General Results on Conditional Symmetry for the Two-Dimensional Nonlinear Wave Equation

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Abstract

We present full classification of $Q$-conditional symmetries for the two-dimensional nonlinear wave equation

$$u_{tt} - u_{xx} = F(t, x, u)$$
1 Introduction

We discuss conditional symmetries of the two-dimensional nonlinear wave equation

\[ u_{tt} - u_{xx} = F(t, x, u) \]  \hspace{1cm} (1)

for the real-valued function \( u = u(t, x) \), \( t \) is the time variable, \( x \) is the space variable. Further we will use the following designations for the partial derivatives:

\[ u_t = -\frac{\partial u}{\partial t}; \quad u_x = -\frac{\partial u}{\partial x}; \quad u_{tt} = \frac{\partial^2 u}{\partial t^2}; \quad u_{xt} = u_{tx} = \frac{\partial^2 u}{\partial t \partial x}; \quad u_{xx} = \frac{\partial^2 u}{\partial x^2}. \]

The general equation in the class (1) is not invariant with respect to any operators, with only particular cases having wide symmetry algebras, see e.g. [1].

The maximal invariance algebra of the equation (1) with \( F = F(u) \) (not depending on \( t, x \) and not equal to zero) is the Poincaré algebra \( AP(1, 1) \) with the basis operators

\[ p_t = \frac{\partial}{\partial t}, \quad p_x = \frac{\partial}{\partial x}, \quad J = tp_x + xp_t. \]

The invariance algebras of the equation (1) will also include dilation operators for \( F = \lambda u^k \) or \( F = \lambda \exp u \).

The symmetry of the linear equation (1) with \( F = 0 \) and \( F = \lambda \exp u \) is infinite-dimensional.

Similarity solutions for the equation (1) can be found by symmetry reduction with respect to non-equivalent subalgebras of its invariance algebras [1 2 3 4].

Here we present the general result on the \( Q \)-conditional invariance of the equation (1).
2 Conditional symmetry

The concept of conditional symmetry in general (additional invariance under arbitrary additional condition) and a narrower concept of the $Q$-conditional invariance (the additional condition has the form $Qu = 0$) was initiated and discussed in the papers [5, 6, 7, 8, 9] and later it was developed by numerous authors into the theory and a number of algorithms for studying symmetry properties of equations of mathematical physics. The importance of investigation of the $Q$-conditional symmetry was presented (see e.g. [10]) where equivalence of the $Q$-conditional invariance and reducibility of the equations by means of ansatzes determined by such operators $Q$ was proved.

Here we will use the following definition of the $Q$-conditional symmetry:

**Definition 1.** The equation $\Phi(x, u, u_1, \ldots, u_l) = 0$, where $u_k$ is the set of all $k$th-order partial derivatives of the function $u = (u_1, u_2, \ldots, u_m)$, is called $Q$-conditionally invariant [3] under the operator

$$Q = \xi^i(x, u)\partial_{x_i} + \eta^r(x, u)\partial_{u^r}$$

if there is an additional condition

$$Qu = 0, \quad (2)$$

such that the system of two equations $\Phi = 0, Qu = O$ is invariant under the operator $Q$. All differential consequences of the condition $Qu = 0$ shall be taken into account up to the order $l - 1$.

This definition of the conditional invariance of some equation is based on what is in reality a Lie symmetry (see e.g. the classical texts [11, 12, 13]) of the same equation with a certain additional condition. Conditional symmetries of the multidimensional nonlinear wave equations are specifically discussed in [15, 19, 20].
3 Previous papers on the problem

Solving of the particular problem we discuss here was started by P. Clarkson and E. Mansfield in [16] (the case $f = f(u)$), where the relevant determining equations were written out but not solved, and continued by M. Euler and N. Euler in [17]. In the latter paper the determining conditions for the $Q$-conditional invariance were taken without consideration of the differential consequences of the condition $Qu = 0$, so the resulting operators did not actually present the solution.

Following [16], we will consider the equation equivalent to (1) of the form

$$u_{yz} = f(y, z, u).$$

(3)

The search for the operators of $Q$-conditional invariance in the form

$$Q = a(y, z, u)\partial_y + b(y, z, u)\partial_z + c(y, z, u)\partial_u.$$

(4)

4 Equivalence Transformations

We will classify the systems of the type (3), (4) under its equivalence transformations, that is write down a set of all such systems that cannot be transformed into each other by means of equivalence transformations. However, here we will not be looking for description of such group, but first limit ourselves by equivalence up to some obvious transformations.

Further it will be expedient to look at equivalence transformations for the special cases as they be different from those in the general class.

The concept of equivalence of $Q$-conditional symmetries was introduced by R. Popovych in [18]
5 Determining Equations

It is obvious that we can consider three inequivalent cases when $a = 0$, $b \neq 0$, then we can take

$$Q = \partial_z + L(y, z, u) \partial_u$$

and $a \neq 0, b \neq 0$, then we can take

$$Q = \partial_y + K(y, z, u) \partial_z + L(y, z, u) \partial_u.$$  

where $K(y, z, u) \neq 0$.

The case $a \neq 0, b = 0$ is equivalent to $a = 0, b \neq 0$.

The case $a = 0, b = 0$ is trivial.

The additional condition $Qu = 0$ will be represented respectively by the equations

$$u_z = L(y, z, u),$$

and

$$u_y + K(y, z, u) u_z = L(y, z, u).$$

The determining equations have the form

$$-K_u^2 + K_{uu} K = 0,$$

$$-K L_{uu} + \frac{K_u K_y}{K} + \frac{K_u^2 L}{K} + K_u (L_u - K_z) - K_{uy} - L K_{uu} + K K_{zu} = 0,$$

$$L_{uy} - L_{uz} K + L_{uu} L - \frac{L_u K_y}{K} + \frac{K_y K_z}{K} - K_{yz}$$

$$-3K_u f - \frac{K_u L}{K} (L_u - K_z) + K_u L_z - K_{zu} L = 0,$$

$$-f_y - K f_z - L f_u + L y_z + L u_z L + L u f - \frac{K_y}{K} (L_z - f) - K_z f - \frac{K_u L}{K} (L_z - f) = 0.$$ 

(12)
6 Main Results

Case 1. $K_u = 0, K \neq 0$. The determining equations have the form

$$-KL_{uu} = 0,$$

$$L_{uy} - L_{uz}K + L_{uu}L - \frac{L_uK_y}{K} + \frac{K_yK_z}{K} - K_yz = 0,$$

$$-f_y - Kf_z - Lf_u + L_{yz} + L_{uz}L + L_u f - \frac{K_y}{K} (L_z - f) - K_z f = 0.$$

We have $K = k(y, z)$, $L = s(y, z)u + d(y, z)$. Using equivalence transformations we can put $d(y, z) = 0$.

From the determining equations we get

$$k(y, z) = \frac{T_y}{T_z},$$

$$s(y, z) = \frac{T_{yz}}{T_z},$$

where $T = T(y, z)$ is an arbitrary function.

In this case the ansatz reducing equation (3) will have the form

$$u = \sigma(y, z)\phi(\omega), \quad \text{(13)}$$

where $\omega = \omega(y, z)$ is a new variable,

$$T_y\omega_z + T_z\omega_y = 0,$$

$$T_y\sigma_z + T_z\sigma_y = \sigma T_{yz}.$$

A reduced equation will have the form:

$$\sigma_{yz}\phi + \phi'(\omega_y\sigma_z + \omega_z\sigma_y + \sigma\omega_{yz}) + \phi''\sigma\omega_y\omega_z = f, \quad \text{(14)}$$

where $f$ satisfies the relevant conditions (12).
Case 2. \( K = 0 \) - the case is equivalent to \( a = 0 \), with the additional condition (7). Here we have equations

\[
    u_y = L, \quad u_{yz} = f.
\]

The determining equations have the form

\[
    L_{uy} + L_{uu}L = 0,
\]

\[
    -f_y - Lf_u + L_{yz} + L_{uz}L + L_u f = 0
\]

This case is actually equivalent to a pair of first-order equations

\[
    u_y = L, \quad u_z = \frac{f - L_z}{L_u}.
\]

Case 3. \( K_u \neq 0 \), then \( K_{uu}K = K_u^2 \), \( K = k(y, z) \exp(l(y, z)u) \). We can put \( k = 1 \) and prove from the resulting determining conditions \( l_y = l_z = 0 \), so we can put \( l = 1 \).

Then we can found that \( L = s(y, z) \exp u + d(y, z) \). It is possible to reduce this case to \( k = 1 \), and we get the following determining equation for \( f \) with arbitrary \( s \) and \( d \):

\[
    f = \frac{1}{3}(s_y + d_z),
\]

so \( f \) in this case depends only on \( y \) and \( z \), and the equation \( u_{yz} = f(y, z) \) is equivalent to the equation \( u_{yz} = 0 \).

The conditions for \( s \) and \( d \) have the form

\[
    2s_{yz} - sd_z + 2s_y s - d_{zz} = 0, \quad -s_{yy} + 2d_{yz} + s_y d - 2d_z d = 0.
\]
7 Conclusions

We have considered the equations

\[ u_{yz} = f(y, z, u) \]

with \( f \) depending on \( y, z, u \).

For such class the only nontrivial case is Case 1, \( K_u = 0, K \neq 0 \).

The case \( f = f(u) \) required special consideration, and has more inequivalent cases.
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