Abstract

In this paper we use the Hall – Dilworth gluing construction to obtain multiple congruences of a lattice \( L \). For any finite lattice \( L \), \( G_m(L, B_n) \), the gluing of \( L \) and \( B_n \) over \( F \) and \( I \), both \( F \) and \( I \) are isomorphic to \( 2 \). For any lattice \( L \), the congruences of \( G_m(L, B_n) \) is \( 2^{(n-1)} \) times the congruences of \( L \) where \( F \) be the filter of \( L \) and \( I \) be an ideal of \( B_n \) and are isomorphic to \( 2 \). We call \( G_m(L, B_n) \) the congruence multiple operator.

Key words : Gluing, filter, ideal, congruence and multiple congruence
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1 Introduction

In\(^{1,2,3}\) Gratzer et al studied congruence lattices of lattices. The gluing construction in the lattice theory started with a paper of M. Hall and R.P. Dilworth\(^4\) to prove that there exists a modular lattices that cannot be embedded in any complemented modular lattice. This construction is as follows.

Let \( K \) and \( L \) be lattices. Let \( F \) be filter of \( K \) and let \( I \) be an ideal of \( L \). If \( F \) is isomorphic to \( I \) with \( \psi \) as the isomorphism, then we can form the gluing of \( K \) and \( L \) over \( F \) and \( I \) with respect to \( \psi \) defined as follows.

We form the disjoint union \( K \cup L \) and identify \( a \in F \), to obtain the set \( G \). We order \( G \) as follows:

\[
\begin{align*}
   a \leq_K b & \text{ if } a, b \in K \\
   a \leq_L b & \text{ if } a, b \in L \\
   a \leq_K x \text{ and } \psi(x) \leq_L b & \text{ if } a \in K \text{ and } b \in L \text{ for some } x \in F
\end{align*}
\]
Lemma 1 [5]:

G is a lattice. The join in G is described by

\[ a \lor_G^b = \begin{cases} 
  a \lor_K b & \text{if } a, b \in K \\
  a \lor_L b & \text{if } a, b \in L \\
  \psi(a \lor_K x) \lor_L b & \text{if } a \in K, b \in L \text{ for any } x \in F \text{ and } x \leq b 
\end{cases} \]

and dually for the meet. If L has a zero, 0_L, then the last clause for the join may be rephrased:

\[ a \lor_G^b = \psi(a \lor_K 0_L) \lor_L b \quad \text{if } a \in K \text{ and } b \in L. \]

G contains K and L as sublattices. Infact, K is an ideal and L is a filter of G.

Lemma 2:

Let K, L, F, I and G be given as above. Let A be a lattice containing K and L as sublattices so that

\[ K \cap L = I = F. \]

Then \( K \cup L \) is a sublattice of A and it is isomorphic to G.

Definition 3:

If \( \theta_k \) is a binary relation on K and \( \theta_L \) is a binary relation on L, the reflexive product \( \theta_k \circ \theta_L \) is defined as \( \theta_k \cup \theta_L \cup (\theta_k \circ \theta_L) \).

Lemma 4:

A congruence \( \theta \) of G can be uniquely written in the form \( \theta = \theta_k \circ \theta_L \), where \( \theta_k \) is a congruence of K and \( \theta_L \) is a congruence of L satisfying the condition that \( \theta_k \) restricted to F equals \( \theta_L \) restricted to I (under the identification of elements by \( \psi \)). Conversely, if \( \theta_k \) is a congruence of K and \( \theta_L \) is a congruence of L satisfying the condition that \( \theta_k \) restricted to F equals \( \theta_L \) restricted to I, then \( \theta = \theta_k \circ \theta_L \) is a congruence of G.

Lemma 5:

Let G be the gluing of lattices K and L over F and I as above. If K and L are modular so is the gluing G of K and L. If K and L are distributive, so is the gluing G.

In this paper, we study about the multiple congruences of a lattice L. For any finite lattice L, \( G_m(L, B_n) \), the gluing of L and B_n over F and I, both F and I are isomorphic to \( C_2 \). For any lattice L, the congruences of \( G_m(L, B_n) \) is \( 2^{m-1} \) times the congruences of L where F be the filter of L and I be an ideal of B_n and are isomorphic to \( C_2 \). We call \( G_m(L, B_n) \) the congruence multiple operator. In [6], we studied about proper modular congruence preserving extensions lattices.

3 Multiple Congruence of a lattice:

Definition 6: The lattice G is called the congruence multiple operator if \( ConG \cong C_2 \times ConL \) for all
L. Equivalently for any lattice L, the congruence of an extension of L is multiple.

Construction 7: Let L be a finite bounded lattice with filter F and let $B_2$ be a nontrivial finite non simple lattice with an ideal I of $B_2$ isomorphic to $C_2$. $B_2$ is not a congruence preserving extension of I. Now glue the lattices L and $B_2$ over F and I under the isomorphism $\psi$. Let us denote it by $G_m(L, B_2)$.

Example 8: Let L be a chain with three elements with a filter F isomorphic to $C_2$ and let $B_2$ be a Boolean algebra with two atoms having an ideal I isomorphic to F. Glue the lattices L and $B_2$ over the filter F of L and an ideal I of $B_2$. The lattices L, $B_2$ and $G_m(L, B_2)$ are shown in figure 1.

![Figure 1](image1)

Also their congruence with the congruence classes are given below. The congruence lattices of $C_3$, $B_2$, and $G(\text{Con}(C_3), \text{Con}(B_2))$ are given in figure 2.

![Figure 2](image2)

The congruence of $C_3$ is $\text{Con}(C_3) = \{ \omega, \theta_1, \theta_2, \tau \}$ where $\omega$ the null congruence, $\theta_1 = \{(0),(a,1)\}$, $\theta_2 = \{(0,a),(1)\}$ and $\tau$ all congruence. The congruence lattice of $C_3$ is isomorphic to $B_2$. Glue the lattices $\text{Con}(C_3)$ and $\text{Con}(B_2)$ over the filter F and an ideal I both isomorphic to $C_2$. 
The congruence of $B_2$ is $\text{Con}(B_2) = \{ \omega, \theta_1, \theta_2, \tau \}$ where $\omega$ the null congruence, $\theta_1 = \{(a,b),(c,1)\}$, $\theta_2 = \{(a,c), (b,1)\}$ and $\tau$ all congruence. The congruence lattice of $B_2$ is isomorphic to $B_2$. The lattice $G_m(C_3, B_2)$ is obtained by gluing $C_3$ and $B_2$ over the filter $F$ of $C_3$ and an ideal $I$ of $B_2$. The congruence of $G_m(C_3, B_2)$ is obtained by the reflexive binary product of congruences of $C_3$ and congruences of $B_2$.

$\text{Con}(G_m(C_3, B_2)) = \{ \omega, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6, \tau \}$, where $\omega$ - the null congruence, $\theta_1 = \{(0,a),(b),(c,1)\}$, $\theta_2 = \{(0),(a,c),(b,1)\}$, $\theta_3 = \{(0),(a,b),(c,1)\}$, $\theta_4 = \{(0),(ab),(c1)\}$, $\theta_5 = \{(0),(ac),(b1)\}$, $\theta_6 = \{(0),(abc1)\}$ and $\tau$ all congruence. The congruence lattice of $G(C_3, B_2)$ is isomorphic to $B_2$. That is, every congruence of $C_3$ is restricted to two times of the congruences of $B_2$.

Example 9: Let $L$ be an ortholattice. Let $G_m(C_3, B_2)$ be the gluing of $L$ and $B_2$ over $F$ and $I$ under the isomorphism $\psi$. The lattice $L$, $G_m(L, B_2)$ are given in figure 3.

![Figure 3](image)

The congruence of $G_m(L, B_2)$ is given by $\text{Con}(G_m(L, B_2))$ isomorphic to $\text{Con}L \times C_2$. $\text{Con}(G_m(L, B_2)) = \{ \omega, \tau, \theta$'s $\}$. Where $\omega$ is the null congruence, $\tau$ is the all congruence and $\theta$'s the non trivial congruence of $G_m(L, B_2)$. They are $\{(0),(a),(b),(c),(d),(e),(f,1)\}, \{(0),(a),(c),(d),(e),(f,1)\}, \{(0),(a),(b),(d),(e),(f,1)\}, \{(0),(a),(b),(c),(e),(f,1)\}, \{(0),(a),(b),(c),(d),(f,1)\}, \{(0),(a),(b),(c),(d),(e),(f,1)\}$. As there are seven congruence of $L$, every nontrivial congruence of $B_2$ restricted to an ideal $I$ equal twice the congruence of $L$ restricted to a filter $F$.

The congruence lattice of $L$ and $G_m(L, B_2)$ is given in figure 4.

Hence by gluing of a Boolean algebra with an ortholattice $L$, whose ideal and filter are simple lattice $(C_2)$ has 14 congruences $(2 \times 7)$ isomorphic to $C_2 \times \text{Con}(L)$.

Hence it is isomorphic to two times the congruence of $L$.

Remark 10: From the construction given in I, the lattice $G_m(L, B_2)$ contains a sub lattice $L$, which is an ideal of $G_m(L, B_2)$.

Remark 11: The number of congruences of $B_n$, Boolean algebra with $n$ atoms is $2^n$. If all the congruences
Theorem 12: Let \( L \) be any lattice with a prime filter \( F \). Then the number of congruences of \( G_n(L) \) is \( 2^{n-1} \) times the number of congruences of \( L \).

Proof. Let \( L \) be any finite lattice with a prime filter \( F \) and \( B_n \) be a Boolean algebra with \( n \) atoms. Let \( I=(a) \) be a prime ideal of \( B_n \) and \( F \) be a filter of \( L \) which is isomorphic to \( I \) under an isomorphism \( \varphi \). Let \( \theta_L \) be a congruence of \( L \). Then \( \theta_L \cap F \) is either equal to \( (0)(a) \) or \( (0a) \). If \( \theta_L \cap F \) is equal to \( 2^{n-1} \) congruences of the gluing of \( B_n \) restricted to \( I \). By definition 3, \( \theta = \theta_L \circ^r \theta_{B_n} \) is a congruence of \( G_n(L) \). Therefore corresponding to each \( \theta_L \), there are \( 2^{n-1} \theta_{B_n} \) such that \( \theta_L \cap F = \theta_{B_n} \cap I \) and \( \theta = \theta_L \circ^r \theta_{B_n} \) is a congruence of \( G_n(L) \). Hence \( |Con(G_n(L))| = 2^{n-1} \times |Con(L)| \).

Conclusion

From the gluing of lattices we obtain that for every finite lattice \( A \) and a Boolean algebra \( B \), the congruence of gluing of \( A \) and \( B \) over \( F \) and \( I \) is isomorphic to \( 2^{n-1} \) times the congruence of \( A \). Congruence of gluing of \( A \) and \( B \) over \( F \) and \( I \) is isomorphic to the congruence of \( \text{con}(A) \) and \( \text{con}(B) \) over \( F \) and \( I \) under \( \varphi \), where \( F \) and \( I \) of \( A, B \) are same as \( F \) and \( I \) of \( \text{con}(A) \) and \( \text{con}(B) \). In \( \text{6} \) I have studied about Distributive congruence preserving extension of chains using relative separator. In \( \text{7} \) I have studied about proper modular congruence preserving extension of lattices by gluing of simple modular lattice. We further extend the gluing concept to get a proper congruence preserving extension of \( A \) if \( B \) is not a congruence preserving extension of \( I \).

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