{*,x,y}$. Hence $M^{t,x_n,y}$ converges in $L^1$ to $M^{*,x,y}$ for each $t$. Consequently, $M^{*,x,y}$ is a martingale. □

In Lemma 3.10 we established a bound on the first moments of those solutions $X$ of the martingale problem (MP$_1$) that arise as limiting points of the approximating processes $\tilde{X}$. In order to show uniqueness of the solution to (MP$_1$), we need to establish a similar bound for any solution to (MP$_1$). In fact we will establish a bit stronger result, but first let us define the notion of the local martingale problem. We say that $X$ solves local martingale problem (MP$_1$) with $X_0 = x \in \mathbb{L}^{\beta,E}$ if for any $y \in \mathbb{L}^{f,E}$, the process $M^{x,y}$ is a local martingale. Now we are ready to prove the following lemma.

**Lemma 4.3** Let $X_0 \in \mathbb{L}^{\beta,E}$ and let $X$ be a solution to the local martingale problem (MP$_1$) with $X_0 = x \in \mathbb{L}^{\beta,E}$. Then

(i) for all $k \in \mathbb{S}$, $t \geq 0$ and for $i = 1, 2$, we have

$$\mathbb{E}[X_{t,i}(k)] \leq \mathbf{S}_i(x_1 + x_2)(k),$$

(ii) and $X$ is a solution to the martingale problem (MP$_1$) with $X_0 = x \in \mathbb{L}^{\beta,E}$.
Proof. (i) Let \( y = (1,1)\mathbf{1}_{\{k\}} \in \mathbb{L}^{f,E} \) be the test function that takes the value \((1,1) \in E\) at \(k\) and is zero otherwise. Fix the stopping time

\[
\tau_n = \inf\{ t > 0 : \|X_{t,1} + X_{t,2}\|_\beta \geq n \}.
\]

Note that

\[
e^{\langle X_{t_\wedge \tau_n,y} \rangle} - e^{\langle x,y \rangle} - \int_0^{t_\wedge \tau_n} e^{\langle X_s,e_y \rangle} \langle AX_s, y \rangle \, ds, \quad t \geq 0,
\]

is a martingale. Then for any \( \varepsilon > 0 \), we have

\[
E\left[ \varepsilon^{-1} (1 - e^{\langle X_{t_\wedge \tau_n,y} \rangle}) \right] = e^{-1} (1 - e^{\langle x,y \rangle}) - E \left[ \int_0^{t_\wedge \tau_n} e^{\langle X_s,e_y \rangle} \langle AX_s, y \rangle \, ds \right].
\]

Note that \( \text{Re}\langle x,e_y \rangle \leq 0 \) for all \( x \in E^S \). Hence

\[
\text{Re} \left( 1 - e^{\langle X_{t_\wedge \tau_n,y} \rangle} \right) \geq 0.
\]

Using Fatou's lemma, we get

\[
E\left[ X_{t_\wedge \tau_n,1}(k) + X_{t_\wedge \tau_n,2}(k) \right] = E \left[ \lim_{\varepsilon \downarrow 0} \text{Re} \varepsilon^{-1} (1 - e^{\langle X_{t_\wedge \tau_n,e_y} \rangle}) \right] \\
\leq \liminf_{\varepsilon \downarrow 0} e^{-1} E \left[ (1 - e^{\langle X_{t_\wedge \tau_n,e_y} \rangle}) \right] \\
= x_1(k) + x_2(k) + \liminf_{\varepsilon \downarrow 0} \text{Re} \left( E \left[ \int_0^{t_\wedge \tau_n} e^{\langle X_s,e_y \rangle} (-\langle AX_s,y \rangle) \, ds \right] \right).
\]

Using dominated convergence (recall \( \tau_n \)), we obtain

\[
E\left[ X_{t_\wedge \tau_n,1}(k) + X_{t_\wedge \tau_n,2}(k) \right] \\
\leq x_1(k) + x_2(k) + \text{Re} \left( E \left[ \int_0^{t_\wedge \tau_n} (-\langle AX_s,y \rangle) \, ds \right] \right) \\
\leq x_1(k) + x_2(k) + 2 E \left[ \int_0^{t_\wedge \tau_n} (AX_{s,1}(k) + AX_{s,2}(k)) \, ds \right] \\
\leq x_1(k) + x_2(k) + 2nMt < \infty.
\]

From (4.10), we get

\[
E\left[ X_{t_\wedge \tau_n,1}(k) + X_{t_\wedge \tau_n,2}(k) \right] \\
\leq x_1(k) + x_2(k) + E \left[ \int_0^t A(X_{s_\wedge \tau_n,1} + X_{s_\wedge \tau_n,2})(k) \, ds \right].
\]

Since both sides are finite by (4.10), standard arguments yield

\[
E\left[ X_{t_\wedge \tau_n,1}(k) + X_{t_\wedge \tau_n,2}(k) \right] \leq S_t(x_1 + x_2)(k) \quad \text{for all} \quad t \geq 0 \quad \text{and} \quad n \in \mathbb{N}.
\]

Letting \( n \to \infty \) and using Fatou's lemma, we obtain

\[
E[X_{t,1}(k) + X_{t,2}(k)] \leq S_t(x_1 + x_2)(k).
\]
This finishes the proof of (i).

(ii) We have to show that the local martingale

$$M_t^{x,y} = H(X_t, y) - H(x, y) - \int_0^t \langle AX_s, y \rangle H(X_s, y) \, ds$$

is in fact a martingale. The argument is similar as in the proof of Lemma 3.13. Therefore, we omit the details. \(\Box\)

Corollary 4.4 Let \(X_0 \in \mathbb{L}^\beta, E\) and \(X\) be a solution to the martingale problem \((\text{MP}_1)\) and let \(\phi \in \mathbb{L}^\beta_\infty\). Then for all \(t \geq 0\) and \(i = 1, 2\), we have

$$\mathbb{E}[\langle X_t, i, \phi \rangle] \leq \langle S_t(x_1 + x_2), \phi \rangle \leq e^{Mt} \langle x_1 + x_2, \phi \rangle < \infty. \quad (4.11)$$

**Proof.** The first inequality is a consequence of the previous lemma, the second is due to (1.11) and the third is due to the very definition of \(\mathbb{L}^\beta_\infty\). \(\Box\)

Corollary 4.5 Let \(X_0 = x \in \mathbb{L}^\beta, E\) and \(X\) be a solution to (1.30). Then \(X\) is a solution to the martingale problem \((\text{MP}_1)\) with \(X_0 = x\).

**Proof.** By Itô’s formula (see (3.31), (3.32) in the proof of Lemma 3.13) we get that \(X\) is a solution to the local martingale problem \((\text{MP}_1)\). Then by Lemma 4.3(ii) we get that it is also a solution to the martingale problem \((\text{MP}_1)\). \(\Box\)

By definition, for any \(x \in \mathbb{L}^\beta, E\) and any solution \(X\) of the martingale problem \((\text{MP}_1)\) with \(X_0 = x\), the process \(M^{x,y}\) is a martingale for any \(y \in \mathbb{L}^\beta, E\). The \(L^1\) estimates we have just established enable us to show that this is true even for \(y \in \mathbb{L}^\beta_\infty\).

Lemma 4.6 For any \(x \in \mathbb{L}^\beta, E\), any solution \(X\) of \((\text{MP}_1)\) with \(X_0 = x\) and any \(y \in \mathbb{L}^\beta_\infty\), the process \(M^{x,y}\) is a martingale.

**Proof.** The proof is similar to Step 2 of Proposition 4.7. For the key estimate of (4.6), here we employ Corollary 4.4 instead of Lemma 3.10. We omit the details. \(\Box\)

Proposition 4.7 (Duality) Let \(Y_0 = y \in \mathbb{L}^\beta, E\) and let \(Y \in D_{\mathbb{L}^\beta, \infty}\) be a solution to the martingale problem \((\text{MP}_1)\). Let \(X_0 = x \in \mathbb{L}^\beta, E\) and let \(X \in D_{\mathbb{L}^\beta, \infty}\) be an a solution to the martingale problem \((\text{MP}_1)\) which is independent of \(Y\). Then \(X\) and \(Y\) are dual with respect to the function \(H:\)

$$\mathbb{E}[H(X_t, Y_0)] = \mathbb{E}[H(X_0, Y_t)] \quad \text{for all } t \geq 0. \quad (4.12)$$

**Proof.** Fix \(t > 0\). For \(r, s \in [0, t]\) define

$$f(s, r) = \mathbb{E}[H(X_s, Y_r)] \quad \text{and} \quad g(s, r) = \mathbb{E}[\langle AX_s, Y_r \rangle H(X_s, Y_r)] = \mathbb{E}[\langle AX_s, \mathcal{A}^*Y_r \rangle H(X_s, Y_r)].$$

By (4.6) and Corollary 4.4 we get

$$\mathbb{E}[M_t^{x,y}] \leq 2 + Me^{Mt} \mathbb{E}[\|X_{s,1} + X_{s,2}\|_\beta] \sum_{k \in S} \frac{y_1(k) + y_2(k)}{\beta(k)} \leq 2 + 2Me^{M(t+s)} \|x_1 + x_2\|_\beta \sum_{k \in S} \frac{y_1(k) + y_2(k)}{\beta(k)} < \infty.$$
Hence we can compute
\[ f(s, r) - f(s, 0) - \int_0^r g(s, u) \, du = E[M_r^{x,Y}] - E[M_0^{x,Y}] = 0 \] (4.13)

since \( M_0^{x,Y} = 0 \). Similarly, we get
\[ f(s, r) - f(0, r) - \int_0^s g(u, r) \, du = E[M_s^{x,Y}] = 0. \] (4.14)

Using the same estimates for \( E[A_{X_0,Y_0}] \), we obtain
\[ \int_0^t \int_0^t |g(r, s)| \, dr \, ds < \infty. \] (4.15)

By (4.13), (4.14), (4.15) and Lemma 4.4.10 of [EK86] (with their \( f_1 \) and \( f_2 \) both equal to our \( g \)), we get \( f(0, t) = f(t, 0) \). \qed

**Proof of Proposition 4.1** Step 1 (One-dimensional distributions). Let \( x \in \mathbb{L}^{\beta,E} \) and let \( X, X' \in D_{1,\beta,E} \) be two solutions to the martingale problem \( (\text{MP}_1) \) with \( X_0 = X'_0 = x \). Let \( y \in \mathbb{L}^{f,E} \) and let \( Y \) be a solution to \( (\text{MP}_1) \) with \( Y_0 = y \). By Proposition 4.1 we have
\[ E[H(X_t, y)] = E[H(X_0, Y_t)] = E[H(X', y)] \quad \text{for all } t \geq 0. \] (4.16)

By Corollary 2.4 of [KM08], the family \( \{H(\cdot, y), y \in \mathbb{L}^{f,E}\} \) is measure determining, hence the one-dimensional marginals of \( X \) and \( X' \) coincide.

**Step 2 (Finite-dimensional distributions).** Now we use a version of the well-known theorem claiming that “uniqueness of one-dimensional distributions for solutions to a martingale problem implies uniqueness of finite-dimensional distributions”\(^8\). More precisely, denote by \( F_s = \sigma(X_s, s \leq t) \) the \( \sigma \)-algebra generated by \( X_s \), \( s \leq t \). Note that \( (\mathbb{L}^{f,E}, \| \cdot \|) \) is a separable Banach space. Hence there exists a regular conditional probability \( Q_s = \mathbb{P}[\{X_{s+t} \geq 0 \} \mid \cdot \mid F_s] \). Arguing as in [B97] Corollary VI.2.2, we see that for almost all \( \omega \), under \( Q_s \) the canonical process is a solution to \( (\text{MP}_1) \) started in \( X_s \).

Now we may argue as in the proof of Theorem VI.3.2 in [B97] to get uniqueness of the finite-dimensional distributions of \( X \).

**Step 3 (Measurability).** For the proof of the existence of a solution to \( (\text{MP}_1) \) we employed a two stage approximation procedure: We constructed processes \( \tilde{X}^{m,n} \) with finitely many jumps from a given noise, showed that as \( n \to \infty \) the processes converge almost surely to some process \( \tilde{X}^m \) and finally established the existence of a convergent subsequence of \( \tilde{X}^m \). Let us denote the corresponding laws (with initial point \( x \)) by \( P_{x}^{m,n}, P_{x}^{m} \) and \( P_x \). By the very construction of \( \tilde{X}^{m,n} \), it is clear that \( x \mapsto P_{x}^{m,n} \) is measurable. Hence also the limit \( x \mapsto P^m \) is measurable. By the uniqueness that we have established in Step 2, we infer that \( P^m \to P_x \) as \( m \to \infty \) (not only along a subsequence). Hence \( x \mapsto P_x \) is measurable. \qed

**Proof of Theorems 1.1 and 3.1** Theorems 1.1 (a) and 3.1 follow immediately from Propositions 3.2 and 4.1. Theorem 1.1 (b) follows from Lemma 4.6

In order to show the strong Markov property of Theorem 1.1 (c), by [EK86] Theorem 4.4.2, it is enough to show that the martingale problem \( (\text{MP}_1) \) is well-posed not only for deterministic points \( x \in \mathbb{L}^{\beta,E} \), but also for probability measures \( \mu \in \mathcal{M}_1(\mathbb{L}^{f,E}) \). The problem is, of course, that for \( X_0 \sim \mu \) and \( y \in \mathbb{L}^{f,E} \), in general
the process $M^{x_0,y}$ is not well-defined, as the integrand $\langle AX_1, y \rangle H(X_1, y)$ is unbounded. Hence, we propose a slight modification of (MP$^1$) and assume that $y \in L^\infty$, where
\[
\mathbb{L}_{E,++} := \{ y \in L^E : \exists c < \infty \text{ with } c^{-1} \beta(k) < y_i(k) < c \beta(k) \forall i = 1, k \in S \} \subset \mathbb{L}_E.
\]
Recall that $\|Au\|_{\beta} \leq M\|u\|_\beta$ for all $u \in ([0,1)^2)^S$. Hence for all $u \in L^\infty$, the map $L^\infty \rightarrow \mathbb{C}$, $x \mapsto \langle\langle Ax, y \rangle\rangle H(x, y)$ is bounded. Hence for $y \in L^\infty$, the process $M^{x_0,y}$ is well-defined, and we say that $X_0$ as a solution to the martingale problem (MP$''$) if $M^{X_0,y}$ is a martingale for all $y \in L^\infty$. Arguing as in the proof of Proposition 4.7, we get the duality
\[
\mathbb{E}[H(X_t, y)] = \mathbb{E}[H(X_0, Y_t)] \text{ for all } y \in L^\infty.
\]
(4.17)
Note that $L^\infty \subset L^\infty$ is dense. Hence (4.17) determines the distribution of $X_t$. By [EK86, Theorem 4.4.2(a)], we infer uniqueness of the finite-dimensional distributions and hence of the solution to (MP$''$). Hence $P_{x_0} := \int \mu(dx) P_x$ is the unique distribution of any solution to (MP$''$) with $X_0 \sim \mu$. That is, the martingale (MP$'$) is well-posed and hence by [EK86, Theorem 4.4.2], $(P_x)_{x \in L^\infty}$ possesses the strong Markov property. \hfill $\square$

5 Proof of Theorem 1.2

First we will show the weak uniqueness for (1.30). Let $X$ be any solution to (1.30) with $X_0 = x \in L^\infty$. Then by Corollary 4.5, we get that $X$ is also a solution to the martingale problem (MP$''$). However, by Theorem 1.1 the solution to (MP$'$) is unique in law. Hence also the solution to (1.30) is unique in law.

Now we will show the existence of $(X, N_0)$ solving (1.30). Let $X$ be the unique in law solution to the martingale problem (MP$''$). By Lemma 3.11 and Theorem 3.1, we get that since $X$ is a unique solution to (MP$'$), then it can be constructed in a way that it also satisfies (3.24). Moreover if we define the point process $N$ by
\[
N_\Delta(\{k\}, dt, A) = \int_E 1_{A \setminus \{0\}}(J(y, X_{t-}(k))) N(\{k\}, dt, dy), \ A \subset \mathbb{R}^2,
\]
it is easy to get that in fact $X$ solves (1.30) with
\[
\mathcal{M} := \mathcal{N} - \mathcal{N}'
\]
To finish the proof of Theorem 1.2 we have just to show that there exists the Poisson point process (PPP) $N_0$ on $S \times \mathbb{R}^+ \times \mathbb{R}^+ \times E$ with intensity measure
\[
N_0' = \ell_S \otimes \lambda \otimes \lambda \otimes \nu,
\]
(5.1)
such that $\mathcal{N}$ is given by (1.28).

The construction of such $N_0$ is rather standard, but we present it here for the sake of completeness.

Let $(k_n, t_n, x_n)_{n \geq 1}$ be arbitrary labeling of the points of the point process $\mathcal{N}$. Let $\mathcal{N}'$ be a Poisson point process on $S \times \mathbb{R}^+ \times \mathbb{R}^+ \times E$ independent of $\mathcal{N}$ and $X$. Also let $(U_n)_{n \geq 1}$ be a sequence of independent random variables uniform on $(0,1)$ which are also independent of $\mathcal{N}$ and $X$.

Define the new point process $\mathcal{N}_0$ on $S \times \mathbb{R}^+ \times \mathbb{R}^+ \times E$ by
\[
\mathcal{N}_0(\{k\}, dt, dr, dx)
\]
\[
= \sum_{n \geq 1} \delta_{(k_n, t_n, U_n I_{t_n}(l_n), x_n)}(dk, dt, dr, dx) + 1_{\{r > t_n(k_n)\}} \mathcal{N}'(dk, dt, dr, dx)
\]
Note that by definition
\[
\mathcal{N}(\{k\}, dt, dx) = \mathcal{N}_0(\{k\}, dt, [0, I_t(l)]), dx).
\]
(5.2)
Hence we have just to show that $\mathcal{N}_0(dk, dt, dr, dx)$ is a Poisson point process on $S \times \mathbb{R}_+ \times \mathbb{R}_+ \times E$ with intensity measure given by (5.1). This and (5.2) will finish the proof of the theorem.

Let $f$ be an arbitrary non-negative measurable test function on $S \times \mathbb{R}_+ \times \mathbb{R}_+ \times E$ with compact support. Then

\begin{align*}
\mathbb{E} \left[ \sum_{n \geq 1} f(k_n, t_n, U_n I_{1_n}(k_n), x_n) \right] &= \mathbb{E} \left[ \sum_{k \in S} \int_0^1 \int_0^\infty \int_E f(k, t, r I_1(k), x) \nu(dx) dt dr \right] \\
&= \mathbb{E} \left[ \sum_{k \in S} \int_0^\infty \int_0^\infty f(k, t, r I_1(k), x) \nu(dx) dt dr \right] = \mathbb{E} \left[ \sum_{k \in S} \int_0^\infty \int_0^\infty f(k, t, r, x) \nu(dx) dt dr \right] (5.3)
\end{align*}

Next we get

\begin{align*}
\mathbb{E} \left[ \sum_{k \in S} \int_0^\infty \int_0^\infty \int_E f(k, t, r, x) 1_{\{r > I_1(k)\}} N_1(dk, dt, dr, dx) \right] &= \mathbb{E} \left[ \sum_{k \in S} \int_0^\infty \int_0^\infty f(k, t, r, x) 1_{\{r > I_1(k)\}} \nu(dx) dt dr \right] (5.4)
\end{align*}

By summing (5.3), (5.4) we get

\begin{align*}
\mathbb{E} \left[ \sum_{k \in S} \int_0^\infty \int_0^\infty \int_E f(k, t, r, x) N_0(dk, dt, dr, dx) \right] &= \mathbb{E} \left[ \sum_{k \in S} \int_0^\infty \int_0^\infty f(k, t, r, x) \nu(dx) dt dr \right].
\end{align*}

Since $f$ was arbitrary we conclude that $\mathcal{N}_0$ is the Poisson point process with intensity measure (5.1) and we are done by (5.2).

\section{Proof of Theorem 1.4}

Recall that $Y^\gamma$ solves the following system of equations

\begin{equation}
Y^\gamma_{t,i}(k) = y_{0,i}(k) + \int_0^t AY^\gamma_{s,1}(k) ds + \int_0^t \gamma^{1/2} \sigma(Y^\gamma_s(k)) dW_{s,i}(k), \quad t \geq 0, k \in S.
\end{equation}

First of all we will get the uniform integrability condition on $Y^\gamma_i$, $i = 1, 2$.

\textbf{Lemma 6.1} For any $T > 0$, $p \in (0, 2)$ and $i = 1, 2$, we have

\begin{equation*}
\sup_{\gamma \geq 0} \mathbb{E} \left[ \sup_{t \leq T} \langle Y^\gamma_{t,i}, \beta \rangle^p \right] < \infty.
\end{equation*}
Proof. By simple stochastic calculus
\[
e^{-tM} \langle Y_{t,i}, \gamma, \beta \rangle = \langle Y_{0,i}, \beta \rangle + \int_0^t \left( \langle AY_{s,i}, \gamma, \beta \rangle - M \langle Y_{s,i}, \gamma, \beta \rangle \right) ds
\]
\[+ \sum_{k \in S} \beta(k) \int_0^t e^{-sM} \gamma^{1/2} \sigma(Y_s(k)) dW_{s,i}(k)
\leq \langle Y_{0,i}, \beta \rangle + \sum_{k \in S} \beta(k) \int_0^t e^{-sM} \gamma^{1/2} \sigma(Y_s(k)) dW_{s,i}(k)
\leq \langle Y_{0,i}, \beta \rangle + \sum_{k \in S} \beta(k) \int_0^t e^{-sM} \gamma^{1/2} \sigma(Y_s(k)) dW_{s,i}(k)
\] where the second inequality follows by (1.10) and \(B_{i}, i = 1, 2,\) are independent Brownian motions. Hence we get that the pair
\[(e^{-tM} \langle Y_{t,1}, \gamma \rangle, e^{-tM} \langle Y_{t,2}, \gamma \rangle)\]
is stochastically bounded by the time-changed planar Brownian motion \(B_{0} = (u, v) := (\langle Y_{0,1}, \gamma \rangle, \langle Y_{0,2}, \gamma \rangle)\)
and evolving until the stopping time
\(\tau = \inf \{t \geq 0 : B_{t,1}B_{t,2} = 0\} \).

For \(p \in (1, 2),\) by Doob's inequality, we have
\[
K_i \equiv E \left[ \sup_{t \leq \tau} (B_{t,i})^p \right] < \left( \frac{p}{p - 1} \right)^p E \left[ (B_{\tau,i})^p \right].
\]
The \((p/2)\)th moment of the exit time of planar Brownian motion from a quadrant is finite if and only if \(p < 2\) (see, e.g., [Bur77, Equation (3.8)] with \(\alpha = \pi/2\)). Hence, using Burkholder's inequality, we get
\[K_i < \infty. \quad (6.4)\]
We can get (6.4) also by an explicit estimate using the density of the distribution of \(\tilde{B}_{\tau}\) from (1.23):
\[
E \left[ (B_{\tau,i})^p \right] \leq |u^2 - v^2|^{p/2} + \frac{2^{p/2} (uv)^{p/2}}{\cos(p\pi/4)}.
\]
This immediately implies that
\[
E \left[ \sup_{t \leq T} (Y_{t,i}, \gamma, \beta)^p \right] < e^{MT} K_i < \infty \quad \text{for all } i = 1, 2,
\]
uniformly in \(\gamma \geq 0. \]

Lemma 6.2 The family \((Y_{\gamma})_{\gamma \geq 0}\) is tight in \(D_{L^{1,2}, E}\) equipped with Meyer-Zheng topology.

Proof. The process \(M_i^\gamma(k)\) defined by
\[
M_i^\gamma(k) := Y_{t,i}^\gamma(k) - Y_{0,i}(k) - \int_0^t AY_{s,i}^\gamma(k) ds
\]
is a martingale. In order to show tightness of \((Y_{\gamma})_{\gamma \geq 0}\), it is enough to show tightness of \((Y_{t,i}^\gamma(k))_{\gamma \geq 0}\) for all \(k \in S\) and \(i = 1, 2.\) By Lemma 6.1 the random variable \((Y_{t,i}^\gamma, \beta)\) has \(p\)-th moment for any \(p \in (0, 2),\) hence we immediately get the tightness of
\[
\int_0^t AY_{s,i}^\gamma(k) ds.
\]
By Theorem 4 Remark 2 of [MZ84], in order to get the tightness of the martingale \( M_{t,i}^\gamma(k) \) it is enough to show that
\[
\sup_{t \leq T} \mathbb{E} \left[ |M_{t,i}^\gamma(k)| \right] \tag{6.6}
\]
is bounded uniformly in \( n \) for any \( T > 0 \). However,
\[
|M_{t,i}^\gamma(k)| \leq |Y_{t,i}^\gamma(k)| + |x_1(k)| + \left| \int_0^t A Y_{s,i}^\gamma(k) \, ds \right| \tag{6.7}
\]
and (6.6) again follows immediately by boundedness of \( p \)-th moments (for \( p < 2 \)) of \( \langle Y_{t,i}^\gamma, \beta \rangle \).

**Lemma 6.3** Let \( X \) be an arbitrary limit point of \( (Y_\gamma^\gamma)_{\gamma \geq 0} \). Then \( X \) solves the martingale problem \( (MP_1) \).

**Proof.** Let \( \gamma_n \to \infty \) be such that \( Y_\gamma^\gamma \) converges to \( X \) as \( n \to \infty \). By Itô’s formula, for \( z \in L^{f,2} \) (recall (1.18)), the process \( M_{t,i}^{\gamma,y,z} \) defined by
\[
M_{t,i}^{\gamma,y,z} = H(Y_\gamma^\gamma, z) - H(Y_0^\gamma, z) - \int_0^t \langle A Y_s^\gamma, z \rangle H(Y_s^\gamma, z) \, ds \tag{6.8}
\]
is a martingale. Since \((Y_\gamma^\gamma)_n \to X\) converges to \( M_{t,i}^{y,z} \). As the \( p \)-th moments \( \langle Y_\gamma^\gamma, \beta \rangle \) (for \( p \in (0,2) \)) are uniformly bounded (in \( \gamma \)), also the \( p \)-th moments of \( M_{t,i}^{\gamma,y,z} \) are uniformly bounded. By [MZ84, Theorem 11], we infer that 
\[
M_{x,z} = \lim_{n \to \infty} M_{t,i}^{\gamma_n,y,z}(k)\]
is a martingale. In other words, \( X \) is a \([0, \infty)^2\)-valued solution to the martingale problem \( (MP_1) \). It remains to show that \( X_t \in E \) for all \( t \geq 0 \) almost surely.

The above lemma finishes the proof of Theorem 1.4.
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