PURITY AND ASCENT FOR GORENSTEIN FLAT COTORSION MODULES

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Abstract. The extension of scalars functor along a finite ring homomorphism is a classic example of a functor which preserves purity and pure injectivity. We consider how this functor behaves when restricted to the class of Gorenstein flat modules over a right coherent rings, and give particular attention to the Frobenius category of Gorenstein flat and cotorsion modules by showing there is an induced triangulated functor on the stable categories. This enables a comparison between pure injectivity for Gorenstein flat modules and pure injectivity in the triangulated categories as well as an investigation in how purity for Gorenstein flat modules is transferred along the homomorphism. Throughout motivating examples from commutative algebra are considered, including over hypersurfaces where a pure injective analogue of Knörrer periodicity for Gorenstein flat modules is developed.

1. Introduction

The transfer of properties of modules along a ring homomorphism \( R \to S \) is a classic concern of homological algebra and dates back to the subject’s inception, with the corresponding extension of scalars functor \( S \otimes_R - \) frequently playing a pivotal role. If one is interested in studying how properties related to purity ascend along this homomorphism one requires an assumption of \( S \) being finitely presented over \( R \), as this ensures \( S \otimes_R - \) will preserve both direct limits and direct products. Functors which have these properties are called interpretation, or definable, functors, and they are precisely the ones that will preserve pure exact sequences and pure injective objects.

Recent developments in the study of Gorenstein flat modules have highlighted the significance of the subclass of modules that are simultaneously Gorenstein flat and cotorsion, where a module is cotorsion if it is right Ext-orthogonal to the flat modules. As originally shown in the case of coherent rings in [Gil17] and expanded to all rings in [CET20], this class has the structure of a Frobenius exact category whose projective objects are the modules that are simultaneously flat and cotorsion. In the case that the ring is coherent, the corresponding stable category is triangle equivalent to the homotopy category of totally acyclic complexes of flat cotorsion modules.

Every pure injective module is cotorsion and the converse is true for flat modules, thus whenever \( S \otimes_R - \) is an interpretation functor it will automatically preserve flat cotorsion modules. With a further finiteness assumption on the flat dimension of \( S \) over \( R \), this functor will also preserve Gorenstein flat modules by results in [CH09]. Using this, we are able to exploit the triangulated equivalence to show that \( S \otimes_R - \) actually also preserves modules that are simultaneously Gorenstein flat and cotorsion. Furthermore, we show that there is an induced functor between the corresponding stable categories:

**Theorem.** Let \( R \to S \) be a finite ring homomorphism of coherent rings such that \( S \) has finite flat dimension over \( R \). Then \( S \otimes_R - \) preserves totally acyclic complexes of flat-cotorsion modules. In particular it yields a functor between the Frobenius exact categories \((G\mathcal{F} \cap C)(R) \to (G\mathcal{F} \cap C)(S)\), which induces a triangulated functor between the corresponding stable categories.

Following this we investigate how properties of \( S \otimes_R - \) can be used to determine information about purity in the class of Gorenstein flat modules (both over \( R \) and \( S \)), with particular emphasis placed on how to deduce information about the Ziegler spectra and indecomposable pure injective Gorenstein flat modules. If interpretation functors are the natural functors to understand the pure structure, then the
natural classes of modules to do the same are definable classes, which are uniquely determined by the (indecomposable) pure injective modules within them.

When the Gorenstein flat modules over \( R \) and \( S \) are definable, such as when both rings are Gorenstein, substantial information about the pure structures on both categories can be deduced by, and transferred via, the extension of scalars. To do this, we first establish an analogous result to [Kra00, 1.16] which enables a direct comparison between pure injectivity in \( \text{G} \mathcal{F}(R) \) and \( (\text{G} \mathcal{F} \cap \mathcal{C})(R) \). Of particular note are the kernels of the functor \( S \otimes_R - \) and we relate the pure injective objects between these two categories. In the case of certain morphisms between commutative rings, we are able to show the following:

**Theorem. (4.6)** Let \( R \) be a commutative noetherian ring, \( I \subset R \) an ideal generated by a regular sequence consider the canonical map \( R \to R/I \). Then the kernel \( \{ M \in \text{G} \mathcal{F}(R) : S \otimes_R M = 0 \} \) is closed under flat covers and Gorenstein cotorsion envelopes. Consequently it is a Frobenius exact category whose stable category is the kernel of the corresponding functor \( (\text{G} \mathcal{F} \cap \mathcal{C})(R) \to (\text{G} \mathcal{F} \cap \mathcal{C})(S) \).

A detailed example is then given for commutative hypersurface rings, establishing an extension of Knörrer periodicity over hypersurfaces to indecomposable pure-injective modules. Last but not least, we consider the special case when \( S \) is a finitely presented projective module over \( R \). In this case \( S \otimes_R - \) becomes an endofunctor on \( (\text{G} \mathcal{F} \cap \mathcal{C})(R) \), and the embedding of the image is the usual restriction of scalars functor.

**Theorem. (5.3)** Let \( R \to S \) be a finite flat ring epimorphism of right coherent rings. Then, \( I \), the image of \( S \otimes_R - : (\text{G} \mathcal{F} \cap \mathcal{C})(R) \to (\text{G} \mathcal{F} \cap \mathcal{C})(S) \), is a Frobenius exact category and there is recollement of triangulated categories

\[
\begin{array}{ccc}
\text{Ker}(S \otimes_R -) & \xleftarrow{\rho} & (\text{G} \mathcal{F} \cap \mathcal{C})(R) \\
\text{inc} & \xrightarrow{\lambda} & (\text{G} \mathcal{F} \cap \mathcal{C})(S) \\
S \otimes_R - & \xleftarrow{\text{res}} & I
\end{array}
\]

where with \( \rho \) a right adjoint and \( \lambda \) a left adjoint to the inclusion.

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## 2. Preliminaries

### 2.1. Purity and definable classes of modules.

For a ring \( R \) the category of left \( R \)-modules will be denoted by \( \text{Mod}(R) \), while \( \text{Mod}(R^\circ) \) will denote the category of right \( R \)-modules, where \( R^\circ \) is the opposite ring. We will similarly let \( \text{mod}(R) \) and \( \text{mod}(R^\circ) \) denote the finitely presented left and right \( R \)-modules. Unless stated otherwise, any reference to ‘a module’ will assume said module is in \( \text{Mod}(R) \). For brevity, the category \( \text{Mod}(\mathbb{Z}) \) will be denoted by \( \text{Ab} \). Let us start with a discussion around purity and definable classes of modules: initially recall that a monomorphism \( f : M \to N \) in \( \text{Mod}(R) \) is pure if for every \( X \in \text{Mod}(R^\circ) \) the induced map \( 1_X \otimes_R f : X \otimes_R M \to X \otimes_R N \) is also a monomorphism, in which case we say \( M \) is a pure submodule of \( N \). A short exact sequence

\[
0 \to L \to M \to N \to 0
\]

is called pure if \( L \to M \) is a pure monomorphism. The map \( M \to N \) is then called a pure epimorphism, and \( N \) is called a pure quotient. A subcategory \( \mathcal{D} \subseteq \text{Mod}(R) \) is definable if it is closed under all direct limits, direct products and pure submodules. As a consequence of these properties, it also follows that definable classes are closed under pure quotients. While being given by closure properties, it is also
possible to realise definable classes as the classes of modules which vanish on certain functors. More precisely, call a functor $F : \text{mod}(R) \to \text{Ab}$ finitely presented if there is a morphism $f : A \to B$ in $\text{mod}(R)$ such that

$$\text{Hom}_R(B, -) \xrightarrow{(f, -)} \text{Hom}_R(A, -) \to F \to 0$$

is exact in the functor category $(\text{mod}(R), \text{Ab})$; correspondingly we will denote by $(\text{mod}(R), \text{Ab})^{\text{fp}}$ the category of all finitely presented functors. Any such functor $F$ in $(\text{mod}(R), \text{Ab})^{\text{fp}}$ can be extended uniquely to a functor $\text{Mod}(R) \to \text{Ab}$, denoted by $\overrightarrow{F}$: let $M = \lim(M_i, f_{ij})_{i < j \in I}$ be expressed as a direct limit of finitely presented $R$-modules, and define $\overrightarrow{F}(M) = \lim_{j \to i} F(M_i) \in \text{Ab}$, which has a value independent of the choice of directed system representing $M$ (see [Pre09, 10.2.8]). There is then a bijective correspondence between the Serre subcategories of $(\text{mod}(R), \text{Ab})^{\text{fp}}$ and definable classes of $R$-modules, given by $S \mapsto \{ X \in \text{Mod}(R) : \overrightarrow{F}(M) = 0 \text{ for all } F \in S \}$ and $D \mapsto \{ F : F(X) = 0 \text{ for all } X \in D \}$. A treatment of the bijection (which holds in a much more general setting than module categories) can be found at [Pre09, 12.4.1].

**Example 2.1.** The class $\mathcal{F}(R)$ of flat $R$-modules is definable if and only if $R$ is a coherent ring. In this case it can be realised as the modules that vanish on the set $\{ \text{Tor}^R_1(R/I, -) : I \subset R \text{ is a finitely generated ideal} \}$. That each of these functors is finitely presented can be seen at [Pre09, 10.2.36]. It is also clear that over any ring the class of flat modules is closed under pure submodules and arbitrary direct limits, and thus is definable if and only if $\mathcal{F}(R)$ is closed under direct products; it is a classic result due to Chase in [Cha60] that this occurs if and only if $R$ is coherent.

A fundamental concept that is intimately related to definable categories is that of a pure-injective module.

**Definition 2.2.** An $R$-module $X$ is pure injective if for every pure-exact sequence $0 \to L \to M \to N \to 0$ the induced sequence $0 \to \text{Hom}_R(N, X) \to \text{Hom}_R(M, X) \to \text{Hom}_R(L, X) \to 0$ is exact.

Following convention, we will let $\mathcal{PI}(R)$ denote the class of pure injective left $R$-modules. There are several equivalent formulations for pure-injectivity, and not solely in module categories, which we will introduce when necessary. Before discussing the relationship between pure-injective modules and definable categories, let us recall some notions about approximations of modules which will be relevant to the discussion. If $C$ is a class of $R$-modules and $M$ is an $R$-module, then a morphism $f \in \text{Hom}_R(C, M)$, with $C \subset C$, is a $C$-precover of $M$ if

$$\text{Hom}_R(\overline{C}, f) : \text{Hom}_R(\overline{C}, C) \to \text{Hom}_R(\overline{C}, M)$$

is a surjective map of abelian groups for every $\overline{C} \in C$. A $C$-precover of $f : C \to M$ is a cover if for every $g \in \text{End}_R(C)$ that satisfies $fg = f$ yields that $g$ is an automorphism. We say that $C$ is (pre)covering if every $R$-module has a $C$-(pre)cover. The dual notion is that of (pre)-envelopes: if $E$ is a class, the $\gamma \in \text{Hom}_R(M, E)$, with $E \in E$, is a pre-envelope if $\text{Hom}_R(\gamma, \overline{E}) : \text{Hom}_R(E, \overline{E}) \to \text{Hom}_R(M, \overline{E})$ is surjective for every $\overline{E} \in E$. A pre-envelope $\gamma : M \to E$ is an envelope if for every $g \in \text{End}_R(E)$ such that $\gamma = g\gamma$ yields that $g$ is an automorphism. The following collates some known results about definable categories, pure-injectives, covers and envelopes.

**Proposition 2.3.** Let $D$ be a definable category. The following hold:

1. $D$ is covering;
2. $D$ is pre-enveloping;
3. The class $\mathcal{PI}$ of pure-injective $R$-modules is enveloping;
4. If $M$ is an $R$-module, then $M \in D$ if and only if $PE(M) \in D$, where $PE(M)$ denotes the pure-injective envelope of $M$. 
Proof. The first item is [CPT10, 2.7], while the remaining three are [Pre09, 3.4.42, 4.3.18, 4.3.21] respectively.

The interaction between definable subcategories and pure-injective modules runs a lot further, and is especially prevalent when one restricts to the set of indecomposable pure-injective modules, which will be denoted by \( \text{Pinj}(R) \). Primarily, definable categories are uniquely determined by the indecomposable pure-injective modules contained within them: if \( D \) and \( D' \) are definable classes of \( R \)-modules, then \( D = D' \) if and only if \( D \cap \text{Pinj}(R) = D' \cap \text{Pinj}(R) \), see [Pre09, 5.1.5]. Secondly, and significantly, the set \( \text{Pinj}(R) \) becomes a topological space which has a basis of closed sets parameterised by definable subcategories: given any definable \( D \) the intersection \( D \cap \text{Pinj}(R) \) is a closed subset [Pre09, 5.1.1]. By the preceding fact it is clear that one can go in the other direction by considering the definable category generated by a closed set, and these operations are mutually inverse. This topological space is called the Ziegler spectrum and is denoted by \( \text{Zg}(R) \) (although we may drop the \( R \) if it is clear which ring and modules are being considered). One can consider the Ziegler spectrum of a class of any \( R \)-modules, say \( C \), by considering \( \langle C \rangle \cap \text{Zg}(R) \), where \( \langle C \rangle \) is the definable category generated by \( C \). The topology on this set is the subset topology, and we will denote the resultant topological space by \( \text{Zg}(C) \).

One may then wonder which functors between definable categories preserve the definable and pure structures contained within, and it is precisely the interpretation functors, elsewhere referred to as definable functors, as described in the introduction. More specifically, if \( C \) and \( D \) are definable categories of modules (for some rings), then a functor \( F : C \to D \) that preserves direct limits and direct products will preserve both pure exact sequences and pure injective modules, as illustrated in [Pre11, 13.4]. However, it does not follow immediately that indecomposability of pure injectives is preserved, so in general one does not obtain a map \( \text{Zg}(C) \to \text{Zg}(D) \); for this to occur a considerably stronger assumption is needed on \( F \), which is for it to be full on pure injectives. In other words, if the canonical morphism \( \text{Hom}_C(X, Y) \to \text{Hom}_D(F(X), F(Y)) \) is surjective for every \( X, Y \in \mathcal{P}(C) \), then \( F \) preserves not only indecomposable pure injectives, but also induces a closed and continuous map \( \text{Zg}(C) \to \text{Zg}(D) \) of topological spaces, which then induces a homeomorphism \( \text{Zg}(C) \setminus K \to I \), where \( K = \{ X \in \text{Zg}(C) : F(X) = 0 \} \) and \( I \) is the subset of \( \text{Zg}(D) \) corresponding to the definable closure of the essential image of \( F \) in \( D \). A proof of these statements, as well as a substantial discussion around functors between definable categories, can be found in [Pre11, §15].

Example 2.4. The following example of an interpretation functor will be used throughout. Let \( R \) and \( S \) be two rings such that \( S \) is endowed with a right \( R \)-module structure, yielding a functor \( S \otimes_R - : \text{Mod}(R) \to \text{Mod}(S) \) which trivially commutes with direct limits. For any collection \( \mathcal{M} = \{ M_i \}_I \) of \( R \)-modules, there is a canonical map

\[ \varphi_{S, \mathcal{M}} : S \otimes_R \prod_I M_i \to \prod_I S \otimes_R M_i \]

which is an isomorphism if and only if \( S \) is finitely presented over \( R \). However, even if one restricts to the case when \( R \) and \( S \) are coherent and considers the restriction to a functor of definable subcategories \( S \otimes_R - : \mathcal{F}(R) \to \mathcal{F}(S) \), one cannot dispense of this finitely presented assumption. Indeed, if one desires \( \varphi_{S, \mathcal{M}} \) to be an isomorphism when \( \mathcal{M} \) is an arbitrary collection of flat modules, then it must be an epimorphism when \( M_i = R \) for each \( i \in I \), which, by [GT12, 3.8], necessitates \( S \) being finitely generated. However, by [Goo72, Theorem 1], it follows that for every finitely generated \( R \)-submodule \( A \subseteq S \), the inclusion map factors through a finitely presented module. In particular, the identity on \( S \) does, and is therefore a quotient of a finitely presented \( R \)-module, so is finitely presented itself.
Given a class \( \mathcal{A} \) of modules over any ring let
\[
\mathcal{A}^\perp = \{ M \in \text{Mod}(R) : \text{Ext}^1_R(A, M) = 0 \text{ for all } M \in \mathcal{A} \}
\]
denote the right Ext-orthogonal class, and similarly define \( \perp \mathcal{A} \) as the left Ext-orthogonal class. A pair \((A, B)\) of classes of modules is a cotorsion pair if \( A^\perp = B \) and \( A = \perp B \). The classic non-trivial example of a cotorsion pair is the flat-cotorsion pair, \((\mathcal{F}(R), \mathcal{C}(R))\), and the modules in \( \mathcal{C}(R) \) are called cotorsion modules. Notice immediately that every pure-injective module is cotorsion, if one takes a short-exact sequence \( 0 \to L \to M \to F \to 0 \) with \( F \) flat, hence the sequence is pure-exact, then \( \text{Ext}^1_R(F, M) = 0 \) for all pure-injective modules \( M \) by definition. The advantage for flat modules over coherent rings is that the converse holds, namely modules that are simultaneously flat and cotorsion are pure-injective, see \([ \text{Xu96}, 3.5.1 \]) and therefore \( \mathcal{P}\text{I} \cap \mathcal{F} \) is extension closed. In fact, over any ring \( R \) a module is flat and cotorsion if and only if it is definable, in which case any injective \( R \)-module is necessarily coherent. As originally shown in \([ \text{Gil17}, 3.4 \]) for right coherent rings, and expanded upon in \([ \text{CET20}, 4.5 \]) to all rings, the class \( (\mathcal{F} \cap \mathcal{C})(R) \), consisting of all modules that are simultaneously Gorenstein flat and cotorsion, is a Frobenius exact category whose projective-injective objects are \((\mathcal{F} \cap \mathcal{C})(R)\). In fact, the latter reference also shows that \( \mathcal{G}\mathcal{F}(R) \) is itself Frobenius if and only if \( R \) is left perfect, which is equivalent to \( \mathcal{G}\mathcal{F}(R) = (\mathcal{G}\mathcal{F} \cap \mathcal{C})(R) \). Before describing the resulting stable category in more detail, we will introduce some more notation. For ease we transfer the notation and terminology from \([ \text{CET20} \]) and similar notations.

2.2. Totally acyclic complexes and Gorenstein flat cotorsion modules. The concepts of the previous section will be applied primarily to the class of Gorenstein flat modules, which are now introduced. If \( R \) is any ring, a complex \( T \) of flat \( R \)-modules is \( \mathcal{F} \)-totally acyclic if it acyclic and \( I \otimes_R T \) is acyclic for every injective \( R \)-module \( I \). A module \( M \) is Gorenstein flat if there is a \( \mathcal{F} \)-totally acyclic complex \( T \) such that \( M = L_0(T) \). The class of Gorenstein flat modules will be denoted \( \mathcal{G}\mathcal{F}(R) \). Like the flat modules, there is a cotorsion pair whose left hand class is \( \mathcal{G}\mathcal{F}(R) \), by \([ \text{SS20b}, 3.12 \]) the right hand side is denoted \( \mathcal{G}\mathcal{C}(R) \), and we refer to these modules as Gorenstein cotorsion. Again similarly to the flat modules \( \mathcal{G}\mathcal{F}(R) \) is in general not closed under products, and it was shown in \([ \text{ˇSˇS20}b, 4.13 \]) that it is product closed if every injective \( R \)-module is definable, see \([ \text{Xu96}, 3.5.1 \]) for left coherent rings, and expanded upon in \([ \text{CET20}, 4.5 \]) to all rings, the class \( \mathcal{G}\mathcal{F}(R) \cap \mathcal{C} \), consisting of all modules that are simultaneously Gorenstein flat and cotorsion, is a Frobenius exact category whose projective-injective objects are \((\mathcal{G}\mathcal{F} \cap \mathcal{C})(R)\). In fact, the latter reference also shows that \( \mathcal{G}\mathcal{F}(R) \) is itself Frobenius if and only if \( R \) is left perfect, which is equivalent to \( \mathcal{G}\mathcal{F}(R) = (\mathcal{G}\mathcal{F} \cap \mathcal{C})(R) \). Before describing the resulting stable category in more detail, we will introduce some more notation. For ease we transfer the notation and terminology from \([ \text{CET20} \]). The following is actually a proposition in its original form, but we shall use it as a definition for brevity.

Definition 2.5. \([ \text{CET20}, 1.3 \]) Let \( R \) be a ring and \( \mathcal{U} \) a class of \( R \)-modules. A complex \( T \) is right \( \mathcal{U} \)-totally acyclic if
\begin{enumerate}
\item \( T \) is acyclic;
\item for every \( I \in \mathcal{Z} \) the module \( T_i \) is in \( \mathcal{U} \cap \mathcal{U}^\perp \);
\item for every \( V \in \mathcal{U} \), the complex \( \text{Hom}_R(V, T) \) is acyclic;
\item for every \( W \in \mathcal{U} \cap \mathcal{U}^\perp \), the complex \( \text{Hom}_R(T, W) \) is acyclic.
\end{enumerate}
We will call a right \((\mathcal{F} \cap \mathcal{C})(R)\)-totally acyclic complex a totally acyclic complex of flat-cotorsion modules. The following theorem relates these complexes to \( \mathcal{F} \)-totally acyclic complexes, as well as right \( \mathcal{F}(R) \)-totally acyclic complexes.

Theorem 2.6. \([ \text{CET20}, 4.4 \]) If \( R \) is a right coherent ring, the following conditions are equivalent for a complex \( T \):
\begin{enumerate}
\item \( T \) is a totally acyclic complex of flat-cotorsion modules;
\item \( T \) is a \( \mathcal{F} \)-totally acyclic complex that is termwise flat-cotorsion;
\item \( T \) is right \( \mathcal{F}(R) \)-totally acyclic.
If $T$ is such a complex then $Z_0(T)$ is in $(\mathcal{GF} \cap \mathcal{C})(R)$, and likewise if $M \in (\mathcal{GF} \cap \mathcal{C})(R)$ there is a complex $T'$ satisfying the above equivalent conditions with $M = Z_0(T')$. The above formulation enables a clear and concrete description of the stable category of $(\mathcal{GF} \cap \mathcal{C})(R)$. The following gives one of many triangulated equivalences with the stable category, but it is the only one that shall be used, albeit extensively, in this text.

**Theorem 2.7.** [CET20, 5.6, 5.7] Let $R$ be a right-coherent ring. Then there is a triangulated equivalence

$$(\mathcal{GF} \cap \mathcal{C})(R) \simeq K_{\text{tac}}((\mathcal{F} \cap \mathcal{C})(R)).$$

Where $K_{\text{tac}}((\mathcal{F} \cap \mathcal{C})(R))$ is the homotopy category of totally acyclic complexes of flat-cotorsion modules.

We will also extensively use the following vanishing result concerning Gorenstein flat modules and modules of finite homological dimensions.

**Lemma 2.8.** [BCIE20, 2.3] If $M$ is a Gorenstein flat $R$-module, then $\text{Tor}^i_R(T, M) = 0$ for all $i > 0$ and all $R$-modules of finite flat or finite injective dimension.

### 3. Transfer of Gorenstein flat-cotorsion modules

Let $R \to S$ be a morphism of rings such that $S$ is finitely presented as a right $R$-module and has finite flat dimension over $R$. The functor $S \otimes_R - : \text{Mod}(R) \to \text{Mod}(S)$ is therefore an interpretation functor, and, as it sends flat $R$-modules to flat $S$-modules, it restricts to a functor $(\mathcal{F} \cap \mathcal{C})(R) \to (\mathcal{F} \cap \mathcal{C})(S)$. With some further assumptions, we also have a functor between Gorenstein flat and cotorsion modules. To this end let $C_{\text{tac}}((\mathcal{F} \cap \mathcal{C})(R))$ denote the category of totally acyclic complexes of flat cotorsion modules, and likewise over $S$.

**Lemma 3.1.** Let $\varphi : R \to S$ be as above, with the additional assumption that both $R$ and $S$ are right coherent. Then $S \otimes_R - : C_{\text{tac}}((\mathcal{F} \cap \mathcal{C})(R)) \to C_{\text{tac}}((\mathcal{F} \cap \mathcal{C})(S))$. In particular, there is a functor $S \otimes_R - : (\mathcal{GF} \cap \mathcal{C})(R) \to (\mathcal{GF} \cap \mathcal{C})(S)$.

**Proof.** Let $T$ be a totally acyclic complex of flat-cotorsion $R$ modules and consider the $S$-complex $S \otimes_R T$. We will show that it satisfies the conditions from (2.5) in the case where $\mathcal{U} = (\mathcal{F} \cap \mathcal{C})(S)$. The second of the criteria is obvious since $(\mathcal{F} \cap \mathcal{C})(S)$ is self-orthogonal and $S \otimes_R -$ is an interpretation functor. For the first item, note that $\text{Tor}^i_R(S, T_j) = 0$ for all $i > 0$ and $j \in \mathbb{Z}$ as $T_j$ is flat; moreover as $Z_j(T) \in \mathcal{GF}(R)$ for each $j \in \mathbb{Z}$, we have $\text{Tor}^i_R(S, Z_j(T)) = 0$ for all $i \geq 1$ by (2.5). In particular $S \otimes_R T$ is an acyclic complex (over both $R$ and $S$). We now turn our attention to the third and fourth items. As $S \otimes_R T$ is an acyclic complex of flat-cotorsion $S$-modules, each of its cyclic modules $Z_j(S \otimes_R T)$ are in $\mathcal{C}(S)$ by [BCIE20, 1.3]. In particular

$$\text{Ext}^i_S(F, (S \otimes_R T)_j) = 0 = \text{Ext}^i_S(F, Z_j(S \otimes_R T))$$

for every flat $S$-module $F$, $j \in \mathbb{Z}$ and $i > 0$, hence $\text{Hom}_S(F, S \otimes_R T)$ is acyclic. As $S \otimes_R - : C(R) \to C(S)$ arises as the extension of a right exact functor between $\text{Mod}(R)$ and $\text{Mod}(S)$, there are isomorphisms $C_i(S \otimes_R T) \simeq S \otimes_R Z_{-i}(T)$ for all $i \in \mathbb{Z}$. By our coherence assumption, $Z_{-i}(T)$ is a Gorenstein flat $R$-module and as $\text{fd}_R(S) < \infty$ it follows that $S \otimes_R Z_{-i}(T)$ is a Gorenstein flat $S$-module by [CH00, 4.6]. In particular, since $\mathcal{GF} \cap \mathcal{GF}^{-1} = \mathcal{F} \cap \mathcal{C}$ by [Gil17, 3.2], it follows that

$$\text{Ext}^i_S(S \otimes_R T_j, F) = 0 = \text{Ext}^i_S(C_j(T), F),$$

for every $F \in (\mathcal{F} \cap \mathcal{C})(S)$, $i \geq 0$ and $j \in \mathbb{Z}$. Therefore the fourth condition of (2.5) also holds, and $S \otimes_R T$ is a totally acyclic complex of flat-cotorsion $S$-modules.

□
Remark. The statement of the above theorem also holds if one replaces \( S \) with an \( S-R \)-bimodule \( F \) such that \( F \) has finitely presented and of finite flat dimension over \( R \), while simultaneously being flat over \( S \), because \( F \otimes_R - : \mathcal{F}(R) \to \mathcal{F}(S) \) is then again an interpretation functor that preserves Gorenstein flat modules by [CH09 4.6].

The following example illustrates a rather natural setting in which the above lemma holds, and will appear throughout.

Example 3.2. Let \( R \) be a commutative noetherian ring. A sequence \( x = x_1, \ldots, x_n \) of elements of \( R \) is called an \( R \)-sequence if \( x_{i+1} \) is a nonzerodivisor on \( R/(x_1, \ldots, x_i)R \) for all \( 0 \leq i < n \) and \( R/xR \neq 0 \). For such a sequence, the quotient ring \( R/x \), viewed as an \( R \)-module, has a projective resolution given by the Koszul complex \( K(x) \), see, for example, [BH93 1.6.14]. In particular, the flat dimension of \( R/x \) as an \( R \)-module is equal to the length of \( x \). Consequently the canonical projection \( R \to R/x \) satisfies the conditions of the above lemma, and thus factoring via a regular sequence preserves Gorenstein flat and cotorsion modules.

Theorem 3.3. Let \( R \) and \( S \) be right coherent rings and \( R \to S \) a ring homomorphism that realises \( S \) as a finitely presented right \( R \)-module of finite flat dimension. Then the functor \( S \otimes_R - : (\mathcal{G} \mathcal{F} \cap \mathcal{C})(R) \to (\mathcal{G} \mathcal{F} \cap \mathcal{C})(S) \) is a functor of Frobenius exact categories which restricts to a triangulated functor
\[
(\mathcal{G} \mathcal{F} \cap \mathcal{C})(R) \to (\mathcal{G} \mathcal{F} \cap \mathcal{C})(S)
\]
between the corresponding stable categories.

Proof. By (3.1) we know that \( S \otimes_R - : (\mathcal{G} \mathcal{F} \cap \mathcal{C})(R) \to (\mathcal{G} \mathcal{F} \cap \mathcal{C})(S) \), and by the assumptions it is an interpretation functor so preserves flat-cotorsion modules, which are exactly the injective objects in these categories. Moreover, the exact structure is also preserved, since \( \text{Tor}^1_R(S, M) = 0 \) for all Gorenstein flat modules by [2.8], so, by [Hap88 2.8], there is an induced functor between the stable categories when viewed as additive categories. Using the equivalence \( (\mathcal{G} \mathcal{F} \cap \mathcal{C})(R) \cong K_{\text{inc}}(\mathcal{F} \cap \mathcal{C})(R) \) from (2.7) and the fact that \( S \otimes_R - : K(R) \to K(S) \) is triangulated, it follows that the restriction \( (\mathcal{G} \mathcal{F} \cap \mathcal{C})(R) \to (\mathcal{G} \mathcal{F} \cap \mathcal{C})(S) \) is also triangulated, which finishes the proof.

Remark. A closely related triangulated category to the above is the singularity category, which is the stable category of the Frobenius category consisting of finitely presented Gorenstein projective objects. The study of how the singularity category moves along ring homomorphisms has been studied in [OPS19], albeit from a different viewpoint.

We will now assume that \( R \to S \) satisfies the condition of the theorem, together with the additional assumption that the classes \( \mathcal{G} \mathcal{F}(R) \) and \( \mathcal{G} \mathcal{F}(S) \) are definable. As previously mentioned, the definability of these classes necessitates both \( R \) and \( S \) to be left coherent, so we will assume that \( R \) and \( S \) are coherent on both sides. In such a setting, \( S \otimes_R - : \mathcal{G} \mathcal{F}(R) \to \mathcal{G} \mathcal{F}(S) \) is an interpretation functor and therefore much of the pure structure is preserved immediately. However, as every pure injective module is cotorsion, the pure injective part of \( \mathcal{G} \mathcal{F}(R) \) is contained within the Frobenius category \( (\mathcal{G} \mathcal{F} \cap \mathcal{C})(R) \), and likewise for \( S \), and thus every non-flat pure injective module is seen in the functor \( S \otimes_R - : (\mathcal{G} \mathcal{F} \cap \mathcal{C})(R) \to (\mathcal{G} \mathcal{F} \cap \mathcal{C})(S) \).

We now show that, with some assumptions that are trivially satisfied by commutative rings, certain information regarding the functor between the stable categories can be lifted to deduce information about the interpretation functor between the definable categories, with the end goal showing that if the functor \( (\mathcal{G} \mathcal{F} \cap \mathcal{C})(R) \to (\mathcal{G} \mathcal{F} \cap \mathcal{C})(S) \) is full, then \( S \otimes_R - : \mathcal{G} \mathcal{F}(R) \to \mathcal{G} \mathcal{F}(S) \) is full on pure injectives, and therefore preserves all the induced structure on \( \mathcal{G} \mathcal{F}(R) \). In order to show this, we need several auxiliary lemmas.
Lemma 3.4. Let $F$ be a flat $S$-module, then $F$ is a pure submodule of a module of the form $S \otimes_R X$ with $X$ a flat cotorsion $R$-module. If $F$ is cotorsion, this is a split embedding.

Proof. As $R$ is coherent, the definable closure of $\{R\}$ is $\mathcal{F}(R)$, and likewise for $S$. Let $\mathcal{I} \subset \text{Mod}(S)$ denote the image of $\mathcal{F}(R)$ under $S \otimes_R -$. While $\mathcal{I}$ is not definable, its closure under pure submodules, $\bar{\mathcal{I}}^p$, is by \cite{Prel}. Moreover $\bar{\mathcal{I}}^p = \mathcal{F}(S)$: as $S \in \bar{\mathcal{I}}^p$ we have $\mathcal{F}(S) \subseteq \bar{\mathcal{I}}^p$, while the other inclusion is clear since $S \otimes_R -$ preserves flat modules and $\mathcal{F}(S)$ is closed under pure submodules. Therefore if $F \in \mathcal{F}(S)$ there is a pure embedding $0 \to F \to S \otimes_R X$ with $X \in \mathcal{F}(R)$. We now show that $X$ can be chosen to additionally be pure-injective. To this end, consider the pure embedding $0 \to X \to PE(X)$ in $\mathcal{F}(R)$, where $PE(X)$ is the pure injective envelope of $X$. It follows that $0 \to S \otimes_R X \to S \otimes_R PE(X)$ is a pure embedding by $S \otimes_R -$ being an interpretation functor. As pure embeddings are closed under composition, we have that $0 \to F \to S \otimes_R PE(X)$ gives the desired pure embedding. For the second claim, if $F$ is cotorsion it is pure injective, and thus the pure embedding $0 \to F \to S \otimes_R PE(X)$ splits. \[\square\]

The following corollary is now essentially immediate.

Corollary 3.5. Let $M$ and $N$ be $S$-modules. A morphism $f \in \text{Hom}_S(M, N)$ factors through a flat cotorsion $S$-module if and only if it factors through a module of the form $S \otimes_R X$ with $X \in (\mathcal{F} \cap \mathcal{C})(R)$.

Proof. Suppose $F \in (\mathcal{F} \cap \mathcal{C})(S)$ and consider the split embedding $0 \to F \to S \otimes_R X$ with $X \in (\mathcal{F} \cap \mathcal{C})(R)$ which exists by the preceding lemma. Clearly if $f$ factors through $F$ it also factors through $S \otimes_R X$ via $M \to F \to S \otimes_R X \to F \to N$, where $S \otimes_R X \to F$ is the canonical projection. The other direction is trivial as $S \otimes_R X$ is a flat cotorsion $S$-module for every flat-cotorsion $R$-module $X$. \[\square\]

Consider the following two 'fullness' conditions for $M \in \mathcal{GF}(R) \cap \mathcal{PL}$ and $X \in \mathcal{F}(R)$:

(C1) The canonical map $\text{Hom}_R(M, X) \to \text{Hom}_S(S \otimes_R M, S \otimes_R X)$ is surjective;

(C2) The canonical map $\text{Hom}_R(X, M) \to \text{Hom}_S(S \otimes_R X, S \otimes_R M)$ is surjective.

The above two restrictions will be necessary in lifting properties from the stable categories to the class of Gorenstein flat modules. While they may initially appear slightly restrictive, they hold in common situations.

Lemma 3.6. Let $R \to S$ be a surjective homomorphism of commutative coherent rings, such that $S \in \text{mod}(R)$ is of finite flat dimension. Then the above two conditions are satisfied.

Proof. As $R \to S$ is surjective, there is a short exact sequence $0 \to \Omega^1_R(S) \to R \to S \to 0$ of $R$-modules. For the first condition, there is the induced exact sequence

\[0 \to \Omega^1_R(S) \otimes_R X \to X \to S \otimes_R X \to 0,\]

and thus it suffices to show that $\text{Ext}_R^1(M, S \otimes_R X) = 0$ for all pure injective Gorenstein flat $R$-modules. In fact, in our setting we can show this for every $M \in (\mathcal{GF} \cap \mathcal{C})(R)$. Indeed, let us first observe that $S \otimes_R X$ is a Gorenstein cotorsion $R$-module. Indeed, since $X$ is flat and $M$ is Gorenstein flat, by the assumptions on $S$ there are isomorphisms $S \otimes_R M \simeq S \otimes_R^M M$ and $S \otimes_R X \simeq S \otimes_R^L X$ in $\text{D}(R)$. Consequently we have

\[\text{RHom}_S(S \otimes_R^M M, S \otimes_R^L X) \simeq \text{RHom}_R(M, S \otimes_R^L X),\]

yet as $S \otimes_R -$ is an interpretation functor, we have $S \otimes_R X \in (\mathcal{GF} \cap \mathcal{C})(S) = \mathcal{GC}(S)$, hence, by the above isomorphism, $\text{Ext}_R^i(M, S \otimes_R X) = 0$ for all $i > 0$, proving the observation. In particular, it follows that the induced exact sequence \[[\square]\] is a resolution of $\Omega^1_R(S) \otimes_R X$ by Gorenstein cotorsion modules, hence it has Gorenstein cotorsion dimension at most one. If it is Gorenstein cotorsion, then we trivially have $\text{Ext}_R^0(M, S \otimes_R X) = 0$, so assume it has Gorenstein cotorsion dimension equal to one. We may, therefore, chose a minimal Gorenstein cotorsion resolution

\[0 \to \Omega^1_R(S) \otimes_R X \to C^0 \to C^1 \to 0\]
where $C^0$ and $C^1$ are Gorenstein cotorsion and $C^0$ is a cotorsion envelope of $\Omega^1_R(S) \otimes_R X$. As $(\mathcal{GF}(R), \mathcal{GF}(R))$ is a complete cotorsion pair, it follows that $C^1 \in \mathcal{GF}(R)$, but is then, by assumption, additionally in $\mathcal{GC}(R)$, so is a flat cotorsion $R$-module and is therefore pure injective. But as $X$ is pure injective and $\Omega^1_R$ is finitely presented, the tensor product $\Omega^1_R(S) \otimes_R X$ is also pure injective, and thus $\text{Ext}^1_R(C^1, \Omega^1_R(S) \otimes_R X) = 0$, so the minimal Gorenstein cotorsion resolution of $\Omega^1_R(S) \otimes_R X$ splits, and thus it is a summand of a Gorenstein cotorsion module so is itself Gorenstein cotorsion. This proves the first condition holds. For the second condition, if $M$ is pure injective then so is $\Omega^1_R(S) \otimes_R M$, and thus $\text{Ext}^1_R(X, \Omega^1_R(S) \otimes_R M) = 0$. For the same reasoning as the first case this is sufficient.

We now show how to use the above conditions to lift a property of the functor $S \otimes_R - : (\mathcal{GF} \cap \mathcal{C})(R) \rightarrow (\mathcal{GF} \cap \mathcal{C})(S)$ to deduce results about purity of Gorenstein flat modules.

**Theorem 3.7.** Let $R \rightarrow S$ be a ring homomorphism between rings that are coherent on both sides that makes $S$ a finitely presented right $R$-module of finite flat dimension and $\mathcal{GF}(R)$ and $\mathcal{GF}(S)$ are definable. If $S \otimes_R - : (\mathcal{GF} \cap \mathcal{C})(R) \rightarrow (\mathcal{GF} \cap \mathcal{C})(S)$ is full and both (C1) and (C2) hold, then $S \otimes_R - : \mathcal{GF}(R) \rightarrow \mathcal{GF}(S)$ is full on pure injectives.

**Proof.** Let $M$ and $N$ be pure-injective Gorenstein flat $R$-modules and let $q_R : (\mathcal{GF} \cap \mathcal{C})(R) \rightarrow (\mathcal{GF} \cap \mathcal{C})(R)$ denote the obvious functor, and likewise for $S$. Suppose that $f \in \text{Hom}_R(S \otimes_R M, S \otimes_R N)$; then $\overline{f} := q_R(f) \in \text{Hom}_S(S \otimes_R M, S \otimes_R N)$ and by the there is a $\overline{\varphi} \in \text{Hom}_R(M, N)$ with $S \otimes_R \overline{\varphi} = \overline{f}$. Let $g \in \text{Hom}_R(M, N)$ be a representative of $\varphi$. Since

$$
\begin{array}{ccc}
(\mathcal{GF} \cap \mathcal{C})(R) & \xrightarrow{S \otimes_R -} & (\mathcal{GF} \cap \mathcal{C})(S) \\
\downarrow{q_R} & & \downarrow{q_S} \\
(\mathcal{GF} \cap \mathcal{C})(R) & \xrightarrow{S \otimes_R -} & (\mathcal{GF} \cap \mathcal{C})(S)
\end{array}
$$

commutes, it is clear that $(q_S \circ S \otimes_R -)(g) = \overline{f}$, and thus $S \otimes_R g = f + \varphi$ for some $\varphi$ that factors through a flat-cotorsion $S$-module. By (55), we can assume that $\varphi$ factors through $S \otimes_R X$ for some $X \in (\mathcal{F} \cap \mathcal{C})(R)$. In particular, there are maps $\alpha : S \otimes_R M \rightarrow S \otimes_R X$ and $\beta : S \otimes_R X \rightarrow S \otimes_R N$ such that $\beta \circ \alpha = \varphi$. Yet by the assumptions (C1) and (C2) it follows that there are maps $\overline{\alpha} \in \text{Hom}_R(M, X)$ and $\overline{\beta} \in \text{Hom}_R(X, N)$ that are preimages of $\alpha$ and $\beta$ under $S \otimes_R -$. Let $\gamma := \beta \circ \overline{\alpha} \in \text{Hom}_R(M, N)$. Then $S \otimes_R \gamma = \varphi$ by construction. In particular we see that $S \otimes_R (g - \gamma) = f$, so the functor is full on pure-injectives. \hfill \Box

Let us recall some of the consequences of $S \otimes_R - : \mathcal{GF}(R) \rightarrow \mathcal{GF}(S)$ being full on pure injectives, which hold whenever the conditions of the above theorem hold.

**Corollary 3.8.** Should the conditions of (3.7) hold, then we have the following:

1. $S \otimes_R -$ preserves pure injective hulls of Gorenstein flat modules: $S \otimes_R PE(X) = PE(S \otimes_R X)$ for all $X \in \mathcal{GF}(R)$;
2. $S \otimes_R -$ induces a homeomorphism $Z_\mathcal{G}(\mathcal{GF}(R)) \setminus K \rightarrow \mathcal{I}$, where $K = \{ X \in Z_\mathcal{G}(\mathcal{GF}(R)) : S \otimes_R X = 0 \}$ and $\mathcal{I}$ is the closed subset of $Z_\mathcal{G}(\mathcal{GF}(S))$ corresponding to the image of $\mathcal{GF}(R)$ under $S \otimes_R -$;
3. as a particular form of the second item, if $X$ is an indecomposable pure injective Gorenstein flat $R$-module, then $S \otimes_R X$ is either indecomposable or zero in $\mathcal{GF}(S)$.

We note that the above results are applications of much more general results regarding the relationships between interpretation functors and the Ziegler spectrum. These can be found, including proofs, at [Pre11] §13.

Throughout we have used the assumption that $R \rightarrow S$ is a morphism of coherent rings. This is because, as seen in (2.6), the notions of being Gorenstein flat and cotorsion coincides with being a cycle
module in a totally acyclic complex of flat-cotorsion modules. However, once the assumption of being coherent is removed, this stops being the case, since objects in the latter class need not be Gorenstein flat. A detailed discussion of the difference is found at [CEL+21, §5]. The consequence of this is that \( S \otimes_R - \) need not preserve totally-acyclic complexes of flat-cotorsion modules.

4. Purity considerations in the stable category

As mentioned, on the level of objects every pure injective Gorenstein flat module appears in \((\mathcal{GF} \cap C)(R)\) whenever \( R \) is a right coherent ring; this notably holds for the indecomposable objects. However, if one wishes to understand the topology on the Ziegler spectrum, it is much more difficult to glean information from \((\mathcal{GF} \cap C)(R)\), since this is really rather far from being definable: in general it will not be closed under either pure submodules nor under coproducts. In fact, closure of \((\mathcal{GF} \cap C)(R)\) under coproducts is a highly restrictive consition on \( R \), as the next result shows. Before stating said result, recall that an \( R \)-module is \( \Sigma \)-cotorsion (resp. pure injective) if and only if every set indexed coproduct of copies of it is also cotorsion (resp. pure injective).

**Proposition 4.1.** Let \( R \) be a right coherent ring. Then the following are equivalent.

1. \((\mathcal{GF} \cap C)(R)\) is closed under arbitrary coproducts;
2. \((\mathcal{F} \cap C)(R)\) is closed under coproducts;
3. every flat \( R \)-module is \((\Sigma -)\) cotorsion;
4. every \( R \)-module is \((\Sigma -)\) cotorsion;
5. \( R \) is a left perfect ring.

**Proof.** For (1 \( \implies \) 2), the inclusion \( \mathcal{FC}(R) \subset (\mathcal{GF} \cap C)(R) \) shows that any coproduct of flat and cotorsion \( R \)-modules is Gorenstein flat and cotorsion by assumption, yet it is then immediate that this coproduct is also flat as coproducts of flat modules are always flat. For (2 \( \implies \) 3) let \( F \) be an arbitrary flat module and consider its pure-injective hull \( PE(F) \). As \( R \) is right coherent \( \mathcal{F}(R) \) is definable and thus \( PE(F) \) is also flat, so by assumption is \( \Sigma \)-cotorsion. Yet it then follows that every pure submodule of \( PE(X) \) is \( \Sigma \)-cotorsion by [ˇSˇt20b, 3.3], and therefore \( F \) is \( \Sigma \)-cotorsion as well. For (3 \( \implies \) 4), let \( M \) be an \( R \)-module and consider the flat cover \( \varphi : F \to M \), which is an epimorphism and \( \text{Ker}(\varphi) \in \mathcal{C}(R) \). Since \( F \) is also cotorsion, so is \( M \). In particular, all \( R \)-modules are then cotorsion, hence also \( \Sigma \)-cotorsion. For (4 \( \implies \) 5), note that \( R \) is itself then \( \Sigma \)-pure-injective, and therefore by [Rot02, 3.2] \( R \) is left perfect. For (5 \( \implies \) 1), we observe that over a right coherent perfect ring \( \mathcal{GF}(R) = (\mathcal{GF} \cap C)(R) \) by [CET20, 4.5].

We can therefore see that \((\mathcal{GF} \cap C)(R)\) is very rarely closed under coproducts; this happens if and only if \( R \) is a left perfect and right coherent. In this case \((\mathcal{GF} \cap C)(R) = \mathcal{GF}(R) \). Consequently it does not make sense to speak of compact objects in \((\mathcal{GF} \cap C)(R)\), which is usually an assumption to discuss purity in triangulated categories. However, in the case that \( \mathcal{GF}(R) \) is closed under products, which as we have seen is equivalent to it being definable, there is an alternative notion of pure injectivity, introduced by Saorín and Štovíček, which only requires the existence of products.

**Definition 4.2.** [SS20a, 5.1] Let \( A \) be an additive category with products. Then \( Y \in A \) is pure-injective if for each set \( I \) there is a morphism \( f : Y^I \to Y \) such that \( f \circ \lambda_i = 1_Y \) for every \( i \in I \), where \( \lambda_i : Y \to Y^I \) is the canonical embedding.

This formulation of pure-injectivity agrees with the notion as used above. We can use it to slightly modify a classic result of Krause to compare pure-injectivity of objects in \((\mathcal{GF} \cap C)(R)\) and \((\mathcal{GF} \cap C)(R)\), at least in the case when \( \mathcal{GF}(R) \) is definable.
Lemma 4.3. Let $R$ be a coherent ring such that $\mathcal{GF}(R)$ is closed under products. Then an object $X \in (\mathcal{GF} \cap \mathcal{C})(R)$ is pure-injective if and only if it is pure-injective in $(\mathcal{GF} \cap \mathcal{C})(R)$.

Proof. The proof is a modification of [Kra00, 1.16]. It is clear that a pure-injective object in $\mathcal{GF}(R)$ is pure-injective in $(\mathcal{GF} \cap \mathcal{C})(R)$. For the converse, suppose $X \in (\mathcal{GF} \cap \mathcal{C})(R)$ is pure-injective. Then for every index set $I$ there is a morphism $f : X^I \to X$ such that $1_i = f \circ \lambda_i$ for each $i \in I$. Let $\tilde{f}$ be the corresponding map $X^I \to X$ in $\text{Mod}(R)$, so $1_X = \tilde{f} \circ \lambda_i = \beta \circ \alpha$, where $\alpha : X \to F$ and $\beta : F \to X$ are morphisms with $F$ flat and cotorsion. Since $F$ is injective in $(\mathcal{GF} \cap \mathcal{C})(R)$ and $\lambda_i$ is a monomorphism, there is a factorisation $\alpha = \gamma \circ \lambda_i$ where $\gamma : X^I \to F$. In particular, we see that $1_X = (\tilde{f} + \beta \circ \gamma)\lambda_i$, hence $X$ is pure-injective in $(\mathcal{GF} \cap \mathcal{C})(R)$. \hfill \Box

The following result is now immediate.

Corollary 4.4. Let $R \to S$ be a morphism of right coherent rings such that $S$ is finitely presented and of finite flat dimension as a right $R$-module. With the assumption that $\mathcal{GF}(R)$ and $\mathcal{GF}(S)$ are definable, the functor $S \otimes_R - : (\mathcal{GF} \cap \mathcal{C})(R) \to (\mathcal{GF} \cap \mathcal{C})(R)$ preserves pure-injectivity.

Let us continue with the assumption that $\mathcal{GF}(R)$ and $\mathcal{GF}(S)$ are definable. In this case, the kernel of the functor $S \otimes_R - : \mathcal{GF}(R) \to \mathcal{GF}(S)$ is of significant interest when attempting to understand the relationship between pure injective objects in $\mathcal{GF}(R)$ and $\mathcal{GF}(S)$ respectively - see, for example, (4.3).

Yet the kernel of $S \otimes_R - : (\mathcal{GF} \cap \mathcal{C})(R) \to (\mathcal{GF} \cap \mathcal{C})(S)$ is also of interest, since this is a thick subcategory of $(\mathcal{GF} \cap \mathcal{C})(R)$. To ease the notation, we introduce the following abbreviations:

$$\mathcal{K} = \text{Ker}(S \otimes_R - : (\mathcal{GF} \cap \mathcal{C})(R) \to (\mathcal{GF} \cap \mathcal{C})(S))$$

and

$$S = \text{Ker}(S \otimes_R - : (\mathcal{GF} \cap \mathcal{C})(R) \to (\mathcal{GF} \cap \mathcal{C})(S)).$$

The quotient functor $(\mathcal{GF} \cap \mathcal{C})(R) \to (\mathcal{GF} \cap \mathcal{C})(R)$ restricts to a functor $\mathcal{K} \to S$. Let us show that, under certain conditions, there is a closer relationship between the two. First, observe that $\mathcal{F}(R) \cap \mathcal{K}$ is an exact category as it is clearly extension closed.

Proposition 4.5. Let $R \to S$ be as above. If for every $M \in \mathcal{K}$ there is an inflation $M \to X$ and deflation $Y \to M$ in $(\mathcal{GF} \cap \mathcal{C})(R)$ with $X$ and $Y$ both in $\mathcal{K}$, then $\mathcal{K}$ is a Frobenius exact category with projective-injective objects $\mathcal{F} \cap \mathcal{K}$. The corresponding stable category $\mathcal{K}^+$ is a thick subcategory of $(\mathcal{GF} \cap \mathcal{C})(R)$ contained in $S$.

Proof. As stated above, both $\mathcal{K}$ and $\mathcal{F}(R) \cap \mathcal{K}$ are exact subcategories. Since each object in $\mathcal{F}(R) \cap \mathcal{K}$ is in $(\mathcal{F} \cap \mathcal{C})(R)$, it follows that each $F \in (\mathcal{F} \cap \mathcal{C})(R)$ is projective and injective in $\mathcal{K}$. Suppose that $P \in \mathcal{K}$ is projective. By assumption there is an exact sequence $0 \to P' \to F \to P \to 0$ with $F \in (\mathcal{F} \cap \mathcal{C})(R)$ such that $P$ is a summand of $F$ as it is projective, hence $P \in (\mathcal{F} \cap \mathcal{C})(R)$. A similar argument shows every injective object lies in $\mathcal{F}(R) \cap \mathcal{K}$. That there are enough projectives and injectives follows immediately from the assumption. Now consider the stable category $\mathcal{K}^+$; clearly every object in it is also in $\mathcal{S}$ and there is an inclusion $\text{Hom}_\mathcal{K}(X,Y) \subseteq \text{Hom}_\mathcal{S}(X,Y)$ for every object $X,Y \in \mathcal{K}$. On the other hand, suppose $X \to Y$ factors through a flat-cotorsion module $F$. By assumption there is an inflation $0 \to X \to G$ with $G$ in $\mathcal{F} \cap \mathcal{K}$ and this is also an inflation in $(\mathcal{GF} \cap \mathcal{C})(R)$. In particular, as $F$ is injective in $(\mathcal{GF} \cap \mathcal{C})(R)$, the map $X \to F$ factors through $G$, and thus $X \to Y$ factors through a module in $\mathcal{F} \cap \mathcal{K}$, hence $\mathcal{K}^+$ is a full subcategory of $(\mathcal{GF} \cap \mathcal{C})(R)$. Clearly $\mathcal{K}$ is closed under (co)syzygies and direct summands. Suppose that $X \xrightarrow{i} Y \to Z \to \Sigma X$ is a standard triangle in $(\mathcal{GF} \cap \mathcal{C})(R)$ with $X, Y \in \mathcal{K}$ with $Z \in (\mathcal{GF} \cap \mathcal{C})(R)$. Then, by the definition of the triangulated structure, $Z$ is a pushout of the maps $i : X \to I(X)$ in $(\mathcal{GF} \cap \mathcal{C})(R)$, where $I(X)$ is an injective. In particular there is then a deflation $Y \oplus I(X) \to Z$ in
(\mathcal{GF} \cap \mathcal{C})(R). However, we may choose \( I(X) \) to be in \( \mathcal{K} \), and thus by right exactness \( S \otimes_R Z = 0 \), hence \( Z \) is also in \( \mathcal{K} \). By rotation it follows that \( \mathcal{K} \) is thick in \((\mathcal{GF} \cap \mathcal{C})(R)\). \( \square \)

The assumption of enough projective/injectives in the above proposition is unpleasant. However, there are common examples of ring homomorphisms where this does hold, namely quotients of commutative noetherian rings; to this end, we will now assume that \( R \) is a commutative noetherian ring. If \( \mathfrak{a} \) is an ideal of \( R \), the \( \mathfrak{a} \)-torsion and \( \mathfrak{a} \)-adic completion functors are defined to be

\[
\Gamma_\mathfrak{a}(-) := \lim_{\leftarrow i} \text{Hom}_R(R/\mathfrak{a}^i, -)
\]

and

\[
\Lambda_\mathfrak{a}(-) := \lim_{\leftarrow i} R/\mathfrak{a}^i \otimes_R -
\]

respectively. We let \( R\Gamma_\mathfrak{a}(-) \) and \( \Lambda_\mathfrak{a}(-) \) denote their derived functors, which are typically referred to as derived torsion and completion. The \( t \)-th local cohomology of an \( R \)-module \( M \) with support in \( \mathfrak{a} \) is \( H^t_\mathfrak{a}(M) := H^t_\mathfrak{a}R\Gamma_\mathfrak{a}(M) \). Recall that a sequence \( x = x_1, \ldots, x_n \subset R \) is \textit{weakly pro-regular} if \( R\Gamma_x(X) \simeq \check{C}_x \otimes^L_R X \) in \( D(R) \), where \( \check{C}_x \) is the \( \check{C} \)-ech complex on \( x \) given by

\[
\check{C}_x = \otimes_{i=1}^n (\cdots \rightarrow 0 \rightarrow R \rightarrow R_{x_i} \rightarrow 0 \rightarrow \cdots)
\]

where \( R \rightarrow R_{x_i} \) is the canonical localisation map (see [Sch03] for more information). A crucial point is that over commutative noetherian rings every finite sequence is weakly proregular, and thus every ideal has a weakly progenerating sequence that generates it. For our purposes, the key relationship between derived torsion and completion is Greenlees-May duality, which states for any complexes \( X, Y \in D(R) \) and ideal \( \mathfrak{a} \) generated by a weakly proregular sequence there is an isomorphism \( R\text{Hom}_R(X, \Lambda_\mathfrak{a}Y) \simeq R\text{Hom}_R(R\Gamma_\mathfrak{a}X, Y) \). For a proof of this adjunction and a historical account of its development see [PSY14].

Let us now recall the structure of flat-cotorsion modules over commutative noetherian rings. For any prime \( \mathfrak{p} \in \text{Spec } R \), define

\[
T_\mathfrak{p} := \text{Hom}_R(E(R/\mathfrak{p}), E(R/\mathfrak{p})[X_\mathfrak{p}])
\]

which is a flat and cotorsion \( R \)-module, that, by Matlis duality [EJ11 3.4.1], can be viewed as the completion of a free \( R_{\mathfrak{p}} \)-module of cardinality \( \text{card}(X_\mathfrak{p}) \). In fact, such \( T_\mathfrak{p} \) build all flat and cotorsion \( R \)-modules, as [EJ11 5.3.28] shows: indeed if \( F \in (\mathcal{F} \cap \mathcal{C})(R) \), then

\[
F \simeq \bigoplus_{\mathfrak{p} \in \text{Spec } R} T_\mathfrak{p}.
\]

If \( M \) is a cotorsion module, then so is \( F(M) \), the flat cover of \( M \); the cardinality of the free \( R_{\mathfrak{p}} \)-module whose completion appears in \( F(M) \) is denoted by \( \pi_0(\mathfrak{p}, M) \), which was shown in [Xu96 5.2.2] to be determined via the formula

\[
t\pi_0(\mathfrak{p}, M) = \dim_{k(\mathfrak{p})} k(\mathfrak{p}) \otimes_{R_{\mathfrak{p}}} \text{Hom}_R(R_{\mathfrak{p}}, M).
\]

We now show that, in the case when \( R \rightarrow S \) is a surjective homomorphism of commutative noetherian rings, the conditions in the above proposition are satisfied.

**Theorem 4.6.** Let \( R \) be a commutative noetherian ring and \( I \subset R \) an ideal generated by a regular sequence \( x = x_1, \ldots, x_n \). Then \( \mathcal{K} \) is closed under flat precovers and Gorenstein cotorsion preenvelopes. In particular it is a Frobenius category whose stable category is \( \mathcal{S} \).

**Proof.** Notice that the claim about the Frobenius category follows immediately from the closure under flat precovers and Gorenstein cotorsion preenvelopes. Let us first show that \( \mathcal{K} \) is closed under \( \mathcal{GC} \)-preenvelopes. Pick \( M \in \mathcal{K} \) and consider the short exact sequence

\[
0 \rightarrow M \rightarrow X \rightarrow N \rightarrow 0
\]
where $M \to X$ is a $\mathcal{G}F$-preenvelope of $M$ and $N \in \mathcal{G}F(R)$ - such a sequence exists as $(\mathcal{G}F(R), \mathcal{G}F(R)^{\perp})$ is a complete cotorsion pair. Since $M \in \mathcal{K}$ and $\mathcal{G}F(R)^{\perp} \subset \mathcal{C}(R)$, it follows that $N \in (\mathcal{G}F \cap \mathcal{C})(R)$, so if $S \otimes_R X = 0$ the above sequence will be a conflation in $\mathcal{K}$. As $\mathcal{G}F(R)$ is extension closed, we see that $X \in \mathcal{G}F(R) \cap \mathcal{G}F(R)^{\perp} = (\mathcal{F} \cap \mathcal{C})(R)$ so we can express it as

$$X \simeq \prod_{p \in \text{Spec } R} T_p$$

where $T_p = \widehat{R(R_p)}^p$ for some set $X_p$. Then we have $\text{Hom}_R(M, X) \simeq \prod_p \text{Hom}_R(M, T_p)$. Yet since $T_p$ is flat and cotorsion, and therefore also Gorenstein flat and Gorenstein cotorsion, there is an isomorphism $\text{Hom}_R(M, T_p) \simeq \mathcal{R}\text{Hom}_R(M, T_p)$. Yet

$$\mathcal{R}\text{Hom}_R(M, T_p) \simeq \mathcal{R}\text{Hom}_R(M, \mathcal{L}A^p(R_p^{X_p}))) \simeq \mathcal{R}\text{Hom}_R(\mathcal{R}Γ_p(M), R_p^{X_p})).$$

Yet if $x \subset p$, then, as $\text{Tor}_k^R(S, M) = 0 = \text{Ext}_k^R(S, M)$ necessitates $\text{Ext}_k^R(R/p, M) = 0$, it follows that $H^i_k(M) = 0$ for all $i \geq 0$. Yet since $H^i_k(M) = 0$ for all $i < 0$, we see that $\mathcal{R}Γ_p(M)$ is acyclic and thus so is $\mathcal{R}\text{Hom}_R(\mathcal{R}Γ_p(M), R_p^{X_p})).$ In particular $\text{Hom}_R(M, T_p) = 0$ for all $p \supset x$.

Consequently, we may assume that $X_p = 0$ for all $p$ containing $x$, so $X \simeq \prod_{x \supset p} T_p$. But in this case we have

$$S \otimes_R T_p \simeq \text{Hom}_R(\text{Hom}_R(S, E(R/p)), E(R/p)^{(X_p)}) = 0$$

as $S$ is finitely generated and $\text{Hom}_R(S, E(R/p)) = 0$ by [Str90 3.2.9].

We now show $\mathcal{K}$ is closed under flat covers. So let $M$ be as above and let $F \to M$ be the flat cover of $M$. Noting that $M$ is cotorsion, we have $F \simeq \prod T_p$, as above, where here we will consider the isomorphism $T_p \simeq \text{Hom}_R(E(R/p), E(R/p)^{(X_p)})$ with

$$X_p = \dim_{k(p)} k(p) \otimes_{R_p} \text{Hom}_R(R_p, M)$$

the zeroth dual Bass numbers of $M$, see [Xu96 5.2.2]. As $S \otimes_R T_p \neq 0$ if and only if $x \subset p$, it is enough to show that $X_p = 0$ for all $p$ containing $x$. Again we turn to the derived category: as $M$ is cotorsion and $R_p$ is flat, it is enough to show that $k(p) \otimes_{R_p} \mathcal{R}\text{Hom}_R(R_p, M)$ is acyclic. Yet by [SWW17 4.4] this is equivalent to showing that $\mathcal{R}\text{Hom}_R(k(p), M)$ is acyclic. Yet as $k(p) \simeq R_p \otimes_{R_p} R/p$, we have isomorphisms $\mathcal{R}\text{Hom}_R(k(p), M) \simeq \mathcal{R}\text{Hom}_R(R_p, R/p \otimes_{R_p} M)$ which is itself isomorphic to $\text{Hom}_R(R_p, R/p \otimes_{R_p} M)$, and this is zero for the same reasoning as above. Consequently we have shown the desired acyclicity of $k(p) \otimes_{R_p} \mathcal{R}\text{Hom}_R(R_p, M)$ and thus $X_p = 0$ for all $p \supset x$. □

Returning to the more general setting where $R \to S$ is a morphism between coherent rings such that $S \otimes_R : \mathcal{G}F(R) \to \mathcal{G}F(S)$ is an interpretation functor between definable categories, it is possible to compare the indecomposable pure injective objects in $\mathcal{K}$ and $\mathcal{S}$ when $S \otimes_R -$ is full on pure injectives.

Let $\mathcal{X} = \{ X \in \mathcal{G}F(R) : S \otimes_R X \in \mathcal{F}(R) \}$. Suppose $S \in \mathcal{S}$ is pure-injective. Then $S$ seen as an object in $\mathcal{G}F(R)$ is also pure-injective, and is in $\mathcal{X}$. On the other hand it is clear that the image in $(\mathcal{G}F \cap \mathcal{C})(R)$ of any pure-injective object in $\mathcal{X}$ is in $\mathcal{S}$, and clearly $\mathcal{K} \subset \mathcal{X}$. Observe that $\mathcal{X}$ is also coherent, since $R$ is coherent and we assumed that $\mathcal{G}F(R)$ is product closed. From the assumption that $S \otimes_R -$ is full on pure-injectives, it induces a homeomorphism $\text{Zg}(\mathcal{X}) \setminus \text{Zg}(\mathcal{K}) \to \text{Zg}(\mathcal{F}(S))$. However there is also a homeomorphism $\text{Zg}(\mathcal{F}(R)) \setminus \text{Zg}(\mathcal{F}(R) \cap \mathcal{K}) \to \text{Zg}(\mathcal{F}(S))$, which is a restriction of the former homeomorphism. In particular, we see that every indecomposable pure-injective module in $\text{Pinj} \cap \mathcal{X}$ that is not in $\mathcal{K}$ must be flat. That is $\text{Pinj} \cap \mathcal{X} \cap \mathcal{K} \subset \text{Pinj} \cap \mathcal{F}(R)$. In particular, if $M \in \mathcal{S}$ is pure-injective such that $M$ is indecomposable in $\text{Mod}(R)$, then $M \in \mathcal{K}$. Consequently we have shown the following:

**Proposition 4.7.** Let $R \to S$ be such that $S \in \text{mod}(R^o)$ is of finite flat dimension and that $\mathcal{G}F(R)$ and $\mathcal{G}F(S)$ are definable. If $S \otimes_R - : \mathcal{G}F(R) \to \mathcal{G}F(S)$ is full on pure injectives, then the indecomposable pure-injective modules that appear in $\mathcal{S}$ are the same as those in $\mathcal{K}$. 
In the case that $R$ is coherent such that $\mathcal{G} \mathcal{F}(R)$ is definable, the inclusion $\mathcal{F}(R) \subset \mathcal{G} \mathcal{F}(R)$ of definable categories induces a further definable category, called the *definable quotient category*. Introduced in [Kra98], we describe the construction of this in more generality for definable classes of modules. Let $\mathcal{D}$ be such a definable class and $X$ be the set of functors in $(\text{mod}(R), \text{Ab})^{\text{fp}}$ defining $\mathcal{D}$. There is a unique corresponding Serre subcategory of $(\text{mod}(R), \text{Ab})^{\text{fp}}$ corresponding to $\mathcal{D}$, denoted $S_{\mathcal{D}}$, which is precisely the Serre subcategory generated by $X$. Given another definable class $\mathcal{C}$ such that $\mathcal{C} \subset \mathcal{D}$, there is a reverse inclusion $S_{\mathcal{D}} \subset S_{\mathcal{C}}$ of Serre subcategories of $(\text{mod}(R), \text{Ab})^{\text{fp}}$. In particular, the corresponding localisation $S'$ is a skeletally small abelian category, and thus $\text{Ex}(S', \text{Ab})$, the category of exact functors $S' \rightarrow \text{Ab}$, is a definable class of modules (over some ring, but not typically $R$), see [Kra98 2.9]. This definable class, which we will denote $\mathcal{Q}_{C \subset D}$ is the definable quotient category. The following collates the results about definable quotients that will be of use; the notation $\mathcal{X}/\mathcal{Y}$ refers to the stable category of $\mathcal{X}$ with respect to morphisms factoring through $\mathcal{Y}$, and thus aligns with the notation used throughout.

**Proposition 4.8.** Let $\mathcal{C} \subset \mathcal{D}$ be definable classes of modules, and $\mathcal{Q}_{C \subset D}$ the corresponding definable quotient category.

1. There is an equivalence $\mathcal{P} \mathcal{I} \cap \mathcal{D}/\mathcal{P} \mathcal{I} \cap \mathcal{C} \rightarrow \mathcal{P} \mathcal{I} \cap \mathcal{Q}_{C \subset D}$.
2. In the case that $\mathcal{D}$ is locally finitely presented and $\mathcal{A} \subset D^{\text{fp}}$ is pre-enveloping in $D^{\text{fp}}$, then $\mathcal{C} := \lim_{\rightarrow} \mathcal{A}$ is definable and $\mathcal{Q}_{C \subset D}$ is locally finitely presented. There is an equivalence $D^{\text{fp}}/\mathcal{A} \rightarrow Q^{\text{fp}}_{C \subset D}$.
3. There is a homeomorphism $Zg(D) \setminus Zg(C) \rightarrow Zg(\mathcal{Q}_{C \subset D})$.

These are [Kra98 5.1, 5.4, 6.3] respectively.

**Lemma 4.9.** If $\mathcal{F}(R)$ and $\mathcal{G} \mathcal{F}(R)$ are definable, then there is a bijection between $\mathcal{P} \mathcal{I} \cap \mathcal{Q}_{F \subset G} \mathcal{F}$ and $\mathcal{P} \mathcal{I} \cap (\mathcal{G} \mathcal{F} \cap \mathcal{C})(R)$.

**Proof.** By (4.3), the pure injectives in $(\mathcal{G} \mathcal{F} \cap \mathcal{C})(R)$ are precisely the non-flat pure-injective Gorenstein flat modules, which, by (4.8) are exactly the pure-injectives in $\mathcal{Q}_{C \subset D}$. \hfill \Box

We will now put definable quotient categories to use to discuss indecomposable pure-injective Gorenstein flat modules over hypersurfaces. For the rest of the section we will let $k$ denote an algebraically closed field of characteristic other than two. Set $S = k[[x_1, \cdots, x_n]]$ to be the power series ring in $n$ variables over $k$, which is a complete regular local ring, whose maximal ideal will be denoted by $\mathfrak{m}$. If $f \in \mathfrak{m}^2$ is non-zero, we will let $R = S/(f)$ denote the corresponding hypersurface ring which is Gorenstein. Recall that a finitely generated $R$-module is *maximal Cohen-Macaulay* if Ext$_R^i(M, R) = 0$ for all $i > 0$, and the full subcategory of mod($R$) comprising of maximal Cohen-Macaulay modules will be denoted $\text{CM}(R)$. The *double branched cover* of $R$ is the ring

$$R^2 := S[[z]]/(f + z^2).$$

The element $z$ is a nonzerodivisor on $R^2$, and $R$ can be obtained from $z$ via the isomorphism $R^2/zR^2 \simeq R$. In particular, $R$ has projective dimension one as an $R^2$-module, with a minimal resolution given by $0 \rightarrow R^2 \rightarrow R^2 \rightarrow R \rightarrow 0$. As $R$ is Gorenstein, $\text{CM}(R)$ is a Frobenius category with the projective-injective objects just being the projective modules, and the stable category is equivalent to $\textsf{K}_{ac}(\text{proj})(R)$ by [Buc86]. Since for every $M \in \text{CM}(R^2)$ the quotient $M/zM$ is in $\text{CM}(R)$, it is clear $R^2/(z)\otimes_{R^2} - : \text{CM}(R^2) \rightarrow \text{CM}(R)$ is a functor of Frobenius exact categories and also a triangulated functor between the stable categories. Both functors are the restriction of the functor $R \otimes_{R^2} - : (\mathcal{G} \mathcal{F} \cap \mathcal{C})(R^2) \rightarrow (\mathcal{G} \mathcal{F} \cap \mathcal{C})(R)$ (and its triangulated version): since $R$ is complete every finitely generated $R$-module is pure-injective (by Matlis duality) and therefore cotorsion. Moreover, the maximal Cohen-Macaulay modules are precisely the finitely generated Gorenstein flat modules. Thus there is a fully faithful embedding $\text{CM}(R) \rightarrow (\mathcal{G} \mathcal{F} \cap \mathcal{C})(R)$ and likewise $\text{CM}(R) \rightarrow (\mathcal{G} \mathcal{F} \cap \mathcal{C})(R)$ (and likewise for $R^2$), and $R \otimes_{R^2} -$ preserves these embeddings.
Let us recall the following classic theorem of Knörrer relating CM(R) and CM(R^{\mathbb{Z}}):

**Theorem 4.10.** [Knö87] There is a triangulated equivalence of categories \( \text{CM}(R) \simeq \text{CM}(R^{\mathbb{Z}}) \). In particular there is a bijection between non-free indecomposable maximal Cohen-Macaulay modules over \( R \) and \( R^{\mathbb{Z}} \).

It is not possible to completely extend this result to Gorenstein flat modules, but we do obtain the following.

**Theorem 4.11.** There is a bijection between the pure-injective non-flat Gorenstein flat modules over \( R \) and \( R^{\mathbb{Z}} \) given by \( R \otimes_{R^{\mathbb{Z}}} - \). This restricts to a homeomorphism

\[
\text{Zg}(GF(R)) \setminus \text{Zg}(F(R)) \simeq \text{Zg}(GF(R^{\mathbb{Z}})) \setminus \text{Zg}(F(R^{\mathbb{Z}})).
\]

In particular, the triangulated functor \( R \otimes_{R^{\mathbb{Z}}} - : (GF \cap C)(R^{\mathbb{Z}}) \to (GF \cap C)(R) \) induces a map between pure injective objects in \((GF \cap C)(R)\) and \((GF \cap C)(R^{\mathbb{Z}})\).

**Proof.** Everything that is said for \( R \) also holds for \( R^{\mathbb{Z}} \). Since \( R \) is Gorenstein, \( GF(R) \) is locally finitely presented with \( GF(R)^{\text{fp}} = \text{CM}(R) \). As \( \text{proj}(R) \) is preenveloping in \( \text{CM}(R) \) (which follows immediately from \( F(R) \) being definable), there are bijections \( PF \cap GF \cap PF \to \text{proj}(G,F \cap C) \) and a homeomorphism \( \text{Zg}(GF(R)) \setminus \text{Zg}(F(R)) \simeq \text{Zg}(GF \cap C) \) by (1.3). Now, by (1.10), there is an equivalence \( \text{CM}(R) \simeq \text{CM}(R^{\mathbb{Z}}) \), and therefore there is an equivalence \( \text{lim} \text{CM}(R) \simeq \text{lim} \text{CM}(R^{\mathbb{Z}}) \). However, \( \text{lim} \text{CM}(R) \) is just \( \text{Q}(GF \cap C)(R) \) by (1.3).2 In particular we have the bijections and homeomorphisms as stated in the theorem. As described above, \( R \) has finite flat dimension over \( R^{\mathbb{Z}} \) hence \( R \otimes_{R^{\mathbb{Z}}} - \) is a functor \((GF \cap C)(R^{\mathbb{Z}}) \to (GF \cap C)(R)\), and this preserves pure injectives.

Let us turn our attention to a question posed by Puninski at [Pun18, 10.3]. If \( A \) is an abelian category, the Krull-Gabriel dimension of \( A \), denoted \( \text{KGdim}(A) \), is the smallest integer \( n \) such that there is a filtration \( A_{-1} \subseteq A_0 \subseteq \cdots \subseteq A_n \) of Serre subcategories of \( A \), with \( A_{-1} = 0 \) and \( A_n = A \), such that the localisation \( A_i/A_{i-1} \) is precisely the full subcategory of finite length objects in \( A/A_{i-1} \). If no such filtration exists, then we say that \( \text{KGdim}(A) = \infty \). Given a definable category \( D \subseteq \text{Mod}(R) \), the corresponding Serre subcategory \( SD \subseteq (\text{mod}(R), \text{Ab})^{\text{fp}} \) bjects with a hereditary torsion theory of finite type on \( (\text{mod}(R), \text{Ab}) \), the localisation corresponding to which is a locally coherent Grothendieck category, denoted \( \text{fun} - D \), and we write \( \text{KGdim}(D) \) for \( \text{KGdim}(\text{fun} - D) \) (see [Pre11] and [Pre09] for substantial discussions). Puninski showed that over the \( \text{A}_\infty \) curve singularity \( k[[x,y]]/x^2 \) that the Krull-Gabriel dimension of \( GF(A_{\infty}) \) was finite, and more specifically 2. He then questioned whether Krull-Gabriel dimension of the Gorenstein flat modules over the \( d \)-dimensional \( A_{\infty} \)-singularity was finite. We partially answer this question when one considers an infinitely generated maximal Cohen-Macaulay over a Gorenstein flat module.

**Proposition 4.12.** Let \( R \) be a hypersurface ring. Then the Krull-Gabriel dimension of \( GF(R) \) is finite if and only if the Krull-Gabriel dimension of \( GF(R^{\mathbb{Z}}) \) is.

**Proof.** For brevity, let \( Q_R := \text{lim} \text{CM}(R) \) denote the definable quotient of \( F(R) \subseteq GF(R) \). Since \( \text{Zg}(F(R)) = \text{Zg}(F(R^{\mathbb{Z}})) \) it is clear that \( \text{KGdim}(F(R)) = \text{KGdim}(F(R^{\mathbb{Z}})) \) and by the above theorem we see that \( \text{KGdim}(Q_R) = \text{KGdim}(Q_{R^{\mathbb{Z}}}) \). The result then follows immediately from the inequalities

\[
\sup \{ \text{KGdim}(F(R)), \text{KGdim}(Q_R) \} \leq \text{KGdim}(GF(R)) \leq \text{KGdim}(Q_R) \oplus \text{KGdim}(F(R)),
\]

and similarly for \( R^{\mathbb{Z}} \), found at [Kra98, 12.2].

\[\square\]
In particular, \(A_{2n+1}^\infty\) has finite Krull-Gabriel dimension for every \(n \geq 0\). To answer the case for even dimension, one would require a description of the indecomposable pure-injective Gorenstein flat modules over \(k[[x, y, z]]/(x^2 + z^2)\), which is a difficult problem in its own right.

5. The case when \(S\) is flat over \(R\)

For this section, we will assume that \(R\) and \(S\) are right coherent rings such that \(S\) is finitely presented and flat over \(R\), and thus projective as an \(R\)-module. The main advantage of this situation is that \(S \otimes_R -\) is now additionally an endofunctor on \((\mathcal{GF} \cap \mathcal{C})(R)\). This is because \(S\) is a summand of a finitely presented free \(R\)-module, and \((\mathcal{GF} \cap \mathcal{C})(R)\) and \((\mathcal{F} \cap \mathcal{C})(R)\) are both closed under finite direct sums and summands.

We will let \(I\) denote the image of the functor \(S \otimes_R - : (\mathcal{GF} \cap \mathcal{C})(R) \to (\mathcal{GF} \cap \mathcal{C})(R)\). Clearly the inclusion \(I \to (\mathcal{GF} \cap \mathcal{C})(R)\) coincides with the restriction of scalars functor \(\text{Mod}(S) \to \text{Mod}(R)\), and is therefore a right adjoint to \(S \otimes_R -\). Yet since we assumed \(S\) was finitely generated and projective over \(R\), the extension of scalars functor also has a left adjoint given by \(S^* \otimes_S -\), where \(S^* := \text{Hom}_R(S, R)\). Indeed, there are isomorphisms

\[
\text{Hom}_R(\text{Hom}_R(S, R) \otimes_R X, Y) \simeq \text{Hom}_S(X, \text{Hom}_R(\text{Hom}_R(S, R), Y))
\]

for every \(S\)-module \(X\) and \(R\)-module \(Y\), and since \(\text{Hom}_R(\text{Hom}_R(S, R), Y) \simeq S \otimes_R Y\), the adjunction is clear. Now, if \(M \in \mathcal{GF}(R)\), then \(\text{Hom}_R(S, R) \otimes_S (S \otimes_R M) \simeq \text{Hom}_R(S, M)\), and since \(S\) is finitely generated projective is module is also Gorenstein flat, hence there is the following diagram of adjoint functors between \(\mathcal{GF}(R)\) and \(\mathcal{J} \subset \mathcal{GF}(S)\):

\[
\begin{array}{ccc}
S^* \otimes_S - & \simeq & \mathcal{GF}(R) \\
\text{res} & \mathcal{J} & S \otimes_R - \to I
\end{array}
\]

Let us now consider the above functors when considered on \((\mathcal{GF} \cap \mathcal{C})(R)\) and \((\mathcal{GF} \cap \mathcal{C})(S)\). If \(\mathcal{J}\) denotes the image of \(S \otimes_R - : (\mathcal{GF} \cap \mathcal{C})(R) \to (\mathcal{GF} \cap \mathcal{C})(S)\), then the restriction of scalars functor gives a triangulated right adjoint \(\mathcal{J} \to (\mathcal{GF} \cap \mathcal{C})(R)\). This is clear on objects, while if \(f : S \otimes_R X \to S \otimes_R Y\) factors through a flat cotorsion \(S\)-module, it necessarily factors through a module of the form \(S \otimes_R F\) with \(F \in (\mathcal{F} \cap \mathcal{C})(R)\). Since restriction preserves flat-cotorsion modules, it is clear that \(\text{res}(f)\) factors through \(\text{res}(S \otimes_R F)\). A similar argument holds for \(S^* \otimes_R -\): by the above reasoning this preserves Gorenstein flat and cotorsion modules in \(I\), and likewise flat-cotorsion modules, since if \(S \otimes_R F \in I\) is flat-cotorsion, then \(S^* \otimes_S (S \otimes_R F) \simeq \text{Hom}_R(S, F)\), and since this is a summand of \(F^{(n)}\) for some finite \(n\), it is clear this is flat-cotorsion. Both functors are clearly triangulated, so we have a comparable diagram to the above:

\[
\begin{array}{ccc}
\mathcal{GF}(R) & \simeq & S^* \otimes_S - \\
\text{res} & \mathcal{J} & S \otimes_R - \to (\mathcal{GF} \cap \mathcal{C})(R)
\end{array}
\]

Although \(\text{res}\) and \(S^* \otimes_R -\) are adjoints to \(S \otimes_R -\) on the entire module category, there is no reason for them to, in general, preserve Gorenstein flat or cotorsion modules (although \(\text{res}\) always preserves cotorsion modules). To ensure that these give functors \((\mathcal{GF} \cap \mathcal{C})(S) \to (\mathcal{GF} \cap \mathcal{C})(R)\) one needs to impose further restrictions.

**Lemma 5.1.** Suppose \(S\) is a finitely generated flat \(R\) module such that \(S^*_R\) is finitely generated as an \(S\)-module and either \(\text{inj dim}_S S^*_R\) or \(\text{flat dim}_S S^*_R\) is finite. Then \(S^*_R \otimes_R - : \text{Mod}(S) \to \text{Mod}(R)\) preserves totally acyclic complexes of flat-cotorsion modules. In particular it induces a functor of Frobenius categories \((\mathcal{GF} \cap \mathcal{C})(S) \to (\mathcal{GF} \cap \mathcal{C})(R)\) and a triangulated functor on the corresponding stable categories.
Proof. The assumption that $S_R^*$ is finitely generated over $S$ ensures that $S_R^* \otimes_S -$ is an interpretation functor, so it will preserve flat-cotorsion modules provided it preserves projective modules. Yet this is true since it is a left adjoint to a right exact functor as $S$ is flat over $R$. Consequently if $T$ is a totally acyclic complex of flat-cotorsion $S$-modules, $(S_R^* \otimes_S T)_i \in (\mathcal{F} \cap \mathcal{C})(R)$ for every $i \in \mathbb{Z}$. Let us now turn our attention to the acyclicity requirements. The requirement on finite flat or injective dimension of $S_R^*$ ensures that $\text{Tor}^i_S(S_R^*, Z_j(T)) = 0$ for all $i > 0$ and $j \in \mathbb{Z}$. In particular, it follows that $S_R^* \otimes_S T$ is acyclic. Since the cycles of $S_R^* \otimes_S T$ are cotorsion, the third condition of (2.5) is satisfied. Lastly we show that $\text{Hom}_R(S_R^* \otimes_S T, F)$ is acyclic for every $F \in (\mathcal{F} \cap \mathcal{C})(R)$. Suppose $F$ is a flat-cotorsion $R$-module, then there are isomorphisms

$$\text{RHom}_R(S_R^* \otimes_S \frac{1}{S} T, F) \simeq \text{RHom}_S(T, \text{RHom}_R(S^*, F))$$

$$\simeq \text{RHom}_S(T, \text{RHom}_R(\text{RHom}_R(S, R), F))$$

$$\simeq \text{RHom}_S(T, S \otimes_R \text{RHom}_R(R, F))$$

$$\simeq \text{RHom}_S(T, S \otimes_R F)$$

$$\simeq 0,$$

where the third isomorphism follows from the fact that $S$ is projective over $R$ and the fifth follows from the fact that $S \otimes_R F$ is a flat-cotorsion $S$-module, and by assumption $\text{RHom}_S(T, \tilde{F}) = 0$ for all flat-cotorsion $S$-modules. In particular, $\text{Hom}_R(S_R^* \otimes_S T, F)$ is acyclic, which finishes the proof. \qed

The assumption on $S_R^*$ having finite flat or injective dimension holds immediately if $R$ is a Gorenstein ring. Over such rings $S$ being projective ensures it has finite injective dimension, and thus the injective dimension over $S$ of $\text{Hom}_R(S, R)$ is also finite by classic results. For restriction of scalars a similar assumption is needed on injective dimension - under the assumption that every injective $R$-module has finite flat dimension, which certainly holds if $R$ is Gorenstein, [BO10, 2.5] shows that the restriction of a Gorenstein flat module is Gorenstein flat, and as restriction is exact it follows that, with this assumption there is then an exact functor of Frobenius categories $(\mathcal{G} \mathcal{F} \cap \mathcal{C})(S) \to (\mathcal{G} \mathcal{F} \cap \mathcal{C})(R)$ which will induce a triangulated functor between the stable categories. Combining these, we have the following diagrams of functors

$$(\mathcal{G} \mathcal{F} \cap \mathcal{C})(S) \xleftarrow{S_R^* \otimes_S \text{res}} S \otimes_R \text{res} \xrightarrow{\text{res}} (\mathcal{G} \mathcal{F} \cap \mathcal{C})(R)$$

and

$$(\mathcal{G} \mathcal{F} \cap \mathcal{C})(S) \xleftarrow{S_R^* \otimes_S \text{res}} S \otimes_R \text{res} \xrightarrow{\text{res}} (\mathcal{G} \mathcal{F} \cap \mathcal{C})(R)$$

whenever the following assumptions are made:

1. Every injective $R$-module has finite flat dimension;
2. At least one of $\text{inj dim}_S S_R^*$ or $\text{flat dim}_S S_R^*$ is finite;
3. $S_R^*$ is finitely generated over $S$.

The first of the two conditions immediately hold whenever $R$ is a Gorenstein ring.

Let us now further consider the case when $R \to S$ is additionally a ring epimorphism, which, since $S$ is finite over $R$, is equivalent to the map being surjective. Recall that $R \to S$ is a ring epimorphism if and only if $S \otimes_R S \simeq S$. The advantage of this is that it enables a much more concrete description of the relationship between $\mathcal{I}$ and $\mathcal{J}$.

Lemma 5.2. With the usual assumption on $R$, if $R \to S$ is a finite flat ring epimorphism, then $\mathcal{I}$ is a Frobenius category. The projective-injective objects are of the form $S \otimes_R F$ with $F \in (\mathcal{F} \cap \mathcal{C})(R)$, the class of which we will denote by $\mathcal{I}_P$. The stable category $\mathcal{I}$ is a full subcategory of $(\mathcal{G} \mathcal{F} \cap \mathcal{C})(R)$ which is precisely $\mathcal{J}$, the image of $S \otimes_R - : (\mathcal{G} \mathcal{F} \cap \mathcal{C})(R) \to (\mathcal{G} \mathcal{F} \cap \mathcal{C})(S)$. 
Proof. If $0 \to S \otimes_R M \to X \to S \otimes_R M \to 0$ is an exact sequence in $\mathcal{I}$, then, viewed as a sequence of $R$-modules it is also exact sequence in $(\mathcal{G} \mathcal{F} \cap \mathcal{C})(R)$. In particular $X_R$ is in $(\mathcal{G} \mathcal{F} \cap \mathcal{C})(R)$. Yet since $R \to S$ is an epimorphism, we have $S \otimes_R X_R \cong X$, hence $X \in \mathcal{I}$. The same argument shows that $\mathcal{I}_p$ is also extension closed, hence both $\mathcal{I}$ and $\mathcal{I}_p$ are exact categories. It is clear that each object in $\mathcal{I}_p$ is projective and injective in $\mathcal{I}_0$.

Proof. If $0 \to L \to F \to M \to 0$ be a conflation with $F \in (\mathcal{F} \cap \mathcal{C})(R)$. Then $0 \to S \otimes_R L \to S \otimes_R F \to S \otimes_R M \to 0$ is exact. In particular, since $S \otimes_R M = X \neq 0$ it follows that $S \otimes_R F \neq 0$ so this is a deflation in $\mathcal{I}$ with $S \otimes_R F \in \mathcal{I}_p$. A similar argument shows enough injectives. Now, suppose that $P \in \mathcal{I}$ is projective. Then $P = S \otimes_R X$ for some $X \in (\mathcal{G} \mathcal{F} \cap \mathcal{C})(R)$. There is then an exact sequence $0 \to S \otimes_R \Omega^1_R(X) \to S \otimes_R F \to P \to 0$ in $\mathcal{I}$ with $F \in (\mathcal{F} \cap \mathcal{C})(R)$. Since $P$ is projective this splits so $P$ is a summand of $S \otimes_R F$. But $\mathcal{I}_p$ is closed under direct summands since restriction of scalars is exact and $(\mathcal{F} \cap \mathcal{C})(R)$ is closed under direct summands. Therefore $P \in \mathcal{I}_p$. An almost identical argument shows every injective object is also in $\mathcal{I}_p$, which proves that $\mathcal{I}$ is Frobenius. In order to show that $\mathcal{I}$ is the image of $S \otimes_R -$ , it is enough to prove that $\mathcal{I}$ is a full subcategory of $(\mathcal{G} \mathcal{F} \cap \mathcal{C})(S)$, since the objects of $\mathcal{I}$ and $\text{Im}(S \otimes_R -)$ are clearly the same. Yet this follows immediately from (3.5). \hfill \Box

Combining the above lemma and the discussion that led to the diagram (2), we obtain the following theorem.

**Theorem 5.3.** Let $R \to S$ be a finite flat ring epimorphism. Then $S \otimes_R - : (\mathcal{G} \mathcal{F} \cap \mathcal{C})(R) \to (\mathcal{G} \mathcal{F} \cap \mathcal{C})(S)$ is a Bousfield localising functor on $(\mathcal{G} \mathcal{F} \cap \mathcal{C})(R)$. Moreover there is recollement of triangulated categories

$$
\begin{array}{cccccc}
S & \leftarrow & \text{inc} & \rightarrow & (\mathcal{G} \mathcal{F} \cap \mathcal{C})(R) & \leftarrow & S \otimes_R - \\
& & \rho & & \eta & & \text{res}
\end{array}
$$

with $\rho$ a right adjoint and $\lambda$ a left adjoint to the inclusion.

**Proof.** Let us first show that $S \otimes_R -$ is a localising functor. We have already $S \otimes_R -$ is an endofunctor on $(\mathcal{G} \mathcal{F} \cap \mathcal{C})(R)$, so define a natural transformation $\eta : \text{Id} \to S \otimes_R -$ via the map $X \mapsto S \otimes_R X$, $x \mapsto 1_S \otimes_R x$.

Since $R \to S$ is a ring epimorphism, the isomorphism $S \otimes_R S \cong S$ shows that $\eta \circ (S \otimes_R -) = (S \otimes_R -) \circ \eta$ and that $S \otimes_R - \circ \eta : S \otimes_R - \to (S \otimes_R S) \otimes_R -$ is invertible, which proves the claim. For the existence of a recollement, it suffices by [Kra10, 4.13.1] to show that $S \otimes_R - : (\mathcal{G} \mathcal{F} \cap \mathcal{C})(R) \to J = \mathcal{I}$ admits a left adjoint, but we have already seen that this happens whenever $S$ is finite and flat over $R$. \hfill \Box

Let us give some examples of ring homomorphisms that satisfy the conditions of the above theorem.

**Example 5.4.**

1. Let $R$ be a coherent ring and $e$ a non-trivial central idempotent of $R$. Then the canonical map $R \to eRe$ is a finite flat epimorphism (as $e$ is central $eRe = e^2R = eR$ which is projective) and therefore the functor $eRe \otimes_R - : \text{Mod}(R) \to \text{Mod}(eRe)$ induces a functor $(\mathcal{G} \mathcal{F} \cap \mathcal{C})(R) \to (\mathcal{G} \mathcal{F} \cap \mathcal{C})(eRe)$ and $(\mathcal{G} \mathcal{F} \cap \mathcal{C})(R) \to (\mathcal{G} \mathcal{F} \cap \mathcal{C})(eRe)$ which is essentially surjective. The kernel of this map, and thus the left hand category in the recollement is $(\mathcal{G} \mathcal{F} \cap \mathcal{C})(fRf)$ where $f = 1 - e$.

2. Let $f : X \to Y$ be a morphism of affine schemes, then [Gro67, 17.9.1] states that $f$ is an open immersion if and only if the corresponding ring homomorphism is a finitely presented flat ring epimorphism. Consequently every ring homomorphism between commutative rings satisfying the conditions of the above theorem arises (or induces) a corresponding open immersion of schemes.
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