Ground state property of Neumann domains on the torus

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Abstract

Neumann domains form a natural counterpart to nodal domains for Laplacian eigenfunctions. It is well-known that the restriction of an eigenfunction to a nodal domain always yields the ground state of the Dirichlet problem on that nodal domain. It was asked in [Zel13] and [BF16] whether this ‘ground state property’ also holds for Neumann domains. Here we show that this holds for half of the Neumann domains for some torus eigenfunctions. Our proof is based on a novel rearrangement method via a reference system, indirectly allowing a comparison between possible ground states.

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1 Introduction and motivation

The study of geometric properties of Laplacian eigenfunctions has a long history, dating back at least to the end of the 18th century when Chladni [Chl87] investigated the structure of nodal lines resp. nodal domains of vibrating plates. In fact Chladni’s analysis involves the biharmonic operator, but may be performed for the Laplacian as well. The nodal domains for an real-valued eigenfunction of a self-adjoint Laplacian defined on some domain are the connected components of the domain with a fixed sign of the eigenfunction. They generate a disjoint decomposition of the domain into subdomains possessing as boundary the nodal lines, i.e. the set of points where the eigenfunction vanishes. The restriction of the eigenfunction onto a nodal domain is always the ground state for the Laplacian with Dirichlet boundary conditions and for the same eigenvalue. This may readily seen invoking Courant’s celebrated nodal domain theorem [CH53 Section VI.1.6].

An alternative decomposition into Neumann domains was introduced by [Zel13, MF14] and builds a natural counterpart to nodal domains for Laplacian eigenfunctions. The restriction of the eigenfunction onto a Neumann domain possesses Neumann boundary conditions rather than Dirichlet boundary conditions. For the Neumann Laplacian the ground state is always the constant function possessing zero as eigenvalue. A more interesting object are properties of the eigenfunction corresponding to the first non-zero and nondegenerate eigenvalue which is hereafter called the ’ground state’. Counterexamples have recently been given [BF16] to the ground state property, the suggestion that the restriction of the eigenfunction to a Neumann domain yields the ground state for this domain.

Arrange and enumerate the eigenvalues according their value i.e.

$$0 < \lambda_0 < \lambda_1 \leq \ldots \leq \lambda_k \leq \ldots,$$

where $\lambda_0$ is the ground state eigenvalue and $\lambda_1$ the eigenvalue for the first excited state. It has even been shown that the eigenvalue for the original Laplacian can be located arbitrarily high in the spectrum for the Neumann Laplacian on a Neumann domain [BF16 Proposition 1.7].

In this paper we will show that the ground state property is satisfied for a specific type of Neumann domain for the Laplace operator defined on the flat 2-torus $\mathbb{T}$. This is eventually obtained by a suitable development of a rearrangement process. It is worth mentioning that our rearrangement tool only provides a reference system and does not compare the potentials ground states directly. The task of our method is then to show that a suitable reference system can be found yielding finally the result.

We briefly review the construction of Neumann domains on subdomains $\Omega \subset \mathbb{R}^2$. The starting point is an eigenfunction $\psi$ of the Laplacian with eigenvalue $\lambda$,

$$-\Delta \psi = \lambda \psi.$$  \hfill (2)

The eigenfunction $\psi$ should be a Morse-Smale function [BH04 Definition 3.1, Definition 6.1]. The gradient flow $\gamma : \mathbb{R} \times M \to M$ w.r.t. $\psi$ is defined by $\dot{\gamma}(t) = -\text{grad } \psi(\gamma(t))$ and
generates the stable and unstable manifolds by

$$W^{s/u}(\tilde{m}) := \left\{ m \in M; \lim_{t \to \pm\infty} \gamma_m(t) = \tilde{m} \right\}, \quad \tilde{m} \in \mathcal{C},$$

where $$\mathcal{C} := \{ c \in M; \text{grad} \psi(c) = 0 \} = \mathcal{M}_+ \cup \mathcal{M}_- \cup \mathcal{S}$$ are the critical points, i.e. the union of maxima, minima and saddle points. Following [Zel13, MF14] the restriction of the Laplacian onto the Neumann domains

$$\Omega(p, q) := W^s(p) \cap W^u(q), \quad p \in \mathcal{M}_+, \quad q \in \mathcal{M}_-,$$

possesses Neumann boundary conditions rather than Dirichlet boundary conditions. Moreover, $$\Omega$$ possesses the disjoint decomposition

$$\Omega = \left\{ \bigcup_{p \in \mathcal{M}_+, q \in \mathcal{M}_-} \Omega(p, q) \right\} \cup \left\{ \bigcup_{r \in \mathcal{S}} W^s(r) \cup W^u(r) \right\}$$

where $$W^s(r) \cup W^u(r)$$ are called Neumann lines [BF16, Proposition 1.3].

A Neumann domain is simply connected, and there exists exactly one nodal line inside going from its boundary to its boundary. Moreover, on the boundary of a generic Neumann domain there exist two saddle points, one maximum and one minimum [BF16, Theorem 1.4], connected by Neumann lines. In this case the angle between neighboring Neumann lines at saddle points is $$\frac{\pi}{2}$$ whereas the corresponding angle at extrema is either zero or $$\pi$$. There are therefore generically three types of Neumann domains: lens-like where both angles at the extrema are $$\pi$$, star-like where both angles are zero and wedge-like where one angle is zero and the other one is $$\pi$$. Interestingly, these three types exhibit different behaviour regarding the ground state property which we are going to describe in the following.

We make some general conventions for our notation.

- For a subspace $$A$$ of a Banach space $$B$$ we denote by $$\overline{A}|_B$$ its completion in $$B$$.
- For $$A \subset \mathbb{R}^n$$, $$n = 1, 2$$, we write $$\kappa_A$$ for its characteristic function.
- For $$A \subset \mathbb{R}^2$$ we denote by $$|A|_d$$ its $$d$$-dimensional Hausdorff measure [Mag12] p. 5.

1.1 A geometric condition for ground states of Neumann domains

We denote by

$$A_{p, q} = |\Omega(p, q)|_2$$

the area and by

$$l_{p, q} = |\partial \Omega(p, q)|_1$$

the following.
the perimeter of a Neumann domain $\Omega_{p,q}$ and define

$$\rho_{p,q} := \frac{A_{p,q}\sqrt{\lambda}}{l_{p,q}}.$$  

(6)

Necessary upper bounds for $\rho_{p,q}$ are provided by ($j_1' \approx 1.8411$ first zero of the derivative of the $J_1$ Bessel function [MOS66, p. 65])

**Proposition 1.** Assume that the eigenfunction restricted to a Neumann domain $\Omega(p,q)$ yields the ground state resp. the first excited state. Then

$$\rho_{p,q} \leq \frac{j'_1}{2} \approx 0.9206,$$  

(7)

resp.

$$\rho_{p,q} \leq \frac{j'_1}{\sqrt{2}} \approx 1.3019.$$  

(8)

**Proof.** We first recall the isoperimetric inequality in two dimensions [Mag12, Theorem 14.1]

$$\frac{1}{2\sqrt{\pi}} \leq \frac{\sqrt{A_{p,q}}}{l_{p,q}}.$$  

(9)

The Szegö-Weinberger inequality [Sze54, Wei56] yields for the ground state the bound

$$A_{p,q}\lambda_0 \leq \pi j_1'^2$$  

(10)

and the Giraud-Nadirashvili-Polterovich inequality [GNP09] gives

$$A_{p,q}\lambda_1 \leq 2\pi j_1'^2.$$  

(11)

Rewriting $\rho_{p,q}$ as

$$\rho_{p,q} = \frac{A_{p,q}\sqrt{\lambda}}{l_{p,q}} = \sqrt{A_{p,q}\lambda} \frac{\sqrt{A_{p,q}}}{l_{p,q}}$$  

(12)

and plugging in (9) and (10) resp. (11) proves the claim. 

Hence, (7) and (8) can serve to disprove the ground state property of a Neumann domain, respectively, to show the restriction can’t be the first excited state. This will be exploited for a numerical analysis of the three possible type of Neumann domains of random waves on the torus.
1.2 Numerical results for random waves

In our setting the events of a random waves take values in an eigenspace $H_\lambda$ of the torus Laplacian for a single eigenvalue $\lambda$. The degeneracy of $\lambda$ is denoted by $d_\lambda := \dim H_\lambda$. An orthonormal set of real valued eigenfunctions is given by,

$$\psi_{n,1} = \sqrt{2} \sin(2\pi(n_1 x_1 + n_2 x_2)), \quad \psi_{n,2} = \sqrt{2} \cos(2\pi(n_1 x_1 + n_2 x_2))$$

(13)

corresponding to the eigenvalue

$$\lambda_n = 4\pi^2(n_1^2 + n_2^2)$$

(14)

and we immediately see that $d_\lambda \geq 2$ for nonconstant eigenfunctions. For a fixed eigenvalue $\lambda$ we define a random wave to be

$$\phi_\lambda(x_1, x_2) = \sum_{4\pi^2|n|^2 = \lambda} a_n \sin \left( 2\pi \left(n_1 x_1 + n_2 x_2 \right) + \theta \right)$$

(15)

where $a_n$ are iid Gaussian random variables and $\theta$ possesses the uniform distribution in $[0, 2\pi]$. These random eigenfunctions of the torus, known as arithmetic random waves [KKW13], take as their high energy limit the statistics of the isotropic random wave model such as would describe chaotic eigenfunctions in the semiclassical limit [Ber77].

We numerically trace the Neumann domains in random wave functions in order to recover statistics of their geometrical properties; the values of a given random wave $\psi$ are calculated on a grid, the positions of the critical points numerically approximated according to the algorithm of [Kui04], and the Neumann lines located by numerical gradient ascent or descent starting at the saddle points. The Neumann domains are robustly recovered as long as the initial numerical resolution is sufficiently small, although the only approximates the true shape of the domains. Errors occasionally occur where the gradient tracing fails, or critical points are located incorrectly. This is normally easy to detect, and excluded from the statistics discussed below.

Using these numerical techniques, the random eigenfunctions are found to exhibit Neumann domains of all three different types, and with a wide range of values for $\rho$. Figure 1 shows a generic region of a random wave, in which domains of all three shapes are common, marked as lens-like, wedge-like or star-like in (b).

Figure 1(c) shows how $\rho$ takes different values for domains with different shapes, with the full probability distribution function shown in Figure 2(a), drawn from 8448822 domains in eigenfunctions with $\lambda = 925$, 1926306 with $\lambda = 325$ and 2218231 with $\lambda = 65$ ($d_\lambda = 12$ in all three cases). The $\rho$ distribution is almost identical in each case, and can clearly exceed the bound $\rho = 0.9206$ from (7). For instance, this happens for $\sim 20.7\%$ of domains at $\lambda = 925$. However, this property is not evenly distributed across different domain types, shown in Figure 2 for $\lambda = 925$. Lens-like domains have $\rho > 0.9206 \ldots$ with probability $\sim 0.64$, while for wedge-like domains this is rarer, with probability $\sim 0.059$. For star-like domains it is unclear from the numerics whether the bound of (7) is exceeded: we recover only a few hundred numerical examples, which are insufficient to rule out numerical error
Figure 1: Neumann domains in an almost-isotropic random wave formed from 100 plane waves with uniform random directions, with different features highlighted. The Neumann lines are shown in purple, maxima as red points, minima as blue points, and saddle points as purple diamonds. (a) shows also the value of the wavefield, (b) distinguishes the different domain types, and (c) shows the value of $\rho$ for each domain. In (b) and (c), the nodal lines are shown as dashed lines.

Figure 2: The probability distribution of $\rho$ values for Neumann domains in random eigenfunctions with $\lambda = 925$. (a) shows the combined PDF for all domains. (b) shows the domains with $\lambda = 925$, but with separately normalised distributions for each of the lens-like, wedge-like and star-like domains. The results are drawn from 2494622 lens-like domains, 2670896 star-like domains and 3283304 wedge-like domains, numerically traced and analysed from approximately 9000 individual eigenfunctions. The vertical lines mark the bound $\rho \sim 0.9206$ from (7).
in the detection of their Neumann line boundaries. All the examples have a highly unusual extended shape that is unstable to perturbations of the random state, as this may cause the creation of new critical points that break up the domain. The goal of this paper is to prove the ground state property of star-like eigenfunctions for a specific class of torus eigenfunctions.

2 Preliminaries

The starting point is the self-adjoint Laplacian \((-\Delta, H^2(T))\) on the torus \(T\) with fundamental domain \(F := [0, 1] \times [0, 1]\). \((-\Delta, H^2(T))\) possesses as domain the second order Sobolev space \(H^2(T)\).

By a slight abuse of notation we denote, as well, by \(H^1(T)\) the image of the canonical embedding \(H^1(T) \rightarrow H^1(F)\).

We define for an arbitrary domain \(\Omega \subset \mathbb{R}^2\) and for \(\psi \in H^1(\Omega)\) the map

\[
q_\Omega(\psi) := \int_\Omega |\text{grad} \psi|^2 \, dx.
\]

It is then easy to see that \((q_F, H^1(T))\) is the quadratic form corresponding to \((-\Delta, H^2(T))\) by [EE87, Theorem 1.9, p. 311]. Generally, the form norm on \(H^1(\Omega)\) induced by \(q_\Omega\) is defined by [RS80, p. 277]

\[
\|\psi\|_{\Omega, +1} := \|\psi\|_{H^1(\Omega)}.
\]

The embedding (16) is continuous which in turn implies the self-adjointness of \((-\Delta, H^2(T))\) by the completeness of \(H^1(F)\) see e.g. [RS80, Theorem VIII.15]. Since (13) gives a complete set of eigenfunction for the Laplacian \((-\Delta, H^2(T))\) we immediately observe that the spectrum of \((-\Delta, H^2(T))\) is purely discrete.

We may use the trigonometric addition formulas and alternatively consider the real valued functions \(f_{n,1,i}(x_1)f_{n,2,j}(x_2), i, j = 1, 2\), where \(f_{n,1,i}(x_1) = \sqrt{2}\cos(2\pi n_1 x_1)\) and \(f_{n,2,i}(x_i) = \sqrt{2}\sin(2\pi n_2 x_i), i = 1, 2\), corresponding to the same eigenvalue (14) and being again an orthonormal basis for \(L^2(T)\). We will study the Neumann domains of

\[
\psi_{n_1,n_2}(x,y) = \cos(2\pi n_2 x_2)\cos(2\pi n_1 x_1),
\]

and due to the identity \(\cos(x) = \sin(x + \frac{\pi}{2})\) the other cases are analogous. Indeed, an easy calculation shows that \(\tilde{\psi}_{n_1,n_2}\) is a Morse function. We will see the Smale transversality condition [BH04, Definition 6.1] as a byproduct in the following.

First, by the following Lemmata we may deduce that in the Neumann domains of such an eigenfunction are either lens-like or star-like and in particular not wedge-like. An example is given in Figure 3. The positions of the extrema are marked as red points.
and the saddle points as black points. It is not hard to see that the lens-like and star-like Neumann domains can be uniquely characterized by the two parameters

\[ a := (4n_1)^{-1} \quad \text{and} \quad b := (4n_2)^{-1}. \]  

and it will turn out that for fixed \( a \) and \( b \) the star-like resp. lens-like Neumann domains are equivalent modulo translations. More important for our considerations is the fact, by Proposition 5, that the the ground state property and the \( \rho \)-value for the Neumann domains only depends on the ratio \( a/b \).

By \( \Omega_{a,b} \) we denote a star-like Neumann domain and by a slight abuse of notation by \( \psi_{a,b} \) the restriction of \( \tilde\psi_{n_1,n_2} \) onto \( \Omega_{a,b} \) and we have

\[ \lambda_n = \lambda_{a,b} := \frac{\pi^2}{4} \left( a^{-2} + b^{-2} \right). \]  

(21)

By Figure 3: Neumann lines (blue) and nodal lines (green) for \( n_1 = 1, n_2 \)

and Figure 4: Numerically calculated \( \rho \) values for lens-like Neumann domains in eigenfunctions with different ratios \( a/b \). The points represent every ratio obtained from \( 0 < m < 100 \) and \( m < n < 100 \), and the horizontal lines mark the \( \rho \) cutoffs of (7) and (8).

\[ \rho = 1.3019 \ldots \]

\[ \rho = 0.9206 \ldots \]
Moreover, we put for convenience the origin of the coordinate charts in the center of the Neumann domain $\Omega_{a,b}$. This convention implies

$$\psi_{a,b}(x_1, x_2) = 2 \sin \left( \frac{\pi}{2a} x_1 \right) \cos \left( \frac{\pi}{2b} x_2 \right).$$

The point at $(a,0)$ is denoted in the following by $c$ and the point at $(0,b)$ by $w$. It will turn out that $c$ is a polynomial cusp and $w$ is a wedge with an angle of $\pi/2$. The set of the two wedges on $\Omega_{a,b}$ is denoted by $\mathcal{W}$ and the set of the two cusps by $\mathcal{C}$.

The star-like Neumann domains allow an explicit parametrization using

$$\gamma_{a,b}(x) := \frac{2b}{\pi} \arcsin \left( \cos \left( \frac{\pi}{2a} x_1 \right) \right)^2.$$

(23)

Lemma 1. The eigenfunctions $\psi_{n_1,n_2}$ is a Morse-Smale function. The star-like Neumann domains can be parametrized by

$$\Omega_{a,b} = \{(x_1, x_2) ; |x_2| < \gamma_{a,b}(x_1), -a < x_1 < a\}.$$  

(24)

Proof. We only consider the star-like case and the other case is analogous. A general gradient flow $\gamma(x_1(t), x_2(t)) : \mathbb{R} \to \mathbb{T}$ line satisfies

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{\pi^2}{2} \begin{pmatrix} a^{-1} \cos \left( \frac{\pi}{2a} x_1 \right) \cos \left( \frac{\pi}{2b} x_2 \right) \\ -b^{-1} \sin \left( \frac{\pi}{2a} x_1 \right) \sin \left( \frac{\pi}{2b} x_2 \right) \end{pmatrix}. $$

(25)

which translates into

$$\frac{dx_2}{dx_1} = \frac{a}{b} \tan \left( \frac{\pi}{2a} x_1 \right) \tan \left( \frac{\pi}{2a} x_2 \right). $$

(26)

Integrating this ordinary differential equation we arrive at

$$x_2(x_1) = \frac{2b}{\pi} \arcsin \left( \sin(g) \left[ \cos \left( \frac{\pi}{2a} x_1 \right) \right]^2 \right)$$

(27)

where every $-\frac{\pi}{2} \leq g \leq \frac{\pi}{2}$ is possible. The extreme values $g = \pm \frac{\pi}{2}$ parametrize the boundary gradient flow lines, which are piecewise smooth Jordan curves. Since (27) depend continuously on $\sin(g)$ and is monotonic in $\sin(g)$ we can infer that through every point lying inside the Jordan curve exactly one gradient flow line crosses this point. This shows that the Smale universality condition [BH04, Definition 6.1] is satisfied. 

We want to emphasize that the ground state problem may be analogously formulated for $\Omega_{a,b}$ with arbitrary $0 < b < a < \infty$ being not necessarily of the form (20). This is true since it will turn out that in this general case the ground state for the Neumann Laplacian on $\Omega_{a,b}$ exist and $\lambda_{a,b}$ in (21) can be evaluated for $0 < b < a < \infty$. One then may simply compare the ground state eigenvalue with $\lambda_{a,b}$. We tacitly consider in the following this more general setting.

We study the boundary at the points $c$ and $w$ in more detail.
Lemma 2. For fixed $a > 0$ we have

$$\left[\cos \left(\frac{\pi}{2a} x\right)\right]^{\left(\frac{\pi}{2a}\right)} = e^{-\frac{x^2}{\pi} \left(1 - R_b(x)\right)},$$

where for all $b, x > 0$

$$R_b(x) > 0, \quad R'_b(x) > 0$$

and for an arbitrary but fixed $\beta > \frac{1}{2}$ we have, $b \to 0$,

$$R_b(x) = \begin{cases} O \left(\frac{x^4}{b^2}\right) = O \left(b^{2(2\beta - 1)}\right), & x \leq b^\beta, \\ O(1) & x > b^\beta. \end{cases}$$

uniformly in $b$.

Proof. Using the expansion [AS64, 4.3.72] and [MOS66, p. 27] we may deduce for $x < a$

$$\left[\cos \left(\frac{\pi}{2a} x\right)\right]^{\left(\frac{\pi}{2a}\right)} = e^{\left(\frac{\pi}{2a}\right)^2 \ln \cos \left(\frac{\pi}{2a} x\right)} = e^{-\frac{x^2}{\pi} \left(\frac{x}{b}\right)^2 + O \left(\frac{x^4}{b^2}\right)},$$

where the error term on the r.h.s. in the exponent of (31) is always negative, which follows from the fact that every term in the Taylor expansion for $\ln \cos \left(\frac{\pi}{2a} x\right)$ is negative. This proves the claim.

We remark here that for $x > b^{1/2}$ a stronger decay than (30) holds but its exact behavior is of minor importance in the following.

Lemma 3. At $w$ we have a wedge with opening angle $\alpha_0 := \frac{\pi}{2}$ i.e.

$$\gamma_{a,b}(x) = b - |x| + O \left(x^3\right), \quad x \to 0.$$  

At $c$ we have a polynomial cusp with exponent $\beta = \left(\frac{\pi}{b}\right)^2$ i.e.

$$\gamma_{a,b}(a - |x|) = \frac{2b}{\pi} \left(\frac{\pi}{2a} |x|\right)^2 + O \left(|x|^3 \left(\frac{\pi}{b}\right)^2\right), \quad x \to a_-.$$  

Proof. We prove the asymptotic expansions for $x > 0$ and then the claim follows by the symmetry $\gamma_{a,b}(-x) = \gamma_{a,b}(x)$. Employing [AS64, p. 81]

$$\arcsin(1 - z) = \frac{\pi}{2} - \sqrt{2z} + O(z^{3/2}), \quad z \to 0,$$

we can deduce from Lemma 28 that

$$\left[\cos \left(\frac{\pi}{2a} x\right)\right]^{\left(\frac{\pi}{2a}\right)} = 1 - \frac{\pi^2}{8} \left(\frac{x}{b}\right)^2 + O \left(x^4\right).$$
Plugging (35) into (34) gives
\[ \gamma_{a,b}(x) = \frac{2b}{\pi} \arcsin \left( 1 - \frac{\pi^2}{8} \left( \frac{x}{b} \right)^2 + O(x^4) \right) = b - x + O(x^3), \quad x \to 0_+ . \] (36)

Using \( \cos \left( \frac{\pi}{2} a \left( a - x \right) \right) = \sin \left( \frac{\pi}{2} a x \right) = \frac{\pi}{2} a x + O \left( \left( \frac{\pi}{2} a x \right)^3 \right), \) \( (1 + x)^\beta = 1 + O(x^3), \) \( \beta > 0, \) and \( \arcsin(x) = x + O(x^3) \) we see that, \( x \to a_-, \)
\[ \frac{2b}{\pi} \arcsin \left( \cos \left( \frac{\pi}{2a} \right)^2 \right) = \frac{2b}{\pi} \left( \frac{\pi}{2a} x \right)^2 + O \left( \left( \frac{\pi}{2a} x \right)^3 \right) . \] (37)

It is worth mentioning that the value of the meeting angle for the Neumann lines may also be deduced by [MF14, Theorem 3.2].

Due to the presence of the cusps \( C \) on \( \partial \Omega_{a,b} \) it is more convenient to introduce the Neumann Laplacian in the weak form on \( \Omega_{a,b} \) and then consider the operator itself. In the following we denote by \( \partial_n \) the outward normal derivative w.r.t. \( \partial \Omega_{a,b} \) of appropriate sets. We are going to exploit the symmetry properties of the domain \( \Omega_{a,b} \) for our analysis. In doing so we emphasize the vertical and horizontal symmetry lines of \( \Omega_{a,b} \).

\[ h := \{(x, 0) ; x \in \mathbb{R} \}, \quad v := \{(0, y) ; y \in \mathbb{R} \} . \] (38)

With an slight abuse of notation we will denote the intersection \( A \cap \gamma \) of any suitable set \( A \) and curve \( \gamma \) in \( \mathbb{R}^2 \) with \( \gamma \) as well if it is clear from the context. Furthermore, we define the horizontal resp. vertical reflection operators \( R_h/v : \Omega_{a,b} \to \Omega_{a,b} \) by

\[ R_h/v (x_1, x_2) := \begin{cases} (-x_1, x_2), & \text{for } h, \\ (x_1, -x_2), & \text{for } v. \end{cases} \] (39)

These operators induce unitary operators on \( L^2(\Omega_{a,b}) \) and maps for subsets \( A \subset \Omega_{a,b} \) by

\[ (R_h/v \psi)(x) := \psi(R_h/v (x)), \quad R_h/v A := \{ R_h/v (x) ; x \in A \} . \] (40)

**Proposition 2.** The quadratic form
\[ q_{\Omega_{a,b}} := (q_{\Omega_{a,b}}, H^1(\Omega_{a,b})) \] (41)
defines a self-adjoint Laplace operator \( \Delta_{a,b} := (-\Delta, D(\Omega_{a,b})) \). The domain \( D(\Omega_{a,b}) \) of \( \Delta_{a,b} \) satisfies
\[ D(\Omega_{a,b}) \subset H^2_N(\Omega_{a,b}) := \left\{ \psi \in H^2(\Omega_{a,b}) ; \partial_n \psi|_{\partial \Omega_{a,b}} = 0 \right\} . \] (42)
Proof. That $q_{\Omega_{a,b}}$ defines a self-adjoint operator follows from [RS80, Theorem VIII.15] since $q_{\Omega_{a,b}}$ is positive and closed. Following the lines of [HP15, Proposition 2.6] and [Gri92, pp. 58-60] we may deduce that for every simply connected Lipschitz subdomain $\tilde{\Omega} \subset \Omega_{a,b}$ such that $C \cap \Omega_{a,b} = \emptyset$ and $|\partial \tilde{\Omega}_{a,b}|_1 \neq 0$, where $\partial \tilde{\Omega}_{a,b} := \partial \tilde{\Omega} \cap \partial \tilde{\Omega}$ we have that $\psi \in H^2(\Omega_{a,b})$ implies
\[ \psi \in H^1(\Omega_{a,b}), \quad -\Delta \psi \in L^2(\Omega_{a,b}), \quad \text{(43)} \]
and
\[ \partial_n \psi|_{\partial \tilde{\Omega}_{a,b}} \equiv 0, \quad \psi|_{\tilde{\Omega}_{a,b}} \in H^2(\tilde{\Omega}_{a,b}). \quad \text{(44)} \]
Since (44) holds for every suitable $\tilde{\Omega}_{a,b} \subset \Omega_{a,b}$ we may infer that in fact
\[ \partial_n \psi|_{\partial \Omega_{a,b}} \equiv 0. \quad \text{(45)} \]
It remains to show $D(\Omega_{a,b}) \subset H^2_N(\Omega_{a,b})$. For this we write
\[ \psi = \psi_{\text{asy}} + \psi_{\text{sym}}, \quad \psi_{\text{sym}} := \frac{1}{2} (\psi + R^v \psi), \quad \psi_{\text{asy}} := \frac{1}{2} (\psi - R^v \psi). \quad \text{(46)} \]
Is easy to see that $\psi \in H^2_N(\Omega_{a,b})$ implies $R^v \psi \in H^2_N(\Omega_{a,b})$ and hence $\psi_{\text{asy}}/\psi_{\text{sym}} \in H^2_N(\Omega_{a,b})$. Conversely, $\psi_{\text{asy}}/\psi_{\text{sym}} \in H^2_N(\Omega_{a,b})$ implies $\psi \in H^2_N(\Omega_{a,b})$. Therefore it suffices to prove (42) for $\psi_{\text{asy}}/\psi_{\text{sym}}$. We observe that $\psi_{\text{sym}}$ satisfies Neumann boundary conditions on $h$ and $\psi_{\text{asy}}$ satisfies Dirichlet boundary conditions. Both cases are covered by [Khc78, Remarque pp. 1115, 1116] owing to Lemma 3.

The following proposition is a requisite to investigate spectral problems on Neumann domains.

**Proposition 3.** $\Delta_{a,b}$ possesses a purely discrete spectrum satisfying $\sigma_d(\Delta_{a,b}) \subset \mathbb{R}^+_0$.

**Proof.** In order to prove the discreteness of the spectrum of $\Delta_{a,b}$ it suffices to prove that the embedding $H^1(\Omega_{a,b}) \to L^2(\Omega_{a,b})$ is compact by [ABHN01, p. 484] and [EE87, Theorem 2.9]. But this follows by Rellich’s theorem [EE87, Theorem 4.17] since the boundary $\partial \Omega_{a,b}$ is of class $C = C^{0,0}$ [EE87, Definition 4.1] and in particular the set $C$ creates no problem. The relation $\sigma_d(\Delta_{a,b}) \subset \mathbb{R}^+_0$ follows by the min – max principle [RS78, Theorem XIII.2] and by the observation that $q_{\Omega_{a,b}}$ is non-negative.

By a similar discussion we may treat the Laplacian equipped with a Dirichlet boundary condition on $\Omega_{a,b}$ summarized in the following remark. By $H^1_0(\Omega)$ we denote here the closure $C^\infty(\Omega)|_{H^1(\Omega)}$ and one may prove the remark using again [RS78, Theorem XI11.73] and Courant’s nodal domain theorem [CH89, p. 452] in combination with [Hör83, Theorem 9.5.1] and [Kra01, Corollary 2.3.8], see the proof of Lemma 1. We only require $\Omega$ to be a subset of $\Omega_{a,b}$ and in particular no regularity properties on $\partial \Omega$ are imposed.

**Remark 1.** The positive and closed quadratic form $(q, H^1_0(\Omega))$, $\Omega \subset \Omega$, yields the Dirichlet Laplacian $\Delta^D(\Omega) := (\Delta, H^2(\Omega) \cap H^1_0(\Omega))$ possessing a purely discrete spectrum. Moreover, the lowest eigenvalue $\lambda^D_1(\Omega)$ is simple.
3 Ground states of star-like Neumann domains

It is easy to see that every constant function is an eigenfunction for the Laplacian $\Delta_{a,b}$ corresponding to the eigenvalue zero. Since $\Delta_{a,b}$ possesses purely discrete spectrum we will consider the eigenspace of the first non-constant eigenvalue $\lambda_1$. Assuming that $\lambda_1$ is a simple eigenvalue we may choose the corresponding eigenfunctions as real-valued. We call the corresponding normalized and real-valued eigenfunction $\psi_{\lambda_1}, \|\psi_{\lambda_1}\|_{L^2(\Omega_{a,b})} = 1$, modulo sign, the ground state of $\Delta_{a,b}$. An analogously definition is made for every self-adjoint Laplacian on suitable subdomains $\Omega \subset \Omega_{a,b}$. A natural generalization of the ground state property of nodal domains to our Neumann domain setting is the following:

**Definition 1.** We say $\Omega_{a,b}$ possesses the ground state property if the ground $\psi_{\lambda_1}$ exists and is given by, modulo sign,

$$\psi_{\lambda_1} = \|\psi_{a,b}\|^{-1}_{L^2(\Omega_{a,b})}\psi_{a,b}. \quad (47)$$

Unlike to lens-like domains the star-like domains, indeed, share the ground state property being content of the paper.

**Theorem.** For every $a > 0$ a $b_a$ exists such that $b < b_a$ implies that the Neumann domain $\Omega_{a,b}$ possesses the ground state property.

For the proof we first provide several Lemmata which are going to be utilized for the final proof in Section 4. But first we want to give a remark about the dependence of $b_a$ on $a$. The numerical results of Figure 2 in Section 1.2 indicates that not every star-like Neumann domain possesses the ground state property. In our setting the threshold must be w.r.t. to the $b$ parameter since $b > a$ corresponds to a rotation of the Neumann domain by $\pi/2$. Moreover, the threshold $b_a$ must be $a$ dependent since Proposition 5 reveals that the violation of the ground state property only depends on the ration $a/b$ which is summarized by the following corollary.

**Corollary 1.** Let $a, \tilde{a} \in \mathbb{R}^+$. Then

$$b_{\tilde{a}} = \frac{\tilde{a}}{a} b_a. \quad (48)$$

**Proof.** This is an easy consequence the above theorem and Lemma 19. \hfill \Box

We will refer to the geometric shape of an eigenfunction as any distinguished pattern of its nodal and Neumann domains. In particular we emphasize three possible shapes assuming in every case that there are exactly two nodal domains, $\tilde{\Omega}_{a,b}$ and $\Omega_{a,b} \setminus \tilde{\Omega}_{a,b}$, where $\cdot$ denotes the interior of a set, such that:

I) $\tilde{\Omega}_{a,b}$ satisfies

$$\mathcal{R}^{h/v}\tilde{\Omega}_{a,b} = \tilde{\Omega}_{a,b}, \quad \text{and} \quad \Omega_{a,b} \setminus \tilde{\Omega}_{a,b} \subset \Omega_{a,b}, \quad (49)$$

II) $v$ is a gradient flow line and there is only one nodal line given by $h$.

III) $h$ is a gradient flow line and there is only one nodal line given by $v$.  

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Figure 5: Neumann lines (blue) and nodal lines (green) for I)-III) from right to left

Figure 5 depicts the above detailed three geometric shapes of an eigenfunction. Indeed the geometric shapes I)-III) are the relevant ones for our problem being the content of the next lemma.

**Lemma 4.** Assume that the ground state for \( \Delta_{a,b} \) exists. Then the only possible geometric shapes of the ground state are given by I)-III).

**Proof.** We assumed that the ground state \( \psi_{\lambda_1} \) exists implying that the corresponding eigenspace is one dimensional. Since \( R^{h/v}_a \Delta_{a,b} = \Delta_{a,b} R^{h/v}_v \) the ground state \( \psi_{\lambda_1} \) is also an eigenfunction for \( R^{h/v} \). The only possible eigenvalues for \( R^{h/v} \) are \( \lambda^{h/v} = \pm 1 \) since \( R^{h/v} \) is an isometry and \( R^{h/v^2} = 1 \). In the case \( R^h \) the eigenvalue \( \lambda^h = -1 \) corresponds to an asymmetric function and Dirichlet w.r.t. \( v \) and \( \lambda^h = 1 \) to a symmetric function and Neumann boundary conditions w.r.t. \( v \). The case \( R^v \) is analogous. Now Courant’s nodal domain theorem [CH89, p. 452] tells us that \( \psi_{\lambda_1} \) possesses at most two nodal domains. Moreover, [Hör83, Theorem 9.5.1] shows that \( \psi_{\lambda_1} \) is real analytic in \( \Lambda_{a,b} \) which implies by [Kra01, Corollary 2.3.8] it can’t vanish on any open subset. Since \( \psi_{\lambda_1} \) must be orthogonal to the constant function we can infer from the previously derived facts that it has exactly two nodal domains. Combining this with the formerly concluded symmetry properties of \( \psi_{\lambda_1} \) we may deduce that the only possible shapes for the ground state are given by the claimed ones.

The case \( i) \) can be excluded to be the ground state by the following lemma. We remark here that for the case \( I) \) the set \( \bar{\Omega}_{a,b} \) corresponds to the inner set in Figure 5.

**Lemma 5.** Assume \( \lambda \) is an eigenvalue of \( \Delta_{a,b} \) with an eigenfunction of shape \( I) \). Then

\[
\lambda > \lambda_1. 
\]  

**Proof.** We prove it by contradiction assuming the converse to (50) which means \( \lambda = \lambda_1 \). By Courant’s nodal domain theorem [CH89, p. 452] we can conclude that \( \lambda = \lambda_D^D(\Omega_{a,b}) \) is the ground state for the Dirichlet Laplacian \( \Delta^D(\Omega_{a,b}) \). Due to (49) we have \( |\Omega_{a,b} \setminus \bar{\Omega}_{a,b}| > 0 \). We can find a suitable injective homotopy from \( h : [0, 1] \times \Omega_{a,b} \to \Omega_{a,b} \) such that:

i) \( h(0) \) is the identity map and for \( h(1) \) the image of \( \Omega_{a,b} \) equals \( \Omega_{a,b} \),

ii) the image of \( h(t') \) is a subset of \( h(t) \) for \( t' < t \),
iii) for an open subinterval \( I \subset [0, 1] \) the map \( h : I \times \tilde{\Omega}_{a,b} \to \Omega_{a,b} \) is smooth and the boundary of the image of \( \Omega_{a,b} \) is a smooth submanifold for every \( t \in I \).

The ground state \( \lambda^D_1(t) \) of the Dirichlet Laplacian on the image of \( h(t) \) exists for every \( t \in \mathbb{R} \), see Remark [1]. Generally, [RS78, p. 270] implies by ii) that \( \lambda^D_1(t') \geq \lambda^D_1(t) \) for \( t > t' \) and Hadamard’s formula [Gri10, (81)] gives that \( \lambda^D_1(t') > \lambda^D_1(t) \) for \( t > t' \) and \( t', t \in I \). We obtain \( \lambda^D_1(1) < \lambda^D_1(0) \). As shown in the the proof to Proposition [3] the embedding \( H^1(\Omega_{a,b}) \to L^2(\Omega_{a,b}) \) is compact. This in turn allows to use the Pólya type inequality [Fil04, Theorem p. 404] which delivers the inequality \( \lambda_1 \leq \lambda^D_1(1) \). Putting everything together we gain the contradiction

\[
\lambda = \lambda^D_1(0) > \lambda^D_1(1) \geq \lambda_1 = \lambda
\]  

proving the claim. \( \square \)

**Corollary 2.** The shape i) is not possible for the ground state of \( \Delta_{a,b} \).

It remains to exclude the case III). As a first step it is convenient to reduce the problem by using the symmetry of \( \Omega_{a,b} \). For that reason we define two auxiliary Laplacians \( \Delta^h_{a,b} \) and \( \Delta^v_{a,b} \) on the first quarter \( \Lambda_{a,b} \) i.e.

\[
\Lambda_{a,b} = \{(x_1, x_2); 0 < x_2 < \gamma(x_1), 0 < x_1 < a\},
\]  

and note that the boundary of \( \Lambda_{a,b} \) is given by

\[
\partial\Lambda_{a,b} = h \cup v \cup \gamma_{a,b}.
\]  

By \( C_\infty^0(v/h)(\Lambda_{a,b}) \) we denote the space of all smooth function on \( \Lambda_{a,b} \) such that for each function \( \psi \in C_\infty^0(v/h)(\Lambda_{a,b}) \) there exists a relatively open set containing \( v/h \) such that the function \( \psi \) vanishes on this set. We are now able to define the domains \( H^1_0(v/h)(\Lambda_{a,b}) \) for the quadratic forms corresponding to our auxiliary operators as the completion

\[
H^1_0(v/h)(\Lambda_{a,b}) = \left[ C_\infty^0(v/h)(\Lambda_{a,b}) \right]_{H^1(\Lambda_{a,b})}.
\]  

Similar to \( (q_f, H^1(\mathbb{T})) \) we define the quadratic form

\[
q_{v/h}^{\Lambda_{a,b}} := (q, H^1_0(v/h)(\Lambda_{a,b})).
\]  

**Lemma 6.** The quadratic form \( q_{v/h}^{\Lambda_{a,b}} \) define as self-adjoint operator

\[
\Delta_{a,b}^{v/h} := (-\Delta, D_{0,v/h}(\Lambda_{a,b}))
\]  

possessing a purely discrete spectrum. The domains \( D_{0,v/h}(\Lambda_{a,b}) \) satisfy

\[
D_{0,v/h}(\Lambda_{a,b}) \subset \left\{ H^2(\Lambda_{a,b}); \psi|_{v/h} = 0, \partial_n \psi|_{h/v/\gamma_{a,b}} = 0 \right\}.
\]
Proof. The proof of \( (57) \) is analogous to the proof of Proposition 2. To prove the discreteness of the spectrum we can directly use [EE87, Theorem 4.17], as in Proposition 3, since the boundary \( \partial \Omega_{a,b} \) is not of class \( C = C^{0,0} \), see [EE87, Definition 4.1] at the cusp \( c \). Nevertheless we may introduce the auxiliary set

\[ \tilde{\Lambda}_{a,b} = \{(x_1, x_2); 0 < |x_2| < \gamma(x_1), 0 < x_1 < a\} \quad (58) \]

and unfold every \( \psi \in H^1_{0,v/h}(\Lambda_{a,b}) \) to \( \tilde{\psi} \in H^1_{0,v/h}(\tilde{\Lambda}_{a,b}) \) by

\[ \tilde{\psi}(x_1, x_2) := \begin{cases} 
\psi(x_1, x_2), & x_2 \geq 0, \\
\psi(R^{h/v}(x_1, x_2)), & x_2 \leq 0,
\end{cases} \quad (59) \]

and we note that by e.g. [Hei07, Theorem 4.3] it is not hard to show that indeed \( \tilde{\psi} \in H^1_{0,v/h}(\tilde{\Lambda}_{a,b}) \). Now take an arbitrary sequence \( \{u_n\}_n \) in \( H^1_{0,v/h}(\Lambda_{a,b}) \) and we obtain a bounded sequence \( \{\tilde{u}_n\}_n \) in \( H^1_{0,v/h}(\tilde{\Lambda}_{a,b}) \). Unless to \( \partial \Lambda_{a,b} \) the boundary \( \partial \tilde{\Lambda}_{a,b} \) is of class \( C = C^{0,0} \), [EE87, Definition 4.1] and we can apply [EE87, Theorem 4.17] eventually yielding a subsequence \( \{u_{n_j}\}_j \) converging in \( L^2(\Lambda_{a,b}) \). This proves the compactness of \( H^1_{0,v/h}(\Lambda_{a,b}) \rightarrow L^2(\Lambda_{a,b}) \) implying the discreteness of the spectrum by [ABHN01, p. 484] and [EE87, Theorem 2.9].

Next we prove that we can indeed compare the ground states of \( \Delta_{a,b}^{v/h} \).

**Lemma 7.** The spectrum of \( \Delta_{a,b}^{v/h} \) is strictly positive and the ground states for \( \Delta_{a,b}^{v/h} \) exist.

**Proof.** That the spectrum is a subset of \( \mathbb{R}_0^+ \) follows by the min – max principle [RS78, Theorem XIII.2] applied for \( q_{\Lambda_{a,b}}^{v/h} \). Hence, that the lowest eigenvalue is strictly positive it is enough to show that zero is not an eigenvalue i.e. no harmonic function exists being an element of \( (57) \). We consider the case \( \Delta_{a,b}^v \) and the other case is analogous. We prove the claim by contradiction. For this we define the unfolding operators \( (\cdot)^{v/h} : L^2(\Lambda_{a,b}) \rightarrow L^2(\Omega_{a,b}) \) by

\[ (\tilde{\psi})^{v/h}(x_1, x_2) := \begin{cases} 
\psi(x_1, x_2), & x_1, x_2 > 0, \\
\psi(R^{h/v}(x_1, x_2)), & x_1 < 0, \ x_2 > 0, \\
-\psi(R^{v/h}(x_1, x_2)), & x_1 > 0, \ x_2 < 0, \\
-\psi(R^{h/v}(R^{v/h}(x_1, x_2))), & x_1, x_2 < 0.
\end{cases} \quad (60) \]

Assume \( \psi \) is such an harmonic functions. Then we can unfold \( \psi \) on \( \Lambda_{a,b} \) to \( \tilde{\psi}^v \) on \( \Omega_{a,b} \) and using [Hei07, Theorem 4.3] it is not hard to see that \( \tilde{\psi}^v \) is an harmonic functions satisfying \( (12) \). Now [AU10, Satz 7.33] tells us that the only harmonic function satisfying this is a constant function on \( \Omega_{a,b} \) and the Dirichlet condition on \( v \) implies then that the function has to be the zero function. That the lowest eigenvalue has to be simple may be deduced by an analogous argumentation as in the proof for Lemma 4. \qed
Definition 2. By \( \psi_{v/h} \) and \( \lambda_{h/v} \) we denote the ground state, modulo sign, and ground state eigenvalue for \( \Delta_{a,b}^{v/h} \).

Lemma 8. The the ground state for \( \Delta_{a,b}^v \) is given by, modulo sign,

\[
\psi_v = \| \psi_{a,b} \rangle \Lambda_{a,b} \| L^2(\Lambda_{a,b}) \psi_{a,b} \rangle \Lambda_{a,b}.
\]  

(61)

Proof. Obviously, \( \tilde{\psi} := \| \psi_{a,b} \rangle \Lambda_{a,b} \| L^2(\Lambda_{a,b}) \psi_{a,b} \rangle \Lambda_{a,b} \) is a eigenfunction of \( \Delta_{a,b}^v \) with \( L^2 \) norm equal to one. Since \( \tilde{\psi} \) possesses only one nodal domain we may employ an analogous argument as in the proof for Lemma 4 using Courant’s nodal domain theorem and the identity property of real analytic functions.

This observation leads to

Lemma 9. The ground state property of \( \Omega_{a,b} \) is equivalent to

\[
\lambda_v < \lambda_h.
\]  

(62)

Proof. Since \( \mathcal{R}^{v/h} \) and \( \Delta_{a,b} \) commute we can find a set of orthonormal eigenfunctions for \( \Delta_{a,b} \) being either symmetric or antisymmetric w.r.t. \( h/v \). For the lowest nonnegative eigenvalue \( \lambda_1 \) denote by \( \{ \psi_1, \ldots, \psi_d \} \) such a set of simultaneous. Using Courant’s nodal domain theorem [CHS9 p. 452] and Lemma 5 we see that we must have \( \mathcal{R}^v \mathcal{R}^h \psi_l = -\psi_l \) for every \( l \in \{ 1, \ldots, d \} \) being the in agreement with two possible nodal domains. So only the shapes II) and III), see Figure 5 are possible. Therefore, the restrictions \( \psi_l \rangle \Lambda_{a,b}, l \in \{ 1, \ldots, d \} \), are eigenfunctions for either \( \Delta_{a,b}^v \) or \( \Delta_{a,b}^h \). Now we observe that by an analogous unfolding for \( \psi_v \) and \( \psi_h \) along the lines of (61) we eventually obtain eigenfunctions \( \psi_{v/h} \) for \( \Delta_{a,b} \) possessing a nodal line at \( v/h \). This in turn implies that \( d \leq 2 \) since the eigenvalues \( \lambda_{1/v}^{v/h} \) are simple by Lemma 7. Obviously, \( d = 1 \) if \( \lambda_v \neq \lambda_h \) and we proved the claim.

The next section is now devoted to prove (62) eventually proving the theorem.

4 An appropriate rearrangement method

We introduce an auxiliary Laplacian which finally shall allow to compare \( \lambda_h \) and \( \lambda_v \) by proving an inequality of the form \( \lambda_v < \tilde{\lambda} < \lambda_h \), where \( \tilde{\lambda} \) is the ground state for the auxiliary Laplacian. In doing so we use as domain for the auxiliary Laplacian the sector

\[
S_{a,R} := \left\{ (r \cos(\phi), r \sin(\phi)) ; 0 < r < R, |\phi| < \frac{\alpha}{2} \right\}
\]  

(63)

parametrized by an opening angle \( \alpha \) and radius \( r \). For further application we introduce the circle segments

\[
\{ r \equiv a \} := \left\{ (r \cos(\phi), r \sin(\phi)) ; r \equiv a, |\phi| < \frac{\alpha}{2} \right\}.
\]  

(64)
The radial lines \( \{ \phi \equiv \varphi \} \) and the set \( \{ r < a \} \) may be defined analogously to (64). We observe that obviously \( \{ r < a \} = S_{a,a} \) holds.

The function space \( C_0^\infty (r \equiv R) \) resp. the Sobolev space \( H^1_0 (r \equiv R) \) is analogously defined to \( \mathcal{H}_1 (\Lambda_{a,b}) \) in Section 3 using the completion \( C_0^\infty (r \equiv R) \) of \( H^1_0 (S_{\alpha,a}) \).

By a slight abuse of notation the quadratic form \( q_{a,R} := \langle q, H^1_0 (S_{\alpha,a}) \rangle \) is now analogously defined to (55).

The ground state \( \psi_{a,R} \) of \( \Delta_{a,R} \) is given by the Bessel function \( \psi_{a,R}(x) := J_0 \left( \frac{\|x\|^2 j_0}{R} \right) \), \( x \in S_{\alpha,a} \) with corresponding eigenvalue \( \lambda_{a,R} = \frac{j_0^2}{R^2} \).

Proof. The relation ‘\( \subset \)’ in (66) may be proven analogous to the proof of Lemma Proposition 2. We prove the relation ‘\( \supset \)’ by realizing that this relation is equivalent with \( \langle \psi, \Delta \phi \rangle_{L^2(S_{\alpha,a})} = \langle \Delta \psi, \phi \rangle_{L^2(S_{\alpha,a})} \) for every \( \psi, \phi \) being an element of the r.h.s. of (66).

The later property, however, can be shown using Greens identity [Gri85, Theorem 1.5.3.1] owing to the fact that the \( \partial S_{\alpha,a} \) is a Lipschitz boundary.

The discreteness of the spectrum of \( \Delta_{a,R} \) and (66) follows by analogous arguments to the proof of Proposition 2. An easy calculation shows that \( \psi_{a,R} \) is an eigenfunction for \( \Delta_{a,R} \). With the same argument as in the proof of Lemma 4 we can infer that \( \psi_{a,R} \) is the ground state since it only possesses one nodal domain.

The analytic tool which will eventually allow us to compare \( \psi_h \) with \( \psi_v \) via \( \psi_{a,R} \) is the rearrangement technique. In our situation a rearrangement \((\cdot)^*\) maps a function \( \psi : \Lambda_{a,b} \to \mathbb{R}_0^+ \) to a function \( \psi^* : S_{\alpha,a} \to \mathbb{R}_0^+ \). It depends therefore from the parameters \( a, b \) and \( \alpha, R \). We emphasize here that, unless to the usual rearrangement methods, our rearrangement connects not directly \( \psi_h \) with \( \psi_v \) but only via the reference state \( \psi_{a,R} \).

In order to define the rearrangement of functions we first have to define the rearrangement of sets. In doing so we demand that the sector \( S_{a,R} \) to have

\[ |S_{a,R}| = |\Lambda_{a,b}| \]  

which translates into a hypersurface condition for \((\alpha, R)\) depending on \((a, b)\). This will eventually ensure that the image of our rearrangement of \( H^1_{0,h} (\Lambda_{a,b}) \) is a subset of \( H^1_{0,r \equiv R} (S_{a,R}) \).
Now, for a Lebesgue measurable subset $\Omega \subset \Lambda_{a,b}$ we define its rearrangement $\Omega^*$ by

$$
\Omega^* := \{ r < a \} = S_{a,a}, \quad \text{such that} \quad |\Omega| = |S_{a,a}|. 
$$

(69)

The requirement (68) ensures that (69) is well defined and implies by

$$
\Lambda_{a,b}^* = S_{a,R}.
$$

(70)

Furthermore, for suitable functions $\psi : \Lambda_{a,b} \to \mathbb{R}_0^+$ we denote the superlevel sets by, $t \in \mathbb{R}_0^+$,

$$
\{ \psi > t \} := \{ x \in \Lambda_{a,b}; \quad \psi(x) > t \}
$$

(71)

and an analogously definition is made for the level set $\{ \psi = t \}$.

**Definition 3.** For a measurable nonnegative function $\psi : \Lambda_{a,b} \to \mathbb{R}_0^+$ the rearranged function $\psi^* : S_{a,R} \to \mathbb{R}_0^+$ is defined by

$$
\psi^*(x) := \int_0^\infty \kappa_{\{\psi > t\}}^*(x) \, dt.
$$

(72)

The next lemma shows that the gradient is well-defined.

**Lemma 11.** Let $\psi : \Lambda_{a,b} \to \mathbb{R}_0^+$ be in $H^1_{0,b} (\Lambda_{a,b})$. Then $\psi^* : S_{a,R} \to \mathbb{R}_0^+$ is in $H^1_{b,R} (S_{a,R})$.

**Proof.** We first introduce the auxiliary domain

$$
\tilde{\Lambda}_{a,b} := \Lambda_{a,b} \cup \{ (x_1, x_2) \in \mathbb{R}^2; \quad 0 < x_1 \leq a, \quad -\delta < x_2 \leq 0 \}
$$

(73)

with some $\delta > 0$ and the extended function $\tilde{\psi} : \tilde{\Lambda}_{a,b} \to \mathbb{R}_0^+$ by

$$
\tilde{\psi}(x) := \begin{cases} 
\psi(x), & x \in \Lambda_{a,b} \\
0, & \text{else}
\end{cases}
$$

(74)

We now rearrange $\tilde{\psi}$ to $\tilde{\psi}_{a,R}^* : S_{a,R_a} \to \mathbb{R}_0^+$ analogously to Definition 3 for a fixed but arbitrary $\alpha$ where $S_{a,R_a}$ is determined by $|\tilde{\Lambda}_{a,b}| = |S_{a,R_a}|$. We remark here that $R_a$ depends on $\alpha$ and $\delta$. It is easy to see that

$$
\tilde{\psi}_{a,R_a}^*(x) = \begin{cases} 
\psi^*(x), & x \in S_{a,R_a} \\
0, & \text{else}
\end{cases}
$$

(75)

Hence, it suffices to consider the statement for $\tilde{\psi}_{a,R_a}^*$. To facilitate the proof we first consider the rearrangement for $\alpha = 2\pi$ i.e. the usual spherical rearrangement. Since $\tilde{\psi}$ satisfies the assumptions of [Bra93, Theorem 1.2] and $\tilde{\Lambda}_{a,b}$ is a Lipschitz domain we can deduce $\tilde{\psi}_{2\pi}^* \in H^1_{0,R_a}(S_{a,R_a})$. Finally the observation that the relation $\tilde{\psi}_{a,R_a}^*(x) = \tilde{\psi}_{2\pi}^*(\frac{x}{2\pi}x)$ for $x \in S_{a,R_a}$ holds proves the claim.
For a subset $A \subset \Lambda_{a,b}$ we denote by $\partial^h A$ the non-Neumann part of the boundary of $A$ w.r.t. $\Delta^h_{a,b}$ i.e. 
\[ \partial^h A := \partial A \setminus \{v \cup \gamma_{a,b}\} . \] (76)

The analogous definition is made for subsets $A \subset S_{\alpha,R}$ on any subsector where the non-Neumann part of $\Delta_{\alpha,R}$ i.e. $\{\phi = \pm \alpha/2\}$ do not contribute. For the following definition we denote by $C^\infty_0(\Lambda_{a,b})$ the space of $C^\infty_0(\Lambda_{a,b})$ functions possessing a uniformly continuous partial derivatives in every order.

**Definition 4.** We call a rearrangement $(\cdot)^*$ admissible if for $\psi \in C^\infty_0(\Lambda_{a,b})$:

i) a measurable function $\rho_\psi : \mathbb{R} \to \mathbb{R}$ exists such that
\[ \|\text{grad } \psi^*\|_{L^2}^2 = \rho_\psi \circ \psi^* , \] (77)

ii) for almost every $t \in \mathbb{R}_+^*$ we have that the perimeter inequality
\[ |\partial^h \{ \psi > t \}|_1 \geq |\partial^h \{ \psi^* > t \}|_1 , \] (78)

is fulfilled and

iii) a $\epsilon_0 > 0$ exists such that the eigenvalue inequality
\[ \lambda_{a,b} < \lambda_{\alpha,R} + \epsilon_0 . \] (79)

holds.

**Remark 2.** Condition (78) in particular means that the corresponding sets possesses finite perimeter.

We recall here an inequality of [Spe74, p. 166,177] and adapt it to our situation. It is well-known that the rearrangement satisfies, see e.g. [Spe74, p. 164],
\[ \|\rho \circ \psi\|_{L^2(\Lambda_{a,b})} = \|\rho \circ \psi^*\|_{L^2(S_{\alpha,R})} \] (80)

for every Borel measurable function $\rho : \mathbb{R}_+^* \to \mathbb{R}$.

**Lemma 12.** Let the rearrangement $(\cdot)^*$ satisfy the conditions (77) and (78) and assume $\psi \in C^\infty_0(\Lambda_{a,b})$. Then
\[ \|\text{grad } \psi^*\|_{L^2(S_{\alpha,R})} \leq \|\text{grad } \psi\|_{L^2(\Lambda_{a,b})} . \] (81)

**Proof.** The proof is analogous to [Spe74] pp. 167,168 replacing only $\mathbb{R}^2$ by $\Lambda_{a,b}$ and putting $p = 2$ using (78), (78) and (80). We remark that the only difference in our setting is that in Federer’s coarea formula [Fed69, 3.2.12. Theorem] only the non Neumann part contributes i.e. $v$ and $\gamma_{a,b}$ are omitted. \qed

As a last axillary lemma we prove the following approximation result.
Lemma 13. The set $C_{0,h}^\infty(\Lambda_{a,b})$ is dense in $H_{0,h}^1(\Lambda_{a,b})$.

Proof. We extend an arbitrary $\psi \in H_{0,h}^1(\Lambda_{a,b})$ to $\tilde{\psi}$ by (74) on $\tilde{\Lambda}_{a,b}$ in (73). Then we observe that $\tilde{\psi}_{a,b}$ possess the segment property by [MP97, Definition 2., p. 18] and [MP97, Theorem 1] together with [MP97, Theorem, p. 18] shows that we can approximate $\tilde{\psi}$ by $C^\infty(\Lambda_{a,b})$-functions in $H^1(\psi_{a,b})$ which proves the claim. \hfill \Box

Proposition 4. Assume that an admissible rearrangement $(\cdot)^*$ exists. Then

$$\lambda_h > \lambda_v. \quad (82)$$

Proof. First since the space $C_{0,h}^\infty(\Lambda_{a,b})$ is dense in $H_{0,h}^1(\Lambda_{a,b})$ and because of Lemma 11, (80) and (81) we may extend (81) from $C_{0,h}^\infty(\Lambda_{a,b})$ to $H_{0,h}^1(\Lambda_{a,b})$. Next we exploit the extended Lemma 13 and we can employ Lemma 11 and (79) and obtain by means of the min-max principle [RS78, Theorem XIII.2] the inequality

$$\lambda_h = \min_{\psi \in H_{0,h}^1(\Lambda_{a,b})} \|\text{grad } \psi\|_{L^2(\Lambda_{a,b})} = \min_{\psi \in C_{0,h}^\infty(\Lambda_{a,b})} \|\text{grad } \psi\|_{L^2(\Lambda_{a,b})} \geq \min_{\psi \in C_{0,h}^\infty(\Lambda_{a,b})} \|\psi^*\|_{L^2(S_{a,R})} \geq \lambda_{a,R} \geq \lambda_{a,b} + \epsilon_0. \quad (83)$$

Finally, Lemma 7 proves the claim. \hfill \Box

The requirement (77) is always satisfied provided by the next lemma.

Lemma 14. For every $\psi \in H_{0,h}^1(\Lambda_{a,b})$ a measurable $\rho_\psi$ exists satisfying (77).

Proof. We first remark that $\tilde{\psi}^*(r) := \psi^*(x), r := \|x\|_\mathbb{R}^2$ is well-defined since $\psi^*$ depends only on $r$. Moreover, by Lemma 11 $\psi^* \in H_{0,R}^1(S_{a,R})$ and, $x \in \mathbb{R}^2, r = \|x\|_{\mathbb{R}^2}$, we may extend $\psi^*$ onto $\mathbb{R}^2$ by

$$\tilde{\psi}_{\mathbb{R}^2}^*(r) := \psi_{\mathbb{R}^2}^*(x) = \begin{cases} \psi^*(\tilde{x}), & \|x\|_\mathbb{R}^2 = \|\tilde{x}\|_\mathbb{R}^2 \leq R, \; \tilde{x} \in S_{a,R}, \\ 0, & \text{else} \end{cases} \quad (84)$$

and then $\tilde{\psi}_{\mathbb{R}^2}^*(x) \in H^1(\mathbb{R}^2)$. Therefore by [Str77, p. 155,156] we may deduce that $\tilde{\psi}^*(r)$ is locally absolutely continuous in $(0,R)$ and hence $\psi^*$ is locally absolutely continuous in $S_{a,R}$. Now the proof may be performed analogously to the proof of [Spe74, Lemma 2] putting $p = 2$. \hfill \Box

We are left to provide a sufficient criterion for (78) and (79). We first treat (79) and in doing so the next lemma turns out to be useful.
Lemma 15. For arbitrary but fixed $a > 0$ the area of the triangle $\Lambda_{a,b}$ possesses the asymptotics, $b \to 0$,

$$|\Lambda_{a,b}| = \gamma b^2 (1 + O(b)),$$

where

$$\gamma = \frac{4\sqrt{2}}{\pi^2} \int_0^\infty \arcsin(e^{-x^2}) \, dx \sim 0.6080.$$  \hspace{1cm} (85)

Proof. Using (52) and Lemma 28 we arrive for the area $|\Lambda_{a,b}|^2$ at

$$|\Lambda_{a,b}|^2 = \int_0^a \frac{2b}{\pi} \arcsin \left[ \cos \left( \frac{\pi}{2a} x \right) \right] \left( \frac{\pi}{2a} x \right)^2 \, dx$$

$$= \int_0^a \frac{2b}{\pi} \arcsin \left[ e^{-\pi \left( \frac{\pi}{2a} x \right)^2} \left( 1 - R_b(x) \right) \right] \, dx.$$  \hspace{1cm} (86)

Now Lemma 28 implies that $e^{-\pi \left( \frac{\pi}{2a} x \right)^2} \left( 1 - R_b(x) \right) = O \left( e^{-\pi \beta(\beta-1)} \right)$ for $x > b^\beta$ for every arbitrary but fixed $\beta > \frac{1}{2}$. For $x \leq b^\beta$ we infer from Lemma 28 that

$$\int_0^{b^\beta} \arcsin \left( e^{-\pi \left( \frac{\pi}{2a} x \right)^2} \left( 1 - R_b(x) \right) \right) \, dx = \sqrt{\frac{2b}{\pi}} \int_0^{\frac{\pi b^\beta - 1}{\sqrt{2\pi}}} \arcsin \left( e^{-x^2} \left( 1 - \tilde{R}_b(x) \right) \right) \, dx$$

where

$$\tilde{R}_b(x) := R_b \left( \frac{b\sqrt{2}x}{\pi} \right) = \begin{cases} O(b^2x^4) = O(b^{2(\beta-1)}) , & x \leq b^{\beta-1} , \\ O(1) , & \text{else} . \end{cases}$$  \hspace{1cm} (87)

An easy calculation yields that for $0 \leq \tau < \eta \leq 1$ we have $\sin \left( \arcsin(\eta) - \arcsin(\tau) \right) = \sqrt{1 - \tau^2 \eta - \sqrt{1 - \eta^2 \tau}}$ which yields using (28) and (89), $0 < x \leq b^{\beta-1}$,

$$\sin \left( \arcsin \left( e^{-x^2} \right) - \arcsin \left( e^{-x^2} \left( 1 - \tilde{R}_b(x) \right) \right) \right)$$

$$= -\sqrt{1 - e^{-2x^2}e^{-x^2}(1 - \tilde{R}_b(x))} + e^{-x^2} \sqrt{1 - e^{-2x^2}} \sqrt{1 - \frac{O(b^2x^4)}{1 - e^{-2x^2}}}$$

$$= O \left( \sqrt{1 - e^{-2x^2}e^{-x^2}b^{2\beta}} \right)$$

uniformly in $b$ and $0 \leq x \leq b^{\beta-1}$. This in turn implies that, $b \to \infty$,

$$\arcsin \left( e^{-x^2} \right) - \arcsin \left( e^{-x^2} \left( 1 - \tilde{R}_b(x) \right) \right) = O \left( \sqrt{1 - e^{-2x^2}e^{-x^2}b^{2\beta}} \right)$$

uniformly in $0 \leq x \leq b^{\beta-1}$ since the sinus is asymptotically linear for small arguments. Moreover, the r.h.s. of (89) as well as (28) es exponentially decaying for large argument uniformly in $b$ and hence (85) follows. The $\gamma$ value is calculated with Matlab QAGI algorithm (QUADPACK). \hfill $\Box$
We define
\[ \alpha_{\text{min}} := \frac{\gamma \pi^2}{2 j_0^2} \sim 0.1652\pi \] (92)
and we are ready to provide a sufficient criterion for (79).

**Lemma 16.** For arbitrary but fixed chosen \( a > a_b > 0 \) exists such that for all \( b < b_a \) the implication
\[ \alpha > \alpha_{\text{min}} \implies (79) \] (93)
holds.

**Proof.** Using Lemma 15 we obtain for the radius \( R, b \) sufficiently small,
\[ R = \sqrt{\frac{2 |S_{\alpha,R}|}{\alpha}} < \sqrt{\frac{2 |\Lambda_{a,b}|}{\alpha_{\text{min}}}} \leq \sqrt{\frac{2 \gamma}{\alpha_{\text{min}}}} b. \] (94)
Now since for fixed \( a \) we have \( \lambda_{a,b} = \frac{\pi^2}{4 b^2} (1 + O(b^2)) \), see (21), we can choose \( b_a \) such that after plugging (94) into (67) we obtain
\[ \lambda_{\alpha,R} + \epsilon_0 < \frac{j_0^2 \alpha_{\text{min}}}{2 \gamma b^2} = \frac{\pi^2}{4 b^2} \sim \lambda_{a,b} \] (95)
for a suitable \( \epsilon_0 > 0 \) which proves the claim. \( \square \)

In order to present a sufficient criterion for (78) we employ well-known facts of geometric measure theory. For this we denote by \( \mathcal{R}_{a,b} \) the set of all subsets of \( \Lambda_{a,b} \) possessing a \( \mathcal{H}^1 \)-rectifiable boundary [Mag12, p. 96]. We remark here that the rectifiability of \( \partial^h A \) and \( \partial A \) for a subset \( A \) may considered to be equivalent by [Mag12, Lemma 12.22] and [Mag12, Corollary 16.1]. For a sets \( A \in \mathcal{R}_{a,b} \) we introduce the functional \( F(A) \) by
\[ F(A) := \frac{|\partial^h A|_1^2}{2 |A|_2} \] (96)
and we denote the infimum of \( F \) by
\[ \alpha_{\text{max}} := \inf_{A \in \mathcal{R}_{a,b}} F(A). \] (97)
The infimum is of the functional \( F(A) \) is a proper one i.e. not a minimum and we are able to determine it. We refer here to Section 5 which illustrates that the functional corresponding to Chegger’s constant is harder to analyze.

**Lemma 17.** The infimum in (97) is a proper one and we have
\[ \alpha_{\text{max}} = \frac{\pi}{4}. \] (98)
Proof. We first observe that in order to find a lower bound for \(\alpha_{\min}\) it is sufficient to consider \(\tilde{\Lambda}_{a,b}\) in (111). For a fixed area \(\eta\) we consider the minimizer \(A_{\min,\eta}\) of Lemma 21. We denote the straight semiline starting at \(w\) and going through \(p\) in Figure 7 by \(g\). Moreover, we denote the angle of \(g\) and \(v\) by \(\varphi(\eta)\). Note that by the Lemmata 20 and 3 we have \(\varphi(\eta) > \frac{\pi}{4}\) in Figure 7 and \(\varphi(\eta)\) is a strictly increasing function w.r.s. to the corresponding area \(\eta\) for \(A_{\min,\eta}\). By a slight abuse of notation we denote the sector generated by \(v\) and \(w\) in Figure 7 and \(\tilde{\eta}\) by \(\tilde{\varphi}\) and \(\tilde{\eta}\) and going through \(p\) in (111). For a fixed area \(\eta\) we consider the minimizer \(\min\) in (111). We denote \(\tilde{\varphi}\) by \(\tilde{\varphi}(\eta)\). Note that by the Lemmata 20 and 3 we have \(\tilde{\varphi}(\eta) > \frac{\pi}{4}\). By Lemma 20 we have \(|\tilde{\varphi}(\eta)| < |A_{\min,\eta}| =: \tilde{\eta}\) and by the above observations we also have \(|\partial^h S_{\varphi(\eta),r(\tilde{\eta})}| < |\partial^h (A_{\min,\eta})|_1\). Plugging everything together we obtain

\[
F(A_{\min,\eta}) = \frac{|\partial^h A_{\min,\eta}|^2}{2 |A_{\min,\eta}|_2} > \frac{2}{2} |\partial^h S_{\varphi(\eta),r(\tilde{\eta})}|_1 = \varphi(\eta) > \frac{\pi}{4}
\]  

(99)

Now using Lemma 3 we see that the angle \(\phi\), the radius \(r\) and the line \(g\) in Figure 7 a) possess for \(x_p \to 0\), denoting \(p = (x_p, \gamma_{a,b}(x_p))\), and small \(x\) the asymptotics

\[
g(x) = -x + b + O(x_p^2) > -x + b,
\]

\[
\phi(x_p) = \pi + O(x_p^2) > \frac{\pi}{4}, \quad r(x_p) = \sqrt{2} x_p + O(x_p^2).
\]

(100)

From (100) it easily follows that for \(x_p \to 0\) we have

\[
|A_{\min}|_2 = \frac{\pi x_p^2}{4} + O(x_p^3), \quad |\partial^h A_{\min}|_1 = \frac{\pi x_p}{\sqrt{2}} + O(x_p^2)
\]

(101)

which immediately gives, \(x_p \to 0\),

\[
\frac{|\partial^h A_{\min}|^2}{2 |A_{\min}|_2} = \frac{4 \pi^2 x_p^2 (1 + O(x_p))}{16 \pi x_p^2 (1 + O(x_p))} \to \frac{\pi}{4}
\]

(102)

proving the claim. \(\square\)

Lemma 18. The implication

\[
\alpha < \alpha_{\max} \implies \quad (78)
\]

holds for all \(\psi \in H^1_{0,h}(\Gamma_{a,b})\) being Lipschitz continuous.
Proof. By Kirszbraun’s theorem [Mag12, Theorem 7.2] we can assume w.l.o.g. that \( \psi \) is the restriction of a Lipschitz function on \( \mathbb{R}^2 \). Then by [Mag12, Corollary 16.1, p. 216] we may w.l.o.g. assume that \( \{ \psi = t \cap \Lambda_{a,b} \} = \partial^h \{ \psi > t \} \) possesses for a.e. \( t \) finite perimeter and is \( \mathcal{H}^1 \)-rectifiable. Denote by \( A_{\min,t} \) the minimizer of \( F \) in (96) among all sets in \( \mathcal{R}_{a,b} \) with fixed area equal to \( \eta_t := | \{ \psi > t \} |_2 \). We first notice that

\[
| \partial^h \{ \psi > t \} |_1 \geq | \partial^h \{ \psi^* > t \} |_1 \quad \Leftrightarrow \quad \frac{| \partial^h \{ \psi > t \} |_1^2}{| \partial^h \{ \psi^* > t \} |_1^2} \geq 1
\]

and calculate, using \( \frac{| \partial^h \{ \psi^* > t \} |_1}{2\eta_t} = \alpha \) and Lemma 17

\[
\frac{| \partial^h \{ \psi > t \} |_1^2}{| \partial^h \{ \psi^* > t \} |_1^2} \geq \frac{| \partial^h A_{\min,t} |_1^2}{| \partial^h \{ \psi^* > t \} |_1^2} = \frac{| \partial^h A_{\min,t} |_1}{2\eta_t} \quad \geq \quad \frac{\alpha_{\max}}{\alpha} > 1
\]

proving the claim. \( \square \)

We now possess every ingredient to prove the theorem.

Proof of Theorem. First we observe that the Lemmata (14), (16) and (18) imply that for every \( a > 0 \) a \( b_0 > 0 \) exists such that for all \( b < b_0 \) one may chose a suitable \( \alpha \) such that the rearrangement in the sense of Definition 3 is admissible. Now Proposition 4 and Lemma 9 finally proves the claim. \( \square \)

5 Outlook via Cheeger’s inequality

Some results of the paper are of a geometric nature like Lemmata 16 and 17. On the contrary some methods rely on an accurate explicit knowledge of the boundary \( \gamma_{a,b} \) of the Neumann domain. In particular, Lemma 15 needs an accurate analytic estimate of \( \gamma_{a,b} \) not only at the wedge \( w \) but on the whole domain. Moreover, we exploited several times the symmetry of our Neumann domain, such as in Lemma 7.

When it comes about to analyse more general Neumann domains an alternative approach would be Cheeger’s inequality [Che70]. Cheeger’s inequality bounds the ground state eigenvalue from below by a purely geometric quantity of the underlying domain for the Laplacian. The Cheeger inequality (\( \lambda_0 \) ground state, see [11]), for our Laplacian \( \Delta^h_{a,b} \) reads [Cha84, pp. 95,109,259]

\[
\lambda_0 \geq \inf_{A \in \mathcal{R}_{a,b}} C(A)
\]

and involves on its geometric side the infimum of Cheeger’s functional defined by

\[
C(A) := \frac{1}{4} \left( \frac{F(A)}{|\partial^h A|_1} \right)^2, \quad A \in \mathcal{R}_{a,b}.
\]
We believe that the Cheeger approach is a natural candidate to prove a ground state property for Neumann domains. However, we drifted in this paper to our developed rearrangement device because of two reasons. First, a numerical test of (106) showed, indeed, that (106) is satisfied but only for a very small $a/b$ ratio.

Second, the infimum (in fact a minimum) of the functional $C$ in (107) is not as convenient located in $\Lambda_{a,b}$ as the infimum of the functional $F$ in (96). By Lemma 21 we know that the minimizer of $C$ among all rectifiable sets with a prescribed area $\eta$ is given by a set of shape $A_{\min,\eta}$ in Figure 7 a) for small $\eta$ and for larger $\eta$ by a suitable adaptation. Figure 6 shows the dependence of $C(A_{\min,\eta})$ on the volume of such minimizers but with the approximation of $\gamma_{a,b}$ by Lemma 28 i.e. for small $b$.

Figure 6: The cross marks the minimizer of Cheeger’s functional

The red circle in Figure 6 corresponds to a minimizer characterized by the requirement it possesses the largest volume whose boundary is a circle going from $v$ to $\gamma_{a,b}$ and touching $h$ at the point $v \cap h$. The cross denotes the minimizer. Its non-Neumann boundary consist of a circle meeting $\gamma_{a,b}$ tangentially and intersecting $h$ orthogonally and a straight line being a subset of $v$.

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A Ground state and $\rho$ dependence on parameters

With a slight abuse of notation we denote by $\rho_{a,b}^{s/l}$ the $\rho$ value of a star-like/lens-like Neumann domain corresponding to the eigenvalue $\lambda_{a,b}$.

Lemma 19. Assume that $\tilde{a} = \gamma a$ and $\tilde{b} = \gamma b$, $\gamma \in \mathbb{R}^+$. Then there is a bijective scaling between the spectrum and eigenfunctions of $\Delta_{a,b}$ and $\Delta_{\tilde{a},\tilde{b}}$ given by

$$\lambda \in \sigma (\Delta_{a,b}) \iff \tilde{\lambda} := \gamma^{-2} \lambda \in \sigma (\Delta_{\tilde{a},\tilde{b}})$$

(108)
and
\[ \Delta_{a,b}\psi = \lambda\psi \iff \tilde{\psi}(x_1, x_2) := \psi(\gamma^{-1}(x_1, x_2)) \text{ satisfies } \Delta_{\tilde{a},b}\tilde{\psi} = \tilde{\lambda}\tilde{\psi}. \tag{109} \]

**Proof.** By an easy calculation it suffices to observe that the Neumann boundary conditions are preserved by the transformation (109). \qed

**Proposition 5.** We have
\[ \rho^s_{a,b} < \rho^l_{a,b}. \tag{110} \]
Moreover, the ground state property the \( \rho \)-values depend only on the ratio \( a/b \).

**Proof.** The first claim follows by observing that the area for the lens is always larger than the area of the star but both share the same perimeter. The second claim directly follows from Lemma [19]. \qed

## B The isoperimetric problem

We introduce an auxiliary domain
\[ \Lambda_{\tilde{a},b} := \Lambda_{a,b} \cup \{(x_1, x_2); \ x_1 \in \mathbb{R}_+, \ x_2 \in \mathbb{R}_-\}, \tag{111} \]
and
\[ \gamma_{\tilde{a},b} := \gamma_{a,b} \cup \{(x_1, 0); \ x_1 \geq a\}. \tag{112} \]

We make an easy observation

**Lemma 20.** The function \( x \to \tilde{\gamma}(x), \ x \in [0, a], \) is convex.

**Proof.** It is enough to show \( (f \circ g)'' \geq 0 \) with \( f := \arcsin(x) \) and \( g = \cos^\alpha(x) \) for all \( 1 \leq \alpha \leq \infty \) and \( 0 \leq x \leq 2\pi \). Denote \( y := \cos(x) \) and a simple calculation yields the equivalence
\[ f'' \circ g'^2 + f' \circ g'' \geq 0 \iff \eta(y) := \alpha(1 - y^2) - (1 - y^{2\alpha}) \geq 0. \tag{113} \]
The r.h.s. of (113) follows by observing that \( \eta(1) = 0 \) and \( \eta' \leq 0 \). \qed

For subsets \( A \subset \Lambda_{\tilde{a},b} \) the non-Neumann part \( \partial^h A \) of its boundary is analogously defined to \([70] \).

**Lemma 21.** Among all \( H^1 \)-rectifiable subset of \( \Lambda_{\tilde{a},b} \) with fixed volume \( \eta \) exactly one subset \( A_{\min;\eta} \) exists with \( \eta = |A_{\min;\eta}|_2 \) minimizing \( |\partial^h(\cdot)|_1 \). \( A \cap \Lambda_{\tilde{a},b} \) is simply connected such that \( w \in \overline{A_{\min;\eta} \cap \Lambda_{a,b}}, \ w = (0, \gamma_{a,b}(0)), \) and \( \partial^h A_{\min;\eta} \) is a part of a circle meeting \( \partial \Lambda_{\tilde{a},b} \) in a right angle going from \( v \) to \( \gamma_{\tilde{a},b} \).

Figure [7](a) shows a minimizing set \( A_{\min;\eta} \).
Proof. It is well-known that the minimizer $A_{\text{min; } \eta}$ exists and possesses constant mean curvature and the intersection property locally [Mag12, SZ99, Proposition 12.30, Theorem 2.1]. We only have to exclude the case b) in Figure 7. By Lemma 20 we can employ [CGR07, (1.1)] and place the set $A_3$ w.l.o.g. on the linear part of $\gamma_{a,b}$ i.e. on $\mathbb{R} \cap \gamma_{a,b}$ and we can choose $A_2 = \emptyset$. Let $A_i$, $i = 1, 3$, be of the shape as in Figure 7 respectively. By a slight abuse of notation we put $\partial h(|A_2|) = \partial h(A)$ and $A$ is of shape $A_i$, $i = 1, 3$ and using [SZ99, Proposition 2] we have that,

$$r_i(|A|) \geq \phi_i(|A|),$$

where $r_i(|A|)$ resp. $\phi_i(|A|)$ is the radius resp angle of the corresponding sector of shape $A_i$ see e.g. Figure 7(a) and we assume here that the variation of the set $s$ is such that is shape, requiring the right angle intersection with $v$ and $\gamma_{a,b}$, is retained. We observe that

$$\phi_1(A_1) \leq \frac{\pi}{2} \leq \pi = \phi_3(A_3)$$

for every choice of the area $A_i$. Assume now that the minimizer is of shape b) and set $A_{\text{min; } \eta} = A_1 \cup A_3$. We calculate setting $\tilde{A}$ to be the set of shape $A_1$, requiring the right angle intersection with $v$ and $\gamma_{a,b}$, with $|\tilde{A}|_2 = |A_{\text{min; } \eta}|_2$

$$|\partial h(A_{\text{min; } \eta})|_2 \geq \sqrt{|\partial h(|A_2| - |A_3|)|_2^2 + |\partial h(|A_3|)|_2^2}$$

$$\geq \sqrt{|\partial h(|A_2|)|_2^2 + |\partial h(|A_3|)|_2^2}$$

where the third line in (116) follows from (114) and (115). \hfill \Box

Remark 3. We emphasize here that we do not use in the proof of Lemma 21 a second variation of the perimeter see e.g. [SZ99, Theorem 2.5].

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