Agnostic Learnability of Halfspaces via Logistic Loss

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Abstract

We investigate approximation guarantees provided by logistic regression for the fundamental problem of agnostic learning of homogeneous halfspaces. Previously, for a certain broad class of “well-behaved” distributions on the examples, Diakonikolas et al. (2020d) proved an $\tilde{\Omega}(\OPT)$ lower bound, while Frei et al. (2021b) proved an $\tilde{O}(\sqrt{\OPT})$ upper bound, where $\OPT$ denotes the best zero-one/misclassification risk of a homogeneous halfspace. In this paper, we close this gap by constructing a well-behaved distribution such that the global minimizer of the logistic risk over this distribution only achieves $\Omega(\sqrt{\OPT})$ misclassification risk, matching the upper bound in (Frei et al., 2021b). On the other hand, we also show that if we impose a radial-Lipschitzness condition in addition to well-behaved-ness on the distribution, logistic regression on a ball of bounded radius reaches $O(\OPT)$ misclassification risk. Our techniques also show for any well-behaved distribution, regardless of radial Lipschitzness, we can overcome the $\Omega(\sqrt{\OPT})$ lower bound for logistic loss simply at the cost of one additional convex optimization step involving the hinge loss and attain $O(\OPT)$ misclassification risk. This two-step convex optimization algorithm is simpler than previous methods obtaining this guarantee, all of which require solving $O(\log(1/\OPT))$ minimization problems.

1 Introduction

In this paper, we consider the fundamental problem of agnostically learning homogeneous halfspaces. Specifically, we assume there is an unknown distribution $P$ over $\mathbb{R}^d \times \{-1, +1\}$ to which we have access in the form of independent and identically distributed samples drawn from $P$. Our goal is to compete with a homogeneous linear classifier $\bar{u}$ (i.e. one that predicts the label sign$(\langle \bar{u}, x \rangle)$ for input $x$) that achieves the optimal zero-one risk of $\OPT$ over $P$. Alternatively, we can think that the labels of the examples are first generated by $\bar{u}$, and then an OPT fraction of the labels are adversarially corrupted.

There have been many algorithmic and hardness results on this topic, see Section 1.1 for a discussion. A very natural heuristic for solving the problem is to use logistic regression. However, the analysis of logistic regression for this problem is still largely incomplete, even though it is one of the most fundamental algorithms in machine learning. One reason for this is that it can return extremely poor solutions in the worst case: Ben-David et al. (2012) showed that the minimizer of the logistic risk may attain a zero-one risk as bad as $1 - OPT$ on an adversarially-constructed distribution.

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As a result, much attention has been devoted to certain “well-behaved” distributions, for which much better results can be obtained. However, even when the marginal distribution on the feature space, $P_x$, is assumed to be isotropic log-concave, in recent work Diakonikolas et al. (2020d) proved an $\Omega(OPT)$ lower bound on the zero-one risk for any convex surrogate including logistic regression. On the positive side, in another recent work, Frei et al. (2021b) proved that vanilla gradient descent on the logistic risk can attain a zero-one risk of $O(\sqrt{OPT})$, as long as $P_x$ satisfies some specific well-behaved-ness conditions. (For the details of such conditions, see Sections 1.1 and 3.)

The above results still leave a big gap between the upper and the lower bounds, raising the question of identifying the fundamental limits of logistic regression for this problem. In this work we study this question and develop the following set of results.

A matching $\sqrt{OPT}$ lower bound. In Section 2 we construct a distribution $Q$ over $\mathbb{R}^2 \times \{-1, 1\}$, and prove a lower bound for logistic regression that matches the upper bound in Frei et al. (2021b), thereby closing the gap in recent works Diakonikolas et al. (2020d; Frei et al. 2021b). Specifically, the marginal distribution $Q_x$ is isotropic and bounded, and satisfies all the well-behaved-ness conditions from the aforementioned papers, but the global minimizer of the logistic risk on $Q$ only attains $\Omega(\sqrt{OPT})$ zero-one risk on $Q$.

An $O(OPT)$ upper bound for radially Lipschitz densities. The lower bound mentioned above shows that one needs to make additional assumptions to prove better bounds. In Section 3 we show that by making a radial Lipschitzness assumption in addition to well-behaved-ness, it is indeed possible to achieve the near-optimal $O(OPT)$ zero-one risk via logistic regression. In particular, our upper bound result holds if the projection of $P_x$ onto any two-dimensional subspace has Lipschitz continuous densities. Moreover, our upper bound analysis is versatile: it can recover the $O(\sqrt{OPT})$ guarantee for general well-behaved distributions shown by Frei et al. (2021b), and it also works for the hinge loss, which motivates a simple and efficient two-phase algorithm, as described next.

An $O(OPT)$ upper bound for general well-behaved distributions with a two-phase algorithm. Motivated by our analysis, in Section 4 we describe a simple two-phase algorithm that achieves $O(OPT)$ error for general well-behaved distributions, without assuming radial Lipschitzness. Thus, we show that the cost of avoiding the radial Lipschitzness condition is simply an additional convex loss minimization. Our two-phase algorithm involves logistic regression followed by stochastic gradient descent with the hinge error for general well-behaved distributions, the first phase can only achieve an $O(\sqrt{OPT})$ guarantee, however we show that the second phase can boost the upper bound to $O(OPT)$.

Previously, for any given $\epsilon > 0$, Diakonikolas et al. (2020d) designed a nonconvex optimization algorithm that can achieve $O(OPT + \epsilon)$ risk using $O(d/\epsilon^4)$ samples. Their algorithm requires guessing $OPT$ within a constant multiplicative factor via binary search and running a nonconvex SGD using each guess as an input. Similarly, prior algorithms achieving $O(OPT + \epsilon)$ error involve solving multiple rounds of convex loss minimization (Awasthi et al., 2014; Daniely, 2015). By contrast, our two-phase algorithm is a simple logistic regression followed by a perceptron algorithm, and the output is guaranteed to have $O(OPT \cdot \ln(1/OPT) + \epsilon)$ zero-one risk using only $O(d/\epsilon^2)$ samples.

1.1 Related work

The problem of agnostic learning of halfspaces has a long and rich history Kearns et al. (1994). Here we survey the results most relevant to our work. It is well known that in the distribution independent setting, even weak agnostic learning is computationally hard Feldman et al. 2006; Guruswami and Raghavendra, 2009; Daniely, 2016). As a result most algorithmic results have been obtained under assumptions on the marginal distribution $P_x$ over the examples.

The work of Kalai et al. (2008) designed algorithms that achieve $OPT + \epsilon$ error for any $\epsilon > 0$ in time $d^{poly}(\epsilon)$ for isotropic log-concave densities and for the uniform distribution over the hypercube. There is also recent evidence that removing the exponential dependence on $1/\epsilon$, even for Gaussian marginals is computationally hard Klivans and Kothari, 2014; Diakonikolas et al. 2020a; Goel et al., 2020).
As a result, another line of work aims to design algorithms with polynomial running time and sample complexity (in $d$ and $\frac{1}{\epsilon}$) and achieve an error of $g(OPT) + \epsilon$, for $g$ being a simple function. Along these lines, Klivans et al. (2009) designed a polynomial-time algorithm that attains $\tilde{O}(OPT^{1/3}) + \epsilon$ zero-one risk for isotropic log-concave distributions. Awasthi et al. (2014) improved the upper bound to $O(OPT) + \epsilon$, using a localization-based algorithm. Balcan and Zhang (2017) further extended the algorithm to more general $s$-concave distributions. The work of Daniely (2015) further provided a PTAS guarantee: an error of $(1 + \eta)OPT + \epsilon$ for any desired constant $\eta > 0$ via an improper learner.

In a recent work, Diakonikolas et al. (2020d) studied the problem for distributions satisfying certain “well-behaved-ness” conditions which include isotropy and certain regularity conditions on the projection of $P_x$ on any 2-dimensional subspace (see Assumption 3.2 for a subset of these conditions). This class of distributions include any isotropic log-concave distribution such as the standard Gaussian. In addition to their nonconvex optimization method discussed above, for any convex, nonincreasing, and nonconstant loss function, they also showed an $\Omega(OPT \ln(1/OPT))$ lower bound for log-concave marginals and an $\Omega(OPT^{1-1/s})$ lower bound for $s$-heavy-tailed marginals.

In another recent work, Frei et al. (2021a) assumed $P_x$ satisfies a “soft-margin” condition: for anti-concentrated marginals such as isotropic log-concave marginals, this assumes $\Pr\left(\langle u, x \rangle \leq \gamma \right) = O(\gamma)$ for any $\gamma > 0$. For sub-exponential distributions with soft-margins, they proved an $O(\sqrt{OPT})$ upper bound for gradient descent on the logistic loss, which can be improved to $O(\sqrt{OPT})$ for bounded distributions. Note that these upper bounds and the lower bounds in (Diakonikolas et al., 2020d) do not match: if $P_x$ is sub-exponential, then Diakonikolas et al. (2020d) only gave an $\Omega(OPT)$ lower bound, while if $P_x$ is $s$-heavy-tailed, then the upper bound in (Frei et al., 2021a) becomes worse.

Finally, some prior works on agnostic learning of halfspaces have considered various extensions of the problem such as active agnostic learning (Awasthi et al., 2014; Yan and Zhang, 2017), agnostic learning of sparse halfspaces with sample complexity scaling logarithmically in the ambient dimensionality (Shen and Zhang, 2021), and agnostic learning under weaker noise models such as the random classification noise (Blum et al., 1998; Dunagan and Vempala, 2008), Massart’s noise model (Awasthi et al., 2015, 2016; Zhang et al., 2020; Diakonikolas et al., 2019, 2020b, 2021; Chen et al., 2020) and the Tsybakov noise model (Diakonikolas et al., 2020c; Zhang and Li, 2021). We do not consider these extensions in our work.

### 1.2 Notation

Let $\| \cdot \|$ denote the $\ell_2$ (Euclidean) norm. Given $r > 0$, let $B(r) := \{ x ||x|| \leq r \}$ denote the Euclidean ball with radius $r$. Given two nonzero vectors $u$ and $v$, let $\varphi(u, v) \in [0, \pi]$ denote the angle between them.

Given a data distribution $P$ over $\mathbb{R}^d \times \{-1, +1\}$, let $P_x$ denote the marginal distribution of $P$ on the feature space $\mathbb{R}^d$. We will frequently need the projection of the input features onto a two-dimensional subspace $V$; in such cases, it will be convenient to use polar coordinates $(r, \theta)$ for the associated calculations, such as parameterizing the density with respect to the Lebesgue measure as $p_V(r, \theta)$.

Given a nonincreasing loss function $\ell : \mathbb{R} \rightarrow \mathbb{R}$, we consider the population risk

$$R_\ell(w) := E_{(x,y) \sim P} \left[ \ell \left( y \langle w, x \rangle \right) \right],$$

and the corresponding empirical risk

$$\hat{R}_\ell(w) := \frac{1}{n} \sum_{i=1}^{n} \ell \left( y_i \langle w, x_i \rangle \right),$$

defined over $n$ i.i.d. samples drawn from $P$. We will focus on the logistic loss $\ell_{log}(z) := \ln(1 + e^{-z})$, and the hinge loss $\ell_h(z) := \max\{-z, 0\}$. Let $R_{log} := R_{\ell_{log}}$ for simplicity, and also define $\hat{R}_{log}$, $R_h$ and $\hat{R}_h$ similarly. Let $R_{0-1}(w) := \Pr_{(x,y) \sim P} (y \neq \text{sign} \left( \langle w, x \rangle \right))$ denote the population zero-one risk.

### 2 An $\Omega(\sqrt{OPT})$ lower bound for logistic loss

In this section, we construct a distribution $Q$ over $\mathbb{R}^2 \times \{-1, +1\}$ which satisfies standard regularity conditions in (Diakonikolas et al., 2020c; Frei et al., 2021a), but the global minimizer $w^*$ of the population logistic risk
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\[ R_{\log} \] on \( Q \) only achieves a zero-one risk of \( \Omega \left( \sqrt{\OPT} \right) \). Our focus on the global logistic optimizer is motivated by the lower bounds from \cite{DiakonikolasF19}; in particular, this means that the large classification error is not caused by the sampling error.

The distribution \( Q \) has four parts \( Q_1, Q_2, Q_3, \) and \( Q_4, \) as described below. It can be verified that if \( \OPT \leq 1/16, \) the construction is valid.

1. The feature distribution of \( Q_1 \) consists of two squares: one has edge length \( \sqrt{\OPT} / 2 \), center \( \left( \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right) \) and density 1, with label \(-1\); the other has edge length \( \sqrt{\OPT} / 2 \), center \( \left( -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right) \), density 1, with label \(+1\).

2. The feature distribution of \( Q_2 \) is supported on

\[
\left( \left[ 0, \sqrt{\OPT} \right] \times [0, 1] \right) \cup \left( \left[ -\sqrt{\OPT}, 0 \right] \times [-1, 0] \right)
\]

with density 1, and the label is given by \( \sign(x_1) \).

3. Let \( q_3 := \sqrt{\OPT}(1 - \OPT) \), then \( Q_3 \) consists of two squares: one has edge length \( \sqrt{\OPT} / 2 \), center \( (1, 0) \), density 1 and label \(+1\), and the other has edge length \( \sqrt{\OPT} / 2 \), center \( (-1, 0) \), density 1 and label \(-1\).

4. The feature distribution of \( Q_4 \) is the uniform distribution over the unit ball \( B(1) := \{ x \mid \|x\| \leq 1 \} \) with density \( q_4 := \frac{1 - \OPT - 2\sqrt{\OPT} - q_3}{16 \OPT} \), and the label is given by \( \sign(x_1) \).

Note that the correct label is given by \( \sign(x_1) \) on \( Q_2, Q_3 \) and \( Q_4 \); therefore \( \bar{u} := (1, 0) \) is our ground-truth solution that is only wrong on the noisy part \( Q_1 \).

Here is our lower bound result.

**Theorem 2.1.** Suppose \( \OPT \leq 1/100 \), and let \( Q_x \) denote the marginal distribution of \( Q \) on the feature space. It holds that \( \mathbb{E}_{x \sim Q_1}[x] = 0, \) and \( \mathbb{E}_{x \sim Q_2}[x_1 x_2] = 0, \) and \( \mathbb{E}_{x \sim Q_3}[x_1^2 - x_2^2] = 0. \) Moreover, the population logistic risk \( R_{\log} \) has a global minimizer \( w^* \), and

\[
R_{0-1}(w^*) = \Pr \left( y \neq \sign \left( (w^*, x) \right) \right) \geq \frac{\sqrt{\OPT}}{60 \pi}.
\]

Note that we can further normalize \( Q_x \) to unit variance and make it isotropic. Then it is easy to check that \( Q_x \) satisfies the “well-behaved-ness” conditions in \cite{DiakonikolasF19}, and the “soft-margin” and “sub-exponential” conditions in \cite{FreiF19}. In particular, our lower bound matches the upper bound in \cite{FreiF19}.

### 2.1 Proof of Theorem 2.1

Here is a proof sketch of Theorem 2.1, the full proof is given in Appendix B.

First, basic calculation shows that \( Q \) is isotropic up to a constant multiplicative factor, and that \( Q \) has bounded density and support. Specifically, \( Q_1, Q_2, \) and \( Q_4 \) are constructed to make the risk lower bound proof work, while \( Q_3 \) is included to make \( Q \) isotropic. It turns out that \( Q_3 \) does not change the risk lower bound proof too much, since it is highly aligned with the ground-truth solution \( \bar{u} := (1, 0). \)

Next we consider the risk lower bound. We only need to show that \( \varphi(\bar{u}, w^*) \), the angle between \( \bar{u} \) and \( w^* \), is \( \Omega \left( \sqrt{\OPT} \right) \), since it then follows that \( w^* \) is wrong on an \( \Omega \left( \sqrt{\OPT} \right) \) fraction of \( Q_4 \), which is enough since \( Q_4 \) accounts for more than half of the distribution \( Q \).

Note that the minimizer of the logistic risk on \( Q_4 \) by itself is infinitely far in the direction of \( \bar{u} \). However, this will incur a large risk on \( Q_1 \). By balancing these two parts, we can show that by moving along the direction of \( \bar{u} \) by a distance of \( \Theta \left( \frac{1}{\sqrt{\OPT}} \right) \), we can achieve a logistic risk of \( O \left( \sqrt{\OPT} \right) \).
Lemma 2.2. Suppose \( \text{OPT} \leq 1/100 \), let \( \tilde{w} := (\tilde{r}, 0) \) where \( \tilde{r} = \frac{3}{\sqrt{\text{OPT}}} \), then \( R_{\text{log}}(\tilde{w}) \leq 5\sqrt{\text{OPT}} \).

Next we consider the global minimizer \( w^* \) of \( R_{\text{log}} \), which exists since \( R_{\text{log}} \) has bounded sub-level sets. Let \((r^*, \theta^*)\) denote the polar coordinates of \( w^* \). We will assume \( \theta^* \in \left[-\frac{\sqrt{\text{OPT}}}{30}, \frac{\sqrt{\text{OPT}}}{30}\right] \), and derive a contradiction. In our construction, \( Q_3 \) and \( Q_4 \) are symmetric with respect to the horizontal axis, and they will induce the ground-truth solution. However, \( Q_1 \) and \( Q_2 \) are skew, and they will pull \( w^* \) above, meaning we actually have \( \theta^* \in \left[0, \frac{\sqrt{\text{OPT}}}{30}\right] \). The first observation is an upper bound on \( r^* \): if \( r^* \) is too large, then the risk of \( w^* \) over \( Q_1 \) will already be larger than \( R_{\text{log}}(\tilde{w}) \) for \( \tilde{w} \) constructed in Lemma 2.2, a contradiction.

Lemma 2.3. Suppose \( \text{OPT} \leq 1/100 \) and \( \theta^* \in \left[0, \frac{\sqrt{\text{OPT}}}{30}\right] \), then \( r^* \leq \frac{10}{\sqrt{\text{OPT}}} \).

However, our next lemma shows that under the above conditions, the gradient of \( R_{\text{log}} \) at \( w^* \) does not vanish, which contradicts the definition of \( w^* \).

Lemma 2.4. Suppose \( \text{OPT} \leq 1/100 \), then for any \( w = (r, \theta) \) with \( 0 \leq r \leq \frac{10}{\sqrt{\text{OPT}}} \) and \( 0 \leq \theta \leq \frac{\sqrt{\text{OPT}}}{30} \), it holds that \( \nabla R_{\text{log}}(w) \neq 0 \).

To prove Lemma 2.4, let us consider an arbitrary \( w = (r, \theta) \) under the conditions of Lemma 2.4. For simplicity, let us first look at the case \( \theta = 0 \). In this case, \( y(w, x) \leq 10 \) on \( Q_2 \), and it follows that \( Q_2 \) induces a component of length \( \frac{C_1}{\sqrt{\text{OPT}}} \) in the gradient \( \nabla R_{\text{log}}(w) \) along the direction of \(-e_2 = (0, -1)\), where \( C_1 \) is a universal constant. Moreover, \( Q_1 \) also induces a component in the gradient along \(-e_2\), while \( Q_3 \) and \( Q_4 \) induce a zero component along \( e_2 \). As a result, \( \langle \nabla R_{\text{log}}(w), e_2 \rangle < 0 \). Now if \( 0 \leq \theta \leq C_2 \sqrt{\text{OPT}} \) for some small enough constant \( C_2 \) (1/30 in our case), we can show that \( Q_3 \) and \( Q_4 \) cannot cancel the effect of \( Q_2 \), and it still holds that \( \langle \nabla R_{\text{log}}(w), e_2 \rangle < 0 \).

3 An \( \widetilde{O}(\text{OPT}) \) upper bound for logistic loss with radial Lipschitz-ness

The \( \Omega(\sqrt{\text{OPT}}) \) lower bound construction in Section 2 shows that further assumptions on the distribution are necessary in order to improve the upper bound on the zero-one risk of the logistic regression solution. In particular, we note that the distribution \( Q \) constructed in Section 2 has a discontinuous density. In this section, we show that if we simply add a very mild Lipschitz continuity condition on the density, then we can achieve \( \widetilde{O}(\text{OPT}) \) zero-one risk using logistic regression.

First, we formally provide the standard assumptions from prior work. Because of the lower bound for \( s \)-heavy-tailed distributions from [Diakonikolas et al., 2020d], to get an \( \widetilde{O}(\text{OPT}) \) zero-one risk, we need to assume \( P_x \) has a light tail. Following [Frei et al., 2021b], we will either consider a bounded distribution, or assume \( P_x \) is sub-exponential as defined below (cf. [Vershynin, 2018, Proposition 2.7.1 and Section 3.4.4]).

Definition 3.1. We say \( P_x \) is \((\alpha_1, \alpha_2)\) sub-exponential for constants \( \alpha_1, \alpha_2 > 0 \), if for any unit vector \( v \) and any \( t > 0 \),

\[
\Pr_{x \sim P_x} \left(|\langle v, x \rangle| \geq t \right) \leq \alpha_1 \exp(-t/\alpha_2).
\]

We also need the next assumption, which is part of the “well-behaved-ness” conditions from [Diakonikolas et al., 2020d].

Assumption 3.2. There exist constants \( U, R > 0 \) and a function \( \sigma : \mathbb{R}_+ \to \mathbb{R}_+ \), such that if we project \( P_x \) onto an arbitrary two-dimensional subspace \( V \), the corresponding density \( p_V \) satisfies \( p_V(r, \theta) \geq 1/U \) for all \( r \leq R \), and \( p_V(r, \theta) \leq \sigma(r) \) for all \( r \geq 0 \), and \( \int_0^\infty \sigma(r) \, dr \leq U \), and \( \int_0^\infty r \sigma(r) \, dr \leq U \).

Note that for a broad class of distributions including isotropic log-concave distributions, the sub-exponential condition and Assumption 3.2 hold with \( \alpha_1, \alpha_2, U, R \) all being universal constants.

Finally, as discussed earlier, the previous conditions are also satisfied by \( Q \) from Section 2 and thus to get the improved \( \widetilde{O}(\text{OPT}) \) risk bound, we need the following radial Lipschitz continuity assumption.
Assumption 3.3. There exists a measurable function $\kappa : \mathbb{R}^+ \to \mathbb{R}^+$ such that for any two-dimensional subspace $V$,

$$|p_V(r, \theta) - p_V(r, \theta')| \leq \kappa(r)|\theta - \theta'|.$$ 

We will see Assumption 3.3 is crucial for the upper bound analysis in Lemma 3.12. For some concrete examples, note that if $p_V$ is radially symmetric, then $\kappa$ is 0, while if $p_V$ is $\lambda$-Lipschitz continuous in the usual sense (under $\ell_2$), then we can let $\kappa(r) := \lambda r$.

Now we can state our main results. In the following, we denote the unit linear classifier with the optimal zero-one risk by $\bar{u}$, with $R_{0-1}(\bar{u}) = \text{OPT} \in (0, 1/e)$. Our first result shows that, with Assumption 3.3 minimizing the logistic risk yields a solution with continuous densities, in which case $R$ under Assumptions 3.2 and 3.3, let $\underbar{u}$ minimize the logistic risk yields a solution with continuous densities, in which case $R$.

Theorem 3.4. Under Assumptions 3.2 and 3.3 let $w^*$ denote the global minimizer of $R_{\log}$.

1. If $\|x\| \leq B$ almost surely, then

$$R_{0-1}(w^*) = O((1 + \kappa)\text{OPT}),$$

where $C_\kappa := \int_0^B \kappa(r) \, dr$.

2. If $P_x$ is $(\alpha_1, \alpha_2)$-sub-exponential, then

$$R_{0-1}(w^*) = O((1 + \kappa)\text{OPT} \cdot \ln(1/\text{OPT})), $$

where $C_\kappa := \int_0^{3\alpha_2 \ln(1/\text{OPT})} \kappa(r) \, dr$.

Note that for bounded distributions, $R_{0-1}(w^*) = O(\text{OPT})$ as long as $C_\kappa$ is a constant, which is true if $p_V$ is Lipschitz continuous. Similarly, for sub-exponential distributions, $C_\kappa$ is still $O(\ln(1/\text{OPT})^2)$ for Lipschitz continuous densities, in which case $R_{0-1}(w^*) = \tilde{O}(\text{OPT})$.

Next we give an algorithmic result. Given a target error $\epsilon \in (0, 1)$, we consider projected gradient descent on the empirical risk with a norm bound of $1/\sqrt{\epsilon}$: let $w_0 := 0$, and

$$w_{t+1} := \Pi_{\|r\| \leq \epsilon} \left[ w_t - \eta \nabla \hat{R}_{\log}(w_t) \right].$$

(1)

Our next result shows that projected gradient descent can give an $\tilde{O}(\text{OPT})$ risk. Note that for the two cases discussed below (bounded or sub-exponential), we use the corresponding $C_\kappa$ defined in Theorem 3.4.

Theorem 3.5. Suppose Assumptions 3.2 and 3.3 hold.

1. If $\|x\| \leq B$ almost surely, then with $\eta = 4/B^2$, and $O\left(\frac{\ln(1/\epsilon^2)}{\epsilon^2}\right)$ samples and $O\left(\frac{1}{\epsilon^2}\right)$ iterations, with probability $1 - \delta$, projected gradient descent outputs $w_t$ satisfying

$$R_{0-1}(w_t) = O\left((1 + C_\kappa)\text{(OPT + } \epsilon)\right).$$

2. On the other hand, if $P_x$ is $(\alpha_1, \alpha_2)$-sub-exponential, then with $\eta = \bar{\Theta}(1/d)$, using $\tilde{O}\left(\frac{d\ln(1/\delta)^2}{\delta \epsilon^2}\right)$ samples and $\tilde{O}\left(\frac{d\ln(1/\delta)^2}{\epsilon^2}\right)$ iterations, with probability $1 - \delta$, projected gradient descent outputs $w_t$ with

$$R_{0-1}(w_t) = \left((1 + C_\kappa)\right(\text{OPT} \cdot \ln(1/\text{OPT}) + \epsilon)\right).$$

Theorems 3.4 and 3.5 rely on the following key lemma which provides a zero-one risk bound on near optimal solutions to the logistic regression problem:

Lemma 3.6. Under Assumptions 3.2 and 3.3 suppose $\hat{w}$ satisfies $R_{\log}(\hat{w}) \leq R_{\log}(\|\hat{w}\|) + \epsilon_\ell$ for some $\epsilon_\ell \in [0, 1)$. 

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1. If \( \|x\| \leq B \) almost surely, then

\[
R_{0-1}(\hat{w}) = O \left( \max \left\{ \text{OPT}, \sqrt{\frac{C}{\|\hat{w}\|^2}}, \sqrt{\frac{C}{\|\hat{w}\|}} \right\} \right).
\]

2. If \( P_x \) is \((\alpha_1, \alpha_2)\)-sub-exponential and \( \|\hat{w}\| = \Omega(1) \), then

\[
R_{0-1}(\hat{w}) = O \left( \max \left\{ \text{OPT} \cdot \ln(1/\text{OPT}), \sqrt{\frac{\epsilon}{\|\hat{w}\|}}, \sqrt{\frac{C}{\|\hat{w}\|}} \right\} \right).
\]

Next we give proof outlines of our results; the full proofs are given in Appendix C. For simplicity, here we focus only on the bounded case, while the sub-exponential case will be handled in Appendix C. Although the proofs of the two cases share some similarity, we want to emphasize that the sub-exponential case does not follow by simply truncating the distribution to a certain radius and reduce it to the bounded case. The reason is that the truncation radius will be as large as \( \sqrt{d} \), while for the bounded case in Lemma 3.6, \( B \) is considered a constant independent of \( d \) and hidden in the \( O \) notation; therefore this truncation argument will introduce a \( \text{poly}(d) \) dependency in the final bound. By contrast, our zero-one risk upper bounds only depend on \( \alpha_1, \alpha_2, U \) and \( R \), but do not depend on \( d \).

3.1 Proof of Theorems 3.4 and 3.5

Here we prove Theorems 3.4 and 3.5; the details are given in Appendix C.1.

We first prove Theorem 3.4. Note that by Lemma 3.6, it suffices to show that \( \|w^*\| = \Omega \left( \frac{1}{\sqrt{\text{OPT}}} \right) \) (since \( \epsilon \ell = 0 \) in this case), which is true due to the next result.

**Lemma 3.7.** Under Assumption 3.2, if \( \|x\| \leq B \) almost surely and \( \text{OPT} < \frac{R^4}{200U^2B} \), then \( \|w^*\| = \Omega \left( \frac{1}{\sqrt{\text{OPT}}} \right) \).

Next we prove Theorem 3.5. Recall that given the target (zero-one) error \( \epsilon \in (0, 1) \), we run projected gradient descent on a Euclidean ball with radius \( \frac{1}{\sqrt{\epsilon}} \) (cf. eq. (1)). Using a standard optimization and generalization analysis, we can prove the following guarantee on \( R_{\log}(w_t) \).

**Lemma 3.8.** Let the target optimization error \( \epsilon \ell \in (0, 1) \) and the failure probability \( \delta \in (0, 1/e) \) be given. If \( \|x\| \leq B \) almost surely, then with \( \eta = \frac{4}{B^2} \), using \( O \left( \frac{B^2 \ln(1/\delta)}{\epsilon \ell} \right) \) samples and \( O \left( \frac{B^2 \ln(1/\delta)}{\epsilon \ell} \right) \) iterations, with probability \( 1 - \delta \), projected gradient descent outputs \( w_t \) satisfying

\[
R_{\log}(w_t) \leq \min_{0 \leq \rho \leq 1/\sqrt{\epsilon}} R_{\log}(\rho \hat{\mu}) + \epsilon \ell. \quad (2)
\]

We also need the following lower bounds on \( \|w_t\| \).

**Lemma 3.9.** Under Assumption 3.2, suppose

\[
\epsilon < \min \left\{ \frac{R^4}{36U^2}, \frac{R^4}{72^2U^4} \right\} \quad \text{and} \quad \epsilon \ell \leq \sqrt{\epsilon},
\]

and that eq. (2) holds. If \( \|x\| \leq B \) almost surely and \( \text{OPT} < \frac{R^4}{5000U^2B} \), then \( \|w_t\| = \Omega \left( \min \left\{ \frac{1}{\sqrt{\epsilon}}, \frac{1}{\sqrt{\text{OPT}}} \right\} \right) \).

Now to prove Theorem 3.5 we simply need to combine Lemmas 3.6, 3.8 and 3.9 with \( \epsilon \ell = \epsilon^{3/2} \).

3.2 Proof of Lemma 3.6

Here we give a proof sketch of Lemma 3.6; the details are given in Appendix C.2. One remark is that some of the lemmas in the proof are also true for the hinge loss, and this fact will be crucial in the later discussion regarding our two-phase algorithm (cf. Section 4).
Let \( \hat{w} := ||\hat{w}||U \), and consider \( \ell \in \{ \ell_{\log}, \ell_h \} \). The first step is to express \( \mathcal{R}_\ell(\hat{w}) - \mathcal{R}_\ell(\bar{w}) \) as the sum of three terms, and then bound them separately. The first term is given by

\[
\mathcal{R}_\ell(\hat{w}) - \mathcal{R}_\ell(\bar{w}) - \mathbb{E} \left[ \ell \left( \text{sign} \left( \langle \hat{w}, x \rangle \right) \langle \hat{w}, x \rangle \right) - \ell \left( \text{sign} \left( \langle \bar{w}, x \rangle \right) \langle \bar{w}, x \rangle \right) \right],
\]

the second term is given by

\[
\mathbb{E} \left[ \ell \left( \text{sign} \left( \langle \hat{w}, x \rangle \right) \langle \hat{w}, x \rangle \right) - \ell \left( \text{sign} \left( \langle \bar{w}, x \rangle \right) \langle \bar{w}, x \rangle \right) \right],
\]

and the third term is given by

\[
\mathbb{E} \left[ \ell \left( \text{sign} \left( \langle \hat{w}, x \rangle \right) \langle \hat{w}, x \rangle \right) - \ell \left( \text{sign} \left( \langle \bar{w}, x \rangle \right) \langle \bar{w}, x \rangle \right) \right],
\]

where the expectations are taken over \( P_z \).

We first bound term (3), which is the approximation error of replacing the true label \( y \) with the label given by \( \bar{u} \). Since \( \ell(-z) - \ell(z) = z \) for the logistic loss and hinge loss, it follows that

\[
\text{term } (3) = \mathbb{E} \left[ 1_{y \neq \text{sign}(\bar{u}, x)} \cdot y(\bar{w} - \hat{w}, x) \right].
\]

The approximation error can be bounded as below, using the tail bound on \( P_z \) and the fact \( \mathcal{R}_{0-1}(\hat{w}) = \text{OPT} \).

**Lemma 3.10.** For \( \ell \in \{ \ell_{\log}, \ell_h \} \), if \( ||x|| \leq B \) almost surely,

\[
|\text{term } (3)| \leq B||\hat{w} - \bar{w}|| \cdot \text{OPT}.
\]

Next we bound term (4).

**Lemma 3.11.** Under Assumption 3.2, for \( \ell \in \{ \ell_{\log}, \ell_h \} \),

\[
\text{term } (4) \geq \frac{4R^3}{3U^2} ||\hat{w}|| \varphi(\hat{w}, \bar{w})^2.
\]

Lastly, we consider term (5). Note that it is 0 for the hinge loss \( \ell_h \), because \( \ell_h(z) = 0 \) when \( z \geq 0 \). For the logistic loss, term (5) is also 0 if \( P_z \) is radially symmetric; in general, we will bound it using Assumption 3.3.

**Lemma 3.12.** For \( \ell = \ell_h \), term (5) is 0. For \( \ell = \ell_{\log} \), under Assumption 3.3, if \( ||x|| \leq B \) almost surely, then

\[
|\text{term } (5)| \leq 12C_\kappa \cdot \varphi(\hat{w}, \bar{w})/||\hat{w}||,
\]

where \( C_\kappa := \int_0^B \kappa(r) \, dr \).

Now we are ready to prove Lemma 3.6. For simplicity, here we let \( \varphi \) denote \( \varphi(\hat{w}, \bar{w}) \). For bounded distributions, Lemmas 3.10 to 3.12 imply

\[
C_1 ||\hat{w}|| \varphi^2 \leq \epsilon_\ell + B||\bar{w} - \hat{w}|| \cdot \text{OPT} + C_2 C_\kappa \cdot \varphi/||\hat{w}|| \leq \epsilon_\ell + B||\hat{w}|| \varphi \cdot \text{OPT} + C_2 C_\kappa \cdot \varphi/||\hat{w}||,
\]

where \( C_1 = 4R^3/(3U^2 \pi^2) \) and \( C_2 = 12 \). It follows that at least one of the following three cases is true:

1. \( C_1 ||\hat{w}|| \varphi^2 \leq 3\epsilon_\ell \), which implies \( \varphi = O\left( \sqrt{\epsilon_\ell/||\hat{w}||} \right) \);
2. \( C_1 ||\hat{w}|| \varphi^2 \leq 3B||\hat{w}|| \varphi \cdot \text{OPT} \), and it follows that \( \varphi = O(\text{OPT}) \);
3. \( C_1 ||\hat{w}|| \varphi^2 \leq 3C_2 C_\kappa \cdot \varphi/||\hat{w}|| \), and it follows that \( \varphi = O\left( C_\kappa /||\hat{w}||^2 \right) \).

This finishes the proof of Lemma 3.6 for \( \hat{w} \), in light of Diakonikolas et al. 2020a, Claim 3.4) which is stated below.

**Lemma 3.13.** Under Assumption 3.2

\[
\mathcal{R}_{0-1}(\hat{w}) - \mathcal{R}_{0-1}(\bar{w}) \leq \Pr \left( \text{sign} \left( \langle \hat{w}, x \rangle \right) \neq \text{sign} \left( \langle \bar{w}, x \rangle \right) \right) \leq 2U \varphi(\hat{w}, \bar{w}).
\]
3.3 Recovering the general \(\tilde{O}(\text{OPT})\) bound

Frei et al. (2021) showed an \(\tilde{O}(\sqrt{\text{OPT}})\) upper bound under the “soft-margin” and “sub-exponential” conditions. Here we give an alternative proof of this result using our proof technique. The result in this section will later serve as a guarantee of the first phase of our two-phase algorithm (cf. Section 4) that achieves \(\tilde{O}(\text{OPT})\) risk.

Recall that the only place we need Assumption 3.3 is in the proof of Lemma 3.12. However, even without Assumption 3.3, we can still prove the following general bound which only needs Assumption 3.2.

**Lemma 3.14.** Under Assumption 3.2 for \(\ell = \ell_{\log}\),

\[
|\text{term (5)}| \leq \frac{12U}{\|w\|}.
\]

Now with Lemma 3.14 we can prove a weaker but more general version of Lemma 3.6 (cf. Theorem C.1). Further invoking Lemmas 3.8 and 3.9 (cf. Lemmas C.5 and C.6 for the corresponding sub-exponential results), and let \(\epsilon = \sqrt{\alpha}\), we can show the next result. We present the bound in terms of the angle instead of zero-one risk for later application in Section 4.

**Lemma 3.15.** Given the target error \(\epsilon \in (0, 1)\) and the failure probability \(\delta \in (0, 1/\epsilon)\), consider projected gradient descent eq. (1). If \(\|x\| \leq B\) almost surely, then with \(\eta = 4/B^2\), using \(O\left(\frac{(B+1)^2 \ln(1/\delta)}{\epsilon^4}\right)\) samples and \(O\left(\frac{B^2}{\epsilon^2 \delta}\right)\) iterations, with probability \(1 - \delta\), projected gradient descent outputs \(w_t\) with

\[
\varphi(w_t, \bar{u}) = O\left(\sqrt{\text{OPT} + \epsilon}\right).
\]

On the other hand, if \(P_x\) is \((\alpha_1, \alpha_2)\)-sub-exponential, then with \(\eta = \Theta(1/d)\), using \(\tilde{O}\left(\frac{d \ln(1/\delta^2)}{\epsilon^2}\right)\) samples and \(\tilde{O}\left(\frac{4 \ln(1/\delta)}{\epsilon^2 \delta}\right)\) iterations, with probability \(1 - \delta\), projected gradient descent outputs \(w_t\) with

\[
\varphi(w_t, \bar{u}) = O\left(\sqrt{\text{OPT} \cdot \ln(1/\text{OPT}) + \epsilon}\right).
\]

The proofs of the results above are given in Appendix C.8.

4 An \(\tilde{O}(\text{OPT})\) upper bound with hinge loss

We now show how to avoid Assumption 3.3 and achieve an \(\tilde{O}(\text{OPT})\) zero-one risk bound using an extra step of hinge loss minimization. The key observation is that the only place where Assumption 3.3 is used is in Lemma 3.12 for bounding term (5) for logistic loss. However, as noted in Lemma 3.12, for hinge loss, term (5) is conveniently 0. So a version of Lemma 3.6 holds for hinge loss, without using Assumption 3.3 and dropping the third term of \(C_{\|w\|}\) in the max. Thus, to get an \(\tilde{O}(\text{OPT})\) upper bound, we need to minimize the hinge loss to find a solution \(\hat{w}\) such that \(\|\hat{w}\| = \Omega(1)\) and \(R_h(\hat{w}) \leq R_h(\|\hat{w}\|\bar{u}) + \epsilon\) for some \(\epsilon = \tilde{O}(\sqrt{\text{OPT} + \epsilon})\). Unfortunately the requirement \(\|\hat{w}\| = \Omega(1)\) is non-convex. However, we can bypass the non-convexity by leveraging the solution of the logistic regression problem, which is guaranteed to make an angle of at most \(\tilde{O}(\sqrt{\text{OPT} + \epsilon})\) with \(\bar{u}\), even without Assumption 3.3 by Lemma 3.15. This solution, represented by a unit vector \(v\), gives us a “warm start” for hinge loss minimization. Specifically, suppose we optimize the hinge loss over the halfspace

\[
\mathcal{D} := \left\{ w \in \mathbb{R}^d \mid \langle w, v \rangle \geq 1 \right\},
\]

then any solution we find must have norm at least 1. Furthermore, using the fact that \(\varphi(v, \bar{u}) \leq \tilde{O}(\sqrt{\text{OPT} + \epsilon})\) and the positive homogeneity of the hinge loss, we can also conclude that the optimizer of the hinge loss satisfies \(R_h(\hat{w}) \leq R_h(\|\hat{w}\|\bar{u}) + \epsilon\), giving us the desired solution.

While the above analysis does yield a simple two-phase polynomial time algorithm for getting an \(\tilde{O}(\text{OPT})\) zero-one risk bound, closer analysis reveals a sample complexity requirement of \(\tilde{O}(1/\epsilon^4)\). We can improve
Lemma 4.2. Suppose Assumption 3.2 holds. Consider an arbitrary of the hinge loss $\ell(x)$. Taking expectation of eq. (8) w.r.t. $(x, y)$ and let $\phi(t) := \mathbb{E}_{(x, y) \sim P} \left[ -\ell'_h \left( y_t(w_t, x_t) \right) \right] = R_{0-1}(w_t)$. Here we give a proof sketch of Theorem 4.1, and we also focus on bounded distributions for simplicity. The full proof is given in Appendix D.

1. Run projected gradient descent under the settings of Lemma 3.15, and find a unit vector $v$ such that $\varphi(v, \bar{u})$ is $O(\sqrt{\text{OPT} + \epsilon})$ for bounded distributions, or $O(\sqrt{\text{OPT} \cdot \ln(1/\text{OPT}) + \epsilon})$ for sub-exponential distributions.

2. Run projected SGD over the domain $D$ defined in eq. (9) starting from $w_0 := v$. At step $t$, we sample $(x_t, y_t) \sim P$, and let

$$w_{t+1} := \Pi_D \left[ w_t - \eta \ell'_h \left( y_t(w_t, x_t) \right) y_t x_t \right].$$

where we make the convention that $\ell'_h(0) = -1$. We show the following result on the expectation; it can also be turned into a high-probability bound by probability amplification by repetition.

**Theorem 4.1.** Given the target error $\epsilon \in (0, 1/e)$, suppose Assumption holds.

1. First, for bounded distributions, with $\eta = \Theta(1/\epsilon)$, for all $T = \Omega(1/\epsilon^2)$,

$$\mathbb{E} \left[ \min_{0 \leq t < T} R_{0-1}(w_t) \right] = O(\text{OPT} + \epsilon).$$

2. On the other hand, for sub-exponential distributions, with $\eta = \Theta \left( \frac{1}{d \ln(d/\epsilon)^2} \right)$, for all $T = \Omega \left( \frac{d \ln(d/\epsilon)^2}{\epsilon^2} \right)$,

$$\mathbb{E} \left[ \min_{0 \leq t < T} R_{0-1}(w_t) \right] = O(\text{OPT} \cdot \ln(1/\text{OPT}) + \epsilon).$$

### 4.1 Proof of Theorem 4.1

Here we give a proof sketch of Theorem 4.1 and we also focus on bounded distributions for simplicity. The full proof is given in Appendix D.

Let $\bar{r} := 1/(v, \bar{u})$, and thus $\bar{r} \bar{u} \in D$. At step $t$, we have

$$\|w_{t+1} - \bar{r} \bar{u}\| \leq \|w_t - \bar{r} \bar{u}\| - 2\eta \left( \ell'_h \left( y_t(w_t, x_t) \right) y_t x_t, w_t - \bar{r} \bar{u} \right) + \eta^2 \ell'_h \left( y_t(w_t, x_t) \right)^2 \|x_t\|^2.$$  

Define

$$\mathcal{M}(w) := \mathbb{E}_{(x, y) \sim P} \left[ -\ell'_h \left( y(x, w) \right) \right] = R_{0-1}(w).$$

Taking expectation of eq. (8) w.r.t. $(x_t, y_t)$, and note that $\|x\| \leq B$ almost surely and $(\ell'_h)^2 = -\ell'_h$, we have

$$\mathbb{E} \left[ \|w_{t+1} - \bar{r} \bar{u}\|^2 \right] - \|w_t - \bar{r} \bar{u}\|^2 \leq -2\eta \left( \nabla \mathcal{R}_h(w_t), w_t - \bar{r} \bar{u} \right) - \eta^2 B^2 \mathcal{M}(w_t)$$

$$\leq -2\eta \left( \mathcal{R}_h(w_t) - \mathcal{R}_h(\bar{r} \bar{u}) \right) + \eta^2 B^2 \mathcal{M}(w_t).$$  

To continue, we note the following lemma, which follows from Lemmas 3.10 to 3.12 and the homogeneity of the hinge loss $\ell_h$.

**Lemma 4.2.** Suppose Assumption 3.2 holds. Consider an arbitrary $w \in D$, and let $\varphi$ denote $\varphi(w, \bar{u})$. If $\|x\| \leq B$ almost surely, then

$$\mathcal{R}_h(\bar{r} \bar{u}) \leq \mathcal{R}_h(\|w\| \bar{u}) + O \left( (\text{OPT} + \epsilon)^2 \right)$$

and

$$\mathcal{R}_h(w) - \mathcal{R}_h(\|w\| \bar{u}) \geq \frac{4R^3}{3U\pi^2} \|w\|^2 \varphi^2 - B \|w\| \varphi \cdot \text{OPT}.$$  

\(^1\)Note that the parameters $\eta$, $T$, etc. in this section are all chosen for the second phase.
The remaining steps of the proof proceed as follows. We will prove the following: for \( \varphi_t := \varphi(w_t, \bar{u}) \),

\[
E \left[ \min_{0 \leq t \leq T} \varphi_t \right] = O(\text{OPT} + \epsilon).
\]  

(10)

First, note that if \( \varphi_t = O(\text{OPT}) \) for some \( t \), then eq. (10) holds vacuously. Hence, we assume that \( \varphi_t \geq C \cdot \text{OPT} \) for all \( t \) for a sufficiently large constant \( C > 0 \). Then Lemma 4.2 ensures that for some constant \( C_1 \),

\[
R_h(w_t) - R_h(\|w_t\| \bar{u}) \geq C_1 \|w_t\| \varphi_t^2 \geq C_1 \varphi_t^2,
\]

where we also use \( \|w\| \geq 1 \) for all \( w \in D \).

Next, note that \( M(w_t) = O(\varphi_t) \), due to our assumption \( \varphi_t \geq C \cdot \text{OPT} \) and Lemma 3.13. If \( \varphi_t \leq \epsilon \), then eq. (10) also holds, otherwise we can assume \( \epsilon \leq \varphi_t \), and let \( \eta = C_2 \epsilon \) for some small enough constant \( C_2 \), such that

\[
\eta B^2 M(w_t) \leq C_1 \varphi_t \leq C_1 \varphi_t^2.
\]

Now eq. (9) implies

\[
E \left[ \|w_{t+1} - \bar{r}\bar{u}\|^2 \right] - \|w_t - \bar{r}\bar{u}\|^2 \leq -2\eta C_1 \varphi_t^2 + \eta \cdot O ((\text{OPT} + \epsilon)^2) + \eta C_1 \varphi_t^2
\]

\[
= -\eta C_1 \varphi_t^2 - \eta \cdot O ((\text{OPT} + \epsilon)^2).
\]

Taking the total expectation and telescoping the above inequality for all \( t \), we have

\[
E \left[ \frac{1}{T} \sum_{t<T} \varphi_t^2 \right] \leq \frac{\|w_0 - \bar{r}\bar{u}\|^2}{\eta C_1 T} + O ((\text{OPT} + \epsilon)^2).
\]

Recall that

\[
\|w_0 - \bar{r}\bar{u}\| = \|v - \bar{r}\bar{u}\| = O (\sqrt{\text{OPT} + \epsilon})
\]

due to the first phase of the algorithm. Since \( \eta = C_2 \epsilon \), we can further let \( T = \Omega(1/\epsilon^2) \) and finish the proof.

5 Open problems

Here are some open problems. First, as shown by Theorem 4.1, we can achieve \( O (\text{OPT} \cdot \ln(1/\text{OPT}) + \epsilon) \) zero-one risk using the two-phase algorithm. However, previous algorithms can reach \( O (\text{OPT} + \epsilon) \) (Awasthi et al., 2014; Diakonikolas et al., 2020). Is it possible to develop an algorithm relying on a small constant number of convex optimization phases that achieves \( O (\text{OPT} + \epsilon) \) risk?

It is also interesting to consider neural networks. Previously, Frei et al. (2021) showed that stochastic gradient descent on a two-layer leaky ReLU network of any width achieves \( \tilde{O} (\sqrt{\text{OPT}}) \) zero-one risk, where \( \text{OPT} \) still denotes the best zero-one risk of a linear classifier. On the other hand, Ji et al. (2021) showed that a wide two-layer ReLU network can even achieve the optimal Bayes risk, but their required width depends on a complexity measure that may be exponentially large in the worst case. Can a network with a reasonable width always reach a zero-one risk of \( O (\text{OPT}) \)?

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A Technical lemmas

Here are some technical results we will need in our analysis.

Lemma A.1. Let $r, \rho > 0$ be given, then

$$
\frac{2}{\rho} (1 - e^{-r\rho}) \leq \int_{0}^{2\pi} \ell_{\log} \left( r\rho |\cos(\theta)| \right) r \, d\theta \leq \frac{8\sqrt{2}}{\rho}.
$$

Proof. First note that by symmetry,

$$
\int_{0}^{2\pi} \ell_{\log} \left( r\rho |\cos(\theta)| \right) r \, d\theta = 4 \int_{0}^{\frac{\pi}{4}} \ell_{\log} \left( r\rho |\cos(\theta)| \right) r \, d\theta.
$$

On the upper bound, note that $\ell_{\log} \left( r\rho |\cos(\theta)| \right)$ is increasing as $\theta$ goes from $0$ to $\frac{\pi}{4}$, and moreover $\sin(\theta) \geq \frac{\sqrt{2}}{2}$ for $\theta \in \left( \frac{\pi}{4}, \frac{\pi}{2} \right)$, therefore

$$
4 \int_{0}^{\frac{\pi}{4}} \ell_{\log} \left( r\rho |\cos(\theta)| \right) r \, d\theta \leq 8 \int_{0}^{\frac{\pi}{4}} \ell_{\log} \left( r\rho |\cos(\theta)| \right) r \, d\theta \leq \frac{8\sqrt{2}}{\rho} \int_{0}^{\frac{\pi}{4}} \ell_{\log} \left( r\rho |\cos(\theta)| \right) r \rho \sin(\theta) \, d\theta.
$$

Also because $\ell_{\log}(z) \leq \exp(-z)$,

$$
\int_{0}^{2\pi} \ell_{\log} \left( r\rho |\cos(\theta)| \right) r \, d\theta \leq \frac{8\sqrt{2}}{\rho} \int_{0}^{\frac{\pi}{4}} \exp \left( -r\rho \cos(\theta) \right) r\rho \sin(\theta) \, d\theta
$$

$$
= \frac{8\sqrt{2}}{\rho} \left( 1 - \exp \left( -\frac{\sqrt{2}r\rho}{2} \right) \right)
$$

$$
\leq \frac{8\sqrt{2}}{\rho}.
$$

On the lower bound, note that $\ell_{\log}(z) \geq \frac{1}{z} \exp(-z)$ for $z \geq 0$, therefore

$$
\int_{0}^{2\pi} \ell_{\log} \left( r\rho |\cos(\theta)| \right) r \, d\theta = 4 \int_{0}^{\frac{\pi}{4}} \ell_{\log} \left( r\rho |\cos(\theta)| \right) r \, d\theta \geq 2 \int_{0}^{\frac{\pi}{4}} \exp \left( -r\rho \cos(\theta) \right) r \, d\theta
$$

$$
\geq \frac{2}{\rho} \int_{0}^{\frac{\pi}{4}} \exp \left( -r\rho \cos(\theta) \right) r \rho \sin(\theta) \, d\theta
$$

$$
= \frac{2}{\rho} \left( 1 - e^{-r\rho} \right).
$$

Lemma A.2. Given $w, w' \in \mathbb{R}^d$, suppose $\Pr_{(x,y) \sim P} \left( y \neq \text{sign} \left( \langle w, x \rangle \right) \right) = \text{OPT}$. If $\|x\| \leq B$ almost surely, then

$$
\mathbb{E}_{(x,y) \sim P} \left[ \mathbb{1}_{y \neq \text{sign} \left( \langle w, x \rangle \right)} \|w', x\| \right] \leq B \|w'\| \cdot \text{OPT}.
$$

If $P_x$ is $(\alpha_1, \alpha_2)$-sub-exponential, and $\text{OPT} \leq \frac{1}{2}$, then

$$
\mathbb{E}_{(x,y) \sim P} \left[ \mathbb{1}_{y \neq \text{sign} \left( \langle w, x \rangle \right)} \|w', x\| \right] \leq \left( 1 + 2\alpha_1 \alpha_2 \|w'\| \cdot \text{OPT} \cdot \ln \left( \frac{1}{\text{OPT}} \right) \right).
$$

Proof. If $\|x\| \leq B$ almost surely, then

$$
\mathbb{E}_{(x,y) \sim P} \left[ \mathbb{1}_{y \neq \text{sign} \left( \langle w, x \rangle \right)} \left| \langle w', x \rangle \right| \right] \leq B \|w'\| \mathbb{E}_{(x,y) \sim P} \left[ \mathbb{1}_{y \neq \text{sign} \left( \langle w, x \rangle \right)} \right] = B \|w'\| \cdot \text{OPT}.
$$
Below we assume $P_x$ is $(\alpha_1, \alpha_2)$-sub-exponential.

Let $\nu_x := \langle w', x \rangle$; we first give some tail bounds for $\nu_x$. Since $P_x$ is $(\alpha_1, \alpha_2)$-sub-exponential, for any $t > 0$, we have

$$\Pr \left( \left\langle \frac{w'}{\|w'\|}, x \right\rangle \geq t \right) \leq \alpha_1 \exp \left( -\frac{t}{\alpha_2} \right), \quad \text{equivalently} \quad \Pr (|\nu_x| \geq t) \leq \alpha_1 \exp \left( -\frac{t}{\alpha_2\|w'\|} \right).$$

Let $\mu(t) := \Pr (|\nu_x| \geq t)$. Given any threshold $\tau > 0$, integration by parts gives

$$\mathbb{E} \left[ 1_{|\nu_x| \geq \tau} |\nu_x| \right] = \int_{\tau}^{\infty} t \cdot (-d\mu(t)) = \tau \mu(\tau) + \int_{\tau}^{\infty} \mu(t) \, dt \leq \alpha_1 \|w'\| + \tau \exp \left( -\frac{\tau}{\alpha_2\|w'\|} \right). \quad (11)$$

Now let $\tau := \alpha_2\|w'\| \ln \left( \frac{1}{\alpha_2\tau} \right)$. Note that

$$\mathbb{E}_{(x,y) \sim P} \left[ 1_{y \neq \text{sign}(\langle w, x \rangle)} |\langle w', x \rangle| \right] = \mathbb{E}_{(x,y) \sim P} \left[ 1_{|\nu_x| \leq \tau} 1_{y \neq \text{sign}(\langle w, x \rangle)} |\nu_x| \right] + \mathbb{E}_{(x,y) \sim P} \left[ 1_{|\nu_x| \geq \tau} 1_{y \neq \text{sign}(\langle w, x \rangle)} |\nu_x| \right].$$

We bound the two parts separately. When $|\nu_x| \leq \tau$, we have

$$\mathbb{E} \left[ 1_{|\nu_x| \leq \tau} 1_{y \neq \text{sign}(\langle w, x \rangle)} |\nu_x| \right] \leq \tau \mathbb{E} \left[ 1_{y \neq \text{sign}(\langle w, x \rangle)} \right] = \tau \cdot \text{OPT} = \alpha_2\|w'\| \cdot \text{OPT} \cdot \ln \left( \frac{1}{\text{OPT}} \right).$$

On the other hand, when $|\nu_x| \geq \tau$, eq. (11) gives

$$\mathbb{E}_{(x,y) \sim P} \left[ 1_{|\nu_x| \geq \tau} 1_{y \neq \text{sign}(\langle w, x \rangle)} |\nu_x| \right] \leq \mathbb{E} \left[ 1_{|\nu_x| \geq \tau} |\nu_x| \right] \leq \alpha_1 \alpha_2\|w'\| \left( 1 + \ln \left( \frac{1}{\text{OPT}} \right) \right) \text{OPT} \leq 2 \alpha_1 \alpha_2\|w'\| \cdot \text{OPT} \cdot \ln \left( \frac{1}{\text{OPT}} \right),$$

where we also use $\text{OPT} \leq \frac{1}{e}$. To sum up,

$$\mathbb{E}_{(x,y) \sim P} \left[ 1_{y \neq \text{sign}(\langle w, x \rangle)} \right] \leq (1 + 2 \alpha_1) \alpha_2\|w'\| \cdot \text{OPT} \cdot \ln \left( \frac{1}{\text{OPT}} \right).$$

\[\square\]

## B Omitted proofs from Section 2

In this section, we will prove Theorem 2.1. First, we bound the density and support of $Q_x$.

**Lemma B.1.** If $\text{OPT} \leq \frac{1}{100}$, then it holds that $q_3 \leq \frac{1}{15}$, and $\frac{1}{25} \leq q_4 \leq \frac{1}{5}$. As a result, $Q_x$ is supported on $B(2) := \{x \mid \|x\| \leq 2 \}$ with its density bounded by $2$.

**Proof.** For $q_3$, we have

$$q_3 = \frac{2}{3} \sqrt{\text{OPT}(1 - \text{OPT})} \leq \frac{2}{3} \sqrt{\frac{1}{100}} \leq \frac{2}{3} \frac{1}{10} = \frac{1}{15}.$$

For $q_4$, its total measure can be bounded as below:

$$1 - \text{OPT} - 2\sqrt{\text{OPT}} - q_3 \geq 1 - \frac{1}{100} - \frac{2}{10} - \frac{1}{15} \geq \frac{1}{2},$$

therefore $q_4 \geq \frac{1}{20}$. The upper bound $q_4 \leq \frac{1}{5}$ is trivial.
On the support of $Q_x$, note that for $Q_1$, the largest $\ell_2$ norm is given by
\[ 1 + \frac{\sqrt{2}}{2} \sqrt{\frac{\text{OPT}}{2}} \leq 1 + \frac{1}{20} \leq 2. \]

For $Q_2$, the largest $\ell_2$ norm can be bounded by
\[ 1 + \sqrt{\text{OPT}} \leq 1 + \frac{1}{10} \leq 2. \]

For $Q_3$, the largest $\ell_2$ norm can be bounded by
\[ 1 + \sqrt{2} \cdot \sqrt{\frac{q_3}{2}} \leq 1 + \frac{1}{2} \sqrt{\frac{1}{15}} \leq 2. \]

Finally, it is easy to verify that if $\text{OPT} \leq \frac{10}{100}$, then $Q_1$, $Q_2$ and $Q_3$ do not overlap, therefore the density of $Q$ is bounded by $1 + \frac{1}{\pi} \leq 2$. \hfill \Box

Next we verify that $Q_x$ is isotropic up to a multiplicative factor. We first note the following fact; its proof is straightforward and omitted.

**Lemma B.2.** It holds that
\[ \int_{a - \frac{1}{2}}^{a + \frac{1}{2}} \int_{b - \frac{1}{2}}^{b + \frac{1}{2}} xy \, dy \, dx = ab \delta^2, \quad \text{and} \quad \int_{a - \frac{1}{2}}^{a + \frac{1}{2}} \int_{b - \frac{1}{2}}^{b + \frac{1}{2}} (x^2 - y^2) \, dy \, dx = (a^2 - b^2) \delta^2. \]

Then we can prove the following result.

**Lemma B.3.** It holds that $\mathbb{E}_{x \sim Q_x} [x] = 0$, and $\mathbb{E}_{x \sim Q_x} [x_1 x_2] = 0$, and $\mathbb{E}_{x \sim Q_x} \left[ x_1^2 - x_2^2 \right] = 0$.

**Proof.** It follows from the symmetry of $Q$ that $\mathbb{E}_{x \sim Q_x} [x] = 0$.

To verify $\mathbb{E}_{x \sim Q_x} [x_1 x_2] = 0$, note that the expectation of $x_1 x_2$ is 0 on $Q_3$ and $Q_4$, and thus we only need to check $Q_1$ and $Q_2$. First, due to Lemma B.2 we have
\[ \mathbb{E}_{(x,y) \sim Q_1} [x_1 x_2] = -\frac{\text{OPT}}{2}. \]

Additionally,
\[ \mathbb{E}_{(x,y) \sim Q_2} [x_1 x_2] = 2 \int_0^{\sqrt{\text{OPT}}} \int_0^1 x_1 x_2 \, dx_2 \, dx_1 = \frac{\text{OPT}}{2}. \]

Therefore $\mathbb{E}_{x \sim Q_x} [x_1 x_2] = 0$.

Finally, note that the expectation of $x_1^2 - x_2^2$ is 0 on $Q_1$ due to Lemma B.2 and also 0 on $Q_4$ due to symmetry; therefore we only need to consider $Q_2$ and $Q_3$. We have
\[ \mathbb{E}_{(x,y) \sim Q_2} \left[ x_1^2 - x_2^2 \right] = 2 \int_0^{\sqrt{\text{OPT}}} \int_0^1 \left( x_1^2 - x_2^2 \right) \, dx_2 \, dx_1 = \frac{2}{3} \text{OPT}^{3/2} - \frac{2}{3} \sqrt{\text{OPT}} = -q_3. \]

Since $\mathbb{E}_{(x,y) \sim Q_3} \left[ x_1^2 - x_2^2 \right] = q_3$ by Lemma B.2, it follows that $\mathbb{E}_{x \sim Q_x} \left[ x_1^2 - x_2^2 \right] = 0$. \hfill \Box

Next, we give a proof of the risk lower bound of Theorem 2.1. For simplicity, in this section we will let $\mathcal{R}$ denote $\mathcal{R}_{\log}$. For $i = 1, 2, 3, 4$, we also let $\mathcal{R}_i(w) := \mathbb{E}_{(x,y) \sim Q_i} \left[ \ell_{\log} \left( y \langle w, x \rangle \right) \right]$; therefore $\mathcal{R}(w) := \sum_{i=1}^4 \mathcal{R}_i(w)$. We first prove Lemma 2.2 showing that there exists a solution $\bar{w}$ with $\|\bar{w}\| = \Theta \left( \frac{1}{\sqrt{\text{OPT}}} \right)$ and $\mathcal{R}(\bar{w}) = O \left( \sqrt{\text{OPT}} \right)$.

**Proof of Lemma 2.2** We consider $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$ and $\mathcal{R}_4$ respectively.
1. For $Q_1$, note that the minimum of $y(\bar{w}, x)$ is
\[
-\left(\frac{\sqrt{2}}{2} + \frac{1}{2} \sqrt{\frac{\OPT}{2}}\right) \bar{r} = -\frac{3\sqrt{2}}{2} \frac{1}{\sqrt{\OPT}} - \frac{3\sqrt{2}}{4}.
\]
Because $\ell_{\log}(z) \leq -z + 1$ when $z \leq 0$, and $\OPT \leq \frac{1}{100}$, we have
\[
\mathcal{R}_1(\bar{w}) \leq \ell_{\log}(\left(-\frac{3\sqrt{2}}{2} \frac{1}{\sqrt{\OPT}} - \frac{3\sqrt{2}}{4}\right) \cdot \OPT \leq \frac{3\sqrt{2}}{2} \sqrt{\OPT} + \left(\frac{3\sqrt{2}}{4} + 1\right) \frac{1}{100} \sqrt{\OPT} \\
\leq \frac{5\sqrt{\OPT}}{2}.
\]
2. For $Q_2$, we have
\[
\mathcal{R}_2(\bar{w}) = 2 \int_0^{\sqrt{\OPT}} \int_0^1 \ell_{\log}(x_1 \bar{r}) dx_2 dx_1 = 2 \int_0^{\sqrt{\OPT}} \ell_{\log}(x_1 \bar{r}) dx_1 \\
\leq 2 \int_0^{\sqrt{\OPT}} \exp(-x_1 \bar{r}) dx_1 \\
= \frac{2}{\bar{r}} \left(1 - \exp\left(-\bar{r} \sqrt{\OPT}\right)\right) \leq \frac{2}{\bar{r}}.
\]
where we use $\ell_{\log}(z) \leq \exp(-z)$.
3. For $Q_3$, the minimum of $y(\bar{w}, x)$ is
\[
\left(1 - \frac{1}{2} \sqrt{\frac{q_3}{2}}\right) \bar{r} \geq \frac{2\bar{r}}{3},
\]
where we use $q_3 \leq \frac{1}{15}$ by Lemma B.1. Further note that $\ell_{\log}(z) \leq 1/z$ when $z > 0$, we have
\[
\mathcal{R}_3(\bar{w}) \leq q_3 \ell_{\log}\left(\frac{2\bar{r}}{3}\right) \leq \frac{1/15}{2\bar{r}/3} \leq \frac{1}{10\bar{r}}.
\]
4. For $Q_4$, 
\[
\mathcal{R}_4(\bar{w}) = \int_0^1 \int_0^{2\pi} \ell_{\log}\left(r \bar{r} |\cos(\theta)|\right) q_4 r d\theta dr \
\leq \frac{1}{\pi} \int_0^1 \int_0^{2\pi} \ell_{\log}\left(r \bar{r} |\cos(\theta)|\right) r d\theta dr,
\]
where we use $q_4 \leq \frac{1}{\pi}$ from Lemma B.1. Lemma A.1 then implies
\[
\mathcal{R}_4(\bar{w}) \leq \frac{1}{\pi} \int_0^1 \frac{8\sqrt{2}}{r} dr = \frac{8\sqrt{2}}{\pi \bar{r}}.
\]
Putting everything together, we have
\[
\mathcal{R}(\bar{w}) = \mathcal{R}_1(\bar{w}) + \mathcal{R}_2(\bar{w}) + \mathcal{R}_3(\bar{w}) + \mathcal{R}_4(\bar{w}) \\
\leq \frac{5\sqrt{\OPT}}{2} + \frac{2}{\bar{r}} + \frac{1}{10\bar{r}} + \frac{8\sqrt{2}}{\pi \bar{r}} \\
\leq \frac{5\sqrt{\OPT}}{2} + \frac{6}{\bar{r}} \leq 5\sqrt{\OPT}.
\]
\[\square\]
Next we prove Lemma 2.3, the upper bound on $\|w^*\|$.

**Proof of Lemma 2.3.** Let
\[ u := \left( \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right), \quad \text{and} \quad v := \left( \frac{\sqrt{2}}{2} - \frac{1}{2} \sqrt{\frac{\text{OPT}}{2}}, -\frac{\sqrt{2}}{2} - \frac{1}{2} \sqrt{\frac{\text{OPT}}{2}} \right). \]

Let $\phi$ denote the angle between $u$ and $v$, then
\[ \phi \leq \tan(\phi) = \frac{\sqrt{2}}{2} \sqrt{\frac{\text{OPT}}{2}} \leq \frac{1}{20} \leq \frac{\pi}{24}. \]

Moreover, note that the maximum of $y(w^*, x)$ on $Q_1$ is given by
\[ -(w^*, v) \leq -r^* \|v\| \cos \left( \frac{\pi}{3} \right) \leq -r^* \cos \left( \frac{\pi}{3} \right) = -\frac{r^*}{2}. \]

Additionally because $\ell - \log(z) > -z$, we have
\[ R(w^*) \geq R_1(w^*) \geq \ell - \log \left( -\frac{r^*}{2} \right) \cdot \text{OPT} > \frac{r^*}{2} \cdot \text{OPT}. \]

If $r^* > \frac{10}{\sqrt{\text{OPT}}}$, then $R(w^*) > 5\sqrt{\text{OPT}}$, which contradicts the definition of $w^*$ in light of Lemma 2.2. Therefore $r^* \leq \frac{10}{\sqrt{\text{OPT}}}$. 

Next we prove Lemma 2.4.

**Proof of Lemma 2.4.** Let $w = (r, \theta)$, where $0 \leq r \leq \frac{10}{\sqrt{\text{OPT}}}$ and $0 \leq \theta \leq \frac{\sqrt{\text{OPT}}}{30}$. We will consider the projection of $\nabla R(w)$ onto the direction $e_2 := (0, 1)$, and show that this projection cannot be zero.

1. For $Q_1$, the gradient of this part has a negative inner product with $e_2$, due to the construction of $Q_1$ and the fact $\ell' - \log < 0$.

2. For $Q_2$, the inner product between $e_2$ and the gradient of this part is given by
\[ 2 \int_0^{\sqrt{\text{OPT}}} \int_1^1 \ell'(x_1 w_1 + x_2 w_2) x_2 \, dx_2 \, dx_1. \]

Note that $x_1 w_1 \leq r x_1$, while
\[ x_2 w_2 = x_2 r \sin(\theta) \leq r \theta \leq \frac{10}{\sqrt{\text{OPT}}} \cdot \frac{\sqrt{\text{OPT}}}{30} = \frac{1}{3}, \]

and that $\ell' - \log$ is increasing, therefore
\[ \ell'(x_1 w_1 + x_2 w_2) \leq \ell' \left( rx_1 + \frac{1}{3} \right). \]

We can then upper bound eq. (12) as follows:
\[ \text{eq. (12)} \leq 2 \int_0^{\sqrt{\text{OPT}}} \int_0^1 \ell'(rx_1 + \frac{1}{3}) x_2 \, dx_2 \, dx_1 \]
\[ = \int_0^{\sqrt{\text{OPT}}} \ell' \left( rx_1 + \frac{1}{3} \right) \, dx_1 \]
\[ = \frac{1}{r} \left( \ell' \left( \frac{1}{3} + r \sqrt{\text{OPT}} \right) - \ell' \left( \frac{1}{3} \right) \right). \]
Now we consider two cases. If \( r \sqrt{\text{OPT}} \leq 2 \), then it follows from the convexity of \( \ell_\log \) that

\[
eq \frac{1}{r} \ell_\log \left( \frac{1}{3} + r \sqrt{\text{OPT}} \right) \leq \ell_\log (3) \sqrt{\text{OPT}} \leq -\frac{\sqrt{\text{OPT}}}{30}.
\]

On the other hand, if \( r \sqrt{\text{OPT}} \geq 2 \), then

\[
eq \frac{1}{r} \ell_\log \left( \frac{7}{3} \right) \leq \frac{\sqrt{\text{OPT}}}{10} \left( \ell_\log \left( \frac{7}{3} \right) - \ell_\log \left( \frac{1}{3} \right) \right) \leq -\frac{\sqrt{\text{OPT}}}{30}.
\]

Therefore, it always holds that eq. (12) \( \leq -\frac{\sqrt{\text{OPT}}}{30} \).

3. For \( Q_3 \), the gradient of this part can have a positive inner product with \( e_2 \). For simplicity, let \( \rho := \frac{1}{4} \sqrt{\text{OPT}} \). To upper bound this inner product, it is enough to consider the region given by

\[
\{(1 - \rho)(1 + \rho) \times [-\rho, 0]) \cup (-1 - \rho, -1 + \rho] \times [0, \rho]\}.
\]

Moreover, note that \( y(w, x) \geq 0 \) on \( Q_3 \), therefore \( \ell_\log (y(w, x)) \geq -\frac{1}{2} \). Therefore the inner product between \( e_2 \) and the gradient of \( Q_3 \) can be upper bounded by (note that \( x_2 \leq 0 \) in the integral)

\[
2 \int_{-1 - \rho}^{1 + \rho} \int_{-\rho}^{0} -\frac{1}{2} x_2 dx_2 dx_1 = \rho^3 = \frac{\sqrt{3}}{16 \sqrt{2}} \leq \frac{\sqrt{\text{OPT}}}{16 \sqrt{2}} \leq \frac{\sqrt{\text{OPT}}}{60}.
\]

where we use \( q_3 \leq \frac{1}{16} \) by Lemma [B.1] and \( q_3 \leq \frac{1}{4} \sqrt{\text{OPT}} \) by its definition.

4. For \( Q_4 \), we further consider two cases.

(a) Consider the part of \( Q_4 \) with polar angles in \((-\frac{\pi}{4} + 2\theta, \frac{\pi}{4}) \cup (\frac{\pi}{4} + 2\theta, \frac{3\pi}{4})\). By symmetry, the gradient of this part is along the direction with polar angle \( \pi + \theta \), and it has a negative inner product with \( e_2 \).

(b) Consider the part of \( Q_4 \) with polar angles in \((-\frac{\pi}{2}, -\frac{\pi}{4} + 2\theta) \cup (\frac{\pi}{4} + 2\theta, \frac{3\pi}{4})\). We can verify that the gradient of this part has a positive inner product with \( e_2 \); moreover, since \( -1 < \ell_\log < 0 \), this inner product can be upper bounded by

\[
2 \int_{0}^{1} \int_{0}^{2\theta} r' \cos(\theta') q_4 r' \ dr' \ dr = 2 q_4 \frac{1}{3} \sin(2\theta) \leq \frac{4\theta}{3\pi} \leq \frac{4}{3\pi} \sqrt{\text{OPT}} \ leq \sqrt{\text{OPT}} \leq \sqrt{\text{OPT}}
\]

where we also use \( q_4 \leq \frac{1}{3} \) and \( \sin(z) \leq z \) for \( z \geq 0 \).

As a result, item 3 and item 4(b) cannot cancel item 2, and thus \( \nabla R(w) \) cannot be 0.

Now we are ready to prove the risk lower bound of Theorem 2.1.

Proof of Theorem 2.1 risk lower bound. It is clear that \( R \) has bounded sub-level sets, and therefore can be globally minimized. Let the polar coordinates of the global minimizer be given by \( (r^*, \theta^*) \), where \( |\theta^*| \leq \pi \).

Assume that \( \theta^* \in \left[-\frac{\sqrt{\text{OPT}}}{60}, \frac{\sqrt{\text{OPT}}}{60}\right] \); due to \( Q_1 \) and \( Q_2 \), it actually follows that \( \theta^* \in \left[0, \frac{\sqrt{\text{OPT}}}{30}\right] \). Lemma 2.3 then implies \( r^* \leq 10 \sqrt{\alpha} \), and then Lemma 2.4 implies \( \nabla R(w^*) \neq 0 \), a contradiction.

It then follows that \( w^* \) is wrong on a \( \frac{\theta^*}{\pi} \) portion of \( Q_4 \). Since the total measure of \( Q_4 \) is more than half due to Lemma [B.1] we have

\[
R_{0-1}(w^*) \geq \frac{1}{2} \theta^* \geq \frac{\sqrt{\text{OPT}}}{60 \pi}.
\]

\[\square\]
C Omitted proofs from Section 3

In this section, we provide omitted proofs from Section 3. First, we prove some general results that will be used later.

**Lemma C.1.** Under Assumption 3.2, for any \( w \in \mathbb{R}^d \),

\[
\mathbb{E} \left[ \ell \log \left( \|w \| \cos(\theta) \right) \right] \leq \frac{12U}{\|w\|}.
\]

**Proof.** Let \( v \) denote an arbitrary vector orthogonal to \( w \), and let \( p \) denote the density of the projection of \( P_x \) onto the space spanned by \( w \) and \( v \). Then we have

\[
\mathbb{E} \left[ \ell \log \left( \|w \| \cos(\theta) \right) \right] = \int_0^\infty \int_0^{2\pi} \ell \log \left( r \|w\| \cos(\theta) \right) p(r, \theta) r d\theta dr.
\]

Invoking Assumption 3.2, we have

\[
\mathbb{E} \left[ \ell \log \left( \|w \| \cos(\theta) \right) \right] \leq \int_0^\infty \sigma(r) \left( \int_0^{2\pi} \ell \log \left( r \|w\| \cos(\theta) \right) r d\theta \right) dr.
\]

Lemma A.1 then implies

\[
\mathbb{E} \left[ \ell \log \left( \|w \| \cos(\theta) \right) \right] \leq \int_0^\infty \sigma(r) \frac{8\sqrt{2}}{\|w\|} dr.
\]

Then it follows from Assumption 3.2 that

\[
\mathbb{E} \left[ \ell \log \left( \|w \| \cos(\theta) \right) \right] \leq \frac{8\sqrt{2}U}{\|w\|} \leq \frac{12U}{\|w\|}.
\]

Next, we note that following the direction of the ground-truth solution \( \bar{u} \) can achieve \( \tilde{O}(\sqrt{\text{OPT}}) \) logistic risk.

**Lemma C.2.** Given \( \rho > 0 \), under Assumption 3.2, if \( \|x\| \leq B \) almost surely, then

\[
\mathcal{R}_{\log}(\rho \bar{u}) \leq \frac{12U}{\rho} + \rho B \cdot \text{OPT}, \quad \text{with} \quad \inf_{\rho > 0} \mathcal{R}_{\log}(\rho \bar{u}) \leq \sqrt{50UB \cdot \text{OPT}},
\]

while if \( P_x \) is \((\alpha_1, \alpha_2)\)-sub-exponential, then

\[
\mathcal{R}_{\log}(\rho \bar{u}) \leq \frac{12U}{\rho} + (1 + 2\alpha_1)\alpha_2 \rho \cdot \text{OPT} \cdot \ln \left( \frac{1}{\text{OPT}} \right),
\]

with

\[
\inf_{\rho > 0} \mathcal{R}_{\log}(\rho \bar{u}) \leq \sqrt{50(1 + 2\alpha_1)\alpha_2 U \cdot \text{OPT} \cdot \ln \left( \frac{1}{\text{OPT}} \right)}.
\]

**Proof.** Note that

\[
\mathcal{R}_{\log}(\rho \bar{u}) = \mathbb{E}_{(x, y) \sim P} \left[ \ell \log \left( y \rho \bar{u}, x \right) \right]
\]

\[
= \mathbb{E}_{x \sim P_x} \left[ \ell \log \left( \|\rho \bar{u}, x \| \right) \right] + \mathbb{E}_{(x, y) \sim P} \left[ \ell \log \left( y \rho \bar{u}, x \right) - \ell \log \left( \|\rho \bar{u}, x \| \right) \right].
\]
Since $\ell_{\log}(-z) - \ell_{\log}(z) = z$, and also invoking Lemma [C.1] we have

$$
R_{\log}(\rho \tilde{u}) = E_{x \sim P_x} \left[ \ell_{\log} \left( \langle \rho \bar{u}, x \rangle \right) \right] + E_{(x,y) \sim P} \left[ \mathbb{1}_{y \neq \text{sign}(\langle \bar{u}, x \rangle)} \cdot (y \langle \rho \bar{u}, x \rangle) \right].
$$

If $\|x\| \leq B$ almost surely, then Lemma [A.2] further implies

$$
R_{\log}(\rho \tilde{u}) \leq \frac{12U}{\rho} + \rho B \cdot \text{OPT},
$$

and thus

$$
\inf_{\rho > 0} R_{\log}(\rho \tilde{u}) \leq \sqrt{12UB \cdot \text{OPT}} \leq \sqrt{50UB \cdot \text{OPT}}.
$$

Next we prove a risk lower bound, that will later be used to prove lower bounds on $\|w^\star\|$ and $\|w_t\|$.

**Lemma C.3.** Under Assumption 3.2, given $w \in \mathbb{R}^d$, if $R\|w\| \leq 2$, then

$$
R_{\log}(w) \geq \frac{R^2}{2U},
$$

while if $R\|w\| \geq 2$, then

$$
R_{\log}(w) \geq \frac{R}{U\|w\|}.
$$

**Proof.** First, since $\ell_{\log}(z) \geq \ell_{\log}(\|z\|)$,

$$
R_{\log}(w) = E_{(x,y) \sim P} \left[ \ell_{\log}(y\langle w, x \rangle) \right] \geq E_{x \sim P_x} \left[ \ell_{\log}(\|w\|) \right].
$$

Let $v$ denote an arbitrary vector that is orthogonal to $w$, and let $p$ denote the density of the projection of $P_x$ onto the space spanned by $w$ and $v$. Without loss of generality, we can assume $w$ has polar angle 0. Then eq. (13) becomes

$$
R_{\log}(w) \geq \int_{0}^{2\pi} \int_{0}^{\infty} \ell_{\log} \left( r\|w\|\cos(\theta) \right) p(r, \theta) r \, d\theta \, dr.
$$

Assumption 3.2 and Lemma [A.1] then imply

$$
R_{\log}(w) \geq \frac{1}{U} \int_{0}^{R} \int_{0}^{2\pi} \ell_{\log} \left( r\|w\|\cos(\theta) \right) r \, d\theta \, dr \geq \frac{1}{U\|w\|^2} \int_{0}^{R} \left( 1 - e^{-r\|w\|} \right) dr = \frac{2}{U\|w\|^2} \left( e^{-R\|w\|} - 1 + R\|w\| \right).
$$
If $R \|w\| \leq 2$, then because $e^{-z} - 1 + z \geq \frac{z^2}{4}$ when $0 \leq z \leq 2$, we have

$$\mathcal{R}_{\log}(w) \geq \frac{2}{U} \frac{1}{\|w\|^2} \frac{R^2 \|w\|^2}{4} = \frac{R^2}{2U}.$$ 

Otherwise if $R \|w\| \geq 2$, then because $e^{-z} - 1 + z \geq \frac{z^2}{2}$ when $z \geq 2$, we have

$$\mathcal{R}_{\log}(w) \geq \frac{2}{U} \frac{1}{\|w\|^2} \frac{R \|w\|}{2} = \frac{R}{U \|w\|}.$$

\[\square\]

C.1 Omitted proofs from Section 3.1

In this section, we prove Theorems 3.4 and 3.5 using Lemma 3.6.

First, we prove the following norm lower bound on $\|w^*\|$, which covers Lemma 3.7 and also the sub-exponential case.

Lemma C.4 (Lemma 3.7, including the sub-exponential case). Under Assumption 3.2, if $\|x\| \leq B$ almost surely and $\text{OPT} < \frac{R^4}{200U^3B}$, then $\|w^*\| = \Omega \left( \frac{1}{\sqrt{\text{OPT}}} \right)$; if $P_x$ is $(\alpha_1, \alpha_2)$-sub-exponential and $\text{OPT} \cdot \ln(1/\text{OPT}) < \frac{R^4}{200(1+2\alpha_1)\alpha_2U^3}$, then $\|w^*\| = \Omega \left( \frac{1}{\sqrt{\text{OPT} \cdot \ln(1/\text{OPT})}} \right)$.

Proof. Suppose $\|x\| \leq B$ almost surely. Since $\text{OPT} < \frac{R^4}{200U^3B}$, Lemma C.2 implies

$$\mathcal{R}_{\log}(w^*) \leq \inf_{\rho > 0} \mathcal{R}_{\log}(\rho \bar{u}) \leq \sqrt{50UB \cdot \text{OPT}} < \frac{R^4}{200U^3B} = \frac{R^2}{2U}.$$ 

Therefore it follows from Lemma C.3 that $R \|w^*\| \geq 2$, and

$$\frac{R}{U \|w^*\|} \leq \mathcal{R}_{\log}(w^*) \leq \inf_{\rho > 0} \mathcal{R}_{\log}(\rho \bar{u}) \leq \sqrt{50UB \cdot \text{OPT}},$$

which implies

$$\|w^*\| \geq \frac{R}{U \sqrt{50UB} \cdot \frac{1}{\sqrt{\text{OPT}}} \cdot 1}.$$

Now suppose $P_x$ is $(\alpha_1, \alpha_2)$-sub-exponential. Since $\text{OPT} \cdot \ln \left( \frac{1}{\text{OPT}} \right) < \frac{R^4}{200(1+2\alpha_1)\alpha_2U^3}$, Lemma C.2 implies

$$\inf_{\rho > 0} \mathcal{R}_{\log}(\rho \bar{u}) \leq \sqrt{50(1+2\alpha_1)\alpha_2U \cdot \text{OPT} \cdot \ln \left( \frac{1}{\text{OPT}} \right)} < \sqrt{50(1+2\alpha_1)\alpha_2U \cdot \frac{R^4}{200(1+2\alpha_1)\alpha_2U^3}} = \frac{R^2}{2U}.$$ 

Therefore it follows from Lemma C.3 that $R \|w^*\| \geq 2$, and

$$\frac{R}{U \|w^*\|} \leq \mathcal{R}_{\log}(w^*) \leq \inf_{\rho > 0} \mathcal{R}_{\log}(\rho \bar{u}) \leq \sqrt{50(1+2\alpha_1)\alpha_2U \cdot \text{OPT} \cdot \ln \left( \frac{1}{\text{OPT}} \right)}$$

which implies

$$\|w^*\| \geq \frac{R}{U \sqrt{50(1+2\alpha_1)\alpha_2U} \cdot \sqrt{\text{OPT} \cdot \ln(1/\text{OPT})} \cdot 1}.$$

\[\square\]

Now we can prove Theorem 3.4.
Proof of Theorem 3.4. If $\|x\| \leq B$ almost surely, Lemma 3.6 implies

$$R_{0-1}(w^*) = O \left( \max \left\{ \text{OPT}, \frac{C_\kappa}{\|w^*\|^2} \right\} \right).$$

If $\text{OPT} \geq \frac{R^4}{200U^3B}$, then Theorem 3.4 holds vacuously; otherwise Lemma C.4 ensures $\|w^*\| = \Omega \left( \frac{1}{\sqrt{\text{OPT}}} \right)$, and thus

$$R_{0-1}(w^*) = O \left( \max \{ \text{OPT}, C_\kappa \cdot \text{OPT} \} \right) = O \left( (1 + C_\kappa) \text{OPT} \right).$$

The proof of the sub-exponential case is similar.

Next, we analyze project gradient descent. First we restate Lemmas 3.8 and 3.9, and also handle sub-exponential distributions.

**Lemma C.5 (Lemma 3.8, including the sub-exponential case).** Let the target optimization error $\epsilon \ell \in (0, 1)$ and the failure probability $\delta \in (0, 1/e)$ be given. If $\|x\| \leq B$ almost surely, then with $\eta = \frac{4}{B^2}$, using $O \left( \frac{(B + 1)^2 \ln(1/\delta)}{\epsilon \ell^2} \right)$ samples and $O \left( \frac{B^2}{\epsilon \ell} \right)$ iterations, with probability $1 - \delta$, projected gradient descent outputs $w_t$ satisfying

$$R_{\log}(w_t) \leq \min_{0 \leq \rho \leq 1/\sqrt{\epsilon}} R_{\log}(\rho \hat{u}) + \epsilon \ell. \quad (14)$$

If $P_x$ is $(\alpha_1, \alpha_2)$-sub-exponential, then with $\eta = \tilde{\Theta}(1/d)$, using $\tilde{O} \left( \frac{d \ln(1/\delta)^3}{\epsilon \ell^2} \right)$ samples and $\tilde{O} \left( \frac{d \ln(1/\delta)^2}{\epsilon \ell} \right)$ iterations, with probability $1 - \delta$, projected gradient descent outputs $w_t$ satisfying eq. (14).

**Lemma C.6 (Lemma 3.9, including the sub-exponential case).** Under Assumption 3.2, suppose

$$\epsilon < \min \left\{ \frac{R^4}{36U^2}, \frac{R^4}{72^2U^4} \right\} \quad \text{and} \quad \epsilon \ell \leq \sqrt{\epsilon},$$

and that eq. (14) holds. If $\|x\| \leq B$ almost surely and $\text{OPT} < \frac{R^4}{900U^2B}$, then $\|w_t\| = \Omega \left( \min \left\{ \frac{1}{\sqrt{\epsilon} \cdot \sqrt{\text{OPT}}}, \frac{1}{\sqrt{\text{OPT} \cdot \text{ln}(1/\text{OPT})}} \right\} \right).$

On the other hand, if $P_x$ is $(\alpha_1, \alpha_2)$-sub-exponential, and $\text{OPT} \cdot \text{ln}(1/\text{OPT}) < \frac{R^2}{10000(1 + 2\alpha_1)\alpha_2}$, then it holds that $\|w_t\| = \Omega \left( \min \left\{ \frac{1}{\sqrt{\epsilon} \cdot \sqrt{\text{OPT} \cdot \text{ln}(1/\text{OPT})}} \right\} \right).$

Next we prove Lemmas C.5 and C.6. We first consider bounded distributions, and then handle sub-exponential distributions. For simplicity, in the rest of this subsection we will use $\mathcal{R}$ and $\hat{\mathcal{R}}$ to denote $R_{\log}$ and $\hat{R}_{\log}$, respectively.

**Bounded distributions.** First, here are some standard optimization and generalization results for projected gradient descent.

**Lemma C.7.** If $\|x_i\| \leq B$ for all $1 \leq i \leq n$, then $\hat{\mathcal{R}}$ is $\frac{B^2}{4}$-smooth. Moreover, if $w_0 := 0$ and $\eta \leq \frac{4}{B^2}$, then for all $t \geq 1$,

$$\hat{\mathcal{R}}(w_t) \leq \min_{w \in B(1/\sqrt{\epsilon})} \mathcal{R}(w) + \frac{1}{2\eta \epsilon \ell}.$$
Proof. Note that $\ell_{\log}$ is $\frac{1}{4}$-smooth. To show $\hat{R}$ is $\frac{B^2}{4}$-smooth, note that given any $w, w' \in \mathbb{R}^d$,

$$
\left\| \nabla \hat{R}(w) - \nabla \hat{R}(w') \right\| = \left\| \frac{1}{n} \sum_{i=1}^{n} \left( \ell'_{\log}(y_i \langle w, x_i \rangle) - \ell'_{\log}(y_i \langle w', x_i \rangle) \right) y_i x_i \right\|
$$

$$
\leq \frac{1}{n} \sum_{i=1}^{n} \left| \ell'_{\log}(y_i \langle w, x_i \rangle) - \ell'_{\log}(y_i \langle w', x_i \rangle) \right| B
$$

$$
\leq \frac{B}{4n} \sum_{i=1}^{n} \left| y_i \langle w, x_i \rangle - y_i \langle w', x_i \rangle \right|
$$

$$
\leq \frac{B}{4n} \sum_{i=1}^{n} \|w - w'\| B = \frac{B^2}{4} \|w - w'\|.
$$

The following analysis basically comes from the proof of (Bubeck, 2014, Theorem 6.3); we include it for completeness, and also handle the last iterate. Let $w^* := \text{arg min}_{w \in B(1/\sqrt{\epsilon})} \hat{R}(w)$. Convexity gives

$$
\hat{R}(w_t) - \hat{R}(w^*) \leq \left\langle \nabla \hat{R}(w_t), w_t - w^* \right\rangle = \left\langle \nabla \hat{R}(w_t), w_t - w_{t+1} \right\rangle + \left\langle \nabla \hat{R}(w_t), w_{t+1} - w^* \right\rangle.
$$

Smoothness implies

$$
\left\langle \nabla \hat{R}(w_t), w_t - w_{t+1} \right\rangle \leq \hat{R}(w_t) - \hat{R}(w_{t+1}) + \frac{B^2}{2} \|w_t - w_{t+1}\|^2
$$

$$
\leq \hat{R}(w_t) - \hat{R}(w_{t+1}) + \frac{1}{2\eta} \|w_t - w_{t+1}\|^2.
$$

On the other hand, the projection step ensures

$$
\left\langle \nabla \hat{R}(w_t), w_{t+1} - w^* \right\rangle \leq \frac{1}{\eta} \left( w_t - w_{t+1}, w_{t+1} - w^* \right)
$$

$$
= \frac{1}{\eta} \left( \|w_t - w^*\|^2 - \|w_{t+1} - w^*\|^2 - \|w_t - w_{t+1}\|^2 \right).
$$

Therefore

$$
\hat{R}(w_t) - \hat{R}(w^*) \leq \hat{R}(w_t) - \hat{R}(w_{t+1}) + \frac{1}{2\eta} \left( \|w_t - w^*\|^2 - \|w_{t+1} - w^*\|^2 \right)
$$

which implies

$$
\hat{R}(w_{t+1}) - \hat{R}(w^*) \leq \frac{1}{2\eta} \left( \|w_t - w^*\|^2 - \|w_{t+1} - w^*\|^2 \right). \tag{15}
$$

Next we show that $\hat{R}(w_{t+1}) \leq \hat{R}(w_t)$. Smoothness implies

$$
\hat{R}(w_{t+1}) - \hat{R}(w_t) \leq \left\langle \nabla \hat{R}(w_t), w_{t+1} - w_t \right\rangle + \frac{B^2}{2} \|w_{t+1} - w_t\|^2
$$

$$
\leq -\frac{1}{\eta} \|w_t - w_{t+1}\|^2 + \frac{B^2}{2} \|w_{t+1} - w_t\|^2
$$

$$
\leq -\frac{1}{\eta} \|w_t - w_{t+1}\|^2 + \frac{1}{2\eta} \|w_{t+1} - w_t\|^2
$$

$$
= -\frac{1}{2\eta} \|w_{t+1} - w_t\|^2,
$$

where we also use the property of the projection step on the second line.

It now follow from eq. (15) and $\hat{R}(w_{t+1}) \leq \hat{R}(w_t)$ that for $t \geq 1$,

$$
\hat{R}(w_t) \leq \hat{R}(w^*) + \frac{\|w_0 - w^*\|^2}{2\eta t} \leq \hat{R}(w^*) + \frac{1}{2\eta \epsilon t}.
$$

\[\square\]
Lemma C.8. If $\|x\| \leq B$ almost surely, then with probability $1 - \delta$, for all $w \in B \left( \frac{1}{\sqrt{\epsilon}} \right)$,

$$\left| \mathcal{R}(w) - \hat{\mathcal{R}}(w) \right| \leq \frac{2B}{\sqrt{\epsilon n}} + 3 \left( \frac{B}{\sqrt{\epsilon}} + 1 \right) \sqrt{\frac{\ln(4/\delta)}{2n}}.$$  

Proof. Note that $\ell_{\log}(z) \leq |z| + 1$, therefore

$$\ell_{\log}(y(w, x)) \leq \|w\| \|x\| + 1 \leq \frac{B}{\sqrt{\epsilon}} + 1.$$  

Since $\ell_{\log}$ is 1-Lipschitz continuous, (Shalev-Shwartz and Ben-David, 2014, Theorem 26.5, Lemma 26.9, Lemma 26.10) imply that with probability $1 - \delta$, for all $w \in B \left( \frac{1}{\sqrt{\epsilon}} \right)$,

$$\mathcal{R}(w) - \hat{\mathcal{R}}(w) \leq \frac{2B}{\sqrt{\epsilon n}} + 3 \left( \frac{B}{\sqrt{\epsilon}} + 1 \right) \sqrt{\frac{\ln(2/\delta)}{2n}}.$$  

Next we can just apply the same technique and get a uniform deviation bound on $\hat{\mathcal{R}}(w) - \mathcal{R}(w)$.

We can now prove Lemma C.5.

Proof of Lemma C.5 for bounded distributions. Lemma C.7 implies that

$$\hat{\mathcal{R}}(w_t) - \min_{0 \leq \rho \leq 1/\sqrt{\epsilon}} \mathcal{R}(\rho \bar{u}) \leq \frac{1}{2\eta t} = \frac{B^2}{8\epsilon t}.$$  

Moreover, Lemma C.8 ensures with probability $1 - \delta$, for all $w \in B_2 \left( \frac{1}{\sqrt{\epsilon}} \right)$,

$$\left| \hat{\mathcal{R}}(w) - \mathcal{R}(w) \right| \leq \frac{2B}{\sqrt{\epsilon n}} + 3 \left( \frac{B}{\sqrt{\epsilon}} + 1 \right) \sqrt{\frac{\ln(4/\delta)}{2n}} = O \left( \frac{B}{\sqrt{\epsilon n}} \sqrt{\ln(1/\delta)} \right).$$  

Therefore, to ensure $\mathcal{R}(w_t) - \min_{0 \leq \rho \leq 1/\sqrt{\epsilon}} \mathcal{R}(\rho \bar{u}) \leq \epsilon_t$, we only need

$$O \left( \frac{B^2}{\epsilon \epsilon_t} \right) \text{ steps, and } O \left( \frac{(B + 1)^2 \ln(1/\delta)}{\epsilon \epsilon_t^2} \right) \text{ samples.}$$  

Next we prove the norm lower bound on $\|w_t\|$.

Proof of Lemma C.6. First, we consider the case $\|x\| \leq B$ almost surely. It follows from Lemma C.2 that

$$\mathcal{R}(\rho \bar{u}) \leq \frac{12U}{\rho} + \rho B \cdot \text{OPT}.$$  

(16)

Let $\bar{\rho} := \sqrt{\frac{12U}{B \cdot \text{OPT}}}$. We consider two cases below, $\bar{\rho} \leq \frac{1}{\sqrt{\epsilon}}$ or $\bar{\rho} \geq \frac{1}{\sqrt{\epsilon}}$.

First, we assume $\bar{\rho} \leq \frac{1}{\sqrt{\epsilon}}$. Then by the conditions of Lemma C.6 and eq. (16), we have

$$\mathcal{R}(w_t) \leq \mathcal{R}(\bar{\rho} \bar{u}) + \epsilon_t \leq 2\sqrt{12UB \cdot \text{OPT}} + \sqrt{\epsilon} < 2\sqrt{12UB \cdot \frac{R^4}{500U^3B}} + \sqrt{\frac{R^4}{36U^2}} < 2\frac{R^2}{6U} + \frac{R^2}{6U} = \frac{R^2}{2U}.$$  

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It then follows from Lemma \[C.3\] that \( R\|w_t\| \geq 2 \), and

\[
\frac{R}{U\|w_t\|} \leq \mathcal{R}(w_t) \leq \mathcal{R}(\bar{\rho} \bar{u}) + \epsilon \leq 2\sqrt{12UB \cdot \text{OPT}} + \sqrt{\epsilon}.
\]

since \( \bar{\rho} \leq \frac{1}{\sqrt{\epsilon}} \),

\[
\sqrt{\epsilon} \leq \frac{1}{\bar{\rho}} = \sqrt{\frac{B \cdot \text{OPT}}{12U}}.
\]

As a result, \( \frac{R}{U\|w_t\|} = O\left(\sqrt{\text{OPT}}\right) \), which implies \( \|w_t\| = \Omega\left(\frac{1}{\sqrt{\epsilon}}\right) \).

Next, assume \( \bar{\rho} \geq \frac{1}{\sqrt{\epsilon}} \), which implies that \( B \cdot \text{OPT} \leq \frac{12U}{12U} \epsilon \).

Moreover, eq. (10) implies

\[
\mathcal{R}\left(\frac{1}{\sqrt{\epsilon}} \bar{u}\right) \leq 12U \sqrt{\epsilon} + \frac{1}{\sqrt{\epsilon}} B \cdot \text{OPT} \leq 12U \sqrt{\epsilon} + \frac{1}{\sqrt{\epsilon}} 12U \epsilon = 24U \sqrt{\epsilon}.
\]

Then because

\[
\mathcal{R}(w_t) \leq \mathcal{R}\left(\frac{1}{\sqrt{\epsilon}} \bar{u}\right) + \epsilon \leq 24U \sqrt{\epsilon} + \sqrt{\epsilon} < 24U \sqrt{\frac{R^2}{72^2U^4}} + \sqrt{\frac{R^4}{36U^2}} = \frac{R^2}{2U},
\]

it further follows from Lemma \[C.3\] that \( R\|w_t\| \geq 2 \), and

\[
\frac{R}{U\|w_t\|} \leq \mathcal{R}\left(\frac{1}{\sqrt{\epsilon}} \bar{u}\right) + \epsilon \leq 24U \sqrt{\epsilon} + \sqrt{\epsilon},
\]

therefore \( \|w_t\| = \Omega\left(\frac{1}{\sqrt{\epsilon}}\right) \).

Now assume \( P_x \) is \((\alpha_1, \alpha_2)\)-sub-exponential. Lemma \[C.2\] implies

\[
\mathcal{R}(\rho \bar{u}) \leq \frac{12U}{\rho} + (1 + 2\alpha_1)\alpha_2 \rho \cdot \text{OPT} \cdot \ln\left(\frac{1}{\text{OPT}}\right).
\]

Let

\[
\bar{\rho} := \sqrt{\frac{12U}{(1 + 2\alpha_1)\alpha_2 \cdot \text{OPT} \cdot \ln(1/\text{OPT})}},
\]

and similarly consider the two cases \( \bar{\rho} \leq \frac{1}{\sqrt{\epsilon}} \) and \( \bar{\rho} \geq \frac{1}{\sqrt{\epsilon}} \), we can finish the proof. \(\square\)

Now we are ready to prove Theorem \[3.5\]

**Proof of Theorem \[3.5\] for bounded distributions.** First, note that if \( \epsilon \) or \( \text{OPT} \) does not satisfy the conditions of Lemma \[C.6\] then Theorem \[3.5\] holds vacuously. Under the conditions of Lemmas \[C.5\] and \[C.6\] let \( \epsilon \leq \epsilon^{3/2} \), we have that projected gradient descent can find \( w_t \) satisfying

\[
\mathcal{R}_{\log}(w_t) \leq \min_{0 \leq \rho \leq 1/\sqrt{\epsilon}} \mathcal{R}_{\log}(\rho \bar{u}) + \epsilon^{3/2},
\]

and

\[
\|w_t\| = \Omega\left(\min\left\{\frac{1}{\sqrt{\epsilon}}, \frac{1}{\sqrt{\epsilon}} \cdot \frac{1}{\sqrt{\text{OPT}}}\right\}\right).
\]

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Now we just need to invoke Lemma 3.6. If $\epsilon \leq \text{OPT}$, then $\|w_t\| = \Omega \left( \frac{1}{\sqrt{\epsilon \text{OPT}}} \right)$, and Lemma 3.6 implies

$$R_{0-1}(w_t) = O \left( \max \left\{ \text{OPT}, \sqrt{\epsilon^{3/2} \text{OPT}}, C_\kappa \cdot \text{OPT} \right\} \right)$$

$$= O \left( (1 + C_\kappa) \text{OPT} \right).$$

If $\epsilon \geq \text{OPT}$, then $\|w_t\| = \Omega \left( \frac{1}{\sqrt{\epsilon}} \right)$, and similarly we can show

$$R_{0-1}(w_t) = O \left( \max \left\{ \text{OPT}, \sqrt{\epsilon^{3/2} \epsilon}, C_\kappa \epsilon \right\} \right)$$

$$= O \left( (1 + C_\kappa) (\text{OPT} + \epsilon) \right).$$

The sample and iteration complexity follow from Lemma C.5 and that $\epsilon_\ell = \epsilon^{3/2}$. □

Sub-exponential distributions. Next we handle $(\alpha_1, \alpha_2)$-sub-exponential distributions. We will prove Lemma C.5 for sub-exponential distributions; the rest of the proof is similar to the bounded case and thus omitted.

Let the target zero-one error $\epsilon$, the target optimization error $\epsilon_\ell$, and failure probability $\delta$ be given. Given $r > 0$, we overload the notation a little bit and let

$$\delta(r) := d\alpha_1 \exp \left( -\frac{r}{\alpha_2 \sqrt{d}} \right).$$

In particular, note that

$$\Pr_{x \sim P_x} \left( \|x\| \geq r \right) \leq \sum_{j=1}^d \Pr \left( |x_j| \geq \frac{r}{\sqrt{d}} \right) \leq d\alpha_1 \exp \left( -\frac{r}{\sqrt{d} \alpha_2} \right) = \delta(r).$$

Let $B > 1$ be large enough such that

$$(1 - \delta(B))^{100(B+1)^2 \ln(4/\delta)/(\alpha_2^2)} \geq 1 - \delta, \quad \text{and} \quad \alpha_1 (\alpha_2 + B) \exp \left( -\frac{B}{\alpha_2} \right) \leq \epsilon_\ell \sqrt{\epsilon}. \quad (17)$$

We have the following bound on $B$.

**Lemma C.9.** To satisfy eq. (17), it is enough to let

$$B = \Omega \left( \sqrt{d \ln \left( \frac{d}{\epsilon \ell \delta} \right)} \right).$$

**Proof.** First, we let $B \geq \alpha_2 \sqrt{d \ln(2d\alpha_1)}$ to ensure $\delta(B) \leq 1/2$. Since for $0 \leq z \leq 1/2$, we have $e^{-z} \geq 1 - z \geq e^{-2z}$, to satisfy the first condition of eq. (17), it is enough to ensure

$$e^{-\delta(B) \cdot 200(B+1)^2 \ln(4/\delta)/(\alpha_2^2)} \geq e^{-\delta}, \quad \text{equivalently} \quad \delta(B) \leq \frac{\delta \epsilon_\ell^2}{200(B+1)^2 \ln(4/\delta)}. $$

Invoking the definition of $\delta(B)$, we only need

$$B \geq \alpha_2 \sqrt{d \ln \left( \frac{200(B+1)^2 d\alpha_1 \ln(4/\delta)}{\delta \epsilon_\ell^2} \right)}. $$
In other words, it is enough if $B = \Omega \left( \sqrt{d \ln \left( \frac{d}{\epsilon \ell \delta} \right)} \right)$.

Similarly, to satisfy the second condition of eq. (17), we only need

$$B \geq \alpha_2 \ln \left( \frac{\alpha_1 (\alpha_2 + B)}{\epsilon \ell \sqrt{\epsilon}} \right),$$

and it is enough if $B = \Omega \left( \sqrt{d \ln \left( \frac{d}{\epsilon \ell \delta} \right)} \right)$. \hfill \Box

Now we define a truncated logistic loss $\ell_\circ \log$ as following:

$$\ell_\circ \log(z) := \begin{cases} 
\ell_\log \left( -\frac{B}{\sqrt{\epsilon}} \right) & \text{if } z \leq -\frac{B}{\sqrt{\epsilon}}, \\
\ell_\log(z) & \text{if } z \geq -\frac{B}{\sqrt{\epsilon}}.
\end{cases}$$

We also let $\mathcal{R}_\circ(w)$ and $\hat{\mathcal{R}}_\circ(w)$ denote the population and empirical risk with the truncated logistic loss. We have the next result.

**Lemma C.10.** Suppose $B > 1$ is chosen according to eq. (17). Using a constant step size $4/B^2$, and

$$\frac{100 (B + 1)^2 \ln(4/\delta)}{\epsilon \ell^2}$$

samples, and $\frac{B^2}{4 \epsilon \ell}$ steps,

with probability $1 - 2\delta$, projected gradient descent can ensure

$$\mathcal{R}_\circ(w_t) \leq \min_{0 \leq \rho \leq 1/\sqrt{\epsilon}} \mathcal{R} (\rho \bar{u}) + \epsilon_\ell.$$

**Proof.** It follows from eq. (17) that with probability $1 - \delta$, it holds that $||x_i|| \leq B$ for all training examples. Therefore Lemma [C.7] implies that

$$\hat{\mathcal{R}}(w_t) \leq \min_{0 \leq \rho \leq 1/\sqrt{\epsilon}} \hat{\mathcal{R}} (\rho \bar{u}) + \frac{B^2}{8 \epsilon \ell}.$$

Since $||x_i|| \leq B$, and the domain is $B(1/\sqrt{\epsilon})$, it follows that

$$\hat{\mathcal{R}}_\circ(w_t) \leq \min_{0 \leq \rho \leq 1/\sqrt{\epsilon}} \hat{\mathcal{R}}_\circ (\rho \bar{u}) + \frac{B^2}{8 \epsilon \ell}.$$

Letting $t = \frac{B^2}{4 \epsilon \ell}$, we get

$$\hat{\mathcal{R}}_\circ(w_t) \leq \min_{0 \leq \rho \leq 1/\sqrt{\epsilon}} \hat{\mathcal{R}}_\circ (\rho \bar{u}) + \frac{\epsilon_\ell}{2}. \tag{18}$$

Note that by the construction of the truncated logistic loss, it holds that

$$\ell_\circ \log(z) \leq \frac{B}{\sqrt{\epsilon}} + 1.$$

Then by invoking the standard Rademacher complexity results ([Shalev-Shwartz and Ben-David, 2014, Theorem 26.5, Lemma 26.9, Lemma 26.10]), and recall that we work under the event $||x_i|| \leq B$ for all training examples, we can show with probability $1 - 2\delta$ that for all $w \in B(1/\sqrt{\epsilon})$,

$$\left| \mathcal{R}_\circ(w) - \hat{\mathcal{R}}_\circ(w) \right| \leq \frac{2B}{\sqrt{\epsilon n}} + 3 \left( \frac{B}{\sqrt{\epsilon}} + 1 \right) \sqrt{\frac{\ln(4/\delta)}{2n}} + \frac{3(B + 1)}{\sqrt{\epsilon}} \sqrt{\frac{\ln(4/\delta)}{2n}} \leq \frac{2(B + 1)}{\sqrt{\epsilon}} \sqrt{\frac{\ln(4/\delta)}{n}} + \frac{3(B + 1)}{\sqrt{\epsilon}} \sqrt{\frac{\ln(4/\delta)}{2n}} \leq 5(B + 1) \sqrt{\frac{\ln(4/\delta)}{en}}.$$
Letting \( n = \frac{100(B+1)^2 \ln(4/\delta)}{\epsilon^2} \), we have
\[
\left| \mathcal{R}^n(w) - \tilde{\mathcal{R}}^n(w) \right| \leq \frac{\epsilon}{2}. \tag{19}
\]

It then follows from eqs. 18 and 19 that with probability \( 1 - 2\delta \),
\[
\mathcal{R}^n(w_t) \leq \min_{0 \leq \rho_0 \leq 1/\sqrt{\epsilon}} \mathcal{R}^n(\rho \tilde{u}) + \epsilon_t \leq \min_{0 \leq \rho_0 \leq 1/\sqrt{\epsilon}} \mathcal{R}(\rho \tilde{u}) + \epsilon_t,
\]
where we use \( \ell_\log \leq \ell_\log \) in the last inequality. \( \square \)

Finally, we show that \( \mathcal{R}^n(w_t) \) is close to \( \mathcal{R}(w_t) \).

**Lemma C.11.** For all \( w \in B(1/\sqrt{\epsilon}) \), it holds that \( \mathcal{R}^n(w) \geq \mathcal{R}(w) - \epsilon_t \).

**Proof.** Note that if \( \ell_\log (y(w,x)) \neq \ell_\log (y(w,x)) \), then \( y(w,x) \leq -B/\sqrt{\epsilon} \), which implies \( |\langle w, x \rangle| \geq B/\sqrt{\epsilon} \). Moreover, in this case
\[
\ell_\log (y(w,x)) - \ell_\log (y(w,x)) \leq \ell_\log (y(w,x)) - \ell_\log (0) \leq |\langle w, x \rangle|.
\]
Therefore
\[
\mathcal{R}(w) - \mathcal{R}^n(w) = E_{x \sim P_x} \left[ \ell_\log (y(w,x)) - \ell_\log (y(w,x)) \right] \leq E_{x \sim P_x} \left[ |\langle w, x \rangle| \mathbb{1}_{\langle w, x \rangle \geq B/\sqrt{\epsilon}} \right].
\]

We can then invoke eq. 11 and get
\[
\mathcal{R}(w) - \mathcal{R}^n(w) \leq \alpha_1 \left( \alpha_2 ||w|| + \frac{B}{\sqrt{\epsilon}} \right) \exp \left( -\frac{B}{\alpha_2 ||w|| \sqrt{\epsilon}} \right). \tag{20}
\]
Note that the right hand side of eq. 20 is increasing with \( ||w|| \), therefore we can let \( ||w|| \) be \( 1/\sqrt{\epsilon} \) and get
\[
\mathcal{R}(w) - \mathcal{R}^n(w) \leq \alpha_1 \frac{\alpha_2 + B}{\sqrt{\epsilon}} \exp \left( -\frac{B}{\alpha_2} \right) \leq \epsilon_t,
\]
where we use eq. 17 in the last inequality. \( \square \)

Now putting everything together, under the conditions of Lemma C.10 with probability \( 1 - 2\delta \), projected gradient descent ensures \( \mathcal{R}(w_t) \leq \min_{0 \leq \rho_0 \leq 1/\sqrt{\epsilon}} \mathcal{R}(\rho \tilde{u}) + 2\epsilon_t \). Moreover, by applying Lemma C.9 to Lemma C.10, we can see the sample complexity is \( \tilde{O} \left( d \ln(1/\delta)^3 / (\epsilon \epsilon_t^2) \right) \), and the iteration complexity is \( \tilde{O} \left( d \ln(1/\delta)^2 / (\epsilon \epsilon_t) \right) \).

### C.2 Omitted proofs from Section 3.2

In this section, we prove Lemma 3.6. We first prove the following approximation bound after we replace the true label with the label given by the ground-truth solution, which covers Lemma 3.10 and sub-exponential distributions.

**Lemma C.12 (Lemma 3.10, including the sub-exponential case).** For \( \ell \in \ell_\log, \ell_h \), if \( ||x|| \leq B \) almost surely,
\[
\text{term (3)} \leq B||\tilde{w} - \tilde{w}|| \cdot \text{OPT}.
\]
If \( P_x \) is \( (\alpha_1, \alpha_2) \)-sub-exponential, then
\[
\text{term (3)} \leq (1 + 2\alpha_1)\alpha_2 ||\tilde{w} - \tilde{w}|| \cdot \text{OPT} \cdot \ln(1/\text{OPT}).
\]
Proof. Note that for both the logistic loss and the hinge loss, it holds that \( \ell(-z) - \ell(z) = z \), therefore

\[
\text{term (3)} = \mathbb{E}_{(x,y) \sim P} \left[ \mathbb{I}_{y \neq \text{sign}(\langle \hat{w}, x \rangle)} \cdot y \langle \hat{w} - \tilde{w}, x \rangle \right],
\]  

(21)

It then follows from the triangle inequality that

\[
|\text{term (3)}| \leq \mathbb{E}_{(x,y) \sim P} \left[ \mathbb{I}_{y \neq \text{sign}(\langle \hat{w}, x \rangle)} |\langle \hat{w} - \tilde{w}, x \rangle| \right]
\]

Now we can invoke Lemma A.2 with \( w = \bar{w} \) and \( w' = \bar{w} - \hat{w} \) to prove Lemma C.12.

Next we prove the lower bound on term (4).

Proof of Lemma 3.11. Note that in term (4), we only care about \( \langle \hat{w}, x \rangle \) and \( \langle \bar{w}, x \rangle \), therefore we can focus on the two-dimensional space spanned by \( \bar{w} \) and \( \hat{w} \). Let \( \phi \) denote the angle between \( \bar{w} \) and \( \hat{w} \). Without loss of generality, we can consider the following graph, where we put \( \bar{w} \) at angle 0, and \( \hat{w} \) at angle \( \phi \).

![Graph](image)

We divide the graph into four parts given by different polar angles: (i) \((-\frac{\pi}{2}, -\frac{\pi}{2} + \phi)\), (ii) \((-\frac{\pi}{2} + \phi, \frac{\pi}{2})\), (iii) \((\frac{\pi}{2}, \frac{\pi}{2} + \phi)\), and (iv) \((\frac{\pi}{2} + \phi, \frac{3\pi}{2})\). Note that term (4) is 0 on parts (ii) and (iv), therefore we only need to consider parts (i) and (iii):

\[
\text{term (4)} = \mathbb{E}_{(i) \text{ and } (iii)} \left[ \ell \left( \text{sign} \left( \langle \hat{w}, x \rangle \right) \langle \hat{w}, x \rangle \right) - \ell \left( \text{sign} \left( \langle \bar{w}, x \rangle \right) \langle \bar{w}, x \rangle \right) \right]
\]

\[
= \mathbb{E}_{(i) \text{ and } (iii)} \left[ -\text{sign} \left( \langle \hat{w}, x \rangle \right) \langle \hat{w}, x \rangle \right].
\]

Here we use the fact that \( \ell(-z) - \ell(z) = z \) for both the logistic loss and the hinge loss.

For simplicity, let \( p \) denote the density of the projection of \( P_x \) onto the space spanned by \( \hat{w} \) and \( \tilde{w} \). Under Assumption 3.2 we have

\[
\text{term (4)} = \mathbb{E}_{(i) \text{ and } (iii)} \left[ -\text{sign} \left( \langle \hat{w}, x \rangle \right) \langle \hat{w}, x \rangle \right]
\]

\[
= \int_{0}^{\infty} \int_{-\frac{\pi}{2}}^{\pi} -r \| \hat{w} \| \cos(\phi - \theta)p(r, \theta)r \, d\theta \, dr + \int_{0}^{\infty} \int_{\frac{\pi}{2}}^{\frac{\pi}{2} + \phi} r \| \hat{w} \| \cos(\phi - \theta)p(r, \theta)r \, d\theta \, dr
\]

\[
\geq \frac{2}{U} \int_{0}^{R} \int_{0}^{\phi} r \| \hat{w} \| \sin(\theta)r \, d\theta \, dr
\]

\[
= \frac{2 R^3 \| \hat{w} \| (1 - \cos(\phi))}{3U} \geq \frac{4 R^3 \| \hat{w} \|^2 \phi^2}{3U \pi^2},
\]

where we use the fact that \( 1 - \cos(\phi) \geq \frac{2\phi^2}{\pi^2} \) for all \( \phi \in [0, \pi] \).

\[\square\]
Next, we prove the following upper bound on term (5), covering Lemma 3.12 and the sub-exponential case.

**Lemma C.13 (Lemma 3.12 including the sub-exponential case).** For $\ell = \ell_h$, term (5) is 0. For $\ell = \ell_{\log}$, under Assumption 3.3, if $\|x\| \leq B$ almost surely, then

$$|\text{term (5)}| \leq 12C_\kappa \cdot \frac{\varphi(\hat{w}, \hat{w})}{\|\hat{w}\|},$$

where $C_\kappa := \int_0^B \kappa(r) \, dr$, while if $P_x$ is $(\alpha_1, \alpha_2)$-sub-exponential, then

$$|\text{term (5)}| \leq 2\alpha_1 \text{OPT}^2 + 12C_\kappa \cdot \frac{\varphi(\hat{w}, \hat{w})}{\|\hat{w}\|},$$

where $C_\kappa := \int_0^{3\alpha_2 \ln(1/\text{OPT})} \kappa(r) \, dr$.

**Proof.** For the hinge loss, term (5) is 0 simply because $\ell_h(z) = 0$ when $z \geq 0$. Next we consider the logistic loss.

Note that term (5) only depends on $\langle \hat{w}, x \rangle$ and $\langle \hat{w}, x \rangle$, therefore we can focus on the subspace spanned by $\hat{w}$ and $\hat{w}$. For simplicity, let $\varphi$ denote the density function of the projection of $P_x$ onto the space spanned by $\hat{w}$ and $\hat{w}$. Moreover, without loss of generality we can assume $\hat{w}$ has polar angle 0 while $\hat{w}$ has polar angle $\varphi$, where we let $\varphi$ denote $\varphi(\hat{w}, \hat{w})$ for simplicity. It then follows that

$$\text{term (5)} = \int_0^\infty \int_0^{2\pi} \ell_{\log} \left( r \|\hat{w}\| |\cos(\theta - \varphi)| \right) \left( \int_0^\pi \ell_{\log} \left( r \|\hat{w}\| |\cos(\theta)| \right) \, d\theta \right) \, r \, dr \, d\theta\right) - \int_0^B \int_0^{2\pi} \ell_{\log} \left( r \|\hat{w}\| |\cos(\theta)| \right) \left( \int_0^\pi \ell_{\log} \left( r \|\hat{w}\| |\cos(\theta)| \right) \, d\theta \right) \, r \, dr \, d\theta\right).$$

First, if $\|x\| \leq B$ almost surely, then

$$|\text{term (5)}| \leq \int_0^B \int_0^{2\pi} \ell_{\log} \left( r \|\hat{w}\| |\cos(\theta)| \right) \left( \int_0^{2\pi} |p(r, \theta + \varphi) - p(r, \theta)| \, r \, d\theta \right) \, dr \, d\theta\right) \leq \varphi \int_0^B \kappa(r) \left( \int_0^{2\pi} \ell_{\log} \left( r \|\hat{w}\| |\cos(\theta)| \right) \, r \, d\theta \right) \, dr.$$

Then Lemma A.1 implies

$$|\text{term (5)}| \leq \varphi \int_0^B \kappa(r) \frac{8\sqrt{2}}{\|\hat{w}\|} \, dr = 8\sqrt{2}C_\kappa \cdot \frac{\varphi}{\|\hat{w}\|} \leq 12C_\kappa \cdot \frac{\varphi}{\|\hat{w}\|}.$$
Invoking the previous bound for bounded distributions, we get
\[
\text{term (5)} \leq 2\alpha_1\text{OPT}^2 + 12\|\hat{w}\| \int_0^{2\sqrt{2}\alpha_2 \ln(\frac{\|\hat{w}\|}{\kappa(r)})} \kappa(r) \, dr \leq 2\alpha_1\text{OPT}^2 + 12C_\kappa \cdot \frac{\varphi}{\|\hat{w}\|},
\]
where \(C_\kappa := \int_0^{\sqrt{2}\alpha_2 \ln(\frac{\|\hat{w}\|}{\kappa(r)})} \kappa(r) \, dr\). Similarly, we can show
\[
-\text{term (5)} \leq 2\alpha_1\text{OPT}^2 + 12C_\kappa \cdot \frac{\varphi}{\|\hat{w}\|}.
\]

Next we prove Lemma 3.13, which is basically (Diakonikolas et al., 2020d, Claim 3.4).

**Proof of Lemma 3.13** Under Assumption 3.2, we have
\[
\Pr\left(\text{sign} \left(\langle \hat{w}, x \rangle\right) \neq \text{sign} \left(\langle \bar{w}, x \rangle\right)\right) \leq 2\varphi(\hat{w}, \bar{w}) \int_0^\infty \sigma(r) \, dr \leq 2U\varphi(\hat{w}, \bar{w}).
\]

Lastly, we prove Lemma 3.6 for sub-exponential distributions.

**Proof of Lemma 3.6, sub-exponential distributions.** For simplicity, let \(\varphi\) denotes \(\varphi(\hat{w}, \bar{w})\). Lemmas 3.11, C.12 and C.13 imply
\[
C_1\|\hat{w}\|\varphi^2 \leq \epsilon_\ell + C_2\|\hat{w} - \hat{w}\| \cdot \text{OPT} \cdot \ln\left(\frac{1}{\text{OPT}}\right) + C_3\text{OPT}^2 + C_4C_\kappa \cdot \frac{\varphi}{\|\hat{w}\|}
\]
\[
\leq \epsilon_\ell + C_2\|\hat{w}\|\varphi \cdot \text{OPT} \cdot \ln\left(\frac{1}{\text{OPT}}\right) + C_3\text{OPT}^2 + C_4C_\kappa \cdot \frac{\varphi}{\|\hat{w}\|},
\]
where \(C_1 = \frac{4R^3}{3\|\hat{w}\|}\), and \(C_2 = (1 + 2\alpha_1)\alpha_2\), and \(C_3 = 2\alpha_1\), and \(C_4 = 12\). It follows that at least one of the following four cases is true:

1. \(C_1\|\hat{w}\|\varphi^2 \leq 4\epsilon_\ell\), which implies \(\varphi = O\left(\sqrt{\epsilon_\ell/\|\hat{w}\|}\right)\).
2. \(C_1\|\hat{w}\|\varphi^2 \leq 4C_2\|\hat{w}\|\varphi \cdot \text{OPT} \cdot \ln\left(\frac{1}{\text{OPT}}\right)\), which implies \(\varphi = O\left(\text{OPT} \ln\left(\frac{1}{\text{OPT}}\right)\right)\).
3. \(C_1\|\hat{w}\|\varphi^2 \leq 4C_3\text{OPT}^2\), which implies \(\varphi = O(\text{OPT})\) since \(\|\hat{w}\| = \Omega(1)\).
4. Lastly,
\[
C_1\|\hat{w}\|\varphi^2 \leq 4C_2C_\kappa \cdot \frac{\varphi}{\|\hat{w}\|}, \quad \text{which implies} \quad \varphi = O\left(\frac{C_\kappa}{\|\hat{w}\|^2}\right).
\]

Finally, we just need to invoke Lemma 3.13 to finish the proof.

**C.3 Omitted proofs from Section 3.3**

We first prove the upper bound of term (5) under Assumption 3.2 without assuming the radially Lipschitz condition.

**Proof of Lemma 3.14** Note that
\[
\text{term (5)} \leq \mathbb{E} \left[\ell_\log \left(\text{sign} \left(\langle \hat{w}, x \rangle\right)\right) \left\langle \hat{w}, x \right\rangle\right] = \mathbb{E} \left[\ell_\log \left(\left|\langle \hat{w}, x \rangle\right|\right)\right] \leq 12U \frac{\|\hat{w}\|}{\|\bar{w}\|},
\]
where we invoke Lemma C.1 at the end. Similarly, we can show
\[
-\text{term (5)} \leq 12U \frac{\|\hat{w}\|}{\|\bar{w}\|} = 12U \frac{\|\hat{w}\|}{\|\bar{w}\|}.
\]
Next we prove a general result similar to Lemma 3.6.

**Theorem C.14.** Under Assumption 3.2, suppose \( \hat{w} \) satisfies \( R_{\log}(\hat{w}) \leq R_{\log}(\|\hat{w}\|) + \epsilon \ell \) for some \( \epsilon \ell \in [0, 1) \). If \( \|x\| \leq B \) almost surely, then

\[
\varphi(\hat{w}, \bar{u}) = O \left( \max \left\{ \text{OPT}, \sqrt{\frac{\epsilon \ell}{\|\hat{w}\|}}, \frac{1}{\|\hat{w}\|} \right\} \right).
\]

If \( P_x \) is \((\alpha_1, \alpha_2)\)-sub-exponential and \( \|\hat{w}\| = \Omega(1) \), then

\[
\varphi(\hat{w}, \bar{u}) = O \left( \max \left\{ \text{OPT}, \sqrt{\frac{\epsilon \ell}{\|\hat{w}\|}}, \frac{1}{\|\hat{w}\|} \right\} \right).
\]

**Proof.** For simplicity, let \( \varphi \) denote \( \varphi(\hat{w}, \bar{u}) \). Consider the case \( \|x\| \leq B \) almost surely. The condition \( R_{\log}(\hat{w}) \leq R_{\log}(\|\hat{w}\|) + \epsilon \ell \), and Lemmas 3.11, 3.14 and C.12 imply

\[
C_1 \|\hat{w}\| \varphi^2 \leq \epsilon \ell \cdot \|\hat{w}\| \cdot \text{OPT} + \frac{C_2}{\|\hat{w}\|},
\]

where \( C_1 = 4R^3/(3U \pi^2) \) and \( C_2 = 12U \). Now at least one of the following three cases is true:

1. \( C_1 \|\hat{w}\| \varphi^2 \leq 3\epsilon \ell \), which implies \( \varphi = O \left( \sqrt{\epsilon \ell/\|\hat{w}\|} \right) \);
2. \( C_1 \|\hat{w}\| \varphi^2 \leq 3B \|\hat{w}\| \varphi \cdot \text{OPT} \), which implies \( \varphi = O(\text{OPT}) \);
3. \( C_1 \|\hat{w}\| \varphi^2 \leq 3C_1/\|\hat{w}\| \), which implies \( \varphi = O(1/\|\hat{w}\|) \).

The proof of the sub-exponential case is similar. \( \square \)

Now we prove Lemma 3.15.

**Proof of Lemma C.15** First, if \( \epsilon \) or OPT does not satisfy the conditions of Lemma C.6, then Lemma C.15 holds vacuously; therefore in the following we consider the settings of Lemmas C.5 and C.6 with \( \epsilon \ell = \sqrt{\epsilon} \).

First, if \( \|x\| \leq B \) almost surely, eq. (14) and Theorem C.14 imply

\[
\varphi(w_t, \bar{u}) = O \left( \max \left\{ \text{OPT}, \sqrt{\frac{\epsilon \ell}{\|w_t\|}}, \frac{1}{\|w_t\|} \right\} \right),
\]

and moreover Lemma C.6 implies

\[
\|w_t\| = \Omega \left( \min \left\{ \frac{1}{\sqrt{\epsilon}}, \frac{1}{\sqrt{\text{OPT}}} \right\} \right).
\]

If \( \epsilon \leq \text{OPT} \), then \( \|w_t\| = \Omega \left( \frac{1}{\sqrt{\text{OPT}}} \right) \), and

\[
\varphi(w_t, \bar{u}) = O \left( \max \left\{ \text{OPT}, \sqrt{\frac{\epsilon \ell}{\sqrt{\text{OPT}}}}, \sqrt{\text{OPT}} \right\} \right) = O \left( \max \left\{ \text{OPT}, \sqrt{\epsilon \ell \sqrt{\text{OPT}}} \right\} \right) = O \left( \sqrt{\text{OPT}} \right).
\]
If $\epsilon \geq \text{OPT}$, then $\|w_t\| = \Omega \left( \frac{1}{\sqrt{\epsilon}} \right)$, and

$$\varphi(w_t, \bar{u}) = O \left( \max \left\{ \text{OPT}, \sqrt{\epsilon}, \sqrt{\epsilon} \right\} \right)$$

$$= O \left( \max \left\{ \text{OPT}, \sqrt{\epsilon}, \sqrt{\epsilon} \right\} \right)$$

$$= O \left( \sqrt{\text{OPT} + \epsilon} \right).$$

The proof for the sub-exponential case is similar. □

## D Omitted proofs from Section 4

In this section, we prove Theorem 4.1. We first prove a bound on $R_h(\bar{u})$.

**Lemma D.1.** If $\|x\| \leq B$ almost surely, then $R_h(\bar{u}) \leq B \cdot \text{OPT}$, while if $P_x$ is $(\alpha_1, \alpha_2)$-sub-exponential, then $R_h(\bar{u}) \leq (1 + 2\alpha_1)\alpha_2 \cdot \text{OPT} \cdot \ln(1/\text{OPT})$.

**Proof.** Note that

$$R_h(\bar{u}) = \mathbb{E}_{(x,y) \sim P} \left[ \ell_h \left( y(\bar{u}, x) \right) \right] = \mathbb{E}_{(x,y) \sim P} \left[ \mathbb{I}_{\text{sign} \ (\bar{u}, x) \neq y} \right] \left( \langle \bar{u}, x \rangle \right).$$

It then follows from Lemma A.2 that if $\|x\| \leq B$ almost surely, then

$$R_h(\bar{u}) \leq B \cdot \text{OPT},$$

while if $P_x$ is $(\alpha_1, \alpha_2)$-sub-exponential, then

$$R_h(\bar{u}) \leq (1 + 2\alpha_1)\alpha_2 \cdot \text{OPT} \cdot \ln \left( \frac{1}{\text{OPT}} \right).$$

□

Next we prove the following result, which covers Lemma 4.2 but also handles sub-exponential distributions.

**Lemma D.2 (Lemma 4.2 including the sub-exponential case).** Suppose Assumption 3.2 holds. Consider an arbitrary $w \in D$, and let $\varphi$ denote $\varphi(w, \bar{u})$. If $\|x\| \leq B$ almost surely, then

$$R_h(\bar{r}) \leq R_h(\|w\| \bar{u}) + O \left( (\text{OPT} + \epsilon)^2 \right)$$

and

$$R_h(w) - R_h(\|w\| \bar{u}) \geq \frac{4R^3}{3U \pi^2} \|w\|^2 - B \|w\| \varphi \cdot \text{OPT}.$$

If $P_x$ is $(\alpha_1, \alpha_2)$-sub-exponential, then

$$R_h(\bar{r}) \leq R_h(\|w\| \bar{u}) + O \left( (\text{OPT} \cdot \ln(1/\text{OPT}) + \epsilon)^2 \right)$$

and

$$R_h(w) - R_h(\|w\| \bar{u}) \geq \frac{4R^3}{3U \pi^2} \|w\|^2$$

$$- (1 + 2\alpha_1)\alpha_2 \|w\| \varphi \cdot \text{OPT} \cdot \ln(1/\text{OPT}).$$
Proof. First assume $\|x\| \leq B$ almost surely. Note that $\ell_t$ is positive homogeneous, and thus for any positive constant $c$, we have $R_h(cw) = cR_h(w)$. Therefore, if $\bar{r} \leq \|w\|$, then

$$R_h(\bar{r} \bar{u}) = \bar{r} \|w\| R_h(\|w\| \bar{u}) \leq R_h(\|w\| \bar{u}).$$

If $\bar{r} \geq \|w\|$, then

$$R_h(\bar{r} \bar{u}) = R_h(\|w\| \bar{u}) + R_h(\bar{u}) (\bar{r} - \|w\|) \leq R_h(\|w\| \bar{u}) + R_h(\bar{u}) (\bar{r} - 1),$$

since $\|w\| \geq 1$ for all $w \in \mathcal{D}$. Recall that

$$\bar{r} := \frac{1}{\langle v, \bar{u} \rangle} = \frac{1}{\cos(\varphi(v, \bar{u}))} \leq \frac{1}{1 - \varphi(v, \bar{u})^2/2},$$

and therefore the first-phase of algorithm ensures $\bar{r} = 1 + O(\text{OPT} + \epsilon)$ for bounded distributions, and $\bar{r} = 1 + O(\text{OPT} \cdot \ln(1/\text{OPT}) + \epsilon)$ for sub-exponential distributions. Then it follows that for bounded distributions,

$$R_h(\bar{r} \bar{u}) \leq R_h(\|w_t\| \bar{u}) + R_h(\bar{u}) \cdot O(\text{OPT} + \epsilon) \leq R_h(\|w_t\| \bar{u}) + B \cdot \text{OPT} \cdot O(\text{OPT} + \epsilon) = R_h(\|w_t\| \bar{u}) + O((\text{OPT} + \epsilon)^2),$$

where we apply Lemma 3.11 at the end. It also follows directly from Lemmas 3.11, 3.12, and 3.13 that

$$R_h(w) - R_h(\|w\| \bar{u}) \geq \frac{4R^3}{3U\pi^2} \|w\| \varphi^2 - B \|w - \|w\| \bar{u}\| \cdot \text{OPT} \geq \frac{4R^3}{3U\pi^2} \|w\| \varphi^2 - B \|w\| \varphi \cdot \text{OPT}.$$ 

The proof for the sub-exponential case is similar.

Next we prove Theorem 4.1. We first consider the bounded case.

Proof of Theorem 4.1. Here we assume $\|x\| \leq B$ almost surely. We will show that under the conditions of Theorem 4.1, then

$$\mathbb{E} \left[ \min_{\theta \in \mathcal{T}} \varphi_t \right] = O(\text{OPT} + \epsilon), \quad \text{where} \quad \varphi_t := \varphi(w_t, \bar{u}).$$

Further invoking Lemma 3.13 finishes the proof.

Recall that at step $t$, after taking the expectation with respect to $(x_t, y_t)$, we have

$$\mathbb{E} \left[ \|w_{t+1} - \bar{r} \bar{u}\|^2 \right] \leq \|w_t - \bar{r} \bar{u}\|^2 - 2\eta \langle \nabla R_h(w_t), w_t - \bar{r} \bar{u} \rangle + \eta^2 B^2 M(w_t) \leq \|w_t - \bar{r} \bar{u}\|^2 - 2\eta \left( R_h(w_t) - R_h(\bar{r} \bar{u}) \right) + \eta^2 B^2 M(w_t).$$

First, Lemma 3.2 implies

$$R_h(w_t) - R_h(\bar{r} \bar{u}) \geq R_h(w_t) - R_h(\|w_t\| \bar{u}) - O((\text{OPT} + \epsilon)^2) \geq 2C_1\|w_t\| \varphi^2 - B\|w_t\| \varphi_t \cdot \text{OPT} - O((\text{OPT} + \epsilon)^2),$$

where $C_1 := 2R^3/(3U\pi^2)$. Note that if $\varphi_t \leq B \cdot \text{OPT}/C_1$, then eq. (23) holds; therefore in the following we assume

$$\varphi_t \geq \frac{B}{C_1} \cdot \text{OPT},$$

which implies

$$R_h(w_t) - R_h(\bar{r} \bar{u}) \geq C_1\|w_t\| \varphi^2 - O((\text{OPT} + \epsilon)^2) \geq C_1 \varphi^2 - O((\text{OPT} + \epsilon)^2),$$

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since \( \|w\| \geq 1 \) for all \( w \in \mathcal{D} \).

On the other hand, eq. (25) and Lemma 3.13 imply

\[
\mathcal{M}(w_t) = \mathcal{R}_{0-1}(w_t) \leq \text{OPT} + 2U \varphi_t \leq \left( \frac{C_1}{B} + 2U \right) \varphi_t.
\]

Let

\[
C_2 := \frac{C_1}{\left( \frac{C_1}{B} + 2U \right) B^2}.
\]

Note that if \( \varphi_t \leq \epsilon \), then eq. (23) is true; otherwise we can assume \( \epsilon \leq \varphi_t \), and let \( \eta = C_2 \epsilon \), we have

\[
\eta B^2 \mathcal{M}(w_t) \leq C_2 \epsilon B^2 \left( \frac{C_1}{B} + 2U \right) \varphi_t = C_1 \epsilon \varphi_t \leq C_1 \varphi_t^2.
\]

(27)

Now eqs. (24), (26) and (27) imply

\[
\mathbb{E} \left[ \|w_{t+1} - \bar{r} u\|^2 \right] \leq \|w_t - \bar{r} u\|^2 - 2\eta C_1 \varphi_t^2 + \eta \cdot O \left( \left( \text{OPT} + \epsilon \right)^2 \right).
\]

Taking the expectation and average, we have

\[
\mathbb{E} \left[ \frac{1}{T} \sum_{t<T} \varphi_t^2 \right] \leq \frac{\|w_0 - \bar{r} u\|^2}{\eta C_1 T} + \frac{O \left( \left( \text{OPT} + \epsilon \right)^2 \right)}{C_1}.
\]

Note that

\[
\|w_0 - \bar{r} u\| = \tan(\varphi_0) = O \left( \sqrt{\text{OPT} + \epsilon} \right),
\]

and also recall \( \eta = C_2 \epsilon \), we have

\[
\mathbb{E} \left[ \frac{1}{T} \sum_{t<T} \varphi_t^2 \right] \leq \frac{O(\text{OPT} + \epsilon)}{C_1 C_2 \epsilon T} + \frac{O \left( \left( \text{OPT} + \epsilon \right)^2 \right)}{C_1}.
\]

Letting \( T = \Omega(1/\epsilon^2) \), we have

\[
\mathbb{E} \left[ \frac{1}{T} \sum_{t<T} \varphi_t^2 \right] \leq O \left( \left( \text{OPT} + \epsilon \right) + O \left( \left( \text{OPT} + \epsilon \right)^2 \right) = O \left( \left( \text{OPT} + \epsilon \right)^2 \right),
\]

and thus eq. (23) holds.

Next we consider sub-exponential distributions. We first prove the following bound on the square of norm.

Lemma D.3. Suppose \( P_x \) is \((\alpha_1, \alpha_2)\)-sub-exponential. Given any threshold \( \tau > 0 \), it holds that

\[
\mathbb{E} \left[ \|x\|^2 \mathbb{I}_{\|x\| \geq \tau} \right] \leq \alpha_1 \left( \tau^2 + 2\sqrt{\alpha_2 \tau} + 2\alpha_2 \right) \exp \left( -\frac{\tau}{\sqrt{\alpha_2} \alpha_1} \right).
\]

Proof. First recall that

\[
\Pr \left( \|x\| \geq \tau \right) \leq \sum_{j=1}^d \Pr \left( |x_j| \geq \frac{\tau}{\sqrt{d}} \right) \leq \alpha_1 \exp \left( -\frac{\tau}{\sqrt{d} \alpha_2} \right) =: \delta(\tau).
\]
Let $\mu(\tau) := \Pr(\|x\| \geq \tau)$. Integration by parts gives

$$
\mathbb{E} \left[ \|x\|^2 1_{\|x\| \geq \tau} \right] = \int_\tau^\infty r^2 \cdot (-d\mu(r)) = \tau^2 \mu(\tau) + \int_\tau^\infty 2r\mu(r) \, dr \leq \tau^2 \delta(\tau) + \int_\tau^\infty 2r\delta(r) \, dr.
$$

Calculation gives

$$
\mathbb{E} \left[ \|x\|^2 1_{\|x\| \geq \tau} \right] \leq d\alpha_1 \left( \tau^2 + 2\sqrt{d\alpha_2} \tau + 2d\alpha_2^2 \right) \exp \left( -\frac{\tau}{\sqrt{d\alpha_2}} \right).
$$

Now we are ready to prove Theorem 4.1 for sub-exponential distributions.

**Proof of Theorem 4.1** sub-exponential distributions. At step $t$, we have

$$
\|w_{t+1} - \bar{r}u\|^2 \leq \|w_t - \bar{r}u\|^2 - 2\eta \left( \ell'_h(y_t(w_t, x_t)) y_t x_t, w_t - \bar{r}u \right) + \eta^2 \ell'_h(y_t(w_t, x_t))^2 \|x_t\|^2
$$

$$
= \|w_t - \bar{r}u\|^2 - 2\eta \left( \ell'_h(y_t(w_t, x_t)) y_t x_t, w_t - \bar{r}u \right) - \eta^2 \ell'_h(y_t(w_t, x_t))^2 \|x_t\|^2,
$$

where we use $(\ell'_h)^2 = -\ell'_h$. Next we bound $\mathbb{E}_{(x_t, y_t)} \left[ -\ell'_h(y_t(w_t, x_t)) \|x_t\|^2 \right]$. Let $\tau := \sqrt{d\alpha_2} \ln(d/\epsilon)$. When $\|x_t\| \leq \tau$, we have

$$
\mathbb{E} \left[ -\ell'_h(y_t(w_t, x_t)) \|x_t\|^2 1_{\|x_t\| \leq \tau} \right] \leq \tau^2 \mathcal{M}(w_t) \leq d\alpha_2 \mathcal{M}(w_t) \cdot \ln(d/\epsilon)^2.
$$

On the other hand, when $\|x_t\| \geq \tau$, Lemma D.3 implies

$$
\mathbb{E} \left[ -\ell'_h(y_t(w_t, x_t)) \|x_t\|^2 1_{\|x_t\| \geq \tau} \right] \leq \mathbb{E} \left[ \|x_t\|^2 1_{\|x_t\| \geq \tau} \right] \leq d\alpha_1 \cdot O \left( \frac{d}{d\ln(d/\epsilon)^2} \right) \cdot \frac{\epsilon}{d} = O \left( \frac{d}{d \ln(d/\epsilon)^2} \right),
$$

where we also use $\ln(1/\epsilon) > 1$, since $\epsilon < 1/e$. To sum up,

$$
\mathbb{E}_{(x_t, y_t)} \left[ -\ell'_h(y_t(w_t, x_t)) \|x_t\|^2 \right] \leq Cd(\mathcal{M}(w_t) + \epsilon) \cdot \ln(d/\epsilon)^2
$$

for some constant $C$.

Now taking the expectation with respect to $(x_t, y_t)$ on both sides of eq. (28), we have

$$
\mathbb{E} \left[ \|w_{t+1} - \bar{r}u\|^2 \right] \leq \|w_t - \bar{r}u\|^2 - 2\eta \left( \mathcal{R}_h(w_t) - \mathcal{R}_h(\bar{r}u) \right) + \eta^2 Cd(\mathcal{M}(w_t) + \epsilon) \cdot \ln(d/\epsilon)^2.
$$

Similarly to the bounded case, we will show that

$$
\mathbb{E} \left[ \min_{0 \leq t < T} \varphi_t \right] = O \left( \text{OPT} \cdot \ln(1/\text{OPT}) + \epsilon \right), \quad \text{where} \quad \varphi_t := \varphi(w_t, \bar{u}).
$$

First, Lemma D.2 implies

$$
\mathcal{R}_h(w_t) - \mathcal{R}_h(\bar{r}u) \geq \mathcal{R}_h(w_t) - \mathcal{R}_h(\|w_t\| \bar{u}) - O \left( (\text{OPT} \cdot \ln(1/\text{OPT}) + \epsilon)^2 \right)
$$

$$
\geq 2C_1 \|w_t\|^2 \varphi_t^2 - 2C_2 \|w_t\| \varphi_t \cdot \text{OPT} \cdot \ln(1/\text{OPT}) - O \left( (\text{OPT} \cdot \ln(1/\text{OPT}) + \epsilon)^2 \right),
$$

where $C_1 := 2R^3/(3U^2\pi^2)$ and $C_2 = (1 + 2\alpha_1)\alpha_2$. Note that if $\varphi_t \leq C_2 \cdot \text{OPT} \cdot \ln(1/\text{OPT})/C_1$, then eq. (30) holds; therefore in the following we assume

$$
\varphi_t \geq \frac{C_2}{C_1} \cdot \text{OPT} \cdot \ln(1/\text{OPT}),
$$

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which implies
\[ \mathcal{R}_h(w_t) - \mathcal{R}_h(\bar{r}u) \geq C_1 \|w_t\|_\varphi^2 - O \left( (\OPT \cdot (1/\OPT) + \epsilon)^2 \right) \]
\[ \geq C_1 \varphi_t^2 - O \left( (\OPT \cdot \ln(1/\OPT) + \epsilon)^2 \right), \tag{32} \]
since \( \|w\| \geq 1 \) for all \( w \in D \).

On the other hand, for \( \OPT \leq 1/e \), eq. (31) and Lemma 3.13 imply
\[ \mathcal{M}(w_t) = \mathcal{R}_{0-1}(w_t) \leq \OPT + 2U \varphi_t \leq \left( \frac{C_1}{C_2} + 2U \right) \varphi_t. \]

Let
\[ C_2 := \frac{C_1}{\left( \frac{C_1}{C_2} + 2U \right) C}. \]

Note that if \( \varphi_t \leq \epsilon \), then eq. (30) is true; otherwise we can assume \( \epsilon \leq \varphi_t \), and let \( \eta = \frac{C \epsilon}{3 \ln(d/\epsilon)} \). We have
\[ \eta C d \left( \mathcal{M}(w_t) + \epsilon \right) \ln(d/\epsilon)^2 = \frac{C_2 \epsilon}{d \ln(d/\epsilon)} C d \mathcal{M}(w_t) \cdot \ln(d/\epsilon)^2 + \frac{C_2 \epsilon}{d \ln(d/\epsilon)} C d \epsilon \cdot \ln(d/\epsilon)^2 \]
\[ \leq C_2 \epsilon C \left( \frac{C_1}{C_2} + 2U \right) \varphi_t + C_2 C \epsilon^2 \]
\[ = C_1 \epsilon \varphi_t + O \left( \left( \OPT \cdot \ln(1/\OPT) + \epsilon \right)^2 \right) \]
\[ \leq C_1 \varphi_t^2 + O \left( \left( \OPT \cdot \ln(1/\OPT) + \epsilon \right)^2 \right). \tag{33} \]

Now eqs. (29), (32) and (33) imply
\[ \mathbb{E} \left[ \|w_{t+1} - \bar{r}u\|^2 \right] \leq \|w_t - \bar{r}u\|^2 - 2\eta C_1 \varphi_t^2 + \eta C_1 \varphi_t^2 + \eta \cdot O \left( \left( \OPT \cdot \ln(1/\OPT) + \epsilon \right) + \epsilon \right) \]
\[ = \|w_t - \bar{r}u\|^2 - \eta C_1 \varphi_t^2 + \eta \cdot O \left( \left( \OPT \cdot \ln(1/\OPT) + \epsilon \right) + \epsilon \right). \]

Taking the expectation and average, we have
\[ \mathbb{E} \left[ \frac{1}{T} \sum_{t<T} \varphi_t^2 \right] \leq \frac{\|w_0 - \bar{r}u\|^2}{\eta C_1 T} + \frac{O \left( \left( \OPT \cdot \ln(1/\OPT) + \epsilon \right) + \epsilon \right)}{C_1}. \]

Note that
\[ \|w_0 - \bar{r}u\| = \tan(\varphi_0) = \epsilon \left( \sqrt{\OPT \cdot \ln(1/\OPT)} + \epsilon \right), \]
and also recall \( \eta = \frac{C \epsilon}{3 \ln(d/\epsilon)} \), we have
\[ \mathbb{E} \left[ \frac{1}{T} \sum_{t<T} \varphi_t^2 \right] \leq \frac{O \left( \OPT \cdot \ln(1/\OPT) + \epsilon \right) d \ln(d/\epsilon)^2}{C_1 C_2 \epsilon T} + \frac{O \left( \left( \OPT \cdot \ln(1/\OPT) + \epsilon \right) + \epsilon \right)}{C_1}. \]

Letting \( T = \Omega \left( \frac{d \ln(d/\epsilon)^2}{\epsilon^2} \right) \), we have
\[ \mathbb{E} \left[ \frac{1}{T} \sum_{t<T} \varphi_t^2 \right] \leq \frac{O \left( \OPT \cdot \ln(1/\OPT) + \epsilon \right) \cdot \epsilon + \epsilon \left( \OPT \cdot \ln(1/\OPT) + \epsilon \right)}{\epsilon} \]
\[ = O \left( \left( \OPT \cdot \ln(1/\OPT) + \epsilon \right) \right), \]
and thus eq. (30) holds. □