NONCOMMUTATIVE WEAK $(1, 1)$ TYPE ESTIMATE FOR A SQUARE FUNCTION FROM ERGODIC THEORY

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Abstract. In this paper, we investigate the boundedness of a square function from ergodic theory on noncommutative $L_p$-spaces. The main result is a weak $(1, 1)$ type estimate of this square function. We also show the $(L_\infty, \text{BMO})$ estimate, and thus strong $(L_p, L_p)$ estimate by interpolation. The main novel difficulty lies in the fact that the kernel of this square function does not enjoy any regularity, which is crucial in showing such endpoint estimates for standard noncommutative Calderón-Zygmund singular integral operators.

1. Introduction

Inspired by quantum mechanics and probability, noncommutative harmonic analysis has become an independent field of mathematical research. By using new functional analytic methods from operator space theory and quantum probability, various problems in noncommutative harmonic analysis have been investigated (see, for instance, [10, 20, 21, 22, 23, 39, 42, 34, 31]). Especially, Parcet et al developed a remarkable operator-valued Calderón-Zygmund theory. More precisely, Parcet [34] formulated a noncommutative version of Calderón-Zygmund decomposition using the theory of noncommutative martingales. Moreover, in the same paper Parcet developed a pseudo-localisation principle for singular integrals which was new even in classical theory (see [14] for more results on this principle). As a result, Parcet obtained the weak $(1, 1)$ type estimates of Calderón-Zygmund operators acting on operator-valued functions. This result played an important role in the perturbation theory [6], where the weak $(1, 1)$ type estimates were exploited to solve the Nazarov-Peller conjecture.

Later on, Mei and Parcet [31] proved a weak $(1, 1)$ type estimate for a large class of noncommutative square functions, see [12] for more related results. However, it seems that Mei and Parcet’s weak type estimate could not be used to get $(L_p, L_p)$ estimate (for $1 < p < 2$) by interpolation, since the decomposition does not linearly depend on the original functions. This drawback could be revised through operator-valued Calderón-Zygmund theory—Proposition 4.3 in [2] where the author proposed a simplified version of Parcet’s argument [34], together with

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the noncommutative Khintchine’s inequality as considered in [37, 38]. Moreover, using the Khintchine’s inequality for weak \( L_1 \) spaces considered in [3], one gets another kind of weak \((1,1)\) type inequality for the Calderón-Zygmund operators with Hilbert valued kernels acting on operator valued functions.

Note that the argument in [34, 31, 2] depend heavily on the Lipschitz’s regularity of the kernel. In this paper, motivated by the study of noncommutative maximal inequality, we establish a weak \((1,1)\) type estimate for a square function from ergodic theory. And this square function is different from the class of Calderón-Zygmund operators considered in previous papers [34, 31, 2]. Indeed, the associated kernel does not enjoy any regularity, which is a crucial assumption in [34, 31, 2].

To illustrate our motivation and present the main results, we need to set up some definitions. Let \( \mathcal{M} \) be a semi-finite von Neumann algebra equipped with a normal semi-finite faithful (n.s.f.) trace \( \tau \) and \( \mathcal{N} = L_\infty(\mathbb{R}^d) \overline{\otimes} \mathcal{M} \) be the tensor von Neumann algebra. Let \( f : \mathbb{R}^d \to \mathcal{M} \) be locally integrable. For \( t > 0 \), denote \( B_t \) to be the the open ball centered at the origin 0 with radius \( r(B_t) = 2^{-t} \). Then we define the central averaging operators on \( \mathbb{R}^d \) as

\[
M_t f(x) = \frac{1}{|B_t|} \int_{B_t} f(x + y) dy = \frac{1}{|B_t|} \int_{\mathbb{R}^d} f(y) 1_{B_t}(x - y) \, dy, \quad x \in \mathbb{R}^d.
\]

Given \( k \in \mathbb{Z} \), \( E_k \) denote the \( k \)-th conditional expectation associated to the sigma algebra generated by the standard dyadic cube with side-length equal to \( 2^{-k} \). We refer the reader to Section 2 for precise definitions of all the notions or notations not explicitly given in this section. The sequence of operators that we are going to investigate in the present paper is defined as follows:

\[
T_k f(x) = (M_k - E_k)f(x) \quad \text{and} \quad T f = (T_k f)_{k \in \mathbb{Z}}. \tag{1.1}
\]

In the scalar-valued case, that is, replacing \( \mathcal{M} \) by the set of complex numbers \( \mathbb{C} \), the square function

\[
L f(x) = \left( \sum_{k \in \mathbb{Z}} |(M_k - E_k)f(x)|^2 \right)^{\frac{1}{2}} \tag{1.2}
\]

plays an important role in deducing variational inequalities for ergodic averages or averaging operators from the ones for martingales.

The variational inequalities are much stronger than the maximal inequalities and imply pointwise convergence immediately without knowing a priori pointwise convergence on a dense subclass of functions, which are absent in some models of dynamical system. Let us recall briefly the history of the development of the variational inequalities. This line of research started with Lépingle’s work [25] on martingales which improved the classical Doob maximal inequality. The first variational inequality for the ergodic averages of a dynamical system proved by Bourgain [1] has opened up a new research direction in ergodic theory and harmonic analysis. Bourgain’s work has been extended to many other kinds of operators in ergodic theory and harmonic analysis. For instance, Campbell et al [4, 5] first proved the variational inequalities associated with singular integrals. The reader is referred to [15, 16, 24, 17, 8, 27, 28, 29, 18, 33] and references
therein for more information on the development of ergodic theory and harmonic analysis in this direction of research.

The square function (1.2) appears in most of the above references on variational inequality, and play important roles. In the present paper, similarly, using noncommutative square function estimate, we provide another proof of the noncommutative Hardy-Littlewood maximal inequality (or ergodic maximal inequality) combined with noncommutative Doob’s maximal inequality, see Corollary 1.3.

The statement of our result on noncommutative square functions below requires the so-called column and row function spaces [40]. Let $1 \leq p \leq \infty$, and $(f_k)$ be a finite sequence in $L_p(\mathcal{N})$. Define

$$
\|f_k\|_{L_p(\mathcal{N}; \ell_p^c)} = \left\| \left( \sum_{j} |f_k^j|^2 \right)^{\frac{1}{2}} \right\|_p, \quad \|f_k\|_{L_p(\mathcal{N}; \ell_p^c)} = \left\| \left( \sum_{j} |f_k^j|^2 \right)^{\frac{1}{2}} \right\|_p.
$$

This procedure is also used to define the spaces

$$
L_{1,\infty}(\mathcal{N}; \ell_p^c) \quad \text{and} \quad L_{1,\infty}(\mathcal{N}; \ell_p^c).
$$

Let $1 \leq p \leq \infty$. We define the spaces $L_p(\mathcal{N}; \ell_p^c)$ as follows:

- If $p \geq 2$,
  $$
  L_p(\mathcal{N}; \ell_p^c) = L_p(\mathcal{N}; \ell_p^c) \cap L_p(\mathcal{N}; \ell_p^c)
  $$
  equipped with the intersection norm:
  $$
  \|f_k\|_{L_p(\mathcal{N}; \ell_p^c)} = \max \{\|f_k\|_{L_p(\mathcal{N}; \ell_p^c)}, \|f_k\|_{L_p(\mathcal{N}; \ell_p^c)}\}.
  $$

- If $p < 2$,
  $$
  L_p(\mathcal{N}; \ell_p^c) = L_p(\mathcal{N}; \ell_p^c) + L_p(\mathcal{N}; \ell_p^c)
  $$
  equipped with the sum norm:
  $$
  \|f_k\|_{L_p(\mathcal{N}; \ell_p^c)} = \inf \{\|g_k\|_{L_p(\mathcal{N}; \ell_p^c)} + \|h_k\|_{L_p(\mathcal{N}; \ell_p^c)}\},
  $$
  where the infimum runs over all decompositions $f_k = g_k + h_k$ with $g_k$ and $h_k$ in $L_p(\mathcal{N})$.

It is obvious that $L_2(\mathcal{N}; \ell_2^c) = L_2(\mathcal{N}; \ell_2^c) = L_2(\mathcal{N}; \ell_2^c)$.

In the following, we recall the definition of the noncommutative analogue of BMO space associated to the von Neumann algebra $\mathcal{R} = \mathcal{N} \otimes \mathcal{B}(\ell_2)$. According to [30, 31], we define the dyadic BMO space $\text{BMO}_d(\mathcal{R})$ as the subset of measurable operators associated to $\mathcal{R}$ with

$$
\|f\|_{\text{BMO}_d(\mathcal{R})} = \max \left\{\|f\|_{\text{BMO}_d(\mathcal{R})}, \|f\|_{\text{BMO}_d(\mathcal{R})} \right\} < \infty,
$$

where the row and column dyadic BMO$_d$ norms are given by

$$
\|f\|_{\text{BMO}_d(\mathcal{R})} = \sup_{Q \in \mathcal{Q}} \left\| \left( \frac{1}{|Q|} \int_Q (f(x) - f_Q) (f(x) - f_Q)^* \, dx \right)^{\frac{1}{2}} \right\|_{\mathcal{M} \otimes \mathcal{B}(\ell_2)},
$$

$$
\|f\|_{\text{BMO}_d(\mathcal{R})} = \sup_{Q \in \mathcal{Q}} \left\| \left( \frac{1}{|Q|} \int_Q (f(x) - f_Q)^* (f(x) - f_Q) \, dx \right)^{\frac{1}{2}} \right\|_{\mathcal{M} \otimes \mathcal{B}(\ell_2)}.
$$

Let $T_k$ and $T$ are defined as (1.1). The following is our main result.

**Theorem 1.1.** Let $1 \leq p \leq \infty$ and $f \in L_p(\mathcal{N})$. Then we have
(i) For $p = 1$,
$$
\inf_{T_k f = g_k + h_k} \left\{ \|g_k\|_{L_1(N; \ell'_2)} + \|h_k\|_{L_1(N; \ell'_2)} \right\} \lesssim \|f\|_1,
$$
where the constant depends only on the dimension $d$.

(ii) For $p = \infty$,
$$
\left\| \sum_{k=1}^{\infty} T_k f \otimes e_{1k} \right\|_{\text{BMO}_d({\mathcal R})} + \left\| \sum_{k=1}^{\infty} T_k f \otimes e_{k1} \right\|_{\text{BMO}_d({\mathcal R})} \lesssim \|f\|_{\infty},
$$
where the constant depends only on the dimension $d$.

(iii) For $1 < p < \infty$,
$$
\|Tf\|_{L_p(N; \ell'_2)} \lesssim \|f\|_p,
$$
where the constant depends on the dimension $d$ and $p$.

If we set $R_k f = (M_k - M_{k-1}) f$ and $R f = (R_k f)_{k \geq 1}$, then together with the noncommutative Bukholder-Gundy inequality [36, 35], Theorem 1.1 finds its first application:

**Corollary 1.2.** For $f \in L_1(N)$, we have
$$
\inf_{R_k f = g_k + h_k} \left\{ \|g_k\|_{L_1(N; \ell'_2)} + \|h_k\|_{L_1(N; \ell'_2)} \right\} \lesssim \|f\|_1,
$$
where the constant depends only on the dimension $d$.

Moreover, together with Cuculescu’s noncommutative weak $(1,1)$ type maximal estimate for martingales [7], this endpoint estimate for the Hardy-Littlewood maximal function, firstly established in [30], follows as a corollary of Theorem 1.1.

**Corollary 1.3.** For $(f, \lambda) \in L_1(N) \times \mathbb{R}_+$, there exists a projection $q \in N$ with
$$
\sup_k \|qM_k f q\|_{\infty} \leq 3\lambda \quad \text{and} \quad \lambda \varphi(1_N - q) \lesssim \|f\|_1.
$$
where the constant depends only on the dimension $d$.

**Remark 1.4.** (i). The $L_p$-version of the two corollaries $(1 < p < \infty)$ also hold true if we appeal to the noncommutative Burkholder-Gundy inequality [39] and Doob’s maximal inequality [19]. Moreover, the $L_p$-version of Corollary 1.2 and Theorem 1.1 (iii) seem new even in the framework of vector-valued harmonic analysis.

(ii). Replacing the domain $\mathbb{R}^d$ by $\mathbb{Z}^d$ in Theorem 1.1, Corollary 1.2 and 1.3, similar results hold also true. Then by noncommutative Calderón’s transference principle [11], we provide another proof of ergodic maximal inequality.

Let us briefly analyse the proof of Theorem 1.1. The result for $p = 2$ follows trivially from the corresponding commutative result, but we prefer to provide a noncommutative proof in the appendix for warming up. For $1 \leq p < 2$, using the noncommutative Khintchine’s inequalities in $L_{1,\infty}$ space [3] and in $L_p$ space [26], we reduce to show the weak $(1,1)$ type and strong $(p,p)$ estimates of the following operator
$$
\tilde{T} f(x) = \sum_k \varepsilon_k (M_k - E_k) f(x),
$$
where \((\varepsilon_k)\) is a Rademacher sequence on a probability space \((\Omega, P)\). Note that the operator \(\widetilde{T}\) is linear which allows us to deduce the result for intermediate \(p\)'s from the weak \((1, 1)\) type and strong \((2, 2)\) estimates by real interpolation. On the other hand, the result for \(2 < p < \infty\) follows by complex interpolation from \((L_\infty, BMO)\) estimate and \((2, 2)\) estimate. But this time the linear operators are \(\sum_{k=1}^{\infty} T_k \otimes e_{1k}\) and \(\sum_{k=1}^{\infty} T_k \otimes e_{k1}\). Thus we reduce to show the two endpoint estimates for \(p = 1, \infty\).

However, with a moment’s thought, there are many difficulties to adapt the arguments in [34, 31, 2] to our setting. Indeed, it is obvious that the kernel associated with \(\widetilde{T}\) (or \(T_k\)) does not enjoy Lipschitz’s regularity and the methods in [34, 31, 2] depend heavily on this smoothness condition of the kernel. This prompted us to look for some new methods. It turns out that the main ingredient in showing strong \((2, 2)\) estimate—the almost orthogonality principle plays important roles in overcoming these difficulties. But numerous modifications are necessary in establishing the noncommutative endpoint estimates.

We end our introduction with a brief description of the organisation of the paper. In Section 2, we present some preliminaries on noncommutative \(L_p\)-space. This section also introduces noncommutative martingales and general notations. In Section 3, we prove conclusion (i) of Theorem 1.1 and we also show Corollary 1.2 and Corollary 1.3 in this section. The \((L_\infty, BMO)\) estimate is proved in Section 4. In Section 5, we give the proof of conclusion (iii) of Theorem 1.1.

2. Preliminaries

This section collects all the necessary preliminaries for the whole paper. The reader is referred to [40] for more information on noncommutative \(L_p\)-spaces and noncommutative martingales.

2.1. Noncommutative \(L_p\) space. Let \(\mathcal{M}\) be a von Neumann algebra equipped with a n.s.f. trace \(\tau\). For \(1 \leq p \leq \infty\), denote by \(L_p(\mathcal{M})\) the noncommutative \(L_p\) space. We define \(L_p\)-norm as follows:

\[
\|x\|_{L_p(\mathcal{M})} = (\tau(|x|^p))^{\frac{1}{p}},
\]

where \(|x| = (x^*x)^{\frac{1}{2}}\). For convenience, we set \(L_\infty(\mathcal{M}) = \mathcal{M}\) equipped with the operator norm \(\|\cdot\|_{\mathcal{M}}\). Like the classical \(L_p\)-spaces, noncommutative \(L_p\)-spaces behave well with respect to duality and interpolation. The most important properties for our purposes are the following:

- Hölder inequality: If \(\frac{1}{r} = \frac{1}{p} + \frac{1}{q}\), then \(\|ab\|_{L_r(\mathcal{M})} \leq \|a\|_{L_p(\mathcal{M})} \|b\|_{L_q(\mathcal{M})}\).
- Interpolation: For \(1 \leq p_0 < p_1 \leq \infty\) and \(0 < \eta < 1\), we have

\[
(L_{p_0}(\mathcal{M}), L_{p_1}(\mathcal{M}))_\eta = L_p(\mathcal{M})
\]

where \(\frac{1}{p} = \frac{1-\eta}{p_0} + \frac{\eta}{p_1}\).
2.2. Noncommutative weak $L_1$-spaces. Let
\[ \mathcal{M}' = \{ b \in \mathcal{B}(\mathcal{H}) \mid ab = ba \text{ for all } a \in \mathcal{M} \} \]
be the commutant of $\mathcal{M}$. A closed densely-defined operator on $\mathcal{H}$ is called affiliated with $\mathcal{M}$ when it commutes with every unitary $u$ in $\mathcal{M}'$. If $a$ is a densely defined self-adjoint operator on $\mathcal{H}$ and $a = \int_{\mathbb{R}} s d\gamma_a(s)$ is its spectral decomposition, $\int_{\mathbb{R}} d\gamma_a(s)$ will be simply denoted by $\chi_{\mathcal{R}}(a)$, where $\mathcal{R}$ is a measurable subset of $\mathbb{R}$. An operator $a$ affiliated with $\mathcal{M}$ is said $\tau$-measurable if there exists $s > 0$ such that
\[ \tau(\chi_{(s,\infty)}(|a|)) = \tau\{ |a| > s \} < \infty. \]
The generalized singular value $\mu(a) : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by
\[ \mu_t(a) = \inf \{ s > 0 \mid \tau\{ |a| > s \} \leq t \}. \]
We refer to [9] for a detailed exposition of the function $\mu(a)$ and the corresponding notion of convergence in measure.

If $L_0(\mathcal{M})$ denotes the $*$-algebra of $\tau$-measurable operators. The noncommutative weak $L_1$-space $L_{1,\infty}(\mathcal{M})$ is defined as the set of all $a$ in $L_0(\mathcal{M})$ for which the quasi-norm
\[ \|a\|_{1,\infty} = \sup_{t > 0} t \mu_t(a) = \sup_{\lambda > 0} \lambda \tau\{ |a| > \lambda \} \]
is finite. The following inequality holds for $a_1, a_2 \in L_{1,\infty}(\mathcal{M})$
\[ \lambda \tau\{ |a_1 + a_2| > \lambda \} \leq \lambda \tau\{ |a_1| > \lambda/2 \} + \lambda \tau\{ |a_2| > \lambda/2 \}. \]

2.3. Noncommutative martingales. Consider a von Neumann subalgebra $\mathcal{M}_k$ of $\mathcal{M}$ such that $\tau|_{\mathcal{M}_k}$ is semi-finite. Then there exists a map $E : \mathcal{M} \to \mathcal{M}_k$ satisfying the following properties:
- $E$ is a normal positive contractive projection from $\mathcal{M}$ onto $\mathcal{M}_k$.
- Bimodule property:
  \[ E(ab) = a E(b) \quad \text{for all} \quad a, b \in \mathcal{M} \]
- Trace preserving: $\tau \circ E = \tau$.

The map $E$ is called the conditional expectation of $\mathcal{M}$ with respect to $\mathcal{M}_k$. We call a filtration of von Neumann subalgebras of $\mathcal{M}$ an increasing sequence $(\mathcal{M}_k)_{k \geq 1}$ such that $\cup_k \mathcal{M}_k$ is weak$^*$ dense in $\mathcal{M}$ and $\tau|_{\mathcal{M}_k}$ is semi-finite for every $k \geq 1$. Note for every $1 \leq p < \infty$ and $k \geq 1$, $E_k$ extends to a positive contraction $E_k : L_p(\mathcal{M}) \to L_p(\mathcal{M}_k)$. It follows that each $E_k$ satisfies
\[ \forall k, j \geq 1, E_k E_j = E_k E_j = E_{\min(k,j)}. \]

A noncommutative martingale with respect to the filtration $(\mathcal{M}_k)_{k \geq 1}$ is a sequence $a = (a_k)_{k \geq 1}$ in $L_1(\mathcal{M})$ such that
\[ E_j(a_k) = a_j \quad \text{for all} \quad 1 \leq j \leq k < \infty. \]
If additionally $a \in L_p(\mathcal{M})$ for some $1 \leq p \leq \infty$ and $\|a\|_p = \sup_{k \geq 1} \|a_k\|_p < \infty$, then $a$ is called an $L_p$-bounded martingale. A noncommutative martingale $a$ is said to
be positive if \( a_k \geq 0 \) for all \( k \). Given a martingale \( a = (a_k)_{k \geq 1} \), we assume the convention that \( a_0 = 0 \). Then, the martingale difference sequence \( da = (dak)_{k \geq 1} \) associated to \( a \) is defined by \( dak = ak - a_{k-1} \).

2.4. General notations. In this subsection, we need to set up some notations that will remain fixed through the paper. Let \( \mathcal{M} \) be a semi-finite von Neumann algebra equipped with a n.s.f. trace \( \tau \). We consider the tensor von Neumann algebra \( \mathcal{N} = L_\infty(\mathbb{R}^d) \otimes \mathcal{M} \) equipped with the tensor n.s.f. trace \( \varphi = \int dx \otimes \tau \), where \( dx \) is Lebesgue measure. Note that for every \( 0 < p < \infty \),

\[
L_p(\mathcal{N}; \varphi) \cong L_p(\mathbb{R}^d; L_p(\mathcal{M})).
\]

The space on the right-hand side is the space of Bochner \( p \)-integrable functions from \( \mathbb{R}^d \) to \( L_p(\mathcal{M}) \). For \( 1 \leq p \leq \infty \), we simply write \( L_p(\mathcal{N}) \) for the noncommutative \( L_p \) space associated to the pairs \( (\mathcal{N}, \varphi) \) and \( || \cdot ||_p \) denotes the norm of \( L_p(\mathcal{N}) \). But if any other \( L_p \)-space appears in a same context, we will precisely mention the respective \( L_p \)-norms in order to avoid possible ambiguity. The lattices of projections are written \( \mathcal{M}_\pi \) and \( \mathcal{N}_\pi \), while \( 1_{\mathcal{M}} \) and \( 1_{\mathcal{N}} \) stand for the unit elements.

Denoted by \( Q \) the set of all dyadic cubes in \( \mathbb{R}^d \). For \( Q \in Q \), denote by \( \ell(Q) \) the side length of the cube \( Q \) and sides parallel to the axis. Given an integer \( k \in \mathbb{Z} \), \( Q_k \) will denote the set of dyadic cubes of side length \( 2^{-k} \). Let \( |Q| = 2^{-dk} \) be the volume of such a cube. If \( Q \in Q \) and \( f : \mathbb{R}^d \to \mathcal{M} \) is integrable on \( Q \), we define its average as

\[
f_Q = \frac{1}{|Q|} \int_Q f(y) dy.
\]

For \( k \in \mathbb{Z} \), let \( \sigma_k \) be the \( k \)-th dyadic \( \sigma \)-algebra, i.e., \( \sigma_k \) is generated by the dyadic cubes with side lengths equal to \( 2^{-k} \). Denote by \( E_k \) the conditional expectation associated to the classical dyadic filtration \( \sigma_k \) on \( \mathbb{R}^d \). We also use \( E_k \) for the tensor product \( E_k \otimes id_\mathcal{M} \) acting on \( \mathcal{N} \). If \( 1 \leq p < \infty \) and \( f \in L_p(\mathcal{N}) \), we have

\[
E_k(f) = \sum_{Q \in Q_k} f_Q 1_Q.
\]

Similarly, \( (\mathcal{N}_k)_{k \in \mathbb{Z}} \) will stand for the corresponding filtration and \( \mathcal{N}_k = E_k(\mathcal{N}) \). For convenience, we will write \( f_k := E_k(f) \) and \( \Delta_k(f) := f_k - f_{k-1} =: df_k \).

For all \( x \in \mathbb{R}^d \), we write \( Q_{x,k} \) for the cube in \( Q_k \) containing \( x \), and its centre is denoted by \( c_{x,k} \). For any positive integer \( i \) and \( Q \) in \( Q_k \), let \( iQ \) be the cube with the same center of \( Q \) such that \( \ell(iQ) = i \ell(Q) \). Notice that for all \( x, y \in \mathbb{R}^d \) and \( k \in \mathbb{Z} \), \( x \in iQ_{x,k} \Leftrightarrow y \in iQ_{y,k} \).

Throughout the paper we use the notation \( X \lesssim Y \) for nonnegative quantities \( X \) and \( Y \) to mean \( X \leq CY \) for some inessential constant \( C > 0 \). Similarly, we use the notation \( X \preceq Y \) if both \( X \lesssim Y \) and \( Y \lesssim X \) hold.

3. Weak type \((1,1)\) boundedness

In this section, we prove conclusion \((i)\) of Theorem 1.1, Corollary 1.2 and Corollary 1.3. By decomposing \( f = f_1 - f_2 + i(f_3 - f_4) \) with positive \( f_j \) (\( j = i \).
1, 2, 3, 4), we assume that \( f \) is positive to avoid unnecessary computations. Let us focus on the following dense subspace

\[
\mathcal{N}_{c,+} = L_1(\mathcal{N}) \cap \left\{ f : \mathbb{R}^d \to \mathcal{M} \mid f \in \mathcal{N}_+, \ \text{supp} f \text{ is compact} \right\} \subset L_1(\mathcal{N}).
\]

Here \( \text{supp} f \) means the support of \( f \) as an operator-valued function in \( \mathbb{R}^d \). That is, we have \( \text{supp} f = \text{supp} \| f \|_{\mathcal{M}} \). We employ this terminology to distinguish from \( \text{supp} f \). The noncommutative analogue of the weak type \((1, 1)\) estimate of Doob’s maximal function is due to the following Cuculescu Theorem.

**Lemma 3.1.** [7] (Cuculescu) Suppose \( f = (f_1, f_2, \ldots) \) is a positive \( L_1 \) martingale relative to the filtration \((\mathcal{N}_k)_{k \geq 1}\) whose union is \( w^* \)-dense in a noncommutative measure space \((\mathcal{N}, \varphi)\) and let \( \lambda \) be a positive number. Then there exists a decreasing sequence of projections

\[
q_1, q_2, q_3, \ldots
\]

in \( \mathcal{N} \) satisfying the following properties

(i) \( q_k \) commutes with \( q_{k-1}f_kq_{k-1} \) for each \( k \geq 1 \).

(ii) \( q_k \) belongs to \( \mathcal{N}_k \) for each \( k \geq 1 \) and \( q_k f_k q_k \leq \lambda q_k \).

(iii) The following estimate holds

\[
\varphi \left( 1_{\mathcal{N}} - \bigwedge_{k \geq 1} q_k \right) \leq \frac{1}{\lambda} \sup_{k \geq 1} \| f_k \|_1.
\]

Explicitly, we set \( q_0 = 1_{\mathcal{N}} \) and define \( q_k = \chi_{[0, \lambda]}(q_{k-1}f_kq_{k-1}) \).

Given \( f \in \mathcal{N}_{c,+} \) and \( \lambda > 0 \), by the assumptions of \( f \), there exists \( k_0 \in \mathbb{Z} \) such that for all \( k \leq k_0 \), \( f_k \leq \lambda \). Hence, by a change of variable, we assume \( k_0 = 0 \). From now on, considering the Cuculescu’s sequence \((q_k)_{k \geq 1}\) associated to \( \lambda \) and \((f_k)_{k \geq 1}\), we set \( q_k = 1 \) for all \( k \leq 0 \) to complete the definition. It is easy to see for all \( k \in \mathbb{Z} \), \( q_k f_k q_k \leq \lambda \). Define the sequence \((p_k)_{k \in \mathbb{Z}}\) of disjoint projections \( p_k = q_{k-1} - q_k \), so that

\[
\sum_{k \in \mathbb{Z}} p_k = 1_{\mathcal{N}} - q \quad \text{with} \quad q = \bigwedge_{k \in \mathbb{Z}} q_k.
\]

### 3.1 Calderón-Zygmund decomposition. [34]

Let \( f \in \mathcal{N}_{c,+} \) and \( \lambda > 0 \). Then \( f \) can be decomposed as the sum of four functions \( f = g_d + g_{\text{off}} + b_d + b_{\text{off}} \) with

\[
\begin{align*}
    g_d &= qfq + \sum_{k \in \mathbb{Z}} p_k f_k p_k, \\
    g_{\text{off}} &= \sum_{i \neq j} p_i f_i p_j + q f q^+ + q^+ f q,
\end{align*}
\]

\[
\begin{align*}
    b_d &= \sum_{k \in \mathbb{Z}} p_k (f - f_k) p_k, \\
    b_{\text{off}} &= \sum_{i \neq j} p_i (f - f_i p_j),
\end{align*}
\]

where \( i \lor j = \max(i, j) \) and \( q^+ = 1_{\mathcal{N}} - q \).

From now on, we collect some fundamental and useful propositions of Calderón-Zygmund decomposition. See [34, 2] for more details.

**Proposition 3.2.** [34] We have the following diagonal estimates

\[
\left\| qfq + \sum_{k \in \mathbb{Z}} p_k f_k p_k \right\|_2^2 \leq 2^d \lambda \| f \|_1 \quad \text{and} \quad \sum_{k \in \mathbb{Z}} \| p_k (f - f_k) p_k \|_1 \leq 2 \| f \|_1. \tag{3.1}
\]
Proposition 3.3. \[34\] We have
\[
  g_{\text{off}} = \sum_{s=1}^{\infty} \sum_{k=1}^{\infty} p_k d f_{k+s} g_{k+s-1} + q_{k+s-1} d f_{k+s} p_k = \sum_{s=1}^{\infty} \sum_{k=1}^{\infty} g_{k,s} = \sum_{s=1}^{\infty} g_{(s)}.
\]
Moreover, it is easy to check that
\[
  \sup_{s \geq 1} \|g_{(s)}\|_2^2 = \sup_{s \geq 1} \sum_{k=1}^{\infty} \|g_{k,s}\|_2^2 \lesssim \lambda \|f\|_1
\]
and that \(dg_{(s)k+s} = g_{k,s} \leq p_k \leq 1_N - q_k\).

For all \(k\) and \(Q \in Q_k\), we denote \(p_Q := p_k(x)\) for any \(x \in Q\).

Proposition 3.4. \[2\] Define
\[
  \zeta = \bigvee_{Q \in \mathcal{Q}} p_Q 1_{5Q}^\perp.
\]
Then
(1) \(\varphi(1 - \zeta) \leq 5^d \frac{\|f\|_1}{\lambda}\)
(2) For all cubes \(Q \in \mathcal{Q}\), we have the following cancellation property:
\[
  x \in 5Q \Rightarrow \zeta(x)p_Q\zeta(x) = 0.
\]

Proposition 3.5. \[2\] Let \(b_{i,j} = p_i(f - f_{i\vee j})p_j\) for all \(i, j \in \mathbb{Z}\). The following cancellation properties hold:
(1) For all \(i, j \in \mathbb{Z}\) and \(Q \in \mathcal{Q}_{i\vee j}\): \(\int_Q b_{i,j} = 0\).
(2) For all \(x, y \in \mathbb{R}^d\) such that \(y \in 5Q_{x, i\vee j}\): \(\zeta(x)b_{i,j}(y)\zeta(x) = 0\).

Before proving weak type \((1, 1)\) boundedness of \(T\), we need a noncommutative Khintchine inequality in \(L_{1,\infty}\) for a Rademacher sequence \((\varepsilon_k)\) on a probability space \((\Omega, P)\) which is established by Cadilhac \[3\].

Lemma 3.6. \[3\] Let \(f = (f_k)\) be any finite sequence in \(L_{1,\infty}(\mathcal{N})\). We have
\[
  \left\| \sum_k f_k\varepsilon_k \right\|_{L_{1,\infty}(L_\infty(\Omega)\boxtimes \mathcal{N})} \preceq \inf_{f_k = g_k + h_k} \left\{ \|g_k\|_{L_{1,\infty}(\mathcal{N}; \ell_2^c)} + \|h_k\|_{L_{1,\infty}(\mathcal{N}; \ell_2^c)} \right\}.
\]
Recall
\[
  \tilde{T}f(x) = \sum_k \varepsilon_k T_k f(x) = \sum_k \varepsilon_k (M_k - E_k)f(x). \tag{3.2}
\]
Using Lemma 3.6, we immediately obtain the following corollary:

Corollary 3.7. Let \(f \in L_1(\mathcal{N})\). Then we have
\[
  \inf_{T_k f = g_k + h_k} \left\{ \|g_k\|_{L_{1,\infty}(\mathcal{N}; \ell_2^c)} + \|h_k\|_{L_{1,\infty}(\mathcal{N}; \ell_2^c)} \right\} \preceq \|\tilde{T}f\|_{L_{1,\infty}(L_\infty(\Omega)\boxtimes \mathcal{N})}.
\]

In the following of this section, let us focus on estimating weak \((1, 1)\) type of the operator \(\tilde{T}\). Thus, if there is no ambiguity, we still denote \(\tilde{T}\) by \(T\) in the rest of this section.

Now we start with the proof of the part (i) of Theorem 1.1.
A couple of remarks are in order. Firstly, Corollary 3.7 yields that it suffices to verify \( \| T f \|_{L_{1, \infty}(L_{\infty}(\Omega) \otimes N)} \lesssim \| f \|_1 \), where \( T f(x) \) defined as in (3.2). Secondly, by applying properties of distribution function, we have

\[
\tilde{\varphi}(|Tf| > \lambda) \lesssim \tilde{\varphi}(|Tg_d| > \frac{\lambda}{4}) + \tilde{\varphi}(|Tg_{off}| > \frac{\lambda}{4}) + \tilde{\varphi}(|Tb_d| > \frac{\lambda}{4}) + \tilde{\varphi}(|Tb_{off}| > \frac{\lambda}{4}),
\]

where \( \tilde{\varphi} = \int_\Omega \otimes \varphi \). Hence, it suffices to prove estimate of the form:

\[
\tilde{\varphi}(|T h| > \lambda) \lesssim \frac{\| f \|_1}{\lambda}
\]

for \( h = g_d, g_{off}, b_d \) and \( b_{off} \).

3.2. **Weak type estimates for bad function.** Using the projection \( \zeta \) introduced in Proposition 3.4, we consider the following decomposition

\[
Tb = (1_N - \zeta)Tb(1_N - \zeta) + \zeta Tb(1_N - \zeta) + (1_N - \zeta)Tb\zeta + \zeta Tb\zeta.
\]

Therefore, Proposition 3.4 gives:

\[
\tilde{\varphi}(|T h| > \lambda) \lesssim \varphi(1_N - \zeta) + \tilde{\varphi}(|\zeta Tb\zeta| > \lambda)
\]

\[
\lesssim \frac{\| f \|_1}{\lambda} + \tilde{\varphi}(|\zeta Tb\zeta| > \lambda).
\]

Hence, our aim is to estimate \( \tilde{\varphi}(|\zeta Tb\zeta| > \lambda) \).

To estimate the diagonal part of \( b \), we use the following lemma—the almost orthogonality principle, which is well-known in classical harmonic analysis, see for instance [16, 13].

**Lemma 3.8.** Let \( S_k \) be a bounded linear map on \( L_2 \) for each \( k \in \mathbb{Z} \) and \( f \in L_2 \). If \( (u_n)_{n \in \mathbb{Z}} \) and \( (v_n)_{n \in \mathbb{Z}} \) are two sequences of functions in \( L_2 \) such that \( f = \sum_n u_n \) and \( \sum_n \| v_n \|_2^2 \leq C \| f \|_2^2 \), then

\[
\sum_k \| S_k f \|_2^2 \leq Cw^2 \| f \|_2^2
\]

provided that there is a sequence \( (\sigma(j))_{j \in \mathbb{Z}} \) of positive numbers with \( w = \sum_j \sigma(j) < \infty \) such that

\[
\| S_k(u_n) \|_2 \leq \sigma(n - k) \| v_n \|_2
\]

for every \( n, k \).

3.2.1. **Weak type estimate for \( T b_d \).** Let us now prove the assertions for \( b_d \). On one hand, Chebychev’s inequality gives:

\[
\tilde{\varphi}(|\zeta Tb_d\zeta| > \lambda) \lesssim \frac{\| \zeta Tb_d\zeta \|_{L_2(L_{\infty}(\Omega) \otimes N)}^2}{\lambda^2}.
\]

Therefore, it suffices to show

\[
\| \zeta Tb_d\zeta \|_{L_2(L_{\infty}(\Omega) \otimes N)}^2 \lesssim \lambda^2 \sum_{n=1}^\infty \| p_n \|_2^2,
\] (3.3)
since we have
\[ \sum_{n=1}^{\infty} \|p_n\|^2 = \sum_{n=1}^{\infty} \|p_n\|_1 \lesssim \frac{\|f\|_1}{\lambda}. \]

To estimate (3.3), we first note that the orthogonality of \( \varepsilon_k \) implies
\[ \|\zeta T b_d \zeta\|_{L^2(L^\infty(\Omega) \otimes \mathcal{N})}^2 = \sum_{k=1}^{\infty} \|\zeta T b_d \zeta\|_2^2 = \sum_{k=1}^{\infty} \|\zeta (M_k - E_k)b_d \zeta\|_2^2. \]

Now we claim that \( \zeta(x)E_kb_d(x)\zeta(x) = 0 \) for every \( k \geq 1 \) and \( x \in \mathbb{R}^d \). Indeed, if we take \( b_n \) to be \( p_n(f - f_n)p_n \), then \( b_d = \sum_{n=1}^{\infty} b_n \). It is clear that \( b_n \) is self-adjoint. For \( k \leq n \), applying the properties of conditional expectations, we have \( E_kb_n(x) = E_kE_n b_n(x) = 0 \); for \( k > n \), Proposition 3.5 gives: for any \( Q \in \mathcal{Q}_k \),
\[ \zeta(x)E_kb_n(x)\zeta(x) = \zeta(x) \frac{1}{|Q|} \int_Q b_n(y)1_Q(x)1_{x \notin \mathcal{Q}_n}(x)dy\zeta(x) = 0. \]

This establishes the claim.

Furthermore, the similar argument allows us to conclude that for \( k > n \) and all \( x \in \mathbb{R}^d \), \( \zeta(x)M_kb_n(x)\zeta(x) = 0 \). To this end, using these observations, we just need to prove for \( k \leq n \)
\[ \|\zeta T b_d \zeta\|_{L^2(L^\infty(\Omega) \otimes \mathcal{N})}^2 = \sum_{k=1}^{\infty} \|M_k \sum_{n,k \leq n} b_n\|_2^2 \lesssim \lambda^2 \sum_{n=1}^{\infty} \|p_n\|_2^2. \tag{3.4} \]

Taking \( S_kg = M_kg \), \( u_n = b_n \) and \( v_n = p_n \) in Lemma 3.8, it suffices to show for \( k \leq n \)
\[ \|M_kb_n\|_2^2 \lesssim 2^{k-n} \lambda^2 \|p_n\|_2^2. \tag{3.5} \]

In turn, the desired estimate (3.5) can be deduced from the following relations:
\[ |M_kb_n(x)|^2 \lesssim 2^{k-n} \lambda^2 M_{k-1}|p_n(x)|. \tag{3.6} \]

To see this, integrating over \( \mathbb{R}^d \), using Fubini’s theorem and the fact \( p_n = p_n^2 \), we get
\[
\|M_kb_n\|_2^2 = \tau \otimes \int_{\mathbb{R}^d} |M_kb_n(x)|^2 dx \\
\lesssim 2^{k-n} \lambda^2 \tau \otimes \int_{\mathbb{R}^d} \frac{1}{|B_{k-1}|} \int_{x+B_{k-1}} |p_n(y)|^2 dy dx \\
= 2^{k-n} \lambda^2 \frac{1}{|B_{k-1}|} \int_{B_{k-1}} dy \otimes \int_{\mathbb{R}^d} |p_n(x + y)|^2 dx \\
= 2^{k-n} \lambda^2 \|p_n\|_2^2.
\]

Thus, in the following let us focus on (3.6). For a given ball \( B \subset \mathbb{R}^d \), we define
\[ \mathcal{I}(B, n) = \bigcup \{Q \cap B | Q \in \mathcal{Q}_n, \partial B \cap Q \neq \emptyset\}. \]

Since \( \{Q | Q \in \mathcal{Q}_n\} \) are pairwise disjoint, we have
\[ M_kb_n(x) = \frac{1}{|B_k|} \int_{\mathcal{I}(x+B_k,n)} b_n(y)dy \]
\begin{align*}
&= \frac{1}{|B_k|} \int_{I(x+B_k, n)} (p_n f p_n)(y) dy \\
&- \frac{1}{|B_k|} \int_{I(x+B_k, n)} (p_n f p_n)(y) dy \\
&\triangleq I(x) - II(x).
\end{align*}

By applying operator convexity inequality of \( x \mapsto |x|^2 \), we obtain
\[
|M_k b_n(x)|^2 \leq 2|I(x)|^2 + 2|II(x)|^2.
\]

Hence, in order to obtain (3.6), it suffices to show
\[
|h(x)|^2 \lesssim 2^{k-n} \lambda^2 M_{k-1}|p_n(x)|
\] (3.7)

for \( h = I \) and \( II \).

We first deal with \( II \). Using the property of \( I(x+B_k, n) \), the fact \( \|p_n f p_n\|_\infty \lesssim \lambda \), and noting that \( k \leq n \), we have
\[
II(x) \leq \frac{1}{|B_k|} \sum_{Q \in Q_n} \int_{x+\partial B_k \cap Q \neq \emptyset} (p_n f p_n)(y) dy \\
\lesssim \frac{\lambda}{|B_k|} \sum_{Q \in Q_n} \int_{x+\partial B_k \cap Q \neq \emptyset} p_n(y) dy \\
\lesssim \lambda M_{k-1} p_n(x).
\]

On the other hand, since the measure of the union of the dyadic cubes in \( Q_n \) which intersects with the boundary of \( x + B_k \) is not more than a constant multiple of \( 2^{-n} 2^{(d-1)(-k)} \), we have another estimate
\[
II(x) \lesssim \frac{\lambda}{|B_k|} \sum_{Q \in Q_n} |Q| \\
\lesssim 2^{kd} 2^{-n} 2^{(d-1)(-k)} \lambda = 2^{k-n} \lambda.
\]

Putting the above estimates together, we deduce
\[
|II(x)|^2 = II(x) II(x) \lesssim 2^{k-n} \lambda^2 M_{k-1} |p_n(x)|.
\]

It is obvious to see that \( I \) can be similarly treated due to the following relation:
\[
\int_Q (p_n f p_n)(y) dy = \int_Q (p_n f p_n)(y) dy
\]

for \( Q \in Q_n \). Thus we get the required estimate (3.7). This completes the proof for \( T b_d \).

**Remark 3.9.** We should point out that the above method using 2-norm estimate seems no longer applicable to the off-diagonal term of \( b \), as we have seen that the diagonal estimate and the positivity play important roles in above argument, while the off-diagonal part of \( b \) does not enjoy these properties.
3.2.2. Weak type estimate for $T_{\text{off}}$. Let us now consider the off-diagonal term $b_{\text{off}}$ determined by $b = b_d + b_{\text{off}}$. As for $T_{\text{off}}$, it suffices to estimate $\zeta T_{\text{off}}$. We set $b_{\text{off}} = \sum_{s=1}^{\infty} \sum_{n=1}^{\infty} b_{n,s}$, where

$$b_{n,s} = p_n (f - f_{n+s}) p_{n+s} + p_{n+s} (f - f_{n+s}) p_n.$$ 

According to Proposition 3.5, the following cancellation properties hold:

- for all $Q \in Q_{n+s}$: $\int_Q b_{n,s} = 0$;
- for all $x, y \in \mathbb{R}^d$ such that $y \in 5Q_{x,n}$: $\zeta(x) b_{n,s}(y) \zeta(x) = 0$.

We use Minkowski and Chebychev’s inequalities to get

$$\lambda \mathcal{F} \left( |\zeta T_{\text{off}}| \right) \leq \| \zeta T_{\text{off}} \|_{L_1(\mathcal{L}_1(\Omega))} \leq \sum_{s=1}^{\infty} \sum_{n=1}^{\infty} \| \zeta T_{n,s} \|_{L_1(\mathcal{L}_1(\Omega))} \leq \sum_{s=1}^{\infty} \sum_{n=1}^{\infty} \| \zeta \sum_{k=1}^{\infty} \varepsilon_k T_{k,b_{n,s}} \|_{L_1(\mathcal{L}_1(\Omega))}.$$ 

Since $(\varepsilon_k)_k$ is a bounded sequence, it is obvious that the last term above is controlled by:

$$\sum_{s=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left\| \zeta T_{k,b_{n,s}} \right\|_1.$$ 

To this end, we claim $\zeta(x) E_k b_{n,s}(x) \zeta(x) = 0$ for every $k \geq 1$, $s \geq 1$ and $n \geq 1$. Indeed, if $k \leq n$, then $E_k b_{n,s}(x) = E_k E_{n+s} b_{n,s}(x) = 0$; if $n < k$, using the cancellation properties of $b_{n,s} \zeta$, we have for any $Q \in Q_k$

$$\zeta(x) E_k b_{n,s}(x) \zeta(x) = \zeta(x) \frac{1}{|Q|} \int_Q b_{n,s}(y) 1_Q(x) 1_{x \notin 5Q_{y,n}}(x) dy \zeta(x) = 0.$$ 

Hence, $\zeta(x) E_k b_{n,s}(x) \zeta(x) = 0$. This is precisely the claim. On the other hand, the same argument gives $\zeta(x) M_k b_{n,s}(x) \zeta(x) = 0$ for $k > n$.

Putting above discussions together, we obtain

$$\sum_{s=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left\| \zeta T_{k,b_{n,s}} \right\|_1 \leq \sum_{s=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=k \leq n} \left\| M_k b_{n,s} \right\|_1.$$ 

We now use the decomposition

$$b_{n,s} = \left( \sum_{r=0}^{s} p_{n+r} \right) (f - f_{n+s}) \left( \sum_{r=0}^{s} p_{n+r} \right) - \left( \sum_{r=0}^{s-1} p_{n+r} \right) (f - f_{n+s}) \left( \sum_{r=0}^{s-1} p_{n+r} \right) - \left( \sum_{r=1}^{s} p_{n+r} \right) (f - f_{n+s}) \left( \sum_{r=1}^{s} p_{n+r} \right).$$
+ \left( \sum_{r=1}^{s-1} p_{n+r} \right) (f - f_{n+s}) \left( \sum_{r=1}^{s-1} p_{n+r} \right) \triangleq b_{n,s}^1 - b_{n,s}^2 - b_{n,s}^3 + b_{n,s}^4.

It is straightforward to see that the four projections above belong to $\mathcal{N}_{n+s}$. Furthermore, since $E_{n+s}(f - f_{n+s}) = 0$, we conclude that for any $Q \in \mathcal{Q}_{n+s}$ and any $1 \leq i \leq 4$

$$\int_Q b_{n,s}^i(y) \, dy = 0.$$ 

Therefore, we find

$$\sum_{s=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k,k \leq n} \|M_k b_{n,s}\|_1 \leq \sum_{s=1}^{\infty} \sum_{i=1}^{4} \sum_{n=1}^{\infty} \sum_{k,k \leq n} \|M_k b_{n,s}^i\|_1.$$ 

We first deal with $M_k b_{n,s}^1$. To this end, we decompose $M_k b_{n,s}^1$ into two parts:

$$M_k b_{n,s}^1(x) = \frac{1}{|B_k|} \int_{\mathcal{I}(x+B_k,n+s)} b_{n,s}^1(y) \, dy$$

$$= \frac{1}{|B_k|} \int_{\mathcal{I}(x+B_k,n+s)} \left( \sum_{r=0}^{s} p_{n+r} \right) f \left( \sum_{r=0}^{s} p_{n+r} \right) (y) \, dy$$

$$- \frac{1}{|B_k|} \int_{\mathcal{I}(x+B_k,n+s)} \left( \sum_{r=0}^{s} p_{n+r} \right) f_{n+s} \left( \sum_{r=0}^{s} p_{n+r} \right) (y) \, dy$$

$$\triangleq III(x) - IV(x).$$

In order to deal with $III$, we use the properties of $\mathcal{I}(x+B_k,n+s)$ and Fubini’s theorem to get

$$\left\| \frac{1}{|B_k|} \int_{\mathcal{I}(x+B_k,n+s)} \left( \sum_{r=0}^{s} p_{n+r} \right) f \left( \sum_{r=0}^{s} p_{n+r} \right) (y) \, dy \right\|_1$$

$$= \tau \otimes \int_{\mathbb{R}^d} \frac{1}{|B_k|} \int_{\mathcal{I}(x+B_k,n+s)} \left( \sum_{r=0}^{s} p_{n+r} \right) f \left( \sum_{r=0}^{s} p_{n+r} \right) (y) \, dy \, dx$$

$$\leq \tau \otimes \int_{\mathbb{R}^d} \frac{1}{|B_k|} \sum_{Q \in \mathcal{Q}_{n+s}} \int_{Q} \left( \sum_{r=0}^{s} p_{n+r} \right) f \left( \sum_{r=0}^{s} p_{n+r} \right) (y) \, dy \, dx$$

$$= 2^{kd} \sum_{Q \in \mathcal{Q}_{n+s}} \tau \otimes \int_{Q} \left( \sum_{r=0}^{s} p_{n+r} \right) f \left( \sum_{r=0}^{s} p_{n+r} \right) (y) \, dy \int_{\mathbb{R}^d} 1_{x \in \partial B_k \cap Q \neq \emptyset} (y) \, dx$$

$$\lesssim 2^{-s} \cdot 2^{k-n} \varphi \left( \left( \sum_{r=0}^{s} p_{n+r} \right) f \left( \sum_{r=0}^{s} p_{n+r} \right) \right).$$

In the same way, we also have

$$\|IV\|_1 \lesssim 2^{-s} \cdot 2^{k-n} \varphi \left( \left( \sum_{r=0}^{s} p_{n+r} \right) f \left( \sum_{r=0}^{s} p_{n+r} \right) \right),$$
since the conditional expectation is trace preserving, more precisely,
\[ \varphi \left( \left( \sum_{r=0}^{s} p_{n+r} \right) f_{n+s} \left( \sum_{r=0}^{s} p_{n+r} \right) \right) = \varphi \left( \left( \sum_{r=0}^{s} p_{n+r} \right) f \left( \sum_{r=0}^{s} p_{n+r} \right) \right). \]

Furthermore, it is obvious to see that \( M_{k} b_{n,s}^{i} \) \((i = 2, 3, 4)\) can be similarly treated. Hence, we conclude that for \( i = 1, 2, 3, 4 \)
\[ \| M_{k} b_{n,s}^{i} \|_{1} \lesssim 2^{-s} \cdot 2^{k-n} \varphi \left( \left( \sum_{r=0}^{s} p_{n+r} \right) f \left( \sum_{r=0}^{s} p_{n+r} \right) \right). \]

Finally, summing over \((s, i, n, k)\) we get
\[
\sum_{s=1}^{\infty} \sum_{i=1}^{4} \sum_{n=1}^{\infty} \sum_{k:k \leq n} \| M_{k} b_{n,s}^{i} \|_{1} \\
\lesssim \sum_{s=1}^{\infty} \sum_{i=1}^{4} \sum_{n=1}^{\infty} \sum_{k:k \leq n} 2^{-s} \cdot 2^{k-n} \varphi \left( \left( \sum_{r=0}^{s} p_{n+r} \right) f \left( \sum_{r=0}^{s} p_{n+r} \right) \right) \\
\lesssim \left( \sum_{s=1}^{\infty} \sum_{i=1}^{4} (s + 1)2^{-s} \right) \| f \|_{1} \\
= 4 \left( \sum_{s=1}^{\infty} \frac{s + 1}{2^{s}} \right) \| f \|_{1} \lesssim \| f \|_{1}.
\]

This completes the argument for the term \( Tb_{\text{off}} \).

3.3. **Weak type estimates for good function.** To give the proof of good function, we need the fact that \( T \) is bounded from \( L_{2}(\mathcal{N}) \) to \( L_{2}(L_{\infty}(\Omega) \otimes \mathcal{N}) \), which will be proved in the Appendix.

**Lemma 3.10.** Let \( f \in L_{2}(\mathcal{N}) \). Then there exists a constant \( C_{d} \) depending only on the dimension \( d \) such that
\[ \| T f \|_{L_{2}(L_{\infty}(\Omega) \otimes \mathcal{N})} \leq C_{d} \| f \|_{2}. \]

3.3.1. **Weak type estimate for \( T g_{d} \).** Note that the estimate for diagonal term of good function can be deduced from Lemma 3.10 and Proposition 3.2. Indeed, Chebychev’s inequality and Hölder’s inequality give
\[ \varphi \left( \| T g_{d} \| > \lambda \right) \lesssim \frac{\| T g_{d} \|_{L_{2}(L_{\infty}(\Omega) \otimes \mathcal{N})}^{2}}{\lambda^{2}} \lesssim \frac{\| g_{d} \|_{1}^{2}}{\lambda^{2}} \lesssim \frac{\| f \|_{1}^{2}}{\lambda^{2}}. \]

This completes the proof of our assertion for \( T g_{d} \).

3.3.2. **A pseudo-localization result.** As mentioned before, to prove the off-diagonal term of \( g \), we need to establish a pseudo-localization principle in our case. We state this principle as the following theorem.

**Theorem 3.11.** Let \( f \in L_{2}(\mathcal{N}) \) and \( s \in \mathbb{N} \). For all \( k \in \mathbb{Z} \), let \( A_{k} \) be projections in \( \mathcal{N}_{k} \) such that \( A_{k}^{\perp} d f_{k+s} A_{k}^{\perp} = 0 \). Write:
\[ A_{k} = \bigvee_{Q \subset Q_{k}} A_{Q} 1_{Q}, \]
\[ A_{Q} \in \mathcal{M} \] and define \( 5 A_{k} = \bigvee_{Q \subset Q_{k}} A_{Q} 1_{5Q} \),
as well as
\[ A_{f,s} = \bigcup_{k \in \mathbb{Z}} 5A_k. \] (3.8)

Then we have
\[ \| A_{f,s}^\perp T f A_{f,s}^\perp \|_{L_2(L_\infty(\Omega) \otimes \mathbb{N})} \lesssim 2^{-\frac{3}{4}} \| f \|_2. \]

Before proving Theorem 3.11, we briefly analyze the construction of \( A_{f,s} \) in (3.8). It is important to note that by construction, for all \( x, y \in \mathbb{R}^d \) with \( x \in 5Q_{y,k} \), we have \( A_k(y) \leq (5A_k(x)) \leq A_{f,s}(x) \) and \( A_k^\perp(y)df_{k+s}(y)A_k^\perp(y) = 0 \). Hence,
\[ A_{f,s}^\perp(x)df_{k+s}(y)A_{f,s}^\perp(x) = A_{f,s}^\perp(x)(5A_k)\perp(x)df_{k+s}(y)(5A_k)\perp(x)A_{f,s}^\perp(x) = 0. \] (3.9)

Now we are ready to prove Theorem 3.11.

**Proof of Theorem 3.11.** Let \( f = \sum_{n \in \mathbb{Z}} df_n \), then
\[ \sum_{n \in \mathbb{Z}} \| df_n \|_2^2 = \| f \|_2^2. \]

By applying the orthogonality of \( \varepsilon_k \), we have
\[ \| A_{f,s}^\perp T f A_{f,s}^\perp \|_{L_2(L_\infty(\Omega) \otimes \mathbb{N})} = \sum_k \| A_{f,s}^\perp(M_k - E_k) \sum_n df_n A_{f,s}^\perp \|_2^2. \]

Moreover, Lemma 3.8 implies that in order to finish the proof of Theorem 3.11, it is enough to show
\[ \| A_{f,s}^\perp(M_k - E_k)df_n A_{f,s}^\perp \|_2^2 \lesssim 2^{-\frac{3}{4}} 2^{-\frac{\| k \|}{2}} \| df_n \|_2^2. \] (3.10)

To this end, we first consider the case \( k \geq n \). In this case, it is enough to show
\[ \| A_{f,s}^\perp(M_k - E_k)df_n A_{f,s}^\perp \|_2^2 \lesssim 2^{-\frac{3}{4}} 2^{-\frac{\| n \|}{2}} \| df_n \|_2^2. \] (3.11)

The proof of (3.11) will be complete if the following equality can be verified: for \( k \geq n \),
\[ A_{f,s}^\perp(x)(M_k - E_k)df_n(x)A_{f,s}^\perp(x) = 0. \] (3.12)

To see this, note that \( E_kdf_n = df_n \) when \( k \geq n \). Due to the support of \( df_n \) and (3.9), it is clear that
\[ A_{f,s}^\perp(x)df_n(x)A_{f,s}^\perp(x) = A_{f,s}^\perp(x)(5A_{n-s})\perp(x)df_n(x)(5A_{n-s})\perp(x)A_{f,s}^\perp(x) = 0. \]

On the other hand, (3.9) implies
\[ A_{f,s}^\perp(x)M_kdf_n(x)A_{f,s}^\perp(x) = A_{f,s}^\perp(x) \frac{1}{|B_k|} \int_{x+B_k} df_n(y)dy A_{f,s}^\perp(x) \]
\[ = A_{f,s}^\perp(x) \frac{1}{|B_k|} \int_{x+B_k} (5A_{n-s})\perp(x)df_n(y)(5A_{n-s})\perp(x)dy A_{f,s}^\perp(x) \]
\[ = A_{f,s}^\perp(x) \int_{\mathbb{R}^d} df_n(y)1_{x \notin 5Q_{y,n-s}} dy A_{f,s}^\perp(x). \]

Therefore,
\[ \int_{x+B_k} 1_{x \notin 5Q_{y,n-s}} df_n(y)dy = 0 \]
since \( k \geq n \) (see Figure I). This is precisely the claim. Hence, we finish the proof of (3.11).

Now we turn to the case \( n > k \). Then \( \mathbb{E}_k df_n = 0 \) in this case. Thus, using again the almost orthogonality principle again—Lemma 3.8, we reduce to show
\[
\|A_{f,s}^\perp M_k df_n A_{f,s}^\perp\|_2^2 \lesssim 2^{-\frac{s}{2}} 2^{\frac{k-n}{2}} \|df_n\|_2^2.
\] (3.13)

To this end, (3.9) allows us to conclude that for \( n \leq k \),
\[
A_{f,s}^\perp(x) M_k df_n(x) A_{f,s}^\perp(x) = A_{f,s}^\perp(x) \frac{1}{|B_k|} \int_{x+B_k} 1_{x \not\in 5Q_{n-s}} df_n(y) dy A_{f,s}^\perp(x).
\]

We make a geometric observation. It is important to note that for \( y \in x + B_k \),
\[
\int_{x+B_k} 1_{x \not\in 5Q_{n-s}} df_n(y) dy
\]
may be nonzero unless \( k \leq n - s - 1 \) (see Figure I).

**Figure I**

We suppose \( \text{supp} df_n \subset Q_{n-s} \).

Hence, \( A_{f,s}^\perp(x) M_k df_n(x) A_{f,s}^\perp(x) = 0 \) for \( n - s - 1 < k < n \). Therefore, we get
\[
\|A_{f,s}^\perp M_k df_n A_{f,s}^\perp\|_2^2 \lesssim 2^{-\frac{s}{2}} 2^{\frac{k-n}{2}} \|df_n\|_2^2
\] (3.14)
when \( n - s - 1 < k < n \).

Now we turn to prove (3.13) in the case \( k \leq n - s - 1 \). In this case, we first claim that
\[
\|M_k df_n\|_2^2 \lesssim 2^{k-n} \|df_n\|_2^2.
\] (3.15)

Indeed, the argument in proving (3.5) implies that (3.15) can be deduced from the following pointwise estimate:
\[
|M_k df_n(x)|^2 \lesssim 2^{k-n} \cdot M_{k-1}|df_n(x)|^2.
\] (3.16)

In order to prove (3.16), we divide \( \mathbb{R}^d \) into all atoms in \( Q_{n-1} \). Taking any \( Q \in Q_{n-1} \), we have \( \int_Q df_n = 0 \). Recall
\[
\mathcal{I}(B, n) = \cup \{Q \cap B | Q \in Q_n, \partial B \cap Q \neq \emptyset \}.
\]
Then we have
\[ \int_{x + B_k} df_n(y)dy = \sum_{Q \in \mathcal{Q}_{n-1}} \int_{x + B_k \cap Q} df_n(y)dy = \int_{\mathcal{I}(x + B_k, n-1)} df_n(y)dy. \]
Since \(|\mathcal{I}(x + B_k, n-1)| \lesssim 2^{-n/2}(d-1)(-k)\) and \(|B_k| \asymp 2^{-kd}\), then by applying Cauchy-Schwarz inequality via the operator convexity of the square function \(x \mapsto |x|^2\), we get
\[
|M_k df_n(x)|^2 = \frac{1}{|B_k|^2} \int_{\mathcal{I}(x + B_k, n-1)} df_n(y)dy^2 \\
\lesssim 2^{kd} 2^{-n(d-1)(-k)} \int_{x + B_{k-1}} |df_n(y)|^2 dy \\
\lesssim 2^{-k-n} \frac{1}{2^{-d(k-1)}} \int_{x + B_{k-1}} |df_n(y)|^2 dy \\
\lesssim 2^{-k-n} \frac{1}{|B_{k-1}|} \int_{x + B_{k-1}} |df_n(y)|^2 dy.
\]
Thus we obtain the desired estimate (3.16). Therefore, for \(k \leq n-s-1\)
\[
\|A_{f,s}^+ M_k df_n A_{f,s}^+\|_2 \leq \|M_k df_n\|^2_2 \lesssim 2^{-k-n} \|df_n\|^2_2 \lesssim 2^{-\frac{k-n}{2}} \|df_n\|^2_2. \tag{3.17}
\]
From now on, (3.14) and (3.17) give the desired estimate (3.13). Finally, combining (3.11) with (3.13), we get (3.10). Thus, the proof of pseudo-localisation theorem of \(T\) is complete. 

**Lemma 3.12.** Let \(\zeta\) be the projection introduced in Proposition 3.4. We have the following estimate:

\[
\|\zeta T g(s)\zeta\|_{L^2(\Omega;\mathbb{R}^N)} \lesssim 2^{-\frac{d}{4}} \|g(s)\|_2
\]

**Proof.** This is where we use pseudo-localization principle, i.e., Theorem 3.11 to \(g(s)\). By applying Proposition 3.3, we take \(A_k = p_k\) since \(p_{k}^+ g_{s,k} p_{k}^+ = 0\). Then the definition of \(A_{f,s}\) yields:

\[
A_{f,s} = \bigvee_{k>0} 5p_k = \zeta^\perp.
\]

Theorem 3.11 gives:

\[
\|A_{f,s}^+ T g(s) A_{f,s}^+\|_{L^2(\Omega;\mathbb{R}^N)} \lesssim 2^{-\frac{d}{4}} \|g(s)\|_2
\]

which is exactly the desired result. 

**3.3.3. Weak type estimate for \(Tg_{\text{off}}\).** We now consider the off-diagonal term \(g_{\text{off}}\) determined by \(g = g_d + g_{\text{off}}\). As usual, we decompose the term \(Tg_{\text{off}}\) into the following four parts

\[
(1_N - \zeta)Tg_{\text{off}}(1_N - \zeta) + \zeta Tg_{\text{off}}(1_N - \zeta) + (1_N - \zeta)Tg_{\text{off}}\zeta + \zeta Tg_{\text{off}}\zeta,
\]

where \(\zeta\) denotes the projection constructed in Proposition 3.4. Hence, we reduce to estimate the last term above. Arguing as for \(g_d\) and applying Lemma 3.12, we
find
\[ \lambda \tilde{\varphi}\left( |\zeta T g_{\text{off}} \zeta| > \lambda \right) \lesssim \frac{1}{\lambda} \left\| \zeta T g_{\text{off}} \zeta \right\|_{L_2(L_\infty(\Omega)^{\otimes N})}^2 \]
\[ \lesssim \frac{1}{\lambda} \left( \sum_{s=1}^{\infty} \left\| \zeta g(s) \zeta \right\|_{L_2(L_\infty(\Omega)^{\otimes N})}^2 \right)^2 \]
\[ \lesssim \frac{1}{\lambda} \left( \sum_{s=1}^{\infty} 2^{-\frac{s}{4}} \|g(s)\|_2 \right)^2 \]
\[ \lesssim \frac{1}{\lambda} \left( \sum_{s=1}^{\infty} 2^{-\frac{s}{4}} \sqrt{\lambda \|f\|_1} \right)^2 \lesssim \|f\|_1, \]
where the penultimate inequality follows from the inequality given in Proposition 3.3. This is the required estimate for \( T g_{\text{off}} \).

3.4. Conclusion. Combining all necessary estimates so far in Section 3, we obtain the desired weak \((1,1)\) type estimate announced in Theorem 1.1. Therefore, we complete the proof of conclusion (i) in Theorem 1.1.

At the end of this section, we are at a position to prove the two corollaries announced in the Introduction.

Proof of Corollary 1.2. We can write
\[ R_k f(x) = (M_k - M_{k-1}) f(x) \]
\[ = (M_k - E_k) f(x) + (E_k - E_{k-1}) f(x) + (E_{k-1} - M_{k-1}) f(x). \]
The first and the third terms of the final expression can be handled by conclusion (i) of Theorem 1.1, and the middle term we refer [36, Theorem 3.1]. Thus, we get the desired estimate of Corollary 1.2. \( \square \)

Proof of Corollary 1.3. We first decompose \( M_k f \) as
\[ M_k f = (M_k - E_k) f + E_k f. \]
Then Cuculescu’s result—Lemma 3.1 implies that we can find a projection \( e_1 \in \mathcal{N} \) such that
\[ \sup_k \|e_1 E_k f e_1\|_{\infty} \leq \lambda \quad \text{and} \quad \lambda \varphi(1_{\mathcal{N}} - e_1) \lesssim \|f\|_1. \]
On the other hand, the conclusion (i) of Theorem 1.1 yields that there exists a decomposition \( T_k f = g_k + h_k \) satisfying
\[ \|(g_k)\|_{L_1(\mathcal{N};c_k^2)} + \|(h_k)\|_{L_1(\mathcal{N};c_k^2)} \lesssim \|f\|_1. \]
We now take \( e_2 = \chi_{(0,\lambda]}\left( \left( \sum_{k=1}^{\infty} |g_k|^2 \right)^{\frac{1}{2}} \right) \) and \( e_3 = \chi_{(0,\lambda]}\left( \left( \sum_{k=1}^{\infty} |h_k|^2 \right)^{\frac{1}{2}} \right) \), then
\[ \left\| \left( \sum_{k=1}^{\infty} |g_k|^2 \right)^{\frac{1}{2}} e_2 \right\|_{\infty} \leq \lambda \quad \text{and} \quad \lambda \varphi(1_{\mathcal{N}} - e_2) \lesssim \|f\|_1. \]
Also for \( e_3 \), we have
\[
\| \left( \sum_{k=1}^{\infty} |h_k^*|^2 \right)^{\frac{1}{2}} e_3 \|_\infty \leq \lambda \quad \text{and} \quad \lambda \varphi (1_N - e_3) \lesssim \| f \|_1.
\]

Let \( e_4 = e_2 \land e_3 \). Then
\[
\lambda \varphi (1_N - e_4) \lesssim \| f \|_1.
\]

It remains to show
\[
\sup_{k \geq 1} \| e_4 (M_k - E_k) f e_4 \|_\infty \leq 2\lambda.
\]

Indeed,
\[
\| e_4 (M_k - E_k) f e_4 \|_\infty \leq \| e_4 g_k e_4 \|_\infty + \| e_4 h_k e_4 \|_\infty \\
= \| e_4 g_k e_4 \|_\infty + \| e_4 h_k^* e_4 \|_\infty \\
= \| e_4 u_k |g_k| e_4 \|_\infty + \| e_4 v_k |h_k^*| e_4 \|_\infty \\
\leq \| |g_k| e_4 \|_\infty + \| |h_k^*| e_4 \|_\infty \\
= \| e_4 |g_k|^2 e_4 \|_\infty^{\frac{1}{2}} + \| e_4 |h_k^*|^2 e_4 \|_\infty^{\frac{1}{2}} \\
\leq \| \sum_{k=1}^{\infty} |g_k|^2 e_4 \|_\infty^{\frac{1}{2}} + \| \sum_{k=1}^{\infty} |h_k^*|^2 e_4 \|_\infty^{\frac{1}{2}} \\
= \| (\sum_{k=1}^{\infty} |g_k|^2)^{\frac{1}{2}} e_2 e_4 \|_\infty + \| (\sum_{k=1}^{\infty} |h_k^*|^2)^{\frac{1}{2}} e_3 e_4 \|_\infty \\
\leq 2\lambda,
\]

where for the second equality we used the polar decomposition. Finally, if we set \( q = e_1 \land e_4 \), then it is easy to see that this is the desired projection. Thus, we finish the proof of Corollary 1.3. \( \square \)

4. \((L_\infty, \text{BMO})\) Estimate

In this section, we examine the \((L_\infty, \text{BMO})\) estimate. The dyadic BMO spaces \( \text{BMO}_d(\mathcal{R}) \) are defined in the introduction. On one hand, it suffices to show
\[
\left\| \sum_{k=1}^{\infty} T_k f \otimes e_k \right\|_{\text{BMO}_d(\mathcal{R})} \lesssim \| f \|_\infty.
\] (4.1)

Indeed, (4.1) is equivalent to
\[
\left\| \sum_{k=1}^{\infty} T_k f \otimes e_k \right\|_{\text{BMO}_d'(\mathcal{R})} \lesssim \| f \|_\infty
\] (4.2)

and
\[
\left\| \sum_{k=1}^{\infty} T_k f \otimes e_k \right\|_{\text{BMO}_d'(\mathcal{R})} \lesssim \| f \|_\infty.
\] (4.3)
Using the fact \( \|g\|_{\text{BMO}_d(\mathcal{R})} = \|g^*\|_{\text{BMO}_d(\mathcal{R})} \) and taking the adjoint of both sides in (4.2), we have

\[
\left\| \sum_{k=1}^{\infty} T_k f \otimes e_{1k} \right\|_{\text{BMO}_d(\mathcal{R})} = \left\| \left( \sum_{k=1}^{\infty} T_k f \otimes e_{1k} \right)^* \right\|_{\text{BMO}_d(\mathcal{R})} = \left\| \sum_{k=1}^{\infty} T_k f^* \otimes e_{1k} \right\|_{\text{BMO}_d(\mathcal{R})} \lesssim \|f^*\|_{\infty} = \|f\|_{\infty}.
\]

Similarly, we use (4.3) to get

\[
\left\| \sum_{k=1}^{\infty} T_k f \otimes e_{1k} \right\|_{\text{BMO}_d(\mathcal{R})} \lesssim \|f\|_{\infty}.
\]

These imply

\[
\left\| \sum_{k=1}^{\infty} T_k f \otimes e_{1k} \right\|_{\text{BMO}_d(\mathcal{R})} \lesssim \|f\|_{\infty}.
\]

On the other hand, as usual, in the definition of the BMO norm of a function \( f \), we may replace \( f_Q \) by any other operator \( \alpha_Q \) depending on \( Q \).

Now we are ready to prove the second part of Theorem 1.1.

**Proof.** For \( f \in L_\infty(\mathcal{N}) \), and a dyadic cube \( Q \), we decompose \( f \) as \( f = f 1_{3Q} + f 1_{\mathbb{R}^d \setminus 3Q} \triangleq f_1 + f_2 \). We shall take \( \alpha_{Q,k} = T_k f_2(c_Q) \) and \( \alpha_Q = \sum_{k=1}^{\infty} \alpha_{Q,k} \otimes e_{1k} \), where \( c_Q \) is the center of \( Q \). Write \( T_k f - \alpha_{Q,k} \) as

\[
T_k f - \alpha_{Q,k} = (T_k f - T_k f_2) + (T_k f_2 - \alpha_{Q,k}) \triangleq B_{k1} f + B_{k2} f.
\]

We first prove (4.2). Using the operator convexity inequality of square function \( x \mapsto |x|^2 \), we obtain

\[
\left| \sum_{k=1}^{\infty} (T_k f - \alpha_{Q,k}) \otimes e_{1k} \right|^2 \leq 2 \left| \sum_{k=1}^{\infty} B_{k1} f \otimes e_{1k} \right|^2 + 2 \left| \sum_{k=1}^{\infty} B_{k2} f \otimes e_{1k} \right|^2.
\]

The first term \( B_{11} = \sum_{k=1}^{\infty} B_{k1} f \otimes e_{1k} \) is easy to estimate. Indeed,

\[
\left\| \left( \frac{1}{|Q|} \int_Q (B_1 f(x))^*(B_1 f(x)) \, dx \right)^{1/2} \right\|_{\mathcal{M} \otimes B(\ell^2)}^2 = \left( \frac{1}{|Q|} \int_Q \sum_{k=1}^{\infty} |T_k f_1(x)|^2 \, dx \right)^{1/2} \left( \frac{1}{|Q|} \int_Q \sum_{k=1}^{\infty} |T_k f_1(x)|^2 \, dx \right)^{1/2} = \frac{1}{|Q|} \left( \sum_{k=1}^{\infty} |T_k f_1(x)|^2 \right) \|_{\mathcal{M}}^2 \leq \frac{1}{|Q|} \sup_{\|a\|_{L_2(\mathcal{M})} \leq 1} \tau \otimes \int_{\mathbb{R}^d} \sum_{k=1}^{\infty} |T_k f_1 a(x)|^2 \, dx \leq \frac{1}{|Q|} \sup_{\|a\|_{L_2(\mathcal{M})} \leq 1} \tau \otimes \int_{\mathbb{R}^d} \sum_{k=1}^{\infty} |T_k f_1 a(x)|^2 \, dx.
\]
\[
= \frac{1}{|Q|} \sup_{\|a\|_{L_2(M)} \leq 1} \|T(fa1_{3Q})\|_{L_2(N', \ell_r')}^2
\]
\[
\lesssim \frac{1}{|Q|} \sup_{\|a\|_{L_2(M)} \leq 1} \|fa1_{3Q}\|_2^2 \lesssim \|f\|_\infty^2,
\]
where in the penultimate inequality we have used Lemma 3.10.

Now we turn to the second term \(B_2f = \sum_{k=1}^\infty B_{k2}f \otimes e_{k1}\). We have
\[
B_2f(x)^*B_2f(x) = \sum_{k=1}^\infty |T_kf_2(x) - T_kf_2(c_Q)|^2
\]
\[
= \sum_{k=1}^\infty |(M_kf_2(x) - \mathbb{E}_kf_2(x) - (M_kf_2(c_Q) - \mathbb{E}_kf_2(c_Q)))|^2
\]
\[
\triangleq \sum_{k=1}^\infty |F_{k,Q}(x)|^2.
\]
We first claim that
\[
\|F_{k,Q}(x)\|_{\mathcal{M}} \lesssim |Q| \cdot 2^{kd}\|f\|_\infty. \quad (4.4)
\]
To this end, we have to estimate \(F_{k,Q}\) in two different conditions for any \(x \in Q\). If \(2^{-k} < \ell(Q)\), then \(\mathbb{E}_kf_2\) is supported in \(\mathbb{R}^d \setminus Q\) and \(x + B_k\) is contained in \(3Q\). Hence, for any \(y \in B_k\)
\[
|x_i + y_i - (c_Q)_i| \leq |x_i - (c_Q)_i| + |y_i| \leq \frac{1}{2}\ell(Q) + 2^{-k} < \frac{3}{2}\ell(Q),
\]
where \(x_i\) denotes the \(i\)-th coordinate of \(x\). Therefore, we get
\[
M_kf_2(x) - \mathbb{E}_kf_2(x) - (M_kf_2(c_Q) - \mathbb{E}_kf_2(c_Q)) = 0, \text{ for any } x \in Q.
\]
Thus it suffices to consider \(2^{-k} \geq \ell(Q)\). Note that \(Q\) should be contained in \(\sigma_k\) for some \(k \in \mathbb{Z}\), so \(\mathbb{E}_kf_2(x) = \mathbb{E}_kf_2(c_Q)\). On the other hand,
\[
\|M_kf_2(x) - M_kf_2(c_Q)\|_{\mathcal{M}} = \frac{1}{|B_k|} \left\| \int_{B_k + x} f_2(y)dy - \int_{B_{2k} + c_Q} f_2(y)dy \right\|_{\mathcal{M}}
\]
\[
= 2^{kd}\left\| \int_{\mathbb{R}^d} f_2(1_{B_k + c_Q \setminus B_k + x} - 1_{B_k + c_Q \setminus B_k + x})(y)dy \right\|_{\mathcal{M}}
\]
\[
\leq 2^{kd}\|f\|_\infty \int_{\mathbb{R}^d} 1_{B_k + c_Q \setminus B_k + x}\|f\|_\infty
\]
\[
\leq 2^{kd}\|B_k + c_Q \setminus B_k + x\| \cdot \|f\|_\infty.
\]
Then the fact that \(|B_k + c_Q \setminus B_k + x| \leq C|x - c_Q|^d \leq C|Q|\) yields
\[
\|M_kf_2(x) - M_kf_2(c_Q)\|_{\mathcal{M}} \leq C|Q| \cdot 2^{kd}\|f\|_\infty.
\]
This is precisely the claim of (4.4). Putting all these observations together, we obtain
\[
\left\| \left( \frac{1}{|Q|} \int_Q (B_2f(x))^*(B_2f(x)) dx \right)^\frac{1}{2} \right\|_{\mathcal{M} \otimes B(\ell_2)}^2
\]
= \left\| \left( \frac{1}{|Q|} \int_Q |Tf_2(x) - \alpha_Q|^2 \, dx \right)^\frac{1}{2} \right\|^2_{\mathcal{M}} \\
= \frac{1}{|Q|} \left\| \int_Q \sum_{k=1}^\infty |F_{k,Q}(x)|^2 \, dx \right\|_{\mathcal{M}} \\
\leq \frac{1}{|Q|} \int_Q \sum_{k=1}^\infty \|F_{k,Q}(x)\|_M^2 \, dx \\
\lesssim |Q|^2 \cdot \|f\|_2^2 \sum_{2^{-k} \geq \ell(Q)} 2^{2kd} \\
\lesssim \|f\|_\infty^2.

This yields the desired estimate.

We now consider (4.3). Here we only need to deal with \( B_1 f = \sum_{k=1}^\infty B_{k1} f \otimes e_{k1} \), while \( B_2 f \) can be treated as before. We note that

\[
\left\| \left( \frac{1}{|Q|} \int_Q (B_1 f(x))(B_1 f(x))^* \, dx \right)^\frac{1}{2} \right\|^2_{\mathcal{M} \otimes B(\ell_2)} \\
= \frac{1}{|Q|} \left\| \int_Q (B_1 f(x))(B_1 f(x))^* \, dx \right\|_{\mathcal{M} \otimes B(\ell_2)} \\
= \frac{1}{|Q|} \left\| \sum_{k_1, k_2=1}^\infty \left[ \int_Q T_{k_1} f_1(x) T_{k_2} f_1^*(x) \, dx \right] \otimes e_{k_1, k_2} \right\|_{\mathcal{M} \otimes B(\ell_2)} \\
\triangleq \frac{1}{|Q|} \left\| \Lambda \right\|_{\mathcal{M} \otimes B(\ell_2)}.
\]

Since \( \Lambda \) is a positive operator acting on \( \ell_2(L_2(\mathcal{M})) \) (\( = L_2(\mathcal{M}; \ell_2^{rc}) \)), we have

\[
\frac{1}{|Q|} \left\| \int_Q (B_1 f(x))(B_1 f(x))^* \, dx \right\|_{\mathcal{M} \otimes B(\ell_2)} \\
= \frac{1}{|Q|} \sup_{\|a\|_{L_2(M, \ell_2^{rc})} \leq 1} \langle \Lambda a, a \rangle \\
= \frac{1}{|Q|} \sup_{\|a\|_{L_2(M, \ell_2^{rc})} \leq 1} \tau \left( \left[ \sum_{k_1=1}^\infty a_{k_1}^* \otimes e_{1k_1} \right] \Lambda \left[ \sum_{k_2=1}^\infty a_{k_2} \otimes e_{k_21} \right] \right) \\
= \frac{1}{|Q|} \sup_{\|a\|_{L_2(M, \ell_2^{rc})} \leq 1} \tau \otimes \int_Q \left| \sum_{k=1}^\infty T_k f_1^*(x) a_k \right|^2 \, dx \\
\leq \frac{1}{|Q|} \sup_{\|a\|_{L_2(M, \ell_2^{rc})} \leq 1} \tau \otimes \int_{\mathbb{R}^d} \left| \sum_{k=1}^\infty T_k f_1^* a_k(x) \right|^2 \, dx \\
\lesssim \frac{1}{|Q|} \sup_{\|a\|_{L_2(M, \ell_2^{rc})} \leq 1} \tau \otimes \int_{\mathbb{R}^d} \sum_{k=1}^\infty |f_1^* a_k(x)|^2 \, dx.
\]
\[
\leq \frac{1}{|Q|} \sup_{\|a\|_{L_2(\mathcal{M},\ell_2^c)} \leq 1} \tau(\sum_{k=1}^{\infty} |a_k|^2) |3Q| \|f\|_{\infty}^2 \lesssim \|f\|_\infty^2.
\]

This proves (4.3). Therefore, the estimates obtained so far and their row analogues give rise to
\[
\max \left\{ \left\| \sum_{k=1}^{\infty} T_k f \otimes e_k \right\|_{\text{BMO}_d(\mathbb{R})}, \left\| \sum_{k=1}^{\infty} T_k f \otimes e_{1k} \right\|_{\text{BMO}_d(\mathbb{R})} \right\} \lesssim \|f\|_\infty.
\]
This completes the BMO estimate. 

5. INTERPOLATION

In this section, we show the strong \((p, p)\) estimates of \(T\) for \(1 < p < \infty\). Let us divide the proof into two parts. And we use the symbol \(\tilde{T}\) defined as (3.2) to distinguish from \(T\) in this section.

**Proposition 5.1.** Let \(1 < p < \infty\). Then \(T\) is bounded from \(L_p(\mathcal{N})\) to \(L_p(\mathcal{N}; \ell_2^c)\).

**Proof.** The result for \(p = 2\) is just Lemma 3.10. The cases \(1 < p < 2\). Using the weak \((1, 1)\) type estimate of \(\tilde{T}\) obtained in Section 3 and Lemma 3.10, we conclude that \(\tilde{T}\) is bounded from \(L_p(\mathcal{N})\) to \(L_p(L_\infty(\Omega) \otimes \mathcal{N})\) by real interpolation. Thus \(T\) is bounded from \(L_p(\mathcal{N})\) to \(L_p(\mathcal{N}; \ell_2^c)\) thanks to noncommutative Khintchine’s inequalities [37, 38].

The cases \(2 < p < \infty\). If we set \(T_c f = \sum_{k=1}^{\infty} T_k f \otimes e_{1k}\) and \(T_r f = \sum_{k=1}^{\infty} T_k f \otimes e_{1k}\), then Lemma 3.10 in conjunction with BMO type estimate yields that \(T_c\) and \(T_r\) are bounded from \(L_p(\mathcal{N})\) to \(L_p(\mathcal{N} \otimes \mathcal{B}(\ell_2))\) by interpolation [32]. Therefore, \(T\) is bounded from \(L_p(\mathcal{N})\) to \(L_p(\mathcal{N}; \ell_2^c)\) for all \(2 \leq p < \infty\). 

APPENDIX. PROOF OF LEMMA 3.10

As mentioned in the introduction, the noncommutative \(L_2\)-boundedness of the square function follows trivially from the corresponding commutative result. However, in this appendix, we provide a proof using similar idea as presented in [13, Theorem 2.3] but not using the commutative result as a black box. It is this proof that inspires us to complete the whole paper.

*Proof of Lemma 3.10.* For any \(f \in L_2(\mathbb{R}^d; L_2(\mathcal{M}))\), without loss of generality, we can assume \(f\) is positive. Let \(f = \sum_{n \in \mathbb{Z}} df_n\), since martingale differences are orthogonal, we have
\[
\sum_{n \in \mathbb{Z}} \|df_n\|_2^2 = \|f\|_2^2.
\]

On the other hand,
\[
\|Tf\|_{L_2(L_\infty(\Omega) \otimes \mathcal{N})}^2 = \sum_k \| (M_k - E_k) \sum_n df_n \|_2^2.
\]
To this end, according to the almost orthogonality principle—Lemma 3.8, it suffices to prove
\[ \| (M_k - E_k)df_n \|_2^2 \lesssim 2^{-[n-k]} \| df_n \|_2^2. \] (5.1)

We first prove (5.1) in the case \( k \geq n \). Note that \( E_k df_n = df_n \) for \( k \geq n \). It is enough to show
\[ \| M_k df_n - df_n \|_2^2 \lesssim 2^{n-k} \| df_n \|_2^2. \] (5.2)

To this end, we write
\[ \| M_k df_n - df_n \|_2^2 = \int_{\mathbb{R}^d} \| M_k df_n(x) - df_n(x) \|_{L_2(M)}^2 dx \]
\[ = \sum_{H \in \mathcal{Q}_n} \int_H \| M_k df_n(x) - df_n(x) \|_{L_2(M)}^2 dx. \]

Since \( df_n \) is a constant operator on \( H \in \mathcal{Q}_n \), we have \( M_k df_n(x) - df_n(x) = 0 \) if \( x + B_k \subset H \). Thus, for \( x \in H \), \( (M_k df_n - df_n)(x) \) may be nonzero only if \( x + B_k \) intersects with the complement of \( H \). Hence, for a given set \( B \) in \( \mathbb{R}^d \), we define
\[ \mathcal{H}(B, H) = \{ x \in H | B \cap H^c \neq \emptyset \}. \]

Noting that \( H \in \mathcal{Q}_n \) and using the property of \( \mathcal{H}(B, H) \), we have
\[ |\mathcal{H}(x + B_k, H)| \lesssim 2^{(d-1)(-n)} \cdot 2^{-k}. \]

Let \( m_H \) be the maximum of \( \| df_n(x) \|_{L_2(M)} \) on \( H \) and the atom in \( \mathcal{Q}_n \) neighboring \( H \). Then \( \| M_k df_n(x) - df_n(x) \|_{L_2(M)} \leq 2m_H \) for every \( x \in H \). Therefore
\[ \int_H \| M_k df_n(x) - df_n(x) \|_{L_2(M)}^2 dx \lesssim 2^{(d-1)(-n)} \cdot 2^{-k} \cdot m_H^2 \lesssim 2^{n-k} \int_H m_H^2 dx. \] (5.3)

Since \( m_H \) is a constant on \( H \), then \( \int_{\mathbb{R}^d} m_H^2 \leq C_d \int_{\mathbb{R}^d} \| df_n(x) \|_{L_2(M)}^2 dx \). Thus, summing over all \( H \in \mathcal{Q}_n \) in (5.3), we obtain the desired estimate (5.2).

Now we refer to (5.1) in the case \( n > k \). Note that \( E_k df_n = 0 \) in this case. Hence, it suffices to prove
\[ \| M_k df_n \|_2^2 \lesssim 2^{k-n} \| df_n \|_2^2, \] (5.4)
while the proof of (5.4) is the same as that of (3.15). Therefore, the proof of Lemma 3.10 is complete. \( \square \)

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