COMPLEXES WITH THE DERIVED DOUBLE CENTRALISER PROPERTY

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Abstract. In representation theory, the double centraliser property is an important property for a module (bimodule). It plays a fundamental role in many theories. In this paper, we extend this property to complexes in derived categories of finite dimensional algebras, under the name derived double centraliser property. Characterizations for complexes with the derived double centraliser property and (two-sided) tilting complexes in derived categories of hereditary algebras are given. In particular, all complexes with this property in the derived categories of lower triangular matrix algebras are classified.

1. Introduction

Let \(A\) be an algebra and \(M\) an \(A\)-module with endomorphism algebra \(B\). Then naturally \(M\) is an \(A\)-\(B\)-bimodule and \(M\) is said to have the double centralizer property if left multiplication by elements in \(A\) is an isomorphism \(A \cong \text{End}_{B^{op}}(M)\).

Modules with the double centralizer property occur frequently in algebra. Simple modules over central simple algebras do satisfy the property, and more generally a Morita bimodule satisfies the double centralizer property. Schur-Weyl duality (cf. [9]) is another prominent occurrence. More recently it was shown (cf. [20, Section 4.1]) that a tilting module over a finite dimensional algebra is another case. Only very recently a systematic classification of modules with the double centralizer property was started by Crawley-Boevey, Ma, Rognerud and Sauter [5] in the special case of lower triangular matrix algebras, i.e., hereditary algebras of type \(A\).

Rickard’s Morita theory for derived categories gives another case. Rickard showed [19] that if \(A\) and \(B\) are derived equivalent as algebras over a commutative ring \(R\), both suppose to be projective over \(R\), then there is a complex \(X^*\) in the bounded derived category \(D^b(A \otimes_R B^{op})\) of \(A\)-\(B\)-bimodules such that \(X^* \otimes_B^L \ : \ D^c(B) \to D^c(A)\) is an equivalence of triangulated categories. Such a complex is called a two-sided tilting complex and Rickard showed that \(X^*\) also has the double centralizer property in the sense that right multiplication by elements in \(B\) gives an isomorphism \(B \cong \text{End}_{D^c(A)}(X^*)^{op}\) and left multiplication by elements of \(A\) gives an isomorphism \(A \cong \text{End}_{D^c(B^{op})}(X^*)\). We call this property under the name derived double centralizer property. See Definition 3.1 for a more precise statement. Moreover, we point out that the notation of derived double centraliser property is different from the property of a complex \(A_X^*\) that \(\text{RHom}_{A}(X^*, X^*)^{op} \cong B\) and \(\text{RHom}_{B^{op}}(X^*, X^*) \cong A\) in the sense of Keller [14], Keller showed that such a complex preserves the Hochschild cohomologies of the algebras, however, complexes with the derived double centraliser property in our sense don’t. A typical example we refer to work [7] by Fang and Miyachi. Note that, if \(B\) is hereditary, then two-sided tilting and double centralizer property in the sense of Keller coincide. We characterize two-sided tilting complexes in the same way as complexes with the derived double centralizer property in this situation in our main Theorem 1.1.
The purpose of this paper is to study the derived double centralizer property systematically. Since this seems to be vast, we start with the case when $A$ and $B$ are finite dimensional algebras and in addition $B$ is hereditary. Note that we need to start with a complex of bimodules rather than with a complex in $D^+(A)$ since the endomorphism algebra in the derived category of $A$ of such a complex does not give an action on the complex itself. In some cases, it is possible to lift this to an action on the complex, but Keller showed that this is a highly non trivial task [13].

Our main results make use of left approximations. If $C$ is a Krull-Schmidt category and $D$ is a full subcategory of $C$, then a left $D$-approximation of an object $X$ of $C$ is a morphism $f : X \to D$ such that $D$ is an object of $D$ and such that $\text{Hom}_C(f, D')$ is an epimorphism for any object $D'$ in $D$. The approximation $f$ is left minimal if $gf = f$ for an endomorphism $g$ implies that $g$ is an isomorphism. See Section 2.3 for more ample details.

We denote by $r_{X^*}$ the right multiplication map, roughly speaking, for the precise definition of $r_{X^*}$, we refer to Definition 3.1.

**Theorem 1.1.** (Theorems 4.2 and 4.6) Let $A$ and $B$ be algebras. Let $X^*$ be a bounded complex of $A$-$B$-bimodules. Assume that $B$ is hereditary and $r_{X^*} : B \to \text{End}_{D^b(A)}(X^*)^{op}$ is an isomorphism. Then $X^*$ has the derived double centraliser property (resp. is two-sided tilting) if and only if for any indecomposable projective $A$-module $Ae$,

1. there is a unique $i \in \mathbb{Z}$ such that $\text{Hom}_{D^b(A)}(Ae[i], X^*) \neq 0$;
2. there exists a minimal left add $X^*$-approximation sequence

$$Ae[i] \to X_0^* \xrightarrow{g} X_1^*$$

of $Ae[i]$ such that $Ae \cong H^{-i-1}(\text{cone}(g))$ as $A$-modules (resp. which forms a distinguished triangle in $D^b(A)$).

In [17], Miyachi and Yekutieli studied the derived Picard group of a hereditary algebra $A$ by associating an automorphism of the Auslander-Reiten quiver of the derived category $D^b(A)$ to each two-sided tilting complex of $A$-bimodules. We give an intrinsic characterization of (two-sided) tilting complexes of hereditary algebras by approximation theory. Combining with complexes with the derived double centraliser property, the difference between them are reflected on the approximation sequences of the shifted of indecomposable projective modules.

**Proposition 1.2.** (Propositions 4.13 and 4.14) Let $A$ be a hereditary algebra and $X^*$ a bounded complex of $A$-modules. Assume $\text{End}_{D^b(A)}(X^*)$ is hereditary. Then $X^*$ has the derived double centraliser property (resp. is isomorphic to a tilting complex in $D^b(A)$) if and only if for any indecomposable projective $A$-module $Ae$,

1. there is a unique $i \in \mathbb{Z}$ such that $\text{Hom}_{D^b(A)}(Ae[i], X^*) \neq 0$;
2. there is an exact minimal left add $H^{-i}(X^*)$-approximation sequence

$$Ae \to X_0 \to X_1$$

(resp. $Ae \to X_0 \to X_1 \to 0$)

of $Ae$ such that $\ker f \in \text{add} H^{-i-1}(X^*)$.

Let $A$ be a lower triangular matrix algebra. Crawley-Boevey, Ma, Rognerud and Sauter classified [5] the $A$-modules with the double centraliser property and related them to some combinatorial objects. In the present paper, we classified all bounded complexes of $A$-bimodules with the derived double centraliser property, see Theorem 5.7.

The paper is organized as follows: In Section 2, we give the necessary notation and definitions. In Section 3, the definition of complexes with the derived double centraliser property is provided.
and the related basic properties are also given. In Section 4, we characterize the complexes with the derived double centraliser property and two-sided tilting complexes over hereditary algebras and describe their homologies. In section 5, complexes with the derived double centraliser property over lower triangular matrix algebras are classified.

2. Notation and definitions

Let $K$ be a field. Throughout, all algebras are assumed to be finite dimensional $K$-algebras and all modules are assumed to be finite dimensional left modules, unless stated otherwise. Let $A$ be an algebra. $A^{op}$ denotes the opposite algebra of $A$, hence $A^{op}$-modules represent right $A$-modules. $A^e$ denotes the enveloping algebra $A \otimes_K A^{op}$. Denote by $A$-$mod$ the category of all $A$-modules and by $proj A$ the full subcategory of $A$-$mod$ which contains all projective $A$-modules.

2.1. Complexes and categories. Let $A$ be an algebra. A complex $X^\bullet = (X^i, d^i_X)$ of $A$-modules, we simply write $A$-complex, is a sequence of $A$-modules $X^i$ and $A$-module morphisms $d^i_X : X^i \rightarrow X^{i+1}$ such that $d^{i+1}_X d^i_X = 0$ for all $i \in \mathbb{Z}$. A morphism $f^\bullet : X^\bullet \rightarrow Y^\bullet$ of $A$-complexes $X^\bullet$ and $Y^\bullet$ is a collection of morphisms $f^i : X^i \rightarrow Y^i$ of $A$-modules such that $d^i_Y f^i = f^{i+1} d^i_X$. The complex $X^\bullet$ is called bounded above if there is a number $n$ such that $X^i = 0$ for all $i > n$, bounded below if there is a number $m$ such that $X^i = 0$ for all $i < m$, and bounded if it is both bounded above and bounded below. $X^\bullet$ is called radical if all its differentials are radical morphisms. Assume $X^i = 0$ for all $i < m$ and $i > n$ and $X^m \neq 0 \neq X^n$, then the width of $X^\bullet$ is defined as $n - m + 1$. Note that, in this paper, we will not distinguish an $A$-module $X$ and the $A$-complex $X^\bullet$ which is concentrated in degree 0 and $X^0 = X$.

We denote by $C(A)$ the category of $A$-complexes, by $K(A)$ (resp. $K^b(A)$, $K^+(A)$, $K^-(A)$) the homotopy category of (resp. bounded, bounded above, bounded below) $A$-complexes, and by $D(A)$ (resp. $D^b(A)$, $D^+(A)$, $D^-(A)$) the derived category of (resp. bounded, bounded above, bounded below) $A$-complexes.

2.2. Derived equivalences and tilting complexes. Let $A$ and $B$ be algebras. Recall, due to Rickard [18] and Keller [15], that the following conditions are equivalent:

- $D^b(A)$ and $D^b(B)$ are equivalent as triangulated categories;
- $K^b(proj A)$ and $K^b(proj B)$ are equivalent as triangulated categories;
- There is a complex $T^\bullet \in K^b(proj A)$ such that $\text{End}_{D^b(proj A)}(T^\bullet)^{op} \cong B$, $T^\bullet$ is self-orthogonal (i.e., $\text{Hom}_{D^b(proj A)}(T^\bullet, T^\bullet[i]) = 0$ unless $i = 0$), and add $T^\bullet$ generates $K^b(proj A)$ as triangulated category.

If one of above conditions holds, algebras $A$ and $B$ are said derived equivalent. An $A$-complex satisfies the third condition is called a tilting complex. Later, in [19], he promoted this setting further. A complex $\Delta^\bullet \in D^b(A \otimes_K B^{op})$ is called two-sided tilting if there is a complex $\Theta^\bullet \in D^b(B \otimes_K A^{op})$ such that $\Delta^\bullet \otimes^L_B \Theta^\bullet \cong_{A^e} \Delta^\bullet \otimes^L_A \Theta^\bullet \cong_{B^e} \Delta^\bullet \otimes^L_B \Theta^\bullet$. Then

$\Theta^\bullet \otimes^L_A \Theta^\bullet : D^b(A) \leftrightarrow D^b(B) : \Delta^\bullet \otimes^L_B \Delta^\bullet$

are mutually inverse triangle equivalences. Moreover, it is shown that for any tilting $A$-complex $T^\bullet$ with $\text{End}_{D^b(A)}(T^\bullet)^{op} \cong B$, there is a two-sided tilting $A$-$B$-complex $X^\bullet$ such that $X^\bullet \cong T^\bullet$ in $D^b(A)$. Combining with the result in [6], a two-sided tilting $A$-$B$-complexes with homology concentrated in degree 0 is isomorphic to an $A$-$B$-bimodule $T$ such that $A^e T$ is tilting and $B \cong \text{End}_A(T)^{op}$ canonically. Such an $A$-$B$-bimodule $T$ we simply called tilting. For the knowledge of tilting module, we refer to the book [10].
2.3. Approximations. Let \( \mathcal{C} \) be a Krull-Schmidt category. Let \( X \) be an object in \( \mathcal{C} \). We say \( X \) is basic if it is isomorphic to a direct sum of pairwise non-isomorphic indecomposable objects in \( \mathcal{C} \). Denote by \( \text{add} X \) the full subcategory of \( \mathcal{C} \) which contains all direct summands of direct sums of copies of \( X \). Let \( \mathcal{D} \) be a full subcategory of \( \mathcal{C} \). Recall that a sequence

\[
\xi : X \xrightarrow{f} D_0 \to D_1 \to \cdots \to D_n
\]

of morphisms in \( \mathcal{C} \) is called a left \( \mathcal{D} \)-approximation sequence of \( X \) if \( D_i \in \mathcal{D}, \)

\[
\text{Hom}_\mathcal{C}(\xi, D') : \text{Hom}_\mathcal{C}(D_n, D') \to \cdots \to \text{Hom}_\mathcal{C}(D_1, D') \to \text{Hom}_\mathcal{C}(D_0, D') \to \text{Hom}_\mathcal{C}(X, D')
\]

is exact and \( \text{Hom}_\mathcal{C}(f, D') \) is an epimorphism, for any \( D' \in \mathcal{D} \). A morphism \( f : X \to Y \) in \( \mathcal{C} \) is called left minimal if any morphism \( g \in \text{End}_\mathcal{C}(Y) \) with \( gf = f \) is an isomorphism. The sequence \( \xi \) is called a minimal left \( \mathcal{D} \)-approximation sequence if it is a left \( \mathcal{D} \)-approximation sequence and every morphism in \( \xi \) is left minimal. If \( n = 0 \), (minimal) left \( \mathcal{D} \)-approximation sequence is abbreviated to (minimal) left \( \mathcal{D} \)-approximation. Dually, there is the notion of right minimal and (minimal) right \( \mathcal{D} \)-approximation (sequence).

3. Definition and basic properties of complexes with the derived double centraliser property

In this section, we will provide the definition of complex of bimodules with the derived double centraliser property and then some related basic results are given.

3.1. Definition. Let \( A \) and \( B \) be two algebras. For an \( A-B \)-bimodule \( X \), the natural left multiplication algebra homomorphism and right multiplication algebra homomorphism defined respectively as:

\[
l_X : A \to \text{End}_{B^\text{op}}(X), \quad a \mapsto (x \mapsto ax), \quad r_X : B \to \text{End}_A(X)^{\text{op}}, \quad b \mapsto (x \mapsto xb).
\]

We say that \( X \) has the double centraliser property if both \( l_X \) and \( r_X \) are isomorphisms. Moreover, we say that \( X \) has the double centraliser property if the bimodule \( A^\text{X}_{\text{End}_A(X)} \) has the double centraliser property. Note that \( l_X \) is an \( A \)-bimodule isomorphism and \( r_X \) is a \( B \)-bimodule isomorphism.

Let \( X^\bullet \) be an \( A-B \)-complex. Then there are natural maps

\[
l_X^\bullet : A \to \text{End}_{C(B^\text{op})}(X^\bullet), \quad a \mapsto (l_X(a))_i, \quad r_X^\bullet : B \to \text{End}_{C(A)}(X^\bullet)^{\text{op}}, \quad b \mapsto (r_X(b))_i.
\]

Note that the differentials of \( X^\bullet \) are \( A-B \)-bimodule morphisms, so \( (l_X(a))_i \in \text{End}_{C(B^\text{op})}(X^\bullet) \) and \( (r_X(b))_i \in \text{End}_{C(A)}(X^\bullet)^{\text{op}} \). It is easy to check that \( l_X^\bullet \) and \( r_X^\bullet \) are algebra homomorphisms. Now let

\[
l_X^\bullet : A \xrightarrow{l_X^\bullet} \text{End}_{C(B^\text{op})}(X^\bullet) \to \text{End}_{K(B^\text{op})}(X^\bullet) \to \text{End}_{D(B^\text{op})}(X^\bullet),
\]

\[
r_X^\bullet : B \xrightarrow{r_X^\bullet} \text{End}_{C(A)}(X^\bullet)^{\text{op}} \to \text{End}_{K(A)}(X^\bullet)^{\text{op}} \to \text{End}_{D(A)}(X^\bullet)^{\text{op}}
\]

where the last two maps of \( l_X^\bullet \) and \( r_X^\bullet \) are the natural homotopic algebra quotients and the localisation algebra homomorphisms.

**Definition 3.1.** Let \( A \) and \( B \) be algebras. For an \( A-B \)-complex \( X^\bullet \), we say that \( X^\bullet \) has the derived double centraliser property if both \( l_X^\bullet \) and \( r_X^\bullet \) are isomorphisms. Moreover, for an \( A \)-complex \( X^\bullet \), we say that \( X^\bullet \) has the derived double centraliser property if there is an \( A-B \)-complex \( Y^\bullet \) for some algebra \( B \) such that \( X^\bullet \cong Y^\bullet \) in \( D(A) \) and \( Y^\bullet \) as \( A-B \)-complex has the derived double centraliser property.
Let $A$ and $B$ be algebras. Let $X^*$ and $Y^*$ be $A$-$B$-complexes. Suppose $s : Y^* \to X^*$ is a quasi-isomorphism of $A$-$B$-complexes, then there is a map

$$S : \text{End}_{D(A)}(X^*)^{op} \to \text{End}_{D(A)}(Y^*)^{op}, \frac{a}{b} \mapsto \frac{as}{bs}$$

where for any $a \in \text{End}_{C(A)}(X^*)$, $\overline{a}$ represents the corresponding element of $a$ in $K(A)$. Note that we used left roofs to represent morphisms in derived categories. It is easy to check that $S$ is an algebra isomorphism.

Next, we show that $r_{Y^*} = r_{X^*}S$. Recall that $r_{X^*}(b) = \overline{r_{X^*}(b)} = 1$ for $b \in B$. So we only need to prove that $\overline{r_{Y^*}(b)s} = \overline{r_{X^*}(b)s}$ in $\text{End}_{D(A)}(Y^*)^{op}$. This is due to the following commutative diagram:

\[
\begin{array}{ccc}
Y^* & \xrightarrow{1} & Y^* \\
\downarrow & & \downarrow \\
X^* & \xrightarrow{r_{X^*}S} & Y^* \\
\uparrow & & \uparrow \\
X^* & \xrightarrow{r_{X^*}} & Y^* \\
\end{array}
\]

where $\overline{r_{Y^*}(b)s} = S\overline{r_{X^*}(b)}$ because $s$ is a morphism of $B^{op}$-complexes.

So in this case, $r_{X^*}$ is an isomorphism if and only if $r_{Y^*}$ is an isomorphism. Similarly, $l_{X^*}$ is an isomorphism if and only if $l_{Y^*}$ is an isomorphism. Conclusively,

**Lemma 3.2.** Let $A$ and $B$ be two algebras. Let $X^*$ and $Y^*$ be two $A$-$B$-complexes. If $X^*$ and $Y^*$ are quasi-isomorphic as $A$-$B$-complexes, assume $S : \text{End}_{D(A)}(X^*)^{op} \to \text{End}_{D(A)}(Y^*)^{op}$ is the isomorphism induced by the quasi-isomorphism, then $r_{Y^*} = r_{X^*}S$. In this case, $X^*$ has the derived double centraliser property if and only if so does $Y^*$.

**Remark 3.3.**

(1) Using Lemma 3.2, a module or a bimodule has the double centraliser property if and only if it has the derived double centraliser property.

(2) It is shown by Rickard in [19] that two-sided tilting complexes have the derived double centraliser property.

(3) Yekutieli showed [22] dualizing complexes have the derived double centraliser property.

(4) More generally, as mention before, $A$-$B$-complexes $X^*$ which satisfy $\text{RHom}_A(X^*, X^*)^{op} \cong B$ and $\text{RHom}_{B^{op}}(X^*, X^*) \cong A$ have the double centraliser property in the sense of Keller [14].

### 3.2. Action on complexes.

Let $A$, $B$ and $C$ be algebras, $X^*$ an $A$-$B$-complex, and $Y^*$ an $A$-$C$-complex. Then $\text{Hom}_{D(A)}(X^*, Y^*)$ has a $B$-$C$-bimodule structure. For $f \in \text{Hom}_{D(A)}(X^*, Y^*)$, $a \in A$ and $b \in B$, the left $B$-module structure is given by $bf = fr_{X^*}(b)$ and the right $C$-module structure is given by $fc = r_{Y^*}(c)f$. In this subsection, we collect some results around this topic.

**Lemma 3.4.** Let $A$ and $B$ be two algebras and $X^*$ an $A$-$B$-bimodule complex. The following conditions are equivalent:

1. $r_{X^*}$ is an isomorphism,
2. there is a $B$-module isomorphism $B \to \text{End}_{D(A)}(X^*)$,
3. there is a $B^{op}$-module isomorphism $B \to \text{End}_{D(A)}(X^*)$.

**Proof.** The proof is similar to the bimodule version in [23, Lemma 2.3]. For any $b, b' \in B$, $r_{X^*}(bb') = r_{X^*}(b')r_{X^*}(b) = br_{X^*}(b')$. Then $r_{X^*}$ is a $B$-module morphism. Similarly, $r_{X^*}$ is also a $B^{op}$-module morphism. Hence if $r_{X^*}$ is an isomorphism, then it is a $B$-module isomorphism and is also a $B^{op}$-module isomorphism. Suppose there is a $B$-module isomorphism $F : B \to \text{End}_{D(A)}(X^*)$. 


Lemma 3.5. Let $A$ and $B$ be algebras and $X^\bullet$ a bounded $A$-$B$-complex. Let $b \in B$ and $f : B \to Bb$ the canonical epimorphism. Then $X^\bullet \otimes^L_B Bb = 0$ in $D^b(A)$ if and only if $X^\bullet \otimes^L_B f = 0$ in $D^b(A)$. Moreover, in this case, $r_{X^\bullet}(b) = 0$.

Proof. If $X^\bullet \otimes^L_B Bb = 0$ in $D^b(A)$, then, obviously, $X^\bullet \otimes^L_B f : X^\bullet \to X^\bullet \otimes^L_B Bb$ is zero morphism in $D^b(A)$. Conversely, suppose that $X^\bullet \otimes^L_B f : X^\bullet \to X^\bullet \otimes^L_B Bb$ is zero morphism in $D^b(A)$. Let $Y^\bullet$ be a projective resolution of $A$-$B$-complex $X^\bullet$. Then $X^\bullet \otimes^L_B f = 0$ in $D^b(A)$ implies $Y^\bullet \otimes^L_B f = 0$ in $K^-(A)$. So there is a homotopy $F : Y^\bullet \to Y^\bullet \otimes^L_B Bb[−1]$ for $Y^\bullet \otimes^L_B f$. Assume $Y^i = 0$ for $i > k$ and $Y^k \neq 0$. Then $Y^k \otimes B = d^{k-1} f^i$, where $d$ is the differential of $Y^\bullet \otimes B b$. Note that $Y^k \otimes B f$ is surjective, then so is $d^{k-1}$. Since $X^\bullet \otimes_B Bb$ is a complex of projective $A$-modules, $d^{k-1}$ is split as a morphism of $A$-modules. Iteratively, we have $Y^\bullet \otimes_B Bb = 0$ in $K^-(A)$, then $X^\bullet \otimes_B Bb = 0$ in $D^b(A)$.

Let $h : B \to Bb \to B$ with $g$ the canonical embedding. To prove $r_{X^\bullet}(b) = 0$, we only need to show that $r_{X^\bullet}(b) = 0$ if and only if $X^\bullet \otimes_B h = 0$, since then $X^\bullet \otimes_B h = X^\bullet \otimes_B g f = (X^\bullet \otimes_B g)(X^\bullet \otimes_B f) = 0$ in $D^b(A)$, so $r_{X^\bullet}(b) = 0$. Indeed, $X^\bullet$ is quasi-isomorphic to $X^\bullet \otimes B Bb$ as $A$-$B$-complexes, by Lemma 3.2, we have $r_{X^\bullet}(b) = 0$ if and only if $r_{X^\bullet \otimes_B Bb}(b) = 0$. We point out that $r_{X^\bullet \otimes_B Bb}(b) = X^\bullet \otimes^L_B h$.

Example 3.6. Let $A$ and $B$ be algebras, and $X^\bullet$ a bounded $A$-$B$-complex. Note that, for $b \in B$, in general, $r_{X^\bullet}(b) = 0$ could not imply $X^\bullet \otimes_B Bb = 0$ in $D^b(A)$.

Set $A = B = K(\langle x \rangle)/(x^2)$ and $X^\bullet$ the $A^e$-complex concentrated in degrees 0 and 1 as $A^e \to A^e$, where $r$ is given by $r(a) = ax$ for $a \in A^e$. Then we can verify that $r_{X^\bullet}(x)$ is homotopic to zero, hence $r_{X^\bullet}(x) = 0$. However, $X^\bullet \otimes_B Ax \cong A' x \otimes A' x[−1]$ in $D^b(A)$ which is not the zero object (note that a two-term complex is homotopic to zero if and only if the connected morphism of the two terms is an isomorphism).

Lemma 3.7. Let $A$ and $B$ be two algebras and $X^\bullet$ a bounded $A$-$B$-complex. Assume $r_{X^\bullet}$ is an isomorphism and $\{e_1, e_2, \cdots, e_n\}$ is a complete set of primitive orthogonal idempotents of $B$, then $\oplus_{i=1}^n (X^\bullet \otimes_B Be_i)$ is a decomposition of $X^\bullet$ into indecomposable objects in $D^b(A)$.

Proof. Note that $X^\bullet \cong X^\bullet \otimes_B^L B \cong X^\bullet \otimes_B^L \bigoplus_{i=1}^n Be_i \cong \bigoplus_{i=1}^n (X^\bullet \otimes_B^L Be_i)$ in $D^b(A)$. Since $r_{X^\bullet}$ is an isomorphism, the number of indecomposable direct summands of $X^\bullet$ in $D^b(A)$ is equal to the cardinal of a complete set of primitive orthogonal idempotents of $B$. So we only need to prove that, for each $e_i$, $X^\bullet \otimes_B^L Be_i \neq 0$ in $D^b(A)$. But this directly follows from Lemma 3.5.

Lemma 3.8. Let $A$ and $B$ be algebras. Let $X^\bullet$ be a bounded $A$-$B$-complex. Assume that $r_{X^\bullet}$ is an isomorphism. A sequence $\xi : Y^\bullet \to X^\bullet \to X^\bullet$ in $D^b(A)$ is a minimal left add $X^\bullet$-approximation sequence of $Y^\bullet$ if and only if

$$\text{Hom}_{D^b(A)}(\xi, X^\bullet) : \text{Hom}_{D^b(A)}(X^\bullet, X^\bullet) \to \text{Hom}_{D^b(A)}(X^\bullet, X^\bullet) \to \text{Hom}_{D^b(A)}(Y^\bullet, X^\bullet)$$

is a minimal projective presentation of $B^\bullet$ module $\text{Hom}_{D^b(A)}(Y^\bullet, X^\bullet)$.

Proof. We abbreviate $(-)^* = \text{Hom}_{D^b(A)}(-, X^\bullet)$. By definition, we may assume $\xi : Y^\bullet \to X^\bullet \to X^\bullet$ is a left add $X^\bullet$-approximation sequence of $Y^\bullet$. $\text{Hom}_{D^b(A)}(\xi, X^\bullet)$ is exact and $f^*$ is an epimorphism.

Then we only need to prove that $f$ (resp. $h$) is left minimal if and only if $f^*$ (resp. $h^*$) is right...
minimal. By Lemma 3.4, \( r_{X^*} : B \to \text{End}_{D^b(A)}(X^*) \) is a \( B^{op} \)-module isomorphism, then there are mutually inverse dualities:

\[
\text{Hom}_{D^b(A)}(-, X^*) : \text{add } X^* \leftrightarrow \text{proj } B^{op} : X^* \otimes_B^L \text{Hom}_{B^{op}}(-, B).
\]

So \( h \) is left minimal if and only if \( h^* \) is right minimal. Suppose that \( f^* \) is right minimal. For any \( g \in \text{End}_{D^b(A)}(X_0^*) \) with \( gf = f^* \), then \( f^* g^* = f^* \), so \( g^* \) is an isomorphism. Since \((-)^* : \text{add } X^* \to \text{proj } B^{op} \) is a duality, \( g \) is an isomorphism. Conversely, suppose that \( f \) is left minimal. For any \( g' \in \text{End}_{D^b(B)}((X_0^*)^*) \) with \( f^* g' = f^* \). Since \( \text{End}_{D^b(A)}(X_0^*) \cong \text{End}_{B^{op}}((X_0^*)^*) \) canonically, there is \( g \in \text{End}_{D^b(A)}(X_0^*) \) such that \( g' = g^* \). Then we have \( (gf)^* = f^* \). Note that \( r_{X^*} \) is an isomorphism, then \( \text{Hom}_{D^b(A)}(Y^*, X_0^*) \cong \text{Hom}_{B^{op}}((X_0^*)^*, (Y^*)^*) \) canonically. We have \( gf = f \), hence \( g \) is an isomorphism, then so is \( g^* \).

\( \square \)

3.3. Several properties related to the derived double centraliser property.

**Lemma 3.9.** [16, Lemma 1.1] Let \( A \) and \( B \) be two algebras. Let \( X \) and \( Y \) be two \( A \)-\( B \)-bimodules. If \( X \cong Y \) as \( A \)-modules and \( r_X \) and \( r_Y \) are isomorphisms, then there is \( \beta \in \text{Aut}(B) \) such that \( X \cong Y_\beta \) as \( A \)-\( B \)-bimodules.

**Proposition 3.10.** Let \( A \) and \( B \) be two algebras and \( X^* \) an \( A \)-\( B \)-complex. If \( X^* \) is isomorphic to a tilting complex in \( D^b(A) \) and \( r_{X^*} \) is an isomorphism, then \( X^* \) is two-sided tilting.

**Proof.** Since \( X^* \) is isomorphic to a two-sided tilting complex in \( D^b(A) \) and \( r_{X^*} : B \to \text{End}_{D^b(A)}(X^*)^{op} \) is an isomorphism, there is a two-sided tilting \( A \)-\( B \)-complex \( \overline{X^*} \) such that \( \overline{X^*} \cong X^* \) in \( D^b(A) \), and there is a two-sided tilting \( B \)-\( A \)-complex \( Y^* \) such that \( \overline{X^*} \otimes_B^L Y^* \cong X^* \) in \( D^b(A) \). Then we have \( \overline{X^*} \otimes_A^L X^* \cong B \) in \( D^b(B) \), hence \( H^0(\overline{X^*} \otimes_A^L X^*) = B \) as \( B \)-modules. Note that \( H^0(\overline{X^*} \otimes_A^L X^*) \) is a \( B \)-module. Then we have the following \( B \)-module isomorphisms:

\[
B \xrightarrow{r_{X^*}} \text{End}_{D^b(A)}(X^*) \cong \text{Hom}_{D^b(A)}(X^*, \overline{X^*} \otimes_B^L Y^* \otimes_A^L X^*) \cong \text{Hom}_{D^b(A)}(X^*, \text{RHom}_B(Y^*, Y^* \otimes_A^L X^*)) \cong \text{End}_{B^{op}}(Y^* \otimes_A^L X^*) \cong \text{End}_{D^b(B)}(Y^* \otimes_A^L X^*) \cong \text{End}_{D^b(B)}(H^0(\overline{Y^*} \otimes_A^L X^*)).
\]

By Lemma 3.4 or [23, Lemma 2.3], \( r_{r_{X^*}(Y^* \otimes_A^L X^*)} \) is an isomorphism. Note that \( B_B \) has the double centraliser property, by Lemma 3.9, there is \( \sigma \in \text{Aut}(B) \) such that \( H^0(\overline{Y^*} \otimes_A^L X^*) \cong B_\sigma \) as \( B \)\(-\)bimodules. Then \( Y^* \otimes_A^L X^* \cong B_\sigma \) in \( D^b(B) \). Hence \( X^* \cong \overline{X^*} \otimes_B^L Y^* \otimes_A^L X^* \cong \overline{X^*} \otimes_B^L B_\sigma \) in \( D^b(A \otimes_k B^{op}) \). Note that both \( \overline{X^*} \) and \( B_\sigma \) are two-sided tilting complexes, by [19, Proposition 4.1], \( X^* \) is a two-sided tilting \( A \)-\( B \)-complex.

\( \square \)

For algebras \( A \) and \( B \), let \( \lambda(A, B) \) be the number of all bounded \( A \)-\( B \)-complexes, up to isomorphism and shift in \( D^b(A \otimes_k B^{op}) \), which have the derived double centraliser property.

**Proposition 3.11.** Let \( A \) and \( B \) be two algebras and \( X^* \) a bounded \( A \)-\( B \)-complex. Let \( \sigma \in \text{Aut}(B) \).

1. \( X^* \) has the derived double centraliser property if and only if so does \( X^* \otimes_B^L B_\sigma \).
2. Assume \( r_{X^*} \) is an isomorphism. Then \( X^* \otimes_B^L B_\sigma \cong X^* \) in \( D^b(A \otimes_k B^{op}) \) if and only if \( \sigma \) is inner.

So both \( |\text{Out}(A)| \) and \( |\text{Out}(B)| \) divide \( \lambda(A, B) \) if \( \lambda(A, B) \) is finite.

**Proof.** (1). Since \( X^* \otimes_B^L B_\sigma \otimes_B^L B_{\sigma^{-1}} \cong X^* \) in \( D^b(A \otimes_k B^{op}) \), we just prove the necessity. Assume \( X^* \) has the derived double centraliser property. Note that \( \sigma^{-1} : B_\sigma \to \sigma^{-1}B \) and \( \sigma : B \to \sigma B_\sigma \) are
B-bimodule isomorphisms. Then we have the following A-bimodule isomorphisms:
\[
\text{End}_{D^b(B^{op})}(X^* \otimes_B^L B_{\sigma}) \cong H^0 \mathbb{R}\text{Hom}_{B^{op}}(X^* \otimes_B^L B_{\sigma}, X^* \otimes_B^L B_{\sigma}) \\
\cong H^0 \mathbb{R}\text{Hom}_{B^{op}}(X^*, \mathbb{R}\text{Hom}_{B^{op}}(B_{\sigma}, X^* \otimes_B^L B_{\sigma})) \\
\cong H^0 \mathbb{R}\text{Hom}_{B^{op}}(X^*, \mathbb{R}\text{Hom}_{B^{op}}(\sigma^{-1} B, X^* \otimes_B^L B_{\sigma})) \\
\cong H^0 \mathbb{R}\text{Hom}_{B^{op}}(X^*, X^*) \cong A.
\]
and B-bimodule isomorphisms: \( \text{End}_{D^b(A)}(X^* \otimes_B^L B_{\sigma}) \cong \text{End}_{D^b(B)}(B_{\sigma}) \cong \sigma B_{\sigma} \cong B. \) By Lemma 3.4 and its dual, \( l_{X^* \otimes^r_B B_{\sigma}} \) and \( r_{X^* \otimes^l_B B_{\sigma}} \) are isomorphisms, hence \( X^* \otimes^l_B B_{\sigma} \) has the derived double centraliser property.

(2). Let \( P(X^*) \) be a projective resolution of \( A\text{-}\text{complex} X^* \). Since \( X^* \otimes^l_B B_{\sigma} \cong X^* \) in \( D^b(A \otimes_K B^{op}) \) if only if \( P(X^*) \otimes_B^L B_{\sigma} \cong P(X^*) \) in \( K^-\bigl(A \otimes_K B^{op}\bigr) \), we only need to prove that \( P(X^*) \otimes_B^L B_{\sigma} \cong P(X^*) \) in \( K^-\bigl(A \otimes_K B^{op}\bigr) \) if and only if \( \sigma \) is inner. 

Suppose \( \sigma \in \text{Aut}(B) \) is inner. Assume for any \( b \in B \), \( \sigma(b) = z^{-1}bz \) for an invertible element \( z \in B \). Then \( r'_{X^*}(z) : P(X^*) \to P(X^*) \otimes^l_B B_{\sigma} \) is an isomorphism in \( C^-\bigl(A \otimes_K B^{op}\bigr) \), so \( P(X^*) \otimes_B^L B_{\sigma} \cong P(X^*) \) in \( K^-\bigl(A \otimes_K B^{op}\bigr) \).

Coversely, suppose that \( P(X^*) \otimes_B^L B_{\sigma} \cong P(X^*) \) in \( K^-\bigl(A \otimes_K B^{op}\bigr) \). By [11, pp. 112 (a)], there is a radical \( A\text{-}\text{complex} Y^* \) such that \( P(X^*) \cong Y^* \) in \( K^-\bigl(A \otimes_K B^{op}\bigr) \). Note that \( Y^* \) is also a complex of projective \( A \otimes_K B^{op}\)-modules. Then \( Y^* \otimes_B^L B_{\sigma} \cong Y^* \) in \( K^-\bigl(A \otimes_K B^{op}\bigr) \). It is easy to check that \( Y^* \otimes^l_B B_{\sigma} \) is also radical. By [11, pp. 113 (b)], \( Y^* \otimes_B^L B_{\sigma} \cong Y^* \) in \( C^-\bigl(A \otimes_K B^{op}\bigr) \).

Let \( F' : Y^* \to Y^* \otimes^l_B B_{\sigma} \) be an isomorphism in \( C^-\bigl(A \otimes_K B^{op}\bigr) \) and \( F \) be the corresponding isomorphism of \( F' \) in \( D^-\bigl(A \otimes_K B^{op}\bigr) \). Note that \( X^* \) is quasi-isomorphic to \( Y^* \), by Lemma 3.2, \( r_{X^*} \) is an isomorphism if and only if so is \( r_{Y^*} \). We identify \( Y^* \otimes_B^L B_{\sigma} \) with \( Y^* \) in \( D^-\bigl(A\bigr) \). Then \( F = r_{Y^*}(z) \in \text{End}_{D^-\bigl(A\bigr)}(Y^*) \) for some \( z \in B \). As \( F \) is an isomorphism in \( \text{End}_{D^-\bigl(A\bigr)}(Y^*) \), \( z \) is invertible in \( B \). Since \( F' \) is a morphism of \( B^{op}\)-complexes, then for any \( b \in B \), \( F'F_{Y^*}(b) = r'_{Y^* \otimes^l_B B_{\sigma}}(b)F' \). Hence \( F_{r_{Y^*}}(b) = r_{Y^* \otimes^l_B B_{\sigma}}(b)F \). So \( r_{Y^*}(z)r_{Y^*}(b) = r_{Y^* \otimes^l_B B_{\sigma}}(b)r_{Y^*}(z) = r_{Y^*}(\sigma(b))r_{Y^*}(z) \). As \( r_{Y^*} \) is an algebra isomorphism, then \( \sigma(b) = zbz^{-1} \). We are done. \( \square \)

We record the following two results which will be used in the sequel.

**Proposition 3.12.** [21, Proposition 2.3] Let \( A \) and \( B \) be two algebras. Let \( X^* \) and \( Y^* \) be two-sided tilting \( A\text{-}\text{complexes} \). Then \( X^* \cong Y^* \) in \( D^b(B^{op}) \) if and only if there is \( \alpha \in \text{Aut}(A) \) such that \( X^* \cong \alpha A \otimes^L_A Y^* \) in \( D^b(A \otimes_K B^{op}) \).

**Proposition 3.13.** [4, Proposition 2.1] Let \( A \) be an algebra and \( M \) an \( A\text{-}\text{module} \). Then \( M \) has the double centraliser property if and only if there exists an exact sequence

\[ 0 \to A \xrightarrow{f} M_0 \to M_1 \]
such that \( M_0, M_1 \in \text{add} M \) and \( f : A \to M_0 \) is a left ad\( M\)-approximation.

4. For hereditary algebras

In this section, explicit characterizations for complexes with the double centraliser property and two-sided tilting complexes over hereditary algebras are given. Hereditary algebras is a well-studied class of algebras with a lot of applications. The structure theory appears in categorifications of many mathematical objects such as cluster algebras and Lie algebras.

Let \( A \) be a hereditary algebra. We will frequently use one of the following fundamental equivalent characterizations for hereditary algebras: (1) \( \text{gl}(A) \leq 1 \), where \( \text{gl}(A) \) denotes the global
dimension of $A$; (2) Any submodule of a projective $A$-module is still projective. Moreover, it is
known that indecomposable objects in $D^b(A)$ is isomorphic to those of the form $X[i]$, for some
indecomposable $A$-module $X$ and $i \in \mathbb{Z}$. Given another $A$-module $Y$ and $j \in \mathbb{Z}$, $\text{Hom}_{D^b(A)}(X[i], Y[j])$
is isomorphic to $\text{Hom}_A(X, Y)$ if $j = i$, to $\text{Ext}^1_A(X, Y)$ if $j = i + 1$, or to 0 otherwise. Meanwhile, for
a bounded $A$-complex $X^*$, $X^* \cong \oplus_{i \in \mathbb{Z}} H^i(X^*)[-i]$ in $D^b(A)$, see [10].

4.1. Characterizations I. In this subsection, we will characterize $A$-$B$-complexes with the
derived double centraliser property and two-sided tilting $A$-$B$-complexes where $A$ and $B$ are algebras
and $B$ is hereditary.

Lemma 4.1. Let $A$ and $B$ be algebras. Let $X^*$ be a bounded $A$-$B$-complex. Assume that $B$
is hereditary and $r_{X^*} : B \to \text{End}_{D^b(A)}(X^*)^{op}$ is an isomorphism. Then for any bounded $A$-complex
$Y^*$, there is a minimal left add $X^*$-approximation sequence

$$Y^* \to X^*_0 \xrightarrow{g} X^*_1$$

of $Y^*$ such that $\text{cone}(g) \cong \text{RHom}_{B^{op}}(\text{Hom}_{D^b(A)}(Y^*, X^*), X^*)[1]$ in $D^b(A)$.

Proof. Since $r_{X^*} : B \to \text{End}_{D^b(A)}(X^*)$ is a $B^{op}$-module isomorphism by Lemma 3.4 and $B$ is hereditary, let

$$(\dagger) \quad 0 \to \text{Hom}_{D^b(A)}(X^*_0, X^*) \to \text{Hom}_{D^b(A)}(X^*_0, X^*) \to \text{Hom}_{D^b(A)}(Y^*, X^*) \to 0$$

be a minimal projective presentation of the $B^{op}$-module $\text{Hom}_{D^b(A)}(Y^*, X^*)$ with $X^*_0, X^*_1 \in \text{add} X^*$.

By Lemma 3.8, we have a minimal left add $X^*$-approximation sequence $Y^* \to X^*_0 \xrightarrow{g} X^*_1$ of $Y^*$
in $D^b(A)$. Applying $\text{RHom}_B(\_ , X^*)$ to the distinguished triangle in $D^b(B^{op})$ induced by the exact
sequence $(\dagger)$, we get a distinguished triangle

$$\text{RHom}_{B^{op}}(\text{Hom}_{D^b(A)}(Y^*, X^*), X^*) \to \text{RHom}_{B^{op}}(\text{Hom}_{D^b(A)}(X^*_0, X^*), X^*) \to \text{RHom}_{B^{op}}(\text{Hom}_{D^b(A)}(Y^*, X^*), X^*)[1]$$

in $D^b(A)$. Since $r_{X^*}$ is an isomorphism, the endofunctor $\text{RHom}_B(\text{Hom}_{D^b(A)}(\_ , X^*), X^*)$ on add $X^*$
is an identity. Then we get the following distinguished triangle

$$\text{RHom}_{B^{op}}(\text{Hom}_{D^b(A)}(Y^*, X^*), X^*) \to X^*_0 \xrightarrow{g} X^*_1 \to \text{RHom}_{B^{op}}(\text{Hom}_{D^b(A)}(Y^*, X^*), X^*)[1].$$

Hence $\text{cone}(g) \cong \text{RHom}_{B^{op}}(\text{Hom}_{D^b(A)}(Y^*, X^*), X^*)[1]$ in $D^b(A)$. \hfill \Box

Theorem 4.2. Let $A$ and $B$ be algebras. Let $X^*$ be a bounded $A$-$B$-complex. Assume that $B$
is hereditary and $r_{X^*} : B \to \text{End}_{D^b(A)}(X^*)^{op}$ is an isomorphism. Then $X^*$ has the derived double
centraliser property if and only if for any indecomposable projective $A$-module $Ae$,

1. there is a unique $i \in \mathbb{Z}$ such that $\text{Hom}_{D^b(A)}(Ae[i], X^*) \neq 0$;
2. there exists a minimal left add $X^*$-approximation sequence

$$Ae[i] \to X^*_0 \xrightarrow{g} X^*_1$$

of $Ae[i]$ in $D^b(A)$ such that $Ae \cong H^{-(i+1)}(\text{cone}(g))$ as $A$-modules.

Proof. Suppose that $X^*$ has the derived double centraliser property. Then for any indecomposable
$A$-module $Ae$ with $e$ an idempotent of $A$, by Lemma 3.7, $eA \otimes^L_A X^*$ is an indecomposable direct
summand of $X^*$ in $D^b(B^{op})$. Since $B$ is hereditary, there exists a unique $i \in \mathbb{Z}$ such that $eA \otimes^L_A X^* \cong H^{-i}(eA \otimes^L_A X^*)[i]$ in $D^b(B^{op})$. Then $\text{Hom}_{D^b(A)}(Ae[i], X^*) \cong H^{-i}\text{RHom}_A(Ae, X^*) \cong H^{-i}(eA \otimes^L_A X^*) \cong$
eA \otimes_{A} X[-i] \neq 0 \text{ in } D^{b}(B^{op}). \text{ Consider the } A\text{-complex } Ae[i], \text{ by Lemma 4.1, there is a minimal left add } X^{*}\text{-approximation sequence}
\begin{align*}
Ae[i] & \rightarrow X_{0}^{*} \xrightarrow{g} X_{1}^{*}
\end{align*}
of Ae[i] such that cone(g) \cong \text{RHom}_{B^{op}}(\text{Hom}_{D^{b}(A)}(Ae[i], X^{*}), X^{*})[1] \text{ in } D^{b}(A). \text{ Then}
\begin{align*}
H^{-i+1}(\text{cone}(g)) & \cong H^{-i}(\text{RHom}_{B^{op}}(\text{Hom}_{D^{b}(A)}(Ae[i], X^{*}), X^{*})) \cong H^{-i}(\text{RHom}_{B^{op}}(eA \otimes_{A} X^{*}[-i], X^{*})) \\
& \cong H^{0}(\text{RHom}_{A^{op}}(eA, \text{RHom}_{B^{op}}(X^{*}, X^{*}))) \cong H^{0}(\text{RHom}_{B^{op}}(X^{*}, X^{*}) \otimes_{A} Ae) \\
& \cong H^{0}(\text{RHom}_{B^{op}}(X^{*}, X^{*})) \otimes_{A} Ae \cong \text{Hom}_{D^{b}(B^{op})}(X^{*}, X^{*}) \otimes_{A} Ae \cong Ae
\end{align*}
as A\text{-modules.}

Conversely, let A \cong \oplus_{j} Ae_{j} be a decomposition of A into indecomposables Ae_{j} with e_{j} idempotents of A. Assume for any indecomposable projective A-module Ae_{j}, there is a unique n(j) \in \mathbb{Z} \text{ such that } \text{Hom}_{D^{b}(A)}(Ae[n(j)], X^{*}) \neq 0. \text{ Then, since } B \text{ is hereditary,}
\begin{align*}
\text{RHom}_{A}(Ae_{j}, X^{*}) & \cong \oplus_{k \in \mathbb{Z}} H^{k}(\text{RHom}_{A}(Ae_{j}, X^{*}))[-k] \cong \oplus_{k \in \mathbb{Z}} \text{Hom}_{D^{b}(A)}(Ae_{j}[-k], X^{*})[-k] \\
& \cong \text{Hom}_{D^{b}(A)}(Ae[n(j)], X^{*})[n(j)]
\end{align*}
in D^{b}(B^{op}). \text{ So } X^{*} \cong \text{RHom}_{A}(A, X^{*}) \cong \text{RHom}_{A}(\oplus_{j} Ae_{j}, X^{*}) \cong \oplus_{j} \text{Hom}_{D^{b}(A)}(Ae_{j}[n(j)], X^{*})[n(j)] \text{ in } D^{b}(B^{op}). \text{ By hypothesis, there exists a minimal left add } X^{*}\text{-approximation sequence}
\begin{align*}
Ae_{j}[n(j)] & \rightarrow X_{0}^{*} \xrightarrow{g} X_{1}^{*}
\end{align*}
of Ae[n(j)] in D^{b}(A) such that Ae_{j} \cong H^{-(n(j)+1)}(\text{cone}(g)) \text{ as } A\text{-modules. \text{ Combining with Lemma 4.1,}}
\begin{align*}
A & \cong \oplus_{j} Ae_{j} \cong \oplus_{j} H^{-n(j)}(\text{RHom}_{B^{op}}(\text{Hom}_{D^{b}(A)}(Ae_{j}[n(j)], X^{*}), X^{*})) \\
& \cong \text{Hom}_{D^{b}(B^{op})}(\oplus_{j} \text{Hom}_{D^{b}(A)}(Ae_{j}[n(j)], X^{*})[n(j)], X^{*}) \\
& \cong \text{Hom}_{D^{b}(B^{op})}(X^{*}, X^{*})
\end{align*}
as A\text{-modules. By Lemma 3.4, } l_{X^{*}} \text{ is an isomorphism. So } X^{*} \text{ has the derived double centraliser property.} \quad \Box

\textbf{Remark 4.3.} \text{ We point out that, when the } A-B\text{-complex } X^{*} \text{ is concentrated as a bimodule, the characterization Theorem 4.2 coincides with the characterization Proposition 3.13 by Auslander and Solberg.}

Now, we are going to characterize two-sided tilting complexes. Before we give the first lemma, let us recall the generating subcategories of triangulated categories. Let A be an algebra and X be an object in K^{b}(A). Denote by thick(X) the smallest triangulated full subcategory of K^{b}(A) which contains add X and is closed under taking direct summand. Note that if X is an object of K^{b}(\text{proj} A), then X generates K^{b}(\text{proj} A) if and only if \text{A} \subset \text{thick}(X).

The following lemma is implicitly in Rickard’s Morita theory for derived categories.

\textbf{Lemma 4.4.} \text{ Let } A \text{ and } B \text{ be algebras. Let } X^{*} \text{ be a bounded } A-B\text{-complex. If } \text{RHom}_{B^{op}}(X^{*}, X^{*}) \cong A \text{ in } D^{b}(A), \text{ then}
\begin{align*}
F & := \text{RHom}_{A}(\cdot, X^{*}) : K^{b}(\text{proj} A) \leftrightarrow \text{thick}(X^{*}_{B}) : \text{RHom}_{B^{op}}(\cdot, X^{*}) \cong: G
\end{align*}
are mutually inverse triangle dualities.
Proof. First, we prove that $F(Y^*)$ is an object of thick($X_B^*$), for any object $Y^*$ of $K^b(\text{proj} A)$. We prove this by induction on the width $t(Y^*)$ of $Y^*$. When $t(Y^*) = 1$, this means that $Y^*$, up to shift, is isomorphic to a projective $A$-module, then $F(Y^*) \in \text{add} X^*$, hence $F(Y^*) \in \text{thick}(X_B^*)$. Assume the assertion holds for $t(Y^*) < n$. Let $t(Y^*) = n$. We may assume without loss of generality that the non-zero components of $Y^*$ are exactly in degrees between 1 and $n$. Let $Y_{\leq n}^*$ be the brutal truncation of $Y^*$ at degree $n$. Then there is a distinguished triangle $Y_{\leq n}[-1] \rightarrow Y_n \rightarrow Y^* \rightarrow$ in $K^b(\text{proj} A)$. Applying $F$ to it we have a distinguished triangle $F(Y_{\leq n}^*) \rightarrow F(Y_n^*) \rightarrow F(Y^*)[1]$ in $K^b(B^{op})$. By the induction hypothesis, $F(Y_{\leq n}^*)$ and $F(Y_n^*)$ are objects of thick($X_B^*$). Since thick($X_B^*$) is a triangulated subcategory of $K^b(B^{op})$, then $F(Y^*) \in \text{thick}(X_B^*)$.

Second, $G(Y^*)$ is an object of $K^b(\text{proj} A)$, for any object $Y^*$ of thick($X_B^*$). The proof is similar to the previous, just by induction on the distance of $Y^*$ to add $X^*$ (the notation is introduced in [10, pp. 71]).

Since the restrictions $F_{\text{proj} A} : \text{proj} A \leftrightarrow \text{add} X_B^* : \text{add} X^* G$ are mutually inverse dualities. To prove that $GF$ is an identity on $K^b(\text{proj} A)$ and $FG$ is an identity on thick($X_B^*$), it is also given respectively by induction on the width of complex in $K^b(\text{proj} A)$ and by induction on the distance of complex in thick($X_B^*$) to add $X^*$. 

\[ \square \]

Lemma 4.5. Let $A$ be a hereditary algebra. Let $X^*$ be a bounded and self-orthogonal $A$-complex. Then $\text{gl}(\text{End}_{D^b(A)}(X^*)) < \infty$.

Proof. We prove this lemma by induction on the width $t(X^*)$ of the complex $X^*$. Let $B = \text{End}_{D^b(A)}(X^*)$. If $t(X^*) = 1$, then $X^*$, up to shift, is isomorphic to a partial tilting $A$-module. By [10, Corollary III.6.5 and Proposition III.3.4], $\text{gl}(B) < \infty$. Assume the assertion holds for $t(X^*) < n$. Let $t(X^*) = n$. We may assume without loss of generality that $X^* \cong X_1[1] \oplus X_2[2] \oplus \cdots \oplus X_n[n]$, where $X_i$ are $A$-modules and $X_1 \neq 0 \neq X_n$. Let $Y^* = X_1[1] \oplus X_2[2] \oplus \cdots \oplus X_{n-1}[n-1]$. Let $\mathcal{I}$ be the set of all morphisms in $B$ factoring through $X_n[n]$. Note that $\mathcal{I}$ is an idempotent ideal of $B$ and it is projective as $B$-module, moreover, $B/\mathcal{I} \cong \text{End}_{D^b(A)}(Y^*)$). By [2, Corollary 5.6], $\text{gl}(B) < \infty$ if and only if $\text{gl}(\text{End}_{D^b(A)}(Y^*)) < \infty$ and $\text{gl}(\text{End}_{D^b(A)}(X_n[n])) < \infty$. Note that $Y^*$ and $X_n[n]$ are self-orthogonal and the widths of them are less than $n$, by the induction hypothesis, we get the assertion. 

\[ \square \]

Theorem 4.6. Let $A$ and $B$ be two algebras. Let $X^*$ be a bounded $A$-$B$-complex. Assume $B$ is hereditary and $r_{X^*} : B \rightarrow \text{End}_{D^b(A)}(X^*)^{op}$ is an isomorphism. Then $X^*$ is two-sided tilting if and only if for any indecomposable projective $A$-module $Ae$, 

1. there is a unique $i \in \mathbb{Z}$ such that $\text{Hom}_{D^b(A)}(Ae[i], X^*) \neq 0$;
2. there is a minimal left add $X^*$-approximation sequence $Ae[i] \rightarrow X_0^* \rightarrow X_1^*$ of $Ae[i]$ which forms a distinguished triangle in $D^b(A)$.

Proof. Suppose $X^*$ is two-sided tilting. Then $X^*$ has the derived double centraliser property. For any indecomposable projective $A$-module $Ae$ with $e$ an idempotent of $A$, the condition (a) holds by Theorem 4.2. By Lemma 4.1, there is a minimal left add $X^*$-approximation sequence $Ae[i] \rightarrow X_0^* \rightarrow X_1^*$ of $Ae[i]$ such that $\text{cone}(g) \cong \text{RHom}_{B^{op}}(\text{Hom}_{D^b(A)}(Ae[i], X^*), X^*)[1]$ in $D^b(A)$. Since $X^*$ is two-sided tilting and $\text{Hom}_{D^b(A)}(Ae[i], X^*) \cong eA \otimes_A X^*[-i]$ in $D^b(B^{op})$, we have $\text{cone}(g) \cong Ae[i+1]$ in $D^b(A)$. This yields a distinguished triangle $Ae[i] \rightarrow X_0^* \rightarrow X_1^* \rightarrow Ae[i+1]$ in $D^b(A)$. Applying
Hom$_{D^b(A)}$($\cdot$, $X^*$) to it, we get an exact sequence $0 \to \text{Hom}_{D^b(A)}(X^*_1, X^*) \to \text{Hom}_{D^b(A)}(X^*_0, X^*) \to \text{Hom}_{D^b(A)}(\text{Ae}[i], X^*) \to 0$ since Hom$_{D^b(A)}(\text{Ae}[i+1], X^*) = 0$ by (a). So $\text{Ae}[i] \rightarrow X^*_0 \rightarrow X^*_1$ is a minimal left add $X^*$-approximation sequence of $\text{Ae}[i]$ in $D^b(A)$, where $f'$ is left minimal since $f$ is left minimal.

Conversely, suppose that for any indecomposable projective $A$-module $\text{Ae}$ with $e$ an idempotent of $A$, there is a unique $i \in \mathbb{Z}$ such that Hom$_{D^b(A)}(\text{Ae}[i], X^*) \neq 0$ and there is a minimal left add $X^*$-approximation sequence $\text{Ae}[i] \rightarrow X^*_0 \rightarrow X^*_1$ of $\text{Ae}[i]$ which forms a distinguished triangle in $D^b(A)$. Note that cone($g$) $\cong \text{Ae}[i+1]$ in $D^b(A)$, using Theorem 4.2. $X^*$ has the derived double centraliser property. Then by Proposition 3.10, we only need to prove that $X^*$ is isomorphic to a tilting complex in $D^b(B^{op})$.

Using Lemma 4.1, $\text{Ae}[i+1] \cong \text{cone}(g) \cong \text{RHom}_{B^{op}}(\text{Hom}_{D^b(A)}(\text{Ae}[i], X^*), X^*)[1]$ in $D^b(A)$. Note that Hom$_{D^b(A)}(\text{Ae}[i], X^*) \cong eA \otimes_A X^*[-i]$ in $D^b(B^{op})$, then $\text{RHom}_{B^{op}}(eA \otimes_A X^*, X^*) \cong \text{Ae}$ in $D^b(A)$. Hence $\text{RHom}_{B^{op}}(X^*, X^*) \cong A$ in $D^b(A)$ and then $X^*$ is self-orthogonal in $D^b(B^{op})$. By Lemma 4.5, gl($A$) $< \infty$. Now by [1, Theorem 16], gl($A \otimes_K B^{op}$) $= \text{gl}(A) + \text{gl}(B) < \infty$. Hence we may assume $X^*$ is a bounded complex of projective $A \otimes_K B^{op}$-modules. Naturally, $X^*$ is an object in $K^b(\text{proj} A)$ and also an object in $K^b(\text{proj} B^{op})$. Consider the triangle functor $F := \text{RHom}_A(\cdot, X^*) : K^b(\text{proj} A) \to K^b(B^{op})$.

Since $r_{X^*} : B \to \text{End}_{D^b(A)}(X^*)$ is a $B^{op}$-module isomorphism by Lemma 3.4 and $\text{End}_{D^b(A)}(X^*)$ is a direct summand of $F(X^*)$ as $B$ is hereditary, we have $B_0$ is isomorphic to a direct summand of $F(X^*)$. Moreover, by Lemma 4.4, im($F$) $= \text{thick}(X^*)$ which is closed under taking direct summand. Then $B_0 \in \text{thick}(X^*)$. This means that $X^*$ generates $K^b(\text{proj} B^{op})$. We are done. \hfill \square

**Remark 4.7.** Without the assumption that $B$ is hereditary, Theorem 4.6 may fail.

Let $A$ be an algebra. Let $X^*$ be an $A$-$B$-bimodule $X$ such that $\mathcal{A}X$ is a non-projective generator of $A$ (i.e., $\text{add}A \subseteq \text{add}X$) and $B = \text{End}_{A}(X)^{op}$. Then $A \xrightarrow{\sim} A \rightarrow 0$ is a minimal left add $X^*$-approximation of $A$. So $X^*$ satisfies the two conditions in Theorem 4.6. But, as $\mathcal{A}X$ is a non-projective generator, $|K_0(A)| \neq |K_0(B)|$, $X$ is not tilting.

As a directly consequence of Theorem 4.6, we have

**Corollary 4.8.** Let $A$ be an algebra and $M$ an $A$-module. Assume $\text{End}_{A}(M)$ is hereditary. Then $A_M \text{End}_{A}(M)$ is tilting if and only if there is an exact sequence $0 \to A \xrightarrow{f} M_0 \to M_1 \to 0$ such that $M_0, M_1 \in \text{add} M$ and $f : A \to M_0$ is a minimal left add $M$-approximation.

An interesting new approach to prove the following well-known property can be given using our result.

**Corollary 4.9.** Let $B$ be a connected hereditary algebra. Let $A$ be another algebra such that $A$ and $B$ are derived equivalent. Then $\text{gl}(A) \leq |K_0(B)| + 1$.

**Proof.** Let $P$ be a tilting $B^{op}$-complex such that $\text{End}_{D^b(B^{op})}(P) \cong A$. Then there is a two-sided tilting $A$-$B$-complex $X^*$ such that $X^* \cong P$ in $D^b(B^{op})$. We may assume $P$ is radical [11, pp. 112 (a)] and let $t$ be the width of $P$. Then by [8, Section 12.5(b)], gl($A$) $\leq \text{gl}(B) + t - 1$. Note that $B$ is connected, by [24, Corollary 6.7.11], $A$ is also connected. Moreover, since $\text{End}_{D^b(B^{op})}(X^*) \cong A$ and $B$ is hereditary, all the degrees of non-zero homologies of $X^*$ are continuous integers. By
Theorem 4.6, the number of the degrees of non-zero homologies of $X^*$ is at most $|K_0(A)|$. Since $B$ is hereditary, $t \leq |K_0(A)| + 1$. Note that by [8, Section 12.5(d)], $|K_0(B)| = |K_0(A)|$. Hence $\text{gl}(A) \leq \text{gl}(B) + t - 1 \leq |K_0(B)| + 1$.  

4.2. Characterizations II. In this subsection, we will further investigate $A$-$B$-complexes with the derived double centraliser property and two-sided tilting $A$-$B$-complexes where algebras $A$ and $B$ are hereditary algebras. The following lemma is crucial.

**Lemma 4.10.** Let $A$ and $B$ be hereditary algebras and $X^*$ a bounded $A$-$B$-complex. Assume $r_{X^*} : B \rightarrow \text{End}_{D^b(A)}(X^*)^{op}$ is an isomorphism. Let $Ae$ be an indecomposable projective $A$-module. Assume there is a unique $i \in \mathbb{Z}$ such that $\text{Hom}_{D^b(A)}(Ae[i], X^*) \neq 0$.

(i) The following conditions are equivalent:

(a) There is a minimal left add $X^*$-approximation sequence

$$Ae[i] \rightarrow X_0^* \rightarrow X_1^*$$

of $Ae[i]$ such that $Ae \cong H^{-(i+1)}(\text{cone}(g))$ as $A$-modules.

(b) There is an exact minimal left add $H^{-i}(X^*)$-approximation sequence

$$Ae \overset{f}{\rightarrow} X_0 \rightarrow X_1$$

of $Ae$ such that $\text{ker} f \in \text{add} H^{-(i+1)}(X^*)$.

(ii) The following conditions are equivalent:

(a) There is a minimal left add $X^*$-approximation sequence

$$Ae[i] \rightarrow X_0^* \rightarrow X_1^*$$

of $Ae[i]$ which forms a distinguished triangle in $D^b(A)$.

(b) There is an exact minimal left add $H^{-i}(X^*)$-approximation sequence

$$Ae \overset{f}{\rightarrow} X_0 \rightarrow X_1 \rightarrow 0$$

of $Ae$ such that $\text{ker} f \in \text{add} H^{-(i+1)}(X^*)$.

**Proof.** (i). Suppose that there exists a minimal left add $X^*$-approximation sequence

$$Ae[i] \overset{f}{\rightarrow} X_0^* \overset{g}{\rightarrow} X_1^*$$

of $Ae[i]$ such that $Ae \cong H^{-(i+1)}(\text{cone}(g))$ as $A$-modules. Because $Ae[i] \overset{f}{\rightarrow} X_0^*$ is a minimal left add $X^*$-approximation of $Ae[i]$ and $A$ is hereditary, $X_0^*$ is concentrated in degree $-i$. Hence, observing from the distinguished triangle $X_0^* \overset{g}{\rightarrow} X_1^* \rightarrow \text{cone}(g) \rightarrow X_0^*[1]$, there is an exact sequence of $A$-modules $0 \rightarrow H^{-(i+1)}(X_1^*) \rightarrow Ae \rightarrow H^{-i}(X_0^*) \rightarrow H^{-i}(X_1^*)$. We only need to prove that the sequence

$$Ae \rightarrow H^{-i}(X_0^*) \rightarrow H^{-i}(X_1^*)$$

defined as above is a minimal left add $H^{-i}(X^*)$-approximation of $Ae$.

We claim that there exists a minimal left add $X^*$-approximation cone($f$) $\rightarrow X_1^*$ of cone($f$) in $D^b(A)$. Indeed, applying $\text{Hom}_{D^b(A)}(-, X^*)$ to the distinguished triangle $Ae[i] \overset{f}{\rightarrow} X_0^* \rightarrow \text{cone}(f) \rightarrow Ae[i+1]$, since $\text{Hom}_{D^b(A)}(Ae[i+1], X^*) = 0$, we have an exact sequence

$$0 \rightarrow \text{Hom}_{D^b(A)}(\text{cone}(f), X^*) \rightarrow \text{Hom}_{D^b(A)}(X_0^*, X^*) \overset{m}{\rightarrow} \text{Hom}_{D^b(A)}(Ae[i], X^*) \rightarrow 0.$$
Since projective dimension of $B^{op}$-module $\text{Hom}_{D^b(A)}(Ae[i], X^*)$ is less or equal to 1 and since
\[ \text{Hom}_{D^b(A)}(X_1^*, X^*) \to \text{Hom}_{D^b(A)}(X_0^*, X^*) \to \text{Hom}_{D^b(A)}(Ae[i], X^*) \to 0 \]
is minimal presentation of $B^{op}$-module $\text{Hom}_{D^b(A)}(Ae[i], X^*)$ by Lemma 3.8, there is an isomorphism
\[ \text{Hom}_{D^b(A)}(X_1^*, X^*) \cong \text{Hom}_{D^b(A)}(\text{cone}(f^*), X^*). \]
Note that since $r_{X^*}$ is an isomorphism, applying $\text{Hom}_{D^b(A)}(-, X^*)$,
\[ \text{Hom}_{D^b(A)}(\text{cone}(f^*), X^*) \cong \text{Hom}_{B^{op}}(\text{Hom}_{D^b(A)}(X_1^*, X^*), \text{Hom}_{D^b(A)}(\text{cone}(f^*), X^*)). \]
Again by Lemma 3.8, there exists a minimal left add $X^*$-approximation $\text{cone}(f^*) \to X_1^*$ of $\text{cone}(f)$ in $D^b(A)$.

Applying $\text{Hom}_{D^b(A)}(-, H^{-i}(X^*)[i])$ to the sequence (1), we have an exact sequence
\[ (3) \quad \text{Hom}_{D^b(A)}(X_1^*, H^{-i}(X^*)[i]) \to \text{Hom}_{D^b(A)}(X_0^*, H^{-i}(X^*)[i]) \to \text{Hom}_{D^b(A)}(Ae[i], H^{-i}(X^*)[i]) \to 0. \]
Note that $X_0^*$ is concentrated in degree $-i$, then $H^{-i}(\text{cone}(f)) = 0$ when $j < i$. As above claim, we have
$H^{-i}(X_1^*) = 0$ when $j < i$. So $\text{Hom}_{D^b(A)}(X_1^*, H^{-i}(X^*)[i]) \cong \text{Hom}_{D^b(A)}(H^{-i}(X_1^*)[i], H^{-i}(X^*)[i])$.
Therefore, the sequence (3) induces an exact sequence
\[ \text{Hom}_{A}(H^{-i}(X_1^*), H^{-i}(X^*)) \to \text{Hom}_{A}(H^{-i}(X_1^*), H^{-i}(X^*)) \to \text{Hom}_{A}(Ae, H^{-i}(X^*)) \to 0. \]
Then we get the sequence (2) is a left add $H^{-i}(X^*)$-approximation of $Ae$. Since the sequence (1) is left minimal and $H^{-i}(X^*)[i]$ is a direct summand of $X^*$, it is not difficult to get that the sequence (2) is left minimal.

Conversely, suppose that there is an exact minimal left add $H^{-i}(X^*)$-approximation sequence $Ae \to X_0 \to X_1$ of $Ae$ such that $\ker f \in \text{add} H^{-i(i+1)}(X^*)$. Let $Ae[i] \to X_0[i] \xrightarrow{(h_1, g_1)} (\ker f)[i + 1] \oplus (\text{coker } f)[i] \to Ae[i + 1]$ be the distinguished triangle in $D^b(A)$ induced by $f[i]$. We will show that the following sequence
\[ (4) \quad Ae[i] \xrightarrow{f[i]} X_0[i] \xrightarrow{(h_1, g_1)} (\ker f)[i + 1] \oplus X_1[i] \]
is a minimal left add $X^*$-approximation sequence of $Ae[i]$ in $D^b(A)$ with $H^{-i(i+1)}(\text{cone}(h_1, g[i])) \cong Ae$ as $A$-modules.

First, we claim $\text{cone}(h_1, g[i]) \cong (\text{coker } g)[i] \oplus Ae[i + 1]$ in $D^b(A)$, hence $Ae \cong H^{-i(i+1)}(\text{cone}(h_1, g[i]))$ as $A$-modules. Indeed, by the octahedral axiom, we get the following commutative diagram

$$
\begin{array}{ccc}
X_0[i] & \xrightarrow{(h_1, h_2)} & (\ker f)[i + 1] \oplus (\text{coker } f)[i] \xrightarrow{1} Ae[i + 1] \xrightarrow{f[i + 1]} X_0[i + 1] \\
\downarrow \cong \quad \downarrow 1 \quad \downarrow \cong \quad \downarrow \\
X_0[i] & \xrightarrow{(h_1, g_1)} & (\ker f)[i + 1] \oplus X_1[i] \xrightarrow{\text{cone}(h_1, g[i])} X_0[i + 1] \\
\downarrow \cong \quad \downarrow \\
(\text{coker } g)[i] & \equiv & (\text{coker } g)[i] \\
\downarrow \cong \quad \downarrow \\
(\ker f)[i + 2] \oplus (\text{coker } f)[i + 1] & \to & Ae[i + 2]
\end{array}
$$

where the distinguished triangle of second column is induced by the canonical short exact sequence $0 \to \text{coker } f \xrightarrow{^s} X_1 \to \text{coker } g \to 0$. Since $A$ is hereditary, $\text{Hom}_{D^b(A)}((\text{coker } g)[i], Ae[i + 2]) = 0$, then the distinguished triangle of third column is split, hence $\text{cone}(h_1, g[i]) \equiv (\text{coker } g)[i] \oplus Ae[i + 1]$ in $D^b(A)$.

Second, we claim that the sequence (4) is a left add $X^*$-approximation sequence of $Ae[i]$ in $D^b(A)$. Indeed, it is easy to check that if we apply $\text{Hom}_{D^b(A)}(-, H^{-i}(X^*)[i])$ to the sequence (4),
we obtain an exact sequence since $Ae \xrightarrow{f} X_0 \xrightarrow{g} X_1$ is a left add $H^{−(i+1)}(X^*)$-approximation sequence of $Ae$. Now applying $\text{Hom}_{D^b(A)}(−, H^{−(i+1)}(X^*)[i + 1])$ to the sequence (4), we have a sequence

$\text{Hom}_{D^b(A)}((\ker f)[i + 1] \oplus X_1[i], H^{−(i+1)}(X^*)[i + 1]) \xrightarrow{F} \text{Hom}_{D^b(A)}(X_0[i], H^{−(i+1)}(X^*)[i + 1]) \rightarrow 0$.

Let $t \in \text{Hom}_{D^b(A)}(X_0[i], H^{−(i+1)}(X^*)[i + 1])$ and $W \rightarrow X_0[i] \xrightarrow{t} H^{−(i+1)}(X^*)[i + 1] \rightarrow W[1]$ the induced distinguished triangle in $D^b(A)$. By the octahedral axiom, we get the following commutative diagram:

\[
\begin{array}{c}
W \xrightarrow{=} X_0[i] \xrightarrow{t} H^{−(i+1)}(X^*)[i + 1] \xrightarrow{=} W[1] \\
\big| \quad (\ker f)[i + 1] \oplus X_1[i] \quad s \quad V \quad W[1] \\
\big| \quad \big| \\
W \xrightarrow{(\ker g)[i] \oplus Ae[i + 1]} (\ker g)[i] \oplus Ae[i + 1] \xrightarrow{=} (\ker g)[i] \oplus Ae[i + 1] \\
\big| \\
X_0[i + 1] \xrightarrow{=} H^{−(i+1)}(X^*)[i + 2]
\end{array}
\]

Since $A$ is hereditary and $Ae$ is projective, $\text{Hom}_{D^b(A)}((\ker g)[i] \oplus Ae[i + 1], H^{−(i+1)}(X^*)[i + 2]) = 0$, then the distinguished triangle of third column is split. Let $u' : V \rightarrow H^{−(i+1)}(X^*)[i + 1]$ such that $u'u = 1$. Then $u's(h_1, g[i]) = u'u't = t$. Consequently, $F$ is an epimorphism. Summarily, applying $\text{Hom}_{D^b(A)}(−, X^*)$ the sequence (4) we have an exact sequence $\text{Hom}_{D^b(A)}((\ker f)[i + 1] \oplus X_1[i], X^*) \rightarrow \text{Hom}_{D^b(A)}(X_0[i], X^*) \rightarrow \text{Hom}_{D^b(A)}(Ae[i], X^*) \rightarrow 0$. Therefore, the sequence (4) is a left add $X^*$-approximation sequence of $Ae[i]$ in $D^b(A)$.

Third, we claim that the sequence (4) is left minimal. Indeed, since $f$ and $g$ are left minimal, then so are $f[i]$ and $g[i]$, we only need to prove that $h_1 : X_0[i] \rightarrow \ker f[i + 1]$ is left minimal. Let $q : \ker f[i + 1] \rightarrow \ker f[i + 1]$ with $h_1 = qh_1$. Then we have a morphism of triangles:

\[
\begin{array}{c}
Ae[i] \xrightarrow{f[i]} X_0[i] \xrightarrow{(h_1, b_2)} \ker f[i + 1] \oplus (\ker f)[i] \xrightarrow{\rho[1]} Ae[i + 1] \\
\big| \quad \big| \\
Ae[i] \xrightarrow{f[i]} X_0[i] \xrightarrow{(h_1, b_2)} \ker f[i + 1] \oplus (\ker f)[i] \xrightarrow{\rho[1]} Ae[i + 1]
\end{array}
\]

Since $Ae$ is indecomposable and $f \neq 0$, $f$ is right minimal and then so is $f[i]$. Hence $p$ is an isomorphism. Then from the morphism of triangles, we have $q$ is an isomorphism.

(ii). Suppose that there is a minimal left add $X^*$-approximation sequence $Ae[i] \rightarrow X_0^* \rightarrow X_1^*$ of $Ae[i]$ which forms a distinguished triangle in $D^b(A)$. Then $X_0^*$ is concentrated in degree $−i$. Hence there is an exact sequence

$0 \rightarrow H^{−(i+1)}(X_1^*) \rightarrow Ae \rightarrow H^{−i}(X_0^*) \rightarrow H^{−i}(X_1^*) \rightarrow 0$.

By a similar proof of (i), $Ae \rightarrow H^{−i}(X_0^*) \rightarrow H^{−i}(X_1^*) \rightarrow 0$ is a minimal left add $H^i(X^*)$-approximation sequence of $Ae$.

Conversely, suppose there is an exact left add $H^{−i}(X^*)$-approximation sequence

\[
Ae \xrightarrow{f} X_0 \xrightarrow{g} X_1 \rightarrow 0
\]
of $A_e$ such that $\ker f \in \text{add } H^{-i-1}(X^*)$. Let $A_e[i] \xrightarrow{f[i]} X_0[i] \xrightarrow{(h_1,h_2)} (\ker f)[i+1] \oplus (\text{coker } f)[i] \rightarrow A_e[i+1]$ be the distinguished triangle in $D^b(A)$ induced by $f[i]$. We have showed in (a) that the following sequence

$$A_e[i] \xrightarrow{f[i]} X_0[i] \xrightarrow{(h_1,g[i])} (\ker f)[i+1] \oplus X_1[i]$$

is a minimal left add $X^*$-approximation sequence of $A_e[i]$, and showed cone$(h_1,g[i]) \cong A_e[i+1] \oplus (\text{coker } g)[i]$. Note that now $\text{coker } g = 0$. Then the second row of the first diagram in the proof of (i) yields the desired distinguished triangle. □

**Lemma 4.11.** Let $A$ and $B$ be two algebras. Let $X^*$ be a bounded $A$-$B$-complex. Assume $B$ is hereditary. Then for $b \in B$, $r_{X^*}(b) \neq 0$ if and only if $X^* \otimes^L_B Bb \neq 0$ in $D^b(A)$.

**Proof.** By Lemma 3.5, we only need to prove the sufficiency. Assume $X^* \otimes^L_B Bb \neq 0$ in $D^b(A)$. Since $B$ is hereditary, the canonical epimorphism $f : B \rightarrow Bb$ is split. Let $g : Bb \rightarrow B$ such that $fg = 1$. Suppose $r_{X^*}(b) = 0$. Then $X^* \otimes^L_B g f = 0$. Hence $X^* \otimes^L_B f = (X^* \otimes^L_B g f)(X^* \otimes^L_B g) = 0$. Again by Lemma 3.5, $X^* \otimes^L_B Bb = 0$ in $D^b(A)$. This is a contradiction. □

Let $A$ be a hereditary algebra and $X^*$ a bounded $A$-complex. Let $B = \text{End}_{D^b(A)}(X^*)^{op}$. We may assume without loss of generality that

$$X^* = X_0 \oplus X_1[1] \oplus \cdots \oplus X_n[n],$$

where $X_i$ are $A$-modules and $X_0 \neq 0 \neq X_n$. Then each $X_i$ is endowed with a $B^{op}$-module structure given by the canonical algebra epimorphism: $B^{op} = \text{End}_{D^b(A)}(X^*) \rightarrow \text{End}_A(X_i)$. Hence each $X_i$ is endowed with an $A$-$B$-bimodule structure. We denote by $A$-$B$-complex

$$T(X^*) := X_0 \oplus X_1[1] \oplus \cdots \oplus X_n[n],$$

where $X_i$ are $A$-$B$-bimodules defined as above.

**Lemma 4.12.** With the notation defined as above, if $B$ is hereditary, then $r_{T(X^*)}$ is an isomorphism.

**Proof.** Since $B = \text{End}_{D^b(A)}(X^*)^{op}$, we only need to prove $r_{T(X^*)}$ is injective. Since $B$ is hereditary, by Lemma 4.11, we only need to prove that for any non-zero idempotent $e \in B$, $T(X^*) \otimes^L_B Be \neq 0$ in $D^b(A)$. Note that $T(X^*) \otimes^L_B Be \cong X_0e \oplus X_1e[1] \oplus \cdots \oplus X_ne[n]$ in $D^b(A)$. Following from the construction of $B$-module structures for $X_i$, there exists $X_i$ for some $i$ such that $X_ie \neq 0$. Hence the assertion holds. □

**Proposition 4.13.** Let $A$ be a hereditary algebra. Let $X^*$ be a bounded $A$-complex. Assume $\text{End}_{D^b(A)}(X^*)$ is hereditary. Then $X^*$ has the derived double centraliser property if and only if for any indecomposable projective $A$-module $A_e$,

1. there is a unique $i \in \mathbb{Z}$ such that $\text{Hom}_{D^b(A)}(A_e[i],X^*) \neq 0$;
2. there is an exact left add $H^{-i}(X^*)$-approximation sequence

$$\begin{array}{c}
A_e \xrightarrow{f} X_0 \rightarrow X_1
\end{array}$$

of $A_e$ such that $\ker f \in \text{add } H^{-i+1}(X^*)$.

**Proof.** Combining Theorem 4.2 with Lemma 4.10(i), we only need to prove the sufficiency. Suppose the conditions (1) and (2) hold for $X^*$, then so do for $T(X^*)$. By Lemma 4.12, $r_{T(X^*)}$ is an isomorphism, again by Theorem 4.2 and Lemma 4.10(i), $T(X^*)$ has the derived double centraliser property, then so does $X^*$. □
The proof of the following proposition is similar, just by Theorem 4.6 and Lemmas 4.10(ii) and 4.12.

**Proposition 4.14.** Let $A$ be a hereditary algebra. Let $X^*$ be a bounded $A$-complex. Assume $\text{End}_{D^b(A)}(X^*)$ is hereditary. Then $X^*$ is isomorphic to a tilting complex in $D^b(A)$ if and only if for any indecomposable projective $A$-module $Ae$,

1. there is a unique $i \in \mathbb{Z}$ such that $\text{Hom}_{D^b(A)}(Ae[i], X^*) \neq 0$;
2. there is an exact left add $H^{-i}(X^*)$-approximation sequence

$$Ae \xrightarrow{f} X_0 \rightarrow X_1 \rightarrow 0$$

of $Ae$ such that $\ker f \in \text{add } H^{-(i+1)}(X^*)$.

**Example 4.15.** Let $A$ be the path algebra $KQ$ of quiver $Q : \bullet_1 \rightarrow \bullet_2 \rightarrow \bullet_3$. Let $P(i)$ (resp. $I(i), S(i)$) be the indecomposable projective (indecomposable injective, simple) $A$-module corresponding to the vertexes. Let $X^* = S(1) \oplus I(2) \oplus S(3)[1]$. It is easy to see that $\text{End}_{D^b(A)}(X^*)$ is hereditary and for all integers $i$ we get

- $\text{Hom}_{D^b(A)}(P(1)[i], X^*) = 0$, unless $i = 0$;
- $\text{Hom}_{D^b(A)}(P(2)[i], X^*) = 0$, unless $i = 0$;
- $\text{Hom}_{D^b(A)}(P(3)[i], X^*) = 0$, unless $i = 1$.

The exact left approximation sequences of projectives are given by the following maps:

- $P(1) \xrightarrow{f_1} I(2) \rightarrow 0$, with $\ker f_1 = S(3)$;
- $P(2) \xrightarrow{f_2} I(2) \rightarrow S(1) \rightarrow 0$, with $\ker f_2 = S(3)$;
- $P(3) \xrightarrow{\varphi} S(3) \rightarrow 0$.

By Proposition 4.14, $X^*$ is tilting.

### 4.3. On homologies of complexes.

After characterized the complexes with the derived double centraliser property and two-sided tilting complexes over hereditary algebras, under the setting in the former subsection, in this subsection, we will turn our attention to the shapes of their homologies. The following lemma is well-known, but we could not find a reference, so we provide the details.

**Lemma 4.16.**

1. Let $A$ be a hereditary algebra and $P$ a projective $A$-module. Then $\text{End}_A(P)$ is hereditary.
2. Let $A$ be an algebra. Let $P$ be a projective $A$-module and $E = \text{End}_A(P)^{\text{op}}$. Let $X$ and $Y$ be two $A$-modules. Assume for any indecomposable projective $A$-module $P' \notin \text{add } P$, $\text{Hom}_A(P', X) = 0 = \text{Hom}_A(P', Y)$. Then $X$ and $Y$ have natural $E$-module structures satisfying $\text{Hom}_E(X, Y) \cong \text{Hom}_A(X, Y)$.

**Proof.** (1) is trivial.

(2). We may assume that $P$ is basic. Then there is an idempotent $e \in A$ such that $P \cong Ae$ and $E \cong eAe$. Since for any projective $A$-module $P' \notin \text{add } P$, $\text{Hom}_A(P', X) = 0 = \text{Hom}_A(P', Y)$, then $(1 - e)X = 0 = (1 - e)Y$. Hence $X$ and $Y$ are endowed with the natural $E$-module structures. Consider the map $eAe \rightarrow A/A(1-e)A, a \mapsto \overline{a}$. It is not difficult to verify that the map is an algebra epimorphism. Hence there are fully faithful embeddings:

$$eAe\text{-mod} \leftrightarrow A/A(1-e)A\text{-mod} \leftrightarrow A\text{-mod}.$$ 

So $\text{Hom}_E(X, Y) \cong \text{Hom}_{A/A(1-e)A}(X, Y) \cong \text{Hom}_A(X, Y)$. \qed
Proposition 4.17. Let $A$ and $B$ be two hereditary algebras. Let $X^*$ be a bounded $A$-$B$-complex with the derived double centraliser property. Assume that $X^*$ has non-zero homology exactly at degree in $I \subseteq \mathbb{Z}$. Then there is a decomposition $\oplus_{i \in I} P_i$ of $A \otimes B$ with $E_i := \text{End}_{A}(P_i)^{op}$ and a decomposition $\oplus_{i \in I} P_i'$ of $B \otimes A$ with $E_i' := \text{End}_{B^{op}}(P_i')$, such that for each $i \in I$, $H^i(X^*)$ endowed with natural $E_i$-$E_i'$-bimodule has the double centraliser property. Moreover, if $X^*$ is two-sided tilting, the $E_i$-$E_i'$-bimodule $H^i(X^*)$ is tilting.

Proof. Let $X^*$ be a bounded $A$-$B$-complex with the derived double centraliser property. Assume that $X^*$ has non-zero homology exactly at degree in $I \subseteq \mathbb{Z}$. Using Proposition 4.13, for any indecomposable projective $A$-module $Ae$, there is a unique $i \in \mathbb{Z}$ such that $\text{Hom}_{\mathbb{D}(A)}(Ae[-i], X^*) \neq 0$. Then there is a decomposition $P_i \oplus Q$ of $A \otimes B$ such that for any $V \in \mathbb{D}(A)$, $\text{Hom}_{\mathbb{D}(A)}(V[-i], X^*) \neq 0$, and for any $V \in \mathbb{D}(Q)$, $\text{Hom}_{\mathbb{D}(A)}(V[-i], X^*) = 0$. With such process, there is a decomposition $\oplus_{i \in I} P_i$ of $A \otimes B$ such that $\text{Hom}_{\mathbb{D}(A)}(P_i[-j], X^*) \neq 0$ if and only if $i = j$. Let $E_i = \text{End}_{A}(P_i)^{op}$. Then $H^i(X^*)$ is endowed with natural $E_i$-module structure by Lemma 4.16(2). Dually, there is a decomposition $\oplus_{i \in I} P_i'$ of $B \otimes A$ such that $\text{Hom}_{\mathbb{D}(B^{op})}(P_i'[-j], X^*) \neq 0$ if and only if $i = j$; and $H^i(X^*)$ is endowed with natural $(E_i')^{op}$-module structure where $E_i' = \text{End}_{B^{op}}(P_i')$. Hence $H^i(X^*)$ is endowed with $E_i$-$E_i'$-bimodule structure which is induced from the $A$-$B$-bimodule structure of $H^i(X^*)$. Note that in this case, $X^* \otimes_B (P_i')^* \approx H^i(X^*)[-i]$ in $\mathbb{D}(A)$, where $(P_i')^*$ means the $B$-module $\text{Hom}_{\mathbb{D}(B^{op})}(P_i', B)$. Then by Lemma 4.16(2), $\text{End}_{E_i}(H^i(X^*)) \cong \text{End}_{A}(H^i(X^*)) \cong \text{End}_{\mathbb{D}(A)}(X^* \otimes_B (P_i')^*) \cong (E_i')^{op}$.

Using Proposition 4.13 again, for each $i \in I$, there is an exact left add $A \otimes H^i(X^*)$-approximation sequence $P_i \rightarrow X_0 \rightarrow X_1$ of $P_i$ such that $\ker f \in \text{add} H^{i-1}(X^*)$. Since $\text{Hom}_{\mathbb{D}(A)}(P_i, H^{i-1}(X^*)) \cong \text{Hom}_{\mathbb{D}(A)}(P_i[-1], X^*) = 0$, applying $\text{Hom}_{\mathbb{D}(A)}(P_i, -)$ to the sequence, we have an exact sequence

$$0 \rightarrow E_i \rightarrow X_0 \rightarrow X_1$$

of $E_i$-modules. It is an exact left add $A \otimes H^i(X^*)$-approximation sequence of $E_i$ by Lemma 4.16(2). By Proposition 3.13, the bimodule $E_i \otimes H^i(X^*)$ has the double centraliser property. Moreover, since $\text{End}_{E_i}(H^i(X^*)) \cong (E_i')^{op}$, we have $H^i(X^*)$ as $E_i$-$E_i'$-bimodule has the double centraliser property.

If $X^*$ is two-sided tilting, by Proposition 4.14, for each $i \in I$, there is an exact left add $H^i(X^*)$-approximation sequence $P_i \rightarrow X_0 \rightarrow X_1 \rightarrow 0$ of $P_i$ such that $\ker f \in \text{add} H^{i-1}(X^*)$. Similar with the discussion above, applying $\text{Hom}_{\mathbb{D}(A)}(P_i, -)$ to the sequence, we have an exact sequence

$$0 \rightarrow E_i \rightarrow X_0 \rightarrow X_1 \rightarrow 0$$

of $E_i$-modules, which is an exact left add $A \otimes H^i(X^*)$-approximation sequence of $E_i$. By Lemma 4.16(1), $E_i' = \text{End}_{B^{op}}(P_i')$ is hereditary. Then by Corollary 4.8, the bimodule $E_i \otimes H^i(X^*)_E_i$ is tilting.

Let $A$ be an algebra. Let $X^*$ be a bounded $A$-complex. Denote by $H(X^*)$ the $A$-complex $\oplus_{i \in \mathbb{Z}} H^i(X^*)[-i]$. $X^*$ is called split if $H(X^*) \cong X^* \otimes_B L_B$ in $\mathbb{D}(A)$. As we know, all bounded $A$-complexes are split if $A$ is hereditary.

Proposition 4.18. Let $A$ and $B$ be hereditary algebras. Let $X^*$ be a bounded $A$-$B$-complex. Then $X^*$ has the derived double centraliser property if and only if so does $H(X^*)$. In particular, if $X^*$ is two-sided tilting, then $X^*$ is split as $A \otimes_B B^{op}$-complex.

Proof. Let $0 \neq b \in B$. Since $B$ is hereditary, assume $B \cdot e = B$ for some idempotent $e \in B$. Note that $X^* \otimes_B L_B Bb = X^* \otimes_B L_B Be = 0$ in $\mathbb{D}(A)$ if and only if $X^* \otimes_B L_B Be$ is acyclic if and only if $H(X^*) \otimes_B L_B Bb = H(X^*) \otimes_B L_B Be$ is acyclic if and only if $H(X^*) \otimes_B L_B Bb = 0$ in $\mathbb{D}(A)$. Then by
Lemma 4.11, $r_\chi(b) \neq 0$ if and only if $r_{H(X)}(b) \neq 0$. Note that $\text{End}_{D^b(A)}(X^\bullet) \cong B$ if and only if $\text{End}_{D^b(A)}(H(X^\bullet)) \cong B$. So $r_\chi$ is an isomorphism if and only if so is $r_{H(X)}$. Dually, $l_X$ is an isomorphism if and only if so is $l_{H(X)}$. Hence $X^\bullet$ has the derived double centraliser property if and only if so does $H(X^\bullet)$.

If $X^\bullet$ is two-sided tilting, then $X^\bullet$ has the derived double centraliser property, so does $H(X^\bullet)$. Note that $X^\bullet$ is isomorphic to a tilting complex in $D^b(A)$, then so is $H(X^\bullet)$ by Proposition 3.10. $H(X^\bullet)$ is two-sided tilting. Note that $X^\bullet \cong H(X^\bullet)$ in $D^b(A)$ by Proposition 3.12, $X^\bullet \cong H(X^\bullet) \otimes B_\sigma^L$ in $D^b(A \otimes_K B^{op})$ for some $\sigma \in \text{Aut}(B)$. Hence $X^\bullet \cong \oplus_{i \in \mathbb{Z}} H(X^\bullet)[\sigma][-i]$ in $D^b(A \otimes_K B^{op})$. So $X^\bullet$ is split as $A \otimes_K B^{op}$-complex.

\textbf{Remark 4.19.} Let $A$ and $B$ be hereditary algebras. We do not know whether all bounded $A$-$B$-complexes with the derived double centraliser property are split as $A \otimes_K B^{op}$-complex.

5. Classifying complexes of bimodules over lower triangular matrices with the derived double centraliser property

In this section, we will classify complexes of bimodules over lower triangular matrix algebras with the derived double centraliser property.

Let $\Lambda_n$ be the algebra of $n \times n$ lower triangular matrices over $K$ with $n \in \mathbb{N}$. Then $\Lambda_n$ is isomorphic to the path algebra $KQ$ of quiver

\[ Q : \bullet_1 \to \bullet_2 \to \cdots \to \bullet_n. \]

Let $P(i), I(i)$ and $S(i)$ be the indecomposable projective $\Lambda_n$-module, the indecomposable injective $\Lambda_n$-module and the simple $\Lambda_n$-module corresponding to the vertex $i$ of $Q$, respectively. Since $\Lambda_n$ is a Nakayama algebra, let $X(i, j)$ be the indecomposable $\Lambda_n$-module with top $S(i)$ and socle $S(j)$.

A class of $\Lambda_n$-bimodules. Let $A = \Lambda_n$. Consider the $A$-module

\[ V_m = P(m) \oplus P(m-1) \oplus \cdots \oplus P(1) \oplus I(n-1) \oplus \cdots \oplus I(m) \]

for some $1 \leq m \leq n$, where $V_1 = ADA$ and $V_n = AA$. Then we can verify $\text{End}_A(V_m)^{op} \cong A$. We still denote by $V_m$ the $A$-bimodule induced by the $A$-module $V_m$. It is easy to check that the exact sequence

\[ A \hookrightarrow P(m)^{n-m+1} \oplus P(m-1) \oplus \cdots \oplus P(1) \to I(n-1) \oplus \cdots \oplus I(m) \]

is a left add $V_m$-approximation of $A$. By Proposition 3.13, $A$-bimodule $V_m$ has the double centraliser property.

A class of $\Lambda_n$-$\Lambda_n$-complexes. Let $A = \Lambda_n$. Let $e_i \in A$ be an idempotent of $A$ such that $Ae_i \cong P(1) \oplus P(2) \oplus \cdots \oplus P(i)$ as $A$-modules, for $1 \leq i \leq n-1$. Consider the $A$-$A$-complex

\[ T_i = A/Ae_{n-i} \oplus D(A/Ae_i)A[1], \]

where $D = \text{Hom}_K(-, K)$ and $e_{n-i} = 1 - e_i$.

Lemma 5.1. The $A$-$A$-complex $T_i$, for $1 \leq i \leq n-1$, is two-sided tilting.

\textbf{Proof.} Note that $A/Ae_{n-i}A \cong X(i, i) \oplus X(i-1, i) \oplus \cdots \oplus X(1, i)$ and $D(A/Ae_i)A \cong X(i+1, n) \oplus X(i+1, n-1) \oplus \cdots \oplus X(i, i+1)$ as $A$-modules. Then $\text{End}_{D^b(A)}(X^\bullet)^{op} \cong A$. For any idempotent $0 \neq e \in A$, $T_i \otimes_A^L Ae \cong (A/Ae_{n-i}A)e \oplus D(A/Ae_i)Ae[1] \neq 0$ in $D^b(A)$, hence for any $0 \neq a \in A$, $T_i \otimes_A^L Ae \neq 0$. By Lemma 4.11, $r_X : A \to \text{End}_{D^b(A)}(X^\bullet)^{op}$ is an isomorphism. Similarly, for any indecomposable projective $A$-module $Ae$, either $\text{Hom}_{D^b(A)}(Ae, T_i) \neq 0$ or $\text{Hom}_{D^b(A)}(Ae[1], T_i) \neq 0$. We point out that $Ae_{n-i} \cong A/Ae_iA$ as $A$-modules and $A/Ae_iA \cong \Lambda_{n-i}$ as algebras. Let $A/Ae_iA \to I_0 \to I_1 \to 0$ be a minimal injective resolution of $A/Ae_iA$-module $A/Ae_iA$. Then the induced
sequence 0 \rightarrow Ae_{n-i} \rightarrow I_0 \rightarrow I_1 \rightarrow 0 gives an exact minimal left add D(A/Ae_i A)-approximation of \( Ae_{n-i} \). A minimal left add \( A/Ae_{n-i} A \)-approximation of \( Ae_i \) is given by the exact sequence:

\[ 0 \rightarrow X(i+1,n) \rightarrow Ae_i \rightarrow A/Ae_{n-i} A \rightarrow 0. \]

By Proposition 4.14, \( T_i \) is two-sided tilting.

Let \( A \) be an algebra. Recall from [3] that a path from an indecomposable module \( M \) to an indecomposable module \( N \) in \( A \)-mod is a sequence of morphisms

\[ M \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \rightarrow \cdots \rightarrow M_{t-1} \xrightarrow{f_t} N \]

between indecomposable modules, where \( t \geq 1 \) and \( f_i \) is not zero and not an isomorphism. In this case, we call the length of the path is \( t \), and say the path is a zero-path if \( f_1 \cdots f_t = 0 \).

**Lemma 5.2.** Let \( A = A_n \). Then all the paths of length \( t \) in \( A \)-mod are zero-paths if \( t \geq n \). Hence if there is a \( A_n \)-\( A_m \)-bimodule with the double centraliser property, then \( m = n \).

**Proof.** Let \( M \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \rightarrow \cdots \rightarrow M_{t-1} \xrightarrow{f_t} N \) be a path in \( A \)-mod with length \( t \). Then it induces the sequence of morphisms: \( M \xrightarrow{g_1} \text{im}(f_1) \xrightarrow{g_2} \text{im}(f_2 f_1) \xrightarrow{g_3} \text{im}(f_3 f_2 f_1) \rightarrow \cdots \rightarrow \text{im}(f_t \cdots f_2 f_1) \). Let \( M = X(i,j) \) and \( N = X(i',j') \). Suppose that the path is nonzero and \( t \geq n \). Then \( \text{im}(f_t \cdots f_2 f_1) \neq 0 \), hence \( j' \leq i' \leq i \). Note that \( g_k \) is an isomorphism if only if \( f_k \) is a proper monomorphism. Then there are at least \( t - (j' - j) \) morphisms in the path which are proper monomorphisms. Therefore, \( i' \geq i + t - (j' - j) = t + j + (i - j') \geq t + j \geq n + j \geq n + 1 \). This is a contradiction.

Let \( X \) be a \( A_n \)-\( A_m \)-bimodule with the double centraliser property. Then \( \text{End}_{A_n}(X) \cong A_m \). Hence there is a non-zero path of length \( m - 1 \) in \( A_n \)-mod. Then \( m \leq n \). Dually, \( n \leq m \). So \( m = n \). □

**Lemma 5.3.** Let \( A = A_n \). If a bounded \( A \)-\( A \)-complex \( X^\bullet \) has the derived double centraliser property, then \( X^\bullet \) has at most two non-zero homologies.

**Proof.** Suppose \( A \)-\( A \)-complex \( X^\bullet \) has the derived double centraliser property. By definition, \( A \cong \text{End}_{D^b(A)}(X^\bullet)^{op} \). This means for each two direct summands \( X_0^\bullet \) and \( X_1^\bullet \) of \( X^\bullet \) in \( D^b(A) \), either \( \text{Hom}_{D^b(A)}(X_0^\bullet, X_1^\bullet) \neq 0 \) or \( \text{Hom}_{D^b(A)}(X_1^\bullet, X_0^\bullet) \neq 0 \). Suppose there exist at least three nonzero homologies of \( X^\bullet \), say in degrees \( i, j \) and \( k \). We may assume \( i < j < k \). Then

\[ \text{Hom}_{D^b(A)}(H^i(X^\bullet)[-i], H^k(X^\bullet)[-k]) = 0 = \text{Hom}_{D^b(A)}(H^k(X^\bullet)[-k], H^i(X^\bullet)[-i]). \]

This is a contradiction. □

**Lemma 5.4.** [12, Theorem A] For any \( n \geq 1 \), every automorphism of algebra \( A_n \) is inner.

**Lemma 5.5.** Let \( A = A_n \). Then an \( A \)-bimodule \( X \) has the double centraliser property if and only if \( X \cong V_m \) as \( A \)-bimodules for some \( 1 \leq m \leq n \).

**Proof.** Let \( X \) be an \( A \)-bimodule with the double centraliser property. Note that \( _A X \) is faithful if and only if \( P(1) \in \text{add} \ X \). Since \( \text{End}_A(X)^{op} \cong A \), any two indecomposable direct summands of \( _A X \) has nonzero morphism. Note that \( \text{Hom}_A(P(1), Y) = 0 = \text{Hom}_A(Y, P(1)) \) for any indecomposable \( A \)-module \( Y \notin \text{add} A \oplus DA \), we have \( _A X \in \text{add} A \oplus DA \). Then it is not difficult to verify that \( X \cong V_m \) as \( A \)-modules for some \( 1 \leq m \leq n \). Now, by Lemmas 3.9 and 5.4, we have \( X \cong V_m \) as \( A \)-bimodules for some \( 1 \leq m \leq n \). □

**Lemma 5.6.** Let \( A = A_n \). Then a bounded \( A-A \)-complex \( X^\bullet \) with exactly two nonzero homologies has the derived double centraliser property if and only if \( X^\bullet \), up to shift, is isomorphic to \( T_i \) in \( D^b(A^\bullet) \) for some \( 1 \leq i \leq n - 1 \).
**Proof.** Let $X^\bullet$ be an $A$-$A$-complex with the derived double centraliser property. We may assume $X^\bullet$ concentrated in degree 0 and 1. Let $M = H^0(X^\bullet)$ and $N = H^1(X^\bullet)$. By Proposition 4.17, there is a decomposition $P_0 \oplus P_1$ of $A$-$A$ with $E_i := \text{End}_A(P_i)^{op}$ and a decomposition $P'_0 \oplus P'_1$ of $A$-$A$ with $E'_i := \text{End}_A(P'_i)^{op}$, such that $M$ endowed with natural $E_0$-$E'_0$-bimodule and $N$ endowed with natural $E_1$-$E'_1$-bimodule have the double centraliser property. Note that $E_i$ or $E'_i$, for $i = 0$ or 1, is isomorphic to $\Lambda_k$ for some $k$. By Lemma 5.2, $E_0 \cong E'_0 \cong \Lambda_i$ and $E_1 \cong E'_1 \cong \Lambda_{n-i}$, for some $1 \leq i \leq n-1$. Note that $\text{End}_{D^b(A)}(X^\bullet)^{op} \cong A$ forces, for any direct summand $M'$ of $A$-$A$ and any direct summand $N'$ of $A$-$A$, Ext$^1$$(M', N') \neq 0$. Then we may assume $P_0 \cong P(1) \oplus P(2) \oplus \cdots \oplus P(i)$ and $P_1 \cong P(i + 1) \oplus P(i + 2) \oplus \cdots \oplus P(n)$ as $A$-modules. By Lemma 5.5,

$$M \cong X(m, i) \oplus X(m - 1, i) \oplus \cdots \oplus X(1, i) \oplus X(1, i - 1) \oplus \cdots \oplus X(1, m)$$

$$N \cong X(i + l, n) \oplus X(i + l - 1, n) \oplus \cdots \oplus X(i + 1, n) \oplus X(i + 1, n - 1) \oplus \cdots \oplus X(i + 1, n - i - l)$$

as $A$-modules, for some $1 \leq m \leq i$ and $1 \leq l \leq n - i$. Note that Ext$^1_A(X(1, m), N) = 0$ if $m \neq i$ and Ext$^1_A(X(i + l, n), M) = 0$ if $l \neq 1$. Then $m = i$ and $l = 1$. Actually, now $M \cong A/Ae_{n-I}A$ and $N \cong D(A/Ae_A)$ as $A$-modules, so $X^\bullet \cong T_i$ in $D^b(A)$. By Lemma 5.1 and Proposition 3.10, $X^\bullet$ is two-sided tilting. By Proposition 3.12 and Lemma 5.4, we have $X^\bullet \cong T_i$ in $D^b(A^e)$. □

Combining Lemmas 5.3, 5.5 and 5.6, we have

**Theorem 5.7.** Let $A = \Lambda_n$. Let $X^\bullet$ be a bounded $A$-$A$-complex. Then $X^\bullet$ has the derived double centraliser property if and only if, in $D^b(A^e)$, $X^\bullet$, up to shift, is isomorphic to one of the following cases:

1. The $A$-bimodules $V_1, V_2, \cdots, V_n$;
2. The $A$-$A$-complexes $T_1, T_2, \cdots, T_{n-1}$.

**Remark 5.8.** Given two algebras $A$ and $B$ which are derived equivalent, there is a derived equivalence $F : D^b(A^e) \to D^b(B^e)$ [19]. Then $F$ may not preserve the derived double centraliser property, in other words, if an $A$-$A$-complex $X^\bullet$ has the derived double centraliser property, then $F(X^\bullet)$ may have not.

Let $A = \Lambda_3$. Let $B$ be the path algebra $KQ$ of quiver $Q : \bullet \to \bullet \leftarrow \bullet$. It is known that $A$ and $B$ are derived equivalent. Recall that for algebras $C$ and $D$, $\lambda(C, D)$ is the number of all bounded $C$-$D$-complexes, up to isomorphism and shift in $D^b(C \otimes_k D^{op})$, which have the derived double centraliser property. Note that $|\text{Out}(B)| = 2$. By Proposition 3.11, $\lambda(B, B)$ is even. However, by Theorem 5.7, $\lambda(A, A) = 5$.

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