SUMSETS CONTAINED IN SETS OF UPPER BANACH DENSITY 1

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Abstract. Every set \( A \) of positive integers with upper Banach density 1 contains an infinite sequence of pairwise disjoint subsets \((B_i)_{i=1}^\infty\) such that \( B_i \) has upper Banach density 1 for all \( i \in \mathbb{N} \) and \( \sum_{i \in I} B_i \subseteq A \) for every nonempty finite set \( I \) of positive integers.

1. Upper Banach density

Let \( \mathbb{N}, \mathbb{N}_0, \) and \( \mathbb{Z} \) denote, respectively, the sets of positive integers, nonnegative integers, and integers. Let \( |S| \) denote the cardinality of the set \( S \). We define the interval of integers

\[ [x, y] = \{ n \in \mathbb{N} : x \leq n \leq y \}. \]

Let \( A \) be a set of positive integers. Let \( n \in \mathbb{N} \). For all \( u \in \mathbb{N}_0 \), we have

\[ |A \cap [u, u + n - 1]| \in [0, n] \]

and so

\[ f_A(n) = \max_{u \in \mathbb{N}_0} |A \cap [u, u + n - 1]| \]

exists. The upper Banach density of \( A \) is

\[ \delta(A) = \limsup_{n \to \infty} \frac{f_A(n)}{n}. \]

Let \( n_1, n_2 \in \mathbb{N} \). There exists \( u_1^* \in \mathbb{N}_0 \) such that, with \( u_1^* = u_1^* + n_1 \),

\[
\begin{align*}
    f_A(n_1 + n_2) &= |A \cap [u_1^*, u_1^* + n_1 + n_2 - 1]| \\
     &= |A \cap [u_1^*, u_1^* + n_1 - 1]| + |A \cap [u_1^* + n_1, u_1^* + n_1 + n_2 - 1]| \\
     &= |A \cap [u_2^*, u_2^* + n_1 - 1]| + |A \cap [u_2^*, u_2^* + n_2 - 1]| \\
     &\leq f_A(n_1) + f_A(n_2).
\end{align*}
\]

It is well known, and proved in the Appendix, that this inequality implies that

\[ \delta(A) = \lim_{n \to \infty} \frac{f_A(n)}{n} = \inf_{n \in \mathbb{N}} \frac{f_A(n)}{n}. \]

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2. An Erdős sumset conjecture

About 40 years ago, Erdős conjectured that if $A$ is a set of positive integers of positive upper Banach density, then there exist infinite sets $B$ and $C$ of positive integers such that $B + C \subseteq A$. This conjecture has not yet been verified or disproved.

The translation of the set $X$ by $t$ is the set

$$X + t = \{x + t : x \in X\}.$$ 

Let $B$ and $C$ be sets of integers. For every integer $t$, if $B' = B + t$ and $C' = C - t$, then

$$B' + C' = (B + t) + (C - t) = B + C.$$ 

In particular, if $C$ is bounded below and $t = \min(C)$, then $0 = \min(C')$ and $B' \subseteq B' + C'$. It follows that if $B$ and $C$ are infinite sets such that $B + C \subseteq A$, then, by translation, there exist infinite sets $B'$ and $C'$ such that $B' \subseteq A$ and $B' + C' \subseteq A$.

However, a set $A$ with positive upper Banach density does not necessarily contain infinite subsets $B$ and $C$ with $B + C \subseteq A$. For example, let $A$ be any set of odd numbers. For all sets $B$ and $C$ of odd numbers, the subset $B + C$ is a set of even numbers, and so $A \cap (B + C) = \emptyset$. Of course, in this example we have $B + C \subseteq A + 1$.

In this note we prove that if $A$ is a set of positive integers with upper Banach density $\delta(A) = 1$, then for every $h \geq 2$ there exist pairwise disjoint subsets $B_1, \ldots, B_h$ of $A$ such that $\delta(B_i) = 1$ for all $i = 1, \ldots, h$ and

$$B_1 + \cdots + B_h \subseteq A.$$ 

Indeed, Theorem 2 states an even stronger result.

There are sets $A$ of upper Banach density 1 for which no infinite subset $B$ of $A$ satisfies $2B \subseteq A + t$ for any integer $t$. A simple example is

$$A = \bigcup_{i=1}^{\infty} [4^i, 4^i + i - 1].$$ 

The set $A$ is the union of the infinite sequence of pairwise disjoint intervals

$$A_i = [4^i, 4^i + i - 1].$$ 

Let $t \in \mathbb{N}_0$. There exists $i_0(t)$ such that $4^i - i > t$ for all $i \geq i_0(t)$. If $b_i \in A_i$ for some $i \geq i_0(t)$, then

$$4^i + i + t < 2 \cdot 4^i \leq 2b_i < 2 \cdot 4^i + 2i < 4^{i+1} - 2t \leq 4^{i+1} - t$$

and so $2b_i \not\in 2A + t$. If $B$ is an infinite subset of $A$, then for infinitely many $i$ there exist integers $b_i \in B \cap A_i$, and so $2B \not\subseteq A + t$ for all $t \in \mathbb{Z}$.

There are very few results about the Erdős conjecture. In 1980, Nathanson [9] proved that if $\delta(A) > 0$, then for every $n$ there is a finite set $C$ with $|C| = n$ and a subset $B$ of $A$ with $\delta(B) > 0$ such that $B + C \subseteq A$. In 2015, Di Nasso, Goldbring, Jin, Leth, Lupini, and Mahlburg [3] used nonstandard analysis to prove that the Erdős conjecture is true for sets $A$ with upper Banach density $\delta(A) > 1/2$. They also proved that if $\delta(A) > 0$, then there exist infinite sets $B$ and $C$ and an integer $t$ such that

$$B + C \subseteq A \cup (A + t).$$ 

It would be of interest to have purely combinatorial proofs of the results of Di Nasso, et al.
For related work, see Di Nasso [1, 2], Gromov [4], Hegyvári [5, 6], Hindman [7], and Jin [8].

3. Results

The following result is well known.

**Lemma 1.** A set of positive integers has upper Banach density 1 if and only if, for every \( d \), it contains infinitely many pairwise disjoint intervals of \( d \) consecutive integers.

**Proof.** Let \( A \) be a set of positive integers. If, for every positive integer \( d \), the set \( A \) contains an interval of \( d \) consecutive integers, then

\[
\max_{u \in \mathbb{N}_0} \left( \frac{|A \cap [u, u + d - 1]|}{d} \right) = 1
\]

and so

\[
\delta(A) = \lim_{d \to \infty} \max_{u \in \mathbb{N}_0} \left( \frac{|A \cap [u, u + d - 1]|}{d} \right) = 1.
\]

Suppose that, for some integer \( d \geq 2 \), the set \( A \) contains no interval of \( d \) consecutive integers. For every \( u \in \mathbb{N}_0 \), we consider the interval \( I_{u,n} = [u, u + n - 1] \). By the division algorithm, there are integers \( q \) and \( r \) with \( 0 \leq r < d \) such that

\[
|I_{u,n}| = n = qd + r
\]

and

\[
q = \frac{n - r}{d} > \frac{n}{d} - 1.
\]

For \( j = 1, \ldots, q \), the intervals of integers

\[
I^{(j)}_{u,n} = [u + (j - 1)d, u + jd - 1]
\]

and

\[
I^{(q+1)}_{u,n} = [u + qd, u + n - 1]
\]

are pairwise disjoint subsets of \( I_{u,n} \) such that

\[
I_{u,n} = \bigcup_{j=1}^{q+1} I^{(j)}_{u,n}.
\]

We have

\[
A \cap I_{u,n} = \bigcup_{j=1}^{q+1} (A \cap I^{(j)}_{u,n})
\]

If \( A \) contains no interval of \( d \) consecutive integers, then, for all \( j \in [1, q] \), at least one element of the interval \( I^{(j)}_{u,n} \) is not an element of \( A \), and so

\[
|A \cap I^{(j)}_{u,n}| \leq |I^{(j)}_{u,n}| - 1.
\]
It follows that
\[|A \cap I_{u,n}| = \sum_{j=1}^{q+1} |A \cap I^{(j)}_{u,n}| \leq \sum_{j=1}^{q} \left( |I^{(j)}_{u,n}| - 1 \right) + |I^{(q+1)}_{u,n}| \]
\[= \sum_{j=1}^{q+1} |I^{(j)}_{u,n}| - q = |I_{u,n}| - q = n - q \]
\[< n - \frac{n}{d} + 1 = \left( 1 - \frac{1}{d} \right) n + 1.\]

Dividing by \(n = |I_{u,n}|\), we obtain
\[\max_{u \in \mathbb{N}_0} \frac{|A \cap I_{u,n}|}{n} \leq 1 - \frac{1}{d} + \frac{1}{n},\]
and so
\[\delta(A) = \lim_{n \to \infty} \max_{u \in \mathbb{N}_0} \frac{|A \cap I_{u,n}|}{n} \leq 1 - \frac{1}{d} < 1\]
which is absurd. Therefore, \(A\) contains an interval of \(d\) consecutive integers for every \(d \in \mathbb{N}\).

To prove that \(A\) contains infinitely many intervals of size \(d\), it suffices to prove that if \([u, u + d - 1] \subseteq A\), then \([v, v + d - 1] \subseteq A\) for some \(v \geq u + d\). Let \(d' = u + 2d\). There exists \(u' \in \mathbb{N}\) such that
\([u', u' + d' - 1] = [u', u' + u + 2d - 1] \subseteq A.\]
Choosing \(v = u' + u + d\), we have \(v \geq u + d\) and
\([v, v + d - 1] \subseteq [u', u' + u + 2d - 1] \subseteq A.\]
This completes the proof. \(\square\)

Let \(\mathcal{F}(S)\) denote the set of all finite subsets of the set \(S\), and let \(\mathcal{F}^*(S)\) denote the set of all nonempty finite subsets of \(S\). We have the fundamental binomial identity
\[(1) \quad \mathcal{F}^*([1, n+1]) = \mathcal{F}^*([1, n]) \cup \{ \{n+1\} \cup J : J \in \mathcal{F}([1, n]) \}.

**Theorem 1.** Let \(A\) be a set of positive integers that has upper Banach density 1. For every sequence \((\ell_j)_{j=1}^{\infty}\) of positive integers, there exists a sequence \((b_j)_{j=1}^{\infty}\) of positive integers such that
\[b_{j+1} \geq b_j + \ell_j\]
for all \(j \in \mathbb{N}\), and
\[\sum_{j \in J} [b_j, b_j + \ell_j - 1] \subseteq A\]
for all \(J \in \mathcal{F}^*(\mathbb{N})\).

**Proof.** We shall construct the sequence \((b_j)_{j=1}^{\infty}\) by induction. For \(n = 1\), choose \(b_1 \in A\) such that \([b_1, b_1 + \ell_1 - 1] \subseteq A\).

Suppose that \((b_j)_{j=1}^{n}\) is a finite sequence of positive integers such that \(b_{j+1} \geq b_j + \ell_j\) for \(j \in [1, n-1]\) and
\[(2) \quad \sum_{j \in J} [b_j, b_j + \ell_j - 1] \subseteq A\]
for all \(J \in \mathcal{F}^*(\mathbb{N})\).
for all \( J \in \mathcal{F}^*([1, n]) \). By Lemma 1 there exists \( b_{n+1} \in A \) such that 
\[
\sum_{j=1}^{n+1} (b_j + \ell_j) - 1 \leq b_{n+1} \geq b_n + \ell_n
\]
and
\[
\left[ b_{n+1}, \sum_{j=1}^{n+1} (b_j + \ell_j) - 1 \right] \subseteq A.
\]
It follows that
\[
[ b_{n+1}, b_{n+1} + \ell_{n+1} - 1 ] \subseteq A.
\]

Let \( J \in \mathcal{F}([1, n]) \). If
\[
a \in \sum_{j \in \{n+1\} \cup J} [b_j, b_j + \ell_j - 1]
\]
then
\[
b_{n+1} \leq a \leq (b_{n+1} + \ell_{n+1} - 1) + \sum_{j \in J} (b_j + \ell_j - 1)
\]
and so \( a \in A \) and
\[
\sum_{j \in \{n+1\} \cup J} [b_j, b_j + \ell_j - 1] \subseteq \left[ b_{n+1}, \sum_{j=1}^{n+1} (b_j + \ell_j) - 1 \right] \subseteq A.
\]

Relations (1), (2), and (3) imply that
\[
\sum_{j \in \{n+1\} \cup J} [b_j, b_j + \ell_j - 1] \subseteq A
\]
for all \( J \in \mathcal{F}^*([1, n+1]) \). This completes the induction. \( \square \)

**Theorem 2.** Every set \( A \) of positive integers that has upper Banach density 1 contains an infinite sequence of pairwise disjoint subsets \( (B_i)_{i=1}^{\infty} \) such that \( B_i \) has upper Banach density 1 for all \( i \in \mathbb{N} \) and
\[
\sum_{i \in I} B_i \subseteq A
\]
for all \( I \in \mathcal{F}^*(\mathbb{N}) \).

**Proof.** Let \( (\ell_j)_{j=1}^{\infty} \) be a sequence of positive integers such that \( \lim_{j \to \infty} \ell_j = \infty \), and let \( (b_j)_{j=1}^{\infty} \) be a sequence of positive integers that satisfies Theorem 1 (For simplicity, we can let \( \ell_j = j \) for all \( j \)). Let \( (X_i)_{i=1}^{\infty} \) be a sequence of infinite sets of positive integers that are pairwise disjoint. For \( i \in \mathbb{N} \), let
\[
B_i = \bigcup_{j \in X_i} [b_j, b_j + \ell_j - 1].
\]
The set \( B_i \) contains intervals of \( \ell_j \) consecutive integers for infinitely many \( \ell_j \), and so \( B_i \) has upper Banach density 1.
Let \( I \in \mathcal{F}^*(\mathbb{N}) \). If
\[
a \in \sum_{i \in I} B_i \subseteq A
\]
then for each \( i \in I \) there exists \( a_i \in B_i \) such that \( a = \sum_{i \in I} a_i \). If \( a_i \in B_i \), then there exists \( j_i \in X_i \) such that
\[
x_i \in [b_{j_i}, b_{j_i} + \ell_{j_i} - 1].
\]
We have \( J = \{ j_i : i \in I \} \in \mathcal{F}^*(\mathbb{N}) \) and
\[
a \in \sum_{j_i \in J} [b_{j_i}, b_{j_i} + \ell_{j_i} - 1] \subseteq A.
\]
This completes the proof. \( \square \)

**Theorem 3.** Let \( A \) be a set of integers that contains arbitrarily long finite arithmetic progressions with bounded differences. There exist positive integers \( m \) and \( r \), and an infinite sequence of pairwise disjoint sets \( (B_i)_{i=1}^\infty \) such that \( B_i \) has upper Banach density 1 for all \( i \in \mathbb{N} \) and
\[
m \ast \sum_{i \in I} B_i + r \subseteq A
\]
for all \( I \in \mathcal{F}^*(\mathbb{N}) \).

**Proof.** If the differences in the infinite set of finite arithmetic progressions contained in \( A \) are bounded by \( m_0 \), then there exists a difference \( m \leq m_0 \) that occurs infinitely often. It follows that there are arbitrarily long finite arithmetic progressions with difference \( m \). Because there are only finitely many congruence classes modulo \( m \), there exists a congruence class \( r \) (mod \( m \)) such that \( A \) contains arbitrarily long sequences of consecutive integers in the congruence class \( r \) (mod \( m \)). Thus, there exists an infinite set \( A' \) such that
\[
m \ast A' + r \subseteq A
\]
and \( A' \) contains arbitrarily long sequences of consecutive integers. Equivalently, \( A' \) has Banach density 1. By Theorem 2 the sequence \( A' \) contains an infinite sequence of pairwise disjoint subsets \( (B_i)_{i=1}^\infty \) such that \( B_i \) has upper Banach density 1 for all \( i \in \mathbb{N} \) and
\[
\sum_{i \in I} B_i \subseteq A'
\]
for all \( I \in \mathcal{F}^*(\mathbb{N}) \). It follows that
\[
m \ast \sum_{i \in I} B_i + r \subseteq m \ast A' + r \subseteq A
\]
for all \( I \in \mathcal{F}^*(\mathbb{N}) \). This completes the proof. \( \square \)

**Appendix A. Subadditivity and Limits**

A real-valued arithmetic function \( f \) is **subadditive** if
\[
f(n_1 + n_2) \leq f(n_1) + f(n_2)
\]
for all \( n_1, n_2 \in \mathbb{N} \).

The following result is sometimes called **Fekete’s lemma**.
Lemma 2. If $f$ is a subadditive arithmetic function, then $\lim_{n \to \infty} f(n)/n$ exists, and

$$\lim_{n \to \infty} \frac{f(n)}{n} = \inf_{n \in \mathbb{N}} \frac{f(n)}{n}.$$ 

Proof. It follows by induction from inequality (4) that

$$f(n_1 + \cdots + n_q) \leq f(n_1) + \cdots + f(n_q)$$

for all $n_1, \ldots, n_q \in \mathbb{N}$. Let $f(0) = 0$. Fix a positive integer $d$. For all $q, r \in \mathbb{N}_0$, we have

$$f(qd + r) \leq qf(d) + f(r).$$

By the division algorithm, every nonnegative integer $n$ can be represented uniquely in the form $n = qd + r$, where $q \in \mathbb{N}_0$ and $r \in [0, d - 1]$. Therefore,

$$\frac{f(n)}{n} = \frac{f(qd + r)}{n} \leq \frac{qf(d) + f(r)}{n} = \frac{f(d)}{d} + \frac{f(r)}{n}.$$

Because the set $\{f(r) : r \in [0, d - 1]\}$ is bounded, it follows that

$$\limsup_{n \to \infty} \frac{f(n)}{n} \leq \limsup_{n \to \infty} \left( \frac{f(d)}{d} + \frac{f(r)}{n} \right) = \frac{f(d)}{d}$$

for all $d \in \mathbb{N}$, and so

$$\limsup_{n \to \infty} \frac{f(n)}{n} \leq \inf_{d \in \mathbb{N}} \frac{f(d)}{d} \leq \liminf_{d \to \infty} \frac{f(d)}{d} = \liminf_{n \to \infty} \frac{f(n)}{n}.$$

This completes the proof. \qed

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