A Fast Heuristic for Exact String Matching
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Abstract
Given a pattern string $P$ of length $n$ consisting of $\delta$ distinct characters and a query string $T$ of length $m$, where the characters of $P$ and $T$ are drawn from an alphabet $\Sigma$ of size $\Delta$, the exact string matching problem consists of finding all occurrences of $P$ in $T$. For this problem, we present a randomized heuristic that in $O(n\delta)$ time preprocesses $P$ to identify sparse($P$), a rarely occurring substring of $P$, and then use it to find all occurrences of $P$ in $T$ efficiently. This heuristic has an expected search time of $O\left(\frac{m}{\min(|\text{sparse}(P)|, \Delta)}\right)$, where $|\text{sparse}(P)|$ is at least $\delta$. We also show that for a pattern string $P$ whose characters are chosen uniformly at random from an alphabet of size $\Delta$, $E[|\text{sparse}(P)|]$ is $\Omega(\Delta \log(\frac{2\Delta}{2\Delta - \delta}))$.

Key words: Keywords: Exact String Matching; Combinatorial Pattern Matching; Computational Biology; Bio-informatics; Analysis of Algorithms; Fast Heuristics.

1 Introduction

Given a pattern string $P$ of length $n$ consisting of $\delta$ distinct characters and a query string $T$ of length $m$, where the characters of $P$ and $T$ are drawn from an alphabet $\Sigma$ of size $\Delta$, the exact string matching problem consists of finding all occurrences of $P$ in $T$. This is a fundamental problem with wide range of applications in Computer Science (used in parsers, word processors, operating systems, web search engines, image processing and natural language processing), Bioinformatics and Computational Biology (Sequence Alignment and Database Searches). The algorithms for exact string matching can be broadly categorized into the following categories: (1) character based comparison algorithms, (2) automata based algorithms, (3) algorithms based on bit-parallelism and (4) constant-space algorithms. In this paper, our focus is on designing efficient character based comparison algorithms for exact string matching. For a comprehensive survey of all categories of exact string matching algorithms, we refer the readers to Baeza-Yates[17], Gusfield[25], Charras and Lecroq [28], Crochemore et al [29] and Faro and Lecroq [30].

A Typical character based comparison algorithm can be described within the following general framework as follows:

(1) First, initialize the search window to be the first $n$ characters of the query string $T$ (i.e. align the $n$ characters of the pattern string $P$ with the first $n$ characters of $T$).

(2) Repeat the following until the search window is no longer contained within the query string $T$:...
inspect the aligned pairs in some order until there is either a mis-match in an aligned pair or there is a complete match among all the \(n\) aligned pairs. Then shift the search window to the right. The order in which the aligned pairs are inspected and the length by which the search window is shifted differs from one algorithm to another.

The mechanism that the above framework provides is usually referred to as the *sliding window mechanism* [30, 31]. The algorithms that employ the sliding window mechanism can be further classified based on the order in which they inspect the aligned pairs into the following broad categories: (1) left to right scan; (2) right to left scan; (3) scan in specific order, and (4) scan in random order or scan order is not relevant. The algorithms that inspect the aligned pairs from left to right are the most natural algorithms; the algorithms that inspect the aligned pairs from right to left generally perform well in practice; the algorithms that inspect the aligned pairs in a specific order yield the best theoretical bounds. For a comprehensive description of the exact string matching algorithms and access to an excellent framework for development, testing and analysis of exact string matching algorithms, we refer the readers to the *SMART tool* (string matching research tool) of Faro and Lecroq [31].

For algorithms that inspect aligned pairs from left to right, Morris and Pratt [1] proposed the first known linear time algorithm. This algorithm was improved by Knuth, Morris and Pratt [4] and requires \(O(n)\) preprocessing time and a worst case search time of at most \(2m - 1\) comparisons. For small pattern strings and reasonable probabilistic assumptions about the distribution of characters in the query string, hashing [2,9] provides an \(O(n)\) preprocessing time and \(O(m)\) worst case search time solution. For pattern strings that fit within a word of main memory, Shift-Or [16,21] requires \(O(n + \Delta)\) preprocessing time and a search time of \(O(m)\) and can also be easily adapted to solve approximate string matching problems. For algorithms that inspects aligned pairs from right to left, Boyer-Moore algorithm [3] is one of the classic algorithms that requires \(O(n\delta)\) preprocessing time and a worst case search time of \(O(nm)\) but in practice is very fast. There are several variants that simplify the Boyer-Moore algorithm and mostly avoid its quadratic behaviour. Among the variants of Boyer-Moore, the algorithms of Apostolico and Giancarlo [7,24], Crochemore et al [13, 23] (Turbo BM), and Colussi (Reverse Colussi) [12, 22] have \(O(m)\) worst case search time and are efficient in minimizing the number of character comparisons, whereas the Quick Search [10], Reverse Factor [19], Turbo Reverse Factor [24], Zhu and Takaoka [8], and Berry and Ravindran [27] algorithms are very efficient in practice. For Algorithms that inspects the aligned pairs in a specific order, Two Way algorithm [13], Colussi [12], Optimal Mismatch and Maximal Shift [10], Galil and Giancarlo [18],...
Skip Search, KMP Skip Search and Alpha Skip Search [26] are some of the well known algorithms. Two way algorithm was the first known linear time optimal space algorithm. The Colussi algorithm improves the Knuth-Morris-Pratt algorithm and requires at most $\frac{3n}{2}$ text character comparisons in the worst case. The Galil and Giancarlo algorithm improves the Colussi algorithm in one special case which enables it to perform at most $\frac{4n}{3}$ text character comparisons in the worst case. For Algorithms that inspects the aligned pairs in any order, the Horspool [5], Quick Search [10], Tuned Boyer-Moore [14], Smith [15] and Raita [20] algorithms are some of the well known algorithms. All these algorithms have worst case search time that is quadratic but are known to perform well in practice.

**Our Results:** In this paper, we present a simple randomized heuristic $R$ that essentially preprocesses $P$ in $O(n\delta)$ time to identify $\text{sparse}(P)$, a rarely occurring substring of $P$ characterized by two characters of $P$, and use it to find occurrences of $P$ in $T$ efficiently. Heuristic $R$ has a worst case search time of $O(mn)$ and an expected search time of $O\left(\frac{m}{\text{min}(|\text{sparse}(P)|,\Delta)}\right)$. The worst case search time of $O(mn)$ is realized due to pathological pattern instances (i.e. periodic patterns consisting of few distinct characters) that are rare and highly unlikely to occur in practice. However, our algorithm has a superior provable sub-linear expected search time when compared to existing algorithms. In addition, we prove that for a pattern string $P$ whose characters are chosen uniformly at random from an alphabet of size $\Delta$, $E[|\text{sparse}(P)|]$ is $\Omega(\Delta \log(\frac{2\Delta}{2\Delta-\delta}))$, where $\delta$ is the number of distinct characters in $P$.

The rest of this paper is organized as follows: In Section 2, we present our randomized heuristic $R$. In Section 3, we present the analysis of our heuristic. In Section 4, we show for a pattern string $P$ whose characters are chosen uniformly at random from an alphabet of size $\Delta$, we lower bound $E[|\text{sparse}(P)|]$. Finally, in Section 5 we present our conclusions and future work.

## 2 A Randomized Heuristic for Exact String Matching

In this section, we present a randomized Heuristic $R$ that given a pattern string $P$ and a query string $T$, finds all occurrences of $P$ within $T$. First, we introduce some definitions that are essential for defining our Heuristic.

**Definitions 2.1** Given a pattern string $P$, and an ordered pair of characters $u, v \in \Sigma$ (not necessarily distinct), we define $\text{sparse}^{(u,v)}(P)$, the 2-sparse pattern of $P$ with respect to $u$ and $v$, to be the rightmost occurrence of a substring of $P$ of longest length that starts with $u$, ends with $v$, but does not contain $u$ or $v$ within it. If such a substring does not exist then $\text{sparse}^{(u,v)}(P)$ is defined to be
the empty string. We define \( \text{sparse}(P) \) to be the right most occurring among the longest 2-sparse patterns of \( P \).

**Definitions 2.2** Given \( \text{sparse}(P) \), the longest 2-sparse pattern of \( P \), we define \( \text{startc}(P) \) and \( \text{endc}(P) \) to be the respective first and last characters of \( \text{sparse}(P) \), and \( \text{startpos}(P) \) and \( \text{endpos}(P) \) be the respective indices of the first and last characters of \( \text{sparse}(P) \) in \( P \). For \( c \in \Sigma \), if \( c \in \text{sparse}(P) \), \( \text{shift}^c(P) \) is the distance between the rightmost occurrence of \( c \) in \( \text{sparse}(P) \) and the last character of \( \text{sparse}(P) \). If \( c \) is not present in \( P \) then \( \text{shift}^c(P) \) is set to \( n \), the length of \( P \). If \( c \) is present in \( P \) but not in \( \text{sparse}(P) \) then \( \text{shift}^c(P) \) is set to \( |\text{sparse}(P)| + 1 \).

**Example 1:** Let \( P = "abcabdacabdbb" \). From definition, we can see \( \text{sparse}^{(a,a)}(P) = "abda" \), \( \text{sparse}(a,b)(P) = "ab" \) (substring starting at location 9), \( \text{sparse}^{(a,c)}(P) = "abc" \), \( \text{sparse}^{(a,d)} = "abd" \) (substring starting at location 9), \( \text{sparse}^{(b,a)}(P) = "bda" \), \( \text{sparse}^{(b,b)}(P) = "bdacab" \), \( \text{sparse}^{(b,c)}(P) = "bdac" \), \( \text{sparse}^{(b,d)}(P) = "bd" \) (substring starting at location 10), \( \text{sparse}^{(c,a)}(P) = "ca" \) (substring starting at location 8), \( \text{sparse}^{(c,b)}(P) = "cab" \) (substring starting at location 8) and \( \text{sparse}^{(c,c)}(P) = "cabdac" \). Therefore, \( \text{sparse}(P) = "bdacab" \), \( \text{startc}(P) = "b" \), \( \text{endc}(P) = "b" \), \( \text{startpos}(P) = 5 \), and \( \text{endpos}(P) = 10 \). \( \text{shift}^a(P) = 1 \), \( \text{shift}^b(P) = 0 \), \( \text{shift}^c(P) = 2 \), and \( \text{shift}^d(P) = 4 \).

**Basic Idea:** First, we preprocess \( P \) to identify \( \text{sparse}(P) \), a rarely occurring substring of \( P \) characterized by two characters in \( P \), and compute statistics of the occurrence of characters of \( P \) in \( \text{sparse}(P) \) relative to \( \text{endc}(P) \) (the last character of \( \text{sparse}(P) \)). Then, during the search phase, we first set the search window to be the first \( n \) characters of \( T \) (i.e. align the \( n \) characters of \( P \) to the first \( n \) characters of \( T \)). Then, until the search window reaches the end of \( T \), we repeatedly check whether there is a match between the first and last characters of \( \text{sparse}(P) \) (i.e. \( \text{startc}(P) \) and \( \text{endc}(P) \)) and their respective aligned characters \( c \) and \( d \) in the search window. The following three situations are possible:

(i) [Type-1] A mismatch between \( \text{endc}(P) \) and its corresponding aligned character \( c \): In this case the search window is shifted by \( \text{shift}^c(P) \) (i.e. the distance in \( \text{sparse}(P) \) between the right most occurrence of \( c \) and \( \text{endc}(P) \)) and then continues.

(ii) [Type-2] A match between \( \text{endc}(P) \) and its corresponding aligned character \( c \) but a mismatch between the \( \text{startc}(P) \) and its corresponding aligned character \( d \): In this case the search window is shifted by at least \( |\text{sparse}(P)| \) and then continues.

(iii) [Type-3] a match between \( \text{endc}(P) \) and \( \text{startc}(P) \) with their respective aligned characters \( c \) and \( d \): In this case \( R \) invokes routine \( \text{Random} - \text{Match} \) to verify an exact match between the \( n \)
characters of $P$ and their respective aligned characters in the search window. $Random-Match$ inspects the $n$ aligned pairs in random order until it encounters a mismatch or finds an exact match (i.e. match in all the $n$ aligned pairs). If it finds an exact match then it reports the starting location of the exact match in $T$ and then shifts the search window by at least $|\text{sparse}(P)|$ and continues. Otherwise, it shifts the search window by at least $|\text{sparse}(P)|$ and continues.

**Randomized Heuristic $R$**

**Input(s):** (1) Pattern string $P$ of length $n$;

(2) Query string $T$ of length $m$;

**Output(s):** The starting positions of the occurrences of $P$ in $T$;

**Preprocessing:** (1) Scan $P$ from left to right and compute

[a] for each $u,v \in P$, $\text{sparse}^{(u,v)}(P)$;

[b] $\text{sparse}(P) = |\text{sparse}^{(a,b)}(P)| = \max_{u,v \in P} |\text{sparse}^{(u,v)}(P)|$.

(2) From $\text{sparse}(P)$, compute $\text{startc}(P),$ $\text{endc}(P),$ $\text{startpos}(P),$ and $\text{endpos}(P)$.

(3) For $c \in \Sigma,$ compute $\text{shift}^c(P)$.

**Search:**

[1] [a] Set $i = 0$ and $j = n$ [Search Window set to $[1..n]$]

[b] Set $\hat{j} = \text{endpos}(P)$ and $\hat{i} = \text{startpos}(P)$; [Indices of last and first characters of $\text{sparse}(P)$ in $P$]

[2] while ($j < m$) [While the search window is contained in $T$]

[a] Let $c = T[i + \hat{j}]$; $d = T[i + \hat{i}]$; [Characters in search window aligned with the last and first characters of $\text{sparse}(P)$]

[i] if ($c \neq \text{endc}(P)$) [Type-1 event]

\[i = i + \text{shift}^c(P); j = j + \text{shift}^c(P);\] [Shift window by $\text{shift}^c(P)$]

[ii] elseif ($d \neq \text{startc}(P)$) [Type-2 event]

if ($\text{startc}(P) == \text{endc}(P)$)

\[i = i + |\text{sparse}(P)|; j = j + |\text{sparse}(P)|\] [Shift window by $|\text{sparse}(P)|$]

else

\[i = i + |\text{sparse}(P)| + 1; j = j + |\text{sparse}(P)| + 1\] [Shift window by $|\text{sparse}(P)| + 1$

[iii] elseif [Type-3 event]

Call $Random-Match(P[1..n], T[i + 1..i + n])$; [Look for $P$ in $T[i + 1..j]$]

if ($\text{startc}(P) == \text{endc}(P)$)

\[i = i + |\text{sparse}(P)|; j = j + |\text{sparse}(P)|\] [Shift window by $|\text{sparse}(P)|$]

else
\[ i = i + |\text{sparse}(P)| + 1; \ j = j + |\text{sparse}(P)| + 1. \]

Subroutine **Random-Match**

Input(s): Strings \( P \) and \( Q \) each of length \( n \);

Output(s): If there is an exact match between the strings \( P \) and \( Q \) then return "TRUE" else "FALSE";

begin

Let \( \pi = (\pi(1), \pi(2), ..., \pi(n)) \) be a permutation drawn randomly from the set of \( n! \) permutations of integers 1 to \( n \);

for \((i = 1; i \leq n; i = i + 1)\)

if \( P[\pi(i)] \neq Q[\pi(i)] \)

return FALSE;

return TRUE;

end

**Remark:** We can view the Heuristic \( R \) as using the first and last characters of \( \text{sparse}(P) \) to decompose \( T \) in \( O\left(\frac{m}{\min(|\text{sparse}(P)|, \Delta)}\right) \) comparisons into at most \( \frac{m}{|\text{sparse}(P)|} \) substrings of \( T \) of length \( n \) that are potential candidates for an exact match with \( P \). Then each of these \( n \) candidate substrings are inspected for an exact match with \( P \) using Random – Match in \( O(1) \) expected number of comparisons.

### 3 Analysis of Algorithm \( R \)

In this section, we present the analysis of the randomized heuristic \( R \). For rest of this paper we assume that the pattern string \( P \) is an arbitrary string of length \( n \) consisting of \( \delta \) distinct characters that are drawn from an alphabet \( \Sigma \) of size \( \Delta \) and query string \( T \) is a string of length \( m \) whose characters are drawn from \( \Sigma \) uniformly at random. We now present the main results and their proofs.

**Theorem 1** Randomized Heuristic \( R \) finds all occurrences of \( P \) in \( T \).

**Proof** The Algorithm \( R \) during its search phase looks for \( P \) by first looking for a match between the last and first characters of \( \text{sparse}(P) \) and the characters of \( T \) in the search window at offsets \( \text{endpos}(P) \) and \( \text{startpos}(P) \) respectively. Let \( c \) and \( d \) be the respective characters in the search window at offsets \( \text{endpos}(P) \) and \( \text{startpos}(P) \). The following three scenarios (events) are possible. (i) Type-1 event happens if there is a mismatch between \( c \) and \( \text{endc}(P) \) (ii) Type-2 event happens if there is a match between \( c \) and \( \text{endc}(P) \) and a mismatch between \( d \) and \( \text{startc}(P) \), and (iii) Type-3 event happens when there is a match between \( c \) and \( \text{endc}(P) \) and a match between \( d \) and \( \text{startc}(P) \).
In the case of Type-1 event, the search window is shifted to the right by $\text{shift}^c(P)$. Recall that if $c$ is present in $\text{sparse}(P)$ then $\text{shift}^c(P)$ is the distance in $\text{sparse}(P)$ between the right most occurrence of $c$ and $\text{endc}(P)$. If $c$ is present in $P$ but not in $\text{sparse}(P)$ then $\text{shift}^c(P)$ is set to $|\text{sparse}(P)|+1$, otherwise $\text{shift}^c(P)$ is set to $n$, the length of $P$. Now, by shifting the search window by $\text{shift}^c(P)$, we will show that no occurrence of $\text{sparse}(P)$ will be skipped and hence no occurrence of $P$ will be skipped. There are two situations possible depending on whether or not $\text{endc}(P)$ occurs in $T$ within the shifted interval. If $\text{endc}(P)$ did not occur within the shifted interval of $T$ then we can see that no occurrence of $\text{sparse}(P)$ can end within the shifted interval. Hence we will not skip $\text{sparse}(P)$ and hence not skip $P$. Now, we consider the situation when $\text{endc}(P)$ occurs within the shifted interval. This implies that $c$ occurs in the search window less than $\text{shift}^c(P)$ positions to the right end of the search window. From definition of $\text{shift}^c(P)$, we know that the right most occurrence of $c$ in $\text{sparse}(P)$ will be $\text{shift}^c(P)$ positions to the left of $\text{endc}(P)$. This implies that no occurrence of $\text{sparse}(P)$ can end within the shifted portion of the search window. Hence we are done.

In the case of Type-2 event, we know that $c == \text{endc}(P)$ but $d \neq \text{startc}(P)$ and the search window is shifted to the right by $|\text{sparse}(P)|$ ($|\text{sparse}(P)|+1$) if $\text{startc}(P) == \text{endc}(P)$ ($\text{startc}(P) \neq \text{endc}(P)$). Notice in this case, from definition of $\text{sparse}(P)$ we know that the character $\text{endc}(P)$ does not occur inside $\text{sparse}(P)$, hence $\text{sparse}(P)$ cannot end within the shifted portion of the search window. Hence we are done.

In the case of Type-3 event, we can observe that $c == \text{endc}(P)$ and $d = \text{startc}(P)$ and after invoking the $\text{Random-Match}$ subroutine we shift the window to the right by either $|\text{sparse}(P)|$ ($|\text{sparse}(P)|+1$) if $\text{startc}(P) == \text{endc}(P)$ ($\text{startc}(P) \neq \text{endc}(P)$). Notice in this case, from definition of $\text{sparse}(P)$ we know that the characters $\text{endc}(P)$ and $\text{startc}(P)$ do not occur inside $\text{sparse}(P)$, hence $\text{sparse}(P)$ cannot end within the shifted portion of the search window. Hence we are done.

Theorem 2 Given any pattern string $P$ of length $n$ with $\delta$ distinct characters and a query string $T$ of length $m$, Heuristic $R$ finds all occurrences of $P$ in $T$ in $O\left(\frac{m}{\min(|\text{sparse}(P)|, \Delta)}\right)$ expected time, where $|\text{sparse}(P)|$ is at least $\delta$.

Proof We bound the expected number of comparisons performed by Heuristic $R$ during its search phase by looking at the expected number of comparisons performed during each of the three type of events it encounters in comparison to the expected length by which the search window is shifted. For Type-1 events the number of character comparisons involving the query string $T$ is 1 and for Type-2
events it is 2. For Type-3 events, after two character comparisons we are invoking the Random-Match subroutine. From Lemma 4, we know the expected number of matches before a mismatch while invoking \textit{Random-Match} is \(O(1)\). Therefore the number of character comparisons for Type-3 event is \(O(1)\). Now since the number of character comparisons involving \(T\) is \(O(1)\) for all three events, the total number of comparisons during the search phase of Heuristic \(R\) is bound by the total number of events. From Lemma 3, we know that the expected length by which the search window is shifted after encountering a Type-1, Type-2 or Type-3 event is at least \(\min(|\text{sparse}(P)|, \Delta)\). Since the total length by which the search window can be shifted is \(m\), the total number of events is therefore \(O(\min(|\text{sparse}(P)|, \Delta))\). From Lemma 5 we know that \(|\text{sparse}(P)|\) is at least \(\delta\), and from definition we know \(\Delta \geq \delta\), therefore \(\min(|\text{sparse}(P)|, \Delta) \geq \delta\). Hence the result.

\textbf{Lemma 3} For any pattern string \(P\), during the search phase of Heuristic \(R\), the expected length of shift of the search window after a Type-1, Type-2 or Type-3 event is at least \(\min(|\text{sparse}(P)|, \Delta)\).

\textbf{Proof} Notice that the query string \(T\) is of length \(m\) and each of its characters are drawn uniformly at random from the alphabet \(\Sigma\) of size \(\Delta\). In the case of a Type-1 event, we can observe that the search window is shifted by \(\text{shift}^c(P)\). Notice that the mismatch character \(c\) is equally likely to be any character in \(\Sigma\) other than \(\text{endc}(P)\). So, we can observe that if \(c \in \text{sparse}(P)\), then \(\text{shift}^c(P)\) is equally likely to be any value in the interval \([1..\delta]\) and if \(c \notin \text{sparse}(P)\) then \(\text{shift}^c(P)\) is at least \(|\text{sparse}(P)|\).

Therefore, the expected shift length will be at least \([(1 + 2 + ... + \delta) + (\Delta - \delta)|\text{sparse}(P)|])/\(\Delta\). If \(\delta < \Delta/2\), then this above sum is \(O(|\text{sparse}(P)|)\), otherwise it is \(O(|\Delta|)\). Therefore, the expected shift length is at least \(O(\min(|\text{sparse}(P)|, \Delta))\). In the case of Type-2 event, we can observe that the search window is shifted by at least \(|\text{sparse}(P)|\). Similarly after a type-3 event, Random-Match subroutine is invoked and the search window is shifted by at least \(|\text{sparse}(P)|\). Hence in all three types of events the expected search window shift is at least \(O(\min(|\text{sparse}(P)|, \Delta))\). Hence the result.

\textbf{Lemma 4} For any given pattern string \(P\) of length \(n\) from \(\Sigma\) and a \(n\) length substring of query string \(T\) whose characters are drawn independently and uniformly from \(\Sigma\), the expected number of matches before a mismatch when invoking Random-Match subroutine is \(O(1)\).

\textbf{Proof} Each character in \(T\) is drawn independently and uniformly from \(\Sigma\). Therefore, the expected number of matches before a mismatch = \(\frac{\Delta-1}{\Delta}(1 + \frac{1}{2} + \frac{2}{3} + ... + \frac{n-1}{n}) = O(1)\).

\textbf{Lemma 5} For any pattern string \(P\), the length of \(\text{sparse}(P)\), the longest \(2\)-sparse pattern of \(P\), is at least \(\delta\), where \(\delta\) is the number of distinct characters in \(P\).

\textbf{Proof} Let \(a\) be the character in \(P\) whose last occurrence has the smallest index and let its index in \(P\) be denoted by \(\text{start}\). From definition of \(a\), we can observe that every character in \(P\) occurs at
least once to the right of index \textit{start} in \( P \). Let \( b \) be the character in \( P \) whose first occurrence in \( P \) to the right of \textit{start} has the highest index and let its position in \( P \) be denoted by \textit{end}. We can easily observe that in the interval \([\textit{start}, \textit{end}]\) all characters in \( P \) are present at least once and the characters \( a \) and \( b \) do not appear in between. Therefore, \(|\text{sparse}(P)| \geq |\text{sparse}^{(a, b)}(P)| \geq \delta. \]

\[ \textbf{Lemma 6} \text{ For any pattern string } P, R \text{ preprocesses } P \text{ in } O(n\delta) \text{ time to determine } \text{sparse}(P), \text{ and } \text{shift}^{c}(P), \text{ for } c \in \Sigma, \text{ where } \delta \text{ is the number of distinct characters in } P. \]

\[ \textbf{Proof} \text{ First, by scanning } P \text{ in } O(n) \text{ time we can compute } \text{shift}^{c}(P), \text{ for } c \in P. \text{ Second, for each pair of characters } a, b \in P, \text{ we initialize } \text{sparse}^{(a, b)}(P) \text{ to 0. Then, as we scan } P, \text{ when we encounter a character } c \in P, (i) \text{ we update the index of its last occurrence, and (ii) based on the index of } c \text{ and the position of the last occurrence of characters } x \text{ in } P \text{ that have occurred earlier, we update } \text{sparse}^{(c, x)}. \text{ This requires } O(n) \text{ time for the scan and and for each character in } P O(\delta) \text{ time for at most } \delta \text{ updates to the length of sparse patterns. Therefore, the total time during scan is } n\delta. \text{ Finally, since there are } \delta^2 \text{ ordered pairs, we can trivially find the maximum length for all pairs of characters in } P \text{ and from them choose the longest in } O(\delta^2) \text{ time. Hence, the total preprocessing time is } O(n\delta + \delta^2) = O(n\delta). \]

### 4 Expected Length of \text{sparse}(P) for a Random String \( P \)

In this section, we show that for a pattern string \( P \) whose characters are chosen uniformly at random from an alphabet of size \( \Delta \), \( E[|\text{sparse}(P)|] \) is \( \Omega(\Delta \log(\frac{2\Delta}{2\Delta - \delta + 1})) \). We will first present the key idea behind our proofs, then we introduce some definitions that are necessary for stating and establishing a lower bound on \( E[|\text{sparse}(P)|] \).

\[ \textbf{Key Idea: } \text{Let } \mathcal{P} \text{ be the set of strings of length } n \text{ whose characters are drawn from the alphabet } \Sigma = \{c_1, c_2, \ldots, c_\Delta\} \text{ and } \mathcal{P}(\delta) \subseteq \mathcal{P} \text{ be the set of strings with at least } \delta \text{ distinct characters. For a pattern string } P \text{ that is chosen uniformly at random from } \mathcal{P}, \text{ we show } E[|\text{sparse}(P)|] = \frac{1}{|\mathcal{P}|} \sum_{P \in \mathcal{P}} |\text{sparse}(P)| \text{ is } \Omega(\Delta \log(\frac{2\Delta}{2\Delta - \delta + 1})) \text{ by first showing } \frac{|\mathcal{P}(\delta)|}{|\mathcal{P}|} = (1 - o(1)) \text{ and then lower bounding } \frac{1}{|\mathcal{P}(\delta)|} \sum_{P \in \mathcal{P}(\delta)} |\text{sparse}(P)| \text{ as follows:} \]

\[ (1) \text{ We first define an onto function } F: \mathcal{P}(\delta) \rightarrow \mathcal{Q}(\delta) \text{ that maps a given string } P = (a_1, a_2, \ldots, a_n) \in \mathcal{P} \text{ to a string } Q = F(P) = (b_1, b_2, \ldots, b_n) \text{ such that (i) there are at least } \frac{\delta}{2} - 1 \text{ distinct characters between the first and second occurrence of } b_1 \text{ in } Q, \text{ (ii) } \mathcal{Q}(\delta) \subseteq \mathcal{P}(\delta), \text{ and (iii) } |\mathcal{Q}(\delta)| \geq \frac{1}{2}|\mathcal{P}(\delta)|. \]

\[ (2) \text{ We then use properties of the mapped string } Q = F(P) \text{ to lower bound } \frac{1}{|\mathcal{P}(\delta)|} \sum_{P \in \mathcal{P}(\delta)} |\text{sparse}(P)| \text{ as follows:} \]
\[
\frac{1}{|P(\delta)|} \sum_{P \in P(\delta)} |\text{sparse}(P)| \geq \frac{1}{2|Q(\delta)|} \sum_{Q \in Q(\delta)} |\text{sparse}(Q)| \geq \frac{1}{2|Q(\delta)|} \sum_{Q \in Q(\delta)} f_{\text{firstpos}}^{\frac{1}{2}+1}(Q)
\]

\[
\geq \frac{1}{2}(1 - o(1))E[W^{\frac{1}{2}+1}] = \Omega(\Delta \log(\frac{2\Delta}{2\Delta - \delta + 1}))
\]

where, \(f_{\text{firstpos}}^{\frac{1}{2}+1}(Q)\) - denotes the position of the first occurrence of the \((\frac{1}{2} + 1)\)th distinct character in \(Q\);

\(W^{\frac{1}{2}+1}\) - the waiting time for the selection of the \((\frac{1}{2} + 1)\)th distinct coupon in a sequence of independent trials where during each trial a coupon is selected uniformly at random from among \(\Delta\) different coupon types.

**Definitions 4.1** Let \(P = (a_1, a_2, ..., a_n)\) be an arbitrary string in \(P(\delta)\). Let \(\text{FIRST}(P) = (a_{\pi(1)}, a_{\pi(2)}, ..., a_{\pi(\frac{\delta}{2})})\) denote the subsequence of \(P\) of length \(\frac{\delta}{2}\) consisting of the first occurrence of the first \(\frac{\delta}{2}\) distinct characters in \(P\) and \(\text{NEXT}(P) = (a_{\pi(\frac{\delta}{2}+1)},..., a_{\pi(\delta)})\) denote the subsequence of the first occurrence of the next \(\frac{\delta}{2}\) distinct characters in \(P\).

**Definitions 4.2** Let \(P = (a_1, a_2, ..., a_n)\) be an arbitrary string in \(P(\delta)\), \(r = \text{rank}\, \text{FIRST}(P)(a_1)\) denote the lexical rank of \(a_1\) among the characters in \(\text{FIRST}(P)\), and \(b_1 = a_{\pi(\frac{\delta}{2}+r)}\) denote the first occurrence of the \((\frac{\delta}{2} + r)\)th distinct character in \(P\). We now define the function \(F : P(\delta) \rightarrow Q(\delta)\) as follows:

\[
Q = F(P) = \begin{cases} 
P & \text{if } a_1 \text{ occurs exactly once in the substring } (a_1, a_2, ..., a_{\pi(\delta/2)}) \\
(b_1, a_2, ..., a_n) & \text{otherwise}
\end{cases}
\]

**Theorem 7** Let \(P\) be the set of strings of length \(n\) whose characters are drawn from the alphabet \(\Sigma = \{c_1, c_2, ..., c_\Delta\}\) and \(P\) be a string in \(P\) whose characters are chosen uniformly at random from \(\Sigma\). Then, \(E[|\text{Sparse}(P)|] = \min(\frac{\Delta}{2}, \Delta \log(\frac{2\Delta}{2\Delta - \delta + 1}))\).

**Proof** From Lemma 5, we know that \(|\text{Sparse}(P)| \geq \delta\). So to establish this theorem, we only need to consider the situation when \(\delta < \frac{\Delta}{2}\). From definition, we know \(E[|\text{Sparse}(P)|] = \frac{1}{|P|} \sum_{P \in P} |\text{Sparse}(P)| \geq \frac{1}{|P|} \sum_{P \in P(\delta)} |\text{Sparse}(P)|\). From Lemma 8, we know \(\frac{|P(\delta)|}{|P|} = (1 - o(1))\). Therefore, \(\frac{1}{|P|} \sum_{P \in P(\delta)} |\text{Sparse}(P)| \geq (1 - o(1)) \frac{|P(\delta)|}{|P|} \sum_{P \in P(\delta)} |\text{Sparse}(P)|\). From Lemma 10, we get \(\frac{1}{|P(\delta)|} \sum_{P \in P(\delta)} |\text{Sparse}(P)| = \Omega(\Delta \log(\frac{2\Delta}{2\Delta - \delta + 1}))\). Hence we are done.

**Lemma 8** Let \(P\) be the set of strings of length \(n\) whose characters are drawn from the alphabet \(\Sigma = \{c_1, c_2, ..., c_\Delta\}\) and \(P(\delta) \subseteq P\) be the set of strings with at least \(\delta\) distinct characters. If \(\delta < \frac{\Delta}{2}\), then \(\frac{|P(\delta)|}{|P|} = (1 - o(1))\).
Proof We establish this lemma by showing that the number of strings of length $n$ in $P$ with less than $\delta$ distinct symbols is a small fraction of $P$. That is, we show that $(\Delta C_\delta)(\frac{1}{\delta})^n < 1/n$. On expanding the left hand side, taking logarithms on both sides and then solving for $n$, we see that the above inequality holds for $n > \log n + \delta$.  

Lemma 9 Let $F : \mathcal{P}(\delta) \rightarrow \mathcal{Q}(\delta)$ be the function defined in Definitions 4.2. Let $P(a_1, a_2, ..., a_n)$ be any string in $\mathcal{P}(\delta)$ that is mapped to string $Q = F(P) = (b_1, b_2, ..., b_n)$. We will show that (i) there are at least $\frac{\delta}{2} - 1$ distinct characters between the first and second occurrence of $b_1$ in $Q$, (ii) $\mathcal{Q}(\delta) \subseteq \mathcal{P}(\delta)$, and (iii) $|\mathcal{Q}(\delta)| \geq \frac{1}{2}|\mathcal{P}(\delta)|$.

Proof From definition of $F$, we know that if the first character in $P$ does not repeat until the first occurrence of the $\frac{\delta}{2}$th distinct character then $Q = P$ and Property (i) is automatically satisfied. Otherwise, we obtain $Q$ from $P$ by replacing the first character in $P$ by the first occurrence of the $\frac{\delta}{2} + r$th distinct character in $Q$, where $r$ is the rank of $\text{FIRST}(P)(a_1) \geq 1$. Hence, Property (i) is satisfied in this situation also. Now, we will show that $\mathcal{Q}(\delta) \subseteq \mathcal{P}(\delta)$. Notice from the definition of $F$, for any $P$, $Q = F(P)$ has the same number of distinct symbols as $P$ and the length of $Q$ is $n$, therefore $Q$ is also an element of $\mathcal{P}(\delta)$. Therefore, $\mathcal{Q}(\delta) \subseteq \mathcal{P}(\delta)$. Also, we can observe that from the definition of $F$ that for each string $Q \in \mathcal{Q}(\delta)$ there are at most two strings in $\mathcal{P}(\delta)$ that map to it. In addition, for each string $Q \in \mathcal{Q}(\delta)$ $Q$ itself is one of the pre-images, so $|\mathcal{Q}(\delta)| \geq \frac{1}{2}|\mathcal{P}(\delta)|$.  

Lemma 10 Let $F : \mathcal{P}(\delta) \rightarrow \mathcal{Q}(\delta)$ be the function defined in Definitions 1.2. We will now establish the following inequalities.

$$\frac{1}{|\mathcal{P}(\delta)|} \sum_{P \in \mathcal{P}(\delta)} |\text{sparse}(P)| \geq \frac{1}{2|\mathcal{Q}(\delta)|} \sum_{Q \in \mathcal{Q}(\delta)} |\text{sparse}(Q)| \geq \frac{1}{2|\mathcal{Q}(\delta)|} \sum_{Q \in \mathcal{Q}(\delta)} \text{firstpos}_{\delta + 1}(Q) \quad (1)$$

$$\frac{1}{|\mathcal{Q}(\delta)|} \sum_{Q \in \mathcal{Q}(\delta)} \text{firstpos}_{\delta + 1}(Q) = (1 - o(1))E[W(\frac{\delta}{2} + 1)] \quad (2)$$

$$\quad E[W(\frac{\delta}{2} + 1)] \geq \Delta \log\left(\frac{2\Delta}{2\Delta - \delta + 1}\right) \quad (3)$$

Proof The first inequality follows from definition of $F$ and the properties of $F$ established in Lemma 9. The second inequality is established as follows. First, we compute $\sum_{Q \in \mathcal{Q}(\delta)} \text{firstpos}_{\delta + 1}(Q)$ by computing the sum of the waiting times for the selection of the $(\frac{\delta}{2} + 1)$th distinct coupon over all possible sequences of trials, where during a trial a coupon is selected uniformly at random from among $\Delta$ different coupon types. Then, from this total we exclude the waiting time of those sequences that require more than $n$ trials to select $\delta$ distinct coupons. More formally,

$$\frac{1}{|\mathcal{Q}(\delta)|} \sum_{Q \in \mathcal{Q}(\delta)} \text{firstpos}_{\delta + 1}(Q) \approx E[W(\frac{\delta}{2} + 1)] - \sum_{i \in [n+1, \infty]} i \cdot \Pr(W(\delta) = i) \quad (4)$$
The expected waiting time \( E[W^{(d+1)}] \) for the selection of \((d+1)th\) distinct coupon can be determined by letting \( X_i \) denote the number of trials following the selection of the \( i^{th} \) distinct coupon until the \((i+1)th\) distinct coupon type is selected. Then, we can see that \( W^{(d+1)} = \sum_{i\in[1,\frac{d}{2}+1]} X_i \) is the waiting time until the \((\frac{d}{2}+1)th\) coupon type is selected. It is easy to observe that the \( X_i \)'s are geometrically distributed random variables with parameter \( p_i = \frac{\Delta - i}{\Delta} \). Therefore,

\[
E[W^{(d+1)}] = E\left[ \sum_{i\in[1,\frac{d}{2}+1]} X_i \right] = \Delta \log\left(\frac{2\Delta}{(2\Delta - \delta + 1)}\right) \tag{5}
\]

For \( i \in [n+1..\infty] \), we bound \( i \cdot Pr(W^{(d)}) = i \) as follows

\[
i \cdot Pr(W^{(d)}) = (\Delta C^d)[\delta^i - \delta^{i-1}] \cdot \frac{i}{\Delta^i} < \frac{\Delta^i}{\delta^i} \cdot \frac{i}{\Delta^i} = o(1) \quad \forall i > \delta + \log(i) \tag{6}
\]

Now, from Equations (4), (5) and (6) we see that the Inequality (2) follows.

\[\square\]

5 Conclusions and Future Work

Given a pattern string \( P \) of length \( n \) consisting of \( \delta \) distinct characters and a query string \( T \) of length \( m \), where the characters of \( P \) and \( T \) are drawn from an alphabet \( \Sigma \) of size \( \Delta \), the exact string matching problem consists of finding all occurrences of \( P \) in \( T \). For this problem, we present a randomized heuristic that in \( O(n\delta) \) time preprocesses \( P \) to identify \( \text{sparse}(P) \), a rarely occurring substring of \( P \), and then use it to find all occurrences of \( P \) in \( T \) efficiently. This heuristic has an expected search time of \( O\left(\frac{m}{\min(|\text{sparse}(P)|,\Delta)}\right) \), where \( |\text{sparse}(P)| \) is at least \( \delta \). We also show that for a pattern string \( P \) whose characters are chosen uniformly at random from an alphabet of size \( \Delta \), \( E[|\text{sparse}(P)|] \) is \( \Omega(\Delta \log(\frac{2\Delta}{2\Delta - \delta})) \). We believe that for a large class of non-trivial pattern strings, our heuristic with better analysis can yield a randomized algorithm that has sublinear run time in the worst case scenario.

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