Energy Contents of a Class of Regular Black Hole Solutions in Teleparallel Gravity

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Abstract
In this paper, we discuss the energy-momentum problem in the realm of teleparallel gravity. The energy-momentum distribution for a class of regular black holes coupled with a non-linear electrodynamics source is investigated by using Hamiltonian approach of teleparallel theory. The generalized regular black hole contains two specific parameters $\alpha$ and $\beta$ (a sort of dipole and quadrupole of non-linear source) on which the energy distribution depends. It is interesting to mention here that our results exactly coincide with different energy-momentum prescriptions in General Relativity.

Keywords: Teleparallel Gravity; Energy-Momentum.
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1 Introduction
The notion of energy-momentum localization has been one of the interesting, attractive but controversial issue since the advent of General Relativity (GR). Many physicists have made painstaking efforts on this problematic issue but it is still an open problem. Einstein [1] himself established the first

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energy-momentum complex of gravitational field. After this superb work, Møller [2], Landau-Lifshitz [3], Papapetrou [4], Bergmann [5], Tolman [6], Weinberg [7] and Komar [8] provided definitions of energy-momentum. All these complexes, except Møller and Komar, gave reasonable results only in Cartesian coordinates. Due to coordinate dependence and non-uniqueness, physicists abandoned this subject for a long time.

Virbhadra [9] re-aminated the subject of energy-momentum and gave marvelous idea about coincidence of some complexes. He turned the attention of researchers in this subject. It was found [10]-[13] that several complexes provided the same meaningful expressions of energy-momentum for some well-known spacetimes. Aguirregabiria et al. [14] proved that Einstein, Landau-Lifshitz, Papapetrou, Tolman, Weinberg prescriptions yielded the same distribution of energy, momentum and angular momentum for any metric of Kerr-Schild class if calculations are performed in Kerr-Schild coordinates.

Later, Virbhadra [15] explored that these complexes provided same results for a more general class than the Kerr-Schild class. Penrose [16] also shared his ideas in this field and suggested the coordinate independent quasi-local mass. Bergqvist [17] applied seven different definitions of quasi-local mass to Reissner-Nordström and Kerr metrics and found inconsistent results. It has been pointed out by many people [18]-[20] that different prescriptions do not give the same results for stringy black holes. Sharif and his co-workers [21]-[23] found that different prescriptions did not provide same results for some well-known spacetimes.

It was argued [24, 25] that telleparallel theory equivalent to General Relativity (TEGR) might provide pointer to the localization of energy. Møller [26] was the pioneer of tetrad theory of gravitational field while Mikhail et al. [24] derived the energy-momentum complex in this theory. Vargas [27] defined Bergmann, Einstein and Landau-Lifshitz complexes in this alternative theory. Sharif and Amir [28] used these energy-momentum complexes and found that results did not coincide with GR for a given spacetime. Sharif and Nazir [29] concluded that energy-momentum turned out to be the same in TEGR and GR for Bell-Szekeres metric.

Andrade et al. [30]-[33] suggested that energy-momentum problem might be settled down in the Hamiltonian framework of TEGR. Maluf and his collaborators [34]-[35] accomplished gigantic amount of work through this approach and studied energy for some well-known spacetimes. Maluf et al. [36] gave the definitions of the gravitational energy, momentum and angular
momentum from the Hamiltonian formulation of TEGR \cite{37}. da Rocha-Neto and Castello-Branco \cite{38} computed gravitational energy for the Kerr and Kerr anti-de Sitter spacetimes. Recently, Sharif and Sumaira \cite{39} have used this procedure to obtain energy and its relevant quantities for some vacuum and non-vacuum spacetimes.

It has been found \cite{40}-\cite{42} that different prescriptions in GR yield the same results for a regular black hole and a class of regular black holes coupled with non-linear electrodynamics source. This paper is focussed to evaluate energy and its related quantities for a class of regular black holes using the Hamiltonian approach in TEGR. The layout of the paper is the following: Section 2 contains expressions of gravitational energy, momentum, angular momentum, gravitational and matter energy-momentum fluxes. In section 3, we evaluate energy and its contents for a class of regular black hole solutions. Section 4 provides discussion on the obtained results.

We shall use the following convention throughout the paper: Spacetime indices \((\mu, \nu, \rho, \ldots)\) and tangent space indices \((a, b, c, \ldots)\) run from 0 to 3. Time and space indices are represented as \(\mu = 0\), \(i\) and \(a = (0), (i)\) respectively.

## 2 Energy-Momentum in Teleparallel Theory using Hamiltonian Approach

The basic ingredient of teleparallel theory is the tetrad field \(e^a_\mu\) which is used to define Weitzenböck connection \cite{43}

\[
\Gamma^\lambda_{\mu\nu} = e^a_\lambda \partial_\nu e^a_\mu
\]

and the torsion tensor

\[
T^a_{\mu\nu} = \partial_\mu e^a_\nu - \partial_\nu e^a_\mu.
\]

The Lagrangian density for the gravitational field endowed with matter in TEGR is described as \cite{37}

\[
L \equiv -\kappa e \Sigma^{abc} T_{abc} - L_M,
\]

where \(\kappa = 1/16\pi\), \(e = \text{det}(e^a_\mu)\) and the anti-symmetric tensor \(\Sigma^{abc}\) on the right two indices is

\[
\Sigma^{abc} = \frac{1}{4}(T^{abc} + T^{bac} - T^{cab}) + \frac{1}{2}(\eta^{a(c}T^{b)} - \eta^{ab}T^{c}).
\]
The corresponding field equations are
\[ e_{a\lambda} e_{b\mu} \partial_{\nu} (e_{\Sigma^{b\lambda\nu}}) - e_{a\mu} T_{b\nu} - \frac{1}{4} e_{a\mu} T_{bcd} \Sigma^{b\lambda\nu} = \frac{1}{4\kappa} e T_{a\mu}, \]
\[ \delta L_M = e T_{a\mu}. \]
(5)

The total Hamiltonian density is
\[ H(e_{ai}, \Pi_{ai}) = e_{a0} C^a + \alpha_{ik} \Gamma^{ik} + \beta_k \Gamma^k + \partial_k (e_{a0} \Pi^{ak}), \]
(6)
where \( C^a \), \( \Gamma^{ik} \), \( \Gamma^k \) and \( \alpha_{ik} \), \( \beta_k \) represent primary constraints and Lagrangian multipliers respectively.

The gravitational energy-momentum over an arbitrary volume \( V \) is defined as
\[ P^a = -\int_V d^3x \partial_i \Pi^{ai}, \]
(7)
where
\[ -\partial_i \Pi^{ai} = \partial_l (4\kappa e \Sigma^{al}) \]
(8)
is the energy-momentum density [36]. The total angular momentum can be written as [45]
\[ M^{ik} = 2\kappa \int_V d^3x \left[-g^{im} g^{kj} T^0_{mj} + (g^{im} g^{0k} - g^{km} g^{0i}) T^j_{mj}\right]. \]
(9)

After some simple manipulations in the field equations (5), one can define the \( a \) component of the gravitational energy-momentum flux and matter energy-momentum flux [46] as
\[ \Phi_g^a = \int_S dS_j \phi_j^a, \quad \Phi_m^a = \int_S dS_j (ee^a_{\mu} T^j_{\mu}), \]
(10)

\( S \) represents the spatial boundary of the volume \( V \). Here the quantity \( \phi_j^a \) describe the \( a \) component of the gravitational energy-momentum flux density in \( j \) direction and its expression is
\[ \phi_j^a = \kappa ee^a_{\mu} (4\Sigma^{bcj} T_{bc\mu} - \delta^a_j \Sigma^{bedr} T_{bcd}). \]
(11)

3 A Class of Regular Black Hole Solutions

One of the burning issues of GR is the global regularity of black hole solutions. Bardeen [47] was the first who discovered astonishing model known as regular
black hole (also called Bardeen model). After this, some more singularity free models [48]-[50] were found also called Bardeen models [51]. These models are not exact solutions of the Einstein fields equations due to unavailability of appropriate physical source. In the mean time, Ayon-Beato and Garcia [52]-[54] found exact singularity free solutions by coupling the EFEs with non-linear electrodynamics. They [54] generalized a regular class of exact black hole solutions of the EFEs coupled with non-linear electrodynamics source [52]. This class of solutions can be converted into Maxwell theory under the restriction of weak field approximations which correspond to asymptotic Reissner-Nordström black hole. The generalized form of these solutions is given by the line element
\[ ds^2 = -F dt^2 + F^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \] (12)
where
\[ F = 1 - \frac{2M(r)}{r} \]
and the function \( M(r) \) for three different models [53, 54] is given by
\[ M_1(r) = \frac{mr^3 e^{-q^2/2mr}}{(r^2 + q^2)^{3/2}}, \] (13)
\[ M_2(r) = m(1 - \tanh \frac{q^2}{2mr}), \] (14)
\[ M_3(r) = \left( \frac{mr^\alpha}{(r^2 + q^2)^{\alpha/2}} - \frac{q^2 r^{\beta-1}}{2(r^2 + q^2)^{\beta/2}} \right). \] (15)
The associated electric field sources are given by
\[ \Xi_1 = \frac{qe^{-q^2/2mr}}{(r^2 + q^2)^{7/2}} \left( r^5 + \frac{(60m^2 - q^2)r^4}{8m} + \frac{q^2 r^3}{2} - \frac{q^4 r^2}{4m} - \frac{q^6}{8m} \right), \] (16)
\[ \Xi_2 = \frac{q}{4mr^3} \left( 1 - \tanh^2 \frac{q^2}{2mr} \right) \left( 4mr - q^2 \tanh \frac{q^2}{2mr} \right), \] (17)
\[ \Xi_3 = q \left( \frac{\alpha m [5r^2 - (\alpha - 3)q^2] r^{\alpha-1}}{2(r^2 + q^2)^{\alpha/2+2}} \right) \left( 4r^4 - (7\beta - 8)q^2 r^2 + (\beta - 1)(\beta - 4)q^4 r^{\beta-2} \right) \] (18)

Here \( m \) and \( q \) represent mass and electric charge respectively and the parameters \( \alpha, \beta \) indicate a sort of dipole and quadrupole moments, respectively, of

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non-linear source due to the presence of asymptotic behavior of electric field. For the choice of \( \alpha \geq 3, \beta \geq 4, q \leq 2s_c m \) \((s = |q|/2m \text{ and } s_c \text{ is the critical value})\), these solutions elaborate regular charged black holes and geometrically its global structure is same as Reissner-Nordström black hole. However, the disturbance occurs at essential singularity, \( r = 0 \), which is taken as origin of spherical coordinates. The corresponding asymptotic behavior of these solutions is given by

\[
\begin{align*}
F_1 &= 1 - \frac{2m}{r} + \frac{q^2}{r^2} + O\left(\frac{1}{r^3}\right), \\
F_2 &= 1 - \frac{2m}{r} + \frac{q^2}{r^2} + O\left(\frac{1}{r^4}\right), \\
F_3 &= 1 - \frac{2m}{r} + \frac{q^2}{r^2} + \alpha \frac{mq}{r^3} - \beta q^4 r^4 + O\left(\frac{1}{r^4}\right). \tag{19}
\end{align*}
\]

We can construct the tetrad components associated with (12) by adopting the procedure [28] as

\[
e^a_\mu(r, \theta, \phi) = \begin{pmatrix}
\sqrt{F} & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{F}} \cos \phi \sin \theta & r \cos \phi \cos \theta & -r \sin \phi \sin \theta \\
0 & \frac{1}{\sqrt{F}} \sin \phi \sin \theta & r \sin \phi \cos \theta & r \cos \phi \sin \theta \\
0 & \frac{1}{\sqrt{F}} \cos \theta & -r \sin \theta & 0
\end{pmatrix} \tag{20}
\]

with \( e = \text{det}(e^a_\mu) = r^2 \sin \theta \). The non-vanishing components of torsion tensor are

\[
\begin{align*}
T_{(0)01} &= \sqrt{F}, \quad T_{(1)12} = (1 - \frac{1}{\sqrt{F}}) \cos \theta \cos \phi, \\
T_{(1)13} &= -(1 - \frac{1}{\sqrt{F}}) \sin \theta \sin \phi, \quad T_{(2)12} = (1 - \frac{1}{\sqrt{F}}) \cos \theta \sin \phi, \\
T_{(2)13} &= (1 - \frac{1}{\sqrt{F}}) \sin \theta \cos \phi, \quad T_{(3)12} = -(1 - \frac{1}{\sqrt{F}}) \sin \theta \tag{21}
\end{align*}
\]

which yield the following non-zero components of the tensor \( T_{\lambda \mu \nu} = e^a_\lambda T_{a \mu \nu} \)

\[
\begin{align*}
T_{001} &= \sqrt{F} \sqrt{F}, \quad T_{212} = r(1 - \frac{1}{\sqrt{F}}), \quad T_{313} = r \sin^2 \theta (1 - \frac{1}{\sqrt{F}}), \tag{22}
\end{align*}
\]

where dot represents derivative with respect to radial component \( r \).
3.1 Energy, Momentum and Angular Momentum

The energy density corresponding to Eq.(12) can be obtained with the help of Eqs.(4) and (8)

\[- \partial_i \Pi^{(0)i} = 4\kappa r \partial_1 (\sin \theta (1 - \sqrt{F})). \tag{23}\]

Consequently, the energy will become

\[P^{(0)} = E = r [1 - \sqrt{F}], \tag{24}\]

\[E = r [1 - \sqrt{1 - 2 \frac{M(r)}{r}}]. \tag{24}\]

Using the binomial expansion with \(r \gg M(r)\), it follows that

\[E \approx M(r). \tag{25}\]

Inserting Eqs.(13)-(15) in the above equation, it follows

\[E_1 = \frac{me^{-q^2/2mr}}{(1 + \frac{q^2}{m^2})^{3/2}}, \tag{26}\]

\[E_2 = m(1 - \tanh(\frac{q^2}{2mr})), \tag{27}\]

\[E_3 = \frac{m}{(1 + \frac{q^2}{m^2})^{a/2}} - \frac{q^2}{2r(1 + \frac{q^2}{m^2})^{b/2}}. \tag{28}\]

Momentum and angular momentum turn out to be constant.

3.2 Energy-Momentum Flux

Here we evaluate energy-momentum flux. Since all the components of gravitational energy flux density \(\phi^{(0)j}\) vanish, hence the gravitational energy flux becomes constant, i.e., for \(a = 0\), we have \(\Phi^0_j = \text{constant}\). The momentum
flux density components are

\[
\begin{align*}
\phi^{(1)}_1 &= 2 \kappa \sin^2 \theta \cos \phi (\sqrt{F} (\sqrt{F} - 1)^2), \\
\phi^{(1)}_2 &= 2 \kappa \sin \theta \cos \theta \cos \phi (\sqrt{F} (\sqrt{F} - 1)), \\
\phi^{(1)}_3 &= -2 \kappa \sin \phi (\sqrt{F} (\sqrt{F} - 1)), \\
\phi^{(2)}_1 &= 2 \kappa \sin^2 \theta \sin \phi (\sqrt{F} (\sqrt{F} - 1)^2), \\
\phi^{(2)}_2 &= 2 \kappa \sin \theta \cos \theta \sin \phi (\sqrt{F} (\sqrt{F} - 1)), \\
\phi^{(2)}_3 &= 2 \kappa \cos \phi (\sqrt{F} (\sqrt{F} - 1)), \\
\phi^{(3)}_1 &= 2 \kappa \sin \theta \cos \theta (\sqrt{F} (\sqrt{F} - 1)^2), \\
\phi^{(3)}_2 &= -2 \kappa \sin^2 \theta (\sqrt{F} (\sqrt{F} - 1)), \\
\phi^{(3)}_3 &= 0.
\end{align*}
\] (29)

The momentum flux \( \Phi_g^{(i)} \) is obtained by replacing \( a = i \) in Eq. (10)

\[
\Phi^{(i)}_g = \int_S dS_j \phi^{ij}.
\] (30)

Inserting the values of momentum flux densities in the above expression for \( i = 1, 2, 3 \), we get the gravitational momentum flux

\[
\begin{align*}
\Phi^{(1)}_g &= -2 \kappa \pi \sin \phi \left( \frac{F}{2} - \sqrt{F} \right) + \text{const}, \\
\Phi^{(2)}_g &= 2 \kappa \pi \cos \phi \left( \frac{F}{2} - \sqrt{F} \right) + \text{const}, \\
\Phi^{(3)}_g &= -4 \kappa \pi \sin^2 \theta \left( \frac{F}{2} - \sqrt{F} \right) + \text{const}.
\end{align*}
\] (31)

These turn out to be constant for \( r \gg M(r) \), i.e.

\[
\frac{F}{2} - \sqrt{F} = (1/2 - M(r)/r) - \sqrt{(1 - 2M(r)/r)} \approx -\frac{1}{2}.
\]

giving rise to

\[
\begin{align*}
\Phi^{(1)}_g &= \kappa \pi \sin \phi + \text{const}, \\
\Phi^{(2)}_g &= -\kappa \pi \cos \phi + \text{const}, \\
\Phi^{(3)}_g &= 2 \kappa \pi \sin^2 \theta + \text{const}.
\end{align*}
\] (32)

Thus the components of momentum flux are free of parameters \( \alpha, \beta, m \) and \( q \) but depend upon spherical coordinates \( \theta \) and \( \phi \).
In order to evaluate matter energy-momentum flux, we have to calculate electromagnetic energy-momentum tensor. Its non-zero components are

\[
T^{00} = -\frac{\Xi^2}{8\pi F}, \quad T^{11} = \frac{F\Xi^2}{8\pi}, \quad T^{22} = -\frac{\Xi^2}{8\pi r^2}, \quad T^{33} = -\frac{\Xi^2}{8\pi r^2 \sin^2 \theta}.
\] (33)

The matter energy flux becomes constant while the components of matter momentum flux are

\[
\Phi_m^{(1)} = \frac{1}{8} \sin \phi \int (r\Xi^2)dr + \text{const}, \\
\Phi_m^{(2)} = -\frac{1}{8} \cos \phi \int (r\Xi^2)dr + \text{const}, \\
\Phi_m^{(3)} = \frac{1}{4} \sin^2 \theta \int (r\Xi^2)dr + \text{const}.
\] (34)

Here \(\Xi\) is the electric field related to each solution and hence matter momentum flux is different for each solution. The non-vanishing values of \(\Phi_g^a\) and \(\Phi_m^a\) represent the transfer of gravitational and matter energy-momentum respectively.

## 4 Summary and Discussion

In this paper, we have investigated gravitational energy and its related quantities such as momentum, angular momentum, gravitational and matter energy-momentum fluxes. These are found for a class of regular black hole solutions of the Einstein equation coupled with a non-linear electrodynamics source. For this purpose, we have used the Hamiltonian approach of TTEGR.

We note that Eq.\((28)\) gives a well-defined energy for all values of \(\alpha, \beta, q, r\) except for \(\alpha = 0 = \beta = r\). For \(\alpha = 3, \beta = 4, \) this yields the results of energy given in [41]. When \(\alpha = 0 = \beta, \) the energy distribution corresponds to the energy of Reissner-Nordström spacetime. For \(q = 0, \) this reduces to the ADM mass which also corresponds to the energy of Schwarzschild solution. The energy vanishes for \(\alpha \geq 3, \beta \geq 4, q \leq 2sm, r = 0.\)

It is interesting to mention here that our results for energy distribution \((26), (27)\) are exactly the same with those found [41, 55] by using Einstein and Bergmann prescriptions in GR. Equation (28) yields exactly the same energy found by Yang et al. [42] evaluated by using Einstein and Weinberg prescriptions in GR. The gravitational and matter energy flux vanish and
the components of momentum flux become independent of mass parameter which turn to be constant for particular values of $\theta$ and $\phi$. The constant gravitational momentum flux indicates that there exist a uniform distribution of matter or no matter in asymptotically flat region. We would like to mention here that this prescription is coordinate independent and best tool in the race of energy localization problem.

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