Enlarged Galilean symmetry of anyons and the Hall effect

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Abstract

Enlarged planar Galilean symmetry, built of both space-time and field variables and also incorporating the “exotic” central extension is introduced. It is used to describe non-relativistic anyons coupled to an electromagnetic field. Our theory exhibits an anomalous velocity relation of the type used to explain the Anomalous Hall Effect. The Hall motions, characterized by a Casimir of the enlarged algebra, become mandatory for some critical value(s) of the magnetic field. The extension of our scheme yields the semiclassical effective model of the Bloch electron.

1 Introduction

The planar Galilei group admits a two-fold “exotic” central extension, labeled with \( m \) (the mass) and a second, “exotic” parameter \( \kappa \) \[1\]. Models which provide a physical realization of this “exotic” symmetry have been presented in \[2, 3\]. Below, we focus our attention on the theory of \[3\], since that of \[2\] is in fact an extended version of the latter. Minimal (symplectic) coupling of the particle to an external electromagnetic field yields the first-order phase space Lagrangian

\[
L = P_i \cdot \dot{X}_i - \frac{\vec{P}^2}{2m} + e(A_i \dot{X}_i + A_0) + \frac{\theta}{2} \varepsilon_{ij} P_i \dot{P}_j,
\]

where \( \theta = -\kappa/m^2 \) is the non-commutative parameter. The Euler-Lagrange equations are

\[
m\dot{X}_i = P_i - em\theta \varepsilon_{ij}E_j,
\]

\[
\dot{P}_i = eB\varepsilon_{ij} \dot{X}_j + eE_i,
\]

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where $E_i$ and $B$ are the electric and magnetic field, respectively, and $m^* = m(1 - e\theta B)$ is an effective mass. These equations can also be obtained in a Hamiltonian framework, using the usual Hamiltonian $H = \vec{P}^2/2m - eA_0$ and the modified Poisson-brackets

$$\{X_i, X_j\} = \frac{m}{m^*} \delta_{ij}, \quad \{X_i, P_j\} = \frac{m}{m^*} \delta_{ij}, \quad \{P_i, P_j\} = \frac{m}{m^*} eB \delta_{ij}. \quad (1.4)$$

The most dramatic prediction of the model is that when $m^* = 0$, i.e., when the magnetic field takes the critical value

$$B = B'_{\text{crit}} = \frac{1}{e\theta}, \quad (1.5)$$

the system becomes singular, and the only allowed motions follow the Hall law. The Poisson-brackets (1.4) are changed for new ones in a reduced phase-space (see [3]). Requiring $B = B'_{\text{crit}}$ amounts to a restriction to the lowest Landau level, and quantization allowed us to recover the “Laughlin” wave functions [3].

In this Letter we generalize this model, and indicate its relation to models used in solid state physics [4, 5].

### 2 Enlarged Galilean symmetry

Adapting the idea of [6] to planar physics, we consider a homogeneous electric field $E_i(t)$ and a constant magnetic field $B$, and view $E_i$ and its canonical conjugate momentum, $\pi_i$, as additional variables on an enlarged phase space. This latter is endowed with the enlarged Lagrangian $L^\text{enl} = L + \pi_i \dot{E}_i$. The $\pi_i$ are Lagrange multipliers and one of the equations of motion is $\dot{E}_i = 0$, i.e., the electric field should actually be constant.

The Galilean symmetry of combined particle+homogeneous field system is readily established: the enlarged Lagrangian is (quasi-)invariant w. r. t. enlarged space translations, rotations and boosts, implemented as

$$\begin{align*}
\delta X_i &= a_i, & \delta P_i &= 0, & \delta E_i &= 0, & \delta \pi_i &= e a_i t, \\
\delta X_i &= -\epsilon \epsilon_{ij} X_j, & \delta P_i &= -\epsilon \epsilon_{ij} P_j, & \delta E_i &= -\epsilon \epsilon_{ij} E_j, & \delta \pi_i &= -\epsilon \epsilon_{ij} \pi_j, \\
\delta X_i &= b_i t, & \delta P_i &= m b_i, & \delta E_i &= -B \epsilon_{ij} b_j, & \delta \pi_i &= e b_i \frac{2}{e\theta}. \quad (2.1)
\end{align*}$$

Now we consider the enlarged Hamiltonian structure. Then the equations of motion (1.2)-(1.3), augmented with $\pi_i = e X_i$ are Hamiltonian, $\dot{Y} = \{Y, H\}$, with the usual Hamiltonian $H = \vec{P}^2/2m - eE_i X_i$, and the fundamental Poisson brackets (1.4), supplemented with $\{E_i, \pi_j\} = \delta_{ij}$. Then conserved quantities are readily constructed. Integration of the equation of motion (1.3) shows that

$$P_i = P_i - e B \epsilon_{ij} X_j - e E_i t \quad (2.2)$$

is a constant of the motion. Using the commutation relations

$$\{X_i, P_j\} = \delta_{ij}, \quad \{P_i, P_j\} = 0, \quad \{P_i, E_j\} = 0 \quad \{\pi_i, P_j\} = e t \delta_{ij}, \quad (2.3)$$

we find furthermore that (2.2) generates enlarged translations, (2.1). Similarly,

$$J = \epsilon_{ij} X_i P_j + \frac{\theta}{2} \vec{P}^2 + \frac{eB}{2} \vec{X}^2 + \epsilon_{ij} E_i \pi_j + s_0 \quad (2.4)$$

$$K_i = mX_i - \left( P_i + e \frac{E_i t}{2} \right) t + m\theta \epsilon_{ij} P_j - B \epsilon_{ij} \pi_j \quad (2.5)$$
are conserved and generate enlarged rotations and boosts, respectively. Note that anyonic spin, represented by the real number $s_0$, has also been included. The generators satisfy the enlarged Poisson relations

\[
\{\mathcal{P}_i, H\} = e E_i, \quad \{\mathcal{K}_i, H\} = \mathcal{P}_i, \quad \{\mathcal{J}, H\} = 0,
\]

\[
\{\mathcal{P}_i, \mathcal{J}\} = -\epsilon_{ij} \mathcal{P}_j, \quad \{\mathcal{K}_i, \mathcal{J}\} = -\epsilon_{ij} \mathcal{K}_j, \quad \{\mathcal{P}_i, \mathcal{K}_j\} = -m \delta_{ij},
\]

\[
(2.6)
\]

A closed algebra is obtained, therefore, if the electromagnetic fields $E_i$ and $B$ are considered as additional elements of an enlarged Galilei algebra $\tilde{g}$, cf. [6]. The (constant) magnetic field, $B$, belongs, together with $m$ and $\kappa = -\theta m^2$, to the center of $\tilde{g}$. The additional nonzero brackets are

\[
\{E_i, \mathcal{J}\} = -\epsilon_{ij} E_j, \quad \{E_i, \mathcal{K}_j\} = B \epsilon_{ij},
\]

\[
(2.7)
\]

Our enlarged Galilei group has two independent Casimirs, namely

\[
\mathcal{C} = e\theta \left( BH - \epsilon_{ij} \mathcal{P}_i E_j + \frac{m}{2B} \vec{E}^2 \right),
\]

\[
\mathcal{C}' = \frac{\vec{P}^2}{2m} - H - \frac{e}{m} \left( \mathcal{K}_i E_i + \mathcal{J} B \right) - \frac{m e \theta}{2B} \vec{E}^2,
\]

\[
(2.8)
\]

\[
(2.9)
\]

In the representation of the enlarged Galilei algebra given in terms of phase-space variables,

\[
\mathcal{C} = \frac{e\theta B}{2m} \left( \mathcal{P}_i - \frac{m}{B} \epsilon_{ij} E_j \right)^2 \quad \text{and} \quad \mathcal{C}' = -\mathcal{C} - \frac{e s_0 B}{m}.
\]

\[
(2.10)
\]

$\mathcal{C}'$ generalizes one of the two Casimirs of the planar Galilei group, namely the internal energy [7]. $\mathcal{C} + \mathcal{C}'$ is in turn proportional to the second Casimir identified as the spin; these two quantities are linked, just like for the model of [2]. The relation of the enlarged and ordinary Galilean algebras can be clarified by a subtle group contraction. Though constants of the motion, our Casimirs are not fixed constant: the representation of our enlarged Galilei group is, in general reducible. If the fields become non-dynamical, those transformations which are consistent with the constant fields remain symmetries. For example, (2.2) becomes the familiar magnetic translation.

3 Algebraic construction of the coupled anyon plus electromagnetic field system and Hall effects

Having established our enlarged planar Galilei algebra, now we build a new theory of anyons interacting with (constant) external fields. Generalizing a formula of Bacry [8] who argued that the Hamiltonian should be constructed from generators of the symmetry group, we consider

\[
H' = \frac{\vec{P}^2}{2m} - \frac{e}{m} \left( \mathcal{K}_i E_i + \mathcal{J} B \right) - \frac{m e \theta}{2B} \vec{E}^2,
\]

\[
(3.1)
\]

where that last term is dictated by boost invariance. $H'$ is indeed $H' = H + \mathcal{C}'$. Choosing a real parameter $g$, $H'$ can be further generalized as

\[
H'_{\text{anom}} = H + \frac{g}{2} \mathcal{C}' = \frac{\vec{P}^2}{2m} \left( 1 - \frac{g}{2} e \theta B \right) - e \vec{E} \cdot \vec{X} - \mu B + \frac{g e \theta}{2} \vec{P} \times \vec{E} - \frac{m g e \theta}{4B} \vec{E}^2.
\]

\[
(3.2)
\]
where $\mu = ge s_0 / 2m$. The kinetic term gets hence a field-dependent factor; our Hamiltonian contains, together with the usual magnetic moment term $\mu B$ [which is here a constant], also an anomalous term proportional to $\theta \vec{P} \times \vec{E}$.

Such a theory is still symmetric w.r.t. the enlarged Galilei group by construction. The equation of motion, reminiscent to Eq. (5.3) of \[9\], is

$$m^* \dot{X}_i = (1 - \frac{g}{2} e \theta B) P_i - \left(1 - \frac{g}{2} \right) e m \theta \epsilon_{ij} E_j,$$  \hspace{1cm} (3.3)

supplemented with the Lorentz force law, \[1.3\].

When $g = 2$ and $e \theta B \neq 1$, $m \dot{X}_i = P_i$, so that our equations describe an ordinary charged particle in an electromagnetic field. For $g = 2$ and $e \theta B = 1$, Eq. \[3.3\] is identically satisfied.

We assume henceforth that $g \neq 2$. Then Eq. \[3.3\] describes an “exotic” particle with anomalous moment coupling with gyromagnetic ratio $g$, cf. \[9\], which generalizes the $g = 0$ theory of Ref. \[3\].

Let us now consider Hall motions, i. e. such that

$$\dot{X}_i = \epsilon_{ij} \frac{E_j}{B},$$  \hspace{1cm} (3.4)

For

$$B \neq B''_{\text{crit}} = \frac{2}{g} \frac{1}{e \theta},$$  \hspace{1cm} (3.5)

this is a solution of the equations of motion \[3.3\] \[1.3\] when the momentum satisfies

$$P_i = m e \epsilon_{ij} \frac{E_j}{B},$$  \hspace{1cm} (3.6)

The constraint \[3.6\] is clearly equivalent to the vanishing of the Casimir, $C = 0$. Conversely, from \[3.6\] we infer that $\dot{P}_i = 0$, so that the Lorentz force on the r.h.s. of \[1.3\] is necessarily zero and the motion follows the Hall law. Thus, when $B \neq B''_{\text{crit}}$ the Hall motions are characterized by the constraint \[3.6\], in turn equivalent to $C = 0$. The condition \[3.6\] is invariant w. r. t. the enlarged Galilei transformations and, when restricted to such motions, the representation of the enlarged Galilei algebra becomes irreducible.

The generic motions have the familiar cycloidal form, made of the Hall drift of the guiding center, composed with uniform rotations with frequency

$$\Omega = \frac{eB}{2m^*} (1 - \frac{g}{2} e \theta B).$$  \hspace{1cm} (3.7)

For $g = 2$ this reduces to the usual Larmor frequency $eB/m$. The Casimir $C$ measures the extent the actual motion fails to be a Hall motion. For $m^* = 0$, i. e. for $B = B'_{\text{crit}} = (e \theta)^{-1}$, \[3.3\] implies \[3.6\], and hence the only allowed motions are the Hall motions \[3\] \[9\].

Interestingly, this is also what happens for $B = B''_{\text{crit}}$ (which plainly requires $g \neq 0$). Then the momentum drops out from \[3.3\]. For $g = 2$, \[3.3\] holds identically, but for $g \neq 2$ it becomes

$$\dot{X}_i = \frac{g}{2} e \theta \epsilon_{ij} E_j,$$  \hspace{1cm} (3.8)

which is once again the Hall law \[3.1\] with $B = B''_{\text{crit}}$ \[9\]. Now the constraint \[3.6\] is not enforced: by \[1.3\], the momentum is an arbitrary constant.

The two critical values correspond to the frequencies $\Omega = \infty$ and $\Omega = 0$, respectively. In the first case $m^* = 0$, only those initial conditions are consistent which satisfy \[3.6\]. The system is singular and requires reduction \[3\]. In the second case the initial momentum can be arbitrary, since it has no influence on the motion. The system acquires an extra translational symmetry in momentum space.
4 Planar Bloch electron in external fields

While our theory may seem to be rather speculative, it has interesting analogies in solid state physics, namely in the theory of a Bloch electron in a crystal. Restricting ourselves to a single band, the band energy and the background fields provide in fact effective terms for the semiclassical dynamics of the electronic wave packet [5]. The mean Bloch wave vector (quasi-momentum) we denote here by \( \vec{P} \) varies in a Brillouin zone. In terms of \( \vec{P} \) and the mean band position coordinates, \( X_i \), the system is described by the effective Lagrangian [5]

\[
L^{\text{Bloch}} = P_i \dot{X}_i - \mathcal{E} + e(A_i(\vec{X}, t) \dot{X}_i + A_0) + A_i(\vec{P}) \dot{P}_i \tag{4.1}
\]

with \( A_i = -\frac{1}{2} \epsilon_{ij} B X_j \), \( A_0 = \vec{E} \cdot \vec{X} \). The expression \( \mathcal{E}(\vec{P}) = \mathcal{E}_0(\vec{P}) - M(\vec{P}) B \), [where \( \mathcal{E}_0(\vec{P}) \) is the energy of the band and \( M(\vec{P}) \) is the mean magnetic moment] yields a kinetic energy term for the effective dynamics.

The (effective) vector potential \( A_i \) is the Berry connection; it can arise, e.g., in a crystal with no spatial inversion symmetry as in GaAs [5]. The last term in (4.1) is indeed analogous to our “exotic” term \( (\theta/2) \epsilon_{ij} P_i \dot{P}_j \) in (1.1), to which it would reduce if the Berry curvature,

\[
\theta(\vec{P}) = \epsilon_{ij} \frac{\partial}{\partial P_i} A_j(\vec{P}), \tag{4.2}
\]

was a constant. \( \dot{X}_i \), physically the group velocity of the Bloch electron, satisfies hence the equation

\[
(1 - e B \theta(\vec{P})) \dot{X}_i = \partial_P \mathcal{E} - e B \theta(\vec{P}) \epsilon_{ij} E_j, \tag{4.3}
\]

supplemented with the Lorentz equation (1.4).

Eq. (4.3) has the same structure as our original \((g = 0)\) velocity relation (1.2). The Berry curvature provides us with a momentum dependent effective non-commutative parameter \( \theta(\vec{P}) \), which yields in turn \( \vec{P} \)-dependent effective mass \( m^* = m(1 - B \theta(P) B) \) and anomalous velocity terms, cf. [5, 10].

In a Hamiltonian framework, the system is described by the Bloch Hamiltonian \( H^{\text{Bloch}} = \mathcal{E}(\vec{P}) - e E_i X_i \), and by formally the same Poisson brackets (1.4), except for the momentum dependence of \( \theta \). In particular, the mean band position coordinates do not commute, as it had been observed a long time ago [10]. The \( \vec{P} \) dependence is consistent with the Jacobi identity, (11), even for a position-dependent \( B \) [3].

Once again, we can consider the enlarged framework of Section 2. This yields the previous equations of motion and commutation relations, supplemented with \( \dot{E}_i = 0 \) and \( \{ E_i, \pi_i \} = \delta_{ij} \), respectively. The \( \vec{P} \)-dependence of NC parameter breaks the Galilean symmetry down to the magnetic translations (times time translation) alone. Assuming \( \theta \) and \( \mathcal{E} \) only depend on \( \vec{P}^2 \), rotational symmetry is restored, though. It is generated by \( \mathcal{J} \) in (2.4), with \( \frac{1}{2} \int \vec{P}^2 \theta(\vec{P}^2) \) replacing the “exotic” contribution \( \theta \vec{P}^2 / 2 \). Hence, we are left with a residual symmetry with generators \( \mathcal{P}_i, H, \mathcal{J}, E_i \) and \( B \); the latter belongs to the center. The enlarged euclidean algebra has the Casimirs \( C_0 = B H^{\text{Bloch}} - \epsilon_{ij} \mathcal{P}_i E_j \) and the infinite tower \( C_n = (\vec{E}^2)^n, n = 1, \ldots \).

Let us now inquire about the Hall motions. Inserting the Hall law into the equation of motion (1.3) generalizes the condition (1.6) as

\[
\partial_P \mathcal{E} = \frac{\epsilon_{ij} E_j}{B}. \tag{4.4}
\]
Then (4.4) satisfies $C = 0$ for the Casimir
\begin{equation}
C = B \left( \mathcal{E}(\vec{P}^2) - \mathcal{E}(P_0^2) \right) - \epsilon_{ij} P_i E_j + |E| P_0
\end{equation}
where $P_0$ is a solution of the equation $P_0^2 (\mathcal{E}'(P_0^2))^2 = (\vec{E}/2B)^2$. That $C = 0$ is equivalent to (4.4) can be proved, e. g., in the generalized parabolic case $\mathcal{E} \sim (\vec{P}^2)^\alpha$, $1/2 < \alpha < 3/2$. It appears, however, that the vanishing of $C$ is a mere coincidence, and (4.4) is better interpreted as the extremum condition $\partial P_i C = 0$ for the Casimir. (For (2.10) we obviously have a minimum). Eq. (4.4) can have more than just one solution. This happens, e. g. , for the energy expression $\mathcal{E} = a^2 \vec{P}^2 + b^2 \sqrt{1 + c^2 \vec{P}^2}$ considered by Culcer et al. in Ref. [4] for the Anomalous Hall Effect. Then the vanishing of $C$ is clearly irrelevant, but the extremum property is still valid.

If the effective non-commutative parameter $\theta = \theta(\vec{P})$ is genuinely momentum-dependent as it happens in Ref. [4, 5], the magnetic field cannot be tuned to either of the critical values $B'_\text{crit}$ or $B''_\text{crit}$. Hence, the Hall motions can not be made mandatory.

Writing either (1.2) or (4.3) as
\begin{equation}
\dot{X}_i = (\text{kinetic term}) - e\theta \epsilon_{ij} \dot{P}_j,
\end{equation}
we recognize the anomalous velocity used in the description of the Anomalous Hall Effect in ferromagnetics [4]. The additional contribution $(g/2)em \theta \epsilon_{ij} E_j$ in eqn. (3.3) comes from the anomalous term $(g/2) \vec{E} \times \vec{P}$ in the Hamiltonian (3.2). This latter can be viewed as the “Jackiw-Nair” [12] limit of the spin-orbit coupling
\begin{equation}\frac{1}{m^2 c^2} \vec{\sigma} \times \vec{E} \cdot \vec{P},
\end{equation}
advocated by Karplus and Luttinger half a century ago [4]. Putting $\vec{\sigma} = s\sigma_3$, $\theta = -s/c^2 m^2$ (4.7) becomes indeed proportional to our anomalous term. Such a term also arises in the non-relativistic limit of charged Dirac particle in a constant electric field [13].

Adding $(ge\theta/2)C$ to $H^{\text{Bloch}}$ yields an anomalous extension of the semiclassical Bloch model with (4.3) generalized to
\begin{equation}
(1 - eB \theta) \dot{X}_i = \left( 1 - \frac{g}{2} e \theta B \right) \partial P_i \mathcal{E} - \left( 1 - \frac{g}{2} \right) e \theta \epsilon_{ij} E_j - \frac{eg}{2} \partial P_i \theta \\quad (4.8)
\end{equation}
$[\theta = \theta(\vec{P})]$, supplemented with the Lorentz equation (1.3). For a Hall motion the momentum is constant, $\vec{P} = \vec{P}^0$. Putting $\theta^0 = \theta(\vec{P}^0)$ and inserting into the equation of motion
\begin{equation}
(1 - \frac{g}{2} e \theta^0 B) \left( \partial_P \mathcal{E} \bigg|_{\vec{P}^0} - \epsilon_{ij} \frac{E_j}{B} \right) \bigg|_{\vec{P}^0} = \frac{eg}{2} C(\vec{P}^0) \partial_P \theta \bigg|_{\vec{P}^0}.
\end{equation}

Let us assume that $\partial_P \theta \neq 0$. Although (4.9) has more general solutions, we observe that if the Hall condition (4.4) holds, then we have again $C = 0$. At last, $1 - (g/2) e \theta^0 B = 0$ can also yield Hall motions of the second type for some particular value of the momentum.

5 Conclusion

In this Letter we presented, following Ref. [6], a framework which unifies phase space and field variables, and used it to introduce “enlarged Galilean symmetry”. Then we built a theory along the lines put forward by Bacry [8]. Adding a Casimir to the Hamiltonian, our theory
can accommodate anomalous moment coupling cf. [9]. We derived an algebraic characterization of the Hall motions and have shown that for $B = B_{\text{crit}}'$ or $B_{\text{crit}}''$ the Hall motions become mandatory. The physical interpretation is provided by the semiclassical theory of a Bloch electron, where a momentum-dependent effective non-commutative parameter is derived by a Berry phase calculation [5][11].

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