One-dimensional Anderson Localization: Devil’s Staircase of Statistical Anomalies.

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The statistics of wavefunctions in the one-dimensional (1d) Anderson model of localization is considered. It is shown that at any energy that corresponds to a rational filling factor $f = \frac{p}{q}$ there is a statistical anomaly which is seen in expansion of the generating function (GF) to the order $q - 2$ in the disorder parameter. We study in detail the principle anomaly at $f = \frac{1}{2}$ that appears in the leading order. The transfer-matrix equation of the Fokker-Planck type with a two-dimensional internal space is derived for GF. It is shown that the zero-mode variant of this equation is integrable and a solution for the generating function is found in the thermodynamic limit.

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**Introduction.** Anderson localization (AL) enjoys an unusual fate of being a subject of advanced research during a half of century. The seminal paper by P.W. Anderson opened up a direction of research on the interplay of quantum mechanics and disorder which is of fundamental interest up to now. The one-dimensional tight-binding model with diagonal disorder—the Anderson model (AM)—which is the simplest and the most studied model of this type, became a paradigm of AL:

$$H = \sum_i \varepsilon_i c_i^\dagger c_i - \sum_i t_i \left( c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i \right). \quad (1)$$

In this model the hopping integral is deterministic $t_i = t$ and the on-site energy $\varepsilon_i$ is a random Gaussian variable uncorrelated at different sites and characterized by the variance $\langle (\delta \varepsilon_i)^2 \rangle = w$. The dimensionless parameter $\alpha^2 = w/t^2$ describes the strength of disorder.

The best studied is the continuous limit of this model in which the lattice constant $a \to 0$ at $ta^2$ remaining finite. There was also a great deal of activity aimed at a rigorous mathematical description of 1d AL. However, despite considerable efforts invested, some subtle issues concerning 1d AM still remain unsolved. One of them is the effects of commensurability between the de-Broglie wavelength $\lambda_E$ (which depends on the energy $E$) and the lattice constant $a$. The parameter that controls the commensurability effects is the filling factor $f = \frac{\alpha^2}{\lambda_E^2}$ (fraction of states below the energy $E$).

It was known for quite a while that the Lyapunov exponent (which is essentially the inverse localization length) takes anomalous values at the filling factors equal to $\frac{1}{2}$ and $\frac{3}{2}$ (compared to those at filling factors $f$ beyond the window of the size $\alpha^2 \ll 1$ around $f = \frac{1}{2}$ and $f = \frac{3}{2}$, see Fig.1). Recently it was found that the statistics of conductance in 1d AM is anomalous at the center of the band $E = 0$ in the Lifshitz model described by Eq.(1) with the deterministic $\varepsilon_i = 0$ and a random hopping integral $t_i$. Thus the statistical anomalies in the 1d AM raise a question about hidden symmetries which do not merely reduce to the two-sublattice division. Finally, the sign of the anomaly is different for the center of the band $f = \frac{1}{2}$ and the filling factor $f = \frac{3}{2}$. All these observations point out to a new phenomenon of the devil’s staircase type which is essentially due to disorder and its interplay with the Bragg scattering off the underlying lattice.

In this Letter we study the generating function (GF) $\Phi(u, \phi; x)$ that allows to compute all local statistical properties of 1d AM. The simplest of them is the statistics of the wavefunction amplitude $|\psi(x)|^2$ characterized

![FIG. 1: Schematic representation of statistical anomalies in the localization radius. The dashed line represents the "bare" localization radius $l_0 = a \frac{\alpha^2}{\lambda_E^2} \sin^2(\pi f)$; the circles give the localization length at a filling factor being a simple fraction and solid lines give a sketch of behavior in the perturbed window.](image-url)
by the moments $I_m = \langle |\psi(x)|^{2m} \rangle f_0^m$:

$$I_m = \frac{2}{(m - 2)!} \int_0^\infty \frac{d\phi}{\pi} \cos^{2m}\phi \times \int_0^\infty du u^{m-2} \Phi(u, \phi; x) \Phi(u, -\phi - 2\pi f; L - x),$$

(2)

where $L$ is the total length of the system and $\ell_0 = a^{2f} \sin^2(\pi f)$ is the "bare" localization length.

We derive the corresponding transfer-matrix equation (TME) for the GF

$$\partial_x \Phi(u, \phi; x) = [\hat{L}_f(u, \phi) - u] \Phi(u, \phi; x),$$

(3)

which zero-mode variant ($\Phi(u, \phi; x) \equiv \Phi(u, \phi)$) is independent of the space coordinate $x$ appears to be a partial differential equation (PDE) depending on two variables. One of them (denoted by $u$) is associated with the amplitude of the wavefunction $\psi \sim \sqrt{u} \cos \phi$ while the other (denoted by $\phi$) has a physical meaning of its phase. The well known TME in the continuous limit $f \ll 1$ can be obtained by the averaging of this PDE over the phase variable $\phi$ thus reducing it to an ODE in the single variable $u$.

We show that there are statistical anomalies at any rational filling factor $f = \frac{p}{q}$. Namely, the operator $\hat{L}_f(u, \phi)$ in Eq.(3) expanded in the disorder parameter $\alpha^2$

$$\hat{L}_f(u, \phi) = \hat{L}_f^{(0)}(u, \phi) + \alpha^2 \hat{L}_f^{(1)}(u, \phi) + \alpha^4 \hat{L}_f^{(2)}(u, \phi) + \ldots$$

(4)

is such that

$$\hat{L}_f^{(n)} = \hat{L}_f^{(n, \text{reg})} + \sum_{p=1}^{n+1} \Delta \hat{L}_p^{(n)} \delta \left( f, \frac{p}{n+2} \right)$$

(5)

contains a regular part $\hat{L}_f^{(n, \text{reg})}$ with a smooth dependence on $f$ and an anomalous part that appears only at $f = \frac{p}{n+2}, \frac{2p}{n+2}, \ldots, \frac{(n+1)p}{n+2}$. In the leading order ($n = 0$) in $\alpha^2$ the anomalous term appears only at $f = \frac{1}{2}$. In the next order one can observe anomalies at $f = \frac{1}{3}$ and $\frac{2}{3}$, etc. Though anomalous terms corresponding to the denominator $q > 2$ are small at weak disorder, they have an abrupt dependence on $f$. This allows to speak about the "devil's staircase of anomalies".

We study in detail the principal anomaly at $f = \frac{1}{2}$. Remarkably, the corresponding zero-mode TME appears to be integrable. We find a unique solution to this equation which describes any local statistics of wavefunctions in the center of the band.

- **Derivation of the TM equation.** The starting point of our analysis is the TM equation for the generating function $\Phi_j(u, \phi)$ on the lattice site $j$:

$$\Phi_{j+1}(u, \phi) = \left( 1 + \frac{2a}{\ell_0} [\mathcal{L}(u, \phi) - c_1(\phi) u] \right) \Phi_j(u, \phi - \pi f),$$

(6)

where $\mathcal{L}(u, \phi) = c_2(\phi) u^2 \partial_u^2 + c_3(\phi) (u \partial_u - 1) + c_4(\phi) u \partial_u \partial_\phi + c_5(\phi) \partial_\phi + c_6(\phi) \partial_\phi^2$. The coefficients $c_i(\phi)$ are all combinations of $\cos(2\phi)$ and $\sin(2\phi)$ which at first glance do not show any nice structure: $c_1(\phi) = \frac{1}{2}(1 + \cos(2\phi))$, $c_2(\phi) = 1 - \cos^2(2\phi)$, $c_3(\phi) = -(1 - \cos(2\phi) - 2 \cos^2(2\phi))$, $c_4(\phi) = \sin(2\phi)(1 + \cos(2\phi))$, $c_5(\phi) = -\frac{3}{2} \sin(2\phi)(1 + \cos(2\phi))$, $c_6(\phi) = \frac{1}{4}(1 + \cos(2\phi))^2$.

This equation has been derived in Ref.13 by expansion to the first order in $\alpha^2$ of the exact integral TME equation obtained by the super-symmetry method14. By construction the function $\Phi_j(u, \phi)$ must be periodic in $\phi$ with the period of $\pi$ which corresponds to the phase factor $\cos \phi$ of the wave function sweeping all possible values in the interval $[0, \pi]$. However, the shift in the argument $\phi$ in the r.h.s. of Eq.(3) is by a fraction $f$ of $\pi$. For a rational $f = \frac{p}{q}$ one has to make $q$ iterations in Eq.(3) in order to get a closed equation for the GF. In the leading order in $\alpha$ we obtain:

$$\Phi_{j+q}(u, \phi) - \Phi_j(u, \phi) = \frac{2a}{\ell_0} \times \left[ \sum_{r=0}^{q-1} \mathcal{L}(\phi - r \pi p/q) - u \sum_{r=0}^{q-1} c_1(\phi - r \pi p/q) \right] \Phi_j(u, \phi).$$

(7)

The reason for the anomaly is the following identity that shows a jump at $q = 2$:

$$\sum_{r=0}^{q-1} e^{2i\phi - 2ir \pi p/q} = 0, \quad \sum_{r=0}^{q-1} e^{4i\phi - 4ir \pi p/q} = \begin{cases} 0, & q > 2 \\ 2e^{i\phi}, & q = 2 \end{cases}$$

(8)

One can see from this identity that for $q > 2$ the summation in Eq.(7) is the same as averaging over $\phi$: all the $\phi$-dependent terms vanish in both cases. Assuming $q \ll \ell_0/a$, expanding the l.h.s. of Eq.(7) and introducing the dimensionless coordinate $x = ja/\ell_0$ we obtain:

$$\partial_x \Phi = \hat{L}_f^{(0, \text{reg})} \Phi = \left[ u^2 \partial_u^2 - u + \frac{3}{4} \partial_u^2 \right] \Phi.$$  

(9)

This equation admits the independent of $\phi$ stationary solution:

$$\Phi(u, \phi) = e^{-\epsilon u} \sqrt{u} K_{\sqrt{1-\epsilon^2}}(2\sqrt{u}).$$

(10)

This solution has been earlier obtained15 in the continuous limit $f \ll 1$. It also arises in the theory of a multichannel disordered wire14,15. For a system of the size $L \rightarrow \infty$ only zero mode solution corresponding to $\epsilon = 0$ is relevant. Substituting Eq.(10) with $\epsilon = 0$ into Eq.(2) we found the following distribution function of the eigenfunction amplitude in a long strictly one-dimensional system (amazingly, this result was not known before):

$$\mathcal{P}(|\psi|^2) = \frac{\ell_0}{L} \exp \left( -\frac{|\psi|^2 \ell_0}{|\psi|^2} \right).$$

(11)

This distribution is valid for $|\psi|^2 \ell_0 \gg e^{-L/\ell_0}$ and should be cut off at very small $|\psi|^2$ to ensure normalizability16.
At $q = 2$ (and only at $q = 2$ in the leading order in $a$) the $\phi$-dependence in Eqs. (3), (7) survives and gives rise to the anomalous term: 

$$\Delta L^{(0)} = \cos(4\phi) \left[ -u^2 \rho_u^2 + 2u \rho_u + \frac{1}{4} \rho_u^3 - 2 \right] + \sin(4\phi) \left[ u \rho_u \rho_v - \frac{3}{2} \rho_u^2 \right].$$ (12)

Again, like in Eq. (9), there is apparently no nice structure in Eq. (12). Moreover, because of the anomalous term the entire operator $L^{(0)}_f(u, \phi)$ acquires an explicit $\phi$-dependence and thus the zero-mode TME becomes a two-variable second-order PDE which no longer admits a $\phi$-independent solution. Yet it appears exactly solvable!

**Separation of variables.** The integrability of the zero mode TME for $f = \frac{1}{2}$ is shown in three steps. The first step is to introduce new set of variables $u$ and $v = u \cos(2\phi)$ instead of $(u, \phi)$ and a new function $\Phi(u, v) = u^{-1} \Phi(u, \frac{1}{2} \arccos(v/u))$. In these variables the stationary TME $[\hat{\Delta} L^{(0)} + \Delta L^{(0)}] \Phi = -\epsilon \Phi$ takes a very symmetric form:

$$[D_1^2 + D_3^2] \Phi = \frac{u}{2} \Phi - \epsilon \Phi,$$ (13)

where the operators $D_1$ and $D_3$ belong to the family of three operators from the representation of the $sl_2$ algebra:

$$D_1 = \sqrt{u^2 - v^2} \partial_u, \quad D_2 = u \partial_v + v \partial_u, \quad D_3 = -\sqrt{u^2 - v^2} \partial_v$$

obeying the commutation relations:

$$[D_1, D_2] = -D_3, \quad [D_3, D_1] = D_2, \quad [D_2, D_3] = D_1.$$ (15)

Now it is clear that there is a hidden order in a set of $\phi$-dependent terms in Eq. (12) and the way they match the regular part $\hat{L}^{(0,reg)}_f$ in r.h.s. of Eq. (3).

The next step is to transform Eq. (13) to the Schrödinger-like equation $-\rho_u^2 \rho_u \Psi + U(u, v) \rho_v = 0$ for the function $\Psi(u, v) = (u^2 - v^2) \Phi$, where

$$U = -\frac{3}{4} \frac{u^2 + v^2}{(u^2 - v^2)^2} + \frac{1}{2} \frac{u}{u^2 - v^2} - \frac{\epsilon}{u^2 - v^2}.$$ (16)

Finally we introduce the variables

$$\xi = \frac{u + v}{2} = u \cos^2 \phi, \quad \eta = \frac{u - v}{2} = u \sin^2 \phi.$$ (17)

It is easy to see that the kinetic energy and the first two terms in Eq. (10) become the sum of two identical Hamiltonians $\hat{H}_\xi + \hat{H}_\eta$ where $\hat{H}_\xi$ is given by:

$$\hat{H}_\xi = -\rho_u^2 - \frac{3}{16} \frac{1}{\xi^2} + \frac{1}{4\xi}.$$ (18)

Thus at $\epsilon = 0$ we explicitly separated the variables in Eq. (13) reducing the two-dimensional PDE to two ODE’s of the Schrödinger type $\hat{H}_\xi \phi_\lambda(\xi) = \lambda \phi_\lambda(\xi)$ and $\hat{H}_\eta \phi_\lambda(\eta) = -\lambda \phi_\lambda(\eta)$. Each of these equations reduces to the well known Weber’s differential equation which solution is given in terms of the hypergeometric functions (Whittaker functions).

Albeit the above procedure does not work for $\epsilon \neq 0$ because of the last term in Eq. (10), the integrability of the zero-mode TME is a remarkable fact that allows to describe anomalous statistics in an infinitely long system.

**Uniqueness of the solution.** The general solution to the "Schrödinger equation" with $\epsilon = 0$ in Eq. (10) is given by the integral over the parameter $\lambda$:

$$\Psi = \int d\lambda \, \lambda \, \phi_\lambda(\xi) \phi_\lambda(\eta),$$ (19)

where integration is generically over the complex plane of $\lambda$ and $c(\lambda, \lambda)$ is an arbitrary function. How does this huge degeneracy comply with the intuitive expectation that the statistics of wavefunctions in an infinite disordered chain should be unique and independent of the boundary conditions? Below we show that the natural physical requirements on $\Phi(u, \phi)$ help to determine GF up to a constant factor which can be further fixed using the wave function normalization $|\Psi|^2 = \frac{1}{2}$.

First of all we note that $\hat{H}(\lambda; \xi, \eta) = \phi_\lambda(\xi) \phi_\lambda(\eta)$ is a holomorphic function of $\lambda$, i.e. it depends only on $\lambda = e^{i\pi}$ but not on $\tilde{\lambda} = e^{-i\pi}$. The idea is to represent the integral over the complex plane as an integral over $\rho$ and $\sigma$ and then rotate the contour of integration $\rho \rightarrow te^{-i\sigma}$ so that the dependence on $\sigma$ remains only in $c(\lambda, \tilde{\lambda})$ and in the integration measure but not in $\hat{F}(\lambda; \xi, \eta)$. Then performing integration over $\sigma$ one obtains a new function $\hat{C}(t) = t \int d\sigma \, e^{-2i\sigma} \hat{C}(t, e^{-2i\sigma})$ which stands for $c(\lambda, \tilde{\lambda})$ in an expression similar to Eq. (19) but involving only a one-dimensional contour integral. This contour can be further rotated to make the expression more symmetric. Thus without loss of generality we write a solution to the zero-mode TM equation Eq. (13) for $f = \frac{1}{2}$:

$$\Phi(\xi, \eta) = \left[ \eta \xi \right]^{1/4} \int_0^\infty d\lambda \, C(\lambda) \times W_{-\lambda, \xi} \left( \left[ \eta \xi \right] \frac{\xi}{4\lambda} \right) W_{-\lambda, \xi} \left( \left[ \eta \xi \right] \frac{\epsilon \eta}{4\lambda} + c.c. \right).$$ (20)

Here $W_{\epsilon, \mu}(z)$ is the Whittaker function; $\epsilon = e^{i\pi}/4$, $\tilde{\epsilon} = e^{-i\pi}/4$, and $\hat{C}(\lambda)$ is a real function yet to be determined.

Before we proceed with determining this function it is important to establish its properties as $\lambda \rightarrow 0$. To this end we note that the integral over $u$ in Eq. (2) with $m = 1$ must be divergent at small $u$. Indeed, if it is convergent then the factor $\frac{1}{(m-1)!} = \frac{1}{(m-1)!}$ makes the first moment equal to zero which contradicts the normalizability of wave function $\int dx |\psi(x)|^2 = 1$. At the same time for $m > 1$ the integral should converge to ensure finite higher moments. This implies that $\Phi(u, \phi)$ must tend to a constant as $u \rightarrow 0$. Given the asymptotic behavior of Whittaker functions this is equivalent to:

$$C(\lambda) = \lambda^{-\frac{1}{2}} \hat{C}(\lambda), \quad \hat{C}(0) = \text{const}. \quad (21)$$
GF defined by Eq.(20) is periodic in $\phi$ with the period $\frac{\pi}{2}$ as it should be for $q = 2$. This is guaranteed by the adding of the c.c term in Eq.(20). What is not automatically guaranteed is that $\Phi(\xi, \eta)$ is smooth as a function of $\phi$ at $\phi = 0$. We will see that it is the requirement of smoothness at $\phi = 0$ which fixes (up to a constant factor) the unknown function $\tilde{C}(\lambda)$.

Indeed, the discontinuity of derivatives at $\phi = 0$ may arise from the branching of the expression in Eq.(20) at a small $\eta$. From the representation of the Whittaker function in terms of the hypergeometric functions one concludes that the general solution Eq.(20) is a sum of a part which is regular in the vicinity of $\eta = 0$ and a part which has a square-root singularity $\sqrt{\eta} \approx \sqrt{u[\phi]}$. The condition that this latter part cancels out in the solution Eq.(20) is the following (in the complex plane):

$$\Im \left[ \frac{\tilde{C}(\iota t)}{\Gamma \left( \frac{1}{4} - it \right)} e^{-\iota t} \right]_{1} F_{1} \left( \frac{3}{4} - it, \frac{3}{2} \iota t, \frac{3}{2} \right) = 0. \quad (22)$$

The crucial fact for the possibility to fulfil this condition is the identity for the hypergeometric functions:12

$$e^{-\iota z/2} \left( \frac{3}{4} - it, \frac{3}{2} \iota z \right) = e^{\iota z/2} \left( \frac{3}{4} + it, \frac{3}{2} - \iota z \right). \quad (23)$$

Now one can immediately guess the solution for $\tilde{C}(\lambda)$:

$$C_{0}(\lambda) = \Gamma \left( \frac{1}{4} + \epsilon \lambda \right) \Gamma \left( \frac{1}{4} + \epsilon \lambda \right). \quad (24)$$

It is easy to see that the general solution to Eq.(22) is

$$\tilde{C}(\lambda) = C_{0}(\lambda) S(\lambda) = C_{0}(\lambda) \sum_{k=0}^{\infty} a_{k} \lambda^{4k}, \quad (25)$$

where the function $S(\lambda)$ must be regular in the entire complex plane of $\lambda$. Now we apply the condition of convergence of the integral over $\lambda$ in Eq.(20) at large $\lambda$ to find the allowed asymptotic behavior of $S(\lambda)$ at $\lambda \to \infty$. Substituting Eq.(25) into Eq.(20) and using the asymptotics of the Whittaker and $\Gamma$-functions we find that the integrable behavior of $\lambda^{-3} S(\lambda)$ at $\lambda \to \infty$. This means that $|S(\lambda)|$ should increase not faster than $\lambda^{2}$. There is only one such entire function with the structure of Eq.(25): this is a constant $S(\lambda) = a_{0} = \text{const.}$

Conclusion and discussion. Eqs.(20), (21), (24) is the main result of the paper. They give an exact and unique (up to a constant factor) solution for the generating function at $f = \frac{1}{q}$ anomaly which can be used to compute all local statistics of the one-dimensional Anderson model at $L \to \infty$. The integrability of TME Eq.(3) suggests that there is a hidden symmetry of the problem at $f = \frac{1}{q}$. We make a conjecture that this symmetry is naturally formulated in the three dimensional space rather than in the two-dimensional space ($\xi, \eta$) and that it has to do with the symmetry of the 3d harmonic oscillator. This conjecture is based on an analogy between our main result Eq.(20) and the expression for the Green’s function of the 3d harmonic oscillator problem.21 This analogy concerns the parameter $\lambda$ in our problem and $k$ in Ref.21 entering both in the argument of the Whittaker functions and in its first index in a mutually reciprocal way, as well as the second index of the Whittaker functions being $\frac{1}{q}$ in both cases. Establishing this symmetry would also be useful for studying the anomalies at $f = \frac{1}{q}$ with $q > 2$. We have obtained17 the operator $F_{f}^{(1)}(u, \phi)$ in Eq.(4) and shown that the mechanism similar to Eq.(5) is responsible for the anomaly at $f = \frac{1}{q}$. The results of this study will be published elsewhere.

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19 Note that the “Hamiltonian” in Eq.(13) is a non-Hermitean operator. This is a consequence of the singular inverse-square potential. To make it Hermitean one has to impose
a condition $\varphi(0) = 0$ assuming a hard wall at $\xi < 0$. There is no such condition in our problem.

20 One can show that $\Phi(u, \phi) = h_0(\phi) + \sum_{k=1}^{\infty} u^k [h_k(\phi) + g_k(\phi) \ln u]$.

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