String Scattering Amplitudes in High Energy Limits

Yi Yang
NCTU@Taiwan

(C.T.Chan, P.M.Ho, S.H, S.L.Ko, J.C.Lee, Y.Mitsuka, K.Takahashi, T.Takimi, S.Terguchi, H.F.Yan ...)

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1. **Outline**

1. About String Scattering Amplitudes

2. High Energy Limits

   (a) $E \to \infty$, fixed angle: hard scattering

   (b) $E \to \infty$, small angle: Regge limit

3. Summary
2. String Scattering Amplitudes

- Amplitudes in field theory and string theory
• QFT amplitude

\[ A_{tree}^{(J)} \sim E^{-2(1-J)} \]

\[ \Rightarrow A_{1-loop}^{(J)} \sim \int d^4p \frac{\left( A_{tree}^{(J)} \right)^2}{(p^2)^2} \sim \int d^4E \frac{E^{-4(1-J)}}{E^4} \]

• Sum over intermedian states

\[ A = \sum_J A_{tree}^{(J)} \sim \sum_J a_J E^{-2(1-J)} \sim e^{-E} \]

• infinite high-spin particles

• perfect coefficients \( a_J \)
• QFT amplitude

\[ A^{(J)}_{tree} \sim E^{-2(1-J)} \]

\[ \Rightarrow A^{(J)}_{1-loop} \sim \int d^4p \frac{(A^{(J)}_{tree})^2}{(p^2)^2} \sim \int d^4E \frac{E^{-4(1-J)}}{E^4} \]

• Sum over intermediate states

\[ A = \sum_J A^{(J)}_{tree} \sim \sum_J a_J E^{-2(1-J)} \sim e^{-E} \]

• infinite high-spin particles ⇔ Regge poles

• perfect coefficients \( a_J \) ⇔ symmetry: Gross’ conjecture
• Veneziano amplitude (4-tachyon)

\[ \mathcal{T} = \left\langle e^{ik_1X(z_1)} \cdot e^{ik_2X(z_2)} e^{ik_3X(z_3)} e^{ik_4X(z_4)} \right\rangle 
\]

\[ = \int \, dz_1 \, dz_2 \, dz_3 \, dz_4 \prod_{1 \leq i < j \leq 4} |z_i - z_j|^{2k_i \cdot k_j} \]

\[ = \int_0^1 \, dx \cdot x^{-\frac{s}{2} - 2} (1 - x)^{-\frac{t}{2} - 2} \]

\[ = B \left( -\frac{s}{2} - 1, -\frac{t}{2} - 1 \right) \]

• \( E \to \infty \), fixed angle

• Exponential fall-off: \( \mathcal{T} \sim e^{-\alpha E} \)
String spectrum

| Mass level | Positive-norm states | Zero-norm states |
|------------|----------------------|------------------|
| $M^2 = -2$ | ●                    | n/a              |
| $M^2 = 0$  |                      | ●                |
| $M^2 = 2$  | [Diagram]            | [Diagram]        |
| $M^2 = 4$  | [Diagram], [Diagram] | [Diagram], 2 × [Diagram] |
3. High Energy Limits

- $E \to \infty$, fixed angle $\phi$
  \[ s \sim E^2 \to \infty, \quad \frac{t}{s} \sim \sin^2 \frac{\phi}{2} \to \text{fixed}. \]

  **Gross conjecture:** symmetry $\to$ linear relations.

- $E \to \infty$, small angle $\phi$
  \[ s \sim E^2 \to \infty, \quad t \sim E^2 \sin^2 \frac{\phi}{2} \to \text{fixed} \]

  **Regge limit**
3.1. $E \to \infty$, Fixed Angle

- $E \to \infty$, fixed angle $\phi$

  \[ \Rightarrow s \sim E^2 \to \infty, \quad \frac{t}{s} \sim \sin^2 \frac{\phi}{2} \to \text{fixed}. \]

- 4-point scattering amplitudes

  at mass level $M^2 = 2(N - 1)$,

  \[ T^{(N,2m,q)} = \langle V_1 V^{(N,2m,q)}(k)V_3 V_4 \rangle, \]

  where the vertex of leading states are

  \[ V^{(N,2m,q)}(k) \sim (\alpha^T_{-1})^{N-2m-2q} (\alpha^L_{-1})^{2m} (\alpha^L_{-2})^q |0; k\rangle, \]
\[ \mathcal{T}^{(N,2m,q)} = \langle V_1 V^{(N,2m,q)}(k)V_3 V_4 \rangle, \]

- Linear relations

\[ \frac{\mathcal{T}^{(N,2m,q)}}{\mathcal{T}^{(N,0,0)}} = \left( -\frac{1}{2M} \right)^q \left( \frac{1}{2M^2} \right)^m (2m - 1)!! \]

1. Decoupling of zero norm states

2. Virasoro constraints

3. Saddle point
Section 3: High Energy Limits

3.2. \( E \to \infty, \text{ Small Angle: Regge Limit} \)

- \( E \to \infty, \text{ small angle } \phi \)

\[ \Rightarrow s \sim E^2 \to \infty, \quad t \sim E^2 \sin^2 \frac{\phi}{2} \to \text{fixed}. \]

- "Leading states" for the mass level \( M^2 = 2(N-1) \),

\[ |N, k_n, q_m\rangle = \prod_{n>0} (\alpha^T_n)^{k_n} \prod_{m>0} (\alpha^L_m)^{q_m} |0\rangle \]

\[ \sum_{n,m} nk_n + mq_m = N \]
• Scattering amplitude in Regge limit,
\[
T^{(N,k_n,q_m)} = \left(-\frac{i}{M^2}\right)^{q_1} U \left(-q_1, \frac{t}{2} + 2 - q_1, \frac{t + M^2 + 2}{2}\right) \\
\cdot B \left(-1 - \frac{s}{2}, -1 - \frac{t}{2}\right) \cdot \prod_{n=1}^{k_n} \left[i\sqrt{-t(n - 1)!}\right]^{k_n} \\
\cdot \prod_{m=2}^{q_m} \left[i \left(t + M^2 + 2\right) (m - 1)! \left(-\frac{1}{2M^2}\right)\right]
\]

• Kummer function of the first kind \(U(a, c, x)\)
\[
xU'''(x) + (c - x)U'(x) - aU(x) = 0
\]
• Reproducing the ratios when $|t| \to \infty$

$$U \left( -2m, \frac{t}{2} + 2 - 2m, \frac{t}{2} \right)$$

$$= \sum_{j=0}^{2m} (-2m)_j \left( -1 - \frac{t}{2} \right)_j \frac{(-2)^j}{j!} t^{2m-j}$$

$$= 0 \cdot t^{2m} + 0 \cdot t^{2m-1} + \cdots + 0 \cdot t^{m+1} + \frac{(2m)!}{(-4)^m m!} t^m + O(t^{m-1})$$

• Universal power-law behavior: $\mathcal{T} \sim E^{\alpha(t)}$
## Section 3: High Energy Limits

- **Compactified space**

| \( \phi \) | \((s, t, N, K)\) | \( E \to \infty \) | \( \tilde{\phi} \) |
|---|---|---|---|
| \( \phi \) fixed | \((s, t) \gg (N, K)\) | \( e^{-E} \) | \( \tilde{\phi} \) fixed |
| \( \phi \) fixed | \((s, t, K) \gg N\) | \( E^{-c} \) | \( \tilde{\phi} \sim 0 \) |
| \( \phi \sim 0 \) | \( s \gg (t, N, K)\) | \( E^{-c} \) | \( \tilde{\phi} \sim 0 \) |
| \( \phi \sim 0 \) | \((s, K) \gg (t, N)\) | \( e^{-E} \) | \( \tilde{\phi} \) fixed |
4. Summary

- High energy, fixed angle $\Rightarrow$ linear relations,
  \[
  \lim_{E \to \infty} \frac{\mathcal{T}^{(n,2p,q)}}{\mathcal{T}^{(n,0,0)}} = \left(-\frac{1}{2M}\right)^q \left(\frac{1}{2M^2}\right)^p (2p - 1)!!
  \]

- High energy, small angle (Regge) $\Rightarrow$ Kummer function
  \[
  \mathcal{T} \sim U\left(-q_1, \frac{t}{2} + 2 - q_1, \frac{t + M^2 + 2}{2}\right)
  \]

- Reproduce the linear relations from Regge limit $t \to \infty$,
  \[
  U\left(-2m, \frac{t}{2} + 2 - 2m, \frac{t}{2}\right) = 0 + \frac{(2m)!}{m!} \left(-\frac{t}{4}\right)^m + \mathcal{O}\left(t^{m-1}\right)
  \]