CHARACTERIZING LIMINAL AND TYPE I GRAPH
\textit{C\textsuperscript{*}-algebras}

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\textbf{Abstract.} We prove that the \textit{C\textsuperscript{*}}-algebra of a directed graph \(E\) is liminal iff the graph satisfies the finiteness condition: if \(p\) is an infinite path or a path ending with a sink or an infinite emitter, and if \(v\) is any vertex, then there are only finitely many paths starting with \(v\) and ending with a vertex in \(p\). Moreover, \(C\textsuperscript{*}(E)\) is Type I precisely when the circuits of \(E\) are either terminal or transitory, i.e., \(E\) has no vertex which is on multiple circuits, and \(E\) satisfies the weaker condition: for any infinite path \(\lambda\), there are only finitely many vertices of \(\lambda\) that get back to \(\lambda\) in an infinite number of ways.

1. Introduction

A directed graph \(E = (E^0, E^1, o, t)\) consists of a countable set \(E^0\) of vertices and \(E^1\) of edges, and maps \(o, t : E^1 \to E^0\) identifying the origin (source) and the terminus (range) of each edge. The graph is row-finite if each vertex emits at most finitely many edges. A vertex is a sink if it is not an origin of any edge. A vertex \(v\) is called singular if it is either a sink or emits infinitely many edges. A path is a sequence of edges \(e_1e_2\ldots e_n\) with \(t(e_i) = o(e_{i+1})\) for each \(i = 1, 2, \ldots, n - 1\). An infinite path is a sequence \(e_1e_2\ldots\) of edges with \(t(e_i) = o(e_{i+1})\) for each \(i\).

For a finite path \(p = e_1e_2\ldots e_n\), we define \(o(p) := o(e_1)\) and \(t(p) := t(e_n)\). For an infinite path \(p = e_1e_2\ldots\), we define \(o(p) := o(e_1)\). We regard vertices as paths of length zero, and hence if \(v \in E^0\), \(o(v) = v = t(v)\).

\[E^* = \bigcup_{n=0}^{\infty} E^n\text{, where } E^n := \{p : p \text{ is a path of length } n\}\]

\[E^{**} := E^* \cup E^\infty\text{, where } E^\infty \text{ is the set of infinite paths.}\]

A Cuntz-Krieger \(E\)-family consists of mutually orthogonal projections \(\{p_v : v \in E^0\}\) and partial isometries \(\{s_e : e \in E^1\}\) satisfying:

\[p_{o(e)} = s_e^* s_e \forall e \in E^1\]

\[\sum_{e \in F} s_\mu s_e^* \leq p_v \forall v \in E^0 \text{ and for any finite subset } F \text{ of } \{e \in E^1 : o(e) = v\}\]

\[\sum_{o(e) = v} s_e s_e^* = p_v \text{ for each non-singular vertex } v \in E^0\]

The graph \textit{C\textsuperscript{*}}-algebra \(C\textsuperscript{*}(E)\) is the universal \textit{C\textsuperscript{*}}-algebra generated by a Cuntz-Krieger \(E\)-family \(\{s_e, p_v\}\).

For a finite path \(\mu = e_1e_2\ldots e_n\), we write \(s_\mu\) for \(s_{e_1} s_{e_2} \ldots s_{e_n}\).

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Since the family \( \{s_\mu s_\nu^* : \mu, \nu \in E^* \} \) is closed under multiplication, we have:

\[
C^*(E) = \overline{\text{span}}(\{s_\mu s_\nu^* : \mu, \nu \in E^* \text{ and } t(\mu) = t(\nu)\})
\]

The outline of the paper is as follows. In section 2 we introduce the basic notations and conventions we will use throughout the paper. Section 3 deals with row-finite graphs with no sinks. We begin the section by defining a property of a graph which we later prove to characterize liminal graph \( C^* \)-algebras when the graph has no singular vertices. Section 4 provides us with a proposition that gives us a method on how to obtain the largest liminal ideal of a \( C^* \)-algebra of a row-finite graph with no sinks. In sections 5, respectively 6, with the use of ‘desingularizing graphs’ of [4], we generalize the results of sections 3, respectively 4, to arbitrary graphs.

In section 7 we give a characterization for type I graph \( C^* \)-algebras. We finish the section with a proposition on how to obtain the largest type I ideal of a graph \( C^* \)-algebra.

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### 2. Preliminaries

Given a directed graph \( E \), we write \( v \geq w \) if there is a directed path from \( v \) to \( w \).

For a directed graph \( E \), we say \( H \subseteq E^0 \) is hereditary if \( v \in H \) and \( v \geq w \) imply that \( w \in H \). We say \( H \) is saturated if \( v \) is not singular and \( \{w \in E^0 : v \geq w\} \subseteq H \) imply that \( v \in H \).

If \( z \in \mathbb{T} \), then the family \( \{zs_v, p_v\} \) is another Cuntz-Krieger \( E \)-family with which generates \( C^*(E) \), and the universal property gives a homomorphism \( \gamma_z : C^*(E) \to C^*(E) \) such that \( \gamma_z(s_v) = zs_v \) and \( \gamma_z(p_v) = p_v \). \( \gamma \) is a strongly continuous action, called gauge action, on \( C^*(E) \). See [1] for details.

Let \( E \) be a row-finite directed graph, let \( I \) be an ideal of \( C^*(E) \), and let \( H = \{v : p_v \in I\} \). In [1] Lemma 4.2 they proved that \( H \) is a hereditary saturated subset of \( E^0 \). Moreover, if \( I_H := \overline{\text{span}}\{s_\alpha s_\beta^* : \alpha, \beta \in E^* \text{ and } t(\alpha) = t(\beta) \in H\} \), the map \( H \mapsto I_H \) is an isomorphism of the lattice of saturated hereditary subsets of \( E^0 \) onto the lattice of closed gauge-invariant ideals of \( C^*(E) \) [1] Theorem 4.1 (a)]. Letting \( F := F(E \setminus H) \) the sub-graph of \( E \) that is gotten by removing \( H \) and all edges that point into \( H \), it is proven in [1] Theorem 4.1(b)] that \( C^*(F) \cong C^*(E)/I_H \). In the event that \( I \) is not a gauge-invariant ideal, we only get \( I_H \not\subseteq I \).

We will use the following notations and conventions:

- Every path we take is a directed path.
- A circuit in a graph \( E \) is a finite path \( p \) with \( o(p) = t(p) \). We save the term loop for a circuit of length 1.
- We say that a circuit is terminal if it has no exits, and a circuit is transitory if it has an exit and no exit of the circuit gets back to the circuit.
- \( \Lambda_E := \{v \in E^0 : v \text{ is a singular vertex}\} \)
- \( \Lambda_E^\circ := t^{-1}(\Lambda_E) \cap E^* \) i.e., \( \Lambda_E^\circ \) is the set of paths ending with a singular vertex. When there are no ambiguities, we will just use \( \Lambda^\circ \).
- We say \( v \) gets to \( w \) (or reaches \( w \)) if there is a path from \( v \) to \( w \).
- We say \( v \) gets to a path \( p \) if \( v \) gets to a vertex in \( p \).
- For a subset \( S \) of \( E^0 \), we write \( S \geq v \) if \( w \geq v \) \( \forall w \in S \).
- For a subset $H$ of $E^0$, we write $\text{Graph}(H)$ to refer to the sub-graph of $E$ whose set of vertices is $H$ and whose edges are those edges of $E$ that begin and end in $H$.
- $V(v) := \{w \in E^0 : v \geq w\}$.
- $E(v) := \text{Graph}(V(v))$. i.e., $E(v)$ is that part of the sub-graph of $E$ that the vertex $v$ can ‘see’. Accordingly we use $F(v)$, etc. when the graph is $F$, etc.
- For $v \in E^0$ let $\Delta(v) := \{e \in E^1 : o(e) = v\}$.
- For a finite subset $F$ of $\Delta(v)$, we write $V(v; F) := \{v\} \cup \bigcup_{e \in \Delta(v) \setminus F} V(t(e))$.
- $E(v; F) := \text{Graph}(V(v; F))$.
- For a hereditary subset $H$ of $E^0$, we write $\overline{H}$ to refer to the saturation of $H = \text{the smallest saturated set containing } H$. Notice that $\overline{H}$ is hereditary and saturated.
- For any path $\lambda$, $\lambda^0$ will denote the vertices of $\lambda$.
- As was used above, $F(E \setminus H)$ will denote the sub-graph of $E$ that is gotten by removing $H$ and all edges that point into $H$.
- We use $K$ to denote the space of compact operators on an (unspecified) separable Hilbert space.

3. Liminal $C^*$-Algebras of Graphs with no singular vertices.

We begin the section by a definition.

**Definition 3.1.** A subset $\gamma$ of $E^0$ is called a maximal tail if it satisfies the following three conditions.

(a) for any $v_1, v_2 \in \gamma$ there exists $z \in \gamma$ s.t. $v_1 \geq z$ and $v_2 \geq z$.
(b) for any $v \in \gamma \exists e \in E^1$ s.t. $o(e) = v$ and $t(e) \in \gamma$.
(c) $v \geq w$ and $w \in \gamma$ imply that $v \in \gamma$.

We will prove a result similar to (one direction of) [II Proposition 6.1] with a weaker assumption on the graph $E$ and a weaker assumption on the ideal.

**Lemma 3.2.** Let $E$ be a row-finite graph with no sinks. If $I$ is a primitive ideal of $C^*(E)$ and $H = \{v \in E^0 : p_v \in I\}$, then $\gamma = E^0 \setminus H$ is a maximal tail.

**Proof.** By [II Lemma 4.2] $H$ is hereditary saturated. The complement of a hereditary set satisfies (c). Since $E$ has no sinks, and $H$ is saturated, $\gamma$ satisfies (b). We prove (a). Let $v_1, v_2 \in \gamma$ and let $H_i = \{v \in \gamma : v_i \geq v\}$. We will first show that $\overline{H_i} \cap \overline{H_j} = \emptyset$.

For each $i$, $I_{\overline{H_i}}$ is a non-zero ideal of $C^*(F) \cong C^*(E)/I_H$, hence is of the form $I_i/I_H$, and $p_v + I_H \in I_{\overline{H_i}}$. Since each $I_{\overline{H_i}}$ is gauge-invariant, so is $\overline{I_{\overline{H_i}}} \cap \overline{I_{\overline{H_j}}}$. Therefore $\overline{I_{\overline{H_i}}} \cap \overline{I_{\overline{H_j}}} = \emptyset$.

Applying [II Lemma 6.2] to $F$ and $v_1$ shows that $\exists x \in E^0 \setminus H$ s.t. $y \geq x$ and $v_1 \geq x$. Since $y \in \overline{H_i}$ and $\overline{H_i}$ is hereditary, $x \in \overline{H_i}$. Applying [II Lemma 6.2] to $F$ and $v_2$ shows that $\exists z \in E^0 \setminus H$ s.t. $y \geq z$ and $v_2 \geq z$. Thus $v_1 \geq z$, and $v_2 \geq z$ as needed. □
Lemma 3.4. Let \( e \)dition (relation defined in [8] Definition 1.8). Let any infinite path \( \alpha \) precisely when the following finiteness condition is satisfied: for any vertex \( v \) and any infinite path \( \lambda \), there is only a finite number of ways to get to \( \lambda \) from \( v \).

To state the finiteness condition more precisely, we will use the equivalence relation defined in [S Definition 1.8].

If \( p = e_1 e_2 \ldots \) and \( q = f_1 f_2 \ldots \in E^\infty \), we say that \( p \sim q \) iff \( \exists j, k \) so that \( e_{j+r} = f_{k+r} \) for \( r \geq 0 \). i.e., iff \( p \) and \( q \) (eventually) share the same tail.

We use \([p]\) to denote the equivalence class containing \( p \).

Definition 3.3. A row-finite directed graph \( E \) that has no sinks is said to satisfy condition (M) if for any \( v \in E^0 \) and any \([p]\) \( \in E^\infty / \sim \) there is only a finite number of representatives of \([p]\) that begin with \( v \).

We note that \( E \) satisfies condition (M) implies that every circuit in \( E \) is terminal.

Lemma 3.4. Let \( E \) be a row-finite directed graph with no sinks that satisfies condition (M). Let \( F \) be a sub-graph of \( E \) so that \( F^0 \) is a maximal tail. If \( F \) has a circuit, say \( \alpha \), then the saturation of \( \alpha^0 \), \( \overline{\alpha}^0 \), is equal to \( F^0 \).

Proof. Let \( v_\alpha \) be a vertex of \( \alpha \). Since \( \alpha \) is terminal, \( v_\alpha \in \overline{\alpha} \) implies that \( z \) is in \( \alpha^0 \). Also, for each \( w \in F^0 \), by (a) of Definition 3.1 there exists \( z \in F^0 \) s.t. \( w \geq z \) and \( v_\alpha \geq z \), but \( z \) is in \( \alpha^0 \) which implies that \( z \) is not in \( \alpha^0 \). Therefore \( w \geq v_\alpha \), i.e., each vertex in \( F^0 \) connects to \( v_\alpha \) (via a directed path).

Now, assuming the contrary, let \( v_1 \notin \overline{\alpha} \). If \( v_1 \) is in a circuit, say \( \beta \). Then, by the previous paragraph, \( v_1 \geq v_\alpha \) hence either \( \beta = \alpha \) or \( \beta \) has an exit. But \( v_1 \notin \overline{\alpha} \), therefore \( \beta = \alpha \) is not possible, and since \( F \) satisfies condition (M), \( \beta \) can not have an exit. Thus \( v_1 \) is not in a circuit. Therefore \( \exists e_1 \in F^1 \) s.t. \( o(e_1) = v_1 \), \( t(e_1) \notin \overline{\alpha} \). Let \( v_2 = t(e_1) \). Inductively, \( \exists e_n \in F^1 \) s.t. \( v_n = o(e_n) \), \( t(e_n) = v_{n+1} \notin \overline{\alpha} \). Look at the infinite path \( e_1 e_2 \ldots \).

Notice that the \( v_i \)'s are distinct and each \( v_i \geq v_\alpha \). Therefore there are infinitely many ways to get to \( \alpha \) from \( v_1 \), i.e., there are infinitely many representatives of \([\alpha]\) that begin with \( v_1 \), which contradicts to the assumption that \( E \) satisfies condition (M). Therefore \( F^0 = \overline{\alpha} \).

Lemma 3.5. Let \( E \) be a row-finite directed graph with no sinks that satisfies condition (M). Let \( F \) be a sub-graph of \( E \) so that \( F^0 \) is a maximal tail. If \( F \) has no circuits then \( F \) has a hereditary infinite path, say \( \lambda \), s.t. \( F^0 = \overline{\lambda}^0 \).

Proof. Since \( F \) has no sinks, it must have an infinite path, say \( \lambda \). Let \( v_\lambda \) be a vertex in \( \lambda \). By condition (M), there are only a finite number of infinite paths that begin with \( v_\lambda \) and share a tail with \( \lambda \). By going far enough on \( \lambda \), there exists \( w \in \lambda^0 \) s.t. \( v_\lambda \geq w \) and \( [\lambda] \) has only one representative that begins with \( w \). By re-selecting \( v_\lambda \) (to be \( w \), for instance) we can assume that there is only one representative of \([\lambda]\) that begins with \( v_\lambda \). We might, as well, assume that \( o(\lambda) = v_\lambda \).

We will now prove that \( \lambda^0 \) is hereditary. Suppose \( u \in F^0 \) s.t. \( v_\lambda \geq u \) and \( u \notin \lambda^0 \). Since \( F^0 \) is a maximal tail and since \( F \) has no circuits, by (b) of Definition 3.1 we can choose \( w_1 \in F^0 \) s.t. \( v_\lambda \geq w_1 \) and \( \lambda \neq w_1 \). By (a) of Definition 3.1 there exists \( z_1 \in F^0 \) s.t. \( u \geq z_1 \), and \( w_1 \geq z_1 \). If \( z_1 \in \lambda^0 \) then we have two ways to get to \( \lambda \) from \( v_\lambda \) (through \( u \) and through \( w_1 \)) which contradicts to the choice of \( v_\lambda \), hence \( z_1 \notin \lambda^0 \).
Let $w_2 \in \lambda^0$ (far enough) so that $w_2 \not\sim z_1$. If such a choice was not possible, we would be able to get to $z_1$ and hence to any path that begins with $z_1$ from $v_\lambda$ in an infinite number of ways, contradicting condition (M).

Again since $F^0$ is a maximal tail, there exists $z_2 \in F^0$ s.t. $w_2 \geq z_2$ and $z_1 \geq z_2$. Notice that there are (at least) two ways to get to $z_2$ from $v_\lambda$. By inductively choosing a $w_n \in \lambda^0$ and a $z_n \in F^0$ s.t. $w_n \not\sim z_{n-1}, w_n \geq z_n$ and $z_{n-1} \geq z_n$, there are at least $n$ ways to get to $z_n$ from $v_\lambda$ (one through $w_n$ and $n - 1$ through $z_{n-1}$).

We now form an infinite path $\alpha$ that contains $z_1, z_2 \ldots$ as (some of) its vertices that we can reach to, from $v_\lambda$, in an infinite number of ways, which is again a contradiction. Hence no such $\alpha$ can exist. Thus $\lambda$ is hereditary.

We will now prove that $F^0 = \lambda^0$. Assuming the contrary, let $v_1 \not\in \lambda^0$. Then $\exists e_1 \in F^1$ s.t. $o(e_1) = v_1$, $t(e_1) \notin \lambda^0$. Inductively, let $v_n = t(e_{n-1})$, then $\exists e_n \in F^1$ s.t. $v_n = o(e_n)$, $t(e_n) = v_{n+1} \notin \lambda^0$. Consider the infinite path $e_1 e_2 \ldots$.

Notice that since $F^0$ is a maximal tail, for each $v_i \exists x_i$ s.t. $v_i \geq x_i$ and $v_\lambda \geq x_i$. But $\lambda^0$ is hereditary hence $x_i \in \lambda^0$, implying that each $v_i$ reaches $\lambda$. Therefore there are infinitely many ways to get to $\lambda$ from $v_1$, i.e., there are infinitely many representatives of $[\lambda]$ that begin with $v_1$ which contradicts to condition (M). Therefore $F^0 = \lambda^0$.

**Lemma 3.6.** Let $I_H$ be a primitive ideal of $C^*(E)$, where $H$ is a hereditary saturated subset of $F^0$. Let $F = F(H \setminus H)$. Then $F$ has no circuits.

**Proof.** Note that $F^0$ is a maximal tail. And $C^*(F) \cong C^*(E)/I_H$. Since $I_H$ is a primitive ideal of $C^*(E)$, $\{0\}$ is a primitive ideal of $C^*(F)$.

Suppose that $F$ has a circuit, say $\alpha$. By Lemma 3.3, $F^0 = \alpha^0$. Hence $C^*(F) \cong I_{\alpha^0}$ is the ideal of $C^*(F)$ generated by $\{\alpha^0\}$. Since $\alpha$ has no exits (is hereditary), by [8] Proposition 2.1 $I_{\alpha^0}$ is Morita equivalent to $C^*(\alpha)$ which is Morita equivalent to $C(T)$. But $\{0\}$ is not a primitive ideal of $C(T)$ implying that $\{0\}$ is not a primitive ideal of $C^*(F)$ which is a contradiction. Hence $F$ has no circuits. □

Hidden in the proofs of Lemma 3.3 and Lemma 3.5 we have proven a (less relevant) fact: if a directed graph $E$ with no singular vertices satisfies condition (M) and $F^0$ is a maximal tail then $F$ has (essentially) one infinite tail, i.e., $F^\infty \sim$ contains a single element.

**Remark 3.7.** Let $E_1$ be a sub-graph of a directed graph $E_2$. Applying [8] Theorem 2.34 and [8] Corollary 2.33, we observe that $C^*(E_1)$ is a quotient of a $C^*$-subalgebra of $C^*(E_2)$. (Letting $S_2 = E_2^+ \setminus \{v \in E_2^+ : v$ is a singular vertex$\}$)

**Lemma 3.8.** Let $E$ be a directed graph. Suppose all the circuits of $E$ are transitory and suppose $\exists \lambda \in E^\infty$ s.t. the number of vertices of $\lambda$ that emit multiple edges that get back to $\lambda$ is infinite. Then $C^*(E)$ is not Type I.

**Proof.** Let $v_1 \in \lambda^0$ s.t. $v_1$ emits (at least) two edges that get back to $\lambda$.

choose a path $\alpha_1^1 = e_1 e_2 \ldots e_{n_1}$, s.t. $e_1$ is not in $\lambda$, $o(e_1) = v_1$, and $t(\alpha_1^1) = v_2 \in \lambda^0$. If $t(e_{n_1}) = v_1$, i.e. $e_1 e_2 \ldots e_{n_1}$ is a circuit we extend $e_1 e_2 \ldots e_{n_1}$ so that $v_2$ is further along $\lambda$ than $v_1$ is.

We might again extend $\alpha_1^1$ along $\lambda$, if needed, and assume that $v_2$ emits (at least) two edges that get back to $\lambda$.

Let $\alpha_2^1$ be the path along $\lambda$ s.t. $o(\alpha_2^1) = v_1$ and $t(\alpha_2^1) = v_2$. Inductively, choose $\alpha_k^1 = e_1 e_2 \ldots e_{n_k}$ s.t. $e_1$ is not in $\lambda$, $o(e_1) = v_k$, $t(\alpha_k^1) = v_{k+1} \in \lambda^0$, by extending
\( \alpha_k^1 \), if needed, we can assume that \( v_{k+1} \) is further along \( \lambda \) than \( v_k \) and emits multiple edges that get back to \( \lambda \). Let \( \alpha_k^2 \) be the path along \( \lambda \) s.t. \( o(\alpha_k^2) = v_k \) and \( t(\alpha_k^2) = v_{k+1} \). Now look at the following sub-graph of \( E \), call it \( F \).

\[
\begin{array}{c}
  v_1 & \sim & v_2 & \sim & v_3 & \sim & v_4 & \sim & \cdots \\
  \alpha_1^2 & \sim & \alpha_2^1 & \sim & \alpha_3^1 & \sim & \alpha_4^1 & \sim & \cdots \\
\end{array}
\]

Now let \( \{ s_e, p_v : e \in F^1, v \in F^0 \} \) be a Cuntz-Krieger \( F \)-family.

Thus \( C^*(F) = \overline{\text{span}} \{ s_\mu s_\nu^* : \mu, \nu \in F^* \text{ and } t(\mu) = t(\nu) \} \).

Let \( F_k := \overline{\text{span}} \{ s_\mu s_\nu^* : \mu, \nu \text{ are paths made up of } \alpha_i^r \text{'s (or just } v_k \text{) s.t. } t(\mu) = v_k = t(\nu) \} \).

By \( [9] \) Corollary 2.3, \( F_k \cong M_{N_k}(\mathbb{C}) \) where \( N_k \) is the number of paths made up of \( \alpha_i^r \text{'s (or just } v_k \text{) ending with } v_k \), which is finite.

Also, if \( s_\mu s_\nu^* \in F_k \) then \( s_\mu s_\nu^* = s_\mu p_{v_k} s_\nu^* = s_\mu s_{\alpha_k^1} s_{\alpha_k^1}^* s_\nu^* + s_{\mu} s_{\alpha_k^2} s_{\alpha_k^2}^* s_\nu^* \in F_{k+1} \).

Hence \( F_k \subseteq F_{k+1} \). Let \( A = \bigcup_{k=1}^\infty F_k \). Then \( A \) is a \( C^* \)-subalgebra of \( C^*(F) \). Since \( A \) is a UHF algebra, it is not Type I. Therefore \( C^*(F) \) has a \( C^* \)-subalgebra that is not Type I and can not be Type I. Since \( F \) is a sub-graph of \( E \), by Remark 3.7 \( C^*(E) \) has a \( C^* \)-subalgebra whose quotient is not Type I. Therefore \( C^*(E) \) is not Type I.

It might be useful to keep following graph in mind when reading Lemma 3.9, it can be viewed as a prototype of a graph that satisfies the assumption of the lemma.

\[
\begin{array}{c}
z_1 & \sim & f_2 & \sim & z_2 & \sim & f_3 & \sim & z_3 & \sim & f_4 & \sim & \cdots \\
\sim & f_3 & \sim & z_3 & \sim & z_3 & \sim & f_4 & \sim & \cdots \\
\sim & e_1 & \sim & e_2 & \sim & e_3 & \sim & \cdots \\
\end{array}
\]

**Lemma 3.9.** Let \( E \) be a directed graph, let \( \lambda = e_1 e_2 \ldots \) be an infinite path in \( E \), and let \( o(\lambda) = v_\lambda \). Suppose:

1. \( E \) has no circuits.
2. The number of representatives of \( [\lambda] \) that begin with \( v_\lambda \) is infinite.
3. \( v_\lambda \) is the only such vertex in \( \lambda^0 \).
4. \( E = E(v_\lambda) \), and
5. \( \forall v \in E^0 \exists w \in \lambda^0 \text{ s.t. } v \geq w \) (i.e., \( E^0 \geq \lambda^0 \)).

Then

i) \( \{0\} \) is a primitive ideal of \( C^*(E) \), and
ii) \( C^*(E) \) is not simple.

**Proof.** We prove i). First note that \( E \) satisfies condition (K) of \( [11] \): every vertex lies on either no circuits or at least two circuits. This is because \( E \) has no circuits. We will show that \( E^0 \) is a maximal tail. Since \( E \) has no sinks, \( E^0 \) satisfies (b) of Definition 3.1 and clearly \( E^0 \) satisfies (c). We will show that \( E \) satisfies (a). Let \( v_1, v_2 \in E^0 \). By (4) above, \( \exists v_1, w_2 \in \lambda^0 \text{ s.t. } v_1 \geq w_1 \). Since \( \lambda \) is an infinite path, either \( w_1 \geq w_2 \) or \( w_2 \geq w_1 \). WLOG let \( w_2 \geq w_1 \). We have \( v_1 \geq w_1 \) and
\(v_2 \geq w_2 \geq w_1\), hence (a) is satisfied. Therefore \(E^0\) is a maximal tail and, by Proposition 6.1, \(I_0 = \{0\}\) is a primitive ideal of \(C^*(E)\).

We will prove ii). Since \(E\) is row finite and since \(v_\lambda\) gets to \(\lambda\) infinitely often, \(\exists f_1 \in E^1\) s.t. \(o(f_1) = v_\lambda\) and \(z_1 := t(f_1)\) gets to \(\lambda\) infinitely often. Moreover there is no vertex in \(\lambda^0\) that gets to \(\lambda\) infinitely often except \(v_\lambda\) and \(E\) has no circuits, therefore \(z_1 \notin \lambda^0\). Inductively, \(\exists f_{n+1} \in E^1\) s.t. \(o(f_{n+1}) = z_n\), \(z_{n+1} := t(f_{n+1})\) gets to \(\lambda\) infinitely often, and \(z_{n+1} \notin \lambda^0\). Notice that the number of representatives of \([\lambda]\) that begin with \(t(e_1)\), by (2) above, is finite. Therefore \(t(e_1)\) does not get to any of the \(z_i\)'s, that is, \(t(e_1)\) does not reach the infinite path \(f_1f_2\ldots\). Thus \(E\) is not co-final. Therefore \(C^*(E)\) is not simple. \(\square\)

We are now ready to prove the first of the measure results.

**Theorem 3.10.** Let \(E\) be a row-finite directed graph with no sinks. \(C^*(E)\) is liminal iff \(E\) satisfies condition \((M)\).

**Proof.** Suppose \(E\) satisfies \((M)\). Let \(I\) be a primitive ideal of \(C^*(E)\), let \(H = \{v : p_v \in I\}\), and let \(F = F(E \setminus H)\). By Lemma 3.10, \(F^0\) is a maximal tail, and Theorem 4.1 (b) implies that \(C^*(F) \cong C^*(E)/I_H\).

Case I. \(I = I_H\).

Then \(I_H\) is a primitive ideal, hence Lemma 3.6 implies that \(F\) has no circuits. Using Lemma 3.5 let \(\lambda\) be a hereditary infinite path s.t. \(F^0 = \lambda^0\). \(C^*(E)/I_H \cong C^*(F) = I_\lambda = \mathcal{M}\text{mac}(s_\alpha s_\beta^* : \alpha, \beta \in F^*, \text{ s.t. } t(\alpha) = t(\beta) \in \lambda^0)\).

By [5] Proposition 2.1 \(I_\lambda\) is Morita equivalent to \(C^*(\lambda) \cong K(t^2(\alpha^0))\). Therefore \(C^*(E)\) is liminal.

Case II. \(I_H \subsetneq I\).

We will first prove that \(F(E \setminus H)\) has a circuit. If \(F = F(E \setminus H)\) has no circuits then by Lemma 3.4 \(F^0 = \lambda^0\) for some hereditary infinite path \(\lambda\). Therefore \(C^*(F)\) is simple, implying that \(C^*(E)/I_H\) is simple. But \(I/I_H\) is a (proper) ideal of \(C^*(E)/I_H\) therefore \(I/I_H = 0\) implying that \(I = I_H\). A contradiction.

Hence \(F\) must have a circuit, say \(\alpha\). Lemma 3.4 implies that \(F^0 = \alpha^0\). Using [5] Proposition 2.1, \(C^*(F)\) is Morita equivalent to \(C^*(\alpha)\) which is Morita equivalent to \(C(\mathcal{T})\) which is liminal. Therefore \(C^*(E)/I_H\) is liminal. Since \(I/I_H\) is a primitive ideal of \(C^*(E)/I_H\) we get \(C^*(E)/I \cong C^*(E)/I_H / I/I_H \cong K\). Hence \(C^*(E)\) is liminal.

To prove the converse, suppose \(E\) does not satisfy condition \((M)\), i.e., there exist an infinite path \(\alpha\) and a \(v_\lambda \in E^0\) s.t. the number of representatives of \([\lambda]\) that begin with \(v_\lambda\) is infinite.

Suppose that \(E\) has a non-terminal circuit, say \(\alpha\). Let \(v\) be a vertex of \(\alpha\) s.t. \(\exists e \in E^1\) which is not an edge of \(\alpha\) and \(o(e) = v\). \(p_v = s_\alpha^* s_\alpha \sim s_\alpha s_\alpha^* \leq s_\alpha s_\alpha^* + s_e s_e^* \leq p_e\). Therefore \(p_v\) is an infinite projection. Hence \(C^*(E)\) cannot be liminal.

Suppose now that all circuits of \(E\) are terminal and that the number of representatives of \([\lambda]\) that begin with \(v_\lambda\) is infinite. We might assume that \(v_\lambda = o(\lambda)\). We want to prove that \(C^*(E)\) is not liminal. If \(v\) is a vertex s.t. \(V(v)\) does not intersect \(\lambda^0\), we can factor \(C^*(E)\) by the ideal generated by \{\(v\)\}. Hence we might assume that \(\forall v \in E^0: v \geq \lambda^0\). Moreover, this process gets rid of any terminal circuits, and hence we may assume that \(E\) has no circuits.

Also, since \(V(v_\lambda)\) is hereditary, by [5] Proposition 2.1, \(I_{V(v_\lambda)}\) is Morita equivalent to \(C^*(E(v_\lambda))\). Therefore it suffices to show that \(C^*(E(v_\lambda))\) is not liminal. Hence we might assume that \(E = E(v_\lambda)\).
If \( \forall v \in \lambda^0 \exists w \in \lambda^0 \) s.t. \( v \geq w \) and \( |\{e \in E^1 : o(e) = w\}| \geq 2 \), then by Lemma 8.8 \( C^*(E) \) is not type I, therefore it is not liminal.

Suppose \( \exists u \in \lambda^0 \) s.t. \( \forall w \in \lambda^0 \) with \( u \geq w \), \( |\{e \in E^1 : o(e) = w\}| = 1 \). Notice that there is exactly one representative of \( [\lambda] \) that begins with \( u \).

By re-selecting \( v_\lambda \) further along on \( \lambda \), we might assume that \( \forall w \in \lambda^0 \setminus \{v_\lambda\} \) the number of representatives of \( [\lambda] \) that begin with \( w \) is finite.

Thus \( E \) satisfies the following conditions:

1. \( E \) has no circuits.
2. The number of representatives of \( [\lambda] \) that begin with \( v_\lambda \) is infinite.
3. \( v_\lambda \) is the only such vertex in \( \lambda^0 \).
4. \( E = E(v_\lambda) \), and
5. \( \forall v \in E^0 \exists w \in \lambda^0 \) s.t. \( v \geq w \) (i.e., \( E^0 \geq \lambda^0 \)).

Therefore by Lemma 8.3 we get:

i) \( \{0\} \) is a primitive ideal of \( C^*(E) \).

ii) \( C^*(E) \) is not simple.

If \( C^*(E) \) is liminal, by i), since \( \{0\} \) is a primitive ideal of \( C^*(E) \), \( C^*(E) \cong C^*(E)/\{0\} \) is *-isomorphic to \( \mathcal{K} \). But from ii) \( C^*(E) \) can not be *-isomorphic to \( \mathcal{K} \) because \( \mathcal{K} \) is a simple \( C^* \)-algebra. Therefore \( C^*(E) \) can not be liminal. This concludes the proof of the theorem. \( \square \)

4. THE LARGEST LIMINAL IDEAL OF \( C^* \)-ALGEBRAS OF GRAPHS WITH NO SINGULAR VERTICES.

In this section we will investigate a method of extracting the largest liminal ideal of the \( C^* \)-algebra of a row finite graph \( E \) with no sinks.

Before we state the proposition, we will extend the definition of the equivalence \( \sim \) from \( E^\infty \) to \( E^{**} = E^\infty \cup E^* \), as it is done in [8] Remark 1.10. For \( p, q \in E^* \), we say \( p \sim q \) if \( t(p) = t(q) \).

The proposition gives a method of extracting the largest liminal ideal of \( C^*(E) \) of a graph \( E \) with no singular vertices by giving a characterization on the set of vertices that generate the ideal. The first part of the proposition, which will eventually be needed, can be proven for a general graph without much complication. Therefore we state that part of the proposition for a general graph.

**Proposition 4.1.** Let \( E \) be a directed graph and \( H = \{v \in E^0 : \forall [\lambda] \in (E^\infty \cup \Lambda^*)/\sim, \text{ the number of representatives of } [\lambda] \text{ that begin with } v \text{ is finite}\}. \) Then

(a) \( H \) is hereditary and saturated.

(b) If \( E \) is row-finite with no sinks then \( I_H \) is the largest liminal ideal of \( C^*(E) \).

**Proof.** Suppose \( v \in H \) and \( v \geq w \). Let \( p \) be a path from \( v \) to \( w \) and let \( \lambda \in E^\infty \cup \Lambda^* \).

If \( \beta \sim \lambda \) and \( o(\beta) = w \) then \( p\beta \sim \lambda \) and \( o(p\beta) = v \). Therefore the number of representatives of \( [\lambda] \) that begin with \( w \) is less than or equal to the number of representatives of \( [\lambda] \) that begin with \( v \). Therefore \( w \in H \). Thus \( H \) is hereditary.

Suppose \( v \in E^0 \) is not singular and \( \{w \in E^0 : v \geq w\} \subseteq H \). Let \( \Delta(v) = \{e \in E^1 : o(e) = v\} \). Note that \( \Delta(v) \) is a finite set and \( \forall e \in \Delta(v), t(e) \in H \). Let \( \lambda \in E^\infty \cup \Lambda^* \) and \( \beta \sim \lambda \) where \( o(\beta) = v \). Then the first edge of \( \beta \) is in \( \Delta(v) \). Therefore the number of representatives of \( [\lambda] \) that begin with \( v \) is equal to the sum of the number of representatives of \( [\lambda] \) that begin with a vertex in \( \{t(e) : e \in \Delta(v)\} \), which is a finite sum of finite numbers. Therefore \( v \in H \). Hence \( H \) is saturated.
To prove (b), suppose \( E \) is row-finite with no sinks. Let \( F = \text{Graph}(H) \). Clearly \( F \) satisfies condition (M). Hence Theorem 3.10 implies that \( C^*(F) \) is liminal. By [5] Proposition 2.1, \( I_H \) is Morita equivalent to \( C^*(F) \). Hence \( I_H \) is a liminal ideal. What remains is to prove that \( I_H \) is the largest liminal ideal of \( C^*(E) \).

Let \( I \) be the largest liminal ideal of \( C^*(E) \). Thus \( I_H \subseteq I \). Since the largest liminal ideal of a \( C^* \)-algebra is invariant under automorphisms, \( I \) is gauge invariant, therefore \( I = I_K \) for some saturated hereditary subset \( K \) of \( E^0 \) which includes \( H \). We will prove that \( K \subseteq H \). Let \( G = \text{Graph}(K) \). Since \( I_K \) is Morita equivalent to \( C^*(G) \), \( C^*(G) \) is liminal hence, by Theorem 5.1, \( G \) satisfies condition (M). Let \( v \in K = G^0 \). If \( \beta \in E^\infty \) with \( o(\beta) = v \), because \( K \) is hereditary, \( \beta^0 \subseteq K \). Therefore \( \beta \in G^\infty \). Now let \( [\lambda] \in E^\infty / \sim \), and let \( \gamma \) be a representative of \( [\lambda] \) that begins with \( v \). (If no such \( \gamma \) exists then the number of representatives of \( [\lambda] \) is zero.) Then \( \{ \beta \in E^\infty : \beta \sim \lambda, o(\beta) = v \} = \{ \beta \in G^\infty : \beta \sim \gamma, o(\beta) = v \} \), i.e., the set of representatives of \( [\lambda] \) that begin with \( v \) is subset of the set of representatives of \( [\gamma] \) (as an equivalence class of \( G^\infty / \sim \)) that begin with \( v \). Since \( G \) satisfies condition (M) the second set is finite. Therefore \( v \in H \), implying that \( K \subseteq H \). Therefore \( I_H = I_K \).

5. Liminal \( C^* \)-Algebras of General Graphs.

In this section we will consider for a general graph \( E \) and give the necessary and sufficient conditions for \( C^*(E) \) to be liminal in terms of the properties of the graph.

In [4] the authors gave a recipe on how to “desingularize a graph \( E \),” that is, obtain a graph \( F \) that has no singular vertices (by adding a tail at every singular vertex of \( E \)) so that \( C^*(E) \) and \( C^*(F) \) are Morita equivalent. Therefore, we will use this idea of desingularizing \( E \) and use the results of the previous sections to get the needed results.

We will begin by extending the definition of condition (M) from row-finite graphs with no sinks to general graphs:

**Definition 5.1.** A graph \( E \) is said to satisfy condition (M) if \( \forall [p] \in (E^\infty \cup \Lambda^*) / \sim \) and any \( v \in E^0 \), the number of representatives of \( [p] \) that begin with \( v \) is finite.

Notice that when \( E \) is a row-finite graph with no sinks, Definition 5.1 and Definition 3.3 say the same thing.

Since we need to use the results of the previous sections, it is important to check that condition (M) is preserved by the desingularization process. We will do that in the next two lemmas.

**Remark 5.2.** [4] Lemma 2.6(a)] states that if \( F \) is a desingularization of a directed graph \( E \) then there are bijective maps.

\[
\phi : E^* \longrightarrow \{ \beta \in F^* : o(\beta), t(\beta) \in E^0 \}
\]

\[
\phi_\infty : E^\infty \cup \Lambda^* \longrightarrow \{ \lambda \in F^\infty : o(\lambda) \in E^0 \}
\]

The map \( \phi \) preserves origin and terminus (and hence preserves circuits). The map \( \phi_\infty \) preserves origin.

**Lemma 5.3.** The map \( \phi_\infty \) preserves the equivalence, in fact, for \( p, q \in E^\infty \cup \Lambda^* \), \( p \sim q \iff \phi_\infty(p) \sim \phi_\infty(q) \).

**Proof.** Observe that if \( \mu \nu \in E^\infty \cup \Lambda^* \) where \( \mu \in E^* \) then \( \phi_\infty(\mu \nu) = \phi(\mu)\phi_\infty(\nu) \) and \( \phi_\infty(\nu) \in F^\infty \).
Now let $p = e_1e_2\ldots, q = f_1f_2\ldots \in E^\infty$ s.t. $p \sim q, \exists i,j$ s.t. $e_{i+r} = f_{j+r} \forall r \in \mathbb{N}$. Thus $p = \mu_1\nu$ and $q = \mu_2\nu$ where $\mu_1 = e_1e_2\ldots e_i$, $\mu_2 = f_1f_2\ldots f_j$ and $\nu = e_{i+1}e_{i+2}\ldots = f_{j+1}f_{j+2}\ldots$. Therefore $\phi_\infty(p) = \phi(\mu_1)\phi_\infty(\nu)$ and $\phi_\infty(q) = \phi(\mu_2)\phi_\infty(\nu)$ implying $\phi_\infty(p) \sim \phi_\infty(q)$.

If $p,q \in \Lambda^*$ s.t. $p \sim q$ then $t(p) = t(q)$ is a singular vertex. Hence $\phi_\infty(t(p)) = \phi_\infty(t(q))$. Moreover $\phi_\infty(p) = \phi(p)\phi_\infty(t(p))$ and $\phi_\infty(q) = \phi(q)\phi_\infty(t(q))$ implying $\phi_\infty(p) \sim \phi_\infty(q)$.

Hence $\phi_\infty(p) \sim \phi_\infty(q)$ whenever $p \sim q$.

To prove the converse, suppose $\phi_\infty(p_1) \sim \phi_\infty(p_2)$ for $p_1, p_2 \in E^\infty \cup \Lambda^*$. Claim: If $p_1 \in \Lambda^*$ then $p_2 \in \Lambda^*$. If $p_1 \in E^\infty$ then $p_2 \in E^\infty$.

We prove the claim. Suppose $p_1 \in \Lambda^*$. Thus $\phi_\infty(p_1) = \phi(p_1)e_1e_2\ldots$ where $e_1e_2\ldots$ is the tail added to $t(p_1)$ in the construction of $F$, i.e., $t(p_1) = o(e_1e_2\ldots)$. Therefore, $\phi_\infty(p_1) \sim e_2e_3\ldots$. Since $\phi_\infty(p_1) \sim \phi_\infty(p_2)$ we get $\phi_\infty(p_2) \sim e_2e_3\ldots$. If $p_2 \in E^\infty$ then $p_2 = f_1f_2\ldots$ for some $f_1, f_2, \ldots \in E^1$ Therefore $\phi_\infty(p_2) = \phi(f_1)\phi(f_2)\ldots$. Implying that $\phi(f_1)\phi(f_2)\ldots \sim e_2e_3\ldots$. But for each $i \geq 1$ we have $o(\phi(f_i)), t(\phi(f_i)) \in E^0$ and by the construction of $F$, for each $i \geq 2 o(e_i), t(e_i) \notin E^0$. Therefore $\phi(f_1)\phi(f_2)\ldots$ can not be equivalent to the path $e_2e_3\ldots$, which is a contradiction. Therefore $p_2 \in \Lambda^*$. The second statement follows from the contrapositive of the first statement by symmetry.

Now suppose $p_1 \in \Lambda^*$. By the above claim, $p_2 \in \Lambda^*$. Thus $\phi_\infty(p_2) = \phi(p_2)g_1g_2\ldots$, where $g_1g_2\ldots$ is the tail added to $t(p_2)$ in the construction of $F$. Hence $\phi_\infty(p_2) \sim g_1g_2\ldots$. Since $\phi_\infty(p_1) \sim e_1e_2\ldots$ we get $e_1e_2\ldots \sim g_1g_2\ldots$. Notice that (by the construction of $F$) $t(p_1)$ is the only entrance of $e_1e_2\ldots$ and $t(p_2)$ is the only entrance to $g_1g_2\ldots$. Therefore either $t(p_1) = o(g_1)$ for some $i$ or $t(p_2) = o(e_i)$ for some $i$. WLOG suppose $t(p_1) = o(g_1), t(e_i) \notin E^0$ and $t(p_1) = t(p_2) = o(g_1)$ is the only vertex in the path $g_1g_2\ldots$ that belongs to $E^0$ and $t(p_1) \in E^0$. Hence $t(p_1) = t(p_2)$. Therefore $p_1 \sim p_2$.

If $p_1 \in E^\infty$ then, by the above claim, $p_2 \in E^\infty$. Notice that $\forall v \in \phi_\infty(p_1)^0$ either $v \in E^0$ (hence in $p_1^0$) or $\exists w \in p_1^0$ s.t. $v \sim w$. Since $\phi_\infty(p_1) \sim \phi_\infty(p_2), \phi_\infty(p_1) = \mu_1\nu$, for some $\mu_1, \mu_2 \in F^\ast$ and some $\nu \in F^\infty$, and $t(\mu_1) = t(\mu_2) = o(\nu)$. Extending $\mu_1$ and $\mu_2$ along $\nu$, if needed, we may assume that $t(\mu_1) \in E^0$, i.e., $\mu_1, \mu_2 \in \{\beta \in F^\ast : o(\beta), t(\beta) \in E^0\}, \nu \in \{\beta \in F^\infty : o(\beta) \in E^0\}$ and $t(\mu_1) = t(\mu_2) = o(\nu)$. Therefore $\mu_i = \phi(\delta_1), \nu = \phi_\infty(\gamma)$ for some $\delta_1 \in E^\ast$ and some $\gamma \in E^\infty \cup \Lambda^\ast$. Implying $p_i = \phi_\infty^{-1}(\mu_i\nu) = \phi_\infty^{-1}(\phi(\delta_1)\phi_\infty(\gamma)) = \phi_\infty^{-1}(\phi(\delta_1\gamma)) = \phi_\infty^{-1}(\phi_\infty(\delta_1\gamma)) = \delta_1\gamma$. Thus $p_1 \sim p_2$.

**Lemma 5.4.** Let $F$ be a disingularization of a directed graph $E$. Then $E$ satisfies condition $(M)$ if $F$ satisfies condition $(M)$.

**Proof.** We will prove the only if side. Recall that $F$ has no singular vertices. Suppose $F$ does not satisfy condition $(M)$. Let $v \in F^0$ and $[\lambda] \in F^\infty / \sim$ s.t. the number of representatives of $[\lambda]$ that begin with $v$ is infinite.

If $v \notin E^0$ then $v$ is on an added tail to a singular vertex $v_0$ of $E$ and there is (only one) path from $v_0$ to $v$. Then the number of representatives of $[\lambda]$ that begin with $v$ (in the graph $F$) is equal to the number of representatives of $[\lambda]$ that begin with $v_0$ (in the graph $F$). If the latter is finite then the first is finite, hence we might assume that $v \in E^0$. Moreover, every path in $F^\infty$ is equivalent to one whose origin lies in $E^0$. Therefore we might choose a representative $\lambda$ with $o(\lambda) \in E^0$.

The set of representatives of $[\lambda]$ that begin with $v$ is $\{\beta \in F^\infty : o(\beta) = v and \lambda \sim \beta\}$. Since $\phi_\infty$ is bijective, $\phi_\infty^{-1}\{\beta \in F^\infty : o(\beta) = v and \lambda \sim \beta\}$ is an infinite subset of $E^\infty \cup \Lambda^\ast$. As $\phi_\infty^{-1}$ preserves origin and the equivalence, $\phi_\infty^{-1}\{\beta \in F^\infty : o(\beta) = v and \lambda \sim \beta\}$
Let $E$ be a directed graph. $C^*(E)$ is liminal iff $E$ satisfies condition (M).

**Proof.** Let $F$ be a desingularization of $E$. $E$ satisfies condition (M) iff $F$ satisfies condition (M) iff $C^*(F)$ is liminal iff $C^*(E)$ is liminal. □

6. THE LARGEST LIMINAL IDEAL OF $C^*$-ALGEBRAS OF GENERAL GRAPHS.

In this section we will identify the largest liminal ideal of $C^*(E)$ for a general graph $E$.

We will, once again, follow the construction in [4]. For a hereditary saturated subset $H$ of $E^0$, define:

$$B_H := \{v \in \Lambda : 0 < |o^{-1}(v) \cap t^{-1}(E^0 \setminus H)| < \infty\}.$$ 

Thus $B_H$ is the set of infinite emitters that point into $H$ infinitely often and out of $H$ at least once but finitely often. In [4] it is proven that the set $\{(H, S) : H$ is a hereditary saturated subset of $E^0$ and $S \subseteq B_H\}$ is a lattice with the lattice structure $(H, S) \leq (H', S')$ iff $H \subseteq H'$ and $S \subseteq H' \cup S'$. Observe that, since $B_H \cap H = \emptyset$, $(H, S) \leq (H, S')$ iff $S \subseteq S'$.

Let $E$ be a directed graph and $F$ be a disingularization of $E$, let $H$ be a hereditary saturated subset of $E^0$, and let $S \subseteq B_H$. Following the construction in [4], define:

$$\tilde{H} := H \cup \{v_n \in F^0 : v_n \text{ is on a tail added to a vertex in } H\}.$$ 

Thus $\tilde{H}$ is the smallest hereditary saturated subset of $F^0$ containing $H$.

Let $S \subseteq B_H$, and let $v_0 \in S$. Let $v_i = t(e_i)$, where $e_1e_2\ldots$ is the tail added to $v_0$ in the construction of $F$. If $N_{v_0}$ is the smallest non-negative integer s.t. $t(e_j) \in H$, $\forall j \geq N_{v_0}$, we have that $\forall j \geq N_{v_0}$, $v_j$ emits exactly two edges: one pointing to $v_{j+1}$ and one pointing to a vertex in $H$. Define

$$T_{v_0} := \{v_n \in F^0 : v_n \text{ is on a tail added to } v_0 \text{ and } n \geq N_{v_0}\}$$

and

$$H_S := \tilde{H} \cup \bigcup_{v_0 \in S} T_{v_0}.$$
Lemma 3.2 states that the above construction defines a lattice isomorphism from the lattice \( \{(H, S) : H \text{ is a hereditary saturated subset of } E^0 \text{ and } S \subseteq B_H \} \) onto the lattice of hereditary saturated subsets of \( F^0 \).

Let \( \{t_e, q_e\} \) be a generating Cuntz-Krieger \( F \)-family and \( \{s_v, p_v\} \) be the canonical generating Cuntz-Krieger \( E \)-family. Let \( p = \sum_{v \in E^0} q_v \). Since \( C^*(E) \) and \( C^*(F) \) are Morita equivalent via the imprimitivity bimodule \( pC^*(F) \) it follows that the Rieffel correspondence between ideals in \( C^*(F) \) and ideals in \( C^*(E) \) is given by the map \( I \mapsto \text{p}I\text{p} \).

Let \( H \) be a hereditary saturated subset of \( E^0 \) and \( S \subseteq B_H \). For \( v_0 \in S \), define

\[
p^{H}_{v_0} := p_{v_0} - \sum_{\alpha(e) = v_0 \atop t(e) \notin H} s_e s_e^*\]

and

\[
I_{(H, S)} := \text{the ideal generated by } \{p_v : v \in H\} \cup \{p^H_{v_0} : v_0 \in S\}.
\]

Proposition 3.3 states that if \( E \) satisfies condition (K): every vertex of \( E \) lies on either no circuits or at least two circuits, then \( pI_{HS}p = I_{(H, S)} \). The assumption that \( E \) satisfies condition (K) was only used to make sure that all the ideals of \( F \) are gauge invariant. Therefore whenever \( I \) is a gauge invariant ideal of \( C^*(E) \), it follows that \( I \) is of the form \( I_{(H, S)} \) for some hereditary saturated subset \( H \) of \( E^0 \) and for some \( S \subseteq B_H \).

To identify the largest liminal ideal of \( C^*(E) \), first recall that the largest liminal ideal of a \( C^* \)-algebra is invariant under automorphisms. Therefore the largest liminal ideal of \( C^*(E) \) has to be of the form \( I_{(H, S)} \) for some hereditary saturated subset \( H \) of \( E^0 \) and a subset \( S \) of \( B_H \). We set \( H_1 = \{v \in E^0 : \forall |\lambda| \in (E^\infty \cup \Lambda^+_E)/\sim, \text{ the number of representatives of } |\lambda| \text{ that begin with } v \text{ is finite}\} \). Since \( \text{Graph}(H_1) \) satisfies condition (M), we see that the ideal \( I_{H_1} = I_{(H_1, \emptyset)} \) is a subset of the largest liminal ideal of \( C^*(E) \). While it is true that \( H = H_1 \), as illustrated in the following example, it is not automatically clear what \( S \) can be.
Example 6.1. Consider the following graphs:

Let $I_{(H,S)}$ denote the largest liminal ideal of $C^*(E)$. It is not hard to see that $H_1 = \{u_2, u_3, \ldots\}$, $H_2 = \{v_2, v_3, \ldots\} \cup \{v\}$, $H_3 = \{w_2, w_3, \ldots\} \cup \{w\}$, $B_{H_1} = \{u_0\}$, $B_{H_2} = \{v_0\}$, and $B_{H_3} = \{w_0\}$. A careful computation shows that $S_1 = \{u_0\}$, $S_2 = \{v_0\}$ where as $S_3 = \emptyset$. Notice that we can reach from $v_0$ to $v$ in an infinite number of ways, but not through $H$. We can reach from $w_0$ to $w$ through $H$ in an infinite number of ways.

For a hereditary and saturated subset $H$ of $E^0$ and $v \in B_H$, we define $D(v,H) := \{e \in \Delta(v) : t(e) \notin H\}$, that is, $D(v,H)$ is the set of all edges that begin with $v$ and point outside of $H$. Notice that $D(v,H)$ is a non empty finite set.

Proposition 6.2. Let $E$ be a directed graph and $H = \{v \in E^0 : \forall [\lambda] \in (E^\infty \cup \Lambda^*_E)/\sim, \text{ the number of representatives of } [\lambda] \text{ that begin with } v \text{ is finite}\}$. Let $S = \{v \in B_H : E(v; D(v,H)) \text{ satisfies condition } (M)\}$. Then $I_{(H,S)}$ is the largest liminal ideal of $C^*(E)$.

Proof. That $H$ is hereditary and saturated is proven in Proposition 6.1. Let $I_{(H',S')}^*$ be the largest liminal ideal of $C^*(E)$ and let $F$ be a desingularization of $E$. In what follows, we will prove that $I_{(H,S)} = I_{(H',S')}$. To do that we will prove: $H \subseteq H'$, $H' \subseteq H$, $S \subseteq S'$ and $S' \subseteq S$, in that order.

We will prove that $H \subseteq H'$. Notice that $I_{H',S'}^*$ is the largest liminal ideal of $C^*(F)$. Using Proposition 2.3 we get that $H''_S = \{v \in F^0 : \forall [\lambda] \in F^\infty / \sim, \text{ the number of representatives of } [\lambda] \text{ that begin with } v \text{ is finite}\}.

Let $G_H = \text{Graph}(H)$. Notice that by [5] Proposition 2.1 $I_H$ is Morita equivalent to $C^*(G_H)$. Since $G_H$ satisfies condition $(M)$, by Theorem 3.10 $C^*(G_H)$ is liminal. Therefore $I_H = I_{(H,\emptyset)}$ is liminal. By the maximality of $I_{(H',S')}$, $I_{(H,\emptyset)} \subseteq I_{(H',S')}$, implying that $H \subseteq H'$.

We will prove that $H' \subseteq H$. Let $G_{H'} = \text{Graph}(H')$. $I_{H'} = I_{(H',\emptyset)} \subseteq I_{(H',S')}$. Hence $I_{H'}$ is liminal. And by [5] Proposition 2.1, $I_{H'}$ is Morita equivalent to $C^*(G_{H'})$. Therefore $G_{H'}$ satisfies condition $(M)$.

Let $v \in H'$. If $\beta \in E'^*$ with $o(\beta) = v$ then, since $H'$ is hereditary, $\beta \in G_{H'}$. Now let $[\lambda] \in (E^\infty \cup \Lambda^*)/\sim$. If $\gamma$ is a representative of $[\lambda]$ with $o(\gamma) = v$ then
\( \gamma \in G_{H}^{0} \cup \Lambda_{H}^{*} \). Therefore the set of representatives of \([\lambda]\) that begin with \(v\) is 
\( \{ \beta \in E^{\infty} \cup \Lambda^{*} : o(\beta) = v, \beta \sim \gamma \} \) is finite, since \(G_{H}\) satisfies condition \((M)\). Therefore \(v \in H\), hence \(H' \subseteq H\).

Next we will prove that \(S \subseteq S'\). Let \(v_{0} \in S\). To show that \(v_{0} \in S'\) we will show that \(v_{n} \in H_{S'}\) whenever \(n \geq N_{v_{0}}\), i.e., \(\forall n \geq N_{v_{0}}\), and \(\forall [\lambda] \in F_{\infty}/\sim\), the number of representatives of \([\lambda]\) that begin with \(v_{n}\) is finite.

Let \(n \geq N_{v_{0}}\) and let \([\lambda] \in F_{\infty}/\sim\). If \([\lambda]\) has no representative that begins with \(v_{n}\) then there is nothing to prove. Let \(\gamma\) be a representative of \([\lambda]\) with \(o(\gamma) = v_{n}\).

First suppose that \(\gamma^{0} = \{v_{n}, v_{n+1}, \ldots\}\), i.e., \(\gamma\) is the part of the tail added to \(v_{0}\) in the construction of \(F\). Then \(\{\beta \in F_{\infty} : o(\beta) = v_{n}, \beta \sim \gamma \} = \{\gamma\}\) since \(\gamma\) has no entry other than \(v_{n}\). Therefore the number of representatives of \([\lambda]\) that begin with \(v_{n}\) is 1.

Now suppose \(\gamma^{0}\) contains a vertex not in \(\{v_{n}, v_{n+1}, \ldots\}\). Recalling that \(\forall k \geq N_{v_{0}}, v_{k}\) emits exactly two edges, one pointing to \(v_{k+1}\) and one pointing to a vertex in \(H\), let \(w \in H\) be the first such vertex, i.e., \(w \in H \cap \gamma^{0}\) is chosen so that whenever \(v \geq w\) and \(v \in \{v_{n}, v_{n+1}, \ldots\}\) then \(v \notin H\). If \(p\) is the (only) path from \(v_{0}\) to \(v_{n}\) and \(q\) is the path from \(v_{n}\) to \(w\) along \(\gamma\), then \(\gamma = q_{\mu}\) for some \(\mu \in F_{\infty}\) with \(o(\mu) = w\). Moreover, \(\phi^{-1}_{\infty}(p_{\gamma}) = \phi^{-1}(pq_{\mu}) = \phi^{-1}(pq)\phi^{-1}_{\infty}(\mu)\) and \(\phi^{-1}(pq)\) is an edge in \(E_{1}\) with \(o(\phi^{-1}(pq)) = v_{0}\) and \(t(\phi^{-1}(pq)) = w\). Therefore \(\phi^{-1}_{\infty}(p_{\gamma}) \in E(v_{0}; D(v_{0}, H))_{\infty} \cup \Lambda_{E_{1}(v_{0}; D(v_{0}, H))}^{*}\). The set of representatives of \([\lambda]\) that begin with \(v_{n}\) is \(\{\beta \in F_{\infty} : o(\beta) = v_{n}, \beta \sim p_{\gamma} \}\). If \(\beta \in F_{\infty}\) is any representative of \([\lambda]\) that begins with \(v_{n}\), then \(\beta \sim p_{\gamma} \sim \mu\). Hence \(\beta^{0}\) has to contain a vertex in \(H\). Applying the same argument on \(p\beta\) we see that \(p_{\beta}\) is a representative of \([\lambda]\), \(o(p_{\beta}) = v_{0}\) and \(\phi^{-1}_{\infty}(p_{\beta}) \in E(v_{0}; D(v_{0}, H))_{\infty} \cup \Lambda_{E_{1}(v_{0}; D(v_{0}, H))}^{*}\).

Hence \(\{|\beta \in F_{\infty} : o(\beta) = v_{n}, \beta \sim p_{\gamma} \}| = |\{p_{\beta} \in F_{\infty} : p_{\beta} \sim p_{\gamma} \}| = |\{\phi^{-1}_{\infty}(p_{\beta}) \in E(v_{0}; D(v_{0}, H))_{\infty} \cup \Lambda_{E_{1}(v_{0}; D(v_{0}, H))}^{*} : \phi^{-1}_{\infty}(p_{\beta}) \sim \phi^{-1}_{\infty}(p_{\gamma})\}|\) which is finite, since \(E(v_{0}; D(v_{0}, H))_{\infty} \cup \Lambda_{E_{1}(v_{0}; D(v_{0}, H))}^{*}\) satisfies condition \((M)\).

In each case, the number of representatives of \([\lambda]\) that begin with \(v_{n}\) is finite, implying that \(v_{n} \in H_{S'}\). Therefore \(v_{0} \in S'\).

Finally we will prove that \(S' \subseteq S\). Let \(v_{0} \in S'\). We will show that \(E(v_{0}; D(v_{0}, H))_{\infty} \cup \Lambda_{E_{1}(v_{0}; D(v_{0}, H))}^{*}\) satisfies condition \((M)\). Let \(\lambda \in E(v_{0}; D(v_{0}, H))_{\infty} \cup \Lambda_{E_{1}(v_{0}; D(v_{0}, H))}^{*}\). If a vertex \(v \neq v_{0}\) is in \(E(v_{0}; D(v_{0}, H))_{0}\) then it is in \(H\), hence, by the definition of \(H\), the number of representatives of \([\lambda]\) that begin with \(v\) is finite. What remains is to show that the number of representatives of \([\lambda]\) that begin with \(v_{0}\) is finite. Noting that \(v_{N_{v_{0}}} \in H_{S'}\), for any \(\gamma \in F_{\infty}\) the set \(\{|\mu \in F_{\infty} : o(\mu) = v_{N_{v_{0}}}, \mu \sim \gamma \}\) is finite. In particular, the set \(\{|\mu \in F_{\infty} : o(\mu) = v_{N_{v_{0}}}, \mu \sim \phi(\lambda)\}|\) is finite.

Let \(\beta = e_{1}e_{2} \ldots \in E(v_{0}; D(v_{0}, H))_{\infty} \cup \Lambda_{E_{1}(v_{0}; D(v_{0}, H))}^{*}\) with \(o(\beta) = v_{0}\). Then \(\phi(\beta) = \phi(e_{1})\phi(\phi(e_{2}e_{3} \ldots )) \in F_{\infty}\) and \(o(\phi(e_{1})) = v_{0}, t(\phi(e_{1})) = o(\phi(e_{2}e_{3} \ldots )) \in H\). Let \(p\) be the path from \(v_{0}\) to \(v_{N_{v_{0}}}\).

We will first show that the set \(\{|e_{1}e_{2} \ldots \sim \lambda and v_{N_{v_{0}}} \in \phi(e_{1})^{0} \}|\) is finite.

If \(v_{N_{v_{0}}} \in \phi(e_{1})^{0}\) then \(\phi(e_{1})^{0} = \mu_{\lambda}\) for some \(\mu \in F_{\infty}\) with \(o(\mu) = v_{N_{v_{0}}}\). Hence \(\{|e_{1}e_{2} \ldots \in E(v_{0}; D(v_{0}, H))_{\infty} \cup \Lambda_{E_{1}(v_{0}; D(v_{0}, H))}^{*} : e_{1}e_{2} \ldots \sim \lambda and v_{N_{v_{0}}} \in \phi(e_{1})^{0} \}|\) is finite. Therefore \(\{|\phi(\phi(e_{1}e_{2} \ldots )) \in F_{\infty} : \phi(\phi(e_{1}e_{2} \ldots )) \sim \phi(\lambda), o(e_{1}) = v_{0}, t(e_{1}) = H and v_{N_{v_{0}}} \in \phi(e_{1})^{0} \}|\) is finite.

\[ \{\mu \in F_{\infty} : o(\mu) = v_{N_{v_{0}}} and \mu \sim \phi(\lambda)\}\]
We will next show that the set \( \{ e_1 e_2 \ldots \in E(v_0; D(v_0, H)) \cup \Lambda^*_E(v_0; D(v_0, H)) : e_1 e_2 \ldots \sim \lambda \text{ and } v_{N_{v_0}} \notin \phi(e_1) \} \) is finite.

Observe that the set \( E := \{ e \in E : t(e) \in H \text{ and } v_{N_{v_0}} \notin \phi(e) \} \) is finite. And \( \forall e \in E \) the set \( \{ \beta \in E^\infty \cup \Lambda^*_E : o(\beta) = t(e), \beta \sim \lambda \} \) is finite, since \( \{ t(e) : e \in E \} \subseteq H \).

Hence \( |\{ e_1 e_2 \ldots \in E(v_0; D(v_0, H)) \cup \Lambda^*_E(v_0; D(v_0, H)) : e_1 e_2 \ldots \sim \lambda, v_{N_{v_0}} \notin \phi(e_1) \}| = |\{ e_1 e_2 \ldots \in E^\infty \cup \Lambda^*_E : e_1 e_2 \ldots \sim \lambda, t(e_1) \in K \}| = |\{ \beta \in E^\infty \cap \Lambda^*_E : o(\beta) \in K, \beta \sim \lambda \}| \) which is finite, as the set is a finite union of finite sets.

Therefore the set \( \{ \beta \in E(v_0; D(v_0, H)) \cup \Lambda^*_E(v_0; D(v_0, H)) : \beta \sim \lambda \} \) is a union of two finite sets, hence is finite. Thus \( v_0 \in S \). It follows that \( S \subseteq S' \) concluding the proof. \( \square \)

7. Type I Graph \( C^*\)-Algebras.

In this section we will characterize Type I graph \( C^*\)-algebras.

We say that an edge \( e \) reaches a path \( p \) if \( t(e) \) reaches \( p \), i.e. if there is a path \( q \) s.t. \( o(q) = t(e) \) and \( q \sim p \).

If \( v \) is a sink then we regard \( \{ v \} \) as a tree.

For an infinite path \( \lambda \), we use \( N_\lambda \) to denote the number of vertices of \( \lambda \) that emit multiple edges that get back to \( \lambda \).

**Lemma 7.1.** Let \( E \) be a directed graph with:

1. Every circuit in \( E \) is either terminal or transitory.
2. For any \( \lambda \in E^\infty \), \( N_\lambda \) is finite.

Then \( \exists v \in E^0 \) s.t. \( E(v) \) is either a terminal circuit or a tree.

**Proof.** Let \( z_1 \in E^0 \). If \( E(z_1) \) is neither a terminal circuit nor a tree, then there exists \( z_2 \neq z_1 \) s.t. \( z_1 \) and \( z_2 \) do not belong to a common circuit, and there are (at least) two paths from \( z_1 \) to \( z_2 \).

Notice that \( \exists w_1 \in E^0 \) s.t. \( z_1 \geq w_1 \geq z_2 \) and \( w_1 \) emits multiple edges that reach \( z_2 \) (perhaps is \( z_1 \) itself). Observe that, by construction, \( z_2 \neq z_1 \).

Inductively: if \( E(z_i) \) is neither a terminal circuit nor a tree, then there exists \( z_{i+1} \neq z_i \) s.t. \( z_i \) and \( z_{i+1} \) do not belong to a common circuit, and there are (at least) two paths from \( z_i \) to \( z_{i+1} \). Again \( \exists w_i \in E^0 \) s.t. \( z_i \geq w_i \geq z_{i+1} \) and \( w_i \) emits multiple edges that reach \( z_{i+1} \). Observe also that \( z_{i+1} \neq z_i \) and hence \( w_{i+1} \neq w_i \).

This process has to end, for otherwise, let \( \lambda \in E^\infty \) be s.t. \( \forall i \in \mathbb{N}, w_i \in \lambda^0 \). Then \( \lambda \) has infinite number of vertices that emit multiple edges that reach \( \lambda \), namely \( w_1, w_2, \ldots \) contradicting the assumption. \( \square \)

**Remark 7.2.** For \( \lambda, \gamma \in E^\infty \), if \( \lambda = p_\gamma \), for some \( p \in E^* \), then \( N_\gamma \leq N_\lambda \leq N_\gamma + |p^0| \), where \( |p^0| \) is the number of vertices in \( p \), which is finite since \( p \) is a finite path. Therefore, \( N_\lambda \) is finite iff \( N_\gamma \) is finite. Moreover, if \( \lambda \sim \mu \) then \( \lambda = p_\gamma, \mu = q_\gamma \) for some \( p, q \in E^* \) and some \( \gamma \in E^\infty \). Hence \( N_\lambda \) is finite iff \( N_\gamma \) is finite iff \( N_\mu \) is finite.

**Theorem 7.3.** Let \( E \) be a graph. \( C^*(E) \) is type I iff:

1. Every circuit in \( E \) is either terminal or transitory.
2. For any \( \lambda \in E^\infty \), \( N_\lambda \) is finite.

We will first prove the following lemma.
Lemma 7.4. Let $E$ be a directed graph and $F$ be a desingularization of $E$. $E$ satisfies (1) and (2) of Theorem 7.3 iff $F$ satisfies (1) and (2) of Theorem 7.3.

Proof. That $E$ satisfies (1) iff $F$ satisfies (1) follows from the fact that the map $\phi$ of Remark 7.2 preserves circuits.

Now we suppose that $E$ satisfies (1), equivalently $F$ satisfies (1).

Suppose $E$ fails to satisfy (2). Let $\lambda \in E^\infty$ s.t. $N_\lambda$ is infinite. Suppose $v \in \lambda^0$ and $p$ is a path s.t. $o(p) = v$, $t(p) \in \lambda^0$. Let $q$ be the path along $\lambda$ s.t. $o(q) = v$, $t(q) = t(p)$, then $\exists \beta \in E^*$ and $\mu \in E^\infty$ s.t. $\lambda = \beta \mu$. Since $\phi$ preserves origin and terminus, $o(\phi(p)) = v = o(\phi(q))$ and $t(\phi(p)) = t(\phi(q))$. And $\phi_\infty(\lambda) = \phi_\infty(\beta \mu) = \phi(\beta) \phi(q) \phi_\infty(\mu)$. Since $\phi$ is bijective, $\phi(p) = \phi(q)$ iff $p = q$. Therefore, if $v$ (as a vertex in $E$) emits multiple edges that get back to $\lambda$ then it (as a vertex in $F$) emits multiple edges that get back to $\phi(\lambda)$, implying that $N_{\phi_\infty(\lambda)}$ is infinite. Hence $F$ does not satisfy (2).

To prove the converse, suppose $E$ satisfies (2). Let $\lambda \in E^\infty$. If $o(\lambda) \notin E^0$, then $o(\lambda)$ is on a path extended from a singular vertex. Using Remark 7.2 we may extend $\lambda$ (backwards) and assume that $o(\lambda) \in E^0$. Let $\gamma = \phi_\infty^{-1}(\lambda) \in E^\infty \cup E^*$. First suppose $\gamma \in \Lambda^*$. Then $v_0 := t(\gamma) \sim \gamma$. Hence $\phi_\infty(v_0) \sim \phi_\infty(\gamma) = \lambda$. Using Remark 7.2 we may assume that $\lambda = \lambda_\infty(v_0)$, that is, $\lambda$ is the path added to $v_0$ in the construction of $F$. Thus each vertex of $\lambda$ emits exactly two edges: one pointing to a vertex in $\lambda$ (the next vertex) and one pointing to a vertex in $E^0$. Since $v_0$ is the only entry to $\lambda$, if a vertex $v$ of $\lambda$ emits multiple edges that get back to $\lambda$ then $v \geq v_0$. And since $F$ satisfies (1), there could be at most one such vector, for otherwise $v_0$ would be on multiple circuits. Hence $N_\lambda$ is at most 1.

Now suppose $\gamma \in E^\infty$. Since $E$ satisfies (2), $N_\gamma$ is finite. Going far enough on $\gamma$, let $w \in \gamma^0$ be s.t. no vertex of $\gamma$ that $w$ can reach to emits multiple edges that get back to $\gamma$. Let $\mu \in E^*$, $\beta \in E^\infty$ be s.t. $\gamma = \mu \beta$ and $t(\mu) = w = o(\beta)$, then $\lambda = \phi(\mu) \phi_\infty(\beta)$. Hence $\lambda \sim \phi_\infty(\beta)$. Moreover, each $v \in \beta^0$ emits exactly one edge that gets to $\beta$, which, in fact, is an edge of $\beta$.

Let $v \in \beta^0$ and $p \in F^*$ be s.t. $o(p) = v$, $t(p) \in \lambda^0$. Extending $p$, if needed, we may assume that $t(p) \in \beta^0$. Let $q \in F^*$ be the path along $\lambda$ s.t. $o(q) = v$ and $t(q) = t(p)$. Since $\phi$ is bijective, $\phi^{-1}(p) = \phi^{-1}(q)$ iff $p = q$. But $v$ can get to $\beta$ in only one way, therefore $\phi^{-1}(p) = \phi^{-1}(q)$, implying that $p = q$. Thus $v$ emits (in the graph $F$) only one edge that gets to $\lambda$. Hence for each vertex $v \in \phi_\infty(\beta)$, if $v \in E^0$ then $v$ emits only one edge that gets to $\lambda$.

Now let $v \in \phi_\infty(\beta) \setminus E^0$. Then $v$ is on a path extended from a singular vertex, say $v_0$. Since $w \geq v_0$ by the previous paragraph, $v_0$ emits only one edge that gets to $\lambda$. Let $p$ be the (only) path from $v_0$ to $v$. Let $\mu, \nu \in F^*$ be s.t. $t(\mu), t(\nu) \in \lambda^0$ and $o(\mu) = o(\nu) = v$. Extending $\mu$ or $\nu$ along $\lambda$, if needed, we can assume that $t(\mu) = t(\nu)$. Again extending them along $\lambda$ we can assume that $t(\mu) = t(\nu) \in \beta^0$. Observe that $o(p \mu) = o(p \nu) = v_0$ and $t(p \mu) = t(p \nu) \in \beta^0$. Therefore $o(\phi^{-1}(p \mu)) = o(\phi^{-1}(p \nu)) = v_0$ and $t(\phi^{-1}(p \mu)) = t(\phi^{-1}(p \nu)) \in \beta^0$. But each vertex in $\beta$ emits exactly one edge that gets to $\beta$, i.e., there is exactly one path from $v_0$ to $t(\phi^{-1}(p \mu))$ hence $p \mu = p \nu$. Therefore, $\mu = \nu$. That is, $v$ emits only one edge that gets to $\lambda$. Therefore $N_{\phi_\infty(\beta)} = 0$. By Remark 7.2 we get $N_\lambda$ is finite. \hfill $\Box$

Remark 7.5. $E$ satisfies (2) of Theorem 7.3 does not imply that its desingularization $F$ satisfies (2) of Theorem 7.3 as illustrated by the following example.
Example 7.6. If $E$ is the $O_{\infty}$ graph (one vertex with infinitely many loops), which clearly satisfies (2) of Theorem 7.3, then its disingularization does not satisfy (2) of Theorem 7.3. The disingularization looks like this:

Proof of Theorem 7.3. We first prove the if side. We will first assume that $E$ is a row-finite graph with no sinks. Let $(I_\rho)_{0 \leq \rho \leq \alpha}$ be an increasing family of ideals of $C^*(E)$ s.t.

(a) $I_0 = \{0\}, \ C^*(E)/I_\alpha$ is antiliminal.
(b) If $\rho \leq \alpha$ is a limit ordinal, $I_\rho = \bigcup_{\beta < \rho} I_\beta$
(c) If $\rho < \alpha$, $I_{\rho+1}/I_\rho$ is a liminal ideal of $C^*(E)/I_\rho$ and is non zero.

We prove that $I_\alpha = C^*(E)$. Since $I_\alpha$ is the largest Type I ideal of $C^*(E)$, it is gauge invariant. Let $H$ be a hereditary saturated subset of $E^0$ s.t. $I_\alpha = I_H$. If $H \neq E^0$ then let $F = F(E \setminus H)$. Clearly $F$ satisfies (1) and (2) of the theorem. Using Lemma 7.1 let $v_0 \in F^0$ be s.t. $K = \{v \in F^0 : v_0 \geq v\}$ is the set of vertices of either a terminal circuit or a tree. Let $G = \text{Graph}(K)$, thus $G$ is either a terminal circuit or a tree. By [5] Proposition 2.1 $I_K$ is Morita equivalent to $C^*(G)$. Moreover $G$ satisfies condition (M), hence by Theorem 3.10 $C^*(G)$ is liminal. And $I_K$ is an ideal of $C^*(F) \cong C^*(E)/I_\alpha$ contradicting the assumption that $C^*(E)/I_\alpha$ is antiliminal. It follows that $I_\alpha = C^*(E)$. Therefore $C^*(E)$ is Type I.

For an arbitrary graph $E$, let $F$ be a desingularization of $E$. By Lemma 7.1 $F$ satisfies (1) and (2) of the theorem. And by the above argument, $C^*(F)$ is Type I. Therefore $C^*(E)$ is Type I.

To prove the converse, suppose $E$ has a non-terminal non-transitory circuit, that is, $E$ has a vertex that is on (at least) two circuits. Let $v_0$ be a vertex on two circuits, say $\alpha$ and $\beta$. Let $F$ be the sub-graph containing (only) the edges and vertices of $\alpha$ and $\beta$.

$$A := \overline{\text{span}}\{s_\mu s_\nu^* : \mu, \nu \text{ are paths made by } \alpha \text{ and } \beta \text{ or just } v_0\}$$

is a $C^*$-subalgebra of $C^*(F)$. But $A \cong O_2$ which is not Type I. Hence $C^*(F)$ is not Type I. By Remark 7.3 $C^*(E)$ has a sub-algebra whose quotient is not Type I therefore $C^*(E)$ is not Type I.

Suppose now that each circuit in $E$ is either terminal or transitory and $\exists \lambda \in E^\infty$ s.t. $N_\lambda$ is infinite. Let $v_\lambda = o(\lambda)$. Let $G = E(v)$. If $v$ is a vertex s.t. $V(v)$ does not intersect $\lambda^0$, we can factor $C^*(G)$ by the ideal generated by $\{v\}$. This process gets rid of any terminal circuits of $G$. By Lemma 7.3 $C^*(G)$ is not Type I, implying that $C^*(E)$ is not Type I.

Next we will identify the largest Type I ideal of the $C^*$-algebra of a graph $E$. For a vertex $v$ of $E$ (respectively $F$), recall that $E(v)$ (respectively $F(v)$) denotes the sub-graph of $E$ (respectively $F$) that $v$ can ‘see’.

We begin with the following lemma.

Lemma 7.7. Let $E$ be a directed graph, $F$ a desingularization of $E$ and $v \in E^0$. Then $F(v)$ is a desingularization of $E(v)$.
Proof. Let \( u \in E(v)^0 = \{ w \in E : v \geq w \} \). Let \( p \) be a path in \( E \) with \( o(p) = v \), and \( t(p) = u \). Then \( \phi(p) \) is a path in \( F \) with \( o(\phi(p)) = v \), and \( t(\phi(p)) = u \). Hence \( u \in F(v)^0 \), implying that \( E(v)^0 \subseteq F(v)^0 \). Clearly \( F(v) \) has no singular vertices. Let \( v_0 \in E(v)^0 \) be a singular vertex. If \( v_0 \) is a vertex on the path added to \( v_0 \) in the construction of \( F \), since \( F(v)^0 \) is hereditary and \( v_0 \in F(v)^0 \), we get \( v_n \in F(v)^0 \). Therefore the path added to \( v_0 \) is in the graph \( F(v) \). To show that \( F(v) \) has exactly the vertices needed to desingularize \( E(v) \), let \( w \in F(v)^0 \). Let \( p \) be a path in \( F(v) \) with \( o(p) = v \) and \( t(p) = w \). If \( w \in E^0 \) then \( \phi^{-1}(p) \in E^* \) and \( o(\phi^{-1}(p)) = v \) and \( t(\phi^{-1}(p)) = w \). Therefore \( v \geq w \) in the graph \( E \). Hence \( w \in E(v)^0 \). If \( w \notin E^0 \) then there is a singular vertex, say \( v_0 \in E^0 \) s.t. \( w \) is on the path added to \( v_0 \) in the construction of \( F \). Since the path from \( v_0 \) to \( w \) has no other entry than \( v_0 \) and since \( v \geq w \), we must have \( v \geq v_0 \). Hence \( w \) is on the the graph obtained when \( E(v) \) is desingularized. Therefore \( F(v) \) is a desingularization of \( E(v) \). □

The following corollary follows from Lemma \[ \ref{lem:desingularization} \] and Lemma \[ \ref{lem:hereditary} \].

**Corollary 7.8.** Let \( E \) be a directed graph, \( F \) a desingularization of \( E \) and \( v \in E^0 \). Then \( E(v) \) satisfies (1) and (2) of Theorem \[ \ref{thm:main} \] iff \( F(v) \) satisfies (1) and (2) of Theorem \[ \ref{thm:main} \].

The next proposition identifies the largest Type I ideal of the \( C^* \)-algebra of a row-finite graph \( E \) with no sinks. The first part of the proposition, which will be needed later, is written for a general graph as it is proven without the need of the property that \( E \) is row-finite and has no sinks.

**Proposition 7.9.** Let \( E \) be a directed graph and

\[
H = \{ v \in E^0 : E(v) \text{ satisfies (1) and (2) of Theorem } \ref{thm:main} \}.
\]

Then

(a) \( H \) is a hereditary saturated subset of \( E^0 \).

(b) If \( E \) is a row-finite graph with no sinks then \( I_H \) is the largest Type I ideal of \( C^*(E) \).

Proof. We first prove (a). That \( H \) is hereditary follows from \( v \geq w \implies E(v) \supseteq E(w) \). We prove that \( H \) is saturated. Suppose \( v \in E^0 \) and \( \{ w \in E^0 : v \geq w \} \subseteq H \). Let \( \Delta(v) = \{ e \in E^1 : o(e) = v \} \). Note that \( v \in \Delta(v) \), \( t(e) \in H \). If there is a circuit at \( v \), i.e., \( v \) is a vertex of some circuit, then \( v \geq v \), implying that \( v \in H \). Suppose there are no circuits at \( v \). If there is a vertex \( w \in E(v)^0 \) on a circuit then it is in \( E(t(e))^0 \) for some \( e \in \Delta(v) \). But \( t(e) \in H \), hence \( w \) can not be on multiple circuits, i.e, \( E(v) \) has no non-terminal and non-transitory circuits. Hence \( E(v) \) satisfies (1) of Theorem \[ \ref{thm:main} \]. Let \( \lambda \in E(v) \) then \( \exists e \in \Delta(v) \) and \( \beta \in E(t(e)) \) s.t. \( \lambda \sim \beta \). Since \( t(e) \in H \), \( N_{\beta} \) is finite. Using Remark \[ \ref{rem:finite} \] we get that \( N_{\lambda} \) is finite. Therefore \( v \in H \). Hence \( H \) is saturated.

To prove (b), suppose \( E \) is row-finite with no sinks. Let \( F = Graph(H) \). Clearly \( F \) satisfies (1) and (2) of Theorem \[ \ref{thm:main} \]. Hence by Theorem \[ \ref{thm:main} \], \( C^*(F) \) is Type I. Moreover, by \[ \ref{prop:hereditary} \] Proposition 2.1], \( I_H \) is Morita equivalent to \( C^*(F) \). Hence \( I_H \) is Type I. Let \( I \) be the largest Type I ideal of \( C^*(E) \), then \( I_H \subseteq I \). Since \( I \) is gauge invariant, \( I = I_K \) for some hereditary saturated subset \( K \) of \( E^0 \) that includes \( H \). We will prove that \( K \subseteq H \). Let \( G = Graph(K) \). Since \( I_K \) is Morita equivalent to \( C^*(G) \), \( C^*(G) \) is Type I, hence \( G \) satisfies (1) and (2) of Theorem \[ \ref{thm:main} \]. Let \( v \in K \), since \( E(v) \subseteq G \), \( E(v) \) satisfies (1) and (2) of Theorem \[ \ref{thm:main} \]. Therefore \( v \in H \), hence \( K \subseteq H \). □
The next proposition generalizes Proposition \[7.9\].

**Proposition 7.10.** Let \( E \) be a directed graph and
\[
H = \{ v \in E^0 : E(v) \text{ satisfies (1) and (2) of Theorem 7.3} \}.
\]
Then \( I_{(H,B_H)} \) is the largest Type I ideal of \( C^*(E) \).

**Proof.** Let \( I_{(H',S')} \) be the largest Type I ideal of \( C^*(E) \) and let \( F \) be a desingularization of \( E \) then \( I_{H',S'} \) is the largest Type I ideal of \( C^*(F) \). From (b) of Proposition \[7.3\] we get that \( H' = \{ v \in E^0 : E(v) \text{ satisfies (1) and (2) of Theorem 7.3} \} \).

We will prove that \( H \subseteq H' \). Let \( G_H = \text{Graph}(H) \). Clearly \( G_H \) satisfies (1) and (2) of Theorem \[7.3\] hence \( C^*(G_H) \) is Type I. By [3] Proposition 2.1 \( I_H \) is Morita equivalent to \( C^*(G_H) \). Therefore \( I_H = I_{(H,\emptyset)} \) is Type I. By the maximality of \( I_{(H',S')} \), \( I_{(H,\emptyset)} \subseteq I_{(H',S')} \), implying that \( H \subseteq H' \).

We will prove that \( H' \subseteq H \). Let \( G_{H'} = \text{Graph}(H') \). \( I_{H'} = I_{(H',\emptyset)} \subseteq I_{(H',S')} \). Hence \( I_{H'} \) is liminal. By [3] Proposition 2.1 \( I_{H'} \) is Morita equivalent to \( C^*(G_{H'}) \), implying that \( C^*(G_{H'}) \) is liminal. Hence \( G_{H'} \) satisfies (1) and (2) of Theorem \[7.3\].

Let \( v \in H' \). Since \( H' \) is hereditary and \( E(v)^0 \subseteq H' \) it follows that \( E(v) \) is a sub-graph of \( G_{H'} \). Thus \( E(v) \) satisfies (1) and (2) of Theorem \[7.3\]. Therefore \( v \in H' \), hence \( H' \subseteq H \).

Since \( S' \subseteq B_H \), as \( H = H' \), it remains to prove that \( B_H \subseteq S' \). Let \( v_0 \in B_H \). To show that \( v_0 \in S' \) we will show that \( \forall n \geq N_{v_0}, v_n \in H^S \), i.e., \( F(v_n) \) satisfies (1) and (2) of Theorem \[7.3\]. Let \( n \geq N_{v_0} \) and suppose \( F(v_n) \) does not satisfy (1) of Theorem \[7.3\]. Let \( \alpha \) be a non-terminal and non-transitory circuit in \( F(v_n) \), and let \( \nu \in \alpha^0 \).

If \( v \) is on the infinite path added to \( v_0 \) in the construction of \( F \) then \( v_0 \) is in the circuit \( \alpha \). Notice that \( v_n \geq v \geq v_0 \). Recall that \( \forall k \geq N_{v_0} \) \( v_k \) emits exactly two edges one pointing to \( v_{k+1} \) and one pointing to a vertex in \( H \). Following along \( \alpha \), we get that \( v \geq w \) for some vertex \( w \in H \) of \( \alpha \). But \( H \) is hereditary, therefore \( v_0 \in H \), which contradicts to the fact that \( H \cap B_H = \emptyset \).

Suppose now that \( v \) is not on the infinite path added to \( v_0 \). Let \( p \) be a path from \( v_n \) to \( v \). \( p \) must contain a vertex, say \( w \), in \( H \). Notice that \( w \geq v \) which implies that \( v \in F(w) \). Since \( F(w)^0 \) is hereditary, \( \alpha \) is in the graph \( F(w) \). Hence \( F(w) \) contains a non-terminal and non-transitory circuit. Since \( w \in H \), \( E(w) \) satisfies (1) and (2) of Theorem \[7.3\]. But this contradicts to Corollary \[7.8\]. Therefore \( F(v_n) \) satisfies (1) of Theorem \[7.3\].

To prove that \( F(v_n) \) satisfies (2) of Theorem \[7.3\] let \( \lambda \in F(v_n)^\infty \). Either \( \lambda \) is on the tail added to \( v_0 \) on the construction of \( F \) or \( \lambda^0 \) contains a vertex in \( H \).

If \( \lambda \) is on the tail added to \( v_0 \) then \( N_\lambda = 0 \). Otherwise let \( w \in \lambda^0 \cap H \). Then \( \lambda = p \mu \) for some \( p \in F(v_n)^\infty \) and some \( \mu \in F(v_n)^\infty \) with \( o(p) = v_n \), \( t(p) = w = o(\mu) \). Implies that \( \lambda \sim \mu \). Since \( w \in H \), \( E(w) \) satisfies (1) and (2) of Theorem \[7.3\]. By Corollary \[7.8\] we get that \( F(w) \) satisfies (1) and (2) of Theorem \[7.3\]. Hence \( N_\mu \) is finite and Remark \[7.2\] implies that \( N_\lambda \) is finite. Therefore \( F(v_n) \) satisfies (2) of Theorem \[7.3\].

We have established that \( F(v_n) \) satisfies (1) and (2) of Theorem \[7.3\]. Therefore \( v_n \in H^S \), and hence \( B_H \subseteq S' \). This concludes the proof. \( \square \)

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