Rigidity and Superfast Signal Propagation in Fluids and Solids in Non-Equilibrium Steady States

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In the 1980s it was theoretically predicted that correlations of various observables in a fluid in a non-equilibrium steady state (NESS) are extraordinarily long-ranged, extending, in a well-defined sense, over the size of the system. This is to be contrasted with correlations in an equilibrium fluid, whose range is typically just a few particle diameters. These NESS correlations were later confirmed by numerous experimental studies. Unlike long-ranged correlations at critical points, these correlations are generic in the sense that they exist for any temperature as long as the system is in a NESS. In equilibrium systems, generic long-ranged correlations are caused by spontaneously broken continuous symmetries and are associated with a generalized rigidity, which in turn leads to a new propagating excitation or mode. For example, in a solid, spatial rigidity leads to transverse sound waves, while in a superfluid, phase rigidity leads to temperature waves known as second sound at finite temperatures, and phonons at zero temperature. More generally, long-ranged spatial correlations imply rigidity irrespective of their physical origin. This implies that a fluid in a NESS should also display a type of rigidity and related anomalous transport behavior. Here we show that this is indeed the case. For the particular case of a simple fluid in a constant temperature gradient, the anomalous transport behavior takes the form of a super-diffusive spread of a constant-pressure temperature perturbation. We also discuss the case of an elastic solid, where we predict a spread that is faster than ballistic.

I. INTRODUCTION

Correlations in a fluid in equilibrium are very short-ranged on a macroscopic scale. For instance, for distances large compared to a molecular diameter the temperature-temperature correlation function (TTCF) is given by

\[ \langle \delta T(r) \delta T(r') \rangle = \frac{k_B T_{eq}^2}{c_V} \delta(r - r') , \tag{1.1a} \]

or, in wave-number space,

\[ \langle |\delta T(k)|^2 \rangle = \frac{k_B T_{eq}^2}{c_V} . \tag{1.1b} \]

Here \( \delta T(r) = T(r) - T_{eq} \) is the temperature fluctuation, with \( T(r) \) the local fluctuating temperature and \( T_{eq} \) the equilibrium temperature, \( k_B \) is Boltzmann’s constant, \( c_V \) is the specific heat per volume at constant volume, and the angular brackets denote an average over an equilibrium ensemble.

In a non-equilibrium steady state (NESS), by contrast, spatial correlations behave dramatically differently. To be specific, consider a simple fluid subject to a constant temperature gradient in the \( \hat{z} \)-direction, as illustrated in Fig. 1. We consider a system that has a spatial extent \( L \) in the \( \hat{z} \)-direction and is infinite in the directions perpendicular to the direction of the gradient. For this case, the TTCF in wave-number space is

\[ \langle |\delta T(k)|^2 \rangle = \frac{k_B T_0^2}{c_V} + (\partial_z T)^2 \frac{k_B T_0}{\rho D_T (\nu + D_T)} \frac{k^2}{k^4} . \tag{1.2} \]

Here the angular brackets denote a non-equilibrium (NE) ensemble, and only the leading small-\( k \) (large distance) terms have been retained for the NE contribution. In Eq. (1.2), \( \rho \) is the mass density, \( D_T \) is the thermal diffusivity, and \( \nu \) is the kinematic viscosity. \( T_0 \) is the spatially averaged temperature of the NE fluid, and \( \partial_z T = \text{const} \) is the constant temperature gradient. \( \hat{k} = k/k_c \) with \( k_c = |k| \) is the unit wave vector, and \( \hat{k}_\perp \) is its component perpendicular to the direction of the temperature gradient, i.e., \( \hat{k}_\perp = (k_x, k_y)/k \). All thermophysical quantities in Eq. (1.2) should be interpreted as spatially averaged. Slip boundary conditions have been used, which leads to \( k_c = N\pi/L \) with \( N \) a positive integer. The \( k^{-4} \) small-\( k \) singularity in the NE term in Eq. (1.2) indicates that very long-ranged correlations result from the tem-

\[ \begin{align*}
T & \\
T_{z=0} & \\
T_0 & \\
T_{z=L} & \\
0 & \quad \frac{L}{2} & \quad L
\end{align*} \]

FIG. 1: A fluid with a linear temperature profile between two parallel confining plates.
perature gradient. This result has been derived by kinetic theory,\textsuperscript{7,5} mode-coupling theory,\textsuperscript{2,5} and fluctuating hydrodynamics.\textsuperscript{3,6} The general equivalence of kinetic theory and fluctuating hydrodynamics for computing long-ranged correlations in a NESS was shown in Ref. 5.

The real-space TTTCF can be written

\[ \langle \delta T(r) \delta T(r') \rangle = \frac{k_B T_0^2}{c V} \delta(r - r') + \frac{k_B T_0}{\rho D_T (\nu + D_T)} G_{NE}(r \perp, z, z') , \quad (1.3) \]

where \( r \perp = \sqrt{(x - x')^2 + (y - y')^2} \). In general, \( G_{NE} \) is a complicated function of its arguments. Two simple limits are (see Sec. 7.5 in Ref. 3)

\[ G_{NE}(r \perp = 0, z = L/2, z' \ll L/2) = \frac{L}{16} \left[ 1 - 2 \frac{|z - z'|}{L} \right] \quad (1.4a) \]

and

\[ G_{NE}(r \perp \ll L, z = 0, z' = 0) = \frac{L}{16} \left[ 1 - 3 \frac{r \perp}{L} \right] . \quad (1.4b) \]

Note that in both limits the real-space correlations scale with the system size \( L \), and the correlations decay on the same scale. For instance, if the system size \( L \) is scaled by a factor of \( b > 1 \), and the distance \( |z - z'| \) in Eq. (1.4a) is scaled by the same factor, then the correlation \( G_{NE} \) increases by a factor of \( b \). However, if \( |z - z'| \) increases at fixed \( L \), then \( G_{NE} \) decreases. This is the real-space manifestation of the \( 1/k^4 \) singularity in Eq. (1.2).

Physically, one expects these long-ranged correlations to have dynamical consequences in a macroscopic description of a fluid, independent of thermal fluctuation effects. That is, if the fluid is so strongly correlated that spatial correlations extend throughout the entire system, which represents a generalized rigidity, then a perturbation at one point in the fluid should propagate infinitely faster, in a scaling sense, than a diffusive process. Indeed, we will show that a temperature perturbation at \( t = 0 \) at a distance \( R \) from the observer is detectable at a time \( t = R/v_0 \), with \( v_0 \) a characteristic velocity, rather than at the much longer diffusive time scale \( t \sim R^2/D_T \) that characterizes an equilibrium fluid. This is analogous to what happens in equilibrium systems if a spontaneously broken continuous symmetry leads to generic long-ranged correlations\textsuperscript{7,8} that endow the system with a generalized rigidity property.\textsuperscript{8,9} In that case the broken symmetry leads to Goldstone modes, which typically are propagating and emerge in addition to any soft modes that may be present in the absence of symmetry breaking. For example, in a solid the long-ranged spatial correlations (namely, displacement fluctuations that scale as \( 1/k^2 \)), and the associated rigidity (represented by a nonzero shear modulus) lead to transverse sound waves.\textsuperscript{10,11} That is, in a fluid in equilibrium the transverse modes are diffusive, while in a solid they are propagating. Another example is second sound in superfluids, where long-ranged phase correlations couple to energy-density fluctuations to form a propagating mode at nonzero temperature, second sound, that is a constant-pressure temperature wave.\textsuperscript{10} In normal fluids, by contrast, temperature perturbations at constant pressure are diffusive.\textsuperscript{7} At zero temperature these second-sound excitations are phonons with the same linear dispersion relation and the same speed of (second) sound as the Goldstone mode, namely, the single-particle Bogoliubov excitations.\textsuperscript{12} Yet another example is the magnon in the magnetically ordered phase of a Heisenberg ferromagnet, which is a propagating spin wave, whereas in the paramagnetic phase the corresponding transverse spin modes are diffusive.\textsuperscript{7,10} In all of these examples, the long-ranged static correlations, and the associated rigidity, lead to a signal propagation that scales linearly with time, as opposed to a diffusive process, where it scales as the square root of time.

II. LANGEVIN EQUATIONS, AND THE TEMPERATURE-TEMPERATURE TIME CORRELATION FUNCTION IN A NESS

In this section we give the Langevin equations that describe temperature and velocity fluctuations about a NESS. These equations are then used to obtain the dynamical fluctuation about a NESS, and Eq. (1.2). Both of these quantities can be directly measured by light scattering experiments. We also present an alternative procedure that derives an effective Langevin equation for temperature fluctuations only.

A. Langevin equations

Ignoring fast sound-mode or pressure-fluctuation effects, the Langevin equations describing fluctuations in a simple fluid in a thermal gradient, see Fig.1, are\textsuperscript{3,13}

\[ \partial_t \delta T(r, t) + v_z(r, t) \partial_z T = D_T \nabla^2 \delta T(r, t) + Q(r, t) \quad (2.1a) \]
and,
\[ \partial_t v_z(r, t) = \nu \nabla^2 v_z(r, t) + P_z(r, t) \]  
(2.1b)
Here \( v_z \) is the \( z \)-component of the fluctuating transverse velocity.\(^{14} \) \( D_T \) is the thermal diffusivity, and \( \nu \) is the kinematic viscosity. \( Q \) and \( P_z \) are Langevin forces that are Gaussian distributed and delta-correlated in space and time,
\[ \langle Q(r, t)Q(r', t') \rangle = \frac{2k_B T_0^2}{c_p} D_T k^2 \delta(r - r') \delta(t - t') \]
\[ \equiv G_{QQ}(r, t; r', t'), \]  
(2.2a)
\[ \langle P_z(r, t)P_z(r', t') \rangle = \frac{2k_B T_0}{\rho} \nu k_\perp^2 \delta(r - r') \delta(t - t') \]
\[ \equiv G_{PP}(r, t; r', t'), \]  
(2.2b)
or, in wave-number space,
\[ \langle Q(k, t)Q(k', t') \rangle = \frac{2k_B T_0^2}{c_p} D_T k^2 \delta_{k, -k'} \delta(t - t'), \]  
(2.3a)
\[ \langle P_z(k, t)P_z(k', t') \rangle = \frac{2k_B T_0}{\rho} \nu k_\perp^2 \delta_{k, -k'} \delta(t - t'). \]  
(2.3b)
Here \( c_p \) is the specific heat at constant pressure. The prefactors on the right-hand sides of these equations reflect the equilibrium correlations of the temperature at constant pressure, \( \langle |\delta T(k)|^2 \rangle = k_B T_0^2 / c_p \), and of the velocity, \( \langle |v_z(k)|^2 \rangle = k_B T_0 / \rho. \)\(^1 \) The cross correlations \( \langle QP \rangle = 0 \) vanish since there is no kinetic coefficient that couples \( \delta T \) and \( v_z \). For the validity of these equations in the context of long-range correlations in a NESS, see the discussion in Sec. IV.

### B. The TTCF

Solving these equations for the TTCF by Fourier transforming in space and time and then transforming back to time gives\(^2,3 \)
\[ \langle \delta T(k, t)\delta T^*(k, 0) \rangle = \frac{k_B T_0^2}{c_p} \left[ (1 + A_T(k)) \exp(-D_T k^2 |t|) - A_v(k) \exp(-\nu k^2 |t|) \right], \]  
(2.4a)
where
\[ A_v(k) = \frac{D_T}{\nu} A_T(k) = \frac{c_p}{T_0} \frac{\hat{k}_\perp^2 (\partial_T)^2}{(\nu^2 - D_T^2) k^4}. \]  
(2.4b)
Note that \( A_T \) and \( A_v \) are singular for \( k \to 0 \) and scale as \( 1/k^4 \). Setting \( t = 0 \) in Eq. (2.4a) gives Eq. (1.2) adapted for the case of constant pressure.\(^15 \)

Equation (2.4a) can be directly measured in small angle light scattering.\(^2,3 \) The results are shown in Fig. 2. There are no adjustable parameters in the fit, all thermophysical properties are taken from other experimental data. Note how large the effect is: For the wave numbers and temperature gradients in the experiment, the NE contribution is much larger than the equilibrium one. The conclusion is that the long-ranged correlations in a NESS are well confirmed by these experiments as well as by many others.\(^16 \)

An interesting aspect of Eqs. (2.4) is the fact that the time dependence is entirely diffusive: A Laplace transform of either of the two terms in Eq. (2.4a) has the form of an ordinary diffusion pole
\[ D(z) = \frac{A(k)}{z + iDk^2} \]  
(2.5a)
with a spectrum
\[ D''(k, \omega) = \text{Im} D(k, \omega + i0) = A(k) \frac{Dk^2}{\omega^2 + D^2k^4}. \]  
(2.5b)
Here \( z \) is a complex frequency with \( \text{Im} z > 0 \) and \( D \) is a diffusivity that in the present context can be either \( D_T \) or \( \nu \). What is anomalous is the prefactor \( A(k) \), which represents a static susceptibility
\[ \int_{-\infty}^{\infty} \frac{d\omega}{\pi} D''(k, \omega) = A(k) \sim 1/k^4 \]  
(2.6)
that is highly singular in the limit \( k \to 0 \), scaling as \( 1/k^4 \), as a result of the non-equilibrium effects, see Eq. (2.4b). The diffusivity is related to a generalized conductivity \( \sigma \) via an Einstein relation \( \sigma = D \chi \), with \( \chi \) another static susceptibility that is not qualitatively affected by the nonequilibrium fluctuations. It is illustrative to consider this structure from another angle by deriving an effective equation for the temperature fluctuations only, as we do in the following subsection.
C. Effective Langevin equation for temperature fluctuations

Let us rewrite the Langevin equations (2.1, 2.2), using a Martin-Siggia-Rose formalism.\textsuperscript{18–20} The starting point is the stochastic ‘partition function’

\begin{equation}
Z[Q, P] = \int D[\delta T, v_z] \delta[\partial_t \delta T + v_z \partial_z T - D_T \nabla^2 \delta T - Q] \\
\times \delta[\partial_t v_z - \nu \nabla^2 v_z - P] \ J \ (2.7)
\end{equation}

Here the integrations and the \( \delta \)-functions are to be understood in a functional sense, and \( J \) is a Jacobian associated with the arguments of the \( \delta \)-functions that ensures that \( Z[Q, P] = 1 \). By adding sources for \( \delta T \) and \( v_z \) one can turn \( Z \) into a generating functional for correlation functions. This will not be important for what follows, and neither will the Jacobian, which for our linearized theory is independent of the fields.\textsuperscript{21} In what follows we will ignore the Jacobian, as well as constant prefactors that arise from Gaussian integrals. The next step is to introduce auxiliary ‘conjugate’ fields \( \delta T \) and \( \tilde{v}_z \) to rewrite the functional \( \delta \)-functions in terms of auxiliary integrals:

\begin{equation}
Z[Q, P] = \int D[\delta T, v_z, \delta T, \tilde{v}_z] \\
\times e^{i(\delta T \partial_t \delta T + v_z \partial_z T - D_T \nabla^2 \delta T - Q)} \\
\times e^{i(\tilde{v}_z \partial_t v_z - \nu \nabla^2 v_z - P)} , \ (2.8a)
\end{equation}

where we have defined a scalar product

\begin{equation}
(A|B) = \int d\rho \sum_k A(k, \omega) B(-k - \omega) . \ (2.8b)
\end{equation}

We next integrate out the Langevin forces \( Q \) and \( P \), using Gaussian distributions with second moments given by Eqs. (2.2, 2.3). It is most convenient to work in Fourier space, which makes the second moments\textsuperscript{22}

\begin{equation}
\langle |Q(k, \omega)|^2 \rangle \equiv G_{QQ}(k, \omega) = \frac{2k_B T_0}{c_p} D_T k^2 , \ (2.9a)
\end{equation}

\begin{equation}
\langle |P_z(k, \omega)|^2 \rangle \equiv G_{PP}(k, \omega) = \frac{2k_B T_0}{\rho} \nu k_z^2 . \ (2.9b)
\end{equation}

The moments are frequency independent due to the delta-correlations in time space. We obtain

\begin{equation}
Z = \int D[Q, P] Z[Q, P] e^{-\frac{i}{2} \langle Q |G_{QQ}^{-1}| Q \rangle - \frac{i}{2} \langle P_z |G_{PP}^{-1} | P_z \rangle} \\
= \int D[\delta T, v_z, \delta T, \tilde{v}_z] e^{i(\delta T \partial_t \delta T + v_z \partial_z T \nu v_z)} \\
\times e^{i(\tilde{v}_z \partial_t v_z - \nu \nabla^2 v_z \nu v_z)} e^{-\frac{1}{2} \langle \delta T |G_{QQ} | \delta T \rangle - \frac{1}{2} \langle \tilde{v}_z |G_{PP} | \tilde{v}_z \rangle} \ (2.10)
\end{equation}

We now integrate out \( \tilde{v}_z \), which produces a term that is quadratic in \( v_z \), and finally we integrate out \( v_z \). This procedure yields

\begin{equation}
Z = \int D[\delta T, \delta T] e^{i(\delta T \partial_t \delta T + v_z \partial_z T \nu v_z) \delta T} \\
\times e^{-\frac{1}{2} \langle \delta T |G_{QQ} | \delta T \rangle - \frac{1}{2} \langle \tilde{v}_z |G_{PP} | \tilde{v}_z \rangle} \ (2.11)
\end{equation}

A comparison with Eq. (2.10) shows that this result is equivalent to a fluctuating diffusion equation for \( \delta T \) only,

\begin{equation}
\partial_t \delta T = D_T \nabla^2 \delta T + Q_r(r, t) , \ (2.12a)
\end{equation}

with a renormalized fluctuating force \( Q_r \) that is Gaussian distributed with a second moment

\begin{equation}
\langle |Q_r(k, \omega)|^2 \rangle = \frac{2k_B T_0}{c_p} D_T k^2 + \frac{2k_B T_0}{\rho} \frac{(\partial_z T)^2 \nu k_z^2}{\omega^2 + \nu^2 k_z^2} . \ (2.12b)
\end{equation}

The salient point is that the fluctuating force gets renormalized, and becomes long ranged. Indeed, the relative scaling of the non-equilibrium term compared to the equilibrium one is \( 1/k^4 \), just as in the TTCF, Eqs. (2.4). The diffusion coefficient, on the other hand, does not get renormalized, in agreement with the discussion in Sec. II B. However, since we have integrated out the velocity fluctuations, this description of effective anomalous temperature diffusion does not reflect the anomalous behavior caused by an initial velocity perturbation discussed in the next section (see Eq. (3.2) below).

It is important to note that in this effective description of temperature fluctuations an assumption of a delta-correlated fluctuating force would be incorrect: the coupling to the, now hidden, velocity degrees of freedom leads to an emergent long-rangedness of the fluctuating force. See also Point 3. in Sec. IV.

III. A DYNAMICAL CONSEQUENCE OF RIGIDITY IN A NESS

In this section we ignore thermal fluctuation effects and simply consider how far a macroscopic perturbation at one point in the fluid travels in a time \( t \). The equations in Sec. II are diffusive, so in equilibrium a macroscopic perturbation diffuses a distance proportional to \( t^{1/2} \). As we will show, in a fluid in a NESS this distance scales as \( t^{3/2} \), i.e., information about the perturbation travels faster than ballistically.

A. Fluids

The equations describing macroscopic perturbations about the NESS in a fluid are Eqs. (2.1) without the
fluctuating forces:
\[ \partial_t \delta T(r, t) + v_z(r, t) \partial_z \delta T = D_T \nabla^2 \delta T(r, t), \tag{3.1a} \]
and
\[ \partial_t v_z(r, t) = \nu \nabla^2 v_z(r, t), \tag{3.1b} \]
where \( \delta T \) and \( v_z \) are macroscopic perturbations specified by initial conditions \( \delta T(r, t = 0) = \delta T^{(0)}(r) \) and \( v_z(r, t = 0) = v_z^{(0)}(r) \).

These equations are easy to solve for \( \delta T \) by using a spatial Fourier transform and a temporal Laplace transform. Transforming back to real time yields
\[ \delta T(k, t) = \delta T^{(0)}(k) e^{-D_T k^2 t} + \frac{v_z^{(0)}(k)}{k^2(\nu - D_T)} \left[ e^{-\nu k^2 t} - e^{-D_T k^2 t} \right]. \tag{3.2} \]
Note that the time dependence of all terms in Eq. (3.2) is diffusive. However, since \( k^2 \) scales as \( 1/t \), the \( 1/k^2 \) factor in the NE part of Eq. (3.2) suggests that the spread of the initial temperature perturbation is effectively faster than diffusive. To make this precise, we assume strongly localized initial perturbations, which in our macroscopic description are represented by \( \delta \)-functions in space. Accordingly, we take
\[ \delta T^{(0)}(k) = \delta T^{(0)} \tag{3.3a} \]
\[ v_z^{(0)}(k) = v_z^{(0)} \tag{3.3b} \]
to be independent of the wave number. We can then perform a Fourier back transform into real space, which yields
\[ \delta T(r, t) = \delta T_E(r, t) + \delta T_{NE}(r, t), \tag{3.4a} \]
where \( r = |r| \). The equilibrium part has the usual diffusive form
\[ \delta T_E(r, t) = \frac{\delta T^{(0)}}{(4 \pi D_T t)^{3/2}} e^{-r^2/4D_T t}. \tag{3.4b} \]
For the non-equilibrium part one finds
\[ \delta T_{NE}(r, t) = \frac{T_0 \text{sgn}((\partial_z T)v_z^{(0)})}{(\nu - D_T) t_0} \]
\[ \times \left[ \text{erf} \left( \frac{r}{2\sqrt{t_0}} \right) - \text{erf} \left( \frac{r}{2\sqrt{D_T t}} \right) \right]. \tag{3.4c} \]
Here \text{erf} is the error function, and
\[ t_0 = 4\pi T_0/|\langle \partial_z T \rangle v_z^{(0)}| \tag{3.4d} \]
is a time scale that characterizes the NESS. Note that \( \delta T_{NE} \) can be positive or negative; this has no physical significance. We see that, for fixed \( r/\sqrt{t} \), \( \delta T_E \) scales as \( 1/t^{3/2} \), whereas \( \delta T_{NE} \) scales as \( 1/t^{1/2} \), consistent with Eq. (3.2). As a result, their spatial moments have different time dependences. In particular,
\[ \langle r^2 \rangle \equiv \int dr \ r^2 \frac{\delta T_E(r, t)}{T_0} + \int dr \ r^2 \frac{\delta T_{NE}(r, t)}{T_0} \]
\[ = 6D_T \frac{\delta T^{(0)}}{T_0} t + \frac{\pi}{4} (\nu + D_T) t_0^2, \tag{3.5} \]
has the surprising property that it grows quadratically as a function of time, as is expected for a propagating mode. In fact, writing it as
\[ \langle r^2 \rangle_{NE} = \frac{\pi}{4} (\nu + D_T) t_0^2, \tag{3.6a} \]
defines a characteristic velocity
\[ v_0 = \sqrt{\pi(\nu + D_T)/4t_0}, \tag{3.7b} \]
that vanishes in the equilibrium limit where \( t_0 \rightarrow \infty \).

As can be seen from the above derivation, this behavior, which is akin to ballistic propagation, is due to the non-equilibrium term proportional to \( \partial_z T/k^2 \) in Eq. (3.2). In the fluctuation calculation of Sec. II this term is effectively squared, which results in a term proportional to \( (\partial_z T)^2/k^4 \). The conclusion is that the long-ranged correlations in a NESS expressed by Eq. (1.2) on one hand, and the anomalous mean-squared spread of a perturbation expressed by Eqs. (3.6, 3.7) on the other, have the same physical origin: They both are manifestations of rigidity in fluids in a NESS.

\section{Solids}

We now extend our discussion to the case of solids, which have rigidity even in equilibrium, as represented by a nonvanishing shear modulus. As we will see, the NE effects induced by a constant temperature gradient \( \partial_z T \) lead to an increased rigidity that leads to a TTCF that scales with the wave number as \( (\partial_z T)^2/k^2 \), and a mean square displacement that scales with time as \( t^3 \). That is, temperature perturbations in a NESS spread faster than ballistically.

For simplicity, we will consider an isotropic solid, and we focus on the coupling between temperature fluctuations and transverse displacement fluctuations. The motivation for the latter is that in solids with a small shear modulus the transverse speed of sound can be substantially less than the longitudinal one, meaning that the coupling is to a relatively soft, if still propagating,
mode. The applicable Langevin equations that replace Eqs. (2.1) now read
\[ \partial_t \delta T(r, t) + \partial_t u_z(r, t) \partial_z T(r, t) = D_T \nabla^2 \delta T(r, t) + Q(r, t), \]  
Eq. (3.8a)
\[ \partial_t^2 u_\perp(r, t) = c_\perp^2 \nabla^2 u_\perp(r, t) + \Gamma \nabla^2 \partial_t u_\perp(r, t) + P(r, t). \]  
Eq. (3.8b)

Here \( u_\perp \) is the transverse displacement field (i.e., \( \partial_t u_\perp \) is the transverse velocity), \( u_z \) is the z-component of \( u_\perp \), \( c_\perp \) is the transverse sound velocity, and \( \Gamma \) is the sound attenuation coefficient. The correlations of \( Q \) are again given by Eqs. (2.2a, 2.3a), and those of \( P \) by
\[ \langle P_i(k, \omega) P_j(-k, -\omega) \rangle = 2\delta_{ij} \langle |\partial_t u_i(k)|^2 \rangle k^2 \Gamma \]
Eq. (3.9a)
\[ = \delta_{ij} \frac{2k_B T_0}{\rho} k^2 \Gamma, \]  
Eq. (3.9b)

as in Eqs. (3.3), and
\[ u_z(k, t = 0) = v_z(0), \]  
Eq. (3.9c)
where \( z \) is the complex frequency. Transforming back to the time domain, we find
\[ \delta T(k, t) = \delta T(0) e^{-D_T k^2 t} - \frac{k_B T_0}{\rho} \sin(c_\perp k) e^{-\Gamma k^2 t/2}. \]  
Eq. (3.11)

In the second, non-equilibrium, term we have kept only the leading contribution for \( k \rightarrow 0 \). For the mean-squared displacement, which can be written
\[ \langle r^2 \rangle = \left. \frac{-1}{T_0} (\nabla_k)^2 \delta T(k, t) \right|_{k=0} \]  
Eq. (3.12a)
this yields
\[ \langle r^2 \rangle = 6D_T \frac{\delta T(0)}{T_0} t + \frac{(\partial_t v_z(0))}{2T_0} c_\perp^2 t^3, \]  
Eq. (3.12b)
This is the result for a solid that is analogous to Eq. (3.5) for a fluid. For the second, non-equilibrium, term only the leading result is shown, corrections are proportional to \( t^2 \). The nonequilibrium contribution grows as the time cubed, and hence faster than what results from ballistic propagation. This is to be contrasted with the corresponding result in a fluid, Eq. (3.5), where the nonequilibrium contribution grows as the time squared.

We finally determine the TTCF in a solid. Performing spatial and temporal Fourier transforms on Eq. (3.8b) yields
\[ u_z(k, \omega) = \frac{-1}{\omega^2 - c_\perp^2 k^2 + i\omega \Gamma k^2} \]  
Eq. (3.13)
Inserting this in Eq. (3.8a) we have
\[ \delta T(k, \omega) = \frac{1}{\omega + iD_T k^2} \left[ \frac{\omega(\partial_z T) P_z(k, \omega)}{(\omega^2 - c_\perp^2 k^2 + i\omega \Gamma k^2)} + Q(k, \omega) \right]. \]  
Eq. (3.14)
This yields
\[ \langle |\delta T(k, \omega)|^2 \rangle = \frac{2k_B T_0^2}{c_p} \frac{D_T k^2}{\rho} \frac{\omega^2 (c_\perp T)^2}{\omega^2 + (D_T k^2)^2} \]
\[ + \frac{2k_B T_0 (\Gamma / \rho) k^2}{\omega^2 - c_\perp^2 k^2} + \omega^2 \Gamma k^2. \]  
Eq. (3.15)
Integrating over the frequency we finally obtain the solid-state analog to the second term in Eq. (1.2):
\[ \langle |\delta T(k)|^2 \rangle = \frac{2k_B T_0^2}{c_p} \frac{D_T k^2}{\rho} \frac{\omega^2 (c_\perp T)^2}{\omega^2 + (D_T k^2)^2} \]
\[ = \frac{k_B T_0^2}{c_p} + (\partial_z T)^2 \frac{k_B T_0}{\rho} \frac{k^2}{k^2}. \]  
Eq. (3.16)
Comparing with Eq. (1.2), we see that the nonequilibrium effect is similar to that in a fluid, but weaker in the sense that the TTCF diverges as \( 1/k^2 \) rather than \( 1/k^4 \). For the difference in the equilibrium term \( (c_p \text{ instead of } c_V) \), see Ref. 15.

IV. DISCUSSION

Consistent with the existence of generic long-ranged correlations in fluids in a NESS, we have shown that there is a novel type of rigidity in the macroscopic fluid equations describing perturbations around a NESS. As a consequence of this, the propagation of temperature perturbations in simple fluids in a temperature gradient is faster than diffusive. In a solid, the corresponding effect is faster than ballistic.

We conclude with a number of additional remarks:

1. For an estimate of \( v_0 \) given by Eq. (3.7b) we use \( v_z(0) = 5 \times 10^4 \text{cm/s} \), a typical thermal velocity, \( \partial_z T/T_0 \approx 0.2 \text{cm}^{-1} \), a typical gradient, and \( \nu + D_T \approx 2 \times 10^{-2} \text{cm}^2/\text{s} \), appropriate for water. This yields
$t_0 \approx 10^{-3}$ s and $v_0 \approx 10$ cm/s. This is more than four orders of magnitude smaller than the speed of sound in water, which validates our approximations, which ignored sound waves, a posteriori. Note, however, that for very viscous supercooled liquids, where $\nu$ is large, $v_0$ can be much bigger.

From Eq. (3.5) we see that the superdiffusive non-equilibrium contribution to $\langle v^2 \rangle$ dominates over the diffusive equilibrium part for times $t > (24/\pi)(D_T/(\nu + D_T))(\delta T(0)/T_0)t_0$. With $\delta T(0)/T_0 \approx 0.01$, and parameters again appropriate for water at room temperature, this time scale is on the order of a few $\mu$s. For larger times the non-equilibrium contribution dominates, and for $t \approx 1$ s the root-mean-squared displacement is on the order of a few cm.

For a semi-quantitative estimate of the magnitude of the effect in a solid, we take again $\partial_z T/T_0 \approx 0.2$ cm$^{-1}$, $v_z(0) \approx 5 \times 10^4$ cm/s, and $\delta T(0)/T_0 \approx 0.01$. With $D_T \approx 1$ cm$^2$/s and $c_L \approx 5 \times 10^5$ cm/s, as appropriate for typical metals, the non-equilibrium term in Eq. (3.12b) dominates over the diffusive term after a few picoseconds. It is also of interest to compare the former with the speed of a sound wave. Suppose the perturbing heat pulse is created at the same time and the same location as a sound wave. Then the root-mean-squared displacement of the heat pulse will overtake the sound displacement as a sound wave. Then the root-mean-squared displacement is on the order of a few cm.

We emphasize that the NE effects are very large, leading to correlations on a scale of centimeters and seconds, and these correlations are generic in the sense that they do not require any fine tuning. By contrast, in order to have the correlation length of an Ising magnet reach 1 cm, one must be within roughly $10^{-5}$ of the critical point.

2. The mechanism for producing the anomalous dynamics is very different from the Goldstone mechanism in an equilibrium system with a broken symmetry. In the latter case, a new soft mode gets created, and often the dynamics of an existing soft mode is altered, viz., the mode becomes faster due to the rigidity. We recall the simplest case of an observable $\mathcal{O}$ that is not conserved and does not couple to any other modes.\footnote{\unrelatedtext} The Kubo function $K$ for that observable then has the structure

$$K(k, z) = \frac{\chi(k)}{z + i\sigma(k, z)/\chi(k)} \quad (4.1)$$

with $z$ the complex frequency. The quantity $\sigma$ is finite in the limit $k \to 0$, $z \to 0$, since $\mathcal{O}$ is not conserved. $\chi$ is the static susceptibility, and if no symmetry is broken, then $\chi(k \to 0)$ is also finite, and there is no soft mode. However, if $\mathcal{O}$ is a broken-symmetry variable, then $\chi(k \to 0) \propto 1/k^2$ and there is a soft mode with $z \sim k^2$. If $\mathcal{O}$ were conserved, then in the absence of a broken symmetry $K$ would have a diffusion pole. Upon breaking the symmetry, an additional soft mode would appear, and the existing diffusive mode would change its nature. As is obvious from Eqs. (2.4) and (2.5), this is not what happens in a NESS. Rather, the nature of the existing diffusive mode is unchanged, but the susceptibility that comprises the residue of the diffusion pole becomes long-ranged as a result of the non-equilibrium fluctuations. This is underscored by the discussion in Sec. II C, which shows that the Langevin force in the fluctuating heat equation gets renormalized, but the dissipative term does not.

3. A long-standing question is whether or not Langevin equations such as Eqs. (2.1) with fluctuating forces that are delta-correlated in space, Eqs. (2.2), can be used to consistently calculate power law correlations of the hydrodynamic variables in a NESS.\footnote{\unrelatedtext} In Ref. 24 it was shown that the effects that lead to the long-ranged correlations of the hydrodynamic variables do not modify the fluctuating heat and stress currents, so Eqs. (2.1) and (2.2) can indeed be consistently used in a NESS. We note, however, that this conclusion no longer holds if the velocity fluctuations are integrated out, which makes the fluctuating force in the remaining temperature equation long-ranged, see the discussion after Eqs. (2.12).

In Sec. III we have effectively shown that the long-ranged behavior in Eqs. (1.2) - (1.4) arises from just the deterministic parts of the Langevin equations. That is, by simply solving the averaged equations and calculating the mean-square displacement, without any reference to fluctuations, one can conclude that the dynamics are anomalous.

4. Similar long-ranged correlation exist in more complex fluids such as binary mixtures with either a concentration gradient or a thermal gradient,\footnote{\unrelatedtext} and in wet active matter.\footnote{\unrelatedtext} These systems therefore also support the super-diffusive propagation of perturbations.

5. In giving Eqs. (3.4) we have for simplicity used a continuous Fourier transform rather than a discrete Fourier series in the $z$-direction. This simplification places an upper limit on the times for which our explicit results are valid, viz., $t \lesssim L/v_0$. The super-diffusive non-equilibrium contribution in Eq. (3.5) dominates over the diffusive equilibrium contribution for times $t \gtrsim \delta T_0 D_T/v_0^2$. With $L = 10$ cm, $v_0 \approx 10$ cm/s as estimated above, $D_T \approx 0.2 \times 10^{-2}$ cm$^2$/s as appropriate for water, and $\delta T_0/T_0 = 0.03$ this yields a large time window $1 \mu$s $\lesssim t \lesssim 1$ s.

6. We emphasize again that in a fluid there is no propagating mode associated with the spread of a temperature perturbation. Rather, the temperature gradient couples the temperature fluctuations to the transverse current fluctuations, see Eqs. (2.1), both of which are diffusive. However, the coupling results in long-ranged correlations that are reflected in the $1/k^2$ prefactor of
the second term on the right-hand side of Eq. (3.2). This can be seen already at the level of the hydrodynamic equations (3.1): Since \( v_z \) is diffusive, it scales as \( 1/k^2 \sim z \), and since \( \partial_x T \) is constant, this effectively introduces an inhomogeneity proportional to \( 1/k^2 \) in the diffusion equation for \( \partial_t T \). Upon a Laplace transform, this multiplies the diffusion pole. As a consequence, a localized temperature perturbation at one point the NE system has a measurable effect at a distance that scales with a higher power of time than in the corresponding equilibrium system. The diffusive dynamics make the divergent prefactor scale as \( t \), and therefore the mean-square displacement carries an extra power of \( t \) compared to the result for the diffusive process. Hence, \( \langle r^2 \rangle \propto t \times t = t^2 \), see Eqs. (3.5) - (3.7).

In a solid, the transverse fluctuations that couple to the temperature fluctuations are propagating, see Eqs. (3.8b), (3.10a), and (3.13). Again, the coupling leads to long-ranged correlations that are reflected in the \( 1/k^2 \) prefactor in the NE term in Eq. (3.11).

The propagating nature of the transverse fluctuations makes this scale as \( t \) again, and as a result the mean-squared displacement scales as \( \langle r^2 \rangle \propto t \times t^2 = t^3 \), Eq. (3.12b). In a solid, the temperature gradient thus has two distinct effects: First, it couples the constant-pressure temperature (i.e., entropy) fluctuations, which are diffusive in the absence of the coupling, to a propagating mode. This is somewhat analogous to the coupling between the energy density and the superfluid velocity that creates the second-sound mode in a superfluid. Second, it leads to long-ranged correlations that make the temperature/entropy fluctuations ‘supersonic’ in the sense that \( \langle r^2 \rangle \propto t^3 \) rather than \( t^2 \).

In the calculation of the TTCF the divergent prefactor of the coupled mode effectively gets squared, and hence the TTCF diverges as \( 1/k^2 \) in a fluid, Eq. (1.2), and as \( 1/k^2 \) in a solid, Eq. (3.16).

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1. L. D. Landau and E. M. Lifshitz, *Statistical Physics Part 1* (Butterworth-Heinemann, Oxford, 1980), Third ed.
2. T. R. Kirkpatrick, E. G. D. Cohen, and J. R. Dorfman, Phys. Rev. A 26, 995 (1982).
3. J. M. Ortiz de Zárate and J. V. Sengers, *Hydrodynamic fluctuations in fluids and fluid mixtures* (Elsevier, Amsterdam, 2007).
4. J. R. Dorfman, T. R. Kirkpatrick, and J. V. Sengers, Ann. Rev. Phys. Chem. 45, 213 (1994).
5. T. R. Kirkpatrick, E. G. D. Cohen, and J. R. Dorfman, Phys. Rev. A 26, 950 (1982).
6. D. Ronis and I. Procaccia, Phys. Rev. A 26, 1812 (1982). Note that Ronis and Procaccia considered only the case \( k_z^2 = 1 \), which makes \( k_z^2 = 1 \).
7. D. Forster, *Hydrodynamic Fluctuations, Broken Symmetry, and Correlation Functions* (Benjamin, Reading, MA, 1975).
8. P. W. Anderson, *Basic Notions of Condensed Matter Physics* (Benjamin, Menlo Park, CA, 1984).
9. The concept of generalized rigidity is more generally associated with long-ranged correlations, see the discussion in Ref. 8. It can be characterized as an ‘action at a distance’, whereby a perturbation at one point in space is felt at a macroscopic distance at a time that seems instantaneous on a macroscopic time scale, and in any case much faster than one would expect in the absence of long-ranged correlations. Long-range order due to a broken symmetry is one avenue to long-ranged correlations and generalized rigidity, but it is not the only one. For instance, the hexatic phase of liquid crystals has nonzero elastic coefficients, and hence rigidity, but no true long-range order. The NESS example discussed here is another manifestation of long-ranged correlations, and hence rigidity, in the absence of long-range order.
10. P. Chaikin and T. C. Lubensky, *Principles of Condensed Matter Physics* (Cambridge University, Cambridge, 1995).
11. P. C. Martin, O. Parodi, and P. S. Peshan, Phys. Rev. A 6, 2401 (1972).
12. J. Gavoret and P. Nozières, Ann. Phys. (NY) 28, 349 (1973).
13. L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon, Oxford, 1966), chap. XVII, Third Revised English ed., many editions are missing this chapter.
14. That is, \( v_z \) is the \( z \)-component of \( \mathbf{v} \), which is the part of the fluctuating velocity \( \mathbf{v}(\mathbf{k}, t) \) that is perpendicular to the wavenumber \( \mathbf{k} \). Equation (2.1b) can be obtained from the general Langevin equation for \( v_{\beta}(\mathbf{k}, t) \), see Ref. 13, by multiplying it by the projector \( P_{\alpha\beta} = \delta_{\alpha\beta} - 2 \mathbf{k}_\alpha \mathbf{k}_\beta/k^2 \), where \( k = k/\sqrt{2} \) and then setting \( \alpha = z \).
15. Note that the equilibrium term in Eq. (2.4a) is Eq. (1.1a) with \( cv \) replaced with \( c_p \). This is because in our treatment we have neglected sound modes effects so that we are effectively examining temperature fluctuations at constant pressure, which in turn are proportional to entropy fluctuations, see Ref. 1. More generally, the TTCF has a contribution from pressure fluctuations even in equilibrium. Furthermore, in a NESS even the entropy fluctuations couple to sound modes and hence to pressure fluctuations. To put it another way, Eqs. (2.1) represent the frequency integration over the central, or Rayleigh, peak in the structure factor only. If pressure fluctuations are kept, then the Brillouin, or sound wave, doublet in the structure factor also contributes to the sum rule, which results in \( c_p \) rather than \( c_v \), entering Eqs. (1.1). The neglected pressure fluctuations do not contribute to the leading long-ranged NE correlations.
16. See Ref. 29 and references therein, esp. Refs. 14 - 16, 20, 21, 24 - 27, 38 - 40.
17. W. B. Li, P. N. Segré, R. W. Gammon, and J. V. Sengers, Physica A 204, 399 (1994).
18. P. C. Martin, E. D. Siggia, and H. A. Rose, Phys. Rev. A 8, 423 (1973).
19. R. Bausch, H. K. Janssen, and H. Wagner, Z. Phys. B 24, 113 (1976).
In nonlinear theories the Jacobian does depend on the fields and is of crucial importance, see Ref. 20. Here we follow the convention of Ref. 1 §122 and write
\[ \langle Q(\omega)Q(\omega') \rangle = \langle |Q(\omega)|^2 \rangle \delta(\omega + \omega'), \]
which serves as the definition of \( \langle |Q(\omega)|^2 \rangle \).

See, e.g., Appendix A in Ref. 5, and Ref. 30.

23 T. R. Kirkpatrick and J. R. Dorfman, Phys. Rev. E 92, 022109 (2015).
24 B. M. Law and J. C. Nieuwoudt, Phys. Rev. A 40, 3880 (1989).
25 J. M. Ortiz de Zárate, F. Peluso, and J. V. Sengers, Eur. Phys. J. E 15, 319 (2004).
26 T. R. Kirkpatrick and J. K. Bhattacharjee, Phys. Rev. Fluids 4, 024306 (2019).
27 D. R. Nelson and B. I. Halperin, Phys. Rev. B 19, 2457 (1979).
28 J. V. Sengers, J. M. Ortiz de Zárate, and T. R. Kirkpatrick, in Non-equilibrium thermodynamics with applications, edited by D. Bedeaux, S. Kjelstrup, and J. V. Sengers (RSC Publishing, Cambridge, 2016), chap. 3, p. 39.
29 D. Ronis, I. Procaccia, and J. Machta, Phys. Rev. A 22, 714 (1980).