Thermal generation of magnetization via orbital degree of freedom in low-symmetry two dimensional systems

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We propose an orbital magnetothermal effect wherein the temperature gradient generates an out-of-plane orbital magnetization of Bloch electrons in low-symmetry two-dimensional (2D) systems. We provide the hitherto absent theoretical approach to attaining the linear thermoelectric responses of the orbital magnetization, which does not correspond to a single-particle operator in extended-state systems thus is hard to fit into the linear response theory. The Mott relation is extended to such thermoelectric responses. The different symmetry requirements and temperature dependencies of intrinsic and extrinsic responses are elucidated. Besides, we propose a phonon-drag orbital magnetization and point out the presence of a peak in its temperature dependence. These results are relevant to various 2D systems with reduced symmetry, opening the door to thermal generation and control of magnetization via the orbital degree of freedom.

TABLE I. The equilibrium (eq) and out-of-equilibrium (oeq) orbital magnetization of intrinsic and extrinsic nature in 2D, their dependence on scattering time $\tau$, and their demands on inversion ($I$), time-reversal ($T$) and $IT$ symmetries.

|                | $\tau$-scaling | $I$ | $T$ | $IT$ |
|----------------|----------------|-----|-----|------|
| eq eq intrinsic/extrinsic | $\tau^0$ | Any | No | No |
| oeq extrinsic | $\tau^1$ | No | Any | No |

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Introduction.— Breaking the space inversion ($I$) symmetry in 2D valley systems underlines intriguing nonequilibrium phenomena [1, 2], such as the valley Hall effect [3, 4], the nonlinear anomalous Hall effect [5–7], and the electrically induced valley magnetization [8, 9]. In particular, in a strained MoS$_2$ monolayer the observed magnetization response to the applied electric field is of orbital nature [8], opening up the possibility of electrical control of magnetization via the orbital degree of freedom. This scenario has been argued [10] to be relevant to the magnetization switch in a recent experiment on the substrated twisted bilayer graphene [11].

From the viewpoint of symmetry, the equilibrium orbital magnetization (OM) can be nonzero only if the time-reversal ($T$) symmetry is broken. However, when the out of equilibrium OM is induced by the spatial gradient of a scalar field, breaking $I$ is necessary instead (we focus on the 2D case). Moreover, the presence of scattering induced dissipative processes even enables an off-equilibrium OM in nonmagnetic metals. The necessary symmetry requirements for the appearance of various contributions addressed in this work are summarized in Table. [4] and explicated later.

Since the OM is a pseudoscalar in 2D, its susceptibility with respect to the driving field is an in-plane pseudovector, which shares the same spatial symmetry properties as the Berry curvature dipole [5]. The largest symmetry allowing for an off-equilibrium OM is a single mirror line, implying only four of the noncentrosymmetric point groups, i.e., $C_1$, $C_2$, $C_{1v}$, and $C_{2v}$. They are realized in low-symmetry valley systems, such as the 1-Td WTe$_2$ [6, 7], the strained 2H MoS$_2$ [8] and the hBN-substrated twisted bilayer graphene [10].

Apart from the electric field $E$, the OM should also arise from an applied temperature ($T$) gradient with the same symmetry requirement. This new effect, referred to as the orbital magnetothermal response in the following, enables thermal generation of magnetization via the orbital degree of freedom, and is of both practical and theoretical values. It combines the thermoelectric and magnetothermal physics with strain engineering in 2D valley systems, and is a basic ingredient in the finite-temperature valley-based applications. Besides, it provides complementary information to the electrical tuning of OM. For instance, as we will reveal, different contributions to this effect possess different $T$-dependencies, helping deducing the nature of the observed OM. Furthermore, the mutual drag between electrons and other excitations such as phonons contributes to the orbital magnetothermal physics, promising information on these excitations and their coupling to electrons.

Notwithstanding the appealing prospect, there is no existing theory indicating how to obtain the thermal response of the OM contributed by Bloch electrons. The main difficulty is that the OM does not correspond to a single-particle operator in extended-state systems, thus its thermoelectric response is hard to fit into the linear response theory. The thermodynamic approach, wherein the OM is given by the derivative of the free energy with respect to magnetic field, works in metals in equilibrium [13, 14], but does not apply to nonequilibrium metals. Another approach extracting the extrinsic OM from the natural optical activity [15, 16] cannot address insulators and $T$-broken metals. Furthermore, both of these approaches do not accommodate the statistical forces given by gradients of temperature and of chemical potential $\mu$. [10]
In this work we propose the hitherto absent theoretical framework to attain the thermally induced OM in low-symmetry 2D systems, thus opens the door to thermal generation and manipulation of magnetization via the orbital degree of freedom. The Einstein and Mott relations are shown for the linear thermoelectric responses of OM. Although these two relations are well known for the transport current electric current density in steady states consists of the transport current and orbital magnetization current: 

\[ j_{\text{loc}} = j^\text{r} + j^\text{mag}, \quad j^\text{mag} = \nabla \times \mathbf{M}. \]  

\hspace{1cm} (1) 

Since \( j^\text{mag} \) is divergence-free, it does not contribute to the net current flow passing through the sample and cannot be measured in standard transport experiments \[20\]. Nevertheless, by identifying the magnetization current one can obtain the OM and its thermoelectric responses. Given that the transport current up to the first order of the combined electric and magnetic fields has been identified \[21, 22\], in the following we find the expression for \( j^\text{loc} \) and hence extract the magnetization current. The OM thus obtained is consistent with the thermodynamic approach whenever the latter works. The uncertainty related to a curl-free term is irrelevant, since at most the linear-in-\( E \) contribution to \( \mathbf{M} \) is considered.

A Bloch electron in crystalline solids is described semiclassically as a wave-packet centered around \((r, k)\) in phase-space, and the physics related to the wave-packet motion under the perturbations of slowly varying external fields can be analyzed by constructing a wave-packet action \( S \). In particular, the electric current density contributed by a wave-packet is shown to be the physical value \((\langle p \rangle \) of the functional derivative of the wave-packet action with respect to the magnetic vector potential \( \mathbf{A} \): 

\[ \frac{\delta S}{\delta \mathbf{A}(r)} |_p \]  

as can be expected. Summing over all wave-packets, one then obtains the local electric current density contributed by the Bloch-electron grand canonical ensemble. This summation can be done with the aids of the semiclassical distribution function \( F \) of wave-packets and the Berry-curvature corrected phase-space measure \( D \) \[14\], yielding

\[ j^\text{loc}(r) = \int [dk_c] dr_c DF \frac{\delta S}{\delta \mathbf{A}(r)} |_p. \]  

\hspace{1cm} (2) 

Here \([dk_c]\) is shorthand for \(\sum_n dk_c/(2\pi)^d\) with \(d\) the spatial dimensionality and \(n\) the band index.

To implement the field variation one needs the explicit form of the wave-packet Lagrangian (set \(\hbar = 1\)) \[21, 22\]

\[ \mathcal{L} = - (\tilde{\epsilon} + e\phi) + \hat{r}_i (p_i + eA_i) + \hat{p}_i \tilde{A}_i^p + \hat{r}_i \tilde{A}_i^t + \tilde{A}_i^t. \]  

\hspace{1cm} (3) 

All the quantities involved in \( \mathcal{L} \) are evaluated at \((r_c, k_c)\), and the subscript \(c\) is suppressed for simplicity. Summation over repeated indices is implied henceforth. \( p = k - e\mathbf{A} \) is the mechanical crystal momentum. We take the gauge \( E_i = -\partial_i \phi \) and \( \mathbf{A} = \frac{1}{2} \mathbf{B} \times \mathbf{r} \) for uniform static electromagnetic fields. The wave-packet energy \( \tilde{\epsilon} \) differs from the usual band energy \( \epsilon \) by coupling to external fields \[23\]. For the present purpose, the Berry connection \( \tilde{A}_i^p = \tilde{A}_i^p + a_i^{p\tau} \) includes the first order correction \( a_i^{p\tau} \) by electromagnetic fields \[21\] as well as the gradient of the other slowly varying static (thus \( A_i^t = 0 \)) mechanical field \[24\], labeled by \( \mathbf{h} \). In the presence of electromagnetic fields alone, one would not get the magnetization current in bulk, since the bulk is uniformly magnetized and the magnetization current only shows up near boundaries of the sample. To remove this subtlety \[27\], we introduce the \( \mathbf{h} \) field to make the bulk nonuniform, thus the spatial gradient acts through the \( \mathbf{h} \) field and the magnetization current in Eq. \[11\] receives a clear meaning in bulk.

The field variation yields \((\partial_i \equiv \partial/\partial r_i)\)

\[ j^\text{loc} = e \int DF \hat{r}_i [p + e\mu_i \partial_i \int DF (\hat{m}_i + \hat{p}_i \partial m_i/\partial \delta E)] |_p, \]  

\hspace{1cm} (4) 

where the contributions pertaining to the OM induced by the magnetic field \[28\] are dropped in the second term on the right hand side (rhs). Hereafter \( \int \) is shorthand for \( \int [dp] \), and \(-\hat{m} \cdot \mathbf{B}\) is the magnetic field induced energy correction up to \( O(EB) \), where \( \hat{m} = m - (\tilde{A}^p)^E \times \partial p \tilde{\epsilon} \), with \( m \) the familiar orbital moment of Bloch electrons \[16\] and \((\tilde{A}^p)^E\) the electric-field induced positional shift \[21\]. While \( \hat{m} \) can be viewed as a result of the wave-packet self-rotation \[24\], the other contribution in the second term stems from the wave-packet dynamics \( \hat{p} \).

The notation \( |_p \) means that the quantity is evaluated for physical wave-packet states which minimize the action, namely, satisfy the Euler-Lagrange equations

\[ \frac{\partial \mathcal{L}}{\partial \lambda} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \lambda} = 0, \quad (\lambda = r, p). \]  

\hspace{1cm} (5) 

Having this in mind, we omit the notation \( |_p \) below. Compared to the equations of motion in Ref. \[21\], Eq. \[5\] is not limited to electromagnetic fields. The pertinent phase-space measure reads \( D = 1 + \Omega^\mathbf{p} i - eB_i \Omega^\mathbf{p}_i \), where \( \Omega^\mathbf{p} i \) are the Berry curvatures derived from unperturbed Berry connections \( \mathcal{A}^\mathbf{p} \) \[23\], and \( \Omega^\mathbf{p}_i = \frac{1}{2} \epsilon_{ijl} \mathcal{A}_j^\mathbf{p} \mathcal{A}_l^\mathbf{p} \). Comparing Eqs. \[11\] and \[4\] may suggest that the OM is given by the integrand of the second term on the rhs of Eq. \[4\]. This is incomplete because, as will be shown later, some other total spatial derivatives emerge from the wave-packet dynamics described in the first term and contribute to the magnetization current as well.

The distribution function is given by \( F = f + \delta f \), where \( \delta f = f (\tilde{\epsilon}) \) is the equilibrium Fermi-Dirac distribution, and the off-equilibrium component \( \delta f \) is determined by the Boltzmann equation incorporating scattering processes.
Hence $j^{\text{loc}}$ [Eq. (4)] can be split into intrinsic and extrinsic parts which are related respectively to $f$ and $\delta f$:

$$j^{\text{loc}} = j^{\text{loc, in}} + j^{\text{loc, ex}},$$

(6)

where $j^{\text{loc, in/ex}} = \nabla \times \tilde{M}^{\text{in/ex}} + j^{\text{tr, in/ex}}$. While $j^{\text{loc, ex}}$ is unique to metals, $j^{\text{loc, in}}$ is present even in insulators.

**Orbital magnetoelectricity in insulators.**—In insulators the statistical forces can be set to zero and the $E$-field induced OM has been studied by fully quantum theories [29, 30], providing a platform to test the validity of the present approach. Making use of the equations of motion [5] we arrive at

$$j_{s}^{\text{loc}} = e_{ik} \partial_{i}((M_{i} + \partial_{Bi} \int e a_{j}^{p} E_{j}) - e^{2} E_{j} \int (\Omega_{jj}^{pp} + \Omega_{jj}^{[pr]p}),$$

(7)

where $\Omega_{jj}^{[pr]p} = \Omega_{jj}^{pp} \Omega_{ij}^{[pr]} - \Omega_{jj}^{pp} \Omega_{ij}^{pp} - \Omega_{jj}^{[pr]} \Omega_{ij}^{pp}$ is the second Chern form of Berry curvatures, and $M = \int (m + e\epsilon\Omega)$ is the OM in the absence of $E$ in insulators [31]. The Berry curvature contribution to $M$ comes from the first term on the rhs of Eq. (4). In 3D insulators, we restrict ourselves to the case where the Chern invariants of the valence band structure vanish identically, as in all previous researches [29, 30]. In such systems $\int \Omega_{jj}^{pp} = 0$, and the second Chern form can be recast under the periodic gauge into a spatial derivative of the Brillouin-zone integral of the abelian Chern-Simons form of Berry connections: $\int \Omega_{jj}^{pp} = \partial_{i} \int (A_{ij}^{p} \partial_{p} A_{i}^{p} + A_{ij}^{p} \partial_{p} A_{i}^{p} + A_{ij}^{p} \partial_{p} A_{i}^{p})$. The antisymmetry properties of the Chern-Simons form further renders $\int \Omega_{jj}^{pp} = -\epsilon_{ijs} \partial_{i} \theta$, involving the ME $\theta$-term [24, 32, 34]. Thus $j^{\text{loc}} = \nabla \times \tilde{M} = j^{\text{mag}}$, where $\tilde{M} = M_{s} + f^{0} F^{\text{occ}} \times \Omega^{p}$, and $M^{\text{loc}}$ is the intrinsic OM in the absence of $E$ and $B$.

**Intrinsic orbital magnetothermoelectricity in 2D.**—Simplifications occur in 2D, where the second Chern form $\Omega_{jj}^{[pr]p}$ vanishes so that the OM can be conveniently pursued even in the presence of nonzero Chern numbers, in both insulators and metals. In the presence of statistical forces, which enter the formulation through the inhomogeneity of the grand potential density for a particular state $g(\epsilon, \mu, T) = -k_{B} T \ln[1 + e^{-(\epsilon - \mu)/k_{B} T}]$, we get

$$j^{\text{loc, in}} = \nabla \times \tilde{M}^{\text{in}} - e \int f^{0} F^{\text{occ}} \times \Omega^{p}$$

$$+ \int \partial_{i} f^{0} F^{\text{FS}} \times [e \epsilon + (m \cdot B) \Omega^{p}],$$

(8)

where $\partial_{i} = \partial / \partial \epsilon$, $f^{0} = f(\epsilon)$, $g^{0} = g(\epsilon)$, and

$$F^{\text{FS}} = e E - \nabla \mu - (\epsilon - \mu) \nabla T,T,$$

$$F^{\text{occ}} = e E - \nabla \mu + [g^{0}/f^{0} - (\epsilon - \mu)] \nabla T/T.$$

(9)

We recognize that the second and third terms on the rhs of Eq. (8) give the intrinsic transport current $j^{\text{tr, in}}$, which is comprised of the linear [13] and magneto-nonlinear anomalous Hall and Nernst currents. The intrinsic magneto-nonlinear current induced by the electric field has been obtained before [21], whereas its thermal counterpart induced by gradients of $(T, \mu)$ is demonstrated here. The generalized driving force $F^{\text{FS}}$ was well known as a basic ingredient in the conventional Boltzmann theromoelectric transport [18] which relies on scattering of electrons around the Fermi surface. However, here it is related to the intrinsic Fermi surface transport, which is characteristic of nonlinear responses in metals.

After identifying the transport current, we conclude that $\tilde{M}^{\text{in}} = M^{\text{in}} \hat{z}$ is the intrinsic OM. Here $\hat{z}$ is the out-of-plane direction,

$$\tilde{M}^{\text{in}} = M_{z} + \int f^{0} F^{\text{occ}} \times \partial_{B} a^{p},$$

(10)

and

$$\partial (\epsilon^{p})_{n} = -2 \text{Re} \sum_{n_{1} \neq n_{2}} A_{n_{1} n_{2}}^{p} m_{n_{1}, n_{2}} + \epsilon_{ij} \frac{e}{2} \partial_{p} g_{s_{ij} n}. $$

(11)

We recover the band indices, as they are necessary here. The first term on the rhs is related to the interband mixing of Bloch states induced by $B$ field [21], where $A_{n_{1} n_{2}}^{p}$ and $m_{n_{1}, n_{2}}$ are the interband Berry connection and interband orbital magnetic moment, respectively [25]. The second term is unique to metallic systems, and involves the momentum space dipole of the quantum metric [32]

$$G_{s_{ij} n}^{p} = \text{Re} \left( \partial_{p} u_{n} | \partial_{p} u_{n} \right) - A_{n_{1} n_{2}}^{p} A_{n_{1} n_{2}}^{n}. $$

We consider the periodic part of the Bloch state.

In 2D metals which break both $I$ and $T$ but preserve their combined symmetry, the OM has only off-equilibrium component and is of intrinsic nature (Table. 1). In the case of large band gap, the quantum metric dipole contribution dominates.

Given that the magnetoelectric susceptibility in a 2D insulator reads $\partial_{E} \tilde{M}_{z} = \int e \partial_{B} A^{p}$, the electric field related contribution in Eq. (10) is a reasonable counterpart in metals. Pictorially, an electron wave-packet under the electric field acquires a positional shift by the magnetic field, resulting in a change in energy. Besides, the statistical and mechanical forces appear in the combined form of $F^{\text{occ}}$ in Eq. (10), as in the intrinsic linear transport electric current. This is a decisive feature dictating the Mott relation in intrinsic linear theromoelectric responses.

**Extrinsic orbital magnetothermoelectricity in 2D.**—The extrinsic local current density is obtained as

$$j^{\text{loc, ex}} = \nabla \times \int \delta f_{p} (m + e\epsilon\Omega^{p})$$

$$+ e \int \delta f_{p} (\epsilon - m \cdot B) - e (\Omega^{p} \cdot \partial_{p} \epsilon) B + D_{B} v_{p},$$

(12)

with $D_{B} = 1 - e\Omega^{p} \cdot B$. The last line is nothing but the extrinsic linear and magneto-nonlinear transport currents.
Besides the intrinsic semiclassical velocity given by Eq. [1], there is a side-jump velocity $v^s_j$ arising from the coordinate-shift of the wave-packet upon scattering [36]. The extrinsic OM is then extracted as
\[
\tilde{M}^{\text{ex}} = \int \delta f_p \left( m + e \epsilon_p \Omega^p \right). \tag{13}
\]
The second term on the rhs arises from wave-packet dynamics [29].

The statistical forces enter $\tilde{M}^{\text{ex}}$ through the linearized Boltzmann equation [18]
\[
F^{\text{FS}} \cdot \left( v^0_p - v^p_p \right) \partial E_0 = -\int_{p'} \omega_{pp'} \left( \delta f_p - \delta f_{p'} \right). \tag{14}
\]
Here only the scattering rate $\omega_{pp'}$ is symmetric in $p$ and $p'$ is retained [37]. The side-jump velocity related effective driving term derives from the coordinate-shift during scattering [36, 38], and is detailed in Ref. [25]. In correspondence to the driving terms, one has $\delta f_p = \delta f^0_p + \delta f^1_p$. Contrary to $\delta f^0_p$, $\delta f^1_p$ does not contribute to $\tilde{M}^{\text{ex}}$ in time-reversal invariant systems. Moreover, scattering effects are secondarily underlined by the momentum-space Berry-curvature [38], such as side-jump, vanish in $\mathcal{IT}$ symmetric systems.

Equation (14) ensures $\delta f_p \propto F^{\text{FS}}$ for elastic scattering, such as the electron-phonon scattering in the high-$T$ equipartition regime with linear-in-$T$ resistivity and the electron-impurity scattering. This feature implies the Mott relation in extrinsic linear thermoelectric responses.

In $\mathcal{I}$-broken but $\mathcal{T}$-symmetric 2D conducting systems, the OM has only out-of-equilibrium component and is of extrinsic nature: $\tilde{M}_z = -\int \delta f^0_p \left( m_0 + e \epsilon_p \Omega^p_0 \right)$. The pertinent minimal model is the tilted massive Dirac model around valleys $\xi = \pm 1$ [5, 8]
\[
\tilde{H}_0 = \xi \beta k_x e \left( k_x \sigma_x + \xi k_y \sigma_y \right) + \Delta \sigma_z, \tag{15}
\]
where $\sigma_i$ are the Pauli matrices, and the Dirac cones are tilted along $x$ direction by strain. When the Fermi level $\epsilon_F$ intersects the upper bands, up to the first order of the small tilt $\beta$ we get $\partial_{E_0} M_z = -e^2 \beta^3 \Delta \left( \epsilon^3_F - \Delta^2 \right) \epsilon^3_F / 2\pi$, with $\tau$ the transport time on the Fermi surface, whereas the statistical force induced contributions are given by the Einstein and Mott relations shown below.

\textbf{Mott relations.}—Our result can be grouped into
\[
\tilde{M}_z = M_z + \sigma_{zi}^M \left( E_i - \partial_i \mu / e \right) - \alpha_{zi}^M \partial_i T. \tag{16}
\]
The Einstein relation is evident, stating that the electric field and the chemical potential gradient are equivalent in inducing the OM. We also prove the generalized Mott relation between the linear orbital magnetothermal and magnetoelastic response coefficients:
\[
\alpha_{zi}^M = \frac{1}{e} \int d\epsilon \partial \epsilon \frac{M_z - \mu}{T} \sigma_{zi}^M \left( \epsilon \right). \tag{17}
\]
Here $\sigma_{zi}^M = \sigma_{zi}^{M,\text{in}} + \sigma_{zi}^{M,\text{ex}}$, $\sigma_{zi}^{M,\text{in}}(\epsilon) = e \int \Theta(\epsilon - \epsilon_p) \partial_B \alpha^p$ is the zero-temperature intrinsic orbital magnetoelastic coefficient with Fermi energy $\epsilon$, and $\sigma_{zi}^{M,\text{ex}}(\epsilon)$ is the energy resolved extrinsic orbital magnetoelectric coefficient: $\sigma_{zi}^{M,\text{ex}} = \int d\epsilon \left( -\partial \epsilon \sigma_{zi}^M \right) \epsilon^2(\epsilon)$. For static impurity scattering, $\sigma_{zi}^{M,\text{ex}}(\epsilon)$ is the zero-temperature value of $\sigma_{zi}^M$.

When the temperature is much less than the distances between the chemical potential and band edges, the Sommerfeld expansion is legitimate [39], yielding the standard Mott relation $\alpha_{zi}^M = -\frac{\pi^2 k^2_B T}{3e} \partial \sigma_{zi}^M(\epsilon) = \mu$. It shows that $\alpha_{zi}^M$ is proportional to the energy derivative of $\sigma_{zi}^M$ around the chemical potential. If the extrinsic response is dominated by static impurity scattering, which is usually the case at lower temperatures in moderately disordered samples, the standard Mott relation indicates that $\alpha_{zi}^M$ is linear in $T$. On the other hand, when the extrinsic response is dominated by electron-phonon scattering in the high-$T$ regime, $\sigma_{zi}^{M,\text{ex}}(\epsilon) \sim 1/T$ due to the $T^{-1}$ scaling of the electron-phonon transport time, and $\alpha_{zi}^M$ becomes $T$-independent. The $T$-dependencies of various contributions to $\alpha_{zi}^M$ are summarized in Table. [4], which helps infer the dominant mechanism when multiple mechanisms are allowed by symmetry.

\textbf{Phonon-drag peak in the T-dependence of OM}—The phonon ensemble is taken to be at thermal equilibrium in the Boltzmann equation [14]. In fact, however, the temperature gradient would induce an off-equilibrium phonon distribution, which in turn drives electrons out of equilibrium through the electron-phonon scattering, giving rise to a phonon-drag OM. This process is embodied in a scattering induced effective driving term in the Boltzmann equation for electrons, which in $\mathcal{I}$-broken but $\mathcal{T}$-symmetric metals reads [25]
\[
V^p_\mathbf{x} \cdot \nabla T = -\int_{p'} \omega_{pp'} \left( \delta f^p_p - \delta f^p_{p'} \right). \tag{18}
\]

The rhs no longer contains off-equilibrium phonons, and $V^p_\mathbf{x}$ is related to the phonon relaxation time $\tau_{ph}$, which consists of two main competing mechanisms: the phonon-boundary and phonon-phonon scattering [40]. At very low temperatures the phonon-boundary scattering dominates and $\tau_{ph}$ is $T$-independent. In this case we consider model (15) as an example, where $V^p_\mathbf{x}$ of $O (\beta^3)$ is sufficient to generate an OM. With details in Ref. [23], we get [11]
\[
V^p_\mathbf{x} = -p \partial_p f^0 \epsilon^2 \tau_{ph} / \tau_{ep}, \tag{18}
\]
where $\epsilon_{ph}$ is the sound velocity and $\tau_{ep}$ is the electron-phonon transport $\mathbf{TABLE II. Temperature (T) dependence of the intrinsic and extrinsic orbital magnetothermal response coefficients (\alpha_{zi}^M) following the standard Mott relation.}$

| $T$            | $T^{-1}$ | $T$^{-1}$ | $T$^{-1}$ |
|----------------|----------|-----------|-----------|
| intrinsic      | $\sigma_{zi}^M$ | $\alpha_{zi}^M$ | $\alpha_{zi}^M$ |
| extrinsic      | $\sigma_{zi}^{M,\text{ex}}$ | $\alpha_{zi}^{M,\text{ex}}$ | $\alpha_{zi}^{M,\text{ex}}$ |

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|}
\hline
$T$ & $T^{-1}$ & $T$^{-1}$ & $T$^{-1}$ & $T$^{-1}$ & $T$^{-1}$ & $T$^{-1}$ & $T$^{-1}$ \\
\hline
\end{tabular}
\end{table}
time. The specific $T$-dependence of $\tau_{\text{ep}}^{-1}$ relies on fine details of electron-phonon coupling, but generally takes a power law $T^n$ in the low-$T$ limit with $n \geq 2$. The phonon-drag OM thus increases with temperature until the phonon-phonon scattering is manifested in phonon relaxation. Since the phonon-phonon process degrades $\tau_{\text{ph}}$ greatly as temperature increases, a peak in the $T$-dependence of the phonon-drag OM is generally anticipated. In fact, the similar phonon-drag induced peak at relatively low temperatures in the $T$-dependence of thermoelectric power has been well confirmed.

The above qualitative and semi-quantitative considerations can be made quantitative in particular systems with adequate knowledge of the electron-impurity, electron-phonon and phonon-phonon couplings. This is left for future work. Besides, attention could be paid to thermally driven off-equilibrium phonons with angular momentum. The phonon angular momentum corresponds to the circular motion of ions, hence contributes directly to the OM. This mechanism has been proposed recently within the constant-$\tau_{\text{ph}}$ approximation. Here, according to the above analysis, we infer that its $T$-dependence is akin to the phonon-drag contribution, i.e., it has a peak at low temperatures and is strongly decreased at high temperatures due to phonon-phonon scattering. Nonetheless, the interplay of this contribution with the electron related one at low temperatures remains to be detailed. Finally, we note that in magnetic systems the role of spin excitations in the orbital magnetothermoelectric physics is waiting to be revealed in the future.

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