TOWARDS RELIABLE CALCULATIONS
OF THE CORRELATION FUNCTION

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The correlation function of two identical pions interacting via Coulomb potential is computed for a general case of anisotropic particle’s source of finite life time. The effect of halo is taken into account as an additional particle’s source of large spatial extension. Due to the Coulomb interaction, the effect of halo is not limited to very small relative momenta but it influences the correlation function in a relatively large domain. The relativistic effects are discussed in detail and it is argued that the calculations have to be performed in the center-of-mass frame of particle’s pair where the (nonrelativistic) wave function of particle’s relative motion is meaningful. The Bowler-Sinyukov procedure to remove the Coulomb interaction is tested and it is shown to significantly underestimate the source’s life time.

1. Introduction

The correlation functions of two particles with ‘small’ relative momenta provide information about space-time characteristics of particle’s sources in high-energy nucleus-nucleus collisions, see the review articles.[1,2,3] Within the standard ‘femtoscopy’ method, one obtains parameters of a particle’s source, which is usually called the fireball, comparing the experimental correlation functions to the theoretical ones which are calculated in a given model. Such an analysis can be performed for pairs of non-identical or identical particles. In the former case, the correlation appears due to inter-particle interaction while in the latter one the interaction is

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combined with the effects of quantum statistics. Since we usually deal with electrically charged particles, observed two-particle correlations are influenced, if not dominated, by the Coulomb interaction. The effect of the Coulomb force is treated as a correction and it is usually eliminated from the experimental data by means of the so-called Bowler-Sinyukov procedure. And then, the correlation function, which is obtained in such a way from experimental data, is compared with the theoretical correlation function of two non-interacting particles. The latter one is computed for a given source function - a distribution of the emission points - parameterized usually in a gaussian form. The comparison with the experimental data provides parameters of the source function.

Within the method of imaging, one obtains the source function not referring to its specific parametrization but directly inverting the functional dependence of the correlation function on the source function. The procedure of inversion involves the effect of quantum statistics as well as that of inter-particle interaction. The method provides essentially model independent information on the source space-time sizes but modeling is still needed to deduce the source life time which is coupled to the spatial size parameters. The method of one-dimensional imaging was successfully applied to experimental data but the three-dimensional imaging is technically very complex and cumbersome. However, very first results of the three-dimensional imaging applied to NA49 data are presented in these proceedings.

The femtoscopy was applied to a large volume of experimental data on nucleus-nucleus collisions at SPS energy. The spatial size of particle’s sources appeared to be comparable to the expected size a fireball created in nucleus-nucleus collisions while the emission time of particles was significantly shorter. It was predicted that at RHIC energies the emission time would be significantly longer due to the long lasting hydrodynamic evolution of the system created at the early stage of nucleus-nucleus collisions. To a big surprise the experimental data obtained at RHIC show a very little, if any, change of the space-time characteristics of a fireball when compared to the SPS data. In particular, the emission time of particles appeared to be as short as 1 fm/c. This surprising result, which is now known as the ‘HBT Puzzle’, demonstrates that either we do not really understand the dynamics of relativistic heavy-ion collisions or the principles of the femtoscopy are doubtful. In any case, the femtoscopy method need to be examined.

In this presentation there is given a preliminary account of our study which is aimed to check three aspects of the standard method. Relativistic effects are, in our opinion, not satisfactory incorporated, as the non-relativistic wave function of two interacting particles is often implicitly treated as a Lorentz scalar. It should be stressed that a transformation law of a wave function under Lorentz transformations is unknown. However, the observed correlation functions are significantly different from unity for a small relative momenta when the relative motion of particles is

\[ \text{Using the Bethe-Salpeter equation, it has been recently shown that the hydrogen atom wave function experiences the Lorentz contraction under the Lorentz boost.} \]
non-relativistic. Therefore, it is fully legitimate to use the non-relativistic wave function in the center-of-mass frame of two particles. However, it requires an explicit transformation of the source function to the center-of-mass frame. In Sec. 4 the relativistic effects are discussed in detail and it is shown that a proper treatment of them leads to a numerically different result when compared with the standard procedure.

The correlation function of two identical non-interacting bosons is expected to be equal to 2 for vanishing relative momentum of the two particles. The correlation function extracted from experimental data by means of the procedure, which is supposed to remove the Coulomb interaction, does not posses this property. The correlation function at zero relative momentum rather equals \(1 + \lambda\) with \(0 < \lambda < 1\). This fact can be explained referring to the concept of halo. It assumes that only a fraction of observed particles, which equals \(\lambda\), comes from the fireball while the rest originates from the long living resonances. Then, we have two sources of particles: the fireball, which is rather small, and the halo with the radius given by the distance traveled by long living resonances. The complete correlation function, which includes particles from the fireball and the halo, equals 2 at exactly vanishing relative momentum. However, the quantum statistical correlation of two particles coming from the halo occurs at the relative momentum which is as small as the inverse radius of the halo. Since experimental momentum resolution is usually much poorer and such small relative momenta are not accessible, the correlation function is claimed to be less than 2 for effectively vanishing relative momentum. While the effect of halo is commonly believed to resolve the problem, the concept has been never examined for the particles which experience Coulomb interaction. In Sec. 6 we show that the effect of halo is not limited to very small relative momenta but due to the Coulomb repulsion the correlation function is modified for larger momenta.

The Bowler-Sinyukov correction procedure, which is used to eliminate the Coulomb interaction from the experimental data, assumes that the Coulomb effects can be factorized out. The correction’s factor is calculated for a particle’s source which is spherically symmetric and has zero life time. We find such an approach rather inconsistent as the objective is to determine the source spatial parameters not assuming any source symmetry. The Bowler-Sinyukov procedure is examined in Sec. 7 where we also comment on a similar test of the procedure performed in the paper.\(^{21}\)

We use the natural units, where \(c = \hbar = 1\), and the metric convention \((+,-,-,-)\).

### 2. Definition

The correlation function \(C(p_1, p_2)\) of two particles with momenta \(p_1\) and \(p_2\) is defined as

\[
C(p_1, p_2) = \frac{dN}{dp_1 dp_2} \cdot \frac{dN}{dp_1 dp_2}.
\]
\[ \frac{dN}{dp_1 dp_2} \text{ and } \frac{dN}{dp_1} \]

is, respectively, the two- and one-particle momentum distribution. The correlation function can be written down in a Lorentz covariant form

\[ C(p_1, p_2) = \frac{E_1 E_2 \frac{dN}{dp_1 dp_2}}{E_1 \frac{dN}{dp_1} E_2 \frac{dN}{dp_2}}, \tag{1} \]

where \( E \frac{dN}{dp} \) is the Lorentz invariant distribution.

The covariant form (1) shows that the correlation function is Lorentz scalar which can be easily transformed from one to another reference frame. If the particle four-momenta, which are on mass-shell, transform as \( p_i \rightarrow p'_i \) with \( i = 1, 2 \), the transformed correlation function equals

\[ C'(p'_1, p'_2) = C(p_1(p'_1), p_2(p'_2)). \]

3. Nonrelativistic Koonin formula

Within the Koonin model\(^{22}\), the correlation function \( C \) can be expressed in the source rest frame as

\[ C(p_1, p_2) = \int d^3r_1 dt_1 d^3r_2 dt_2 D(r_1, t_1) D(r_2, t_2) |\Psi(r'_1, r'_2)|^2, \tag{2} \]

where \( r'_i \equiv r_i + v_i t_i \), \( \Psi(r'_1, r'_2) \) is the wave function of the two particles and \( D(r, t) \) is the single-particle source function which gives the probability to emit the particle from the space-time point \((t, r)\). The source function is normalized as

\[ \int d^3r \ dt \ D(r, t) = 1. \tag{3} \]

After changing the variables \( r' \leftrightarrow r \), the correlation function can be written in the form

\[ C(p_1, p_2) = \int d^3r_1 dt_1 d^3r_2 dt_2 D(r_1 - v_1 t_1, t_1) D(r_2 - v_2 t_2, t_2) |\Psi(r_1, r_2)|^2. \]

Now, we introduce the center-of-mass coordinates

\[ r = r_2 - r_1, \quad R = \frac{1}{M}(m_1 r_1 + m_2 r_2), \]
\[ t = t_2 - t_1, \quad T = \frac{1}{M}(m_1 t_1 + m_2 t_2), \]
\[ q = \frac{1}{M}(m_2 p_1 - m_1 p_2), \quad P = p_1 + p_2, \]

where \( M \equiv m_1 + m_2 \). Using the center-of-mass variables, one gets

\[ C(q) = \int d^3r \ dt \ D_r(r - vt, t) |\varphi_q(r)|^2, \tag{4} \]

where the ‘relative’ source function is defined as

\[ D_r(r, t) \equiv \int d^3R \ dT \ D(R - \frac{m_2}{M} R, T - \frac{m_2}{M} t) D(R + \frac{m_1}{M} R, T + \frac{m_1}{M} t). \tag{5} \]
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To get Eq. (4), the wave function was factorized as
\[ \Psi(r_1, r_2) = e^{iPR\phi_q(r)} \]
with \( \phi_q(r) \) being the wave function of the relative motion in the center-of-mass frame. Deriving Eq. (4), it has been assumed that the particles move with equal velocities i.e. \( v_1 = v_2 = v \) which requires, strictly speaking, \( q = 0 \). However, one observes that \( |v_1 - v_2| \ll |v| \) if \( |q| \ll \mu |p_i|/m_i \) where \( \mu \equiv m_1m_2/M \). Thus, the approximation \( v_1 = v_2 \) holds for a sufficiently small center-of-mass momentum.

We choose the gaussian form of the single-particle source function \( D(r, t) \)
\[
D(r, t) = \frac{1}{16\pi^2 R_x R_y R_z \tau} \exp \left[ -\frac{x^2}{2R_x^2} - \frac{y^2}{2R_y^2} - \frac{z^2}{2R_z^2} - \frac{t^2}{4\tau^2} \right],
\]
where \( r = (x, y, z) \) and the parameters \( \tau, R_x, R_y \) and \( R_z \) characterize the life time and sizes of the source. Specifically, the parameters \( \tau \) and \( R_x \) give, respectively,
\[
\tau^2 = \langle t^2 \rangle = \int d^3r \int dt \ t^2 D(r, t), \quad R_x^2 = \langle x^2 \rangle = \int d^3r \int dt \ x^2 D(r, t).
\]

The relative source function computed from Eq. (5) with the single-particle source (6) is
\[
D_r(r, t) = \frac{1}{16\pi^2 R_x R_y R_z \tau} \exp \left[ -\frac{x^2}{4R_x^2} - \frac{y^2}{4R_y^2} - \frac{z^2}{4R_z^2} - \frac{t^2}{4\tau^2} \right].
\]
We note that the particle’s masses, which are present in the definition (5), disappear completely in the formula (7). This is the feature of the gaussian parameterization (6).

In the case of non-interacting identical bosons, the two-particle symmetrized wave function is
\[
\Psi(r_1, r_2) = \frac{1}{\sqrt{2}} [e^{i(p_1r_1 + p_2r_2)} + e^{i(p_1r_1 + p_1r_2)}] = \frac{1}{\sqrt{2}} [e^{iqr} + e^{-iqr}] e^{iPR}.
\]
It gives the modulus square of the wave function of relative motion \( |\phi_q(r)|^2 = 1 + \cos(2qr) \) which in turn provides the correlation function equal to
\[
C(q) = 1 + \exp \left[ -4(\tau^2(qv)^2 + R_x^2q_x^2 + R_y^2q_y^2 + R_z^2q_z^2) \right],
\]
where \( q \equiv (q_x, q_y, q_z) \). We note that the ‘cross terms’ such as \( q_x q_z \) do not show up as the source function (6) obeys the mirror symmetry \( D(r, t) = D(-r, t) \). We also note that \( q \) often denotes the relative momentum \( p_1 - p_2 \) not the center-of-mass momentum, which for equal mass nonrelativistic particles equals \( \frac{1}{2}(p_1 - p_2) \), and then, the factor 4 does not show up in the correlation function (8) of identical free bosons. However, we believe that using the center-of-mass momentum is physically better motivated.
4. Relativistic formulations

There are two natural ways to ‘relativize’ the Koonin formula (2). The first one provides explicitly Lorentz covariant correlation function but it is applicable only for the non-interacting particles. The second one holds only in a specific reference frame but it is applicable for interacting particles as well. Below, we consider the two methods. We start, however, with the discussion of the Lorentz covariant form of the source function.

4.1. Lorentz covariant source function

The Lorentz covariant form of the gaussian parameterization of the source function (6) is

\[ D(x) = \frac{\sqrt{\det \Lambda}}{4\pi^2} \exp\left[-\frac{1}{2} x_\mu \Lambda^{\mu\nu} x_\nu\right], \]

where \( x_\nu \) is the position four-vector and \( \Lambda^{\mu\nu} \) is the Lorentz tensor characterizing the source which in the source rest frame is

\[ \Lambda^{\mu\nu} = \begin{bmatrix} \frac{1}{\tau^2} & 0 & 0 & 0 \\
0 & \frac{1}{\tau^2} & 0 & 0 \\
0 & 0 & \frac{1}{\tau^2} & 0 \\
0 & 0 & 0 & \frac{1}{\tau^2} \end{bmatrix}. \]

The source function as written in Eq. (9) obeys the normalization condition (3) not only for the diagonal matrix \( \Lambda \) but for non-diagonal as well. The covariant relative source function (7) is given by

\[ D_r(x) = \frac{\sqrt{\det \Lambda}}{16\pi^2} \exp\left[-\frac{1}{4} x_\mu \Lambda^{\mu\nu} x_\nu\right]. \]

4.2. Explicitly covariant ‘relativization’

As follows from Eq. (1), the correlation function is a Lorentz scalar. Therefore, the Koonin formula (2) can be ‘relativized’ demanding its Lorentz covariance. Let us write the formula

\[ C(p_1, p_2) = \int d^4x_1 d^4x_2 D(x_1) D(x_2) |\Psi(x_1, x_2)|^2, \]

where \( p_i \) and \( x_i \) is, respectively, the four-momentum and four-position. Since the source function \( D(x) \) and the four-volume element \( d^4x_i \) are both the Lorentz scalars,
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the whole formula (12) is covariant if the wave function \( \Psi(x_1, x_2) \) is covariant as well. In the case of non-interacting bosons the relativistic wave function \( \Psi(x_1, x_2) \) is

\[
\Psi(x_1, x_2) = \frac{1}{\sqrt{2}} (e^{ip_1 x_1 + ip_2 x_2} + e^{ip_1 x_2 + ip_2 x_1}).
\] (13)

As the function depends on the scalar products of two four-vectors, this is the Lorentz scalar. We note that the function (13) depends on two time arguments.

Our further considerations are limited to pairs of identical particles and thus, we introduce the relative coordinates as

\[
x = x_2 - x_1, \quad X = \frac{1}{2} (x_1 + x_2),
\]

\[
q = \frac{1}{2} (p_1 - p_2), \quad P = p_1 + p_2.
\] (14)

We note that in the non-relativistic treatment the three-vectors \( r \) and \( q \), which are given by the four-vectors \( x = (t, r) \) and \( q = (q_0, q) \), correspond to the inter-particle separation and the particle’s momentum in the center-of-mass of the particle pair. This is, however, not the case in the relativistic domain. To get the center-of-mass variables, the four-vectors need to be Lorentz transformed.

With the variables (14), the wave function (13) equals

\[
\Psi(x, X) = \frac{1}{\sqrt{2}} (e^{iqx} + e^{-iqx}) e^{-iPX},
\]

and the correlation function is found in the form

\[
C(q) = 1 + \exp \left[ -4q_\mu (\Lambda^{\mu\nu})^{-1} q_\nu \right],
\]

which is explicitly Lorentz covariant. For the source matrix (10), the correlation function equals

\[
C(q) = 1 + \exp \left[ -4 (q_0^2 \tau^2 + q_x^2 R_x^2 + q_y^2 R_y^2 + q_z^2 R_z^2) \right].
\] (15)

If \( |q| \ll |p_i| \) with \( i = 1, 2 \), then \( q_0 \approx q v \), and the correlation function (15) exactly coincides with the non-relativistic expression (8). This coincidence is not completely obvious as the time variables enter differently in the Koonin formula (2) and in the covariant one (12).

Let us consider the correlation function in the center-of-mass frame of the particle’s pair. We assume that the velocity of the center-of-mass frame in the source rest frame is along the axis \( x \). Then, \( v = (v, 0, 0) \) and \( q_0 = qv = q_x v \). The correlation function (15), which holds in the source rest frame, equals

\[
C(q) = 1 + \exp \left[ -4 (v^2 \tau^2 + R_x^2 q_x^2 + R_y^2 q_y^2 + R_z^2 q_z^2) \right].
\] (16)

As seen, the effective source radius in the direction \( x \) is \( \sqrt{R_x^2 + v^2 \tau^2} \). We now transform the source function to the center-of-mass frame where the quantities are labeled with the index *. The center-of-mass source matrix (10), which is computed as

\[
\Lambda_{\mu}^{\alpha} = L_{\mu}^{\alpha} \Lambda^{\sigma \rho} L_{\rho}^{\nu},
\]
where
\[
L_{\sigma}^{\mu} = \begin{pmatrix}
\gamma & -v\gamma & 0 & 0 \\
-\gamma v & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]
with \(\gamma \equiv (1 - v^2)^{-1/2}\), equals
\[
\Lambda_{\sigma}^{\mu\nu} = \begin{pmatrix}
\gamma^2 (1 + \frac{v^2}{R_1^2}) & -\gamma^2 v (1 + \frac{1}{R_2^2}) & 0 & 0 \\
-\gamma^2 v (1 + \frac{1}{R_2^2}) & \gamma^2 (1 + \frac{v^2}{R_1^2}) & 0 & 0 \\
0 & 0 & \frac{1}{R_1^2} & 0 \\
0 & 0 & 0 & \frac{1}{R_2^2}
\end{pmatrix}.
\]

Then, the correlation function in the center-of-mass frame is found as
\[
C(q_*) = 1 + \exp[-4q_\mu(\Lambda_{\sigma}^{\mu\nu})^{-1}q_\nu]
\]
\[
= 1 + \exp[-4(\gamma^2(v^2r^2 + R_1^2)q_{xx}^2 + R_2^2q_{yy}^2 + R_2^2q_{zz}^2)].
\]
As seen, the effective source radius along the direction of the velocity is elongated, not contracted as one can naively expect, by the factor \(\gamma\).

4.3. Non-covariant relativization

The quantum mechanical description of two relativistic interacting particles faces serious difficulties. The problem is greatly simplified when the relative motion of two particles is non-relativistic (with the center-of-mass motion being fully relativistic). Since the correlation functions usually differ from unity only for small relative momenta of particles, it is reasonable to assume that the relative motion is non-relativistic. We further discuss the correlation functions taking into account the relativistic effects of motion of particles with respect to the source but the particle’s relative motion is treated non-relativistically. In such a case, the wave function of relative motion is a solution of the non-relativistic Schrödinger equation. Thus, we compute the correlation function directly from the Koonin formula \((2)\) but we use it in the center-of-mass frame of the pair. For this reason we first transform the source function to this frame and then, after performing the integrations over \(x_1\) and \(x_2\), we transform the whole correlation function, which is known to be a Lorentz scalar, to the source rest frame.

As already stressed, we compute the correlation function in the center-of-mass frame of the pair and we use the relative variables \((14)\). Since the source function has the gaussian form \((9)\) (with the non-diagonal matrix \(\Lambda\)), the integration over \(X\) can be easily performed and the correlation function equals
\[
C(q_*) = \int d^4x_* D_r(x_*) |\varphi_{q_*}(r'_*)|^2,
\]
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Fig. 1. The Coulomb correlation function computed directly in the source rest frame (‘fake covariance’) and the correlation function computed in the center-of-mass frame of the pair and then transformed to the source rest frame (‘CM calculations’).

Fig. 2. The Coulomb correlation function $C(q_0, 0, 0)$ as a function of $q_0$ for the source parameters: $R_x = 2$ fm, $R_y = 2$ fm, $R_z = 3$ fm, $\tau = 2.5$ fm and $v = (0.9, 0, 0)$.

where $D_r(x_*)$ is the relative source function (11) and $\varphi_{q_0}(r'_*)$ with $r'_* \equiv r_* + v_\star t_\star$ is the non-relativistic wave function of relative motion.

Although our aim is to compute the correlation functions of interacting particles, we start the discussion with the free identical bosons for the sake of comparison with the results of the previous section 4.2 where the covariant ‘relativisation’ was presented. We again assume that $v = (v, 0, 0)$, and then the source matrix is given by Eq. (17). The correlation function, which follows from Eq. (19), exactly coincides with the formula (18). To get the correlation function in the source rest frame, one performs the Lorentz transformation and obtains the formula (16). Thus, the two ways of ‘relativization’ give the same result for non-interacting particles. This is not quite trivial as the time dependence of the Koonin formula (19) and of the explicitly covariant one (12) is rather different.

In the following sections we compute the correlation function of identical pions interacting due to Coulomb force, using the Koonin formula (19). Thus, we first compute the correlation function in the center-of-mass frame of the pair, and then, we transform it to the source rest frame. The correlation function of interacting particles is often calculated following the explicitly covariant method described in Sec. 4.2. And then, the non-relativistic wave function is treated as a Lorentz scalar function. Sometimes one argues that it must be a Lorentz scalar to guarantee that the right-hand side of Eq. (12) is the Lorentz scalar. We believe that such a procedure is incorrect and we refer to it as ‘fake covariance’. In principle, one can argue that the function $\Psi(x_1, x_2)$ from Eq. (12) has to be a Lorentz scalar but it is unjustified to identify it with the non-relativistic wave function. The wave function is well defined in the center-of-mass of the pair and in the reference frames which move non-relativistically with respect to it. But properties of the wave function under Lorentz transformations are unknown, and thus, there is no reliable way to
transform it from one frame to another. We can also put it differently. The correlation function as defined by Eq. (1) is certainly a Lorentz scalar but there is no guarantee that the theoretical model (2) gives the correlation function which is a Lorentz scalar. However, we expect that the model works well in the center-of-mass frame of the pair where the non-relativistic wave function is well defined. Thus, we can compute the correlation function in this frame and then, we can transform it to an arbitrary frame, knowing that the correlation function is a Lorentz scalar. In this way, the non-covariant procedure circumvents the problem of unknown transformation properties of the non-relativistic wave function.

Although the problem of transformation properties looks somewhat academic, it leads to a numerically significant effect. In Fig. 1 we show the correlation function of two identical charged pions computed in the center-of-mass frame of the pair and then transformed to the source rest frame. The function is compared to the correlation function which is directly computed in the source rest frame, treating the wave function as a Lorentz scalar. The source is assumed here to be spherically symmetric with \( R = 2 \) fm and \( \tau = 0 \). The pair velocity with respect to the source chosen to be \( v = 0.9 \). (Details of the calculations of the Coulomb correlation functions are discussed in the next section.) As seen, the assumption of ‘fake covariance’ noticeably distorts the correlation function.

5. Coulomb correlation functions

In this section we compute, using Eq. (19), the correlation functions of two identical pions interaction via Coulomb potential. The calculations are performed for the anisotropic gaussian source of finite emission time \([9, 10]\). We use the Bertsch-Pratt coordinates \([24, 25]\) out, side, long. These are the Cartesian coordinates, where the direction long is chosen along the beam axis (z), the out is parallel to the component of the pair momentum \( \mathbf{P} \) which is transverse to the beam. The last
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Fig. 5. The Coulomb correlation function which includes the effect of halo for three values of $\lambda$ equal 1.0 (upper line), 0.9 (middle line), 0.7 (lower line).

Fig. 6. The free correlation function (upper line), the Coulomb correlation function (middle line), the Bowler-Sinyukov correction factor (lower line).

direction - side - is along the vector product of the out and long versors. So, the vector $\mathbf{q}$ is decomposed into the $q_o$, $q_s$, and $q_l$ components. If the particle’s velocity is chosen along the axis $x$, the out direction coincides with the direction $x$, the side direction with $y$ and the long direction with $z$. We note that the correlation function of two identical free bosons in the Bertsch-Pratt coordinates in the source rest frame is

$$C(\mathbf{q}) = 1 + \exp \left[ - \frac{4}{q^2} \left( q_o^2 R_o^2 + q_s^2 R_s^2 + q_l^2 R_l^2 \right) \right],$$

where $R_o = \sqrt{R_x^2 + v^2 r^2}$, $R_s = R_y$ and $R_l = R_z$. As seen, the source life time is mixed up with the size parameter $R_x$.

As well known, the Coulomb problem is exactly solvable within the non-relativistic quantum mechanics. The exact wave function of two non-identical particles interacting due to repulsive Coulomb force is given as

$$\varphi_q(\mathbf{r}) = e^{-\frac{2\pi}{q}} \Gamma(1 + i \frac{\eta}{q}) e^{i \frac{\pi}{q} F(-i \frac{\eta}{q}, 1, i(q \mathbf{r} - \mathbf{q} \mathbf{r})),$$

(20)

where $q \equiv |\mathbf{q}|$, $\eta \equiv \mu e^2/8\pi$ with $\mu$ being the reduced mass of the two particles and $\pm e$ is the charge of each of them; $F$ denotes the hypergeometric confluent function. The wave function for the attractive interaction is obtained from (20) by means of the substitution $\eta \rightarrow -\eta$. When one deals with identical particles, the wave function $\varphi_q(\mathbf{r})$ should be (anti-)symmetrized. The modulus of the symmetrized Coulomb wave function equals

$$|\varphi_q(\mathbf{r})|^2 = \frac{1}{2} G(q) \left[ |F(-i \frac{\eta}{q}, 1, i(q \mathbf{r} - \mathbf{q} \mathbf{r})))|^2 + |F(-i \frac{\eta}{q}, 1, i(q \mathbf{r} + \mathbf{q} \mathbf{r})))|^2 \right)^2$$

$$+ 2 \text{Re} \left( e^{i q \mathbf{r}} F(-i \frac{\eta}{q}, 1, i(q \mathbf{r} - \mathbf{q} \mathbf{r})) F^*(-i \frac{\eta}{q}, 1, i(q \mathbf{r} + \mathbf{q} \mathbf{r}))) \right),$$

(21)
where $G(q)$ is the so-called Gamov factor defined as

$$G(q) = \frac{2\pi\eta}{q} \frac{1}{\exp\left(\frac{2\pi\eta}{q}\right) - 1}.$$  \hfill (22)

Substituting the modulus \[21\] in Eq. \[19\], one finds the correlation function in the center-of-mass frame which is further transformed to the source rest frame. In Figs. 2, 3 and 4 we show the correlation functions $C(q_o, 0, 0)$, $C(0, q_s, 0)$ and $C(0, 0, q_l)$, respectively, which are calculated for the following values of the source parameters $R_x = 2$ fm, $R_y = 2$ fm, $R_z = 3$ fm, $\tau = 2.5$ fm. The velocity of the particle’s pair equals $v = 0.9$ and it is along the axis $x$. As seen, the correlation function $C(q_o, 0, 0)$ is qualitatively different than $C(0, q_s, 0)$ and $C(0, 0, q_l)$. We note that except the Monte Carlo calculations presented in the paper \[21\], where the source functions were generated according to the so-called blast-wave models, the results shown in Figs. 2, 3, 4 represent the first, as far as we know, relativistic calculations of the Coulomb correlation function for an anisotropic source of finite life-time.

6. The Halo

As mentioned in the introduction, the halo \[21\] was introduced to explain the fact that, after removing the Coulomb effect, the experimentally measured correlation functions are smaller than 2 at vanishing relative momentum. The idea of halo assumes that only a fraction $\lambda$ of particles contributing to the correlation function comes from the fireball while the remaining fraction $(1 - \lambda)$ originates from long living resonances. Then, we have two sources of the particles: the small one - the fireball and the big one corresponding to the long living resonances. The source function has two contributions with $\lambda$ as a relative weight that is

$$D(x) = \lambda D_f(x) + (1 - \lambda) D_h(x),$$
where $D_f(x)$ and $D_h(x)$ represent the fireball and halo, respectively. For non-interacting identical bosons, the correlation function is

$$C(q) = 1 + \lambda e^{-4R_f^2q^2} + (1 - \lambda) e^{-4R_h^2q^2},$$

(23)

where both the fireball and halo are assumed to be spherically symmetric sources of zero lifetime; $R_f$ and $R_h$ are the radii of, respectively, the fireball and the halo. If $R_h$ is so large that $R_h^{-1}$ is below an experimental resolution of the relative momentum $q$, the second term of the correlation function (23) is effectively not seen, and one claims that $C(q = 0) = 1 + \lambda$.

We have included the halo in the calculations of the Coulomb correlation functions. The exemplary result is shown in Fig. 4 for three values of $\lambda$: 1.0, 0.9, and 0.7. For simplicity, the fireball and halo are spherically symmetric and have zero lifetime; $R_f = 2$ fm and $R_h = 60$ fm.

7. The Bowler-Sinyukov procedure

As mentioned in the Introduction, the Coulomb effect is usually treated as a correction and it is subtracted from the experimentally measured correlation functions by means of the Bowler-Sinyukov procedure. We first note that the Coulomb effect is far not small. In Fig. 6 we show the Coulomb and free correlation functions computed for the spherically symmetric source of zero lifetime with $R = 2$ fm. As seen, the correlation functions are qualitatively different from each other in the domain of small momenta $q$. Therefore, a method to subtract the Coulomb effect should be carefully tested.

The Bowler-Sinyukov procedure assumes that the Coulomb effect can be factorized out, that is the correlation function can be expressed as

$$C(q) = A(q) C_{\text{free}}(q),$$

(24)
where $C_{\text{free}}(q)$ is the free correlation function and $A(q)$ is the correction factor which depends only on $q \equiv |q|$.

The correction factor is actually the Coulomb correlation function which neglects the effect of quantum statistics, i.e., the wave function is not symmetrized. The computation, which is, in particular, described in detail in the Appendix to the paper, is performed in the center-of-mass frame where the particle’s source is assumed to be symmetric and of zero live time. Thus, the correction factor is given by the formula

$$A(q_s) = \int d^3r D_r(r) |\varphi_{q_s}(r)|^2 = G(q_s) \int d^3r D_r(r) |F(-i\eta, 1, i(q_s r - q, r))|^2, \quad (25)$$

where $\varphi_{q_s}(r)$ is the Coulomb wave function and $D_r(r)$ describes the spherically symmetric gaussian source of zero life time and of the ‘effective’ radius $R = \frac{2}{3} \sqrt{R_o^2 + R_s^2 + R_l}$ where $R_o$, $R_s$ and $R_l$ are the actual source radii; the Gamov factor $G(q)$ is given by Eq. (22). Using the parabolic coordinates, the double integration can be easily performed analytically in Eq. (25), and one is left with the one-dimensional integral which has to be taken numerically. Finally, the transformation to the source rest frame should be performed. The exemplary result for $R = 2$ fm is shown in Fig. 6. The pair velocity with respect to the source is assumed here to be so small that $q = q_s$. As seen, the factor $A(q)$ vanishes as $q$ tends to zero. Thus, the (multiplicative) correction to $C_{\text{free}}(q)$ is infinite for $q = 0$.

Once we are able to compute exact Coulomb correlation functions for an anisotropic source of finite life time, we can test whether the Bowler-Sinyukov procedure correctly subtracts the Coulomb effect for such a source and properly reproduces the free correlation functions. For this purpose, we first calculate the exact Coulomb correlation function in the pair center-of-mass frame, then we divide
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it by the correction factor $A(q_\ast)$ computed according to Eq. (25) for a given effective source radius, and finally we transform the function to the source rest frame. To get source parameters, which are contained in the extracted ‘free’ correlation function, the extracted function is fitted with the gaussian parameterization of the free function. The source parameters, which are used to compute the exact Coulomb correlation function, are called ‘input parameters’ while those which are obtained by fitting the extracted correlation function as the ‘output parameters’.

7.1. No halo

We have first tested the Bowler-Sinyukov procedure for the case when all particles come from the fireball - there is no halo. Examples of the correlation functions extracted from the exact correlation functions by means of the Bowler-Sinyukov

| Table 1. The input and output source parameters, the radii and life time are given in fm. |
|-----------------------------------|-------|-------|-------|-------|
| $R_o$ | $R_s$ | $R_l$ | $\tau$ | $\lambda$ |
|-------|-------|-------|-------|-------|
| input | 3.68  | 2.50  | 3.00  | 3.00  | 1.00  |
| output| 3.48  | 2.49  | 2.93  | 2.69  | 1.26  |

| Table 2. The input and output source parameters, the radii and life time are given in fm. |
|-----------------------------------|-------|-------|-------|
| $R_o$ | $R_s$ | $R_l$ | $\tau$ | $\lambda$ |
|-------|-------|-------|-------|-------|
| input | 4.39  | 4.00  | 6.00  | 2.00  | 0.70  |
| output| 3.56  | 3.60  | 5.09  | -     | 0.51, 0.72, 0.64 |
procedure are shown in Figs. 7, 8, 9. The extracted functions are compared to the expected correlation functions of noninteracting bosons for the given source. The input and output parameters are given in Table 1. Actually, we have obtained three values of $\lambda$ as the extracted functions $C(q_o, 0, 0)$, $C(0, q_s, 0)$ and $C(0, 0, q_l)$ are fitted independently from each other. However, all three output values of $\lambda$ are very close to each other, and we give only one value in the table. As seen, the correction procedure reproduces the source radii quite well. However, $R_o$ is underestimated, and consequently, the source life time is reduced. We also note that the parameter $\lambda$ is increased by about 20%.

### 7.2. Halo included

The situation changes significantly when the halo is included in the consideration. In Fig. 10 we show the correlation function extracted from the Coulomb correlation function which includes the halo. In this illustrative example the fireball and halo are spherically symmetric and have zero life times; $R_f = 2$ fm and $R_h = 60$ fm. The correction factor is computed for $R = 2$ fm. For comparison we also show in Fig. 10 the free correlation function computed for the double source of the fireball and halo with $R_f = 2$ fm and $R_h = 60$ fm. We see that the extracted correlation function is badly distorted and the distortion extends far beyond $R^{-1}_h$.

The problem we face here is known and it can be partially resolved by extracting the ‘free’ correlation function not as in Eq. (24) but according to the formula

$$C(q) = \left(1 + \lambda(A(q) - 1)\right)C_{\text{free}}(q).$$

The correction factor is ‘diluted’ that is only the contribution from the fireball is corrected. The correlation function extracted from the Coulomb correlation function by means of the formula (26) is shown in Fig. 11 together with the expected free function. The extracted function is less distorted than that one shown in Fig. 10 but the distortion is still dramatic at small $q$ and it extends beyond $R^{-1}_h$.

We have applied the ‘diluted’ correction procedure to the Coulomb correlation function computed for an anisotropic source of finite life time with halo. Examples of the extracted correlation function are shown in Figs. 12, 13, and 14. The extracted functions are presented together with the respective expected free functions and with the gaussian parameterizations of free function fitted to the extracted functions. When the latter functions were fitted with the gaussian parameterizations, the momenta smaller than $q_{\text{max}}$, where $q_{\text{max}}$ is the momentum for which the extracted function has a maximum, were cut-off. The input and output parameters

| $R_o$ | $R_s$ | $R_0$ | $\tau$ | $\lambda$ |
|-------|-------|-------|--------|----------|
| input | 5.38  | 4.00  | 6.00   | 4.00     | 0.70     |
| output| 4.07  | 3.70  | 5.08   | 1.88     | 0.38, 0.60, 0.70 |
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The velocity of the particle’s pair equals \( v = 0.9 \) and it is along the axis \( x \). Since \( \lambda \) is, as already mentioned, extracted separately from \( C(q_o, 0, 0) \), \( C(0, q_s, 0) \) and \( C(0, 0, q_t) \), there are three output values of \( \lambda \) which differ from each other. As also seen, all extracted source radii are significantly reduced. Since the output \( R_o \) is smaller than the output \( R_s \), the output life time vanishes! A set of the input and output parameters with an extended input life time is presented in Table 3. The velocity of the particle’s pair, as previously, equals \( v = 0.9 \) and it is along the axis \( x \). We again observe a significant reduction of the source life time.

An accuracy of the Bowler-Sinyukov procedure was also tested applying the Monte Carlo calculations presented in the paper [21]. And it was concluded that the source radii are reproduced quite well. However, the source functions were generated according to the so-called blast-wave models where the emission time is rather short. Then, as our findings also show, the Bowler-Sinyukov procedure works indeed very well.

8. Conclusions

We have presented here the preliminary account of our study of the two-particle correlation functions. We have argued that the calculations must be performed in the center-of-mass frame of the pair where a nonrelativistic wave function of the particle’s relative motion is meaningful. We have computed the Coulomb correlation function of two pions coming from an anisotropic source of finite life time. The effect of halo has been also taken into account, and it has been shown that due to the Coulomb force the effect of halo extends for the particle’s relative momenta far beyond the inverse halo radius.

Having exact Coulomb correlation functions, the Bowler-Sinyukov procedure to remove Coulomb effect was tested. It was shown that the procedure works rather well when the halo is absent but with the halo the source radii are significantly reduced when compared to the original one. Since \( R_o \) is reduced more that \( R_s \), the extracted life time of the source can be reduced even to zero. It might explain the ‘HBT puzzle’ but a firm conclusion requires further systematic analysis which is still in progress.

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