Limit behaviour of $\mu$—equicontinuous cellular automata

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January 26, 2022

Abstract

The concept of $\mu$—equicontinuity was introduced in [12] to classify cellular automata. We show that under some conditions the sequence of Cesaro averages of a measure $\mu$, converge under the actions of a $\mu$—equicontinuous CA. We address questions raised in [3] on whether the limit measure is either shift-ergodic, a uniform Bernoulli measure or ergodic with respect to the CA. Many of our results hold for CA on multidimensional subshifts.

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1 Introduction

Cellular automata (CA) are discrete systems that depend on local rules. Hedlund [13] characterized CA using dynamical properties: $\phi : \{1, 2, \ldots n\}^Z \rightarrow \{1, 2, \ldots n\}^Z$ is a cellular automaton if and only if it is continuous (with respect to the Cantor product topology) and shift-commuting. This means every CA is a topological dynamical system (TDS), i.e. a continuous transformation $\phi$ on a compact metric space $X$. The dynamical behaviour of these systems
can range from very predictable to very chaotic. Equicontinuity represents predictability. A TDS is equicontinuous if the family \( \{ \phi^i \} \) is equicontinuous, that is, whenever two points \( x, y \in X \) are close, then \( \phi^i(x), \phi^i(y) \) stay close for all \( i \in \mathbb{Z}_+ \). Sensitivity (or sensitivity to initial conditions) is considered a weak form of chaos. There are different classifications of cellular automata and TDS using equicontinuity and sensitivity (see [1] and [10]).

Equicontinuity is a very strong property particularly for cellular automata; different attempts have been made to define weaker but similar properties. Using shift-ergodic probability measures, Gilman [12][11] introduced the concept of \( \mu \)-equicontinuity for cellular automata: \( x \in X \) is a \( \mu \)-equicontinuity point if for most \( y \) close to \( x \) we have that \( \phi^i(x), \phi^i(y) \) stay close for all \( i \in \mathbb{Z}_+ \). A CA is \( \mu \)-equicontinuous if almost every point is \( \mu \)-equicontinuous. He also introduced \( \mu \)-sensitivity (\( \mu \)-expansivity) and showed that a CA is \( \mu \)-equicontinuous if and only if it is not \( \mu \)-sensitive.

The study of long term behaviour is a main topic of interest of dynamical systems/ergodic theory. Long term behaviour can be studied for points, sets, or measures. In particular one may ask if the orbit of a measure converges weakly. Limit behaviour of measures under CA have been studied mainly for two subclasses, linear and \( \mu \)-equicontinuous.

In [17] Lind studied the limit behaviour of the CA on the binary full-shift defined by adding the value of two consecutive positions mod 2. He concluded that the weak limit of the Cesaro average of every Bernoulli measure is the uniform Bernoulli measure. This result has been generalized to other linear expansive CA, and it has been shown that the Cesaro weak limit of an ergodic Markov measure is the uniform Bernoulli (or in a more general setting the measure of maximal entropy)[22][19].

In [3] Blanchard and Tisseur studied CA and measures that give equicontinuity points full measure. These CA are \( \mu \)-equicontinuous but \( \mu \)-equicontinuous CA may not have any equicontinuity point (like in Example 2.26). They showed that the Cesaro weak limit exists and they asked questions about the dynamical behaviour of the limit measure. In particular they asked when the limit measure is shift-ergodic, a measure of maximal entropy or \( \phi \)-ergodic. In this paper we address those questions; we show that those three conditions are very strong.

We characterize \( \mu \)-equicontinuity on shifts of finite type using locally periodic behaviour; \( \mu \)-LEP (Proposition 2.20). We present a natural generalization of Blanchard-Tisseur’s result (Theorem 3.7). In section 3.2 we present the main results of this paper. Let \( \phi \) be a CA and \( \mu \) a \( \sigma \)-ergodic measure that gives equicontinuity points full measure. We show the limit measure is of maximal entropy (Theorem 5.16) if and only if \( \phi \) is surjective and the original measure is the measure of maximal entropy. We show that if \( \phi \) is surjective then the limit measure is shift-ergodic if and only \( \mu \) is \( \phi \)-invariant (Theorem 3.15). Finally we show that if the limit measure is ergodic with respect to \( \phi \) then the system is isomorphic (measurably) to a cyclic permutation on a finite set (Corollary 3.24).

Some of our results hold for CA on multidimensional subshifts, in the cases where they don’t we present weaker analogous results. We also present several
results for $\mu - LEP$ and $\mu$-equicontinuous systems, which may not have any equicontinuity points (like in Example 2.26).

Acknowledgement 1.1 I would like to thank Brian Marcus and Tom Meyerovitch for their suggestions and comments.

2 Equicontinuity and local periodicity

2.1 Definitions

A topological dynamical system (TDS) is a pair $(X, \phi)$ where $X$ is a compact metric space and $\phi : X \rightarrow X$ is a continuous transformation.

The $n$–window, $W_n \subset \mathbb{Z}^d$ is defined as the cube of radius $n$ centred at the origin; a window is an $n$–window for some $n$. For any set $W \subset \mathbb{Z}^d$ and $x \in \mathcal{A}^{\mathbb{Z}^d}$, $x_W \in \mathcal{A}^W$ is the restriction of $x$ to $W$. We will endow $\mathcal{A}^{\mathbb{Z}^d}$ with the Cantor (product) topology; this is the same topology obtained by the metric given by $d(x, y) = \frac{1}{2^n}$, where $m$ is the largest integer such that $x_{W_m} = y_{W_m}$.

We denote the balls with $B_n(x) := \{z | d(x, z) \leq \frac{1}{2^n}\}$.

We define the full $\mathcal{A}$-shift as the metric space $\mathcal{A}^{\mathbb{Z}^d}$. For $i \in \mathbb{Z}^d$ we will use $\sigma_i : \mathcal{A}^{\mathbb{Z}^d} \rightarrow \mathcal{A}^{\mathbb{Z}^d}$ to denote the shift maps (the maps that satisfy $x_{i+j} = (\sigma_i x)_j$ for all $x \in X$ and $i, j \in \mathbb{Z}^d$). The algebra of sets generated by balls and their shifts is called the algebra of cylinder sets. A subset $X \subset \mathcal{A}^{\mathbb{Z}^d}$ is a subshift (or shift space) if it is closed and $\sigma_i$–invariant for all $i \in \mathbb{Z}^d$. If $d = 1$ we say the space is 1D. A cellular automaton (CA) is a pair $(X, \phi)$ where $X$ is a subshift and $\phi(\cdot) : X \rightarrow X$ is a continuous $\sigma$–commuting map, i.e. $\phi$ commutes with all the $\mathbb{Z}^d$ shifts. We say $(X, \phi)$ is a 1D CA if $X$ is a 1D subshift. In most of the literature cellular automata is studied only on full-shifts. In our definition a 1D subshift itself is a CA. Cellular automata of this kind are also known as shift endomorphisms.

A one sided subshift is a set $X \subset \mathcal{A}^\mathbb{N}$ that is closed and $\sigma$–invariant (i.e $\sigma(X) \subset X$).

The following theorem was established in [13] for 1D CA on full-shifts. The same result holds for CA on higher dimensional subshifts.

Theorem 2.1 (Curtis-Hedlund-Lyndon) Let $X$ be a shift space, and $\phi(\cdot) : X \rightarrow X$ a function. The map $\phi$ is a CA if and only if there exists a non-negative integer $n$, and a function $\Phi[ \cdot ] : \mathcal{A}^W \rightarrow \mathcal{A}$, such that $(\phi(x))_i = \Phi[(\sigma_i x)_{W_n}]$.

The radius of the CA is the smallest possible $n$.

Definition 2.2 We say $X \subset \mathcal{A}^{\mathbb{Z}^d}$ is a shift of finite type (SFT) if there exists $n \in \mathbb{N}$ and a finite list of forbidden patterns $\{B_i\} \subset \mathcal{A}^W$ such that $x \in X$ if and only if none of the elements of $\{B_i\}$ appear in $x$.

Example 2.3 The 1D SFT obtained by the forbidden word $\{11\}$ (i.e. the set of doubly infinite sequences that never have two 1’s together) is commonly known as the golden mean shift.
2.2 Topological equicontinuity and local periodicity

Definition 2.4 Given a CA \((X, \phi)\), we define the orbit metric \(d_\phi\) on \(X\) as
\[
d_\phi(x, y) := \sup_{i \geq 0} \{d(\phi^i x, \phi^i y)\},
\]
and the orbit balls as
\[
O_m(x) := \left\{ y \mid d_\phi(x, y) \leq \frac{1}{2^m} \right\} = \left\{ y \mid d(\phi^i(x), \phi^i(y)) \leq \frac{1}{2^m} \forall i \in \mathbb{N} \right\}.
\]

A point \(x\) is an equicontinuity point of \(\phi\) if for all \(m \in \mathbb{N}\) there exists \(n \in \mathbb{N}\) such that \(B_n(x) \subset O_m(x)\). The transformation \(\phi\) is equicontinuous if every \(x \in X\) is an equicontinuity point.

We have that \(\phi\) is equicontinuous if and only if the family \(\{\phi^i\}_{i \in \mathbb{N}}\) is equicontinuous.

If \(X\) is a subshift with dense periodic points, then a CA \((X, \phi)\) is equicontinuous if and only if it is eventually periodic (i.e. there exists \(p\) and \(p'\) such that \(\phi^{p+p'} = \phi^p\))\cite{16,8}.

A weaker notion of periodicity that is related to equicontinuity is local periodicity.

Definition 2.5 Let \((X, \phi)\) be a CA. The set of \(m\)-locally periodic points of \(\phi\), \(\text{LP}_m(\phi)\), is the set of points \(x\) such that \((\phi^i x)_{W_m}\) is periodic (with respect to \(i\)); the set of locally periodic points is defined as \(\text{LP}(\phi) := \bigcap \text{LP}_m(\phi)\). Similarly, the set of \(m\)-locally eventually periodic points of \(\phi\), \(\text{LEP}_m(\phi)\), is the set of points \(x\) such that \((\phi^i x)_{W_m}\) is eventually periodic (with respect to \(i\)); the set of locally eventually periodic points is defined as \(\text{LEP}(\phi) := \bigcap \text{LEP}_m(\phi)\).

A transformation \(\phi\) is locally eventually periodic (\(\text{LEP}\)) if \(\text{LEP}(\phi) = X\) and locally periodic (\(\text{LP}\)) if \(\text{LP}(\phi) = X\).

For \(x \in \text{LEP}_m(\phi)\) the smallest period will be denoted as \(p_m(x)\), and then the smallest preperiod as \(pp_m(x)\).

It is easy to see that \(x \in \text{LEP}(\phi)\) if and only if \((\phi^n x)_i\) is eventually periodic for all \(i \in \mathbb{Z}\).

Proposition 2.6 Let \((X, \phi)\) be a CA. If \((X, \phi)\) is equicontinuous then it is \(\text{LEP}\).

Proof. Let \(m \in \mathbb{N}\). Since \(X\) is compact equicontinuity implies uniform equicontinuity, hence there exists \(n \in \mathbb{N}\) such that if \(y \in B_n(x)\) then \(y \in O_m(x)\).

Let \(x \in X\). There exist \(j > j'\) such that \((\phi^j x)_i = (\phi^{j'} x)_i\) for \(|i| \leq n\). Thus \(\phi^j x \in B_n(\phi^j x)\) and hence \(\phi^j x \in O_m(\phi^j x)\). This implies the orbit ball is eventually periodic so \(x \in \text{LEP}_m(\phi)\); hence \(\phi\) is \(\text{LEP}\). ■

The converse is not true.
Example 2.7 Let $X \subset \{0,1\}^\mathbb{Z}$ be the subshift that contains the points that contain at most one 1. We have that $(X,\sigma)$ is LEP but $0^\infty$ is not an equicontinuity point. Hence LEP does not imply equicontinuity.

Nonetheless we will see that LP implies equicontinuity.

The following is an unpublished result by Chandgotia [4]. We give the proof for completeness.

Proposition 2.8 Let $(X,\sigma)$ be a one-sided subshift. If $X$ has infinitely many periodic points then $X$ contains a non-periodic point.

Proof. For $x \in A^\mathbb{N}$ a periodic point, the set $B_r(x)$ denotes all the possible words of size $r$ that appear in $x$; we also define $B(x) := \cup B_r(x)$; the minimal period of the word $w \in B(x)$ in $x$ (i.e. the minimal "space" in $x$ between consecutive occurrences of $w$) is denoted by $p_x(w)$. We say $X$ has bounded periodic words if for all $w \in \cup \sigma$ periodic $B(x)$ there exists $n_w$ such that $p_x(w) \leq n_w$ for all periodic points $x$.

If $x \in A^\mathbb{N}$ is periodic with minimal period $n$, then $|B_n(x)| = n$ and $|B_r(x)| > r$ for $1 \leq r < n$.

Suppose $X$ is not a subshift with bounded periodic words. So there exists $w \in B(x)$ and $x^n \in X$ a sequence of periodic points such that $p_x^n(w) \geq n$ and $x^n$ begins with $w$. Any limit point of the sequence $x^n$ contains $w$ only once, hence it is not periodic.

Now suppose $X$ has bounded periodic words. Let $x_{1,n}$ be a sequence of periodic points such that $p(x_{1,n}) > n$ and $B_1(x_{1,n})$ is constant. Inductively, define $x_{i,n}$ to be a subsequence of $x_{i-1,n}$ such that $B_i(x_{i,n})$ is constant. We can then find a sequence of points $(y^n) \in X$ such that $p(y^n) > n$ (and hence $\lim_{n \to \infty} |B_r(y^n)| > r$ for all $r$) and the sequence of sets $(B_r(y^n))$ is eventually constant for all $r$. Let $y$ be a subsequential limit of $y^n$ and $w \in \lim_{n \to \infty} B_r(y^n)$.

There exists $n_w$ such that $p_{y^n}(w) \leq n_w$ for all $n$, so $w \in B_r(y^n)$. This means that $|B_r(y)| \geq \lim_{n \to \infty} |B_r(y^n)| > r$ for all $r$ and hence $y$ is not periodic. $\blacksquare$

Corollary 2.9 Let $X$ be a one-sided subshift that contains only $\sigma$ periodic points. Then $X$ is finite and hence $(X,\sigma)$ is equicontinuous.

Proposition 2.10 Let $(X,\phi)$ be a CA. If $(X,\phi)$ is LP then it is equicontinuous.

Proof. Let $X_j = \{ y \in A^\mathbb{N} | y_i = (\phi^i)x \}$, i.e. the space of all sequences that appear as the $j-th$ column of a spacetime diagram. We have that $(X_j,\sigma)$ is a one-sided subshift that contains only periodic points, hence there are only finitely many. This means $\phi$ is equicontinuous. $\blacksquare$

A point $x$ is recurrent if for every open neighbourhood $U$ the orbit of $x$ under $\phi$ intersects $U$ infinitely often; the set of recurrent points is denoted by $R(\phi)$. The following lemma will be useful later.

Lemma 2.11 Let $(X,\phi)$ be a CA. Then $R(\phi) \cap LEP(\phi) = LP(\phi)$. 

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Proof. Using the definitions it is easy to see that $LP(\phi) \subset R(\phi) \cap LEP(\phi)$.

Let $x \in R(\phi) \cap LEP(\phi)$, $m \in \mathbb{N}$, and $q := pp_m(x)$ (see Definition 2.14). Suppose $x \notin LP(\phi)$. This means that $q > 0$, $(\phi^{i+q}x)_{m} \text{ is periodic for } i \geq 0$, and $(\phi^{q-1}x)_{m} \neq (\phi^{q+p_m(x)-1}x)_{m}$. Using the continuity of $\phi$, we know there exists $m' \geq m$ such that for every $y \in B_{m'}(\phi^{q-1}x)$, $(\phi^{i+q}x)_{m} = (\phi^{i+q}y)_{m}$ for $0 \leq i \leq m'(x)$. Using the fact that $\phi^{q-1}x$ is recurrent we obtain that there exists $N > pp_m(x) + m'(x)$ such that $\phi^N x \in B_{m'}(\phi^{q-1}x)$. This means that $(\phi^N x)_m \neq (\phi^{q+p_m(x)-1}x)_m$ and $(\phi^{N+i}x)_m = (\phi^{q+p_m(x)+i-1}x)_m$ for $0 < i \leq m'(x)$. This is a contradiction since the second condition and the fact that $p_m(x)$ is the smallest period implies that there exists $j > 0$ such that $N = q + j p_m(x) - 1$, and hence $(\phi^N x)_m = (\phi^{q+p_m(x)-1}x)_m$. $\blacksquare$

2.3 Measure theoretical equicontinuity and local periodicity

We will use $\mu$ to denote Borel probability measures on $X$. These do not need to be invariant under $\phi$.

Definition 2.12 Let $(X, \phi)$ be a CA and $\mu$ a Borel probability measure on $X$. A point $x \in X$ is a $\mu$-equicontinuity point of $\phi$ if for all $m \in \mathbb{N}$, one has

$$\lim_{n \to \infty} \frac{\mu(B_m(x) \cap O_m(x))}{\mu(B_m(x))} = 1.$$ 

$(X, \phi)$ is $\mu$-equicontinuous if almost every $x \in X$ is a $\mu$-equicontinuity point.

In this case we also say $\mu$ is $\phi$-equicontinuous.

The concept of $\mu$-equicontinuity first appeared in [12] [11], and it was used to classify cellular automata using Bernoulli measures. There exist CAs that have no equicontinuity points and that are $\mu$-equicontinuous for every ergodic Markov chain (see Example 2.26).

Cellular automata with $\mu$-equicontinuous directional dynamics (e.g. when $\sigma \circ \phi$ is $\mu$-equicontinuous) were studied in [23].

The following result is a consequence of Corollary 7 and Theorem 9 in [9].

Theorem 2.13 ([9]) Let $(X, \phi)$ be a CA and $\mu$ a Borel probability measure. The following are equivalent:

1) $(X, \phi)$ is $\mu$-equicontinuous

2) For every $\varepsilon > 0$ there exists a compact set $M$ such that $\mu(M) > 1 - \varepsilon$ and $\phi|_M$ is equicontinuous.

3) There exists $X' \subset X$ such that $X'$ is $d_\phi$-separable and $\mu(X') = 1$.

Definition 2.14 Let $(X, \phi)$ be a CA. If $\mu(LEP(\phi)) = 1$, we say $(X, \phi)$ is $\mu$-locally eventually periodic ($\mu$-LEP) and $\mu$ is $\phi$-locally eventually periodic ($\phi$-LEP). We define $\mu$-LP and $\phi$-LP analogously.

The concept of $\mu$-LEP is new in the literature nonetheless it was motivated by Proposition 5.2 in [12].
**Definition 2.15** Let $m \in \mathbb{N}$ and $\varepsilon > 0$. We define

$$Y^m_\varepsilon := \{ x \mid x \in \text{LEP}(\phi), \text{ with } p_m(x) \leq p^m_\varepsilon \text{ and } pp_m(x) \leq pp^m_\varepsilon \}.$$ 

**Remark 2.16** On the set $Y^m_\varepsilon$ we are only considering a finite possibility of preperiods and periods. This implies that $Y^m_\varepsilon$ is equal to a finite union of orbit balls of size $m$; hence it is Borel.

**Lemma 2.17** Let $m \in \mathbb{N}$ and $\varepsilon > 0$. If $(X, \phi)$ is $\mu$-LEP then there exist positive integers $p^m_\varepsilon$ and $pp^m_\varepsilon$ such that $\mu(Y^m_\varepsilon) > 1 - \varepsilon$.

**Proof.** Let

$$Y := \bigcup_{s,k \in \mathbb{N}} \{ x \mid x \in \text{LEP}(\phi), \text{ with } p_m(x) \leq s \text{ and } pp_m(x) \leq k \}.$$ 

Since $(X, \phi)$ is $\mu$-LEP we have that $\mu(Y) = 1$. Monotonicity of the measure gives the desired result. ■

Given a $\mu$-LEP transformation $\phi$ and $m, \varepsilon > 0$, we will use $p^m_\varepsilon$ and $pp^m_\varepsilon$ to denote a particular choice of integers that satisfy the conditions of the previous lemma and that satisfy that $p^m_\varepsilon \to \infty$ and $pp^m_\varepsilon \to \infty$, as $\varepsilon \to 0$.

Given a subshift $X$, we denote the $\sigma$-periodic points by $P_X(\sigma)$. The following result was proved in [11] when $X$ is a full shift, but the same result holds when $X$ is an SFT.

**Lemma 2.18** Let $X$ be a 1D SFT with forbidden words of size $q$, $(X, \phi)$ a CA with radius $r$. If there is a point $x$ and an integer $m \neq 0$ such that $O_i(x) \cap \sigma^{-m}O_i(x) \neq \emptyset$ with $i \geq q, r$ then $O_i(x) \cap P_X(\sigma) \neq \emptyset$.

The following proposition is proven in greater generality in [9].

**Proposition 2.19** ([9]) Let $(X, \phi)$ be a CA. If $(X, \phi)$ is $\mu$-LEP then it is $\mu$-equicontinuous.

The converse of this proposition is not true in general (see counter-example in [9]). We obtain the converse with an extra hypothesis.

**Proposition 2.20** Let $X$ be a 1D SFT, $(X, \phi)$ a CA, and $\mu$ a $\sigma$-invariant probability measure on $X$. Then $(X, \phi)$ is $\mu$-equicontinuous if and only if it is $\mu$-LEP.

**Proof.** Let $X$ be a 1D SFT with forbidden words of size $q$, $(X, \phi)$ a CA with radius $r$ and $p \geq q, r$. If $x$ is a $\mu$-equicontinuous point then

$$\lim_{n \to \infty} \frac{\mu(B_n(x) \cap O_p(x))}{\mu(B_n(x))} = 1.$$ 

Using $\mu(O_p(x)) > 0$ and Poincare’s recurrence theorem we obtain that

$$\{ y \mid \sigma^i(y) \in O_p(x) \text{ i.o.} \}$$
is not empty. Using $p \geq q, r$ and Lemma 2.18 we conclude that every orbit ball with positive measure contains a $\sigma-$periodic point and hence $(\phi^nx)_j$ is eventually periodic for $-p \leq j \leq p$. The reverse implication is obtained with Proposition 2.19. 

Proposition 2.20 shows that $\mu-$equicontinuity and $\mu-$LEP are equivalent if $X$ is a 1D SFT and $\mu$ a $\sigma-$invariant measure. We do not know if this result holds for cellular automata on multidimensional SFTs. We can show a weaker result (Proposition 2.22) by strengthening the $\mu-$equicontinuity hypothesis.

Let $(X, \rho)$ be a metric space, and $A \subset X$. The closure of $A$ is denoted with $\text{cl}_\rho(A)$. 

Recall that $d_\phi$ denotes the orbit metric (Definition 2.4).

Lemma 2.21 Let $(X, \phi)$ be a CA. If $\mu(\text{cl}_{d_\phi}(P_X(\sigma))) = 1$, then $(X, \phi)$ is $\mu-$LEP.

Proof. Using the fact that the $\phi-$image of a $\sigma-$periodic point is $\sigma-$periodic with at most the same period one can see that any point in $P_X(\sigma)$ is eventually periodic for $\phi$. Let $O_m$ be an orbit ball of size $m$. This means that if $O_m \cap P_X(\sigma) \neq \emptyset$ and $x \in O_m$ then $x \in \text{LEP}_m(\phi)$. Hence if $\mu(\text{cl}_{d_\phi}(P_X(\sigma))) = 1$ then $\phi$ is $\mu-$LEP. 

We represent the change of metric identity map by $\Gamma : (X, d) \to (X, d_\phi)$. A point $x \in X$ is an equicontinuity point of $\phi$ if and only if it is a continuity point of $\Gamma$. Hence $\phi$ is equicontinuous if and only if $\Gamma$ is continuous.

Let $(X, \phi)$ be a CA. We denote the set of equicontinuity points with $\text{EQ}(\phi)$. 

In \cite{3} $\sigma-$ergodic measures that give full measure to the equicontinuity points of a CA were studied. As a consequence of Lemma 3.1 (in that paper) we can see that if $X$ is a 1D subshift, $(X, \phi)$ a CA, and $\mu$ a $\sigma-$ergodic probability measure on $X$ with $\mu(\text{EQ}(\phi)) = 1$ then $(X, \phi)$ is $\mu-$LEP.

If we assume the subshift has dense periodic points (or more generally are dense in a set of full measure) then we obtain that result for multidimensional subshifts.

Proposition 2.22 Let $(X, \phi)$ be a CA and $\mu$ a measure such that $\mu(\text{cl}_{d}(P_X(\sigma))) = 1$. If $\mu(\text{EQ}(\phi)) = 1$ then $(X, \phi)$ is $\mu-$LEP.

Proof. Since equicontinuity points have full measure, then $\Gamma$ is continuous on a set of full measure. This means that for almost every $x \in \text{cl}_{d}(P_X(\sigma))$, we have $x \in \text{cl}_{d_\phi}(P_X(\sigma))$. So $\mu(\text{cl}_{d_\phi}(P_X(\sigma))) = \mu(\text{cl}_{d}(P_X(\sigma))) = 1$. Using Lemma 2.21 we conclude that $(X, \phi)$ is $\mu-$LEP. \n
We already noted that if $X$ is a subshift with dense $\sigma-$periodic points, and $(X, \phi)$ is an equicontinuous CA then $(X, \phi)$ is eventually periodic. Proposition 2.22 is a measure theoretic analogous result.

When $X$ is an SFT then this result is also a consequence of Proposition 2.20.

Now we present some examples.

Example 2.23 Let $x$ be a non $\sigma-$periodic point and let $\mu$ be the delta measure supported on $\{x\}$. Then $\phi = \sigma$ is $\mu-$equicontinuous but not $\mu-$LEP.
Definition 2.24 A 1D CA \((X, \phi)\) has **right radius 0** if there exist \(L \in \mathbb{N}\) and a function \(\phi(\cdot) : A^L \to A\), such that \((\phi(x))_i = \phi(x_{-L}...x_{i-1}x_i)

Example 2.25 ([12]) Let \(X = \{-1,0,1\}^\mathbb{Z}\), and \((X, \phi)\) a radius 1 CA with right radius 0 defined as follows: \(\phi[11] = 1, \phi[10] = 1, \phi[1-1] = 0, \phi[01] = 0\) if \(a \neq 1\), \(\phi[ab] = b\) if \(a, b \neq 1\). The reader can picture the \(-1s\) and 0s as not moving \((\phi[ab] = b\) if \(a, b \neq 1\)\) and the 1s moving to the right at speed one until they encounters a \(-1\) and the position converts into a 0 \((\phi[1-1] = 0)\). It is easy to see that this CA does not contain equicontinuity points. For Bernoulli measures this CA is \(\mu\)-equicontinuous when \(\mu(-1) > \mu(1)\) [12].

We define the set \(E_i := \{x \in \{0,1\}^\mathbb{Z} \mid x_j = 1 \text{ for } 0 \leq j \leq i\}\).

Example 2.26 There exists a CA on the full 2-shift with no equicontinuity points that is \(\mu\)-equicontinuous for every \(\sigma\)-invariant measure that satisfies \(\sum_{i \geq 0} \mu(E_i) < \infty\) (in particular every non-trivial ergodic Markov chain).

Proof. On \(\{0,1\}^\mathbb{Z}\) we define \(\phi(x) = y\) as \(y_i = x_{i-1}x_{i-2}\).

One can check that for every \(i > 0\), \((\phi^i x)_0 = \prod_{j=-i}^{2i} x_j\). Let \(a \in \{0,1\}\). If there exists \(n > 0\) such that \(x_i = a\) for \(i \leq -n\), then \((\phi^i x)_0 = a\) for \(i \geq 2n\). This means that for every ball \(B\) there exists \(x, y \in B\) such that \(O_0(x) \neq O_0(y)\), so the CA has no equicontinuity points. Let \(E_i := \{x \mid x_j = 1 \text{ for } -2i \leq j \leq -i\}\). We have that \(\sum_{i \geq 0} \mu(E_i) = \sum_{i \geq 0} \mu(E_i) < \infty\). By the Borel-Cantelli Lemma we have that \(\mu(E_i)\) infinitely often) = 0. This means that the probability that \((\phi^i x)_0\) has infinitely many ones is zero; since the same argument can be given for \((\phi^i x)_m\) we conclude \(\phi\) is \(\mu\)-\(\text{LEP}\) and hence \(\mu\)-equicontinuous.

For every non-trivial ergodic Markov chain \(\mu(E_i)\) decreases exponentially so \(\sum_{i \geq 0} \mu(E_i) < \infty\). \(\blacksquare\)

Note that in the trivial case (i.e. when \(\mu(1) = 1\) or 0), the hypothesis is not satisfied but we also conclude \(\phi\) is \(\mu\)-equicontinuous.

Q: Does there exists a CA with no equicontinuity points that is \(\mu\)-equicontinuous for every \(\sigma\)-invariant \(\mu\)?

For more examples of \(\mu\)-equicontinuous CA see [24].

The following diagrams illustrate how the different properties relate on the topological and measure theoretical level.

**Topological**

\[ LP \Rightarrow \text{Equicontinuous} \Rightarrow \text{LEP} \]

**Measure theoretical**

\[ \mu - LP \Rightarrow \mu - \text{LEP} \Rightarrow \mu - \text{equicontinuous} \]

If \(X\) is a 1D SFT and \(\mu\) \(\sigma\)-invariant then

\[ \mu - \text{LEP} = \mu - \text{equicontinuous} \]
3 Limit behaviour

3.1 Weak convergence

A sequence of measures $\mu_n$ (on $X$) converges weakly to $\mu_\infty$ (denoted as $\mu_n \to^w \mu_\infty$) if for every continuous function $f : X \to \mathbb{R}$,

$$\int f \, d\mu_n \to \int f \, d\mu_\infty.$$ 

This form of convergence is called weak convergence in the Probability literature and weak* convergence in the Functional Analysis literature.

One can study limit behaviour of dynamical systems by studying the long term behaviour of $\phi_n \mu$ ($\phi \mu$ is the push-forward of the measure) or of its Cesaro averages:

$$\mu_c^n := \frac{1}{n} \sum_{i=1}^{n} \phi_i(\mu).$$

In particular we may ask if $\phi_n \mu$ or $\mu_c^n$ converges weakly, and which are the properties of the limit measure.

**Theorem 3.1 (Portmanteau [2] pg. 15)** We have $\mu_n \to^w \mu_\infty$ if and only if for every open set $U$, $\mu_\infty(U) \leq \liminf \mu_n(U)$ if and only if $\mu_n(E) \to \mu_\infty(E)$ for every set $E$ with zero boundary measure.

We will see that orbit balls form a weak convergence determining class when the limit measure is $\phi-$equicontinuous (Lemma 3.2). Note that even when $\mu$ is $\phi-$LEP, the measure of the boundary of an orbit ball is not necessarily zero. For example, one can check that the orbit balls of Example 2.25 are each contained in their own boundary.

**Lemma 3.2** Let $(X, \phi)$ be a CA, $\mu_n$ be a sequence of measures, and $\mu_\infty$ a $\phi-$equicontinuous measure. If for every orbit ball $A$ we have that $\mu_n(A) \to \mu_\infty(A)$ then $\mu_n \to^w \mu_\infty$. Also, if $\mu$ and $\mu'$ are $\phi-$equicontinuous and $\mu(A) = \mu'(A)$ for every orbit ball $A$ then $\mu = \mu'$.

**Proof.** If we have two orbit balls $O$ and $O'$, then either $O \cap O' = \emptyset$ or one is contained in the other. This implies we have convergence for finite unions of orbit balls (since they can be written as unions of disjoint orbit balls). Let $U$ be an open set. We have that $U$ is the countable union of balls. From Theorem 2.13 we know there exists a $d_\phi-$separable set $X'$ such that $\mu_\infty(X') = 1$. This means that for every $\delta > 0$ there exist a finite number of orbit balls $O_i$ such that $\bigcup_{i=1}^{N} O_i \subset U$ and $\mu_\infty(U) \leq \mu_\infty(\bigcup_{i=1}^{N} O_i) + \delta$. We have that

$$\mu_\infty(U) - \delta \leq \mu_\infty(\bigcup_{i=1}^{N} O_i) = \lim \mu_n(\bigcup_{i=1}^{N} O_i) \leq \liminf \mu_n(U),$$

and hence

$$\mu_\infty(U) \leq \liminf \mu_n(U).$$

Therefore $\mu_n \to^w \mu_\infty$.

Now suppose $\mu(A) = \mu'(A)$ for every orbit ball $A$. Let $X'$ be the $d_\phi-$separable set with $\mu(X') = 1$. This means $\mu(X') = \mu'(X') = 1$ (that is because $X'$ is equal
to the union of countably many orbit balls, i.e. the balls of the topology of $d_{\phi}$.
Thus $\mu$ and $\mu'$ agree on a $\Gamma$--system (family closed under finite intersections)
that generate the Borel sigma algebra (intersected with $X'$); we conclude $\mu = \mu'$.

**Definition 3.3** For $x \in LEP_m(\phi)$ we define
$$O_m^{-q}(x) := \left\{ y \mid \exists i \in \mathbb{N} \text{ s.t. } \phi^{p_m(x)+q}(x) \in O_m(x) \right\}.$$ We denote the Cesaro average of $\phi^i\mu$ with $\mu^c_n$, i.e. $\mu^c_n := \frac{1}{n} \sum_{i=1}^{n} \phi^i(\mu)$.

In the following proposition we show Cesaro convergence holds for orbit balls.
Note in the proof that the convergence is actually stronger than Cesaro; there
is convergence along periodic subsequences.

In some of the following proofs we will use $Y^m_\varepsilon$; see Definition 2.15.

**Lemma 3.4** Let $(X, \phi)$ be a $\mu$--LEP CA, and $m \in \mathbb{N}$. If $x \in LP_m(\phi)$ then
$$\mu^c_n(O_m(x)) \to \frac{1}{p_m(x)} \sum_{q=0}^{p_m(x)-1} \mu(O_m^{-q}(x)).$$
Furthermore if $x \notin LP_m(\phi)$ then $\mu^c_n(O_m(x)) \to 0$.

**Proof.** We have
$$\bigcup_{n \geq 0} \phi^{-p_m(x)n-q}(O_m(x)) = O_m^{-q}(x).$$
For every $0 \leq q < p_m(x)$ and $n \geq 0$
$$\phi^{-p_m(x)n-q}(O_m(x)) \subset \phi^{-p_m(x)(n+1)-q}(O_m(x)).$$
This implies $\mu(\phi^{-p_m(x)n-q}(O_m(x)))$ is non-decreasing and
$$\lim_{n \to \infty} \mu(\phi^{-p_m(x)n-q}(O_m(x))) = \mu(O_m^{-q}(x)). \tag{1}$$
Since we have convergence along periodic subsequences we have that
$$\lim_{n \to \infty} \mu^c_n(O_m(x)) = \frac{1}{p_m(x)} \sum_{q=0}^{p_m(x)-1} \mu(O_m^{-q}(x)).$$
Let $\varepsilon > 0$ and $x \notin LP_m(\phi)$. If $np^l > pq^m$, then
$$\phi^{-p^l n-s}(O_m(x)) \cap Y^m_\varepsilon = \emptyset,$$ so $\mu(\phi^{-p^l n-s}(O_m(x))) < \varepsilon$. ■

**Proposition 3.5** Let $(X, \phi)$ be a $\mu$--LEP CA. If $\phi\mu = \mu$ then $(X, \phi)$ is $\mu$--LP.
Proof. Using the invariance of $\mu$ and Poincaré’s recurrence theorem we obtain that the set of recurrent points has full measure, i.e. $\mu(R(\phi)) = 1$. By Lemma 2.11 we have that

$$\mu(LP(\phi)) = \mu(LEP(\phi) \cap R(\phi)) = 1.$$ 

\[ \square \]

Remark 3.6 Let $B$ be a finite union of balls (thus $B$ is compact). If $B = \cup_{i=1}^{\infty} B_{i}$, where $\{B_{i}\}$ is a disjoint family of balls, then there exists $K$ such that $B = \cup_{i=1}^{K} B_{i}$. From this fact we get that any premeasure on the algebra generated by the balls can be extended to a measure on the Borel sigma-algebra.

The existence of a limit measure in the following result is a natural generalization of a result for 1D CA in [3] ($X$ is multidimensional and $\phi$ may not have any equicontinuity points). Here we also show that the limit measure is $\phi - LP$.

Theorem 3.7 Let $(X, \phi)$ be a $\mu - LEP$ CA. The sequence of measures $\mu_{n}^{c}$ converges weakly to a $\phi - LP$ measure $\mu_{\infty}$.

Proof. Let $B_{m}$ be a ball. For every $\varepsilon > 0$ and $n \in \mathbb{N}$ we have that

$$\left| \frac{1}{n} \sum_{i=1}^{n} \mu(\phi^{-i}(B_{m} \cap Y_{\varepsilon}^{m})) - \frac{1}{n} \sum_{i=1}^{n} \mu(\phi^{-i}(B_{m})) \right| \leq \frac{n \varepsilon}{n} = \varepsilon.$$ 

Consequently

$$\lim_{\varepsilon \to 0} \frac{1}{n} \sum_{i=1}^{n} \mu(\phi^{-i}(B_{m} \cap Y_{\varepsilon}^{m})) = \frac{1}{n} \sum_{i=1}^{n} \mu(\phi^{-i}(B_{m}))$$

uniformly on $n$.

On the other hand for every $\varepsilon > 0$ there exists a finite set of disjoint orbit balls $\{O_{m_{k}}(x_{k})\}$ such that the $x_{k}$ are $LEP$ and $B_{m} \cap Y_{\varepsilon}^{m} = \cup_{k=1}^{K} O_{m_{k}}(x_{k})$. This implies that

$$\frac{1}{n} \sum_{i=1}^{n} \mu(\phi^{-i}(B_{m} \cap Y_{\varepsilon}^{m})) = \frac{1}{n} \sum_{i=1}^{n} \mu(\phi^{-i}(\cup_{k=1}^{K} O_{m_{k}}(x_{k}))).$$

By Lemma 3.4 $\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \mu(\phi^{-i}(\cup_{k=1}^{K} O_{m_{k}}(x_{k})))$ exists. Let

$$F(n, \varepsilon) := \frac{1}{n} \sum_{i=1}^{n} \mu(\phi^{-i}(B_{m} \cap Y_{\varepsilon}^{m})).$$
We have shown that
\[
\lim_{n \to \infty} F(n, \varepsilon) \text{ exists, and}
\lim_{\varepsilon \to 0} F(n, \varepsilon) = \frac{1}{n} \sum_{i=1}^{n} \mu(\phi^{-i}(B_m)) \text{ uniformly on } n.
\]

Thus we obtain that \( \lim_{n \to \infty} \lim_{\varepsilon \to 0} F(n, \varepsilon) \) exists, and
\[
\lim_{n \to \infty} \lim_{\varepsilon \to 0} \frac{1}{n} \sum_{i=1}^{n} \mu(\phi^{-i}(B_m \cap Y_{\varepsilon}^m)) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \mu(\phi^{-i}(B_m)) = \lim_{n \to \infty} \mu_n^{c}(B_m).
\]

The proof of the previous statement is common in analysis (see for example Theorem 1 in [14]).

We define \( \mu_\infty \) as the measure that satisfies \( \mu_\infty(B_m) := \lim_{n \to \infty} \mu_n^{c}(B_m) \) (see Remark 4.10) for every ball \( B_m \). Every open set \( U \) can be approximated by a finite disjoint union of balls. This implies \( \mu_\infty(U) \leq \liminf \mu_n^{c}(U) \), and hence \( \mu_\infty \rightarrow_{w}^{m} \mu_\infty \).

Since \( \phi^{-1}(LEP(\phi)) = LEP(\phi) \) we have that \( \mu_\infty^{c} \) is \( \phi - LEP \). For every \( m \in \mathbb{N} \) and \( \varepsilon > 0 \) we have that \( Y_{\varepsilon}^m \) is a finite union of orbit balls and hence
\[
\mu_\infty(Y_{\varepsilon}^m) = \lim \mu_n^{c}(Y_{\varepsilon}^m) \geq 1 - \varepsilon.
\]
Since \( Y_{\varepsilon}^m \subset LEP(\phi) \) we obtain that \( \mu_\infty = \phi - LEP \). Considering that \( \phi \mu_\infty = \mu_\infty \) and Proposition 4.5, we obtain that \( (X, \phi) \) is \( \mu - LEP \).

There is a more general definition of \( \mu - LEP \) for topological dynamical systems (see [9]). It is possible to check that the previous result holds for topological dynamical systems on zero dimensional spaces.

### 3.2 Behaviour of \( \mu_\infty \)

In this section we prove the main results of the paper. In this section we will use \( \mu_\infty \) to denote the weak limit of \( \mu_n^{c} \).

**Proposition 3.8** Let \( (X, \phi) \) be a \( \mu - LEP \) CA. Then \( \phi^{n} \mu \rightarrow_{w}^{m} \mu_\infty \) if and only if \( \phi^{n} \mu(O) \rightarrow \mu_\infty(O) \) for all orbit balls \( O \), and \( \mu_n^{c} \rightarrow_{w}^{m} \mu_\infty \) if and only if \( \mu_n^{c}(O) \rightarrow \mu_\infty(O) \) for all orbit balls \( O \).

**Proof.** Assume that \( \phi^{n} \mu \rightarrow_{w}^{m} \mu_\infty \).

Let \( m \in \mathbb{N} \) and \( x \in LEP(\phi) \). Take \( \varepsilon > 0 \) so that \( p_m^{c} > p_m(x) \).

Let \( O_m(x) = \cap_{i \in \mathbb{N}} \phi^{-i}G_i \), where every \( G_i \) is a ball defined by the \( i \)th row of the spacetime diagram of \( O_m(x) \). There exists \( k_1 \) such that
\[
|\mu_\infty(\cap_{i=1}^{k} \phi^{-i}G_i) - \mu_\infty(O_m(x))| \leq \varepsilon \text{ for } k \geq k_1.
\]

Fix \( k \geq 2p_m^{c} + k_1 \). If \( n \geq pp_m^{c} \) then
\[
\phi^{-n}(\cap_{i=1}^{k} \phi^{-i}G_i) \cap Y_{\varepsilon}^m = \phi^{-n}(O_m(x)) \cap Y_{\varepsilon}^m.
\]
Since \( \mu(Y^m_\varepsilon) > 1 - \varepsilon \) we obtain

\[
|\phi^n \mu(\cap_{i=1}^k G_i) - \phi^n \mu(O_m(x))| \leq \varepsilon \text{ for } n \geq pp^m_\varepsilon.
\]

Since \( \cap_{i=1}^k \phi^{-1}G_i \) has no boundary, there exists \( N \) such that

\[
|\phi^n \mu(\cap_{i=1}^k \phi^{-1}G_i) - \mu_\infty(\cap_{i=1}^k \phi^{-1}G_i)| \leq \varepsilon \text{ for } n \geq N.
\]

Using the inequalities we obtain

\[
|\phi^n \mu(O_m(x)) - \mu_\infty(O_m(x))| \leq 3\varepsilon \text{ for } n \geq N, pp^m_\varepsilon.
\]

Hence \( \phi^n \mu(O_m(x)) \to \mu_\infty(O_m(x)). \)

The other direction is a corollary of Lemma 3.2. The proof for \( \mu_c^n \) is analogous.

**Definition 3.9** For \( a \in \mathbb{R} \) and \( E \subset X \) Borel, we define

\[
A_a^E := \{ y : \lim_{n \to \infty} \frac{1}{|W_n|} \sum_{i \in W_n} 1_E(\sigma^i(y)) = a \},
\]

where \( W_n = [-n, n]^d \).

Note that if \( \mu \) is \( \sigma \)-ergodic then by the pointwise ergodic theorem

\[
\mu(A_a^E) = \begin{cases} 
1 & \text{if } a = \mu(E) \\
0 & \text{otherwise }
\end{cases}.
\]

**Lemma 3.10** Let \( \mu \) be a \( \sigma \)-ergodic measure, \( (X, \phi) \) a \( \mu - \) LEP CA, \( m \in \mathbb{N} \), and \( x \in LP_m(\phi) \). If for every \( 0 \leq q < p_m(x) \) there exists \( N_q \in \mathbb{N} \) such that

\[
\mu(\phi^{-p_m(x)n-q}(O_m(x))) = \mu(O_{m-q}(x))
\]

then

\[
\mu_c^n(A_{a}^{O_m(x)}) \to \mu_\infty(A_{a}^{O_m(x)}) = \frac{1}{p_m(x)} \sum_{q=0}^{p_m(x)-1} \mu(A_{a}^{O_{m-q}(x)}).
\]

**Proof.** By hypothesis we have that for \( n \geq N_q \)

\[
\phi^{p_m(x)n+q}\mu(O_m(x)) = \mu(O_{m-q}(x)).
\]

Note that since \( \mu \) is \( \sigma \)-ergodic then \( \phi^n \mu \) is \( \sigma \)-ergodic for every \( n \geq 1 \). Let \( 0 \leq q < p_m(x) \).

If \( n \geq N_q \) then

\[
\phi^{p_m(x)n+q}\mu(A_{a}^{O_m(x)}) = \begin{cases} 
1 & \text{if } a = \phi^{p_m(x)n+q}\mu(O_m(x)) \\
0 & \text{otherwise }
\end{cases} = \begin{cases} 
1 & \text{if } a = \mu(O_{m-q}(x)) \\
0 & \text{otherwise }
\end{cases} = \mu(A_{a}^{O_{m-q}(x)}).
\]

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This implies we have convergence along periodic subsequences, thus

\[ \mu_n^c(A_{\alpha}^{O_m(x)}) \to \mu_\infty(A_{\alpha}^{O_m(x)}) = \frac{1}{p_m(x)} \sum_{q=0}^{p_m(x)-1} \mu(A_{\alpha}^{O_m^{-q}(x)}). \]

Using the fact that \( \mu(\phi^{-p_m(x)n-q}(O_m(x))) \to \mu(O_m^{-q}(x)) \) one can check that one of the hypotheses of this result is always satisfied if we assume the CA is \( \mu - LP \).

**Lemma 3.11** Let \((X, \phi)\) be a \( \mu - LP \) CA, \( m \in \mathbb{N} \), and \( x \in LP_m(\phi) \). Then for every \( 0 \leq q < p_m(x) \) we have

\[ \mu(\phi^{-p_m(x)i-q}(O_m(x))) = \mu(O_m^{-q}(x)) \text{ for all } i \in \mathbb{N}. \]

**Proof.** This comes from the fact that \( LP(\phi) \cap \phi^{-p_m(x)i-q}(O_m(x)) = LP(\phi) \cap O_m^{-q}(x) \) for all \( i \in \mathbb{N} \).

**Theorem 3.12** Let \( \mu \) be a \( \sigma \)-ergodic measure and \((X, \phi)\) a \( \mu - LP \) CA. Then \( \mu_\infty \) is \( \sigma \)-ergodic if and only if \( \mu \) is \( \phi \)-invariant.

**Proof.** If \( \phi \mu = \mu \) then \( \mu = \mu_\infty \) and hence it is \( \sigma \)-ergodic.

Suppose \( \mu_\infty \) is \( \sigma \)-ergodic and \( \mu \) is not \( \phi \)-invariant. By Lemma 3.2 we know there exists \( x \in LP(\phi) \) and \( m \in \mathbb{N} \) such that

\[ \mu(O_m(x)) \neq \mu(\phi^{-1}O_m(x)). \]

This implies that that \( p_m(x) \geq 2 \).

We have that

\[ O_m(x) \cap LP(\phi) = O_m^0(x) \cap LP(\phi), \]

\[ \phi^{-1}O_m(x) \cap LP(\phi) = O_m^{-1}(x) \cap LP(\phi). \]

Since \((X, \phi)\) is \( \mu - LP \) we obtain that

\[ \mu(O_m^0(x)) \neq \mu(O_m^{-1}(x)). \]

Using Lemma 3.10 and Lemma 3.11 we get

\[ \mu_\infty(A_{\alpha}^{O_m(x)}) = \frac{1}{p_m(x)} \sum_{q=0}^{p_m(x)-1} \mu(A_{\alpha}^{O_m^{-q}(x)}). \]

We reach a contradiction because \( A_{\alpha}^{O_m(x)}(\mu(O_m^0(x))) \) is \( \sigma \)-invariant, \( \mu_\infty \) is \( \sigma \)-ergodic but

\[ \frac{1}{p_m(x)} \leq \mu_\infty(A_{\alpha}^{O_m(x)}(\mu(O_m^0(x)))) \leq \frac{p_m(x) - 1}{p_m(x)}. \]
Every subshift has at least one **measure of maximal entropy** (MME), i.e. a measure whose entropy is the same as the topological entropy of the subshift. A 1D subshift is **irreducible** if for every pair of balls $U, V$ there exists $j \in \mathbb{Z}$ such that $\sigma^j U \cap V \neq \emptyset$. A well known result of Shannon and Parry states that every irreducible 1D SFT admits a unique MME, and it always has full support [21]. The MME of a fullshift is the uniform Bernoulli measure.

We note that we are only discussing measures of maximal entropy with respect to the shift not to $\phi$.

**Theorem 3.13 (Coven-Paul [5])** Let $X$ be a 1D irreducible SFT with a unique MME, and $(X, \phi)$ a CA. Then $(X, \phi)$ is surjective if and only if it preserves the MME. In particular if $\phi : \mathbb{A}^\mathbb{N} \to \mathbb{A}^\mathbb{N}$ is a CA, then $(X, \phi)$ is surjective if and only if it preserves the uniform Bernoulli measure.

**Lemma 3.14** Let $(X, \phi)$ be CA. Assume that $\phi$ preserves a measure with full support. If $x$ is an equicontinuity point then $x$ is recurrent ($x \in R(\phi)$).

**Proof.** Let $x$ be an equicontinuity point and $m \in \mathbb{N}$. There exists $n \geq 2m$ such that $B_n(x) \subset O_{2m}(x)$. We have that $\phi$ preserves a fully supported measure. Using Poincare’s recurrence theorem we conclude there exists $j \in \mathbb{N}$ such that $\phi^j B_n(x) \cap B_n(x) \neq \emptyset$. This implies that $\phi^j x \subset B_m(x)$, and thus $x \in R(\phi)$. □

In [3] it was asked under which conditions the limit measure, under a 1D CA, of $\sigma$-ergodic measures that give full measure to equicontinuity points i.e. $\mu(EQ(\phi)) = 1$, is $\sigma$-ergodic, a measure of maximal entropy or $\phi$-ergodic. We address those questions.

**Theorem 3.15** Let $X$ be a 1D irreducible SFT, $(X, \phi)$ a surjective CA, and $\mu$ a $\sigma$-ergodic measure with $\mu(EQ(\phi)) = 1$. Then $\mu_\infty$ is $\sigma$-ergodic if and only if $\mu$ is $\phi$-invariant.

**Proof.** If $\mu = \phi \mu$, then $\mu_\infty = \mu$ is $\sigma$-ergodic.

Since $\phi$ is surjective by Shannon-Parry and Coven-Paul we obtain that $\phi$ preserves a fully supported measure. By Proposition 2.22 we have that $\mu(LEP(\phi)) = 1$. Using Lemma 3.14 and Lemma 2.11 we get that $\mu(LP(\phi)) = \mu(LEP(\phi) \cap R(\phi)) = 1$; and hence $(X, \phi)$ is $\mu = LP$. Using Theorem 3.12 we obtain $\mu = \phi \mu$. □

The proof of the following result is similar.

**Theorem 3.16** Let $X$ be a 1D irreducible SFT, $(X, \phi)$ a CA, and $\mu$ a $\sigma$-ergodic measure with $\mu(EQ(\phi)) = 1$. Then $\mu_\infty$ is the MME if and only if $\mu$ is the MME and $(X, \phi)$ is surjective.

**Proof.** If $\mu$ is the MME and $\mu = \phi \mu$, then $\mu_\infty = \mu$ is the MME.

Assume $\mu_\infty$ is the MME; hence it is $\sigma$-ergodic and has full support. By Proposition 2.22 we have that $\mu(LEP(\phi)) = 1$. Using Lemma 3.14 and Lemma 2.11 we get that $\mu(LP(\phi)) = \mu(LEP(\phi) \cap R(\phi)) = 1$; and hence $(X, \phi)$ is $\mu = LP$. Using Theorem 3.12 we obtain $\mu = \phi \mu$. □
we get that $\mu(LP(\phi)) = \mu(LEP(\phi) \cap R(\phi)) = 1$; and hence $(X, \phi)$ is $\mu-LP$. Using Theorem 3.12 we obtain $\mu = \phi \mu = \mu_\infty$. ■

There is interest in dynamical systems such that the orbit of the measure will converge in some sense to a measure of maximal entropy or an equilibrium markov measure (for example see Section 4.4 of [15]). It has been shown that the markov measures under 1D linear permutive CA converge in Cesaro sense to the measure of maximal entropy (e.g. [17] [22] [19]). Theorem 3.16 shows that if a measure $\mu$ is not the measure of maximal entropy and the equicontinuity points have full measure then the limit measure will not be the measure of maximal entropy.

Under some conditions these results hold for multidimensional subshfits.

One of the implications of the mentioned theorem by Coven-Paul has been generalized. Let $X$ be a subshift with a unique MME, and $(X, \phi)$ a CA. If $(X, \phi)$ is surjective then it preserves the MME (Theorem 3.3 in [20]). The proof of the following results are almost the same as the proofs for Theorems 3.16 and 3.15.

**Theorem 3.17** Let $X$ be an subshift with dense periodic points and a unique and fully supported MME, $(X, \phi)$ a surjective CA, and $\mu$ a $\sigma-$ergodic measure with $\mu(EQ(\phi)) = 1$. Then $\mu_\infty$ is $\sigma-$ergodic if and only if $\mu$ is $\phi-$invariant.

**Theorem 3.18** Let $X$ be an subshift with dense periodic points and a unique and fully supported MME, $(X, \phi)$ a CA, and $\mu$ a $\sigma-$ergodic measure with $\mu(EQ(\phi)) = 1$. Then $\mu_\infty$ is the MME if and only if $\mu$ is the MME and $\mu = \phi \mu$.

For $\mu-LPE$ systems we can show sufficient conditions for the $\sigma-$ergodicity of $\mu_\infty$.

**Lemma 3.19** Let $\mu$ be a $\phi-LPE$ measure. Then $\mu_\infty$ is $\sigma-$ergodic if and only if for every $x \in LP(\phi)$.

$$
\mu_\infty(A^O_m(x)) = \begin{cases} 
1 & \text{if } a = \mu_\infty(O_m(x)) \\
0 & \text{otherwise}
\end{cases}
$$

**Proof.** The $\Rightarrow$ implication is given by the pointwise ergodic theorem.

If the equation is satisfied then the pointwise ergodic theorem conclusion holds for all sets of the form $O_m(x)$ with $x \in LP_m(\phi)$. By Proposition 3.5 $\mu_\infty(LP(\phi)) = 1$. Since $\{O_m(x) \mid m \in \mathbb{N} \text{ and } x \in LP_m(\phi)\}$ generates the Borel sigma algebra (intersected with $LP(\phi)$), we conclude $\mu_\infty$ is $\sigma-$ergodic (see [25] pg.41.) ■

**Theorem 3.20** Let $\mu$ be a $\phi-LPE$, $\sigma-$ergodic measure. If for every orbit ball $O$, with $\mu_\infty(O) > 0$, there exists $N_O$ such that

$$
\phi^n \mu(O) = \mu_\infty(O) \text{ for } n \geq N_O,
$$

then $\mu_\infty$ is $\sigma-$ergodic.
Proof. Let \( m \in \mathbb{N}, x \in L^p_m(\phi) \) and \( 0 < q < p_m(x) \). From the proof of equation (1) of Lemma 3.4 one can see that for every \( 0 < q < p_m(x) \), \( \phi^{p_m(n+q)}(O_m(x)) \) is non-decreasing and converges (as \( n \to \infty \)). Using this and the hypothesis we have that there exists \( N \) such that

\[
\phi^{p_m(n+q)}(O_m(x)) = \mu(\mathcal{O}_m^n(x)) \text{ for } n \geq N.
\]

This implies

\[
\mu(O_m^q(x)) = \mu(\mathcal{O}_m(x))
\]

Using Lemma 3.10 we obtain

\[
\mu(\mathcal{O}_m^n(x)) = \frac{1}{p_m(x)} \sum_{r=0}^{p_m(x)-1} \mu(A_a^{O_m^n(x)}) \text{ for every } a \in \mathbb{R}.
\]

This implies

\[
\mu(\mathcal{O}_m^n(x)) = \mu(A_a^{O_m^n(x)}) \text{ for every } a \in \mathbb{R}.
\]

Using the \( \sigma \)-ergodicity of \( \mu \) we get

\[
\mu(A_a^{O_m^n(x)}) = \begin{cases} 1 & \text{if } a = \mu(O_m^q(x)) \\ 0 & \text{otherwise} \end{cases}
\]

Hence, we obtain

\[
\mu(\mathcal{O}_m^n(x)) = \begin{cases} 1 & \text{if } a = \mu(\mathcal{O}_m(x)) \\ 0 & \text{otherwise} \end{cases}
\]

Using the previous lemma we conclude that \( \mu \) is \( \sigma \)-ergodic. \( \blacksquare \)

We will now study measure preserving dynamical systems. We say \((M, T, \mu)\) is a measure preserving transformation if \((M, \mu)\) is a measure space, \(T : M \to M\) is measurable and \(T\mu = \mu\). When we say \(\mu\) is ergodic we also assume it is invariant under \(T\).

Two measure preserving transformations \((M_1, T_1, \mu_1)\) and \((M_2, T_2, \mu_2)\) are isomorphic (measurably) if there exists an invertible measure preserving transformation \(f : (X_1, \mu_1) \to (X_2, \mu_2)\), such that the inverse is measure preserving and \(T_2 \circ f = f \circ T_1\).

The spectral theory for dynamical systems (TDS and measure preserving transformations) is useful for studying rigid transformations. We will give the definitions and state the most important results. For more details and proofs see [25].

A measure preserving transformation \(T\) on a measure space \((M, \mu)\) generates a unitary linear operator on the Hilbert space \(L^2(M, \mu)\), by \(U_T : f \mapsto f \circ T\), known as the Koopman operator. The spectrum of the Koopman operator is called the spectrum of the measure preserving transformation. The spectrum is pure point or discrete if there exists an orthonormal basis for \(L^2(M, \mu)\) which
consists of eigenfunctions of the Koopman operator. The spectrum is \textbf{rational} if the eigenvalues are complex roots of unity. Classical results by Halmos and Von Neumann state that two ergodic measure preserving transformation with discrete spectrum have the same group of eigenvalues if and only if they are isomorphic, and that an ergodic measure preserving transformation has pure point spectrum if and only if it is isomorphic to a rotation on a compact metric group. The eigenfunctions of a rotation on a compact group are generated by the characters of the group. Discrete spectrum can be characterized for topological dynamical systems using a weak forms of \(\mu\)-equicontinuity \cite{10}.

**Example 3.21** Let \(S = (s_0, s_1, \ldots)\) be a finite or infinite sequence of integers larger or equal than 1. The \(S\)-adic odometer is the \(+\text{(1, 0, \ldots)}\) (with carrying) map defined on the compact set \(D = \prod_{i \geq 0} \mathbb{Z}_{s_i}\) (for a survey on odometers see \cite{7}).

These transformations are also called adding machines. An ergodic measure preserving transformation has discrete rational spectrum if and only if it is isomorphic to an odometer.

Any odometer can be embedded in a CA \cite{6}.

The following result was proved in \cite{9}.

**Proposition 3.22** Let \((X, \phi)\) be a CA and \(\mu\) a \(\phi\)-invariant measure. If \((X, \phi)\) is \(\mu\)–LEP then \((X, \phi, \mu)\) has discrete rational spectrum.

This implies that every ergodic \(\mu\)–LEP CA is isomorphic to an odometer.

We obtain a stronger result if the measure is \(\sigma\)-invariant.

**Proposition 3.23** Let \((X, \phi)\) be a \(\mu\)-ergodic, \(\mu\)-LEP CA. If \(\mu\) is \(\sigma\)-invariant then \((X, \phi, \mu)\) is isomorphic to a cyclic permutation on a finite set.

**Proof.** By Proposition 3.22 \(\mu(\text{LP} (\phi)) = 1\). Since \(\mu\)-LP CA are \(\mu\)-equicontinuous, there exists \(x \in \text{LP} (\phi)\) such that \(\mu(O_0(x)) > 0\). Let \(O_0^\infty\) be the \(\phi\)-orbit of \(O_0(x)\). We have that \(\mu(O_0^\infty) > 0\). Since \(\phi(O_0^\infty) = O_0^\infty\) and \(\mu\) is \(\phi\)-ergodic, then \(\mu(O_0^\infty) = 1\).

We have that \(p_0(O_0^\infty) = \{p_0(x)\}\). This implies the 0th column of almost every point is periodic with period \(p_0(x)\). Since \(\mu\) is \(\sigma\)-invariant we have that almost every point is \(\phi\)-periodic (with period \(p_0(x)\)). \(\blacksquare\)

Using this and other assumptions we can characterize when a limit measure of a \(\mu\)-equicontinuous CA is \(\phi\)-ergodic.

**Corollary 3.24** Let \(X\) be a 1D SFT, \(\mu\) be a \(\sigma\)-invariant measure, \((X, \phi)\) be a \(\mu\)-equicontinuous CA. We have that \((X, \phi, \mu_\infty)\) is isomorphic to a cyclic permutation on a finite set if and only if \(\mu_\infty\) is \(\phi\)-ergodic.

The following corollary combines the main results of this paper.
Corollary 3.25 Let $X$ be a 1D irreducible SFT, $(X, \phi)$ CA, and $\mu$ a shift-ergodic $\mu$–equicontinuous measure. Let $\mu_\infty$ be the weak limit of the Cesaro average of $\phi^n \mu$. We have that

- $\mu_\infty$ is the measure of maximal entropy if and only if $\mu$ is the measure of maximal entropy and $\phi \mu = \mu = \mu_\infty$.
- $\mu_\infty$ is $\phi$–ergodic if and only if $(X, \mu_\infty, \phi)$ is isomorphic (measurably) to a cyclic permutation on a finite set.

Furthermore if we assume $\phi$ is surjective then
- $\mu_\infty$ is $\sigma$–ergodic if and only if $\mu$ is $\phi$–invariant.

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