A feasible interpolation for random resolution

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Abstract

We show how to apply the general feasible interpolation theorem for semantic derivations from [6] to random resolution defined by [3]. As a consequence we get a lower bound for random resolution refutations of the clique-coloring formulas.

Assume $A_1, \ldots, A_m, B_1, \ldots, B_\ell$ is an unsatisfiable set of clauses in variables partitioned into three disjoint sets $p, q$ and $r$, with clauses $A_i$ containing only variables from $p$ and $q$ while clauses $B_j$ contain only variables from $p$ or $r$.

Feasible interpolation for resolution [6, Thm.6.1] says that if the set has a resolution refutation with $k$ clauses then there is a circuit of size $kn^{O(1)}$, where $n$ is the number of variables $p$, with inputs $p$ that outputs 1 on all $p := a \in \{0, 1\}^n$ for which $\bigwedge_i A_i(a, q)$ is satisfiable and 0 on all $a$ for which $\bigwedge_j B_j(a, r)$ is satisfiable. Moreover, if variables $p$ occur only positively in clauses $A_i$ then such a circuit exists monotone.

The monotone version can then be applied to the clique-coloring clauses [6, Def.7.1] where there are $\binom{n}{2}$ variables $p$ indexed by unordered pairs $i, j$ of different elements from $[n] := \{1, \ldots, n\}$, $\omega \cdot n$ variables $q$ index by elements of $[\omega] \times [n]$ and $n \cdot \xi$ variables $r$ index by elements of $[n] \times [\xi]$, with $n \geq \omega > \xi \geq 1$:

1. $\{q_{u1}, \ldots, q_{un}\}$, for each $u \in [\omega]$
2. $\{-q_{ui}, -q_{vi}\}$, for $u \neq v \in [\omega]$ and $i \in [n]$
3. $\{-q_{ui}, -q_{vj}, p_{ij}\}$, for $u \neq v \in [\omega]$ and $i \neq j \in [n]$
4. $\{r_{i1}, \ldots, r_{i\xi}\}$, for each $i \in [n]$
5. $\{-r_{iu}, -r_{iv}\}$, for each $u \neq v \in [\xi]$ and $i \in [n]$
6. $\{-r_{iv}, -q_{jv}, -p_{ij}\}$, for $v \in [\xi]$ and $i \neq j \in [n]$

The clauses in the first three items comprise the set $\text{Clique}_{n,\omega}$ and the clauses in the last three items comprise the set $\text{Color}_{n,\xi}$. They have only variables $p$ in common and these occur only positively in $\text{Clique}_{n,\omega}$. The assignments $a$ to $p$ for which $\text{Clique}_{n,\omega}(a, q)$ is satisfiable can be identified with undirected graphs
on \([n]\) without loops and having a clique of size at least \(\omega\) while those \(a\) for which \(Color_{n,\xi}(a, r)\) is satisfiable are \(\xi\)-colorable graphs. Hence \(Clique_{n,\omega} \cup Color_{n,\xi}\) is unsatisfiable as \(\xi < \omega\) and the monotone feasible interpolation combined with Alon-Boppana \([1]\) exponential lower for monotone circuits separating the two classes of graphs implied that all resolution refutations of the set must have an exponential number of clauses, cf.\([5\text{, Sec.7}]\).

Buss, Kolodziejczyk and Thapen \([3\text{, Sec.5.2}]\) defined the notion of \(\delta\)-random resolution (the definition is attributed in \([3]\) to S. Dantchev). A \(\delta\)-random resolution refutation distribution of a set of clauses \(\Psi\) \([3]\) considers only narrow clauses because of the specific problem studied there) is a random distribution \((\pi_s, \Delta_s)_s\) such that \(\pi_s\) is a resolution refutations of \(\Psi \cup \Delta_s\), and where any fixed truth assignment to all variables satisfies the set of clauses \(\Delta_s\) with the probability at least \(1 - \delta\). The number of clauses in such a random refutation is the maximal number of clauses among all \(\pi_s\). (Variants of this definition and properties of the resulting systems are studied in \([10]\).)

The presence of clauses \(\Delta_s\) spoils the separation of the \(q\) and \(r\) variables in initial clauses and this seems to prohibit any application of the feasible interpolation method. The point of this note is to show that, in fact, the construction behind the general feasible interpolation theorem \([6]\) for semantic derivations based on communication complexity does apply here fairly straightforwardly.

We recall some feasible interpolation preliminaries from \([6]\) in Section 1. In Section 2 we prove monotone feasible interpolation for random resolution and this will yield the following lower bound for random resolution refutations of the clique-coloring clauses.

**Theorem 0.1** Let \(n \geq \omega > \xi \geq 1\) and \(\xi^{1/2} \omega \leq 8n/\log n\). Assume \(\delta < 1\) and let \((\pi_s, \Delta_s)_s\) be a \(\delta\)-random resolution refutation distribution of \(Clique_{n,\omega} \cup Color_{n,\xi}\) with \(k\) clauses. Put \(d := \max_s |\Delta_s|\).

Then:

1. If \(d\delta < 1\) then \(k \geq (1 - d\delta^{1/2})n^{\Omega(\xi^{1/2})}\).
2. \(k \geq \min(1/(2\delta^{1/2}), n^{\Omega(\xi^{1/2})})\).

The proof of this theorem will be given at the end of Section 2. We only remark that for tree-like refutations a feasible interpolation via ordinary randomized Karchmer-Wigderson protocols follows from \([6]\) immediately and it yields an exponential lower bound for formulas formalizing Hall’s theorem as described in \([7\text{, Sec.4}]\).

We will give below a detailed formulation of constructions from \([6]\) needed here but we will not repeat the arguments from that paper. For more general background on proof complexity the reader may consult \([5\text{, 9}]\).

### 1 Feasible interpolation via protocols

We review the needed material from \([6]\) just for the case of monotone interpolation and the clique-coloring clauses (but it is quite representative). Identify
undirected graphs without loops on $[n]$ with strings from $\{0, 1\}$

Let $U \subseteq \{0, 1\}^{\binom{n}{2}}$ be the set of graphs having a clique of size at least $\omega$ and let $V \subseteq \{0, 1\}^{\binom{n}{2}}$ be the set of $\xi$-colorable graphs. Let $KW^m(u, v)$ be a multi-function defined on $U \times V$ whose valid value on a pair $(u, v) \in U \times V$ is any edge (i.e. unordered pair $i \neq j \in [n]$) that is present in $u$ but not in $v$.

The method in [6] extracts from a resolution refutation of $\text{Clique}_{n, \omega} \cup \text{Color}_{n, \xi}$ a protocol for a communication between two players, one holding $u$ and the other one $v$, who want to find a valid value for $KW^m(u, v)$. The protocols in [6] are, however, more complex than just binary trees as in the ordinary communication complexity set-up of [4].

A monotone protocol for computing $KW^m$ in the sense of [6, Def.2.2] is a 4-tuple $(G, \text{lab}, F, S)$ satisfying the following conditions:

1. $G$ is a directed acyclic graph that has one root (the in-degree 0 node) denoted $\emptyset$.

2. The nodes with the out-degree 0 are leaves and they are labelled by mapping $\text{lab}$ by elements of $\binom{[n]}{2}$ (i.e. by potential edges).

3. $S(u, v, x)$ is a function (called the strategy) that assigns to a node $x \in G$ and a pair $u \in U$ and $v \in V$ node $S(u, v, x)$ reachable from the node $x$ by one edge.

4. For every $u \in U$ and $v \in V$, $F(u, v) \subseteq G$ is a set (called the consistency condition) satisfying:
   
   (a) $\emptyset \in F(u, v)$, 
   
   (b) $x \in F(u, v) \implies S(u, v, x) \in F(u, v)$, 
   
   (c) if $x \in F(u, v)$ is a leaf and $\text{lab}(x) = \{i, j\}$, then $u_{i,j} = 1 \land v_{i,j} = 0$ holds.

The size of $(G, \text{lab}, F, S)$ is the cardinality of $G$ and its communication complexity is the minimal $t$ such that for every $x \in G$ the communication complexity for the players (one knowing $u$ and $x$, the other one $v$ and $x$) to decide $x \in F(u, v)$ or to compute $S(u, v, x)$ is at most $t$.

Put $s := n \cdot \omega$ and identify strings from $\{0, 1\}^s$ with assignments to $q$-variables, and similarly put $t := n \cdot \xi$ and identify strings from $\{0, 1\}^t$ with assignments to $r$-variables. For any $u \in U$ fix $q^u \in \{0, 1\}^s$ such that $(u, q^u)$ satisfies all clauses from $\text{Clique}_{n, \omega}$ and for $v \in V$ fix $r^v \in \{0, 1\}^t$ such that $(v, r^v)$ satisfies all clauses of $\text{Color}_{n, \xi}$.

Now we are ready to recall a particular fact about the existence of protocols from the proof of [6 Thm.5.1 and Thm.6.1] (again we restrict ourselves to the clique-coloring formulas and the monotone case).
Theorem 1.1 ([6])

Assume that \( \pi \) is a resolution refutation of the set \( \text{Clique}_{n,\omega} \cup \text{Color}_{n,\xi} \) having \( k \) clauses. Then there is a protocol \((G, \text{lab}, F, S)\) for \( \text{KW}^m \) of size \( k + \binom{n}{2} \) whose strategy has the communication complexity at most \( 2 + 2 \log n \) and whose consistency condition has the communication complexity \( 2 \).

In particular, the vertices of \( G \) are the clauses of \( \pi \) (inner nodes) together with \( \binom{n}{2} \) extra nodes that are leaves labelled by the \( \binom{n}{2} \) possible values of the multi-function. The consistency condition \( C \in F(u, v) \) for a clause in \( \pi \) is defined by the condition that the assignment \((v, q, r^v)\) falsifies \( C \), and for a leaf by the condition that the label is a valid value for the pair \((u, v)\).

Further, the existence of a protocol for \( \text{KW}^m \) on \( U' \times V' \subseteq U \times V \) of size \( k' \) and monotone communication complexity \( O(\log n) \) implies the existence of a monotone circuit of size at most \( k' \cdot n^{O(1)} \) separating \( U' \) from \( V' \).

The part about the existence of a circuit is in [6] proved using a result from [11]; a stand alone proof can be found in [8, Sec.2.4].

2 The lower bound

For \( X \subseteq U \) and \( Y \subseteq V \) define \( W(X, Y) \subseteq \{0, 1\}^{\binom{n}{2}} \times \{0, 1\}^s \times \{0, 1\}^t \) to be the set of all tuples \((v, q^n, r^v)\) for \((u, v) \in X \times Y\).

Assume \((\pi_s, \Delta_s)\) is a \( \delta \)-random resolution refutation distribution of clauses \( \text{Clique}_{n,\omega} \cup \text{Color}_{n,\xi} \) having \( k \) steps. For a sample \( s \) define the set \( \text{Bad}_s \subseteq \{0, 1\}^{\binom{n}{2}} \times \{0, 1\}^s \times \{0, 1\}^t \) to be the set of all assignments from \( W(U, V) \) that falsify \( \Delta_s \). An averaging argument implies the following statement.

Lemma 2.1 There exists sample \( s \) such that \(|\text{Bad}_s| < \delta |U \times V|\).

Fix for the rest of the paper one such \( s \). Denote by \((G, \text{lab}, F, S)\) the protocol-like object constructed from \( \pi_s \) as described in Theorem 1.1 but with a different treatment of clauses of \( \Delta_s \):

- the inner nodes are the clauses of \( \pi_s \) except the initial clauses from \( \Delta_s \);
- the leaves are the \( \binom{n}{2} \) labelled extra nodes as in Theorem 1.1 and also the clauses from \( \Delta_s \);
- the strategy \( S \) is the same as in the protocol in Theorem 1.1 (we do not need its particular definition here) and has the communication complexity at most \( 2 + 2 \log n \);
- the consistency condition \( F \) is defined by the condition from Theorem 1.1.

Put \( d := |\Delta_s| \).

Lemma 2.2 There exists \( U' \subseteq U \) and \( V' \subseteq V \) such that:

1. \( W(U', V') \cap \text{Bad}_s = \emptyset \).
2. $|U'| \geq (1 - d\delta^{1/2})|U|$ and $|V'| \geq (1 - d\delta^{1/2})|V|.$

**Proof:**
Take the protocol-like 4-tuple $(G, \text{lab}, F, S)$ constructed above.

**Claim 1:** The set $\text{Bad}_s \cap U \times V$ is a union of at most $d$ rectangles $U_i \times V_i \subseteq U \times V$, for $i \leq d' \leq d$.

We have that

$$\text{Bad}_s = \bigcup_{D \in \Delta_s} \{(u, v) \in U \times V \mid D \in F(u, v)\}.$$ 

But for each of at most $d$ possible $D$ the set $\{(u, v) \in U \times V \mid D \in F(u, v)\}$ is a combinatorial rectangle corresponding to one particular communication of the players, each asserting that the literals known to him are all false. (Note that if we did not know the particular definition of $F$ but only that its communication complexity is 2, it would follow that $\{(u, v) \in U \times V \mid D \in F(u, v)\}$ is a union of at most 4 rectangles and that would suffice too.)

Let $\mu_i$ be the measure of $U_i \times V_i$ in $U \times V$. The following is obvious.

**Claim 2:** For each $i \leq d'$, either $|U_i| \leq \mu_i^{1/2}|U|$ or $|V_i| \leq \mu_i^{1/2}|V|$.

Consider the following process. For $i = 1, \ldots, d'$ delete from $U$ all elements in $U_i$, if $|U_i| \leq \mu_i^{1/2}|U|$, otherwise delete from $V$ all elements of $V_i$. Let $U'$ and $V'$ be what remains of $U$ and $V$, respectively. Because we deleted one side of every rectangle $U_i \times V_i$, all of them have the empty intersection with $U' \times V'$.

The measure of $U \setminus U'$ in $U$, as well as the measure of $V \setminus V'$ in $V$, is bounded above by $\sum_{i \leq d'} \mu_i^{1/2} < d\delta^{1/2}$.

**q.e.d.**

**Lemma 2.3** There exists a monotone protocol for $KW^m$ on $U' \times V'$ of size at most $k + \binom{n}{2}$ and of communication complexity at most $O(\log n)$.

**Proof:**
Take the protocol-like 4-tuple $(G, \text{lab}, F, S)$ described earlier and the sets $U'$ and $V'$ from Lemma 2.2. By the definition of these two sets no clauses from $\Delta_s$ are in $F(u, v)$ for any pair $(u, v) \in U' \times V'$. Hence if we delete these leaves from $(G, \text{lab}, F, S)$ we get a proper protocol for $KW^m$ on $U' \times V'$.

**q.e.d.**

**Proof of Theorem 0.1**

The proof of the $n^{O(\xi^{1/2})}$ lower bound from [1] for monotone circuits separating $U$ from $V$ culminates by comparing two quantities with the sizes of $U$ and $V$, respectively (see the elementary presentation in [2, Sec.4.3]). The same argument applies also to separations of any $U' \subseteq U$ from any $V' \subseteq V$ and the
resulting lower bound just gets multiplied by the smaller of the two measures $|U'|/|U|$ and $|V'|/|V|$.

By Lemmas 2.2 and 2.3 we have two sets $U', V'$ of relative measures at least $(1 - \frac{d\delta}{2})$ and a monotone protocol for $KW^m$ on them of the size at most $k + \binom{n}{2}$ and communication complexity $O(\log n)$. By Theorem 1.1 this yields a monotone circuit separating $U'$ from $V'$ of size $kn^{O(1)}$. Hence it must hold:

\[ kn^{O(1)} \geq (1 - d\delta^{1/2})n^{\Omega(\xi^{1/2})} \]

which entails the first inequality in Theorem 0.1. The second follows from the first one by estimating $d \leq k$: if $k \leq 1/(2\delta^{1/2})$ then the factor $(1 - d\delta^{1/2})$ is at least $1/2$ and the lower bound $n^{\Omega(\xi^{1/2})}$ follows.

q.e.d.

We shall present in a forthcoming paper a fairly general randomized feasible interpolation for semantic derivations. A feasible interpolation for random resolution can be also obtained from it but different then the one given here.

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