Pulling a polymer out of a potential well and the mechanical unzipping of DNA

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Motivated by the experiments on DNA under torsion, we consider the problem of pulling a polymer out of a potential well by a force applied to one of its ends. If the force is less than a critical value, then the process is activated and has an activation energy proportional to the length of the chain. Above this critical value, the process is barrierless and will occur spontaneously. We use the Rouse model for the description of the dynamics of the peeling out and study the average behaviour of the chain, by replacing the random noise by its mean. The resultant mean-field equation is a nonlinear diffusion equation and hence rather difficult to analyze. We use physical arguments to convert this into a moving boundary value problem, which can then be solved exactly. The result is that the time $t_{po}$ required to pull out a polymer of $N$ segments scales like $N^2$. For models other than the Rouse, we argue that $t_{po} \sim N^{1+\nu}$

I. INTRODUCTION

Considerable attention has been paid to the replication of the DNA molecule [1]. The first step in the replication is the unzipping of the two strands. Usually, this is caused by enzymes and mechanical force at the molecular level is involved in the action of these enzymes. Recently, micromanipulation techniques have been developed to study single molecules of DNA.
DNA under stress \[2\]. Most such single molecule experiments have been on the stretching properties of DNA, but very recently, the response of the DNA to external torques has also been studied \[3,4\]. Mechanical unzipping of the two strands of the DNA, in the absence of enzymes has been carried out \[6\]. There have been interesting investigations in to the theory of stretching the DNA \[5\] and its denaturation under a torque (see Fig. \[1\]) \[7,8\]. In an interesting recent e-print, Bhattacharjee \[7\] has suggested a minimal model to study the unzipping. He treats the DNA as consisting of two flexible interacting elastic strings which are bound together by an attractive interaction. The two are tied together at one end and at the other end, there are forces acting on the two strands, trying to separate them. Winding is ignored in his analysis. He assumes that in relative coordinates, the Hamiltonian for the unzipping problem is equivalent to that of for a single chain, subject to an attractive short range potential well with a force acting at one of its ends. It is shown that the polymer can be pulled out only if the force exceeds a critical value. The model is of interest in other contexts too. For example, one can imagine constructing micro-fabricated channels having different depths in different regions to trap the polymer by entropic barriers (see the recent experiments by Han et. al. \[9\]) and then pulling it out from the traps by the application of a force at one end. Another possible application is the desorption of a long chain molecule adsorbed on a surface by pulling at one end. Our aim in this paper is to investigate the dynamics of the escape of a chain from a potential well as a result of applying a force at one of its ends. We consider only the one dimensional problem and our approach is much less sophisticated than that of \[7\], as we do not take in to account the excluded volume interaction. We bring out the physical reason for the existence of the critical force. For describing the dynamics, we use the simple Rouse model. We find that \(t_{po}\), the time required to pull out a chain molecule of \(N\) segments scales as \(N^2\).

In the section \[II\] we give an outline of the statistical mechanics of a polymer, trapped by an attractive potential, with a force acting at one end. Section \[III\] outlines the Rouse model that is applicable for a polymer subject to a force, and analyze why the force has to exceed a critical force for pulling it out. In section \[IV\], we argue that the problem can be
approximated by a moving boundary value problem and then solve the problem. Finally, section V summarizes our conclusions.

II. STATISTICAL MECHANICS OF A TRAPPED POLYMER WITH A FORCE AT ONE OF ITS ENDS

In this section we consider the statistical mechanics of a polymer in one dimension, trapped in an attractive potential $V(R)$, that is located near $R = 0$ (see 3 for the nature of the potential) and subject to a force $F$ at one of its ends. We take the number of segments in the polymer to be $N$. The partition function for a free polymer, that is not subject to any force, but having one end at $R = 0$ may be calculated as $Z = \int dRG(R,0,N)$ where $G(R,0,N)$ is the propagator for the unforced molecule, which obeys the differential equation

$$\left(\frac{\partial}{\partial N} - \frac{l^2}{6} \frac{\partial^2}{\partial R^2} + \frac{1}{k_B T} V(R)\right)G(R,R_0,N) = \delta(N)\delta(R - R_0)$$

(see [10]). $l$ is the Kuhn length for the chain and $R$ and $R_0$ denote the positions of the two ends of the chain.

In the presence of a force $F$ acting at the end at $R$, the partition function becomes

$$Z = \int_{-\infty}^{\infty} dRG(R,0,N)e^{RF/k_B T}$$

Using the ground state dominance approximation for the propagator, valid for long polymers [10], we can write the above as:

$$Z = \int_{-\infty}^{\infty} dR\psi_0^*(R)\psi_0(0)e^{RF/k_B T}$$

(2) where $\psi_0(R)$ is the lowest eigen-function of the operator $H = -\frac{l^2}{6} \frac{\partial^2}{\partial R^2} + \frac{1}{k_B T} V(R)$, having the eigenvalue $E_0/(k_B T)$. Now, for large values of $R$, the eigen-function $\psi_0(R) \sim \exp(-\sqrt{\frac{6V_0}{l^2 k_B T}}R) \sim \exp(-\sqrt{\frac{6V_0}{l^2 k_B T}}R)$, where in the last step we have neglected the "zero point energy" contribution to the lowest eigenvalue and approximated it by $-V_0$, where $V_0$ is the depth of the potential well. On using this in the equation (2), we realize that for $F > F_c = \sqrt{\frac{6k_B T V_0}{l^2}}$, the contribution from large values of $R$ make the integral diverge, indicating that for $F > F_c$, the polymer gets pulled out of the hole. For $F < F_c$, the partition function is finite, indicating
that the molecule remains trapped. In this analysis, we have neglected the zero-point energy, which in the polymer problem, is equivalent to neglecting the effect of thermal fluctuations. Thermal fluctuations make $F_c$ slightly lower [7].

III. THE ROUSE MODEL AND THE FREE ENERGY PROFILE FOR THE PULLED POLYMER

The process that we study is shown in the Fig. 2. For its description, we use the continuum limit of the Rouse model, discussed in detail by Doi and Edwards [10], which has recently been used to study the related problem of escape of a polymer over a barrier [11]. The chain is approximated as a string, with segments (beads) labelled by their position $n$ along the chain. $n$ is taken to be a continuous variable, having values ranging from $-N$ to 0 (this is convenient for the present problem). The position of the $n^{th}$ segment in space (one dimension) is denoted by $R(n, t)$, where $t$ is time. In the Rouse model, this position undergoes overdamped Brownian motion and its time development is described by the equation

$$\zeta \frac{\partial R(n, t)}{\partial t} = m \frac{\partial^2 R(n, t)}{\partial n^2} - V'(R(n, t)) + f(n, t).$$

In the above, $\zeta$ is a friction coefficient for the $n^{th}$ segment. The term $m\frac{\partial^2 R(n, t)}{\partial n^2}$ comes from the fact that stretching the chain can lower its entropy and hence increase its free energy. Consequently, the parameter $m = 3k_B T/l^2$ (see Doi and Edwards [10], Eq. (4.5). They use the symbol $k$ for the quantity that we call $m$). $V(R)$ is the free energy of a segment of chain, located at the position $R$ and $V'(R) = \partial V(R)/\partial R$. $f(n, t)$ are random forces acting on the $n^{th}$ segment and have the correlation function $\langle f(n, t)f(n_1, t_1) \rangle = 2\zeta k_B T \delta(n - n_1) \delta(t - t_1)$ (see [10], Eq. (4.12)). The deterministic part of the Eq. (3), which will play a key role in our analysis, is obtained by replacing the random noise term in (3), by its mean. As we are applying a force at one end of the polymer (at $n = 0$) and as the other end is free, the boundary conditions to be satisfied are

$$\left\{ \frac{\partial R(n, t)}{\partial n} \right\}_{n=0} = \frac{F}{m}$$

(4)
\[ \left\{ \frac{\partial R(n, t)}{\partial n} \right\}_{n=-N} = 0. \quad (5) \]

The above equations may equivalently be written as
\[ \zeta \frac{\partial R}{\partial t} = -\frac{\delta}{\delta R(n)} E[R] + f(n, t). \quad (6) \]

where we have defined the free energy functional
\[ E[R] = \int_{-N}^{0} dn \left\{ \frac{m}{2} \left( \frac{\partial R}{\partial n} \right)^2 + V(R) - FR(n) \delta(n + \epsilon) \right\} \quad (7) \]

\( \epsilon \to 0 \), is a small positive number. The process of pulling out may be made physically clearer by analyzing this free energy functional. For this, we consider the free energy hypersurface that results from the Eqs. (4) (5) and (7). We find the extrema on this surface by putting \( \frac{\delta}{\delta R} E[R] = 0 \), which leads to
\[ m \frac{\partial^2 R}{\partial n^2} = V'(R) \text{ for } n < 0 \quad (8) \]

This has to be solved, subject to the conditions of the Eqs. (4) and (5). Imagining and speaking of \( n \) as time for the rest of this section, we realize that this is just Newton’s equation for a particle of mass \( m \), moving in the upside down potential \(-V(R)\). In the Fig. 3, we show this potential. The boundary condition of Eq. (5) means that the particle has to start at the time \( n = -N \), with zero velocity and end at the time \( n = 0 \), with the velocity \( F/m \). This means that at the end of its path, its kinetic energy is \( \frac{F^2}{2m} \). At \( n = -N \), the particle can at the most start at the top of the potential, where its total energy is \( V_0 \). If it obeyed the equation (8), then at \( n = 0 \), its kinetic energy has to be less than this amount. Thus we get the condition that \( V_0 > \frac{F^2}{2m} \) (or equivalently, \( F_c > F \) for the free energy surface to have a local minimum). If this condition is satisfied, then for any finite value of \( N \), we can find two solutions to the equation (8) satisfying the boundary conditions (this is true for potentials of the shape shown in the figure). These two solutions are shown in the Fig. 3. In the first solution (shown as line AB in the figure), the particle starts near the maximum of \(-V(R)\), at the time \( n = -N \) and takes a very long time to move away from the maximum,
and eventually arrives at the point B at the final time \( n = 0 \) with exactly the right kinetic energy of \( \frac{F^2}{2m} \). As this corresponds to a configuration for the polymer where almost all its \( N \) (assumed to be large) segments are near the minimum of the potential well \( V(R) \), we can estimate the free energy for this configuration to be \( \sim -NV_0 \). The second solution, also shown in the Fig. 3, CD, corresponds to the particle starting from rest from an appropriate value of \( R \) (point C in the Fig. 3) at the time \( n = -N \) and arriving at \( n = 0 \) with the kinetic energy \( \frac{F^2}{2m} \). The particle spends only relatively small amount of time in the region where the potential energy \( -V(R) \) is non-zero. One easily estimates the free energy of the polymer to be \( \sim -N\frac{F^2}{2m} \). Thus, for a polymer of \( N \) segments, there is a barrier of height \( N(V_0 - \frac{F^2}{2m}) = N(F_c^2 - F^2)/2m \), to be overcome for the pulling the polymer out. As \( F \to F_c \) this barrier is reduced and if \( F > F_c \) then the free energy surface has no local minimum - its minimum is at infinity - this means that the pulling out is barrierless and would happen spontaneously. A pictorial representation of this scenario is given in the Fig. 4.

IV. THE DYNAMICS

We now consider the case with \( F > F_c \) so that the pulling out of the chain has no activation barrier. We are interested in only the average behavior of the segments, in the process of pulling out. This may be obtained by analyzing the deterministic equation

\[
\zeta \frac{\partial R(n,t)}{\partial t} = m\frac{\partial^2 R(n,t)}{\partial n^2} - V'(R(n,t))
\]

where we have replaced the random term \( f(n,t) \) by its average value, viz., \( 0 \). We now wish to analyze this equation subject to the two boundary conditions of the Eqs. ([4]) and ([3]). This equation, is just a nonlinear diffusion equation.

The equation ([3]) is very difficult to solve for any realistic \( V(R) \). However, we can use the following physical picture. We take \( V(R) \) to be a short ranged well having a depth of \( V_0 \), and to have a rather sharp boundary like the one in the figure [2]. The change over from \( -V_0 \) to the flat region where \( V(R) = 0 \) occurs over a region of width \( w \), which is assumed to

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be small. This means that $V'(R(n, t))$ resembles a Dirac delta function, which implies that $rac{\partial R(n, t)}{\partial n}$ has to change very rapidly, across this region, and would resemble a step function. If we consider the process of escape from the well, then at any time, the units of the polymer with $n < n_i(t)$ would have $R(n, t) = 0$, and the segments from $n_i(t)$ to $n_o(t)$ would be spread across the region of width $w$. We shall refer to $n_o(t)$ as the peeling point. On multiplying the equation (9) by $\frac{\partial R(n, t)}{\partial n}$ and integrating from $n_i(t)$ to $n_o(t)$, we get

$$\frac{m}{2} \left(\frac{\partial R(n, t)}{\partial n}\right)^2_{n_o(t)} - V_0 = \zeta \int_{n_i(t)}^{n_o(t)} \frac{\partial R(n, t)}{\partial t} \frac{\partial R(n, t)}{\partial n} dn$$

(10)

In arriving at the Eq. (10) we have used the fact that $\left(\frac{\partial R(n, t)}{\partial n}\right)_{n_i(t)} = 0$ and $V(R(n_i(t))) = -V_0$. If the width $w$ is small, then we are justified in neglecting the right hand side of the Eq. (10) and then we find that $\left(\frac{\partial R(n, t)}{\partial n}\right)^2_{n_o(t)} = \frac{2V_0}{m}$, which on taking the square root may be written as

$$\left(\frac{\partial R(n, t)}{\partial n}\right)_{n_o(t)} = \frac{F_c}{m}$$

(11)

where $F_c = \sqrt{2mV_0}$. (If we picture the process as in Fig. 4, then we need not worry about the case where $F_c = -\sqrt{2mV_0}$). Now, all that we have to do is to solve the diffusion equation (4), outside the well region, subject to the above boundary condition. Outside the well, the equation (4) becomes the simpler diffusion equation

$$\zeta \frac{\partial R(n, t)}{\partial t} = m \frac{\partial^2 R(n, t)}{\partial n^2}$$

(12)

which is a linear equation. As the range of the potential well is small in comparison with the total length of the polymer, we can take

$$R(n, t) = 0 \quad \text{for} \quad n < n_o(t)$$

(13)

Thus we now have a moving boundary value problem: we have to solve the Eq. (12) for $R(n, t)$, and determine $n_o(t)$, such that the conditions (4), (11) and (13) are satisfied (see Fig. 5 for a pictorial representation of the process). In thus simplifying the equation, our assumption of neglecting the right hand side of the Eq. (10) is crucial. Physically, what
is being done is that near the point of breaking away from the potential, the segments are assumed to be at equilibrium - that means the time scale for peeling the polymer away from the well is slow in comparison with the time scale for establishment of local equilibrium near the peeling point.

Usually, moving boundary value problems are difficult to solve. However, in this case, it is possible to find similarity solutions to these equations. For this, we put \( R(n, t) = \sqrt{t} \rho(z) \), where \( z = n/(\alpha \sqrt{t}) \), with \( \alpha = \sqrt{2m/\zeta} \).

\[
\frac{d^2 \rho(z)}{dz^2} + z \frac{d\rho(z)}{dz} - \rho(z) = 0
\]

(14)

Note that \( z \in (-\infty, 0) \). The boundary conditions become

\[
\left( \frac{d\rho(z)}{dz} \right)_{z=0} = \frac{\alpha F}{m}
\]

(15)

\[
\left( \frac{d\rho(z)}{dz} \right)_{z=z_o} = \frac{\alpha F_c}{m}
\]

(16)

and

\[
\rho(z_o) = 0
\]

(17)

where \( z_o \) is related to \( n_o(t) \) by

\[
n_o(t) = z_o \alpha \sqrt{t}
\]

These equations can be solved to find \( \rho(z) \) and \( z_o \). The result is:

\[
\rho(z) = \frac{\alpha}{m \text{Erf}(\frac{z_o}{\sqrt{2}})} \left[ (F_c - F) \sqrt{\frac{2}{\pi}} e^{-\frac{z^2}{2}} + z \left( (F_c - F) \text{Erf}(\frac{z}{\sqrt{2}}) + F \text{Erf}(\frac{z_0}{\sqrt{2}}) \right) \right]
\]

(18)

where \( z_o(<0) \) is the solution of the equation

\[
\sqrt{\frac{\pi}{2}} \text{Erf}(\frac{z_0}{\sqrt{2}}) e^{\frac{z_o^2}{2}} z_0 = \frac{(F - F_c)}{F_c}
\]

(19)

Note that \( z_0 \) is a function of \( \frac{(F - F_c)}{F_c} \) only. Using all the above results, we get:
\[ \frac{R(n, t)}{F} \sqrt{\frac{\zeta m}{2t}} = n \sqrt{\frac{\zeta}{2mt}} - \left( \frac{F - F_c}{F} \right) \left( \sqrt{\frac{2}{\pi}} e^{-\frac{n^2 \zeta}{4mt}} + n \sqrt{\frac{\zeta}{2mt}} \text{Erf} \left( \frac{1}{\sqrt{2}} n \sqrt{\frac{\zeta}{2mt}} \right) \right) \]  

(20)

To demonstrate the nature of this solution, we note that the right hand side is a function of \( \sqrt{n \zeta m t} \) alone and make a plot of the left hand side against this variable, in the Fig. 3. The important conclusion that we can draw from the above is that the point of detachment of the polymer from the well is given by the equation \( \eta_0(t) = z_0 \sqrt{\frac{2mt}{\zeta}} \), where \( z_0 \) is the solution of the equation (19) and is \( t \) independent. As the left hand side of the equation (19) is positive, one has a solution for \( z_0 \) only if \( \frac{(F - F_c)}{F_c} > 0 \) and again this just means the existence of the critical force (a similar conclusion is arrived at by Bhattacharjee (7)). His answer is slightly different, as his analysis takes the noise term too into account. The pulling out of the polymer is complete when \( \eta_0(t) = -N \). Hence the time required to pull out the chain is given by

\[ t_{po} = \left( \frac{N}{z_0 \alpha} \right)^2 \]

For forces slightly exceeding the critical force, one can solve the Eq. (19) and get \( z_0 = -\left( \frac{(F - F_c)}{F_c} \sqrt{\frac{2}{\pi}} \right)^{1/2} \). Hence we conclude: \( t_{po} \sim N^2 \). Further, for small \((F - F_c)\),

\[ t_{po} \sim (F - F_c)^{-1} \].

(21)

V. CONCLUSIONS

We now summarize our results and point out the drawbacks of our analysis. Using the Rouse model, we find that for a polymer trapped in a potential well, subject to a force at one of its ends, there exists a critical force \( F_c \) and below this critical force, the pulling out is an activated process, with an activation energy proportional to the number of segments in the chain. So the pulling out would not happen. For forces higher than \( F_c \), there is no barrier and the process would occur spontaneously. The dynamics of the pulling out is described by a non-linear Rouse equation, which is rather difficult to solve. We find an approximate solution by (a) confining ourselves to the average behavior of the positions of the segments
and (b) introducing a local equilibrium assumption, which enables us to convert the non-linear equation to a moving boundary value problem. This moving boundary value problem has a similarity solution and this leads to the conclusion that the time required to pull the chain out scales like $N^2$.

Obviously, there is scope for improvement of our analysis. One would like to perform a more detailed analysis, including the effects of noise. Also, the effects of hydrodynamic interactions and the excluded volume interactions have to be included. Inclusion of these may be done by modifying the equation 3 to

$$\zeta \frac{\partial R(n, t)}{\partial t} = m \int dn_1 h(n - n_1) \frac{\partial^2 R(n_1, t)}{\partial n_1^2} - V'(R(n, t)) + f(n, t)$$

(see the equation (4.50) of Ref. [10]). In general $h(n) \sim n^{-\nu}$ with $\nu < 1$ (different values can be used for $\nu$ to account for Zimm dynamics or excluded volume interactions - see equations (4.49) and (4.69) of Ref. [10]). With the boundary conditions same as in our equations (4), (11) and (13), then one finds that $t_{po} \sim N^{1+\nu}$.

Our results above, however, are only the first few steps in the analysis of this very interesting problem. The most important next step, obviously, is to go beyond our mean field description by including the fluctuations. It seems likely that fluctuations can modify the long time behavior qualitatively.

VI. ACKNOWLEDGMENTS

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[1] A. Kornberg and T.A. Baker, DNA Replication, W.H. Freeman and Co., (1992).
[2] R.H. Austin, J.P. Broady, E.C. Cox, T. Duke and W. Volkmutth, Phys. Today, 50, No.2, 32 (1997) and the references therein.

[3] T.R. Strick, F. Allemand, D. Bensimon, R. Lavery and V. Croquette, Physica (Amsterdam) 263A, 392 (1999)

[4] J.F. Léger, G. Romano, A. Sarkar, J. Robert, I. Bordieu, D. Chatenay, and J.F. Marko, Phys. Rev. Lett. 83, 1066 (1999).

[5] J.F. Marko and E.D. Siggia, Macromolecules, 28, 8759 (1995). J.F. Marko, Europhys. Lett. 38, 183 (1997).

[6] B. Essevaz-Roulet, U. Bockelmann and F. Heslot, Proc. Natl. Acad. Sci. 94, 11935 (1997).

[7] S.M. Bhattacharjee, cond-mat/9912297 LANL archive (1999).

[8] S.Cocco and R. Monasson, Phys. Rev. Lett., 83, 5178 (1999).

[9] J. Han, S.W. Turner and H.G. Craighead, Physical Review Letters, 83, 1688 (1999).

[10] M. Doi and S.F. Edwards, The theory of polymer dynamics, Clarendon Press, Oxford, (1986).

The model has the defect of not taking excluded volume interactions into account.

[11] K.L. Sebastian, Physical Review E61, 3245 (2000). cond-mat/9907003 LANL e-print archive (1999)); K.L. Sebastian, J. Am. Chem. Soc. 122, 2972 (2000). See the more detailed version: K.L. Sebastian and A.K.R. Paul, cond-mat/0001043 (2000) (to appear in Phys. Rev. E).
FIG. 1. The forced unzipping of DNA.
FIG. 2. The polymer trapped in a potential well, subject to a force $F$. 
FIG. 3. The upside down potential \(-V(R)\). The lines AB and CD represent the solutions of the Eq. 8 corresponding to the initial and the transition states respectively. The corresponding configurations of the polymer are represented by the coiled curves above them. TS stand for transition state. The kinetic energy at the \(n = 0\) end is \(\frac{F^2}{2m}\) and this must be less than the height of the potential \(V_0\), if the extrema are to exist.
FIG. 4. The free energy hypersurface. For $F < F_c$ it has both a local maximum and a minimum, while for $F > F_c$ no local extrema exist. $E_{act}$ is the activation energy and is equal to $\frac{N(F^2 - F_c^2)}{2m}$. 
FIG. 5. The moving boundary value problem. The force $F$ is applied at $n = 0$. $n$ varies from $-N$ to 0. At a given time $t$, the portion with $0 > n > n_o(t)$ has been pulled out from the well.
FIG. 6. Plots of $\sqrt{m_\gamma^2 t} R(n,t)/(F)$ against $n\sqrt{\xi/(2mt)}$.

$F/F_c = 2$