On a connection between a class of $q$-deformed algebras and the Hausdorff derivative in a medium with fractal metric

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Abstract

Over the recent decades, diverse formalisms have emerged that are adopted to approach complex systems. Amongst those, we may quote the $q$-calculus in Tsallis’ version of Non-Extensive Statistics with its undeniable success whenever applied to a wide class of different systems; Kaniadakis’ approach, based on the compatibility between relativity and thermodynamics; Fractional Calculus (FC), that deals with the dynamics of anomalous transport and other natural phenomena, and also some local versions of FC that claim to be able to study fractal and multifractal spaces and to describe dynamics in these spaces by means of fractional differential equations.

The question we might ask is whether or not there are common aspects that connect these alternative approaches. In this short communication, we discuss a possible relationship between $q$–deformed algebras in two different contexts of Statistical Mechanics, namely, the Tsallis’ framework and the Kaniadakis’ scenario, with local form of fractional-derivative operators defined in fractal media, the so-called Hausdorff derivatives, mapped into a continuous medium with a fractal measure. This connection opens up new perspectives for theories that satisfactorily describe the dynamics for the transport in media with fractal metrics, such as porous or granular media. Possible connections with other alternative definitions of FC are also contemplated. Insights on complexity connected to concepts like coarse-grained space-time and physics in general are pointed out.

Keywords: Hausdorff derivative, Fractal, Local fractional calculus, $q$-deformed algebra, $k$-deformed algebra

1. Introduction

The calculus with fractional derivatives and integrals of non-integer orders started more than three centuries ago, when Leibniz proposed a derivative of order $\frac{1}{2}$ in response to a letter from l’Hôpital \cite{1}. This subject was also considered by several mathematicians like Euler, Fourier, Liouville, Grünwald, Letnikov, Riemann and others up to nowadays. Although

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the fractional calculus is almost as old as the usual integer order calculus, only over the last three decades it has raised more attention due to its applications to various fields of science (see \cite{2, 3, 4, 5, 6, 7} for a review). As an example, Fractional Calculus (FC) is historically applied to study non-local or time-dependent processes, as well as to model phenomena involving coarse-grained and fractal spaces \cite{8, 9, 10, 11, 12, 13, 14, 15, 16} and frictional systems.

Presently, areas such as field theory and gravitational models demand new conceptions and approaches which might allow us to understand new systems and could help in extending well-known results. Interesting problems may be related to the quantization of field theories for which new approaches have been proposed \cite{17, 18, 19, 20}.

Fractional systems are described as being dissipative \cite{21, 22}. The use of FC is also justified here based on our proposition that there exists an intimate relationship between dissipation, coarse-grained media and the some limit scale of energy for the interactions. Since we are dealing with open systems, as already commented, the particles are in fact dressed particles or pseudo-particles that exchange energy with other particles and the environment. Depending on the energy scale an interaction may change the geometry of space-time, disturbing it at the level of its topology. A system composed by particles and the surrounding environment may be considered nonconservative due to the possible energy exchange. This energy exchange may be the responsible for the resulting non-integer dimension of space-time, giving rise then to a coarse-grained medium. This is quite reasonable since, even standard field theory, comes across a granularity in the limit of Planck scale. So, some effective limit may also exist in such a way that it should be necessary to consider a coarse-grained space-time for the description of the dynamics for the system, in this scale. Also, another perspective that may be proposed is the previous existence of an nonstandard geometry, e.g., near a cosmological black hole or even in the space nearby a pair creation, that imposes a coarse-grained view to the dynamics of the open system. Here, we argue that FC allows us to describe and emulate this kind of dynamics without explicit many-body, dissipation or geometrical terms in the dynamical governing equations. In some way, FC may yield an effective theory, with some statistical average without imposing any specific nonstandard statistics. So, FC may be the tool that could describe, in a softer way, connections between coarse-grained medium and dissipation at a certain energy scale.

It seems that a reasonable way to probe the classical framework of physics is to highlight that, in the space of our real world, a generic point is not infinitely small (or thin), it rather has a thickness. In a coarse-grained space, a point is not infinitely thin, and here, this feature is modeled by means of a space in which the generic differential is not $dx$, but rather $(dx)\alpha$, and likewise for the time variable $t$. It is noteworthy to emphasize that the notion of fractal space-time was introduced in the 70s with the seminal work by Stillinger \cite{26}, where the axiomatic basis for spaces with non-integer dimension was set up. Later, the concepts associated to a possible non-integer dimension were reinforced with the work by Zeilinger and Svozil \cite{27}, were they take into account the intrinsically unavoidable finite resolution of any physical experiment, and also the works in ref. \cite{28, 29, 30} should be quoted here. Along this line, we also point out the work by Nottale \cite{31}, where the notion of fractal space-time is revisited. Non-integer differentiability and randomness \cite{32} are mutually related in their nature, in such a way that studies on fractals on the one hand, and fractional Brownian motion
on the other hand, are often parallel as in the work of ref. [31]. A function continuous everywhere, but nowhere integer-differentiable, necessarily exhibits random-like or pseudo-random features, in that various samplings of these functions, in the same given interval, will be different. This may explain the huge amount of literature extending the theory of stochastic differential equations to describe stochastic dynamics driven by fractional Brownian motion [25, 33, 34]. Regarding the anomalous properties of space-time with multifractal structure, we highlight the interesting work in ref. [35] and references therein. Also, we call attention to the efforts to build up a solid foundation for the construction of a geometry and field theory in fractional spaces [22], multifractals [36] and multi-scale [37] space-times.

The majority of actual classical systems is nonconservative but, in spite of that, the most advanced formalisms of classical mechanics deal only with conservative systems [39]. Dissipation [38], for example, is present even at the microscopic level. There is dissipation in every non-equilibrium or fluctuating process, including dissipative tunneling [40] and electromagnetic cavity radiation [41], for instance. In ref. [38], we adopt that one way to suitably treat nonconservative systems is through FC, since it can be shown that, for example, a friction force has its form stemming from a Lagrangian that contains a term proportional to the fractional derivative, which may be a derivative of any non-integer order [39].

Parallel to the standard FC, there are the local fractional calculus with certain definitions called local fractional derivatives, as the ones introduced by Kolwankar and Gandal [56, 57, 58] with several works related to this approach; for example, the works of refs. [59, 60, 61], the related approaches with Hausdorff derivative and also the formulation with the so-called fractal derivative [62, 63]. All of the mentioned approaches seem to be applicable to power-law phenomena. There is also the recently developed $\alpha$–derivative formalism by Kobelev [64].

Also parallel to the attempt for the description of complex systems by FC, the $q$–calculus, in a non-extensive statistic context, has its formal development based on the definition of deformed expressions for the logarithm and exponential [78], namely, the $q$–logarithm and the $q$–exponential. In this context, an interesting algebra emerges and the formalism of a deformed derivative opened new possibilities for, besides the thermodynamical, other treatment of complex systems, specially those with fractal or multifractal metrics and presenting long-range dynamical interactions. The deformation parameter or entropic index, $q$, occupying an important place in the description of those complex systems, describes deviations from standard Lie symmetries and the formalism aimed to accommodate scale invariance in a system with multifractal properties to the thermodynamic formalism. For $q \to 1$, the formalism reverts to the standard one.

Here, we show that a definition of local fractional derivative by means of mathematical limit operation is comparable to the definition of the $q$–derivative approach and have similar rules.

Our paper is outlined as follows: In Section 2, we present the mathematical aspects, In Section 3, we develop the main subject of this communication and we cast our Conclusions and Outlook in Section 4.
2. Mathematical Aspects

Following the works of [76, 77], once the solution to the differential equation \( \frac{dy}{dx} = y \), is the exponential function \( e^x \), the author starts off from the following the equation,

\[
\frac{dy}{dx} = y^q,
\]

whose solution leads to the q-exponential, \( y = e^{x_q} \).

The \( q \)-derivative sets up a deformed algebra and takes into account that the \( q \)-exponential is eigenfunction of \( D(q) \) [78]. Borges proposed the operator for \( q \)-derivative as below:

\[
D(q)f(x) \equiv \lim_{y \to x} \frac{f(x) - f(y)}{x \ominus_q y} = [1 + (1 - q)x] \frac{df(x)}{dx}.
\]

(2)

Here, \( \ominus_q \) is the deformed difference operator, \( x \ominus_q y \equiv \frac{x - y}{1 - qy} \) (\( y \neq 1/(q - 1) \)).

The differential equation, with the Hausdorff derivative proposed in ref. [79], is:

\[
\frac{d^H y}{dx^α} = y,
\]

(3)

and leads to the stretched exponential solution \( y = e^{x^α} \).

The fractional differential equation

\[
\frac{d^α y}{dx^α} = y,
\]

(4)

with the Caputo fractional derivative or the Modified Riemann-Liouville approach, yields the solution in terms of the Mittag-Leffler function \( y = E_α(x^α) \).

Now, that we have worked out these fundamental expressions, we are ready to carry out the calculations of main interest.

3. Fractal Continuum and the Hausdorff derivative connections

A model that maps hydrodynamics continuum flow in a fractal coarse-grained (fractal porous) space, which is essentially discontinuous in the embedding Euclidean space, into a continuous flow governed by conventional partial differential equations was suggested in ref. [80]. Using non-conventional partial differential equations based on the model of a fractal continuous flow employing local fractional differential operators, Balankin [73] has suggested essentially that the discontinuous fractal flow in a fractally permeable medium can be mapped into the fractal continuous flow, which is describable within a continuum framework, indicating also that the geometric framework of fractal continuum model is the three-dimensional Euclidean space with a fractal metric. For more details, the reader may consult refs. [73, 74].
Employing the local fractional differential operators in connection with the Hausdorff derivative \[79\], the latter derivative can be written as \[74\]:

\[
\frac{d^H}{dx^\zeta} f(x) = \lim_{x \to 0} \frac{f(x') - f(x)}{x' - x} = \left( \frac{x}{l_0} + 1 \right)^{1-\zeta} \frac{d}{dx} f \equiv \frac{l_0^{\zeta-1}}{c_1} \frac{d}{dx} f = \frac{d}{dx} f,
\]

(5)

where \( l_0 \) is the lower cutoff along the Cartesian \( x \)-axis and the scaling exponent, \( \zeta \), characterizes the density of states along the direction of the normal to the intersection of the fractal continuum with the plane, as defined in the work \[74\].

Notice that the differential equation

\[
\left( \frac{x}{l_0} + 1 \right)^{1-\zeta} \frac{d}{dx} y = y
\]

leads to the stretched exponential solution

\[
y = e^{\frac{l_0}{\zeta} \left( \frac{x}{l_0} + 1 \right)^{\zeta}}.
\]

Now, to achieve our main purposes, using the well-known gamma function, we can remember the binomial expansion:

\[
(1 + x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha + 1)}{\Gamma(k + 1)\Gamma(\alpha - k + 1)} x^k = 1 + \alpha x + \frac{\alpha(\alpha - 1)}{2!} x^2 + \cdots,
\]

(7)

where \( \alpha \) can be a noninteger rational number.

Developing eq. (5) by the binomial series (7), with fractional exponent \( (1 - \zeta) \) to the second order, yields what follows below:

\[
\frac{d^H}{dx^\zeta} f(x) = \left( 1 + \frac{x}{l_0} \right)^{-\zeta} \frac{d}{dx} f \equiv \left[ 1 + \frac{(1 - \zeta)}{l_0} x + \frac{(1 - \zeta)(-\zeta)}{2 l_0^2} x^2 + \cdots \right] \frac{d}{dx} f.
\]

(8)

Comparing the first two terms of eq. (8) with eq. (2), we can see (the nontrivial solution for \( \zeta \neq 0 \)) that the entropic index, \( q \), and the scaling exponent, \( \zeta \), are related as follows:

\[
1 + \frac{(1 - \zeta)}{l_0} x \frac{d}{dx} = 1 + (1 - q)x,
\]

(9)

so,

\[
1 - q = \frac{(1 - \zeta)}{l_0}.
\]

(10)
We can see that if \( q \to 1 \Rightarrow \zeta \to 1 \); if \( q \to 0 \), than \( \zeta \to 1 - l_0 \) and if \( l_0 \to \infty \Rightarrow q \to 1 \). So, we conclude that the deformed \( q \)-derivative is the first order expansion of the Hausdorff derivative and that there is a strong connection between these formalism by means of a fractal metric.

Now, let us examine the Kaniadakis definition of \( k \)-deformed derivative \[81\]:

\[
\frac{d f(x)}{dx_{(\kappa)}} = \sqrt{1 + \kappa^2 x^2} \frac{d f(x)}{dx}.
\]

\[
(1 + \kappa^2 x^2)^{1/2} \cong 1 + \frac{1}{2!} \kappa^2 x^2 - \frac{1}{3!} \kappa^4 x^4 + \cdots.
\]

We observe that only even powers appear.

Kaniadakis \( k \)-deformed algebra is in a sense more complete than the \( q \)-deformed algebra in the Tsallis context of non-extensive statistics, by the fact that the former leads to a relativistic theory and is based upon physical foundations of the principle of kinetic interactions. In spite of that, the \( k \)-deformed derivative connection with Hausdorff derivative is not so clear by a simple comparison, but we may conjecture that there may exist some relationship between the \( k \)- and the \( \zeta \)- parameters. The appearance of only even powers in the Kaniadakis \( k \)-deformed derivative expansion may indicate, whenever compared to the Hausdorff expansion, that the latter is still more complete than the former since it contains all orders, even and odd in the expansion. Some kind of \( k \)-complex number may complete the connection with Hausdorff derivative.

Let us now examine other versions of local fractional derivative (LFD): The controversial version of LFD in ref. \[82\], that has many similarities with the older one Jumarie’s definition \[25\], is

\[
f^{(\alpha)}(x) = \frac{d^\alpha f(x)}{dx^\alpha} = \lim_{x \to x_0} \frac{\Delta^\alpha (f(x) - f(x_0))}{(x - x_0)^\alpha},
\]

where there was taken the approximation \( \Delta^\alpha (f(x) - f(x_0)) \cong \Gamma(\alpha + 1) \Delta (f(x) - f(x_0)) \).

Also, with the Jumarie’s version of the fractional Fourier transform as \( f(x + h) = \sum_{k=0}^{\infty} \frac{h^{\alpha k}}{\Gamma(\alpha + 1)} f^{(\alpha k)}(x) \), we have for \( f = x^\alpha \),

\[
(x + h)^\alpha = f^{(0)}(x) + \frac{h^\alpha}{\Gamma(\alpha + 1)} f^{(\alpha)}(x) + ... \cong x^\alpha + \frac{h^\alpha}{\Gamma(\alpha + 1)} \Gamma(\alpha + 1),
\]

where we have used that the fractional derivative of a function \( x^\gamma \) is \( D^\alpha x^\gamma = \frac{\Gamma(\alpha + 1) x^{\gamma - \alpha}}{\Gamma(\gamma - \alpha + 1)} \).

Now, with \( (x + h)^\alpha = x'^\alpha \), we can write \( x'^\alpha - x^\alpha \cong (x' - x)^\alpha \).

With these approximations, the above definition of LFD \[12\] seems to be nothing but the Balankian version of Hausdorff derivative with \( \left( \frac{x}{x_0} + 1 \right)^{1-\zeta} \cong \Gamma(\alpha + 1) \left( \frac{x}{x_0} + 1 \right)^{1-\alpha}. \) So, we can conclude that this version is identical to the Balankin
version of Hausdorff derivative with a dilatation constant depending on the $\alpha$–fractional parameter, and represents a mapping, similarly to the Balankin’s version.

Jumarie’s definition for an alternative fractional derivative is

$$f^{(\alpha)}(x) = \frac{d^\alpha f(x)}{dx^\alpha} \bigg|_{x_0} = \lim_{x \to x_0} \frac{\Delta^\alpha(f(x) - f(0))}{h^\alpha} = \lim_{x \to 0} h^{-\alpha} \sum_{k=0}^{\infty} \binom{\alpha}{k} \left[ f(x + (\alpha - k)h) - f((\alpha - k)h) \right].$$

(14)

(15)

Without any prior approximation, it seems to borrow similarities from the Gründwald-Letnikov definition and may be thought of as a weighted sum in a linear chain of points, from $x_0$ to $x$, with different weights $\binom{\alpha}{k} h^{-\alpha}$, for each point indexed by $k$. An attempt to interpret this is to consider again a mapping into a continuous medium with a fractal measure, so that the final result for the derivative is a sum of local Hausdorff derivatives, one for each point in a linear chain. So, the non-local character is maintained by this weighted sum. Each local derivative in the weighted sum is mapped into the continuous. Since in this deformed (mapped) space the modified Leibniz product and the chain rules hold for each local Hausdorff derivative, Jumarie’s approach seems to also satisfy those rules or, at least locally, it remains valid.

Recently, a promising new definition of LFD, called conformable fractional derivative, has been proposed by the authors in ref. [86] that preserves classical properties and given by

$$T_\alpha f(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon}.$$  

(16)

If the function is differentiable in a classical sense, the definition above yields

$$T_\alpha f(t) = t^{1-\alpha} \frac{df(t)}{dt}.$$  

(17)

Changing the variable $t \to 1 + \frac{x}{t_0}$, we should write (17) as $l_0 \left(1 + \frac{x}{t_0}\right)^{1-\alpha} \frac{d}{dx} f$, that is nothing but the Hausdorff derivative up to a constant and valid for differentiable functions.

4. Conclusions and Outlook

In this short communication, we have shown that the Hausdorff derivative and the $q$–derivative, in the context of non-extensive statistics share possible connections. They open up possibilities for a better understanding of both formalisms, specially within complex systems dynamics and with fractals and multifractals media.

The physical basis involved in this the connection is the mapping into the fractal continuum. Physically, this connection also justifies the construction of some non-linear theories, where long-range interactions between particles are present, giving a basis in the realm of complex systems and fractals and so allowing to re-visit well-established theories, but now with the possibility for the substitution of the standard derivatives by the non-linear one. This allows to introduce modifications into the equations that describe the dynamics of such systems and gives rise to potential applications, for
example, the non-linear classical and quantum field theories, non-linear electrodynamics and so on. Also, the entropic index that appears in the context of non-extensive statistical theories may be connected with the fractal dimension of the medium. The importance of this connection is thus evident. I also indicates that local fractional derivatives should be really relevant for studying complex systems, interactions into the Cantor sets, porous media and so on. Going further, we could hypothesize that the definition of the derivative, in the appropriate context, plays an important role in the description of complex systems, and thus, alternative definitions of local and non-local derivative operators are relevant for the understanding of such systems. The suitable definition of derivatives, for example, by the substitution of the simple translations in the independent variable by a dilatation or power scale factor instead, may yield more effective operators to the study of the dynamical systems with long-range interactions. Examples of such derivatives, considering different contexts, are the Gateaux derivative, the Fréchet derivative, the Jackson derivative, local and non-local fractional derivatives, the Borges-Derivative operator and so on.

We can also conjecture that these connections indicate that there may exist some general definitions of entropy that include the Boltzmann-Gibbs and the non-extensive statistics in a superstatistics.

An attempt to connect with Kaniadakis formalism has also been contemplated.

Finally, it seems to us that the Hausdorff fractal derivatives and all the alternative versions of fractional calculus should be more deeply investigated, under both the mathematical and physical points of view. A better understanding of the exact differences and similarities with respect to the traditional fractional calculus based on the Riemann-Liouville or the Caputo definitions, and those with local fractional calculus or even with fractional $q-$calculus, might be more thoroughly investigated to also determine the scope of the applicability of them.

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