ON ERGODIC PROPERTIES OF SOME LÉVY-TYPE PROCESSES

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Abstract. In this note we prove some sufficient conditions for ergodicity of a Lévy-type process, such that on the test functions the generator of the respective semigroup is of the form

\[ Lf(x) = a(x)f'(x) + \int_{\mathbb{R}} (f(x + u) - f(x) - \nabla f(x) \cdot u 1_{|u| \leq 1}) \nu(x, du), \quad f \in C^2_c(\mathbb{R}). \]

Here \( \nu(x, du) \) is a Lévy-type kernel and \( a(\cdot) : \mathbb{R} \to \mathbb{R} \). We consider the case when the tails are of polynomial decay as well as the case when the decay is (sub)-exponential. For the proof the Foster-Lyapunov approach is used.

1. Introduction

Consider an operator

\[ Lf(x) = a(x)f'(x) + \int_{\mathbb{R}} (f(x + u) - f(x) - \nabla f(x) \cdot u 1_{|u| \leq 1}) \nu(x, du), \quad f \in C^2_c(\mathbb{R}), \]

where \( C^2_c(\mathbb{R}) \) is the class of twice continuously differentiable functions with compact support, \( a(\cdot) : \mathbb{R} \to \mathbb{R} \) and the kernel \( \nu(x, du) \) satisfies

\[ \sup_x \int_{\mathbb{R}} (1 \wedge |u|^2) \nu(x, du) < \infty. \]

Under certain conditions (see Section 2 below) one can show that the martingale problem for \( (L, C^2_c(\mathbb{R})) \) is well posed and its solution is a Markov process possessing the Feller property (i.e. the respective semigroup of operators \( (P_t)_{t \geq 0} \), \( P_tf(x) := \mathbb{E}^x f(X_t), \quad f \in C^\infty_\infty(\mathbb{R}) \), preserves the space \( C^\infty_\infty(\mathbb{R}) \) of continuous functions vanishing at infinity). The respective process \( X \) is often called a Lévy-type process, because locally its structure resembles the structure of a Lévy process, e.g. a process with independent stationary increments.

The aim of this note is to investigate the ergodicity in total variation of the process \( X_t \) related to (1.1) and to describe the speed of convergence to the respective invariant measure. In order to investigate this problem, we need a certain set up. Following the framework of [Ku17], if the transition probability kernel satisfies the local Dobrushin condition and if the so-called extended generator of \( (P_t)_{t \geq 0} \) satisfies the Lyapunov type condition with some Lyapunov function, then one can reduce the continuous time setting to the discrete one (cf. [Ku17, Th.2.8.9, Th.3.2.3]), and apply the ergodic theorems for the skeleton chain \( X^h := \{X_{nh}, n \in \mathbb{Z}_+\} \). Since the ergodicity of the skeleton chain implies the ergodicity of the initial Markov process with the

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same rates, we derive the desired statement. This problem in the context of Lévy-driven SDEs is investigated in [Ku17], see also [KP19]. On the other hand, not any Lévy-type process can be derived easily as a solution to some Lévy-driven SDE, but nevertheless such process are of interests, which motivated the current research. Knowing the speed of convergence to the ergodic distribution allows us to use simulation methods (see, for example, [B10]) in order to approximate this distribution in a “long run” having the distribution of the initial Lévy-type process.

In this note we consider for simplicity the one-dimensional situation. We believe that under certain control on the distortion in different directions of the Lévy kernel similar result should hold in the multi-dimensional settings (see [Ku17] for the respective results on the Lévy-driven SDEs). However, when the Lévy type kernel \( \nu(x, du) \) is not regular enough, the choice of the Lyapunov function and the proof of the Lyapunov type inequality is not so obvious. Therefore we postpone this problem for further investigations.

The paper is organized as follows. In Section 2 we recall the notion of ergodicity and discuss the assumptions. Then we give the definition of the full generator and explain which functions are in its domain. Further, we recall the Dobrushin condition and the definitions of the Lyapunov function and of the Lyapunov type equation. This is the framework necessary for the investigation of ergodicity. In Section 3 we formulate our results, e.g. we provide the conditions under which our process is ergodic and get the speed of convergence to the invariant measure. The speed of convergence heavily relies on the order of the moments of the kernel \( \nu(x, du) \), therefore we consider two types of conditions: when we have polynomial decay of the tails and the “(sub)-exponential” decay. Proofs are given in Section 4.

2. Settings

In order to make the paper self-contained, we provide below the necessary notions and give some overview of the existing results.

Recall that a Markov process \((X_t)_{t \geq 0}\) on \(\mathbb{R}\) with transition probability kernel \(P_t(x, dy)\) is called \textit{ergodic} if there exists an invariant probability measure \(\pi(\cdot)\) such that

\[
\lim_{t \to \infty} \|P_t(x, \cdot) - \pi(\cdot)\|_{TV} = 0
\]

for any \(x \in \mathbb{R}\). Here \(\|\cdot\|_{TV}\) is the total variation norm.

There are classical results on Markov chains (cf. [Ku17, Ch.2]) establishing ergodicity under some additional assumptions. As we mentioned in Section 1, having a continuous time Markov process one can construct the so-called \textit{skeleton chain}, establish the ergodicity for this chain, which in turn implies the ergodicity of the initial Markov process with the same convergence rates. Therefore, we formulate a few definitions and results for a Markov chain and then indicate the “return” route to the initial process.

Let \(X\) be a Markov chain on a state metric space \((X, \rho)\), where \(\rho\) is the metric on \(X\), with transition kernel \(P(x, dy)\).

The first condition we need for the ergodicity is a \textit{local Dobrushin condition} for \(X\). We say that a chain \(X\) satisfies the local Dobrushin condition (cf. [Ku17] p. 42) on a measurable set \(B \subset X \times X\) if

\[
\sup_{x,y \in B} \|P(x, \cdot) - P(y, \cdot)\|_{TV} < 2.
\]
Condition (2.2) is not always easy to check, but there are sufficient conditions which are much more clear. Suppose that the probability kernel $P(x, dy)$ is of the form

$$P(x, dy) = P_c(x, dy) + P_d(x, dy),$$

where

i) $P_c(x, dy), P_d(x, dy)$ are non-negative kernels;

ii) there exists a measure $m$ on $X$ such that $P_c(x, dy) \ll m(dy)$;

iii) for a given point $x^* \in X$ the mapping

$$x \mapsto \frac{P_c(x, dy)}{m(dy)} \in L_1(X, m)$$

is continuous at $x^*$ and $P_c(x^*, X) > 0$.

Assumptions i)--ii) are sufficient for the Dobrushin condition (2.2) to hold true, see [Ku17, Prop.2.9.1].

The second condition we need is the Lyapunov type condition. Suppose that there exist

- a norm-like function $V : X \to [1, \infty)$ (i.e. $V(x) \to \infty$ as $|x| \to \infty$), $V$ is bounded on a compact $K$,
- some function $f : [1, \infty) \to (0, \infty)$, which admits a non-negative increasing and concave extension to $[0, \infty)$,
- a constant $C > 0$,

such that the following relations hold:

\begin{align}
(2.3) & \quad E_x V(X_1) - V(x) \leq -f(V(x)) + C, \quad x \in X, \\
(2.4) & \quad f \left(1 + \inf_{x \in K} V(x)\right) > 2C.
\end{align}

Then (cf. [Ku17 Th.2.8.6]) the chain $X$ satisfies the so-called coupling condition, which is the second counterpart for the required ergodicity. Such a function $V$ is called a Lyapunov function.

Dobrushin and Lyapunov conditions allow to prove that the chain $X$ is ergodic (cf. [Ku17 Th.2.7.2]), and find the convergence rates. Namely, the following result holds true (cf. [Ku17, 2.8.8]): The invariant probability measure $\pi$ is unique, satisfies

$$\int_R f(V(x)) \mu(dx) < \infty,$$

and for any $\delta \in (0, 1)$ there exist $c, \gamma > 0$ such that

\begin{align}
(2.5) & \quad \|P_n(x, \cdot) - \pi(\cdot)\|_{TV} \leq \psi^\delta(n) \left(f^\delta(V(x)) + \int_R f(V(x)) \mu(dx)\right).
\end{align}

Here

\begin{align}
(2.6) & \quad \psi(n) := \frac{1}{f(F^{-1}(\gamma n))}, \quad F(t) := \int_1^t dw f(w).
\end{align}
Therefore, in order to follow the strategy described above we need to find the assumptions on the continuous time Markov process $X_t$ related to (1.1), which guarantee (2.3) and (2.4).

Let us come back to the operator defined by (1.1). The problem how to relate an operator (1.1) and a (strong) Feller process was investigated a lot; see [Ja01], [Ja02], [G68], [Ko73], [Ko84], [MP92a], [MP92b], [Ja92], [Ja94], [JJ93], [Ne94], [KN97], [Ho98], [Ho00], [Ku19], [BKS20], [KK17]–[KKS20]. In [KKS20] we give an extensive overview of this problem. For example, in the series of papers [KK17]–[KKS20], it is shown (for “bounded coefficients”) that under certain regularity and boundedness assumptions on the kernel $\nu(x,du)$ the operator admitting the representation (1.1) extends to the generator of a strong Markov process $X = (X_t)_{t \geq 0}$, which has a strictly positive transition probability density, continuous in $x$. Hence, the assumptions i)–iii) are satisfied, and the respective transition probability kernel $P_t(x,dy)$ satisfies the Dobrushin condition. The case of unbounded drift is more complicated.

In the case of a Lévy-driven SDE we refer to [Ku17, Prop.3.4.7] for the Dobrushin condition; see also the recent paper [MZ22] (and the references therein) where the transition probability density of the process is constructed and the Dobrushin condition follows. Unfortunately, at the moment we do not know the reference to the result, which allows effectively check the Dobrushin condition for a general Lévy-type process.

At this point we would like to comment on the other standard references for the ergodicity conditions, e.g. on the works [MT93a, MT93d]. Namely, in this framework it is often required that the process $X_t$ is the Lebesgue irreducible $T$-model, see [MT93a, Ch.6], [T94, Th.7.1], also [BSW13, Cor.6.26]. This assumption is satisfied, if the process $X_t$ admits a strictly positive transition probability density and the respective semigroup possesses a $C_b$–Feller property. By the $C_b$-Feller property we mean that the respective semigroup $(T_t)_{t \geq 0}$ maps the space $C_b(\mathbb{R})$ of bounded continuous functions into $C_b(\mathbb{R})$. Note that if $X$ is a Feller process and $T_t \mathbb{1}_B \in C_b$, then $X$ has the $C_b$-Feller property, see [BSW13, Th.1.9]. This property follows, for example, from the results, proved in [KK17]–[KKS20], [BKS20], but again we account the problem how to show the existence of a positive continuous density in the case when the drift coefficient is unbounded.

_Through the paper we assume that $a(x)$ and $\nu(x,du)$ are such that the martingale problem for $(\mathcal{L},D_0)$ is well posed, and the respective Markov Feller process such that for any $t > 0$ the transition kernel $P_t(x,dy)$ satisfies the Dobrushin condition._

Let us discuss now the Lyapunov type condition for the initial process. Our aim is to construct the Lyapunov function such that (2.3) holds true. Since we start with a Markov process related to (1.1), the inequality we check is not actually (2.3), but

\[
\mathcal{L}V(x) \leq -f(V(x)) + C,
\]

where $V$ and $f$ have the same meaning as above, and $\mathcal{L}$ is the full generator of $(T_t)_{t \geq 0}$. Recall the “stochastic version” of a full generator, see [BSW13, (1.50)].

**Definition 2.1.** The full generator of a measurable contraction semigroup $(T_t)_{t \geq 0}$ on $C_\infty(\mathbb{R})$ is

\[
\hat{L} := \left\{ (f,g) \in (B(\mathbb{R}),B(\mathbb{R})) : M_t^{[f,g]} := f(X_t) - f(x) = \int_0^t g(X_s)ds, \ x \in \mathbb{R}, \ t \geq 0, \right. \\
\left. \text{is a local } \mathcal{F}_t^X := \sigma(X_s, s \leq t) \text{ martingale} \right\}.
\]
Since for $f \in D(L)$

$$T_t f(x) - f(x) = \int_0^t T_s Lf(x) \, ds \quad \forall t \geq 0, \ x \in \mathbb{R},$$

see [EK86, Ch.1, Prop.1.5], then by the strong Markov property and the Dynkin formula $(f,Lf) \subset \hat{L}$ for any $f \in D(L)$. Since for $f \in C^2(\mathbb{R})$

$$\left| \int_{|u| \leq 1} (f(x + u) - f(x) - f'(x)u) \nu(x,du) \right| \leq \max_{|u| \leq 1} f''(x + u) \int_{|u| \leq 1} u^2 \nu(x,du) < \infty,$$

we have $Lf(x) < \infty$ for $f \in \mathcal{D}_0$, where

$$\mathcal{D}_0 := \left\{ f \in C^2(\mathbb{R}) : \int_{|u| \geq 1} (f(x + u) - f(x)) \nu(x,du) < \infty \quad \forall x \in \mathbb{R} \right\}.$$

Using the localization procedure, Ito formula and taking the expectation, we derive that in this $(f,Lf) \in \hat{L}$. We denote this extension of $L$ to $\mathcal{D}_0$ by $\mathcal{L}$, and denote also by $\mathcal{L}_0$ the extension of the integral part of $L$ to $\mathcal{D}_0$.

Thus, if we manage to prove (2.7), then by [Ku17, Th.3.2.3] (2.3) holds true for the skeleton chain, and thus we have (2.5). Moreover, the same convergence rate will still hold for the initial Markov process:

$$\|P_t(x,\cdot) - \pi(\cdot)\|_{TV} \leq C_{\text{erg}}(x) \psi(t), \quad t \to \infty,$$

where

$$\psi(t) := \frac{1}{f(F^{-1}(\gamma t)))},$$

$f$ is the function for which the Lyapunov condition (2.7) holds true, $F$ is defined by (2.6), and $C_{\text{erg}}(x)$ is the respective constant (depending on $x$). In this case we call the initial process $X$ $\psi$-ergodic.

Let us briefly discuss some other existing results. Ergodicity of Markov processes is studied a lot and under various assumptions. We quote here only several results which are in a similar context. The results in some sense similar to ours were obtained in Wang [W08], see also [W13], and in the papers of Sandric [Sa13, Sa16a]. See Barky, Cattiaux, Guillin [BCG08] for the approach relying on functional inequalities, and Douc, Fort, Guillin [DFG09] for the condition of positive Harris recurrence and ergodicity, which in turn relies on Fort, Roberts [FR05], and Douc, Fort, Moulines, Soulier [DFMS04] for the case Markov chains, see also [DFMS18].

3. Results

Through the paper we assume that

(S) \quad $\nu(x,\cdot)$ is symmetric in the sense that $\nu(x,A) = \nu(x,-A)$, $A \in \mathcal{B}(\mathbb{R})$, $\{0\} \notin A$.

Denote by $N(x,u)$ the tails of $\nu$:

$$N(x,u) := \nu(x,(-\infty,u]), \quad u < 0.$$
We consider three cases. First we assume that the tails of $\nu$ have polynomial decay and

\[(N1)\quad \lambda^{-\delta(z)} \leq \liminf_{x \to \infty} \frac{N(z, \lambda x)}{N(z, x)} \leq \limsup_{x \to \infty} \frac{N(z, \lambda x)}{N(z, x)} \leq \lambda^{-\sigma(z)} \quad \forall \lambda \geq 1,
\]

for some $0 < \sigma \leq \sigma(z) \leq \delta(z) \leq \delta < \infty$. Note that this relation is equivalent to

\[(N1')\quad \lambda^{-\sigma(z)} \leq \liminf_{x \to \infty} \frac{N(z, \lambda x)}{N(z, x)} \leq \limsup_{x \to \infty} \frac{N(z, \lambda x)}{N(z, x)} \leq \lambda^{-\delta(z)} \quad 0 < \lambda < 1.
\]

Let

\[(3.2)\quad N^*(x, \lambda) := \limsup_{u \to \infty} \frac{N(x, \lambda u)}{N(x, u)} , \quad N_*(x, \lambda) := \liminf_{u \to \infty} \frac{N(x, \lambda u)}{N(x, u)} .
\]

We assume that

\[(N2)\quad \text{The functions } N_*(x, \lambda) \text{ and } N^*(x, \lambda) \text{ are continuous in } x \text{ for all } \lambda \geq 1 \text{ and } |x| \geq x_0.
\]

By \((N1)\) and \((N2)\) we have

\[(3.3)\quad \frac{1}{\lambda^\delta} \leq \inf_z N_*(z, \lambda) \leq \sup_z N^*(z, \lambda) \leq \frac{1}{\lambda^\sigma} .
\]

On the other hand, \((N1)\) implies the following estimates on $N(x, x)$: there exists $N_\sigma, N_\delta > 0$, such that

\[(3.4)\quad \frac{N_\delta}{x^\delta} \leq N(x, x) \leq \frac{N_\sigma}{x^\sigma} , \quad |x| \geq c .
\]

We need to set some additional notation. Let

\[(3.5)\quad N(x) := N(x, x) , \quad |x| \geq c ,
\]

\[(3.6)\quad N_{\text{max}} := \max_{|x| \geq 1} N(x, 1) .
\]

\[(3.7)\quad \nu_{\text{small}}(x) := \int_{|u| \leq 1} |u|^2 \nu(x, du) , \quad |x| \geq 1 ,
\]

\[(3.8)\quad \nu_{\text{small}} := \sup_{|x| \geq 1} \nu_{\text{small}}(x) < \infty .
\]

Denote

\[(3.9)\quad \begin{cases} C^{(1)} := 2 \text{if } \delta = \sigma & \sum_{k=1}^\infty C^{2k}_p \left( \frac{N_\delta}{2k-\delta} - \frac{N_\delta(2k-p)}{2k-p+\delta} \right) + \frac{2pN_\sigma}{\sigma-p} , \quad 1 < \sigma \leq \delta < 2 , \\ C^{(2)} := 2p(p-1)N_\delta , \quad \sigma = \delta = 2 \\ C^{(3)} := \frac{p(p-1)}{2} \left( \nu_{\text{small}} + 2N_{\text{max}} + \frac{4N_\delta}{\delta-2} \right) + \frac{2pN_\sigma}{\sigma-p} , \quad \sigma \geq 2 , \ \delta \neq 2 , \\ C^{(4)} := \frac{2pN_\delta}{2\sigma-p} , \quad 1 < \sigma < 2 , \ \delta \geq 2 , \end{cases}
\]

where $N_\sigma, N_\delta$ and $N_{\text{max}}$ are defined in \((3.4)\) and \((3.6)\), respectively. Finally, define the following scaling functions:
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(3.10) $\phi^{(1)}(x) := |x|^{p-\sigma}$, \hspace{1cm} $1 < \sigma \leq \delta < 2$;

(3.11) $\phi^{(2)}(x) := |x|^{p-2} \ln(1 + |x|)$, \hspace{1cm} $\sigma = \delta = 2$;

(3.12) $\phi^{(3)}(x) := |x|^{p-2}$, \hspace{1cm} $\sigma \geq 2$, $\delta > 2$;

(3.13) $\phi^{(4)}(x) := \phi^{(1)}(x)$, \hspace{1cm} $\sigma < 2$, $\delta \geq 2$.

In what follows $f$ and $\psi$ are related by (2.6), and $f$ varies from theorem to theorem.

**Theorem 3.1.** Suppose that $p \in (1, \sigma)$, and $f(x) \geq 1$, $x \in \mathbb{R}$, and one of the conditions below holds true with some $f$ (which may vary from case to case):

(3.14) $\limsup_{|x| \to \infty} \left( \frac{pa(x) \text{ sign}(x)|x|^{p-1} + f(|x|^p)}{\phi^{(i)}(x)} + C^{(i)} \right) < 0$, \hspace{1cm} $1 \leq i \leq 4$,

where $\phi^{(i)}(x)$ and $C^{(i)}$ are defined in (3.10)–(3.12) and (3.9), respectively. Then the respective process $X$ is $\psi$-ergodic.

In order to write the ergodicity condition in a simpler form, we impose one more assumption. Assume the following condition on the drift $a(x)$ (see also [Ku17], cf. (3.3.4)):

(3.15) $\limsup_{|x| \to \infty} a(x) \text{ sign}(x)|x|^\kappa \leq -A_\kappa \in (-\infty, 0)$.

**Corollary 3.2.** Suppose that the assumptions of Theorem 1 hold true and

(3.16) $\kappa + \min(\sigma, 2) > 1$.

Then the process $X$ is $\psi$-ergodic, where $\psi$ is related to $f(x) = Cx^{1+(\kappa-1)/p}$, $C > 0$, for $\kappa \in [-1, 1)$, and $f(x) = Ct$ for $\kappa \geq 1$, by (2.12).

**Remark 3.3.**

i) Observe that for $t \geq 1$ we have

(3.17) $F(t) = \begin{cases} \frac{\ln t}{C}, & \kappa \geq 1, \\ \frac{p}{C(1-\kappa)} \left( \frac{1}{t^{\frac{1}{p}}} - 1 \right), & \kappa \in [-1, 1). \end{cases}$

Consequently, for any $\gamma > 0$

(3.18) $\psi(t) = \begin{cases} C^{-1} e^{-C\gamma t}, & \kappa \geq 1, \\ C^{-1} \left( \frac{C^{\kappa}}{p} t + 1 \right)^{1-\frac{p}{\kappa}}, & \kappa \in [-1, 1). \end{cases}$

Thus, the rate of convergence in (2.11) is exponential, if $\kappa \geq 1$, and is of order $t^{\frac{p}{\kappa}-1}$, if $\kappa \in [-1, 1)$, provided that (3.10) holds true.

ii) Note that for $\kappa > 1$ the function $\frac{p}{C(1-\kappa)} \left( \frac{1}{t^{\frac{1}{p}}} - 1 \right)$ is bounded.

Consider now the light-tail case. We assume that there exist $\alpha > 0$, $\zeta \in (-1, 0]$ such that

(3.19) $\nu_{\alpha, \zeta, \text{large}}(x) := \int_{|u| \geq 1} e^{\alpha |u|^{1+\zeta}} \nu(x, du) < \infty$ \hspace{0.5cm} for any $x \in \mathbb{R}$. 

Suppose that
\begin{equation}
\nu_{\alpha,\zeta,\text{large}} := \sup_{|x| \geq 1} \nu_{\alpha,\zeta,\text{large}}(x) < \infty.
\end{equation}
Let
\begin{equation}
\phi^{(5)}(x) = |x|^\kappa + \zeta e^{|x|}, \quad C^{(5)} := 2^{-1/2} (1 + \zeta)^2 \left( e^{\beta \nu_{\text{small}}} + c_0 \nu_{\alpha,\zeta,\text{large}} \right),
\end{equation}
where \( c_0 \) is such that \( x^2 e^{\beta x^{1+\zeta}} \leq c_0 e^{\alpha x^{1+\zeta}}, x > 1 \).

**Theorem 3.4.** Suppose that \( \beta > 0, \zeta \in (-1,0) \) are such that (3.20) is satisfied. Suppose that \( f \) satisfies the inequality
\begin{equation}
\limsup_{|x| \to \infty} \left( \frac{\beta (1 + \zeta)a(x) \text{sign}(x)}{|x|^{\kappa}} + \frac{f(e^{|x|^{1+\zeta}})}{\phi^{(5)}(x)} + C^{(5)} \right) < 0.
\end{equation}
Then the process \( X \) is \( \psi \)-ergodic.

In order to write this result in a more convenient form assume (3.15).

**Corollary 3.5.** i) Suppose that \( \zeta = 0 \) and (3.15) holds with \( \kappa \geq 0 \). Then for \( \kappa > 0 \) the process \( X \) is \( \psi \)-ergodic with \( f(x) = Cx \), where \( C > 0 \) is arbitrary.

And for \( \kappa = 0 \) the process \( X \) is \( \psi \)-ergodic with the same \( f \) as above, provided that
\begin{equation}
- \beta A_0 + C + C^{(5)} < 0.
\end{equation}

ii) Suppose that \( \zeta \in (-1,\kappa] \) and (3.15) holds with \( \kappa \in (-1,0) \) and \( f(x) = Cx \left( \frac{1}{\beta} \ln x \right)^{\kappa+\zeta} \).

Then the process \( X \) is \( \psi \)-ergodic, provided that
\begin{equation}
- \beta (1 + \zeta) A_\kappa + C + C^{(5)} < 0.
\end{equation}
It is possible to build the corresponding function \( \psi \) from (2.11). For \( f(x) = Cx \) we have
\begin{equation*}
F(t) = C^{-1} \ln t, \quad \psi(t) = C^{-1} e^{-C\gamma t},
\end{equation*}
and for \( f(x) = C (\beta^{-1} \ln x)^{\kappa+\zeta} x \) we have
\begin{equation*}
F(t) = \frac{1 + \zeta}{C(1 - \kappa)} \beta^{\kappa+\zeta} (\ln t)^{\frac{\kappa+\zeta}{1+\zeta}},
\end{equation*}
and
\begin{equation*}
\psi(t) = C^{\frac{\kappa+\zeta}{\kappa-1}} \beta^{\kappa+\zeta} \left( \frac{\gamma(1 - \kappa)}{1 + \zeta} t \right)^{\frac{\kappa+\zeta}{1+\zeta}} e^{-\left( c_\beta^{-\frac{\kappa+\zeta}{\kappa-1}} \left( (1 - \kappa) t \right)^{\frac{\kappa+\zeta}{1+\zeta}} \right)^{\frac{1+\zeta}{\kappa+\zeta}}}.
\end{equation*}

**Remark 3.6.** It is possible to choose \( \zeta \) optimally. Note that \( \kappa < 0 \) and \( 1 + \zeta > 0 \), thus the power in the exponent is positive. It will be maximal, if we choose \( \zeta \) maximal possible, e.g. \( \zeta = \kappa \). Note also, that \( \frac{\kappa+\zeta}{\kappa-1} > 0 \), thus the decay is determined by the exponent.
4. Proofs

We begin with the proof of Theorem 3.1.

Recall the following series decomposition. Let $x, y$ be real, $p \in \mathbb{R}$. If $|x| > |y|$, then

\[(x + y)^p = \sum_{k=0}^{\infty} C_p^k x^{p-k} y^k.\]

Let $\phi : \mathbb{R} \to \mathbb{R}_+$, $\phi \in C^2(\mathbb{R})$, be defined as follows:

\[(4.1) \quad \phi(x) = |x| \quad \text{for } |x| > 1 \quad \text{and} \quad \phi(x) \leq |x| \quad \text{for } |x| \leq 1.\]

Define

\[(4.2) \quad V(x) = \phi^p(|x|), \quad p > 0, \quad x \in \mathbb{R};\]

note that in both cases $V \in D_0$.

The proof of Theorem 3.1 relies on the following lemma.

**Lemma 4.1.** Let $p \in (1, \sigma)$ in the definition (4.3) of $V$. Then

\[(4.4) \quad \limsup_{x \to \infty} \frac{\mathcal{L}_0 V(x)}{\phi^{(i)}(x)} \leq C^{(i)}, \quad 1 \leq i \leq 4,\]

where the functions $\phi^{(i)}$ are defined for the respective $\sigma$ and $\delta$ in (3.10) – (3.13).

**Proof.** Case (1). Note that in this part we assumed that $\delta < \frac{1}{2}$.

\[
\mathcal{L}_0 V(x) = \left( \int_{|u| > 1} + \int_{|u| \leq 1} \right) \left( V(x + u) - V(x) - V'(x) \mathbb{1}_{|u| \leq 1} \right) \nu(x, du) = I_1(x) + I_2(x).
\]

For $I_2(x)$ we have for $x$ large enough

\[(4.5) \quad I_2(x) = \frac{p(p - 1)}{2} \int_{|u| \leq 1} (u(1 - \zeta))^2 \left[ \int_0^1 (x + \zeta u)^{p-2} d\zeta \right] \nu(x, du) \leq \frac{C(x)}{x^{2-p}}.\]

Since $\delta < 2$, by (3.13) we get

\[(4.6) \quad \lim_{x \to \infty} \frac{I_2(x)}{\phi^{(1)}(x)} = 0.\]

Let us estimate $I_1(x)$. Let

\[\tilde{I}_1(x) := \int_{|u| > 1} (|x + u|^p - |x|^p) \nu(x, du),\]

and split

\[I_1(x) = \tilde{I}_1(x) + \int_{|u| > 1} (\phi(|x + u|))^p - |x + u|^p \nu(x, du)\]

\[+ \int_{|u| > 1, |x + u| < 1} (\phi(|x + u|))^p - |x + u|^p \nu(x, du)\]

\[= I_{11}(x) + I_{12}(x) + I_{13}(x) + I_{14}(x) + I_{15} + I_{16}.\]

In order to estimate $I_{1k}(x)$, $1 \leq k \leq 5$, we use (4.1).
For $I_{16}(x)$ we have

$$|I_{16}(x)| \leq 2 \int_{|u|>1,|x+u|<1} \nu(x,du) \leq 2\nu(x, B(x,1)),$$

implying

$$\limsup_{x \to \infty} \frac{|I_{16}(x)|}{x^{p-\sigma}} \leq \limsup_{x \to \infty} \frac{2N(x, x-1)}{x^{p-\sigma}} = 0.$$

For $I_{11}(x)$ we have due to the symmetry of $\nu(x,du)$ in $u$

$$I_{11}(x) = x^p \int_x^\infty (1+\frac{u}{x})^p \nu(x,du) = x^p \int_x^\infty \sum_{k=0}^\infty C_p^k \left(\frac{u}{x}\right)^{p-k} (-1)^k \nu(x,du),$$

where in the second equality we used (4.1). Similarly,

$$I_{12}(x) = x^p \int_x^\infty \sum_{k=0}^\infty C_p^k \left(\frac{u}{x}\right)^{p-k} \nu(x,du),$$

implying after interchanging the sum and in the integral, the equality

$$I_{11}(x) + I_{12}(x) = 2x^p \sum_{k=0}^\infty \int_x^\infty C_p^{2k} \left(\frac{u}{x}\right)^{p-2k} \nu(x,du),$$

Since $p < \sigma$,

$$u^{p-2k} N(x,u) \leq \frac{N(x,1)}{u^{2k+\sigma-p}} \to 0, \quad u \to \infty,$$

for any fixed $x, |x| \geq c$, and thus

$$\int_x^\infty u^{p-2k} \nu(x,du) = x^{p-2k} N(x) + (p-2k) \int_x^\infty u^{p-2k-1} N(x,u)du, \quad k \geq 0.$$

Therefore, interchanging the series and the integral, we get

$$I_{11}(x) + I_{12}(x) = 2x^p \sum_{k=0}^\infty C_p^{2k} (x^{p-2k} N(x) + (p-2k) \int_x^\infty \frac{N(x,u)}{u^{2k+1-p}}du)$$

$$= 2x^p N(x) \sum_{k=0}^\infty C_p^{2k} \left(1 + (p-2k)x^{2k-p} \int_x^\infty \frac{1}{u^{2k+1-p}} \frac{N(x,u)}{N(x)}du\right)$$

$$= 2x^p N(x) \left(\sum_{k=0}^\infty C_p^{2k} + (p-2k) \int_1^\infty \frac{1}{u^{2k+1-p}} \frac{N(x,ux)}{N(x)}du\right).$$

Consider now $I_{13}(x)$. As above, due to the symmetry of $\nu(x,du)$ and (4.1),

$$I_{13} = x^p \int_1^x \left(1-\frac{u}{x}\right)^p \nu(x,du) = x^p \int_1^x \sum_{k=0}^\infty C_p^k \left(\frac{u}{x}\right)^k (-1)^k \nu(x,du).$$

Analogously,

$$I_{14} = x^p \int_1^x \left(1+\frac{u}{x}\right)^p \nu(x,du) = x^p \int_1^x \sum_{k=0}^\infty C_p^k \left(\frac{u}{x}\right)^k \nu(x,du),$$
implying that

\[ I_{13} + I_{14} = 2x^p \int_1^x \sum_{k=0}^{\infty} C_p^k \left( \frac{u}{x} \right)^{2k} \nu(x, du) = 2x^p \int_1^x \sum_{k=1}^{\infty} C_p^k \left( \frac{u}{x} \right)^{2k} \nu(x, du) + 2x^p \int_1^x \nu(x, du), \]

which leads to

\[ I_{13} + I_{14} + I_{15} = 2x^p \int_1^x \sum_{k=1}^{\infty} C_p^k \left( \frac{u}{x} \right)^{2k} \nu(x, du) - 2x^p N(x). \]

Using integration by parts, we get

\[ I_{13} + I_{14} + I_{15} = 2x^p \sum_{k=1}^{\infty} C_p^{2k} x^{-2k} \left( -x^{2k} N(x) + N(x, 1) + 2k \int_1^x u^{2k-1} N(x, u) du \right) - 2x^p N(x) \]

\[ = 2x^p N(x) \left( -\sum_{k=0}^{\infty} C_p^{2k} + \sum_{k=1}^{\infty} C_p^{2k} \left( \frac{N(x, 1)}{N(x)x^{2k}} + \frac{2k}{x^{2k}} \int_1^x u^{2k-1} N(x, u) du \right) \right) \]

\[ = 2x^p N(x) \left( -\sum_{k=0}^{\infty} C_p^{2k} + \sum_{k=1}^{\infty} C_p^{2k} \left( \frac{N(x, 1)}{N(x)x^{2k}} + 2k \int_1^{1/x} u^{2k-1} N(x, xu) du \right) \right). \]

Thus,

\[ I_1(x) \leq 2x^p \sum_{k=1}^{\infty} C_p^{2k} \left( \frac{N(x, 1)}{x^{2k}} + (p - 2k) \int_1^{\infty} \frac{N(x, xu)}{u^{2k+1-p}} du + 2k \int_1^{1/x} u^{2k-1} N(x, xu) du \right) \]

\[ + 2x^p \int_1^{\infty} u^{p-1} N(x, xu) du. \]

Note that for \( p \in (1, 2) \) we have \( C_p^{2k} > 0 \). Then

\[ \limsup_{x \to \infty} \frac{I_1(x)}{\phi^{(1)}(x)} \leq \limsup_{x \to \infty} 2 \sum_{k=1}^{\infty} C_p^{2k} \left( (p - 2k) \int_1^{\infty} \frac{x^{\delta} N(x, xu)}{u^{2k+1-p}} du + 2k \int_1^{1/x} u^{2k-1} x^{\delta} N(x, xu) du \right) \]

\[ + 2pN_\sigma \int_1^{\infty} u^{p-1} du. \]

If \( \sigma < \delta < 2 \), then all the terms in the sum \( \sum_{k=1}^{\infty} \) vanish, and we have

\[ \limsup_{x \to \infty} \frac{I_1(x)}{\phi^{(1)}(x)} \leq \frac{2pN_\sigma}{\sigma - p}. \]

If \( \sigma = \delta < 2 \), we have

\[ \limsup_{x \to \infty} \frac{I_1(x)}{\phi^{(1)}(x)} \leq 2 \sum_{k=1}^{\infty} C_p^{2k} \left( \frac{N_\delta^{2k}}{2k - \delta} - \frac{N_\delta(2k - p)}{2k - p + \delta} \right) + \frac{2pN_\sigma}{\sigma - p}. \]

Thus, we get

\[ \limsup_{x \to \infty} \frac{I_1(x)}{\phi^{(1)}(x)} \leq C^{(1)}, \]

which implies (4.4) for \( i = 1 \).
\textbf{Case (2).} If $\sigma = 2 = \delta$ we have

\begin{equation}
\lim_{x \to \infty} \frac{I_2(x)}{\phi^{(2)}(x)} = 0.
\end{equation}

Consider now $I_1(x)$. Due to the $\ln x$ in the definition of $\phi^{(2)}$, all terms vanish except the following for $k = 1$:

$$
\limsup_{x \to \infty} \frac{x^2}{\ln x} \int_{1/x}^{1} u^{2k-1} N(x, xu) du \leq N_\delta.
$$

Therefore, in this case we have

\begin{equation}
\limsup_{x \to \infty} \frac{I_1(x)}{\phi^{(2)}(x)} \leq 4C_p^2 N_\delta = 2p(p - 1)N_\delta.
\end{equation}

\textbf{Case (3).} Recall that in this case $\phi^{(3)}(x) = x^{p-2}$. Then

\begin{equation}
\limsup_{x \to \infty} \frac{I_2(x)}{\phi^{(3)}(x)} \leq \limsup_{x \to \infty} \frac{p(p - 1)\nu_{\text{small}}(x)x^{p-2}}{2\phi^{(3)}(x)} = \frac{p(p - 1)\nu_{\text{small}}}{2}.
\end{equation}

Consider now $I_1(x)$. Since $\delta > 2$, we have

$$
\limsup_{x \to \infty} x^2 \int_{1}^{\infty} N(x, xu) \frac{1}{u^{2k+1-p}} du = 0.
$$

If $\delta \neq 2m$ for some $m \geq 2$, then

$$
\limsup_{x \to \infty} x^2 \int_{1/x}^{1} u^{2k-1} N(x, xu) du \leq \frac{N_\delta}{2k - \delta} \limsup_{x \to \infty} (x^{2-\delta} - x^{2-2k}) = \frac{N_\delta}{\delta - 2}.
$$

If $\delta = 2m$, we also have

$$
\limsup_{x \to \infty} x^2 \int_{1/x}^{1} u^{2k-1} N(x, xu) du \leq N_\delta \lim_{x \to \infty} x^{2-2m} \ln x = 0.
$$

Thus, from above

$$
\limsup_{x \to \infty} \frac{I_1(x)}{\phi^{(3)}(x)} \leq p(p - 1) \left( N_{\max} + \frac{2N_\delta}{\delta - 2} \right) + \limsup_{x \to \infty} 2pN_\sigma x^{2-\sigma} \int_{1}^{\infty} u^{p-1-\sigma} \frac{du}{\sigma - p}.
$$

$$
= p(p - 1) \left( N_{\max} + \frac{2N_\delta}{\delta - 2} \right) + \frac{2pN_\sigma}{\sigma - p}.
$$

\textbf{Case (4).} Finally, from above it follows that in the case $\sigma < 2$, $\delta \geq 2$ we have

$$
\limsup_{x \to \infty} \frac{I_2(x)}{\phi^{(3)}(x)} \leq 0
$$

and

$$
\limsup_{x \to \infty} \frac{I_1(x)}{\phi^{(1)}(x)} \leq \frac{2pN_\sigma}{\sigma - p}.
$$
Proof of Theorem 3.4. Consider $V(x) = e^{\beta x^{1+\zeta}}$ for some $\zeta \in (-1, 0]$, $\beta < \alpha$, where $\phi(x)$ is defined in (4.2). Since we assumed (3.19), we modify a bit the operator $\mathcal{L}$:

$$
\mathcal{L}V(x) = \left( a(x) + \int_{|u| \geq 1} u\nu(x, du) \right) V'(x) + \int_{\mathbb{R}} (V(x + u) - V(x) - V'(x)u) \nu(x, du).
$$

Recall that we assumed that $\nu(x, du)$ is symmetric in $u$; therefore, the integral term in the coefficient near $V'(x)$ is 0.

First we show that

$$
\limsup_{|x| \to \infty} \mathcal{L}_0 V(x) \leq C^{(5)},
$$

where $\phi^{(5)}$ and $C^{(5)}$ are defined in (3.21), and $\mathcal{L}_0$ now is defined by

$$
\mathcal{L}_0 V(x) = \int_{\mathbb{R}} (V(x + u) - V(x) - V'(x)u) \nu(x, du).
$$

Assume that $x > 1$ is big enough, and let for $g(u) = e^{\beta |u|^{1+\zeta}}$

$$
I_1(x) = \int_{|u| \geq 1} (g(x + u) - g(x) - g'(x)u) \nu(x, du)
$$

and split

$$
\mathcal{L}_0 V(x) = \left( \int_{|u| < 1} + \int_{|u| \geq 1} \right) (V(x + u) - V(x) - V'(x)u) \nu(x, du) \pm I_1(x)
$$

$$
= \left( \int_{|u| < 1} + \int_{1 \leq |u| \leq \epsilon x} + \int_{|u| > \epsilon x} \right) (g(x + u) - g(x) - g'(x)u) \nu(x, du)
$$

$$
+ \int_{|u| > \epsilon x} (V(x + u) - e^{\beta |x+u|^{1+\zeta}}) \nu(x, du)
$$

$$
= J_1(x) + J_2(x) + J_3 + J_4(x),
$$

where $\epsilon > 0$ is such that $\beta + \epsilon < \alpha$. Using the Taylor formula for $g(u)$ we get for some $\theta \in (0, 1)$

$$
|g(x + u) - g(x) - g'(x)u| \leq 2^{-1} |g''(x + \theta u)| u^2
$$

$$
= 2^{-1} \beta (1 + \zeta) (x + u\theta)^{\zeta} (\beta (1 + \zeta) + \zeta (x + \theta u)^{-1-\zeta}) \int_{|u| \leq 1} u^2 \nu(x, du),
$$
implying for $|u| \leq 1$

$$J_1(x) \leq 2^{-1}(1 + \zeta)(x - 1)^{2\zeta}V(x + 1) \left(\beta(1 + \zeta) + \zeta(x - 1)^{-1-\zeta}\right) \int_{|u| \leq 1} u^2 \nu(x, du)$$

(4.16)

$$\leq C_1(x - 1)^{2\zeta}V(x + 1)(1 + o(1)).$$

where in the last line we used that $|x + u|^{1+\zeta} \leq x^{1+\zeta} + |u|^{1+\zeta} \leq x^{1+\zeta} + 1$ (cf. (3.8))

$$C_1 = 2^{-1}\beta^2(1 + \zeta)^2 \nu_{small}.$$

The argument for $J_2$ is similar. Using (4.15) and that $1 < |u| \leq \epsilon x$ we derive

$$|g(x + u) - g(x) - g'(x)u| \leq 2^{-1}(1 - \epsilon)^{2\zeta}\beta^2(1 + \zeta)^2 x^{2\zeta}V(x)e^{\beta |u|^{1+\zeta}}u^2(1 + o(1)),$$

implying that

(4.17)

$$J_2(x) \leq C_2 x^{2\zeta}V(x)(1 + o(1)),$$

where

$$C_2 := 2^{-1}\beta^2(1 + \zeta)^2c_0\nu_{\alpha,\zeta,large},$$

where $c_0$ is such that $x^2e^{\beta x^{1+\zeta}} \leq c_0 e^{\alpha x^{1+\zeta}}, x > 1,$ and we used that $(1 - \epsilon)^{2\zeta} \leq 1$. For $J_3$ we have

$$J_3(x) \leq \int_{|u| \geq \epsilon x} e^{\beta |x+u|^{1+\zeta}}\nu(x, du) \leq V(x) \int_{|u| \geq \epsilon x} e^{-(\alpha - \beta)|u|^{1+\zeta} + \alpha |u|^{1+\zeta}}\nu(x, du)$$

$$\leq V(x)e^{-(\alpha - \beta)(\epsilon x)^{1+\zeta}}\int_{|u| \geq 1} e^{\alpha |u|^{1+\zeta}}\nu(x, du)$$

$$\leq \nu_{\alpha,\zeta,large}V(x)e^{-(\alpha - \beta)(\epsilon x)^{1+\zeta}}.$$

Finally,

$$J_4(x) \leq \int_{|u| \geq \epsilon x} e^{\beta |x+u|^{1+\zeta}}\nu(x, du) \leq \nu_{\alpha,\zeta,large}V(x)e^{-(\alpha - \beta)(\epsilon x)^{1+\zeta}}.$$

Thus, we have (4.13) with

$$C^{(5)} = C_1 + C_2.$$

Consider now the drift term. We have

(4.18)

$$\limsup_{x \to \infty} \frac{a(x)V'(x)}{\phi(x)} = \limsup_{x \to \infty} \frac{a(x)\beta(1 + \zeta)x^{\zeta}V(x)}{\phi(x)} \leq -\beta(1 + \zeta)A_{\kappa}.$$

Summarizing, we get (3.22). The restriction $\zeta \in (-1, \kappa]$ comes from the following argument. The orders of the drift, the function $f(V(x))$ and of the integral terms are, respectively, $x^{\kappa+\zeta}V(x)$, $x^{\kappa+\zeta}V(x)$ and $x^{2\zeta}V(x)$. Since for the integral term we have the positive bound and the function $f(V(x))$ is positive, we need the drift to dominate. Therefore, we need the restriction $\kappa + \zeta > 2\zeta$, which implies the necessary bound for $\zeta$. \hfill \Box

The proof of Corollary 3.5 follows in the same way as that of Corollary 3.2.
5. Application

Consider one simple application of Theorems 3.1 and 3.4. Suppose we want to estimate the value of the functional
\[ \pi_x(u) := \mathbb{E} \pi(u(X)), \]
where \( X \) is a random variable with probability distribution \( \pi \). Knowing the order of convergence, we can estimate \( \pi_x(u) \) with a given precision choosing \( t \) large enough according to (2.11):
\[
|T_t u(x) - \pi_x(u)| \leq \sup_{z \in \mathbb{R}} |u(z)| \|P_t(x, \cdot) - \pi(\cdot)\|_{TV} \leq C_u C_{\text{erg}}(x) \psi(t).
\]

In [BS09] it was suggested to approximate \( X_t \) by a Markov chain \( (Y_n(t))_{n \geq 1} \). The advantage of this approach is that the chain \( (Y_n(t))_{n \geq 1} \) is relatively easy to simulate. Let us briefly explain the construction.

Suppose that \( x \) in (1.1) is fixed and define the operator \( L \) from (1.1) by \( L_x \). Then \( (L_x, C^\infty_0(\mathbb{R})) \) extends to the generator of a Lévy process \( Y_x \) with the characteristic function
\[
\mathbb{E} e^{i\xi Y_x t} = e^{-tq(x,\xi)}, \quad \xi \in \mathbb{R},
\]
where (see the monograph [Ja01] for the detailed explanations)
\[
q(x,\xi) := -ia(x)\xi + \int_{\mathbb{R} \setminus [0]} \left(1 - e^{i\xi u} + i\xi u 1_{|u| \leq 1}\right) \nu(x, du).
\]

Define the family of probability measures \( (\mu_{x,\frac{1}{n}})_{n \geq 1} \) by
\[
\int_{\mathbb{R}} e^{i\xi u} \mu_{x,\frac{1}{n}}(dy) = e^{i\xi x - \frac{1}{n}q(x,\xi)}.
\]

Then we define a Markov chain \( (Y_n(k))_{n \geq 1} \) with the transition kernel \( (\mu_{x,\frac{1}{n}})_{n \geq 1} \), and let
\[
W_{\frac{1}{n}} u(x) := \int_{\mathbb{R}} u(y) \mu_{x,\frac{1}{n}}(dy).
\]

Then (cf. [BS09])
\[
\sup_{t \in [a,b]} \|W_{\frac{1}{n}}^{[nt]} u - T_t u\|_\infty = 0,
\]
where \( \|u\|_\infty := \sup_x |u(x)| \), and the rate of convergence is of order \( n^{-\frac{1}{2}} \), which follows from Lemma 6.4 and Theorems 6.1 and 6.5 [EK86].

Note that \( W_{\frac{1}{n}}^{[nt]} u(x) = \mathbb{E} u(Y_n([nt])) \), and finally applying the Monte-Carlo method, we derive
\[
\pi^x(u) \approx \frac{1}{N} \sum_{i=1}^N u \left( Y_n^{(i)}([nt]) \right),
\]
where \( Y_n^{(i)}([nt]) \), \( 1 \leq i \leq N \) are independent copies of \( Y_n([nt]) \), and \( Y_n \) starts at \( x \). Thus, the approach reduces to the simulation of the Markov chain with a given kernel. Some particular examples were treated in [B10], but the methodology of simulation of a general Lévy process heavily depends on the structure of the kernel \( \nu(x_0, du) \) (here \( x_0 \) is fixed). We postpone this problem for future research.
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