A modified harmonic balance method for solving forced vibration problems with strong nonlinearity

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Abstract
In this paper, a modified harmonic balance method is presented to solve nonlinear forced vibration problems. A set of nonlinear algebraic equations appears among the unknown coefficients of harmonic terms and the frequency of the forcing term. Usually a numerical method is used to solve them. In this article, a set of linear algebraic equations is solved together with a nonlinear one. The solution obtained by the proposed method has been compared to those obtained by variational and numerical methods. The results show good agreement with the results obtained by both methods mentioned above.

Keywords
Harmonic balance, variational method, nonlinear oscillations, forced vibration

Introduction
Nonlinear vibration is an important issue in science and engineering. Most of the differential equations involving physical phenomena are nonlinear. Therefore, the analysis of nonlinear vibration problems has attracted intensive research attention in the past few decades.\textsuperscript{1–6} Generally, it is very hard to find the exact or closed-form solution of such a nonlinear problem. Therefore, many researchers have paid attention to both numerical and analytical methods. Numerical methods are comparatively easy but require heavy computational effort and a proper initial gauge value to obtain the desired results. Moreover, numerical methods cannot provide an overall view of the behavior of the nonlinear dynamical systems. Consequently, the quest for accurate behavior of the nonlinear dynamical systems led to the development of many analytical approximations. In the literature, several analytical approximate methods are found such as perturbation,\textsuperscript{7,8} homotopy analysis,\textsuperscript{9,10} homotopy perturbation,\textsuperscript{11,12} variational iteration,\textsuperscript{13,14} harmonic balance (HBM),\textsuperscript{15–18} etc.

The perturbation methods\textsuperscript{19–23} are a widely used technique for handling nonlinear problems which were originally developed for weak nonlinear problems. Furthermore, the classical perturbation methods have been extended or modified by several authors to investigate strong nonlinear problems. Jones\textsuperscript{24} introduced an approximate method to enhance the range of validity of the classical perturbation where the parameter is not small. Later, Cheung et al.\textsuperscript{25} modified the Lindstedt–Poincare method using the concept of Jones.\textsuperscript{24} Recently, Alam, Yeasmin and Ahamed\textsuperscript{26} have generalized the modified Lindstedt–Poincare method\textsuperscript{25} which covers a wide variety of nonlinear oscillators.

The HBM is another powerful technique for determining periodic solutions of nonlinear differential equations where a truncated Fourier series is chosen as the solution of the differential equations. In the classical HBM, a set of nonlinear algebraic equations is solved by a numerical method to determine the unknown coefficients. Furthermore, the classical HBM has been modified by some researchers. For example, Rahman et al.\textsuperscript{27} used...
the HBM to investigate the nonlinear Van der Pol equation. Lau and Cheung \cite{28} introduced an incremental variational principle for nonlinear vibration of an elastic system. Azrar et al. \cite{29} presented a semi-analytical approach based on the concept of harmonic balance to analyze the nonlinear response of a large-amplitude beam. Rahman and Lee \cite{30} introduced a modified multi-level residue HBM for solving nonlinearly vibrating double-beam problems. Belendez et al. \cite{31} applied a modified HBM to solve a class of strongly nonlinear oscillators including a rational nonlinear term. Wagner and Lentz \cite{32} used a HBM to investigate a Duffing oscillator with excitation force. They also studied an extended Duffing oscillator.

Recently, variational and Hamiltonian principles are two interesting approaches to solve strong nonlinear problems. Earlier, He \cite{33} used the semi-inverse method to find the variational principle for handling some nonlinear partial differential equations with variable coefficients. Furthermore, He \cite{34} applied this method to solve some nonlinear oscillators with fractional nonlinear terms. He \cite{35} developed a new variational method for nonlinear oscillators using the Hamiltonian approach. Akbarzade and Kargar \cite{36} also applied the Hamiltonian approach to nonlinear free and forced vibrating systems without any damping effect. Then Yildirim et al. \cite{37} utilized the same method to solve nonlinear oscillators with rational and irrational elastic terms. Sadeghzadeh and Kabiri \cite{38} presented a higher-order Hamiltonian approach to the nonlinear vibration of micro electro-mechanical systems without any damping effect. From the above literature, it is seen that in most cases, variational and Hamiltonian principles are applied for nonlinear free vibration problems. Yet for the higher-order approximate solution, a set of nonlinear algebraic equations is raised and generally solved by the numerical method.

In this paper, a modified harmonic balance is presented. The advantage of the proposed method is that a set of linear algebraic equations together with a nonlinear one are solved, thereby reducing the computational effort and requiring less computational time than other HBMs. The method is applied to solve nonlinear forced vibration problems. The results have been compared to those obtained by numerical and a variational method to verify the accuracy of the method.

### The method

Consider a nonlinear forced vibration equation of the following form

\[
\ddot{x} + \omega_0^2 x + \mu \dot{x} + \varepsilon f(x) = E \cos(\omega t) \tag{1}
\]

where over dots denote differentiation with respect to \(t\), \(\omega_0\) is the natural frequency, \(\mu\) is the damping coefficient, \(f(x)\) is a given nonlinear function of \(x\), \(\varepsilon\) is a parameter, \(E\) is the excitation force amplitude, and \(\omega\) is the excitation frequency.

The solution of equation (1) is chosen as follows

\[
x = A \cos(\omega t) + B \sin(\omega t) + a_3 \cos(3\omega t) + b_3 \sin(3\omega t) + \cdots \tag{2}
\]

Substituting equation (2) into equation (1) and expanding \(f(x)\) in Fourier series and equating the coefficients of equal harmonics, the following equations are obtained

\[
a(-\omega^2 + \omega_0^2) + b \mu \omega + \varepsilon C_1(a, b, a_3, b_3, \cdots) = E \tag{3a}
\]

\[
b(-\omega^2 + \omega_0^2) - a \mu \omega + \varepsilon S_1(a, b, a_3, b_3, \cdots) = 0 \tag{3b}
\]

\[
a_3(-9\omega^2 + \omega_0^2) + 3b_3 \mu \omega + \varepsilon C_3(a, b, a_3, b_3, \cdots) = 0 \tag{3c}
\]

\[
b_3(-9\omega^2 + \omega_0^2) - 3a_3 \mu \omega + \varepsilon S_3(a, b, a_3, b_3, \cdots) = 0 \tag{3d}
\]

Eliminating \(\omega^2\) from equations (3b–d) with the help of equation (3a), and neglecting the terms whose responses are negligible, the above equations can be re-written as follows

\[
\omega^2 = \omega_0^2 + \varepsilon C_1(a, b, a_3, b_3, \cdots) - E/a \tag{4a}
\]
\[-a_1\omega + \varepsilon C_1(a, b, a_3, b_3, \cdots) + \varepsilon S_1(a, b, a_3, b_3, \cdots) - bE/a = 0 \quad (4b)\]

\[-8\omega_0^2a_3 + a_3\varepsilon C_1(a, b, a_3, b_3, \cdots) + \varepsilon C_3(a, b, a_3, b_3, \cdots) - a_3E/a = 0 \quad (4c)\]

\[-8\omega_0^2b_3 + b_3\varepsilon C_1(a, b, a_3, b_3, \cdots) + \varepsilon S_3(a, b, a_3, b_3, \cdots) - 3a_3\mu\omega - bE/a = 0 \quad (4d)\]

Now using equation (4b), eliminating \(\omega\) from equations (4c and d) and considering only linear terms of \(a_3, b_3\) and neglecting the terms whose responses are negligible, a system of linear algebraic equations of \(a_3, b_3\) is obtained. Solving these two equations, \(a_3, b_3\) are determined in terms of \(a, b\). Then the values of \(a_3, b_3\) are substituted into equation (4b) and \(b\) is expressed in powers of small parameter \(\lambda(\mu, \omega, E)\) as follows

\[b = l_0 + l_1\lambda + l_2\lambda^2 + l_3\lambda^3 + \cdots\]

where \(l_0, l_1, l_2, \ldots\) are functions of \(a\). Finally, substituting \(a_3, b_3,\) and \(b\) into equation (4a) and solving, the value of \(a\) is obtained. Consequently, \(a, a_3,\) and \(b_3\) are obtained.

**Solution obtained by proposed method**

**Example 1**

Consider a damped forced vibration equation of the form

\[\ddot{x} + x + \mu\dot{x} + \varepsilon x^3 = E\cos(\omega t)\]

The solution of equation (6) is considered as follows

\[x = a\cos(\omega t) + b\sin(\omega t) + a_3\cos(3\omega t) + b_3\sin(3\omega t)\]

Substituting equation (7) into equation (6) and equating the coefficients of equal harmonics from both sides and ignoring the terms whose responses are negligible, the following equations were found

\[a(-\omega^2 + 1) + b\mu\omega + \frac{3a\omega}{4} \left( a^2 + aa_3 + b^2 + 2a_3^2 + 2bb_3 - \frac{a_3b_3^2}{a} \right) = E \quad (8a)\]

\[b(-\omega^2 + 1) - a\mu\omega + \frac{3b\omega}{4} \left( a^2 - 2aa_3 + b^2 - bb_3 + \frac{a_3^2b_3}{b} \right) = 0 \quad (8b)\]

\[a_3(-9\omega^2 + 1) + 3b_3\mu\omega + \frac{\omega a}{4} \left( a^2 + 6aa_3 - 3b^2 + 6b_3^2a_3 \right) = 0 \quad (8c)\]

\[b_3(-9\omega^2 + 1) - 3a_3\mu\omega + \frac{\omega b}{4} \left( 3a^2 - b^2 + \frac{6a_3b_3}{b} + 6bb_3 \right) = 0 \quad (8d)\]

Eliminating \(\omega^2\) from equations (8b–d) with the help of equation (8a), and neglecting the terms whose responses are negligible, the above equations can be written as follows

\[4bE - 9\omega a_3 b + 3aa_3b^3 + 3a_3^3b_3 - 9a_3b_3^2 - 4(a^2 + b^2)\mu\omega = 0 \quad (9a)\]

\[-32aa_3 + 36a_3E + \omega a - 21aa_3a_3 - 3aa_3b^2 - 21aa_3b^2 + (3ab_3 - 9a_3b)\mu\omega = 0 \quad (9b)\]

\[-32ab_3 + 36b_3E + 3aa_3b - cabb - 21aa_3b_3 - 21aa_3b_3 - (3aa_3 + 9bb_3)\mu\omega = 0 \quad (9c)\]
Now using equation (9a), eliminating \( \omega \) from equations (9c and d) and considering only linear terms of \( a_3, b_3 \), and ignoring the terms whose responses are negligible, the following equations are obtained
\[
-32aa_3 + 36a_3E + xa^4 - 21Ea^3a_3 - 2xa^2b^2 - 42Eaa_3b^2 - 3eb^4 = 0
\]  
(10a)
\[
-32ab_3 + 36b_3E + 3Ea^3b + 2xab^3 - 21Ea^3b_3 - 42Eab^2b_3 - ab^5/a = 0
\]  
(10b)

Solving equations (10a and b), \( a_3 \) and \( b_3 \) are obtained as follows
\[
a_3 = \frac{E(a^4 - 2a^2b^2 - 3b^4)}{32a - 36E + 21a^3 + 42Eab^2}, \quad b_3 = \frac{E(a^4b + 2a^2b^3 - b^5)}{a(32a - 36E + 21a^3 + 42Eab^2)}
\]  
(11)

Substituting the values of \( a_3 \) and \( b_3 \) into equation (9a), and then expanding in powers of the small parameter \( \lambda \) as follows
\[
b = l_0 + l_1\lambda + l_2\lambda^2 + l_3\lambda^3 + \cdots
\]  
(12)

where
\[
\lambda = \frac{\mu \omega}{E}, \quad l_0 = \frac{a^2\mu \omega}{E}, \quad l_1 = \frac{a^4\mu^2 \omega^2}{E^2}, \quad l_2 = \frac{2a^6\mu^3 \omega^3}{E^3}, \quad l_3 = \frac{5a^8\mu^4 \omega^4}{E^4}, \ldots
\]

Finally, after substituting \( a_3, b_3, \) and \( b \) into equation (8a) and solving, the value of \( a \) is determined.

**Example 2**

Consider a damped forced vibration equation of the form
\[
\ddot{x} + x + \mu \dot{x} + cx^2 = E \cos(\omega t)
\]  
(13)

The solution of equation (13) is taken as follows
\[
x = a_0 + a_1 \cos(\omega t) + b_1 \sin(\omega t) + a_2 \cos(2\omega t) + b_2 \sin(2\omega t)
\]  
(14)

Substituting equation (14) into equation (13) and equating the coefficients of equal harmonics from both sides and neglecting the terms whose responses are negligible, the following equations are found
\[
ea_0^2 + a_0 + \frac{1}{2}e(a_1^2 + b_1^2) = 0
\]  
(15a)
\[
a_1(-\omega^2 + 1) + b_1 \mu \omega + \epsilon(2a_0a_1 + a_1a_2 + b_1b_2) = E
\]  
(15b)
\[
b_1(-\omega^2 + 1) - a_1 \mu \omega + \epsilon(2a_0b_1 - a_2b_1 + a_1b_2) = 0
\]  
(15c)
\[
a_2(-4\omega^2 + 1) + 2b_2 \mu \omega + \frac{1}{2} \epsilon(a_1^2 + 2a_0a_2 - b_1^2) = 0
\]  
(15d)
\[
b_2(-4\omega^2 + 1) - 2a_2 \mu \omega + \epsilon(a_1b_1 + 2a_0b_2) = 0
\]  
(15e)

Eliminating \( \omega \) from equations (15c–e) and neglecting the terms whose responses are negligible, from equations (15a, c, and d) one can obtain the following
\[
a_0 = -\frac{ea_0^2}{4} \left( (2 + e^2a_1^2 + e^4a_1^4) + a_1^2(2 + 2e^2a_1^2 + 3e^4a_1^4) \lambda^2 + a_1^4(4 + 5e^2a_1^2 + 9e^4a_1^4) \lambda^4 + a_1^6(5 + 7e^2a_1^2 + 14e^4a_1^4) \lambda^6 \right)
\]  
(16a)
\[ b_1 = a_1^2 \lambda + a_1^4 \lambda^3 + 2a_1^6 \lambda^5 \quad (16b) \]

\[ a_2 = \frac{e(a_1^4 - b_1^4)}{2a_1(3a_1 - 4E + 6a_0a_1)} \quad (16c) \]

\[ b_2 = \frac{e(a_1^4 + b_1^4)}{3a_1 - 4E + 6a_0a_1} \quad (16d) \]

where

\[ \lambda = \frac{\mu \omega}{E} \]

Finally, after substituting \( a_0, a_2, b_1, \) and \( b_2 \) into equation (15b) and solving, the value of \( a_1 \) is determined.

**Solution obtained by the variational approach**

Pierre de Fermat (1601–1665)\(^{39}\) developed the variational principles in physics which played a key role in the variational principle. Furthermore, Gottfried Leibniz (1646–1716) made significant contributions to the development of variational principles in classical mechanics. In the past few decades, variational principles have been used for various differential equations. Several methods are used to obtain the variational principles such as Lagrange multiplier,\(^{40,41}\) Semi-inverse,\(^{33}\) Noether’s theorem,\(^{42}\) etc. Earlier, the Lagrange multiplier was the most convenient way to develop generalized variational principles. In 1997, He\(^{43}\) proposed a more effective way to establish the generalized variational principles using the semi-inverse method. Later, the method became popular to the researcher and was used to solve various types of differential equations. In this section, a variational approach based on the semi-inverse method\(^{33}\) to solve the nonlinear forced vibration problem with damping is presented as follows.

Consider a damped forced vibration equation of the form

\[ \ddot{x} + x + \mu \dot{x} + \varepsilon x^3 = E \cos(\omega t) \quad (17) \]

The variational parameter of equation (17) can be written as\(^{33,36}\)

\[ J(x) = \int_{T} e^\mu \left(-\frac{1}{2} \dot{x}^2 + \frac{1}{2} x^2 + \frac{1}{4} \varepsilon x^4 - x E \cos(\omega t) \right) dt \quad (18) \]

where \( T = 2\pi/\omega \) is the period of oscillation.

The second approximation solution of equation (17) is chosen as follows

\[ x = A \cos(\omega t) + B \sin(\omega t) + C \cos(3\omega t) + D \sin(3\omega t) \quad (19) \]

where \( \omega \) is frequency, and \( A, B, C, \) and \( D \) are constants to be determined.

Substituting equation (19) into equation (18) results in

\[ J(A, \ B, \ C, \ D) = \int_{T} e^\mu \left(-\frac{1}{2} \omega^2(\cos(\omega t) + B \cos(3\omega t) - 3D \cos(\omega t) + 3D \cos(3\omega t)) \right)^2 \]

\[ \quad + \frac{1}{2} (A \cos(\omega t) + B \sin(\omega t) + C \cos(3\omega t) + D \sin(3\omega t))^2 \]

\[ \quad + \frac{1}{4} e^\mu (A \cos(\omega t) + B \sin(\omega t) + C \cos(3\omega t) + D \sin(3\omega t))^4 \]

\[ \quad - (A \cos(\omega t) + B \sin(\omega t) + C \cos(3\omega t) + D \sin(3\omega t)) E \cos(\omega t) \right) dt \quad (20) \]
The values of $A$, $B$, $C$, and $D$ are obtained by solving the following four equations

\[
\frac{\partial J(A, B, C, D)}{\partial A} = 0 \quad (21a)
\]
\[
\frac{\partial J(A, B, C, D)}{\partial B} = 0 \quad (21b)
\]
\[
\frac{\partial J(A, B, C, D)}{\partial C} = 0 \quad (21c)
\]
\[
\frac{\partial J(A, B, C, D)}{\partial D} = 0 \quad (21d)
\]

**Results and discussion**

In this section, a comparison among the results obtained by the proposed method, numerical method, and a variational approach has been presented and graphically presented in Figure 1(a) to (d) and Figure 2(a) and (b). From the figures, it is observed that our result agrees reasonably well with those obtained by the variational approach and numerical method.

Moreover, both solutions have been compared in phase plane together with the numerical solution in Figure 3 (a) and (b). These figures also indicate that our solution shows good agreement with those calculated by the variational approach and numerical method.
Conclusion

In this paper, a modified HBM has been presented. The main advantage of the proposed method is that only one nonlinear algebraic equation together with a set of linear algebraic equations have been solved with less effort. However, in the classic HBM and variational approach, a set of nonlinear algebraic equations is solved by the numerical method which needs more computational effort than the proposed method.

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References

1. Ansari R, Pourashraf T and Gholami R. An exact solution for the nonlinear forced vibration of functionally graded nanobeams in thermal environment based on surface elasticity theory. Thin Walled Struct 2015; 93: 169–176.
He JH. The simplest approach to nonlinear oscillators. *Results Phys* 2019; 15: 102546.

Uddin MA, Sattar MA and Alam MS. An approximate technique for solving strongly nonlinear differential systems with damping effects. *Indian J Math* 2011; 53: 83–98.

He JH. Variational principle for the generalized KdV-burgers equation with fractal derivatives for shallow water waves. *J Appl Comput Mech* 2020; 6: 735–740.

Lee YY, Poon WY and Ng CF. Anti-symmetric mode vibration of a curved beam subject to auto parametric excitation. *J Sound Vib* 2006; 290: 48–64.

Lee YY, Su RKL, Ng CF, et al. The effect of modal energy transfer on the sound radiation and vibration of a curved panel: theory and experiment. *J Sound Vib* 2009; 324: 1003–1015.

Wickert JA. Non-linear vibration of a traveling tensioned beam. *Int J Non Linear Mech* 1992; 27: 503–517.

Nayfeh AH. *Introduction to perturbation techniques*. Hoboken, NJ: John Wiley, 1993.

Fooladi M, Abaspour SR, Kimiaeifar A, et al. On the analytical solution of Kirchhoff simplified model for Beam by using of homotopy analysis method. *World Appl Sci J* 2009; 6: 297–302.

Liao SJ. *The proposed homotopy analysis technique for the solution of nonlinear problems*. PhD Thesis, Shanghai Jiao Tong University, China, 1992.

Wu Y and He JH. Homotopy perturbation method for nonlinear oscillators with coordinate dependent mass. *Results Phys* 2018; 10: 270–271.

Uddin MA, Alom MA and Ullah MW. An analytical approximate technique for solving a certain type of fourth order strongly nonlinear oscillatory differential system with small damping. *Far East J Math Sci* 2012; 67: 59–72.

Wazwaz AM. The variational iteration method: a reliable analytic tool for solving linear and nonlinear wave equations. *Comput Math Appl* 2007; 54: 926–932.

Lee YY, Su RKL, Ng CF, et al. The effect of modal energy transfer on the sound radiation and vibration of a curved panel: theory and experiment. *J Sound Vib* 2009; 324: 1003–1015.

Nayfeh AH. Perturbation method. New York, NY: John Wiley & Sons, 1973.

Nayfeh AH and Mook DT. *Nonlinear oscillations*. New York, NY: John Wiley & Sons, 1979.

Krylov NN and Bogolyubov NN. *Introduction to nonlinear mechanics*. Princeton, NJ: Princeton University, 1947.

Alam MS. A unified Krylov-Bogoliubov-Mitropolskii method for solving some strongly nonlinear oscillators. *J Frank Inst* 2002; 339: 239–248.

Alam MS, Azad MAK and Hoque MA. A general Struble’s technique for solving an nth order weakly non-linear differential system with damping. *Int J Non Linear Mech* 2006; 41: 905–918.

Cheung YK, Chen SH and Lau SL. A modified Lindstedt-Poincare method for certain strongly nonlinear oscillators. *Int J Non Linear Mech* 1991; 26: 367–378.

Alam MS, Yeasmin IA and Ahmed MS. Generalization of the modified Lindstedt-Poincare method for solving some strongly nonlinear oscillators. *J. Ain Shams* 2019; 10: 195–201.

Rahman MS, Haque ME and Shanta SS. Harmonic balance solution of nonlinear differential equation (non-conservative). *Ganit: J Bangladesh Math Soc* 2010; 9: 345–356.

Lau SL and Cheung YK. Amplitude incremental variational principle for nonlinear structural vibrations. *J Appl Mech* 1981; 48: 959–964.

Azzar L, Benamar R and White RG. A semi-analytical approach to the nonlinear dynamic response problem of beams at large vibration amplitudes, Part II: multimode approach to the steady state forced periodic response. *J Sound Vib* 2002; 255: 1–4.

Rahman MS and Lee YY. New modified multi-level residue harmonic balance method for solving nonlinearily vibrating double-beam problem. *J Sound Vib* 2017; 406: 295–327.

Wagner UV and Lentz L. On the detection of artifacts in harmonic balance solutions of nonlinear oscillators. *Appl Math Modell* 2019; 65: 408–414.

He JH. Variational principles for some nonlinear partial differential equations with variable coefficients. *Chaos Solution Fract* 2004; 19: 847–851.

He JH. Variational approach for nonlinear oscillators. *Chaos Solution Fract* 2007; 34: 1430–1439.

He JH. Hamiltonian approach o nonlinear oscillators. *Phys Lett A* 2010; 374: 2312–2314.

Akbarzade M and Kargar A. Application of the Hamiltonian approach to nonlinear vibrating equations. *Math Comput Modell* 2011; 54: 2504–2514.
37. Yildirim A, Saadatnia Z and Askari H. Application of the Hamiltonian approach to nonlinear oscillators with rational and irrational elastic terms. *Math Comput Modell* 2011; 54: 697–703.

38. Sadeghzadeh S and Kabiri A. Application of higher order Hamiltonian approach to the nonlinear vibration of micro electro-mechanical systems. *Lat Am J Solids Struct* 2016; 13: 478–497.

39. Cline D. *Variational principles in classical mechanics*. Rochester, NY: University of Rochester Campus Libraries, 2018.

40. Chien WZ. Method of high-order Lagrange multiplier and generalized variational principles of elasticity with more general forms of functions. *Appl Math Mech* 1983; 4: 137–150.

41. Liu GL. A symmetric approach to the research and transformation for variational principles in fluid mechanics with emphasis on inverse and hybrid problems. In: *Experimental and Computational Aerothermodynamics of Internal Flows: Proceedings of the First International Symposium (1st ISAIF)*, Beijing, 8-12 July 1990, pp.128–135.

42. Khater AH, Moussa MHM and Abdul-Aziz SF. Invariant variational principles and conservation laws for some nonlinear partial differential equations with constant coefficients–II. *Chaos Soliton Fract* 2003; 15: 1–13.

43. He JH. Semi-inverse method of establishing generalized variational principles for fluid mechanics with emphasis on turbomachinery aerodynamics. *Int J Turbo Jet Eng* 1997; 14: 23–28.