The Sum over Topologies in Three-Dimensional Euclidean Quantum Gravity

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Abstract

In Hawking’s Euclidean path integral approach to quantum gravity, the partition function is computed by summing contributions from all possible topologies. The behavior such a sum can be estimated in three spacetime dimensions in the limit of small cosmological constant. The sum over topologies diverges for either sign of \( \Lambda \), but for dramatically different reasons: for \( \Lambda > 0 \), the divergent behavior comes from the contributions of very low volume, topologically complex manifolds, while for \( \Lambda < 0 \) it is a consequence of the existence of infinite sequences of relatively high volume manifolds with converging geometries. Possible implications for four-dimensional quantum gravity are discussed.

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In Hawking’s Euclidean path integral approach to quantum gravity \([1]\), the partition function is computed as a weighted sum over topologies, with the contribution of each four-manifold \(M\) determined by a path integral over Riemannian (positive definite) metrics. This program has not been easy to implement — general relativity is nonrenormalizable, the action is apparently unbounded below, and the relevant topologies cannot be classified — but it seems as plausible an approach as any for quantizing general relativity. It would therefore be useful to have a simple model in which more explicit calculations could be made.

Three-dimensional gravity provides such a model. Three-dimensional solutions of the empty space Einstein equations necessarily have constant Riemann curvature, so the classical theory is dramatically simplified. At the quantum level, the three-dimensional model is renormalizable, and its relationship to Chern-Simons theories makes a systematic perturbation expansion possible. The three-dimensional path integral may also have a more direct \((3+1)\)-dimensional interpretation in terms of the finite temperature partition function.

The goal of this paper is to evaluate the sum over three-dimensional topologies to first order in the loop expansion. We shall see that the behavior of this sum depends strongly on the sign of the cosmological constant. For \(\Lambda > 0\), the sum is dominated by very low volume manifolds with complicated topologies, giving a highly nonclassical behavior. For \(\Lambda < 0\), on the other hand, the main contributions come from infinite sequences of relatively high volume manifolds whose geometries converge to those of incomplete (cusped) manifolds. The sensitivity to the sign of \(\Lambda\) comes as something of a surprise, and suggests that even a very small cosmological constant can have dramatic quantum mechanical effects.

1. The Partition Function

For a compact three-manifold \(M\), the “Wick rotated” gravitational path integral takes the form

\[
Z_M = \int [dg] \exp \left\{ \frac{1}{16\pi G} \int_M d^3x \sqrt{\bar{g}} (R[g] - 2\Lambda) \right\},
\]

where the integration is over all Riemannian metrics \(g\) on \(M\), \(R[g]\) is the scalar curvature, and \(\Lambda\) is the cosmological constant. The extrema of the action are Einstein spaces, that is, manifolds and metrics for which

\[
R_{ik}[\bar{g}] = 2\Lambda \bar{g}_{ik}.
\]

In three dimensions, the Ricci tensor completely determines the full curvature; in particular, an Einstein space always has constant curvature. This means that an extremum \((M, \bar{g})\) of the action must be an elliptic \((\Lambda > 0)\), hyperbolic \((\Lambda < 0)\), or flat \((\Lambda = 0)\) manifold. For now, we shall take \(\Lambda \neq 0\); the flat case will be discussed briefly in the conclusion.

In the neighborhood of an extremum, standard techniques may be used to evaluate the path integral \((1.1)\) in the saddle point (or “semiclassical”) approximation. One obtains an expression of the form

\[
Z_M = |\pi_0(Diff M)|^{-1} \sum_{\text{extrema}} D_M \exp \left\{ \text{sign}(\Lambda) \frac{\text{Vol}(M)}{4\pi G |\Lambda|^{1/2}} \right\},
\]

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where \( \text{Vol}(M) \) is the volume of \( M \) with the constant curvature metric rescaled to have curvature \( \pm 1 \). \( |\pi_0(\text{Diff}M)| \) is the order of the mapping class group of \( M \), that is, the number of equivalence classes of diffeomorphisms of \( M \) that cannot be smoothly deformed to the identity. This factor is needed to compensate for the overcounting of physically equivalent metrics that differ by such “large” diffeomorphisms; in the saddle point approximation, it can be omitted as long as we include only one extremum from each diffeomorphism class of metrics. The prefactor \( D_M \) is a combination of determinants coming from small fluctuations around \( \bar{g} \) and from gauge-fixing, and can be computed explicitly by taking advantage of the connection between three-dimensional gravity and Chern-Simons theory \([2]\). The result may be summarized as follows.

Any compact, constant curvature three-manifold \( M \) can be expressed as a quotient space
\[
M = \widetilde{M}/\Gamma ,
\]
where the universal covering space \( \widetilde{M} \) is either the three-sphere (\( \Lambda > 0 \)) or hyperbolic three-space (\( \Lambda < 0 \)), and \( \Gamma \) is a discrete group of isometries of \( \widetilde{M} \). Denote the full group of isometries of \( \widetilde{M} \) by \( G \); \( G \) will be either \( \text{SO}(4) \) (\( \Lambda > 0 \)) or \( \text{SL}(2, \mathbb{C}) \) (\( \Lambda < 0 \)). The group \( \Gamma \) acts on \( \widetilde{M} \) by isometries and on \( G \) by the adjoint action, and one can construct a \( G \)-bundle
\[
E_M = (G \times \widetilde{M})/\Gamma , \tag{1.5}
\]
where the quotient is by this combined action. This bundle arises in the theory of geometric structures \([3]\), and admits a natural flat connection \( \bar{A} \) with a holonomy group isomorphic to \( \Gamma \). The covariant derivative \( D_{\bar{A}} \) then determines a Ray-Singer torsion
\[
T(E_M) = \frac{(\det' \Delta_0)^{3/2}}{(\det' \Delta_1)^{1/2}} , \tag{1.6}
\]
where \( \Delta_k \) is the Laplacian formed from \( D_{\bar{A}} \) acting on \( (\text{Ad} G) \)-valued \( k \)-forms. The definition of such a Laplacian requires a metric; we choose the extremal metric on \( M \) scaled to curvature \( \pm 1 \). Denote the centralizer of \( \Gamma \) in \( G \) by \( Z(\Gamma) \), and let \( \text{Vol}(Z(\Gamma)) \) be its volume with respect to the Haar measure. Up to an overall constant independent of the topology of \( M \), the prefactor is then
\[
D_M = \left( G|\Lambda|^{1/2}|\Gamma| \right)^{\dim Z(\Gamma)/2} \text{Vol}(Z(\Gamma))^{-1} T(E_M)^{1/2} , \tag{1.7}
\]
where \( |\Gamma| \) denotes the order of the group \( \Gamma \).

The torsion dependence of (1.7) follows from two results of Witten. In reference \([2]\), it is shown that three-dimensional gravity can be rewritten as a Chern-Simons theory for the gauge group \( G \). An extremum of the action thus determines a flat connection, with a corresponding bundle given by (1.5). The saddle point approximation can then be extracted from the results obtained in \([1]\) for arbitrary Chern-Simons theories (see also \([3]\) for a careful extension to noncompact gauge groups, and \([4]\) for an application to gravity). Note that for

*The Ray-Singer torsion is independent of this metric only when none of the Laplacians appearing in its definition have zero-modes.
Euclidean quantum gravity, there is no imaginary contribution to the effective action, i.e., no complex phase in (1.7). This is most easily understood by noting that the phase in a Chern-Simons path integral comes from regulating integrals of the form
\[ \int dx \frac{1}{\sqrt{2\pi}} e^{-i\lambda x^2/2}, \]
while for Euclidean gravity the exponent in (1.1) is real and the corresponding integrals are absolutely convergent.

The remaining terms in (1.7) come from ghost zero-modes, that is, from the subgroup \( G_0 \) of gauge transformations that leave the extremal connection \( \bar{A} \) fixed. These factors are of the form \( \text{Vol}(G_0)^{-1} \), appropriately normalized, and have their origin in the standard Faddeev-Popov gauge-fixing procedure: the gauge orbit through \( \bar{A} \) is parameterized by \( G/G_0 \) rather than the full gauge group \( G \), so when the volume \( \text{Vol}(G) \) is factored out of the partition function, a term \( \text{Vol}(G_0) \) remains in the denominator. In our case, \( G_0 \) is isomorphic to the centralizer \( Z(\Gamma) \). Indeed, gauge transformations act on the holonomy by conjugation, so only transformations that commute with every \( \gamma \in \Gamma \) can leave \( \bar{A} \) invariant, while conversely, any \( g \in Z(\Gamma) \) can be extended to all of \( M \) by parallel transport to define a gauge transformation that fixes \( \bar{A} \).

To obtain the proper normalization of the volume \( \text{Vol}(G_0) \), recall that the path integral measure is fixed by the condition
\[ \int [d\epsilon] \exp \left\{ -k \int_M \text{Tr} \epsilon \wedge \ast \epsilon \right\} = 1. \] 
(1.8)
Here \( k = 1/(64\pi G|\Lambda|^{1/2}) \) is the constant multiplying the Chern-Simons action, and the Hodge dual \( \ast \) is defined in terms of the metric of constant curvature \( \pm 1 \) on \( \tilde{M} \) to ensure consistency with (1.6). Let us choose as a basis an orthonormal set of \( (\text{Ad} G) \)-valued eigenfunctions of the Laplacian on \( \tilde{M} \),
\[ \Delta_0 u_\alpha = \lambda_\alpha u_\alpha, \quad \int_{\tilde{M}} \text{Tr} (u_\alpha \wedge \ast u_\beta) = \delta_{\alpha\beta}. \]
(1.9)
A corresponding basis for sections of \( E_M \) is given by
\[ \left\{ u_\alpha \mid u_\alpha(\gamma x) = \gamma^{-1} u_\alpha(x) \gamma \quad \forall \gamma \in \Gamma \right\}. \]
(1.10)
If we now write \( \epsilon = \epsilon^\alpha u_\alpha \), it is evident from (1.8) that the measure must take the form
\[ [d\epsilon] = \prod_\alpha \left( \frac{k}{2\pi|\Gamma|} \right)^{1/2} d\epsilon^\alpha. \]
(1.11)
The factor \( |\Gamma| \) comes from the fact that we are integrating over \( M = \tilde{M}/\Gamma \) rather than \( \tilde{M}/\Gamma \). The generators of \( G_0 \approx Z(\Gamma) \) are now the zero-modes \( D_{\bar{A}} u_\alpha = 0 \). Integrating the measure (1.11) over this space of zero-modes and using the definition of \( k \), we obtain
\[ \text{Vol}(G_0) = \text{const.} \left( G|\Lambda|^{1/2}|\Gamma| \right)^{-\dim Z(\Gamma)/2} \text{Vol}(Z(\Gamma)), \]
(1.12)
\[ ^{\dagger} \]
Up to an overall constant, \( |\Gamma| \) can be replaced by \( \text{Vol}(M)^{-1} \). This permits a simple generalization to the case in which \( \Gamma \) has infinite order.
giving the corresponding factor in (1.7).

In principle, the Chern-Simons formulation also makes it possible to calculate higher order corrections to the saddle point approximation. Such calculations have not yet been carried out (although exact results for SU(2) Chern-Simons theory might be extendible to SO(4)), but we know from (3) that these corrections will be suppressed by factors of $1/k \sim G|\Lambda|^{1/2}$.

For small cosmological constant, the semiclassical approximation (1.7) should therefore be reliable.

2. Positive Cosmological Constant

Let us now consider the case $\Lambda > 0$ in more detail. We are faced with several tasks: we must classify the possible extremal manifolds and metrics $(M, \bar{g})$, or equivalently the discrete groups of isometries $\Gamma$; for each such manifold, we must evaluate the volume $\text{Vol}(M)$ and the Ray-Singer torsion $T(E_M)$; and we must compute the remaining normalization factors in (1.7). Fortunately, all of these tasks are manageable.

Elliptic three-manifolds — three-manifolds of constant positive curvature — were classified by Seifert and Threlfall in 1930 [8]. A recent summary is given by Wolf [9] (see also [10, 11]). Reference [9] contains an explicit description of each admissible group $\Gamma$ of isometries of $S^3$. Given such a group, the volume $\text{Vol}(M)$ is simply

$$\text{Vol}(M) = 2\pi^2/|\Gamma|. \tag{2.1}$$

($2\pi^2$ is the volume of the three-sphere of constant curvature $+1$.) Note that for topologically complicated three-manifolds, the order $|\Gamma|$ becomes large, $\text{Vol}(M)$ is small, and the exponent in (1.3) approaches zero. We shall see later that the behavior of manifolds with $\Lambda < 0$ is dramatically different.

The Ray-Singer torsion for $M$ can be calculated from the results of Ray [12]:

$$T(E_M) = \prod_{k=1}^{|\Gamma|} \left| 1 - e^{2\pi i k/|\Gamma|} \right|^{\frac{1}{2}A_k}, \tag{2.2}$$

where

$$A_k = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} \text{Tr} \rho_A(g) \text{Tr} \rho_0(g^k). \tag{2.3}$$

In this last expression, $\rho_A(g)$ is the adjoint representation of $g$, while $\rho_0(g)$ is the representation of $g$ as a group of rotations in $\mathbb{R}^4$. In particular, if $g$ is a rotation of period $p$, one can pick complex coordinates $z_1$ and $z_2$ such that

$$g z_j = e^{2\pi i \nu_j/p} z_j; \tag{2.4}$$

*Strictly speaking, Ray’s paper only deals with the torsion for one-dimensional representations of $\Gamma$, but our case — in which the adjoint representation is needed — is an easy generalization; see, for instance, [13]. Note that Ray’s normalization differs from that of (1.3).
then
\[ \text{Tr} \rho_0(g^k) = 2 \cos \frac{2\pi k \nu_1}{p} + 2 \cos \frac{2\pi k \nu_2}{p}. \] (2.5)
We also need the Haar volume \( Vol(Z(\Gamma)) \); this is easily calculated from Wolf’s explicit representations of \( \Gamma \).

To obtain some more explicit results, let us work out the path integral for the case of cyclic fundamental groups \( \Gamma \approx \mathbb{Z}_p \). The corresponding three-manifolds are lens spaces \( L_{p,q} \), where \( q \) is an integer relatively prime to \( p \) with \( 1 \leq q \leq p \). Two lens spaces \( L_{p,q} \) and \( L_{p,\overline{q}} \) are topologically distinct unless \( q\overline{q} \equiv 1 \pmod{p} \). In this aspect, lens spaces are exceptional among elliptic three-manifolds, which are usually uniquely determined by their fundamental groups. One might therefore expect lens spaces to dominate the partition function, since for a typical volume \( Vol(M) = 2\pi^2/p \), most of the saddle points contributing to the sum over topologies will be lens spaces. Moreover, as \( p \) becomes large, the exponent in (1.3) will become unimportant, while the number of lens spaces will grow. One might thus anticipate that manifolds with large fundamental groups dominate. This will indeed prove to be the case.

In the representation (2.4), the holonomy group \( \Gamma_{p,q} \) of \( L_{p,q} \) is generated by an element \( g \) for which
\[ \nu_1 = 1, \quad \nu_2 = q, \] (2.6)
while of course \( |\Gamma| = p \). The evaluation of the traces in (2.3) is straightforward, and the sum over elements in \( \Gamma \) reduces to a sum over powers of \( g \). Using the identity
\[ \sum_{j=0}^{p-1} e^{2\pi i m j/p} = \begin{cases} p & \text{if } m = 0 \pmod{p}, \\ 0 & \text{otherwise} \end{cases}, \] (2.7)
we find a Ray-Singer torsion (for \( q \neq \pm 1 \pmod{p} \))
\[ T(E_{p,q})^{1/2} = 4 \left( \cos \frac{2\pi}{p} - \cos \frac{2\pi q}{p} \right) \left( \cos \frac{2\pi}{p} - \cos \frac{2\pi \overline{q}}{p} \right), \] (2.8)
where \( 1 \leq \overline{q} \leq p \) is the integer determined by \( q\overline{q} = 1 \pmod{p} \). Moreover, for \( q \neq \pm 1 \pmod{p} \) the centralizer \( Z(\Gamma_{p,q}) \) consists of rotations that can be represented as
\[ g z_j = e^{2\pi i \theta_j} z_j, \] (2.9)
so \( \dim Z(\Gamma_{p,q}) = 2 \). The Haar volume \( Vol(Z(\Gamma_{p,q})) \) is a constant independent of \( p \) and \( q \), and the factor \( |\pi_0(Diff L_{p,q})| \) can be omitted from (1.3) as long as we count only one saddle point contribution per lens space.

Combining these results, we find a partition function
\[ Z_{L_{p,q}} = c G \Lambda^{1/2} / p \left( \cos \frac{2\pi}{p} - \cos \frac{2\pi q}{p} \right) \left( \cos \frac{2\pi}{p} - \cos \frac{2\pi \overline{q}}{p} \right) \exp \left\{ \frac{\pi}{2 G \Lambda^{1/2} p} \right\} \] (2.10)
where \( c \) is a constant independent of \( p \) and \( q \). Equation (2.10) will fail to hold when \( q = \pm 1 \pmod{p} \); this case could be dealt with separately, but it is unimportant in understanding the qualitative behavior of the sum over topologies, so we shall simply ignore it.
To study the asymptotic behavior of the sum over topologies, let us first sum (2.10) over integers $1 \leq q \leq p$ relatively prime to $p$. Such a set of integers is called a reduced system of residues for $p$, and will be denoted, with a slight abuse of notation, by $\mathbb{Z}_p^*$. This sum will overcount topologies by approximately a factor of two, since $L_{p,q} \approx L_{p,\bar{q}}$. (It is easily checked that as $q$ ranges over a reduced system of residues, so does $\bar{q}$, so $L_{p,q}$ will appear twice unless $q = \bar{q}$, i.e., unless $q^2 = 1 \pmod{p}$.)

The sum over $q$ can be written in terms of three standard functions in number theory. Euler’s function $\phi(p)$ is the number of positive integers $1 \leq q \leq p$ relatively prime to $p$. Ramanujan’s sum is

$$c_p(m) = \sum_{q \in \mathbb{Z}_p^*} \exp\{2\pi i mq/p\},$$

and Kloosterman’s sum is

$$S(m, n, p) = \sum_{q \in \mathbb{Z}_p^*} \exp\{2\pi i (mq + n\bar{q})/p\},$$

with $\bar{q}$ defined as above. It is now easy to check that

$$\sum_{q \in \mathbb{Z}_p^*} Z_{L_{p,q}} = cG\Lambda^{1/2}p$$

(2.13)

$$\cdot \left( \phi(p) \cos^2 \frac{2\pi}{p} - 2c_p(1) \cos \frac{2\pi}{p} + 2(S(1, 1, p) + S(1, -1, p)) \right) \exp \left\{ -\frac{\pi}{2GA^{1/2}p} \right\}.$$

For large $p$, the asymptotic behavior of these quantities is known [14, 13]:

$$\phi(p) \sim \frac{6p}{\pi^2}$$

$$c_p(1) \sim 1$$

$$S(1, \pm 1, p) \sim p^{1/2+\epsilon}.$$  

Hence

$$\sum_{q \in \mathbb{Z}_p^*} Z_{L_{p,q}} \sim c'G\Lambda^{1/2}p^2$$

(2.15)

for large $p$. As anticipated, the sum over topologies is dominated by manifolds with arbitrarily large fundamental groups, and correspondingly small volumes.

There should be no serious difficulty in extending this analysis to other elliptic manifolds. Some preliminary observations support the suggestion that the lens spaces give the most important contribution. For instance, if $\Gamma$ is a binary dihedral group $D_{4n}^*$, $S^3/\Gamma$ is a “prism manifold.” The centralizer $Z(\Gamma)$ is then three-dimensional, so the $\text{Vol}(\mathcal{G}_0)$ term in the prefactor (1.7) goes as $|\Gamma|^{3/2}$, but there is only one prism manifold for each value of $|\Gamma|$, so the overall growth is slower than (2.15). For $\Gamma = \mathbb{Z}_m \times D_{2n+1}^*$, there is more than one relevant representation of $\Gamma$ for each pair $(m, n)$, and may therefore be more than one inequivalent manifold (I do not know which representations give diffeomorphic manifolds). In this case, however, the centralizer $Z(\Gamma)$ is only one-dimensional, so the growth is again likely to be slower than (2.15).
3. Negative Cosmological Constant

We next turn to the case $\Lambda < 0$. Classical extrema of the Einstein action are now hyperbolic manifolds, manifolds of constant negative curvature. Unlike elliptic manifolds, hyperbolic manifolds have not been completely classified, so a systematic computation is not yet possible. Nevertheless, some useful qualitative conclusions can be drawn.

Let us first consider the exponential term in the partition function (1.3). For $\Lambda < 0$, manifolds with high volumes are exponentially suppressed. In contrast to the elliptic case, more complicated fundamental groups now generally give larger volumes: $\Gamma$ can be viewed as a set of gluing instructions for tetrahedral cells of $M$, and more complicated groups require more “building blocks.” One might therefore expect the partition function to be dominated by low volume, topologically simple manifolds.

Next, the $\text{Vol}(\mathcal{G}_0)$ factor in $D_M$ drops out of (1.7) when $\Lambda$ is negative. Geometrically, the centralizer $Z(\Gamma)$ is a measure of the size of the isometry group of $M$, and generic hyperbolic three-manifolds admit no continuous isometries. Thus $\dim Z(\Gamma)$ usually vanishes, and $D_M$ depends only on the Ray-Singer torsion.

But while large volumes tend to suppress the contributions of manifolds with complicated topologies, there is a competing effect coming from statistical weight, or “entropy” — the number of manifolds contributing to the partition function may peak near certain volumes, possibly overcoming the exponential suppression. This phenomenon is discussed in [6] in a slightly different context; it is demonstrated there that the path integral for manifolds with a single boundary component is dominated by such peaks.

To compare the effects of “action” and “entropy” in (1.3), we must first understand the distribution of volumes of hyperbolic three-manifolds. The crucial result is due to Thurston [16]. We start with a definition: a cusp of a hyperbolic three-manifold $M$ is a neighborhood of an embedded circle that is “infinitely far away” from the rest of the manifold in the hyperbolic metric. Topologically, a neighborhood of a cusp is diffeomorphic to $T^2 \times [t_0, \infty)$, where $T^2$ is a two-dimensional torus. (The circle itself is not part of $M$, which is complete but not compact.) Metrically, we can take the upper half space model for $\mathbb{H}^3$, with the standard constant negative curvature metric

$$ds^2 = t^{-2}(dx^2 + dy^2 + dt^2) ;$$

a neighborhood of a cusp then looks like a region

$$N = \{(x, y, t) : t > t_0, z \sim z + 1, z \sim z + \tau\} .$$

Although cusped manifolds are incomplete, their volumes are finite, since the area of a toroidal cross section of $N$ shrinks exponentially as the proper distance $(\log t)$ goes to infinity.

Thurston has shown that the set of volumes of compact hyperbolic three-manifolds has accumulation points precisely at the volumes of cusped manifolds. Given any cusped manifold $M_\omega$, one can construct an infinite family of hyperbolic manifolds $M_{p,q}$, where $p$ and $q$ are relatively prime integers, by means of a technique known as hyperbolic Dehn surgery.[7]

*Strictly speaking, a finite number of relatively prime pairs $(p, q)$ must be excluded. Without loss of generality, we can also take $p$ to be positive.*
For $p^2 + q^2$ large, there is a reasonable sense in which the geometry of the manifolds $M_{p,q}$ can be said to converge to that of $M_\omega$. In particular, the volumes $Vol(M_{p,q})$ converge (from below) to $Vol(M_\omega)$. Thurston shows that every accumulation point in the set of volumes can be obtained in this manner. Any important peak in the partition function due to “entropy” is thus likely to occur near a cusped manifold.

The behavior of the Ray-Singer torsion $T(E_M)$ under hyperbolic Dehn surgery has been investigated in [6]. The results may be summarized as follows. Given relatively prime integers $p$ and $q$, define two new integers $r$ and $s$ by the condition $ps - qr = 1$. Then for $p$ and $q$ sufficiently large, the torsion $T_{p,q}$ of $M_{p,q}$ is

$$T_{p,q}^{1/2} = T_\omega^{1/2} \sin^2 \left( \frac{2\pi r}{p} \right) + O\left( \frac{1}{p} \right),$$

(3.3)

where $T_\omega$ is a suitably defined torsion for the cusped manifold $M_\omega$. The behavior of the volumes $Vol(M_{p,q})$ is also known [17]:

$$Vol(M_{p,q}) = Vol(M_\omega) + O\left( \frac{1}{p^2 + q^2} \right).$$

(3.4)

The partition function for the sum over topologies $M_{p,q}$ obtained by hyperbolic Dehn surgery can thus be approximated as

$$Z \sim \sum_{(p,q)=1} T_{\omega}^{1/2} \sin^2 \frac{2\pi r}{p} \exp \left\{ -\frac{Vol(M_\omega)}{4\pi G|\Lambda|^{1/2}} \right\}.$$  

(3.5)

As in the elliptic case, this sum can be partially evaluated in terms of functions that occur in number theory. Let us write $q = np + q'$, where $1 \leq q' \leq p$ is relatively prime to $p$, i.e., $q' \in \mathbb{Z}_p^\ast$. It may be shown that as $q'$ varies a reduced set of residues for $p$, so does $r$. The sum over $q$ in (3.3) thus becomes

$$\sum_n \sum_{r \in \mathbb{Z}_p^\ast} \sin^2 \frac{2\pi r}{p},$$

which can be expressed in terms of $\phi(p)$ and $c_p(2)$. One obtains an asymptotic behavior of the form

$$Z \sim \sum_{n,p} cpT_{\omega}^{1/2} \exp \left\{ -\frac{Vol(M_\omega)}{4\pi G|\Lambda|^{1/2}} \right\},$$

(3.6)

which clearly diverges.

In contrast to the elliptic case, divergences in the sum over topologies may now come from manifolds with fairly large volumes. The smallest cusped hyperbolic manifold has a volume $v_0 \approx 1.01494$, and manifolds with $m$ cusps have volumes

$$Vol(M) \geq mv_0;$$

(3.7)

the inequality is strict for $m > 2$ [18]. It is not entirely clear how to insert a cut-off in the sum (3.6), so it is difficult to compare divergences coming from different cusped manifolds. But hyperbolic Dehn surgery can be performed independently on each cusp of an $m$-cusped manifold, giving $m$ independent divergent sums, so it can be plausibly argued that the main contributions will come from manifolds with large numbers of cusps, and hence large volumes.
4. Conclusions

We have now seen that the sum over topologies in three-dimensional Euclidean quantum gravity depends dramatically on the sign of the cosmological constant. For $\Lambda > 0$, the behavior of the partition function is highly nonclassical — the main contributions come from manifolds with very small volumes and very complex topologies. For $\Lambda < 0$, the dominant contributions come from infinite sequences of relatively high volume manifolds, although still with complicated topologies. In both cases, it is clear that the “leading” saddle point is relatively unimportant.

The particular form of these partition functions certainly depends strongly on the fact that we are working in three dimensions. But the qualitative conclusions may well generalize to four dimensions. The saddle points that contribute to the sum over topologies will again depend strongly on the sign of $\Lambda$, and as in three dimensions, the exponential dependence on the classical action will tend to suppress complicated topologies much more strongly when $\Lambda < 0$. Moreover, as in three dimensions, the statistical weight will be of key importance when $\Lambda < 0$, since the number of topologically distinct Einstein spaces with negative curvature grows very rapidly with volume \[19\].

Three important approximations have been made in these calculations. First, only the one-loop contribution to the prefactor (1.7) has been computed. For large cosmological constant, this is a significant omission. But for $\Lambda$ small, higher loop corrections are suppressed by powers of $G|\Lambda|^{1/2}$, and should be unimportant for any given topology. Note, however, that the prefactor can involve powers of $|\Gamma|$, so the two-loop contribution of a topologically complex manifold could be comparable to the one-loop contribution of a simpler one. This can only be determined by a careful evaluation of the $|\Gamma|$ dependence of higher loop corrections.

Second, we have not yet summed over all saddle points. For $\Lambda$ positive, there should be no serious difficulty in extending these results to arbitrary elliptic manifolds, and it is likely that the lens space contributions will dominate. For $\Lambda$ negative, the problem is more difficult, since hyperbolic three-manifolds have not been classified. It would be particularly useful to find a way to compare the contributions of the series (3.6) for neighborhoods of different cusped manifolds.

Third, we have not taken into account the contributions of topologies for which the Einstein equations have no classical solution. In Yang-Mills theory, the contributions of approximate “multi-instanton” saddle points are very important. The analog here would be the contribution of connected sums $M = M_1 \# \ldots \# M_n$. (The connected sum of $M_1$ and $M_2$ is formed by cutting out a three-ball from each manifold and identifying the resulting spherical boundaries.) In pure Chern-Simons theory, the partition function for a connected sum is \[4\]

$$Z(M_1 \# \ldots \# M_n) = \frac{Z(M_1) \cdot \ldots \cdot Z(M_n)}{Z(S^3)^{n-1}}. \quad (4.1)$$

Gravity is not quite a pure Chern-Simons theory, however — because of the mapping class group dependence of (1.3), we need an additional factor of

$$\frac{|\pi_0(Diff M_1)| \cdots |\pi_0(Diff M_n)|}{|\pi_0(Diff (M_1 \# \ldots \# M_n))|}, \quad (4.2)$$
which counts the new diffeomorphisms that appear when one forms a connected sum. Such diffeomorphisms include permutations of identical summands $M_i$ and $M_j$, giving a factor of $1/m!$ for each set of $m$ identical manifolds in the connected sum. If this were the only contribution, the effect of “multi-instantons” would be simply to exponentiate the “one-instanton” partition function we have already investigated. An assumption similar to this is made in wormhole dynamics [20].

In fact, however, connected summation introduces a large number of new diffeomorphisms, and the factor (4.2) is generally much smaller than $1/m!$. In addition to permutations of summands, a connected sum admits “slide diffeomorphisms,” diffeomorphisms in which one factor is dragged around a closed curve in another factor [21,22,23]. As the order $|\Gamma_i|$ of the fundamental group of $M_i$ grows, the number of such diffeomorphisms increases rapidly. It is therefore plausible that the “multi-instanton” contribution will fall off quickly as the topology of each instanton becomes more complicated. This effect was considered for $M = S^2 \times S^1$ in [24], although in the somewhat different framework of the Lorentzian path integral; a similar phenomenon lies behind the “sewing problem” in string theory [25]. Clearly, a more careful analysis would be desirable.

Finally, let us briefly address the case of vanishing cosmological constant. The saddle points will now be flat three-manifolds, and the exponential in (1.3) will vanish. Flat three-manifolds are completely classified [4] — all are finitely covered by the torus — and there is nothing to prevent us from repeating the calculations above. In fact, we can do better: when $\Lambda = 0$, the the partition function $Z_M$ is given exactly in terms of an integral over Ray-Singer torsions [26]. However, as Witten discusses in [26], the resulting integral typically diverges, so it is not easy to compare contributions from different topologies.

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