HAMILTONIAN ANALYSIS OF \(SL(2, \mathbb{R})\) SYMMETRY IN LIOUVILLE THEORY *

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Abstract

We consider a Hamiltonian analysis of the Liouville theory and construction of symmetry generators using Castellani’s method. We then discuss the light-cone gauge fixed theory. In particular, we clarify the difference between Hamiltonian approaches based on different choices of time, \(\xi^0\) and \(\xi^+\). Our main result is the construction of \(SL(2,\mathbb{R})\) symmetry generators in both cases.

1. Introduction

Einstein’s theory of gravity in 2d is dynamically trivial, as the corresponding action is a topological invariant. By studying the quantum fluctuations of matter fields coupled to the 2d metric (treated as an independent dynamical variable), Polyakov [1] obtained the nontrivial effective theory of gravity in the form

\[
W[g] = \kappa \int d^2 \xi \sqrt{-g} \left( \frac{1}{2} R - \mu^2 \right).
\]

The original theory of matter couplings is invariant under general coordinate transformations, as well as Weyl rescalings. If the quantum fluctuations of matter fields are regulated so as to preserve general coordinate invariance, the Weyl invariance is lost and the effective action is related to the notion of the Weyl anomaly.

Polyakov and his collaborators [2] demonstrated that in the light-cone gauge the n-point functions of the effective theory can be explicitly found. Although the gauge is fixed, these solutions display an \(SL(2,\mathbb{R})\) symmetry. The origin of the symmetry, in the case where the theory is also Weyl invariant, has been traced by Dass and Summittra [3] to the residual symmetry transformations that leave the light-cone gauge intact. They showed that after fixing the light-cone gauge there exists a combination of general coordinate transformations and Weyl transformations that is still a symmetry of the theory — the \(SL(2,\mathbb{R})\) symmetry alluded to before. The results of the light-cone analysis inspired the investigation of the 2d gravity in the conformal gauge (David, Distler and Kawai, [4]).

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Interesting Lagrangian treatment of the (light-cone and conformal) symmetries of the theory was given in [5], while the BRST quantization was discussed in [6].

Although many important properties of the 2d gravity have been obtained in the path-integral quantization scheme, it is also important to understand the Hamiltonian structure of the theory. Egorian and Manvelian [7] noted that the correct approach to understanding the residual symmetry is to use the light-cone coordinate $\xi^+$ as the time variable in the Hamiltonian approach. They showed that the constraints appearing in the total Hamiltonian satisfy the $SL(2, R)$ algebra. Although the $SL(2, R)$ symmetry is the symmetry of the light-cone gauge, Abdalla et al. [8] were able to find a canonical description of the theory in terms of gauge independent variables — $SL(2, R)$ currents, which is very important for understanding the general structure of the theory, and, in particular, the relation to the conformal gauge. The currents are defined in a gauge independent way, but they are conserved only in the light-cone gauge. These variables were used by Miković [9] to derive some exact results in 2d gravity in a gauge independent way. Ghosh and Mukherjee [10] used the improved Hamiltonian formalism in which the generators of reparametrizations are corrected by certain surface terms, and showed that these terms represent the $SL(2, R)$ currents.

The residual $SL(2, R)$ symmetry is not the standard local symmetry: it is characterized by parameters $\omega_a$ which are not arbitrary functions of both coordinates, but depend only on $\xi^+$. The fact that this symmetry appears in the gauge-fixed theory seems to contradict the basic principle of the Dirac formalism [11], since there is a symmetry defined by parameters $\omega_a(\xi^+)$, but there are no first-class constraints, as the gauge is fixed. Barcelos-Neto [12] noted that the contradiction is only an apparent one, but his arguments were not complete.

The subject of the present paper is a complete analysis of the $SL(2, R)$ symmetry of the effective 2d gravity as a (nonstandard) Hamiltonian local symmetry, including the construction of the corresponding generators as well as the comparison with the Noether method and the improved Hamiltonian approach. In particular, we clarify the difference between the Hamiltonian approaches based on times $\xi^0$ and $\xi^+$, respectively. In Sec. 2 we develop the general Hamiltonian description of the covariant theory and describe the light-cone gauge. In Sec. 3 we discuss the $SL(2, R)$ symmetry of the gauge fixed theory in the light-front formalism with $\xi^+$ as the time variable, and construct the related gauge generators by using Castellani’s algorithm [13]. In Sec. 4 previous results are translated into the standard formalism based on the time $\xi^0$. Section 5 is devoted to conclusions. Some technical details and a modification of Castellani’s method appropriate to the case of residual symmetries are discussed in the Appendix.

2. The standard Hamiltonian analysis

A. Constraints and the Hamiltonian

The Liouville action $I = -W/\kappa$ can be written as a local functional by introducing
an auxiliary field $F$ [14]:

$$ I[F, g] = - \int d^2 \xi \sqrt{-g} \left[ \frac{1}{2} g^{\alpha \beta} \partial_\alpha F \partial_\beta F + \frac{\alpha}{2} F R + \mu^2 \right]. $$ (1a)

The elimination of $F$ with the help of the equations of motion leads back to the nonlocal theory. After disregarding a surface term the Liouville action takes the following form [7,12]:

$$ I[F, g_{\alpha \beta}] = \int d^2 \xi \frac{1}{\sqrt{-g}} \left\{ \frac{1}{2} (g_{11} \dot{F}^2 + g_{00} F'') - g_{01} F F' 
+ \frac{\alpha}{2} \left[ \frac{g_{01}}{g_{11}} (\dot{F} g_{11} - F' g_{11}) + \dot{g}_{11} \dot{F} + g_{00} F' - 2g_{01} F \right] + g\mu^2 \right\}. $$ (1b)

The basic Lagrangian dynamical variables are $(F, g_{\alpha \beta})$. The corresponding canonical momenta $(\pi_F, \pi_{\alpha \beta})$ are easily obtained in the form

$$ \pi_F = \frac{1}{\sqrt{-g}} \left[ g_{11} \dot{F} - g_{01} F' + \frac{\alpha}{2} \left( \frac{g_{01}}{g_{11}} g_{11}' + \dot{g}_{11} - 2g_{01}' \right) \right], $$

$$ \pi^{11} = \frac{\alpha}{2 \sqrt{-g}} \left( \dot{F} - \frac{g_{01}}{g_{11}} F' \right), $$

$$ \pi^{01} = 0, \quad \pi^{00} = 0, $$

leading to the following primary constraints:

$$ \phi^{01} \equiv \pi^{01} \approx 0, \quad \phi^{00} \equiv \pi^{00} \approx 0. $$

Having found all the primary constraints we now proceed to find the Hamiltonian. It is convenient to write the Lagrangian in a condensed form as

$$ \mathcal{L} = a \dot{F}^2 + b \dot{F} \dot{g}_{11} + c \dot{F} + d \dot{g}_{11} + e. $$

The corresponding canonical Hamiltonian density is

$$ \mathcal{H}_c = \pi_F \dot{F} + \pi^{11} \dot{g}_{11} - \mathcal{L} = -\frac{a}{b^2} (\pi^{11} - d)^2 + \frac{1}{b} (\pi^{11} - d) (\pi_F - c) - e, $$

or, more explicitly,

$$ \mathcal{H}_c = -\frac{\sqrt{-g}}{g_{11}} \mathcal{H}_0 + \frac{g_{01}}{g_{11}} \mathcal{H}_1, $$ (3a)

where

$$ \mathcal{H}_0 = \frac{1}{2} \left[ \frac{4}{\alpha^2} (g_{11} \pi^{11})^2 - \frac{4}{\alpha} g_{11} \pi^{11} \pi_F - F'' - 2\alpha F'' - 2g_{11} \mu^2 \right], $$

$$ \mathcal{H}_1 = \pi_F F' - 2g_{11} (\pi^{11})' - \pi^{11} g_{11}'. $$ (3b)
The general Hamiltonian dynamics of the system is described by the total Hamiltonian
density,

\[ H_T = H_c + u_{01}\phi^{01} + u_{00}\phi^{00}, \]

where \( u_{01} \) and \( u_{00} \) are, at this stage, arbitrary multipliers.

In order to have a consistent Hamiltonian theory we shall demand that the constraints
do not change during the general time evolution of the dynamical system governed by the
total Hamiltonian \( H_T \equiv \int d^2\xi H_T \). Using the usual Poisson brackets between the basic
Hamiltonian variables, the consistency of the primary constraints leads to

\[
\dot{\phi}^{01} = \{\phi^{01}, H_T\} = \frac{1}{g_{11}} \left( -H_1 + \frac{g_{01}}{\sqrt{-g}} H_0 \right),
\]

\[
\dot{\phi}^{00} = \{\phi^{00}, H_T\} = -\frac{1}{2\sqrt{-g}} H_0.
\]

Therefore, the secondary constraints are just \( H_0 \) and \( H_1 \).

A direct calculation yields the following Poisson brackets between \( H_0 \) and \( H_1 \):

\[
\{H_0(\sigma), H_0(\sigma')\} = [H_1(\sigma) + H_1(\sigma')]\partial_\sigma \delta,
\]

\[
\{H_0(\sigma), H_1(\sigma')\} = [H_0(\sigma) + H_0(\sigma')]\partial_\sigma \delta,
\]

\[
\{H_1(\sigma), H_1(\sigma')\} = [H_1(\sigma) + H_1(\sigma')]\partial_\sigma \delta,
\]

where \( \delta \equiv \delta(\sigma, \sigma') \). It is now easy to see that the consistency of \( H_0 \) and \( H_1 \) does not produce further constraints.

B. The reparametrization symmetry

All the constraints \( \phi^{00}, \phi^{01}, H_0 \) and \( H_1 \) are of the first class, which implies the existence of a gauge symmetry. One can study more explicitly the nature of the symmetry by constructing the corresponding gauge generators, and calculating the related gauge transformations of dynamical variables.

A general method of constructing the generators of a local symmetry in the Hamiltonian approach has been given by Castellani [13]. Here we shall limit ourselves to physically important situations where gauge transformations are given in terms of arbitrary parameters \( \varepsilon \) and their first time derivatives \( \dot{\varepsilon} \). In that case the gauge generators take the form

\[
G[\varepsilon] = \int d\xi^1 (\varepsilon G_0 + \dot{\varepsilon} G_1),
\]

where \( G_0 \) and \( G_1 \) are phase-space functions satisfying the conditions

\[
G_1 = C_{PFC},
\]

\[
G_0 + \{G_1, H_T\} = C_{PFC},
\]

\[
\{G_0, H_T\} = C_{PFC},
\]

and \( C_{PFC} \) means primary first-class (PFC) constraint.
Starting with the PFC constraints $2g_{00}\pi^{00} + g_{01}\pi^{01}$ and $g_{11}\pi^{01} + 2g_{01}\pi^{00}$ as $G_1$'s, Castellani’s method yields the following expressions for the gauge generators:

$$
G[\varepsilon^0] = \int d\xi^1 \left\{ \varepsilon^0[H_T - (g_{00}\pi^{01})'] + \varepsilon^0(2g_{00}\pi^{00} + g_{01}\pi^{01}) \right\},
$$

$$
G[\varepsilon^1] = \int d\xi^1 \left\{ \varepsilon^1[H_1 + g_{00}'\pi^{00} - g_{01}(\pi^{01})'] + \varepsilon^1(g_{11}\pi^{01} + 2g_{01}\pi^{00}) \right\},
$$

where $\varepsilon^\alpha = \varepsilon^\alpha(\xi^0, \xi^1)$. The corresponding gauge transformations are defined by

$$
\delta_0 X = \{X, G\}, \quad G \equiv G[\varepsilon^0] + G[\varepsilon^1].
$$

Their form on the fields $(F, g_{\alpha\beta})$ is easily seen to coincide with the reparametrizations:

$$
\delta_0 g_{\alpha\beta} = \varepsilon^{\gamma, \alpha} g_{\gamma\beta} + \varepsilon^{\gamma, \beta} g_{\gamma\alpha} + \varepsilon^{\gamma} g_{\alpha\beta, \gamma},
$$

$$
\delta_0 F = \varepsilon^\alpha \partial_\alpha F.
$$

C. The light-cone gauge

The theory (1a) simplifies significantly in the light-cone gauge, defined by

$$
g_{+-} = 1, \quad g_{-+} = 0,
$$

$$
ds^2 = 2h(\sqrt{2}d\xi^+)^2 + 2d\xi^+ d\xi^-,
$$

where $g_{ab}$ ($a, b = +, -$) are the components of the metric tensor in the light-cone coordinates $\xi^\pm = (\xi^0 \pm \xi^1)/\sqrt{2}$:

$$
g_{+-} \equiv \frac{1}{2}(g_{00} - g_{11}), \quad g_{-+} \equiv \frac{1}{2}(g_{00} + g_{11}) - g_{01},
$$

$$
2h \equiv g_{++} = \frac{1}{2}(g_{00} + g_{11}) + g_{01}.
$$

After finding the inverse metric

$$
g^{++} = 0, \quad g^{+-} = g^{-+} = 1, \quad g^{--} = -2h,
$$

the calculation of the metric connection $\Gamma^\alpha_{\beta\gamma}$ yields the following nonvanishing components:

$$
\Gamma_{++}^+ = -\partial_- h, \quad \Gamma_{++}^- = \partial_- h,
$$

$$
\Gamma_{+-}^+ = \partial_+ h + 2h\partial_- h,
$$

where $\partial_\pm = (\partial_0 \pm \partial_1)/\sqrt{2}$. The curvature components and the laplacian are of the simple form

$$
R_{++} = 2h\partial_-^2 h, \quad R_{+-} = \partial_-^2 h, \quad R = 2\partial_-^2 h,
$$

$$
\Box = 2\partial_-(\partial_+ - h\partial_-).
$$
The equation of motion for $F$ is given by
\[ \Box F = \frac{\alpha}{2} R \quad \implies \quad \partial_-(\partial_+ - h\partial_-)F = \frac{\alpha}{2} \partial_-^2 h. \]

The equations of motion for $g_{\alpha\beta}$ can be obtained from the energy-momentum corresponding to the action (1a). By using the relation
\[ \delta \int d^2 \xi \sqrt{-g} \mathcal{L} = - \int d^2 \xi \sqrt{-g} \delta g^{\alpha\beta}(\nabla_\alpha \nabla_\beta - g_{\alpha\beta} \nabla^2)F, \]
one easily finds
\[ T_{\alpha\beta} = -\frac{1}{2} \partial_\alpha F \partial_\beta F + \frac{\alpha}{2}(\nabla_\alpha \nabla_\beta - g_{\alpha\beta} \nabla^2)F + \frac{1}{2} g_{\alpha\beta}(\frac{1}{2} g^{\gamma\delta}\partial_\gamma F \partial_\delta F + \mu^2). \]

In the light-cone gauge they are reduced to
\[ T_{--} = -\frac{1}{2}(\partial_- F)^2 + \frac{\alpha}{2} \partial_-^2 F, \]
\[ T_{+-} = hT_{--} - \frac{1}{4}(\alpha^2 \partial_-^2 h - 2\mu^2), \]
\[ T_{++} = h^2 T_{--} - h(\alpha^2 \partial_-^2 h - 2\mu^2) + \frac{\alpha^2}{8} [(\partial_- h)^2 - 2h \partial_-^2 h + 2\partial_- \partial_+ h], \]
after using the equation of motion for $F$. From the Eq.(9) and $T_{--} = 0$ follows the important result
\[ \partial_-^3 h = 0. \]

Now we shall focus our attention to the gauged-fixed action, which has the following form:
\[ I = \int d^2 \xi [-\partial_+ F \partial_- F + h(\partial_- F)^2 - \alpha F \partial_-^2 h - \mu^2]. \]

As a result we have only two equations of motion: Eq.(9) and $T_{--} = 0$ and we shall show that they are $SL(2,R)$ invariant, which is not the case for the whole set of equations of motion [5].

For the beginning, let us observe that the gauge-fixed action (12) is nondegenerate:
\[ \pi_h = \frac{\alpha}{2}(\dot{F} - F'), \]
\[ \pi = -\dot{F} + h(\dot{F} - F') + \frac{\alpha}{2}(\dot{h} - h'). \]

This can be also seen from the relations (2) restricted to the light-cone gauge:
\[ \pi^{11} + \frac{\alpha}{2} \frac{F'}{g_{11}} = \frac{\alpha}{2}(\dot{F} - F'), \]
\[ \pi_F - \frac{\alpha}{2} \frac{g_{11}'}{g_{11}} = -\dot{F} + h(\dot{F} - F') + \frac{\alpha}{2}(\dot{h} - h'). \]
These two equations show that the transition $(\pi^{11}, \pi_F) \to (\pi_h, \pi)$ is achieved by a canonical transformation

$$\pi_h = \pi^{11} + \frac{\alpha}{2} \frac{F'}{g_{11}}, \quad \pi = \pi_F - \frac{\alpha}{2} \frac{g'_{11}}{g_{11}}.$$  

The reason for this lies in the fact that the action (1b) is obtained after disregarding a surface terms in (1a), whereas Eq.(12) is obtained directly from (1a).

The nondegeneracy of the action (12) is a natural consequence of the gauge fixing procedure, and implies the absence of first class constraints. Consequently, we can conclude that there are no gauge symmetries of the Hamiltonian equations of motion.

The true meaning of this assertion is the following. It is well known that gauge symmetries in the Hamiltonian framework are related to the presence of arbitrary multipliers in the total Hamiltonian. Let us consider a dynamical evolution of a system described by a phase-space trajectory starting from a given point at time $t = 0$. For different choices of arbitrary multipliers we can solve the Hamiltonian equations of motion and obtain different trajectories, all starting from the same point and describing the same physical state. At any time $t > 0$ we can pass from one trajectory to the other, without changing the physical state. This unphysical transition from one trajectory to the other at a given time $t$ is called the gauge transformation. It is clear that the Hamiltonian definition of gauge symmetries is based on a definite choice of time. The absence of gauge symmetries in a given Hamiltonian formalism based on one specific choice of time does not mean that these symmetries are also absent for any other choice. We shall see in the next section how the hidden symmetries of the Liouville theory are detected by using the light-cone time variable $\xi^+$.

3. SL(2,R) symmetry in the light-front formalism

There are several reasons to study relativistic field theories at fixed light-cone time $\xi^+$. Dirac [15] showed that in the light-front form of the Hamiltonian formalism a maximum number of Poincare generators becomes independent of the dynamics. The same approach was used to develop a practical method of performing non-perturbative calculations in quantum field theory, and to study the problem of vacuum structure [16]. Here, the light-front formalism is used to clarify the nature of the residual symmetries in the Liouville theory.

A. Hamiltonian analysis

We have seen in the standard Hamiltonian approach that the Liouville action in the light-cone gauge is not degenerate. However, if we choose $\xi^+$ as the time variable, then the action (12) becomes degenerate. The definition of the momenta $(\pi, \pi_h)$ corresponding to the Lagrangian variables $(F, h)$ leads to the following primary constraints:

$$\varphi_1 \equiv \pi_h \approx 0, \quad \varphi_2 \equiv \pi + \partial_- F \approx 0. \quad (13)$$

The canonical Hamiltonian density is

$$\mathcal{H}_c = -h(\partial_- F)^2 - \alpha \partial_- F \partial_- h + \mu^2,$$
while the total Hamiltonian density takes the form
\[ H_T = H_c + u_1 \varphi_1 + u_2 \varphi_2. \]

The consistency requirements are calculated by using the Poisson brackets taken at the same time \( \xi^+ \), and lead to further constraint. By demanding \( \{ \varphi_1, H_T \} = 0 \) one obtains the secondary constraint
\[ \chi_1 \equiv (\partial_- F)^2 - \alpha \partial_-^2 F \approx 0, \]
the consistency of \( \chi_1 \) yields the tertiary constraint
\[ \theta_1 \equiv -\frac{\alpha^2}{\delta_0} \partial_3^2 h \approx 0, \]
while the consistency of \( \theta_1 \) leads to the condition on \( u_1 \):
\[ \partial_3^2 u_1 = 0. \]
The last relation can be solved in the form
\[ u_1(\xi^+, \xi^-) = u_-(\xi^+) + \xi^- u_0(\xi^+) + (\xi^ -)^2 u_+(\xi^+), \]
where \( u_- , u_0, u_+ \) are arbitrary functions of the time \( \xi^+ \).

On the other hand, the requirement \( \{ \varphi_2, H_T \} = 0 \) leads to
\[ u_2 = \dot{u}_2 + v(\xi^+) , \quad \dot{u}_2 \equiv h \partial_\pi F + \frac{\alpha}{2} \partial_\pi h, \]
where \( v \) is an arbitrary multiplier depending on \( \xi^+ \) only.

Now, the total Hamiltonian can be written in the form
\[ H_T = H' + u_- \varphi^- + u_0 \varphi^0 + u_+ \varphi^+ + v \rho, \quad (14) \]
where
\[ H' = \int d^2 \xi^- (H_c + \dot{u}_2 \varphi_2) = \int d\xi^- \left[ -\frac{\alpha}{2} \partial_- h \partial_- F + \dot{u}_2 \pi + \mu^2 \right], \quad (15a) \]
and
\[ \varphi^a \equiv \int d\xi^- (\xi^-)^{a+1} \varphi_1 = \int d\xi^- (\xi^-)^{a+1} \pi_h, \quad (15b) \]
with \( a = (-1, 0, +1) \). The algebra of the constraints is
\[ \{ \varphi_1(\xi^-_1), \theta_1(\xi^-_2) \} = \frac{\alpha^2}{\delta_0} \partial_\xi^3 \delta, \]
\[ \{ \varphi_2(\xi^-_1), \chi_1(\xi^-_2) \} = -2 \partial_- F \partial_- \delta + \alpha \partial_-^2 \delta, \]
\[ \{ \varphi_2(\xi^-_1), \varphi_2(\xi^-_2) \} = 2 \partial_- \delta, \quad (16) \]
where \( \delta \equiv \delta(\xi^-_1 - \xi^-_2) \), and all other Poisson brackets vanish. Although these constraints are not of the first class their combinations, given by Eq.(15b), might be, which can be seen from Eq.(14). Since we are now dealing with nonlocal quantities we have to check whether they have well defined functional derivatives.
B. Asymptotic behaviour and surface terms

The solution of the problem is given by the following considerations [17,11]. In field theory the Hamiltonian and the gauge generators are expressed as functionals of the phase-space variables,

\[ G[q, p] = \int dx \mathcal{G}[q(x), \partial_\alpha q(x), p(x), \partial_\alpha p(x)]. \]

Since \( G[q, p] \) is a nonlocal expression that acts on phase-space variables via the Poisson brackets, one has to check whether this quantity has well defined functional derivatives.

The first step in that direction is to define precisely the phase space in which all the nonlocal quantities act. This is achieved by defining the asymptotic behaviour of the basic dynamical variables.

The constraints \( \chi_1 \) and \( \theta_1 \) can be easily solved leading to

\[
\begin{align*}
F(\xi^+, \xi^-) &= -(\alpha/2) \ln \partial_- f, \quad f \equiv (a\xi^- + b)/(c\xi^- + d), \\
h(\xi^+, \xi^-) &= J^+ - 2\xi^- J^0 + (\xi^-)^2 J^- ,
\end{align*}
\]

where \( a, b, c, d \) and \( J^a \) are arbitrary functions of \( \xi^+ \), and \( ad - bc = 1 \). From these solutions we find the following asymptotic behaviour of the field variables \( (F, h) \):

\[
\begin{align*}
F &= -\alpha \ln |\xi^-| + A(\xi^+) + O_1, \\
h &= J^+(\xi^+) - 2\xi^- J^0(\xi^+) + (\xi^-)^2 J^-(\xi^+) ,
\end{align*}
\]

where \( O_n \) denotes a term that decreases like \( (\xi^-)^{-n} \) or faster for large \( \xi^- \), i.e. \( (\xi^-)^n O_n \) remains finite when \( \xi^- \rightarrow \infty \).

It should be noted that for those expressions that vanish on shell one can demand an arbitrarily fast decrease, as no solution of the equations of motion is thereby lost. In accordance with this remark the asymptotic behaviour of the momentum variables is determined by requiring

\[
p - \partial L \partial q = \hat{O},
\]

where \( \hat{O} \) denotes a term that decreases sufficiently fast, e.g. like \( O_3 \). By using the definitions of momenta (13) and the accepted asymptotic behaviour of the fields, one finds

\[
\pi_h = \hat{O}, \quad \pi = \frac{\alpha}{\xi^-} + O_2.
\]

Keeping in mind the above asymptotic relations we now wish to check whether various nonlocal expressions in the theory have functional derivatives. A functional \( G[q, p] \) has well defined functional derivatives if its variation can be written as

\[
\delta_0 G = \int dx [A(x)\delta_0 q(x) + B(x)\delta_0 p(x)],
\]

where \( \delta_0 q, \alpha \) and \( \delta_0 p, \alpha \) are absent. In general, when the derivatives of fields are present in \( G \), this requirement will not be satisfied. This will lead us to redefine \( G \) by adding
certain surface terms, obtaining in this way correctly defined quantities. If the surface terms happen to vanish, then the original functional does not need any modification.

Let us proceed by demonstrating the above procedure in the case of the canonical Hamiltonian. The variation of $H_c$ yields

$$\delta_0 H_c = \int d\xi^- \left[ -2h \partial_- f(\partial_- \delta_0 F) - \alpha (\partial_- \delta_0 h) \partial_- F - \alpha \partial_- h (\partial_- \delta_0 F) \right] + R$$

where those terms that contain the unwanted variations of fields or momenta are explicitly displayed, and the remaining terms of the correct form (19) are denoted by $R$. The second equality is obtained by performing the integration by parts, and this brings in the surface terms. Denoting these terms by $S$ and using the asymptotics (17b) we have

$$S = \left[ -2(J^+ - 2\xi^0) + (\xi^-)^2 J^- \right] \delta_0 F \bigg|^{+\infty}_{-\infty} - \alpha \left( \delta_0 J^+ - 2\xi^- \delta_0 J^0 + (\xi^-)^2 \delta_0 J^- \right) \bigg|^{+\infty}_{-\infty}.$$

From the asymptotic behaviour of $F$ and $h$ it follows

$$\delta_0 F(+\infty) = \delta_0 F(-\infty),$$

$$\delta_0 J^a(+\infty) = \delta_0 J^a(-\infty) \quad (a = 0, +, -),$$

and we easily obtain

$$S = \alpha^2 \xi^- \delta_0 J^- \bigg|^{-\infty}_{+\infty}.$$

It is also easy to see that $H_c$ is not even finite but that it can be made finite with well defined functional derivatives by adding a surface term

$$\tilde{H}_c = H_c - \int d\xi^- \left( \frac{\alpha^2}{2} \partial_-^2 h + \mu^2 \right) = \int d\xi^- \left[ -h(\partial_- F)^2 - \alpha \partial_- F \partial_- h - \frac{\alpha^2}{2} \partial_-^2 h \right].$$

Similar considerations apply to $\rho$ and $\varphi^a$, with the conclusion that the whole total Hamiltonian $\tilde{H}_T$, where $H_c$ is replaced with $\tilde{H}_c$, has correctly defined functional derivatives, as all the surface terms vanish. Now, one can verify that constraints $\rho$ and $\varphi^a$, given by Eq.(15b), are first-class since they have vanishing Poisson brackets with all "linear combinations" of constraints

$$\varphi[\lambda] \equiv \int d\xi^- \lambda(\xi^+, \xi^-) \varphi(\xi^+, \xi^-)$$

which are well defined (this requirement gives certain conditions on parameters $\lambda$).
C. Construction of gauge generators

It is clear from Eq.(14) that the arbitrary multipliers in \( H_T \) are functions of the time \( \xi^+ \) only, in contrast to the general case where they depend on both \( \xi^- \) and \( \xi^+ \). The gauge generators will be of the general form (6a), but the parameters must be of the same type as the multipliers in \( H_T \), \( \omega = \omega(\xi^+) \):

\[
G[\omega] = \omega(\xi^+)G_0 + \partial_+\omega(\xi^+)G_1 .
\] (20a)

A detailed analysis of Castellani’s conditions shows that they might be slightly changed in this case. Instead of (6b) we found that \( G_0 \) and \( G_1 \) should satisfy the relations

\[
\begin{align*}
G_1 &= \tilde{C}_{PFC}, \\
G_0 + \{G_1, H_T\} &= \tilde{C}_{PFC}, \\
\{G_0, H_T\} &= \tilde{C}_{PFC},
\end{align*}
\] (20b)

where \( \tilde{C}_{PFC} \) denotes a quantity multiplying an arbitrary multiplier in \( H_T \). We see that in principle it may happen that \( \tilde{C}_{PFC} \) is not even a constraint. In our case surface terms are absent, so \( \tilde{C}_{PFC} \) is a constraint.

Let us start with \( G_1 = \varphi^a \) or \( \rho \) in (20b) and try to find the corresponding \( G_0 \)’s. The calculation of \( \chi^a \equiv \{\varphi^a, H_T\} \) leads to

\[
\chi^a = \int d\xi^- (\xi^-)^{a+1}[-\pi \partial_- F - \frac{\alpha}{2} \partial_- (\partial_- F - \pi)] \approx 0,
\] (21)

while \( \{\rho, H_T\} = 0 \).

Before continuing, let us check on the differentiability of \( \chi^a \). By varying \( \chi^a \) one finds

\[
\delta_0 \chi^a = \int d\xi^- (\xi^-)^{a+1}[-\pi \partial_- \delta_0 F - (\alpha/2) \partial_-^2 \delta_0 F + (\alpha/2) \partial_- \delta_0 \pi] + R
\]

\[
= (\xi^-)^{a+1}[-\pi \delta_0 F - (\alpha/2) \partial_- \delta_0 F + (\alpha/2) \delta_0 \pi] + (\alpha/2) \partial_- (\xi^-)^{a+1} \delta_0 F \bigg|_{-\infty}^{+\infty} + R .
\]

It is now easy to see that for \( a = -1, 0, 1 \) the above surface term vanishes and, therefore, \( \chi^a \) is differentiable.

The algebra of the constraints has the form

\[
\begin{align*}
\{\chi^-, \chi^0\} &= -\chi^- , \\
\{\chi^-, \chi^+\} &= -2\chi^0 , \\
\{\chi^0, \chi^+\} &= -\chi^+ ,
\end{align*}
\] (22a)

while the remaining Poisson brackets vanish. One also finds

\[
\begin{align*}
\{\chi^-, H_T\} &= -2J^0 \chi^- + 2J^- \chi^0 , \\
\{\chi^0, H_T\} &= -J^+ \chi^- + J^- \chi^+ , \\
\{\chi^+, H_T\} &= -2J^+ \chi^0 + 2J^0 \chi^+ .
\end{align*}
\] (22b)
The above relations represent the proof that $\phi^a$ and $\rho$ are effectively PFC.

It should be noted that the constraints $\chi^a$ satisfy the $SL(2, R)$ algebra, which is closely related to the residual symmetry of the theory; as we shall see soon.

Now one can find $G_0$. Starting with $\phi^a$ as $G^a_1$, one obtains

$$G_0^- = -\chi^- - 2J^0 \phi^- + 2J^- \phi^0, \quad G_0^0 = -\chi^0 - J^+ \phi^- + J^- \phi^+, \quad G_0^+ = -\chi^+ - 2J^+ \phi^0 + 2J^0 \phi^+, \quad (23a)$$

or, equivalently,

$$G^a_0 = -\chi^a + f^{abc} J_b \phi_c, \quad (23b)$$

where $f^{abc}$ are the structure constants of $SL(2, R)$ ($f^{abc}$ is totally antisymmetric, and $f^{+-0} = 1$).

The complete gauge generator $G = G[\omega_-] + G[\omega_0] + G[\omega_+]$ has the form

$$G = \int d\xi^- \left[ (\partial_+ \varepsilon + \varepsilon \partial_- h - h \partial_- \varepsilon) \pi_h + (\varepsilon \partial_- F + \frac{\alpha}{2} \partial_- \varepsilon) \pi + \frac{\alpha}{2} (\partial_-^2 \varepsilon) F \right], \quad (24)$$

where we introduced the parameter

$$\varepsilon(\xi^+, \xi^-) = \omega_-(\xi^+) + \xi^- \omega_0(\xi^+) + (\xi^-)^2 \omega_+(\xi^+), \quad (25a)$$

satisfying the relation

$$\partial_-^3 \varepsilon = 0. \quad (25b)$$

The gauge transformations of the fields take the form

$$\delta_0 h \equiv \{ h, G \} = \partial_+ \varepsilon + \varepsilon \partial_- h - h \partial_- \varepsilon, \quad \delta_0 F \equiv \{ F, G \} = \varepsilon \partial_- F + \frac{\alpha}{2} \partial_- \varepsilon, \quad (26)$$

which is easily recognized as the $SL(2, R)$ symmetry.

We note that the gauge generator obtained from $\rho$ has the form

$$G = \lambda(\xi^+) \rho(\xi^+),$$

and produces the trivial transformations of the fields:

$$\delta_0 h = 0, \quad \delta_0 F = \lambda(\xi^+).$$

In this way, the $SL(2, R)$ symmetry of the Liouville theory in the light-cone gauge is consistently described as a kind of gauge symmetry in the light-front Hamiltonian formalism, based on the time $\xi^+$. The symmetry is described by three parameters $\omega_a(\xi^+)$, which are functions of only one coordinate — the time $\xi^+$. The situation differs from the case of standard gauge symmetries, where the gauge parameters depend on both variables $\xi^+$ and $\xi^-$. This property is closely related to the the fact that the symmetry is a residual symmetry of the theory.
4. SL(2,R) symmetry in the standard formalism

Now we shall return to the time $\xi_0$ and try to understand the existence of the $SL(2,R)$ symmetry in the Hamiltonian formalism based on this "real" time. Although the gauge-fixed action is invariant under the $SL(2,R)$ transformations, these transformations are not the gauge symmetries in the sense of the Hamiltonian formalism based on $\xi_0$, as we already explained at the end of Sec. 2 (the existence of a gauge symmetry would contradict the fact that the Liouville theory in the light-cone gauge is not degenerate). In this situation it is instructive to use a modification of Castellani’s method and obtain certain conditions that the generators of a global or residual (=not local) symmetry of the Hamiltonian equations of motion should satisfy. Unlike the case of local symmetries, these conditions do not give a prescription for the construction of the generators. We shall also analyse the problem from the point of view of the Noether currents.

The gauge fixed action (9) is invariant under the residual $SL(2,R)$ transformations (23) followed by the coordinate transformations $\delta \xi^\mu = -\varepsilon^\mu$, where $\varepsilon \equiv \varepsilon^-$. The total variation of the Lagrangian density is given by

$$\Delta L \equiv \delta L - \partial_\mu (L \varepsilon^\mu) = \partial_\mu \Lambda^\mu.$$

The variation of the action $\hat{I}$ with $\varepsilon^-$ satisfying the condition $\partial_3 \varepsilon^- = 0$ yields

$$\Lambda^0 = \frac{1}{\sqrt{2}} \left[ \frac{\alpha}{2} \partial_\varepsilon (\partial_- F - \partial_+ F) + \frac{\alpha^2}{2} \hbar \partial^2 \varepsilon + \varepsilon \mu^2 \right].$$

On the other hand, after using the equations of motion one obtains

$$\Delta L = \partial_\mu K^\mu, \quad K^\mu = \frac{\partial L}{\partial \Phi^i} \delta \Phi^i_j - L \varepsilon^\mu,$$

so that the elimination of $F$ leads to

$$K^0 = \frac{1}{\sqrt{2}} \left\{ -\varepsilon \left[ (1 - \hbar) T_- + \frac{\alpha^2}{2} \partial^2 \hbar - \mu^2 \right] + \frac{\alpha}{2} \partial_\varepsilon (\partial_- F - \partial_+ F + Q \partial_- h) \right\}.$$

The Noether current takes the form

$$N^\mu = K^\mu - \Lambda^\mu,$$

which, after an explicit calculation, leads to

$$N^0 = \frac{1}{\sqrt{2}} \left\{ \varepsilon \left[ (h - 1) T_- - \frac{\alpha^2}{2} \partial^2 \hbar \right] + \frac{\alpha^2}{2} \partial_\varepsilon \partial_- \hbar - \frac{\alpha^2}{2} (\partial^2 \varepsilon') \hbar \right\}. \quad (27)$$

After decomposing $\varepsilon$ as in Eq.(25a), $N^0$ can be written in the form

$$N^0 = -\frac{\alpha^2}{\sqrt{2}} (\omega_- j^- + \omega_0 j^0 + \omega_+ j^+), \quad (28a)$$

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where

\[
\begin{align*}
    j^- &= (1 - h)T_- + \frac{1}{2} \partial_-^2 h, \\
    j^0 &= -\frac{1}{2} \partial_- h + \xi^- j^-, \\
    j^+ &= h + 2\xi^- j^0 - (\xi^-)^2 j^-.
\end{align*}
\]  

The last equation yields

\[
h = j^+ - 2\xi^- j^0 + (\xi^-)^2 j^-.
\]

These results will be helpful in studying the Hamiltonian form of the $SL(2, R)$ symmetry. The generators $J^a$ of the global or residual symmetry of the Hamiltonian equations of motion have to satisfy the following conditions:

- After the elimination of momenta they should be reduced to $j^a$, i.e.

  \[
  J^a[q, p(q)] = j^a(q, \dot{q}).
  \]

- As discussed in details in the Appendix B, when the symmetry parameters depend only of $\xi^+$, $\omega_a = \omega_a(\xi^+)$, then the generators satisfy the relation

  \[
  \{J^a, H_T\} + \frac{\partial J^a}{\partial t} = \frac{\partial J^a}{\partial \sigma} \quad \text{i.e.} \quad \partial_- J^a = 0.
  \]

We shall now try to find the Hamiltonian currents $J^a$ by using the Hamiltonian equations of motion analogous to (28b), with $j^a \rightarrow J^a$. These objects will automatically satisfy the condition (A). Let us, therefore, define the Hamiltonian currents $J^a$ by

\[
\begin{align*}
    J^- &= \frac{1}{2} \partial_-^2 h, \\
    J^0 &= -\frac{1}{2} \partial_- h + \xi^- J^-, \\
    J^+ &= h + 2\xi^- J^0 - (\xi^-)^2 J^-,
\end{align*}
\]

where

\[
\partial_- X \equiv \frac{1}{\sqrt{2}} \left[ \{X, H_T\} - X' \right].
\]

An explicit calculation leads to the following result:

\[
\begin{align*}
    J^- &= \frac{1}{\alpha^2(h - 1)}(\mathcal{H}_0 - \mathcal{H}_1) + \frac{1}{\alpha^2} \mu^2 \\
    &= \frac{1}{\alpha^2} \left[ \frac{2}{\alpha^2} (h - 1) \pi_h^2 - \frac{2}{\alpha} F' \pi_h - \frac{2}{\alpha} \pi_h \pi + 2\pi_h' \right], \\
    J^0 &= \xi^- J^- + \frac{\sqrt{2}}{\alpha^2} \left[ (h - 1) \pi_h - \frac{\alpha}{2} F' - \frac{\alpha}{2} \pi \right], \\
    J^+ &= h + 2\xi^- J^0 - (\xi^-)^2 J^-.
\end{align*}
\]

Although these objects were obtained in the light-cone gauge, they can be easily written in the gauge invariant form by making the transition from $(h, \pi_h, F, \pi)$ to $(g_{11}, \pi_{11}, F, \pi_F)$. Their algebra is gauge independent, but their dynamics is not.
If we now construct the generator $J$ in analogy with Eq. (28a), i.e.

$$ J = -\frac{\alpha^2}{\sqrt{2}} \int d\sigma \left[ \omega_-(\xi^+)J^- + \omega_0(\xi^+)J^0 + \omega_+(\xi^+)J^+ \right], $$

we obtain the result

$$ J = \int d\sigma \left\{ -\frac{1}{\sqrt{2}} \varepsilon \left[ \frac{2}{\alpha^2} (h-1)\pi_h^2 - \frac{2}{\alpha} F'\pi_h \right] + (\partial_+ \varepsilon - h\partial_- \varepsilon)\pi_h \right. $$

$$ + \left. \left( \frac{\sqrt{2}}{\alpha} \varepsilon \pi_h + \frac{\alpha}{2}\partial_- \varepsilon \right)\pi + \frac{\alpha}{2}(\partial_- \varepsilon)F' - \frac{\alpha^2}{2\sqrt{2}}(\partial^2_- \varepsilon)h \right\}. $$

This quantity generates the transformations

$$ \delta_0 h = \left\{ h, J \right\}, \quad \delta_0 F = \left\{ F, J \right\}, $$

that coincide with (26) after the elimination of momenta.

The generator $J$ was found starting from the Noether current $N_0$ and replacing velocities with momenta using the Hamiltonian equations of motion, in accordance with the condition (A). Once we have found $J$, we can derive the corresponding transformations of all dynamical variables. Although these transformations are the Hamiltonian analogue of the Noether symmetries of the action, it is instructive to check whether they are the symmetries of the Hamiltonian equations of motion. An analysis of Castellani’s type (Appendix B) leads to the consistency requirement (B) on $J$. It can be easily verified that the condition (B) is also fulfilled, so that $J$ is indeed the generator of the residual symmetry of the Hamiltonian equations of motion.

The importance of this result becomes evident when the currents $J^a$ are defined in a gauge independent way. In that case one can also find the related $SL(2, \mathbb{R})$ transformations of the dynamical variables, but the consistency condition (B) tells us that these transformations are the symmetries of the theory only in the light-cone gauge.

Appendix: The conditions on the generators

The general method for studying the generators of local symmetries in the Hamiltonian approach has been developed by Castellani. The method yields an algorithm to construct the corresponding gauge generators. A slight modification of the method can be applied to study the generators of gauge symmetries in the case when the gauge parameters are not arbitrary functions of space–time variables, but depend only on $\xi^-$. In that case, there are no PFC constraints, so that the equations of motion for $q^i$ and $p_i$ are

$$ \dot{q}^i = \left\{ q^i, H \right\}, \quad \dot{p}_i = \left\{ p_i, H \right\}. $$

If we demand that varied trajectories

$$ \tilde{q}^i \equiv q^i + \delta_0 q^i, \quad \tilde{p}_i \equiv p_i + \delta_0 p_i, $$

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also satisfy the equations of motions, then
\[
\delta_0 \dot{q}_i = \frac{\partial^2 H}{\partial p_i \partial p_j} \delta_0 p_j + \frac{\partial^2 H}{\partial p_i \partial q^j} \delta_0 q^j , \\
\delta_0 \dot{p}_i = - \frac{\partial^2 H}{\partial q^i \partial p_j} \delta_0 p_j - \frac{\partial^2 H}{\partial q^i \partial q^j} \delta_0 q^j .
\]

On the other hand, if the variations of \( p \) and \( q \) are produced by the gauge generator,
\[
\delta_0 q_i(\xi) = \{q_i(\xi), G\}, \quad G \equiv \int d\tilde{\sigma} \omega_a(\tilde{\xi}^+) G^a(\tilde{\xi})
\]

[where \( \tilde{\xi} = (\xi^0, \tilde{\sigma}) \)] it follows that
\[
\delta_0 \dot{q}_i = \int d\tilde{\sigma} \left[ \dot{\omega}_a \{q_i, G^a\} + \omega_a \{\{q_i, G^a\}, H_T\} + \omega_a \{q_i, \frac{\partial G^a}{\partial t}\} \right],
\]

An analogous result holds for \( \delta_0 \dot{p}_i \). Now, we can use the fact that \( \dot{\omega} = \omega' \) and perform the integration by parts in the first term in (A3). If the surface term vanishes, we easily obtain the relation
\[
\partial_- G^a \equiv \{G^a, H\} + \frac{\partial G^a}{\partial t} - (G^a)' = 0 .
\]

References:

1. A. M. Polyakov, Phys. Lett. B103 (1981) 207.
2. A. M. Polyakov, Mod. Phys. Lett. A2 (1987) 893; V. G. Knizhnik, A. M. Polyakov and A. B. Zamolodchikov, Mod. Phys. Lett. A3 (1988) 819.
3. N. D. Hari Dass and R. Sumitra, Int. J. Mod. Phys. A4 (1988) 2245.
4. F. David, Mod. Phys. Lett. A3 (1988) 1651; J. Diestler and H. Kawai, Nucl. Phys. B321 (1989) 509.
5. Kai-Wen Xu and Chuan-Jie Zhu, Int. J. Mod. Phys. A6 (1991) 2331; Chang-Jun Ahn, Young-Jai Park, Kee Yong Kim and Yongduk Kim, Phys. Rev D42 (1990) 1144; J. A. Helayel-Neto, S. Mokhtari and A. W. Smith, Phys. Lett. B236 (1990) 12; Chang-Jun Ahn, Won-Tae Kim, Young-Jai Park, Kee Yong Kim and Yongduk Kim, Mod. Phys. Lett. A7 (1992) 2263.
6. T. Kuramoto, Phys. Lett. B233 (1989) 363; Y. Tanii, Int. J. Mod. Phys. 6A (1991) 4639.
7. Ed. Sh. Egorian and R. P. Manvelian, Mod. Phys. Lett. A5 (1990) 2371.
8. E. Abdalla, M. C. B. Abdalla, J. Gamboa and A. Zadra, Phys. Lett. B273 (1991) 222.
9. A. Miković, Canonical quantization of 2d gravity coupled to \( c < 1 \) matter, Queen Mary and Westfield College preprint QMW/PH/91/22 (1992); Canonical quantization approach to 2d gravity coupled to \( c < 1 \) matter, Imperial College preprint Imperial-TP/92-93/15 (1993).
10. S. Ghosh and S. Mukherjee, *SL(2, R)* currents in 2D-gravity are generators of improper
gauge transformations, Saha Institute preprint (1993).
11. P. A. M. Dirac, *Lectures on quantum mechanics*, Yeshiva University — Belfer Graduate
  School of Science (Academic, New York, 1964); A. Hanson, T. Regge and C. Teitelboim,
  *Constrained Hamiltonian Dynamics*, (Academia Nationale del Lincei, Rome, 1976);
  K. Sundermeyer, *Constrained Dynamics*, (Springer, Berlin, 1982).
12. J. Barcelos-Neto, Constraints and hidden symmetry in 2D-gravity, Univ. Rio de
    Janeiro preprint IF/UFRJ/92/21 (1992).
13. L. Castellani, Ann. Phys. (N. Y.) **143** (1982) 357.
14. R. Marnelius, Nucl. Phys. **B211** (1983) 14. S. Hwang, Phys. Rev. **D28** (1983) 2614.
15. P. A. M. Dirac, Rev. Mod. Phys. **21** (1948) 392.
16. K. G. Wilson, Nucl. Phys. **B** (Proc. Suppl.) **17** (1990); Prem. P. Srivastava,
    Constraints and Hamiltonian in Light-Front Quantized Field Theory, preprint DFPF/9/TH/58
    (1992).
17. T. Regge and C. Teitelboim, Ann. of Phys. (NY) **88** (1974) 286; R. Benguria, P.
    Cordero and C. Teitelboim, Nucl. Phys. **B122** (1977) 61; P. J. Steinhardt, Ann. of
    Phys. (NY) **128** (1980) 425.