A new constraint on mean-field galactic dynamo theory

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ABSTRACT
Appealing to an analytical result from mean-field theory, we show, using a generic galaxy model, that galactic dynamo action can be suppressed by small-scale magnetic fluctuations. This is caused by the magnetic analogue of the Rädler or $\mathbf{D} \times \mathbf{J}$ effect, where rotation-induced corrections to the mean-field turbulent transport result in what we interpret to be an effective reduction of the standard $\alpha$ effect in the presence of small-scale magnetic fields.

Key words: magnetic fields – dynamo – galaxies: magnetic fields – MHD

1 INTRODUCTION

Astrophysical dynamos can be loosely divided into small-scale (or fluctuation) dynamos, which amplify the field on scales up to the outer scale, and large-scale (or mean-field) dynamos, which amplify the field on larger scales up to the system size. Both are expected to occur simultaneously in turbulent astronomical bodies such as stars and galaxies. There have been some attempts to understand these in a single unified framework (Subramanian 1999; Subramanian & Brandenburg 2014; Bhat et al. 2016). Nevertheless, given sufficient scale-separation, it seems reasonable to treat them as distinct, yet interconnected, entities (Brandenburg & Subramanian 2005a; Brandenburg et al. 2012a).

The small-scale dynamo operates much faster than the large-scale dynamo. In galaxies, the former exponentiates the field on a timescale of order the shortest eddy turnover time, whilst the latter has an e-folding time that is probably limited from below by the galactic rotation period. As the small-scale magnetic field saturates near energy equipartition with turbulence, it could significantly influence already growing large-scale magnetic field by modifying mean-field transport coefficients (e.g. Rädler et al. 2003; Brandenburg & Subramanian 2005a) (hereafter RKR; BS05). This is expected to affect not only the growth/decay rates, but also the saturation level of the large-scale magnetic field. Importantly, such effects are often proportional to the second moment of the small-scale magnetic field, so relevant even for non-helical magnetic field.

Part of the reason such effects have been mostly ignored in models may be that they tend to be associated with anisotropy in the turbulent transport coefficients, which is often neglected for simplicity (but see e.g. Gressel et al. 2008b; Pipin & Seehafer 2009). On the other hand, numerical studies that have calculated the transport coefficients using the test field method (Schrinner et al. 2005, 2007; Rheinhardt & Brandenburg 2010) have so far been restricted to the regime where large- and small-scale magnetic field components are weak (Sur et al. 2008; Gressel et al. 2008a,b; Brandenburg et al. 2008, 2012a).

A different numerical approach was taken by Squire & Bhattacharjee (2015b). They concluded that the shear-current effect (Rogachevskii & Kleerorin 2003, 2004) can drive dynamo action in the presence of magnetic fluctuations, for moderate values of the magnetic Reynolds number $R_m$. Likewise, a magnetic contribution to the $\mathbf{D} \times \mathbf{J}$ effect (Rädler 1969) was found by Squire & Bhattacharjee (2015b) to play a role. Their results were supported with analytical calculations using quasilinear theory (Squire & Bhattacharjee 2015a). However, it is not clear what the implications are for the large $R_m$ and fluid Reynolds number $R_e$ (along with Strouhal number $\sim 1$) regime relevant for galaxies.

In the present work, we apply the basically equivalent results of RKR; BS05, who calculated the mean electromotive force (emf) for turbulence with slow rotation and weak stratification, to a simple mean-field galactic dynamo model. We show that a magnetic Rädler effect competes with, and partially suppresses, the $\alpha$ effect responsible for the generation of poloidal mean-field from toroidal.

2 MEAN EMF

The mean induction equation is

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (U \times \mathbf{B} + \mathbf{E}),$$

(1)

where $\mathbf{E} = \mathbf{u} \times \mathbf{b}$ is the mean emf and Ohmic terms have been neglected. Here bar represents mean, and we use uppercase (lowercase) to designate large-scale (small-scale) fields. We adopt the expression for $\mathbf{E}$ from Sect. 10.3 of BS05, where helicity is induced by slow rotation and weak stratification of the turbulence, and $\rho$ is assumed to be constant (the incompressible regime). Large-scale
shear is neglected. The mean emf can then be expanded as
\[ E_{\text{t}} = a_{ij} B_j - \eta_{ij} J_j + \langle \gamma \times B \rangle + \langle \delta \times J \rangle + \kappa_{ijk} B_{jk}, \]
where comma denotes partial differentiation, and \( \mu_0 J = \nabla \times B \).
Henceforth we adopt units such that \( \rho = 1 \) and \( \mu_0 = 1 \) (for details see RKR; BS05). BS05 find
\[ \alpha_{ij} = \frac{1}{2} \tau \delta_{ij} b^2 - \frac{1}{2} \tau^2 \left[ \delta_{ij} \omega \cdot \nabla \left( u^2 - \frac{1}{2} b^2 \right) \right. \]
\[ - \frac{1}{2} \tau (\omega \cdot \nabla \omega), \]
(3)
\[ \eta_{ij} = \frac{1}{2} \tau \delta_{ij} u^2, \]
(4)
\[ \gamma = - \frac{1}{3} \tau \nabla \left( u^2 - b^2 \right), \]
(5)
\[ \delta = \frac{1}{3} \tau^2 \omega \left( u^2 - b^2 \right), \]
(6)
\[ \kappa_{ij} = \frac{1}{3} \tau^2 \left( \omega \delta_{ij} + \omega k_{ij} \right) \left( u^2 + \frac{5}{2} b^2 \right), \]
(7)
where \( u = \langle u \rangle \) is the rms turbulent velocity, \( b = \langle b \rangle \) is the rms small-scale magnetic field (or Alfvén speed in our units), and \( \omega \) is the angular velocity.

One effect of rotation is to induce anisotropy of the turbulence through the action of the Coriolis force. Strictly speaking \( \omega \) characterizes a presumed initial turbulent state with low magnetic fields. There is no such restriction on \( b \) (BS05).

Note that \( \eta_{ij}, \delta \), and \( \kappa_{ij} \) are independent of stratification, where \( \eta_{ij} \) describes the isotropic turbulent diffusion, the \( \delta \)-term is often also known as the \( \Omega \times J \) or Rädler effect (Rädler 1969), and the tensorial \( \kappa \)-term involves a nontrivial diffusion of the mean magnetic field. Both \( \delta \) and \( \kappa \) effects, exhibiting a generalized diffusion, lead to a cross-coupling of different components of the mean magnetic field and thus share this property with the standard \( \alpha \) effect.

Expressions (3)–(7) were derived using the minimal \( \tau \) approximation (MTA) (Blackman & Field 2002) with relaxation time \( \tau \) assumed to be scale-independent, but we consider the effects of relaxing this assumption (the spectral \( \tau \) approximation) in Sect. 3.6. For simplicity we solve the mean-field dynamo equations under the quasilinear approximation, which, unlike the MTA, neglects nonlocality in time. This is not expected to make a difference when the e-folding time of mean-field dynamo growth or decay is much larger than the relaxation time \( \tau \) (e.g. Chamandy et al. 2013).

We adopt cylindrical coordinates \((r, \phi, z)\) with \( \omega = \Omega \hat{z} \) and apply the slab approximation suitable for a thin disk, which renders the problem 1D in \( z \). That is, we ignore all derivatives except \( \partial / \partial z \), with the exception of the radial derivative of \( \Omega \), which leads to the \( \Omega \) effect (which depends on the shear). All turbulent transport coefficients then reduce to scalars, and we obtain
\[ E_{\text{r}} = \alpha B_r + \eta B_{\phi} / \partial \tau = \gamma B_r - (\delta - \kappa) B_{\phi} / \partial \tau, \]
(8)
\[ E_{\phi} = \alpha B_{\phi} - \gamma B_r / \partial \tau + \gamma B_{\phi} - (\delta - \kappa) B_{\phi} / \partial \tau, \]
(9)

1 To maintain consistency with results of the quasilinear approximation, \( \tau \) is interpreted to be equal to the turbulent time scale (see also Brandenburg & Subramanian 2005b).

2 The \( \phi \)-component of equation (1) has a part \( - \partial E_{\phi} / \partial r \) which contains a term proportional to \( \partial \Omega / \partial r \). The ratio of the magnitude of this contribution to that of the \( \Omega \) effect is \( - \langle \Omega \rangle / \langle r \rangle / b^2 \), where \( b \) is the scale height. Therefore we need not consider \( E_{\text{r}} \) in what follows.

with
\[ \alpha = \alpha_{rr} = \alpha_{\phi \phi} = \frac{1}{3} \tau \Omega b^{2} - \frac{1}{2} \tau^{2} \Omega b^{2} \]
\[ \eta = \eta_{rr} = \eta_{\phi \phi} = \frac{1}{2} \tau b^{2}, \]
\[ \gamma = - \frac{1}{3} \tau \nabla \left( u^2 - b^2 \right), \]
\[ \delta = \frac{1}{3} \tau^2 \omega \left( u^2 - b^2 \right), \]
\[ \kappa = \kappa_{rr} = \kappa_{\phi \phi} = \frac{1}{3} \tau^2 \omega \left( u^2 + \frac{5}{2} b^2 \right). \]

Combining equations (13) and (14) we find
\[ \delta' = \delta - \kappa = - \frac{1}{3} \tau^2 \Omega b^2; \]
that is, the \( \delta \) and \( \kappa \) effects reduce to one effect in the regime considered (Brandenburg et al. 2008), so below we use \( \delta' \) for ease of notation. Note that the component of \( \delta' \) proportional to \( b^2 \) cancels out, leading to effects that depend on \( b^2 \) but not on \( u^2 \), and therefore we term it as the magnetic Rädler effect. In the slab approximation, \( (\nabla \times E)_{\phi} = - \frac{\partial E_{\phi}}{\partial z} = - \frac{\partial \alpha B_{\phi}}{\partial z} - \frac{\partial \gamma B_{r}}{\partial z} - (\alpha - \delta') \frac{\partial B_{\phi}}{\partial z} \]
\[ = \frac{\partial \alpha B_{\phi}}{\partial z} + \frac{\partial \gamma B_{r}}{\partial z} + \left( \alpha - \delta' \right) \frac{\partial B_{\phi}}{\partial z} \]
\[ + \frac{\partial \delta}{\partial z} - \frac{\partial B_{\phi}}{\partial z}, \]
(16)
\[ (\nabla \times E)_{\phi} = \frac{\partial E_{\phi}}{\partial z} = \frac{\partial \alpha}{\partial z} B_r - \frac{\partial \gamma}{\partial z} B_{\phi} + \left( \alpha - \delta' \right) \frac{\partial B_{\phi}}{\partial z} \]
\[ + \frac{\partial \delta}{\partial z} - \frac{\partial B_{\phi}}{\partial z}, \]
(17)

Expressions (10)–(15) can now be substituted into equations (16)–(17). For simplicity we assume \( \tau \) to be independent of \( z \), and we also assume \( \partial \Omega / \partial z = 0 \), which is reasonable for a thin galactic disk. The diamagnetic pumping term \( \gamma \times \hat{B} \) in equation (2) is of the same form as the term in equation (1) involving the vertical mean velocity \( U_z \hat{z} \), and may lead to an effective reduction of \( U_z \) (e.g. Gressel et al. 2008a); here we make the simplifying assumption that \( \gamma = 0 \). Likewise we take \( \partial \eta / \partial z = 0 \). Finally, we neglect the terms proportional to \( \partial \delta' / \partial z \) for reasons explained in Sect. 3.1 below, so that equation (1) reduces to
\[ \frac{\partial B_{\phi}}{\partial t} = - \frac{\partial}{\partial z} \left( \alpha B_{\phi} \right) + \frac{\delta}{\partial z} \frac{\partial^{2} B_{\phi}}{\partial z^{2}} + \frac{\partial^{2} B_{\phi}}{\partial z^{2}} \]
\[ = - q \Omega B_{r} - \frac{\partial \alpha}{\partial z} \left( \alpha B_{\phi} \right) - \frac{\delta}{\partial z} \frac{\partial^{2} B_{\phi}}{\partial z^{2}} + \frac{\partial^{2} B_{\phi}}{\partial z^{2}} \]
(18)
\[ \frac{\partial B_{\phi}}{\partial t} = \frac{\partial}{\partial z} \left( \alpha B_{\phi} \right) + \frac{\delta}{\partial z} \frac{\partial^{2} B_{\phi}}{\partial z^{2}} + \frac{\partial^{2} B_{\phi}}{\partial z^{2}} \]
(19)

with \( B_{\phi} \) equal to a constant since solenoidality requires that \( \partial \phi / \partial z = 0 \). Terms involving \( B_{\phi} \) are small, so neglected (Ruzmaikin et al. 1988). Here \( q = -d \ln Q / d \ln r \) is the local shear parameter, equal to unity for a flat rotation curve.

3 EFFECTS OF MAGNETIC FLUCTUATIONS

Below, we make use of both analytic and numerical solutions of equation (1) for a slab geometry and with vacuum boundary conditions at the disc surfaces. The analytic solution neglects the \( \alpha \) and \( \delta' \) terms in the \( \phi \)-component, and also makes use of the ‘no-z’ approximation (Subramanian & Metel 1993; Phillips 2001).

The numerical method solves the full equations (18) and (19) for \( B_{\phi}(z, t) \) and \( B_{\phi}(z, t) \) with \( \alpha \propto \sin(z/\pi)/h \) and other parameters being constants. Details of the galactic dynamo model as well as analytical and numerical methods can be found in Chamandy et al. (2014) (hereafter CSSS) and Chamandy & Taylor (2015). Simulations are already well-converged at the low resolution of 51 grid points.
3.1 $\alpha$ effect and $(\partial B^\prime/\partial z)\partial B_\phi/\partial z$ term

Defining $a_\xi$ to be the part of $\alpha$ that is not explicitly dependent on $b$ we find from equation (10),

$$a_\xi = -\frac{4}{3}\Omega^2 \frac{\partial u^2}{\partial z} \sim \Omega^2 u^2 / h,$$

(20)

with scale height $h$. This is the standard estimate (Krause & Rädler 1980, Ch. VI of Ruzmaikin et al. 1988). Invoking the ‘no-$z’ approximation we obtain

$$(\nabla \times \mathcal{E})_\nu = \cdots - \frac{\partial}{\partial z}(a_\xi B_\phi) \sim \cdots - \frac{2\Omega^2 u^2}{\pi h^2} B_\phi,$$

(21)

where dots represent other terms in the equation.

Now consider how $\alpha$ is affected by terms involving $b^2$, excluding the current helicity term $a_m$. The latter term governs the so-called dynamical quenching (Blackman & Field 2002; Shukurov et al. 2006). Substituting equations (10) and (15) into equation (16), we obtain

$$(\nabla \times \mathcal{E})_\nu = \cdots + \frac{4}{3}\Omega^2 \left[ \frac{\partial^2}{\partial z^2} (u^2 - \frac{1}{3}b^2) B_\phi \right. + \left. \frac{\partial}{\partial z} \right] \frac{\partial B_\phi}{\partial z}. $$

(22)

The coefficients of $b^2$ for the two terms are different ($\sim 1/3$ vs. $\sim 5/6$) because of the contribution of the term proportional to $\partial u^2 / \partial z$.

In galactic dynamo models such as CSSS the two terms in equation (22) have opposite sign, and the first term must out-compete the second to obtain growing solutions. Both terms in equation (22) get reduced when $b^2 \sim u^2$. However, the second term is reduced more, which helps the first term to win. This would tend to effectively enhance the $\alpha$ effect in certain galactic dynamo models, but the overall effect is model-dependent, since it depends on, e.g., the stratification of $u^2$. In any case, the role of magnetic fluctuations may not be very important here given that in the first (dominant) term $b^2$ is multiplied by the factor $1/3$. Hence we do not consider these contributions further, though they may be important in some cases, and we intend to explore them in detail in a subsequent work.

3.2 $\delta^r(\partial^2 B_\phi/\partial z^2)$ term

More interesting and hence the focus of the present work is the contribution of the small-scale magnetic field to the term in equation (16) involving $\delta^r$ and proportional to $\partial^2 B_\phi / \partial z^2$. (This term has the same form as the term involving $\eta_{\delta^r} \partial^2 B_\phi / \partial z^2$ would have had if $\eta_{\delta^r}$ had been finite.) Now to gain insight into the effect of this term, assume, for the moment, that the vertical profile (but not necessarily amplitude) of the mean magnetic field in the linear regime is comparable to that of standard galactic dynamo solutions. Then the ‘no-$z’ approximation can be used to estimate the contribution of this term. We obtain

$$(\nabla \times \mathcal{E})_\nu = \cdots + \delta \frac{\partial^2 B_\phi}{\partial z^2} = \cdots - \frac{2}{3} \Omega^2 \frac{\partial^2 B_\phi}{\partial z^2}$$

$$\sim \cdots - \frac{2}{3} \Omega^2 u^2 \left[ -\frac{\partial^2 B_\phi}{4h^2} \right] = \cdots + \frac{\pi^3}{10h^2} B_\phi,$$

(23)

which can be compared with the term involving $a_\xi$ in equation (21). The contributions of equations (21) and (23) do not have the same form in $z$, in general. However, they do have the same form under the ‘no-z’ approximation. We see then that for $b^2 \sim u^2$, the above term involving $\delta^r$ can suppress, or even cause an effective sign reversal of $\alpha$. Results reported below show that this ‘no-z’ prediction is borne out in numerical solutions.

3.3 When is the effect important?

It is convenient to define the parameter $\xi = b^2 / u^2$, the ratio of mean small-scale magnetic and kinetic energy densities. Let us also define an effective $\alpha$, called $\tilde{\alpha}$, such that

$$\frac{\partial}{\partial z}(\tilde{\alpha} B_\phi) = \frac{\partial}{\partial z}(a_\xi B_\phi) + \frac{\partial^2}{\partial z^2} B_\phi. $$

(24)

Solving for $\tilde{\alpha}$, using relation (24) along with estimates (21) and (23) (with $\partial / \partial z \approx 2/(\pi h)$), we obtain

$$\tilde{\alpha} \sim a_\xi + \frac{\pi^3}{8h^4} \sim \frac{\pi^3}{20} \xi. $$(25)

This gives an estimate of the threshold value of $\xi$, $\xi_0 \approx 20 / \pi^3 = 0.65$, such that $\tilde{\alpha} = 0$ if $\xi = 0$, and for $\xi > \xi_0$, the sign of $\tilde{\alpha}$ is opposite to that of $a_\xi$. At this point it is useful to define the dimensionless Reynolds numbers: $R_\Omega = -\alpha \Omega h^2 / \eta$ and $R_\delta = a_\delta h / \eta$. In the $\alpha$ limit of the $\alpha^2$ dynamo with a purely toroidal mean velocity field $U = (0, r, 0)$, the kinematic growth rate depends only on the dynamo number $D = R_\Omega R_\delta$. Let us now estimate the threshold value $\xi_c < \xi_0$ for obtaining a supercritical dynamo. Let us assume, for simplicity, that vertical outflows are too small to affect the dynamo. The value we obtain for $\xi_c$ is then an upper limit because such outflows generally weaken mean-field dynamo action in the (linear in $B$) regime.

In the $\alpha \Omega$ approximation, the condition for a supercritical dynamo is $|D| > |D_c|$, where $D_c$ can be determined numerically or estimated analytically. The analytic solution (CSSS) gives $D_c \approx -(\pi^2/3)$. Defining $D$ in a way analogous to $\tilde{\alpha}$ in equation (25), we get $D = D(1 - \xi / \xi_0)$. Then $\xi_c$ is obtained (under the ‘no-$z’ and $\alpha \Omega$ approximations) by setting $D = D_c = D_c$ and solving for $\xi$, which gives

$$\xi_c \approx \xi_0(1 - D_c / D) \sim \frac{20}{\pi} \left( 1 - \frac{\pi^3}{288h^4} \xi_0^2 \right). $$(26)

Here we have made use of equations (11) and (20) so that $D \approx -9q \Omega^2 b^2 / u^2$. With these assumptions we also obtain the useful relations $R_\delta = 3\Omega^2$ and $R_\Omega = -3(\Omega^2) h^2 / (\pi u^2)$, where $\Omega^2$ is the Coriolis (inverse Rossby) number, and $h / (\pi u)$ is a dimensionless scale height.

3.4 Estimating $\xi$

What is a best estimate for $\xi$ in spiral galaxies? Observations currently constrain this parameter to only moderate precision, but $\xi$ is measured to be of order unity (e.g. Beck 2007, 2015). It is important to emphasize, however, that in the regime of interest, the large-scale magnetic field is still small $B \ll b$, whilst observations of nearby galaxies generally find $B \leq b$ (Fletcher 2010).

Recent detailed ISM simulations by Kim & Ostriker (2015) support the value $\xi \sim 0.4$ in the fully saturated state. Gent et al.
(2013) obtain $\xi \sim 0.3$ to 0.6 as the field approaches saturation, depending on how $b$ is defined. Again, here we are more interested in the theoretically predicted transitory but long-lived regime for which $b$ has saturated while $B$ is still small. This regime is inaccessible in these simulations. In Kim & Ostriker (2015), the initial large-scale field has a strength of at least $\sim 0.1$ times the value corresponding to equipartition with turbulent kinetic energy density. The small-scale field will thus always be affected by the presence of the large-scale field, e.g. by tangling of the latter to produce the former. Gent et al. (2013) employ a much weaker large-scale seed field, but the exponential growth rate of $b$ is lower than would be expected, and even lower than that of $B$ for at least part of the simulation. The saturation value of $B$ is greater than that of $b$, contrary to observational estimates. As the authors note, the fluctuation dynamo could be unrealistically weak due to a lack of resolution, which leads to Reynolds numbers that are too small. Such simulations have a resolution of a few pc, which is much greater than the scales predicted to be most relevant for fluctuation dynamo growth in the linear regime of $b$. Thus, obtaining a value of $\xi$ from ISM simulations that is directly applicable to our model is likely to remain out of reach into the foreseeable future.

One can also turn to results of more generic MHD simulations, carried out inside a periodic box. For example, Federrath et al. (2011) obtain $\xi \sim 0.2$–0.4 at the end of the exponential growth stage in solenoidally forced turbulence simulations with Mach number $M \sim 1$, while the value is smaller by about an order of magnitude when the turbulence is forced compressively. These simulations are forced at the scale of the box, so that the magnetic field can be thought of as small-scale. They used $Rm = 1500$ and magnetic Prandtl number $Pm = Rm/F \approx 2$ (for galaxies $Pm \gg 1$). Several other simulations of fluctuation dynamo action have been carried out, with saturated values of $\xi$ generally falling somewhere in the range $\sim$ a few $\times 0.01$ to 1, depending on the values of $M$, $Pm$, etc. (e.g. Haugen et al. 2004; Schekochihin et al. 2004; Cho et al. 2009; Brandenburg et al. 2012a; Bhat & Subramanian 2013; Federrath et al. 2014; Tricco et al. 2016, see also Schober et al. (2015) for a model). Note that $u^2$ in the analytical theory characterizes the original turbulence unaffected by the Lorentz force, which tends to be somewhat larger than the saturated value of $u^2$, leading to a somewhat smaller $\xi$. Taken together, all of these results suggest that the best estimate of $\xi$ may be somewhat smaller than 0.4, but we consider 0.4 to be plausible. Below we sometimes adopt this value for illustration; $\xi$ likely also varies within and between galaxies.

How do these estimates compare with the expected value of $\xi$? Two realistic parameter sets are explored in Models $A_\xi$ and $B_\xi$ of Table 1. Model $A_\xi$ has $R_a \approx 0.92$ and $R_D \approx -14.1$, obtained, for example, by setting $q = 1$, $u = 10\, km\, s^{-1}$, $h = 0.4$ kpc, and $\tau = 10\, Myr$, canonical values for the solar neighbourhood (Ch. VI of Ruzmaikin et al. 1988). Model $B_\xi$ assumes $R_a \approx 1.38$ and $R_D \approx -21.1$. These values are obtained, for instance, by setting $q = 1$, $u = 10\, km\, s^{-1}$, $h = 0.2$ kpc, and $\tau = 5\, Myr$, as might be appropriate closer to the Galactic centre. For both models, $\Omega \tau = R_a / 3 < 0.5$. This ensures self-consistency with the estimates of the turbulent transport coefficients, since those calculations assumed slow rotation, ignoring quadratic and higher order terms in $\Omega \tau$. For these models, our analytic estimate (26) yields $\xi_A \approx 0.2$ for Model $A_\xi$ and $\xi_B \approx 0.4$ for Model $B_\xi$. This would imply that the $\alpha$ effect can effectively be reduced by a factor of order unity in a real galaxy setting.

\begin{align}
\lambda_d \approx \frac{\pi^2}{4} \left( \frac{D}{\Omega_{\xi}} - 1 \right) = 3 \left( \frac{h}{\tau u} \right)^2 \lambda \tau,
\end{align}

where $\lambda_d = b^2/\eta$ is the turbulent diffusion time, and we have made use of equation (11) for the rightmost relation. In Fig. 1 we illustrate the dependence of $\lambda$ on $\xi$ obtained for Models $A_\xi$ (red) and $B_\xi$ (blue). Solid lines show numerical solutions of the full $\overline{\alpha^2}\Omega$ dynamo, whilst dashed lines show solutions that neglect the $\alpha$ and $\delta'$ terms in the $\Phi$-component of the mean induction equation (the $\overline{\alpha\Omega}$ approximation). Dashed-dotted lines show the analytic solution (27). All lines end where the solution becomes oscillatory (but retains quadrupole-like symmetry), or, for the analytic solution, at $\xi = \xi_0$. The left axis shows $\lambda_d$ while the right axis shows $\lambda \tau$. Right axis labels assume $q = 1$. Note that $\lambda \tau \ll 1$; thus our implicit assumption to neglect non-locality in time is self-consistent, since

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{Figure1.pdf}
\caption{Kinematic local growth rate $\lambda$, normalized to the inverse local turbulent diffusion time $\lambda_d^{-1}$, for Models $A_\xi$ (red, bottom set of curves) and $B_\xi$ (blue, top set of curves). Solutions are shown as a function of $\xi = b^2/\eta$, sampled in increments of 0.005 for (\varepsilon-dependent) numerical solutions. Full $\overline{\alpha^2}\Omega$ numerical solutions (solid), $\overline{\alpha\Omega}$ numerical solutions (dashed), and (no-$\varepsilon$) analytic solutions (dashed-dotted). The right axis shows $\lambda$ normalized to the inverse turbulence correlation time $\tau$, assuming a locally flat rotation curve $q = 1$. Curves terminate where solutions become oscillatory (numerical) or at $\xi = \xi_0$ (analytical).}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{Figure2.pdf}
\caption{Kinematic local growth rate $\lambda$, normalized to the inverse local turbulent diffusion time $\lambda_d^{-1}$, as a function of the Reynolds number $R_D$, for Models $A_\xi$ (orange), $A_{0.4}$ (red, bottom set of curves), $B_\xi$ (light blue, top set of curves), and $B_{0.4}$ (dark blue). Line styles same as Fig. 1.}
\end{figure}

3.5 Local growth rate

The mean-field induction equation can be solved in the linear regime under the slab approximation to yield a local (in radius and azimuth) mean magnetic field (eigenfunction, which depends on $z$), and corresponding exponential growth rate $\lambda$ (eigenvalue, which is independent of $z$). The growth rate can be estimated as (CSSS)

\begin{align}
\lambda_d \approx \frac{\pi^2}{4} \left( \frac{D}{\Omega_{\xi}} - 1 \right) = 3 \left( \frac{h}{\tau u} \right)^2 \lambda \tau,
\end{align}

where $\lambda_d = b^2/\eta$ is the turbulent diffusion time, and we have made use of equation (11) for the rightmost relation. In Fig. 1 we illustrate the dependence of $\lambda$ on $\xi$ obtained for Models $A_{\xi}$ (red) and $B_{\xi}$ (blue). Solid lines show numerical solutions of the full $\overline{\alpha^2}\Omega$ dynamo, whilst dashed lines show solutions that neglect the $\alpha$ and $\delta'$ terms in the $\Phi$-component of the mean induction equation (the $\overline{\alpha\Omega}$ approximation). Dashed-dotted lines show the analytic solution (27). All lines end where the solution becomes oscillatory (but retains quadrupole-like symmetry), or, for the analytic solution, at $\xi = \xi_0$. The left axis shows $\lambda_d$ while the right axis shows $\lambda \tau$. Right axis labels assume $q = 1$. Note that $\lambda \tau \ll 1$; thus our implicit assumption to neglect non-locality in time is self-consistent, since
non-local effects enter with the factor \((1 + \lambda \tau)\) in axisymmetric models (Chamandy et al. 2013).

Numerical solutions yield slightly larger growth rates than analytic solutions (c.f. CSSS), but show similar dependence on \(\xi\). For the parameter values chosen, the \(\delta^2\) effect plays only a minor role, as can be seen by the proximity of the solid and dashed curves. Clearly, the magnetic Rädler effect is important, especially for expected solar neighbourhood parameters, where \(\xi_c \approx 0.3\). The global growth rate is governed by parameter values near the radius at which \(\lambda\) peaks (e.g. Moss et al. 1998); values typically assumed to be closer to those of Model \(B_\xi\) than Model \(A_\xi\). Even in this case, the \(\delta^2\) term is important, reducing \(\lambda\) by more than half for \(\xi = 0.4\).

We can explore the parameter space by varying \(R_\Omega\). This is done while holding \(\xi\) constant: either \(\xi = 0\) in Models \(A_0\) and \(B_0\), or \(\xi = 0.4\) in Models \(A_{0.4}\) and \(B_{0.4}\). Results are presented in Fig. 2, where \(\xi = 0\) models are shown in lighter colours (orange and light blue). Evidently, the effect continues to be important at other realistic values of \(R_\Omega\).

3.6 Spectral \(\tau\) approximation

The MTA assumes \(\tau\) to be independent of the wavenumber \(k\). Relaxing this assumption leads to the spectral \(\tau\) approximation. Under the latter, terms proportional to \(\tau^2\) get multiplied by the factor 4/3, the \(\alpha_m\) term gets modified, and \(\kappa_{ijk}\) gains an extra term depending on the spectral index \(s\). Of relevance here, equations (6) and (7) become (Appendix G of BS05)

\[
\delta = \frac{2}{3} \tau^2 \omega \left( u^2 - b^2 \right),
\]

(28)

\[
\kappa_{ijk} = \frac{2}{3} \tau^2 \left( \omega_j \delta_{ik} + \omega_k \delta_{ij} \right) + \frac{1}{3} (s-1) (u^2 + b^2).
\]

(29)

For \(s = 1\), we get back MTA, whereas \(s = 5/3\) corresponds to Kolmogorov turbulence. Using relations (28) and (29) we obtain, for the slab case, \(\delta^\prime = -\left(4/45\right) \Omega^2 \tau^2 \left[\left(-1 + s \right) u^2 + \left(3 + s \right) b^2 \right]\), which gives \(\approx -\left(8/135\right) \Omega^2 \tau^2 \left[1 + 10\xi \right]\) for \(s = 5/3\). That is, we obtain a somewhat stronger effect, including a (small) part that is independent of \(b^2\). Unfortunately, the \(s\)-dependent term in \(\kappa_{ijk}\) is the one term for which BS05 and RKR do not agree: the latter derive an extra factor \(\tau^2\), which would effectively suppress the \(\alpha_{ij}\) term for realistic parameter values. A non-trivial partial suppression from this effect depends on the presence of rotation and magnetic fluctuations at a level of \(<\sim 1\times 10^{-1}\%\) of the turbulent kinetic energy density. Such conditions are expected to exist in the early stages of galaxies, after saturation of the small-scale dynamo. This magnetic analogue of the Rädler or \(\mathbf{Q} \times \mathbf{J}\) effect thus presents a challenge to classical galactic dynamo theory.

Measuring \(\xi\) from appropriate direct numerical simulations would be valuable to help confirm or falsify the effect discussed, though this will require significant advances beyond standard methods (Rheinhardt & Brandenburg 2010). Brandenburg et al. (2008) and Brandenburg et al. (2012b) measured the appropriate coefficients using the test-field method for statistically homogeneous turbulence with rotation, with or without density stratification. However, those studies did not include the feedback of the magnetic field onto the turbulence from the Lorentz force, which would be essential. Nevertheless, Brandenburg et al. (2012b) obtained the expected sign and rough linear dependence on \(\Omega\) expected for \(\delta\) and \(k\) for the case of homogeneous turbulence with rotation. In our notation, they found \(\left(\kappa_{ijkl} \eta^2 \right) \approx 1/3\). Thus, if \(\kappa_{ijkl} \eta^2 \approx 1/3\), then the dynamical quenching theory, the expected magnetic analogue of the Rädler or \(\mathbf{Q} \times \mathbf{J}\) effect thus presents a challenge to classical galactic dynamo theory.

Our work points to a need to better constrain the ratio of turbulent magnetic to kinetic energy density \(\xi\). If, as seems likely, \(\xi = 0.4\) for any galaxy known to harbour a large-scale magnetic field, there would be a contradiction between theory and observation. It should be emphasized, however, that the mean-field dynamo theory used and the underlying galaxy model are rather approximate, and results are uncertain by factors of order unity. Thus, it could simply be the case that \(\xi\) is never large enough for the effect to be important. An alternative remedy to this apparent inconsistency is a suppression of the fluctuation dynamo, e.g. by shear (Tobias & Cattaneo 2013); but such a scenario seems unlikely (Kolokolov et al. 2011; Singh et al. 2016). In any case, mean-field dynamo models sometimes appeal to a small-scale dynamo with saturation strength \(\xi \sim 1\) to provide sufficient large-scale seeds of \(\sim 0.1\)–\(1\) nG at high redshift (e.g. Beck et al. 1994; Brandenburg & Upin 1998), in which case a negation of the small-scale dynamo would only replace one problem with another.

A related question that remains to be addressed is whether a magnetic shear-current effect could be important. Such an effect was explored by Squire & Bhattacharjee (2015a), but not for the high \(R_\kappa\) and \(R_{\text{m}}\) regime relevant for galaxies. It would therefore be interesting to incorporate shear into the derivation of the mean emf presented by RKR and BS05. New contributions to the small-scale magnetic helicity flux could also have important effects (Subramanian & Brandenburg 2006; Vishniac 2010; Vishniac & Shapovalov 2014), and should be investigated. More generally, our work highlights the need to clarify the influence of small-scale magnetic fluctuations on the evolution of large-scale magnetic fields in galaxies and other objects.

4 SUMMARY AND DISCUSSION

Starting with an analytical result for the mean emf from BS05 and RKR, we explore the implications for a generic mean-field galactic dynamo model in the linear regime of the mean field \(\mathbf{B}\). We find that the combination of two terms, involving turbulent transport coefficients \(\delta\) and \(\kappa_{ijk}\), effectively suppresses the \(\alpha_{ij}\) term for realistic parameter values. A non-trivial partial suppression from this effect depends on the presence of rotation and magnetic fluctuations at a level of \(<\sim 1\times 10^{-1}\%\) of the turbulent kinetic energy density. Such conditions are expected to exist in the early stages of galaxies, after saturation of the small-scale dynamo. This magnetic analogue of the Rädler or \(\mathbf{Q} \times \mathbf{J}\) effect thus presents a challenge to classical galactic dynamo theory.
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