HOW TO DEMONSTRATE A POSSIBLE EXISTENCE OF A MASS GAP IN QCD

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We propose to realize a mass gap in QCD by not imposing the transversality condition on the full gluon self-energy, while preserving the color gauge invariance condition for the full gluon propagator. This is justified by the nonlinear and nonperturbative dynamics of QCD. None of physical observables/processes in low-energy QCD will be directly affected by such a temporary violation of color gauge invariance/symmetry. No truncations/approximations and no special gauge choice are made for the regularized skeleton loop integrals, contributing to the full gluon self-energy, which enters the Schwinger-Dyson equation for the full gluon propagator. In order to make the existence of a mass gap perfectly clear the corresponding subtraction procedure is introduced. All this allows one to establish the general structure of the full gluon propagator and the corresponding gluon Schwinger-Dyson equation in the presence of a mass gap. It is mainly generated by the nonlinear interaction of massless gluon modes. The physical meaning of the mass gap is to be responsible for the large-scale (low-energy/momentum), i.e., nonperturbative structure of the true QCD vacuum.

In the presence of a mass gap two different types of solutions for the full gluon propagator are possible. The massive solution leads to an effective gluon mass, which depends on the gauge-fixing parameter explicitly. This solution becomes smooth at small gluon momentum in the Landau gauge. The general iteration solution is always severely singular at small gluon momentum, i.e., the gluons remain massless, and this does not depend on the gauge choice.

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I. INTRODUCTION

Today there is no doubt left that color confinement and other dynamical effects, such as spontaneous breakdown of chiral symmetry, bound-state problems, etc., being essentially nonperturbative (NP) effects, are closely related to the large-scale (low-energy/momentum) structure of the true QCD ground state and vice-versa [1, 2] (and references therein). The perturbation theory (PT) methods in general fail to investigate them. If QCD itself is a confining theory then a characteristic scale has to exist. It should be directly responsible for the above-mentioned structure of the true QCD vacuum in the same way as \( \Lambda_{QCD} \) is responsible for the nontrivial perturbative dynamics there (scale violation, asymptotic freedom (AF) [3]).

The Lagrangian of QCD [3, 4] does not contain explicitly any of the mass scale parameters which could have a physical meaning even after the corresponding renormalization program is performed. The main goal of this paper is to show how a characteristic scale (the mass gap, for simplicity) responsible for the NP dynamics in the infrared (IR) region may explicitly appear in QCD. This becomes an imperative especially after Jaffe and Witten have formulated their theorem "Yang-Mills Existence And Mass Gap" [5]. We will show that the mass gap is dynamically generated mainly due to the nonlinear (NL) interaction of massless gluon modes.

The propagation of gluons is one of the main dynamical effects in the true QCD vacuum. It is described by the corresponding quantum equation of motion, the so-called Schwinger-Dyson (SD) equation [1] (and references therein) for the full gluon propagator. The importance of this equation is due to the fact that its solutions reflect the quantum-dynamical structure of the true QCD ground state. The color gauge structure of this equation is the main subject of our investigation in order to find a way how to realize a mass gap in QCD. Also we will discuss at least two possible types of solutions of the gluon SD equation in the presence of a mass gap, making no approximations/truncations and no special gauge choice for the skeleton loop integrals contributing to it. So they can be considered as the generalizations of the explicit solutions because the latter ones are necessarily based on the above-mentioned specific approximations/truncations schemes.
II. QED

It is instructive to begin with a brief explanation why a mass gap does not occur in quantum electrodynamics (QED). The photon SD equation can be symbolically written down as follows:

\[ D(q) = D^0(q) + D^0(q)\Pi(q)D(q), \quad (2.1) \]

where we omit, for convenience, the dependence on the Dirac indices, and \(D^0(q)\) is the free photon propagator. \(\Pi(q)\) describes the electron skeleton loop contribution to the photon self-energy (the so-called vacuum polarization tensor). Analytically it looks

\[ \Pi(q) \equiv \Pi_{\mu\nu}(q) = -g^2 \int \frac{id^4p}{(2\pi)^4} Tr[\gamma_\mu S(p-q)\Gamma_\nu(p-q)S(p)], \quad (2.2) \]

where \(S(p)\) and \(\Gamma_\mu(p-q, q)\) represent the full electron propagator and the full electron-photon vertex, respectively. Here and everywhere below the signature is Euclidean, since it implies \(q^2 \rightarrow 0\) when \(q^2 \rightarrow 0\) and vice-versa. This tensor has the dimensions of a mass squared, and therefore it is quadratically divergent. To make the formal existence of a mass gap (the quadratically divergent constant, so having the dimension of a mass squared) perfectly clear, let us now, for simplicity, subtract its value at zero. One obtains

\[ \Pi^*(q) \equiv \Pi^*_{\mu\nu}(q) = \Pi_{\mu\nu}(q) - \Pi_{\mu\nu}(0) = \Pi_{\mu\nu}(q) - \delta_{\mu\nu}\Delta^2(\lambda). \quad (2.3) \]

The explicit dependence on the dimensionless ultraviolet (UV) regulating parameter \(\lambda\) has been introduced into the mass gap \(\Delta^2(\lambda)\), given by the integral (2.2) at \(q^2 = 0\), in order to assign a mathematical meaning to it. In this connection, a few remarks are in order in advance. The dependence on \(\lambda\) (when it is not shown explicitly) is assumed in all divergent integrals here and below in the case of the gluon self-energy as well (see next section). This means that all the expressions are regularized (including photon/gluon propagator), and we can operate with them as with finite quantities. \(\lambda\) should be removed on the final stage only after performing the corresponding renormalization program (which is beyond the scope of the present investigation, of course). Whether the regulating parameter \(\lambda\) has been introduced in a gauge-invariant way (though this always can be achieved) or not, and how it should be removed is not important for the problem if a mass gap can be “released/liberated” from the corresponding vacuum. We will show in the most general way (not using the PT and no special gauge choice will be made) that this impossible in QED and might be possible in QCD.

The tensor structure of the subtracted photon self-energy can be written as follows:

\[ \Pi^*_{\mu\nu}(q) = T_{\mu\nu}(q)q^2\Pi^*_1(q^2) + q_{\mu}q_{\nu}(q)\Pi^*_2(q^2), \quad (2.4) \]

where both invariant functions \(\Pi^*_n(q^2)\) at \(n = 1, 2\) are, by definition, dimensionless and regular at small \(q^2\), since \(\Pi^*(0) = 0\); otherwise they remain arbitrary. From this relation it follows that \(\Pi^*(q) = O(q^2)\), i.e., it is always of the order \(q^2\). Also, here and everywhere below

\[ T_{\mu\nu}(q) = \delta_{\mu\nu} - q_{\mu}q_{\nu}/q^2 = \delta_{\mu\nu} - L_{\mu\nu}(q). \quad (2.5) \]

Taking into account the subtraction (2.3), the photon SD equation becomes

\[ D(q) = D^0(q) + D^0(q)\Pi^*(q)D(q) + D^0(q)\Delta^2(\lambda)L(q). \quad (2.6) \]

Its subtracted part can be summed up into the geometric series, so one has

\[ D(q) = \tilde{D}^0(q) + \tilde{D}^0(q)\Delta^2(\lambda)L(q), \quad (2.7) \]

where the modified photon propagator is
\[
\mathcal{D}^0(q) = \frac{D^0(q)}{1 - \Pi^0(q)D^0(q)} = D^0(q) + D^0(q)\Pi^0(q)D^0(q) - D^0(q)\Pi^0(q)D^0(q)\Pi^0(q)D^0(q) + \ldots .
\] (2.8)

Since \( \Pi^0(q) = O(q^2) \) and \( D^0(q) \sim (q^2)^{-1} \), the IR singularity of the modified photon propagator is determined by the IR singularity of the free photon propagator, i.e., \( \mathcal{D}^0(q) = O(D^0(q)) \) with respect to the behavior at small photon momentum (in Eqs. (2.6), (2.7) and (2.8) we again omit the tensor indices, for simplicity).

Similar to the subtracted photon self-energy, the photon self-energy (2.2) in terms of independent tensor structures is

\[
\Pi_{\mu\nu}(q) = T_{\mu\nu}q^2\Pi_1(q^2) + q_\mu q_\nu\Pi_2(q^2),
\] (2.9)

where both invariant functions \( \Pi_n(q^2) \) at \( n = 1, 2 \) are dimensionless and remain arbitrary. Due to the transversality of the photon self-energy

\[
q_\mu \Pi_{\mu\nu}(q) = q_\nu \Pi_{\mu\nu}(q) = 0,
\] (2.10)

which comes from the current conservation condition in QED, one then has \( \Pi_2(q^2) = 0 \), i.e., this tensor should be purely transversal. On the other hand, from the subtraction (2.3) and transversality condition (2.10) it follows that

\[
\Pi^*_2(q^2) = -(\Delta^2(\lambda)/q^2).
\] (2.11)

However, this is impossible, since \( \Pi^*_2(q^2) \) is a regular function of \( q^2 \), so the mass gap should be zero and consequently \( \Pi_2(q^2) = 0 \) as well, i.e.,

\[
\Pi_2(q^2) = \Delta^2(\lambda) = 0.
\] (2.12)

This means that the photon self-energy coincides with its subtracted counterpart and both of them are purely transversal. Moreover, this means that the photon self-energy does not have a pole in its invariant function \( \Pi_1(q^2) = \Pi_1^0(q^2) \). As mentioned above, in obtaining these results neither the PT has been used nor a special gauge has been chosen.

So there is no place for quadratically divergent constants in QED, while logarithmic divergence still can be present in the invariant function \( \Pi_1(q^2) = \Pi_1^0(q^2) \). It is to be included into the electric charge through the corresponding renormalization program (for these detailed gauge-invariant derivations explicitly done in lower order of the PT see Refs. [4, 6, 7, 8, 9]).

In fact, the current conservation condition (2.10) lowers the quadratical divergence of the corresponding integral (2.2) to a logarithmic one. That is the reason why in QED logarithmic divergences survive only. Thus in QED there is no mass gap and the relevant photon SD equation is shown in Eq. (2.8), simply identifying the full photon propagator with its modified counterpart. In QED we should replace \( \Pi(q) \) by its subtracted counterpart \( \Pi^0(q) \) from the very beginning \( (\Pi(q) \to \Pi^0(q)) \), totally discarding the quadratically divergent constant \( \Delta^2(\lambda) \) from all the equations and relations. The current conservation condition for the photon self-energy (2.10), i.e., its transversality, and for the full photon propagator \( q_\mu q_\nu D_{\mu\nu}(q) = i\xi \), where \( \xi \) is the gauge-fixing parameter, are consequences of gauge invariance. They should be maintained at every stage of the calculations, since the photon is a physical state. In other words, at all stages the current conservation plays a crucial role in extracting physical information from the S-matrix elements in QED. For example, if some QED process includes the full photon propagator, then the corresponding S-matrix element is proportional to the combination \( j_{\mu}^1(q)D_{\mu\nu}(q)j_{\nu}^2(q) \). The current conservation condition \( j_{\mu}^1(q)q_\mu = j_{\nu}^2(q)q_\nu = 0 \) implies that the unphysical (longitudinal) component of the full photon propagator does not change the physics of QED, i.e., only its physical (transversal) component is important. In its turn this means that the transversality condition imposed on the photon self-energy is important, since \( \Pi_{\mu\nu}(q) \) itself is a correction to the amplitude of the physical process, for example such as electron-electron scattering.

\section{III. QCD}

Due to color confinement in QCD the gluon is not a physical state. Still, color gauge invariance should also be preserved, so the color current conservation takes place in QCD as well. However, in this theory it plays no role
in the extraction of physical information from the $S$-matrix elements for the corresponding physical processes and quantities. So in QCD there is no such physical amplitude to which the gluon self-energy may directly contribute (for example, quark-quark/antiquark scattering is not a physical process). The lesson which comes from QED is that if one preserves the transversality of the photon self-energy at every stage, then there is no mass gap. Thus, in order to realize a mass gap in QCD, our proposal is not to impose the transversality condition on the gluon self-energy, but preserving the color gauge invariance condition for the full gluon propagator (see below). As mentioned above, no QCD physics will be directly affected by this. So color gauge symmetry will be violated at the initial stage (at the level of the gluon self-energy) and will be restored at the final stage (at the level of the full gluon propagator).

### A. Gluon SD equation

The gluon SD equation symbolically is

$$D_{\mu\nu}(q) = D^0_{\mu\nu}(q) + D^0_{\mu\nu}(q) i \Pi_{\rho\sigma}(q; D) D_{\sigma\nu}(q),$$

where $D^0_{\mu\nu}(q)$ is the free gluon propagator. $\Pi_{\rho\sigma}(q; D)$ is the gluon self-energy, and in general it depends on the full gluon propagator due to the non-Abelian character of QCD (see below). Thus the gluon SD equation is highly NL, while the photon SD equation (2.1) is a linear one. In what follows we omit the color group indices, since for the gluon propagator (and hence for its self-energy) they are reduced to the trivial $\delta$-function, for example $D^{\rho\nu}_{\mu\nu}(q) = D_{\mu\nu}(q) \delta^{ab}$. Also, for convenience, we introduce $i$ into the gluon SD equation (3.1).

The gluon self-energy $\Pi_{\rho\sigma}(q; D)$ is the sum of a few terms, namely

$$\Pi_{\rho\sigma}(q; D) = -\Pi^g_{\rho\sigma}(q) - \Pi^{gh}_{\rho\sigma}(q) + \Pi^t_{\rho\sigma}(D) + \Pi^{(1)\rho\sigma}(q; D) + \Pi^{(2)\rho\sigma}(q; D) + \Pi'_{(2)\rho\sigma}(q; D),$$

where $\Pi^g_{\rho\sigma}(q)$ describes the skeleton loop contribution due to quark degrees of freedom (it is an analog of the vacuum polarization tensor in QED, see Eq. (2.2)), while $\Pi^{gh}_{\rho\sigma}(q)$ describes the skeleton loop contribution due to ghost degrees of freedom. Both skeleton loop integrals do not depend on the full gluon propagator $D$, so they represent the linear contribution to the gluon self-energy. $\Pi^t_{\rho\sigma}(D)$ represents the so-called constant skeleton tadpole term. $\Pi^{(1)\rho\sigma}(q; D)$ represents the skeleton loop contribution, which contains the triple gluon vertices only. $\Pi^{(2)\rho\sigma}(q; D)$ and $\Pi'_{(2)\rho\sigma}(q; D)$ describe topologically independent skeleton two-loop contributions, which combine the triple and quartic gluon vertices. The last four terms explicitly contain the full gluon propagators in different powers, that is why they form the NL part of the gluon self-energy. The explicit expressions for the corresponding skeleton loop integrals (in which the corresponding symmetry coefficients can be included) are of no importance here. Let us note that like in QED these skeleton loop integrals are in general quadratically divergent, and therefore they should be regularized (see remarks above and below).

### B. A temporary violation of color gauge invariance/symmetry (TVCGI/S)

The color gauge invariance condition for the gluon self-energy (3.2) can be reduced to the three independent transversality conditions imposed on it. It is well known that the quark contribution can be made transversal independently of the pure gluon contributions within any regularization scheme which preserves gauge invariance, for example such as the dimensional regularization method (DRM) [2][3][6][11]. So, one has

$$q_\rho \Pi^g_{\rho\sigma}(q) = q_\rho \Pi^g_{\rho\sigma}(q) = 0,$$

indeed. In the same way the sum of the gluon contributions can be done transversal by taking into account the ghost contribution, so again one has

$$q_\rho \left[ \Pi^{(1)\rho\sigma}(q; D) + \Pi^{(2)\rho\sigma}(q; D) + \Pi'_{(2)\rho\sigma}(q; D) - \Pi^{gh}_{\rho\sigma}(q) \right] = 0.$$

The role of ghost degrees of freedom is to cancel the unphysical (longitudinal) component of gauge bosons in every order of the PT, i.e., going beyond the PT and thus being general. The previous relation just demonstrate this, since it contains the corresponding skeleton loop integrals.
However, there is no such regularization scheme (preserving or not gauge invariance) in which the transversality condition for the constant skeleton tadpole term could be satisfied, i.e., \( q_0 \Pi^s_{\rho\sigma}(D) = q_0 \delta_{\rho\sigma} \Delta^2(D) = q_\sigma \Delta^2(D) \neq 0 \), indeed. This means that in any NP approach the transversality condition imposed on the gluon self-energy may not be valid, i.e., in general

\[
q_\rho \Pi_{\rho\sigma}(q; D) = q_\sigma \Pi_{\rho\sigma}(q; D) \neq 0.
\] (3.5)

In the PT, when the full gluon propagator is always approximated by the free one, the constant tadpole term is set to be zero within the DRM\[8, 11\], i.e., \( \Pi^t_{\rho\sigma}(D^0) = 0 \). So in the PT the transversality condition for the gluon self-energy is always satisfied.

The relation (3.5) justifies our proposal not to impose the transversality condition on the gluon self-energy. The special role of the constant skeleton tadpole term in the NP QCD dynamics should be emphasized. It explicitly violates the transversality condition for the gluon self-energy (3.5). The second important observation is that now ghosts themselves cannot automatically provide the transversality of the gluon propagator in NP QCD. However, this does not mean that we need no ghosts at all. Of course, we need them in other sectors of QCD, for example in the quark-gluon Ward-Takahashi identity, which contains the so-called ghost-quark scattering kernel explicitly\[3\].

**C. Subtractions**

As we already know from QED, the regularization of the gluon self-energy can be started from the subtraction its value at the zero point (see, however, remarks below). Thus, quite similarly to the subtraction (2.3), one obtains

\[
\Pi^s_{\rho\sigma}(q; D) = \Pi_{\rho\sigma}(q; D) - \Pi_{\rho\sigma}(0; D) = \Pi_{\rho\sigma}(q; D) - \delta_{\rho\sigma} \Delta^2(\lambda; D).
\] (3.6)

Let us remind once more that for our purpose, namely to demonstrate a possible existence of a mass gap \( \Delta^2(\lambda; D) \) in QCD, it is not important how \( \lambda \) has been introduced and how it should be removed at the final stage. The mass gap itself is mainly generated by the nonlinear interaction of massless gluon modes, slightly corrected by the linear contributions coming from the quark and ghost degrees of freedom, namely

\[
\Delta^2(\lambda; D) = \Pi^t(D) + \sum_\alpha \Pi^\alpha(0; D) = \Delta^2_t(D) + \sum_\alpha \Delta^2_\alpha(0; D),
\] (3.7)

where index “\( \alpha \)” runs as follows: \( \alpha = -q, -gh, 1, 2, 2', \) and \( -q, -gh \) mean that both terms enter the above-mentioned sum with minus sign (here, obviously, the tensor indices are omitted). In these relation all the divergent constants \( \Pi^t(D) \) and \( \Pi^\alpha(0; D) \), having the dimensions of a mass squared, are given by the corresponding skeleton loop integrals at \( q^2 = 0 \). Thus these constants summed up into the mass gap squared (3.8) cannot be discarded like in QED, since the transversality condition for the gluon self-energy is not satisfied, see Eq. (3.5). In other words, in QCD in general the quadratical divergences of the corresponding loop integrals cannot be lowered to logarithmic ones, and therefore the mass gap (3.7) should be explicitly taken into account in this theory. The transversality condition for the gluon self-energy can be satisfied partially, i.e., if one imposes it on quark and gluon (along with ghost) degrees of freedom as it follows from above. Then the mass gap is to be reduced to \( \Pi^t(D) \), since all other constants \( \Pi^\alpha(0; D) \) can be discarded in this case (see Eq. (3.7)). However, we will stick to our proposal not to impose the transversality condition on the gluon self-energy, and thus to deal with the mass gap on account of all possible contributions.

The subtracted gluon self-energy

\[
\Pi^s_{\rho\sigma}(q; D) \equiv \Pi^s(q; D) = \sum_\alpha \Pi^\alpha(0; D)
\] (3.8)

is free from the tadpole contribution, because \( \Pi^t_s(D) = \Pi^t(D) - \Pi^t(D) = 0 \), by definition, at any \( D \), while in the gluon self-energy it is explicitly present through the mass gap (see Eqs. (3.7) and (3.6)). The general decomposition of the subtracted gluon self-energy into the independent tensor structures can be written down as follows:

\[
\Pi^s_{\rho\sigma}(q; D) = T_{\rho\sigma}(q)q^2\Pi(q^2; D) + q_\rho q_\sigma \tilde{\Pi}(q^2; D),
\] (3.9)
where both invariant functions $\Pi(q^2; D)$ and $\tilde{\Pi}(q^2; D)$ are dimensionless and regular at small $q^2$. Since the subtracted gluon self-energy does not contain the tadpole contribution, we can now impose the color current conservation condition on it, i.e., to put

$$q_\mu \Pi^*_\mu\nu(q; D) = q_\nu \Pi^*_\mu\nu(q; D) = 0,$$

which implies $\Pi(q^2; D) = 0$, so that the subtracted gluon self-energy finally becomes purely transversal

$$\Pi^*_\mu\nu(q; D) = T_{\mu\nu}(q)q^2\Pi(q^2; D),$$

and it is always of the order $q^2$ at any $D$, since the invariant function $\Pi(q^2; D)$ is regular at small $q^2$ at any $D$. Thus the subtracted quantities are free from the quadratic divergences, but logarithmic ones can be still present in $\Pi(q^2; D)$ like in QED.

### D. General structure of the gluon SD equation

Our strategy is not to impose the transversality condition on the gluon self-energy in order to realize a mass gap despite whether or not the tadpole term is explicitly present. To show that this works, it is instructive to substitute the subtracted gluon self-energy (3.9) (and not its transversal part (3.11)) into the initial gluon SD equation (3.1), on account of the subtraction (3.6). Then one obtains

$$D_{\mu\nu}(q) = D_{\mu\nu}^0(q) + D_{\mu\nu}^0(q)i[T_{\mu\sigma}(q)q^2\Pi(q^2; D) + q_\rho q_\sigma \tilde{\Pi}(q^2; D)]D_{\sigma\nu}(q) + D_{\mu\sigma}^0(q)i\Delta^2(\lambda; D)D_{\sigma\nu}(q).$$

Let us now introduce the general tensor decompositions of the full and auxiliary free gluon propagators $D_{\mu\nu}(q) = i[T_{\mu\nu}(q)d(q^2) + L_{\mu\nu}(q)d_1(q^2)](1/q^2)$ and

$$D_{\mu\nu}^0(q) = i[T_{\mu\nu}(q) + L_{\mu\nu}(q)d_0(q^2)](1/q^2),$$

respectively. The form factor $d_0(q^2)$ introduced into the unphysical part of the auxiliary free gluon propagator $D_{\mu\nu}^0(q)$ is needed in order to explicitly show that the longitudinal part of the subtracted gluon self-energy $\tilde{\Pi}(q^2; D)$ plays no role. The color gauge invariance condition imposed on the full gluon propagator

$$q_\mu q_\nu D_{\mu\nu}(q) = i\xi,$$

implies $d_1(q^2) = \xi$, so that the full gluon propagator becomes

$$D_{\mu\nu}(q) = i\{T_{\mu\nu}(q)d(q^2) + \xi L_{\mu\nu}(q)\} \frac{1}{q^2}.$$  

Substituting all these decompositions into the gluon SD equation (3.12), one obtains

$$d(q^2) = \frac{1}{1 + \Pi(q^2; D) + (\Delta^2(\lambda; D)/q^2)},$$

and

$$d_0(q^2) = \frac{\xi}{1 - \xi[\Pi(q^2; D) + (\Delta^2(\lambda; D)/q^2)].}$$

However, the auxiliary free gluon propagator defined in Eqs. (3.13) and (3.17) is to be equivalently replaced as follows:

$$D_{\mu\nu}^0(q) \rightarrow D_{\mu\nu}^0(q) + i\xi L_{\mu\nu}(q)d_0(q^2)\left[\Pi(q^2; D) + \frac{\Delta^2(\lambda; D)}{q^2}\right] \frac{1}{q^2},$$

(3.18)
where \( D^0_{\mu\nu}(q) \) is the right-hand-side is the standard free gluon propagator, i.e.,

\[
D^0_{\mu\nu}(q) = i \{ T_{\mu\nu}(q) + \xi L_{\mu\nu}(q) \} \frac{1}{q^2}.
\] (3.19)

Then the gluon SD equation in the presence of the mass gap (3.12), on account of the explicit expression for the auxiliary free gluon form factor (3.17), and doing some tedious algebra, is also to be equivalently replaced as follows:

\[
D_{\mu\nu}(q) = D^0_{\mu\nu}(q) + D^0_{\mu\nu}(q)iT_{\mu\nu}(q)q^2\Pi(q^2; D)D_{\sigma\nu}(q)
+ D^0_{\sigma\nu}(q)i\Delta^2(\lambda; D)D_{\sigma\nu}(q) + i\xi^2L_{\mu\nu}(q)\frac{\Delta^2(\lambda; D)}{q^2}.
\] (3.20)

Here and below \( D^0_{\mu\nu}(q) \) is the free gluon propagator (3.19). The gluon SD equation (3.20) does not depend on \( d_0(q^2) \) and \( \Pi(q^2; D) \), i.e., they played their role and then retired from the scene. So, our derivation explicitly shows that the longitudinal part of the subtracted gluon self-energy \( \tilde{\Pi}(q^2; D) \) plays no role and can be put to zero without losing generality, and thus making the subtracted gluon self-energy purely transversal in accordance with Eq. (3.11).

Using now the explicit expression for the free gluon propagator (3.19) this equation can be further simplified to

\[
D_{\mu\nu}(q) = D^0_{\mu\nu}(q) - T_{\mu\nu}(q)\left[ \Pi(q^2; D) + \frac{\Delta^2(\lambda; D)}{q^2} \right]D_{\sigma\nu}(q).
\] (3.21)

It is easy to check that the full gluon propagator satisfies the color gauge invariance condition (3.14), indeed. So the full gluon propagator is the expression (3.15) with the full gluon form factor given in Eq. (3.16), which obviously satisfies Eq. (3.21). The only price we have paid by violating color gauge invariance is the gluon self-energy, while the full and free gluon propagators and the subtracted gluon self-energy always satisfy it. Let us emphasize that the full gluon propagator (3.15) with explicit expression for the free gluon propagator (3.19) this equation can be further simplified to

\[
D_{\mu\nu}(q) = D^0_{\mu\nu}(q) - T_{\mu\nu}(q)\left[ \Pi(q^2; D) + \frac{\Delta^2(\lambda; D)}{q^2} \right]D_{\sigma\nu}(q).
\] (3.21)

IV. MASSIVE SOLUTION

An immediate consequence of the explicit presence of the mass gap in the full gluon propagator is that a massive-type solution for it becomes possible. In other words, in this case the gluon may indeed acquire an effective mass. From Eq. (3.16) it follows that

\[
\frac{1}{q^2}d(q^2) = \frac{1}{q^2 + q^2\Pi(q^2; \xi) + \Delta^2(\lambda; \xi)}.
\] (4.1)

where instead of the dependence on \( D \) the dependence on \( \xi \) is explicitly shown. The full gluon propagator (3.15) may have a pole-type solution at the finite point if and only if the denominator in Eq. (4.1) has a zero at this point \( q^2 = -m^2_g \) (Euclidean signature), i.e.,

\[
-m^2_g - m^2_g\Pi(-m^2_g; \xi) + \Delta^2(\lambda, \xi) = 0,
\] (4.2)

where \( m^2_g = m^2_g(\lambda, \xi) \) is an effective gluon mass, and the previous equation is a transcendental equation for its determination. Excluding the mass gap, one obtains that the denominator in the full gluon propagator becomes

\[
q^2 + q^2\Pi(q^2; \xi) + \Delta^2(\lambda, \xi) = q^2 + m^2_g + q^2\Pi(q^2; \xi) + m^2_g\Pi(-m^2_g; \xi).
\]

Let us now expand \( \Pi(q^2; \xi) \) in a Taylor series near \( m^2_g \):
\[ \Pi(q^2; \xi) = \Pi(-m_g^2; \xi) + (q^2 + m_g^2)\Pi'(-m_g^2; \xi) + O\left((q^2 + m_g^2)^2\right). \] (4.3)

Substituting this expansion into the previous relation and after doing some tedious algebra, one obtains \(q^2 + m_g^2 + q^2 \Pi(q^2; \xi) + m_g^2 \Pi(-m_g^2; \xi) = (q^2 + m_g^2)(1 + \Pi(-m_g^2; \xi) - m_g^2 \Pi(-m_g^2; \xi))[1 + \Pi^R(q^2; \xi)]\), where \(\Pi^R(q^2; \xi) = 0\) at \(q^2 = -m_g^2\) (otherwise it remains arbitrary). Thus the full gluon propagator (3.15) now looks

\[ D_{\mu\nu}(q) = iT_{\mu\nu}(q) + \frac{Z_3}{(q^2 + m_g^2)[1 + \Pi^R(q^2; m_g^2)]} + \frac{\xi L_{\mu\nu}(q)}{q^2}, \] (4.4)

where, for future purpose, in the invariant function \(\Pi^R(q^2; m_g^2)\) instead of \(\xi\) we introduced the dependence on the gluon effective mass squared \(m_g^2\) which depends on \(\xi\) itself. The gluon renormalization constant is \(Z_3 = [1 + \Pi(-m_g^2; \xi) - m_g^2 \Pi(-m_g^2; \xi)]^{-1}\). In the formal PT limit \(\Delta^2(\lambda, \xi) = 0\), an effective gluon mass is also zero, \(m_g^2(\lambda, \xi) = 0\), as it follows from Eq. (4.2). So an effective gluon mass is the NP effect. At the same time, it cannot be interpreted as the "physical" gluon mass, since it remains explicitly gauge-dependent quantity. The gluon renormalization constant in this limit becomes a standard one, namely \([1 + \Pi(0; \xi)]^{-1}\). The massive-type solution (4.4) becomes smooth in the IR \((q^2 \to 0)\) in the Landau gauge \(\xi = 0\) only (the ghosts now cannot guarantee the cancellation of the longitudinal part of the full gluon propagator as mentioned above). In this connection let us point out that Landau gauge smooth (even vanishing in the IR) gluon propagator at the expense of more singular (than the free one) in the IR ghost propagator has been obtained and discussed (see, for example Refs. [12, 13] and references therein). As mentioned above, however, these results are necessarily based on different approximations/truncations for the skeleton loop integrals contributing to the gluon self-energy.

V. ITERATION SOLUTION

In order to perform a formal iteration of the gluon SD equation (3.21), much more convenient to address to its "solution" for the full gluon form factor (3.16), nevertheless, and rewrite it as follows:

\[ d(q^2) = 1 - \left[\Pi(q^2; d) + \frac{\Delta^2(\lambda; d)}{q^2}\right]d(q^2) = 1 - P(q^2; d)d(q^2), \] (5.1)

i.e., in the form of the corresponding transcendental (i.e., not algebraic) equation suitable for the formal nonlinear iteration procedure. Here we replace the dependence on \(D\) by the equivalent dependence on \(d\). For future purposes, it is convenient to introduce short-hand notations as follows:

\[ \Delta^2(\lambda; d = d^{(0)} + d^{(1)} + d^{(2)} + \ldots + d^{(m)} + \ldots) = \Delta^2_m = \Delta^2 c_m(\lambda, \alpha, \xi, g^2), \]

\[ \Pi(q^2; d = d^{(0)} + d^{(1)} + d^{(2)} + \ldots + d^{(m)} + \ldots) = \Pi_m(q^2) = [P_m(q^2) - (\Delta^2_m/q^2)]. \] (5.2)

In these relations \(\Delta^2_m\) are the auxiliary mass squared parameters, while \(\Delta^2\) is the mass gap itself (see, however, remarks in Conclusions). The dimensionless constants \(c_m\) via the corresponding subscripts depend on which iteration for the gluon form factor \(d\) is actually done. They may depend on the dimensionless coupling constant squared \(g^2\), as well as on the gauge-fixing parameter \(\xi\). We also introduce the explicit dependence on the dimensionless finite (slightly different from zero) subtraction point \(\alpha\), since the initial subtraction at the zero point may be dangerous. The dependence of \(\Delta^2\) on all these parameters is not shown explicitly, and if necessary can be restored any time. Let us also remind that all the invariant functions \(\Pi_m(q^2)\) are regular at small \(q^2\). If it were possible to express the full gluon form factor \(d(q^2)\) in terms of these quantities then it would be the formal solution for the full gluon propagator. In fact, this is nothing but the skeleton loops expansion, since the regularized skeleton loop integrals, contributing to the gluon self-energy, have to be iterated. This is the so-called general iteration solution. No truncations/approximations and no special gauge choice have been made. This formal expansion is not a PT series. The magnitude of the coupling constant squared and the dependence of the regularized skeleton loop integrals on it is completely arbitrary.

It is instructive to describe the general iteration procedure in some details. Evidently, \(d^{(0)} = 1\) and doing the first iteration in Eq. (5.1), one thus obtains

\[ d(q^2) - 1 = P_0(q^2) + \ldots = 1 + d^{(1)}(q^2) + \ldots, \] (5.3)
where obviously \( d^{(1)}(q^2) = -P_0(q^2) \). Doing the second iteration, one obtains
\[
d(q^2) = 1 - P_1(q^2)[1 + d^{(1)}(q^2)] + ... = 1 + d^{(1)}(q^2) + d^{(2)}(q^2) + ..., \tag{5.4}
\]
where \( d^{(2)}(q^2) = -d^{(1)}(q^2) - P_1(q^2)[1 - P_0(q^2)] \). Doing the third iteration, one further obtains
\[
d(q^2) = 1 - P_2(q^2)[1 + d^{(1)}(q^2) + d^{(2)}(q^2)] + ... = 1 + d^{(1)}(q^2) + d^{(2)}(q^2) + d^{(3)}(q^2) + ..., \tag{5.5}
\]
where \( d^{(3)}(q^2) = -d^{(1)}(q^2) - d^{(2)}(q^2) - P_2(q^2)[1 - P_1(q^2)(1 - P_0(q^2))] \), and so on for the next iterations. Thus up to the third iteration, one finally obtains
\[
d(q^2) = \sum_{m=0}^{\infty} d^{(m)}(q^2) = 1 - [\Pi_2(q^2) + \frac{\Delta_2}{q^2}] [1 - [\Pi_1(q^2) + \frac{\Delta_1}{q^2}] [1 - \Pi_0(q^2) - \frac{\Delta_0}{q^2}]] + ... . \tag{5.6}
\]
We restrict ourselves to the third iterated term, since this already allows to show explicitly some general features of such kind of the nonlinear iteration procedure.

### A. Splitting/shifting procedure

Doing some tedious algebra, the previous expression can be rewritten as follows:
\[
d(q^2) = [1 - \Pi_2(q^2) + \Pi_1(q^2)\Pi_2(q^2) - \Pi_0(q^2)\Pi_1(q^2)\Pi_2(q^2) + ...]
+ \frac{1}{q^2} [\Pi_2(q^2)\Delta_2^2 + \Pi_1(q^2)\Delta_2^2 - \Pi_0(q^2)\Pi_1(q^2)\Delta_2^2 - \Pi_0(q^2)\Pi_2(q^2)\Delta_2^2 - \Pi_1(q^2)\Pi_2(q^2)\Delta_2^2 + ...]
- \frac{1}{q^2} [\Pi_0(q^2)\Delta_0^2\Delta_2 + \Pi_1(q^2)\Delta_0^2\Delta_2 + \Pi_2(q^2)\Delta_0^2\Delta_2 + ...]
- \frac{1}{q^2} [\Delta_2^2 - \frac{\Delta_2^2\Delta_2}{q^2} + \frac{\Delta_2^2\Delta_2}{q^2} + ...], \tag{5.7}
\]
so that this formal expansion contains three different types of terms. The first type are the terms which contain only different combinations of \( \Pi_m(q^2) \) (they are not multiplied by inverse powers of \( q^2 \)); the third type of terms contains only different combinations of \( (\Delta_m^2/q^2) \). The second type of terms contains the so-called mixed terms, containing the first and third types of terms in different combinations. The two last types of terms are multiplied by the corresponding powers of \( 1/q^2 \). Evidently, such structure of terms will be present in each iteration term for the full gluon form factor. However, any of the mixed terms can be split exactly into the first and third types of terms by keeping the necessary number of terms in the Taylor expansions in powers of \( q^2 \) for \( \Pi_m(q^2) \), which are regular functions at small \( q^2 \). Thus the IR structure of the full gluon form factor (which is just our primary goal to establish) is determined not only by the third type of terms. It gains contributions from the mixed terms as well.

Let us present the above-mentioned Taylor expansions as follows:
\[
\Pi_m(q^2) = \Pi_m(0) + (q^2/\mu^2)\Pi^{(1)}_m(0) + (q^2/\mu^2)^2\Pi^{(2)}_m(0) + O_m(q^6), \tag{5.8}
\]
since for the third iteration we need to use the Taylor expansions up to this order (here \( \mu^2 \) is some fixed mass squared (not to be mixed up with the tensor index)). For example, the mixed term \( (1/q^2)\Pi_2(q^2)\Delta_2^2 \) should be split as
\[
\frac{\Delta_2^2}{q^2} \Pi_2(q^2) = \frac{\Delta_2^2}{q^2} \left[ \Pi_2(0) + (q^2/\mu^2)\Pi^{(1)}_2(0) + O(q^4) \right] = \frac{\Delta_2^2}{q^2} \Pi_2(0) + a_1\Pi^{(1)}_2(0) + O(q^2). \tag{5.9}
\]
Here and everywhere below \( a_m = (\Delta_m^2/\mu^2), \ m = 0, 1, 2, 3, ... \) are the dimensionless constants. The first term now is to be shifted to the third type of terms and combined with the term \( (-1/q^2)\Delta_2^2 \), while the second term \( a_1\Pi^{(1)}_2(0) + O(q^2) \) is to be shifted to the first type of terms. All other mixed terms of similar structure should be treated absolutely in the same way. For the mixed term \( (-1/q^2)\Pi_0(q^2)\Delta_2^2 \), one has
\[-\frac{\Delta^2 \Delta^2}{q^4} \Pi_0(q^2) = -\frac{\Delta^2 \Delta^2}{q^4} \left[ \Pi_0(0) + \frac{(q^2/\mu^2) \Pi_0^{(1)}(0)}{O(q^6)} + (q^2/\mu^2)^2 \Pi_0^{(2)}(0) + O(q^6) \right] \]

\[= -\frac{\Delta^2 \Delta^2}{q^4} \Pi_0(0) - \frac{\Delta^2 \Delta^2}{q^4} \omega_2 \Pi_0^{(1)}(0) - \omega_1 \omega_2 \Pi_0^{(2)}(0) - O(q^2). \quad (5.10)\]

Again the first and second terms should be shifted to the third type of terms and combined with terms containing there the same powers of $1/q^2$, while the last two terms should be shifted to the first type of terms.

Similar to the Taylor expansion (5.8), one has

\[\Pi_m(q^2) \Pi_n(q^2) = \Pi_{mn}(q^2) = \Pi_{mn}(0) + \frac{(q^2/\mu^2) \Pi_{mn}^{(1)}(0)}{O(q^6)} + (q^2/\mu^2)^2 \Pi_{mn}^{(2)}(0) + O_{mn}(q^6). \quad (5.11)\]

Then, for example the mixed term $-1/q^2 \Pi_0(q^2) \Pi_1(q^2) \Delta^2$ can be split as

\[-\frac{\Delta^2}{q^2} \Pi_0(q^2) \Pi_1(q^2) = -\frac{\Delta^2}{q^2} \left[ \Pi_0(0) + \frac{(q^2/\mu^2) \Pi_0^{(1)}(0)}{O(q^4)} + O(q^6) \right] \]

\[= -\frac{\Delta^2}{q^2} \Pi_0(0) - \omega_2 \Pi_0^{(1)}(0) + O(q^2), \quad (5.12)\]

so again the first term should be shifted to the third type of terms and combined with the terms containing the corresponding powers of $1/q^2$, while other terms are to be shifted to the first type of terms.

Completing this exact splitting/shifting procedure in the expansion (5.7), one can in general represent it as follows:

\[d(q^2) = (\Delta^2/q^2) B_1(\lambda, \alpha, \xi, g^2) + (\Delta^2/q^2)^2 B_2(\lambda, \alpha, \xi, g^2) + (\Delta^2/q^2)^3 B_3(\lambda, \alpha, \xi, g^2) + f_3(q^2) + ..., \quad (5.13)\]

where we used notations (5.2), since the coefficients of the above-used Taylor expansions depend in general on the same set of parameters: $\lambda, \alpha, \xi, g^2$. The invariant function $f_3(q^2)$ is dimensionless and regular at small $q^2$; otherwise it remains arbitrary. The generalization on the next iterations is almost obvious. Let us only note that in this case more terms in the corresponding Taylor expansions should be kept "alive".

B. The exact structure of the general iteration solution

Substituting the generalization of the expansion (5.13) on all iterations and omitting the tedious algebra, the general iteration solution of the gluon SD equation (3.21) for the regularized full gluon propagator (3.15) can be exactly decomposed as the sum of the two principally different terms as follows:

\[D_{\mu \nu}(q; \Delta^2) = D^{\text{NP}}_{\mu \nu}(q; \Delta^2) + D_{\mu \nu}^{\text{PT}}(q) = i T_{\mu \nu}(q) \sum_{k=0}^{\infty} (\Delta^2/q^2)^k \sum_{m=0}^{\infty} \Phi_{k,m}(\lambda, \alpha, \xi, g^2) \]

\[+ \frac{1}{q^2}, \quad (5.14)\]

where the superscript "NP" stands for the intrinsically NP part of the full gluon propagator. We distinguish between the two terms in Eq. (5.14) by the character of the corresponding IR singularities and the explicit presence of the mass gap (see below). Let us emphasize that the general problem of convergence of the formally regularized series (5.14) is irrelevant here. Anyway, the problem how to remove all types of the UV divergences (overlapping and overall) is a standard one. Our problem will be how to deal with severe IR singularities due to their novelty and genuine NP character. Fortunately, there already exists a well-elaborated mathematical formalism for this purpose, namely the distribution theory (DT) to which the DRM should be correctly implemented (see also Ref. [10]).

The NP part of the full gluon propagator is characterized by the presence of severe power-type (or equivalently NP) IR singularities $(q^2)^{-2-k}$, $k = 0, 1, 2, 3, ...$. So these IR singularities are defined as more singular than the power-type IR singularity of the free gluon propagator $(q^2)^{-1}$, which thus can be defined as the PT IR singularity. The INP
part depends only on the transversal degrees of freedom of gauge bosons. Though its coefficients $\Phi_{k,m}(\lambda, \alpha, \xi, g^2)$ may explicitly depend on the gauge-fixing parameter $\xi$, the structure of this expansion itself does not depend on it. It vanishes as the mass gap goes formally to zero, while the PT part survives. The INP part of the full gluon propagator in Eq. (5.14) is nothing but the corresponding Laurent expansion in integer powers of $g^2$ accompanied by the corresponding powers of the mass gap squared and multiplied by the sum over the $q^2$-independent factors, the so-called residues $\Phi_k(\lambda, \alpha, \xi, g^2) = \sum_{m=0}^{\infty} \Phi_{k,m}(\lambda, \alpha, \xi, g^2)$. The sum over $m$ indicates that an infinite number of iterations (all iterations) of the corresponding regularized skeleton loop integrals invokes each severe IR singularity labelled by $k$. It is worth emphasizing that this Laurent expansion cannot be summed up into anything similar to the initial Eq. (3.16), since its residues at poles gain additional contributions due to the splitting/shifting procedure, i.e., they become arbitrary. However, this arbitrariness is not a problem, because severe IR singularities should be treated by the DRM correctly implemented into the DT. For this the dependence of the residues on their arguments is all that matters and not their concrete values. The PT part of the full gluon propagator, which has only the PT IR singularity, remains undetermined. This is the price we have paid to fix the functional dependence of its INP part. In Refs. [10, 16] we came to the same structure (5.14) but in a rather different way. Concluding, it is worth emphasizing that both terms in the general iteration solution (5.13) are valid in the whole energy/momentum range, i.e., they are not asymptotics. At the same time, we achieved the exact separation between the two terms responsible for the NP (dominating in the IR ($q^2 \to 0$)) and the nontrivial PT (dominating in the UV ($q^2 \to \infty$)) dynamics in the true QCD vacuum. This separation is unique as well, since for severe (i.e., NP) IR singularities there exists a special regularization expansion within the DT, complemented by the DRM, while for the PT IR singularity it does not exist [10, 11].

VI. CONCLUSIONS

Our consideration at this stage is necessarily formal, since the mass gap $\Delta^2$ remains neither IR (within the INP solution) nor UV renormalized yet. At this stage it has been only regularized, i.e., $\Delta^2 \equiv \Delta^2(\lambda, \alpha, \xi, g^2)$. However, there is no doubt that it will survive both multiplicative renormalization (MR) programs (which include the corresponding removal of both $\lambda$ and $\alpha$ parameters). For some preliminary aspects of the IRMR program see Refs. [10, 16]. Anyway, how to conduct the UVMR program is not our problem (as mentioned above, it is a standard one, and its description can be found in Refs. [3, 4, 7, 8, 9, 14]). It is worth noting that the mass gap which appears in the gluon SD equation cannot be in principle the same one which appears in the INP part of the general iteration solution, though we have identified them, for simplicity.

It is important to emphasize that a mass gap has not been introduced by hand. It is hidden in the skeleton loop integrals, contributing to the gluon self-energy, and dynamically generated mainly due to the NL interaction of massless gluon modes. No truncations/approximations and no special gauge choice are made for the above-mentioned regularized skeleton loop integrals. An appropriate subtraction scheme has been applied to make the existence of a mass gap perfectly clear. Within the general iteration solution the mass gap shows up when the gluon momentum goes to zero. The Lagrangian of QCD does not contain a mass gap, while it explicitly appears in the corresponding gluon SD equation. This once more underlines the importance of investigation of the SD system of equations and identities [3, 4, 8, 9] for understanding the true structure of the QCD ground state. We have established the general structure of the regularized full gluon propagator (see Eqs. (3.15) and (3.16)), and the corresponding SD equation (3.12) in the presence of a mass gap.

In order to realize a mass gap, we propose not to impose the transversality condition on the gluon self-energy (see Eq. (3.5)), while preserving the color gauge invariance condition (3.14) for the full gluon propagator. This proposal is justified by the NL and NP dynamics of QCD. Such a temporary violation of color gauge invariance/symmetry (TVCGI/S) is completely NP effect, since in the PT limit $\Delta^2 = 0$ this effect vanishes. Let us emphasize that we would propose this even if there were no explicit violation of the transversality of the gluon self-energy by the constant skeleton tadpole term. In other words, whether this term is explicitly present or not, but just color confinement (the gluon is not a physical state) gives us a possibility not to impose the transversality condition on the gluon self-energy. The existence of this term is a hint that the above-mentioned transversality might be temporary violated. Since gluon is not a physical state because of color confinement as mentioned above, the TVCGI/S in QCD has no direct physical consequences. None of physical observables in QCD will be directly affected by this proposal. For their calculations from first principles in low-energy QCD we need the full gluon propagator, which transversality has been sacrificed in order to realize a mass gap (despite their general role the ghosts cannot guarantee its transversality in this case). However, we have already formulated a general method how the transversality of the gluon propagator relevant for NP QCD is to be restored at the final stage. In accordance with our prescription [17] it becomes automatically transversal, free of the PT contributions ("contaminations"), and it regularly depends on the mass gap, so that it vanishes when the mass gap goes to zero.
On the general ground (no truncations/approximations and no special gauge have been made) we have established the existence at least of two different types of solutions for the full gluon propagator in the presence of a mass gap. The so-called general iteration solution (5.14) is always severely singular in the IR \((q^2 \rightarrow 0)\), i.e., the gluons always remain massless, and this does not depend on the gauge choice (this behavior of the full gluon propagator in different approximations and gauges has been earlier obtained and investigated in many papers, see, for example Ref. [10] and references therein). The massive-type solution (4.4) leads to an effective gluon mass, which explicitly depends on the gauge-fixing parameter, and it cannot be directly identified with the mass gap. Moreover, we were unable to make an effective gluon mass a gauge-invariant as a result of the renormalization, and therefore to assign to it a physical meaning. This solution becomes smooth at \(q^2 \rightarrow 0\) in the Landau gauge \(\xi = 0\) only. Both types of solutions are independent from each other and should be considered on equal footing, since the gluon SD equation is highly NL system. For such kind of systems the number of solutions is not fixed \textit{a priori}. The UV behavior \((q^2 \rightarrow \infty)\) of all solutions should be fixed by AF [3]. Due to unsolved yet confinement problem, the IR behavior \((q^2 \rightarrow 0)\) is not fixed. Only solution of the color confinement problem will decide which type of formal solutions really takes place. At the present state of arts none of them can be excluded.

In summary, the behavior of QCD at large distances is governed by a mass gap, possibly realized in accordance with our proposal. The dynamically generated mass gap is usually related to breakdown of some symmetry (for example, the dynamically generated quark mass is an evidence of chiral symmetry breakdown). Here a mass gap is an evidence of the TVCGI/S. Thus there is no breakdown of \(U(1)\) gauge symmetry in QED, since the photon is a physical state. At the same time, a temporary breakdown of \(SU(3)\) color gauge symmetry in QCD is possible, since the gluon is not a physical state (color confinement). In the presence of a mass gap the coupling constant becomes play no role. This is also a direct evidence of the "dimensional transmutation", \(g^2 \rightarrow \Delta^2(\lambda, \alpha, \xi, g^2)\) [3, 18, 19], which occurs whenever a massless theory acquires masses dynamically. It is a general feature of spontaneous symmetry breaking in field theories. The mass gap has to play a crucial role in the realization of the quantum-dynamical mechanism of color confinement [3].

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[1] Confinement, Duality, and Nonperturbative Aspects of QCD, edited by P. van Baal, NATO ASI Series B: Physics, vol. 368 (Plenum, New York, 1997).
[2] Non-Perturbative QCD, Structure of the QCD vacuum, edited by K-I. Aoki, O. Miymura and T. Suzuki, Prog. Theor. Phys. Suppl. 131 (1998) 1.
[3] W. Marciano, H. Pagels, Phys. Rep. C 36 (1978) 137.
[4] M.E. Peskin, D.V. Schroeder, An Introduction to Quantum Field Theory (AW, Advanced Book Program, 1995).
[5] A. Jaffe, E. Witten, Yang-Mills Existence and Mass Gap, http://www.claymath.org/prize-problems/ [http://www.arthurjaffe.com]
[6] V.N. Gribov, J. Nyiri, Quantum Electrodynamics. (Cambridge University Press, 2001).
[7] J.D. Bjorken, S.D. Drell, Relativistic Quantum Fields, (Mc Graw-Hill Book Company, 1978).
[8] C. Itzykson, J.-B. Zuber, Quantum Field Theory, (Mc Graw-Hill Book Company, 1984).
[9] T. Muta, Foundations of QCD, (Word Scientific, 1987).
[10] V. Gogohia, Phys. Lett. B 618 (2005) 103; V. Gogohia, hep-ph/0311061
[11] G. ’t Hooft, M. Veltman, Nucl. Phys. B 44 (1972) 189.
[12] L. von Smekal, A. Hauck, R. Alkofer, Ann. Phys. 267 (1998) 1; R. Alkofer, C.S. Fischer, F.J. Llanes-Estrada, Phys. Lett. B 611 (2005) 279.
[13] D.V. Shirkov, hep-ph/0208082.
[14] M. Baker, Ch. Lee, Phys. Rev. D 15 (1977) 2201.
[15] I.M. Gel’fand, G.E. Shilov, Generalized Functions, (Academic Press, New York, 1968), Vol. I.
[16] V. Gogohia, Phys. Lett. B 584 (2004) 225.
[17] V. Gogokhia, hep-ph/0606010.
[18] S. Coleman, E. Weinberg, Phys. Rev. D 7 (1973) 1888.
[19] D.J. Gross, A. Neveu, Phys. Rev. D 10 (1974) 3235.