IMPROVING THE “NO–HAIR” THEOREM FOR THE PROCA FIELD

Eloy Ayón–Beato

Departamento de Física, Centro de Investigación y Estudios Avanzados del IPN
Apdo. Postal 14–740, 07000 México DF, MEXICO
ayon@fis.cinvestav.mx

Abstract

This paper reconsider the problem of a Proca field in the exterior of a static black hole. The original Bekenstein’s demonstration on the vanishing of this field, based on an integral identity, is improved by using more natural arguments at the event horizon. In particular, the use of the so–called standard integration measure in the horizon is fully justified. Accordingly, the horizon contribution to the Bekenstein integral identity is more involved and its vanishing can be only established using the related Einstein equations. With the new reasoning the “no–hair” theorem for the Proca field now rest on better founded grounds.

Keywords: Black holes, “no–hair” theorems, horizon measure, Proca field.

1. Introduction

Massive fields are forbidden in the exterior of any stationary black hole. Such statement rest in the fact that the strongest version of the “no–hair” conjecture establishes that a stationary black hole is uniquely determined by global charges (conserved Gauss–like surface integrals at spatial infinity) [1], but massive fields exponentially fall–off at infinity and made no contributions to the corresponding surface integrals; there are no global charges associated with them. This kind of unobserved–from–infinity configuration is what is called “hair” in the literature.

The first results explicitly showing the nonexistence of massive “hair” were due to Bekenstein who studied massive scalar fields, Proca–massive spin–1 fields, and massive spin–2 fields [2]. This paper intents to improve the original demonstration of Bekenstein for Proca fields in the presence of static black holes. Such demonstration depends on an integral identity only built from matter field equations; no Einstein equation is used. The fundamental changes introduced in the proof are related to the
arguments concerning the event horizon, which are usually the more involved. On the one hand, an essentially appropriate integration measure is introduced on the event horizon, which is a degenerate hypersurface. As a consequence the horizon contribution to the cited integral identity is more elaborate. On the other hand, the Bekenstein proofs are usually announced as independent of the particular metric theory of gravity (see [3]); due to they involve only matter field equations. However, we shall show that in order to vanishing properly the horizon contributions to the Bekenstein’s identity, it is imperative also the use of Einstein’s equations. We would like to point out that the appropriate justification of the “no–hair” conjecture for massive vector fields has been a useful tool for excluding the existence of new black hole configurations from very complicated theories as metric–affine gravity, where a relevant sector of this theory reduces to an effective Einstein–Proca system [4, 5]. Secondly, but not least important, the methods developed here has also been a start point in the exclusion of “hair” for more complex system where the mass terms appears dynamically through spontaneous symmetry breaking [6].

In the following Sec. 2 the Einstein–Proca system is introduced and the consequences of considering this system in the exterior of a static black hole are highlighted. In Sec. 3 the “no–hair” theorem for the Proca field is established using the Bekenstein argument but with a different integration measure in the horizon, and helping us of the Einstein equations in the reasoning. Section 4 is devoted to the relevant conclusions. The final Appendix is dedicated to properly justify the use of the standard integration measure in the horizon mentioned above.

2. Proca Fields on Static Spacetimes

In this section we introduce some fundamental properties of a Proca field lying in the domain of outer communications $\mathcal{I} \ll I$ of a static black hole. The Einstein–Proca action describing such interaction is given by

$$S = \int \left( \frac{1}{2\kappa} R - \frac{1}{16\pi} H_{\mu\nu} H^{\mu\nu} - \frac{m^2}{8\pi} B_\mu B^\mu \right) dv,$$

where $R$ stands for scalar curvature, and $H_{\mu\nu} \equiv 2\nabla_{[\mu} B_{\nu]}$ is the field strength of the Proca field $B_\mu$. From (1) the Einstein and Proca equations are established

$$\frac{4\pi}{\kappa} R_{\mu\nu} = H_\mu^\alpha H_{\nu\alpha} + m^2 B_\mu B_\nu - \frac{1}{4} g_{\mu\nu} H_{\alpha\beta} H^{\alpha\beta},$$

$$\nabla_\beta H^{\beta\alpha} = m^2 B^\alpha.$$
Improving the “No–Hair” Theorem for the Proca Field

In a static black hole the Killing field $k$ coincides with the null generator of the event horizon $\mathcal{H}^+$. At the same time this field is timelike and hypersurface orthogonal in all the domain of outer communications $\mathcal{I}$. These properties of the Killing field together with the simply connectedness of $\mathcal{I}$ allow us to choose a global coordinate system $(t, x^i)$, $i = 1, 2, 3$, in all $\mathcal{I}$ such that $k = \partial/\partial t$ and
\[
g = -V dt^2 + \gamma_{ij} dx^i dx^j, \tag{4}
\]
where $V$ and $\gamma$ are $t$–independent, $\gamma$ is positive definite in all $\mathcal{I}$, and the function $V$ is positive in all $\mathcal{I}$ and vanishes in $\mathcal{H}^+$. From (4) it can be note that staticity is equivalent to the existence of a time–reversal isometry $t \mapsto -t$ in all $\mathcal{I}$.

We shall assume that the Proca field shares the same symmetries of the metric; firstly, that it is stationary $\mathcal{L}_k B = 0$. Secondly, that the staticity of the metric is also extended to the Proca field $B^\alpha$ and its field equations (3) in the sense of requiring they are all invariant under time–reversal transformations (electromagnetic staticity). The condition of time–reversal invariance for Proca equations (3) written in the coordinates of Eq. (4) demands that the components $B^t$ and $H^{ti}$ remain unchanged while $B^i$ and $H^{ij}$ change sign, or the opposite scheme, i.e., $B^t$ and $H^{ti}$ change sign as long as $B^i$ and $H^{ij}$ remain unchanged under time reversal [2]. Therefore, for a time–reversal invariant Proca field the components $B^i$ and $H^{ij}$ must vanish in the first case mentioned above, and the components $B^t$ and $H^{ti}$ vanish in the second one. Hence, time–reversal invariance implies the existence of two separated cases: a purely electric case (I) and a purely magnetic case (II).

3. The “No–Proca–Hair” Theorem

Now we are ready to proof the “no–hair” theorem for the Proca field and we start by obtaining the corresponding integral identity mentioned in the introduction. Let $\mathcal{V} \subset \mathcal{I}$ be the open region bounded by the spacelike hypersurface $\Sigma$, the spacelike hypersurface $\Sigma'$, and the pertinent portions of the horizon $\mathcal{H}^+$, and the spatial infinity $i^\circ$. The spacelike hypersurface $\Sigma'$ is obtained by shifting each point of $\Sigma$ a unit parametric value along the integral curves of the Killing field $k$. Multiplying the Proca equations (3) by $B_\alpha$ and integrating by parts over $\mathcal{V}$ using the Gauss law, one obtains
\[
\left[ \int_{\Sigma'} - \int_{\Sigma} + \int_{\mathcal{H}^+ \cap \mathcal{V}} + \int_{\partial \mathcal{V}} \right] B_\alpha H^{\beta \alpha} d\Sigma_\beta
= \int_{\mathcal{V}} \left( \frac{1}{2} H_{\alpha \beta} H^{\alpha \beta} + m^2 B_\alpha B^\alpha \right) dv. \tag{5}
\]
The boundary integral over \( \Sigma' \) cancels out the corresponding one over \( \Sigma \), since \( \Sigma' \) and \( \Sigma \) are isometric hypersurfaces taken with reversed normals in the Gauss law. The boundary integral over the infinity \( i^0 \cap \mathcal{I}^+ \) vanishes by the usual Yukawa fall–off of massive fields asymptotically.

We will show that the integrand of the remaining boundary integral at the portion of the horizon \( \mathcal{H}^+ \cap \mathcal{I}^+ \) also vanishes. To achieve this goal we use the standard measure at the horizon [9],

\[
\text{d}\Sigma_\beta = 2n_\beta l_\mu l^\mu \text{d}\sigma ,
\]

(6)

where \( l \) is the null generator of the horizon, \( n \) is the other future–directed null vector \( (n_\mu l^\mu = -1) \), orthogonal to the spacelike cross sections of the horizon, and \( \text{d}\sigma \) is the surface element. We shall justify the use of the standard measure on the horizon in the final Appendix, see Eq. (A.4).

By using the quoted measure the horizon integrand can be written as

\[
B_\alpha H_\beta^\alpha \text{d}\Sigma_\beta = (B_\alpha H_\beta^\alpha l_\beta + B_\alpha H_\beta^\alpha n_\beta l_\mu l^\mu) \text{d}\sigma .
\]

(7)

In order to show that the last integrand is vanishing it is sufficient to prove that the quantities inside the parenthesis at the right–hand side of Eq. (7) satisfy the following conditions: \( B_\alpha H_\beta^\alpha l_\beta \) vanishes and \( B_\alpha H_\beta^\alpha n_\beta \) remains bounded at the horizon. The behavior of these quantities at the horizon can be established by studying some invariants constructed from the curvature. Using Einstein equations (2), we obtain,

\[
\frac{16\pi^2}{\kappa^2} R_{\mu\nu} R^{\mu\nu} = 3H^2 + 4I^2 + (H - m^2 B_\mu B^\mu)^2 + 2m^2 H_\mu^\alpha B^\mu B^\nu H_\nu^\alpha B^\nu ,
\]

(8)

where \( H \equiv H_{\alpha\beta} H^{\alpha\beta}/4, I \equiv ^*H_{\alpha\beta} H^{\alpha\beta}/4 \), and \( ^*H_{\alpha\beta} = \eta_{\mu\nu\alpha\beta} H^{\mu\nu} / 2 \) is the usual Hodge dual. Since the horizon is a smooth surface curvature invariants are bounded there, from which it follows first that \( B_\mu B^\mu \) is bounded at the horizon. The last term in Eq. (8) is nonnegative in both cases (I) and (II), the remaining terms are also nonnegative, and consequently each one is bounded at the horizon, in particular the invariants \( H \) and \( I \). Other invariants can be built from the Ricci curvature (2) by means of \( l \) and \( n \), which are well–defined smooth vector fields on the horizon. The first invariant reads

\[
\frac{4\pi}{\kappa} R_{\mu\nu} n^\mu n^\nu = J_\mu J^\mu + m^2 (B_\mu n^\mu)^2 - n_\mu n^\mu H ,
\]

(9)

where \( J^\mu \equiv H^{\mu\nu} n_\nu \). The last term above vanishes because the bounded behavior of the invariant \( H \). Since \( J \) is orthogonal to the null vector \( n \) it must be spacelike or null \( (J_\mu J^\mu \geq 0) \), therefore each one of the
remaining terms in the right–hand side of Eq. (9) must be bounded. The next invariant to be considered, which vanishes at the horizon by applying the Raychaudhuri equation to the null generator \[10\], reads

\[
0 = 4\pi \kappa R_{\mu\nu} l^\mu l^\nu = D_\mu D^\mu + m^2 (B_\mu l^\mu)^2 - l_\mu l^\mu H ,
\]

where \( D^\mu \equiv H^\mu l^\nu \) is the electric field at the horizon. Once again the bounded behavior of the invariant \( H \) can be used to vanishing the last term of relations (10). The vector \( D \) is orthogonal to the null generator \( l \) hence must be spacelike or null \( (D_\mu D^\mu \geq 0) \). Consequently each term on the right–hand side of Eq. (10) vanishes independently, which implies that \( B_\mu l^\mu = 0 \) and that \( D \) is proportional to the null generator \( l \) at the horizon, i.e., \( D = -(D_\alpha n^\alpha) l \). The following relation arise from the last invariant to be studied

\[
4\pi \kappa R_{\mu\nu} n^\mu n^\nu - H = (D_\mu n^\mu)^2 + m^2 (B_\mu n^\mu)(B_\nu l^\nu) ,
\]

where it has been used that \( D = -(D_\alpha n^\alpha) l \). Since \( B_\mu l^\mu = 0 \) and \( B_\mu n^\mu \) is bounded at the horizon, it follows that the second term on the right–hand side of Eq. (11) vanishes. Therefore, \( B_\mu n^\mu \) is bounded at the horizon as consequence of the bounded behavior of the related left–hand side in Eq. (11).

Summarizing, the study of the horizon behavior of all the above invariants leads to the following conclusions: the quantities \( D_\mu n^\mu \), \( B_\mu n^\mu \), \( B_\mu B^\mu \), and \( J_\mu J^\mu \) are bounded at the horizon, and the relations \( B_\mu l^\mu = 0 \), and \( D = -(D_\alpha n^\alpha) l \) are satisfied in the same region.

Now we are in position to show the fulfillment of the sufficient conditions for the vanishing of the integrand (7) over the horizon, i.e., that \( B_\alpha H^\alpha l_\beta \) vanishes and \( B_\alpha H^\alpha n_\beta \) remains bounded at the horizon. Using the definition \( D^\mu \equiv H^\mu l^\nu \) and that \( D = -(D_\alpha n^\alpha) l \), we obtain for the first quantity at the horizon

\[
B_\alpha H^\alpha l_\beta = (D_\mu n^\mu)(B_\nu l^\nu) = 0 ,
\]

where the vanishing follows from the fact that, as we just establish, \( D_\mu n^\mu \) is bounded and \( B_\nu l^\nu \) vanishes at the horizon.

For the second quantity we note that \( B \) and \( J \) are orthogonal to the null vectors \( l \) and \( n \), respectively. Therefore, \( B \) must be spacelike or proportional to \( l \), and \( J \) must be spacelike or proportional to \( n \). Using a null tetrad basis constructed with \( l, n \), and a pair of linearly independent spacelike vectors, spanning the spacelike cross sections of the horizon, the \( B \) and \( J \) vectors can be written as

\[
B = -(B_\alpha n^\alpha) l + B^\perp , \quad J = -(J_\alpha l^\alpha) n + J^\perp ,
\]
where $B^\perp$ and $J^\perp$ are the projections, orthogonal to $l$ and $n$, on the spacelike cross sections of the horizon. Using expressions (13) it is clear that $B_\mu B^\mu = B^\perp_\mu B^{\perp\mu}$ and $J_\mu J^\mu = J^\perp_\mu J^{\perp\mu}$, i.e., the contribution to these bounded magnitudes comes only from the spacelike sector orthogonal to $l$ and $n$. With the help of Eqs. (13) the other quantity appearing in the integrand (7) can be written as

$$B_\alpha H^{\beta\alpha} n_\beta = -B_\alpha J^\alpha = -(B_\alpha n^\alpha)(D^\beta n_\beta) - B^\perp_\alpha J^{\perp\alpha},$$

where the identity $J_\alpha l^\alpha = -D_\alpha n^\alpha$ has been used. The first term in (14) is bounded because $B_\alpha n^\alpha$ and $D^\beta n_\beta$ are bounded. For the second term we can apply the Schwarz inequality since $B^\perp$ and $J^\perp$ belong to a spacelike subspace. Thus, $(B^\perp_\alpha J^{\perp\alpha})^2 \leq (B^\perp_\mu B^{\perp\mu})(J^{\perp\nu} J^{\perp\nu}) = (B_\mu B^\mu)(J^\nu J^\nu)$ and since $B_\mu B^\mu$ and $J^\nu J^\nu$ are bounded at the horizon we conclude that the second term of Eq. (14) is also bounded.

Finally, the vanishing of the term (12) and the bounded behavior of the other term (14), together with the null character of $l$ at the horizon lead to the vanishing of the integrand (7) over the event horizon.

With no contribution from boundary integrals in the identity (5) we shall write the volume integral, using the coordinates from Eq. (4), for each one of the different cases discussed at the beginning of this section.

For the purely electric case (I) we have

$$\int_{\mathcal{V}} -V \left( \frac{1}{2} \gamma_{ij} H^{ti} H^{tj} + m^2 (B^t)^2 \right) dv = 0.$$

The nonpositiveness of the above integrand, which is minus the sum of squared terms, implies that the integral is vanishing only if $H^{ti}$ and $B^t$ vanish everywhere in $\mathcal{V}$, and hence in all $\ll \mathcal{I} \gg$.

For the purely magnetic case (II) the volume integral reads as

$$\int_{\mathcal{V}} \left( \frac{1}{2} \gamma_{ik} \gamma_{jl} H^{kl} H^{ij} + m^2 B_i B^j \right) dv = 0,$$

in this case the non–negativeness of the above integrand is responsible for the vanishing of $H^{ij}$ and $B^i$ in all $\ll \mathcal{I} \gg$.

4. Conclusions

Concluding, we have proved that the Proca field $B$ is trivial in the presence of a static black hole. It must be pointed that we improve the original proof of Bekenstein on the subject by using an appropriate integration measure on the event horizon of the black hole, and also making explicit use of the gravitational field equations. The vanishing of $B$ implies that the action (1) reduces to the Einstein–Hilbert one, for
which the only static black hole is the Schwarzschild solution (see [11, 12] for references on improvements to the original proofs). The existence of static soliton (particle-like) configurations can be also excluded using similar arguments, since the only change in the proof is that in this case the boundary of the volume $\mathcal{Y}$ only consists of the isometric surfaces $\Sigma$ and $\Sigma'$, and a portion of the spatial infinity $i^o$, i.e., there is no interior boundary corresponding to the event horizon.

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Appendix: On the Suitable Integration Measure of the Horizon

In this appendix we justify the use of the standard integration measure (6) in the boundary integrals on the event horizon. In the derivation of the basic identity (5) we make use of Gauss’s law, which is a well-known particular form of Stokes’s theorem

$$\int_{\mathcal{V}} d\alpha = \int_{\partial \mathcal{V}} \alpha \quad \implies \quad \int_{\mathcal{V}} \nabla \beta v^\beta dv = \int_{\partial \mathcal{V}} v^\beta d\Sigma_\beta, \quad (A.1)$$

for some volume $\mathcal{V}$ with boundary $\partial \mathcal{V}$. The relation between both theorems rest on that the three-form $\alpha$ is the Hodge dual of the vector field $v$ [10], we shall explore such relation in order to find the horizon integration measure. In the Stokes version we can write the boundary integrand as $\alpha = h \eta_3$ using that the three-form $\alpha$ must be proportional to the volume three-form $\eta_3$ of the boundary $\partial \mathcal{V}$.

For example, in the case of a boundary consisting of non-null surfaces the induced metric there is nondegenerate, and we can choose as volume three-form on $\partial \mathcal{V}$ the one associated with the induced metric. It is given by $\eta_{\mu\nu\lambda} = \pm \tilde{n}_{\mu\nu\lambda} \equiv \pm \eta_{\mu\nu\lambda} \tilde{n}^\mu$, where $\eta$ is the four-dimensional volume form, $\tilde{n}$ is the unit normal to $\partial \mathcal{V}$, and we use the plus sign if $\tilde{n}$ is spacelike and the minus one if is timelike; in both cases the normal is chosen to be "outward pointing" in the volume $\mathcal{V}$ in order to keep the orientation needed in Stokes’s theorem [10]. For this election of the boundary volume three-form we have the relation $\alpha = \star \alpha = \pm h \star \tilde{n}$, from which it follows that $h = \star \alpha^\beta \tilde{n}_\beta$, and we recover the Gauss form of the boundary integral

$$\int_{\partial \mathcal{V}} \alpha = \int_{\partial \mathcal{V}} \star \alpha^\beta \tilde{n}_\beta \eta_3 = \int_{\partial \mathcal{V}} v^\beta d\Sigma_\beta, \quad (A.2)$$

here $v^\beta = \star \alpha^\beta = \eta^{\mu\nu\lambda\beta} \alpha_{\mu\nu\lambda}/3!$, and the integration measure at the boundary is the traditional one for non-null surfaces $d\Sigma_\beta = \tilde{n}_\beta \eta_3 = \tilde{n}_\beta d\sigma$, where $d\sigma$ stands for the volume element ($d\sigma = \eta_3$) following the usual notation.

The above situation does not apply to null surfaces, which is our case of interest when we try to integrate on the horizon. In this case the induced metric is degenerate, and a priori there is no natural choice for the volume form. However, in the case of the
event horizon we can use other geometrical objects naturally defined on it to specify a volume three–form. Let \( l \) be the null generator of the horizon, and \( n \) be the other linearly independent and future–directed null vector orthogonal to the spacelike cross sections of the horizon, and normalized in such a way that \( n_\mu l^\mu = -1 \). For smooth event horizon they are well–defined smooth vector fields along it. In this case the volume three–form must not be orthogonal to \( l \) since such vector is tangent to the horizon, in fact, their interior product must coincide with the two–form expanding the area of spacelike cross sections of horizon which is obviously given by \( *(l \wedge n) \).

Hence, the volume three–form \( \eta_3 \) must satisfy the relation

\[
\eta_3 \alpha_\beta \gamma l_\mu n_\nu l_\gamma = 3 \eta \rho_\alpha \beta \gamma n_\rho l_\mu n_\nu l_\gamma = 3 l^\mu ,
\]

and expanding the left–hand side above using that \( \alpha = ** \alpha \) we have finally

\[
h = \frac{1}{2} \eta_\rho \alpha \beta \gamma ^* \alpha^\rho n_\mu n_\nu n_\gamma = 2 ^* \alpha^\rho n_\mu l_\gamma ,
\]

(A.3)

Hence the boundary integral can be expressed in the Gauss form as

\[
\int_{\partial V} \alpha = \int_{\partial \Sigma} ^* \alpha^\beta 2 n_\beta l_\mu l^\mu = \int_{\partial \Sigma} v^\beta d\Sigma_\beta ,
\]

(A.4)

where again \( v^\beta = ^* \alpha^\beta \), but this time the boundary integration measure is expressed as \( d\Sigma_\beta = 2 n_\beta l_\mu l^\mu d\sigma \), and we use the standard notation for the volume element \( d\sigma = \eta_3 \). This is the boundary integration measure introduced in Eq. (6) for the boundary integral at the event horizon and also used in previous references [9, 4, 6].

References

[1] P. Bizoń, Acta Phys. Polon. B25 (1994) 877.
[2] J.D. Bekenstein, Phys. Rev. Lett. 28 (1972) 452; Phys. Rev. D5 (1972) 1239; D5 (1972) 2403.
[3] J.D. Bekenstein, “Black Holes: Classical Properties, Thermodynamics and Heuristic Quantization,” in: Proceedings of the 9th Brazilian School of Cosmology and Gravitation, Rio de Janeiro, Brazil (1998) gr–qc/9808028.
[4] E. Ayón–Beato, A. García, A. Macías and H. Quevedo, Phys. Rev. D61 (2000) 084017; D64 (2001) 024026.
[5] Y.N. Obukhov, E.J. Vlachynsky, Annals Phys. 8 (1999) 497; M. Toussaint, Gen. Rel. Grav. 32 (2000) 1689.
[6] E. Ayón–Beato, Phys. Rev. D62 (2000) 104004.
[7] P.T. Chruściel and R.M. Wald, Class. Quant. Grav. 11 (1994) L147.
[8] B. Carter, in: Gravitation in Astrophysics (Cargèse Summer School 1986), eds. B. Carter, J.B. Hartle (Plenum, New York 1987).
[9] T. Zannias, J. Math. Phys. 36 (1995) 6970; 39 (1998) 6651.
[10] R.M. Wald, General Relativity (Univ. of Chicago Press, Chicago 1984).
[11] M. Heusler, Black Hole Uniqueness Theorems (Cambridge Univ. Press, Cambridge 1996); Living Rev. Rel. 1, 1998–6, http://www.livingreviews.org/Articles/Volume1/1998–6heusler.
[12] P.T. Chruściel, “Black Holes,” gr-qc/0201053.